Complex paths for regular-to-chaotic tunnelling rates

N. Mertig$^{1,2}$, S. L"ock$^{1,2,3}$, A. B"acker$^{1,2}$, R. Ketzmerick$^{1,2}$ and A. Shudo$^{2,4}$

$^1$ Institut f"ur Theoretische Physik, Technische Universit"at Dresden - 01062 Dresden, Germany, EU
$^2$ Max-Planck-Institut f"ur Physik komplexer Systeme - N"othnitzer Straße 38, 01187 Dresden, Germany, EU
$^3$ OncoRay - National Center for Radiation Research in Oncology, Technische Universit"at Dresden Fetscherstraße 74, 01307 Dresden, Germany, EU
$^4$ Department of Physics, Tokyo Metropolitan University - Minami-Osawa, Hachioji, Tokyo 192-0397, Japan

received 30 January 2013; accepted in final form 21 March 2013
published online 22 April 2013

PACS 05.45.Mt – Quantum chaos; semiclassical methods
PACS 03.65.Sq – Semiclassical theories and applications

Abstract – For generic non-integrable systems we show that a semiclassical prediction of tunnelling rates between regular and chaotic phase-space regions is possible. Our prediction is based on complex paths which can be constructed despite the obstacle of natural boundaries. The semiclassically obtained tunnelling rates are in excellent agreement with numerical tunnelling rates for the standard map where few complex paths dominate. This gives a semiclassical foundation of the long-conjectured and often-observed exponential scaling with Planck’s constant of regular-to-chaotic tunnelling rates.

Copyright © EPLA, 2013

Tunnelling is a fascinating manifestation of wave mechanics, which allows waves to explore regions, that are inaccessible to classical trajectories. In textbooks on quantum mechanics [1], this phenomenon is paradigmatically treated for potential barriers using the WKB method: Here, a metastable state which localizes on a torus of energy $E$ that is classically confined in a potential well, tunnels through the barrier with a rate

$$\gamma \propto \exp\left(-\frac{2\text{Im}S}{\hbar}\right).$$

This rate depends exponentially on Planck’s constant $\hbar = 2\pi \hbar$ and the classical action $S = \int p(q) dq$ of a single complex path. This path goes through the barrier region, where the potential $V(q)$ is higher than the energy $E$ of the confined state such that the momentum $p(q) = \sqrt{2m(E - V(q))}$ is imaginary. This gives a path $(q, p(q))$ in the complexified phase space which is situated on the complexified torus of energy $E$.

In contrast to this integrable barrier tunnelling, non-integrable systems typically exhibit dynamical tunnelling [2,3] between classically disjoint regions of regular and chaotic motion in phase space. This regular-to-chaotic tunnelling is the essential building block of chaos-assisted tunnelling [4]. It is important for current applications in atomic and molecular physics [5–7], ultracold atoms [8–10], optical cavities [11–15] and microwave resonators [16–18]. Here, the rate $\gamma$ of regular-to-chaotic tunnelling describes the inverse life-time of metastable states, initialized on a regular torus to the chaotic region. This rate not only determines the mean tunnel splittings in chaos-assisted tunnelling but is also needed to describe tunnelling effects on spectral statistics [19–22] and the structure of eigenfunctions [23–25] in generic non-integrable systems.

In many applications regular-to-chaotic tunnelling rates can be determined numerically. Moreover, quantum-mechanical predictions exist and show [26] for the regime $\hbar \lesssim A$, where Planck’s constant $\hbar$ is smaller but comparable to the area of the regular region $A$, that these rates arise from direct regular-to-chaotic tunnelling [11,27,28], while for $\hbar \ll A$ resonance-assisted tunnelling [29–31] occurs additionally. A deep understanding of regular-to-chaotic tunnelling, however, requires a semiclassical theory similar to eq. (1). Such theories exist for integrable [1,32,33] and near-integrable systems [34–39] only.

For generic non-integrable systems it is conjectured [40] that a complex-path prediction like eq. (1) should exist. This conjecture was put forward by Hanson, Ott and Antonsen in 1984 to explain the exponential $h$-scaling of numerical tunnelling rates in the standard map. Such scaling was numerically observed for cavities [11,12,16,18], ultracold atoms [10] and quantum maps [28] in the regime $\hbar \lesssim A$. A semiclassical theory, however, still does not...
exist. Up to now it is not even clear where to search for such a complex path: i) WKB-like methods are bound to fail, as pointed out in refs. [34,35,39]. This is due to natural boundaries [41,42] at which the complexification of regular tori ends, see fig. 1. ii) While tunnelling contributions in long-time propagators can be predicted successfully using time-evolution paths in complex phase space [43,44], this approach has not been used to discuss tunnelling rates.

