THE FROBENIUS-VIRASORO ALGEBRA AND EULER EQUATIONS

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ABSTRACT. We introduce an \( \mathfrak{F} \)-valued generalization of the Virasoro algebra, called the Frobenius-Virasoro algebra \( \text{vir}_{\mathfrak{F}} \), where \( \mathfrak{F} \) is a Frobenius algebra over \( \mathbb{R} \). We also study Euler equations on the regular dual of \( \text{vir}_{\mathfrak{F}} \), including the \( \mathfrak{F} \)-KdV equation and the \( \mathfrak{F} \)-CH equation and the \( \mathfrak{F} \)-HS equation, and discuss their Hamiltonian properties.

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1. INTRODUCTION

Let \( \mathfrak{G} \) be a Lie algebra and \( \mathfrak{G}^\ast \) (the regular part of) its dual, and let \( \langle \ , \ \rangle^\ast \) denote a natural pairing between \( \mathfrak{G} \) and \( \mathfrak{G}^\ast \).

Definition 1.1. The Euler equation on \( \mathfrak{G}^\ast \) is defined by the following system (e.g., [2, 7]):

\[
\frac{dm}{dt} = -ad_{A^{-1}}^x m,
\]

(1.1)

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as an evolution of a point \( m \in \mathfrak{g}^* \), where \( A : \mathfrak{g} \to \mathfrak{g}^* \) is an invertible self-adjoint operator, called the inertia operator.

It is well known that the KdV equation

\[
    u_t + 3uu_x + cu_{xxx} = 0
\]

and the Camassa-Holm (CH in brief) equation

\[
    m_t + 2mu_x + m_xu + cu_{xxx} = 0, \quad m = u - u_{xx}
\]

and the Hunter-Saxton (HS in brief) equation

\[
    m_t + 2mu_x + m_xu + cu_{xxx} = 0, \quad m = -u_{xx}
\]

could be regarded as Euler equations on the dual of Virasoro algebra \( \mathfrak{vir} \) with different inner products (\([7, 9, 11, 12]\)). Let us remark that V.I.Arnold in \([1]\) suggested a general framework for the Euler equation on an arbitrary Lie group \( G \), which is useful to characterize a variety of conservative dynamical systems, please see e.g., \([2, 6, 7, 8, 9, 11, 12, 15, 17, 18]\) and references therein. If the corresponding Lie algebra is \( \mathfrak{g} \), then the Euler equation (1.1) on \( \mathfrak{g}^* \) could describe a geodesic flow w.r.t a suitable one-side invariant Riemannian metric on Lie group \( G \).

In our recent works \([16, 19]\), we studied the relation between Frobenius manifolds and Frobenius algebra-valued integrable systems.

**Definition 1.2.** A Frobenius algebra \((\mathfrak{F}, g_\mathfrak{F}, 1_\mathfrak{F}, \circ)\) over \( \mathbb{R} \) is a free \( \mathbb{R} \)-module \( \mathfrak{F} \) of finite rank \( l \), equipped with a commutative and associative multiplication \( \circ \) and a unit \( 1_\mathfrak{F} \), and a \( \mathbb{R} \)-bilinear symmetric nondegenerate form \( g_\mathfrak{F} : \mathfrak{F} \times \mathfrak{F} \to \mathbb{R} \) satisfying

\[
    g_\mathfrak{F}(a \circ b, c) = g_\mathfrak{F}(a, b \circ c).
\]

Having this nondegenerate form \( g_\mathfrak{F} \) is equivalent to having a linear form \( \text{tr}_\mathfrak{F} : \mathfrak{F} \to \mathbb{R} \) whose kernel contains no trivial ideas. This linear form is often called a trace map. Indeed, given \( g_\mathfrak{F} \), we put \( \text{tr}_\mathfrak{F}(a) := g_\mathfrak{F}(a, 1_\mathfrak{F}) \). Conversely, given \( \text{tr}_\mathfrak{F} \), we could define \( g_\mathfrak{F}(a, b) := \text{tr}_\mathfrak{F}(a \circ b) \).

Observe that an \( \mathfrak{F} \)-valued KdV (\( \mathfrak{F} \)-KdV) equation

\[
    u_t + 3u \circ u_x + \zeta \circ u_{xxx} = 0, \quad \zeta \in \mathfrak{F}
\]

has been derived in \([16, 19]\), where \( u \) is a smooth \( \mathfrak{F} \)-valued function. A natural question is to ask:

“Could the \( \mathfrak{F} \)-KdV equation (1.2) be regarded as a Euler equation on the regular dual of an infinite-dimensional Lie algebra \( \mathfrak{g} \) ?”
Our work is inspired by this question. This paper is to give an affirmative answer and organized as follows. Firstly, we introduce an infinite dimensional Lie algebra, called the Frobenius-Virasoro algebra $\text{vir}_{\mathfrak{F}}$, which is an $\mathfrak{F}$-valued generalization of the Virasoro algebra. Afterwards, we compute Euler equations on the regular dual $\text{vir}_{\mathfrak{F}}^*$ of $\text{vir}_{\mathfrak{F}}$ for certain products, including the $\mathfrak{F}$-KdV equation, the $\mathfrak{F}$-CH equation and the $\mathfrak{F}$-HS equation. Moreover we show that all resulted Euler equations for the inner product $P_{\alpha,\beta}$ are local bihamiltonian. Let us remark that in order to define the Euler equation on $\text{vir}_{\mathfrak{F}}^*$, it is enough to require a commutative and associative algebra $(\mathfrak{F}, 1_{\mathfrak{F}}, \circ)$. In other words, we don’t require the existence of trace map $\text{tr}_{\mathfrak{F}}$. An interesting fact (also noted in [16]) is that if on $(\mathfrak{F}, 1_{\mathfrak{F}}, \circ)$, there are many different trace maps, then the corresponding $\mathfrak{F}$-valued Euler equation has many different (bi)hamiltonian structures. Finally we discuss some examples to illustrate our construction.

