ON EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS OF AN ELLIPTIC SYSTEM WITH COUPLED TERMS

YAYUN LI
Institute of Mathematics, School of Mathematical Sciences
Nanjing Normal University, Nanjing, Jiangsu, 210023, China

YUTIAN LEI
Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences
Nanjing Normal University, Nanjing, 210023, China

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Abstract. This paper is concerned with the elliptic system
\[-\Delta u = (q + 1)u^q v^{p+1}, \quad u > 0 \text{ in } \mathbb{R}^n,\]
\[-\Delta v = (p + 1)v^p u^{q+1}, \quad v > 0 \text{ in } \mathbb{R}^n,\]
where \(n \geq 3, \ p, q > 0 \text{ and } \max\{p, q\} \geq 1\). We discuss the nonexistence of positive solutions in subcritical case and stable solutions in supercritical case, the necessary and sufficient conditions of classification in the critical case, and the Joseph-Lundgren-type condition for existence of local stable solutions.

1. Introduction. In this paper, we study the following elliptic system with coupled terms
\[-\Delta u = (q + 1)u^q v^{p+1}, \quad u > 0 \text{ in } \mathbb{R}^n,\]
\[-\Delta v = (p + 1)v^p u^{q+1}, \quad v > 0 \text{ in } \mathbb{R}^n,\]
(1.1)
where \(n \geq 3, \ p, q > 0 \text{ and } \max\{p, q\} \geq 1\).

This elliptic system and the corresponding parabolic problem appear in the study of static Schrödinger theory and Bose Einstein condensate with two components. They can also be used to describe competition of biological population. In 1996, Bidaut-Véron and Raoux [1] investigated comprehensively the mathematical theory. In addition, such an elliptic system is a representative example in the class of systems with homogeneous right hand side terms because of its variational structure. The corresponding functional is
\[E(u, v) = \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla u|^2 + |\nabla v|^2) dx - \int_{\mathbb{R}^n} u^{q+1} v^{p+1} dx.\]

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Let \( w_1 = c_1 u, w_2 = c_2 v \) in (1.1) with \( c_1^{p+q} = (p + 1)^{\frac{q+1}{n-2}} (q + 1)^{-\frac{1}{n-2}} \), \( c_2^{p+q} = (p + 1)^{-\frac{1}{n-2}} (q + 1)^{\frac{1}{n-2}} \). Then \((w_1, w_2)\) satisfies
\[
\begin{cases}
-\Delta w_1 = w_1^q w_2^{p+1}, & w_1 > 0 \text{ in } \mathbb{R}^n, \\
-\Delta w_2 = w_2^p w_1^{q+1}, & w_2 > 0 \text{ in } \mathbb{R}^n.
\end{cases}
\]

As a consequence of Theorem 1.4(i) in [21], it follows that any classical solution \((u, v)\) of (1.3) satisfies \( u > v \) or \( u \leq v \). In the special case of \( w_1 = w_2 = w \), the system is reduced to the following Lane-Emden equation
\[
-\Delta w = w^{p+q+1}, \quad w > 0 \text{ in } \mathbb{R}^n.
\]

Eq. (1.4) is an important equation in the study of conformal geometry. It has been the object of a huge literature and has been studied extensively. Some foundational work has been done by Lane, Emden, Fowler, Serrin, and Chandrasekar from 1898 to 1967 (cf. [22] and the references therein). In 1981, Serrin, Ni proved foundational work has been done by Lane, Emden, Fowler, Serrin, and Chandrasekar.

