Conservation laws for dynamical black holes

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An essentially complete new paradigm for dynamical black holes in terms of trapping horizons is presented, including dynamical versions of the physical quantities and laws which were considered important in the classical paradigm for black holes in terms of Killing or event horizons. Three state functions are identified as surface integrals over marginal surfaces: irreducible mass, angular momentum and charge. There are three corresponding conservation laws, expressing the rate of change of the state function in terms of flux integrals, or equivalently as divergence laws for associated conserved currents. The currents of energy and angular momentum include the matter energy tensor in a physically appropriate way, plus terms attributable to an effective energy tensor for gravitational radiation. Four other state functions are derived: an effective energy, surface gravity, angular speed and electric potential. There follows a dynamical version of the so-called first law of black-hole mechanics. A corresponding zeroth law holds for null trapping horizons.

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Introduction. Black holes are now generally regarded as astrophysical realities, which are expected to be major sources of gravitational waves, prompting extensive studies of dynamical, strong-field processes such as binary mergers. The textbook theory of black holes, however, mostly concerns stationary black holes or physically unlocatable event horizons [1, 2]. In recent years, a new paradigm for dynamical black holes has been developed in terms of trapping horizons [3, 4, 5, 6, 7, 12], hypersurfaces where light is momentarily caught by the gravitational field, which locate the black hole in a practical way. For physical reasons, unique measures of mass and angular momentum for such locally defined black holes are desired, together with conservation laws describing how they change in terms of the fluxes of energy and angular momentum of the infalling matter and gravitational radiation, so as to describe how a black hole grows and spins up or down.

This Letter reports a generically unique definition of angular momentum, obtained directly from the Komar integral [13], satisfying a conservation law with a similar form to the energy conservation law [7]. Adding charge conservation for generality, this allows general definitions of all the key physical quantities of the classical paradigm, plus dynamical versions of the so-called first and zeroth laws of black-hole mechanics [1]. A preliminary report was given previously [11] and a more detailed description is given in a longer article [12].

Geometry. General Relativity will be assumed, with space-time metric $g$. A one-parameter family of spatial surfaces $S$ locally generates a foliated hypersurface $H$. Labelling the surfaces by a coordinate $x$, they are generated by a vector $\xi = \partial/\partial x$, which can be taken to be normal to the surfaces, $\bot \xi = 0$, where $\bot$ denotes projection onto $S$. A duality operation on normal vectors $\eta$, $\bot \eta = 0$, yields a dual normal vector $\eta^*$ defined by $\bot \eta^* = 0$, $g(\eta^*, \eta) = 0$, $g(\eta^*, \eta^*) = -g(\eta, \eta)$, and in particular $\tau = \xi^*$ is normal to $H$ (Fig 1). The coordinate freedom is $x \mapsto \tilde{x}(x)$ and choice of angular coordinates on $S$, under which all the key formulas will be invariant.

The expansion $\theta_\eta = L_\eta \log(d^2A)$ along a normal vector $\eta$, where $L$ denotes the Lie derivative and $d^2A$ the area form of $S$, can be expressed in terms of the expansion 1-form $\theta = d \log(d^2A)$ as $\theta_\eta = \theta(\eta)$. It is convenient to use two future-pointing null normal vectors $l^+$ and $l^-$ to $S$, $g(l^+, l^+) = 0$, $\bot l^\pm = 0$, since their directions are unique. Then a normal vector $\eta$ has components $\eta^\pm$ along $l^\pm$, in particular $\xi = \xi^+ l^+ + \xi^- l^-$, $\tau = \xi^+ l^+ - \xi^- l^-$ and $g^{-1}(\eta) = -e^\tau (\theta_{-l^+} + \theta_{+l^-})$, where $f$ is a normalization function defined by $e^{-f} = -g(l^+, l^-)$ and $\theta_{\pm l^\pm}$ are the null expansions. One may adapt $l^\pm$ to $H$ via $l_A(dx^B) = \delta_A^B$, where $x^\pm$ are coordinates labelling the null hypersurfaces generated from $S$ in the normal directions $[3, 4, 5, 6, 7, 12]$.

A trapping horizon [3, 4, 7] is a hypersurface $H$ foliated by marginal surfaces, where $S$ is marginal if one of the null expansions, $\theta_+ or \theta_-$, vanishes everywhere on $S$. A confusing multiplicity of names have been proposed for trapping horizons under various extra conditions, but such conditions will be largely irrelevant here, since all the equations and results, except where specifically noted, hold for any trapping horizon with compact $S$.

