Some analyses on optimal energy-extraction efficiency in a free-electron laser

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Abstract
In this paper, the energy-extraction efficiencies of the electron beam are studied for several cases of free-electron laser (FEL). For an initially cold electron beam, the optimal initial detuning for the maximum energy-extraction efficiency and the corresponding saturation length are given. A scheme of the ‘top up’ is proposed to enhance the efficiency: after the electron energy being modulated somewhat, a phase shift and a step-down of the phase bucket are introduced, so that all electrons are located near the upper separatrix of the bucket in the phase space. Finally, the energy-extraction efficiency can be increased by 30% compared with that of the normal undulator case. For a linear tapering undulator with an initially cold electron beam, the simple scaling laws for the maximum energy-extraction efficiency and the corresponding optical power gain are obtained. For a tapered undulator with the pre-bunched electron beam, our analysis gives that the energy-extraction efficiency reaches the maximum when the phase bucket height of the tapered undulator is equal to the amplitude of the detuning modulation. The numerical results validated this reasoning and show that the efficiency has a large increase compared with the case of the unbunched electron beam. The single stair-step undulator is investigated, the optimal step and the corresponding saturation power and saturation length are given by analysis, they agree well with the numerical simulation results.

1. Introduction

In a free-electron laser (FEL), a relativistic electron beam interacts with the radiation field in a periodic magnetic field, and transforms its kinetic energy into the electromagnetic radiation. For the relativistic electron, only a small portion of its energy is transformed into the radiation in a FEL process, namely, the energy extraction efficiency is not high. In high gain FELs, the saturation efficiency is roughly given by the Pierce parameter $\rho$ [1], which is typically on the order of 10^{-4} in the x-ray wavelengths.

There are many investigations on the improvement of FEL efficiency [2–8]. Various strategies have been studied in theory and some of them have been demonstrated in experiments. The FEL performance depends on the parameter relations of the electron beam, the undulator, and the optical field. Therefore, optimizing initial parameters can enhance FEL efficiency, such methods as optimizing the electron initial energy detuning [9], prebunching the electron beam [7, 10, 11]. Normally, the FEL saturation occurs as the power reaches a certain level with the qualities of electrons deteriorated severely, namely, the parameter relations of the electron beam and the optical field have deteriorated. To increase the FEL efficiency, an appropriate way is the use of a tapered undulator [2–6, 12–14]. Around the FEL saturation, the undulator parameters are tapered along the undulator to maintain the resonant condition and the energy transfer from the electrons to the radiation. An equivalent way is ramping the electron energy post-saturation [8]. The optimal taper strategy highly depends on the electron beam parameters and FEL parameters. The optimization of the taper profile typically relies on numerical simulations, such as multidimensional scanning schemes [12], the GINGER self-design taper algorithm [13], the modified KMR model [14], and so on.
In this paper, we analyze the optimal energy-extraction efficiency for several typical cases. We confine our studies to low gain cases, such as oscillator FEL, and the tapered undulator schemes, which generally work in the low gain mode, including the situation they being used beyond the saturation of high gain FEL. In the analyses, we use the normalized variables for the most general conditions, and we limit the discussion to the one-dimensional regime so that analytical expressions for the optimizations could be obtained. We validate the analytical results by numerically solving the pendulum equation or by numerical simulation with FEL code GENESIS\[15\].

The paper is organized as follows: we first study the relations of initial detuning to the energy-extraction efficiency and the saturation power for normal undulator (next section). In section 3, we propose and analyze the ‘top up’ scheme for efficiency enhancement. Then, in section 4, we analyze the optimal energy-extraction efficiency for tapered undulators, including the linear tapered undulator, the tapered undulator with the pre-bunched electron beam and the single stair-step undulator. Finally, a summary is given.

2. The optimal initial detuning for uniform undulator

The energy exchange between the optical field and the electron is dependent on the phase of the electron in the optical field plus the undulator magnetic field: $\psi = (k_z + k_y)y - \omega_t f$, where $k_z$, $k_y$ are the corresponding wavenumbers of the optical field and the undulator magnetic field, respectively. For the resonant electron, its phase is fixed, namely, it has $\psi' = 0$. Therefore, the phase velocity of the electron $\psi'$ describes the resonance offset of the electron, and is called the detuning parameter, which can be expressed as the relative energy deviation from the resonant energy

$$\psi' = 2ka_y \frac{\gamma - \gamma_{\gamma}}{\gamma_{\gamma}}$$

where $\gamma$ is the normalized energy of the electrons, $\gamma_{\gamma}$ is the resonant energy.

