Third-order cosmological perturbations of zero-pressure multi-component fluids: pure general relativistic non-linear effects

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Abstract. The present expansion stage of the universe is believed to be mainly governed by the cosmological constant, collisionless dark matter and baryonic matter. The latter two components are often modeled as zero-pressure fluids. In our previous work we have shown that to second order in the cosmological perturbations, the relativistic equations for the density and velocity perturbations of the zero-pressure, irrotational, multi-component fluids in a spatially near flat background without gravitational waves effectively coincide with the Newtonian equations. As the Newtonian equations only have quadratic order non-linearity, it is of practical interest to derive the third-order perturbation terms in a general relativistic treatment, which correspond to pure general relativistic corrections. In our previous work we have shown that even in a single-component fluid there exist a substantial number of pure relativistic third-order correction terms. We have, however, shown that those correction terms are independent of the horizon scale, and are quite small ($\sim 5 \times 10^{-5}$) smaller compared with the relativistic/Newtonian second-order terms) near the horizon scale due to the weak level anisotropy of the cosmic microwave background radiation. Here, we present pure general relativistic correction terms appearing in the third-order perturbations of the multi-component zero-pressure fluids. The forms of the pure general relativistic correction terms are quite similar to the ones in a single-component situation. The third-order correction terms involve only the `linear
order spatial curvature perturbation in the comoving gauge’ $\varphi_v$ which has the order of the ‘perturbed Newtonian gravitational potential divided by $c^2$’, and thus is small on nearly all scales. Consequently, we show that, as in a single-component situation, the third-order correction terms are quite small ($\sim 5 \times 10^{-5}$ smaller) near the horizon, and independent of the horizon scale. We emphasize that these results are based on our proper choice of perturbation variables and gauge conditions for describing the relativistic perturbations. Still, there do exist a substantial number of pure general relativistic correction terms in third-order perturbations which could potentially become important in future development of precision cosmology. Although $\varphi_v$ is small on nearly all scales, our third-order corrections are applicable only in weakly non-linear regimes where perturbation analysis is viable. We include the cosmological constant in all our analyses.

**Keywords:** cosmological perturbation theory, classical tests of cosmology

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1. **Introduction**

Recently, we have been presenting a series of works based on our theoretical study of relativistic non-linear cosmological perturbations [1]–[4]. We have shown that to the second-order perturbations, general relativistic equations for the density and velocity perturbations of a zero-pressure, irrotational fluid in a spatially flat background are exactly the same as the equations in Newton’s gravity except for the presence of the gravitational wave contribution [1, 2]: we call this a ‘relativistic/Newtonian correspondence to second
order’ which applies to the density and velocity perturbation equations only without the gravitational potential.

Recently in [5] we have relaxed all the assumptions we made in the second-order perturbations in [1,2], and have derived pure general relativistic effects from the pressures, rotation, spatial curvature, and multi-component cases. In that work, we showed that except for the multi-component situation, relaxing any of the other three assumptions leads to pure general relativistic correction terms appearing in the second order. Pressures are intrinsically general relativistic even in the background and the linear order perturbations. The presence of background curvature leads to first non-vanishing relativistic correction terms appearing in the second-order perturbations. The rotational perturbations generally lead to relativistic correction terms which become negligible in the small-scale (sub-horizon scale) limit, thus having relativistic/Newtonian correspondence in that limit. In the case of zero-pressure, irrotational multi-component fluids in a flat background, effectively we have exact relativistic/Newtonian correspondence even in the multi-component situation; this will be summarized in section 3 of this work.

The relativistic/Newtonian correspondence in the background world model was known in the zero-pressure medium from combining the work of Friedmann in 1922 [6] in the context of Einstein’s gravity and the work of Milne and McCrea in 1934 [7] in the context of Newton’s gravity; the latter Newtonian derivation was later known to have been guided by the already derived Einstein gravity result [8]. In the case of linear perturbations, the relativistic/Newtonian correspondence in the zero-pressure medium was also known from combining the work of Lifshitz in 1946 [9] in the context of Einstein’s gravity and the work of Bonnor in 1957 [10] in the context of Newton’s gravity. The fully non-linear perturbation equations in the context of Newtonian cosmology in a zero-pressure medium were known from a textbook by Peebles from 1980 [11]. The Einstein gravity counterpart of the non-linearly perturbed cosmological medium, especially the continued relativistic/Newtonian correspondence even to the second order, was first shown only recently in our works in [1,2].

Here, we would like to state clearly that our relativistic/Newtonian correspondence to the second order applies to the equations for the density and velocity perturbations only without resorting to the Newtonian gravitational potential; more precisely, the relativistic equations are identified with the continuity equation and with the divergence of the momentum conservation equation replacing the Newtonian gravitational potential using Poisson’s equation [1,2].

At this point it would be interesting to comment on the origins of Poisson’s equation in the Newtonian limit, and in the linear perturbation theory, of the Einstein gravity in the cosmological context. The Poisson equation appears in the Newtonian limit of the post-Newtonian approximation from the time–time component of Einstein’s equation (the energy or Hamiltonian constraint equation); in the cosmological context the relativistic equation subtly differs from (and is better than!) the known Newtonian Poisson equation; see equation (88) of [12]. To the first post-Newtonian order, the Poisson equation acquires a substantial number of post-Newtonian correction terms; see equations (85) and (119) of [12]. In the cosmological linear perturbation theory, the Hamiltonian constraint equation and the momentum constraint equation (space–time component of Einstein’s equation) give a gauge-invariant form of the Newtonian Poisson equation for vanishing background spatial curvature; see equation (4.3) of [13], or equation (171)
of [5]. Therefore, even in the linear perturbation theory we have an exact Poisson-type equation only for vanishing background spatial curvature, and consequently the gravitational potential has a relativistic/Newtonian correspondence only in that limited case; see [14]. In the context of the second-order perturbation, in the first reference of [2] we have checked these equations (see equations (17) and (37) in that paper) and other equations without a trace of the Poisson-type equation available in Einstein’s gravity. We regard our ‘relativistic/Newtonian correspondence of density and velocity perturbation equations to second order’ as a mere coincidence. And it is not surprising that we could not find a similar correspondence including the Newtonian potential because we already know historically very well the difference of the two gravity theories when involving the potential; e.g., remember the factor 2 difference in predicted light bending between the two gravity theories.

Recently, in [4], we derived pure general relativistic correction terms appearing in the third-order single-component, zero-pressure, irrotational fluid in a flat background. In [5], we also have shown that the relativistic/Newtonian correspondence to second order continues even in the multi-component situation. Thus, now it is a natural step to find the potential third-order pure general relativistic correction terms appearing in the multi-component, zero-pressure, irrotational fluids in a flat background. As the Newtonian system has only quadratic non-linearity even in the multi-component situation, see section II of [5], any non-vanishing third-order terms can be regarded as pure general relativistic corrections. The situation is also of practical importance because the current stage of the universe is supposed to be dominated by two zero-pressure components (the baryon and the cold dark matter) and the cosmological constant. We will include the effect of the cosmological constant in all our analyses and equations in this work, which is also the case in our previous works in [1]–[5].

In this work we will present the pure general relativistic third-order correction terms which will turn out to be substantial. Our results will show that even in the multi-component situation the third-order corrections are effectively the same as in the single-component case. Thus, although we have a substantial number of pure third-order general relativistic correction terms, such corrections are independent of the presence of the horizon and depend only on the ‘linear order spatial curvature perturbation in the comoving gauge’ ($\phi_v$) which has several interesting properties. The variable $\phi_v$ is quite small ($\sim 5 \times 10^{-5}$) near the horizon scale because of the low level anisotropies of the cosmic microwave background radiation. The value of linear order $\phi_v$ is also similar to the ‘perturbed Newtonian gravitational potential divided by $c^2$ ($\delta \Phi/c^2$, thus dimensionless) which is also quite small ($\sim 10^{-6}$) on nearly all celestial scales from stars to galaxies and clusters of galaxies. Although the third-order correction terms are quite small (compared with the relativistic/Newtonian second-order terms) on nearly all scales, our third-order perturbation equations are applicable only in weakly non-linear regimes. Thus, as long as the perturbation approach is applicable (i.e., in weakly non-linear regimes) our third-order general relativistic equations are valid and the pure third-order corrections are small with $\phi_v \sim (5/3)\delta \Phi/c^2$. We note that even in the fully non-linear regimes like in the stars and galaxies we have $\phi_v \sim \delta \Phi/c^2 \sim 10^{-6}$ whereas the density contrast $\delta$ could be huge. Our perturbation equations are not necessarily applicable in the fully non-linear stage. In order to handle the small-scale fully non-linear but weakly relativistic (i.e., $\delta \Phi/c^2 \ll 1$) structures, a post-Newtonian approach can serve in a complementary way
compared with our present perturbative approach which can handle fully relativistic but weakly non-linear structures; see our work in [12] for the cosmological post-Newtonian equations.

In section 2 we present the metric and fluid quantities perturbed to the third order which will be required in our calculation. We present fluid quantities for most general fluids with pressures and stresses which will turn out to be important even in the zero-pressure situation in our main analysis. As in the single-component case in [4], under our proper choice of variables and gauges we do not need third-order perturbations of the connection or curvature tensor. In section 3 we summarize the relativistic equations to second order, and their effective correspondence with the Newtonian ones even in the multi-component situation. In section 4 we derive the general relativistic perturbation equations valid to third order, and present equations in the context of Newtonian gravity with the third-order terms as pure general relativistic corrections. We compare our equations for the multi-component case with the previously derived ones for a single component, and present implications of our results. Section 5 is a discussion. We often set $c \equiv 1$, but recover $c$ in the Newtonian context presentation. If the reader is interested in the third-order equations presented in the Newtonian context, and their implications, she/he may go directly to sections 4.1, 4.2, and 5.

2. Third-order perturbations

2.1. The covariant and ADM equations

In the following we summarize the basic sets of covariant equations and ADM equations that we need in our analysis. These equations, except for the covariant equations for individual components, are presented section II of [1]; the notation can be found in that work and section II of [5]. For original studies of the covariant and the ADM equations, see [15], and [16], respectively. Although we will use the ADM equations in our calculation, the covariant equations show other aspects of the same fully non-linear system of Einstein’s equation.

The energy–momentum tensor of a fluid can be decomposed into fluid quantities as

$$\tilde{T}_{ab} = \tilde{\mu} \tilde{u}_a \tilde{u}_b + \tilde{p} \left( \tilde{u}_a \tilde{u}_b + \tilde{g}_{ab} \right) + \tilde{q}_a \tilde{u}_b + \tilde{q}_b \tilde{u}_a + \tilde{\pi}_{ab},$$

where $\tilde{q}_a \tilde{u}^a = 0 \equiv \tilde{\pi}_{ab} \tilde{u}^b$, $\tilde{\pi}_{ab} = \tilde{\pi}_{ba}$ and $\tilde{\pi}^a_a \equiv 0$. Without losing generality, we take the energy frame, and thus set $\tilde{q}_a \equiv 0$. This decomposition is valid even in the multi-component fluids; in such a case the above fluid quantities can be regarded as collective fluid quantities. In the multi-component case we introduce the energy–momentum tensor and fluid quantities of individual components as

$$\tilde{T}_{ab} \equiv \sum_j \tilde{T}_{(j)ab},$$

$$\tilde{T}_{(i)ab} = \tilde{\mu}_{(i)} \tilde{u}_{(i)a} \tilde{u}_{(i)b} + \tilde{p}_{(i)} \left( \tilde{u}_{(i)a} \tilde{u}_{(i)b} + \tilde{g}_{ab} \right) + \tilde{\pi}_{(i)ab},$$

where, without losing generality, we also took the energy frame condition for each component, and thus set $\tilde{q}_{(i)a} \equiv 0$. For interactions among components we introduce

$$\tilde{T}^{(i)}_{(i)a} \equiv \tilde{I}_{(i)a}, \quad \sum_j \tilde{I}_{(j)a} = 0.$$
In a single-component situation, taking the energy frame (and thus, setting \( \tilde{q}_a \equiv 0 \)), the energy conservation equation, the momentum conservation equation, and the Raychaudhury equation are

\[
\begin{align*}
\tilde{\mu} + (\tilde{\mu} + \tilde{p}) \tilde{\theta} + \tilde{\pi}^{ab} \tilde{\sigma}_{ab} &= 0, \\
(\tilde{\mu} + \tilde{p}) \tilde{\alpha}_a + \tilde{h}_a^b (\tilde{p}_b + \tilde{\pi}_b^c) &= 0, \\
\tilde{\theta} + \frac{1}{3} \tilde{\theta}^2 - \tilde{a}_a^a + \tilde{\sigma}^{ab} \tilde{\sigma}_{ab} - \tilde{\omega}^{ab} \tilde{\omega}_{ab} + 4\pi G (\tilde{\mu} + 3\tilde{p}) - \Lambda &= 0,
\end{align*}
\]

where \( \tilde{\mu} \equiv \tilde{\mu}_a \tilde{u}^a \) is a covariant derivative along \( \tilde{u}^a \) flow, and \( \tilde{\theta} \equiv \tilde{u}^a_{,a} \) is the expansion scalar; the shear tensor \( \tilde{\sigma}_{ab} \) and the rotation tensor \( \tilde{\omega}_{ab} \) are based on \( \tilde{u}_a \); for the notation, see section 2 of [1, 5]. These equations are valid for the collective fluid quantities even in the multi-component situation. In the multi-component case, taking the energy frame for individual components (and thus, setting \( \tilde{q}^{a}_{(i)a} \equiv 0 \)) we have

\[
\begin{align*}
\tilde{\mu}^{(i)} + (\tilde{\mu}^{(i)} + \tilde{p}^{(i)}) \tilde{\theta}^{(i)} + \tilde{\pi}^{ab} \tilde{\sigma}^{(i)ab}_{(i)} &= - \tilde{u}^{a}_{(i)}, \\
(\tilde{\mu}^{(i)} + \tilde{p}^{(i)}) \tilde{\alpha}^{(i)}_{a} + \tilde{h}^{(i)}_a^b (\tilde{p}^{(i)}_b + \tilde{\pi}^{(i)}_b^c) &= \tilde{h}^{(i)}_a^b \tilde{I}^{(i)}_a, \\
\tilde{\theta}^{(i)} + \frac{1}{3} \tilde{\theta}^{(i)}_a - \tilde{a}^{a}_{(i)a} + \tilde{\sigma}^{ab} \tilde{\sigma}^{(i)ab}_{(i)} - \tilde{\omega}^{ab} \tilde{\omega}^{(i)ab}_{(i)} + 4\pi G \left( \tilde{\mu}^{(i)} + 3\tilde{p}^{(i)} + 2\tilde{T}_{ab}^{(i)} \tilde{u}^{a}_{(i)} \tilde{u}^{b}_{(i)} \right) - \Lambda &= 0,
\end{align*}
\]

where \( \tilde{\mu}^{(i)} \equiv \tilde{\mu}^{(i)}_a \tilde{u}^{a}_{(i)} \), \( \tilde{\theta}^{(i)} \equiv \tilde{u}^{a}_{(i)a} \), and \( \tilde{\sigma}^{(i)ab}_{(i)} \) \( \tilde{\omega}^{ab} \tilde{\omega}^{(i)ab}_{(i)} \) are based on \( \tilde{u}^{a}_{(i)a} \); see [5]. In order to handle cosmological perturbations to third order, we also need the momentum constraint equation for a collective fluid

\[
\tilde{h}^{ab}_{a} \left( \tilde{\omega}^{bc}_{(i)c} - \tilde{\sigma}^{bc}_{(i)c} + \frac{2}{3} \tilde{\theta}^{(i)b} \right) + (\tilde{\omega}_{ab} + \tilde{\sigma}_{ab}) \tilde{a}^{b} = 0.
\]

A complete set of the covariant equations can be found in section 2 of [1] and [15].