Angular momentum. The standard definition of angular momentum for an axial Killing vector $\psi$ and at spatial infinity is the Komar integral [13]

$$J[\psi] = -\frac{1}{16\pi} \oint_S \epsilon_{\alpha\beta} \nabla^\alpha \psi^\beta d^2A$$

(1)

where $\epsilon$ is the binormal to $S$ and Newton’s constant is set.
to unity. Here $(\alpha, \beta \ldots)$ denote general indices, $(a, b \ldots)$ will denote transverse indices and $(A, B \ldots) = \pm$ will denote normal indices, e.g., $\epsilon^{AB} = e^J(1_{+}^{AB} - 1_{-}^{AB})$.

Now consider $\psi$ to be a general transverse vector, $\perp \psi = \psi$ (Fig. 2). Since $\epsilon_{\alpha\beta} \psi^\beta = 0$, the Komar integral can be rewritten as

$$J[\psi] = \frac{1}{8\pi} \oint_S \psi^a \omega_a d^2A$$

(2)

where the twist $\omega_a = \frac{1}{2} \epsilon^{J}_{\alpha\beta} h_{ab}[l_-, l_+]^{\beta}$ is a transverse 1-form, $\perp \omega = \omega$, and $h$ is the induced metric on $S$. The twist encodes the non-integrability of the normal space, thereby providing a geometrical measure of rotational frame-dragging. It is an invariant of a non-null foliated hypersurface $H,$ so $J[\psi]$ is also an invariant. It can be checked to recover the standard definition of angular momentum for a weak-field metric $\mathcal{E}$, with $\omega$ determining the precessional angular velocity of a gyroscope due to the Lense-Thirring effect.

There are several similar definitions of angular momentum, as clarified by Gourgoulhon and in the longer article. Ashtekar & Krishnan gave a definition for dynamical horizons, involving a 1-form which coincides with $\omega$ in that case. Ashtekar et al. earlier gave a definition for isolated horizons, involving a 1-form which does not generally coincide with $\omega$. However, it can be made to coincide if the dual-null gauge is fixed in a natural way. An earlier definition of angular momentum by Brown & York involves a 1-form which does not generally coincide with $\omega$, but does so if re-interpreted as adapted to the horizon.

In all cases, there remains the question of choosing $\psi$ with properties appropriate to an axial vector. Ashtekar & Krishnan proposed that $\psi$ has vanishing transverse divergence, $D_a \psi^a \equiv 0$, where $D$ is the covariant derivative of $h$ and $\cong$ denotes equality on $H$. This condition holds if $\psi$ is an axial Killing vector, and can be understood as a weaker condition, equivalent to $\psi$ generating a symmetry of the area form rather than of the whole metric, since $L_\psi \mathcal{D}A = D_\alpha \psi^\alpha d^2A$. Alternatively, assuming that the integral curves of $\psi$ are closed, as expected for an axial vector, it can be satisfied by choice of scaling of $\psi$, as discussed by Booth & Fairhurst.

Spherical topology will be assumed henceforth for $S$, which follows from the topology law for outer trapping horizons, assuming the dominant energy condition. If there exist angular coordinates $(\theta, \phi)$ on $S$ with $\psi = \partial/\partial \phi$, completing coordinates $(x, \theta, \phi)$ on $H$, then $L_\xi \psi \cong 0$, as proposed by Gourgoulhon. Noting the commutator identity $L_\xi (D_\alpha \psi^a) - D_\alpha (L_\xi \psi)^a = \psi^a D_\alpha \theta_\xi$, assuming both conditions on $\psi$ forces $\psi^a D_\alpha \theta_\xi \cong 0$. This is automatic if $D \theta_\xi \cong 0$, as in spherical symmetry or along a null trapping horizon. However, generically one expects $D \theta_\xi \neq 0$ almost everywhere. It must vanish somewhere on a sphere, by the hairy ball theorem, but the simplest generic situation is that there are curves $\gamma$ of constant $\theta_\xi$ which form a smooth foliation of circles, covering the surface except for two poles (Fig. 3). Assuming so, since $\psi$ is tangent to $\gamma$, one can find a unique $\psi$, up to sign, in terms of the unit tangent vector $\psi$ and arc length $ds$ along $\gamma$: $\psi \cong \psi \frac{d}{ds}/2\pi$, where the scaling ensures that the axial coordinate $\phi$ is identified at 0 and $2\pi$.

Then the angular momentum becomes unique up to sign, $J[\psi] = J$, the sign being naturally fixed by $J \geq 0$ and continuity of $\psi$. This construction, if unique, will yield the axial Killing vector if one exists, in particular for a Kerr black hole. The definition can be applied in any situation where $D \theta_\xi \neq 0$ almost everywhere, though the physical interpretation as angular momentum seems to be safest in the case of two poles, which locate the axis of rotation. Then $J$ is proposed to measure the angular momentum about that axis.

