Research Article

Continuous Dependence on the Forchheimer Coefficient of the Forchheimer Fluid Interfacing with a Darcy Fluid

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1. Introduction

In this paper, we study two fluids in one bounded domain when they interface with each other. We want to know what effect they can give to each other. Let an appropriate part of the plane \( z = x_3 = 0 \) denote the boundary between a porous medium which occupies a bounded region \( \Omega_1 \subseteq \mathbb{R}^3 \) and a nonlinear viscous fluid which occupies a bounded region \( \Omega_2 \subseteq \mathbb{R}^3 \). The interface is denoted by \( L \). The remaining parts of the boundaries of \( \Omega_1 \) and \( \Omega_2 \) are denoted, respectively, by \( \Gamma' \) and \( \Gamma'' \). We also denote \( \partial \Omega_1 = \Gamma' \cup L \) and \( \partial \Omega_2 = \Gamma'' \cup L \). We also note that \( \Omega_1 \) is above the plane \( z = x_3 = 0 \) and \( \Omega_2 \) is below the plane \( z = x_3 = 0 \).

Let \( (u_1, T, p) \) and \( (v_1, \theta, q) \) denote the velocity, temperature, and pressure in \( \Omega_1 \) and \( \Omega_2 \), respectively. The Forchheimer system consists of the partial differential equations (see [1])

\[
b|u|u_i + (1 + \gamma T)u_i - g_i T + \frac{\partial p}{\partial x_i} = 0, \quad \text{in} \quad \Omega_1 \times [0, \tau], \tag{1}
\]

\[
\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \Delta T, \quad \text{in} \quad \Omega_1 \times [0, \tau], \tag{2}
\]

\[
\frac{\partial u_i}{\partial x_i} = 0, \quad \text{in} \quad \Omega_1 \times [0, \tau], \tag{3}
\]

where \( g_i \) is the gravity force function. The coefficient \( b \) is a positive constant which is named as the Forchheimer coefficient. The viscosity variation in (1) is accounted for by the term \( 1 + \gamma T \), i.e., we are considering a viscosity \( \mu \) like \( \mu = \mu_1 (1 + \gamma T), \gamma > 0 \).

The Darcy equations are (see Nield and Bejan [2])

\[
v_i - g_i \theta + \frac{\partial q}{\partial x_i} = 0, \quad \text{in} \quad \Omega_2 \times [0, \tau], \tag{4}
\]

\[
\frac{\partial \theta}{\partial t} + v_i \frac{\partial \theta}{\partial x_i} = \Delta \theta, \quad \text{in} \quad \Omega_2 \times [0, \tau], \tag{5}
\]

\[
\frac{\partial v_i}{\partial x_i} = 0, \quad \text{in} \quad \Omega_2 \times [0, \tau], \tag{6}
\]

where \( \Omega_1 \) and \( \Omega_2 \) are the bounded, simply connected, and star-shaped domains and \( \tau \) is a given number satisfying \( 0 \leq \tau < \infty \). We impose the following boundary conditions:

\[
u_i n_i^{(1)} = 0, \quad T = \hat{G}(x, t), \quad \text{on} \quad \Gamma_1 \times [0, \tau],
\]

\[
\nu_i n_i^{(2)} = 0, \quad \theta = \hat{G}(x, t), \quad \text{on} \quad \Gamma_2 \times [0, \tau],
\]

for prescribed functions \( \hat{G}(x, t) \) and \( \hat{G}(x, t) \), and \( n_i^{(2)} \) denotes the unit outward normal of \( \partial \Omega_2 \). We also let \( n_i^{(1)} \) be the unit...
outward normal of \( \partial \Omega \). Obviously, \( n_3^{(1)} = -n_3^{(2)} \) on \( L \). The initial conditions are written as
\[
\begin{align*}
    u_t(x, 0) &= f_t(x), & T(x, 0) &= T_0(x), & \text{in } \Omega_1, \\
    \theta(x, 0) &= \theta_0(x), & \text{in } \Omega_2,
\end{align*}
\]  
(8)
for prescribed functions \( f_t, T_0, \) and \( \theta_0 \). On the interface \( L \), the conditions are
\[
\begin{align*}
    u_3 &= v_3, \\
    T &= \theta, \\
    T_3 &= \theta_3, \\
    q &= p.
\end{align*}
\]  
(9)

The purpose of this paper is to study the continuous dependence on the coefficients of problems (1)–(5). This type of stability is often called structural stability to distinguish it from continuous dependence on the initial data, on the boundary data, or even on the partial differential equation themselves. In continuum mechanics problems, it is necessary to be able to establish continuous dependence on the model; this is discussed in terms of differential equations by Hirsch and Smale [3]. Such stability estimates are fundamental in that one wishes to know if a small change in a coefficient in an equation or boundary data, or a small change in the equations themselves, will lead to a drastic change in the solution. When we study the continuous dependence or convergence, structural stability expresses the changes in the model itself rather than the original data. Many references to work of this nature are discussed in the monograph of Ames and Straughan [4] and the monograph of Straughan [5].

In the area of porous media, there have been many studies of continuous dependence, or structural stability, including those of Scott and Straughan [6, 7], Franchi and Straughan [8], Hoang and Ibragimov [9], Lin and Payne [10, 11], Liu [12, 13], Liu et al. [14–16], Scott [6], Scott and Straughan [7], Li [17], Cichon et al. [18], Ma and Liu [19], Ciarletta et al. [20], and Payne et al. [18–21]. These results show structural stability mainly focus on one fluid in the domain. But in reality, there are usually two or more fluids in the domain. They can interact with each other. So the study of such type of problems may be interesting and meaningful. It is worth drawing attention to this type of study. Several quantitative results in physical problems were obtained in [22–25].

