AN OPERATOR EQUALITY INVOLVING A CONTINUOUS FIELD OF OPERATORS AND ITS NORM INEQUALITIES

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Abstract. Let $A$ be a $C^*$-algebra, $T$ be a locally compact Hausdorff space equipped with a probability measure $P$ and let $(A_t)_{t \in T}$ be a continuous field of operators in $A$ such that the function $t \mapsto A_t$ is norm continuous on $T$ and the function $t \mapsto \|A_t\|$ is integrable. Then the following equality including Bouchner integrals holds

$$\int_T \left| A_t - \int_T A_s dP \right|^2 dP = \int_T |A_t|^2 dP - \left| \int_T A_t dP \right|^2.$$  

(0.1)

This equality is related both to the notion of variance in statistics and to a characterization of inner product spaces. With this operator equality, we present some uniform norm and Schatten $p$-norm inequalities.

1. Introduction and preliminaries

Many interesting characterizations of inner product spaces have been introduced (see, e.g., [1]). It is shown by Th. M. Rassias [8] that a normed space $X$ (with norm $\| \cdot \|$) is an inner product space if and only if for any finite set of vectors $x_1, \cdots, x_n \in X$,

$$\sum_{i=1}^n \left\| x_i - \frac{1}{n} \sum_{j=1}^n x_j \right\|^2 = \sum_{i=1}^n \| x_i \|^2 - n \left\| \frac{1}{n} \sum_{j=1}^n x_j \right\|^2.$$  

(1.1)

This equality is of fundamental importance in the study of normed spaces and inner product spaces since it reveals a basic relation between the two sorts of spaces. From statistical point of view, let $(X, \mu)$ be a probability measure space and $f$ be a random variable, i.e., $f$ is an element of $L^2(X, \mu)$. With the variance of $f$ defined by

$$\text{Var}(f) = E(|f - E(f)|^2),$$

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where $E(f) = \int_X f d\mu$ denotes the expectation of $f$, (1.1) resembles (or vice versa) the well-known equality

$$\text{Var}(f) = E(|f|^2) - |E(f)|^2.$$ 

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$ endowed with inner product $\langle \cdot, \cdot \rangle$ and the operator norm $\| \cdot \|_{\infty}$. We denote the absolute value of $A \in \mathcal{B}(\mathcal{H})$ by $|A| = (A^*A)^{1/2}$. For $x, y \in \mathcal{H}$, the rank one operator $x \otimes y$ is defined on $\mathcal{H}$ by $(x \otimes y)(z) = \langle z, y \rangle x$.

Let $A$ be a $C^*$-algebra and let $T$ be a locally compact Hausdorff space. A field $(A_t)_{t \in T}$ of operators in $A$ is called a continuous field of operators if the function $t \mapsto A_t$ is norm continuous on $T$. If $\mu(t)$ is a Radon measure on $T$ and the function $t \mapsto \|A_t\|$ is integrable, one can form the Bochner integral $\int_T A_t d\mu(t)$, which is the unique element in $\mathfrak{A}$ such that

$$\varphi \left( \int_T A_t d\mu(t) \right) = \int_T \varphi(A_t) d\mu(t)$$

for every linear functional $\varphi$ in the norm dual $\mathfrak{A}^*$ of $\mathfrak{A}$; cf. [6, Section 4.1].

Let $A \in \mathcal{B}(\mathcal{H})$ be a compact operator and let $0 < p < \infty$. The Schatten $p$-norm ($p$-quasi-norm) for $1 \leq p < \infty$ ($0 < p < 1$) is defined by $\|A\|_p = (\text{tr}|A|^p)^{1/p}$, where tr is the usual trace functional. Clearly, for $p, q > 0$,

$$\|A\|_p^q = \|A\|^{q/p}_2.$$ 

For $p > 0$, the Schatten $p$-class, denoted by $C_p$, is defined to be the two-sided ideal in $\mathcal{B}(\mathcal{H})$ of those compact operators $A$ for which $\|A\|_p$ is finite. In particular, $C_1$ and $C_2$ are the trace class and the Hilbert-Schmidt class, respectively. For more information on the theory of the Schatten $p$-classes, the reader is referred to [2, 9].

