RESIDUAL INTERSECTIONS AND SOME APPLICATIONS

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ABSTRACT. We give a new residual intersection decomposition for the refined intersection products of Fulton-MacPherson. Our formula refines the celebrated residual intersection formula of Fulton, Kleiman, Laksov, and MacPherson. The new decomposition is more likely to be compatible with the canonical decomposition of the intersection products and each term in the decomposition thus has simple geometric meaning. Our study is motivated by its applications to some geometric problems. In particular, we use the decomposition to find the distribution of limiting linear subspaces in degenerations of hypersurfaces. A family of identities for characteristic classes of vector bundles is also obtained as another consequence.

1. Introduction

Given a closed regular embedding of codimension $d$

$$i : X \to Y$$

and a morphism

$$f : V \to Y$$

with $V$ a purely $k$-dimensional scheme, the fundamental construction of Fulton-MacPherson [F] defines the refined intersection product

$$X \cdot V \in A_{k-d}(W), \quad W = f^{-1}(X),$$

where $A_{k-d}(W)$ is the $(k - d)$-th Chow group of $W$. Let $N$ be the pull-back of the normal bundle $N(X,Y)$ of $X$ in $Y$ to $W$ and $c(N)$ be its Chern class. Then $X \cdot V$ can be expressed in terms of $c(N)$ and the Segre class $s(W,V)$ of $W$ in $V$ by

$$X \cdot V = \{c(N) \cap s(W,V)\}_{k-d}.$$ 

Furthermore, there is a canonical decomposition

$$(1.1) \quad X \cdot V = \sum_j \alpha_j,$$

where $\alpha_j$ are classes supported on the so-called distinguished varieties $Z_j$ of $X \cdot V$. It is well-known that every irreducible component of $W$ is a distinguished variety of $X \cdot V$. The canonical decomposition gives all the important information about $X \cdot V$. However, it is in general very difficult to find such a decomposition.

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There are many other ways to decompose $X \cdot V$ and in many applications a less comprehensive decomposition will be enough. Given a closed subscheme $Z$ of $W$, a basic problem of residual intersections is to decompose $X \cdot V$ into the sum of a class supported on $Z$ and a class supported on a "residual" set of $W$ with respect to $Z$. The celebrated residual intersection formula of Fulton, Kleiman, Laksov, and MacPherson gives such a decomposition

$$X \cdot V = \{ (c(N) \cap s(Z, V)) \}_{k-d} + \mathbb{R} =: M_Z + \mathbb{R},$$

where $s(Z, V)$ is the Segre class of $Z$ in $V$. The construction of the residual intersection class $\mathbb{R}$ is more involved and one can refer to [F] for details. This decomposition is much easier to compute than the canonical decomposition and the formula has many important applications. Please see Chapter 9 of [F] for further references on the extensive literature and numerous results on residual intersections.

If $Z$ is a connected component of $W$, then $M_Z$ in (1.2) is nothing but the term (or the sum of terms) in the canonical decomposition (1.1) supported on $Z$ and $\mathbb{R}$ is the sum of all other terms in the right-hand side of (1.1). However, those two decompositions are not compatible in general. For example, assume that $W$ is the union of two irreducible components $Z$ and $R$ and there are no contributions to $X \cdot V$ other than those from these two components. In this case, both decompositions have two terms which are supported on the same varieties $Z$ and $R$, respectively. However, they are generally not the same unless $Z \cap R$ has a dimension less than $k-d$. In fact, there seems to be no good coordination between two decompositions in general. Consider the case when $N$ is generated by its sections. Then it is well-known that all the terms in (1.1) must be represented by non-negative cycles. Furthermore, they can be interpreted geometrically as limits by using dynamic intersections and deformation theory. On the other hand, as we will see in Section 4 (Example 4.14), both terms in the right-hand side of (1.2) can be negative (of course, not at the same time). Therefore, (1.2) cannot be obtained by shuffling terms of (1.1) and it is hard to interpret the formula geometrically. One cause of this is easy to see, since two terms in the right-hand side of (1.2) are not equally weighted. The formula for $M_Z$ uses no information about the residual set $R$ while construction of $\mathbb{R}$ depends on structures of both $Z$ and $R$. In a sense, (1.2) packs all the complication into $\mathbb{R}$ to make $M_Z$ as simple as possible. While this is very useful in many applications, it also makes the formula less favorable in other situations.

In this paper, we propose another way to decompose $X \cdot V$ into the sum of a class supported on $Z$ and a class supported on a "residual" subscheme of $W$ with respect to $Z$. Our decomposition is somehow "symmetric". This makes it more likely to be compatible with the canonical decomposition and the decomposition can thus be used to find the equivalence of $Z$ for $X \cdot V$. In fact, in the case that (1.1) has only two terms as mentioned in the last paragraph, our decomposition gives the actual canonical decomposition. In particular, it has nice geometric meaning as the distribution of the limits. Of course, there is a price to be paid for this. Mainly, both terms in the decomposition are now equally complicated. However, it is still manageable and can be computed (with the help of computers) in many cases. We are particularly encouraged by its applications for some geometric problems that initially motivated this study.

This paper is organized as follows. In Section 2, one can find basic formulas for our new decomposition (Theorem 2.4 and Corollary 2.10). The idea behind is very
simple. If $W$ is the sum of two divisors in $V$, then there is a very natural way to decompose $X \cdot V$ symmetrically into the sum of two classes that are supported on the divisors. In general, if a closed subscheme $Z$ of $W$ is given, one can define a residual scheme $R(Z)$ to $Z$ in $W$ such that

\begin{equation}
W = Z \cup R(Z).
\end{equation}

We then decompose $X \cdot V$ into the sum of a class supported on $Z$ and a class supported on $R(Z)$ by reducing to the case of divisors using blow-ups. Our formulas are given in such a way so that they can be easily compared with the standard residual intersection formula (1.2) of Fulton, Kleiman, Laksov, and MacPherson. In particular, we see that the new decomposition refines the standard one in a sense that will be made clear in Section 2.

A closer look of the new decomposition is given in Section 3. In general, $R(R(Z))$ may not be equal to $Z$. Therefore, (1.3) and hence the decomposition of $X \cdot V$ given in Section 2 will not be symmetric in general. Even in the case that (1.3) is symmetric, our decomposition of $X \cdot V$ still depends on the order of two blow-ups needed in the process (see Example 3.1). In fact, as in the standard decomposition (1.2), it can still happen that neither term in the new one is related to the scheme structure of $R(Z)$. In spite of all of that, the new intersection decomposition does behave well for many interesting decompositions

\begin{equation*}
W = Z_1 \cup Z_2
\end{equation*}

of $W$ (Theorem 3.4). In particular, we obtain some new formulas which express our decomposition of $X \cdot V$ in a truly symmetric form in terms that are related to the scheme structures of $Z_1$ and $Z_2$ under some modest hypotheses (Theorem 3.6 and Corollary 3.16).

The main motivation that inspired our search for a refined residual intersection decomposition is its geometric meaning. This and some applications of the new decomposition are studied in Section 4. If we interpret the refined intersection product $X \cdot V$ as the class of the limits derived from deformation theory, then our decomposition tells us how the limits are distributed in different components. In particular, we use it to study degenerations of hypersurfaces in $\mathbb{P}^n$ and their limiting $\mathbb{P}^r$'s (Proposition 4.1). This not only provides a much simpler and intrinsic way to recapture some results of [W1], [W2], and [W3], but also yields new results which would be very difficult to obtain from deformation theory directly. Actual calculations of some examples can also be found there. As another application, we obtain a family of identities for characteristic classes of vector bundles.

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**Blanket Conventions.** To simplify the notation, we will make the following conventions for this paper. Hopefully, the meaning will be clear from the content.

1. A class in the Chow group of a subspace will often be identified with its image in the Chow group of a bigger space without warning.
2. The same notation will be used to denote a vector bundle and its restriction to any subspace.
(3) Similarly, the same notation will be used to denote a morphism and its restriction to any subspace.

(4) Finally, the same notation will be used to denote a divisor, the corresponding line bundle, and the first Chern class of the bundle.

