EXPLICIT BOUNDS FOR $L$-FUNCTIONS ON THE EDGE OF THE CRITICAL STRIP

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Abstract. Assuming GRH and the Ramanujan-Petersson conjecture we prove explicit bounds for $L(1, f)$ for a large class of $L$-functions $L(s, f)$, which includes $L$-functions attached to automorphic cuspidal forms on $GL(n)$. The proof generalizes work of Lamzouri, Li and Soundararajan. Furthermore, the main results improve the classical bounds of Littlewood

$$(1 + o(1)) \left( \frac{12e^\gamma}{\pi^2} \log \log C(f) \right)^{-d} \leq |L(1, f)| \leq (1 + o(1)) \left( 2e^\gamma \log \log C(f) \right)^d,$$

where $C(f)$ is the analytic conductor of $L(s, f)$.

1. Introduction

In analytic number theory, and increasingly in other surprising places, $L$-functions show up as a tool for describing interesting algebraic and geometric phenomena. In particular, understanding the value of $L$-functions on the 1-line has a number of applications. For example, the non-vanishing of the Riemann zeta function for $\zeta(1 + it)$, $t \in \mathbb{R}$, proves the celebrated Prime Number Theorem. Additionally, understanding the value $L(1, \chi)$ for certain Dirichlet characters, provides us with insight to the order of the class group of imaginary quadratic fields through Dirichlet’s Class Number Formula.

Unconditionally, for any non-trivial Dirichlet character $\chi$ with conductor $q$, we have

$$\frac{1}{q^2} \ll |L(1, \chi)| \ll \log q.$$

In fact, we can improve the lower bound to $(\log q)^{-1}$, excluding some exceptional cases related to Landau-Siegel zeros (see [2, Chapter 14]). Louboutin [8] proves an explicit upper bound of this shape. Under the assumption of the Generalized Riemann Hypothesis (GRH), we have the much stronger bounds due to Littlewood [7]:

$$\frac{\zeta(2)(1 + o(1))}{2e^\gamma \log \log q} \leq |L(1, \chi)| \leq (2e^\gamma + o(1)) \log \log q,$$

where $o(1)$ tends to 0 as $q \to \infty$. Recently, Lamzouri, Li and Soundararajan gave the following explicit refinement

**Theorem 1.1.** [6, Theorem 1.5] Assume GRH. Let $q$ be a positive integer and $\chi$ be a primitive character modulo $q$. For $q \geq 10^{10}$ we have

$$|L(1, \chi)| \leq 2e^\gamma \left( \log \log q - \log 2 + \frac{1}{2} + \frac{1}{\log \log q} \right),$$

and

$$\frac{1}{|L(1, \chi)|} \leq \frac{12e^\gamma}{\pi^2} \left( \log \log q - \log 2 + \frac{1}{2} + \frac{1}{\log \log q} + \frac{14 \log \log q}{\log q} \right).$$

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The goal of this paper is to provide explicit upper and lower bounds for a large class of \( L \)-functions, including \( L \)-functions attached to automorphic cuspidal forms on \( GL(n) \). More precisely, we bound the quantity \(|L(1, f)|\), where \( L \) is a degree \( d \geq 1 \) \( L \)-function and \( f \) is some arithmetic or geometric object. The results will be valid under the assumption of GRH and the Ramanujan-Petersson conjecture. Additionally, we improve on the bound that comes from generalizing Littlewood’s technique, which under both GRH and Ramanujan-Petersson conjecture provides

\[
(1 + o(1)) \left( \frac{12e^\gamma \log \log C(f)}{\pi^2} \right)^{-d} \leq |L(1, f)| \leq (1 + o(1)) \left( 2e^\gamma \log \log C(f) \right)^d,
\]

where \( o(1) \) is a quantity that tends to 0 as \( C(f) \to \infty \). Here \( C(f) \) denotes the analytic conductor of the \( L \)-function. A precise definition of \( C(f) \) along with what the term \( L \)-function describes will be provided after another example. Other works discussing explicit bounds for higher degree \( L \)-functions focus on bounding \( L(\frac{1}{2}, f) \), we refer the reader to [1] for details.

We provide a degree 2 example before appealing to the precise definitions. Let \( k, q \geq 1 \) be integers and let \( \chi \) be a Dirichlet character modulo \( q \). Take \( f \) to be a Hecke cusp form of weight \( k \), level \( q \), and character \( \chi \), with the following Fourier expansion at the cusp \( \infty \),

\[
f(z) = \sum_{n \geq 1} \lambda_f(n) n^{(k-1)/2} e(nz), \quad e(z) = e^{2\pi iz}.
\]

Then

\[
L(s, f) = \prod_p \left( 1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}} \right)^{-1} = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s},
\]

is a degree 2 \( L \)-function. By works of Deligne [3] and Deligne and Serre [4], it is known that \( L(s, f) \) satisfies Ramanujan-Petersson for all weights \( k \geq 1 \). In this situation, the analytic conductor is given by

\[
C(f) = \frac{q}{\pi^2} \left( \frac{1 + (k - 1)/2}{2} \right) \left( \frac{1 + (k + 1)/2}{2} \right) \asymp qk^2.
\]

We deduce the following corollary from our main results Thereom 2.1 and Theorem 2.2 below.

**Corollary 1.2.** Under the assumption of GRH, if \( \log C(f) \geq 46 \), we have

\[
|L(1, f)| \leq (2e^\gamma)^2 \left( (\log \log C(f))^2 - (2 \log 4 - 1) \log \log C(f) + (\log 4)^2 - \log 4 + 2.51 \right),
\]

and

\[
\frac{1}{|L(1, f)|} \leq \left( \frac{12e^\gamma}{\pi^2} \right)^2 \left( (\log \log C(f))^2 - (2 \log 4 - 1) \log \log C(f) + (\log 4)^2 - \log 4 + 2.67 \right.
\]

\[
+ \left. \frac{89.40((\log \log C(f))^2 - 2 \log 4 \log \log C(f) + 2 \log 4)}{\log C(f)} \right) .
\]

1.1. **Definitions and Notation.** To begin, let \( d \geq 1 \) be a fixed positive integer, and let \( L(s, f) \) be given by the Dirichlet series and Euler product

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \prod_{j=1}^d \left( 1 - \frac{\alpha_{i,f}(p)}{p^s} \right)^{-1},
\]
where \( \lambda_f(1) = 1 \), and both the series and product are absolutely convergent in \( \text{Re}(s) > 1 \). We shall assume that \( L(s, f) \) satisfies the Ramanujan-Petersson conjecture which states that \( |\alpha_{j, f}(p)| \leq 1 \) for all primes \( p \) and \( 1 \leq j \leq d \). Further, we define the gamma factor

\[
\gamma(s, f) = \pi^{-ds/2} \prod_{j=1}^{d} \Gamma \left( \frac{s + \kappa_j}{2} \right),
\]

where \( \kappa_j \) are complex numbers. These \( \kappa_j \) are called the local parameters at infinity and may be referred to as such throughout. In general, it is assumed that \( \text{Re}(\kappa_j) > -1 \), in our case the Ramanujan-Petersson conjecture guarantees that \( \text{Re}(\kappa_j) \geq 0 \). This last condition ensures that \( \gamma(s, f) \) has no pole in \( \text{Re}(s) > 0 \). Furthermore, there exists a positive integer \( q(f) \) (called the conductor of \( L(s, f) \)), such that the completed \( L \)-function,

\[
\xi(s, f) = q(f)^{s/2} \gamma(s, f) L(s, f),
\]

has an analytic continuation to the entire complex plane, and has finite order. This completion satisfies a functional equation

\[
\xi(s, f) = \epsilon(f) \xi(1 - s, \overline{f}),
\]

where \( \epsilon(f) \) is a complex number of absolute value 1, and \( \xi(s, \overline{f}) = \overline{\xi(s, f)} \) (\( \overline{f} \) is called the dual of \( f \)). Uniform estimates for analytic quantities associated to \( L(s, f) \), when \( L(s, f) \) is varying rely on a number of parameters, it is therefore convenient to state the results in terms of the analytic conductor which we define as follows: For \( s \in \mathbb{C} \),

\[
C(f, s) := \frac{q(f)}{\pi^d} \prod_{j=1}^{d} \left| \frac{s + \kappa_j}{2} \right|.
\]

In this article we are interested in studying the value of \( L() \)

\[
C(f) := C(f, 1) = \frac{q(f)}{\pi^d} \prod_{j=1}^{d} \left| \frac{1 + \kappa_j}{2} \right|.
\]

We note that in [1] the author uses \( C(f) = C(f, 1/2) \). This definition is very similar to the one given in Iwaniec and Kowalski [5] and only differs by a constant factor to the power of the degree of the \( L \)-function. To help orient the reader, we give an example in the form of the analytic conductor of a Dirichlet \( L \)-function. Let \( \chi \) be a Dirichlet character modulo \( q \) then the associated \( L \)-function has analytic conductor:

\[
C(\chi) = q^{1 + a} / 2\pi, \text{ where } a = \begin{cases} 1 & \text{if } \chi(-1) = -1 \\ 0 & \text{if } \chi(-1) = 1. \end{cases}
\]

2. Results

Here we detail the theorems and make some remarks about how they fit into the general context of what is already known.

