Geometric Transitions and $\mathcal{N} = 1$ Quiver Theories

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Abstract

We construct $\mathcal{N} = 1$ supersymmetric theories on worldvolumes of $D5$ branes wrapped around 2-cycles of threefolds which are A-D-E fibrations over a plane. We propose large $N$ duals as geometric transitions involving blowdowns of two cycles and blowups of three-cycles. This yields exact predictions for a large class of $\mathcal{N} = 1$ supersymmetric gauge systems including $U(N)$ gauge theories with two adjoint matter fields deformed by superpotential terms, which arise in A-D-E fibered geometries with non-trivial monodromies.

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1. Introduction

Geometric transitions in the context of D-branes have been shown to be related to large $N$ dualities. This has been seen in the context of topological strings [1] and more recently in the context of superstrings [2][3][4][5]. See also some recent works in this direction [6]. The aim of this paper is to enlarge this class of dualities by considering a class of $\mathcal{N} = 1$ supersymmetric gauge theories in the context of wrapped branes for A-D-E fibered Calabi-Yau 3-folds. These can be viewed as natural generalization of the dualities of CY 3-folds studied in [5]. These include $\mathcal{N} = 2$ gauge theories with gauge groups given by the quiver diagrams of A-D-E with bi-fundamental matter dictated by the diagram, deformed by superpotential terms for the adjoint fields, breaking the supersymmetry to $\mathcal{N} = 1$. Some examples of this kind were studied in [5] and the exact results for gauge theory were recovered more naturally from the geometric large $N$ dual.

We also construct, by considering A-D-E fibered geometries with monodromy, $\mathcal{N} = 1$ $U(N)$ gauge theories with two adjoint fields with certain superpotential terms. In particular we geometrically engineer theories with superpotential

$$W = P_{p+2}(X) + P_{q+2}(Y) + P_{r+2}(X + Y)$$

where $X, Y$ are adjoint fields and $P_i$ denotes traces of polynomials of degree $i$ in the variable. The $p, q, r$ denote the number of nodes of the D-E Dynkin diagrams in each of the three disconnected components after removing the trivalent node. Obtaining exact results for the vacuum structure of these theories has been difficult with the available techniques for dealing with supersymmetric gauge theories. This is because theories with two adjoint fields is neither close to being an $\mathcal{N} = 4$ system which has 3 adjoint fields, nor close to being an $\mathcal{N} = 2$ system which has only 1 adjoint field. Nevertheless geometric duals we propose provide exact information for the vacuum structure of these theories with two adjoint fields.

The organization of this paper is as follows: In section 2, we review A-D-E singularities and their deformations. We also review the worldvolume theory of branes wrapped around cycles of these geometries. In section 3 we consider 3-fold geometries obtained by fibering the A-D-E singularities over a plane. The corresponding wrapped branes yield $\mathcal{N} = 1$ quiver theories. In section 4 and 5 we study the Higgs branches of these theories. In section 6 we consider the quiver theories in case the fibration involves non-trivial Weyl group monodromies. In section 7 we present detailed examples of the monodromic cases, involving $\mathcal{N} = 1$ gauge theories with one or two adjoint fields. Finally in section 8 we propose the corresponding large $N$ dual geometries.

1
2. A-D-E singularities in dimension 2

Consider ALE spaces with A-D-E singularities at the origin. These spaces are constructed as the quotient of $\mathbb{C}^2$ by a discrete subgroup of $SU(2)$, $\Gamma$. The correspondence between the groups $\Gamma$ and the geometry is cyclic $\leftrightarrow A$, dihedral $\leftrightarrow D$ and exceptional $\leftrightarrow E$. As is well known these geometries are singular and can be viewed as the hypersurface $f(x, y, z) = 0$ of $\mathbb{C}^3$:

- \( A_r : \quad f = x^2 + y^2 + z^{r+1} \)
- \( D_r : \quad f = z^{r-1} + y^2 + x^2 \)
- \( E_6 : \quad f = y^3 + z^4 + x^2 \)
- \( E_7 : \quad f = y^3 + yz^3 + x^2 \)
- \( E_8 : \quad f = y^3 + z^5 + x^2 \)

Note that these spaces are singular at the origin $x = y = z = 0$. There are two ways to desingularize these spaces. One way is by deforming the defining equation $f = 0$ by adding relevant deformations. The other way is by blowing up the singularity.

Consider deforming these by relevant deformations, so that $f = df = 0$ has no solutions. Then the resulting space is non-singular. Let $r$ denote the rank of the corresponding A-D-E. There are $r$ such deformations for each singularity type. It is convenient, as will be explained below, to introduce coordinates in the deformation space $t_i$ with $i = 1, \ldots, r$ for $A_{r-1}$, $D_r$ and $E_r$. For $A_{r-1}$ one has to impose one constraint. The deformed equations $f(x, y, x, t_i)$ are given by,

- \( A_{r-1} : \quad x^2 + y^2 + \prod_{i=1}^{r-1} (z + t_i) \sum_{i=1}^{r} t_i = 0 \) \quad (2.1)
- \( D_r : \quad x^2 + y^2 z + \frac{\prod_{i=1}^{r-1} (z + t_i)}{z} - \frac{\prod_{i=1}^{r-1} t_i^2}{z} + 2 \prod_{i=1}^{r} t_i y \) \quad (2.2)

- \( E_6 : \quad x^2 + z^3 + y^2 + \epsilon_2 y z^2 + \epsilon_5 y z + \epsilon_6 z^2 + \epsilon_8 y + \epsilon_9 z + \epsilon_{12} \)
- \( E_7 : \quad x^2 + y^3 + y z^3 + \epsilon_2 y z^2 + \epsilon_6 y^2 + \epsilon_8 y z + \epsilon_{10} z^2 + \epsilon_{12} y + \epsilon_{14} z + \epsilon_{18} \)
- \( E_8 : \quad x^2 + y^3 + z^5 + \epsilon_2 y z^3 + \epsilon_8 y z^2 + \epsilon_{12} z^3 + \epsilon_{14} y z + \epsilon_{18} z^2 + \epsilon_{20} y + \epsilon_{24} z + \epsilon_{30} \)

where $\epsilon_i$ are complicated homogeneous polynomials of $t_i$'s of degree $i$ and invariant under the permutation of $t_i$'s. The importance of the choice of canonical coordinates $t_i$ is that

1 The explicit form of these for $E_6$ and $E_7$ can be found in appendix 1 and 2 of [7] respectively.
roughly speaking they measure the holomorphic volume of \( S^2 \)'s in the geometry: Upon a generic such deformation the space \( f = 0 \) admits \( r \) non-vanishing \( S^2 \)'s. Moreover a basis can be chosen so that the intersection of these \( S^2 \)'s is the same as that of the corresponding Dynkin diagram. In fact there are as many deformation parameters as \( S^2 \)'s and we can relate them to the “holomorphic volume” of the corresponding 2-cycle. This we define as the integral of the holomorphic 2-form over the corresponding \( S^2 \):

\[
\alpha_i = \int_{S^2_i} \omega = \int_{S^2_i} \frac{dxdy}{z}
\]

Thus the set of \( \alpha_i \) will be identified with the simple roots of the corresponding Dynkin system (Figure 1). It is also natural to use \( \alpha_i \) as the \( r \) deformation parameters for \( f \) and write \( f(x, y, z; \alpha_i) \). In fact the \( \alpha_i \) are very simply related to the \( t_i \). Namely, up to a constant factor we have

\[
A_r : \quad \alpha_i = t_i - t_{i+1} \quad i = 1, \ldots, r \quad \text{(2.3)}
\]

\[
D_r : \quad \alpha_i = t_i - t_{i+1} \quad i = 1, \ldots, r - 1 \quad \text{and} \quad \alpha_r = t_{r-1} + t_r \quad \text{(2.4)}
\]

\[
E_r : \quad \alpha_i = t_i - t_{i+1} \quad i = 1, \ldots, r - 1 \quad \text{and} \quad \alpha_r = -t_1 - t_2 - t_3 \quad \text{(2.5)}
\]

In the above parametrization the root lattice of \( A_r \) and \( E_r \) are realized as a hyperplane in \( R^{r+1} \) and that of \( D_r \) is realized in \( R^r \). Moreover the identification of the root vectors with vectors in this space can be read off from the above equations. For example for \( A_r \) they are identified with \( \hat{u}_i - \hat{u}_{i+1} \), where \( \hat{u}_i \) denote unit vectors of \( R^{r+1} \). Moreover in the \( A_r, D_r \) case the canonical Euclidean inner product on \( R^{r+1} \) and \( R^r \) induce the Cartan inner product on the roots. In the \( E_r \) case the \( R^{r+1} \) has signature \((r, 1)\) with \( \hat{u}_{r+1} \cdot \hat{u}_{r+1} = -1 \) and \( \hat{u}_i \cdot \hat{u}_i = 1 \) for \( i = 1, \ldots, r \). In the \( E_r \) case the roots are identified with \( \hat{u}_i - \hat{u}_{i+1} \) for \( i = 1, \ldots, r - 1 \) and the \( r \)-th root is identified with \( \hat{u}_r - \hat{u}_1 - \hat{u}_2 - \hat{u}_3 \).

The degrees of polynomials \( \epsilon_i \) as a function of \( t \)'s follows from the quasi-homogeneity of \( f \). The choice of \( t \)'s as opposed to \( \alpha_i \) may appear unmotivated. The reason that we choose \( t_i \)'s as the basic variables has to do with the fact that the corresponding Weyl group of the singularity has a subgroup given by the permutation group \( S_r \) which in the \( t \)-basis simply act as permutation of the \( t \)'s (and in the \( A_r \) case it is \( S_{r+1} \) and we have one more \( t \) with one extra constraint as given above). The action of the Weyl group on the parameters turns out to be very important for our considerations in this paper.
Instead of complex deformation, one can also blow up the singularity to get an $S^2$. Indeed one can consider both complex deformation and blow up, which gives a three dimensional space of deformation of the ALE metric for each $S^2$. Let us denote the corresponding real Kahler parameter by $r_i$. In other words

$$r_i = \int_{S^2_i} k$$

where $k$ is the Kahler form. The metric one obtains is hyperKahler and under the $SO(3)$ rotation the $r_i$ and $t_i$ mix. Once we consider fiberering this geometry over a plane, the rotation mixing them will no longer be a symmetry. In string theory one can also turn on $B$ fields. In the context of type IIB theory that we will be considering in this paper, there are two choices the $NS$ field $B^{NS}$ and the $R$ field $B^R$, and we can turn on both of them. Thus for each $S^2$ we have altogether a 5 parameter family of deformations in type IIB string theory. The “stringy” volume of the i-th $S^2$ is given by [8]

$$V_i = \left( (B_i^{NS})^2 + r_i^2 + |\alpha_i|^2 \right)^{1/2} \quad (2.6)$$

### 2.1. Wrapped D5 Branes

Now consider wrapping $N_i$ D5 branes around an $S^2_i$ of the deformed ALE space, and occupying an $R^4 \subset R^6$ subspace of the non-compact spacetime. This gives rise to an $\mathcal{N} = 2$ supersymmetric $U(N_i)$ gauge theory on $R^4$. The coupling constant of this gauge group is given by

$$\frac{1}{(g_i^YM)^2} = \frac{V_i}{g_s}$$
where $V_i$ is the quantum volume of the $S_i^2$ given by (2.6) and $g_s$ denotes the string coupling constant. The $B^R_i$ field plays the role of the theta angle for this gauge theory. The diagonal components of the scalar field $\Phi_i$ in the adjoint representation of $U(N)$ correspond to moving the branes on the $R^2 \subset R^6$ transverse subspace to the branes. Below we will parametrize this $R^2$ subspace by the complex coordinate $t$. The parameters $\alpha_i$ and $r_i$ form a triplet of $\mathcal{N} = 2$ supersymmetric FI terms \[9\], for the gauge theory on the brane. In an $\mathcal{N} = 1$ superspace formalism the $\alpha_i$ appear in the superpotential term

$$\delta W = \int d^2 \theta [\alpha_i \text{Tr} \Phi_i]$$

and the $r_i$ term is the $\mathcal{N} = 2$ supersymmetric completion of this term which appears as an ordinary $\mathcal{N} = 1$ FI term.

If we try to find the critical points of $W$ we might think there is a contradiction with $dW/d\Phi_i = 0$ as that would lead to $\alpha_i = 0$. However we know that there are supersymmetric wrapped branes even if $\alpha_i \neq 0$. This puzzle was noted in \[10\] where it was pointed out that since $W$ is linear in $\Phi_i$ in this case it just trivially adds a constant energy to the action, and we can still preserve the supersymmetry for non-zero holomorphic 2-volume $\alpha_i$. However if the superpotential was not linear in $\Phi_i$, as will not be the case when we consider fibering this geometry over the plane, the condition for supersymmetry will require being at the critical points of $W$.

Now consider wrapping branes about all $S_i^2$’s corresponding to the i-th node of the Dynkin diagram. Note that wrapping around any other $S^2$ can be viewed as a bound state of the configuration of this basis of $H_2$ homology of ALE. If $N_i$ denotes the number of the corresponding branes we obtain an $\mathcal{N} = 2$ gauge theory system with gauge group

$$G = \prod_{i=1}^{r} U(N_i)$$

Moreover for each neighboring (i.e. intersecting) $S_i^2, S_j^2$ we get an $\mathcal{N} = 2$ hypermultiplet, which can be viewed in $\mathcal{N} = 1$ superfield terminology as two chiral multiplets $Q_{ij}$ and $Q_{ji}$ which transform as $(N_i, \overline{N}_j)$ and $(N_j, \overline{N}_i)$ respectively. The superpotential term for this theory can be written in the $\mathcal{N} = 1$ superspace notation as

$$W = \int d^2 \theta (\sum_{i,j} s_{ij} \text{Tr} Q_{ij} \Phi_j Q_{ji} - \alpha_i \text{Tr} \Phi_i) \quad (2.7)$$
where \(|s_{ij}| = 1\) for intersecting \(S^2\)'s and zero otherwise. Moreover the sign assignment on the \(s_{ij}\) is such that it is an anti-symmetric matrix \(S\). In other words if the intersection form of the Dynkin diagram is given as \(A + A^t\) where \(A\) is an upper triangular matrix with 1 on the diagonal, \(S = A - A^t\). There are various branches for these theories, which are obtained by considering the critical points of \(W\), leading to

\[
s_{ij}Q_{ij}Q_{ji} = \alpha_i \quad Q_{ij}\Phi_j = \Phi_iQ_{ji}
\]

One should also consider the D-term equations. However, if one is just interested in parameterizing the space of solutions this latter constraint, together with gauge invariance under \(G\), is equivalent to considering the space of solutions to (2.8) subject to the equivalence given by the complex gauge transformation \(G^c = \prod_i GL(N_i)\) (see e.g. [11]) as long as FI terms \(r_i = 0\) which we will mainly assume (if \(r_i \neq 0\) one typically obtains a blow up of the moduli space). The space of solutions to (2.8) modulo \(G^c\) has been known for a long time by mathematicians [12][13] and the answer is as follows: There is one irreducible representation of the above algebra for each positive root \(\rho^k \in \Delta^+\) where \(\Delta^+\) denotes the set of positive roots of A-D-E. Moreover if we expand

\[
\rho^k = \sum_{i=1}^r n_i^k e^i
\]

where \(e^i\) denote the simple roots, then the dimension of the \(i\)-th vector space is \(n_i^k\) dimensional. For each branch there is a condition that

\[
\sum_i n_i^k \alpha_i = 0,
\]

which however is more than necessary to preserve the supersymmetry for a single brane as was discussed before. However if we wish to use all such branches at the same time, if we had more than one brane, then these conditions will generically become also necessary conditions for preserving supersymmetry. Also these conditions become necessary conditions, even if we consider a single brane, when we consider fibering these geometries, which leads to superpotentials which are not linear in \(\Phi\).

This result implies that our gauge theory will have various branches given by how many copies of each of these irreducible representations we have. Let \(M_k\) denote the
number of times the $k$-th branch appears. Then, since the total dimension of the $i$-th space should be given by $N_i$, then we must have

$$N_i = \sum_k M_k n_i^k.$$ 

Moreover this Higgses the gauge group to

$$\prod_i U(N_i) \rightarrow \prod_k U(M_k).$$

This result can be easily understood in terms of primitive 2-cycles of the A-D-E geometry. There is one primitive 2-cycle for every positive root of the corresponding A-D-E. In fact, in the context of type IIA superstring theory, wrapping D2 branes about these cycles give rise to the degrees of freedom for the gauge multiplet of the corresponding A-D-E gauge group. Thus the above classification simply implies that with a total number of branes we can wrap them about all the primitive 2-cycles subject only to the condition that the $H_2$ class of the internal charge of branes are preserved.

