Affine subspace concentration conditions for centered polytopes

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Abstract
Recently, K.-Y. Wu introduced affine subspace concentration conditions for the cone volumes of polytopes and proved that the cone volumes of centered, reflexive, smooth lattice polytopes satisfy these conditions. We extend the result to arbitrary centered polytopes.

MSC 2020
26B15, 28A75, 52A23, 52A38, 52B11 (primary)

1 | INTRODUCTION AND RESULTS

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space equipped with the standard inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\|x\| = \sqrt{\langle x, x \rangle}$, $x \in \mathbb{R}^n$. Let $P_0^n$ be the family of all $n$-dimensional polytopes $P \subset \mathbb{R}^n$ containing the origin in its interior, that is, $0 \in \text{int} P$. Given such a polytope $P \in P_0^n$, it admits a unique representation as

$$P = \{ x \in \mathbb{R}^n : \langle a_i, x \rangle \leq 1, 1 \leq i \leq m \},$$

where the vectors $a_i \in \mathbb{R}^n \setminus \{0\}$ are pairwise different and $F_i = P \cap \{ x \in \mathbb{R}^n : \langle a_i, x \rangle = 1 \}, 1 \leq i \leq m$, are the facets of $P$. Then the volume of $P$ (i.e., the $n$-dimensional Lebesgue measure of $P$) can be written as

$$\text{vol}(P) = \frac{1}{n} \sum_{i=1}^{m} \text{vol}_{n-1}(F_i) \frac{1}{\|a_i\|},$$

where, in general, for a $k$-dimensional set $S \subseteq \mathbb{R}^n$, $\text{vol}_k(S)$ denotes the $k$-dimensional Lebesgue measure with respect to the space $\text{aff} S$, the affine hull of $S$. This identity is also known as the...
pyramid formula, as it sums up the volumes of the pyramids (cones)

\[ C_i = \text{conv}(\{0\} \cup F_i), \]

where \( \text{conv}S \) denotes the convex hull of the set \( S \). Observe that

\[ \text{vol}(C_i) = \frac{1}{n} \frac{1}{\|a_i\|} \text{vol}_{n-1}(F_i), \quad 1 \leq i \leq m. \]

These cone volumes are the geometric base of the cone-volume measure of an arbitrary convex body, which is a finite positive Borel measure on the \((n - 1)\)-dimensional unit sphere \( S^{n-1} \subset \mathbb{R}^n \). The cone-volume measure is the subject of the well-known and important log-Minkowski problem in modern Convex Geometry, see, for example, \([3–9, 13, 16, 17]\).

In the discrete setting, that is, the polytopal case, the cone-volume measure \( V_P(\cdot) \) associated to \( P \) is the discrete measure

\[ V_P(\eta) = \sum_{i=1}^{m} \text{vol}(C_i) \delta_{u_i}(\eta), \]

where \( \eta \subseteq S^{n-1} \) is a Borel set, and \( \delta_{u_i}(\cdot) \) denotes the delta measure concentrated on \( u_i \). In analogy to the classical Minkowski problem, the discrete log-Minkowski problem asks for sufficient and necessary conditions such that a discrete Borel measure \( \mu = \sum_{i=1}^{m} \gamma_i \delta_{u_i}(\cdot), \gamma_i \in \mathbb{R}_{>0}, u_i \in S^{n-1}, \) is the cone-volume measure of a polytope.

Böröczky, Lutwak, Yang, and Zhang settled the general (i.e., not necessarily discrete) log-Minkowski problem for arbitrary finite even Borel measures. Here even means that \( \mu(A) = \mu(-A) \) holds for all Borel sets \( A \subseteq S^{n-1} \). This assumption corresponds to the case of origin-symmetric convex bodies; reduced to the discrete setting their result may be stated as follows.

**Theorem I** (Böröczky–Lutwak–Yang–Zhang [7]). A discrete even Borel measure \( \mu : S^{n-1} \to \mathbb{R}_{\geq 0} \) given by \( \mu = \sum_{i=1}^{m} \gamma_i \delta_{u_i}, \gamma_i \in \mathbb{R}_{>0}, u_i \in S^{n-1}, \) is the cone-volume measure of an origin-symmetric polytope \( P \in P_n^o \) if and only if the subspace concentration condition is fulfilled, that is, (i) for every linear subspace \( L \subseteq \mathbb{R}^n \) it holds

\[ \mu(L \cap S^{n-1}) = \sum_{i: u_i \in L} \gamma_i \leq \frac{\dim L}{n} \sum_{i=1}^{m} \gamma_i = \frac{\dim L}{n} \mu(S^{n-1}), \tag{1.1} \]

and (ii), equality holds in (1.1) for a subspace \( L \) if and only if there exists a complementary subspace \( L' \) such that \( \mu \) is concentrated on \( L \cup L' \).

In the non-even case, even in the discrete setting, a complete characterization is still missing, see [9] for the state of the art. The main problem here is to find the right position of the origin.

A polytope \( P \in P_n^o \) is called centered if its centroid \( c(P) \) is at the origin, that is,

\[ c(P) = \text{vol}(P)^{-1} \int_{P} x \, dx = 0. \]

It is known that centered polytopes satisfy the subspace concentration condition.
**Theorem II** (Henk–Linke [13]). Let \( P = \{ x \in \mathbb{R}^n : \langle a_i, x \rangle \leq 1, 1 \leq i \leq m \} \) be a centered polytope and let \( L \subseteq \mathbb{R}^n \) be a linear subspace. Then, (1.1) holds true, that is,

\[
V_P(L \cap S^{n-1}) \leq \frac{\dim L}{n} V_P(S^{n-1}).
\]

Equality is obtained if and only if there exists a complementary linear subspace \( L' \subseteq \mathbb{R}^n \) to \( L \) such that \( \{ a_i : 1 \leq i \leq m \} \subseteq L \cup L' \).

