RADON TRANSFORMS OVER LOWER-DIMENSIONAL HOROSPHERES IN REAL HYPERBOLIC SPACE

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Abstract. We study horospherical Radon transforms that integrate functions on the $n$-dimensional real hyperbolic space over horospheres of arbitrary fixed dimension $1 \leq d \leq n - 1$. Exact existence conditions and new explicit inversion formulas are obtained for these transforms acting on smooth functions and functions belonging to $L^p$. The case $d = n - 1$ agrees with the well-known Gelfand-Graev transform.

1. Introduction

Let $\mathbb{H}^n$ be the $n$-dimensional real hyperbolic space. We will be dealing with the hyperboloid model of this space, when $\mathbb{H}^n$ is identified with the upper sheet of the two-sheeted hyperboloid in the pseudo-Euclidean space $E^{n,1} \sim \mathbb{R}^{n+1}$. The term horosphere (or orisphere), which means a sphere of infinite radius, was introduced by Lobachevsky. In the hyperboloid model, the $(n - 1)$-dimensional horosphere is a cross-section of the hyperboloid $\mathbb{H}^n$ by the hyperplane whose normal lies in the asymptotic cone.

In the present article, we study horospherical Radon-like transforms $f \to \hat{f}$ that integrate functions on $\mathbb{H}^n$ over $d$-dimensional horospheres for arbitrary $1 \leq d \leq n - 1$. Our main objective is explicit definition of these transforms, their properties, and inversion formulas on $L^p$ and smooth functions.

In the case $d = n - 1$, the corresponding horospherical transforms are also known as the Gelfand-Graev transforms; see [7, p. 290], [33, p. 532], [34, p. 162]. In these publications, a compactly supported smooth function $f$ was reconstructed from $\hat{f}$ in terms of certain integrals that should be understood in the sense of distributions.

Our approach essentially differs from that in [7, 33, 34] and consists of two parts. The first part deals with arbitrary continuous and $L^p$ functions and relies on the properties of some mean value operators.

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This idea dates back to the classical works by Funk, Radon, and Helgason; see historical notes in [16, 29]. We show that it is applicable to horospherical transforms over horospheres of arbitrary dimension.

The second part deals with compactly supported smooth functions. Here the reconstruction of $f$ reduces to inversion of certain operators of the potential type by means of polynomials of the Beltrami-Laplace operator. Operators of this kind are hyperbolic counterparts of the classical Riesz potentials (see, e.g., Stein [32]) and might be of independent interest. This approach was applied by Helgason to totally geodesic Radon transforms of smooth functions on constant curvature spaces and extended by Rubin [29] to horospherical transforms over $(n-1)$-dimensional horospheres.

It was surprising that Radon transforms over lower-dimensional horospheres in the hyperbolic space were not considered in the literature (to the best of our knowledge). Our aim is to complete this gap.

It is worth noting that horospherical transforms play an important role in the representation theory and appear in the general context of symmetric spaces under the name “horocycle transform”. More information on this subject can be found in the works by Gelfand and Graev [6], Helgason [13, 15, 16, 17], Gindikin [8, 9, 10], Gonzalez [11], Gonzalez and Quinto [12], Hilgert, Pasquale, and Vinberg [18, 19]; see also Berenstein and Casadio Tarabusi [1], Bray and Rubin [3] for the case $d = n - 1$. The methods and results of these publications essentially differ from those in the present article.

**Plan of the paper.** Section 2 contains auxiliary facts related to analysis on $\mathbb{H}^n$. In Section 3 we define the horospherical transform $\hat{f}(\xi)$ for $d$-dimensional horospheres $\xi$. In particular, we show that if $f \in L^p(\mathbb{H}^n)$, then $\hat{f}(\xi)$ is finite for almost all $\xi$ provided $1 \leq p < 2(n-1)/d$ and this bound is sharp. Section 4 is devoted to inversion formulas for $\hat{f}$ on functions $f \in C^\infty_c(\mathbb{H}^n)$ and $f \in L^p(\mathbb{H}^n)$. The main results are stated in Theorems 4.10, 4.12, and 4.16. We conclude the paper by Section 5, in which some open problems are formulated.

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to Edmund Kelly who kindly sent us his 1974 preprint [20]. The methods of our paper and the results are essentially different from those in [20, 23, 24].

2. Preliminaries

2.1. Basic Definitions. The pseudo-Euclidean space $\mathbb{E}^{n,1}$, $n \geq 2$, is the $(n + 1)$-dimensional real vector space of points in $\mathbb{R}^{n+1}$ with the inner product

$$[x, y] = -x_1y_1 - \ldots - x_ny_n + x_{n+1}y_{n+1}. \tag{2.1}$$

We denote by $e_1, \ldots, e_{n+1}$ the coordinate unit vectors in $\mathbb{E}^{n,1}$; $S^{n-1}$ is the unit sphere in the coordinate plane $\mathbb{R}^n = \{ x \in \mathbb{E}^{n,1} : x_{n+1} = 0 \}$; $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of $S^{n-1}$. For $\theta \in S^{n-1}$, $d\theta/s$ is the surface element on $S^{n-1}$ (a similar notation will be used for normalized surface elements of lower-dimensional spheres).

Given a set $X$ and a group $G$, the group action of $G$ on $X$ is a function $G \times X \to X$ with $(g, x) \to gx$ where $g \in G$ and $x \in X$. We also write $Gx$ for the set $\{ y \in X : y = gx \text{ for some } g \in G \}$; cf. Knapp [21, p. 159].

The $n$-dimensional real hyperbolic space $\mathbb{H}^n$ is realized as the upper sheet of the two-sheeted hyperboloid in $\mathbb{E}^{n,1}$, that is,

$$\mathbb{H}^n = \{ x \in \mathbb{E}^{n,1} : [x, x] = 1, \ x_{n+1} > 0 \}.$$ 

In the following, the points of $\mathbb{H}^n$ will be denoted by the non-boldfaced letters, unlike the generic points in $\mathbb{E}^{n,1}$. The point $x_0 = (0, \ldots, 0, 1) \sim e_{n+1}$ serves as the origin of $\mathbb{H}^n$;

$$\Gamma = \{ x \in \mathbb{E}^{n,1} : [x, x] = 0, \ x_{n+1} > 0 \}.$$ 

is the asymptotic cone for $\mathbb{H}^n$. The notation

$$G = SO_0(n, 1)$$

is used for the identity component of the special pseudo-orthogonal group $SO(n, 1)$ preserving the bilinear form $[x, y]$.

The geodesic distance between the points $x$ and $y$ in $\mathbb{H}^n$ is defined by $d(x, y) = \cosh^{-1}[x, y]$, so that

$$[x, a] = \cosh r$$

is the equation of the $(n - 1)$-dimensional geodesic sphere in $\mathbb{H}^n$ of radius $r$ with center at $a \in \mathbb{H}^n$. 
We will be using different coordinate systems on $\mathbb{H}^n$. Every point $x \in \mathbb{H}^n$ is represented in the hyperbolic coordinates $(\theta, r) \in S^{n-1} \times [0, \infty)$ as

\begin{equation}
(2.2) \quad x = \theta \sinh r + e_{n+1} \cosh r.
\end{equation}

In the horospherical coordinates $(v, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we have

\begin{equation}
(2.3) \quad x = n_v a_t x_0 = a_t n_{e^{-t}v} x_0
\end{equation}

\begin{equation}
(2.4) \quad = (e^{-t}v, \sinh t + \frac{|v|^2}{2} e^{-t}, \cosh t + \frac{|v|^2}{2} e^{-t}).
\end{equation}

Here $v \in \mathbb{R}^{n-1}$ (a column vector),

\begin{equation}
(2.5) \quad n_v = \begin{bmatrix}
I_{n-1} & -v & v \\
v^T & 1 - |v|^2/2 & |v|^2/2 \\
v^T & -|v|^2/2 & 1 + |v|^2/2
\end{bmatrix};
\end{equation}

\begin{equation}
(2.6) \quad a_t = \begin{bmatrix}
I_{n-1} & 0 & 0 \\
0 & \cosh t & \sinh t \\
0 & \sinh t & \cosh t
\end{bmatrix}, \quad t \in \mathbb{R},
\end{equation}

is the hyperbolic rotation in the coordinate plane $(x_n, x_{n+1})$; cf. [34, p. 13]. Changing variable $e^{-t}v \to v$, we also have

\begin{equation}
(2.7) \quad x = a_t n_v x_0 = n_{e^{-t}v} a_t x_0 = (v, \sinh t + \frac{|v|^2}{2} e^t, \cosh t + \frac{|v|^2}{2} e^t).
\end{equation}

A straightforward matrix multiplication yields

\begin{equation}
(2.8) \quad n_v n_{v_2} = n_{v_1+v_2} \quad \text{for all} \quad v_1, v_2 \in \mathbb{R}^{n-1}.
\end{equation}

Let

\begin{equation}
(2.9) \quad K = \left\{ \begin{bmatrix}
k & 0 \\
0 & 1
\end{bmatrix} : k \in SO(n) \right\}.
\end{equation}

Abusing notation, we identify $k \in SO(n)$ with the corresponding matrix $\begin{bmatrix}
k & 0 \\
0 & 1
\end{bmatrix} \in G$. The subgroups of matrices $a_t$ and $n_v$ will be denoted by $A$ and $N$, respectively. Since the stabilizer of $x_0$ in $G$ coincides with $K$, (2.3) implies that every $g \in G$ is representable in the form

\begin{equation}
(2.10) \quad g = n_v a_t k.
\end{equation}

This representation is unique and agrees with the Iwasawa decomposition $G = NAK$.