From above phase equation and the energy equation of the electron, the FEL pendulum equation can be obtained (for convenience, denoting $\psi = \phi + \varphi_2 + \pi/2$, $\varphi_2$ is the phase of the radiation field)[16]:

$$\psi'' = -\Omega^2 \sin \psi$$

where $\Omega = \sqrt{2k_0 a_u a_z f_s / \gamma_0 a_z} = eE_0 / (mc^2 k_z)$ and $a_u = eB_{0y} / (mc^2 k_z)$ are dimensionless vector potential of the $rms$ radiation optical field $E_0$ and undulator magnetic field $B_{0y}$ respectively; $f_s$ is the undulator coupling factor: for circularly polarized helical undulator $f_s = 1$; for linearly polarized planar undulator it is a difference of the two Bessel function $f_s = J_0(\zeta) - J_1(\zeta)$, $\zeta = a_z^2 / 2(1 + a_u^2)$. From the pendulum equation, one can get the first integral

$$\frac{\psi'^2}{2} - \Omega^2 \cos \psi = U$$

where $U$ is a constant determined by the initial conditions. Here, compared with the variation of the phase and of the detuning parameter, the variation of $\Omega$ is neglected under the standard lowly varying envelope approximation (SVEA), while $\Omega$ is proportional to the square root of the optical field amplitude, i.e. the fourth root of the optical power.

There are two classes of electron trajectory in the phase space: the bounded and the unbounded. The separatrix separating the two of them is given by equation (3) with $U = \Omega^2$, namely $\psi' = \pm 2\theta \cos (\psi/2)$. The region enclosed by the separatrix is called the ponderomotive bucket. Within the bucket, the electrons are trapped and rotate clockwise along the closed trajectories; and outside of the bucket, the electrons are untrapped, their trajectories are unbounded. The maximum value of $\psi'$ along the separatrix gives the half-height of the bucket: $\psi'_{\text{max}} = 2\Omega$.

For the phase near zero $\psi \approx 0$, the pendulum equation (equation 2) can be approximately written as $\psi'' \approx -\Omega^2 \psi$. Thus, the electrons near the zero phase make a simple harmonic vibration in the phase space, the corresponding period of the synchrotron oscillation is $L_{\text{syn}} = 2\pi / \Omega$. Therefore, $\Omega$ is the wavenumber of the synchrotron oscillation for the zero phase electrons. The synchrotron oscillation period for the general trapped electrons can be given from the second integral of equation (2)

$$L_{\text{syn}} = \frac{2}{\pi} K(\chi^2) L_{\text{syn}}$$

here $K$ is a complete elliptic integral of the first kind, $\chi^2 = (1 + U/\Omega^2) / 2$. When $\psi' = 0$, it has $\chi^2 = \sin^2(\psi_z / 2)$, $\psi_z$ is the intersection point of the closed trajectory and the $\psi$ coordinate axis in the positive direction.

For low gain case, the bucket can be regarded as invariant, such as amplification of the light at each time passing through the undulator with the electron beam in an oscillator FEL. To extract the energy from the...
electrons to the optical field, the electrons initially are injected at the top of the bucket, i.e. the initial energy of the electron beam is larger than the resonant energy. When the trapped electron moves down to the bottom of the phase bucket and begins to move upward in the phase space, namely to gain energy from the optical field, the distance it traveled down the undulator is a half of its synchrotron oscillation period. When the most trapped electrons located at the bottom of the phase bucket, the FEL saturation occurs. One can see that the larger initial detuning, the more energy can be extracted from the trapped electrons, but the fewer electrons are trapped, the smaller will be the trapping fraction. Therefore, there is an optimal initial detuning for the largest energy-extraction efficiency. By contrast, for a high gain FEL, the amplification of the optical field is maximum when the electrons are injected at the resonant energy, i.e. the initial detuning is zero. In fact, in a high gain FEL the phase bucket grows quickly, and the variation of the optical field phase also cannot be neglected, that makes the phase bucket transverse moving in the phase space.

Let $\tau = z/L_{sy0}$ be the interaction distance normalized with the synchrotron oscillation period of the zero phase electron, $x = \psi/\pi$, and $y = \psi'/2\Omega$ be the detuning parameter normalized with the half-height of the phase bucket, then the pendulum equation becomes as

$$\ddot{\xi} = -4\pi \sin(\pi x)$$

Here, we use $\langle \cdot \rangle$ to denote the derivative with respect to the normalized interaction distance $\tau = z/L_{sy}$, and to differentiate it from the derivative with respect to the interaction distance $z$ ($\dot{\cdot}$). The energy-extraction efficiency can be calculated according to

$$\eta(e, y_0) = -\frac{\Omega}{k_u} \langle \delta y \rangle \approx -\frac{\Omega}{k_u} x_{0m} \langle \delta y \rangle_t$$

where $\delta \gamma$ is the electron energy variation due to the interaction, $\langle \cdot \rangle$ represents the average over all electrons, $\langle \cdot \rangle_t$ represents the average over all the trapped electrons, and $x_{0m}$ is the maximum initial phase of the trapped electrons. We neglect the contribution of the untrapped electrons to the energy-extraction efficiency here, and consider an initially unbunched cold electron beam, then trapping fraction of the electrons is $f_i = \psi_{0m}/\pi = x_{0m}$. Owing to $x_{0m} = 2\arccos(y_0)/\pi$ and $\langle -\delta y \rangle_t \propto y_0$, we have

$$\eta(e, y_0) \propto \arccos(y_0)y_0$$

Hence, I get the optimal initial detuning $y_0 = 0.65$.

