Approximate quantum error correction for generalized amplitude damping errors

Carlo Cafaro\textsuperscript{1,2} and Peter van Loock\textsuperscript{2}
\textsuperscript{1}Max-Planck Institute for the Science of Light, 91058 Erlangen, Germany and \textsuperscript{2}Institute of Physics, Johannes-Gutenberg University Mainz, 55128 Mainz, Germany

We present analytic estimates of the performances of various approximate quantum error correction schemes for the generalized amplitude damping (GAD) qubit channel. Specifically, we consider both stabilizer and nonadditive quantum codes. The performance of such error-correcting schemes is quantified by means of the entanglement fidelity as a function of the damping probability and the non-zero environmental temperature. The recovery scheme employed throughout our work applies, in principle, to arbitrary quantum codes and is the analogue of the perfect Knill-Laflamme recovery scheme adapted to the approximate quantum error correction framework for the GAD error model. We also analytically recover and/or clarify some previously known numerical results in the limiting case of vanishing temperature of the environment, the well-known traditional amplitude damping channel. In addition, our study suggests that degenerate stabilizer codes and self-complementary nonadditive codes are especially suitable for the error correction of the GAD noise model. Finally, comparing the properly normalized entanglement fidelities of the best performant stabilizer and nonadditive codes characterized by the same length, we show that nonadditive codes outperform stabilizer codes not only in terms of encoded dimension but also in terms of entanglement fidelity.

PACS numbers: 03.67.Pp (quantum error correction)

I. INTRODUCTION

Quantum computers are especially sensitive to noise. Therefore, any technological implementation of such a machine requires the use of suitable error mitigation techniques. Quantum error correction (QEC) represents one of the most efficient available techniques capable of giving us the realistic hope of building practical quantum computers \cite{1}. For a detailed overview of the basic working principles of QEC, we refer to \cite{2}.

In general, it is a highly nontrivial task to design quantum codes for any given noise model. In the majority of cases, researchers have focused on the error correction of Pauli-type errors. This type of error model certainly constitutes the worst possible scenario to be considered and a quantum code that can correct all Pauli errors \{I, X \equiv \sigma_x, Y \equiv \sigma_y, Z \equiv \sigma_z\},

\begin{equation}
I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{equation}

can also provide protection against arbitrary qubit noise, since the Pauli operators form a basis of the $2 \times 2$ matrices. However, the worst possible scenario is not necessarily the most realistic one in actual experimental laboratories. Furthermore, aiming at designing quantum codes capable of error-correcting general noise errors may not be the most fruitful way to combat decoherence and noise in quantum computers.

A very common type of noise that appears in realistic settings is the so-called amplitude damping (AD) noise model \cite{3}. For instance, the AD noise model is employed to describe the photon loss in an optical fiber. The AD channel is the simplest channel whose Kraus operators cannot be described by unitary Pauli operations. The two non-unitary Kraus operators for the qubit AD channel are given by \cite{3},

\begin{equation}
A_0 \equiv \frac{1}{2} \left[ (1 + \sqrt{1 - \gamma}) I + (1 - \sqrt{1 - \gamma}) \sigma_z \right] \quad \text{and} \quad A_1 \equiv \frac{\sqrt{\gamma}}{2} (\sigma_x + i \sigma_y) = \sqrt{\gamma} |0 \rangle \langle 1|,
\end{equation}

where $\gamma$ denotes the amplitude damping probability parameter. For the sake of completeness, we point out that AD channels acting on states characterized by higher photon numbers in combination with a finite number of modes can also be considered. For instance, qubits living in a two-dimensional Hilbert space can be replaced by bosonic states of higher photon numbers in a finite number of optical modes \cite{4}. In general, the operator-sum decomposition of such higher-dimensional noise models is characterized by error operators $A_k$ with $k = 1, \ldots, N$ and, in principle, $N$ can approach infinity \cite{3}. We observe that there is no simple way of reducing $A_1$ in Eq. (2) to one Pauli error operator, since $|0 \rangle \langle 1|$ is not normal. The Pauli operators are remarkable in that they are unitary and Hermitian at the same time and, in addition, both unitary and Hermitian operators are normal. Although the five-qubit code \cite{5, 6} can be successfully used for the error correction of AD errors, since it is a universal 1-error correcting quantum code, Leung \textit{et al.} were capable of designing a four-qubit quantum code especially suitable for the error correction of arbitrary single-AD errors with a higher encoding rate equal to $1/4$ (greater than $1/5$) \cite{7}. A quantum code with
higher encoding rate is very welcome, since it would require less resources for its implementation. The two main points advocated in [7] were the following: first, when dealing with specific error models, better codes may be uncovered; second, the fulfillment of the approximate (relaxed) QEC conditions enlarges the realm of possible useful quantum error correcting codes. Therefore, it can simplify the code construction process.

Both the five-qubit and four-qubit codes are nondegenerate stabilizer (additive) quantum codes [3]. However, the scientific literature accommodates a fairly wide variety of additional AD error correction schemes where neither additive nor nondegenerate quantum codes are employed. For instance, one of the very first QEC scheme used to combat AD errors was quite unconventional, since it used bosonic states of higher photon numbers in a finite number of optical modes (the so-called bosonic quantum codes, [4]). More conventional and recent works are inspired by the seminal work presented in [7]. In [8], using the stabilizer formalism, generalizations of the Leung’s et al. four-qubit code for higher rates were constructed. Specifically, a class of [[2(m + 1), m]] channel-adapted quantum codes for integers \( m \geq 1 \) with encoding rate arbitrarily close to 1/2 are generated. In [9], the performance of QEC schemes for the AD model has been investigated via semidefinite programs (that is, numerical convex optimization methods [10]). Specifically, the optimal recovery operation to maximize the entanglement fidelity [11] for a given encoding and noise process was uncovered. Unfortunately, numerically computed recovery maps are difficult to describe and understand analytically. Furthermore, recovery operations generated through convex optimization methods suffer two significant drawbacks. First, the dimensions of the optimization problem grow exponentially with the length of the code, limiting the technique to short codes. Second, the optimal recovery operation may be quite difficult to implement, despite being physically legitimate. However, this exponential growth can be mitigated in two manners. First, it is possible to reduce the high dimensionality of the convex optimization procedures for generating recovery operations in QEC by transforming the problem in a suitable manner. Consider a quantum noisy channel \( \Lambda : \mathcal{H}_1 \to \mathcal{H}_2 \) with dim \( \mathcal{H}_i = d_i \) for \( i = 1, 2 \). By embedding the encoding into the noise process and redefining \( \Lambda \) as a quantum spreading channel with \( d_1 \) strictly less than \( d_2 \), the dimensionality of the convex optimization decreases from \( d_1^2d_2^2 \to d_1^3 \). For instance, for the [[5, 1, 3]] five-qubit code, the dimensionality reduces from \( 2^{20} \) to \( 2^{12} \) in terms of the optimization variables. For more details, we refer to [8]. Second, if the focus is on near-optimal rather than on optimal recovery schemes, it turns out that it is possible to compute such recovery operations with less computationally intensive numerical algorithms that are more scalable than those involved in semidefinite programs. Briefly speaking, regarding the algorithm in terms of eigenanalysis, it is possible to significantly reduce the size of the eigenvector problem and this has a significant effect on the computational cost of these new algorithms. For illustrative purposes, consider the AD channel and an \([n, k, d]\) quantum code. In this case, the dimensionality of the full optimal semidefinite program grows as \( 4^n \). Instead, if one considers the semidefinite programs for the first- and second-order subspaces, the dimensionality of the approximate programs only grows as \( n^2 \) and \( n^4 \), respectively. For instance, using the [[7, 1, 3]] CSS seven-qubit code, the full optimal semidefinite program requires 65536 optimization variables. However, the first-order semidefinite program requires 1024 variables and the second-order semidefinite program has 7056 optimization variables. For further details, we refer to [12]. In particular, in [12], to mitigate such drawbacks of the optimal recovery, a structured near-optimal channel-adapted recovery procedure was determined and applied to the AD noise model. In [13], an analytical approach to channel-adapted recovery based on the pretty-good measurement and the average entanglement fidelity appeared. Following [13], a simple analytical approach to approximate QEC based on the transpose channel was derived and used for the AD noise model in [14]. In particular, it was shown in [14] that the transpose channel is a recovery map that coincides with the perfect recovery map for codes satisfying the perfect Knill-Laflamme QEC conditions [15]. Very recently, it was also proved that the transpose channel works nearly as well as the optimal recovery channel, with optimality defined in terms of worst-case fidelity over all code states [16]. As a side remark, we underline that no definitive choice for the best figure of merit in quantum information processing tasks has been made yet [17] and this fact becomes especially relevant when quantifying the performance of quantum codes [18–22]. In [18], focusing on the AD noise model, it was shown that fidelity alone may be not sufficient to compare the efficiency of different error correction codes. In [23], a numerical search based upon a greedy algorithm [10] was employed to construct a family of high rate nonadditive quantum codes adapted to the AD noise model that outperform (in terms of encoded dimension) the stabilizer codes presented in [8]. In [24], families of high performance nonadditive quantum codes of the codeword stabilized (CWS, [25]) type for single AD-errors that outperform, in terms of encoded dimension, the best possible additive codes were presented. These code families were built from nonlinear error-correcting codes for classical asymmetric channels or classical codes over \( GF(3) \). Finally, for the sake of completeness, we also point out that a method for the construction of good multi-error-correcting AD codes that are both degenerate and additive can be found in [26].

Taking into account all these above-mentioned facts, we emphasize that there is:

- no clear understanding of the role played by degeneracy in the analysis of the performance of stabilizer quantum codes for AD errors, [12];
- no explicit evidence of the relevance of the role played by self-complementarity in the analysis of the performance
(quantified by means of the entanglement fidelity) of nonadditive quantum codes, [24];
- no explicit analytical computation of the entanglement fidelity of arbitrary quantum codes for AD errors for simple-to-construct recovery maps, [10];
- no explicit performance comparison in terms of the entanglement fidelity between additive and nonadditive quantum codes for AD errors, [23].

In this article, following the working methodology advocated by one of the Authors in [27, 28], we seek to advance our quantitative understanding via simple analytical computations of the role played by degeneracy, self-complementarity, additivity and nonadditivity of quantum codes for the GAD error model [3]. Specifically, we present the analysis of the performance of various approximate quantum error correction schemes for the GAD channel. We consider both stabilizer and nonadditive quantum codes. The performance of such error-correcting schemes is quantified by means of the entanglement fidelity as a function of the damping probability and the non-zero environmental temperature. The recovery scheme employed throughout our work applies, in principle, to arbitrary quantum codes and is the analogue of the perfect Knill-Laflamme recovery scheme adapted to the approximate quantum error correction framework for the GAD error model. We also analytically recover and/or clarify some previously known numerical results in the limiting case of vanishing temperature of the environment. In addition, our extended analytical investigation suggests that degenerate stabilizer codes and self-complementary nonadditive codes are especially suitable for the error correction of the GAD noise model. Finally, comparing the properly normalized entanglement fidelities of the best performant stabilizer and nonadditive codes characterized by the same length, we show that nonadditive codes outperform stabilizer codes not only in terms of encoded dimension but also in terms of fidelity.

This article is organized as follows. In Section II, we describe the GAD noise model. In Section III, we present some preliminary material concerning exact and approximate QEC conditions, recovery maps, and entanglement fidelity. In Section IV, we analyze the performances of various stabilizer codes, both degenerate and nondegenerate. In Section V, we quantify the performances of various nonadditive codes, both self-complementary and non-self-complementary. Finally, our conclusions are presented in Section VI. A number of appendices with technical details of calculations are also provided.

II. THE GAD NOISE MODEL

The AD quantum operation can characterize the behavior of different types of dissipative open quantum systems [9]: the spontaneous emission of a single atom coupled to a single mode of the electromagnetic radiation, the gradual loss of energy from a principal system to the environment where both systems are modeled by simple harmonic oscillators or the scattering of a photon via a beam splitter represent physical processes modeled by an AD channel.

It can be shown that the GAD qubit channel can be realized by considering the evolution of a two-level quantum system (that is, a qubit) in a dissipative interaction, in the Born-Markov rotating-wave approximation [29], with a bath of harmonic oscillators taken to be initially in a thermal state [30, 31].

The Lindblad form of the master equation that describes the evolution that generates the GAD channel reads [31],

$$\frac{d\rho^S(t)}{dt} = \sum_{j=1}^{2} \left(2R_j \rho^S R_j^\dagger - R_j^\dagger R_j \rho^S - \rho^S R_j R_j^\dagger \right),$$  

(3)

where the operators $R_1$, $R_2$ and $R$ are given by,

$$R_1 \equiv \left[\frac{\gamma_0}{2} (N_{th} + 1)\right]^\frac{1}{2} R, \quad R_2 \equiv \left(\frac{\gamma_0 N_{th}}{2}\right)^\frac{1}{2} R^\dagger, \quad R \equiv \sigma_-, \tag{4}$$

with $\gamma_0$, $N_{th}$ and $\sigma_-$ defined as,

$$\gamma_0 \equiv \frac{4\omega^3 |d|^2}{3\hbar c^3}, \quad N_{th} \equiv \frac{1}{e^{\hbar \omega T} - 1}, \quad \sigma_- \equiv \sigma_x - i\sigma_y = |1\rangle \langle 0|.$$

(5)

In Eq. (4), $\gamma_0$ denotes the spontaneous emission rate, $\omega$ is the photonic frequency, $\hbar$ is the Planck constant divided by $2\pi$, $c$ is the speed of light, $d$ is the transition matrix of the atomic dipole operator describing the interaction between the two-level quantum system with the bath of harmonic oscillators, $k_B$ is the Boltzmann constant, $\sigma_-$ is the lowering operator, $N_{th}$ is the Planck distribution that gives the number of thermal photons at frequency $\omega$ and, finally, $T$ denotes the temperature of the environment. The operator $\rho^S$ denotes the reduced density matrix operator of the
two-level quantum system interacting with a thermal bath in the weak Born-Markov rotating-wave approximation \[31\]. We remark that when \( T = 0 \), then \( N_{th} = 0 \) and \( R_2 = 0 \). Therefore, when the temperature of the environment is zero, a single Lindblad operator is sufficient to describe the master equation.

The evolution of the density operator \( \rho^S \) in Eq. \[3\] can be given a Kraus operator-sum decomposition. The Kraus representation is useful, because it provides an intrinsic description of the principal system, without explicitly considering the detailed properties of the environment. The essential features of the problem are contained in the Kraus error operators \( A_k \). This not only simplifies calculations, but often provides theoretical insight.

Following \[31\], it turns out that the Kraus decomposition of the GAD channel becomes,

\[
\Lambda_{GAD}(\rho) = \sum_{k=0}^{3} A_k \rho A_k^\dagger,
\]

where the Kraus error operators \( A_k \) read,

\[
A_0 = \sqrt{\gamma} \begin{bmatrix} 0 & \sqrt{\gamma} \\ \sqrt{\gamma} & 0 \end{bmatrix}, \quad A_1 = \sqrt{\gamma} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \sqrt{1-p} \begin{bmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \sqrt{1-p} \begin{bmatrix} 0 & \sqrt{\gamma} \\ \sqrt{\gamma} & 0 \end{bmatrix},
\]

and their action on the computational basis vectors \(|0\rangle\) and \(|1\rangle\) of the complex Hilbert space \( \mathcal{H}_2 \) (the Hilbert space of 1-qubit quantum states) reads,

\[
A_0 |0\rangle = \sqrt{\gamma} |0\rangle, \quad A_0 |1\rangle = \sqrt{\gamma} |1\rangle, \quad A_1 |0\rangle = 0, \quad A_1 |1\rangle = \sqrt{\gamma} |0\rangle,
\]

\[
A_2 |0\rangle = \sqrt{1-p} \sqrt{1-\gamma} |0\rangle, \quad A_2 |1\rangle = \sqrt{1-p} |1\rangle, \quad A_3 |0\rangle = \sqrt{1-p} |1\rangle, \quad A_3 |1\rangle = 0.
\]

Notice that for \( p = 1 \), the Kraus operator-sum decomposition of the AD channel can be recovered. The GAD channel generalizes the AD channel in that it allows transitions from \(|0\rangle \rightarrow |1\rangle\) as well as from \(|1\rangle \rightarrow |0\rangle\). For an alternative and explicit derivation of the operator-sum decomposition of the GAD channel in the context of scattering of a photon via a beam splitter, we refer to Appendix A.

We emphasize that the GAD channel is particularly noisy and, unlike the AD channel, is characterized by a two-dimensional parametric region \((\gamma, p(\gamma))\) where it exhibits entanglement breaking features (for details, see Appendix B). From \[31\], it also follows that the two GAD channel parameters \( \gamma \) and \( p \) are formally given by,

\[
\gamma(t) \overset{\text{def}}{=} 1 - e^{-\gamma_0(2N_{th}+1)t} \quad \text{and} \quad p \overset{\text{def}}{=} \frac{N_{th} + 1}{2N_{th} + 1}.
\]

Observe that \( p = 1 \) when \( T = 0 \) and \( p = \frac{1}{2} \) when \( T \) approaches infinity. For the sake of future convenience, we also introduce a new additional parameter \( \varepsilon \) defined as,

\[
\varepsilon(T) \overset{\text{def}}{=} 1 - p(T) = \frac{e^{-\frac{\gamma_0}{\varepsilon(T)}T} - e^{-\frac{\gamma_0}{\varepsilon(T)}}}{1 + e^{-\frac{\gamma_0}{\varepsilon(T)}}} \geq 1 - e^{-\frac{\gamma_0}{\varepsilon(T)}} \approx e^{-\frac{\gamma_0}{\varepsilon(T)}}.
\]

Combining Eqs. \[10\] and \[11\], we get

\[
\gamma(t) \overset{\text{def}}{=} \gamma_\varepsilon(t) = 1 - \exp \left[ -\left( \frac{1}{1-2\varepsilon} \right) \gamma_0 t \right] \overset{\varepsilon \ll 1}{\approx} 1 - e^{-\gamma_0(1+2\varepsilon)t}.
\]

From Eq. \[12\], we conclude that \( \gamma_\varepsilon \) is a monotonic increasing function of \( \varepsilon \) for \( \varepsilon \ll 1 \) and fixed values of \( \gamma_0 \) and \( t \).
III. QEC CONDITIONS, RECOVERY MAPS AND ENTANGLEMENT FIDELITY

A. QEC conditions

1. Exact QEC

Sufficient conditions for approximate QEC were introduced by Leung et al. in [7]. They showed that quantum codes can be effective in the error correction procedure even though they violate the traditional (exact) Knill-Laflamme QEC conditions [15]. However, these violations, characterized by small deviations from the standard error-correction conditions are allowed provided that they do not affect the desired fidelity order.

For the sake of reasoning, let us consider a binary quantum stabilizer code $C$ with code parameters $[[n, k, d]]$ encoding $k$-logical qubits in the Hilbert space $H^k_2$ into $n$-physical qubits in the Hilbert space $H^n_2$ with distance $d$. Assume that the noise model after the encoding procedure is $\Lambda (\rho)$ and can be described by an operator-sum representation,

$$\Lambda (\rho) \overset{\text{def}}{=} \sum_{k \in K} A_k \rho A_k^\dagger,$$

where $K$ is the index set of all the enlarged Kraus operators $A_k$ that appear in the sum. The noise channel $\Lambda$ is a CPTP (completely positive and trace preserving) map. The codespace of $l$ for any $i$ is equivalent to the traditional orthogonality and non-deformation conditions (see Eq. (16)) for a nondegenerate code, where Eq. (15) is replaced by

$$\Lambda' (\rho) \overset{\text{def}}{=} \sum_{k \in K'_{\text{reversible}}} A_k \rho A_k^\dagger,$$

and the requirement that all diagonal elements of $A_k \rho A_k^\dagger$ must vanish when $i \neq j$ (orthogonality condition); and the diagonal condition which, since $\alpha_{lm}$ are entries of a positive Hermitian complex matrix, is nothing but the requirement that all diagonal elements of $A_k \rho A_k^\dagger$ (inside the coding space) be identical (non-deformability condition). The fulfillment of Eq. (15) for some subset of enlarged error operators $A_k$ that characterize the operator-sum representation of the noise model $\Lambda$ implies that there exists a new operator-sum decomposition of $\Lambda$ such that $\Lambda' (\rho)$ in Eq. (14) becomes,

$$\Lambda' (\rho) \overset{\text{def}}{=} \sum_{k \in K'_{\text{reversible}}} A_k \rho A_k^\dagger,$$

where Eq. (15) is replaced by

$$P_C A_m^\dagger A_m P_C = \alpha_{lm} P_C,$$

for any $l, m \in K'_{\text{reversible}}$ with the error detection probabilities $p_m$ non-negative $c$-numbers. We remark that Eq. (18) is equivalent to the traditional orthogonality and non-deformation conditions (see Eq. (16)) for a nondegenerate code,

$$\langle i_L | A_m^\dagger A_m | j_L \rangle = \delta_{ij} \delta_{lm} p_m$$

for any $i, j$ labelling the logical states and $l, m \in K_{\text{reversible}}$. Observe that for any linear operator $A'_k$ on a vector space $V$ there exists a unitary $U_k$ and a positive operator $J \overset{\text{def}}{=} \sqrt{A_k A'_k}$ such that [3],

$$A'_k = U_k J = U_k \sqrt{A_k A'_k}.$$
We stress that $J$ is the unique positive operator that satisfies Eq. (20). As a matter of fact, multiplying $A'_k = U_k J$ on the left by the adjoint equation $A_k'^\dagger = JU_k'^\dagger$ gives

$$A_k'^\dagger A'_k = JU_k'^\dagger U_k J = J^2 \Rightarrow J = \sqrt{A_k'^\dagger A'_k}. \tag{21}$$

Furthermore, if $A'_k$ is invertible (that is, $\det A'_k \neq 0$), $U_k$ is unique and reads,

$$U_k \overset{\text{def}}{=} A_k'^\dagger J^{-1} = A_k' \left( \sqrt{A_k'^\dagger A'_k} \right)^{-1}. \tag{22}$$

How do we choose the unitary $U_k$ when $A'_k$ is not invertible? The operator $J$ is a positive operator and belongs to a special subclass of Hermitian operators such that for any vector $|v\rangle \in V$, $(|v\rangle, \langle v|)$ is a real and non-negative number. Therefore, $J$ has a spectral decomposition

$$J \overset{\text{def}}{=} \sqrt{A_k'^\dagger A'_k} = \sum l \lambda_l \langle l| l \rangle, \tag{23}$$

where $\lambda_l \geq 0$ and $\{|l|\}$ denotes an orthonormal basis for the vector space $V$. Define the vectors $|\psi_l\rangle \overset{\text{def}}{=} A_k'|l\rangle$ and notice that,

$$\langle \psi_l | \psi_l \rangle = \langle l| A_k'^\dagger A'_k | l \rangle = \langle l| J^2 | l \rangle = \lambda_l^2. \tag{24}$$

For the time being, consider only those $l$ for which $\lambda_l \neq 0$. For those $l$, consider the vectors $|e_l\rangle$ defined as

$$|e_l\rangle \overset{\text{def}}{=} \frac{\langle \psi_l |}{\lambda_l} = \frac{A_k'|l\rangle}{\lambda_l}, \tag{25}$$

with $\langle e_l | e_{l'} \rangle = \delta_{l l'}$. For those $l$ for which $\lambda_l = 0$, extend the orthonormal set $\{|e_l\rangle\}$ in such a manner that it forms an orthonormal basis $\{|E_l\rangle\}$. Then, a suitable choice for the unitary operator $U_k$ such that

$$A_k'|l\rangle = U_k J |l\rangle, \tag{26}$$

with $\{|l\rangle\}$ an orthonormal basis for $V$ reads,

$$U_k \overset{\text{def}}{=} \sum l |E_l\rangle \langle l| \tag{27}.$$  

In summary, the unitary $U_k$ is uniquely determined by Eq. (22) when $A'_k$ is invertible or Eq. (27) when $A'_k$ is not necessarily invertible. We finally stress that the non-uniqueness of $U_k$ when $\det A'_k = 0$ is due to the freedom in choosing the orthonormal basis $\{|l\rangle\}$ for the vector space $V$.

