Casimir Scaling and String Breaking in $G_2$ Gluodynamics

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We study the potential energy between static charges in $G_2$ gluodynamics in three and four dimensions. Our work is based on an efficient local hybrid Monte-Carlo algorithm and a multi-level Lüscher-Weisz algorithm with exponential error reduction to accurately measure expectation values of Wilson- and Polyakov loops. Both in three and four dimensions we show that at intermediate scales the string tensions for charges in various $G_2$-representations scale with the second order Casimir. In three dimensions Casimir scaling is confirmed within one percent for charges in representations of dimensions 7, 14, 27, 64, 77, 77', 182 and 189 and in 4 dimensions within 5 percent for charges in representations of dimensions 7, 14, 27 and 64. In three dimensions we detect string breaking for charges in the two fundamental representations. The scale for string breaking agrees very well with the mass of the created pair of glue-lumps.

I. INTRODUCTION

There is compelling experimental evidence that the fundamental constituents of QCD, quarks and gluons, never show up as asymptotic states of strong interaction – rather they are confined in mesons and baryons. Understanding the dynamics of this confinement mechanism is one of the challenging problems in strongly coupled gauge theories. There are convincing analytical and numerical arguments to believe that confinement is a property of pure gauge theories (gluodynamics) alone and that the underlying mechanism should not depend on the number $N$ of colours. Confinement is lost at high temperatures and for gauge groups with a non-trivial center the trace of the Polyakov loop

$$P(\vec{x}) = \text{tr} \mathcal{P}(\vec{x}), \quad \mathcal{P}(\vec{x}) = \frac{1}{N} \text{tr} \left( \exp i \int_0^{\beta_T} A_0(\tau, \vec{x}) \, d\tau \right), \quad \beta_T = \frac{1}{T},$$

vanishes in the confined low-temperature phase and is close to an element of the center in the deconfined high-temperature phase. In gluodynamics or gauge theories with matter in the adjoint representation the action and measure are both invariant under center transformations, whereas the Polyakov loop transforms non-trivially and hence serves as order parameter for the global center symmetry. This means that the center symmetry is realized in the confined phase and spontaneously broken in the deconfined phase.

In the vicinity of the transition point the dynamics of the Polyakov loop is successfully described by effective 3d scalar field models for the characters of the Polyakov loop [1–4]. If one further projects the scalar fields onto the center of the gauge group then one arrives at generalized Potts models describing the effective Polyakov-loop dynamics [5]. The temperature dependent couplings constants of these effective theories have been calculated ab initio by inverse Monte Carlo methods in [3].

With dynamical quarks in the fundamental representation the center symmetry is explicitly broken and the Polyakov loop points always in the direction of a particular center element. In a strict sense the Polyakov loop ceases to be an order parameter. This is attributed to breaking of the string connecting a static ‘quark anti-quark pair’ when one tries to separate the charges. It breaks via the spontaneous creation of dynamical quark anti-quark pairs which in turn screen the individual static charges.

The pivotal role of the center for confinement also follows from a recent observation relating the Polyakov loop with center averaged spectral sums of the Dirac operator [6–8]. More precisely, for gauge groups with non-trivial center one can relate the expectation value of the Polyakov loop to dual condensates. This result could finally explain why for gauge groups with a non-trivial center and fundamental matter the transition temperatures for the deconfinement and chiral phase transitions coincide. On the contrary, for gauge theories with adjoint matter the two transition temperatures can be very different [9, 10].

To clarify the relevance of the center for confinement it suggests itself to study pure gauge theories whose gauge groups have a trivial center. For such theories the string connecting external charges can break via the spontaneous creation of dynamical ‘gluons’ such that the Polyakov loop acquires a non-vanishing expectation value for all temperatures, similarly as it does in QCD with dynamical fermions. Here the simple gauge group $SO(3)$ suggests itself and indeed the $SO(3)$ gauge theory has been studied in great detail on the lattice, see for example [11]. Unfortunately, via the non-trivial first homotopy group $\pi_1(SO(3)) = \mathbb{Z}_2$ the lattice gauge theory ‘detects’ its simply connected universal covering group $SU(2)$. To avoid the resulting lattice artifacts one should investigate theories with simply connected gauge groups with trivial center.

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and we shall use this decomposition in our simulations. The short exact sequence shows that there are at least  

\[ \pi_3(S^6) \rightarrow \pi_3(SU(3)) \rightarrow \pi_3(G_2) \rightarrow \pi_3(S^6) = 0 \]

\[ \pi_4(S^6) = \mathbb{Z} \]  

and hence there should exist \( G_2 \)-instantons of any integer topological charge. In the charge \( k \)-sector there are at least \( 3k \) magnetically charged defects [12].

From Tab. I, taken from [12], one reads off that the smallest simple Lie group with these properties is the 14-dimensional exceptional Lie group \( G_2 \). This is one reason why the group in Bern investigated \( G_2 \) gauge theories with and without Higgs fields in series of papers [13–15]. In their pioneering works it has been convincingly demonstrated that \( G_2 \)-gluodynamics shows a first order finite temperature phase transition without order parameter from a confining to a deconfining phase. In this context confinement refers to confinement at intermediate scales, where a Casimir scaling of string tensions has been reported [16]. On large scales strings will finally break due to spontaneous gluon production and the static inter-quark potential is expected to flatten [17]. However, the threshold energy for string breaking in \( G_2 \)-gauge theory is rather high and all previous attempts to detect this flattening have been without success. In the present paper we shall demonstrate that string breaking for charges in the fundamental and adjoint representations of \( G_2 \) takes place at the expected scales. To that aim we implemented a slightly modified Lüscher-Weisz multistep algorithm for high-precision measurements of the static inter-quark potential.

The present paper deals with \( G_2 \)-gluodynamics in 3 and 4 dimensions. The simulations are performed with an efficient and fast implementation of a local HMC algorithm. Below we shall calculate the potentials at intermediates scales for static charges in the 7, 14, 27, 64, 77, 77', 182 and 189-dimensional representations. We show that in 3 and 4 dimensions the string tensions on intermediate scales are proportional to the second order Casimir of the representations. The high-precision measurements in 3 dimensions confirm Casimir scaling within 1 percent. In 4 dimensions Casimir scaling for the lowest 4 representations is fulfilled within 5 percent. In 3 dimensions we also calculated the static potential for widely separated charges in the two fundamental 7-dimensional representations. In both cases we see a flattening of the potential which signals the breaking of the connecting string. The energy where string breaking sets in is in full agreement with the independently calculated masses of the glue lumps formed after string breaking.

### II. THE GROUP \( G_2 \)

The exceptional Lie-Group \( G_2 \) is the automorphism group of the octonion algebra or, equivalently, the subgroup of \( SO(7) \) that preserves any vector in its 8-dimensional real spinor representation. This means that the 8-dimensional real spinor representation of \( \text{Spin}(7) \) branches into the trivial representation and the 7-dimensional fundamental representation of \( G_2 \). The 14-dimensional fundamental representation of \( G_2 \), which at the same time is the adjoint representation, arises in the branching of the adjoint of \( SO(7) \) according to \( 21 \rightarrow 7 \oplus 14 \). The 27-dimensional representations of \( SO(7) \) acting on symmetric traceless 2-tensors remains irreducible under \( G_2 \). In this work we need the following branchings of \( SO(7) \)-representations to \( G_2 \):

\[ 7 \rightarrow 7, \quad 21 \rightarrow 14 \oplus 7, \quad 27 \rightarrow 27, \quad 35 \rightarrow 27 \oplus 7 \oplus 1, \quad 77 \rightarrow 77. \]  

For explicit calculations it is advantageous to view the elements of the 7-dimensional representation of \( G_2 \) as matrices in the defining representation of \( SO(7) \), subject to seven independent cubic constraints [15]:

\[ T_{abc} = T_{def} g_{da} g_{eb} g_{fc}. \]  

Here \( T \) is a total antisymmetric tensor given by

\[ T_{127} = T_{154} = T_{163} = T_{235} = T_{264} = T_{374} = T_{576} = 1. \]