By numerically solving the pendulum equation, we calculate the variation of the energy-extraction efficiency with the distance traveled by the electron in the undulator for different initial detuning (figure 1). We can see that the energy-extraction efficiency oscillates with the interaction length, and the first maximum energy-extraction efficiency appeared at about $0.6L_{sy0}$, which does not change much with initial detuning. That means the optimal undulator length for the maximum energy-extraction efficiency is insensitive to the initial energy spread.

Then, for different initial detuning, we give the corresponding variation of the maximum energy-extraction efficiency (figure 2). It is shown that the optimal initial detuning is $\psi_0' = 1.3\Omega$ and the corresponding maximum
energy-extraction efficiency is \((-\delta y)_m = 0.7\). The result agrees with that from the previous analysis, in which only the contribution of the trapped electrons is considered, while the curve in figure 2 are obtained from all the first peaks of the curves in figure 1, which are obtained by numerically solving the pendulum equation (equation 5) for all electrons. This shows that the contribution of the untrapped electron to the energy-extraction efficiency can be neglected indeed.

Our new result of optimal initial detuning \(\psi'_{m} = 1.3\Omega\) is applicable to the general strong optical field in the low gain region, as the normalized quantities are used. Note that the optimal initial detuning for the low gain case of the weak optical field (the small signal gain) is \(\psi'_{m} = 2.6/L\) \([16]\). \(L\) is the length of the undulator, it is independent on the optical field.

For free-electron laser oscillators, the optical field develops from spontaneous radiation. Initially, most of the electrons are outside of the phase bucket in the phase space. As the optical field intensity increase through multi-pass amplification, the phase bucket grows larger, and the trapped portion of the electron bunch becomes larger. At about saturation, we can apply our previous result to here: we change the value of the electron beam detuning at the entrance of the undulator to get the larger energy extraction efficiency. Actually, this has been demonstrated already in the simulation and the experiment by EAFEL in Israel \([8]\), which shows that the energy extraction efficiency is increased by around 50\% through increasing the optical field after the saturation. From the simulation results of \([8]\), the energy extraction \(\Delta \gamma\) is increased from 0.0552 to 0.0882, a 60 percent increase when the initial energy is changed from 0.0406 to 0.0795. Correspondingly, the initial detuning \(\psi'_{m}/2\Omega\) is changed about from 0.24 to 0.44, then according to our previous study (figure 2), the energy extraction efficiency is increased by about \((\psi'_{m}/2\Omega) = 0.35\) to 0.57, a 63 percent increase, agree with the results of \([8]\). Besides, according to our analysis (see figure 2), the energy extraction efficiency can be further increased by optimizing the initial detuning.

From the relation of the optical field power and the phase bucket height (or the synchrotron oscillation period of the zero phase electron), we can get

\[ P = \left( \frac{\Omega^2}{8k_{e}^2 \rho^2} \right)^2 \rho \rho_{e} = \left( \frac{\sqrt{6} \pi L_{g}}{\lambda_{0}} \right)^4 \rho \rho_{e} \]  

where \(P\) is the power of the electron beam, \(L_{g} = 1/(2\sqrt{3} k_{e} \rho)\) is the optical power gain length, \(\rho\) is the FEL parameter, and \((2\gamma_{0})^3 = 2\pi r_{e} n_{e} f_{g}^2 / k_{e}^2 \rho_{e}\). \(r_{e}\) is the classical electron radius and \(n_{e}\) is the peak electron density. Using the previous result, the saturation length for the maximum energy-extraction efficiency is \(L = 0.6 L_{sa}\), then the saturation power is

\[ P_{s} \approx \left( \frac{4.6 L_{g}}{L} \right)^4 \rho \rho_{e} \]  

\[ (1) \] From the definitions, one can obtain: \(\Omega = 2\sqrt{3} k_{e} \rho (p_{e}/\rho_{e})^{3/4}\), where \(p_{e} = cE_{e}^2/4\pi = \pi c (\rho_{e}/\tau)\lambda_{0}^{3}\), \(p_{e}\) is the power density of the radiation and the electron beam, respectively. We assumed that the radiation beam size is equal to the electron beam size here, then, equation (1) can be obtained. For the more rigorous expression, the right hand side of equation (1) should be multiplied by a scale factor of beam size.
For low gain FELs considered here, it has $L < 3L_g$ [17], and therefore the saturation power in the resonator of oscillator FEL can be larger than that of high gain FEL (for SASE: $P_s \sim \rho P_l$). The ‘low gain’ refers to the gain of each single pass amplification, but the whole gain for multi-pass amplification is high. The SASE FEL also develops from spontaneous radiation, but the interaction length of it is larger than four gain length $L > 4L_g$, i.e. working in the high gain regime [17]. High gain FEL is a single amplification process, while oscillator FEL works in the low gain region and is a multiple amplification process with the mode competition.

