Magnetohydrodynamic Equilibria of a Cylindrical Plasma with Poloidal Mass Flow and Arbitrary Cross Section Shape

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Abstract

The equilibrium of a cylindrical plasma with purely poloidal mass flow and cross section of arbitrary shape is investigated within the framework of the ideal MHD theory. For the system under consideration it is shown that only incompressible flows are possible and, consequently, the general two dimensional flow equilibrium equations reduce to a single second-order quasilinear partial differential equation for the poloidal magnetic flux function $\psi$, in which four profile functionals of $\psi$ appear. Apart from a singularity occurring when the modulus of Mach number associated with the Alfvén velocity for the poloidal magnetic field is unity, this equation is always elliptic and permits the construction of several classes of analytic solutions. Specific exact equilibria for a plasma confined within a perfectly conducting circular cylindrical boundary and having i) a flat current density and ii) a peaked current density

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I. INTRODUCTION

Plasma rotation of the order of sound speed has been observed in many tokamaks heated by neutral beams, such as JET \(^1\) and TFTR \(^2\). Early reported data relates to purely toroidal rotation \(^3\); more recent measurements show that poloidal rotation can occur as well \(^4\). It has also been found that poloidal flows in the outer region of tokamaks driven by strong electric fields play an important role in the L-H transition (e.g. \(^5\)).

Although theoretical studies of flowing plasmas began in the mid fifties \(^7\)-\(^9\), since the early eighties there has been an increasing interest (motivated from the above mentioned observations) mainly in the investigation of equilibrium and stability of plasmas with mass flow \(^10\)-\(^22\). In particular, the symmetric equilibrium states of a plasma with mass flow in a two dimensional geometry are governed by a second-order partial semilinear differential equation for the poloidal magnetic flux function \(\psi\), which contains five free surface quantities (i.e. quantities solely dependent on \(\psi\)) in conjunction with a nonlinear algebraic Bernoulli equation (e.g. \(^10\)-\(^17\)). Unlike in static equilibria, the above mentioned differential equation is not always elliptic; there are three critical values of the modulus of Mach number \(|M|\) of the poloidal flow with respect to the poloidal magnetic field, at which the type of this equation changes, i.e. it becomes alternatively elliptic and hyperbolic. To solve the equilibrium problem in the two elliptic regions, several computer codes have been developed (e.g. \(^15\)-\(^17\)). For purely toroidal flow the partial differential equation becomes elliptic and can be solved analytically; analytic solutions were obtained in Refs. \(^23\)-\(^25\). Owing to the complexity of the equilibrium equations, the construction of analytic solutions in the case of arbitrary flows (toroidal and poloidal) is difficult. To our best knowledge the only analytic solution of this kind is that obtained by Mashke and Perrin \(^26\) in the specific case of small ratio of poloidal to toroidal magnetic field and small \(\beta\) defined as the ratio of the kinetic energy to the magnetic energy.

In the present paper we study the equilibrium of a cylindrical plasma with purely poloidal mass flow in the framework of ideal MHD theory. Several physics aspects of a similar system
has been often considered in the literature, e.g. i) the numerical study of Ref. [21] concerning the current sheet formation in a cylindrical plasma with rectangular cross section and purely poloidal flow and ii) the analytic study of Ref. [22] on nonlinear interactions of tearing modes in plane geometry in the presence of sheared poloidal flow. The plasma cross section of the system under consideration is arbitrary and therefore the configuration is two-dimensional. No restriction is imposed either on the components of the magnetic field or $\beta$. We show that only incompressible equilibrium flows are possible and that several classes of analytic equilibrium solutions can be obtained.

The equilibrium equations of the system under consideration are presented in Section II. In Section III we derive exact equilibrium solutions. Our conclusions are summarized in Section IV.