We fix a $G$-invariant measure $dx$ on $\mathbb{H}^n$, which has the following form in the coordinates (2.2):

\begin{equation}
(2.11) \quad dx = \sinh^{n-1} r \, dr d\theta.
\end{equation}
If $f$ is $K$-invariant, that is, $f(x) \equiv f_0(x_{n+1})$, then

\begin{equation}
(2.12) \int_{\mathbb{H}^n} f(x) \, dx = \sigma_{n-1} \int_1^\infty f_0(s)(s^2 - 1)^{n/2 - 1} \, ds.
\end{equation}

The Haar measure $dg$ on $G$ will be normalized in a consistent way by the formula

\begin{equation}
(2.13) \int_G f(gx_0) \, dg = \int_{\mathbb{H}^n} f(x) \, dx.
\end{equation}

Using (2.10), we also have

\begin{equation}
(2.14) dg = e^{(1-n)t} \, dt \, dv \, dk,
\end{equation}

where $dk$ is the normalized Haar measure on $K$, $dv$ and $dt$ are the standard Euclidean measures on $\mathbb{R}^{n-1}$ and $\mathbb{R}$, respectively; cf. [34, p. 23]. Thus,

\begin{equation}
(2.15) \int_G f(gx_0) \, dg = \int_{\mathbb{R}} e^{(1-n)t} \, dt \int_{\mathbb{R}^{n-1}} f(n_v a_t x_0) \, dv = \int_{\mathbb{H}^n} f(x) \, dx
\end{equation}

or, by (2.3),

\begin{equation}
(2.16) \int_G f(gx_0) \, dg = \int_{\mathbb{R}} dt \int_{\mathbb{R}^{n-1}} f(a_t n_v x_0) \, dv = \int_{\mathbb{H}^n} f(x) \, dx;
\end{equation}

cf. [31, Lemma 3.1]. The equality (2.16) agrees with the representation

\begin{equation}
(2.17) g = a_t n_v k,
\end{equation}

in terms of which $dg = dt dv dk$ and $G = ANK$.

Replacing $g$ by $kg, k \in K$, in (2.16), we obtain

\begin{equation}
(2.18) \int_G f(gx_0) \, dg = \int_K dk \int_{\mathbb{R}} dt \int_{\mathbb{R}^{n-1}} f(k a_t n_v x_0) \, dv = \int_{\mathbb{H}^n} f(x) \, dx.
\end{equation}

This equality (2.18) agrees with the representation

\begin{equation}
(2.19) g = k a_t n_v,
\end{equation}

in terms of which $dg = dk dt dv$ and $G = KAN$.

**More notation.** In the following, $u \cdot v = u_1 v_1 + \ldots + u_n v_n$ denotes the usual inner product of the vectors $u, v \in \mathbb{R}^n$; $I_m$ is the identity $m \times m$ matrix; $C(\mathbb{H}^n)$ is the space of continuous functions on $\mathbb{H}^n$; $C_0(\mathbb{H}^n)$ denotes the space of continuous functions on $\mathbb{H}^n$ vanishing at infinity. We also set

\begin{equation}
(2.20) C_\mu(\mathbb{H}^n) = \{ f \in C(\mathbb{H}^n) : f(x) = O(x_{n+1}^{-\mu}) \}.
\end{equation}
Let \( \Omega = \{ \mathbf{x} \in \mathbb{R}^{n,1} : [\mathbf{x}, \mathbf{x}] > 0, x_{n+1} > 0 \} \) be the interior of the cone \( \Gamma \). We denote by \( C^\infty_c(\mathbb{H}^n) \) the space of infinitely differentiable compactly supported functions on \( \mathbb{H}^n \). This space is formed by the restrictions onto \( \mathbb{H}^n \) of functions belonging to \( C^\infty_c(\Omega) \).

We say that an integral under consideration exists in the Lebesgue sense if it is finite when the corresponding integrand is replaced by its absolute value. The letter \( c \) (sometimes with subscripts) denotes a constant that may vary at each occurrence.

2.2. Horospheres.

2.2.1. The case \( d = n - 1 \). An \((n-1)\)-dimensional horosphere in \( \mathbb{H}^n \) is defined as the cross-section of the hyperboloid \( \mathbb{H}^n \) by the hyperplane \( [\mathbf{x}, \mathbf{b}] = 1 \), where \( \mathbf{b} \) is a point of the cone \( \Gamma \). The correspondence between the set \( \Xi_{n-1} \) of all \((n-1)\)-horospheres and the set of all points in \( \Gamma \) is one-to-one. One can equivalently define \( \Xi_{n-1} \) as the set of all \( G \)-orbits

\begin{equation}
\Xi_{n-1} = \{ g\xi^0_{n-1} : g \in G \}
\end{equation}

of the “basic” horosphere \( \xi^0_{n-1} \) corresponding to the point

\[ \mathbf{b}_0 = (0, \ldots, 0, 1, 1) \in \Gamma. \]

The stabilizer \( G^0_{n-1} \) of \( \mathbf{b}_0 \) (and therefore, of \( \xi^0_{n-1} \)) in \( G \) is the semidirect product

\begin{equation}
G^0_{n-1} = N \rtimes M
\end{equation}

where \( N \) is the group of transformations (2.5) and

\[ M = \left\{ \begin{bmatrix} m & 0 \\ 0 & I_2 \end{bmatrix} : m \in SO(n-1) \right\}. \]

We observe that \( N \) is a normal subgroup of \( G^0_{n-1} \). For the sake of simplicity, we write (2.22) as

\begin{equation}
G^0_{n-1} = \text{Stab}_G(\xi^0_{n-1}) = \text{Stab}_G(\mathbf{b}_0) = MN
\end{equation}

(cf. [15, p. 60]). Note also that \( MN = NM \).

2.2.2. The case \( d < n - 1 \). We set

\begin{equation}
\mathbb{R}^{n-d-1} = \mathbb{R}_e^{1} \oplus \cdots \oplus \mathbb{R}_e^{n-d-1} \quad \text{(if} \ d = n-1 \text{this set is empty)},
\end{equation}

\begin{equation}
\mathbb{R}^d = \mathbb{R}_e^{n-d} \oplus \cdots \oplus \mathbb{R}_e^{n-1}, \quad \mathbb{R}^{d+1} = \mathbb{R}^d \oplus \mathbb{R}_e^1,
\end{equation}

\begin{equation}
\mathbb{E}^{d+1,1} \simeq \mathbb{R}^{d+2} = \mathbb{R}^{d+1} \oplus \mathbb{R}_e^{n+1}, \quad \mathbb{H}^{d+1} = \mathbb{H}^n \cap \mathbb{R}^{d+2};
\end{equation}

\begin{equation}
\xi_0 = \mathbb{H}^{d+1} \cap \{ \mathbf{x} \in \mathbb{H}^n : [\mathbf{x}, \mathbf{b}_0] = 1 \}, \quad \mathbf{b}_0 = (0, \ldots, 0, 1, 1).
\end{equation}
The last formula defines a $d$-dimensional horosphere in $\mathbb{H}^{d+1}$. We call it the basic one. The set $\Xi_d$ of $d$-dimensional horospheres in $\mathbb{H}^n$ ($d$-horospheres, for short) is defined as the collection of all $G$-orbits

$$\Xi_d = \{ g\xi_0 : g \in G \}. \quad (2.28)$$

Let $M_d \subset G$ be the subgroup of matrices of the form

$$m_{\alpha,\beta} = \begin{bmatrix} \tilde{m}_{\alpha, \beta} & 0 \\ 0 & I_2 \end{bmatrix}$$

where

$$\tilde{m}_{\alpha, \beta} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \in S(O(n-d-1) \times O(d)).$$

Let also

$$N_d = \{ n_v \in N : v_1 = \ldots = v_{n-1-d} = 0 \}, \quad (2.29)$$

$$N_{n-1-d} = \{ n_v \in N : v_d = \ldots = v_{n-1} = 0 \}, \quad (2.30)$$

be the subgroups of $N$ (cf. (2.5)) generated by vectors $v \in \mathbb{R}^{n-1}$ with the corresponding zero coordinates. A straightforward matrix multiplication yields

$$m_{\alpha,\beta} n_v = n_{\beta v} m_{\alpha,\beta} \quad (2.31)$$

for all $m_{\alpha,\beta} \in M_d$ and $n_v \in N_d$, so that we can write $M_d N_d = N_d M_d$.

