EXPONENTIAL CONVERGENCE OF WEIGHTED BIRKHOFF AVERAGE

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ABSTRACT. In this paper, we consider the polynomial and exponential convergence rate of weighted Birkhoff averages of irrational rotations on tori. It is shown that these can be achieved for finite and infinite dimensional tori which correspond to the quasiperiodic and almost periodic dynamical systems respectively, under certain balance between the nonresonant condition and the decay rate of the Fourier coefficients. Diophantine rotations with finite and infinite dimensions are provided as examples. For the first time, we prove the universality of exponential convergence and arbitrary polynomial convergence in the quasiperiodic case and almost periodic case under analyticity respectively.

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1. INTRODUCTION

The classical Birkhoff ergodic theorem asserts that for ergodic dynamical systems, the time average of a function $f$ evaluated along a trajectory of length $N$ converges to the space average, i.e., the integral of $f$ over the space. Namely, assume $T : X \to X$ is a map on a topological space $X$ with a probability measure $\mu$ for which $T$ is invariant. Then for a fixed point $x \in X$ and a function $f$ on $X$, we define the long time average of $f$ as

$$B_N(f)(x) := \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)),$$

which we call the Birkhoff average of $f$. Actually, it has a long history to study the convergence of (1.1), see survey articles [11, 13]. The von Neumann ergodic theorem shows that (1.1) converges to the integral $\int_X f \, d\mu$ in the $L^2$ norm, if $f \in L^2(X, \mu)$, $\mu$ is a probability measure on $X$, $T$ preserves $\mu$ and is ergodic, see Theorem 4.5.2 given in [3]. The Birkhoff ergodic theorem weakens the restriction of the former on $f$, only $f \in L^1(X, \mu)$ is required, then (1.1) converges to $\int_X f \, d\mu$, $\mu$-a.e. on $X$. These theorems are of great importance both in mathematics and statistical mechanics. However, the convergence rate of the Birkhoff average may be very slow. It can be proved that for any non-constant $f$, there exists a constant independent of $N$, such that

$$\left| B_N(f)(x) - \int_X f \, d\mu \right| \geq \frac{C}{N}$$

holds for infinitely many $N$, see [5]. In fact, many mathematicians who have worked on Birkhoff ergodic theorem know that it is not possible to prove any general positive result about the speed of convergence in (1.1), and later it has been shown in [9] that for any null-sequence $\{\omega_n\}_{n=1}^\infty$ of positive reals, there exists a continuous function $f$ such that

$$\limsup_{N \in \mathbb{N}^+} \omega_N^{-1} \left| B_N(f)(x) - \int_X f \, d\mu \right| = +\infty, \text{ a.e.}$$

And the analogous result holds also for norm-convergence.

Obviously, the slow rate of the convergence of the Birkhoff average (1.1) makes numerical computations in real problems extremely difficult, although the convergence is guaranteed in theory. Aiming to get high precision numerical results, some computations may even take billions of years to complete, see Subsection 1.9 in [5] and [4]. This forces ones to find a faster convergence method, from which some weighted Birkhoff averages have been derived.

Recently, a weighted method of non-uniform distribution was proposed in [5] to study ergodicity in quasiperiodic dynamical systems, which surprisingly confirms that when $f : \mathbb{T}^d \to E$ is sufficiently smooth, provided $d \in \mathbb{N}^+$ and $\dim E < +\infty$, and the rotation vector on $\mathbb{T}^d$ satisfies the Diophantine condition, then the weighted Birkhoff average could converge at an arbitrarily polynomial rate which they called...
super-convergence. This is indeed a breakthrough. See [4] for numerical simulation of some physical models. At this point, it is therefore natural that ones should consider the following questions step by step:

(Q1) How about the convergence type in the almost periodic case?
(Q2) Could faster convergence than arbitrary polynomial’s type be achieved, such as exponential’s type?
(Q3) Can we show certain universality of arbitrary polynomial convergence and exponential convergence via analyticity?

These questions are quite nontrivial. On the one hand, the almost periodic case is fundamentally different from the quasiperiodic case in that the rotations of the former are infinite-dimensional vectors, while the latter only deals with finite-dimensional rotations, see [7] and [8] for relevant work on these two aspects. Their topological properties are completely different, such as the infinite-dimensional torus has no compactness. Additionally, the data processing is even more different, that is, the almost periodic case may lead to Curse of Dimensionality. On the other hand, (Q1), (Q2) and (Q3) are crucial both theoretically and applicability, and they also explain Laskar’s simulation results [10] (Remark 2 in Appendix, p.146) on quasiperiodic flows, that is, convergence faster than arbitrary polynomial’s type. In this paper, we make further developments following [5] and answer these questions.

This paper is organized as follows. In Section 2, we show that the weighted Birkhoff average converges at an arbitrary polynomial rate for the quasiperiodic case ($\mathbb{T}^d$) and the almost periodic case ($\mathbb{T}^\infty$), as long as the nonresonance of the irrational rotation vector and the Fourier coefficients of $f$ satisfy certain conditions, i.e., (H1) and (H2), respectively. Roughly speaking, the rotating vector might satisfy weaker nonresonance than the usual Diophantine one, particularly involving with infinite-dimensional cases. In Section 3, we present our main results in this paper, which further show that the weighted Birkhoff average can indeed converge at an exponential rate, by requiring stronger conditions than that before. Diophantine rotations are constructed as examples at this point, including finite and infinite dimensional cases. As a corollary to the above results, we show that under the assumption of analyticity, exponential convergence and arbitrary polynomial convergence are universal in the case of quasiperiodic and almost periodic, respectively. It is worth mentioning that some assumptions can be removed in the cases without small divisors, in dealing with exponential convergence. It seems inevitable, however, that the difficulty of analyzing exponential convergence and circumventing the limitation of dimensionality lead to technical complications.

2. CONVERGENCE OF ARBITRARY POLYNOMIAL RATE TYPE

2.1. Finite-dimensional case $\mathbb{T}^d$. Das and Yorke [5] proved arbitrary polynomial convergence in the quasiperiodic case via Diophantine rotations and $C^\infty$ regularity. Following their idea, we extend the results to the general nonresonant conditions and regularity of functions, as well as the continuous case. To introduce the results we first give some notions, which are basic to our discussion.
Definition 2.1. A function $\Delta : [1, +\infty) \to [1, +\infty)$ is said to be an approximation function, if it is continuous, strictly monotonic increasing, and satisfies $\Delta(+\infty) = +\infty$.

Definition 2.2 (Finite-dimensional nonresonant condition). An irrational vector $\rho \in \mathbb{T}^d$ is said to be nonresonant if there exist $\alpha > 0$ and an approximation function $\Delta$ such that

(a) The discrete case
$$|k \cdot \rho - n| \geq \frac{\alpha}{\Delta(||k||)}, \quad \forall 0 \neq k \in \mathbb{Z}^d, \forall n \in \mathbb{Z};$$

(b) The continuous case
$$|k \cdot \rho| \geq \frac{\alpha}{\Delta(||k||)}, \quad \forall 0 \neq k \in \mathbb{Z}^d,$$

where $||k|| = |k_1| + \cdots + |k_d|$.

Remark 2.3. Obviously (2.1) implies (2.2), which means that in the continuous case one has a lower restriction on rotation vectors and thus we state them separately.

Remark 2.4. We say that $\rho \in \mathbb{T}^d$ satisfies the Finite-dimensional Diophantine condition, if
$$\Delta(x) = x^\tau, \quad \tau > d - 1.$$

Definition 2.5. Assume $(\mathcal{B}, || \cdot ||_{\mathcal{B}})$ is a Banach function space (could be infinite-dimensional), and $f : \mathbb{T}^d \to \mathcal{B}$ with
$$f = \sum_{k \in \mathbb{Z}^d} \hat{f}_k e^{2\pi ik \cdot \theta}, \quad \hat{f}_k = \int_{\mathbb{T}^d} f(\hat{\theta}) e^{-2\pi ik \cdot \hat{\theta}} d\hat{\theta},$$

where the first “=” represents equality in the sense of the norm $|| \cdot ||_{\mathcal{B}}$. Now we define the following space
$$\mathcal{B}_{\tilde{\Delta}} := \left\{ f : \mathbb{T}^d \to \mathcal{B} : f \text{ satisfies (2.4) and } \sup_{0 \neq k \in \mathbb{Z}^d} \tilde{\Delta} (||k||) ||\hat{f}_k||_{\mathcal{B}} < +\infty \right\}$$

for a given approximation function $\tilde{\Delta}$.

For a given map $T_\rho : \mathbb{T}^d \to \mathbb{T}^d$ with $T_\rho(\theta) = \theta + \rho \mod 1$ in each coordinate ($\rho$ is an irrational nonresonant vector) and a function $f \in \mathcal{B}_{\tilde{\Delta}}$, define the weighted Birkhoff average as
$$WB_N (f) (\theta) := \frac{1}{A_N} \sum_{n=0}^{N-1} w\left(\frac{n}{N}\right) f \left( T_\rho^n(\theta) \right), \quad A_N = \sum_{n=0}^{N-1} w\left(\frac{n}{N}\right), \quad \theta \in \mathbb{T}^d,$$

where $w$ is a $C^m_0 ([0, 1])$ weighting function with $2 \leq m \leq \infty$, that is, $w \in C^\infty ([0, 1]), w^{(k)} (0) = w^{(k)} (1) = 0$ for all $0 \leq k \leq m, w(x) > 0$ for $x \in (0, 1)$, and $\int_0^1 w (s) ds = 1$.

