LYAPUNOV FUNCTIONALS FOR THE ENSKOG EQUATION

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Abstract. Two Lyapunov functionals are presented for the Enskog equation. One is to describe interactions between particles with various velocities and another is to measure the $L^1$ distance between two classical solutions. The former yields the time-asymptotic convergence of global classical solutions to the collision free motion while the latter is applied into the verification of the $L^1$ stability of global classical solutions.

1. Introduction

We are concerned with two Lyapunov functionals to describe the time-asymptotic behaviour and the $L^1$ stability of global classical solutions to the Enskog equation without any external force for a hard sphere gas. In the absence of any external force, the Enskog equation is as follows:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = Q(f)$$  \hspace{1cm} (1.1)

where $f = f(t, x, v)$ is a one-particle distribution function that depends on the time $t \in \mathbb{R}_+$, the position $x \in \mathbb{R}^3$ and the velocity $v \in \mathbb{R}^3$, and $Q$ is the Enskog collision operator whose form will be addressed below. Here and throughout this paper, $\mathbb{R}_+$ represents the positive side of the real axis including its origin and $\mathbb{R}^3$ denotes a three-dimensional Euclidean space.

The collision operator $Q$ is expressed by the difference between the gain and loss terms respectively defined by

$$Q^+(f)(t, x, v) = a^2 \int_{\mathbb{R}^3 \times S^2_+} f(t, x, v') f(t, x - a \omega, w') B(v - w, \omega) d\omega dw,$$

$$Q^-(f)(t, x, v) = a^2 \int_{\mathbb{R}^3 \times S^2_+} f(t, x, v) f(t, x + a \omega, w) B(v - w, \omega) d\omega dw$$  \hspace{1cm} (1.2)

In equations (1.2) and (1.3), $S^2_+ = \{ \omega \in S^2 : \omega(v - w) \geq 0 \}$ is a subset of a unit sphere surface $S^2$ in $\mathbb{R}^3$, $a$ is a diameter of hard sphere ($a \geq 0$), $\omega$ is a unit vector along the line passing through the centers of the spheres at their interaction, $(v', w')$ are velocities after collision of two particles have precollisional velocities $(v, w)$ and $B(v - w, \omega) = \max(0, (v - w)\omega)$ is the collision kernel.

This equation (1.1) is indeed the commonly known Enskog-Boltzmann equation and it is a modification of the original work of Enskog [5]. There are other different versions of the Enskog equation in order that they formally satisfy some such properties as entropy bound and consistence with irreversible thermodynamics (see [1], [9], [13], [15]). It is worth mentioning that equation (1.1) still obeys the conservation laws of mass, momentum and energy [12].

As for the Boltzmann equation, two colliding particles obey the conservation laws of both kinetic momentum and energy, as follows.

$$v + w = v' + w', \quad v^2 + w^2 = v'^2 + w'^2.$$  \hspace{1cm} (1.4)
This results in their velocity relations
\[ v' = v - [(v - w)\omega]_\omega, \quad w' = w + [(v - w)\omega]_\omega, \]
where \( \omega \in S^2_\pm \). This implies that \( B(v' - w', -\omega) = B(v - w, \omega) \).

The Boltzmann equation models dilute gases successfully but it is no longer suitable for gases with high-density effects. The Enskog equation, a partial differential integral equation of the hyperbolic type, is a model first proposed in 1922 by Enskog [5] as the generalization of the Boltzmann equation describing the dynamical behavior of the density of a moderately dense or high-density gas. There are many results about the global existence and uniqueness of the solutions to the initial value problem for the Boltzmann and the Enskog equations without external forces. Concerning the Boltzmann equation in the absence of external forces, a global solution existence is obtained by DiPerna & Lions [4] for the large data but one cannot yet know whether the solution to the problem is unique or not; however, a global existence and uniqueness result is shown by Illner & Shinbrot [7] about the solutions to the initial value problem to the Boltzmann equation for small initial data in the infinite vacuum. Concerning the Enskog equation without external forces, a global existence and uniqueness proof is given by Polewczak [9] for near-vacuum data and another one shown by Arkeryd [1] for the large data. Some different existence results are also given by Cercignani (see [2], [3]). With an analysis of the well-posedness of the initial value problem in unbounded domains, some global existence and uniqueness theorems is obtained by Toscani & Bellomo [13] about the solutions to the Enskog equation in the absence of external forces for small initial data with suitable decay to zero at infinity in the phase space, and the asymptotic stability of the solutions and the influence of the external field have been discussed. For the Enskog equation in the absence of external forces, the \( L^1 \) stability of solutions is first given by Cercignani [8] and the time-asymptotic behaviour of solutions in the weighted \( L^\infty \) is then provided by Polewczak (see [2], [10], [11]). Many other results about this subject can be found in the references in the papers mentioned above.

Two so-called Lyapunov functionals recently constructed by Ha [6] mathematically neither yield the time-asymptotic behaviour in the \( L^1 \) norm nor recover the \( L^1 \) stability for global classical solutions to the Enskog equation without external forces (see Appendix A). Now there are not yet any Lyapunov functionals to restore the the time-asymptotic behaviour and the stability of solutions to the Enskog equation in the absence of external forces. The aim of this paper is to build two Lyapunov functionals for the Enskog equation. One is to describe interactions between particles with various velocities and another is to measure the \( L^1 \) distance between two classical solutions. The former yields the time-asymptotic convergence of global classical solutions to the collision free motion while the latter is applied into the verification of the \( L^1 \) stability of global classical solutions.

The rest of this paper is arranged as follows. In Section 2 some properties of the collision operator \( Q \) of the Enskog equation (1.1) are introduced including both the entropy identity and the nonincreasing property of the entropy functional. Then in Section 3 a Lyapunov functional is constructed and its time-decay property is given together with an application of the time-asymptotic behaviour of any nonnegative solution to the Enskog equation (1.1). A different Lyapunov functional is defined and its time-decay property, together an application of the \( L^1 \) stability of solution to the Enskog equation (1.1), is provided in Section 4. A counterexample of inequalities given by Ha is finally shown in Appendix A.

