On the random nature of (prime) number distribution

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Abstract

Let \( \pi(x) \) denote the number of primes smaller or equal to \( x \). We compare \( \sqrt{\pi(x)} \) with \( \sqrt{R(x)} \) and \( \sqrt{\ell_i(x)} \), where \( R(x) \) and \( \ell_i(x) \) are the Riemann function and the logarithmic integral, respectively. We show a regularity in the distribution of the natural numbers in terms of a phase related to \( (\sqrt{\pi} - \sqrt{R}) \) and indicate how \( \ell_i(x) \) can cross \( \pi(x) \) for the first time.

1 Introduction

1.1 Preliminaries

The function \( \pi(x) \) is the function counting the number of primes smaller or equal to \( x \). For example, \( \pi(2) = 1, \pi(3) = 2, \pi(4) = 2, \pi(5) = 3, \ldots \). In 1792, when he was 15 years old, Gauss proposed

\[
\frac{x}{\ln x}
\]

as an approximation to \( \pi(x) \), which he refined afterwards [1] to

\[
\ell_i(x) = PV \int_0^x \frac{dt}{\ln t}
\]

where \( PV \) means the integral principal value. The function \( \ell_i(x) \) can also be written as \( \ell_i(x) = \int_\mu^x dt/\ln t \), with \( \mu = 1.4513692348 \ldots \)

Later, Riemann [2] improved the approximation with his Riemann function \( R(x) \) defined as

\[
R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \ell_i(x^{1/n})
\]

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where $\mu$ is the Möbius function $[3]$, given by

\[
\mu(n) = \begin{cases} 
0 & \text{if } n \text{ has one or more primes repeated} \\
1 & \text{if } n = 1 \\
(-1)^k & \text{if } n \text{ is a product of } k \text{ different primes}
\end{cases}
\]

Riemann also proposed that $[4]$

\[
\pi(x) - R(x) = -\sum_{\rho} R(x^\rho)
\]

where $\rho$ are the trivial and non trivial zeroes of the Riemann zeta function, $\zeta$, which is defined as

\[
\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}
\]

for $\Re(s) > 1$. Although Riemann did the analytical continuation of $\zeta$ to all the complex plane excepting the point $s = 1$, an easier expression is given by $[5]$

\[
\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^n+1} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!(k+1)^s}
\]

The trivial zeroes of $\zeta$ are found easily from the relation $[6]$

\[
\zeta(1 - s) = 2(2\pi)^{-s} \cos \left(\frac{s\pi}{2}\right) \Gamma(s) \zeta(s)
\]

because when $s = 2n + 1$, with $n$ an integer, $\zeta(-2n) = 0$.

With respect to the non trivial zeroes, the Riemann hypothesis $[2]$ says that all of them lie on the “critical” line, $\rho(t) = 1/2 + it$. It is one of the most important problems of mathematics today.

The prime number theorem, proved independently by de la Vallée-Poussin $[7]$ and Hadamard $[8]$, assures that

\[
\lim_{x \to \infty} \frac{\pi(x)}{\ell i(x)} = \lim_{x \to \infty} \frac{\pi(x)}{R(x)} = \lim_{x \to \infty} \frac{\pi(x) \ln x}{x} = 1
\]

Currently $\pi(x)$ has been computed up to $x \sim 10^{23}$. All the computed values of $\pi(x)$ today satisfy the inequality $\ell i(x) > \pi(x)$. However, in 1914 Littlewood $[9]$ showed that this inequality changes its sign infinitely often for very large $x$ $[11]$.