II. EQUILIBRIUM EQUATIONS

The system under consideration is a cylindrical plasma with mass flow and arbitrary cross section shape. For this configuration convenient coordinates are $\xi$, $\eta$ and $z$ with unit basis vectors $e_\xi$, $e_\eta$, $e_z$, where $e_z$ is parallel to the axis of symmetry and $\xi$, $\eta$ are generalized coordinates pertaining to the poloidal cross section. Specific examples are circular and elliptical cylindrical coordinates. The equilibrium quantities do not depend on $z$. The equations governing the equilibrium states can be obtained from the general two dimensional equilibrium equations with flow, e.g. Refs. [11], [12], [15]. In particular, the components of the momentum conservation equation

$$\rho (v \cdot \nabla) v = j \times B - \nabla P$$

along the magnetic field and perpendicular to a magnetic surface, respectively, are put in the form

$$B \cdot \left[ \nabla \left( \frac{v^2}{2} + \frac{|\nabla F|^2}{2\rho^2} \right) + \frac{\nabla P}{\rho} \right] = 0$$

and

$$4$$
\[ \nabla \cdot \left[ \left( 1 - \frac{(F')^2}{\rho} \right) \nabla \psi \right] + \frac{F'' F' |\nabla \psi|^2}{\rho \nabla \psi^2} + \frac{B_z \nabla B_z \cdot \nabla \psi}{|\nabla \psi|^2} + \rho \frac{\nabla \psi}{|\nabla \psi|^2} \cdot \nabla \left( \frac{(F')^2 B^2}{2 \rho^2} \right) + \frac{\nabla P}{\rho} = 0. \]  

(2)

Here, \( \psi(\xi, \eta), F(\xi, \eta), B_z(\xi, \eta) \) and \( v_z(\xi, \eta) \) are stream functions in terms of which the divergence free fields, i.e. the magnetic field \( B \), the current density \( j \) and the mass flow \( \rho v \), are expressed as

\[ B = B_z e_z + e_z \times \nabla \psi, \]  

(3)

\[ j = \nabla^2 \psi e_z - e_z \times \nabla B_z \]  

(4)

and

\[ \rho v = \rho v_z e_z + e_z \times \nabla F; \]  

(5)

the function \( F \) and the electrostatic potential \( \Phi \) are surface quantities satisfying the equations

\[ B_z - F' (\psi) v_z = X(\psi) \]  

(6)

(\( X \) also is a surface quantity.) and

\[ \frac{B_z}{\rho} F' (\psi) - v_z = \Phi' (\psi); \]  

(7)

the prime denotes derivative with respect to \( \psi \); SI units are used and the vacuum magnetic permeability \( \mu_0 \) is set to unity. It is noted here that a particular equation of state, e.g. either isentropic or isothermal magnetic surfaces, has not been employed in order to express in Eq. (2) the pressure gradient \( P' \) in terms of surface quantities; thus, Eq. (2) holds for any equation of state.

We now restrict attention to purely poloidal flow and therefore set \( v_z = 0 \). Then, Eq. (4) implies that the function \( B_z \) is a surface quantity, i.e. \( B_z = B_z(\psi) \). From equation (7) it follows then that
\[ \rho = \rho(\psi), \]  

(8)

and therefore the mass density and the magnetic field share also the same flux surfaces. Using Eq. (8), the mass conservation equation \( \nabla \cdot (\rho \mathbf{v}) = 0 \) yields \( \nabla \cdot \mathbf{v} = 0 \). Thus, it turns out that for a cylindrical plasma and purely poloidal mass flow only incompressible equilibrium flows are possible.

With the use of Eq. (8), Eq. (1) can now be readily integrated, yielding an expression for the pressure, i.e.

\[ P = P_s(\psi) - \frac{F'^2}{2\rho} |\nabla \psi|^2. \]  

(9)

We note here that, unlike in static equilibria, in the presence of flow magnetic surfaces do not coincide with isobaric surfaces because the momentum balance equation implies that \( \mathbf{B} \cdot \nabla P \) in general differs from zero. In this respect, the term \( P_s(\psi) \) is the static part of the pressure which does not vanish when \( F' \) is set to zero; thus, the second term makes the isobaric surfaces to detach from magnetic surfaces.