3. The ‘top up’ scheme for the efficiency enhancement

From the previous analysis, we can see that the initial detuning requirement for the maximum trapping fraction is in contradiction with that for the maximum extracted energy. To enhance the efficiency, I propose a novel scheme: for an initial electron beam with resonant energy in a FEL amplifier, first, its energy is modulated by the optical field in the undulator

$$\psi' = -\Delta \psi_m' \sin(\psi_0)$$

where $\Delta \psi_m'$ is the amplitude of the detuning modulation. Then we introduce a phase shift $\delta \psi = \pi/2$ and a step-down of the phase bucket $\delta \psi' = 2\pi - \Delta \psi_m'$, which can be achieved by changing the undulator parameters. In figure 3, the shift in position and the energy is visualized and the orientation of the phase-space distribution of the electrons with respect to the buckets corresponding to the first and second undulator can be seen. As a result, almost all electrons are located near the upper separatrix of the phase space (figure 3), i.e. the detuning of every electron is near its maximum value in the bucket for the phase, and no electron detrapping from the bucket occurs. Therefore, we call it the ‘top up’ scheme.

We chose the amplitude of the detuning modulation to be equal to the one-fourth height of the phase bucket $\Delta \psi_m' = \Omega$. The corresponding length of the modulation undulator is $L_1 = L_{Sy0}/2\pi$ [18]. In this case, the shift downward of the phase bucket is also equal to its one-fourth height $\delta \psi' = \Omega$. The corresponding required magnetic field parameter of the next undulator has a change relative to the previous undulator

$$\frac{\Delta a_u}{a_u} = \left(1 + \frac{1}{a_u^2}\right) \frac{\delta \psi'}{2k_u} = -\left(1 + \frac{1}{a_u^2}\right) \sqrt{2} \rho \left(\frac{P}{\rho P_l} \right)^\frac{3}{4}$$

Numerically solving the pendulum equation (equation (5)) for the scheme of the ‘top up’, we obtain the evolution of the energy-extraction efficiency with the interaction length (figure 4), the maximum energy-extraction efficiency is $(-\delta \psi)_m' = 0.9$ at $L = 0.825L_{Sy0}$, which is increased by 30% compared with the normal undulator case (see figure 1). The evolution of the energy-extraction efficiency for the linear-tapering undulator (see next section, figure 7) is also presented in figure 4. We can see that within about one synchrotron oscillation period $L_{Sy0}$, the energy-extraction efficiency of the ‘top up’ scheme is also larger than that of the linear-tapering undulator. Therefore, the ‘top up’ scheme can increase the trapping fraction and the energy-extraction efficiency by using relatively short undulator for an amplifier FEL with a fresh electron bunch, or for the amplification of the strong optical field in an oscillator FEL.

Figure 3. The energy modulated electron beam with a phase shift and a step down of the bucket. The open circle and solid circle: the electron distribution before and after the manipulation. The dash line and solid line: the phase buckets for the first and second undulator.
At the maximum energy-extraction efficiency, the distribution of the electrons in the phase space is shown in figure 5. It exhibits that the electrons have moved from initially near the upper separatrix (see figure 3) to near the lower separatrix. That is, the electron energy changes from the maximum to the minimum in the bucket.

4. The optimal energy extraction efficiency for tapered undulator

4.1. Linear-tapering undulator

In the FEL process, as the energy is transferred to the optical field, the electron deviates too much from the resonance, which eventually leads to the cessation of the interaction, i.e. the saturation of the optical field. To heighten the FEL power, the tapered undulator was proposed [2–6], tapering the undulator parameters along the undulator to maintain the resonance condition and the interaction.

From the resonance condition

$$\lambda_s = \frac{\lambda_w}{2\gamma^2 n} (1 + a_n^2)$$

(11)

Figure 4. The energy-extraction efficiency with the interaction length for the 'top up' scheme compared with that of the normal undulator (figure 1) and the linear-tapering undulator (figure 7).