### 1.2 Motivation

In general the absolute value of the difference between the function $\pi(x)$ and its approximations, $\ell i(x)$ or $R(x)$, although it is smaller than $\sim \sqrt{x}$, is a number much greater than the unity for large $x$. However, the absolute value of the difference between the square roots of $\pi(x)$ and of $\ell i(x)$ or between the square roots of $\pi(x)$ and of $R(x)$ are
smaller than 1. Then these ones are what we will consider in order to have a better scope of the approximations to \( \pi(x) \). In Figure 1a, it is shown the difference \( \sqrt{\pi(x)} - \sqrt{R(x)} \) and the maximal difference between these functions is \( \sqrt{\pi(2)} - \sqrt{R(2)} = -0.244906 \) when \( x = 2 \). We see that \( \sqrt{R(x)} \) averages very well \( \sqrt{\pi(x)} \). In Figure 1b it is shown the difference \( \sqrt{\ell(x)} - \sqrt{\pi(x)} \), whose maximal difference corresponds to the point \( x = 28 \), where \( \sqrt{\ell(28)} - \sqrt{\pi(28)} = 0.525426 \). The gross line represents the function \( \sqrt{\ell(x)} - \sqrt{R(x)} \), which is the “average” of the points \( \sqrt{\ell(x)} - \sqrt{\pi(x)} \). In both figures not all the points are shown, there is a higher density in the center, a lot of external points are included to make the border explicit. The points were calculated with Mathematica until \( 10^{12} \) and the rest were taken from the tables of [10], which give values of \( \pi(x) \) for numbers with three or four significant digits, and so, the points shown in the border after \( 10^{12} \) are not necessarily the points with the biggest difference \( |\sqrt{\pi(x)} - \sqrt{R(x)}| \).

In section 2, our plan is to delimit the function \( \sqrt{\pi(x)} - \sqrt{R(x)} \) from above and below with a tight function, in such a way that all the points remain inside the bounds, then, to delimit the functions \( \sqrt{\ell(x)} - \sqrt{\pi(x)} \) and \( \ell(x) - \pi(x) \), and finally to discuss the statistical distribution of a phase defined in terms of the functions previously mentioned.

2 Discussion

2.1 \( \sqrt{\pi} - \sqrt{R} \)

One can study the general characteristics of the function \( \sqrt{\pi(x)} - \sqrt{R(x)} \). The absolute value of this function is bounded with its maximal value \( |\sqrt{\pi(2)} - \sqrt{R(2)}| = 0.244906 \). So, we can propose that \( \sqrt{\pi(x)} \) is given by

\[
(i) \quad \sqrt{\pi(x)} = \sqrt{R(x)} + \eta(x) \cos \delta(x) \quad \eta(x) > 0
\]

where \( \eta(x) \) is the envelope, and all the points of Figure 1a are delimited by this one.

Other parameterization is

\[
(ii) \quad a(x) = \sqrt{R(x)} + \eta(x)e^{i\delta(x)} \quad \eta(x) > 0, \quad |a(x)|^2 = \pi(x)
\]

this last one puts in evidence the parameterization in terms of an amplitude \( \eta(x) \) and a phase \( \delta(x) \). Equation 3 implies

\[
\pi(x) = R(x) + 2\eta(x) \cos \delta \sqrt{R(x)} + \eta^2(x)
\]

Observe that, when \( \delta(x) = 0 \) or \( \pi \), Equations 2 and 3 coincide. The first proposal for \( \eta(x) \) is the function

\[
\eta_1(x) = \frac{0.2595}{\ln \ln(x + 15.9)}
\]

However, from the work of [11] we know that the first zero of the function \( \sqrt{\ell(x)} - \sqrt{\pi(x)} \) happens before \( x = 1.3982 \times 10^{316} \), and may be much earlier. The function of Equation 4
crosses x axis around \( x = 10^{65} \). A function that crosses x axis around \( x = 1.3982 \times 10^{316} \), is

\[
\eta_2(x) = \frac{0.315647}{\ln(x + 4.07206)^{0.430202}}
\]  

If \( \sqrt{\ell_1(x)} - \sqrt{\pi(x)} \) crossed the axis before, \( \eta(x) \) would be a function between the ones defined in Equation (5) and Equation (6). In Figure 2 it is shown the points \( \sqrt{\pi(x)} - \sqrt{\ell_1(x)} \) with the two bounds and in Figure 3 it is shown the points \( \sqrt{\ell_1(x)} - \sqrt{\pi(x)} \), with its “average” function \( \sqrt{\ell_1(x)} - \sqrt{\pi(x)} \), where the borders are given by

\[
(\sqrt{\ell_1} - \sqrt{\pi})_{\text{max, min}} = \sqrt{\ell_1} - \sqrt{\pi} \pm \eta
\]

### 2.2 \( \ell_1 - \pi \)

We can delimit \( \ell_1 - \pi \) from above and below.