2. Preliminaries

In this section some properties of the collision operator \( Q \) of the Enskog equation (1.1) are introduced including both the entropy identity and the nonincreasing property of the entropy functional, and the conservation law of mass is shown for the Enskog equation.
We consider the Enskog equation (1.1) with (1.2) and (1.3). Notice that $B(v - w, \omega) = \max(0, (v - w)\omega)$. Then we have the following lemma.

**Lemma 2.1.** Suppose that $Q$ is a collisional operator as defined by (1.2) and (1.3). Let $\psi(x, v)$ be a measurable function on $\mathbb{R}^3 \times \mathbb{R}^3$. If $f \in C_0(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$, then

$$
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \psi(x, v)Q(f)dvdx = \frac{a^2}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2_+} B(v - w, \omega)f(t, x, v)f(t, x + a\omega, w)\omega dv dw dx
$$

and

$$
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \psi(x, v)Q(f)dvdx = \frac{a^2}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2_+} B(v - w, \omega)f(t, x - a\omega, w')f(t, x, v')\omega dv dw dx.
$$

**Proof.** Denote $I_g$ and $I_l$ by

$$
I_g = a^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times S^2_+} B(v - w, \omega)\psi(x, v)f(t, x, v')f(t, x - a\omega, w')\omega dv dw dx
$$

and

$$
I_l = a^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times S^2_+} B(v - w, \omega)\psi(x, v)f(t, x, v)f(t, x + a\omega, w)\omega dv dw dx,
$$

respectively. By (1.2) and (1.3), we have

$$
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \psi(x, v)Q(f)dvdx = I_g - I_l.
$$

Let us first consider the loss integral $I_l$. By exchanging $v$ and $w$ and replacing $\omega$ with $-\omega$ in the integral on the right side of (2.4), we can get

$$
I_l = a^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times S^2_+} B(v - w, \omega)\psi(x, v)f(t, x, v)f(t, x - a\omega, w)\omega dv dw dx.
$$

Replacing $x$ with $x + a\omega$ in the integral on the right side of (2.6) gives

$$
I_l = a^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times S^2_+} B(v - w, \omega)\psi(x + a\omega, w)f(t, x, v)f(t, x + a\omega, w)\omega dv dw dx.
$$

Combining (2.4) and (2.7), we have

$$
I_l = \frac{a^2}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times S^2_+} B(v - w, \omega)[\psi(x, v) + \psi(x + a\omega, w)]f(t, x, v)f(t, x + a\omega, w)\omega dv dw dx.
$$

Then the gain integral $I_g$ is below considered. By using the properties that $B(v' - w', -\omega) = B(v - w, \omega)$ and that $dw dv = dw' dv'$, (2.4) can be rewritten as

$$
I_g = a^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times S^2_+} B(v' - w', -\omega)\psi(x, v)f(t, x, v')f(t, x - a\omega, w')\omega dv dw' dv' dx.
$$

By exchanging $v$ and $v'$, $w$ and $w'$, and replacing $\omega$ with $-\omega$ in the integral on the right side of (2.9), we have

$$
I_g = a^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times S^2_+} B(v - w, \omega)\psi(x, v')f(t, x, v)f(t, x + a\omega, w)\omega dv dw dx.
$$

(2.10)
By repeating the analysis of the loss integral $I_l$ discussed above, \((2.10)\) can be rephrased as

\[
I_g = a^2 \int_{R^3 \times R^3 \times R^3 \times S^2_+} B(v - w, \omega) \psi(x + \alpha \omega, v') f(t, x, v) f(t, x + \alpha \omega, w) d\omega dv dx. \tag{2.11}
\]

Combining \((2.10)\) and \((2.11)\), we can get

\[
I_g = \frac{a^2}{2} \int_{R^3 \times R^3 \times R^3 \times S^2_+} B(v - w, \omega) [\psi(x, v') + \psi(x + \alpha \omega, v')] f(t, x, v) f(t, x + \alpha \omega, w) d\omega dv dx. \tag{2.12}
\]

Inserting \((2.8)\) and \((2.12)\) into \((2.5)\) gives \((2.1)\).

We below prove \((2.2)\). By replacing $x$ with $x - \omega$ in the integral on the right side of \((2.1)\), we know that

\[
\int_{R^3 \times R^3} \psi(x, v) Q(f) dv dx = \frac{a^2}{2} \int_{R^3 \times R^3 \times R^3 \times S^2_+} B(v - w, \omega) f(t, x - \omega, v) f(t, x, w)
\]

\[
\times [\psi(x - \omega, v') - \psi(x - \omega, v) + \psi(x, w') - \psi(x, w)] d\omega dv dx. \tag{2.13}
\]

Exchanging $v$ and $w$ and replacing $\omega$ with $-\omega$ in the integral on the right side of \((2.13)\), we can obtain

\[
\int_{R^3 \times R^3} \psi(x, v) Q(f) dv dx = \frac{a^2}{2} \int_{R^3 \times R^3 \times R^3 \times S^2_+} B(v - w, \omega) f(t, x + \omega, w) f(t, x, v)
\]

\[
\times [\psi(x + \omega, v') - \psi(x + \omega, v) + \psi(x, v') - \psi(x, v)] d\omega dv dx. \tag{2.14}
\]

Notice that $B(v - w, \omega) = B(v' - w', -\omega)$ and that $d\omega dv = dw' dv'$. Then \((2.13)\) gives

\[
\int_{R^3 \times R^3} \psi(x, v) Q(f) dv dx = \frac{a^2}{2} \int_{R^3 \times R^3 \times R^3 \times S^2_+} B(v' - w', -\omega) f(t, x + \omega, w) f(t, x, v)
\]

\[
\times [\psi(x + \omega, v') - \psi(x + \omega, w) + \psi(x, v') - \psi(x, v)] d\omega dw' dv' dx. \tag{2.15}
\]

By exchanging $v$ and $v'$, $w$ and $w'$, and replacing $\omega$ with $-\omega$ in the integral on the right side of \((2.15)\), \((2.2)\) follows.

