Computation of the maximal invariant set of
discrete-time systems subject to quasi-smooth
non-convex constraints

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Abstract—In this paper, we consider the problem of computing the maximal invariant set of linear systems and certain types of nonlinear systems that include switched linear systems subject to a class of non-convex constraints with quadratic relaxations. Special cases of such non-convex constraints include semialgebraic constraints and other smooth constraints with Lipschitz gradient. With these quadratic relaxations, a sufficient condition for set invariance is derived and it can be formulated as a set of linear matrix inequalities. Based on the sufficient condition, a new algorithm is presented with finite convergence to the exact maximal invariant set under mild assumptions. The performance of this algorithm is demonstrated on several numerical examples.

Index Terms—Invariant sets, non-convex constraints, switched linear systems, semi-algebraic sets

I. INTRODUCTION

Invariant set theory is an important tool for stability analysis and control design of constrained dynamical systems and it has been successfully used to solve various problems in systems and control; see, for instance, [1]–[4] and the references therein. An invariant set of a dynamical system refers to a region where the trajectory will never leave once it enters. One well-known application is in Model Predictive Control (MPC) [5], where invariant sets are often used to ensure recursive feasibility and stability.

Given the extensive applications of invariant sets in systems and control, significant attention has been paid to their characterization and computation. In [6]–[8], recursive algorithms have been proposed to compute polyhedral invariant sets of linear systems. For linear systems with bounded disturbances, robust invariant sets are introduced and computable using different algorithms in [9]–[14]. For linear systems with control, the computation of (control) invariant sets is more complicated and a few algorithms have been proposed to compute inner or outer approximations [15]–[17]. Algorithms for computing invariant sets of different types of nonlinear systems are also available in the literature, see, e.g., [18]–[23]. The concept of set invariance can be even extended to hybrid systems. For instance, invariant sets can be defined for switched systems, which constitute an important family of hybrid systems, and the computation of such sets have been extensively studied, see, e.g., [24]–[29].

Among various invariant sets, the maximal invariant set is of particular interest. The standard algorithm for computing the maximal invariant set of linear systems with polytopic constraints is presented in [6], [9] with sufficient conditions for finite convergence. Recently, necessary and sufficient conditions for finite convergence are concretely discussed in [30]. Even though the literature on set invariance of linear systems is large, computing the exact maximal control invariant set is still challenging, especially when the constraints are non-convex, see, e.g., recent works [16], [17] for inner or outer approximations. For switched linear systems, algorithms to compute the maximal invariant set are also provided in the cases of polytopic/convex constraints [24], [27], [28], [31] and semialgebraic constraints [26]. Although there are some algorithms for estimating the maximal invariant sets of certain types of nonlinear systems, see, e.g., [19], [22], [23], computing the exact maximal invariant set is still an open problem for general nonlinear systems. When the constraints are non-convex, the computation will be even more challenging. In fact, in the presence of non-convex constraints, to the best of our knowledge, the exact computation of the maximal invariant set is only addressed in [26] for switched linear systems with semialgebraic constraints by lifting the original system into a higher dimension. For general non-convex constraints, computing the exact maximal invariant set is an unsolved problem even for linear systems.

This paper is focused on the exact computation of the maximal invariant set of linear systems, switched linear systems, and some special nonlinear systems in the presence of a broad class of non-convex constraints that admit quadratic relaxations. We will give formal assumptions on such non-convex constraints which include semialgebraic constraints and smooth constraints with Lipschitz gradient. Using the quadratic relaxations, a sufficient condition for set invariance is derived from the S-procedure [32] and can be expressed as a set of Linear Matrix Inequalities (LMI). Based on this sufficient condition, we present a new algorithm that solves a set of Linear Matrix Inequalities (LMI) at each iteration. The tightness of the sufficient condition largely depends on the conservatism of the S-procedure [33]. Under mild assump-
tions, finite convergence to the exact maximal invariant set can be established. We show that this proposed algorithm can be extended to switched linear systems and some nonlinear systems that can be linearized via state transformation. In the case of semialgebraic constraints, a similar lifting method as \cite{26} is used while the dimension of the lifted space in this paper is significantly lower for the same setting.

A preliminary version of this paper appears as a conference paper in \cite{34}, which is only focused on linear systems. In this paper, we provide complete detailed proofs of all lemmas and theorems, the discussion on the extensions to switched linear systems and some special nonlinear systems, and additional numerical results.

The rest of the paper is organized as follows. This section ends with the notation, followed by the next section on the review of preliminary results on the invariant sets of linear systems. Section \[\text{III}\] presents the proposed approach for computing the maximal invariant set of linear systems with non-convex constraints. Section \[\text{IV}\] discusses semi-algebraic constraints and the extensions to switched linear systems and some special nonlinear systems. Several numerical examples are provided Section \[\text{V}\]. The last section concludes the work.

The notation used in this paper is as follows. Non-negative and positive integer sets are indicated respectively by $\mathbb{N}_0$ and $\mathbb{Z}^+$ with $\mathbb{Z}_0^M := \{1, 2, \ldots, M\}$ and $\mathbb{Z}_0^M := \{L, L + 1, \ldots, M\}, M \geq L, M, L \in \mathbb{Z}_0^+$. Similarly, $\mathbb{R}_0^+$ and $\mathbb{R}^+$ refer respectively to the sets of non-negative and positive real numbers. $\mathbb{S}^n$ denotes the set of symmetric matrices in $\mathbb{R}^{n \times n}$. $I_n$ (the subscription is omitted when the dimension is clear from the context) is the $n \times n$ identity matrix and $I_n$ denote the vector of $n$ ones. For a square matrix $Q$, $Q \succ (\succeq) 0$ means $Q$ is positive definite (semi-definite). The $p$-norm of $x \in \mathbb{R}^n$ is $\|x\|_p$ while $\|x\|_2^2 = x^T Q x$ for $Q \succeq 0$. Given a set of vectors, $x_1, x_2, \ldots, x_M$, the collection of vectors, $(x_1, x_2, \ldots, x_M)$ also refers to the stack vector of $[(x_1)^T \ (x_2)^T \ \cdots \ (x_n)^T]^T \in \mathbb{R}^{\sum_{i=1}^M n_i}$ for notational simplicity. Additional notation is introduced as required in the text.

## II. Preliminaries

This section reviews some known results on the invariant sets of constrained discrete-time linear systems. We consider the linear system

\[ x(t + 1) = Ax(t), \quad \forall t \in \mathbb{Z}_0^+, \tag{1} \]

where $x(t) \in \mathbb{R}^n$ is the state vector. The system is subject to state constraints

\[ x(t) \in \Omega, \quad \forall t \in \mathbb{Z}_0^+ \tag{2} \]

where $\Omega \subseteq \mathbb{R}^n$ is a quadratic set in the form of

\[ \Omega = \{x \in \mathbb{R}^n : x^T Q_i x + 2 q_i^T x \leq 1, i \in \mathbb{Z}_0^p\} \tag{3} \]

where $Q_i \in \mathbb{S}^n$, $q_i \in \mathbb{R}^n$ and $p$ is the number constraints. When $Q_i = 0$, for all $i \in \mathbb{Z}_0^p$, $\Omega$ becomes a polyhedron. More generally, other nonlinear constraints may also be imposed on the system:

\[ x(t) \in \Theta := \{x \in \mathbb{R}^n : H_i(x) \leq 1, i \in \mathbb{Z}_0^+\}, \forall t \in \mathbb{Z}_0^+ \tag{4} \]

where $H_i : \mathbb{R}^n \to \mathbb{R}$ is a continuous nonlinear function and $m \in \mathbb{Z}_0^+$ is the number of other nonlinear constraints. The overall state constraint set is the intersection of $\Omega$ and $\Theta$:

\[ x(t) \in X := \Omega \cap \Theta, \forall t \in \mathbb{Z}_0^+. \tag{5} \]

For computational reasons, we treat quadratic constraints and general nonlinear constraints differently. The following assumptions are made.

**Assumption 1:** The matrix $A$ is Schur stable, i.e., for any eigenvalue $\lambda$ of $A$, $|\lambda|$ is smaller than one.

**Assumption 2:** The set $\Omega$ is compact and contains the origin in its interior. For any $i \in \mathbb{Z}_0^m$, $H_i : \mathbb{R}^n \to \mathbb{R}$ is a continuous nonlinear function with $H_i(0) = 0$.

**Assumption 3:** For all $i \in \mathbb{Z}_0^m$, $H_i : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and there exist a vector $H_i^\nabla \in \mathbb{R}^n$ and a scalar $L_i \geq 0$ such that

\[ |H_i(x) - H_i(0) - (H_i^\nabla)^T x| \leq \frac{L_i}{2} \|x\|^2 \tag{6} \]

for all $x \in X$.