In this paper, we derive an \textit{a priori} convergence result which compares the solution to the Forchheimer system of partial differential equations with that of the Darcy equations. The purpose of this paper is to study the continuous dependence of a solution to the Forchheimer system to a solution to Darcy equations on the Forchheimer coefficient \( b \) and the effective viscosity coefficient \( \gamma \). Different from [14, 26], there are two nonlinear terms \( \gamma T u_i \) and \( |u| u_i \), and there is no Laplace term in (1). So, some Sobolev inequalities do not hold for our problem. This will bring great difficulty. To get our result, we must seek a new method to overcome the difficulty. In the next section, we derive a number of \textit{a priori} bounds which will be used in establishing the continuous dependence result in Section 3.

2. A \textit{A Priori} Bounds

In this section, we want to drive bounds for the various norms of \( u, \theta, T, \) and \( \theta \).

2.1. \textit{Bounds for} \( \|u\|_{L^2(\Omega)} \) \textit{and} \( \|\theta\|_{L^2(\Omega)} \). Multiplying (1.1), with \( u \) and integrating over \( \Omega \), we obtain
\[
\begin{align*}
    b\|u\|_{L^3(\Omega)}^3 + \left\| \sqrt{1 + \gamma T}u \right\|_{L^2(\Omega)}^2 = \int_{\Omega} g_i T u_i dx - \int_{\Omega} \frac{\partial p}{\partial x_i} u_i dx.
\end{align*}
\]  
(10)

Integrating by parts and using Young’s inequality and the arithmetic-geometric mean inequality now lead to
\[
\begin{align*}
    b\|u\|_{L^2(\Omega)}^3 + \left\| \sqrt{1 + \gamma T}u \right\|_{L^2(\Omega)}^2 = \int_{\Omega} g_i T u_i dx \\
    - \int_L \mu s n_3^{(1)}(x) dx \leq \frac{1}{2} \|u\|_{L^2(\Omega)}^2 + \frac{1}{2} \theta \|T\|_{L^2(\Omega)}^2 \\
    + \int_L q \theta n_i^{(2)}(x) dx,
\end{align*}
\]  
(11)
where \( \theta = \max \{ g_i \theta \} \). By using the divergence theorem and Equation (2), we get
\[
\begin{align*}
    2b\|\theta\|_{L^2(\Omega)}^3 + \left\| \sqrt{1 + \gamma T}u \right\|_{L^2(\Omega)}^2 &\leq \theta \|T\|_{L^2(\Omega)}^2 \\
    + 2 \int_{\Omega} v_i (g_i \theta - v_i) dx \leq \theta \|T\|_{L^2(\Omega)}^2
\end{align*}
\]  
(12)

So, we have
\[
\begin{align*}
    2b\|\theta\|_{L^2(\Omega)}^3 + \left\| \sqrt{1 + \gamma T}u \right\|_{L^2(\Omega)}^2 + \frac{\theta}{2} \|\theta\|_{L^2(\Omega)}^2 \leq \theta \|T\|_{L^2(\Omega)}^2 \\
    + \theta \|\theta\|_{L^2(\Omega)}^2.
\end{align*}
\]  
(13)

2.2. \textit{Bounds for} \( \|T\|_{L^2(\Omega)}^2 \) \textit{and} \( \|\theta\|_{L^2(\Omega)}^2 \). Payne et al. [27] have obtained the following result:
\[
\sup_{[0,T]} \|T\|_{\infty} \leq T_M,
\]  
(14)
with \( T_M = \max \{ \|T\|_{\infty}, \sup_{[0,\tau]} G, T_{LM} \} \) and \( T_{LM} \) is the maximum temperature on the interface \( L \). Similarly, in the \( \Omega_2 \times [0, \tau] \),
\[
\sup_{[0,T]} \|\theta\|_{\infty} \leq \theta_M,
\]  
(15)
with \( \theta_M = \max \{ \| \theta_0 \|_{\infty}, \sup_{[0,T]} \| \hat{G} \|_{\infty}, T_{LM} \} \). However, in the area \( \Omega_1 \cup \Omega_2 \times [0,T] \), the maximum temperature cannot be reached on the interface \( L \). Therefore, we observe that

\[
\max \left\{ \sup_{[0,T]} \| T \|_{\infty}, \sup_{[0,T]} \| \theta \|_{\infty}, T_{LM} \right\} \leq N_M, \tag{16}
\]

where \( N_M = \max \{ \| T_0 \|_{\infty}, \sup_{[0,T]} \| G \|_{\infty}, \| \theta_0 \|_{\infty}, \sup_{[0,T]} \| \hat{G} \|_{\infty} \} \). To derive the bounds for \( \| T \|^2_{L^2(\Omega_2)} \) and \( \| \theta \|^2_{L^2(\Omega_2)} \), we introduce another two functions \( \varphi \) and \( \tilde{\varphi} \) which for each \( t \) satisfy

\[
\begin{align*}
\varphi_{\eta} - \Delta \varphi &= 0, \quad \text{in } \Omega_1, \\
\tilde{\varphi}_{\eta} - \Delta \tilde{\varphi} &= 0, \quad \text{in } \Omega_2, \\
\varphi &= G, \quad \text{on } \Gamma_1, \\
\tilde{\varphi} &= \hat{G}, \quad \text{on } \Gamma_2, \\
\varphi(x,0) &= T_0, \quad \text{in } \Omega_1, \\
\tilde{\varphi}(x,0) &= \theta_0, \quad \text{in } \Omega_2, \\
\varphi &= \tilde{\varphi}, \quad \varphi_1 = \tilde{\varphi}_3, \quad \text{on } L.
\end{align*}
\tag{17}
\]

