Abstract. We give a comprehensive account of Chern’s Theorem that $S^6$ admits no $\omega$-compatible almost complex structures. No claim to originality is being made, as the paper is mostly an expanded version of material in differential sources already in the literature.

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1. Introduction

Following Bryant’s exposition [Br2], we present a theorem of Chern that there is no integrable almost complex structures on $S^6$ compatible with the standard 2-form $\omega$ on $S^6$. It is determined by the octonionic almost complex structure $J_{\text{can}}$, see (13), and the round metric $g_{\text{can}}$ on $S^6$ through

$$\omega(u, v) := g_{\text{can}}(J_{\text{can}}u, v).$$

Definition 1.1. An almost complex structure $J$ on $S^6$ is $\omega$-compatible if

$$\omega(u, v) = \omega(Ju, Jv) \quad \forall u, v. \quad (1)$$

We have left out the usual condition $\omega(u, Jv) > 0$ for $u \neq 0$. Instead, the $\omega$-index of $J$ is defined as the index $(2p, 2q)$ of the non-degenerate symmetric bilinear form $g_J := \omega(\cdot, J\cdot)$. The main result is as follows.

Theorem 1.2 (Chern). There are no $\omega$-compatible complex structures on $S^6$.

The reader may also wish to refer to [Da] for a related proof.

2. The exceptional Lie group $G_2$

2.1. $G_2$-action on $S^6$. For the understanding of this paper, a shortcut definition of the exceptional Lie group $G_2$ suffices. More information may be found in [Ag].
Let $V := \mathbb{R} \oplus \mathbb{C}^3$ with basis $e_1, \ldots, e_7$ and the standard inner product. We identify $S^6$ with the unit sphere in $V$. Define a basis of the complexification $V_{\mathbb{C}} = \mathbb{C}^7$ by

$$
(2) \quad e_1, F_1 = \frac{e_2 - ie_3}{2}, F_2 = \frac{e_4 - ie_5}{2}, F_3 = \frac{e_6 - ie_7}{2}, \tilde{F}_1, \tilde{F}_2, \tilde{F}_3.
$$

Then $|e_1| = 1$ and $|F_k| = |\tilde{F}_k| = 1/\sqrt{2}$. We use this basis to identify endomorphisms of $V_{\mathbb{C}}$ with matrices. Let

$$
(3) \quad \mathfrak{g}_2 = \left\{(0, -ia^*, ia^t), \begin{pmatrix}-2ia & D & [a] \vline D \end{pmatrix} \in \mathbb{C}^{3 \times 3}, D \in \mathfrak{su}(3) \right\} \subset \mathbb{C}^{7 \times 7},
$$

using the notation $a^* = a^t$ and

$$
[a] := \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} \in \mathbb{C}^{3 \times 3}, \quad a \in \mathbb{C}^3.
$$

Then $\mathfrak{g}_2 \subset \mathfrak{so}(V) \subset \mathfrak{su}(V_{\mathbb{C}})$ since by normalizing $F_k$ to unit length the matrix in (3) becomes skew-Hermitian. It is easy to check that (3) is closed under the matrix Lie bracket. According to Lie’s Theorems there exists a unique simply-connected Lie group $G_2$ with this Lie algebra and a smooth monomorphism $G_2 \to SO(V)$. Since the Killing form is negative definite on $\mathfrak{g}_2$, the group $G_2$ is compact so that

$$
(4) \quad G_2 \subset SO(V)
$$

is topologically embedded. Using (2) we write this faithful representation as

$$
\begin{array}{ccc}
G_2 & \longrightarrow & SO(V) \cong SO(7) \\
\rho^C & \downarrow \rho \cong \rho \downarrow \rho^C & \hspace{1cm} \rho_g = (g_1, \ldots, g_7), \; \rho^C_g = (x, f_1, f_2, f_3, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3), \\
SU(\mathbb{C}^7) & \cong SU(V_{\mathbb{C}})
\end{array}
$$

using column notation for the matrices $\rho_g, \rho^C_g$. Thus $g_1 = \rho_g(e_1)$ and $x = g_1$, $f_1 = \rho_g^C(F_1) = \frac{1}{2}(g_2 - ig_3)$ and so on. The functions $x, f_i, \tilde{f}_i : G_2 \to \mathbb{C}^7$ are called the moving frame on $G_2$.