2. A refined residual intersection formula

Let us first recall the basic construction of refined intersection products of Fulton-MacPherson from [F]. Consider the fiber square

\[
\begin{array}{ccc}
W & \longrightarrow & V \\
\downarrow f & & \downarrow f \\
X & \longrightarrow & Y,
\end{array}
\]

where

\[i : X \to Y\]

is a closed regular embedding of codimension \(d\),

\[f : V \to Y\]

is a morphism with \(V\) a purely \(k\)-dimensional scheme, and \(W\) is the inverse image scheme \(f^{-1}(X)\). From this, the fundamental construction of Fulton-MacPherson defines the refined intersection product \(X \cdot V\) in the \((k-d)\)-th Chow group \(A_{k-d}(W)\) of \(W\) by embedding the normal cone \(C_{WV}\) of \(W\) in \(V\) into the vector bundle \(N\), where \(N\) is the pull-back of the normal bundle \(N(X,Y)\) of \(X\) in \(Y\) to \(W\). Let \(c(N)\) be the Chern class of \(N\) and \(s(W,V)\) be the Segre class of \(W\) in \(V\). Then \(X \cdot V\) can be expressed as

\[
X \cdot V = \{c(N) \cap s(W,V)\}_{k-d}.
\]

Given a closed subscheme \(Z\) of \(W\), our goal in this section is, for a suitable decomposition

\[W = Z \cup Z'\]

of \(W\), to decompose \(X \cdot V\) in a natural way as the sum of two classes \(R_Z\) and \(R_{Z'}\) that are supported on \(Z\) and \(Z'\), respectively. To simplify the notation, we will assume that \(W\) has at least codimension 1 in \(V\). The case of \(W = V\) is simple since we then have

\[s(W,V) = [V] \quad \text{and} \quad X \cdot V = c_{top}(N) \cap [V].\]

Consider first the case that \(Z\) is a divisor in \(V\). Let \(R(Z)\) be the residual scheme to \(Z\) in \(W\). To be more precise, in general, given a closed subscheme \(Z\) of \(W\), we define the residual subscheme \(R(Z)\) to \(Z\) in \(W\) with respect to \(V\) to be the subscheme of \(V\) defined by the ideal sheaf

\[\mathcal{I}_{R(Z)} = \text{Ann}(\mathcal{I}_Z/\mathcal{I}_W),\]

where \(\mathcal{I}_Z\) and \(\mathcal{I}_W\) are ideal sheaves of \(Z\) and \(W\) in \(V\), respectively. We hence have set-theoretically

\[
W = Z \cup R(Z).
\]
Moreover, in the case that $Z$ is a divisor in $V$, (2.3) gives actually a scheme-theoretic decomposition of $W$ in the sense that

$$I_W = I_Z \cdot I_{R(Z)}.$$

For a given vector bundle $E$, the standard notation $c_i(E)$ for the $i$-th Chern class of $E$ and $s_i(E)$ for the $i$-th Segre class of $E$ will be used. Recall also that we will use the same notation to denote a divisor and its corresponding line bundle. We are now ready to propose the following residual intersection decomposition of $X \cdot V$.

**Theorem 2.4.** Consider the following expansion of the fiber square (2.1)

$$\begin{array}{ccc}
R & \rightarrow & W \\
\downarrow & & \downarrow f \\
D & \rightarrow & V \\
\downarrow f & & \downarrow \\
X & \rightarrow & Y,
\end{array}$$

where the subscheme $D$ of $W$ is a divisor in $V$ and $R$ is the residual scheme to $D$ in $W$. Let $R_D$ and $R_R$ be two classes supported on $D$ and $R$ defined by

(2.6)

$$R_D = \{ c(N) \cap s(D, V) \}_{k-d} + \sum_{i=0}^{d-2} \sum_{j=1}^{d-1-i} \left( \binom{d-1-i}{j} c_i(N) s_{d-i-j}(D) \cap s_{k-j}(R, V) \right)$$

and

(2.7)

$$R_R = \{ c(N) \cap s(R, V) \}_{k-d} + \sum_{i=0}^{d-2} \sum_{j=1}^{d-1-i} \left( \binom{d-1-i}{j} c_i(N) s_j(D) \cap s_{k-d+i+j}(R, V) \right),$$

respectively. Then the refined intersection product $X \cdot V$ can be decomposed into

(2.8)

$$X \cdot V = R_D + R_R.$$

**Remark.** There are many different ways to express $R_D$ and $R_R$ and they may serve some particular needs better than the forms given above. In the proof given below, one can find some different formulas of $R_D$ and $R_R$ that are less messy. We choose the formulas (2.6) and (2.7) not only because they are very explicit but also to compare our decomposition with the standard residual intersection decomposition (1.2) as given in [F].

**Proof of Theorem 2.4.** By our assumption on $W$, $R$ is a subscheme of codimension at least 1 in $V$. Consider the blow-up of $V$ along $R$

$$b: \widetilde{V} \rightarrow V$$

Let

$$\widetilde{W} = b^{-1}(W), \quad \widetilde{D} = b^{-1}(D) = b^*(D), \quad \text{and} \quad \widetilde{R} = b^{-1}(R).$$
be the exceptional divisor. Consider the following fiber diagram:

\[
\begin{array}{ccc}
\tilde{W} & \longrightarrow & \tilde{V} \\
\downarrow b & & \downarrow b \\
W & \longrightarrow & V \\
\downarrow f & & \downarrow f \\
X & \longrightarrow & iY,
\end{array}
\]

where the low half of the diagram is just the fiber square (2.1). By the push-forward formula [Theorem 6.2, F], we have

\[b^*(X \cdot \tilde{V}) = b^!(i^!(\tilde{V})) = i^!(b^!(\tilde{V})) = i^!(\tilde{V}) = X \cdot V,\]

where \(i^!\) is the refined Gysin homomorphism defined from refined intersections. Notice that \(\tilde{W}\) is now a divisor in \(\tilde{V}\) and there is a very natural way to decompose \(X \cdot \tilde{V}\) symmetrically with respect to two components \(\tilde{D}\) and \(\tilde{R}\) of \(\tilde{W}\). In fact,

\[
X \cdot V = b^*(X \cdot \tilde{V}) \\
= b^*(\{c(b^*N) \cap s(\tilde{D} + \tilde{R}, \tilde{V})\}_{k-d}) \\
= b^*(\sum_{i=0}^{d-1} c_i(b^*N) \cap s_{k-d-i}(\tilde{D} + \tilde{R}, \tilde{V})) \\
= b^*(\sum_{i=0}^{d-1} c_i(b^*N)(-\tilde{D} - \tilde{R})^{d-1-i} \cap ([\tilde{D} + \tilde{R}]) \\
= \sum_{i=0}^{d-1} c_i(N) \cap b^*((-\tilde{D} - \tilde{R})^{d-1-i} \cap [\tilde{D}]) + \sum_{i=0}^{d-1} c_i(N) \cap b^*((-\tilde{D} - \tilde{R})^{d-1-i} \cap [\tilde{R}]) \\
=: \mathbb{R}_D + \mathbb{R}_R.
\]

Before going further, let us point out that the formulas above are already in computable forms in many cases. They are truly symmetric with respect to \(\tilde{D}\) and \(\tilde{R}\). The drawback is that they are expressed indirectly by push forwards.
We will now examine each term more closely. Let us start with $\mathbb{R}_R$.

\[
\mathbb{R}_R = \sum_{i=0}^{d-1} c_i(N) \cap b_* ((-\tilde{D} - \tilde{R})^{d-1-i} \cap [\tilde{R}])
\]

\[
= \sum_{i=0}^{d-1} c_i(N) \cap b_* \left( \sum_{j=0}^{d-1-i} \binom{d-1-i}{j} (-\tilde{D})^j (-\tilde{R})^{d-1-i-j} \cap [\tilde{R}] \right)
\]

\[
= \sum_{i=0}^{d-1} c_i(N) \cap b_* \left( \sum_{j=0}^{d-1-i} \binom{d-1-i}{j} (-b^* D)^j \cap (-\tilde{R})^{d-1-i-j} \cap [b^* R] \right)
\]

\[
= \sum_{i=0}^{d-1} \sum_{j=0}^{d-1-i} \binom{d-1-i}{j} c_i(N) s_j(D) \cap s_{k-d+i+j}(R, V)
\]

\[
= \sum_{i=0}^{d-1} c_i(N) \cap s_{k-d+i}(R, V) + \sum_{i=0}^{d-2} \sum_{j=1}^{d-1-i} \binom{d-1-i}{j} c_i(N) s_j(D) \cap s_{k-d+i+j}(R, V)
\]

\[
= \{c(N) \cap s(R, V)\}_{k-d} + \sum_{i=0}^{d-2} \sum_{j=1}^{d-1-i} \binom{d-1-i}{j} c_i(N) s_j(D) \cap s_{k-d+i+j}(R, V).
\]

