On Symmetric Solutions to Linear Matrix Time-Varying Differential Equations

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Abstract. In this paper, we discuss when the solution to the initial value problem for a linear matrix time-varying differential equation is symmetric on a given interval. By symmetry, we mean that the solution does not change when transposed. Throughout the paper, we assume that the equation has coefficients of finite order of smoothness. We demonstrate that, in order to verify whether the solution to the initial value problem is symmetric on a given interval, it can be useful to construct two matrix sequences associated to the equation. Using these sequences, we prove a sufficient condition for the solution symmetry on a given interval. Assuming that the initial value problem for a linear matrix time-varying differential equation satisfies this condition, we derive a formula for a symmetric solution to this problem.

1. Introduction. Problem formulation
Linear matrix differential equations play an important role in both mathematical and applied problems. Various theoretical aspects of solving linear matrix differential equations can be found in [1–5]. Our work is motivated by a relationship between linear matrix differential equations and boundary value problems for nonlinear control systems. More precisely, recently it has been shown that linear matrix differential equations appear as part of solving boundary value problems for nonlinear control systems. In this regard, we should mention the covering method for solving a boundary value problem (see the description of the method given in [6] and its development presented in [7–9]). In some cases [6], a construction of covering for a control system can be reduced to solving a sequence of initial value problems for linear matrix time-varying differential equations. A qualitative analysis of linear matrix time-varying differential equations is often required [10] when solving a steering point-to-point problem for nonlinear control systems by the approach proposed in [11–13]. In this paper, we consider a linear matrix time-varying differential equation

\[ X' = A(t)X + B(t) \]  

(1)

where \( t \in \mathbb{R} \) is the independent variable, \( X = X(t) \) is an unknown \( n \times n \)-matrix, \( A(t) \), \( B(t) \in C^l(a,b) \) are given \( n \times n \)-matrices, \( l \in \mathbb{N} \). Let \( X(t) \) be a solution to (1) satisfying the initial condition \( X(t_0) = X_0 \) where \( t_0 \in \mathbb{R} \) and \( X_0 \) is a given \( n \times n \)-matrix. It is well-known [14] that if \( A(t) \), \( B(t) \in C^l(a,b) \) and \( t_0 \in (a,b) \), then \( X(t) \) belongs to \( C^l(a,b) \). Denote by \( X_0^{(i)} \) the \( i \)th derivative of \( X(t) \) calculated at \( t_0 \), \( i \in \{0, \ldots, l\} \). We are going to establish conditions under which there exists \( m \in \{1, \ldots, l\} \) such that the symmetry of \( X_0^{(0)}, X_0^{(1)}, \ldots, X_0^{(m)} \) yields the
symmetry of \(X(t)\) on \((a,b)\). It should be noted that by symmetry of \(X(t)\) we mean that the transposed matrix \(X^T(t)\) coincides with \(X(t)\). The problem in question was solved in [15] for the case when \(A(t)\) and \(B(t)\) are analytical. In the present paper, we deal with the case when \(A(t)\) and \(B(t)\) are of finite order of smoothness.

2. Condition for the solution symmetry
Firstly, we need to establish the following property of solutions to the equation (1).

**Lemma 1** Let \(A(t), B(t) \in C^l(a,b)\) and \(Y(t)\) be an arbitrary solution to (1) on \((a,b)\). Then the \(i\)th derivative of \(Y(t)\) is given on \((a,b)\) by the relation

\[
Y^{(i)}(t) = W_i(t)Y(t) + V_i(t), \quad i = 0, \ldots, l
\]

where

\[
W_0(t) = E, \quad W_{i+1}(t) = W_i'(t) + W_i(t)A(t), \quad i = 0, \ldots, l - 1,
\]

\[
V_0(t) = 0, \quad V_{i+1}(t) = V_i'(t) + W_i(t)B(t), \quad i = 0, \ldots, l - 1,
\]

\(E\) is the identity \(n \times n\)-matrix.

**Proof.** We prove Lemma 1 by induction on \(i\). For \(i = 0\) the statement is clearly true. Assume that for a particular \(i \in \{1, \ldots, l\}\), the equality

\[
Y^{(i-1)}(t) = W_{i-1}(t)Y(t) + V_{i-1}(t)
\]

holds. Let us show that \(Y^{(i)}(t) = W_i(t)Y(t) + V_i(t)\). Differentiating (5) with respect to \(t\) and taking into account the relation \(Y'(t) = A(t)Y(t) + B(t)\), we get

\[
Y^{(i)}(t) = W_{i-1}'(t)Y(t) + W_{i-1}(t)Y'(t) + V_{i-1}'(t)
\]

\[
= W_i'(t)Y(t) + W_{i-1}(t)(A(t)Y(t) + B(t)) + V_{i-1}'(t)
\]

\[
= (W_{i-1}'(t) + W_{i-1}(t)A(t))Y(t) + V_{i-1}'(t) + W_{i-1}(t)B(t) = W_i(t)Y(t) + V_i(t).
\]

The proof is complete.

