QCD$_2$ in the modified Fock–Schwinger gauge

Yu.S. Kalashnikova,* A.V. Nefediev†

Institute of Theoretical and Experimental Physics, 117218, Moscow, Russia

Abstract

The system of light quark and heavy antiquark source is studied in 1+1 QCD in the large $N_C$ limit. The situation is demonstrated to be the two dimensional analogue of the problem in 3+1 QCD corresponding to the essentially nonlocal case of zero vacuum correlation length, which does not allow perturbative treatment. The modified Fock–Schwinger gauge condition is employed to derive the effective Dirac-type equation for the spectrum, which is proved to be equivalent to the 't Hooft equation in the limit of one heavy mass. The chiral condensate is shown to appear naturally in the given approach and to coincide with the standard value.

Theory of quarks in 1+1 dimensions interacting via gluons in the large $N_C$ limit has a long history. It was in 70-th when the celebrated 't Hooft equation for the meson spectrum in such a model was derived in the light-cone gauge ($A_\perp = 0$) [1] and then re-derived in the axial gauge ($A_1 = 0$) [2]. Since that time it has done it credit to every new approach to the realistic 3+1 QCD to reproduce this equation when applied to a much more simple and physically clear 1+1 't Hooft model. Many theories have passed this examination with honour.

Recently there was suggested a new approach to the theory of light spinning quarks in the confining vacuum [3], that allowed, in the large $N_C$ limit, to obtain a nonlinear Dirac-type equation for the light quark motion in the nontrivial vacuum of QCD in presence of heavy antiquark source. Despite of its nonlocality, this equation allows a systematic analysis, which was started in our previous work [4] and led to the conclusion that the two cases of small and large vacuum correlation length $T_g$, value which defines the distances where vacuum gluonic fields are correlated, differ drastically, viz. for the case of large $T_g$ ($T_g \gg \frac{1}{m}$, where $m$ is the mass of the light quark) the nonlocality can be treated perturbatively, whereas the opposite limiting case of small $T_g$ ($T_g \ll \frac{1}{m}$) turned out sufficiently

*e-mail: yulia@vxitep.itep.ru
†e-mail: nefediev@vxitep.itep.ru
nonlocal and demanded extra ideas for its considering. Fortunately the ’t Hooft model is of great help at this point.

Indeed, in 3+1 the interaction of the light quark with the heavy antiquark is driven by the irreducible correlators $< A_{\mu_1}^a(x_1) \ldots A_{\mu_n}^a(x_n) >$, where $a_1, \ldots, a_n$ are the colour indices in the adjoint representation. No exact solutions derived directly from the QCD are known for such correlators, and the strategy [3] is to employ a kind of the Fock–Schwinger gauge which allows to express correlator $< A \ldots A >$ in terms of the gauge–invariant field strength one, $< F \ldots F >$, and to insert some theoretical input for the latter. Namely, it was shown in the framework of the Vacuum Background Correlators method [7] that with the Euclidean bilocal field strength correlator

\[ < F_{a \mu \nu}^a(x_0, x) F_{b \lambda \rho}^b(y_0, y) > = \delta^{ab} \frac{2N_C}{N_C^2 - 1} D(x - y)(\delta_{\mu \lambda} \delta_{\nu \rho} - \delta_{\mu \rho} \delta_{\nu \lambda}) \]  

(1)

normalized by the condition

\[ \sigma = 2 \int_0^\infty d\tau \int_0^\infty d\lambda D(\tau, \lambda) \]

(2)

the area law asymptotic for the isolated Wilson loop average holds true, $< W(C) > \to N_C \exp(-\sigma S)$. Expression [2] for the string tension implies that function $D$ decreases rapidly in all directions of the Euclidean space, and this decrease is governed by gluonic correlation length $T_g$.

We are interested in the limit $T_g \to 0$. It is easy to see that quite the same situation is realised in 1+1 QCD, where the Coulomb force is confining and in any axial gauge the only nontrivial correlator of gluonic fields is the gluon propagator which is bilocal and effectively corresponds to the case of $T_g$ equal to zero.