The gauge group \( SU(3) \) of strong interaction is a subgroup of \( G_2 \) and the corresponding coset space is a sphere [18],

\[ G_2/SU(3) \sim S^6. \]  

This means that every element \( U \) of \( G_2 \) can be factorized as

\[ U = S \cdot V \quad \text{with} \quad V \in SU(3) \quad \text{and} \quad S \in G_2/SU(3), \]  

and we shall use this decomposition in our simulations. The short exact sequence

\[ 0 = \pi_4(S^6) \rightarrow \pi_4(SU(3)) \rightarrow \pi_4(G_2) \rightarrow \pi_4(S^6) = 0 \]

shows that \( \pi_3(G_2) = \mathbb{Z} \) and hence there should exist \( G_2 \)-instantons of any integer topological charge. In the charge \( k \)-sector there are at least \( 3k \) magnetically charged defects [12].

| group | \( A_2 \) | \( B_2 \) | \( C_2 \) | \( D_4 \) r even | \( D_4 \) r odd | \( E_6 \) | \( E_7 \) | \( E_8 \) | \( F_4 \) | \( G_2 \) |
|-------|--------|--------|--------|--------------|--------------|--------|--------|--------|--------|--------|
| center \( Z \) | \( Z_{r+1} \) | \( Z_2 \) | \( Z_2 \) | \( Z_2 \times Z_2 \) | \( Z_4 \) | \( Z_3 \) | \( Z_2 \) | \( \| \) | \( \| \) | \( \| \) |
Below we also use the physics-convention and denote a representation by its dimension. For example, the dimension of an arbitrary irreducible representation $R = [p, q]$ can be calculated with the help of Weyl’s dimension formula and is given by

$$d_R = \text{dim}_{p,q} = \frac{1}{120}(1 + p)(1 + q)(2 + p + q)(3 + p + 2q)(4 + p + 3q)(5 + 2p + 3q).$$  \hspace{1cm} (8)$$

Below we also use the physics-convention and denote a representation by its dimension. For example, the fundamental representations are $[1, 0] = 7$ and $[0, 1] = 14$. However, this notation is ambiguous, since there exist different representations with the same dimension. For example $[3, 0] = 77$ and $[0, 2] = 77'$ have the same dimension. An irreducible representation of $G_2$ can also be characterized by the values of the two Casimir operators of degree 2 and 6. Below we shall need the values of the quadratic Casimir in a representation $[p, q]$, given by

$$C_R \equiv C_{p,q} = 2p^2 + 6q^2 + 6pq + 10p + 18q.$$  \hspace{1cm} (9)$$

For an easy comparison we normalize these ‘raw’ Casimir values with respect to the defining representation by $C_{p,q} = C_{p,q}/C_{1,0}$. The normalized Casimir values for the eight non-trivial representations with smallest dimensions are given in Tab. II.

Quarks and gluons in $G_2$ are in the fundamental representations 7 and 14, respectively. To better understand $G_2$-gluodynamics we recall the decomposition of tensor products of these representations,

$$7 \otimes 7 = 1 \oplus 7 \oplus 14 \oplus 27$$

$$7 \otimes 14 = 7 \oplus 27 \oplus 64$$

$$14 \otimes 14 = 1 \oplus 14 \oplus 27 \oplus 77 \oplus 77'$$

$$7 \otimes 7 \otimes 7 = 1 \oplus 4 \cdot 7 \oplus 2 \cdot 14 \oplus 3 \cdot 27 \oplus 2 \cdot 64 \oplus 77'$$

$$14 \otimes 14 \otimes 14 = 1 \oplus 7 \oplus 5 \oplus 14 \oplus 3 \cdot 27 \oplus \cdots$$  \hspace{1cm} (10)$$

The decompositions (10) show that, similarly as in QCD, two or three quarks or two or three gluons can build colour singlets – mesons, baryons or glueballs. Since three gluons can screen the charge of a single (static) quark,

$$7 \otimes 14 \otimes 14 \otimes 14 = 1 \oplus \cdots,$$  \hspace{1cm} (11)$$

one expects that the string between two static quarks will break for large charge separations. The two remnants are two glue-lumps – charges screened by (at least) 3 gluons. The same happens for charges in the adjoint representation. Each adjoint charge can be screened by one gluon.

**Construction of characters from tensor products**

The character $\chi_R = \text{tr} \ R$ of any irreducible representation $R$ is a polynomial of the characters $\chi_7$ and $\chi_{14}$ of the two fundamental representations 7 and 14. For example, the first two decompositions in (10) imply

$$\chi_{27} = \chi_7 \cdot \chi_7 + \chi_7 - \chi_7 - \chi_{14}$$

$$\chi_{64} = \chi_7 \cdot \chi_{14} - \chi_7 - \chi_{27} = \chi_7 \chi_{14} - \chi_7^2 + \chi_1 + \chi_{14}$$  \hspace{1cm} (12)$$

and yield the characters of the representations 27 and 64 as polynomials of $\chi_7$ and $\chi_{14}$. From further tensor products of irreducible representations one can calculate the polynomial in $\chi_R = \text{Pol}_R(\chi_7, \chi_{14})$ for any irreducible representation $R$. For a fast implementation of our algorithms we also need reducible representations. In particular we use

$$(7 \otimes 7)_s, \quad (7 \otimes 7 \otimes 7)_s, \quad (7 \otimes 7 \otimes 7 \otimes 7)_s, \quad (7 \otimes 7)_s \otimes 14$$  \hspace{1cm} (13)$$
where the subscript ‘s’ denotes the symmetrized part of the respective tensor product. Comparing the reduction of representations for $SO(7)$ and $G_2$ and mapping representations from $SO(7)$ to $G_2$ the following characters of reducible representations can be computed

\begin{align}
\chi_{(7 \otimes 7)_0} &= \chi_{27} + \chi_1, \\
\chi_{(7 \otimes 7 \otimes 7)_1} &= \chi_{77} + \chi_7, \\
\chi_{(7 \otimes 7 \otimes 7 \otimes 7)_0} &= \chi_{182} + \chi_{77} + \chi_{27} + \chi_{64} + 2 \chi_{14} + \chi_7, \\
\chi_{(7 \otimes 7)_0} \otimes 14 &= \chi_{189} + \chi_{27} + \chi_1.
\end{align}

(14)

III. CASIMIR SCALING AND STRING BREAKING FOR $SU(N)$ GAUGE THEORIES

In QCD quarks and anti-quarks can only be screened by particles with non-vanishing 3-ality, especially not by gluons. Thus, in zero-temperature gluodynamics the potential energy for two static color charges is linearly rising up to arbitrary large separations of the charges. The potentials for charges in a representation $\mathcal{R}$ can be extracted from the 2-point correlator of Polyakov loops or the expectation values of Wilson loops with time-extent $T$ according to

\begin{equation}
\langle P_\mathcal{R}(0) P_\mathcal{R}(R) \rangle = e^{-\beta TV_\mathcal{R}(R)}, \quad \langle W_\mathcal{R}(R, T) \rangle = e^{\kappa_{\mathcal{R}} - TV_\mathcal{R}(R)}.
\end{equation}

(15)

With dynamical quarks the string should break at a characteristic length $r_b$ due to the spontaneous creation of quark-anti-quark pairs from the energy stored in the flux tube connecting the static charges. However, for intermediate separations $r < r_b$ the string cannot break since there is not enough energy stored in the flux tube.