Then, from the identities

\[
\begin{align*}
\int_0^T \int_{\Omega_1} (T - \varphi) \left[ \left( T - \varphi \right)_{\eta} + u_j T_{ij} - \Delta (T - \varphi) \right] \, dx \, dn &= 0, \\
\int_0^T \int_{\Omega_2} (\theta - \tilde{\varphi}) \left[ (\theta - \tilde{\varphi})_{\eta} + v_j \theta_{ij} - \Delta (\theta - \tilde{\varphi}) \right] \, dx \, dn &= 0,
\end{align*}
\tag{18}
\]

we have

\[
\begin{align*}
\| T - \varphi \|^2_{L^2(\Omega_1)} + \| \theta - \tilde{\varphi} \|^2_{L^2(\Omega_2)} + 2 \int_0^T \| \nabla (T - \varphi) \|^2_{L^2(\Omega_1)} \, dn \\
+ 2 \int_0^T \| \nabla (\theta - \tilde{\varphi}) \|^2_{L^2(\Omega_2)} \, dn \\
+ 2 \int_0^T \int_{\Omega_1} (T - \varphi)_{\eta} u_j T_{ij} \, dx \, dn \\
+ 2 \int_0^T \int_{\Omega_2} (\theta - \tilde{\varphi})_{\eta} v_j \theta_{ij} \, dx \, dn.
\end{align*}
\tag{19}
\]

So, (20) leads to that

\[
\begin{align*}
\| T - \varphi \|^2_{L^2(\Omega_1)} + \| \theta - \tilde{\varphi} \|^2_{L^2(\Omega_2)} \\
+ 2 \int_0^T \| \nabla (T - \varphi) \|^2_{L^2(\Omega_1)} \, dn \\
+ 2 \int_0^T \| \nabla (\theta - \tilde{\varphi}) \|^2_{L^2(\Omega_2)} \, dn \\
+ 2 \int_0^T \int_{\Omega_1} (T - \varphi)_{\eta} u_j T_{ij} \, dx \, dn \\
+ 2 \int_0^T \int_{\Omega_2} (\theta - \tilde{\varphi})_{\eta} v_j \theta_{ij} \, dx \, dn \\
\leq 2N_M \left( \int_0^T \| \nabla (T - \varphi) \|^2_{L^2(\Omega_1)} \, dn \right)^{1/2} \int_0^T \| u \|^2_{L^2(\Omega_1)} \, dn \\
+ 2N_M \left( \int_0^T \| \nabla (\theta - \tilde{\varphi}) \|^2_{L^2(\Omega_2)} \, dn \right)^{1/2} \int_0^T \| v \|^2_{L^2(\Omega_2)} \, dn
\end{align*}
\tag{20}
\]

It follows by Lemma 1 that

\[
\| T - \varphi \|^2_{\Omega_1} + \| \theta - \tilde{\varphi} \|^2_{\Omega_2} \leq N_M \left( \int_0^T \| u \|^2_{L^2(\Omega_1)} \, dn + \int_0^T \| v \|^2_{L^2(\Omega_2)} \, dn \right),
\tag{21}
\]

where

\[
\begin{align*}
\| T - \varphi \|^2_{\Omega_1} &= \| T - \varphi \|^2_{L^2(\Omega_1)} + \int_0^T \| \nabla (T - \varphi) \|^2_{L^2(\Omega_1)} \, dn, \\
\| \theta - \tilde{\varphi} \|^2_{\Omega_2} &= \| \theta - \tilde{\varphi} \|^2_{L^2(\Omega_2)} + \int_0^T \| \nabla (\theta - \tilde{\varphi}) \|^2_{L^2(\Omega_2)} \, dn.
\end{align*}
\tag{22}
\]

Since

\[
\| T \|^2_{\Omega_1} \leq \| T - \varphi \|^2_{\Omega_1} + \| \varphi \|^2_{\Omega_1},
\tag{23}
\]

we have

\[
\begin{align*}
\| T \|^2_{\Omega_1} + \| \theta \|^2_{\Omega_2} &\leq N_M \left( \int_0^T \| u \|^2_{L^2(\Omega_1)} \, dn + \int_0^T \| v \|^2_{L^2(\Omega_2)} \, dn \right) \\
&\quad + \| \varphi \|^2_{\Omega_1} + \| \tilde{\varphi} \|^2_{\Omega_2}.
\end{align*}
\tag{24}
\]

In computing the bounds for \( \| \varphi \|^2_{\Omega_1} \) and \( \| \tilde{\varphi} \|^2_{\Omega_2} \), we introduce another two functions \( h \) and \( \tilde{h} \) which for each \( t \) satisfy

\[
\begin{align*}
\Delta h &= 0, \quad \text{in } \Omega_1, \\
\Delta \tilde{h} &= 0, \quad \text{in } \Omega_2, \\
h &= G, \quad \text{on } \Gamma_1, \\
\tilde{h} &= \hat{G}, \quad \text{on } \Gamma_2, \\
h &= \tilde{h}, \quad \text{on } L.
\end{align*}
\tag{25}
\]

Then, from the identities

\[
\begin{align*}
\int_0^T \int_{\Omega_1} (\varphi - h) \left[ (\varphi - h)_{\eta} - \Delta (\varphi - h) + h_{\eta} \right] \, dx \, dn &= 0, \tag{26}
\
\int_0^T \int_{\Omega_2} (\tilde{\varphi} - \tilde{h}) \left[ (\tilde{\varphi} - \tilde{h})_{\eta} - \Delta (\tilde{\varphi} - \tilde{h}) + \tilde{h}_{\eta} \right] \, dx \, dn &= 0,
\end{align*}
\tag{27}
\]
it follows from (26) that
\[
\|
\phi - h \|_{L^2_1(\Omega_1)}^2 + \|
\tilde{\phi} - \tilde{h} \|_{L^2_1(\Omega_1)}^2 + 2 \int_0^t \|
\nabla (\phi - h) \|_{L^2_1(\Omega_1)}^2 \, d\eta
\]
\[
+ 2 \int_0^t \|
\nabla (\tilde{\phi} - \tilde{h}) \|_{L^2_1(\Omega_1)}^2 \, d\eta
\]
\[
+ \|
\theta_0 - \tilde{h}(x,0) \|_{L^2_1(\Omega_1)}^2
\]
\[
+ 2 \left( \int_0^t \|
\phi - h \|_{L^2_1(\Omega_2)}^2 \, d\eta \right)^{1/2}
\]
\[
+ 2 \left( \int_0^t \|\tilde{\phi} - \tilde{h} \|_{L^2_1(\Omega_2)}^2 \, d\eta \right)^{1/2}.
\]