Theorem 2.1. Let \( d \geq 1 \) be a fixed positive integer and let \( L(s, f) \) be an \( L \)-function of degree \( d \) with conductor \( q(f) \) and analytic conductor \( C(f) \). Suppose that GRH and Ramanujan-Petersson hold for
Remark 2. If we take $J(2.2)$ where
\begin{equation}
\frac{d}{2}(\log \log C(f) - \log 2d)^d - \frac{d}{2} \left( \frac{d}{2} \right)(\log \log C(f))^{d-1} + O_d((\log \log C(f))^{d-2})
\end{equation}
where
\begin{equation}
K(d) = 2.31 + \frac{22.59}{d}(e^{0.31d} - 1 - 0.31d).
\end{equation}

Remark 1. This result is asymptotically better than the classical bound as it has the shape
\begin{equation}
|L(1, f)| \leq (2e^gamma)^d \left( \left( \log \log C(f) \right)^d - d \log(2d) \right) + O((\log \log C(f))^{d-2}),
\end{equation}
and $(d \log(2d) - \frac{d}{2}) > 0$ for all $d \geq 1$.

Remark 2. If we take $d = 1$, we may take $C(f) \geq 10^{10}$ and we obtain $K(1)/4 \leq 0.88$ which gives essentially Theorem 1.1

\begin{equation}
|L(1, \chi)| \leq 2e^gamma \left( \log \log C(f) - \log 2 + \frac{1}{2} + \frac{0.88}{\log \log C(f) - \log 2} \right).
\end{equation}

Theorem 2.2. Let $d \geq 1$ be a fixed positive integer and let $L(s, f)$ be an $L$-function of degree $d$ with conductor $q(f)$ and analytic conductor $C(f)$. Suppose that GRH and Ramanujan-Petersson hold for $L(s, f)$. Then for $C(f)$ chosen such that $\log C(f) \geq 23d$ we have

\begin{equation}
\frac{1}{|L(1, f)|} \leq \left( \frac{12e^gamma}{\pi^2} \right)^d \left( \log \log C(f) - \log 2d \right)^d + \left( \frac{d}{2} \log \log C(f) - \log 2d \right)^{d-1}
\end{equation}

where
\begin{equation}
J_1(d) \leq 2 + \frac{4.18}{d}(e^{0.69d} - 1 - 0.69d).
\end{equation}

and

\begin{equation}
J_2(d) = 9 + \frac{16.74}{d}(e^{0.69d} - 1 - 0.69d).
\end{equation}

We notice that lower bound also provides something asymptotically better as in Remark 1.

Remark 3. If $d = 1$ we may take $C(f) \geq 10^{10}$ then $J_1(1)/4 \leq 0.82$ and $J_2(1) \leq 14.09$ this provides essentially Theorem 1.1

\begin{equation}
\frac{1}{|L(1, \chi)|} \leq \left( \frac{12e^gamma}{\pi^2} \right)^d \left( \log \log C(f) + \frac{1}{2} - \log 2 + \frac{0.82}{\log \log C(f) - \log 2} \right) + 14.09\log \log C(f) - \log 2 \right).
\end{equation}

As an easy corollary to these theorems we may a bound degree $d$ $L$-functions in the $t$ aspect as follows. Let $t$ be a real number and define $L_t(1, f) := L(1 + it, f)$, then the analytic conductor of $L_t(s, f)$ is given by

\begin{equation}
C_t(f) := \frac{q(f)}{\pi^d} \prod_{j=1}^{d} \left| \frac{1 + it + \kappa_j}{2} \right| \geq |t|^d.
\end{equation}
Corollary 2.3. Let $d \geq 1$ be a fixed positive integer and let $L(s, f)$ be an $L$-function of degree $d$ with conductor $q(f)$ and analytic conductor $C(f)$. Suppose that GRH and Ramanujan-Petersson hold for $L(s, f)$. If $\log C_t(f) \geq 23d$ then

$$|L(1+it, f)| \leq (2e^\gamma)^d \left( \log \log C_t(f) - \log 2d \right)^d + \frac{d}{2} \left( \log \log C_t(f) - \log 2d \right)^{d-1} \frac{dK(d)}{4} \left( \log \log C_t(f) - \log 2d \right)^{d-2},$$

and

$$\frac{1}{|L(1+it, f)|} \leq \frac{12e^\gamma}{\pi^2} \left( \log \log C_t(f) - \log 2d \right)^d + \frac{d}{2} \left( \log \log C_t(f) - \log 2d \right)^{d-1} \frac{dJ_1(d)}{4} \left( \log \log C_t(f) - \log 2d \right)^{d-2} + \frac{d^2J_2(d)}{2} \left( \log \log C_t(f) - \log 2d \right)^d.$$

The definitions of $K(d)$, $J_1(d)$ and $J_2(d)$ are given by equations (2.1), (2.2) and (2.3) respectively.

3. Lemmata

In this section we will outline a number of results which are necessary for proving the final bound. Additionally, we will disclose a few more properties of the $L$-functions we are studying. First, the logarithmic derivative of $L(s, f)$ is given by

$$-\frac{L'(s, f)}{L(s, f)} = \sum_{n \geq 2} \frac{a_f(n)\Lambda(n)}{n^s} \text{ for } \text{Re}(s) > 1,$$

where $a_f(n) = 0$ unless $n = p^k$ is a prime power in which case $a_f(n) = \sum_{j=1}^{d} \alpha_{j, f} p^{j-k}$. Since $L(s, f)$ satisfies the Ramanujan-Petersson conjecture, then $|a_f(n)| \leq d$. Further, let $\{\rho_f\}$ be the set of the nontrivial zeros of $L(s, f)$. Then we have the Hadamard factorization formula (3.1)

$$\xi(s, f) = e^{A(f)+sB(f)} \prod_{\rho_f} \left(1 - \frac{s}{\rho_f}\right) e^{s/\rho_f},$$

where $A(f)$ and $B(f)$ are constants. We note that $\text{Re}(B(f)) = -\text{Re}(\sum_{\rho_f} 1/\rho_f)$ and taking the logarithmic derivatives of both sides of (3.1) gives

$$\text{Re} \frac{\xi'}{\xi}(s, f) = \text{Re} \sum_{\rho_f} \frac{1}{s - \rho_f}.$$

3.1. Explicit Formulas for $\log |L(1, f)|$ and $|\text{Re}(B(f))|.$

Lemma 3.1. Let $d \geq 1$ be a fixed positive integer and let $L(s, f)$ be an $L$-function of degree $d$ with conductor $q(f)$. Suppose that GRH and Ramanujan-Petersson hold for $L(s, f)$. For any $x \geq 2$ there exists a real number $|\theta| \leq 1$ such that

$$\log |L(1, f)| = \text{Re} \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n \log n} \log \left( \frac{x}{n^s} \right) + \frac{1}{2} \log x \left( \log \frac{q(f)}{\pi^d} + \text{Re} \sum_{j=1}^{d} \frac{\Gamma'(1+\kappa_j)}{\Gamma(1+\kappa_j)} \left( \frac{1+\kappa_j}{2} \right) \right)$$

$$- \left( \frac{1}{\log x} - \frac{2\theta}{\sqrt{x} \log^2 x} \right) |\text{Re}(B(f))| + \frac{2d\theta}{x \log x}.$$

Proof. We have for any fixed $\sigma \geq 1$ that

$$\frac{1}{2\pi i} \int_{2-\infty}^{2+i\infty} L'(s + \sigma, f) \frac{x^s}{s^2} ds = \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n^{\sigma}} \log(\frac{2}{n}).$$
Shifting the contour to the left, we see this integral is also equal to
\[- \left( \frac{L'}{L} \right)'(\sigma, f) - \frac{L'}{L}(\sigma, f) \log x - \sum_{\rho_f} \frac{x^\rho_f - \sigma}{(\rho_f - \sigma)^2} - \sum_{j=1}^{d} \sum_{m=0}^{\infty} \frac{x^{-2m-\kappa_j-\sigma}}{(2m + \kappa_j + \sigma)^2}.\]