For pure gauge theories we expect the following qualitative behavior of the static potential: At short distances perturbation theory applies and the interaction is dominated by gluon exchange giving rise to a Coulomb-like potential, $V \sim -\alpha/r$, the strength $\alpha$ being proportional to the value $C_\mathcal{R}$ of the quadratic Casimir operator in the given representation $\mathcal{R}$ of the charges; at intermediate distances, from the onset of confinement to the onset of color screening at $r_b$, the potential is expected to be linearly rising, $V \sim \sigma r$, and the corresponding string tension is again proportional to the quadratic Casimir; at asymptotic distance scales (partial) screening sets in such that the string tension typically decreases and only depends on the $N$-ality of the representation. In particular for center-blind color charges or gauge groups without center the potential flattens. The characteristic length $r_b$ where the intermediate confinement regime turns into the asymptotic screening regime is determined by the masses of the debris left after string breaking. The Casimir scaling hypothesis, according to which the string tension at intermediate scales is proportional to the quadratic Casimir of the representation [19], is exact for two dimensional continuum and lattice gauge theories and dimensional reduction arguments support that it also holds in higher dimensions. Within the Hamiltonian approach to Yang-Mills theories in $2+1$ dimensions the following prediction for the string tensions has been derived [20]

\begin{equation}
\sigma_{\mathcal{R}} = \frac{g^4}{4\pi} C_4 C_{\mathcal{R}}.
\end{equation}

(16)

For pure $SU(2)$ and $SU(3)$ gauge theories in three and four dimensions there is now conclusive numerical evidence for Casimir scaling from Monte-Carlo simulations: for $SU(2)$ in 3 dimensions [19, 21] and in 4 dimensions [22–25] as well as for $SU(3)$ in 4 dimensions at finite temperature [26] and zero temperature [27–30]. In particular the simulations for $SU(3)$ gluodynamics in [29] confirm Casimir scaling within $5\%$ for separations up to 1 fm of static charges in representations with Casimirs (normalized by the Casimir of $\{3\}$) up to 7. String breaking for charges in the adjoint representation has been found in several simulations: In 3-dimensional $SU(2)$-gluodynamics with improved action and different operators in [31, 32] and in 4-dimensional $SU(2)$-gluodynamics in [33] with the help of a variational approach involving string and glueball operators. For a critical discussion of the various approaches we refer to [34], where string breaking in a simple setting but with an improved version of the Lässcher-Weisz algorithm has been analyzed and compared with less sophisticated approaches. There is a number of works in which a violation of Casimir scaling on intermediate scales has been reported. For example, it has been claimed that in 4-dimensional $SU(N)$-gluodynamics with larger $N = 4, 6$ the numerical data favor the sin-formula, as suggested by supersymmetry, in place of the Casimir scaling formula [35]. The differences between the Casimir scaling law and sin-formula are tiny and it is very difficult to discriminate between the two predictions in numerical simulations. Indeed, in [36] agreement with Casimir scaling and sin-formula in 4-dimensions and disagreement in 3-dimensions has been claimed. In addition the high precision simulation based on the Lässcher-Weisz algorithm in [37] point to a violation of the Casimir scaling law in 3-dimensional $SU(2)$ gluodynamics. In a very recent paper Pepe and Wiese [38] reanalyzed the static potential for $SU(2)$-gluodynamics in 3 dimensions with the help of the Lässcher-Weisz algorithm and confirmed Casimir scaling at intermediate scales and 3-ality scaling at asymptotic scales.

For gauge theories with matter we expect a similar qualitative behavior: a Coulomb-like potential at short distances, Casimir scaling at intermediate distances and (partial) screening at asymptotic distances. The string tension at asymptotic scales depends both on the $N$-alities of the static color charges and of the dynamical matter. In particular, if dynamical quarks or scalars can
form center blind composites with the static charges then the potential is expected to flatten at large separations. To see any kind of screening between fundamental charges requires a full QCD simulation with sea quarks, which is demanding. Thus the earlier works dealt with gauge theories with scalars in the fundamental representation. For example, in [39] clear numerical evidence for string breaking in the 3-dimensional \( SU(2) \)-Yang-Mills-Higgs model via a mixing analysis of string and two-meson operators has been presented. Probably the first observation of hadronic string breaking in simulation of QCD with two flavors of dynamical staggered fermions using only Wilson loops have been reported in [40, 41]. Despite extensive searches for colour screening in 4-dimensional gauge theories with dynamical fermions the results are still preliminary at best. First indications for string breaking in two-flavor QCD, albeit only at temperatures close to or above the critical deconfinement temperature, have been reported in [42]. More recently Bali et al. used sophisticated methods (e.g. optimized smearing, improved action, stochastic estimator techniques, hopping parameter acceleration) to resolve string breaking in 2-flavor QCD at a value of the lattice spacing \( a^{-1} \approx 2.37 \) GeV and of the sea quark mass slightly below \( m_s \) [43]. By extrapolation they estimate that in real QCD with light quarks the string breaking should happen at \( r_b \approx 1.13 \) fm.

To measure the static potential and study string breaking three approaches have been used: correlations of Polyakov loops at finite temperature, variational ansaetze using two types of operators (for the string-like states and for the broken string state) and Wilson loops. Most results on Casimir scaling and string breaking have been obtained with the first two methods. This is attributed to the small overlap of the Wilson loops with the broken-string state. To measure Polyakov or Wilson loop correlators for charges in higher representations or to see screening at asymptotic scales one is dealing with extremely small signals down to \( 10^{-40} \). In order to measure such small signals one needs to improve existing algorithms considerably or/and use improved versions of the Lüscher-Weisz multistep algorithm.

For \textit{gauge groups with trivial centers} like \( G_2, F_4 \) or \( E_8 \) the flux tube between static charges in any representation will always break due to gluon production. The potential flattens for large separations and expectation values of the Polyakov loop never vanish [13]. However, for \( G_2 \) it changes rapidly at the phase transition temperature and is very small in the low-temperature confining phase, see Fig. 1. Similarly as in QCD we characterize confinement as the absence of free colour charges in the physical spectrum [16, 44].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{phase_transition}
\caption{Phase transition on a \( 16^3 \times 6 \) lattice in terms of the Polyakov loop in the fundamental representation.}
\end{figure}

IV. \textbf{ALGORITHMIC CONSIDERATIONS}

A. \textbf{Local hybrid Monte-Carlo}

In simulations of gauge field theories different algorithms are in use. For \( SU(N) \)-gluodynamics heat-bath algorithms based on the Cabibbo-Marinari \( SU(2) \) subgroup updates, often improved by over-relaxation steps, have proven to be fast and reliable. For QCD with dynamical fermions a hybrid Monte-Carlo (HMC) scheme is preferable. Based on [45] also local versions of HMC algorithms are available where single links are evolved in a HMC style. According to [46] the cost for the local hybrid Monte-Carlo (LHMC) is about three times more than for a combined heat-bath and overrelaxation (HOR) scheme for the case of \( SU(N) \)-gluodynamics.

For the exceptional gauge group \( G_2 \) there exists a modification of the heat-bath update [13] which combines the heat-bath update for a \( SU(3) \)-subgroup with randomly distributed \( G_2 \) gauge transformations to rotate the \( SU(3) \) subgroup through \( G_2 \). In the present work we instead use a LHMC algorithm for several good reasons: First, the formulation is given entirely in terms of Lie-group and Lie-algebra elements and there is no need to back-project onto \( G_2 \). The autocorrelation time can be controlled (in certain ranges) by the integration time in the molecular dynamics part of the HMC algorithm. Furthermore, one can use
a real representation of $G_2$ and relatively simple analytical expressions for the two involved exponential maps to obtain a fast implementation of the algorithm. Finally, the inclusion of a (normalized) Higgs field is straightforward and does not suffer from a low Metropolis acceptance rate (even for large hopping parameters).

