A note on non-broken-circuit sets and the chromatic polynomial

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The purpose of this note is to demonstrate several generalizations of Whitney’s theorem \([\text{BlaSag86}]\) – a classical formula for the chromatic polynomial of
a graph. The directions in which we generalize this formula are the following:

- Instead of summing over the sets which contain no broken circuits, we can sum over the sets which are “\(R\)-free” (i.e., contain no element of \(R\) as a subset), where \(R\) is some fixed set of broken circuits (in particular, \(R\) can be \(\emptyset\), yielding another well-known formula for the chromatic polynomial).

- Even more generally, instead of summing over \(R\)-free subsets, we can make a weighted sum over all subsets, where the weight depends on the broken circuits contained in the subset.

- Analogous (and more general) results hold for chromatic symmetric functions.

- Analogous (and more general) results hold for matroids instead of graphs.

Note that, to my knowledge, the last two generalizations cannot be combined: Unlike graphs, matroids do not seem to have a well-defined notion of a chromatic symmetric function.

We shall explore these generalizations in the note below. We shall also use them to prove an apparently new formula for the chromatic polynomial of a graph obtained from a transitive digraph by forgetting the orientations of the edges (Proposition 4.2). This latter formula was suggested to me as a conjecture by Alexander Postnikov, during a discussion on hyperplane arrangements on a space with a bilinear form; it is this formula which gave rise to this whole note. The subject of hyperplane arrangements, however, will not be breached here.

Acknowledgments

I thank Alexander Postnikov and Richard Stanley for discussions on hyperplane arrangements that led to the results in this note.

1. Definitions and a main result

1.1. Graphs and colorings

This note will be concerned with finite graphs. While some results of this note can be generalized to matroids, we shall not discuss this generalization here. Let us start with the definition of a graph that we shall be using:
Definition 1.1. (a) If $V$ is any set, then \( \binom{V}{2} \) will denote the set of all 2-element subsets of $V$. In other words, if $V$ is any set, then we set
\[
\binom{V}{2} = \{ S \in \mathcal{P}(V) \mid |S| = 2 \} = \{ \{s, t\} \mid s \in V, t \in V, s \neq t \}
\]
(where $\mathcal{P}(V)$ denotes the powerset of $V$).

(b) A graph means a pair $(V, E)$, where $V$ is a set, and where $E$ is a subset of \( \binom{V}{2} \). A graph $(V, E)$ is said to be finite if the set $V$ is finite. If $G = (V, E)$ is a graph, then the elements of $V$ are called the vertices of the graph $G$, while the elements of $E$ are called the edges of the graph $G$. If $e$ is an edge of a graph $G$, then the two elements of $e$ are called the endpoints of the edge $e$. If $e = \{s, t\}$ is an edge of a graph $G$, then we say that the edge $e$ connects the vertices $s$ and $t$ of $G$.

Comparing our definition of a graph with some of the other definitions used in the literature, we thus observe that our graphs are undirected (i.e., their edges are sets, not pairs), loopless (i.e., the two endpoints of an edge must always be distinct), edge-unlabelled (i.e., their edges are just 2-element sets of vertices, rather than objects with “their own identity”), and do not have multiple edges (or, more precisely, there is no notion of several edges connecting two vertices, since the edges form a set, nor a multiset, and do not have labels).

Definition 1.2. Let $G = (V, E)$ be a graph. Let $X$ be a set.

(a) An $X$-coloring of $G$ is defined to mean a map $V \rightarrow X$.

(b) An $X$-coloring $f$ of $G$ is said to be proper if every edge $\{s, t\} \in E$ satisfies $f(s) \neq f(t)$.

1.2. Symmetric functions

We shall now briefly introduce the notion of symmetric functions. We shall not use any nontrivial results about symmetric functions; we will merely need some notations.

In the following, $\mathbb{N}$ means the set \{0, 1, 2, \ldots\}. Also, $\mathbb{N}_+$ shall mean the set \{1, 2, 3, \ldots\}.

A partition will mean a sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots) \in \mathbb{N}^\infty$ of nonnegative integers such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ and such that all sufficiently high integers $i \geq 1$ satisfy $\lambda_i = 0$. If $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ is a partition, and if a positive integer $n$ is such that all integers $i \geq n$ satisfy $\lambda_i = 0$, then we shall identify the

\[\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) \equiv (\lambda_1, \lambda_2, \lambda_3, \ldots) = (\lambda_1 + n, \lambda_2 + n, \lambda_3 + n, \ldots).\]

[1] For an introduction to symmetric functions, see any of [Stan99] Chapter 7, [Martin15] Chapter 9 and [GriRei14] Chapter 2 (and a variety of other texts).
partition \( \lambda \) with the finite sequence \((\lambda_1, \lambda_2, \ldots, \lambda_{n-1})\). Thus, for example, the sequences \((3, 1)\) and \((3, 1, 0)\) and the partition \((3, 1, 0, 0, 0, \ldots)\) are all identified. Every weakly decreasing finite list of positive integers thus is identified with a unique partition.

Let \( k \) be a commutative ring with unity. We shall keep \( k \) fixed throughout the paper. The reader will not be missing out on anything if she assumes that \( k = \mathbb{Z} \).

We consider the \( k \)-algebra \( k[[x_1, x_2, x_3, \ldots]] \) of (commutative) power series in countably many distinct indeterminates \( x_1, x_2, x_3, \ldots \) over \( k \). It is a topological \( k \)-algebra. A power series \( P \in k[[x_1, x_2, x_3, \ldots]] \) is said to be bounded-degree if there exists an \( N \in \mathbb{N} \) such that every monomial of degree \( > N \) appears with coefficient 0 in \( P \). A power series \( P \in k[[x_1, x_2, x_3, \ldots]] \) is said to be symmetric if and only if \( P \) is invariant under any permutation of the indeterminates. We let \( \Lambda \) be the subset of \( k[[x_1, x_2, x_3, \ldots]] \) consisting of all symmetric bounded-degree power series \( P \in k[[x_1, x_2, x_3, \ldots]] \). This subset \( \Lambda \) is a \( k \)-subalgebra of \( k[[x_1, x_2, x_3, \ldots]] \), and is called the \( k \)-algebra of symmetric functions over \( k \).

We shall now define the few families of symmetric functions that we will be concerned with in this note. The first are the power-sum symmetric functions:

**Definition 1.3.** Let \( n \) be a positive integer. We define a power series \( p_n \in k[[x_1, x_2, x_3, \ldots]] \) by

\[
p_n = x_1^n + x_2^n + x_3^n + \cdots = \sum_{j \geq 1} x_j^n. \tag{1}
\]

This power series \( p_n \) lies in \( \Lambda \), and is called the \( n \)-th power-sum symmetric function.

We also set \( p_0 = 1 \in \Lambda \). Thus, \( p_n \) is defined not only for all positive integers \( n \), but also for all \( n \in \mathbb{N} \).

**Definition 1.4.** Let \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) \) be a partition. We define a power series \( p_\lambda \in k[[x_1, x_2, x_3, \ldots]] \) by

\[
p_\lambda = \prod_{i \geq 1} p_{\lambda_i}.
\]

This is well-defined, because the infinite product \( \prod_{i \geq 1} p_{\lambda_i} \) converges (indeed, all but finitely many of its factors are 1 (because every sufficiently high integer \( i \) satisfies \( \lambda_i = 0 \) and thus \( p_{\lambda_i} = p_0 = 1 \)).

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\(^2\)See [GriRei14, Section 2.6] or [Grinbe16] §2 for the definition of its topology. This topology makes sure that a sequence \( (P_n)_{n \in \mathbb{N}} \) of power series converges to some power series \( P \) if and only if, for every monomial \( m \), all sufficiently high \( n \in \mathbb{N} \) satisfy

\[
(\text{the } m\text{-coefficient of } P_n) = (\text{the } m\text{-coefficient of } P)
\]

(where the meaning of “sufficiently high” can depend on the \( m \)).
We notice that every partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) (written as a finite list of nonnegative integers) satisfies
\[
p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}.
\]

1.3. Chromatic symmetric functions

The next symmetric functions we introduce are the actual subject of this note; they are the chromatic symmetric functions and originate in [Stanle95, Definition 2.1]:

**Definition 1.5.** Let \( G = (V, E) \) be a finite graph. For every \( \mathbb{N}_+ \)-coloring \( f : V \to \mathbb{N}_+ \), we let \( x_f \) denote the monomial \( \prod_{v \in V} x_{f(v)} \) in the indeterminates \( x_1, x_2, x_3, \ldots \). We define a power series \( X_G \in k[[x_1, x_2, x_3, \ldots]] \) by
\[
X_G = \sum_{f : V \to \mathbb{N}_+ \text{ is a proper } \mathbb{N}_+\text{-coloring of } G} x_f.
\]

This power series \( X_G \) is called the chromatic symmetric function of \( G \).

We have \( X_G \in \Lambda \) for every finite graph \( G = (V, E) \); this will follow from Theorem 1.8 further below (but is also rather obvious).

We notice that \( X_G \) is denoted by \( \Psi [G] \) in [GriRei14, §7.3.3].

1.4. Connected components

We shall now briefly recall the notion of connected components of a graph.

**Definition 1.6.** Let \( G = (V, E) \) be a finite graph. Let \( u \) and \( v \) be two elements of \( V \) (that is, two vertices of \( G \)). A walk from \( u \) to \( v \) in \( G \) will mean a sequence \( (w_0, w_1, \ldots, w_k) \) of elements of \( V \) such that \( w_0 = u \) and \( w_k = v \) and
\[\{w_i, w_{i+1}\} \in E \quad \text{for every } i \in \{0, 1, \ldots, k - 1\}.\]
We say that \( u \) and \( v \) are connected (in \( G \)) if there exists a walk from \( u \) to \( v \) in \( G \).

**Definition 1.7.** Let \( G = (V, E) \) be a graph.

(a) We define a binary relation \( \sim_G \) (written infix) on the set \( V \) as follows: Given \( u \in V \) and \( v \in V \), we set \( u \sim_G v \) if and only if \( u \) and \( v \) are connected (in \( G \)). It is well-known that this relation \( \sim_G \) is an equivalence relation. The \( \sim_G \)-equivalence classes are called the connected components of \( G \).

(b) Assume that the graph \( G \) is finite. We let \( \lambda (G) \) denote the list of the sizes of all connected components of \( G \), in weakly decreasing order. (Each connected component should contribute only one entry to the list.) We view
\( \lambda (G) \) as a partition (since \( \lambda (G) \) is a weakly decreasing finite list of positive integers).

Now, we can state a formula for chromatic symmetric functions:

**Theorem 1.8.** Let \( G = (V, E) \) be a finite graph. Then,

\[
X_G = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(V,F)}.
\]

(Here, of course, the pair \((V, F)\) is regarded as a graph, and the expression \( \lambda (V, F) \) is understood according to Definition 1.7(b).)

This theorem is not new; it appears, e.g., in [Stanley95, Theorem 2.5]. We shall show a far-reaching generalization of it (Theorem 1.11) soon.

### 1.5. Circuits and broken circuits

Let us now define the notions of cycles and circuits of a graph:

**Definition 1.9.** Let \( G = (V, E) \) be a graph. A cycle of \( G \) denotes a list \( (v_1,v_2, \ldots, v_{m+1}) \) of elements of \( V \) with the following properties:

- We have \( m > 1 \).
- We have \( v_{m+1} = v_1 \).
- The vertices \( v_1, v_2, \ldots, v_m \) are pairwise distinct.
- We have \( \{v_i,v_{i+1}\} \in E \) for every \( i \in \{1,2,\ldots,m\} \).

If \( (v_1,v_2, \ldots, v_{m+1}) \) is a cycle of \( G \), then the set \( \{\{v_1,v_2\},\{v_2,v_3\}, \ldots, \{v_m,v_{m+1}\}\} \) is called a circuit of \( G \).

**Definition 1.10.** Let \( G = (V, E) \) be a graph. Let \( X \) be a totally ordered set. Let \( \ell : E \to X \) be a function. We shall refer to \( \ell \) as the labeling function. For every edge \( e \) of \( G \), we shall refer to \( \ell (e) \) as the label of \( e \).

A broken circuit of \( G \) means a subset of \( E \) having the form \( C \setminus \{e\} \), where \( C \) is a circuit of \( G \), and where \( e \) is the unique edge in \( C \) having maximum label (among the edges in \( C \)). Of course, the notion of a broken circuit of \( G \) depends on the function \( \ell \); however, we suppress the mention of \( \ell \) in our notation, since we will not consider situations where two different \( \ell \)'s coexist.

Thus, if \( G \) is a graph with a labeling function, then any circuit \( C \) of \( G \) gives rise to a broken circuit provided that among the edges in \( C \), only one attains the maximum label. (If more than one of the edges of \( C \) attains the maximum label, then \( C \) does not give rise to a broken circuit.) Notice that two different circuits may give rise to one and the same broken circuit.
Theorem 1.11. Let $G = (V, E)$ be a finite graph. Let $X$ be a totally ordered set. Let $\ell : E \to X$ be a function. Let $\mathcal{R}$ be some set of broken circuits of $G$ (not necessarily containing all of them). Let $a_K$ be an element of $k$ for every $K \in \mathcal{R}$. Then,

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} \left( \prod_{K \in \mathcal{R}; K \subseteq F} a_K \right) p_{\lambda(V,F)},$$

(Here, of course, the pair $(V,F)$ is regarded as a graph, and the expression $\lambda(V,F)$ is understood according to Definition 1.7(b).)

Before we come to the proof of this result, let us explore some of its particular cases. First, a definition is in order:

Definition 1.12. Let $E$ be a set. Let $\mathcal{R}$ be a subset of the powerset of $E$ (that is, a set of subsets of $E$). A subset $F$ of $E$ is said to be $\mathcal{R}$-free if $F$ contains no $K \in \mathcal{R}$ as a subset. (For instance, if $\mathcal{R} = \emptyset$, then every subset $F$ of $E$ is $\mathcal{R}$-free.)

Corollary 1.13. Let $G = (V, E)$ be a finite graph. Let $X$ be a totally ordered set. Let $\ell : E \to X$ be a function. Let $\mathcal{R}$ be some set of broken circuits of $G$ (not necessarily containing all of them). Then,

$$X_G = \sum_{\substack{F \subseteq E; \ F \text{ is } \mathcal{R}\text{-free}}} (-1)^{|F|} p_{\lambda(V,F)}.$$

Corollary 1.14. Let $G = (V, E)$ be a finite graph. Let $X$ be a totally ordered set. Let $\ell : E \to X$ be a function. Then,

$$X_G = \sum_{\substack{F \subseteq E; \ F \text{ contains no broken circuit of } G \text{ as a subset}}} (-1)^{|F|} p_{\lambda(V,F)}.$$

Corollary 1.14 appears in [Stanle95, Theorem 2.9], at least in the particular case in which $\ell$ is supposed to be injective.

Let us now see how Theorem 1.8 Corollary 1.13 and Corollary 1.14 can be derived from Theorem 1.11:

Proof of Corollary 1.13 using Theorem 1.11. For every subset $F$ of $E$, we have

$$\prod_{K \in \mathcal{R}; K \subseteq F} 0 = \begin{cases} 1, & \text{if } F \text{ is } \mathcal{R}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathcal{R}\text{-free} \end{cases}$$ (3)
(because if \( F \) is \( \mathcal{S} \)-free, then the product \( \prod_{K \in \mathcal{S}; \ K \subseteq F} 0 \) is empty and thus equals 1; otherwise, the product \( \prod_{K \in \mathcal{S}; \ K \subseteq F} 0 \) contains at least one factor and thus equals 0). Now, Theorem 1.11 (applied to 0 instead of \( a_K \)) yields

\[
X_G = \sum_{F \subseteq E} (-1)^{|F|} \left( \prod_{K \in \mathcal{S}; \ K \subseteq F} 0 \right) p_{\lambda(V,F)} = \begin{cases} 1, & \text{if } F \text{ is } \mathcal{S}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathcal{S}\text{-free} \end{cases}
\]

(by (3))

\[
p_{\lambda(V,F)} = \sum_{\substack{F \subseteq E; \ F \text{ is } \mathcal{S}\text{-free}}} (-1)^{|F|} p_{\lambda(V,F)}.
\]

This proves Corollary 1.13. \( \square \)

**Proof of Corollary 1.14 using Corollary 1.13.** Corollary 1.14 follows from Corollary 1.13 when \( \mathcal{S} \) is set to be the set of all broken circuits of \( G \). \( \square \)

**Proof of Theorem 1.8 using Theorem 1.11.** Let \( X \) be the totally ordered set \( \{1\} \), and let \( \ell : E \to X \) be the only possible map. Let \( \mathcal{S} \) be the empty set. Clearly, \( \mathcal{S} \) is a set of broken circuits of \( G \). For every \( F \subseteq E \), the product \( \prod_{K \in \mathcal{S}; \ K \subseteq F} 0 \) is empty (since \( K \) is the empty set), and thus equals 1. Now, Theorem 1.11 (applied to 0 instead of \( a_K \)) yields

\[
X_G = \sum_{F \subseteq E} (-1)^{|F|} \left( \prod_{K \in \mathcal{S}; \ K \subseteq F} 0 \right) p_{\lambda(V,F)} = \sum_{\substack{F \subseteq E; \ F \text{ is } \mathcal{S}\text{-free}}} (-1)^{|F|} p_{\lambda(V,F)}.
\]

This proves Theorem 1.8. \( \square \)

### 2. Proof of Theorem 1.11

We shall now prepare for the proof of Theorem 1.11 with some notations and some lemmas. Our proof will imitate [BlaSag86, proof of Whitney’s theorem].

#### 2.1. Eqs \( f \) and basic lemmas
Definition 2.1. Let \( V \) and \( X \) be two sets. Let \( f : V \rightarrow X \) be a map. We let \( \text{Eqs } f \) denote the subset

\[
\left\{ \{s,t\} \mid (s,t) \in V^2, s \neq t \text{ and } f(s) = f(t) \right\}
\]

of \( \binom{V}{2} \). (This is well-defined, because any two elements \( s \) and \( t \) of \( V \) satisfying \( s \neq t \) clearly satisfy \( \{s,t\} \in \binom{V}{2} \).)