Before proceeding to obtain the final form of the equilibrium equation we examine a particular case, i.e. when the functions \( F \) and \( \rho \) satisfy the relation

\[ \frac{(F')^2}{\rho} = 1. \]  

(10)

Then, the second-order differential operator in Eq. (2) is singular and since \( F \) and \( \rho \) are surface quantities this singularity occurs uniformly over a magnetic surface. On the basis of Eq. (5) for \( \rho \mathbf{v} \) and the definitions

\[ \frac{v^2}{\mathcal{A}_p} \equiv \frac{|\nabla \psi|^2}{\rho} \]  

for the Alfvén velocity associated with the poloidal magnetic field and the Mach number

\[ M^2 \equiv \frac{\nu^2}{v^2_{\mathcal{A}p}}, \]  

Eq. (10) can be written as \( M^2 = 1 \). Assuming now \( \frac{(F')^2}{\rho} \neq 1 \), inserting Eq. (4) into Eq. (2) and setting \( v_z = 0 \), Eq. (4) after some straightforward algebra, reduces to the elliptic differential equation

\[ \left[ 1 - \frac{(F')^2}{\rho} \right] \nabla^2 \psi + \frac{F'}{\rho} \left( \frac{F' \rho'}{2 \rho} - F'' \right) |\nabla \psi|^2 + \left( P_s + \frac{B_z^2}{2} \right)' = 0. \]  

(11)

The absence of any hyperbolic regime in Eq. (11) can be understood by noting that, as is well known from the gass dynamics, the flow must be compressible to allow the governing partial
differential equation to depart from ellipticity. It may also be noted that in axisymmetric equilibria with arbitrary flows, e.g. Eq. (2), \(|M| = 1\) is not a transition point. Finally, we note that replacing the stream functions by the corresponding field quantities, Eq. (11) is put in the form

\[
\frac{d}{dr} \left( P + \frac{|B|^2}{2} \right) + \frac{B_0^2}{r} - \rho v^2 = 0, \tag{12}
\]

the terms of which have a readily understandable physical meaning.

III. EXACT SOLUTIONS

In this section, Eq. (11) is solved for a specific choice of the flux functions \(\rho(\psi)\) and \(F(\psi)\), i.e. when these functions satisfy the relation

\[
\frac{\rho'}{\rho} = 2 \frac{F''}{F'}. \tag{13}
\]

This ansatz makes the second term in Eq. (11) to vanish. Eq. (13) is an ordinary differential equation for \(\psi\), which can easily be integrated. Its solution is

\[
\frac{(F')^2}{\rho} = M_c^2, \tag{14}
\]

where \(M_c^2 = \text{const.} \neq 1\). The choice (13) leads therefore to flows with constant Mach numbers. Solution (14) makes the factor multiplying the second-order differential operator in Eq. (11) constant; thus, Eq. (11) reduces to

\[
\nabla^2 \psi + \frac{1}{1 - M_c^2} \left( P_s + \frac{B_0^2}{2} \right)' = 0. \tag{15}
\]

This is similar in form to the equation governing static equilibria; the only explicit reminiscent of flow is the presence of \(M_c\). Eq. (15), which is still a semilinear partial differential equation, can be linearized for several choices of \(P_s + \frac{B_0^2}{2}\), e.g. a) for \(P_s + \frac{B_0^2}{2}\) linear in \(\psi\) becomes an inhomogeneous Laplace equation with constant inhomogeneity and b) for \(P_s + \frac{B_0^2}{2}\) quadratic in \(\psi\) becomes i) a homogeneous Helmholtz equation for \(M_c^2 > 1\) and ii) a homogeneous diffusion equation for \(0 < M_c^2 < 1\). Several classes of analytic solutions
can therefore be derived. In the following, specific exact equilibria will be constructed for a circular cylindrical plasma having i) a flat current density profile and ii) a peaked current density profile.