Assumptions 1 and 2 are standard and necessary for the problem to be well-posed, see \cite{6}. From the continuity of the nonlinear functions $(H_i(x))_{i=1}^m$, $\Theta$ contains the origin in its interior, and thus $X$ is compact and contains the origin in its interior. Assumption 3 requires all the nonlinear functions to have quadratic lower and upper bounds. However, these functions are not necessarily Lipschitz continuous or differentiable. Clearly, for functions with Lipschitz continuous gradient, the condition in Assumption 3 will be satisfied. Indeed, suppose that, for any $i \in \mathbb{Z}_0^m$, $H_i$ is a continuously differentiable function with Lipschitz gradient:

\[ \|\nabla H_i(x) - \nabla H_i(y)\| \leq L_i \|x - y\|, \forall x, y \in \Omega, \tag{7} \]

then, Assumption 3 is satisfied with $H_i^\nabla = \nabla H_i(0)$ (see, e.g., Lemma 6.9.1 in \cite{33}). We will refer to a function satisfying (6) as a quasi-smooth function. All the polynomial functions satisfy (6). For notational simplicity, let

\[ q := [q_1 \ q_2 \ \cdots \ q_p] \tag{8} \]

\[ H(x) := (H_1(x), H_2(x), \ldots, H_m(x)), \tag{9} \]

where $q \in \mathbb{R}^{n \times p}$ and $H(x) \in \mathbb{R}^m$.

We now define some central concepts of this paper.

**Definition 1:** \cite{2}, \cite{5} The nonempty set $Z \subseteq X$ is a CA-invariant (Constraint Admissible invariant) set for System (1) if and only if for any $x \in Z$ one has that $Ax \in Z$.

With Assumptions 1 and 2 there often exist multiple CA-invariant sets. In many applications, it is desirable to compute the maximal CA-invariant set, which is defined below.

**Definition 2:** The nonempty set $O_\infty$ is the maximal CA-invariant set for system (1) if and only if $O_\infty$ is a CA-invariant set and contains all CA-invariant sets in $X$.

It is a standard result that the maximal CA-invariant set exists (see \cite{6} for general conditions guaranteeing its existence), and that it can be computed recursively by the following iteration:

\[ O_0 := X \tag{10} \]

\[ O_{k+1} := O_k \cap \{x \in \mathbb{R}^n : Ax \in O_k\}, k \in \mathbb{Z}_0^+. \tag{11} \]
With these iterates, it is can be verified that
\[ O_k = \{ x \in \mathbb{R}^n : A^T x \in X, \ell \in \mathbb{Z}_+^k \} \].
(12)

Thus, the maximal CA-invariant set can be expressed as
\[ O_{\infty} := \bigcap_{k \in \mathbb{Z}_+^O} O_k = \{ x \in \mathbb{R}^n : A^k x \in X, k \in \mathbb{Z}_+^O \} \].
(13)

From Assumptions 1 and 2 the set \( O_{\infty} \) defined in (13) has the following properties:
(i) if \( Z \subseteq \mathbb{R}^n \) is a CA-invariant set of system (1), \( Z \subseteq O_{\infty} \); (ii) there exists a finite \( k^* \) such that \( O_{k^*+1} = O_{k^*} \); (iii) for any \( k \) satisfying (ii), it can be shown that \( O_k = O_{k^*} \) for all \( k \geq k^* \) and \( O_{\infty} = O_{k^*} \).

From the properties above, the problem of computing \( O_{\infty} \) boils down to the search for a \( k^* \) such that \( O_{k^*+1} = O_{k^*} \). The standard procedure is to increase \( k \) from 0 until \( O_{k+1} = O_k \), which is equivalent to
\[ O_k \subseteq \{ x \in \mathbb{R}^n : A^{k+1} x \in X \} \],
(14)
see [6] for details. This condition can be treated as a stopping criterion for the algorithm in (10)-(11). Observe that \( \{ x \in \mathbb{R}^n : A^{k+1} x \in X \} \) can be rewritten as \( \{ x \in \mathbb{R}^n : (A^{k+1} x)^T Q_i A^{k+1} x + 2q_i (A^{k+1} x) \leq 1, i \in \mathbb{Z}^P, H(A^{k+1} x) \leq 1 \} \). During the computational procedure, we aim to find the minimal \( k \) that satisfies (14).

Let \( k_{\min} := \arg \min_{k \in \mathbb{Z}_+^O} \{ k : (14) \} \). As shown in Property (iii), \( O_{k+1} = O_{k_{\min}} = O_{\infty} \) for any \( k \geq k_{\min} \). With this property, we can determine \( O_{\infty} \) for any upper bound on \( k_{\min} \). For the verification of (14), we essentially need to solve a set of nonlinear optimization problems. For general nonlinear constraints in (5), these problems are non-convex and it is computationally expensive to reach the global optimality. For this reason, we will aim to derive a sufficient condition that can be efficiently verified.

III. THE PROPOSED APPROACH

This section discusses the computation of the exact maximal CA-invariant set with nonlinear constraints. An algorithm will be presented to compute an upper bound on \( k_{\min} \) and it can be determined in a finite number of iterations under mild assumptions.

For the quadratic (or linear) constraints, the following nonlinear optimization problem is defined at the \( k \)-th iteration of (11):
\[ g_i^k := \max_{x \in O_k} (A^{k+1} x)^T Q_i A^{k+1} x + 2q_i (A^{k+1} x) - 1 \]
(15a)
s.t. \( x \in O_k \)
(15b)
for \( i \in \mathbb{Z}^P \). Let \( g_{\max} := \max_{i \in \mathbb{Z}^P} g_i^k \) for all \( k \in \mathbb{Z}_+^O \). If \( g_{\max} \leq 0 \) for some \( k \in \mathbb{Z}_+^O \), \( O_k \subseteq \{ x \in \mathbb{R}^n : (A^{k+1} x)^T Q_i A^{k+1} x + 2q_i x \leq 1, i \in \mathbb{Z}^P \} \). Similarly, for the other nonlinear constraints, the following nonlinear optimization problem is defined at the \( k \)-th iteration of (11):
\[ h_i^k := \max_{x \in O_k} H_i(A^{k+1} x) - 1 \]
(16a)
s.t. \( x \in O_k \)
(16b)
for \( i \in \mathbb{Z}^m \). Let \( h_{\max} := \max_{i \in \mathbb{Z}^m} h_i^k \) for all \( k \in \mathbb{Z}_+^O \). If \( h_{\max} \leq 0 \) for some \( k \in \mathbb{Z}_+^O \), \( \{ x \in \mathbb{R}^n : H(A^{k+1} x) \leq 1 \} \subseteq O_k \). Using (15) and (16), \( k_{\min} \) can be determined via \( \min_{k \in \mathbb{Z}_+^O} \{ k : g_{\max}^k \leq 0, h_{\max}^k \leq 0 \} \). To do so, we need in principle to solve (15) and (16) and get their global optimal solutions. However, for general nonlinear constraints, both (15) and (16) are nonlinear non-convex problems. Even if \( \Omega \) and \( \Theta \) are convex sets, (15) and (16) may not be convex problems. Therefore, we do not attempt to solve (15) and (16) directly, only upper bounds on the optimal values of \{\( g_i^k \)\} and \{\( h_i^k \)\}.

A. Quadratic constraints

Consider the case where only quadratic constraints exist, i.e., \( \Theta = \mathbb{R}^n \) and \( \Theta = \mathbb{R} \). Let
\[ \bar{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \]
(17)
\[ \bar{Q}_i = \begin{pmatrix} Q_i & q_i \\ q_i^T & -1 \end{pmatrix}, \forall i \in \mathbb{Z}^P. \]
(18)
Following the iteration in (10)-(11), we define:
\[ Q_0 := \{ \bar{Q}_i, i \in \mathbb{Z}^P \} \]
(19)
\[ Q_{k+1} := \bar{Q}_k \bigcup \{ \bar{A}^T \bar{Q} \bar{A} : \bar{Q} \in Q_k, k \in \mathbb{Z}_+^O \} \]
(20)
with \( Q_k(i) \) being the \( i \)-th element of \( Q_k \) for \( i \in \mathbb{Z}^{|Q_k|} \). From the construction of \( Q_k \), it can be shown that \( Q_k = \{ \bar{Q}_1, \ldots, \bar{Q}_p, \bar{A}^T \bar{Q}_1 \bar{A}, \ldots, \bar{A}^T \bar{Q}_p \bar{A}, \ldots, (\bar{A}^k)^T \bar{Q}_1 \bar{A}^k, \ldots, (\bar{A}^k)^T \bar{Q}_p \bar{A}^k \}, k \in \mathbb{Z}_+^O \),
(21)
with the cardinality being \( |Q_k| = (k+1)p \). For all \( k \in \mathbb{Z}_+^O \), let
\[ \Delta Q_k := Q_{k+1} \setminus Q_k = \{ (\bar{A}^{k+1})^T \bar{Q}_i \bar{A}^{k+1}, \forall i \in \mathbb{Z}^P \}
(22)
with \( |\Delta Q_k| = p \). Using the notation above, \( O_k \) defined in (10)-(11) can be rewritten as
\( O_k = \{ x \in \mathbb{R}^n : (x)^T \bar{Q} (x) \leq 0, \forall Q_k \in \bar{Q}_k \} \).
(23)
for all \( k \in \mathbb{Z}_+^O \). From the S-procedure, see Section 2.6.3 in [2], the following lemma can be obtained.

