Reparametrization invariance of the classical metric

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Abstract

There is a statement on the parametrization dependence of the classical metric in the recent paper of N.E.J. Bjerrum-Bohr, J.F. Donoghue, B.R. Holstein, gr-qc/0610096. I completely disagree with this statement. Here I show reparametrization invariance of the classical metric.

1 General consideration of the reparametrization transformation

The statement made in the paper [1] is that using the following parametrization of the metric

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - \frac{a}{4} h_{\rho\sigma} h_{\nu}^\rho, \]

(1)
to solve the Einstein equations, where \( a \) is an arbitrary constant, one derives the metric, which depends on \( a \).

Let us consider this statement in detail. Expansion of the Schwarzschild metric in \( r_g/r \) can be represented in the form

\[ g_{\mu\nu} = \eta_{\mu\nu} + g_{\mu\nu} + \mathcal{O}(r_g^3/r^3), \]

(2)
where \( g_{\mu\nu} \) and \( g_{\mu\nu} \) are the first and the second order corrections, respectively. In harmonic gauge,

\[ \partial_\mu(\sqrt{-g} g^{\mu\nu}) = 0, \]

(3)
the corrections of first order have the form

\[ g_{00} = -\frac{r_g}{r}, \quad g_{ij} = -\frac{r_g}{r} \delta_{ij}. \]

(4)

Using these corrections to find a perturbative solution of the Einstein equations, one can derive the second order corrections:

\[ g_{00} = \frac{r_g^2}{2r^2}, \quad g_{ij} = -\frac{r_g^2}{4r^2} (\delta_{ij} + n^in^j). \]

(5)

It follows from the expression (1) that \( g_{\mu\nu} = h_{\mu\nu} \), where \( h_{\mu\nu} \) is the correction of first order in \( r_g/r \) to \( h_{\mu\nu} \).

The first aspect to be clarified is that the parametrization of metric perturbations does not change the gauge conditions for total metric. If one uses the harmonic gauge \( \partial_\mu(\sqrt{-g} g^{\mu\nu}(h_{\alpha\beta})) = 0, \)

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where $\sqrt{-g} \, g^{\mu \nu}(h_{\alpha \beta})$ is an analytical function of $h_{\alpha \beta}$, then after reparametrization $h_{\alpha \beta} \rightarrow f(h_{\alpha \beta})$, where $f(h_{\alpha \beta})$ is an analytical function of $h_{\alpha \beta}$, the gauge conditions become $\sqrt{-g} \, g^{\mu \nu}(f(h_{\alpha \beta})) = 0$. The gauge conditions for total metric (3) are the same, whatever parametrization for perturbations is used.

The statement of the paper [1] can be reduced to the following: if, in the Einstein equations, one uses the following expansion

$$g_{\mu \nu} = \eta_{\mu \nu} + g_{\mu \nu} + (h_{\mu \nu} + w_{\mu \nu}) + O(r^3/r^3),$$

instead of Eq. (2), where $w_{\mu \nu}$ is the known function

$$w_{\mu \nu} = -\frac{a}{4} \, h_{\mu \alpha} h_{\nu}^{\alpha} = -\frac{a}{4} \, g_{\mu \nu} g^{\alpha}_{\alpha},$$

finds $h_{\mu \nu}$ and inserts it into Eq. (6), then one derives the corrections of second order which differ from those of Eq. (5).

The statement can be repeated once more in a different form. Assume that one can find $g_{\mu \nu}$ from the equation

$$F(g_{\mu \nu}) = 0,$$

where $F$ is a certain functional. Let us substitute $g_{\mu \nu}$ by $h_{\mu \nu} + w_{\mu \nu}$ in this equation, where $w_{\mu \nu}$ is a known function and $h_{\mu \nu}$ is a new function to be found:

$$F(h_{\mu \nu} + w_{\mu \nu}) = 0.$$  

The statement of the paper [1] is equivalent to the following one: if one finds $h_{\mu \nu}$ from the equation (9), then $h_{\mu \nu} + w_{\mu \nu}$ would be different from the solution of the equation (8). It is obviously incorrect. In the following section, I will demonstrate the complete series of the arithmetical mistakes that led the authors of Ref. [1] to the false conclusions.