In the ADM formulation, the energy conservation, momentum conservation, and trace of ADM propagation equations are

\[
\begin{align*}
E_{a} N^{-1} - E_{a} N^{a} N^{-1} - K^{a}_{a} (E + \frac{1}{3} S) - \bar{S}^{a\beta} \bar{K}_{a\beta} + N^{-2} \left( N^{2} J^{a} \right)_{,a} &= 0, \\
J_{a,0} N^{-1} - J_{a0} N^{a} N^{-1} - J_{\beta\alpha} N^{a} N^{-1} - K^{\beta}_{\beta} J^{a}_{a} + EN_{a} N^{-1} + S^{\alpha}_{a;\beta} + S^{\beta}_{a} N_{\beta} N^{-1} &= 0, \\
K^{\alpha}_{a,0} N^{-1} - K^{\alpha}_{a0} N^{a} N^{-1} + N^{a}_{\alpha} N^{-1} - \bar{K}^{\alpha\beta} \bar{K}_{a\beta} - \frac{1}{3} (K^{a})^{2} - 4\pi G \left( E + S \right) + \Lambda &= 0.
\end{align*}
\]

In the multi-component case we have

\[
\begin{align*}
E^{(i)}_{a} N^{-1} - E^{(i)}_{a} N^{a} N^{-1} - K^{a^{(i)}}_{a} \left( E^{(i)} + \frac{1}{3} S^{(i)} \right) - \bar{S}^{a^{(i)}}_{a} \bar{K}^{(i)}_{a\beta} + N^{-2} \left( N^{2} J^{a} \right)_{,a}^{(i)} &= 0, \\
J^{(i)}_{a,0} N^{-1} - J^{(i)}_{a0} N^{a} N^{-1} - J^{(i)}_{a} N^{a} N^{-1} - K^{\beta}_{\beta} J^{(i)}_{a} + E^{(i)} N_{a} N^{-1} + S^{\alpha}_{a;\beta}^{(i)} + S^{\beta}_{a} N_{\beta} N^{-1} &= \tilde{I}^{(i)}_{a}.
\end{align*}
\]

The momentum constraint equation is

\[
\bar{K}^{\alpha\beta} - \frac{2}{3} K^{\beta}_{\alpha} = 8\pi G J^{a}_{\alpha}.
\]
A complete set of the ADM equations together with notation can be found in section 2 of [1] and [16].

The above sets of equations are only parts of the covariant and the ADM equations; for complete sets, see [15, 16, 13, 17, 1]. We will show that, by making the proper choice of gauges, the scalar-type perturbations to third order can be derived from either of the above sets of equations. We will present the derivation based on the ADM equations, because the covariant formalism often requires lengthier calculation in our particular case; of course, the covariant equations also give the same result. As we will show, however, we use the two formulations simultaneously depending on what is convenient.

2.2. Metric

Our metric convention is the same as in [1]:

\[ \tilde{g}_{00} = -a^2 (1 + 2A), \quad \tilde{g}_{0\alpha} = -a^2 B_{\alpha}, \quad \tilde{g}_{\alpha\beta} = a^2 \left( g^{(3)}_{\alpha\beta} + 2C_{\alpha\beta} \right), \quad (18) \]

where tensor indices of \( B_{\alpha} \) and \( C_{\alpha\beta} \) are based on \( g^{(3)}_{\alpha\beta} \). To the third order, the inverse metric becomes

\[ \tilde{g}^{00} = -\frac{1}{a^2} (1 - 2A + 4A^2 - B^{\alpha}B_{\alpha} - 8A^3 + 4AB^{\alpha}B_{\alpha} + 2B^{\alpha}B^{\beta}C_{\alpha\beta}) , \]
\[ \tilde{g}^{0\alpha} = -\frac{1}{a^2} \left[ B^{\alpha} - 2AB^{\alpha} - 2B^{\beta}C^{\beta}_{\alpha} + B^{\alpha} \left( 4A^2 - B^{\beta}B_{\beta} \right) + 4C^{\alpha}_{\beta} \left( AB^{\beta} + B^{\gamma}C^{\gamma}_{\beta} \right) \right] , \]
\[ \tilde{g}^{\alpha\beta} = \frac{1}{a^2} \left( g^{(3)\alpha\beta} - 2C^{\alpha\beta} - B^{\gamma}B^{\beta} + 4C^{\alpha}_{\gamma}C^{\beta\gamma} + 2AB^{\alpha}B^{\beta} + 2B^{\alpha}B^{\beta}C^{\gamma}_{\beta} \right. \]
\[ \left. + 2B^{\beta}B^{\gamma}C^{\alpha}_{\gamma} - 8C^{\alpha}_{\gamma}C^{\beta\gamma}C^{\gamma\delta} \right) . \]

In order to derive perturbation equations to the third order, we need the connections only to the second order. These are presented in equation (52) of [1].

The ADM metric variables follow from equation (2) of [1] as

\[ N = a \left( 1 + A - \frac{1}{2} A^2 + \frac{1}{2} B^{\alpha}B_{\alpha} + \frac{1}{2} A^3 - \frac{1}{2} AB^{\alpha}B_{\alpha} - B^{\alpha}B^{\beta}C_{\alpha\beta} \right) , \]
\[ N_{\alpha} = -a^2 B_{\alpha}, \quad N^{\alpha} = -B^{\alpha} + 2B^{\beta}C^{\alpha}_{\beta} - 4B^{\beta}C^{\alpha}_{\gamma}C^{\gamma}_{\beta} , \]
\[ h_{\alpha\beta} = a^2 \left( g^{(3)}_{\alpha\beta} + 2C_{\alpha\beta} \right) , \quad (20) \]
\[ h^{\alpha\beta} = \frac{1}{a^2} \left( g^{(3)\alpha\beta} - 2C^{\alpha\beta} + 4C^{\alpha}_{\gamma}C^{\beta\gamma} - 8C^{\alpha}_{\gamma}C^{\beta\gamma}C^{\gamma\delta} \right) , \]

where the tensor index of \( N_{\alpha} \) is based on \( h_{\alpha\beta} \) as the metric; \( h^{\alpha\beta} \) is an inverse metric of \( h_{\alpha\beta} \).

2.3. Fluid quantities

We introduce perturbations of fluid quantities as

\[ \tilde{\mu} = \mu + \delta \mu, \quad \tilde{p} = p + \delta p, \quad \tilde{u}_{\alpha} \equiv av_{\alpha}, \quad \tilde{\pi}_{\alpha\beta} \equiv a^2 \Pi_{\alpha\beta} , \quad (21) \]

where tensor indices of \( v_{\alpha} \) and \( \Pi_{\alpha\beta} \) are based on \( g^{(3)}_{\alpha\beta} \). The above fluid quantities can be regarded as collective fluid quantities in the case of multi-component fluids; see below...
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equation (27). Although we will consider zero-pressure fluids, it is important to keep the
perturbed pressure ($\delta p$) and anisotropic stress ($\Pi_{\alpha\beta}$) because the collective pressure and
anisotropic stress do not vanish to non-linear order in the multi-component fluids even
in the zero-pressure case; see equation (45). In any case, for later convenience, in this
section we will present fluid quantities for most general fluids.

The components of the 4-vector $\tilde{u}_a$ are

$$\tilde{u}_a = av_a,$$
$$\tilde{u}_a = -a[1 + A - \frac{1}{2} A^2 + \frac{1}{2} (v^\alpha + B^\alpha) (v_\alpha + B_\alpha) + \frac{1}{2} A^3
\quad + \frac{1}{2} A (v^\alpha v_\alpha - B^\alpha B_\alpha) - C_{\alpha\beta} (v^\alpha + B^\alpha) (v^\beta + B^\beta)],$$
$$\tilde{u}_a = \frac{1}{a} \left[ v^\alpha + B^\alpha - AB^\alpha - 2C_{\alpha\beta} (v_\beta + B_\beta) + \frac{3}{2} A^2 B^\alpha + 2AB^\alpha C_{\beta}^\alpha
\quad + \frac{1}{2} B^\alpha (v^\beta v_\beta - B^\beta B_\beta) + 4C_{\beta}^\alpha C_{\gamma}^\beta (v^\gamma + B^\gamma) \right],$$
$$\tilde{u}_0 = \frac{1}{a} \left[ 1 - A + \frac{3}{2} A^2 + \frac{1}{2} (v^\alpha v_\alpha - B^\alpha B_\alpha) - 5 A^3 - \frac{1}{2} A (v^\alpha v_\alpha - 3B^\alpha B_\alpha)
\quad - C_{\alpha\beta} (v^\alpha v^\beta - B^\alpha B^\beta) \right].$$

In [1], instead of $v_a$, we used $V_a$ defined as

$$\tilde{u}_a ^\alpha \equiv \frac{1}{a} V^\alpha.$$ 

Thus, we have

$$v_\alpha = V_\alpha - B_\alpha + AB_\alpha + 2C_{\alpha\beta} V^\beta - \frac{3}{2} A^2 B_\alpha - B_\alpha V_\beta \left( \frac{1}{2} V^\beta - B^\beta \right).$$

The components of $\tilde{\pi}_{ab}$ are

$$\tilde{\pi}_{a\beta} ^\alpha \equiv a^2 \Pi_{a\beta} ^\alpha, \quad \tilde{\pi}_{0a} = -a^2 \Pi_{a\beta} \left[ v^\beta + B^\beta + Av^\beta - 2C_{\beta\gamma} (v_\gamma + B_\gamma) \right],$$
$$\tilde{\pi}_{00} = a^2 \Pi_{a\beta} (v^\alpha + B^\alpha) (v^\beta + B^\beta).$$

From $\tilde{\pi}_c ^\alpha \equiv 0$ we have

$$\Pi_\alpha ^\alpha - 2C_{\alpha\beta} \Pi_{\alpha\beta} + \left( 4C_{\gamma} ^\alpha C_{\beta\gamma} - v^\alpha v^\beta \right) \Pi_{\alpha\beta} = 0.$$ 

To the third order, the energy–momentum tensor becomes

$$\tilde{T}_0 ^\alpha = -\mu - \delta \mu - (\mu + p) v^\alpha (v_\alpha + B_\alpha) + (\mu + p) \left[ AB_\alpha + 2C_{\alpha\beta} (v^\beta + B^\beta) \right] v^\alpha
\quad - (\delta \mu + \delta p) v^\alpha (v_\alpha + B_\alpha) - \Pi_{\alpha\beta} v^\alpha (v^\beta + B^\beta),$$
$$\tilde{T}_\alpha ^\alpha = (\mu + p) v_\alpha - (\mu + p) Av_\alpha + (\mu + \delta \mu + \delta p) v_\alpha + \Pi_{\alpha\beta} v^\beta - (\delta \mu + \delta p) Av_\alpha
\quad + \frac{1}{2} (\mu + p) \left( 3A^2 + v^\beta v_\beta - B^\beta B_\beta \right) v_\alpha - \Pi_{\alpha\beta} \left( Av^\beta + 2C_{\beta\gamma} v_\gamma \right),$$
$$\tilde{T}_\beta ^\alpha = p\delta_\alpha ^\beta + \delta \rho \delta_\beta ^\alpha + \Pi_\beta ^\gamma + (\mu + p) (v^\alpha + B^\alpha) v_\beta - 2C_{\alpha\gamma} \Pi_{\beta\gamma} + (\delta \mu + \delta p) (v^\alpha + B^\alpha) v_\beta
\quad - (\mu + p) \left[ AB_\alpha + 2C_{\alpha\gamma} (v_\gamma + B_\gamma) \right] v_\beta + \Pi_{\beta\gamma} \left( B^\alpha v^\gamma + 4C_{\beta} ^\delta C_{\gamma} ^\delta \right).$$
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The above relations for 4-vectors, the energy–momentum tensor, and fluid quantities are derived for a single-component fluid. However, these are also valid as the collective components in the case of multi-component fluids. We can easily show that, by replacing these quantities with those for individual components, the same relations are valid for individual components as well. That is, for an individual component, say the $i$th component, on replacing

$$ T_{ab}, \, \tilde{\mu}, \, \tilde{p}, \, \tilde{u}_a, \, \tilde{v}(i)_{ab}, \, \mu, \, p, \, \delta \mu, \, \delta p, \, V_\alpha, \, v_\alpha, \, \Pi_{\alpha\beta}, $$

with

$$ \tilde{T}_{(i)ab}, \, \tilde{\mu}(i), \, \tilde{p}(i), \, \tilde{u}_{(i)a}, \, \tilde{v}(i)_{ab}, \, \mu(i), \, p(i), \, \delta \mu(i), \, \delta p(i), \, V_{(i)\alpha}, \, v_{(i)\alpha}, \, \Pi_{(i)\alpha\beta}, $$

respectively, equations (21)–(27) are valid for the $i$th component.

Using equations (2) and (27) we can express the collective fluid quantities in terms of the individual ones. In our perturbation approach we assume that the spatially homogeneous and isotropic Friedmann world model is valid in the background. Thus, to the background order we have

$$ \mu = \sum_j \mu(j), \quad p = \sum_j p(j). \quad (30) $$

To the third order in perturbations, we can show

$$ \delta \mu + \left[ (\mu + p) v^\alpha + (\delta \mu + \delta p) v^\alpha + \Pi^\alpha_{\beta} v^\beta \right] v_\alpha - 2 (\mu + p) C_{\alpha\beta} v^\alpha v^\beta $$

$$ = \sum_j \left\{ \delta \mu(j) + \left[ (\mu(j) + p(j)) v^\alpha_{(j)} + (\delta \mu(j) + \delta p(j)) v^\alpha_{(j)} + \Pi(j)_{\alpha\beta} v^\beta_{(j)} \right] v_{(j)\alpha} \right. $$

$$ - 2 \left( \mu(j) + p(j) \right) C_{\alpha\beta} v^\alpha_{(j)} v^\beta_{(j)} \right\}, $$

$$ \delta p + \frac{1}{3} \left[ (\mu + p) v^\alpha + (\delta \mu + \delta p) v^\alpha + \Pi^\alpha_{\beta} v^\beta \right] v_\alpha - \frac{2}{3} (\mu + p) C_{\alpha\beta} v^\alpha v^\beta $$

$$ = \sum_j \left\{ \delta p(j) + \frac{1}{3} \left[ (\mu(j) + p(j)) v^\alpha_{(j)} + (\delta \mu(j) + \delta p(j)) v^\alpha_{(j)} + \Pi(j)_{\alpha\beta} v^\beta_{(j)} \right] v_{(j)\alpha} \right. $$

$$ - \frac{2}{3} \left( \mu(j) + p(j) \right) C_{\alpha\beta} v^\alpha_{(j)} v^\beta_{(j)} \right\}, $$

$$ (\mu + p) v_\alpha + \left[ \delta \mu + \delta p + \frac{4}{3} (\mu + p) v^\beta v_\beta \right] v_\alpha + \left[ \Pi^\beta_{\alpha\gamma} + (\mu + p) \left( v_\alpha v^\beta - \frac{1}{3} \delta^\beta\gamma v_\gamma \right) \right] v_\beta $$

$$ - \frac{2}{3} (\mu + p) v_\alpha v^\beta v_\beta - 2 \Pi_{\alpha\beta\gamma} v_\gamma = \sum_j \left\{ (\mu(j) + p(j)) v_{(j)\alpha} \right. $$