2.2. Stringy Orbifolds and Affine A-D-E

In the context of string theory when one studies $\mathbb{C}^2/\Gamma$ one obtains $r + 1$ choices for the basis of wrapped D-branes, where $r$ is the rank of the corresponding A-D-E [9]. The extra charge in this case arises from the $H_0$ class of ALE. The choice of the basis for the cycles are now in 1-1 correspondence with the nodes of affine A-D-E. The classes corresponding to $H_2$ can be read off from the usual correspondence with the affine Dynkin diagram. In particular the $H_2$ class of

$$H_2(\delta = e_0 + \sum_{i=1}^r d_i e_i = \sum_{i=0}^r d_i e_i) = 0 \quad (2.9)$$

where $d_i$ denotes the Dynkin index of the $i$-th node and $d_0 = 1$ and $e_0$ corresponds to the extended node of the affine Dynkin diagram. Moreover the $H_0$ class of ALE is generated by the class of the brane $\delta$ given above. In the context of type IIB string theory that we are considering we have $r + 1$ classes of wrapped brane charges: $r$ types correspond to D5 branes wrapped over two cycles in $H_2$ and 1 type correspond to D3 brane which is a point on the ALE and corresponds to $H_0$. If we wrap $N_i$ branes around the $e_i$ ( which we now include $N_0$ branes wrapped around $e_0$) , then we get the gauge theory corresponding to the affine A-D-E quiver [9]. Again we end up as in the A-D-E quiver theories with the gauge
group $\prod_{i=0}^r U(N_i)$ and a pair of chiral multiplets $Q_{ij}$ and $Q_{ji}$ for each link of the affine Dynkin diagram, with the superpotential given in (2.7). Note that since the $H_2$ class of $\delta$ is zero, and $\alpha_i$ are the holomorphic volumes of the 2-cycles, we learn that

$$\int_{\delta} \omega = \sum_{i=0}^r d_i \alpha_i = 0 \quad (2.10)$$

Finding the Higgs branches by finding the critical points of (2.7) for the affine A-D-E case leads to the existence of one irreducible representation of the algebra for each positive root of the affine A-D-E [13] (see the appendix of [14] for a review of this result). The roots of affine A-D-E can be represented by vectors in an $r+1$ dimensional lattice which is an extension of the root lattice of A-D-E by a one dimensional lattice, of the form $(R, n)$ where $R$ is a vector in the root lattice of A-D-E and $n$ is an integer. In this basis $e_i$ for the affine case correspond to $(e_i, 0)$ for $i > 0$ and $e_0 = (-\rho, 1)$, where $\rho = \sum_{i=1}^r d_i e_i$. The positive roots of affine A-D-E are of two types: “real” and “imaginary”. The real roots correspond to rigid representations with no moduli and the imaginary roots correspond to representations with moduli. The real positive roots are given by the vectors

$$(\Delta, n^+) \quad \text{and} \quad (\Delta^+, 0) \quad (2.11)$$

where $\Delta$ denotes the set of all roots of A-D-E and $n^+ > 0$ is a positive integer. Any real positive root can be written as

$$\rho^k = \sum_{i=0}^r n_i^k e_i = n(e_0 + \sum_{i=1}^r d_i e_i) + (\beta, 0)$$

for some $n_i^k \geq 0$, $n \geq 0$ and where $\beta \in \Delta$. The dimension of the $i$-th vector space associated with $\rho^k$ is given by $n_i^k$. Moreover for this branch we would get a necessary condition that

$$\sum_{i=0}^r n_i^k \alpha_i = \alpha(\beta) = 0$$

where $\alpha(\beta)$ denotes the holomorphic volume of the 2-cycle associated with $\beta$. Note again, as discussed before, this condition which is sufficient for supersymmetry preservation is not strictly necessary for a single brane but is necessary for a collection of branes or if we consider fibrations of this geometry, as we will be interested to do later in this paper.

The “imaginary” positive roots of the affine A-D-E are of the form $N\delta = N(0, 1)$, with $N > 0$ [13]. Note that for these branches the dimension of the vector space for the
$i$-th gauge group is $Nd_i$, where $d_i$ is the Dynkin number associated with the $i$-th node. These correspond to branches with moduli. Moreover for any decomposition $N = \sum N_k$, this branch includes in its moduli the subbranches corresponding to $\sum N_k \delta$. So for any representation corresponding to imaginary roots we will only have to label the total number $N$.

Let $M_k$ denote the number of times that $\rho^k$ appears in a given Higgs branch. Let $N$ denote the total number of times the imaginary root $\delta$ appears. The latter representation has a moduli which is isomorphic to the $N$ fold symmetric product of the ALE space [15]. Then by brane charge conservation we have

$$N_i = \sum_k M_k n_i^k + N d_i$$

Moreover we get the Higgsing

$$\prod_{i=0}^{r} U(N_i) \to U(N) \times \prod_k U(M_k)$$

(where for a generic point on the moduli of the $U(N)$ theory it further higgses to $U(1)^N$). Just as in the non-affine case, we can explain the existence of these branches by the fact that there are as many non-trivial primitive element of $H_2$ as the root lattice $\Delta$, and they can have arbitrary number of internal $H_0$ class bound to them. Up to a choice of an overall $Z_2$ reflection, having to do with choice of brane versus anti-brane, this explains the formula (2.11) for the choices of the branches. Moreover the branch corresponding to the imaginary roots correspond to having only the $H_0$ class. Thus if we have $N$ of them, it should have a Higgs branch moduli corresponding to the $N$-fold symmetric product of the ALE space, as expected.

3. Fibration of A-D-E Spaces

So far we have only talked about the 2-fold internal geometry. We wish to consider the fibration of these spaces over a complex plane denoted by $t$ (when we include the D5 branes this is identified with the complex dimension one space transverse to the brane). Similar kinds of fibrations have been considered in the physics [15] and math literature [7]. We fix the complexified Kahler class of the 2-fold geometry and vary the complex moduli of the ALE space. We consider $t_i = t_i(t)$ or equivalently, $\alpha_i = \alpha_i(t)$. Having a
well defined fibration allows \( \alpha_i \) not to be single valued functions of \( t \). This is because any global diffeomorphism, which is given by the Weyl group of the corresponding A-D-E leaves the geometry invariant but acts on \( \alpha_i \) by the corresponding Weyl group action. The generic fibrations will involve such monodromic actions on \( \alpha_i \). These we will call \textit{monodromic fibrations}. One can also consider the case of fibrations where \( \alpha_i(t) \) are single valued functions of \( t \). This we will call \textit{non-monodromic fibrations}.

Consider first the \( A_1 \) case. In this case we have an equation for the local 3-fold given by

\[
f = x^2 + y^2 + (z + t_1)(z + t_2) = 0 \quad \text{with} \quad t_1 + t_2 = 0
\]

using that \( \alpha = t_1 - t_2 = 2t_1 \) and a trivial rescaling of the equation we get the familiar form, and recalling that \( \alpha \) is a function of \( t \) we have

\[
f = x^2 + y^2 + z^2 + \alpha(t)^2 = 0. \tag{3.1}
\]

Over each point \( t \), there is a non-trivial \( S^2 \) whose holomorphic volume is given as \( \int_{S^2} \omega = \alpha(t) \). For a well defined fibered geometry we must have \( \alpha^2(t) \) to be a well defined function of \( t \). However this leaves room for \( \alpha(t) \) to have branch cuts in the \( t \)-plane, which would correspond to monodromic fibrations.

\[\textit{3.1. Adding in the Branes}\]

Now consider wrapping a D5 brane around \( S^2 \) fiber, just as before and consider the total fibered geometry including the D5 brane. As noted before, the modulus \( t \) corresponds to the scalar field \( \langle \Phi \rangle = t \) on the D5 brane worldvolume, which is in the adjoint representation of the gauge group on the D5 brane (for \( N \) D5 branes it will be an \( N \times N \) matrix). Using the relation between BPS tension of domain walls and the value of the superpotential, the computation of the superpotential as a function of the modulus in this case boils down to the computation of the integral of holomorphic 3-form \( \Omega = \omega \wedge dt \) over a 3-chain ending on \( S^2 \), which is parametrized by \( t \) (as in \[17][18][19]). In other words we have

\[W(\Phi) = \int_{S^2(t) \times I} \omega \wedge dt = \int_I \alpha(t) dt\]

where \( I \) is an interval in the \( t \)-plane. shifting the origin of the interval shifts \( W \) by an irrelevant constant. From this we see that

\[
\frac{dW}{d\Phi} = \alpha(\Phi)
\]
This in particular means that if we wish to induce a superpotential $P(\Phi)$, we have to define

$$\alpha(t) = \frac{dP(t)}{dt}.$$  

(3.2)

Note that the condition that we are at the critical point of the superpotential means that $\alpha(t) = 0$, i.e., this corresponds to holomorphically shrunk $S^2$. Now consider blowing up the quantum volume of $S^2$. Note that the above superpotential makes sense only when the fibration is non-monodromic, i.e. the $H_2$ class where the brane is wrapped is invariant, which in this case means that $\alpha(t)$ is a single valued function of $t$. We will assume this is the case. In this way we obtain a $U(N)$ gauge theory with $N = 1$. Moreover the blowup mode does not affect the superpotential (the Kahler deformation does not affect the superpotential computation above), and so we end up with a gauge theory with superpotential $W = P(\Phi)$. Of course if we have $N$ such D-branes we obtain the same superpotential for the $U(N)$ gauge theory where we now have to insert the trace. The quantum volume of the blown up $S^2$, for $\alpha = 0$ is given by $V = (r^2 + B_{NS}^2)^{1/2}$, which determines the coupling constant of the $U(N)$ gauge theory to be $1/g_{YM}^2 = V/g_s$ as discussed before. The $N = 1$ FI term is given by $r$ (which in this case is superfluous as there is no field charged under the $U(1)$). If we wish to have no $N = 1$ FI term, then $r = 0$ and $V = B_{NS}$. Note that the gauge theory we have geometrically engineered for this case was already studied in [18]. The observation above for the derivation of the relation between the 3-fold geometry and the gauge theory superpotential was also made independently by Aganagic [20].

Note that the $N = 2$ FI term in the context of trivial fibering discussed in the previous section is a special case of this formula where $\alpha$ is a constant and we get

$$\frac{dW}{d\Phi} = \alpha \rightarrow W = \alpha Tr \Phi.$$

For the same $A_1$ case we could consider the affine situation. In this case the geometry of the underlying manifold will be the same as we have discussed. The only difference is in the choice of the branes we consider, i.e. we also allow D5 branes bound to D3 branes. Now we will have the gauge group $U(N_0) \times U(N_1)$ where $N_0, N_1$ denote the number of branes wrapped around the classes corresponding to each one of the two nodes of the affine $A_1$ Dynkin diagram. In addition we will have two pairs of chiral multiplets $Q^i_{01}, Q^i_{10}$ where $i = 1, 2$ correspond to the two links of $A_1$. There will also be a superpotential of the form

$$W = \int d^2\theta \sum_i [\text{Tr} Q^i_{01} \Phi_1 Q^i_{10} - \text{Tr} Q^i_{10} \Phi_0 Q^i_{01}] + W_0(\Phi_0) + W_1(\Phi_1)$$
where we can identify $W_i(\Phi_i) = P_i(\Phi_i)$ where $P_i'(t) = \alpha_i(t)$. Moreover using the condition that $\sum_{i=0}^{r} d_i \alpha_i = 0$ we learn that $W_1(\Phi_1) = P_1(\Phi_1)$ and $W_0(\Phi_0) = -P_1(\Phi_0)$. The special case where $\alpha(t) = mt$ corresponding to addition of mass terms to the adjoint fields was studied in [21] and further elaborated in [22] from the point of view we are pursuing here.

So far we have presented the construction for the case of $A_1$ and affine $A_1$. The extension of this construction to general A-D-E theories with superpotential terms is straightforward. We consider non-monodromic fibrations so that we can wrap branes about non-trivial 2-cycles in the fibered geometry. In this case $t_i(t)$ or equivalently $\alpha_i(t)$ are single valued functions of $t$ and we take each one to be a polynomial in $t$. Moreover we define $\alpha_i(t) = dP_i(t)/dt$, leading to a superpotential term

$$W = \int d^2 \theta \sum_{ij} s_{ij} \text{Tr} Q_{ij} \Phi_j Q_{ji} + \sum_i \text{Tr} P_i(\Phi_i). \quad (3.3)$$

This formula is equally valid for the affine as well as the non-affine case. In the affine case one will have a constraint from $\sum_i d_i \alpha_i = 0$ which gives rise to

$$\sum_{i=0}^{r} d_i P_i(t) = 0$$

as a functional equation satisfied by the superpotentials $P_i(\Phi_i)$. This class of theories we call $\mathcal{N} = 1$ A-D-E quiver theories, and we shall now turn to the study of their Higgs branch. The affine A-D-E case with mass terms, where $P_i'(t) = m_i t$ has already been geometrically engineered in exactly this way in [22] for the case where $N_i = N d_i$ (and for the affine $A_1$ generalized to arbitrary ranks with quadratic superpotential in [23]).

4. Higgs Branches of $\mathcal{N} = 1$ A-D-E Quiver Theories: The Non-mondromic Case

Let us recapitulate what these theories are: Given an affine or ordinary Dynkin diagram of rank $r + 1$ or $r$ with some labelling of the nodes $i = 0, 1, \ldots, r$ or $i = 1, \ldots, r$ we assign a class of field theories with the following field content. Each node corresponds to an $\mathcal{N} = 1$ vector multiplet with gauge group $U(N_i)$ and an adjoint chiral field $\Phi_i$. Each arrow connecting the i-th node to the j-th node corresponds to a bifundamental chiral field $Q_{ij}$ transforming in the fundamental of $U(N_i)$ and antifundamental of $U(N_j)$. We only consider non-chiral quivers due to the $\mathcal{N} = 2$ origen of this theories. This means that $Q_{ji}$ and $Q_{ij}$ are both present or both absent.
The total gauge group of the theory is then given by $G = \prod_i U(N_i)$. The tree level superpotential is (3.3), which we write here for convenience,

$$W = \int d^2 \theta \sum_{ij} s_{ij} \text{Tr}(Q_{ij} \Phi_j Q_{ji}) + \sum_i \text{Tr} P_i(\Phi_i),$$

(4.1)

where we take $P_i(x)$ to be a polynomial of degree $s_i + 1$, and $s_{ij}$ is the anti-symmetric form associated to the corresponding Dynkin diagram. In the affine case in addition we have the constraint $\sum_{i=0}^r d_i P_i(x) = 0$ where $d_i$ are the affine Dynkin indices. As noted in the previous section these theories can be geometrically engineered by fibering wrapped branes around cycles of a fibered A-D-E geometry over a complex plane. Moreover, to obtain this structure we assumed that the fibration does not induce monodromies on the 2-cycles of A-D-E. The coupling constant for the $i$-th gauge group is given by

$$\tau_i = \theta_i + \frac{i}{g_i^2} = B_i^R + i \frac{g_s}{g_i^2} (r_i^2 + B_i^{NS^2})^{1/2}$$

and the $i$-th $\mathcal{N} = 1$ FI terms is given by $r_i$. Note that if we set these FI terms to zero, then the coupling constants $\tau_i$ are given by

$$\tau_i = B_i^R + i \frac{B_i^{NS}}{g_s} = B_i^{R} + B_i^{NS} \tau$$

where $\tau$ is the coupling constant of type IIB strings (here we also include the possibility of turning on the RR scalar of type IIB strings as part of $\tau$). Note that in the affine case, due to the fact that $\delta$ has a trivial $H_2$ class,

$$\sum_{i=0}^r d_i \tau_i = 0 \mod (n_1 + n_2 \tau)$$

where the mod condition on the right arises because $B$’s are defined up to addition of an integer. It is convenient to restrict the domain of $B$’s such that $\tau$.

$$\sum_{i=0}^r d_i \tau_i = \tau.$$

---

Aspects of this will be discussed in detail in [24]. Note that a particular solution to the above equation is $\tau_i = d_i/|\Gamma|$ which was identified in [25] as the one corresponding to the stringy orbifold $\mathbb{C}^2/\Gamma$. 

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Note that these theories can be viewed as deformations of $\mathcal{N} = 2$ supersymmetric gauge theories consisting of $\prod U(N_i)$, with bifundamental hypermultiplets according to the edges of the Dynkin diagram. The deformation to $\mathcal{N} = 1$ is achieved by adding the superpotential $W = \sum_i \text{Tr} P_i(\Phi_i)$ (the first term in (4.1) is already induced by the $\mathcal{N} = 2$ supersymmetry).