2. Euler equations on $\text{vir}_{\mathfrak{F}}^*$

Throughout this paper, we assume that the Frobenius algebra $\mathfrak{F} := (\mathfrak{F}, \text{tr}_{\mathfrak{F}}, 1_{\mathfrak{F}}, \circ)$ has the basis $e_1 = 1_{\mathfrak{F}}, e_2, \cdots, e_l$.

2.1. The Frobenius-Virasoro algebra $\text{vir}_{\mathfrak{F}}$. We begin with some definitions.

Definition 2.1. We define an infinite-dimensional Lie algebra $(\mathfrak{X}, [\ , \ ])$ over $\mathbb{R}$ by

$$\mathfrak{X} := \{ u(x)\frac{d}{dx} | u \in C^\infty (\mathbb{S}^1, \mathfrak{F}) \} , \quad [u\partial, v\partial] := (u \circ v_x - u_x \circ v)\partial, \quad \partial = \frac{d}{dx}.$$  \hfill (2.1)

We remark that $\mathfrak{X}$ is different from the loop algebra $L\mathfrak{F}$ of $\mathfrak{F}$. As vector spaces, they are isomorphic under the map

$$\Psi : L\mathfrak{F} \to \mathfrak{X}, \quad \Psi(u) = u\partial.$$  

But as Lie algebras, $\Psi$ is not a Lie algebra homomorphism.

Lemma 2.2. The map $\omega_{\mathfrak{F}} : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{F}$ defined by

$$\omega_{\mathfrak{F}}(u\partial, v\partial) = \int_{\mathbb{S}^1} u \circ v_{xxx} dx$$  \hfill (2.2)

is a nontrivial 2-cocycle on $\mathfrak{X}$, called the $\mathfrak{F}$-valued Gelfand-Fuchs cocycle.

Proof. Observe that the Frobenius algebra $\mathfrak{F}$ is commutative and associative, then we have

(i). $\omega_{\mathfrak{F}}(u\partial, v\partial) = -\omega_{\mathfrak{F}}(v\partial, u\partial); \quad$ (ii). $\omega_{\mathfrak{F}}(u\partial, [v\partial, w\partial]) + c.p. = 0,$

which follow the desired result.  \hfill \square
Definition 2.3. The central extension of \( \mathfrak{X} \) is called the Frobenius-Virasoro algebra, denoted by \( \mathfrak{vir}_{\mathfrak{F}} \) with the Lie bracket
\[
[(u\partial, a), (v\partial, b)] := (u\partial(v\partial), \omega_{\mathfrak{F}}(u\partial, v\partial)).
\] (2.3)

When one chooses the Frobenius algebra \( \mathfrak{F} \) to be \( \mathbb{R} \), \( \mathfrak{vir}_{\mathfrak{F}} \) is exactly the Virasoro algebra and \( \mathfrak{X} = \text{Vect}(S^1) \). It is well known (e.g. [8, 13]) that the second continuous cohomology group \( H^2(\text{Vect}(S^1), \mathbb{R}) \cong \mathbb{R} \) is generated by the Gefland-Fuchs cocycle. Generally when \( \dim \mathfrak{F} > 1 \), \( H^2(\mathfrak{X}, \mathfrak{F}) \) is not generated by the \( \mathfrak{F} \)-valued Gelfand-Fuchs cocycle \( \omega_{\mathfrak{F}} \). An interesting problem is to compute \( H^2(\mathfrak{X}, \mathfrak{F}) \).

2.2. Euler equations on \( \mathfrak{vir}_{\mathfrak{F}}^\ast \). We denote the regular dual of the Frobenius-Virasoro algebra \( \mathfrak{vir}_{\mathfrak{F}} \) by \( \mathfrak{vir}_{\mathfrak{F}}^\ast = \{ (m(x, t)(dx)^2, \zeta(t)) | m(x, t) \text{ and } \zeta(t) \text{ are smooth } \mathfrak{F}-\text{valued functions} \} \) with respect to the paring
\[
\langle (mdx^2, \zeta), (u\partial, a) \rangle^\ast = \text{tr}_{\mathfrak{F}} \int_{S^1} m \circ ud\xi + \text{tr}_{\mathfrak{F}} (\zeta \circ a).
\] (2.4)

