Contact spectral invariants and persistence

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Abstract

This sketch shows that the usual generating function based capacities have an interpretation in the language of persistent homology as persistences of certain homology classes in the persistence module formed by the corresponding generating function homology groups. This viewpoint suggests various new invariants, in particular a $\mathbb{Z}_k$-equivariant capacity which can be used to prove orderability of lens spaces, proved by Milin [Mi08] using contact homology and by Sandon [San11] using equivariant generating function homology. These are informal notes originally circulated in January 2014.

1 Generating-function based capacities

The usual Viterbo/Sandon capacity [Vit92, San09] is defined for $\phi \in \text{Ham}(\mathbb{R}^{2n})$, resp. $\text{Cont}_0(\mathbb{R}^{2n} \times S^1)$, by first assuming a quadratic at infinity generating function $S : E \to \mathbb{R}$ for the (exact Lagrangian, resp. Legendrian) “graph” of $\phi$. Here $E$ is a vector bundle over $M = S^{2n}$ resp. $S^{2n} \times S^1$, the original manifold compactified, and may even be taken to be trivial. One then considers sub-level sets $E^a := \{S \leq a\} \subseteq E$ and puts

$$c(\phi) := \inf\{a \in \mathbb{R} : \theta_a(\mu) \neq 0\}$$

where $\mu \in H^{2n}(M)$ is the orientation class of the base $M$ and for any $a \in \mathbb{R}$, $\theta_a : H^*(M) \to H^*(E^a, E^-)$ is the composition $\theta_a = i^*_a \circ \theta$ of the natural isomorphism (c.f. Thom isomorphism and Excision theorem) $\theta : H^*(M) \to H^*(E, E^-)$ with the homomorphism $i^*_a : H^*(E, E^-) \to H^*(E^a, E^-)$ induced by inclusion $i_a : E^a \to E$.

Assume coefficients in a field. Note:

• $c(\phi)$, resp. $[c(\phi)]$ in the contact case is conjugation-invariant

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1 Technically, the degree $*$ is shifted by $j$, the index of $Q_\infty$, the quadratic at infinity part of $S$. Indeed by the Thom isomorphism, $H^*(M) \cong H^{*-*}(DE_-, SE_-)$, where $E_-$ is any rank $j$ real vector bundle over $M$. This is isomorphic to $H^{*-*}(E_-, CE_-)$, letting $CE_-$ be the closure of $(DE_-)^c \subseteq E_-$, and excising $(DE_-)^c$. On the other hand, taking $E_-$ to be the the maximal sub-bundle on which $Q_\infty$ is negative definite, $H^{*-*}(E_-, CE_-) \cong H^{*-*}(E, E^-)$ because for sufficiently negative $b$, $E^b$ and $E^{-\infty}$ are homotopy equivalent and the pair $(E, E^b)$ fiberwise deformation retracts to the pair $(E_-, \{e \in E_- : |e| \geq B\})$ for some $B \in \mathbb{R}$.

2 Here $[x]$ denotes the largest $a \in \mathbb{Z}$ s.t. $a < x$; any $c(\phi, \mu)$, resp. $[c(\phi, \mu)]$ is also conjugation-invariant.
• An analogous $c(\phi) = c(\phi, \mu)$ can be defined using any non-trivial $\mu \in H^*(M)$. This does not give any more information about $\phi$ in the symplectic case. Nor does it in the contact case for $\phi \in Cont_0(\mathbb{R}^{2n} \times S^1)$ which is a lift $\phi = \bar{\sigma}$ of $\sigma \in \text{Ham} (\mathbb{R}^{2n})$ because, as may be checked, $c_{\text{contact}}(\bar{\sigma}) = [c_{\text{symp}}(\sigma)]$.

• By letting $\eta = \theta(\mu) \in H^p(E, E^{-\infty})$ ($p = 2n$ resp. $2n + 1$) we may equivalently write

$$c(\phi) := \inf \{ a \in \mathbb{R} : \iota^*_a(\eta) \neq 0 \}.$$  

Alternatively,

$$c(\phi) := \sup \{ a \in \mathbb{R} : (j_a)_*(\eta) \neq 0 \}$$

where $(j_a)_*: H_p(E, E^{-\infty}) \to H_p(E, E^a)$ induced by inclusion $j_a : E^{-\infty} \to E^a$ with $\eta = \theta([M])$, $\theta$ re-defined analogously for homology, and $[M]$ being the fundamental class of the base, i.e. Kronecker dual to $\mu$.