A. Flat Current Density Profile

To describe a plasma which is confined within a circular infinitely conducting cylinder of radius \( r_0 \), the coordinates \( \xi, \eta, z \) are specified to be the circular cylindrical coordinates \( r, \theta, z \) with unit basis vectors \( e_r, e_\theta, e_z \). The equilibrium quantities depend just on \( r \). To obtain equilibria having a flat current density profile, we employ the ansatz \( \left( P_\psi + \frac{B_z^2}{2} \right)' = \text{const} \). The simplest non trivial solution of Eq. (13) is written then as

\[
\psi \propto \tau^2, \quad (16)
\]

where \( \tau \equiv \frac{r}{r_0} \). Solution (16) yields a singly peaked parabolic pressure profile

\[
P(\rho) = P(0)(1 - \tau^2), \quad (17)
\]
a flat “toroidal” current density

\[
j_z = -\left( \frac{\alpha}{1 - M_c^2} \right) \frac{\sqrt{P(0)}}{r_0}, \quad (18)
\]

\[
B_z(\rho) = \left\{ B_z^2(0) + 2P(0) \left[ 1 - \left( \frac{\alpha}{1 - M_c^2} \right)^2 \right] \tau^2 \right\}^{1/2}, \quad (19)
\]

\[
B_\theta(\rho) = -\left( \frac{\alpha}{1 - M_c^2} \right) \sqrt{P(0)} \tau, \quad (20)
\]

\[
\mathbf{v} = \frac{F'}{\rho} (\mathbf{e}_z \times \nabla \psi) = \pm |M_c| \frac{\sqrt{P(0)}}{1 - M_c^2} \frac{1}{\sqrt{\rho}} \mathbf{e}_\theta \quad (21)
\]

and

\[
\mathbf{E} = -B_z \frac{F'}{\rho} \nabla \psi = \pm |M_c| \frac{\sqrt{P(0)} B_z(\rho)}{1 - M_c^2} \frac{1}{\sqrt{\rho}} \mathbf{e}_r. \quad (22)
\]
Here, $\alpha$ is a parameter which, together with $M_c$, describes the magnetic properties of the plasma, i.e. the plasma is diamagnetic for \( \left( \frac{\alpha}{1 - M_c^2} \right)^2 < 1 \) and paramagnetic for \( \left( \frac{\alpha}{1 - M_c^2} \right)^2 > 1 \). For vanishing flow ($M_c = 0$) the equilibrium reduces to the exact axisymmetric equilibrium solution, in the limit of infinite aspect ratio, which was suggested by Shafranov [28] and has been extensively used by Solov’ev [29]. In Eqs. (21) and (22), the density $\rho$ remains a free function of $\tau$ and therefore a variety of velocity and electric field profiles can be derived. As an example, we consider an equilibrium having a singly peaked density profile:

$$\rho = \frac{\rho(0)}{k^\nu} (k - \rho^2)^\nu,$$  \hspace{1cm} (23)

where $\nu > 0$ and $k > 1$. The parameter $k$ here is necessary in order to make $\rho$ non-vanishing on plasma surface ($\rho = 1$). (Note that $\rho$ appears in the denominator of Eqs. (21) and (22)).

This agrees qualitatively with experimental results [4, 27], according to which finite densities and very low temperatures were measured in the edge region of plasmas with mass flow. For the density profile (23), the velocity profile becomes hollow and the temperature profile, consistent with Eq. (17), is given by

$$T = T(0) k^\nu \frac{(1 - \rho^2)}{(k - \rho^2)^\nu}.$$  \hspace{1cm} (24)

### B. Peaked current density profile

For simplicity we ignore here diamagnetic effects, viz. we consider $\beta_p = 1$ equilibria. (The construction of $\beta_p \neq 1$ equilibria is, however, possible). Accordingly, for $B_z = B_0 = \text{const.}$ and

$$P_z \alpha (1 - M_c^2) \psi^2$$

the simplest solution of Eq. (13) is

$$\psi = \psi(0) J_0(\zeta).$$  \hspace{1cm} (25)
$J_0$ and $J_1$ below are zeroth- and first-order Bessel functions, respectively; $\zeta \equiv \frac{\lambda_0 r_0}{r}$, where $\lambda_0$ is the first zero of $J_0$. Solution (25) yields a peaked pressure profile