In the scenario being considered, when Eq. (18) is satisfied, the enlarged error operators $A'_m$ admit polar decompositions,

$$A'_m P_C = \sqrt{p_m} U_m P_C, \tag{28}$$

with $m \in K_{\text{reversible}}$. From Eqs. (18) and (28), we get

$$p_m \delta_{lm} P_C = P_C A_l'^\dagger A'_m P_C = \sqrt{p_m p_l} P_C U_l'^\dagger U_m P_C, \tag{29}$$

that is,

$$P_C U_l'^\dagger U_m P_C = \delta_{lm} P_C. \tag{30}$$

We stress that Eq. (30) is needed for an unambiguous syndrome detection since, as a consequence of the orthogonality of different $R_m \overset{\text{def}}{=} U_m P_C$, the recovery operation $R$ is trace preserving. This can be shown as follows.

Let $V^{i L}$ be the subspace of $H_2^n$ spanned by the corrupted images $\{A'_k|i_L\}$ of the codewords $|i_L\rangle$. Let $\{|v^{i L}\rangle\}$ be an orthonormal basis for $V^{i L}$. We define such a subspace $V^{i L}$ for each of the codewords. Because of the traditional Knill-Laflamme QEC conditions [15],

$$\langle i_L | A_k'^\dagger A_{k'} | i_L \rangle = \langle j_L | A_k'^\dagger A_{k'} | j_L \rangle, \forall i, j$$

$$\langle i_L | A_k'^\dagger A_{k'} | j_L \rangle = 0, \forall i \neq j, \tag{31}$$
the subspaces $\mathcal{V}_i^L$ and $\mathcal{V}_j^L$ with $i \neq j$ are orthogonal subspaces. If $\mathcal{V}_i^L \oplus \mathcal{V}_j^L$ is a proper subset of $\mathcal{H}_2^n$ with $\mathcal{V}_i^L \oplus \mathcal{V}_j^L \neq \mathcal{H}_2^n$, we denote its orthogonal complement by $\mathcal{O}$,

$$\mathcal{H}_2^n \overset{\text{def}}{=} (\mathcal{V}_i^L \oplus \mathcal{V}_j^L) \oplus \mathcal{O},$$

(32)

where,

$$\mathcal{O} \overset{\text{def}}{=} (\mathcal{V}_i^L \oplus \mathcal{V}_j^L)^\perp.$$

(33)

Let $\{ |o_k \rangle \}$ be an orthonormal basis for $\mathcal{O}$. Then, the set of states $\{ |v_r^L \rangle, |o_k \rangle \}$ constitutes an orthonormal basis for $\mathcal{H}_2^n$. Notice that, since $|v_r^L \rangle$ are mutually orthogonal, there exist unitary $V_r$ such that $V_r |v_r^L \rangle = |i_L \rangle$ ($V_r$ is an isometry which returns $|v_r^L \rangle$ to the corresponding $|i_L \rangle$). We introduce the quantum recovery operation $\mathcal{R}$ with operation elements

$$\mathcal{R} \overset{\text{def}}{=} \{ R_1, ..., R_r, ..., \hat{O} \},$$

(34)

with,

$$\mathcal{R} (\rho) = \sum_{k \in K_{\text{reversible}}} R_k \rho R_k^\dagger + \hat{O} \rho \hat{O}^\dagger,$$

(35)

where $[15]$,

$$R_r \overset{\text{def}}{=} V_r \sum_i |v_r^L \rangle \langle v_r^L | = \sum_i |i_L \rangle \langle v_r^L |,$$

(36)

and $\hat{O}$ (with $\hat{O} = \hat{O}^\dagger = \hat{O}^\dagger \hat{O}$) is a projector onto the subspace $\mathcal{O}$ in Eq. (33),

$$\hat{O} \overset{\text{def}}{=} \sum_k |o_k \rangle \langle o_k |.$$

(37)

We remark that the recovery operation $\mathcal{R}$ is a trace preserving quantum operation since,

$$\sum_r R_r^\dagger R_r + \hat{O}^\dagger \hat{O} = \sum_r \left[ \left( \sum_i |i_L \rangle \langle v_r^i | \right)^\dagger \left( \sum_j |j_L \rangle \langle v_r^j | \right) \right] + \left( \sum_k |o_k \rangle \langle o_k | \right)^\dagger \left( \sum_{k'} |o_{k'} \rangle \langle o_{k'} | \right)$$

$$= \sum_{r, i, j} |v_r^i \rangle \langle v_r^j | \delta_{ij} + \sum_{k, k'} |o_k \rangle \langle o_{k'} | \delta_{kk'}$$

$$= \sum_{r, i} |v_r^i \rangle \langle v_r^i | + \sum_k |o_k \rangle \langle o_k |$$

$$= \mathcal{I}_{2^n \times 2^n},$$

(38)

because $\mathcal{B}_{\mathcal{H}_2^n} \overset{\text{def}}{=} \{ |v_r^L \rangle, |o_k \rangle \}$ is an orthonormal basis for $\mathcal{H}_2^n$. We emphasize that $\mathcal{R}$ is indeed a CPTP superoperator (whose recovery operators $R_k$ can be regarded as projective measurements followed by unitary rotations), since it is a sum of orthogonal projections followed by unitary operators where the projections span the Hilbert space $\mathcal{H}_2^n$. Furthermore, we point out that the recovery scheme $\mathcal{R}$ in Eq. (34) applies, in principle, to any quantum code satisfying the QEC conditions independent of the stabilizer formalism $[15]$. For more technical details, we refer to $[15]$ and $[33]$. 
2. Approximate QEC

In general, approximate QEC becomes useful when the operator-sum representation of the noise model is defined by errors parametrized by a certain number of small parameters such as the coupling strength between the environment and the quantum system. For the sake of simplicity, suppose the error model is characterized by a single small parameter $\delta$ and assume the goal is to uncover a quantum code for the noise model $\Lambda'$ with fidelity,

$$ F \geq 1 - O(\delta^{2k+1}). $$

(39)

How strong can be the violation of the traditional perfect Knill-Laflamme QEC conditions in order to preserve the desired fidelity order in Eq. (39)? In other words, how relaxed can the approximate error correction conditions be so that the inequality in Eq. (39) is satisfied? The answer to this important question was provided by Leung et al. in [7].

It turns out that for both exact and approximate QEC conditions, it is necessary that

$$ P_{\text{detection}} \defeq \sum_{k \in K_{\text{reversible}}} p_k \geq F, $$

(40)

where $P_{\text{detection}}$ denotes the total error detection probability. Eq. (40) requires that all the enlarged error operators $A'_i$ with maximum detection probability must be included in $A'_{\text{reversible}}$,

$$ \max_{|\psi_{in}\rangle \in C} \text{Tr} \left( |\psi_{in}\rangle \langle \psi_{in} | A'^\dagger_i A'_i \right) \approx O(\delta^n) \text{ with } \alpha \leq \beta. $$

(41)

The important point is that a good overlap between the input and output states is needed while it is not necessary to recover the exact input state $|\psi_{in}\rangle \langle \psi_{in} |$, since we do not require $F = 1$. In terms of the enlarged error operators restricted to the codespace, this means that such errors need to be only approximately unitary and mutually orthogonal. These considerations lead to the relaxed sufficient QEC conditions.

In analogy to Eq. (28), assume that the polar decomposition for $A'_i$ is given by,

$$ A'_i P_C = U_i \sqrt{P_C A'^\dagger_i A'_i P_C}. $$

(42)

Since $P_C A'^\dagger_i A'_i P_C$ restricted to the codespace $C$ have different eigenvalues, the exact error correction conditions are not fulfilled. Let us say that $\lambda_{i}^{(\text{max})} \defeq p_l$ and $\lambda_{i}^{(\text{min})} \defeq \lambda_i p_l$ are the largest and the smallest eigenvalues, respectively, where both $p_l$ and $\lambda_i$ are $\alpha$-numbers. Furthermore, let us define the so-called residue operator $\pi_l$ as [7],

$$ \pi_l \defeq \sqrt{P_C A'^\dagger_i A'_i P_C} - \sqrt{\lambda_i p_l P_C}, $$

(43)

where,

$$ 0 \leq |\pi_l| \defeq \left( \pi_i^\dagger \pi_l \right)^{\frac{1}{2}} \leq \sqrt{p_l} - \sqrt{\lambda_i p_l}. $$

(44)

Substituting Eq. (43) into Eq. (42), we obtain

$$ A'_i P_C = U_i \left( \sqrt{\lambda_i p_l I + \pi_l} \right) P_C. $$

(45)

From Eq. (45) and imposing that $P_C U_i^\dagger U_m P_C = \delta_{lm} P_C$, the analog of Eq. (18) becomes

$$ P_C A'^\dagger_i A'_m P_C = \left( \sqrt{\lambda_i p_l I + \pi_l} \right) \left( \sqrt{\lambda_m p_m I + \pi_m} \right) P_C \delta_{lm}, $$

(46)

where,

$$ \lambda_i^{(\text{max})} - \lambda_i^{(\text{min})} \equiv p_l (1 - \lambda_l) \leq O(\delta^{2k+1}), \forall l \in K_{\text{reversible}}. $$

(47)

We point out that when the exact QEC conditions are satisfied, $\lambda_l = 1$ and $\pi_l = 0$, thus, Eqs. (18) and (46) coincide. Finally, we point out that an approximate recovery operation $R \defeq \{ R_1, \ldots, R_r, \ldots, \hat{O} \}$ with $R_k$ defined in Eq. (36) and $\hat{O}$ formally defined just as in the exact case can be employed in this new scenario as well. However, extra care in the explicit computation of the unitary operators $U_k$ is needed in view of the fact that the polar decomposition in Eq. (28) is replaced by the one in Eq. (42). More details can be found in [7].
B. Recovery maps

In general, numerically constructed recovery maps do not exhibit any practical implementation structure while the perfect Knill-Laflamme recovery map can be implemented simply using syndrome measurements and conditional unitary gates [15]. For these reasons, our intention here is to pursue an analytical approach to the recovery scheme that reduces to the perfect Knill-Laflamme recovery scheme in the limiting case of small deviations from the exact QEC conditions.

We point out, as mentioned in [13] and explicitly shown in [14], that the transpose channel projector on the codespace, a special case of the near-optimal reversal operation in Eq. (49) was introduced in [34]. Following [13] and assuming \( \rho \) is the dimension of the codespace and \( P_c \) the projector on the codespace, a special case of the near-optimal reversal operation in Eq. (49) was introduced in [13]. Such reversal operation is the so-called transpose channel recovery map,

\[
\mathcal{R}_{TC} \overset{\text{def}}{=} \mathcal{R}_{\Lambda, P_c/d} \sim \left\{ P_c A_k^\dagger \Lambda (P_c)^{-\frac{1}{2}} \right\},
\]

(50)

The transpose channel recovery map \( \mathcal{R}_{TC} \) is a simple-to-construct recovery map built from the noise channel \( \Lambda \) and the code \( C \). In particular it works nearly as well as the optimal recovery channel, with optimality defined in terms of worst-case fidelity over all input states (the fidelity between any two states \( \rho \) and \( \sigma \) is given by \( f(\rho, \sigma) \overset{\text{def}}{=} \text{Tr}\left( \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}^\dagger} \right) \)). We point out, as mentioned in [13] and explicitly shown in [13], that the transpose channel \( \mathcal{R}_{TC} \) in Eq. (50) reduces to the perfect Knill-Laflamme recovery operation \( \mathcal{R}_{\text{Knill-Laflamme}} \) when the traditional exact QEC conditions are satisfied.

The recovery scheme that we choose to use in this work is formally defined in terms of recovery operators that are just like the operators \( R_k \) in Eq. (36),

\[
R_k \overset{\text{def}}{=} V_k P_k \equiv V_k \sum_i |v_k^{0i}\rangle \langle v_k^{0i}| = \sum_i |v_k^{Li}\rangle \langle v_k^{Li}| = \frac{|0_L \rangle \langle 0_L| A_k^\dagger}{\sqrt{\langle 0_L | A_k^\dagger A_k | 0_L \rangle}} + \frac{|1_L \rangle \langle 1_L| A_k^\dagger}{\sqrt{\langle 1_L | A_k^\dagger A_k | 1_L \rangle}},
\]

(51)

where, however, we must now take into account that \( \langle 0_L | A_k^\dagger A_k | 0_L \rangle \) may only be approximately equal to \( \langle 1_L | A_k^\dagger A_k | 1_L \rangle \) in the approximate QEC framework. Thus, the set of errors \( \{A_k\} \) that appear in Eq. (51) has
to be considered correctable in the approximate sense specified in the previous subsection. For this reason, the recovery operators $R_k$ in Eq. (51) cannot assume the simple expression they exhibit in the case of exact fulfillment of the QEC conditions. In the optimal (exact) case, the superoperator $\mathcal{R}$ with elements $R_k$ in Eq. (51) becomes

$$
\mathcal{R}_{\text{perfect}}(\text{Knill-Laflamme}) \sim \left\{ \frac{PC_A^k}{\sqrt{p_k}} \right\},
$$

with $p_k \equiv \langle 0_L | A_k^\dagger A_k | 0_L \rangle \equiv \langle 1_L | A_k^\dagger A_k | 1_L \rangle$. Explicit analytical investigations in the framework of exact QEC where the superoperator in Eq. (52) was employed can be found in [19, 27, 28].

Before describing the concept of entanglement fidelity, we wish to hint at what happens with our recovery scheme in the traditional AD noise model when error correction is performed via the Leung et al. four-qubit code [7]. For the sake of clarity, we only consider the recovery operator for the enlarged error $A_{0000} = A_0 \otimes A_0 \otimes A_0 \otimes A_0$ with $A_0$ defined as in Eq. (2). In this case, we have

$$
R_{A_{0000}, A_{0000}} \rho = \lambda_{00} (\gamma) P_C + P_C \hat{B}_{00} (\gamma) P_C,
$$

with,

$$
\lambda_{00} (\gamma) = 1 - 2\gamma \quad \text{and} \quad \hat{B}_{00} (\gamma) = \left( 3\gamma^2 - 2\gamma^3 + \frac{1}{2}\gamma^4 \right) |0_L\rangle \langle 0_L| + \gamma^2 |1_L\rangle \langle 1_L|,
$$

while $P_C$ is the projector on the codespace of the code $C$. For more details on this point, we refer to [35].

C. Entanglement fidelity

Entanglement fidelity is a useful performance measure of the efficiency of quantum error correcting codes. It is a quantity that keeps track of how well the state and entanglement of a subsystem of a larger system are stored, without requiring the knowledge of the complete state or dynamics of the larger system. More precisely, the entanglement fidelity is defined for a mixed state

$$
\rho \equiv \sum_i p_i \rho_i = \text{Tr}_{\mathcal{H}_R} |\psi\rangle \langle \psi|,
$$

in terms of a purification $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}_R$ to a reference system $\mathcal{H}_R$. The purification $|\psi\rangle$ encodes all of the information in $\rho$. Entanglement fidelity is a measure of how well the channel $\Lambda$ preserves the entanglement of the state $\mathcal{H}$ with its reference system $\mathcal{H}_R$. The entanglement fidelity is defined as follows [11],

$$
\mathcal{F} (\rho, \Lambda) \equiv \langle \psi | (\Lambda \otimes \mathcal{I}_{\mathcal{H}_R}) (|\psi\rangle \langle \psi|) |\psi\rangle,
$$

where $|\psi\rangle$ is any purification of $\rho$, $\mathcal{I}_{\mathcal{H}_R}$ is the identity map on $M(\mathcal{H}_R)$ (the space of all linear operators on the Hilbert space $\mathcal{H}_R$) and $\Lambda \otimes \mathcal{I}_{\mathcal{H}_R}$ is the evolution operator extended to the space $\mathcal{H} \otimes \mathcal{H}_R$, the space on which $\rho$ has been purified. If the quantum operation $\Lambda$ is written in terms of its Kraus error operators $\{ A_k \}$ as $\Lambda (\rho) \equiv \sum_k A_k \rho A_k^\dagger$, then it can be shown that [36],

$$
\mathcal{F} (\rho, \Lambda) = \sum_k \text{Tr} (A_k \rho) \text{Tr} (A_k^\dagger \rho) = \sum_k |\text{Tr} (\rho A_k)|^2.
$$

This expression for the entanglement fidelity is very useful for explicit calculations. Finally, assuming that

$$
\Lambda : M(\mathcal{H}) \ni \rho \mapsto \Lambda (\rho) = \sum_k A_k \rho A_k^\dagger \in M(\mathcal{H}), \dim_{\mathcal{C}} \mathcal{H} = N
$$
and choosing a purification described by a maximally entangled unit vector for the mixed state $\rho = I_H/\dim C_H$, we obtain

$$\mathcal{F}\left(\frac{1}{N}I_H, A\right) = \frac{1}{N^2} \sum_k |\text{Tr} A_k|^2.$$  \hspace{1cm} (60)

The expression in Eq. (60) represents the entanglement fidelity when no error correction is performed on the noisy channel $A$ defined in Eq. (59).

Finally, for the sake of completeness, we point out that there exists a relation between the fidelity of a recovery operation $R$ and the worst-case error probability parameter $p$, $p \overset{\text{def}}{=} 1 - \mathcal{F}(R, C, E)$, \hspace{1cm} (61)

where $C$ and $E$ denote the code and the noise channel, respectively. The meaning of the above relation can be described as follows. Consider a quantum state $|\psi\rangle$ encoded into the state $U_{\text{enc}}|\psi\rangle|00...0\rangle$, then subjected to some noise (corresponding to the $E_i$ operators), then subjected to a recovery operation (corresponding to the $R_j$ operators). Finally, the ancilla work-space is discarded giving back some state $\rho_\psi$ on the original Hilbert space,

$$\rho_\psi = \text{Tr}_{\text{ancilla}} \left[ \sum_j R_j U_{\text{enc}}^\dagger \left( \sum_i E_i U_{\text{enc}} |\psi\rangle |00...0\rangle \langle 0...00| \langle \psi| U_{\text{enc}}^\dagger E_i^\dagger \right) U_{\text{enc}} R_j^\dagger \right].$$  \hspace{1cm} (62)

We are interested in how close $\rho_\psi$ is to the original state $|\psi\rangle\langle \psi|$. The probability $p_\psi = \langle \psi| \rho_\psi |\psi\rangle$ can be regarded as the probability of no error on the encoded state and the fidelity of a recovery operation $R$ is defined as,

$$\mathcal{F}(R, C, E) \overset{\text{def}}{=} \min |\psi\rangle p_\psi,$$  \hspace{1cm} (63)

the minimum of all such probabilities $p_\psi$ over all encoded states $|\psi\rangle$. Thus, the probability parameter $p$ gives an upper bound on the probability with which a generic encoded state will end up in the wrong state.

IV. ADDITIVE CODES

We denote by $[[n, k, d]]$ a stabilizer (or, additive) code that encodes $k$ logical qubits into $n$ physical qubits correcting $\left\lfloor \frac{d-1}{2} \right\rfloor$-qubit errors where $d$ is the distance of the code and $\lfloor x \rfloor$ denotes the largest integer less than $x$. Additive quantum codes are characterized by a codespace, the space spanned by the so-called codewords, which is a simultaneous eigenspace of an Abelian subgroup of the Pauli group. For more details on the stabilizer formalism, we refer to [38].

A. Nondegenerate codes

Formally speaking, a quantum code for which the positive Hermitian matrix $\alpha$ in Eq. (15) is non-singular are called nondegenerate codes. Instead, codes for which $\alpha$ is singular are called degenerate. For nondegenerate codes, each error is individually identifiable and, for a given choice of error operators, the quantum code is transformed into a set of distinct orthogonal subspaces by applying the errors. In short, for nondegenerate codes, all the errors acting on the codewords produce linearly independent quantum states.