Figure 5. The phase distribution at the maximum energy-extraction efficiency ($L = 0.825L_{sy0}$). For the initial distribution, see figure 3.)
one can get
\[
\frac{\gamma'_r}{\gamma'_f} = \frac{a_n^2}{1 + a_n^2} + \frac{\lambda'_u}{\lambda_u} = \frac{2a_n^2}{1 + a_n^2} \frac{B'_u}{B_u} + \frac{1 + 3a_n^2}{1 + a_n^2} \frac{\lambda'_u}{\lambda_u}
\]  
(12)

The field or the period of tapered undulator equation may be changed, for the technical reason, usually the undulator magnetic field is changed.

For tapered undulator, the pendulum equation (equation 2) becomes [2]
\[
\psi'' = -\Omega^2 [\sin \psi - \sin \psi_r]
\]  
(13)

where
\[
\sin \psi_r = -\frac{k_u \gamma'_r}{\Omega^2} \approx -\frac{2k_u \delta \gamma_r}{\Omega^2 \gamma_r}
\]  
(14)

the \(\psi_r\) is the synchronous phase, for positive tapered undulator \(0 < \psi_r < \pi/2\). For the electron with the synchronous phase, the detuning of it is invariant.

Usually, the change of the parameters of the tapered undulator is very small, namely, the taper ratio is small. Thus, the taper is approximate to linear synchronous phase, the detuning of it is invariant.

\[
\psi'' = -\Omega^2 \cos \psi - \psi \sin \psi_r = U
\]  
(15)

The ponderomotive potential:
\[
V = -\Omega^2 \cos \psi + \psi \sin \psi_r
\]  
(16)

There are still two classes of electron trajectory, the bounded and the unbounded, in the phase space. The closed orbit region \(\psi_2 < \psi < \psi_1\), where \(\psi_1 = \pi - \psi_r\), \(\psi_2\) is given by \(V(\psi_2) = V(\psi_1)\). The critical orbit is given by the first integral formula \(U = V(\psi_r)\) as follows:
\[
\psi'' = \pm \sqrt{2[V(\psi) - V(\psi_r)]} \pm \Omega \sqrt{2[\cos \psi + \cos \psi_r - (\pi - \psi_r) \sin \psi_r]}\]  
(17)

The stable oscillation range \(\psi_1 - \psi_2\) corresponds to the width of the phase bucket or the width of the ponderomotive dynamic potential well.

The height of the phase bucket is obtained by taking \(\psi = \psi_r\) in equation (17):
\[
\psi'_m = \pm 2\Omega \sqrt{\cos \psi_r - (\text{sgn}(\psi_r) \pi/2 - \psi_r) \sin \psi_r} = \pm 2\Omega \Gamma(\psi_r)
\]  
(18)

where \(\Gamma(\psi_r) < 1\).

We plot the variation of \(\psi_1\), \(\psi_2\), and \(\psi_r\) with the function \(\Gamma(\psi_r)\) (figure 6). It should be noted that I take \(\Gamma(\psi_r)\) as the independent variable and the longitudinal coordinate in figure 6, thus the variation of the height and the width of the phase bucket were shown visually. It can be seen that the larger the \(\psi_r\), the smaller the \(\Gamma(\psi_r)\) and the closer the \(\psi_2\) to the \(\psi_r\), namely, the lower the height and the narrower the width of the phase bucket (i.e. the smaller the area of the phase bucket), or the shallower the depth and the narrower the width of the ponderomotive potential well.

For the electrons near the synchronous phase \(\psi_r\), the equation (13) can be approximated by
\[
\psi'' \approx -\Omega^2 \cos \psi_r (\psi - \psi_r),
\]  
the corresponding synchrotron period is
\[
L_{syn} = 2\pi / (\Omega \sqrt{\cos \psi_r})
\]  
(19)

For the general trapped electrons, I find that when \(\psi_r < 1\), the synchrotron oscillation period can be approximately written as [18]
\[
L_{syn} \approx \frac{2}{\pi} K\left(\sin^2\left(\frac{\psi_x + \psi_r}{2}\right)\right) L_{y0}
\]  
(20)

Namely, the \(\psi_x\) in equation (4) is replaced with \(\psi_x + \psi_r\), and \(\psi_x \leq \psi_x \leq \pi - \psi_r\).

We designate \(\eta_r = (\gamma - \gamma_0)/\gamma_0\) as the relative deviation of the energy with respect to the instantaneous resonance energy, and \(\eta = (\gamma - \gamma_0)/\gamma_0\) as the relative deviation of the energy with respect to the initial resonance energy. Then the detuning can be written as
\[
\psi' = 2k_u \eta_r = 2k_u \eta - 2k_u \frac{\delta \gamma_r}{\gamma_r}
\]  
(21)
If taking $\eta = \phi' + 2k_u \delta \gamma_r / \gamma_r$ as the vertical coordinate in the phase space, then the coordinate value corresponding to the resonance energy ($\psi' = 0$) is zero only at the beginning, it decreases with time evolution, i.e. the phase bucket moves downward together with the trapped electrons.