### 2 Solution of the Einstein equations

The first incorrect formula in the paper [1] is (Eq. 4 of Ref. [1])

$$h_{\mu \nu}(x) = -16\pi G \int d^3 y \, D(x - y) \left( T_{\mu \nu}(y) - \frac{1}{2} \eta_{\mu \nu} T(y) \right).$$

I would like to stress here that the authors of Ref. [1] use harmonic gauge (3) and they derive the second order corrections from Eq. (10). Putting $a = 0$ yields $h_{\mu \nu} = g_{\mu \nu}$, but, as follows from Eq. (10)

$$\partial_\mu \left( h^{\mu \nu} - \frac{1}{2} \eta^{\mu \nu} h \right) = \partial_\mu \left( g^{\mu \nu} - \frac{1}{2} \eta^{\mu \nu} g \right) = 0,$$

which is obviously incorrect (see the explicit form of $g_{\mu \nu}$ in Eq. (5), consequently the expression (10) is incorrect. The correct formula is Eq. (A12) given in the paper [2] by the same authors. It is useful to write this equation in the form

$$\Box \left( g_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} g \right) - 2 P^{\lambda \sigma}_{\mu \nu} \partial_\lambda \left( \partial^\lambda g_{\lambda \sigma} - \frac{1}{2} \partial_\sigma g \right) = -16\pi G T_{\mu \nu}^{grav}(g),$$

$$P^{\lambda \sigma}_{\mu \nu} = \frac{1}{2} \left( \delta^\lambda_\mu \delta^\sigma_\nu + \delta^\lambda_\nu \delta^\sigma_\mu - \eta^{\lambda \sigma} \eta_{\mu \nu} \right),$$

where $T_{\mu \nu}^{grav}(g)$ is defined by the expression (A14) of Ref. [2]. I have to write out the expression (12) because the corresponding formula (A13) in Ref. [2] is incorrect. However, it was pointed out correctly that (in the case when $a = 0$) the expression (12) can be reduced to formula (A16) of Ref. [2].
It is easy to derive the analogue of the formula (A16) of Ref. [2] in the case when $a \neq 0$. Actually, the substitution $g_{\mu \nu} = h_{\mu \nu} + w_{\mu \nu}$ in the equation (12) yields

$$\Box \left( h_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} h \right) - 2P^\lambda_\mu h_\lambda \partial_\lambda \left( \partial^\alpha h_{\alpha \sigma} - \frac{1}{2} \partial_\sigma h \right) = -16\pi G \left( T_{\mu \nu}^{\text{grav}}(w) + \tilde{T}_{\mu \nu}(w) \right),$$  \hspace{1cm} (14)

where

$$\tilde{T}_{\mu \nu} = \frac{1}{16\pi G} \left( \Box \left( w_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} w \right) - 2P^\lambda_\mu h_\lambda \partial_\lambda \left( \partial^\alpha w_{\alpha \sigma} - \frac{1}{2} \partial_\sigma w \right) \right).$$  \hspace{1cm} (15)

It is easy to notice that $\tilde{T}_{\mu \nu}$ is exactly the $a$-dependent correction calculated in [1], formula (3) of Ref. [1]. In fact, $\partial_\mu \tilde{T}^\mu_\nu = 0$ for any $w_{\mu \nu}$ by virtue of the contracted Bianchi identity [1], therefore, this correction can be included in the transverse form factors (Eq. 2 of Ref. [1]). In fact, $\tilde{T}_{\mu \nu}$ can contribute only to the form factor $F_2(q^2)$ in the formula (2) of Ref. [1] because $w_{\mu \nu}$ consists of the product of the fields $h_{\mu \nu}$ in our particular case (7). It is easy to see that $\tilde{T}_{\mu \nu}$ is the correction induced by the structure $a\chi^\alpha_\gamma_\delta$ in my comment [3].

To simplify the term, that is proportional to $P^\lambda_\mu h_\lambda$ in the lhs of the equation (14), we insert the definition (1) in the gauge condition $\partial_\mu(\sqrt{-g}g^{\mu \nu}) = 0$. As a result, we have:

$$0 = - \left( \partial_\mu h^{\mu \nu} - \frac{1}{2} \partial^\nu h \right)$$

$$= - \left( \partial_\mu h^{\mu \nu} - \frac{1}{2} \partial^\nu h \right) + \frac{a}{4} \partial_\mu P^{\mu \nu}_\gamma_\delta h_{\gamma \delta} + \frac{h^{\mu \sigma}}{2} \partial_\mu L_\sigma - \frac{1}{2} \partial_\nu h_{\mu \sigma}.$$  \hspace{1cm} (16)

