ABSTRACT

We find the transformation properties of the prepotential $F$ of $N = 2$ SUSY gauge theory with gauge group $SU(2)$. In particular we show that $G(a) = \pi i \left( F(a) - \frac{1}{2} a \partial_a F(a) \right)$ is modular invariant. This function satisfies the non-linear differential equation $(1 - G^2) G'' + \frac{1}{4} a G'^3 = 0$, implying that the instanton contribution are determined by recursion relations. Finally, we find $u = u(a)$ and give the explicit expression of $F$ as function of $u$. These results can be extended to more general cases.

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1. Recently the low-energy limit of $N = 2$ super Yang-Mills theory with gauge group $G = SU(2)$ has been solved exactly [1]. This result has been generalized to $G = SU(n)$ in [2] whereas the large $n$ analysis has been investigated in [3]. Other interesting results concern the generalization to $SO(2n + 1)$ [4] and non-locality at the cusp points in moduli spaces [5].

The low-energy effective action $S_{\text{eff}}$ is derived from a single holomorphic function $F(\Phi_k)$ [6]

$$S_{\text{eff}} = \frac{1}{4\pi} \text{Im} \left( \int d^2 \theta d^2 \bar{\theta} \Phi_D \Phi_i + \frac{1}{2} \int d^2 \theta \tau^{ij} W_i W_j \right),$$

(1)

where $\Phi_D \equiv \partial F/\partial \Phi_i$ and $\tau^{ij} \equiv \partial^2 F/\partial \Phi_i \partial \Phi_j$. Let us denote by $a_i \equiv \langle \phi^i \rangle$ and $a_D^i \equiv \langle \phi_D^i \rangle$ the vevs of the scalar component of the chiral superfield. For $SU(2)$ the moduli space of quantum vacua, parametrized by $u = \langle \text{tr} \phi^2 \rangle$, is the Riemann sphere with punctures at $u_1 = -\Lambda, u_2 = \Lambda$ (we will set $\Lambda = 1$) and $u_3 = \infty$ and a $Z_2$ symmetry acting by $u \leftrightarrow -u$. The asymptotic expansion of the prepotential has the structure [1]

$$F = i \frac{a^2}{2\pi} \log a^2 + \sum_{k=0}^{\infty} F_k a^{2-4k}. \quad \text{(2)}$$

In [1] the vector $(a_D, a)$ has been considered as a holomorphic section of a flat bundle. In particular in [1] the monodromy properties of $(a_D(u), a(u))$ have been identified with $\Gamma(2)$

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \mapsto \begin{pmatrix} \tilde{a}_D \\ \tilde{a} \end{pmatrix} = M_{u_i} \begin{pmatrix} a_D \\ a \end{pmatrix}, \quad i = 1, 2, 3, \quad \text{(3)}$$

where

$$M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$ 

The asymptotic behaviour of this section, derived in [1], and the geometrical data above completely determine $(a_D(u), a(u))$. In particular the explicit expression of the section $(a_D, a)$ has been obtained by first constructing tori parametrized by $u$ and then identifying a suitable meromorphic differential [1].

Before considering the framework of uniformization theory, we find the explicit expression of $F$ in terms of $u$. Next we will find the modular properties of $F$ by solving a linear differential equation which arises from defining properties. We will use uniformization theory in order to explicitly find $u = u(a)$ and to derive the (non-linear) differential equation satisfied by $F$ as a function of $a$. This equation furnishes, as expected, recursion relations which determine the instanton contributions to $F$. Our general formula is in agreement with the results in [4] where the first six terms of the instanton contribution have been computed.
Let us start with the explicit expression of $F$ as function of $u$. Let us recall that

$$a_D = \frac{\sqrt{2}}{\pi} \int_1^u \frac{dx \sqrt{x-u}}{\sqrt{x^2-1}}, \quad a = \frac{\sqrt{2}}{\pi} \int_{-1}^1 \frac{dx \sqrt{x-u}}{\sqrt{x^2-1}}. \quad (4)$$

