Temporal interference of relativistic bosons

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Abstract

In quantum mechanics, time is introduced as a non-measurable quantity, as there is no possibility to build a hermitian operator canonically conjugated to the Hamiltonian. We cannot have, therefore, the time operator, which means that the temporal structure of the evolution of quantum systems is ill-defined. We present an extension to the model, in which the time evolution is based on the projection postulate rather than the unitary operator. This approach is in agreement with all other aspects of quantum mechanics and allows to discuss time as an observable. Using this framework we present a description of the temporal double-slit experiment in which a single particle interferes with itself from a different instant of time. Such behaviour has already been observed experimentally but lacked a consistent theoretical explanation.

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I. INTRODUCTION

In the standard formulation of the quantum theory all observables are represented as hermitian operators. Basing on this assumption Pauli showed, that it is impossible to construct an operator of time, canonically conjugated to the Hamiltonian \([1–3]\). At present we know, that one can perform measurements on quantities described by positive operator-valued measures (POVM), which is a broader construction than the hermitian operators. It was proven in Ref. \([4]\) that a POVM observable representing time is possible to construct.

It has been intensively discussed how to introduce time as an observable in the theory, as this affects the construction of the arrow of time and clocks (see \([5, 6]\) for recent developments). Related topics include also the problem of time in entangled systems, the time of decoherence and the role of the energy-time uncertainty relation. The process of quantization can be performed in different ways and tested using specially designed experiments, like it has been shown in the example of the time of arrival operator \([7, 8]\). Since time is connected with the energy operator, thermodynamics of quantum processes started to be of interest \([9]\), especially in the connection with the future quantum computers. It has already been shown that due to the quantum correlations, heat may spontaneously flow from the colder to the hotter subsystem \([10]\), which is not observed in the macroscopic scale. Entanglement and the immediate change of state of both entangled particles rises also the question how to describe \([11]\) and experimentally investigate \([12]\) this process.

The problem of time appears also in systems performing quantum computation. Most quantum protocols silently assumes that we can neglect the time delays introduced by quantum gates and connections in the system. Another problem arises with the theoretically proposed quantum gates with feedback \([13]\), which are impossible to describe using standard tools. Understanding the time structure of quantum operations is also vital for constructing future quantum neural networks \([14]\).

Treating time as an observable leads to the problem of time measurements \([15]\), also in the context of quantum cosmology \([16]\). As time becomes a variable, new phenomena start to be possible, like dark matter described by fields evolving backwards in time \([17]\).

In this paper we discuss the interference of states in the temporal version of the double-slit experiment. The phenomenon of interference in time has most probably been already observed \([18, 19]\), and a proper theoretical description requires time as an observable \([20]\).
Following this line of research we discuss a relativistic charged pion passing a slit which opens two times in a fixed spatial location. This leads to the self-interference of the particle along the time axis. Instead of the Schrödinger evolution, we base our description on the more general projection postulate.

II. TIME IN PHYSICS

Time is introduced in physics in two ways. In the relativity theory it is regarded as a coordinate, forming with the spatial coordinates the spacetime. In all other approaches, including the standard formulation of quantum mechanics, it is introduced as a background parameter. In both situations, time is always present and can be measured by a suitable clock. According to the Pauli theorem, this type of time cannot be represented as a hermitian operator canonically conjugated to the Hamiltonian. This suggests, that the macroscopic time cannot be transferred directly to the microscopic scale.

One of the assumptions leading to the Pauli theorem is, that the time evolution must be unitary. A unitary operator is invertible, from which follows that the time evolution can be reversed, as there is no energy dissipation during the process. We know, however, that the time evolution of a physical system is invertible only under special circumstances. On the quantum level, every measurement which destroys the quantum superposition is not invertible. This suggests, that in general we should formulate the evolution of the quantum systems in a non-unitary way.

