Determination of low energy parameters for NN-scattering at N^{4}LO in all partial waves with \( j \leq 5 \).

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(Dated: November 10, 2018)

The Variable S-matrix approach offers a unique way to extract low energy threshold parameters for a given NN potential. We extract those parameters for the np system from the NijmII and Reid93 potentials, to all partial waves with total angular momentum \( j \leq 5 \), using the generalized effective range expansion

\[
(f^{s}_{j})_{l,l'}k^l k^{l'} = -(a^{s}_{j-1})_{l,l'} + \frac{1}{2}(r^{s}_{j})_{l,l'} k^2 + (v_2^{s}_{j})_{l,l'} k^4 \quad \text{and} \quad (v_3^{s}_{j})_{l,l'} k^6 \quad \text{and} \quad (v_4^{s}_{j})_{l,l'} k^8 \quad \text{and} \quad \ldots \quad -i k^{l+l'+1}
\]

where \( f^{s}_{j} = (S^{s}_{j} - 1)/(2i) \) is the scattering amplitude and \( S^{s}_{j} \) is the unitary S-matrix in coupled channel space with total spin \( s \) and total angular momentum \( j \). Our calculation includes all the relevant contributions of the full amplitude to order \( O(k^8) \) in the CM momentum. We also discuss the validity of the generalized effective range expansion in the region of analyticity \( k \leq m_{e}/2 \).

PACS numbers: 03.65.Nk,13.75.Cs,21.30.Fe

Keywords: NN-interaction, Variable S-matrix, Effective Range Expansion, Coupled Channels

I. INTRODUCTION

The study of the NN interaction has played a dominant role in the theory of nuclear structure. At low energies the corresponding scattering phase shifts can be best parameterized in terms of an effective range expansion (ERE) \( \| \). This expansion can be suitably generalized for all partial waves and coupled channels (for a review see e.g. Ref. \( 4 \) and references therein.). The calculation of the low energy threshold parameters is straightforward in potential scattering, and can be extracted from the asymptotic form of the wave function computed order by order in a low energy expansion, but unlike potentials the ERE parameters are not subjected to off-shell ambiguities. Benchmark descriptions of NN scattering data have been obtained by the phase shift partial wave analysis (PWA) of the Nijmegen group carried out a decade ago \( 2 \) and further parameterizations through high quality potentials \( 5 \). For channels involving central waves low energy threshold parameters have been determined routinely and for the high quality potentials this calculation has been undertaken in Ref. \( 6 \). However, a detailed and systematic determination of these threshold parameters in non central partial waves is, to our knowledge, still missing.

Almost parallel with the previous developments there has been in the last decade a renewed interest on the NN interaction based on the application of Lagrangian effective field theories (EFT) methods, motivated by Weinberg’s ideas \( 8,9 \), and pursued by many others \( 10,11,12,13,14,15,16,17,18,19,20,21,22,23,24 \) to attempt a realistic description of np scattering data. This approach has open promising perspectives in the theory of nuclear systems (for a recent review see e.g. Ref. \( 25 \) and references therein.). Actually, the reference database most modern EFT calculations are confronted with are those of the Nijmegen group \( 26,27 \). Moreover, the determination of the low energy constants (LEC’s) appearing in the EFT Lagrangian depends on the regularization scale and also on the short distance behaviour of the long distance chiral potential due to One, Two and higher Pion exchanges. The results for the scattering matrix and in particular for the low energy threshold parameters should, of course, be independent on the renormalization prescription. In the absence of explicit pion exchanges the EFT approach can be mapped into an effective range expansion \( 13 \), and the LEC’s can, in principle, be deduced from the physical threshold parameters for a given renormalization scale. When long distance chiral potentials generated by successive pion exchanges are included the situation becomes much more involved, particularly if those potentials cannot be treated perturbatively. Recently, we have proposed a regularization scheme \( 34,35 \) where the LEC’s can also be deduced directly from the low energy threshold parameters beyond perturbation theory within a variable S-matrix approach \( 36 \). The implementation of such an approach requires an explicit knowledge of the threshold parameters as input. In practice, most EFT calculations fit their results to the Nijmegen database \( 5,6 \) in a given energy window in terms of the LEC’s within a given regularization scheme and at a given scale, and low energy threshold parameters could be determined afterwards. Under these circumstances we think it of interest to provide in any case these low energy threshold parameters as deduced \( \text{directly} \) from the high quality potentials \( 5,6 \) in all partial waves.
In this paper we determine the low energy parameters of the NijmII and Reid93 potentials for all partial waves with $j \leq 5$. To this end we use the variable S-matrix approach (for a review with many references see e.g. Ref. 32). An application of the variable phase approach in the context of EFT in the singlet $1 S_0$ channel can be found in Ref. 32, where the S-matrix can be computed by solving a non-linear system of first order differential equations representing the variation of the S-matrix under a continuous switching on of the potential starting from a trivial potential (see below for a sketchy derivation). By implementing a low energy expansion one obtains a set of first order coupled differential equations from which the low energy threshold parameters can be obtained as asymptotic values at long distances for any partial wave. The method is stable and whenever comparison can be made reproduces previous known results for the ERE parameters in central waves. Partial aspects of the present paper have already appeared in connection with the $1 S_0$ and $\bar{3} S_1 - 3 D_1$ channels and in the context of the renormalization of the singular OPE potential.

The paper is organized as follows. In Sect. II we introduce our notation for scattering amplitudes, partial wave expansion M-matrix and scaled M-matrix, which enable to take the low energy expansion in a particularly transparent and clean way. We also discuss the interplay between low energy expansion, rotational covariance and unitarity. In Sect. III we briefly outline the derivation of the variable S-matrix for coupled channels, following the method of adiabatically switching on the potential from the origin. Translating this equation to the variable scaled M-matrix we write down in Sect. IV the coupled differential equations which determine the final condition to the low energy threshold parameters. Our numerical results are presented in Sect. V. In Sect. VI A we show the deduced threshold parameters for the NijmII and Reid93 potentials for all partial waves with total angular momentum $j \leq 5$. In Sect. VI B we examine the validity of the expansion for these partial waves. In Sect. VII we study the poles and residues of the scattering matrix, mainly in connection with the $1 S_0$ virtual state and the $\bar{3} S_1 - 3 D_1$ bound state deuteron within the coupled channel effective range expansion. Finally, in Sect. VIII we present some conclusions and outlook for further work. In Appendix A we provide some raison d’être for the present calculation and show the difficulties we have encountered when a direct fit to the database is attempted. In the Appendix B we investigate further the virtual and bound state properties in an expansion of the inverse scattering length in the $s$–wave channel.