In order to solve the problem we use the integrability of the 1-differential

$$\eta(u) = a \partial_u a_D - a_D \partial_u a = \frac{1}{\pi^2} \int_1^u dx \int_{-1}^1 dy \frac{y-x}{\sqrt{(x^2-1)(y^2-1)}}. \quad (5)$$

We have

$$g(u) = \int_1^u dz \eta(z) = \frac{1}{\pi^2} \int_1^u dx \int_{-1}^1 dy \frac{y-x}{\sqrt{(x^2-1)(y^2-1)}} \log \left[ \frac{2u - x - y + 2\sqrt{(u-x)(u-y)}}{x-y} \right]. \quad (6)$$

On the other hand notice that

$$\partial_u F = a_D \partial_u a = \frac{1}{2} \left[ \partial_u (aa_D) - \eta(u) \right],$$

so that, up to an additive constant, we have

$$F(a(u)) = \frac{1}{2\pi^2} \int_1^u dx \int_{-1}^1 dy \frac{4\sqrt{(x-u)(y-u)} - (y-x) \log \left[ \frac{2u - x - y + 2\sqrt{(u-x)(u-y)}}{x-y} \right]}{\sqrt{(x^2-1)(y^2-1)}}. \quad (7)$$

Later, in the framework of uniformization theory, we will show that $\eta$ is a constant (in the $u$-patch), so that $g$ is proportional to $u$.

We now find the transformation properties of $F(a)$. By (15), we have

$$\frac{\partial^2 \tilde{F}(\tilde{a})}{\partial \tilde{a}^2} = \frac{A\partial^2 F(a)}{\partial a^2} + B \frac{C\partial^2 F(a)}{\partial a^2} + D,$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(2)$ and $\tilde{a} = Ca_D + Da$. On the other hand

$$\frac{\partial^2 \tilde{F}(\tilde{a})}{\partial \tilde{a}^2} = \left[ - \left( \frac{\partial \tilde{a}}{\partial a} \right)^2 \frac{\partial^2 \tilde{a}}{\partial a^2} \partial a + \left( \frac{\partial \tilde{a}}{\partial a} \right)^{-2} \left( \frac{\partial^2 \tilde{a}}{\partial a^2} \right) \left[ \tilde{F}(\tilde{a}) \right. \right.$$

Eqs.(8) imply that

$$(C F^{(2)} + D) \partial^2 \tilde{a} \tilde{F}(\tilde{a}) - C F^{(3)} \partial \tilde{a} \tilde{F}(\tilde{a}) - (A F^{(2)} + B)(C F^{(2)} + D)^2 = 0, \quad (10)$$
where $\mathcal{F}^{(k)} \equiv \partial^k \mathcal{F}(a)$, whose solution is
\[
\tilde{\mathcal{F}}(\tilde{a}) = \mathcal{F}(a) + \frac{AC}{2}a^2_D + \frac{BD}{2}a^2 + BCaa_D.
\] (11)

This means that the function
\[
\tilde{G}(\tilde{a}) = \mathcal{G}(a)
\] (13)

By (2) we have asymptotically
\[
\mathcal{G} = \sum_{k=0}^{\infty} G_k a^{2-k}, \quad G_0 = \frac{1}{2}, \quad G_k = 2\pi i k \mathcal{F}_k.
\] (14)

2. In order to find $u = u(a)$ and $\mathcal{F}$ as function of $a$, we need few facts about uniformization theory. Let us denote by $\hat{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}$ the Riemann sphere and by $H$ the upper half plane endowed with the Poincaré metric $ds^2 = |dz|^2/(\text{Im } z)^2$. It is well known that $n$-punctured spheres $\Sigma_n \equiv \hat{\mathbb{C}} \setminus \{u_1, \ldots, u_n\}$, $n \geq 3$, can be represented as $H/\Gamma$ with $\Gamma \subset \text{PSL}(2, \mathbb{R})$ a parabolic (i.e. with $|\text{tr } \gamma| = 2$, $\gamma \in \Gamma$) Fuchsian group. The map $J_H : H \to \Sigma_n$ has the property $J_H(\gamma \cdot z) = J_H(z)$, where $\gamma \cdot z = (Az + B)/(Cz + D)$, $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$. It follows that after winding around nontrivial loops the inverse map transforms as
\[
J_H^{-1}(u) \longrightarrow \tilde{J}_H^{-1}(u) = \frac{AJ_H^{-1}(u) + B}{CJ_H^{-1}(u) + D}.
\] (15)