We can also know that if $\psi(x, v) = 1$, $v$, $v^2/2$ in Lemma \((2.1)\) then

\[
\int_{R^3 \times R^3} \psi(x, v) Q(f) dv dx = 0.
\]

This formally implies the conservation laws of mass, momentum and energy hold for the Enskog equation in the absence of external forces, i.e.,

\[
\frac{d}{dt} \int_{R^3 \times R^3} \left(1, v, \frac{v^2}{2}\right) f(t, x, v) dv dx = 0.
\]

We now introduce a notation $f^\#(t, x, v) = f(t, x + vt, v)$. Then the Enskog equation \((1.1)\) can be also written as

\[
\frac{\partial f^\#(t, x, v)}{\partial t} = Q(f^\#)(t, x, v). \tag{2.16}
\]

Then let us consider two functionals $H_B(t)$ and $I(t)$ defined as follows:

\[
H_B(t) = \int_{R^3 \times R^3} f(t, x, v) \ln f(t, x, v) dx dv \tag{2.17}
\]
and
\[ I(t) = \frac{a^2}{2} \int_{\mathbb{R}^3 \times S^2_+} B(v - w, \omega)[f(t, x, v)f(t, x + a\omega, w) - f(t, x, v)f(t, x - a\omega, w)] d\omega dw dx dv. \] (2.18)

They have the following property.

**Lemma 2.2** \([10]\). Assume that \( H(t) = H_B(t) + \int_0^t I(s) ds \) where \( H_B(t) \) and \( I(t) \) are given by (2.17) and (2.18). If \( f = f(t, x, v) \) is a nonnegative classical solution to the Enskog equation (1.1) with (1.2) and (1.3), then \( \frac{dH(t)}{dt} \leq 0. \)

**Proof.** By (2.17), we have
\[ \frac{dH_B(t)}{dt} = \int_{\mathbb{R}^3 \times \mathbb{R}^3} [1 + \ln f^\#(t, x, v)] \frac{\partial f^\#(t, x, v)}{\partial t} dxdv. \] (2.19)

Inserting (2.16) into (2.19) gives
\[ \frac{dH_B(t)}{dt} = \int_{\mathbb{R}^3 \times \mathbb{R}^3} [1 + \ln f^\#(t, x, v)] Q(f)^\#(t, x, v) dxdv. \] (2.20)

By integrating over the variable \( x \) on the right side of (2.20), it follows that
\[ \frac{dH_B(t)}{dt} = \int_{\mathbb{R}^3 \times \mathbb{R}^3} [1 + \ln f(t, x, v)] Q(f)(t, x, v) dxdv. \] (2.21)

By Lemma 2.1 (2.21) can be rechanged as
\[ \frac{dH_B(t)}{dt} = \frac{a^2}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} B(v - w, \omega) f(t, x, v)f(t, x + a\omega, w) \]
\[ \times \ln \left[ \frac{f(t, x, v)f(t, x + a\omega, w')}{f(t, x, v)f(t, x + a\omega, w)} \right] d\omega dw dx dv. \] (2.22)

Equation (2.22) is also called entropy identity. Since \( \ln x \leq x - 1 \) for \( x > 0 \), the estimation of the integral on the right side of the entropy identity (2.22) reads
\[ \frac{dH_B(t)}{dt} \leq \frac{a^2}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} B(v - w, \omega) \]
\[ \times [f(t, x, v')f(t, x + a\omega, w') - f(t, x, v)f(t, x + a\omega, w)] d\omega dw dx dv, \] (2.23)
i.e., \( \frac{dH_B(t)}{dt} \leq -I(t) \). Therefore \( \frac{dH_B(t)}{dt} \leq -I(t) + I(t) = 0 \). The proof of this lemma is finished. □

3. **Asymptotic Behaviour**

In this section two new functionals are built. One of them is a Lyapunov functional and its time-decay property is related to another functional. This time-decay property can be applied into our description of the time-asymptotic behaviour of any nonnegative solution to the Enskog equation (1.1) with (1.2) and (1.3).

Let us begin with two new functionals \( \mathcal{D} = \mathcal{D}^+ + \mathcal{D}^- \) and \( \mathfrak{F} = \mathfrak{F}^+ + \mathfrak{F}^- \), where \( \mathcal{D}^\pm \) and \( \mathfrak{F}^\pm \) are defined as follows:
\[ \mathcal{D}^\pm[f](t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} f^\#(t, x, v) \]
\[ \times \left[ \int_{\mathbb{R}_+ \times S^2_+} f^\#(t, x + (v - w)t + a\omega + \tau \frac{v - w}{(v - w)\omega}, w) d\tau d\omega \right] dwdxdv, \] (3.1)
\[ \mathfrak{S}^\pm[f](t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times S^2_+} |(v - w)\omega| f^\#(t, x, v) f^\#(t, x + (v - w)t \mp a\omega, w) dw \, d\omega d\nu d\nu. \] (3.2)

Obviously, by (1.2) and (1.3), we know that \( \int_{\mathbb{R}^3 \times \mathbb{R}^3} Q^\pm(f)(t, x, v) dv \, dx \leq a^2 \mathfrak{S}^\pm[f](t) \). For any nonnegative solution \( f \) to the Enskog equation, \( \mathcal{D}[f](t) \) describes interactions between particles with various velocities. It can be below found that the time-decay property of \( \mathcal{D}[f](t) \) for any nonnegative solution \( f \) to the Enskog equation leads directly to the time-asymptotic behaviour of this solution in \( L^1(\mathbb{R}^3 \times \mathbb{R}^3) \) and so \( \mathcal{D} \) is called Lyapunov functional. To estimate the time decay of \( \mathcal{D}[f](t) \), we first have to show the following lemma.

**Lemma 3.1.** Let \( F(x) \) be an integrable function on \( \mathbb{R}^3 \) and \( v \) a vector in \( \mathbb{R}^3 \). Assume that \( S^2_+ = \{\omega|\nu \geq 0, \omega \in S^2\} \) where \( S^2 \) is a unit sphere surface in \( \mathbb{R}^3 \). Then

\[ \int_{\mathbb{R}^3} F(x) dx = a^2 \int_{\mathbb{R} \times S^2_+} F(a\omega + \tau v/\nu) d\tau d\omega \] (3.3)

where \( a \neq 0 \).

