An Open Problem on Sparse Representations in Unions of Bases

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Abstract

We consider sparse representations of signals from redundant dictionaries which are unions of several orthonormal bases. The spark introduced by Donoho and Elad plays an important role in sparse representations. However, numerical computations of sparks are generally combinatorial. For unions of several orthonormal bases, two lower bounds on the spark were established in previous work. We constructively prove that both of them are tight. Our main results give positive answers to Gribonval and Nielsen’s open problem on sparse representations in unions of orthonormal bases. Constructive proofs rely on a family of mutually unbiased bases which first appears in quantum information theory.

Index Terms

Sparse Approximation, Spark, Mutual Coherence, Latin Squares, Mutually Unbiased Bases.

I. INTRODUCTION

Given a redundant dictionary, the problem of representing vectors (also referred to as signals) with linear combinations of small numbers of atoms from the dictionary is called the sparse representation [1], [2], [3]. Two fundamental concepts defined in [1], [4] are core issues of sparse representations. One is the mutual coherence, the other is the spark. The matrix notation $D$ is used for a dictionary. The spark denoted by $\eta(D)$ is defined to be the smallest number of columns from matrix $D$ that are linearly dependent. The mutual coherence denoted by $\mu(D)$ is defined to be the largest absolute normalized inner product between different columns from the matrix $D$. The value of spark is difficult to evaluate, the mutual coherence is used to estimate the spark in [4], [5], [6]. Our interest in this paper centers around tightness of two lower bounds for the spark obtained in the previous work.

A. Background

By $\|x\|_0$ we refer to the number of nonzero entries of the vector $x$. A vector is said to be $q$ sparse if $\|x\|_0 \leq q$. By definition of the spark, we see that

$$\eta(D) = \min_{x \in \ker(D), x \neq 0} \|x\|_0$$

where $\ker(D)$ denotes the null space of $D$. The spark is useful to bound the sparsity of the uniqueness of sparse solutions. If a linear system $y = Dx$ has a solution $x$ obeying $\|x\|_0 < \eta(D)/2$, then this solution is necessarily the sparsest possible [4]. Therefore, large values of spark are expected in applications. To estimate the spark, lower bounds depending on the mutual coherence were obtained in [4], [5], [6].

For any given arbitrary dictionary $D$, Elad and Donoho proved in [4] that

$$\eta(D) \geq 1 + \frac{1}{\mu(D)}$$

(I.1)

If $D$ is assumed to be a union of two orthonormal bases, then a tighter estimate

$$\eta(D) \geq \frac{2}{\mu(D)}$$

(I.2)

was obtained by Elad and Bruckstein in [5]. If the dictionaries are Dirac/Fourier matrix pairs, then the inequality (I.2) reduces to the support uncertainty principle obtained in [7], [8]. Extensions of the support uncertainty principle to the Fourier transform on abelian groups are referred to [9], [10], [11], [12] and references therein. Robust uncertainty principles were proved to hold for most supports in time and frequency [13], [14]. The case that dictionaries are concatenations of several orthonormal bases were studied in [4], [7], [6], [15]. Let $q$ denote a positive integer. If dictionaries $D$ are unions of $q+1$ orthonormal bases, then Gribonval and Nielsen proved in [6] Lemma 3 that

$$\eta(D) \geq \left(1 + \frac{1}{q}\right) \frac{1}{\mu(D)}$$

(I.3)
B. Motivations

The inequality (I.3) is a natural generalization of the inequality (I.2) from \( q = 1 \) to \( q \geq 2 \). The estimate (I.2) is tight, since there exist Dirac/Fourier matrix pairs that meet the bound with equality [4], [8], [5]. However, the question whether the estimate (I.3) for \( q \geq 2 \) is tight remains open. This open question was further discussed by Gribonval and Nielsen in [15]. For general dictionary, the tightness of the estimate (I.1) has been studied in [16, Section III]. For unions of several orthonormal bases, to the best of our knowledge, the tightness of the estimate (I.1) is also unknown. Motivated by these open questions, we study the tightness of the estimate (I.1) and the estimate (I.3).

C. Observations

Before going further, we discuss some observations. Following the line in [6], [15], we are concerned with unions of three or more orthonormal bases. Suppose that the number of orthonormal bases is \( q + 1 \). We discuss two cases.

1) Let the mutual coherence of \( D \) be \( 1/q \). Then
   - The right-hand side of (I.1) is equal to \( q + 1 \).
   - The right-hand side of (I.3) is equal to \( q + 1 \).
   - If \( x \) is in \( \text{Ker}(D) \), then it follows from the bound (I.3) that \( \|x\|_0 \geq q + 1 \).

Based on those observations, for \( q = 2^m \), \( m = 1, 2, 3, \ldots \), the goal is to find a dictionary \( D \) and a corresponding sparse vector \( x \) that satisfy the following three conditions
\[
Dx = 0, \quad \|x\|_0 = q + 1, \quad \mu(D) = \frac{1}{q}.
\]

2) Let the mutual coherence of \( D \) be \( 1/q^2 \). Then
   - The right-hand side of (I.1) is equal to \( q^2 + 1 \).
   - The right-hand side of (I.3) is equal to \( q^2 + q \).
   - If \( x \) is in \( \text{Ker}(D) \), then it follows from the bound (I.3) that \( \|x\|_0 \geq q^2 + q \).

Based on those observations, for \( q = 2^m \), \( m = 1, 2, 3, \ldots \), the goal is to find a dictionary \( D \) and a corresponding sparse vector \( x \) that satisfy the following three conditions
\[
Dx = 0, \quad \|x\|_0 = q^2 + q, \quad \mu(D) = \frac{1}{q^2}.
\]

D. Contributions

The main contributions of this paper are summarized as follows.

**Theorem 1.1.** For any \( q = 2^m \), \( m = 1, 2, 3, \ldots \), there exists a dictionary \( D \) which is a union of \( q + 1 \) orthonormal bases satisfies \( \eta(D) = q + 1 \) and \( \mu(D) = 1/q \). Then
\[
\eta(D) = 1 + \frac{1}{\mu(D)} \quad \text{and} \quad \eta(D) = \left(1 + \frac{1}{q}\right) \frac{1}{\mu(D)}.
\]

**Theorem 1.2.** For any \( q = 2^m \), \( m = 1, 2, 3, \ldots \), there exists a dictionary \( D \) which is a union of \( q + 1 \) orthonormal bases satisfies \( \eta(D) = q^2 + q \) and \( \mu(D) = 1/q^2 \). Then
\[
\eta(D) > 1 + \frac{1}{\mu(D)} \quad \text{and} \quad \eta(D) = \left(1 + \frac{1}{q}\right) \frac{1}{\mu(D)}.
\]

Therefore, for unions of several orthonormal bases, both the inequality (I.1) and the inequality (I.3) are achievable. Theorem (I.2) implies that the estimate (I.3) is sharper than the estimate (I.1) for some special dictionaries. Now we can answer Gribonval and Nielsen’s open problem in [6] positively:

“There exist examples of \( q + 1 \) orthonormal bases for which \( \eta(D)\mu(D) = 1 + 1/q \).”

E. Flowchart

Our constructive proofs base on techniques from discrete mathematics and quantum information theory [17], [18], [19], [20]. Two families of dictionaries are constructed by using the *mutually unbiased bases* (MUBs) obtained in [20]. The process of construction is illustrated in Figure[1] step by step. Symbols’ meanings are listed in Table I. The existence theorem of MUBs is clear, see e.g. [19], [20]. To find specific sparse vectors in the null space of such dictionaries, however, more detailed properties need to be established.
F. Outline

The rest of this paper is organized as follows. Section II defines two kinds of matrices by using elements in Galois fields. One of them contains mutual orthogonal Latin Squares, the other is used for theoretical analysis. Section III defines a class of real Hadamard matrices. Section IV obtains mutual unbiased bases in square dimension with explicitly structures. Section V proves main results and answers the open problem using the mutual unbiased bases constructed in section IV. Section VI presents three examples to illustrate our constructions and theoretical proofs. Section VII gives conclusions and further remarks.