1. The five-qubit code

The $[[5, 1, 3]]$ code is the smallest single-error correcting quantum code [5, 6]. Of all QECCs that encode 1 qubit of data and correct all single-qubit errors, the $[[5, 1, 3]]$ is the most efficient, saturating the quantum Hamming bound. It encodes $k = 1$ qubit in $n = 5$ qubits. The cardinality of its stabilizer group $S$ is $|S| = 2^{n-k} = 16$ and the set $B_S^{[[5,1,3]]}$ of $n-k = 4$ stabilizer group generators is given by [33],

$$B_S^{[[5,1,3]]} \overset{\text{def}}{=} \{ X^1 Z^2 Z^3 X^4, X^2 Z^3 Z^4 X^5, X^1 X^3 Z^4 Z^5, Z^1 X^2 X^4 Z^5 \},$$  \hspace{1cm} (64)
with \( X \equiv \sigma_x \), \( Y \equiv \sigma_y \) and \( Z \equiv \sigma_z \) and \( \{\sigma_x, \sigma_y, \sigma_z\} \) as given in Eq. (1). For the sake of notational clarity, we emphasize that when describing stabilizer generators as tensor products of Pauli operators, we may omit to use the symbol \( \otimes \). In addition, the superscripts on the RHS of Eq. (64) label the qubits 1,..., 5. The distance of the code is \( d = 3 \) and therefore the weight of the smallest enlarged error operators of the form \( A^\dagger_i A_k \) that cannot be detected by the code is 3. Finally, we recall that it is a nondegenerate code, since the smallest weight for elements of \( S \) (other than identity) is 4 and therefore it is greater than the distance \( d = 3 \). The encoding for the \([5, 1, 3]\) code is given by (65).

\[
|0_L\rangle \equiv \frac{1}{\sqrt{8}} [|-00000\rangle + |01111\rangle - |10011\rangle + |11100\rangle + |00110\rangle + |01001\rangle + |10101\rangle + |11010\rangle],
\]

\[
|1_L\rangle \equiv \frac{1}{\sqrt{8}} [|-11111\rangle + |10000\rangle + |01100\rangle - |00111\rangle + |11001\rangle + |10110\rangle - |01011\rangle - |00011\rangle].
\]

The enlarged GAD quantum channel after performing the encoding defined by means of Eq. (65) reads,

\[
A^{[[5,1,3]]}_{\text{GAD}}(\rho) \equiv \sum_{i, j, k, l, m=0}^{2^{10}-1} A_i^\dagger A_i A_j \otimes A_k \otimes A_l \otimes A_m = A_{ijklm} \rho A_{ijklm}^\dagger,
\]

where to any of the \( 2^{10} \) values of \( r \) we can associate a set of indices \( (i, j, k, l, m) \) (and vice-versa) such that,

\[
A_i^\dagger \leftrightarrow A_{ijklm} \equiv A_i \otimes A_j \otimes A_k \otimes A_l \otimes A_m \equiv A_i A_j A_k A_l A_m.
\]

(67) The errors \( A_i \) with \( i \in \{0, 1, 2, 3\} \) are defined in Eq. (8) and \( \rho \in \mathcal{M}(\mathcal{C}) \) with \( \mathcal{C} \subset \mathcal{H}_2^5 \). In particular, the number of weight-\( q \) enlarged error operators \( A_i^\dagger \) is given by \( 3^q \binom{5}{q} \) and,

\[
2^{10} = \sum_{q=0}^{5} 3^q \binom{5}{q}.
\]

(68) We point out that consistency requires that the sum of the probabilities \( P(\cdot) \) that an \( A_i^\dagger \) error occurs must sum up to unity. For clarity of exposition, consider the limiting case with \( \varepsilon = 0 \) where only \( 2^5 = 32 \) enlarged errors \( A_i^\dagger \) are present. In this case, we have

\[
1 \sum_{i, j, k, l, m=0}^{2^5-1} P(\cdot) = \frac{1}{2} \sum_{a=0}^{2^5-1} \text{Tr} (A_a^\dagger \rho A_a) = \frac{1}{2} \sum_{a=0}^{2^5-1} [\langle 0_L | A_a^\dagger A_a | 0_L \rangle + \langle 1_L | A_a^\dagger A_a | 1_L \rangle] = 1,
\]

(69) that is,

\[
1 \sum_{i, j, k, l, m=0}^{2^5-1} P(\cdot) = P_{\text{weight-0}} + P_{\text{weight-1}} + P_{\text{weight-2}} + P_{\text{weight-3}} + P_{\text{weight-4}} + P_{\text{weight-5}} = 1,
\]

(70) where,

\[
P_{\text{weight-0}} = 1 - \frac{5}{2} \gamma^2 + \frac{5}{2} \gamma^2 - \frac{5}{4} \gamma^3 + \frac{3}{8} \gamma^4 - \frac{1}{16} \gamma^5,
\]

\[
P_{\text{weight-1}} = \frac{5}{2} \gamma^2 - 5 \gamma^2 + \frac{15}{4} \gamma^3 - \frac{3}{2} \gamma^4 + \frac{5}{16} \gamma^5,
\]

\[
P_{\text{weight-2}} = \frac{5}{2} \gamma^2 - \frac{15}{4} \gamma^3 + \frac{9}{4} \gamma^4 - \frac{5}{8} \gamma^5,
\]

\[
P_{\text{weight-3}} = \frac{5}{4} \gamma^3 - \frac{3}{2} \gamma^4 + \frac{5}{8} \gamma^5,
\]

\[
P_{\text{weight-4}} = \frac{3}{8} \gamma^4 - \frac{5}{16} \gamma^5,
\]

(71) \[ P_{\text{weight-5}} = \frac{1}{16} \gamma^5. \]

Using the brute-force approach, it would be fairly straightforward, though very tedious, to check the approximate QEC conditions for all the \( 2^{10} \) enlarged errors. Fortunately, this is not necessary. Indeed, we aim at finding an analytical estimate of the entanglement fidelity of the code such that,

\[
\mathcal{F}^{[[5,1,3]]}(\gamma, \varepsilon) \geq 1 - \mathcal{O}(2),
\]

(72)
where \( \mathcal{O}(2) \sim \mathcal{O}(\gamma n^1 \varepsilon n^2) \); that is, the pair \((n_1, n_2)\) is such that,

\[
\lim_{\gamma, \varepsilon \to 0} \frac{\mathcal{O}(2)}{\mathcal{O}(\gamma n^1 \varepsilon n^2)} = \text{constant.} \tag{73}
\]

For instance, we may have \(\{(n_1, n_2)\} = \{(2, 0), (0, 2), (1, 1)\}\). Observe that the codewords that span the code belong to the \(2^5 = 32\)-dimensional complex Hilbert space \(\mathcal{H}_2^5\). Thus, a basis of orthonormal vectors for \(\mathcal{H}_2^5\) requires 32 elements. For \(\varepsilon = 0\), it turns out that none of the \(\binom{5}{2} = 10\) weight-2 errors is correctable. Specifically, errors \(A_{01100}, A_{00011}, A_{01010}\) and \(A_{00101}\) are not compatible with \(A_{00000}; A_{00110}\) and \(A_{01001}\) are not compatible with \(A_{10000}; A_{11000}\) is not compatible with \(A_{01000}; A_{01010}\) is not compatible with \(A_{00100}\) and, finally, \(A_{10001}\) is incompatible with \(A_{00001}\). For the general case, it can be shown that all weight-0 (1 error) and weight-1 (15 errors) enlarged error operators satisfy the approximate QEC conditions up to the sought order. However, the action on the codewords of five weight-1 enlarged errors (specifically, \(A_{20000}, A_{02000}, A_{00200}, A_{00020}\) and \(A_{00002}\)) leads to vectors that are not orthogonal to those obtained from the action of the weight-0 error \(A_{00000}\) on the codewords. Thus, we omit them from the construction of our recovery scheme. In view of these considerations, we construct our recovery operation \(\mathcal{R}\) as follows,

\[
\mathcal{R} \overset{\text{def}}{=} \left\{ R_0, R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10}, \hat{O} \right\}, \tag{74}
\]

where,

\[
R_r \overset{\text{def}}{=} |L_L\rangle \langle v_r^0 L | + |1_L\rangle \langle v_r^1 L | \text{ with, } |v_r^i L \rangle \overset{\text{def}}{=} \frac{A'_{kL} |i_L\rangle}{\sqrt{\langle i_L | A''_{kL} A_k | i_L\rangle}}, \tag{75}
\]

for \(i \in \{0, 1\}\) and \(\langle v_r^i L | v_r^j L \rangle = \delta_{r r'} \delta_{i j}.\) To be clear \(R_0\) is associated with the weight-0 error \(A_{00000}; R_k\) with \(k = 1, \ldots, 5\) are associated with the five weight-1 errors where single-qubit errors of type \(A_1\) occur; finally, \(R_k\) with \(k = 6, \ldots, 10\) are associated with the five weight-1 errors where single-qubit errors of type \(A_3\) occur. The construction of these 11 recovery operators \(R_k\) is described in terms of 22 orthonormal vectors in \(\mathcal{H}_2^5\). The missing 10 orthonormal vectors can be uncovered using the rank-nullity (dimension) theorem together with the Gram-Schmidt orthonormalization procedure (for more details, see Appendix C). They define the operator \(\hat{O}\) in Eq. (74),

\[
\hat{O} = \sum_{j=1}^{10} |o_j\rangle \langle o_j|. \tag{76}
\]

For the sake of convenience, we put \(R_{11} \overset{\text{def}}{=} \hat{O}.\) Finally, the estimate of the entanglement fidelity of the five-qubit code, when the recovery operation \(\mathcal{R}\) in Eq. (74) is employed, becomes (for more details, see Appendix D),

\[
\mathcal{F}^{[[5,1,3]]} (\gamma, \varepsilon) \overset{\text{def}}{=} \frac{1}{(\dim_2 C)^2} \sum_{k=0}^{10-1} \sum_{l=0}^{11} | \text{Tr} (R_l A_k) |_C^2 \approx 1 - \frac{5}{2} \gamma^2 - 10 \varepsilon^2 \left(1 + \frac{\gamma}{\varepsilon}\right) + \mathcal{O}(3). \tag{77}
\]

We point out that in the limiting case of \(\varepsilon = 0\), the AD noise model is recovered and our analytical estimate in Eq. (77) reduces to the numerically obtained truncated series expansion appeared in [9] and [39] (specifically, see page 48 in [39]). The effect of the non-zero environmental temperature on the five-qubit code is illustrated in Fig. 1.

2. The CSS seven-qubit code

The Calderbank-Shor-Steane (CSS) codes are constructed from two classical binary codes \(C\) and \(C'\) that have the following properties [40, 41]: 1) \(C\) and \(C'\) are \([n, k, d]\) and \([n, k', d']\) codes, respectively; 2) \(C' \subset C; 3) C\) and \(C'\) (the dual code of \(C'\)) are both \(t\)-error correcting codes. For instance, in case of the seven-qubit code, the two classical codes are the \([7, 4, 3]\) binary Hamming code \((C)\) and the \([7, 3, 4]\) binary simplex code \((C')\). The dual code \(C'\) is the \([7, 4, 3]\) binary Hamming code. Thus \(C\) and \(C'\) are both 1-error correcting codes. In this case, \(n = 7, k = 4, k' = 3, k - k' = 1\) so that 1 qubit is mapped into 7 qubits. The seven-qubit code is the simplest example of a CSS code. Although the seven-qubit code is ostensibly more complicated than the five-qubit code, it is actually more useful in certain situations by virtue of being a CSS code. The CSS codes are a particularly interesting class of codes for two reasons.
First, they are built using classical codes which have been more heavily studied than quantum codes, so it is fairly easy to construct useful quantum codes simply by looking at lists of classical codes. Second, because of the form of generators, the CSS codes are precisely those for which a CNOT applied between every pair of corresponding qubits in two blocks performs a valid fault-tolerant operation. This makes them particularly good candidates in fault-tolerant computation.

The CSS seven-qubit code encodes \( k = 1 \) qubit in \( n = 7 \) qubits. The cardinality of its stabilizer group \( S \) is \( |S| = 2^n - k = 64 \) and the set \( B_S^{[7,1,3]} \) of \( n - k = 6 \) stabilizer group generators reads \[ 33 \],

\[
B_S^{[7,1,3]} \overset{\text{def}}{=} \left\{ X^4 X^5 X^6 X^7, X^2 X^3 X^6 X^7, X^1 X^3 X^5 X^7, Z^4 Z^5 Z^6 Z^7, Z^2 Z^3 Z^6 Z^7, Z^1 Z^3 Z^5 Z^7 \right\}. \tag{78}
\]

The distance of the code is \( d = 3 \) and therefore the weight of the smallest error \( A^r_1 A^r_1 \) that cannot be detected by the code is 3. Finally, we recall that it is a nondegenerate code, since the smallest weight for elements of \( S \) (other than identity) is 4 and therefore it is greater than the distance \( d = 3 \). The encoding for the \([7,1,3]\) code is given by \[ 33 \],

\[
|0\rangle \rightarrow |0_L\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \left[ |0000000\rangle + |0110011\rangle + |1010101\rangle + |1100110\rangle + \right] + \left[ |0001111\rangle + |0111100\rangle + |1011010\rangle + |1101001\rangle \right], \tag{79}
\]

and,

\[
|1\rangle \rightarrow |1_L\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \left[ |1111111\rangle + |1001100\rangle + |0101010\rangle + |0011001\rangle + \right] + \left[ |1110000\rangle + |1000011\rangle + |0100101\rangle + |0010110\rangle \right]. \tag{80}
\]

The enlarged GAD quantum channel after performing the encoding defined by means of Eqs. \([79] \) and \([80] \) reads,

\[
A_{\text{GAD}}^{[7,1,3]} (\rho) \overset{\text{def}}{=} \sum_{r=0}^{2^{14}-1} A^r_i \rho A^r_i = \sum_{i,j,k,l,m,n,s=0} 3^{3} A_{ijklmns} \rho A^\dagger_{ijklmns}, \tag{81}
\]

where to any of the \( 2^{14} \) values of \( r \) we can associate a set of indices \((i,j,k,l,m,n,s)\) (and vice-versa) such that,

\[
A^r_i \leftrightarrow A_{ijklmns} \overset{\text{def}}{=} A_i \otimes A_j \otimes A_k \otimes A_l \otimes A_m \otimes A_n \otimes A_s = A_i A_j A_k A_l A_m A_n A_s. \tag{82}
\]

The errors \( A_i \) with \( i \in \{0,1,2,3\} \) are defined in Eq. \([5]\) and \( \rho \in \mathcal{M}(\mathcal{C}) \) with \( \mathcal{C} \subset \mathcal{H}_2^\otimes \). In particular, the number of weight-\( q \) enlarged error operators \( A^r_i \) is given by \( 3^q \binom{7}{q} \) and,

\[
2^{14} = \sum_{q=0}^{7} 3^q \binom{7}{q}. \tag{83}
\]
For $\varepsilon = 0$, it can be shown that for any of the $\binom{7}{2} = 21$ weight-2 errors, there exists at least one of the 7 weight-1 errors for which the correctability conditions are not satisfied. For the general case, it can be shown that all weight-0 (1 error) and weight-1 (21 errors) enlarged error operators satisfy the approximate QEC conditions up to the sought order. In this case, the recovery scheme $R$ that we use can be described as follows: $R_0$ is associated with the weight-0 error $A_{0000000}$; $R_k$ with $k = 1, \ldots, 7$ are associated with the seven weight-1 errors where single-qubit errors of type $A_1$ occur; finally, $R_k$ with $k = 8, \ldots, 14$ are associated with the five weight-1 errors where single-qubit errors of type $A_3$ occur. The construction of these 15 recovery operators $R_k$ is described in terms of 30 orthonormal vectors in $H_7$. In analogy to the case of the five-qubit code, the action on the codewords of seven weight-1 enlarged errors (specifically, $A_{2000000}$, $A_{0200000}$, $A_{0020000}$, $A_{0002000}$, $A_{0000200}$, $A_{0000020}$, and $A_{0000002}$) leads to vectors that are not orthogonal to those obtained from the action of the weight-0 error $A_{0000000}$ on the codewords. Thus, we omit them from the construction of our recovery scheme. The missing 98 orthonormal vectors (needed to obtain an orthonormal basis of $H_7$ and to construct $R_{15} = \hat{O}$) can be formally computed by using the rank-nullity theorem together with the Gram-Schmidt orthonormalization procedure. Omitting further technical details (for more details, see Appendix D) but using the very same line of reasoning presented for the five-qubit code, our analytical estimate of the entanglement fidelity of the CSS seven-qubit code reads,

$$F^{[7,1,3]}(\gamma, \varepsilon) \overset{\text{def}}{=} \frac{1}{(\dim C)^2} \sum_{k=0}^{2^{14}} \sum_{l=0}^{15} \left| \text{Tr}(R_l A'_k)_{\mathcal{C}} \right|^2 \approx 1 - \frac{21}{4} \gamma^2 - 21\varepsilon^2 \left( 1 + \frac{\gamma}{\varepsilon} \right) + O(3).$$

(84)

To the best of our knowledge and unlike the case of the five-qubit code, no truncated series expansion of $F^{[7,1,3]}(\gamma, \varepsilon)$ with $\varepsilon = 0$ is available in the literature. However, we emphasize that in the special case of $\varepsilon = 0$, our analytical estimate in Eq. (84) appears to exhibit a fairly good agreement with the numerical plot presented in [12] (specifically, see Figure 9 in [12]). For $\varepsilon = 0$, we compared our non-truncated analytical estimate in Eq. (84) to the single-qubit baseline performance (entanglement fidelity when no QEC is performed) given by,

$$F^{1\text{-qubit baseline}}(\gamma) \overset{\text{def}}{=} 2^{-2} \left( 1 + \sqrt{1 - \gamma} \right)^2.$$

(85)

Then, we checked the good overlap between our results (non-truncated fidelity expressions with and without error correction) and those plotted in [12]. For some more details, see Appendix E. The robustness against non-zero environmental temperature of the CSS seven-qubit code is compared to that of the Shor nine-qubit code in Fig. 2.

### 3. The eight-qubit concatenated code

The notion of correctability depends on all the errors in the error set that one is considering and, unlike detectability, cannot be applied to individual errors. However, for a given code $C$, both sets of detectable and correctable errors are closed under linear combinations. Within the stabilizer formalism, the error correction conditions can be described
as follows [3, 33]: an \([n, k, d]\) quantum code with stabilizer \(S\) and generators \(g_j\) where \(j = 1, \ldots, n - k\), corrects an error set \(A\) if every error pair \(A^i_1 A^i_m \in A\) either anticommutes with at least one stabilizer generator,

\[
\exists g_j \in S : \{g_j, A^i_1 A^i_m \} = 0,
\]

or is in the stabilizer, \(A^i_1 A^i_m \in S\).

The two Kraus operators for the AD noise model are given by \(A_0 = I - \mathcal{O}(\gamma)\) and \(A_1 \propto \sigma_x + i \sigma_y\). In the GAD noise model, the error \(A_3 \propto \sigma_x - i \sigma_y\) appears as well. The linear span of \(A_1\) and \(A_3\) equals the linear span of \(\sigma_x\) and \(\sigma_y\). Thus, if a code is capable of correcting \(t \sigma_x\) - and \(t \sigma_y\)-errors, it can also correct \(t A_1 \) and \(t A_3\) errors. The stabilizer of the Leung et al. four-qubit code,

\[
\begin{align*}
|0_L\rangle &\defeq \frac{|0000\rangle + |1111\rangle}{\sqrt{2}}, \quad |1_L\rangle \defeq \frac{|0011\rangle + |1100\rangle}{\sqrt{2}}, \quad (86)
\end{align*}
\]

is given by \(S \defeq \langle \sigma_x^4 \rangle\). According to the above-mentioned considerations, it follows that the error set \(\{I, \sigma_x^i, \sigma_y^j\}\) with \(i = 1, 2, 3, 4\) is not correctable. For instance, the set \(\{\sigma_x^1, \sigma_y^2\}\) is not correctable because \(\sigma_x^1 \sigma_x^2\) commutes with all the stabilizer generators.

To construct a quantum code capable of error-correcting the set \(\{I, \sigma_x^i, \sigma_y^j\}\) with \(i = 1, 2, 3, 4\), we concatenate the quantum dual rail code \(C_{\text{QDR}}\) (inner code) with the perfect 1-erasure correcting code \(C_{\text{erasure}}\) (outer code) given by,

\[
|0_L\rangle \defeq |01\rangle, \quad |1_L\rangle \defeq |10\rangle, \quad (87)
\]

and [42],

\[
|0_L\rangle \defeq \frac{|0000\rangle + |1111\rangle}{\sqrt{2}}, \quad |1_L\rangle \defeq \frac{|0110\rangle + |1001\rangle}{\sqrt{2}}, \quad (88)
\]

resulting both \(C_{\text{QDR}}\) and \(C_{\text{erasure}}\) are stabilizer codes with stabilizer groups given by

\[
S \defeq \langle \sigma_x^4 \rangle, \quad (89)
\]

and,

\[
S \defeq \langle \sigma_x^1 \sigma_x^2 \sigma_x^3 \sigma_x^4, \sigma_x^1 \sigma_y^1 \sigma_x^2 \sigma_y^1, \sigma_x^2 \sigma_y^4 \rangle, \quad (90)
\]

respectively. We recall that, as pointed out in [43], minus signs do not really matter when the stabilizers are specified. Erasures are errors at known positions and a \(t\)-error correcting code is a \(2t\)-erasure correcting code. It can be shown that the perfect 1-erasure correcting code is also a single AD-error correcting and is local permutation equivalent to the Leung et al. four-qubit code. Using Eqs. (87) and (88), the concatenated code \(C_{\text{conc.}} \defeq C_{\text{QDR}} \odot C_{\text{erasure}}\) is spanned by the following codewords,

\[
|0_L\rangle \defeq \frac{|00001110\rangle + |00010101\rangle + |11110110\rangle + |11111100\rangle}{\sqrt{4}}, \quad (91)
\]

\[
|1_L\rangle \defeq \frac{|01100000\rangle + |01101111\rangle + |10010000\rangle + |10011111\rangle}{\sqrt{4}}.
\]

The stabilizer generators of the concatenated code can be obtained as follows. The concatenated code uses 8 qubits that parse two blocks, each containing 4 qubits. Qubits 1-4 belong to block 1; qubits 5-8 belong to block 2. To each block we associate a copy of the generators of \(C_{\text{erasure}}\). This gives the following six generators,

\[
g_1 \defeq \sigma_x^1 \sigma_x^2 \sigma_x^3 \sigma_x^4, \quad g_2 \defeq \sigma_x^5 \sigma_x^6 \sigma_x^7 \sigma_x^8, \quad g_3 \defeq \sigma_x^9, \quad g_4 \defeq \sigma_x^3 \sigma_x^7, \quad g_5 \defeq \sigma_x^5 \sigma_x^6, \quad g_6 \defeq \sigma_x^2 \sigma_x^5. \quad (92)
\]

The remaining generator \(g_7\) is the encoded version of \(-\sigma_x^4\), that is \(g_7 \defeq -\sigma_x^4\). Summing up, the stabilizer group for the concatenated code reads,

\[
\mathcal{S}_{\text{conc.}} \defeq \langle \sigma_x^1 \sigma_x^2 \sigma_x^3 \sigma_x^4, \sigma_x^5 \sigma_x^6 \sigma_x^7 \sigma_x^8, \sigma_x^1 \sigma_x^4, \sigma_x^5 \sigma_x^7, \sigma_x^2 \sigma_x^3, \sigma_x^6 \sigma_x^8, -\sigma_x^4 \rangle. \quad (93)
\]
It turns out that the concatenated code with stabilizer structure defined in Eq. (93) is a nondegenerate code of distance 2. Furthermore, it can be explicitly checked that the error set \{\sigma^i_1, \sigma^i_2, \sigma^i_3\} with \(i = 1, \ldots, 8\) is a set of linearly independent errors with unequal error syndromes, a property of correctable errors by means of nondegenerate codes\[14\]. We recall that the syndrome \(s(E)\) for an error \(E\) in the Pauli group \(\mathcal{P}_{\mathcal{H}_2}\) is the bit string \(l = l_1 \ldots l_{n-k}\) where the component bits \(l_i\) are given by,

\[
l_i \equiv \begin{cases} 0, & \text{if } [E, g_i] = 0 \\ 1, & \text{if } [E, g_i] = 0, \end{cases}
\]

with \(i = 1, \ldots, n-k\) and \(S \equiv \{g_i\}\) the stabilizer group of the quantum code.

Before discussing the computation of the entanglement fidelity, recall that the quantum Hamming bound places an upper bound on the number of errors \(t\) that an \([n, k, d]\) nondegenerate code can correct for given \(n\) and \(k\),

\[
2^k \sum_{j=0}^t 3^j \binom{n}{j} \leq 2^n. \tag{95}
\]

Since the code distance \(d\) equals \(2t + 1\), it also places an upper bound on the code distance. At present it is unknown whether a degenerate code might allow a violation of the Hamming bound\[14\]. For the sake of completeness, we also point out that the Hamming bound for \(q\)-dimensional systems reads

\[
K \sum_{j=0}^t (q^2 - 1)^j \binom{n}{j} \leq q^n, \tag{96}
\]

where \(k = \log_q K\) and \(\dim_{\mathbb{C}} \mathcal{H} = q^n\) with \(\mathcal{H} = (\mathbb{C}^q)^n\).