Now we wish to classify all possible classical solutions of the field equations. For simplicity let us consider the case where all the FI-terms $r_i = 0$. For this purpose we should consider the set of solutions to $dW = 0$ modulo complex gauge transformations $[11]$. It turns out that given the analysis we already did for the trivially fibered case, one can already read off the answer for the fibered case as well, as we will now argue.

The field equations obtained from varying the superpotential (4.1) are given by,

$$\sum_i (s_{ij} Q_{ji} Q_{ij}) + P_j'(\Phi_j) = 0 \quad Q_{ij} \Phi_j = \Phi_i Q_{ij} \quad \Phi_j Q_{ji} = Q_{ji} \Phi_i,$$

(4.2)

for all $j$ and where $P'_k(x)$ is the derivative with respect to $x$ of $P_k(x)$. This is essentially the same algebra we already encountered in the case of the trivially fibered geometry where now we have $P'_j$ playing the role of $\alpha_j$. However note that $P'_j$ commutes with all the elements acting on the $j$-th vector space. It clearly commutes with $\Phi_j$. Moreover from (4.2) it follows that $\Phi_j$ and therefore $P'_j(\Phi_j)$ commutes with any chain of the form $Q_{ji}Q_{ik} \ldots Q_{lj}$ and this gives the totality of operators acting on the $j$-th vector space. This implies that $P'_j$ is a c-number in any irreducible representation of this algebra. Thus we are back to classifying exactly the same algebra as we encountered in the trivially fibered case. We thus borrow the same results we already mentioned: Let $\rho^k$ be a positive root of the corresponding ADE Dynkin diagram (ordinary or affine as the case may be). For the affine case we also have to choose how many branches $N$ we choose for the imaginary direction $(0,1)$. Recall that $\rho^k$, by definition, has an expansion in simple roots $e^i$ with positive integer coefficients,

$$\rho^k = \sum_i n_i^k e^i.$$

Moreover all the allowed $\Phi_j$ in a branch have the same diagonal vev $x$ (due to (4.2)) satisfying

$$\alpha(\rho^k) = \sum_i n_i^k P'_i(x) = 0$$

(4.3)

---

3 We could also consider the case where all the $B^NS_i = 0$ and $r_i \neq 0$ which would also lead to a similar classification of branches.
(for the affine case the number \( N \) of the branches in the imaginary direction do not enter as an extra condition because \( \sum_i d_i P'_i \) is identically zero). This last equation is a polynomial equation and generically has \( d_k \) distinct solutions, where \( d_k \) is the maximum degree of the \( P'_i \)'s involved in the equation. Let \((a, k)\) denote the solutions of (4.3) with \( a \in \{1, \ldots d_k\} \).

The gauge group \( G \) will generically break as follows,

\[
G = \prod_i U(N_i) \rightarrow \prod_{(a,k)} U(M_{(a,k)}),
\]

where \( M_{(a,k)} \) denotes the number of times a given branch appears (with an extra \( U(N) \), or its Higgsings, in the affine case for the branches which contain \( N \) pure \( D3 \) branes). This statement can be made more precise by saying that each time a branch appears each \( \Phi_j \) will have \( n^k_j \) eigenvalues equal to the \((a, k)\) solution of (4.3). Therefore, \( M_{(a,k)} n^k_j \) is the total number of eigenvalues equal to the \((a, k)\) solution. Also the condition that the total dimension of the \( i \)-th vector space is \( N_i \) leads to the statement

\[
\sum_{(a,k)} M_{(a,k)} n^k_i = N_i \quad (4.4)
\]

(where the left hand side of the above equation has an extra \( Nd_i \) in the affine case). An example of the above general analysis is the case considered in [3] corresponding to \( A_1 \). The gauge group is \( U(N) \) and the theory has an adjoint field \( \Phi \) with superpotential \( W(\Phi) = P(\Phi) \) where \( P(x) \) is a polynomial of degree \( k + 1 \). The classical equation for the eigenvalues of \( \Phi \) is \( P'(x) = g_{k+1} \prod_{i=1}^r (x - a_i) \). Choosing \( M_i \) eigenvalues of \( \Phi \) to be equal to \( a_i \) breaks the gauge group to \( U(M_1) \times \ldots \times U(M_k) \) with \( \sum_{i=1}^k M_i = N \).

Here we used a general analysis of the representation of A-D-E quivers. For the \( A_r \) and \( D_r \) cases the direct analysis is simple enough so that the classification can be done very explicitly using very simple arguments that we will present below. This analysis turns out to be crucial also for the case of monodromic fibration, and we will heavily rely on it. For \( E_6 \), \( E_7 \) and \( E_8 \) it is more complicated and it requires the use of a classification theorem of indecomposable representations of quivers proven in [12] [13].

Let us show how the branch analysis works explicitly in the \( A_r \) and \( D_r \) cases.
4.1. Branches of \( A_r \)

The field equations (1.2) are given by,

\[
Q_{12}Q_{21} + P'_1(\Phi_1) = 0 \quad -Q_{21}Q_{12} + Q_{23}Q_{32} + P'_2(\Phi_2) = 0 \tag{4.5}
\]

\[
\vdots
\]

\[
-Q_{r-1,r-2}Q_{r-2,r-1} + Q_{r-1,r}Q_{r,r-1} + P'_{r-1}(\Phi_{r-1}) = 0 \quad -Q_{r,r-1}Q_{r-1,r} + P'_{r}(\Phi_r) = 0,
\]

and

\[
Q_{i,i+1}\Phi_{i+1} = \Phi_i Q_{i,i+1} \quad \Phi_{i+1}Q_{i+1,i} = Q_{i+1,i}\Phi_i \quad \text{for} \quad i = 1, \ldots, r - 1. \tag{4.6}
\]

Let us define an operation that we call conjugation by \( Q_{i,i+1} \) as follows. **Given any adjoint field \( F \) of \( U(N_{i+1}) \) we construct a new adjoint field of \( U(N_i) \) given by \( Q_{i,i+1}FQ_{i+1,i} \).** A very important case will be when \( F \) is a polynomial in \( \Phi_{i+1} \). In this case, conjugation will have a simple action, namely, from (4.6) we have

\[
Q_{i,i+1}F(\Phi_{i+1})Q_{i+1,i} = F(\Phi_i)Q_{i,i+1}Q_{i+1,i}. \tag{4.7}
\]

At each node \((j)\) we have three natural adjoint fields, namely, \( \Phi_j \), \( Q_{j,j-1}Q_{j-1,j} \) and \( Q_{j,j+1}Q_{j+1,j} \). For \( j = 1 \) we define \( Q_{1,0}Q_{0,1} \equiv 0 \) and for \( j = r \) we define \( Q_{r,r+1}Q_{r+1,r} \equiv 0 \). The idea is to find a set of three equations for each node that will only involve adjoint fields at that node. This will provide enough information about the eigenvalues of each of them since the three adjoint fields commute among themselves. To see this remember that we showed in the general discussion that \( P'_j(\Phi_j) \) commutes with any chain of bifundamentals, in particular, with \( Q_{j,j-1}Q_{j-1,j} \) and with \( Q_{j,j+1}Q_{j+1,j} \). Finally, considering the field equation, \(-Q_{j,j-1}Q_{j-1,j} + Q_{j,j+1}Q_{j+1,j} + P'_j(\Phi_j) = 0 \) is simple to see that \( Q_{j,j+1}Q_{j+1,j} \) commutes with \( Q_{j,j-1}Q_{j-1,j} \).

The first step in getting the equations at the \((j)\)-th node is to conjugate the first equation in (4.3) by \( Q_{21} \). Doing so and using (4.7) we get,

\[
Q_{21}Q_{12}(Q_{21}Q_{12} + P'_1(\Phi_2)) = 0,
\]

using now the second equation in (4.7) to replace \( Q_{21}Q_{12} \) by \( Q_{23}Q_{32} + P'_2(\Phi_2) \) we get,

\[
(Q_{23}Q_{32} + P'_2(\Phi_2))(Q_{23}Q_{32} + P'_1(\Phi_2) + P'_2(\Phi_2)) = 0.
\]
Conjugating next by $Q_{32}$ and repeating the same procedure until reaching the $(j - 1)$-th node we get,

\[ X_j (X_j + P'_{j-1}(\Phi_j)) (X_j + P'_{j-2}(\Phi_j)) + \ldots + P'_{j-1}(\Phi_j)) = 0, \quad (4.8) \]

where we have denoted $Q_{j,j-1}Q_{j-1,j}^\dagger$ by $X_j$ to simplify the equation.

The second equation can be obtained by repeating the same procedure as before but starting at the last node and conjugating by $Q_{r-1,r}$. Repeating this until reaching the $(j + 1)$-th node will produce the following equation,

\[ Y_j (Y_j + P'_{j+1}(\Phi_j)) (Y_j + P'_{j+2}(\Phi_j)) \ldots + P'_{r}(\Phi_j)) = 0, \quad (4.9) \]

where we have denoted $-Q_{j,j+1}Q_{j+1,j}^\dagger$ by $Y_j$.

Finally, the last equation we need is the original field equation coming from the variation w.r.t $\Phi_j$ of the superpotential.

\[ X_j + Y_j = P'_{j}(\Phi_j). \quad (4.10) \]

It turns out to be very useful to remember the identification of $P'_i$ with the simple root $\alpha_i$ and introduce the $t_i$ coordinates as in (2.3). This gives $P'_i(\Phi_j) = t_i(\Phi_j) - t_{i+1}(\Phi_j)$ for $i = 1, \ldots, r$. In terms of these and shifting $X_j \to X_j + t_j$ and $Y_j \to Y_j - t_{j+1}$ we get field equations of the form,

\[ \prod_{i=1}^{j} (X_j + t_i) = 0 \quad \prod_{i=j+1}^{r+1} (Y_j - t_i) = 0 \quad X_j + Y_j = 0 \quad (4.11) \]

Since the three operators commute, we can choose basis in which all three are diagonal. Let $\vec{v}$ be an eigenvector of $\Phi_j$, $X_j$ and $Y_j$ with eigenvalues $\phi$, $x_j$ and $y_j$ respectively. These three eigenvalues will have to solve (4.11) when each of the operators is replaced by its eigenvalue.

The possible solutions are then given by choosing $x_j$ to be $-t_k$ for some $k = 1, \ldots, j$ and $y_j$ to be $t_l$ for some $l = k + 1, \ldots, r$. However, the last equation in (4.11) implies that,

\[ x_j + y_j = -t_k + t_l = 0 \]

or equivalently,

\[ \sum_{m=k}^{l-1} P'_m(\phi) = 0 \]
Notice that these are exactly the solutions given in (4.3) since the positive roots for $A_r$ are $\sum_{k=n}^{m} e^k$ with $n \leq m$.

Finally, we want to show that $\phi$ is an eigenvalue only of $\Phi_m$ for $m = k, \ldots, l - 1$ and moreover, that the dimension of the corresponding eigenspaces are all the same.

From (4.4) it is easy to see that the $\phi$ eigenspace of $\Phi_i$ is either sent to the $\phi$ eigenspace of $\Phi_{i+1}$ or belongs to the kernel of $Q_{i+i,i}$. This implies that we can consider the restriction of the field equations (4.5) to $\phi$ eigenspaces. Finally, we only have to take the trace of each of the restrictions in (4.5) to get,

$$\text{Tr} P_1'(\Phi_1) + \text{Tr} P_2'(\Phi_2) + \ldots + \text{Tr} P_r'(\Phi_r) = \sum_{i=1}^{r} n_i P_i'(\phi) = 0 \quad (4.12)$$

where $n_i$ is the dimension of the $\phi$ eigenspace of $\Phi_i$. But $\phi$ cannot satisfy two polynomial equations simultaneously for generic $P_i'(\phi)$. Therefore, we are led to conclude that $n_i = 0$ for $i \neq k, \ldots, l - 1$ and $n_i = n$ for $i = k, \ldots, l - 1$ with $n$ the number of times we want the given branch to appear.

4.2. Branches of $D_r$

Let us use the labels for the nodes given in Figure 1 for $D_r$. Clearly, the novelty in this case corresponds to the node $(r-2)$. All branches that do not contain this node are simply $A_{r-3}$ and two $A_1$’s. For this reason we will only study the field equations concentrated at the $(r-2)$ node.

The field equations are now given in terms of 3 bifundamentals and one adjoint field. These are $X = Q_{r-2,r-3}Q_{r-3,r-2}$, $Z = Q_{r-2,r-1}Q_{r-1,r-2}$, $Y = Q_{r-2,r}Q_{r,r-2}$ and $\Phi_{r-2}$.

Conjugating from node $(1)$ to node $(r-2)$ we get,

$$X(X + P_{r-3}'(\Phi_{r-2})) \ldots (X + P_{r-3}'(\Phi_{r-2}) + \ldots + P_1'(\Phi_{r-2})) = 0$$

Conjugating from nodes $(r-1)$ and $(r)$ to node $(r-2)$ we get,

$$Z(Z + P_{r-1}'(\Phi_{r-2})) = 0 \quad Y(Y + P_r'(\Phi_{r-2})) = 0$$

Finally, the equation at node $(r-2)$ is,

$$X + Y + Z = P_{r-2}'(\Phi_{r-2})$$
As for the \( A_r \) case, it turns out to be useful to use (2.4) to write \( P'_i = t_i - t_{i+1} \) for \( i = 1, \ldots, r - 1 \) and \( P'_r = t_{r-1} + t_r \). The field equations are very simple if we shift, \( X \rightarrow X + t_r - 2 \), \( Y \rightarrow Y - \frac{1}{2}(t_{r-1} + t_r) \) and \( Z \rightarrow Z - \frac{1}{2}(t_{r-1} - t_r) \). Then the equations are given by,

\[
\prod_{i=1}^{r-2} (X + t_k) = 0 \quad Y^2 = \frac{1}{4}(t_{r-1} + t_r)^2 \quad Z^2 = \frac{1}{4}(t_{r-1} - t_r)^2 \quad X + Y + Z = 0 \quad (4.13)
\]

In this case, \( X, Y \) and \( Z \) may not commute and we have to be more careful since some branches might correspond to higher dimensional representations of these algebra. Let us study each possibility stating from the one dimensional.

The one dimensional (i.e, \( X, Y \) and \( Z \) are c-numbers) solutions are easily found. Clearly, \( Y + Z \) can only be \( \pm t_{r-1} \) or \( \pm t_r \). Using this in the equation for \( X \) we get that all the solutions can be collected in a single equation,

\[
\prod_{i=1}^{r-2} (t_{r-1}^2 - t_i^2)(t_r^2 - t_i^2) = 0
\]

Going back to the language of the simple roots \( P'_j \)'s we see that the solutions are in one to one correspondence with the positive roots containing \( P'_{r-2} \) with coefficient one.

Now we can look for two dimensional representations. For this we have to promote \( X, Y \) and \( Z \) to \( 2 \times 2 \) matrices. Notice that if any of them is proportional to the identity, then using the last equation of (1.13) we get that the other two commute and we are back to the one dimensional case. Therefore, in order to get 2 dimensional representations we need each of them to have two distinct eigenvalues. For \( Y \) and \( Z \), notice that this immediately implies that \( \text{Tr} Y = \text{Tr} Z = 0 \). This follows from the fact that the second and third equations in (1.13) imply that \( Y \) and \( Z \) can each only have two eigenvalues that only differ in sign.

Using that \( X + Y + Z = 0 \) we can conclude that \( \text{Tr} X \) is also zero. But in some basis \( X = -\text{diag}(t_i, t_j) \) for some \( i, j \in \{1, \ldots, r - 2\} \) with \( i \neq j \). Therefore the tracelessness condition is simply the following polynomial equation \( t_i(\phi) + t_j(\phi) = 0 \) where \( \phi \) is an eigenvalue of \( \Phi_{r-2} \).

Going back to the simple root notation, namely, \( P'_i \) we see that \( t_i + t_j \) with \( i, j = 1, \ldots, r - 2 \) and \( i \neq j \) precisely produce all the positive roots were \( P'_{r-2} \) enters with coefficient two.
Using the same argument as for the $A_r$ case, we can prove that for $\phi$ of a given branch associated to a root $\rho^k$ the corresponding $\Phi_i$ eigenspace will have dimension $M_kn_i^k$ where $n_i^k$ is given by

$$\rho^k = \sum_j n_j^k e^j$$

and $M_k$ denotes the number of times we choose that branch to appear.

It is also easy to show that no irreducible representation with dimension greater than two exists (similar to the arguments we present in the context of Laufer’s example in section 7.2), and that this analysis completes the list of all irreducible representations and shows that they are in one to one correspondence with positive roots of $D_r$.

5. Higgs Branch for Pure D3 Branes

The case where we only have D3 branes, i.e. where we have $N_i = N d_i$ for some fixed $N$, gives rise to a theory with $N$ D3 branes. In particular the case with $N = 1$ should have a moduli space given by the moduli of a D3 brane transverse to the threefold geometry. This space should in fact be isomorphic to the space itself. Let us see how the moduli space of Higgs branch in this case gives back the threefold in this case for the affine $A_r$ case.