In this paper a semiclassical prediction of regular-to-chaotic tunnelling rates \( \gamma \) for generic non-integrable systems is achieved in the regime \( h \lesssim A \), i.e. in the experimentally relevant regime where \( \gamma \) is large. For this prediction we use specific complex paths, whose construction overcomes the fundamental difficulties caused by natural boundaries. Our prediction, eq. (7), formally resembles eq. (1), but is based on complex paths which contain a time-evolution segment in addition to WKB-like segments, see fig. 1. We demonstrate how to predict tunnelling rates from our approach and find excellent agreement to numerical tunnelling rates for the standard map, see fig. 3, where few complex paths dominate. These results provide a semiclassical foundation of the conjectured [40] exponential \( h \)-scaling of regular-to-chaotic tunnelling rates.

We now derive our complex-path prediction for regular-to-chaotic tunnelling rates \( \gamma \), eq. (7). For simplicity the presentation focuses on time-periodic quantum systems, described by a time-evolution operator \( \hat{U} \) over one period of the driving. The corresponding classical system is a symplectic map \( U \) in phase space \( (q,p) \). The starting point of our derivation is the quantum prediction of regular-to-chaotic tunnelling rates [27,28]

\[
\gamma_m = \sum_{I_{ch}\geq I_0} |\langle I_{ch}|\hat{U}|I_m\rangle|^2.
\]

Here, \( \gamma_m \) is given by the quantum-mechanical probability transfer during one time period between the regular basis state \( |I_m\rangle \) to all basis states \( |I_{ch}\rangle \) of the chaotic region. As discussed in detail in ref. [28], it is essential that the basis states are not the eigenstates of \( \hat{U} \), since those predominantly concentrate in either the regular or the chaotic region but also have tunnelling admixtures in the complementary region. To provide the basis states one can, e.g., use the eigenstates of a fictitious integrable system \( H_{\text{reg}}(q,p) \), which resembles the regular region of the non-integrable system \( U \) as closely as possible (ignoring resonance chains) and extends it beyond its boundary, see fig. 2(c). As these eigenstates localize on tori with quantised action \( I_m = \frac{1}{\pi} \oint p dq = \hbar (m + \frac{1}{2}) \), we can distinguish between regular basis states \( |I_m\rangle \) with \( I_m < I_0 \) and chaotic basis states \( |I_{ch}\rangle \) with \( I_{ch} \equiv I_m \geq I_0 \). Here, \( I_0 \) is the (not necessarily quantised) action of the first torus of \( H_{\text{reg}} \) that is entirely located outside the regular region of \( U \), see fig. 2. Starting with eq. (2), we derive a semiclassical prediction of \( \gamma \), eq. (7), using standard semiclassical methods [45,46] for complex paths. To this end we generalize the complex-path technique for propagators [43] to the tunnelling matrix elements \( \langle I_{ch}|\hat{U}|I_m\rangle \). Choosing the position representation gives

\[
\langle I_{ch}|\hat{U}|I_m\rangle = \int dq' \int dq \langle I_{ch}|q'\rangle \langle q'|\hat{U}|q\rangle \langle q|I_m\rangle.
\]

To transform this into a semiclassical expression the initial basis state \( \langle q|I_m\rangle \) and the final basis state \( \langle I_{ch}|q'\rangle \) are replaced by WKB-like states \([1,32,33,47]\) on the quantizing tori \( I_m \) and \( I_{ch} \). The propagator \( \langle q'|\hat{U}|q\rangle \) is also expressed semiclassically [43,45]. The arising integrals over \( q \) and \( q' \) are evaluated by a saddle-point method [46]. This results in semiclassical tunnelling matrix elements