The enlarged GAD quantum channel after performing the encoding defined by means of Eq. (91) reads,

\[
A_{\text{GAD}}^{[8,1]} (\rho) \equiv \sum_{r=0}^{2^{16}-1} A'_r \rho A'_r^\dagger = \sum_{i,j,k,l,m,n,s,t=0}^{3} A_{ijklmnst} \rho A_{ijklmnst}^\dagger, \tag{97}
\]

where to any of the \(2^{16}\) values of \(r\) we can associate a set of indices \((i, j, k, l, m, n, s, t)\) (and vice-versa) such that,

\[
A'_r \leftrightarrow A_{ijklmnst} \equiv A_i \otimes A_j \otimes A_k \otimes A_l \otimes A_m \otimes A_n \otimes A_s \otimes A_t \equiv A_i A_j A_k A_l A_m A_n A_s A_t. \tag{98}
\]

The errors \(A_i\) with \(i \in \{0, 1, 2, 3\}\) are defined in Eq. (8) and \(\rho \in \mathcal{M}(\mathcal{C})\) with \(\mathcal{C} \subset \mathcal{H}_2^8\). In particular, the number of weight-\(q\) enlarged error operators \(A'_r\) is given by \(3^q(8)\) and,

\[
2^{16} = \sum_{q=0}^8 3^q \binom{8}{q}. \tag{99}
\]

In this case, the recovery scheme \(R\) that we use can be described as follows: \(R_0\) is associated with the weight-0 error \(A_{00000000}\); \(R_k\) with \(k = 1, \ldots, 8\) are associated with the eight weight-1 errors where single-qubit errors of type \(A_1\) occur; \(R_k\) with \(k = 9, \ldots, 16\) are associated with the eight weight-1 errors where single-qubit errors of type \(A_3\) occur. In analogy to the case of the CSS seven-qubit code, the action on the codewords of seven weight-1 enlarged errors (specifically, \(A_{20000000}, A_{02000000}, A_{00200000}, A_{00020000}, A_{00002000}, A_{00000200}, A_{00000020}, A_{00000002}\)) leads to vectors that are not orthogonal to those obtained from the action of the weight-0 error \(A_{00000000}\) on the codewords. Thus, we omit them from the construction of our recovery scheme. We also choose to recover the weight-2 errors that are more likely to occur where the likelihood can be expressed in terms of the perturbation parameters \(\gamma\) and \(\epsilon\). For instance, in the limiting case of \(\epsilon = 0\), it can be shown that the sum of all the probabilities for errors of weight-\(k\) to occur reads,

\[
P_{\text{weight-}k} \equiv \frac{1}{2} \left[ \binom{2}{k} \gamma^k (1 - \gamma)^{2-k} + \binom{6}{k} \gamma^k (1 - \gamma)^{6-k} \right], \tag{100}
\]

with the normalization constraint,

\[
\sum_{k=0}^{8} P_{\text{weight-}k} = 1. \tag{101}
\]
FIG. 3: Ranking additive codes. The truncated series expansions of the entanglement fidelity $F(\gamma)$ vs. the amplitude damping parameter $\gamma$ for $\varepsilon = 0$ and $0 \leq \gamma \leq 10^{-1}$: the Shor nine-qubit code (dotted line), the degenerate six-qubit code (dashed line), the five-qubit code (thin solid line), and the CSS seven-qubit code (thick solid line).

Therefore, to the 17 recovery operators $R_k$ constructed so far, we add the additional 20 out of the possible $28 = \binom{8}{2}$ recovery operators constructed by means of weight-2 errors where errors of type $A_1$ occur. We point out that we only consider 20 recovery operators, since the following eight errors are not correctable,

$$A_{11000000}, A_{10100000}, A_{01100000}, A_{00010100}, A_{00001010}, A_{00000101}, A_{00000011}.$$ (102)

We finally arrive at the construction of 37 recovery operators $R_k$ with $k = 0, \ldots, 36$ described in terms of 74 orthonormal vectors in $H_2^8$. The missing 182 orthonormal vectors (needed to obtain an orthonormal basis of $H_2^8$ and to construct $R_{37}$ can be formally computed using the rank-nullity theorem together with the Gram-Schmidt orthonormalization. Omitting further technical details but using the very same line of reasoning presented for the CSS seven-qubit code, our analytical estimate of the entanglement fidelity of the eight-qubit concatenated code reads,

$$F^{[8,1]}(\gamma, \varepsilon) \overset{\text{def}}{=} \frac{1}{\dim C_{\text{conc}}} \sum_{k=0}^{2^{16}-1} \sum_{l=0}^{37} \left| \text{Tr}(R_l A_k') C \right| \approx 1 - 2\gamma^2 - 28\varepsilon^2 \left( 1 + \frac{\gamma}{\varepsilon} \right) + O(3).$$ (103)

We remark that in the limiting case of $\varepsilon = 0$, the GAD noise model reduces to the traditional AD model and the concatenated code $C_{\text{conc.}}$ applied to AD errors works as follows: the inner code $C_{\text{QDR}}$ transforms AD errors into erasures which are then corrected by the outer code $C_{\text{erasure}}$. We also point out that in the limit of $\varepsilon = 0$ our estimated series expansion of the entanglement fidelity of the eight-qubit concatenated code coincides with that obtained by means of the traditional Leung et al. four-qubit code applied to AD errors,

$$F^{[4,1]}(\gamma) \approx 1 - 2\gamma^2 + O(3) \approx F_{\text{Leung}}^{[4,1]}(\gamma),$$ (104)

where we assume to use recovery schemes with the same structure as in Eq. [36]. This finding is not unexpected and is in agreement with the fact that, as pointed out earlier, $C_{\text{erasure}}$ and $C_{\text{Leung}}$ are local permutation equivalent quantum codes.

Finally, we compare the performances of the additive codes employed in our error correction schemes in Fig. 3.

B. Degenerate codes

Degeneracy is a property of quantum codes which has no analog for classical error correcting codes and it arises from the fact that two different error patterns can have indistinguishable effects on a coded quantum state [45]. A degenerate code has linearly independent matrices that act in a linearly dependent way on the codewords, while in a nondegenerate code, all the errors acting on the codewords produce linearly independent quantum states. For instance, the Shor nine-qubit code is a degenerate code, since phase errors within a group of 3 qubits act the same way. A striking feature of degenerate quantum codes is that they can be used to correct more errors than they can uniquely identify [16]. Also, strictly speaking, degeneracy is not a property of a code, but a property of a code together with a family of errors it is designed to correct [33, 46].
1. The six-qubit code

Calderbank et al. discovered two distinct six-qubit degenerate quantum codes encoding one logical qubit into six physical qubits [47]. The first of these codes was discovered by trivially extending the five-qubit code while the other one through an exhaustive search of the encoding space. In particular, in [47] it was argued that this second example is unique up to equivalence. The example that we consider here was originally introduced by Bilal et al. in [48]. They argue that, since their example is not reducible to the trivial six-qubit code because every one of its qubits is entangled with the others, their code is equivalent to the second nontrivial six-qubit code according to the arguments of Calderbank et al. The codespace of this nontrivial [[6, 1, 3]] six-qubit code is spanned by the codewords \(|0_L\rangle\) and \(|1_L\rangle\) defined as [48],

\[
|0_L\rangle \equiv \frac{1}{\sqrt{8}} \left( |000000\rangle - |100111\rangle + |001111\rangle - |101000\rangle - |010010\rangle + |110101\rangle + |011101\rangle - |111010\rangle \right),
\]

and,

\[
|1_L\rangle \equiv \frac{1}{\sqrt{8}} \left( |001010\rangle + |101101\rangle + |000101\rangle + |100010\rangle - |011000\rangle - |111111\rangle + |010111\rangle + |110000\rangle \right),
\]

respectively. The five stabilizer generators for this code can be written as,

\[
g_1 \equiv Y^1 Z^3 X^4 X^5 Y^6, \quad g_2 \equiv Z^1 X^2 Z^5 Z^6, \quad g_3 \equiv Z^2 X^3 X^4 X^5 Z^6, \quad g_4 \equiv Z^4 Z^6, \quad g_5 \equiv Z^1 Z^2 Z^3 Z^5.
\]

Within the quantum stabilizer formalism, an \([n, k, d]\) code is degenerate if the stabilizer group contains elements of weight less than \(d\) (other than the identity) [33]. Thus, it appears evident from Eq. (107) that this distance \(d = 3\) code is degenerate.

The enlarged GAD quantum channel after performing the encoding defined by means of Eqs. (105) and (106) reads,

\[
A^{[6,1,3]}_{GAD}(\rho) = \sum_{r=0}^{2^{12}-1} A_r^i \rho A_r^i = \sum_{i,j,k,l,m,n=0}^3 A_{ijklmn} \rho A_{ijklmn}^\dagger,
\]

where to any of the \(2^{12}\) values of \(r\) we can associate a set of indices \((i, j, k, l, m, n)\) (and vice-versa) such that,

\[
A_r^i \leftrightarrow A_{ijklmn} \equiv A_i \otimes A_j \otimes A_k \otimes A_l \otimes A_m \otimes A_n \equiv A_1 A_2 A_3 A_4 A_5 A_6.
\]

The errors \(A_i\) with \(i \in \{0, 1, 2, 3\}\) are defined in Eq. (8) and \(\rho \in \mathcal{M}(\mathcal{C})\) with \(\mathcal{C} \subset \mathcal{H}_2^{6}\). In particular, the number of weight-\(q\) enlarged error operators \(A_r^i\) is given by \(3^q \binom{6}{q}\) and,

\[
2^{12} = \sum_{q=0}^{6} 3^q \binom{6}{q}.
\]

For \(\varepsilon = 0\), it can be shown that among the \(\binom{6}{2} = 15\) weight-2 errors, the set of five weight-2 errors given by \(\{A_{100000}, A_{100010}, A_{010000}, A_{001010}, A_{000101}\}\) is not correctable, since they are not compatible with \(A_{000000}\). In addition, the action of the two weight-2 errors \(A_{101000}\) and \(A_{010010}\) on the codewords leads to state vectors that are not orthogonal to \(A_{000000} \vert i_L\rangle\) with \(i \in \{0, 1\}\). All weight-1 errors are correctable, of course. In view of these considerations, we construct the recovery scheme \(\mathcal{R}\) for the general case as follows: \(\mathcal{R}_0\) is associated with the weight-0 error \(A_{000000}\); \(\mathcal{R}_k\) with \(k = 1, ..., 6\) are associated with the six weight-1 errors where single-qubit errors of type \(A_1\) occur; \(\mathcal{R}_k\) with \(k = 7, ..., 12\) are associated with the six weight-1 errors where single-qubit errors of type \(A_3\) occur; finally, \(\mathcal{R}_k\) with \(k = 13, ..., 20\) are associated with the eight correctable weight-2 errors where errors of type \(A_1\) occur. The construction of these 21 recovery operators \(\mathcal{R}_k\) is described in terms of 42 orthonormal vectors in \(\mathcal{H}_2^{6}\). As pointed out earlier, the missing 22 orthonormal vectors (needed to obtain an orthonormal basis of \(\mathcal{H}_2^{6}\) and to construct \(\mathcal{R}_{21}\)) can be formally computed by using the rank-nullity theorem together with the Gram-Schmidt orthonormalization procedure. Our analytical estimate of the entanglement fidelity of the nontrivial six-qubit degenerate code reads,

\[
\mathcal{F}^{[6,1,3]}(\gamma, \varepsilon) = \frac{1}{\dim(\mathcal{C})^2} \sum_{k=0}^{2^{12}-1} \sum_{l=0}^{21} \left| \text{Tr}(R_k A_k^l) \right|^2 \approx 1 - 2\gamma^2 - 15\varepsilon^2 \left(1 + \frac{2}{\varepsilon}\right) + \mathcal{O}(3) .
\]
Comparing Eqs. [77], [84] and [103] to Eq. (111), we observe that the six-qubit degenerate code outperforms the five-, CSS seven-, and eight-qubit concatenated nondegenerate codes. Our findings show that despite the fact that the CSS seven- and eight-qubit concatenated codes have larger Hilbert spaces for encoding than that allowed for the six-qubit code, their error-correcting capability is smaller, given the noise model considered and the recovery schemes employed. Our finding in Eq. (111) strengthens the suspect advanced in [12] where it was conjectured that thanks to their degenerate structure, such codes can outperform nondegenerate codes despite their shorter length.

2. The Shor nine-qubit code

We consider here the [[9, 1, 3]] Shor nine-qubit code [1], the code that gave birth to the subject of quantum error correcting codes. The codespace of such a code is spanned by the following two codewords [33],

\[ |0_L\rangle \defeq \frac{1}{\sqrt{8}} (|000\rangle + |111\rangle) (|000\rangle + |111\rangle) (|000\rangle + |111\rangle), \]  

and,

\[ |1_L\rangle \defeq \frac{1}{\sqrt{8}} (|000\rangle - |111\rangle) (|000\rangle - |111\rangle) (|000\rangle - |111\rangle). \]  

This degenerate code can be constructed by concatenating two nondegenerate [[3, 1, 1]] codes and its eight stabilizer generators can be written as,

\[ g_1 \defeq Z^1Z^2, g_2 \defeq Z^1Z^3, g_3 \defeq Z^4Z^5, g_4 \defeq Z^4Z^6, g_5 \defeq Z^7Z^8, g_6 \defeq Z^7Z^9, \]

\[ g_7 \defeq X^1X^2X^3X^4X^5X^6, g_8 \defeq X^1X^2X^3X^7X^8X^9. \]  

The enlarged GAD quantum channel after performing the encoding defined by means of Eqs. (112) and (113) reads,

\[ \mathcal{A}_{GAD}^{[[9,1,3]]} (\rho) \defeq \sum_{r=0}^{2^{18}-1} A_r' \rho A_r'^\dagger = \sum_{i, j, k, l, m, n, s, t, u=0}^{3} A_{ijklmnstu} \rho A_{ijklmnstu}^\dagger, \]  

where to any of the \(2^{18}\) values of \(r\) we can associate a set of indices \((i, j, k, l, m, n, s, t, u)\) (and vice-versa) such that,

\[ A_i' \leftrightarrow A_{ijklmnstu} \defeq A_i \otimes A_j \otimes A_k \otimes A_l \otimes A_m \otimes A_n \otimes A_s \otimes A_t \otimes A_u \equiv A_i A_j A_k A_l A_m A_n A_s A_t A_u. \]  

The errors \(A_i\) with \(i \in \{0, 1, 2, 3\}\) are defined in Eq. (8) and \(\rho \in \mathcal{M}(C)\) with \(C \subset \mathcal{H}_2^3\). In particular, the number of weight-\(q\) enlarged error operators \(A_i'\) is given by \(3^q\binom{3}{q}\) and,

\[ 2^{18} = \sum_{q=0}^{9} 3^q \binom{9}{q}. \]  

As a side remark, we recall that a quantum code has distance \(d\) if all errors of weight less than \(d\) satisfy the QEC conditions \(\langle j_L | A_i^\dagger A_m | j_L \rangle = \alpha_{im} \delta_{ij}\) and at least one error of weight \(d\) exists that violates it. Otherwise stated, the distance of a code is the weight of the smallest error \(A_i^\dagger A_m\) that cannot be detected by the code. For instance, using the three-qubit bit-flip repetition code [[3, 1, 1]] to correct bit-flip errors, it turns out that the weight-1 error \(\sigma^J_0\) cannot be detected. However, this code of distance \(d = 1\) also detects errors of weight-2 such as, for instance, \(\sigma^J_0 \sigma^S_0\).

For \(\varepsilon = 0\), it turns out that all the nine weight-1 and thirty-six weight-2 errors are correctable. In addition, all weight-3 errors can be recovered as well, except for \(A_{111000000}, A_{000110000}\) and \(A_{000000111}\). These three errors could be potentially recovered by means of the recovery operator constructed with the weight-0 error \(A_{000000000}\). However, it can be checked that their contributions is null. For the general case, the recovery scheme \(\mathcal{R}\) that we use can be described as follows: \(R_0\) is associated with the weight-0 error \(A_{000000000}\); \(R_k\) with \(k = 1, ..., 9\) are associated with the nine weight-1 errors where single-qubit errors of type \(A_1\) occur; \(R_k\) with \(k = 10, ..., 18\) are associated with the nine weight-1 errors where single-qubit errors of type \(A_3\) occur; \(R_k\) with \(k = 19, ..., 54\) are associated with the thirty-six weight-2 errors where errors of type \(A_1\) occur; finally, \(R_k\) with \(k = 55, ..., 135\) are associated with the eighty-one
weight-3 errors where errors of type $A_1$ occur. The construction of these 136 recovery operators $R_k$ is described in terms of 272 orthonormal vectors in $H^g_2$. The missing 240 orthonormal vectors, needed to obtain an orthonormal basis of $H^g_2$ and to construct $R_{136} \equiv \hat{O}$, can be formally computed by using the rank-nullity theorem together with the Gram-Schmidt orthonormalization procedure. Our analytical estimate of the entanglement fidelity of the Shor nine-qubit code reads,
\[
\mathcal{F}^{[[9,1,3]]} (\gamma, \varepsilon) \overset{\text{def}}{=} \frac{1}{(\dim C)^2} \sum_{k=0}^{2^8-1} \sum_{l=0}^{136} \left| \text{Tr} (R_k A_L) \right|^2 \approx 1 - \frac{3}{2} \gamma^3 - 36 \varepsilon^2 \left( 1 + \frac{\gamma^2}{\varepsilon} \right) + \mathcal{O} \left( \frac{1}{\varepsilon} \right).
\]

To the best of our knowledge and unlike the case of the five-qubit code, no truncated series expansion of $\mathcal{F}^{[[9,1,3]]} (\gamma, \varepsilon)$ with $\varepsilon = 0$ is available in the literature. However, we emphasize that in the special case of $\varepsilon = 0$, our analytical estimate in Eq. (118) appears to exhibit a fairly good agreement with the numerical plot presented in [8] (specifically, see Figure 12 in [8]). For $\varepsilon = 0$, we compared our non truncated analytical estimate in Eq. (118) to the baseline performance of a single qubit given by Eq. (119). Then, we checked the good overlap between our results (non truncated fidelity expressions with and without error correction) and the ones plotted in [8]. For some more details, see Appendix F.

V. NONADDITIVE CODES

There are codes that do not exhibit stabilizer structures. Such codes are known as nonadditive quantum codes. A quantum code is nonadditive if it is not additive, that is if it cannot be constructed within the stabilizer framework.

Examples of nonstabilizer codes can be found when one of the two following requirements are satisfied [32]:

- It exists a state $|\psi\rangle \notin C$ such that $g |\psi\rangle = |\psi\rangle$ for any operator $g$ that belongs to the group stabilizer $S_C$ of the code $C$. This happens when $C$ is not maximal;
- It exists $g \notin S_C$ such that $g |\psi\rangle = |\psi\rangle$ for any state in the codespace of $C$. This happens when $S_C$ is not maximal.

We point out it can also occur that the code $C$ does not allow any stabilizer group $S_C$ at all, neither maximal nor minimal. For instance, consider the code $C$ defined as,
\[
C \overset{\text{def}}{=} \text{Span} \left\{ |0_L\rangle \overset{\text{def}}{=} \frac{|01\rangle + |10\rangle}{\sqrt{2}}, |1_L\rangle \overset{\text{def}}{=} |11\rangle \right\}.
\]

The code $C$ in Eq. (119) is not the joint +1-eigenspace of any operator in the Pauli group $P_{H^2}$. Therefore, this code is not a stabilizer code, since it does not have the standard stabilizer structure.

A nonadditive $((n, K, d))$ code is a $K$-dimensional subspace of a $n$-qubit Hilbert space correcting $\lfloor \frac{d-1}{2} \rfloor$-qubit errors and $d$ is the distance of the code. The first example of nonadditive code was a $((5,6,2))$ code presented in [49]. This code was constructed numerically by building a projector operator with a given weight distribution. It encodes six logical qubits into five physical qubits and can correct single-qubit erasure. This code outperforms any known stabilizer code in terms of encoded dimension $\log_2 6/5$. In [50], a family of nonadditive codes of distance $d = 2$ capable of detecting any single-qubit error (or, equivalently, correct any single-qubit erasure) with high encoded dimensions was introduced. The simplest example of a nonadditive code in such a family is represented by a self-complementary $((5,5,2))$ nonadditive quantum code that is not a subcode of the $((5,6,2))$ code in [49]. Necessary and sufficient conditions for the error correction of amplitude damping errors with self-complementary nonadditive quantum codes were presented in [23, 24]. In particular, in [23] a numerical investigation of a $((8,12))$ self-complementary nonadditive quantum code with a high encoding rate $\log_2 12/8$ was presented for the correction of amplitude damping errors.

The performance of this code was quantified by means of the entanglement fidelity and evaluated numerically for a maximum likelihood recovery scheme that corrects all the first-order amplitude damping errors. We stress that for nonadditive codes, the decoding procedure does not have the syndrome-diagnosis and the recovery structure of the stabilizer codes. However, like the additive case, recovery schemes for nonadditive codes may exhibit a projection nature as well. Furthermore, thanks to the graph-state formalism [51], many nonadditive codes can be characterized by a stabilizer-like structure [25]. The stabilizer-like structure of some classes of nonadditive codes simplify significantly the encoding and decoding procedures for these codes [52]. Nonadditive codes of distance $d = 2$ can detect but cannot correct arbitrary single-qubit errors. To achieve this task, codes of distance $d = 3$ are needed. In [53], sufficient general conditions for the existence of nonadditive codes were given. In particular, an example of a strongly nonadditive $((11,2,3))$ quantum code was presented. However, the question of whether the nonadditive codes correcting errors beyond erasures are more efficient (in terms of the encoded dimension) than the corresponding stabilizer codes remained open. The very first example of a 1-error correcting nonadditive code capable of outperforming the optimal stabilizer code (the $[[9,3,3]]$ code) of the same length was the nondegenerate $[[9,12,3]]$ code in [24].
A. Non-self-complementary codes

1. The \( ((11, 2, 3)) \) code

According to [53], two quantum codes \( C_1 \) and \( C_2 \) in \( \mathbb{C}^{2^n} \) are \textit{locally equivalent} if there is a transversal operator \( U \defeq u_1 \otimes \ldots \otimes u_n \) with \( u_j \in SU(2, \mathbb{C}) \), mapping \( C_1 \) into \( C_2 \). Instead, two codes are \textit{globally equivalent}, or simply equivalent, if \( C_1 \) is locally equivalent to a code obtained from \( C_2 \) by a permutation on qubits. A quantum code \( C \subset \mathbb{C}^{2^n} \) is called nonadditive if it is not equivalent to any additive code; moreover, \( C \) is strongly nonadditive if the only additive code that contains any code equivalent to \( C \) is the trivial code \( \mathbb{C}^{2^n} \) (in other words, if \( \pm X_\alpha Z_\beta \) with \( \alpha, \beta \in \{0, 1\}^n \) is in the stabilizer of any code equivalent to \( C \)). Also, the generalized stabilizer \( GS_C \) of a code \( C \subset \mathbb{C}^{2^n} \) is the set of all unitary operators \( V \) on \( \mathbb{C}^{2^n} \) such that \( V|c\rangle = |c\rangle \) for every \( |c\rangle \in C \). Then, the stabilizer \( S_C \) of a code \( C \) is \( S_C = P_{\mathbb{C}^{2^n}} \cap GS_C \) where \( P_{\mathbb{C}^{2^n}} \) is the \( n \)-qubit Pauli group.