Restricting to unit vectors, (4) defines a smooth $G_2$-action on $S^6$ and $x$ is simply the orbit map at $e_1 \in S^6$. To proceed, we next need the differential of $x$.

### 2.2. Structure Equations.

**Definition 2.1.** The Maurer–Cartan form $\phi \in \Omega^1(G_2; \mathfrak{g}_2)$ is the matrix-valued form $\phi = g^{-1}dg$. Thus $\phi(X \in T_gG_2) = g^{-1} \cdot X$ (matrix multiplication).

The wedge product of matrix-valued differential forms is given by the usual formula, using matrix multiplication instead of the product of numbers. In terms of (3) we write the components of $\phi$ as

$$
\phi = \begin{pmatrix} 0 & -i\theta^* & i\theta \\ -2i\theta & \kappa & [\theta] \\ 2i\bar{\theta} & [\bar{\theta}] & \bar{\kappa} \end{pmatrix}, \quad \theta \in \Omega^1(G_2; \mathbb{C}^3), \kappa \in \Omega^1(G_2; \mathfrak{su}(3)).
$$
Theorem 2.2 (Bryant [Br1]). We have the first structure equations (where \( f = (f_1, f_2, f_3) \) in the obvious vector notation)

\[
d(x, f, \bar{f}) = (x, f, \bar{f}) \cdot \begin{pmatrix}
0 & -i\theta^* & i\theta^t \\
-2i\theta & \kappa & [\bar{\theta}] \\
2i\bar{\theta} & [\theta] & \bar{\kappa}
\end{pmatrix}.
\]

Also, the second structure equations hold:

\[
d\theta = -\kappa \wedge \theta + [\bar{\theta}] \wedge \bar{\theta}
\]
\[
d\kappa = -\kappa \wedge \kappa + 2\theta \wedge \theta^* - [\bar{\theta}] \wedge [\theta]
\]

**Proof.** In our matrix notation \( g = (x, f, \bar{f}) \). So (5) is just Definition 2.1 multiplied by \( g \). The second structure equations follow by reading off matrix entries on both sides of the so-called Maurer–Cartan equation

\[
d\phi = d(g^{-1}dg) = -g^{-1}dg \wedge g^{-1}dg = -\phi \wedge \phi.
\]

\( \square \)

2.3. \( S^6 \) as a homogeneous space.

**Lemma 2.3.** The action of \( G_2 \) of \( S^6 \) is transitive with isotropy group \( SU(3) \). Hence the orbit map \( x = \rho_g(e_1) \) restricts to a principal \( SU(3) \)-bundle

\[
x: G_2 \to S^6,
\]

where \( SU(3) \) is embedded in \( G_2 \subset SU(C^7) \) as

\[
\begin{pmatrix}
1 & A \\
\bar{A}
\end{pmatrix}, \quad \forall A \in SU(3).
\]

**Proof.** By (5) the differential is, where the notation indicates a matrix-vector multiplication \( f \cdot \theta = f_1\theta_1 + f_2\theta_2 + f_3\theta_3 \),

\[
dx = -2if \cdot \theta + 2if \cdot \bar{\theta}.
\]

Hence \( x \) is a submersion. The image is therefore open and closed, so all of \( S^6 \). By the long exact sequence of homotopy groups of the fibration \( u \) together with the fact that \( G_2 \) is connected and \( S^6 \) is simply-connected, the stabilizer must be simply connected and is therefore \( SU(3) \). \( \square \)

**Lemma 2.4.** For the right translation we have

\[
R_A^* \theta = A^{-1} \cdot \theta \quad \forall A \in SU(3).
\]