Lemma 1: Suppose \( \Omega = \mathbb{R}^n \) and \( X = \mathbb{R} \). Let \( O_k \) be defined by the procedure in (10)-(11). \( Q_k \) be defined in (19)-(20), and \( \Delta Q_k \) be given in (22) for all \( k \in \mathbb{Z}_+^O \). If, for some \( k \in \mathbb{Z}_+^O \), there exists a non-negative sequence \( \tau_{|Q_k|} := \{ \tau_{k, \ell} \geq 0, \ell \in \mathbb{Z}^{|Q_k|} \} \) such that
\[ \Delta Q_k(i) \leq \sum_{\ell=1}^{|Q_k|} \tau_{k, \ell} Q_k(\ell), \forall i \in \mathbb{Z}^{|\Delta Q_k|} \]
(24)
where \( \Delta Q_k(i) \) is the \( i \)-th element of \( \Delta Q_k \), then, \( O_{k+1} = O_k \).
From (23), \[ \begin{pmatrix} x \\ 1 \end{pmatrix}^T Q_k(\ell) \begin{pmatrix} x \\ 1 \end{pmatrix} \leq 0, \forall \ell \in \mathbb{Z}^{|Q_k|}, \] for any \( x \in O_k \). Hence, (24) implies \[ \begin{pmatrix} x \\ 1 \end{pmatrix}^T \Delta Q_k(i) \begin{pmatrix} x \\ 1 \end{pmatrix} \leq 0, \] for all \( i \in \mathbb{Z}^{|\Delta Q_k|} \) and \( x \in O_k \). Considering the fact that
\[ O_{k+1} = O_k \cap \{ x : \begin{pmatrix} x \\ 1 \end{pmatrix}^T \Delta Q_k(i) \begin{pmatrix} x \\ 1 \end{pmatrix} \leq 0, \forall i \in \mathbb{Z}^{|\Delta Q_k|} \}, \]
we can claim that \( x \in O_{k+1} \) for all \( x \in O_k \), which implies \( O_{k+1} = O_k \). \( \square \)

As we have seen, under Assumptions 1 and 2, the formal algorithm described in (10)-(11) always terminates in finite time. This algorithm is easily implementable when \( X \) is a polyhedron, see (23), (36). In many cases, it is not directly implementable in the presence of nonlinear constraints. Even if \( X \) is convex, the optimization problem (15) is still non-convex. However, the same algorithm with (24) is practically implementable, since these inequalities are LMI, which can be efficiently solved using interior point methods (32). To recover the nice finite termination property of the formal algorithm, an additional assumption is needed.

**Assumption 4**: There exists \( D_x > 0 \) such that \( \| x \|^2 \leq D_x \) for all \( x \in \Omega \).

This assumption in fact always holds under Assumption 1. Indeed, without loss of generality, we can always add the redundant ball constraint of the form \( \| x \|^2 \leq D_x \) to \( \Omega \). With this additional assumption, we can let \( Q_1 = \frac{1}{2} I \) and \( q_1 = 0 \) in (3). We now show that the finiteness property of the former algorithm in (10)-(11) still holds for the LMI version.

**Lemma 2**: Suppose Assumptions 1, 2 and 4 hold, \( \Theta = \mathbb{R}^n \), and \( X = \Omega \) with \( Q_1 = \frac{1}{2} I \) and \( q_1 = 0 \) in (3). Let \( O_k \) be defined by the procedure in (10)-(11) for all \( k \in \mathbb{Z}^+ \). There always exists some finite \( k \) such that \( O_{k+1} = O_k \).

**Proof of Lemma 2**. From (17), (18) and (22), we have
\[ \Delta Q_k(i) = (A^{k+1})^T Q_i A^{k+1} = -Q_i \]
for all \( i \in \mathbb{Z}^{|\Delta Q_k|} \) and \( k \in \mathbb{Z}^+ \). From Assumption 1, \( A^{k+1} \) goes to 0 as \( k \) increases. Hence, for any \( 0 < \beta < 1 \), there always exists a \( k \) such that
\[ \Delta Q_k(i) \leq \beta \left( \begin{array}{cc} \frac{1}{2} I & 0 \\ 0 & -1 \end{array} \right). \] (25)
for all \( i \in \mathbb{Z}^{\Delta Q_k} \). This implies that the LMI (24) is satisfied with \( \tau^*_\ell = \beta \) and \( \tau^*_\ell = 0, \ell \neq 1 \) for all \( i \in \mathbb{Z}^{|\Delta Q_k|} \). Therefore, \( O_{k+1} = O_k \). \( \square \)

Based on Lemma 2, the following LMI optimization problem is defined for all \( i \in \mathbb{Z}^{|\Delta Q_k|} \) and \( k \in \mathbb{Z}^+ \):
\[ r^*_\ell := \min_{r, \tau_{Q_k|}^*} \quad \text{s.t.} \quad \tau^*_\ell \geq 0, \ell \in \mathbb{Z}^{|Q_k|}, \]
\[ \Delta Q_k(i) \leq \sum_{\ell=1}^{|Q_k|} \tau^*_\ell Q_k(\ell) + r I. \] (26c)
where \( \tau_{Q_k|}^* := \{ \tau^*_\ell \geq 0, \ell \in \mathbb{Z}^{|Q_k|} \} \). Some properties of the LMI problem above are stated in the following lemma.

**Lemma 3**: Suppose Assumptions 1, 2 and 4 hold, \( \Theta = \mathbb{R}^n \), and \( X = \Omega \). Let \( Q_k \) be defined in (19)-(20), and \( \Delta Q_k \) be given in (24) for all \( k \in \mathbb{Z}^+ \). The optimal of Problem (26) is denoted by \( r^*_k \) for all \( i \in \mathbb{Z}^{|\Delta Q_k|} \) and \( k \in \mathbb{Z}_+^* \). Then, for all \( i \in \mathbb{Z}^{|\Delta Q_k|} \), there exists a finite \( \tilde{k}_i \) such that \( r^*_k \leq 0 \) and \( r^*_k \leq 0 \) for all \( k \geq \tilde{k}_i \).

**Proof of Lemma 3**. The fact that there exists a finite \( \tilde{k}_i \) such that \( r^*_k \leq 0 \) for all \( i \in \mathbb{Z}^{\Delta Q_k} \) is a direct consequence of Lemma 2. Hence, we only need to show that \( r^*_k \leq 0 \) implies \( r^*_k \leq 0 \) for all \( k \geq \tilde{k}_i \). The proof goes by induction. First, we show that \( r^*_k \leq 0 \) implies \( r^*_{k+1} \leq 0 \) for any \( i \in \mathbb{Z}^{\Delta Q_k} \). Suppose \( r^*_k \leq 0 \) and the optimal solution is \((\tilde{r}^*_k, \tilde{\tau}^*_{Q_k|})\), we have
\[ \Delta Q_k(i) = (A_i^{k+1})^T Q_i A_i^{k+1} \leq \sum_{\ell=1}^{|Q_k|} \tilde{r}^*_\ell Q_k(\ell), \]
which implies that
\[ \Delta Q_{k+1}(i) = (A_i^{k+1})^T Q_i A_i^{k+1} \]
\[ = (A_i^{k+1})^T Q_i (A_i^{k+1})^T A_i^{k+1} \max_{|Q_k|} \]
\[ = \sum_{\ell=1}^{|Q_k|} \tilde{r}^*_\ell Q_k(\ell). \] (26b)
with \( \tilde{r}^*_{k+p} = \tilde{r}^*_k \) for all \( \ell \in \mathbb{Z}^{Q_k} \) and \( \tilde{r}^*_{k+p} = 0 \) for all \( \ell \in \mathbb{Z}^P \). This means that \((0, \tilde{\tau}^*_{Q_k|+1})\) is a feasible solution to (26) at \( k = \tilde{k}_i + 1 \). From the optimality, we can get \( r^*_{k+1} \leq 0 \). Finally, we conclude that \( r^*_k \leq 0 \) for all \( k \geq \tilde{k}_i \). \( \square \)

In the following theorem, we show that the LMI problem (26) can be used to establish a stopping criterion for the algorithm summarized in (10)-(11).

**Theorem 1**: Suppose Assumptions 1, 2 and 4 hold, \( \Theta = \mathbb{R}^n \), and \( X = \Omega \). Let \( O_k \) be defined by the procedure in (10)-(11), \( Q_k \) be defined in (19)-(20), and \( \Delta Q_k \) be given in (22) for all \( k \in \mathbb{Z}^+ \). For all \( i \in \mathbb{Z}^{\Delta Q_k} \) and \( k \in \mathbb{Z}^+ \), define \( r^*_k \) as in (26) and let \( r^*_k = \max_{i \in \mathbb{Z}^{|\Delta Q_k|}} r^*_k \). Then, there exists some finite \( k^* \) such that \( r^*_k \leq 0 \) and \( O_\infty = O_{k^*} \).