We make the following assumption:
(H1) The approximation functions given in (2.1), (2.2) and (2.5) satisfy the integrability condition:

$$\int_{1}^{+\infty} \frac{r^{d-1} \Delta^m(r)}{\Delta(r)} dr < +\infty.$$  

**Theorem 2.6.** Give $f \in B_{\tilde{\Delta}}$, and $\rho$ satisfies the Finite-dimensional nonresonant condition in Definition 2.2. Assume (H1). Then there hold

$$\left\| WB_N (f) (\theta) - \int_{T^d} f(\hat{\theta}) d\hat{\theta} \right\|_{B} \leq \frac{C_1}{N^m}, \quad N \geq 1, \quad (2.7)$$

and

$$\left\| \frac{1}{T} \int_{0}^{T} w(t/T) f(\rho(t + \theta)) dt - \int_{T^d} f(\hat{\theta}) d\hat{\theta} \right\|_{B} \leq \frac{C_1}{T^m}, \quad T \geq 1, \quad (2.8)$$

where the positive constant $C_1 > 0$ only depends on $f, \Delta, \tilde{\Delta}, w, \alpha, m, d.$

Let us make some comments.

(1) As long as the weighting function $w$ is sufficiently smooth, the convergence rate of weighted Birkhoff average can reach arbitrarily polynomial convergence. For example, we could take

$$\tilde{w}(x) := \begin{cases} \left( \int_{0}^{1} \exp \left( -s^{-p}(1-s)^{-q} \right) ds \right)^{-1} \exp \left( -x^{-p}(1-x)^{-q} \right), & x \in (0, 1), \\ 0, & x = 0, 1, \end{cases}$$

for any $p, q > 0$. One easily verify that $\tilde{w}$ is $C_0^\infty([0, 1])$. It turns out that the relationship between the computational convergence rate and $p, q$ is not particularly clear, see [4].

(2) Spatial structure (2.5) of $B_{\tilde{\Delta}}$ and condition (H1) together guarantee the uniform convergence of weighted Birkhoff average (2.7). However, after removing (2.4) in (2.5), then (2.7) may only hold a.e. on $\mathbb{T}^d$, because for $f$ in general, the Fourier series of $f$ does not necessarily pointwise converge to $f$, so we cannot obtain uniform convergence of the weighted Birkhoff average with respect to all $\theta \in \mathbb{T}^d$. If $f \in \mathbb{R}^l$ with $l \in \mathbb{N}^+$ is continuous and satisfies $f(0) = f(1)$ in $\mathbb{R}^l$ (here $0 = (0, \ldots, 0), 1 = (1, \ldots, 1)$ in $\mathbb{T}^d$), then (2.7) could indeed converge uniformly. At this point, if we further assume that:

(i) $f$ is $C^M$ smooth, then $|\hat{f}_k| \leq C_{f,M} ||k||^{-M}$ can be obtained by integration by parts for all $0 \neq k \in \mathbb{Z}^d$, that is, $\hat{\Delta}(x) := x^M$ and $f \in B_{\tilde{\Delta}}$. Obviously, if the rotation vector $\rho$ is Diophantine, i.e., $\Delta(x) := x^\tau$ with $\tau > d - 1$, then (H1) can be satisfied as long as $M > d + m\tau$, because

$$\int_{1}^{+\infty} \frac{r^{d-1} \Delta^m(r)}{\Delta(r)} dr = \int_{1}^{+\infty} \frac{1}{r^{M + 1 - d - m\tau}} dr < +\infty.$$  

This is the case given in [5].

(ii) $f$ is Gevrey smooth in some neighbourhood of $\mathbb{T}^d$ in $\mathbb{C}^d$, i.e., there exist $c_f, \mu > 0$ and $\nu \in (0, 1]$ such that $|\hat{f}_k| \leq c_f e^{-\mu||k||^\nu}$ for all $0 \neq k \in \mathbb{Z}^d$.  


In particular, $f$ is analytic when $\nu = 1$. This leads to $\tilde{\Delta} (x) := e^{\mu x \nu}$. Therefore, if the rotation vector $\rho$ satisfies the nonresonant conditions (2.1), (2.2) with $\Delta (x) = e^{\tilde{\nu} x}$, $0 < \tilde{\nu} < m^{-1} \mu$ (weaker than the Diophantine type), then one can verify (H1) as:

$$\int_{1}^{+\infty} \frac{r^{d-1} \Delta^m (r)}{\Delta (r)} dr \leq \int_{1}^{+\infty} \frac{r^{d-1} e^{m \tilde{\nu} r}}{e^{\mu r \nu}} dr \leq c_{d, \mu, m, \tilde{\nu}, \nu} \int_{1}^{+\infty} \frac{1}{e^{(\mu-m \tilde{\nu})r/2}} dr < +\infty.$$  

For two cases given above, the uniform convergence of weighted Birkhoff average can be obtained by applying Theorem 2.6, and the convergence rate is polynomial.

(3) It should be pointed out that we generalize the Diophantine condition for the irrational rotation vector $\rho$ since the rapid convergence of the Fourier coefficients of $f$ could overcome the nonresonance of $\rho$, as shown in (ii). For example, for a given approximation function $\Delta (x)$ in (2.1) and (2.2), we can require that the Fourier coefficients of $f$ to converge rapidly to

$$\tilde{\Delta} (x) \sim \Delta^m (x) x^d (\log x) (\log \log x) \cdots (\log \cdots \log x)^{1+\zeta}, \quad x \to +\infty$$

with $\ell \in \mathbb{N}^+$ and $\zeta > 0$ in (2.5). Then (H1) holds because

$$\int_{1}^{+\infty} \frac{r^{d-1} \Delta^m (r)}{\Delta (r)} dr = o \left( \int_{1}^{+\infty} \frac{1}{r (\log r) (\log \log r) \cdots (\log \cdots \log r)^{1+\zeta}} dr \right) < +\infty.$$  

(4) For the continuous case (2.8), we only require that the nonresonant condition (2.1) holds for $n = 0$, i.e., (2.2), since we just have to integrate by parts directly with respect to $\int_{0}^{T} w (t/T) e^{2\pi i t k \cdot \rho} dt$ in proof.

(5) The dependence of the universal constant $C_1 > 0$ on the spatial dimension $d$ is actually caused by the integrability assumption (H1) which is somewhat easier to verify. As to the subsequent infinite dimensional cases (e.g., Theorem 2.9), one has to only require the boundedness for series (e.g., (H2)) to eliminate the influence of dimension, which is not essential. Additionally, one observes that $C_1$ might tend to infinite (e.g., it can be verified that $\lim_{m \to +\infty} \| \tilde{\omega}^{(m)} \|_{L^1 (0,1)} = +\infty$ with the weighting function $\tilde{\omega}$ in (3.1)), so if we want to achieve the exponential convergence, some special techniques are needed, as shown in Section 3 and Subsection 4.4.

2.2. **Infinite-dimensional case** $\mathbb{T}^{\infty}$. However, when considering the weighted Birkhoff average (2.6) on the infinite-dimensional torus $\mathbb{T}^{\infty} := \mathbb{T}^{\mathbb{N}}$, some spatial structure
has to be required. For convenience, we use the Diophantine condition for the irrational vectors $\rho$ proposed by Bourgain and the corresponding metric, see [2, 12]. More precisely, our set of irrational vectors $\rho$ is the infinite-dimensional cube $[1,2]^\mathbb{N}$ (equal to $\mathbb{T}^\infty$), endowed with the probability measure $P$ induced by the product measure of the infinite-dimensional cube $[1,2]^\mathbb{N}$. Next, for fixed $2 \leq \eta \in \mathbb{N}^+$, we define the set of infinite integer vectors with finite support

$$Z_\ast^\infty := \left\{ k \in \mathbb{Z}^\mathbb{N} : |k|_\eta := \sum_{j \in \mathbb{N}} \langle j \rangle^\eta |k_j| < +\infty, \langle j \rangle := \max\{1, |j|\} \right\}. \tag{2.10}$$

At this point, $k_j \neq 0$ only for finitely many indices $j \in \mathbb{N}$. It can be seen later that the such a metric like $|k|_\eta$ is necessary for the infinite-dimensional case since it determines the boundedness of the summation in proof. Besides, the infinite-dimensional analyticity also depends on the above framework, see Corollary 2.11.