We shall now state some first properties of this notion:

Lemma 2.2. Let \( G = (V, E) \) be a graph. Let \( X \) be a set. Let \( f : V \rightarrow X \) be a map. Then, the \( X \)-coloring \( f \) of \( G \) is proper if and only if \( E \cap \text{Eqs } f = \emptyset \).

Proof of Lemma 2.2. The set \( E \cap \text{Eqs } f \) is precisely the set of edges \( \{s,t\} \) of \( G \) satisfying \( f(s) = f(t) \); meanwhile, the \( X \)-coloring \( f \) is called proper if and only if no such edges exist. Thus, Lemma 2.2 becomes obvious.

Lemma 2.3. Let \( G = (V, E) \) be a graph. Let \( X \) be a set. Let \( f : V \rightarrow X \) be a map. Let \( C \) be a circuit of \( G \). Let \( e \in C \) be such that \( C \setminus \{e\} \subseteq \text{Eqs } f \). Then, \( e \in E \cap \text{Eqs } f \).

Proof of Lemma 2.3. The set \( C \) is a circuit of \( G \). Hence, we can write \( C \) in the form

\[
C = \left\{ \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{m-1}, v_m\} \right\}
\]

for some cycle \((v_1, v_2, \ldots, v_{m+1})\) of \( G \). Consider this cycle \((v_1, v_2, \ldots, v_{m+1})\). According to the definition of a “cycle”, the cycle \((v_1, v_2, \ldots, v_{m+1})\) is a list of elements of \( V \) having the following properties:

- We have \( m > 1 \).
- We have \( v_{m+1} = v_1 \).
- The vertices \( v_1, v_2, \ldots, v_m \) are pairwise distinct.
- We have \( \{v_i, v_{i+1}\} \in E \) for every \( i \in \{1, 2, \ldots, m\} \).

Recall that \( e \in C \). We can thus WLOG assume that \( e = \{v_m, v_{m+1}\} \) (since otherwise, we can simply relabel the vertices along the cycle \((v_1, v_2, \ldots, v_{m+1})\)). Assume this. Since \( \{v_m, v_{m+1}\} = e \), we have

\[
C \setminus \{e\} = \left\{ \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{m-1}, v_m\} \right\}
\]
Proof of Lemma 2.4. Let $$\{v_i, v_{i+1}\} \subseteq \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{m-1}, v_m\}\} = C \setminus \{e\} \subseteq \text{Eqs } f$$.

). Hence, $$f(v_1) = f(v_2) = \cdots = f(v_m)$$, so that $$f(v_m) = f \left( \frac{v_1}{v_{m+1}} \right) = f(v_{m+1})$$.

Thus, $$\{v_m, v_{m+1}\} \in \text{Eqs } f$$. Thus, $$e = \{v_m, v_{m+1}\} \in \text{Eqs } f$$. Combined with $$e \in E$$, this yields $$e \in E \cap \text{Eqs } f$$. This proves Lemma 2.3.

\[\square\]

**Lemma 2.4.** Let $$(V, B)$$ be a finite graph. Then,

$$\sum_{f: V \rightarrow \mathbb{N}_+; \ B \subseteq \text{Eqs } f} x_f = p_{\lambda(V, B)}$$

(Here, $$x_f$$ is defined as in Definition 1.5 and the expression $$\lambda(V, B)$$ is understood according to Definition 1.7(b).)

**Proof of Lemma 2.4.** Let $$(C_1, C_2, \ldots, C_k)$$ be a list of all connected components of $$(V, B)$$, ordered such that $$|C_1| \geq |C_2| \geq \cdots \geq |C_k|$$.\(^3\) Then, $$\lambda(V, B) = (|C_1|, |C_2|, \ldots, |C_k|)$$ (by the definition of $$\lambda(V, B)$$). Hence, (2) (applied to $$\lambda(V, B)$$ and $$|C_i|$$ instead of $$\lambda$$ and $$\lambda_i$$) shows that

$$p_{\lambda(V, B)} = p_{|C_1|} p_{|C_2|} \cdots p_{|C_k|} = \prod_{i=1}^k p_{|C_i|}. \quad (4)$$

But for every $$i \in \{1, 2, \ldots, k\}$$, we have $$p_{|C_i|} = \sum_{s \in \mathbb{N}_+} x_s^{|C_i|}$$ (by the definition of $$p_{|C_i|}$$). Hence, (4) becomes

$$p_{\lambda(V, B)} = \prod_{i=1}^k p_{|C_i|} = \prod_{i=1}^k \sum_{s \in \mathbb{N}_+} x_s^{|C_i|} = \sum_{s \in \mathbb{N}_+} x_s^{|C_1|} \cdots \sum_{s \in \mathbb{N}_+} x_s^{|C_k|} \quad (5)$$

(by the product rule).

The list $$(C_1, C_2, \ldots, C_k)$$ contains all connected components of $$(V, B)$$, each exactly once. Thus, $$V = \bigsqcup_{i=1}^k C_i$$.

We now define a map

$$\Phi : (\mathbb{N}_+)^k \rightarrow \{f : V \rightarrow \mathbb{N}_+ \mid B \subseteq \text{Eqs } f\}$$

\(^3\)Every connected component of $$(V, B)$$ should appear exactly once in this list.
as follows: Given any \((s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k\), we let \(\Phi(s_1, s_2, \ldots, s_k)\) be the map \(V \to \mathbb{N}_+\) which sends every \(v \in V\) to \(s_i\), where \(i \in \{1, 2, \ldots, k\}\) is such that \(v \in C_i\). (This is well-defined, because for every \(v \in V\), there exists a unique \(i \in \{1, 2, \ldots, k\}\) such that \(v \in C_i\); this follows from \(V = \bigcup_{i=1}^k C_i\).) This map \(\Phi\) is well-defined, because for every \((s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k\), the map \(\Phi(s_1, s_2, \ldots, s_k)\) actually belongs to \(\{f : V \to \mathbb{N}_+ \mid B \subseteq \text{Eqs } f\}\).

A moment’s thought reveals that the map \(\Phi\) is injective. Let us now show that the map \(\Phi\) is surjective.

In order to show this, we must prove that every map \(f : V \to \mathbb{N}_+\) satisfying \(B \subseteq \text{Eqs } f\) has the form \(\Phi(s_1, s_2, \ldots, s_k)\) for some \((s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k\). So let us fix a map \(f : V \to \mathbb{N}_+\) satisfying \(B \subseteq \text{Eqs } f\). We must find some \((s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k\) such that \(f = \Phi(s_1, s_2, \ldots, s_k)\).

We have \(B \subseteq \text{Eqs } f\). Thus, for every \(\{s, t\} \in B\), we have \(\{s, t\} \in B \subseteq \text{Eqs } f\) and thus

\[
f(s) = f(t).
\]

(6)

Now, if \(x\) and \(y\) are two elements of \(V\) lying in the same connected component of \((V, B)\), then

\[
f(x) = f(y)
\]

(7)\footnote{In other words, the map \(f\) is constant on each connected component of \((V, B)\). Thus, the map \(f\) is constant on \(C_i\) for each \(i \in \{1, 2, \ldots, k\}\) (since \(C_i\) is a connected component of \((V, B)\)). Hence, for each \(i \in \{1, 2, \ldots, k\}\), we can define a positive integer \(s_i \in \mathbb{N}_+\) to be the image of any element of \(C_i\) under \(f\) (this is well-defined, because \(f\) is constant on \(C_i\) and thus the choice of the element does not matter). Define \(s_i \in \mathbb{N}_+\) for each \(i \in \{1, 2, \ldots, k\}\) this way. Thus, we have defined a \(k\)-tuple \((s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k\). Now, \(f = \Phi(s_1, s_2, \ldots, s_k)\) (this follows immediately by recalling the definitions of \(\Phi\) and \(s_i\)).}

Proof. We just need to check that \(B \subseteq \text{Eqs } (\Phi(s_1, s_2, \ldots, s_k))\). But this is easy: For every \((u, v) \in B\), the vertices \(u\) and \(v\) of \((V, B)\) lie in one and the same connected component \(C_i\) of the graph \((V, B)\), and thus (by the definition of \(\Phi(s_1, s_2, \ldots, s_k)\)) the map \(\Phi(s_1, s_2, \ldots, s_k)\) sends both of them to \(s_i\); but this shows that \((u, v) \in \text{Eqs } (\Phi(s_1, s_2, \ldots, s_k))\).

In fact, we can reconstruct \((s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k\) from its image \(\Phi(s_1, s_2, \ldots, s_k)\), because each \(s_i\) is the image of any element of \(C_i\) under \(\Phi(s_1, s_2, \ldots, s_k)\) and this allows us to compute \(s_i\), since \(C_i\) is nonempty.)

Proof of \(\Phi\): Let \(x\) and \(y\) be two elements of \(V\) lying in the same connected component of \((V, B)\). Then, the vertices \(x\) and \(y\) are connected by a walk in the graph \((V, B)\) (by the definition of a “connected component”). Let \((v_0, v_1, \ldots, v_j)\) be this walk (regarded as a sequence of vertices); thus, \(v_0 = x\) and \(v_j = y\). For every \(i \in \{0, 1, \ldots, j-1\}\), we have \(\{v_i, v_{i+1}\} \in B\) (since \((v_0, v_1, \ldots, v_j)\) is a walk in the graph \((V, B)\) and thus \(f(v_i) = f(v_{i+1})\) (by \(\Phi\), applied to \((s, t) = (v_i, v_{i+1})\)). In other words, \(f(v_0) = f(v_1) = \cdots = f(v_j)\). Hence, \(f(v_0) = f(v_j)\), so that \(f\left(\begin{array}{c} x \\
0 \end{array}\right) = f(v_0) = f\left(\begin{array}{c} v_i \\
0 \end{array}\right) = f(y)\), qed.\footnote{Proof of \(\Phi\): Let \(x\) and \(y\) be two elements of \(V\) lying in the same connected component of \((V, B)\). Then, the vertices \(x\) and \(y\) are connected by a walk in the graph \((V, B)\) (by the definition of a “connected component”). Let \((v_0, v_1, \ldots, v_j)\) be this walk (regarded as a sequence of vertices); thus, \(v_0 = x\) and \(v_j = y\). For every \(i \in \{0, 1, \ldots, j-1\}\), we have \(\{v_i, v_{i+1}\} \in B\) (since \((v_0, v_1, \ldots, v_j)\) is a walk in the graph \((V, B)\) and thus \(f(v_i) = f(v_{i+1})\) (by \(\Phi\), applied to \((s, t) = (v_i, v_{i+1})\)). In other words, \(f(v_0) = f(v_1) = \cdots = f(v_j)\). Hence, \(f(v_0) = f(v_j)\), so that \(f\left(\begin{array}{c} x \\
0 \end{array}\right) = f(v_0) = f\left(\begin{array}{c} v_i \\
0 \end{array}\right) = f(y)\), qed.}
that \( f = \Phi(s_1, s_2, \ldots, s_k) \). In other words, the map \( \Phi \) is surjective. Since \( \Phi \) is both injective and surjective, we conclude that \( \Phi \) is a bijection.

Moreover, it is straightforward to see that every map \((s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k\) satisfies

\[
x_{\Phi(s_1, s_2, \ldots, s_k)} = \prod_{i=1}^{k} x_{s_i}^{[C_i]}(8)
\]

(by the definitions of \( x_{\Phi(s_1, s_2, \ldots, s_k)} \) and of \( \Phi \)). Now,

\[
\sum_{f: V \rightarrow \mathbb{N}_+; \quad B \subseteq \text{Eqs } f} x_f = \sum_{(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k} x_{\Phi(s_1, s_2, \ldots, s_k)} = \prod_{i=1}^{k} x_{s_i}^{[C_i]}(by \ (8))
\]

here, we have substituted \( \Phi(s_1, s_2, \ldots, s_k) \) for \( f \) in the sum, since the map \( \Phi: (\mathbb{N}_+)^k \rightarrow \{f: V \rightarrow \mathbb{N}_+ \mid B \subseteq \text{Eqs } f\} \) is a bijection

\[
= \sum_{(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k} \prod_{i=1}^{k} x_{s_i}^{[C_i]} = p_{\lambda(V, B)} \quad (by \ (5)).
\]

This proves Lemma 2.4.

**Lemma 2.5.** Let \( G = (V, E) \) be a finite graph. Let \( X \) be a totally ordered set. Let \( \ell : E \rightarrow X \) be a function. Let \( K \) be a broken circuit of \( G \). Then, \( K \neq \emptyset \).

**Proof of Lemma 2.5** The set \( K \) is a broken circuit of \( G \), and thus is a circuit of \( G \) with an edge removed (by the definition of a broken circuit). Thus, the set \( K \) contains at least 1 edge (since every circuit of \( G \) contains at least 2 edges). This proves Lemma 2.5. \( \square \)

### 2.2. Alternating sums

We shall now come to less simple lemmas.

**Definition 2.6.** We shall use the so-called Iverson bracket notation: If \( S \) is any logical statement, then \([S]\) shall mean the integer

\[
1, \quad \text{if } S \text{ is true;}
0, \quad \text{if } S \text{ is false}
\]

The following lemma is probably the most crucial one in this note:

**Lemma 2.7.** Let \( G = (V, E) \) be a finite graph. Let \( X \) be a totally ordered set. Let \( \ell : E \rightarrow X \) be a function. Let \( \mathcal{R} \) be some set of broken circuits of \( G \) (not necessarily containing all of them). Let \( a_K \) be an element of \( k \) for every \( K \in \mathcal{R} \).
Let $Y$ be any set. Let $f : V \rightarrow Y$ be any map. Then,
\[
\sum_{B \subseteq \text{E} \cap \text{Eqs} f} (-1)^{|B|} \prod_{K \in \mathcal{R} : K \subseteq B} a_K = [E \cap \text{Eqs} f = \emptyset].
\]

**Proof of Lemma 2.7.** We WLOG assume that $E \cap \text{Eqs} f \neq \emptyset$ (since otherwise, the claim is obvious). Thus, $[E \cap \text{Eqs} f = \emptyset] = 0$.

Pick any $d \in E \cap \text{Eqs} f$ with maximum $\ell(d)$ (among all $d \in E \cap \text{Eqs} f$). (This is clearly possible, since $E \cap \text{Eqs} f \neq \emptyset$.) Define two subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{P}(E \cap \text{Eqs} f)$ as follows:
\[
\mathcal{U} = \{ F \in \mathcal{P}(E \cap \text{Eqs} f) \mid d \notin F \};
\]
\[
\mathcal{V} = \{ F \in \mathcal{P}(E \cap \text{Eqs} f) \mid d \in F \}.
\]

Thus, we have $\mathcal{P}(E \cap \text{Eqs} f) = \mathcal{U} \cup \mathcal{V}$, and the sets $\mathcal{U}$ and $\mathcal{V}$ are disjoint. Now, we define a map $\Phi : \mathcal{U} \rightarrow \mathcal{V}$ by
\[
(\Phi(B) = B \cup \{d\} \quad \text{for every } B \in \mathcal{U}).
\]

This map $\Phi$ is well-defined (because for every $B \in \mathcal{U}$, we have $B \cup \{d\} \in \mathcal{V}$ and a bijection.[8] Moreover, every $B \in \mathcal{U}$ satisfies
\[
(\Phi(B) = B \cup \{d\} \quad \text{for every } B \in \mathcal{U}).
\]

Now, we claim that, for every $B \in \mathcal{U}$ and every $K \in \mathcal{R}$, we have the following logical equivalence:
\[
(K \subseteq B) \iff (K \subseteq \Phi(B)).
\]

---

7In (slightly) more detail: If $E \cap \text{Eqs} f = \emptyset$, then the sum
\[
\sum_{B \subseteq E \cap \text{Eqs} f} (-1)^{|B|} \prod_{K \in \mathcal{R} : K \subseteq B} a_K
\]
has only one addend (namely, the addend for $B = \emptyset$), and thus simplifies to
\[
(-1)^{0} \prod_{K \in \mathcal{R} : K \subseteq \emptyset} a_K = \prod_{K \in \mathcal{R} : K \subseteq \emptyset} a_K = (\text{empty product}) = 1 = [E \cap \text{Eqs} f = \emptyset].
\]

8This follows from the fact that $d \in E \cap \text{Eqs} f$.

9Its inverse is the map $\Psi : \mathcal{V} \rightarrow \mathcal{U}$ defined by $\Psi(B) = B \setminus \{d\}$ for every $B \in \mathcal{V}$.

10Proof. Let $B \in \mathcal{U}$. Thus, $d \notin B$ (by the definition of $\mathcal{U}$). Now, $\Phi(B) = B \cup \{d\} = |B \cup \{d\}| = |B| + 1$ (since $d \notin B$), so that $(-1)^{|\Phi(B)|} = -(-1)^{|B|}$, qed.
Proof of (10): Let \( B \in \mathcal{U} \) and \( K \in \mathcal{K} \). We must prove the equivalence (10). The definition of \( \Phi \) yields \( \Phi(B) = B \cup \{d\} \supseteq B \), so that \( B \subseteq \Phi(B) \). Hence, if \( K \subseteq B \), then \( K \subseteq B \subseteq \Phi(B) \). Therefore, the forward implication of the equivalence (10) is proven. It thus remains to prove the backward implication of this equivalence. In other words, it remains to prove that if \( K \subseteq \Phi(B) \), then \( K \subseteq B \). So let us assume that \( K \subseteq \Phi(B) \).

We want to prove that \( K \subseteq B \). Assume the contrary. Thus, \( K \nsubseteq B \). We have \( K \in \mathcal{K} \). Thus, \( K \) is a broken circuit of \( G \) (since \( \mathcal{K} \) is a set of broken circuits of \( G \)). In other words, \( K \) is a subset of \( E \) having the form \( C \setminus \{e\} \), where \( C \) is a circuit of \( G \), and where \( e \) is the unique edge in \( C \) having maximum label (among the edges in \( C \)) (because this is how a broken circuit is defined). Consider these \( C \) and \( e \). Thus, \( K = C \setminus \{e\} \).

The element \( e \) is the unique edge in \( C \) having maximum label (among the edges in \( C \)). Thus, if \( e' \) is any edge in \( C \) satisfying \( \ell(e') \geq \ell(e) \), then

\[ e' = e. \]  

(11)

But \( \{d\} \subseteq (B \cup \{d\}) \setminus \{d\} \subseteq B \).