The energy extraction efficiency is

$$\eta = -\langle \delta \eta \rangle = -\frac{\langle \delta \psi' \rangle}{2k_u} - f_t \frac{k_u \delta \gamma_r}{\Omega \gamma} \leq f_t \frac{\Omega}{2k_u} [2\Gamma (\psi_r) + \Omega z \sin \psi_r]$$

(22)

Here, we consider the linear tapered undulator, $f_t$ is the trapping fraction of the electrons. The first term in equation (22) corresponds to the synchrotron oscillation of the trapped electrons in the phase bucket, and the second term corresponds to the descending down of the phase bucket (the tapering). The contribution of the first term is important only at the beginning, and soon the contribution of the second term is dominant. Then the energy extraction efficiency becomes

$$\eta / k_u = -\langle \delta y \rangle - f_t \frac{k_u \delta \gamma_r}{\Omega \gamma} \approx -f_t \frac{k_u \delta \gamma_r}{\Omega \gamma} = f_t \pi \sin \psi_r \frac{z}{L_{\eta 0}}$$

(23)

It is proportional to the product of the number of the trapped particles and the change of the resonant energy (the taper strength). For the larger synchronous phase $\psi_r$, the variation of resonant energy $\delta \gamma_r (\sim \sin \psi_r)$ is larger, but the bucket width and height are smaller (see figure 6), thus the trapping fraction $f_t$ is lower. Therefore, there exist an optimal synchronous phase for the maximum energy extraction efficiency.

For an initially unbunched monoenergetic electron beam, the trapping fraction $f_t$ is proportional to the distribution range of the initial phase in the phase bucket:

$$f_t \leq \frac{|\psi_1 - \psi_2|}{2\pi} = \frac{\cos \psi_2 + \cos \psi_r}{2\pi \sin \psi_r}$$

(24)

The equal sign in the above formula holds for the resonant electron beam, in that case the optimal synchronous phase for the maximum energy extraction efficiency is $\psi_r \approx 40^\circ$ [20]. From equations (23) and (24), we obtain the maximum energy extraction efficiency:

$$\eta / k_u \approx \frac{1}{2} \left( \cos \psi_2 + \cos \psi_r \right) \frac{z}{L_{\eta 0}} \leq \frac{\sqrt{3}}{2} \frac{z}{L_{\eta 0}}$$

(25)

Using equation (25) and the relation between the synchrotron oscillation wavenumber $\Omega$ and the optical field power $P$ (equation 7), we can calculate the corresponding gain of the optical field:

$$g = \frac{\Delta P}{P} = \eta k_u P / P = \frac{1}{6\pi} \left( \frac{L_{\eta 0}}{\pi L_g} \right)^2 \frac{L_g}{L_g} (L > L_{\eta 0}/2)$$

(26)

here $L$ is the length of the tapered undulator. By taking the original saturation power of SASE $P \sim \rho P_e$ in equation (7), we get the synchrotron oscillation period $L_{\eta 0} \approx 2\pi L_g$ at the original saturation of SASE, then substitute it into equation (26), we have

$$g \approx \frac{\Delta P}{P} \sim 2 \frac{L}{3\pi L_g} (L > \pi L_g)$$

(27)
We numerically solve the pendulum equation (equation 13)

$$\ddot{x} = -4\pi [\sin(\pi x) - \sin(\pi x')]$$

The results (figure 7) agree well with the analytical results (equation 25). The maximum energy extraction efficiency is much larger than that of the uniform undulator case (see figure 2).

The equations (25)–(27) I obtained give the simple scaling laws for the maximum energy-extraction efficiency and the optical power gain for a linear-tapering undulator with a cold initial electron beam. When the energy spread of the electron beam must be considered, the trapping fraction scales as the bucket area, which decreases faster than the bucket width as a function of $\psi_r$ because both the width and the height of the bucket decreases with the increase of $\psi_r$ (figure 6). Consequently, the optimal synchronous phase is smaller (about $25^\circ$).

**4.2. The tapered undulator with the pre-bunched electron beam**

One method to increase the fraction of the particles trapped in the bucket is to inject a prebunched electron beam into a tapered undulator. We consider such a prebunched electron beam, which is just like that before the radiator section of the undulator in a harmonic generation FEL, e.g. the coherent harmonic generation (CHG) [21] or the high gain harmonic generation (HGHG) [22]. In a shorter undulator called the modulator, which tuned at the wavelength of the seeding laser, the energy of the electron beam is modulated due to the interaction with the optical field of the seeding laser. Afterward, in a dispersive magnetic field, the energy modulation is transformed into the density modulation, i.e. the electrons are pre-bunched. Therefore, the phase and the detuning of the pre-bunched electron beam at the entrance of the tapered undulator are [23]

$$\psi = \psi_0 - \Delta \psi_m D \sin(\psi_0)$$

$$\psi' = -\Delta \psi_m' \sin(\psi_0)$$

where $\psi_0$ is the initial phase at the entrance of the modulation section, $\Delta \psi_m$ is the amplitude of the detuning modulation induced by the seeding laser in the modulation section, $D$ is the scale dispersive parameter. Here, we consider that the electron beam is initially with a uniform phase distribution and the resonant energy $\psi_0' = 0$.