The strong nonadditivity is guaranteed by the fulfillment of two conditions [53]: i) the identity operator is the only operator in the stabilizer of the code; ii) there is no element in \( GS_C \) by the following two codewords [53], \( u \)

\[
1 \text{ (1)} \;
\text{is the all-1 vector of length 11 and } r_i \text{ denotes the } i \text{th row of the following (12 } \times 11 \text{)-matrix } H \text{ defined as,}
\]

\[
H \defeq \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{pmatrix}.
\]

(123)

The enlarged GAD quantum channel after performing the encoding defined by means of Eqs. (121) and (122) reads,

\[
A_{GAD}^{((11,2,3))} \frac{\text{(} \rho \text{)}}{\text{(} \rho \text{)}} \defeq \sum_{r=0}^{2^{22}-1} A_r^{\dagger} \rho A_r^{\dagger} = \sum_{a_1, \ldots, a_{11} = 0}^{3} A_{a_1 a_2 \ldots a_{10} a_{11}} \rho A_{a_1 a_2 \ldots a_{10} a_{11}}^{\dagger},
\]

(124)

where to any of the \( 2^{22} \) values of \( r \) we can associate a set of indices \( (a_1, \ldots, a_{11}) \) (and vice-versa) such that,

\[
A_r \leftrightarrow A_{a_1 a_2 \ldots a_{10} a_{11}} \defeq A_{a_1} \otimes A_{a_2} \otimes \ldots \otimes A_{a_{10}} \otimes A_{a_{11}} \equiv A_{a_1} A_{a_2} \ldots A_{a_{10}} A_{a_{11}}.
\]

(125)
The errors $A_i$ with $i \in \{0, 1, 2, 3\}$ are defined in Eq. (5) and $\rho \in \mathcal{M} (\mathcal{C})$ with $\mathcal{C} \subset \mathcal{H}_2^{11}$. In particular, the number of weight-$q$ enlarged error operators $A'_i$ is given by $3^q \binom{11}{q}$ and,

$$2^{22} = \sum_{q=0}^{11} 3^q \binom{11}{q}.\quad (126)$$

We assume to focus on the recovery of weight-1 errors only and pay no attention to the possible recovery of higher-order enlarged errors. For instance, this working hypothesis is especially plausible for values of the perturbation parameters $\gamma$ and $\varepsilon$ with $0 \leq \varepsilon \ll \gamma \ll 1$. In this case, the recovery scheme $\mathcal{R}$ that we use can be described as follows: $R_0$ is associated with the weight-0 error; $R_k$ with $k = 1, \ldots, 11$ are associated with the eleven weight-1 errors where single-qubit errors of type $A_1$ occur; finally, $R_k$ with $k = 12, \ldots, 22$ are associated with the eleven weight-1 errors where single-qubit errors of type $A_3$ occur. The construction of these 23 recovery operators $R_k$ is described in terms of 46 orthonormal vectors in $\mathcal{H}_2^{11}$. The missing orthonormal vectors needed to obtain an orthonormal basis of $\mathcal{H}_2^{11}$ and to construct $R_{23} \equiv \hat{O}$ can be formally computed by using the rank-nullity theorem together with the Gram-Schmidt orthonormalization procedure. Finally, our analytical estimate of the entanglement fidelity of the $((11,2,3))$ code reads,

$$\mathcal{F}^{((11,2,3))} (\gamma, \varepsilon) \equiv \frac{1}{(\dim \mathcal{C})^2} \sum_{k=0}^{2^{14} - 1} \sum_{l=0}^{2^{22}} \left| \text{Tr} \left( R_l A'_k \right) \right|_C^2 \approx 1 - \frac{55}{4} \gamma^2 - 55 \varepsilon^2 \left( 1 + \frac{\gamma}{\varepsilon} \right) + \mathcal{O} (3),\quad (127)$$

with,

$$\mathcal{F}^{((11,2,3))} (\gamma, \varepsilon) \equiv \frac{1}{(\dim \mathcal{C})^2} \sum_{k=0}^{2^{14} - 1} \sum_{l=0}^{2^{23}} \left| \text{Tr} \left( R_l A'_k \right) \right|_C^2.\quad (128)$$

From Eq. (127), we conclude that the performance of this strongly nonadditive code whose encoding rate equals 1/11 is not especially good for GAD errors. After all, this code was originally introduced in [53] for conceptual reasons without any claim about its error-correcting capabilities against any specific noise model. This code lacks two essential features: high encoding rate and self-complementarity [50]. The nonadditive code we consider next, although not self-complementary, has a very high encoding rate.

2. The $((9,12,3))$ code

We consider next the nondegenerate $((9,12,3))$ code [54], the first example of a 1-error correcting nonadditive code capable of outperforming (in terms of encoded dimension) the optimal stabilizer code with the same length, namely, the $[[9,3,3]]$ code. This is trivially a nonadditive code, since it encodes a fractional number of qubits, $k = \log_2 12 \approx 3.6$. This code was constructed by means of the graph-state formalism [51]. Therefore, in order to justify the structure of the codewords spanning this $K = 12$-dimensional codespace of this code, a subspace of the $2^9$-dimensional complex Hilbert space $\mathcal{H}_2^9$, we introduce first the basic ingredients of the graph-state formalism and we refer to [51] and [54] for more details on this specific point.

The starting point in the graph-state formalism is the notion of graph. An undirected simple graph $G \equiv G (V, \Gamma)$ with $n = |V|$ vertices is characterized by the so-called adjacency matrix $\Gamma$. This is a $n \times n$ symmetric matrix with vanishing diagonal elements such that $\Gamma_{ij} = 1$ if vertices $i$ and $j$ are connected and $\Gamma_{ij} = 0$ otherwise. The graph-state $|G\rangle$ associated with the graph $G$ reads,

$$|G\rangle \equiv \frac{1}{\sqrt{2^n}} \sum_{\vec{\mu}} (-1)^{\frac{1}{2} \vec{\mu} \cdot \vec{\mu}} |\vec{\mu}\rangle_z,\quad (129)$$

where $|\vec{\mu}\rangle_z$ are the simultaneous eigenstates of $\{Z_j\}_{j \in V}$ with $(-1)^{\rho_j}$ as eigenvalues and $Z_j$ the Pauli operator acting on qubit $j \in V$. 
The \((9, 12, 3)\) code is associated with the so-called loop graph \(L_9\) whose \(9 \times 9\) adjacency matrix reads,

\[
\Gamma_{L_9} \overset{\text{def}}{=} \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 
\end{pmatrix},
\]

(130)

and its corresponding graph-state is denoted as \(|L_9\rangle\). In terms of \(|L_9\rangle\), the codespace of the code is spanned by the following states

\[
|i_L\rangle \overset{\text{def}}{=} Z_{V_i} |L_9\rangle,
\]

(131)

where \(i = 1, \ldots, 12\) and,

\[
Z_{V_i} \overset{\text{def}}{=} \prod_{a \in V_i} Z_a,
\]

(132)

with the set of vertices \(V_i\) defined as,

\[
V_1 \overset{\text{def}}{=} \emptyset, \quad V_2 \overset{\text{def}}{=} \{2, 6, 7\}, \quad V_3 \overset{\text{def}}{=} \{4, 5, 9\}, \quad V_4 \overset{\text{def}}{=} \{2, 3, 6, 8\}, \quad V_5 \overset{\text{def}}{=} \{3, 5, 8, 9\}, \quad V_6 \overset{\text{def}}{=} \{2, 3, 4, 5, 6, 7, 8, 9\},
\]

\[
V_7 \overset{\text{def}}{=} \{1, 4, 7\}, \quad V_8 \overset{\text{def}}{=} \{1, 2, 4, 6\}, \quad V_9 \overset{\text{def}}{=} \{1, 5, 7, 9\}, \quad V_{10} \overset{\text{def}}{=} \{1, 2, 3, 4, 6, 7, 8\}, \quad V_{11} \overset{\text{def}}{=} \{1, 3, 4, 5, 7, 8, 9\},
\]

\[
V_{12} \overset{\text{def}}{=} \{1, 2, 3, 5, 6, 8, 9\}.
\]

(133)

To be explicit, the 12 codewords are given by

\[
|1_L\rangle \overset{\text{def}}{=} |L_9\rangle, \quad |2_L\rangle \overset{\text{def}}{=} Z^2 Z^6 Z^7 |L_9\rangle, \quad |3_L\rangle \overset{\text{def}}{=} Z^4 Z^5 Z^9 |L_9\rangle, \quad |4_L\rangle \overset{\text{def}}{=} Z^2 Z^3 Z^6 Z^8 |L_9\rangle, \quad |5_L\rangle \overset{\text{def}}{=} Z^3 Z^5 Z^8 Z^9 |L_9\rangle,
\]

\[
|6_L\rangle \overset{\text{def}}{=} Z^2 Z^3 Z^4 Z^5 Z^6 Z^7 Z^8 Z^9 |L_9\rangle, \quad |7_L\rangle \overset{\text{def}}{=} Z^1 Z^4 Z^7 |L_9\rangle, \quad |8_L\rangle \overset{\text{def}}{=} Z^1 Z^2 Z^4 Z^6 |L_9\rangle, \quad |9_L\rangle \overset{\text{def}}{=} Z^1 Z^5 Z^7 Z^9 |L_9\rangle,
\]

\[
|10_L\rangle \overset{\text{def}}{=} Z^1 Z^2 Z^3 Z^4 Z^6 Z^7 Z^8 |L_9\rangle, \quad |11_L\rangle \overset{\text{def}}{=} Z^1 Z^3 Z^5 Z^7 Z^8 Z^9 |L_9\rangle, \quad |12_L\rangle \overset{\text{def}}{=} Z^1 Z^2 Z^3 Z^5 Z^6 Z^8 Z^9 |L_9\rangle.
\]

(134)

We stress that each codeword is the sum of 512 state vectors,

\[
\sum_{k=0}^9 \binom{9}{k} = 2^9 = 512,
\]

where \(\binom{9}{k}\) denotes the number of state vectors of length 9 in this sum with \(k\)-1s in their definition. For each codeword, the sign distribution of these 512 state vectors changes according to the action of \(Z_{V_k}\) with \(k \in \{1, \ldots, 12\}\).

The enlarged GAD quantum channel after performing the encoding defined by means of Eq. \((131)\) reads,

\[
A_{\text{GAD}}^{(9,12,3)}(\rho) \overset{\text{def}}{=} \sum_{r=0}^{2^{18}-1} A_r \rho A_r^\dagger = \sum_{a_1, \ldots, a_9 = 0}^3 A_{a_1 a_2 \ldots a_8 a_9} \rho A_{a_1 a_2 \ldots a_8 a_9}^\dagger,
\]

(135)

where to any of the \(2^{18}\) values of \(r\) we can associate a set of indices \((a_1, \ldots, a_9)\) (and vice-versa) such that,

\[
A_r \leftrightarrow A_{a_1 a_2 \ldots a_8 a_9} \overset{\text{def}}{=} A_{a_1} \otimes A_{a_2} \otimes \ldots \otimes A_{a_8} \otimes A_{a_9} \equiv A_{a_1} A_{a_2} \ldots A_{a_8} A_{a_9}.
\]

(136)
The errors $A_i$ with $i \in \{0, 1, 2, 3\}$ are defined in Eq. (8) and $\rho \in \mathcal{M}(\mathcal{C})$ with $\mathcal{C} \subset \mathcal{H}_2^9$. In particular, the number of weight-$q$ enlarged error operators $A'_i$ is given by $3^q\binom{9}{q}$ and,

$$\sum_{q=0}^{9} 3^q \binom{9}{q} = 2^{18}. \quad (137)$$

Before describing our recovery scheme, let us make two remarks. First, the codespace of this nonadditive code is a 12-dimensional subspace of $\mathcal{H}_2^9$ spanned by twelve codewords. Each codeword is the sum-decomposition of 512 vector states in $\mathcal{H}_2^9$. The number of vector states in such decomposition with $m$ non-zero components is given by $\binom{9}{m}$. This binomial factor can be regarded as the cardinality of vector states of Hamming weight $m$ that appear in the sum-decomposition of the codewords. The normalization condition requires,

$$2^9 = \sum_{q=0}^{9} \binom{9}{q} = 2^9 = 512. \quad (138)$$

Second, after some thinking, it can be shown that the action of any weight-1 enlarged error operator (where single-qubit errors of type $A_3$ may occur) on each of these codewords with the above-mentioned structure leads to quantum states in $\mathcal{H}_2^9$ described in terms of a sum-decomposition of 162 vector states which give rise to the following Hamming weight distribution: 1 vector with Hamming weight $m = 1$, 8 vectors with $m = 2$, 28 vectors with $m = 3$, 56 vectors with $m = 4$, 35 vectors with $m = 5$, 20 vectors with $m = 6$, 10 vectors with $m = 7$, 3 vectors with $m = 8$ and, finally, 1 vector with $m = 9$.

As stated earlier, we assume to focus on the recovery of weight-1 errors only and pay no attention to the possible recovery of higher-order enlarged errors. In this case, the recovery scheme $\mathcal{R}$ that we use can be described as follows: $R_0$ is associated with the weight-0 error; $R_k$ with $k = 1, ..., 9$ are associated with the eleven weight-1 errors where single-qubit errors of type $A_1$ occur; finally, $R_k$ with $k = 10, ..., 18$ are associated with the eleven weight-1 errors where single-qubit errors of type $A_3$ occur. The construction of these 19 recovery operators $R_k$ is described in terms of $19 \times 12 = 228$ orthonormal vectors in $\mathcal{H}_2^9$. The missing orthonormal vectors needed to obtain an orthonormal basis of $\mathcal{H}_2^9$ and to construct $R_{19} \doteq \tilde{O}$ can be formally computed by using the rank-nullity theorem together with the Gram-Schmidt orthonormalization procedure. Finally, our analytical estimate of the entanglement fidelity of the $((9, 12, 3))$ code reads,

$$\tilde{F}^{(9,12,3)}_{\text{first-order}}(\gamma, \varepsilon) \overset{\text{def}}{=} \frac{1}{(\dim \mathcal{C})^2} 2^{18} \sum_{k=0}^{18} \sum_{l=0}^{18} \left| \text{Tr} (R_l A'_k) \right|^2 \approx 1 - 9\gamma^2 - 36\gamma^2 \left(1 + \frac{\gamma}{\varepsilon}\right) + \mathcal{O}(3), \quad (139)$$

with,

$$\tilde{F}^{(9,12,3)}(\gamma, \varepsilon) \overset{\text{def}}{=} \frac{1}{(\dim \mathcal{C})^2} 2^{18} \sum_{k=0}^{18} \sum_{l=0}^{19} \left| \text{Tr} (R_l A'_k) \right|^2. \quad (140)$$

From Eqs. (127) and (139), we conclude that the nondegenerate $((9,12,3))$ code not only outperforms the $((11,2,3))$ in terms of encoded dimension but also in terms of entanglement fidelity with recovery schemes limited to first-order recovery. Furthermore from Eq. (118), it can be shown that,

$$\tilde{F}^{([9,1,3])}_{\text{first-order}}(\gamma, \varepsilon = 0) \approx 1 - \frac{45}{4} \gamma^2 + \mathcal{O}(\gamma^3) \lesssim 1 - \frac{9}{\log_2 12} \gamma^2 + \mathcal{O}(\gamma^3) \approx \tilde{F}^{(9,12,3)}(\gamma, \varepsilon = 0), \quad (141)$$

where $\tilde{F}$ denotes the entanglement fidelity normalized with exponentiation by $1/k$ where $k \overset{\text{def}}{=} \log_2 K$ is the number of encoded logical qubits ($k = 1$ for the Shor nine-qubit code) [8]. From the comparison of the first-order entanglement-based performances of these two codes in Eq. (141), we are lead to the conclusion that the nondegenerate and nonadditive code $((9,12,3))$ outperforms the degenerate and additive code $[[9,1,3]]$ not only in terms of encoded dimension. The comparison between these two codes is shown in Fig. 4.

### B. Self-complementary codes

In what follows, we consider single-AD error correcting codes. For this reason, we set $\varepsilon = 0$ and limit our considerations to the approximate QEC of AD errors by means of self-complementary nonadditive quantum codes. An
FIG. 4: Additive vs. nonadditive codes of length nine. The truncated series expansions of the normalized entanglement fidelity $F(\gamma)$ vs. the amplitude damping parameter $\gamma$ for $\varepsilon = 0$ and $0 \leq \gamma \leq 10^{-1}$: the Shor nine-qubit code (thin solid line) and the $((9,12,3))$-code (dashed line).

$((n, K, d))$ code is called self-complementary if its codespace is spanned by codewords $\{|c_a\rangle\}$ defined as [50],

$$|c_a\rangle \equiv |a\rangle + |\bar{a}\rangle \sqrt{2},$$

where $a$ is a binary string of length $n$ and $\bar{a}$ def $1 \oplus a$ is the complement of $a$. The suitability of self-complementary nonadditive codes for the error correction of AD errors was first proposed in [23, 24]. However, no explicit comparison of the performances quantified by means of analytical estimates of the entanglement fidelity of self-complementary and non-self-complementary nonadditive codes for AD errors was presented. Furthermore, no similar comparison between nonadditive self-complementary and stabilizer codes for AD errors exists as well. In view of these considerations, we seek to provide some useful insights into these unexplored issues.

1. The $((6, 5))$ code

The first self-complementary single-AD error-correcting code that we consider is the $((6, 5))$ code whose codespace is spanned by the following five codewords in $\mathcal{H}_2^6$,

$$|0_L\rangle \equiv \frac{|000000\rangle + |111111\rangle}{\sqrt{2}}, \quad |1_L\rangle \equiv \frac{|110000\rangle + |001111\rangle}{\sqrt{2}}, \quad |2_L\rangle \equiv \frac{|001100\rangle + |110011\rangle}{\sqrt{2}},$$

$$|3_L\rangle \equiv \frac{|000011\rangle + |111100\rangle}{\sqrt{2}}, \quad |4_L\rangle \equiv \frac{|010101\rangle + |101010\rangle}{\sqrt{2}}.$$ (143)

We point out that the existence of such a code was originally proposed in [23], although it was neither explicitly shown nor used for error correction of single-AD errors.

The enlarged AD quantum channel after performing the encoding defined by means of Eq. (143) reads,

$$\Lambda^{((6,5))}_{AD}(\rho) \equiv \sum_{r=0}^{2^6-1} A'_r \rho A'_r = \sum_{a_1,\ldots, a_6=0}^1 A_{a_1a_2\ldots a_5a_6} \rho A_{a_1a_2\ldots a_5a_6}^\dagger,$$ (144)

where to any of the $2^6$ values of $r$ we can associate a set of indices $(a_1,\ldots, a_6)$ (and vice-versa) such that,

$$A'_r \leftrightarrow A_{a_1a_2\ldots a_5a_6} \equiv A_{a_1} \otimes A_{a_2} \otimes \ldots \otimes A_{a_5} \otimes A_{a_6} \equiv A_{a_1} A_{a_2}A_{a_5}A_{a_6}.$$ (145)

The errors $A_i$ with $i \in \{0, 1\}$ are defined in Eq. (5) and $\rho \in \mathcal{M}(\mathcal{C})$ with $\mathcal{C} \subset \mathcal{H}_2^6$. In particular, the number of
weight-$q$ enlarged error operators $A_r'$ is given by $\binom{6}{q}$ and,
\[
\sum_{q=0}^{6} \binom{6}{q} = 2^6.
\] (146)

The recovery scheme $\mathcal{R}$ that we use can be described as follows: $R_0$ is associated with the weight-0 error; $R_k$ with $k = 1, \ldots, 6$ are associated with the six weight-1 errors where single-qubit errors of type $A_1$ occur. The construction of these 7 recovery operators $R_k$ is described in terms of $7 \times 5 = 35$ orthonormal vectors in $\mathcal{H}_2^5$. The missing orthonormal vectors needed to obtain an orthonormal basis of $\mathcal{H}_2^5$ and to construct $R_7 \equiv \hat{O}$ can be formally computed by using the rank-nullity theorem together with the Gram-Schmidt orthonormalization procedure. Finally, our analytical estimate of the entanglement fidelity of the $((6, 5))$ code reads,
\[
\mathcal{F}_{\text{first-order}}^{((6, 5))} (\gamma) \equiv \frac{1}{(\dim \mathcal{C})^2} \sum_{k=0}^{2^6-1} \sum_{l=0}^{6} \left| \text{Tr} (R_l A_k') \right|_{\mathcal{C}}^2 \approx 1 - \frac{21}{5} \gamma^2 + \mathcal{O} (\gamma^3),
\] (147)

with,
\[
\mathcal{F}_{\text{first-order}}^{((6, 5), \ell)} (\gamma) \equiv \frac{1}{(\dim \mathcal{C})^2} \sum_{k=0}^{2^6-1} \sum_{l=0}^{7} \left| \text{Tr} (R_l A_k') \right|_{\mathcal{C}}^2.
\] (148)

From Eqs. (111) and (147), it follows that
\[
\mathcal{F}_{\text{first-order}}^{((6, 5))} (\gamma) \approx 1 - \frac{21}{5 \log_2 5} \gamma^2 + \mathcal{O} (\gamma^3) \geq 1 - 2 \gamma^2 + \mathcal{O} (\gamma^3) \approx \mathcal{F}^{[[6, 1, 3]]} (\gamma, \varepsilon = 0) \geq \mathcal{F}_{\text{first-order}}^{((6, 5), \ell)} (\gamma, \varepsilon = 0).
\] (149)

Eq. (149) is an explicit manifestation of the superiority, in terms of both encoded dimension and entanglement fidelity, of nonadditive over additive codes.