Since $C_2$ is a Hilbert space under the inner product $\langle A, B \rangle = \text{tr}(B^*A)$, it follows from (1.1) that if $A_1, \cdots, A_n \in C_2$, then

$$\sum_{i=1}^n \left\| A_i - \frac{1}{n} \sum_{j=1}^n A_j \right\|_2^2 = \sum_{i=1}^n \left\| A_i \right\|_2^2 - n \left\| \frac{1}{n} \sum_{j=1}^n A_j \right\|_2^2. \quad (1.2)$$

In this paper, we establish a general operator version of equality (1.1), from which we deduce an extension of (1.1). We present some inequalities concerning various norms such as Schatten $p$-norms that form natural generalizations (in inequalities) of the identity (1.2) (see, e.g., [7]). It seems that the inequalities related to Schatten $p$-norms are useful in operator theory and mathematical physics and are interesting in their own right.
2. Main results

We begin by establishing an operator version of equality (1.1) involving continuous fields of operators and integral means of operators.

**Theorem 2.1.** Let \( \mathfrak{A} \) be a \( C^* \)-algebra, \( T \) be a locally compact Hausdorff space equipped with a probability measure \( P \) and let \((A_t)_{t \in T}\) be a continuous field of operators in \( \mathfrak{A} \) such that the function \( t \mapsto A_t \) is norm continuous on \( T \) and the function \( t \mapsto \|A_t\| \) is integrable. Then

\[
\int_T \left| A_t - \int_T A_s dP(s) \right|^2 dP(t) = \int_T |A_t|^2 dP(t) - \left\| \int_T A_t dP(t) \right\|^2 \tag{2.1}
\]

**Proof.**

\[
\begin{align*}
\int_T |A_t - \int_T A_s dP(s)|^2 dP(t) &= \int_T \left( A_t - \int_T A_s dP(s) \right)^* \left( A_t - \int_T A_s dP(r) \right) dP(t) \\
&= \int_T |A_t|^2 dP(t) - \int_T \left( A_t^* \int_T A_s dP(r) \right) dP(t) \\
&\quad - \int_T \left( \left( \int_T A_s dP(s) \right)^* A_t \right) dP(t) \\
&\quad + \left( \int_T A_s dP(s) \right)^* \int_T A_t dP(r) \int_T dP(t) \\
&= \int_T |A_t|^2 dP(t) - \left( \int_T A_s dP(s) \right)^* \int_T A_t dP(r) \\
&\quad \left( \text{by } \int_T dP(t) = 1 \text{ and } \int_T A_t^* dP(t) = \left( \int_T A_t dP(t) \right)^* \right) \\
&= \int_T |A_t|^2 dP(t) - \left\| \int_T A_t dP(t) \right\|^2.
\end{align*}
\]

\( \square \)

**Corollary 2.2.** Let \( A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H}) \). Then

(i) for any set of nonnegative numbers \( t_1, \ldots, t_n \) with \( \sum_{i=1}^n t_i = 1 \),

\[
\sum_{i=1}^n t_i \left| A_i - \sum_{j=1}^n t_j A_j \right|^2 = \sum_{i=1}^n t_i |A_i|^2 - \left| \sum_{j=1}^n t_j A_j \right|^2; \tag{2.2}
\]
Proof. (i) Take $T = \{1, \cdots, n\}$ and $P(\{i\}) = t_i$ in Theorem 2.1.

(ii) Put $t_i = 1/n$ in (i). \hfill \qed

As an immediate consequence of (2.2), we get the following known AM-QM operator inequality; cf. [10, Theorem 7].

**Corollary 2.3.** Let $A_1, \cdots, A_n \in \mathbb{B}(H)$. Then for any set of nonnegative numbers $t_1, \cdots, t_n$ with $\sum_{i=1}^{n} t_i = 1$,

$$\left| \sum_{i=1}^{n} t_i A_i \right|^2 \leq \sum_{i=1}^{n} t_i |A_i|^2.$$  \hfill (2.3)

The following result compares the mean of the squares of the operators to the square of the mean, and gives some bounds of their difference.

**Corollary 2.4.** Let $A_1, \cdots, A_n \in \mathbb{B}(H)$ be positive operators such that $0 \leq m_i I \leq A_i \leq M_i I$ for some nonnegative scalars $m_i$ and $M_i$ and all $1 \leq i \leq n$, and let $t_1, \cdots, t_n$ be nonnegative numbers such that $\sum_{i=1}^{n} t_i = 1$. Then

$$\sum_{i=1}^{n} t_i \alpha_i^2 I \leq \sum_{i=1}^{n} t_i A_i^2 - \left( \sum_{i=1}^{n} t_i A_i \right)^2 \leq \sum_{i=1}^{n} t_i \beta_i^2 I,$$

where

$$\alpha_i = \max \left\{ \left| M_i - \sum_{i=1}^{n} t_i m_i \right|, \left| m_i - \sum_{i=1}^{n} t_i M_i \right| \right\}$$

and

$$\beta_i = \min \left\{ \left| M_i - \sum_{i=1}^{n} t_i m_i \right|, \left| m_i - \sum_{i=1}^{n} t_i M_i \right| \right\}.$$