Write \( \hat{m} = (mdx^2, \zeta) \in \mathfrak{vir}_{\mathfrak{F}}^\ast \) and \( \hat{u} = (u\partial, a) \), \( \hat{v} = (v\partial, b) \in \mathfrak{vir}_{\mathfrak{F}}^\ast \). By the definition,
\[
\langle \text{ad}^\ast_{\hat{u}}(\hat{m}), \hat{v} \rangle^\ast = -\langle \hat{m}, [\hat{u}, \hat{v}] \rangle^\ast = \text{tr}_{\mathfrak{F}} \int_{S^1} (2m \circ u_x + m_x \circ u + \zeta \circ u_{xxx}) \circ v d\xi
\]
which yields that the coadjoint action of \( \mathfrak{vir}_{\mathfrak{F}} \) on \( \mathfrak{vir}_{\mathfrak{F}}^\ast \) is given by
\[
\text{ad}^\ast_{\hat{u}} \hat{m} = \left( (2m \circ u_x + m_x \circ u + \zeta \circ u_{xxx})(dx)^2, 0 \right).
\] (2.5)

On \( \mathfrak{vir}_{\mathfrak{F}} \), we introduce a two-parameter family of inner product \( P_{\alpha, \beta}, \alpha, \beta \in \mathfrak{F} \) defined by
\[
\langle \hat{u}, \hat{v} \rangle = \text{tr}_{\mathfrak{F}} \int_{S^1} (a \circ u_x + \beta \circ u_x \circ v_x) d\xi + \text{tr}_{\mathfrak{F}} (a \circ b).
\] (2.6)

Observe that for the \( P_{\alpha, \beta} \), the inertia operator \( \mathcal{A} : \mathfrak{vir}_{\mathfrak{F}} \rightarrow \mathfrak{vir}_{\mathfrak{F}}^\ast \) is defined by \( \langle \hat{u}, \hat{v} \rangle = \langle \mathcal{A}(\hat{u}), \hat{v} \rangle^\ast \). In other words, \( \mathcal{A}(\hat{a}) = (\Lambda(u), a) \), where \( \Lambda = \alpha - \beta \partial^2 \) is an \( \mathfrak{F} \)-valued differential operator. So we have

Proposition 2.4. The Euler equation (1.1) on \( \mathfrak{vir}_{\mathfrak{F}}^\ast \) for \( P_{\alpha, \beta} \) reads
\[
m_t + 2m \circ u_x + m_x \circ u + \zeta \circ u_{xxx} = 0, \quad \zeta_t = 0,
\] (2.7)
where \( m = \Lambda(u) = \alpha \circ u - \beta \circ u_{xx} \).
When $\zeta = 0$, the system (2.7) could be regarded as the Euler equation on $\mathfrak{X}^*$. When the Frobenius algebra $\mathfrak{F}$ is one-dimensional, i.e., $\mathbb{R}$, the system (2.7) is the Euler equation on $\text{vir}^*$ (e.g., [7]). Generally, when $\alpha \neq 0$, $\beta = 0$ and $0 \neq \zeta \in \mathfrak{F}$, the system (2.7) reads the $\mathfrak{F}$-KdV equation
\begin{equation}
\alpha \circ u_t + 3\alpha \circ u_x + \zeta \circ u_{xxx} = 0.
\tag{2.8}
\end{equation}
When $\alpha \neq 0$, $\beta \neq 0$ and $\zeta \in \mathfrak{F}$, the system (2.7) becomes the $\mathfrak{F}$-CH equation
\begin{equation}
m_t + 2m \circ u_x + m_x \circ u + \zeta \circ u_{xxx} = 0, \quad m = \alpha \circ u - \beta \circ u_{xx}.
\tag{2.9}
\end{equation}
When $\alpha = 0$, $\beta \neq 0$ and $\zeta \in \mathfrak{F}$, the system (2.7) reduces to the $\mathfrak{F}$-HS equation
\begin{equation}
\beta \circ (u_{xx} + 2u_{xx} \circ u_x + u_{xxx} \circ u) - \zeta \circ u_{xxx} = 0, \quad m = -\beta \circ u_{xx}.
\tag{2.10}
\end{equation}

Example 2.5. Let $\mathcal{Z}_2^\varepsilon$ be a 2-dimensional commutative and associative algebra over $\mathbb{R}$ with the basis $e_1 = 1_\mathfrak{F}, e_2$ satisfying
\begin{align*}
e_1 \circ e_1 &= e_1, \quad e_1 \circ e_2 = e_2, \quad e_2 \circ e_2 = \varepsilon e_1, \quad \varepsilon \in \mathbb{R}.
\end{align*}
Thus for any $A \in \mathcal{Z}_2^\varepsilon$, we could write $A = a_1 e_1 + a_2 e_2$, $a_k \in \mathbb{R}$ and define two “basic” trace-type maps as follows
\begin{equation}
\text{tr}_2 \circ(A) = a_k + a_2 (1 - \delta_{k,2}) \delta \varepsilon, \quad k = 1, 2.
\tag{2.11}
\end{equation}
So $(\mathcal{Z}_2^\varepsilon, \text{tr}_2 \circ, 1_\mathfrak{F}, \circ)$ for $k = 1, 2$ are the Frobenius algebras ([16]). The $\mathcal{Z}_2^\varepsilon$-valued Euler equation with $\zeta \in \mathcal{Z}_2^\varepsilon$ is given by
\begin{equation}
m_t + 2m \circ u_x + m_x \circ u + \zeta \circ u_{xxx} = 0, \quad m = \alpha \circ u - \beta \circ u_{xx}.
\tag{2.12}
\end{equation}
(i). When $\alpha = \zeta = 1_\mathfrak{F}$ and $\beta = 0$, the system (2.12) reduces to the $\mathcal{Z}_2^\varepsilon$-KdV equation ([16] [19])
\begin{equation}
u_t + 3\nu \circ \nu_x + \nu_{xxx} + 3\varepsilon \nu \circ \nu_x = 0, \quad \nu = ve_1 + we_2
\tag{2.13}
\end{equation}
equivalently in componentwise forms,
\begin{equation}
v_t + 3vv_x + v_{xxx} + 3\varepsilon vw_x = 0, \quad w_t + 3(vw)_x + w_{xxx} = 0.
\tag{2.13}
\end{equation}
When $\varepsilon = 0$, the system (2.13) is the coupled KdV equation in [3] [15]. When $\varepsilon = -1$, the system (2.13) is a complexification of the KdV equation.