$$ + \left[ \delta \mu(j) + \delta p(j) + \frac{4}{3} (\mu(j) + p(j)) v^\beta_{(j)} v_{(j)\beta} \right] v_{(j)\alpha} $$

$$ + \left[ \Pi(j)_{\alpha\beta} + (\mu(j) + p(j)) \left( v_{(j)\alpha} v^\beta_{(j)} - \frac{1}{3} \delta^\beta\gamma_{(j)} v_{(j)\gamma} \right) \right] v_{(j)\beta} $$

$$ - \frac{2}{3} (\mu(j) + p(j)) v_{(j)\alpha} v^\beta_{(j)} v_{(j)\beta} - 2 \Pi_{(j)\alpha\beta\gamma} v_{(j)\gamma} \right\}, $$
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\[ \Pi^\alpha_\beta + \left[ (\mu + p) v^\alpha + (\delta \mu + \delta p) v^\alpha + \Pi^\alpha_\gamma v^\gamma \right] v^\beta - \frac{2}{3} (\mu + p) C^\alpha_\beta v^\gamma v^\nu - \Pi^\alpha_\beta v^\gamma v^\nu \]

\[ - \frac{i}{3} \delta^\alpha_\beta \left\{ \left[ (\mu + p) v^\gamma + (\delta \mu + \delta p) v^\gamma + \Pi^\alpha_\gamma v^\nu \right] v^\gamma - 2 (\mu + p) C^\gamma_\delta v^\nu v^\delta \right\} \]

\[ = \sum_j \left\{ \Pi^{\alpha}_{(j)\beta} + \left[ (\mu_{(j)} + p_{(j)}) v^{\alpha}_{(j)} + (\delta \mu_{(j)} + \delta p_{(j)}) v^{\alpha}_{(j)} + \Pi^{\alpha}_{(j)\gamma} v^{\gamma}_{(j)} \right] v^{(j)\beta} \right\}, \]

\[ - \frac{2}{3} (\mu_{(j)} + p_{(j)}) C^\alpha_\beta v^\gamma_{(j)} v^\gamma_{(j)\beta} - \Pi^{\alpha}_{(j)\gamma} v^\gamma_{(j)} v^\gamma_{(j)\beta} - \frac{i}{3} \delta^\alpha_\beta \left\{ \left[ (\mu_{(j)} + p_{(j)}) v^\gamma_{(j)} \right] v^\gamma_{(j)} \right\} + \left( \delta \mu_{(j)} + \delta p_{(j)} \right) v^\gamma_{(j)} + \Pi^{\alpha}_{(j)\gamma} v^\delta_{(j)} \left\{ v^{(j)\gamma} v^\delta_{(j)} \right\}. \] \tag{31} \]

These relations follow from equation (2) using equations (27) for the collective component and its individual component counterpart constructed using the prescription in equations (28) and (29). Equation (31) can be rearranged to give

\[ \delta \mu = \sum_j \left\{ \delta \mu_{(j)} + (\mu_{(j)} + p_{(j)}) v^{(j)\alpha} \right\} (v^{(j)\alpha} - v_{\alpha}) \]

\[ + \left[ (\delta \mu_{(j)} + \delta p_{(j)}) v^{\alpha}_{(j)} + \Pi^{\alpha}_{(j)\beta} v^\beta_{(j)} - 2 (\mu_{(j)} + p_{(j)}) C^\alpha_\beta v^\beta_{(j)} \right] \left\{ v^{(j)\alpha} - v_{\alpha} \right\} \]

\[ \delta p = \sum_j \left\{ \delta p_{(j)} + \frac{i}{3} (\mu_{(j)} + p_{(j)}) v^{\alpha}_{(j)} \right\} (v^{(j)\alpha} - v_{\alpha}) \]

\[ + \frac{1}{3} \left\{ (\delta \mu_{(j)} + \delta p_{(j)}) v^{\alpha}_{(j)} + \Pi^{\alpha}_{(j)\beta} v^\beta_{(j)} - 2 (\mu_{(j)} + p_{(j)}) C^\alpha_\beta v^\beta_{(j)} \right\} \left\{ v^{(j)\alpha} - v_{\alpha} \right\} \]

\[ (\mu + p) v_{\alpha} = \sum_j \left\{ (\mu_{(j)} + p_{(j)}) v^{(j)\alpha} + (\delta \mu_{(j)} + \delta p_{(j)}) \left\{ v^{(j)\alpha} - v_{\alpha} \right\} \right. \]

\[ + \Pi^{\beta}_{(j)\alpha} \left( v^{(j)\beta} - v_{\beta} \right) + (\mu_{(j)} + p_{(j)}) v^\beta_{(j)} v^{(j)\beta} \left\{ v^{(j)\alpha} - v_{\alpha} \right\} \]

\[ - \left\{ \frac{1}{2} (\mu_{(j)} + p_{(j)}) v^{(j)\alpha} \left( v^\beta_{(j)} + 3v^\beta \right) + 2 \Pi^{\beta}_{(j)\alpha} C^\beta_\gamma \right\} \left\{ v^{(j)\beta} - v_{\beta} \right\} \right\} \]

\[ \Pi^{\alpha}_{\beta} = \sum_j \left\{ \Pi^{\alpha}_{(j)\beta} + (\mu_{(j)} + p_{(j)}) \left\{ v^{\alpha}_{(j)} \left( v^{(j)\beta} - v_{\beta} \right) - \frac{i}{3} \delta^\alpha_\beta v^\gamma_{(j)} \left( v^{(j)\gamma} - v_{\gamma} \right) \right\} \right. \]

\[ + \left[ (\delta \mu_{(j)} + \delta p_{(j)}) v^{\alpha}_{(j)} - \Pi^{\alpha}_{(j)\gamma} v^\gamma_{(j)} \right] \left\{ v^{(j)\beta} - v_{\beta} \right\} \]

\[ - \frac{2}{3} (\mu_{(j)} + p_{(j)}) C^\alpha_\beta v^\gamma_{(j)} \left( v^{(j)\gamma} - v_{\gamma} \right) - \frac{i}{3} \delta^\alpha_\beta \left\{ (\delta \mu_{(j)} + \delta p_{(j)}) v^\gamma_{(j)} + \Pi^{\alpha}_{(j)\gamma} v^\delta_{(j)} \right. \]

\[ - 2 (\mu_{(j)} + p_{(j)}) C^\gamma_\delta v^\delta_{(j)} \left\{ v^{(j)\gamma} - v_{\gamma} \right\} \right\}. \]

These relations follow from the relations in equation (31). For example, the \( \delta \mu \) relation above can be derived by moving terms other than \( \delta \mu \) in the first relation of equation (31) to the right-hand side using the third relation of equation (31) perturbatively. The ADM
fluid quantities are introduced as

\[ E \equiv \tilde{n}_a \tilde{n}_b \tilde{T}^{ab}, \quad J_\alpha \equiv -\tilde{n}_b \tilde{T}^b_\alpha, \]

\[ S_{\alpha\beta} \equiv \tilde{T}_{\alpha\beta}, \quad S \equiv h^{\alpha\beta} S_{\alpha\beta}, \quad \bar{S}_{\alpha\beta} \equiv S_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta} S, \]  

(33)

where tensor indices of \( J \) and \( S_{\alpha\beta} \) are based on \( h_{\alpha\beta} \) as the metric. The 4-vector \( \tilde{n}_a \) is a normal frame 4-vector with \( \tilde{n}_a \equiv 0 \). Thus by setting \( v_\alpha \equiv 0 \) we have \( \bar{n}_a = \tilde{n}_a \), and equation (22) gives

\[ \tilde{n}_a \equiv 0, \]

\[ \tilde{n}_0 = -a \left( 1 + A - \frac{1}{2} A^2 + \frac{1}{2} B^\alpha B_\alpha + \frac{1}{2} A^3 - \frac{1}{2} AB^\alpha B_\alpha - C_{\alpha\beta} B^\alpha B^\beta \right), \]

\[ \tilde{n}^\alpha = \frac{1}{a} \left( B^\alpha - AB^\alpha - 2 C_{\alpha\beta} B^\alpha + \frac{1}{2} A^2 B^\alpha + 2 AB^\alpha C_{\alpha\beta} - \frac{1}{2} B^\alpha B^\beta B_\beta + 4 C_{\alpha\beta} C_{\gamma} B^\gamma \right), \]

(34)

\[ \tilde{n}^0 = \frac{1}{a} \left( 1 - A + \frac{3}{2} A^2 - \frac{1}{2} B^\alpha B_\alpha - \frac{5}{2} A^3 + \frac{3}{2} AB^\alpha B_\alpha + C_{\alpha\beta} B^\alpha B^\beta \right). \]

Using equations (27) and (34), equation (33) gives

\[ E = \mu + \delta \mu + (\mu + p) v^\alpha v_\alpha + (\delta \mu + \delta p) v^\alpha v_\alpha - 2 (\mu + p) C_{\alpha\beta} v^\alpha v^\beta + \Pi_{\alpha\beta} v^\alpha v^\beta, \]

\[ J_\alpha = a \left( (\mu + p) v_\alpha + (\delta \mu + \delta p) v_\alpha + \Pi_{\alpha\beta} v^\beta v_\beta + \frac{1}{2} (\mu + p) v_\alpha v^\beta v_\beta - 2 \Pi_{\alpha\beta} C_{\gamma} B^\gamma \right), \]

\[ S_{\alpha\beta} = a^2 \left[ pg_{\alpha\beta}^{(3)} + \delta pg_{\alpha\beta}^{(3)} + \Pi_{\alpha\beta} + 2 p C_{\alpha\beta} + (\mu + p) v_\alpha v_\beta + 2 \delta p C_{\alpha\beta} + (\delta \mu + \delta p) v_\alpha v_\beta \right], \]

\[ S = 3 p + 3 \delta p + (\mu + p) v^\alpha v_\alpha + (\delta \mu + \delta p) v^\alpha v_\alpha + v^\alpha v^\beta \Pi_{\alpha\beta} - 2 (\mu + p) C_{\alpha\beta}, \]

\[ \bar{S}_{\alpha\beta} = a^2 \left\{ \Pi_{\alpha\beta} + (\mu + p) \left( v_\alpha v_\beta - \frac{1}{3} g_{\alpha\beta} v^\gamma v_\gamma \right) + (\delta \mu + \delta p) v_\alpha v_\beta - \frac{2}{3} (\mu + p) C_{\alpha\beta} v^\gamma v_\gamma - \frac{1}{3} g_{\alpha\beta}^{(3)} \left[ (\delta \mu + \delta p) v^\gamma v_\gamma + \Pi_{\gamma\delta} v^\gamma v_\delta \right] - 2 (\mu + p) C_{\gamma\delta} v^\gamma v_\delta \right\}. \]

The individual ADM fluid quantities can be found by replacing

\[ E, \quad J_\alpha, \quad S_{\alpha\beta}, \quad S, \quad \bar{S}_{\alpha\beta}, \]

(36)

with

\[ E_{(i)}, \quad J_{(i)\alpha}, \quad S_{(i)\alpha\beta}, \quad S_{(i)}, \quad \bar{S}_{(i)\alpha\beta}, \]

(37)

and similarly for the fluid quantities and the energy–momentum tensor as in equations (28) and (29). From equation (3) we have

\[ E = \sum_j E_{(j)}, \quad J_\alpha = \sum_j J_{(j)\alpha}, \]

\[ S_{\alpha\beta} = \sum_j S_{(j)\alpha\beta}, \quad S = \sum_j S_{(j)}, \quad \bar{S}_{\alpha\beta} = \sum_j \bar{S}_{(j)\alpha\beta}. \]

(38)

2.4. Decomposition

We decompose the metric into three perturbation types [1, 18]:

\[ A \equiv \alpha, \quad B_\alpha \equiv \beta_\alpha + B^{(v)}_\alpha, \quad C_{\alpha\beta} \equiv \varphi g_{\alpha\beta}^{(3)} + \gamma_{\alpha\beta} + C_{(\alpha\beta)}^{(v)} + C_{(\alpha\beta)}^{(t)} \]

(39)
where superscripts \((v)\) and \((t)\) indicate the transverse vector-type, and transverse-tracefree tensor-type perturbations, respectively. Only to the linear order perturbations in the homogeneous isotropic background do these three types of perturbation decouple and evolve independently. We introduce

\[
\chi \equiv a \left( \beta + c^{-1} a^2 \right), \quad \Psi_a^{(v)} \equiv B^{(v)}_a + c^{-1} a \dot{C}^{(v)}_a, \quad (40)
\]

which are spatially gauge-invariant combinations to the linear order \([13]\). We set

\[
K^{(v)}_a \equiv -3H + \kappa. \quad (41)
\]

Thus, \(\kappa\) is the perturbed part of the trace of the extrinsic curvature; \(\kappa\) can be also regarded as the perturbed part of expansion scalar \(\dot{\theta}\) based on the normal frame 4-vector \(\tilde{n}_a\) with a minus sign, i.e., \(\dot{\theta} \equiv \tilde{n}_a^{(\nu)} \equiv 3H - \kappa\). By using \(\kappa\) we can avoid third-order expansion of the trace of extrinsic curvature \(K^{(v)}_a\). Identifying \(\kappa\) with a divergence of a Newtonian velocity perturbation later will be an important step in our analysis; see equations \((64)\) and \((102)\).

For the fluid quantities we make the decomposition

\[
v_\alpha \equiv -v_\alpha + v_\alpha^{(v)}, \quad \Pi_{\alpha \beta} \equiv \frac{1}{a^2} \left( \Pi_{(i)\alpha \beta} - \frac{1}{3} g_{\alpha \beta} \Delta \Pi \right) + \frac{1}{a} \Pi^{(v)}_{(i)\alpha \beta} + \Pi^{(t)}_{(i)\alpha \beta},
\]

\[
v^{(v)}_{(i)\alpha} \equiv -v^{(v)}_{(i)\alpha} + v^{(v)}_{(i)\alpha},
\]

\[
\Pi^{(v)}_{(i)\alpha \beta} \equiv \frac{1}{a^2} \left( \Pi^{(i)}_{(i)\alpha \beta} - \frac{1}{3} g^{i(i)}_{\alpha \beta} \Delta \Pi^{(i)} \right) + \frac{1}{a} \Pi^{(v)}_{(i)\alpha \beta} + \Pi^{(i)}_{(i)\alpha \beta},
\]

\[
\delta I^{(v)}_{(i)\alpha} \equiv \delta I^{(i)}_{(i)\alpha} + \delta I^{(v)}_{(i)\alpha}.
\]

The perturbed fluid velocity variables \(v\) and \(v^{(v)}\) subtly differ from the ones introduced in \([1]\); see equations \((84)\) and \((110)\) in \([5]\).

