CANONICAL HEIGHT FUNCTIONS ON THE AFFINE PLANE ASSOCIATED WITH POLYNOMIAL AUTOMORPHISMS

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Abstract. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a polynomial automorphism of dynamical degree $\delta \geq 2$ over a number field $K$. (This is equivalent to say that $f$ is a polynomial automorphism that is not triangularizable.) Then we construct canonical height functions defined on $\mathbb{A}^2(K)$ associated with $f$. These functions satisfy the Northcott finiteness property, and an $K$-valued point on $\mathbb{A}^2(K)$ is $f$-periodic if and only if its height is zero. As an application of canonical height functions, we give an estimate on the number of points with bounded height in an infinite $f$-orbit.

Introduction and the statement of the main results

One of the basic tools in Diophantine geometry is the theory of height functions. On Abelian varieties defined over a number field, Néron and Tate developed the theory of canonical height functions that behave well relative to the $[n]$-th power map (cf. [9, Chap. 5]). On certain K3 surfaces with two involutions, Silverman [14] developed the theory of canonical height functions that behave well relative to the two involutions. For the theory of canonical height functions on some other projective varieties, see for example [1], [16], [7]. In this paper, we show the existence of canonical height functions on the affine plane associated with polynomial automorphisms of dynamical degree $\geq 2$.

Consider a polynomial automorphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ given by
$$f \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} p(x, y) \\ q(x, y) \end{array} \right),$$
where $p(x, y)$ and $q(x, y)$ are polynomials in two variables. The degree $d$ of $f$ is defined by $d := \max\{\deg p, \deg q\}$. The dynamical degree $\delta$ of $f$ is defined by
$$\delta := \lim_{n \to +\infty} (\deg f^n)^{\frac{1}{n}},$$
which is an integer with $1 \leq \delta \leq d$. We let $d \geq 2$.

Polynomial automorphisms with $\delta = d$ are exactly regular polynomial automorphisms. Here a polynomial automorphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ is said to be regular if the unique point of indeterminacy of $f$ is different from the unique point of indeterminacy of $f^{-1}$, where the birational map $\overline{f} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ (resp. $\overline{f}^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$) is the extension of $f$ (resp. $f^{-1}$). In the moduli of polynomial automorphisms of degree $d$, regular polynomial automorphisms constitute general members, including Hénon maps.

The other extreme is polynomial automorphisms of dynamical degree $\delta = 1$, and they are exactly triangularizable automorphisms. Here a polynomial automorphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ is
said to be triangularizable if it is conjugate, in the group of polynomial automorphisms, to a polynomial automorphism of the form

\[ f(x, y) = \left(\frac{ax + P(y)}{by + c}\right), \]

where \(ab \neq 0\) and \(P(y)\) is a polynomial in \(y\). For more details, see the survey of Sibony [12] and the references therein. See also §3.

Over a number field, Silverman [15] studied arithmetic properties of quadratic Hénon maps, and then Denis [2] studied arithmetic properties of Hénon maps and some classes of polynomial automorphisms. Marcello [10], [11] studied arithmetic properties of some other classes of polynomial automorphisms of the affine spaces, including regular polynomial automorphisms.

Our first result shows the existence of height functions that behave well relative to polynomial automorphisms of \(\mathbb{A}^2\).

**Theorem A.** Let \(f: \mathbb{A}^2 \to \mathbb{A}^2\) be a polynomial automorphism of dynamical degree \(\delta \geq 2\) over a number field \(K\). (This is equivalent to say that \(f\) is a polynomial automorphism that is not triangularizable.) Then there exists a function \(\hat{h}: \mathbb{A}^2(\overline{K}) \to \mathbb{R}\) with the following properties:

(i) \(h_{nv} \gg \ll \hat{h}\) on \(\mathbb{A}^2(\overline{K})\) (Here \(h_{nv}\) is the logarithmic naive height function, and \(h_{nv} \gg \ll \hat{h}\) means that there are positive constants \(a_1, a_2\) and constants \(b_1, b_2\) such that \(a_1h_{nv} + b_1 \leq \hat{h} \leq a_2h_{nv} + b_2\));

(ii) \(\hat{h} \circ f + \hat{h} \circ f^{-1} = (\delta + \frac{1}{\delta}) \hat{h}\).

Moreover, \(\hat{h}\) enjoys the following uniqueness property: if \(\hat{h}'\) is another function satisfying (i) and (ii) such that \(\hat{h}' = \hat{h} + O(1)\), then \(\hat{h}' = \hat{h}\). We call a function \(\hat{h}\) satisfying (i) and (ii) a canonical height function associated with the polynomial automorphism \(f\).

It follows from (i) that \(\hat{h}\) satisfies the Northcott finiteness property. Namely, for any positive number \(M\) and positive integer \(D\), the set \(\{x \in \mathbb{A}^2(\overline{K}) \mid [K(x) : K] \leq D, \hat{h}(x) \leq M\}\) is finite. This leads to the following corollary, which shows that the set of \(\overline{K}\)-valued \(f\)-periodic points is not only a set of bounded height but also characterized as the set of height zero with respect to a canonical height function associated with \(f\).

**Corollary B.** With the notation and assumption in Theorem A,

1. \(\hat{h}(x) \geq 0\) for all \(x \in \mathbb{A}^2(\overline{K})\).
2. \(\hat{h}(x) = 0\) if and only if \(x\) is \(f\)-periodic. (Here, \(x \in \mathbb{A}^2(\overline{K})\) is said to be \(f\)-periodic if \(f^m(x) = x\) for some positive integer \(m\).)

As an application of canonical height functions, we obtain an estimate on the number of points with bounded height in an infinite \(f\)-orbit. First we introduce some notation and terminology. For a canonical height function \(\hat{h}\) associated with \(f\), we set

\[
\hat{h}^+(x) = \frac{\delta^2}{\delta^4 - 1} \left(\frac{1}{\delta} \hat{h}(f(x)) - \frac{1}{\delta} \hat{h}(f^{-1}(x))\right), \quad \hat{h}^-(x) = \frac{\delta^2}{\delta^4 - 1} \left(\frac{1}{\delta} \hat{h}(f^{-1}(x)) - \frac{1}{\delta} \hat{h}(f(x))\right).
\]
Then \( \hat{h}^+ \geq 0 \) and \( \hat{h}^- \geq 0 \), and \( \hat{h}^+(x) = 0 \) if and only if \( \hat{h}^-(x) = 0 \) if and only if \( x \) is \( f \)-periodic (cf. Lemma 5.1). For a point \( x \in \mathbb{A}^2(K) \), let \( O_f(x) := \{ f^l(x) \mid l \in \mathbb{Z} \} \) denote the \( f \)-orbit of \( x \). For a non \( f \)-periodic point \( x \in \mathbb{A}^2(K) \), we set

\[
\hat{h}(O_f(x)) = \frac{\log(\hat{h}^+(y)\hat{h}^-(y))}{\log \delta}
\]

for any \( y \in O_f(x) \). Then \( \hat{h}(O_f(x)) \) is well-defined, i.e., \( \hat{h}(O_f(x)) \) is independent of the choice of \( y \in O_f(x) \). Moreover, as a function of \( x \), we have \( \hat{h}(O_f(x)) \gg \min_{y \in O_f(x)} \log \hat{h}(y) \) on \( \mathbb{A}^2(K) \setminus \{ f \text{-periodic points} \} \) (cf. Lemma 5.2).

For regular polynomial automorphisms of degree \( d \geq 2 \), it is known that, for a non \( f \)-periodic point \( x \in \mathbb{A}^2(K) \), one has

\[
\lim_{T \to +\infty} \frac{\#\{ y \in O_f(x) \mid h_{nv}(y) \leq T \}}{\log T} = \frac{2}{\log d} \quad ([15, \text{Theorem C}], [2, \text{Théorème 2}], \text{and} \ [11, \text{Théorème A}]).
\]

The next theorem gives its refinement and generalization.

**Theorem C.** Let \( f : \mathbb{A}^2 \to \mathbb{A}^2 \) be a polynomial automorphism of dynamical degree \( \delta \geq 2 \) over a number field \( K \). Suppose \( x \in \mathbb{A}^2(K) \) is not an \( f \)-periodic point. Then,

\[
(0.1) \quad \#\{ y \in O_f(x) \mid h_{nv}(y) \leq T \} = \frac{2}{\log \delta} \log T - \hat{h}(O_f(x)) + O(1) \quad \text{as} \ T \to +\infty,
\]

where the \( O(1) \) constant depends only on \( f \) and the choice of \( \hat{h} \).

It seems interesting that the dynamical degree of \( f \) appears in the left-hand side of (0.1). We remark that, when \( f \) is not regular, i.e., \( (2 \leq) \delta < \deg f \), even a weaker estimate

\[
\lim_{T \to +\infty} \frac{\#\{ y \in O_f(x) \mid h_{nv}(y) \leq T \}}{\log T} = \frac{2}{\log \delta} \quad \text{seems new.}
\]

The contents of this paper is as follows. In §1 we briefly review the properties of height functions. In §2 we show that if \( f \) is a regular polynomial automorphism of degree \( d \geq 2 \) then there is a constant \( c \) such that

\[
(0.2) \quad h_{nv}(f(x)) + h_{nv}(f^{-1}(x)) \geq \left( d + \frac{1}{d} \right) h_{nv}(x) - c
\]

for all \( x \in \mathbb{A}^2(K) \). In §3 we recall Hénon maps, Friedland–Milnor’s theorem on the conjugacy classes of polynomial automorphisms, and some properties of dynamical degrees of polynomial automorphisms. In §4 we prove Theorem A and Corollary B in a more general setting of polynomial automorphisms of \( \mathbb{A}^n \) whose conjugates satisfy an inequality similar to (0.2). In §5 we prove Theorem C in this more general setting. On certain K3 surfaces, Silverman counted the number of points with bounded height in a given infinite chain ([14, §3]). Our method of proof of Theorem C is inspired by his method.

**Acknowledgments.** The author expresses his sincere gratitude to Prof. Noboru Nakayama for simplifying the proof of (0.2).