If we had \( d \notin K \), then we would have \( K \setminus \{d\} = K \) and therefore \( K \subseteq B \); this would contradict \( K \nsubseteq B \). Hence, we cannot have \( d \notin K \). We thus must have \( d \in K \). Hence, \( d \in K = C \setminus \{e\} \). Hence, \( d \in C \) and \( d \neq e \).

But \( C \setminus \{e\} = K \subseteq \Phi(B) \subseteq E \cap \text{Eqs } f \) (since \( \Phi(B) \in \mathcal{P}(E \cap \text{Eqs } f) \)), so that \( C \setminus \{e\} \subseteq E \cap \text{Eqs } f \subseteq \text{Eqs } f \). Hence, Lemma 2.3 (applied to \( Y \) instead of \( X \)) shows that \( e \in E \cap \text{Eqs } f \). Thus, \( \ell(d) \geq \ell(e) \) (since \( d \) was defined to be an element of \( E \cap \text{Eqs } f \) with maximum \( \ell(d) \) among all \( d \in E \cap \text{Eqs } f \)).

Also, \( d \in C \). Since \( \ell(d) \geq \ell(e) \), we can therefore apply (11) to \( e' = d \). We thus obtain \( d = e \). This contradicts \( d \neq e \). This contradiction proves that our assumption was wrong. Hence, \( K \subseteq B \) is proven. Thus, we have proven the backward implication of the equivalence (10); this completes the proof of (10).

Now, recall that we have \( \mathcal{P}(E \cap \text{Eqs } f) = \mathcal{U} \cup \mathcal{V} \), and the sets \( \mathcal{U} \) and \( \mathcal{V} \) are disjoint. Hence, the sum \( \sum_{B \subseteq E \cap \text{Eqs } f} (-1)^{|B|} \prod_{K \subseteq B} a_K \) can be split into two sums as

\[ 14 \]
follows:

\[
\sum_{B \subseteq E \cap \text{Eqs } f} (-1)^{|B|} \prod_{K \in \mathcal{K} : K \subseteq B} a_K
\]

\[
= \sum_{B \in \mathcal{U}} \left( -(-1)^{|\Phi(B)|} \right) \prod_{K \in \mathcal{K} : K \subseteq \Phi(B)} a_K + \sum_{B \in \mathcal{V}} (-1)^{|\Phi(B)|} \prod_{K \in \mathcal{K} : K \subseteq \Phi(B)} a_K
\]

\[
= - \sum_{B \in \mathcal{U}} (-1)^{|\Phi(B)|} \prod_{K \in \mathcal{K} : K \subseteq \Phi(B)} a_K + \sum_{B \in \mathcal{V}} (-1)^{|\Phi(B)|} \prod_{K \in \mathcal{K} : K \subseteq \Phi(B)} a_K
\]

\[
= 0 = [E \cap \text{Eqs } f = \emptyset] \quad \text{(since } [E \cap \text{Eqs } f = \emptyset] = 0). \quad (12)
\]

This proves Lemma 2.7. \qed

We now finally proceed to the proof of Theorem 1.11.
Proof of Theorem 1.11. The definition of $X_G$ shows that

$$X_G = \sum_{f: V \to \mathbb{N}_+ \text{ is a proper } \mathbb{N}_+ \text{-coloring of } G} x_f$$

$$= \sum_{f: V \to \mathbb{N}_+} \left( \text{if } f \text{ is a proper } \mathbb{N}_+ \text{-coloring of } G \right) x_f$$

$$\iff (\text{the } \mathbb{N}_+ \text{-coloring } f \text{ of } G \text{ is proper})$$

$$\iff (E \cap \text{Eqs } f = \emptyset)$$

(by Lemma 2.2 applied to $X$)

$$= \sum_{f: V \to \mathbb{N}_+} \left( \text{if } E \cap \text{Eqs } f = \emptyset \right) x_f$$

(by Lemma 2.2 applied to $Y = \mathbb{N}_+$)

$$= \sum_{f: V \to \mathbb{N}_+} \sum_{B \subseteq E \cap \text{Eqs } f} (-1)^{|B|} \left( \prod_{K \subseteq B} a_K \right) x_f = \sum_{f: V \to \mathbb{N}_+} \sum_{B \subseteq E \cap \text{Eqs } f} (-1)^{|B|} \left( \prod_{K \subseteq B} a_K \right) x_f$$

(by Lemma 2.4)

$$= \sum_{B \subseteq E} \sum_{f: V \to \mathbb{N}_+} (-1)^{|B|} \left( \prod_{K \subseteq B} a_K \right) x_f = \sum_{B \subseteq E} (-1)^{|B|} \left( \prod_{K \subseteq B} a_K \right) \sum_{f: V \to \mathbb{N}_+} x_f$$

$$= \sum_{B \subseteq E} (-1)^{|B|} \left( \prod_{K \subseteq B} a_K \right) p_{\lambda(V,B)} = \sum_{F \subseteq E} (-1)^{|F|} \left( \prod_{K \subseteq F} a_K \right) p_{\lambda(V,F)}$$

(here, we have renamed the summation index $B$ as $F$). This proves Theorem 1.11.

Thus, Theorem 1.11 is proven; as we know, this entails the correctness of Theorem 1.8, Corollary 1.13 and Corollary 1.14.
3. The chromatic polynomial

3.1. Definition

We have so far studied the chromatic symmetric function. We shall now apply the above results to the chromatic polynomial. The definition of the chromatic polynomial rests upon the following fact:

**Theorem 3.1.** Let $G = (V, E)$ be a finite graph. Then, there exists a unique polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies

\[ P(q) = \text{(the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G). \]

**Definition 3.2.** Let $G = (V, E)$ be a finite graph. Theorem 3.1 shows that there exists a polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies $P(q) =$ \text{(the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G).$ This polynomial $P$ is called the chromatic polynomial of $G$, and will be denoted by $\chi_G$.

We shall later prove Theorem 3.1 (as a consequence of something stronger that we show). First, we shall state some formulas for the chromatic polynomial which are analogues of results proven before for the chromatic symmetric function.

3.2. Formulas for $\chi_G$

Before we state several formulas for $\chi_G$, we need to introduce one more notation:

**Definition 3.3.** Let $G$ be a finite graph. We let $\text{conn } G$ denote the number of connected components of $G$.

The following results are analogues of Theorem 1.8, Theorem 1.11, Corollary 1.13 and Corollary 1.14, respectively:

**Theorem 3.4.** Let $G = (V, E)$ be a finite graph. Then,

\[ \chi_G = \sum_{F \subseteq E} (-1)^{|F|} \chi_{\text{conn}(V,F)}. \]

(Here, of course, the pair $(V, F)$ is regarded as a graph, and the expression $\text{conn}(V,F)$ is understood according to Definition 3.3)

**Theorem 3.5.** Let $G = (V, E)$ be a finite graph. Let $X$ be a totally ordered set. Let $\ell : E \to X$ be a function. Let $\mathcal{R}$ be some set of broken circuits of $G$ (not
necessarily containing all of them). Let \( a_K \) be an element of \( k \) for every \( K \in \mathcal{K} \). Then,

\[
\chi_G = \sum_{F \subseteq E, F \text{ is } \mathcal{K}-free} (-1)^{|F|} \left( \prod_{K \in \mathcal{K} : K \subseteq F} a_K \right) x^{\text{conn}(V,F)}.
\]

(Here, of course, the pair \((V,F)\) is regarded as a graph, and the expression \( \text{conn}(V,F) \) is understood according to Definition 3.3)

**Corollary 3.6.** Let \( G = (V,E) \) be a finite graph. Let \( X \) be a totally ordered set. Let \( \ell : E \to X \) be a function. Let \( \mathcal{K} \) be some set of broken circuits of \( G \) (not necessarily containing all of them). Then,

\[
\chi_G = \sum_{F \subseteq E; F \text{ is } \mathcal{K}-free} (-1)^{|F|} x^{\text{conn}(V,F)}.
\]

**Corollary 3.7.** Let \( G = (V,E) \) be a finite graph. Let \( X \) be a totally ordered set. Let \( \ell : E \to X \) be a function. Then,

\[
\chi_G = \sum_{F \subseteq E; F \text{ contains no broken circuit of } G \text{ as a subset}} (-1)^{|F|} x^{\text{conn}(V,F)}.
\]

### 3.3. Proofs

There are two approaches to these results: One is to derive them similarly to how we derived the analogous results about \( X_G \); the other is to derive them from the latter. We shall take the first approach, since it yields a proof of the classical Theorem 3.1 “for free”. We begin with an analogue of Lemma 2.4:

**Lemma 3.8.** Let \((V,B)\) be a finite graph. Let \( q \in \mathbb{N} \). Then,

\[
\sum_{f:V \to \{1,2,...,q\}; B \subseteq \text{Eqs } f} 1 = q^{\text{conn}(V,B)}.
\]

(Here, the expression \( \text{conn}(V,B) \) is understood according to Definition 1.7 (b).)

One way to prove Lemma 3.8 is to evaluate the equality given by Lemma 2.4 at \( x_k = \begin{cases} 1, & \text{if } k \leq q \\ 0, & \text{if } k > q \end{cases} \). Another proof can be obtained by mimicking our proof of Lemma 2.4.
Proof of Lemma 3.8 Define \((C_1, C_2, \ldots, C_k)\) as in the proof of Lemma 2.4. Thus, \(\text{conn} (V, B) = k\). Define a map \(\Phi\) as in the proof of Lemma 2.4 but with \(\mathbb{N}_+\) replaced by \(\{1, 2, \ldots, q\}\). Then,

\[
\Phi : \{1, 2, \ldots, q\}^k \rightarrow \{f : V \rightarrow \{1, 2, \ldots, q\} \mid B \subseteq \text{Eqs } f\}
\]

is a bijection. Now,

\[
\sum_{f : V \rightarrow \{1, 2, \ldots, q\}; B \subseteq \text{Eqs } f} \frac{1}{(-1)^{|F|}} \prod_{K \in \mathbb{N}_+; K \subseteq F} a_K q^{\text{conn}(V, F)}.
\]

(Here, of course, the pair \((V, F)\) is regarded as a graph, and the expression \(\text{conn} (V, F)\) is understood according to Definition 3.3.)

This proves Lemma 3.8.

We shall now show a weaker version of Theorem 3.5 (as a stepping stone to the actual theorem):

Lemma 3.9. Let \(G = (V, E)\) be a finite graph. Let \(X\) be a totally ordered set. Let \(\ell : E \rightarrow X\) be a function. Let \(\mathfrak{G}\) be some set of broken circuits of \(G\) (not necessarily containing all of them). Let \(a_K\) be an element of \(\mathbb{K}\) for every \(K \in \mathfrak{G}\). Let \(q \in \mathbb{N}\). Then,

\[
(\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G)
\]

\[
= \sum_{F \subseteq E} (-1)^{|F|} \left( \prod_{K \in \mathfrak{G}; K \subseteq F} a_K \right) q^{\text{conn}(V, F)}.
\]

(Here, of course, the pair \((V, F)\) is regarded as a graph, and the expression \(\text{conn} (V, F)\) is understood according to Definition 3.3.)

\[\text{[This can be shown in the same way as for the map } \Phi \text{ in the proof of Lemma 2.4; we just have to replace every } \mathbb{N}_+ \text{ by } \{1, 2, \ldots, q\}.]\]
Proof of Lemma 3.9 We have \(^\text{12}\)

\[
\sum_{f : V \to \{1, 2, \ldots, q\}} \begin{cases} \text{the number of all proper } \{1, 2, \ldots, q\} \text{-colorings of } G \\ \text{if } f \text{ is a proper } \{1, 2, \ldots, q\} \text{-coloring of } G \\ \iff (E \cap \text{Eqs } f = \emptyset) \\ \iff (E \cap \text{Eqs } f = \emptyset) \\ \text{(by Lemma } 2.2 \text{ applied to } \{1, 2, \ldots, q\} \text{ instead of } X) \\ \end{cases}
\]

\[
= \sum_{f : V \to \{1, 2, \ldots, q\}} \sum_{B \subseteq E \cap \text{Eqs } f} (-1)^{|B|} \prod_{K \subseteq B} a_K
\]

\[
= \sum_{f : V \to \{1, 2, \ldots, q\}} \sum_{B \subseteq E} \sum_{\text{Eqs } f} (-1)^{|B|} \prod_{K \subseteq B} a_K
\]

\[
= \sum_{B \subseteq E} \sum_{f : V \to \{1, 2, \ldots, q\}} (-1)^{|B|} \prod_{K \subseteq B} a_K
\]

\[
= \sum_{B \subseteq E} (-1)^{|B|} \prod_{K \subseteq B} a_K
\]

\[
= \sum_{F \subseteq E} (-1)^{|F|} q_{\text{conn}}(V, F)
\]

(here, we have renamed the summation index \(B\) as \(F\)). This proves Theorem 1.11 \(\square\)

From Lemma 3.9 we obtain the following consequence:

\textbf{Lemma 3.10.} Let \(G = (V, E)\) be a finite graph. Let \(q \in \mathbb{N}\). Then,

\[
\sum_{F \subseteq E} (-1)^{|F|} q_{\text{conn}}(V, F)
\]

(Here, of course, the pair \((V, F)\) is regarded as a graph, and the expression conn \((V, F)\) is understood according to Definition 3.3) \text{12}

\text{We are again using the Iverson bracket notation, as defined in Definition 2.6.}
Proof of Lemma 3.10. This is derived from Lemma 3.9 in the same way as Theorem 1.8 was derived from Theorem 3.5.

Next, we recall a classical fact about a polynomials over fields: Namely, if a polynomial (in one variable) over a field has infinitely many roots, then this polynomial is 0. Let us state this more formally:

**Proposition 3.11.** Let $K$ be a field. Let $P \in K[x]$ be a polynomial over $K$. Assume that there are infinitely many $\lambda \in K$ satisfying $P(\lambda) = 0$. Then, $P = 0$.

We shall use the following consequence of this proposition:

**Corollary 3.12.** Let $R$ be an integral domain. Assume that the canonical ring homomorphism from the ring $\mathbb{Z}$ to the ring $R$ is injective. Let $P \in R[x]$ be a polynomial over $R$. Assume that $P(q \cdot 1_R) = 0$ for every $q \in \mathbb{N}$ (where $1_R$ denotes the unity of $R$). Then, $P = 0$.

**Proof of Corollary 3.12.** Let $K$ denote the fraction field of the integral domain $R$. We regard $R$ and $R[x]$ as subrings of $K$ and $K[x]$, respectively. By assumption, we have $P(q \cdot 1_R) = 0$ for every $q \in \mathbb{N}$. But the elements $q \cdot 1_R$ of $R$ for $q \in \mathbb{N}$ are pairwise distinct (since the canonical ring homomorphism from the ring $\mathbb{Z}$ to the ring $R$ is injective). Hence, there are infinitely many $\lambda \in K$ satisfying $P(\lambda) = 0$ (namely, $\lambda = q \cdot 1_R$ for all $q \in \mathbb{N}$). Thus, Proposition 3.11 shows that $P = 0$. This proves Corollary 3.12.

We can now prove the classical Theorem 3.1:

**Proof of Theorem 3.1.** We need to show that there exists a unique polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies

$$P(q) = (\text{the number of all proper } \{1, 2, \ldots, q\} \text{-colorings of } G).$$

To see that such a polynomial exists, we notice that $P = \sum_{F \subseteq E} (-1)^{|F|} x^{|\text{conn}(V, F)|}$ is such a polynomial (by Lemma 3.10). It remains to prove that such a polynomial is unique. This follows from the fact that if two polynomials $P_1 \in \mathbb{Z}[x]$ and $P_2 \in \mathbb{Z}[x]$ satisfy

$$P_1(q) = P_2(q) \quad \text{for all } q \in \mathbb{N},$$

then $P_1 = P_2$. \[13\] Theorem 3.1 is therefore proven.

Next, it is the turn of Theorem 3.5.

\[13\] This fact follows from Corollary 3.12 (applied to $R = \mathbb{Z}$ and $P = P_1 - P_2$.)
Proof of Theorem 3.5. Let $R$ be the polynomial ring $\mathbb{Z}[y_K \mid K \in \mathfrak{r}]$, where $y_K$ is a new indeterminate for each $K \in \mathfrak{r}$.

The claim of Theorem 3.5 is a polynomial identity in the elements $a_K$ of $k$. Hence, we can WLOG assume that $k = R$ and $a_K = y_K$ for each $K \in \mathfrak{r}$. Assume this. Thus, $k$ is an integral domain, and the canonical ring homomorphism from the ring $\mathbb{Z}$ to the ring $k$ is injective.

For every $q \in \mathbb{N}$, we have

$$\chi_G(q) = (\text{the number of all proper } \{1, 2, \ldots, q\} \text{-colorings of } G)$$

(by the definition of the chromatic polynomial $\chi_G$)

$$= \sum_{F \subseteq E} (-1)^{|F|} \left( \prod_{K \in \mathfrak{r}; K \subseteq F} a_K \right) q^{\text{conn}(V,F)} \quad (13)$$

(by Lemma 3.9). Define a polynomial $P \in k[x]$ by

$$P = \chi_G - \sum_{F \subseteq E} (-1)^{|F|} \left( \prod_{K \in \mathfrak{r}; K \subseteq F} a_K \right) x^{\text{conn}(V,F)} \quad (14)$$

Then, for every $q \in \mathbb{N}$, we have

$$P \left( q \cdot 1_k \right) = P(q) = \chi_G(q) - \sum_{F \subseteq E} (-1)^{|F|} \left( \prod_{K \in \mathfrak{r}; K \subseteq F} a_K \right) q^{\text{conn}(V,F)} \quad (by \ (14))$$

$$= 0 \quad (by \ (13)).$$

Thus, Corollary 3.12 (applied to $R = k$) shows that $P = 0$. In light of (14), this rewrites as follows:

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} \left( \prod_{K \in \mathfrak{r}; K \subseteq F} a_K \right) x^{\text{conn}(V,F)}.$$ 

This proves Theorem 3.5. 

Now that Theorem 3.5 is proven, we could derive Theorem 3.4, Corollary 3.6 and Corollary 3.7 from it in the same way as we have derived Theorem 1.8, Corollary 1.13 and Corollary 1.14 from Theorem 1.11. We leave the details to the reader.