**Definition 2.7 (Infinite-dimensional nonresonant condition).** An irrational vector $\rho \in \mathbb{T}^\infty$ is said to satisfy the Infinite-dimensional nonresonant condition if there exist $\gamma > 0$ and an approximation function $d$ such that

(c) The discrete case

$$|k \cdot \rho - n| > \frac{\gamma}{d(|k|_\eta)}, \quad \forall 0 \neq k \in Z_\ast^\infty, \quad \forall n \in \mathbb{Z}; \tag{2.11}$$

(d) The continuous case

$$|k \cdot \rho| > \frac{\gamma}{d(|k|_\eta)}, \quad \forall 0 \neq k \in Z_\ast^\infty. \tag{2.12}$$

**Remark 2.8.** In particular, if

$$d(|k|_\eta) = \prod_{j \in \mathbb{N}} (1 + |k_j|^{\langle j \rangle^\mu}), \quad \forall 0 \neq k \in Z_\ast^\infty \tag{2.13}$$

in (2.11) with some $\mu > 1$, then we say that the irrational vector $\rho$ satisfies the Infinite-dimensional Diophantine condition. Define the set

$$D_{\gamma,\mu} := \left\{ \rho \in [1,2]^\mathbb{N} : \rho \text{ satisfies the Infinite-dimensional Diophantine condition} \right\}.$$

Then there exists a positive constant $C(\mu)$ such that $P\left( [1,2]^\mathbb{N} \setminus D_{\gamma,\mu} \right) \leq C(\mu) \gamma$, as proved in [1, 2].

Under the above spatial structure, we introduce Fourier expansions of functions $f \in B$ on the infinite-dimensional torus $\mathbb{T}^\infty$ below, see [12] for details:

$$f = \sum_{k \in Z_\ast^\infty} \hat{f}_k e^{2\pi i k \cdot \theta}, \quad \hat{f}_k = \int_{\mathbb{T}^\infty} f(\theta) e^{-2\pi i k \cdot \hat{\theta}} d\hat{\theta}. \tag{2.14}$$

Now we define the function space with rapid convergence

$$B_{\Delta_\infty} := \left\{ f : \mathbb{T}^\infty \to B : f \text{ satisfies (2.14), and} \sup_{0 \neq k \in Z_\ast^\infty} \Delta_\infty\left(|k|_\eta\right) \|\hat{f}_k\|_B < +\infty \right\}, \tag{2.15}$$
for an approximation function $\tilde{\Delta}_\infty$. In order to establish a weighted Birkhoff aver-
gage theorem on $T_\infty$, we have to make an assumption like ($H1$):

(H2) The approximation functions given in (2.11), (2.12) and (2.15) satisfy the
following boundedness condition:

$$
\sum_{0 \neq k \in \mathbb{Z}^N} \frac{a^n}{\tilde{\Delta}_\infty(|k|_\eta)} < +\infty.
$$

After the above preparation, we are in a position to give the following theorem.

**Theorem 2.9.** Give $f \in B_{\tilde{\Delta}_\infty}$, and $\rho$ satisfies the Infinite-dimensional nonresonant
condition in Definition 2.7. Assume ($H2$). Then there hold

$$
\left\| WB_N (f) (\theta) - \int_{T_\infty} f(\hat{\theta}) d\hat{\theta} \right\|_B \leq \frac{C_2}{N^m}, \quad N \geq 1, \quad (2.16)
$$

and

$$
\left\| \frac{1}{T} \int_0^T w(t/T) f(\rho t + \theta) dt - \int_{T_\infty} f(\hat{\theta}) d\hat{\theta} \right\|_B \leq \frac{C_2}{T^m}, \quad T \geq 1, \quad (2.17)
$$

where the positive constant $C_2$ only depends on $f, a, \tilde{\Delta}_\infty, \eta, w, \gamma, m$.

**Remark 2.10.** For the selected spatial structure and ($H2$), we eliminate the depen-
dence of the universal constant on the dimension of the domain. This is extremely
surprising because it avoids the Curse of Dimensionality.

As an application of Theorem 2.9, we give the following corollary based on the
Infinite-dimensional Diophantine condition.

**Corollary 2.11** (Universality of arbitrary polynomial convergence via analyticy
in the almost periodic case). Give $f \in B_{\tilde{\Delta}_\infty}$. Assume that the irrational vector
$\rho$ satisfies the Infinite-dimensional Diophantine condition in Definition 2.7 with
(2.13), and $f$ is analytic in some neighbourhood of $T_\infty$ in $\mathbb{C}^N$. Then (2.16) and
(2.17) hold with arbitrary given $2 \leq m \in \mathbb{N}^+$ and a positive constant $C_3$ that only
depends on $f, \tilde{\Delta}_\infty, \eta, \mu, w, \gamma, m$.

**Remark 2.12.** Analyticity in the almost periodic case implies that the Fourier co-
efficients of $f$ converge at an exponential rate under the spatial structure (sim-
ilar to (ii) in Comment (2)), i.e., the approximation function in (2.15) satisfies
$\tilde{\Delta}_\infty(x) = \exp(x)$ without loss of generality, see also [12].

**Remark 2.13.** Corollary 2.11 shows that arbitrary polynomial convergence is in-
deed universal via analyticity in the almost periodic case, as long as the weighting
function considered is $C^\infty_0 ([0, 1])$, since the Diophantine rotations form a set of
full Lebesgue measure, see Remark 2.8.
3. Convergence of exponential rate type

As mentioned in (1.2) and (1.3), the classical Birkhoff average might converge at an arbitrarily slow rate. Surprisingly, if we choose a weighting function good enough and require that the Fourier coefficients of \( f \) to converge more rapidly, then the corresponding weighted Birkhoff average could converge at an exponential rate. We also provide an intuitive explanation of why the exponential rate could be indeed achieved, see the proof of Theorem 3.3 in Subsection 4.4.

Here we choose the new weighting function as
\[
\bar{w}(x) := \left( \int_{0}^{1} \exp\left( -s^{-1}(1-s)^{-1} \right) ds \right)^{-1} \cdot \exp\left( -x^{-1}(1-x)^{-1} \right) \tag{3.1}
\]
on \((0,1)\), i.e., \( p = q = 1 \) in (2.9), and let \( \bar{w}(0) = \bar{w}(1) = 0 \). Denote by \( \mathbb{W}B_{\bar{w}}(f)(\theta) \) the corresponding weighted Birkhoff average at this point. According to Lemma 5.3, we have the following \( L^1 \) norm estimates for the higher derivatives of \( \bar{w} \):
\[
\int_{0}^{1} \left| \bar{w}^{(n)}(x) \right| dx \leq C_{*} n^{\beta n}, \quad n \geq 2, \quad \tag{3.2}
\]
provided with \( C_{*} > 0 \) that only depends on \( \bar{w} \), and \( \beta > 1 \) is an absolute constant. As we will see later, (3.2) and the truncation technique will play an important role in dealing with exponential convergence for quasiperiodic and almost periodic cases. In fact, the resulting convergence rate will be faster if one can improve the upper bound in (3.2) or find a better weighting function. However, we suspect that the hyperexponential convergence rate (e.g., \( \exp(-\exp(N)) \)) cannot be achieved through this approach, because higher derivatives in (3.2) generally have coefficients such as \( n! \sim \sqrt{2\pi n(e/n)^n} \), and the former seems to require that \( \| \bar{w}^{(n)} \|_{L^1(0,1)} = O((\log n)^n) \).

**Definition 3.1 (Adaptive function).** A function \( \varphi(x) \) defined on \([1, +\infty)\) is called an adaptive function, if it is nondecreasing, satisfies that \( \varphi(+\infty) = +\infty \) and \( \varphi(x) = o(x) \) as \( x \to +\infty \).

**Remark 3.2.** For example, \( \varphi_1(x) = \log^u(1 + x) \) with \( u > 0 \) and \( \varphi_2(x) = x^v \) with \( 0 < v < 1 \) are all adaptive functions. The selection of an adaptive function is important for the analysis of convergence rate below.

We are now in a position to establish the exponential convergence theorems through a given adaptive function \( \varphi \) and under certain assumptions.

3.1. Finite-dimensional case \( \mathbb{T}^d \). We make the following assumption:

**\( H3 \)** Let an adaptive function \( \varphi \) be given. The approximation functions given in (2.1), (2.2) and (2.5) satisfy the smallness condition with some \( c > 0 \):
\[
\int_{0}^{+\infty} r^{d-1} \frac{\Delta^2(r)}{\Delta(r)} dr = O\left( e^{-cr} \right), \quad x \to +\infty. \tag{3.3}
\]

**Theorem 3.3.** Give an adaptive function \( \varphi \), let \( f \in B_{\Delta} \), and \( \rho \) satisfy the Finite-dimensional nonresonant condition in Definition 2.2. Assume (\( H3 \)). Then there
exist an absolute constant $\beta_\ast > 0$, and a positive constant $C_4$ that only depends on $f, \alpha, d, \Delta, \tilde{\Delta}, \varphi, c$ such that the following hold with $N, T$ sufficiently large

$$
\left\| \nabla B_N (f) (\theta) - \int_{\mathbb{T}^d} f(\hat{\theta}) d\hat{\theta} \right\|_B \leq C_4 \exp \left( - (\varphi(N))^{\beta_\ast} \right),
$$

and

$$
\left\| \frac{1}{T} \int_0^T \bar{w}(t/T) f (\rho t + \theta) dt - \int_{\mathbb{T}^d} f(\hat{\theta}) d\hat{\theta} \right\|_B \leq C_4 \exp \left( - (\varphi(T))^{\beta_\ast} \right). \tag{3.4}
$$

**Remark 3.4.** It can be obviously seen from the L’Hospital’s rule that if the divergence rate of $\tilde{\Delta}(x)$ is rapid enough, then the convergence of the weighted Birkhoff average can indeed be of exponential rate type (because $\Delta(x)$ is fixed at this point). In fact, the smallness of (H3) can be further weakened, we do not pursue that.