Thus, since \(\text{Re}(\kappa_j) \geq 0\) we have
\[- \frac{L'}{L}(\sigma, f) = \sum_{n \leq x} \frac{a_f(n)\Lambda(n) \log(\frac{x}{n})}{n^\sigma} \log x + \frac{1}{\log x} \left( \frac{L'}{L} \right)'(\sigma, f)\]
\[+ \frac{\theta x^{1-\sigma}}{\log x} \sum_{\rho_f} \frac{1}{|\rho_f|^2} + \frac{\theta x^{-\sigma}}{\log x} \sum_{j=1}^{d} \sum_{m=0}^{\infty} \frac{x^{-2m}}{(2m + 1)^2}.\]

We integrate both sides with respect to \(\sigma\) from 1 to \(\infty\), then take real parts to obtain
\[\log |L(1, f)| = \text{Re} \sum_{n \leq x} \frac{a_f(n)\Lambda(n) \log(\frac{x}{n})}{n \log n} \log x - \frac{1}{\log x} \text{Re} \left( \frac{L'}{L} \right)(1, f) + \frac{\theta}{\sqrt{x} \log^2 x} \sum_{\rho_f} \frac{1}{|\rho_f|^2} + \frac{2d\theta}{x \log^2 x}\]

We note that \(\sum_{\rho_f} \frac{1}{|\rho_f|^2} = 2|\text{Re}B(f)|\). Now we have
\[- \frac{L'}{L}(1, f) = \frac{1}{2} \log q(f) - \frac{1}{2} \log \pi + \frac{d}{2} \sum_{j=1}^{d} \Gamma' \left( \frac{1 + \kappa_j}{2} \right) - \frac{\xi'}{\xi}(1, f)\]

Hence, after taking real parts we have the desired result. \(\square\)

**Lemma 3.2.** Let \(d \geq 1\) be a fixed positive integer and let \(L(s, f)\) be an \(L\)-function of degree \(d\) with conductor \(q(f)\). Suppose that GRH and Ramanujan-Petersson hold for \(L(s, f)\). Define \(0 \leq l(f) \leq d\) to be the number of \(\kappa_j\) in the gamma factor of \(L(s, f)\) which equal 0. For any \(x > 1\) there exists a real number \(|\theta| \leq 1\) such that
\[- \frac{\xi'}{\xi}(0, f) - \frac{1}{x} \frac{\xi'}{\xi}(0, f) + \frac{2\theta}{\sqrt{x}} |\text{Re}(B(f))| = \frac{1}{2} \log \left( \frac{q(f)}{\pi^d} \right) \left( 1 - \frac{1}{x} \right) - \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n} \left( 1 - \frac{n}{x} \right) + E(f, x),\]

where
\[E(f, x) = l(f) \left( -\log 2 - \frac{\gamma}{2} \left( 1 - \frac{1}{x} \right) + \frac{\log x + 1}{x} - \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)} \right)\]
\[+ \sum_{i=1}^{d-l(f)} \left( \frac{1}{2} \Gamma' \left( \frac{1+\kappa_i}{2} \right) - \frac{1}{2x} \Gamma' \left( \frac{\kappa_i}{2} \right) - \sum_{n=0}^{\infty} \frac{x^{-2n-\kappa_i-1}}{(2n+\kappa_i)(2n+\kappa_i+1)} \right),\]

In particular, \(\left( 1 + \frac{1}{x} + \frac{2\theta}{\sqrt{x}} \right) |\text{Re}(B(f))|\) equals
\[\frac{1}{2} \left( 1 - \frac{1}{x} \right) \left( \log \left( \frac{q(f)}{\pi^d} \right) + \text{Re} \sum_{j=1}^{d} \Gamma' \left( \frac{1+\kappa_j}{2} \right) \right) - \text{Re} \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n} \left( 1 - \frac{n}{x} \right)\]
\[-(\theta - (1+\theta)l(f)) \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)} + l(f) \log x + 1 + \frac{(d-2l(f)) \log 2}{x} + \frac{d-l(f)}{x} \sum_{i=1}^{d-l(f)} \text{Re} \left( \sum_{n=1}^{\infty} \left( \frac{2}{\kappa_i+1+2n} - \frac{1}{\kappa_i+1+n} \right) + \frac{x^{-\kappa_i-1}}{\kappa_i(\kappa_i+1)} \right).\]
In both of the above expressions, the terms inside \( \sum_{i=1}^{d-1(f)} \) are ranging over the local parameters at infinity, \( \kappa_i \neq 0 \).

Proof. We consider

\[
I(f) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\zeta'(s, f)}{\zeta(s, f)} \frac{x^{s-1}}{s(s-1)} ds.
\]

Pulling the contour to the left we collect the residues of the poles at \( s = 0, 1 \) and \( \rho_f \) the nontrivial zeros of \( L(s, f) \). Hence,

\[
I(f) = \frac{\zeta'(1, f)}{\zeta(1, f)} - \frac{1}{x} \frac{\zeta'(0, f)}{\zeta(0, f)} + \sum_{\rho_f} \frac{x^{\rho_f-1}}{\rho_f(\rho_f - 1)}.
\]

Thus applying GRH we have for some \( |\theta| \leq 1 \)

\[
I(f) = -\frac{\zeta'(0, f)}{\zeta(0, f)} - \frac{1}{x} \frac{\zeta'(0, f)}{\zeta(0, f)} + \frac{2\theta}{\sqrt{x}} |\text{Re}(B(f))|.
\]

On the other hand, we can also write

\[
I(f) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left( \frac{1}{2} \log(q(f)) + \frac{\zeta'(s, f)}{s} + \frac{L'(s, f)}{L(s, f)} \right) \frac{x^{s-1}}{s(s-1)} ds
\]

\[
= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{1}{2} \log \left( \frac{q(f)}{\pi^d} \right) \frac{x^{s-1}}{s(s-1)} ds + \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{1}{2} \sum_{j=1}^{d} \frac{\Gamma'(s + \kappa_j)}{\Gamma(s + \kappa_j)} \frac{x^{s-1}}{s(s-1)} ds
\]

\[
= I_1 + I_2 + I_3.
\]

The contribution from \( I_1 \) and \( I_3 \) is

\[
\frac{1}{2} \log \left( \frac{q(f)}{\pi^d} \right) \left( 1 - \frac{1}{x} \right) - \sum_{n \leq x} a_f(n) \Lambda(n) \frac{1}{n} \frac{1}{x}.
\]

We rewrite \( I_2 \) as

\[
I_2 = \sum_{j=1}^{d} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{1}{2} \frac{\Gamma'(s + \kappa_j)}{\Gamma(s + \kappa_j)} \frac{x^{s-1}}{s(s-1)} ds.
\]

Fix \( j \), if \( \kappa_j \neq 0 \) then the \( j \)-th term of the summand will have simple poles at \( s = 0, 1 \) and \( s = -2n - \kappa_j \) for \( n \geq 0 \). Thus the contribution will be

\[
\frac{1}{2} \frac{\Gamma'(1 + \kappa_j)}{\Gamma(1 + \kappa_j)} - \frac{1}{2} \frac{\Gamma'(\kappa_j)}{\Gamma(\kappa_j)} - \sum_{n=0}^{\infty} \frac{x^{-2n-\kappa_j-1}}{(2n + \kappa_j)(2n + 1 + \kappa_j)}.
\]

On the other hand, if \( \kappa_j = 0 \) then the \( j \)-th term of the summand will have simple poles at \( s = 1 \) and \( s = -2n \) for \( n \geq 1 \), which contribute

\[
\frac{1}{2} \frac{\Gamma'(1/2)}{\Gamma(1/2)} - \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)}.
\]

Additionally, we know that

\[
\frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} = -\frac{1}{s} + \frac{1}{2} \frac{\Gamma'(s/2 + 1)}{\Gamma(s/2 + 1)},
\]

so the residue of the double pole at \( s = 0 \) is given by

\[
\frac{1 + \log x + \frac{1}{2} \frac{\Gamma'(1)}{\Gamma(1)}}{x}.
\]
Using the fact that $\frac{\Gamma'}{\Gamma}(1) = -\gamma$ and $\frac{\Gamma'}{\Gamma}(1/2) = \log 2 - \gamma$ we see the overall contribution will be

$$- \log 2 - \frac{\gamma}{2} \left(1 - \frac{1}{x}\right) = \frac{\log x + 1}{x} - \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)}.$$ 

Let $l(f)$ be as in the statement of the lemma. Then, reordering the $\kappa_j$ so that $\kappa_1, \kappa_2, \ldots, \kappa_{d-l(f)}$ are all nonzero and summing over $j$ we get the desired expression for $E(f, x)$.