3.4. Special case: Whitney’s Broken-Circuit Theorem

Corollary 3.7 is commonly stated in the following simplified (if less general) form:
Corollary 3.13. Let $G = (V, E)$ be a finite graph. Let $X$ be a totally ordered set. Let $\ell : E \to X$ be an injective function. Then,

$$\chi_G = \sum_{F \subseteq E; F \text{ contains no broken circuit of } G \text{ as a subset}} (-1)^{|F|} x^{|V| - |F|}.$$

Corollary 3.13 is known as Whitney’s Broken-Circuit theorem (see, e.g., [BlaSag86]). Notice that $\ell$ is required to be injective in Corollary 3.13, the purpose of this requirement is to ensure that every circuit of $G$ has a unique edge $e$ with maximum $\ell(e)$, and thus induces a broken circuit of $G$. The proof of Corollary 3.13 relies on the following standard result:

Lemma 3.14. Let $(V, F)$ be a finite graph. Assume that $(V, F)$ has no circuits. Then, $\text{conn}(V, F) = |V| - |F|.$

(A graph which has no circuits is commonly known as a forest.) Lemma 3.14 is both extremely elementary and well-known; for example, it appears in [Bona11, Proposition 10.6] and in [Bollob79 §I.2, Corollary 6]. Let us now see how it entails Corollary 3.13.

Proof of Corollary 3.13. Corollary 3.13 follows from Corollary 3.7. Indeed, the injectivity of $\ell$ shows that every circuit of $G$ has a unique edge $e$ with maximum $\ell(e)$, and thus contains a broken circuit of $G$. Therefore, if a subset $F$ of $E$ contains no broken circuit of $G$ as a subset, then $F$ contains no circuit of $G$ either, and therefore the graph $(V, F)$ contains no circuits; but this entails that $\text{conn}(V, F) = |V| - |F|$ (by Lemma 3.14). Hence, Corollary 3.7 immediately yields Corollary 3.13.

4. Application to transitive directed graphs

We shall now see an application of Corollary 3.6 to graphs which are obtained from certain directed graphs by “forgetting the directions of the edges”. Let us first introduce the notations involved:

Definition 4.1. (a) A digraph means a pair $(V, A)$, where $V$ is a set, and where $A$ is a subset of $V^2$. Digraphs are also called directed graphs. A digraph $(V, A)$ is said to be finite if the set $V$ is finite. If $D = (V, A)$ is a digraph, then the elements of $V$ are called the vertices of the digraph $D$, while the elements of $A$ are called the arcs (or the directed edges) of the digraph $D$. If $a = (v, w)$ is an arc of a digraph $D$, then $v$ is called the source of $a$, whereas $w$ is called the target of $a$.
(b) A digraph \((V, A)\) is said to be loopless if every \(v \in V\) satisfies \((v, v) \notin A\). (In other words, a digraph is loopless if and only if it has no arc whose source and target are identical.)

(c) A digraph \((V, A)\) is said to be transitive if it has the following property: For any \(u \in V\), \(v \in V\) and \(w \in V\) satisfying \((u, v) \in A\) and \((v, w) \in A\), we have \((u, w) \in A\).

(d) A digraph \((V, A)\) is said to be 2-path-free if there exist no three elements \(u, v\) and \(w\) of \(V\) satisfying \((u, v) \in A\) and \((v, w) \in A\).

(e) Let \(D = (V, A)\) be a loopless digraph. Define a map set : \(A \to \binom{V}{2}\) by setting
\[
\{v, w\} \quad \text{for every} \quad (v, w) \in A.
\]

(It is easy to see that set is well-defined, because \((V, A)\) is loopless.) The graph \((V, \text{set } A)\) will be denoted by \(\mathcal{D}\).

We can now state our application of Corollary 3.6, answering a question suggested by Alexander Postnikov:

**Proposition 4.2.** Let \(D = (V, A)\) be a finite transitive loopless digraph. Then,
\[
\chi_D = \sum_{\text{the digraph } (V,F) \text{ is 2-path-free}} (-1)^{|F|} \chi_{\text{conn}(V,\text{set } F)}.
\]

**Proof of Proposition 4.2.** Let \(E = \text{set } A\). Then, the definition of \(\mathcal{D}\) yields \(\mathcal{D} = \left(\begin{array}{c} V \\ \text{set } A \\ =E \end{array}\right) = (V, E)\).

The map set : \(A \to \binom{V}{2}\) (which sends every arc \((v, w) \in A\) to \(\{v, w\} \in \binom{V}{2}\)) restricts to a surjection \(A \to E\) (since \(E = \text{set } A\)). Let us denote this surjection by \(\pi\). Thus, \(\pi\) is a map from \(A\) to \(E\) sending each arc \((v, w) \in A\) to \(\{v, w\} \in E\). We shall soon see that \(\pi\) is a bijection.

We define a partial order on the set \(V\) as follows: For \(i \in V\) and \(j \in V\), we set \(i < j\) if and only if \((i, j) \in A\) (that is, if and only if there is an arc from \(i\) to \(j\) in \(D\)). This is a well-defined partial order[14] Thus, \(V\) becomes a poset. For every \(i \in V\) and \(j \in V\) satisfying \(i \leq j\), we let \([i, j]\) denote the interval \(\{k \in V \mid i \leq k \leq j\}\) of the poset \(V\).

There exist no \(i, j \in V\) such that both \((i, j)\) and \((j, i)\) belong to \(A\) (because if such \(i\) and \(j\) would exist, then they would satisfy \(i < j\) and \(j \leq i\), but this would contradict the fact that \(V\) is a poset). Hence, the projection \(\pi : A \to E\) is injective,

[14]Indeed, the relation < that we have just defined is transitive (since the digraph \((V, A)\) is transitive) and antisymmetric (since the digraph \((V, A)\) is loopless).
and thus bijective (since we already know that $\pi$ is surjective). Hence, its inverse map $\pi^{-1} : E \to A$ is well-defined. For every subset $F$ of $E$, we have

$$F = \pi \left( \pi^{-1} (F) \right) \quad \text{(since $\pi$ is bijective)}$$

$$= \text{set} \left( \pi^{-1} (F) \right)$$

(since $\pi$ is a restriction of the map set).

For any $(u, v) \in A$ and any subset $F$ of $E$, we have the following logical equivalence:

$$\left\{ (u, v) \in F \right\} \iff \left( (u, v) \in \pi^{-1} (F) \right)$$

Define a function $\ell' : A \to \mathbb{N}$ by

$$\ell' (i, j) = ||i, j|| \quad \text{for all} \ (i, j) \in A.$$  

Define a function $\ell : E \to \mathbb{N}$ by $\ell = \ell' \circ \pi^{-1}$. Thus, $\ell \circ \pi = \ell'$. Therefore,

$$\ell \left( \left\{ i, j \right\} \right) = (\ell \circ \pi) (i, j) = \ell' (i, j) = ||i, j||$$

for all $(i, j) \in A$.

Let $\mathcal{R}$ be the set

$$\left\{ \left\{ i, k \right\}, \left\{ k, j \right\} \mid (i, k) \in A \text{ and } (k, j) \in A \right\}.$$  

Each $K \in \mathcal{R}$ is a broken circuit of $D$.  

Proof of (16): Let $(u, v) \in A$, and let $F$ be a subset of $E$. We need to prove the equivalence (16).

From $(u, v) \in A$, we see that $\pi (u, v)$ is well-defined. The definition of $\pi$ shows that $\pi (u, v) = (u, v)$. Hence, we have the following chain of equivalences:

$$\left\{ (u, v) \in F \right\} \iff (\pi (u, v) \in F) \iff (u, v) \in \pi^{-1} (F).$$

This proves (16).

Proof. Let $K \in \mathcal{R}$. Then, $K = \left\{ \left\{ i, k \right\}, \left\{ k, j \right\} \right\}$ for some $(i, k) \in A$ and $(k, j) \in A$ (by the definition of $\mathcal{R}$). Consider these $(i, k)$ and $(k, j)$. Since $(V, A)$ is transitive, we have $(i, j) \in A$. Thus, $\left\{ i, k \right\}, \left\{ k, j \right\}$ and $\left\{ i, j \right\}$ are edges of $D$. These edges form a circuit of $D$. In particular, $i, j$ and $k$ are pairwise distinct.

Applications of (17) yield $\ell \left( \left\{ i, j \right\} \right) = ||i, j||$, $\ell \left( \left\{ i, k \right\} \right) = ||i, k||$ and $\ell \left( \left\{ k, j \right\} \right) = ||k, j||$.

But we have $i < k$ (since $(i, k) \in A$) and $k < j$ (since $(k, j) \in A$). Hence, $||i, k||$ is a proper subset of $||i, j||$. (It is proper because it does not contain $j$, whereas $||i, j||$ does.) Hence, $||i, k|| < ||i, j||$. Thus, $\ell \left( \left\{ i, j \right\} \right) = ||i, j|| > ||i, k|| = \ell \left( \left\{ i, k \right\} \right)$. Similarly, $\ell \left( \left\{ i, j \right\} \right) > \ell \left( \left\{ k, j \right\} \right)$. The last
A subset $F$ of $E$ is $\mathcal{K}$-free if and only if the digraph $(V, \pi^{-1}(F))$ is 2-path-free.\(^{17}\)

\(^{17}\)Proof. Let $F$ be a subset of $E$. Then, we have the following equivalence of statements:

\begin{align*}
\text{($F$ is $\mathcal{K}$-free)} & \iff (\{\{i, k\}, \{k, j\}\} \not\subseteq F \text{ whenever } (i, k) \in A \text{ and } (k, j) \in A) \\
& \quad \text{(by the definition of $\mathcal{K}$)} \\
& \iff (\text{no } (i, k) \in A \text{ and } (k, j) \in A \text{ satisfy } \{\{i, k\}, \{k, j\}\} \subseteq F) \\
& \iff (\text{no } (i, k) \in A \text{ and } (k, j) \in A \text{ satisfy } \{i, k\} \in F \text{ and } \{k, j\} \in F) \\
& \iff (\text{no } (i, k) \in A \text{ and } (k, j) \in A \text{ satisfy } (i, k) \in \pi^{-1}(F) \text{ and } \{k, j\} \in F) \\
& \quad \text{(because for } (i, k) \in A \text{, we have } \{i, k\} \in F \text{ if and only if } (i, k) \in \pi^{-1}(F) \text{)} \\
& \quad \text{(by \text{\ref{16}}, applied to } u = i \text{ and } v = k) \\
& \iff (\text{no } (i, k) \in A \text{ and } (k, j) \in A \text{ satisfy } (i, k) \in \pi^{-1}(F) \text{ and } (k, j) \in \pi^{-1}(F)) \\
& \quad \text{(because for } (k, j) \in A \text{, we have } \{k, j\} \in F \text{ if and only if } (k, j) \in \pi^{-1}(F) \text{)} \\
& \quad \text{(by \text{\ref{16}}, applied to } u = k \text{ and } v = j) \\
& \iff (\text{the digraph } (V, \pi^{-1}(F)) \text{ is 2-path-free}) \quad \text{(by the definition of “2-path-free”),}
\end{align*}

\text{qed.}
Now, Corollary 3.6 (applied to $X = N$ and $G = D$) shows that

$$\chi_D = \sum_{F \subseteq E; \ F \text{ is } \mathcal{K} \text{-free}} (-1)^{|F|} \chi_{\text{conn}(V,F)}$$

(since $\pi$ is bijective)

$$= (-1)^{|\pi^{-1}(F)|} \chi_{\text{conn}(V,\pi^{-1}(F))}$$

(by (15))

the digraph $(V,\pi^{-1}(F))$ is 2-path-free

and we have just shown that a subset $F$ of $E$ is $\mathcal{K}$-free if and only if the digraph $(V,\pi^{-1}(F))$ is 2-path-free

$$= \sum_{B \subseteq A; \ (V,B) \text{ is 2-path-free}} (-1)^{|B|} \chi_{\text{conn}(V,B)}$$

here, we have substituted $B$ for $\pi^{-1}(F)$ in the sum, since the map $\pi : A \to E$ is bijective and thus induces a bijection from the subsets of $E$ to the subsets of $A$

sending each $F \subseteq E$ to $\pi^{-1}(F)$

$$= \sum_{F \subseteq A; \ (V,F) \text{ is 2-path-free}} (-1)^{|F|} \chi_{\text{conn}(V,F)}$$

(here, we have renamed the summation index $B$ as $F$). This proves Proposition 4.2.

5. A matroidal generalization

5.1. An introduction to matroids

We shall now present a result that can be considered as a generalization of Theorem 3.5 in a different direction than Theorem 1.11: namely, a formula for the characteristic polynomial of a matroid. Let us first recall the basic notions from the theory of matroids that will be needed to state it.

First, we introduce some basic poset-related terminology:

**Definition 5.1.** Let $P$ be a poset.

(a) An element $\nu$ of $P$ is said to be maximal (with respect to $P$) if and only if every $w \in P$ satisfying $w \geq \nu$ must satisfy $w = \nu$.

(b) An element $\nu$ of $P$ is said to be minimal (with respect to $P$) if and only if every $w \in P$ satisfying $w \leq \nu$ must satisfy $w = \nu$.
Definition 5.2. For any set $E$, we shall regard the powerset $\mathcal{P}(E)$ as a poset (with respect to inclusion). Thus, any subset $\mathcal{S}$ of $\mathcal{P}(E)$ also becomes a poset, and therefore the notions of “minimal” and “maximal” elements in $\mathcal{S}$ make sense. Beware that these notions are not related to size; i.e., a maximal element of $\mathcal{S}$ might not be a maximum-size element of $\mathcal{S}$.

Now, let us define the notion of “matroid” that we will use:

Definition 5.3. (a) A matroid means a pair $(E, \mathcal{I})$ consisting of a finite set $E$ and a set $\mathcal{I} \subseteq \mathcal{P}(E)$ satisfying the following axioms:

- **Matroid axiom 1:** We have $\emptyset \in \mathcal{I}$.

- **Matroid axiom 2:** If $Y \in \mathcal{I}$ and $Z \in \mathcal{P}(E)$ are such that $Z \subseteq Y$, then $Z \in \mathcal{I}$.

- **Matroid axiom 3:** If $Y \in \mathcal{I}$ and $Z \in \mathcal{I}$ are such that $|Y| < |Z|$, then there exists some $x \in Z \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

(b) Let $(E, \mathcal{I})$ be a matroid. A subset $S$ of $E$ is said to be independent (for this matroid) if and only if $S \in \mathcal{I}$. The set $E$ is called the ground set of the matroid $(E, \mathcal{I})$.

Different texts give different definitions of a matroid; these definitions are (mostly) equivalent, but not always in the obvious way. \(^{18}\) Definition 5.3 is how a matroid is defined in [Schrij13, §10.1] and in [Martin15, Definition 3.15] (where it is called a “(matroid) independence system”). There exist other definitions of a matroid, which turn out to be equivalent. The definition of a matroid given in Stanley’s [Stanley06, Definition 3.8] is directly equivalent to Definition 5.3 with the only differences that

- Stanley replaces Matroid axiom 1 by the requirement that $\mathcal{I} \neq \emptyset$ (which is, of course, equivalent to Matroid axiom 1 as long as Matroid axiom 2 is assumed), and

- Stanley replaces Matroid axiom 3 by the requirement that for every $T \in \mathcal{P}(E)$, all maximal elements of $\mathcal{I} \cap \mathcal{P}(T)$ have the same cardinality.\(^{19}\) (this requirement is equivalent to Matroid axiom 3 as long as Matroid axiom 2 is assumed).

\(^{18}\)I.e., it sometimes happens that two different texts both define a matroid as a pair $(E, U)$ of a finite set $E$ and a subset $U \subseteq \mathcal{P}(E)$, but they require these pairs $(E, U)$ to satisfy non-equivalent axioms, and the equivalence between their definitions is more complicated than just “a pair $(E, U)$ is a matroid for one definition if and only if it is a matroid for the other”.

\(^{19}\)Here, as we have already explained, we regard $\mathcal{I} \cap \mathcal{P}(T)$ as a poset with respect to inclusion. Thus, an element $Y$ of this poset is maximal if and only if there exists no $Z \in \mathcal{I} \cap \mathcal{P}(T)$ such that $Y$ is a proper subset of $Z$.  

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We now introduce some terminology related to matroids:

**Definition 5.4.** Let $M = (E, I)$ be a matroid.

(a) We define a function $r_M : \mathcal{P}(E) \to \mathbb{N}$ by setting

$$\forall S \subseteq E, \quad r_M(S) = \max \{|Z| \mid Z \in I \text{ and } Z \subseteq S\} \quad (18)$$

(Note that the right hand side of (18) is well-defined, because there exists at least one $Z \in I$ satisfying $Z \subseteq S$ (namely, $Z = \emptyset$).) If $S$ is a subset of $E$, then the nonnegative integer $r_M(S)$ is called the rank of $S$ (with respect to $M$). It is clear that $r_M$ is a weakly increasing function from the poset $\mathcal{P}(E)$ to $\mathbb{N}$.

(b) If $k \in \mathbb{N}$, then a $k$-flat of $M$ means a subset of $E$ which has rank $k$ and is maximal among all such subsets (i.e., it is not a proper subset of any other subset having rank $k$). (Beware: Not all $k$-flats have the same size.) A flat of $M$ is a subset of $E$ which is a $k$-flat for some $k \in \mathbb{N}$. We let Flats $M$ denote the set of all flats of $M$; thus, Flats $M$ is a subposet of $\mathcal{P}(E)$.

(c) A circuit of $M$ means a minimal element of $\mathcal{P}(E) \setminus I$. (That is, a circuit of $M$ means a subset of $E$ which is not independent (for $M$) and which is minimal among such subsets.)

(d) An element $e$ of $E$ is said to be a loop (of $M$) if $\{e\} \notin I$. The matroid $M$ is said to be loopless if no loops (of $M$) exist.

Notice that the function that we called $r_M$ in Definition 5.4 (a) is denoted by $\text{rk}$ in Stanley’s [Stanley06, Lecture 3].