From (28), we have
\[
\|
\phi - h \|_{L^2_1(\Omega_1)}^2 + \|
\tilde{\phi} - \tilde{h} \|_{L^2_1(\Omega_2)}^2
\]
\[
\leq \int_0^t \left( \|
\phi - h \|_{L^2_1(\Omega_1)}^2 + \|
\tilde{\phi} - \tilde{h} \|_{L^2_1(\Omega_2)}^2 \right) \, d\eta + m_1(t),
\]
where
\[
m_1(t) = \|
T_0 - h(x,0) \|_{L^2_1(\Omega_1)}^2 + \|
\theta_0 - \tilde{h}(x,0) \|_{L^2_1(\Omega_1)}^2
\]
\[
+ \int_0^t \|
\tilde{h} \|_{L^2_1(\Omega_2)}^2 \, d\eta + \int_0^t \|
\tilde{\eta} \|_{L^2_1(\Omega_2)}^2 \, d\eta.
\]

Upon integration of (29), we have
\[
\int_0^t \left( \|
\phi - h \|_{L^2_1(\Omega_1)}^2 + \|
\tilde{\phi} - \tilde{h} \|_{L^2_1(\Omega_2)}^2 \right) \, d\eta \leq \int_0^t m_1(\eta) e^{-\gamma \eta} \, d\eta.
\]

Inserting (31) into (28), we get
\[
\|
\phi - h \|_{L^2_1(\Omega_1)}^2 + \|
\tilde{\phi} - \tilde{h} \|_{L^2_1(\Omega_2)}^2 \leq \|
T_0 - h(x,0) \|_{L^2_1(\Omega_1)}^2
\]
\[
+ \|
\theta_0 - \tilde{h}(x,0) \|_{L^2_1(\Omega_1)}^2 + \int_0^t m_1(\eta) e^{-\gamma \eta} \, d\eta
\]
\[
+ \int_0^t \|
\tilde{h} \|_{L^2_1(\Omega_2)}^2 \, d\eta + \int_0^t \|
\tilde{\eta} \|_{L^2_1(\Omega_2)}^2 \, d\eta.
\]

So
\[
\|
\phi \|_{L^2_1(\Omega_1)}^2 + \|
\tilde{\phi} \|_{L^2_1(\Omega_2)}^2 \leq \|
\phi - h \|_{L^2_1(\Omega_1)}^2 + \|
\tilde{\phi} - \tilde{h} \|_{L^2_1(\Omega_2)}^2 + \|
\theta_0 - \tilde{h}(x,0) \|_{L^2_1(\Omega_1)}^2
\]
\[
+ \int_0^t \|
\theta_0 - \tilde{h}(x,0) \|_{L^2_1(\Omega_1)}^2 + \int_0^t m_1(\eta) e^{-\gamma \eta} \, d\eta
\]
\[
+ \int_0^t \|
\tilde{h} \|_{L^2_1(\Omega_2)}^2 \, d\eta + \int_0^t \|
\tilde{\eta} \|_{L^2_1(\Omega_2)}^2 \, d\eta.
\]

The terms \( \|h\|_{L^2_1(\Omega_1)}^2, \|\tilde{h}\|_{L^2_1(\Omega_2)}^2, \) \( \|\theta_0 - \tilde{h}(x,0)\|_{L^2_1(\Omega_1)}^2 \) may be bounded by boundary data by using a Rellich identity, cf. [6, 7]. Combining (13), (22), (24), and (33), we may have
\[
\|
T \|_{L^2_1(\Omega_1)}^2 + \|\theta\|_{L^2_1(\Omega_1)}^2 + \int_0^t \|\nabla T\|_{L^2_1(\Omega_1)}^2 \, d\eta
\]
\[
+ \int_0^t \|\nabla \theta\|_{L^2_1(\Omega_1)}^2 \, d\eta \leq N_\delta g^2 \int_0^t \left( \|T\|_{L^2_1(\Omega_1)}^2 + \|\theta\|_{L^2_1(\Omega_1)}^2 \right) \, d\eta
\]
\[
+ m_2(t),
\]
where
\[
m_2(t) = \|h\|_{L^2_1(\Omega_1)}^2 + \|\tilde{h}\|_{L^2_1(\Omega_2)}^2 + \|T_0 - h(x,0)\|_{L^2_1(\Omega_1)}^2
\]
\[
+ \|\theta_0 - \tilde{h}(x,0)\|_{L^2_1(\Omega_1)}^2 + \int_0^t m_1(\eta) e^{-\gamma \eta} \, d\eta
\]
\[
+ \int_0^t \|h\|_{L^2_1(\Omega_1)}^2 \, d\eta + \int_0^t \|\tilde{h}\|_{L^2_1(\Omega_2)}^2 \, d\eta.
\]

After integrating (34), we have
\[
\int_0^t \left( \|T\|_{L^2_1(\Omega_1)}^2 + \|\theta\|_{L^2_1(\Omega_1)}^2 \right) \, d\eta \leq \int_0^t m_2(\eta) e^{N_\delta g^2(t-\eta)} \, d\eta.
\]

Inserting (36) back into (34), we have the following lemma.

**Lemma 1.** Let \( (u_0, T_0, p) \) and \( (v_0, \theta, q) \) be the solutions of (1)-(5) with \( T_0, \theta_0, G, \tilde{G} \in L_{\infty} \). Then,
\[
\|
T \|_{L^2_1(\Omega_1)}^2 + \|\theta\|_{L^2_1(\Omega_1)}^2 + \int_0^t \|\nabla T\|_{L^2_1(\Omega_1)}^2 \, d\eta
\]
\[
+ \int_0^t \|\nabla \theta\|_{L^2_1(\Omega_1)}^2 \, d\eta \leq P_1(t),
\]
where
\[
P_1(t) = N_\delta g^2 \int_0^t m_2(\eta) e^{N_\delta g^2(t-\eta)} \, d\eta + m_2(t).
\]

Combining (13) and (34), we can obtain the following lemma.

