Fixed-time formation tracking for multiple nonholonomic wheeled mobile robots based on distributed observer

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Abstract This paper studies the distributed fixed-time formation tracking problem of multiple nonholonomic wheeled mobile robots system over directed fixed and switching topologies. Through a classical nonlinear transformation, the formation control problem is transformed into a consensus problem. New control protocols based on a distributed observer are proposed. The directed communication topology between multiple nonholonomic wheeled mobile robots is considered. Some sufficient conditions of multiple robots achieving the desired formation shape are given. All follower robots can form the desired formation shape within a fixed settling time and make the leader in the geometric center of the formation. By adopting graph theory and fixed-time stability theory, an upper bound of settling time that is independent of the system’s initial states is obtained. Finally, two examples are presented to illustrate the correctness of the main results.

Keywords Fixed-time formation · Multiple nonholonomic robots · Distributed observer

1 Introduction

Cooperative control of multiple robots has been greatly concerned over the past few years, and nonholonomic wheeled mobile robots were widely used [1–10]. The formation control of multi-robot systems can achieve many intricate tasks, such as target tracking, security and military operations, which cannot be achieved by only one robot [1–4]. There are many ways to keep the formation shape, such as leader–follower [6–8], behavior-based [11], and virtual structure [12,13]. The convergence speed is a vital performance index for the formation tracking control. However, most prior studies did not consider the convergence speed, which indicated that the formation shape is formed when time approaches infinity [14,15]. A protocol that the convergence speed depends on the algebraic connectivity was proposed in [16]. Olfati-Saber presented some methods...
to improve the algebraic connectivity, but it was still only asymptotically convergent [17]. Then, the finite-time control technology is introduced to make the states converge quickly [18–20], and a distributed finite-time control law was proposed to solve the consensus tracking problem for nonlinear multi-agent systems. Zhao et al. proposed a saturation protocol, which makes the consensus achieve for a second-order multi-agent system within a limited time [21]. In [22,23], the consensus problem of the multi-agent system in the form of high-order chain structure is considered, and a finite-time cooperative control protocol is designed to ensure that the state consensus is reached within a limited time.

In most of the existing results, the settling time depends on the initial state of the system. However, in practice, the initial state usually cannot be accurately obtained. Sometimes, it is even unable to be obtained under certain circumstances. Therefore, if the initial state is unavailable, the result that depends on the initial state does not make much sense. In this case, the fixed-time stability theory is introduced, which makes the settling time independent of the initial conditions [24,25]. A fixed-time consensus control is designed for multi-agent systems with linear and nonlinear state measurement [26]. Wang et al. [27] proposed a fixed-time formation control protocol with variable delay. In addition, to resolve the problem of uncertain disturbances in multi-agent system, Hong et al. [28] proposed two fixed-time consensus controller for two cases: leaderless and leader–follower. Chu et al. [29] studied the robust fixed-time consensus tracking problem of second-order multi-agent systems under undirected topology. However, in the control protocols, they did not consider the angular velocity of the robot. In contrast to the existing works on fixed-time cooperative control [26–29] for multi-agent systems, in which the systems were regarded as linear systems, in this paper, each wheeled mobile robot is considered as a nonholonomic and nonlinear constrained system. Many previous studies about the formation tracking problem of multi-robot systems are based on undirected topologies [15,30]. However, in practice, most communication between two robots is directed. Distributed fixed-time formation tracking protocols was studied in [31]. The mobile robots are nonholonomic, but the control protocol in [31] was based on the assumption that the precise information of the leader is available to the followers. However, because of the complex of the environment, usually, the precise information of the agent is not available. Hence, the control method based on an observer is very important [32,33].

As far as the authors know, there are few results on the fixed-time control of nonholonomic constrained multi-robot systems based on an observer. So this paper studies the distributed fixed-time formation tracking problem for multiple wheeled mobile robots based on distributed observer. The main novelty of this paper is as follows: (i) The fixed-time formation tracking for multiple nonholonomic wheeled mobile robots is investigated. (ii) The desired formation is not unique and can be arbitrary. (iii) The observer method is adopted.

2 Preliminaries

Throughout this work, denote \( \mathbb{R} \) the real number set, \( \mathbb{R}_+ \) the nonnegative real number set, \( \mathbb{R}^N \) the \( N \)-dimensional real column vector space, and \( \mathbf{1}_m = [1, ..., 1]^T \), \( \mathbf{0}_m = [0, ..., 0]^T \) the \( m \)-dimensional column vectors, \( \max(\cdot) \) presents the maximum value of a function, and \( \text{sign}(\cdot) \) is the sign function. For a vector \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_m)^T \), \( \text{diag}(\alpha) = \text{diag}(\alpha_1, \alpha_2, ..., \alpha_m) \).

The considered robot system is over the directed communication topology. A directed graph \( G \) is used to describe the communication topology of multi-robot system. Here, \( G \) is defined as a 3-tuple \( \{Y, \xi, A\} \), where \( Y = \{v_1, ..., v_m\} \) represents the set of notes, \( \xi \subseteq Y \times Y \) is the set of edges, and \( A = [a_{ij}]_{m \times m} \) represents the weighted adjacency matrix, where \( a_{ij} > 0 \) if the robot \( j \) can receive the information from the robot \( i \) directly, otherwise \( a_{ij} = 0 \), and \( a_{ij} = 0 \) for any \( i, j = 1, 2, ..., m \). The edge \( (v_i, v_j) \in \xi \) means that the robot \( i \) is a neighbor of the robot \( j \), and the robot \( j \) can receive the state information from the robot \( i \) directly. \( N_j = \{i|v_i \in Y, a_{ji} \neq 0\} \) denotes the neighbor set of robot \( j \). \( D = \text{diag}[d_1, d_2, ..., d_m] \) represents the degree matrix, and \( d_j = \sum_{i=1}^{m} a_{ij} \). The Laplacian matrix of the graph \( G \) is \( L = D - A \) with appropriate dimension. Let the diagonal matrix \( B = \text{diag}[b_1, b_2, ..., b_m] \) with \( b_j \) being the adjacency weight between agent \( j \) and the leader, and \( b_j > 0 \) if the information of the leader is accessible to the \( j \)th follower directly, otherwise \( b_j = 0 \). Let \( \text{sign}(z)^S = |z|^S \text{sign}(z) \), where \( \text{sign}(z) = z/|z| \) for \( z \neq 0 \), \( \text{sign}(0) = 0 \), and matrix \( H = L + B \).

The following lemmas are introduced to develop the main results.
Lemma 1 [16,34] For matrix $H = L + B$, $H$ is a positive stable matrix. If the graph $G$ is connected or contains a directed spanning tree, and $B = \text{diag}\{b_1, b_2, \ldots, b_m\} \geq 0$, then matrix $H$ is positive definite.