We take such a dispersive strength to get the largest bunching:

$$\Delta \psi_m D \approx \frac{\pi}{2}$$

and make a phase shift to set the bunching phase equal to the synchronous phase. In this case, the trapping fraction is a function of the detuning modulation amplitude and the phase bucket height of the tapered undulator

$$f_t = \frac{1}{2} \min[\Delta \gamma_m, \Gamma(\psi_0)]$$

where $\Delta \gamma_m$ is the modulation amplitude of the normalized detuning $\Delta \gamma_m = \Delta \psi_m / 2\Omega$, and the factor $1/2$ comes from that we assume that only the bunched electrons can be trapped, which were distributed over the length about one-half the period in each phase bucket before the density being modulated. Then I give the energy-extraction efficiency
The corresponding gain can be given in the same way as obtaining equation (27). The energy-extraction efficiency reaches the maximum when the height of the phase bucket of the tapered undulator is equal to the detuning modulation amplitude: \( \Gamma(\psi_r) = \Delta y_m \). We take the detuning modulation amplitude to be the one-fourth height of the phase bucket in the modulation undulator \( \Delta \psi_m' = \Omega \) (namely \( \Delta y_m = 0.5 \)), then the optimal synchronous phase is \( \psi_r = 0.6 \).

We numerically solve the pendulum equation (equation 28) with the initial phase and the initial detuning

\[
\begin{align*}
    x &= x_0 - \frac{1}{2} \sin(x_0\pi - x_r\pi), \\
    \dot{x} &= -4\Delta y_m \sin(x_0\pi - x_r\pi)
\end{align*}
\]

The result (figure 8) shows that the synchronous phase for the maximum energy-extraction efficiency is \( \psi_r = 0.6 \) as given by the analytical analysis equation (33), but the maximum energy-extraction efficiency is larger than that from the theoretical analysis; this is because some electrons near bunching phase also are trapped. Compared with the unbunched case (figure 7), we can see that the energy-extraction efficiency has a large increase, from \( \eta_e / (\Omega/k_u) \approx (\sqrt{3}/2) z/L_{sy0} \) (equation 25) to \( 1.2z/L_{sy0} \), increased near 40%.

### 4.3. Single stair-step undulator

For the uniform undulator, the FEL saturation occurs when the most trapped electrons move down to the bottom of the ponderomotive bucket in the phase space and begin to move upward, namely to gain energy from the optical field. We consider a single stair-step undulator. Starting from the point nearby the original saturation, we use a new undulator with a strength parameter having a step relative to the previous undulator: \( a_{u} - \Delta a_{u} \). So that the corresponding resonance energy has a change that shifts the phase bucket downward to locate the electrons to the upper of the new phase bucket, whereby they can continue to move downward and impart the energy to the optical field.

Therefore, we chose the shift downward of the phase bucket to be its half-height \( 2\Omega \)

\[
\Delta \psi' = -2\Omega = -4\sqrt{2} k_u \rho \left( \frac{P}{\rho P_c} \right)^{1/2} \approx -4\sqrt{2} k_u \rho
\]

where the original saturation power for SASE is \( P \sim \rho P_c \). The required change of the undulator strength (and the resonant electron energy) is given by (see equation 12)

\[
\frac{\Delta a_u}{a_u} = \left( 1 + \frac{1}{a_u} \right) \frac{\Delta \gamma_r}{\gamma_r} = \left( 1 + \frac{1}{a_u} \right) \frac{\Delta \psi'}{2k_u} = \left( 1 + \frac{1}{a_u} \right) \frac{2\sqrt{2}}{\rho}
\]
The increase of the saturation power can be given as

\[ D = D \approx \frac{P}{P_f} \]

Hence, the saturation power can be increased by three-fold or more. Using the results in section II, the new saturation occurs at

\[ L \approx 0.6 \]

The synchrotron oscillation period at the saturation of SASE is

\[ L_{sy} \approx \frac{2\pi}{L_g}, \]

and then we give the length estimation for the stepped undulator

\[ \sim L_4. \]

By contrast, a three-fold increase of the power for the previous linear tapered undulator case needs more than twice this length (from equation (27), the \( L \sim 9.4 L_g \)). For FEL facilities with limited available space, the single step-tapering scheme can enhance the FEL performance compared with other kind tapering schemes that need long undulators.