One of the most classical examples of a matroid is the graphical matroid of a graph:

**Example 5.5.** Let $G = (V, E)$ be a finite graph. Define a subset $I$ of $\mathcal{P}(E)$ by

$$I = \{T \in \mathcal{P}(E) \mid T \text{ contains no circuit of } G \text{ as a subset} \}.$$ 

Then, $(E, I)$ is a matroid; it is called the graphical matroid (or the cycle matroid) of $G$. It has the following properties:

- The matroid $(E, I)$ is loopless.
- For each $T \in \mathcal{P}(E)$, we have

$$r_{(E,I)}(T) = |V| - \text{conn}(V, T)$$

(where $\text{conn}(V, T)$ is defined as in Definition 3.3).
- The circuits of the matroid $(E, I)$ are precisely the circuits of the graph $G$. 

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The flats of the matroid \((E, \mathcal{I})\) are related to colorings of \(G\). More precisely: For each set \(X\) and each \(X\)-coloring \(f\) of \(G\), the set \(E \cap \text{Eqs}\ f\) is a flat of \((E, \mathcal{I})\). Every flat of \((E, \mathcal{I})\) can be obtained in this way when \(X\) is chosen large enough; but often, several distinct \(X\)-colorings \(f\) lead to one and the same flat \(E \cap \text{Eqs}\ f\).

We recall three basic facts that are used countless times in arguing about matroids:

**Lemma 5.6.** Let \(M = (E, \mathcal{I})\) be a matroid. Let \(T \in \mathcal{I}\). Then, \(r_M(T) = |T|\).

**Proof of Lemma 5.6.** We have \(T \in \mathcal{I}\) and \(T \subseteq T\). Thus, \(T\) is a \(Z \in \mathcal{I}\) satisfying \(Z \subseteq T\). Therefore, \(|T| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}\), so that

\[
|T| \leq \max \{ |Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}
\]

(since any element of a set of integers is smaller or equal to the maximum of this set).

On the other hand, the definition of \(r_M\) yields

\[
r_M(T) = \max \{ |Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}.
\]

Hence, (19) rewrites as follows:

\[
|T| \leq r_M(T).
\]

Also,

\[
r_M(T) = \max \{ |Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \quad \text{(by the definition of } r_M)\]

\[
\in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}
\]

(since the maximum of any set belongs to this set). Thus, there exists a \(Z \in \mathcal{I}\) satisfying \(Z \subseteq T\) and \(r_M(T) = |Z|\). Consider this \(Z\). From \(Z \subseteq T\), we obtain \(|Z| \leq |T|\), so that \(r_M(T) = |Z| \leq |T|\). Combining this with \(|T| \leq r_M(T)\), we obtain \(r_M(T) = |T|\). This proves Lemma 5.6.

**Lemma 5.7.** Let \(M = (E, \mathcal{I})\) be a matroid. Let \(Q \in \mathcal{P}(E) \setminus \mathcal{I}\). Then, there exists a circuit \(C\) of \(M\) such that \(C \subseteq Q\).

**Proof of Lemma 5.7.** We have \(Q \in \mathcal{P}(E) \setminus \mathcal{I}\). Thus, there exists at least one \(C \in \mathcal{P}(E) \setminus \mathcal{I}\) such that \(C \subseteq Q\) (namely, \(C = Q\)). Thus, there also exists a minimal such \(C\). Consider this minimal \(C\). We know that \(C\) is a minimal element of \(\mathcal{P}(E) \setminus \mathcal{I}\) such that \(C \subseteq Q\). In other words, \(C\) is an element of \(\mathcal{P}(E) \setminus \mathcal{I}\) satisfying \(C \subseteq Q\), and moreover,

\[
every \ D \in \mathcal{P}(E) \setminus \mathcal{I} \text{ satisfying } D \subseteq Q \text{ and } D \subseteq C \ \text{must satisfy } D = C.
\]

(20)
Thus, $C$ is a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$. In other words, $C$ is a circuit of $M$ (by the definition of a “circuit”). This circuit $C$ satisfies $C \subseteq Q$. Thus, we have constructed a circuit $C$ of $M$ satisfying $C \subseteq Q$. Lemma 5.7 is thus proven. □

Lemma 5.8. Let $M = \langle E, \mathcal{I} \rangle$ be a matroid. Let $T$ be a subset of $E$. Let $S \in \mathcal{I}$ be such that $S \subseteq T$. Then, there exists an $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$ and $|S'| = r_M(T)$.

Proof of Lemma 5.8: Clearly, there exists at least one $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$ (namely, $S' = S$). Hence, there exists a maximal such $S'$. Let $Q$ be such a maximal $S'$. Thus, $Q$ is an element of $\mathcal{I}$ satisfying $S \subseteq Q \subseteq T$.

Recall that

$$r_M(T) = \max \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}$$

(by the definition of $r_M$)

(since the maximum of any set must belong to this set). Hence, there exists some $Z \in \mathcal{I}$ satisfying $Z \subseteq T$ and $r_M(T) = |Z|$. Denote such a $Z$ by $W$. Thus, $W$ is an element of $\mathcal{I}$ satisfying $W \subseteq T$ and $r_M(T) = |W|$.

We have $|Q| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}$ (since $Q \in \mathcal{I}$ and $Q \subseteq T$). Since any element of a set is smaller or equal to the maximum of this set, this entails that $|Q| \leq \max \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} = r_M(T) = |W|$.

Now, assume (for the sake of contradiction) that $|Q| \neq |W|$. Thus, $|Q| < |W|$ (since $|Q| \leq |W|$). Hence, Matroid axiom 3 (applied to $Y = Q$ and $Z = W$) shows that there exists some $x \in W \setminus Q$ such that $Q \cup \{x\} \in \mathcal{I}$. Consider this $x$. We have $x \in W \setminus Q \subseteq W \subseteq T$, so that $Q \cup \{x\} \subseteq T$ (since $Q \subseteq T$). Also, $x \notin Q$ (since $x \in W \setminus Q$).

Recall that $Q$ is a maximal $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$. Thus, if some $S' \in \mathcal{I}$ satisfies $S \subseteq S' \subseteq T$ and $S' \supseteq Q$, then $S' = Q$. Applying this to $S' = Q \cup \{x\}$, we obtain $Q \cup \{x\} = Q$ (since $S \subseteq Q \subseteq Q \cup \{x\} \subseteq T$ and $Q \cup \{x\} \supseteq Q$). Thus, $x \in Q$. But this contradicts $x \notin Q$. This contradiction shows that our assumption (that $|Q| \neq |W|$) was wrong. Hence, $|Q| = |W| = r_M(T)$. Thus, there exists an $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$ and $|S'| = r_M(T)$ (namely, $S' = Q$). This proves Lemma 5.8. □

5.2. The lattice of flats

We shall now show a lemma that can be regarded as an alternative criterion for a subset of $E$ to be a flat:

---

20Proof. We need to show that every $D \in \mathcal{P}(E) \setminus \mathcal{I}$ satisfying $D \subseteq C$ must satisfy $D = C$ (since we already know that $C \in \mathcal{P}(E) \setminus \mathcal{I}$).

So let $D \in \mathcal{P}(E) \setminus \mathcal{I}$ be such that $D \subseteq C$. Then, $D \subseteq C \subseteq Q$. Hence, 20 shows that $D = C$. This completes our proof.
Lemma 5.9. Let \( M = (E, I) \) be a matroid. Let \( T \) be a subset of \( E \). Then, the following statements are equivalent:

- **Statement \( \exists_1 \):** The set \( T \) is a flat of \( M \).
- **Statement \( \exists_2 \):** If \( C \) is a circuit of \( M \), and if \( e \in C \) is such that \( C \setminus \{e\} \subseteq T \), then \( C \subseteq T \).

**Proof of Lemma 5.9.** Proof of the implication \( \exists_1 \implies \exists_2 \): Assume that Statement \( \exists_1 \) holds. We must prove that Statement \( \exists_2 \) holds.

Let \( C \) be a circuit of \( M \). Let \( e \in C \) be such that \( C \setminus \{e\} \subseteq T \). We must prove that \( C \subseteq T \).

Assume the contrary. Thus, \( C \not\subseteq T \). Combining this with \( C \setminus \{e\} \subseteq T \), we obtain \( e \not\in T \). Hence, \( T \) is a proper subset of \( T \cup \{e\} \).

We have assumed that Statement \( \exists_1 \) holds. In other words, the set \( T \) is a flat of \( M \). In other words, there exists some \( k \in \mathbb{N} \) such that \( T \) is a \( k \)-flat of \( M \). Consider this \( k \).

The set \( T \) is a \( k \)-flat of \( M \), thus a subset of \( E \) which has rank \( k \) and is maximal among all such subsets. In other words, \( r_M(T) = k \), but every subset \( S \) of \( E \) for which \( T \) is a proper subset of \( S \) must satisfy

\[
r_M(S) \neq k. \tag{21}
\]

Applying (21) to \( S = T \cup \{e\} \), we obtain \( r_M(T \cup \{e\}) \neq k \). Since \( T \cup \{e\} \supseteq T \) (and since the function \( r_M : \mathcal{P}(E) \to \mathbb{N} \) is weakly increasing), we have \( r_M(T \cup \{e\}) \geq r_M(T) = k \). Combined with \( r_M(T \cup \{e\}) \neq k \), this yields \( r_M(T \cup \{e\}) > k \). Thus, \( r_M(T \cup \{e\}) \geq k + 1 \).

Notice that \( C \setminus \{e\} \) is a proper subset of \( C \) (since \( e \in C \)). The set \( C \) is a circuit of \( M \), thus a minimal element of \( \mathcal{P}(E) \setminus I \) (by the definition of a “circuit”). Hence, no proper subset of \( C \) belongs to \( \mathcal{P}(E) \setminus I \) (because \( C \) is minimal). In other words, every proper subset of \( C \) belongs to \( I \). Applying this to the proper subset \( C \setminus \{e\} \) of \( C \), we conclude that \( C \setminus \{e\} \) belongs to \( I \). Hence, Lemma 5.8 (applied to \( S = C \setminus \{e\} \)) shows that there exists an \( S' \in I \) satisfying \( C \setminus \{e\} \subseteq S' \subseteq T \) and \(|S'| = r_M(T)\). Denote this \( S' \) by \( S \). Thus, \( S \) is an element of \( I \) satisfying \( C \setminus \{e\} \subseteq S \subseteq T \) and \(|S| = r_M(T)\).

Furthermore, \( S \subseteq T \subseteq T \cup \{e\} \). Thus, Lemma 5.8 (applied to \( T \cup \{e\} \) instead of \( T \)) shows that there exists an \( S' \in I \) satisfying \( S \subseteq S' \subseteq T \cup \{e\} \) and \(|S'| = r_M(T \cup \{e\})\). Consider this \( S' \).

We have \(|S'| = r_M(T \cup \{e\}) > r_M(T)\). Hence, \( S' \not\subseteq T \). Combining this with \( S' \subseteq T \cup \{e\} \), we obtain \( e \in S' \). Combining this with \( C \setminus \{e\} \subseteq S' \), we find

\[
r_M(T) = \max\{|Z| \mid Z \in I \text{ and } Z \subseteq T\} \geq |S'|
\]

(since \(|S'| \in \{|Z| \mid Z \in I \text{ and } Z \subseteq T\}\)). This contradicts \(|S'| > r_M(T)\). This contradiction proves that our assumption was wrong, qed.
that \((C \setminus \{e\}) \cup \{e\} \subseteq S'\). Thus, \(C = (C \setminus \{e\}) \cup \{e\} \subseteq S'\). Since \(S' \in \mathcal{I}\), this entails that \(C \in \mathcal{I}\) (by Matroid axiom 2). But \(C \in \mathcal{P}(E) \setminus \mathcal{I}\) (since \(C\) is a minimal element of \(\mathcal{P}(E) \setminus \mathcal{I}\), so that \(C \notin \mathcal{I}\). This contradicts \(C \in \mathcal{I}\). This contradiction shows that our assumption was wrong. Hence, \(C \subseteq T\) is proven. Therefore, Statement \(\mathcal{F}_2\) holds. Thus, the implication \(\mathcal{F}_1 \implies \mathcal{F}_2\) is proven.

Proof of the implication \(\mathcal{F}_2 \implies \mathcal{F}_1\): Assume that Statement \(\mathcal{F}_2\) holds. We must prove that Statement \(\mathcal{F}_1\) holds.

Let \(k = r_M(T)\). We shall show that \(T\) is a \(k\)-flat of \(M\).

Let \(W\) be a subset of \(E\) which has rank \(k\) and satisfies \(T \subseteq W\). We shall show that \(T = W\).

Indeed, assume the contrary. Thus, \(T \neq W\). Combined with \(T \subseteq W\), this shows that \(T\) is a proper subset of \(W\). Thus, there exists an \(e \in W \setminus T\). Consider this \(e\). We have \(e \notin T\) (since \(e \in W \setminus T\)).

We have

\[
k = r_M(T) = \max \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}
\]

(by the definition of \(r_M\))

\[
\in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}
\]

(since the maximum of a set must belong to that set). Hence, there exists some \(Z \in \mathcal{I}\) satisfying \(Z \subseteq T\) and \(k = |Z|\). Denote this \(Z\) by \(K\). Thus, \(K\) is an element of \(\mathcal{I}\) and satisfies \(K \subseteq T\) and \(k = |K|\). Notice that \(e \notin T\), so that \(e \notin K\) (since \(K \subseteq T\)).

We have \(r_M(W) = k\) (since \(W\) has rank \(k\)). Hence, \(K \cup \{e\} \notin \mathcal{I}\). In other words, \(K \cup \{e\} \in \mathcal{P}(E) \setminus \mathcal{I}\). Hence, Lemma 5.7 (applied to \(Q = K \cup \{e\}\)) shows that there exists a circuit \(C\) of \(M\) such that \(C \subseteq K \cup \{e\}\). Consider this \(C\). From \(C \subseteq K \cup \{e\}\), we obtain \(C \setminus \{e\} \subseteq K \subseteq T\).

From \(C \setminus \{e\} \subseteq K\), we conclude (using Matroid axiom 2) that \(C \setminus \{e\} \in \mathcal{I}\) (since \(K \in \mathcal{I}\)). On the other hand, \(C\) is a circuit of \(M\). In other words, \(C\) is a minimal element of \(\mathcal{P}(E) \setminus \mathcal{I}\) (by the definition of a “circuit”). Hence, \(C \in \mathcal{P}(E) \setminus \mathcal{I}\), so that \(C \notin \mathcal{I}\). Hence, \(e \in C\) (since otherwise, we would have \(C \setminus \{e\} = C \notin \mathcal{I}\), which would contradict \(C \setminus \{e\} \in \mathcal{I}\)). Now, Statement \(\mathcal{F}_2\) shows that \(C \subseteq T\). Hence, \(e \in C \subseteq T\), which contradicts \(e \notin T\).

This contradiction shows that our assumption was wrong. Hence, \(T = W\) is proven.

Now, forget that we fixed \(W\). Thus, we have shown that if \(W\) is a subset of \(E\) which has rank \(k\) and satisfies \(T \subseteq W\), then \(T = W\). In other words, \(T\) is a subset of \(E\) which has rank \(k\) and is maximal among all such subsets (because we already know that \(T\) has rank \(r_M(T) = k\)). In other words, \(T\) is a \(k\)-flat of \(M\) (by the definition of a “\(k\)-flat”). Thus, \(T\) is a flat of \(M\). In other words, Statement \(\mathcal{F}_1\) holds. This proves the implication \(\mathcal{F}_2 \implies \mathcal{F}_1\).

\[\text{Proof:}\] Assume the contrary. Thus, \(K \cup \{e\} \in \mathcal{I}\). Thus, \(r_M(K \cup \{e\}) = |K \cup \{e\}|\) (by Lemma 5.6). Thus, \(r_M(K \cup \{e\}) = |K \cup \{e\}| > |K|\) (since \(e \notin K\)).

But \(K \cup \{e\} \subseteq W\) (since \(K \subseteq T \subseteq W\) and \(e \in W \setminus T \subseteq W\)). Since the function \(r_M\) is weakly increasing, this yields \(r_M(K \cup \{e\}) \leq r_M(W) = k = |K|\). This contradicts \(r_M(K \cup \{e\}) > |K|\). This contradiction proves that our assumption was wrong.

qed.
Lemma 5.9. Let $\mathcal{F} = (E, \mathcal{I})$ be a matroid. Let $F_1, F_2, \ldots, F_k$ be flats of $\mathcal{F}$. Then, $F_1 \cap F_2 \cap \cdots \cap F_k$ is a flat of $\mathcal{F}$ of $\mathcal{F}$. (Notice that $k$ is allowed to be 0 here; in this case, the empty intersection $F_1 \cap F_2 \cap \cdots \cap F_k$ is to be interpreted as $E$.)

Proof of Lemma 5.9: Lemma 5.9 gives a necessary and sufficient criterion for a subset $T$ of $E$ to be a flat of $\mathcal{F}$. It is easy to see that if this criterion is satisfied for $T = F_1$, for $T = F_2$, etc., and for $T = F_k$, then it is satisfied for $T = F_1 \cap F_2 \cap \cdots \cap F_k$. In other words, if $F_1, F_2, \ldots, F_k$ are flats of $\mathcal{F}$, then $F_1 \cap F_2 \cap \cdots \cap F_k$ is a flat of $\mathcal{F}$. This proves Corollary 5.10. □

Corollary 5.10 (a well-known fact, which is left to the reader to prove in [Stanley06 §3.1]) allows us to define the closure of a set in a matroid:

Definition 5.11. Let $\mathcal{F} = (E, \mathcal{I})$ be a matroid. Let $T$ be a subset of $E$. The closure of $T$ is defined to be the intersection of all flats of $\mathcal{F}$ which contain $T$ as a subset. In other words, the closure of $T$ is defined to be $\bigcap_{T \subseteq F} F$. The closure of $T$ is denoted by $\overline{T}$.

The following proposition gathers some simple properties of closures in matroids:

Proposition 5.12. Let $\mathcal{F} = (E, \mathcal{I})$ be a matroid.

(a) If $T$ is a subset of $E$, then $\overline{T}$ is a flat of $\mathcal{F}$ satisfying $T \subseteq \overline{T}$.

(b) If $G$ is a flat of $\mathcal{F}$, then $\overline{G} = G$.