Lemma 2 [25] (Fixed-time Stability Theory) Consider the following system

$$\dot{z}(t) = g(z, t),$$

where $z \in \mathbb{R}^N$ and $g(z, t) : \mathbb{R}^N \times \mathbb{R}_+ \rightarrow \mathbb{R}^N$ is a nonlinear function. Assume that the origin is contained in the equilibrium point of system (1). If there is a continuous radial unbounded function $V(z) : \mathbb{R}^N \rightarrow \mathbb{R}_+ \cup \{0\}$ such that $V(z) = 0$ if and only if $z = 0$ and the inequality $D^+ V(z(t)) \leq -c_1 V^{\varsigma_3}(z(t)) - c_2 V^{\varsigma_4}(z(t))$ for $c_1 > 0, c_2 > 0, 0 < \varsigma_3 < 1, \varsigma_4 > 1$, then the origin is the global fixed-time stable equilibrium point of system (1) and the settling time is limited by

$$T = \frac{1}{c_1(1-\varsigma_3)} + \frac{1}{c_2(\varsigma_4-1)}.$$

Lemma 3 [35] If $p_1, p_2, \ldots, p_v \geq 0$, then there is

$$\sum_{j=1}^v p_j^\varsigma \geq \left(\sum_{j=1}^v p_j\right)^\varsigma, \quad 0 < \varsigma \leq 1,$$

and

$$\sum_{j=1}^v p_j^{1-\varsigma} \geq \left(\sum_{j=1}^v p_j\right)^{1-\varsigma}, \quad 1 < \varsigma \leq \infty.$$

Consider a system with $m$ robots, which are nonholonomic wheeled mobile robots and have the same kinematics (see Fig. 1). The kinematics of the wheeled mobile robot $j$ ($j = 1, 2, \ldots, m$) are described as follows:

$$\dot{x}_j(t) = v_j(t) \cos \theta_j(t),$$
$$\dot{y}_j(t) = v_j(t) \sin \theta_j(t),$$
$$\dot{\theta}_j(t) = \omega_j(t).$$

where $x_j(t) : R \rightarrow R, y_j(t) : R \rightarrow R, \theta_j(t) : R \rightarrow R$, denote the state of the horizontal axis, the state of longitudinal axis and the orientation of the wheeled mobile robot $j$, respectively, and $v_j(t) : R \rightarrow R$ and $\omega_j(t) : R \rightarrow R$ are the linear velocity and angular velocity of the robot $j$, respectively.

The control goal of this paper is to design suitable velocity $v_j(t)$ and angular velocity $\omega_j(t), j = 1, 2, \ldots, m$, such that the $m$ followers achieve a desired formation shape $F$ in a fixed time, and the leader is at the center of the formation shape $F$. Meanwhile, all the followers track the leader in the fixed time.

Use the orthogonal coordinates $(r_{jx}, r_{jy})$ to define the desired formation shape $F$, and $(r_{0x}, r_{0y})$ is the leader’s orthogonal coordinate. Without loss of generality, choose $(r_{0x}, r_{0y}) = (0, 0)$. If the position of the leader in the desired formation shape is not $(0, 0)$, it can be converted to $(0, 0)$ by mathematical transformation. The kinematics of the leader are described as follows:

$$\dot{x}_0(t) = v_0(t) \cos \theta_0(t),$$
$$\dot{y}_0(t) = v_0(t) \sin \theta_0(t),$$
$$\dot{\theta}_0(t) = \omega_0(t).$$

Definition 1 For the multi-robot system (2), if for any given bounded initial states, there exists a constant $T_{\text{max}} > 0$, such that for any $j = 1, 2, \ldots, m$, there is

$$\lim_{t \rightarrow T_{\text{max}}} \begin{bmatrix} x_j(t) - x_i(t) \\ y_j(t) - y_i(t) \end{bmatrix} = \begin{bmatrix} r_{jx} - r_{ix} \\ r_{jy} - r_{iy} \end{bmatrix},$$
$$\lim_{t \rightarrow T_{\text{max}}} (\theta_j(t) - \theta_0(t)) = 0,$$
$$\lim_{t \rightarrow T_{\text{max}}} \frac{1}{m} \sum_{j=1}^m x_j(t) - x_0(t) = 0,$$
$$\lim_{t \rightarrow T_{\text{max}}} \frac{1}{m} \sum_{j=1}^m y_j(t) - y_0(t) = 0.$$
then system (2) is said to accomplish the fixed-time formation tracking, and the settling time is $T_{\text{max}}$.

Suppose that the following assumptions hold throughout the paper.

**Assumption 1** The angular velocity $\omega_j$ of the robot $j$ is bounded. That is there exists $\kappa > 0$ such that $|\omega_j| \leq \kappa$, $j = 0, 1, ..., m$.

**Assumption 2** The state of the leader is only available to some of the followers.

**Assumption 3** Communication graph $G$ has a directed spanning tree with the leader as the root node, i.e., $B > 0$.

The wheeled mobile robot system is more complex than a linear system due to its nonholonomic characteristic. Therefore, the robot system needs to be partially linearized. There is a transformation in [35] as follows:

\begin{align*}
p_{1j} &= \theta_j, \\
p_{2j} &= (x_j - r_{jx}) \sin \theta_j - (y_j - r_{jy}) \cos \theta_j, \\
p_{3j} &= (x_j - r_{jx}) \cos \theta_j + (y_j - r_{jy}) \sin \theta_j, \\
u_{1j} &= \omega_j, \quad u_{2j} = v_j - u_{1j} p_{2j}.
\end{align*}

After the conversion (5), the inputs are converted to $u_{1j}$ and $u_{2j}$, $j = 0, 1, ..., m$, and the formation tracking control problem of system (2) becomes a state consensus problem. Then system (2) can be described as

\begin{align*}
p_{1j} &= u_{1j}, \\
p_{2j} &= u_{1j} p_{3j}, \\
p_{3j} &= u_{2j}.
\end{align*}

According to transformation (5), the original control objective (4) can be transformed to the consensus of system (6) as

\begin{align*}
\lim_{t \to T_{\text{max}}} (p_{1j} - p_{10}) &= 0, \\
\lim_{t \to T_{\text{max}}} (p_{2j} - p_{20}) &= 0, \\
\lim_{t \to T_{\text{max}}} (p_{3j} - p_{30}) &= 0,
\end{align*}

for $j = 1, 2, ..., m$.

**Lemma 4** [31] If the equations in (7) hold for $j = 1, 2, ..., m$, the fixed-time formation tracking control objectives in (4) can be achieved in the fixed time $T_{\text{max}}$. That is the $m$ mobile robots form the desired formation shape in the fixed time $T_{\text{max}}$.

**Assumption 4** Suppose that the leader’s inputs $w_0, v_0$ are unknown to any follower, then $u_{10}, u_{10} p_{30}, u_{20}$ are also unknown to the followers.

**Remark 1** Assumption 3 is a standard assumption, which is widely used in the existing results, such as Assumption 3 in [36] and Assumption 4 in [37].