### 2.5. Zero-pressure irrotational fluids in the comoving gauge

The zero-pressure condition sets

\[
p_{(i)} \equiv 0, \quad \delta p_{(i)} \equiv 0 \equiv \Pi_{(i)\alpha \beta}. \quad (43)
\]

The comoving gauge condition \((\nu \equiv 0)\) and irrotational condition \((v^{(v)} = 0)\) give

\[
v_\alpha \equiv 0. \quad (44)
\]

Under these conditions, equation \((32)\) becomes

\[
\delta \mu = \sum_j \left( \delta \mu_{(j)} + \mu_{(j)} v^{(v)}_{(j)\alpha} v^{(v)}_{(j)\alpha} + \delta \mu_{(j)} v^{(v)}_{(j)\alpha} v^{\beta}_{(j)\alpha} - 2 \mu_{(j)} C_{\alpha \beta} v^{\alpha}_{(j)\beta} v^{\beta}_{(j)\alpha} \right),
\]

\[
\delta p = \frac{1}{3} \sum_j \left( \mu_{(j)} v^{(v)}_{(j)\alpha} v^{(v)}_{(j)\alpha} + \delta \mu_{(j)} v^{(v)}_{(j)\alpha} v^{\beta}_{(j)\alpha} - 2 \mu_{(j)} C_{\alpha \beta} v^{\alpha}_{(j)\beta} v^{\beta}_{(j)\alpha} \right),
\]

\[
0 = \sum_j \left( \mu_{(j)} v^{(v)}_{(j)\alpha} + \delta \mu_{(j)} v^{(v)}_{(j)\alpha} + \frac{1}{2} \mu_{(j)} v^{(v)}_{(j)\alpha} v^{(v)}_{(j)\beta} \right),
\]

\[
\Pi^{(v)}_{\alpha \beta} = \sum_j \left[ \delta \mu_{(j)} v^{(v)}_{(j)\beta} - \frac{1}{3} \delta^{(v)}_{\beta} v^{(v)}_{(j)\gamma} \right] + \delta \mu_{(j)} (v^{(v)}_{(j)\beta} v^{(v)}_{(j)\gamma} - \frac{1}{3} \delta^{(v)}_{\beta} v^{(v)}_{(j)\gamma})
\]

\[
- \frac{2}{3} \mu_{(j)} \left( C^{(v)}_{\alpha \beta} v^{(v)}_{(j)\gamma} - \delta^{(v)}_{\beta} C^{(v)}_{\alpha \gamma} v^{(v)}_{(j)\gamma} \right).
\]
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Notice that for the non-linear perturbations, the collective pressure $\delta p$ and anisotropic stress $\Pi_{\alpha\beta}$ no longer vanish even for zero-pressure fluids; also, the collective energy density $\delta \mu$ is no longer a simple sum of individual components.

For the ADM fluid quantities, equation (35) becomes

$$E = \mu + \delta \mu, \quad J_\alpha = 0,$$

$$S_{\alpha\beta} = a^2 \left( \delta p_{3\alpha\beta}^{(3)} + \Pi_{\alpha\beta} + 2 \delta p C_{\alpha\beta} \right), \quad S = 3 \delta p, \quad \tilde{S}_{\alpha\beta} = a^2 \Pi_{\alpha\beta}. \quad (46)$$

For the individual component, equation (35) gives

$$E_{(i)} = \mu_{(i)} + \delta \mu_{(i)} + \mu_{(i)} v_{(i)}^\alpha v_{(i)\alpha} + \delta \mu_{(i)} v_{(i)}^\alpha v_{(i)\alpha} - 2 \mu_{(i)} C_{\alpha\beta} v_{(i)}^\alpha v_{(i)}^\beta,$$

$$J_{(i)\alpha} = a \left( \mu_{(i)} v_{(i)\alpha} + \delta \mu_{(i)} v_{(i)\alpha} + \frac{1}{2} \mu_{(i)} v_{(i)\alpha} v_{(i)\beta} \right),$$

$$S_{(i)\alpha\beta} = a^2 \left( \mu_{(i)} v_{(i)\alpha} v_{(i)\beta} + \delta \mu_{(i)} v_{(i)\alpha} v_{(i)\beta} \right),$$

$$S_{(i)} = \mu_{(i)} v_{(i)}^\alpha v_{(i)\alpha} + \delta \mu_{(i)} v_{(i)}^\alpha v_{(i)\alpha} - 2 \mu_{(i)} C_{\alpha\beta} v_{(i)}^\alpha v_{(i)}^\beta,$$

$$\tilde{S}_{(i)\alpha\beta} = a^2 \left[ \delta \mu_{(i)} v_{(i)\alpha} v_{(i)\beta} - \frac{1}{3} g_{3\alpha\beta} (\gamma_{(i)} v_{(i)\gamma}) + \delta \mu_{(i)} v_{(i)\alpha} v_{(i)\beta} + \frac{2}{3} \mu_{(i)} C_{\alpha\beta} v_{(i)}^\gamma v_{(i)\gamma} - \frac{1}{3} g_{3\alpha\beta} \left( \delta \mu_{(i)} v_{(i)}^\gamma v_{(i)\gamma} - 2 \mu_{(i)} C_{\gamma\delta} v_{(i)}^\gamma v_{(i)}^\delta \right) \right]. \quad (47)$$

3. Second-order equations: summary

In this section we summarize our previous work on second-order perturbations in the zero-pressure, irrotational, but multi-component fluids [5]. The results show that, to the second order, effectively the relativistic equations for the density and velocity perturbations coincide with the Newtonian ones even in the multi-component situation. This provides a reason to go to the third order in relativistic perturbation in order to find pure general relativistic deviations from Newton’s theory. Since the zero-pressure Newtonian perturbation equations have only quadratic order non-linearity, any non-vanishing third-order terms in relativistic analysis can be regarded as pure general relativistic correction terms. In [4] we presented such third-order correction terms in a single-component case. In the next section we will derive the third-order correction terms appearing in the multi-component situation. As the third-order analysis closely follows the second-order case, in the following we will derive the second-order equations directly from the ADM equations presented in section 2.1.

We consider zero-pressure multi-component fluids; thus

$$p_{(i)} \equiv 0, \quad \delta p_{(i)} \equiv 0 \equiv \Pi_{(i)\alpha\beta}. \quad (48)$$

The irrotational condition and the temporal comoving gauge condition lead to $v_\alpha = 0$; thus from equation (35) we have $J_\alpha = 0$. As the spatial gauge condition we take $\gamma \equiv 0$, and thus $\beta = \chi/a$. To the second-order perturbations, equation (13) gives

$$\alpha = -\frac{1}{2a^2} \chi^\alpha \chi_\alpha - \sum_j \frac{\mu_{(j)}}{\mu} \left[ \frac{1}{2} v_{(j)}^\alpha v_{(j)\alpha} + \Delta^{-1} \nabla_\alpha \left( v_{(j)}^\alpha v_{(j)\beta} \right) \right], \quad (49)$$
and thus equation (20) gives

\[ N = a - a \sum_j \frac{\mu(j)}{\mu} \left[ \frac{1}{2} v^{\alpha}_j v_{(j)\alpha} + \Delta^{-1} \nabla_\alpha \left( v^{\alpha}_j v^{\beta}_{(j)\beta} \right) \right], \tag{50} \]

Using equations (46) and (47), equations (12), (14), (15), and (16), respectively, give

\[ \dot{\delta} - \kappa = -\frac{c}{a^2} \delta \alpha \chi - \delta \kappa + \frac{1}{2} H \sum_j \mu(j) v^{\alpha}_j v_{(j)\alpha} + 3H \sum_j \frac{\mu(j)}{\mu} \Delta^{-1} \nabla_\alpha \left( v^{\alpha}_j v^{\beta}_{(j)\beta} \right), \tag{51} \]

\[ \dot{\kappa} + 2H \kappa - 4\pi G \rho \delta = -\frac{c}{a^2} \kappa \chi + \frac{1}{3} \kappa^2 + \left( \dot{C}^{(t)\alpha \beta} + \frac{c}{a^2} \chi \alpha \beta \right) \left( \dot{C}^{(t)\alpha \beta} + \frac{c}{a^2} \chi \alpha \beta \right) \]

\[ - \frac{1}{3} \left( \frac{\Delta}{a^2} \right)^2 + \frac{1}{2} \left( 3\dot{H} + 8\pi G \rho + \frac{c^2 \Delta}{a^2} \right) \sum_j \mu(j) v^{\alpha}_j v_{(j)\alpha} \]

\[ + \left( 3\dot{H} + \frac{c^2 \Delta}{a^2} \right) \sum_j \frac{\mu(j)}{\mu} \Delta^{-1} \nabla_\alpha \left( v^{\alpha}_j v^{\beta}_{(j)\beta} \right), \tag{52} \]

\[ \delta_{(i)} - \kappa + \frac{a}{2} \left[ (1 + \delta_{(i)}) v^{\alpha}_{(i)\alpha} \right] = -\frac{c}{a^2} \delta_{(i)\alpha} \chi - \delta_{(i)\alpha} \kappa + H v^{\alpha}_{(i)\alpha} v^{\alpha}_{(i)\alpha} \]

\[ - \frac{c}{a} \left( \varphi \alpha \beta v^{\alpha}_{(i)\alpha} - 2\varphi \alpha \beta v^{\alpha}_{(i)\alpha} - 2 \varphi_{(i)\beta} C^{(t)\alpha \beta} \right) + \frac{3}{2} H \sum_j \mu(j) v^{\alpha}_j v_{(j)\alpha} \]

\[ + 3H \sum_j \frac{\mu(j)}{\mu} \Delta^{-1} \nabla_\alpha \left( v^{\alpha}_j v^{\beta}_{(j)\beta} \right), \tag{53} \]

\[ \frac{1}{a^2} \left[ a^2 \left( \delta + \frac{c}{a^2} \delta \alpha \chi \alpha \right) \right] - 4\pi G \rho \delta (1 + \delta) = -\frac{c}{a^2} \kappa \chi - \frac{4}{3} \kappa^2 \]

\[ + \left( \dot{C}^{(t)\alpha \beta} + \frac{c}{a^2} \chi \alpha \beta \right) \left( \dot{C}^{(t)\alpha \beta} + \frac{c}{a^2} \chi \alpha \beta \right) - \frac{1}{3} \left( \frac{\Delta}{a^2} \right)^2 \]

\[ + \left( 2\dot{H} + 4\pi G \rho + \frac{c^2 \Delta}{a^2} \right) \sum_j \mu(j) v^{\alpha}_j v_{(j)\alpha} \]

\[ + \left( 6\dot{H} + \frac{c^2 \Delta}{a^2} \right) \sum_j \frac{\mu(j)}{\mu} \Delta^{-1} \nabla_\alpha \left( v^{\alpha}_j v^{\beta}_{(j)\beta} \right), \tag{54} \]

From equations (51) and (52), and equations (52)–(54), respectively, we can derive

\[ \frac{1}{a^2} \left[ a^2 \left( \delta + \frac{c}{a^2} \delta \alpha \chi \alpha \right) \right] - 4\pi G \rho \delta (1 + \delta) = -\frac{c}{a^2} \kappa \chi + \frac{4}{3} \kappa^2 \]

\[ + \left( \dot{C}^{(t)\alpha \beta} + \frac{c}{a^2} \chi \alpha \beta \right) \left( \dot{C}^{(t)\alpha \beta} + \frac{c}{a^2} \chi \alpha \beta \right) - \frac{1}{3} \left( \frac{\Delta}{a^2} \right)^2 \]

\[ + \left( 2\dot{H} + 4\pi G \rho + \frac{c^2 \Delta}{a^2} \right) \sum_j \mu(j) v^{\alpha}_j v_{(j)\alpha} \]

\[ + \left( 6\dot{H} + \frac{c^2 \Delta}{a^2} \right) \sum_j \frac{\mu(j)}{\mu} \Delta^{-1} \nabla_\alpha \left( v^{\alpha}_j v^{\beta}_{(j)\beta} \right), \tag{55} \]
Newtonian velocity perturbation variables are introduced as \[14,19\] and thus the dimensions are
\[
\sum \Delta \left( v_{(i)\alpha} \chi^{\alpha} \right) + \frac{c^2}{a^2} \left( v^\alpha_{(i)} v^\beta_{(i)} \right)_{\alpha\beta} + \frac{2c}{a} v^\alpha_{(i)} \dot{C}^\alpha_{\alpha\beta} + \frac{\mu}{\mu^2} \Delta^{-1} \nabla \chi \left( v^\alpha_{(j)} v^\beta_{(j)} \right)_{\alpha\beta}.
\]
(56)

We have recovered \( c \) using
\[
\left[ g_{ab} \right] = \left[ g^{ab} \right] = \left[ \bar{u} \right] = \left[ \bar{a} \right] = \left[ a \right] = \left[ \varphi \right] = \left[ C^{(\ell)}_{\alpha\beta} \right] = 1, \quad \beta = \left[ \chi \right] = L,
\]
(57)

where \( R^{(3)} \equiv 6K \) is the scalar curvature of the 3-space metric \( g^{(3)}_{\alpha\beta} \).

Without the rotational mode, we introduce
\[
v_{(i)\alpha} \equiv -v_{(i),\alpha}.
\]
(58)

Since we used the comoving gauge condition, to the linear order, we have
\[
v_{(i)} = v_{(i)v} \equiv v_{(i)} - v = v_{(i)\chi} - v_{\chi},
\]
(59)

where \( v_{\chi} \equiv v - \chi/a \) and \( v_{(i)\chi} \equiv v_{(i)} - \chi/a \) to the linear order. To the linear order, Newtonian velocity perturbation variables are introduced as \[14,19\]
\[
u \equiv \nabla u \equiv -c \nabla v_{\chi}, \quad u_i \equiv \nabla u_i \equiv -c \nabla v_{(i)\chi},
\]
(60)

and thus
\[
v_{(i)} \equiv -\nabla v_{(i)} = \frac{1}{c} \left( u_i - u \right).
\]
(61)

The dimensions are
\[
[v] = [v_{(j)}] = L, \quad [u] = [u_i] = [c] = L/T, \quad [u] = [u_i] = L[c].
\]
(62)

Thus, we have
\[
\frac{\Delta \rho}{\mu} = \frac{1}{3} \sum_j \frac{\rho_j}{\rho} \frac{1}{c^2} \left| u_j - u \right|^2,
\]
\[
\sum_j \frac{\mu_j}{\mu} \Delta^{-1} \nabla \chi \left( v^\alpha_{(j)} v^\beta_{(j)} \right)_{\alpha\beta} = \sum_j \frac{\rho_j}{\rho} \frac{1}{c^2} \Delta^{-1} \nabla \cdot \left[ (u_j - u) \nabla \cdot (u_j - u) \right].
\]
(63)

3.1. Newtonian correspondence

We assume a flat background, and thus \( K = 0 \). In this case we have \( \varphi_v = 0 \) to the linear order even in the presence of the cosmological constant and multiple components \[14,19\]; see equation (100). To the second order, we identify the Newtonian perturbation variables \( \delta, \delta_i, u, \) and \( u_i \) as
\[
\kappa_v \equiv -\frac{1}{a} \nabla \cdot u, \quad u \equiv \nabla u, \quad v_{(i)v} \equiv \frac{1}{c} \left( u_i - u \right), \quad \delta \equiv \delta_v, \quad \delta_i \equiv \delta_{(i)v}.
\]
(64)

Comparison of the consequent relativistic equations with the Newtonian equations will show apparently why these identifications are the proper ones. Examination of
equations (51)–(56) shows that we need $\chi(= \chi_v)$ only to the linear order. From the momentum constraint equation in equation (17), see equation (197) of [1], we have

$$\kappa_v = -\frac{\Delta}{a^2} \chi_v,$$

and thus

$$\chi_v \equiv \frac{a}{c} u,$$

to the linear order.