The projection of the Poincaré metric onto $\Sigma_n \cong H/\Gamma$ is
\[
ds^2 = e^\varphi|du|^2 = \frac{|J_H^{-1}(u)|^2}{(\text{Im } J_H^{-1}(u))^2}|du|^2,
\] (16)

which is invariant under $SL(2, \mathbb{R})$ fractional transformations of $J_H^{-1}$. The fact that $e^\varphi$ has constant curvature $-1$ means that $\varphi$ satisfies the Liouville equation
\[
\partial_u \partial_{\bar{u}} \varphi = \frac{e^\varphi}{2}.
\] (17)

Near a puncture we have $\varphi \sim -\log \left(|u - u_i|^2 \log^2 |u - u_i|\right)$. For the Liouville stress tensor we have the following equivalent expressions
\[
T(u) = \partial_u \partial_{\bar{u}} \varphi - \frac{1}{2} (\partial_u \varphi)^2 = \left\{J_H^{-1}, u\right\} = \sum_{i=1}^{n-1} \left( \frac{1}{2(u - u_i)}^2 + \frac{c_i}{u - u_i} \right).
\] (18)
where \( \{J_H^{-1}, u\} \) denotes the Schwarzian derivative of \( J_H^{-1} \) and the \( c_i \)'s, called accessory parameters, satisfy the constraints
\[
\sum_{i=1}^{n-1} c_i = 0, \quad \sum_{i=1}^{n-1} c_i u_i = 1 - \frac{n}{2}.
\]

Let us now consider the covariant operators introduced in the formulation of the KdV equation in higher genus \[8\]. We use \( 1/J_H^{-1} \) as covariantizing polymorphic vector field \[9\].

\[
S_{J_H^{-1}}^{(2k+1)} = (2k+1) J_H^{-1} \frac{1}{J_H^{-1}} \frac{1}{J_H^{-1}} \ldots \frac{1}{J_H^{-1}} \partial_u J_H^{-1},
\]

where the number of derivatives is \( 2k+1 \) and \( ' \equiv \partial_u \). Univalence of \( J_H^{-1} \) implies holomorphicity of \( S_{J_H^{-1}}^{(2k+1)} \). An interesting property of the equation \( S_{J_H^{-1}}^{(2k+1)} \cdot \psi = 0 \) is that its projection on \( H \) reduces to the trivial equation \( (2k+1) z^{k+1} \partial_z^{k+1} \tilde{\psi} = 0 \), where \( z = J_H^{-1}(u) \). Operators \( S_{J_H^{-1}}^{(2k+1)} \) are covariant, holomorphic and \( SL(2, \mathbb{C}) \) invariant, which by \( (15) \) implies singlevaluedness of \( S_{J_H^{-1}}^{(2k+1)} \). Furthermore, Möbius invariance of the Schwarzian derivative implies that \( S_{J_H^{-1}}^{(2k+1)} \) depends on \( J_H^{-1} \) only through the stress tensor \( (18) \) and its derivatives.

For \( k = 1/2 \), we have the uniformizing equation
\[
\left( J_H^{-1} \right)^{\frac{1}{2}} \partial_u J_H^{-1} \partial_u \left( J_H^{-1} \right)^{\frac{1}{2}} \cdot \psi = \left( \partial^2 + \frac{T}{2} \right) \cdot \psi = 0,
\]

that, by construction, has the two linearly independent solutions
\[
\psi_1 = \left( J_H^{-1} \right)^{-\frac{1}{2}} J_H^{-1}, \quad \psi_2 = \left( J_H^{-1} \right)^{-\frac{1}{2}},
\]

so that
\[
J_H^{-1} = \psi_1/\psi_2.
\]

By \( (13) \) and \( (22) \) it follows that
\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix} \longrightarrow \begin{pmatrix}
\tilde{\psi}_1 \\
\tilde{\psi}_2
\end{pmatrix} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}.
\]