Finally, since $-\text{Re} \frac{j\pi}{2} \log \left| \frac{\Gamma'(0, t)}{\Gamma'(\frac{1}{2})} \right| = -\text{Re} \frac{j\pi}{2} \log |\text{Re}(B(f))|$, we see that taking real parts of the established identity we obtain

$$\left(1 + \frac{1}{x} + \frac{2\theta}{\sqrt{x}}\right) |\text{Re}(B(f))| = \frac{1}{2} \log \left(\frac{q(f)}{\pi^d}\right) \left(1 - \frac{1}{x}\right) - \text{Re} \sum_{n \leq x} \frac{a(f)\Lambda(n)}{n} \left(1 - \frac{n}{x}\right) + \text{Re}(E(f, x)).$$

We find an explicit expression for the right hand side as follows:

Start by noting that for $\kappa_j = 0$ we have

$$\frac{\Gamma'}{\Gamma} \left(\frac{1}{2}\right) = \frac{\Gamma'}{\Gamma} \left(\frac{1 + \kappa_j}{2}\right),$$

so that

$$\frac{1}{2} \log \left(\frac{q(f)}{\pi^d}\right) \left(1 - \frac{1}{x}\right) + \frac{E(f, x)}{2} = \frac{1}{2} \log \left(\frac{q(f)}{\pi^d}\right) \left(1 - \frac{1}{x}\right) + \sum_{j=1}^{d} \frac{1}{2} \frac{\Gamma'(\kappa_j)}{\Gamma} \left(\frac{1 + \kappa_j}{2}\right)$$

$$+ \frac{l(f)}{2} \left(\frac{\log x + 1 + \gamma/2}{x} - \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)}\right) - \frac{d-l(f)}{2} \sum_{i=1}^{\infty} \left(\frac{1}{2x} \frac{\Gamma'(\kappa_i)}{\Gamma} \left(\frac{\kappa_i}{2}\right) + \sum_{n=0}^{\infty} \frac{x^{-2n-\kappa_i-1}}{(2n+\kappa_i)(2n+\kappa_i+1)}\right).$$

We note that for some $|\theta| \leq 1$

$$\sum_{n=0}^{\infty} \frac{x^{-2n-\kappa_i-1}}{(2n+\kappa_i)(2n+\kappa_i+1)} = \frac{x^{-\kappa_i}}{x\kappa_i(\kappa_i+1)} + \theta \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)},$$

hence

(3.3)

$$\frac{1}{2} \log \left(\frac{q(f)}{\pi^d}\right) \left(1 - \frac{1}{x}\right) + \frac{E(f, x)}{2} = \frac{1}{2} \log \left(\frac{q(f)}{\pi^d}\right) \left(1 - \frac{1}{x}\right) + \sum_{j=1}^{d} \frac{1}{2} \frac{\Gamma'(\kappa_j)}{\Gamma} \left(\frac{1 + \kappa_j}{2}\right)$$

$$- (d\theta - (1 + \theta)l(f)) \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)} + \frac{l(f)}{2} \frac{\log x + 1 + \gamma/2 + 1}{x} - \frac{1}{x} \sum_{i=1}^{\infty} \left(\frac{1}{2x} \frac{\Gamma'(\kappa_i)}{\Gamma} \left(\frac{\kappa_i}{2}\right) + \frac{x^{-\kappa_i}}{\kappa_i(\kappa_i+1)}\right).$$

Now, from the functional equation of $\Gamma(s)$ we see that

$$\frac{1}{2} \frac{\Gamma'(\kappa_i/2)}{\Gamma} = \frac{1}{2} \frac{\Gamma' \left(\frac{\kappa_i}{2} + 1\right)}{\Gamma} - \frac{1}{\kappa_i},$$

we recall Legendre’s duplication formula

$$\Gamma(s)\Gamma(s + \frac{1}{2}) = 2^{1 - 2s} \log(\sqrt{\pi})\Gamma(2s),$$

so we have

$$\frac{1}{2} \frac{\Gamma' \left(\frac{\kappa_i}{2} + 1\right)}{\Gamma} = - \log 2 + \frac{\Gamma'}{\Gamma} \left(\frac{\kappa_i + 1}{2}\right) - \frac{1}{2} \frac{\Gamma' \left(\frac{\kappa_i}{2}\right)}{\Gamma}.$$ 

Finally, we note

$$\frac{\Gamma'}{\Gamma}(s) = -\gamma - \frac{1}{s} - \sum_{n=1}^{\infty} \left(\frac{1}{s + n} - \frac{1}{n}\right),$$

so that

$$\left[\frac{\Gamma' \left(\frac{\kappa_i}{2}\right)}{\Gamma} + - \frac{1}{2} \frac{\Gamma' \left(\frac{\kappa_i + 1}{2}\right)}{\Gamma} - \frac{1}{2} \frac{\Gamma' \left(\frac{\kappa_i + 1}{2}\right)}{\Gamma} \right] = \frac{1}{\kappa_i + 1} + \sum_{n=1}^{\infty} \frac{2}{\kappa_i + 1 + 2n} - \frac{1}{\kappa_i + 1 + n}.$$
(3.4) \[ \frac{1}{2} \log \left( \frac{q(f)}{\pi x} \right) \left( 1 - \frac{1}{x} \right) + E(f, x) = \frac{1}{2} \log \left( \frac{q(f)}{\pi x} \right) \left( 1 - \frac{1}{x} \right) + \sum_{j=1}^{d} \frac{\Gamma'(1 + \kappa_j)}{2 \Gamma} \left( 1 + \frac{1}{2} \right) \\
- (d \theta - (1 - \theta)l(f)) \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)} + l(f) \frac{\log x + \gamma/2 + 1}{x} \\
- \frac{1}{x} \sum_{i=1}^{d-l(f)} \left( \frac{1}{\Gamma} \left( \frac{\kappa_i + 1}{2} \right) - \log 2 + \sum_{n=1}^{\infty} \left( \frac{2}{\kappa_i + 1 + 2n} - \frac{1}{\kappa_i + 1 + n} + \frac{x^{-\kappa_i} - 1}{\kappa_i(\kappa_i + 1)} \right) \right). \]

Then since \( \frac{\Gamma'(1 + \frac{1}{2})}{\Gamma} = \frac{\Gamma'(1)}{\Gamma} = -\log 2 - \gamma/2 \) when \( \kappa_i = 0 \), we add \( -l(f) \frac{\Gamma'(1)}{2x} \) so that the RHS of (3.3) is given by

\[ \frac{1}{2} \left( 1 - \frac{1}{x} \right) \left( \log \left( \frac{q(f)}{\pi x} \right) + \sum_{j=1}^{d} \frac{\Gamma'(1 + \frac{1}{2})}{\Gamma} \left( \frac{1 + \kappa_j}{2} \right) \right) - (d \theta - (1 + \theta)l(f)) \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)} + l(f) \frac{\log x + 1}{x} \\
+ \frac{(d - 2l(f)) \log 2}{x} \sum_{i=1}^{d-l(f)} \left( \sum_{n=1}^{\infty} \left( \frac{2}{\kappa_i + 1 + 2n} - \frac{1}{\kappa_i + 1 + n} + \frac{x^{-\kappa_i} - 1}{\kappa_i(\kappa_i + 1)} \right) \right). \]

Taking real parts gives the desired result.

3.2. Bounds for the Digamma Function. The following are some technical lemmas which help to shorten the proof of the main results. The first is taken from V. Chandee.