Consider the case where all $N_i = 1$ i.e, the gauge group is $U(1)^{r+1}$. Let us recall the field equations for the affine $A_r$.

$$-Q_{0,r}Q_{r,0} + Q_{01}Q_{10} + P_0'(\Phi_0) = 0 \quad -Q_{10}Q_{01} + Q_{12}Q_{21} + P_1'(\Phi_1) = 0$$

$$-Q_{21}Q_{12} + Q_{23}Q_{32} + P_2'(\Phi_2) = 0 \quad \ldots \quad -Q_{r-1,r}Q_{r-1,r} + Q_{r,0}Q_{0,r} + P_r'(\Phi_r) = 0$$

In this case $\Phi_i = \phi$ is independent of $i$. Let us use that $P_i' = t_i - t_{i+1}$ and recall that we have an extra condition given by $\sum_{i=0}^r P_i'(\phi) = 0$ or in terms of $t_i$’s we get, $t_0 = t_{r+1}$.

Let us define the following coordinates for the moduli space,

$$u \equiv Q_{0,r}Q_{r,r-1}Q_{r-1,1}Q_{1,0} \quad v \equiv Q_{0,1}Q_{1,2}Q_{r-1,r}Q_{r,0} \quad \text{and} \quad x = Q_{r,0}Q_{0,r} - t_0$$

The equations of motion imply that we can write $Q_{i-1,i}Q_{i,i-1} = x + t_i$ for $i = 0, \ldots, r$.

It is easy to see that $u, v, x, t$ form a complete set of holomorphic gauge invariant observables. These three variables are not independent. In order to see the relation, consider,

$$uv = (Q_{0,r}Q_{r,r-1}Q_{r-1,1}Q_{1,0})(Q_{0,1}Q_{1,2}Q_{r-1,r}Q_{r,0}) = (Q_{0,1}Q_{1,0}) \ldots (Q_{r,0}Q_{0,r})$$
where we have used the fact that we are considering abelian gauge theories in reshuffling the position of the $Q'$s.

Using the field equations we get,

$$uv = \prod_{i=1}^{r+1}(x + t_i)$$

that is the equation describing the $A_r$ fibration over the $t$-plane (2.1). Thus we see that the moduli space of the Higgs branch for the D3 brane is exactly the same as the $A_r$ fibered geometry, as expected. This is a further confirmation of the picture we have developed. This generalizes the same analysis done in [22] for the case of quadratic superpotentials.

6. N=1 A-D-E Quiver Theories in the Monodromic Case

In this section we will begin our discussion of the monodromic case. Recall that this is the case where the A-D-E fibration undergoes Weyl reflections for the 2-cycles of the geometry. For simplicity we consider the non-affine case. In general the relevant monodromy will generate a subgroup of the corresponding Weyl group, which could also be the full group. We will discuss examples of all of these cases, but our main focus will be on the cases corresponding to the monodromy group being the subgroup of the full Weyl group generated by Weyl reflections of all the nodes except for one. In these cases, as we will discuss, there is a single blow up mode, corresponding to blowing up the node with no monodromy. This we study in detail, as it is the simplest case leading to some new gauge theories, corresponding to wrapping D5 branes around the blown up $S^2$, leading to $U(N)$ gauge theory with some matter content. It is also the case where quite a bit is known about the geometry of the local 3-fold [7].

More generally, consider a local Calabi-Yau threefold with a single $H_2$ element, represented by a $\mathbb{P}^1$. An interesting mathematical question is to classify all such $\mathbb{P}^1$'s that can shrink inside a Calabi-Yau. The normal bundle to $\mathbb{P}^1$ in the blown up CY can be only one of three cases: i) $O(-1) \oplus O(-1)$, ii) $O(0) \oplus O(-2)$ and iii) $O(1) \oplus O(-3)$. The case i) is rather trivial. The more interesting cases correspond to ii) and iii). It turns out that all such $\mathbb{P}^1$'s can be obtained by considering monodromic fibrations of A-D-E, where only one class can be blown up to give rise to $\mathbb{P}^1$, just as in the above case. This has been studied in detail in [7], where case ii) arises from A quiver theories and case iii) arise in D and E quiver theories. In particular case ii) is exemplified by an A-fibration where the
monodromy group is generated by Weyl reflection about all nodes except for one (any of the nodes). For case iii) the inequivalent cases can be chosen so that the special node is the trivalent node of the D and E Dynkin diagrams—except for one extra $E_8$ case where the special node is one with Dynkin number 5.\footnote{The inequivalent local geometries can be classified by the Dynkin numbers of the corresponding A-D-E node, which lead to $A_1$ (1), $D_4$ (2), $E_6$ (3), $E_7$ (4), $E_8$ (5), $E_8$ (6).}

Now consider wrapping N D5 branes around such a $\mathbb{P}^1$. This gives rise to an $\mathcal{N} = 1$ $U(N)$ gauge theory. Moreover the number of adjoint matter fields depend on the number of normal deformations of $\mathbb{P}^1$ in the Calabi-Yau, i.e. it should be a holomorphic section of the normal bundle $[16]$. This implies that in case ii) there is one and in case iii) there are two. This matches nicely the cases one would expect based on asymptotic freedom for the gauge theory. We find the adjoint matter fields satisfy some constraints. In some cases, these constraints correspond to critical points of a superpotential, and the corresponding gauge theory can be presented as a $U(N)$ theory with one or 2 adjoint fields with some superpotential.

The organization of our discussion for the monodromic case is as follows: We first present examples of the monodromic A-D-E fibrations. We then state the mathematical conditions needed for the fibration to have a single $S^2$ blow up for cases ii) and iii). We then analyze the gauge theory of the wrapped branes around the corresponding $S^2$. Finally in section 7 we present some detailed examples.

6.1. Simple Examples of Monodromic Fibration

We start with an A-D-E geometry, and we fiber this over the complex plane. However, we consider a situation where the fibration induces monodromy action on the A-D-E 2-cycles. In other words, as we go around various loops in the base of the fibration the total geometry comes back to itself, but not each individual 2-cycle. They get reshuffled, according to an element of the Weyl group of A-D-E. The monodromy group we obtain will be a subgroup of the Weyl group. If the monodromy group is the full Weyl group, then this means that there is no invariant 2-cycle in the fiber geometry, which also implies that we cannot blow up any 2-cycle in the full 3-fold geometry. In fact this would be the generic case of A-D-E fibration. In such a case we will not have any possibility of wrapping 5-branes around 2-cycles of the Calabi-Yau.
However we can also consider situations where the monodromy only partially mixes the 2-cycles. In such cases there would be left-over 2-cycles in the full 3-fold geometry for which we can wrap branes around and obtain non-trivial $\mathcal{N} = 1$ supersymmetric gauge theories.

Let us first give examples of both of these types. Consider the $A_1$ case fibered over the $t$ plane:

$$x^2 + y^2 + (z + t_1(t))(z + t_2(t)) = 0$$

As noted before, with no loss of generality we can take $t_1 = -t_2$. If $t_1(t)$ is a single valued function of $t$ we are back to the non-monodromic case we have studied. However if $t_1$ has branch cuts as a function of $t$ then we are in the monodromic case. Suppose for example

$$t_1(t) = -t_2(t) = t^{n + \frac{1}{2}}$$

In this case as we go around the origin in $t$ plane we exchange the $t_1 \leftrightarrow t_2$, which corresponds to $A_1$ Weyl reflection. The geometry in this case is given by

$$x^2 + y^2 + z^2 - t^{2n + 1} = 0$$

To begin with we have a 2-cycle corresponding to $e_1$ node of $A_1$. However this cycle does not survive the fibration because as we go around the origin $e_1 \rightarrow -e_1$. Thus in the full 3-fold geometry there is no non-trivial 2-cycle. More precisely what we mean is that the above complex geometry cannot be blown up to yield a 2-cycle. In such a situation we cannot wrap D5 branes around any non-trivial 2-cycles as there are none. One could try to “force” a D5 brane in this situation as follows: Consider $t \neq 0$. Then there is a non-trivial 2-cycle over this point, as we just have an $A_1$ geometry. Wrap the D5 branes around this cycle. Then as discussed before this yields a deformed $\mathcal{N} = 2$ gauge theory with superpotential $W(\Phi) \sim \text{Tr}\Phi^{n + \frac{2}{3}}$. However this is only heuristic: As we mentioned the 2-cycle cannot be blown up, which means that the quantum 2-volume is zero, so the gauge coupling is infinite. The fact that the we are getting branch cuts in the superpotential is another sign that forcing a D5 brane in this situation is not allowed.

Let us give one more example along these lines: Consider again the $A_1$ case, but now take $t_1 = -t_2 = \sqrt{t(t - 2a)}$ with $a \neq 0$. We thus have two points along the $t$ plane where we get a Weyl reflections $t = 0$ and $t = 2a$. Thus again we would have no non-trivial 2-cycles in the total geometry. This geometry is given by

$$x^2 + y^2 + z^2 - t(t - 2a) = 0$$

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which by a shift $t \to t + a$ is equivalent to

$$x^2 + y^2 + z^2 - t^2 + a^2 = 0$$

This is the deformed conifold, and clearly there are no non-trivial 2-cycles in this geometry as expected.

A more interesting case, from the gauge theory viewpoint, is when we have a partial monodromy. Consider for example an $A_2$ geometry fibered over the plane:

$$x^2 + y^2 + (z - t_1(t))(z - t_2(t))(z - t_3(t)) = 0$$

Let us suppose that as we go around $t = 0$ the $t_2 \leftrightarrow t_3$, and $t_1$ is invariant. In other words this generates a $Z_2$ subgroup of the Weyl group of $A_2$ which is given by permutation group of three elements $S_3$. For example let us take the case

$$t_1(t) = bt \quad t_2(t) = -at + t^{n+\frac{1}{2}} \quad t_3(t) = -at - t^{n+\frac{1}{2}}$$

This yields the threefold geometry

$$x^2 + y^2 + (z - bt)((z + at)^2 - t^{2n+1}) = 0$$

which as expected is single valued over the $t$ plane. The full threefold geometry does admit a blow up where we obtain exactly one non-trivial 2-cycle\[5\]. To see this, note that if we identify the nodes $e_1, e_2$ of $A_2$ with $\alpha(e_1) = (t_1 - t_2)$ and $\alpha(e_2) = (t_2 - t_3)$ then the Weyl group as we go around $t = 0$ acts by

$$e_1 \to e_1 + e_2$$

$$e_2 \to -e_2$$

and the combination $2e_1 + e_2$ is invariant. So we can blow up this class and wrap $N$ D5 branes around it and ask what $\mathcal{N} = 1$ gauge theory it corresponds to?

Note that when we blow up, the quantum volume of $e_2$ is zero, (as $e_2 \to -e_2$ under the Weyl reflection) so the corresponding gauge coupling constant is infinite. On the other hand the volume of $e_1$ is finite and is the same as the quantum volume of $e_1 + 

\[5\] In this case only for the case with $n = 0$ the blow up geometry is smooth. The general condition for the blow up to be smooth will be spelled out in the next subsection.
This implies that the gauge group for the node 1 will be conventional, whereas for the second node is unconventional. This suggests that we should get some insight into this geometry by concentrating on the gauge theory description associated to node 1 in the geometry. In other words from the corresponding quiver theory we only keep the observables which are trivial under all the other gauge groups, i.e., geometrically speaking are monodromy invariant. This means that we promote certain composite fields involving chiral multiplets which are neutral under all the would be gauge groups corresponding to nodes with monodromy and which transform in the adjoint representation of the gauge group corresponding to the blown up node, as fundamental fields. We will see that this approach makes sense, and yields results which we check with other methods as well.

Let us specialize to a simple example where $A_2 \rightarrow A_1$: Consider the geometry

$$x^2 + y^2 + (z^2 + az + t)(z - t) = 0$$

This corresponds to

$$t_1(t) = t, \quad t_{2,3} = \frac{-a}{2} \pm \frac{\sqrt{a^2 - 4t}}{2}$$

If we consider the $\mathcal{N} = 1$ $A_2$ quiver theory associated to this case, as studied in section 4, we have two bifundamental fields $Q_{12}$ and $Q_{21}$ and if we define

$$X = Q_{12}Q_{21}$$

writing the equations we get at the first node, where the gauge theory has finite coupling constant we have from (4.8) and (4.9)

$$X = P'_1(\Phi_1) = t_1(\Phi_1) - t_2(\Phi_1)$$

$$X(X + P'_2(\Phi_1)) = X(X + t_2(\Phi_1) - t_3(\Phi_1)) = 0$$

Note that $t_1$ is a well defined field as it is single valued. However $t_2, t_3$ are not good fields as they are not single valued functions of $\Phi_1$. This implies that $X$ is not a good field. To this end we define

$$\tilde{X} = X + t_2$$

in terms of which we obtain

$$\tilde{X} = t_1(\Phi_1) = \Phi_1$$

$$(\tilde{X} - t_2)(\tilde{X} - t_3) = (\tilde{X}^2 + a\tilde{X} + \Phi_1) = 0$$
From these equations we see that we can eliminate \( \Phi_1 \) using the first equation and obtain

\[
\tilde{X}^2 + (a + 1)\tilde{X} = 0
\]

Identifying this as \( W'(\tilde{X}) = \tilde{X}^2 + (a + 1)\tilde{X} \), this can be obtained from a theory with superpotential

\[
W(\tilde{X}) = \text{Tr} \left( \frac{\tilde{X}^3}{3} + (a + 1)\frac{\tilde{X}^2}{2} \right)
\]

Thus we seem to come to the conclusion that we have a single gauge group with adjoint field \( \tilde{X} \) with the above superpotential. Indeed if we redefine the coordinates of the CY 3-fold geometry (6.1) by

\[
\rho = z - t - \frac{1}{2}z(z + a + 1)
\]

This corresponds to the geometry

\[
x^2 + y^2 - \rho^2 + \frac{1}{2}W'(z)^2 = 0
\]

which up to rescalings is exactly the theory corresponding to a single adjoint field with superpotential \( W' \), discussed before. This gives us confidence that the idea we have proposed for extracting the gauge theory in situations with non-trivial monodromies is sound.

Before dealing with the general case using the same methods, we turn to a mathematical discussion of the monodromic fibrations for which only one node can be blown up.

6.2. Mathematical Description of Monodromic Fibrations

Let us describe the threefold geometry in the monodromic case where the monodromy group is generated by Weyl reflections about all nodes except for one. This discussion parallels that of [7] but our viewpoint is slightly different.

We start with a collection of positive simple roots of the A-D-E geometry, corresponding to a subcollection of the holomorphic spheres. These roots generate a subdiagram \( \Gamma \) of the Dynkin diagram. Monodromy is introduced by simultaneously taking these spheres to zero size and taking the subgroup of the Weyl group generated by the associated reflections. This A-D-E geometry is singular at the points \( p_i \) which have replaced these curves. These are in one to one correspondence with the connected components \( \Gamma_i \) of \( \Gamma \).
Fibering this geometry over the \( t \)-plane in a generic fashion will smooth out the singularities, but the threefold geometry can be singular at the \( p_i \) for a non-generic fibering. We will specify the condition for obtaining a smooth threefold.

The key technical result we use to simplify computations is the assertion that the geometry near the singularity \( p_i \) can be computed by the following simple procedure: take the equation of the deformation of \( p_i \) (in one of the forms (2.1) (2.2) given in Section 2), then make a substitution given by the projection onto the root subspace \( V_i \) generated by the component \( \Gamma_i \) of \( \Gamma \) corresponding to \( p_i \). A precise statement is given in Theorem 3 of [7]. Instead of setting up the machinery required for the general case, we illustrate with examples and refer the reader to [7] for more details.

![Diagrams of A, D, E types](image)

**Figure 2**: We consider fibrations of A-D-E geometry where the monodromy group is generated by Weyl reflection about all nodes except for the black node.

\[ A_n \rightarrow A_1 \text{ via a partial resolution} \]

We illustrate with the case of an \( A_n \) singularity where only one vertex is blown up.

Recall the general deformation of an \( A_n \) for which all \( n \) vertices of the Dynkin diagram get blown up.

\[
x^2 + y^2 + \prod_{i=1}^{n+1} (z + t_i) = 0 \quad \sum_{i=1}^{n+1} t_i = 0
\]

(6.2)

If we only want to blow up vertex number \( k \), we are left with a surface \( Y_0 \) with a single \( \mathbb{P}^1 \) with two singularities, an \( A_{k-1} \) and an \( A_{n-k} \) (we modify conventions accordingly if
$k = 1$ or $k = n$; in those cases there is only one singularity). Denote by $Z_0$, the blown up geometry of $n$ curves in an $A_n$ configuration. As discussed in section 2, the space of complex deformations $\text{Def}(Z_0)$ is the hyperplane $\sum t_i = 0$ in the affine space with coordinates $t_1, \ldots, t_{n+1}$, the corresponding deformation being described by (6.2).