Here is an argument in slightly more detail:

For every $i \in \{1, 2, \ldots, k\}$, the following statement holds: If $C$ is a circuit of $\mathcal{F}$, and if $e \in C$ is such that $C \setminus \{e\} \subseteq F_i$, then

$$C \subseteq F_i.$$ (22)

Proof of (22): Let $i \in \{1, 2, \ldots, k\}$. Then, the set $F_i$ is a flat of $\mathcal{F}$. In other words, Statement $\overline{3}_1$ of Lemma 5.9 is satisfied for $T = F_i$. Therefore, Statement $\overline{3}_2$ of Lemma 5.9 must also be satisfied for $T = F_i$ (since Lemma 5.9 shows that the Statements $\overline{3}_1$ and $\overline{3}_2$ are equivalent). In other words, if $C$ is a circuit of $\mathcal{F}$, and if $e \in C$ is such that $C \setminus \{e\} \subseteq F_i$, then $C \subseteq F_i$. This proves (22).

Now, let $C$ be a circuit of $\mathcal{F}$, and let $e \in C$ be such that $C \setminus \{e\} \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$. For every $i \in \{1, 2, \ldots, k\}$, we have $C \setminus \{e\} \subseteq F_1 \cap F_2 \cap \cdots \cap F_k \subseteq F_i$, and therefore $C \subseteq F_i$ (by (22)). So we have shown the inclusion $C \subseteq F_i$ for each $i \in \{1, 2, \ldots, k\}$. Combining these $k$ inclusions, we obtain $C \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$.

Now, forget that we fixed $C$. We thus have shown that if $C$ is a circuit of $\mathcal{F}$, and if $e \in C$ is such that $C \setminus \{e\} \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$, then $C \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$. In other words, Statement $\overline{3}_2$ of Lemma 5.9 is satisfied for $T = F_1 \cap F_2 \cap \cdots \cap F_k$. Therefore, Statement $\overline{3}_1$ of Lemma 5.9 must also be satisfied for $T = F_1 \cap F_2 \cap \cdots \cap F_k$ (since Lemma 5.9 shows that the Statements $\overline{3}_1$ and $\overline{3}_2$ are equivalent). In other words, the set $F_1 \cap F_2 \cap \cdots \cap F_k$ is a flat of $\mathcal{F}$.

Qed.
(c) If $T$ is a subset of $E$ and if $G$ is a flat of $M$ satisfying $T \subseteq G$, then $\overline{T} \subseteq G$.

(d) If $S$ and $T$ are two subsets of $E$ satisfying $S \subseteq T$, then $\overline{S} \subseteq \overline{T}$.

(e) If the matroid $M$ is loopless, then $\emptyset = \emptyset$.

(f) Every subset $T$ of $E$ satisfies $r_M (T) = r_M (\overline{\emptyset})$.

(g) If $T$ is a subset of $E$ and if $G$ is a flat of $M$, then the conditions ($\overline{T} \subseteq G$) and ($T \subseteq G$) are equivalent.

Proof of Proposition 5.12 (a) The set Flats $M$ is a subset of the finite set $P (E)$, and thus itself finite.

Let $T$ be a subset of $E$. The closure $\overline{T}$ of $T$ is defined as $\bigcap_{F \in \text{Flats } M; \ T \subseteq F} F$. Now, Corollary 5.10 shows that any intersection of finitely many flats of $M$ is a flat of $M$. Hence, $\bigcap_{F \in \text{Flats } M; \ T \subseteq F} F$ (being an intersection of finitely many flats of $M$) is a flat of $M$. In other words, $\overline{T}$ is a flat of $M$ (since $\overline{T} = \bigcap_{F \in \text{Flats } M; \ T \subseteq F} F$).

Also, $T \subseteq F$ for every $F \in \text{Flats } M$ satisfying $T \subseteq F$. Hence, $T \subseteq \bigcap_{F \in \text{Flats } M; \ T \subseteq F} F = \overline{T}$. This completes the proof of Proposition 5.12 (a).

(c) Let $T$ be a subset of $E$, and let $G$ be a flat of $M$ satisfying $T \subseteq G$. Then, $G$ is an element of Flats $G$ satisfying $T \subseteq G$. Hence, $G$ is one term in the intersection $\bigcap_{F \in \text{Flats } M; \ T \subseteq F} F$. But the definition of $\overline{T}$ yields $\overline{T} = \bigcap_{F \in \text{Flats } M; \ T \subseteq F} F \subseteq G$. This proves Proposition 5.12 (c).

(b) Let $G$ be a flat of $M$. Proposition 5.12 (b) (applied to $T = G$) yields $\overline{G} \subseteq G$. But Proposition 5.12 (a) (applied to $T = G$) shows that $\overline{G}$ is a flat of $M$ satisfying $G \subseteq \overline{G}$. Combining $G \subseteq \overline{G}$ with $\overline{G} \subseteq G$, we obtain $\overline{G} = G$. This proves Proposition 5.12 (b).

(d) Let $S$ and $T$ be two subsets of $E$ satisfying $S \subseteq T$. Proposition 5.12 (a) shows that $\overline{T}$ is a flat of $M$ satisfying $T \subseteq \overline{T}$. Now, $S \subseteq T \subseteq \overline{T}$. Hence, Proposition 5.12 (b) (applied to $S$ and $\overline{T}$ instead of $T$ and $G$) shows $\overline{S} \subseteq \overline{T}$. This proves Proposition 5.12 (d).

(e) Assume that the matroid $M$ is loopless. In other words, no loops (of $M$) exist.

The definition of $r_M$ quickly yields $r_M (\emptyset) = 0$. In other words, the set $\emptyset$ has rank 0. We shall now show that $\emptyset$ is a 0-flat of $M$.

Indeed, let $W$ be a subset of $E$ which has rank 0 and satisfies $\emptyset \subseteq W$. We shall show that $\emptyset = W$.

Assume the contrary. Thus, $\emptyset \neq W$. Hence, $W$ has an element $w$. Consider this $w$. The element $w$ of $E$ is not a loop (since no loops exist). In other words, we cannot have $\{w\} \notin I$ (since $w$ is a loop if and only if $\{w\} \notin I$ (by the

24“Finitely many” since the set Flats $M$ is finite.
definition of a loop). In other words, we must have \( \{w\} \in \mathcal{I} \). Clearly, \( \{w\} \subseteq W \) (since \( w \in W \)). Thus, \( \{w\} \) is a \( Z \in \mathcal{I} \) satisfying \( Z \subseteq W \). Thus, \( |\{w\}| \in \{ |Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq W \} \).

But \( W \) has rank 0. In other words,

\[
0 = r_M(W) = \max \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq W\} \quad \text{(by the definition of } r_M) \\
\geq |\{w\}| \quad \text{(since } |\{w\}| \in \{ |Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq W \}) \\
= 1,
\]

which is absurd. This contradiction shows that our assumption was wrong. Hence, \( \emptyset = W \) is proven.

Let us now forget that we fixed \( W \). We thus have proven that if \( W \) is any subset of \( E \) which has rank 0 and satisfies \( \emptyset \subseteq W \), then \( \emptyset = W \). Thus, \( \emptyset \) is a subset of \( E \) which has rank 0 and is maximal among all such subsets (because we already know that \( \emptyset \) has rank 0). In other words, \( \emptyset \) is a 0-flat of \( M \) (by the definition of a “0-flat”). Thus, \( \emptyset \) is a flat of \( M \). Thus, Proposition 5.12 (b) (applied to \( G = \emptyset \)) yields \( \emptyset = \emptyset \). This proves Proposition 5.12 (e).

(f) Let \( T \) be a subset of \( E \). We have \( T \subseteq T \) (by Proposition 5.12 (a)), and thus \( r_M(T) \leq r_M(\overline{T}) \) (since the function \( r_M \) is weakly increasing).

Let \( k = r_M(T) \). Thus, there exists a \( Q \in \mathcal{P}(E) \) satisfying \( T \subseteq Q \) and \( k = r_M(Q) \) (namely, \( Q = T \)). Hence, there exists a maximal such \( Q \). Denote this \( Q \) by \( R \). Thus, \( R \) is a maximal \( Q \in \mathcal{P}(E) \) satisfying \( T \subseteq Q \) and \( k = r_M(Q) \). In particular, \( R \) is an element of \( \mathcal{P}(E) \) and satisfies \( T \subseteq R \) and \( k = r_M(R) \).

Now, \( R \) is a subset of \( E \) (since \( R \in \mathcal{P}(E) \)) and has rank \( r_M(R) = k \). Thus, \( R \) is a subset of \( E \) which has rank \( k \). Furthermore, \( R \) is maximal among all such subsets.\(^{25}\) Thus, \( R \) is a \( k \)-flat of \( M \) (by the definition of a “\( k \)-flat”), and therefore a flat of \( M \). Now, Proposition 5.12 (c) (applied to \( G = R \)) shows that \( \overline{T} \subseteq R \). Since the function \( r_M \) is weakly increasing, this yields \( r_M(\overline{T}) \leq r_M(R) = k \). Combining this with \( k = r_M(T) \leq r_M(\overline{T}) \), we obtain \( r_M(T) = k = r_M(T) \). This proves Proposition 5.12 (f).

(g) Let \( T \) be a subset of \( E \). Let \( G \) be a flat of \( M \). Proposition 5.12 (a) shows that \( T \subseteq \overline{T} \). Hence, if \( \overline{T} \subseteq G \), then \( T \subseteq \overline{T} \subseteq G \). Thus, we have proven the implication \( (\overline{T} \subseteq G) \implies (T \subseteq G) \). The reverse implication (i.e., the implication \( (T \subseteq G) \implies (\overline{T} \subseteq G) \)) follows from Proposition 5.12 (c). Combining these two implications, we obtain the equivalence \( (\overline{T} \subseteq G) \iff (T \subseteq G) \). This proves Proposition 5.12 (g).

We shall now recall a few more classical notions related to posets:

\(^{25}\) **Proof.** Let \( W \) be any subset of \( E \) which has rank \( k \) and satisfies \( W \supseteq R \). We must prove that \( W = R \).

We have \( W \in \mathcal{P}(E) \), \( T \subseteq R \subseteq W \) and \( k = r_M(W) \) (since \( W \) has rank \( k \)). Thus, \( W \) is a \( Q \in \mathcal{P}(E) \) satisfying \( T \subseteq Q \) and \( k = r_M(Q) \). But recall that \( R \) is a maximal such \( Q \). Hence, if \( W \supseteq R \), then \( W = R \). Therefore, \( W = R \) (since we know that \( W \supseteq R \)). Qed.
**Definition 5.13.** Let $P$ be a poset.

(a) An element $p \in P$ is said to be a global minimum of $P$ if every $q \in P$ satisfies $p \leq q$. Clearly, a global minimum of $P$ is unique if it exists.

(b) An element $p \in P$ is said to be a global maximum of $P$ if every $q \in P$ satisfies $p \geq q$. Clearly, a global maximum of $P$ is unique if it exists.

(c) Let $x$ and $y$ be two elements of $P$. An upper bound of $x$ and $y$ (in $P$) means an element $z \in P$ satisfying $z \geq x$ and $z \geq y$. A join (or least upper bound) of $x$ and $y$ (in $P$) means an upper bound $z$ of $x$ and $y$ such that every upper bound $z'$ of $x$ and $y$ satisfies $z' \geq z$. In other words, a join of $x$ and $y$ is a global minimum of the subposet $\{w \in P \mid w \geq x \text{ and } w \geq y\}$ of $P$. Thus, a join of $x$ and $y$ is unique if it exists.

(d) Let $x$ and $y$ be two elements of $P$. A lower bound of $x$ and $y$ (in $P$) means an element $z \in P$ satisfying $z \leq x$ and $z \leq y$. A meet (or greatest lower bound) of $x$ and $y$ (in $P$) means a lower bound $z$ of $x$ and $y$ such that every lower bound $z'$ of $x$ and $y$ satisfies $z' \leq z$. In other words, a meet of $x$ and $y$ is a global maximum of the subposet $\{w \in P \mid w \leq x \text{ and } w \leq y\}$ of $P$. Thus, a meet of $x$ and $y$ is unique if it exists.

(e) The poset $P$ is said to be a lattice if and only if it has a global minimum and a global maximum, and every two elements of $P$ have a meet and a join.

**Proposition 5.14.** Let $M = (E, \mathcal{I})$ be a matroid. The subposet Flats $M$ of the poset $P(E)$ is a lattice.

**Proof of Proposition 5.14.** By the definition of a lattice, it suffices to check the following four claims:

Claim 1: The poset Flats $M$ has a global minimum.

Claim 2: The poset Flats $M$ has a global maximum.

Claim 3: Every two elements of Flats $M$ have a meet (in Flats $M$).

Claim 4: Every two elements of Flats $M$ have a join (in Flats $M$).

Proof of Claim 1: Applying Proposition 5.12(a) to $T = \emptyset$, we see that $\emptyset$ is a flat of $M$ satisfying $\emptyset \subseteq \emptyset$. In particular, $\emptyset$ is a flat of $M$, so that $\emptyset \in \text{Flats } M$. If $G$ is a flat of $M$, then $\emptyset \subseteq G$ (by Proposition 5.12(c), applied to $T = \emptyset$). Hence, $\emptyset$ is a global minimum of the poset Flats $M$. Thus, the poset Flats $M$ has a global minimum. This proves Claim 1.

Proof of Claim 2: Applying Proposition 5.12(a) to $T = E$, we see that $\overline{E}$ is a flat of $M$ satisfying $E \subseteq \overline{E}$. From $E \subseteq \overline{E}$, we conclude that $\overline{E} = E$. Thus, $E$ is a flat of $M$ (since $\overline{E}$ is a flat of $M$). In other words, $E \in \text{Flats } M$. If $G$ is a flat of $M$, then $E \supseteq G$ (obviously). Hence, $E$ is a global maximum of the poset Flats $M$. Thus, the poset Flats $M$ has a global maximum. This proves Claim 2.

Proof of Claim 3: Let $F$ and $G$ be two elements of Flats $M$. We have to prove that $F$ and $G$ have a meet.

We know that $F$ and $G$ are elements of Flats $M$, thus flats of $M$. Hence, Corollary 5.10 shows that $F \cap G$ is a flat of $M$. In other words, $F \cap G \in \text{Flats } M$. Clearly, $F \cap G \subseteq F$ and $F \cap G \subseteq G$; thus, $F \cap G$ is a lower bound of $F$ and $G$ in
Flats $M$. Also, every lower bound $H$ of $F$ and $G$ in Flats $M$ satisfies $H \subseteq F \cap G$. Hence, $F \cap G$ is a meet of $F$ and $G$. Thus, $F$ and $G$ have a meet. This proves Claim 3.

Proof of Claim 4: Let $F$ and $G$ be two elements of Flats $M$. We have to prove that $F$ and $G$ have a join.

We know that $F$ and $G$ are elements of Flats $M$, thus flats of $M$. Proposition 5.12 (applied to $T = F \cup G$) shows that $F \cup G$ is a flat of $M$ satisfying $F \cup G \subseteq F \cup G$ and $G \subseteq F \cup G \subseteq F \cup G$; thus, $F \cup G$ is an upper bound of $F$ and $G$ in Flats $M$. Also, every upper bound $H$ of $F$ and $G$ in Flats $M$ satisfies $H \supseteq F \cup G$. Hence, $F \cup G$ is a join of $F$ and $G$. Thus, $F$ and $G$ have a join. This proves Claim 4.

We have now proven all four Claims 1, 2, 3, and 4. Thus, Proposition 5.14 is proven.

Definition 5.15. Let $M = (E, \mathcal{I})$ be a matroid. Proposition 5.14 shows that the subposet Flats $M$ of the poset $\mathcal{P}(E)$ is a lattice. This subposet Flats $M$ is called the lattice of flats of $M$. (Beware: It is a subposet, but not a sublattice of $\mathcal{P}(E)$, since its join is not a restriction of the join of $\mathcal{P}(E)$.)

The lattice of flats Flats $M$ of a matroid $M$ is denoted by $L(M)$ in Stanley06 §3.2.

Next, we recall the definition of the Möbius function of a poset (see, e.g., Stanley06 Definition 1.2) or Martin15 §5.2):

Definition 5.16. Let $P$ be a poset.

(a) If $x$ and $y$ are two elements of $P$ satisfying $x \leq y$, then the set \{ $z \in P \mid x \leq z \leq y$ \} is denoted by $[x, y]$.

(b) A subset of $P$ is called a closed interval of $P$ if it has the form $[x, y]$ for two elements $x$ and $y$ of $P$ satisfying $x \leq y$.

(c) We denote by Int $P$ the set of all closed intervals of $P$.

(d) If $f : \text{Int } P \to \mathbb{Z}$ is any map, then the image $f([x, y])$ of a closed interval $[x, y] \in \text{Int } P$ under $f$ will be abbreviated by $f(x, y)$.

(e) Assume that every closed interval of $P$ is finite. The Möbius function of the poset $P$ is defined to be the unique function $\mu : \text{Int } P \to \mathbb{Z}$ having the following two properties:

- We have
  $$\mu(x, x) = 1 \quad \text{for every } x \in P.$$  \hspace{1cm} (23)

Proof. Let $H$ be a lower bound of $F$ and $G$ in Flats $M$. Thus, $H \subseteq F$ and $H \subseteq G$. Combining these two inclusions, we obtain $H \subseteq F \cap G$, qed.

Proof. Let $H$ be an upper bound of $F$ and $G$ in Flats $M$. Thus, $H \supseteq F$ and $H \supseteq G$. Combining these two inclusions, we obtain $H \supseteq F \cup G$. But $H \in $ Flats $M$; thus, $H$ is a flat of $M$. Since $H$ satisfies $F \cup G \subseteq H$, we therefore obtain $F \cup G \subseteq H$ (by Proposition 5.12 (c), applied to $F \cup G$ and $H$ instead of $T$ and $G$). In other words, $H \supseteq F \cup G$, qed.
We have
\[ \mu(x, y) = -\sum_{z \in P; x \leq z < y} \mu(x, z) \quad \text{for all } x, y \in P \text{ satisfying } x < y. \tag{24} \]

(It is easy to see that these two properties indeed determine \( \mu \) uniquely.) This Möbius function is denoted by \( \mu \).