**Proof of Theorem 1**. From Lemma 3, there always exists some \( \tilde{k}_i \) such that \( r^*_k \leq 0 \) for all \( k \geq \tilde{k}_i \). For all \( i \in \mathbb{Z}^{\Delta Q_k} \), we can get \( O_{k^*+1} = O_{k^*} \). Finally, it holds that \( O_\infty = O_{k^*} \). \( \square \)

Based on the discussion above, the algorithm to compute the maximal \( CA\)-invariant set with quadratic constraints is summarized in Algorithm 1.

Since \( |Q_k| = (k+1)p \) and \( |\Delta Q_k| = p, k \in \mathbb{Z}^+ \), at the \( k \)-th iteration in Algorithm 1 we solve \( p \) LMI problems with \((k+1)p+1\) variables and one LMI constraint. As \( k \) increases, \( Q_k \) may have some redundant elements, which can be removed.
Algorithm 1 Computation of the maximal CA-invariant set with quadratic constraints

Input: $A$ and $\{Q_i, q_i\}_{i=1}^P$ as in (3)

Output: $O_k$

1: Initialization: let $X := \{x \in \mathbb{R}^n : x^TQ_ix + 2q_i^Tx \leq 1, i \in \mathbb{Z}^P\}$, set $k = 0$ and $O_0 = X$, and construct $Q_0$ as in (19):

2: Let $Q_{k+1}$ be updated according to (20).
3: Obtain $r_k^*$ from (26) for all $i \in \mathbb{Z}[Q_0]$.
4: Let $r_{k_{\max}} := \max_{i \in \mathbb{Z}[Q_0]} r_k^*$. If $r_{k_{\max}} \leq 0$, let $k^* = k$ and terminate; otherwise, let $O_{k+1} := O_k \bigcap \{x \in \mathbb{R}^n : Ax \in O_k\}$, set $k \leftarrow k + 1$ and go to Step 2.

using a similar formulation as (20):

\[
\begin{align*}
\min_{r} & \quad r \\
\text{s.t.} & \quad \tau_k^\ell \geq 0, \ell \in \mathbb{Z}[Q_0], \\
& \quad Q_k(i) \leq \sum_{\ell \neq i} \tau_k^\ell Q_k(\ell) + rI.
\end{align*}
\]

for any $i \in \mathbb{Z}[Q_0]$. If, for some $i \in \mathbb{Z}[Q_0]$ at the $k^{th}$ iteration, the optimal of the problem above is non-positive, then, $Q_k(i)$ is redundant and can be removed from $Q_k$.

As (15) is not directly solved, the $k^*$ obtained from Algorithm 1 is an upper bound on $k_{\min}$. For a loose upper bound $k^*$, the description of $O_{k^*}$ may not be tight enough though it is still true that $O_{k^*} = O_{\infty}$. However, in some cases, $k^*$ is not necessarily a loose upper bound. It can be close or equal to $k_{\min}$. One example is the case with only linear constraints, i.e., $\Theta = \mathbb{R}^n$ and $Q_i = 0$ for all $i \in \mathbb{Z}^P$. The proposition below shows that the $k^*$ obtained from Algorithm 1 is exactly equal to $k_{\min}$ in the case of linear constraints. Without quadratic constraints, Assumption 4 is not needed.

Proposition 1: Suppose Assumption 1 holds, $\Theta = \mathbb{R}^n$ and $Q_i = 0$ for all $i \in \mathbb{Z}^P$. The constraint set $X$ can be expressed as $\{x \in \mathbb{R}^n : 2q_i^Tx \leq 1\}$ without any nonlinear constraint. Let $\{r_{k_{\max}}^i, O_k\}$ be generated by Algorithm 1. For any $i \in \mathbb{Z}_0^P$, $r_{k_{\max}}^i \leq 0$ if and only if $O_{k+1} = O_k$.

From Proposition 1 we can see that Algorithm 1 is eventually equivalent to the standard algorithm for linear systems with linear constraints. Generally speaking, the conservatism of $k^*$ obtained from Algorithm 1 depends on the conservatism of the S-procedure in Lemma 1. If the LMI (24) is a necessary and sufficient condition of the set inclusion in (13), the S-procedure is lossless and $k^*$ is exactly equal to $k_{\min}$. However, for general quadratic constraints, this is not true. A detailed discussion on the conservatism of S-procedure can be found in (33). More precisely, $k^*$ can be larger than $k_{\min}$ in most of the cases. However, the size of the resulting $O_{\infty}$ is not affected although there are redundant constraints in the description of the set. With Assumption 4 another possibility to determine a $k$ that satisfies (14) is to find a $k$ such that $A^{k+1}x$ enters an open ball inside $X$ for any $x \in \{x : \|x\|^2 \leq D_x\}$. However, this is usually very conservative and such a $k$ can be much larger than the $k^*$ obtained from Algorithm 1.

B. Quasi-smooth nonlinear constraints

In the rest of this section, the proposed approach will be generalized to handle general nonlinear constraints that satisfy Assumption 3. This is possible by making use of the quadratic upper and lower bounds in (36). With these quadratic bounds, we are able to establish the quadratic relaxations of (16) and apply the idea above. For notational simplicity, let

\[
H_i^u(x) = H_i(0) + (H_i^V)^T x + \frac{L_i}{2} x^2
\]

for all $i \in \mathbb{Z}^m$. Similar to (19)-(20), we define:

\[
H_i^l(x) = H_i(0) + (H_i^V)^T x - \frac{L_i}{2} x^2
\]

The sets $\{\mathcal{H}_i^u\}$ and $\{\mathcal{H}_i^l\}$ are updated differently because $\{\mathcal{H}_i^l\}$ is used in the cost function while $\{\mathcal{H}_i^u\}$ is used in the constraints as we will see soon. With additional definitions above, a relaxed quadratic constraint set of $O_k$ can be obtained for all $k \in \mathbb{Z}_0^+$:

\[
\tilde{O}_k := \{x : \begin{pmatrix} x^T \tilde{Q} & x \end{pmatrix} \leq 0, \tilde{Q} \in Q_k \bigcup \mathcal{H}_k^l\}
\]

Based on this relaxed constraint set, a modification of (15) is given by

\[
\tilde{g}_i^k := \max(A^{k+1}x)^T Q_i A^{k+1} x + 2q_i^T A^{k+1} x - 1
\]

s.t. $x \in \tilde{O}_k$

for any $i \in \mathbb{Z}^P$ and $k \in \mathbb{Z}_0^+$. As $O_k \subseteq \tilde{O}_k$, $\tilde{g}_i^k \geq g_i^k$ for all $i \in \mathbb{Z}^P$ and $k \in \mathbb{Z}_0^+$. Similarly, we can also modify (16) using the relaxed set. Since the cost function of (16) is also nonlinear, we will replace it by its quadratic upper bound (28). With the relaxed set and the quadratic upper bound of the cost function, the corresponding modification of (16) is given by

\[
\tilde{h}_i^k := H_i^u(A^{k+1}x) - 1
\]

s.t. $x \in \tilde{O}_k$

for all $i \in \mathbb{Z}^m$. Again, we can see that $\tilde{h}_i^k \geq h_i^k$ for all $i \in \mathbb{Z}^m$ and $k \in \mathbb{Z}_0^+$. Using the S-procedure, the following lemma can be obtained immediately.

Lemma 4: Suppose Assumption 3 holds. Let the set $O_k$ be defined by the procedure in (10)-(11) and the relaxed quadratic set $\tilde{O}_k$ be defined in (34) using the quadratic lower bounds (29) for all $k \in \mathbb{Z}_0^+$. Consider the sets $\{Q_k, \mathcal{H}_k^u, \mathcal{H}_k^l\}$ defined in (19)-(20) and (30)-(33), the following results hold.
Proof of Lemma 5: The proof follows the same arguments in (ii) Similarly, from the S-procedure, (39) implies that holds, then, for any \( x \in O_k \).

Proof of Lemma 4: (i) An immediate consequence of the S-procedure is that for any \( x \in O_k \) if (37) holds for some \( k \in Z^+ \). Taking into account that \( O_k \subseteq \tilde{O}_k \), property (i) hold true.

(ii) Similarly, from the S-procedure, (39) implies that for any \( x \in \tilde{O}_k \). Since \( H_i(A^{k+1}x) \leq H_i(A^{k+1}x) \) for any \( x \in \Omega \) and \( O_k \subseteq \tilde{O}_k \), property (ii) is proved. □

From the lemma above, we can see that it is also possible to implement the formal algorithm in (10)-(11) using the LMIs in (37)-(39) for general nonlinear constraints that satisfy Assumption 3. The finite termination of the algorithm is discussed in the next lemma.