Equations (51)–(56) become

$$\dot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{u} = -\frac{1}{a} \nabla \cdot (\delta \mathbf{u})
$$

+ $H \sum_j \frac{\theta_j}{c^2} \left\{ \frac{1}{2} |\mathbf{u}_j - \mathbf{u}|^2 + 3 \Delta^{-1} \nabla \cdot [(\mathbf{u}_j - \mathbf{u}) \nabla \cdot (\mathbf{u}_j - \mathbf{u})] \right\},$ (67)

$$\frac{1}{a} \nabla \cdot (\dot{\mathbf{u}} + H \mathbf{u}) + 4\pi G \rho \delta = -\frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) - \dot{C}^{(t)\alpha\beta} \left( \frac{2}{a} \dot{u}_{\alpha|\beta} + C^{(t)}_{\alpha\beta} \right)
$$

+ $\sum_j \frac{\theta_j}{c^2} \left\{ \frac{1}{2} (4\pi G \rho - c^2 \frac{\Delta}{a^2}) |\mathbf{u}_j - \mathbf{u}|^2
$$

+ 12\pi G \rho - c^2 \frac{\Delta}{a^2} \right) \Delta^{-1} \nabla \cdot [(\mathbf{u}_j - \mathbf{u}) \nabla \cdot (\mathbf{u}_j - \mathbf{u})],$ (68)

$$\dot{\delta}_i + \frac{1}{a} \nabla \cdot \mathbf{u}_i = -\frac{1}{a} \nabla \cdot (\delta_i \mathbf{u}_i) + \frac{1}{a} \left[ 2\varphi \nabla \cdot (\mathbf{u}_i - \mathbf{u}) - (\mathbf{u}_i - \mathbf{u}) \cdot \nabla \varphi + 2 (u_i^\alpha - u^\alpha)^{\beta} C^{(t)}_{\alpha\beta} \right]
$$

+ $H \frac{1}{c^2} |\mathbf{u}_i - \mathbf{u}|^2
$$

+ 3H \sum_j \frac{\theta_j}{c^2} \left\{ \frac{1}{2} |\mathbf{u}_j - \mathbf{u}|^2 + \Delta^{-1} \nabla \cdot [(\mathbf{u}_j - \mathbf{u}) \nabla \cdot (\mathbf{u}_j - \mathbf{u})] \right\},$ (69)

$$\frac{1}{a} \nabla \cdot (\dot{\mathbf{u}}_i + H \mathbf{u}_i) + 4\pi G \rho \delta = -\frac{1}{a^2} \nabla \cdot (\mathbf{u}_i \cdot \nabla \mathbf{u}_i) - \dot{C}^{(t)\alpha\beta} \left( \frac{2}{a} \dot{u}_{i|\beta} + C^{(t)}_{\alpha\beta} \right)
$$

+ $4\pi G \sum_j \frac{\theta_j}{c^2} \left\{ \frac{1}{2} |\mathbf{u}_j - \mathbf{u}|^2 + 3 \Delta^{-1} \nabla \cdot [(\mathbf{u}_j - \mathbf{u}) \nabla \cdot (\mathbf{u}_j - \mathbf{u})] \right\},$ (70)

$$\frac{1}{a^2} (a^2 \dot{\delta}) - 4\pi G \rho \delta = -\frac{1}{a^2} \left[ a \nabla \cdot (\delta \mathbf{u}) \right] + \frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \dot{C}^{(t)\alpha\beta} \left( \frac{2}{a} \dot{u}_{\alpha|\beta} + C^{(t)}_{\alpha\beta} \right)
$$

- $\sum_j \frac{\theta_j}{c^2} \left\{ (4\pi G \rho - c^2 \frac{\Delta}{a^2}) |\mathbf{u}_j - \mathbf{u}|^2
$$

+ \left( 24\pi G \rho - c^2 \frac{\Delta}{a^2} \right) \Delta^{-1} \nabla \cdot [(\mathbf{u}_j - \mathbf{u}) \nabla \cdot (\mathbf{u}_j - \mathbf{u})],$ (71)
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\[
\frac{1}{a^2} \left( a^2 \dot{\delta} \right) - 4 \pi G \rho \delta = - \frac{1}{a^2} \left[ a \nabla \cdot (\delta_i u_i) \right] + \frac{1}{a^2} \nabla \cdot (u_i \cdot \nabla u_i) + \dot{C}^{(t)}_{\alpha \beta} \left( \frac{2}{a} u_i^{\alpha} + \dot{C}^{(t)}{\alpha \beta} \right)
\]

\[+ \frac{1}{a^2} \left\{ \Delta \left[ u_i \cdot (u_i - u) \right] - \nabla \cdot \left[ \left( u_i - u \right) \cdot \nabla u + u \cdot \nabla (u_i - u) \right] \right\} - \frac{4 \pi G \rho}{c^2} \left| u_i - u \right|^2 \]

\[+ 8 \pi G \sum_j \frac{\rho_j}{c^2} \left\{ \left| u_j - u \right|^2 + 3 \Delta^{-1} \nabla \cdot \left[ (u_j - u) \cdot \nabla (u_j - u) \right] \right\}.
\]

In section VII.B.1 of [5] we have shown that, to the linear order, \((u_i - u)\) simply decays:

\[
\frac{1}{a} \nabla \cdot u_i \equiv \frac{1}{a} \nabla \cdot (\delta_i u_i) = \frac{1}{a} \left[ 2 \nabla \cdot (u_i - u) \right] - \nabla \cdot \left[ (u_i - u) \cdot \nabla \varphi + 2 \left( u_i^{\alpha} - u^{\alpha} \right) C^{(t)}_{\alpha \beta} \right],
\]

\[
\frac{1}{a} \nabla \cdot (u_i + H u_i) + 4 \pi G \rho \delta = - \frac{1}{a^2} \nabla \cdot (u_i \cdot \nabla u_i) - \dot{C}^{(t)}{\alpha \beta} \left( \frac{2}{a} u_i^{\alpha} + \dot{C}^{(t)}{\alpha \beta} \right),
\]

\[
\frac{1}{a^2} \left( a^2 \dot{\delta} \right) - 4 \pi G \rho \delta = - \frac{1}{a^2} \left[ a \nabla \cdot (\delta_i u_i) \right] + \frac{1}{a^2} \nabla \cdot (u_i \cdot \nabla u_i) + \dot{C}^{(t)}{\alpha \beta} \left( \frac{2}{a} u_i^{\alpha} + \dot{C}^{(t)}{\alpha \beta} \right),
\]

and

\[
\frac{1}{a} \nabla \cdot (u_i + H u_i) + 4 \pi G \rho \delta = - \frac{1}{a^2} \nabla \cdot (u_i \cdot \nabla u_i) - \dot{C}^{(t)}{\alpha \beta} \left( \frac{2}{a} u_i^{\alpha} + \dot{C}^{(t)}{\alpha \beta} \right),
\]

\[
\frac{1}{a^2} \left( a^2 \dot{\delta} \right) - 4 \pi G \rho \delta = - \frac{1}{a^2} \left[ a \nabla \cdot (\delta_i u_i) \right] + \frac{1}{a^2} \nabla \cdot (u_i \cdot \nabla u_i) + \dot{C}^{(t)}{\alpha \beta} \left( \frac{2}{a} u_i^{\alpha} + \dot{C}^{(t)}{\alpha \beta} \right)
\]

\[+ \frac{1}{a^2} \left\{ \Delta \left[ u_i \cdot (u_i - u) \right] - \nabla \cdot \left[ (u_i - u) \cdot \nabla u + u \cdot \nabla (u_i - u) \right] \right\}.
\]

Equations (74)–(76) coincide with the density and velocity perturbation equations of a single-component medium [2]; thus, except for the contribution from gravitational waves, these equations coincide with ones in the Newtonian context. In the Newtonian context, equations (74)–(76) without the gravitational waves are valid to fully non-linear order [11]. To the linear order, equation (76) was derived in the synchronous gauge by Lifshitz [9], and in the comoving gauge by Nariai [20]. In the single-component zero-pressure medium a free-falling object is also comoving; thus we can impose both the synchronous gauge and the comoving gauge simultaneously. In [3] we compared subtle differences of the second-order perturbation equations in the synchronous gauge with the ones in the comoving gauge.
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If we further ignore \((u_i - u)\) terms appearing in the pure second-order combinations, equations \((77)\)–\((79)\) become

\[
\dot{\delta}_i + \frac{1}{a} \nabla \cdot (u_i) = -\frac{1}{a} \nabla \cdot (\delta_i u_i),
\]

\[
\frac{1}{a} \nabla \cdot (\dot{u}_i + H u_i) + 4\pi G \varrho \delta = -\frac{1}{a^2} \nabla \cdot (u_i \cdot \nabla u_i) - \dot{C}^{(t)}_{\alpha\beta} \left( \frac{2}{a} u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta} \right),
\]

\[
\frac{1}{a^2} \left( a^2 \dot{\delta}_i \right) - 4\pi G \varrho \delta = -\frac{1}{a^2} \left[ a \nabla \cdot (\delta_i u_i) \right] + \frac{1}{a^2} \nabla \cdot (u_i \cdot \nabla u_i) + \dot{C}^{(t)}_{\alpha\beta} \left( \frac{2}{a} u_i^2_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta} \right),
\]

which coincide with the Newtonian equations except for the contributions from the gravitational waves. In this context, except for the contribution from gravitational waves, the above equations coincide with ones in the Newtonian context even in the multi-component case. Therefore, we have shown the relativistic/Newtonian correspondence, except for the contributions from the gravitational waves, to the second-order perturbations in the case of multi-component, zero-pressure, irrotational fluids assuming a flat background. In the Newtonian context, equations \((80)\)–\((82)\) without the gravitational waves are valid to fully non-linear order \([5]\). We note that, as in the single-component case, our present relativistic/Newtonian correspondence in the multi-component situation applies to the density and velocity perturbations only without involving the gravitational potential.

4. Third-order equations

We consider irrotational fluids, and thus ignore all vector-type perturbations. As all three types of perturbation are generally coupled in non-linear perturbations, apparently this is an important assumption that we make in this work. In an expanding phase, however, the linear order rotational perturbations have only decaying mode due to the angular momentum conservation. Effects of rotational perturbations to the second-order perturbations are considered in our work in \([5]\). Pure general relativistic effects of rotational perturbations appear already in the second order, but in \([5]\) we have shown that we recover the relativistic/Newtonian correspondence in the small-scale (sub-horizon) limit.

We take the temporal comoving gauge

\[
v \equiv 0.
\]

Combining this with the irrotational condition \(v_i^{(v)} \equiv 0\) we have \(v_\alpha = 0\). Equation \((35)\) shows that this leads to \(J_\alpha = 0\) for general fluids. As the spatial gauge condition we take

\[
\gamma \equiv 0.
\]

In \([1]\) we have shown that these gauge conditions fix the space–time gauge transformation properties completely to all orders in perturbations. Thus, each perturbation variable under these gauge conditions has a corresponding unique gauge-invariant combination, and can be equivalently regarded as a gauge-invariant one to all orders in perturbations.

We consider zero-pressure fluids; thus

\[
p_{(i)} = 0, \quad \delta p_{(i)} \equiv 0 \equiv \Pi_{(i)\alpha\beta}.
\]

The collective fluid quantities are non-trivial and are presented in equation \((45)\).
In our previous study of second-order perturbations, summarized in a previous section, we showed that \( O(|v_0|^2) = O(|u - u|^2/c^2) \) terms simply correspond to a pure decaying mode in an expanding phase. We showed that by ignoring these terms we recover complete relativistic/Newtonian correspondence even in the multi-component case. On the basis of this observation, in our third-order calculation in the following we will ignore \( O(|v_0|^2) \) terms. If we ignore \( O(|v_0|^2) \) terms, the collective fluid quantities in equation (45) give

\[
\delta \mu = \sum_j \delta \mu_j, \quad \delta p = 0, \quad 0 = \sum_j \left( \mu_j v_{(j)\alpha} + \delta \mu_{(j)} v_{(j)\alpha} \right), \quad \Pi_{\alpha\beta}^\alpha = 0, \quad (86)
\]

and the ADM fluid quantities in equations (46) and (47) become

\[
E = \mu + \delta \mu, \quad J_\alpha = 0, \quad S_{\alpha\beta} = 0, \quad S = 0, \quad \bar{S}_{\alpha\beta} = 0, \quad (87)
\]

Equation (13) gives

\[
N = a(t), \quad (88)
\]

thus,

\[
\alpha = -\frac{1}{2a^2} \chi^\alpha \chi_\alpha (1 - 2\varphi) + \frac{1}{a^2} \chi^\alpha \chi^\beta C^{(t)}_{\alpha\beta}. \quad (89)
\]

Using equation (87), equations (12), (14), (15), and (16), respectively, give

\[
\left( \frac{\dot{\mu}}{\mu} + 3H \right) (1 + \delta) + \ddot{\delta} - \kappa = -\frac{c}{a^2} \delta_{,\alpha} \chi^\alpha (1 - 2\varphi) + \kappa \delta + \frac{2c}{a^2} \delta_{,\alpha} \chi^\beta C^{(t)}_{\alpha\beta}, \quad (90)
\]

\[-3\dot{H} + 3H^2 + 4\pi G \rho - \Lambda c^2] + \kappa + 2HK - 4\pi G \rho \delta 
\]

\[
= -\frac{c}{a^2} \delta_{,\alpha} \chi^\alpha (1 - 2\varphi) + \frac{\chi^\alpha \alpha^\beta}{c^3} \delta_{,\alpha} \chi^\beta C^{(t)}_{\alpha\beta} 
+ \frac{c}{a^2} \delta_{,\alpha} \chi^\beta C^{(t)}_{\alpha\beta} \left[ \left( \frac{c}{a^2} \chi_{,\alpha} \chi_{,\beta} + \dot{C}^{(t)}_{\alpha\beta} \right) (1 - 4\varphi) 
- \frac{4c}{a^2} \chi_{,\alpha} \varphi_{,\beta} - \frac{2c}{a^2} \chi^\gamma \left( 2C_{\gamma\alpha|\beta}^{(t)} - C^{(t)}_{\alpha\beta|\gamma} \right) - 4C_{\beta\gamma}^{(t)} \left( \frac{c}{a^2} \chi^\gamma \chi^\alpha + \dot{C}^{(t)}_{\gamma\alpha} \right) \right] 
- 4\dot{\varphi} C^{(t)}_{\alpha\beta}, \quad (91)
\]

\[
\left( \frac{\dot{\mu}_{(i)}}{\mu_{(i)}} + 3H \right) (1 + \delta_{(i)}) + \ddot{\delta}_{(i)} - \kappa \left[ (1 + \delta_{(i)}) v_{(i)\alpha} \right]_{\alpha} (1 - 2\varphi + 4\varphi^2) 
= -\frac{c}{a^2} \delta_{(i),\alpha} \chi^\alpha (1 - 2\varphi) + \kappa \delta_{(i)} - \frac{c}{a} \varphi_{,\alpha} v_{(i)\alpha} (1 - 4\varphi) + \frac{2c}{a} C^{(t)}_{\alpha\beta} v_{(i)\alpha} (1 - 4\varphi) 
+ \frac{2c}{a^2} \delta_{(i),\alpha} \chi^\beta C^{(t)}_{\alpha\beta} + \frac{c}{a} \left[ -\varphi_{,\alpha} \left( \delta_{(i)} v_{(i)\alpha} + 2C^{(t)}_{\alpha\beta} v_{(i)\beta} \right) + 2C_{\beta\gamma} \chi^\gamma v_{(i)\alpha} \right] 
+ 2 \left( \delta_{(i)} C^{(t)}_{\alpha\beta} v_{(i)\beta} - 2C^{(t)}_{\alpha\beta} C^{(t)}_{\beta\gamma} v_{(i)\gamma} \right)_{\alpha}, \quad (92)
\]
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\[
\frac{1}{a} \left[ a (1 + \delta (i)) v (i) \alpha \right] = \left( 1 + \delta (i) \right) \kappa v (i) \alpha - \frac{c}{a^2} \left\{ \left( v (i) \beta \chi ^{\beta} \right) \alpha \left( 1 + \delta (i) \right) + \delta (i, \beta) \chi ^{\beta} v (i) \alpha \right\} - 2 \left[ v (i) \beta \left( \chi ^{\beta} \varphi + \chi ^{\gamma} C ^{(i)} \beta \gamma \right) \right] \alpha ,
\]

where we recovered \( c \). Equation (93) can be written as

\[
\frac{1}{a} \left( av (i) \right) = - \frac{c}{a^2} \left[ v (i) \beta \chi ^{\beta} (1 - 2 \varphi) - 2 v (i) \beta \chi ^{\gamma} C ^{(i)} \beta \gamma \right] \alpha .
\]

Equations (90) and (91) are the same as equations (20) and (21) in [4] which were derived in the single-component situation.