If there exists a compact subset \( K \) such that the above inequality holds for any \( \varphi \in C_0^\infty(\mathbb{R}^n \setminus K) \), then \( w \) is called a finite Morse index solution. Introduce a Joseph-Lundgren exponent (cf. [15])
\[
p_{jl} = 1 + \frac{4}{n - 4 - 2\sqrt{n - 1}} \quad \text{when } n \geq 11,
\]
\[
p_{jl} = \infty \quad \text{when } n \leq 10.
\]

Farina gave the following results
1. Eq. (1.4) has stable solutions if and only if \( p + q + 1 \geq p_{jl} \);
2. Eq. (1.4) has finite Morse index solutions if and only if \( p + q + 1 \geq p_{jl} \) or \( p + q + 1 = \frac{n+2}{n-2} \).
We introduce an integral system which is helpful to study PDE system (1.1).

**Proposition 1.1** ([8]). Let \( n \geq 3 \) and \( \min\{p, q\} > 0 \). If \( (u, v) \) solves (1.1), then it also solves the integral system

\[
\begin{align*}
    u(x) &= C_1 \int_{\mathbb{R}^n} \frac{u^q(y)\nu^{p+1}(y)}{|x - y|^{n-2}} \, dy, \quad u > 0 \text{ in } \mathbb{R}^n, \\
    v(x) &= C_2 \int_{\mathbb{R}^n} \frac{v^{q+1}(y)\nu^{p}(y)}{|x - y|^{n-2}} \, dy, \quad v > 0 \text{ in } \mathbb{R}^n,
\end{align*}
\]  

(1.7)

for some positive constants \( C_1, C_2 \).

In this paper, we study existence (nonexistence) of positive solutions of (1.1) and (1.7). Here, the Serrin exponent \( \frac{n}{n-2} \), the Sobolev exponent \( \frac{n+2}{n-2} \) and the Joseph-Lundgren exponent come into play.

1.1. **Subcritical case: Liouville theorems.** We will prove two Liouville theorems for (1.7) in section 2. The Serrin exponent \( \frac{n}{n-2} \) plays a key role.

**Theorem 1.2.** Let \( n \geq 3, \ p, q > 0 \) and \( \max\{p, q\} \geq 1 \). When \( p + q + 1 \leq \frac{n}{n-2} \), (1.7) has no positive solution.

As a corollary of Proposition 1.1 and Theorem 1.2, we know that (1.1) has no positive solution if \( 1 < p + q + 1 \leq \frac{n}{n-2} \). In fact, Theorem 1.2 in [21] shows that \( u = v \) and hence (1.1) is reduced to (1.4) in the case of \( 1 < p + q + 1 \leq \frac{n}{n-2} \). By the nonexistence result of the Lane-Emden equation (cf. [18]), we also see this Liouville theorem of (1.1).

Next we consider the case of \( \frac{n}{n-2} < p + q + 1 < \frac{n+2}{n-2} \).

**Theorem 1.3.** Let \( n \geq 3 \). If \( \frac{n}{n-2} < p + q + 1 < \frac{n+2}{n-2} \), then integral system (1.7) has no positive \( C^1(\mathbb{R}^n) \)-solution satisfying \( u^{q+1}v^{p+1} \in L^1(\mathbb{R}^n) \).

It is a direct corollary of the following proposition.

**Proposition 1.4.** Let \( n \geq 3, \ p + q + 1 > \frac{n}{n-2} \), and \( (u, v) \) be a solution of (1.7). If \( u, v \in C^1(\mathbb{R}^n) \) and \( u^{q+1}v^{p+1} \in L^1(\mathbb{R}^n) \), then \( p + q + 1 = \frac{n+2}{n-2} \).

Come back to (1.1), we have the following Liouville theorem.

**Corollary 1.5.** Let \( n \geq 3 \), if \( p + q + 1 < \frac{n+2}{n-2} \), then (1.1) has no positive classical solution.

Finally, we consider a problem on the bounded domain

\[
\begin{align*}
    -\Delta u &= (q + 1)u^q v^{p+1}, \quad u > 0 \text{ in } \Omega, \\
    -\Delta v &= (p + 1)v^p u^{q+1}, \quad v > 0 \text{ in } \Omega,
\end{align*}
\]  

(1.8)

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain. By using the Liouville theorem of (1.1), we will estimate boundary blowing-up rate. To obtain the estimate, we quote the following doubling lemma in [20] by Polacik, Quittner and Souplet.