**Proof.** We have

\[
(R_A^* \phi)(X) = (gA)^{-1}X \cdot A = A^{-1} \cdot \phi(X) \cdot A \quad \forall X \in T_gG_2.
\]

Now perform the matrix multiplication and compare entries in

\[
R_A^* \phi_{G_2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & A^{-1} & 0 \\
0 & 0 & A^t
\end{pmatrix} \cdot \begin{pmatrix}
0 & -i\theta^* & i\theta^t \\
-2i\theta & \kappa & [\bar{\theta}] \\
2i\bar{\theta} & [\theta] & \bar{\kappa}
\end{pmatrix} \cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & A & 0 \\
0 & 0 & \bar{A}
\end{pmatrix}.
\]

\( \square \)
3. The Standard Almost Complex Structure on $S^6$

Let $y \in S^6$. Fix also a $g \in G_2$ with $x(g) = y$. The submersion $\mathbf{8}$ induces an exact sequence

$$0 \to T_y(gSU(3)) \to T_yG_2 \xrightarrow{dx} T_yS^6 \to 0.$$  \hfill (12)

According to $\mathbf{11}$, the forms $\bar{\theta}_g^i, \bar{\theta}_g^j$ vanish on the kernel of $dx$ and thus descend to a basis of $T_y^*S^6 \otimes \mathbb{C}$. Note the dependence on $g$, but by $\mathbf{11}$ the spanned subspaces $\langle \theta^1, \theta^2, \theta^3 \rangle$ and $\langle \bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3 \rangle$ are invariant under $SU(3)$ and hence determine a well-defined subspace of $T_yS^6 \otimes \mathbb{C}$. We may therefore define:

**Definition 3.1.** The octonionic complex structure $J_{\text{can}}$ is defined for any choice of $g \in G_2$ with $x(g) = y$ by the decomposition

$$T_{J_{\text{can}}}(T^*_yS^6) = \langle \bar{\theta}_g^1, \theta^2_g, \theta^3_g \rangle, \quad T_{J_{\text{can}}}(T^*_yS^6) = \langle \bar{\theta}_g^1, \bar{\theta}_g^2, \bar{\theta}_g^3 \rangle.$$  \hfill (13)

This is in fact a nearly Kähler structure (see also $\mathbf{Da}$):

**Proposition 3.2.** There exists a complex 3-form $\Upsilon$ on $S^6$ such that

i) $x^*g_{\text{can}} = 4\theta^i \circ \bar{\theta}$ where $g_{\text{can}}$ denotes also the $\mathbb{C}$-bilinear extension of the round metric to $TS^6 \otimes \mathbb{C}$.

ii) $x^*\omega = 2i\theta^i \wedge \bar{\theta}$

iii) $d\omega = -3\text{Im}(\Upsilon)$

iv) $x^*\Upsilon = 8\theta^i \wedge \theta^2 \wedge \theta^3$ and $\Upsilon$ has $J_{\text{can}}$-type $(3,0)$.

**Proof.** i) Both sides are $G_2$-invariant, so we check equality at $1 \in G_2$. Write $A \in g_2$ as in $\mathbf{3}$. Then using $|f_k| = 1/\sqrt{2}$

$$g_{\text{can}}(dx(A), dx(A)) = g_{\text{can}}(-2iaf + 2i\bar{a}f, -2iaf + 2i\bar{a}f) = 4||a||^2 = 4(\theta^i \circ \bar{\theta})(A, A)$$

(the $(1,0)$ and $(0,1)$-subspaces are isotropic for the $\mathbb{C}$-bilinear extension.)