We denote

$$G_d^0 = M_d N_d = N_d M_d. \quad (2.32)$$

**Proposition 2.1.**

(i) The basic $d$-horosphere $\xi_0$ is the $N_d$-orbit of $x_0$. Moreover,

$$\xi_0 = G_d^0 x_0. \quad (2.33)$$

(ii) The subgroup $G_d^0 = M_d N_d$ is the stabilizer of $\xi_0$ in $G$, so that the set $\Xi_d$ of all $d$-horospheres in $\mathbb{H}^n$ is isomorphic to the quotient space $G/G_d^0$.

**Proof.** The first statement in (i) is a consequence of the similar fact for $d = n - 1$. Then

$$G_d^0 x_0 = N_d M_d x_0 = N_d x_0 = \xi_0.$$

To prove (ii), we observe that by (2.27), $\xi_0$ can be identified with the pair $(\mathbb{R}^{n-1-d}, b_0)$. Thus it suffices to show that $M_d N_d$ is the stabilizer of this pair, i.e.,

$$\text{Stab}_G(\mathbb{R}^{n-1-d}, b_0) = M_d N_d. \quad (2.34)$$
Because
\[ \text{Stab}_G(\mathbb{R}^{n-1-d}, b_0) = \text{Stab}_G(\mathbb{R}^{n-1-d}) \cap \text{Stab}_G(b_0) \]
and \( \text{Stab}_G(b_0) = MN \) (see (2.23)), it remains to prove that
\[ (2.35) \quad \text{Stab}_{MN}(\mathbb{R}^{n-1-d}) = M_dN_d. \]
The embedding \( M_dN_d \subset \text{Stab}_{MN}(\mathbb{R}^{n-1-d}) \) is obvious because \( m_{\alpha,\beta} \in M_d \) and \( n_v \in N_d \) preserve \( \mathbb{R}^{n-1-d} \). Conversely, suppose that \( g \in MN \) preserves \( \mathbb{R}^{n-1-d} \) and let \( g = \gamma n_v \) with
\[ \gamma = \begin{bmatrix} \tilde{\gamma} & 0 \\ 0 & I_2 \end{bmatrix}, \quad \tilde{\gamma} = \begin{bmatrix} \alpha & \mu \\ \nu & \beta \end{bmatrix} \in SO(n-1), \]
and \( v = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1} \). If \( e_j (j = 1, \ldots, n-1-d) \) are coordinate unit vectors, then
\[ ge_j = \gamma n_v e_j = \begin{bmatrix} \alpha & \mu & 0 & 0 \\ \nu & \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_j \\ 0 \\ v_j \\ v_j \end{bmatrix} = \begin{bmatrix} \alpha e_j \\ \nu e_j \\ v_j \\ v_j \end{bmatrix}. \]
The vector on the right-hand side belongs to \( \mathbb{R}^{n-1-d} \) if and only if \( \nu \) is a zero matrix and \( v_j = 0 \) for all \( j = 1, \ldots, n-1-d \). Thus, if \( g \in MN \) preserves \( \mathbb{R}^{n-1-d} \), then, necessarily, \( g = \begin{bmatrix} \tilde{\gamma} & 0 \\ 0 & I_2 \end{bmatrix} n_v \) with \( n_v \in N_d \) and \( \tilde{\gamma} = \begin{bmatrix} \alpha & \mu \\ 0 & \beta \end{bmatrix} \). Because \( \tilde{\gamma} \in SO(n-1) \), we have \( \tilde{\gamma}^T \tilde{\gamma} = \tilde{\gamma} \tilde{\gamma}^T = I_{n-1} \).

Multiplying matrices, we obtain
\[ \alpha^T \alpha = I_{n-1-d}, \quad \alpha^T \mu = 0, \quad \mu^T \mu + \beta^T \beta = I_d, \]
\[ \alpha \alpha^T + \mu \mu^T = I_{n-1-d}, \quad \mu \beta^T = 0, \quad \beta \beta^T = I_d. \]
It follows that
\[ \alpha^T \alpha = \alpha \alpha^T = I_{n-1-d}, \quad b^T \beta = \beta \beta^T = I_d, \]
and therefore, \( \alpha \in O(n-1-d) \), \( \beta \in O(d) \). Since \( \tilde{\gamma} \in SO(n-1) \), we obtain \( \tilde{\gamma} = \text{diag}(\alpha, \beta) \in S(O(n-d-1) \times O(d)) \). The latter means that \( \text{Stab}_{MN}(\mathbb{R}^{n-1-d}) \subset M_dN_d \) which completes the proof. \( \square \)

According to the Iwasawa decomposition \( G = KAN \), every \( g \in G \) is uniquely represented as \( g = ka_t n_v \), where \( k \in K \), \( a_t \in A \), and \( n_v \in N \). We write \( v \in \mathbb{R}^{n-1} \) as an orthogonal sum
\[ (2.36) \quad v = u + w, \quad u \in \mathbb{R}^{n-1-d}, \quad w \in \mathbb{R}^d. \]
Noting that \( n_w \xi_0 = \xi_0 \) for all \( w \in \mathbb{R}^d \), we conclude that every \( d \)-horosphere \( \xi = g \xi_0 \) can be represented as
\[
(2.37) \quad \xi = k a_t n_u \xi_0, \quad k \in K, \ a_t \in A, \ n_u \in N_{n-1-d}.
\]
Following this equality, we equip \( \Xi_d \) with the measure \( d\xi \) by setting
\[
(2.38) \quad \int_{\Xi_d} \varphi(\xi) \, d\xi = \int_{K} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1-d}} \varphi(k a_t n_u \xi_0) \, du.
\]

**Proposition 2.2.** Let \( \Xi_d \) be the set (2.28) of \( d \)-horospheres in \( \mathbb{H}^n \) with the basic horosphere \( \xi_0 \) (the “origin” of \( \Xi_d \)) and the stabilizer \( G^0_d = \text{Stab}_G(\xi_0) \). The set \( \Xi_d \) coincides with the set of orbits of conjugates of \( G^0_d \). Specifically,
\[
(2.39) \quad \Xi_d = \{ g G^0_d g^{-1} x : g \in G, \ x \in \mathbb{H}^n \}.
\]

**Proof.** Denote the right-hand side of (2.39) by \( \Xi'_d \). Let \( \xi \in \Xi_d \), that is, \( \xi = g \xi_0 \) for some \( g \in G \). Setting \( g = k a_t n_u = k a_t n_u n_w \) (cf. (2.36)), we obtain
\[
\xi = k a_t n_u \xi_0 = k a_t n_u G^0_d x_0 = (k a_t n_u) G^0_d (k a_t n_u)^{-1} (k a_t n_u) x_0 = \tilde{G}^0_d x,
\]
where \( \tilde{G}^0_d = (k a_t n_u) G^0_d (k a_t n_u)^{-1} \) and \( x = (k a_t n_u) x_0 \). Hence \( \xi \in \Xi'_d \).

Conversely, let \( \xi \in \Xi'_d \), that is, \( \xi = g G^0_d g^{-1} x \) for some \( g \in G \) and \( x \in \mathbb{H}^n \). We write \( g^{-1} x \) in horospherical coordinates as \( g^{-1} x = n_u a_t x_0 \) (cf. (2.3)). Then
\[
\xi = g G^0_d n_u a_t x_0 = g M_d N_d n_u a_t x_0 = g M_d n_u N_d a_t x_0.
\]
By (2.3) and (2.7), \( N_d a_t x_0 = a_t N_d x_0 \). Hence
\[
\xi = g M_d n_u a_t N_d x_0 = g M_d n_u a_t \xi_0 \subset G \xi_0 = \Xi_d.
\]

**Remark 2.3.** Proposition 2.2 shows that our definition of \( d \)-horospheres agrees with Helgason’s definition of horocycles in symmetric spaces; see [15, p. 60].