2. The $((8, 12))$ code

The second self-complementary single-AD error-correcting code that we consider is the $((8, 12))$ code whose codespace is spanned by the following twelve codewords in $\mathcal{H}_2^8$ [23],
\[
|0_L\rangle \equiv \frac{|00000000\rangle + |11111111\rangle}{\sqrt{2}}, \quad |1_L\rangle \equiv \frac{|00000111\rangle + |11111000\rangle}{\sqrt{2}}, \quad |2_L\rangle \equiv \frac{|00011000\rangle + |11101111\rangle}{\sqrt{2}},
\]
\[
|3_L\rangle \equiv \frac{|01001100\rangle + |10110111\rangle}{\sqrt{2}}, \quad |4_L\rangle \equiv \frac{|11000000\rangle + |00111111\rangle}{\sqrt{2}}, \quad |5_L\rangle \equiv \frac{|10101000\rangle + |01010111\rangle}{\sqrt{2}},
\]
\[
|6_L\rangle \equiv \frac{|01011000\rangle + |10100111\rangle}{\sqrt{2}}, \quad |7_L\rangle \equiv \frac{|01100100\rangle + |10011011\rangle}{\sqrt{2}}, \quad |8_L\rangle \equiv \frac{|10010100\rangle + |01101011\rangle}{\sqrt{2}},
\]
\[
|9_L\rangle \equiv \frac{|11100000\rangle + |00011111\rangle}{\sqrt{2}}, \quad |10_L\rangle \equiv \frac{|11001100\rangle + |00110011\rangle}{\sqrt{2}}, \quad |11_L\rangle \equiv \frac{|00111100\rangle + |11000011\rangle}{\sqrt{2}}.
\] (150)

The enlarged AD quantum channel after performing the encoding defined by means of Eq. (150) reads,
\[
A_{\text{AD}}^{((8, 12))} (\rho) \equiv \sum_{r=0}^{2^8-1} A_r' \rho A_r' = \sum_{a_1, \ldots, a_8 = 0}^{1} A_{a_1 a_2 \ldots a_7 a_8} \rho A_{a_1 a_2 \ldots a_7 a_8}^\dagger,
\] (151)

where to any of the $2^8$ values of $r$ we can associate a set of indices $(a_1, \ldots, a_8)$ (and vice-versa) such that,
\[
A_r' \leftrightarrow A_{a_1 a_2 \ldots a_7 a_8} \equiv A_{a_1} \otimes A_{a_2} \otimes \ldots \otimes A_{a_7} \otimes A_{a_8} = A_{a_1} A_{a_2} \ldots A_{a_7} A_{a_8}.
\] (152)
The errors $A_i$ with $i \in \{0, 1\}$ are defined in Eq. [5] and $\rho \in {\mathcal{M}(C)}$ with $C \subset {\mathcal{H}}^2_q$. In particular, the number of weight-$q$ enlarged error operators $A'_i$ is given by $\binom{8}{q}$ and,

$$
\sum_{q=0}^{8} \binom{8}{q} = 2^8.
$$

(153)

The recovery scheme $R$ that we use can be described as follows: $R_0$ is associated with the weight-0 error; $R_k$ with $k = 1, ..., 8$ are associated with the eight weight-1 errors where single-qubit errors of type $A_1$ occur. The construction of these 9 recovery operators $R_k$ is described in terms of $9 \times 12 = 108$ orthonormal vectors in $H_q$. The missing orthonormal vectors needed to obtain an orthonormal basis of $H^3_q$ and to construct $R_0 \equiv \hat{O}$ can be formally computed by using the rank-nullity theorem together with the Gram-Schmidt orthonormalization procedure. Finally, our analytical estimate of the entanglement fidelity of the $((8, 12))$ code reads,

$$
F^{((8,12))}_{\text{first-order}} (\gamma) \overset{\text{def}}{=} \frac{1}{(\dim C)^2} \sum_{k=0}^{2^{8-1}} \sum_{l=0}^{8} \left| \text{Tr} (R_l A'_k) \right|^2 \approx 1 - \frac{15}{2} \gamma^2 + \mathcal{O} (\gamma^3),
$$

(154)

with,

$$
F^{((8,12))} (\gamma) \overset{\text{def}}{=} \frac{1}{(\dim C)^2} \sum_{k=0}^{2^{8-1}} \sum_{l=0}^{9} \left| \text{Tr} (R_l A'_k) \right|^2.
$$

(155)

It is convenient to compare the performance of this code with that of a multi-qubit encoding stabilizer code with the same length. For instance, the $[[8, 3, 3]]$ code is a special case of a class of $[[2^j, 2^j - j - 2, 3]]$ codes [56]. It encodes three logical qubits into eight physical qubits and corrects all single-qubit errors. The five stabilizer generators are given by [56],

$$
g_1 \overset{\text{def}}{=} X^1 X^2 X^3 X^4 X^5 X^6 X^7 X^8, \quad g_2 \overset{\text{def}}{=} Z^1 Z^2 Z^3 Z^4 Z^5 Z^6 Z^7 Z^8, \quad g_3 \overset{\text{def}}{=} X^2 X^3 Y^5 Z^6 Y^7 Z^8,
$$

$$
g_4 \overset{\text{def}}{=} X^2 Z^3 Y^4 X^5 Z^6 Y^8, \quad g_5 \overset{\text{def}}{=} Y^2 X^3 Z^4 X^5 Z^6 Y^8,
$$

and a suitable choice for the logical operations $\hat{X}_i$ and $\hat{Z}_i$ with $i \in \{1, 2, 3\}$ reads,

$$
\hat{X}_1 \overset{\text{def}}{=} X^1 X^2 Z^6 Z^8, \quad \hat{X}_2 \overset{\text{def}}{=} X^1 X^3 Z^4 Z^7, \quad \hat{X}_3 \overset{\text{def}}{=} X^1 Z^4 X^5 Z^6, \quad \hat{Z}_1 \overset{\text{def}}{=} Z^2 Z^4 Z^6 Z^8, \quad \hat{Z}_2 \overset{\text{def}}{=} Z^3 Z^4 Z^7 Z^8, \quad \hat{Z}_3 \overset{\text{def}}{=} Z^5 Z^6 Z^7 Z^8.
$$

(156)

A convenient choice for the basis codewords reads,

$$
|\overrightarrow{g} \rangle \overset{\text{def}}{=} (\hat{X}_1)^i (\hat{X}_2)^j (\hat{X}_3)^k \sum_{g \in S} g |00000000\rangle.
$$

(157)

For an explicit representations of the codewords of the $[[8, 3, 3]]$ code, we refer to [56] (specifically, see Table III in [56]). Following our line of reasoning presented for the error correction of AD errors by means of stabilizer codes and omitting technical details, it turns out that the entanglement fidelity of the $[[8, 3, 3]]$ with our recovery up to first-order errors becomes,

$$
F^{[[8,3,3]]}_{\text{first-order}} (\gamma) \approx 1 - 7\gamma^2 + \mathcal{O} (\gamma^3).
$$

(159)

Unlike the case of single-qubit encoding, in this case the entanglement fidelity when no QEC is performed is represented by the three-qubit baseline performance given by,

$$
F^{\text{3-qubit}}_{\text{baseline}} (\gamma) \overset{\text{def}}{=} 8^{-2} \left[ 1 + 3 \sqrt{1 - \gamma} + 3 (1 - \gamma) + (1 - \gamma)^{3} \right]^2.
$$

(160)

In addition, from Eqs. (154) and (159), we get

$$
F^{((8,12))}_{\text{first-order}} (\gamma) \approx 1 - \frac{15}{2 \log_2 12} \gamma^2 + \mathcal{O} (\gamma^3) \geq 1 - \frac{7}{3} \gamma^2 + \mathcal{O} (\gamma^3) \approx F^{[[8,3,3]]}_{\text{first-order}} (\gamma).
$$

(161)

Eq. (161) is yet another clear fingerprint of the superiority, in terms of both encoded dimension and entanglement fidelity, of nonadditive over additive codes. The comparison between these two codes is shown in Fig. 5. Finally, we compare the performances of the nonadditive codes employed in our error correction schemes in Fig. 6.
FIG. 5: Additive vs. nonadditive codes of length eight. The truncated series expansions of the normalized entanglement fidelity $F(\gamma)$ vs. the amplitude damping parameter $\gamma$ for $\varepsilon = 0$ and $0 \leq \gamma \leq 10^{-1}$: Gottesman’s [[8, 3, 3]]-code (thin solid line) and the ((8, 12))-code (dashed line).

FIG. 6: Ranking nonadditive codes. The truncated series expansions of the normalized entanglement fidelity $F(\gamma)$ vs. the amplitude damping parameter $\gamma$ for $\varepsilon = 0$ and $0 \leq \gamma \leq 10^{-1}$: the ((6, 5))-code (dotted line), the ((8, 12))-code (dashed line), the ((9, 12, 3))-code (thin solid line), and the ((11, 2, 3))-code (thick solid line).

VI. FINAL REMARKS

In this article, we presented the first analytic analysis of the performance of various approximate QEC codes for GAD errors. Specifically, we considered both stabilizer and nonadditive quantum codes. The performance of such codes was quantified by means of the entanglement fidelity as a function of the damping probability and the non-zero environmental temperature. We analytically recovered and clarified some previously known numerical results in the limiting case of the AD channel (zero environmental temperature). In addition, our extended investigation suggested that degenerate stabilizer codes and self-complementary nonadditive codes are especially suitable for the error correction of GAD errors. Finally, comparing the properly normalized entanglement fidelities of the best performant stabilizer and nonadditive codes characterized by the same length, we showed that, in general, nonadditive codes outperform stabilizer codes not only in terms of encoded dimension but also in terms of entanglement fidelity.

Our main findings may be summarized as follows:

1. We have explicitly shown that in the presence of non-zero environmental temperature, the performance of both additive and nonadditive quantum codes decreases (with respect to zero environmental temperature case). In particular, degenerate stabilizer codes seem to be more robust than the nondegenerate ones against this effect, as evident from Eqs. (111) and (118);

2. In the limiting case of $\varepsilon = 0$ and considering the [[5, 1, 3]]-code, our analytic estimate in Eq. (77) reduces to the
We wish to emphasize three aspects of our analysis:

1. From Eqs. (127) and (139), we have explicitly shown that the nonadditive ([9, 1, 3])-code, our estimate in Eq. (84) reduces to the numeric one in [12]; finally, considering the Shor [[9, 1, 3]]-code, our estimate in Eq. (118) reduces to that numerically obtained in [8];

3. We have constructed a nondegenerate eight-qubit concatenated stabilizer code (see Eq. (91)), a natural generalization of the Leung et al. four-qubit code, suitable for the QEC of GAD errors. We have also checked that in the limit of $\varepsilon = 0$, our estimated performance for such a code reduces to that of the four-qubit code applied to AD errors (see Eq. (104));

4. We have provided further evidence (see Eq. (111) for the [[6, 1, 3]]-code) in support of the suspect advanced in [12] where it was conjectured that thanks to their degenerate structure, such codes can outperform nondegenerate codes despite their shorter length;

5. From Eqs. (127) and (139), we have explicitly shown that the nonadditive ((9, 12, 3))-code not only outperforms the ((11, 2, 3))-code in terms of encoded dimension but also in terms of entanglement fidelity with recovery schemes limited to first-order recovery;

6. We have shown that, to first-order recovery, self-complementary nonadditive codes can outperform non-self-complementary nonadditive codes despite exhibiting smaller encoded dimension (see Eqs. (139) and (147) for the ((6, 5))- and ((9, 12, 3))-codes, respectively);

7. From the comparison of the first-order entanglement-based performances between additive and nonadditive quantum codes with identical lengths (see Eqs. (141), (149), and (161)), we concluded that nonadditive codes outperform, in general, additive codes not only in terms of encoded dimension. In particular, nonadditivity seems to matter more than degeneracy in the approximate QEC of both GAD and AD errors (see Eq. (141)).

We wish to emphasize three aspects of our analysis:

- First, despite the great variety of quantum codes employed in this work, we have limited our attention to qubit channels only. The AD and GAD channels can, of course, be extended to quantum systems of dimension greater than two, leading to the so-called qudit channels. For instance, a qutrit is a three-state quantum system. In such higher-dimensional cases, the physical processes that we may consider are described by the modeling of atoms as having more than two states (multilevel atoms) interacting with environments modeled by a bath of harmonic oscillators which can be initially in the vacuum state or, more generally, in a thermal state with temperature greater than zero. The study of the effectiveness of QEC schemes suitable for such additional realistic scenarios will be the object of our attention in future efforts;

- Second, the QEC strategy that we have employed for our analytic estimates may be considered mildly conservative, since we might have slightly underestimated the quantum codes performances to avoid false overestimations. However, we have tried to maintain the same degree of conservativeness in all our estimates in order to preserve the fairness of the comparisons between pairs of different quantum codes. After all, exact analytical calculations of the entanglement fidelities can be quite intimidating. They are conceptually straightforward but computationally extremely tedious if performed by hands to avoid the drawbacks of numerical results [12, 14]. We also emphasize that conservative estimates are not uncommon in QEC since, after all, we are lead to deal with approximations. For instance, conservativeness appears in analytic estimates of error thresholds in topological QEC when crude combinatorics arguments are employed [57] as well as in the numerical estimation of the error threshold to depolarization of toric codes by means of Monte Carlo simulations [58].

- Third, despite the variety of quantum codes employed in the approximate QEC of GAD errors presented in this work, we emphasize that we cannot state without a doubt that for any given arbitrary noise model, nonadditive and degenerate codes are more efficient, in general, than additive and nondegenerate codes. In all honesty, we feel uncomfortable in extrapolating a general statement from a large, but yet limited, number of special cases. Perhaps, this is the reason why QEC is an art: general statements cannot be made a priori, each scenario has to be taken into consideration separately. It is a matter of finding a good matching between the noise model and the quantum code. Furthermore, we are not able to quantify exactly the effect on the code performances provided by fractional encoding, a feature of nonadditive codes, or by the possibility of correcting a number of errors greater than the one that is uniquely identified, a feature of degenerate codes. Also, we cannot rule out the possibility that the mathematical structure behind the sign pattern that characterizes the various codewords spanning the codespaces may play a relevant role in determining the performance of the error correction schemes. In conclusion, the merit of achieving higher performances cannot be ascribed to specific properties of a code, but rather to the properties of a code together with a family of errors it is designed to correct. This final remark
is in excellent agreement with the fact that, for instance, degeneracy must not be regarded as a property of a quantum code alone, but rather a property of a code together with a class of errors it is designed to recover.

In conclusion, we feel we have truly challenged ourselves in this work from an analytical standpoint and we tried to advance our understanding of approximate QEC of GAD errors for various qubit codes, both additive and nonadditive, as much as we could. We are confident we have succeeded in this regard. However, in all fairness, it is very likely that quantitative results in QEC will always require the help of numerical investigations. In future efforts, we wish to further sharpen our analytic estimations by using, if possible, both additional analytical and numerical scrutiny.

Acknowledgments

We thank the ERA-Net CHIST-ERA project HIPERCOM for financial support. CC also thanks Marcel Bergmann for numerically testing Eq. (139) and for helpful conversations.

Appendix A: The Kraus operator-sum decomposition of the GAD channel

In this Appendix, we provide an explicit derivation of the (2 × 2)-matrix representation of the Kraus error operators for the GAD channel in Eq. (8).

Let us consider AD from the scattering of a photon via a beam splitter [3]. Consider a single optical mode that contains the quantum state $|\psi\rangle \equiv \alpha|0\rangle + \beta|1\rangle$, where $|0\rangle$ is the vacuum state while $|1\rangle$ denotes the single photon state. The scattering of a photon from this mode can be modeled by inserting a beam splitter in the path of the photon. The beam splitter acts on two modes: it performs the unitary operation $U \equiv e^{i\chi(a^\dagger b - b^\dagger a)}$ and allows the photon to couple to another single optical mode that represents the environment. The operators $a$, $a^{\dagger}$ and $b$, $b^{\dagger}$ are the annihilation and creation operators for photons in the two modes. Assuming the environment starts out with no photons, $\rho_{\text{env.}} \equiv |0\rangle \langle 0|$, (A1)

it can be shown that the quantum operation that describes this process reads,

$$\Lambda_{\text{AD}} (\rho) \equiv A_0 \rho A_0^{\dagger} + A_1 \rho A_1^{\dagger},$$

with $A_k \equiv \langle k|U|0\rangle$ and,

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \gamma} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix},$$

where $\gamma \equiv \sin^2 \chi$ denotes the probability of losing a photon. If we assume that the environment starts out in a linear superposition of zero and one photons,

$$\rho_{\text{env.}} \equiv \sum_{j=0}^{1} q_j |j\rangle \langle j|,$$

where $q_0 \equiv p$, $q_1 = 1 - p$, and $0 \leq p \leq 1$, it turns out that the quantum operation that describes this process reads,

$$\Lambda_{\text{GAD}} (\rho) \equiv A_{00} \rho A_{00}^{\dagger} + A_{01} \rho A_{01}^{\dagger} + A_{11} \rho A_{11}^{\dagger} + A_{10} \rho A_{10}^{\dagger},$$

with $A_{jk} \equiv \sqrt{q_j} \langle k|U|j\rangle$. Observe that $A_{0k} \equiv \sqrt{p} \langle k|U|0\rangle$ and $A_{1k} \equiv \sqrt{1 - p} \langle k|U|1\rangle$ with $k = 0, 1$. Let us first focus on $A_{1k}$ which can be written as,

$$A_{1k} \equiv \sum_{m,n} (A_{1k})_{mn} |m\rangle \langle n|,$$

where the coefficients $(A_{1k})_{mn}$ are defined as,

$$(A_{1k})_{mn} \equiv \sqrt{1 - p} \langle m, k|U|n, 1\rangle = \sqrt{1 - p} \langle (m, k)|(U|n, 1)\rangle.$$
Before computing an explicit expression for \( U |n, 1\) in Eq. \([A7]\), observe that using the Baker-Campbell-Hausdorff formula,

\[
e^{\chi A} B e^{-\chi A} = B + \chi [A, B] + \frac{\chi^2}{2!} [A, [A, B]] + \frac{\chi^3}{3!} [A, [A, [A, B]]] + O(3),
\]

we obtain,

\[
U a U^\dagger \overset{\text{def}}{=} e^{\chi (a^\dagger b - b^\dagger a) a} e^{-\chi (a^\dagger b - b^\dagger a)}
\]

\[
= a + \chi [a^\dagger b - b^\dagger a, a] + \frac{\chi^2}{2!} [a^\dagger b - b^\dagger a, [a^\dagger b - b^\dagger a, a]] + \frac{\chi^3}{3!} [a^\dagger b - b^\dagger a, [a^\dagger b - b^\dagger a, [a^\dagger b - b^\dagger a, a]]] + O(3). \tag{A9}
\]

Notice that the commutator \([a^\dagger b - b^\dagger a, a]\) in Eq. \([A9]\) can be written as,

\[
[a^\dagger b - b^\dagger a, a] = -b, \quad [a^\dagger b - b^\dagger a, [a^\dagger b - b^\dagger a, a]] = a, \quad [a^\dagger b - b^\dagger a, [a^\dagger b - b^\dagger a, [a^\dagger b - b^\dagger a, a]]] = -a,
\]

therefore \(U a U^\dagger\) becomes,

\[
U a U^\dagger = a - \chi b - \frac{\chi^2}{2!} a + \frac{\chi^3}{3!} b + ... = a \left(1 - \frac{\chi^2}{2!} + ...\right) - b \left(\chi - \frac{\chi^3}{3!} + ...\right) = (\cos \chi) a - (\sin \chi) b. \tag{A11}
\]

Following this line of reasoning, we can also show that \(U b U^\dagger\) equals,

\[
U b U^\dagger = (\cos \chi) b - (\sin \chi) a. \tag{A12}
\]

Finally, recalling that \((n + m)^l\) can be written as,

\[
(n + m)^l = \sum_{p=0}^l \binom{l}{p} n^p m^{l-p}, \tag{A13}
\]

we have that \(U |n, 1\) in Eq. \([A7]\) becomes,

\[
U |n, 1\rangle = U \left(\frac{a^\dagger}{\sqrt{n!}}\right)^n b^\dagger |00\rangle = \left[U \left(\frac{a^\dagger}{\sqrt{n!}}\right)^n U^\dagger\right] \left[U b U^\dagger\right] |00\rangle \tag{A14}
\]

\[
= \frac{1}{\sqrt{n!}} \left[(\cos \chi) a^\dagger - (\sin \chi) b^\dagger\right]^n \left[(\cos \chi) b^\dagger - (\sin \chi) a^\dagger\right] |00\rangle
\]

\[
= \frac{1}{\sqrt{n!}} \left[\sum_{p=0}^n \binom{n}{p} (\cos \chi)^p (\sin \chi)^{n-p} (a^\dagger)^p (b^\dagger)^{n-p}\right] \left[(\cos \chi) |01\rangle - (\sin \chi) |10\rangle\right]
\]

\[
= \frac{1}{\sqrt{n!}} \left[\sum_{p=0}^n \binom{n}{p} \left[(\cos \chi)^{1+p} (\sin \chi)^{n-p} (a^\dagger)^p (b^\dagger)^{n-p} |01\rangle + (\cos \chi)^p (\sin \chi)^{n-p+1} (a^\dagger)^p (b^\dagger)^{n-p} |10\rangle\right]
\]

\[
= \frac{1}{\sqrt{n!}} \left[\sum_{p=0}^n \binom{n}{p} \left[(\cos \chi)^{1+p} (\sin \chi)^{n-p} (a^\dagger)^p |0\rangle (b^\dagger)^{n-p} |1\rangle + (\cos \chi)^p (\sin \chi)^{n-p+1} (a^\dagger)^p |1\rangle (b^\dagger)^{n-p} |0\rangle\right]
\]

\[
= \frac{1}{\sqrt{n!}} \left[\sum_{p=0}^n \binom{n}{p} \left[(\cos \chi)^{1+p} (\sin \chi)^{n-p} \sqrt{p!} |p\rangle (b^\dagger)^{n-p+1} |0\rangle + (\cos \chi)^p (\sin \chi)^{n-p+1} (a^\dagger)^{1+p} |0\rangle \sqrt{(n-p)!} |n-p\rangle\right],
\]
Therefore, substituting Eq. (A15) into Eq. (A7), we obtain

\[
U |n, 1\rangle = \frac{1}{\sqrt{n!}} \sum_{p=0}^{n} \binom{n}{p} \left[ (\cos \chi)^{1+p} (\sin \chi)^{n-p} \sqrt{p!} \sqrt{(n-p+1)!} |p, n-p+1\rangle + (\cos \chi)^{n-p+1} \sqrt{(1+p)!} \sqrt{(n-p)!} |1+p, n-p\rangle \right]
\]

\[
= \sum_{p=0}^{n} \left[ \frac{1}{\sqrt{p!}} \binom{n}{p} \sqrt{(n-p+1)!} (\cos \chi)^{1+p} (\sin \chi)^{n-p} |p, n-p+1\rangle + \frac{1}{\sqrt{(1+p)!}} \sqrt{(n-p)!} (\cos \chi)^{n-p+1} |1+p, n-p\rangle \right]
\]

\[
= \sum_{p=0}^{n} \left[ \sqrt{\binom{n}{p}} \sqrt{(n-p+1)} (\cos \chi)^{1+p} (\sin \chi)^{n-p} |p, n-p+1\rangle + \sqrt{\binom{n}{p}} \sqrt{(1+p)} (\cos \chi)^{n-p+1} |1+p, n-p\rangle \right]. \tag{A15}
\]

Therefore, substituting Eq. \[\text{(A15)}\] into Eq. \[\text{(A7)}\], we obtain

\[
\frac{(A_{1k})_{mn}}{\sqrt{1-p}} \overset{\text{def}}{=} \langle m, k | U | n, 1 \rangle = \sum_{p=0}^{n} \left[ \sqrt{\binom{n}{p}} \sqrt{(n-p+1)} (\cos \chi)^{1+p} (\sin \chi)^{n-p} \langle m, k | p, n-p+1 \rangle + \sqrt{\binom{n}{p}} \sqrt{(1+p)} (\cos \chi)^{n-p+1} \delta_{m, 1+p} \delta_{k, n-p} \right]
\]

\[
= \sum_{p=0}^{n} \left[ \sqrt{\binom{n}{p}} \sqrt{(n-p+1)} (\cos \chi)^{1+p} (\sin \chi)^{n-p} \delta_{m, p} \delta_{k, n-p+1} + \sqrt{\binom{n}{p}} \sqrt{(1+p)} (\cos \chi)^{n-p+1} \delta_{m, 1+p} \delta_{k, n-p} \right]
\]

\[
= \left[ \frac{\sqrt{\binom{n}{n-k+1}}}{\sqrt{n-k+1}} (\cos \chi)^{n-k+2} (\sin \chi)^{n-k-1} \delta_{m, n-k+1} \right]
\]

\[
\left( \frac{\sqrt{\binom{n}{n-k}}}{\sqrt{n-k}} (\cos \chi)^{n-k} (\sin \chi)^{n-k} | n-k + 1 \rangle \langle n | \right). \tag{A16}
\]

Using Eq. \[\text{(A16)}\], the Kraus operators \[A_{1k}\] in Eq. \[\text{(A6)}\] become

\[
A_{1k} = \sum_{m, n} (A_{1k})_{mn} |m \rangle \langle n| = \sqrt{1-p} \left[ \left( \frac{\sqrt{\binom{n}{n-k+1}}}{\sqrt{n-k+1}} (\cos \chi)^{n-k+2} (\sin \chi)^{n-k-1} | n-k + 1 \rangle \langle n | + \frac{\sqrt{\binom{n}{n-k}}}{\sqrt{n-k}} (\cos \chi)^{n-k} (\sin \chi)^{n-k} | n-k + 1 \rangle \langle n | \right) \right]. \tag{A17}
\]

Therefore, the error operators \[A_{11}\] and \[A_{10}\] read

\[
A_{11} = \sqrt{1-p} \sum_{n=0}^{1} \left[ \sqrt{\binom{n}{n}} (\cos \chi)^{1+n} + \sqrt{\binom{n}{n}} (\cos \chi)^{1-n} (\sin \chi)^{2} \right] |n \rangle \langle n|
\]

\[
= \sqrt{1-p} \left\{ (\cos \chi) | 0 \rangle \langle 0| + [(\cos \chi)^{2} + (\sin \chi)^{2}] | 1 \rangle \langle 1| \right\} = \sqrt{1-p} [(1) \langle 1| + (\cos \chi) | 0 \rangle \langle 0|]
\]

\[
= \sqrt{1-p} \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right), \tag{A18}
\]

where \(\gamma \overset{\text{def}}{=} \sin \chi \equiv 1 - \cos ^{2} \chi\) and,

\[
A_{10} = \sqrt{1-p} (\sin \chi) | 1 \rangle \langle 0| = \sqrt{1-p} \left( \begin{array}{cc}
0 & 0 \\
0 & 1
\end{array} \right), \tag{A19}
\]

respectively. Following the same line of reasoning employed for computing \[A_{11}\] and \[A_{10}\] and noticing that

\[
U |00\rangle = \left[ I + \chi (a^{\dagger}b - b^{\dagger}a) + \frac{\chi^{2}}{2!} (a^{\dagger}b - b^{\dagger}a)^{2} + \ldots \right] |00\rangle = |00\rangle, \tag{A20}
\]
we can also compute the Kraus errors $A_{0k}$, where

$$A_{0k} \overset{\text{def}}{=} \sum_{m, n} (A_{0k})_{mn} |m\rangle \langle n|,$$

and the coefficients $(A_{0k})_{mn}$ read,

$$(A_{0k})_{mn} \overset{\text{def}}{=} \sqrt{p} \langle m, k | U | n, 0 \rangle = \sqrt{p} (\langle m, k | (U | n, 0) \rangle).$$