**Lemma 2.** Let \( (u_0, T_0, p) \) and \( (v_0, \theta, q) \) be the solutions of (1)-(5) with \( T_0, \theta_0, G, \tilde{G} \in L_{\infty} \). Then,
\[
2b \|u\|_{L^2_1(\Omega_1)}^3 + \|\sqrt{1+\gamma T} u\|_{L^2_1(\Omega_1)}^2 + \|v\|_{L^2_1(\Omega_1)}^2 \leq g^2 P_1(t) + P_2(t).
\]
3. Continuous Dependence on the Forchheimer Coefficient

In this section, we want to derive an a priori estimate showing how \((u, T, p)\) and \((v, \theta, q)\) depend continuously on the Forchheimer coefficient \(b\). Let \((u, T, p)\) and \((v, \theta, q)\) be solutions of (1)-(5) with \(b = b_1\), and \((u^*_1, T^*, p^*)\) and \((v^*_1, \theta^*, q^*)\) be solutions of (1)-(5) with \(b = b_2\), respectively. We define

\[
\begin{align*}
\omega_i &= u_i - u^*_1, \\
\Sigma &= T - T^*, \\
\pi &= p - p^*, \\
\sigma &= b_1 - b_2, \\
\omega_i^m &= v_i - v^*_1, \\
\Sigma^m &= \theta - \theta^*, \\
\pi^m &= q - q^*,
\end{align*}
\]

Then, \((\omega_i, \Sigma, \pi)\) satisfy the following equations

\[
\begin{align*}
\sigma|u_i| + b_1|u_i| - u^*_1|u^*_1| + (1 + \gamma T^*)\omega_i \\
+ \gamma \Sigma u_i - g_1 \Sigma + \pi = 0, \quad \text{in} \quad \Omega_1 \times [0, T], \\
\frac{\partial \Sigma}{\partial t} + \omega_i T_{,j} + u^*_1 (\Sigma)_{,j} = \Delta \Sigma, \quad \text{in} \quad \Omega_1 \times [0, T], \\
\frac{\partial \omega_i}{\partial x_j} = 0, \quad \text{in} \quad \Omega_1 \times [0, T],
\end{align*}
\]

and \((\omega^m_i, \Sigma^m, \pi^m)\) satisfy equations

\[
\begin{align*}
\omega_i^m - g_1 \Sigma^m + \pi^m = 0, \quad \text{in} \quad \Omega_2 \times [0, T], \\
\frac{\partial \Sigma^m}{\partial t} + \omega^m_i T_{,j} + v^*_i (\Sigma^m)_{,j} = \Delta \Sigma^m, \quad \text{in} \quad \Omega_2 \times [0, T], \\
\frac{\partial \omega^m_i}{\partial x_j} = 0, \quad \text{in} \quad \Omega_2 \times [0, T].
\end{align*}
\]

The boundary conditions are

\[
\begin{align*}
\omega_i = 0, \quad \Sigma = 0, \quad \text{on} \quad \Gamma_1 \times [0, T], \\
\omega^m_i |_{\Gamma_1} p = 0, \quad \Sigma^m = 0, \quad \text{on} \quad \Gamma_2 \times [0, T].
\end{align*}
\]

The initial conditions can be written as

\[
\begin{align*}
\omega_i(x, 0) = 0, \quad \Sigma(x, 0) = 0, \quad \text{in} \quad \Omega_1, \\
\Sigma^m(x, 0) = 0, \quad \text{in} \quad \Omega_2.
\end{align*}
\]

The interface \(L\) conditions are

\[
\begin{align*}
\omega_j &= \omega^m_j, \\
\Sigma &= \Sigma^m, \\
\Sigma_j &= \Sigma^m_j, \\
\pi^m &= \pi.
\end{align*}
\]

We observe for later convenience that (4.3) may be rearranged as

\[
\sigma|u^*_1| + b_1|u_i| - u^*_1|u^*_1| + (1 + \gamma T)\omega_i + \gamma \Sigma u_i - g_1 \Sigma + \pi = 0.
\]

Our main result is the following theorem.

**Theorem 3.** Let \((u, T, p)\) and \((v, \theta, q)\) be the classical solutions to the initial-boundary value problem (1)–(5) corresponding to \(b_1\), and \((u^*_1, T^*, p^*)\) and \((v^*_1, \theta^*, q^*)\) also be the classical solutions to the initial-boundary value problem (1)–(5) but corresponding to \(b_2\). Then, for any \(t > 0\), we have

\[
(u_i, T, p) \longrightarrow (u^*_1, T^*, p^*), \quad (v_i, \theta, q) \longrightarrow (v^*_1, \theta^*, q^*),
\]

as \(b_1 \longrightarrow b_2\). The differences of velocities satisfy

\[
\int_0^t \left( \|u_i\|^2_{L_1(\Omega_1)} + \|u^*_1\|^2_{L_1(\Omega_1)} \right) dt \leq \frac{\sigma^2}{b_1 b_2} \int_0^t \left( P_2(s) e^{\int_0^s a_i(t) \, dt} \right) ds.
\]

Furthermore, there is a positive function \(a_i(t)\), given specifically in (58), such that

\[
\|\Sigma\|^2_{L_1(\Omega_1)} + \|\Sigma^m\|^2_{L_1(\Omega_2)} \leq \frac{\sigma^2}{b_1 b_2} \int_0^t \left( P_2(s) e^{\int_0^s a_i(t) \, dt} \right) ds,
\]

where \(\omega, w^m, \Sigma, \Sigma^m, \) and \(\sigma\) have been defined in (40) and (41).