**Remark 2** After transformation (5), if the new control objective (7) of system (6) is achieved, then the control objective (4) of system (2) is achieved. Therefore, next, distributed fixed-time consensus protocols for the system (6) will be designed.

### 3 Main Results

The main results are given in this section.

For any $j = 1, 2, ..., m$, denote $\hat{p}_{1j}, \hat{p}_{2j}, \hat{p}_{3j}$ the estimate of leader’s state $p_{10}, p_{20}, p_{30}$ by the $j$th follower. Then adopt the following fixed-time observers

\begin{align*}
\dot{\hat{p}}_{1j} &= \beta_3 \text{sign} \left( \sum_{i=1}^{N} a_{ji} (\hat{p}_{1i} - \hat{p}_{1j}) + b_j (p_{10} - \hat{p}_{1j}) \right) \\
&\quad + \beta_1 \left( \sum_{i=1}^{N} a_{ji} (\hat{p}_{1i} - \hat{p}_{1j}) + b_j (p_{10} - \hat{p}_{1j}) \right)^2, \\
\dot{\hat{p}}_{2j} &= \beta_4 \text{sign} \left( \sum_{i=1}^{N} a_{ji} (\hat{p}_{2i} - \hat{p}_{2j}) + b_j (p_{20} - \hat{p}_{2j}) \right) \\
&\quad + \beta_2 \left( \sum_{i=1}^{N} a_{ji} (\hat{p}_{2i} - \hat{p}_{2j}) + b_j (p_{20} - \hat{p}_{2j}) \right)^2, \\
\dot{\hat{p}}_{3j} &= \beta_5 \text{sign} \left( \sum_{i=1}^{N} a_{ji} (\hat{p}_{3i} - \hat{p}_{3j}) + b_j (p_{30} - \hat{p}_{3j}) \right) \\
&\quad + \beta_3 \left( \sum_{i=1}^{N} a_{ji} (\hat{p}_{3i} - \hat{p}_{3j}) + b_j (p_{30} - \hat{p}_{3j}) \right)^2,
\end{align*}

where $\beta_1 = \frac{\varepsilon}{\sqrt{m}}$, $\beta_2 = \sqrt{\frac{\lambda_{\text{max}}(H)}{2\lambda_{\text{min}}(H)}}, \beta_3 = \beta_2 + \max(u_{10}), \beta_4 = \beta_2 + \max(u_{10} p_{30}), \beta_5 = \beta_2 + \max(u_{20})$, and $\lambda_{\text{min}}(H), \lambda_{\text{max}}(H)$ is the minimum, maximum eigenvalue of matrix $H$. 

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Remark 3 In practice, due to the complexity of the actual environment, as well as the diversity and uncertainty of robots own nature, usually, it is very difficult to get the exact information of robots states in practice. Because of this problem, one cannot design the control protocol based on the exact states of the robots. Therefore, compared with the existing results, the distributed observer is introduced to achieve state estimation for the leader in a fixed time. The control protocol based on the fixed-time observer is designed. The detailed thought of the method is as follows:

(i) Based on the fixed-time stability theory, the distributed fixed-time observers are proposed for each follower robot to ensure that the estimated state can converge to the desired state in a fixed time.

(ii) Based on the technique of fixed-time control, a fixed-time tracking controller/protocol is proposed for each follower robot such that the estimated state of the leader can be tracked in a fixed time.

(iii) Combining the previous fixed-time observer and fixed-time controller/protocol, the fixed-time consensus problem for the considered multiple robots systems is solved.

Therefore, based on the thoughts as above, the observers in Eqs. (8)–(10) are proposed to make the state of the leader observed by the followers, that is the estimated state of the leader, and approach the actual state of the leader within a fixed time. Meanwhile, the estimated state of the leader can be tracked by the followers in a fixed time.

Lemma 5 [38] For any $\varepsilon > 0$, if Assumptions 2 and 4 hold, then based on the distributed observers (8)–(10), the estimate of the leader’s position state $\hat{p}_{1j}$, $\hat{p}_{2j}$, $\hat{p}_{3j}$ by agent $j$ converges to the leader’s real position state $p_{10}$, $p_{20}$, $p_{30}$, $j = 1, 2, \ldots, m$, and the settling time $T_0 = \frac{\pi}{\varepsilon}$.

For system (6), define the estimates of tracking errors between the robot $j$ and the leader as:

$$\hat{p}_{1j} = p_{1j} - \hat{p}_{1j}, \hat{p}_{2j} = p_{2j} - \hat{p}_{2j}, \hat{p}_{3j} = p_{3j} - \hat{p}_{3j}. \quad (11)$$

Then, the estimate of the local relative tracking errors of the robot $j$ ($j = 1, 2, \ldots, m$) are defined as:

$$e_{1j} = \sum_{i \in N_j} a_{ji}(\hat{p}_{1i} - \hat{p}_{1j}) + b_j(p_{1j} - \hat{p}_{1j}),
$$

$$e_{2j} = \sum_{i \in N_j} a_{ji}(\hat{p}_{2i} - \hat{p}_{2j}) + b_j(p_{2j} - \hat{p}_{2j}), \quad (12)$$

$$e_{3j} = \sum_{i \in N_j} a_{ji}(\hat{p}_{3i} - \hat{p}_{3j}) + b_j(p_{3j} - \hat{p}_{3j}).$$

Let

$$p_1 = [p_{11}, \ldots, p_{1m}]^T, \quad p_2 = [p_{21}, \ldots, p_{2m}]^T,$$

$$p_3 = [p_{31}, \ldots, p_{3m}]^T, \quad \hat{p}_1 = [\hat{p}_{11}, \ldots, \hat{p}_{1m}]^T,$$

$$\hat{p}_2 = [\hat{p}_{21}, \ldots, \hat{p}_{2m}]^T, \quad \hat{p}_3 = [\hat{p}_{31}, \ldots, \hat{p}_{3m}]^T,$$

$$\hat{p}_1 = [\hat{p}_{11}, \ldots, \hat{p}_{1m}]^T, \quad \hat{p}_2 = [\hat{p}_{21}, \ldots, \hat{p}_{2m}]^T,$$

$$\hat{p}_3 = [\hat{p}_{31}, \ldots, \hat{p}_{3m}]^T, \quad e_1 = [e_{11}, \ldots, e_{1m}]^T,$$

$$e_2 = [e_{21}, \ldots, e_{2m}]^T, \quad e_3 = [e_{31}, \ldots, e_{3m}]^T.$$ Then (12) can be written into the vector form as

$$e_1 = H \hat{p}_1, \quad e_2 = H \hat{p}_2, \quad e_3 = H \hat{p}_3. \quad (13)$$

Define the auxiliary local relative tracking error as:

$$\tilde{e}_2 = e_2 - H\text{diag}(p_3)\hat{p}_1.$$

Denote $\bar{e}_2 = [\bar{e}_{21}, \ldots, \bar{e}_{2m}]^T$ and

$$\eta = k_1\bar{e}_2^T\text{sig}(\bar{e}_2)^l + k_3\bar{e}_2^T\text{sig}(\bar{e}_2)^l.$$