**Lemma 1.6** (Doubling Lemma). Let \((X, d)\) be a complete metric space and let \( \emptyset \neq D \subset \Sigma \subset X \), with \( \Sigma \) closed. Set \( \Gamma = \Sigma/D \). Finally let \( M : D \to (0, \infty) \) be bounded on compact subsets of \( D \) and fix a real \( K > 0 \). If \( y \in D \) is such that \( M(y) \text{dist}(y, \Gamma) > 2K \), then there exists \( x \in D \) such that

\[
M(x) \text{dist}(x, \Gamma) > 2K, \quad M(x) \geq M(y),
\]

\[
M(z) \leq 2M(x) \text{ for all } z \in D \cap \overline{B_{K/M(x)}(x)}.
\]
Based on this lemma, we first establish the equivalence between the Liouville theorem of (1.1) and the estimate of boundary blowing-up rate for solutions of (1.8) (cf. Theorem 2.1). Combining with the nonexistence of (1.1) we can obtain that if $p + q + 1 < \frac{n + 2}{n - 2}$, then

$$u(x), v(x) \leq C\text{dist}^{-\frac{2}{n+2}}(x, \partial \Omega), \quad \forall x \in \Omega. \quad (1.9)$$

Here $(u, v)$ solves (1.8).

1.2. Critical case: classification. In 2009, Chen and Li [6] proved that if $u, v \in L_{loc}^{\infty}(R^n)$ solve (1.7), then the critical condition $p + q + 1 = \frac{n + 2}{n - 2}$ implies a classification result. Namely, both $u$ and $v$ are given by (1.5).

Here, we will search for other equivalent conditions of the classification result.

**Theorem 1.7.** Let $(u, v)$ be a pair of positive solution of (1.7) with $C_1 = C_2 = 1$. Then the following items are equivalent

(i) $u, v \in L_{loc}^{\infty}(R^n)$ and $p + q + 1 = \frac{n + 2}{n - 2}$;
(ii) $u(x) \equiv v(x)$ and they are both given by the radial form (1.5);
(iii) $u, v \in L_{loc}^{\frac{(n+2)}{n-2}}(R^n)$;
(iv) $u, v \in L^t(R^n)$ for all $t > \frac{n}{n-2}$;
(v) $(u, v)$ is bounded and $u(x), v(x) = O(|x|^{2-n})$ as $|x| \to \infty$.

According to Proposition 1.1, the result above holds for (1.3). And hence for (1.1), the corresponding result is still true.

1.3. Supercritical case: stable solutions.

**Definition 1.8.** A pair of solution $(u, v)$ of (1.1) is $u$-stable, if for any $\psi \in C_0^\infty(R^n)$ there holds

$$\int_{R^n} |\nabla \psi|^2 dx \geq q(q + 1) \int_{R^n} u^{q-1} v^{p+1} \psi^2 dx. \quad (1.10)$$

If there exists some compact subset $\mathcal{K}$ of $R^n$ such that (1.10) holds for any $\psi \in C_0^\infty(R^n \setminus \mathcal{K})$, $(u, v)$ is called a pair of local $u$-stable solution. If (1.10) is replaced by

$$\int_{R^n} |\nabla \psi|^2 dx \geq p(p + 1) \int_{R^n} u^{q+1} v^{p-1} \psi^2 dx,$$

for any $\psi \in C_0^\infty(R^n)$ and $\psi \in C_0^\infty(R^n \setminus \mathcal{K})$ respectively, we call $(u, v)$ $v$-stable solution and local $v$-stable solution correspondingly.