ii) By i) $\sqrt{2}\theta^i$ is an orthonormal basis of $(1,0)$-forms. Hence by $\mathbf{22}$

$$x^*\omega = i\sqrt{2}\theta^i \wedge \sqrt{2}\bar{\theta} = 2i\theta^i \wedge \overline{\theta}.$$

iv) $\theta_1 \wedge \theta_2 \wedge \theta_3$ is invariant under $SU(3)$ since by $\mathbf{11}$ for $A^{-1} = (a^{ij})$

$$(a^{11}\theta_1 + a^{12}\theta_2 + a^{13}\theta_3) \wedge (a^{21}\theta_1 + a^{22}\theta_2 + a^{23}\theta_3) \wedge (a^{31}\theta_1 + a^{32}\theta_2 + a^{33}\theta_3)$$

$$= \det(A^{-1})\theta_1 \wedge \theta_2 \wedge \theta_3 = \theta_1 \wedge \theta_2 \wedge \theta_3.$$

This proves the existence of $\Upsilon$. It is clearly a $(3,0)$-form.

iii) Using $(\alpha \wedge \beta)^t = (-1)^{\alpha \beta} \beta^t \wedge \alpha^t$, $\kappa^t = -\bar{\kappa}$, $[\theta]^t = -[\theta]$ and $\mathbf{6}$ we find

$$d(\theta^i \wedge \bar{\theta}) = (d\theta)^t \wedge \theta - \theta^t \wedge d\Theta$$

$$= (-\kappa \wedge \theta + [\bar{\theta}] \wedge \bar{\theta})^t \wedge \theta - \theta^t \wedge (-\bar{\kappa} \wedge \bar{\theta} + [\theta] \wedge \theta)$$

$$= \theta^t \wedge \kappa^t \wedge \theta - \bar{\theta}^t \wedge [\bar{\theta}]^t \wedge \theta + \theta^t \wedge \kappa \wedge \bar{\theta} - \theta^t \wedge [\theta] \wedge \theta$$

$$= -\theta^t \wedge [\theta] \wedge \theta + \overline{\theta}^t \wedge [\theta] \wedge \overline{\theta}$$

$$= 6(\theta_1 \wedge \theta_2 \wedge \theta_3 - \overline{\theta}_1 \wedge \overline{\theta}_2 \wedge \overline{\theta}_3) = 12i \text{Im}(\theta_1 \wedge \theta_2 \wedge \theta_3) \quad \square$$
Lemma 4.2. Two cases can be ruled out topologically: \( \omega \)

Proof. (15)

Also, the matrix \( \begin{pmatrix} r & s \\ \bar{s} & \bar{r} \end{pmatrix} \) has non-zero determinant.

Proposition 4.1. There are unique smooth maps \( r, s : B_J \to \mathbb{C}^{3 \times 3} \) with

Also, the matrix \( \begin{pmatrix} r & s \\ \bar{s} & \bar{r} \end{pmatrix} \) has non-zero determinant.

Proof. Let \( (g, u) \in B_J \) with \( y = x(g) = \pi(u) \). Thus \( u : T_y S^6 \to \mathbb{C}^3 \) is a \( (J, i) \)-complex linear isomorphism. In particular,

is a complex basis of \( T^*_{y} S^6 \). Hence we have an expansion \( \theta = ru + s\bar{u} \). Recall from (13) that \( \theta^q_g, \theta^q_{\bar{g}} \) is a second basis of \( (T^* S^6) \otimes \mathbb{C} \). We thus get a change of basis matrix with non-zero determinant. \( \square \)

4.2. The bundles \( \mathbb{I}_1(\omega, S^6) \) and \( \mathbb{I}_2(\omega, S^6) \). Let \( \mathbb{I}(M, \omega) \subset \text{End}(TM) \) be the bundle of \( \omega \)-compatible almost complex structures on a smooth manifold \( M \). Its fiber at \( p \in M \) are all \( J : T_p M \to T_p M \) with \( J^2 = -1 \) and satisfying (11) for \( \omega|_{T_p M} \). Then

where \( \mathbb{I}_q(M, \omega) \subset \mathbb{I}(M, \omega) \) is the subbundle of almost complex structures of \( \omega \)-index \( (2n - 2q, 2q) \). Here the dimension of \( M \) is \( 2n \). Thus, in the case \( M = S^6 \) we get