For a generalization to centered convex bodies we refer to [5] (Figure 1).

Apart from the study of the log-Minkowski problem, the subspace concentration inequalities have been recently reinterpreted in the context of toric geometry in [15], exploiting the deep connection between lattice polytopes and toric varieties. This lead K.-Y. Wu to prove an elegant variant to Theorem II in which the linear subspace concentration condition (1.1) is replaced by an affine subspace concentration condition.

**Theorem III** (K.-Y. Wu [19]). Let \( P = \{ x \in \mathbb{R}^n : \langle a_i, x \rangle \leq 1, 1 \leq i \leq m \} \) be a centered reflexive smooth polytope and let \( A \subset \mathbb{R}^n \) be a proper affine subspace. Then,

\[
V_P(\overline{A}) \leq \frac{\dim A + 1}{n + 1} V_P(S^{n-1}),
\]

where \( \overline{A} = \{ \frac{a_i}{|a_i|} : a_i \in A, 1 \leq i \leq m \} \). Equality is obtained if and only if there exists a complementary affine subspace \( A' \), that is, \( A \cap A' = \emptyset \) and \( \text{af}(A \cup A') = \mathbb{R}^n \), such that \( \{ a_i : 1 \leq i \leq m \} \subseteq A \cup A' \).

Here a polytope \( P \) is reflexive if the vectors \( a_i, 1 \leq i \leq m \), as well as the vertices of \( P \) are points of \( \mathbb{Z}^n \). In other words, \( P \) and \( P^* \), the polar of \( P \), are both lattice polytopes. A lattice polytope \( P \) is said to be smooth if it is simple, that is, each vertex of \( P \) is contained in exactly \( n \) facets \( F_{j_1}, \ldots, F_{j_n} \), say, and the corresponding normals \( a_{j_1}, \ldots, a_{j_n} \) form a lattice basis of \( \mathbb{Z}^n \), that is, \( (a_{j_1}, \ldots, a_{j_n}) \mathbb{Z}^n = \mathbb{Z}^n \).

The purpose of this paper is to generalize K.-Y. Wu's affine subspace concentration inequalities to arbitrary centered polytopes.
Theorem 1.1. Let $P = \{ x \in \mathbb{R}^n : \langle x, a_i \rangle \leq 1, 1 \leq i \leq m \}$ be a centered polytope and let $A \subseteq \mathbb{R}^n$ be an affine subspace. Then,

$$V_P(A) \leq \dim A + 1 \frac{\dim A + 1}{n + 1} V_P(S^{n-1}), \quad (1.2)$$

where $\bar{A} = \{ \frac{a_i}{|a_i|} : a_i \in A, 1 \leq i \leq m \}$.

The statement of Theorem III and Theorem 1.1 relies on the normalization of the vectors $a_i$ in the representation of $P$. Therefore, in contrast to the linear case, (1.2) does not solely depend on the measure $V_P$.

Since the total mass of the cone-volume measure $V_P$ is the volume of $P$, (1.2) may be rewritten equivalently as

$$\sum_{i : a_i \in A} \text{vol}(C_i) \leq \frac{\dim A + 1}{n + 1} \text{vol}(P). \quad (1.3)$$

Unlike Theorem III, which covers the case of reflexive smooth polytopes, our proof does not give us insight into the characterization of the equality case. We are, however, able to treat the equality case in two special cases.

Theorem 1.2. Let $P = \{ x \in \mathbb{R}^n : \langle x, a_i \rangle \leq 1, 1 \leq i \leq m \}$ be a centered polytope.

(i) If $A = \{a_i\}$ for some $1 \leq i \leq m$, then equality holds in (1.2) if and only if $P$ is a pyramid with base $F_i$.

(ii) If $A$ is the hyperplane spanned by the points $a_i$ corresponding to all the facets containing a vertex $v$ of $P$, then equality holds in (1.2) if and only if $P$ is a pyramid with apex $v$. 
As a byproduct of the proof of Theorem 1.2, we will see alternative proofs of (1.2) in these special cases. The first case of Theorem 1.2 slightly generalizes a former result by Zhou and He [20, Theorem 1.2]. There an additional technical assumption on \( P \) is made. We also point out that for simple polytopes Theorem 1.2 implies the following corollary.

**Corollary 1.3.** Let \( P = \{ x \in \mathbb{R}^n : \langle x, a_i \rangle \leq 1, 1 \leq i \leq m \} \) be a centered simple polytope. Let \( A \subset \mathbb{R}^n \) be an affine subspace spanned by points \( a_i \) corresponding to all the facets containing a \( k \)-face of \( P \) with \( 1 \leq k \leq n - 1 \). Then we have equality in (1.2) if and only if \( P \) is a centered simplex.

In contrast to the description of the equality case in Theorem III, the descriptions of the equality cases in Theorem 1.2 do not explicitly refer to the normal vectors \( a_i \). The following proposition gives an equivalent formulation of the equality case in Theorem III; it shows that the two conditions in Theorem 1.2 are indeed special cases of the general description in terms of the normal vectors \( a_i \).

**Proposition 1.4.** Let \( P = \{ x \in \mathbb{R}^n : \langle x, a_i \rangle \leq 1, 1 \leq i \leq m \} \). Then there exist a proper affine subspace \( A \) and a complementary affine subspace \( A' \) such that \( \{ a_i : 1 \leq i \leq m \} \subseteq A \cup A' \) if and only if \( P \) can be written as

\[
P = \text{conv}(Q_1 \cup Q_2),
\]

where \( Q_1, Q_2 \subset \mathbb{R}^n \) are polytopes with \( \dim Q_1 + \dim Q_2 = n - 1 \) and \( \text{aff} Q_1 \cap \text{aff} Q_2 = \emptyset \).