The deformation space $\text{Def}(Y_0)$ is a quotient of $\text{Def}(Z_0)$ by $S_k \times S_{n-k+1}$, i.e. the Weyl group generated by all nodes except for the $k$-th node. The coordinates on $\text{Def}(Z_0)$ are naturally given by appropriate symmetric functions of the $t_j$:

$$\sigma_i(t_1, \ldots, t_k), \ \ i = 1, \ldots, k, \ \ \tilde{\sigma}_i(t_{k+1}, \ldots, t_{n+1}), \ \ i = 1, \ldots, n - k + 1.$$ 

where

$$\sigma_i(x_1, \ldots, x_n) = \sum_{j_1, j_2, \ldots, j_i \text{ distinct}} x_{j_1} \cdots x_{j_i}.$$ 

Only $n$ of these are independent coordinates on $\text{Def}(Z_0)$ due to the relation

$$\sigma_1(t_1, \ldots, t_k) + \tilde{\sigma}_1(t_{k+1}, \ldots, t_{n+1}) = 0.$$ 

The equation (6.2) becomes

$$x^2 + y^2 + (z^k + \sum_i \sigma_i z^{k-i}) (z^{n-k+1} + \sum_i \tilde{\sigma}_i z^{n-k-i+1}) = 0. \quad (6.3)$$

The CY geometry is obtained by letting the $\sigma_i, \tilde{\sigma}_i$ be holomorphic functions of $t$. If they all vanish at $t = 0$ (which we will not in general require) it was shown in [7] that the condition for the single blow up to be smooth near $t = 0$ is that $\sigma_k(t)$ and $\tilde{\sigma}_2^{n-k+1}(t)$ vanish to first order in $t$.

**$D_r \rightarrow \text{trivalent vertex}$**

If we blow down all but the trivalent vertex of the $D_r$, we get an $A_{r-3} \times A_1 \times A_1$ subdiagram with Weyl subgroup $S_{r-2} \times Z_2 \times Z_2$. With reference to the coordinate system $t_1, \ldots, t_r$ given in equation (2.2), the $S_{r-2}$ acts on the first $r - 2$ coordinates, and the $Z_2$ reflections are

$$(t_{r-1}, t_r) \mapsto (t_r, t_{r-1})$$

$$(t_{r-1}, t_r) \mapsto (-t_r, -t_{r-1}).$$

The invariant coordinates are $\sigma_i = \sigma_i(t_1, \ldots, t_{r-2})$ as well as $\tilde{\sigma}_1^2 = (t_{r-1} + t_r)^2$ and $\tilde{\sigma}_2 = t_{r-1} t_r$.  

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The projections of Theorem 3 can be described in terms of two projections, first onto the space \( V_1 \) spanned by the first \( r - 2 \) roots and second onto the space \( V_2 \) spanned by the last 2 roots. The first describes an \( A_{r-2} \) singularity, and the second gives the \( A_1 \times A_1 \) (which we conveniently think of as a \( D_2 \)). Near the \( A_r \) the geometry is given by

\[
x^2 + y^2 + z^{r-1} + \sum_{i=1}^{r-1} \sigma_i(t)z^{r-i} = 0.
\]

This is smooth at \((x, y, z, t)\) unless \(x = y = 0\) and \(z^{r-1} + \sum_{i=1}^{r-1} \sigma_i(t)z^{r-i} = 0\) has a multiple root at \((z, t)\). Near the \( D_2 \), we find the local equation by putting \(r = 2\) in (2.2) and rewriting in terms of the invariant coordinates. We get

\[
x^2 + y^2z + \frac{(z - t^2_{r-1})(z - t^2_r) - t^2_{r-1}t^2_r}{z} + 2t_{r-1}t_ry = 0
\]

or

\[
x^2 + y^2z + z + (2\tilde{\sigma}_2(t) - \tilde{\sigma}_1^2(t)) + 2\tilde{\sigma}_2(t)y = 0. \tag{6.4}
\]

The geometry (6.4) is smooth at \((x, y, z, t) = (0, 0, 0, 0)\) when \(\tilde{\sigma}_2^2(t)\) and \((\tilde{\sigma}_1^2 - 4\tilde{\sigma}_2)(t)\) have simple zeros at \(t = 0\). Thus, assuming that the \(t_i\) vanish at \(t = 0\) (which we will not necessarily assume) then the condition to have a smooth blow up is that \(\tilde{\sigma}_2^2(t)\) and \((\tilde{\sigma}_1^2 - 4\tilde{\sigma}_2)(t)\) have simple zeros at \(t = 0\).

If desired, the \( D_2 \) can be replaced by two local calculations near each of the \( A_1 \)s. The result is of course the same.

**\( E_n \rightarrow \text{trivalent vertex} \)**

We now consider the \( E_n \) case with \( n = 6, 7, 8 \) where the blown up sphere corresponds to the trivalent vertex. Suppose we blow down all holomorphic spheres except the one corresponding to the trivalent vertex of \( E_n \). Then \( \Gamma = A_2 \times A_{n-4} \times A_1 \).

We present the roots and consequently the Weyl group action as in [7]. For convenience of the reader, the \( W(A_{n-1}) = S_n \) subgroup acts on \((t_1, \ldots, t_n)\) in the usual way, while the reflection in the remaining root is given by sending \((t_1, \ldots, t_n)\) to

\[
\left( t_1 - \frac{2}{3}T, t_2 - \frac{2}{3}T, \ldots, t_3 - \frac{2}{3}T, t_4 + \frac{1}{3}T, \ldots, t_n + \frac{1}{3}T \right) \tag{6.5}
\]

where \(T = t_1 + t_2 + t_3\).

The \( W(A_2 \times A_{n-4} \times A_1) \) invariant coordinates are
\[
\sigma_j = \sigma_j \left( t_1 - \frac{T}{3}, t_2 - \frac{T}{3}, t_3 - \frac{T}{3} \right), \quad j = 2, 3
\]
\[
\tilde{\sigma}_j = \sigma_j \left( t_4 + \frac{T}{6}, \ldots, t_n + \frac{T}{6} \right) \quad j = 1, \ldots, n-3
\]
\[
\rho_2 = T^2.
\]

Near the \(A_2\) singularity, the local geometry is given by
\[
x^2 + y^2 + z^3 + \sigma_2(t)z + \sigma_3(t) = 0.
\]
Near the \(A_{n-4}\) singularity, the local geometry is given by
\[
x^2 + y^2 + z^{n-3} + \tilde{\sigma}_1(t)z^{n-4} + \ldots + \tilde{\sigma}_{n-3}(t) = 0.
\]
Near the \(A_1\) singularity, the equation is
\[
x^2 + y^2 + z^2 + \rho_2(t) = 0.
\]

So the condition for smoothness at \(z = t = 0\) is that the expressions \(\sigma_3\), \(\tilde{\sigma}_{n-3}\), and \(\rho_2\) all have simple zeros at \(t = 0\). General values of \((z, t)\) are similar.

**Second \(E_8\) case**

Suppose we blow down all holomorphic spheres except the one corresponding to the vertex of \(E_8\) with Dynkin number 5. Then \(\Gamma = A_4 \times A_3\). We have already described the Weyl group action in the last section.

The first four \(W(A_4 \times A_3)\) invariant coordinates are
\[
\sigma_j = \sigma_j (s_1, s_2, s_3, s_4, s_5), \quad j = 2, 3, 4, 5,
\]
where
\[
s_1 = t_1 - \frac{2}{5} (t_1 + t_2 + t_3 + t_4)
\]
\[
s_2 = t_2 - \frac{2}{5} (t_1 + t_2 + t_3 + t_4)
\]
\[
s_3 = t_3 - \frac{2}{5} (t_1 + t_2 + t_3 + t_4).
\]
\[
s_4 = t_4 - \frac{2}{5} (t_1 + t_2 + t_3 + t_4)
\]
\[
s_5 = \frac{3}{5} (t_5 + t_6 + t_7 + t_8)
\]
The remaining four coordinates are
\[ \tilde{\sigma}_j = \sigma_j \left( t_5 + \frac{1}{5} \tilde{T}, t_6 + \frac{1}{5} \tilde{T}, t_7 + \frac{1}{5} \tilde{T}, t_8 + \frac{1}{5} \tilde{T} \right) \quad j = 1, \ldots, 4 \]
where \( \tilde{T} = t_1 + t_2 + t_3 + t_4 \). The geometry near the \( A_4 \) is given by
\[ x^2 + y^2 + z^5 + \sigma_2(t)z^3 + \sigma_3(t)z^2 + \sigma_4(t)z + \sigma_5(t) = 0 \]
and near the \( A_3 \) it is
\[ x^2 + y^2 + z^4 + \tilde{\sigma}_1(t)z^3 + \ldots + \tilde{\sigma}_4(t) = 0. \]

So the condition for smoothness at \( z = t = 0 \) is that the expressions \( \sigma_5 \) and \( \tilde{\sigma}_4 \) have simple zeros at \( t = 0 \). General values of \((z, t)\) are similar.

### 6.3. Gauge theory description

We would now like to describe the gauge theory corresponding to D5 branes wrapping the blown up \( S^2 \) in the A-D-E fibered geometries described above. We will consider all cases, except the last \( E_8 \) case with the special node corresponding to Dynkin number 5.

The strategy to find the gauge theories obtained when we wrap \( N \) D5 branes on these 2-cycles is to consider the corresponding quiver theory for the full A-D-E, but restrict attention to the fields which are charged only under the \( U(N) \) gauge group associated to blown up node, and neutral under all other groups. This in particular entails promoting certain composite fields to fundamental fields of the \( U(N) \) gauge theory. Algebraically the description of the neutral fields and the equations they satisfy parallels the mathematical description given above, and this is related to the fact that both involve monodromic invariant data.

Let us start with \( A_r \) quiver theories with the special node corresponding to the \( k \)-th node as in Figure 2. Then \( Q_{k,k-1}Q_{k-1,k} \) and \( Q_{k,k+1}Q_{k+1,k} \) and \( \Phi \) (the adjoint field corresponding to the \( k \)-th node) generate all the fields invariant with respect to other gauge groups, as can be verified from the equations discussed in section 4. Moreover, as discussed in section 4, denoting a (a suitably shifted) \( Q_{k,k-1}Q_{k-1,k}, Q_{k,k+1}Q_{k+1,k} \) by \( X, Y \) respectively, we find that they satisfy some particular field equations given by (4.11). Let us rewrite (4.11) (changing the sign in the definition of \( Y \)),

\[ \prod_{i=1}^{k}(X + t_i) = 0 \quad \prod_{j=k+1}^{r+1}(Y + t_j) = 0 \quad X - Y = 0 \]
where \( t_i \) should be viewed as functions of the field \( \Phi \) given by \( t_i(\Phi) \). These can now be written as,

\[
X^k + \sum_i \sigma_i X^{k-i} = 0 \quad Y^{r-k+1} + \sum_i \tilde{\sigma}_i Y^{r-k+i+1} = 0 \quad X - Y = 0
\]

where \( \sigma_i, \tilde{\sigma}_i, X, Y \) are invariant under the Weyl reflections we are interested in. This is very similar to the equation describing the geometry (6.3). The first two equations, substituted by \( X = Y \) from the third equation and identifying \( X = Y = z \), appended with \( x = y = 0 \) of equation (6.3) give precisely the condition for singular points in the geometry. Solutions to these equations are precisely the positions where we can have a blown up \( \mathbb{P}^1 \) from the geometry analysis. From the gauge theory analysis this is very natural as this is also classifying the possible choices of deformation of the gauge theory, by giving vev to \( X = Y \), and should correspond to supersymmetric vacua, which are in 1-1 correspondence with allowed \( \mathbb{P}^1 \)'s.

Consider now the \( D_r \) case. In this case, as can be seen from the analysis of section 4, the fields generating the relevant chiral ring is generated by the adjoint field \( \Phi \) and the composite adjoint fields \( X, Y, Z \) (suitably shifted with functions of \( \Phi \)) corresponding to the three edges adjacent to the trivalent node. We wish to write the chiral ring relations for these fields. These equation were also worked out in section 4 (4.13). Let us rewrite them here,

\[
\prod_{i=1}^{r-2} (X + t_k) = 0 \quad Y^2 = \frac{1}{4}(t_{r-1} + t_r)^2 \quad Z^2 = \frac{1}{4}(t_{r-1} - t_r)^2 \quad X + Y + Z = 0
\]

These can now be written as,

\[
\sum_{i=0}^{r-2} \sigma_i X^{r-2-i} = 0 \quad Y^2 = \frac{1}{4}(\tilde{\sigma}_1)^2 \quad Z^2 = \frac{1}{4}(\tilde{\sigma}_1)^2 - \tilde{\sigma}_2 \quad X + Y + Z = 0 \quad (6.6)
\]

where \( \sigma_i, \tilde{\sigma}_i, X, Y, \) and \( Z \) are invariant under the Weyl reflections.

Finally, let us consider the \( E_r \) cases with the special node being the trivalent vertex. Again the chiral fields are generated by the three relevant meson fields together with \( \Phi \). These cases were not studied in section 4 therefore we will have to start from the original

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\[6\] The extra condition \( x = y = 0 \) from (6.3) can be viewed, perhaps, as a trivial addition of two additional massive adjoint fields to the gauge theory, which have been integrated out.
form of the equations at the trivalent node. We will use the label for the nodes given in Figure 1.

The field equations at the trivalent node are given by,

\[ X(X + P_2')(X + P_2' + P_1') = 0 \quad Y \prod_{i=4}^r (Y + P_i' + \ldots + P_i') = 0 \]

\[ Z(Z + P_r') = 0 \quad X + Y + Z = P_3' \]

Shifting \( X \rightarrow X - \frac{1}{3}(P_1' + 2P_2') \), \( Z \rightarrow Z - \frac{1}{2}P_r' \) and \( Y \rightarrow Y + \frac{1}{6}(3P_r' + 2P_1' + 4P_2' + 6P_3') \)

and using the coordinates given in (2.5) we can write the equations in terms of invariant fields as follows,

\[ X^3 + \sigma_2 X + \sigma_3 = 0 \quad Z^2 - \frac{1}{4}\rho_2 = 0 \quad Y^{r-3} + \tilde{\sigma}_1 Y^{r-4} + \ldots + \tilde{\sigma}_{r-3} = 0 \quad X + Y + Z = 0 \]

where \( \sigma_i, \tilde{\sigma}_i \) and \( \rho_2 \) turn out to be exactly equal to those defined in the mathematical analysis for \( E_r \) at the trivalent node.

Let us now try to identify the field theories that will give rise to these equations. We will study first the examples with trivalent nodes. The \( A_r \) case will be studied in full detail as the first example in the next section.

We have seen that for all \( D_r \) and \( E_r \) cases, the field equations in terms of invariant fields can be written as follows,

\[ X^{r+1} + \sigma_1 X^r + \ldots + \sigma_{r+1} = 0 \quad Y^{q+1} + \hat{\sigma}_1 Y^q + \ldots + \hat{\sigma}_{q+1} = 0 \]

\[ Z^{p+1} + \tilde{\sigma}_1 Z^p + \ldots + \tilde{\sigma}_{p+1} = 0 \quad X + Y + Z = 0 \quad (6.7) \]

where \( p, q, r \) denote the total number of extra nodes on each of the three legs of the Dynkin diagram emanating from the trivalent vertex. Note that in addition the coefficients of the constraints are functions of the adjoint field \( \Phi \). Furthermore, the quiver analysis given in section 4 implies that, in addition to the above equations we have to impose

\[ [X, \Phi] = [Y, \Phi] = [Z, \Phi] = 0 \quad (6.8) \]

Note that the constraints satisfied by the fields parallels the mathematical description of the geometry. As noted above this is not surprising as both parameterize the position of the blow up sphere. The local geometry discussion we reviewed earlier in the paper,
provided three local patch descriptions, which we naturally identify with the three polynomial constraints above. However, the geometry description of \[7\] does not provide a global description of the blown up geometry in terms of these patches. The gauge theory construction above suggests, however, that there must exist a simple global description of the blown up geometry where the three local coordinate patches get identified with \(X, Y, Z\) which in turn should be identified with three sections of \(O(1)\) over the \(\mathbb{P}^1\). Moreover the fact that they satisfy a linear condition such as \(X + Y + Z = 0\) is natural given that there are only two independent holomorphic sections for \(O(1)\). We will verify that the geometry does satisfy this description at least in some cases that we present as examples in the next section.