In order to make the fixed-time formation tracking control for the wheeled mobile robot, for any $j = 1, 2, \ldots, m$, design the following control protocols

$$u_{1j} = \frac{1}{\sum_{i \in N_j} a_{ji} + b_j}(-k_1\text{sig}(e_{1j})^l - k_3\text{sig}(e_{1j})^l
$$

$$+ b_j \hat{p}_{1j} + \sum_{i \in N_j} a_{ji}u_{1i} + u_{1i}^*), \quad (14)$$

$$u_{2j} = \frac{1}{\sum_{i \in N_j} a_{ji} + b_j}(-k_1\text{sig}(e_{3j})^l - k_3\text{sig}(e_{3j})^l
$$

$$+ b_j \hat{p}_{3j} + \sum_{i \in N_j} a_{ji}u_{2i} - k_2\text{sign}(e_{3j})(\bar{e}_{2j}), \quad (15)$$

and

$$u_{1i}^* = \frac{1}{\sum_{i \in N_j} a_{ji} + b_j} \left( a_{ij} \left( \sum_{i \in N_j} a_{ji}(\bar{e}_{2i} - \bar{e}_{2i}) + b_j \bar{e}_{2i} \right) \right)$$

$$+ \sum_{i \in N_j} a_{ij}a_{ii}^*, \quad (16)$$
where $k_1$, $k_2$, $k_3$, $l_1$ and $l_2$ are constant parameters of the controllers, and satisfy $k_1 > 0$, $k_2 > \kappa + \frac{\eta}{|e_3|^*|w_2|}$, $k_3 > 0$, $0 < l_1 < 1$ and $l_2 > 1$. From (14), one can get

$$
\left( \sum_{i \in N_j} a_{ij} + b_j \right) u_{1j} = -k_1 \text{sign}(e_{1j})^l_{1} - k_3 \text{sign}(e_{1j})^l_{2} + b_j \hat{p}_{1j} + \sum_{i \in N_j} a_{ij} u_{ii} + u_{1j}^*. \tag{17}
$$

And (17) can be rewritten as:

$$
\sum_{i \in N_j} a_{ij}(u_{1j} - u_{ii}) + b_j u_{1j} = -k_1 \text{sign}(e_{1j})^l_{1} - k_3 \text{sign}(e_{1j})^l_{2} + b_j \hat{p}_{1j} + u_{1j}^*. \tag{18}
$$

Denote

$$
u_1 = (u_{11}, u_{12}, ..., u_{1m})^T,
$$

$$
u_2 = (u_{21}, u_{22}, ..., u_{2m})^T.
$$

Then (18) can be written in the vector form as:

$$
Hu_1 = -k_1 \text{sign}(e_{1})^l_{1} - k_3 \text{sign}(e_{1})^l_{2} + B \hat{p}_1 + u_1^*, \tag{19}
$$

where $u_1^* = [u_{11}^*, u_{12}^*, ..., u_{1m}^*]^T$. According to Assumption 3 and Lemma 1, $H$ and $H^T$ are reversible. Then left multiplying the matrix $H^{-1}$ at both sides of (19), one can obtain

$$
u_1 = H^{-1}[-k_1 \text{sign}(e_{1})^l_{1} - k_3 \text{sign}(e_{1})^l_{2} + B \hat{p}_1 + u_1^*].
$$

Similarly, there is

$$
u_2 = H^{-1}[-k_1 \text{sign}(e_{3})^l_{1} - k_3 \text{sign}(e_{3})^l_{2} + B \hat{p}_3 - k_2 \text{diag}(\text{sign}(e_{3}))[\bar{\epsilon}_2]].
$$

From (16), one can get

$$
\left( \sum_{i \in N_j} a_{ij} + b_j \right) u_{1j}^* = u_{2j} \left( \sum_{i \in N_j} a_{ij}(\bar{e}_{2j} - \bar{\epsilon}_2) + b_j \bar{\epsilon}_2 \right) + \sum_{i \in N_j} a_{ij} u_{1i}. \tag{20}
$$

Then (20) can be rewritten as:

$$
\sum_{i \in N_j} a_{ij}(u_{1j}^* - u_{1i}^*) + b_j u_{1j}^* = u_{2j} \left( \sum_{i \in N_j} a_{ij}(\bar{e}_{2j} - \bar{\epsilon}_2) + b_j \bar{\epsilon}_2 \right), \tag{21}
$$

And the vector form of (21) is:

$$
H^T u_1^* = \text{diag}(u_2) H^T \bar{\epsilon}_2. \tag{22}
$$

Left multiplying the matrix $(H^T)^{-1}$ at both sides of (22), one can obtain

$$
u_1^* = (H^T)^{-1} \text{diag}(u_2) H^T \bar{\epsilon}_2.
$$

Based on the above analysis, the control protocols (14) and (15) can be written in the vector form as:

$$
u_1 = H^{-1}[-k_1 \text{sign}(e_{1})^l_{1} - k_3 \text{sign}(e_{1})^l_{2} + B \hat{p}_1 + (H^T)^{-1} \text{diag}(u_2) H^T \bar{\epsilon}_2],
$$

$$
u_2 = H^{-1}[-k_1 \text{sign}(e_{3})^l_{1} - k_3 \text{sign}(e_{3})^l_{2} + B \hat{p}_3 - k_2 \text{diag}(\text{sign}(e_{3}))[\bar{\epsilon}_2]],
$$

where $0 < l_1 < 1$, $l_2 > 1$, $k_1 > 0$, $k_2 > \kappa + \frac{\eta}{|e_3|^*|w_2|}$, and $k_3 > 0$.

3.1 System over fixed topology

**Theorem 1** Suppose that Assumptions 1–4 hold. Then under the control protocols (14)–(15) with $0 < l_1 < 1$, $l_2 > 1$, $k_1 > 0$, $k_3 > 0$, systems (2) and (3) achieve the fixed-time formation tracking control. Particularly, $p_{1j} = p_{10}$, $p_{2j} = p_{20}$, $p_{3j} = p_{30}$, $j = 1, ..., m$, for $T \geq T_{\max}$. And the settling time $T_{\max}$ satisfies

$$
T_{\max} = T_0 + \frac{2}{2\frac{l_1 + 1}{l_1} k_1 (1 - l_1)} + \frac{2}{2\frac{l_3 + 1}{l_3} k_3 m (\frac{1-l_2}{l_2} - 1)}
$$

where $T_0 = \frac{\pi}{T}$ is as that in Lemma 5.