Similarly, we can get the formula for $\mathbb{R}_D$ as follows.

\[
\mathbb{R}_D = \sum_{i=0}^{d-1} c_i(N) \cap b_* ((-\tilde{D} - \tilde{R})^{d-1-i} \cap [\tilde{D}])
\]

\[
= \sum_{i=0}^{d-1} c_i(N) \cap b_* \left( \sum_{j=0}^{d-1-i} \binom{d-1-i}{j} (-\tilde{R})^j (-\tilde{D})^{d-1-i-j} \cap [\tilde{D}] \right)
\]

\[
= \sum_{i=0}^{d-1} c_i(N) \cap s_{k-d+i}(D, V) + \sum_{i=0}^{d-2} \sum_{j=1}^{d-1-i} \binom{d-1-i}{j} c_i(N) b_* ((-\tilde{R})^{j-1} (-\tilde{D})^{d-i-j} \cap [\tilde{D}])
\]

\[
= \{c(N) \cap s(D, V)\}_{k-d} + \sum_{i=0}^{d-2} \sum_{j=1}^{d-1-i} \binom{d-1-i}{j} c_i(N) s_{d-i+j}(D) \cap s_{k-j}(R, V).
\]

This completes our proof. \(\square\)

In general, if a closed subscheme $Z$ of $W$ is given, we can blow up $V$ along $Z$ to reduce to the case of a divisor in the blow-up $\tilde{V}$. We then obtain our decomposition by pushing forward the decomposition of $X \cdot \tilde{V}$ as stated in Theorem 2.4.

**Corollary 2.10.** Let $X \cdot V$ be the refined intersection product defined from the fiber square (2.1). Given a closed subscheme $Z$ of $W$, let

\[
\pi : \tilde{V} \to V
\]

be the blow-up of $V$ along $Z$ and $D$ be the exceptional divisor. Furthermore, let $\tilde{R}$ be the residual scheme to $D$ in $\tilde{V}$ and $R$ be the image of $\tilde{R}$ under $\pi$. Then $X \cdot V$ can be decomposed into

\[
X \cdot V = \mathbb{R}_D + \mathbb{R}_R
\]
where \( \mathbb{R}_Z \) and \( \mathbb{R}_R \) be two classes supported on \( Z \) and \( R \) defined by

\[
(2.12) \quad \mathbb{R}_Z = M_Z + A_Z,
\]

\[
(2.13) \quad M_Z = \{ c(N) \cap s(Z, V) \}_{k-d} = \sum_{i=0}^{d} c_i(N) \cap s_{k-d+i}(Z, V),
\]

\[
(2.14) \quad A_Z = \sum_{i=0}^{d-2} \sum_{j=1}^{d-1-i} \binom{d-1-i}{j} c_i(N) \cap \pi_*(s_{d-i-j}(D) \cap s_{k-j}({\tilde{R}}, {\tilde{V}}))
\]

and

\[
(2.15) \quad \mathbb{R}_R = M_R + A_R,
\]

\[
(2.16) \quad M_R = \{ c(N) \cap \pi_*(s({\tilde{R}}, {\tilde{V}})) \}_{k-d} = \sum_{i=0}^{d} c_i(N) \cap \pi_*(s_{k-d+i}({\tilde{R}}, {\tilde{V}}),
\]

\[
(2.17) \quad A_R = \sum_{i=0}^{d-2} \sum_{j=1}^{d-1-i} \binom{d-1-i}{j} c_i(N) \cap \pi_*(s_j(D) \cap s_{k-d+j}({\tilde{R}}, {\tilde{V}}))
\]

respectively.

**Remark.** Let \( R(Z) \) be the residual scheme to \( Z \) in \( W \) as defined earlier. Then \( R(Z) \) may not be equal to \( R \) even as a set. However, \( R \) is always contained in \( R(Z) \) and \( \mathbb{R}_R \) can hence be considered as a class supported on \( R(Z) \) as well. A more detailed study on this matter will be conducted in the next section.

In closing of this section, we observe that each term in our decomposition as stated in Corollary 2.10 is expressed in two terms itself. In particular, we see that \( \mathbb{R}_Z \) contains two parts. One part \( M_Z \) is nothing but the very term that appears in the standard residual intersection formula (1.2) and we will call it the main term. The other more complicated one \( A_Z \) is a class supported on \( Z \cap R \) and we will call it the adjunct term. While this term is packed in the residual intersection class \( \mathbb{R} \) in the standard residual decomposition (1.2), it is now squeezed out in the new decomposition as we try to somehow even up the distribution. In this sense, the new decomposition gives a refinement of the standard decomposition (1.2).

### 3. The decomposition in symmetric forms

The idea behind the residual decomposition given in the last section is very simple. We see that if \( W \) is the sum of two divisors of \( V \), then there is a very natural way to decompose \( X \cdot V \) into the sum of two classes that are supported on the divisors. The decomposition in general cases is then obtained by reducing to the case of divisors using blow-ups. In general, we need two blow-ups to do that and the order matters. In other words, although the new decomposition has a more...
balanced distribution between its terms, it is still not symmetric with respect to the residual decomposition (2.3)

\[ W = Z \cup R(Z) \]

of \( W \). This is not surprising, of course, since the above decomposition of \( W \) itself is not symmetric. We do not in general have that \( R(R(Z)) \) is equal to \( Z \). However, that is not the main obstacle. In fact, even if the decomposition of \( W \) above is symmetric, our decomposition of \( X \cdot V \) may still depends on which component to be chosen first. Indeed, let \( \pi \) be the blow-up of \( V \) along \( Z \) and \( D \) be the exceptional divisor. In general, the image \( \pi(R(D)) \) is not equal to \( R(Z) \) even as a set, where \( R(D) \) is the residual scheme to \( D \) in \( \pi^{-1}(W) \). Therefore, the second term \( \mathbb{R}_R \) in our decomposition is not expressed in terms of (and may indeed not be related to) the scheme structure of \( R(Z) \). The following simple example illustrates that.

**Example 3.1.** Let \( A \) and \( B \) be lines in \( \mathbb{P}^2 \) defined by

\[ x = 0 \quad \text{and} \quad y = 0, \]

respectively. Consider the following fiber square

\[
\begin{array}{ccc}
W & \xrightarrow{f} & V = \mathbb{P}^2 \\
\downarrow{f} & & \downarrow{f} \\
X = 2A \times 2B & \xrightarrow{i} & Y = \mathbb{P}^2 \times \mathbb{P}^2,
\end{array}
\]

where \( f \) is the diagonal embedding. Then

\[ \mathcal{I}_W = \{x^2, y^2\} \quad \text{and} \quad X \cdot V = 4p, \]

where

\[ p = A \cap B \quad \text{and} \quad \mathcal{I}_p = \{x, y\}. \]

It is easy to see that the residual scheme \( R(p) \) to \( p \) in \( W \) is defined by

\[ \mathcal{I}_{R(p)} = \{x^2, y^2, xy\} = \mathcal{I}_p^2. \]

Furthermore, we do have

\[ R(R(p)) = p. \]

Therefore, if we take

\[ Z_1 = p \quad \text{and} \quad Z_2 = R(p) \]

then we have a “symmetric” decomposition

\[
W = Z_1 \cup Z_2
\]

of \( W \). (Notice that the multiplicity of \( W, Z_1, \) and \( Z_2 \) are 4, 1, and 3, respectively.) However, our decomposition of \( X \cdot V \) into the sum of two classes that are supported on \( Z_1 \) and \( Z_2 \) will depend on which \( Z_i \) to be blown up first. Let \( \pi_i \) be the blow-ups of \( W \) along \( Z_i \) and \( D_i \) be the exceptional divisors. Then it is easy to see that the
residual scheme to $D_1$ in $\pi_1^{-1}(W)$ is $D_1$ itself and the residual scheme to $D_2$ in $\pi_2^{-1}(W)$ is the empty set. The corresponding decompositions are therefore

$$X \cdot V = 2p + 2p \quad \text{and} \quad X \cdot V = 4p + 0,$$

respectively. (As a comparison, the corresponding decompositions using the standard residual intersection formula (1.2) are

$$X \cdot V = p + 3p \quad \text{and} \quad X \cdot V = 4p + 0,$$

respectively.)