The definition of these stable solutions is a little ‘partial’. In fact, $E(u, v)$ is a functional of two-variables. Stability often comes from nonnegative definite of hessian matrix of two order Frechet derivatives. However, this nonnegative definite property is too strong to calculate easily those ‘total’ derivatives. Here, Definition 1.8 can be employed to deduce a sufficient condition analogous as the Joseph-Lundgren type, because (1.10) is just like to calculate the ‘partial’ derivatives of $E(u, v)$. Other definitions of stable solutions on the semilinear elliptic system can be found in [4] and [11].

**Lemma 1.9.** Let $u, v$ be the positive solutions of (1.1) in $R^n$ and set

$$\overline{\pi}(r) = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} u(x) ds, \quad \overline{\nu}(r) = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} v(x) ds.$$

Then both $\lim_{r \to \infty} \overline{\pi}(r)$ and $\lim_{r \to \infty} \overline{\nu}(r)$ exist. We denote them by $a$ and $b$ respectively. Moreover, $ab = 0$. 

Theorem 1.10. Assume \( n \geq 3, \min\{p, q\} > 0 \).

(i) If \( a = 0 \) and \( q > 1 \), then (1.1) has no \( u \)-stable solution when \( n - 2 < \frac{4q+4\sqrt{q(q-1)}}{p+q} \).

(ii) If \( b = 0 \) and \( p > 1 \), then (1.1) has no \( v \)-stable solution when \( n - 2 < \frac{4p+4\sqrt{p(p-1)}}{p+q} \).

Theorem 1.11. Assume \( n \geq 3, \min\{p, q\} > 0 \).

(i) If \( a = 0 \) and \( q > 1 \), then (1.1) has a local \( u \)-stable solution when \( n - 2 \geq \frac{4q+4\sqrt{q(q-1)}}{p+q} \) or \( p + q + 1 = \frac{n+2}{2} \).

(ii) If \( b = 0 \) and \( p > 1 \), then (1.1) has a local \( v \)-stable solution when \( n - 2 \geq \frac{4p+4\sqrt{p(p-1)}}{p+q} \) or \( p + q + 1 = \frac{n+2}{2} \).

Remark 1.12. (i) In view of Lemma 1.9, it is natural to suppose \( a = 0 \) or \( b = 0 \) in Theorems 1.10 and 1.11. If \( a = b = 0 \), then Lemma 3.1 in [21] shows that \( v = \left(\frac{p+1}{q+1}\right)^{\frac{1}{2}} u \). It is also deduced by applying the maximum principle to \( v - \left(\frac{p+1}{q+1}\right)^{\frac{1}{2}} u \).

Now, (1.1) is reduced to a single equation

\[-\triangle u = (p+1)^{\frac{q+1}{q+1}} (q+1)^{\frac{p+1}{p+1}} \ u^{p+q+1} \quad \text{in} \quad \mathbb{R}^n. \]  

(1.11)

It has been studied by Farina [12].

(ii) Since the local stable solution is not necessarily stable, it is not verified that the conditions \( n - 2 < \frac{4q+4\sqrt{q(q-1)}}{p+q} \) and \( n - 2 < \frac{4p+4\sqrt{p(p-1)}}{p+q} \) are critical for nonexistence of stable solutions and local stable solutions. When \( p + q + 1 > p_j \), we can find a stable solution (cf. Remark 4.2). Thus, there is a gap between \( p + q + 1 > p_j \) and \( n - 2 < \frac{4q+4\sqrt{q(q-1)}}{p+q} \) (or \( n - 2 < \frac{4p+4\sqrt{p(p-1)}}{p+q} \)) for the existence of stable solutions.