TABLE I
NOTATION IN FIGURE 1

| Notation                          | Description                                      |
|----------------------------------|--------------------------------------------------|
| $F_q$                            | Galois Field of order $q$                        |
| $G(q)$                           | Matrices of order $q$ over $F_q$                 |
| $\{L^r\}_{r \in F_q}$           | Latin Squares over $F_q$                        |
| $\{B_b\}_{b \in F_q \cup \{\infty\}}$ | Mutually Unbiased Bases for $\mathbb{R}^q$  |
| $H(q)$                           | Hadamard matrix of order $q$                    |
| $\tilde{H}(q)$                   | Permutated Hadamard matrix of order $q$         |
| $D$                              | Dictionary of size $(q^2, q^2(q+1))$             |
| $x$                              | Sparse vector in $\mathbb{R}^{q^2(q+1)}$        |

II. FAMILIES OF MATRICES

We briefly recall the Galois Field and Latin Squares in discrete mathematics \[21\]. Then we construct two families of matrices by using operations of Galois field $F_q$ and establish some properties.

A. Galois fields

Write $F_2 = \{0, 1\}$ for the prime field of order 2. Let $m$ be a positive integer. In the rest of this paper, we assume that $q = 2^m$. The Galois Field of order $q$ is a finite field of characteristic 2, denoted by $F_q$. As a vector space over $F_2$, $F_q$ is $m$-dimensional, and so the elements of $F_q$ have a one to one correspondence to ones of $F_2^m$.

More precisely, for any $i$ in $F_q$, there exist $\omega_1, \omega_2, \cdots, \omega_m$ in $F_2$ such that $i$ can be represented as follows

$$i = \omega_1 \omega_2 \cdots \omega_m,$$

with respect to some basis of the vector space $F_q$ over $F_2$. In particular, 0 $\in F_q$ can represented by

$$\underbrace{0 \cdots 0}_m.$$

Naturally, there is an $F_q$-indexed square matrix over $F_q$ arisen by the multiplication table of $F_q$, denoted by $G(q)$, that is, the $(i, j)$-th entry is

$$g^{(q)}_{i,j} = ij$$ (II.1)

for any $i$ and $j$ in $F_q$.

**Lemma II.1.** For any given $q = 2^m$, the diagonal of $G(q)$ is a permutation of the elements of $F_q$. 
Proof. It is an immediate consequence of the fact that the Frobenius map of $\mathbb{F}_q$ is a bijection. For completeness, we give a whole proof.

Write $Z$ for the set $\{a^2 \mid a \in \mathbb{F}_q\}$. It suffices to show that the set $Z$ equals $\mathbb{F}_q$. Suppose $a^2 = b^2$ where $a, b \in \mathbb{F}_q$. Since the characteristic of the Galois field $\mathbb{F}_q$ is 2, we have $b^2 = -b^2$, $2ab = 0$, and

$$(a - b)^2 = a^2 + b^2 = a^2 - b^2 = 0.$$ 

Hence, $a = b$. It implies there are $q$ elements in $Z$, and $Z = \mathbb{F}_q$. \qed

B. Two families of matrices

A Latin square of order $q$ is a square matrix of order $q$ with entries from a set of cardinality $q$ such that each element occurs once in each row and each column. Two Latin squares $L$ and $L'$ are said to be orthogonal if all the ordered pairs are different. A collection of Latin squares of order $q$, any pair of which is orthogonal, is called a set of mutually orthogonal Latin squares. Any Galois field $\mathbb{F}_q$ generates $q - 1$ different orthogonal Latin squares of $q$ symbols.

We define a family of $\mathbb{F}_q$-indexed square matrices $\{L^r\}_{r \in \mathbb{F}_q}$ over $\mathbb{F}_q$, where the $(i, j)$-th entry of $L^r$ is

$$l^r_{i,j} = g^{(q)}_{i,r} + j,$$  \hspace{1cm} (II.2)

for any $i, j, r$ in $\mathbb{F}_q$.

Remark II.2. It is not hard to check that $\{L^r\}_{r \in \mathbb{F}_q}$ is a family of $q - 1$ different orthogonal Latin squares, where $\mathbb{F}_q^* = \mathbb{F}_q - \{0\}$.

There is a useful property of $\{L^r\}_{r \in \mathbb{F}_q}$.

Proposition II.3. Let $\{L^r\}_{r \in \mathbb{F}_q}$ be the family of squares matrices defined in (II.2).

1) For any distinct $j_1, j_2 \in \mathbb{F}_q$ and any $i, r \in \mathbb{F}_q$, $l^r_{i,j_1} \neq l^r_{i,j_2}$.
2) For any distinct $r, r' \in \mathbb{F}_q$ and any $j_1, j_2 \in \mathbb{F}_q$, there exists a unique $i \in \mathbb{F}_q$ such that

$$l^r_{i,j_1} = l^{r'}_{i,j_2}.$$ 

Proof. 1) Since $j_1 \neq j_2$, we have

$$g^{(q)}_{i,r} + j_1 \neq g^{(q)}_{i,r} + j_2$$

for any $i, r \in \mathbb{F}_q$. It implies that $l^r_{i,j_1} \neq l^r_{i,j_2}$ by (II.2).

2) Since $r \neq r'$, it is easy to deduce that

$$\mathbb{F}_q = \{(r - r')i \mid i \in \mathbb{F}_q\}.$$ 

For any $j_1, j_2 \in \mathbb{F}_q$, there exits a unique $i \in \mathbb{F}_q$ such that $(r - r')i = j_2 - j_1$. Then

$$l^r_{i,j_1} = g^{(q)}_{i,r} + j_1 = ri + j_1 = r'i + j_2 = g^{(q)}_{i,r'} + j_2 = l^{r'}_{i,j_2}.$$ 

The proof is completed. \qed

We choose one column from each of Latin squares $\{L^r \mid r \in \mathbb{F}_q\}$ to define a new $\mathbb{F}_q$-indexed square matrix $A^{(q)}$ over $\mathbb{F}_q$ such that the $j$-th column $a^{(q)}_j$ of $A^{(q)}$ satisfies

$$a^{(q)}_j = l^j_{i,j}, \quad j \in \mathbb{F}_q,$$  \hspace{1cm} (II.3)

where $l^j_{i,j}$ is the $j$-th column vector of $L^i$ for any $j \in \mathbb{F}_q$. More precisely, the $(i, j)$-th entry $a^{(q)}_{i,j}$ of $A^{(q)}$ is

$$a^{(q)}_{i,j} = ij + j^2 = l^{ij}_{i,j}, \quad i, j \in \mathbb{F}_q.$$  \hspace{1cm} (II.4)

Theorem II.4. For any given $q = 2^m$, the $0$-th row of $A^{(q)}$ is a permutation of the elements of $\mathbb{F}_q$, while $a^{(q)}_{i,j} = a^{(q)}_{i,j}$ if and only if $j_1 + j_2 = i$ for any $i, j_1, j_2 \in \mathbb{F}_q$ and $j_1 \neq j_2$.