Using the FEL code GENESIS [11], we simulate SASE FELs with a single stair-step undulator based on the Linac Coherent Light Source (LCLS) [24] and the Tesla Test Facility (TTF) [25] like parameters, as shown in table 1. The corresponding results are shown in figures 9 and 10, respectively. The start of the stepped undulator is the last undulator segment before the original saturation. We have scanned the stepped undulator strength with simulations in steady-state mode.

For LCLS like parameters, from figure 9, the radiation has the highest saturation power and the shortest saturation length when \( \Delta a / a_0 = 0.375\% \), while according to the analysis, it is calculated to be 0.4%; the saturation power increase is \( \Delta P / P = 1.89\% \) and the saturation length increase is \( \Delta L_s = 7 \text{ m} \), (the 1D and 3D gain length are 1.12 m and 1.90 m, respectively). Similarly, for TTF like parameters, the optimized \( \Delta a / a_0 \) from the simulation is 1.05% while 1.17% from the analytical calculation; the increase of the saturation power and the saturation length are \( \Delta P / P = 1.92\% \) and \( \Delta L_s = 4.4 \text{ m} \), respectively, (the 1D and 3D gain length are 0.86m and 1.20m, respectively). Clearly, our theory analyses agree with the simulation results very well.

5. Summary

In this paper, we studied the optimal energy-extraction efficiency of the electron beam for several cases of free electron laser.
For a uniform undulator and an initially cold electron beam, our study shows that the optimal initial detuning for the maximum energy-extraction efficiency is $\Psi_0^* = 1.3 \, \Omega$ and the corresponding saturation length is $L_s = 0.6\,L_{sy}$. The method can be used to enhance the efficiency of oscillator FEL, and can be implemented by the voltage ramping in an electrostatic accelerator or the radiofrequency (RF) field ramping in a RF accelerator.

To enhance the efficiency we propose a scheme of the 'top up', we introduced a phase shift and a step-down of the phase bucket after the electron energy being modulated somewhat, so that all electrons are located near the upper separatrix of the bucket in the phase space. Consequently, the energy-extraction efficiency can be increased by 30% compared with that of the normal undulator case. The 'top up' scheme can increase the trapping fraction and the energy-extraction efficiency with a shorter undulator, and be used to the case of optical amplification with a fresh electron bunch, such as the amplification of the strong optical field in an oscillator FEL. In practice, both 'phase shift' and 'step down' are easy to implement, the hardwares of them are simpler than that of the tapered undulator.

For a linear-tapering undulator with an initially cold and unbunched electron beam, corresponding to the optimal synchronous phase, we gave the simple scaling laws for the maximum energy-extraction efficiency and the optical power gain.

We analyzed a tapered undulator with the pre-bunched electron beam. We find that when the phase bucket height of the tapered undulator is equal to the detuning modulation amplitude of the electron beam, the energy-extraction efficiency reaches the maximum. This reasoning is validated by the numerical results, which show that the efficiency can be increased by near 40% compared with the unbunched case. Using the pre-bunched electron beam can achieve large energy-extraction efficiency in a long tapered undulator.

We investigated the scheme of the single stair-step undulator and gave the optimal step amplitude. Starting from the point nearby the original saturation of SASE, we get that the length of the stair-step undulator is about four gain length and the increase of the saturation power is three times or more. The numerical simulation results agree well with our analyses. The single stair-step undulator scheme can significantly increase the saturation power in a relatively short interaction distance and is especially useful for the FEL facilities whose available space is limited. The scheme can readily be applied since it is feasible and straightforward to implement in the existing or planned FEL facilities: it only needs the undulator modules after the original saturation working with a different magnetic gap from that of the previous undulator.

Actually, stepping down of the phase bucket and increasing the initial energy of the electron beam are physically equivalent, because instead of tapering undulator, one also can increase the initial energy of the electron beam to enhance the energy-extraction efficiency. In this paper, the optimization of the initial detuning is used in oscillator FEL, the single step-down of the phase bucket is used in SASE FEL.

In our analyses, an initially cold electron beam is assumed. Consider the energy spread, the optimal initial detuning for the maximum energy-extraction efficiency will become smaller, while its effect on the corresponding saturation length is small. For tapered undulator schemes, the energy spread will have some effect on the trapping fraction, and thereby on the energy extraction efficiency. Then, a modification coefficient may be introduced in the results. In fact, our simulation of the single stair-step undulator is carried out for the electron beam with an energy spread.
The results obtained in this paper are not constrained to a particular set of parameters only, because the normalized variables are used, and they will contribute to relevant analyses, design and the parameters optimization, as well as insight into the physical processes of FEL.

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