Proposition 1.4 appears to be well known, but since we are not aware of a proof in the literature, we provide one in Section 4. We note that the corresponding equality statement in the case of linear subspaces (see Theorem II) gives \( P = Q_1 + Q_2 \) where \( Q_1, Q_2 \subset \mathbb{R}^n \) are polytopes with \( \dim Q_1 + \dim Q_2 = n \) and \( \text{lin} Q_1 \cap \text{lin} Q_2 = \{0\} \) (see, e.g., [13, Section 3]). Here \( \text{lin} S \) denotes the linear hull of a subset \( S \subset \mathbb{R}^n \).

The rest of the paper is organized as follows: Section 2 contains some preliminaries. In Section 3, we prove Theorems 1.1 and 1.2. We give two proofs for Theorem 1.2(ii), one geometric and one analytic. Finally, in Section 4, we discuss the geometric meaning of the equality case in Theorem III and prove Proposition 1.4.

## 2 | Preliminaries

In this section we give a brief overview of the concepts that are necessary for the understanding of the paper. We refer to [21] for a detailed introduction into the theory of polytopes and their face structure, and to [1, 10, 11, 18] for exhaustive background information on Convex Geometry.

### 2.1 | Polytopes

A polytope \( P \subset \mathbb{R}^n \) is, by definition, the convex hull of a finite set \( X \subset \mathbb{R}^n \). By the Minkowski–Weyl theorem, \( P \) may be represented as

\[
P = \{ x \in \mathbb{R}^n : \langle a_i, x \rangle \leq b_i, \ 1 \leq i \leq m \},
\]
for certain \(a_1, \ldots, a_m \in \mathbb{R}^n\) and \(b_1, \ldots, b_m \in \mathbb{R}\). Conversely, the right-hand side in the above equation defines a polytope, whenever the set is bounded. We say that this description is irredundant if none of the constraints \(\langle a_i, x \rangle \leq b_i\) may be omitted without changing the polytope. In this case, the set \(F_i = P \cap \{x \in \mathbb{R}^n : \langle x, a_i \rangle = b_i\}\) is an \((n-1)\)-dimensional polytope in the boundary of \(P\) and it is called a facet of \(P\). More generally, a convex subset \(F \subseteq P\) with the property that \(\lambda x + (1 - \lambda)y \in F\), for some \(\lambda \in (0, 1)\) and \(x, y \in P\), implies that \(x, y \in F\) is called a face of \(P\). This is equivalent to the existence of a hyperplane \(H\) such that \(F = P \cap H\) and \(P\) is contained in one of the closed half spaces defined by \(H\).

For \(P \in P_0^n\) with an irredundant description

\[
P = \{x \in \mathbb{R}^n : \langle x, a_i \rangle \leq 1, \ 1 \leq i \leq m\},
\]

one defines the polar polytope of \(P\) as

\[
P^* = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall x \in P\} = \text{conv}\{a_1, \ldots, a_m\}.
\]

While the inequality description is certainly redundant, the convex hull description is not; the points \(a_i\) are precisely the vertices of \(P\).

There is an even stronger duality between the faces of \(P\) and \(P^*\). For a face \(F \subseteq P\) of dimension \(d \in \{0, \ldots, n-1\}\) one defines its polar face as

\[
F^\circ = \{y \in P^* : \langle x, y \rangle = 1, \forall x \in F\}.
\]

It turns out that \(F^\circ\) is indeed an \((n - d - 1)\)-face of \(P^*\) and that any \((n - d - 1)\)-faces of \(P^*\) arises this way. Moreover, we have \((F^\circ)^\circ = F\) and \(F^\circ \supseteq G^\circ\) for \(F \subseteq G\).

## 2.2 Volume and centroids

For a \(k\)-dimensional polytope \(P \subset \mathbb{R}^n\) we denote by \(\text{vol}_k(P)\) its \(k\)-dimensional volume within its affine hull. Likewise, we define \(c(P) = \frac{\text{vol}_k(P)}{k} \int_P x \, d^k x\) as its centroid. We will need the following formula in order to compute the centroid of a pyramid.

**Lemma 2.1.** Let \(P \subset \mathbb{R}^n\) be an \((n-1)\)-dimensional polytope and \(v \not\in \text{aff } F\). Then,

\[
c(\text{conv}(F \cup \{v\})) = \frac{n}{n+1} c(F) + \frac{1}{n+1} v.
\]

**Proof.** As \(c(\cdot)\) is affinely equivariant, it is enough to consider the case where \(F \subseteq \{x \in \mathbb{R}^n : x_n = 0\}\), \(c(F) = 0\), and \(v = e_n\), where \(e_n\) denotes the \(n\)th standard unit vector. Let \(H_t = \{x \in \mathbb{R}^n : x_n = t\}\). Using Fubini’s theorem, we have

\[
c(P) = \frac{1}{\text{vol}(P)} \int_0^1 \text{vol}_{n-1}(P \cap H_t) \, c(P \cap H_t) \, dt.
\]
In our setting, we have $P \cap H_t = (1 - t)F + te_n$. Thus, it follows that

$$c(P) = \frac{\text{vol}_{n-1}(F)}{\text{vol}(P)} \left( \int_0^1 (1 - t)^{n-1} t \, dt \right) e_n$$

$$= \frac{\text{vol}_{n-1}(F)}{\text{vol}(P)} \frac{1}{n(n+1)} e_n = \frac{1}{n+1} e_n,$$

where the last equality follows from the fact that $P$ is a pyramid with height one over $F$ and therefore $\text{vol}(P) = \text{vol}_{n-1}(F)/n$. Given our assumptions, the proof of the lemma is finished. □

Moreover, we are going to make use of the following additivity property of the centroid. Consider a finite family of convex bodies $K_1, \ldots, K_m \subseteq \mathbb{R}^n$ whose union $K = K_1 \cup \cdots \cup K_m$ is again a convex body and suppose that $K_i \cap K_j$ is a set of Lebesgue measure zero for all $i \neq j$. Then we have

$$c(K) = \frac{1}{\text{vol}(K)} (\text{vol}(K_1)c(K_1) + \cdots + \text{vol}(K_m)c(K_m)). \quad (2.3)$$

Finally, the following lemma will be used in the proof of Theorem 1.2. Here and in the following, for $u \in \mathbb{R}^n$, $u^\perp$ denotes the orthogonal complement of $\text{lin}\{u\}$.