• This is in fact the persistence of the class $\eta$, in the sense of persistent homology (see next Section), for the persistence module

$$(V_a \mid a \in \mathbb{R}), \quad V_a := H_p(E, E^a)$$

(with linear map $V_a \to V_b$ induced by inclusion whenever $a \leq b$) and moreover many more such persistences and persistent homology groups can be defined; under certain circumstances they too will be conjugation-invariant.

## 2 Persistence

A persistence module $\mathbb{V}$ over $\mathbb{R}$ is an indexed family of vector spaces

$$(V_t \mid t \in \mathbb{R})$$

and a doubly-indexed family of linear maps

$$(v^t_s : V_s \to V_t \mid s \leq t)$$

which satisfy the composition law

$$v^t_r \circ v^r_s = v^t_s$$

whenever $r \leq s \leq t$ and where $v^t_t$ is the identity map on $V_t$. For more on persistent homology see Weinberger [W11]. For the framework of persistent modules see Chazal et al [CSGO13] who introduced this useful formalism. Equivalently, a persistence module can be viewed as a functor from $\mathbb{R}$ (as a category with a unique morphism $s \to t$ whenever $s \leq t$) to the category of vector spaces. Let us call the maps $v^t_s$ persistence maps (not standard terminology).

Any functor from topological spaces to vector spaces, for example singular homology in a fixed degree $H(\cdot) := H_p(X, K)$, $K$ a field, produces a persistence module when applied to a topological space with an $\mathbb{R}$-filtration, $X = \bigcup_{t \in \mathbb{R}} X^t, s \leq t \Rightarrow X^s \subseteq X^t$ using as linear
maps \( v_t^s \) the homomorphisms induced by inclusion. A contravariant functor, such as singular cohomology could be used instead; this produces a persistence module if the indexing of the filtration is first reversed (so \( s \leq t \Rightarrow X^s \supseteq X^t \)).

**Remark:** In principle, equivariant (co)homology groups could also be used (when sets \( X^t \) of the filtration are invariant under an ambient group action). To my knowledge this has not been done in topological data analysis but a possible application in computational geometry is proposed in Section 4.

Note that defining a persistence module via singular homology with coefficients in a field also produces an intermediate persistence module, singular chain groups \( C_p(X^t), t \in \mathbb{R} \) which form an \( \mathbb{R} \)-family of chain complexes \( C_\bullet(X^t), \ t \in \mathbb{R} \) where the chain groups are vector spaces and for any \( s, t \in \mathbb{R} \) the persistence maps \( v^s_t := (i_s)_t \) (in all degrees, induced by inclusion \( i^s_t : X^s \rightarrow X^t \)) define a chain map between the complexes at level \( s \) and \( t \).

\[
\begin{array}{cccc}
\ldots & C_{p+1}(X^s) & \overset{d_{p+1}}{\longrightarrow} & C_p(X^s) \\
\downarrow v^s_t & & v^s_t & \downarrow v^s_t \\
\ldots & C_{p+1}(X^t) & \overset{d_{p+1}}{\longrightarrow} & C_p(X^t)
\end{array}
\]

Such an object can also be used as starting point. Taking homology one then obtains persistence modules \( H_p(X^t), \ t \in \mathbb{R} \) whose union over all \( p \in \mathbb{Z} \) forms what can be thought of as a \( \mathbb{Z} \)-graded persistence module (non-standard terminology).

**Remark:** Suppose there is an abelian group \( G \) acting on each of the chain complexes in the \( \mathbb{R} \)-family of chain complexes \( C_\bullet(X^t), \ t \in \mathbb{R} \); i.e., for each \( t \in \mathbb{R} \), and each \( p \in \mathbb{Z} \), \( G \) acts on the vector space \( C_p(X^t) \) and the differential \( d_{p+1} : C_{p+1}(X^t) \rightarrow C_p(X^t) \) is \( G \)-equivariant. Then we can define \( G \)-equivariant homology groups \( H_{G,p}(X^t), \ t \in \mathbb{R} \). If, in addition to the \( G \)-action on each chain complex \( C_\bullet(X^t) \) for \( t \in \mathbb{R} \) we require that persistence maps \( v^s_t \) define morphisms in the category of chain complexes with \( G \)-actions, i.e. commute not only with the differential but also the group action, then the \( G \)-equivariant homology groups \( H_{G,p}(X^t), \ t \in \mathbb{R} \) will form a persistence module in each degree \( p \in \mathbb{Z} \). This construction has not been considered in topological data analysis but a possible application in computational geometry is work in progress.