Because \( S^6 \) is connected an the \( \omega \)-index is a continuous pair of integers, every \( \omega \)-compatible almost complex structure \( J \) is a section of one of these subbundles. Two cases can be ruled out topologically:

Lemma 4.2. \( \mathbb{I}_1(\omega, S^6) \) and \( \mathbb{I}_2(\omega, S^6) \) do not admit a global continuous section.
Proof. Assume that \( J \) is a continuous section of \( \mathbb{J}_1(S^6, \omega) \) or of \( \mathbb{J}_2(S^6, \omega) \). Then the positive and negative definite subspaces of \( g = \omega(\cdot, \cdot) \) yield a decomposition

\[
TS^6 = E_4 \oplus E_2
\]

into two vector subbundles of ranks 4 and 2. However, as is well known, the Euler class (and characteristic) of \( TS^6 \) is nontrivial: \( e(TS^6) \neq 0 \). On the other hand, rank 4 and rank 2 vector bundles over \( S^6 \) have trivial Euler classes, since \( H^2(S^6, \mathbb{Z}) = H^4(S^6, \mathbb{Z}) = 0 \). Using the formula for the Euler classes of the Whitney sum one obtains the contradiction

\[
0 \neq e(TS^6) = e(E_4) \cup e(E_2) = 0. \tag{\ref{16}} \]

4.3. Chern’s identity. Putting \( (14) \) into Proposition \( \ref{3.2} \) ii) and using that \( \eta \) has \( J \)-type \((1, 0)\) gives

\[
\omega^{1,1}_J = 2i\eta^t \wedge (r^t\bar{\eta} - s^t\bar{s})\bar{\eta}. \tag{16}
\]

The assumption that \( J \) is \( \omega \)-compatible means that \( \omega = \omega^{1,1}_J \) has \( J \)-type \((1, 1)\).

**Proposition 4.3.** For any integrable \( \omega \)-compatible complex structure on \( S^6 \) we have Chern’s identity

\[
\det(s) = \det(r). \tag{17}
\]

**Proof.** Putting \( (14) \) into Proposition \( \ref{3.2} \) iii) gives

\[
\Upsilon^{3,0}_J = 8 \det(r)\eta_1 \wedge \eta_2 \wedge \eta_3, \quad \Upsilon^{0,3}_J = 8 \det(s)\bar{\eta}_1 \wedge \bar{\eta}_2 \wedge \bar{\eta}_3. \tag{18}
\]

When \( J \) is integrable, Lemma \( \ref{A.4} \) implies that \( d\omega \) has type \((2, 1) + (1, 2)\). Hence its \((3, 0)\)-part with respect to \( J \) vanishes. Recalling also \( d\omega = 31\text{Im}(\Upsilon) \) we calculate

\[
0 = (d\omega)^{3,0}_J = (3\text{Im}(\Upsilon))^{3,0}_J = \frac{3}{2i}(\Upsilon - \bar{\Upsilon})^{3,0}_J
\]

\[
\text{\ref{19}} \quad 12i(\det(s) - \det(r))\eta_1 \wedge \eta_2 \wedge \eta_3 \tag{19}
\]

4.4. Proof of Chern’s Theorem. Before giving the proof, recall that for two Hermitian matrices \( A, B \) we say \( A > B \) (resp. \( A \geq B \)) if \( A - B \) has only positive (resp. non-negative) eigenvalues. \( A > B \) is equivalent to

\[
\langle Ax, x \rangle > \langle Bx, x \rangle, \quad \forall x \neq 0.
\]

For example, for an arbitrary matrix \( C \) we have \( C^*C \geq 0 \) and \( CC^* \geq 0 \). Moreover \( C^*C > 0 \) and \( CC^* > 0 \) precisely when \( C \) is invertible.