(ii). When $\alpha = \beta = 1_\mathfrak{F}$ and $\zeta = 0$, the system (2.12) reduces to the $\mathcal{Z}_2^\varepsilon$-CH equation
\begin{equation}
m_t + 2m \circ u_x + m_x \circ u = 0, \quad m = u - u_{xx}, \quad u = ve_1 + we_2,
\tag{2.12}
\end{equation}
equivalently in componentwise forms,

\[ p_t + 2pv_x + pxv + \varepsilon (2qw_x + qxw) = 0, \quad p = v - v_{xx}, \quad (2.14) \]
\[ q_t + 2qv_x + qxv + 2pw_x + pxw = 0, \quad q = w - w_{xx}. \]

When \( \varepsilon = -1 \), the system (2.14) is the complex-CH equation (e.g., [14]).

(iii). When \( \alpha = 0 \) and \( \beta = \zeta = 1_3 \), the system (2.12) reduces to the \( \mathbb{Z}_2^\varepsilon \)-HS equation

\[ m_t + 2m \circ u_x + m_x \circ u = 0, \quad m = -u_{xx}, \quad u = ve_1 + we_2, \]
equivalently in componentwise forms,

\[ p_t + 2pv_x + pxv + \varepsilon (2qw_x + qxw) = 0, \quad p = -v_{xx}, \quad (2.15) \]
\[ q_t + 2qv_x + qxv + 2pw_x + pxw = 0, \quad q = -w_{xx}. \]

2.3. Hamiltonian structures of the Euler equation (2.7).

Let us take two arbitrary smooth functionals

\( \tilde{F}_i : \text{vir}_{\mathbb{S}}^* \rightarrow \mathbb{R}, \quad \tilde{F}_i(\hat{m}) = \int_{S^1} \text{tr}_\mathbb{S} F_i(m) dx = \int_{S^1} f_i(m_1, \ldots, m_l) dx, \quad i = 1, 2, \)

where \( m = \sum_{k=1}^l m_k e_k \). The variational derivative \( \frac{\delta \tilde{F}_i}{\delta m} \) is defined as

\[ \delta \tilde{F}_i = \left( \frac{\delta F_i}{\delta m} \partial, 0 \right) \in \text{vir}_{\mathbb{S}}, \quad (2.16) \]

where \( \frac{\delta F_i}{\delta m} \) is implicitly determined by

\[ \tilde{F}_i(m + \delta m) - \tilde{F}_i(m) = \int_{S^1} \text{tr}_\mathbb{S} \left( \delta F_i \circ \delta m + o(\delta m) \right) dx \]
\[ = \int_{S^1} \left( \sum_{k=1}^l \frac{\delta f_i}{\delta m_k} \delta m_k + o(\delta m) \right) dx \quad (2.17) \]

and \( \frac{\delta f_i}{\delta m_k} \) is the usual variational derivative. This formula (2.17) is very crucial to construct the bihamiltonian representation of the Euler equation. On \( \text{vir}_{\mathbb{S}}^* \), there is a canonical Lie-Poisson bracket

\[ \mathcal{P}_2 := \{ \tilde{F}_1, \tilde{F}_2 \}(\hat{m}) = \langle m, \left[ \frac{\delta \tilde{F}_1}{\delta m}, \frac{\delta \tilde{F}_2}{\delta m} \right] \rangle^* = \text{tr}_\mathbb{S} \int_{S^1} \frac{\delta F_1}{\delta m} \circ J_2 \circ \frac{\delta F_2}{\delta m} dx \quad (2.18) \]
where $J_2 = -(m \partial + \partial m + \zeta \partial^3)$ and $\hat{m} = (mdx^2, \zeta) \in \text{vir}_\mathfrak{g}^*$. Taking a fixed point $\hat{m}_0 = (\frac{\alpha}{2} dx^2, -\beta)$, we get another compatible Poisson bracket denoted by

$$P_1 = \{\bar{F}_1, \bar{F}_2\}_1(\hat{m}) := \text{tr}_3 \int_{S^1} \frac{\delta F_1}{\delta \hat{m}} \circ J_1 \circ \frac{\delta F_2}{\delta \hat{m}} dx,$$

i.e., $P_1 = P_2|\hat{m} = \hat{m}_0$, (2.19)

where $J_1 := J_2|\hat{m} = \hat{m}_0 = \beta \partial^3 - \alpha \partial = -\partial \Lambda$.