Proof. Clearly, each entry of $0$-th row of $A^{(q)}$ is $a^{(q)}_{0,j} = j^2 = g^{(q)}_{j,j}$ for $j \in \mathbb{F}_q$. It is exactly the diagonal of $G^{(q)}$, then the first result follows from Lemma II.1

For any given $i \in \mathbb{F}_q$, assume $j_1 + j_2 = i$ where $j_1, j_2 \in F_q$ and $j_1 \neq j_2$. We have

$$a^{(q)}_{i,j_1} = ij_1 + j_1^2 = ((j_1 + j_2)j_1) + j_1^2 = j_2j_1,$$

and

$$a^{(q)}_{i,j_2} = ij_2 + j_2^2 = ((j_1 + j_2)j_2) + j_2^2 = j_1j_2.$$
Proof. We fix a basis of the vector space \( \mathbb{F}_q \) and \( j_1, j_2 \in \mathbb{F}_q \) and \( j_1 \neq j_2 \), that is, 
\[
i j_1 + j_1^2 = i j_2 + j_2^2.
\]
Adding \( ij_2 + j_2^2 \) to both sides of the equation above, one obtains that 
\[
i(j_1 + j_2) = j_1^2 + j_2^2 = (j_1 + j_2)^2.
\]
Then \( i = j_1 + j_2 \) holds, since \( j_1 + j_2 \neq 0 \) in \( \mathbb{F}_q \). \( \Box \)

III. Hadamard Matrix

The construction of MUBs in \( \mathbb{F}_2^m \) are intimately linked to Hadamard matrices. For the desired construction of sparse vectors in the null space, this section focuses on a special family of real Hadamard matrices. Recall that a Hadamard matrix \( H \) is a square matrix whose entries are either +1 or −1 and whose columns are mutually orthogonal. Let 
\[
H(2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]
be a Hadamard matrix of order 2. In a natural way, the Hadamard matrix \( H(q) \) can be viewed as an \( \mathbb{F}_2^m \)-indexed matrix, that is, the entries of \( H(q) \) are \( h_{0,0}^{(2)} = h_{0,1}^{(2)} = h_{1,0}^{(2)} = 1 \) and \( h_{1,1}^{(2)} = -1 \).

For any given positive integer \( m \), by repeating used of the Hadamard matrix \( H(2) \), a Hadamard matrix of order \( q = 2^m \) can be obtained as follows \[22]\n\[
H(q) = H(2) \otimes \cdots \otimes H(2).
\]
Clearly, \( H(q) \) is an \( \mathbb{F}_2^m \)-indexed matrix. To be precise, the \(((\omega_1 \omega_2 \cdots \omega_m), (\nu_1 \nu_2 \cdots \nu_m))\)-th entry of \( H(q) \) is 
\[
h_{\omega_1, \nu_1}^{(2)} h_{\omega_2, \nu_2}^{(2)} \cdots h_{\omega_m, \nu_m}^{(2)},
\]
for any \( \omega_1, \omega_2, \ldots, \omega_m, \nu_1, \nu_2, \ldots, \nu_m \in \mathbb{F}_2 \). Such Hadamard matrices have many applications in computer science and quantum information \[23\].

Let \( \varphi \) be a bijective self-mapping of \( \mathbb{F}_2^m \) such that \( \varphi(0) = 0 \), and \( H(\varphi) \) an \( \mathbb{F}_2^m \)-indexed matrix satisfying the \((\omega_1 \cdots \omega_m)\)-th row of \( H(\varphi) \) is the \( \varphi(\omega_1 \cdots \omega_m) \)-th row of \( H(q) \) for any \( \omega_1, \cdots, \omega_m \in \mathbb{F}_2 \). It is clear that \( H(\varphi) \) is also a Hadamard matrix. To our end, we choose such a bijection \( \sigma \) as follows 
\[
\sigma : \mathbb{F}_2^m \rightarrow \mathbb{F}_2^m,
\]
where if \( k \) is the largest integer such that \( \omega_k \) is nonzero, \( \omega'_1 \omega'_2 \cdots \omega'_m \) satisfies 
\[
\begin{cases}
\omega'_s + \omega_s = 1, & s = 1, \ldots, k - 1, \\
\omega'_k = \omega_k = 1, & s = k + 1, \ldots, m.
\end{cases}
\]

In the sequel, we denote the permuted Hadamard matrix \( H(q) \) by \( \hat{H}(q) \).

Since each element of \( \mathbb{F}_q \) has a representation by one of \( \mathbb{F}_2^m \) with respect to some basis of the vector space \( \mathbb{F}_q \) over \( \mathbb{F}_2 \), the Hadamard matrix \( H(q) \) and \( \hat{H}(q) \) are also \( \mathbb{F}_q \)-indexed matrices. We establish useful properties of the permuted Hadamard matrix \( \hat{H}(q) \).

**Theorem III.1.** For any given \( q = 2^m \), the permuted Hadamard matrix \( \hat{H}(q) \) satisfies the following conditions,

1) the entries in 0-th row and 0-th column of \( \hat{H}(q) \) are 1;
2) \( h_{i,j}^{(q)} = -\hat{h}_{i,j}^{(q)} \) for \( i \neq j \), \( j_1 + j_2 = i \) and \( i, j_1, j_2 \in \mathbb{F}_q \),

where \( \hat{h}_{i,j}^{(q)} \) is \((i,j)\)-th entry of \( \hat{H}(q) \) for any \( i, j \in \mathbb{F}_q \).

**Proof.** We fix a basis of the vector space \( \mathbb{F}_q \) over \( \mathbb{F}_2 \) in this proof. Write \( \hat{h}_{i,j}^{(q)} \) for \((i,j)\)-th entry of \( \hat{H}(q) \) for any \( i, j \in \mathbb{F}_q \).

Since \( \sigma(0) = 0 \), the 0-th row of \( \hat{H}(q) \) is the 0-th row of \( H(q) \) whose entries are all 1. For any \( j \in \mathbb{F}_q \), it is not hard to check that \( \hat{h}_{j,0}^{(q)} \) is a multiplication of elements of 0-th column of \( H(2) \), and so it equals 1.

Now we assume \( i = \omega_1 \omega_2 \cdots \omega_m \) is a nonzero element of \( \mathbb{F}_q \), where \( \omega_1, \omega_2, \ldots, \omega_m \in \mathbb{F}_2 \). Let \( i' = \sigma(\omega_1 \omega_2 \cdots \omega_m) = \omega'_1 \omega'_2 \cdots \omega'_m \in \mathbb{F}_q \) where \( \omega'_1, \omega'_2, \ldots, \omega'_m \in \mathbb{F}_2 \) satisfies (III.1). By the definition of the permuted Hadamard matrix, the \( i \)-th row of \( \hat{H}(q) \) is exactly the \( i' \)-th row of \( H(q) \). For any \( j_1, j_2 \in \mathbb{F}_q \) such that \( j_1 + j_2 = i \), \( \hat{h}_{i,j_1}^{(q)} = -\hat{h}_{i,j_2}^{(q)} \) is equivalent to \( h_{j_1,j_2}^{(q)} = -h_{j_1,j_2}^{(q)} \).
Write \( j_1 = \nu_1 \nu_2 \cdots \nu_m \) and \( j_2 = \nu'_1 \nu'_2 \cdots \nu'_m \) where \( \nu_1, \cdots, \nu_m, \nu'_1, \cdots, \nu'_m \in \mathbb{F}_2 \). Then

\[
\begin{align*}
 h^{(q)}_{i,j_1} &= h^{(2)}_{\omega'_1 \nu'_1} h^{(2)}_{\omega'_2 \nu'_2} \cdots h^{(2)}_{\omega'_m \nu'_m}, \\
 h^{(q)}_{i,j_2} &= h^{(2)}_{\omega'_1 \nu'_1} h^{(2)}_{\omega'_2 \nu'_2} \cdots h^{(2)}_{\omega'_m \nu'_m}
\end{align*}
\]

and

\[
\begin{align*}
 h^{(q)}_{i,j_1 + j_2} &= h^{(2)}_{\omega'_1 \nu'_1} h^{(2)}_{\omega'_2 \nu'_2} \cdots h^{(2)}_{\omega'_m \nu'_m}
\end{align*}
\]

The condition is

\[
\begin{align*}
 j_1 + j_2 &= \nu_1 \nu_2 \cdots \nu_m + \nu'_1 \nu'_2 \cdots \nu'_m = \omega_1 \omega_2 \cdots \omega_m.
\end{align*}
\]

Notice that the addition of \( \nu_1 \nu_2 \cdots \nu_m \) and \( \nu'_1 \nu'_2 \cdots \nu'_m \) in the vector space \( \mathbb{F}_q \) over \( \mathbb{F}_2 \) is a kind of the binary XOR operation.

Let \( k \) be the largest integer such that \( \omega_k \) is nonzero. There are three cases.

- If \( s = 1, \ldots, k-1 \), then \( \omega'_s + \omega_s = 1 \), and either \( \omega_s = 0, \omega'_s = 1 \) or \( \omega_s = 1, \omega'_s = 0 \).
  