It is natural to ask whether all these constraints arise from a physical gauge system with a superpotential. The answer to this question appears to be in the negative, except for a very special class, that we will identify. The explanation of this is as follows: In the superpotential the mixing between \(X, Y, Z\) and \(\Phi\) is fixed by the original quiver theory to be

\[
W = \text{Tr}(X + Y + Z)\Phi + \ldots
\]

which suggests that we could interpret \(X + Y + Z = 0\) equation as the \(dW/d\Phi = 0\) equation. If so, this would imply that the other three equations should be viewed as \(dW/dX, dW/dY, dW/dZ\) respectively. However, this is possible only if

\[
\frac{\partial^2 W}{\partial \Phi \partial X} = \frac{\partial^2 W}{\partial X \partial \Phi} \text{ etc.}
\]

which implies that the equation involving the polynomial in \(X\) should have only an additive linear dependence in \(\Phi\), and the same goes for the polynomial equations in \(Y\) and \(Z\). In other words \(\sigma_{r+1}, \hat{\sigma}_{q+1}, \tilde{\sigma}_{p+1}\) are linear in \(\Phi\), and all the other \(\sigma\)’s are constants. Note that in this case the condition of commutativity \((\ref{eq:commutativity})\) is automatic. In particular if we commute any of the polynomial equations with the corresponding field we get the corresponding commutativity condition \((\ref{eq:commutativity})\). This is as it had to be if we were to be able to find a superpotential giving the constraints, because we cannot obtain 7 equations by considering gradient of a superpotential involving only 4 fields. There should have been three redundant equations\(^7\). It is not difficult to see that this condition alone would essentially force us to

\[\text{\footnotesize{\(\text{\(7\) Actually there is always one redundant equation. The commutativity of } X \text{ and } \Phi \text{ follows from the other two commutativity conditions, given the constraint } X + Y + Z = 0.}\)}}\]
the above structure on the form of the Φ dependence of σ’s for the compatibility of the above constraints with the existence of the superpotential. It is interesting to note that the condition of linearity of the highest indexed σ with respect to Φ near the blow up point was exactly the condition for the blow up to be smooth, as discussed in the previous subsection.

So now let us determine the superpotential for the case where all σᵢ, ˜σᵢ and ˆσᵢ are constants except for σᵣ₊₁, ˆσₚ₊₁ and ˜σₚ₊₁ that are taken to be linear functions of Φ. This implies that the field equations can be written as,

\[ R'_{r+2}(X) = Φ \quad Q'_{q+2}(Y) = Φ \quad P'_{p+2}(Z) = Φ \quad \text{and} \quad X + Y + Z = 0 \] (6.9)

where \( R_{r+2}, Q_{q+2} \) and \( P_{p+2} \) are polynomials of degree \( r + 2, q + 2 \) and \( p + 2 \) respectively.

Now it is clear that this equations can be obtained from a superpotential of the form,

\[ W = \text{Tr} \left( R_{r+2}(X) + Q_{q+2}(Y) + P_{p+2}(Z) - Φ(X + Y + Z) \right) \]

(In the A case we obtain the same except we set \( Z = 0 \) in the above).

Now we can write an effective superpotential in terms of two adjoint fields by integrating out Φ and Z. This results in,

\[ W(X, Y) = \text{Tr} \left( R_{r+2}(X) + Q_{q+2}(Y) + P_{p+2}(-X - Y) \right) \] (6.10)

In the next section we will explore in detail some examples that illustrate the general relations found in this section.

7. Examples of Monodromic Fibrations

In this section we provide several examples where the gauge theory obtained from wrapping a D5 brane around the \( P^1 \) that survives the monodromy induced in the geometry can be described very explicitly. As we saw, the gauge theory description suggests a rather simple global geometric description of the blown up \( P^1 \) for all cases. However such a mathematical construction is not currently known in the generality suggested by the gauge theory. Instead only some explicit blown up geometries are known in detail and we shall discuss them and see that they match the gauge theory description precisely.

First, we will study some examples of monodromic \( A_r \) fibrations whose geometry is globally equal to that of an \( A_1 \) case, where the blown up node is the first node of the
$A_r$. The last two examples are related to monodromic $D_r$ fibrations. We first analyze the geometry introduced by Laufer \[26\] as an example of a $P^1$ with normal bundle $O(1) \oplus O(-3)$ and then we generalize to all $D_r$ monodromic fibrations. The mathematical discussion is rather involved, but we present it for the sake of completeness. We also specialize the gauge theory discussion of the previous section to these cases and see a perfect match between the geometric description and the gauge theoretic description.

7.1. $A_r \to A_1$

Let us start with the gauge theory construction. Let us assume that the monodromic $A_r$ fibration is such that the geometry dictates a Weyl reflection on $A_{r-1} \subset A_r$, where we identify the nodes of $A_{r-1}$ with the nodes $2, \ldots, r$.

Using the general analysis of the previous section we can write the equations at the first node as,

$$X = t_1 \prod_{i=2}^{r+1} (X - t_i) = \sum_{j=0}^{r} (-1)^j \sigma_j X^{r-j} = 0 \tag{7.1}$$

where $\sigma_j$ are the symmetric functions of $t_2, \ldots, t_{r+1}$.

We will consider the special case where $\sigma_i$ are constants for $i = 1, \ldots, r-1$ except for $\sigma_r$ and $t_1$ to be linear in $\Phi$ plus a constant. Let us take $t_1 = \Phi$ and $\sigma_r = -\alpha \Phi$. From the first equation in (7.1) we get $X = \Phi$ and using this we can eliminate $\Phi$ from the second equation in (7.1) to yield

$$\sum_{j=0}^{r-1} (-1)^j \sigma_j X^{r-j} - \alpha X = 0$$

which we interpret as $W'(X) = 0$, i.e. the critical points of the superpotential $W(X)$.

The CY geometry for the case where $\sigma_i$ are constant can be shown to be globally the same as an $A_1$ geometry. Let us now show that explicitly. The geometry is given in general by,

$$z^2 + y^2 + \left( \sum_{j=0}^{r} (-1)^j \sigma_j x^{r-j} \right)(x - t_1) = 0$$

Writing the geometry in an expanded form

$$z^2 + y^2 + (x^r + \sigma_1 x^{r-1} + \ldots + \sigma_{r-2} x^2 + (\sigma_{r-1} - \alpha)x + \alpha(x - t))(x - t) = 0$$

Let $\omega = (x - t)$ and $W'(x) = x^r + \sigma_1 x^{r-1} + \ldots + \sigma_{r-2} x^2 + (\sigma_{r-1} - \alpha)x$. Then,

$$z^2 + y^2 + (W'(x) + \alpha \omega) \omega = 0$$
by letting $\rho = \omega + \frac{1}{2\alpha} W'(x)$ we get the $A_1$ geometry expected for this superpotential

$$z^2 + y^2 + \rho^2 - \frac{1}{\alpha^2} W'^2(x) = 0.$$  

This shows that our identification of the field theory was correct because D5-branes wrapping the non trivial two cycles of this geometry has already been engineered in [18] and studied in detail in [5]. Note that the more general form of the dependence of $\sigma$’s on $\Phi$ will not yield a threefold geometry in the above form, further strengthening the argument that in these cases the gauge theory with a single adjoint field cannot be described by a superpotential term.

7.2. Laufer’s Example: the D-case

Consider the local geometry described in [26]. This is the classical example of an exceptional $P^1$ with normal bundle $O(1) \oplus O(-3)$. We will see that the blown down geometry corresponds to a monodromic $D_r$ fibration where only the trivalent node can be blown up and corresponds to the $P^1$. The deformation theory can be worked out explicitly and a superpotential can be obtained. We study the $\mathcal{N} = 1$ gauge theory with two adjoint fields obtained by wrapping D5-branes in the $P^1$ and work out the analysis of critical points of the superpotential. The field equations give an algebra that admits one and two dimensional representations (as we expect from the fibering of $D$ geometries), and we find a prefect match between these representations and the degree one and two curves in the 3-fold geometry.

Resolved Geometry

In this section we present a detailed mathematical description of a blown up $\mathbb{P}^1$ with normal bundle being $O(1) \oplus O(-3)$.

We glue together two copies of $\mathbb{C}^3$. The first $\mathbb{C}^3$ has coordinates $(x, y_1, y_2)$ and the second has coordinates $(w, z_1, z_2)$. The glueing data is

$$z_1 = x^3 y_1 + y_2^2 + x^2 y_2 y_2^{2n+1}$$
$$z_2 = x^{-1} y_2$$
$$w = x^{-1}.$$  

Consider the curve $C$ given in the first patch by $y_1 = y_2 = 0$ and in the second patch by $z_1 = z_2 = 0$, with $x$ and $w$ being the coordinates along $C$. The curve $C$ has normal bundle
\(O(1) \oplus O(-3)\) as can be seen from the terms in the glueing data which are linear in the equations of \(C\).

The deformation theory of the curve has already been worked out in [27]. If we consider the deformation of the equation \(y_2 = 0\) for \(C\) to \(y_2 = ax + b\) and deform \(y_1 = 0\) to \(y_1 = -(ax + b)^{2n+1} - b^{2n+1}/x\), we obtain

\[
z_1 = b^2 + 2abx + (a^2 + b^{2n+1})x^2, \quad z_2 = a + bx^{-1}.
\]

Note that the deformation parameters \(a, b\) can be identified with vev of massless adjoint scalars if we consider wrapping \(D5\) branes around the curve \(C\). This deformation of \(z_2 = 0\) is automatically holomorphic in \(w = x^{-1}\) (and in fact fixed the general choice of deformation of \(y_2 = 0\)). The deformation of \(z_1 = 0\) is holomorphic in \(w\) precisely when \(2ab = a^2 + b^{2n+1} = 0\). The condition for having a holomorphic deformation should translate to the condition of satisfying critical points of a superpotential. In fact, these equations are precisely the critical points of the holomorphic function

\[
W(a, b) = a^2b + b^{2n+2}/2n + 2
\]

which we therefore identify with a superpotential (including a trace in front, if we consider more than one wrapped brane).

The only solution of (7.2) is \(a = b = 0\), but this is a degenerate solution. We deform to a generic situation by adding terms to (7.2) modulo its partial derivatives. The deformed geometry is conveniently written as

\[
\begin{align*}
z_1 &= x^3y_1 + y_2^2 + \alpha x + x^2(y_2P(y_2^2) + Q(y_2^2)) \\
z_2 &= x^{-1}y_2 \\
w &= x^{-1}.
\end{align*}
\]

where \(P\) has degree exactly \(n\) and \(Q\) has degree at most \(n\).

This geometry is blown down by four holomorphic functions.

\[
\begin{align*}
v_1 &= z_1 = x^3y_1 + y_2^2 + \alpha x + x^2(y_2P(y_2^2) + Q(y_2^2)) \\
v_2 &= w^2z_1 - z_2^2 - \alpha w = xy_1 + y_2P(y_2^2) + Q(y_2^2) \\
v_3 &= wv_2 - z_2P(z_1) - wQ(z_1) \\
v_4 &= z_2v_2 - wz_1P(z_1) - z_2Q(z_1)
\end{align*}
\]
The last two equations are linear in $z_2$ and $w$, and can be solved as

$$z_2 = \frac{-v_3 v_1 P(v_1) - v_2 v_4 + v_4 Q(v_1)}{v_1 P(v_1)^2 - (v_2 - Q(v_1))^2}, \quad w = \frac{-v_4 P(v_1) - v_2 v_3 + v_3 Q(v_1)}{v_1 P(v_1)^2 - (v_2 - Q(v_1))^2}. \quad (7.3)$$

We substitute (7.3) and $v_1 = z_1$ into the equation for $v_2$ and obtain

$$v_2(v_1 P(v_1)^2 - (v_2 - Q(v_1))^2) = v_4^2 - v_1 v_3^2 + \alpha(v_4 P(v_1) + v_2 v_3 - v_3 Q(v_1)). \quad (7.4)$$

This is the blown-down geometry. The degree 1 curves are obtained by solving

$$2xy + \alpha = 0$$
$$x^2 + yP(y^2) + Q(y^2) = 0,$$

which leads to the superpotential

$$W = yx^2 + R(y) + \alpha x \quad (7.5)$$

where $R'(y) = yP(y^2) + Q(y^2)$. This superpotential has been obtained from the geometry [27]. Suppose that we have a solution $(x, y) = (-\alpha/2d, d)$ for $d \neq 0$. There are generically $2n + 3$ such solutions. If $d = 0$, then $\alpha = 0$, $x = 0$, and $Q$ has no constant term, which implies from the equations that $\alpha = O(d)$. In any case, we get the equation

$$\alpha^2 + 4y^2(yP(y^2) + Q(y^2)). \quad (7.6)$$

The deformed curve is

$$y_1 = \ldots$$
$$y_2 = d - \frac{\alpha}{2d} x$$

where we do not need the exact form of $y_1$, only needing to observe that its form suffices to guarantee that the equation $z_1 = 0$ deforms holomorphically. Substituting into the $v_i$ gives for the location of the singular curves

$$v_1 = d^2$$
$$v_2 = \frac{\alpha^2}{4d^2}$$
$$v_3 = \frac{\alpha}{d} P(d)$$
$$v_4 = \frac{\alpha^3}{8d^3} + \frac{\alpha}{2d} Q(d)$$
In this case and all subsequent cases, it can be checked directly from the blown-down equation that these singular points are conifolds for generic deformations.

The degree 2 curve has equations \( y_1 = y_2^2 = 0 \) in the first patch, and \( z_1 = z_2^2 = 0 \) in the second patch. Now for the deformed degree 2 curves, we let \( c \) be a root of \( P \). There are \( n \) such choices. The deformed curve is at

\[
\begin{align*}
y_1 &= \ldots \\
y_2^2 &= c - \alpha x - Q(c)x^2.
\end{align*}
\]

If \( \alpha = Q(c) = 0 \), one might naively think that this has split up into two disjoint curves \( y_2 = \pm c^{1/2} \), but they meet in the other patch so are a connected curve. Substituting into the \( v_i \) we get

\[
\begin{align*}
v_1 &= c \\
v_2 &= Q(c) \\
v_3 &= 0 \\
v_4 &= 0
\end{align*}
\]

The variables here should be matched to [28] or equivalently to [29]. A partial match identifies \( x_1 \) of [28] with our \( P \), and the defining equation of \( D_1 \) in [28] with our (7.6).

The geometry given in (7.4) can be rewritten as a \( D_r \) fibration with monodromies. In order to see this let us introduce the following change of variables. Let \( x = v_4 + \frac{1}{2} \alpha P(v_1) \), \( y = v_3 \), \( z = v_1 \) and \( t = v_2 - Q(v_1) \). Then (7.4) can be written as,

\[
x^2 - zy^2 + (t + Q(z))(t^2 - zP^2(z)) - \frac{1}{4} \alpha^2 P^2(z) + \alpha t y = 0 \quad (7.7)
\]

This will lead us to an alternative derivation of the above result for the superpotential based on the quiver analysis we have discussed in this paper.

We have seen above that this geometry has \( 2n + 3 \) isolated non-trivial degree 1 curves and \( n \) isolated degree 2 curves. What this should imply in the corresponding gauge theory is that if we consider a \( U(N) \) version of this theory, the inequivalent choices of Higgs branch will have \( 2n + 3 \) inequivalent one dimensional representations and \( n \) two dimensional representations. We will verify this claim in the next section, which is rather non-trivial for the relation of degree two curves and irreducible representations of the \( dW = 0 \) equations.

**The Monodromic Quiver Construction and Laufer’s Example**
We will construct now a gauge theory that will correspond to the geometry in (7.4). The idea is to use a particular monodromic $D_{r}$ fibration. In the previous section we studied the most general case. So we only have to borrow the field equations (6.6) in terms of invariant fields,

$$
\sum_{i=0}^{r-2} \sigma_i X^{r-2-i} = 0 \quad Y^2 = \frac{1}{4} (\tilde{\sigma}_1)^2 \quad Z^2 = \frac{1}{4} (\tilde{\sigma}_1)^2 - \sigma_2 \quad X + Y + Z = 0. \quad (7.8)
$$

In order to induce the monodromy we want we have to impose that $\sigma_{r-2}, \tilde{\sigma}_{1}^2$ and $\tilde{\sigma}_2$ be linear functions of $\Phi$. The rest of them will be taken to be constants independent of $\Phi$.