Proof. It is sufficient to notice that for each $A_i$, by the functional calculus,

$$\min \left\{ \left| M_i - \sum_{i=1}^{n} t_i m_i \right|^2, \left| m_i - \sum_{i=1}^{n} t_i M_i \right|^2 \right\} \leq \left| A_i - \sum_{j=1}^{n} t_j A_j \right|^2 \leq \max \left\{ \left| M_i - \sum_{i=1}^{n} t_i m_i \right|^2, \left| m_i - \sum_{i=1}^{n} t_i M_i \right|^2 \right\}$$

and use (2.2). \hfill \qed
By a $Q$-norm on a subspace $D$ of $\mathcal{B}(\mathcal{H})$ we mean a norm $\| \cdot \|_Q$ for which there exists a norm $\| \cdot \|_{\tilde{Q}}$ defined on a subspace $\tilde{D}$ such that $\| A \|_{\tilde{Q}}^2 = \| A^* A \|_{\tilde{Q}}$ ($A \in D$) (we implicitly assume that $A^* A \in \tilde{D}$ if and only if $A \in D$). The operator norm $\| \cdot \|_{\infty}$ and $\| \cdot \|_p$ (for $2 \leq p < \infty$) are examples of $Q$-norms on $\mathcal{B}(\mathcal{H})$; see, e.g., [2, p. 89] or [4].

**Corollary 2.5.** Let $\| \cdot \|_Q$ be a $Q$-norm on a subspace $D$ of $\mathcal{B}(\mathcal{H})$, and let $t_1, \ldots, t_n$ be nonnegative numbers such that $\sum_{i=1}^n t_i = 1$. Then for any $A_1, \ldots, A_n \in D$ with $A_i^* A_j = 0$ for $1 \leq i \neq j \leq n$,

$$\left\| \sum_{i=1}^n \sqrt{t_i} A_i \right\|_Q^2 \leq \sum_{i=1}^n t_i \left\| A_i \right\|_Q^2 - \sum_{i=1}^n t_i \sum_{j=1}^n t_j A_j \right\|_Q^2.$$

**Proof.**

$$\left\| \sum_{i=1}^n \sqrt{t_i} A_i \right\|_Q^2 = \left\| \sum_{i=1}^n \sqrt{t_i} A_i \right\|_{\tilde{Q}}^2 = \left\| \sum_{i=1}^n t_i A_i^2 \right\|_{\tilde{Q}}$$

$$\leq \sum_{i=1}^n \left\| t_i A_i - \sum_{j=1}^n t_j A_j \right\|_{\tilde{Q}}^2 + \left\| \sum_{j=1}^n t_j A_j \right\|_{\tilde{Q}}^2.$$

Remark 2.6. The operators $A_i$ acting on a Hilbert space having the orthogonal property $A_i^* A_j = 0$ for $1 \leq i \neq j \leq n$ are not uncommon. For instance, let $(e_i)$ be an orthogonal family (not containing zero) in $\mathcal{H}$ and define the operators $A_i : \mathcal{H} \to \mathcal{H}$ by $A_i = \frac{e_i \otimes e_i}{\| e_i \|}$, $1 \leq i \leq n$. Then $A_i$'s are positive operators in $\mathcal{B}(\mathcal{H})$ with $\| A_i \|_{\infty} = \| e_i \|$ for all $i$ and $\| A_i A_j \|_{\infty} = |\langle e_i, e_j \rangle|$ for all $1 \leq i, j \leq n$ (for details see [5]).

In the setting of Hilbert spaces, the known equality (1.1) can be proved directly. In what follows we show that an extension of it can also be obtained from equality (2.2).