II. S-MATRIX, PHASE SHIFTS AND THRESHOLD PARAMETERS

As it is well known (see e.g. Ref. 32), the requirement of all relevant symmetries in the NN-interaction implies that for the two nucleon system in a total spin $s$ and total isospin $t$ state, the elastic scattering amplitude for the transition $\hat{k}, m_s \rightarrow \hat{k}', m_s'$ with fixed CM energy $E = k^2/2\mu_{np}$ reads

$$f_{m_s^\prime m_s}(\hat{k}', \hat{k}) = \sum_{j l' l} C_{l' l}^{s j}(\hat{k}', \hat{k}) [1 - (-i)^{l + s + 1}] f_{l', l}^{s j} \quad (1)$$

where

$$C_{l' l}^{s j}(\hat{k}', \hat{k}) = \sum_{m} i^{j - l'} C(l, s, j | m - m_s, m_s, m) Y_{l m - m_s}(\hat{k})$$

$$\times C(l', s, j | m - m_s', m_s', m) Y_{l' m - m_s'}(\hat{k})^* \quad (2)$$

and $f_{l', l}^{s j}$ is the scattering amplitude for the $2s + 1 l_j$ state with a total angular momentum $j$, total spin $s$ and orbital angular momentum $l$ and is given by

$$f_{l', l}^{s j} = \frac{1}{2i k} \left( S_{l', l}^{s j} - \delta_{l', l} \right) \quad (3)$$

in terms of the symmetric and unitary S-matrix, $S_{l', l}^{s j} = S_{l, l}^{s j} - \delta_{l', l}$. Due to parity conservation, for $j > 0$ the states with $l = j$ cannot couple to states with $l = j \pm 1$. Thus, for the spin singlet state, $s = 0$, one has $l = j$ and hence the state is uncoupled

$$S_{j j}^{0 j} = e^{2 i \delta_{j j}^0} \quad (4)$$

whereas for the spin triplet state $s = 1$, one has the uncoupled $l = j$ state

$$S_{j j}^{1 j} = e^{2 i \delta_{j j}^1}, \quad (5)$$

and the two channel coupled states $l, l' = j \pm 1$ states which we use Stapp-Ypsilantis-Metropolis (SYM or Nuclear bar) 40 parameterization for

$$S_{j j}^{1} = \begin{pmatrix}
S_{j + 1, j}^{1 j} & S_{j - 1, j + 1}^{1 j} \\
S_{j - 1, j}^{1 j} & S_{j + 1, j + 1}^{1 j}
\end{pmatrix}$$

$$= \begin{pmatrix}
\cos (2 \epsilon_j) e^{2 i \delta_{j j}^{-1}} & i \sin (2 \epsilon_j) e^{(i \delta_{j j}^{-1} + \epsilon_j)} \\
i \sin (2 \epsilon_j) e^{(i \delta_{j j}^1 + \epsilon_j)} & \cos (2 \epsilon_j) e^{2 i \delta_{j j}^1}
\end{pmatrix}$$

In the discussion of low energy properties it is also interesting to use the Blatt-Biedenharn (BB or Eigen phase) parameterization 44 that is given by

$$S_{j j}^{1} = \begin{pmatrix}
\cos \epsilon_j & -\sin \epsilon_j \\
\sin \epsilon_j & \cos \epsilon_j
\end{pmatrix} \begin{pmatrix}
e^{2 i \delta_{j j}^{-1}} & 0 \\
0 & e^{2 i \delta_{j j}^1}
\end{pmatrix} \begin{pmatrix}
\cos \epsilon_j & \sin \epsilon_j \\
-\sin \epsilon_j & \cos \epsilon_j
\end{pmatrix} \quad (6)$$

1 We use the normalization for the differential cross section

$$\frac{d\sigma(k, m_s \rightarrow k', m_s')}{d\Omega} = |f_{m_s^\prime m_s}(k', k)|^2$$
The relation between the BB and SYM phase shifts is given by
\begin{align}
\delta_{ij}^{1j} + \delta_{j-1}^{1j} &= \delta_{i+1}^{1j} + \delta_{j-1}^{1j}, \tag{7}
\sin(\delta_{j-1}^{1j} - \delta_{j-1}^{1j}) &= \frac{\tan(2\bar{\varepsilon}_j)}{\tan(2\bar{\varepsilon}_j)}. \tag{8}
\end{align}
In order to take the low energy limit we define the standard symmetric and real coupled channel scaled M-matrix
\[ S = \left( \hat{M} + ikD^2 \right) \left( \hat{M} - ikD^2 \right)^{-1}, \tag{9} \]
with \( k \) the CM up momentum and \( D = \text{diag}(k^1, \ldots, k^N) \). Due to unitarity of the S-matrix in the low energy limit, \( k \to 0 \) we have
\[ (S - 1)f_{l,j} = -2i\alpha_{l,j}k^{l+i+1} + \ldots, \tag{10} \]
with \( \alpha_{l,j} \) the (hermitian) scattering length matrix. The threshold behaviour acquires its simplest form in the SYM representation,
\begin{align}
\delta_{ij}^{0j} &\to -\bar{\alpha}_{ij}^0 k^{2j+1}, \tag{11}
\delta_{ij}^{1j} &\to -\bar{\alpha}_{ij}^{-1} k^{2j+1}, \tag{12}
\delta_{j-1}^{1j} &\to -\bar{\alpha}_{j-1}^{1j} k^{2j+1}, \tag{13}
\delta_{j+1}^{1j} &\to -\bar{\alpha}_{j+1}^{1j} k^{2j+3}, \tag{14}
\delta_{j}^{2j} &\to -\bar{\alpha}_{j}^{2j} k^{2j+1}. \tag{15}
\end{align}
defining a natural hierarchy for phase-shifts (see also below) \(^2\). The scaled M-matrix, \( \hat{M} \), has a good low energy behaviour \(^4\) and admits the coupled channel analog of the effective range expansion
\[ \hat{M} = -\bar{a}^{-1} + \frac{1}{2} \bar{r} k^2 + \bar{v}_2 k^4 + \bar{v}_3 k^6 + \bar{v}_4 k^8 + \ldots, \tag{16} \]
where \( a, r \) and \( v_2 \) are the scattering length matrix, effective range and curvature parameters respectively. Higher order matrix parameters will be denoted for notational simplicity \( v_3, v_4 \), and so on. Within the ERE our notation is as follows: \( \text{LO} \) means including order \( k^0 \) terms, \( \text{NLO} \) including \( k^2 \) terms and \( \text{NNLO} \) including \( k^4 \) terms, and so on. The expansion in \( k^2 \) around the origin holds up to the next singularity, which in the case of NN interaction corresponds to the pion left cut, so \( |k| \lesssim \pm m_{\pi}/2 \sim 70 \text{MeV} \). One of the advantages of the low energy expansion at the level of the scaled M-matrix is that unitarity is preserved exactly at any order in the expansion.

If one diagonalizes the M-matrix using the orthogonal transformation, which we call \( \Omega \), in Eq. \( \text{[6]} \)
\[ \hat{m} = \Omega D \Omega^{-1} \hat{M} \Omega \Omega^{-1} D \tag{17} \]
the corresponding eigenvalues are related to the BB eigenphaseshifts as follows
\[ \hat{m}_l = k^{2l+1} \cot \delta^{l}_j = -\frac{1}{\alpha^{l}_j} + \frac{1}{2} \varepsilon^{l}_j k^2 + \nu^{l}_j k^4 + \ldots \tag{18} \]
for \( l = j \pm 1 \).