$$P = P_s(0) \left[ (1 - M_c^2) J_0(\zeta) + M_c^2 \left( J_1^2(\lambda_0) - J_1^2(\zeta) \right) \right], \quad (26)$$

a singly peaked current density profile

$$J_z = J_z(0) J_0(\zeta), \quad (27)$$

$$B_\theta = -\frac{\lambda_0 \psi(0)}{r_0} J_1(\zeta), \quad (28)$$

$$\mathbf{v} = \pm |M_c| \frac{\lambda_0 \psi(0)}{r_0} \frac{J_1(\zeta)}{\sqrt{\rho}} \mathbf{e}_\theta \quad (29)$$

and

$$\mathbf{E} = \pm |M_c| \frac{\lambda_0 \psi(0) B_0 J_1(\zeta)}{r_0} \frac{\sqrt{\rho}}{\rho} \mathbf{e}_r. \quad (30)$$

Here, $P_s(0) = P(0)|_{M_c=0}$ is the static pressure at the plasma center. The above equilibrium has the following characteristics:

1. Both the pressure and the current density profiles vanish on the plasma surface.

2. The magnetic field is independent of flow.

We note that the requirement $P(0) > 0$ imposes a restriction on the values of Mach number, i.e. $M_c^2 < [1 - J_1(\lambda_0)]^{-1}$. By inspection of Eq. (26) it turns out that as flow increases, i.e. when $|M_c|$ takes higher values, the pressure $P$ takes lower values and its profile becomes flatter; the negative term $\frac{dP}{dr}$ takes therefore higher values. On the other hand, as $|\mathbf{v}|$ increases, the flow term $-\rho \frac{\mathbf{v}^2}{r}$ (for a given density profile) becomes more negative. Thus, the quantity $\frac{dP}{dr} - \rho \frac{\mathbf{v}^2}{r}$ becomes flow independent and therefore guarantees the equilibrium equation (12), in which the magnetic field terms (for the solution (25)) are independent of flow.
IV. CONCLUSIONS

Using the general two dimensional ideal MHD equilibrium equations with flow, we investigated the stationary states of a cylindrical plasma with purely poloidal flow and arbitrary cross-section shape and showed that the flows are necessarily incompressible. Consequently: (i) The pressure profile is expressed in terms of a static pressure surface quantity $P_s$ and a flow dependent term not remaining constant on a magnetic surface (Eq. (9)), which is consistent with the fact that for equilibria with mass flow the magnetic surfaces are detached from the isobaric surfaces. (ii) The flow equations reduce to a single second-order quasilinear partial differential equation for $\psi$, (Eq. (11)), which contains four surface quantities, i.e. $P_s(\psi)$, the flow function $F(\psi)$, the “toroidal” magnetic field $B_z(\psi)$ and the density $\rho(\psi)$. One has to specify these quantities together with the boundary conditions for $\psi$ in order to determine the equilibrium completely. A singularity occurs in the differential equation when the modulus of the Mach number $|M|$, associated with the Alfvén velocity for the poloidal magnetic field, is unity. Apart from this case, this equation is always elliptic. This is consistent with the fact that the existence of hyperbolic regimes in the governing equilibrium differential equation is possible only in compressible fluids. For velocities satisfying the relation $|M| = \text{const.} \neq 1$, the equilibrium equation takes a simpler form (Eq. (13)), which is amendable to several classes of analytic solutions. We constructed and studied particular exact equilibria for a plasma confined within a circular cylindrical perfectly conducting boundary having i) a flat current density profile and ii) a peaked current density.

The analytic equilibria with poloidal flow we constructed in the present paper can be the basis of stability and transport investigations, which would be of relevance to magnetic confinement systems. In particular, they may help in understanding the physics involved in the transition from the low confinement mode to the high confinement mode in tokamaks, which is crucially related to poloidal flow. The derivation of new analytic solutions in more realistic situations, e.g. axisymmetric equilibria, helically symmetric equilibria (even in specific cases such as in rotating plasmas with a single velocity component or/and small $\beta$),
might help to easily get physical insight of plasmas with flows. Solutions of this kind might also be used for checking relevant equilibrium codes.
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