We can now define the characteristic polynomial of a matroid \( M \), following Stanley (22)\(^{28}\).

**Definition 5.17.** Let \( M = (E, \mathcal{I}) \) be a matroid. Let \( m = r_M(E) \). The characteristic polynomial \( \chi_M \) of the matroid \( M \) is defined to be the polynomial
\[ \sum_{F \in \text{Flats } M} \mu(\emptyset, F) x^{m-r_M(F)} \in \mathbb{Z}[x] \]
(where \( \mu \) is the Möbius function of the lattice Flats \( M \)). We further define a polynomial \( \tilde{\chi}_M \in \mathbb{Z}[x] \) by \( \tilde{\chi}_M = [\emptyset = \emptyset] \chi_M \). Here, we are using the Iverson bracket notation (as in Definition 2.6). If the matroid \( M \) is loopless, then
\[ \tilde{\chi}_M = \sum_{i=1}^{m} \chi_M = \chi_M. \]
(by Proposition 5.12(e))

**Example 5.18.** Let \( G = (V, E) \) be a finite graph. Consider the graphical matroid \( (E, \mathcal{I}) \) defined as in Example 5.5. Then, the characteristic polynomial \( \chi_{(E, \mathcal{I})} \) of this matroid is connected to the chromatic polynomial \( \chi_G \) of the graph \( G \) as follows:
\[ x^{\text{conn } G} \cdot \chi_{(E, \mathcal{I})}(x) = \chi_G(x). \]

**5.3. Generalized formulas**

Let us next define broken circuits of a matroid \( M = (E, \mathcal{I}) \). Stanley, in [Stanley06 §4.1], defines them in terms of a total ordering \( \mathcal{O} \) on the set \( E \), whereas we shall use a “labeling function” \( \ell : E \rightarrow X \) instead (as in the case of graphs); our setting is slightly more general than Stanley’s.

\(^{28}\)Our notation slightly differs from that in [Stanley06 (22)]. Namely, we use \( x \) as the indeterminate, while Stanley instead uses \( t \). Stanley also denotes the global minimum \( \emptyset \) of Flats \( M \) by \( 0 \).
**Definition 5.19.** Let $M = (E, \mathcal{I})$ be a matroid. Let $X$ be a totally ordered set. Let $\ell : E \to X$ be a function. We shall refer to $\ell$ as the labeling function. For every $e \in E$, we shall refer to $\ell (e)$ as the label of $e$.

A broken circuit of $M$ means a subset of $E$ having the form $C \setminus \{e\}$, where $C$ is a circuit of $M$, and where $e$ is the unique element of $C$ having maximum label (among the elements of $C$). Of course, the notion of a broken circuit of $M$ depends on the function $\ell$; however, we suppress the mention of $\ell$ in our notation, since we will not consider situations where two different $\ell$’s coexist.

We shall now state analogues (and, in light of Example 5.18, generalizations, although we shall not elaborate on the few minor technicalities of seeing them as such) of Theorem 3.5, Theorem 3.4, Corollary 3.6, Corollary 3.7 and Corollary 3.13:

**Theorem 5.20.** Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M (E)$. Let $X$ be a totally ordered set. Let $\ell : E \to X$ be a function. Let $\mathfrak{K}$ be some set of broken circuits of $M$ (not necessarily containing all of them). Let $a_K$ be an element of $k$ for every $K \in \mathfrak{K}$. Then,

$$
\tilde{\chi}_M = \sum_{F \subseteq E} (-1)^{|F|} \left( \prod_{K \in \mathfrak{K}_F} a_K \right) x^{m - r(M)(F)}.
$$

**Theorem 5.21.** Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M (E)$. Then,

$$
\tilde{\chi}_M = \sum_{F \subseteq E} (-1)^{|F|} x^{m - r(M)(F)}.
$$

**Corollary 5.22.** Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M (E)$. Let $X$ be a totally ordered set. Let $\ell : E \to X$ be a function. Let $\mathfrak{K}$ be some set of broken circuits of $M$ (not necessarily containing all of them). Then,

$$
\tilde{\chi}_M = \sum_{F \subseteq E; F \text{ is } \mathfrak{K}-\text{free}} (-1)^{|F|} x^{m - r(M)(F)}.
$$

**Corollary 5.23.** Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M (E)$. Let $X$ be a totally ordered set. Let $\ell : E \to X$ be a function. Then,

$$
\tilde{\chi}_M = \sum_{F \subseteq E; F \text{ contains no broken circuit of } M \text{ as a subset}} (-1)^{|F|} x^{m - r(M)(F)}.
$$
Corollary 5.24. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let $X$ be a totally ordered set. Let $\ell : E \to X$ be an injective function. Then,

$$\tilde{\chi}_M = \sum_{F \subseteq E; F \text{ contains no broken circuit of } M \text{ as a subset}} (-1)^{|F|} x^{m-|F|}.$$ 

We notice that Corollary 5.24 is equivalent to [Stanley06, Theorem 4.12] (at least when $M$ is loopless).

Before we prove these results, let us state a lemma which will serve as an analogue of Lemma 2.7:

Lemma 5.25. Let $M = (E, \mathcal{I})$ be a matroid. Let $X$ be a totally ordered set. Let $\ell : E \to X$ be a function. Let $K$ be some set of broken circuits of $M$ (not necessarily containing all of them). Let $a_K$ be an element of $k$ for every $K \in \mathcal{K}$. Let $F$ be any flat of $M$. Then,

$$\sum_{B \subseteq F} (-1)^{|B|} \prod_{K \in \mathcal{K}; K \subseteq B} a_K = \left[ F = \emptyset \right]. \quad (25)$$

(Again, we are using the Iverson bracket notation as in Definition 2.6)

Proof of Lemma 5.25. Our proof will imitate the proof of Lemma 2.7 much of the time (with $E \cap \text{Eqs } f$ replaced by $F$); thus, we will allow ourselves some more brevity.

We WLOG assume that $F \neq \emptyset$ (since otherwise, the claim is obvious). Thus, $[F = \emptyset] = 0$.

Pick any $d \in F$ with maximum $\ell(d)$ (among all $d \in F$). (This is clearly possible, 

Proof. Assume that $F = \emptyset$. We must show that the claim is obvious.

Let us first show that no $K \in \mathcal{K}$ satisfies $K = \emptyset$. Indeed, assume the contrary. Thus, there exists a $K \in \mathcal{K}$ satisfying $K = \emptyset$. In other words, $\emptyset \in \mathcal{K}$. Thus, $\emptyset$ is a broken circuit of $M$ (since $\mathcal{K}$ is a set of broken circuits of $M$). Therefore, $\emptyset$ is obtained from a circuit of $M$ by removing one element (by the definition of a broken circuit). This latter circuit must therefore be a one-element set, i.e., it has the form $\{e\}$ for some $e \in E$. Consider this $e$. Thus, $\{e\}$ is a circuit of $M$.

But $F$ is a flat of $M$. In other words, Statement $\tilde{\delta}_1$ (of Lemma 5.9) holds for $T = F$. Hence, Statement $\tilde{\delta}_2$ (of Lemma 5.9) also holds for $T = F$ (since Lemma 5.9 shows that these two statements are equivalent). Applying Statement $\tilde{\delta}_2$ to $T = F$ and $C = \{e\}$, we thus obtain $\{e\} \subseteq F$ (because $\{e\} \setminus \{e\} = \emptyset \subseteq F$). Thus, $e \in \{e\} \subseteq F = \emptyset$, which is absurd. This contradiction proves that our assumption was wrong.

Hence, we have shown that no $K \in \mathcal{K}$ satisfies $K = \emptyset$. But from $F = \emptyset$, we see that the sum $\sum_{B \subseteq F} (-1)^{|B|} \prod_{K \in \mathcal{K}; K \subseteq B} a_K$ has only one addend (namely, the addend for $B = \emptyset$), and thus simplifies
since \( F \neq \emptyset \). Define two subsets \( \mathcal{U} \) and \( \mathcal{V} \) of \( \mathcal{P}(F) \) as follows:

\[
\mathcal{U} = \{ T \in \mathcal{P}(F) \mid d \notin T \}; \\
\mathcal{V} = \{ T \in \mathcal{P}(F) \mid d \in T \}.
\]

Thus, we have \( \mathcal{P}(F) = \mathcal{U} \cup \mathcal{V} \), and the sets \( \mathcal{U} \) and \( \mathcal{V} \) are disjoint. Now, we define a map \( \Phi : \mathcal{U} \to \mathcal{V} \) by

\[
(\Phi(B) = B \cup \{d\}) \quad \text{for every } B \in \mathcal{U}.
\]

This map \( \Phi \) is well-defined (because for every \( B \in \mathcal{U} \), we have \( B \cup \{d\} \in \mathcal{V} \)) and a bijection\(^{31}\). Moreover, every \( B \in \mathcal{U} \) satisfies \((−1)^{|\Phi(B)|} = −(−1)^{|B|}\) (26).

Now, we claim that, for every \( B \in \mathcal{U} \) and every \( K \in \mathcal{F} \), we have the following logical equivalence:

\[
(K \subseteq B) \iff (K \subseteq \Phi(B)).
\]

Proof of (27): Let \( B \in \mathcal{U} \) and \( K \in \mathcal{F} \). We must prove the equivalence (27). The definition of \( \Phi \) yields \( \Phi(B) = B \cup \{d\} \supseteq B \), so that \( B \subseteq \Phi(B) \). Hence, if \( K \subseteq B \), then \( K \subseteq B \subseteq \Phi(B) \). Therefore, the forward implication of the equivalence (27) is proven. It thus remains to prove the backward implication of this equivalence. In other words, it remains to prove that if \( K \subseteq \Phi(B) \), then \( K \subseteq B \). So let us assume that \( K \subseteq \Phi(B) \).

We want to prove that \( K \subseteq B \). Assume the contrary. Thus, \( K \nsubseteq B \). We have \( K \in \mathcal{F} \). Thus, \( K \) is a broken circuit of \( M \) (since \( \mathcal{F} \) is a set of broken circuits of \( M \)). In other words, \( K \) is a subset of \( E \) having the form \( C \setminus \{e\} \), where \( C \) is a circuit of \( M \), and where \( e \) is the unique element of \( C \) having maximum label (among the elements of \( C \)) (because this is how a broken circuit is defined). Consider these \( C \) and \( e \). Thus, \( K = C \setminus \{e\} \).

The element \( e \) is the unique element of \( C \) having maximum label (among the elements of \( C \)). Thus, if \( e' \) is any element of \( C \) satisfying \( \ell(e') \geq \ell(e) \), then

\[
e' = e.
\]

Thus, Lemma 5.25 is proven.

\(^{30}\)This follows from the fact that \( d \in F \).

\(^{31}\)Its inverse is the map \( \Psi : \mathcal{V} \to \mathcal{U} \) defined by \( \Psi(B) = B \setminus \{d\} \) for every \( B \in \mathcal{V} \).

\(^{32}\)Proof. This is proven exactly like we proved (4).
But \( K \setminus \{ d \} \subseteq (B \cup \{ d \}) \setminus \{ d \} \subseteq B. \)

If we had \( d \notin K \), then we would have \( K \setminus \{ d \} = K \) and therefore \( K = K \setminus \{ d \} \subseteq B \); this would contradict \( K \nsubseteq B \). Hence, we cannot have \( d \notin K \). We thus must have \( d \in K \). Hence, \( d \in C \subseteq B \).

But \( C \setminus \{ e \} = K \subseteq \Phi (B) \subseteq F \) (since \( \Phi (B) \in \mathcal{P} (F) \)). On the other hand, Statement 3 of Lemma 5.9 holds for \( T = F \) (since \( F \) is a flat of \( M \)). Hence, Statement 3 of Lemma 5.9 also holds for \( T = F \) (since Lemma 5.9 shows that these two statements are equivalent). Thus, from \( C \setminus \{ e \} \subseteq F \), we obtain \( C \subseteq F \). Thus, \( e \in C \subseteq F \). Consequently, \( \ell (d) \geq \ell (e) \) (since \( d \) was defined to be an element of \( F \) with maximum \( \ell (d) \) among all \( d \in F \)).

Also, \( d \in C \). Since \( \ell (d) \geq \ell (e) \), we can therefore apply (28) to \( e' = d \). We thus obtain \( d = e \). This contradicts \( d \neq e \). This contradiction proves that our assumption was wrong. Hence, \( K \subseteq B \) is proven. Thus, we have proven the backward implication of the equivalence (27); this completes the proof of (27).

Now, proceeding as in the proof of (12), we can show that

\[
\sum_{B \subseteq F} (-1)^{|B|} \prod_{K \in \mathcal{P} (B) \setminus B} a_K = [F = \emptyset].
\]

This proves Lemma 5.25.

We shall furthermore use a classical and fundamental result on the Möbius function of any finite poset:

\begin{prop}
Let \( P \) be a finite poset. Let \( \mu \) denote the Möbius function of \( P \).

\begin{enumerate}
\item[(a)] For any \( x \in P \) and \( y \in P \), we have

\[
\sum_{z \in P; \ x \leq z \leq y} \mu (x, z) = [x = y].
\]  \hspace{1cm} (29)

\item[(b)] For any \( x \in P \) and \( y \in P \), we have

\[
\sum_{z \in P; \ x \leq z \leq y} \mu (z, y) = [x = y]. \]  \hspace{1cm} (30)

\item[(c)] Let \( k \) be a \( \mathbb{Z} \)-module. Let \( (\beta_x)_{x \in P} \) be a family of elements of \( k \). Then, every \( z \in P \) satisfies

\[
\beta_z = \sum_{y \in P; \ y \leq z} \mu (y, z) \sum_{x \in P; \ x \leq y} \beta_x.
\]
\end{enumerate}
\end{prop}
For the sake of completeness, let us give a self-contained proof of this proposition (slicker arguments appear in the literature\textsuperscript{33}):

Proof of Proposition 5.26 (a) Let \( x \in P \) and \( y \in P \). We must prove the equality (29). We are in one of the following three cases:

Case 1: We have \( x = y \).

Case 2: We have \( x < y \).

Case 3: We have neither \( x = y \) nor \( x < y \).

Let us first consider Case 1. In this case, we have \( x = y \). Hence, the sum

\[
\sum_{z \in P; \ x \leq z \leq y} \mu(x, z) = \mu(x, x) = 1 \quad \text{(by the definition of the Möbius function)}
\]

\[= [x = y] \quad \text{(since } x = y) .
\]

Thus, (29) is proven in Case 1.

Let us now consider Case 2. In this case, we have \( x < y \). Hence, \( x \neq y \), so that \([x = y] = 0\). Now, \( y \) is an element of \( P \) satisfying \( x \leq y \leq y \). Thus, the sum

\[
\sum_{z \in P; \ x \leq z \leq y} \mu(x, z) \text{ contains an addend for } z = y .
\]

Splitting off this addend, we obtain

\[
\sum_{z \in P; \ x \leq z \leq y} \mu(x, z) = \sum_{z \in P; \ x \leq z \leq y; z \neq y} \mu(x, z) + \mu(x, y) = -\sum_{z \in P; \ x \leq z < y} \mu(x, z) \quad \text{(by (24))}
\]

\[= \sum_{z \in P; \ x \leq z < y} \mu(x, z) + \left( - \sum_{z \in P; \ x \leq z < y} \mu(x, z) \right) = 0 = [x = y] .
\]

Hence, (29) is proven in Case 2.

Finally, let us consider Case 3. In this case, we have neither \( x = y \) nor \( x < y \). Thus, we do not have \( x \leq y \). Hence, there exists no \( z \in P \) satisfying \( x \leq z \leq y \). Thus,

\[
\sum_{z \in P; \ x \leq z \leq y} \mu(x, z) = \text{empty sum} = 0 = [x = y]
\]

(since we do not have \( x = y \)). Thus, (29) is proven in Case 3.

Hence, (29) is proven in all three cases. This proves Proposition 5.26 (a).

\textsuperscript{33}For example, Proposition 5.26 (c) is equivalent to the \( \implies \) implication of [Martin15 (5.1a)].
For any two elements $u$ and $v$ of $P$, we define a subset $[u,v]$ of $P$ by

$$[u,v] = \{ w \in P \mid u \leq w \leq v \}.$$ 

Thus subset $[u,v]$ is finite (since $P$ is finite), and thus its size $|[u,v]|$ is a nonnegative integer.

We shall now prove Proposition 5.26 (b) by strong induction on $|[x,y]|$:

**Induction step:** Let $N \in \mathbb{N}$. Assume that Proposition 5.26 (b) holds whenever $|[x,y]| < N$. We must now prove that Proposition 5.26 (b) holds whenever $|[x,y]| = N$.

We have assumed that Proposition 5.26 (b) holds whenever $|[x,y]| < N$. In other words, we have assumed the following claim:

**Claim 1:** For any $x \in P$ and $y \in P$ satisfying $|[x,y]| < N$, we have

$$\sum_{z \in P; x \leq z \leq y} \mu(z,y) = [x = y].$$

Now, let $x$ and $y$ be two elements of $P$ satisfying $|[x,y]| = N$. We are going to prove that

$$\sum_{z \in P; x \leq z \leq y} \mu(z,y) = [x = y].$$

We are in one of the following three cases:

Case 1: We have $x = y$.

Case 2: We have $x < y$.

Case 3: We have neither $x = y$ nor $x < y$.

In Case 1 and in Case 3, we can prove (31) in exactly the same way as (in our above proof of Proposition 5.26 (a)) we have proven (29). Thus, it remains only to prove (31) in Case 2. In other words, we can WLOG assume that we are in Case 2.

Assume this. Hence, $x < y$, so that $[x = y] = 0$.

**Proof of (32):** Let $t \in P$ be such that $x \leq t < y$. We shall proceed in several steps:

- We have

$$[x,t] = \{ w \in P \mid x \leq w \leq t \} = [x,y]$$

(by the definition of $[x,y]$)

because every $w \in P$ satisfying $w \leq t$ must also satisfy $w \leq y$ (since $t < y$)

(by the definition of $[x,y]$).