**Corollary 2.7.** Let $x_1, \ldots, x_n \in \mathcal{H}$. Then

$$\sum_{i=1}^n t_i \left\| x_i - \sum_{j=1}^n t_j x_j \right\|^2 = \sum_{i=1}^n t_i \left\| x_i \right\|^2 - \sum_{j=1}^n t_j \left\| x_j \right\|^2.$$
Proof. Let $e$ be a non-zero vector of $\mathcal{H}$ and set $A_i = x_i \otimes e$. It follows from the elementary properties of rank one operators and equality (2.2) that

\[
\sum_{i=1}^{n} t_i \| x_i - \sum_{j=1}^{n} t_j x_j \|_2^2 e \otimes e = \sum_{i=1}^{n} t_i \left( x_i - \sum_{j=1}^{n} t_j x_j \right) \otimes e^2
\]
\[
= \sum_{i=1}^{n} t_i |A_i - \sum_{j=1}^{n} t_j A_j|^2
\]
\[
= \sum_{i=1}^{n} t_i |A_i|^2 - \left| \sum_{j=1}^{n} t_j A_j \right|^2
\]
\[
= \sum_{i=1}^{n} t_i |x_i \otimes e|^2 - \left| \left( \sum_{j=1}^{n} t_j x_j \right) \otimes e \right|^2
\]
\[
= \left( \sum_{i=1}^{n} t_i \| x_i \|^2 - \left| \sum_{j=1}^{n} t_j x_j \right|^2 \right) e \otimes e,
\]

from which we conclude the result. $\square$

From now on we restrict ourselves to (2.3).

A special case (for $X = \mathbb{C}$) of equality (1.1) is

\[
\sum_{i=1}^{n} \left| z_i - \frac{1}{n} \sum_{j=1}^{n} z_j \right|^2 = \sum_{i=1}^{n} |z_i|^2 - \frac{1}{n} \sum_{j=1}^{n} z_j^2 \quad (z_1, \cdots, z_n \in \mathbb{C}).
\]

This equality is in turn a special case (when $A = \text{diag}(z_1, \cdots, z_n)$ and $\text{tr}(a_{ij}) := \sum_{i=1}^{n} a_{ii}$) of the next equality concerning the usual normalized trace functional.

**Theorem 2.8.** Let $\hat{\text{tr}}(A) = \text{tr}(A)/\text{tr}(I)$ be the normalized trace on $M_n(\mathbb{C})$. Then

\[
\| A - \hat{\text{tr}}(A) \|_2^2 = \| A \|^2_2 - \text{tr}(I) |\hat{\text{tr}}(A)|^2.
\]
Proof.
\[
\|A - \hat{\text{tr}}(A)\|_2^2 = \text{tr} \left| A - \frac{\text{tr}(A)}{\text{tr}(I)} I \right|^2 \\
= \text{tr} \left( \left( A^* - \frac{\text{tr}(A)}{\text{tr}(I)} I \right) \left( A - \frac{\text{tr}(A)}{\text{tr}(I)} I \right) \right) \\
= \text{tr}(A^*A) - \frac{\text{tr}(A^*)\text{tr}(A)}{\text{tr}(I)} - \frac{\text{tr}(A)\text{tr}(A)}{\text{tr}(I)} + \frac{|\text{tr}(A)|^2}{\text{tr}(I)^2} \text{tr}(I) \\
= |\text{tr}(A)|^2 - \text{tr}(I) \left| \frac{\text{tr}(A)}{\text{tr}(I)} \right|^2 \\
= \|A\|_2^2 - \text{tr}(I)|\hat{\text{tr}}(A)|^2.
\]

\[\square\]

We need the following lemma which can be deduced from [4, Lemma 4] and [9, p. 20] (see also [7, Lemma 2.1]).

Lemma 2.9. Let \(A_1, \ldots, A_n\) be positive operators in \(C_p\) for some \(p > 0\).

(i) If \(0 < p < 1\), then
\[
n^{p-1} \sum_{i=1}^{n} \|A_i\|_p^p \leq \left\| \sum_{i=1}^{n} A_i \right\|_p^p \leq \sum_{i=1}^{n} \|A_i\|_p^p.
\]

(ii) If \(1 \leq p < \infty\), then
\[
\sum_{i=1}^{n} \|A_i\|_p^p \leq \left\| \sum_{i=1}^{n} A_i \right\|_p^p \leq n^{p-1} \sum_{i=1}^{n} \|A_i\|_p^p.
\]

Note that the commutative version of Lemma 2.9 for scalars follows from the well-known Hölder inequality (see, e.g., [2, p. 88]).

The next theorem is our second main result. It can be regarded as a generalization of (1.2) in inequalities.

Theorem 2.10. Let \(A_1, \ldots, A_n \in C_p\) for some \(p > 0\).