**Theorem 2.6.** The $\mathfrak{g}$-valued Euler equation (2.7) with $\zeta \in \mathfrak{g}$ is local bihamiltonian with the freezing point $\hat{m}_0 = (\frac{\alpha}{2} dx^2, -\beta) \in \text{vir}_\mathfrak{g}^*$.

**Proof.** Setting

$$H_1 = \frac{1}{2} \text{tr}_3 \int_{S^1} m \circ udx, \quad H_2 = \frac{1}{2} \text{tr}_3 \int_{S^1} \left( \zeta \circ u \circ u_{xx} + \alpha \circ u^3 - \frac{1}{2} \beta \circ u^2 \circ u_{xx} \right) dx.$$

With the formula (2.16), we get

$$\frac{\delta H_1}{\delta u} = \Lambda(u), \quad \frac{\delta H_2}{\delta u} = \zeta \circ u_{xx} + 3 \alpha \circ u^2 - \frac{1}{2} \beta \circ u_x - \beta \circ u \circ u_{xx}.$$

By using $m = \Lambda(u)$, then

$$\frac{\delta H_1}{\delta m} = \Lambda^{-1} \circ \frac{\delta H_1}{\delta u} = u, \quad \frac{\delta H_2}{\delta m} = \Lambda^{-1} \circ \frac{\delta H_2}{\delta u}.$$

So the system (2.7) could be written as

$$m_t = J_1 \circ \frac{\delta H_2}{\delta m} = J_2 \circ \frac{\delta H_1}{\delta m}. \quad (2.20)$$

Furthermore using the formula (2.17), in componentwise forms the Euler equation (2.7) has the following bihamiltonian representation

$$m_{k,t} = \{m_k, H_2\}_1 = \{m_k, H_1\}_2, \quad k = 1, \ldots, l, \quad (2.21)$$

where two compatible Poisson brackets $\{\ , \}_i, i = 1, 2$ are defined in (2.18) and (2.19) respectively.

**Remark 2.7.** Let us remark that when choose $\mathfrak{g}$ as the Frobenius algebra $(\mathcal{Z}_l, \text{tr}_l)$ ([3, 10, 19]), the Frobenius-Virasoro algebra $\text{vir}_\mathfrak{g}$ coincides with the polynomial Virasoro algebra introduced by P. Casati and G. Ortenzi in [4]. They also computed Euler equations on $\text{vir}_\mathfrak{g}^*$ and proved that they admitted a local bihamiltonian structure by using the trace-type map $\text{tr}_l$. Actually in [16], it has been shown that there are at least $l$ “basic” different ways to regard the algebra $\mathcal{Z}_l$ as the Frobenius algebra $(\mathcal{Z}_l, \omega_k)$ for $k = 0, \ldots, l - 1$. We want to mention that the trace map $\text{tr}_l$ is a linear combination
of “basic” trace maps given by \( \text{tr}_l = \sum_{k=0}^{l-1} \omega_k - (l-1) \omega_{l-1} \). Using Theorem 2.6 we thus obtain

**Corollary 2.8.** The \( \mathbb{Z}_l \)-valued Euler equation (2.7) has at least \( l \) “basic” local bihamiltonian structures.

### 2.4. Examples

According to Example 2.5, \((\mathbb{Z}_2, \text{tr}_2, \iota, \circ)\) for \( k = 1, 2 \) are the Frobenius algebras. We thus have

**Corollary 2.9.** The \( \mathbb{Z}_2 \)-valued Euler equation (2.12) has at least two kinds of “basic” local bihamiltonian structures.

Naturally, we know that the \( \mathbb{Z}_2 \)-CH equation (2.9) and the \( \mathbb{Z}_2 \)-HS equation (2.10) have at least two kinds of “basic” local bihamiltonian structures. For the \( \mathbb{Z}_2 \)-KdV equation (2.9), two kinds of “basic” local bihamiltonian structures have been obtained in [16, 19] by other methods. Based on our construction, more precisely we have

**Example 2.10.** We consider the case: \([\varepsilon \neq 0]\).

(i). The \( \mathbb{Z}_2 \)-KdV equation (2.13)

\[
v_t + 3vv_x + v_{xxx} + 3\varepsilon ww_x = 0, \quad w_t + 3(vw)_x + w_{xxx} = 0
\]

could be rewritten as

\[
\left( \begin{array}{c}
v \\
w
\end{array} \right)_t = - \left( \begin{array}{cc} 0 & \partial \\ \partial & 0 \end{array} \right) \left( \begin{array}{c} \frac{\delta H_2}{\delta v} \\ \frac{\delta H_2}{\delta w} \end{array} \right) = - \left( \begin{array}{cc} \varepsilon J_1 & J_0 \\ J_0 & J_1 \end{array} \right) \left( \begin{array}{c} \frac{\delta H_1}{\delta v} \\ \frac{\delta H_1}{\delta w} \end{array} \right)
\]

with Hamiltonians

\[
H_1 = \int_{\mathbb{S}^1} vwdx, \quad H_2 = \frac{1}{2} \int_{\mathbb{S}^1} (3v^2w + \varepsilon w^3 + 2vw_{xx})dx
\]