  (1) If \( \omega_s = 0, \omega'_s = 1 \), then \( \nu_s = \nu'_s \). Hence,
  
  \[
  h^{(2)}_{\omega'_s, \nu'_s} = h^{(2)}_{\omega'_s, \nu'_s}.
  \]

  (2) If \( \omega_s = 1, \omega'_s = 0 \), one obtains that
  
  \[
  h^{(2)}_{\omega'_s, \nu'_s} = 1 = h^{(2)}_{\omega'_s, \nu'_s}.
  \]

- If \( s = k \), then \( \omega'_s = \omega_s = 1 \), and either \( \nu_s = 1, \nu'_s = 0 \) or \( \nu_s = 0, \nu'_s = 1 \). We have
  
  \[
  h^{(2)}_{\omega'_s, \nu'_s} = -1, \quad h^{(2)}_{\omega'_s, \nu'_s} = 1
  \]

  or

  \[
  h^{(2)}_{\omega'_s, \nu'_s} = 1, \quad h^{(2)}_{\omega'_s, \nu'_s} = -1.
  \]

  It implies that

  \[
  h^{(2)}_{\omega'_s, \nu'_s} = -h^{(2)}_{\omega'_s, \nu'_s}.
  \]

- If \( s = k+1, \ldots, m \), then \( \omega'_s = \omega_s = 0 \), and
  
  \[
  h^{(2)}_{\omega'_s, \nu'_s} = h^{(2)}_{\omega'_s, \nu'_s} = 1.
  \]

In conclusion, \( h^{(q)}_{i,j_1 + j_2} = -h^{(q)}_{i,j_1} \), \( \blacksquare \)

IV. MUBs in Square Dimensions

The concept of MUBs plays an important role in quantum information theory [19], [24]. MUBs are uniform tight frames which have been well studied in computational harmonic analysis [25], [26]. There exist numerous ways of constructing sets of MUBs, see e.g. [24], [19], [20] and references therein. This section recalls the method introduced in [20]. Following the line in [20], we construct \((q+1,q)-\text{net}\) from mutually orthogonal Latin squares obtained in subsection II-B. Then we define a family of MUBs using \((k,q)-\text{nets}\) and permuted Hadamard matrices obtained in section III.

In the sequel, write \( \mathbf{1}_q \) for the \( \mathbb{F}_q \)-indexed column vector over \( \mathbb{R} \) with all entries that are equal to one, and \( e_i \) for the \( i \)-th column vector of \( \mathbb{F}_q \)-indexed identity matrix \( I \) over \( \mathbb{R} \) for any \( i \in \mathbb{F}_q \).

A. \((q+1,q)-\text{net}\)

A column vector \( \mathbf{m} := (m_1, \ldots, m_d)^T \) of size \( d \) is called to be an incidence vector if its entries take only the values 0 and 1. Nets are collections of incidence vectors satisfying special properties. The definition of \((q+1,q)-\text{net}\) is from the design theory [18].

**Definition IV.1 \(((k,q)-\text{net})\), [20] Definition I** Let \( K \) be an indexed set with \( k \) elements, and \( \{ m_{b,j} \mid j \in \mathbb{F}_q \} \) be a set of incidence vectors of size \( d = q^2 \) for any \( b \in K \). The collection of incidence vectors

\[
\{ \{ m_{b,j} \} \mid j \in \mathbb{F}_q, b \in K \}
\]

is called a \((k,q)-\text{net}\), if the following conditions hold

1. \( \langle m_{b,i}, m_{b,j} \rangle = 0 \), for any distinct \( i, j \in \mathbb{F}_q \) and \( b \in K \);  
2. \( \langle m_{b,i}, m_{c,j} \rangle = 1 \), for any distinct \( b, c \in K \) and \( i, j \in \mathbb{F}_q \).

The relationship between mutually orthogonal Latin squares and nets has been discussed in [20]. Let \( K = \mathbb{F}_q \cup \{ \infty \} \). Using the square matrices \( \{ L^r \}_{r \in \mathbb{F}_q} \) over \( \mathbb{F}_q \) obtained in (II.2), we construct a collection of incidence vectors to be a \((q+1,q)-\text{net}\) as follows,

1. If \( b \in \mathbb{F}_q \), \( m_{b,j} \) is an \( \mathbb{F}_q^2 \)-indexed vector consisting of \( q \)-blocks \( \{ e_{u,j} \mid u \in \mathbb{F}_q \} \), that is, 

\[
 m_{b,j} = \left( e_{u,j} \right)_{u \in \mathbb{F}_q}, \quad j \in \mathbb{F}_q.
\]

(IV.1)
2) If \( b = \infty \), \( m_{b,j} \) is an \( \mathbb{F}_q^2 \)-indexed vector satisfying
\[
m_{\infty,j} = e_j \otimes 1_q, \quad j \in \mathbb{F}_q.
\] (IV.2)

Similar to the case of \( b \in \mathbb{F}_q \), \( m_{\infty,j} \) also consists of \( q \)-blocks, where \( (m_{\infty,j})_j = 1_q \) and \( (m_{\infty,j})_u = 0_q \) if \( u \neq j \), for any \( j \in \mathbb{F}_q \).

**Theorem IV.2.** The collection of incidence vectors
\[
\{ \{ m_{b,j} \} \in \mathbb{F}_q \mid b \in \mathbb{F}_q \cup \{ \infty \} \}
\]

obtained in (IV.1) and (IV.2) is a \((q + 1, q)\)-net.

**Proof.** Let \( b \) be a given element of \( \mathbb{F}_q \). For any distinct \( j_1, j_2 \in \mathbb{F}_q \), then \( t_{u,j_1}^b \neq t_{u,j_2}^b \) by Proposition [I.3] Hence,
\[
(\langle m_{b,j_1}, m_{b,j_2} \rangle = \sum_{u \in \mathbb{F}_q} \langle e_{u,j_1}^b, e_{u,j_2}^b \rangle = 0.
\]

For any distinct \( b, b' \in \mathbb{F}_q \) and any \( j_1, j_2 \in \mathbb{F}_q \), there exists a unique \( i \in \mathbb{F}_q \) such that \( t_{i,j_1}^b = t_{i,j_2}^{b'} \) by Proposition [I.3]. One obtains
\[
(\langle m_{b,j_1}, m_{b',j_2} \rangle = \sum_{u \neq i} \langle e_{u,j_1}^b, e_{u,j_2}^{b'} \rangle + \langle e_{i,j_1}^b, e_{i,j_2}^{b'} \rangle = 1.
\]

It’s easy to verify that for any distinct \( j_1, j_2 \in \mathbb{F}_q \),
\[
(\langle m_{\infty,j_1}, m_{\infty,j_2} \rangle = 0.
\]

For any \( b, j_1, j_2 \in \mathbb{F}_q \), we have
\[
(\langle m_{\infty,j_1}, m_{b,j_2} \rangle = \sum_{u \in \mathbb{F}_q} \langle (m_{\infty,j_1})_u, e_{u,j_2}^b \rangle = \langle 1_q, e_{i,j_2}^b \rangle = 1.
\]

The proof is completed. \( \Box \)

**B. MUBs**

Let
\[
B_0 = (\phi_1 \phi_2 \cdots \phi_d)
\]

and
\[
B_1 = (\psi_1 \psi_2 \cdots \psi_d)
\]

denote orthonormal bases in the \( d \)-dimensional space. Then they are said to be mutual unbiased if and only if
\[
|\langle \phi_i, \psi_j \rangle | = \frac{1}{\sqrt{d}} \quad (IV.3)
\]

for all \( i, j \). The quantity (IV.3) is called the mutual coherence in compressed sensing [7]. A set \( \{ B_0, \ldots, B_d \} \) of orthonormal bases is said to be a set of mutually unbiased bases (MUB) if and only if every pair of bases in the set is mutually unbiased.

