The Volume of some Non-spherical Horizons and the AdS/CFT Correspondence

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Abstract

We calculate the volumes of a large class of Einstein manifolds, namely Sasaki-Einstein manifolds which are the bases of Ricci-flat affine cones described by polynomial embedding relations in $\mathbb{C}^n$. These volumes are important because they allow us to extend and test the AdS/CFT correspondence. We use these volumes to extend the central charge calculation of Gubser (1998) to the generalized conifolds of Gubser, Shatashvili, and Nekrasov (1999). These volumes also allow one to quantize precisely the D-brane flux of the AdS supergravity solution. We end by demonstrating a relationship between the volumes of these Einstein spaces and the number of holomorphic polynomials (which correspond to chiral primary operators in the field theory dual) on the corresponding affine cone.

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1 Introduction

The AdS/CFT correspondence is motivated by comparing a stack of elementary branes with the metric it produces (for reviews, see, for example, [1, 2]). In order to break some of the supersymmetry, we may place the stack at a conical singularity [3, 4, 5]. Consider, for instance, a stack of D3-branes placed at the apex of a Ricci-flat six dimensional cone $Y_6$ whose base is a five dimensional Einstein manifold $X_5$. Comparing the metric with the D-brane description leads one to conjecture that type IIB string theory on $AdS_5 \times X_5$ is dual to the low-energy limit of the worldvolume theory on the D3-branes at the singularity. One may also consider a stack of M2-branes placed at the apex of a Ricci-flat eight dimensional cone $Y_8$ whose base is a seven dimensional Einstein manifold $X_7$. There is a similar conjectured correspondence between M-theory on an $AdS_4 \times X_7$ background and the low-energy limit of the worldvolume theory of the M2-branes at the singularity.

In the simplest example of AdS/CFT correspondence, the D3-branes form a stack in ten dimensional space. The Einstein manifold is $S^5$, and the theory preserves $\mathcal{N} = 4$ supersymmetry (SUSY). Subsequently, it was realized that certain orbifolds of $S^5$, $S^5/\Gamma$ where $\Gamma$ is a discrete subgroup of $SU(2)$, preserve $\mathcal{N} = 2$ supersymmetry [6, 7]. The manifold $X_5 = T^{1,1}$ was examined in [8] and gives $\mathcal{N} = 1$ supersymmetry. (This example and a number of others were also examined in [10].) Unfortunately, $S^5, T^{1,1}$, and their orbifolds just about exhaust the mathematical literature of five dimensional Einstein spaces for which explicit metrics are known. In these cases, it was explicit knowledge of the metric which allowed for many tests of the AdS/CFT correspondence.

Because of extensive work on compactifying eleven dimensional supergravity to four dimensions in the eighties [11] and because seven dimensions allow for a larger variety of homogeneous spaces than do five, the metric situation is slightly better for the AdS/CFT correspondence with M2-branes. Metrics are known for the seven dimensional Einstein spaces $Q^{1,1,1}, M^{1,1,1}, N^{0,1,0}, V_{5,2}$, and of course $S^7$ among a few others as well. Moreover, many tests of the AdS/CFT correspondence have been carried out using these metrics.

The set of spaces described above is somewhat limited, and it would be useful to broaden the number of examples for which concrete calculations can be carried out which test and strengthen these correspondences. Encouragingly, many papers have appeared in the literature recently (for example [12, 13]) where explicit knowledge of the metric was not needed, and valuable information about the corresponding gauge theory duals was extracted from the geometry in other ways. Continuing this trend, we show how to calculate the volume of a large class of Einstein spaces without knowing an explicit metric on them. We then use these volumes to extend and test the AdS/CFT correspondence.

The large class of Einstein spaces we are concerned with are bases of certain Ricci-flat affine cones. Consider a weighted homogeneous polynomial in $\mathbb{C}^{n+1}$, by which we mean a polynomial $F(z)$ which satisfies

$$F(\lambda^{w_0}z_0, \lambda^{w_1}z_1, \ldots, \lambda^{w_n}z_n) = \lambda^d F(z_0, z_1, \ldots, z_n),$$

1
where $\lambda \in \mathbb{C}^*$ and $w_i \in \mathbb{Z}^+$, and the degree $d$ is a positive integer. These spaces are cones because of the scaling with respect to $\lambda$. Thus, we can write the metric on these spaces as

$$ds_\mathbb{C}^2 = dr^2 + r^2 ds_X^2.$$

The tensor $ds_X^2$ gives a metric on the intersection of this cone with the unit sphere in $\mathbb{C}^{n+1}$.

Our formula gives the volume of the intersection manifold endowed with this metric.

These volumes are important for at least two reasons. First, they allow us to determine the central charge of the dual gauge theory. It was conjectured in [8] that the gauge theory corresponding to an $AdS_5 \times T^{1,1}$ background can be obtained as the IR fixed point of a renormalization group (RG) flow from the $S^5/\mathbb{Z}_2$ orbifold theory. Gubser realized [14] that this flow had calculable consequences for the central charge of the two theories, namely,

$$\frac{c_{IR}}{c_{UV}} = \frac{1/\text{Vol}(T^{1,1})}{1/\text{Vol}(S^5/\mathbb{Z}_2)}.$$

Later, extending this work, the authors of [15] conjectured that the $N = 2$ orbifolds $S^5/\Gamma$ flow to certain generalized conifolds $Y_{\Gamma}$. However, as the volumes of these generalized conifolds were unknown, the same central charge calculation could not be repeated. Our volume formula applies to these generalized conifolds. We show that the ratio of the central charges for these generalized conifolds is exactly as predicted by the AdS/CFT correspondence.

Second, the volumes allow us to quantize precisely the flux in the supergravity solutions. It is known that for a stack of D3-branes placed at the conical singularity of $Y_6$, the supergravity solution is

$$ds^2 = h(r)^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + h(r)^{1/2}(dr^2 + r^2 ds_X^2),$$

$$h(r) = 1 + \frac{4\pi^4 g_s N \alpha'^2}{\text{Vol}(X_5)r^4},$$

$$F_5 = \mathcal{F}_5 + *\mathcal{F}_5, \quad \mathcal{F}_5 = 16\pi \alpha'^2 N \frac{\text{Vol}(S^5)}{\text{Vol}(X_5)}\text{vol}(X_5),$$

where all the other field strengths vanish and $N$ is the number of D3-branes. With this notation, vol is the volume differential form. Thus

$$\int_{X_5} \text{vol}(X_5) = \text{Vol}(X_5).$$

Presumably, other uses for these volumes can be found.

We finish with an intriguing relationship between the volume of our Einstein spaces and the number of holomorphic monomials of a given total degree $L$ on the corresponding affine cone. For affine cones of a given complex dimension $n$ and a given index which is the sum of the weights $w_i$ minus the degree $d$, the volume and number of these holomorphic monomials

1The cones for the $A_k$ groups were also derived in [16].
are directly proportional to each other. Although the relationship is generally true, we have only directly shown “why” it is true in the smooth case. The proof should generalize, but we have not done so here. The relationship is intriguing because these holomorphic monomials play an important role in AdS/CFT correspondence. They are chiral primary operators, and supersymmetry protects their dimension allowing for direct comparison between the gauge theory and the supergravity dual.

This paper is organized as follows. We begin with the derivation of our general volume formula. In section 3, we apply this formula to the case of the generalized conifolds of \[15\] and show that the AdS/CFT correspondence gives the correct result for the ratio of central charges. Finally, we discuss the relation between the volume of these spaces and the number of holomorphic monomials and the mathematical reason why this conjecture is true.

2 The Volume of a Large Class of Einstein Manifolds

In order to understand the volume computation, it will be useful to keep the following picture in mind. We recall from above that there is a natural $\mathbb{C}^*$ action on our cones. If we quotient out by the $\mathbb{R}^+$ portion of $\mathbb{C}^*$, we get a manifold that is termed the base of the cone and which we denote $X$. If we further quotient out by the remaining $U(1)$ part of $\mathbb{C}^*$, we obtain the weighted projective variety defined by $F(z) = 0$ in the weighted projective space $\mathbb{WP}(w)$. We will call this manifold $V$.