**Proof** From Lemma 5, one can get that the estimate of the leader’s position state $\hat{p}_{1j}$, $\hat{p}_{2j}$, $\hat{p}_{3j}$ by agent $j$ converges to the leader’s real position state $p_{10}$, $p_{20}$, $p_{30}$ when $t \geq T_0$. That is $\bar{p}_{1j} = p_{10}$, $\bar{p}_{2j} = p_{20}$, $\bar{p}_{3j} = p_{30}$. Denote $\bar{p}_{1j}^0$, $\bar{p}_{2j}^0$, $\bar{p}_{3j}^0$, the tracking errors between the robot $j$ and the leader for $t \geq T_0$. That is

$$
\begin{align*}
\bar{p}_{1j}^0 &= p_{1j} - p_{10}, \\
\bar{p}_{2j}^0 &= p_{2j} - p_{20}, \\
\bar{p}_{3j}^0 &= p_{3j} - p_{30}.
\end{align*} \tag{23}
$$

The tracking errors in (23) can be derived as:

$$
\begin{align*}
\dot{\bar{p}}_{1j}^0 &= \bar{p}_{1j} - u_{1}, \\
\dot{\bar{p}}_{2j}^0 &= \bar{p}_{2j} + p_{3j}(u_{1j} - u_{10}), \\
\dot{\bar{p}}_{3j}^0 &= \bar{p}_{3j} - u_{2}.
\end{align*} \tag{24}
$$

Then, (24) can be written into the vector form as:

$$
\begin{align*}
\dot{\bar{p}}_{1j}^0 &= u_{1} - 1_m u_{10}, \\
\dot{\bar{p}}_{2j}^0 &= u_{10} \bar{p}_{3j} + \text{diag}(p_3)(u_1 - 1_m u_{10}), \\
\dot{\bar{p}}_{3j}^0 &= u_{2} - 1_m u_{20}.
\end{align*} \tag{25}
$$
Similarly, for any \( t \geq T_0 \), the local relative tracking errors of the robot \( j \) \(( j = 1, 2, \ldots, m)\) can be written as:

\[
e_j^0 = \sum_{i \in N_j} a_{ji}(p_{1j} - p_{1i}) + b_j(p_{1j} - p_{10}),
\]

\[
e_j^2 = \sum_{i \in N_j} a_{ji}(p_{2j} - p_{2i}) + b_j(p_{2j} - p_{20}),
\]

\[
e_j^3 = \sum_{i \in N_j} a_{ji}(p_{3j} - p_{3i}) + b_j(p_{3j} - p_{30}).
\]

Then (26) can be written into vector form as:

\[
e_1^0 = H \bar{p}_1^0, \quad e_2^0 = H \bar{p}_2^0, \quad e_3^0 = H \bar{p}_3^0.
\]

The local relative tracking errors (27) can be derived as:

\[
e_1^0 = H u_1 - B_1 m u_{10},
\]

\[
\dot{e}_2^0 = u_{10} e_3^0 + H \text{diag}(p_3)(u_1 - 1_m u_{10}),
\]

\[
\dot{e}_3^0 = H u_2 - B_1 m u_{20}.
\]

For \( t \geq T_0 \), the auxiliary local relative tracking error can be written as:

\[
\ddot{e}_2^0 = \ddot{e}_2^0 - H \text{diag}(p_3) \dot{p}_1^0,
\]

with \( \ddot{e}_2^0 = (\ddot{e}_{21}^0, \ddot{e}_{22}^0, \ldots, \ddot{e}_{2m}^0)^T \). Then (29) can be derived as:

\[
\ddot{e}_2^0 = \ddot{e}_2^0 - H \text{diag}(p_3) \dot{p}_3^0 - H \text{diag}(\dot{p}_3) \dot{p}_1^0.
\]

Substituting (6), (25) and (28) into (30), one can get

\[
\ddot{e}_2^0 = u_{10} e_3^0 - H \text{diag}(u_2) \dot{p}_1^0.
\]

Choose the Lyapunov function candidate

\[
V = \frac{1}{2}(e_1^0)^T e_1^0 + (e_2^0)^T \ddot{e}_2^0 + (e_3^0)^T \ddot{e}_3^0.
\]

The derivative of \( V \) along (28) and (31) is:

\[
\dot{V} = (e_1^0)^T [-k_1 \text{diag}(e_1^0)] - k_3 \text{diag}(e_3^0) + (H^T)^{-1} \text{diag}(u_2) H \ddot{e}_2^0
\]

\[
+ (e_2^0)^T [u_{10} e_3^0 - H \text{diag}(u_2) \dot{p}_1^0]
\]

\[
+ (e_3^0)^T [-k_1 \text{diag}(e_3^0)] - k_3 \text{diag}(e_3^0) + k_2 \text{diag}(\text{sign}(e_3^0)) \dot{e}_2^0]
\]

\[
= -k_1 (e_1^0)^T \text{diag}(e_1^0) - k_3 (e_1^0)^T \text{diag}(e_1^0)
\]

\[
- k_1 (e_3^0)^T \text{diag}(e_3^0) - k_3 (e_3^0)^T \text{diag}(e_3^0)
\]

\[
+ u_{10}^2 (e_3^0)^T e_3^0 - k_2 |e_3^0|^2 - k_2 |e_2^0| - \ddot{e}_2^0 H \text{diag}(u_2) \ddot{p}_1^0
\]

\[
+ (\ddot{p}_1^0)^T \text{diag}(u_2) H \ddot{e}_2^0.
\]

By Assumption 1, one can get \(|u_{10}| < \kappa \).

Let \( k_2 > \kappa + \frac{\eta}{|e_3^0|^2} \), \( k_3 > 0 \), and \((p_1^0)^T \text{diag}(u_2) H^T \ddot{e}_2^0 = (\ddot{e}_2^0)^T H \text{diag}(u_2) \ddot{p}_1^0 \). Then (32) can be simplified as:

\[
\dot{V} \leq -k_1 (e_1^0)^T \text{diag}(e_1^0) - k_3 (e_1^0)^T \text{diag}(e_1^0)
\]

\[
- k_1 (e_3^0)^T \text{diag}(e_3^0) - k_3 (e_3^0)^T \text{diag}(e_3^0)
\]

\[
+ k_2 \text{diag}(\text{sign}(e_3^0)) \dot{e}_2^0]
\]

\[
= -k_1 \sum_{j=1}^m \frac{|e_{1j}^0|^2}{l_j + 2} - k_3 \sum_{j=1}^m \frac{|e_{1j}^0|^2}{l_j + 2}
\]

\[
- k_1 \sum_{j=1}^m \frac{|e_{3j}^0|^2}{l_j + 2} - k_3 \sum_{j=1}^m \frac{|e_{3j}^0|^2}{l_j + 2}.
\]