Combining equations (90) and (91), and equations (91)–(93), respectively, we can derive

\[
\frac{1}{a^2} \left[ a^2 \delta (i) + c \delta (i, \alpha) \chi ^{\alpha} (1 - 2 \varphi) - 2 c \delta (i, \alpha) \chi ^{\alpha} C ^{(i)} \alpha \beta \right] - 4 \pi G \varphi \delta (1 + \delta )
\]

\[
= \frac{4}{3} \kappa ^2 (1 + \delta ) \left\{ \left( \frac{c}{a^2} \chi ^{\alpha} \chi ^{\beta} \alpha \beta \right) \right\} - 4 \pi G \varphi \delta (1 + \delta )
\]

\[
= \frac{4}{3} \kappa ^2 \left( \frac{c}{a^2} \chi ^{\alpha} \chi ^{\beta} \alpha \beta \right) \left\{ \left( \frac{c}{a^2} \chi ^{\alpha} \chi ^{\beta} \alpha \beta \right) \right\} - 4 \pi G \varphi \delta (1 + \delta )
\]

\[
= \frac{4}{3} \kappa ^2 \left( \frac{c}{a^2} \chi ^{\alpha} \chi ^{\beta} \alpha \beta \right) \left\{ \left( \frac{c}{a^2} \chi ^{\alpha} \chi ^{\beta} \alpha \beta \right) \right\} - 4 \pi G \varphi \delta (1 + \delta )
\]

\[
= \frac{4}{3} \kappa ^2 \left( \frac{c}{a^2} \chi ^{\alpha} \chi ^{\beta} \alpha \beta \right) \left\{ \left( \frac{c}{a^2} \chi ^{\alpha} \chi ^{\beta} \alpha \beta \right) \right\} - 4 \pi G \varphi \delta (1 + \delta )
\]

\[
= \frac{4}{3} \kappa ^2 \left( \frac{c}{a^2} \chi ^{\alpha} \chi ^{\beta} \alpha \beta \right) \left\{ \left( \frac{c}{a^2} \chi ^{\alpha} \chi ^{\beta} \alpha \beta \right) \right\} - 4 \pi G \varphi \delta (1 + \delta )
\]
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All terms in the RHS of equation (96) contain \( v_{(i)}^\alpha \) which decays to the linear order as

\[
v_{(i)} = \frac{1}{c} (u_i - u) \propto \frac{1}{a};
\]  

(97)

see equations (61) and (73), and [5]. On setting \( \delta_{(i)} = \delta \) the LHS of equation (96) is the same as equation (95). On setting \( \delta_{(i)} = \delta \) and \( v_{(i)}^\alpha = 0 \), equation (96) becomes equation (95). Notice, however, that the first six terms in the RHS of equation (96) contain \( v_{(i)}^\alpha \) to the second order; these terms will be either absorbed into second-order terms in the LHS or become pure third order in our Newtonian context presentation in the next section; compare with equation (111).

In order to complete equation (96) we need equations for \( \dot{\phi} \), \( v_{(i)}^\alpha \), and \( C_{\alpha\beta}^{(t)} \) terms to the second-order perturbations. Equation for \( v_{(i)}^\alpha \) is in equation (94) which gives

\[
\frac{1}{a} (av_{(i)}^\alpha) = -\frac{c}{a^2} (v_{(i)}^\beta \chi^{\beta})_{,\alpha},
\]  

(98)

to the second order. The relation between \( \kappa \) and \( \chi \) can be found in equation (23) of [4]. Recovering the background curvature, from the momentum constraint equation in equation (17), we can derive

\[
\kappa + \frac{c}{a^2} (\Delta + 3K) \chi = \frac{c}{a^2} (2\varphi \Delta \chi - \chi^\alpha \varphi_{,\alpha}) + C_{\alpha\beta}^{(t)} (\frac{2c}{a^2} \chi_{,\alpha|\beta} - \dot{C}_{\alpha\beta}^{(t)})
\]

\[
+ \frac{3}{2} \Delta^{-1} \nabla^\alpha \left\{ \frac{c}{a^2} [\chi^{\beta} \varphi_{,\alpha|\beta} + \chi_{,\alpha} (\Delta + 4K) \varphi] + \chi^{\beta} \frac{c}{a^2} (\Delta - 2K) C_{\alpha\beta}^{(t)} - \varphi^{\beta} \dot{C}_{\alpha\beta}^{(t)} + 2C_{\beta\gamma}^{(t)} \dot{C}_{\alpha\beta|\gamma}^{(t)} + C_{\beta\gamma|\alpha}^{(t)} \dot{C}_{\alpha\beta}^{(t)} \right\},
\]  

(99)

to the second order. Equation for \( \varphi \) to the second order follows from equation (99) in [1]. Ignoring \( O(|v_{(i)}|^2) \), using equation (89) for \( a \) we can thus derive

\[
3\dot{\varphi} = - \left( \kappa + \frac{c}{a^2} \Delta \chi \right) + \frac{c}{a^2} (2\varphi \Delta \chi - \chi^\alpha \varphi_{,\alpha}) + 2C_{\alpha\beta}^{(t)} (\frac{c}{a^2} \chi_{,\alpha|\beta} + \dot{C}_{\alpha\beta}^{(t)}).
\]  

(100)

In a flat background, the RHS is the second order; thus, to the linear order we have \( \dot{\varphi} = 0 \), see equation (112). The equation for \( C_{\alpha\beta}^{(t)} \) to the second order is presented in equation (43) of [4].

4.1. Pure general relativistic corrections

Now, we assume

\[
K = 0,
\]  

(101)

and thus \( \dot{\varphi} = 0 \) to the linear order. On the basis of the apparent success in second-order perturbations, we continue to use the identifications made in equation (64) now valid to
the third order; thus
\[ \kappa_v \equiv -\frac{1}{a} \nabla \cdot \mathbf{u}, \quad \mathbf{u} \equiv \nabla u, \quad \mathbf{v}_{(i)\nu} \equiv \frac{1}{c} (\mathbf{u}_i - \mathbf{u}), \quad \delta \equiv \delta_v, \quad \delta_i \equiv \delta_{(i)\nu}. \] (102)

Our results will show why these identifications are likely to be the best choice. In the following we consider pure scalar-type perturbations, and thus set \( C_{a\beta}^{(t)} \equiv 0 \). Contributions from the gravitational waves will be considered later in section 4.3.

We need \( \chi_v \) only to the second order. Equation (99) gives
\[ \chi_v \equiv \frac{a}{c} (u + \Delta^{-1} X), \] (103)
where
\[ X \equiv 2\varphi \nabla \cdot u - u \cdot \nabla \varphi + \frac{2}{3} \Delta^{-1} \nabla \cdot [u \cdot \nabla (\nabla \varphi) + u \Delta \varphi]. \] (104)

Equations (90)–(93) become
\[ \dot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{u} = -\frac{1}{a} \nabla \cdot (\delta \mathbf{u}) + \frac{1}{a} (2\varphi \mathbf{u} - \nabla \Delta^{-1} X) \cdot \nabla \delta, \] (105)
\[ \frac{1}{a} \nabla \cdot (\dot{\mathbf{u}} + H \mathbf{u}) + 4\pi G \rho \delta = -\frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) - \frac{\Delta}{a^2} [\mathbf{u} \cdot \nabla (\Delta^{-1} X)] 
+ \frac{1}{a^2} \left( \mathbf{u} \cdot \nabla X + \frac{2}{3} X \nabla \cdot \mathbf{u} \right) 
- \frac{2}{3a^2} \varphi \mathbf{u} \cdot \nabla (\nabla \cdot \mathbf{u}) + \frac{4}{a^2} \nabla \left[ \varphi \left( \mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{3} \mathbf{u} \nabla \cdot \mathbf{u} \right) \right], \] (106)
\[ \dot{\delta}_i + \frac{1}{a} \nabla \cdot \mathbf{u}_i = \frac{1}{a} \nabla \cdot (\delta_i \mathbf{u}_i) + \frac{1}{a} [2\varphi \mathbf{u}_i - \nabla (\Delta^{-1} X)] \cdot \nabla \delta_i 
+ \frac{1}{a} [2\varphi \nabla \cdot (\mathbf{u}_i - \mathbf{u}) - (\mathbf{u}_i - \mathbf{u}) \cdot \nabla \varphi] 
+ \frac{2}{a} \varphi \left( \delta_i \nabla \cdot (\mathbf{u}_i - \mathbf{u}) + (\mathbf{u}_i - \mathbf{u}) \cdot \nabla \varphi - 2\varphi \nabla \cdot (\mathbf{u}_i - \mathbf{u}) \right) 
- \frac{1}{a} \delta_i (\mathbf{u}_i - \mathbf{u}) \cdot \nabla \varphi, \] (107)
\[ \frac{1}{a} \nabla \cdot (\dot{\mathbf{u}}_i + H \mathbf{u}_i) + 4\pi G \rho \delta = -\frac{1}{a^2} \nabla \cdot (\mathbf{u}_i \cdot \nabla \mathbf{u}_i) - \frac{\Delta}{a^2} [\mathbf{u}_i \cdot \nabla (\Delta^{-1} X)] 
+ \frac{1}{a^2} \left( \mathbf{u}_i \cdot \nabla X + \frac{2}{3} X \nabla \cdot \mathbf{u}_i \right) 
- \frac{2}{3a^2} \varphi \mathbf{u}_i \cdot \nabla (\nabla \cdot \mathbf{u}_i) 
+ \frac{4}{a^2} \nabla \left[ \varphi \left( \mathbf{u}_i \cdot \nabla \mathbf{u}_i - \frac{1}{3} \mathbf{u}_i \nabla \cdot \mathbf{u}_i \right) \right] + 2 \frac{\Delta}{a^2} [\varphi \mathbf{u}_i \cdot (\mathbf{u}_i - \mathbf{u})]. \] (108)

Equations (125) and (126) coincide with equations (39) and (40) in [4]. Equation (94) gives
\[ \frac{1}{a} (a (\mathbf{u}_i - \mathbf{u}))' = -\frac{1}{a} \nabla \left\{ (1 - 2\varphi) (\mathbf{u}_i - \mathbf{u}) \cdot [\mathbf{u} + \nabla (\Delta^{-1} X)] \right\}. \] (109)
Equations (95) and (96) give
\[
\frac{1}{a^2} \left\{ a^2 \delta + a \nabla \cdot (\delta \mathbf{u}) - a [2 \varphi \mathbf{u} - \nabla (\Delta^{-1} X)] \cdot \nabla \delta \right\} - 4\pi G \rho \delta \\
= \frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \frac{\Delta}{a^2} [\mathbf{u} \cdot \nabla (\Delta^{-1} X)] - \frac{1}{a^2} \left( \mathbf{u} \cdot \nabla X + \frac{2}{3} X \nabla \cdot \mathbf{u} \right) \\
+ \frac{2}{3a^2} \varphi \mathbf{u} \cdot \nabla (\nabla \cdot \mathbf{u}) - \frac{4}{a^2} \nabla \cdot \left[ \varphi \left( \mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{3} \mathbf{u} \nabla \cdot \mathbf{u} \right) \right],
\]
(110)
\[
\frac{1}{a^2} \left\{ a^2 \dot{\delta}_i + a \nabla \cdot (\delta_i \mathbf{u}_i) - a [2 \varphi \mathbf{u}_i - \nabla (\Delta^{-1} X)] \cdot \nabla \delta_i \right\} - 4\pi G \rho \delta - \frac{1}{a^2} \nabla \cdot (\mathbf{u}_i \cdot \nabla \mathbf{u}_i) \\
- \frac{\Delta}{a^2} [\mathbf{u}_i \cdot \nabla (\Delta^{-1} X)] + \frac{1}{a^2} \left( \mathbf{u} \cdot \nabla X + \frac{2}{3} X \nabla \cdot \mathbf{u} \right) - \frac{2}{3a^2} \varphi \mathbf{u} \cdot \nabla (\nabla \cdot \mathbf{u}) \\
+ \frac{4}{a^2} \nabla \cdot \left[ \varphi \left( \mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{3} \mathbf{u} \nabla \cdot \mathbf{u} \right) \right] \\
= \frac{1}{a} [2 \varphi \nabla \cdot (\mathbf{u}_i - \mathbf{u}) - (\mathbf{u}_i - \mathbf{u}) \cdot \nabla \varphi] - 2\Delta \left[ \varphi \mathbf{u} \cdot (\mathbf{u}_i - \mathbf{u}) \right] \\
- \frac{2}{a^2} \varphi \left\{ \Delta [\mathbf{u} \cdot (\mathbf{u}_i - \mathbf{u})] + (\nabla \cdot \mathbf{u}_i) \nabla \cdot (\mathbf{u}_i - \mathbf{u}) \right\} \\
+ \frac{1}{a^2} (\nabla \varphi) \cdot \{ \nabla [\mathbf{u} \cdot (\mathbf{u}_i - \mathbf{u})] + (\nabla \cdot \mathbf{u}_i) (\mathbf{u}_i - \mathbf{u}) \}.
\]
(111)
Equation (100) becomes
\[
\varphi = \frac{1}{3a} [-X + 2\varphi \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \varphi] = -\frac{1}{2a} \Delta^{-1} \nabla [\mathbf{u} \cdot \nabla (\nabla \varphi) + \mathbf{u} \Delta \varphi].
\]
(112)
We note again that all our relativistic variables are gauge invariant, and are the same as the variables in the comoving gauge ($v \equiv 0$). For convenience, in table 1 we summarize symbols used in this section.

4.2. Implications

A close examination of our third-order perturbation equations reveals the following.

(i) Equations (105), (106), and (110) are the same as equations (25), (26), and (28) in [4] which are valid for a single-component fluid. We note, however, that we have ignored $O(|v|^{i2})$ correction terms, which simply decay in an expanding phase. These terms appear even in the second-order perturbations; see equations (67), (68), and (71) above, or [5].

(ii) If we ignore the $i$-indices, equations (107), (108), and (111) are identical to equations (105), (106), and (110), respectively. We note, however, that the presence of $O(|v|^{i2})$ correction terms causes differences even in the second-order perturbations; see equations (51)–(56).