In connection with the low energy expansion note that the total amplitude \( \hat{T} \) satisfies both the off-forward optical theorem from the unitarity of the S-matrix (\( \hat{q} \) represents solid angle interaction in the unit vector \( \hat{q} \) direction),
\[ f_{m',m}^{st}(\hat{k}', \hat{k}) = f_{m',m}^{st}(\hat{k}', \hat{k})^* = \frac{i}{2\pi} \sum_{m''} \int dq f_{m''}^{st} \left( \hat{k}', \hat{q} \right) f_{m''}^{st} \left( \hat{k}, \hat{q}^* \right) \tag{19} \]

as well as rotational covariance, i.e. under rotations \( \hat{k} \to R\hat{k} \) one has \(^3\)
\[ f_{m',m}^{st}(\hat{k}', \hat{k}) = \sum_{m''} D^*_{m',m''}(R) f_{m',m''}^{st}(\hat{R}\hat{k}', \hat{R}\hat{k}) \tag{20} \]
with \( D_{m',m''}^{st}(R) \) the rotation Wigner matrices corresponding to spin \( s \). This induces a mixing of the coupled channel states, since they are irreducible under rotations. The latter holds for any truncated sum in the partial wave expansion in Eq. \( \text{[1]} \). On the other hand, the low energy expansion of the amplitude only satisfies this properties perturbatively; the coefficients of \( \hat{k}' \) and \( \hat{k} \) in a Taylor expansion of the amplitude \( f_{m',m}^{st}(\hat{k}', \hat{k}) \) are neither rotational covariance nor do they satisfy exact unitarity. This can be seen from the coupled channel amplitude,
\[ f = D \left( \hat{M} - ikD^2 \right)^{-1} D \tag{21} \]
which yields after taking the \( k \to 0 \) expansion yields
\[ f = -\bar{D} \bar{a} D - \frac{1}{2} \bar{D} \bar{a} \bar{r} D k^2 + \ldots \tag{22} \]
This threshold behaviour sets up a natural low energy hierarchy for the \( 2s+1l_j \) states
\begin{align}
\mathcal{O}(k^0) &\quad 1S_0, 3S_1 \\
\mathcal{O}(k^2) &\quad 1P_1, 3P_0, 3P_1, 3P_2, E_1 \\
\mathcal{O}(k^4) &\quad 1D_1, 3D_1, 3D_2, 3D_3, E_2 \\
\mathcal{O}(k^6) &\quad 1F_3, 3F_2, 3F_3, 3F_4, E_3 \\
\mathcal{O}(k^8) &\quad 1G_4, 3G_3, 3G_4, 3G_5, E_4 \tag{23} \\
\end{align}

\(^2\) In the BB form one has similar behaviours for the \( \delta \)’s but \( \varepsilon_j \to -\bar{\alpha}_j^{1j} k^2 \).

\(^3\) Corresponding to the invariance of \( f(\hat{k}', \hat{k}) \).
where we use the notation $E_1 = -S_1 - D_1$, $E_2 = -P_2 - F_2$, $E_3 = -D_3 - G_3$, $E_4 = -F_4 - H_4$, and so on. (in general $E_j = -(j + 1)^2 - (j - 1)$). This means that for a complete calculation at $\alpha = 2$ parameters, at $N U$ one has to include the $\alpha$’s for the states $1G_4, 3G_L, 3G_T$, and $E_4$, $\alpha$’s and $r$’s for $1F_3, 3F_3, 3F_4$ and $E_3$, $\alpha$’s, $r$’s and $v_2$’s for $1D_1, 3D_1, 3D_2, 3D_3$ and $E_2$ and $\alpha$’s, $r$’s $v_2$’s and $v_3$’s for $1P_1, 3P_1, 3P_2, E_1$, and $\alpha$’s, $r$’s, $v_2$’s, $v_3$’s and $v_4$’s for $1S_1$ and $3S_1$. In general, to $O(k^{2N})$ one needs
\begin{equation}
N = 2 \times (n + 1) + 5 \times \frac{1}{2} n(n + 1)
\end{equation}

independent parameters. Thus, at $O(k^0)$ we need only $N = 2$ parameters, at $O(k^2)$ one has $N = 9$ parameters, at $O(k^3)$, $N = 21$, at $O(k^4)$, $N = 38$ and at $O(k^6)$ we have $N = 60$ parameters, etc. Although these seem rather large numbers, note that the declared number of independent adjustable parameters to NN data in the high quality data of Ref. [6] is 41. The very recent EFT calculation NNLO in the Weinberg counting [31] declares 26 adjustable parameters (actually LEC’s) to the NN database of Ref. [6]. This calculation corresponds to go to order $O(k^6)$ for the theory without explicit pions, i.e. for $k \ll m_{\pi}/2$.

In this paper we use the low energy expansion in a way that both unitarity [13] and rotational covariance [20] are simultaneously satisfied. As a consequence, the low energy expansion for the amplitude is not complete in the sense that starting at a given order on $k$ not all the contributions of higher order are taken into account. Nevertheless, up to order $O(k^6)$ we compute all the 60 parameters. The highest angular momentum involving this order is $j = 5$. For completeness we present all the effective range parameters up to $v_3$ in all partial waves with $j \leq 5$. We will see below, however, that presenting the data in terms of the scaled M-matrix provides some interesting insights regarding the validity of the effective range expansion.

III. VARIABLE S-MATRIX

In this Section we re-derive the variable S-matrix equation using a continuous deformation of the potential. Alternative derivations are based on Jost functions [37] and inner boundary conditions [34, 39]. The coupled channel Schrödinger equation for the relative motion reads
\begin{equation}
- u''(r) + \left[ U(r) + \frac{1}{r^2} \right] u(r) = k^2 u(r),
\end{equation}
where $U(r) = 2\mu_{np} V(r)$ is the coupled channel matrix potential which can be written as for $j > 0$,
\begin{equation}
U^{aj}_j(r) = U^{aj}_j(r), \quad U^{1j}_j(r) = \begin{pmatrix}
U^{1j}_{j-1,j-1}(r) & 0 & U^{1j}_{j-1,j+1}(r) \\
0 & U^{1j}_j(r) & 0 \\
U^{1j}_{j+1,j+1}(r) & 0 & U^{1j}_{j+1,j+1}(r)
\end{pmatrix}
\end{equation}
We will take for these potentials the ones available in Ref. [3]. In Eq. (26), $U = \text{diag}(l_1(l_1 + 1), \ldots, l_N(l_N + 1))$ is the angular momentum, $u(r)$ is the reduced matrix wave function and $k$ the C.M. momentum. In the case at hand $N = 1$ for the spin singlet channel with $l = j$ and $N = 3$ for the spin triplet channel with $l_1 = j - 1$, $l_2 = j$ and $l_3 = j + 1$. For ease of notation we will keep the compact matrix notation of Eq. (26). The potentials in Ref. [6] are regular at the origin, so the regular solution is given by the boundary condition at the origin
\begin{equation}
u(0) = 0,
\end{equation}

At long distances, we assume the asymptotic normalization condition
\begin{equation}
u(r) \to \hat{h}^{(-)}(r) - \hat{h}^{(+)}(r) S
\end{equation}
with $S$ the standard coupled channel unitary S-matrix. The corresponding out-going and in-going free spherical waves are given by
\begin{equation}
\hat{h}^{(\pm)}(r) = \text{diag}(\hat{h}^{(\pm)}_1(x), \ldots, \hat{h}^{(\pm)}_j(x))
\end{equation}
with $\hat{h}^{(\pm)}_j(x)$ the reduced Hankel functions of order $l$, $\hat{h}^{(\pm)}_0(x) = x H^{(\pm)}_{1/2}(x)$, and satisfy the free Schrödinger’s equation for a free particle.