### 1. Quick review on height theory

In this section, we briefly review the properties of height functions that we will use in this paper.
Let $K$ be a number field and $O_K$ its ring of integers. For $x = (x_0: \cdots: x_n) \in \mathbb{P}^n(K)$, the logarithmic naive height of $x$ is defined by
\[
h_{nv}(x) = \frac{1}{[K: \mathbb{Q}]} \left[ \sum_{P \in \text{Spec}(O_K) \setminus \{0\}} \max_{0 \leq i \leq n} \{-\text{ord}_P(x_i)\} \log |O_K/P| + \sum_{\sigma: K \to \mathbb{C}} \max_{0 \leq i \leq n} \{\log |\sigma(x_i)|\} \right].
\]

This definition naturally extends to all points $x \in \mathbb{P}^n(\overline{\mathbb{Q}})$ as to give the logarithmic naive height function $h_{nv}: \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{R}$.

We begin by the following two basic properties of height functions.

**Theorem 1.1** (Northcott’s finiteness theorem, [13] Corollary 3.4). For any positive number $M$ and positive integer $D$, the set
\[
\{ x \in \mathbb{P}^n(\overline{\mathbb{Q}}) \mid [\mathbb{Q}(x) : \mathbb{Q}] \leq D, \ h_{nv}(x) \leq M \}
\]
is finite.

**Theorem 1.2** ([13] Theorem 3.3, [9] Chap. 4, Prop. 5.2). (1) (Height machine) There is a unique way to attach, for any projective variety $X$ defined over $\overline{\mathbb{Q}}$, a map
\[
h_X : \text{Pic}(X) \longrightarrow \frac{\{\text{real-valued functions on } X(\overline{\mathbb{Q}})\}}{\{\text{real-valued bounded functions on } X(\overline{\mathbb{Q}})\}}, \quad L \mapsto h_{X,L}
\]
with the following properties:
(i) $h_{X,L \otimes M} = h_{X,M} + h_{X,M} + O(1)$ for any $L, M \in \text{Pic}(X)$;
(ii) If $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$, then $h_{\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1)} = h_{nv} + O(1)$;
(iii) If $f : X \to Y$ is a morphism of projective varieties and $L$ is a line bundle on $X$, then $h_{X,f^*L} = h_{Y,L} \circ f + O(1)$.

(2) (Positivity of height) Let $X$ be projective variety defined over $\overline{\mathbb{Q}}$ and $L$ a line bundle on $X$. We set $B = \text{Supp}(\text{Coker}(H^0(X, L) \otimes \mathcal{O}_X \to L))$. Then there exists a constant $c_1$ such that $h_{X,L}(x) \geq c_1$ for all $x \in (X \setminus B)(\overline{\mathbb{Q}})$.

A rational map $f = [F_0 : F_1 : \cdots : F_n] : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ defined over $\overline{\mathbb{Q}}$ is said to be of degree $d$ if the $F_i$’s are homogeneous polynomials of degree $d$ over $\overline{\mathbb{Q}}$, with no common factors. Let $I_f \subset \mathbb{P}^n(\overline{\mathbb{Q}})$ denote the locus of indeterminacy.

**Theorem 1.3** ([9] Chap. 4, Lemma 1.6). Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be rational map of degree $d$ defined over $\overline{\mathbb{Q}}$. Then there exists a constant $c_2$ such that
\[
h_{nv}(f(x)) \leq d \ h_{nv}(x) + c_2
\]
for all $x \in \mathbb{P}^n(\overline{\mathbb{Q}}) \setminus I_f$.

## 2. Geometric properties of regular polynomial automorphisms

In this section, we show (0.2) for regular polynomial automorphisms of $\mathbb{A}^2$. First we recall the definition of regular polynomial automorphisms of $\mathbb{A}^2$. Consider a polynomial automorphism of degree $d \geq 2$ of the form
\[
f \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} p(x,y) \\ q(x,y) \end{array} \right),
\]
where $p(x, y)$ and $q(x, y)$ are polynomials in two variables, and $d$ is the maximum of $\deg p$ and $\deg q$. Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be the extension of $f$ given in homogeneous coordinates as

$$
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} =
\begin{bmatrix}
Z^d p(X/Z, Y/Z) \\
Z^d q(X/Z, Y/Z) \\
Z^d
\end{bmatrix}.
$$

Let $H$ denote the line at infinity. Then $f$ has a unique point of indeterminacy on $H$, denoted by $p$. Let $f^{-1} : \mathbb{A}^2 \to \mathbb{A}^2$ be the inverse of $f$, and $f^{-1} : \mathbb{P}^2 \to \mathbb{P}^2$ be its extension. Then $f^{-1}$ has a unique point of indeterminacy on $H$, denoted by $q$. A polynomial automorphism of $\mathbb{A}^2$ is said to be regular if $p \neq q$.

By elimination of indeterminacy, by successively blowing up points starting from $p \in \mathbb{P}^2$, we obtain a projective surface $W$ and a composite of blow-ups $\pi_W : W \to \mathbb{P}^2$ such that $f \circ \pi_W : W \to \mathbb{P}^2$ becomes a morphism. We take $W$ so that the number of blow-ups needed for elimination of indeterminacy is minimal. Noting that $\pi_W$ induces an isomorphism $\pi_W^{-1}(\mathbb{P}^2 \setminus \{p\}) \to \mathbb{P}^2 \setminus \{p\}$, we take $q' \in W$ with $\pi_W(q') = q$. In a parallel way as for $p$, $f^{-1} \circ \pi_W : W \to \mathbb{P}^2$ becomes a morphism after a finite number of blow-ups starting at $q'$.

To summarize, there is a projective surface $V$ obtained by successive blow-ups of $\mathbb{P}^2$ at $p$ and then successive blow-ups at $q$ in a parallel way as for $p$ such that, if $\pi : V \to \mathbb{P}^2$ denotes the morphism of blow-ups, $f \circ \pi$ extends to a morphism $\varphi : V \to \mathbb{P}^2$ and $f^{-1} \circ \pi$ extends to a morphism $\psi : V \to \mathbb{P}^2$. As for $W$, we take $V$ so that the number of blow-ups needed for elimination of indeterminacy is minimal.

$$(2.1)$$

Before stating the next theorem, we fix some notation and terminology. Let $\rho : Y \to X$ be a morphism of smooth projective surfaces. For an irreducible curve $C$ on $Y$, its push-forward is defined by

$$
\rho_*(C) := \begin{cases} 
\deg(\rho|_C : C \to f(C)) \cdot f(C) & \text{(if } f(C) \text{ is a curve)}, \\
0 & \text{(if } f(C) \text{ is a point)}. 
\end{cases}
$$

This extends linearly to a homomorphism $\rho_*$ from divisors on $Y$ to divisors on $X$. For two divisors $Z_1, Z_2$, we write $Z_1 \geq Z_2$ if $Z_1 - Z_2$ is effective.

**Theorem 2.1.** Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a regular polynomial automorphism of degree $d \geq 2$. Let $H$ denote the line at infinity. Let $V$ be as in (2.1). Then, as a $\mathbb{Q}$-divisor on $V$,

$$
D := \varphi^*H + \psi^*H - \left( d + \frac{1}{d} \right) \pi^*H
$$

is effective.

**Proof.** The proof we present here, which simplifies the proof we gave in the initial draft, is due to Noboru Nakayama.

As above, let $\pi_W : W \to \mathbb{P}^2$ be a composite of blow-ups of $\mathbb{P}^2$ starting at $p$ such that $\varphi_W := f \circ \pi_W : W \to \mathbb{P}^2$ becomes a morphism. Let $H_W$ be the proper transform of $H$ by
$\pi_W$, and $E_W$ the exceptional curve on $W$ given by the last blow-up of $\pi_W$. Since $\varphi_W$ is a morphism and $W$ is taken so that the number of blow-ups is minimal, we see that $\varphi_W$ sends $E_W$ to $H$ isomorphically.

We consider $\pi_W^*H$ and $\varphi_W^*H$. We write $\pi_W^*H = aH_W + bE_W + M_W$ and $\varphi_W^*H = a'H_W + b'E_W + I_W$, where $a, b, a', b'$ are non-negative integers, and $M_W, I_W$ are effective divisors on $W$ with $\text{Supp}(E_W) \not\subseteq \text{Supp}(M_W), \text{Supp}(E_W) \not\subseteq \text{Supp}(I_W)$ such that $M_W, I_W$ are contracted to $p$ by $\pi_W$.

We determine $a, b, a', b'$. Since $\pi_W$ is a birational morphism, $\pi_W^*\pi_W^*H = H$. It follows that $a = 1$. Similarly, $\varphi_W^*\varphi_W^*H = H$ yields $b' = 1$. On the other hand, let $[H]$ denotes the cohomology class of $H$ in $H^2(\mathbb{P}^2, \mathbb{Z})$. Since the degree of $f : \mathbb{A}^2 \to \mathbb{A}^2$ is $d$, we get $\varphi_W^*\pi_W^*[H] = d[H] \in H^2(\mathbb{P}^2, \mathbb{Z})$. It follows that $\varphi_W^*\pi_W^*H = dH$ and $b = d$. Since the degree of $f^{-1} : \mathbb{A}^2 \to \mathbb{A}^2$ is also $d$, we get $\pi_W^*\varphi_W^*H = dH$ and $a' = d$. Putting together, we have

$$\pi_W^*H = H_W + dE_W + M_W,$$
$$\varphi_W^*H = dH_W + E_W + I_W.$$  

Since the effective divisor $\pi_W^*H$ is nef, Lemma 2.2 below yields that

$$\varphi_W^*(dH) = \varphi_W^*(\varphi_W^*\pi_W^*H) = (\varphi_W^*\varphi_W^*)\pi_W^*H \geq \pi_W^*H.$$  

We thus get

$$(2.2) \quad dI_W \geq M_W.$$  

In a parallel way as for $p$, let $\pi_U : U \to \mathbb{P}^2$ be a composite of blow-ups of $\mathbb{P}^2$ starting at $q$ such that $\psi_U := f^{-1} \circ \pi_U : U \dashrightarrow \mathbb{P}^2$ becomes a morphism. Let $H_U$ be the proper transform of $H$ by $\pi_U$, and $F_U$ the exceptional curve on $U$ given by the last blow-up of $\pi_U$. The morphism $\psi_U$ sends $F_U$ to $H$ isomorphically. In a parallel way, we get

$$\pi_U^*H = H_U + dF_U + N_U,$$
$$\psi_U^*H = dH_U + F_U + J_U,$$
$$dJ_U \geq N_U,$$  

(2.3)

where $N_U, J_U$ are effective divisors on $U$ with $\text{Supp}(F_U) \not\subseteq \text{Supp}(N_U), \text{Supp}(F_U) \not\subseteq \text{Supp}(J_U)$ such that $N_U, J_U$ are contracted to $q$ by $\pi_U$.