\[
\langle I_{ch}|\hat{U}|I_m\rangle = \sum_{\nu} \sqrt{\frac{\hbar}{2\pi}} \frac{\partial^2 S_{\nu}(I_{ch}, I_m)}{\partial I_{ch} \partial I_m} \exp \left( i \frac{S_{\nu}(I_{ch}, I_m)}{\hbar} + i \phi_{\nu} \right).
\]
They are constructed from classical paths $\nu$, with action $S_{\nu}$ and a Maslov-phase shift $\phi_{\nu}$. The paths $\nu$ have to connect the initial torus $I_m$ to the final torus $I_{ch}$. Since there are no such paths in real phase space, see fig. 2, the above propagator is exclusively constructed from paths in the complexified phase space. Such a complex path $\nu$ consists of three segments: i) The first segment originates from the WKB-like eigenfunction $(q|U|q)$. It is the curve $C_{m,\nu}$ on the analytic continuation of the initial torus $I_m$ into the complexified phase space. This curve connects a reference point of the real torus $I_m$ to a specific point $(q_{\nu},p_{\nu})$ whose location is determined by the next segment. ii) The second segment originates from the semiclassical propagator corresponding to $(q'|U|q)$. This time-evolution segment is obtained by once applying the complexified map $U$ of the non-integrable system. It has to connect $(q_{\nu'},p_{\nu'})$ on the initial complexified torus $I_m$ to a point $(q_{\nu''},p_{\nu''})$ on the final complexified torus $I_{ch}$. This requirement determines the possible endpoints $(q_{\nu},p_{\nu})$ of the first segment. iii) The final segment originates from the WKB-like eigenfunction $(I_{ch}|q|)$. It is the curve $C_{ch,\nu}$ along the complexified final torus $I_{ch}$. It connects the point $(q_{\nu''},p_{\nu''})$ to a reference point of the real final torus $I_{ch}$. The complex parts of the three segments are sketched in fig. 1 (green curves). Their explicit construction will be described below.

The action $S_{\nu}$ of such complex paths is given by

$$S_{\nu}(I_{ch}, I_m) = \int_{C_{m,\nu}} p(q) \, dq + S^U_{\nu}(q_{\nu}, q_{\nu'}) + \int_{C_{ch,\nu}} p(q) \, dq.$$  \hspace{1cm} (5)$$

Here, the first and the last contribution originate from the WKB-like segments i) and iii), respectively. The action $S^U_{\nu}(q_{\nu}, q_{\nu'})$ originates from the time-evolution segment ii). From the possibly infinite number of paths, which contribute to the sum in eq. (4), we select the dominant ones which have the smallest positive imaginary action. Note, that this is sufficient for our purposes, while in general a Stokes analysis [32,33,48] would be required. Using these paths $\nu$ in eq. (4) and summing over $I_{ch}$ according to eq. (2) gives a semiclassical prediction of the tunnelling rates

$$\gamma_m = \sum_{I_{ch} \geq b} \left| \sum_{\nu} \sqrt{\frac{h}{2\pi}} \frac{\partial^2 S_{\nu}(I_{ch}, I_m)}{\partial I_{ch} \partial I_m} \exp \left( i \frac{S_{\nu}(I_{ch}, I_m)}{\hbar} + i \phi_{\nu} \right) \right|^2.$$  \hspace{1cm} (6)$$

We evaluate this expression further in terms of a diagonal approximation for the modulus $|...|^2 = \sum_{\nu} h/2\pi |\partial I_{ch} |\partial I_m S_{\nu}| \exp (-2 \text{Im} S_{\nu}/\hbar)$. This approximation assumes that the cross terms cancel because of random interference between different paths.\footnote{The diagonal approximation is typically applicable for a large number $N$ of paths. For a small number $N$ of paths the Jensen inequality ensures that the diagonal approximation at worst underestimates the tunneling rate, eq. (6), by the small factor $N$. Note that destructive interference of different paths, which would lead the diagonal terms over the quantizing actions $I_{ch}$ still requires to determine new complex paths to each of the quantizing tori $I_{ch}$. This effort can be drastically reduced by approximating the sum over discrete quantizing actions $I_{ch}$ by an integral over continuous $I_{ch}$, giving $\gamma_m = \sum_{\nu} \int_{b}^{\infty} dI_{ch}/2\pi |\partial I_{ch} |\partial I_m S_{\nu}| \exp (-2 \text{Im} S_{\nu}/\hbar)$. Numerically, we observe for various systems that the integrand is maximal at the boundary where $I_{ch} = b$ and decreases exponentially with increasing $I_{ch}$, i.e. $\partial I_{ch} \text{Im} S_{\nu}(I_{ch}, I_m) > 0$. Under this assumption we linearize the action in the exponential and evaluate the edge contribution of the integral. In contrast to eq. (6) only paths to the boundary torus $I_b$ are needed and the Maslov phase $\phi_{\nu}$ is no longer required. Our final semiclassical result for regular-to-chaotic tunnelling rates is}

$$\gamma_m = \sum_{\nu} \frac{\hbar}{4\pi} \text{Im} \left( \frac{\partial^2 S_{\nu}(I_{ch}, I_m)}{\partial I_{ch} \partial I_m} \right) \exp \left( -\frac{2 \text{Im} S_{\nu}(I_b, I_m)}{\hbar} \right).$$  \hspace{1cm} (7)$$