The LHMC algorithm has been essential for obtaining the results in the present work. Since we developed the first implementation for $G_2$ it is useful to explain the technical details for this exceptional group. As any (L)HMC algorithm for gauge theories it is based on a fictitious dynamics for the link-variables on the gauge group manifold. The “free evolution” on a semisimple group is the Riemannian geodesic motion with respect to the Cartan-Killing metric

$$ds^2_G = \kappa \text{tr} (\text{d}U^{-1} \otimes \text{d}U^{-1}) .$$

(17)

In the fictitious dynamics the interaction term is given by the Yang-Mills action of the underlying lattice gauge theory and hence it suggests itself to derive the dynamics from the Lagrangian

$$L = \frac{1}{2} \sum_{x,\mu} \text{tr} \left( i \dot{U}_{x,\mu} U^{-1}_{x,\mu} \right)^2 - S_{\text{YM}}[U],$$

(18)

where ‘dot’ denotes the derivative with respect to the fictitious time parameter $\tau$ and

$$S_{\text{YM}}[U] = \frac{\beta}{2 N_c} \sum_{x,\mu} \text{tr} \left( 2 N_c - U_{x,\mu \nu} - U^1_{x,\mu \nu} \right)$$

(19)

is the Wilson action. The Lie algebra valued fictitious conjugated link momentum is given by

$$\mathcal{P}_{x,\mu} = i \frac{\partial L}{\partial \left( U_{x,\mu} U^{-1}_{x,\mu} \right)} = i U_{x,\mu} \frac{\partial L}{\partial U_{x,\mu}} = -i U_{x,\mu} \dot{U}_{x,\mu}^{-1},$$

(20)

and via a Legendre transform yields the pseudo-Hamiltonian

$$H = \frac{1}{2} \sum_{x,\mu} \text{tr} \mathcal{P}^2_{x,\mu} + S_{\text{YM}}[U].$$

(21)

The equations of motion for the momenta are obtained by varying the Hamiltonian. The variation of the Wilson action $S_{\text{YM}}[U]$ with respect to a fixed link variable $U_{x,\mu}$ is given by the corresponding staple variable $R_{x,\mu}$, the sum of triple products of elementary link variables closing to a plaquette with the chosen link variable. Hence we obtain

$$\delta H = \sum_{x,\mu} \text{tr} \left\{ \mathcal{P}_{x,\mu} \delta \mathcal{P}_{x,\mu} - \frac{\beta}{2 N_c} \dot{U}_{x,\mu} \mathcal{P}^1_{x,\mu} \left( U_{x,\mu} R_{x,\mu} - R^1_{x,\mu} U_{x,\mu} \right) \right\}$$

$$= \sum_{x,\mu} \text{tr} \mathcal{P}_{x,\mu} \left\{ \mathcal{P}_{x,\mu} - F_{x,\mu} \right\} d\tau , \quad F_{x,\mu} = \frac{i \beta}{2 N_c} \left( U_{x,\mu} R_{x,\mu} - R^1_{x,\mu} U_{x,\mu} \right) .$$

(22)

The variational principle implies that the projection of the term between curly brackets onto the Lie algebra $\mathfrak{g}_2$ vanishes,

$$\mathcal{P}_{x,\mu} = F_{x,\mu} \big|_{\mathfrak{g}_2} .$$

(23)

Choosing a trace-orthonormal basis $\{T_a\} \subseteq \mathfrak{g}_2$ the equations for the (L)HMC dynamics can be written as follows,

$$\dot{U}_{x,\mu} = \sum_a \text{tr} \left( F_{x,\mu} T_a \right) T_a \quad \text{and} \quad \dot{\mathcal{P}}_{x,\mu} = i \mathcal{P}_{x,\mu} U_{x,\mu}$$

(24)

with the “force” $F_{x,\mu}$ defined in (22). Now a LHMC sweep consists of the following steps:

1. Gaussian draw of the momentum variable on a given link.

2. Integration of the equations of motion for the given link.

3. Metropolis accept/reject step.

4. Repeat these steps for all links of the lattice.

This local version of the HMC does not suffer from an extensive $\delta H \propto V$ problem such that already a second order symplectic (leap frog) integrator allows for sufficiently large timesteps $\delta \tau$. In condensed form the integration for a link variable yields

$$U(t + \delta \tau) = \exp \left( i \mathcal{P} (t + \delta \tau/2) \delta \tau \right) U(t) .$$

(25)

For a large range of Wilson couplings $\beta$ in our simulations an integration length of $T = 0.75$ with a step size of $\delta \tau = 0.25$ is optimal for minimal autocorrelation times and a small number of thermalisation sweeps. Acceptance rates of more than 99% are reached. Nevertheless, the most time consuming part of the calculations involves the exponential maps. A calculation for $G_2$ can be implemented fast and exact up to a given order in $\delta \tau$ as will be shown in the next section.
B. The exponential map $\mathfrak{g}_2 \rightarrow G_2$

For an efficient and fast computation of the exponential map we use the real embedding of the $SU(3)$-representation $3 \oplus \bar{3}$ into $G_2$, given by

$$\mathcal{V}(W) = \Omega \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & W^* \end{array} \right) \Omega \in G_2, \quad \text{with} \quad W \in SU(3).$$

(26)

One can choose the unitary matrix $\Omega$ to have block diagonal form with $\Omega_{11} = 1$. A possible choice for $\Omega$ is

$$\Omega = \left( \begin{array}{cc} 1 & 0 \\ 0 & VQ \end{array} \right) \quad \text{with} \quad Q = \left( \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right), \quad V = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} i & 1 \\ i & 1 \end{array} \right) \otimes \mathbb{I}_3.$$

(27)

Every element of $G_2$ can be factorized as

$$\mathcal{U} = \mathcal{S} \cdot \mathcal{V}(W) \quad \text{with} \quad \mathcal{S} \in G_2/SU(3).$$

(28)

For a given timestep $\delta \tau$ in the molecular dynamics this factorization will be expressed in terms of the Lie algebra elements with the help of the exponential maps,

$$\exp \{ \delta \tau \mathcal{U} \} = \exp \{ \delta \tau \mathfrak{s} \} \cdot \exp \{ \delta \tau \mathfrak{v} \} \quad \text{with generators} \quad \mathcal{U} \in \mathfrak{g}_2, \quad \mathfrak{v} \in \mathcal{V}_\ast(\mathfrak{su}(3))$$

(29)

fulfilling the commutation relations

$$[\mathfrak{v}, \mathfrak{v}'] = \mathfrak{v}'', \quad [\mathfrak{v}, \mathfrak{s}] = \mathfrak{s}' \quad \text{and} \quad [\mathfrak{s}, \mathfrak{s}'] = \mathfrak{v}' + \mathfrak{s}''.$$

(30)

The generators $\mathfrak{s}$ are orthogonal to the generators of the really embedded $SU(3)$-subgroup. To simplify the notation we absorb the time step $\delta \tau$ in the Lie algebra elements.

The last exponential map in (29) can be calculated with the help of the embedding (26) and the exponential map for $SU(3)$, $\mathcal{W} = \exp(\mathfrak{w})$, which follows from the Cayley-Hamilton theorem for $SU(3)$-generators, see [47]. The result can be expressed in terms of the imaginary eigenvalues $w_1, w_2, w_3$ of $\mathfrak{w}$ and the differences $\delta_1 = w_2 - w_3, \delta_2 = w_3 - w_1$ and $\delta_3 = w_1 - w_2$ as follows:

$$\mathcal{W} = \exp(\mathfrak{w}) = -\frac{1}{\delta_1 \delta_2 \delta_3} (\alpha_1 \mathbb{1} + \alpha_{\mathfrak{w}} \mathfrak{w} + \alpha_{\mathfrak{w}^2} \mathfrak{w}^2)$$

(31)

with expansion coefficients

$$\alpha_1 = \sum_{i=1}^{3} \delta_i w_{i+1} w_{i+2} e^{w_i}, \quad \alpha_{\mathfrak{w}} = \sum_{i=1}^{3} \delta_i w_i e^{w_i}, \quad \alpha_{\mathfrak{w}^2} = \sum_{i=1}^{3} \delta_i e^{w_i},$$

(32)

wherein one identifies $w_{3+i}$ and $w_i$.