III. PROJECTION EVOLUTION OF QUANTUM SYSTEMS

We treat the evolution of a quantum system as a series of projections of the density matrix on the space of possible new states. This process is spontaneous and to some extent random, as the new state is chosen randomly according to some probability distribution. For our convenience the subsequent projections will be numbered by an ordered parameter $\tau$. The evolution on the quantum level is not time ordered, although we may expect some correspondence between the order of the projections and the evolution of the system in the macroscopic scale. The $\tau$ parameter allows to keep an order in our description, but it is not an observable and does not represent any physical quantity.
To be more precise, the projection evolution operator at the evolution step $\tau_n$, where $n \in \mathbb{Z}$, is a family of mappings between the space of quantum states at the evolution step $\tau_n$ and the space of quantum states at the evolution step $\tau_{n+1}$. The state spaces are assumed to be some subspaces of the trace class operators (the space of operators with finite trace, i.e., the set of density operators) defined on a given Hilbert space. In this case the Hilbert space of a spinless particle is not contained in $L^2(\mathbb{R}^3, d^3x)$ but rather in $L^2(\mathbb{R}^4, d^4x)$, where the fourth dimension is time, treated here on the same footing as the positions in the 3D-space.

The mappings between the spaces of states can be written in terms of the so-called quantum operations or their generalizations. The formalism of quantum operations was invented around 1983 by Krauss [23], who relied on the earlier mathematical works of Choi [24]. They showed that there exists a family of operators which map the density operator space onto another density operator space. Let us denote such operator by $E(\tau_n; \nu_n)$, where $\nu_n$ represents the set of quantum numbers at the evolution step $\tau_n$. Following Lüders [25], we postulate that the density matrix $\rho(\tau_n; \nu_n)$ at the evolution step $\tau_n$ can be obtained from $\rho(\tau_{n-1}; \nu_{n-1})$ using the evolution operators:

$$\rho(\tau_n; \nu_n) = \frac{E(\tau_n; \nu_n)\rho(\tau_{n-1}; \nu_{n-1})E(\tau_n; \nu_n)^\dagger}{\text{Tr} \left[ E(\tau_n; \nu_n)\rho(\tau_{n-1}; \nu_{n-1})E(\tau_n; \nu_n)^\dagger \right]},$$

where the denominator normalizes the expression.

Many realizations of $\Phi$ are possible. For example, if we assume a Schrödinger type of evolution, the operator takes the form

$$\Phi(\tau; \nu) = U(\tau),$$

where $\nu$ is fixed and $U(\tau)$ is a unitary operator. In this case the next step of the evolution is chosen uniquely with the probability equal to 1. One needs to note that the unitary operator (2) is not parametrized by time but by the evolution parameter $\tau$, even though, in general, it is time dependent.

Another realization of the $\Phi$ operator is the resolution of unity, for which the following conditions hold:

$$\Phi(\tau; \nu)^\dagger = \Phi(\tau; \nu),$$

$$\Phi(\tau; \nu)\Phi(\tau; \nu') = \delta_{\nu\nu'}\Phi(\tau; \nu),$$

$$\sum_{\nu} \Phi(\tau; \nu) = I,$$
where $\mathbb{I}$ is the unit operator. This type of evolution is more general than the unitary evolution and will be used in our description of the time interference.

For more details on the projection evolution formalism, including the construction of the time operator and various forms of the evolution operators, see Ref. [26].

IV. INTERFERENCE IN TIME

A. Interference on time slits

Assume that a relativistic spinless Klein-Gordon particle is emitted from a source and propagates towards the detector. On its way it meets a wall in which a slit opens for a limited time. Let the slit open two times in a fixed spatial location. When the slit is closed, the path to the detector is blocked. The particle, if not observed, will be, after successfully passing the slit, in a superposition of states corresponding to the two time intervals of the opened slit. As a result, the energy spectrum measured by the detector will have the form of an interference pattern, as was reported in Ref. [18].