It follows from (2) that if \(X(t)\) is the solution to the equation (1) satisfying the initial condition \(X(t_0) = X_0\), then the matrices \(X^{(i)}_0\), \(i = 0, \ldots, l\), are given by

\[
X^{(i)}_0 = W_i(t_0)X_0 + V_i(t_0), \quad i = 0, \ldots, l.
\]

The main result of this section is the following theorem.

**Theorem 1.** Let \(A(t), B(t) \in C^l(a,b)\) and \(X(t)\) be a solution to (1) satisfying the condition \(X(t_0) = X_0\). Assume that there exists \(m \in \{1, \ldots, l\}\) such that:

(i) the matrices \(X_0^{(0)}, \ldots, X_0^{(m-1)}\) are symmetric,

(ii) there exist functions \(\lambda_0(t), \ldots, \lambda_{m-1}(t) \in C(a,b)\) such that the equalities

\[
W_m(t) = \sum_{i=0}^{m-1} \lambda_i(t)W_i(t)
\]

and \(V^T(t) = V(t)\) hold on \((a,b)\) where

\[
V(t) = V_m(t) - \sum_{i=0}^{m-1} \lambda_i(t)V_i(t).
\]
Then $X(t)$ is symmetric on $(a, b)$.

**Proof.** Let us consider the matrix $Z(t) = X(t) - X^T(t)$. It follows from (2) that the higher order derivatives of $X^T(t)$ are given by $[X^T(t)]^{(i)} = X^T(t)W_i^T(t) + V_i(t)$, $i = 0, \ldots, l$. Subtracting the latter from (2), we get the following relation for the higher order derivatives of $Z(t)$:

$$Z^{(i)}(t) = W_i(t)X(t) - X^T(t)W_i^T(t) + V_i(t) - V_i^T(t), \quad i = 0, \ldots, l. \quad (8)$$

Putting $i = m$, we conclude that the $m$th derivative of $Z(t)$ is given by

$$Z^{(m)}(t) = W_m(t)X(t) - X^T(t)W_m^T(t) + V_m(t) - V_m^T(t). \quad (9)$$

Substituting (7) into (9), we obtain

$$Z^{(m)}(t) = \sum_{i=0}^{m-1} \lambda_i(t)W_i(t)X(t) - X^T(t)\sum_{i=0}^{m-1} \lambda_i(t)W_i^T(t) + V_m(t) - V_m^T(t)$$

$$= \sum_{i=0}^{m-1} \lambda_i(t)\left[|W_i(t)X(t) - X^T(t)W_i^T(t)| + V_m(t) - V_m^T(t)\right]$$

$$= \sum_{i=0}^{m-1} \lambda_i(t)\left[|W_i(t)X(t) - X^T(t)W_i^T(t)| + V_i(t) - V_i^T(t)\right]$$

$$+ V_m(t) - \sum_{i=0}^{m-1} \lambda_i(t)V_i(t) - V_m^T(t) + \sum_{i=0}^{m-1} \lambda_i(t)V_i^T(t).$$

Taking into account (8) and the definition of $V(t)$, we have

$$Z^{(m)}(t) = \sum_{i=0}^{m-1} \lambda_i(t)Z^{(i)}(t) + V(t) - V^T(t). \quad (10)$$

According to the condition of Theorem 1, the matrix $V(t)$ is symmetric, hence $Z(t)$ satisfies the linear matrix differential equation

$$Z^{(m)} = \sum_{i=0}^{m-1} \lambda_i(t)Z^{(i)}. \quad (11)$$

Since $Z^{(i)}(t_0) = X_0^{(i)} - [X_0^{(i)}]^T, i = 0, \ldots, m - 1$ and the matrices $X_0^{(i)}, i = 0, \ldots, m - 1$, are symmetric, we have $Z^{(i)}(t_0) = 0, i = 0, \ldots, m - 1$. It is obvious that the zero matrix is a solution to the equation (11) satisfying the initial conditions $Z^{(i)}(t_0) = 0, i = 0, \ldots, m - 1$. According to the Cauchy theorem, such a solution is unique. Therefore, $Z(t) = 0$ for all $t \in (a, b)$. As a consequence, we finally get that $X(t) = X^T(t)$ for all $t \in (a, b)$. The proof is complete.