Because there exists a Calabi-Yau metric on our original cone, there is an Sasaki-Einstein metric on $X$ and an Kähler-Einstein metric on $V$. In addition, $X$ is a $U(1)$ fibration over $V$. We will use the Kähler-Einstein condition to determine the volume of $V$, and we will use the metric on $X$ to determine the length of the fiber and thus the total volume of $X$. Unfortunately, $V$ may not in general be a manifold. In fact, it will be an orbifold.\footnote{An orbifold is a space which looks locally like the quotient of $\mathbb{R}^n$ by a finite group. In other words, over each open set in a sufficiently fine open cover of the orbifold, there exists a covering subset of $\mathbb{R}^n$ and a finite group such that we identify the set in the orbifold with the quotient of the covering set by the action of the group. The open cover of the orbifold and the collection of covering sets and groups is termed the local uniformizing system of the orbifold. A $V$-bundle generalizes the concept of a fiber bundle to orbifolds and consists of a fiber bundle over each covering set and certain equivariant gluing relations. See the references for more details.}

In order to accommodate this difficulty, we take a brief excursion to introduce some aspects of weighted projective spaces and hypersurfaces in them.

2.1 Weighted Projective Spaces

A weighted projective space is defined in analogy to ordinary projective space: instead of a uniform weighting, the $\mathbb{C}^*$ action on $\mathbb{C}^{n+1}$ is weighted by a vector of weights $w$ as above. We denote this space $\mathbb{WP}(w) \equiv \mathbb{WP}(w_0,\ldots,w_n)$. Let us also write $|w| = \sum w$ and $w = \prod w_i$.\footnote{An orbifold is a space which looks locally like the quotient of $\mathbb{R}^n$ by a finite group. In other words, over each open set in a sufficiently fine open cover of the orbifold, there exists a covering subset of $\mathbb{R}^n$ and a finite group such that we identify the set in the orbifold with the quotient of the covering set by the action of the group. The open cover of the orbifold and the collection of covering sets and groups is termed the local uniformizing system of the orbifold. A $V$-bundle generalizes the concept of a fiber bundle to orbifolds and consists of a fiber bundle over each covering set and certain equivariant gluing relations. See the references for more details.}
Any polynomial that is homogeneous under the weighted action defines a weighted projective variety. Under certain conditions, one can treat weighted projective varieties similarly to ordinary projective varieties. For a review, see, for example, [17, 18]. In these references (see also [14, 20]), one finds conditions for the hypersurface to be well-formed and quasismooth. The former requires that

\[ \gcd(w_0, \ldots, \hat{w}_i, \ldots, w_n) \mid d \]  
\[ \gcd(w_0, \ldots, \hat{w}_i, \ldots, w_n) = 1 \]  

where a hat means to omit an element. This condition ensures that the singularities of the hypersurface are of complex codimension 2 or greater. Note that any weighted projective space is always isomorphic to one for which (6) holds [17, 18]. The conditions on the weights for quasismoothness are technical and not particularly elucidating, so we will refer the reader to the references for a full discussion. However, one can formulate the condition of quasismoothness in a manner that conforms to our physical expectation. Basically, a hypersurface is quasismooth if the affine cone over it is smooth at all points except the vertex [18]. In other words, this condition is just the statement that the only nonsmooth point of the cone on which the D-branes live is its tip where the D-branes are placed. This is altogether reasonable. All the spaces that we will deal with in this paper satisfy these conditions.

The main consequence of these conditions is that the standard adjunction formula from algebraic geometry holds for these hypersurfaces:

\[ \mathcal{O}(K^{-1}) = \mathcal{O}(|w| - d) \]  

where \( K^{-1} \) is the dual of the canonical sheaf, and \( d \) is the degree of the weighted homogeneous polynomial. The quantity \(|w| - d\) coincides with the index of the orbifold. It is a theorem of [21] that there exists a line V-bundle, \( H \), such that \( H^{|w| - d} = K^{-1} \). We will call \( H \) the hyperplane V-bundle. A degree \( d \) hypersurface is the zero locus of the \( d \)th power.

By working on the local uniformizing system of the orbifold, we can also define a connection on a given V-bundle. This allows us to define characteristic classes by means of the Chern-Weil homomorphism (see, for example, [22]) which defines characteristic classes in terms of symmetric polynomials in the curvature of a connection on the given bundle. Because we are in the orbifold category, however, these will be defined over the rationals rather than over the integers. There is a definition of orbifold cohomology due to Haefliger [23], but we will not refer to it here. Orbifold and ordinary cohomology are, in fact, isomorphic over the rationals.

### 2.2 The Geometry of Calabi-Yau Cones

We now review the features of Calabi-Yau cones that we will use in the sequel. Because of the \( \mathbb{R}^+ \) action on the cones, we can write the Calabi-Yau metric as

\[ ds^2 = dr^2 + r^2 g_{ab} \, dx^a \, dx^b. \]
The base of the cone, \( X \), is simply the intersection of the cone with the unit sphere in \( \mathbb{C}^{n+1} \). The tensor \( g = g_{ab} \, dx^a \, dx^b \) defines a metric on \( X \). This is one definition of a Sasaki manifold. Sasaki manifolds have many special properties and are reviewed in \([19]\). Because our cone is Calabi-Yau and, hence, Ricci-flat, it is easy to see that the metric \( g \) must be Einstein with scalar curvature \( s = 2(n - 1)(2n - 1) \) where \( n - 1 \) is the complex dimension of \( V \) and \( 2n - 1 \) is the real dimension of \( X \). Sasaki-Einstein manifolds are reviewed in \([19]\). Recall that an Einstein manifold satisfies the relation

\[
R = \frac{s}{\dim \mathbb{R}} g \tag{8}
\]

where \( R \) is the Ricci tensor on the manifold.

The main result we will use from the above papers is that there is a canonical foliation of \( X \) by circles and that the space of leaves, \( V \), is an \( n - 1 \) complex dimensional complex orbifold with a Kähler-Einstein metric, \( h \), of scalar curvature \( 4n(n - 1) \).

In fact, we have more structure here. Our cones are cut out by weighted homogeneous polynomials in complex space. This situation has been extensively studied in \([19, 20]\). Here, the space of leaves, \( V \), is exactly the weighted projective variety cut out by the polynomial in \( \mathbb{WP}(w) \). The foliation is just the \( U(1) \) action on \( X \) inherited from the original \( \mathbb{C}^* \) action on the cone. The inversion theorem (theorem 2.8) of \([21]\) tells us that \( X \) is a \( U(1) \) \( V \)-bundle over \( V \) with the Sasaki-Einstein metric

\[
g = \pi^* h + \eta \otimes \eta \tag{9}
\]

where \( \pi^* \) denotes the pullback from the base to the fibration, and \( \eta \) is a connection 1-form on the fibration with curvature \( d\eta = 2\pi^* \omega \) where \( \omega \) is the Kähler form of the Kähler-Einstein metric on the base. For more information on how these definitions generalize to the orbifold category, see, for example, any of the above papers or the original papers \([25, 26, 27, 28]\).

### 2.3 Computing the Volume

#### 2.3.1 The Volume of \( V \)

Before determining the volume of \( X \), we will first determine the volume of \( V \). The Einstein relation will be the key feature that allows us to determine the volume without an explicit knowledge of the metric.

First, we recall a few definitions. As we are working on a complex manifold, we can write our metric in complex coordinates \( h_{ab} \). Thus, the Kähler form is given by

\[
\omega = i \ h_{ab} \ dz^a \wedge d\bar{z}^b .
\]

Also, we denote the volume form on the manifold by \( \star 1 \). A simple calculation gives that

\[
\star 1 = \omega^{n-1}/(n-1)!. \]

The first Chern class of a manifold, denoted by \( c_1(V) \), is given in local
coordinates by:
\[ c_1(V) = i \frac{R_{ab}}{2\pi} d\bar{z}^a \wedge d\bar{z}^b \]  
(10)

where \( R_{ab} \) is the Ricci tensor in complex coordinates.