Let $\sigma_{r-2} = \Phi + b, \tilde{\sigma}_1^2 = 4\Phi$ and $\tilde{\sigma}_2 = \alpha$. The equations are now given by,

$$
Y^2 = \Phi \quad Z^2 + \alpha = \Phi
$$

$$
- \sum_{i=0}^{r-3} \sigma_i X^{r-2-i} - b = \Phi \quad X + Y + Z = 0,
$$

which we have written it in the same form as presented in (6.3). Using this we can immediately borrow the results from the previous section and write the superpotential from (6.10) which gives

$$
W = \text{Tr} \left( \frac{1}{3} Y^3 - \left( \sum_{i=0}^{r-3} \frac{1}{r-1-i} \sigma_i X^{r-1-i} + bX \right) - \frac{1}{3} (X + Y)^3 - \alpha (X + Y) \right).
$$

Note that for the choice of coefficients leading to the Laufer geometry the $Y^3$ piece above cancels out. After a shift $Y \to Y - \frac{1}{2} X$ and removing an irrelevant overall minus sign we obtain

$$
W(X,Y) = \text{Tr} \left( XY^2 + R(X) + \alpha Y \right).
$$

where $R'(X) = \sum_{i=0}^{r-3} \sigma_i X^{r-2-i} + \frac{1}{4} X^2 + b + \frac{1}{2} \alpha$. Note that this is identical with the superpotential arrived at by geometric reasoning leading to (7.5), with the change in notation ($x \leftrightarrow Y, y \leftrightarrow X$) and the definition of $P$ and $Q$ by the identification $R'(X) = XP(X^2) + Q(X^2)$. This gives

$$
P(z) = z^n + \sigma_2 z^{n-1} + ... + \sigma_{2n-2} z + \sigma_{2n}
$$

$$
Q(z) = \sigma_1 z^n + \sigma_3 z^{n-1} + ... + (\sigma_{2n-1} + \frac{1}{4}) z + (b + \frac{1}{2} \alpha), \quad (7.9)
$$
where we have assumed odd $r$ (to obtain Laufer’s geometry) with $r = 2n + 3$. Now that we have succeeded in reproducing the superpotential we have to show that the geometry of this fibration matches (7.7).

**Recovering the blown-down geometry**

We now have discussed which monodromic $D_r$ quiver theory gives rise to the superpotential expected for Laufer’s example. Here we show that this threefold is indeed the one corresponding to Laufer’s example given by (7.7).

Recall that the $D_r$ fibration geometry is given by (2.2),

$$x^2 + y^2z + \prod_{i=1}^{r} (z - t_i^2) + (-1)^r \prod_{i=1}^{r} t_i^2 + 2 \prod_{i=1}^{r} t_i y$$  \(7.10\)

Notice that we have rescaled $t_k$ by a factor of $i$ as well as $y$ and $z$ in order to get the above equation.

Now we want to write this in terms of the invariant coordinates in the deformation space. For this notice that,

$$r^{-2} \prod_{i=1}^{r} (z - t_i^2) = \begin{cases} (z^n + \sigma_2 z^{n-1} + \ldots + \sigma_{2n})^2 - z(\sigma_1 z^{n-1} + \ldots + \sigma_{2n-1})^2 & \text{for } r = 2n + 2 \\ z(z^n + \sigma_2 z^{n-1} + \ldots + \sigma_{2n})^2 - (\sigma_1 z^n + \ldots + \sigma_{2n+1})^2 & \text{for } r = 2n + 3 \end{cases}$$ \(7.11\)

It turns out that Laufer’s geometry is reproduced if we take the odd case ($r = 2n + 3$).

We also need the remaining two factors in product that appears in (7.10),

$$(z - t_{2n+2}^2)(z - t_{2n+3}^2) = z^2 + (2\tilde{\sigma}_2 - \tilde{\sigma}_1^2)z + \tilde{\sigma}_2^2$$

The gauge theory fibration tells us that we simply have to replace $\Phi$ by $t$ to get the new geometry. This implies that $\tilde{\sigma}_1^2 = 4t$, $\tilde{\sigma}_2 = \alpha$ and $\sigma_{2n+1} = t + b$.

Using all this and the two polynomials $P(z)$ and $Q(z)$ defined in (7.9) we get that the geometry (7.10) can be written as,

$$x^2 + zy^2 + 2\alpha (t + b)y + \left(zP^2(z) - (Q(z) - \frac{1}{4}z - \frac{1}{2}\alpha + t)^2\right) + \frac{\alpha}{z} \left((zP^2(z) - (Q(z) - \frac{1}{4}z - \frac{1}{2}\alpha + t)^2)(2z + \alpha) + \alpha(t + b)^2\right) = 0$$ \(7.12\)

Now we only have to use some change of variables. Let us shift $t \rightarrow t - Q(z) + \frac{1}{4}z + \frac{1}{2}\alpha$ and $y \rightarrow y - \frac{\alpha}{z}(Q(z) - \frac{1}{2}\alpha - b - \frac{1}{4}z)$. Notice that the shift in $y$ is well defined for $z = 0$ since $Q(z) - \frac{1}{2}\alpha - b$ has at least a simple zero at $z = 0$. Then (7.12) is given by,

$$-4(zP^2(z) - t^2)(t - Q(z)) + \alpha^2 P^2(z) + zy^2 - 2\alpha ty + x^2 = 0$$
Finally, we only need to rescale $y \to -2y$, $x \to 2ix$ and $t \to -t$ to get,

$$x^2 - zy^2 + (t + Q(z))(t^2 - zP^2(z)) - \frac{1}{4}\alpha^2 P^2(z) + \alpha ty = 0$$

This is in perfect agreement with (7.7).

**Physical Analysis of Critical Points of the Superpotential**

The gauge theory that is proposed above is an $\mathcal{N} = 1$ theory with two chiral superfield $X$ and $Y$. We will now analyze the choices of the branches of this theory, by finding the critical points of $W$ and check that they beautifully match the structure of holomorphic curves of degree 1 and 2 expected in the Laufer's geometry.

The superpotential is given by,

$$W(X, Y) = \text{Tr} \left( XY^2 + R(X) + \alpha Y \right).$$

The field equations are,

$$XY + YX = -\alpha \quad \text{and} \quad Y^2 + R'(X) = Y^2 + XP(X^2) + Q(X^2) = 0 \quad (7.13)$$

We want to find all possible irreducible representations of this algebra. For this one has to realize that $X^2$ and $Y^2$ are Casimirs. This is easy to see since the first equation in (7.13) implies that $[X^2, Y] = 0$ and $[Y^2, X] = 0$.

Let $X^2 = x^2 \mathbb{I}$ and $Y^2 = y^2 \mathbb{I}$. Then the second equation in (7.13) can be written as,

$$y^2 \mathbb{I} + P(x^2)X + Q(x^2) = 0. \quad (7.14)$$

Clearly, if $P(x^2) \neq 0$ then $X$ is a c-number and this implies that $Y$ is also a c-number. These are the 1-dimensional representations. To see how many of then are there, let us multiply (7.14) by $x^2$, and use $2xy = \lambda$ to write it as follows,

$$P(x^2)x^3 + Q(x^2)x^2 + \frac{1}{4}\lambda^2 = 0$$

This equation has $2n + 3$ solutions and that is the number of 1-dimensional representations.

Two dimensional representations, if any, should correspond to $P(x^2) = 0$. There are $n$ such solutions. Choose any of these solutions which correspond to a fixed $x^2$. This also fixes $y^2$ from (7.14). From the first equation in (7.13) one can shift $X \to X + aY$ and $Y \to Y + bX$ such that $X^2 = Y^2 = 0$ and $XY + YX = c$. There is only one irreducible representation of this algebra, and that is a two dimensional representation corresponding to the fock space for the realization of a single fermionic creation/annihilation algebra.

Note that the solutions for one (two) dimensional representations match the location of the degree one (two) curves in the geometry as they should.
7.3. Most general $D_r$ geometry

Laufer’s geometry is not the most general Calabi-Yau with a blow up $P^1$ corresponding to a trivalent node of a $D_r$ monodromic fibration. As we have seen, Laufer’s example is (essentially) the only such $D_r$ case which admits a superpotential description. However, we could consider more general monodromic $D_r$ fibrations which do not admit a superpotential description. We would like to check that also in these cases the geometry description matches the gauge theory analysis based on quivers.

We will first display the mathematical computation for the most general $D_4$ for simplicity and then show how the result generalizes to $D_r$ for any $r$.

General Geometric Construction of Monodromic $D_r$

We want to construct a Calabi-Yau threefold as a deformation of the partial blowup of a $D_4$ singularity where only the curve corresponding to the trivalent vertex of the Dynkin diagram has been blown up.

We recall the general deformation of the minimal resolution of the $D_4$ singularity (see [7] for more details and references). The deformation parameters are $t_1, \ldots, t_4$, and the general deformation is a blowup of

$$x^2 + \frac{(yz + \hat{\sigma}_4)^2 - ((z^2 - \hat{\sigma}_2 z + \hat{\sigma}_4)^2 + z(-\hat{\sigma}_1 z + \hat{\sigma}_3)^2)}{z} = 0. \quad (7.15)$$

Here the $\hat{\sigma}_i$ are elementary symmetric functions of $t_1, \ldots, t_4$. In fact, the full Weyl group of $D_4$ acts on the $t_i$ in the usual way.

The blowup is achieved in 2 steps. We recall the first step completely but only partially do the second step since we do not need the full blowup. The equation can be rewritten as

$$(x + \hat{\sigma}_1 z - \hat{\sigma}_3)(x - \hat{\sigma}_1 z + \hat{\sigma}_3) + (y - z + \hat{\sigma}_2)(yz + z^2 - \hat{\sigma}_2 z + 2\hat{\sigma}_4) = 0.$$ 

Now blow up the ideal $J = (x + \hat{\sigma}_1 z - \hat{\sigma}_3, y - z + \hat{\sigma}_2)$.\footnote{There are 12 branches of the singular locus, lying over the respective loci $t_i = \pm t_j$ in $t$-space. These are in a natural one to one correspondence with the positive roots of $D_4$. The ideal $J$ vanishes on the part of the singular locus $t_3 + t_4 = 0, y = t_1 t_2, z = -t_3^2$ corresponding to one of the exterior vertices of the Dynkin diagram.}

The interesting part of the blowup is in the patch $U = (x + \hat{\sigma}_1 z - \hat{\sigma}_3)/(y - z + \hat{\sigma}_2)$. The equation becomes after eliminating $x$

$$U \left( U(y - z + \hat{\sigma}_2) - 2\hat{\sigma}_1 z + 2\hat{\sigma}_3 \right) + yz + z^2 - \hat{\sigma}_2 z + 2\hat{\sigma}_4$$

$$= (y - z + \hat{\sigma}_2)(z + U^2) + 2U(-\hat{\sigma}_1 z + \hat{\sigma}_3) + 2(z^2 - \hat{\sigma}_2 z + \hat{\sigma}_4)$$

$$= (y - z + \hat{\sigma}_2 + 2(z - \hat{\sigma}_2 - \hat{\sigma}_1 U - U^2))(z + U^2) + 2(U^4 + \hat{\sigma}_1 U^3 + \hat{\sigma}_2 U^2 + \hat{\sigma}_3 U + \hat{\sigma}_4) \quad (7.16)$$
This is visibly a deformed $A_3$. We can complete the blowup by blowing up the $A_3$ as in (7) to get the general deformation of the fully resolved $D_4$.

As we are only interested in blowing up the middle vertex of the $A_3$ (which corresponds to the central vertex of the $D_4$), we only blow up partially, and introduce a new set of variables. We let $\sigma_i$ be the elementary symmetric functions of $t_1, t_2$ and $\tilde{\sigma}_i$ be the elementary symmetric functions of $t_3, t_4$. This corresponds to the change of variables

\[
\begin{align*}
\hat{\sigma}_1 &\rightarrow \sigma_1 + \tilde{\sigma}_1 \\
\hat{\sigma}_2 &\rightarrow \sigma_2 + \tilde{\sigma}_2 + \sigma_1 \tilde{\sigma}_1 \\
\hat{\sigma}_3 &\rightarrow \sigma_1 \hat{\sigma}_2 + \sigma_2 \tilde{\sigma}_1 \\
\hat{\sigma}_4 &\rightarrow \sigma_2 \tilde{\sigma}_2,
\end{align*}
\]

or said differently, to the factorization

\[
U^4 + \hat{\sigma}_1 U^3 + \hat{\sigma}_2 U^2 + \hat{\sigma}_3 U + \hat{\sigma}_4 = (U^2 + \sigma_1 U + \sigma_2)(U^2 + \tilde{\sigma}_1 U + \tilde{\sigma}_2).
\]

The middle vertex is blown up by blowing up the ideal $(y - z + \hat{\sigma}_2 + 2(z - \hat{\sigma}_2 - \hat{\sigma}_1 U - U^2), U^2 + \sigma_1 U + \sigma_2)$. So we introduce homogeneous coordinates $(r, s)$ on a $P^1$ and write

\[
s(y - z + \hat{\sigma}_2 + 2(z - \hat{\sigma}_2 - \hat{\sigma}_1 U - U^2)) = r(U^2 + \sigma_1 U + \sigma_2).
\]

There are two patches, obtained by putting $s = 1$ and $r = 1$ respectively. In the first patch, we get

\[
(y - z + \hat{\sigma}_2 + 2(z - \hat{\sigma}_2 - \hat{\sigma}_1 U - U^2)) = r(U^2 + \sigma_1 U + \sigma_2) \\
r(z + U^2) + 2(U^2 + \tilde{\sigma}_1 U + \tilde{\sigma}_2) = 0,
\]

where the second equation in (7.17) is obtained by substituting the first equation into (7.16). Note that the first equation of (7.17) can be used to eliminate $y$, so that the second patch can be completely described by the second equation of (7.17), obtaining a hypersurface in the variables $r, z, U, t_i$.

Since the exceptional set of the first blowup is not desired, we blow down its proper transform obtained after the second blowup. This exceptional set is located at $\tilde{\sigma}_1 = 0, z = \hat{\sigma}_2$; it then follows from (7.17) that $r = -2$. We blow this down by the change of variables

\[
a = \hat{\sigma}_1 + (r + 2)U,
\]

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which leads after a little algebra to

\[ a^2 + (r(r + 2))z + 2\tilde{\sigma}_2(r + 2) - (\tilde{\sigma}_1)^2 = 0. \] (7.18)

A nice check is that \( \tilde{\sigma}_1 \) only appears to an even power. This was required because the reflection

\[(t_1, t_2, t_3, t_4) \mapsto (t_1, t_2, -t_4, -t_3)\]

in the Weyl group required for the blowdown takes \( \tilde{\sigma}_1 \) to \(-\tilde{\sigma}_1\). In fact, \( \sigma_1, \sigma_2, \tilde{\sigma}_2, (\tilde{\sigma}_1)^2 \) generate the ring of invariants of the \( Z_2^3 \) subgroup of \( W(D_4) \) generated by the 3 reflections in the 3 exterior roots, and may be used as coordinates on the space of deformations of the partially blown up \( D_4 \). These are essentially the same as the coordinates used in \[7\] and \[28\], differing by a linear change of variables.

The blowdown is realized by

\[
x = U(y - z + \hat{\sigma}_2) - \hat{\sigma}_1 z + \hat{\sigma}_3
\]
\[
= -\hat{\sigma}_1 z + \hat{\sigma}_3 - z(a - \hat{\sigma}_1) + (a\sigma_1 + (r + 2)\sigma_2 + \sigma_1\tilde{\sigma}_1) \cdot \frac{a - \hat{\sigma}_1}{r + 2}
\]
\[
= -\hat{\sigma}_1 z + \hat{\sigma}_3 - z(a - \hat{\sigma}_1) - \sigma_1(rz + 2\tilde{\sigma}_2) + \sigma_2(a - \tilde{\sigma}_1)
\]
\[
y = z - \hat{\sigma}_2 - 2(z - \hat{\sigma}_2 - \hat{\sigma}_1 U - U^2) + r(U^2 + \sigma_1 U + \sigma_2)
\]
\[
= -z + \hat{\sigma}_2 + (a - \hat{\sigma}_1)\sigma_1 - rz - 2\tilde{\sigma}_2 + \sigma_2 r
\]
\[
= -(r + 1)z + a\sigma_1 + (r + 1)\sigma_2 - \hat{\sigma}_2
\]
\[
z = z
\]

Similarly, in the second patch we get the equations

\[
s(y - z + \hat{\sigma}_2 + 2(z - \hat{\sigma}_2 - \hat{\sigma}_1 U - U^2)) = U^2 + \sigma_1 U + \sigma_2
\]
\[
z + U^2 + 2s(U^2 + \sigma_1 U + \sigma_2) = 0. \] (7.20)

We could similarly calculate the blowdown in the \( s \) patch, but we will see that we can get away with setting \( t = 0 \) in (7.20). Setting \( b = (2s + 1)U \) we get the equation

\[
(s + 1)b^2 = s(2s + 1)y. \] (7.21)

\[9\] The form of this equation could have been deduced immediately from the results of \[29\]. We can use the identity \( r(r + 2) = (r + 1)^2 - 1 \) and a change of variables to realize this as a local \( D_2 \) deformation.
The blown-down $r$-patch (7.18) and corresponding blown-down $s$-patch glue together to give a space $\mathcal{X}$ describing the general deformation of the partially blown-up $D_4$.