While the examples such as one given above make things more interesting, they also suggest that the decomposition for $X \cdot V$ might behave “better” if the decomposition of $W$ itself is “nicer”. As it turns out, this is indeed the case and the decompositions of $W$ in terms of residual schemes are misleading. In the following, we will show that, for some reasonable decompositions of $W$ as the union of two subschemes $Z_1$ and $Z_2$, our decomposition of $X \cdot V$ as given in Section 2 is actually independent of which component $Z_i$ to be chosen first. We will derive our formulas in truly symmetric forms in terms that are related to the scheme structures of $Z_1$ and $Z_2$.

In many applications, one is mainly interested in finding the contributions to $X \cdot V$ from certain components of $W$. In this case, the most direct way to reduce to the case of divisors is to blow up $V$ along $W$. In fact, this is basically how $X \cdot V$ is defined. This simple observation motivates the following theorem.

**Theorem 3.4.** Let

$$\pi: \tilde{V} \to V$$

be the blow-up of $V$ along $W$. Assume that a decomposition of

$$W = Z_1 \cup Z_2$$

is given such that the exceptional divisor

$$E = \pi^{-1}(W) = \tilde{E}_1 + \tilde{E}_2,$$

where $\tilde{E}_1$ and $\tilde{E}_2$ are inverse image schemes of $Z_1$ and $Z_2$, respectively, then our refined residual decomposition of $X \cdot V$ as defined in Section 2 is independent of the choice of which $Z_i$ to be considered first.

In fact, we will show something stronger. Let

$$\pi_1: V_1 \to V$$

be the blow-up of $V$ along $Z_1$ and

$$E_1 = \pi_1^{-1}(Z_1)$$

be the exceptional divisor. Furthermore, let

$$b: V' \to V$$

be the blow-up of $V_1$ along $R_1$, where $R_1$ is the residual scheme to $E_1$ in $\pi_1^{-1}(W)$, and

$$b_1^{-1}(\pi_1^{-1}(W)) = b_1^{-1}(E_1 \cup R_1) = E_1' + R_1',$$

where $E_1'$ is the inverse image of $E_1$ under $b_1$ and $R_1'$ is the exceptional divisor. Similarly, we define blow-ups

$$\pi_2: V_2 \to V \quad \text{and} \quad b_2: V_2' \to V_2$$

and $E_2, R_2, E_2'$, and $R_2'$, respectively. Putting all those blow-ups together, we consider the following fiber diagram:

$$\begin{align*}
E_1' + R_1' &\longrightarrow V_1' \\
E_1 \cup R_1 &\longrightarrow V_1 \\
E_2' + R_2' &\longrightarrow V_2' \\
E_2 \cup R_2 &\longrightarrow V_2
\end{align*}$$

$$\begin{array}{c}
\tilde{E}_1 + \tilde{E}_2 \longrightarrow \tilde{V} \\
\pi \downarrow \\
Z_1 \cup Z_2 \longrightarrow V \\
f \downarrow \\
X \longrightarrow Y
\end{array}$$

(3.5)

where $p$ and $q$ are the unique morphisms determined by the universal property of blow-ups for $\pi$. From the diagram above, there are three different ways to decompose $X \cdot V$ into the sum of two classes supported on $Z_1$ and $Z_2$ corresponding to three push-forwards of the natural decompositions of $X \cdot V_1', X \cdot \tilde{V}$, and $X \cdot V_2'$ as defined in Section 2. We can now state the following stronger version of Theorem 3.4

**Theorem 3.6.** All three procedures as described above give rise to the same decomposition of $X \cdot V$. In particular, the decomposition of $X \cdot V$ as defined in Section 2 can be expressed in the following symmetric form

(3.7) $$X \cdot V = \mathbb{R}_{Z_1} + \mathbb{R}_{Z_2}$$

and $\mathbb{R}_{Z_1}$ and $\mathbb{R}_{Z_2}$ are the classes supported on $Z_1$ and $Z_2$ defined by

(3.8) $$\mathbb{R}_{Z_l} = \{c(N) \cap s(Z_l, V)\}_{k-d} + A_{Z_l},$$

where $A_{Z_l}$ are the adjunct terms given by

(3.9) $$A_{Z_l} = \sum_{i=0}^{d-2} \sum_{j=1}^{d-1-i} \binom{d-1-i}{j} c_i(N) \cap \pi_{l*}(s_{d-i-j}(E_l) \cap s_{k-j}(R_l, V_l)),$$

for $l$ equal to 1 and 2, respectively.
Remarks. (1) It is not required that \( Z_1 \) and \( Z_2 \) are residual schemes to each other in \( W \). However, it is easy to see that the hypothesis of the theorem above does imply that, for \( l \) equal to 1 and 2, the residual schemes \( R_l \) to \( E_l \) in \( \pi_l^{-1}(W) \) are equal to the inverse image schemes \( \pi_l^{-1}(Z_l) \), respectively, where

\[
\hat{1} = 2 \quad \text{and} \quad \hat{2} = 1.
\]

The above notation \( \hat{l} \) will be use from time to time in this paper.

(2) Theorem 3.6 allows us to write down many different formulas for our residual decomposition. The form given in the statement above is a mixture of the formulas obtained from two of the three procedures. A more natural way to express the decomposition symmetrically is to use the following formulæ obtained directly using the blow-up \( \pi \). For \( l \) equal to 1 and 2,

\[
\begin{align*}
R_{Z_l} & = b_*(c_{d-1}(\pi^*N - \tilde{E}_1 \otimes \tilde{E}_2) \cap [\tilde{E}_l]) \\
& = \sum_{i=0}^{d-1} \sum_{j=0}^{d-1-i} \binom{d-1-i}{j} c_i(N) \cap \pi_*((-\tilde{E}_1)^j(-\tilde{E}_2)^{d-1-i-j} \cap [\tilde{E}_l]) \\
& = \{c(N) \cap s(Z_l, V)\}_{k-d} + A_{Z_l},
\end{align*}
\]

where \( A_{Z_l} \) are adjunct terms defined by

\[
(3.12) \quad A_{Z_l} = \sum_{i=0}^{d-2} \sum_{j=1}^{d-1-i} \binom{d-1-i}{j} c_i(N) \cap \pi_*((-\tilde{E}_l)^j(-\tilde{E}_l)^{d-1-i-j} \cap [\tilde{E}_l]).
\]

Since they are somehow derived directly from the definition of \( X \cdot V \), they might be more useful in some applications.