Based on Theorem 3.3, we give the following corollary to the case where $f$ is analytic and $\rho$ is Diophantine. The Gevrey smooth situation is in fact similar, which we omit here.

**Corollary 3.5** (Universality of exponential convergence via analyticity in the quasiperiodic case). Assume that the irrational vector $\rho$ satisfies the Finite-dimensional Diophantine condition (2.3), and $f$ is analytic in some neighbourhood of $\mathbb{T}^d$ in $\mathbb{C}^d$. Then Theorem 3.3 holds with $N, T$ sufficiently large and a universal constant $C_5 > 0$ independent of them, and the convergence rate is indeed exponential, i.e., $O(\exp(-\ddot{c}N^\xi))$ and $O(\exp(-\ddot{c}T^\xi))$ with some $\ddot{c} > 0, \xi = \beta_\ast (1 + \tau \beta_\ast)^{-1} > 0$.

**Remark 3.6.** This corollary shows that exponential convergence is indeed universal in the quasiperiodic case via analyticity, since the Diophantine rotations form a set of full Lebesgue measure.

### 3.2. Infinite-dimensional case $\mathbb{T}^\infty$

We make the following assumption:

(H4) Let an adaptive function $\varphi$ be given. The approximation functions given in (2.11), (2.12) and (2.15) satisfy the following smallness condition with some $c > 0$:

$$
\frac{d^2 \left(\left| k_\eta \right| \right)}{\bar{\Delta}_{\infty} \left(\left| k_\eta \right| \right)} = O \left( e^{-cx} \right), \quad x \to +\infty.
$$

**Theorem 3.7.** Give an adaptive function $\varphi$, let $f \in B_{\tilde{\Delta}_{\infty}},$ and $\rho$ satisfy the Infinite-dimensional nonresonant condition in Definition 2.7. Assume (H4). Then there exist an absolute constant $\beta_\ast > 0$, and a positive constant $C_6$ that only depends on $f, d, \Delta_{\infty}, \eta, \gamma, \varphi, c$ such that the following hold with $N, T$ sufficiently large

$$
\left\| \nabla B_N (f) (\theta) - \int_{\mathbb{T}^\infty} f(\hat{\theta}) d\hat{\theta} \right\|_B \leq C_6 \exp \left( - (\varphi(N))^{\beta_\ast} \right),
$$

and

$$
\left\| \frac{1}{T} \int_0^T \bar{w}(t/T) f (\rho t + \theta) dt - \int_{\mathbb{T}^\infty} f(\hat{\theta}) d\hat{\theta} \right\|_B \leq C_6 \exp \left( - (\varphi(T))^{\beta_\ast} \right).
$$
The convergence in Theorem 3.7 can be of indeed exponential as long as the adaptive function \( \varphi(x) = \sqrt{x} \) is chosen and the divergence rate of \( \tilde{\Delta}_\infty(x) \) is rapid enough, similar to Remark 3.4 and Corollary 3.5. We present the following corollary via Diophantine rotation as an example, which is a special case in Corollary 2.11 as we forego.

**Corollary 3.8.** Give \( f \in B_{\tilde{\Delta}_\infty} \) with \( \tilde{\Delta}_\infty(x) = \exp(\exp(x)) \), and assume that \( \rho \) satisfies the Infinite-dimensional Diophantine condition (2.13) with \( 2 \leq \mu = \eta \in \mathbb{N}^+ \). Then Theorem 3.7 holds with exponential convergence rate, i.e., \( O(\exp(-N^\hat{c})) \) and \( O(\exp(-T^\hat{c})) \) with some \( \hat{c} > 0 \), as long as \( N, T \) are sufficiently large.

### 3.3. Cases without small divisors.

In fact, small divisors appear in the proof of Theorem 2.6 to Theorem 3.7 due to integration by parts, which not only brings difficulties to the proof, but also requires additional assumptions (such as (H1) to (H4), etc.), and even affects the convergence rate, e.g., if the divergence speed of \( \Delta(x) \) is so slow that the order of the integral in (H3) is only polynomial’s type \( (N^{-m} \text{ with some } m > 0) \), then Theorem 3.3 might not admit exponential convergence. If we can avoid the small divisors, then the above problems are solved and the resulting rate of convergence is certainly exponential. It should be pointed out that, for the discrete case with \( 1 \leq d \leq \infty \) and for the continuous case with \( 2 \leq d \leq \infty \), to avoid small divisors, one has to restrict \( f \) to trigonometric polynomials, namely considering the following spaces

\[
B_{\tilde{\Delta},K} := \left\{ f \in B_{\tilde{\Delta}} : \hat{f}_k = 0 \text{ for all } \|k\| > K \in \mathbb{N}^+ \right\},
\]

and

\[
B_{\tilde{\Delta}_\infty,K} := \left\{ f \in B_{\tilde{\Delta}_\infty} : \hat{f}_k = 0 \text{ for all } |k|_\eta > K \in \mathbb{N}^+ \right\},
\]

provided a \( K \in \mathbb{N}^+ \). As to the continuous case with \( d = 1 \), naturally there are no small divisors. The analysis becomes simpler than that before (in fact part of Theorem 3.3) in the absence of small divisors, we present Theorems 3.9 and 3.10 as follows.

**Theorem 3.9.** Give \( f \in B_{\tilde{\Delta},K} \) (or \( f \in B_{\tilde{\Delta}_\infty,K} \)). Then there exist some \( \hat{c} > 0 \) independent of \( N, T \), such that

\[
\left\| \overline{\mathcal{W}}_N(f)(\theta) - \int_{T^d} f(\hat{\theta})d\hat{\theta} \right\|_B \leq C_7 \exp\left(-N^\hat{c}\right), \quad 1 \leq d \leq \infty
\]

and

\[
\left\| \frac{1}{T} \int_0^T \bar{w}(t/T) f(\rho t + \theta) dt - \int_{T^d} f(\hat{\theta})d\hat{\theta} \right\|_B \leq C_7 \exp\left(-T^\hat{c}\right), \quad 2 \leq d \leq \infty
\]

for \( N, T \) sufficiently large.

**Theorem 3.10.** Give \( f \in B_{\tilde{\Delta}} \) with \( d = 1 \). Assume that

\[
\sum_{k \neq 0} \frac{1}{\Delta(|k|)} < +\infty. \quad (3.5)
\]
Then there exist some \( \hat{c} > 0 \) and \( C_8 > 0 \) independent of \( T \), such that

\[
\left\| \frac{1}{T} \int_0^T \bar{w}(t/T) f(\rho t + \theta) \, dt - \int_{T^{-1}} f(\hat{\theta}) \, d\hat{\theta} \right\|_B \leq C_8 \exp \left( -T^{\hat{c}} \right)
\]

for \( T \) sufficiently large.

4. PROOF OF RESULTS

4.1. Proof of Theorem 2.6. We first prove the discrete case (2.7), and some useful estimates should be provided.

Note that \( w \in C^m_0((0, 1)) \), then there exists \( C_w > 0 \) such that

\[
N_{A_N} = \left( \frac{1}{N} \sum_{n=0}^{N-1} w \left( \frac{n}{N} \right) \right)^{-1} \leq C_w, \quad \forall N \in \mathbb{N}^+.
\]

(4.1)

Integrating by parts \( m \) times yields that

\[
\left| \int_0^1 w(y) e^{2N\pi i (k \cdot \rho - n) y} \, dy \right| = \frac{1}{2N\pi} \left| k \cdot \rho - n \right| \left| \int_0^1 w(y) e^{2N\pi i (k \cdot \rho - n) y} \, dy \right| \\
= \frac{1}{2N\pi} \left| k \cdot \rho - n \right| \left| \int_0^1 w^{(1)}(y) e^{2N\pi i (k \cdot \rho - n) y} \, dy \right| \\
\cdots \\
= \frac{1}{(2N\pi \left| k \cdot \rho - n \right|)^m} \left| \int_0^1 w^{(m)}(y) e^{2N\pi i (k \cdot \rho - n) y} \, dy \right| \\
\leq \frac{1}{(2N\pi \left| k \cdot \rho - n \right|)^m} \| w^{(m)} \|_{L^1((0, 1))},
\]

(4.2)

where we use the fact \( w \in C^m_0((0, 1)) \) to eliminate the boundary terms.