In order to deduce a variable $S$-matrix equation, we determine first the infinitesimal change of the $S$ matrix under a general deformation of the potential $U(r) \to U(r) + \Delta U(r)$. Using Schrödinger’s equation (26) and the standard Lagrange’s identity adapted to this particular case, we get
\begin{equation}
[u(r) \Delta u'(r) - u'(r) \Delta u(r)]' = u'(r) \Delta U(r) u(r)
\end{equation}
which, after integration from the origin to infinity and using the asymptotic form of the matrix wave function, Eq. (28), as well as the regular condition at the origin, Eq. (27) yields
\begin{equation}
2ik S^l \Delta S = \int_0^\infty dr u(r) \Delta U(r) u(r)
\end{equation}
In particular, for the parametric family of potentials $U(r, R) = \delta(R - r) U(r)$ we get $\Delta U(r, R) = \delta(R - r) U(r) \Delta R$ and hence
\begin{equation}
2ik S^l(R) \Delta S^l(R) = u(R)^l U(R) u(R)
\end{equation}
and using the value of the wave function at the outer boundary, Eq. (25)
\begin{equation}
 u(R) = \hat{h}^{(-)}(R) - \hat{h}^{(+)}(R) S(R),
\end{equation}
we finally get the variable S-matrix equation,
\begin{equation}
2ik \frac{dS^l(R)}{dR} = \left[ S(R) \hat{h}^{(+)}(R) - \hat{h}^{(-)}(R) \right] U(R)
\times \left[ \hat{h}^{(-)}(R) - \hat{h}^{(+)}(R) S(R) \right].
\end{equation}
This is a first order non-linear matrix differential equation which can be solved by standard means, provided the S-matrix is known at one given scale. Note that for any value of the boundary radius we have a different on-shell scattering problem. In the case of a regular potential, Eq. (34) has to be supplemented with an initial condition at the origin, namely the trivial one (corresponding to the absence of a potential), and its asymptotic value yields the full S-matrix:

\[ S(0) = 1, \quad S = S(\infty) \]  

(35)

IV. EVOLUTION OF LOW ENERGY PARAMETERS

In order to take the low energy limit of Eq. (34) and corrections there-off, we introduce the variable or running M-matrix, in analogy with Eq. (9)

\[ S(R) = \left( \hat{M}(R) + i kD^2 \right)^{-1} \left( \hat{M}(R) - i kD^2 \right)^{-1}, \]  

(36)
as well as the reduced Bessel functions

\[ \hat{j} = \frac{1}{2i} \left( \hat{h}^+ - \hat{h}^- \right), \]  

(37)

\[ -\hat{y} = \frac{1}{2} \left( \hat{h}^+ + \hat{h}^- \right), \]  

(38)

with

\[ \hat{j}_l(x) = xj_l(x), \quad \hat{y}_l(x) = xy_l(x). \]  

(39)

Thus, we get

\[ \hat{M}'(k, R) = \left( \hat{M}(R, k) + \frac{1}{k} \hat{j}(kR)D^{-1} - \hat{y}(kR)D \right) U(R) \]

\[ \times \left( \frac{1}{k} \hat{j}(kR)D^{-1} \hat{M}(R, k) - \hat{y}(kR)D \right). \]  

(40)

The scaled variable M-matrix admits the analog of the effective range expansion

\[ \hat{M}(R) = -a(R)^{-1} + \frac{1}{2} r(R)k^2 + v_2(R)k^4 + v_3(R)k^6 + \ldots, \]  

(41)

where \( a(R), r(R), v_2(R), v_3(R), v_4(R), \ldots \) are the corresponding running parameters. In this form the low energy limit can be easily taken. Defining the matrix functions

\[ A_k(R) = \text{diag} \left( \frac{j_{2l}(kR)}{k^{2l+1}}, \ldots, \frac{j_{2N}(kR)}{k^{2N+1}} \right) \]

\[ B_k(R) = \text{diag} \left( \frac{y_{2l}(kR)}{k^{2l}}, \ldots, \frac{y_{2N}(kR)}{k^{2N+1}} \right), \]  

(42)

and their low energy expansion

\[ A_k(R) = \frac{j_{2l}(kR)}{k} D^{-1} = A_0 + kA_2 + k^2A_4 + \ldots, \]

\[ B_k(R) = \frac{y_{2l}(kR)}{k} D = B_0 + kB_2 + k^2B_4 + \ldots, \]  

(43)

we get the system of coupled equations

\[ \frac{d}{dR} \left[ a(R) \right]^{-1} = - \left[ \left[ a(R) \right]^{-1} A_0 + B_0 \right] U(R) \left( A_0 \left[ a(R) \right]^{-1} + B_0 \right), \]

\[ \frac{d}{dR} r(R) = \left[ \left[ a(R) \right]^{-1} A_0 + B_0 \right] U(R) \left( r(R)A_0 + 2 \left[ a(R) \right]^{-1} A_2 + 2B_2 \right) \]

\[ + \left( r(R)A_0 + 2 \left[ a(R) \right]^{-1} A_2 + 2B_2 \right) U(R) \left( \left[ a(R) \right]^{-1} A_0 + B_0 \right), \]  

(44)

\[ \frac{d}{dR} v_2(R) = \left[ \left[ a(R) \right]^{-1} A_0 + B_0 \right] U(R) \left( \frac{1}{2} r(R)A_2 + v_2(R)A_0 - B_4 \right) \]

\[ + \left( \frac{1}{2} r(R)A_2 - \left[ a(R) \right]^{-1} A_2 - B_2 \right) U(R) \left( \frac{1}{2} r(R)A_2 - \left[ a(R) \right]^{-1} A_2 - B_2 \right), \]  

and similar equations for \( v_3(R) \) and \( v_4(R) \). These equations generalize to the coupled channel case those already found in Ref. 34 and have to be supplemented with the initial conditions,

\[ a(0) = 0, \quad r(0) = 0, \quad v_2(0) = 0, \ldots \]  

(45)

\[ a(\infty) = a, \quad r(\infty) = r, \quad v_2(\infty) = v_2, \ldots \]  

(46)

The physical threshold parameters correspond to the values at infinite,
In the triplet coupled channel case the SYM threshold parameters matrices are

\[ a^{ij} = \begin{pmatrix} \bar{a}_i^{1j} & \bar{a}_i^{j1} \\ a_j^{1i} & a_j^{j1} \end{pmatrix}, \]  
\[ r^{ij} = \begin{pmatrix} \bar{r}_i^{1j} & \bar{r}_i^{j1} \\ r_j^{1i} & r_j^{j1} \end{pmatrix}, \]  
\[ v^{ij} = \begin{pmatrix} \bar{v}_i^{1j} & \bar{v}_i^{j1} \\ v_j^{1i} & v_j^{j1} \end{pmatrix}, \]

and so on. Similar definitions hold for the uncoupled channels.