In the case of \( \Sigma_3 \simeq H/\Gamma(2) \), Eq.\((19)\) gives \( c_1 = -c_2 = 1/4 \) and the uniformizing equation \( (21) \) becomes\(^1\)
\[
\frac{\partial^2}{\partial u^2} + \frac{3 + u^2}{4(1-u^2)^2} \psi = 0,
\]

\(^1\)This equation has been considered also in \[10\].
which is solved by Legendre functions
\[ \psi_1 = \sqrt{1 - u^2} P_{-1/2}, \quad \psi_2 = \sqrt{1 - u^2} Q_{-1/2}. \] (26)

These solutions define a holomorphic section that by (24) has monodromy \( \Gamma(2) \).

In order to find \( (a, a_D) \) we observe that by (22) \( \psi_1 \) and \( \psi_2 \) are (polymorphic) \(-1/2\)-differentials whereas both \( a_D \) and \( a \) are 0-differentials. This fact and the asymptotic behaviour of \( (a_D, a) \) given in [1] imply that
\[ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1 - u^2} \partial_u a_D \\ \sqrt{1 - u^2} \partial_u a \end{pmatrix}, \] (27)
where \( \sqrt{1 - u^2} \) is considered as a \(-3/2\)-differential. Comparing with (26) we get (4).

3. By Eqs.(25) and (27) it follows that \( a_D \) and \( a \) are solutions of the third-order equation
\[ \left( \partial^2_u + \frac{3 + u^2}{4(1 - u^2)^2} \right) \sqrt{1 - u^2} \partial_u \phi = 0. \] (28)

Let us consider some aspects of this equation. First of all note that, as observed in [7],
\[ \left( \partial^2_u + \frac{3 + u^2}{4(1 - u^2)^2} \right) \sqrt{1 - u^2} \partial_u \phi = \frac{1}{\sqrt{1 - u^2}} \partial_u \left[ (1 - u^2) \partial_u^2 - \frac{1}{4} \right] \phi = 0. \] (29)

It follows that \( \left[ (1 - u^2) \partial_u^2 - \frac{1}{4} \right] \phi = c \) with \( c \) a constant. A check shows that \( a_D \) and \( a \) in (4) satisfy this equation with \( c = 0 \)
\[ \left[ (1 - u^2) \partial_u^2 - \frac{1}{4} \right] a_D = \left[ (1 - u^2) \partial_u^2 - \frac{1}{4} \right] a = 0. \] (30)

As noticed in [7], this explains also why, despite of the fact that \( a \) and \( a_D \) satisfy the third-order differential equation (28), they have two-dimensional monodromy. Eq.(30) is the crucial one to find \( u = u(a) \) and to determine the instanton contributions. In our framework the problem of finding the form of \( F \) as a function of \( a \) is equivalent to the following general basic problem which is of interest also from a mathematical point of view:

Given a second-order differential equation with solutions \( \psi_1 \) and \( \psi_2 \) find the function \( F_1(\psi_1) \) \( (F_2(\psi_2)) \) such that \( \psi_2 = \partial F_1 / \partial \psi_2 \) \( (\psi_1 = \partial F_2 / \partial \psi_2) \).

We show that such a function satisfies a non-linear differential equation. The first step is to observe that by (31) it follows that
\[ aa_D' - a_D a' = c. \] (31)
Since \((a_D, a)\) are (polymorphic) 0-differentials, it follows that in changing patch the constant \(c\) in \((31)\) is multiplied by the Jacobian of the coordinate transformation. Another equivalent way to see this, is to notice that Eq.\((30)\) gets a first derivative under a coordinate transformation. Therefore in another patch the r.h.s. of \((31)\) is no longer a constant. As we have seen, covariance of the equation such has
\[
(\partial_z^2 + F(z)/2)\psi(z) = 0,
\]
is ensured if and only if \(\psi\) transforms as a \(-1/2\)-differential and \(F\) as a Schwarzian derivative.