### 3. Definition and Basic Properties of the \( d \)-Horospherical Transform

**Definition 3.1.** Let \( 1 \leq d \leq n - 1 \). Given \( \xi = g \xi_0 \in \Xi_d, \ g \in G \), the \( d \)-horospherical transform of a sufficiently good function \( f \) on \( \mathbb{H}^n \) is
defined by

\[ \hat{f}(\xi) = \int_{\mathbb{R}^d} f(gn_wx_0) \, dw. \]  

\[ \hat{f}(gM_dN_d) = \int_{N_d} f(gn_dK) \, dn_d. \]  

Remark 3.2. The definition can be put in group-theoretic terms as follows. Since \( H^d \) is identified as the homogeneous space \( G/K \), a function \( f \) on \( H^d \) becomes the right \( K \)-invariant function \( f(gK) \) on \( G \). Denoting group elements in \( N_d \) by \( n_d \) and the Haar measure on \( N_d \) by \( dn_d \), we can write the defining formula as

\[ \hat{f}(gM_dN_d) = \int_{N_d} f(gn_dK) \, dn_d. \]  

The case when \( g \) is the identity map, corresponds to the integral of \( f \) over \( \xi_0 \). If \( g = k a t n_v = k a t n_w + w \), then, by (2.8),

\[ \hat{f}(gM_dN_d) = \int_{N_d} f(gn_wx_0) \, dw. \]  

By (3.1), the map \( f \to \hat{f} \) is \( G \)-equivariant. Indeed, for all \( \gamma \in G \),

\[ \hat{f}(\gamma \xi) \equiv \hat{f}(g\xi_0) = \int_{\mathbb{R}^d} f(\gamma gn_wx_0) \, dw \]

\[ = \int_{\mathbb{R}^d} (f \circ \gamma)(gn_wx_0) \, dw = (f \circ \gamma)^\wedge(g\xi_0) \equiv (f \circ \gamma)^\wedge(\xi). \]

In particular, if \( f \) is \( K \)-invariant (or zonal), then so is \( \hat{f} \). The \( d \)-horospherical transform of a \( K \)-invariant function expresses through the Riemann-Liouville fractional integral

\[ (I^\alpha_\psi)(r) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{\psi(s) \, ds}{(s-r)^{1-\alpha}}, \quad \alpha > 0, \]

as follows.

Lemma 3.3. Let \( \varphi(\xi) = \hat{f}(\xi) \equiv \hat{f}(ka_t n_v \xi_0), f(x) \equiv f_0(x_{n+1}). \) Then

\[ \varphi(\xi) = ce^{-td/2} (I^{d/2}_\varphi f_0)(\eta), \quad \eta = \cos t + \frac{|u|^2}{2} e^t, \]

where

\[ c = 2^{d/2-1} \sigma_{d-1} \Gamma(d/2). \]

It is assumed that the integrals in (3.5) exist in the Lebesgue sense.
Proof. Because $f$ is $K$-invariant, from (3.1) and (2.7) we have

$$\hat{f}(\xi) = \int f_0(\cosh t + \frac{|u|^2 + |w|^2}{2} e^t) dw$$

$$= \sigma_{d-1} \int_0^\infty f_0(\cosh t + \frac{|u|^2 + |r|^2}{2} e^t) r^{d-1} dr.$$  

This gives (3.5). \hfill \Box

Our next goal is to establish conditions under which $\hat{f}$ exists as an absolutely convergent integral. We restrict our consideration to two cases: $f \in C_\mu(\mathbb{H}^n)$ and $f \in L^p(\mathbb{H}^n)$.

Lemma 3.4. If $f \in L^1(\mathbb{H}^n)$, then the integral (3.3) is finite for all $k \in K$, almost all $t \in \mathbb{R}$ and almost all $u \in \mathbb{R}^{n-1-d}$. Moreover, for all $k \in K$,

$$(3.6) \quad \int dt \int_{\mathbb{R}^{n-1-d}} \hat{f}(ka_t u \xi_0) du = \int f(x) dx \quad \text{if} \quad d < n - 1,$$

$$(3.7) \quad \int \hat{f}(ka_t \xi_0) dt = \int f(x) dx \quad \text{if} \quad d = n - 1.$$  

Proof. Let $d < n - 1$. Then

$$l.h.s. = \int dt \int_{\mathbb{R}^{n-1-d}} du \int f(ka_t u n_w x_0) dw \quad \text{(set} \quad u + w = v)$$

$$= \int dt \int_{\mathbb{R}^{d}} f(ka_t v x_0) dv \quad \text{(use} \quad (2.16))$$

$$= \int_{\mathbb{H}^n} f(kx) dx = r.h.s.$$

If $d = n - 1$, the proof is similar. \hfill \Box

Proposition 3.5. If $f \in C_\mu(\mathbb{H}^n)$, $\mu > d/2$, then $\hat{f}(\xi)$ is finite for every $\xi \in \Xi_d$. 

Proof. By (3.3) and (2.7),
\[
|\hat{f}(\xi)| \leq \int_{\mathbb{R}^d} |f(ka_nt_nw, x_0)| \, dv \leq c \int_{\mathbb{R}^d} [a_nt_nw, x_0]^{-\mu} \, dw
\]
\[
= c \int_{\mathbb{R}^d} \left[ \cosh t + \frac{|u|^2 + |w|^2}{2} e^t \right]^{-\mu} \, dw.
\]

The last integral is finite if \(\mu > d/2\). \(\square\)

Remark 3.6. The condition \(\mu > d/2\) is sharp. There is a function \(f \in C_\mu(\mathbb{H}^n), \mu \leq d/2\), for which \(\hat{f}(\xi) \equiv \infty\). An example of such a function can be constructed by making use of the Abel type representation (3.5); see also Remark 3.9.

The question about the existence of \(\hat{f}\) for \(f \in L^p(\mathbb{H}^n)\) requires some preparation.

Lemma 3.7. If
\[
(3.8) \quad \int_{y_{n+1}>1+\varepsilon} |f(y)| y_{n+1}^{d/2-n+1} \, dy < \infty \quad \text{for some } \varepsilon > 0,
\]
then the integral \(\hat{f}(\xi) \equiv \hat{f}(ka_nt_nw_0)\) is finite for almost all \(k \in K\), almost all \(|t| > \cosh^{-1}(1+\varepsilon)\) and all \(u \in \mathbb{R}^{n-1-d}\).

Proof. We use the fact that the map \(f \rightarrow \hat{f}\) is \(K\)-equivariant. Then
\[
\int_{K} \hat{f}(k\xi) \, dk = \check{f}_z(\xi), \quad f_z(y) = \int_{K} f(ky) \, dk \equiv f_0(y_{n+1}).
\]

Hence, Lemma 3.3 yields
\[
\int_{K} \hat{f}(k\xi) \, dk = c e^{-td/2} (I_{d/2} f_0)(\eta), \quad \eta = \cosh t + \frac{|u|^2}{2} e^t.
\]

By Lemma 2.12 from [29], the last integral is finite for almost all \(\eta > 1+\varepsilon\) provided
\[
I_\varepsilon = \int_{1+\varepsilon}^{\infty} s^{d/2-1} |f_0(s)| \, ds < \infty.
\]
However, by (2.12) and (3.8),
\[
I_\varepsilon = \frac{1}{\sigma_{n-1}} \int_{y_{n+1} > 1+\varepsilon} \frac{y_{n+1}^{d/2-1} \, dy}{(y_{n+1}^2 - 1)^{n/2-1}} \int K f(y) \, dk \\
\leq c_\varepsilon \int_{y_{n+1} > 1+\varepsilon} |f(y)| y_{n+1}^{d/2-n+1} \, dy < \infty.
\]
This completes the proof. \(\Box\)

Lemma 3.7 implies the following proposition that extends the existence result of Lemma 3.7 to \(f \in L^p(\mathbb{H}^n)\).

**Proposition 3.8.** If \(f \in L^p(\mathbb{H}^n), 1 \leq p < 2(n-1)/d\), then the integral \(\hat{f}(\xi) \equiv \hat{f}(ka_t u \xi_0)\) is finite for almost all \(k \in K\), almost all \(t \in \mathbb{R}\) and all \(u \in \mathbb{R}^{n-1-d}\).