V. NUMERICAL RESULTS

A. Determination of low energy parameters

Low energy scattering data can directly be described in terms of threshold parameters, like \( \alpha, r, \) etc., defined through Eq. (16). Unfortunately, besides \( \alpha \) and \( r_0 \) in the \( ^1S_0 \) and \( ^3S_1 \) channels, the partial wave analysis data base \[^5\] does not provide values for them except for \( v_2, \bar{v}_3, v_4 \) in Ref. \[^7\] in the deuteron channel. In Appendix A we show that a direct fit to the database turns out to be numerically unreliable. The NN data base provides explicit potentials like the NijmII and Reid93 potentials can be summarized in Table I. To obtain these results we looked for a stable plateau in the large \( R \) region in each threshold parameter separately. So we quote in Table II the stable digits in this flat region. This is the main result of this work. In general we see a rather good agreement between both potentials never worse than 5% and frequently much better. We also observe, as expected, that numerical accuracy worsens by going to higher orders in the ERE in a given partial wave or going to higher partial waves.

In the interesting case of the \( ^3S_1 \) eigen-channel, one has \(^4\)

\[ k \cot \delta_{3S1} = -\frac{1}{\alpha_{3S1}} + \frac{1}{2} r_{3S1} k^2 + v_{3S1} k^4 + \ldots \]  

where we get for the effective range parameters the relations,

\[ \alpha_{3S1} = \bar{\alpha}_{3S1} \]  
\[ r_{3S1} = \bar{r}_{3S1} + \frac{2 \bar{r}_{E1} \bar{\alpha}_{E1}}{\alpha_{3S1}^2} + \frac{\bar{r}_{3D1} \bar{r}_{E1}^2}{\alpha_{3S1}^4} \]  
\[ v_{3S1} = \bar{v}_{3S1} + \frac{1}{4} \bar{\alpha}_{E1} (2 \bar{a}_{3D1} \bar{r}_{E1} - \bar{a}_{E1} \bar{r}_{E1}^2 + 8 \bar{v}_{E1}) \]  
\[ + \frac{\bar{a}_{E1}^2}{4 \alpha_{3S1}^2} (\bar{a}_{3D1} \bar{r}_{E1}^2 - 2 \bar{a}_{E1} \bar{r}_{E1} \bar{r}_{E1} + 4 \bar{v}_{3D1}) \]  
\[ + \frac{1}{4 \alpha_{3S1}^4} (4 \bar{a}_{E1}^2 - \bar{a}_{E1} \bar{r}_{E1}^2) \]

and so on. Using the numerical values for the NijmII (Reid93) potentials from Table II we get

\[ \alpha_{3S1} = 5.41896 (5.42293) \text{ fm} \]  
\[ r_{3S1} = 1.75334 (1.75556) \text{ fm} \]  
\[ v_{3S1} = 0.04531 (0.03266) \text{ fm}^3 \]  
\[ v'_{3S1} = 0.65831 (0.65797) \text{ fm}^5 \]  
\[ v''_{3S1} = -4.19144 (-4.19262) \text{ fm}^7 \]

in agreement with previous findings \[^7\] \[^31\].

B. Validity of the effective range expansion for partial waves

Once the low energy threshold parameters have been determined we can readily check the validity of the effective range expansion given by Eq. (16). On theoretical grounds the full expansion around the origin should be convergent within the analyticity domain, i.e. up to the next singularity, which for NN interaction happens at about \( k = \pm \imath m / 2 \), i.e. \( E_{\text{LAB}} \sim -10 \text{ MeV} \) where it develops a logarithmic branch cut due to the One Pion Exchange Potential. At \( k = \pm \imath m / 2 \) a Two Pion Exchange dilogarithmic branch cut sets in at about \( E_{\text{LAB}} \sim -40 \text{ MeV} \), and so on. From this viewpoint any realistic description of the phase shifts should be considerably unreliable for energies beyond the analyticity domain \( \sim 10 \text{ MeV} \). On the other hand, this does not necessarily imply that the polylogarithmic corrections are numerically large. So one may directly learn from the “data” how important these corrections are in practical terms. There are two possible ways to do this, either by using the SYM phase shifts given in Ref. \[^7\] or by direct evaluation of the scaled M-matrix defined. In terms of the LAB energy, \( E_{\text{LAB}} = k^2 / \mu_{np} \) the scaled \( M \) matrix can be represented as a constant, straight line and a parabola if we keep the scattering lengths, effective ranges and curvature parameters. The results of such a comparison for the scaled M-matrix is given in the series

\(^4\) We use the notation \( v = v_2, v' = v_3 \) and \( v'' = v_4 \) for simplicity.
of figures Fig. 1, 2, 3, 4, 5 and 6 for all partial waves with 0 \leq j \leq 5. In table III we present the effective range parameters for each partial wave in units of powers of \( m_p/2 \), so that they correspond to the contribution of different terms of the ERE at the maximum value of \( k \) within the domain of analyticity of \( \hat{M}(k) \).

On the light of these figures, two conclusions may be drawn. In the first place, for a fixed LAB energy as we increase the angular momentum, the relative error to the full scaled M-matrix increases. This is expected since higher angular momenta are more sensitive to the long distance physics, which is pion dominated. In the limit of large angular momentum peripheral waves, the low energy threshold parameters should be described solely in terms of pion dynamics. The second observation is that the absolute error decreases for increasing angular momentum as it corresponds for a suppression of large scattering angles.

### C. Poles and Residues of the scattering matrix

The scattering amplitude has poles for negative energies \(^5\)

\[
S_{j,k}^{s} \rightarrow \frac{A_{j}^{s} A_{j'}^{s}}{\gamma + i k} \tag{54}
\]

where \( \gamma > 0 \) corresponds to a bound state (first Riemann sheet in \( E \)) and \( \gamma < 0 \) (second Riemann sheet in \( E \)) to a virtual state. The coefficients \( A_{j}^{s} \) correspond to the asymptotic bound state wave function

\[
u_{j}^{s}(r) \rightarrow A_{j}^{s} \hat{h}^{(+)}(i\gamma r) \tag{55}
\]

\(^5\) We assume that there are no degenerate poles. In the two channel case this condition is automatically satisfied.
TABLE II: Contributions $\hat{M}_0$, $\hat{M}_1$, $\hat{M}_2$, $\hat{M}_3$ and $\hat{M}_4$ for all partial waves in np scattering for the NijmII and Reid 93 (in brackets) potentials at a center of mass momentum of $m_s/2$ (equivalent to a laboratory energy about 10 MeV). From them one can compute the M-matrix at $k_{cm} = m_s/2$ using $\hat{M}_1, \hat{M}_2, \hat{M}_3, \hat{M}_4$ and $\hat{M}_3$ and $\hat{M}_4$ and $\hat{M}_5$ and $\hat{M}_6$ and $\hat{M}_7$ and $\hat{M}_8$. The unitary S-matrix in coupled channel space and D = $\text{diag}(k_1^0, \ldots, k_N^0)$ is the coupled channel centrifugal factor. $M_0 = -a_1^{-1}$, $M_1 = D_{\eta}(m_s/2)^2$, $M_2 = v_2(m_s/2)^2$, $M_3 = v_3(m_s/2)^3$, $M_4 = v_4(m_s/2)^4$, and so on. We have taken $m_s = 138.0$ MeV. A long dash (—) stands for cases where numerical accuracy was outraged and no value could be reliably deducted.