Let
\[
\{ \{ m_{b,j} \} \in \mathbb{F}_q \mid b \in \mathbb{F}_q \cup \{ \infty \} \}
\]

be the \((q + 1, q)\)-net obtained in (IV.1) and (IV.2). For any \( \mathbb{F}_q \)-indexed column vector \( h \) over \( \mathbb{R} \), the embedding of \( h \) into \( \mathbb{R}^q \) controlled by \( m_{b,j} \) is an \( \mathbb{F}_q^2 \)-indexed vector, denoted by \( h \uparrow m_{b,j} \), satisfying
\[
( h \uparrow m_{b,j} := (h_u e_{u,j}^b)_{u \in \mathbb{F}_q}, \quad b, j \in \mathbb{F}_q,
\]
\[
( h \uparrow m_{\infty,j} := e_j \otimes h, \quad j \in \mathbb{F}_q,
\]

where \( h_u \) is the \( u \)-th entry of \( h \). Using the permuted Hadamard matrix \( \tilde{H}^{(q)} \), we construct a collection of \( \mathbb{F}_q^2 \times \mathbb{F}_q \)-indexed matrices
\[
\{ (B_b)_u \mid b \in \mathbb{F}_q \cup \{ \infty \}, u \in \mathbb{F}_q \}
\]
such that the \( v \)-th column of \((B_b)_u \) is \( \tilde{h}_v^{(q)} \uparrow m_{b,u} \), or equivalently
\[
( B_b )_u = ( \tilde{h}_v^{(q)} \uparrow m_{b,u} )_{v \in \mathbb{F}_q},
\]
where $\tilde{h}_i^{(q)}$ is the $v$-th column of $\tilde{H}^{(q)}$ for any $v \in \mathbb{F}_q$. Then there is a collection of $\mathbb{F}^2_q$-indexed square matrices $\{B_b \mid b \in \mathbb{F}_q \cup \{\infty\}\}$ over $\mathbb{R}$, where $B_b$ consists of $q$-blocks $\{(B_b)_{u} \mid u \in \mathbb{F}_q\}$, that is

$$B_b = \frac{1}{\sqrt{q}} (B_b)_{u\in\mathbb{F}_q}. \quad \text{(IV.4)}$$

The following result follows from Theorem [IV.2] and [20] Theorem 3.

**Theorem IV.3.** The orthonormal bases $\{B_b \mid b \in \mathbb{F}_q \cup \{\infty\}\}$ constructed in (IV.4) are $q + 1$ mutually unbiased bases for $\mathbb{R}^{q^2}$.

V. PROOFS OF MAIN RESULTS

A. Proof of Theorem [IV.1]

Using the orthonormal bases $\{B_b \mid b \in \mathbb{F}_q \cup \{\infty\}\}$ obtained in (IV.4), we construct a dictionary by

$$D = (B_b)_{b\in\mathbb{F}_q\cup\{\infty\}} \in \mathbb{R}^{q^2 \times q^2(q+1)}. \quad \text{(V.1)}$$

The dictionary $D$ constructed in (V.1) consists of $q + 1$ block matrices indexed by $\mathbb{F}_q \cup \{\infty\}$. It follows from Theorem [IV.3] that

$$\mu(D) = 1/q.$$ 

We define a collection of one sparse vectors as follows

$$\begin{align*}
\{x^b = e_b \otimes e_b \in \mathbb{R}^{q^2}, \quad b \in \mathbb{F}_q, \}
\{x^\infty = -e_0 \otimes e_0 \in \mathbb{R}^{q^2}. \quad \text{(V.2)}\end{align*}$$

Then we obtain a $q + 1$ sparse column vector

$$x = (x^b)_{b\in\mathbb{F}_q\cup\{\infty\}} \in \mathbb{R}^{q^2(q+1)}. \quad \text{(V.3)}$$

**Theorem V.1.** The $q + 1$ sparse vector $x$ constructed in (V.2) is in the null space of $D$ in (V.1), i.e.,

$$\sqrt{q}Dx = \sum_{b\in\mathbb{F}_q\cup\{\infty\}} \sqrt{q}B_b x^b = 0_{q^2}.$$ 

**Proof.** The proof is divided into two cases.

1) For $b \in \mathbb{F}_q$, we have

$$\sqrt{q}B_b x^b = ((B_b)_{b\in\mathbb{F}_q}) e_b = \left(\tilde{h}_i^{(q)} \uparrow m_{b,b^2}\right)_{v\in\mathbb{F}_q} e_b = \tilde{h}_i^{(q)} \uparrow m_{b,b^2}$$

$$= \left(\tilde{h}_i^{(q)} e_{a_{\infty}^{b}}\right)_{i\in\mathbb{F}_q} = \left(\tilde{h}_i^{(q)} e_{a_{\infty}^{b}}\right)_{i\in\mathbb{F}_q},$$

where the last equality follows from (II.4). For any $i \in \mathbb{F}_q$, there are two subcases

a) If $i = 0$,

$$\sum_{b\in\mathbb{F}_q} \tilde{h}_i^{(q)} e_{a_{\infty}^{b}} = \sum_{b\in\mathbb{F}_q} e_{a_{\infty}^{b}} = 1_q,$$

where the first equality follows from Theorem [III.1] and the second one follows from Theorem [II.4]

b) If $i \neq 0$,

$$\sum_{b\in\mathbb{F}_q} \tilde{h}_i^{(q)} e_{a_{\infty}^{b}} = \sum_{b\in\mathbb{F}_q} \tilde{h}_i^{(q)} e_{a_{\infty}^{b}} = \sum_{b\in\mathbb{F}_q} \tilde{h}_i^{(q)} e_{a_{\infty}^{b}},$$

where the second equality follows from Theorem [III.1] and Theorem [II.4] So

$$\sum_{b\in\mathbb{F}_q} \tilde{h}_i^{(q)} e_{a_{\infty}^{b}} = 0_q.$$ 

Hence,

$$\sum_{b\in\mathbb{F}_q} \sqrt{q}B_b x^b = \sum_{b\in\mathbb{F}_q} \left(\tilde{h}_i^{(q)} e_{a_{\infty}^{b}}\right)_{i\in\mathbb{F}_q} = e_0 \otimes 1_q.$$
2) For \( b = \infty \), one obtains
\[
\sqrt{q}B_{\infty}x^{\infty} = -((B_{\infty})_{0})e_{0}
\]
\[
= -\left(\tilde{h}_{0}^{(q)} \uparrow m_{\infty,0}\right)_{\forall q \in \mathbb{F}_q} e_{0}
\]
\[
= -\tilde{h}_{0}^{(q)} \uparrow m_{\infty,0}
\]
\[
= -1_{q} \uparrow m_{\infty,0}
\]
\[
= -e_{0} \otimes 1_{q}.
\]

Combining two cases, we have
\[
\sqrt{q}Dx = \sum_{b \in \mathbb{F}_q \cup \{\infty\}} \sqrt{q}B_{b}x^{b} = (e_{0} \otimes 1_{q} - e_{0} \otimes 1_{q}) = 0_{q^{2}}.
\]

Theorem \( \text{I.1} \) follows from Theorem \( \text{V.1} \) directly.

**B. Proof of Theorem \( \text{I.2} \)**

In this subsection, we focus on the field extension \( \mathbb{F}_q \subseteq \mathbb{F}_q^2 \). Then we have a quotient group \( \mathbb{F}_q^2/\mathbb{F}_q \). For any \( i \in \mathbb{F}_q^2 \), write \([i]\) for the coset of \( i \) in \( \mathbb{F}_q^2/\mathbb{F}_q \).

Obviously, \( \mathbb{F}_q \) and \( \mathbb{F}_q^2/\mathbb{F}_q \) are both 1-dimensional vector spaces over \( \mathbb{F}_q \), and they are isomorphic to each other as vector spaces over \( \mathbb{F}_q \). Write \( \xi \) for an isomorphism from \( \mathbb{F}_q \) to \( \mathbb{F}_q^2/\mathbb{F}_q \).