(iii) We already showed that, to the second order, even in the multi-component case, the general relativistic equations for the density and velocity perturbations are identical to the Newtonian ones, thus having relativistic/Newtonian correspondence. The presence of $O(|v|^{i2})$ correction terms may cause differences, but we have shown that these corrections are simply decaying in the expanding phase.
Table 1. Symbols used in the relativistic third-order perturbation equations.

| Symbol | Definition | Equation |
|--------|------------|----------|
| $\mu$  | Energy density of collective component ($\equiv \rho c^2$) | (1), (21) |
| $\delta \mu$ | Perturbed energy density of collective component ($\equiv \delta \rho c^2$) | (1), (21) |
| $\mu^{(i)}$ | Energy density of $i$th component ($\equiv \rho^{(i)} c^2$) | (3), (30) |
| $\delta \mu^{(i)}$ | Perturbed energy density of $i$th component ($\equiv \delta \rho^{(i)} c^2$) | (3), (30) |
| $\delta$ | Relative density perturbations of collective component ($\equiv \delta \mu/\mu$); Newtonian relative density perturbations of collective component ($\equiv \delta \rho/\rho$) to the second order | (21), (102) |
| $\delta^{(i)}$ | Relative density perturbations of $i$th component ($\equiv \delta \mu^{(i)}/\mu^{(i)}$) | (21), (29) |
| $\delta_i$ | Newtonian relative density perturbations of $i$th component ($\equiv \delta \rho_i/\rho_i$) to the second order | (102) |
| $\kappa$ | Perturbed part of trace of extrinsic curvature; negative of perturbed expansion | (41) |
| $\chi$ | Metric perturbation variable | (40) |
| $\alpha$ | Metric perturbation variable | (39) |
| $\gamma$ | Metric perturbation variable; as a spatial gauge condition we set $\gamma \equiv 0$ | (39) |
| $v_\alpha$ | Spatial component of the collective 4-vector $\tilde{u}_\alpha$ | (22) |
| $v$ | Scalar part of $v_\alpha$; Temporal comoving gauge sets $v \equiv 0$ | (42) |
| $v^{(i)\alpha}$ | Spatial component of the $i$th 4-vector $\tilde{u}^{(i)\alpha}$ | (22), (29) |
| $\varphi$ | Metric perturbation variable, perturbed spatial curvature | (39), (113) |
| $\varphi_v$ | A gauge-invariant combination using $\varphi$ and $v$ which becomes $\varphi$ in the comoving gauge ($v \equiv 0$) | (114), (119) |
| $\varphi_\chi$ | A gauge-invariant combination using $\varphi$ and $\chi$ which becomes $\varphi$ in the zero-shear gauge ($\chi \equiv 0$) | (118) |
| $\psi_\chi$ | A gauge-invariant combination using $v$ and $\chi$ which becomes $v$ in the zero-shear gauge ($\chi \equiv 0$) | (118) |
| $\kappa_v$ | A gauge-invariant combination using $\kappa$ and $v$ which becomes $\kappa$ in the comoving gauge ($v \equiv 0$) | (118) |
| $\chi_v$ | A gauge-invariant combination using $\chi$ and $v$ which becomes $\chi$ in the comoving gauge ($v \equiv 0$) | (118) |
| $\delta \Phi$ | Perturbed Newtonian gravitational potential | (118) |
| $u_i$ | Relativistic velocity perturbation of collective component identified from $\kappa_v$; coincides with Newtonian velocity perturbation of collective component to the second order | (102), (105), (106) |
| $u^{(i)}$ | Relativistic velocity perturbation of $i$th component identified from $v^{(i)\alpha}$ and $u_i$; coincides with Newtonian velocity perturbation of $i$th component to the second order | (102), (107), (108) |
| $X$ | Collection of terms contributing to pure third order | (104), (132) |
| $C_{\alpha \beta}^{(f)}$ | Transverse-tracefree part of metric perturbation variable | (39) |
Table 2. Comparison of the second-order and third-order terms in equation (105) (top), and equation (106) (bottom). Notice that all third-order terms simply have $\varphi$ multiplied in diverse convolutions compared with the second-order term. Similar comparisons can be made for equations (107), (108), (110), and (111).

| 2nd order | 3rd order |
|-----------|-----------|
| $\nabla \cdot (\delta u)$ | $\nabla \Delta^{-1} (\varphi \nabla \cdot u) \cdot \nabla \delta$ |
| $\nabla \Delta^{-1} (u \cdot \nabla \varphi) \cdot \nabla \delta$ | $\nabla \Delta^{-1} \{ \Delta^{-1} \nabla [u \cdot \nabla (\nabla \varphi)] \} \cdot \nabla \delta$ |
| $\nabla \Delta^{-1} [\Delta^{-1} \nabla (u \delta \varphi)] \cdot \nabla \delta$ | |

(iv) The pure third-order correction terms in the above equations all involve $\varphi$-term to the linear order in various forms of convolution with the second-order terms; we summarized these pure third-order terms in table 2. This result is far from what we could have anticipated, and, in fact, curious and surprising in some respects. It is remarkable that although we have a lot of correction terms, these do not involve zero-pressure fluids. The other notable consequences of our third-order equations involve quite a relief from the Newtonian perspective. Before our series of works, even in the second order one might have anticipated some pure general relativistic effects associated with the presence of the horizon in relativistic cosmology. Our work shows that we do not have such correction terms even to the third-order perturbations in the multi-component zero-pressure fluids. The other notable consequences of our third-order equations involve several remarkable properties of the perturbation variable $\varphi$ which will be explained next.

(v) As we took the comoving gauge $v \equiv 0$, the variable $\varphi$ is the same as a gauge-invariant combination $\varphi_v$. To the linear order, we have $\varphi_v \equiv \varphi - aHv/c$; for $\varphi_v$ to the second order; see equation (281) of [1]. To the linear order, the metric perturbation variable $\varphi$ is related to the perturbed spatial curvature as [13]

$$R^{(h)} = \frac{1}{a^2} \left[ 6K - 4 (\Delta + 3K) \varphi \right],$$

where $R^{(h)}$ is the intrinsic scalar curvature. To the second order, $R^{(h)}$ was derived in equation (265) of [1].

(vi) We note that, to the linear order, $\varphi_v$ is one of the well known conserved quantities on the large scale [21, 22]. For $K = 0$, but considering general $\Lambda$, we have $\dot{\varphi}_v = 0$, and thus

$$\varphi_v = C(x),$$

where $C(x)$ is a constant.
with vanishing decaying mode (in an expanding phase) to the leading order in the large-scale expansion [14]. A complete set of linear order exact solutions in the matter dominated era with cosmological constant is presented in tables in [23]. From table 1 of [23] we have the general solutions

\[
\varphi_v = C \left( 1 + c^2 KH \int_0^t \frac{dt}{a^2} \right), \quad \varphi_\chi = C 4\pi G \varrho a^2 H \int_0^t \frac{dt}{a^2},
\]

\[ v_\chi = -\frac{C}{aH} \left( 1 + a^2 H \dot{H} \int_0^t \frac{dt}{a^2} \right), \quad (115) \]

where the lower bounds of the integrations give the decaying mode in an expanding phase; compared with notation used in [23] we have \( \Psi_\chi \equiv -a(\mu + p)v_\chi \). For \( K = 0 = \Lambda \), from Table 2 of [23] we have

\[
\varphi_v = C, \quad \varphi_\chi = \frac{3}{5} C + \frac{4}{9} dt^{-5/3}, \quad v_\chi = -\frac{c}{aH} \left( \frac{2}{5} C - \frac{4}{9} dt^{-5/3} \right), \quad (116)\]

where \( d(x) \) is a coefficient of the transient (decaying) mode in an expanding phase. To the linear order, the above gauge-invariant combinations are (see equation (255) of [1])

\[
\varphi_v \equiv \varphi - \frac{aH}{c} v, \quad \varphi_\chi \equiv \varphi - \frac{H}{c} \chi, \quad v_\chi \equiv v - \frac{1}{a} \chi. \quad (117)\]

Notice that the solutions in equation (115) do not involve the horizon scale \( (a/k \sim c/H) \).

(vii) To the linear order, the relativistic variables are identified with the Newtonian perturbation variables as [24,13,23,14]

\[
\varphi_\chi = \frac{-\delta \Phi}{c^2}, \quad u = -c \nabla v_\chi, \quad (118)\]

where \( \delta \Phi \) is the Newtonian perturbed gravitational potential. Thus, we have [4]

\[
\varphi_v \equiv \varphi - \frac{aH}{c} v = \varphi_\chi - \frac{aH}{c} v_\chi = -\frac{\delta \Phi}{c^2} + \frac{aH}{c^2} \Delta^{-1} \nabla \cdot u. \quad (119)\]

For \( K = 0 = \Lambda \) we have

\[
\varphi_v = \frac{5}{3} \varphi_\chi = -\frac{5}{3} \frac{\delta \Phi}{c^2}. \quad (120)\]

Thus, the value of \( \varphi_v \) on the large scale (super-sound-horizon scale) is of the same order as the perturbed Newtonian gravitational potential divided by \( c^2 \).

(viii) Near the horizon scale, the gravitational potential fluctuation \( \varphi_\chi \) is of the same order as the relative temperature fluctuations \( \delta T/T \) of the cosmic microwave background radiation (CMB). The temperature anisotropy of CMB, in a flat background without the cosmological constant, gives [25,26]

\[
\frac{\delta T}{T} \sim \frac{1}{3} \varphi_\chi = \frac{1}{3} \frac{\delta \Phi}{c^2} \sim \frac{1}{5} \varphi_v \sim \frac{1}{5} C, \quad (121)\]

to the linear order; this is a part of the Sachs–Wolfe effect [25]. The COBE observations of CMB give \( \delta T/T \sim 10^{-5} \) [27], thus

\[
\varphi_v \sim 5 \times 10^{-5}, \quad (122)\]
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in the large-scale limit near the horizon scale. Therefore, near the horizon scale, the
correct terms are independent of the presence of the horizon scale and are smaller by factor $5 \times 10^{-5}$ compared with the second-order
relativistic/Newtonian terms due to the low level anisotropies of CMB.

(ix) Even on small scales where the structures are in the non-linear stage we have

$$\varphi_v \sim \frac{\delta \Phi}{c^2} \sim \frac{GM}{Rc^2} \sim 10^{-6} - 10^{-5},$$

which is true even on a galactic scale or near main sequence stars. This small number
simply means that general relativistic correction to Newtonian gravity is small in these
cosmological systems; we have $GM/(Rc^2)$ to be $\sim 10^{-9}$ near the Earth, $\sim 10^{-6}$ near the Sun,
and $\sim 10^{-4}$ near a white dwarf star. It looks as if the small value of $\varphi_v$ on nearly all scales
ensures the smallness of our pure general relativistic third-order correction terms on
such scales. This, in fact, is not the case because our equations are based on perturbation
expansion which assumes weak non-linearity; for example, although $\varphi_v$ remains small even
in the fully non-linear stage, $\delta$ could be huge which violates the assumption of weak non-
linearity. That is, although our perturbation analysis is fully relativistic, it can handle only
weakly non-linear process, and we cannot apply it to the fully non-linear stage. However,
as long as the perturbation approximation is valid, our third-order equations show that
the pure general relativistic corrections are by the factor $\varphi_v \sim 10^{-5}$ smaller compared
with the second-order relativistic/Newtonian terms.

(x) In situations where the relativistic effects are small we can derive general
relativistic correction terms in the context of fully non-linear Newtonian equations. The
post-Newtonian approximation considers the general relativistic effects as the expansion
of a small parameter:

$$\frac{GM}{Rc^2} \sim \frac{v^2}{c^2},$$

which is quite small on nearly all scales inside the horizon. A complete set of the first-order
cosmological post-Newtonian equations is presented in [12]. Although the post-Newtonian
approximation considers only weakly relativistic situations, it is applicable to a fully non-
linear system. Thus, the post-Newtonian approximation and relativistic perturbation
approach provide complementary ways to handle general relativistic non-linear processes
in the context of large-scale cosmological structure formation.

### 4.3. Contributions from tensor-type perturbations

Here we present a set of equations describing the scalar-type perturbation equations to the
third order, now including the contributions from the tensor-type perturbations. We
continue to use the Newtonian variables identified in equation (102).

Equations (90)–(93) become

$$\dot{\delta} + \frac{1}{a} \nabla \cdot u = -\frac{1}{a} \nabla \cdot (\delta u) + \frac{1}{a} (2\varphi u - \nabla \Delta^{-1} X) \cdot \nabla \delta + \frac{2}{a} \delta^{a\alpha} u^\beta C^{(t)}_{a\alpha\beta},$$

$$\frac{1}{a} \nabla \cdot (\dot{u} + Hu) + 4\pi G \rho \delta = -\frac{1}{a^2} \nabla \cdot (u \cdot \nabla u) - \dot{C}^{(t)\alpha\beta} \left( \frac{2}{a} u_{\alpha\beta} + \dot{C}^{(t)}_{\alpha\beta} \right)$$

$$- \frac{\Delta_{a\beta}^{\alpha} [u \cdot \nabla (\Delta^{-1} X)] + \frac{1}{a^2} \left( u \cdot \nabla X + \frac{2}{3} X \nabla \cdot u \right)}{2 \Delta_{a\beta}^{\alpha} \varphi u \cdot \nabla (\nabla \cdot u)}$$
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\[
\delta_i + \frac{1}{a} \nabla \cdot \mathbf{u} = -\frac{1}{a} \nabla \cdot (\delta_i \mathbf{u}) + \frac{1}{a} \left[ 2 \varphi \mathbf{u} - \nabla (\Delta^{-1} X) \right] \cdot \nabla \delta_i + \frac{2}{a} \delta_i \nabla \varphi C_{\alpha \beta}^{(t)}
\]

\[
\begin{aligned}
+ & \frac{1}{a} \left[ 2 \varphi \nabla \cdot (\mathbf{u} - \mathbf{u}) - (\mathbf{u} - \mathbf{u}) \cdot \nabla \varphi + 2 C_{\alpha \beta}^{(t)} (u_i^\alpha - u_i^\beta) \right] \\
+ & \frac{2}{a} \varphi \delta_i \nabla \cdot (\mathbf{u} - \mathbf{u}) + 2 (\mathbf{u} - \mathbf{u}) \cdot \nabla \varphi - 2 \varphi \nabla \cdot (\mathbf{u} - \mathbf{u}) \\
- & \frac{1}{a} \delta_i (\mathbf{u} - \mathbf{u}) \cdot \nabla \varphi \\
+ & \frac{2}{a} C_{\alpha \beta}^{(t)} \left\{ (\delta_i - 4 \varphi) (u_{i\alpha} - u_{i\beta}) - \varphi, (u_{i\beta} - u_{i\beta}) + C_{\alpha \beta \gamma}^{(t)} (u_i^\gamma - u_i^\gamma) \right\} \\
- & 2 \left\{ C_{\beta \gamma}^{(t)} (u_i^\gamma - u_i^\gamma) \right\}_{|\alpha},
\end{aligned}
\]

\[
\begin{aligned}
\frac{1}{a} \nabla \cdot (\dot{\mathbf{u}}_i + H \mathbf{u}_i) + 4 \pi G \varrho \delta = -\frac{1}{a^2} \nabla \cdot (\mathbf{u}_i \cdot \nabla \mathbf{u}_i) - \dot{C}_{\alpha \beta}^{(t)} \left( \frac{2}{a} u_{i|\alpha \beta} + \dot{C}_{\alpha \beta}^{(t)} \right) \\
- \frac{\Delta}{a^2} \left[ u_i \cdot \nabla (\Delta^{-1} X) \right] + \frac{1}{a^2} \left( \mathbf{u} \cdot \nabla X + \frac{2}{3} X \nabla \cdot \mathbf{u} \right) \\
- \frac{2}{3a^2} \varphi \mathbf{u} \cdot \nabla \nabla \cdot \mathbf{u} + \frac{4}{a^2} \nabla \left[ \nabla \left( \mathbf{u} \cdot \nabla u - \frac{1}{3} \nabla \cdot \mathbf{u} \right) \right] \\
+ \frac{2}{a^2} \left[ \varphi \mathbf{u} \cdot (\mathbf{u} - \mathbf{u}) + u^\alpha (u^\alpha - u^\beta) \right] C_{\alpha \beta}^{(t)} + \frac{2}{a^2} u^\alpha \nabla \beta \left( \nabla \cdot \mathbf{u} \right) C_{\alpha \beta}^{(t)} \\
+ 2 \dot{C}_{\alpha \beta}^{(t)} \left( \frac{4}{a} \varphi u_{i|\alpha \beta} + \frac{2}{a} u_{i|\alpha \beta} \nabla \varphi + 2 \varphi \dot{C}_{\alpha \beta}^{(t)} - \frac{1}{a} (\Delta^{-1} X)_{|\alpha |\beta} \right) \\
+ 2 \left( \frac{1}{a} u_{i|\alpha \beta} + \dot{C}_{\alpha \beta}^{(t)} \right) \left[ -\frac{2}{3a} (\nabla \cdot \mathbf{u}) C_{\alpha \beta}^{(t)} + \frac{1}{a} u^\gamma (2 C_{\gamma \alpha \beta}^{(t)} - C_{\alpha \beta |\gamma \gamma}^{(t)}) \right] \\
+ 2 C_{\beta \gamma}^{(t)} \left( \frac{1}{a} u_{i|\gamma}^{\alpha} + \dot{C}_{\alpha \beta}^{(t)} \right),
\end{aligned}
\]
Equations (125) and (126) coincide with equations (39) and (40), respectively, in [4]. Equation (94) gives

\[
\frac{1}{a^2} \left\{ a^2 \dot{\delta} + a \nabla \cdot (\delta \mathbf{u}) - a \left[ 2 \varphi \mathbf{u} - \nabla \left( \Delta^{-1} \mathbf{X} \right) \right] \cdot \nabla \delta - 2a^2 \delta_{i} \mathbf{u} \right\} - 2a^2 \dot{\varphi} (\mathbf{u} \cdot \nabla) \mathbf{u} - 2a^2 \left[ \nabla \cdot \left( \Delta^{-1} \mathbf{X} \right) \right] - 2a^2 \left[ \mathbf{u} \cdot \nabla \left( \Delta^{-1} \mathbf{X} \right) \right] = 0.
\]