The main advance of this result is that a construction of complex paths for regular-to-chaotic tunnelling rates in non-integrable systems is now possible. These paths connect WKB-like segments i) and iii) on the complexified tori $I_m$ and $I_{ch}$ via a time-evolution segment ii) of the non-integrable system $U$; see fig. 1. With these paths eq. (7) provides a semiclassical foundation of the long-conjectured exponential $\hbar$-scaling of regular-to-chaotic tunnelling rates. This generalizes the WKB-prediction (1) of integrable systems to non-integrable systems. The essential point that allows for the first time to find complex paths for regular-to-chaotic tunnelling rates in non-integrable systems is the use of approximate complexified tori $I_m$ and $I_b$. These tori do not have natural boundaries, such that the WKB-segment i) can be extended sufficiently deep into complexified phase space to provide a time-evolution segment ii) which maps to $I_b$. This would not be possible with the corresponding invariant tori of the non-integrable system $U$, because there the WKB-segment i) can only be extended up to the natural boundary and the time-evolution segment ii) would remain on $I_m$. For WKB-paths in near-integrable systems similar ideas to overcome natural boundaries were used [29,36,38,39]. Note that also for a quantum-mechanical evaluation of eq. (2) it is essential to use basis states $|I_m\rangle$ and $|I_{ch}\rangle$, e.g. eigenstates of an approximate integrable system $H_{reg}$ and not the eigenstates of $\hat{U}$ [27,28]. This exactly corresponds to our semiclassical use of the approximate tori $I_m$ and $I_b$. For applying our prediction, eq. (7), one has to A) construct the complexified tori $I_m$ and $I_b$, e.g. by using a fictitious integrable system $H_{reg}$; B) find the complex paths $\nu$ between $I_m$ and $I_b$ composed of the segments i)–iii) and C) compute their action $S_{\nu}$ using eq. (5) to smaller tunneling rates, is not accounted for by the diagonal approximation. In the semiclassical limit, for which just the path with the smallest imaginary action dominates, the diagonal approximation becomes exact.
and select the dominant paths, which have the smallest positive imaginary action.

We now discuss the successful predictions of eq. (7) for the paradigmatic example of non-integrable systems, the standard map [49]. It is obtained from the one-dimensional kicked Hamiltonian $H(q,p) = T(p) + V(q)\sum_n\delta(t-n)$ with $T(p) = p^2/2$ and $V(q) = \kappa/(2\pi)\cos(2\pi q)$. The stroboscopic time evolution is given by the map $U: q' = q + T'(p)$; $p' = p - V'(q')$. The corresponding quantum time-evolution operator $\hat{U}$ is given by $\hat{U} = \exp(-iV(q)/\hbar)\exp(-iT(p)/\hbar)$ [50,51], where $\hbar = 2\pi\hbar$ is the effective Planck constant. At $\kappa = 2.9$ we determine the numerical tunnelling rates $\gamma_m$ (see dots in fig. 3) by introducing an absorber in position space ([28], sect. (III.A)) tangential to the regular region (inset in fig. 3). The semiclassical prediction according to eq. (7) lines) accurately describes the numerical tunnelling rates (within a factor of 2). The deviations at small $h$ (and small $\gamma$) occur due to the onset of resonance-assisted tunnelling [26,29–31]. Beyond the natural boundary our construction A), B) gives a huge number of paths. In step C) we select four paths (and their symmetry partners) which dominantly contribute to eq. (7), see fig. 1. Paths which have less than 5% contribution to the tunnelling rate are neglected. The four contributing paths have comparable imaginary actions $\text{Im} S_\nu$, which explains the exponential $h$-scaling of the numerical tunnelling rates. Generically a regular island is surrounded by a hierarchical region with partial barriers [4], which leads to a small additional decrease of regular-to-chaotic tunnelling rates. For the standard map at $\kappa = 2.9$ this is observed in the numerical tunnelling rates, when placing the absorber at the boundary of the hierarchical region. Our semiclassical prediction can describe this effect successfully by choosing the boundary torus $I_b$ tangential to the hierarchical region [52]. Note, that semiclassical predictions of similar quality can also be obtained for other approximations $H_{\text{reg}}$ or other parameters $\kappa$ [52].