For the generators $\{u_1, \ldots, u_{14}\}$ of $G_2$ we use the real representation given in [44]. The $\mathfrak{su}(3)$-subalgebra formed by the elements $\{u_1, \ldots, u_8\}$ generates the really embedded $3 \oplus \bar{3}$ of $SU(3)$ and the remaining generators $\{u_9, \ldots, u_{14}\}$ generate the coset-elements $\mathcal{S}$ in the factorization (28). With this choice for the generators the real embedding (26) reads

$$\mathcal{V}(W) = \left( \begin{array}{cc} 1 & 0 \\ 0 & V_\perp \end{array} \right), \quad V_\perp = \left( \begin{array}{ccc} a_{33} & -b_{33} & a_{32} & -b_{32} & -b_{31} & a_{31} \\ b_{33} & a_{33} & b_{32} & a_{32} & a_{31} & b_{31} \\ a_{23} & -b_{23} & a_{22} & -b_{22} & -b_{21} & a_{21} \\ b_{23} & a_{23} & b_{22} & a_{22} & a_{21} & b_{21} \\ b_{13} & a_{13} & b_{12} & a_{12} & a_{11} & b_{11} \\ a_{13} & -b_{13} & a_{12} & -b_{12} & -b_{11} & a_{11} \end{array} \right).$$

(33)
where the entries are the real and imaginary parts of the elements of the $SU(3)$-matrix, $W_{ij} = a_{ij} + ib_{ij}$.

Finally, to parametrize the elements of the coset space we calculate the remaining exponential map

$$S = \exp \{ s \} \quad \text{with} \quad s = \sum_{i=1}^{6} s_i u_{8+i}.$$  

(34)

The result depends on the real parameter $\sigma = \| \vec{s} \|$ and the 6-dimensional unit-vector $\hat{s} = \vec{s}/\|\vec{s}\|$. In a $1 \times 6$-block notation the map takes the form

$$S = \begin{pmatrix} 
\cos 2\sigma & -\sin 2\sigma \; \hat{s}^T \\
\sin 2\sigma \; \hat{s} & S_\perp 
\end{pmatrix}$$  

(35)

with 6-dimensional matrix

$$S_\perp = \cos \sigma \; \mathbb{1} + \sin \sigma \; \hat{s}_\perp + (\cos 2\sigma - \cos \sigma) \hat{s}\hat{s}^T + (1 - \cos \sigma) \hat{v}\hat{v}^T.$$  

(36)

The matrix $\hat{s}_\perp$ is the $6 \times 6$ right-lower block of $s$ in (34). The unit-vector $\hat{v}^T = (\hat{s}_2, -\hat{s}_1, \hat{s}_4, -\hat{s}_3, -\hat{s}_6, \hat{s}_5)$ defining the last projector in (36) is orthogonal to the unit-vector $\hat{s}$ defining the projector $\hat{s}\hat{s}^T$.

In the numerical integration we need the exponential map for elements $u$ in $\mathfrak{g}_2$. They are related to the generators used in the factorization by the Baker-Campbell-Hausdorff formula,

$$\delta \tau \; u = \delta \tau \; (s + v) + \frac{1}{2} \delta \tau^2 \; [s, v] + \cdots$$  

(37)

Depending on the order of the symplectic integrator we must solve this relation for $s$ and $v$ up to the corresponding order in $\delta \tau$. For a second order integrator used in this work this can be done analytically since the commutator $[s, v]$ does not contain any contribution of the sub-algebra $\mathfrak{su}(3)$. The integrator used in the (L)HMC algorithm must be time reversible. It can be checked that time reversibility holds to every order in this expansion. To summarize, for a second order integrator the approximation (37) may be used in the exponentiations needed to calculate $V$ and $S$. This approximation leads to a violation of energy conservation which is of the same order as the violation one finds with a second order integrator. In comparison to the exponentiation via the spectral decomposition the method based on the factorization (28) is more than ten times faster. It is also much faster than computing the exponential map for $SO(7)$ via the Cayley-Hamilton theorem.

C. Exponential error reduction for Wilson loops

In the confining phase the rectangular Wilson loop scales as $W(L, T) \propto \exp(-\sigma L \cdot T)$. In order to estimate the string tension $\sigma$ we probe areas $LT$ ranging from 0 up to 100 and thus $W$ will vary by approximately 40 orders of magnitude. A brute force approach where statistical errors for the expectation value of Wilson or Polyakov loops decrease with the inverse square root of the number of statistically independent configurations by just increasing the number of generated configurations will miserably fail. Nevertheless, convincing results on $G_2$ Casimir scaling on intermediate scales for representations with relative Casimirs $C'_{R} \leq 5$ have been obtained in [16] with a variant of the smearing procedure. When reproducing these results we observed that the calculated string tensions depend sensitively on the smearing parameter$^1$. Thus to obtain accurate and reliable numbers for the static potential and to detect string breaking we implemented the multi-step L"uscher-Weisz algorithm with exponential error-reduction for the time transporters of the Wilson-loops [48]. With this method the absolute errors of Wilson lines decrease

$^1$ This is not the case for the ratios of string tensions.
exponentially with the temporal extent $T$ of the line. This is achieved by subdividing the lattice into $n_t$ sublattices $V_1, \ldots, V_{n_t}$ containing the Wilson loop and separated by time slices plus the remaining sublattice, denoted by $V$, see figure on the right. At the first level in a two-level algorithm the time extent of each sublattice $V_n$ is 4 such that $n_t$ is the smallest natural number with $4n_t \geq T+2$. In the figure on the right $T = 14$ and the lattice is split into four sublattices $V_1, V_2, V_3, V_4$ containing the Wilson loop plus the complement $V$. The Wilson loop is the product of parallel transporters $W = T_2^i T_3^j T_1^i T_2^j T_1^i$. If a sublattice $V_n$ contains only one connected piece of the Wilson loop (as $V_1$ and $V_4$ do) then one needs to calculate the sublattice expectation value

$$\langle T_n \rangle_n = \frac{1}{Z_n} \int DU T_n e^{-S}, \quad (38)$$

if $V_n$ contains two connected pieces (as $V_2$ and $V_3$) then one needs to calculate $\langle T_n \otimes T_n' \rangle_n$. The updates in each sublattice are done with fixed link variables on the time-slices bounding the sublattice. Calculating the expectation value of the full Wilson loop reduces to averaging over the links in the $n_t + 1$ time slices,

$$\langle W \rangle = \left\langle C \left( \langle T_1 \rangle_t \langle T_2 \rangle_t \cdots \langle T_{n_t} \rangle_t \right)^{n_t-1} \langle T_{n_t} \rangle_t \right\rangle_{\text{boundaries}} \quad (39)$$

Here $C$ is that particular contraction of indices that leads to the trace of the product $W = T_2^i \cdots T_{n_t-1}^i T_{n_t}^i T_{n_t-1}^i \cdots T_2 T_1$. In a two-level algorithm each sublattice $V_n$ is further divided into two sublattices $V_{n,1}$ and $V_{n,2}$, see right panel in the above figure, and the sublattice updates are done on the small sublattices $V_{n,k}$ with fixed link variables on the time slices separating the sublattices $V_{n,k}$. This way one finds two levels of nested averages. Iterating this procedure gives the multilevel algorithm. Since the dimensions $d_R$ grow rapidly with the Dynkin labels $[p, q]$ – for example, below we shall verify Casimir scaling for charges in the 189-dimensional representation $[2, 1]$ – it is difficult to store the many expectation values of tensor products of parallel transporters. Thus we implemented a slight modification of the Lüscher-Weisz algorithm where the lattice is further split by a space slice with hyperplane orthogonal to the plane defined by the Wilson loop, see figure on the right. The sublattice updates are done with fixed link variables on the same time slices as before and in addition on the newly introduced space slice. Instead of $n_t$ sublattices containing the Wilson loop we now have $2n_t - 2$ sublattices. But now every sublattice contains only one connected part of the Wilson loop and (39) is replaced

$$\langle W \rangle = \left\langle \text{tr} \prod_{n=1}^{2n_t-2} \langle T_n \rangle_n \right\rangle_{\text{boundaries}} \quad (40)$$

An iteration of this procedure by additional splittings of the time slices leads again to a multilevel algorithm. In the present work we use a two level algorithm with time slices of length 4 on the first and length 2 on the second level. We calculate $\langle W \rangle$ for Wilson loops (and hence transporters $T_n$) of varying sizes and in different representations. To avoid the storage of tensor products of large representations we implemented the modified algorithm as explained above.