**Lemma 3.3.** [Lemma 2.3] Let \( z = x + iy \), where \( x \geq \frac{1}{4} \). Then

\[ \Re \frac{\Gamma'}{\Gamma}(z) \leq \log |z|. \]

**Lemma 3.4.** Let \( \kappa = \sigma + it \) such that \( \sigma \geq 0 \), then

\[ \Re \left( \sum_{n=1}^{\infty} \left( \frac{2}{\kappa + 1 + 2n} - \frac{1}{\kappa + 1 + n} \right) \right) = \frac{1}{2} \log 4 + \frac{\sigma^2 + 3\sigma + 2 + t^2}{(\sigma + 2)^2 + t^2} - \frac{\sigma^2 + 4\sigma + 3 + t^2}{(\sigma + 3)^2 + t^2} + \frac{1}{2} \log \left( \frac{(\sigma + 2)^2 + t^2}{(\sigma + 3)^2 + t^2} \right). \]

**Proof.** We take the real part inside the sum and focus on the individual partial sums given by

\[ \sum_{n=1}^{N} \frac{2(\sigma + 1 + 2n)}{(\sigma + 1 + 2n)^2 + t^2} \text{ and } \sum_{n=1}^{N} \frac{2(\sigma + 1 + n)}{(\sigma + 1 + n)^2 + t^2}. \]

Using partial summation we find

\[ \sum_{n=1}^{N} \frac{2(\sigma + 1 + 2n)}{(\sigma + 1 + 2n)^2 + t^2} = \frac{2N(\sigma + 1 + 2N) + \sigma^2 + 2\sigma(N + 1) + 2Y + 1 + t^2}{(\sigma + 1 + 2N)^2 + t^2} - \frac{\sigma^2 + 4\sigma + 3 + t^2}{(\sigma + 3)^2 + t^2} + \frac{1}{2} \log((\sigma + 1 + 2N)^2 + t^2) - \frac{1}{2} \log((\sigma + 3)^2 + t^2) \]

and

\[ \sum_{n=1}^{N} \frac{2(\sigma + 1 + n)}{(\sigma + 1 + n)^2 + t^2} = \frac{N(\sigma + 1 + N) + \sigma^2 + \sigma(N + 1) + \sigma + N + 1 + t^2}{(\sigma + 1 + N)^2 + t^2} - \frac{\sigma^2 + 3\sigma + 2 + t^2}{(\sigma + 2)^2 + t^2} + \frac{1}{2} \log((\sigma + 1 + N)^2 + t^2) - \frac{1}{2} \log((\sigma + 2)^2 + t^2). \]

Taking the limit as \( N \to \infty \) we see

\[ \Re \left( \sum_{n=1}^{\infty} \left( \frac{2}{\kappa + 1 + 2n} - \frac{1}{\kappa + 1 + n} \right) \right) = \frac{1}{2} \log 4 + \frac{\sigma^2 + 3\sigma + 2 + t^2}{(\sigma + 2)^2 + t^2} - \frac{\sigma^2 + 4\sigma + 3 + t^2}{(\sigma + 3)^2 + t^2} + \frac{1}{2} \log \left( \frac{(\sigma + 2)^2 + t^2}{(\sigma + 3)^2 + t^2} \right), \]
as was claimed. □

Lemma 3.5. Let \( \kappa = s + it \) such that \( \sigma \geq 0 \), and \( x > 1 \) then

\[
\frac{|x^{-\kappa} - 1|}{\kappa(\kappa + 1)} \leq \frac{2 \log x}{\log 3}.
\]

Proof. We consider two cases.

First suppose \( |\kappa| \geq \frac{c}{\log x} \) then we can trivially bound the norm to obtain

\[
\frac{|x^{-\kappa} - 1|}{\kappa(\kappa + 1)} \leq \frac{2 \log x}{c}.
\]

If \( |\kappa| < \frac{c}{\log x} \) then

\[
x^{-\kappa} - 1 = \sum_{k=1}^{\infty} \frac{(-\kappa \log x)^k}{k!},
\]

so that

\[
\frac{|x^{-\kappa} - 1|}{\kappa(\kappa + 1)} \leq \log x \sum_{k=1}^{\infty} \frac{c^{k-1}}{k!} = \frac{e^c - 1}{c} \log x.
\]

The choice of \( c = \log 3 \) gives the desired result. □

3.3. Relevant Results from [6]. Let

\[
B = -\sum_{\rho} \text{Re} \frac{1}{\rho} = \frac{1}{2} \log(4\pi) - 1 - \frac{\gamma}{2},
\]

where the sum is taken over the non-trivial zeros of the Riemann zeta function.

Lemma 3.6. [6 Lemma 2.4] Assume the Riemann Hypothesis. For \( x > 1 \) we have, for some \( |\theta| \leq 1 \),

\[
\sum_{n \leq x} \Lambda(n) n \left( 1 - \frac{n}{x} \right) = \log x - (1 + \gamma) + \frac{2\pi}{x} - \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)} + \frac{2|B|\theta}{\sqrt{x}}.
\]

Lemma 3.7. [6 Lemma 2.6] Assume the Riemann Hypothesis. For all \( x \geq e \) we have

\[
\sum_{n \leq x} \frac{\Lambda(n)}{n \log n} \frac{\log(x/n)}{\log x} = \log \log x - \gamma - 1 + \frac{\gamma}{\log x} + \frac{2|B|\theta}{\sqrt{x} \log^2 x} + \frac{\theta}{3x \log^3 x}.
\]

We also prove the following lemma which is a slight generalization of [6 Lemma 5.1].

Lemma 3.8. Assume the Ramanujan-Petersson conjecture. Then for \( x \geq 100 \) we have

(3.5) \[
\text{Re} \sum_{n \leq x} \alpha_{j,f}(n) \Lambda(n) \left( \frac{1}{n \log n} - \frac{1}{x \log x} \right) \geq \sum_{p^k \leq x} \Lambda(p^k)(-1)^k \left( \frac{1}{p^k \log p^k} - \frac{1}{x \log x} \right).
\]

In particular, we have

(3.6) \[
\text{Re} \sum_{n \leq x} a_f(n) \Lambda(n) \left( \frac{1}{n \log n} - \frac{1}{x \log x} \right) \geq d \sum_{p^k \leq x} \Lambda(p^k)(-1)^k \left( \frac{1}{p^k \log p^k} - \frac{1}{x \log x} \right).
\]

Proof. Note that if \( x \) is a prime power then the summand at \( x \) on both sides of the inequality (3.5) contribute 0, so we assume \( x \) is not a prime power. We begin by recalling that \( a_f(n) = 0 \) unless \( n = p^k \) is a prime power in which case \( a_f(n) = \sum_{j=1}^{d} \alpha_{j,f}(p)^j \). So that (3.6) follows immediately, once we prove (3.5).

Fix \( j \) and consider each \( \alpha_{j,f} \) separately. From the definition we see \( \alpha_{j,f}(n) \) is only nonzero if \( n = p^k \) for some prime power. If \( \alpha_{j,f}(p) = 0 \) then the contribution is 0 while the value on the right
hand side < 0. If \( \alpha_{j,f}(p) \neq 0 \) then, from Ramanujan-Petersson we have that \( |\alpha_{j,f}(p)| \leq 1 \), so we express \( \alpha_{j,f}(p) = -re(\theta) \), for \( 0 < r \leq 1 \) where \( e(\theta) = e^{2\pi i \theta} \). Consider the difference of the left and right side of (3.5):

\[
(3.7) \quad \log(p) \sum_{p^k \leq x} (-1)^{k-1}(1 - r^k \cos(k\theta)) \left( \frac{1}{p^k \log p^k} - \frac{1}{x \log x} \right).
\]

If we establish this is non-negative, then we are finished.