Indeed, the bilocal field strength correlator in 1+1 case, $< 0|TF_{10}^a(x_0, x) F_{10}^b(y_0, y)|0 >$, is gauge–invariant and can be calculated for example in the axial gauge $A_0^a = 0$ with the result [8]

\[ i < 0|TF_{10}^a(x_0, x) F_{10}^b(y_0, y)|0 > = -g^2 \delta^{ab} \delta^{(2)}(x - y). \]

(3)

Then there exists the relation in the Euclidean space, which is similar to (1):

\[ < F_{10}^a(x) F_{10}^b(y) > = \delta^{ab} \frac{2N_C}{N_C^2 - 1} \tilde{D}(x - y), \]

(4)

where

\[ \tilde{D}(x - y) = \frac{N_C^2 - 1}{2N_C} g^2 \delta^{(2)}(x - y) = \lim_{T_g \to 0} D \left( \frac{x_0 - y_0}{T_g}, \frac{x - y}{T_g} \right) \]

(5)

is normalized by the condition

\[ \sigma = 2 \int_0^\infty d\tau \int_0^\infty d\lambda \tilde{D}(\tau, \lambda) = \frac{g^2}{4} \left( N_C - \frac{1}{N_C} \right) \to \frac{g^2 N_C}{4} \]  \[ N_C \to \infty \]

(6)

\[ ^1\text{We have included the coupling constant } g \text{ into the definition of the correlator.} \]
The latter observation allows us to kill two birds with one stone, when applying the given approach to the two-dimensional QCD. First, we prove the selfconsistency of the general method under consideration. On the other hand, dealing with nonlocal interaction in 1+1 we gain valuable experience which may be extremely useful for solving the problem for the physical four dimensional case in the most interesting limit of small $T_g$. The QCD string is believed to be formed in this limit, and we plan to highlight this question in the future publications.

As in the 3+1 case [3, 5] we start from the Green function $S_{\bar{q}Q}$ for the $q\bar{Q}$ system in Minkowski space-time

$$S_{q\bar{Q}}(x, y) = \frac{1}{N_C} \int D\psi D\psi^+ DA \exp \left\{ -\frac{1}{4} \int d^2xF_{\mu\nu}^a - \int d^2x \bar{\psi}^+(i\partial - m - \hat{A})\psi \right\} \times$$

$$\times \psi^+(x)S_{Q}(x,y)\psi(y),$$

where we have placed the static antiquark at the origin, and $S_{Q}(x,y)$ denotes its propagator. In the one–body limit we can considerably simplify the situation making use of the modified Fock–Schwinger gauge [7], having set

$$A_0^a(x_0,0) = 0, \quad A_1^a(x_0, x) = 0.$$  

In such a gauge the gluon field is expressed in terms of the field strength tensor as

$$A_0^a(x_0, x) = \int_0^1 d\alpha F_{10}^a(x_0, \alpha x), \quad A_1^a(x_0, x) = 0,$$

and the gluon propagator takes the form

$$K_{11}^{ab}(x_0 - y_0, x, y) = K_{01}^{ab}(x_0 - y_0, x, y) = 0$$

$$K_{00}^{ab}(x_0 - y_0, x, y) = xy \int_0^1 d\alpha \int_0^1 d\beta < 0|TF_{10}^a(x_0, \alpha x)F_{10}^a(y_0, \beta y)|0 > =$$

$$= -\delta^{ab}g^2/2\delta(x_0 - y_0)(|x| + |y| - |x - y|) \equiv \delta^{ab}K(x, y)$$  

In what follows we use expression [11] for the gluon propagator, bearing in mind that the same results could be recovered in the limit of small vacuum correlation length in the theory with function $D$ instead of $\tilde{D}$ (see equation [1]), which is evidently the two dimensional analogue of the ordinary QCD.

Substituting the heavy antiquark Green function $S_{Q}(x, y)$ in the form

$$S_{Q}(t, x, y) = \left( -i\frac{1 + \gamma_0}{2} \theta(-t) e^{iMt} - i\frac{1 - \gamma_0}{2} \theta(t) e^{-iMt} \right) \delta(x - y)$$

\[\text{We use the following } \gamma\text{-matrix convention: } \gamma_0 = \sigma_3, \gamma_1 = i\sigma_2, \gamma_5 = \sigma_1.\]

\[\text{In } 3+1 \text{ this gauge is usually introduced via conditions } A_5^a(x_0, 0) = 0 \text{ and } \vec{x} \vec{A}^a(x_0, \vec{x}) = 0, \text{ which obviously reduce to } [3] \text{ in } 1+1.\]
into (7) and performing the integration over $A_\mu$ we arrive at the following Schwinger–Dyson equation in the large $N_C$ limit (see e.g. [3] for details):

$$
(i\partial_x - m)S(x,y) + \frac{iN_C}{2} \int d^2z \gamma_0 S(x,z)\gamma_0 K(x,z)S(z,y) = \delta^{(2)}(x-y),$$

(13)

where $S(x,y) = \frac{1}{N_C} S^\alpha(x,y)$ is the colour trace of the light quark Green function. Note that as the Green function (12) of the heavy antiquark is unity in the colour space in gauge (8), quantity $S(x,y)$ completely defines the propagation of the colourless $q\bar{Q}$ system.

