On the comparison theorem for multidimensional SDEs with jumps

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Abstract

In this note, we give a necessary and sufficient condition under which the comparison theorem holds for multidimensional stochastic differential equations (SDEs) with jumps and for matrix-valued SDEs with jumps.

Keywords: Viability property; Viscosity solution; Comparison theorem.

1 Introduction

The comparison theorem for real-valued SDEs turns out to be one of the classic results of this theory. We can refer the reader to [1], [2], [6], [7] and so on. It allows to compare the solutions of two real-valued SDEs whenever we can compare the drift and the diffusion coefficients. Thus they are all sufficient conditions.

Until in [4], Peng and Zhu originally studied comparison theorem of 1-dimensional SDEs with jumps through the viability theory (see Peng and Zhu [5]) and got the necessary and sufficient condition. In the manuscript of [3], Hu and Peng studied the multidimensional situation without jumps applying the viability theory.

The objective of this paper is to give a necessary and sufficient condition under which the comparison theorem holds for multidimensional SDEs with jumps. For this, we still apply the stochastic viability property (SVP) for SDEs with jumps studied in [5] and combine the technique used in [3].

The paper is organized as follows: in the next section, we recall the viability criteria for SDEs with jumps; in section 3, we study the comparison theorem for multidimensional SDEs with jumps and for matrix-valued SDEs with jumps.
2 A characterization for SDEs with jumps under state constraint

Let \((\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})\) be a complete stochastic basis such that \(\mathcal{F}_0\) contains all \(P\)-null elements of \(\mathcal{F}\), and \(\mathcal{F}_t := \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t, t \geq 0\), and \(\mathcal{F} = \mathcal{F}_T\), and suppose that the filtration is generated by the following two mutually independent processes:

(i) a \(d\)-dimensional standard Brownian motion \((W_t)_{0 \leq t \leq T}\), and
(ii) a stationary Poisson random measure \(N\) on \((0, T] \times E\), where \(E \subset \mathbb{R}^l \setminus \{0\}\), \(E\) is equipped with its Borel field \(\mathcal{B}_E\), with compensator \(\hat{N}(dtde) = dt n(de)\), such that \(n(E) < \infty\), and \(\{\tilde{N}((0, t] \times A) = (N - \hat{N})((0, t] \times A)\}_{0 < t \leq T}\) is an \(\mathcal{F}_t\)-martingale, for each \(A \in \mathcal{B}_E\).

By \(T > 0\) we denote the finite real time horizon.

We consider a jump diffusion process as follows:

\[
X^{t,x}_s = x + \int_t^s b(r, X^{t,x}_r)dr + \int_t^s \sigma(r, X^{t,x}_r)dW_r + \int_t^s \int_E \gamma(r, X^{t,x}_{r-}, e)\tilde{N}(drde), s \in [t, T],
\]

(2.1)

where

\[
b : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}^m, \gamma : [0, \infty) \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^m,
\]

\[
\sigma = \{\sigma^i\} : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}^{m \times d}, i = 1, 2, ..., m, \alpha = 1, 2, ..., d.
\]

**Definition 2.1.** The SDE (2.1) enjoys the stochastic viability property (SVP in short) in a given closed set \(K \subset \mathbb{R}^m\) if and only if: for any fixed time interval \([0, T]\), for each \((t, x) \in [0, T] \times K\), there exists a probability space \((\Omega, \mathcal{F}, P)\), a \(d\)-dimensional Brownian motion \(W\), a stationary Poisson process \(N\), such that

\[
X^{t,x}_s \in K, \quad \forall \ s \in [t, T] \quad P-a.s..
\]