Before we proceed we see that for all \( k \geq 1 \)

\[
(3.8) \quad 1 - r^k \cos(k\theta) \leq k^2 (1 - r \cos \theta).
\]

The case \( k = 1 \) is trivial, for the remaining \( k \geq 2 \), the inequality follows from

\[
k^2 - 1 \geq 3 > r^k + r \geq r^k \cos(k\theta) - r \cos(\theta).
\]

If \( p \geq 3 \), then by (3.8) we have (3.7) is greater than

\[
\log(p)(1 - r \cos \theta) \left( \frac{1}{p \log p} - \frac{1}{x \log x} - \sum_{j=1}^{\infty} \frac{(2j)^2}{p^{2j} \log p^{2j}} \right) \geq 0.
\]

For \( p = 2 \), when \( k \geq 6 \) we apply (3.8) again. Otherwise, when \( 1 \leq k \leq 5 \) we compute the trigonometric polynomial exactly. A little computer computation completes the result. \( \square \)

4. Proof of Theorems 2.1 and 2.2

4.1. Upper bounds for \( L(1,f) \). Let \( C(f) \geq 10^{10} \) and \( x \geq 132 \), be a real number to be chosen later. Lemma 3.1 says

\[
\log |L(1,f)| \leq \text{Re} \sum_{n \leq x} \frac{a_f(n)\Lambda(n) \log \frac{x/n}{\log x} + \frac{1}{2 \log x} \left( \log \frac{q(f)}{x^d} + \text{Re} \sum_{j=1}^{d} \frac{\Gamma'(\kappa_j + 1/2)}{\Gamma} \right) + \frac{2d}{x \log^2 x} - \left( \frac{1}{\log x} - \frac{2}{\sqrt{x \log^2 x}} \right) |\text{Re}B(f)|}.
\]

Applying Lemma 3.2 with the conditions on \( x \) as above, we see

\[
|\text{Re}B(f)| \geq \left( 1 + \frac{1}{\sqrt{x}} \right)^{-2} \left( \frac{1}{2} \left( 1 - \frac{1}{x} \right) \left( \log \frac{q(f)}{x^d} + \frac{d}{x} \sum_{j=1}^{d} \frac{\Gamma'(\kappa_j + 1/2)}{\Gamma} \right) - \text{Re} \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n} \left( 1 - \frac{n}{x} \right) \right)
\]

\[
+ \left( d - 2l(f) \right) \log^2 x - \left( d - 2l(f) \right) \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)}
\]

\[
- \frac{1}{x} \sum_{i=1}^{d-l(f)} \text{Re} \left( \sum_{n=1}^{\infty} \frac{\kappa_i}{\kappa_i + 1 + 2n} - \frac{1}{\kappa_i + 1 + n} + \frac{x^{-\kappa_i} - 1}{\kappa_i(\kappa_i + 1)} \right).
\]
For $x \geq 132$ we bound
\[ -(\frac{1}{\log x} - \frac{2}{\sqrt{x \log^2 x}}) \left( 1 + \frac{1}{\sqrt{x}} \right)^{-2} \left( l(f) \frac{\log x + 1}{x} + \frac{(d - 2l(f)) \log 2}{x} \right) \]
\[ -(d - 2l(f)) \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n(2n+1)} - \frac{d - l(f)}{x} \sum_{i=1}^{\infty} \left( \sum_{n=1}^{\infty} \left( \frac{2}{\kappa_i + 1 + 2n} - \frac{1}{\kappa_i + 1 + n} \right) + \frac{x^{-\kappa_i} - 1}{\kappa_i(\kappa_i + 1)} \right) + \frac{2d}{x \log^2 x} \]
\[ = - \left( \frac{1}{\log x} - \frac{2}{\sqrt{x \log^2 x}} \right) \left( 1 + \frac{1}{\sqrt{x}} \right)^{-2} (A_1 + A_2 - A_3 - A_4 - A_5) + \frac{2d}{x \log^2 x}. \]

First, we consider
\[ A_4 = \frac{1}{x} \sum_{i=1}^{d-1(f)} \text{Re} \left( \sum_{n=1}^{\infty} \left( \frac{2}{\kappa_i + 1 + 2n} - \frac{1}{\kappa_i + 1 + n} \right) \right). \]

Fix $i$ and study the inner sum, writing $\kappa_i = \sigma + it$, and noting that Ramanujan-Petersson gives us $\sigma \geq 0$, we apply Lemma 3.4 so that
\[ \text{Re} \left( \sum_{n=1}^{\infty} \left( \frac{2}{\kappa_i + 1 + 2n} - \frac{1}{\kappa_i + 1 + n} \right) \right) = \frac{1}{2} \log 4 + \frac{\sigma^2 + 3\sigma + 2 + t^2}{(\sigma + 2)^2 + t^2} - \frac{\sigma^2 + 4\sigma + 3 + t^2}{(\sigma + 3)^2 + t^2} + \frac{1}{2} \log \left( \frac{(\sigma + 2)^2 + t^2}{(\sigma + 3)^2 + t^2} \right) \]
\[ \leq \log 2 - \frac{4}{5}(\sqrt{3} - 2) \leq 1. \]

The inequality comes from the following facts. First, the last term is negative. Next, taking $\sigma \geq 0$, a maple calculation finds that $-\frac{4}{5}(\sqrt{3} - 2)$ is a global maximum for
\[ \frac{\sigma^2 + 3\sigma + 2 + t^2}{(\sigma + 2)^2 + t^2} - \frac{\sigma^2 + 4\sigma + 3 + t^2}{(\sigma + 3)^2 + t^2}. \]

Thus we may combine the terms $A_2$ and $A_4$ to obtain
\[ - \left( \frac{1}{\log x} - \frac{2}{\sqrt{x \log^2 x}} \right) \left( 1 + \frac{1}{\sqrt{x}} \right)^{-2} (A_2 - A_4) \leq \frac{(d - l(f))(1 - \log 2) + l(f) \log 2}{(1 + \sqrt{x})^2 \log x}. \]

For $A_5$, fix $i$, then writing $\kappa_i = \sigma + it$, since we have $\sigma \geq 0$, we apply Lemma 3.3 to obtain
\[ \text{Re} \left( \frac{x^{-\kappa_i} - 1}{\kappa_i(\kappa_i + 1)} \right) \leq \frac{2 \log x}{\log 3}. \]

Thus combining $A_1$ and $A_5$ we have
\[ - \left( \frac{1}{\log x} - \frac{2}{\sqrt{x \log^2 x}} \right) \left( 1 + \frac{1}{\sqrt{x}} \right)^{-2} (A_1 - A_5) \leq \frac{(2d/\log 3 - l(f)(1 + 2/\log 3))}{(1 + \sqrt{x})^2} - \frac{l(f)}{(1 + \sqrt{x})^2 \log x}. \]

Finally, for $x \geq 132$ we have
\[ - \left( \frac{1}{\log x} - \frac{2}{\sqrt{x \log^2 x}} \right) \left( 1 + \frac{1}{\sqrt{x}} \right)^{-2} (A_1 + A_2 - A_3 - A_4 - A_5) + \frac{2d}{x \log^2 x} \]
\[ \leq \frac{1}{(1 + \sqrt{x})^2} \left( 2d/\log 3 - l(f)(1 + 2/\log 3) + \frac{(d - 2l(f))(1 - \log 2 + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n(2n+1)}) + 2d(1 + \sqrt{x})^2}{x \log^2 x} \right) \]
\[ \leq \frac{2d}{(1 + \sqrt{x})^2}. \]
Hence,
\[
\log |L(1, f)| \leq \Re \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n \log n} \frac{\log(x/n)}{\log x} + \frac{1}{2 \log x} \left( \log \frac{q(f)}{\pi^d} + \Re \sum_{j=1}^d \frac{\Gamma'(\kappa_j + 1/2)}{\Gamma} \right) \\
+ \left( \frac{1}{\log x} - \frac{2}{\sqrt{x} \log^2 x} \right) \left( 1 + \frac{1}{\sqrt{x}} \right)^{-2} \Re \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n} \left( 1 - \frac{n}{x} \right) + \frac{2d}{(1 + \sqrt{x})^2} \\
- \left( \frac{1}{\log x} - \frac{2}{\sqrt{x} \log^2 x} \right) \left( 1 + \frac{1}{\sqrt{x}} \right)^{-2} \frac{1}{2} \left( 1 - \frac{1}{x} \right) \left( \log \frac{q(f)}{\pi^d} + \Re \sum_{j=1}^d \frac{\Gamma'(\kappa_j + 1/2)}{\Gamma} \right).
\]

Next, note that
\[
0 \leq \frac{1}{2 \log x} - \left( \frac{1}{\log x} - \frac{2}{\sqrt{x} \log^2 x} \right) \left( 1 + \frac{1}{\sqrt{x}} \right)^{-2} \frac{1}{2} \left( 1 - \frac{1}{x} \right) \leq \frac{1}{(\sqrt{x} + 1) \log x} \left( 1 + \frac{1}{\log x} \right),
\]
and Lemma 3.3 gives
\[
\Re \frac{\Gamma'}{\Gamma} \left( \frac{\kappa_j + 1/2}{2} \right) \leq \log \left| \frac{1 + \kappa_j}{2} \right|,
\]
so
\[
\log \frac{q(f)}{\pi^d} + \Re \sum_{j=1}^d \frac{\Gamma'(\kappa_j + 1/2)}{\Gamma} \leq \log C(f).
\]