**Terminology:** Consider a persistence module defined by homology in degree \( p \). The image \( v^s_{s+a}(H_p(X^s)) \subseteq H_p(X^{s+a}) \) for \( a > 0 \) is called the \( \alpha \)-**persistent homology group** of \( X^s \) in degree \( k \). An analogous object for a general persistence module would also make sense (though no general name exists). Further terminology which does apply to general persistence modules, but which we give here in terms of homology for simplicity is as follows. For any non-trivial class \( \mu \in H_p(X^s) \) its **persistence** is

\[
\rho(\mu) := \sup\{ a > 0 : v^s_{s+a}(\mu) \neq 0 \}.
\]

More commonly, this information is collected for all \( \mu \in H_p(X^s), \ s \in \mathbb{R} \) and recorded as \( P_p \), a \footnote{Consider \( C_\bullet(X^t) \) as a chain complex of \( K[G] \)-modules, where \( K[G] \) denotes the group ring of \( G \) (in our case the group algebra, since \( K \) is a field). \( G \)-equivariant homology of \( C_\bullet(X^t) \) is computed by taking a projective resolution \( (E_\bullet, \delta) \) of \( K \) as a \( K[G] \)-module (with trivial \( G \)-action), tensoring \( (E_\bullet, \delta) \) with \((C_\bullet, d)\) and taking the homology of \( E_\bullet \otimes_{K[G]} C_\bullet \) (with suitable differential).}
multiset of pairs, each pair specifying birth and death for some $\mu$:

$$\mathbb{P}_p := \{(s, a + s) \in \mathbb{R}^2 : (\exists \mu \in H_p(X^s)) \\
\mu \notin v_r^\phi (H_p(X^r)) \text{ for } r < s, \\
\rho(\mu) = a\}. $$

Arranged as a set of points in the plane with multiplicities, $\mathbb{P}_p$ is referred to as the persistence diagram in degree $p$ of $X$. Alternatively this information is sometimes recorded as a barcode: a collection of horizontal bars stacked (in no particular order) above the $x$-axis in the plane, with one bar for each $\mu$, having left endpoint $s$ and right endpoint $s + a$ as specified above. We think of $s$ and $s + a$ as the birth and death of $\mu$ and the length of the bar thus represents the lifespan of $\mu$. Stability theorems exist in various settings. Roughly speaking, they show that when the function $f$ defining the filtration is changed by less than $C$, classes with lifespans longer than $C$ will continue to exist (their endpoints will not be shifted by more than $C$). See [W11] for an interesting application of this (proving a Theorem of Gromov).

3 GF-based capacities as persistences

I) Extension of GF homology group definition Traynor [Tr94] defines generating function homology groups $G^{(a,b]}(\phi)$, $\phi \in \text{Ham}(\mathbb{R}^{2n})$ for action windows $(a, b]$, $a < b \in \mathbb{R}$ both nonzero and requires the generating function for $\phi$ not to have $a$ or $b$ as critical value. In the contact setting, for $\phi \in \text{Cont}_0(\mathbb{R}^{2n} \times S^1)$, Sandon [San09, San11] makes the same requirement on critical values (but does not explicitly state that $a, b$ be nonzero, just that they be integer in order to obtain conjugation-invariance). In any case, in all these works, GF homology groups are only computed for action windows $(a, \infty]$ where $a > 0$. The non-zero condition is needed since generating functions for compactly supported $\phi$ always have zero as a critical value, but $G^{(a,b]}(\phi)$ is defined as relative homology $H_s(\{S \leq b\}, \{S \leq a\})$ of sub-level sets of a generating function $S$ for $\phi$ having $a$ and $b$ as regular values.

For use in the constructions below, it will be convenient to define a surrogate $G^{(0,\infty]}(\phi)$ for $G^{(0,\infty]}(\phi)$. In fact, $G^{(-\infty,\infty]}(\phi) = G^{(a,\infty]}(\phi)$ coincides for all $\phi$ and all $a < 0$, and is nonzero in degrees 0 and $2n$ only. We denote the generator in degree 0 by $\kappa$, the generator in degree $2n$ by $\eta$, as in Section 1. Note that $c(\phi, \kappa) = 0$. We therefore quotient out the subspace $\langle \kappa \rangle$ from the $\mathbb{Z}$-graded vector space $G^{(-\infty,\infty]}(\phi)$, to define the surrogate, $G^{(0,\infty]}(\phi) := G^{(-\infty,\infty]}(\phi)/\langle \kappa \rangle$. Since $c(\phi, \mu)$ is conjugation invariant for any $\mu$, the vector space $G^{(0,\infty]}(\phi)$ is by definition conjugation-invariant. We denote $\nu_\kappa$ the projection to the quotient. In the next Section we will consider another system of GF homology groups and create a similar surrogate but in that case there will be more generators such that $c(\phi, \mu) = 0$, and we quotient all of them out to define the surrogate.