For any fixed \( 0 \neq k \in \mathbb{Z}^d \), denote by \( n_k \in \mathbb{N} \) the closest integer to the number \( k \cdot \rho \). Note that \( m \geq 2 \). Therefore by (2.1) and \( \Delta(1) = 1 \) we have

\[
\sum_{n=-\infty}^{+\infty} \frac{1}{\left| k \cdot \rho - n \right|^m} = \frac{1}{\left| k \cdot \rho - n_k \right|^m} + \sum_{n \neq n_k} \frac{1}{\left| k \cdot \rho - n \right|^m} \\
\leq \frac{\Delta^m(\|k\|)}{\alpha^m} + 2 \sum_{n=0}^{+\infty} \frac{1}{(|n| + 1/2)^m} \\
\leq C_{\alpha, m} \Delta^m(\|k\|),
\]

(4.3)

because after being far away from the fixed number \( k \cdot \rho \), the rest summation of series naturally converges and is independent of the small divisor.

Note that

\[
\hat{f}_0 = \int_{\mathbb{T}^d} f(\hat{\theta}) \, d\hat{\theta} = \text{WB}_N(f_0).
\]

Then it follows that

\[
\varepsilon_N(\theta) : = \left\| \text{WB}_N(f)(\theta) - \int_{\mathbb{T}^d} f(\hat{\theta}) \, d\hat{\theta} \right\|_B
\]
\[
\leq \sum_{0 \neq k \in \mathbb{Z}^d} \left\| \tilde{f}_k \right\|_B \left| WB_N \left( e^{2\pi ik \cdot \theta} \right) \right|
\]
\[
\leq C_{f,\Delta} \sum_{0 \neq k \in \mathbb{Z}^d} \frac{1}{\Delta(||k||)} \left| \frac{1}{A_N} \sum_{n=0}^{N-1} w \left( \frac{n}{N} \right) e^{2\pi ik \cdot (\theta + n \rho)} \right|
\]
\[
= C_{f,\Delta} \sum_{0 \neq k \in \mathbb{Z}^d} \frac{1}{\Delta(||k||)} \left| \frac{1}{A_N} \sum_{n=0}^{N-1} w \left( \frac{n}{N} \right) e^{2\pi i k \cdot \rho} \right|
\]
\[
= C_{f,\Delta} \sum_{0 \neq k \in \mathbb{Z}^d} \frac{1}{\Delta(||k||)} \left| \sum_{n=-\infty}^{+\infty} w \left( \frac{n}{N} \right) e^{2\pi i k \cdot \rho} \right|
\]
\[
= C_{f,\Delta} \sum_{0 \neq k \in \mathbb{Z}^d} \frac{1}{\Delta(||k||)} \left| \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} w \left( \frac{t}{N} \right) e^{2\pi i k \cdot \rho} e^{-2\pi i n t} dt \right|
\]
\[
= C_{f,\Delta} \sum_{0 \neq k \in \mathbb{Z}^d} \frac{N}{A_N} \frac{1}{\Delta(||k||)} \left| \int_{0}^{1} w \left( y \right) e^{2\pi i k \cdot \rho} \sum_{n=-\infty}^{+\infty} \int_{0}^{1} w \left( y \right) e^{\pi i n t} dt \right|
\]
\[
\leq C_{f,\Delta,\omega} \sum_{0 \neq k \in \mathbb{Z}^d} \frac{1}{\Delta(||k||)} \left| \sum_{n=-\infty}^{+\infty} \frac{1}{(2\pi n)^{m}} \left| w^{(m)} \right|_{L^1(0,1)} \right|
\]
\[
\leq C_{f,\Delta,\omega,\alpha,m} \sum_{0 \neq k \in \mathbb{Z}^d} \frac{1}{\Delta(||k||)} \left| \sum_{n=-\infty}^{+\infty} \frac{1}{|k \cdot \rho - n|^m} \right|
\]
\[
\leq C_{f,\Delta,\omega,\alpha,m} \sum_{0 \neq k \in \mathbb{Z}^d} \frac{\Delta^m(||k||)}{\Delta(||k||)}
\]
\[
\leq C_{1} \int_{1}^{+\infty} \frac{r^{d-1} \Delta^m(r)}{\Delta(r)} dr
\]
\[
\leq C_{1} \frac{1}{N^m}
\]

provided a universal constant \(C_1 = C_{f,\Delta,\omega,\alpha,m,d} > 0\). Here (2.5) is used in (4.4), (4.5) is because \(w(n/N) = 0\) for \(n \in \mathbb{Z}\setminus\{0,1,\cdots,N-1\}\) (note that \(w \in C^m_0((0,1))\)), the Poisson summation formula in Lemma 5.1 is used in (4.6), (4.1) is used in (4.8), (4.2) is used in (4.9), (4.3) is used in (4.10), and finally, (4.11) is because of (H1). This finishes the proof of the discrete case (2.7).

As to the continuous case (2.8), the analysis is similar. This proves Theorem 2.6.
4.2. **Proof of Theorem 2.9.** We first prove the discrete case (2.16). Note that
\[
\left| \frac{1}{A_N} \sum_{n=0}^{N-1} w\left(\frac{n}{N}\right) e^{2\pi i n k \cdot \rho} \right| \leq \frac{N}{A_N} \sum_{n=-\infty}^{+\infty} \frac{1}{(2N \pi |k \cdot \rho - n|)^m} \left\| w^{(m)} \right\|_{L^1(0,1)} \\
\leq C_{w,m} \sum_{n=-\infty}^{+\infty} \frac{1}{|k \cdot \rho - n|^m} \\
\leq \frac{C_{w,m,\gamma}}{N^m} d^\gamma (|k|_\eta).
\]
Then recalling the proof of Theorem 2.6 and (H2), we obtain that
\[
\varepsilon_N (\theta) := \left\| W B_N (f) (\theta) - \int_{-\infty}^{T_\infty} f(\hat{\theta}) d\hat{\theta} \right\|_B \\
\leq \sum_{0 \neq k \in \mathbb{Z}^\infty} \| \hat{f}_k \|_B \left\| W B_N \left( e^{2\pi i k \cdot \theta} \right) \right\| \\
= \sum_{0 \neq k \in \mathbb{Z}^\infty} \| \hat{f}_k \|_B \left| \frac{1}{A_N} \sum_{n=0}^{N-1} w\left(\frac{n}{N}\right) e^{2\pi i k \cdot (\theta + n \rho)} \right| \\
\leq C_{f,\hat{\Delta}_\infty} \sum_{0 \neq k \in \mathbb{Z}^\infty} \Delta_{\infty} \left( |k|_\eta \right) \left| \frac{1}{A_N} \sum_{n=0}^{N-1} w\left(\frac{n}{N}\right) e^{2\pi i n k \cdot \rho} \right| \\
\leq C_{f,\hat{\Delta}_\infty} \frac{C_{w,m,\gamma}}{N^m} \sum_{0 \neq k \in \mathbb{Z}^\infty} \frac{d^m (|k|_\eta)}{\Delta_{\infty} (|k|_\eta)} \\
\leq \frac{C_2}{N^m},
\]
where the positive constant $C_2$ only depends on $f, d, \hat{\Delta}_{\infty}, \eta, w, \gamma, m$. We therefore finish the proof of the discrete case (2.16).

As to the continuous case (2.17), the analysis is similar. This proves Theorem 2.9.

4.3. **Proof of Corollary 2.11.** One just needs to verify (H2).

Note that for all $\mu > 1$ and fixed $\rho_* = 1/2m > 0$, we have
\[
d \left( |k|_\eta \right) = \prod_{j \in \mathbb{N}} (1 + |j|^\mu |j|^\mu) \leq \exp \left( \frac{\tau}{\rho_* |\eta|} \log \left( \frac{\tau}{\rho_*} \right) \right) \cdot e^{\rho_* |k|_\eta}
\]
with some $\tau = \tau (\eta, \mu) > 0$ for all $0 \neq k \in \mathbb{Z}^\infty_\eta$, see Lemma 5.2. Recall $2 \leq \eta \in \mathbb{N}^+$, then
\[
|k|_\eta = \sum_{j \in \mathbb{N}} (j)^\eta |j| \in \mathbb{N}^+.
\]
(4.12)
Thus
\[
\sum_{0 \neq k \in \mathbb{Z}^*_\infty} \frac{d^0}{\tilde{\Delta}_\infty} \left( |k|_\eta \right) \leq C_{\eta, \mu, m} \sum_{0 \neq k \in \mathbb{Z}^*_\infty} \frac{e^{m \rho_\ast |k|_\eta}}{\tilde{\Delta}_\infty (|k|_\eta)}
\]
\[
= C_{\eta, \mu, m} \sum_{\nu=1}^{\infty} \left( \sum_{0 \neq k \in \mathbb{Z}^*_\infty, |k|_\eta = \nu} \frac{1}{\tilde{\Delta}_\infty (|k|_\eta)} e^{m \rho_\ast |k|_\eta} \right)
\]
\[
= C_{\eta, \mu, m} \sum_{\nu=1}^{\infty} \left( \frac{1}{\tilde{\Delta}_\infty (\nu)} e^{m \rho_\ast \nu} \sum_{0 \neq k \in \mathbb{Z}^*_\infty, |k|_\eta = \nu} 1 \right). \quad (4.13)
\]