(i) If \(0 < p < 2\), then
\[
\sum_{i=1}^{n} \left\| A_i - \frac{1}{n} \sum_{j=1}^{n} A_j \right\|_p^p \geq n^{\frac{p}{2}-1} \sum_{i=1}^{n} \|A_i\|_p^p - n \left\| \frac{1}{n} \sum_{j=1}^{n} A_j \right\|_p^p.
\]

(ii) If \(2 < p < \infty\), then
\[
n^{\frac{p}{2}-1} \sum_{i=1}^{n} \|A_i\|_p^p - n \left\| \frac{1}{n} \sum_{j=1}^{n} A_j \right\|_p^p \geq \sum_{i=1}^{n} \left\| A_i - \frac{1}{n} \sum_{j=1}^{n} A_j \right\|_p^p.
\]
Proof. Let $0 < p < 2$. Then

$$
\sum_{i=1}^{n} \left\| A_i - \frac{1}{n} \sum_{j=1}^{n} A_j \right\|_p^p + n \left\| \frac{1}{n} \sum_{j=1}^{n} A_j \right\|_p^p
$$

$$
= \sum_{i=1}^{n} \left\| A_i - \frac{1}{n} \sum_{j=1}^{n} A_j \right\|_{p/2}^{p/2} + n \left\| \frac{1}{n} \sum_{j=1}^{n} A_j \right\|_{p/2}^{p/2}
$$

$$
\geq \sum_{i=1}^{n} \left\| A_i - \frac{1}{n} \sum_{j=1}^{n} A_j \right\|_{p/2}^{p/2} + n \left\| \frac{1}{n} \sum_{j=1}^{n} A_j \right\|_{p/2}^{p/2}
$$

(by the second inequality of Lemma 2.9(i))

$$
= \sum_{i=1}^{n} |A_i|^2 \left\| \frac{p}{p/2} \right\|_p^{p/2} \quad \text{(by equality (2.3))}
$$

$$
\geq n^{\frac{p}{2}-1} \sum_{i=1}^{n} |A_i|^2 \left\| \frac{p}{p/2} \right\|_p^{p/2}
$$

(by the first inequality of Lemma 2.9(i))

$$
= n^{\frac{p}{2}-1} \sum_{i=1}^{n} \| A_i \|_p^p.
$$

This proves the first part of the theorem. By a similar argument, one can prove the second part. \square

The next theorem may be compared to Theorem 2.10. It can also be viewed as a variance of (1.2).

**Theorem 2.11.** Let $A_1, \ldots, A_n \in C_p$ for some $p > 0$. Then

$$
\sum_{i=1}^{n} \left\| A_i - \frac{1}{n} \sum_{j=1}^{n} A_j \right\|_p^2 \geq n^{\frac{p}{2}-1} \sum_{i=1}^{n} \| A_i \|_p^2 - n \left\| \frac{1}{n} \sum_{j=1}^{n} A_j \right\|_p^2
$$

for $0 < p < 2$; and

$$
n^{\frac{p}{2}-1} \sum_{i=1}^{n} \| A_i \|_p^2 - n \left\| \frac{1}{n} \sum_{j=1}^{n} A_j \right\|_p^2 \geq \sum_{i=1}^{n} \left\| A_i - \frac{1}{n} \sum_{j=1}^{n} A_j \right\|_p^2
$$

for $2 \leq p < \infty$. 

Proof. Let $0 < p < 2$. Then

\[
\sum_{i=1}^{n} \left\| A_i - \frac{1}{n} \sum_{j=1}^{n} A_j \right\|^2_p + n \left\| \frac{1}{n} \sum_{j=1}^{n} A_j \right\|^2_p \\
= \sum_{i=1}^{n} \left\| A_i - \frac{1}{n} \sum_{j=1}^{n} A_j \right\|^2_{p/2} + n \left\| \frac{1}{n} \sum_{j=1}^{n} A_j \right\|^2_{p/2} \\
\leq \left( \sum_{i=1}^{n} \left\| A_i \right\|^2_{p/2} \right)^{2/p} + n \left( \left\| \frac{1}{n} \sum_{j=1}^{n} A_j \right\|^2_{p/2} \right)^{2/p} \\
= \left( \sum_{i=1}^{n} \left\| A_i \right\|^p \right)^{2/p} \\
\leq n^{\frac{2}{p} - 1} \sum_{i=1}^{n} \left( \left\| A_i \right\|^p \right)^{2/p} \\
= n^{\frac{2}{p} - 1} \sum_{i=1}^{n} \left\| A_i \right\|^2_p.
\]

This proves the first part of the theorem. The second part of the theorem follows from a similar argument. The second author thanks the NSU Farquhar College of Arts and Sciences for a Mini-grant.

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