By the construction of $V$, there are birational morphisms $\alpha : V \to W$ and $\beta : V \to U$ such that the following diagram is commutative.

Let $H^*$ on $V$ be the proper transform of $H$ by $\pi$. Let $E, M, I$ on $V$ be the proper transforms of $E_W, M_W, I_W$ by $\alpha$, respectively. Let $F, N, J$ be the proper transforms of
Lemma 2.2. Let $Y$ be an effective divisor on $F$ the exceptional curve on $Y$. Since $dI = \rho x$ for all distinct irreducible and reduced curves, and $a(2.4)$ formula yields ($m$ each $\rho$)

\[ H = H^\# + dE + dF + M + N, \]
\[ \varphi^* H = d(H^\# + dF + N) + E + I, \]
\[ \psi^* H = d(H^\# + dE + M) + F + J. \]

By (2.4)–(2.6), we get

\[ D = \varphi^* H + \psi^* H - \left( d + \frac{1}{d} \right) \pi^* H \]
\[ = \left( d - \frac{1}{d} \right) H^\# - \frac{1}{d} M + I - \frac{1}{d} N + J. \]

Since $dI \geq M$ and $dJ \geq N$ by (2.2) and (2.3), we see that $D$ is effective.

Lemma 2.3. Let $\rho : Y \to X$ be a birational morphism of smooth projective surfaces. Let $Z$ be an effective divisor on $Y$. If $Z$ is nef, then $\rho^* \rho_* Z \geq Z$.

Proof. First we treat a case when $\rho$ is the blow-up of $X$ at a point $x \in X$. Let $E$ denote the exceptional curve on $Y$. We write $Z = a_1 C_1 + \cdots + a_k C_k + bE$, where $C_1, \ldots, C_k, E$ are distinct irreducible and reduced curves, and $a_1, \ldots, a_k, b$ are non-negative integers. Then $\rho_* Z = a_1 \rho(C_1) + \cdots + a_k \rho(C_k)$. Hence $\rho^* \rho_* Z = a_1 (C_1 + m_1 E) + \cdots + a_k (C_k + m_k E)$, where $m_i$ is the multiplicity of the curve $\rho(C_i)$ at $x$. Note that $m_i = C_i \cdot E$.

Since $Z$ is nef, we get

\[ Z \cdot E = a_1 (C_1 \cdot E) + \cdots + a_k (C_k \cdot E) + b(E \cdot E) \]
\[ = a_1 m_1 + \cdots + a_k m_k - b \geq 0. \]

Hence $a_1 m_1 + \cdots + a_k m_k \geq b$ and we get $\rho^* \rho_* Z \geq Z$.

In general, we decompose $\rho$ into a composite of blow-ups: $\rho = \rho_1 \circ \cdots \circ \rho_2 \circ \rho_1$, where each $\rho_i$ is a blow-up at a point. Put $\rho' := \rho_1 \circ \cdots \circ \rho_2$, and $Z' := \rho_1_* Z$. Since the projection formula yields $(\rho_1 Z) \cdot C = Z \cdot (\rho_1^* C)$ for any curve, we see that $Z'$ is nef. Then, by induction, $\rho^* \rho'_* Z' \geq Z'$. Pulling back by $\rho_1$, we get $\rho_1^* (\rho^* \rho'_* Z') \geq \rho_1^* (\rho_1_* Z) \geq Z$.

Now we prove (0.2).

Theorem 2.3. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a regular polynomial automorphism of degree $d \geq 2$ defined over a number field $K$. Then, there exists a constant $c$ such that

\[ h_{nv}(f(x)) + h_{nv}(f^{-1}(x)) \geq \left( d + \frac{1}{d} \right) h_{nv}(x) - c \]

for all $x \in \mathbb{A}^2(K)$. 

Proof. We can prove Theorem 2.3 as in [15, Theorem 3.1]. We take \( x \in \mathbb{A}^2(\overline{K}) \). Since \( \pi : V \rightarrow \mathbb{P}^2 \) gives an isomorphism \( \pi|_{\pi^{-1}(\mathbb{A}^2)} : \pi^{-1}(\mathbb{A}^2) \rightarrow \mathbb{A}^2 \), there is a unique point \( \tilde{x} \in V \) with \( \pi(\tilde{x}) = x \). By Theorem 2.1, we have
\[
h_{V,O_V(\varphi^*H)}(\tilde{x}) + h_{V,O_V(\psi^*H)}(\tilde{x}) = \left(d + \frac{1}{d}\right) h_{V,O_V(\varphi^*H)}(\tilde{x}) + h_{V,O_V(D)}(\tilde{x}) + O(1).
\]
It follows from Theorem 1.2(1) that
\[
h_{V,O_V(\varphi^*H)}(\tilde{x}) = h_{\mathbb{P}^2,O_V(H)}(\varphi(\tilde{x})) + O(1) = h_{\mathbb{P}^2,O_V(H)}(f(x)) + O(1).
\]
We similarly have
\[
h_{V,O_V(\psi^*H)}(\tilde{x}) = h_{\mathbb{P}^2,O_V(H)}(f^{-1}(x)) + O(1),
\]
\[
h_{V,O_V(\varphi^*H)}(\tilde{x}) = h_{\mathbb{P}^2,O_V(H)}(x) + O(1).
\]
On the other hand, since \( \pi(\text{Supp}(D)) \subseteq \text{Supp}(H) \), we have \( \tilde{x} \not\in \text{Supp}(D) \). Since \( D \) is effective by Theorem 2.1, it follows from Theorem 1.2(2) that there is a constant \( c_2 \) independent of \( \tilde{x} \) such that \( h_{V,O_V(D)}(\tilde{x}) \geq c_2 \). Hence we get the assertion. \( \square \)

3. Hénon Maps, Conjugacy Classes of Polynomial Automorphisms, and Dynamical Degrees

In this section, we review Hénon maps, Friedland–Milnor’s theorem on the conjugacy classes of polynomial automorphisms, and some properties of dynamical degrees of polynomial automorphisms, which will be used in §4. We also give explicit forms of \( \varphi^*H \), \( \psi^*H \) and \( \pi^*H \) in Theorem 2.1 for Hénon maps.

A Hénon map is a polynomial automorphism of the form
\[
(\varphi^*H) \quad f(x, y) = \left(\begin{array}{c}
p(x) - ay \\
x
\end{array}\right),
\]
where \( a \neq 0 \) and \( p \) is a polynomial of degree \( d \geq 2 \). Let \( \overline{f} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \) (resp. \( \overline{f}^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \)) be the birational extension of \( f \) (resp. \( f^{-1} \)). Then \( \overline{f} \) has the unique point of indeterminacy \( p = t[0, 1, 0] \), and \( \overline{f}^{-1} \) has the unique point of indeterminacy \( q = t[1, 0, 0] \). In particular, Hénon maps are examples of regular polynomial automorphisms.

We recall Friedland–Milnor’s theorem [4, §2], which is based on Jung’s theorem [6]. Let
\[
E = \left\{ f : \mathbb{A}^2 \rightarrow \mathbb{A}^2, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax + P(y) \\ by + c \end{pmatrix} \mid a, b \in \overline{\mathbb{Q}}^*, c \in \overline{\mathbb{Q}}, P(y) \in \overline{\mathbb{Q}}[Y] \right\}
\]
be the group of triangular automorphisms (also called de Jonquères automorphisms).

**Theorem 3.1** ([4, §2]). Let \( f : \mathbb{A}^2 \rightarrow \mathbb{A}^2 \) be a polynomial automorphism over \( \overline{\mathbb{Q}} \). Then there is a polynomial automorphism \( \gamma : \mathbb{A}^2 \rightarrow \mathbb{A}^2 \) over \( \overline{\mathbb{Q}} \) such that \( g := \gamma^{-1} \circ f \circ \gamma \) is one of the following types:

(i) \( g \) is a triangular automorphism;

(ii) \( g \) is a composite of Hénon maps.
Note that Friedland–Milnor proved the theorem over $\mathbb{C}$, but the theorem holds over $\overline{\mathbb{Q}}$ by the specialization argument in [2, Lemme 2].

A polynomial automorphism $f$ is said to be triangularizable if it is conjugate to a triangular automorphism.

Here we recall dynamical degrees of polynomial automorphisms $f : \mathbb{A}^2 \to \mathbb{A}^2$. The dynamical degree of $f$ is defined by

$$\delta(f) := \lim_{n \to +\infty} (\deg f^n)^{-\frac{1}{n}}$$

(cf. [12, Définition 1.4.4]). Suppose $g = \gamma^{-1} \circ f \circ \gamma$ is conjugate to $f$. Then, since $g^n = \gamma^{-1} \circ f^n \circ \gamma$, we have $\deg f^n - 2 \deg \gamma \leq \deg g^n \leq \deg f^n + 2 \deg \gamma$. It follows that $\delta(f) = \delta(g)$. Thus dynamical degrees depend only on conjugacy classes of polynomial automorphisms.

For polynomial automorphisms $g_1, g_2 : \mathbb{A}^2 \to \mathbb{A}^2$ with degree $\deg g_1, \deg g_2 \geq 2$ and their extensions $\overline{g_1}, \overline{g_2} : \mathbb{P}^2 \to \mathbb{P}^2$, one has

$$(3.3) \quad \deg(g_1 \circ g_2) \leq (\deg g_1)(\deg g_2).$$

with equality if and only if the unique point $q_{g_1}$ of indeterminacy of $\overline{g_1}$ is different from the unique point $p_{g_2}$ of indeterminacy of $\overline{g_2}$ (cf. [12, Proposition 1.4.3]). We remark that a composite $g$ of Hénon maps is a regular polynomial automorphism, because the indeterminacy point of $\overline{g}$ is $t[0, 1, 0]$ while the indeterminacy point of $\overline{g^{-1}}$ is $t[1, 0, 0]$.

The following proposition is well-known.

**Proposition 3.2.** Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a polynomial automorphism. Let $d$ be the degree of $f$ and $\delta$ the dynamical degree of $f$.

1. $\delta$ is an integer with $1 \leq \delta \leq d$.
2. $\delta = 1$ if and only if $f$ is triangularizable.
3. Suppose $d \geq 2$. Then $\delta = d$ if and only if $f$ is a regular polynomial automorphism.