Denote
\[
\sum_{0 \neq k \in \mathbb{Z}^*_\infty, |k|_\eta = \nu} 1 = \# \left\{ k : 0 \neq k \in \mathbb{Z}^*_\infty, |k|_\eta = \nu \in \mathbb{N}^+ \right\}. \quad (4.14)
\]

Hence, in view of (4.12), the largest non-zero integer \( j_{\text{max}} \) in (4.14) satisfies
\[
j_{\text{max}} \leq \lfloor \nu^{1/\eta} \rfloor,
\]
and that’s why we need a certain spatial structure. Therefore, we have
\[
\# \left\{ k : 0 \neq k \in \mathbb{Z}^*_\infty, |k|_\eta = \nu \in \mathbb{N}^+ \right\}
\leq \# \left\{ k : 0 \neq k \in \mathbb{Z}^*_\infty, |k_0| + |k_1| + \cdots + |k_{\lfloor \nu^{1/\eta} \rfloor}| = \nu \in \mathbb{N}^+ \right\}
\leq 2^{\lfloor \nu^{1/\eta} \rfloor + 1} \cdot \# \left\{ k : 0 \neq k \in \mathbb{Z}^*_\infty, k_j \in \mathbb{N} \text{ for all } j \in \mathbb{N}, k_0 + k_1 + \cdots + k_{\lfloor \nu^{1/\eta} \rfloor} = \nu \in \mathbb{N}^+ \right\}
= 2^{\lfloor \nu^{1/\eta} \rfloor + 1} \cdot C_{\eta, \nu^{1/\eta}}
\leq 2^{\lfloor \nu^{1/\eta} \rfloor + 1} \cdot C_{\eta, \nu^{1/\eta}} \cdot \nu^{(1-1/\eta)(\nu^{1/\eta}+1)} \cdot e^{[\nu^{1/\eta}]} \quad (4.15)
\leq C_{\eta, \nu^{1/\eta}}. \quad (4.16)
\]

Here (4.15) uses the following fact:
\[
C_{\nu+\lfloor \nu^{1/\eta} \rfloor}^\nu
\]
\[
= \frac{\nu! \lfloor \nu^{1/\eta} \rfloor \nu + \lfloor \nu^{1/\eta} \rfloor}{\nu! \lfloor \nu^{1/\eta} \rfloor} \approx \sqrt{2\pi \nu \left( \nu + \lfloor \nu^{1/\eta} \rfloor \right) \left( \frac{\nu + \lfloor \nu^{1/\eta} \rfloor}{e} \right)^{\nu + \lfloor \nu^{1/\eta} \rfloor}}
\]
\[
\approx \frac{1}{\sqrt{2\pi \nu^{1/\eta}}} \cdot \left( 1 + \frac{\lfloor \nu^{1/\eta} \rfloor}{\nu} \right) \nu^{\nu^{1/\eta}} \cdot \left( \frac{\nu}{\nu^{1/\eta}} \right)^{\lfloor \nu^{1/\eta} \rfloor} \cdot \left( 1 + \frac{\lfloor \nu^{1/\eta} \rfloor}{\nu} \right) \nu^{1/\eta}
\]
\[
= \frac{1}{\sqrt{2\pi \nu^{1/\eta}}} \cdot \left( \frac{\nu}{\nu^{1/\eta}} \right)^{\lfloor \nu^{1/\eta} \rfloor} \cdot \exp \left( \nu \log \left( 1 + \frac{\nu^{1/\eta}}{\nu} \right) \right)
\]
\[
\leq 1.
\]
\[
\exp\left(\frac{\nu^{1/\eta}}{\eta} \log\left(1 + \frac{\nu^{1/\eta}}{\nu}\right)\right)
\]
\[
= \frac{1}{\sqrt{2\pi\nu^{1/\eta}}} \cdot \left(\frac{\nu}{\nu^{1/\eta}}\right)^{[\nu^{1/\eta}]} \cdot \exp\left(\nu \left(\frac{[\nu^{1/\eta}]}{\nu} - \frac{[\nu^{1/\eta}]^2}{2\nu} + \cdots\right)\right)
\]
\[
= \frac{1}{\sqrt{2\pi\nu^{1/\eta}}} \cdot \left(\frac{\nu}{\nu^{1/\eta}}\right)^{[\nu^{1/\eta}]} \cdot \exp\left(\frac{[\nu^{1/\eta}]^2}{2\nu} + \cdots\right)
\]
\[
\leq C_\eta \frac{1}{\sqrt{\nu^{1/\eta}}} \cdot \nu^{(1-1/\eta)(\nu^{1/\eta}+1)} \cdot e^{[\nu^{1/\eta}]}. 
\]

Finally, combining (4.13), (4.14) and (4.16) we arrive at
\[
\sum_{0 \neq k \in \mathbb{Z}_+^n} \frac{d^m}{|k|_\eta} \leq C_{\eta,m} \sum_{\nu=1}^{\infty} \left(\frac{1}{\Delta_\infty (\nu)} e^{m\rho_\nu} \sum_{0 \neq k \in \mathbb{Z}_+^\infty, |k|_\eta=\nu} 1\right)
\]
\[
\leq C_{\eta,m} \sum_{\nu=1}^{\infty} \left(\frac{1}{\Delta_\infty (\nu)} e^{m\rho_\nu} \cdot C_{\eta,\nu^{1/\eta}}\right)
\]
\[
\leq C_{\eta,m} \sum_{\nu=1}^{\infty} \left(\frac{\nu^{1/\eta}}{e^{\nu(1-m\rho_\nu)}}\right)
\]
\[
= C_{\eta,m} \sum_{\nu=1}^{\infty} \left(\frac{1}{e^{\nu/2-\nu^{1/\eta} \log \nu}}\right)
\]
\[
\leq C_{\eta,m} \sum_{\nu=1}^{\infty} \left(\frac{1}{e^{\nu/4}}\right)
\]
\[
\leq C_3 
\]
\[
< +\infty 
\]
(4.17)

for a universal positive constant \(C_3\), since \(\Delta_\infty (x) = \exp(x)\) (see Remark 2.12), i.e., (H2) holds. Then we finish the proof by applying Theorem 2.9.

4.4. **Proof of Theorem 3.3.** Intuitively, let’s first present an explanation for why the exponential rate can be achieved. For cases without small divisors, let us take \(f\) be a trigonometric polynomial as an example. In view of the estimates (3.2) of the higher order derivatives of the weighting function \(\tilde{w}\), we could change the times of integration by parts to achieve the fastest convergence rate under this approach (monotonicity analysis is sufficient), that is, an exponential convergence. Therefore
for a general $f$, if its Fourier coefficients converge rapidly enough, then intuitively it behaves like a trigonometric polynomial. One only needs to truncate the Fourier series into the principal and remainder terms with respect to the given $N \in \mathbb{N}^+$ and the chosen adaptive function $\varphi(x)$ at this point. Specifically, for the principal term we could perform the above operation (integration by parts of varying times), and for the remainder we just employ the analysis of Theorem 2.6 (integration by parts of fixed times).

We first prove the discrete case (3.3), and the proof is divided into four steps.

**Step1:** For given adaptive function $\varphi$ and $N \in \mathbb{N}^+$ sufficiently large, define

$$
\Lambda_1 := \left\{ k : 0 \neq k \in \mathbb{Z}^d, ||k|| \leq \Delta^{-1} (2\pi \alpha N / \varphi(N)) \right\},
$$

$$
\Lambda_2 := \left\{ k : 0 \neq k \in \mathbb{Z}^d, ||k|| > \Delta^{-1} (2\pi \alpha N / \varphi(N)) \right\}.
$$

This gives $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $\Lambda_1 \cup \Lambda_2 = \left\{ k : 0 \neq k \in \mathbb{Z}^d \right\}$. Further, one notices that $|\Lambda_1|, |\Lambda_2| \to +\infty$ when $N \to +\infty$ because $\varphi(x) = o(x)$ and $\Delta^{-1}(+\infty) = +\infty$. At this point, we have

$$
\left\| \nabla \text{B}_N (f)(\theta) - \int_{\mathbb{T}^d} f(\hat{\theta}) d\hat{\theta} \right\|_B \leq C_{f,\Delta} \left( \sum_{0 \neq k \in \mathbb{Z}^d} \frac{1}{\Delta(||k||)} \sum_{n=-\infty}^{+\infty} \int_0^1 \tilde{w}(y) e^{2\pi i N (k \cdot \rho - n)y} dy \right) + \left( \sum_{k \in \Lambda_2} \frac{1}{\Delta(||k||)} \sum_{n=-\infty}^{+\infty} \int_0^1 \tilde{w}(y) e^{2\pi i N (k \cdot \rho - n)y} dy \right)
$$

$$
\leq C_{f,\Delta} \left( \sum_{k \in \Lambda_1} \frac{1}{\Delta(||k||)} \sum_{n=-\infty}^{+\infty} \int_0^1 \tilde{w}(y) e^{2\pi i N (k \cdot \rho - n)y} dy \right) + \left( \sum_{k \in \Lambda_2} \frac{1}{\Delta(||k||)} \sum_{n=-\infty}^{+\infty} \int_0^1 \tilde{w}(y) e^{2\pi i N (k \cdot \rho - n)y} dy \right)
$$

$$
:= C_{f,\Delta} (S_1 + S_2)
$$

(4.19)

according to (4.7) in the proof of Theorem 2.6, where $S_1$ and $S_2$ represent the principal term and the remainder term, respectively.