Proof of Theorem 3.6. As defined in Theorem 2.4, let

\[
X \cdot V_1' = \mathbb{R}_1 + \mathbb{R}_1'
\]

be the natural decomposition of \( X \cdot V_1' \) with respect to \( E_1' \) and \( R_1' \),

\[
X \cdot \tilde{V} = \tilde{\mathbb{R}}_1 + \tilde{\mathbb{R}}_2
\]

be the natural decomposition of \( X \cdot \tilde{V} \) with respect to \( \tilde{E}_1 \) and \( \tilde{E}_2 \), and

\[
X \cdot V_2' = \mathbb{R}_2 + \mathbb{R}_2'
\]

be the natural decomposition of \( X \cdot V_2' \) with respect to \( E_1' \) and \( R_2' \). Since

\[
X \cdot V = (\pi_1 b_1)_*(X \cdot V_1') = \pi_*(X \cdot \tilde{V}) = (\pi_2 b_2)_*(X \cdot V_2')
\]

and

\[
p_*(X \cdot V_1') = X \cdot \tilde{V} = q_*(X \cdot V_2'),
\]

it is enough to show that

\[
p_*(\mathbb{R}_1) = \tilde{\mathbb{R}}_1 \quad \text{and} \quad q_*(\mathbb{R}_2) = \tilde{\mathbb{R}}_2.
\]
By the symmetric, it is enough to show one of them as follows.

$$p_*(\mathbb{R}_{Z_1}) = p_*(\{c((\pi_1 b_1)^* N - E'_1 \otimes R'_1) \cap [E'_1]_{k-d}\})$$

$$= p_*(\{c((\pi p)^* N - p^*(\tilde{E}_1 \otimes \tilde{E}_2)) \cap [E'_1]_{k-d}\})$$

$$= p_*(\{c(p^*(\pi^* N - \tilde{E}_1 \otimes \tilde{E}_2)) \cap [E'_1]_{k-d}\})$$

$$= \{c(\pi^* N - \tilde{E}_1 \otimes \tilde{E}_2) \cap [E'_1]_{k-d}\}$$

$$= \{c(\pi^* N - \tilde{E}_1 \otimes \tilde{E}_2) \cap [E'_1]_{k-d}\} = \mathbb{R}_1.$$

\[\square\]

The hypothesis of Theorem 3.4 (or Theorem 3.6) is not unreasonable and is often satisfied in many applications. Given a subscheme $Z_1$ of $W$, we are free to pick up a $Z_2$ such that the hypothesis are satisfied, since they do not have to be the residual schemes to each other. For instance, for $Z_1$ equal to $p$ or $R(p)$ in Example 3.1, we can take $Z_2$ to be $p$ or $\emptyset$, respectively. Theorem 3.6 can then be applied. Another useful case is when $Z_1$ and $Z_2$ are complementary components in $W$. Roughly speaking, if a component $Z_1$ of $W$ is given, we then take $Z_2$ to be the union of all irreducible components of $W$ not appeared in $Z_1$. It is easy to see that $Z_1$ and $Z_2$ do form the residual subschemes with respect to each other in $W$ in this case.

As an immediate consequence of the theorem, we see that the new decomposition behaves well with respect to the canonical decomposition of $X \cdot V$. For example, the following corollary will be useful for our applications in Section 4.

**Corollary 3.13.** If $Z_1$ and $Z_2$ are irreducible components of $W$ and there is no contribution to $X \cdot V$ from $Z_1 \cap Z_2$, then the refined residual intersection decomposition coincides with the canonical decomposition.

Another interesting case is when dimensions of intersections of distinguished varieties are small. Since the adjunct term in each $\mathbb{R}_{Z_i}$ is a class supported on the intersection of $Z_1$ and $Z_2$, that complicated part will be zero if the dimension of the intersection is less than $k-d$. In this case, only $M_{Z_i}$ are left and our decomposition becomes very simple. In other words, when the dimension of the intersection of two components of an intersection product is less than the dimension of the product itself, we may treat them as they were connected components disjoint to each other. In general, the following corollary follows easily by the induction.

**Corollary 3.14.** Let $X \cdot V$ be the refined intersection product defined from the fiber square (2.1)

$$\begin{array}{ccc}
W & \longrightarrow & V \\
\downarrow f & & \downarrow f \\
X & \longrightarrow & Y \\
\end{array}$$

and $W_i$’s be irreducible components of $W$. If for any pair of components $W_i$ and $W_j$, $i \neq j$,

$$\text{codim}(W_i \cap W_j, V) > \text{codim}(X, Y) = d,$$

then the canonical decomposition of $X \cdot V$ is given by

$$X \cdot V = \sum_{i} \{c(N) \cap s(W_i, V)\}_{k-d}. \quad (3.15)$$
Remarks. (1) In general, \( X \cdot V \) may have many additional distinguished varieties other than \( W_i \)’s. Just assuming all contributions not from \( W_i \)’s to be zero is not sufficient for formula (3.15) to hold since that does not imply the vanishing of the adjunct terms as we will see from examples in Section 4.

(2) The formula (3.15) also follows easily from the fiber diagram below:

\[
\begin{array}{ccc}
W' & \longrightarrow & V' \\
p\downarrow & & \downarrow p \\
W & \longrightarrow & V \\
f\downarrow & & \downarrow f \\
X & \longrightarrow_i & Y,
\end{array}
\]

where

\[ V' = V - \cup_{i \neq j} (W_i \cap W_j), \quad W' = W - \cup_{i \neq j} (W_i \cap W_j), \]

and \( p \) is the open embedding.

If both \( Z_1 \) and \( Z_2 \) are regularly embedded in \( V \), then we can further reduce our formulas into a more explicit form that uses no push-forwards. Those formulas are useful in actual computations, since they involve mostly the characteristic classes of vector bundles.

**Corollary 3.16.** In the set-up of Theorem 3.6, assume further that \( Z_1 \) and \( Z_2 \) intersect properly (transversely). If both \( Z_1 \) and \( Z_2 \) are regularly embedded in \( V \) of codimensions \( r_1 \) and \( r_2 \) with the normal bundles \( N_1 \) and \( N_2 \), respectively, then we have the following new formulas for \( M_{Z_l} \) and \( A_{Z_l} \). For \( l \) equal to 1 and 2,

\[ M_{Z_l} = \{ c(N) \cap s(Z_l, V) \}_{k-d} = c_{d-r_1} (N - N_l) \cap [Z_l], \]

\[ A_{Z_l} = - \sum_{i=0}^{d-r_1-r_2} \sum_{j=r_l} \begin{pmatrix} d-1-i \\ j \end{pmatrix} c_i(N) s_j-r_l(N_l) s_{d-r_l-i-j}(N_l) \cap [Z_{int}], \]

where \( Z_{int} \) is the intersection of \( Z_1 \) and \( Z_2 \).

**Proof.** By the symmetry, it is enough to verify the new formulas for \( M_{Z_1} \) and \( A_{Z_1} \). The formula (3.17) for \( M_{Z_1} \) is easy to see, since

\[ s(Z_1, V) = s(N_1) \cap [Z_1] = c^{-1}(N_1) \cap [Z_1]. \]

To see the new formula for the adjunct term \( A_{Z_1} \), we will start from its formula given in Theorem 3.6. Notice from (3.9) that \( \pi_{1*} \) pushes a class supported on \( E_1 \) into a class supported on \( Z_{int} \). We hence may consider \( \pi_1 \) as the projection from the projective bundle \( E_1 = \mathbb{P}(N_1) \) to \( Z_1 \). We claim that

\[ E_1 \cap s(P_1, V) = \pi^*(s(N_1) \cap [Z_{int}]). \]
To see how to get the formula from the claim above, notice that \( Z \) has codimension \( r_1 + r_2 \) in \( V \). We will assume that it is less than \( d \), since \( A_{Z_1} \) is equal to zero otherwise. We thus have from (3.9) that

\[
A_{Z_1} = \sum_{i=0}^{d_2} \sum_{j=1}^{d_1-i} \binom{d-1-i}{j} c_i(N) \cap \pi_1^*(s_{d-i-j}(E_1) \cap s_{k-j}(R_1, V_1))
\]

\[= - \sum_{i=0}^{d_2} \sum_{j=1}^{d_1-i} \binom{d-1-i}{j} c_i(N) \cap \pi_1^*((-E_1)^{d-i-j-1} \cap (E_1 \cap s(R_1, V_1))_{k-j-1})
\]

\[= - \sum_{i=0}^{d_2} \sum_{j=1}^{d_1-i} \binom{d-1-i}{j} c_i(N) \cap \pi_1^*((-E_1)^{d-i-j-1} \cap \pi_1^*(s_{j-r_2}(N_2) \cap [Z_{int}]))
\]

\[= - \sum_{i=0}^{d_2} \sum_{j=1}^{d_1-i} \binom{d-1-i}{j} c_i(N)s_{d-r_1-i-j}(N_1)s_{j-r_2}(N_2) \cap [Z_{int}].
\]

Dropping the terms which contain any factor with a negative index, we then obtain the formula (3.18). We hence need only to verify the claim (3.19). To see this, notice from the hypotheses that

\[
\pi_1^*(s(N_2) \cap [Z_{int}]) = \pi_1^*(s(Z_{int}, Z_1)) = s(R_1 \cap E_1, E_1).
\]