We now will use the Einstein relation to relate \( c_1(V) \) to \( \omega \). As the scalar curvature of \( V \) is \( 4n(n - 1) \), the Einstein relation takes the form

\[ R = 2n \omega \]

which implies that

\[ \omega = \pi \frac{c_1(V)}{n}. \]  
(11)

So, finally, we have

\[ \text{Vol}(V) = \int_V *1 = \frac{\pi^{n-1}}{(n-1)! n^{n-1}} \int_V c_1(V)^{n-1}. \]  
(12)

It finally remains to determine the integral of the Chern classes. This requires a bit more investment than the rest of the calculation, so we will consign the derivation to an appendix. The end result is that

\[ \int_V c_1(V)^{n-1} = \frac{d}{w(|w| - d)^{n-1}}. \]  
(13)

Combining this with (12), we can now calculate the volume of \( V \):

\[ \text{Vol}(V) = \frac{d}{w(n - 1)!} \left( \frac{\pi(|w| - d)}{n} \right)^{n-1}. \]  
(14)

### 2.3.2 The Volume of \( X \)

The last step that remains is to determine the length of the fiber. While we will write as if the base, \( V \), were a manifold, all the following steps can be justified by working in the uniformizing charts of the orbifold. We recall from above that we have a partially explicit form of the metric on \( X \), (9). Let \( \phi \) be a coordinate along the fiber. Then, we can write

\[ \eta = d\phi - \sigma \]
where \( d\sigma = 2\omega = (2\pi/n)c_1(V) \). An elementary fact is that iff \( \phi \in [0, 2\pi] \) then

\[ d\sigma = 2\pi c_1 \]  
(15)

where \( c_1 \) is the first Chern class of the circle fibration.

In order to determine \( c_1 \) of the fibration, we recall our picture of the cone with a \( \mathbb{C}^* \) action. The weighted projective space that \( V \) lives in is just the quotient of the ambient space that the cone lives in by the same action. Thus, we see that the circle fibration must have the same first Chern class as the dual of the hyperplane V-bundle.³ From the adjunction

³It is worthwhile to note that one can make other choices for this Chern class. In the smooth case, the allowed values can be determined from the Gysin sequence of the fibration. In the five dimensional case, as all four dimensional Kähler-Einstein manifolds are known, one obtains a complete classification of five dimensional regular Sasaki-Einstein manifolds. For more details, see [29, 30]. The role of these spaces in the AdS/CFT correspondence is treated in some detail in [10].
formula (7), we have \( c_1 = -c_1(H) = -c_1(V)/(|w| - d) \). Therefore, it is clear that the length of the fiber is not \( 2\pi \). In order to remedy this, we will rescale the coordinate along the fiber to make the relation (15) hold. In particular, let \( \theta = -\phi n/(|w| - d) \). Then, the metric (9) takes the form

\[
g = \pi^* h + \left( \frac{|w| - d}{n} \right)^2 (d\theta - \sigma')^2
\]

Now, we have

\[
d\sigma' = \frac{-n}{|w| - d} d\sigma = \frac{-2\pi}{|w| - d} c_1(V) = 2\pi c_1
\]

which means that the coordinate length of the fiber is now \( 2\pi \). We can easily do the integration and determine the geodesic length of the fiber to be \( 2\pi \frac{|w| - d}{n} \). Combining this with (14), we obtain a general formula for the volume of the base of an affine cone over a weighted projective variety:

\[
\text{Vol}(X) = 2\pi \frac{|w| - d}{n} \text{Vol}(V) = \frac{2d}{w(n-1)!} \left( \frac{\pi (|w| - d)}{n} \right)^n.
\]

This formula should generalize in a straightforward manner to complete intersections, but we have not done so here. The adjunction formula does hold for complete intersections in weighted projective space [17].

### 2.4 Checks of the Volume Formula

It is interesting to check that we get the expected volumes from (16) in some simple cases. Consider first the hypersurface defined by

\[
F = \sum_{i=0}^{n} z_i = 0.
\]

A moment’s reflection should convince the reader that \( F \) cuts out a copy of \( \mathbb{C}^n \) inside \( \mathbb{C}^{n+1} \). As a result, the corresponding Einstein manifold is a \( 2n - 1 \) dimensional sphere. The volume formula in this case gives

\[
\text{Vol}(S^{2n-1}) = \frac{2\pi^n}{(n-1)!}
\]

which is indeed the correct answer.

Some less trivial examples to consider are the Stenzel manifolds [32]

\[
F = \sum_{i=0}^{n} z_i^2 = 0.
\]

In the case \( n = 2 \), note that the Ricci flat metric on \( x^2 + y^2 + z^2 = \epsilon^2 \) where \( \epsilon \in \mathbb{R} \) is the Eguchi-Hanson metric. The case \( \epsilon = 0 \) is well known to correspond to \( \mathbb{R}^4/\mathbb{Z}_2 \). Indeed, from (16), we find that

\[
\text{Vol}(S^3/\mathbb{Z}_2) = \pi^2
\]
as it should be.

The next Stenzel manifold, \( n = 3 \), is the well known conifold. As a result, \( X^5 = T^{1,1} \). From the explicit metric on \( T^{1,1} \), or from \((16)\), one may calculate that

\[
\text{Vol}(T^{1,1}) = \frac{16\pi^3}{27}.
\]

For \( n = 4 \), \( X_5 = V_{5,2} \), one of the Stiefel manifolds. Again, explicit metrics on \( V_{5,2} \) are known \([31, 32]\). One may calculate from the metric, as we have done in the appendix, or from \((16)\) that

\[
\text{Vol}(V_{5,2}) = \frac{27\pi^4}{128}.
\]

So far, we have only checked the volume formula for the smooth cases, \textit{i.e.}, the cases where \( F(z) \) is a homogeneous polynomial. There is one set of simple singular example that we can also check. Consider the weighted polynomials \( F(x, y, z) \) in the complex variables \( x \), \( y \), and \( z \) of total degree \( h \). Let these variables transform with the weights \( \alpha \), \( \beta \), and \( h/2 \) given in \((22)\). While the weights here do not satisfy \((3)\), as noted above, the spaces are isomorphic to ones for which the condition holds. The polynomials describe ALE spaces and have an ADE classification (see for example \([15, 17]\)). The Einstein manifolds at the bases of these cones are well known to be orbifolds of \( S^3 \). The orbifold groups \( \Gamma \) are discrete subgroups of \( SU(2) \), and each group \( \Gamma \) has \( 2\alpha\beta \) elements. As a result

\[
\text{Vol}(S^3/\Gamma) = \frac{\pi^2}{\alpha\beta}.
\]

Let us compare this result with the volume formula \((16)\). Note from \((22)\) that the index \( |w| - d = \alpha + \beta + h/2 - h \) is in every case equal to one. The products of the weights in \((16)\), \( \alpha\beta h/2 \), cancels with the total degree \( h \) to leave the required factor of \( \alpha\beta \) in the denominator of \((17)\). We get precisely the correct answer.

\section{A Central Charge Calculation}

We now wish to use our formula to check the ratio of central charges for the RG flows of \([13]\). We first compute the answer on the field theory side and then compare with the prediction of our volume formula.

\subsection{A Gauge Theory Perspective}

Let us begin by reviewing the gauge theory on the worldvolume of a collection of \( N \) D-branes placed at the orbifold singularity of \( C^2/\Gamma \), where \( \Gamma \) is a discrete subgroup of \( SU(2) \) of ADE type\footnote{Much of this discussion is drawn from \([15]\).}. The field theory has \( \mathcal{N} = 2 \) supersymmetry. Its gauge group is the product

\[
G = \times_{i=0}^{r} U(N_i)
\]
where $i$ runs through the set of vertices of the extended Dynkin diagram of the corresponding ADE type (see figure 1). We have also introduced $N_i = N n_i$, where $n_i$ is the index of the $i$th vertex of the Dynkin diagram. Equivalently, one may think of $i$ as running through the irreducible representations $r_i$ of $\Gamma$, in which case $n_i$ can be thought of as the dimension of $r_i$. The dual Coxeter number of the corresponding ADE Lie group is $h = \sum n_i$.

The number of elements of $\Gamma$ is $\text{ord}(\Gamma) = \sum n_i^2$.

The field content of the gauge theory can be summarized conveniently with a quiver diagram which for the simple cases under consideration here is nothing but the corresponding extended Dynkin diagram. For each vertex in the Dynkin diagram, we have an $\mathcal{N} = 2$ vector multiplet transforming under the adjoint of $U(N_i)$. For each line in the diagram, there is a bifundamental hypermultiplet $a_{ij}$ in the representation $(N_i, \bar{N}_j)$.