We also applied the Lüscher-Weisz algorithm to calculate the correlators of two Polyakov loops $\langle P_R(0) P_R(R) \rangle$ on larger lattices. In this case the complete lattice is divided into sublattices separated by time slices, hence there is no complement $V$. Since the Polyakov loops are only used for lower-dimensional representations we have not split the lattice by a spatial slicing but used tensor products similar to Eq. (39). Actually for the calculations of Polyakov loop correlators we used the three-step Lüscher-Weisz algorithm.
V. STRING TENSION AND CASIMIR SCALING IN G2 GLUODYNAMICS

The static inter-quark potential is linearly rising on intermediate distances and the corresponding string tension will depend on the representation of the static charges. We expect to find Casimir scaling where the string tensions for different representations R and R' scale according to

\[ \frac{\sigma_R}{c_R} = \frac{\sigma_{R'}}{c_{R'}} \]  \tag{41}

with quadratic Casimir c_R. Although all string tensions will vanish at asymptotic scales it is still possible to check for Casimir scaling at intermediate scales where the linearity of the inter-quark potential is nearly fulfilled.

To extract the static quark anti-quark potential two different methods are available. The first makes use of the behavior of rectangular Wilson loops in representation R, with mass scale set by the string tension in the fundamental 7-representation, we plotted the potentials in ‘physical’ units, with mass scale set by the string tension in the 7-representation, \( \mu = \sqrt{\sigma_7} \). \tag{49}

\[ \langle W_R(R, T) \rangle = \exp(\kappa_R(R) - V_R(R)T) \quad \text{with} \quad V_R(R) = \gamma_R - \frac{\alpha_R}{R} + \sigma_R R. \]  \tag{42}

The potential can be extracted from the ratio of two Wilson loops with different time-extent according to

\[ V_R(R) = \frac{1}{\tau} \ln \frac{\langle W_R(R, T) \rangle}{\langle W_R(R, T + \tau) \rangle}. \]  \tag{43}

We calculated the expectation values of Wilson loops with the two-level Lüscher-Weisz algorithm and fitted the right hand side of (43) with the potential \( V_R(R) \) in (42). The fitting has been done for external charges separated by one lattice unit up to separations \( R \) with acceptable signal to noise ratios. From the fits we extracted the constants \( \gamma_R, \alpha_R \) and \( \sigma_R \) entering the static potential. For an easier comparison of the numerical results on lattices of different size and for different values of \( \beta \) we subtracted the constant contribution to the potentials and plotted

\[ \tilde{V}_R(R) = V_R(R) - \gamma_R \]  \tag{44}

in the figures. The statistical errors are determined with the Jackknife method. In addition we determined the local string tension

\[ \sigma_{\text{loc},R} \left( R + \frac{\rho}{2} \right) = \frac{V_R(R + \rho) - V_R(R)}{\rho}, \]  \tag{45}

given by the Creutz ratio

\[ \sigma_{\text{loc},R} \left( R + \frac{\rho}{2} \right) = \frac{1}{\tau \rho} \ln \frac{\langle W_R(R + \rho, T) \rangle \langle W_R(R, T + \tau) \rangle}{\langle W_R(R, T + \tau) \rangle \langle W_R(R, T) \rangle} = \frac{\alpha_R}{R(R + \rho)} + \sigma_R. \]  \tag{46}

The second method to calculate the string tensions uses correlators of two Polyakov loops,

\[ V_R(R) = -\frac{1}{\beta_T} \ln \langle P_R(0) P_R(R) \rangle. \]  \tag{47}

The correlators are calculated with the three-level Lüscher-Weisz algorithm and are fitted with the static potential \( V_R(R) \) with fit parameters \( \gamma_R, \alpha_R \) and \( \sigma_R \). Now the local string tension takes the form

\[ \sigma_{\text{loc},R} \left( R + \frac{\rho}{2} \right) = -\frac{1}{\beta_T \rho} \ln \frac{\langle P_R(0) P_R(R + \rho) \rangle}{\langle P_R(0) P_R(R) \rangle}. \]  \tag{48}

A. Casimir scaling in 3 dimensions

Most LHMC simulations are performed on a 28^3 lattice with Wilson loops of time-extent \( T = 12 \). To extract the static potentials from the ratio of Wilson loops in (43) we chose \( \tau = 2 \). The fits to the static potential (42) for charges in the fundamental 7-representation and for values \( \beta = 30, 35 \) and 40 yield the lattice parameters \( \alpha, \gamma \) and \( \beta \) given in Tab. III. To check for scaling we plotted the potentials in ‘physical’ units, \( V/\mu \), with mass scale set by the string tension in the 7-representation,
rescaled potentials fall on top of each other within error bars. This implies that the clearly visible, even for charges in the defined in (49). The distance of the charges is measured in the same system of units. The linear rise at intermediate scales is separations of the static charges show Casimir scaling.

\[ \rho = 1 \] and not only for \( R = 0, 1, 2 \) as in Tab. V. The horizontal lines are the values predicted by the Casimir scaling hypothesis. Clearly we see no sign of Casimir scaling violation on a lattice near the continuum at \( \beta = 40 \). Of course, for widely

\[ \frac{1}{\beta} = 0.15, 0.20, 0.25, 0.30 \] as tabulated in [18].

\[ \frac{V}{\mu} \] for short and intermediate separations of the static charges show Casimir scaling.

\[ \frac{C_R}{C_7} \] is given in the last row of that table.

\[ \alpha \] and \( \sigma \) of the potential (42) for the eight smallest representations are given in Tab. IV. The Casimir scaling of coefficients becomes apparent when they are divided by the corresponding coefficients of the static potential in the 7-representation.

The local string tensions extracted from the Creutz ratio can be determined much more accurately as the global string tensions extracted from fits to the static potentials. Tab. V contains the local string tensions for static charges in the eight smallest representations for \( \rho = 1 \) and different \( \gamma \). Of course, for widely

\[ \frac{C_R}{C_7} \] given in the last row of that table.

\[ \gamma \rho \sigma \] is given in the last row of that table.

\[ \frac{C_R}{C_7} \] is given in the last row of that table.

\[ \gamma \rho \sigma \] is given in the last row of that table.
TABLE V. Scaled local string tension.

| $\mathcal{R}$ | 7   | 14  | 27  | 64  | 77  | 77$'$ | 182 | 189 |
|---------------|-----|-----|-----|-----|-----|-------|-----|-----|
| $\sigma_{\mathcal{R}(1/2)/\sigma_{\mathcal{F}(1/2)}}$ | 1.9996(3) | 2.3327(5) | 3.4981 | 3.997(2) | 4.996(3) | 5.991(5) | 5.328(4) |
| $\sigma_{\mathcal{R}(3/2)/\sigma_{\mathcal{F}(3/2)}}$ | 1.99897 | 2.3311 | 3.495(5) | 3.994(4) | 4.9897 | 5.991 | 5.321(9) |
| $\sigma_{\mathcal{R}(5/2)/\sigma_{\mathcal{F}(5/2)}}$ | 1.9961 | 2.3271 | 3.484(5) | 3.9807 | 4.961 | 5.94(2) | 5.291 |
| $C'_{\mathcal{R}}$ | 2.0000 | 2.3333 | 3.5000 | 4.0000 | 5.0000 | 6.0000 | 5.333 |

FIG. 3. Unscaled potential with $\beta = 40$ on a $28^3$ lattice.

separated charges in higher dimensional representations the error bars are not negligible even for an algorithm with exponential error reduction.