The process of the emission does not happen in zero time, so the mass $m_0$ of the particle will be distributed around some mean value $\bar{m}_0$,

$$m_0 \in \Delta_{\bar{m}_0} = \left[ \bar{m}_0 - \frac{\Gamma}{2}, \bar{m}_0 + \frac{\Gamma}{2} \right],$$

where $\Gamma/2 < \bar{m}_0$. It follows from the Klein-Gordon equation $k_\mu k^\mu = m_0^2$ that to fulfill (6) the particle’s four-momentum must belong to the set $B_{\bar{m}_0}$ for which $(\bar{m}_0 - \frac{\Gamma}{2})^2 \leq k^2 \leq (\bar{m}_0 + \frac{\Gamma}{2})^2$. We write the initial state of the particle in the form

$$|\psi_0\rangle = \int_{B_{\bar{m}_0}} d^4k \ a(k)|k\rangle,$$

where $\langle x|k \rangle = \exp(-ik_\mu x^\mu)/(4\pi^2)$ and $a(k)$ denotes the momentum distribution function. We omit for simplicity the normalization factor and introduce the overall normalization in the final formula (23).

The slit is open at certain spatial location during two time periods. We denote the spacetime regions of the opened slit by $\Delta_1$ and $\Delta_2$. The evolution operator describing the passing through the slit takes the form of a projection of the state onto the region $\Delta_T = \Delta_1 \cup \Delta_2$,

$$\mathcal{E}_S(\tau) = \int_{\Delta_T} d^4x \ |x\rangle\langle x|.$$
The operator $E_S(\tau_1)$ will have a different form for particles which do not manage to pass the slit but in what follows we are interested in the form (8) only. The action of (8) on $|\psi_0\rangle$ is given by

$$E_S(\tau_1)|\psi_0\rangle = \int_{\Delta_T} d^4x \int_{B_{m_0}} d^4k a(k) \langle x | k \rangle. \quad (9)$$

In the next step the particle propagates freely from the slits to the detector. During this step the Klein-Gordon equation is fulfilled, so the evolution operator projects onto the momentum space $B_{m_0}$,

$$E_F(\tau_2) = \int_{B_{m_0}} d^4k' |k'\rangle \langle k'|. \quad (10)$$

The unnormalized state of the particle at this step reads

$$E_F(\tau_2)E_S(\tau_1)|\psi_0\rangle = \int_{B_{m_0}} d^4k' \int_{B_{m_0}} d^4k a(k) \int_{\Delta_T} d^4x \langle k'|x \rangle \langle x | k \rangle. \quad (11)$$

The detector measures the four-momentum $\kappa$ of the particle, so the last evolution operator will be the projection

$$E_D(\tau_3) = |\kappa\rangle \langle \kappa|, \quad (12)$$

which results in the final state:

$$E_D(\tau_3)E_F(\tau_2)E_S(\tau_1)|\psi_0\rangle = |\kappa\rangle \int_{B_{m_0}} d^4k' \delta^4(\kappa - k') \int_{B_{m_0}} d^4k a(k) \int_{\Delta_T} d^4x \langle k'|x \rangle \langle x | k \rangle, \quad (13)$$

where the Dirac delta appears due to the orthogonality of the momenta.

To evaluate the expression (13) we notice, that the integration over $k'$ is equal to zero if $\kappa \not\in B_{m_0}$ and is equal to one if $\kappa \in B_{m_0}$. We account for this fact introducing the function $id_{B_{m_0}}(\kappa)$ defined as

$$id_{B_{m_0}}(\kappa) = \int_{B_{m_0}} d^4k' \delta^4(\kappa - k') = \begin{cases} 
1 & \text{if } \kappa \in B_{m_0} \\
0 & \text{if } \kappa \not\in B_{m_0}
\end{cases}. \quad (14)$$