**Proof.** Denote \( x \) by \( x = (x_1, x_2, x_3) \). Using a transformation \( x = a\omega + \tau v/(\nu \omega) \) where \( \omega = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \), \( \theta \) is an angle between \( v \) and \( \omega \), \( 0 \leq \theta \leq \frac{\pi}{2} \), \( 0 \leq \varphi \leq 2\pi \). we can know that the Jacobian determinant of this transformation is \( \frac{\partial(x_1, x_2, x_3)}{\partial(\tau, \theta, \varphi)} = a^2 \sin \theta \). Lemma 3.1 therefore holds.

Combining (3.1) and Lemma 3.1, we can easily deduce that \( \mathcal{D}^\pm[f](t) \leq \frac{1}{a^2} ||f||_{L^1}^2 \) for any nonnegative integrable function \( f = f(t, x, v) \). Furthermore, we also obtain the following time-decay property of \( \mathcal{D}[f](t) \).

**Theorem 3.2.** Let \( \mathcal{D} \) and \( \mathfrak{S} \) be defined by (1.4) and (1.5). Assume that \( f = f(t, x, v) \) is a nonnegative classical solution to the Enskog equation (1.1) with (1.2) and (1.3) through an initial datum \( f_0 = f_0(x, v) \) and that \( f \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \). then

\[ \frac{d\mathcal{D}[f](t)}{dt} + (1 - 4||f||_{L^1}) \mathfrak{S}[f](t) \leq 0 \] (3.4)

for any \( t \in \mathbb{R}_+ \).

**Proof.** We first calculate \( \frac{d\mathcal{D}^\pm[f](t)}{dt} \). To do this, we have to deduce that

\[ \partial_t \left[ f^\#(t, x, v) f^\#(t, x + (v - w)t - a\omega + \tau v/(v - w)\omega, w) \right] \]

\[ = \partial_t \left[ (v - w)\omega f^\#(t, x, v) f^\#(t, x + (v - w)t - a\omega + \tau v/(v - w)\omega, w) \right] \]

\[ + Q(f)^\#(t, x, v) f^\#(t, x + (v - w)t - a\omega + \tau v/(v - w)\omega, w) \]

\[ + f^\#(t, x, v) Q(f)^\#(t, x + (v - w)t - a\omega + \tau v/(v - w)\omega, w) \] (3.5)

for any fixed variables \( (x, v, w, \omega) \). In fact, (3.5) results from (2.16) and the two identities as follows:

\[ \partial_t \left[ f^\#(t, x, v) f^\#(t, x + (v - w)t - a\omega + \tau v/(v - w)\omega, w) \right] \]

\[ = (v - w)\partial_x f^\#(t, x + (v - w)t - a\omega + \tau v/(v - w)\omega, w) \]
\[ + Q(f)^\#(t, x + (v - w)t - a\omega + \tau \frac{v - w}{(v - w)\omega}, w) \]  
\[ \text{(3.6)} \]

and
\[ (v - w)\partial_x f^\#(t, x + (v - w)t - a\omega + \tau \frac{v - w}{(v - w)\omega}, w) = (v - w)\omega \partial_t \left[ f^\#(t, x + (v - w)t - a\omega + \tau \frac{v - w}{(v - w)\omega}, w) \right]. \]  
\[ \text{(3.7)} \]

Denote two functionals \( \mathcal{I}_i^+ (i = 1, 2) \) by
\[ \mathcal{I}_1^+[f](t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2 \times \mathbb{R}_+} Q(f)^\#(t, x, v) \times f^\#(t, x + (v - w)t \equiv a\omega + \tau \frac{v - w}{(v - w)\omega}, w) d\tau d\omega dw dv dx \]  
\[ \text{(3.8)} \]

and
\[ \mathcal{I}_2^+[f](t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2 \times \mathbb{R}_+} f^\#(t, x, v) \times Q(f)^\#(t, x + (v - w)t \equiv a\omega + \tau \frac{v - w}{(v - w)\omega}, w) d\tau d\omega dw dv dx. \]  
\[ \text{(3.9)} \]

By integrating (3.5) with respect to variables \((\tau, \omega, w, v, x)\), we can obtain the following identity
\[ \frac{d\mathfrak{D}^+[f](t)}{dt} = -\mathfrak{F}^+[f](t) + \mathcal{I}_1^+[f](t) + \mathcal{I}_2^+[f](t) \]  
\[ \text{(3.10)} \]
where \( \mathfrak{F}^+ \) is a functional as given by (3.2) and \( \mathcal{I}_i^+ (i = 1, 2) \) are two functionals defined by (3.8) and (3.9). We also have a similar identity
\[ \frac{d\mathfrak{D}^-[f](t)}{dt} = -\mathfrak{F}^-[f](t) + \mathcal{I}_1^-[f](t) + \mathcal{I}_2^-[f](t) \]  
\[ \text{(3.11)} \]
where \( \mathfrak{F}^- \) is a functional as given by (3.2) and \( \mathcal{I}_i^- (i = 1, 2) \) are two functionals defined by (3.8) and (3.9). Estimation of (3.8) and (3.9) gives
\[ \mathcal{I}_1^+[f](t) + \mathcal{I}_2^+[f](t) \leq 2||f_0||_{L^1} \mathfrak{F}^+[f](t) \]  
\[ \text{(3.12)} \]
since \( Q(f)^\#(t, x, v) \leq Q^+(f)^\#(t, x, v) \). Combining (3.10), (3.11) and (3.12), we hence know that (3.4) holds. This completes the proof of this theorem. \( \square \)

By Theorem 3.2 we then get

**Theorem 3.3.** Put \( f_\infty(x, v) = f_0(x, v) + \int_{\mathbb{R}_+} Q(f)^\#(s, x, v) ds \). Assume that \( f = f(t, x, v) \) is a nonnegative classical solution to the Enskog equation (4.1) with (1.2) and (1.3) through an initial datum \( f_0 = f_0(x, v) \) satisfying \( ||f_0||_{L^1} < 1/4 \) and that \( f \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \). Then \( f(t, x, v) \) converges in \( L^1(\mathbb{R}^3 \times \mathbb{R}^3) \) to \( f_\infty(x - vt, v) \) as \( t \to 0 \).