**Lemma 2.2.** Let $P \subset \mathbb{R}^n$ be an $n$-dimensional polytope, $u \in S^{n-1}$ and let $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(t) = \text{vol}_{n-1}((tu + u^\perp) \cap P)^{\frac{1}{n-1}}.$$ 

Let $[\alpha, \beta] = \text{supp}(f)$. If $f$ is affine on $[\alpha, \beta]$ and $f(\beta) = 0$, then $P$ is a pyramid with base $(\alpha u + u^\perp) \cap P$ and apex $(\beta u + u^\perp) \cap P$.

**Proof.** Let $S = (\alpha u + u^\perp) \cap P$ and $T = (\beta u + u^\perp) \cap P$. Since $f$ is affine, $f(\beta) = 0$ and $\text{vol}(P) = \int_{\mathbb{R}} f(x)^{n-1} \, dx > 0$, we know that $\text{vol}_{n-1}(S) = f(\alpha) > 0$. Let $\lambda \in [0, 1]$. By the convexity of $P$, we have

$$\lambda T + (1 - \lambda)S \subseteq [(\lambda \beta + (1 - \lambda)\alpha)u + u^\perp) \cap P =: P_\lambda.$$ 

Combining this with the Brunn–Minkowski inequality [18, Theorem 7.1.1], we obtain

$$f(\lambda \beta + (1 - \lambda)\alpha) \geq \text{vol}_{n-1}(\lambda T + (1 - \lambda)S)^{\frac{1}{n-1}}$$

$$\geq \lambda \text{vol}_{n-1}(T)^{\frac{1}{n-1}} + (1 - \lambda) \text{vol}_{n-1}(S)^{\frac{1}{n-1}}$$

$$= \lambda f(\beta) + (1 - \lambda) f(\alpha). \quad (2.4)$$

Since $f$ is affine, both inequalities hold with equality. The equality in the Brunn–Minkowski inequality implies that $S$ and $T$ are homothetic (the other equality case being ruled out by the fact that $\text{vol}_{n-1}(S) > 0$). Because $\text{vol}_{n-1}(T) = 0$, this shows that $T$ is a singleton. Finally, since the polytope $P_\lambda$ contains the polytope $\lambda T + (1 - \lambda)S$, the first equality in (2.4) implies that $P_\lambda = \lambda T + (1 - \lambda)S$. Since $\lambda \in [0, 1]$ was arbitrary, it follows that $P$ is a pyramid with $S$ as its base. □
3 PROOFS OF THE THEOREMS

We start with Theorem 1.1. The basic idea of the proof is to reduce the problem to the linear case by replacing $P$ by a certain pyramid $\text{pyr}(P)$ one dimension higher. Rather than performing this replacement step once, we do it recursively, leading to an infinite sequence of pyramids $\text{pyr}(P)$, $\text{pyr}(\text{pyr}(P))$, etc. The crucial observation is that the reduction to the linear case becomes stronger in higher dimensions, with the desired estimate as the limiting case.

To this end we define for $k$-dimensional polytope $Q \subset \mathbb{R}^k$ the pyramid $\text{pyr}(Q)$ by

$$\text{pyr}(Q) = \text{conv}((Q \times \{1\}) \cup \{-(k + 1)e_{k+1}\}) \subset \mathbb{R}^{k+1}.$$ 

We will need the following properties of this embedding.

Lemma 3.1. Let $P \in \mathcal{P}^n_o$ be given as in Theorem 1.1, let $P^{(1)} = \text{pyr}(P)$. Then the following holds.

(i) $P^{(1)} = \left\{ x \in \mathbb{R}^{n+1} : \left\langle \left(\frac{n+2}{n+1}a_i - \frac{1}{n+1}\right) , x \right\rangle \leq 1, 1 \leq i \leq m, x_{n+1} \leq 1 \right\}$.

(ii) $\text{vol}_{n+1}(P^{(1)}) = \frac{n+2}{n+1} \text{vol}_n(P)$.

(iii) $P^{(1)}$ is centered, that is, $c(P^{(1)}) = 0$.

(iv) Let $C_i^{(1)}$ be the cone given by the facet of $P^{(1)}$ corresponding to the outer normal vector $(\frac{n+2}{n+1}a_i, -\frac{1}{n+1})^T$ and the origin. Then for $1 \leq i \leq m$

$$\text{vol}_{n+1}(C_i^{(1)}) = \text{vol}_n(C_i).$$

Proof. (i) and (ii) follow directly from the fact that $P^{(1)}$ is indeed a pyramid; (iii) is a consequence of (2.2). For (iv), let $\overline{C}_i = C_i \times \{1\}$, $G_1 = \text{conv}(\overline{C}_i \cup \{-(n + 1)e_{n+1}\})$, and $G_2 = \text{conv}(\overline{C}_i \cup \{0\}) \subseteq G_1$. Then we have $C_i^{(1)} = G_1 \setminus G_2$ and therefore