We assume that, there exists a sufficiently large constant \(\mu > 0\) and a function \(\rho : \mathbb{R}^l \to \mathbb{R}_+\) with

\[
\int_E \rho^2(e)n(de) < \infty,
\]

such that

(A1) \(b, \sigma, \gamma\) are continuous in \((t, x)\),

(A2) for all \(x, x' \in \mathbb{R}^m, t \in [0, +\infty)\)

\[
|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq \mu| x - x'|,
\]

\[
|b(t, x)| + |\sigma(t, x)| \leq \mu(1 + |x|),
\]

\[
|\gamma(t, x, e) - \gamma(t, x', e)| \leq \rho(e)| x - x'|, \forall e \in E,
\]

\[
|\gamma(t, x, e)| \leq \rho(e)(1 + |x|), \forall e \in E.
\]
Here $\langle \cdot, \cdot \rangle$ and $| \cdot |$ denote, respectively, the Euclidian scalar product and norm. Obviously under the above assumptions there exists a unique strong solution to SDE (2.1). $C$ is a constant such that

$$C \geq 1 + 2\mu + \mu^2 + \int_E \rho^2(e)n(de).$$

We denote by $C_2([0, T] \times R^n)$ (resp., $C^1,2([0, T] \times R^n)$) the set of all functions in $C([0, T] \times R^n)$ (resp., $C^1,2([0, T] \times R^n)$) with quadratic growth in $x$. In fact, the SVP in $K$ is related to the following PDE:

$$\begin{align*}
\mathcal{L}u(t, x) + \mathcal{B}u(t, x) - Cu(t, x) + d^2_K(x) &= 0, \quad (t, x) \in (0, T) \times R^n, \\
u(T, x) &= d^2_K(x),
\end{align*}$$

(2.2)

where we denote, for $\varphi \in C^1,2([0, T] \times R^n)$,

$$\mathcal{L}\varphi(t, x) := \frac{\partial \varphi(t, x)}{\partial t} + \langle D\varphi(t, x), b(t, x) \rangle + \frac{1}{2}tr[D^2\varphi(t, x)\sigma\sigma^T(t, x)],$$

$$\mathcal{B}\varphi(t, x) := \int_E [\varphi(t, x + \gamma(t, x, e)) - \varphi(t, x) - \langle D\varphi(t, x), \gamma(t, x, e) \rangle]n(de).$$

**Definition 2.2.** We say a function $u \in C_2([0, T] \times R^n)$ is a viscosity supersolution (resp., subsolution) of (2.2) if, $u(T, x) \geq d^2_K(x)$ (resp., $u(T, x) \leq d^2_K(x)$) and for any $\varphi \in C^1,2([0, T] \times R^n)$ and any point $(t, x) \in [0, T] \times R^n$ at which $u - \varphi$ attains its minimum (resp., maximum),

$$\mathcal{L}\varphi(t, x) + \mathcal{B}\varphi(t, x) - C\varphi(t, x) + d^2_K(x) \leq 0, \quad \text{resp.,} \quad \geq 0.$$

$u$ is called a viscosity solution if it is both viscosity supersolution and subsolution.

Now let us recall the characterization of SVP of SDE (2.1) in $K$ (see [5]):

**Lemma 2.3.** We assume (A1) and (A2). Then the following claims are equivalent:

(i) SDE (2.1) enjoys the SVP in $K$;
(ii) $d^2_K(\cdot)$ is a viscosity supersolution of PDE (2.2).

## 3 Comparison theorem for SDEs with jumps

### 3.1 multidimensional SDEs

Let $S^2_{[0, T]}$ denote the set of $\mathcal{F}_t$-adapted càdlàg $m$-dimensional processes $\{X_t, 0 \leq t \leq T\}$ which are such that

$$\|X\|_{S^2_{[0, T]}} := (E[\sup_{0 \leq t \leq T} |X_t|^2])^{\frac{1}{2}} < \infty.$$
Consider the following two SDEs: $i = 1, 2$,

$$X^i_s = x^i + \int_t^s b^i(r, X^i_r) dr + \int_t^s \sigma^i(r, X^i_r) dW_r + \int_E \int_0^s \gamma^i(r, X^i_{r-}, e) \tilde{N}(dr de), \quad (3.1)$$

where $(b^i, \sigma^i, \gamma^i), i = 1, 2$ satisfy (A1) and (A2), and $x^1, x^2 \in \mathbb{R}^m$. The objective of this section is to study when the comparison theorem holds for two SDEs with jumps of type (3.1). We will find that the comparison theorem can be transformed to a viability problem in $\mathbb{R}^m_+ \times \mathbb{R}^m$ of $(X^1 - X^2, X^2)$.