**Proof.** We begin with the identity

\[
\int_{\Omega_1} \left[ \sigma|u_i| + b_1|u_i| - u^*_1|u^*_1| \right] \\
+ (1 + \gamma T)\omega_i + \gamma \Sigma u_i - g_1 \Sigma + \pi = 0.
\]

From (51), it follows that

\[
\frac{1}{2} \int_{\Gamma_1} \frac{1}{b_2} \left( \|u_i + u^*_1\|_{L_1(\Omega_1)}^2 + \|\sqrt{(1 + \gamma T^*)}w_i\|_{L_1(\Omega_1)}^2 \\
- \int_{\Omega_1} \pi_0 u_i w_i dx - \sigma \int_{\Omega_1} |u_i| u_i w_i dx - \gamma \int_{\Omega_1} \Sigma u_i w_i dx \\
+ \int_{\Omega_1} g_1 \omega_i \Sigma dx.
\]

**Abstract and Applied Analysis**
Using the divergence theorem, we have

\[ \frac{1}{2} b_2 \left\| \sqrt{|u| + |u^*|} \right\| _{L^1(\Omega)}^2 + \left\| \sqrt{1 + \gamma T} w \right\| _{L^1(\Omega)}^2 \]
\[ = -\sigma \int_{\Omega} |u| u w dx - \int_{\Omega} \pi w \Sigma dx - \gamma \int_{\Omega} \Sigma u w dx \]
\[ + \int_{\Omega} g_w \Sigma dx = -\sigma \int_{\Omega} |u| u w dx + \int_{\Omega} \pi w \Sigma dx - \gamma \int_{\Omega} \Sigma u w dx \]
\[ + \int_{\omega} (-w^{m} + g \Sigma w) w^{m} dx - \gamma \int_{\Omega} \Sigma u w dx \]
\[ + \int_{\Omega} g_w \Sigma dx. \]

We use the Cauchy-Schwarz inequality to have

\[ -\sigma \int_{\Omega} |u| u w dx \leq \sigma \left\| \sqrt{|u|} \right\| _{L^1(\Omega)} \left\| u \right\| _{L^1(\Omega)}^{3/2} \]
\[ \leq \frac{b_1 + b_2}{4} \left\| \sqrt{|u|} \right\| _{L^1(\Omega)}^2 + \frac{\sigma^2}{2b_1(b_1 + b_2)} P_2(t). \]  

Using Hölder’s inequality and the A-G mean inequality, we have

\[ \int_{\omega} (-w^{m} + g \Sigma w) w^{m} dx \leq -\frac{1}{2} \left\| w^m \right\| _{L^2(\Omega)}^2 + \frac{1}{2} g^2 \left\| \Sigma \right\| _{L^2(\Omega)}^2. \]

We find that the result given in Appendix B of Liu [13] for \( \|u\|_{L^2(\Omega)}^2 \)

\[ \|u\|_{L^2(\Omega)}^2 \leq k \left( \|u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^{1/2} \|\nabla u\|_{L^2(\Omega)}^{1/2} \right), \quad k > 0, \]

or

\[ \|u\|_{L^2(\Omega)}^2 \leq k \left[ \left(1 + \frac{\delta}{4}\right) \|u\|_{L^2(\Omega)}^2 + \frac{3}{4} \delta^{-1/2} \|\nabla u\|_{L^2(\Omega)} \right], \quad k > 0, \]

for arbitrary positive constant \( \delta > 0 \). We use this inequality for \( \|\Sigma\|_{L^2(\Omega)}^2 \) with \( \delta = 1 \) and (2.30), (2.32) in (55) to get