**Lemma 4.4.** For \( A > B > 0 \) we have \( \det A > \det B > 0 \).

**Proof.** By replacing \( x = A^{-1/2}y \) in \( (19) \) we see that \( A > B > 0 \) is equivalent to \( E > A^{-1/2}BA^{-1/2} > 0 \) for the identity matrix \( E \). Let \( C := A^{-1/2}BA^{-1/2} > 0 \) have eigenvalues \( \lambda_i > 0 \). Then \( E - C > 0 \) has eigenvalues \( 1 - \lambda_i > 0 \). Hence \( \det(A^{-1}) \det(B) = \det(C) \in (0, 1) \).

**Proof of Theorem \( \ref{1.2} \).** Assume by contradiction that \( J \) is both integrable and \( \omega \)-compatible. Then Lemma \( \ref{A.2} \) shows that \( J \) must be a section of \( \mathbb{J}_0(S^6, \omega) \) or of \( \mathbb{J}_3(\omega, S^6) \). Hence the bilinear form \( \omega(\cdot, \cdot, J) \), which according to \( (16) \) is represented by twice the matrix \( H := r^t\bar{\eta} - s^t\bar{s} \), is either positive definite or negative definite.

Assume that \( H \) is positive definite, so \( r^t\bar{\eta} > s^t\bar{s} \). Since \( s^t\bar{s} \geq 0 \) this implies \( r^t\bar{\eta} \geq H > 0 \) and so \( r^t\bar{\eta} \) is invertible. Hence \( 0 \neq \det(r) = \det(s) \) by \( (17) \) and so
Lemma A.1. Suppose \( r \bar{r} > \bar{s}^2 s > 0 \), contradicting Lemma \( \text{[12]} \). The case when \( H \) is negative definite is analogous. So we have reached a contradiction in every case. \( \square \)

A.1. Linear algebra. Let \((V, J)\) be a complex vector space. The complexification \( V_\mathbb{C} := V \otimes_{\mathbb{R}} \mathbb{C} \) carries two commuting complex structures \( J_\mathbb{C} := J \otimes \text{id}_\mathbb{C}, \ i = 1 \otimes i \) which gives a splitting into the \((\pm i)\)-eigenspaces of \( J_\mathbb{C} \)

\[
J_\mathbb{C} = V^{1,0} \oplus V^{0,1}.
\]

By convention \( V_\mathbb{C}, V^{1,0}, V^{0,1} \) are equipped with the complex structure \( i \). We identify \( V \hookrightarrow V_\mathbb{C} \) by \( v \mapsto v \otimes 1 \) with image the real subspace, the subspace of \( V_\mathbb{C} \) fixed by complex conjugation \( \overline{\cdot} : \mathbb{C} \to \mathbb{C} \).

These definitions apply to \( V = \mathbb{R}^{2n} \) or to the tangent space \( T_p M \) at a point of an almost complex manifold \((M, J)\). Note that complex structures on \( V \) are equivalent to complex structures \( J^* \) on the dual space. Then \((J^*)^{1,0} \) is isomorphic to the \((J, i)\)-complex linear maps \( V \to \mathbb{C} \) and similarly \((J^*)^{0,1} \) are the complex anti-linear maps. However, both determine complex-linear maps \( V_\mathbb{C} \) with respect to the complex structure \( i \). We may decompose the complex \( n \)-forms as

\[
\Lambda^n(V^*) = \bigoplus_{p+q=n} \Lambda^{p,q}(V^*), \quad \Lambda^{p,q}(V^*) = \Lambda^p(V^*)^{1,0} \otimes \Lambda^q(V^*)^{0,1}
\]

and we denote the corresponding projection by \( \alpha \mapsto \alpha^{p,q} \).

Conversely, a splitting \( V_\mathbb{C} = V^{1,0} \oplus V^{0,1} \) of the complexification of a real vector space \( V \) into two complex subspaces satisfying \( V^{1,0} = V^{0,1} \) defines a unique complex structure on \( V \) with given type decomposition: decompose \( \omega = v \in V \) as \( v \otimes 1 = v^{1,0} + v^{0,1} \) and define \( J(v) = iv^{1,0} - iv^{0,1} \) (which again belongs to the real subspace).