**Proof.** By Theorem 3.2 we know that \( \frac{d\mathfrak{D}[f](t)}{dt} + (1 - 4||f_0||_{L^1})\mathfrak{F}[f](t) \leq 0 \) where \( \mathfrak{D}[f](t) \) and \( \mathfrak{F}[f](t) \) are the same as in Lemma 3.3. It follows that
\[ \mathfrak{D}[f](t) + (1 - 4||f_0||_{L^1}) \int_0^t \mathfrak{F}[f](s) ds \leq \mathfrak{D}[f](0) \]
for any \( t \in \mathbb{R}_+ \), which implies that \( \int_{\mathbb{R}^3 \times \mathbb{R}^3 } Q^\pm (f)(t, x, v) dxdvdt < +\infty \). Hence it can be easily found that

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3 } |f^\#(t, x, v) - f_\infty(x, v)| dxdv \leq \int_{t}^{+\infty} \int_{\mathbb{R}^3 \times \mathbb{R}^3 } [Q^+(f)(s, x, v) + Q^-(f)(s, x, v)] dxdvds \to 0
\]
as \( t \to +\infty \). Thus \( f(t, x, v) \to f_\infty(x - vt, v) \) in \( L^1 \) as \( t \to +\infty \). This completes our proof of Theorem 3.3. \( \square \)

In Theorem 3.3, the time-asymptotic behaviour of any solution to the Enskog equation is in fact the time-asymptotic convergence of this solution in the \( L^1 \) norm to the free motion as \( t \) trends to infinity. The time-asymptotic convergence in the \( L^\infty \) norm is shown by Polewczak [9].

4. \( L^1 \) Stability

In this section some new functionals are constructed for the \( L^1 \) stability of global classical solutions to the Enskog equation \( (1.1) \) with \( (1.2) \) and \( (1.3) \). One of them is a Lyapunov functional and it is equivalent to the \( L^1 \) distance functional. The time-decay property of the Lyapunov functional is also shown for the \( L^1 \) stability.

Let us begin by constructing two functionals \( \mathcal{L}[f, g](t) \) and \( \mathcal{F}_d[f, g](t) \) as follows. \( \mathcal{L}[f, g](t) \) is denoted by

\[
\mathcal{L}[f, g](t) = \{1 + k_1(\mathcal{D}[f](t) + \mathcal{D}[g](t))\} \mathcal{L}_d[f, g](t) + k_2 \mathcal{D}_d[f, g](t)
\]

where \( k_1 \) and \( k_2 \) are positive constants to be determined later, \( \mathcal{D} \) is the same as given by \( 3.1 \), \( \mathcal{L}_d \) is denoted by

\[
\mathcal{L}_d[f, g](t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3 } |f - g|^\#(t, x, v) dxdv
\]

and \( \mathcal{D}_d[f, g](t) = \mathcal{D}_d^+[f, g](t) + \mathcal{D}_d^-[f, g](t) \) with

\[
\mathcal{D}_d^\pm[f, g](t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3 } |f - g|^\#(t, x, v) \times \left[ \int_{\mathbb{R}^3 \times \mathbb{R}^3 } (f + g)^\#(t, x + t(v - w) \mp a\omega + \tau \frac{v - w}{(v - w)\omega}, w) d\tau d\omega dw \right] dxdv.
\]

\( \mathcal{F}_d[f, g](t) \) is defined by \( \mathcal{F}_d[f, g](t) = \mathcal{F}_d^+[f, g](t) + \mathcal{F}_d^-[f, g](t) \) with

\[
\mathcal{F}_d^\pm[f, g](t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 } (v - w)\omega |f - g|^\#(t, x, v) \times (f + g)^\#(t, x + t(v - w) \mp a\omega, w) d\omega dw dxdv.
\]

This functional \( \mathcal{L} \) is here called Lyapunov functional. Then we have the following property of the equivalence between the Lyapunov and \( L^1 \) distance functionals \( \mathcal{L} \) and \( \mathcal{L}_d \).

**Lemma 4.1.** Let \( \mathcal{L} \) and \( \mathcal{L}_d \) be defined by \( 4.1 \) and \( 4.2 \) respectively. Assume that \( f = f(t, x, v) \) and \( g = g(t, x, v) \) are two nonnegative classical solutions to the Enskog equation \( (1.1) \) with \( (1.2) \) and \( (1.3) \) through initial data \( f_0 = f_0(x, v) \) and \( g_0 = g_0(x, v) \) respectively, and that \( f \) and \( g \) are elements in \( L^1(\mathbb{R}^3 \times \mathbb{R}^3) \). Then

\[
\mathcal{L}_d[f, g](t) \leq \mathcal{L}[f, g](t) \leq C_0 \mathcal{L}_d[f, g](t)
\]

for any \( t \in \mathbb{R}_+ \), where \( C_0 = 1 + 2k_1(\|f_0\|_{L^1}^2 + \|g_0\|^2_{L^1})/\alpha^2 + 2k_2(\|f_0\|_{L^1} + \|g_0\|_{L^1}) \).
Proof. The definition \((4.1)\) of \(\mathcal{L}\) can be rewritten as

\[
\mathcal{L}[f, g](t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} |f - g|^2(t, x, v) W^*(t, x, v) dx dv
\]

where \(W(t, x, v)\) is defined by

\[
W(t, x, v) = 1 + k_1[D[f](t) + D[g](t)]
\]

\[
+k_2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} (f + g)(t, x, v, \omega) d\omega dv
\]

\[
+k_2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} (f + g)(t, x, v, \omega) d\omega dv
\]

and by Lemma \(3.1\) it can be found that \(W(t, x, v)\) is bounded by the \(L^1\) norms of \(f_0\) and \(g_0\) as follows:

\[
1 \leq W(t, x, v) \leq 1 + 2k_1(||f_0||_{L^1}^2 + ||g_0||_{L^1}^2) + 2k_2(||f_0||_{L^1} + ||g_0||_{L^1}).
\]

Our proof of this lemma hence ends up. \(\square\)

For any solutions \(f = f(t, x, v)\) and \(g = g(t, x, v)\) to the Enskog equation \((1.1)\) with \((1.2)\) and \((1.3)\), the time-decay properties of the two functionals \(D[f, g](t)\) and \(L_d[f, g](t)\) can be also obtained as follows.