**Theorem 3.1.** Suppose that $(b^i, \sigma^i, \gamma^i), i = 1, 2$ satisfy (A1) and (A2). Then the following are equivalent:

(i) For any $t \in [0, T]$, $x^1, x^2 \in \mathbb{R}^m$ such that $x^1 \geq x^2$, the unique adapted solutions $X^1$ and $X^2$ in $\mathcal{S}^2_{[t, T]}$ to the SDE (3.1) over time interval $[t, T]$ satisfy:

$$X^1_s \geq X^2_s, s \in [t, T], P - a.s.;$$

(ii) $\sigma^1 \equiv \sigma^2$, and for any $t \in [0, T], k = 1, 2, \ldots, m$,

$$
\begin{cases}
\sigma^1_k & \text{depends only on } x_k, \\
\forall x' \in \mathbb{R}^m, \forall x \geq 0, x_k + \gamma^1_k(t, x + x', e) - \gamma^2_k(t, x', e) \geq 0, n(de) - a.s., \\
\forall x', \delta^k x \in \mathbb{R}^m, \text{ such that } \delta^k x \geq 0, \delta^k x_k = 0,
\end{cases}
$$

$$b^1_k(t, \delta^k x + x') - \int_E \gamma^1_k(t, \delta^k x + x', e) n(de) \geq b^2_k(t, x') - \int_E \gamma^2_k(t, x', e) n(de).$$

**Proof:** Set $\bar{X}_s = (X^1_s - X^2_s, X^2_s)$, then (i) is equivalent to the following:

For any $t \in [0, T], \forall \bar{x} = (x^1, x^2)$ such that $x^1 \geq 0$, the unique solution $\bar{X}$ to the following SDE over time interval $[t, T]$:

$$\bar{X}_s = \bar{x} + \int_t^s \bar{b}(r, \bar{X}_r) dr + \int_t^s \bar{\sigma}(r, \bar{X}_r) dW_r + \int_t^s \int_E \bar{\gamma}(r, \bar{X}_{r-}, e) \tilde{N}(dr de), \quad (3.2)$$

satisfies $\bar{X} \geq 0, s \in [t, T], P - a.s.,$ where for $\bar{x} = (\bar{x}^1, \bar{x}^2)$,

$$
\bar{b}(s, \bar{x}) = (b^1(s, \bar{x}^1 + \bar{x}^2) - b^2(s, \bar{x}^2), b^2(s, \bar{x}^2)), \\
\bar{\sigma}(s, \bar{x}) = (\sigma^1(s, \bar{x}^1 + \bar{x}^2) - \sigma^2(s, \bar{x}^2), \sigma^2(s, \bar{x}^2)), \\
\bar{\gamma}(s, \bar{x}, e) = (\gamma^1(s, \bar{x}^1 + \bar{x}^2, e) - \gamma^2(s, \bar{x}^2, e), \gamma^2(s, \bar{x}^2, e)).
$$

So we can apply Lemma 2.3 to SDE (3.2) and the convex closed set $K := \mathbb{R}^m_+ \times \mathbb{R}^m$, i.e., (i) is equivalent to that $d^2_K(\cdot)$ is a viscosity supersolution of PDE (2.2).

we can see that $\forall x = (x^1, x^2) \in \mathbb{R}^{2m},$

$$\Pi_K(x) = \begin{pmatrix} (x^1)^+ \\ x^2 \end{pmatrix}, x - \Pi_K(x) = \begin{pmatrix} -(x^1)^- \\ 0 \end{pmatrix}.$$
So
\[ d^2_K(x) = |x^1|^2 = \sum_{k=1}^m I_{\{x_k^1 < 0\}} |x_k^1|^2. \]