A more simple version of this approach was suggested in [4], where the full quark Green function in the kernel of equation (13) was replaced by the free one, $\gamma_0 S\gamma_0 \rightarrow \gamma_0 S_0 \gamma_0$. As we shall see below, such replacement can be justified only for the case of heavy quarks.

In what follows we shall demonstrate that equation (13) is equivalent to the well–known ’t Hooft one taken in the limit of one heavy mass. The correct value of the chiral condensate will also be recovered in the given approach.

The usual way of the ’t Hooft equation derivation in any gauge consists of two steps. First the quark self energy part should be calculated to be substituted into the exact quark propagator. Then a Bethe–Salpeter equation for the vertex function should be written out, which is in fact the ultimate answer up to some algebraic transformations. Such a way of acting hints us that as we managed to combine these both steps together when deriving equation (13), a kind of separation of the interaction should be present in it. This idea becomes most clear if one looks more carefully at correlator (11). Indeed, $K$ can be naturally broken into two parts

$$
K = K^{(1)} + K^{(2)},
$$

(14)

where $K^{(1)}(x_0 - y_0, x - y) = \frac{g^2}{2} \delta(x_0 - y_0)|x - y|$ describes the local part of interaction and $K^{(2)}(x_0 - y_0, x, y) = -\frac{g^2}{2} \delta(x_0 - y_0)(|x| + |y|)$ is nonlocal. As we shall work in a more convenient momentum space, we obtain for $K^{(1)}$ and $K^{(2)}$ correspondingly

$$
K^{(1)}(p_0 - q_0, p - q) = -\frac{(2\pi)^2 g^2}{p^2} \delta^{(2)}(p - q),
$$

(15)

$$
K^{(2)}(p_0 - q_0, p, q) = (2\pi)^2 g^2 \delta(p_0 - q_0) \left( \frac{\delta(p)}{q^2} + \frac{\delta(q)}{p^2} \right).
$$

(16)

It is easy to see that equation (13) with $K^{(1)}$ substituted instead of the full kernel $K$ is nothing but the self energy equation in the Coulomb gauge $A_1^a = 0$ [2].

Indeed, with the definition

$$
S^{(1)}(p_{10}, p_1, p_{20}, p_2) = (2\pi)^2 \delta^{(2)}(p_1 - p_2) \tilde{S}(p_{10}, p_1),
$$

$$
\tilde{S}^{-1}(p_0, p) = \gamma_0 p_0 - \gamma_1 p - m - \Sigma(p),
$$

(17)

Note that the self energy is a gauge–variant value.
and the parametrization of the polarization operator $\Sigma(p)$ as

$$\Sigma(p) = [E(p)\cos\theta(p) - m] + \gamma_1 [E(p)\sin\theta(p) - p]$$

(18)
equation (13) can be reduced to the system of two coupled equations [2]

$$E(p)\cos\theta(p) = m + \frac{f}{2} \int \frac{dk}{(p-k)^2} \cos\theta(k)$$

(19)

$$E(p)\sin\theta(p) = p + \frac{f}{2} \int \frac{dk}{(p-k)^2} \sin\theta(k),$$

(20)

where

$$f = \frac{g^2 N_C}{4\pi} = \frac{\sigma}{\pi}.$$  

Solutions of system (19), (20) are known [2, 8], and we shall return to them to calculate the chiral condensate. Note that in the large mass limit these equations can be solved perturbatively, defining in the leading order the dispersion law of the free quark, $E(p) = \sqrt{p^2 + m^2}$, $\cos\theta(p) = \frac{m}{E(p)}$, so that Green function $S^{(1)}$ becomes the free Green function. Such a procedure is obviously legitimate only for the heavy quark.

Unfortunately, the full equation (13) is nonlinear, so any separation in it is not straightforward. Our present goal is to justify the separate consideration of $K^{(1)}$ we have carried out, and to obtain an equation on the eigenenergies which is induced by $K^{(2)}$.

The Green function (17) together with (15), (16) gives rise to the set of propagators presented at Fig.1, where all $K^{(1)}$ loops are already absorbed by quark propagator (17).