Lemma 5: Suppose Assumptions 1-4 hold and consider the relaxed quadratic set \( \tilde{O}_k \) defined in (34) using the quadratic lower bounds (29) for all \( k \in Z^+ \). Consider the sets \( \{ Q_k, H_k^u, H_k^f \} \) defined in (19)-(20) and (30)-(33) for all \( k \in Z^+ \), the following results hold.

(i) For any \( i \in Z^{|\Delta Q_k|} \), there exists some finite \( k \) such that (37) holds for some non-negative sequences \( \pi^i_{Q_k} := \{ \tau^i_\ell \geq 0, \ell \in Z^{ |Q_k| } \} \) and \( \pi^i_{|H^u_k|} := \{ \pi^i_\ell \geq 0, \ell \in Z^{ |H^u_k| } \} \).

(ii) For any \( i \in Z^{|H^u_k|} \), there exists some finite \( k \) such that (39) holds for some non-negative sequences \( \pi^i_{Q_k} := \{ \tau^i_\ell \geq 0, \ell \in Z^{ |Q_k| } \} \) and \( \pi^i_{|H^u_k|} := \{ \pi^i_\ell \geq 0, \ell \in Z^{ |H^u_k| } \} \).

Proof of Lemma 5: The proof follows the same arguments in Lemma 4 and thus is omitted. □

Based on Lemma 5, we can define LMI problems for both quadratic and nonlinear constraints. For the quadratic constraints, let us define:

\[
\begin{align*}
\tau^*_k := \min_{r, \tau^i_{Q_k}, \pi^i_{|H^u_k|}} r \\
\text{s.t.} \quad \tau^*_k \geq 0, \ell \in Z^{ |Q_k| }, \\
\pi^*_k \geq 0, \ell \in Z^{ |H^u_k| }, \\
\Delta Q_k(i) \leq \sum_{\ell=1}^{M} \tau^*_k Q_k(\ell) + \sum_{\ell=1}^{M} \pi^*_k H^u_k(\ell) + rI 
\end{align*}
\]

for all \( i \in Z^{|\Delta Q_k|} \) and \( k \in Z^+_0 \). For the nonlinear constraints, let us define:

\[
\begin{align*}
\tilde{r}^*_k := \min_{r, \tau^i_{Q_k}, \pi^i_{|H^u_k|}} & \tilde{r} \\
\text{s.t.} \quad \tau^*_k \geq 0, \ell \in Z^{ |Q_k| }, \\
\pi^*_k \geq 0, \ell \in Z^{ |H^u_k| }, \\
H^u_{k+1}(i) \leq \sum_{\ell=1}^{M} \tau^*_k Q_k(\ell) + \sum_{\ell=1}^{M} \pi^*_k H^u_k(\ell) + rI 
\end{align*}
\]

for all \( i \in Z^{|H^u_{k+1}|} \) and \( k \in Z^+_0 \). Some properties of these LMI problems are given in the following lemma.

Lemma 6: Suppose Assumptions 1-4 hold. Let the sets \( \{ Q_k, H^u_k, H^f_k \} \) be defined in (19)-(20) and (30)-(33) for all \( k \in Z^+_0 \). The LMI problems defined in (40) and (41) have the following properties.

(i) For all \( i \in Z^{|\Delta Q_k|} \) and \( k \in Z^+_0 \), let \( r^*_k \) be defined in (40), then there exists a finite \( k^*_i \) such that \( r^*_k \leq 0 \) and \( r^*_k \leq 0 \) for all \( k \geq k^*_i \).

(ii) For all \( i \in Z^{|H^u_{k+1}|} \) and \( k \in Z^+_0 \), let \( \tilde{r}^*_k \) be defined in (41), then there exists a finite \( k^*_i \) such that \( \tilde{r}^*_k \leq 0 \) and \( \tilde{r}^*_k \leq 0 \) for all \( k \geq k^*_i \).

Proof of Lemma 6: The proof follows the same arguments in Lemma 5 and hence is omitted. □

Based on Lemmas 4-6, the algorithm for computing the maximal CA-invariant set with nonlinear constraints is summarized in Algorithm 2. At each iteration \( k \) of Algorithm 1 for \( k \in Z^+_0 \), we solve \( p + m \) LMI problems with \( (k+1)(p+m) + 1 \) variables and one LMI constraint. Similar to Algorithm 1, Algorithm 2 will also terminate after a finite time as stated in Theorem 2.

Theorem 2: Suppose Assumptions 1-4 hold, let \( \{ k_{\max}^*, r_{\max}^* \} \) be generated from Algorithm 2. Then, there exists some finite \( k^* \) such that \( r_{\max}^* \leq 0 \) and \( r_{\max}^* \leq 0 \) and \( O_{\infty} = \tilde{O}_k \).

Proof of Theorem 2: From property (i) of Lemma 6 we know that there exists a finite \( k^*_i \) such that \( r^*_k \leq 0 \) for all \( k \leq k^*_i \) for all \( i \in \mathbb{Z}^p \). Similarly, from property (ii) of Lemma 6 we know that then there exists a finite \( k^*_i \) such that \( \tilde{r}^*_k \leq 0 \) for all \( k \leq k^*_i \) for all \( i \in \mathbb{Z}^m \). Hence, for any \( k \geq \max_{i \in \mathbb{Z}^p} \{ k^*_i \} \), \( r_{\max}^* \leq 0 \); for any \( k \geq \max_{i \in \mathbb{Z}^m} \{ k^*_i \} \), \( \tilde{r}^*_{\max} \leq 0 \). Let \( k^* = \max\{ \max_{i \in \mathbb{Z}^p} \{ k^*_i \}, \max_{i \in \mathbb{Z}^m} \{ k^*_i \} \} \), we can easily see that \( r_{\max}^* \leq 0 \) and \( r_{\max}^* \leq 0 \). □
Algorithm 2 Computation of the maximal constraint admissible invariant set with nonlinear constraints

**Input:** \( A_i, \{ Q_i, q_i \}_{i=1}^P \), and \( \{ H_i(x), H_i^\top, L_i \}_{i=1}^m \)

**Output:** \( O_k \).

1. **Initialization:** let \( X := \{ x \in \mathbb{R}^n : (x)^T Q_i x + 2q_i^T x \leq 1, i \in \mathbb{Z}^p, H(x) \leq 0 \} \), set \( k = 0 \) and \( O_k = X \). Construct \( Q_i, H_i^\top \), and \( H_i \) as in (19), (30), and (31) respectively;
   2. **Update** \( Q_{k+1}, H_{k+1}^\top \), and \( H_{k+1} \) according to (20), (32) and (33) respectively;
   3. Obtain \( x^k \) from (40) for all \( i \in \mathbb{Z}^m \Delta Q_i \{ \}
   4. Obtain \( r^k \) from (41) for all \( i \in \mathbb{Z}^m H_i \{ \}
   5. Let \( r^k \) and \( z^k \) be the maximal \( \max z \) and \( \sum z \) respectively. If \( r^k \) and \( z^k \) are both even and \( z^k \leq 0 \), let \( O_{k+1} := O_k \) and terminate; otherwise, let \( O_{k+1} := O_k \) and go to Step 2.

IV. SEMI-ALGEBRAIC CONSTRAINTS AND SPECIAL NONLINEAR SYSTEMS

In this section, we discuss in particular semi-algebraic constraints and certain types of nonlinear systems.

A. Semi-algebraic constraints

We consider the case when \( \Theta \) is a semi-algebraic constraint set and \( \{ H_i(x) \}_{i=1}^m \) are polynomial functions of the maximal degree \( d \). Since quadratic constraints are handled separately, we assume that \( d \geq 3 \). This is a special class of nonlinear constraints that satisfy Assumption 3 with \( H_i^\top = \nabla H_i(0) \) being the Lipschitz constant in \( \Omega \) for all \( i \in \mathbb{Z}^m \). Although semi-algebraic constraints can be handled by Algorithm 2, the Lipschitz constants \( \{ L_i \}_{i=1}^m \) can be conservative for high-order polynomial functions. For this reason, we present an alternative method for handling semi-algebraic constraints. In [26], a lifting method is used to convert semi-algebraic constraints into linear constraints. In this paper, we use a similar lifting method that converts semi-algebraic constraints into quadratic constraints. For the same degree \( d \), the dimension of the lifted space in our method will be lower than the one used in [26].

The lifting method is described as follows. For any \( x \in \mathbb{R}^n \) and \( i \in \mathbb{Z}^+ \), let \( x[i] \in \mathbb{R}^{d+1} \) denote the vector of all the monomials of degree \( i \) and \( A[i] : x[i] \rightarrow (Ax)[i] \) denote the lifted linear map of System (1). With these definitions, the polynomial functions \( \{ H_i(x) \}_{i=1}^m \) can be rewritten as

\[
H_i(x) = \begin{pmatrix} x[1] \\ x[2] \\ \vdots \\ x[d] \end{pmatrix}^T P_i \begin{pmatrix} x[1] \\ x[2] \\ \vdots \\ x[d] \end{pmatrix} + 2F^T x \tag{42}
\]

where \( d = \lceil d/2 \rceil \), \( P_i \in \mathbb{R}^{N \times N} \), and \( F_i \in \mathbb{R}^n \) with \( N = \sum_{t=1}^{d} (n+t-1) \). The lifted system becomes

\[
z(t+1) = \tilde{A}z(t), \quad t \in \mathbb{Z}_0^+
\]

where \( z \in \mathbb{R}^N \) and \( \tilde{A} = \text{diag}(A[1], A[2], \ldots, A[d]) \in \mathbb{R}^{N \times N} \). From [26], [37], \( \tilde{A} \) is also Schur stable if \( A \) is Schur stable.