Finally, we explain the construction steps A)–C): A) We start by constructing a fictitious integrable system $H_{\text{reg}}(q,p)$, fig. 2(c), based on the frequency analysis for the main regular island ([28], sect. (II.B.2)). Since $H_{\text{reg}}(q,p)$ is integrable, there exists a canonical transformation to action-angle coordinates $(I,\theta)$, see fig. 2, which we will exploit numerically. We choose a reference point $(q_i,p_i)$ for each torus, in case of the standard map $q_i \geq 0$, $p_i = 0$. Its action $I(q_i) := \oint p dq/(2\pi)$ and frequency $\omega(q_i)$ are calculated by numerically integrating Hamilton’s equations for $H_{\text{reg}}(q,p)$. The boundary torus $I_b$ is determined by searching for the smallest $q_b$ such that the corresponding torus of $H_{\text{reg}}$ encloses the regular region of the non-integrable system $U$. B) To search for the complex paths $\nu$, we use a shooting algorithm giving segments i) and ii), such that the endpoint of segment ii) is located on the torus $I_b$. For segment i) the torus $I_m$ is analytically continued by complexifying the angle $\theta$, which parametrizes the initial torus. This gives a plane in the complexified action-angle representation, see fig. 4. To obtain the $(q,p)$-coordinates for a given $\theta$, we numerically integrate Hamilton’s equations for $H_{\text{reg}}(q,p)$ up to complex time $t = \theta/\omega_m$ starting from the reference point of $I_m$ first in real and then in imaginary time. For segment ii) we map the point $q''(\theta), p''(\theta)$ with the complexified map $U$ to a point $q'(\theta), p'(\theta)$. This point is on the complexified boundary torus $I_b$, only if its complex final energy $E''(\theta) := H_{\text{reg}}(q''(\theta), p''(\theta))$ is equal to the real energy $E_b$ of the boundary torus $I_b$. Therefore the final step of the shooting algorithm is a numerical root search.
of $E'(\theta) - E_b$ giving the angles $\theta_i$, see white dots in fig. 4 and thus the initial point $q_i(\theta_i), p_i(\theta_i)$ and the final point $q'(\theta_i), p'(\theta_i)$ of segment ii). To find a good initial guess for $\theta_0$ we visualize $E'(\theta) - E_b$ on the complexified initial torus, see the grey lines and the boundary between red and blue areas in fig. 4. For segment iii) we numerically integrate Hamilton’s equations for $H_{\text{reg}}$, first from the point $q_0', p_0'$ in negative imaginary time until the curve has reached real $q, p$-coordinates and then in negative real time until the curve has reached the reference point of the torus $I_b$. Combining segments i–iii) gives the path $\nu$. C) Its action $S_{\nu}$ is evaluated according to eq. (5). For the standard map the second contribution is given by $S^U(q_0', p_0') = (q_0 - q_0')^2/2 - V(q_0')$ [50,51]. The first and the third contribution of eq. (5) are obtained by a numerical integration. We select the dominant paths $\nu$ with the smallest positive imaginary action, see arrows in fig. 4. We observe that the dominant paths $\nu$ have the smallest imaginary angle. Specifically for the standard map we incorporate the parity by choosing one symmetry partner and doubling its contribution in eq. (4), which leads to an additional factor of four in eq. (7). For $\gamma_1$ at $h = 1/50$ the four dominant paths $\nu$ contribute 20%, 6%, 33% and 41% (left to right) to the tunnelling rate.

For higher-dimensional systems the tunnelling rate $\gamma_{\nu}$ is still given by complex paths from a torus $I_b$ to tori $I_{\nu}$. This follows from a generalization of eq. (6) with the prefactor replaced by the determinant of the stability matrix. However, several new challenges arise:

a) Regular regions are formed by a collection of regular tori interspersed by the Arnold web [49,53]. Even when ignoring the complications due to the Arnold web, the boundary of the regular region is now given by a family of boundary tori. b) The construction of approximate tori requires new methods. c) The search algorithm for complex paths has to be generalized to higher dimensions. d) It is not clear, if a simplification analogous to eq. (7) is possible.