And
\[
(D^2 d^2_K)(x) \begin{cases} 
0_{2m \times 2m}, & \text{when } x \in K^c, \\
does not exist, & \text{when } x \in \partial K, \\
(a_{ij})_{2m \times 2m}, & \text{when } x \in R^{2m} \setminus K, 
\end{cases}
\]

where
\[ a_{ij} = 0, \text{ when } i \neq j, \quad a_{ii} = \begin{cases} 
0, & m < i \leq 2m, \\
1, & 1 \leq i \leq m, x_i^1 \geq 0, \\
2, & 1 \leq i \leq m, x_i^1 < 0.
\end{cases} \]

From the above analysis and the Lipschitz condition of \( b^1 \) w.r.t. \( x \), we can easily check that: \( d^2_K(\cdot) \) is a viscosity supersolution of PDE (2.2) if and only if,

\[(ii) \forall t \in [0, T], \forall (x, x') \in R^m \times R^m,
\[
-2\langle x^-, b^1(t, x^+ + x') - b^2(t, x') \rangle + \sum_{k=1}^m I_{\{x_k < 0\}} |\sigma_k^1(t, x^1 + x') - \sigma_k^2(t, x')|^2
\]
\[
+ \sum_{k=1}^m I_{\{x_k < 0\}} \int_E |(x_k + \gamma_k^1(t, x + x', e) - \gamma_k^2(t, x', e)) - |x_k|^2 - 2x_k(\gamma_k^1(t, x + x', e) - \gamma_k^2(t, x', e))|] n(de)
\]
\[
+ \sum_{k=1}^m I_{\{x_k > 0\}} \int_E |(x_k + \gamma_k^1(t, x + x', e) - \gamma_k^2(t, x', e)) - |x_k|^2| n(de)
\]
\[ \leq C^* |x^-|^2, \]

where \( C^* \geq 4\mu + \mu^2 + \int E \rho^2(e)n(de) \) is a constant which does not depend on \( t, x, x' \). Then the left thing we need to do is to prove: \( (ii) \iff (ii)' \).

(ii)' \Rightarrow (ii): If we pick \( x \geq 0 \), we can immediately get (b) in (ii) from (ii)'.

Pick \( x < 0 \), by (ii)' we have
\[
\sum_{k=1}^m 2x_k[b_k^1(t, x') - b_k^2(t, x')] + |\sigma_k^1(t, x^1 + x') - \sigma_k^2(t, x')|^2 \leq C^* |x^-|^2.
\]

Let \( x \) tend to \( 0^- \), we get that \( \sigma^1 \equiv \sigma^2 \).

And \( \forall \delta^k x \in R^m \), such that, \( \delta^k \geq 0, (\delta^k x)_k = 0 \). Pick \( x = \delta^k x - \varepsilon e_k, \varepsilon > 0 \).

From (ii)' we get:
\[
-2\varepsilon[b_k^1(t, \delta^k x + x') - b_k^2(t, x')] + |\sigma_k^1(t, x + x') - \sigma_k^1(t, x')|^2 \leq C^* \varepsilon^2.
\]

Let \( \varepsilon \) tend to 0, we have
\[
\sigma_k^1(t, \delta^k x + x') = \sigma_k^1(t, x').
\]
We deduce quickly that $\sigma_k^1$ depends only on $x_k$.

With $x = \delta^k x - \varepsilon e_k, \varepsilon > 0$ again, from (ii)' we can also get

$$-2\varepsilon[b_k^1(t, \delta^k x + x') - b_k^2(t, x')] + \int_E [\varepsilon^2 + 2\varepsilon(\gamma_k^1(t, x + x', e) - \gamma_k^2(t, x', e))]n(de) \leq C^*\varepsilon^2.$$  

Dividing by $-2\varepsilon$ and letting $\varepsilon$ tend to 0, we have

$$b_k^1(t, \delta^k x + x') - \int E \gamma_k^1(t, \delta^k x + x', e)n(de) \geq b_k^2(t, x') - \int E \gamma_k^2(t, x', e)n(de).$$