For any pair \( X \) in \( Y \), let \( N(X, Y) \) be the normal cone to \( X \) in \( Y \). Restricting everything to \( R_1 \cap E_1 \) and notice that both \( N(E_1, V_1) \) and \( N(R_1 \cap E_1, R_1) \) are isomorphic to \( E_1 \), we have the following exact sequences

\[
0 \rightarrow N(R_1 \cap E_1, E_1) \rightarrow N(R_1 \cap E_1, V_1) \rightarrow E_1 \rightarrow 0
\]

and

\[
0 \rightarrow E_1 \rightarrow N(R_1 \cap E_1, V_1) \rightarrow N(R_1, V_1) \rightarrow 0.
\]

Therefore, continuing from (3.20), we get

\[
s(R_1 \cap E_1, E_1) = s(N(R_1 \cap E_1, E_1))
\]

\[= c(E_1) \cap s(N(R_1 \cap E_1, V_1))
\]

\[= c(E_1) \cap (s(E_1) \cap s(N(R_1, V_1)))
\]

\[= E_1 \cap s(R_1, V_1).
\]

This completes the proof. \( \square \)

4. Geometric meaning and applications — degenerations of hypersurfaces and their limiting \( \mathbb{P}^r \)

An important reason that inspired our study of the new residual intersection is its application to some geometric problems. As we have mentioned in the introduction, using dynamic intersections, we can interpret the refined intersection product \( X \cdot V \) as a limit derived from deformation theory. In this point of view, our decomposition has nice geometric meaning and tells us how the limit is distributed in different components. Instead of doing it in more general terms, we will study the case of...
the limiting $\mathbb{P}^r$’s in hypersurfaces of $\mathbb{P}^n$. The same method can be applied to many other settings.

We will first recall some definitions and facts from [W2]. Let $X$ be a hypersurface of degree $d$ in $\mathbb{P}^n$ and $F_X$ be the scheme of $\mathbb{P}^r$’s contained in $X$. If $X$ is generic, then $F_X$ will have the expected dimension (or empty) and its class in the Chow group of $G$ is given by the top Chern class of the vector bundle $\text{Sym}^dU^*$, where $G$ is the Grassmannian of $\mathbb{P}^r$’s in $\mathbb{P}^n$ and $U$ is the universal subbundle on $G$. When we deform hypersurfaces into a degenerate $X$, the dimension of $F_X$ can jump. In this case, there is a subscheme $F_{lim}$ of $F_X$ with the expected dimension which consists of the limiting $\mathbb{P}^r$’s in $X$ with respect to a general deformation. In general, the structure of $F_{lim}$ is much more complicated than the structure of $F_X$. However, to find the class of $F_{lim}$ and its distribution in different components of $F_X$, one needs only (at least in theory) to understand the structure of $F_X$. To see this, we consider the following fiber square:

$$
\begin{array}{ccc}
F_X & \longrightarrow & G \\
\downarrow & & \downarrow s_X \\
G & \longrightarrow & \text{Sym}^dU^*
\end{array}
$$

where $i$ is the zero-section embedding and $s_X$ is the section of $\text{Sym}^dU^*$ induced by $X$. As it turns out, the class of $F_{lim}$ is equal to the refined intersection product $G \cdot G$ defined from the fiber square above and the distribution of the limiting $\mathbb{P}^r$’s in different components of $F_X$ is given by the canonical decomposition of $G \cdot G$. For details, please refer to [W2] ($G \cdot G$ is called $R_X$ there.) Therefore, the problem is reduced into finding such a decomposition. For the obvious reason, we did not carry out the program along this line in [W2]. Instead, we found the decomposition by conducting a direct study of $F_{lim}$ in the given cases. Unfortunately, the methods used there are not easy to be generalized. However, it did give some hints about the possibility of a general formula and this paper is a result of the search that follows. With the results in the previous sections, we are ready to apply the refined residual intersection formulas to compute the distribution of limiting $\mathbb{P}^r$’s in $F_X$.

The point is that, since our formulas use only information on $F_X$, we now have a general way to find the distribution of the limits without doing any study about $F_{lim}$ itself. To simplify the notation, we will omit $\cap[G]$ in our formulas to identify an element in $A^*(G)$ with its dual in $A_*(G)$. Furthermore, for any vector bundle $E$ and any integer $m$, we will use $E(m)$ to denote a vector bundle of the same rank such that its Chern class is given by

$$
c(E(m)) = \text{Adams}(m, c(E)) = \sum_i m^i c_i(E).
$$

where $\text{Adams}(\ast, \ast)$ is the Adams operator of $K$-theorem. Notice that the Segre class of $E(m)$ is thus determined by the Segre class of $E$ via

$$
s(E(m)) = \text{Adams}(m, s(E)) = \sum_i m^i s_i(E).
$$

For any positive integer $m$, we will also use the following notation

$$
r_m = \text{rank}(\text{Sym}^mU^*) = \binom{m + r}{r}.
$$
Proposition 4.1. Let $X^e_k$ be the $e$-fold of a generic hypersurface of degree $k$ in $\mathbb{P}^n$ and $X^f_l$ be the $f$-fold of a generic hypersurface of degree $l$ in $\mathbb{P}^n$. If we deform a generic hypersurface of degree $d$ in $\mathbb{P}^n$ into the union of $X^e_k$ and $X^f_l$ (with $ke + lf = d$), then the class $[F_{\lim}(X^e_k)]$ of the limiting $\mathbb{P}^r$’s in $X^e_k$ can be computed by the following formulas

\begin{equation}
[F_{\lim}(X^e_k)] = \mathbb{R} F_{X^e_k} = M_{F_{X^e_k}} + A_{F_{X^e_k}},
\end{equation}

where

\begin{equation}
M_{F_{X^e_k}} = c_r (\text{Sym}^k U^*(e)) c_{r_d - r_k} (\text{Sym}^d U^* - \text{Sym}^k U^*(e))
\end{equation}

\begin{equation}
= c_r (\text{Sym}^k U^*(e)) \sum_{i=0}^{r_d - r_k} c_i (\text{Sym}^d U^*) c_{r_d - r_k - i} (\text{Sym}^k U^*(e)),
\end{equation}

and

\begin{equation}
A_{F_{X^e_k}} = - c_r (\text{Sym}^k U^*(e)) c_r (\text{Sym}^l U^*(f)) \sum_{i=0}^{r_d - r_k - r_l} \sum_{j=r_k}^{r_d - r_k - i} \binom{r_d - 1 - i}{j} 
\end{equation}

\begin{equation}
\times c_i (\text{Sym}^d U^*) s_{j - r_k} (\text{Sym}^l U^*(f)) s_{r_d - r_k - i - j}(\text{Sym}^k U^*(e)).
\end{equation}

Similarly, the class $[F_{\lim}(X^f_l)]$ of the limiting $\mathbb{P}^r$’s in $X^f_l$ is given by

\begin{equation}
[F_{\lim}(X^f_l)] = \mathbb{R} F_{X^f_l} = M_{F_{X^f_l}} + A_{F_{X^f_l}},
\end{equation}

where

\begin{equation}
M_{F_{X^f_l}} = c_r (\text{Sym}^l U^*(f)) c_{r_d - r_l} (\text{Sym}^d U^* - \text{Sym}^l U^*(f))
\end{equation}

\begin{equation}
= c_r (\text{Sym}^l U^*(f)) \sum_{i=0}^{r_d - r_l} c_i (\text{Sym}^d U^*) c_{r_d - r_l - i} (\text{Sym}^l U^*(f))
\end{equation}

and

\begin{equation}
A_{F_{X^f_l}} = - c_r (\text{Sym}^l U^*(f)) c_r (\text{Sym}^k U^*(e)) \sum_{i=0}^{r_d - r_k - r_l} \sum_{j=r_k}^{r_d - r_k - i} \binom{r_d - 1 - i}{j} 
\end{equation}

\begin{equation}
\times c_i (\text{Sym}^d U^*) s_{j - r_k} (\text{Sym}^k U^*(e)) s_{r_d - r_k - i - j}(\text{Sym}^l U^*(f)).
\end{equation}