Note that \( 0 < \frac{l_1 + 1}{2} < 1, \frac{l_2 + 1}{2} > 1 \). Then according to Lemma 3, (33) gives

\[
\dot{V} \leq -k_1 \left( \sum_{j=1}^m \frac{|e_{1j}^0|^2}{l_j + 2} + \sum_{j=1}^m \frac{|e_{3j}^0|^2}{l_j + 2} + \sum_{j=1}^m \frac{|e_{2j}^0|^2}{l_j + 2} \right)^{\frac{l_1 + 1}{2}}
\]

\[
- k_3 m \frac{l_3}{2} \left( \sum_{j=1}^m \frac{|e_{1j}^0|^2}{l_j + 2} + \sum_{j=1}^m \frac{|e_{3j}^0|^2}{l_j + 2} + \sum_{j=1}^m \frac{|e_{2j}^0|^2}{l_j + 2} \right)^{\frac{l_2 + 1}{2}}
\]

\[
\leq -k_1 (2V)^{\frac{l_1 + 1}{2}} - k_3 m \frac{l_3}{2} (2V)^{\frac{l_2 + 1}{2}},
\]

which implies

\[
\dot{V} \leq -2 \frac{l_1 + 1}{2} k_1 V^{\frac{l_1 + 1}{2}} - 2 \frac{l_2 + 1}{2} k_3 m \frac{l_3}{2} V^{\frac{l_2 + 1}{2}}.
\]

By Lemma 2, \( V \) will reach zero in a fixed time, and the settling time is bounded and satisfies:

\[
T_1 = \frac{2}{l_1 + 2} k_1 (1 - l_1) + \frac{2}{l_2 + 2} k_3 m \frac{l_3}{2} (l_2 - 1).
\]

Then \( T_{\max} = T_0 + T_1 \), which implies that

\[
\lim_{t \to T_{\max}} e_1(t) = 0, \quad \lim_{t \to T_{\max}} \dot{e}_2(t) = 0, \quad \lim_{t \to T_{\max}} e_3(t) = 0.
\]

(34)

Note that if \( \lim_{t \to T_{\max}} \dot{e}_2(t) = 0 \), then \( \lim_{t \to T_{\max}} e_2(t) = 0 \). This together with (34) gives that \( \lim_{t \to T_{\max}} e_h(t) = 0 \), \( h = 1, 2, 3 \). That is

\[
\lim_{t \to T_{\max}} p_{hj} = p_{h0}, \quad h = 1, 2, 3, \quad j = 1, 2, \ldots, m.
\]
Hence under the distributed protocols (14) and (15), systems (2) and (3) complete the formation tracking with the desired formation.

The proof is completed. □

3.2 System over switching topologies

For a network of multiple mobile robots, the original communication connection may be broken due to obstacles between two robots. Similarly, since the robots reach an effective detection range between each other, some new links can be created between nearby robots. Hence, it is very significant to study the fixed-time formation tracking for systems over switching topology. Let \( G_{\delta(t)} = \{ \Upsilon, \epsilon, A_{\delta(t)} \} \) be a directed graph set. Let \( \delta(t) : R^+ \to \Upsilon \) be a switching signal, and \( \Upsilon = \{1, 2, ..., N\} \) the index set of the graph \( G_{\delta(t)} \).

Denote \( t_0 = 0 < t_1 < \cdots < t_r, \cdots \), the switching time sequence, at which the communication topology changes. For any \( t \in [t_r, t_{r+1}) \), the topology \( G_{\delta(t)} = G_{\gamma} \in G_{\Upsilon} = \{ G_1, ..., G_N \} \) with \( B_{\delta(t)} = B_{\gamma} \) is active. Simultaneously, the adjacency weight between agent \( j \) and \( i \) is \( a_{ji}^{\gamma} \), and the Laplacian matrix is \( L^\gamma, \gamma \in \Upsilon \). Then, there is the following result.

**Theorem 2** Suppose that Assumptions 1–4 hold. Then under the control protocols (14) and (15) with \( 0 < l_1 < 1, l_2 > 1, k_1^Y > 0, k_2^Y > 0 \), system (6) over directed switching topologies forms the formation as (7) with the settling time \( T_{\text{max}}^Y = \max\{T_{\text{max}}^\gamma, \gamma \in \Upsilon\} \), and

\[
T_{\text{max}}^Y = T_0 + \frac{2}{2^{\frac{l_2}{k_2} - \frac{l_1 - l_2}{k_1^Y} (1 - l_1)}} + \frac{2}{2^{\frac{l_2}{k_2} - \frac{l_1 - l_2}{k_2^Y} m^{-\frac{l_2}{k_2}} (l_2 - 1)}},
\]

where \( T_0 = \frac{\pi}{\epsilon} \) is as that in Lemma 5.

**Proof** For any \( t \in [t_r, t_{r+1}) \), \( r = 0, 1, ..., \) there is a \( G_{\gamma} \) with \( B_{\gamma} \), \( \gamma \in \Upsilon \), acts. Then the fixed-time observer of agent \( j \) (\( j = 1, 2, ..., m \)) can be written as:

\[
\hat{p}_{1j} = \beta_5^Y \text{sign} \left( \sum_{i=1}^N a_{ji}^Y (\hat{p}_{i1} - \hat{p}_{1j}) + b_j^Y (p_{10} - \hat{p}_{1j}) \right) + \beta_1^Y \left( \sum_{i=1}^N a_{ji}^Y (\hat{p}_{i1} - \hat{p}_{1j}) + b_j^Y (p_{10} - \hat{p}_{1j}) \right)^2,
\]

\[
\hat{p}_{3j} = \beta_5^Y \text{sign} \left( \sum_{i=1}^N a_{ji}^Y (\hat{p}_{3i} - \hat{p}_{3j}) + b_j^Y (p_{30} - \hat{p}_{3j}) \right) + \beta_3^Y \left( \sum_{i=1}^N a_{ji}^Y (\hat{p}_{3i} - \hat{p}_{3j}) + b_j^Y (p_{30} - \hat{p}_{3j}) \right)^2,
\]

where \( \beta_1^Y = \frac{\epsilon \sqrt{m}}{(2\lambda_{\min}(L^Y + B^Y))^2} \), \( \beta_2^Y = \epsilon \sqrt{\frac{\lambda_{\max}(L^Y + B^Y)}{2\lambda_{\min}(L^Y + B^Y)}} \),

\[
\beta_5^Y = \beta_2^Y + \max(u_{10}), \beta_3^Y = \beta_2^Y + \max(u_{10}p_{30}), \beta_5^Y = \beta_2^Y + \max(u_{20}).
\]

The estimate of the local relative tracking errors of the robot \( j = 1, 2, ..., m \):

\[
e_{1j} = \sum_{i \in N_j} a_{ji}^Y (\hat{p}_{1j} - \hat{p}_{1i}) + b_j^Y (p_{10} - \hat{p}_{1j}),
\]

\[
e_{2j} = \sum_{i \in N_j} a_{ji}^Y (\hat{p}_{2j} - \hat{p}_{2i}) + b_j^Y (p_{20} - \hat{p}_{2j}),
\]

\[
e_{3j} = \sum_{i \in N_j} a_{ji}^Y (\hat{p}_{3j} - \hat{p}_{3i}) + b_j^Y (p_{30} - \hat{p}_{3j}).
\]