and

\[
\left( \begin{array}{c}
v \\
w
\end{array} \right)_t = - \left( \begin{array}{cc} \partial & 0 \\ 0 & \frac{1}{\varepsilon} \partial \end{array} \right) \left( \begin{array}{c} \frac{\delta \tilde{H}_2}{\delta v} \\ \frac{\delta \tilde{H}_2}{\delta w} \end{array} \right) = - \left( \begin{array}{cc} J_0 & J_1 \\ J_1 & \frac{1}{\varepsilon} J_0 \end{array} \right) \left( \begin{array}{c} \frac{\delta \tilde{H}_1}{\delta v} \\ \frac{\delta \tilde{H}_1}{\delta w} \end{array} \right)
\]

with Hamiltonians

\[
\tilde{H}_1 = \frac{1}{2} \int_{\mathbb{S}^1} (v^2 + \varepsilon w^2)dx, \quad \tilde{H}_2 = \frac{1}{2} \int_{\mathbb{S}^1} (v^3 + vv_{xx} + 3\varepsilon vw^2 + \varepsilon w_{xxx})dx
\]

where \( J_0 = \partial^3 + v\partial + \partial v \) and \( J_1 = w\partial + \partial w \).
(ii). The $\mathbb{Z}_2$-CH equation (2.14)

\[
p_t + 2pv_x + px_v + \varepsilon(2qw_x + qx_w) = 0, \quad p = v - v_{xx},
\]

\[
q_t + 2qv_x + qx_v + 2pw_x + px_w = 0, \quad q = w - w_{xx},
\]
could be rewritten as

\[
\begin{pmatrix}
  p \\
  q
\end{pmatrix}_t = \begin{pmatrix}
  0 & \partial^3 - \partial \\
  \partial^3 - \partial & 0
\end{pmatrix} \begin{pmatrix}
  \frac{\delta H_2}{\partial p} \\
  \frac{\delta H_2}{\partial q}
\end{pmatrix} = - \begin{pmatrix}
  \varepsilon K_1 & K_0 \\
  K_0 & K_1
\end{pmatrix} \begin{pmatrix}
  \frac{\delta H_1}{\partial p} \\
  \frac{\delta H_1}{\partial q}
\end{pmatrix}
\]

with Hamiltonians

\[
H_1 = \frac{1}{2} \int_{\mathbb{S}^1} (qv + pw) dx, \quad H_2 = \frac{1}{4} \int_{\mathbb{S}^1} (2vw_{xx} + 2wv_{xx} - 2wvw_{xx} - \varepsilon w^2_{xx}) dx,
\]

and

\[
\begin{pmatrix}
  p \\
  q
\end{pmatrix}_t = \begin{pmatrix}
  \partial^3 - \partial & 0 \\
  0 & 1/\varepsilon (\partial^3 - \partial)
\end{pmatrix} \begin{pmatrix}
  \frac{\delta H_2}{\partial p} \\
  \frac{\delta H_2}{\partial q}
\end{pmatrix} = - \begin{pmatrix}
  K_0 & K_1 \\
  1/\varepsilon K_0 & K_1
\end{pmatrix} \begin{pmatrix}
  \frac{\delta H_1}{\partial p} \\
  \frac{\delta H_1}{\partial q}
\end{pmatrix}
\]

with Hamiltonians

\[
\tilde{H}_1 = \frac{1}{2} \int_{\mathbb{S}^1} (pv + \varepsilon qw) dx, \quad \tilde{H}_2 = \frac{1}{4} \int_{\mathbb{S}^1} (2vw + \varepsilon pq + \varepsilon w^2 q) dx,
\]

where $K_0 = p\partial + \partial p$ and $K_1 = q\partial + \partial q$.

(iii). The $\mathbb{Z}_2$-HS equation (2.15)

\[
p_t + 2pv_x + px_v + \varepsilon(2qw_x + qx_w) = 0, \quad p = -v_{xx},
\]

\[
q_t + 2qv_x + qx_v + 2pw_x + px_w = 0, \quad q = -w_{xx},
\]
could be rewritten as

\[
\begin{pmatrix}
  p \\
  q
\end{pmatrix}_t = \begin{pmatrix}
  \partial^3 & 0 \\
  0 & \partial^3
\end{pmatrix} \begin{pmatrix}
  \frac{\delta H_2}{\partial p} \\
  \frac{\delta H_2}{\partial q}
\end{pmatrix} = - \begin{pmatrix}
  \varepsilon K_1 & K_0 \\
  K_0 & K_1
\end{pmatrix} \begin{pmatrix}
  \frac{\delta H_1}{\partial p} \\
  \frac{\delta H_1}{\partial q}
\end{pmatrix}
\]

with Hamiltonians

\[
H_1 = \frac{1}{2} \int_{\mathbb{S}^1} (qv + pw) dx, \quad H_2 = \frac{1}{4} \int_{\mathbb{S}^1} (2wvp + v^2 q + \varepsilon w^2 q) dx,
\]

and

\[
\begin{pmatrix}
  p \\
  q
\end{pmatrix}_t = \begin{pmatrix}
  0 & 1/\varepsilon \partial^3 \\
  \partial^3 & 0
\end{pmatrix} \begin{pmatrix}
  \frac{\delta H_2}{\partial p} \\
  \frac{\delta H_2}{\partial q}
\end{pmatrix} = - \begin{pmatrix}
  K_0 & K_1 \\
  1/\varepsilon K_0 & K_1
\end{pmatrix} \begin{pmatrix}
  \frac{\delta H_1}{\partial p} \\
  \frac{\delta H_1}{\partial q}
\end{pmatrix}
\]