Depending on the polynomial functions \( \{ H_i(x) \}_{i=1}^m \), we may only need a subset of \( \{ x[1], x[2], \ldots, x[d] \} \) in the expression (42). Hence, the lower and upper bounds on the dimension of the lifted system are \( (n+d-1) \) and \( \sum_{t=1}^{d} (n+t-1) \) respectively, while the lower and upper bounds in [26] are \( (n+d-1) \) and \( \sum_{t=1}^{d} (n+t-1) \) respectively. The quadratic expression in (42) allows us to significantly reduce the dimension of the lifted space. In fact, it can be verified that our upper bound \( \sum_{t=1}^{d} (n+t-1) \) is even much smaller than the lower bound \( (n+d-1) \) in [26] when \( n > 2 \), as depicted in Figure 1.

![Fig. 1: Lower and upper bounds on the dimension of the lifted space for different values of \( n \) and \( d \).](image)

In the rest of this section, for ease of discussion and notational simplicity, we consider the whole vector \( (x[1], x[2], \ldots, x[d]) \). As a result, the original quadratic constraints in (3) can be expressed as

\[
z^T [I_n, 0]^T Q_i [I_n, 0] z + 2q_i^T [I_n, 0] z \leq 1, \quad i \in \mathbb{Z}^p, \tag{44}
\]

and the semi-algebraic constraints in (3) become

\[
z^T P_i z + 2F^T [I_n, 0] z \leq 1, \quad i \in \mathbb{Z}^m. \tag{45}
\]

Since \( \Omega \) is bounded under Assumption 2 without loss of generality, we can always add the redundant constraint of the form \( \| z \|^2 \leq D_z \) for some sufficiently large \( D_z > 0 \) such that \( \| (x[1], x[2], \ldots, x[d]) \|^2 \leq D_z \) for all \( x \in \Omega \). Hence, the overall constraint set of the lifted system can be expressed as

\[
X_z := \{ z \in \mathbb{R}^N : (44), (45), \text{ and } 1/D_z \| z \|^2 \leq 1 \} \tag{46}
\]

Then, we can use Algorithm 1 to compute the maximal CA-invariant set of the lifted system, denoted by \( O_\infty \). The following proposition shows that the maximal CA-invariant set of the original system can be characterized by \( O_\infty \).

Proposition 2: Suppose Assumptions 1 and 2 hold and \( \{ H_i(x) \}_{i=1}^m \) are polynomial functions of the maximal degree \( d \). Let \( O_\infty \) be the maximal CA-invariant set of System (1) with the constraint set \( X \) in (5) and \( O_z \) be the maximal CA-invariant set of the lifted system (44) with the constraint set \( X_z \) in (46).

**Proof of Proposition 3** First, we show that \( O_\infty \subseteq \{ x \in \mathbb{R}^n : (x[1], x[2], \ldots, x[d]) \in O_z \} \). For any \( x \in O_\infty \), we know that \( A^k x \in \mathbb{R}^n \) for all \( k \in \mathbb{Z}_0^+ \). From the definition of the lifted system in (43) and (45), \( (A^k x)[1], (A^k x)[2], \ldots, (A^k x)[d] = \)
As shown in (24), (26), the maximal CA-invariant set of System (49) exists if \( \rho(A) < 1 \) and Assumptions 2 and 3 hold. For its computation, we need to adjust the procedure in (10)-(11) as follows:

\[
O_0 := X
\]
\[
O_{k+1} := O_k \cap \{ x \in \mathbb{R}^n : Ax \in O_k, A \in \mathcal{A} \}, k \in \mathbb{Z}_n^+
\]

Let \( \mathcal{A} := \{ \tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_M \} \) with

\[
\tilde{A}_i = \begin{pmatrix} A_i & 0 \\ 0 & 1 \end{pmatrix}, \forall i \in \mathbb{Z}_M.
\]

The update in (19)-(20) becomes

\[
Q_0 := \{ \tilde{Q}_i, i \in \mathbb{Z}^p \}
\]
\[
Q_{k+1} := Q_k \cup \{ \tilde{A}^T \tilde{Q} \tilde{A} : \tilde{Q} \in Q_k, \tilde{A} \in \mathcal{A} \}, k \in \mathbb{Z}_n^+
\]

with \(|Q_k| = \sum_{\ell=0}^k M^\ell p\). Similarly, the update in (30)-(32) is also adjusted as follows:

\[
\mathcal{H}_0^u = \left\{ \begin{pmatrix} \frac{\ell_2}{2} I \bar{H}_0 \end{pmatrix}^T \bar{H}_0^u \right\}, i \in \mathbb{Z}^m
\]
\[
\mathcal{H}_0^l = \left\{ \begin{pmatrix} -\frac{\ell_1}{2} I \bar{H}_0^l \end{pmatrix}^T \bar{H}_0 - 1 \right\}, i \in \mathbb{Z}^m
\]
\[
\mathcal{H}_{k+1}^u := \{ \tilde{A}^T \tilde{Q} \tilde{A} : \tilde{Q} \in \mathcal{H}_k^u, \tilde{A} \in \mathcal{A} \}, k \in \mathbb{Z}_n^+
\]
\[
\mathcal{H}_{k+1}^l := \mathcal{H}_k^l \cup \{ \tilde{A}^T \tilde{Q} \tilde{A} : \tilde{Q} \in \mathcal{H}_k^l, \tilde{A} \in \mathcal{A} \}, k \in \mathbb{Z}_n^+
\]

with \(|\mathcal{H}_k^u| = M^k m\) and \(|\mathcal{H}_k^l| = \sum_{\ell=0}^k M^\ell m\). Due to the complications arise from the switched system, at each iteration \( k \in \mathbb{Z}_n^+ \), we need to solve \( M^{k+1} p \) LMI problems with \( \sum_{\ell=0}^k M^\ell p \) variables at Algorithm 1 and \( M^{k+1} (p + m) \) LMI problems with \( \sum_{\ell=0}^k M^\ell (p + m) \) +1 variables at Algorithm 2. In this case, as \( k \) increases, it becomes necessary to remove redundancy using the formulation in (27).

### C. Special nonlinear systems

The proposed approach can be also extended to other special nonlinear systems. Consider the following nonlinear system

\[
x(t + 1) = f(x(t)), \forall t \in \mathbb{Z}_n^+
\]

where \( x(t) \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \to \mathbb{R}^n \) is continuous with \( f(0) = 0 \). The state is subject to

\[
x(t) \in X := \{ x \in \mathbb{R}^n : H_i(x) \leq 1, i \in \mathbb{Z}^m \}, \forall t \in \mathbb{Z}_n^+.
\]

In the case of nonlinear systems, quadratic constraints are also included in (62). Similar to the linear case, the following assumptions are made.

**Assumption 5:** System (61) is asymptotically stable at the origin in \( X \) and \( f : \mathbb{R}^n \to \mathbb{R}^n \) is continuous with \( f(0) = 0 \).

**Assumption 6:** For all \( i \in \mathbb{Z}^m \), \( H_i : \mathbb{R}^n \to \mathbb{R} \) is a continuous function with \( H_i(0) = 0 \). In addition, \( X \) is compact.
The maximal CA-invariant set of nonlinear systems can be defined in a similar way as shown in Section III although the computation is more complicated and difficult. Let the maximal CA-invariant set of system (61) be denoted by $O_{∞}^{nl}$, the same iterates can be used to compute $O_{k}^{nl}$.

$$O_{0}^{nl} := X$$

$$O_{k+1}^{nl} := O_{k}^{nl} \cap \{x \in \mathbb{R}^{n} : f(x) \in O_{k}^{nl}, k \in \mathbb{Z}^{+}\}$$

With Assumptions 5 and 6, the existence of $O_{∞}^{nl}$ can be guaranteed and the algorithm above terminates in a finite time, similar to the linear case in Theorem 4.1 in [6]. This is formally stated in the following proposition.

**Proposition 4:** Suppose Assumptions 5 and 6 hold. Let $O_{k}^{nl}$ be defined in (63)-(64) for any $k \in \mathbb{Z}^{+}$. The following properties hold: (i) $O_{∞}^{nl}$ exists and is nonempty. (ii) There exists a finite $k^{*}$ such that $O_{k}^{nl} = O_{k^{*}}^{nl}$ for all $k \geq k^{*}$ and $O_{∞}^{nl} = O_{k^{*}}^{nl}$, (iii) For any $k \in \mathbb{Z}^{+}, O_{k}^{nl}$ is compact and contains the origin in its interior.