**Lemma V.2.** Let \( i \) be an element of \( \mathbb{F}_q^2 \) and \( b_* \) the element in \( \mathbb{F}_q \) such that \( \xi(b_*) = [i] \). Then
\[
\{t_{i,j}^{l} \mid [j] = i(i^2), j \in \mathbb{F}_q^2 \} = \{t_{i,j}^{l_{i,j}} \mid [j] = \xi(b_*^2) \}, \forall j \in \mathbb{F}_q^2
\]

for any \( b \in \mathbb{F}_q \).

**Proof.** Write \( S_b = \{t_{i,j}^{l} \mid [j] = \xi(b^2), j \in \mathbb{F}_q^2 \} \) for any \( b \in \mathbb{F}_q \). For any given \( b \in \mathbb{F}_q \) and any \( j \in \mathbb{F}_q^2 \) such that \([j] = \xi(b^2)\), we have
\[
[j + b_*^2] = [j] + [b_*] = \xi(b^2) + b_*i = \xi(b_*^2) + [b_*] = \xi(b_*^2) + [b_*] = \xi((b_* - b)^2).
\]
in \( \mathbb{F}_q^2/\mathbb{F}_q \), and
\[
t_{i,j}^{l} = bi + j = (b_* - b)i + j + b_*i = t_{i,j}^{l_{i,j}} + b_*i.
\]

Hence \( S_b \subseteq S_{b_* - b} \). One also obtains that \( S_{b_* - b} \subseteq S_{b_* - (b_* - b)} = S_b \). The result follows.

Recall that each element of \( \mathbb{F}_q \) (resp. \( \mathbb{F}_q^2 \)) can be represented by one of \( \mathbb{F}_q^{2n} \) (resp. \( \mathbb{F}_q^{2m} \)) with respect to some basis of the vector space \( \mathbb{F}_q \) (resp. \( \mathbb{F}_q^2 \)) over \( \mathbb{F}_2 \). Note that \( \mathbb{F}_q \) is a subfield of \( \mathbb{F}_q^2 \). In the sequel, we fix a basis of the vector space \( \mathbb{F}_q^2 \) over \( \mathbb{F}_2 \) such that any element of \( \mathbb{F}_q \) in \( \mathbb{F}_q^2 \) has the form
\[
\omega_1 \omega_2 \cdots \omega_m \underbrace{00 \cdots 0}_m.
\]

where \( \omega_1, \omega_2, \cdots, \omega_m \in \mathbb{F}_2 \). Then one obtains that
\[
\mathbb{F}_q^2/\mathbb{F}_q = \{(00 \cdots 0\omega_1 \omega_2 \cdots \omega_m) \mid \omega_1, \omega_2, \cdots, \omega_m \in \mathbb{F}_2\}.
\]

We define a map
\[
\iota : \mathbb{F}_q^2/\mathbb{F}_q \to \mathbb{F}_q^2/\mathbb{F}_q \quad \iota(b) = 00 \cdots 0 \omega_1 \cdots \omega_m.
\]

where \([00 \cdots 0\omega_1 \cdots \omega_m] = \xi(b)\).

**Lemma V.3.** Let \( b \) be any given element in \( \mathbb{F}_q \subseteq \mathbb{F}_q^2 \) and \( i \) an element in \( \mathbb{F}_q^2 \).

1) If \( i \in \mathbb{F}_q \), then \( \iota(h_{i,i}(q^2)) = 1 \).

2) If \( i \notin \mathbb{F}_q \), then \( \iota(h_{i,i}(q^2)) + \iota(h_{i,i}(b_* - b)) = 0 \) where \( b_* \) is the unique nonzero element in \( \mathbb{F}_q \) such that \( \xi(b_*) = [i] \).

**Proof.** 1) Since \( i \in \mathbb{F}_q \), we have
\[
i = \omega_1 \omega_2 \cdots \omega_m 00 \cdots 0,
\]
for some \( \omega_1, \omega_2, \ldots, \omega_m \in \mathbb{F}_q \). Let \( i' = \omega_1' \omega_2' \cdots \omega_m' (0) \cdots 0 \) be the element in \( \mathbb{F}_q^2 \) satisfying (III.1). Then \( \tilde{h}_{i, t}^{(q^2)} \) is the element \( h_{i', t}^{(q^2)} \) in the Hadamard matrix \( H^{(q^2)} \). Write

\[
i(b) = 00 \cdots 0 \nu_1 \nu_2 \cdots \nu_m,
\]

for some \( \nu_1, \nu_2, \ldots, \nu_m \in \mathbb{F}_q \). Then

\[
\tilde{h}_{i, t}^{(q^2)} = h_{i', t}^{(q^2)} = h_{\omega_1'0} \cdots h_{\omega_m'0} h_{00} h_{0v_1} h_{0v_2} \cdots h_{0v_m} = 1.
\]

2) Since \( i \notin \mathbb{F}_q \), we have

\[
i = \omega_1 \omega_2 \cdots \omega_m \omega_{m+1} \omega_{m+2} \cdots \omega_{2m},
\]

for some \( \omega_1, \omega_2, \ldots, \omega_{2m} \in \mathbb{F}_q \) and \( \omega_{m+1}, \omega_{m+2}, \ldots, \omega_{2m} \) are not all zero. Let \( i' \) be the element in \( \mathbb{F}_q^2 \) such that the \( i' \)-th row of the Hadamard matrix \( H^{(q^2)} \) is the \( i \)-th row of the permuted Hadamard matrix \( H^{(q^2)} \), denoted by \( i' = \omega_1' \omega_2' \cdots \omega_m' \omega_{m+1}' \omega_{m+2}' \cdots \omega_{2m}' \). Write

\[
i(b) = 00 \cdots 0 \nu_1 \nu_2 \cdots \nu_m,
\]

and

\[
i(b \ast - b) = 00 \cdots 0 \nu_1 \nu_2 \cdots \nu_m \nu_1' \nu_2' \cdots \nu_m',
\]

for some \( \nu_1, \nu_2, \ldots, \nu_m, \nu_1', \nu_2', \ldots, \nu_m' \in \mathbb{F}_q \). One obtains that

\[
[00 \cdots 0 \nu_1 \nu_2 \cdots \nu_m] + [00 \cdots 0 \nu_1' \nu_2' \cdots \nu_m'] = [i] = [00 \cdots 0 \omega_{m+1} \omega_{m+2} \cdots \omega_{2m}].
\]

We denote

\[
\begin{align*}
j_1 &= \nu_1 \nu_2 \cdots \nu_m, \\
j_2 &= \nu_1' \nu_2' \cdots \nu_m, \\
k &= \omega_{m+1} \omega_{m+2} \cdots \omega_{2m}, \\
k' &= \omega_{m+1} \omega_{m+2} \cdots \omega_{2m},
\end{align*}
\]

in \( \mathbb{F}_q \). It is not hard to check that \( k \) and \( k' \) satisfy the condition (III.1) and \( j_1 + j_2 = k \). By Theorem (III.1)

\[
\begin{align*}
\tilde{h}_{i, t}^{(q^2)} + \tilde{h}_{i, t}^{(q^2)} &= h_{i', t}^{(q^2)} + h_{i', t}^{(q^2)} \\
&= h_{\omega_1'0} \cdots h_{\omega_m'0} h_{00} h_{0v_1} h_{0v_2} \cdots h_{0v_m} \\
&= h_{\omega_{m+1}'0} \cdots h_{\omega_{2m}'0} + h_{\omega_{m+1}'0} h_{\omega_{2m}'0} + h_{\omega_{m+1}'0} h_{\omega_{2m}'0} \\
&= h_{k,j_1} + h_{k,j_2} \\
&= \tilde{h}_{k,j_1} + \tilde{h}_{k,j_2} \\
&= 0.
\end{align*}
\]

Using the orthonormal bases

\[
\{ B_c \mid c \in \mathbb{F}_q \cup \{ \infty \} \}
\]

obtained in (IV.4), we construct a dictionary by

\[
D = (B_b)_{b \in \mathbb{F}_q \cup \{ \infty \} } \in \mathbb{R}^q \times q^{(q+1)}.
\]

The dictionary \( D \) constructed in (IV.4) consists of \( q + 1 \) block matrices indexed by \( \mathbb{F}_q \cup \{ \infty \} \). It follows from Theorem (IV.3) that

\[
\mu(D) = 1/q^2.
\]

Write

\[
\begin{align*}
y^b &= \sum_{[j] = \xi(b^2)} e_j \otimes e_\iota(b) \in \mathbb{R}^q, \quad b \in \mathbb{F}_q \\
y^\infty &= -\sum_{b \in \mathbb{F}_q} e_b \otimes e_0 \in \mathbb{R}^q.
\end{align*}
\]

Then we have a \( q^2 + q \) sparse column vector

\[
y = (y^b)_{b \in \mathbb{F}_q \cup \{ \infty \} } \in \mathbb{R}^{q \times (q+1)}.
\]
The proof is completed.