**Proof.** We rely on the results of Furter [3] to give a quick proof. We put $\tau = \frac{\deg(f^n)}{\deg f}$. Then Furter showed that either (i) $\tau \leq 1$ or (ii) $\tau$ is an integer greater than or equal to 2. Moreover, (i) occurs if and only if $f$ is triangularizable ([3, Proposition 5]). In the case (ii), one has $\deg f^n = \tau^n \cdot \deg f$ ([3, Proposition 4]).

1. In the case (i), $f$ is triangular, and then its definition (3.2) yields that $\deg f^n \leq \deg f$, whence $\delta(f) = 1$. In the case (ii), the dynamical degree of $f$ is equal to an integer $\tau \geq 2$.
2. (ii) follows from the above proof of (1).
3. Since $d$ is assumed to be $\geq 2$, (3.3) shows that $f$ is a regular polynomial automorphism if and only if $\tau = \deg f$ ($\geq 2$). Since $\tau = \delta(f)$ if $\tau \geq 2$, we get the assertion. \(\square\)

Since Hénon maps are basic objects in the dynamics of polynomial automorphisms of $\mathbb{A}^2$ (cf. Theorem 3.1), it would be worth giving explicit forms of $\varphi^*H$, $\psi^*H$ and $\pi^*H$ in Theorem 2.1 for Hénon maps of degree $d \geq 2$, as Silverman [15] did for quadratic Hénon maps. In particular, this gives a different proof of Theorem 2.1 in case of Hénon maps.

For this, we need an explicit description of blow-ups at (infinitely near) points on $\mathbb{P}^2$ that resolve the point of indeterminacy of a Hénon map $\overline{f}$. The case $\deg g = 2$ was carried out by Silverman [15, §2], and the general case by Hubbard–Papadopol–Veselov [5, §2] in their compactification of Hénon maps in $\mathbb{C}^2$ as dynamical systems. Let us put together their
results in the following theorem. (Note that, for the next theorem, the field of definition of $f$ can be any field, and $p(x)$ need not be monic.)

**Theorem 3.3** ([5], §2). (1) Let $f$ be a Hénon map in (3.1), and $\overline{f} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ its birational extension. Then $\overline{f}$ becomes well-defined after a sequence of $2d - 1$ blow-ups. Explicitly, blow-ups are described as follows:

(i) First blow-up at $p$;
(ii) Next blow up at the unique point of indeterminacy, which is given by the intersection of the exceptional divisor and the proper transform of $H$;
(iii) For the next $d - 2$ times after (ii), blow-up at the unique point of indeterminacy, which is given by the intersection of the last exceptional divisor and the proper transform of the first exceptional divisor;
(iv) For the next $d - 1$ times after (iii), blow-up at the unique point of indeterminacy, which lies on the last exceptional divisor but not on the proper transform of the other exceptional divisors.

(2) Let $f_{2d-1} : W \rightarrow \mathbb{P}^2$ be the extension of the Hénon map after the sequence of $2d - 1$ blow-ups. Let $E_i'$ denote the proper transform of $i$-th exceptional divisor ($i = 1, \cdots, 2d - 1$). Then $f_{2d-1}$ maps $E_i' (i = 1, \cdots, 2d - 2)$ to $q$, while $E_{2d-1}'$ is mapped to $H$ by an isomorphism.

(3) $E_i'^2 = -d, E_{2d-2}' = -2 (i = 2, \cdots, 2d - 2)$, and $E_{2d-1}'^2 = -1$.

In particular, for Hénon maps, $V$ in (2.1) is the projective surface obtained by successive $2d - 1$ blow-ups of $\mathbb{P}^2$ at $p$ as in Theorem 3.3 and then successive $2d - 1$ blow-ups at $q$ in a parallel way as in Theorem 3.3.

Let $E_i (1 \leq i \leq 2d - 1)$ be the proper transform of $i$-th exceptional divisor on $V$ on the side of $p$, and $F_j (1 \leq j \leq 2d - 1)$ be the proper transform of $j$-th exceptional divisor on $V$ on the side of $q$. Let $H^\#$ be the proper transform of $H$. The configuration of $H^\#, E_i$ and $F_j$ is illustrated in Figure 1.

**Proposition 3.4.** Let $f : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ be a Hénon map of degree $d \geq 2$. Let the notation be as above.

(1) As divisors on $V$, we have

$$
\pi^* H = H^\# + \sum_{i=1}^{d} iE_i + \sum_{i=d+1}^{2d-1} dE_i + \sum_{j=1}^{d} jF_j + \sum_{j=d+1}^{2d-1} dF_j,
$$

$$
\varphi^* H = dH^\# + E_1 + \sum_{i=2}^{d} dE_i + \sum_{i=d+1}^{2d-1} (2d - i)E_i + \sum_{j=1}^{d} jF_j + \sum_{j=d+1}^{2d-1} d^2 F_j,
$$

$$
\psi^* H = dH^\# + \sum_{i=1}^{d} idE_i + \sum_{i=d+1}^{2d-1} d^2 E_i + F_1 + \sum_{j=2}^{d} dF_j + \sum_{j=d+1}^{2d-1} (2d - j)F_j.
$$
Figure 1. The configuration after blow-ups. The line $H^\#$ has the self-intersection number $-3$. The lines $E_1$ and $F_1$ have the self-intersection numbers $-d$. The lines $E_2, E_3, \cdots, E_{2d-2}$ and $F_2, F_3, \cdots, F_{2d-2}$ have the self-intersection numbers $-2$. The lines $E_{2d-1}$ and $F_{2d-1}$ have the self-intersection numbers $-1$.

(2) The effective $\mathbb{Q}$-divisor $D$ in Theorem 2.1 is expressed as

$$D = \frac{d^2 - 1}{d} H^\# + \frac{d - 1}{d} E_1 + \sum_{i=2}^{d} \frac{d^2 - i}{d} E_i + \sum_{i=d+1}^{2d-1} (2d - i - 1) E_i$$

$$+ \frac{d - 1}{d} F_1 + \sum_{j=2}^{d} \frac{d^2 - j}{d} F_j + \sum_{j=d+1}^{2d-1} (2d - j - 1) F_j.$$}

Proof. We will show the expression for $\varphi^* H$. Since $\varphi$ maps $H^\#, E_i (1 \leq i \leq 2d - 2)$ and $F_j (1 \leq j \leq 2d - 1)$ to the point $q$, we have

$$\varphi^* H \cdot H^\# = 0, \quad \varphi^* H \cdot E_i = 0, \quad \varphi^* H \cdot F_j = 0$$

for $1 \leq i \leq 2d - 2$ and $1 \leq j \leq 2d - 1$. Since $\varphi$ maps $E_{2d-1}$ to $H$ isomorphically, we have

$$\varphi^* H \cdot E_{2d-1} = 1.$$
of the configuration after blow-ups (cf. Figure 1), we have the system of linear equations
\[
-3a + b_2 + c_2 = 0, \\
\begin{cases}
-db_1 + b_d = 0 \\
a - 2b_2 + b_3 = 0 \\
\quad b_{i-1} - 2b_i + b_{i+1} = 0 \\
\quad b_1 + b_d - 2b_d + b_{d+1} = 0 \\
\quad b_{2d-2} - b_{2d-1} = 1,
\end{cases}
\begin{cases}
-dc_1 + c_d = 0 \\
a - 2c_2 + c_3 = 0 \\
\quad c_{j-1} - 2c_j + c_{j+1} = 0 \\
\quad c_1 + c_{d-1} - 2c_d + c_{d+1} = 0 \\
\quad c_{2d-2} - c_{2d-1} = 0,
\end{cases}
\]
where \(i = 3, \cdots, d - 1, d + 1, \cdots, 2d - 2\) and \(j = 3, \cdots, d - 1, d + 1, \cdots, 2d - 2\). By solving this system, we obtain the expression for \(\varphi^*H\). Similarly we obtain the formula for \(\psi^*H\). The formula for \(\pi^*H\) follows from the construction of \(V\). (We can also show this by using \(\pi^*H \cdot H^\# = 1, \pi^*H \cdot E_i = 0\) and \(\pi^*H \cdot F_j = 0\) for all \(i\) and \(j\).) The assertion (2) follows from (1).

\[\square\]

**Remark 3.5.** Using classical results of Jung [6] and van der Kulk [8], it is possible to explicitly compute \(D\) for any regular polynomial automorphisms \(f\) of degree \(d \geq 2\), as in Proposition 3.4 for Hénon maps. In this case, coefficients of \(D\) are expressed in terms of the polydegree \((d_1, \ldots, d_l)\) of \(f\) (cf. [4, §3]). Note that, for Hénon maps \(f\) of degree \(d \geq 2\), its polydegree is \((d)\), i.e., \(l = 1\) and \(d_1 = d\).

4. Canonical height functions

In this section, we will prove Theorem A and Corollary B by showing Theorem 4.1. We first fix some notation and terminology. We refer to the survey [12] for more details about the dynamics of polynomial automorphisms.

Let \(f : \mathbb{A}^n \to \mathbb{A}^n\) be a polynomial automorphism over a number field \(K\). We use the notation \(\overline{f}\) to denote the birational extension of \(f\) to \(\mathbb{P}^n\). Let \(f^{-1} : \mathbb{A}^n \to \mathbb{A}^n\) denote the inverse of \(f\), and we use the notation \(\overline{f^{-1}}\) to denote the birational extension of \(f^{-1}\) to \(\mathbb{P}^n\). Note that the degree of \(f\) and the degree of \(f^{-1}\) may not be the same when \(n \geq 3\) (cf. [12, Chapitre 2]).

Let \(S\) be a set and \(T\) a subset of \(S\). Two real-valued functions \(\lambda\) and \(\lambda'\) on \(S\) are said to be equivalent on \(T\) if there exist positive constants \(a_1, a_2\) and constants \(b_1, b_2\) such that \(a_1\lambda(x) + b_1 \leq \lambda'(x) \leq a_2\lambda(x) + b_2\) for all \(x \in T\). We use the notation \(\lambda \gg \ll \lambda'\) to denote this equivalence. (Note that our notation \(\gg \ll\) is different from that in [9, Chap. 4, §1] where \(b_1 = b_2 = 0\).