Let us consider the class of rainbow diagrams shown at Fig.2(a) with an arbitrary number of $K^{(2)}$ inner loops. Simple but tedious calculations show that all such diagrams vanish (see Appendix for the details). The same conclusion holds true for the diagrams with $K^{(1)}$ covering line (see Fig.2(b)), as the only role played by the covering line is providing the integration over $k_0$. The latter observation means that the contribution of the term $K^{(2)}$ is described by the only diagram depicted at Fig.3, so that equation (13) takes the form

$$(\hat{q}_1 - m - \Sigma(q_1))S(q_1, q_2) +$$

$$+ \frac{ig^2 N_C}{2} \int \frac{d^2p}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \gamma_0 S^{(1)}(p)\gamma_0 S(p + k, q_2)K^{(2)}(q_1 - p_1, k) = (2\pi)^2 \delta^{(2)}(q_1 - q_2),$$

(21)

that can be rewritten as the Dirac equation for the light quark wave function $\varphi(p_0, p)$.

$$p_0 \varphi(p_0, p) = \gamma_5 E(p)\sin\theta(p) + \gamma_0 E(p)\cos\theta(p) -$$

$$- \frac{f}{2} \gamma_0 \int dk [\cos\theta(k) + \gamma_1 \sin\theta(k)] \frac{\varphi(p_0, k)}{(p - k)^2} -$$

(22)
\[-\frac{f}{2} \gamma_0 \int dk \{\cos(\theta(p)) + \gamma_1 \sin(\theta(p))\} \frac{\varphi(p_0, k)}{(p-k)^2}.\]

After the Foldy–Wouthoysen transformation

\[\tilde{\varphi}(p_0, p) = T(p)\varphi(p_0, p), \quad T(p) = e^{\frac{i}{2}\theta(p)\gamma_1}\]

we arrive at the Schrödinger–type equation

\[\varepsilon_n \varphi_n^0(p) = E(p)\varphi_n^0(p) - \frac{f}{2} \int\frac{dk}{(p-k)^2} \cos \theta(p) - \frac{\theta(k)}{2} \varphi_n^0(k),\]

which defines the spectrum of the bound states of light quark and heavy antiquark, and wave functions of the positive- and negative-energy states are given by

\[\tilde{\varphi}_n^+(p) = \varphi_n^0(p) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{\varphi}_n^-(p) = \varphi_n^0(p) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.\]

Equation (24) coincides with the one-body limit of the equation obtained in [2], which in turn is proved to be equivalent to the 't Hooft one on the grounds of the Poincaré invariance [2], numerically [8] and in the framework of discrete light-cone quantization [9].

Now, when the formal correctness of equation (13) is proved, we are ready to perform a next step and to show how the chiral condensate appears in quite a natural way in the given technique.

The chiral condensate can be calculated as

\[<\bar{q}q> = -i \text{Tr}_{x\to y+} S(x, y),\]

where the trace is taken both over colour and spinor indices.

At first glance it has nothing to do with the presence of the heavy antiquark source, but due to the peculiarities of the modified Fock–Schwinger gauge we can reformulate our spectroscopic problem (13) as a problem for the Green function of the light quark in gauge (8). The colour trace of this Green function can be constructed from the solutions of equation (24) as

\[N_C S_{nk}(x_0 - y_0, x, y) = -i N_C \sum_n \psi_{ni}^{(+)}(x) \tilde{\psi}_{nk}^{(+)}(y) e^{-i\varepsilon_n(x_0 - y_0)} \theta(x_0 - y_0) +
\]

\[+ i N_C \sum_n \psi_{ni}^{(-)}(x) \tilde{\psi}_{nk}^{(-)}(y) e^{i\varepsilon_n(x_0 - y_0)} \theta(y_0 - x_0),\]

where \(i\) and \(k\) are spinor indices and

\[\psi_{ni}^{(\pm)}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \varphi_{ni}^{(\pm)}(p) e^{ipx},\]
Substituting solutions (29) into (27) and making use of the completeness condition for the set \( \{ \varphi^0_n(p) \} \),
\[
\sum_n \varphi^0_n(p) \varphi^{0*}_{n'}(p') = \delta(p - p'),
\]
we straightforwardly calculate the chiral condensate to be
\[
\langle \bar{q}q \rangle = -\frac{N_C}{\pi} \int_0^\infty dp \cos(\theta(p)).
\]

Such way the question whether the chiral symmetry is broken or not depends on the solution for \( \theta(p) \) of system (19), (20). Solution found in [2] for the case of \( m = 0 \) reads
\[
\theta(p) = \begin{cases} 
\pi/2 & \text{for } p > 0 \\
-\pi/2 & \text{for } p < 0,
\end{cases}
\]
yielding zero value of the condensate.