\[ -\gamma \int_{\Omega} \Sigma u w dx \leq \gamma \left\| \sqrt{|u|} \right\| _{L^1(\Omega)} \left\| \sum \right\| _{L^1(\Omega)} \left\| u \right|_{L^2(\Omega)}^{1/2} \]
\[ \leq \frac{b_1 + b_2}{4} \left\| \sqrt{|u|} \right\| _{L^1(\Omega)}^2 + \frac{ky^2}{b_1 + b_2} P_2(t) \left( \left\| \sum \right\| _{L^2(\Omega)}^2 \right) \]
\[ + 3 \delta \|\nabla \sum\|_{L^2(\Omega)}^2 \left( \frac{ky^2 b_1 + b_2}{64(b_1 + b_2)} P_2(t) \right) \]
\[ \leq \frac{1}{2} \left\| |u| + |u^*| \right\| _{L^1(\Omega)}^2 + \left\| (1 + \gamma T) w \right\| _{L^1(\Omega)}^2 \]
\[ \leq \frac{b_1 + b_2}{4} \left\| \sqrt{|u|} \right\| _{L^1(\Omega)}^2 + \frac{ky^2}{b_1 + b_2} P_2(t) \left( \left\| \sum \right\| _{L^2(\Omega)}^2 \right) \]
\[ + \left( \frac{ky^2}{b_1 + b_2} P_2(t) + \frac{\delta_1}{64(b_1 + b_2)} P_2(t) + \frac{3}{4} \delta_1 \|\nabla \sum\|_{L^2(\Omega)}^2 \right) \]
\[ \leq \frac{1}{2} \left\| |u| + |u^*| \right\| _{L^1(\Omega)}^2 + \left\| (1 + \gamma T) w \right\| _{L^1(\Omega)}^2 \]
\[ \leq \frac{b_1 + b_2}{4} \left\| \sqrt{|u|} \right\| _{L^1(\Omega)}^2 + \frac{ky^2}{b_1 + b_2} P_2(t) \left( \left\| \sum \right\| _{L^2(\Omega)}^2 \right) \]
\[ + \left( \frac{ky^2}{b_1 + b_2} P_2(t) + \frac{\delta_1}{64(b_1 + b_2)} P_2(t) + \frac{3}{4} \delta_1 \|\nabla \sum\|_{L^2(\Omega)}^2 \right) \]
\[ \leq \frac{1}{2} \left\| |u| + |u^*| \right\| _{L^1(\Omega)}^2 + \left\| (1 + \gamma T) w \right\| _{L^1(\Omega)}^2 \]
\[ \leq \frac{b_1 + b_2}{4} \left\| \sqrt{|u|} \right\| _{L^1(\Omega)}^2 + \frac{ky^2}{b_1 + b_2} P_2(t) \left( \left\| \sum \right\| _{L^2(\Omega)}^2 \right) \]
\[ + \left( \frac{ky^2}{b_1 + b_2} P_2(t) + \frac{\delta_1}{64(b_1 + b_2)} P_2(t) + \frac{3}{4} \delta_1 \|\nabla \sum\|_{L^2(\Omega)}^2 \right) \]
\[ \leq \frac{1}{2} \left\| |u| + |u^*| \right\| _{L^1(\Omega)}^2 + \left\| (1 + \gamma T) w \right\| _{L^1(\Omega)}^2 \]
\[ \leq \frac{b_1 + b_2}{4} \left\| \sqrt{|u|} \right\| _{L^1(\Omega)}^2 + \frac{ky^2}{b_1 + b_2} P_2(t) \left( \left\| \sum \right\| _{L^2(\Omega)}^2 \right) \]
\[ + \left( \frac{ky^2}{b_1 + b_2} P_2(t) + \frac{\delta_1}{64(b_1 + b_2)} P_2(t) + \frac{3}{4} \delta_1 \|\nabla \sum\|_{L^2(\Omega)}^2 \right) \]
\[ \leq \frac{1}{2} \left\| |u| + |u^*| \right\| _{L^1(\Omega)}^2 + \left\| (1 + \gamma T) w \right\| _{L^1(\Omega)}^2 \]
\[ \leq \frac{b_1 + b_2}{4} \left\| \sqrt{|u|} \right\| _{L^1(\Omega)}^2 + \frac{ky^2}{b_1 + b_2} P_2(t) \left( \left\| \sum \right\| _{L^2(\Omega)}^2 \right) \]
\[ + \left( \frac{ky^2}{b_1 + b_2} P_2(t) + \frac{\delta_1}{64(b_1 + b_2)} P_2(t) + \frac{3}{4} \delta_1 \|\nabla \sum\|_{L^2(\Omega)}^2 \right) \]
\[ \leq \frac{1}{2} \left\| |u| + |u^*| \right\| _{L^1(\Omega)}^2 + \left\| (1 + \gamma T) w \right\| _{L^1(\Omega)}^2 \]
\[ \leq \frac{b_1 + b_2}{4} \left\| \sqrt{|u|} \right\| _{L^1(\Omega)}^2 + \frac{ky^2}{b_1 + b_2} P_2(t) \left( \left\| \sum \right\| _{L^2(\Omega)}^2 \right) \]
\[ + \left( \frac{ky^2}{b_1 + b_2} P_2(t) + \frac{\delta_1}{64(b_1 + b_2)} P_2(t) + \frac{3}{4} \delta_1 \|\nabla \sum\|_{L^2(\Omega)}^2 \right) \]
\[ \leq \frac{1}{2} \left\| |u| + |u^*| \right\| _{L^1(\Omega)}^2 + \left\| (1 + \gamma T) w \right\| _{L^1(\Omega)}^2 \]
where

\[ a_1(t) = \frac{2ky^2}{b_1 + b_2} P_1^{1/2}(t) + \frac{k^4y^8}{32(b_1 + b_2)^4} \delta_1 P_2^2(t) + g^2. \] (63)

To bound \( \| \Sigma \|^2_{L^2(\Omega_1)} \) and \( \| \Sigma^m \|^2_{L^2(\Omega_1)} \), we multiply (3.3) by \( \Sigma \) and \( \Sigma^m \), respectively, and integrate by parts to find

\[
\begin{align*}
\left\| \Sigma \right\|^2_{L^2(\Omega_1)} + \left\| \Sigma^m \right\|^2_{L^2(\Omega_1)} & \\
= -2 \int_0^t \int_{\Omega_1} \nabla \Sigma \cdot \nabla d\eta - 2 \int_0^t \int_{\Omega_1} \nabla \Sigma^m \cdot \nabla d\eta \\
+ 2 \int_0^t \int_{\Omega_1} \Sigma \Sigma_T \Sigma_T dx \, d\eta + 2 \int_0^t \int_{\Omega_1} \Sigma \Sigma^m \Sigma_T dx \, d\eta \\
& + N^2_m \int_0^t \int_{\Omega_1} \Sigma^m \cdot \Sigma^m dx \, d\eta
\end{align*}
\]

or

\[
\begin{align*}
\left\| \Sigma \right\|^2_{L^2(\Omega_1)} + \left\| \Sigma^m \right\|^2_{L^2(\Omega_1)} & \\
= -2 \int_0^t \int_{\Omega_1} \nabla \Sigma \cdot \nabla d\eta - 2 \int_0^t \int_{\Omega_1} \nabla \Sigma^m \cdot \nabla d\eta \\
+ \int_0^t \int_{\Omega_1} \nabla \Sigma^m \cdot \nabla d\eta \leq N^2_m \int_0^t \left\| \Sigma \right\|^2_{L^2(\Omega_1)} \, d\eta \\
+ N^2_m \int_0^t \left\| \Sigma^m \right\|^2_{L^2(\Omega_1)} \, d\eta.
\end{align*}
\] (64)

Combining (62) and (4.26), we find that

\[
\begin{align*}
\left\| \Sigma \right\|^2_{L^2(\Omega_1)} + \left\| \Sigma^m \right\|^2_{L^2(\Omega_1)} & \leq \frac{\sigma^2}{2b_1 b_2} P_2(t) + \frac{3}{2} \delta_1 \left\| \nabla \Sigma \right\|^2_{L^2(\Omega_1)} \\
+ a_1(t) N^2_M \left( \int_0^t \left\| \Sigma \right\|^2_{L^2(\Omega_1)} \, d\eta + \int_0^t \left\| \Sigma^m \right\|^2_{L^2(\Omega_1)} \, d\eta \right).
\end{align*}
\] (65)

or

\[
\begin{align*}
\Xi(t) & \leq \frac{\sigma^2}{2b_1 b_2} P_2(t) + \frac{3}{2} \delta_1 \left\| \nabla \Sigma \right\|^2_{L^2(\Omega_1)} + a_1(t) N^2_M \Xi(t),
\end{align*}
\] (66)

where

\[
\Xi(t) = \int_0^t \left\| \Sigma \right\|^2_{L^2(\Omega_1)} \, d\eta + \int_0^t \left\| \Sigma^m \right\|^2_{L^2(\Omega_1)} \, d\eta.
\] (67)