**Step2:** For the principal term $S_1$, we choose

$$
L_1 = L_1(k, N) := \left[ e^{-1} \left( \frac{\Delta(||k||)}{2\pi \alpha N} \right)^{-1/\beta} \right] \geq 2
$$

for fixed $k \in \Lambda_1$ and $N \in \mathbb{N}^+$ sufficiently large, where $\beta > 0$ is the absolute constant given in (3.2). One can verify that $\inf_{k \in \Lambda_1, N \in \mathbb{N}^+} L_1 = +\infty$, which implies that the times of integration by parts become infinite when $N \to +\infty$. Further, it follows that

$$
\left( \frac{L_1^\beta \Delta(||k||)}{2\pi \alpha N} \right)^{L_1} \leq C_{\alpha,\varphi,\Delta} \exp \left( -\left( \varphi(N) \right)^{\beta^*} \right)
$$

(4.20)

The proof of Theorem 2.6 is complete.
with $\beta^* = (2\beta)^{-1} > 0$ (also an absolute constant) for all $k \in \Lambda_1$. Note that $L_1 \geq 2$ as long as $N$ sufficiently large. We therefore derive that

$$S_1 \leq \sum_{k \in \Lambda_1} \frac{1}{\Delta(||k||)} \left( \left\| \bar{w}^{(L_1)} \right\|_{L^1(0,1)} \left( \frac{\Delta(||k||)}{2\pi \alpha N} \right)^{L_1} \right)$$

(4.21)

$$\leq \sum_{k \in \Lambda_1} \frac{1}{\Delta(||k||)} \left( \left\| \bar{w}^{(L_1)} \right\|_{L^1(0,1)} \left( \frac{\Delta(||k||)}{2\pi \alpha N} \right)^{L_1} \right)$$

(4.22)

$$\leq C_{\alpha, \varphi, \Delta} \sum_{k \in \Lambda_1} \frac{1}{\Delta(||k||)} \left( \frac{L_1^\Delta ||k||}{2\pi \alpha N} \right)^{L_1}$$

(4.23)

$$\leq C_{\alpha, \varphi, \Delta} \sum_{k \in \Lambda_1} \frac{1}{\Delta(||k||)} \cdot \exp \left( -\varphi(N) \right)^{\beta^*}$$

(4.24)

Here (4.21) is same as (4.2) and (4.3) in the proof of Theorem 2.6. (4.23) uses (3.2), and it shows that the second term in parentheses in (4.22) is relatively small compared to the first one. (4.24) uses (4.20), and finally (4.25) is because

$$\int_1^{+\infty} \frac{r^{d-1}}{\Delta(r)} dr = O \left( \int_1^{+\infty} \frac{r^{d-1} \Delta^2(r)}{\Delta(r)} dr \right) = O(1)$$

due to (H3) and Cauchy’s Theorem.

**Step 3:** As to the remainder term $S_2$, similar to (4.2) and (4.3) with $m = 2$ we arrive at

$$S_2 \leq \sum_{||k|| > \Delta^{-1}(2\pi \alpha N/\varphi(N))} \frac{1}{\Delta(||k||)} \sum_{n=-\infty}^{+\infty} \left| \int_0^1 \bar{w}(y) e^{2\pi i N(k \cdot \rho - n)y} dy \right|$$

$$\leq C_1 \cdot \sum_{||k|| > \Delta^{-1}(2\pi \alpha N/\varphi(N))} \frac{\Delta^2(||k||)}{\Delta(||k||)}$$

(4.25)

$$\leq C_1 \cdot \frac{C_{\alpha, \Delta, \varphi}}{N^2} \int_{\Delta^{-1}(2\pi \alpha N/\varphi(N))}^{+\infty} \frac{r^{d-1} \Delta^2(r)}{\Delta(r)} dr$$

(4.26)
\begin{equation}
C_1 \cdot C_{\alpha, \Delta, \bar{\Delta}, \varphi} \int_{\Delta^{-1}(2\pi \alpha x / \varphi(x))}^{+\infty} \frac{x^{d-1}\Delta^2(r)}{\Delta(r)} dr 
\leq C_1 \cdot C_{\alpha, \Delta, \bar{\Delta}, \varphi} \exp(-cN),
\end{equation}

where \( C_1 > 0 \) is the universal constant in Theorem 2.6, and (H3) is used in (4.28).

**Step 4:** By substituting (4.25) and (4.28) into (4.19) we immediately have

\[
\left\| \overline{B}_N(f) (\theta) - \int_{\mathbb{T}^d} f(\hat{\theta}) d\hat{\theta} \right\|_B \leq C_4 \exp \left( -\langle \varphi(N) \rangle^{\beta^*} \right)
\]

for \( N \) sufficiently large, provided a positive constant \( C_4 > 0 \) that only depends on \( f, \alpha, d, \Delta, \bar{\Delta}, \varphi, c \). Then we finish the proof of the discrete case (3.3).

The analysis is similar for the continuous case (3.4), and this proves Theorem 3.3.

### 4.5. Proof of Corollary 3.5.

At this point, we have \( \Delta(x) = x^\tau \) with \( \tau > d - 1 \) and \( \Delta^{-1}(x) = x^{1/\tau} \). In view the analyticity of \( f \), there exist \( c_f, \mu > 0 \) such that \( |\hat{f}_k| \leq c_f e^{-2\mu||k||} \) for all \( k \neq 0 \in \mathbb{Z}^d \), i.e., \( \bar{\Delta}(x) = e^{2\mu x} \). However, (H3) does not hold. Recall Remark 3.4, we could choose an appropriate adaptive function \( \varphi \) and slightly modify the proof of Theorem 3.3 (i.e., the analysis of \( S_1 \) and \( S_2 \)) to obtain exponential convergence. Let \( \varphi(x) = x^\tilde{\omega} \) with \( \tilde{\omega} = (1 + \tau \beta_*)^{-1} \in (0, 1) \), where \( \beta_* > 0 \) is the constant given in Theorem 3.3. Then it follows that

\[
\exp \left( -\langle \varphi(x) \rangle^{\beta^*} \right) = \exp \left( -x^\tilde{\omega} \beta^* \right) = \exp \left( -x^\xi \right)
\]

in (4.25), where \( \xi = \beta^* (1 + \tau \beta_*)^{-1} \in (0, 1) \), and

\[
\int_{\Delta^{-1}(2\pi \alpha x / \varphi(x))}^{+\infty} \frac{x^{d-1}\Delta^2(r)}{\Delta(r)} dr = \int_{(2\pi \alpha x / \varphi(x))^{1/\tau}}^{+\infty} \frac{x^{d+2\tau-1}}{e^{2\mu x}} dr 
\]

\[
= O \left( \int_{(2\pi \alpha x / \varphi(x))^{1/\tau}}^{+\infty} e^{-\mu r} dr \right) 
= O \left( \exp \left( -\tilde{c} x (1-x)^{-1} \right) \right)
= O \left( \exp \left( -\tilde{c} x^\xi \right) \right)
\]

in (4.27) with some \( \tilde{c} > 0 \). Therefore Corollary 3.5 is proved by (4.25), (4.28) and (4.19).

### 4.6. Proof of Theorem 3.7.

Recall (4.26), then the proof is the same as Theorem 3.3 due to (H4).

### 4.7. Proof of Corollary 3.8.

One just needs to verify (H4). We omit a few calculations here for brevity. Recall \( \bar{\Delta}_\infty(x) = \exp(\exp(x)) \) and (2.13). Note Lemma 5.2 with \( 2 \leq \mu = \eta \in \mathbb{N}^+ \) implies that \( d(x) = O(e^x) \), and thus
\[
\log x = O \left( d^{-1} (x) \right). \text{ Let } \varphi(x) = \sqrt{x}, \text{ we therefore derive that}
\]
\[
\sum_{|k|_\eta \geq d^{-1}(2\pi \gamma x/\varphi(x))} \frac{d^2(|k|_\eta)}{\Delta_\infty \left( |k|_\eta \right)} = \sum_{\nu \geq d^{-1}(2\pi \gamma x/\varphi(x))} \frac{d^2(\nu)}{\Delta_\infty \left( \nu \right)} \left( \sum_{0 \neq k \in \mathbb{Z}^d, |k|_\eta = \nu} 1 \right)
\]
\[
= O \left( \sum_{\nu \geq \tilde{c} \log x} \frac{e^{2\nu}}{\exp \left( e^\nu \right)} \cdot \nu^{1/\eta} \right)
\]
\[
= O \left( \sum_{\nu \geq \tilde{c} \log x} \frac{1}{\exp \left( e^\nu / 2 \right)} \right)
\]
\[
= O \left( e^{-\tilde{c} x} \right),
\]
provided a universal constant \( \tilde{c} > 0 \), and (4.14), (4.16) are used here. One notices that the convergence rate at this point is of exponential’s type, i.e., \( O(\exp(-N^\nu)) \) and \( O(\exp(-T^\nu)) \) with some \( \nu > 0 \) according to Theorem 3.7, we therefore finish the proof.