To write a superpotential for this gauge theory, it is convenient to decompose the fields into $\mathcal{N} = 1$ multiplets. Each $a_{ij}$ will give rise to a pair of chiral multiplets, $(B_{ij}, B_{ji})$, where $B_{ij}$ is a complex matrix transforming in the $(N_i, \bar{N}_j)$ representation. Moreover, there is a chiral multiplet $\phi_i$ for each vector multiplet in the theory.

The superpotential is then

$$W = \sum_i \text{Tr} \mu_i \phi_i$$

(18)
where $\mu_i$ is the “complex moment map”

$$
\mu_i^{\alpha_i \beta_i} = \sum_j s_{ij} B_{ij}^{\alpha_i \gamma_j} B_{ji}^{\gamma_j \beta_i}.
$$

(19)

Although the indices are confusing, essentially all we have done is construct a cubic polynomial in the $\mathcal{N} = 1$ superfields consistent with $\mathcal{N} = 2$ SUSY and the gauge symmetry. The factor $s_{ij}$ is the antisymmetric adjacency matrix for the Dynkin diagram: $s_{ij} = \pm 1$ when $i$ and $j$ are adjacent nodes and zero otherwise. The upper index $\alpha_i$ indicates a fundamental representation of $U(N_i)$, while a lower index $\beta_i$ indicates an anti-fundamental representation of $U(N_i)$. There is a relation among the $\mu_i$

$$
\sum_i \text{Tr} \mu_i = 0 .
$$

(20)

which holds because the trace gives something symmetric in $i$ and $j$ summed against $s_{ij}$ which is antisymmetric.

This $\mathcal{N} = 2$ gauge theory is superconformal and thus must have an $R$ symmetry. As a result, the superpotential term in the action $\int d^2\theta W$ will have $R$ charge zero. We take the convention that the spinor $\theta$ has $R$ charge 1. Therefore $W$ must have $R$ charge 2. Conveniently, the $B_{ij}$ and the $\phi_i$ have $R$ charge $2/3$ and the superpotential, as noted above, is cubic.

In the large $N$ limit in the case of D3-branes, we can invoke the AdS/CFT correspondence for this gauge theory [6, 7]. The correspondence tells us that the gauge theory described above is dual to type IIB supergravity (SUGRA) on an $AdS_5 \times S^5/\Gamma$ background. To see how the orbifolding works, consider $S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : \sum_i |z_i|^2 = 1\}$. The group $\Gamma$ acts only on $(z_1, z_2) \in \mathbb{C}^2$. As a result, there is an $S^1$ of the $S^5$ which is left invariant under $\Gamma$.

We can add a term to the superpotential (18) that will give masses $m_i$ to the $\phi_i$. Such a term will eliminate the $\phi_i$ from the theory at energies below the mass scale set by the $m_i$ and break the supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. In particular, we add the term

$$
W' = W - \frac{1}{2} \sum_i m_i \text{Tr} \phi_i^2 .
$$

To see what happens at low energies, let us look at the equations of motion $dW' = 0$. By varying with respect to the matter fields that define $\mu_i$, we see that

$$
\phi_i = \phi \text{Id}_{N_i} ,
$$

(21)

where $\phi$ is the Lagrange multiplier used to ensure that the constraint (20) is satisfied. Varying with respect to $\phi_i$ and employing (21), one gets

$$
\mu_i = m_i \phi \text{Id}_{N_i} .
$$
From (20), it follows that

\[ \sum_i n_i m_i = 0. \]

Assuming that none of the \( m_i = 0 \), we can eliminate \( \phi_i \) from the action to find an effective low energy superpotential:

\[ W_{\text{eff}} = \sum_i \frac{1}{2m_i} \text{Tr} \mu_i^2. \]

Notice that \( W_{\text{eff}} \) is quartic in the superfields \( B_{ij} \) (see (19)). We would like the endpoint of the RG flow generated by adding these mass terms to be an IR conformal fixed point. Superconformal theories preserve the \( R \) symmetry. However, if the fields \( B_{ij} \) are given their naive \( R \) charges of \( 2/3 \), this quartic superpotential will explicitly break our \( R \) symmetry. By giving the \( B_{ij} \) anomalous dimensions, we find that after flowing to the IR, the \( R \) charge of the \( B_{ij} \) can be adjusted to \( 1/2 \), and the \( R \) symmetry is preserved.

In the context of AdS/CFT correspondence, the authors of [15] generalized an argument of [8] for the \( A_1 \) case, arguing that the IR endpoint of this RG flow is dual to type IIB SUGRA in an \( AdS_5 \times X_\Gamma \) background where the \( X_\Gamma \) are the level surfaces of certain “generalized conifolds”. The generalized conifolds are three complex dimensional Calabi-Yau manifolds with a conical scaling symmetry. The conifolds can be described by a polynomial embedding relation \( F_\Gamma = 0 \) in \( \mathbb{C}^4 \). To conform with the notation of [15], we use the coordinates \( (x,y,z,\phi) \in \mathbb{C}^4 \). The polynomial \( F_\Gamma \) is invariant under a \( \mathbb{C}^* \) action, the real part of which is the conical scaling symmetry while the imaginary part corresponds to an \( R \) symmetry transformation in the dual gauge theory. \( F_\Gamma \) transforms under this \( \mathbb{C}^* \) action with weight \( h \), the dual Coxeter number. The coordinate \( \phi \) will transform with weight one, and the remaining coordinates \( x, y, \) and \( z \) transform with weights:

\[
\begin{array}{cccc}
\Gamma & \alpha = [x] & \beta = [y] & h/2 = [z] & h \\
A_k & 1 & \frac{k+1}{2} & \frac{k+1}{2} & k + 1 \\
D_k & 2 & k - 2 & k - 1 & 2(k - 1) \\
E_6 & 3 & 4 & 6 & 12 \\
E_7 & 4 & 6 & 9 & 18 \\
E_8 & 6 & 10 & 15 & 30
\end{array}
\]

(22)

We will not need the explicit form of the polynomials \( F_\Gamma \) in what follows; we refer the interested reader to [15] for more details. To make things more concrete, however, we give

\[ F_{A_k} = \prod_{i=0}^{k} (x - \xi_i \phi) + y^2 + z^2, \]

(23)

\footnote{One should really separate out the \( A \) series into two series so that one does not have fractional weights. The series for \( A_{2k} \) would have weights \( (2, 2k + 1, 2k + 1, 2k + 2) \). The volume formula gives the same answer when one plugs in the fractional weights, so, for conciseness of notation, we will not separate the series.}
\[ F_{D_k} = \prod_{i=0}^{k-2} (x - \xi_i \phi^2) + t_0 \phi^k y + xy^2 + z^2, \tag{24} \]

where \( \xi_i \) and \( t_0 \) are free constants transforming with weight zero under the \( \mathbb{C}^* \) action.

### 3.2 The Central Charge

Conformal field theories are characterized by a number called the central charge \( c \) which appears in many correlation functions. From the AdS/CFT dictionary, for a conformal field theory dual to an \( AdS_5 \times X_5 \) background, we know that \( c \sim 1/\text{Vol}(X_5) \) \cite{34}. If we can calculate all the relevant volumes and central charges independently, we can make a check of the AdS/CFT correspondence. In particular, in the UV we have the orbifolded theory, \( X_5 = S^5/\Gamma \), and in the IR we have the generalized conifold theory, \( X_5 = X_\Gamma \). It ought to be true that

\[ \frac{c_{IR}}{c_{UV}} = \frac{1/\text{Vol}(X_\Gamma)}{1/\text{Vol}(S^5/\Gamma)}. \]

In the case \( \Gamma = A_1 \), this calculation was done in \cite{14}. We now attempt to check this formula for arbitrary \( \Gamma \). The volume of \( S^5/\Gamma \) is straightforward to compute. Indeed, \( \text{Vol}(S^5) = \pi^3 \), and to find the volume of \( S^5/\Gamma \) we just divide by the number of elements of \( \Gamma \), which is nothing but \( \sum_i n_i^2 = 2\alpha\beta \) (see (22)). \( \text{Vol}(X_5) \) can be calculated from (16), but, for suspense, we will leave this step to the very end.