Let the spacetime coordinates of the opened slit be in the form

$$\Delta_i = \left( t_i - \frac{\delta_T}{2}, t_i + \frac{\delta_T}{2} \right) \times \left( x_i^1 - \frac{\delta_1}{2}, x_i^1 + \frac{\delta_1}{2} \right) \times \left( x_i^2 - \frac{\delta_2}{2}, x_i^2 + \frac{\delta_2}{2} \right) \times \left( x_i^3 - \frac{\delta_3}{2}, x_i^3 + \frac{\delta_3}{2} \right). \quad (15)$$
Since the scalar product $\langle x|k \rangle = \exp(-i k_{\mu} x^{\mu})/(4\pi^2)$, the integration over $x$ in (13) takes the form

$$\int_{\Delta_T} d^4x \, \langle x|k \rangle \langle k|x \rangle k = \left( \frac{1}{\sqrt{2\pi}} \right)^4 \int_{\Delta T} d^4x \, e^{-i(k_{\mu} - \kappa_{\mu})x^{\mu}}$$
$$= \left( \frac{1}{2\pi} \right)^4 \left( \int_{\Delta_1} d^4x \, e^{-i(k_{\mu} - \kappa_{\mu})x^{\mu}} + \int_{\Delta_2} d^4x \, e^{-i(k_{\mu} - \kappa_{\mu})x^{\mu}} \right). \tag{16}$$

The integrals in (16) can be evaluated using (15),

$$\int_{\Delta_i} d^4x \, e^{-i(k_\mu - \kappa_\mu)x^{\mu}} = \delta_T \delta_1 \delta_3 e^{-i(k_0 - \kappa_0)t_1} e^{-i(k_1 - \kappa_1)x^1} e^{-i(k_2 - \kappa_2)x^2} e^{-i(k_3 - \kappa_3)x^3} \times j_0 \left( \frac{k_0 - \kappa_0}{2} \delta_T \right) j_0 \left( \frac{k_1 - \kappa_1}{2} \delta_1 \right) j_0 \left( \frac{k_2 - \kappa_2}{2} \delta_2 \right) j_0 \left( \frac{k_3 - \kappa_3}{2} \delta_3 \right), \tag{17}$$

where $j_0(z) = \sin(z)/z$ is the spherical Bessel function of the first kind.

The probability density $\text{Prob}(\kappa)$ of detecting a particle with the four-momentum $\kappa$ is given by the modulus squared of the expression (13). Using equations (14), (16) and (17) we obtain

$$\text{Prob}(\kappa) = (\delta_T \delta_1 \delta_3)^2 \left( \frac{1}{2\pi} \right)^8 \text{id}_{B_{\kappa_0}}(\kappa)$$
$$\times \left| \int_{B_{\kappa_0}} d^4k \, a(k) \left( e^{-i(k_0 - \kappa_0)t_1} + e^{i(k_0 - \kappa_0)t_2} \right) e^{i(k - \kappa)\bar{x}_s} \right|^2. \tag{18}$$

The interference term $e^{-i(k_0 - \kappa_0)t_1} + e^{i(k_0 - \kappa_0)t_2}$ can be rewritten after the variable change,

$$t_1 = T_s - \frac{\epsilon_T}{2}, \quad t_2 = T_s + \frac{\epsilon_T}{2}, \tag{19}$$

which leads to

$$\text{Prob}(\kappa) = 4(\delta_T \delta_1 \delta_3)^2 \left( \frac{1}{2\pi} \right)^8 \text{id}_{B_{\kappa_0}}(\kappa)$$
$$\times \left| \int_{B_{\kappa_0}} d^4k \, a(k) e^{-i(k_0 - \kappa_0)T_s} e^{i(k - \kappa)\bar{x}_s} \cos \left( \frac{k_0 - \kappa_0}{2} \epsilon_T \right) j_0 \left( \frac{k_0 - \kappa_0}{2} \delta_T \right) \right|^2.$$