From Equation (2) and Equation (4) and using \( \ell_1 - \pi = \ell_1 - R + R - \pi \) one has that

\[
\ell_1 - (\sqrt{R} + \eta)^2 \leq \ell_1 - \pi \leq \ell_1 - (\sqrt{R} - \eta)^2
\]

Using the fact that in the limit of large \( x \), \( \ell_1(x) - R(x) \to \sqrt{x}/(\ln x) \), \( \sqrt{R} \approx \sqrt{x}/\ln x \) and that \( \eta^2 \) is negligible, one has

\[
\frac{\sqrt{x}}{\ln x} - 2\eta\sqrt{\frac{\sqrt{x}}{\ln x}} < \ell_1 - \pi < \frac{\sqrt{x}}{\ln x} + 2\eta\sqrt{\frac{\sqrt{x}}{\ln x}}
\]

and then, if there are values where \( \ell_1(x) \) is smaller than \( \pi(x) \), then \( \eta(x) \) must decrease in a slower way than \( 1/(2\sqrt{\ln x}) \), as it happens with Equation (3) and Equation (4).

In Figures 4.b and 4.c it is shown \( \ell_1(x) - \pi(x) \) using for their bounds Equation (8), with \( \eta(x) \) given by Equation (5) and Equation (6). The bounds of Equation (8) only work for large \( x \), when \( R(x) \approx \ell_1(x) - (1/2)\ell_1(x^{1/2}) \). For small \( x \), Equation (8) is not valid, and we use directly the bounds (7), and in Figure 4.a we show the later ones in the interval \( x \in (2, 10^4) \). The gross line corresponds to the “average” function \( (\ell_1(x) - R(x)) \).

### 2.3 \( \cos \delta \)

With a sample of the first natural numbers one averages the functions \( \sqrt{x} - \sqrt{R} \) and \( \pi - R \). The values of Table 1 are obtained for different sample sizes. In this table, \( \sigma(f) \) is the standard deviation, \( \sigma \equiv \sqrt{\langle f^2 \rangle - (\langle f \rangle)^2} \), with \( f \) equal to \( \sqrt{\pi} - \sqrt{R} \) or to \( (\pi - R) \). We see that \( \langle \sqrt{x} - \sqrt{R} \rangle \) is a small number bigger than zero and has a small variation in the different intervals.

Working out the value of \( \cos \delta \) in both cases, Equations (2) and (4), one has

\[
\cos \delta = \frac{\sqrt{\pi(x)} - \sqrt{R(x)}}{\eta(x)} \quad \text{and} \quad \cos \delta = \frac{\pi(x) - R(x) - \eta^2(x)}{2\sqrt{R(x)}\eta(x)}
\]
respectively and taking the first definition of $\eta(x) \equiv \eta_1(x)$, Equation (5), one has the averages of Table 2 in the intervals $x \in (2, 100), \ldots, x \in (2, 10^6)$.

The results of Table 2 show that the averages remain approximately constant. With respect to the width of $\sigma$ of the distribution, as to the average of $\cos \delta$ absolute value, the difference in the parameterizations of Equation (2) and Equation (3) is negligible. Also, although for the first intervals the difference in the average $\langle \cos \delta \rangle$ is bigger, as $x$ grows the averages in the two parameterizations get closer, because in general the ratio $\eta^2 / |\pi - R| \ll 1$. From now on, we will keep the parameterization of Equation (2).

Taking the other proposal of $\eta(x) \equiv \eta_2(x)$, Equation (6), the averages of Table 3 are found. In this table, the average value of $\cos \delta$ is not very different from the previous parameterization, being consistent with a small positive number.

In order to see the weight of the different sets of numbers with respect to $\cos \delta$, in Table 4 we give the average of $\cos \delta$ for natural, prime, even and odd (without primes) numbers. We see that as $x$ grows, the prime distribution, which has a higher $\cos \delta$ average, has a smaller weight, because the ratio of prime to natural numbers decreases approximately as $\pi(x)/x \sim 1/\ln x$. So, the average of $\cos \delta$ for the even and odd natural numbers will be approximately the same for large $x$.