First, let us show that

\[ \frac{c_{IR}}{c_{UV}} = \frac{27}{32} \tag{25} \]

independent of \( \Gamma \). We begin by recalling some basic facts about conformal field theories. First, we define the central charge \( c \) and another anomaly coefficient \( a \) in terms of the one point function of the stress energy tensor

\[ \langle T^\mu_\nu \rangle = -aE_4 - cI_4 \]

where \( E_4 \) and \( I_4 \) are scalars quadratic in and depending only on the Riemann curvature. We will not try to calculate \( a \) and \( c \) explicitly, but rather just the ratio \( c_{IR}/c_{UV} \). Let \( R_\mu \) be the \( R \) symmetry current. It was shown in \cite{33} that

\[ \langle (\partial_\mu R^\mu) T_{\alpha\beta} T_{\gamma\delta} \rangle \sim (a - c) \sim \sum_\psi r(\psi), \tag{26} \]

\[ \langle (\partial_\mu R^\mu) R_\alpha R_\beta \rangle \sim (5a - 3c) \sim \sum_\psi r(\psi)^3. \tag{27} \]

The sum is over all the fermions \( \psi \) in the gauge theory, and \( r(\psi) \) is the \( R \) charge.

To proceed, we need to classify all of the fermions in the gauge theories of interest along with their \( R \) charges. As noted above, in the UV orbifold theory the chiral superfields \( B_{ij} \)
and $\phi_i$ have $R$ charge $2/3$. As a result, the fermions in these chiral multiplets will have $R$ charge $-1/3$. Thus, for each line in the extended Dynkin diagram, we have $2N_iN_j$ fermions with $R$ charge $-1/3$, and for each vertex of the diagram, we have $N_i^2$ fermions with $R$ charge $-1/3$. There are also for each vertex $N_i^2$ gluinos in the theory, the superpartners of the gauge fields. As the gauge fields are uncharged under the $R$ symmetry, the gluinos will have $R$ charge $1$. We can now compute the sum over the $R$ charges in the UV:

$$a_{UV} - c_{UV} \sim \sum_\psi r(\psi) = \frac{2N^2}{3} \left( \sum_i n_i^2 - \sum_{(ij)} n_i n_j \right),$$

(28)

$$5a_{UV} - 3c_{UV} \sim \sum_\psi r(\psi)^3 = \frac{26N^2}{27} \sum_i n_i^2 - \frac{2N^2}{27} \sum_{(ij)} n_i n_j .$$

(29)

The sum over $\langle ij \rangle$ is limited to nearest neighbor nodes of the Dynkin diagram, i.e., nodes connected by a line. Typically, for conformal field theories with AdS duals, we expect that $a = c$. Indeed, it is a property of the simply laced Dynkin diagrams we are considering that

$$\sum_i n_i^2 = \sum_{(ij)} n_i n_j .$$

Thus, $5a_{UV} - 3c_{UV} = 2c_{UV} \sim \frac{8N^2}{9} \sum n_i^2$.

When we flow to the IR, the $\phi_i$ fields will get a mass and disappear from the theory. As a result, the $N_i^2$ fermions with $R$ charge $-1/3$ will disappear from the sum. Above, we noted that the $R$ charge of the superfields $B_{ij}$ changes to $1/2$. Thus the $2N_iN_j$ fermions for each line in the Dynkin diagram will now have $R$ charge $-1/2$. The $R$ charge of the gluinos is unchanged. Repeating the above calculation in these slightly different circumstances, we find that

$$a_{IR} - c_{IR} \sim \sum_\psi r(\psi) = N^2 \left( \sum_i n_i^2 - \sum_{(ij)} n_i n_j \right),$$

(30)

$$5a_{IR} - 3c_{IR} \sim \sum_\psi r(\psi)^3 = N^2 \sum_i n_i^2 - \frac{N^2}{4} \sum_{(ij)} n_i n_j .$$

(31)

Thus, $a_{IR} = c_{IR}$ as expected, and, moreover, $5a_{IR} - 3c_{IR} = 2c_{IR} \sim \frac{3N^2}{4} \sum n_i^2$. Dividing our two results yields (25).

At this point, one can say that it is a prediction of the AdS/CFT correspondence that

$$\text{Vol}(X_\Gamma) = \frac{\text{Vol}(T^{1,1})\text{ord}(A_1)}{\text{ord}(\Gamma)} = \frac{16\pi^3}{27\alpha\beta} .$$

(32)

As the products of weights for all these conifolds is $\alpha\beta h/2$ and the index is always 2, we see that our formula (13) gives precisely the correct answer.
4 A Somewhat Strange Conjecture

4.1 Motivation

In attempting to prove the volume formula \( (16) \), an interesting fact about the holomorphic monomials on an affine cone cut out by a weighted homogeneous polynomial was noticed. The initial observation that began the set of ideas which follows was a recollection of Weyl’s Law for the growth of the eigenvalues of the Laplacian. Let \( \nabla^2 f = Ef \), where \( \nabla^2 \) is the Laplacian on some compact \( D \) dimensional manifold \( M_D \) and \( E \) the corresponding energy or eigenvalue. Weyl’s Law states that the number of eigenfunctions with energy less than \( E \) scales as

\[
N(E) \sim \gamma \text{Vol}(M_D)E^{D/2}
\]

for large \( E \) where \( \gamma \) is some constant of proportionality.

Of course, without a metric, we can no sooner derive an expression for \( \nabla^2 \) than calculate the volume. However, in the context of AdS/CFT correspondence, it is not the eigenfunctions of the Laplacian, or equivalently the harmonic functions on the cone over \( X \), which play the most important role. Most of these harmonic functions have unprotected dimensions and energies which can change when the coupling constant in the gauge theory changes. Instead it is the chiral primary operators (CPOs) which are the most important. The CPOs have the maximum possible \( R \) charge for a given dimension, and the SUSY algebra protects their dimension when the coupling constant is changed.

As was noted above, the \( U(1) \) \( R \) symmetry corresponds to the imaginary part of the \( \mathbb{C}^* \) action on \( X \). Let us consider the spaces \( X_F \). The dimension of an operator such as \( x^m \bar{x}^\bar{m} y^n \bar{y}^\bar{n} z^p \bar{z}^\bar{p} \phi^q \bar{\phi}^\bar{q} \phi^q \bar{\phi}^\bar{q} \) is just

\[
L = \alpha(m + \bar{m}) + \beta(n + \bar{n}) + \frac{h}{2}(p + \bar{p}) + q + \bar{q}.
\]

On the other hand, the \( R \) charge is

\[
r = \alpha(m - \bar{m}) + \beta(n - \bar{n}) + \frac{h}{2}(p - \bar{p}) + q - \bar{q}.
\]

Clearly, the operators of this type that maximize the magnitude of the \( R \) charge are the purely holomorphic or purely antiholomorphic monomials. Let \( P_L^F \) be the space of holomorphic degree \( L \) monomials quotiented by the relation \( F = 0 \) where \( F \) is the defining polynomial of the cone. In the spirit of the Weyl scaling law, we conjecture that for large \( L \),

\[
\dim(P_L^F) \sim \gamma L^{n-1} \text{Vol}(X) \quad (33)
\]

where \( \gamma \) is a constant of proportionality.

\[\footnote{We would like to thank Steve Gubser suggesting this possibility.}\]
To calculate \( \dim(\mathcal{P}^F_L) \), note that

\[
\dim(\mathcal{P}^F_L) = \dim(\mathcal{P}_L) - \dim(\mathcal{P}_{L-d})
\] (34)

where \( \dim(\mathcal{P}_L) \) is the number of holomorphic polynomials of degree \( L \) on \( \mathbb{C}^n \) and \( d \) is the degree of \( F \). To leading order,

\[
\dim(\mathcal{P}_L) = \frac{d\mathcal{P}_{<L}}{dL}
\]

where \( \mathcal{P}_{<L} \) is the number of polynomials with degree less than or equal to \( L \). Also to leading order, \( \mathcal{P}_{<L} \) is the volume of a \( n+1 \) dimensional pyramid with apex at the origin and legs along the positive axes of \( \mathbb{R}^{n+1} \). Each leg will have length \( L/w_i \). Thus

\[
\dim(\mathcal{P}_L) = \frac{L^n}{n!w}
\]

where, as before, \( w \) is the product of the weights.

Now, we apply (34) to obtain

\[
\dim(\mathcal{P}^F_L) = L^{n-1} \frac{d}{(n-1)!w} + \mathcal{O}(L^{n-2})
\] (35)

Comparing with (34), we find that

\[
\gamma = \frac{1}{2} \left( \frac{n}{\pi(|w| - d)} \right)^n
\] (36)

Thus, the constant depends only on the index and the dimension of the Einstein space.