**Proof.** By Hölder’s inequality, the integral (3.8) is dominated by \(c_\varepsilon \|f\|_p\) where
\[
c_\varepsilon' = \int_{y_{n+1} > 1+\varepsilon} y_{n+1}^{(d/2-n+1)p'} \, dy \leq \sigma_{n-1} \int_{1}^{\infty} s^{(d/2-n+1)p'} (s^2 - 1)^{n/2-1} \, ds < \infty
\]
provided \(1 \leq p < 2(n-1)/d\). \(\Box\)

**Remark 3.9.** The condition \(1 \leq p < 2(n-1)/d\) is sharp. If \(p \geq 2(n-1)/d\), then the function
\[
f(x) = \frac{(x_{n+1}^2 - 1)^{1-n/2)}{p}}{(x_{n+1} + 1)^{1/p} \log(x_{n+1} + 1)}
\]
belongs to \(L^p(\mathbb{H}^n)\), however, the corresponding integral (3.5) diverges.

The next auxiliary statement, in which \(\hat{f}(\xi) \equiv \hat{f}(ka_t u \xi_0)\) is restricted to \(u = 0\), plays an important role in derivation of the inversion formulas in Section 4.

**Lemma 3.10.** Given a function \(\varphi\) on \((1, \infty)\), let
\[
(3.9) \quad \psi(s) = 2^{d/2} \sigma_{d-1} \int_{1}^{s} \frac{\varphi(r)}{\sqrt{r^2 - 1}} (s-r)^{d/2-1} \, dr, \quad s > 1.
\]
Then
\[
(3.10) \quad \int_{\mathbb{R}} \varphi(\cos t) e^{t d/2} \, dt \int \hat{f}(ka_t \xi_0) \, dk = \frac{1}{\sigma_{n-1}} \int_{\mathbb{H}^n} \frac{\psi(x_{n+1})}{(x_{n+1}^2 - 1)^{n/2-1}} \, dx.
\]
It is assumed that $f$ and $\varphi$ are good enough, so that integrals in (3.9) and (3.10) exist in the Lebesgue sense.

**Proof.** We denote the left-hand side of (3.10) by $I$. Then

$$I = \int_{\mathbb{R}} \varphi(\cosh t) e^{td/2} \left[ \int_{K} f(k(\cdot)) \, dk \right] (a_1 \xi_0) \, dt.$$

The function in square brackets is zonal and we denote

$$f(x) = f_0(x_{n+1}).$$

Then, by Lemma 3.3,

$$I = 2^{d/2-1} \sigma_{d-1} \int_{\mathbb{R}} \varphi(\cosh t) \, dt \int_{\cosht}^{\infty} f_0(s) (s-\cosht)^{d/2-1} \, ds$$

$$= 2^{d/2} \sigma_{d-1} \int_{1}^{\infty} \varphi(r) \frac{dr}{\sqrt{r^2-1}} \int_{r}^{\infty} f_0(s-r)^{d/2-1} \, ds$$

$$= \int_{1}^{\infty} f_0(s) \psi(s) \, ds.$$  \hfill (3.12)

Using (3.11) and (2.11), we continue:

$$I = \int_{1}^{\infty} \psi(s) \, ds \int_{K} f(k(e_n \sinhr + se_{n+1})) \, dk$$

$$= \frac{1}{\sigma_{n-1}} \int_{0}^{\infty} \psi(\cosh r) \sinhr \, dr \int_{s_{n-1}} f(\theta \sinhr + e_{n+1} \cosh r) \, d\theta$$

$$= \frac{1}{\sigma_{n-1}} \int_{\mathbb{H}^n} f(x) \psi(x_{n+1}) \, dx$$

$$\left( x_{n+1}^2 - 1 \right)^{n/2-1},$$

as desired. \hfill $\Box$

**Example 3.11.** Let $\varphi(r) = (r-1)^{\alpha/2-1} \sqrt{r^2-1}$, $\alpha > 0$. Then

$$\psi(s) = 2^{d/2} \sigma_{d-1} \int_{1}^{s} (s-r)^{d/2-1} (r-1)^{\alpha/2-1} \, dr$$

$$= \frac{2^{d/2} \sigma_{d-1} \Gamma(\alpha/2) \Gamma(d/2)}{\Gamma((\alpha+d)/2)} (s-1)^{(\alpha+d)/2-1}$$
and we have
\[
\int_{\mathbb{R}} e^{\frac{t}{d/2}} (\cosh t - 1)^{\alpha/2 - 1} |\sinh t| \, dt \int_{K} \hat{f}(k a_{t} \xi_{0}) \, dk \\tag{3.13}
\]
\[
= c_{1} \int_{\mathbb{H}^{n}} \frac{(x_{n+1} - 1)^{(\alpha + d - n)/2}}{(x_{n+1} + 1)^{n/2 - 1}} f(x) \, dx,
\]
\[
c_{1} = \frac{2^{d/2} \sigma_{d-1} \Gamma(\alpha/2) \Gamma(d/2)}{\sigma_{\alpha} \Gamma((\alpha + d)/2)}.
\]

4. Inversion Formulas

In this section we obtain main results of the paper. The proofs rely on the properties of hyperbolic convolutions and spherical means which are reviewed below.

4.1. Hyperbolic Convolutions and Spherical Means. All details related to this subsection can be found in [29, Section 6.1.2] and [31].

Given a measurable function \( k \) on \([1, \infty)\), the corresponding hyperbolic convolution on \( \mathbb{H}^{n} \) is defined by
\[
(Kf)(x) = \int_{\mathbb{H}^{n}} k([x, y]) f(y) \, dy, \quad x \in \mathbb{H}^{n}.
\\tag{4.1}
\]
If this integral exists in the Lebesgue sense, then, by Fubini’s theorem,
\[
\int_{\mathbb{H}^{n}} k([x, y]) f(y) \, dy = \sigma_{\alpha - 1} \int_{0}^{\infty} k(\cosh r) (M_{x} f)(\cosh r) \sinh^{n-1} r \, dr,
\\tag{4.2}
\]
where \( M_{x} f \) is the spherical mean
\[
(M_{x} f)(s) = \frac{(s^2 - 1)^{(1-n)/2}}{\sigma_{\alpha - 1}} \int_{\{y \in \mathbb{H}^{n}: [x, y] = s\}} f(y) \, d\sigma(y), \quad s > 1,
\\tag{4.3}
\]
\( d\sigma(y) \) being the relevant induced measure. We can also write (4.3) in the “more geometric” form as
\[
(M_{x} f)(\cosh t) = \int_{K} f(r_{x} a_{t} x_{0}) \, dk,
\\tag{4.4}
\]
where \( r_{x} \in G \) takes \( x_{0} \) to \( x \) and \( a_{t} \) is the matrix (2.6).

**Lemma 4.1.** ([29, p. 370], [22, pp. 131-133]).

Let \( f \in L^{p}(\mathbb{R}^{n}), \ 1 \leq p \leq \infty \). Then
\[
\sup_{s > 1} \|(M_{s} f)(s)\|_{p} \leq \|f\|_{p}.
\\tag{4.5}
\]
If $1 \leq p < \infty$, then $(M_x f)(s)$ is a continuous $L^p$-valued function of $s \in [1, \infty)$ and
\[
\lim_{s \to 1} \|(M_x f)(s) - f\|_p = 0.
\]

If $f \in C_0(\mathbb{H}^n)$, then $(M_x f)(s)$ is a continuous function of $(x, s) \in \mathbb{H}^n \times (1, \infty)$ and $(M_x f)(s) \to f(x)$ as $s \to 1$, uniformly on $\mathbb{H}^n$.

An important example of convolutions (4.1) is the analytic family of the potential type operators
\[
(Q^\alpha f)(x) = \zeta_{n, \alpha} \int_{\mathbb{H}^n} f(y) \frac{([x, y] - 1)^{(\alpha-n)/2}}{([x, y] + 1)^{n/2 - 1}} dy,
\]
where
\[
\zeta_{n, \alpha} = \frac{\Gamma((n-\alpha)/2)}{2^{\alpha/2 + 1} \pi^{n/2} \Gamma(\alpha/2)}, \quad \text{Re} \alpha > 0, \quad \alpha - n \neq 0, 2, 4, \ldots.
\]

This analytic family naturally arises in [31] in the study of the horospherical transforms.

**Proposition 4.2.** [29, p. 385] If $f \in L^p(\mathbb{H}^n)$, $1 \leq p < \infty$, $0 < \alpha < 2(n-1)/p$, then $(Q^\alpha f)(x)$ exists as an absolutely convergent integral for almost all $x \in \mathbb{H}^n$.