Thus, after integration, we may derive from (67) the estimate:

\[
\Xi(t) \leq \frac{\sigma^2}{2b_1 b_2} \int_0^t P_2(s) e^{N^2_M \int_0^t a_1(\eta) d\eta} \, ds \\
+ \frac{3}{2} \delta_1 e^{N^2_M \int_0^t a_1(\eta) d\eta} \int_0^t \left\| \nabla \Sigma \right\|^2_{L^2(\Omega_1)} \, d\eta.
\] (69)

Using (65) and (66) and choosing \( \delta_1 \) in (69), we have

\[
\begin{align*}
\int_0^t \left\| \Sigma \right\|^2_{L^2(\Omega_1)} \, d\eta & + \int_0^t \left\| \Sigma^m \right\|^2_{L^2(\Omega_1)} \, d\eta \leq \frac{\sigma^2}{b_1 b_2} \int_0^t P_2(s) e^{N^2_M \int_0^t a_1(\eta) d\eta} \, ds,
\end{align*}
\] (70)

\[
\int_0^t \left\| \Sigma \right\|^2_{L^2(\Omega_1)} \, d\eta \leq \frac{\sigma^2}{b_1 b_2} N^2_M \int_0^t P_2(s) e^{N^2_M \int_0^t a_1(\eta) d\eta} \, ds.
\] (71)

Inequalities (70) and (71) are a priori bounds demonstrating continuous dependence of the solution on the Forchheimer coefficient \( b \).

**Data Availability**

All data generated or analyzed during this study are included within the article.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest.

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**References**

[1] L. E. Payne, J. C. Song, and B. Straughan, “Continuous dependence and convergence results for Brinkman and Forchheimer models with variable viscosity,” *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 455, no. 1986, pp. 2173–2190, 1999.

[2] D. A. Nield and A. Bejan, *Convection in Porous Media*, Springer-Verlag, New York, NY, USA, 1992.

[3] M. W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1974.

[4] K. A. Ames and B. Straughan, “Some further improperly posed problems,” *Mathematics in Science and Engineering*, vol. 194, 1997.

[5] B. Straughan, “The Energy Method, Stability and Nonlinear Convection,” *Applied Mathematical Sciences*, vol. 91, 2004.

[6] N. L. Scott, “Continuous dependence on boundary reaction terms in a porous medium of Darcy type,” *Journal of Mathematical Analysis and Applications*, vol. 399, no. 2, pp. 667–675, 2013.

[7] N. L. Scott and B. Straughan, “Continuous dependence on the reaction terms in porous convection with surface reactions,” *Quarterly of Applied Mathematics*, vol. 71, no. 3, pp. 501–508, 2013.

[8] F. Franchi and B. Straughan, “Continuous dependence and decay for the Forchheimer equations,” *Proceedings of the Royal...*
Society A: Mathematical, Physical and Engineering Sciences, vol. 459, no. 2040, pp. 3195–3202, 2003.

[9] L. Hoang and A. Ibragimov, "Structural stability of generalized Forchheimer equations for compressible fluids in porous media," Nonlinearity, vol. 24, no. 1, pp. 1–41, 2011.

[10] C. Lin and L. E. Payne, "Structural stability for a Brinkman fluid," Mathematical Methods in the Applied Sciences, vol. 30, no. 5, pp. 567–578, 2007.

[11] C. Lin and L. E. Payne, "Continuous dependence of the Soret coefficient for double diffusive convection in Darcy flow," Journal of Mathematical Analysis and Applications, vol. 342, no. 1, pp. 311–325, 2008.

[12] Y. Liu, "Convergence and continuous dependence for the Brinkman-Forchheimer equations," Mathematical and Computer Modelling, vol. 49, no. 7-8, pp. 1401–1415, 2009.

[13] Y. Liu, "Continuous dependence on the Soret coefficient for double diffusive convection in Darcy flow," Journal of Mathematical Analysis and Applications, vol. 325, no. 2, pp. 1479–1490, 2007.

[14] Y. Liu, "Convergence results for Forchheimer's equations for fluid flow in porous media," Mathematics and Computers in Simulation, vol. 150, pp. 66–82, 2018.

[15] Y. Liu, Y. Du, and C. Lin, "Convergence and continuous dependence results for the Brinkman equations," Applied Mathematics and Computation, vol. 215, no. 12, pp. 4443–4455, 2010.

[16] Y. Liu, S. Xiao, and Y. Lin, "Continuous dependence for the Brinkman-Forchheimer fluid interfacing with a Darcy fluid in a bounded domain," Mathematics and Computers in Simulation, vol. 150, pp. 66–82, 2018.

[17] Y. F. Li, "Convergence results on heat source for 2D viscous primitive equations of ocean dynamics," Acta Mathematica Scientia, vol. 41, pp. 339–352, 2020.

[18] M. Ciarletta, B. Straughan, and V. Tibullo, "Structural stability for a thermal convection model with temperature-dependent solubility," Nonlinear Analysis: Real World Applications, vol. 77, no. 4, pp. 317–354, 1998.

[19] L. E. Payne and B. Straughan, "Analysis of the boundary condition at the interface between a viscous fluid and a porous medium and related modelling questions," Journal de Mathématiques Pures et Appliquées, vol. 77, no. 4, pp. 317–354, 1998.

[20] B. Straughan, M. Ciarletta, and V. Tibullo, "Effect of anisotropic permeability on Darcy's law," Mathematical Methods in the Applied Sciences, vol. 24, no. 6, pp. 427–438, 2001.