Lemma 4.2. Let \(\mathcal{L}\) and \(\mathcal{L}_d\) be defined by \((4.1)\) and \((4.2)\) respectively. Assume that \(f = f(t, x, v)\) and \(g = g(t, x, v)\) are two nonnegative classical solutions to the Enskog equation \((1.1)\) with \((1.2)\) and \((1.3)\) through initial data \(f_0 = f_0(x, v)\) and \(g_0 = g_0(x, v)\) respectively, and that \(f\) and \(g\) are elements in \(L^1(\mathbb{R}^3 \times \mathbb{R}^3)\). Then

\[
\frac{d\mathcal{L}_d[f, g](t)}{dt} \leq a^2 \tilde{\mathcal{L}}_d[f, g](t) \quad (4.6)
\]

and

\[
\frac{dL_d[f, g](t)}{dt} + [1 - 2(||f_0||_{L^1} + ||g_0||_{L^1})] \mathcal{L}_d[f, g](t) \leq 2(\tilde{\mathcal{L}}[f](t) + \tilde{\mathcal{L}}[g](t)) \mathcal{L}_d[f, g](t) \quad (4.7)
\]

for any \(t \in \mathbb{R}_+\), where \(\tilde{\mathcal{L}}\) and \(\tilde{\mathcal{L}}_d\) are given by \((3.2)\) and \((4.4)\) respectively.

**Proof.** Let us first prove \((4.6)\). By \((2.10)\), we have the following inequalities

\[
\frac{\partial}{\partial t} [f^*(t, x, v) - g^*(t, x, v)] \leq \mathcal{R}[f, g]^*(t, x, v) \quad (4.8)
\]

and

\[
\frac{\partial}{\partial t} [g^*(t, x, v) - f^*(t, x, v)] \leq \mathcal{R}[f, g]^*(t, x, v) \quad (4.9)
\]

for any nonnegative solutions \(f\) and \(g\) to the Enskog equation, where \(\mathcal{R}\) is denoted by

\[
\mathcal{R}[f, g](t, x, v) = \frac{a^2}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (v - w)\omega ||f - g||^2(t, x, v') (f + g)(t, x, v')
\]

\[
+ |f - g|(t, x, v') (f + g)(t, x, v')
\]

\[
+ |f - g|(t, x, v) (f + g)(t, x + \omega, w) + |f - g|(t, x + \omega, v) (f + g)(t, x, v)] d\omega dv. \quad (4.10)
\]

Using \((4.8)\) and \((4.9)\), we have

\[
\frac{d\mathcal{L}_d[f, g](t)}{dt} \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mathcal{R}[f, g]^*(t, x, v) dx dv = a^2 \tilde{\mathcal{L}}_d[f, g](t).
\]
\[ \partial_t \left( (f-g)^\#(t, x, v) (f+g)^\#(t, x + (v-w)t) - \omega + \tau \frac{v-w}{(v-w)\omega}, w \right) \]
\[
\leq \partial_r \left[ (v-w)\omega (f-g)^\#(t, x, v) (f+g)^\#(t, x + (v-w)t) - \omega + \tau \frac{v-w}{(v-w)\omega}, w \right] \\
+ \Re[f, g]^\#(t, x, v) (f+g)^\#(t, x + (v-w)t) - \omega + \tau \frac{v-w}{(v-w)\omega}, w) \\
+ (f-g)^\#(Q(f) + Q(g))^\#(t, x + (v-w)t) - \omega + \tau \frac{v-w}{(v-w)\omega}, w) \] (4.11)

and
\[ \partial_t \left[ (g-f)^\#(t, x, v) (f+g)^\#(t, x + (v-w)t) - \omega + \tau \frac{v-w}{(v-w)\omega}, w \right] \]
\[
\leq \partial_r \left[ (v-w)\omega (g-f)^\#(t, x, v) (f+g)^\#(t, x + (v-w)t) - \omega + \tau \frac{v-w}{(v-w)\omega}, w \right] \\
+ \Re[f, g]^\#(t, x, v) (f+g)^\#(t, x + (v-w)t) - \omega + \tau \frac{v-w}{(v-w)\omega}, w) \\
+ (f-g)^\#(Q(f) + Q(g))^\#(t, x + (v-w)t) - \omega + \tau \frac{v-w}{(v-w)\omega}, w) \] (4.12)

for any fixed variables \((x, v, w, \omega)\). In fact, (4.11) and (4.12) result from (2.16), (3.6), (3.7), (4.8), (13) and the two identities as follows:
\[ \partial_t \left[ g^\#(t, x + (v-w)t) - \omega + \tau \frac{v-w}{(v-w)\omega}, w \right] \]
\[
= (v-w)\partial_x g^\#(t, x + (v-w)t) - \omega + \tau \frac{v-w}{(v-w)\omega}, w) \\
+ Q(g)^\#(t, x + (v-w)t) - \omega + \tau \frac{v-w}{(v-w)\omega}, w) \]

and
\[ (v-w)\partial_x g^\#(t, x + (v-w)t) - \omega + \tau \frac{v-w}{(v-w)\omega}, w) \]
\[
= (v-w)\omega \partial_r \left[ g^\#(t, x + (v-w)t) - \omega + \tau \frac{v-w}{(v-w)\omega}, w) \right]. \]