In terms of the solutions \(\psi_1, \psi_2\) one can construct the 0-differential \(\psi'_1 \psi_2 - \psi_1 \psi'_2\) that, by the structure of the equation, is just a constant \(c\). In another patch we have \((\partial_w^2 + \tilde{F}(w)/2)\tilde{\psi}(w) = 0\), so that \(\psi_1(z)\partial_z \psi_2(z) - \psi_2(z)\partial_z \psi_1(z) = \tilde{\psi}_1(w)\partial_w \tilde{\psi}_2(w) - \tilde{\psi}_2(w)\partial_w \tilde{\psi}_1(w) = c\).

This discussion shows that flatness of \(a_D\) and \(a\) is the reason of the reduction mechanism from the third-order to second-order equation.

By \((3)\) \((4)\) \((12)\) and \((31)\) it follows that
\[
Au + B = G(a),
\]
where \(B\) is a constant which we will show to be zero. To determine the constant \(A\), we note that asymptotically \(a \sim \sqrt{2}u\), therefore by \((14)\) one has \(A = 1\). By \((4)\) and \((32)\) it follows that
\[
a_D = \sqrt{2} \sqrt{\frac{G(a) + B}{x} - \frac{1}{\sqrt{x^2 - 1}}} \right] = \frac{\sqrt{2}}{\pi} \int_1^1 \frac{dx}{\sqrt{x^2 - 1}}, \quad a = \frac{\sqrt{2}}{\pi} \int_1^1 \frac{dx}{\sqrt{x^2 - 1}}.
\]

Apparently to solve these two equivalent integro-differential equations seems a difficult task. However we can use the following trick. First notice that
\[
\left[1 - a^2\right] \partial_a \psi = 0 = \left[1 - (G + B)\right] \left(G' \partial_a^2 + G'' \partial_a - \frac{1}{4}G^3\right) \phi = 0,
\]
where now \(\partial_a\) is the time-like partial derivative. Then, since \(\phi = a\) (or equivalently \(\phi = a_D = \partial_a F\)) is a solution of \((34)\), it follows that \(G(a)\) satisfies the non-linear differential equation \([1 - (G + B)] G'' + \frac{1}{4}aG^3 = 0\).

Inserting the expansion \((14)\) one can check that the only way to compensate the \(a^{-2(2k+1)}\) terms is to set \(B = 0\). Therefore
\[
(1 - G^2) G'' + \frac{1}{4}aG^3 = 0,
\]
which is equivalent to the following recursion relations for the instanton contribution (recall that \(G = 2\pi ikF_k\))
\[
G_{n+1} = \frac{1}{8G_0^2(n + 1)^2}.
\]
\[
\cdot \left\{ (2n-1)(4n-1)G_n + 2G_0 \sum_{k=0}^{n-1} G_{n-k}G_{k+1}c(k,n) - 2 \sum_{j=0}^{n-1} \sum_{k=0}^{j+1} G_{n-j}G_{j+1-k}d(j,k,n) \right\},
\]

where \( n \geq 0, \ G_0 = 1/2 \) and

\[
c(k,n) = 2k(n-k-1) + n - 1, \quad d(j,k,n) = [2(n-j) - 1][2n - 3j - 1 + 2k(j-k+1)].
\]

The first few terms are \( G_0 = \frac{1}{2}, \ G_1 = \frac{1}{2^2}, \ G_2 = \frac{5}{2^3}, \ G_3 = \frac{9}{2^4} \), in agreement with the results in \cite{7} where the first terms of the instanton contribution have been computed by first inverting \( a(u) \) as a series for large \( a/\Lambda \) and then inserting this in \( a_D \).

Finally let us notice that the inverse of \( a = a(u) \) is

\[
u = G(a),
\]

and

\[
a a_D' - a a_D' = \frac{2i}{\pi},
\]

which is useful to explicitly determine the critical curve on which \( \text{Im} a_D / a = 0 \), whose structure has been considered in \cite{1} \cite{11} \cite{12}.

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\[\text{Notice that we are using different normalizations, thus to compare with } F^{KLT}_k \text{ in } [3] \text{ one should check the invariance of the quantity } \frac{F^K_{k}}{F^{KLT}_{k+1}}.\]
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