We can now at last construct the most general Calabi-Yau threefold geometry of this type. We express the deformation parameters $\sigma_1, \sigma_2, \tilde{\sigma}_2, (\tilde{\sigma}_1)^2$ as holomorphic functions of a parameter $t$ defined in a neighborhood of $t = 0$. With this substitution, (7.15) describes a singular hypersurface $Y$ in a space with coordinates $(x, y, z, t)$, exhibited as deformation of $D_4$. The desired Calabi-Yau threefold $X$ is obtained from $\mathcal{X}$ by performing the same substitution for $\sigma_1, \ldots, (\tilde{\sigma}_1)^2$ in terms of $t$. The blowdown map $f : X \to Y$ is computed by making the same substitution in (7.19). We now focus attention on the curve $C$ which is blown down by $f$.

The exceptional curve $C$ in the blown-down $r$-patch is given by $a = z = t = 0$, and in the blow-down $s$-patch by $b = y = t = 0$.

We now compute the deformation space (Hilbert scheme) of $C \subset X$, starting with first order deformations of $C$, which are described by $\text{Hom}(I/I^2, \mathcal{O}/I)$, where $I$ is the ideal sheaf of $C$ in $X$. It is straightforward to check from (7.18) and (7.21) that $z$ and $y$ are torsion classes modulo $I^2$, so must map to 0 under a homomorphism corresponding to a first order deformation. Furthermore, from the equation $b = a/r$, we conclude that the space of first order deformations is

$$
\begin{align*}
    a &\mapsto \epsilon_1 + \epsilon_2 (r + 1) \\
    z &\mapsto 0 \\
    b &\mapsto \epsilon_1 s + \epsilon_2 (1 + s) \\
    y &\mapsto 0
\end{align*}
$$

(7.22)

where the quadratic terms in $\epsilon_1, \epsilon_2$ are set to 0 so that we are describing a first order deformation. As a check, (7.18) is satisfied after the first two substitutions of (7.22), and (7.21) is satisfied after the last two substitutions of (7.22). We have shifted $r$ to $r + 1$ for convenience in subsequent computations. We identify the $\epsilon_i$ with the values of the 2 chiral fields, and compute deformations to deduce the critical points of the superpotential.

We deform $C$ by expressing $a, z, t$ as holomorphic functions of $\epsilon_1, \epsilon_2, r$ subject to the constraint (7.18). Since deformations of $C$ must blow down via (7.19), we see that $x, y, z, t$ must be independent of the local coordinate $r$ on $C$. In particular $z = z(\epsilon), t = t(\epsilon)$.

Now express $a$ as a power series in $(r + 1)$ and substitute into (7.18). We see that $(r + 1)$ can only occur linearly. So redefining the local coordinates $\epsilon_1$ and $\epsilon_2$ if necessary,
we can assume without loss of generality that \( a = a(\epsilon, r) \) is still given by the first equation in (7.22).

Now expanding (7.18) as power series in \((r + 1)\) and collecting terms, we get

\[
\begin{align*}
\epsilon_1^2 - z(\epsilon) + 2\tilde{\sigma}_2(t(\epsilon)) - (\tilde{\sigma}_1)^2(t(\epsilon)) &= 0 \\
\epsilon_1\epsilon_2 + \tilde{\sigma}_2(t(\epsilon)) &= 0 \\
\epsilon_2^2 + z(\epsilon) &= 0
\end{align*}
\]

(7.23)

These are the conditions that the deformation of the affine part of \( C \) in the \( r \)-patch stays in \( X \). The conditions that the curve blows down under (7.19) comes from the \( y \) and \( x \) equations:

\[
\begin{align*}
-z + \epsilon_2\sigma_1 + \sigma_2 &= 0 \\
-z\epsilon_2 - \sigma_1 z + \sigma_2\epsilon_2 &= 0
\end{align*}
\]

(7.24)

where we have suppressed the \( t(\epsilon) \) dependence of some of the terms. If (7.23) and (7.24) are both satisfied, then the compact curve \( C \) deforms without having to check in the \( s \)-patch, since the fibers of the blowdown map, of which (7.19) is an affine piece, are compact. From the last equation of (7.23) we get \( z = -\epsilon_2^2 \). Then (7.23) and (7.24) reduce to the three equations

\[
\begin{align*}
\epsilon_1^2 + \epsilon_2^2 + 2\tilde{\sigma}_2 - (\tilde{\sigma}_1)^2 &= 0 \\
\epsilon_1\epsilon_2 + \tilde{\sigma}_2 &= 0 \\
\epsilon_2^2 + \epsilon_2\sigma_1 + \sigma_2 &= 0
\end{align*}
\]

(7.25)

The equations (7.23) are interpreted as the equations of the natural correspondence \((C_\epsilon, t)\) consisting of pairs of the deformed curve \( C_\epsilon \) and \( t \) such that \( C_\epsilon \) is contained in the deformation of the partial blowup of \( D_4 \) corresponding to parameter \( t \). So the deformation space (Hilbert scheme) of \( C \) in \( X \) is locally defined by eliminating \( t \).

Now we generalize the result (7.25) to \( D_r \) with \( r = 2n + 3 \). This is achieved by simply replacing the last of the three equations by a polynomial of degree \( 2n + 1 \) in \( \epsilon_2 \). The result

---

10 Eliminating \( \epsilon_1, \epsilon_2 \) yields the discriminant. This corrects an error in [28]. There it was claimed that the Hilbert scheme was defined by the discriminant. We see from the above that the correspondence projects to the Hilbert scheme, and to the scheme in the disc defined by the discriminant under the other projection. In particular, if the Hilbert scheme is discrete, then its multiplicity coincides with the multiplicity of the discriminant, as claimed in [28].
can then be given by,

\[ \epsilon_1^2 + \epsilon_2^2 + 2\bar{\sigma}_2 - (\bar{\sigma}_1)^2 = 0 \]
\[ \epsilon_1 \epsilon_2 + \bar{\sigma}_2 = 0 \]  \hspace{1cm} (7.26)
\[ \epsilon_2^{2n+1} + \sigma_1 \epsilon_2^{2n} + \ldots + \sigma_{2n+1} = 0 \]

These equations are the ones which we compare with the gauge theory analysis in the next section.

**Comparison to Gauge Theory**

The gauge theory obtained from the geometry just discussed when we wrap \( D_5 \) branes is as in Laufer’s example a \( D_r \) quiver theory with monodromies.

The equations at the trivalent node were obtained in (7.8) and are given by,

\[ \sum_{i=0}^{r-2} \sigma_i X^{r-2-i} = 0 \]
\[ Y^2 = \frac{1}{4} (\bar{\sigma}_1)^2 \]
\[ Z^2 = \frac{1}{4} (\bar{\sigma}_1)^2 - \bar{\sigma}_2 \]
\[ X + Y + Z = 0 \]  \hspace{1cm} (7.27)

We want to show that these equations are equivalent to (7.26). For this we only have to notice that \( X + Y + Z = 0 \) implies that we can parametrize those three fields in terms of two new fields \( \epsilon_1 \) and \( \epsilon_2 \). Let us define the following parametrization,

\[ Y = \frac{1}{2} (\epsilon_1 - \epsilon_2) \]
\[ Z = -\frac{1}{2} (\epsilon_1 + \epsilon_2) \cdot \]
\[ X = \epsilon_2 \]  \hspace{1cm} (7.28)

With this the first three equations in (7.27) can easily be seen to be identical to (7.26). This proves that the structure of one dimensional representations agree with that of the geometry. The higher dimensional representations arise by promoting these to \( N \times N \) fields, and it would be interesting to verify that they match the geometric description for enumeration of higher degree curves.

This success encourages us to propose that also for \( E_r \) cases the gauge theory result based on monodromic quivers holds. It would be interesting to construct the corresponding blown up geometries and check that the deformation spaces match the physical prediction given in (6.7).

**8. Large N duals of \( \mathcal{N} = 1 \) A-D-E quiver theories**

For a generic \( \mathcal{N} = 1 \) A-D-E quiver theory, each branch is equivalent in the IR to a pure \( N = 1 \) gauge theory involving a product of some unitary groups. As such one expects to
have gaugino condensation in each gauge group. For a single $U(N)$ theory, it was proposed that the large $N$ description leads to a geometry were the $S^2$ has shrunk and $S^3$ has grown together with some flux.

For the more general A-D-E quiver theory we thus also expect a similar thing to happen. The geometries studied in section 3 become singular after blowing down the $S^2$'s. In order to recover a smooth space one can deform the complex structure and this will generate non-trivial 3-cycles with the topology of $S^3$, which should have the interpretation of appropriate gaugino condensation.

The simplest example $A_1$ was studied in where the equation of the singular space was given in (3.1),

$$x^2 + y^2 + z^2 + W'^2(t) = 0.$$ 

In that case it was argued that the most general deformation of the geometry was,

$$x^2 + y^2 + z^2 + W'^2(t) + g_{p-1}(t) = 0,$$

where $W$ is a polynomial of degree $p$ in $t$ and where $g_{p-1}(t)$ is a polynomial of degree $p-1$. These deformations depend on $p$ parameters, which get identified with the choice of the $p$ parameters denoting the branches of the theory (i.e. as to how we distribute the branes among various vacua). This map was found by extremizing the superpotential in this dual geometry where

$$W = \int H \wedge \Omega$$

for suitable $H$ flux through the 3-cycles. It is natural to ask how one can generalize these transitions to the general case of A-D-E quiver theories under study. For simplicity we first consider non-monodromic A-D-E quiver theories, and discuss the generalization to monodromic cases in the context of Laufer's example. Moreover in the context of non-monodromic A-D-E theories, we first comment on the ordinary A-D-E case and at the end discuss the generalization to the affine case, which involves only a minor modification of the discussion.

For simplicity we consider the case where the superpotentials $W_i(\Phi_i)$ are all polynomials of degree $p + 1$. All the other cases can be viewed as deformations of this case by adjusting coefficients of the superpotentials. As discussed in section 4, for the case where the superpotential is a polynomial of degree $p + 1$ in all the adjoint fields, we expect to have $pR_+$ branches, where $R_+$ is the number of positive roots:

$$2R_+ + r = dim(G)$$
This corresponds to the fact that we can break the \( \prod U(N_i) \) to \( \prod U(M_\alpha) \) where \( \alpha \) runs over \( pR_+ \) branches and \( M_\alpha \) denotes how many of those branches we have.

We should now show that desingularization of the \( W \) will allow exactly as many coefficients as these branches, and as many \( S^3 \)'s. The latter fact is simple to establish: There are exactly as many shrunk \( S^2 \)'s as the branches. That is how we found the branches, namely by the condition of a vanishing \( S^2 \). Thus each one of them can be deformed to an \( S^3 \). We will now show that there are also exactly as many normalizable deformations of the geometry as the number of branches. One can easily deduce, from our discussion in section 2 and the relation of the gradient of superpotential to \( \alpha_i \), that for degree \( p+1 \) superpotentials that we are considering the 3-fold geometry is given by

\[
f = f_{A-D-E}(x, y, z) + at^{pc_2} + \ldots = 0
\]

where \( c_2 \) is the dual coxeter number of the corresponding A-D-E, and we have indicated explicitly the term with the highest power in \( t \). The number of normalizable deformations of this geometry (including the log normalizable ones) is given by \( \[30\] \[31\] \)

\[
D = \frac{1}{2} \dim \mathcal{R} + \frac{1}{2} d_{\hat{c}/2}
\]

where \( \dim \mathcal{R} \) denotes the dimension of the singularity ring

\[
\mathcal{R} : \quad \mathbb{C}[x, y, z, t]/[dW = 0]
\]

and \( d_{\hat{c}/2} \) is the number of fields of charge \( \hat{c}/2 \), where we assign charge 1 to \( W \) and charge to the variables \( x, y, z, t \) compatible with the highest power having charge 1, and \( \hat{c} \) is the highest charge of the ring \( \mathcal{R} \).

From the above equation, one sees that \( \dim \mathcal{R} = (r)(pc_2 - 1) \). This follows from the fact that A-D-E singularity corresponding to rank \( r \) gauge group has \( r \) ring elements, and \( t^{pc_2} \) has \( pc_2 - 1 \) ring elements. Also \( d_{\hat{c}/2} = r \) because for each element of the A-D-E ring there is a unique monomial \( t^\alpha \) which if we multiply it with, makes it have charge \( \hat{c}/2 \). Thus we have

\[
D = \frac{1}{2} r(pc_2 - 1) + \frac{r}{2} = \frac{rpc_2}{2}
\]

We now use the identity

\[
\dim G = (1 + c_2)r
\]

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to write

\[ R_+ = \frac{1}{2} (\text{dim}G - r) = \frac{1}{2} (rc_2) \]

which shows that \( pR_+ = rpc_2/2 = D \) which is indeed the number of inequivalent branches we have. The story from this point onward is identical to the analysis of [9]. Namely we have \( D \) resolved conifolds which can have the topology of \( S^3 \). These are in one to one correspondence with \( \gamma = (a, k) \) where \( k \) runs over positive roots and \( a \) runs from 1 to \( p \). If we have \( M_\gamma \) branes in the \( \gamma \) branch, this corresponds to turning \( M_\gamma \) units of RR flux through the corresponding \( S^3 \). Moreover we have to turn on the dual B-fluxes to correspond to the coupling constant of the corresponding gauge field. Moreover the strength of the corresponding flux for branch \( \gamma \) is given by

\[ H_{B-cycles} = \rho^k \cdot \left( \sum_{i=1}^{r} \tau_i e_i \right) \]

where \( \tau_i = \frac{1}{g_i} + \theta_i \) denotes the coupling constant of the gauge group corresponding to the \( e_i \) node of A-D-E, and can be identified with the dual \( B = B_R + \tau B_{NS} \) through the \( i \)-th \( S^2 \) cycle. Again, as in [3] one expects the running of the coupling constant and this introduces an IR cutoff in geometry, playing the role of UV cutoff in gauge theory.

We thus obtain the superpotential \( W = \int H \wedge \Omega \) [2] [3] in terms of \( M_\gamma \) which when extremized determines the coefficients of the deformed geometry. Dynamical aspects of these theories as well as the relation of Seiberg-like dualities to Weyl reflections of A-D-E (and affine A-D-E) will appear in [24].

As for the affine case the story is very similar to what we discussed above. The main difference is that here we have an additional integrally indexed label for the choice of branches, related to the number of D3 branes. Even though this does affect the geometry by giving rise to the corresponding 5-form field strength in the internal geometry, it does not affect the complex geometry and the superpotential, which depends only on the \( H \) fluxes. Thus for each branch of affine A-D-E, we consider its projection to the A-D-E, which is labeled by a root of A-D-E \( \pm \rho^k \) and a choice of an integer from 1, ..., \( p \) for each such root. The main difference from the previous case, as far as the superpotential is concerned is that now we obtain not only the positive roots of A-D-E but also the negative roots. This simply translates to the statement that the corresponding fluxes can be positive or negative and we only have to include the net flux. The rest of the discussion is parallel to the A-D-E case above. Thus as far as the total possibilities of superpotentials for various branches, we get for the affine versus the non-affine case exactly the same, except that what plays the role of the rank of the gauge group in the non-affine case, can be positive or negative in the affine case.
8.1. Extension to Large $N$ Dualities for Monodromic Quiver Theories

The same ideas should work in the more general case of monodromic A-D-E fibration. For concreteness let us discuss the case of Laufer’s example studied in this paper. In this case the large $N$ dual should correspond to deforming the complex geometry given by (7.7). Recall that the Higgs branch was characterized by the splitting of the branes in one dimensional representations for which there are $2n + 3$ inequivalent choices, and two dimensional representations for which there are $n$ inequivalent choices. Thus we expect to have $(2n + 3) + n = 3n + 3$ normalizable (or log normalizable) deformations for (7.7). Setting $P(z) = z^n, Q(z) = \alpha = 0$ as an example, we see that (7.7) reduces to

$$x^2 - y^2 + t^3 - t z^{2n+1} = 0$$

We wish to find the number of normalizable deformations of this geometry. This is given by the number of elements of the singularity ring, with charge less than or equal to half the maximum value in the ring. One can readily check that this is $3n + 3$ as expected. Thus these $3n + 3$ parameters can be fixed by extremizing the superpotential as in other cases. This provides new result for gauge theories, where one can obtain exact information about the quantum corrected superpotential for an $\mathcal{N} = 1$ $U(N)$ gauge theory with two adjoint superfields, with the above superpotential (i.e. the deformation of D-type superpotential). Similarly one can analyze other E-type cases as well.

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