### 4.2 Why does it Work?

As has often been said, there are no coincidences in mathematics. Thus, we would like to understand a deeper reason why the conjecture turns out to be correct. We will see that, at least in the smooth case, it is a consequence of the Hirzebruch-Riemann-Roch (HRR) theorem. We will not attempt to generalize to the orbifold case, except to note that the HRR theorem has been generalized by Kawasaki to orbifolds in [36].

In order to proceed, we will first express the dimension of \( \mathcal{P}^F_L \) in an algebraic geometric form. Recall that there exists a map

\[
\text{Sym}^L(\mathbb{C}^{n+1}^*) \rightarrow H^0(\mathbb{P}^n, \mathcal{O}(H^L)).
\]

where \( H \) is the hyperplane bundle over the projective space. Thus, we can compute the dimension of \( \mathcal{P}_L \) by computing the dimension of the space of sections of a line bundle over \( \mathbb{P}^n \). All that is left to do is to impose the relation \( F(z) = 0 \). However, \( F \) defines a section of \( H^d \), and its zero locus defines the variety \( \mathcal{V} \) in \( \mathbb{P}^n \). We can impose the relation by restricting the line bundle \( H^L \) to the variety \( \mathcal{V} \) and computing its dimension:

\[
\dim(\mathcal{P}^F_L) = H^0(\mathcal{V}, \mathcal{O}(H^L)).
\] (37)
A useful reference for the algebraic geometry in this section is [37]. A nice introduction to the Atiyah-Singer index theorem and its application to the Dolbeault complex is contained in [38].

We will now convert (37) into an integral formula through the use of the HRR theorem. We first recall the definition of the Euler character of the twisted Dolbeault complex on a manifold $M$ with line bundle $L$:

$$\chi \equiv \sum_q (-1)^q \dim H^{0,q}_\partial(M, L) = \sum_q (-1)^q h^{0,q}(L).$$  \hspace{1cm} (38)

Here, the forms take values in the line bundle $L$. The HRR theorem states that

$$\chi = \int_M ch(L)td(M)$$

where $ch$ and $td$ are characteristic classes to be defined later.

Because we are on a Kähler manifold, the twisted Hodge numbers obey $h^{p,q}(L) = h^{q,p}(L)$, and we can interchange the indices in (38). The next thing we need is the Dolbeault theorem which relates the Dolbeault cohomology groups of a manifold to certain sheaf cohomology groups:

$$H^{p,q}_\partial(M, L) \cong H^q(M, \Omega^p(L)).$$

$\Omega^p(L)$ denotes the sheaf of holomorphic $p$-forms on $M$ valued in $L$. Putting all this together, we obtain the identity:

$$\chi = \chi(\mathcal{O}(L)) \equiv \sum q (-1)^q \dim H^q(M, \mathcal{O}(L)).$$

Next, we recall another fact from algebraic geometry, the Kodaira vanishing theorem. This states that, for any positive line bundle $L$, not necessarily the same $L$ as before, over a manifold of complex dimension $m$,

$$H^q(M, \Omega^p(L)) = 0 \quad \text{when} \quad p + q > m.$$  

Specialize to $M = V$ and let $p = n - 1$. Recall that $\Omega^{n-1} = K_V$, the canonical bundle on $V$. Then, we have

$$H^q(V, \mathcal{O}(K_V \otimes L)) = 0 \quad \text{for} \quad q > 0.$$  

Let $L = K^{-1}_M \otimes H^L$. This is positive because $K^{-1}_M$ and the hyperplane bundle $H$ are both positive. Hence, all $H^q(V, \mathcal{O}(H^L))$ vanish for $q > 0$, and

$$\chi(\mathcal{O}(H^L)) = \dim H^0(M, \mathcal{O}(H^L)).$$

This gives us our integral,

$$\dim(\mathcal{P}_L) = \dim H^0(V, \mathcal{O}(H^L)) = \chi(\mathcal{O}(H^L)) = \int_V ch(H^L)td(V).$$
The Chern character and Todd class are given by infinite series that begin as follows (see, for example [22, 38] and note that all higher Chern classes of $H$ vanish as it is a line bundle):

$$
\text{ch}(H) = 1 + c_1(H) + \frac{1}{2}c_1(H)^2 + \cdots
$$

$$
\text{td}(V) = 1 + \frac{1}{2}c_1(V) + \frac{1}{12}(c_1(V)^2 + c_2(V)) + \cdots
$$

From the adjunction formula we have

$$c_1(H) = \frac{1}{n+1-d}c_1(V).$$

Thus, everything can be expressed as integrals over various Chern classes of $V$. The key point to notice is that the series for $\text{ch}(H)$ contains only first Chern classes of $V$. Thus, when we multiply out the two series and take the $2(n-1)$th degree form, the answer will be of the form:

$$\dim(\mathcal{P}_L^F) = Q(L) \int c_1(V)^{n-1} + O(L^{n-2})$$

where $Q(L)$ is some $(n-1)$th degree polynomial in $L$ with rational coefficients. It is straightforward to compute the leading term in $L$ as it only comes from the expression for the Chern character. If we write $c_1(H) = x$, then, by definition, $\text{ch}(H) = e^x$. Therefore, $\text{ch}(H^L) = e^{Lx}$ and the $n$th order term is $L^n x^n/n!$. As $x = c_1(V)/(n+1-d)$, we can see that the leading term in $Q(L)$ must be $L^{n-1}/((n-1)!(n+1-d)^{n-1})$.

Now, recall formula (12) for the volume of $V$ in terms of the integral of Chern classes. We multiply this by the result for the length of the fiber to obtain

$$\text{Vol}(X) = \frac{2(n+1-d)}{(n-1)!} \left(\frac{\pi}{n}\right)^n \int c_1(V)^{n-1}. \quad (40)$$

Combining (10) with (39), we get

$$\dim(\mathcal{P}_L^F) \sim \frac{L^{n-1}}{2} \left(\frac{n}{\pi(n+1-d)}\right)^n \text{Vol}(X) \quad (41)$$

for large $L$. Thus, we have explained the observation (35) for all Einstein manifolds constructed from $F(z) = 0$ where $F(z)$ is a homogeneous polynomial. In the weighted case, we expect $n+1-d$ would be replaced by the index $|w| - d$. A similar explanation involving the generalized HRR theorem ought to hold for these more general examples.

5 Discussion

We have seen that given remarkably little knowledge about a cone, we can compute the volume of the base. If the cone is smooth except at the tip and admits a Calabi-Yau metric, we can usually compute the volume in terms of the dimension, the degree of the
defining polynomial and the index, \(|w| - d\). This statement is impressive considering how many cones of a given degree, index and dimension exist. For example, the \(A_k\) series from (23) in general also depends on \(k + 1\) variables, \(\xi_i\), but the volume is independent of these deformations. Essentially, we have shown that the volume can be determined almost solely in terms of topological numbers of the manifold. We have also seen that CPOs have an intriguing relationship to sections of a line bundle. It would be interesting to explore this relationship further and to see what additional elements of the gauge theory can be related to the topology of the base.

Given recent results about the existence of Kähler-Einstein metrics on various weighted projective varieties [39, 40] and Sasaki-Einstein manifolds over them [19, 20], there exists a vast new array of spaces on which to examine the AdS/CFT correspondence, extending the list of [10]. It would be very interesting to see if the techniques such as those used in this paper can be extended to give more information about the correspondence for these cones. For example, understanding in precise and quantitative detail the homology structure of the base would allow one to investigate fractional branes on these spaces.

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A The Integral of Chern Classes

We will now compute the integral of Chern classes [13]. We would like to thank János Kollár for giving us this argument. Any mistakes are our own.