**Conservation of angular momentum.** Assuming the above conditions $L_\xi \psi \cong 0, D_\alpha \psi^a \cong 0$, then

$$L_\xi J \cong - \int_S (T_{aB} + \Theta_{ab}) \psi^a \tau^B d^2A$$

(3)

holds along a trapping horizon, where $T$ is the matter energy tensor,

$$\Theta_{\pm} = - \frac{1}{16\pi} h^{cd} D_d \sigma_{\pm ac}$$

(4)

is the transverse-normal block of an effective energy tensor for gravitational radiation, and $\sigma_{\pm ab} = \pm L_{\pm} h_{ab} - \theta_{\pm} h_{ab}$ are the null shears, which are transverse, $\perp \sigma_{\pm} = \sigma_{\pm}$, and traceless, $h_{ab} \sigma_{\pm ab} = 0$. The proof is a calculation using the Einstein equations, the stated conditions and the Gauss divergence theorem.

The null shears have previously been identified in the corresponding energy conservation law as encoding the ingoing and outgoing transverse gravitational radiation, via the energy densities $\Theta_{\pm} = |\sigma_{\pm}|^2/32\pi$, which agree with expressions in other limits, such as the Bondi energy density at null infinity and the Isaacson energy.
density of high-frequency linearized gravitational waves. So the result implies that gravitational radiation with a transversely differential waveform will generally possess angular momentum density. In the absence of such terms, the conservation law is the standard surface-integral form of conservation of angular momentum, were \( \psi \) an axial Killing vector, thereby describing the increase or decrease of angular momentum due to infall of co-rotating or counter-rotating matter respectively.

The identification of the transverse-normal block of \( \Theta \) appears to be new. Previous versions of angular momentum flux laws for dynamical black holes contain different terms, which are not in energy-tensor form, i.e. some tensor contracted with \( \psi \) and \( \tau \). They can be recovered by removing a transverse divergence from \( \Theta_{AB} \psi^a \tau^B \), yielding \( \sigma_{\tau} \psi \sqrt{g} / 16\pi = \sigma_{\psi} \psi^a \sqrt{g} / 32\pi \), where \( \sigma_{\tau} = \tau_B \sigma_{AB}^B \) gives the shear along \( \tau \). Such terms have been derived by analogy with viscosity.

**Conservation of energy.** The recently derived energy conservation law will be stated here for comparison, modifying some notation. Introduce the area \( A = \int_S d^2A \), the area radius \( R = \sqrt{A/4\pi} \), the canonical time vector \( k = (g^{-1}(dR))^* \) and the Hawking mass \( M = \frac{R}{2} \left( 1 - \frac{1}{16\pi} \int_S \sqrt{g^{-1}(-\Theta, -\Theta)} \right) \).

Assuming the null energy condition, this is the irreducible mass \( M \cong R/2 \) of a future outer trapping horizon, \( L_\xi M \geq 0 \), by the area law, which implies

\[
L_\xi A \geq 0. 
\]

Then the energy conservation law has a similar form to that of angular momentum. Of the ten conservation laws in flat-space physics, they are the two independent laws expected for an astrophysical black hole, which defines its own spin axis and centre-of-mass frame, in which its momentum vanishes.

It is appropriate to compare with other approaches, particularly that of Ashtekar & Krishnan, who derived a flux law for any transverse vector \( \psi \). Partly this reflects a different viewpoint, that classes of flux laws were desired, different choices of the vectors \( (k, \psi) \) yielding different quantities \( (M, J) \). Here the aim has been to find unique, physically meaningful conserved quantities for a given black hole, respectively the irreducible mass \( M \), which has a clear physical interpretation, and the angular momentum \( J \) about the axis of rotation, which has been obtained generically from natural restrictions on \( \psi \). Secondly, the conservation laws apply to any trapping horizon, whereas the Ashtekar-Krishnan formalism applies only to spatial trapping horizons, with null trapping horizons having been treated by the separate isolated-horizons formalism, some connections having been drawn between the two, such as for slowly evolving horizons by Booth & Fairhurst. Null trapping horizons remain a degenerate case of the general framework, but there is a natural way to fix the additional gauge freedom, consistently with weakly isolated horizons. Thirdly, the non-matter terms have here been identified as arising from an effective energy tensor \( \Theta \) for gravitational radiation, which allows the physical interpretation as conservation laws rather than just flux laws. Fourthly, charge has not yet been included, except where angular momentum vanishes or for isolated horizons, as remedied below.

**Conservation of charge.** The surface-integral form of conservation of charge \( Q \) is

\[
L_\xi Q = - \int_S g(j, \tau) d^2A 
\]

where the vector \( j \) is the charge-current density. The conservation laws for energy and angular momentum take the same form

\[
L_\xi M \cong - \int_S g(j, \tau) d^2A, \quad L_\xi J \cong - \int_S g(j, \tau) d^2A \]

by identifying current vectors \( j^{AB} = A^A B^B \), \( j^B = \psi_a (T_{aB} + \Theta^{aB}) \).