(ii) $\Rightarrow$ (ii)'. For $x \geq 0$, from (b) in (ii), we know that (ii)' holds true. If there exist some $1 \leq k \leq n$, such that $x_k < 0$, then from (ii) we have

$$-2(x^-, b^1(t, x^+ + x') - b^2(t, x')) + \sum_{k=1}^m I_{\{x_k < 0\}}|\sigma_k^1(t, x + x') - \sigma_k^1(t, x')|^2$$

$$+ \sum_{k=1}^m I_{\{x_k < 0\}} \int E \left[|x_k + \gamma_k^1(t, x + x', e) - \gamma_k^2(t, x', e)|^2ight.$$ 

$$- |x_k|^2 - 2x_k(\gamma_k^1(t, x + x', e) - \gamma_k^2(t, x', e))]n(de)$$

$$+ \sum_{k=1}^m I_{\{x_k \geq 0\}} \int E \left[|x_k + \gamma_k^1(t, x + x', e) - \gamma_k^2(t, x', e)|^2n(de)ight.$$ 

$$\leq \sum_{k=1}^m I_{\{x_k < 0\}}|\sigma_k^1(t, x + x') - \sigma_k^1(t, x')|^2$$

$$+ \sum_{k=1}^m I_{\{x_k < 0\}} 2x_k(b_k^1(t, \delta^k x + x') - b_k^2(t, x')) - \int E [\gamma_k^1(t, \delta^k x + x', e) - \gamma_k^2(t, x', e)]n(de))$$

$$+ \sum_{k=1}^m I_{\{x_k < 0\}} \int E \left[|x_k + \gamma_k^1(t, x + x', e) - \gamma_k^1(t, \delta^k x + x', e)|^2 - |x_k|^2ight.$$ 

$$- 2x_k(\gamma_k^1(t, x + x', e) - \gamma_k^1(t, \delta^k x + x', e))]n(de)$$

$$+ \sum_{k=1}^m I_{\{x_k \geq 0\}} \int E \left[|\gamma_k^1(t, x + x', e) - \gamma_k^1(t, \delta^k x + x', e)|^2n(de)\right.$$ 

$$\leq \sum_{k=1}^m I_{\{x_k < 0\}}|\sigma_k^1(t, x + x') - \sigma_k^1(t, x')|^2$$

$$+ \sum_{k=1}^m \int E \left[|\gamma_k^1(t, x + x', e) - \gamma_k^1(t, \delta^k x + x', e)|^2n(de)\right.$$ 

$$\leq (\mu^2 + \int E \rho^2(e)n(de))|x^-|^2$$

$$\leq C^*|x^-|^2.$$

\[\square\]

**Remark 3.2.** For the holding of comparison theorem for multidimensional SDEs with jumps, condition (ii) in Theorem 3.1 is very natural. $\sigma^1 \equiv \sigma^2$ and condition (a) are the results of that the sign of $dW$ is not always positive or negative; To consider condition (b) and (c), let us transform our SDEs to the
following forms:

\[ X^i_s = x^i + \int^s_t [b^i(r, X^i_r) - \int_E \gamma^i(r, X^i_{r-}, e)n(de)]dr + \int^s_t \sigma^i(r, X^i_r)dW_r + \int^s_t \gamma^i(r, X^i_{r-}, e)N(drde). \]

So condition (b) implies that jumps should occur in the way of holding the advantage. While condition (c) display the form that the new drift coefficients should satisfy. In the classical real-valued case without jumps, we are very familiar with this form.