In the meantime it is easy to see that there is another solution of system (19), (20) which has the following asymptotical forms
\[
\theta(p) \rightarrow \begin{cases} 
\sqrt{m^2 + 4\sigma - m^2} p, & p \rightarrow 0 \\
\frac{\pi}{2} \text{sign}(p), & |p| \rightarrow \infty
\end{cases}
\]

This solution was investigated numerically in [8, 10], and in the limit \( m \rightarrow 0 \) it gives for the condensate
\[
\langle \bar{q}q \rangle_{m=0} = -0.29 N_C \sqrt{2f}.
\]

Again we refer here to the discrete light-cone quantization treatment [9], where it was shown that expression (31) as well as the numerical estimate (34) coincide with the result obtained in [11] via operator expansion.

We would like to conclude with a brief comment on the quantum mechanical aspects of equations (22), (24). With these equations we resolve the long-standing problem of the Lorentz structure of the effective confining force. The quasiclassical behaviour of the spectrum given by equation (24) reproduces a Regge trajectory with a slope specific for the time-like vector confinement, \( M^2 = \pi \sigma n \), but it is known that for the Dirac particle in the time-like external vector field the ill-starred Klein paradox takes place. Still there is no puzzle in such a situation: the interaction in the underlying Dirac equation (22) does not contain the time-like vector part at all, there are only scalar and space-like vector pieces which enter in a certain combination in the essentially nonlocal way. We are
able to perform the Foldy–Wouthoysen transformation (23) in a closed form, and this fact indicates that there is no dangerous light quark Zitterbewegung and therefore there are no reasons to be afraid of the Klein paradox.

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Appendix

Anticipating zero result for all diagrams shown at Fig.2 let us omit all inessential multipliers keeping only the terms containing $\gamma$-matrices and $k_0$ in order to perform the integration over $k_0$ explicitly. It is easy to see that the general form of the given integrals is

$$I_n \sim \int_{-\infty}^{+\infty} dk_0 \gamma_0 S^{(1)}(k_0, p_1) M(p_1, p_2) S^{(1)}(k_0, p_2) \ldots S^{(1)}(k_0, p_{n-1}) M(p_{n-1}, p_n) S^{(1)}(k_0, p_n) \gamma_0,$$

where $p_1 \equiv q$, $p_N \equiv p$ and $M(p, q)$ is defined by the diagram from Fig.3.

Quark propagator (17) can be written in the form

$$S^{(1)}(p_0, p) = \frac{1}{p_0 \gamma_0 - \Omega(p)}, \quad \Omega(p) = -E(p)e^{-\theta(p)\gamma_1},$$

where $\Omega(p)$ possesses the following properties

$$\Omega^+(p)\Omega(p) = E^2(p), \quad \gamma_0\Omega(p) = \Omega^+(p)\gamma_0.$$

Using these properties of the function $\Omega(p)$ and omitting all odd powers of $k_0$ in the nominator, which obviously give no contribution to $I_n$, we arrive at the following expression for $I_n$:

$$I_n \sim \int_{-\infty}^{+\infty} dk_0 \frac{k_0^n + k_0^{n-2} \sum_{i \neq j} E_i E_j \ldots + \Pi_{i=1}^n E_i}{(k_0^2 - (E_1 - i\varepsilon)^2)(k_0^2 - (E_2 - i\varepsilon)^2) \ldots (k_0^2 - (E_n - i\varepsilon)^2)},$$

for even $n$, or

$$I_n \sim \int_{-\infty}^{+\infty} dk_0 \frac{k_0^{n-1} \sum_{i=1}^n E_i + k_0^{n-3} \sum_{i \neq j} E_i E_j \ldots + \Pi_{i=1}^n E_i}{(k_0^2 - (E_1 - i\varepsilon)^2)(k_0^2 - (E_2 - i\varepsilon)^2) \ldots (k_0^2 - (E_n - i\varepsilon)^2)},$$

for odd $n$, where $E_i \equiv E(p_i)$.

As odd powers of $k_0$ give no contribution to the integral, we can add the appropriate terms to the nominator in order to come either to $\Pi_{i=1}^n (k_0 + E_i)$ or to $\Pi_{i=1}^n (k_0 - E_i)$. This means that only one of two series of poles survives, either in the upper, or in the lower half-plane. Closing the contour of integration in the complex plane via the half-plane without poles we arrive at the zero result for $I_n$ for any $n$.  

8
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\[ S^{(1)}(p_0, p) = \frac{1}{p_0 \gamma_0 - \Omega(p)} \]

\[ K^{(1)}(p_0 - q_0, p - q) \]

\[ K^{(2)}(p_0 - q_0, p, q) \]

Figure 1: Propagators for effective theory (13).

Figure 2: Diagrams which mix \( K^{(1)} \) and \( K^{(2)} \) interactions.

Figure 3: The only diagram contributing to the interaction with kernel \( K^{(2)} \).