Let us take $\eta(x)$ given by Equation (5): if we divide $\cos \delta$ in the intervals $(1, -0.95), (-0.95, -0.85), \ldots, (0.85, 0.95), (0.95, 1)$, we find distributions of Table 5 They give the number of positive integers whose $\cos \delta$ falls in one of these intervals, we count them in 4 different sample sizes: $(2, 10^3), (2, 10^4), (2, 10^5)$ and $(2, 10^6)$.

In Figures 5 and 6 it is shown distributions of $\cos \delta$ as explained in the previous paragraph. We have normalized them to have the total area of the bars equal to one. For example, for the natural numbers between $(2, 10^3)$, there are 47 numbers whose $\cos \delta$ falls in the interval $(-0.45, -0.35)$. We divide these 47 numbers by the sample total number, 999, to obtain the relative frequency and multiply by 10, because the size of each interval is 0.1 (except for the intervals $(-1, -0.95)$ and $(0.95, 1)$).

The distribution is gaussian, and from Table 2, the width appears to have the same value, $\sigma = 0.28$, it does not matter the number of positive integers with which we take the average. The average seems to stabilize around $\langle \cos \delta \rangle = 0.014$. In all the figures, Figures 4 and 6 we used the same Gaussian with width $\sigma = 0.28$, average $\langle \cos \delta \rangle = 0.014$ and height $1/(\sqrt{2\pi\sigma}) = 1.425$, and the fit of the Gaussian is in a very good agreement with the data.

Finally, from Equation (2) and Equation (4)

$$\pi - R \approx 2\sqrt{R}\eta \cos \delta$$

then $(\pi(x) - R(x))/(2\sqrt{R(x)}\eta(x))$ follows the same Gaussian distribution.

3 Conclusions

With two parameters, one amplitude $\eta$ and a phase $\delta$, we study the properties of the roots of the functions $\pi$, $\ell_1$ and $R$, using Equations (2) and (3). With $\eta$, we delimit the differences
\((\sqrt{\pi} - \sqrt{R}), (\sqrt{\ell_i} - \sqrt{\pi})\) and \((\ell_i - \pi)\). Concerning the last one, we know from the data that \((\ell_i - \pi < \sqrt{\pi})\), and in Equation (7) we give a more precise relation. We find that \(\cos \delta\), follows a Gaussian distribution, that shows a stable random behavior of the function \(\pi(x)\), see Figures 5 and 6. Taking different sample sizes, \(\cos \delta\) distribution remains constant. The question is if the Gaussian shape remains constant as \(x\) grows.

Appendix

To see how the natural numbers accommodate in the different \(\cos \delta\) intervals, we give as an example the first hundred in Table 6, where the prime numbers have been underlined.

We can see that, each time there is a new prime number, \(\cos \delta\) increases, and meanwhile \(\pi(x)\) remains constant, until the next prime number, the following integers accommodate in intervals with smaller \(\cos \delta\). So, Table 5 and Figures 5 and 6 show that the way of appearance of the prime numbers implies the randomness of the natural numbers with respect to \(\cos \delta\). In Figure 7 we give a pictorial representation of how the first one hundred natural numbers (except 1) are accommodated, where the lines join points with the same \(\pi(x)\)

That \(\cos \delta\) decreases each time \(\pi(x)\) remains constant, while a new prime number does not appear, it is because the function \(\sqrt{R}(x)\) is a monotone growing function. With the appearance of the new prime number, \(\cos \delta\) increases and the cycle is repeated. The rate with which \(\cos \delta\) decreases is given by its derivative, and as the derivative of \(R(x)\) is

\[
\frac{dR}{dx} = \sum_{n=1}^{\infty} \mu(n) \frac{x^{(n-1)/n}}{n.x^{(n-1)/n} \ln x} = \frac{1}{\ln x} \left(1 - \frac{1}{2x^{1/2}} - \frac{1}{3x^{2/3}} - \ldots\right)
\]