5 An alternate definition $G^{(0,\infty]}(\phi) := G^{(\epsilon\phi,\infty]}(\phi)$ for sufficiently small $\epsilon_\phi > 0$ was stated in the first draft of these notes but to do so properly requires some care with the allowed classes of $\phi$ in order to ensure generic generating functions; the current definition avoids these technicalities.
II) Viterbo/Sandon capacities re-expressed

For each $\phi$, and fixed $p \in \mathbb{Z}$, let

$$V_a(\phi) := \begin{cases} 
G_p^{(a, \infty)}(\phi) & \text{if } a < 0 \\
G_p^{(0, \infty)}(\phi) & \text{if } a = 0 \\
G_p^{(a, \infty)}(\phi) & \text{if } a > 0 
\end{cases}$$

Then $(V_a(\phi) \mid a \in \mathbb{R})$ is a persistence module with persistence maps $(v_a^b) : V_a(\phi) \to V_b(\phi)$, $a < b$, induced by the inclusion map $i_a^b : (E, E^a) \to (E, E^b)$ on pairs with the modification that $v_a^b$ is defined as $(i^b_{\epsilon})_*$, the linear map induced on the quotient by the linear map $(i_{\epsilon})_*$ with $-\epsilon \in (-\infty, 0)$ arbitrarily chosen, and $v_0^a$ is defined as $\nu \circ (i^{-\epsilon})_*$ with $-\epsilon \in (a, 0)$ arbitrarily chosen.

Moreover, the Viterbo/Sandon capacity $c(\phi)$ defined in the first Section can equivalently be expressed as the persistence

$$c(\phi) := \rho(\eta')$$

of $\eta' := v_{\infty}^0(\eta) \in V_0$. This is because $c(\phi, \eta) > 0$ for all $\phi$ (recall the definition of $\eta$ in (1)) and $v_{\infty}^0(\eta') \neq 0$ if and only if $(i_{-\epsilon})_*(\eta') \neq 0$ which in the notation of first Section is equivalent to $(i_0)_*(\eta) \neq 0$.

4 $\mathbb{Z}_k$-equivariant capacity

In this section we discuss how to define a $\mathbb{Z}_k$-equivariant capacity by using the $\mathbb{Z}_k$-equivariant GF homology groups of Sandon [San11] to define a persistence module as above.

I) Contact version

The equivariant GF homology groups

$$G_{Z_k,p}^{(a,b)}(\phi) := H_{Z_k,p}(E^a, E^b),$$

defined by Sandon in $\mathbb{R}^{2n} \times S^1$ [San11] for $\phi \in \text{Cont}_{Z_k}(\mathbb{R}^{2n} \times S^1)$ (the identity component of the group of $Z_k$-equivariant contactomorphisms with compact support) are invariant under conjugation by $\psi \in \text{Cont}_{Z_k}(\mathbb{R}^{2n} \times S^1)$. On the other hand, it is readily checked that $C_0$, the critical submanifold of $S$ with critical value 0, has homology

$$H_{Z_k,p}(C_0) = H_{Z_k,p}(pt) = H_p(BZ_k) = \mathbb{Z}_k$$

for all $p \geq 0$ and any $\phi \in \text{Cont}^{Z_k}(\mathbb{R}^{2n} \times S^1)$. This is because the $Z_k$-action in this example fixes the point at $\infty$ (South pole in the base $S^{2n}$) and by perturbing the generating function $S \geq 0$ for $\phi \in \text{Cont}_0(\mathbb{R}^{2n} \times S^1)$ one can assume a single critical point at $\infty$ with critical value 0. Thus, as Sandon computes for certain special $\phi$ supported in $\tilde{B}(R)$

$$G_{Z_k,p}^{(\epsilon \phi, \infty)}(\phi) = \mathbb{Z}_k$$

for all $p \geq 2n$, when $\epsilon \phi$ is sufficiently small that a generic generating function $S$ for $\phi$ has no nonzero critical values less than or equal to $\epsilon \phi$. In fact in this setting, for all compactly