Proof. Since $X$ is the union of $X^e_k$ and $X^f_l$, $F_X$ consists of two components $F_{X^e_k}$ and $F_{X^f_l}$. We can therefore apply the refined residual intersection decomposition to

\begin{equation}
G \cdot G = [F_{\lim}]
\end{equation}

with respect to the decomposition

\begin{equation}
F_X = F_{X^e_k} \cup F_{X^f_l}.
\end{equation}
of $F_X$. As sets, $F_{X_k}^e$ and $F_{X_l}^f$ are equal to $F_{X_k}$ and $F_{X_l}$, respectively. Furthermore, the Segre classes of $F_{X_k}^e$ and $F_{X_l}^f$ in $G$ is closely related to the normal bundles of $F_{X_k}$ and $F_{X_l}$ in $G$, respectively. In fact, by the same method used in the proof of Theorem 1.4 of [W3], we can easily get the following:

$$s(F_{X_k}^e, G) = \text{Adams}(e, s(F_{X_k}, G)) = s(\text{Sym}^kU^*(e)) \cap [F_{X_k}^e]$$

and

$$s(F_{X_l}^f, G) = \text{Adams}(f, s(F_{X_l}, G)) = s(\text{Sym}^lU^*(f)) \cap [F_{X_l}^f].$$

Therefore, $F_{X_k}^e$ and $F_{X_l}^f$ can be considered as being regularly embedded in $G$ with normal bundles $\text{Sym}^kU^*(e)$ and $\text{Sym}^lU^*(f)$, respectively. Since $X_k^e$ and $X_l^f$ are generic, $F_{X_k}^e$ and $F_{X_l}^f$ do intersect properly and the class of their intersection is given by

$$[F_{X_k}^e \cap F_{X_l}^f] = c_{\text{top}}(\text{Sym}^kU^*(e) \oplus \text{Sym}^lU^*(f)) \cap [G]$$

$$= c_{\text{top}}(\text{Sym}^kU^*(e))c_{\text{top}}(\text{Sym}^lU^*(f)) \cap [G].$$

Now, apply Corollary 3.16. □

Proposition 4.1 generalizes some earlier results on the limiting $\mathbb{P}^r$’s from [W1], [W2], and [W3]. Notice that the corresponding formulas in [W1], [W2], and [W3] are obtained from very different methods.

(a) If $r = 1$ and $e = f = 1$, then the adjunct parts are equal to zero by the dimension count. Our formulas hence are reduced to and coincide with the ones given in [W1] in the case of the limiting lines.

(b) More general, if just set $e = f = 1$, we then have the case studied in [W2]. Notice that our formulas in this case are different from (and simpler than) the corresponding formulas in [W2]. Of course, they must be equal to each other but that does not seem to be obvious if one just looks at the formulas themselves.

(c) If $f = 0$, we then have the case studied in [W3] and our formulas is thus a generalization of the formulas obtained there.

(d) The set-ups and the formulas are new if $ef > 1$. Notice that geometric and infinitesimal methods used in [W1] and [W2] will be difficult to apply in such cases, since we are dealing with non-reduced scheme structures and $F_{lim}$ will be very hard to understand.

The formulas in Proposition 4.1 are not as complicated as their look and can be calculated with routine procedures in intersection theory. Our examples below will be mainly for the new case (d) and the case (b) using the new formulas. We will write $X_k$ for $X_k^1$. Most of the calculations are done on computers using the Maple package schubert [KS] written by Katz and Strømme. It is quite easy to write schubert code for our formulas and we will be happy to send the copies upon request. But first, we will check a simple example by hand.

**Example 4.8.** Degenerations of cubic hypersurfaces in $\mathbb{P}^n$, $n \geq 3$, and their lines

The Chow ring of $G$ of lines in $\mathbb{P}^n$ is generated by $c_1(U^*)$ and $c_2(U^*)$. Let

$$c_1(U^*) = x, \quad \text{and} \quad c_2(U^*) = y.$$
It is an easy computation by the splitting principle that

\[
\begin{aligned}
c_1(\text{Sym}^3 U^*) &= 6x, \\
c_2(\text{Sym}^3 U^*) &= 11x^2 + 10y, \\
c_3(\text{Sym}^3 U^*) &= 6x^3 + 30xy, \\
c_4(\text{Sym}^3 U^*) &= 18x^2y + 9y^2.
\end{aligned}
\]

and

\[
\begin{aligned}
s_1(U^*) &= -x, \\
s_2(U^*) &= x^2 - y, \quad \cdots.
\end{aligned}
\]

Therefore, the class of lines on a generic cubic \(X_3\) is equal to

\[
[F_{X_3}] = c_4(\text{Sym}^3 U^*) = 18x^2y + 9y^2.
\]

To compute the distribution of the limiting lines when we deform \(X_3\) into the union of a \((n-1)\)-plane \(X_1\) and a double-\((n-1)\)-plane \(X_1\), we set

\[
r = 1, \quad d = 3, \quad l = k = e = 1, \quad \text{and} \quad f = 2
\]

in the formulas in Proposition 4.1. This together with (4.9) and (4.10) give us

\[
\begin{aligned}
[F_{\text{lim}}(X_1)] &= M_{F_{X_1}} + A_{F_{X_1}} \\
&= c_2(U^*)c_{4-2}(\text{Sym}^3 U^* - U^*) - c_2(U^*)c_2(U^*(2)) \left( \frac{4-1}{2} \right) \\
&= c_2(U^*) \sum_{i=0}^2 c_i(\text{Sym}^3 U^*)s_{2-i}(U^*) - 12c_2(U^*)c_2(U^*) \\
&= y(x^2 - y + 6x(-x) + 11x^2 + 10y - 12y^2) \\
&= 6x^2y - 3y^2
\end{aligned}
\]

and

\[
\begin{aligned}
[F_{\text{lim}}(X_1^2)] &= M_{F_{X_1^2}} + A_{F_{X_1^2}} \\
&= c_2(U^*(2))c_{4-2}(\text{Sym}^3 U^* - U^*(2)) - c_2(U^*)c_2(U^*(2)) \left( \frac{4-1}{2} \right) \\
&= c_2(U^*(2)) \sum_{i=0}^2 c_i(\text{Sym}^3 U^*)s_{2-i}(U^*(2)) - 12c_2(U^*)c_2(U^*) \\
&= 4y(4(x^2 - y) + 6x(-2x) + 11x^2 + 10y) - 12y^2 \\
&= 12x^2y + 12y^2.
\end{aligned}
\]

Notice that we do have

\[
[F_{\text{lim}}(X_1)] + [F_{\text{lim}}(X_1^2)] = 18x^2y + 9y^2 = [F_{X_3}]
\]

as expected.
In particular, if \( n = 3 \), then
\[
x^2 y = y^2 = \text{[point]}
\]
Therefore, we see that 24 of the 27 lines in a generic cubic surface go to the double-plane \( X_1^2 \) and other 3 lines go to the plane \( X_1 \) as one takes the limit. In fact, it is easy to identify those lines geometrically. Let
\[
C = X_1 \cap D \subset X_1
\]
be the elliptic curve in the plane \( X_1 \), where \( D \) is a cubic surface determined infinitesimally by the degeneration. It intersects \( X_1^2 \) at three points. Three limiting lines in \( X_1 \) are exactly the tangent lines in \( X_1 \) to \( C \) at those three points. Similarly, we have 3 tangent lines in \( X_1^2 \). That gives 24 limiting lines in \( X_1^2 \) since now each lines has the multiplicity of 8 on account of the non-reduced scheme structure of \( X_1^2 \).

Another simple example that can be easily checked by hand is the case of the limiting \( \mathbb{P}^2 \)'s in a degeneration of quadrics. Interested readers may try this example as a comparison to the computations made in [W2] [Example 3, Section 5] in which more complicated formulas are used.

**Example 4.13. Degenerations of quintic threefolds and their lines.**

It is well-known that there are 2875 lines on a generic generic quintic threefold. There are seven different degenerations of a generic \( X_5 \) into \( X_5^c \cup X_1^f \). Two of such cases are studied in [K] and [W1] so we will only list below the results for five new cases with at least one non-reduced component. It will be interesting to have those numbers checked geometrically. To compare with the standard residual intersection formula, we will also list values for both the main terms and the adjunct terms.