**Corollary 3.3.** Let \( m = 1 \) and suppose that \((b^i, \sigma^i, \gamma^i), i = 1, 2\) satisfy (A1)and (A2). Then the following are equivalent:

(i) For any \( t \in [0, T] \), \( x^1, x^2 \in \mathbb{R} \) such that \( x^1 \geq x^2 \), the unique adapted solutions \( X^1 \) and \( X^2 \) in \( S^2_{[t,T]} \) to the SDE (3.1) over time interval \([t, T]\) satisfy:

\[ X^1_s \geq X^2_s, s \in [t, T], P - a.s.; \]

(ii)For any \( t \in [0, T], x \in \mathbb{R} \),

\[
\begin{align*}
&\sigma^1(t, x) \equiv \sigma^2(t, x), \\
&b^1(t, x) - \int_E \gamma^1(t, x, e)n(de) \geq b^2(t, x) - \int_E \gamma^2(t, x, e)n(de), \\
&x_1 + \gamma^1(t, x_1, e) \geq x_2 + \gamma^2(t, x_2, e), \forall x_1 \geq x_2, n(de) - a.s..
\end{align*}
\]

This has already been established in [4].

**Corollary 3.4.** Let \( m = 1 \). When \( \gamma^1 \equiv \gamma^2 \neq 0 \) and suppose that \((b^i, \sigma^i, \gamma^i), i = 1, 2\) satisfy (A1)and (A2). Then the following are equivalent:

(i) For any \( t \in [0, T], x^1, x^2 \in \mathbb{R} \) such that \( x^1 \geq x^2 \), the unique adapted solutions \( X^1 \) and \( X^2 \) in \( S^2_{[t,T]} \) to the SDE (3.1) over time interval \([t, T]\) satisfy:

\[ X^1_s \geq X^2_s, s \in [t, T], P - a.s.; \]

(ii)For any \( t \in [0, T], x \in \mathbb{R} \),

\[
\begin{align*}
&\sigma^1(t, x) \equiv \sigma^2(t, x), \\
&b^1(t, x) \geq b^2(t, x), \\
&x_1 + \gamma^1(t, x_1, e) \geq x_2 + \gamma^2(t, x_2, e), \forall x_1 \geq x_2, n(de) - a.s..
\end{align*}
\]

**Corollary 3.5.** Let \( m = 1 \). When \( \gamma^1 \equiv \gamma^2 \equiv 0 \) and suppose that \((b^i, \sigma^i), i = 1, 2\) satisfy (A1) and (A2). Then the following are equivalent:
(i) For any \( t \in [0,T], \ x^1, x^2 \in R \) such that \( x^1 \geq x^2 \), the unique adapted solutions \( X^1 \) and \( X^2 \) in \( \mathcal{S}^2_{[t,T]} \) to the SDE (3.1) over time interval \([t,T]\) satisfy:

\[
X^1_s \geq X^2_s, s \in [t,T], P-a.s.;
\]

(ii) For any \( t \in [0,T], x \in R \),

\[
\sigma^1(t,x) \equiv \sigma^2(t,x), \quad b^1(t,x) \geq b^2(t,x).
\]

This is the classical result in SDEs without jump.

Although when \( \gamma^1 \equiv \gamma^2 \), it is very convenient for us to get the comparison theorem. But in fact, it’s not necessary for the holding of comparison theorem. The following is an counter-example, where \( \gamma^1 \) is not necessarily equal to \( \gamma^2 \), but the comparison theorem can still hold true.

Example 3.6. Let \( m = 1 \). Set

\[
b^i(t,x) = \int_E \gamma^i(t,x,e)n(de), \quad \sigma^i \equiv 0, i = 1, 2.
\]

We have the following two SDEs:

\[
X^i_s = x^i + \int_t^s b^i(r, X^i_r)dr + \int_t^s \sigma^i(r, X^i_r)dW_r + \int_t^s \int_E \gamma^i(r, X^i_{r-}, e)\tilde{N}(drde), \quad (3.3)
\]

Then we can immediately see that as long as

\[
x_1 + \gamma^1(t,x_1,e) \geq x_2 + \gamma^2(t,x_2,e), \quad \forall x_1 \geq x_2, n(de) - a.s.,
\]

the comparison theorem: (i) in Corollary 3.3 holds true. While in this case, we only need, for all \( t \in [0,T], x \in R \),

\[
\gamma^1(t,x,e) \geq \gamma^2(t,x,e), n(de) - a.s..
\]

3.2 Matrix-valued SDEs

Since the first and the second derivatives of the function \( d_{S^m_+}^2(y), y \in S^m \) have been studied due to Hu and Peng [3], where \( S^m \) is the space of symmetric real \( m \times m \) matrices, and \( S^m_+ \) is the subspace of \( S^m \) containing the nonnegative elements in \( S^m \). So at the end of this paper, we can study the comparison theorem for matrix-valued SDEs with jumps. Without loss of generality, we set \( d = 1 \).