**Theorem 4.1.** Let \(f : \mathbb{A}^n \to \mathbb{A}^n\) be a polynomial automorphism over a number field \(K\). Let \(\gamma : \mathbb{A}^n \to \mathbb{A}^n\) be a polynomial automorphism over \(K\), and we define the polynomial automorphism \(g : \mathbb{A}^n \to \mathbb{A}^n\) by \(g := \gamma^{-1} \circ f \circ \gamma\). Let \(\delta\) and \(\delta_\gamma\) denote the degrees of \(g\) and \(g^{-1}\), respectively. We assume that \(\delta \geq 2\) and that there exists a constant \(c\) such that

\[
\frac{1}{\delta} h_{nv}(g(x)) + \frac{1}{\delta_\gamma} h_{nv}(g^{-1}(x)) \geq \left(1 + \frac{1}{\delta \delta_\gamma}\right) h_{nv}(x) - c
\]

for all \(x \in \mathbb{A}^n(\overline{K})\). Then there exists a function \(\hat{h} : \mathbb{A}^n(\overline{K}) \to \mathbb{R}\) with the following properties:

(i) \(h_{nv} \gg \ll \hat{h}\) on \(\mathbb{A}^n(\overline{K})\);
Moreover, \( \hat{h} \) enjoys the following uniqueness property: if \( \hat{h}' \) is another function satisfying (i) and (ii) such that \( \hat{h}' = \hat{h} + O(1) \), then \( \hat{h}' = \hat{h} \). Furthermore, \( \hat{h}(x) \geq 0 \) for all \( x \in \mathbb{A}^n(K) \), and \( \hat{h}(x) = 0 \) if and only if \( x \) is \( f \)-periodic.

**Proof of Theorem A and Corollary B.** Admitting Theorem 4.1, we will prove Theorem A and Corollary B. We may replace \( K \) by a finite extension field. Since the dynamical degree \( \delta \) is greater than or equal to 2, Theorem 3.1 and Proposition 3.2 yield that there is a polynomial automorphism \( \gamma \) such that \( g := \gamma \circ f \circ \gamma^{-1} \) is a composite of Hénon maps. Since a composite of Hénon maps is a regular polynomial automorphism (cf. lines before Proposition 3.2), it follows from Theorem 2.3 that \( g \) satisfies (4.1). Then, noting that the dynamical degrees of \( f \) and \( g \) are the same, Theorem A and Corollary B follows from Theorem 4.1.

**Proof of Theorem 4.1.**

**Step 1.** We show the existence of a function \( \hat{h}_g : \mathbb{A}^n(K) \to \mathbb{R} \) with the following properties:

(iii) \( h_{nv} \gg \gg \hat{h}_g \) on \( \mathbb{A}^n(K) \);

(iv) \( \frac{1}{\delta} \hat{h}_g \circ g + \frac{1}{\delta} \hat{h}_g \circ g^{-1} = \left( 1 + \frac{1}{\delta} \right) \hat{h}_g \).

For \( x \in \mathbb{A}^n(K) \), we define

\[
\hat{h}_g^+(x) = \limsup_{t \to +\infty} \frac{1}{\delta} h_{nv}(g^t(x)), \quad \hat{h}_g^-(x) = \limsup_{t \to +\infty} \frac{1}{\delta} h_{nv}(g^{-t}(x)),
\]

*a priori* in \( \mathbb{R} \cup \{ \infty \} \), but we will show in the next claim that this value is finite. We define

\[
\hat{h}_g(x) = \hat{h}_g^+(x) + \hat{h}_g^-(x).
\]

Note that this definition of \( \hat{h}_g^\pm \) has some similarity to the definition of Green currents on \( \mathbb{A}^n(\mathbb{C}) \) associated with \( g \) (cf. [12, Définition 2.2.5]), and to Silverman’s definition of canonical heights on certain K3 surfaces [14, §3]. Let us show \( \hat{h}_g \) satisfies the properties (iii) and (iv).

**Claim 4.1.1.** There exist constants \( c^\pm \) such that \( \hat{h}_g^\pm(x) \leq h_{nv}(x) + c^\pm \) for all \( x \in \mathbb{A}^n(K) \).

**Proof.** By Theorem 1.3, there exists a constant \( c_2 \) such that \( \frac{1}{\delta} h_{nv}(g^i(x)) \leq h_{nv}(x) + \frac{2}{\delta} \) for all \( x \in \mathbb{A}^n(K) \). We show

\[
\frac{1}{\delta} h_{nv}(g^i(x)) \leq h_{nv}(x) + \left( \sum_{i=1}^{l} \frac{1}{\delta^i} \right) c_2
\]

by the induction on \( l \). Indeed, since \( \frac{1}{\delta} h_{nv}(g^i(x)) \leq h_{nv}(g^i(x)) + \frac{c_2}{\delta^i} \), we have

\[
\frac{1}{\delta^i+1} h_{nv}(g^{i+1}(x)) \leq \frac{1}{\delta} h_{nv}(g^i(x)) + \frac{c_2}{\delta^{i+1}} \leq h_{nv}(x) + \left( \sum_{i=1}^{i+1} \frac{1}{\delta^i} \right) c_2.
\]

By putting \( c^+ = c_2 \sum_{i=1}^{+\infty} \frac{1}{\delta^i} = \frac{c_2}{\delta-1} \), we obtain \( \hat{h}_g^+(x) = \limsup_{t \to +\infty} \frac{1}{\delta} h_{nv}(g^t(x)) \leq h_{nv}(x) + c^+ \). The estimate for \( \hat{h}_g^- \) is shown similarly. (Note that it follows from \( \delta \geq 2 \) that \( \delta^- \geq 2 \).)
Claim 4.1.2. We have
\[ \hat{h}_g(x) \geq h_{nv}(x) - \frac{\delta \delta_-}{(\delta - 1)(\delta_- - 1)} c \]
for all \( x \in \mathbb{A}^n(\overline{K}) \), where \( c \) is the constant given in (4.1).

Proof. We set \( h' = h_{nv} - \frac{\delta \delta_-}{(\delta - 1)(\delta_- - 1)} c \). Then we have for all \( x \in \mathbb{A}^n(\overline{K}) \)
\begin{equation}
(4.2) \quad \frac{1}{\delta} h'(g(x)) + \frac{1}{\delta_-} h'(g^{-1}(x)) \geq \left(1 + \frac{1}{\delta \delta_-}\right) h'(x).
\end{equation}

Then we have \( \frac{1}{\delta} h'(g^2(x)) + \frac{1}{\delta_-} h'(x) \geq \left(1 + \frac{1}{\delta \delta_-}\right) \frac{1}{\delta} h'(g(x)) \) and \( \frac{1}{\delta} h'(x) + \frac{1}{\delta_-} h'(g^{-2}(x)) \geq \left(1 + \frac{1}{\delta \delta_-}\right) \frac{1}{\delta} h'(g^{-1}(x)) \). Adding these two inequalities and using (4.2) again, we obtain
\[ \frac{1}{\delta^2} h'(g^2(x)) + \frac{1}{\delta_-^2} h'(g^{-2}(x)) \geq \left(1 + \frac{1}{(\delta \delta_-)^2}\right) h'(x). \]

Inductively, we obtain
\[ \frac{1}{\delta^l} h'(g^l(x)) + \frac{1}{\delta_-^l} h'(g^{-l}(x)) \geq \left(1 + \frac{1}{(\delta \delta_-)^l}\right) h'(x). \]
(Though not necessary for the proof, one can also show \( \frac{1}{\delta^m} h'(g^m(x)) + \frac{1}{\delta_-^m} h'(g^{-m}(x)) \geq \left(1 + \frac{1}{(\delta \delta_-)^m}\right) h'(x) \) for every \( m \in \mathbb{Z}_{>0} \).) By letting \( l \to +\infty \), it follows that
\begin{equation}
(4.3) \quad \limsup_{l \to +\infty} \frac{1}{\delta^l} h'(g^l(x)) + \limsup_{l \to +\infty} \frac{1}{\delta_-^l} h'(g^{-l}(x))
\geq \limsup_{l \to +\infty} \left( \frac{1}{\delta^l} h'(g^l(x)) + \frac{1}{\delta_-^l} h'(g^{-l}(x)) \right) \geq h'(x).
\end{equation}

Since
\[ \hat{h}_g^+(x) = \limsup_{m \to +\infty} \frac{1}{\delta^m} h_{nv}(g^m(x)) \]
\[ = \limsup_{m \to +\infty} \frac{1}{\delta^m} \left( h'(g^m(x)) + \frac{\delta \delta_-}{(\delta - 1)(\delta_- - 1)} c \right) \geq \limsup_{l \to +\infty} \frac{1}{\delta^l} h'(g^l(x)) \]
and similarly \( \hat{h}_g^-(x) \geq \limsup_{l \to +\infty} \frac{1}{\delta_-^l} h'(g^{-l}(x)) \), the left-hand-side of (4.3) is less than or equal to \( \hat{h}_g(x) \), while the right-hand-side is \( h_{nv}(x) - \frac{\delta \delta_-}{(\delta - 1)(\delta_- - 1)} c \). Thus we get the desired inequality. \( \Box \)

The property (iii) follows from Claim 4.1.1 and Claim 4.1.2. Indeed we have
\begin{equation}
(4.4) \quad h_{nv}(x) - \frac{\delta \delta_-}{(\delta - 1)(\delta_- - 1)} c \leq \hat{h}_g(x) \leq 2h_{nv}(x) + c^+ + c^-.
\end{equation}
The property (iv) is checked by the following equations:
\[
\hat{h}_g^+(f(x)) = \delta \hat{h}_g^+(x), \quad \hat{h}_g^+(f^{-1}(x)) = \frac{1}{\delta} \hat{h}_g^+(x); \\
\hat{h}_g^-(f(x)) = \frac{1}{\delta_-} \hat{h}_g^-(x), \quad \hat{h}_g^-(f^{-1}(x)) = \delta_- \hat{h}_g^-(x).
\]
Thus \(\hat{h}_g : \mathbb{A}^n(K) \to \mathbb{R}\) satisfies the properties (iii) and (iv).

**Step 2.** We show the existence of a function \(\hat{h}_o : \mathbb{A}^n(K) \to \mathbb{R}\) with the properties (i) and (ii). We define \(\hat{h}_o\) by
\[
\hat{h}_o(x) := \hat{h}_g(\gamma^{-1}(x))
\]
for all \(x \in \mathbb{A}^n(K)\).