**Lemma 4.3.** [29, p. 386] Let $f \in C^\infty_c(\mathbb{H}^n)$,
\[
D_\alpha = -\Delta_H - \alpha(2n - 2 - \alpha)/4, \quad \alpha \geq 2,
\]
where $\Delta_H$ is the Beltrami-Laplace operator on $\mathbb{H}^n$. If $\alpha - n \neq 0, 2, 4, \ldots$, then
\[
D_\alpha Q^\alpha f = Q^{\alpha-2} f \quad (Q^0 f = f).
\]

In particular, if $\alpha = 2\ell$ is even, $2\ell - n \neq 0, 2, 4, \ldots$, and $\mathcal{P}_\ell(\Delta_H) = D_2 D_4 \ldots D_{2\ell}$, then
\[
\mathcal{P}_\ell(\Delta_H) Q^{2\ell} f = f.
\]

We will need an extension of Lemma 4.3 to the case $\alpha = n$. For $f \in C^\infty_c(\mathbb{H}^n)$, we define $Q^n f$ as a limit
\[
(Q^n f)(x) = \lim_{\alpha \to n} \zeta_{n, \alpha} \int_{\mathbb{H}^n} f(y) \frac{([x, y] - 1)^{(\alpha-n)/2} - 1}{([x, y] + 1)^{n/2 - 1}} dy
\]
\[
= \zeta'_n \int_{\mathbb{H}^n} f(y) \frac{\log([x, y] - 1)}{([x, y] + 1)^{n/2 - 1}} dy, \quad \zeta'_n = -\frac{2^{-1-n/2}}{\pi^{n/2} \Gamma(n/2)}.
\]

The following statements were proved in [29, 31].
Lemma 4.4. Let $f \in C_c^\infty(\mathbb{H}^n)$, $D_n = -\Delta_H - n(n-2)/4$, $n \geq 2$. Then
\begin{equation}
D_n Q^n f = Q^{n-2} f + B f \quad (Q^0 f = f),
\end{equation}
where
\begin{equation}
(B f)(x) = \zeta_n' \int_{\mathbb{H}^n} f(y) \frac{dy}{([x,y]+1)^{n/2-1}}.
\end{equation}

Lemma 4.5. Let $f \in C_c^\infty(\mathbb{H}^n)$, $n > 2$. Then $(B f)(x)$ is an eigenfunction of the Beltrami-Laplace operator $\Delta_H$, so that
\begin{equation}
-\Delta_H B f = \frac{n(n-2)}{4} B f
\end{equation}
and
\begin{equation}
D_n B f = D_{n-2} B f = 0.
\end{equation}

Proposition 4.6. Let $f \in C_c^\infty(\mathbb{H}^n)$, where $n$ is even. If $n = 2$, then
\begin{equation}
-\Delta_H Q^2 f = f - \frac{1}{4\pi} \int_{\mathbb{H}^2} f(y) \, dy.
\end{equation}
If $n \geq 4$, then
\begin{equation}
\mathcal{P}_{n/2}(\Delta_H) Q^n f = f,
\end{equation}
\begin{equation}
\mathcal{P}_{n/2}(\Delta_H) = (-1)^{n/2} \prod_{i=1}^{n/2} (\Delta_H + i(n-1-i)).
\end{equation}

4.2. The Method of Mean Value Operators. An idea of this inversion method is to average $\hat{f}(\xi)$ over all $d^i$-horospheres $\xi$ at a fixed positive distance from a given point $x \in \mathbb{H}^n$. Inverting a simple Abel type fractional integral, we then obtain the spherical mean (4.3) that gives $f(x)$ after passing to the limit according to Lemma 4.1.

For $\xi = ka_t n_u \xi_0$, the relation $\xi \ni x_0$ is equivalent to $a_{-t} n_{-u} x_0 \in \xi_0$. By (2.7),
\begin{align*}
a_{-t} n_{-u} x_0 &= (u,0,\ldots,0,-\sinh t + \frac{|u|^2}{2} e^{-t}, \cosh t + \frac{|u|^2}{2} e^{-t}) \in \xi_0
\end{align*}
(with $d$ zeros) if and only if $t = 0$ and $u = 0$. These two parameters contribute to the distance between $\xi$ and the origin $x_0$. We can work with one of them or with both. Suppose $u = 0$ and consider the mean value
\begin{equation}
\int_{\mathbb{K}} \hat{f}(r_x ka_t \xi_0) \, dk = \int_{\mathbb{K}} \hat{f}_x(k a_t \xi_0) \, dk, \quad t > 0.
\end{equation}
Here \( r_x \in G \) is an arbitrary transformation satisfying \( r_x x_0 = x \) and \( f_x(y) = f(r_x y) \). Note that \( r_x \) can be moved under the sign of the horospherical transform because the latter is \( G \)-equivariant.

We introduce the mean value operator

\[
\hat{\varphi}_x(t) = \int \varphi(r_x k a t \xi_0) \, dk, \quad x \in \mathbb{H}^n, \quad t \in \mathbb{R}.
\]

**Lemma 4.7.** If \( \varphi = \hat{f} \), then

\[
\hat{\varphi}_x(t) = c \, e^{-td/2} (I^{d/2}_d M_x f)(\cosh t), \quad c = 2^{d/2-1} \sigma_{d-1} \Gamma(d/2),
\]

where \( M_x f \) is the spherical mean (4.3). It is assumed that the integral on the right-hand side of (4.18) exists in the Lebesgue sense.

**Proof.** Fix \( x \in \mathbb{H}^n \) and let \( f_x(y) = f(r_x y) \), \( y \in \mathbb{H}^n \). By \( G \)-invariance,

\[
\hat{\varphi}_x(t) = \int \hat{f}(r_x k a t \xi_0) \, dk = \left[ \int f_x(ky) \, dk \right] (a t \xi_0).
\]

The function \( y \to \int f_x(ky) \, dk \) is zonal, so that there is a single-variable function \( f_{0,x}(\cdot) \) such that

\[
f_{0,x}(y_{n+1}) = \int f_x(ky) \, dk.
\]

By (3.5) with \( u = 0 \),

\[
\hat{\varphi}_x(t) = 2^{d/2-1} \sigma_{d-1} \, e^{-td/2} (I^{d/2}_d f_{0,x})(\cosh t),
\]

where, by (4.19),

\[
f_{0,x}(s) = \int f_x(k(e_n \sqrt{s^2 - 1} + e_{n+1} s)) \, dk = (M_x f)(s).
\]

This completes the proof. \( \square \)

We denote

\[
g_x(s) = (M_x f)(s), \quad \psi_x(r) = c^{-1} e^{td/2} (\hat{\varphi}_x(t) \big|_{t=\cosh^{-1} r}),
\]

c being the same as in (4.18). Then (4.18) can be written as

\[
(I^{d/2}_d g_x)(r) = \psi_x(r).
\]

By Lemma 4.7, to reconstruct \( f \), we first need to find \( g_x(s) = (M_x f)(s) \) from the Abel equation (4.21) by using the tools of fractional differentiation [29]. Then \( f \) will be obtained as a limit \( f(x) = \lim_{s \to 1} (M_x f)(s) \) in accordance with Lemma 4.1.
The proof of the following statements is omitted because it is an almost verbatim copy of the reasoning from [29] for \( d = n - 1 \). Everywhere in these statements, we assume \( 1 \leq d \leq n - 1 \), \( f \in C_\mu(\mathbb{H}^n) \), \( \mu > d/2 \), or \( f \in L^p(\mathbb{H}^n) \), \( 1 \leq p < 2(n - 1)/d \).

**Lemma 4.8.** (cf. Corollary 6.77 in [29]) The integral \( (I_{d/2}^d g_x)(r) \) exists in the Lebesgue sense for almost all \( r > 1 \) and all \( x \in \mathbb{H}^n \). If, moreover, \( d \geq 2 \), then \( I_{d/2}^d g_x \) is a continuous function on \((1, \infty)\) for all \( x \in \mathbb{H}^n \).

**Lemma 4.9.** (cf. Corollary 6.78 in [29]) Let \( g_x(s) = (M_x f)(s) \). If \( I_{d/2}^d g_x = \psi_x \), as in (4.21), then

\[
(4.22) \quad g_x(s) = (\mathcal{D}_{-d}^d \psi_x)(s) \quad \forall s > 1,
\]

where the derivative \( \mathcal{D}_{-d}^d \psi_x \) is defined as follows.

(i) If \( d \) is even, \( d = 2m \), then

\[
(4.23) \quad (\mathcal{D}_{-d}^d \psi_x)(s) = (-1)^m \left( \frac{d}{ds} \right)^m \psi_x(s).
\]

(ii) If \( d \) is odd, \( d = 2m - 1 \), then

\[
(4.24) \quad (\mathcal{D}_{-d}^d \psi_x)(s) = (-1)^m s^{1/2} \left( \frac{d}{ds} \right)^m \left[ s^{m-1/2} I_{1/2}^{1/2} s^{-m} \psi_x \right],
\]

\[
(4.25) \quad = (-1)^m s^{1/2} \frac{d}{ds} \left[ s^{1/2} I_{1/2}^{1/2} s^{-1} \psi_x^{(m-1)} \right].
\]

The equalities (4.23)-(4.25) hold for all \( x \in \mathbb{H}^n \), if \( f \in C_\mu(\mathbb{H}^n) \), and for almost all \( x \in \mathbb{H}^n \), if \( f \in L^p(\mathbb{H}^n) \).