| Wave | $M_0$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|------|-------|-------|-------|-------|-------|
| 1S0  | 0.0422(0.0412) | 0.1633(0.1683) | -0.0071(-0.0074) | 0.0072(0.0067) | -0.0044(-0.0041) |
| 3P0  | 0.4052(0.4050) | 0.2353(0.2366) | 0.0164(0.0144) | 0.0070(0.0068) | -0.0017(-0.0017) |
| 3P1  | -0.3575(-0.3655) | -0.3912(-0.4039) | -0.0236(-0.0274) | 0.0007(0.0019) | 0.0017(0.0019) |
| 3P2  | -0.6539(-0.6538) | -0.5245(-0.5230) | -0.0003(-0.0002) | -0.0167(-0.0017) | -0.0001(-0.00005) |
| 3S1  | -0.1999(-0.1999) | 0.1121(0.1121) | 0.0020(-0.0021) | 0.0027(0.0026) | -0.0017(-0.0017) |
| 3D1  | -0.1666(-0.1680) | -0.2154(-0.2180) | -0.0553(-0.0569) | 0.0021(0.0019) | -0.0006(-0.0006) |
| $E_1$ | 0.0506(0.05097) | 0.02467(0.02524) | -0.00410(-0.00395) | 0.002666(0.002601) | -0.000164(-0.00016) |
| 4D2  | 0.720(0.726) | 0.909(0.919) | 0.245(0.250) | -0.024(-0.024) | 0.008(0.008) |
| 4D3  | 0.1350(0.1349) | 0.1747(0.1743) | 0.0538(0.0534) | -0.0018(-0.0018) | 0.0005(0.0005) |
| 4P1  | -0.4222(-0.4257) | -0.5056(-0.511) | -0.1633(-0.1665) | -0.011(-0.012) | -0.005(-0.005) |
| 4P2  | -0.1230(-0.127) | -0.3448(-0.356) | -0.3433(-0.356) | -0.1450(-0.152) | -0.026(-0.029) |

In the $1S_0$ channel one has $\gamma = -\gamma_\nu$ and the asymptotic wave function is

$$u_{1S_0}(r) \rightarrow A_{1S_0} e^{\gamma r}$$

(56)

In the case of the deuteron one has $\gamma = \gamma_d$

$$u_{3S1}(r) \rightarrow A_{3S1} e^{-\gamma_d r}$$

(57)

$$u_{3D1}(r) \rightarrow A_{3D1} e^{-\gamma_d r} \left(1 + \frac{3}{\gamma_d r} + \frac{3}{(\gamma_d r)^2}\right)$$

(58)

The ratio s to d wave is defined as

$$\eta_d = \frac{A_{3D1}}{A_{3S1}}$$

(59)

The non-relativistic deuteron binding energy reads

$$B_d^{NR} = \frac{\gamma_d^2}{2\mu_{np}}$$

(60)

whereas the relativistic expression is given by

$$B_d^R = M_p + M_n - \sqrt{M_p^2 - \gamma_d^2} - \sqrt{M_n^2 - \gamma_d^2}$$

(61)

The calculation of threshold properties in the $3S_1 - 3D_1$ channel from given deuteron properties is a standard procedure (see e.g. Ref. [7]). Here we do just the opposite, i.e. compute deuteron properties from threshold parameters within the the effective range expansion. In practice this means solving the equation

$$\text{Det}(\hat{M}(k) - i\hbar D^2)|_{k = \gamma_\nu} = 0$$

(62)

using the LO, NLO and NNLO approximations. Residues are evaluated by numerical integration using Cauchy’s theorem. The numerical results for the smallest poles in the singlet $1S_0$ and triplet $3S_1 - 3D_1$ channels and their corresponding residues are presented in Table III.
be seen the expansion is convergent but even to N^4LO in the amplitude is not sufficient to reproduce the observables in a completely satisfactory way within experimental uncertainties. The difference should in principle be attributed to higher orders in the momentum expansion as well as other effects. In appendix B we analyze these results further on the light of an expansion around the limit of large s-wave scattering lengths. Our conclusion is that such an expansion does not improve on the description of the bound state observables.

VI. CONCLUSIONS AND OUTLOOK

In this paper we have extracted the low energy threshold parameters for all np partial waves up to states with total angular momentum \( j \leq 5 \) taking into account the coupled channel nature of the problem for realistic NN potentials. Our description entails up to order \( \mathcal{O}(k^8) \) in the CM momentum of the full np scattering amplitude. An adequate framework is to use the effective range expansion of the scaled coupled channel M-matrix, where the kinematic centrifugal factors have been factorized out. These low energy parameters comprise the scattering length matrix \( v \), the curvature matrix \( v_2 \) and higher like \( v_3 \) and \( v_4 \) and are relevant for a shape independent description of the scattering data in terms of the scaled M-matrix in a region of analyticity in the complex energy plane around the origin which radius extends up to the left partial wave logarithmic cut generated by One Pion Exchange intermediate states. The practical determination of these parameters from the solutions of the Schrödinger equation may be cumbersome, so we have found extremely convenient to use the variable S-matrix formalism. The low energy threshold parameters can be directly determined from the asymptotic solution of a set of coupled non-linear differential equations which correspond to an adiabatic switching on of the NN potential using trivial initial conditions. Finally, we have found that the coupled channel effective range expansion works well within the expected region of analyticity, namely \( k \leq m_\pi/2 \) and in fact a clear trend to convergence is observed. However, beyond the region of analyticity we do not expect the low energy expansion to be realistic regardless on how many terms are included in the expansion. Actually, the analyticity domain of the amplitude can be enlarged if the proper left cut singularities, corresponding to OPE, TPE, etc. are implemented. For such a program the method presented in our previous work \(^{35, 36}\) looks particularly promising. Finally, one problem with the use of high quality potentials has to do with the determination of errors on the potentials and hence on the low energy parameters. In this paper we have used two such potentials to assess those errors, but it would be rather interesting to make the error analysis directly based on the coupled channel effective range expansion.

Acknowledgments

We thank J. Nieves for reading the manuscript. This work is supported in part by funds provided by the Spanish DGI with grant no. BMF2002-03218, Junta de Andalucía grant no. FM-225 and EURIDICE grant number HPRN-CT-2003-00311.

APPENDIX A: DIFFICULTIES IN EXTRACTING THE LOW ENERGY PARAMETERS FROM A FIT

At first sight one might think that the effective range parameters could be determined directly from a fit to the Nijmegen data base \(^2\) and avoid the use of the corresponding potentials. In this appendix we want to elaborate on the problems we have encountered while fitting that data base within a generalized coupled channel effective range expansion of Eq. \(^1\). Unfortunately, this data base does not provide error estimates for their phase shifts (although 8 significant digits are given), nor the typical energy resolution where these data should be trusted, so some compromise must be made. We use the NN-data and define the \( \chi^2 \) as

\[
\chi^2 = \sum_{i=1}^{N} \left( \frac{\hat{M}_{ER} - \hat{M}_{NN}}{\Delta M_{NN}} \right)^2 \frac{M}{4k} \tag{A1}
\]

where we take \( \Delta E_{LAB} = 0.01 \text{MeV} \), and \( \hat{M}_{NN} \) and \( \Delta M_{NN} \) are the mean value and the standard deviation of the six potentials listed in the NN-data base \(^2\), which can be taken as independent uncorrelated primary data. The factor \( M/(4k) \) is the Jacobian of the transformation between the Lab-energy and the C.M. momentum, \( E_{LAB} = 2k^2/M \), and would correspond to make an equidistant sampling in \( p \), in the limit \( \Delta E_{LAB} \to 0 \) (this is why we take a small energy spacing). This weight factor is introduced in order to enhance the region at low momenta. On the other hand, very low momenta must be excluded since the resulting mean value \( M \)-matrix is incompatible within the attributed errors with the expected theoretical behaviour, Eq. \(^1\), so we take \( E_{LAB} \geq 0.5 \text{MeV} \). This is partly due to the poor accuracy of the data at low energies; the calculation of the scaled M-matrix requires increasing accuracy at low energies. Also, the fit goes up to \( E_{LAB} \leq 10 \text{MeV} \), which corresponds to a C.M. momentum about \( k = m_\pi/2 \) where we expect the finite polynomial of the scaled \( M \)-matrix to truly represent an analytical function within the convergence radius up to the branch cut singularity located at \( k = \pm i m_\pi/2 \). The form of the fitting function is

\[
\hat{M}_{ER} = v_0 + v_1 k^2 + v_2 k^4 + v_3 k^6 + v_4 k^8 + \ldots \tag{A2}
\]