### B. Lüscher term

In Tab. IV we have seen that the dimensionless coefficient $\alpha_{\mathcal{R}}$ in the static potential scales with the quadratic Casimir, similarly to the string tension. The corresponding term, if measured at distances where the flux tube has already formed, is referred to as Lüscher term. Its value has been calculated by Lüscher for charges in the fundamental representation, in $d$ dimensions $\alpha = (d - 2)\pi/24$, and it is believed to be universal [49]. The value $\alpha = \pi/24$ in 3 dimensions is off the results in Tab. III. However, since the coefficients in this table are fitted to the static potential from $\mathcal{R} = 1$ to values of $\mathcal{R}$ with acceptable signal to noise ratio, they contain contributions from the short range Coulombic tail. To calculate $\alpha_{\mathcal{R}}$ at intermediate distances we better

FIG. 4. Scaled potential with $\beta = 40$ on a $28^3$ lattice.
FIG. 5. Ratio of the local string Tension with $\beta = 40$ scaled on a $28^3$ lattice for the eight smallest representations.

use the (local) Lüscher term

\[
\alpha_{\text{loc}}(R) = \frac{R^3_{\text{loc}}}{2\beta T R^2} \ln \left( \frac{\langle P_R(0) P_R(R + \rho) \rangle \langle P_R(0) P_R(R - \rho) \rangle}{\langle P_R(0) P_R(R) \rangle^2} \right) = \frac{\alpha_R R^2}{R^2 - \rho^2},
\]

with $\rho = 1$. In Fig. 6 we plotted the local Lüscher term for charges in the 7-representation on a larger $48^3$-lattice with $\beta = 30$. Our data at intermediate distances are in agreement with the theoretical prediction $\alpha_7 = \pi/24 \approx 0.131$.

FIG. 6. Local Lüscher term on a $48^3$ lattice at $\beta = 30$.

C. String breaking and glue-lumps in 3 dimensions

To observe the breaking of strings connecting static charges at intermediate scales when one further increases the separation of the charges we performed high statistics LHMC simulations on a $48^3$ lattice with $\beta = 30$. We calculated expectation values of Wilson loops and products of Polyakov loops for charges in the two fundamental representations of $G_2$. When a string breaks then each static charge in the representation $R$ at the end of the string is screened by $N(R)$ gluons to form a colour blind glue lump. We expect that the dominant decay channel for an over-stretched string is string $\rightarrow$ gluelump + gluelump. For a string to decay the energy stored in the string must be sufficient to produce two glue-lumps. According to (11) it requires at least 3 gluons to screen a static charge in the 7-representation, one gluon to screen a charge in the 14-representation and two gluons to screen a charge in the 27-representation. We shall calculate the separations of the charges where string breaking sets in and the masses of the produced glue-lumps. The mass of such a quark-gluon bound state can be obtained from the correlation function

\[
C_R(T) = \left\langle \left( \bigotimes_{n=1}^{N(R)} F_{\mu\nu}(y) \right)_{R,a} \left( \bigotimes_{n=1}^{N(R)} F_{\mu\nu}(x) \right)_{R,b} \right\rangle \propto \exp(-m_R T),
\]
FIG. 7. Glue-lump correlator (lattice size $48^3$, $\beta = 30$).

where $\mathcal{R}(U_{yx})$ is the temporal parallel transporter in the representation $\mathcal{R}$ from $x$ to $y$ of length $T$. It represents the static sources in the representation $\mathcal{R}$. The vertical line means projection of the tensor product onto that linear subspace on which the irreducible representation $\mathcal{R}$ acts,

$$ (14 \otimes 14 \otimes \cdots \otimes 14) = \mathcal{R} \oplus \cdots. \quad (52) $$

For example, for charges in the $14$-representation the projection is simply

$$ F_{\mu \nu}(x) \bigg|_{14,a} = F^a_{\mu \nu}(x), \quad \text{where} \quad F^a_{\mu \nu} T^a = F_{\mu \nu}. \quad (53) $$

For charges in the $7$-representation we must project the reducible representation $14 \otimes 14 \otimes 14$ onto the irreducible representation $7$. Using the embedding of $G_2$ into $SO(7)$ representations one shows that this projection can be done with the help of the totally antisymmetric $\varepsilon$-tensor with $7$ indices,

$$ F_{\mu \nu}(x) \otimes F_{\mu \nu}(x) \otimes F_{\mu \nu}(x) \bigg|_{7,a} \propto F_{\mu \nu}(x) F^a_{\mu \nu}(x) F^r_{\mu \nu}(x) \varepsilon_{abcdefg} T^a_{bc} T^b_{de} T^r_{fg}. \quad (54) $$

Fig. 7 shows the logarithm of the glue-lump correlator (51) as function of the separation of the two lumps for static charges in the fundamental representations $7$ and $14$. The linear fits to the data yield the glue-lump masses

$$ m_7 = 0.46(4), \quad m_{14} = 0.767(5). \quad (55) $$

Thus we expect that the subtracted static potentials approach the asymptotic values

$$ \tilde{V}_R \rightarrow 2m_R - \gamma_R. \quad (56) $$

With the fit-values $\gamma_7 = 0.197(1)$ and $\gamma_{14} = 0.381(2)$ we find

$$ \tilde{V}_7/\mu \rightarrow 3.46, \quad \tilde{V}_{14}/\mu \rightarrow 5.52. \quad (57) $$

Fig. 8 shows the rescaled potentials for charges in the fundamental representations together with the asymptotic values (57) extracted from the glue-lump correlators. Within error bars both potentials flatten exactly at separations of the charges where the energy stored in the flux tube is twice the glue-lump energy.

A good approximation for the string breaking distance is then given by $V_R(R^c) \approx 2m_R$. Assuming Casimir scaling for the coefficients $\alpha_R$, $\gamma_R$ and $\sigma_R$ in the static potential we obtain

$$ \mu R_0^c_R = \left( \sqrt{\alpha_7 + \frac{1}{4} \left( \frac{\gamma_7}{\mu} - M_R \right)^2} - \frac{1}{2} \left( \frac{\gamma_7}{\mu} - M_R \right) \right), \quad M_R = \frac{2m_R}{\mu C_R}. \quad (58) $$

Inserting the result from the last row in Tab. III and the glue-lump masses we find $\mu R_0^c_7 = 4.00$ and $\mu R_0^c_{14} = 3.28$. These values agree well with the separations $\mu R$ in Fig. 8 where the static potentials flatten such that string breaking sets in at scales predicted by formula (58). Fig. 9 shows the local string tensions in the two fundamental representation and Fig. 10 their ratios. Especially the last plot makes clear that the string connecting charges in the adjoint representation break earlier than the string connecting charges in the $7$-representation. The formula (58) predicts $R_{14}^c = 9.40$ and just above this separation the ratio of local string tensions $\sigma_{14}(R)/\sigma_7(R)$ shows indeed a pronounced knee.
FIG. 8. Potential for both fundamental representations (lattice size $48^3$, $\beta = 30$) and corresponding glue-lump mass.

FIG. 9. Local string tension ($48^3$ lattice, $\beta = 30$).

FIG. 10. Casimir scaling of local string tension ($48^3$ lattice, $\beta = 30$).
TABLE VI. Parameters of the quark anti-quark potential in 4 dimensions.

|       | $\beta = 9.7, L = 14$ | $\beta = 10, L = 14$ | $\beta = 9.7, L = 20$ |
|-------|-----------------------|----------------------|-----------------------|
| $\gamma a$ | 0.83(8)               | 0.74(4)              | 0.68(9)               |
| $\alpha$ | 0.40(7)               | 0.33(3)              | 0.28(8)               |
| $\sigma a^2$ | 0.07(2)              | 0.042(9)             | 0.11(1)               |

D. Casimir scaling in 4 dimensions

In this last section we present our results for the static potential in 4 dimensions. The local HMC-simulations have been performed on a small $14^4$ and a larger $20^4$ lattice for different values of $\beta$. The static potentials and local string tensions have been extracted from (43) and (46), where the expectation values have been calculated with a two-step Lüscher-Weisz algorithm. Tab. VI contains the fits to the parameters in the potential for static charges in the 7-representation for these lattices and values for $\beta$.