Proof of Proposition 4: The proof is adapted from the proof of Theorem 4.1 in [6]. (i) This property holds trivially since $0 \in O_{∞}^{nl}$. (ii) From Assumptions 5, there exists a $k^{*}$ such that $f^{k^{*}}(x) \in X$ for any $x \in X$. We claim that $O_{k^{*}}^{nl}$ is an invariant set of System (61). We have to show that for any $x' \in O_{k^{*}}^{nl}$, $f(x') \in O_{k^{*}}^{nl}$. From the definition of $O_{k^{*}}^{nl}$, we can see that $x \in O_{k^{*}}^{nl}$ implies $f^{k}(x) \in X$ for all $k \in \mathbb{Z}^{+}$. As the system is time-invariant, we know that $f^{k}(f(x')) \in X$ for $k \in \mathbb{Z}^{+}$. From the fact that $f^{k}(x) \in X$ for any $x \in X$, we can see that $f^{k}(f(x')) \in X$, which implies that $f(x') \in O_{k^{*}}^{nl}$. This means that $O_{k^{*}}^{nl}$ is an invariant set and $O_{k}^{nl} = O_{k^{*}}^{nl}$, (iii) From Assumptions 5 and 6, it can be shown that $O_{k}^{nl}$ is closed and bounded for any finite $k \in \mathbb{Z}^{+}$. According to the Heine-Borel theorem, they are also compact. From the continuity of the function $f(x)$, there always exists an open ball $B \in O_{k}^{nl}$ with $0 \in B$ for any finite $k \in \mathbb{Z}^{+}$. This completes the proof. □

Even though the existence of $O_{∞}^{nl}$, computing the exact $O_{∞}^{nl}$ can be very challenging for general nonlinear systems, even when the nonlinear constraints satisfy Assumption 3. For this reason, we only consider a class of nonlinear systems that can be linearized by state transformation, see, e.g. 40-43. An additional assumption is made on the linearizability of System (61).

**Assumption 7:** There exists a diffeomorphism $T : \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ such that System (61) can be transformed into a linear system

$$y(t + 1) = Ay(t), \forall t \in \mathbb{Z}^{+}$$

for some $A \in \mathbb{R}^{n \times n}$, $y(t) = T(x(t))$, with $T(0) = 0$ and $f(x(t)) = T^{-1}(AT(x(t)))$.

An example of nonlinear systems that satisfy Assumption 7 will be given in the next section. The linearized system (65) is subject to the following constraints

$$y(t) \in Y := T(X), \forall t \in \mathbb{Z}^{+}$$

with $T(X) = \{y \in \mathbb{R}^{n} : H_{i}(T^{-1}(y)) \leq 0, i \in \mathbb{Z}^{m}\}$. With the state transformation, it is possible to compute the maximal CA-invariant of System (61) by computing the maximal CA-invariant set of the linearized system (65). Let $O_{∞}^{Y}$ denote the maximal CA-invariant set of the linearized system (65). Suppose $Y$ satisfies Assumption 3, $O_{∞}^{Y}$ can be computed using Algorithm 2. The equivalence between the invariant sets of System (61) and System (65) can be easily established. In this case, $Y$ will often satisfy Assumption 3 (when $T^{-1}(y)$ is continuously differentiable with Lipschitz gradient), although it is not guaranteed.

V. ILLUSTRATIVE EXAMPLES

**Example 1:** We consider the linear system studied in [26] Example 1 with $A = [1.0216 \ 0.3234; -0.6597 \ 0.5226]$. The constraint set is the unit circle given by $\Omega_{1} := \{x \in \mathbb{R}^{2} : x^T x \leq 1\}$ and $\Theta = \mathbb{R}^{n}$. Algorithm 1 is used to obtain the maximal CA-invariant set and the result is given in Figure 2. It can be seen from Figure 2 that Algorithm 1 takes 3 iterations to obtain this set. For the same setting, the algorithm in [26] takes 6 iterations.

![Fig. 2: The maximal CA-invariant set $O_{∞}(O_{6})$ of Example 1 with $\Omega = \Omega_{1}$ and $\Theta = \mathbb{R}^{n}$.](image)

We consider the same dynamical system in Example 1 with additional quadratic constraints. Let the quadratic constraint set be $\Omega_{2} := \{x \in \mathbb{R}^{2} : x_{1}^{2} - x_{2}^{2} + 0.4x_{1}x_{2} \leq 1, (x_{1} + 0.5)^2 + x_{2} \geq \frac{1}{16}, (x_{1} - 0.5)^2 + x_{2} \geq \frac{1}{16}\}$. Note that there are 4 quadratic constraints and that this set is nonconvex. Again, we use Algorithm 1 to compute the maximal CA-invariant set and it can be obtained within 8 iterations as shown in Figure 3. Trajectories are also shown to verify the discrete invariance of the disconnected regions.

![Fig. 3: The maximal CA-invariant set of Example 1 with $\Omega = \Omega_{2}$ and $\Theta = \mathbb{R}^{n}$: (a) shows the set $\Omega$, and (b) shows the maximal CA-invariant set $O_{∞}(O_{6})$.](image)
Additionally, we also consider a nonlinear constraint, which is beyond the class of constraints that the approach in \cite{26} is able to handle. Let $\Theta = \Theta_1 := \{x \in \mathbb{R}^2 : H_1(x) := \sqrt{x_1^2 + x_2^2 + 1 + 2x_1 + 2x_2 - 2} \leq 0\}$. It is easy to verify that Assumption 3 is satisfied with $H_1^\ast = [2 \ 2]^T$ and $L_1 = 1$. Using Algorithm 2 the maximal CA-invariant set can be obtained within 8 iterations as shown in Figure 4.

\[ \begin{align*}
\Omega = \Theta = \Theta_2 \text{ and } \Theta = \Theta_1; \quad \text{(a) shows the set } \Omega \cap \Theta, \text{ and (b) shows the maximal CA-invariant set } O_{\infty}(O_0). 
\end{align*} \]

Fig. 4: The maximal CA-invariant set of Example 1 with $\Omega = \Omega_2$ and $\Theta = \Theta_1$: (a) shows the set $\Omega \cap \Theta$, and (b) shows the maximal CA-invariant set $O_{\infty}(O_0)$.

**Example 2:** We consider an autonomous Wiener system, which consists of a linear system followed by a nonlinear static system (see \cite{44} for details on autonomous Wiener systems), as shown in Figure 5 with $A = [0.5 \ 0.7; -0.7 \ 0.5]$, $C = [1 \ -1]$ and $g(v) = v + v^2 + v^3 - v^4$. The constraints are given by: $\Omega = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\}$ and $\Theta = \{x \in \mathbb{R}^2 : -2 \leq g(Cx) \leq 2\}$.

\[ \begin{align*}
x(t + 1) &= Ax(t) \\
v(t) &= Cx(t) \\
g(v(t)) &= y(t)
\end{align*} \]

Fig. 5: A discrete-time autonomous Wiener model

The output $g(Cx)$ can be rewritten as

\[ g(Cx) = \begin{pmatrix} x_1 \\ x_2 \\ x_1x_2 \\ x_1^2 \\ x_2^2 \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \\ x_1x_2 \\ x_1^2 \\ x_2^2 \end{pmatrix} + 2F^Tx \quad (67) \]

with

\[ P = \begin{pmatrix} 1 & -1 & -1.5 & 0.5 & 1.5 \\ -1 & 1 & 0 & 0 & -0.5 \\ -1.5 & 0 & -6 & 2 & 2 \\ 0.5 & 0 & 2 & -1 & 0 \\ 1.5 & -0.5 & 2 & 0 & -1 \end{pmatrix} \]

\[ F = [0.5 \ -0.5]^T. \] The lifted system $\hat{A}$ in \cite{43} becomes

\[ \hat{A} = \begin{pmatrix} 0.5 & 0 & 0 & 0 & 0 \\ -0.7 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & -0.24 & -0.35 & 0.35 \\ 0 & 0 & 0.7 & 0.25 & 0.49 \\ 0 & 0 & -0.7 & 0.49 & 0.25 \end{pmatrix}. \]

With the inequality $x_1^2 + x_2^2 \leq 4$, it can be easily verified that

\[ \begin{pmatrix} x_1 \\ x_2 \\ x_1x_2 \\ x_1^2 \\ x_2^2 \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \\ x_1x_2 \\ x_1^2 \\ x_2^2 \end{pmatrix} \leq 20 \]

Then, the constraint set for the lifted system is $X = \{z \in \mathbb{R}^5 : z^T[I_2 \ 0]^T[I_2 \ 0]z \leq 4, z^TPz + 2F^T[I_2 \ 0]z \leq 2, -z^TPz - 2F^T[I_2 \ 0]z \leq 2, z^Tz \leq 20\}$. Finally, the maximal CA-invariant set $O_{\infty}$ can be obtained using Algorithm 1 which terminates at $k = 7$. According to Proposition 3, the maximal CA-invariant set of the original system can be given by $O_{\infty} = \{x \in \mathbb{R}^2 : (x_1, x_2, x_1x_2, x_1^2, x_2^2) \in O_{\infty}\}$, which is shown in Figure 6. Again, a trajectory is given to verify set invariance of the disconnected regions.

\[ \begin{align*}
\end{align*} \]

Fig. 6: The maximal CA-invariant set $O_{\infty}$ of the Wiener system.