**Example.** Consider the equation (1) where

$$A(t) = \begin{pmatrix} 1 & e^{\frac{\sqrt{7}t}{2}} \\ 0 & 1 \end{pmatrix}, \quad B(t) = \begin{pmatrix} e^{\frac{\sqrt{7}t}{2}} & 1 \\ 1 & 0 \end{pmatrix}.$$ 

Observe that $A(t), B(t) \in C^2(\mathbb{R})$. We are going to show that the solution $X(t)$ satisfying the initial condition $X(0) = 0$ is symmetric for all $t \in \mathbb{R}$. Calculating the matrices $W_i(t), V_i(t)$ and $X_0^{(i)}$ by the formulas (3), (4) and (6), we get

$$W_0(t) = E, \quad V_0(t) = 0, \quad X_0^{(0)} = 0,$$
It is easily seen that:

1. The matrices $X^{(0)}_0$ and $X^{(1)}_0$ are symmetric;
2. The matrix $W_2(t)$ can be represented in the form $W_2(t) = \lambda_0(t)W_0(t) + \lambda_1(t)W_1(t)$ where
   \[\lambda_0(t) = -\frac{7}{3}\sqrt[3]{t^2} - 1, \quad \lambda_1(t) = \frac{7}{3}\sqrt[3]{t^2} + 2;\]
3. The matrix $V_2(t)$ be a solution to (1) satisfying the condition $X(t_0) = X_0$.

Thus, $X(t)$ is symmetric for all $t \in \mathbb{R}$.

3. Formula for symmetric solutions

In this section, we suppose that the conditions of Theorem 1 are met. Our main goal is to establish a formula for symmetric solutions $X(t)$ to the equation (1) satisfying the condition $X(t_0) = X_0$.

**Lemma 2.** Let $A(t), B(t) \in C^1(a, b)$ and $X(t)$ be a solution to (1) satisfying the condition $X(t_0) = X_0$. Assume that there exists $m \in \{1, \ldots, l\}$ such that the conditions (i) and (ii) hold. Then $X(t)$ is a solution to the initial value problem

$$X^{(m)} + \sum_{i=0}^{m-1} \lambda_i(t)X^{(i)} = V(t), \quad X^{(i)}(t_0) = X^{(i)}_0, \quad i = 0, \ldots, m-1. \quad (12)$$

**Proof.** According to Theorem 1, the matrix $X(t)$ is symmetric for all $t \in (a, b)$. Setting $i = m$ in (2), we obtain $X^{(m)}(t) = W_m(t)X(t) + V_m(t)$. Using (7), we rewrite the latter as

$$X^{(m)}(t) = \sum_{i=0}^{m-1} \lambda_i(t)W_i(t)X(t) + V_m(t).$$

It follows from (2) that $W_i(t)X(t) = X^{(i)}(t) - V_i(t)$. Hence,

$$X^{(m)}(t) = \sum_{i=0}^{m-1} \lambda_i(t)[X^{(i)}(t) - V_i(t)] + V_m(t) = \sum_{i=0}^{m-1} \lambda_i(t)X^{(i)}(t) + V_m(t) - \sum_{i=0}^{m-1} \lambda_i(t)V_i(t)$$

$$= \sum_{i=0}^{m-1} \lambda_i(t)X^{(i)}(t) + V(t).$$

Thus, $X(t)$ is a solution to the initial value problem (12). The proof is complete.

Denote by $V_{jk}(t)$ and $X^{(i)}_{0,jk}$, $j, k = 1, \ldots, n$, $i = 0, \ldots, m - 1$, the entries of the matrices $V(t)$ and $X^{(i)}_0$, respectively. Then each entry $X_{jk}(t)$ of $X(t)$ is a solution to the initial value problem

$$\varphi^{(m)} - \sum_{i=0}^{m-1} \lambda_i(t)\varphi^{(i)} = V_{jk}(t), \quad \varphi^{(i)}(t_0) = X^{(i)}_{0,jk}, \quad i = 0, \ldots, m - 1. \quad (13)$$
Let \( \varphi_j(t, \tau), j = 0, \ldots, m - 1, \) be a solution to the linear homogeneous differential equation

\[
\varphi^{(m)} - \sum_{i=0}^{m-1} \lambda_i(t) \varphi^{(i)} = 0
\]  

satisfying the condition

\[
\left. \frac{\partial^i}{\partial t^i} \varphi_j(t, \tau) \right|_{t=\tau} = \begin{cases} 1, & j = i, \\ 0, & j \neq i, \end{cases} \quad i = 0, \ldots, m - 1.
\]

**Theorem 2.** Let \( A(t), B(t) \in C^l(a, b) \) and \( X(t) \) be a solution to (1) satisfying the condition \( X(t_0) = X_0. \) Assume that there exists \( m \in \{1, \ldots, l\} \) such that the conditions (i) and (ii) hold. Then

\[
X(t) = \sum_{i=0}^{m-1} \varphi_i(t, t_0) X_0^{(i)} + \int_{t_0}^{t} \varphi_{m-1}(t, \tau) V(t) d\tau.
\]  