Recall that the integral of a product of Chern classes of ordinary line bundles is simply the intersection numbers of the varieties defined by their zero loci. Thus, we would like to convert the integral into something we can compute in terms of intersections. If \(I = |w| - d\), then we are dealing with \(V\)-bundles that are the \(I\)th power of the hyperplane \(V\)-bundle. The Chern classes in our integral are defined in terms of the curvature of a connection on the bundle that lives on the uniformizing chart of the orbifold. Thus, following the usual arguments, it easy to see that, for any line \(V\)-bundle \(\mathcal{L}\), \(c_1(\mathcal{L}^n) = nc_1(\mathcal{L})\). This gives

\[
\int_V c_1(V)^{n-1} = I^{n-1} \int_V c_1(H)^{n-1}.
\]
We can convert this into an integral over the entire space by multiplying by the Poincaré dual of the hypersurface, $c_1(H^d) = dc_1(H)$, giving

$$\int_V c_1(V)^{n-1} = dI^{N-1} \int_{\mathbb{P}(w)} c_1(H)^n.$$

Unfortunately, $H$ is still not a line bundle. However, its $w$th power (recall that $w$ is the product of the weights) is one as all the weights divide $w$. Thus, we can compute

$$\int_{\mathbb{P}(w)} c_1(H)^n = \frac{1}{w^n} \int_{\mathbb{P}(w)} c_1(H^w)^n.$$

Now, we can relate this to the intersection number of a generic section of $H^w$. This is simple enough to compute explicitly. Let us examine the following sections of the line bundle:

$$z_{i}^{w/w_i} - z_{0}^{w/w_0}$$

where the $z_i$ are the weighted homogeneous coordinates on the weighted projective space. To determine the zero locus of this section, we first use the $\mathbb{C}^*$ action to set $z_0 = 1$. Then, we see that the solutions are given by $z_i = \zeta_{w/w_i,a}$ where $\zeta_{n,a}$, $a \in 0 \ldots n - 1$, are the $n$th roots of unity. Thus, the points on the common intersection of the zero locus of all these sections are given by

$$(1, \zeta_{w/w_1,a_1}, \ldots, \zeta_{w/w_n,a_n}).$$

As there are $w/w_i$ possibilities for each $a_i$, this gives $w_0 w^{n-1}$ solutions. However, we have overcounted because this choice of coordinates is not unique. In fact, we can act with $\zeta_{w_0,a_0}$ under the weighted $\mathbb{C}^*$ action and keep a one in the first position. Thus, we divide the number of solutions by the factor $w_0$ giving the total number of intersection points as $w^{n-1}$. Finally, we put this all together and obtain

$$\int_V c_1(V)^{n-1} = \frac{d}{w} I^{n-1}$$

which is what we sought to prove.

One can also see this result by looking at the projective cover of weighted projective space. Given our set of weights, we define the following new set of variables: $t_i = z_i^{1/w_i}$. Under the weighted action, the $t_i$ transform uniformly and, as such, are coordinates on the normal projective space $\mathbb{P}^n$. However, the map we have defined here is not 1-1. As we circle the origin in the plane of one of the $t_i$, we circle the origin in the $z_i$ plane $w_i$ times. Thus, in order to obtain the weighted projective space, we have to quotient by the group $\mathbb{Z}_{w_0} \times \cdots \times \mathbb{Z}_{w_n}$. The order of this group is simply the product of the weights, $w$. However, the degree of the cover is this value divided by the greatest common divisor of the weights because that is the order of the element $(1, \cdots, 1)$ in the group. In all the situations we will deal with, the gcd will be 1 as, otherwise, the weighted projective space would not be well-formed.

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If we now look at the inverse image of the hypersurface, it is a degree $d$ hypersurface in an ordinary projective space. Unfortunately the metric is no longer Kähler-Einstein. However, we can still write the volume as the integral of the $n$th power of a differential form. If one accepts that this is $I[\varpi]$ in cohomology where $\varpi$ is the pullback of the Kähler form of the Fubini-Study metric to the projective variety, then we can compute the volume of the covering hypersurface. Then, we divide by the degree of the cover, $w$, to give the same result as above.

**B The Volume of $V_{5,2}$**

We begin by deriving a Ricci-flat metric on the cone over $V_{5,2}$:

$$\sum_{i=1}^{5} w_i^2 = 0. \quad (42)$$

The $w_i$ are assumed to be complex variables. A reason we can even hope to find an explicit metric is that $V_{5,2}$ is a coset manifold, $SO(5)/SO(3)$. We proceed by generalizing an argument contained in [8]. Note, metrics on $V_{5,2}$ have appeared in [31, 32].

As (42) is symmetric under $SO(5)$ rotations of the five complex variables, we assume the Kähler potential $K$ also possesses this $SO(5)$ symmetry. There is only one $SO(5)$ invariant length in the problem,

$$\rho = \sum_{i=1}^{5} |w_i|^2,$$

and $K = f(\rho)$ can be a function only of $\rho$. Indeed as we will now see, we can further assume that $K = \rho^a/2$ because $K$ must transform homogeneously under the scaling $z_i \rightarrow \lambda z_i$ where $\lambda \in \mathbb{C}^\ast$. The factor of 1/2 is added for later convenience.

On a Calabi-Yau manifold of complex dimension 4, there exist nonvanishing holomorphic and antiholomorphic 4-forms whose wedge product is proportional to the volume form on the manifold. For our 4-fold, the holomorphic 4-form is

$$\Omega = \frac{dw_1 \wedge dw_2 \wedge dw_3 \wedge dw_4}{w_5}.$$

On the other hand, the volume form may also be computed from the wedge product of the Kähler form $\omega = \partial\bar{\partial}K$:

$$\omega \wedge \omega \wedge \omega \wedge \omega \sim \Omega \wedge \bar{\Omega}.$$

Counting powers, we find that $a = 3/4$. This gives us the metric on $V_{5,2}$:

$$g_{ij} = \partial_i \bar{\partial}_j \rho^{3/4}/2.$$
The next step is to rewrite the metric in a way that makes the volume calculation easier. In order to write the metric in conical form

\[ ds^2 = dr^2 + r^2 ds^2_{V_{5,2}} , \]

a scaling argument necessitates that we define the radius of the cone as \( r^2 = \rho^{3/4} \). Inspired by the change of variables needed to write the conifold as a cone over \( T^{1,1} \), we have found a change of variables that isolates the angular part of the metric. In particular

\[
egin{align*}
    w_1 &= \frac{1}{\sqrt{2}} r^{4/3} e^{i\psi} \left( \Lambda_+ \cos \left( \frac{\theta_1 + \theta_2}{2} \right) \cos \left( \frac{\phi_1 + \phi_2}{2} \right) + \Lambda_- \cos \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\phi_1 + \phi_2}{2} \right) \right) \\
    w_2 &= \frac{1}{\sqrt{2}} r^{4/3} e^{i\psi} \left( -\Lambda_+ \cos \left( \frac{\theta_1 + \theta_2}{2} \right) \sin \left( \frac{\phi_1 + \phi_2}{2} \right) + \Lambda_- \cos \left( \frac{\theta_1 - \theta_2}{2} \right) \cos \left( \frac{\phi_1 + \phi_2}{2} \right) \right) \\
    w_3 &= \frac{1}{\sqrt{2}} r^{4/3} e^{i\psi} \left( -\Lambda_+ \sin \left( \frac{\theta_1 + \theta_2}{2} \right) \cos \left( \frac{\phi_1 - \phi_2}{2} \right) + \Lambda_- \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\phi_1 - \phi_2}{2} \right) \right) \\
    w_4 &= \frac{1}{\sqrt{2}} r^{4/3} e^{i\psi} \left( -\Lambda_+ \sin \left( \frac{\theta_1 + \theta_2}{2} \right) \sin \left( \frac{\phi_1 - \phi_2}{2} \right) - \Lambda_- \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \cos \left( \frac{\phi_1 - \phi_2}{2} \right) \right) \\
    w_5 &= -\frac{1}{\sqrt{2}} r^{4/3} e^{i\psi} \sin(\alpha)
\end{align*}
\]

where

\[
\Lambda_\pm = \cos \alpha \cos \left( \frac{\beta}{2} \right) \pm i \sin \left( \frac{\beta}{2} \right).
\]

At the locus of points \( \alpha = 0 \) and \( \beta = 0 \), we recover the standard Euler angle parameterization on \( T^{1,1} \). The allowed ranges of the new set of variables are

\[
0 < r , \ 0 \leq \theta_i < \pi , \ 0 \leq \phi_i < 2\pi , \ 0 \leq \psi < 2\pi , \ 0 \leq \alpha < \pi/2 , \ 0 \leq \beta < 4\pi
\]