4.8. Proof of Theorems 3.9 and 3.10. Consider Theorem 3.9. Note that there are no small divisors at this point, we therefore could slightly modify the proof of Theorem 3.3. Specifically, \( \Lambda_1 \) is a finite set and \( \Lambda_2 = \phi \) as long as \( N \) is sufficiently large, we thus only need to estimate the principal term \( S_1 \). The convergence rate is indeed exponential through the same technique of integration by parts, see (4.20). This gives the proof.

As to Theorem 3.10, the proof is similar since the estimates obtained by integration by parts are exponentially small for all \( 0 \neq k \in \mathbb{Z} \) as long as \( T \) is sufficiently large, and the universal coefficient will be guaranteed boundedness by (3.5), we therefore finish the proof.

5. Appendix

Lemma 5.1 (Poisson summation formula). For each \( h(x) \in L^2(\mathbb{R}) \), there holds
\[
\sum_{n \in \mathbb{Z}} h(n) = \sum_{n \in \mathbb{Z}} \int_{-\infty}^{+\infty} h(x) e^{-2\pi nix} dx.
\]

Proof. See Chapter 3 in [6] for details.

Lemma 5.2. For arbitrary given \( \rho_* > 0 \) and \( \mu \in \mathbb{N}^+ \), there exists \( \tau = \tau(\eta, \mu) > 0 \) such that
\[
\prod_{j \in \mathbb{N}} (1 + |k_j|^{\mu(j)}^{\mu}) \lesssim \exp \left( \frac{\tau}{\rho_* \eta} \log \left( \frac{\tau}{\rho_*} \right) \right) \cdot e^{\rho_* |k|_\eta}.
\]

Proof. See details in Lemma B.1 in [12] and Lemma 7.2 in [1].
Lemma 5.3. Define
\[ \bar{w}(x) := \left( \int_0^1 \exp\left(-s^{-1}(1-s)^{-1}\right) ds \right)^{-1} \cdot \exp\left(-x^{-1}(1-x)^{-1}\right) \]
on (0, 1). Then the following holds with \( C_* = \left( \int_0^1 \exp\left(-s^{-1}(1-s)^{-1}\right) ds \right)^{-1} > 0 \), where \( \beta > 1 \) is some universal absolute constant:
\[ \int_0^1 |\bar{w}^{(n)}(x)| \, dx \leq C_* n^{\beta n}, \quad n \geq 2. \] (5.1)

**Proof.** We’re going to prove (5.1) in four steps.

**Step1:** Define \( P(x) := e^{-\frac{x^2}{2}} \). Then it follows that \( \bar{w}(x) = C_* P(x) P(1-x) \) and \((P(x))^{(1)} = \frac{1}{x^2} e^{-\frac{1}{2} x^2} \). One can verify the following by induction:
\[ (P(x))^{(n)} = \left( \frac{1}{x^{2n}} + \frac{a_{2n-1}^{(n)}}{x^{2n-1}} + \cdots + \frac{a_1^{(n)}}{x^n} \right) e^{-\frac{1}{2} x^2}, \quad n \geq 1, \]
(5.2)
where \( a_j^{(n)} \in \mathbb{Z} \) for all \( 1 \leq j \leq 2n-1 \), and we define \( a_{2n}^{(n)} := 1 \). At this point, denote \( b_n := \max_{1 \leq j \leq 2n} |a_j^{(n)}| \in \mathbb{N}^+ \), then \( b_1 = 1 \). In view of (5.2), we get
\[ b_{n+1} \leq 2n \cdot 4n \cdot \max_{1 \leq j \leq 2n} |a_j^{(n)}| = 8n^2 b_n, \]
since when taking the derivative of (5.2), there are 4n terms that haven’t been combined yet. Therefore, we have
\[ b_n \leq \prod_{j=1}^{n} (8j)^2 \cdot b_1 = 8^n (n!)^2, \quad n \geq 1. \] (5.3)

**Step2:** Note that
\[ \sup_{1/2<s<1} \left| P(s)^{(0)} \right| = \sup_{1/2<s<1} e^{-\frac{1}{2} x^2} = e^{-1}. \]
For all \( n \geq 1 \), by using (5.3) we get
\[ \sup_{1/2<s<1} \left| (P(s))^{(n)} \right| = \sup_{1/2<s<1} \left| \left( \frac{1}{x^{2n}} + \frac{a_{2n-1}^{(n)}}{x^{2n-1}} + \cdots + \frac{a_1^{(n)}}{x^n} \right) e^{-\frac{1}{2} x^2} \right| \leq \sup_{1/2<s<1} 2n \cdot \frac{b_n}{8n} e^{-\frac{1}{2} x^2} \leq 2n \cdot 4^n \cdot 8^n (n!)^2 \leq 2^{6n} (n!)^2. \] (5.4)

Then we arrive at
\[ \sup_{1/2<s<1} \left| (P(s))^{(n)} \right| \leq 2^{6n} (n!)^2, \quad n \geq 0. \] (5.5)
**Step 3:** For $n = 0$, we have

$$\int_0^{\frac{1}{2}} |(P(x))^{(0)}| \, dx = \int_0^{\frac{1}{2}} e^{-\frac{1}{2}} \, dx = \int_2^{+\infty} \frac{1}{y^2} e^{-y} \, dy \leq 4 \int_2^{+\infty} e^{-y} \, dy = \frac{4}{e^{2}}.$$ (5.6)

As to $n \geq 1$, by using (5.3) we have

$$\begin{align*}
\int_0^{\frac{1}{2}} &|(P(x))^{(n)}| \, dx = \int_0^{\frac{1}{2}} \left| \frac{1}{x^{2n}} + \frac{a^{(n)}_{2n-1}}{x^{2n-1}} + \cdots + \frac{a^{(n)}_{1}}{x^{1}} \right| e^{-\frac{1}{2}} \, dx \\
&\leq \int_0^{\frac{1}{2}} \frac{2n \cdot b_n}{x^{2n}} e^{-\frac{1}{2}} \, dx \\
&= n2^{3n+1}(nl)^2 \int_2^{+\infty} y^{2n-2} e^{-y} \, dy \\
&\leq n2^{3n+1}(nl)^2 \int_0^{+\infty} y^{2n-2} e^{-y} \, dy \\
&= n2^{3n+1}(nl)^2 \cdot (2n - 2)! \\
&\leq 2^{3n+1}(nl)^2 (2n)!.
\end{align*}$$ (5.7)

**Step 4:** In view of (5.5), (5.6), (5.7) and the Stirling formula $n! \sim \sqrt{2\pi n}(n/e)^n$, we finally arrive at

$$\begin{align*}
\int_0^1 &|\bar{u}^{(n)}(x)| \, dx = 2 \int_0^{\frac{1}{2}} |\bar{u}^{(n)}(x)| \, dx \\
&= 2C \int_0^{\frac{1}{2}} \left| \sum_{i=0}^{n} C_n^i (P(x))^{(i)} (P(1-x))^{(n-i)} \right| \, dx \\
&\leq 2C \sum_{i=0}^{n} C_n^i \int_0^{\frac{1}{2}} \left| (P(x))^{(i)} \right| \cdot \left| (P(1-x))^{(n-i)} \right| \, dx \\
&\leq 2C \sum_{i=0}^{n} C_n^i \int_0^{\frac{1}{2}} \left| (P(x))^{(i)} \right| \sup_{1/2 < s < 1} \left| (P(s))^{(n-i)} \right| \, dx \\
&\leq C2^{6n+1}(nl)^2 \sum_{i=0}^{n} C_n^i \int_0^{\frac{1}{2}} \left| (P(x))^{(i)} \right| \, dx \\
&\leq C2^{6n+1}(nl)^2 \left( \frac{4}{e^{2}} + \sum_{i=1}^{n} C_n^i \int_0^{\frac{1}{2}} \left| (P(x))^{(i)} \right| \, dx \right) \\
&\leq C2^{6n+1}(nl)^2 \left( \left( \sum_{i=0}^{n} C_n^i \right) \cdot 2^{3n+1}(nl)^2 (2n)! \right) \\
&= C2^{10n+2}(nl)^4 (2n)!
\end{align*}$$
\[ \leq C^* 2^{10n+2} \left( \frac{n^4}{2^{4n}} \right) \left( \frac{2^{2n} n^{2n}}{2^{2n}} \right) \]
\[ \leq C^* 2^{6n+2} n^{6n} \]
\[ \leq C^* n^{3n}. \]

This proves (5.1) as long as if we choose $\beta > 6$ independent of $\bar{w}$ sufficiently large. $\square$

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