**Proof.** Consider a linear differential equation

\[
\varphi^{(m)} - \sum_{i=0}^{m-1} \lambda_i(t) \varphi^{(i)} = v(t)
\]

where \( v(t) \in C(a, b) \). It is well-known that its solution \( \varphi(t) \) satisfying the conditions \( \varphi^{(i)}(t_0) = p_i, \) \( i = 0, \ldots, m - 1, \) is given by

\[
\varphi(t) = \sum_{i=0}^{m-1} \varphi_i(t) p_i + \int_{t_0}^{t} \varphi_{m-1}(t, \tau) v(\tau) d\tau.
\]

Applying this formula to the initial value problem (13), we obtain that each entry \( X_{jk}(t) \) of \( X(t) \) is given by

\[
X_{jk}(t) = \sum_{i=0}^{m-1} \varphi_i(t) X_0^{(i)jk} + \int_{t_0}^{t} \varphi_{m-1}(t, \tau) V_{jk}(\tau) d\tau.
\]

As a consequence, the matrix \( X(t) \) has the form (15). The proof is complete.

**Example.** Consider the equation (1) where

\[
A(t) = \begin{pmatrix} -1 & 2e^t \\ -e^{-t} & 1 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 2e^t & 2 \\ 2 & e^{-t} \end{pmatrix}.
\]

Let \( X(t) \) be the solution to (16) satisfying the initial condition \( X(0) = 0. \) Firstly, let us show that \( X(t) \) is symmetric for all \( t \in \mathbb{R}. \) Constructing matrices \( W_i(t), V_i(t) \) and \( X_0^{(i)} \) by the formulas (3), (4) and (6), we obtain

\[
W_0(t) = E, \quad V_0(t) = 0, \quad X_0^{(0)} = 0,
\]

\[
W_1(t) = A(t), \quad V_1(t) = B(t), \quad X_0^{(1)} = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix},
\]

\[
W_2(t) = \begin{pmatrix} -1 & 2e^t \\ e^{-t} & -1 \end{pmatrix}, \quad V_2(t) = \begin{pmatrix} 4e^t & 0 \\ 0 & -2e^{-t} \end{pmatrix}, \quad X_0^{(2)} = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix},
\]

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It is easily seen that the conditions of Theorem 1 are met. Indeed, the matrix \( W_3(t) = \begin{pmatrix} -1 & 2e^t \\ -e^{-t} & 1 \end{pmatrix}, \quad V_3(t) = \begin{pmatrix} 6e^t & 0 \\ 0 & 3e^{-t} \end{pmatrix} \).

Observe that, unlike the original equation (1), the equation in (17) is an equation with constant coefficients. Direct calculations lead to the following result:

\[
V(t) = V_3(t) - \lambda_2(t)V_2(t) - \lambda_1(t)V_1(t) - \lambda_0(t)V_0(t) = V_3(t) - V_1(t) = \begin{pmatrix} 4e^t & -2 \\ -2 & 2e^{-t} \end{pmatrix}
\]

is symmetric for all \( t \in \mathbb{R} \). Hence, \( X(t) \) is symmetric for all \( t \in \mathbb{R} \).

Let us find \( X(t) \). According to Lemma 2, \( X(t) \) can be found as a solution to the initial value problem

\[
X''' - X' = V(t), \quad X(0) = 0, \quad X'(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad X''(0) = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}.
\] (17)

Observe that, unlike the original equation (1), the equation in (17) is an equation with constant coefficients. Direct calculations lead to the following result:

a) the function \( X_{11}(t) = 2te^t \) is the solution to the initial value problem

\[
X_{11}''' - X_{11} = 4e^t, \quad X_{11}(0) = 0, \quad X_{11}'(0) = 2, \quad X_{11}''(0) = 4,
\]

b) the function \( X_{12}(t) = 2t \) is the solution to the initial value problem

\[
X_{12}''' - X_{12} = -2, \quad X_{12}(0) = 0, \quad X_{12}'(0) = 2, \quad X_{12}''(0) = 0,
\]

c) the function \( X_{22}(t) = te^{-t} \) is the solution to the initial value problem

\[
X_{22}''' - X_{22} = 2e^{-t}, \quad X_{22}(0) = 0, \quad X_{22}'(0) = 1, \quad X_{22}''(0) = -2.
\]

Hence, \( X(t) \) acquires the form

\[
X(t) = \begin{pmatrix} 2te^t & 2t \\ 2t & te^{-t} \end{pmatrix}.
\]

Note, as a final remark, that the formula (15) can be used to estimate bounds of solutions to the equation (1). It can be shown that, in some cases, estimates obtained by the proposed approach are more precise than those obtained by the classical techniques.

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