We evaluate the exact expression (20) assuming that the spatial momentum of the particle is directed along the $z$ axis, $\vec{k} = (0, 0, -k_z)$. The profile $a(k)$ takes in this case the form

$$a(k) = \tilde{a}(m^2) \delta(k_1) \delta(k_2) \delta(k_3 - k_z). \tag{21}$$
By changing the integration variables, the modulus squared in (20) reduces to
\[
\left[ j_0\left(\frac{\kappa_1}{2}\delta_1\right) j_0\left(\frac{\kappa_2}{2}\delta_2\right) j_0\left(\frac{k_z + \kappa_3}{2} - \delta_3\right) \right]^2 \int \left(\frac{\bar{m}_0 + \frac{\epsilon_T}{2}}{2}\right)^2 \frac{d(m^2)}{2\sqrt{m^2 + k_z^2}} \\
\times e^{-i\sqrt{\bar{m}_0^2 + k_z^2} T_s} \cos \left(\frac{\sqrt{\bar{m}_0^2 + k_z^2} - \kappa_0}{2} \epsilon_T\right) j_0\left(\frac{\sqrt{\bar{m}_0^2 + k_z^2} - \kappa_0}{2} \delta_T\right)^2.
\]
(22)

This expression can be simplified further if we assume that the mass spread is small, i.e., \(\Gamma \approx 0\). In this case the integration can be approximated using the mean value theorem. As a result we obtain
\[
\text{Prob}(\kappa) \approx \mathcal{N} \left[ j_0\left(\frac{\kappa_1}{2}\delta_1\right) j_0\left(\frac{\kappa_2}{2}\delta_2\right) j_0\left(\frac{k_z + \kappa_3}{2} - \delta_3\right) \right]^2 \\
\times \text{id}_{B\bar{m}_0}(\kappa) \left[ \cos \left(\frac{\sqrt{\bar{m}_0^2 + k_z^2} - \kappa_0}{2} \epsilon_T\right) j_0\left(\frac{\sqrt{\bar{m}_0^2 + k_z^2} - \kappa_0}{2} \delta_T\right) \right]^2,
\]
(23)

where we have introduced \(\mathcal{N}\) as the overall normalization factor. In our construction this factor does not normalize (23) to 1 because not all particles are assumed to pass the slit, see (8). We may however normalize (23) for all particles measured by the detector, in which case we get
\[
\int_{\mathbb{R}^4} d^4\kappa \text{Prob}(\kappa) = 1.
\]
(24)

### B. A numerical example

As an example, let us discuss the \(\pi^+\) particle in a setup with no sources of the electromagnetic interactions. During the numerical calculations the natural units \(c = \hbar = 1\) will be used.

The particle is produced with the initial three-momentum \(\vec{k} = (0, 0, -k_z)\). On its way to the detector it has to pass a slit which opens twice in the same spatial location. The slit has spatial widths \(\delta_1\) and \(\delta_2\) in the plane perpendicular to the direction of motion. The width \(\delta_3\) is less important, because the highest probability for the detector to register the pion is for the incoming momentum \(\kappa_3 = -k_z\), for which the term \(j_0((k_z + \kappa_3)\delta_3/2) = 1\) drops out.

We denote the time width of the opened slit by \(\delta_T\), and the time between the two openings by \(\epsilon_T\).

The mass of \(\pi^+\) is \(m_\pi \approx 139\) MeV. Its half-life is \(t_\pi = 3.95 \cdot 10^7\) eV\(^{-1}\), which implies the mass spread of the order of \(\Gamma \sim 1/t_\pi \approx 2.5 \cdot 10^{-8}\) eV. Because \(m_\pi\) and \(\Gamma\) differ by sixteen
FIG. 1: The detection probability as a function of $\kappa_1$ and $\kappa_2$ for different values of the opening times.

Orders of magnitude, the pion is almost exactly on its mass shell; as a consequence, $B_{\bar{m}_0}$ is just the Klein-Gordon condition, and the function $\text{id}_{B_{\bar{m}_0}}(\kappa) = 1$ becomes trivial. If the initial Klein-Gordon state of the particle is given by $k_0^2 = m_\pi^2 + k_z^2$, the state seen by the detector will be $\kappa_0^2 = m_\pi^2 + \kappa_1^2 + \kappa_2^2 + k_z^2$. Taking all this into account, the unnormalized probability, as a function of $\kappa_1$ and $\kappa_2$, is given by the expression:

$$
\text{Prob}(\kappa_1, \kappa_2) \approx j_0^2 \left( \frac{\kappa_1}{2} \delta_1 \right) j_0^2 \left( \frac{\kappa_2}{2} \delta_2 \right) \times \cos^2 \left( \frac{\sqrt{m_\pi^2 - k_z^2} - \sqrt{m_\pi^2 + \kappa_1^2 + \kappa_2^2 - k_z^2}}{2} \epsilon_T \right) \times j_0^2 \left( \frac{\sqrt{m_\pi^2 - k_z^2} - \sqrt{m_\pi^2 + \kappa_1^2 + \kappa_2^2 - k_z^2}}{2} \delta_T \right).
$$