Then (35) can be written into the vector form as:

\[
e_1 = (L^Y + B^Y) \hat{p}_1,
\]

\[
e_2 = (L^Y + B^Y) \hat{p}_2,
\]

\[
e_3 = (L^Y + B^Y) \hat{p}_3.
\]

The auxiliary local relative tracking error can be written as:

\[
\tilde{e}_2 = e_2 - (L^Y + B^Y) \text{diag}(p_3) \hat{p}_1.
\]

Then, the distributed controllers are:

\[
u_{1j} = \frac{1}{\sum_{i \in N_j} a_{ji}^Y + b_j^Y} \left( -k_1^Y \text{sign}(e_{1j})^l_1 - k_2^Y \text{sign}(e_{1j})^l_2 
\right.
\]

\[
+ b_j^Y \hat{p}_{1j} + \sum_{i \in N_j} a_{ji}^Y u_{1i} + u_{1j}),
\]

\[
u_{2j} = \frac{1}{\sum_{i \in N_j} a_{ji}^Y + b_j^Y} \left( -k_1^Y \text{sign}(e_{2j})^l_1 - k_2^Y \text{sign}(e_{2j})^l_2 
\right.
\]

\[
+ b_j^Y \hat{p}_{3j} + \sum_{i \in N_j} a_{ji}^Y u_{2i} - k_2^Y \text{sign}(e_{2j}) |\tilde{e}_{2j}|),
\]
Similarly, for any \( t \geq T_0 \), there is a \( G_\gamma \) with \( B_\gamma \), \( \gamma \in \Upsilon \), active. Then the local relative tracking errors of the robot \( j \) (\( j = 1, 2, \ldots, m \)) can be written as:

\[
e^0_{1j} = \sum_{i \in N_j} a^0_{ij}(p_{1j} - p_{1i}) + b^0_{ij}(p_{1j} - p_{10}),
\]

\[
e^0_{2j} = \sum_{i \in N_j} a^0_{ij}(p_{2j} - p_{2i}) + b^0_{ij}(p_{2j} - p_{20}), \tag{36}
\]

\[
e^0_{3j} = \sum_{i \in N_j} a^0_{ij}(p_{3j} - p_{3i}) + b^0_{ij}(p_{3j} - p_{30}).
\]

And system (36) can be written as the following compact form:

\[
e^0_1 = (L^\gamma + B^\gamma)\hat{p}^0_1,
\]

\[
e^0_2 = (L^\gamma + B^\gamma)\hat{p}^0_2,
\]

\[
e^0_3 = (L^\gamma + B^\gamma)\hat{p}^0_3.
\]

For any \( t \geq T_0 \), the auxiliary local relative tracking error is:

\[
\dot{\bar{e}}^0_2 = \bar{e}^0_2 - (L^\gamma + B^\gamma)\text{diag}(p_3)\hat{p}^0_1. \tag{37}
\]

Then (37) can be derived as:

\[
\dot{\bar{e}}^0_2 = u_{10}\bar{e}^0_2 - (L^\gamma + B^\gamma)\text{diag}(u_2)\hat{p}^0_1.
\]

Choose the Lyapunov function candidate

\[
V = \frac{1}{2}(e^0_1)^T e^0_1 + (e^0_2)^T e^0_2 + (e^0_3)^T e^0_3.
\]

Then over any topology \( G_\gamma \), \( \gamma \in \Upsilon \), similar to the proof of Theorem 1, from Lemma 2 one can get the settling time

\[
T^\gamma_{\text{max}} = T_0 + \frac{2}{k_1 + 1} + \frac{2}{k_2 + 1}m \frac{1 - l}{l - 1}.
\]

Then over the switching topology \( G_{\delta(t)} \), \( \delta(t) \in \Upsilon \), the maximum settling time is

\[
T_{\text{max}} = \max\{T^\gamma_{\text{max}}, \gamma \in \Upsilon\}.
\]

This completes the proof.  \( \square \)

**Remark 4** Under the switching topology, the settling time should depend on the switching sequence and the dwell time. However, this work only analyzes the upper bound of the maximum convergence time in the switching topology. Due to the complexity of the switching topology and the difficult of the fixed-time problem, no suitable method has been found to obtain more accurate results till now. The systematic and in-depth analysis on the related problem will be given in our future work.

**Remark 5** As we all know, a certain subset of nodes might lose communication with all nodes. Such an “outage” frequently occurs in communication systems/networks due to various reasons, which leads to the nullification of Assumption 3 for a short duration of time. In fact, even if Assumption 3 is nullification in a short time, the results in this work are still true, but the upper bound of settling time will be larger. That is the settling time should be the time in the case of effective network plus the dwell time of invalid network. That’s why Assumption 3 is needed for the main results.

### 4 Simulation result

In this section, two simulation examples are given to demonstrate the effectiveness of the fixed-time formation tracking control protocols. For simplicity, for any \( j \), denote \( E_{1j} = p_{1j} - p_{10}, E_{2j} = p_{2j} - p_{20}, E_{3j} = p_{3j} - p_{30} \) in the following examples and simulations.

**Example 1** Consider a multiple mobile robots system contains six followers \( F_j \), \( j=1,2,\ldots,6 \), and one leader \( L \). For the system over the fixed directed topology as in Fig. 2, under the control protocols (14)–(15) with \( \epsilon = 1, k_1 = 8, k_2 = 26, k_3 = 4, l_1 = 0.9 \) and \( l_2 = 1.1 \), one can calculate that \( T_{\text{max}} = 7.0768 \). Choose the initial state \( \theta(0) = \left[ \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}, \frac{\pi}{6}, \frac{\pi}{7} \right]^T \), \( x(0) = [13, 11, 8, 3, 6, 10]^T \), and \( y(0) = [-15, -12, -18, -20, -23, -21]^T \).

For the system with six followers and one leader, the desired formation shapes \( F_1 \) and \( F_2 \), in which the black star is the leader and the others are followers, are given in Figs. 3 and 13. The orthogonal coordinates of the followers are:

\[(r_{1x}, r_{1y}) = (2, 0), \quad (r_{2x}, r_{2y}) = (1, \sqrt{3}), \]
\[(r_{3x}, r_{3y}) = (-1, \sqrt{3}), \quad (r_{4x}, r_{4y}) = (-2, 0), \]

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Fig. 2 Communication graph $G_1$.

$$(r_{5x}, r_{5y}) = (-1, -\sqrt{3}), \quad (r_{6x}, r_{6y}) = (1, -\sqrt{3}),$$

and the orthogonal coordinates of the leader is:

$$(r_{0x}, r_{0y}) = (0, 0).$$

The leader’s position is:

$$x_0 = 10\sin(2t + 1), \quad y_0 = -20\cos(t + 1/2).$$

Figure 4 shows the initial positions of all agents. The positions of all robots over the communication topology $G_1$ at time instants 0.88s, 2s, 7s, and the position trajectory of the leader are given in Figs. 5, 6, and 7, which illustrate that the six followers constitute the desired formation shape $F_1$, and the leader L is at the center of the formation shape $F_1$. The trajectories of the errors $E_{1j}, E_{2j}, E_{3j}$ ($j = 1, 2, ..., 6$) are given in Figs. 8, 9, and 10, from which one can find that $E_{1j}, E_{2j}, E_{3j}$ ($j = 1, 2, ..., 6$) converge to zero, and the convergence time is about 1.3s, which illustrate that the desired formation shape can be achieved in a fixed time.