with Hamiltonians

\[
\tilde{H}_1 = \frac{1}{2} \int_{\mathbb{S}^1} (pv + \varepsilon qw) dx, \quad \tilde{H}_2 = \frac{1}{4} \int_{\mathbb{S}^1} (pv^2 + \varepsilon pw^2 + 2\varepsilon vwpq) dx,
\]

where $K_0 = p\partial + \partial p$ and $K_1 = q\partial + \partial q$. 
Example 2.11. We consider another case: \( \varepsilon = 0 \).

(i) The \( \mathbb{Z}_2^0 \)-KdV equation \((2.13)\)

\[
v_t + 3vv_x + v_{xxx} = 0, \quad w_t + 3(vw)_x + w_{xxx} = 0
\]

could be rewritten as

\[
\begin{pmatrix}
v \\
w
\end{pmatrix}_t = - \begin{pmatrix}
0 & \partial \\
\partial & 0
\end{pmatrix} \begin{pmatrix}
\frac{\delta H_2}{\delta v} \\
\frac{\delta H_2}{\delta w}
\end{pmatrix} = - \begin{pmatrix}
0 & J_0 \\
J_0 & J_1 - J_0
\end{pmatrix} \begin{pmatrix}
\frac{\delta H_1}{\delta v} \\
\frac{\delta H_1}{\delta w}
\end{pmatrix}
\]

with Hamiltonians

\[
H_1 = \int_{S^1} vwdx, \quad H_2 = \frac{1}{2} \int_{S^1} (3v^2w + 2vw_{xx})dx
\]

and

\[
\begin{pmatrix}
v \\
w
\end{pmatrix}_t = - \begin{pmatrix}
0 & \partial \\
\partial & -\partial
\end{pmatrix} \begin{pmatrix}
\frac{\delta H_2}{\delta v} \\
\frac{\delta H_2}{\delta w}
\end{pmatrix} = - \begin{pmatrix}
0 & J_0 \\
J_0 & J_1 - J_0
\end{pmatrix} \begin{pmatrix}
\frac{\delta \tilde{H}_1}{\delta v} \\
\frac{\delta \tilde{H}_1}{\delta w}
\end{pmatrix}
\]

with Hamiltonians

\[
\tilde{H}_1 = \frac{1}{2} \int_{S^1} (v^2 + 2vw)dx, \quad \tilde{H}_2 = \frac{1}{2} \int_{S^1} (v^3 + vv_{xx} + 3v^2w + 2vw_{xx})dx
\]

where \( J_0 = \partial^3 + v\partial + \partial v \) and \( J_1 = w\partial + \partial w \).

(ii) The \( \mathbb{Z}_2^0 \)-CH equation \((2.14)\)

\[
p_t + 2pv_x + p_x v = 0, \quad p = v - v_{xx}, \quad p = v - v_{xx},
\]

\[
q_t + 2qv_x + q_x v + 2pw_x + p_x w = 0, \quad q = w - w_{xx}
\]

could be rewritten as

\[
\begin{pmatrix}
p \\
q
\end{pmatrix}_t = \begin{pmatrix}
0 & \partial^3 - \partial \\
\partial^3 - \partial & 0
\end{pmatrix} \begin{pmatrix}
\frac{\delta H_2}{\delta p} \\
\frac{\delta H_2}{\delta q}
\end{pmatrix} = - \begin{pmatrix}
0 & K_0 \\
K_0 & K_1 - K_0
\end{pmatrix} \begin{pmatrix}
\frac{\delta H_1}{\delta p} \\
\frac{\delta H_1}{\delta q}
\end{pmatrix}
\]

with Hamiltonians

\[
H_1 = \frac{1}{2} \int_{S^1} (qv + pw)dx, \quad H_2 = \frac{1}{4} \int_{S^1} (2vww_{xx} + 2vww_{xx} - 2wvw_{xx} - v^2w_{xx})dx
\]

and

\[
\begin{pmatrix}
p \\
q
\end{pmatrix}_t = \begin{pmatrix}
0 & \partial^3 - \partial \\
\partial^3 - \partial & 0
\end{pmatrix} \begin{pmatrix}
\frac{\delta \tilde{H}_2}{\delta p} \\
\frac{\delta \tilde{H}_2}{\delta q}
\end{pmatrix} = - \begin{pmatrix}
0 & K_0 \\
K_0 & K_1 - K_0
\end{pmatrix} \begin{pmatrix}
\frac{\delta \tilde{H}_1}{\delta p} \\
\frac{\delta \tilde{H}_1}{\delta q}
\end{pmatrix}
\]

with Hamiltonians

\[
\tilde{H}_1 = \frac{1}{2} \int_{S^1} (pv + qv + pw)dx
\]
and
\[ \tilde{H}_2 = \frac{1}{4} \int_{S^1} (2vw_{xx} + 2wv_{xx} - 2wvv_{xx} - v^2w_{xx} + 2vv_{xx} - v^2v_{xx}) \, dx, \]
where \( K_0 = p\partial + \partial p \) and \( K_1 = q\partial + \partial q \).