**Theorem 4.10.** (cf. Theorem 6.79 in [29]) Let

\[
\varphi = \hat{f}, \quad \psi_x(r) = c^{-1} e^{td/2} \varphi_x(t) \big|_{t = \cosh^{-1} r}, \quad c = 2^{d/2 - 1} \sigma_{d-1} \Gamma(d/2).
\]

Then

\[
(4.26) \quad f(x) = \lim_{s \to 1} (\mathcal{D}_{-d}^d \psi_x)(s),
\]

where \( (\mathcal{D}_{-d}^d \psi_x)(s) \) is defined by (4.23)-(4.25). The limit in (4.26) is uniform for \( f \in C_\mu(\mathbb{H}^n) \) and is understood in the \( L^p \)-norm if \( f \in L^p(\mathbb{H}^n) \).

4.3. Inversion of the Horospherical Transforms by Polynomials of the Beltrami-Laplace Operator.
4.3.1. Local Inversion Formulas for $d$ Even. We consider the mean value operator (4.17) with $t = 0$ and denote

$$\tilde{\varphi}(x) = \int_{K} \varphi(r_x k \xi_0) \, dk.$$  

This operator integrates a function $\varphi$ on $\Xi_d$ over all $d$-horospheres passing through a given point $x \in \mathbb{H}^n$. The next lemma can be considered as a modification of Lemma 4.7 corresponding to $t = 0$. It establishes an important connection between the $d$-horospherical transform, the mean value operator (4.27) and the analytic family (4.7). This lemma is a horospherical analogue of the celebrated Fuglede result for $d$-plane Radon-John transforms [4]. Similar statements are known for all totally geodesic Radon transforms on constant curvature spaces [16, 29, 26, 27].

**Lemma 4.11.** The following equality holds provided that either side of it exists in the Lebesgue sense:

$$\hat{\varphi}(x) = c(Q^d f)(x), \quad c = \frac{2^d \pi^{d/2} \Gamma(n/2)}{\Gamma((n - d)/2)}.$$  

**Proof.** Setting $t = 0$ in (4.18), and using (4.2), we obtain

$$\hat{f}(x) = 2^{d/2 - 1} \sigma_{d-1} \int_{1}^{\infty} (s - 1)^{d/2 - 1} (M_x f)(s) \, ds$$

$$= 2^{d/2 - 1} \sigma_{d-1} \int_{0}^{\infty} (\cosh r - 1)^{d/2 - 1} (M_x f)(\cosh r) \sinh r \, dr$$

$$= \frac{\Gamma((n - d)/2)}{2^{d/2 + 1} \pi^{n/2} \Gamma(d/2)} \int_{\mathbb{H}^n} f(y) \frac{([x, y] - 1)^{(d-n)/2}}{([x, y] + 1)^{n/2 - 1}} \, dy = c(Q^d f)(x).$$

□

Lemmas 4.11 and 4.3 imply the following inversion result.

**Theorem 4.12.** Let $\varphi = \hat{f}$, $f \in C^\infty_c(\mathbb{H}^n)$, $1 \leq d \leq n - 1$. If $d$ is even, then

$$f = c \mathcal{P}(\Delta_H) \tilde{\varphi},$$

where

$$\mathcal{P}(\Delta_H) = \prod_{i=1}^{d/2} \left[ \Delta_H + i(n - 1 - i) \right], \quad c = \frac{(-1)^{d/2} \Gamma((n - d)/2)}{2^d \pi^{d/2} \Gamma(n/2)}.$$
4.3.2. Inversion Formulas for Arbitrary $d$. If $d$ is odd then a local inversion formula, like (4.29), is unavailable in principle. Both even and odd cases can be treated in the framework of a certain analytic family of operators generalizing the mean value operator $\varphi \to \hat{\varphi}$. This approach is inspired by our previous works; cf. [26, Theorem 1.2], [27, Theorem A], [31, Theorem 4.13].

We replace $f$ in (3.13) by the shifted function

$$f(x) = f(r_x y)$$

where $x \in \mathbb{H}^n$ is a new exterior variable and $r_x \in G$ satisfies $r_x x_0 = x$. Denote

$$h_\alpha(t) = e^{td/2} \left( \cos t - 1 \right)^{\alpha/2 - 1} |\sinh t|, \quad \alpha > 0,$$

and write (3.13) as

$$\int_{K \times \mathbb{R}} \hat{f}(r_x k \alpha t \xi_0) h_\alpha(t) \, dk dt = c_1 \int_{\mathbb{H}^n} \frac{(y_{n+1} - 1)^{(\alpha + d - n)/2}}{(y_{n+1} + 1)^{n/2 - 1}} f(y) \, dy$$

(4.30)

$$= c_1 \int_{\mathbb{H}^n} f(y) \frac{([x, y] - 1)^{(\alpha + d - n)/2}}{([x, y] + 1)^{n/2 - 1}} \, dy,$$

$$c_1 = \frac{2^{d/2} \sigma_{d-1} \Gamma(\alpha/2) \Gamma(d/2)}{\sigma_n \Gamma((\alpha + d)/2)}, \quad \alpha > 0.$$

In particular, for $\alpha = n - d$,

$$\int_{K \times \mathbb{R}} \hat{f}(r_x k \alpha t \xi_0) h_{n-d}(t) \, dk dt = \tilde{c}_1 \int_{\mathbb{H}^n} f(y) \, dy$$

$$= \tilde{c}_1 \int_{\mathbb{H}^n} f(y) \frac{([x, y] - 1)^{(n-d)/2 - 1} |\sinh t|}{([x, y] + 1)^{n/2 - 1}} \, dy,$$

(4.31)

$$h_{n-d}(t) = e^{td/2} (\cos t - 1)^{(n-d)/2 - 1} |\sinh t|,$$

(4.32)

$$\tilde{c}_1 = 2^{d/2} \pi^{(d-n)/2} \Gamma((n - d)/2).$$

An expression in (4.30) is a constant multiple of the convolution $Q^{\alpha+d} f$; cf. (4.7). Changing normalization, we obtain the following statement.

**Lemma 4.13.** Let

$$\left( \hat{\mathbf{\delta}^\alpha} \varphi \right)(x) = c_{\alpha} \int_{K \times \mathbb{R}} \varphi(r_x k \alpha t \xi_0) h_\alpha(t) \, dk dt,$$

$$c_{\alpha} = \frac{\Gamma((n - \alpha - d)/2)}{2^{\alpha/2 + d + 1} \pi^{d/2} \Gamma(\alpha/2) \Gamma(n/2)}, \quad \alpha > 0, \quad \alpha + d - n \neq 0, 2, 4, \ldots.$$

Then

$$\hat{\mathbf{\delta}^\alpha} \hat{f} = Q^{\alpha+d} f$$

provided that the integral on the right-hand side exists in the Lebesgue sense.
The next proposition shows that the mean value operator (4.27) is a constant multiple of the limit of the operators (4.33) as $\alpha \to 0$.

**Proposition 4.14.** If $\varphi$ is a compactly supported continuous function on $\Xi_d$, then

$$
\lim_{\alpha \to 0} \hat{\mathcal{S}}^\alpha \varphi = c^{-1} \hat{\varphi}
$$

where $c$ is a constant from (4.28).