Next, consider the system over the switching topology, which switches in the sequence $G_1, G_2, G_3, G_1, G_2, G_3, ...$ with the switching period 0.5s. The initial state is:

$$x(0) = [0, 3, 6, 10, 12, 8]^T,$$

$$y(0) = [-12, -20, -22, -23, -15]^T,$$

$$\theta(0) = [-\pi/2, \pi/3, -\pi/4, \pi/5, -\pi/6, \pi/7]^T.$$

And the desired formation shape $F_2$ is as that in Fig. 13. The orthogonal coordinates of the followers are:

$$(r_{1x}, r_{1y}) = (-2, 2), \quad (r_{2x}, r_{2y}) = (-2, 0), \quad (r_{3x}, r_{3y}) = (-2, -2), \quad (r_{4x}, r_{4y}) = (2, 2), \quad (r_{5x}, r_{5y}) = (2, 0),$$

$$(r_{6x}, r_{6y}) = (2, -2),$$

and the orthogonal coordinates of the leader are $(r_{0x}, r_{0y}) = (0, 0).$ Figure 14 shows
Fig. 6  Shape of formation $F_1$ in 2s and the trajectory of leader

Fig. 7  Shape of formation $F_1$ in 7s and the trajectory of leader

Fig. 8  Trajectories of the error $E_{1j}$ in formation $F_1$

Fig. 9  Trajectories of the error $E_{2j}$ in formation $F_1$

Fig. 10  Trajectories of the error $E_{3j}$ in formation $F_1$

the initial positions of all agents. Figures 15, 16, and 17 show the position of all robots at time instants 0.88s, 2s, and 7s, respectively, and the position trajectory of the leader. Figures 18, 19, and 20 show the trajectories of error between $p_{1j}$ and $p_{10}$, between $p_{2j}$ and $p_{20}$, and between $p_{3j}$ and $p_{30}$, that is $E_{1j}$, $E_{2j}$, and $E_{3j}$ ($j = 1, 2, ..., 6$), respectively, from which one can see that the robots form the desired formation $F_2$ with the leader being at the center, and the formation time is less than $T_{\text{max}}$, which verifies the efficiency of the methods.
Fig. 11 Communication graph $G_2$.

Fig. 12 Communication graph $G_3$.

Fig. 13 Desired shape of formation $F_2$.

Fig. 14 Initial shape of $F_2$.

Fig. 15 Shape of formation $F_2$ in 0.88s and the trajectory of leader.

Fig. 16 Shape of formation $F_2$ in 2s and the trajectory of leader.
Fig. 17 Shape of formation $F_2$ in 7s and the trajectory of leader

Fig. 18 Trajectories of the error $E_{1j}$ in formation $F_2$

Fig. 19 Trajectories of the error $E_{2j}$ in formation $F_2$

Fig. 20 Trajectories of the error $E_{3j}$ in formation $F_2$

Fig. 21 Communication graph $G_4$.

Fig. 22 Desired shape of formation $F_3$. 
Example 2 In order to further verify the effectiveness of the method proposed in this paper, a system with 20 followers and one leader is considered in the following. Firstly, consider the system over the fixed communication topology $G_4$ as that in Fig. 21. The initial positions of all mobile robots are shown in Fig. 23, and the desired formation shape $F_3$ is given in Fig. 22, in which the black star is the leader and the other blue stars are followers. Under the control protocols (14) and (15), choose $\varepsilon = 1$, $l_1 = 0.9$, $l_2 = 1.1$, $k_1 = 10$, $k_3 = 5$. Then it can be obtained that $T_{\text{max}} = 5.8775s$. Figures 24, 25, and 26 show the position of all robots at time instants 0.88s, 2s, and 7s, respectively, and the position’s trajectory of the leader. Figures 27, 28, and 29 show the trajectories of error between $p_{1j}$ and $p_{10}$, between $p_{2j}$ and $p_{20}$, and between $p_{3j}$ and $p_{30}$, i.e., $E_{1j}$, $E_{2j}$ and $E_{3j}$ ($j = 1, 2, \ldots, 20$), respectively. Figures 24, 25, 26, 27, 28, and 29 illustrate that all robots form the desired formation shape $F_3$ in a fixed time, and the center of the formation tracks the leader.

Next, consider the system with 20 followers and one leader over the switching topology, which is switched in the sequence $G_4, G_5, G_6, G_7, G_4, G_5, G_6, G_7, \ldots$, with the switching period 0.25s. And graphs $G_4, G_5, G_6, G_7$ are as that shown in Figs. 21, 30, 31, and 32. The desired formation shape $F_4$ is given in Fig. 33, and the initial positions of all mobile robots are shown in Fig. 34. Under the control protocol with the same parameters as that over the fixed topology, the position of all robots and the position’s trajectory of the leader are shown in Figs. 35, 36, and 37, and the trajectories of the error $E_{1j}$, $E_{2j}$ and $E_{3j}$ ($j = 1, 2, \ldots, 20$) are shown in Figs. 38, 39, and 40. And Figs. 35, 36, 37, 38, 39, and 40 illustrate that all robots form the desired formation shape in a fixed time, and the errors $E_{1j}$, $E_{2j}$, and $E_{3j}$ ($j = 1, 2, \ldots, 20$) tend to zero, which verify the effectiveness of the method proposed in this work.

5 Conclusion

This paper studies the distributed fixed-time formation tracking based on distributed observer for multiple nonholonomic wheeled mobile robots over directed topology. The stability of the distributed controllers is
Fig. 26  Shape of formation $F_3$ in 7s and the trajectory of leader

Fig. 27  Trajectories of error $E_{1j}$ in formation $F_3$

Fig. 28  Trajectories of error $E_{2j}$ in formation $F_3$

Fig. 29  Trajectories of error $E_{3j}$ in formation $F_3$

Fig. 30  Communication graph $G_5$.

Fig. 31  Communication graph $G_6$. 
Also proved by graph theory and stability theory. Some sufficient conditions of multiple robots achieving the desired formation shape are given. Several numerical simulations are presented to verify the efficiency of the proposed method. However, the communication delay between robots and the impact of noisy/lossy communication between neighbors are ignored, which are the future work to be done. Adopting the event-
triggered control technique to reduce the communication between robots also needs to be further studied.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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