Therefore, for every $t \in P$ satisfying $x \leq t < y$, we have

$$\sum_{z \in P; x \leq z \leq t} \mu(z,t) = [x = t]$$

(by the definition of $[x,t]$)
(by Claim 1, applied to \( t \) instead of \( y \)). Also, for every \( u \in P \) and \( v \in P \), we have
\[
\sum_{t \in P; \ u \leq t \leq v} \mu(u, t) = [u = v]
\] (34)

Now,
\[
\sum_{(z,t) \in P^2; \ x \leq z \leq t \leq y} \mu(z,t)
\]
\[
= \sum_{z \in P; \ t \in P; \ x \leq z \leq t \leq y} \sum_{z \leq t \leq y} \mu(z,t)
\]
\[
= \sum_{z \in P; \ x \leq z \leq y} \sum_{z \leq t \leq y} \mu(z,t) = \sum_{z \in P; \ x \leq z \leq y} [z = y]
\]
(by \( \text{by (34)} \))
\[
= \sum_{z \in P; \ x \leq z \leq y \text{ and } z = y} [z = y] + \sum_{z \in P; \ x \leq z \leq y \text{ and } z \neq y} [z = y]
\]
(since every \( z \in P \) satisfies either \( z = y \) or \( z \neq y \) (but not both))
\[
= \sum_{z \in P; \ x \leq z \leq y \text{ and } z = y} 1 + \sum_{z \in P; \ x \leq z \leq y \text{ and } z \neq y} 0 = \sum_{z \in \{w \in P \mid x \leq w \leq y \text{ and } w = y\}} 1
\]
\[
= \left| \{w \in P \mid x \leq w \leq y \text{ and } w = y\} \right| = |\{y\}| = 1.
\]

- We have \( t < y \). Thus, we do not have \( y \leq t \). Hence, we do not have \( x \leq y \leq t \). Hence, \( y \notin [x, t] \). But \( y \in [x, y] \) (since \( x \leq y \leq y \)). Hence, the sets \([x, t]\) and \([x, y]\) are distinct (since the latter contains \( y \) but the former does not). Combining this with \([x, t] \subseteq [x, y]\), we conclude that \([x, t]\) is a proper subset of \([x, y]\). Hence, \(|[x, t]| < |[x, y]| = N\). This proves (32).

\[35\text{Proof of (34)}: \text{Let } u \in P \text{ and } v \in P. \text{Proposition 5.26(a) (applied to } x = u \text{ and } y = v) \text{ shows that}
\]
\[
\sum_{t \in P; \ u \leq t \leq v} \mu(u, t) = [u = v].
\]
Now,
\[
\sum_{t \in P; \ u \leq t \leq v} \sum_{z \in P; \ u \leq z \leq v} \mu(u, t)
\]
\[
= \sum_{z \in P; \ u \leq z \leq v} \mu(u, t) \quad \text{(here, we have substituted } z \text{ for } t \text{ in the sum)}
\]
\[
= [u = v].
\]
This proves (34).
Hence,

\[
1 = \sum_{(z,t) \in P^2; \ x \leq z \leq t \leq y} \mu(z,t) = \sum_{t \in P; \ x \leq t \leq y} \sum_{z \in P; \ x \leq \ y \leq z \leq t} \mu(z,t)
\]

\[
= \sum_{t \in P; \ x \leq t \leq y} \sum_{z \in P; \ x \leq t \leq y \leq z \leq t} \mu(z,t)
\]

\[
= \sum_{t \in \{w \in P | x \leq w \leq y \text{ and } w = y\}; \ x \leq t \leq y} \mu(z,t) + \sum_{t \in \{y\}; \ x \leq t \leq y} \mu(z,t)
\]

\[
= \sum_{t \in \{y\}; \ x \leq t \leq y} \mu(z,t) + \sum_{t \in \{y\}; \ x \leq t \leq y \text{ and } t \neq y} \mu(z,t)
\]

\[
= \sum_{t \in \{y\}; \ x \leq t \leq y} \mu(z,t) + [x = t] \sum_{t \in \{y\}; \ x \leq t \leq y \text{ and } t \neq y} \mu(z,t)
\]

\[
= \sum_{t \in \{y\}; \ x \leq t \leq y} \mu(z,t) + [x = t] \sum_{t \in \{y\}; \ x \leq t \leq y \text{ and } t \neq y} \mu(z,t)
\]
Subtracting $\sum_{z \in P; \ x \leq z \leq y} \mu (z, y)$ from both sides of this equality, we obtain

$$1 - \sum_{z \in P; \ x \leq z \leq y} \mu (z, y)$$

$$= \sum_{t \in P; \ x \leq t \leq y \text{ and } t \neq x} [x = t]$$

$$= \sum_{t \in \{z \in P \mid x \leq z \leq y \text{ and } z = x \text{ and } z \neq y\}} [x = t]$$

$$+ \sum_{t \in P; \ x \leq t \leq y \text{ and } t \neq x \text{ and } t \neq y} [x = t]$$

$$= \sum_{t \in \{x\}} 1 + \sum_{t \in P; \ x \leq t \leq y \text{ and } t \neq x \text{ and } t \neq y} [x = t]$$

$$= \sum_{t \in \{x\}} 1 + 0 = \sum_{t \in \{x\}} 1 = 1.$$  

Solving this equality for $\sum_{z \in P; \ x \leq z \leq y} \mu (z, y)$, we obtain

$$\sum_{z \in P; \ x \leq z \leq y} \mu (z, y) = 1 - 1 = 0 = [x = y]$$

(since $x < y$). Thus, (5.1) is proven.

Let us now forget that we fixed $x$ and $y$. We thus have proven that for any $x \in P$ and $y \in P$ satisfying $|[x, y]| = N$, we have

$$\sum_{z \in P; \ x \leq z \leq y} \mu (z, y) = [x = y].$$

In other words, Proposition 5.26 (b) holds whenever $|[x, y]| = N$. This completes the induction step. Thus, Proposition 5.26 (b) is proven by induction.
For every $v \in P$, we have

$$
\sum_{y \in P; \ y \leq v} \mu(y, v) \sum_{x \in P; \ x \leq y} \beta_x
= \sum_{z \in P; \ z \leq v} \mu(z, v) \sum_{x \in P; \ x \leq z} \beta_x
$$

(here, we have renamed the summation index $y$ as $z$ in the outer sum)

$$
= \sum_{x \in P; \ x \leq z} \mu(z, v) \beta_x
= \sum_{x \in P; \ x \leq z \leq v} \mu(z, v) \beta_x
= \sum_{x \in P; \ x \leq z \leq v} \mu(z, v) \beta_x
$$

(by Proposition 5.26(b) (applied to $y = v$))

$$
= \sum_{x \in P; \ x \leq z \leq v} \mu(z, v) \beta_x
= \sum_{x \in P; \ x \leq z \leq v} \mu(z, v) \beta_x
$$

(since every $x \in P$ satisfies either $x = v$ or $x \neq v$ (but not both))

$$
= \sum_{x \in P; \ x = v} \beta_x + \sum_{x \in P; \ x \neq v} \beta_x
= \sum_{x \in P; \ x = v} \beta_x + \sum_{x \in P; \ x \neq v} \beta_x
= \beta_v
$$

(since $v \in P$).

Renaming $v$ as $z$ in this result, we obtain precisely Proposition 5.26(c). □

**Proof of Theorem 5.20** If $T$ is a subset of $E$, then $\overline{T}$ is a flat of $M$ (by Proposition 5.12(a)). In other words, if $T$ is a subset of $E$, then $\overline{T} \in \text{Flats } M$. Renaming $T$ as $B$ in this statement, we conclude that if $B$ is a subset of $E$, then $\overline{B} \in \text{Flats } M$.

For every $F \in \text{Flats } M$, define an element $\beta_F \in k$ by

$$
\beta_F = \sum_{B \subseteq E; \ \overline{B} = F} (-1)^{|B|} \left( \prod_{K \in \mathcal{R}; \ K \subseteq B} a_K \right).
$$

Now, using Lemma 5.25 we can easily see that

$$
\sum_{G \in \text{Flats } M; \ G \subseteq F} \beta_G = [F = \emptyset] \quad \text{for every } F \in \text{Flats } M \quad (35)
$$

Proof of (35): Let $F \in \text{Flats } M$. Thus, $F$ is a flat of $M$.

If $B$ is a subset of $E$, then the statements $(\overline{B} \subseteq F)$ and $(B \subseteq F)$ are equivalent. (This follows from Proposition 5.12(g), applied to $T = B$ and $G = F$.)
Let \( \mu \) be the Möbius function of the lattice Flats \( M \). The element \( \emptyset \) is the global minimum of the poset Flats \( M \). In particular, \( \emptyset \in \text{Flats } M \) and \( \emptyset \subseteq F \). Hence, \( \mu (\emptyset, F) \) is well-defined.

Now, fix \( F \in \text{Flats } M \). Proposition 5.26 (c) (applied to \( P = \text{Flats } M \) and \( z = F \))

Now,

\[
\sum_{G \in \text{Flats } M; \ G \subseteq F} \beta_G \left( \prod_{K \in B \setminus B \subseteq \emptyset} a_K \right)
\]

\( = \sum_{B \subseteq E; \ \emptyset \subseteq F} (-1)^{|B|} \left( \prod_{K \in R \setminus K \subseteq B} a_K \right) \]

(by the definition of \( \beta_G \))

\( = \sum_{B \subseteq E; \ \emptyset \subseteq F} \left( \prod_{K \in R \setminus K \subseteq B} a_K \right) \]

(because if \( B \) is a subset of \( E \), then \( B \in \text{Flats } M \))

\( = \sum_{B \subseteq E; \ \emptyset \subseteq F} (-1)^{|B|} \left( \prod_{K \in R \setminus K \subseteq B} a_K \right) \]

(because if \( B \) is a subset of \( E \), then the statements \( \emptyset \subseteq F \) and \( B \subseteq F \) are equivalent)

\( = \sum_{B \subseteq F} (-1)^{|B|} \left( \prod_{K \in R \setminus K \subseteq B} a_K \right) = |F = \emptyset| \quad \text{(by 25).} \)

This proves (35).

\[37\text{This was proven during our proof of Proposition 5.14.}\]
shows that
\[ \beta_F = \sum_{y \in \text{Flats} M; \ y \subseteq F} \mu (y, F) \sum_{x \in \text{Flats} M; \ x \subseteq y} \beta_x \]
(since the relation \( \leq \) of the poset Flats \( M \) is \( \subseteq \))
\[ = \sum_{H \in \text{Flats} M; \ H \subseteq F} \mu (H, F) \sum_{G \in \text{Flats} M; \ G \subseteq H} \beta_G \]
\[ = \sum_{H \in \text{Flats} M; \ H \subseteq F} \mu (H, F) [H = \emptyset] \]
\[ = \sum_{H \in \text{Flats} M; \ H \subseteq F} \mu (H, F) [H = \emptyset] + \sum_{H \in \text{Flats} M; \ H \subseteq F} \mu (H, F) [H = \emptyset] \]
\[ = \sum_{H \in \text{Flats} M; \ H \subseteq F} \mu (H, F) \]
\[ = \sum_{H \in \text{Flats} M; \ H = \emptyset} \mu (H, F) \]
(here, we renamed the summation indices \( y \) and \( x \) as \( H \) and \( G \))
\[ = \sum_{H \in \text{Flats} M; \ H = \emptyset} \mu (H, F) . \] (36)

Now, we shall prove that
\[ \beta_F = [\emptyset = \emptyset] \mu (\emptyset, F) . \] (37)

Proof of (37): We are in one of the following two cases:
Case 1: We have \( \emptyset = \emptyset \).
Case 2: We have \( \emptyset \neq \emptyset \).

Let us consider Case 1 first. In this case, we have \( \emptyset = \emptyset \). Hence, \( \emptyset = \emptyset \in \text{Flats} M \). Thus, the sum \( \sum_{H \in \text{Flats} M; \ H = \emptyset} \mu (H, F) \) has exactly one addend: namely, the addend for \( H = \emptyset \). Thus, \( \sum_{H \in \text{Flats} M; \ H = \emptyset} \mu (H, F) = \mu \left( \emptyset, F \right) = \mu (\emptyset, F) \).

Thus, (36) becomes \( \beta_F = \sum_{H \in \text{Flats} M; \ H = \emptyset} \mu (H, F) = \mu (\emptyset, F) \). Comparing this with
\( \emptyset = \emptyset \) \( \mu (\emptyset, F) = \mu (\emptyset, F) \), we obtain \( \beta_F = [\emptyset = \emptyset] \mu (\emptyset, F) \). Thus, (37) is proven in Case 1.

Let us now consider Case 2. In this case, we have \( \emptyset \neq \emptyset \). Thus, there exists no \( H \in \text{Flats } M \) such that \( H = \emptyset \). Hence, the sum \( \sum_{H \in \text{Flats } M; H = \emptyset} \mu (H, F) \) is empty. Thus, \( \sum_{H \in \text{Flats } M; H = \emptyset} \mu (H, F) = (\text{empty sum}) = 0 \), so that (36) becomes \( \beta_F = \sum_{H \in \text{Flats } M; H = \emptyset} \mu (H, F) = 0 \). Comparing this with \( [\emptyset = \emptyset] \mu (\emptyset, F) = 0 \), we obtain \( \beta_F = [\emptyset = \emptyset] \mu (\emptyset, F) \). Thus, (37) is proven in Case 2.

Now, we have proven (37) in both possible Cases 1 and 2. Thus, (37) always holds.

Now, let us forget that we fixed \( F \). We thus have proven (37) for each \( F \in \text{Flats } M \).

---

38Proof. Assume the contrary. Thus, there exists some \( H \in \text{Flats } M \) such that \( H = \emptyset \). In other words, \( \emptyset \in \text{Flats } M \). Hence, \( \emptyset \) is a flat of \( M \). Proposition 5.12 (b) (applied to \( G = \emptyset \)) thus shows that \( \emptyset = \emptyset \). This contradicts \( \emptyset \neq \emptyset \). This contradiction proves that our assumption was wrong, qed.
Now,

\[
\sum_{F \subseteq E} (-1)^{|F|} \left( \prod_{K \in \mathcal{F}; K \subseteq F} a_K \right) x^{m-r_M(F)}
\]

\[
= \sum_{B \subseteq E} (-1)^{|B|} \left( \prod_{K \in \mathcal{F}; K \subseteq B} a_K \right) x^{m-r_M(B)}
\]

\[
= \sum_{F \in \text{Flats } M} \sum_{B \subseteq E; B = F} (-1)^{|B|} \left( \prod_{K \in \mathcal{F}; K \subseteq B} a_K \right) x^{m-r_M(F)}
\]

\[
= \sum_{F \in \text{Flats } M} \sum_{B \subseteq E; B = F} (-1)^{|B|} \left( \prod_{K \in \mathcal{F}; K \subseteq B} a_K \right) x^{m-r_M(F)}
\]

\[
= \sum_{F \subseteq E} (-1)^{|F|} \left( \prod_{K \in \mathcal{F}; K \subseteq F} a_K \right) x^{m-r_M(F)}
\]

(here, we have renamed the summation index \(F\) as \(B\))

\[
= \sum_{B \subseteq E} (-1)^{|B|} \left( \prod_{K \in \mathcal{F}; K \subseteq B} a_K \right) x^{m-r_M(B)}
\]

(by Proposition 5.12 (applied to \(T = B\) shows that \(r_M(B) = r_M(\mathcal{F})\))

But the definition of \(\chi_M\) yields \(\chi_M = \sum_{F \in \text{Flats } M} \mu(\overline{\emptyset}, F) x^{m-r_M(F)}\). The definition of \(\tilde{\chi}_M\) yields

\[
\tilde{\chi}_M = [\overline{\emptyset} = \emptyset] \sum_{F \in \text{Flats } M} \mu(\overline{\emptyset}, F) x^{m-r_M(F)}
\]

\[
= [\overline{\emptyset} = \emptyset] \sum_{F \in \text{Flats } M} \mu(\overline{\emptyset}, F) x^{m-r_M(F)}
\]

\[
= \sum_{F \subseteq E} (-1)^{|F|} \left( \prod_{K \in \mathcal{F}; K \subseteq F} a_K \right) x^{m-r_M(F)}
\]

(by 38).

This proves Theorem 5.20.

\[\square\]

**Proof of Corollary 5.22** Corollary 5.22 can be derived from Theorem 5.20 in the same way as Corollary 1.13 was derived from Theorem 1.11. \[\square\]
Proof of Theorem 5.21. Theorem 5.21 can be derived from Theorem 5.20 in the same way as Theorem 1.8 was derived from Theorem 1.11.

Proof of Corollary 5.23. Corollary 5.23 follows from Corollary 5.22 when $\mathfrak{A}$ is set to be the set of all broken circuits of $M$.

Proof of Corollary 5.24. If $F$ is a subset of $E$ such that $F$ contains no broken circuit of $M$ as a subset, then

$$r_M(F) = |F|$$

(39)

Now, Corollary 5.23 yields

$$\tilde{\chi}_M = \sum_{F \subseteq E; \ F \text{ contains no broken circuit of } M \text{ as a subset}} (-1)^{|F|} x^{m-r_M(F)} = \sum_{F \subseteq E; \ F \text{ contains no broken circuit of } M \text{ as a subset}} (-1)^{|F|} x^{m-|F|}.$$  

This proves Corollary 5.24.

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Proof of (39): Let $F$ be a subset of $E$ such that $F$ contains no broken circuit of $M$ as a subset.

We shall show that $F \in \mathcal{I}$. Indeed, assume the contrary. Thus, $F \notin \mathcal{I}$, so that $F \notin P(E) \setminus \mathcal{I}$. Hence, there exists a circuit $C$ of $M$ such that $C \subseteq F$ (according to Lemma 5.7 applied to $Q = F$). Consider this $C$. The set $C$ is a circuit, and thus nonempty (because the empty set is in $\mathcal{I}$). Let $e$ be the unique element of $C$ having maximum label. (This is clearly well-defined, since the labeling function $\ell$ is injective). Then, $C \setminus \{e\}$ is a broken circuit of $M$ (by the definition of a broken circuit). Thus, $F$ contains a broken circuit of $M$ as a subset (since $C \setminus \{e\} \subseteq C \subseteq F$). This contradicts the fact that $F$ contains no broken circuit of $M$ as a subset.

This contradiction shows that our assumption was wrong. Hence, $F \in \mathcal{I}$ is proven.

Thus, Lemma 5.6 (applied to $T = F$) shows that $r_M(F) = |F|$, qed.

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