$$\text{vol}_{n+1}(C_i^{(1)}) = \text{vol}_{n+1}(G_1) - \text{vol}_{n+1}(G_2)$$

$$= \frac{n+2}{n+1} \text{vol}_n(C_i) - \frac{1}{n+1} \text{vol}_n(C_i)$$

$$= \text{vol}_n(C_i).$$

Proof of Theorem 1.1. Let $A \subseteq \mathbb{R}^n$ be a proper affine space, $d = \text{dim} A$, $I = \{ i \in [m] : a_i \in A \}$ and we may assume $\text{dim}\{a_i : i \in I\} = d$. For any $k$, let

$$\varphi_k : \mathbb{R}^k \to \mathbb{R}^{k+1}, \ x \mapsto \left( \frac{k+2}{k+1}x \right)_{k+1}.$$
For \( j \geq 1 \) and \( i \in [m] \) we set

\[
a^{(j)}_i = (\varphi_{n+j-1} \circ \cdots \circ \varphi_n)(a_i) \in \mathbb{R}^{n+j},
\]

and let \( L^{(j)} = \text{lin}\{a^{(j)}_i : i \in I\} \subseteq \mathbb{R}^{n+j} \). Observe that the vectors \( a^{(j)}_i \) have the form

\[
a^{(j)}_i = \begin{pmatrix}
\frac{n+j+1}{n+1} a_i \\
c_{n+1} \\
\vdots \\
c_{n+j}
\end{pmatrix},
\]

where

\[
c_{n+k} = -\frac{n + j + 1}{(n + k)(n + k + 1)}, \quad 1 \leq k \leq j.
\]

The numbers \( c_{n+k} \) only depend on \( n \) and \( j \), but not on \( a_i \). Therefore, \( L^{(1)} \) is a \((d + 1)\)-dimensional linear space and since the matrix \((a^{(j+1)}_i : i \in I)\) differs from \((a^{(j)}_i : i \in I)\) only by an additional constant row and a multiplication of the first \( n + j - 1 \) rows, we have \( \dim L^{(j)} = d + 1 \) for all \( j \geq 1 \).

Consider the pyramids \( P^{(j)} = \text{pyr}(P^{(j-1)}) \) with \( P^{(0)} = P \). A repeated application of Lemma 3.1(i) and (iii) shows that each \( P^{(j)} \) is a centered pyramid that has the vectors \( \{a^{(j)}_i : i \in [m]\} \) among its normal vectors, and from Lemma 3.1(ii) we get

\[
\text{vol}_{n+j}(P^{(j)}) = \left( \prod_{k=1}^{j} \frac{n+k+1}{n+k} \right) \text{vol}_n(P) = \frac{n + j + 1}{n + 1} \text{vol}_n(P).
\]

Let \( C^{(j)}_i \) be the cone of \( P^{(j)} \) corresponding to \( a^{(j)}_i \). Lemma 3.1(iv) shows that \( \text{vol}_{n+j}(C^{(j)}_i) = \text{vol}_n(C_i) \), and so by Theorem II applied to \( P^{(j)} \) and \( L^{(j)} \) we obtain

\[
\sum_{i \in I} \text{vol}_n(C_i) = \sum_{i \in I} \text{vol}_{n+j}(C^{(j)}_i) \\
\leq \frac{d + 1}{n + j} \text{vol}_{n+j}(P^{(j)}) \\
= \frac{\dim A + 1}{n + 1} \frac{n + j + 1}{n + j} \text{vol}_n(P).
\]

The claim follows from letting \( j \to \infty \). \( \Box \)

Before we come to the proofs of Theorem 1.2(i) and (ii), we observe that equality holds in (1.2) and thus equivalently in (1.3), whenever \( \{a_i : 1 \leq i \leq m\} \subseteq A \cup A' \), where \( A' \) is complementary to \( A \); to see this, it suffices to apply (1.3) to both \( A \) and \( A' \) and obtain

\[
\text{vol}(P) = \sum_{i : a_i \in A} \text{vol}(C_i) + \sum_{i : a_i \in A'} \text{vol}(C_i) \leq \text{vol}(P).
\]
Thus, we have equality in (1.3) for $A$ (and also for $A'$). Turning to the cone-volume measures, this is equivalent to equality in (1.2) for $A$. In view of Proposition 1.4, we thus only need to show the “only if” parts for the equality cases in Theorem 1.2.

We start with case of $A$ being a singleton. Our proof is inspired by the proof of Grünbaum’s theorem on central sections of centered convex bodies [12].

**Proof of Theorem 1.2(i).** Without loss of generality, we assume that $F_i = P \cap \{ x \in \mathbb{R}^n : \langle e_1, x \rangle = -\alpha \}$ for an appropriately chosen $\alpha > 0$. Let $Q = \text{conv}(F_i \cup \{ \beta e_1 \})$, where $\beta > -\alpha$ is chosen such that $\text{vol}(Q) = \text{vol}(P)$. We define two functions $\mathbb{R} \to \mathbb{R}$ via

$$f(t) = \text{vol}_{n-1}((te_1 + e_1^\perp) \cap P)^{\frac{1}{n-1}}, \quad g(t) = \text{vol}_{n-1}((te_1 + e_1^\perp) \cap Q)^{\frac{1}{n-1}}.$$

If $\langle e_1, c(Q) \rangle \geq \langle e_1, c(P) \rangle$, then by Lemma 2.1 it would follow that

$$\text{vol}(C_i) \leq \text{vol}(\text{conv}(F_i \cup \{ c(Q) \})) = \frac{1}{n+1} \text{vol}(Q),$$

as desired. Recalling that $P$ is centered, we have to show for $\gamma = \langle e_1, c(Q) \rangle$ that

$$\gamma = \langle e_1, c(Q) - c(P) \rangle = \int_{-\infty}^{\infty} t[ g(t)^{n-1} - f(t)^{n-1}] dt \geq 0,$$

with equality if and only if $P$ is a pyramid.