then, with the parameterization of Equation (5) the derivative of \(\cos \delta(x)\) for a constant \(\pi(x)\) is

\[
\frac{d\cos \delta}{dx} = \frac{1}{0.2561} \left\{ \frac{(\sqrt{\pi} - \sqrt{R})}{(x + 15.55) \ln(x + 15.55)} - \frac{\ln \ln(x + 15.55)}{2\sqrt{R} \ln x} \left(1 - \frac{1}{2x^{1/2}} - \ldots\right) \right\}
\]

while for the parameterization of Equation (6) is

\[
\frac{d\cos \delta}{dx} = \frac{[\ln(x + 4.07)]^{0.43}}{0.3156} \left\{ \frac{(\sqrt{\pi} - \sqrt{R})}{(x + 4.07) \ln(x + 4.07)} - \frac{1}{2\sqrt{R} \ln x} \left(1 - \frac{1}{2x^{1/2}} - \ldots\right) \right\}
\]

in both cases the derivative is dominated by the negative term, as it is expected, and decreases in absolute value as \(x\) grows. In Figures 8 and 9, some other intervals of 100 numbers are compared for larger \(x\) where it is seen that \(\cos \delta\) gets more horizontal, this is because there is a bigger number of points with the same \(\pi(x)\), also, although at the beginning there are “jumps” when one goes from \(\pi(p - 1)\) to \(\pi(p)\), whose difference is one, as \(x\) increases, \(\cos \delta\) turns into a softer function, because \((\pi(p) - \pi(p - 1))/\pi(p) \to 0\).
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Tables

|                | $\langle \sqrt{\pi} - \sqrt{R} \rangle$ | $\sigma(\sqrt{\pi} - \sqrt{R})$ | $\langle \pi - R \rangle$ | $\sigma(\pi - R)$ |
|----------------|----------------------------------------|---------------------------------|---------------------------|-----------------|
| $2 - 10^2$     | 0.001889                               | 0.062256                        | 0.033137                  | 0.403334        |
| $2 - 10^3$     | 0.001363                               | 0.042803                        | 0.013466                  | 0.714523        |
| $2 - 10^4$     | 0.001302                               | 0.035624                        | 0.050812                  | 1.72635         |
| $2 - 10^5$     | 0.001529                               | 0.031321                        | 0.25608                   | 4.23254         |
| $2 - 10^6$     | 0.001405                               | 0.028509                        | 0.705741                  | 11.1907         |

Table 1: averages $\langle \sqrt{\pi} - \sqrt{R} \rangle$ and $\langle \pi - R \rangle$ in 5 intervals, $\sigma$ is the standard deviation

|                | $\langle \cos \delta \rangle = \langle \sqrt{\pi} - \sqrt{R} \rangle $ | $\sigma$ | $\langle | \cos \delta | \rangle$ | $\langle \cos \delta \rangle = \langle \pi - R \rangle$ | $\sigma$ | $\langle | \cos \delta | \rangle$ |
|----------------|-------------------------------------------------------------------------|---------|----------------|------------------------------------------------|---------|----------------|
| $2 - 10^2$     | 0.014402                                                                | 0.315325| 0.254145       | -0.010719                                             | 0.315370| 0.253362       |
| $2 - 10^3$     | 0.008304                                                                | 0.280109| 0.223332       | -0.000662                                             | 0.280161| 0.223363       |
| $2 - 10^4$     | 0.009965                                                                | 0.283603| 0.224534       | 0.006999                                              | 0.282839| 0.224425       |
| $2 - 10^5$     | 0.014043                                                                | 0.281287| 0.222306       | 0.013073                                              | 0.281312| 0.222302       |
| $2 - 10^6$     | 0.014057                                                                | 0.278975| 0.227005       | 0.013740                                              | 0.278979| 0.226989       |