It is noteworthy that the local differential form of charge conservation, \( \nabla_\nu j^\nu = 0 \), generally does not hold for \( j \) or \( j \). Instead one can obtain

\[
\int_S \nabla_\nu j^\nu d^2A \cong \int_S \nabla_\nu j^\nu d^2A \cong 0. \]

This subtly confirms the view that energy and angular momentum in General Relativity cannot be localized, but might be quasi-localized, as surface integrals. The corresponding conservation laws have indeed been obtained in surface-integral but not local form.

**State space.** There are now three conserved quantities \((M, J, Q)\), forming a state space for dynamical black holes. Following various authors, related quantities may then be defined by formulas satisfied by Kerr-Newman black holes, specifically those for the ADM energy

\[
E \cong \sqrt{(2M)^2 + Q^2)^2 + (2J)^2} \quad (11)
\]

the surface gravity

\[
\kappa \cong \frac{2(M)^2 - (2J)^2 - Q^4}{2(2M)^3 \sqrt{((2M)^2 + Q^2)^2 + (2J)^2}} \quad (12)
\]

the angular speed

\[
\Omega \cong \frac{J}{M \sqrt{((2M)^2 + Q^2)^2 + (2J)^2}} \quad (13)
\]

and the electric potential

\[
\Phi \cong \frac{(2M)^2 + Q^2)Q}{2M \sqrt{((2M)^2 + Q^2)^2 + (2J)^2}}. \quad (14)
\]
In the dynamical context, $E \geq M$ is not the ADM energy, but can be interpreted as the effective energy of the black hole, including irreducible mass $M$, rotational kinetic energy $\approx \frac{1}{2}I\Omega^2$ and electrostatic energy $\approx \frac{1}{2}Q^2/R$, by expanding $E \approx M + \frac{1}{2}I\Omega^2 + \frac{1}{2}Q^2/R$ for $J \ll M^2$ and $Q \ll M$, where $J = I\Omega$ defines the moment of inertia $I \approx M\sqrt{(2M)^2 + Q^2} + (2J)^2 \approx ER^2$.

The state-space formulas

$$\kappa \approx 8\pi \frac{\partial E}{\partial A} \approx \frac{1}{4M} \frac{\partial E}{\partial M}, \quad \Omega \approx \frac{\partial E}{\partial J}, \quad \Phi \approx \frac{\partial E}{\partial Q} \quad \text{(15)}$$

then yield a dynamic version of the so-called first law of black-hole mechanics $\text{[1]}$:

$$L_\xi E \approx \frac{\kappa}{8\pi} L_\xi A + \Omega L_\xi J + \Phi L_\xi Q. \quad \text{(16)}$$

As desired, the state-space perturbations in the classical law for Killing horizons $\text{[4]}$, or the version for isolated horizons $\text{[12]}$, have been replaced by the derivatives along the trapping horizon, thereby promoting it to a genuine dynamical law.

Equilibrium. When a growing black hole ceases to grow, the generically spatial trapping horizon becomes null, leading to a non-uniqueness in the twist. However, preservation of angular momentum suggests a natural way to restore uniqueness by fixing $Df \equiv 0 \text{[12]}$, thereby closing another gap in the paradigm. Then the dominant energy condition implies

$$g(\mathbf{j}, \tau) \approx g(\mathbf{j}, \tau) \equiv g(\mathbf{j}, \tau) \approx 0 \quad \text{(17)}$$

and the conserved quantities are actually preserved:

$$L_\xi M \approx L_\xi J \approx L_\xi Q \approx 0. \quad \text{(18)}$$

This indicates that local equilibrium is indeed attained when a trapping horizon becomes null. Then $L_\xi \kappa \approx 0$, so that the surface gravity, which satisfies $D\kappa \approx 0$ by definition $\text{[12]}$, is constant where a trapping horizon becomes null. This is a quite general zeroth law. By the area law $\text{[3]}$, this also shows that a black hole cannot change its angular momentum or charge without increasing its area.

Conclusion. Of the classical four laws of black-hole mechanics $\text{[1]}$, the optimal form of the third law is still not clear. Generalized versions of the zeroth and first $\text{[10]}$ laws have been given above. The analogue of the second law is the area law derived previously $\text{[3]}$, which can be regarded as a consequence of energy conservation and horizon type.

The new paradigm for black holes in terms of trapping horizons thereby makes appropriate contact with the classical paradigm, while shifting emphasis to more fundamental conservation laws for energy $\text{[4]}$ and angular momentum $\text{[3]}$, which include plausible contributions from gravitational radiation. Thus three areas in General Relativity which are intuitively important as physics but have been conceptually elusive, namely energy, black holes and gravitational radiation, appear to make sense quite generally and are profoundly interrelated.

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