By (4.4), we have \(\hat{h}_g(\gamma^{-1}(x)) \leq 2h_{nv}(\gamma^{-1}(x)) + c^+ + c^-\). Theorem 1.3 yields that there is a constant \(c_{\gamma^{-1}}\) such that \(h_{nv}(\gamma^{-1}(x)) \leq (\deg \gamma^{-1}) h_{nv}(x) + c_{\gamma^{-1}}\) for all \(x \in \mathbb{A}^n(K)\). Thus
\[
\hat{h}_o(x) \leq 2(\deg \gamma^{-1}) h_{nv}(x) + (2c_{\gamma^{-1}} + c^+ + c^-).
\]
On the other hand, Theorem 1.3 yields that there is a constant \(c_\gamma\) such that \(h_{nv}(\gamma(x)) \leq (\deg \gamma) h_{nv}(x) + c_{\gamma}\) for all \(x \in \mathbb{A}^n(K)\). Hence
\[
h_{nv}(\gamma^{-1}(x)) \geq (\deg \gamma)^{-1} h_{nv}(x) - (\deg \gamma)^{-1} c_{\gamma}.
\]
Then by (4.4), we get
\[
\hat{h}_o(x) \geq (\deg \gamma)^{-1} h_{nv}(x) - (\deg \gamma)^{-1} c_{\gamma} - \frac{\delta \delta_-}{(\delta - 1)(\delta_- - 1)} c
\]
for all \(x \in \mathbb{A}^n(K)\). Now the property (i) follows from (4.5) and (4.6).

The property (iv) follows from
\[
\hat{h}_o(f(x)) + \hat{h}_o(f^{-1}(x)) = \hat{h}_g(\gamma^{-1}(f(x))) + \hat{h}_g(\gamma^{-1}(f^{-1}(x)))
\]
\[
= \hat{h}_g(g(\gamma^{-1}(x))) + \hat{h}_g(g^{-1}(\gamma^{-1}(x)))
\]
\[
= \left(1 + \frac{1}{\delta \delta_-}\right) \hat{h}_g(\gamma^{-1}(x)) = \left(1 + \frac{1}{\delta \delta_-}\right) \hat{h}_o(x),
\]
where we used (iv) in the third equality.

**Step 3.** We will show uniqueness property of \(\hat{h}\). In what follows, let \(\hat{h}\) denote a function with the properties (i) and (ii), not necessarily being equal to \(\hat{h}_o\).

Suppose \(\hat{h}'\) is another function with the properties (i) and (ii) such that \(\lambda := \hat{h}' - \hat{h}\) is bounded on \(\mathbb{A}^n(K)\). Set \(M := \sup_{x \in \mathbb{A}^n(K)} |\lambda(x)|\). Then
\[
\left(1 + \frac{1}{\delta \delta_-}\right) M = \left(1 + \frac{1}{\delta \delta_-}\right) \sup_{x \in \mathbb{A}^n(K)} |\lambda(x)|
\]
\[
= \sup_{x \in \mathbb{A}^n(K)} \left|\frac{1}{\delta} \lambda(f(x)) + \frac{1}{\delta_-} \lambda(f^{-1}(x))\right| \leq \left(1 + \frac{1}{\delta \delta_-}\right) M.
\]
Since \(1 + \frac{1}{\delta \delta_-} - \frac{1}{\delta} - \frac{1}{\delta_-} = \frac{(\delta - 1)(\delta_- - 1)}{\delta \delta_-} > 0\), we have \(M = 0\), hence \(\hat{h} = \hat{h}'\).
To show $\hat{h} \geq 0$, we assume the contrary, so that there exists $x_0 \in \mathbb{A}^n(\overline{K})$ with $\hat{h}(x_0) =: a < 0$. Then $\frac{1}{\delta} \hat{h}(f(x_0)) + \frac{1}{\delta} \hat{h}(f^{-1}(x_0)) = \left(1 + \frac{1}{\delta_0}\right) \hat{h}(x_0) = \left(1 + \frac{1}{\delta_0}\right) a$. Thus we have

$$\frac{1}{\delta} \hat{h}(f(x_0)) \leq \frac{1 + \delta \delta_+}{\delta + \delta_+} a \quad \text{or} \quad \frac{1}{\delta} \hat{h}(f^{-1}(x_0)) \leq \frac{1 + \delta \delta_+}{\delta + \delta_+} a.$$  

Since $\frac{1 + \delta \delta_+}{\delta + \delta_+} > 1$, this shows that $\hat{h}$ is not bounded from below. Since $h_{nv}$ is bounded from below and $h_{nv} \gg \hat{h}$, this is a contradiction.

Finally we will show that $x \in \mathbb{A}^n(\overline{K})$ is $f$-periodic if and only if $\hat{h}(x) = 0$.

Suppose $\hat{h}(x_1) = 0$. Then by (ii) and the non-negativity of $\hat{h}$, we have $\hat{h}(f(x_1)) = 0$ and $\hat{h}(f^{-1}(x_1)) = 0$. Take an extension field $L$ of $K$ such that $x_1$ is defined over $L$. Since $\hat{h} \gg h_{nv}$, $\hat{h}$ satisfies the Northcott finiteness property. Thus the set

$$\{f^l(x_1) \mid l \in \mathbb{Z}\} \quad \left(\subseteq \{x \in \mathbb{A}^n(L) \mid \hat{h}(x) = 0\}\right)$$

is finite. Hence $x_1$ is $f$-periodic.

On the other hand, suppose $\hat{h}(x_2) =: b > 0$. Then it follows from (ii) that

$$\frac{1}{\delta} \hat{h}(f(x_2)) \geq \frac{1 + \delta \delta_+}{\delta + \delta_+} b \quad \text{or} \quad \frac{1}{\delta} \hat{h}(f^{-1}(x_2)) \geq \frac{1 + \delta \delta_+}{\delta + \delta_+} b.$$  

This shows that the set $\{f^l(x_2) \mid l \in \mathbb{Z}\}$ is not a set of bounded height. Thus $x_2$ cannot be $f$-periodic. \hfill $\square$

In the remainder of this section, we would like to discuss the condition (4.1) in Theorem 4.1. The next proposition shows that the constant $(1 + \frac{1}{\delta \delta_-})$ in (4.1) is the largest number one can hope for.

**Proposition 4.2.** Let $g : \mathbb{A}^n \rightarrow \mathbb{A}^n$ a polynomial automorphism of degree $\delta \geq 2$ over a number field $K$. Let $\delta_-$ denote the degree of $g^{-1}$. Let $a \in \mathbb{R}$. Suppose there exists a constant $c$ such that

$$\frac{1}{\delta} h_{nv}(g(x)) + \frac{1}{\delta_-} h_{nv}(g^{-1}(x)) \geq ah_{nv}(x) - c$$  

for all $x \in \mathbb{A}^n(\overline{K})$. Then $a \leq 1 + \frac{1}{\delta \delta_-}$.

**Proof.** To lead a contradiction, we assume that $a > 1 + \frac{1}{\delta \delta_-}$. Noting $a > 1 + \frac{1}{\delta \delta_-} \geq \frac{1}{\delta} + \frac{1}{\delta_-}$, we set $c' := \left(a - \frac{1}{\delta} - \frac{1}{\delta_-}\right)^{-1} c$ and $h' := h_{nv} - c'$. Then $h'$ satisfies

$$\frac{1}{\delta} h'(g(x)) + \frac{1}{\delta_-} h'(g^{-1}(x)) \geq ah'(x)$$

for all $x \in \mathbb{A}^n(\overline{K})$. As in the proof of Claim 4.1.2, we get

$$\frac{1}{\delta^2} h'(g^2(x)) + \frac{1}{\delta_-^2} h'(g^{-2}(x)) \geq \left(a^2 - \frac{2}{\delta \delta_-}\right) h'(x).$$

We set $a_1 = a^2 - \frac{2}{\delta \delta_-}$. Since $a_1 - 1 - \frac{1}{(\delta \delta_-)^2} = a^2 - \frac{2}{\delta \delta_-} - 1 - \frac{1}{(\delta \delta_-)^2} > (1 + \frac{1}{\delta \delta_-})^2 - \frac{2}{\delta \delta_-} - 1 - \frac{1}{(\delta \delta_-)^2} = 0$, we have $a_1 > 1 + \frac{1}{(\delta \delta_-)^2}$. Thus, if we define a sequence $\{a_l\}_{l=0}^{\infty}$ by $a_0 = a$ and $a_{l+1} = a_l^2 - \frac{2}{(\delta \delta_-)^2}$,
then we get inductively
\[ \frac{1}{\delta^2} h'(g^2(x)) + \frac{1}{\delta^2} h'(g^{-2}(x)) \geq a_l h'(x). \]

On the other hand, it follows from Theorem 1.3 and the argument in Claim 4.1.1 that there is a constant \( c'' \) independent of \( l \in \mathbb{Z} \) such that for all \( x \in \mathbb{A}^2(K) \),
\[ 2h'(x) + c'' \geq \frac{1}{\delta^2} h'(g^2(x)) + \frac{1}{\delta^2} h'(g^{-2}(x)). \]

Thus \( 2h' + c'' \geq a_l h' \). Since \( h' = h_{nv} - c' \) and \( \lim_{l \to +\infty} a_l = +\infty \) follows from Lemma 4.3(1), this is a contradiction. \( \square \)

**Lemma 4.3.** Let \( D \geq 4 \). Let \( \{a_l\}_{l=0}^{+\infty} \) be a sequence defined by \( a_0 = a \) and \( a_{l+1} = a_l^2 - 2D^{-2} \).

1. If \( a > 1 + \frac{1}{D} \), then \( \lim_{l \to +\infty} a_l = +\infty \).
2. If \( a = 1 + \frac{1}{D} \), then \( \lim_{l \to +\infty} a_l = 1 \).
3. If \( 1 \leq a < 1 + \frac{1}{D} \), then \( \lim_{l \to +\infty} a_l = 0 \).

**Proof.** We show (1). Set \( \varepsilon_l = a_l - 1 - D^{-2} \). In particular \( \varepsilon_0 = a - 1 - D^{-1} > 0 \). Since \( \varepsilon_{l+1} = a_{l+1} - 1 - D^{-2} \varepsilon_{l+1} = 2\varepsilon_l(1 + D^{-2}) + \delta^2 \), we get \( \varepsilon_{l+1} > 2\varepsilon_l > \cdots > 2^{l+1}\varepsilon_0 \). Hence \( \lim_{l \to +\infty} \varepsilon_l = +\infty \) and thus \( \lim_{l \to +\infty} a_l = +\infty \).