Table 2: averages of $\cos \delta$ defined by Equation (2) and of $\cos \delta$ given by Equation (4), where $\eta_1$ is given by Equation (3)

|                | $\langle \cos \delta \rangle = \langle \sqrt{\pi} - \sqrt{R} \rangle $ | $\sigma$ | $\langle | \cos \delta | \rangle$ |
|----------------|-------------------------------------------------------------------------|---------|----------------|
| $2 - 10^2$     | 0.015587                                                                | 0.327365| 0.264286       |
| $2 - 10^3$     | 0.008606                                                                | 0.279154| 0.222093       |
| $2 - 10^4$     | 0.009728                                                                | 0.274722| 0.217424       |
| $2 - 10^5$     | 0.013529                                                                | 0.270749| 0.213973       |
| $2 - 10^6$     | 0.013608                                                                | 0.269512| 0.219318       |

Table 3: averages of $\cos \delta$ defined in Equation (2), with $\eta_2$ given in Equation (6)
Table 4: average of $\cos \delta$, according to the set of positive integers under which the average is taken, the numbers in parentheses are the number of positive integers in the given set

|                | all primes without 1 | primes even without 2 | odd without 1 and without primes | odd without 1 |
|----------------|----------------------|-----------------------|----------------------------------|---------------|
| $2 - 10^2$     | 0.014402             | 0.256581 (25)         | -0.073010                        | -0.056451 (25)|
| $2 - 10^3$     | 0.008304             | 0.182444 (168)        | -0.027533                        | -0.025953 (332)|
| $2 - 10^4$     | 0.009965             | 0.099122 (1229)       | -0.002571                        | -0.002474 (3771)|
| $2 - 10^5$     | 0.014043             | 0.051776 (9592)       | 0.009936                         | 0.010169 (40408)|
| $2 - 10^6$     | 0.014057             | 0.028670 (78498)      | 0.012749                         | 0.012886 (421502)|

Table 5: number of positive integers with $\cos \delta$ in the intervals $(-1, -0.95), (-0.95, -0.85), \ldots$ for different samples: from $(2, 10^3)$ to $(2, 10^6)$

| $\cos \delta$ | $2 - 10^3$ | $2 - 10^4$ | $2 - 10^5$ | $2 - 10^6$ |
|----------------|-----------|-----------|-----------|-----------|
| $(-1, -0.95)$  | 1         | 2         | 3         | 3         |
| $(-0.95, -0.85)$| 0         | 3         | 26        | 331       |
| $(-0.85, -0.75)$| 0         | 14        | 120       | 2230      |
| $(-0.75, -0.65)$| 5         | 67        | 522       | 5024      |
| $(-0.65, -0.55)$| 11        | 186       | 1303      | 15247     |
| $(-0.55, -0.45)$| 31        | 370       | 2504      | 30391     |
| $(-0.45, -0.35)$| 47        | 490       | 4630      | 55051     |
| $(-0.35, -0.25)$| 78        | 657       | 7490      | 78559     |
| $(-0.25, -0.15)$| 116       | 880       | 11776     | 94341     |
| $(-0.15, -0.05)$| 144       | 1389      | 13740     | 114888    |
| $(-0.05, 0.05)$| 136       | 1530      | 15040     | 138262    |
| $(0.05, 0.15)$  | 130       | 1387      | 13006     | 138171    |
| $(0.15, 0.25)$  | 106       | 1038      | 10645     | 115027    |
| $(0.25, 0.35)$  | 76        | 760       | 6816      | 92749     |
| $(0.35, 0.45)$  | 57        | 575       | 5082      | 68886     |
| $(0.45, 0.55)$  | 36        | 390       | 3835      | 34856     |
| $(0.55, 0.65)$  | 11        | 187       | 1915      | 10700     |
| $(0.65, 0.75)$  | 6         | 52        | 871       | 3833      |
| $(0.75, 0.85)$  | 5         | 17        | 397       | 1086      |
| $(0.85, 0.95)$  | 2         | 4         | 267       | 373       |
| $(0.95, 1)$     | 1         | 1         | 11        | 11        |
| $\cos \delta$ |   |
|-----------|---|
| $(-1, -0.95)$ | 2 |
| $(-0.65, -0.55)$ | 4, 10 |
| $(-0.55, -0.45)$ | 28, 36, 40, 58, 96 |
| $(-0.45, -0.35)$ | 16, 57, 66, 95, 100 |
| $(-0.35, -0.25)$ | 9, 27, 35, 39, 52, 70, 94, 99 |
| $(-0.25, -0.15)$ | 6, 12, 56, 60, 65, 98 |
| $(-0.15, -0.05)$ | 15, 22, 26, 30, 34, 38, 42, 51, 55, 64, 69, 78, 88, 93 |
| $(0.05, 0.05)$ | 3, 18, 46, 50, 59, 68, 82, 87, 92, 97 |
| $(0.05, 0.15)$ | 8, 11, 25, 29, 33, 37, 41, 45, 54, 63, 72, 77, 86, 91 |
| $(0.15, 0.25)$ | 5, 14, 21, 49, 53, 62, 67, 71, 76, 81, 90 |
| $(0.25, 0.35)$ | 17, 24, 32, 44, 48, 75, 80, 85 |
| $(0.35, 0.45)$ | 20, 61, 79, 84, 89 |
| $(0.45, 0.55)$ | 7, 13, 31, 43, 47, 74, 83 |
| $(0.55, 0.65)$ | 23, 73 |
| $(0.65, 0.75)$ | 19 |