Since $Q$ is a pyramid with base orthogonal to $e_1$, $g$ is affine on $\text{supp}(g) = [-\alpha, \beta]$. By Brunn’s concavity principle [1, Theorem 1.2.1], $f$ is concave on $\text{supp}(f)$. Hence, $g - f$ is convex on $\text{supp}(f) \cap \text{supp}(g)$. In fact, we have $\text{supp}(f) \subseteq \text{supp}(g)$: If there was a $t > \beta$ with $f(t) > 0$, then the concavity of $f$ would imply $f > g$ on $\text{supp}(g)$, in contradiction to $\text{vol}(Q) = \text{vol}(P)$. Hence, $g - f$ is convex on $\text{supp}(f)$ and the sublevel set

$$\text{supp}(f) \cap \{g - f \leq 0\} = \text{supp}(f) \cap \{g^{n-1} - f^{n-1} \leq 0\}$$

is convex. Since $f(-\alpha) = g(-\alpha)$, it follows that $\text{supp}(f) \cap \{g - f \leq 0\} = [-\alpha, \tau]$ for a $\tau \leq \beta$. On $[\tau, \beta]$ we have $g \geq f$, leading to the desired estimate

$$\gamma = \int_{-\alpha}^{\tau} t[ g(t)^{n-1} - f(t)^{n-1}] dt + \int_{\tau}^{\beta} t[ g(t)^{n-1} - f(t)^{n-1}] dt \geq \int_{-\alpha}^{\tau} \tau[ g(t)^{n-1} - f(t)^{n-1}] dt + \int_{\tau}^{\beta} \tau[ g(t)^{n-1} - f(t)^{n-1}] dt = \tau \left( \int_{-\alpha}^{\beta} [ g(t)^{n-1} - f(t)^{n-1}] dt \right) = \tau (\text{vol}(Q) - \text{vol}(P)) = 0.$$

Equality holds if and only if $g = f$ on $[-\alpha, \beta]$. It is clear that this is the case if $P$ is a pyramid with base $F_i$; the other direction follows from Lemma 2.2.
Next, we give two proofs of Theorem 1.2(ii), corresponding to two different perspectives on the problem. The first proof has a more geometric flavor, whereas the second proof is of a probabilistic nature.

**Geometric proof of Theorem 1.2(ii).** Let $I \subseteq [m]$ be the set of indices such that $\langle v, a_i \rangle = 1$, that is, $A = \text{aff}\{a_i : i \in I\}$. Since $P$ is centered, we have $-\frac{1}{n} v \in P$ (see [2, Section 34]). For $i \in I$, we consider the cones $\overline{C}_i = \text{conv}(F_i \cup \{-\frac{1}{n} v\}) \subseteq P$, where $F_i$ is the facet of $P$ with normal $a_i$. By the volume formula for pyramids, we have $\text{vol}(\overline{C}_i) = \frac{n+1}{n} \text{vol}(C_i)$. As the cones $\overline{C}_i$ intersect in a set of measure zero, we obtain

$$\text{vol}(P) \geq \sum_{i \in I} \text{vol}(\overline{C}_i) = \frac{n+1}{n} \sum_{i \in I} \text{vol}(C_i).$$

(3.1)

So we have reproven Theorem 1.1 in this case. In order to have equality in the above, we must have $P = \bigcup_{i \in I} \overline{C}_i$. Let $J = [m] \setminus I$. Then we have

$$\langle -(1/n)v, a_j \rangle = 1, \forall j \in J,$$

(3.2)

since otherwise, the cone $C_j$ would have a positive volume and we could not achieve equality in (3.1).

For $j \in J$, let $Q_j = \text{conv}(F_j \cup \{v\}) \subseteq P$. Just like the cones $\overline{C}_i$, the cones $Q_j$ subdivide $P$, that is, $P = \bigcup_{j \in J} Q_j$ and the pyramids intersect in sets of measure zero. By (2.2), we have $c(Q_j) = \frac{n}{n+1} c(F_j) + \frac{1}{n} v$ and in view of (2.3) we may write

$$0 = c(P) = \sum_{j \in J} \frac{\text{vol}(Q_j)}{\text{vol}(P)} \left( \frac{n}{n+1} c(F_j) + \frac{1}{n+1} v \right).$$

Multiplying with $(n+1)/n$ and rearranging yields

$$-\frac{1}{n} v = \sum_{j \in J} \frac{\text{vol}(Q_j)}{\text{vol}(P)} \left( -\frac{1}{n} v \right) = \sum_{j \in J} \frac{\text{vol}(Q_j)}{\text{vol}(P)} c(F_j).$$

Hence, (3.2) gives for any $j \in J$

$$1 = \langle -(1/n)v, a_j \rangle = \sum_{k \in J} \frac{\text{vol}(Q_k)}{\text{vol}(P)} \langle c(F_k), a_j \rangle.$$

Towards a contradiction, assume that $J$ contains more than one element. Then there is a $k \in J \setminus \{j\}$. Since $c(F_k) \in \text{relint} F_k$, we have $\langle c(F_k), a_j \rangle < 1$. It follows that $1 < \sum_{k \in J} \text{vol}(Q_k)/\text{vol}(P) = 1$. Therefore, $J$ can contain only one element, which corresponds to the case that $P$ is a pyramid with apex $v$. □

We now come to the second proof of Theorem 1.2(ii) via a probabilistic approach.