The detection probability (25) is plotted on Fig. 1. The slits are 0.01 mm wide in the $x$ and $y$ directions. The time parameter $\epsilon_T$ takes values from $10^{-7}$ s to $10^{-12}$ s whereas $\delta_T$ is set to $\delta_T = \epsilon_T / 3$. The momentum $k_z$ is a constant number and does not play any significant role. For long opening times, the detection is possible for very small perpendicular momenta $\kappa_1$ and $\kappa_2$ only. For smaller $\epsilon_T$ the inner region widens and starting from $\epsilon_T = 10^{-10}$ s higher order maxima start to appear. They are clearly visible for $\epsilon_T = 10^{-10}$ s and shorter times.

The interference effect present on Fig. 1 comes from two sources – the spatial diffraction...
FIG. 2: The temporal part of $\text{Prob}(\kappa_1, \kappa_2)$ for different values of the opening times $\epsilon_T$.

on the slits and the temporal interference. On Fig. 2 we have drawn density plots of the temporal part of Eq. (25). For shorter opening times the axes have been rescaled to show, that the maxima appear for higher values of the momenta. Even though we have not used the Heisenberg-like condition for the energy and time, high momenta, and thus high energy, are needed in the case of short times. It follows from Eq. (25) that the temporal part is dependent on $\kappa_1^2 + \kappa_2^2$, therefore circles appear on the plots on Fig. 2. The comparison of these diagrams with those on Fig. 1 reveals, that the spatial part forces the maxima to appear along the $\kappa_1 = 0$ and $\kappa_2 = 0$ directions, dominating the temporal effects.

One may make the temporal effect visible by manipulating the spatial widths of the slit. Looking at Eq. (25) one sees, that for small $\delta_1$ and $\delta_2$ the Bessel functions are close to one and do not suppress the temporal part. We show this on Fig. 3 where the temporal parameters are set to $\epsilon_T = 10^{-14}$ s and $\delta_T = \epsilon_T/3$. The spatial size of the slit is $\delta_1 = \delta_2 = d$. For $d = 10^{-8}$ mm the circular maxima are clearly visible. For larger slits, $d = 10^{-7}$ mm, this changes into the four-fold shape dictated by the Bessel functions.
FIG. 3: The detection probability as a function of $\kappa_1$ and $\kappa_2$ for different spatial widths of the slit. Here $d = \delta_1 = \delta_2$. The time between the openings is set to $\epsilon_T = 10^{-14}$ s and the slit is open for $\delta_T = \epsilon_T/3$.

V. CONCLUDING REMARKS

A collection of papers devoted to different aspects of the physical time can be found, among others, in [21, 27]. In Ref. [21] the paper by P. Busch mentions three types of time. The most popular one is the time considered as a parameter which is measured by an external laboratory clock, uncoupled from the measured system. This time is called the external time. Time can be defined also through the dynamics of the observed quantum systems, in which case we deal with the dynamical (or intrinsic) time. Lastly, time can be considered on the same footing as other quantum observables, especially as positions in space. This is called by P. Busch the observable (or event) time and it represents the approach used in the present paper.
In the experimental practice the external time is usually used. This approach turned out not to be correct for pure quantum systems. The second possibility is the intrinsic time or, to be more precise, times. They are determined by any arbitrary set of dynamical variables. This definition is compatible with the projection evolution postulate, i.e., that changes of states or observables are more fundamental than the time itself. However, because in our approach the physical time is a quantum observable, the required characteristic times (intrinsic times) for a given physical process can be directly calculated. In this context, the intrinsic times are not fundamental but derivable temporal observables.