Consider the following two SDEs (i=1,2):

\[
X^i_s = x^i + \int_t^s b^i(r, X^i_r)ds + \int_t^s \sigma^i(r, X^i_r)dW_r + \int_t^s \int_E \gamma^i(r, X^i_{r-}, e)\tilde{N}(drde), \quad (3.3)
\]
where, for $i = 1, 2$,

\[ b^i : [0, \infty) \times \mathbb{S}^m \rightarrow \mathbb{S}^m, \quad \sigma : [0, \infty) \times \mathbb{S}^m \rightarrow \mathbb{S}^m, \quad \gamma : [0, \infty) \times \mathbb{S}^m \times \mathbb{R}^l \rightarrow \mathbb{S}^m. \]

**Theorem 3.7.** Suppose that $b^i, \sigma^i, \gamma^i (i=1,2)$ satisfy (A1) and (A2). Then the following are equivalent:

(i) For any $t \in [0,T], x^1, x^2 \in \mathbb{S}^m$ such that $x^1 \geq x^2$, the unique adapted solutions $X^1$ and $X^2$ in $\mathcal{S}^2_{[t,T]}(\mathbb{S}^m)$ to the SDE (3.3) over time interval $[t, T]$ satisfy:

\[ X^1_s \geq X^2_s, \quad s \in [t, T], \quad P - a.s.; \]

(ii) For any $t \in [0,T], \forall (x, x') \in \mathbb{S}^m \times \mathbb{S}^m$,

\[
-4\langle x^-, b^1(t, x^+ + x') - b^2(t, x') \rangle \\
+ \langle D^2d_{\mathbb{S}^m}(y)(\sigma^1(t, x + x') - \sigma^2(t, x')), (\sigma^1(t, x + x') - \sigma^2(t, x')) \rangle \\
+ 2 \int_{E} \{[(x + \gamma^1(t, x + x', e) - \gamma^2(t, x', e)) - x^-]^2 - \|x^--\|^2 \\
+ 2\langle x^-, \gamma^1(t, x + x', e) - \gamma^2(t, x', e) \rangle \} n(de) \]

\[ \leq C^* \|x^-\|^2, \]

where $C^* \geq 4\mu + \mu^2 + \int_{E} \rho^2(e)n(de)$ is a constant which does not depend on $t, x, x'$.

**Proof:** We can see from the appendix of [3] that, for any $y \in \mathbb{S}^m$, $y$ has an expression:

\[ y(\lambda, A) = e^{A} \sum_{i=1}^{m} \lambda_i e_i e_i^T e^{-A}, \]

where $A$ is an antisymmetric real $m \times m$ matrix ($A^T = -A$), $\lambda_i \in \mathbb{R}$, $\{e_1, e_2, ..., e_m\}$ is the standard basis of $\mathbb{R}^m$.

If we set

\[ y^+(\lambda, A) = e^{A} \sum_{i=1}^{m} \lambda^+_i e_i e_i^T e^{-A}, \quad y^-(\lambda, A) = e^{A} \sum_{i=1}^{m} \lambda^-_i e_i e_i^T e^{-A}. \]

Then from [3], we have

\[ d_{\mathbb{S}^m}^2(y) = \|y^-\|^2, \quad \Pi_{\mathbb{S}^m}(y) = y^+, \quad \text{and} \quad \nabla d_{\mathbb{S}^m}^2(y) = -2y^-, \]

where $\|y\| = (tr(y^2))^{1/2}$. This with Lemma 2.3, we can use the same method as Theorem 3.1 to finish the proof of the theorem. We omit it.
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