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\(a\) (see footnote 5 - all references to the surrogate in Sections 3 and 4 were updated accordingly Feb. 2015)
supported $\phi$, by the equivariant Thom isomorphism, $G_{\eta, p}^{[a, \infty]}(\phi)$ is given by $H_{\mathbb{Z}_k}^*(S^{2n})$ and so has a single generator in degrees $p = 0, \ldots, 2n - 1$ and two generators in degrees $p \geq 2n$. Computation with the long exact sequence for the triple $(E, E^\alpha, E^{-1})$ shows that a single generator $\kappa_p$ in all degrees $p \geq 0$ is lost when passing action threshold 0, hence $c(\phi, \kappa_p) = 0$ for all $p \geq 0$; denote their span $P := \langle \kappa_p : p \geq 0 \rangle \subset G_{\eta, p}^{[a, \infty]}(\phi)$. Now define the surrogate $G_{\eta, p}^{[a, \infty]}(\phi) := G_{\eta, p}^{[a, \infty]}(\phi)/P$. By construction this vector space is conjugation-invariant. Then for fixed positive integer $p$ put

$$W_a(\phi) := \begin{cases} G_{\eta, p}^{[a, \infty]}(\phi) & \text{if } a < 0 \\ G_{\eta, p}^{[a, \infty]}(\phi) & \text{if } a = 0 \\ G_{\eta, p}^{[a, \infty]}(\phi) & \text{if } a > 0. \end{cases}$$

This is a persistence module with persistence maps $(w^b_a)_\phi : W_a(\phi) \rightarrow W_b(\phi)$ for $a < b$, defined as before. Recall that $W_0$ has a single generator in each degree $p \geq 2n$; denote these $\eta^\phi$ and define

$$c_{\eta, p}^\phi(\phi) := \rho(\eta^\phi),$$

the persistence of $\eta^\phi$. The conjugation-invariance of the groups $W_a(\phi)$ for $a \in \mathbb{Z}$ implies $[c_{\eta, p}^\phi(\phi)]$ is conjugation-invariant. By taking the supremum over all $\phi$ supported in a bounded domain $W \subset \mathbb{R}^{2n} \times S^1$ one obtains a contact invariant $[c_{\eta, p}^\phi(W)]$ which is by definition monotone. From the results in Sandon [San11] one computes

$$[c_{\eta, p}^\phi(B(R))] = [\ell R] \text{ when } p = 2n\ell$$

so $[c_{\eta, p}^\phi]$ is non-trivial and may be viewed as a $\mathbb{Z}_k$-equivariant contact capacity. Moreover, this capacity is sufficient to establish that squeezing via (compactly supported) $\mathbb{Z}_k$-equivariant contactomorphisms is impossible in $\mathbb{R}^{2n} \times S^1$ and thus to prove orderability of lens spaces which Milin [Mil08] and Sandon [San11] proved by means of contact homology and equivariant GF homology groups respectively.

**Remark:** The interest of $[c_{\eta, p}^\phi]$ is primarily in confirming that the simpler capacity-based non-squeezing argument of the non-equivariant setting is also possible in the equivariant one and to view such invariants more generally from a persistence viewpoint. This argument is only simpler in a minor way, as it makes use of the existing framework of Sandon [San11]: the existence and uniqueness of $\mathbb{Z}_k$-equivariant generating functions for this setting (which she establishes by a $\mathbb{Z}_k$-equivariant adaptation of Chaperon and Théret’s arguments and uses to show her $\mathbb{Z}_k$-equivariant GF homology groups are well-defined) and also the functorial properties of these groups which in our case imply they form a $\mathbb{Z}_k$-invariant persistence module. Regarding Sandon’s computation of the groups $W_a, a \in \mathbb{R}$ for each $p \in \mathbb{N}$ which we use in (3) there is primarily one difficult case, namely when $a$ passes critical points of GF index $2n\ell$ and $2n(\ell - 1)$. She handles that case by perturbing the generating function to obtain a true Morse function and arguing directly with the Morse complex. A similar technique is also used by Milin.
II) Symplectic version  In fact Sandon also defines $\mathbb{Z}_k$-equivariant generating functions for Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ for the same $\mathbb{Z}_k$-action. As above we may define symplectic capacities $c^p_{\mathbb{Z}_k}(\cdot), p \in \mathbb{N}$. Unlike the contact version just defined, these do not give anything beyond the Viterbo capacity $c_V$ at least for balls (because $c_V$ is a real-valued symplectic invariant which already separates all balls). However, we do have

$$c^p_{\mathbb{Z}_k, \text{contact}}(\mathring{U}) = c^p_{\mathbb{Z}_k, \text{symp}}(U)$$

for $\mathring{U} := U \times S^1, U \subset \mathbb{R}^{2n}$ which was used in computing (3).

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