We have presented the description of the temporal self-interference of a relativistic quantum particle. The model is based on the projection evolution of quantum states. In this formalism one can construct the time operator \[20\] and include the time dependence of the quantum states in a consistent way. We have worked out a numerical example of a relativistic pion in a temporal double-slit experiment and showed the difference between the spatial and the temporal interference pattern.

The observable time can, in a natural way, account for some quantum mechanical effects regarded as paradoxes. It is also important that it allows to calculate the temporal characteristics of a quantum system as it can be done for other observables. It introduces the time-energy uncertainty relation on the same basis as for the position–momentum observables. The time operator and the corresponding temporal momentum operator are the very natural complements of the covariant relativistic four-position and four-momentum operators, as discussed in Ref. \[26\]. A few examples of processes analyzed in terms of the observable time can be found in \[28\].

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[1] W. Pauli in: H. Geiger, K. Scheel (editors), *Handbuch der Physik* 23, 1-278, Springer (1926).
[2] W. Pauli in: H. Geiger, K. Scheel (editors), *Handbuch der Physik* 24, 83-272, Springer (1926).
[3] W. Pauli in: S. Fludge (editor), *Handbuch der Physik* 5, 1-168, Springer (1958).
[4] E.A. Galapon, Proc. R. Soc. Lond. A 458, 451 (2002).
[5] L. Loveridge, T. Miyadera, Found. Phys. 49 (2019) 549; M. Vogl, P. Laurell, A.D. Barr, G.A. Fiete, Phys. Rev. A 100 (2019) 012132; J. Ashmead, J. Phys. Conf. Ser. 1239 (2019) 012015; A. Schild, Phys. Rev. A 98 (2018) 052113; N. Argaman, Entropy 20 (2018) 294; A.R.H. Smith, M.
Ahmadi, Quantum 3 (2019) 160; M. Lienert, S. Petrat, R. Tumulka, Found. Phys. 47 (2017) 1582; D.E. Bruschi, Ann. Phys. 394 (2018) 155; J. Dressel, A. Chantasri, A.N. Jordan, A.N. Korotkov, Phys. Rev. Lett. 119 (2017) 220507; S. Khorasani, Comm. Theor. Phys. 68 (2017) 35; H. Kitada, J. Jeknic-Dugic, M. Arsenijevic, M. Dugic, Phys. Lett. A 380 (2016) 3970; P. Aniello, F.M. Ciaglia, F. Di Cosmo, G. Marmo, J.M. Pérez-Pardo, Ann. Phys. 373 (2016) 532; E.O. Dias, F. Parisio, Phys. Rev. A 95 (2017) 032133; A. Sudbery in: D. Aerts, C. de Ronde, H. Freytes, R. Giuntini (editors), Probing the Meaning and Structure of Quantum Mechanics: Superpositions, Semantics, Dynamics and Identity, World Scientific, 2017; V.R. Overbeck, H. Weimer, Phys. Rev. A 93 (2016) 012106; S. Banerjee, S. Bera, T.P. Singh, Int. J. Mod. Phys. 24 (2015) 1544011; T. Miyadera, Found. Phys. 46 (2016) 1522; V. Giovannetti, S. Lloyd, L. Maccone, Phys. Rev. D 92 (2015) 045033; J.S. Briggs, Phys. Rev. A 91 (2015) 052119; V.S. Olkhovsky, E. Recami, Int. J. Mod. Phys. A 22 (2007) 5063; J. Jing, H.R. Ma, Phys. Rev. E 75 (2007) 016701; R. de la Madrid, J.M. Isidro, Adv. Stud. Theor. Phys. 2 (2008) 281.

[6] D. Geiger, Z.M. Kedem, arXiv:1906.11712; D. Buchholz, K. Fredenhagen, arXiv:1905.02711; K.L.H. Bryan, A.J.M. Medved, arXiv:1811.09660; M. Bauer, arXiv:1606.02618; E.O. Dias, F. Parisio, arXiv:1507.02899; G. Bacciagaluppi, arXiv:quant-ph/0701225.