In Fig. 7 we show as an illustration the \( v_2 \) parameter determined from a fit to the low energy region of the NN data base \(^2\) as a function of the maximal LAB-energy considered in the fit. As we see, instead of a
plateau within some energy window, we observe an ever changing value. We observe no stability depending on the number of terms considered in Eq. (13) either. For comparison we also plot the values we obtained by integrating the Eqs. (13) with the NijmII potential in Sect. IV, which where quite stable numerically. As we see, the values obtained from the fit, in the chosen energy window are hardly compatible. The deceptive features extend to other channels, and non diagonal low energy threshold parameters such as the matrix elements of $a$ and $r$.

Finally, we have also tried, with no success, other methods for the determination of the low energy threshold parameters, like evaluation of derivatives within several algorithms. The reason for the failure has to do with round-off errors generated by the relatively small number of digits provided in the NN database. Actually, at very low energies these round-off errors make the construction of the scaled M matrix itself rather unstable numerically, since large centrifugal factors $1/k^4$ are involved.

APPENDIX B: THE LIMIT OF LARGE SCATTERING LENGTHS

In this appendix we discuss some interesting aspects of the limit of large scattering lengths. For a $s$-wave eigen phase shift, i.e. $^1S_0$ and $^3S_1$ eigen channels, the effective range expansion can be written as

$$k \cot \delta = -\frac{1}{\alpha_0} + \frac{1}{2} r_0 k^2 + v_2 k^4 + v_3 k^6 + v_4 k^8 + \ldots $$

(B1)

The poles in the S-matrix are given by the solutions of the equation

$$f_0(i\gamma)^{-1} = k \cot \delta - ik \bigg|_{k=i\gamma} = 0$$

(B2)

For a given order of the truncated effective range expansion the previous equation reduces to an algebraic equation with real coefficients and it has at least one real solution, which can be taken as the analytical continuation of the lower order ones by taking the limit of the effective range parameters $r_0$, $v_2$, $\ldots \rightarrow 0$, or alternatively as the limit $\alpha_0 \rightarrow \infty$. All other roots either wander to infinity or accumulate to build, in the limit of infinitely many terms of the expansion, a branch cut. Expanding into powers of $1/\alpha_0$ one gets

$$\gamma = \frac{1}{\alpha_0} + \frac{r_0}{2\alpha_0^2} + \frac{r_0^2}{2\alpha_0^3} + \left(\frac{5}{8}\alpha_0 - v_2\right) \frac{r_0^3}{\alpha_0^4} + \ldots$$

(B3)

The neglected terms contain the parameters $v_4, \ldots$. Within the same approximation the residue for the pole of the S-matrix in this (eigen)channel defined in analogy to Eq. (13), $S_0^2 \rightarrow A_0^2/(\gamma_0 + i k)$ becomes

$$A_0^2 = \frac{2}{\alpha_0} + \frac{3r_0}{\alpha_0^2} + \frac{5r_0^2}{\alpha_0^3} + \frac{35}{8} r_0^3 - 10v_2 \frac{1}{\alpha_0^4} + \frac{63}{8} r_0^4 - 42r_0^2 v_2 \frac{1}{\alpha_0^5} + \frac{231}{8} r_0^5 - 126v_2 r_0^2 + 2v_3 \frac{1}{\alpha_0^6} + \frac{429}{8} - 330v_2 r_0^3 + 12v_2 r_0^2 + 72v_2^2 \frac{1}{\alpha_0^7} + \ldots$$

(B4)

The expansions Eq. (B3) and Eq. (B4) are convergent provided $\alpha_0$ is within the domain of analyticity of the exact solution of Eq. (13). Otherwise the expansion is asymptotic, i.e. we can use it to evaluate numerically the value of $\gamma$ in the limit $\alpha_0 \rightarrow \infty$, up to a given order where the remainder starts increasing basically due to the presence of large factorials. Nevertheless one can use the expansion with a given error estimate. For instance, at NLO the domain of analyticity is given by the condition of a vanishing discriminant of a second order algebraic equation yielding $\alpha_0 = 2r_0 = 3.50652$ for the analyticity inner boundary. The numerical values for $\alpha_0$ and $r_0$ lie within the boundary and one can use the series expansion to any order to evaluate $\gamma$ with increasing accuracy. In the NNLO case the analyticity boundary is given by an analogous discriminant condition. Numerically we find the inner boundaries for $\alpha_0$ located at lower points $\alpha_0 = 3.444, -0.777, -0.133$ for the NijmII potential.
TABLE IV: Poles and residues of the scattering amplitude for the NijmII potential based on the effective range expansion, in the large scattering length limit $\alpha \to \infty$ to $O(1/\alpha^5)$ with $\alpha = \bar{\alpha}_{3S1}$ in the $^1S_0$ channel, $\alpha = \bar{\alpha}_{3\bar{S}1}$ in the $\bar{\alpha}_{3S1}$ channel ($^3S_1$ eigen channel), and $\alpha = \bar{\alpha}_{3\bar{S}1}$ in the $\bar{\alpha}_{3\bar{S}1}$. This calculation is complete to $N^4$LO (see main text).

|                      | $O(1/\alpha)$ | $O(1/\alpha^2)$ | $O(1/\alpha^3)$ | $O(1/\alpha^4)$ | $O(1/\alpha^5)$ |
|----------------------|---------------|-----------------|-----------------|-----------------|-----------------|
| $\gamma_n$ (fm$^{-1}$) |               |                 |                 |                 |                 |
| NijmII (Reid93)      | -0.04215(-0.4213) | -0.03977(-0.03969) | -0.0400(-0.03997) | -0.0400(-0.03997) | -0.0400(-0.03997) |
| $\gamma_d$ (fm$^{-1}$) | 0.18454(0.18440) | 0.21439(0.21425) | 0.22405(0.22391) | 0.22791(0.2279) | 0.22962(0.22952) |
| $B_{\gamma}^{SR}$ (MeV) | 0.0(0.0) | 1.42272(1.42020) | 1.89222(1.86733) | 2.05462(2.05199) | 2.13682(2.13549) |
| $B_{\gamma}^{SR}$ (MeV) | 0.0(0.00) | 1.42272(1.42020) | 1.89222(1.86733) | 2.05462(2.05199) | 2.13682(2.13549) |
| $\eta_d$             | 0.06(0.0)    | 0.06(0.0)       | 0.1035(0.01032)  | 0.01218(0.01212) | 0.01671(0.01663) |
| $(A_{3S1})^2$ (fm$^{-1}$) | 0.36908(0.36880) | 0.54820(0.54780) | 0.64479(0.64542) | 0.69896(0.69889) | 0.73010(0.73027) |