We show (2). In this case, we have \( a_l = 1 + D^{-2} \). Thus \( \lim_{l \to +\infty} a_l = 1 \).

Finally we show (3). On one hand, we get by induction \( a_l \geq 2D^{-2} - 1 \) for \( l \geq 1 \), and in particular \( a_l \geq 0 \) for \( l \geq 1 \). On the other hand, we claim for sufficiently large \( l \) that \( a_l < 1 \). Indeed, we assume the contrary and suppose \( a_l \geq 1 \) for all \( l \). By induction, we get \( a_l < 1 + D^{-2} \). We set \( \lambda_l = 1 + D^{-2} - a_l \), and so \( 0 < \lambda_l \leq D^{-2} \). Then \( a_{l+1} = a_{l+1}^2 - 2D^{-2} \lambda_l = (1 + D^{-2} - \lambda_l) \lambda_l - 2D^{-2} = 1 + D^{-2} \lambda_l - 2D^{-2} \lambda_l (1 + D^{-2}) + \lambda_l^2 \). Hence we get \( \lambda_{l+1} = 2\lambda_l (1 + D^{-2}) - \lambda_l^2 \geq 2\lambda_l \), which says that \( \lim_{l \to +\infty} \lambda_l = +\infty \). This is a contradiction. Hence there is an \( l_0 \) with \( a_{l_0} < 1 \). Since \( 0 \leq \frac{a_{l_0}}{a_l} \leq a_{l_0}^{2^{l_0}} \), we get \( \lim_{l \to +\infty} a_l = 0 \). \( \square \)

Let \( a_{\sup} \) denote the supremum of \( a \in \mathbb{R} \) that satisfies the inequality in Proposition 4.2. It follows from Theorem 2.3 that, if \( g \) is a regular polynomial automorphism of \( \mathbb{A}^2 \) of degree \( \delta \geq 2 \), then \( \delta = \delta_- \) and \( a_{\sup} = 1 + \frac{1}{\delta_-} \). We remark that Marcello \([11, \text{Théorème 3.1}] \) showed that, if \( g \) is a regular polynomial automorphism of \( \mathbb{A}^n \) (this means the set of indeterminacy \( I_{\mathbb{T}} \) and \( I_{g^{-1}} \) are disjoint, cf. \([12, \text{Définition 2.2.1}] \)), then \( a_{\sup} \geq 1 \). It would be interesting to know what polynomial automorphisms \( g \) on \( \mathbb{A}^n \) satisfy (4.1).

5. The number of points with bounded height in an \( f \)-orbit

In this section, we will prove Theorem C. As in \S 4 we will show Theorem C in a more general setting. The arguments below are inspired by those of Silverman on certain K3 surfaces \([14, \text{§3}] \).

Throughout this section, let \( f : \mathbb{A}^n \to \mathbb{A}^n \) be a polynomial automorphism of over a number field \( K \) satisfying the conditions in Theorem 4.1. Let \( \tilde{h} \) be a height function constructed in Theorem 4.1.
We define functions \( \hat{h}^\pm : \mathbb{A}^n(\overline{K}) \to \mathbb{R} \) to be
\[
\hat{h}^+(x) = \frac{\delta_\delta - (\delta_\delta)^2}{(\delta_\delta)^2 - 1} \left( \delta_\delta \hat{h}(f(x)) - \frac{1}{\delta_\delta} \hat{h}(f^{-1}(x)) \right),
\]
\[
\hat{h}^-(x) = \frac{\delta_\delta - (\delta_\delta)^2}{(\delta_\delta)^2 - 1} \left( \delta_\delta \hat{h}(f^{-1}(x)) - \frac{1}{\delta_\delta} \hat{h}(f(x)) \right)
\]for \( x \in \mathbb{A}^n(\overline{K}) \). We remark that, in the notations of the proof of Theorem 4.1, if \( \hat{h} = \hat{h}_g \), then \( \hat{h}^+ = \hat{h}_g^+ \) and \( \hat{h}^- = \hat{h}_g^- \).

**Lemma 5.1.**

1. \( \hat{h} = \hat{h}^+ + \hat{h}^- \).
2. \( \hat{h}^+ \circ f = \delta \hat{h}^+ \), and \( \hat{h}^- \circ f^{-1} = \delta_- \hat{h}^- \).
3. \( \hat{h}^+ \geq 0 \) and \( \hat{h}^- \geq 0 \).
4. For \( x \in \mathbb{A}^n(\overline{K}) \), \( \hat{h}^+(x) = 0 \) if and only if \( \hat{h}^-(x) = 0 \) if and only if \( \hat{h}(x) = 0 \) if and only if \( x \) is \( f \)-periodic.

**Proof.** By the property (ii) in Theorem 4.1, we readily see (1). Let us see (2). By the property (ii), we have \( \delta_- \hat{h}(f^2(x)) + \hat{h}(x) = (1 + \delta \delta_-)\hat{h}(f(x)) \) and \( \left( \frac{1}{\delta_-} + \delta \right) \hat{h}(x) = \hat{h}(f(x)) + \frac{\delta_\delta}{\delta_-} \hat{h}(f^{-1}(x)) \). Taking the difference, we have
\[
\delta_- \hat{h}(f^2(x)) - \frac{1}{\delta_-} \hat{h}(x) = \delta \left( \delta_- \hat{h}(f(x)) - \frac{1}{\delta_-} \hat{h}(f^{-1}(x)) \right).
\]
This shows \( \hat{h}^+(f(x)) = \delta \hat{h}^+(x) \). Similarly we have \( \hat{h}^+(f^{-1}(x)) = \delta_- \hat{h}^-(x) \). Next let us see (3). Since \( \hat{h} \geq 0 \) by Theorem 4.1, we have \( \hat{h}^+(f^l(x)) + \hat{h}^-(f^l(x)) = \hat{h}(f^l(x)) \geq 0 \) for any \( l \in \mathbb{Z} \) and \( x \in \mathbb{A}^n(\overline{K}) \). This is equivalent to
\[
\hat{h}^+(x) \geq -\frac{1}{(\delta_\delta)^2} \hat{h}^-(x).
\]
By letting \( l \to +\infty \), we have \( \hat{h}^+(x) \geq 0 \). Similarly we have \( \hat{h}^-(x) \geq 0 \).

Next we will show (4). The assertion that “\( \hat{h}(x) = 0 \) if and only if \( x \) is \( f \)-periodic” is shown in Theorem 4.1. Since \( \hat{h}^+ \geq 0 \) and \( \hat{h}^- \geq 0 \), \( \hat{h}^+(x) = \hat{h}^+(x) + \hat{h}^-(x) \) implies \( \hat{h}^+(x) = 0 \) and \( \hat{h}^-(x) = 0 \). We will see that \( \hat{h}^+(x) = 0 \) implies \( \hat{h}(x) = 0 \). A key observation here is that \( \hat{h} \) satisfies Northcott’s finiteness property, which is a consequence of the property (i) of \( \hat{h} \) in Theorem 4.1. Suppose \( \hat{h}^+(x) = 0 \). Then
\[
\hat{h}(f^l(x)) = \hat{h}^+(f^l(x)) + \hat{h}^-(f^l(x)) = \delta \hat{h}^+(x) + \frac{1}{\delta_-} \hat{h}^-(x) = \frac{1}{\delta_-} \hat{h}^-(x).
\]
Let \( L \) be a finite extension of \( K \) over which \( x \) is defined. Then
\[
\{ f^l(x) \in \mathbb{A}^n(\overline{K}) \mid l \geq 0 \} \subseteq \{ y \in \mathbb{A}^n(\overline{K}) \mid \hat{h}(y) \leq \hat{h}^-(x) \}
\]
is finite. Hence \( x \) is \( f \)-periodic. Similarly we see that \( \hat{h}^-(x) = 0 \) implies \( \hat{h}(x) = 0 \). \( \square \)

For \( x \in \mathbb{A}^n(\overline{K}) \), we define the \( f \)-orbit of \( x \) to be
\[
O_f(x) := \{ f^l(x) \mid l \in \mathbb{Z} \}.
\]
Note that $O_f(x)$ is a finite set if and only if $x$ is $f$-periodic.

For an $f$-orbit $O_f(x)$, we define the canonical height of $O_f(x)$ to be

$$
\hat{h}(O_f(x)) = \frac{\log \hat{h}^+(y)}{\log \delta} + \frac{\log \hat{h}^-(y)}{\log \delta_-} \in \mathbb{R} \cup \{-\infty\}
$$

for any $y \in O_f(x)$.

**Lemma 5.2.** (1) $\hat{h}(O_f(x))$ is well-defined, i.e., $\hat{h}(O_f(x))$ is independent of the choice of $y \in O_f(x)$. Moreover, $\hat{h}(O_f(x)) = -\infty$ if and only if $O_f(x)$ is a finite set.

(2) Assume $\#O_f(x) = +\infty$. Then we have

$$
\hat{h}(O_f(x)) + \epsilon_1 \leq \left( \frac{1}{\log \delta} + \frac{1}{\log \delta_-} \right) \min_{y \in O_f(x)} \log \hat{h}(y) \leq \hat{h}(O_f(x)) + \epsilon_2,
$$

where the constants $\epsilon_1$ and $\epsilon_2$ are given by

$$
\epsilon_1 = \frac{1}{\log \delta} \log \left( 1 + \frac{\log \delta}{\log \delta_-} \right) + \frac{1}{\log \delta_-} \log \left( 1 + \frac{\log \delta_-}{\log \delta} \right),
$$

$$
\epsilon_2 = \epsilon_1 + \left( \frac{1}{\log \delta} + \frac{1}{\log \delta_-} \right) \log \max\{\delta, \delta_-\}.
$$

**Proof.** (1) follows from Lemma 5.1. To prove (2), set

$$
p = 1 + \frac{\log \delta}{\log \delta_-} \quad \text{and} \quad q = 1 + \frac{\log \delta_-}{\log \delta}.
$$

Then $p > 1$, $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then we have

$$
\hat{h}(y) = \hat{h}^+(y) + \hat{h}^-(y) = \frac{1}{p} \left( p^{\frac{1}{p}} \hat{h}^+(y) \right)^p + \frac{1}{q} \left( q^{\frac{1}{q}} \hat{h}^-(y) \right)^q \geq p^{\frac{1}{p}} q^{\frac{1}{q}} \hat{h}^+(y) \hat{h}^-(y).
$$