Fig. 11 shows the static potentials in 'physical units' $\mu = \sqrt{\sigma_7}$ for charges in the 7, 14, 27 and 64-dimensional representations and coupling $\beta = 9.7$ as function of the distance between the charges in physical units. The corresponding value for $\sigma_7$ is taken from Tab. VI. The same coupling has been used in [16] on an asymmetric $14^3 \times 28$ lattice. After normalizing the potential with the quadratic Casimirs they are identical within error bars, as can be seen in Fig. 12. Our findings are in complete agreement with the results in [16] on Casimir scaling in 4-dimensional $G_2$-gluodynamics at $\beta = 9.7$ and our accurate results on Casimir scaling on intermediate scales in 3-dimensional $G_2$-gluodynamics.

Figs. 11 and 12 show the corresponding results for a weaker coupling $\beta = 10$ closer to the continuum limit. For this small coupling we can measure the potential only up to separations $\mu R \approx 1.5$ of the charges. But we can do this with high precision.
and for higher-dimensional representations. As for $\beta = 9.7$ we find that the potentials normalized with the second order Casimirs fall on top of each other. This confirms Casimir scaling for $G_2$-gluodynamics in 4 dimensions for charges in representations with dimensions $7, 14, 27, 64, 77, 77', 182$ and $189$.

![Graph 13](image13.png)

**FIG. 13.** Unscaled potential at $\beta = 10$ on a $14^4$ lattice.

![Graph 14](image14.png)

**FIG. 14.** Scaled potential at $\beta = 10$ on a $14^4$ lattice.

Finally we simulated on a much larger $20^4$ lattice at $\beta = 9.7$ in order to calculate the static potential for larger separations of the static quarks. Unfortunately the distance $\mu R \approx 3$ is still not sufficient to detect string breaking, see Fig. 15. But again the potentials normalized with the quadratic Casimirs shown in Fig. 16 are equal within error bars.

![Graph 15](image15.png)

**FIG. 15.** Unscaled potential at $\beta = 9.7$ on a $20^4$ lattice.
In Tab. VII we have listed the fit-values for the parameters of the potentials on the larger $20^4$ lattice for static charges in the representations with dimensions $7, 14$, and $27$. For all representation we find Casimir scaling of all three parameters in the potential. Unfortunately the fit-parameters cannot be determined reliably in the $64$-representation with the present data. This is attributed to larger errors for the potentials at intermediate scales, see Fig. 15, so that the parameters can only be determined from the ultraviolet part of the potential for this representation ($R < 3$) which is rather Coulomb-like than linearly rising. Much more conclusive are the local string tensions calculated on the larger lattice (now up to the $64$-representation). Tab. VIII contains the local string tensions divided by the local string tensions in the $7$-representation. These normalised values are constant up to separations of the charges where the statistical errors are under control. Compared to the corresponding numbers in $3$ dimensions, see Tab. V, we now see a slight dependence of the local string tensions from Eq. 45 on the distance $R$. Despite of the lower precision of the results in $4$ dimensions compared to the corresponding results in $3$ dimensions we again confirm Casimir scaling on intermediate scales within $5$ percent.

All our simulation results for the local string tensions $\sigma_R(R)$ normalized by $\sigma_7(R)$ on a $14^4$-lattice with $\beta \in \{9.7, 10\}$ and on a $20^4$-lattice with $\beta = 9.7$ and for $\mu R \leq 1.5$ are collected in Fig. 17. The horizontal lines in this figure show the prediction of the Casimir scaling hypothesis. The normalised data points are compatible with each other and with the hypothesis.

| $\mathcal{R}$ | $\gamma_{\mathcal{R}a}$ | $\gamma_{\mathcal{R}a}/c'_{\mathcal{R}}$ | $\alpha_{\mathcal{R}}$ | $\alpha_{\mathcal{R}}/c'_{\mathcal{R}}$ | $\sigma_{\mathcal{R}a^2}$ | $\sigma_{\mathcal{R}a^2}/c'_{\mathcal{R}}$ |
|---------------|-----------------|-------------------------------|-----------------|------------------------------|-----------------|----------------------------------|
| $7$           | 0.68 (9)        | 0.68                          | 0.28 (8)        | 0.28                          | 0.11 (1)        | 0.11                            |
| $14$          | 1.39 (4)        | 0.695                         | 0.60 (2)        | 0.30                          | 0.21 (1)        | 0.105                           |
| $27$          | 1.61 (3)        | 0.690                         | 0.69 (2)        | 0.295                         | 0.251 (9)       | 0.107                           |

| $\mathcal{R}$ | $\sigma_{\mathcal{R}(1/2)/\sigma_7(1/2)}$ | $\sigma_{\mathcal{R}(3/2)/\sigma_7(3/2)}$ | $\sigma_{\mathcal{R}(5/2)/\sigma_7(5/2)}$ | $c'_{\mathcal{R}}$ |
|---------------|---------------------------------|---------------------------------|---------------------------------|-----------------|
| $7$           | 1                               | 1.973 (1)                       | 1.92 (1)                       | 2.0000          |
| $14$          | 1.973 (1)                       | 1.987 (3)                       | 1.92 (1)                       | 2.0033          |
| $27$          | 2.294 (1)                       | 2.303 (4)                       | 2.28 (3)                       | 2.3333          |
| $64$          | 3.396 (8)                       | 3.44 (2)                        | —                               | 3.5000          |
VI. CONCLUSIONS

In the present work we implemented an efficient and fast LHMC algorithm to simulate $G_2$ gauge theory in three and four dimensions. With only a slight modification we can include a (normalized) Higgs field in the 7-representation. The corresponding results for the phase diagram of $G_2$-Yang-Mills-Higgs theory will soon be presented in a companion paper. The algorithm has been optimized with the help of the coset decomposition of group elements and the analytic expressions for the exponential maps for the two factors. In addition we implemented a slightly modified Lüscher-Weisz multi-step algorithm with exponential error reduction to measure the static potentials for charges in various $G_2$-representations. The accurate results in 3 dimensions show that all parameters of the fitted static potentials show Casimir scaling, see Tab. III. The global string tensions extracted from these fits show that possible deviations from Casimir scaling, if they exist, must be less than 4 percent. We also extracted the local string tensions from the Creutz ratios to obtain even more precise data. This way we confirm Casimir scaling with 1 percent accuracy. Thus we conclude that in 3-dimensional $G_2$-Gluodynamics the string tensions show Casimir scaling for all charges in the representations with dimensions $7, 14, 27, 64, 77, 77'$, $182$ and $189$. In passing we can check the scaling formula (16) for the string tension $\sigma_R(\beta)$ as function of the coupling $\beta \propto 1/g^2$ [20]. On a fixed lattice this formula implies that the product $\beta^2 \sigma_R(\beta)$ should be independent of $\beta$. Using the values for the string tension $\sigma_7$ in Tab. III we obtain

| $\beta$ | $\sigma_7$ | $\beta^2 \sigma_7$ |
|---------|------------|-------------------|
| 30      | 0.046(1)   | 41.4              |
| 35      | 0.0340(8)  | 41.7              |
| 40      | 0.024(1)   | 38.4              |

The numbers in the last row show that the scaling $\sigma \propto 1/\beta^2$ is almost fulfilled. In the present work we did not attempt to further clarify this interesting point by simulating at many $\beta$-values and using the more accurate local string tensions.

For charges in the two fundamental representations we performed LHMC simulations on larger lattices to detect string breaking at asymptotic scales. In 3 dimensions we observe that string breaking indeed sets in at the expected scale where the energy stored in the flux tube is sufficient to create two glue lumps. To confirm this expectation we calculated masses of glue lumps associated with static charges in the fundamental representations. In 4-dimensional $G_2$-gluodynamics we found Casimir scaling for charges in the representations $7, 14, 27$ and $64$, similarly as we did in 3 dimensions, although the uncertainties are of course larger. But within error bars we see no violation of Casimir scaling and this confirms the corresponding results in [16], obtained with a variant of the smearing procedure. To see the expected string breaking in 4 dimensions one would need larger lattices than those used in the present work.

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