[7] E.A. Galapon, J.J.P. Magadan, Ann. Phys. 397 (2018) 278.

[8] R. Ximenes, F. Parisio, E.O. Dias, Phys. Rev. A 98 (2018) 032105.

[9] A.Y. Klimenko, Phys. Scr. 94 (2019) 034002; W.F. Wreszinski, arXiv:1902.07591.

[10] K. Micadei, J.P.S. Peterson, A.M. Souza, R.S. Sarthour, I.S. Oliveira, G.T. Landi, T.B. Batalhao, R.M. Serra, E. Lutz, Nature Comm. 10 (2019) 2456.

[11] M. Nowakowski, AIP Conf. Proc. 1841 (2017) 020007; M. Nowakowski, arXiv:1604.03976.

[12] E. Moreva, G. Brida, M. Gramegna, V. Giovannetti, L. Maccone, M. Genovese, Phys. Rev. A 89 (2014) 052122.

[13] A.L. Grimsmo, Phys. Rev. Lett. 115 (2015) 060402; S. Mamataj, D. Saha, N. Banu, American Journal of Engineering Research 3 (2014) 151; T. Koike, Y. Okudaira, Phys. Rev. A 82 (2010) 042305; J.E. Rice, Comput. J. 51 (2008) 700.

[14] A. Dendukuri, B. Keeling, A. Fereidouni, J. Burbridge, K. Luu, H. Churchill, arXiv:1905.10912.

[15] V.P. Belavkin, M.G. Perkins, Int. J. Theor. Phys. 37 (1998) 219.

[16] N. Kajuri, Int. J. Mod. Phys. D 26 (2017) 1743011.
[17] E. Alvarez, Presented at the workshop Voyages Beyond the Standard Model II, 2018, arXiv:1803.08531.

[18] U. Houser, W. Neuwirth, N. Thesen, Phys. Lett. A 49, 57 (1974).

[19] F. Lindner et al., Phys. Rev. Lett. 95, 040401 (2005).

[20] P. Busch, Found. Phys. 20, 1 (1990).

[21] P. Busch in: J.G. Muga, R. Sala Mayato, I.L. Egusquiza (editors), *Time in Quantum Mechanics*, Springer (2002).

[22] D.G. Arbó, E. Persson, J. Burgdörfer, Phys. Rev. A 74 (2006) 063407.

[23] K. Krauss, *States, Effects and Operations: Fundamental Notions of Quantum Theory*, Springer (1983).

[24] M.D. Choi, Lin. Alg. App. 10, 285 (1975).

[25] G. Lüders, Ann. Phys. (Leipzig) 8, 322 (1951); reprinted in: Ann. Phys. (Leipzig) 15, 663 (2006).

[26] A. Góźdz, M. Góźdz, arXiv: 1910.11198.

[27] J.G. Muga, A. Ruschhaupt, A. Del Campo (editors), *Time in Quantum Mechanics – vol.2*, Springer (2009).

[28] A. Góźdz, K. Stefańska, Int. J. Mod. Phys. E 15, 500 (2006); M. Dębicki, A. Góźdz, Int. J. Mod. Phys. E 15, 437 (2006); M. Dębicki, A. Góźdz, K. Stefańska, Int. J. Mod. Phys. E 16, 616 (2007); A. Góźdz, M. Dębicki, K. Stefańska, Phys. Atom. Nucl. 71, 1 (2008); A. Góźdz, K. Stefańska, J. Phys.: Conf. Ser. 104, 012007 (2008); A. Góźdz and K. Stefańska, Int. J. Mod. Phys. E 17, 217 (2008); A. Góźdz, M. Góźdz, Acta Phys. Polon. B Proc. Suppl. 10, 85 (2017); A. Góźdz, M. Góźdz, Phys. Atom. Nucl. 80, 373 (2017); M. Góźdz et al., Phys. Atom. Nucl. 81, 853 (2018).