Equations (95) and (96) give

\[
\frac{1}{a^2} \left\{ a^2 \delta + a \nabla \cdot (\delta \mathbf{u}) - a \left[ 2 \varphi \mathbf{u} - \nabla \left( \Delta^{-1} \mathbf{X} \right) \right] \cdot \nabla \delta - 2a^2 \delta_{i} \mathbf{u} \right\} - 2a^2 \dot{\varphi} (\mathbf{u} \cdot \nabla) \mathbf{u} - 2a^2 \left[ \nabla \cdot \left( \Delta^{-1} \mathbf{X} \right) \right] - 2a^2 \left[ \mathbf{u} \cdot \nabla \left( \Delta^{-1} \mathbf{X} \right) \right] = 0.
\]
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\[
+ \frac{1}{a^2} (\nabla \varphi) \cdot \{ \nabla [u \cdot (u_i - u)] + (\nabla \cdot u_i) (u_i - u) \}
- \frac{2}{a^2} C^{(i)\alpha\beta} \left\{ [u \cdot (u_i - u)]_{\alpha|\beta} + (\nabla \cdot u_i) (u_{i\alpha} - u_\alpha)_{|\beta} - a\dot{C}^{(i)}_{\alpha\beta\gamma} (u_\gamma - u^\gamma) \right\}
+ 2a \left[ C^{(i)}_{\beta\gamma} (u_\gamma - u^\gamma) \right]_{|\alpha} + \frac{2}{a} C^{(i)\alpha\beta} \left\{ (\delta_i - 4\varphi) (u_{i\alpha} - u_\alpha)_{|\beta} - \varphi_{,\alpha} (u_{i\beta} - u_\beta) \right\}
+ C^{(i)}_{\alpha\beta\gamma} (u_\gamma - u^\gamma) - 2 \left[ C^{(i)}_{\beta\gamma} (u_\gamma - u^\gamma) \right]_{|\alpha}.
\]

Equations (99) and (100) become

\[
X \equiv 2\varphi \nabla \cdot u - u \cdot \nabla \varphi + C^{(i)\alpha\beta} \left( 2u_{i\alpha|\beta} - a\dot{C}^{(i)}_{\alpha\beta} \right) + \frac{2}{a} \Delta^{-1} \nabla^\alpha \left[ u \cdot \nabla (\nabla \varphi) + u_\alpha \Delta \varphi \right]
+ u_\beta \Delta C^{(i)}_{\alpha\beta} - a\varphi^\alpha \dot{C}^{(i)}_{\alpha\beta} + 2a C^{(i)\beta\gamma} \dot{C}^{(i)}_{\alpha\beta|\gamma} + aC^{(i)}_{\beta\gamma|\alpha} \dot{C}^{(i)}_{\alpha\beta\gamma},
\]

\[
\dot{\varphi} = \frac{1}{3a} \left[ -X + 2\varphi \nabla \cdot u - u \cdot \nabla \varphi + 2C^{(i)\alpha\beta} \left( u_{i\alpha|\beta} + a\dot{C}^{(i)}_{\alpha\beta} \right) \right]
= C^{(i)\alpha\beta} \dot{C}^{(i)}_{\alpha\beta} - \frac{1}{2} \Delta^{-1} \nabla^\alpha \left[ \frac{1}{a} u \cdot \nabla (\nabla \varphi) + u_\alpha \Delta \varphi + u_\beta \Delta C^{(i)}_{\alpha\beta} \right]
- \varphi^\beta \dot{C}^{(i)}_{\alpha\beta} + 2C^{(i)\beta\gamma} \dot{C}^{(i)}_{\alpha\beta|\gamma} + C^{(i)}_{\beta\gamma|\alpha} \dot{C}^{(i)}_{\alpha\beta\gamma}.
\]

Equation for \( C^{(i)}_{\alpha\beta} \) to the second order can be found in equation (43) of [4].

5. Discussion

In this work we have successfully derived pure general relativistic correction terms appearing in the third-order perturbations of the zero-pressure irrotational multi-component fluids in a flat background. We have ignored \( \mathcal{O}(\mathbf{u} - \mathbf{u_i})^2 \) correction terms which simply decay in an expanding phase. Our main results are presented in equations (102)–(112) for pure scalar-type perturbations, and in equations (125)–(133) in the presence of the gravitational waves. The equations for the collective component are identical to the ones in the single-component case. If we further ignore \( \mathcal{O}(\mathbf{u} - \mathbf{u_i}) \) correction terms (appearing in pure third-order combinations), which are again simply decaying in an expanding phase (this is because we need these terms only to the linear order), the forms of relativistic correction terms in the individual component are the same as the ones in the collective component.

Our results show that, even in the multi-component situation, all the pure general relativistic third-order correction terms are smaller than the relativistic/Newtonian second-order terms by a factor \( \varphi_v \) without any further spatial gradient. Thus, the correction terms are independent of the presence of horizon scale. The pure third-order relativistic correction terms depend only on the linear order \( \varphi_v \) which remains constant in time even in the presence of cosmological constant; see equation (114). To the linear order \( \varphi_v \) is of the same order as the Newtonian gravitational potential divided by \( c^2 \):

\[
\varphi_v \sim \frac{5}{3} \frac{\delta \Phi}{c^2}.
\]
Thus, near the horizon scale we have

$$\varphi_v \sim \frac{5}{3} \frac{\delta \Phi}{c^2} \sim 5 \frac{\delta T}{T} \sim 5 \times 10^{-5},$$

(135)

due to extreme low level anisotropies of the cosmic microwave background radiation. Even far inside the horizon we have

$$\varphi_v \sim \frac{5}{3} \frac{\delta \Phi}{c^2} \sim \frac{GM}{Rc^2} \sim 10^{-6} - 10^{-5},$$

(136)

which is quite small. Thus, our relativistic third-order perturbation equations are applicable as long as the structures are in a weakly non-linear stage, and in such cases, the pure third-order terms are small. On small scales where the structures are in a fully non-linear stage, however, we cannot apply our perturbative method. Thus, in the non-linear stage it is not guaranteed to be correct to use our pure third-order terms as the general relativistic corrections; notice, however, that even in such regimes we have that the pure general relativistic correction terms are small because $\varphi_v \sim \delta \Phi/c^2 \ll 1$. In such a stage, instead, we should use the post-Newtonian correction terms as the general relativistic (weakly relativistic) corrections to the Newtonian gravity applicable in the fully non-linear stage; see [12]. Although the usage is limited to the weakly non-linear stage, it is still curious that, compared with the second-order relativistic/Newtonian terms, our third-order correction terms all involve only $\varphi_v$ without $\delta, u$ or further spatial gradients. In summary, our third-order perturbation equations are applicable on all scales as long as perturbations are in a weakly non-linear stage; this might cover quite vast scales even far inside the horizon.

By taking different temporal gauge (hypersurface) conditions or making different identifications we can easily introduce apparently horizon dependent terms with arbitrarily huge amplitudes. This is true even in the linear order and the second-order perturbations; for example, the uniform expansion gauge ($\kappa \equiv 0$) introduces abrupt change of the linear order perturbation variables near the horizon scale [23,14]. The exact relativistic/Newtonian correspondence of the density and velocity perturbations to the second-order perturbations was available essentially due to our proper choice of gauge conditions and correct identifications of relativistic variables with the Newtonian ones. In our third-order extension we have assumed that the same identification holds even in the third-order perturbations, which might not be necessarily the unique choice. However, the properties (the smallness of the amplitudes and independence from the horizon scale) of our third-order correction terms assure that our choice of the gauge and identifications are very likely to be the correct and best ones even to the third order.

At this point it would be interesting to present, from another viewpoint, the importance of our third-order perturbation analysis and the results. The perturbation analysis based on weak non-linearity does not necessarily require the third-order terms to be very small like the ones we have discovered. Instead, the perturbation approach only implies that the approach is not reliable if the third-order corrections are comparable to the second-order ones. For example, if our third-order correction terms somehow involved the density perturbation or velocity perturbation terms, the pure general relativistic
corrections terms could approach a tenth of the second-order terms or even larger. Our result shows that we do have a substantial number of pure general relativistic third-order correction terms. A close examination of these correction terms, however, shows that all of these terms are convolutions of $\varphi_v$ with the relativistic/Newtonian second-order perturbation terms. Thus, the third-order terms are generically smaller than the second-order relativistic/Newtonian terms by a factor $\varphi_v \sim \delta\Phi/c^2 \sim 10^{-5}$ on nearly all scales including the near horizon scale. Further, we emphasize that the third-order terms do not involve any further spatial gradient; thus the resulting corrections are in a sense scale invariant like $\varphi_v$ which, on the large scale, is directly related to the curvature perturbations generated during the inflation theoretically, and to the low level temperature anisotropy of the CMB observationally. This leads to our conclusion that the third-order (second-order as well) correction terms are independent of the presence of a horizon. As consequences, the third-order terms are independent of the presence of a horizon in the relativistic cosmology, and are quite small on nearly all scales. It is important to recall that these are highly non-trivial results and are far from what we would have naively anticipated without the actual calculations. We also note that these two points (the smallness and the horizon independence of the third-order correction terms) are from the pure general relativistic results only revealed by the concrete calculations based on our proper choices of the relativistic perturbation variables and the gauges. Our result assures that the fourth-order and higher order perturbations are practically not necessary. This result, however, does not rule out the possibility that these pure relativistic corrections will become important in future development of physical cosmology.

Our results may have practically important implications in currently favored cosmological pursuits by assuring the use of Newtonian physics in the large-scale non-linear processes which often involve two-component zero-pressure fluids (say, dust and cold dark matter) and the cosmological constant. As we have shown the exact relativistic/Newtonian correspondence to the second order and small horizon independent third-order correction terms, it is now more secure to use Newtonian physics on weakly non-linear scales including near (and even beyond) the horizon scale, which is indeed a noticeable trend in current cosmological simulations based on Newtonian gravity [28]. Our third-order perturbation equations are applicable on nearly all scales which are in a weakly non-linear stage. Considering that $\varphi_v$ is small on nearly all scales, our results guarantee the Newtonian physics (based on the density and velocity perturbations without resorting to the gravitational potential) on such scales as long as we can ignore general relativistic corrections $\sim 10^{-5}$ times smaller.

Here, we would like to discuss the completeness of our relativistic/Newtonian correspondence to the second order. As we have clearly stated, our relativistic/Newtonian correspondence refers to the coincidence of the density and velocity perturbation equations without the direct presence of the (Newtonian) potential. In this sense our correspondence is incomplete: i.e., the exact form of Poisson’s equation does not have a relativistic counterpart to the second order. There are, however, several points we would like to clarify. First, our third-order perturbation equations derived in this work do not depend on the completeness of the relativistic/Newtonian correspondence. Previously we have shown that to the second order the relativistic density and the velocity perturbation equations coincide with the Newtonian ones without directly involving the gravitational potential, and here we presented pure general relativistic third-order correction terms
in those relativistic/Newtonian equations. The second point is related to the reliability of using the Newtonian gravitational potential in Newtonian simulations in spite of the absence of the relativistic/Newtonian correspondence for the gravitational potential to the second order. As we are considering the non-linear perturbations of relativistic hydrodynamic equations, the $N$-body simulation which directly involves the Newtonian gravitational potential is not relevant for our work. Our correspondence is more relevant to the hydrodynamic simulation, and to the authors it is unclear whether introducing the Newtonian potential is necessary in such simulations. Finally, in the Newtonian quasi-linear analytic studies in [29] the gravitational potential appearing in the momentum conservation equation is removed using Poisson’s equation at the beginning stage of the analysis. Indeed, in the irrotational situation the gravitational potential can always be removed using Poisson’s equation mathematically without losing any generality. Thus, in this context our relativistic/Newtonian correspondence can be regarded as a complete one for calculating the density and velocity perturbations.

At this point it might be also worth emphasizing the effects of rotational perturbations. In [5] we have shown that the rotational perturbations to second order also have relativistic/Newtonian correspondence of the density and velocity perturbations in the small-scale (sub-horizon scale) limit. As the numerical simulations naturally include the rotational mode, this might be further good news to the cosmology community based on Newtonian physics. In this work we have shown that the general relativistic third-order correction terms are small near (and beyond) the horizon scale due to the low level temperature anisotropies of the CMB. Furthermore, we have shown that as long as the perturbative approach is applicable, the pure general relativistic third-order corrections are small because $\varphi_v$ is small on nearly all scales. Thus, our third-order perturbation equations are applicable on all scales not in the fully non-linear stage, which might cover quite vast scales even far inside the horizon. Although our results in [5] show that the second-order rotational perturbations have relativistic/Newtonian correspondence only far inside the horizon, due to the pure decaying nature of the rotational perturbations, we do not expect the rotational perturbations to be important near the horizon scale [5].

Such comforts with Newtonian physics will be assured, however, only as long as our pure general relativistic third-order or second-order correction terms are negligible compared with the current state of the art of observations and numerical simulations. Although we have estimated that the third-order pure general relativistic correction terms are small in the current large-scale structures, it would be still interesting to see whether cosmology could reach a stage where correction terms $\varphi_v$ ($\approx 5 \times 10^{-5}$) factor smaller could have noticeable effects on the large-scale structure formation processes. The third-order terms that we have presented in this work are the first non-vanishing general relativistic correction terms in Newtonian non-linear equations, and may have practical as well as historical importance in cosmology. Analytic studies in the Newtonian context of a single-component zero-pressure, irrotational fluid in the flat background have been actively investigated in [29]. Whether or not our general relativistic corrections appearing in the third order (both in the single-component and multi-component cases), and corrections appearing even in the second order because of effects of pressure, rotation, and background curvature, will have noticeable roles in the large-scale structure formation process is left for future investigations.
Third-order cosmological perturbations of zero-pressure multi-component fluids

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