**Theorem V.4.** The $q^2 + q$ sparse vector $y$ constructed in (V.5) is in the null space of $D$ defined in (V.4), i.e.,

$$qDy = \sum_{b \in \mathbb{F}_q \cup \{\infty\}} qB_b y^b = 0_{q^2}. $$

**Proof.** We consider two cases.

1) For $b \in \mathbb{F}_q$, we have

$$qB_b y^b = \sum_{[j] = \xi(b^2)} ((B_b)_j)e_{i(b)} = \sum_{[j] = \xi(b^2)} \tilde{h}_{i, i(b)}(q^2) \uparrow m_{b, j} = \sum_{[j] = \xi(b^2)} \left( \tilde{h}_{i, i(b)}(q^2) e_{i, j} \right)_{i \in \mathbb{F}_{q^2}}.$$

For any $i \in \mathbb{F}_{q^2}$, there are two subcases.

a) If $i \in \mathbb{F}_q$,

$$\sum_{b \in \mathbb{F}_q} \sum_{[j] = \xi(b^2)} \tilde{h}_{i, i(b)}(q^2) e_{bi + j} = \sum_{b \in \mathbb{F}_q} \sum_{[j] = \xi(b^2)} e_j = 1_{q^2},$$

where the first equality follows from Lemma V.3 and the second follows from the fact cosets determining a partition of $\mathbb{F}_{q^2}$ and Lemma V.1.

b) If $i \notin \mathbb{F}_q$, then $[i] = \xi(b_s)$ for some nonzero $b_s \in \mathbb{F}_q$. One obtains that

$$\sum_{b \in \mathbb{F}_q} \sum_{[j] = \xi(b^2)} \tilde{h}_{i, i(b)}(q^2) e_{bi} = \sum_{b \in \mathbb{F}_q} \sum_{[j] = \xi(b^2)} \tilde{h}_{i, i(b)}(q^2) e_{b i - j} = \sum_{b \in \mathbb{F}_q} \sum_{[j] = \xi(b^2)} \tilde{h}_{i, i(b)}(q^2) e_{j},$$

where the second equality follows from Lemma V.2 and Lemma V.3. So

$$\sum_{b \in \mathbb{F}_q} \sum_{[j] = \xi(b^2)} \tilde{h}_{i, i(b)}(q^2) e_{j} = 0_{q^2}.$$

Hence,

$$\sum_{b \in \mathbb{F}_q} qB_b y^b = \sum_{b \in \mathbb{F}_q} \sum_{[j] = \xi(b)} \left( \tilde{h}_{i, i(b)}(q^2) e_{i, j} \right)_{i \in \mathbb{F}_{q^2}} = \sum_{b \in \mathbb{F}_q} e_b \otimes 1_{q^2}. $$

2) For $b = \infty$, one obtains

$$qB_{\infty} y^\infty = -\sum_{b \in \mathbb{F}_q} ((B_{\infty})_b)e_0$$

$$= -\sum_{b \in \mathbb{F}_q} \left( \tilde{h}_{i, i(b)}(q^2) \uparrow m_{\infty, b} \right) e_0$$

$$= -\sum_{b \in \mathbb{F}_q} \tilde{h}_{i, i(b)}(q^2) \uparrow m_{\infty, b}$$

$$= -\sum_{b \in \mathbb{F}_q} e_b \otimes \tilde{h}_{i, i(b)}(q^2)$$

$$= -\sum_{b \in \mathbb{F}_q} e_b \otimes 1_{q^2}.$$

The proof is completed. 

Theorem I.2 follows from Theorem V.4 directly.

**VI. Examples**

This section provides three examples to illustrate the process showed in Figure 1. In order to more easily visualize the dictionaries and sparse vectors in the null space, we map the +1 elements as red squares, the −1 elements as blue squares and 0 elements as gray squares. Figure 2, Figure 3 and Figure 4 show the dictionaries and vectors in Example VI.1, Example VI.2 and Example VI.3 respectively.

| - 1 0 1 |
| 0 0 -1 |
| 1 0 1 |

**TABLE II**

**MULTIPLICATION TABLE FOR $\mathbb{F}_2$**
Example VI.1. Let \( q = 2 \). It follows that
\[
\mathbb{F}_2 = \{0, 1\}.
\]
The multiplication table and addition table for \( \mathbb{F}_2 \) are shown in Table II and Table III. The matrix \( G^{(2)} \) defined in (II.1) is
\[
G^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Then the Latin squares (see (II.2)) are
\[
L^0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad L^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
The matrix \( A^{(2)} \) defined in (II.3) (or (II.4)) is
\[
A^{(2)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]
Six incidence vectors of \((3,2)\)-net constructed in (IV.1) and (IV.2) are as follows:
\[
m_{0,0} = (e_{0}^T e_{0}^T)^T = (1 \ 0 \ 0)^T, \\
m_{0,1} = (e_{1}^T e_{1}^T)^T = (0 \ 1 \ 0)^T; \\
m_{1,0} = (e_{0}^T e_{1}^T)^T = (1 \ 0 \ 0)^T, \\
m_{1,1} = (e_{1}^T e_{0}^T)^T = (0 \ 1 \ 0)^T; \\
m_{\infty,0} = e_{0} \otimes 1_q = (1 \ 1 \ 0)^T, \\
m_{\infty,1} = e_{1} \otimes 1_q = (0 \ 0 \ 1)^T.
\]
In this case, the permuted Hadamard matrix \( \tilde{H}^{(2)} \) equals \( H^{(2)} \). Two column vectors of the permuted Hadamard matrix \( \tilde{H}^{(2)} \) are
\[
\tilde{h}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \tilde{h}_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
\]
It follows from (IV.4) that MUBs consists of
\[
B_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{h}_0 \uparrow m_{0,0} & \tilde{h}_1 \uparrow m_{0,0} & \tilde{h}_0 \uparrow m_{0,1} & \tilde{h}_1 \uparrow m_{0,1} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},
\]
\[
B_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{h}_0 \uparrow m_{1,0} & \tilde{h}_1 \uparrow m_{1,0} & \tilde{h}_0 \uparrow m_{1,1} & \tilde{h}_1 \uparrow m_{1,1} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},
\]
and
\[
B_\infty = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{h}_0 \uparrow m_{\infty,0} & \tilde{h}_1 \uparrow m_{\infty,0} & \tilde{h}_0 \uparrow m_{\infty,1} & \tilde{h}_1 \uparrow m_{\infty,1} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.
\]
We remark that the above MUBs has been constructed in [20]. The dictionary \( D \) (see (VI.1)) is given by
\[
\sqrt{2}D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.
\]
By (V.2) and (V.3), the three sparse vector is given by
\[ x = \begin{pmatrix} x^0 \\ x^1 \\ x^\infty \end{pmatrix} \in \mathbb{R}^{12} \]
where three block vectors are
\[ x^0 = e_{g_{0,0}} \otimes e_0 = (1 0 0 0)^T, \]
\[ x^1 = e_{g_{1,1}} \otimes e_1 = (0 0 0 1)^T, \]
\[ x^\infty = -e_0 \otimes e_0 = (-1 0 0 0)^T. \]
By Theorem [VI], one can check such three sparse vector \( x \) is in the null space of \( D \), i.e.,
\[ \sqrt{2} Dx = 0_4. \]
Hence,
\[ \eta(D) = 1 + \frac{1}{\mu(D)} = 3, \]
and
\[ \mu(D)\eta(D) = \frac{1}{2} \times 3 = \left( 1 + \frac{1}{q} \right)_{q=2}. \]

Example VI.2. Let \( q = 2^2 \). By the common construction, the Galois field \( \mathbb{F}_{2^2} \) of order 4 is \( \mathbb{F}_2[x]/(x^2 + x + 1) \), and consists of the cosets of elements \( 0, 1, x, x^2 \). For convenience, we write \( 0, 1, 2 \) and \( 3 \) for the cosets of \( 0, 1, x \) and \( x^2 \) respectively in this example, that is,
\[ \mathbb{F}_{2^2} = \{ 0, 1, 2, 3 \}. \]
The multiplication table and addition table for \( \mathbb{F}_{2^2} \) are shown in Table IV and Table V.