(iii). The \( Z^0_2\)-HS equation (2.15)
\[ p_t + 2pv_x + p_x v = 0, \quad p = -v_{xx}, \]
\[ q_t + 2qv_x + q_x v + 2pw_x + p_x w = 0, \quad q = -w_{xx} \]
could be rewritten as
\[ \begin{pmatrix} p \\ q \end{pmatrix}_t = \begin{pmatrix} 0 & \partial^3 \\ \partial^3 & 0 \end{pmatrix} \begin{pmatrix} \delta H_2 \\ \delta p \\ \delta H_2 \\ \delta q \end{pmatrix} = - \begin{pmatrix} 0 & K_0 \\ K_0 & 0 \end{pmatrix} \begin{pmatrix} \delta H_1 \\ \delta p \\ \delta H_1 \\ \delta q \end{pmatrix} \]
with Hamiltonians
\[ H_1 = \frac{1}{2} \int_{S^1} (qv + pw) \, dx, \quad H_2 = \frac{1}{4} \int_{S^1} (2wp + v^2q) \, dx, \]
and
\[ \begin{pmatrix} p \\ q \end{pmatrix}_t = \begin{pmatrix} 0 & \partial^3 - \partial \\ \partial^3 - \partial & 0 \end{pmatrix} \begin{pmatrix} \delta H_2 \\ \delta p \\ \delta H_2 \\ \delta q \end{pmatrix} = - \begin{pmatrix} 0 & K_0 \\ K_0 & 0 \end{pmatrix} \begin{pmatrix} \delta H_1 \\ \delta p \\ \delta H_1 \\ \delta q \end{pmatrix} \]
with Hamiltonians
\[ \tilde{H}_1 = \frac{1}{2} \int_{S^1} (pv + qv + pw) \, dx, \quad \tilde{H}_2 = \frac{1}{4} \int_{S^1} (pv + 2wp + v^2q) \, dx, \]
where \( K_0 = p\partial + \partial p \) and \( K_1 = q\partial + \partial q \).

2.5. Euler equations on \( \text{vir}_\delta^* \) for general product \( P_{\alpha_0, \ldots, \alpha_n} \). To end up this section, on \( \text{vir}_\delta \) we introduce a general product \( P_{\alpha_0, \ldots, \alpha_n} \) given by
\[ \langle \hat{u}, \hat{v} \rangle = \text{tr}_\delta \int_{S^1} \left( \alpha_0 \circ u \circ v + \sum_{k=1}^{n} \alpha_k \circ u^{(k)} \circ v^{(k)} \right) \, dx + \text{tr}_\delta (a \circ b), \quad u^{(k)} = \frac{d^k u}{dx^k}. \] (2.22)
By analogy with the above discussions, we have

**Proposition 2.12.** The Euler equation (1.1) on \( \text{vir}_\delta^* \) for \( P_{\alpha_0, \ldots, \alpha_n} \) reads
\[ m_t + 2m \circ u_x + m_x u + \zeta \circ u_{xxx} = 0, \quad \zeta_t = 0, \] (2.23)
where \( m = \alpha_0 \circ u + \sum_{k=1}^{n} (-1)^k \alpha_k \circ u^{(2k)} \). Moreover, the system (2.23) with \( \zeta \in \mathfrak{g} \) could be written as
\[ m_{k,t} = \{ m_k, H_1 \}_2, \quad H_1 = \frac{1}{2} \text{tr}_\delta \int_{S^1} m \circ u \, dx \] (2.24)
where \( \{ , \}_2 \) is defined in (2.18).
Generally, when \( n \geq 2 \), the system (2.23) isn’t a bihamiltonian system. But if there are many different ways to realize the algebra \((\mathfrak{g}, \mathfrak{g}, \circ)\) as the Frobenius algebras, then it follows from Proposition 2.12 that the system (2.23) has many different Hamiltonian structures. For instance,

**Corollary 2.13.** The \( \mathbb{Z}_2^\varepsilon \)-valued Euler equation (2.23) with \( \zeta \in \mathbb{Z}_2^\varepsilon \) admits at least two “basic” local Hamiltonian structures.

3. Conclusion

In order to understand Eulerian nature of the \( \mathfrak{g} \)-valued KdV equation, we have introduced the Frobenius-Virasoro algebra \( \text{vir}_\mathfrak{g} \) and also described Euler equations on \( \text{vir}_\mathfrak{g}^* \) under the product \( P_{\alpha_0, \ldots, \alpha_n} \) and proved that all resulted Euler equations for \( P_{\alpha, \beta} \) are local bihamiltonian systems. Here we only studied the Euler equation associated with \( \text{vir}_\mathfrak{g} \). In subsequent publications we hope to address those problems related to algebraic properties of \( \text{vir}_\mathfrak{g} \), such as

**Q1.** What is the second continuous cohomology group \( H^2(\mathfrak{g}, \mathfrak{g}) \)?

**Q2.** How about the representation theory of \( \text{vir}_\mathfrak{g} \)?

**Q3.** If exists, what is the corresponding Lie group \( G_\mathfrak{g} \) of \( \text{vir}_\mathfrak{g} \)? For instance, \( G_\mathbb{R} \) is the Bott-Virasoro group.

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