In summary, we have presented a complex-path construction which overcomes natural boundaries and allows for predicting regular-to-chaotic tunnelling rates of non-integrable systems. We have successfully applied this method to predict tunnelling rates of the standard map, where few paths dominate. In the future it is desirable to include resonance-assisted tunnelling into our approach. Finally, we believe that our complex-paths prediction will be important for higher-dimensional systems like billiards, optical microcavities, atoms and molecules.

***

We thank S. Creagh, P. Schlagheck and S. Tomsovic for comments on the manuscript and inspiring discussions, also with K. Ikeda and the participants of the Advanced Study Group “Towards a Semi-classical Theory of Dynamical Tunnelling”. We further thank M. Richter for help with the Mayavi [54] based 3D visualizations. We acknowledge support from the DFG within Forschergruppe 760 “Scattering Systems with Complex Dynamics”. AB thanks the Japan Society for the Promotion of Science for supporting his stay in Japan.

REFERENCES

[1] Landau L. D. and Lifshitz E. M., Course of Theoretical Physics, Vol. 3: Quantum Mechanics (Pergamon Press, New York) 1991; Merzbacher E., Quantum Mechanics (Wiley, New York) 1998.
[2] Davis M. J. and Heller E. J., J. Chem. Phys., 75 (1981) 246.
[3] Keshavamurty S. and Schlagheck P., Dynamical Tunnelling: Theory and Experiment (CRC Press, New York) 2011.
[4] Bohigas O., Tomsovic S. and Ullmo D., Phys. Rep., 223 (1993) 43; Tomsovic S. and Ullmo D., Phys. Rev. E, 50 (1994) 145.
[5] Zakrzewski J., Delande D. and Buchleitner A., Phys. Rev. E, 57 (1998) 1458.
[6] Buchleitner A., Delande D. and Zakrzewski J., Phys. Rep., 368 (2002) 409.
[7] Wimberger S., Schlagheck P., Eltschka C. and Buchleitner A., Phys. Rev. Lett., 97 (2006) 043001.
[8] Hensinger W. K. et al., Nature, 412 (2001) 52.
[9] Steck D. A., Oskay W. H. and Raizen M. G., Science, 293 (2001) 274.
[10] Mouchet A., Miniatura C., Kaiser R., Grémaud B. and Delande D., Phys. Rev. E, 64 (2001) 016221.
[11] Podolsky V. A. and Narimanov E. E., Phys. Rev. Lett., 91 (2003) 263601.
[12] Bäcker A., Ketzmerick R., Löck S., Wiersig J. and Hentschel M., Phys. Rev. A, 79 (2009) 063804.
[13] Shinohara S., Harayama T., Fukushima T., Hentschel M., Sasaki T. and Narimanov E. E., Phys. Rev. Lett., 104 (2010) 163902.
[14] Yang J., Lee S.-B., Moon S., Kim S. W., Daq T. A., Lee J.-H. and An K., Phys. Rev. Lett., 104 (2010) 243601.
[15] Song Q., Ge L., Redding B. and Cao H., Phys. Rev. Lett., 108 (2012) 243902.
[16] Doron E. and Frischat S. D., Phys. Rev. Lett., 75 (1995) 3661.
[17] Dembowski C., Gräf H.-D., Heine A., Hoffertber R., Rehfeld H. and Richter A., Phys. Rev. Lett., 84 (1995) 2000.
[18] Bäcker A., Ketzmerick R., Löck S., Robnik M., Vidmar G., Höhmann R., Kuhl U. and Stöckmann H.-J., Phys. Rev. Lett., 100 (2008) 174103.
[19] Vidmar G., Stöckmann H.-J., Robnik M., Kuhl U., Höhmann R. and Grossmann S., J. Phys. A, 40 (2007) 13883.
[20] Batistic B. and Robnik M., J. Phys. A, 43 (2010) 215101.
[21] Bäcker A., Ketzmerick R., Löck S. and Mertig N., Phys. Rev. Lett., 106 (2011) 024101.
[22] Rudolf T., Mertig N., Löck S. and Bäcker A., Phys. Rev. E, 85 (2012) 036213.
[23] Hufnagel L., Ketzmerick R., Otto M.-F. and Schanz H., Phys. Rev. Lett., 89 (2002) 154101.