It turns out that the errors $A_{00}$ and $A_{01}$ become,

$$A_{00} \overset{\text{def}}{=} \sqrt{p} \left| 0 \right\rangle \left\langle 0 \right| + \sqrt{1 - \gamma} \left| 1 \right\rangle \left\langle 1 \right| = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \gamma} \end{pmatrix},$$

and,

$$A_{01} \overset{\text{def}}{=} \sqrt{p} \left| U \right\rangle \left\langle 0 \right| = \sqrt{p} [\sqrt{\gamma}\left| 0 \right\rangle \left\langle 1 \right|] = \sqrt{p} \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix},$$

respectively. In conclusion, the GAD channel $\Lambda_{GAD}$ is given by

$$\Lambda_{GAD} (\rho) \overset{\text{def}}{=} A_{00} \rho A_{00}^\dagger + A_{01} \rho A_{10}^\dagger + A_{10} \rho A_{10}^\dagger + A_{11} \rho A_{11}^\dagger,$$

where,

$$A_{00} \overset{\text{def}}{=} \sqrt{p} \langle 0 | U | 0 \rangle = \sqrt{p} \left[ 0 \right\rangle \left\langle 0 \right| + \sqrt{1 - \gamma} \left| 1 \right\rangle \left\langle 1 \right| = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \gamma} \end{pmatrix},$$

$$A_{01} \overset{\text{def}}{=} \sqrt{p} \langle 1 | U | 0 \rangle = \sqrt{p} [\sqrt{\gamma}\left| 0 \right\rangle \left\langle 1 \right|] = \sqrt{p} \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix},$$

$$A_{10} \overset{\text{def}}{=} \sqrt{1 - p} \langle 0 | U | 1 \rangle = \sqrt{1 - p} [\sqrt{\gamma}\left| 0 \right\rangle \left\langle 1 \right|] = \sqrt{1 - p} \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix},$$

$$A_{11} \overset{\text{def}}{=} \sqrt{1 - p} \langle 1 | U | 1 \rangle = \sqrt{1 - p} [\sqrt{1 - \gamma}\left| 0 \right\rangle \left\langle 0 \right| + \left| 1 \right\rangle \left\langle 1 \right|] = \sqrt{1 - p} \begin{pmatrix} \sqrt{1 - \gamma} & 0 \\ 0 & 1 \end{pmatrix}.$$ (A26)

Finally, provided that we relabel $A_0 \overset{\text{def}}{=} A_{00},$ $A_1 \overset{\text{def}}{=} A_{01},$ $A_2 \overset{\text{def}}{=} A_{11},$ $A_3 \overset{\text{def}}{=} A_{10},$ the Kraus operators in Eq. (8) are obtained.

For the sake of completeness, we also observe that the analysis provided for this model is formally equivalent to that for the model employed in the main text. For instance, the Hamiltonian of the beam splitter, $H_{\text{int}}^{(bs)} \overset{\text{def}}{=} i \chi (ab^\dagger - a^\dagger b),$ should be replaced by the interaction Hamiltonian between the two-level atom and the bath of harmonic oscillators, $H_{\text{int}}^{(\text{atom-HO})} \overset{\text{def}}{=} g \left( a^\dagger \sigma_+ + a \sigma_- \right).$ In the definition of $H_{\text{int}}^{(\text{atom-HO})},$ the Pauli raising ($\sigma_+$) and lowering ($\sigma_-)$ operators act on the two-level atom. The creation ($a^\dagger$) and annihilation ($a$) operators are associated with the harmonic oscillator, instead. Finally, $g$ is the coupling constant for the interaction between the atom and the oscillator.

Appendix B: Qubit entanglement-breaking and the GAD channel

A quantum channel $\Lambda$ is called entanglement-breaking if $(\Lambda \otimes I) (\rho)$ is always separable, i.e., any entangled density matrix $\rho$ is mapped to a separable one [59]. In order to check if a channel is entanglement-breaking, it is sufficient to look at the separability of the output state corresponding just to an input maximally entangled state. In other words, $\Lambda$ is entanglement-breaking if and only if $(\Lambda \otimes I) (|\beta\rangle \langle \beta|)$ is separable for $|\beta\rangle$ defined as,

$$|\beta\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle \otimes |j\rangle,$$ (B1)
\(d\) being the dimension of the Hilbert space. A simple way to check the separability of density matrices is by means of the Peres-Horodecki positive partial transpose (PPT) criterion [60, 61] which provides a necessary and sufficient condition for the joint density matrix \(\rho\) of two \(d = 2\)-dimensional systems \(A\) and \(B\) to be separable. Alternatively, the entanglement of a mixed state \(\rho\) described by a probabilistic mixture of an ensemble of pure states of quantum systems of dimension \(2 \times 2\) can be quantified by means of the so-called concurrence [62]. For systems with this dimensionality, the concurrence \(C(\rho)\) of \(\rho\) reads

\[
C(\rho) \equiv \max \left\{ 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right\},
\]

where \(\lambda_i\) are the non-negative real eigenvalues of the non-Hermitian matrix \(\rho_\sigma \equiv \rho (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)\) where \(\tilde{\rho}\) is the spin-flipped state, \(\lambda_1\) is the largest eigenvalue, and the complex conjugation is taken with respect to the product basis of eigenvectors of \(\sigma_z\) given by \(\{|\uparrow\rangle, \{\downarrow\rangle, \{\uparrow\downarrow\rangle, \{\downarrow\uparrow\rangle\}\}.\)

For the GAD channel, it can be shown that the concurrence of \(\rho \equiv (\Lambda_{GAD} \otimes I) (|\beta\rangle \langle \beta|)\) with \(|\beta\rangle \equiv \frac{|00\rangle + |11\rangle}{\sqrt{2}}\) reads,

\[
C_{\text{GAD}}(\gamma, p) \equiv \frac{1}{2} \left[ \sqrt{\left( 2(1 - \gamma) + p\gamma^2 (1-p) \right) + 2\sqrt{(1 - \gamma)(1 - \gamma) + p\gamma^2 (1-p)}} + \right.
\]

\[
- \sqrt{\left[ 2(1 - \gamma) + p\gamma^2 (1-p) \right] - 2\sqrt{(1 - \gamma)(1 - \gamma) + p\gamma^2 (1-p)}} + \left. -2\sqrt{p(1-p)\gamma^2} \right].
\]

Furthermore, the eigenvalues of the partial transpose of \(\rho\) are given by,

\[
\lambda_1(\gamma, p) = \frac{1}{2} \gamma p + \frac{1}{2} (1 - \gamma), \quad \lambda_2(\gamma, p) = \frac{1}{2} (1 - p\gamma),
\]

\[
\lambda_3(\gamma, p) = \frac{1}{4} \gamma - \frac{1}{2} \sqrt{\frac{1}{4} \gamma^2 - \gamma - p\gamma^2 + p^2\gamma^2 + 1}, \quad \lambda_4(\gamma, p) = \frac{1}{4} \gamma + \frac{1}{2} \sqrt{\frac{1}{4} \gamma^2 - \gamma - p\gamma^2 + p^2\gamma^2 + 1}.
\]

The eigenvalues \(\lambda_1, \lambda_2\) and \(\lambda_4\) are positive. The eigenvalue \(\lambda_3\) is positive provided that,

\[
p^2\gamma^2 - p\gamma^2 + 1 - \gamma \leq 0. \quad (\text{B5})
\]

The inequality in Eq. [B5] is fulfilled in the two-dimensional parametric region \((\gamma, p(\gamma))\) where \(p_{\text{min}}(\gamma) \leq p(\gamma) \leq p_{\text{max}}(\gamma)\) with,

\[
p_{\text{min}}(\gamma) \equiv \frac{1}{\gamma^2} \left( \frac{1}{2} \gamma^2 - \frac{1}{2} \sqrt{\frac{1}{4} \gamma^4 + 4\gamma^3 - 4\gamma^2} \right) \text{ and, } p_{\text{max}}(\gamma) \equiv \frac{1}{\gamma^2} \left( \frac{1}{2} \gamma^2 + \frac{1}{2} \sqrt{\frac{1}{4} \gamma^4 + 4\gamma^3 - 4\gamma^2} \right).
\]

It turns out that for pairs \((\gamma, p(\gamma))\) within this two-dimensional parametric region, \(C_{\text{GAD}}(\gamma, p)\) equals zero and the GAD channel becomes entanglement-breaking.

**Appendix C: Explicit construction of an orthonormal basis**

In general, we can proceed as follows. The rank-nullity theorem allows us with an algorithmic procedure for enlarging a set of \(m\) linearly independent vectors in \(\mathbb{C}^n\) with \(m \leq n\) to a set of \(n\) linearly independent vectors in \(\mathbb{C}^n\). The Gram-Schmidt orthonormalization procedure, instead, can be used to construct an orthonormal basis from this set of \(n\) linearly independent vectors.

The rank-nullity theorem states that given a \((m \times n)\)-matrix \(A\), it turns out that \(n = \text{rank } (A) + \text{nullity } (A)\). The nullity of a \((m \times n)\)-matrix \(A\) representing a linear map \(A : \mathbb{C}^n \rightarrow \mathbb{C}^m\) is the dimension of its null-space (or kernel),

\[
\ker A \equiv \left\{ \vec{x} \in \mathbb{C}^n : A\vec{x} = \vec{0} \right\}.
\]

The rank of a matrix is the maximum number of linearly independent columns (or rows).
Let us suppose we wish to construct an orthonormal basis \( \{|e_k\}\) with \( k = 1, \ldots, 32 \) for the 32-dimensional complex Hilbert space \( \mathcal{H}_2^3 \). The first two orthonormal basis vectors can be constructed from the action of the weight-0 error operators \( A_{00000} \) on the codewords \( |0_L\rangle \) and \( |1_L\rangle \). They read,

\[
|e_1\rangle \overset{\text{def}}{=} N_1(\gamma) \left[ -|00000\rangle + (1-\gamma)^2 |01111\rangle - (1-\gamma)^2 |10011\rangle + (1-\gamma)^2 |11100\rangle + (1-\gamma) |00110\rangle + (1-\gamma) |01001\rangle + (1-\gamma) |10101\rangle + (1-\gamma)^2 |11010\rangle \right],
\]

and,

\[
|e_2\rangle \overset{\text{def}}{=} N_2(\gamma) \left[ - (1-\gamma)^2 |11111\rangle \sqrt{1-\gamma} |10000\rangle + (1-\gamma) |01100\rangle - (1-\gamma) |00111\rangle + (1-\gamma)^2 |11010\rangle - (1-\gamma) |01010\rangle - (1-\gamma) |00101\rangle \right],
\]

respectively. The normalization factors \( N_1(\gamma) \) and \( N_2(\gamma) \) are given by,

\[
N_1(\gamma) \overset{\text{def}}{=} \frac{1}{\sqrt{1 + (1-\gamma)^4 + 4(1-\gamma)^3 + 2(1-\gamma)^2}} \quad \text{and} \quad N_2(\gamma) \overset{\text{def}}{=} \frac{1}{\sqrt{(1-\gamma)^5 + 2(1-\gamma)^3 + 4(1-\gamma)^2 + (1-\gamma)}}.
\]

The next ten orthonormal vectors can be constructed by taking into consideration the action of the five weight-1 error operators \( A_{01000}, A_{10010}, A_{00100}, A_{00010}, A_{00001} \) on the codewords. The action of \( A_{01000} \) on the codewords \( |0_L\rangle \) and \( |1_L\rangle \) leads to,

\[
|e_3\rangle \overset{\text{def}}{=} \frac{-|00011\rangle + |01100\rangle + |00101\rangle + |01010\rangle}{\sqrt{4}},
\]

and,

\[
|e_4\rangle \overset{\text{def}}{=} \frac{- (1-\gamma)^2 |01111\rangle + |00000\rangle + (1-\gamma) |01001\rangle + (1-\gamma) |00110\rangle}{\sqrt{1 + (1-\gamma)^4 + 2(1-\gamma)^2}},
\]

respectively. The action of \( A_{01000} \) on the codewords \( |0_L\rangle \) and \( |1_L\rangle \) yields,

\[
|e_5\rangle \overset{\text{def}}{=} \frac{(1-\gamma)^2 |01111\rangle + (1-\gamma) |10100\rangle + \sqrt{1-\gamma} |00001\rangle + (1-\gamma) |10010\rangle}{\sqrt{(1-\gamma)^3 + 2(1-\gamma)^2 + (1-\gamma)}},
\]

and,

\[
|e_6\rangle \overset{\text{def}}{=} \frac{- (1-\gamma)^2 |10111\rangle + \sqrt{1-\gamma} |01000\rangle + (1-\gamma) |00100\rangle - \sqrt{1-\gamma} |00010\rangle}{\sqrt{(1-\gamma)^4 + (1-\gamma)^2 + 2(1-\gamma)}},
\]

respectively. The action of \( A_{00100} \) on the codewords \( |0_L\rangle \) and \( |1_L\rangle \) leads to,

\[
|e_7\rangle \overset{\text{def}}{=} \frac{(1-\gamma)^2 |01011\rangle + (1-\gamma) |11000\rangle + (1-\gamma) |10001\rangle + \sqrt{1-\gamma} |00010\rangle}{\sqrt{(1-\gamma)^3 + 2(1-\gamma)^2 + (1-\gamma)}},
\]

and,

\[
|e_8\rangle \overset{\text{def}}{=} \frac{- (1-\gamma)^2 |11011\rangle + \sqrt{1-\gamma} |10000\rangle + (1-\gamma) |10010\rangle - \sqrt{1-\gamma} |00001\rangle}{\sqrt{(1-\gamma)^4 + (1-\gamma)^2 + 2(1-\gamma)}},
\]

respectively. The action of \( A_{00010} \) on the codewords \( |0_L\rangle \) and \( |1_L\rangle \) yields,

\[
|e_9\rangle \overset{\text{def}}{=} \frac{(1-\gamma)^2 |01101\rangle - (1-\gamma) |10001\rangle + \sqrt{1-\gamma} |00100\rangle + (1-\gamma) |11000\rangle}{\sqrt{(1-\gamma)^3 + 2(1-\gamma)^2 + (1-\gamma)}},
\]
and,
\[
|e_{10}\rangle \overset{\text{def}}{=} \frac{-(1 - \gamma)^2 |11101\rangle - \sqrt{1 - \gamma} |00001\rangle + (1 - \gamma) |10100\rangle - \sqrt{1 - \gamma} |01000\rangle}{\sqrt{(1 - \gamma)^4 + (1 - \gamma)^2 + 2(1 - \gamma)}},
\]
(C12)
respectively. Finally, the action of \(A_{000001}\) on the codewords \(|0_L\rangle\) and \(|1_L\rangle\) gives,
\[
|e_{11}\rangle \overset{\text{def}}{=} \frac{(1 - \gamma)^2 |01110\rangle - (1 - \gamma) |10010\rangle + \sqrt{1 - \gamma} |01000\rangle + (1 - \gamma) |10100\rangle}{\sqrt{(1 - \gamma)^3 + 2(1 - \gamma)^2 + (1 - \gamma)}},
\]
(C13)
and,
\[
|e_{12}\rangle \overset{\text{def}}{=} \frac{-(1 - \gamma)^2 |11110\rangle - \sqrt{1 - \gamma} |00010\rangle + (1 - \gamma) |11000\rangle - \sqrt{1 - \gamma} |00100\rangle}{\sqrt{(1 - \gamma)^4 + (1 - \gamma)^2 + 2(1 - \gamma)}},
\]
(C14)
respectively. The next ten orthonormal vectors can be constructed by considering the action of the five weight-1 error operators \(A_{30000}, A_{03000}, A_{00300}, A_{00030}, A_{00003}\) on the codewords. The action of \(A_{30000}\) on the codewords \(|0_L\rangle\) and \(|1_L\rangle\) leads to,
\[
|e_{13}\rangle \overset{\text{def}}{=} \frac{|10000\rangle + (1 - \gamma)^2 |11111\rangle + (1 - \gamma) |10110\rangle + (1 - \gamma) |11001\rangle}{\sqrt{1 + (1 - \gamma)^2 + 2(1 - \gamma)^2}},
\]
(C15)
and,
\[
|e_{14}\rangle \overset{\text{def}}{=} \frac{|11100\rangle - |10011\rangle - |11010\rangle - |10101\rangle}{\sqrt{4}},
\]
(C16)
respectively. The action of \(A_{03000}\) on the codewords \(|0_L\rangle\) and \(|1_L\rangle\) yields,
\[
|e_{15}\rangle \overset{\text{def}}{=} \frac{|01000\rangle - (1 - \gamma)^2 |11011\rangle + (1 - \gamma) |01110\rangle + (1 - \gamma)^2 |11101\rangle}{\sqrt{1 + 2(1 - \gamma)^3 + (1 - \gamma)^2}},
\]
(C17)
and,
\[
|e_{16}\rangle \overset{\text{def}}{=} \frac{\sqrt{1 - \gamma} |11000\rangle - (1 - \gamma) |01011\rangle + (1 - \gamma)^2 |11110\rangle - (1 - \gamma) |01101\rangle}{\sqrt{(1 - \gamma)^3 + 2(1 - \gamma)^2 + (1 - \gamma)}},
\]
(C18)
respectively. The action of \(A_{00300}\) on the codewords \(|0_L\rangle\) and \(|1_L\rangle\) leads to,
\[
|e_{17}\rangle \overset{\text{def}}{=} \frac{|00100\rangle - (1 - \gamma)^2 |10111\rangle + (1 - \gamma) |00110\rangle + (1 - \gamma)^2 |11110\rangle}{\sqrt{1 + 2(1 - \gamma)^3 + (1 - \gamma)^2}},
\]
(C19)
and,
\[
|e_{18}\rangle \overset{\text{def}}{=} \frac{\sqrt{1 - \gamma} |10100\rangle - (1 - \gamma) |00111\rangle + (1 - \gamma)^2 |11101\rangle - (1 - \gamma) |01110\rangle}{\sqrt{(1 - \gamma)^3 + 2(1 - \gamma)^2 + (1 - \gamma)}},
\]
(C20)
respectively. The action of \(A_{00030}\) on the codewords \(|0_L\rangle\) and \(|1_L\rangle\) yields,
\[
|e_{19}\rangle \overset{\text{def}}{=} \frac{|00010\rangle + (1 - \gamma)^2 |11110\rangle + (1 - \gamma) |01011\rangle + (1 - \gamma)^2 |10111\rangle}{\sqrt{1 + 2(1 - \gamma)^3 + (1 - \gamma)^2}},
\]
(C21)
and,

\[|e_{20}\rangle \equiv \frac{\sqrt{1 - \gamma} |10010\rangle + (1 - \gamma) |01110\rangle + (1 - \gamma)^2 |11011\rangle - (1 - \gamma) |00111\rangle}{\sqrt{(1 - \gamma)^3 + 2 (1 - \gamma)^2 + (1 - \gamma)}} \]

respectively. Finally, The action of \(A_{00003}\) on the codewords \(|0_L\rangle\) and \(|1_L\rangle\) leads to,

\[|e_{21}\rangle \equiv -|00001\rangle + (1 - \gamma)^2 |11010\rangle + (1 - \gamma) |00111\rangle + (1 - \gamma)^2 |10111\rangle \]

and,

\[|e_{22}\rangle \equiv \frac{\sqrt{1 - \gamma} |10001\rangle + (1 - \gamma) |01101\rangle + (1 - \gamma)^2 |10111\rangle - (1 - \gamma) |01011\rangle}{\sqrt{(1 - \gamma)^3 + 2 (1 - \gamma)^2 + (1 - \gamma)}} \]

respectively. We have 22 orthonormal vectors and we need 10 more. From Eqs. (C2) and (C3), it turns out that the following six linearly independent orthonormal vectors are orthogonal to both \(|e_1\rangle\) and \(|e_2\rangle\),

\[|e_{23}\rangle \equiv \frac{|11101\rangle - |11001\rangle}{\sqrt{2}}, \quad |e_{24}\rangle \equiv \frac{|11100\rangle + |10011\rangle}{\sqrt{2}}, \quad |e_{25}\rangle \equiv \frac{|11010\rangle - |10101\rangle}{\sqrt{2}} \]

\[|e_{26}\rangle \equiv \frac{|00011\rangle + |01100\rangle}{\sqrt{2}}, \quad |e_{27}\rangle \equiv \frac{|00101\rangle - |01010\rangle}{\sqrt{2}}, \quad |e_{28}\rangle \equiv \frac{|01001\rangle - |00110\rangle}{\sqrt{2}} \]

Indeed, it can be explicitly checked that \(\langle e_k e_{k'} \rangle = \delta_{kk'}\) for any \(k, k'\) in \(\{1, \ldots, 28\}\). The last four vectors are uncovered as follows: first, we add four linearly independent vectors in such a manner to have a basis for \(\mathcal{H}_2^5\) and then we apply the Gram-Schmidt orthonormalization procedure to obtain an orthogonal basis. It finally turns out that the remaining four vectors needed are given by,

\[|e_{29}\rangle \equiv \frac{|11000\rangle - (1 - \gamma) |e_7\rangle - (1 - \gamma) |e_9\rangle - (1 - \gamma) |e_{12}\rangle - \sqrt{1 - \gamma} |e_{16}\rangle}{\sqrt{1 - 3 (1 - \gamma)^2}} \]

\[|e_{30}\rangle \equiv \frac{|10100\rangle - (1 - \gamma) |e_5\rangle - (1 - \gamma) |e_{10}\rangle - (1 - \gamma) |e_{11}\rangle - \sqrt{1 - \gamma} |e_{18}\rangle}{\sqrt{1 - 3 (1 - \gamma)^2}} \]

\[|e_{31}\rangle \equiv \frac{|10010\rangle - (1 - \gamma) |e_5\rangle - (1 - \gamma) |e_6\rangle + (1 - \gamma) |e_{11}\rangle - \sqrt{1 - \gamma} |e_{20}\rangle}{\sqrt{1 - 3 (1 - \gamma)^2}} \]

\[|e_{32}\rangle \equiv \frac{|10001\rangle - (1 - \gamma) |e_6\rangle - (1 - \gamma) |e_7\rangle + (1 - \gamma) |e_9\rangle - \sqrt{1 - \gamma} |e_{22}\rangle}{\sqrt{1 - 3 (1 - \gamma)^2}} \]

In conclusion, \(\{|e_k\rangle\}\) with \(k \in \{1, \ldots, 32\}\) is a suitable orthonormal basis for \(\mathcal{H}_2^5\).