- **Case 1**, \( X_5 \to X_1^4 \cup X_1 \).
  \[
  [F_{\text{lim}}(X_1^4)] = \mathbb{R}_{F_{X_1^4}} = M_{F_{X_1^4}} + A_{F_{X_1^4}} = 2400 + 320 = 2720; \\
  [F_{\text{lim}}(X_1)] = \mathbb{R}_{F_{X_1}} = M_{F_{X_1}} + A_{F_{X_1}} = 1275 - 1120 = 155.
  \]
- **Case 2**, \( X_5 \to X_1^3 \cup X_2 \).
  \[
  [F_{\text{lim}}(X_1^3)] = \mathbb{R}_{F_{X_1^3}} = M_{F_{X_1^3}} + A_{F_{X_1^3}} = 3195 - 540 = 2655; \\
  [F_{\text{lim}}(X_2)] = \mathbb{R}_{F_{X_2}} = M_{F_{X_2}} + A_{F_{X_2}} = 1300 - 1080 = 220.
  \]
- **Case 3**, \( X_5 \to X_1^3 \cup X_1^2 \).
  \[
  [F_{\text{lim}}(X_1^3)] = \mathbb{R}_{F_{X_1^3}} = M_{F_{X_1^3}} + A_{F_{X_1^3}} = 3195 - 1080 = 2115; \\
  [F_{\text{lim}}(X_1^2)] = \mathbb{R}_{F_{X_1^2}} = M_{F_{X_1^2}} + A_{F_{X_1^2}} = 2920 - 2160 = 760.
  \]
- **Case 4**, \( X_5 \to X_1^2 \cup X_3 \).
  \[
  [F_{\text{lim}}(X_1^2)] = \mathbb{R}_{F_{X_1^2}} = M_{F_{X_1^2}} + A_{F_{X_1^2}} = 2920 - 540 = 2380; \\
  [F_{\text{lim}}(X_3)] = \mathbb{R}_{F_{X_3}} = M_{F_{X_3}} + A_{F_{X_3}} = 1575 - 1080 = 495.
  \]
- **Case 5**, \( X_5 \to X_1^2 \cup X_1 \).
  \[
  [F_{\text{lim}}(X_1^2)] = \mathbb{R}_{F_{X_1^2}} = M_{F_{X_1^2}} + A_{F_{X_1^2}} = 2880 - 640 = 2240; \\
  [F_{\text{lim}}(X_1)] = \mathbb{R}_{F_{X_1}} = M_{F_{X_1}} + A_{F_{X_1}} = 1275 - 640 = 635.
  \]
Notice that in each of the cases above we do have that

\[ [F_{lim}(X_k^i)] + [F_{lim}(X_f^i)] = R_{F_{X_k^i}} + R_{F_{X_f^i}} = 2875 \]

as expected.

**Example 4.14.** Degenerations of quartic hypersurfaces in \( \mathbb{P}^7 \) and their \( \mathbb{P}^2 \)'s.

As we have computed in [W2], there are 3,297,280 \( \mathbb{P}^2 \)'s on a generic quartic hypersurface in \( \mathbb{P}^7 \). There are five different degenerations for which Proposition 4.1 can be applied. We will list results of our computation for all five cases. Notice that the values of the main terms can range from negative ones to those that are bigger than the whole class. In particular, we see that the adjunct terms are not equal to zero even though there is no contribution from the variety on which they are supported (see [W2]).

Case 1, \( X_4 \to X_3 \cup X_1 \).

\[ [F_{lim}(X_4)] = R_{F_{X_3}} = M_{F_{X_3}} + A_{F_{X_3}} = 3,304,098 - 2,820,258 = 483,840; \]
\[ [F_{lim}(X_1)] = R_{F_{X_1}} = M_{F_{X_1}} + A_{F_{X_1}} = 3,656,569 - 843,129 = 2,813,440. \]

Case 2, \( X_4 \to X_2 \cup X_2 \).

\[ [F_{lim}(X_2)] = R_{F_{X_2}} = M_{F_{X_2}} + A_{F_{X_2}} = 3,087,616 - 1,438,976 = 1,648,640. \]

Case 3, \( X_4 \to X_3 \cup X_1 \).

\[ [F_{lim}(X_3)] = R_{F_{X_3}} = M_{F_{X_3}} + A_{F_{X_3}} = -20,855 + 205 + 24,000,165 = 3,144,960; \]
\[ [F_{lim}(X_1)] = R_{F_{X_1}} = M_{F_{X_1}} + A_{F_{X_1}} = 3,656,569 - 3,504,249 = 152,320. \]

Case 4, \( X_4 \to X_1 \cup X_1 \).

\[ [F_{lim}(X_1)] = R_{F_{X_1}} = M_{F_{X_1}} + A_{F_{X_1}} = 2,645,888 - 997,248 = 1,648,640. \]

Case 5, \( X_4 \to X_1 \cup X_2 \).

\[ [F_{lim}(X_2)] = R_{F_{X_2}} = M_{F_{X_2}} + A_{F_{X_2}} = 3,087,616 - 2,998,016 = 89,600. \]

Notice that the results in case (1) and case (2) do coincide with the ones given in Example 2 of [W2] obtained from different methods and formulas.

We have calculated many other examples. In the cases that other methods and formulas are available, such as the case of Example 4 of [W2] as interested readers may check, the results do coincide as expected.

Our last application is about a family of identities for the characteristic classes of vector bundles. For any vector bundle \( E \), we will formally extend the definition of \( s_i(E) \) to negative \( i \) by

\[ s_i(E) = 0, \quad -\text{rank}(E) < i < 0, \quad s_i(E) = -1/\text{ctop}(E), \quad i = -\text{rank}(E). \]
Corollary 4.15. Let $E$ be a vector bundle of rank $r + 1$ on a purely dimensional scheme $X$. If $E$ is generated by its sections, then the following family of identities holds for any set of non-negative integers $k$, $l$, and $d$ with $d = k + l$.

$$c_r d (\text{Sym}^d E) = - c_r k (\text{Sym}^k E)c_r l (\text{Sym}^l E) \times \left\{ \begin{array}{l} \sum_{i=0}^{r d - r k} \sum_{j=0}^{r d - r k - i} \left( r_d - 1 - i \right) c_i (\text{Sym}^d E) s_{j-r_l} (\text{Sym}^l E) s_{r_d-r_k-i-j} (\text{Sym}^k E) \\ + \sum_{i=0}^{r d - r l} \sum_{j=0}^{r d - r l - i} \left( r_d - 1 - i \right) c_i (\text{Sym}^d E) s_{j-r_k} (\text{Sym}^k E) s_{r_d-r_l-i-j} (\text{Sym}^l E) \end{array} \right\}. \tag{4.16}$$

Remark. It is pointed out to the author by the referee that Corollary 4.15 actually holds for any vector bundle $E$ of rank $r + 1$: Taking an ample line bundle $H$ and replacing $E$ by $E \otimes H^t$, we can then consider both sides of (4.16) as a polynomial in $t$. Since they agree for $t$ sufficiently large, they must agree for all $t$. In particular, they agree for $t = 0$.

Proof of Corollary 4.15. Consider the following fiber square

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow f \\ X & \longrightarrow & \text{Sym}^d E \end{array}$$

where $i$ is the zero-section embedding and $f$ is any section of $\text{Sym}^d E$. Therefore, $W$ is the zero-scheme of $f$. Since $d = k + l$, a section $s_k$ of $\text{Sym}^k E$ and a section $s_l$ of $\text{Sym}^l E$ induce a section of $\text{Sym}^d E$. Let $f$ be such a section and we hence have

$$W = Z_1 \cup Z_2,$$

where $Z_1$ and $Z_2$ are zero-schemes of $s_k$ and $s_l$, respectively. Since $E$ is generated by its sections, if we take $s_k$ and $s_l$ to be generic then $Z_1$ and $Z_2$ are regularly embedded in $X$ with normal bundle $\text{Sym}^k E$ and $\text{Sym}^l E$, respectively. Notice the normal bundle of $X$ in $\text{Sym}^d E$ is $\text{Sym}^d E$ itself and it is well-known that the image of $X \cdot X$ in the Chow group of $X$ is equal to $c_{top} (\text{Sym}^d E) \cap [X]$. From this, the family of identities follows directly from our residual intersection formulas. □

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