Define the one forms:

\[
\begin{align*}
    e^{\psi} &= (d\psi + \frac{1}{2} \cos \alpha (d\beta - \cos \theta_1 d\phi_1 - \cos \theta_2 d\phi_2)) \\
    e^{\beta} &= (d\beta - \cos \theta_1 d\phi_1 - \cos \theta_2 d\phi_2) \\
    e^{\phi_i} &= \sin \theta_i d\phi_i
\end{align*}
\]

Letting Maple handle the dirty work, we find that the metric on \( V_{5,2} \) may be written in angular coordinates as

\[
\begin{align*}
    ds^2_{V_{5,2}} &= \frac{9}{16} (e^{\psi})^2 + \frac{3}{8} \alpha^2 + \frac{3}{32} \sin^2 \alpha (e^{\beta})^2 \\
    &+ \frac{3}{32} (1 + \cos^2 \alpha)((e^{\phi_1})^2 + (e^{\phi_2})^2 + d\theta_1^2 + d\theta_2^2) + \frac{3}{16} \sin^2 \alpha \cos \beta e^{\phi_1} e^{\phi_2} \\
    &- \frac{3}{16} \sin^2 \alpha \cos \beta d\theta_1 d\theta_2 + \frac{3}{16} \sin^2 \alpha \sin \beta (d\theta_1 e^{\phi_2} + d\theta_2 e^{\phi_1}).
\end{align*}
\]

The determinant of this metric is

\[
\sqrt{\det g} = \frac{3^4}{2^{14}} \sin \alpha \cos^2 \alpha \sin \theta_1 \sin \theta_2.
\]
As this metric is the base of a Calabi-Yau cone, we expect it to be Einstein. In fact, $R_{ij} = 6g_{ij}$. Integrating the volume form, we find that the volume of $V_{5,2}$ is

$$\text{Vol}(V_{5,2}) = \frac{27}{128}\pi^4.$$ 

References

[1] J. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200.

[2] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B428 (1998) 105, hep-th/9802109.

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998) 253, hep-th/9802150.

[4] O. Aharony, S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, “Large N Field Theories, String Theory and Gravity,” Phys. Rept. 323 (2000) 183, hep-th/9905111.

[5] I.R. Klebanov, “TASI Lectures: Introduction to the AdS/CFT Correspondence,” hep-th/0009139.

[6] A. Lawrence, N. Nekrasov, and C. Vafa, “On conformal field theories in four-dimensions,” Nucl. Phys. B533 (1998) 199, hep-th/9803013.

[7] S. Kachru and E. Silverstein, “4-D conformal theories and strings on orbifolds,” Phys. Rev. Lett. 80 (1998) 4855, hep-th/9802183.

[8] I. R. Klebanov and E. Witten, “Superconformal field theory on three-branes at a Calabi-Yau singularity,” Nucl. Phys. B536 (1998) 199, hep-th/9807080.

[9] A. Kehagias, “New Type IIB Vacua and Their F-Theory Interpretation,” Phys. Lett. B435 (1998) 337, hep-th/9805131.

[10] D. Morrison and R. Plesser, “Non-Spherical Horizons, I,” Adv. Theor. Math. Phys. 3 (1999) 1, hep-th/9810201.

[11] L. Castellani, L. J. Romans, and N. P. Warner, “A Classification of Compactifying Solutions for d=11 Supergravity,” Nucl. Phys. B241 (1984) 429.

[12] S. B. Giddings, S. Kachru and J. Polchinski, “Hierarchies from fluxes in string compactifications,” hep-th/0105097.

[13] F. Cachazo, K. Intriligator and C. Vafa, “A large N duality via a geometric transition,” Nucl. Phys. B603 (2001) 3, hep-th/0103067.
[14] S. S. Gubser, “Einstein Manifolds and Conformal Field Theories,” Phys. Rev. D 59 (1999) 025006, hep-th/9807164.

[15] S. G. Gubser, N. Nekrasov, and S. Shatashvili, “Generalized Conifolds and Four Dimensional $\mathcal{N} = 1$ Superconformal Theories,” JHEP 9905 (1999) 003, hep-th/9811230.

[16] E. Lopez, “A family of $N = 1$ SU(N)**k theories from branes at singularities,” JHEP 9902, 019 (1999) hep-th/9812028.

[17] I. Dolgachev, “Weighted Projective Varieties” in Proceedings, Group Actions and Vector Fields (1981) LNM 956, 34-71.

[18] A. R. Fletcher, “Working with Weighted Complete Intersections”, Preprint MPI/89-85, available at http://cbel.cit.nih.gov/~arif/CV.html.

[19] C. P. Boyer and K. Galicki, “New Einstein Metrics in Dimension Five,” math.dg/0003174.

[20] C. P. Boyer, K. Galicki and M. Nakamaye, “On the Geometry of Sasakian-Einstein 5-Manifolds,” math.dg/0012047.

[21] C. P. Boyer and K. Galicki, “On Sasakian-Einstein Geometry,” Int. J. Math. 11 (2000) 873, math.dg/9811093.

[22] M. Nakahara, Geometry, Topology and Physics, Institute of Physics Publishing, 1990.

[23] A. Haefliger, “Groupoides d’holonomie et classifiants”, Astérisque 116 (1984), 70-97.

[24] C. P. Boyer and K. Galicki, “3-Sasakian Manifolds,” Surveys in Differential Geometry: essays on Einstein Manifolds, 123–184, Surv. Diff. Geom., VI, Int. Press, Boston, MA, 1999. hep-th/9810250.

[25] I. Satake, “On a Generalization of the Notion of Manifold”, Proc. Nat. Acad. Sci. USA 42 (1956) 359-363.

[26] I. Satake, “The Gauss-Bonnet Theorem for V-Manifolds”, J. Math. Soc. Japan V. 9 No. 4 (1957) 464-476.

[27] W. L. Baily, “The Decomposition Theorem for V-Manifolds”, Amer. J. Math. 78 (1956) 862-888.

[28] W. L. Baily, “On the Imbedding of V-Manifolds in Projective Space”, Amer. J. Math. 79 (1957) 403-430.

[29] T. Friedrich and I. Kath, “Einstein Manifolds of Dimension Five with Small First Eigenvalue of the Dirac Operator”, J. Diff. Geom 29 (1989) 263-279.
[30] T. Friedrich and I. Kath, “Compact Seven-Dimensional Manifolds with Killing Spinors”, *Comm. Math. Phys.* **133** (1990) 263-279.

[31] M. Cvetić, G. W. Gibbons, H. Lu and C. N. Pope, “Ricci-flat metrics, harmonic forms and brane resolutions,” [hep-th/0012011](http://arxiv.org/abs/hep-th/0012011).

[32] M. B. Stenzel, “Ricci-flat metrics on the complexification of a compact rank one symmetric space,” *Manuscripta Mathematica* **80** (1993) 151.

[33] M. Douglas and G. Moore, “D-Branes, Quivers and ALE Instantons,” [hep-th/9603167](http://arxiv.org/abs/hep-th/9603167).

[34] M. Henningson and K. Skenderis, “The holographic Weyl anomaly,” JHEP **9807**, 023 (1998) [hep-th/9806087](http://arxiv.org/abs/hep-th/9806087).

[35] D. Anselmi, D. Z. Freedman, M. T. Grisaru, and A. A. Johansen, “Nonperturbative Formulas for Central Functions of Supersymmetric Gauge Theories,” *Nucl. Phys.* **B526** (1998) 543, [hep-th/9708042](http://arxiv.org/abs/hep-th/9708042).

[36] T. Kawasaki, “The Riemann-Roch Theorem for complex V-Manifolds”, *Osaka J. Math* **16** (1979) 151-159.

[37] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley and Sons, 1994.

[38] C. Nash, *Differential Topology and Quantum Field Theory*, Academic Press, 1991.

[39] J. M. Johnson and J. Kollár, “Kähler-Einstein Metrics on log del Pezzo Surfaces in Weighted Projective 3-spaces,” *Ann. Inst. Fourier (Grenoble)* **51** (2001), no. 1 69–79, [math.AG/0008129](http://arxiv.org/abs/math.AG/0008129).

[40] J. M. Johnson and J. Kollár, “Fano Hypersurfaces in Weighted Projective 4-spaces,” *Experiment. Math.* **10** (2001), no. 1, 151–158, [math.AG/0008189](http://arxiv.org/abs/math.AG/0008189).