**Analytic proof of Theorem 1.2(ii).** Again, we only show the “only if” part of the equality case. To this end, we assume that $\text{vol}(P) = 1$, which is not a restriction as both sides of (1.3) are
n-homogeneous. By definition, we have $c(P) = \mathbb{E}[X]$, where $X$ is a uniformly distributed random vector in $P$. We consider the functional

$$f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = \frac{1}{n} \sum_{i : a_i \in A} \text{dist}(x, \text{af} \mathcal{F}_i) \text{vol}_{n-1}(\mathcal{F}_i),$$

where $\text{dist}(x, \text{af} \mathcal{F}_i)$ is the signed Euclidean distance to $\text{af} \mathcal{F}_i$, oriented such that it is non-negative inside $P$. Note that for $x \in P$ one has

$$f(x) = \sum_{i : a_i \in A} \text{vol}(\text{conv}(\mathcal{F}_i \cup \{x\})).$$

As $f$ is an affine map, we have

$$\sum_{i : a_i \in A} \text{vol}(C_i) = \mathbb{E}[f(X)] = \int_0^1 \mathbb{P}_X(f \geq t)dt = 1 - \int_0^1 \mathbb{P}_X(f < t)dt. \quad (3.3)$$

We consider the function $p : [0, 1] \to [0, 1], \ t \mapsto \mathbb{P}_X(f < t)$. We have $p(0) = 0$ and $p(t) = 1$, for $t \geq m = \max f(P) \leq 1$. Let $H(t) = \{x \in \mathbb{R}^n : f(x) \leq t\}$ be the half-space where $f \leq t$. Since the vertex $v$ is the unique point that is contained in all facets $F_i$, where $a_i \in A$, we have $0 \in f(P)$ and $f(x) = 0$ for $x \in P$, if and only if $x = v$. Thus, $P \cap H(0) = \{v\}$. Using the inclusion

$$P \cap H(t) \supseteq \frac{t}{m}(P \cap H(m)) + \frac{m-t}{m}v, \quad (3.4)$$

we deduce that, for any $t \in [0, m]$,

$$p(t) = \text{vol}(P \cap H(t))^\frac{1}{n} \geq \text{vol} \left( \frac{t}{m}(P \cap H(m)) + \frac{m-t}{m}v \right)^\frac{1}{n}$$

$$= \frac{t}{m} \text{vol}(P \cap H(m))^\frac{1}{n} = \frac{t}{m}p(m) = \frac{t}{m}. \quad (3.5)$$

Applying this to (3.3), we have

$$\sum_{i : a_i \in A} \text{vol}(C_i) = 1 - \int_0^m p(t)^n dt - (1-m)$$

$$\leq m - \int_0^m \left( \frac{t}{m} \right)^n dt = \frac{mn}{n+1} \leq \frac{n}{n+1}. \quad (3.6)$$

By our assumption that $\text{vol}(P) = 1$, this is (1.3). In order to have equality, we must have $m = 1$ and equality in (3.5), that is, $\text{vol}(P \cap H(t)) = t^n$ for $t \in [0, 1]$. This is equivalent to $\text{vol}_{n-1}(P \cap \{x \in \mathbb{R}^n : f(x) = t\}) = nt^{n-1}$ for $t \in [0, 1]$. By Lemma 2.2, this implies that $P$ is a pyramid with apex $v$. \hfill \square

**Remark 3.2.** It is natural to ask whether the assumption in Theorem 1.2(ii) that $v$ is a vertex of $P$ can be removed. In other words, is it possible to adapt our proofs to the situation where the hyperplanes $\{x \in \mathbb{R}^n : \langle a_i, x \rangle = 1\}, a_i \in A$, intersect in a single point $v$ that is not necessarily
Both proofs of Theorem 1.2(ii) make use of the assumption that \( v \in P \). In the first proof, we use it to derive \(-\frac{1}{n} v \in P\); in the second proof, it ensures that \( p \) is concave on \([0, \max f(P)]\). It is not clear how the first proof could be modified to dispense with the assumption. In the second proof, a suitable upper bound on \( \max f(P) \) in terms of \( \min f(P) \) would be sufficient: The concavity of \( p \) on \([\min f(P), \max f(P)]\) leads to the desired estimate if we additionally assume that \( \max f(P) \leq 1 - \frac{\min f(P)}{n} \).

4 | COMPLEMENTARY AFFINE SUBSPACES

To conclude, let us have a closer look at the characterization of the equality case as it has been formulated by K.-Y. Wu: A smooth and reflexive polytope \( P = \{ x \in \mathbb{R}^n : \langle x, a_i \rangle \leq 1, 1 \leq i \leq m \} \) satisfies the affine subspace concentration condition (1.3) for an affine \( d \)-subspace \( A \) with equality if and only if the normal vectors \( \{a_i : 1 \leq i \leq m\} \) of \( P \) are contained in \( A \cup A' \), where \( A' \) is an affine \((n - d - 1)\)-subspace complementary to \( A \).

At first glance, this condition may appear rather technical, but, in fact, it has a strong geometric interpretation for the polytope \( P \): Since the points \( a_i \) are the vertices of \( P^\star \), the condition that \( \{a_1, \ldots, a_m\} \) is contained in \( A \cup A' \) is equivalent to \( P^\star = \text{conv}(P_1 \cup P_2) \), where \( P_1 \) is a \( d \)-polytope and \( P_2 \) is an \((n - d - 1)\)-polytope and \( \text{aff}P_1 = A \) and \( \text{aff}P_2 = A' \) are complementary affine spaces. In general, a polytope that can be expressed as the convex hull of two polytopes \( Q_1 \) and \( Q_2 \) in complementary affine subspaces is also called the join of \( Q_1 \) and \( Q_2 \) [14, p. 390]. Therefore, the statement of Proposition 1.4 can be reformulated as that an \( n \)-dimensional polytope \( P \) is the join of a \( d \)-polytope \( Q_1 \) and an \((n - d - 1)\)-polytope \( Q_2 \) if and only if \( P^\star \) is. Since we could not find a reference for this certainly well-known fact we add a proof.

**Proof of Proposition 1.4.** By polarity, it is enough to prove that \( P \) being the join of \( Q_1 \) and \( Q_2 \) implies that \( P^\star \) is the join of two polytopes \( P_1 \) and \( P_2 \) of appropriate dimension.