Table 6: distribution of the first one hundred natural numbers (without 1) in the different intervals of $\cos \delta$, the primes are underlined.
Figures

Figure 1: (a) $\sqrt{\pi(x)} - \sqrt{R(x)}$ vs $\ln x$ and (b) $\sqrt{li(x)} - \sqrt{\pi(x)}$ vs $\ln x$, in $x \in (2, 10^{23})$, where the gross line is the function $\sqrt{li(x)} - \sqrt{R(x)}$

Figure 2: $\sqrt{\pi(x)} - \sqrt{R(x)}$ vs $\ln x$, envelopes $\eta(x) = 0.2595/\ln \ln(x + 15.9)$ (continuous line) and $\eta(x) = 0.315647/\left[\ln(x + 4.07206)\right]^{0.430202}$ (dashed line)
Figure 3: $\sqrt{\ell i(x)} - \sqrt{\pi(x)}$ vs $\ln x$, envelopes $\eta(x) = 0.2595 / \ln \ln(x + 15.9)$ (continuous line) and $\eta(x) = 0.315647 / [\ln(x + 4.07206)]^{0.430202}$ (dashed line), and the function $\sqrt{\ell i(x)} - \sqrt{R(x)}$ (gross line).

Figure 4: $\ell i(x) - \pi(x)$ vs $\ln x$, in (a) $x \in (2, 10^4)$, (b) $x \in (5 \times 10^8, 2 \times 10^{17})$ and (c) $x \in (2 \times 10^{17}, 8 \times 10^{23})$, with $\eta(x) = 0.2595 / \ln \ln(x + 15.9)$ (continuous line), $\eta(x) = 0.315647 / [\ln(x + 4.07206)]^{0.430202}$ (dashed line) and $\ell i(x) - R(x)$ (gross line).
Figure 5: distribution of $\cos \delta$ with $\eta(x) = 0.2595/\ln \ln(x + 15.9)$, where the relative frequency of $\cos \delta$ has been counted in the intervals $(-1, 0.95), (-0.95, -0.85), \ldots$, the Gaussian is represented by the continuous line, (a) for the first $10^3$ natural numbers (except 1) and (b) for the first $10^4$ ones.

Figure 6: the same as in the figure 5, (a) for the first $10^5$ natural numbers (except 1) and (b) for the first $10^6$ ones.
Figure 7: $\cos \delta = \left( \sqrt{\pi(x)} - \sqrt{R(x)} \right) / \eta(x)$ vs $x$, $x \in (2, 100)$, $\eta(x) = 0.2595/\ln \ln(x + 15.9)$

Figure 8: $\cos \delta = \left( \sqrt{\pi(x)} - \sqrt{R(x)} \right) / \eta(x)$ vs $x$, $x \in (15000, 15100)$, $\eta(x) = 0.2595/\ln \ln(x + 15.9)$

Figure 9: $\cos \delta = \left( \sqrt{\pi(x)} - \sqrt{R(x)} \right) / \eta(x)$ vs $x$, $x \in (1000000, 1000100)$, $\eta(x) = 0.2595/\ln \ln(x + 15.9)$