The matrix \( G^{(4)} \) defined in (II.1) is
\[ G^{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 3 & 1 & 2 \end{pmatrix}. \]
Then the Latin squares (see (II.2)) are

\[
L_0 = \begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 \\
\end{pmatrix},
L_1 = \begin{pmatrix}
0 & 1 & 2 & 3 \\
1 & 0 & 3 & 2 \\
2 & 3 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{pmatrix},
L_2 = \begin{pmatrix}
0 & 1 & 2 & 3 \\
2 & 3 & 0 & 1 \\
3 & 2 & 1 & 0 \\
1 & 0 & 3 & 2 \\
\end{pmatrix},
L_3 = \begin{pmatrix}
0 & 1 & 2 & 3 \\
3 & 2 & 1 & 0 \\
1 & 0 & 3 & 2 \\
2 & 3 & 0 & 1 \\
\end{pmatrix}.
\]

The matrix \( A^{(4)} \) defined in (II.3) (or (II.4)) is

\[
A^{(4)} = \begin{pmatrix}
0 & 1 & 3 & 2 \\
0 & 0 & 1 & 1 \\
0 & 3 & 0 & 3 \\
0 & 2 & 2 & 0 \\
\end{pmatrix}.
\]

Analogue to the conversion between decimal and binary system, we write

\[
0 = 00, \quad 1 = 01, \quad 2 = 10, \quad 3 = 11,
\]
for the elements of \( \mathbb{F}_2^{2} \) with respect to some basis of the vector space \( \mathbb{F}_2^{2} \) over \( \mathbb{F}_2 \). Then the Hadamard matrix \( H^{(4)} \) and the permuted Hadamard matrix \( \tilde{H}^{(4)} \) satisfy the conditions in Theorem III.1 are

\[
H^{(4)} = \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\]

and

\[
\tilde{H}^{(4)} = \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

Consequently, we can obtain the orthonormal bases by (IV.1), (IV.2) and (IV.4),

\[
B_b = \frac{1}{2} \left( h_0 \uparrow m_{b,0}, \cdots, h_3 \uparrow m_{b,0}, \cdots, h_0 \uparrow m_{b,3}, \cdots, h_3 \uparrow m_{b,3} \right)
\]

where \( b = 0, 1, 2, 3, \infty \). The dictionary \( D \) (see (VI.1)) is

\[
D = (B_0 \ B_1 \ B_2 \ B_3 \ B_{\infty}) \in \mathbb{R}^{4^2 \times (4^2 \times (4+1))}.
\]

Following (V.2) and (V.3), five block vectors of \( x \) are defined as follows

\[
x^0 = e_0 \otimes e_0, \\
x^1 = e_1 \otimes e_1, \\
x^2 = e_3 \otimes e_2, \\
x^3 = e_2 \otimes e_3, \\
x^\infty = -e_0 \otimes e_0.
\]

By Theorem VI or direct calculation, we have

\[
2Dx = 0_{16}.
\]

Hence,

\[
\eta(D)|_{q=2^2} = 1 + \frac{1}{\mu(D)}|_{q=2^2} = 5,
\]

and

\[
\mu(D)\eta(D) = \frac{1}{4} \times 5 = \left( 1 + \frac{1}{q} \right)|_{q=2^2}.
\]
Example VI.3. We keep the notations in Example VI.1 and Example VI.2 and consider the field extension $\mathbb{F}_2 \subseteq \mathbb{F}_2^2$ in this example. Clearly the vector space $\mathbb{F}_2^2 / \mathbb{F}_2 = \{[0], [2]\}$, where $[0] = \{0, 1\}$ and $[2] = \{2, 3\}$. We define an isomorphism of vector spaces over $\mathbb{F}_2$ as follows,

$$\xi : \mathbb{F}_2 \rightarrow \mathbb{F}_2^2 / \mathbb{F}_2$$

$$0 \mapsto [0], \quad 1 \mapsto [2].$$

To our end, we choose an appropriate basis of the vector space $\mathbb{F}_2^2$ over $\mathbb{F}_2$ such that the elements of $\mathbb{F}_2^2$ has the following representations

$$0 = 00, \quad 1 = 10, \quad 2 = 01, \quad 3 = 11.$$

In this case, the Hadamard matrix $H^{(4)}$ and the permuted Hadamard matrix $\tilde{H}^{(4)}$ are

$$H^{(4)} = \begin{pmatrix}
00 & 10 & 01 & 11 \\
00 & 1 & 1 & 1 & 1 \\
10 & 1 & -1 & 1 & -1 \\
01 & 1 & 1 & -1 & -1 \\
11 & 1 & -1 & -1 & 1
\end{pmatrix},$$

and

$$\tilde{H}^{(4)} = \begin{pmatrix}
00 & 10 & 01 & 11 \\
00 & 1 & 1 & 1 & 1 \\
10 & 1 & -1 & 1 & -1 \\
01 & 1 & -1 & -1 & 1 \\
11 & 1 & 1 & -1 & -1
\end{pmatrix}.$$

Then we define a map

$$\iota : \mathbb{F}_2 \rightarrow \mathbb{F}_2^2$$

$$0 \mapsto 00, \quad 1 \mapsto 01.$$

Consequently, we can obtain the orthonormal bases,

$$B_b = \frac{1}{2} \left( \begin{array}{c}
h_0 \uparrow m_{b,0}, \cdots, h_3 \uparrow m_{b,0}, \cdots, h_0 \uparrow m_{b,3}, \cdots, h_3 \uparrow m_{b,3} \end{array} \right)$$

where $b = 0, 1, 2, 3, \infty$. Differing from Example VI.2 the dictionary $D$ is

$$D = (B_0 \ B_1 \ B_\infty) \in \mathbb{R}^{(2^2)^2 \times ((2^2)^2 \times (2+1))}.$$  (VI.2)

Following (VI.5) and (VI.6), three block vectors of $y$ are defined by

$$y^0 = (e_0 + e_1) \otimes e_0, \quad y^1 = (e_2 + e_3) \otimes e_0, \quad y^\infty = -(e_0 + e_1) \otimes e_0.$$

By Theorem VI.4

$$2Dy = 0_{16}.$$

We see that the dictionary defined in (VI.2) satisfies

$$\eta(D) = 6 > 1 + \frac{1}{\mu(D)} = 5.$$
Fig. 4. The dictionary and the sparse vector in Example VII.3. Blue: $-1$, Red: $1$, Gray: $0$

and

$$\mu(D)\eta(D) = \frac{1}{4} \times 6 = \left(1 + \frac{1}{q}\right)|_{q=2}.$$  

VII. Conclusion

In this paper, we have studied the tightness of two estimates in literatures on sparse representations. Two classes of redundant dictionaries that are unions of several orthonormal bases were constructed by using the mutually unbiased bases. To accurately calculate the sparks of such dictionaries, the clear structures of sparse vectors in the null space are necessary. Our results imply that two well-known estimates for the spark are indeed tight. Therefore, Gribonval and Nielsen’s open problem is answered positively. The proofs of the above results are based on the techniques in discrete mathematics and quantum information theory. Further generalization of Theorem 12 could be considered. For example, the mutual coherence is $1/2^{q_1}$ and the number of orthonormal bases is $2^{q_2} + 1$, where $q_2$ is a factor of $q_1$.

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