So let \( P = \text{conv}(Q_1 \cup Q_2) \) with \( Q_1 \) and \( Q_2 \) as in the statement of the proposition. First, we show that \( Q_1 \) and \( Q_2 \) are faces of \( P \). Let \( x_1 \in Q_1, x_2 \in Q_2 \) and \( L = \text{lin}((Q_1 - x_1) \cup (Q_2 - x_2)) \). Since \( \dim Q_1 + \dim Q_2 = n - 1 \), we have \( \dim L \leq n - 1 \). Choosing a vector \( u \in L^\perp \setminus \{0\} \), the linear functional \( f : \mathbb{R}^n \to \mathbb{R}, x \mapsto \langle u, x \rangle \) satisfies \( f(Q_1) = \{\alpha\} \) and \( f(Q_2) = \{\beta\} \) for certain \( \alpha, \beta \in \mathbb{R} \). Since \( P \) is \( n \)-dimensional and of the form \( P = \text{conv}(Q_1 \cup Q_2) \), we have \( \alpha \neq \beta, f(P) = \text{conv}\{\alpha, \beta\} \) and

\[
\begin{align*}
\{\alpha\} \cap P &= Q_1, \\
\{\beta\} \cap P &= Q_2.
\end{align*}
\]

This shows that \( Q_1 \) and \( Q_2 \) are faces of \( P \).

The notion of a polar face was introduced in Section 2.1. We consider the polar faces \( P_i = Q_i^\circ \subseteq P^\star, i \in \{1, 2\} \), of the two faces \( Q_1, Q_2 \subset P \). Note that \( \dim P_1 = n - d - 1 \) and \( \dim P_2 = d \). Clearly, we have \( \text{conv}(P_1 \cup P_2) \subseteq P^\star \). If the inclusion was strict, we find a vertex \( v \) of \( P^\star \) which is neither a vertex of \( P_1 \), nor of \( P_2 \). Consider the corresponding facet \( F = v^\circ \) of \( P \). Since \( v \) is not contained in \( P_1 \cup P_2 \), it follows by polarity that neither \( Q_1 \), nor \( Q_2 \) is contained in \( F \). But \( F_1 = Q_1 \cap F \) is a face of \( Q_1 \). Thus, we have \( \dim F_1 \leq d - 1 \) and \( \dim F_2 \leq n - d - 2 \). Due to the assumption \( P = \text{conv}(Q_1 \cup Q_2) \), the vertices of \( F \) are contained in \( Q_1 \cup Q_2 \), that is, \( F = \text{conv}(F_1 \cup F_2) \). It follows that

\[
\dim F \leq 1 + \dim F_1 + \dim F_2 = 1 + (d - 1) + (n - d - 2) = n - 2,
\]

a contradiction. So we have proven \( P^\star = \text{conv}(P_1 \cup P_2) \). Since \( \dim(P^\star) = n \) and \( P^\star \subseteq \text{aff}(P_1 \cup P_2) \), we have \( \text{aff}(P_1 \cup P_2) = \mathbb{R}^n \), so the affine hulls of \( P_1 \) and \( P_2 \) are indeed complementary.  \( \Box \)
We recall that an \( n \)-polytope \( P \) is called simple if every vertex \( v \) of \( P \) is contained in exactly \( n \) edges, or, equivalently, in exactly \( n \) facets of \( P \). For a simple polytope \( P = \{ x \in \mathbb{R}^n : \langle x, a_i \rangle \leq 1, \ 1 \leq i \leq m \} \) the property \( \{a_1, \ldots, a_m\} \subseteq A \cup A' \), for some pair of complementary proper affine subspaces of \( \mathbb{R}^n \), is equivalent to the fact that \( P \) is a simplex. Indeed, we obtain from this that \( P^* = \text{conv}(P_1 \cup P_2) \), where the affine hull of \( P_1 \) is \( A \) and the affine hull of \( P_2 \) is \( A' \). Since \( P \) is simple, \( P^* \) is simplicial, that is, all faces of \( P^* \) are simplices. As we saw in the proof of Proposition \( 1.4 \), the polytopes \( P_1 \) and \( P_2 \) are faces of \( P^* \), so they are simplices of dimension \( \dim A \) and \( n - 1 - \dim A \), respectively. Hence, \( P^* \) is a simplex, which implies that \( P \) is a simplex as well.

As smooth polytopes are simple by definition, we see that simplices are the only equality cases in Theorem \( III \).

We conclude by providing a proof of Corollary \( 1.3 \).

**Proof of Corollary 1.3.** Let \( F \) be a \( k \)-face of \( P \). Since \( P \) is simple, there are exactly \( n - k - 1 \) vectors among the vectors \( a_i \) that satisfy \( F \subseteq F_i \). Without loss of generality, we assume that \( a_1, \ldots, a_{n-k-1} \) are these vectors. In view of Theorem \( 1.2(1) \), we obtain

\[
\text{vol}(C_i) \leq \frac{1}{n+1} \text{vol}(P), \quad \text{for all } 1 \leq i \leq n - k - 1.
\]  

(4.1)

Summing up these inequalities gives (1.3) for \( P \) and \( A \) where equality holds, if and only if equality holds in each of the inequalities in (4.1). In particular, equality holds, only if \( P \) is a pyramid with base \( F_1 \). Since \( P \) is simple, this implies that \( P \) is a simplex. \( \square \)

**ACKNOWLEDGEMENTS**

Christian Kipp is supported by the Deutsche Forschungsgemeinschaft (DFG), Graduiertenkolleg “Facets of Complexity” (GRK 2434). Moreover, we thank the anonymous referee for their helpful comments.

**JOURNAL INFORMATION**

*Mathematika* is owned by University College London and published by the London Mathematical Society. All surplus income from the publication of *Mathematika* is returned to mathematicians and mathematics research via the Society’s research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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