TABLE V: Same as table IV but for higher orders.

|                      | $O(1/\alpha^6)$ | $O(1/\alpha^7)$ | $O(1/\alpha^8)$ | $O(1/\alpha^9)$ | $O(1/\alpha^{10})$ | $N^5$LO |
|----------------------|-----------------|-----------------|-----------------|-----------------|-----------------|----------|
| $\gamma_n$ (fm$^{-1}$) | -0.04001(-0.03994) | -0.04001(-0.03994) | -0.04001(-0.03994) | -0.04001(-0.03994) | -0.04001(-0.03994) |         |
| $\gamma_d$ (fm$^{-1}$) | 0.23047(0.23038) | 0.23091(0.23083) | 0.23110(0.23109) | 0.23130(0.23129) | 0.23148(0.23141) |         |
| $B_{\gamma}^{SR}$ (MeV) | 2.17666(2.17462) | 2.19969(2.19515) | 2.20784(2.20620) | 2.21396(2.21245) | 2.22215(2.22088) |         |
| $B_{\gamma}^{SR}$ (MeV) | 2.17773(2.17568) | 2.19812(2.19632) | 2.20176(2.20107) | 2.21532(2.21371) | 2.22347(2.22219) |         |
| $\eta_d$             | 0.01848(0.01838) | 0.02053(0.02042) | 0.02167(0.02156) | 0.01953(0.01942) | 0.02521(0.02512) |         |
| $(A_{3S1})^2$ (fm$^{-1}$) | 0.74457(0.74097) | 0.76000(0.75659) | 0.77617(0.77290) | 0.78726(0.78393) | 0.79782(0.79393) |         |

If we instead make the large scattering length expansion in the determinant bound state condition, Eq. (52), for a truncated scaled $M$-matrix we get an algebraic equation in the deuteron pole $k = i\gamma$ which order is determined by the level of approximation. In the SYM parameterization the LO, NLO, NNLO, NNNLO effective range parameters start contributing at order $\gamma$, $\gamma^2$, $\gamma^4$, and $\gamma^6$ respectively and also contribute to a maximal order $\gamma^6$, $\gamma^7$, $\gamma^9$ and $\gamma^{12}$ respectively. This means that the coefficients of $\gamma$ in an expansion in $1/\bar{\alpha}_{3S1}$ are complete to $O(\bar{\alpha}_{3S1})$ at LO, $O(\bar{\alpha}_{3S1}^2)$ at NLO, and $O(\bar{\alpha}_{3S1}^3)$ at NNLO. This can also be seen by applying the large scattering length expansion of Eq. 53 to the eigen phase shift, Eq. 54, with the identifications 52, yielding for the pole

$$\gamma_d = \frac{1}{\bar{\alpha}_{3S1}} + \frac{1}{2\bar{\alpha}_{3S1}^2} + \frac{1}{2} (\bar{\alpha}_{3S1}^3 + 2\bar{\alpha}_{3S1} \bar{\alpha}_{3S1}^3) \bar{\alpha}_{3S1}^3 + \frac{1}{4} (2\bar{\alpha}_{3S1}^2 \bar{\alpha}_{3S1} + \bar{\alpha}_{3S1}^3 + 8\bar{\alpha}_{3S1} \bar{\alpha}_{3S1}^3 \bar{\alpha}_{3S1}^3) \bar{\alpha}_{3S1}^3 + \ldots$$

(B5)

We also get for the residues

$$A_{3S1}^d A_{3S1}^{-1} = \frac{2\bar{\alpha}_{3S1}}{\bar{\alpha}_{3S1}^2} - \frac{\bar{\alpha}_{3S1} \bar{\alpha}_{3S1}^2}{\bar{\alpha}_{3S1}^4} + \ldots$$

(B6)

Finally, for the mixing coefficient $\eta_d$ we get

$$\eta_d = \frac{\bar{\alpha}_{3S1}^3}{\bar{\alpha}_{3S1}^4} + \frac{2\bar{\alpha}_{3S1} \bar{\alpha}_{3S1}^3 - \bar{\alpha}_{3S1} \bar{\alpha}_{3S1}^3}{\bar{\alpha}_{3S1}^2 + \bar{\alpha}_{3S1}^2} + \ldots$$

(B7)

and so on. Note that the leading contributions in the inverse scattering length behave differently for any observable. According to Eq. 52, making $\alpha_{3S1}$ large makes also $\bar{\alpha}_{3S1}$ large. Thus one may study separately the convergence of observables in both cases. The numerical values for the pole position and corresponding residue evaluated in the inverse scattering length expansion are presented in Tables IV for the expansion up to order $1/\alpha^5$. 

- $\bar{\alpha}_{3S1} \bar{\alpha}_{3S1}^3 \bar{\alpha}_{3S1}^3$
and $1/\alpha^5$. This order involves for pole position only up to $N^4\text{LO}$ in the amplitude (for the binding energy and mixing parameter the situation is somewhat different, see e.g. Eq. (B7)). As we have already discussed the converse is not true, and $N^4\text{LO}$ contains a series of higher power corrections in the inverse scattering length. We present those corresponding only to $N^4\text{LO}$ in Table V up to ninth order in the inverse scattering length. As we see, and one could have anticipated, the convergence for the virtual state pole is faster than for the deuteron bound state. Actually, we see that the expansion in the inverse scattering length for bound state properties is less convergent than the ERE pursued in Sect. V C.

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FIG. 1: np scaled M-matrix on the effective range expansion for states with total angular momentum $j = 0$. We construct the NijmII scales M-matrix corresponding to the generalized effective range expansion of Eq. (16) to a given order as a function of the LAB energy. LO means including order $k^0$ terms, NLO including $k^2$ terms and NNLO including $k^4$ terms with the low energy threshold parameters obtained from solving the evolution equations for the threshold parameters, Eq. (44) with the NijmII potential. Data are obtained from the partial waves phase shift analysis from Ref. [5].
FIG. 2: np scaled M-matrix based on the effective range expansion for total angular momentum $j = 1$. For notation see Fig. [1]
FIG. 3: np scaled M-matrix based on the effective range expansion for total angular momentum $j = 2$. For notation see Fig. [1]
FIG. 4: np scaled M-matrix based on the effective range expansion for total angular momentum $j = 3$. For notation see Fig. [1]
FIG. 5: np scaled M-matrix based on the effective range expansion for total angular momentum $j = 4$. For notation see Fig. 1.
FIG. 6: np scaled M-matrix based on the effective range expansion for total angular momentum \( j = 5 \). For notation see Fig. 1.
FIG. 7: The $v_2$ parameter for the $^1S_0$ (left) and the $^3S_1$ (right) channels determined from a fit to the low energy date of the NN data base [5] (see Eq. A11) and main text, as a function of the maximal LAB-energy considered in the fit. $p^n$ means a fit including up to $\mathcal{O}(k^n)$ terms in the effective range expansion Eq. A2. “Database” means a fit to the average value of the corresponding scaled $M$-matrix. “Reid93” and “NijmII” means a fit to only this data. The values we obtained by integrating the Eqs. NijmII and Reid93 potentials.