Hence, $\frac{1}{p} \log p + \frac{1}{q} \log q + \frac{1}{p} \log \hat{h}^+(y) + \frac{1}{q} \log \hat{h}^-(y) \leq \log \hat{h}(y)$. Since

$$
\frac{1}{p} \log \hat{h}^+(y) + \frac{1}{q} \log \hat{h}^-(y) = \left( \frac{1}{\log \delta} + \frac{1}{\log \delta_-} \right)^{-1} \hat{h}(O_f(x)),
$$

we obtain

$$
\hat{h}(O_f(x)) + \epsilon_1 \leq \left( \frac{1}{\log \delta} + \frac{1}{\log \delta_-} \right) \min_{y \in O_f(x)} \log \hat{h}(y).
$$

On the other hand, we have $\hat{h}(f^l(x)) = \delta^l \hat{h}^+(x) + \delta_-^l \hat{h}^-(x)$ for $l \in \mathbb{Z}$. We set $g(t) = \delta^l \hat{h}^+(x) + \delta_-^l \hat{h}^-(x)$ for $t \in \mathbb{R}$, and

$$
t_0 := \frac{\log \hat{h}(x) \log \delta_- - \log \hat{h}^+(x) \log \delta}{\log \delta \log \delta_-}.
$$

Then one sees that $g$ takes its minimum at $t_0$, with $g(t_0) = p^{t_0} q^{t_0} \hat{h}^+(x) \hat{h}^-(x)$. Consequently as a function of $l \in \mathbb{Z}$, $\hat{h}(f^l(x))$ takes its minimum at $l = [t_0]$ or $l = [t_0] + 1$, where $[t_0]$ denotes the largest integer less than or equal to $t_0$. Then we get

$$
\hat{h}(f^{[t_0]}(x)) = \delta^{[t_0]} \hat{h}^+(x) + \delta_-^{[t_0]} \hat{h}^-(x) = \delta^{-[t_0]-[t_0]} \delta^{[t_0]} \hat{h}^+(x) + \delta_-^{[t_0]} \delta^{[t_0]} \hat{h}^-(x) < \max\{\delta, \delta_-\} \left( \delta^{[t_0]} \hat{h}^+(x) + \delta_-^{[t_0]} \hat{h}^-(x) \right) = \max\{\delta, \delta_-\} p^{[t_0]} q^{[t_0]} \hat{h}^+(x) \hat{h}^-(x).$$
Similarly we get
\[\hat{h}(f^{[t_0]}(x)) = \delta^{1+[t_0]-t_0}\hat{h}^+(x) + \delta^{-(1+[t_0]-t_0)}\hat{h}^-(x)\]
\[< \max\{\delta, \delta_0\}\min_{y \in O_f(x)} \log \hat{h}(y) \leq \hat{h}(O_f(x)) + e_2.\]

This shows \(\left(\frac{1}{\log \delta} + \frac{1}{\log \delta_-}\right) \min_{y \in O_f(x)} \log \hat{h}(y) \leq \hat{h}(O_f(x)) + e_2.\) □

**Theorem 5.3.** Let \(f : \mathbb{A}^n \to \mathbb{A}^n\) be a polynomial automorphism over a number field \(K\) satisfying the conditions in Theorem 4.1, and \(\hat{h}\) a height function constructed in Theorem 4.1. Let \(x\) be an element of \(\mathbb{A}^n(K)\) such that \(\#O_f(x) = +\infty\). Then we have the following.

1. If \(\left(\frac{1}{\log \delta} + \frac{1}{\log \delta_-}\right) \log T \geq \hat{h}(O_f(x))\), then

   \[\#\{y \in O_f(x) \mid \hat{h}(y) \leq T\} - \left(\frac{1}{\log \delta} + \frac{1}{\log \delta_-}\right) \log T + \hat{h}(O_f(x)) \leq \frac{\log 2}{\log \delta} + \frac{\log 2}{\log \delta_-} + 1.\]

   Note that if \(\left(\frac{1}{\log \delta} + \frac{1}{\log \delta_-}\right) \log T \leq \hat{h}(O_f(x))\), it follows from Lemma 5.2(2) that

   \[\#\{y \in O_f(x) \mid \hat{h}(y) \leq T\} = 0.\]

2. \(\#\{y \in O_f(x) \mid h_{nv}(y) \leq T\} = \left(\frac{1}{\log \delta} + \frac{1}{\log \delta_-}\right) \log T - \hat{h}(O_f(x)) + O(1)\) as \(T \to +\infty\),

   where the \(O(1)\) constant depends only on \(f\) and the choice of \(\hat{h}\).

**Proof.** Since \(\#O_f(x) = +\infty\), the map \(Z \ni l \mapsto f^l(x) \in \mathbb{A}^n(K)\) is one-to-one. Then

\[\#\{y \in O_f(x) \mid \hat{h}(y) \leq T\} = \#\{l \in Z \mid \hat{h}(f^l(x)) \leq T\}\]
\[= \#\{l \in Z \mid \delta^+\hat{h}^+(x) + \delta^-\hat{h}^-(x) \leq T\}.\]

Then it follows from Lemma 5.4 that

\[-1 + \frac{\log \frac{T}{\hat{h}^+(x)}}{\log \delta} + \frac{\log \frac{T}{\hat{h}^-(x)}}{\log \delta_-} \leq \#\{y \in O_f(x) \mid \hat{h}(y) \leq T\} \leq 1 + \frac{\log \frac{T}{\hat{h}^+(x)}}{\log \delta} + \frac{\log \frac{T}{\hat{h}^-(x)}}{\log \delta_-},\]

for \(T \geq \hat{h}^+(x)\) or equivalently \(\left(\frac{1}{\log \delta} + \frac{1}{\log \delta_-}\right) \log T \geq \hat{h}(O_f(x))\).

On the other hand, we have

\[-1 + \frac{\log \frac{T}{\hat{h}^+(x)}}{\log \delta} + \frac{\log \frac{T}{\hat{h}^-(x)}}{\log \delta_-} = -1 - \frac{\log 2}{\log \delta} - \frac{\log 2}{\log \delta_-} + \left(\frac{1}{\log \delta} + \frac{1}{\log \delta_-}\right) \log T - \hat{h}(O_f(x))\]
\[1 + \frac{\log \frac{T}{\hat{h}^+(x)}}{\log \delta} + \frac{\log \frac{T}{\hat{h}^-(x)}}{\log \delta_-} = 1 + \left(\frac{1}{\log \delta} + \frac{1}{\log \delta_-}\right) \log T - \hat{h}(O_f(x)).\]

Thus we obtain (1). Next, we will show (2). Since \(h_{nv} \gg \gg \hat{h}\) by the property (i) of Theorem A, there exist a positive constant \(a_2\) and a constant \(b_2\) such that \(\hat{h} \leq a_2h_{nv} + b_2\).
Then we have
\[
\#\{y \in O_f(x) \mid h_{nv}(y) \leq T\} \\
\leq \#\{y \in O_f(x) \mid \hat{h}(y) \leq a_2T + b_2\} \\
\leq \left(\frac{1}{\log \delta} + \frac{1}{\log \delta_-}\right) \log(a_2T + b_2) - \hat{h}(O_f(x)) + 1 + \frac{\log 2}{\log \delta} + \frac{\log 2}{\log \delta_-} \\
\leq \left(\frac{1}{\log \delta} + \frac{1}{\log \delta_-}\right) \log T - \hat{h}(O_f(x)) + O(1) \quad \text{as } T \to +\infty.
\]

Using \(a_1h_{nv} + b_1 \leq \hat{h}\) for some positive constant \(a_1\) and constant \(b_1\), we have \(\#\{y \in O_f(x) \mid h_{nv}(y) \leq T\} \geq \left(\frac{1}{\log \delta} + \frac{1}{\log \delta_-}\right) \log T - \hat{h}(O_f(x)) + O(1)\) as \(T \to +\infty\). \(\square\)

**Lemma 5.4.** Let \(A, B, T > 0\) be positive numbers. If \(T \geq A^{\frac{\log \delta_-}{\log \delta}} B^{\frac{\log \delta}{\log \delta_-}}\), then we have
\[
-1 + \frac{\log T}{\log \delta} + \frac{\log T}{\log \delta_-} \leq \#\{l \in \mathbb{Z} \mid \delta^lA + \delta^{-l}B \leq T\} \leq 1 + \frac{\log T}{\log \delta} + \frac{\log T}{\log \delta_-}.
\]

**Proof.** If \(l \in \mathbb{Z}\) satisfies \(\delta^lA + \delta^{-l}B \leq T\), then \(\delta^lA \leq T\) and \(\delta^{-l}B \leq T\). Note that \(\frac{\log \delta}{\log \delta_-} \leq \frac{\log T}{\log \delta}\) is equivalent to \(T \geq A^{\frac{\log \delta_-}{\log \delta}} B^{\frac{\log \delta}{\log \delta_-}}\). Then, for \(T \geq A^{\frac{\log \delta_-}{\log \delta}} B^{\frac{\log \delta}{\log \delta_-}}\), we have
\[
\#\{l \in \mathbb{Z} \mid \delta^lA + \delta^{-l}B \leq T\} \leq \#\left\{l \in \mathbb{Z} \mid \frac{\log B}{\log \delta_-} \leq l \leq \frac{\log T}{\log \delta}\right\} \leq 1 + \frac{\log T}{\log \delta} + \frac{\log T}{\log \delta_-}.
\]

On the other hand, if \(l \in \mathbb{Z}\) satisfies \(\delta^lA \leq \frac{T}{2}\) and \(\delta^{-l}B \leq \frac{T}{2}\), then \(\delta^lA + \delta^{-l}B \leq T\). Thus,
\[
\#\{l \in \mathbb{Z} \mid \delta^lA + \delta^{-l}B \leq T\} \geq \#\left\{l \in \mathbb{Z} \mid \frac{\log \delta}{\log \delta_-} \leq l \leq \frac{\log T}{\log \delta}\right\} \geq -1 + \frac{\log T}{\log \delta} + \frac{\log T}{\log \delta_-}.
\]
\(\square\)

**Proof of Theorem C.** As we saw in the proof of Theorem A and Corollary B, polynomial automorphisms on \(\mathbb{A}^2\) of dynamical degree \(\geq 2\) satisfy the conditions in Theorem 4.1. Then Theorem C follows from Theorem 5.3. \(\square\)

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