Denote \( \mathcal{J}_i^+[f, g](t) \) \((i = 1, 2)\) by
\[ \mathcal{J}_1^+[f, g](t) = \int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2_+ \times \mathbb{R}^+} \Re[f, g]^\#(t, x, v) \]
\[
\times (f+g)^\#(t, x + (v-w)t) \pm \omega + \tau \frac{v-w}{(v-w)\omega}, w) d\tau d\omega dw dv dx \] (4.13)

and
\[ \mathcal{J}_2^+[f, g](t) = \int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2_+ \times \mathbb{R}^+} |f-g|^\#(t, x, v) \]
\[
\times (Q(f) + Q(g))^\#(t, x + (v-w)t) \pm \omega + \tau \frac{v-w}{(v-w)\omega}, w) d\tau d\omega dw dv dx. \] (4.14)
Then, using (4.11) and (4.12), we can obtain the following inequality
\[
\frac{d\mathcal{J}_d^+[f,g](t)}{dt} \leq -\mathcal{F}_d[f,g](t) + \mathcal{J}_d^+[f,g](t) + \mathcal{J}_d^-[f,g](t).
\] (4.15)

Similarly calculating \( \frac{d\mathcal{J}_d^-[f,g](t)}{dt} \), we also have
\[
\frac{d\mathcal{J}_d^-[f,g](t)}{dt} \leq -\mathcal{F}_d[f,g](t) + \mathcal{J}_d^+[f,g](t) + \mathcal{J}_d^-[f,g](t).
\] (4.16)

Combining (4.15) and (4.16), we get
\[
\frac{d\mathcal{J}_d[f,g](t)}{dt} \leq -\mathcal{F}_d[f,g](t) + \mathcal{J}_d^+[f,g](t) + \mathcal{J}_d^-[f,g](t) + \mathcal{J}_d^0[f,g](t).
\] (4.17)

By (4.13) and (4.14), the estimates of \( \mathcal{J}_d^+[f,g](t) + \mathcal{J}_d^-[f,g](t) \) can be obtained as follows:
\[
\mathcal{J}_d^+[f,g](t) + \mathcal{J}_d^-[f,g](t) \leq 2(||f_0||_{L^1} + ||g_0||_{L^1})\mathcal{F}_d[f,g](t)
\] (4.18)
and
\[
\mathcal{J}_d^0[f,g](t) \leq 2(\mathcal{F}[f](t) + \mathcal{F}[g](t))\mathcal{L}_d[f,g](t).
\] (4.19)

By applying (4.18) and (4.19) into the estimation of the right side of (4.17), (4.17) follows. \( \square \)

It can be further shown that the functional \( \mathcal{L} \) has the following time-decay property.

**Theorem 4.3.** Let \( \mathcal{L} \) and \( \mathcal{F}_d \) be defined by (4.1) and (4.4) respectively. Assume that \( f = f(t,x,v) \) and \( g = g(t,x,v) \) are two nonnegative classical solutions to the Enskog equation \( \text{(1.1)} \) with \( \text{(1.2)} \) and \( \text{(1.3)} \) through initial data \( f_0 = f_0(x,v) \) and \( g_0 = g_0(x,v) \) satisfying \( ||f_0||_{L^1} < (2 - \sqrt{2})/4 \) and \( ||g_0||_{L^1} < (2 - \sqrt{2})/4 \) respectively, and that \( f \) and \( g \) are elements in \( L^1(\mathbb{R}^3 \times \mathbb{R}^3) \). Then there exists a positive constant \( \alpha \in (0,2\sqrt{2}) \) such that
\[
\frac{d\mathcal{L}[f,g](t)}{dt} \leq a^2 \left[ \frac{\alpha^2 - 8}{4(2 + \alpha)} \right] k_1 \mathcal{F}_d[f,g](t)
\] (4.20)
with \( k_1 = (2 + \alpha)k_2 \) and \( k_2 = ka^2 \) for any \( t \in \mathbb{R}_+ \), where \( k > \frac{4(2+\alpha)}{8-a^2} \). Furthermore, the Lyapunov type estimate holds as follows:
\[
\mathcal{L}[f,g](t) + C \int_0^t \mathcal{F}_d[f,g](s)ds \leq \mathcal{L}[f,g](0)
\] (4.21)
for any \( t \in \mathbb{R}_+ \), where \( C \) is a positive constant independent of \( t \).

**Proof.** By the definition (4.1) of the functional \( \mathcal{L} \), using Theorem 3.3 and Lemma 4.2 we have
\[
\frac{d\mathcal{L}[f,g](t)}{dt} = [1 + k_1(\mathcal{D}[f](t) + \mathcal{D}[g](t))] \frac{d\mathcal{F}_d[f,g](t)}{dt}
\]
\[
+ k_1 \left( \frac{d\mathcal{F}[f](t)}{dt} + \frac{d\mathcal{F}[g](t)}{dt} \right) \mathcal{L}_d[f,g](t) + k_2 \frac{d\mathcal{F}_d[f,g](t)}{dt}
\]
\[
\leq \left[ a^2 k_1(\mathcal{D}[f](t) + \mathcal{D}[g](t)) + 2k_2(||f_0||_{L^1} + ||g_0||_{L^1}) - k_2 + a^2 \right] \mathcal{F}_d[f,g](t)
\]
\[
+ [4k_1||f_0||_{L^1} - k_1 + 2k_2] \mathcal{F}[f](t)\mathcal{L}_d[f,g](t) + [4k_1||g_0||_{L^1} - k_1 + 2k_2] \mathcal{F}[g](t)\mathcal{L}_d[f,g](t).
\]
Since \( \mathcal{D}[f](t) \leq \frac{2}{a^2} ||f_0||_{L^1}^2 \) and \( \mathcal{D}[g](t) \leq \frac{2}{a^2} ||g_0||_{L^1}^2 \), it follows that
\[
\frac{d\mathcal{L}[f,g](t)}{dt} \leq \left[ 2k_1(||f_0||_{L^1} + ||g_0||_{L^1}) + 2k_2(||f_0||_{L^1} + ||g_0||_{L^1}) - k_2 + a^2 \right] \mathcal{F}_d[f,g](t)
\]
\[
+ [4k_1||f_0||_{L^1} - k_1 + 2k_2] \mathcal{F}[f](t)\mathcal{L}_d[f,g](t) + [4k_1||g_0||_{L^1} - k_1 + 2k_2] \mathcal{F}[g](t)\mathcal{L}_d[f,g](t).
\]
Notice that \( ||f_0||_{L^1} < (2 - \sqrt{2})/4 \) and \( ||g_0||_{L^1} < (2 - \sqrt{2})/4 \). There exists a positive constant \( \alpha \in (0, 2\sqrt{2}) \) such that \( ||f_0||_{L^1} \leq \frac{\alpha}{4(2+\alpha)} \) and \( ||g_0||_{L^1} \leq \frac{\alpha}{4(2+\alpha)} \). Let us choose \( k_1 = (2 + \alpha)k_2 \) and \( k_2 = ka^2 \) where \( k > \frac{4(2+\alpha)}{8\alpha^2} \). By estimating the right side of the above inequality, it can be found that