**Example 3:** Now, we evaluate the proposed approach on systems of different sizes. Consider a switched linear system \cite{49} with $A = \{A_1, A_2\}$, which are randomly generated. To make sure that $\rho(A) < 1$ is satisfied, we first generate matrices $A_1$ and $A_2$ whose elements are sampled independently and identically from the uniform distribution between $-1$ and $1$. Then, we compute the JSR $\rho(\{\hat{A}_1, \hat{A}_2\})$ (or an upper bound) using the JSR toolbox \cite{45}. Finally, we let

\[ \begin{align*}
A_1 &= \frac{A_1}{\rho(\{\hat{A}_1, \hat{A}_2\}) + \epsilon}, \quad A_2 &= \frac{\hat{A}_2}{\rho(\{\hat{A}_1, \hat{A}_2\}) + \epsilon},
\end{align*} \]

where $\epsilon > 0$. With this choice of $\{A_1, A_2\}$, the condition that $\rho(A) < 1$ can be ensured for any $\epsilon > 0$. In the simulation, we set $\epsilon = 0.1$. The constraint set is given by $X = \{x \in \mathbb{R}^n : x^TQ_aox + 2q_b^Tx \leq 1, x^TQ_ox + 2q_b^Tx \leq 1\}$, where the symmetric matrices $Q_a, Q_b \in S^n$ and the vectors $q_a, q_b \in \mathbb{R}^n$ are also randomly generated. We then use Algorithm 1 with the modifications in (51)-(52) and (53)-(55) to compute $O_{\infty}$. Let $k^*$ denote the number of iterations and $\nu$ denote the number of constraints in the expression of $O_{\infty}$ (or equivalently $O_{\infty}$) after removing redundancy by solving (27). The comparison with the lifting approach in \cite{26} is made in terms of these two indices. Similarly, let $k^*_L$ and $\nu_L$ denote the number of iterations and the number of constraints respectively in \cite{26}. Note that the dimension of the lifted space is $\frac{(n+3)n}{2}$ while our approach does not have to lift the system as the constraints are quadratic. The difference will even be more significant as $n$ and $d$ increase, see Figure 4. We take 20 realizations of the
dynamics and the constraints and compute the mean values of $k^*, \nu, k^*_L$ and $\nu_L$. The results are shown in Table I. When $n > 5$, the approach in [26] is not conducted as it takes too much time. As we can see in Table I the proposed approach converges faster and produces a better expression of $O_{\infty}$ with a smaller number of constraints.

| $n$ | mean($k^*$) | mean($\nu$) | mean($k^*_L$) | mean($\nu_L$) |
|-----|-------------|-------------|---------------|--------------|
| 2   | 2.25        | 6.65        | 6.45          | 30.65        |
| 3   | 3.85        | 15.1        | 7.9           | 70.6         |
| 4   | 5.55        | 29.5        | 11.45         | 208.15       |
| 5   | 6.55        | 40.2        | 12.55         | 328.25       |
| 6   | 7.85        | 64.65       | -             | -            |
| 10  | 9.6         | 130.65      | -             | -            |
| 20  | 13.4        | 467.15      | -             | -            |

TABLE I: Comparison simulation for Example 3 of different sizes with 20 realizations.

Example 4: In the rest of this section, we consider the following nonlinear system

$$
\begin{align*}
x_1(t+1) &= 2(x_1(t))^2 + x_2(t), \\
x_2(t+1) &= -2(2(x_1(t))^2 + x_2(t))^2 - 0.8x_1(t),
\end{align*}
$$

(68)

The state constraint set is given by $X := \{x \in \mathbb{R}^2: |x_1| \leq 1, |x_2| \leq 1\}$. There exists a diffeomorphism $y = T(x)$,

$$
T(x) = \begin{pmatrix}
x_1 \\
2x_1^2 + x_2
\end{pmatrix},
$$

(69)

such that the nonlinear system can be linearized into

$$
y(t+1) = \begin{pmatrix}
0 \\
-0.81
\end{pmatrix} y(t)
$$

(70)

With the state transformation $T(x)$, the state constraint set of the linearized system can be given by $Y := \{y \in \mathbb{R}^2: |y_1| \leq 1, y_2 - 2y_1^2 \leq 1, 2y_1^2 - y_2 \leq 1\}$. As a result, we get a linear system with quadratic constraints and the constraint set $Y$ is bounded. Using Algorithm I the maximal $CA$-invariant set of the linearized system can be computed and it takes 3 iterations. The set is shown in Figure 7.

![Figure 7](image_url)

Fig. 7: The maximal $CA$-invariant set of the linearized system of Example 4 (a) shows the set $Y$ and (b) shows the maximal $CA$-invariant set $O_{\infty}$.

Using the inverse mapping $x = T^{-1}(y)$,

$$
T^{-1}(y) = \begin{pmatrix}
y_1 \\
y_2 - 2y_1^2
\end{pmatrix},
$$

(71)

the maximal $CA$-invariant set of the original nonlinear system can be obtained and is shown in Figure 8.

![Figure 8](image_url)

Fig. 8: The maximal $CA$-invariant set of Example 4 (a) shows the set $X$ and (b) shows the maximal $CA$-invariant set $O_{\infty}$.

VI. Conclusions

We have studied the exact computation of the maximal $CA$-invariant set of linear systems, switched linear systems and some special nonlinear systems subject to a class of non-convex constraints that admit quadratic lower and upper bounds. By the use of these quadratic bounds, we have derived a sufficient condition for set invariance, which can be expressed as a set of LMIs. Based on this sufficient condition, a new algorithm is presented by solving a number of convex problems with only one LMI constraint at every iteration. Under mild assumptions, finite convergence to the exact maximal $CA$-invariant set can be guaranteed. To illustrate the proposed algorithm, we have presented several numerical examples and made comparison with an existing approach, which is capable of computing the exact maximal $CA$-invariant set of switched linear systems subject to semi-algebraic constraints. For the same setting, we show that our approach converges faster with a tighter expression of the maximal $CA$-invariant set.

APPENDIX

**Proof of Proposition 7**

In the case of linear constraints, $O_k$ is a polyhedral set for any $k \in \mathbb{Z}_0^+$. It is clear from Lemma 7 that $r_{\text{max}}^k \leq 0$ implies $O_{k+1} = O_k$. We only need to show $O_{k+1} = O_k$ implies $r_{\text{max}}^k \leq 0$. From (12), $O_{k+1} = O_k$ if and only if $O_k \subseteq \{x \in \mathbb{R}^n : A^{k+1}x \in X\}$. From the extended Farkas’ lemma [2], [46], for any $k \in \mathbb{Z}_0^+$, $O_k \subseteq \{x \in \mathbb{R}^n : A^{k+1}x \in X\}$ if and only if there exists a non-negative matrix $S \in \mathbb{R}^{p \times (k+1)p}$ such
that,

\[
S \left( \begin{array}{c}
q^T \\
q^T A^k \\
\vdots \\
q^T A^p \\
\end{array} \right) = q^T A^{k+1}
\]

(72)

\[
\sum_{i,j=1}^{p(k+1)} S_{i,j} = 1, \forall i \in \mathbb{Z}^p
\]

(73)

Suppose there exists a non-negative matrix \( S \in \mathbb{R}^{p \times (k+1)p} \) satisfying (72) and (73) for some \( k \in \mathbb{Z}^+ \), for any \( \forall i \in \mathbb{Z}^p \), let \( \tilde{z}_i = S_{i,\ell} \) for all \( \ell \in \mathbb{Z}^{(k+1)p} \). As \( q_i^T A^{k+1} = \sum_{\ell=0}^k \sum_{j=1}^p \tilde{z}_i^{p+\ell} q_j^T A^\ell \) and \( \sum_{\ell=0}^k \sum_{j=1}^p \tilde{z}_i^{p+\ell} \leq 1 \), we can see that

\[
\left( \begin{array}{c}
0 \\
q_i^T A^{k+1} \\
\vdots \\
q_i^T A^p \\
\end{array} \right) \left( \begin{array}{c}
A^{k+1} \tilde{z}_i \\
\vdots \\
A^p \tilde{z}_i \\
\end{array} \right) = 0
\]

(74)

for any \( i \in \mathbb{Z}^p \). This means that \( (0, \tilde{z}_{(k+1)p}) \) is a feasible solution to (26) for any \( i \in \mathbb{Z}^p \). Hence, \( r^k_{\max} \leq 0 \). □

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