Therefore,
\[
\log |L(1, f)| \leq \Re \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n \log n} \frac{\log(x/n)}{\log x} + \frac{\log C(f)}{(\sqrt{x} + 1) \log x} \left( 1 + \frac{1}{\log x} \right) \\
+ \left( \frac{1}{\log x} - \frac{2}{\sqrt{x} \log^2 x} \right) \left( 1 + \frac{1}{\sqrt{x}} \right)^{-2} \Re \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n} \left( 1 - \frac{n}{x} \right) + \frac{2d}{(1 + \sqrt{x})^2}.
\]
The right hand side of the above is largest when $a_f(p) = d$ for all $p \leq x$, thus
\[
\log |L(1, f)| \leq d \Re \sum_{n \leq x} \frac{\Lambda(n)}{n \log n} \frac{\log(x/n)}{\log x} + \frac{d}{\log x} \Re \sum_{n \leq x} \frac{\Lambda(n)}{n} \left( 1 - \frac{n}{x} \right) + \frac{\log C(f)}{(\sqrt{x} + 1) \log x} \left( 1 + \frac{1}{\log x} \right) + \frac{2d}{(1 + \sqrt{x})^2}.
\]
So applying Lemmas 3.3 and 3.7 and choosing $x = \frac{\log^2 C(f)}{4d^2}$ (which implies $\frac{\log C(f)}{\sqrt{x}} = 2d$ and allows us to factor $d$ from each term) we obtain
\[
\log |L(1, f)| \leq d \left( \log \log x + \gamma - \frac{1}{\log x} + \frac{2}{(1 + \sqrt{x})^2} \right) + \log C(f) \frac{1}{\sqrt{x} \log x} \left( 1 + \frac{1}{\log x} \right).
\]
Thus for $x \geq 132$ we have
\[
\log |L(1, f)| \leq d \left( \log \log x + \gamma + \frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{2}{(1 + \sqrt{x})^2} \right) \\
\leq d \left( \log \log x + \gamma + \frac{1}{\log x} + \frac{2.31}{\log^2 x} \right).
\]
Therefore,
\[
|L(1, f)| \leq e^{d\gamma} \log^d x \left( 1 + \frac{d}{\log x} + \frac{dK(d)}{\log^2 x} \right)
\]
where $K(d) = 2.31 + (1 + \frac{4.62}{\log x} + \frac{(2.31)^2}{\log^2 x}) \sum_{k=0}^{\infty} \frac{d^{k+1} k^{(k+1)/2}}{(k+1)!} \log^k x$. Replacing $x$ gives
\[
|L(1, f)| \leq 2^d e^{d\gamma} \left( \log \log C(f) - \log 2d \right)^d + \frac{d}{2} \left( \log \log C(f) - \log 2d \right)^{d-1} + \frac{dK(d)}{4} \left( \log \log C(f) - \log 2d \right)^{d-2},
\]
which proves the result.
4.2. Lower bounds for $L(1, f)$. The argument proceeds similarly. As before we let $C(f)$ be chosen such that $x = \frac{\log^3 C(f)}{4\sigma^2} \geq 132$, then from Lemma [3.1] we have

\[
\log |L(1, f)| \geq \Re \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n \log n} \log x + \frac{1}{2 \log x} \left( \log \frac{q(f)}{\pi^d} + \Re \sum_{j=1}^d \Gamma' \left( \frac{1 + \kappa_j}{2} \right) \right)
- \left( \frac{1}{2 \log x} + \frac{2}{\sqrt{x \log^2 x}} \right) |\Re B(f)| - \frac{2d}{x \log^2 x}.
\]

Applying Lemma [5.2] we see

\[
|\Re(B(f))| \leq \left( 1 - \frac{1}{\sqrt{x}} \right)^{-2} \left( \frac{1}{2} \left( 1 - \frac{1}{x} \right) \left( \log \frac{q(f)}{\pi^d} + \Re \sum_{j=1}^d \Gamma' \left( \frac{1 + \kappa_j}{2} \right) \right) - \Re \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n} \left( 1 - \frac{n}{x} \right) \right) - d \sum_{n=1}^\infty \frac{x^{-2n-1}}{2n(2n+1)} + i(f) \frac{\log x + 1}{x} + \frac{(d - 2l(f)) \log 2}{x} - \frac{1}{x} \sum_{i=1}^{d-l(f)} \Re \left( \sum_{n=1}^\infty \left( \frac{2}{\kappa_i + 1 + 2n} - \frac{1}{\kappa_i + 1 + n} + x^{-\kappa_i - 1} \right) \right). \]

For $x \geq 132$ we bound

\[
- \left( \frac{1}{\log x} + \frac{2}{\sqrt{x \log^2 x}} \right) \left( 1 - \frac{1}{\sqrt{x}} \right)^{-2} \left( i(f) \frac{\log x + 1}{x} + \frac{(d - 2l(f)) \log 2}{x} - d \sum_{n=1}^\infty \frac{x^{-2n-1}}{2n(2n+1)} \right) - d \sum_{n=1}^{d-l(f)} \frac{x^{-\kappa_i - 1}}{\kappa_i (\kappa_i + 1)} - \frac{2d}{x \log^2 x}.
\]

First, we consider

\[
A_4 = \frac{1}{x} \sum_{i=1}^{d-l(f)} \Re \left( \sum_{n=1}^\infty \left( \frac{2}{\kappa_i + 1 + 2n} - \frac{1}{\kappa_i + 1 + n} \right) \right).
\]

Fix $i$ and study the inner sum, writing $\kappa_i = \sigma + it$, and noting that Ramanujan-Petersson gives us $\sigma \geq 0$, we apply Lemma [3.4] so that

\[
\Re \left( \sum_{n=1}^\infty \left( \frac{2}{\kappa_i + 1 + 2n} - \frac{1}{\kappa_i + 1 + n} \right) \right) = \frac{1}{2} \log 4 + \frac{\sigma^2 + 3\sigma + 2 + t^2}{(\sigma + 2)^2 + t^2} - \frac{\sigma^2 + 4\sigma + 3 + t^2}{(\sigma + 3)^2 + t^2} + \frac{1}{2} \log \left( \frac{(\sigma + 2)^2 + t^2}{(\sigma + 3)^2 + t^2} \right) \geq 2 \log 2 - \log(3).
\]

The inequality comes from the following facts. First, the combination of the second and third term is positive since $\sigma \geq 0$, and the last term has a global minimum at the point $(0, 0)$ which gives $\log(2/3)$. Thus we may combine the terms $A_2$ and $A_4$ to obtain

\[
- \left( \frac{1}{\log x} + \frac{2}{\sqrt{x \log^2 x}} \right) \left( 1 - \frac{1}{\sqrt{x}} \right)^{-2} (A_2 - A_4) \geq 1.04 \frac{d(\log 2 - \log 3) + l(f) \log 3}{(\sqrt{x} - 1)^2 \log x}.
\]

For $A_5$, fix $j$, then writing $\kappa_j = \sigma + it$ and invoking Ramanujan-Petersson, we can apply Lemma [3.5] to obtain

\[
\Re \left( \frac{x^{-\kappa_j} - 1}{\kappa_j (\kappa_j + 1)} \right) \geq - \frac{2 \log x}{\log 3}.
\]

Thus combining the terms $A_1$ and $A_5$ we have

\[
- \left( \frac{1}{\log x} + \frac{2}{\sqrt{x \log^2 x}} \right) \left( 1 - \frac{1}{\sqrt{x}} \right)^{-2} (A_1 - A_5) \geq -1.04 \frac{2d/\log 3 + l(f)(2/\log(3) - 1)}{(\sqrt{x} - 1)^2} + \frac{l(f)}{(\sqrt{x} - 1)^2 \log x}.
\]
Finally, for $x \geq 132$ we have
\[
- \left( \frac{1}{\log x} + \frac{2}{\sqrt{x} \log^2 x} \right) \left( 1 - \frac{1}{\sqrt{x}} \right)^{-2} \left( A_1 + A_2 - A_3 - A_4 - A_5 \right) - \frac{2d}{x \log^2 x} \\
\geq \frac{1}{(\sqrt{x} - 1)^2} 1.04 \left( -2d/\log 3 - l(f)(2d/\log(3) - 1) \right) - \frac{2d}{1.04 \log^2 x (\sqrt{x} - 1)^2} \\
+ \frac{d(\log 2 - \log 3) + l(f)(\log(3) - 1) - d \sum_{n=1}^{\infty} \frac{x^{-n-1}}{2n(2n+1)}}{\log x} \geq -2.05d \\
(\sqrt{x} - 1)^2.
\]