**Proof.** We write (4.33) as

$$
(\hat{\mathcal{S}}^\alpha \varphi)(x) = \int_{\mathbb{R}} e^{t/d/2} |\sinh t|^{-1} (\cosh t + 1)^{1-\alpha/2} dt \int_{K} \varphi(k a t \xi_0) dk
$$

(4.36)

$$
= \frac{1}{\gamma_1(\alpha)} \int_{\mathbb{R}} |t|^{a-1} \Omega_\alpha(t) dt,
$$

where

$$
\gamma_1(\alpha) = \frac{2^\alpha \pi^{1/2} \Gamma(\alpha/2)}{\Gamma((1-\alpha)/2)},
$$

$$
\Omega_\alpha(t) = \frac{c_\alpha}{\gamma_1(\alpha)} \left( \frac{\sinh t}{t} \right)^{\alpha-1} (\cosh t + 1)^{1-\alpha/2} e^{t/d/2} \int_{K} \varphi(k a t \xi_0) dk.
$$

By the well-known properties of Riesz kernels (see, e.g. [29, Lemma 3.2]), the limit of the expression (4.36) as $\alpha \to 0$ is

$$
\Omega_\alpha(0)|_{\alpha=0} = c^{-1} \hat{\varphi},
$$

where $c$ is a constant from (4.28). \qed

We will need an analogue of Lemma 4.13 for the case $\alpha = n - d$, which was excluded in (4.34) because of the pole of the gamma function in $c_a$. Starting from (4.33), we define

$$
(\hat{\mathcal{S}}^{n-d} \varphi)(x) = c_{n,d} \int_{K \times \mathbb{R}} \varphi(r_x k a t \xi_0) \tilde{h}_{n-d}(t) dk dt,
$$

(4.37)

where

$$
c_{n,d} = \frac{1}{2^{(n+d)/2+1} \pi^{d/2} \Gamma((n/2) (n-d)/2)},
$$

$$
\tilde{h}_{n-d}(t) = e^{t/d/2} |\sinh t| (\cosh t - 1)^{(n-d)/2-1} \log(\cosh t - 1).
$$

We also use the notation $Q^n$ and $\tilde{c}_1$ from (4.10) and (4.32), respectively.
Proposition 4.15. Let $\varphi = \hat{f}$, $f \in C^\infty_c(\mathbb{H}^n)$, $1 \leq d \leq n - 1$. Then

\begin{equation}
\ast H^{n-d}\varphi = Q^n f + \Phi,
\end{equation}

where

\begin{equation}
\Phi(x) = \gamma_{n,d} \int_{\mathbb{H}^n} \frac{f(y) \, dy}{([x, y] + 1)^{n/2 - 1}}
= \tilde{\gamma}_{n,d} \int_{K \times \mathbb{R}} \varphi(r_x k a_t \xi_0) \tilde{h}_{n-d}(t) \, dk dt,
\end{equation}

\begin{align*}
\gamma_{n,d} &= \psi(n/2) - \psi((n-d)/2) - 2^{n/2 + 1} \pi^{n/2} \Gamma(n/2), \\
\psi(z) &= \frac{\Gamma'(z)}{\Gamma(z)}, \\
\tilde{\gamma}_{n,d} &= \tilde{\gamma}_{n,d} = \frac{\psi(n/2) - \psi((n-d)/2)}{2^{(n+d)/2 + 1} \pi^{d/2} \Gamma(n/2) \Gamma((n-d)/2)}.
\end{align*}

Proof. For $\alpha \neq n - d$, but close to $n - d$, we write (4.34) as

\begin{align*}
c_\alpha \int \int_{K \times \mathbb{R}} \hat{f}(r_x k a_t \xi_0) e^{td/2} \\
\times [(\cosh t - 1)^{\alpha/2 - 1} - (\cosh t - 1)^{(n-d)/2 - 1}] |\sinh t| \, dk dt + c_\alpha I_1
= \zeta_{n,\alpha+d} \int_{\mathbb{H}^n} f(y) \frac{([x, y] - 1)^{(\alpha+d-n)/2} - 1}{([x, y] + 1)^{n/2 - 1}} \, dy + \zeta_{n,\alpha+d} I_2,
\end{align*}

where

\begin{align*}
I_1 &= \int_{K \times \mathbb{R}} \hat{f}(r_x k a_t \xi_0) e^{td/2} (\cosh t - 1)^{(n-d)/2 - 1} |\sinh t| \, dk dt, \\
I_2 &= \int_{\mathbb{H}^n} f(y) \frac{dy}{([x, y] + 1)^{n/2 - 1}}.
\end{align*}

By (4.31), $I_1 = \tilde{c}_1 I_2$. Moving $c_\alpha I_1 = c_\alpha \tilde{c}_1 I_2$ to the right-hand side and passing to the limit as $\alpha \to n - d$, we obtain (4.38). \hfill \square

Now we can formulate the inversion result for $\hat{f}$ in the most general form.

Theorem 4.16. Let $\varphi = \hat{f}$, $f \in C^\infty_c(\mathbb{H}^n)$,

\begin{equation}
\mathcal{P}_\ell(\Delta_H) = (-1)^\ell \prod_{i=1}^{\ell} \left[ \Delta_H + i(n - 1 - i) \right], \quad \ell \in \mathbb{N}.
\end{equation}
(i) If \( n \) is odd, then
\[
(4.41) \quad f = \mathcal{P}_\ell(\Delta_H)\delta_{2\ell-d}^* \varphi \quad \forall \ell \geq d/2.
\]
(ii) If \( n = 2 \), then
\[
(4.42) \quad f = -\Delta_H \delta^*_1 \varphi + \frac{1}{4\pi} \int_{\mathbb{R}} \varphi(a_t \xi_0) \, dt.
\]
(iii) If \( n = 4, 6, \ldots \), then
\[
(4.43) \quad f = \mathcal{P}_{n/2}(\Delta_H)\delta_{n-d}^* \varphi.
\]

Proof. If \( n \) is odd, then (4.34) with \( \alpha = 2\ell - d > 0 \) gives \( \delta_{2\ell-d}^* \varphi = Q^{2\ell} f \) and the result follows by (4.9). If \( 2\ell - d = 0 \), the desired statement was obtained in Theorem 4.12; cf. Proposition 4.14.

If \( n = 2 \), then, by (4.38),
\[
\delta^*_1 \varphi = Q^2 f + \Phi, \quad \Phi = \gamma \int_{\mathbb{H}^2} f(y) \, dy, \quad \gamma = \frac{\psi(1) - \psi(1/2)}{4\pi}.
\]
Applying \(-\Delta_H\) to both sides of this equality, owing to (4.15) and (3.7), we obtain
\[
-\Delta_H \delta^*_1 \varphi = f - \frac{1}{4\pi} \int_{\mathbb{H}^2} f(y) \, dy = f - \frac{1}{4\pi} \int_{\mathbb{R}} \varphi(a_t \xi_0) \, dt.
\]
This gives (4.42).

If \( n = 4, 6, \ldots \), then, by (4.38) and (4.12),
\[
\mathcal{P}_{n/2}(\Delta_H)\delta_{n-d}^* \varphi = \mathcal{P}_{n/2}(\Delta_H)Q^n f + \mathcal{P}_{n/2}(\Delta_H)\Phi = f + \gamma_{n,d} B f.
\]
By Lemma 4.5, \( \mathcal{P}_{n/2}(\Delta_H) B f = D_2 \cdots D_{n-2} B f = 0 \). Hence, we are done. \( \square \)

5. Conclusion

In the present paper, we studied the Radon-like transform over \( d \)-dimensional horospheres in the hyperbolic space \( \mathbb{H}^n \) for any \( 0 < d < n \). Our main concern was explicit inversion formulas for this transform acting on continuous and \( L^p \) functions. The set \( \Xi_d \) of all \( d \)-horospheres was defined in the group-theoretic terms as a \( G \)-orbit of the basic \( d \)-horosphere lying in the cross-section of \( \mathbb{H}^n \) by a fixed \((d+2)\)-dimensional coordinate plane containing the \( x_{n+1} \)-axis. One can give an alternative, “more geometric” definition of the set of \( d \)-horospheres as the set \( \tilde{\Xi}_d \) of
all cross-sections of the hyperboloid $\mathbb{H}^n$ by $(d + 1)$-dimensional planes having the orthogonal $(n - d)$-dimensional normal frames lying in the asymptotic cone $\Gamma$.

Note also that $\dim \Xi_d = (n - d)(d + 2) - 1$. This formula is a consequence of the isomorphism $\Xi_d \sim G/M_dN_d$ (see Proposition 2.1) and known dimensions\footnote{The formula for the dimension of $G$ is immediate, e.g., from the Iwasawa decomposition $G = KAN$.}

$$\dim G = \dim SO_0(n, 1) = \frac{n(n + 1)}{2}, \quad \dim O(n) = \frac{n(n - 1)}{2}.$$ 

It follows that $\dim \Xi_d > \dim \mathbb{H}^n = n$ if and only if $1 \leq d < n - 1$.

The following open problems arise.

**Problem 1.** Investigate the relationship between $\Xi_d$ and $\tilde{\Xi}_d$.

**Problem 2.** Reduce the overdeterminism of the $d$-horospherical Radon transform in the case $d < n - 1$.

In Problem 2, our aim is to define an $n$-dimensional submanifold $\Xi_{d,0}$ of $\Xi_d$, so that a function $f(x)$ could be recovered from its $d$-horospherical transform $\hat{f}(\xi)$ when the values of $\hat{f}(\xi)$ are known only for $\xi \in \Xi_{d,0}$. Integral-geometric problems of this kind amount to I.M. Gelfand [5]. See also [30] and references therein.

We plan to address these problems in the future.

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