Now suppose that \( g \) is a Euclidean metric on \( V \) such that \( J \) is \( g \)-orthogonal. Then we obtain a Hermitian form on \( V_\mathbb{C} \) by

\[
h(v_1 \otimes z_1, v_2 \otimes z_2) := g(v_1, v_2) \otimes z_1 \overline{z_2},
\]

for which \( V^{1,0} \oplus V^{0,1} \) is orthogonal. One may also complexify \( g \) to a real \( \mathbb{C} \)-bilinear form \( g_\mathbb{C}(v_1 \otimes z_1, v_2 \otimes z_2) = g(v_1, v_2) \otimes z_1 \overline{z_2} \) for which \( V^{1,0} \) and \( V^{0,1} \) are isotropic. Since \( J \) is skew-symmetric for \( g \) we have also a 2-form on \( V \)

\[
\omega(X, Y) = g(JX, Y).
\]

Let \( \{z_\alpha\}_{\alpha=1, \ldots, \dim V} \) be a complex basis of \( V^{1,0} \) with dual basis \( z^\alpha \). Then \( \bar{z}_\alpha \) is a basis of \( V^{0,1} \). Letting \( h_{\alpha\beta} := h(z_\alpha, z_\beta) = g_\mathbb{C}(z_\alpha, \bar{z}_\beta) \) the complexification of \( \omega \) is

\[
\omega = ih_{\alpha\beta} z^\alpha \wedge \bar{z}^\beta.
\]

A.2. Almost complex manifolds. An almost complex structure is an endomorphism \( J : TM \to TM \) satisfying \( J^2 = -1 \). For example, an complex manifold is almost complex, since the derivative of local holomorphic coordinates gives real linear isomorphisms \( \mathbb{C}^n \to T_p M \) along which we may transport the standard complex structure \( i \) to get \( J \). An almost complex structure of this type is called integrable.

Let \( \mathcal{A}^{p,q}(M) \) be the global sections of the bundle \( \Lambda^{p,q}(T^* M) \).

Lemma A.1. Suppose \( J \) is integrable and let \( \eta \in \mathcal{A}^{p,q}(M) \). Then

\[
d\eta \in \mathcal{A}^{p+1,q}(M) \oplus \mathcal{A}^{p,q+1}(M).
\]
Proof. When $J$ is integrable we may use the coordinates to get an exact local frame $dz^\alpha, d\bar{z}^\beta$ of the $(1,0)$ and $(0,1)$-forms. By definition a $(p,q)$-form has a local expression
\[ \eta = \eta_{\alpha_1 \cdots \alpha_p, \bar{\beta}_1 \cdots \bar{\beta}_q} dz^{\alpha_1} \cdots dz^{\alpha_p} d\bar{z}^{\bar{\beta}_1} \cdots d\bar{z}^{\bar{\beta}_q}. \]
Now apply $d$ and the fact that for a complex-valued function $f$ we have a splitting $df = \frac{\partial f}{\partial z^\alpha} dz^\alpha + \frac{\partial f}{\partial \bar{z}^\beta} d\bar{z}^\beta$ into the complex linear and anti-linear part. \qed

The converse of the lemma is the difficult Newlander-Nirenberg Theorem.

A.3. Lie groups.

Theorem A.2 (Lie’s Second Theorem). Let $G, H$ be Lie groups with $G$ simply connected. Taking the derivative at the unit sets up a bijection between Lie homomorphisms $G \to H$ and Lie algebra homomorphisms $\mathfrak{g} \to \mathfrak{h}$.

Theorem A.3 (Lie’s Third Theorem). For every finite-dimensional real Lie algebra $\mathfrak{g}$ there exists a unique simply-connected Lie group $G$ whose Lie algebra is $\mathfrak{g}$. Any connected Lie group with that Lie algebra is isomorphic to $G/\Gamma$ for a discrete subgroup $\Gamma \subset Z(G)$ of the center.

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