Appendix D: Estimating the entanglement fidelity

The entanglement fidelity \(\mathcal{F}(\gamma, \varepsilon)\) of a \((n, K, d)\) quantum code \(\mathcal{C}\) for enlarged GAD errors \(A'_a\) with \(a \in \{0, \ldots, 2^n - 1\}\) and recovery operation \(\mathcal{R} \equiv \{R_1, \ldots, R_r, R_s, R_{s+1} \equiv \tilde{O}\} \) reads,

\[\mathcal{F}(\gamma, \varepsilon) \equiv \frac{1}{K^2} \left[ \sum_{a=0}^{2^n-1} \sum_{r=1}^s |\text{Tr}(R_r A'_a)|^2 + \sum_{a=0}^{2^n-1} |\text{Tr}(\tilde{O} A'_a)|^2 \right] \]
where the recovery operator \( \hat{O} \) is defined as,

\[
\hat{O} \equiv \sum_{b=1}^{2^n - 2s} |o_b \rangle \langle o_b |. \tag{D2}
\]

The 2s orthonormal vectors employed to construct the recovery operators \( R_r \) in \( R \setminus \{ \hat{O} \} \) together with the \( 2^n - 2s \) orthonormal vectors used to define the recovery operator \( \hat{O} \) form an orthonormal basis of the complex Hilbert space \( \mathcal{H}_2^n \). We observe that the entanglement fidelity \( F (\gamma, \varepsilon) \) in Eq. \( [D1] \) can be regarded as the sum of two contributions,

\[
F (\gamma, \varepsilon) \equiv F_{R \setminus \{ \hat{O} \}} (\gamma, \varepsilon) + F_{\hat{O}} (\gamma, \varepsilon), \tag{D3}
\]

where,

\[
F_{R \setminus \{ \hat{O} \}} (\gamma, \varepsilon) \equiv \frac{1}{K^2} \sum_{a=0}^{2^{2n} - 1} \sum_{r=1}^{s} \left| \text{Tr} (R_r A_a') \right|^2 \text{ and } F_{\hat{O}} (\gamma, \varepsilon) \equiv \frac{1}{K^2} \sum_{a=0}^{2^{2n} - 1} \left| \text{Tr} (\hat{O} A_a') \right|^2. \tag{D4}
\]

In what follows, we shall describe our reasoning for the analytic estimates of entanglement fidelities.

1. Part A

Let us first consider \( F_{R \setminus \{ \hat{O} \}} (\gamma, \varepsilon) \) in Eq. \( [D4] \). Recall that for \( r \in \{ 1, \ldots, s \} \), the recovery operators \( R_r \) read

\[
R_r \equiv \sum_{j=0}^{K-1} \frac{|j_L \rangle \langle j_L | A_r^t}{\sqrt{\langle j_L | A_r^t | j_L \rangle}}, \tag{D5}
\]

Using Eq. \( [D5] \), \( F_{R \setminus \{ \hat{O} \}} (\gamma, \varepsilon) \) becomes

\[
F_{R \setminus \{ \hat{O} \}} (\gamma, \varepsilon) \equiv \frac{1}{K^2} \sum_{a=0}^{2^{2n} - 1} \sum_{r=1}^{s} \sum_{i=0}^{K-1} \frac{\langle i_L | A_r^t A_a' | i_L \rangle}{\sqrt{\langle i_L | A_r^t A_a' | i_L \rangle}}. \tag{D6}
\]

Let us denote with \( K \) the index set of all the enlarged GAD error operators. This set has cardinality \( 2^{2n} \) and can be decomposed in two parts,

\[
K \equiv K_{R \setminus \{ \hat{O} \}} \oplus K'. \tag{D7}
\]

The set \( K_{R \setminus \{ \hat{O} \}} \) is the index set of all the enlarged GAD errors \( \{ A_a' \} \) recovered by means of the recovery operators \( \{ R_r \} \) in \( R \setminus \{ \hat{O} \} \) and it has cardinality \( s \). The cardinality of the index set \( K' \) is \( 2^{2n} - s \) and denotes the number of all the remaining potentially contributing enlarged error operators. Since we only aim at finding estimates of \( F (\gamma, \varepsilon) \) with \( F (\gamma, \varepsilon) \gtrsim 1 - O(2) \) and assuming as working hypothesis \( 0 \leq \varepsilon \ll \gamma \ll 1 \), we only need to consider a subset of \( K' \) for the estimation of \( F_{R \setminus \{ \hat{O} \}} (\gamma, \varepsilon) \). In the worst scenario, we have to consider the subset \( K_{\text{weight-2}} \subset K' \) of cardinality \( 3^2 \binom{n}{2} \) of all the weight-2 enlarged GAD errors, and

\[
F_{R \setminus \{ \hat{O} \}} (\gamma, \varepsilon) \approx \frac{1}{K^2} \sum_{a \in K_{\text{weight-2}}}^{2^{2n}-1} \sum_{r=1}^{s} \sum_{i=0}^{K-1} \frac{\langle i_L | A_r^t A_a' | i_L \rangle}{\sqrt{\langle i_L | A_r^t A_a' | i_L \rangle}}^2. \tag{D8}
\]

with \( K'' \equiv K_{R \setminus \{ \hat{O} \}} \cup K_{\text{weight-2}} \). In addition, among these \( 3^2 \binom{n}{2} \) errors, some of them are more likely to occur than others. Denote with \( [ik] \) the set of cardinality \( \binom{n}{2} \) with weight-2 enlarged error operators acting on \( n \)-qubit quantum states defined by means of single-qubit errors of type \( A_i \) and \( A_k \),

\[
[ik] \equiv \{ A_{ik0...0}, A_{0k0...0}, \ldots, A_{00...0ik} \}. \tag{D9}
\]
It turns out that the probabilities of occurrence \( \Pr([ik]) \) of errors in the set \([ik]\) scale in terms of the perturbation parameters \( \gamma \) and \( \varepsilon \) as follows,

\[
\begin{pmatrix}
\Pr_{11} & \Pr_{12} & \Pr_{13} \\
\Pr_{21} & \Pr_{22} & \Pr_{23} \\
\Pr_{31} & \Pr_{32} & \Pr_{33}
\end{pmatrix} \approx \begin{pmatrix}
\gamma^2 & \varepsilon \gamma & \varepsilon^2 \\
\varepsilon \gamma & \varepsilon^2 & \varepsilon^2 \\
\varepsilon^2 & \varepsilon^2 & \varepsilon^2
\end{pmatrix}.
\]

(D10)

Therefore, in addition to limit our attention to these 3 weight-2 errors in \( K_{\text{weight-2}} \), we also rank the relevance of each of these nine possible subsets \([ik]\) of cardinality \((\frac{n}{2})\) according to the \( \gamma \) - and \( \varepsilon \)-dependencies of the probabilities of occurrence of their errors.

2. Part B

Let us now take into consideration \( F_{\hat{O}}(\gamma, \varepsilon) \) in Eq. (D4). We notice that,

\[
F_{\hat{O}}(\gamma, \varepsilon) \overset{\text{def}}{=} \frac{1}{K^2} \sum_{a=0}^{2^n-1} \left| \text{Tr} \left( \hat{O} A'_a \right) \right|^2 = \frac{1}{K^2} \sum_{a=0}^{2^n-1} \sum_{i=0}^{K-1} \left| \langle i_L | \hat{O} A'_a | i_L \rangle \right|^2 \leq \frac{2^n}{K} \left| \left\langle i_L | \hat{O} A'_a | i_L \right\rangle \right|^2,
\]

(D11)

with,

\[
\left\langle i_L | \hat{O} A'_a | i_L \right\rangle \overset{\text{def}}{=} \max_{i,a} \left\{ \left\langle i_L | \hat{O} A'_a | i_L \right\rangle \right\}.
\]

(D12)

If the index \( \bar{a} \) labels an enlarged GAD error recovered by a recovery operation \( R_r \) in \( \mathcal{R} \setminus \{ \hat{O} \} \), then \( F_{\hat{O}}(\gamma, \varepsilon) \) is identically zero by construction. Therefore, let us assume that \( \bar{a} \) is not such an index. Using Eq. (D2), \( \langle i_L | \hat{O} A'_a | i_L \rangle \) becomes,

\[
\left\langle i_L | \hat{O} A'_\bar{a} | i_L \right\rangle = \sum_{b=1}^{2^n-2s} \left\langle i_L | o_b \right\rangle \left\langle o_b | A'_\bar{a} | i_L \right\rangle.
\]

(D13)

We observe that \( \langle i_L | o_b \rangle \) would be identically zero for any \( b \in \{1, \ldots, 2^n - 2s\} \) if the enlarged identity error operator belonged to the error model. Unfortunately, this is not our case. Nevertheless, it turns out that

\[
\langle i_L | o_b \rangle \approx \gamma \langle i_L | T^i | o_b \rangle,
\]

(D14)

where the operator \( T^i \) acting on \( n \)-qubit quantum states is defined as,

\[
T^i \overset{\text{def}}{=} T^1 \otimes I^2 \otimes \ldots \otimes I^{n-1} \otimes I^n + I^1 \otimes T^2 \otimes \ldots \otimes I^{n-1} \otimes I^n + I^1 \otimes I^2 \otimes \ldots \otimes I^{n-1} \otimes T^n,
\]

(D15)

with,

\[
T^k \overset{\text{def}}{=} \frac{1}{4} (I^k - \sigma_z^k).
\]

(D16)

Substituting Eq. (D14) into Eq. (D13), we get

\[
\left\langle i_L | \hat{O} A'_\bar{a} | i_L \right\rangle \approx \gamma \sum_{b=1}^{2^{n-2s}} \langle i_L | T^i | o_b \rangle \left\langle o_b | A'_\bar{a} | i_L \right\rangle.
\]

(D17)

We remark that both \( \langle i_L | \hat{O} A'_\bar{a} | i_L \rangle \) and \( T^i \) are \( \gamma \)-independent quantities while the states \( | o_b \rangle \) are the sum-decomposition of \( n \)-qubit quantum states where \( \gamma \)-dependent expansion coefficients may appear. However, as we have noticed from our explicit construction in the previous appendix, these coefficients do not exhibit nontrivial \( \gamma \)-dependence in the limit of \( \gamma \) approaching zero. Therefore, terms like \( \langle i_L | T^i | o_b \rangle \) do not possess relevant scaling laws in the damping probability parameter \( \gamma \) approaching zero. On the contrary, terms like \( \langle o_b | A'_\bar{a} | i_L \rangle \) do exhibit important \( \gamma \)-scaling laws,

\[
\langle o_b | A'_\bar{a} | i_L \rangle \approx \gamma^{\omega(A'_\bar{a})} \left\langle o_b | A'_\bar{a} | i_L \right\rangle,
\]

(D18)
where \( \text{wt}(A'_c) \) denotes the weight of the operator \( A'_c \) and \( \langle o_b \mid A'_c \mid i_L \rangle \) is redefined in such a manner to have no relevant \( \gamma \)-dependence. Substituting Eqs. (D17) and (D18) into Eq. (D11), we finally obtain

\[
F_\hat{O} (\gamma, \varepsilon) \lesssim \frac{2^{2n}}{K} \gamma^{2+\text{wt}(A'_c)} \left| \sum_{b=1}^{2^n-2s} \langle i_L \mid T' \mid o_b \rangle \langle o_b \mid A'_c \mid i_L \rangle \right|^2.
\]  

(D19)

From Eq. (D19), we conclude that while \( F_\hat{O} (\gamma, \varepsilon) \) could be, in principle, nonvanishing and contribute to the computation of the entanglement fidelity \( F (\gamma, \varepsilon) \), its contribution is negligible given to order of approximations chosen for our analytic estimates.

In what follows, we report a more explicit example. For the sake of reasoning, consider the CSS seven-qubit code and AD errors. Does any weight-2 enlarged error operator contribute to the computation of the entanglement fidelity? In general, errors \( A_l \) for which no corresponding recovery operator \( R_{A_l} \) is constructed can contribute to the computation of the entanglement fidelity via the expression given by,

\[
F_{\hat{O}^{[7,1,3]}} (\gamma) \overset{\text{def}}{=} \frac{1}{4} \sum_{l,k} \left( \langle 0_L \mid v_k \rangle \langle v_k \mid A_l \mid 0_L \rangle + \langle 1_L \mid v_k \rangle \langle v_k \mid A_l \mid 1_L \rangle \right)^2,
\]  

(D20)

where the operator \( \hat{O} \) reads,

\[
\hat{O} \overset{\text{def}}{=} \sum_k |v_k \rangle \langle v_k |.
\]  

(D21)

We notice that state vectors \( \{|v_k\} \in Eq. (D20) \) lead to non-vanishing contributions provided that: i) \( |v_k\rangle \) has some non-zero component along \( |0_L\rangle \) and/or \( |1_L\rangle \); ii) \( |v_k\rangle \) has some non-zero component along \( A_l |0_L\rangle \) and/or \( A_l |1_L\rangle \); iii) \( \{|v_k\}, A_l^{\text{correctable}} |i_L\rangle \) forms an orthonormal basis of \( \mathcal{H}_2^7 \).

Before proceeding along this line of reasoning, we would like to emphasize that we could have proceeded along an alternative route. Observe that \( D \overset{\text{def}}{=} \dim_{\mathbb{C}} \mathcal{H}_2^7 = 2^7 = 128 \). Furthermore, the cardinality \( c' \) of the set \( S' \) of linearly independent and orthogonal vectors in the sum-decomposition of the correctable weight-0 and weight-1 enlarged errors is given by,

\[
c' \overset{\text{def}}{=} 2 \times \binom{7}{0} \times 8 + 2 \times \binom{7}{1} \times 4 = 72.
\]  

(D22)

Therefore, there also exists a new set \( S'' \) that consists of \( c'' \overset{\text{def}}{=} D - c' = 56 \) linearly independent and orthogonal vectors such that vectors in \( S' \) and \( S'' \) form an orthonormal basis of \( \mathcal{H}_2^7 \). If any of these 56 vectors is in the vector sum-decomposition of any of the \( \binom{7}{2} = 21 \) weight-2 errors, then some weight-2 error could contribute to the entanglement fidelity provided that the selected vector in \( S'' \) has some non-zero component along \( |0_L\rangle \) and/or \( |1_L\rangle \) (see Eq. (D20)). We checked that all the \( 2 \times \binom{7}{2} = 84 \) state vectors in the sum-decomposition of \( A_l |i_L\rangle \) with \( A_l \) any weight-2 error operator are orthogonal to \( |i_L\rangle \) with \( i \in \{0, 1\} \). Therefore, we cannot take this short-cut and are forced to proceed in a more general manner. As a consequence, constructing these vectors \( |v_k\rangle \) is going to be more involved, since we wish to provide a constructive explanation avoiding numerics that could obscure the construction itself.

Let us then return to the more general approach. For the sake of clarity, focus on the possible partial recovery of the weight-2 error \( A_{1100000} \). This error is not correctable because it is incompatible with the weight-1 error \( A_{0010000} \). Having observed this and recalling the three conditions for good state vectors \( \{|v_k\} \rangle \), it turns out that a suitable vector \( |v_k\rangle \) in the definition of \( \hat{O} \) so that \( |v_k\rangle \langle v_k | \) can partially recover \( A_{1100000} \),

\[
|v_k\rangle \overset{\text{def}}{=} \frac{\{0000110\} - |1100000\rangle + |0110011\rangle - (1 - \gamma)^2 |0000000\rangle}{\sqrt{3 + (1 - \gamma)^4}}.
\]  

(D23)

This contribution of \( A_{1100000} \) to the computation of the entanglement fidelity becomes,

\[
\frac{\mathcal{F}_{\hat{O}}^{[7,1,3]} (\gamma)}{\langle v_k | v_k \rangle} = \frac{1}{4} \left( \langle 0_L | v_k \rangle \langle v_k | A_{1100000} | 0_L \rangle \right)^2 \approx \frac{1}{32} \left( \frac{1 - (1 - \gamma)^2}{\sqrt{3 + (1 - \gamma)^4}} \times \frac{\gamma (1 - \gamma)}{\sqrt{3 + (1 - \gamma)^4}} \right)^2 \approx \gamma^4.
\]  

(D24)

From Eq. (D24), we conclude that although \( A_{1100000} \) contributes to the computation of the entanglement fidelity, its contribution is negligible given our chosen order of approximation. Analogously, for each weight-2 enlarged error operator, a similar line of reasoning can be carried out.
FIG. 7: Performance of the CSS seven-qubit code. The non-truncated expression of the entanglement fidelity $F(\gamma)$ vs. the amplitude damping parameter $\gamma$ in the presence (thin solid line) and absence (dashed line) of quantum error correction.

FIG. 8: Performance of the Shor nine-qubit code. The non-truncated expression of the entanglement fidelity $F(\gamma)$ vs. the amplitude damping parameter $\gamma$ in the presence (thin solid line) and absence (dashed line) of quantum error correction.

Appendix E: On the CSS seven-qubit code

For $\varepsilon = 0$, the non-truncated expression for the entanglement fidelity reads,

$$F_{\text{non-truncated}}^{[7,1,3]}(\gamma) \equiv \frac{1}{4} \left( \sqrt{\frac{1 + 7 (1 - \gamma)^4}{8}} + \sqrt{\frac{(1 - \gamma)^7 + 7 (1 - \gamma)^3}{8}} \right)^2 + \frac{7}{4} \left( \sqrt{\frac{4 \gamma (1 - \gamma)^3}{8}} + \sqrt{\frac{\gamma (1 - \gamma)^6 + 3 \gamma (1 - \gamma)^2}{8}} \right)^2.$$  \hspace{1cm} (E1)

The Taylor-expansion of $F_{\text{non-truncated}}^{[7,1,3]}(\gamma)$ up to the 10th-order is given by,

$$F_{\text{non-truncated}}^{[7,1,3]}(\gamma) \approx 1 - \frac{21}{4} \gamma^2 + \frac{35}{4} \gamma^3 - \frac{63}{8} \gamma^4 + \frac{609}{128} \gamma^5 - \frac{315}{256} \gamma^6 - \frac{51}{256} \gamma^7 - \frac{63}{256} \gamma^8 + \frac{1701}{8192} \gamma^9 + O(\gamma^{10}),$$  \hspace{1cm} (E2)

while the 1-qubit baseline performance is given by,

$$F_{\text{baseline}}^{1\text{-qubit}}(\gamma) \equiv 2^{-2} \left( 1 + \sqrt{1 - \gamma} \right)^2.$$  \hspace{1cm} (E3)

We checked the good overlap between our results (non truncated fidelity expressions with and without error correction) and the ones plotted in [12]. See also Fig. 7.
Appendix F: On the Shor nine-qubit code

For $\varepsilon = 0$, the non-truncated expression for the entanglement fidelity reads,

$$F_{\text{non-truncated}}^{[9,1,3]}(\gamma) \overset{\text{def}}{=} 1 - \frac{3}{2} \gamma^3 - \frac{135}{8} \gamma^4 + \frac{513}{8} \gamma^5 - \frac{201}{2} \gamma^6 + \frac{675}{8} \gamma^7 - \frac{297}{8} \gamma^8 + \frac{53}{8} \gamma^9,$$  \hspace{1cm} (F1)

while the 1-qubit baseline performance is given by,

$$F_{\text{baseline}}^{1\text{-qubit}}(\gamma) \overset{\text{def}}{=} 2^{-2} \left(1 + \sqrt{1 - \gamma}\right)^2.$$  \hspace{1cm} (F2)

We checked the good overlap between our results (non-truncated fidelity expressions with and without error correction) and the ones plotted in [5]. See also Fig. 8.

[1] P. W. Shor, Scheme for reducing decoherence in quantum computer memory, Phys. Rev. A52, R2493 (1995).
[2] D. Gottesman, An introduction to quantum error correction and fault-tolerant quantum computation, in Quantum Information Science and Its Contributions to Mathematics, Proceedings of Symposia in Applied Mathematics 68, pp. 13-58, Amer. Math. Soc., Providence, Rhode Island, USA (2010).
[3] M. A. Nielsen and I. L. Chuang, Quantum Computation and Information, Cambridge University Press (2000).
[4] I. L. Chuang, D. W. Leung, and Y. Yamamoto, Bosonic quantum codes for amplitude damping, Phys. Rev. A56, 1114 (1997).
[5] R. Laflamme, C. Miquel, J. P. Paz, and W. H. Zurek, Perfect quantum error correcting code, Phys. Rev. Lett. 77, 198 (1996).
[6] C. H. Bennett, D. P. Di Vincenzo, J. A. Smolin, and W. K. Wootters, Mixed-state entanglement and quantum error correction, Phys. Rev. A54, 3824 (1996).
[7] D. W. Leung, M. A. Nielsen, I. L. Chuang, and Y. Yamamoto, Approximate quantum error correction can lead to better codes, Phys. Rev. A56, 2567 (1997).
[8] A. S. Fletcher, P. W. Shor, and M. Z. Win, Channel-adapted quantum error correction for the amplitude damping channel, IEEE Transactions on Information Theory 54, 5705 (2008).
[9] A. S. Fletcher, P. W. Shor, and M. Z. Win, Optimum quantum error recovery using semidefinite programming, Phys. Rev. A75, 012338 (2007).
[10] C. H. Papadimitriou and K. Steiglitz, Combinatorial Optimization: Algorithms and Complexity, Dover Publications (1998).
[11] B. Schumacher, Sending entanglement through noisy quantum channels, Phys. Rev. A54, 2615 (1996).
[12] A. S. Fletcher, P. W. Shor, and M. Z. Win, Structured near-optimal channel-adapted quantum error correction, Phys. Rev. A77, 012320 (2008).
[13] H. Barnum and E. Knill, Reversing quantum dynamics with near-optimal quantum and classical fidelity, J. Math. Phys. 43, 2097 (2002).
[14] H. K. Ng and P. Mandayam, Simple approach to approximate quantum error correction based on the transpose channel, Phys. Rev. A81, 062342 (2010).
[15] E. Knill and R. Laflamme, Theory of quantum error-correcting codes, Phys. Rev. A55, 900 (1997).
[16] P. Mandayam and H. K. Ng, Towards a unified framework for approximate quantum error correction, Phys. Rev. A86, 012335 (2012).
[17] A. Gilchrist, N. K. Langford, and M. A. Nielsen, Distance measures to compare real and ideal quantum processes, Phys. Rev. A71, 062310 (2005).
[18] I. Sainz and G. Bjork, Quantum error correction may delay, but also cause, entanglement sudden death, Phys. Rev. A77, 052307 (2008).
[19] C. Cafaro, S. L’Innocente, C. Lupo, and S. Mancini, Quantifying the performance of quantum codes, Open Syst. & Inf. Dyn. 18, 1 (2011).
[20] R. Wickert, N. K. Bernardes, and P. van Loock, Entanglement properties of optical coherent states under amplitude damping, Phys. Rev. A81, 062344 (2010).
[21] R. Wickert and P. van Loock, Quantum error correction and detection: quantitative analysis of a coherent-state amplitude damping code, arXiv:quant-ph/1303.0273 (2013).
[22] R. Wickert and P. van Loock, Assessing quantum error correction: fidelity and entanglement measures with application to photonic codes, arXiv:quant-ph/1303.0279 (2013).
[23] R. Lang and P. W. Shor, Nonadditive quantum error correcting codes adapted to the amplitude damping channel, arXiv:quant-ph/0712.2586 (2007).
[24] P. W. Shor, G. Smith, J. A. Smolin, and B. Zeng, High performance single-error correcting quantum codes for amplitude damping, IEEE Transactions on Information Theory 57, 7180 (2011).
[25] A. Cross, G. Smith, J. A. Smolin, and B. Zeng, Codeword stabilized quantum codes, IEEE Transactions on Information Theory 55, 433 (2009).