Thus, for $x \geq 132$
\[
\log |L(1, f)| \geq \Re \sum_{n \leq x} \frac{a_f(n)\Lambda(n) \log \left( \frac{n}{n!} \right)}{n \log n} \log x + \frac{1}{2 \log x} \left( \log \frac{q(f)}{n^d} + \Re \sum_{j=1}^{d} \Gamma' \left( \frac{1 + \kappa_j}{2} \right) \right) \\
- \left( \frac{1}{\log x} + \frac{2}{\sqrt{x} \log^2 x} \right) \left( 1 - \frac{1}{\sqrt{x}} \right)^{-2} \left( \frac{1}{2} \left( 1 - \frac{1}{x} \right) \left( \log \left( \frac{q(f)}{n^d} \right) + \Re \sum_{j=1}^{d} \Gamma' \left( \frac{1 + \kappa_j}{2} \right) \right) \right) \\
- \Re \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n} \left( 1 - \frac{n}{x} \right) - \frac{2.05d}{(\sqrt{x} - 1)^2}.
\]

We note that
\[
\left( \log \frac{q(f)}{n^d} + \Re \sum_{j=1}^{d} \Gamma' \left( \frac{1 + \kappa_j}{2} \right) \right) \left( \frac{1}{2 \log x} - \left( \frac{1}{\log x} + \frac{2}{\sqrt{x} \log^2 x} \right) \left( 1 - \frac{1}{\sqrt{x}} \right)^{-2} \left( \frac{1}{2} \left( 1 - \frac{1}{x} \right) \right) \right) \\
\geq - \left( \log \frac{q(f)}{n^d} + \Re \sum_{j=1}^{d} \Gamma' \left( \frac{1 + \kappa_j}{2} \right) \right) \frac{1}{(\sqrt{x} - 1) \log x} \left( 1 + \frac{1 + 1/\sqrt{x}}{\log x} \right) \\
\geq - \frac{\log C(f)}{(\sqrt{x} - 1) \log x} \left( 1 + \frac{1 + 1/\sqrt{x}}{\log x} \right).
\]

Where the last inequality comes from Lemma 8.33. So far, we have proven
\[
(4.1) \quad \log |L(1, f)| \geq \Re \sum_{n \leq x} \frac{a_f(n)\Lambda(n) \log \left( \frac{n}{n!} \right)}{n \log n} \log x - \frac{\log C(f)}{(\sqrt{x} - 1) \log x} \left( 1 + \frac{1 + 1/\sqrt{x}}{\log x} \right) \\
+ \left( \frac{1}{\log x} + \frac{2}{\sqrt{x} \log^2 x} \right) \left( 1 - \frac{1}{\sqrt{x}} \right)^{-2} \Re \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n} \left( 1 - \frac{n}{x} \right) - \frac{2.05d}{(\sqrt{x} - 1)^2}.
\]

To continue, we see from Lemma 8.6 if $x \geq 132$ we have
\[
\Re \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n} \left( 1 - \frac{n}{x} \right) \geq d(1 - \log x),
\]
thus as in [6, pg 18 line 11] we have
\[
\left( \left( 1 - \frac{1}{\sqrt{x}} \right)^{-2} \left( \frac{1}{\log x} + \frac{2}{\sqrt{x} \log^2 x} \right) - \frac{1}{\log x} \right) \Re \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n} \left( 1 - \frac{n}{x} \right) \geq - \frac{2d}{\sqrt{x}}.
\]

Using this in (4.1) we have
\[
\log |L(1, f)| \geq \Re \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n \log n} \left( 1 - \frac{1}{x \log x} \right) - \frac{\log C(f)}{(\sqrt{x} - 1) \log x} \left( 1 + \frac{1 + 1/\sqrt{x}}{\log x} \right) - \frac{2d}{\sqrt{x}} - \frac{2.05d}{(\sqrt{x} - 1)^2} \\
\geq \Re \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n \log n} \left( 1 - \frac{1}{x \log x} \right) - \frac{\log C(f)}{(\sqrt{x} - 1) \log x} \left( 1 + \frac{1 + 1/\sqrt{x}}{\log x} \right) - \frac{9d}{4\sqrt{x}}.
\]
We apply Lemma 3.8 thus guaranteeing that the first term in the right hand side is smallest when \( a_f(p) = -d \) for every prime \( p \leq x \). Therefore, we have

\[
\log |L(1, f)| \geq d \sum_{p^k \leq x} \Lambda(p^k)(-1)^k \left( \frac{1}{p^k \log p^k} - \frac{1}{x \log x} \right) - \frac{\log C(f)}{(\sqrt{d} - 1) \log x} \left( 1 + \frac{1 + \sqrt{d}}{\log x} \right) - \frac{9d}{4\sqrt{x}}.
\]

Following the discussion after [6, Equation 5.3] we see that

\[
\log |L(1, f)| \geq d \left( -\log x - \gamma + \log \zeta(2) + \frac{1}{\log x} - \frac{8}{5\sqrt{x}} \right) - \frac{\log C(f)}{(\sqrt{d} - 1) \log x} \left( 1 + \frac{1 + \sqrt{d}}{\log x} \right) - \frac{9d}{4\sqrt{x}}.
\]

Our choice of \( x = \frac{\log^2 C(f)}{4d} \) gives \(-2d\sqrt{x} - 1\) so that with a little calculation one obtains

\[
\log |L(1, f)| \geq d \left( -\log x - \gamma + \log \zeta(2) - \frac{1}{\log x} - \frac{2}{\log^2 x} - \frac{9}{2\sqrt{x}} \right).
\]

Exponentiating both sides gives

\[
\frac{1}{|L(1, f)|} \leq \left( e^\gamma \frac{6}{\pi^2} \right)^d \log^d x \exp \left( \frac{d}{\log x} + \frac{2d}{\log^2 x} + \frac{9d}{2\sqrt{x}} \right)
\]

\[
\leq \left( e^\gamma \frac{6}{\pi^2} \right)^d \log^d x \left( 1 + \frac{d}{\log x} + \frac{dJ_1(d)}{\log^2 x} + \frac{dJ_2(d)}{2\sqrt{x}} \right),
\]

where

\[
J_1(d) = 2 + \left( 1 + \frac{4}{\log x} + \frac{4}{\log^2 x} \right) \sum_{k=0}^{\infty} \frac{d^{k+1}}{(k+2)!} \left( \frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{9}{2\sqrt{x}} \right)^k,
\]

and

\[
J_2(d) = 9 + \left( \frac{18}{\log x} + \frac{18}{\log^2 x} + \frac{81}{2\sqrt{x}} \right) \sum_{k=0}^{\infty} \frac{d^{k+1}}{(k+2)!} \left( \frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{9}{2\sqrt{x}} \right)^k.
\]

Replacing \( x \) we get

\[
\frac{1}{|L(1, f)|} \leq \left( 2e^\gamma \frac{6}{\pi^2} \right)^d \left( \log \log C(f) - \log 2d \right)^d + \frac{d}{2} \left( \log \log C(f) - \log 2d \right)^{d-1} + \frac{dJ_1(d)}{4} \left( \log \log C(f) - \log 2d \right)^{d-2} + \frac{d^2 J_2(d)}{\log C(f)} \left( \log \log C(f) - \log 2d \right)^d.
\]

Thus the theorem is proven.

5. Acknowledgements

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References

[1] V. Chandee, Explicit Upper Bounds for L-functions on the critical line. Proc. Amer. Math. Soc. 137 (2009), 4049-4063.
[2] H. Davenport. Multiplicative Number Theory. Springer, third edition, 2000.
[3] P. Deligne. La conjecture de Weil, I. Inst. Hautes Études Sci. Publ. Math. 43, (1972), 206-226.
[4] P. Deligne and J.P. Serre. Formes modulaire de poids 1. Ann. Sci. Ec. Norm. Sup. 4, 7 (1974), 507-530.
[5] H. Iwaniec & E. Kowalski, Analytic Number Theory American Mathematical Society, 2004
[6] Y. Lamzouri, X. Li & K. Soundararajan, Conditional bounds for the least quadratic non-residue and related problems. Mathematics of Computation Vol. 84 (2015), no. 295, 2391-2412.
[7] J.E. Littlewood. On the class-number of the corpus \( P(\sqrt{F}) \). Proc. London Math. Soc., v.28, 1928, pp. 358-372.
[8] S. Louboutin. Explicit upper bounds for residues of Dedekind zeta functions and values of L-functions at s = 1, and explicit lower bounds for relative class numbers of CM-fields. Can. J. Math. 53, No. 6, (2001), 1194-1222.