Using Eq. (17) yields

$$\Box \left( h_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} h \right) = -16\pi GT_{\mu \nu} + 2P^\lambda_\mu h_\lambda \partial_\lambda \left( \frac{h^{\alpha \beta}}{2} \partial_\alpha h_{\beta \sigma} - \frac{1}{2} \partial_\sigma h_{\alpha \beta} \right) + \frac{a}{2} P^{\mu \sigma}_\gamma_\delta \partial_\lambda P^\lambda_\beta P^{\beta \sigma}_\gamma_\delta h_{\gamma \delta}.$$  \hspace{1cm} (18)

where $T_{\mu \nu} = T_{\mu \nu}^{\text{grav}} + \tilde{T}_{\mu \nu}$.

Using the exact form of $h_{\mu \nu} = g_{\mu \nu}$ (see Eq. (4)), we transform Eq. (18) to

$$\Box h_{\mu \nu} = -16\pi G \left( T_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} T \right) - \partial_\mu (f \partial_\nu f) - \partial_\nu (f \partial_\mu f) - \frac{a}{2} \partial_\mu \partial_\nu f^2,$$  \hspace{1cm} (19)

where $f = -r_g/r$. The expression (19) is the analogue of the formula (A16) of Ref. [2], for an arbitrary $a$.

Solution of the equation (19) has the form

$$h_{\mu 0} = (2 + a) \frac{r_g^2}{4r^2},$$  \hspace{1cm} (20)

$$h_{ij} = -\frac{r_g^2}{4r^2} (\delta_{ij} + n_in_j) + \left[ -\frac{r_g^2}{4r^2} a \left( 3\delta_{ij} - 4n_in_j \right) \right] + \frac{r_g^2}{4r^2} 2a (\delta_{ij} - 2n_in_j)$$

$$= -\frac{r_g^2}{4r^2} (\delta_{ij} + n_in_j) - \frac{a}{4} \frac{r_g^2}{r^2} \delta_{ij}. $$  \hspace{1cm} (21)

The correction (20) is derived in Ref. [3]. In expression (21) we separate by the square brackets the term which the authors of Ref. [1] would have obtained from the expression (10) if they had made the Fourier transform correctly.

Inserting $h_{\mu \nu} = \Box h_{\mu \nu}$ in the equation (11), we find the corrections (4) and (5) exactly.

\footnotetext{1}{the following identity $\eta^{\mu \nu} \partial_\mu (R_{\nu \mu} - \eta_{\mu \nu} R/2) = 0$ is meant, where $R_{\mu \nu}$ is the first order term in the expansion of the Ricci tensor in $h_{\mu \nu}$}
3 Conclusion

Reparametrization invariance of the classical metric is a triviality. It is important to understand that a parametrization is only a way of calculation. All conclusions of the paper [1] are based on the mistaken calculations, therefore, they are incorrect as a whole.

A nontrivial requirement appears when one considers quantum corrections to the Schwarzschild metric [3]. After averaging over the short-wavelength modes of gravitational field one derives an effective action (a non-local effective action, in general case) which depends only on long-range modes of gravitational field. All information about the short-wavelength fluctuations (i.e. their parametrization or even their gauge) is lost after these fluctuations are integrated out (after the averaging).

Following are the several points concerning terminology. Since the corrections to $h_{\mu\nu}$ are not the corrections to the metric (but related to them by the “complicated” equation (1)), I would not even call the quantity $T_{\mu\nu}$ (in the rhs of Eq.[19] “energy momentum tensor for the gravitational field”. Correspondingly, $\tilde{T}_{\mu\nu}$ (see Eq.[15]) is not a correction to the energy momentum tensor, because $w_{\mu\nu}$ can be any arbitrary function which is not related to the source of gravitational field. Terminology is matter of taste though. In any case one can see that $\tilde{T}_{\mu\nu}$ is a total derivative and the statement of the paper [1] that ”the amount of energy and momentum that carried in the classical field also varies with the parametrization” is incorrect.

References

[1] N.E.J. Bjerrum-Bohr, J.F. Donoghue, B.R. Holstein, gr-qc/0610096.

[2] N.E.J. Bjerrum-Bohr, J.F. Donoghue, B.R. Holstein, Quantum corrections to the Schwarzschild and Kerr metric, Phys. Rev. D68, (2003) 084005; hep-th/0211071.

[3] G.G. Kirilin, Quantum corrections to the Schwarzschild metric and reparametrization transformations; gr-qc/0601020