\[
\frac{d\mathcal{L}[f,g](t)}{dt} \leq a^2 \left[ \frac{\alpha^2 - 8}{4(2 + \alpha)}k + 1 \right] \mathcal{S}_a[f,g](t).
\]

(4.20) and (4.21) then follows. The proof of Theorem 4.3 is hence finished. □

By Theorem 4.3, we further have

**Theorem 4.4.** Assume that \( f = f(t,x,v) \) and \( g = g(t,x,v) \) are two nonnegative classical solutions to the Enskog equation (1.1) with (1.2) and (1.3) through initial data \( f_0 = f_0(x,v) \) and \( g_0 = g_0(x,v) \) satisfying \( ||f_0||_{L^1} < (2 - \sqrt{2})/4 \) and \( ||g_0||_{L^1} < (2 - \sqrt{2})/4 \) respectively, and that \( f \) and \( g \) are elements in \( L^1(\mathbb{R}^3 \times \mathbb{R}^3) \). Then

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} |f - g|(t,x,v)dxdv \leq C \int_{\mathbb{R}^3 \times \mathbb{R}^3} |f_0 - g_0|(x,v)dxdv
\]

(4.22)

for any \( t \in \mathbb{R}_+ \), where \( C \) is a positive constant independent of \( t \).

**Proof.** By Theorem 4.3 there exists a positive constant \( \alpha \in (0, 2\sqrt{2}) \) such that \( \mathcal{L}[f,g](t) \leq \mathcal{L}[f,g](0) \) with \( k_1 = (2 + \alpha)k_2 \) and \( k_2 = ka^2 \), where \( k > \frac{4(2+\alpha)}{8\alpha^2} \). By using Lemma 4.1 it follows that

\[
\mathcal{L}_a[f,g](t) \leq \mathcal{L}[f,g](t) \leq \mathcal{L}[f,g](0) \leq C_0 \mathcal{L}_a[f,g](0).
\]

This hence completes the proof of Theorem 4.4. □

Theorem 4.4 can be here regarded as an application of the Lyapunov functional \( \mathcal{L} \) into recovering a result of Cercignani 3 about the \( L^1 \) stability of classical solutions to the Enskog equation under the assumption of the suitably small initial data.

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Proof. Let \( S \) be an integrable function on \( \mathbb{R}^3 \) and \( n \) a unit vector in \( \mathbb{R}^3 \). Assume that \( S^2 = \{ \omega : n\omega \geq 0, \omega \in S^2 \} \) where \( S^2 \) is a unit sphere surface in \( \mathbb{R}^3 \). Then

\[
\int_{\mathbb{R}^3} F(x)dx = a^2 \int_{\mathbb{R}^3} F(aw + \tau n)n\omega d\tau d\omega \tag{A.2}
\]

where \( a \neq 0 \).

Proof. Without loss of generality, we assume that \( a > 0 \). Let \( \omega = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \), where \( \theta \) is an angle between \( v \) and \( \omega \), \( 0 \leq \theta \leq \frac{\pi}{2} \), \( 0 \leq \varphi \leq 2\pi \). Denote \( n \) and \( x \) by \( n = (n_1, n_2, n_3) \) and \( x = (x_1, x_2, x_3) \) respectively. Take a transformation \( x = aw + \tau n \), i.e.,

\[
x_1 = \tau n_1 + a\sin \theta \cos \varphi, \quad x_2 = \tau n_2 + a\sin \theta \sin \varphi, \quad x_3 = \tau n_3 + a\cos \theta.
\]

By direct calculation, it then follows that the Jacobian determinant of this transformation is

\[
\frac{\partial(x_1, x_2, x_3)}{\partial(\tau, \theta, \varphi)} = a^2n\omega \sin \theta. \tag{A.3}
\]

hence holds. Our proof of this lemma is thus finished. \( \square \)

Lemma A.1 not only implies that \( F(x) \) must be integrable in \( \mathbb{R}^3 \) if \( F(aw + \tau n) \) is integrable in \( \mathbb{R} \times S^2_+ \), but also indicates that \( F(aw + \tau n) \) might not be integrable in \( \mathbb{R} \times S^2_+ \) even if \( F(x) \) is integrable in \( \mathbb{R}^3 \) since it can be known from A.3 that \( n\omega = (xn - \tau)/a \) and \( (\tau - xn)^2 = (xn)^2 + a^2 - x^2 \). For example, let us choose \( n = (1, 0, 0) \) and \( F(x) = \frac{F(x)}{\sqrt{x_3^2 + x_3^2 - a^2}} \), where \( F(x) \in C_0^\infty(\Omega) \), \( F(x) = 1 \) as \( x \in B_{2a} \), \( 0 \leq \tilde{F}(x) \leq 1 \) as \( x \in \Omega, B_{2a} \subset \Omega \). It is easy to see that \( \tilde{F}(aw + \tau n) \) is not integrable in \( \mathbb{R} \times S^2_+ \) but \( F(x) \) is integrable in \( \mathbb{R}^3 \). This is a good counterexample of inequalities (A.1) and so the two inequalities do not hold mathematically.

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