2. Liouville theorems in subcritical case.

Proof of Theorem 1.2. The idea comes from [3] and [17]. Without loss of generality, suppose \( p \geq q \). Assume that \((u, v)\) is a pair of solution of the integral system (1.7). Then,

\[ u(x) \geq \frac{1}{(|x|+R)^{n-2}} \int_{B_R} u^q(y) v^{p+1}(y) dy, \]  

(2.1)

\[ v(x) \geq \frac{1}{(|x|+R)^{n-2}} \int_{B_R} u^{q+1}(y) v^p(y) dy. \]  

(2.2)

Take the power \( p \) for (2.2) and multiply by \( u^{q+1} \). Integrating on \( B_R \), we get

\[ \int_{B_R} u^{q+1}(x) v^p(x) dx \geq \left( \int_{B_R} \frac{u^{q+1}(x) dx}{(|x|+R)^p(n-2)} \right) \left( \int_{B_R} u^{q+1}(y) v^p(y) dy \right)^p. \]  

(2.3)
Inserting (2.1) into (2.3) we see
\[
\int_{B_R} u^{q+1}(x)v^p(x)dx \geq \left( \int_{B_R} \frac{dx}{(|x| + R)^{(p+q+1)(n-2)}} \right)^{p-1} \cdot \left( \int_{B_R} u^{q+1}(y)v^p(y)dy \right)^{q+1} \geq CR^{n-(p+q+1)(n-2) - n}.
\] (2.4)

This implies
\[
\left( \int_{B_R} u^q(y)v^{p+1}(y)dy \right)^{q+1} \left( \int_{B_R} u^{q+1}(y)v^p(y)dy \right)^{p-1} \leq CR^{(p+q+1)(n-2) - n}.
\] (2.5)

When \( p + q + 1 < \frac{n}{n-2} \), letting \( R \to \infty \) and noting \( p \geq 1 \), we can obtain a contradiction.

When \( p + q + 1 = \frac{n}{n-2} \), from (2.5) we have
\[
\int_{R^n} u^q(y)v^{p+1}(y)dy < +\infty.
\] (2.6)

Take the power \( q \) for (2.1) and multiply by \( v^{p+1} \). Integrating on \( B_R \setminus B_{\frac{R}{2}} \) we get
\[
\int_{B_R \setminus B_{\frac{R}{2}}} u^q(x)v^{p+1}(x)dx \geq \int_{B_R \setminus B_{\frac{R}{2}}} \frac{v^{p+1}(x)dx}{(|x| + R)^{q(n-2)}} \left( \int_{B_R} u^q(y)v^{p+1}(y)dy \right)^q.
\] (2.7)

Inserting (2.2) into (2.7), we have
\[
\int_{B_R \setminus B_{\frac{R}{2}}} u^q(x)v^{p+1}(x)dx \geq C \left( \int_{B_R} u^{q+1}(y)v^p(y)dy \right)^{p-1} \left( \int_{B_R} u^q(y)v^{p+1}(y)dy \right)^q.
\] (2.8)

By taking the limit as \( R \to \infty \) in (2.8) and using (2.6), we obtain \( u \equiv v \equiv 0 \) at last. Theorem 1.2 is proved. \( \square \)

Next, we consider the case of \( \frac{n}{n-2} < p + q + 1 < \frac{n+2}{n-2} \).

**Proof of Proposition 1.4.** By an analogous derivation of (1.3), we can assume \( C_1 = C_2 = 1 \) in (1.7). Suppose that \( (u, v) \) is a pair of classical solution of (1.7) which satisfies \( u^{q+1}v^{p+1} \in L^1(R^n) \). Thus, we can find \( R_m \to \infty \) such that
\[
R_m \int_{\partial B_{R_m}(0)} u^{q+1}v^{p+1}ds \to 0.
\] (2.9)

Therefore, integrating by parts yields
\[
\int_{R^n} x \cdot \nabla [u^{q+1}(x)v^{p+1}(x)]dx = -n \int_{R^n} u^{q+1}(x)v^{p+1}(x)dx.
\] (2.10)
For $\lambda > 0$, $v(\lambda x) = \lambda^2 \int_{R^n} \frac{u^{q+1}(\lambda y)v^p(y)}{|x-y|^{n-2}} dy$. Differentiate both sides with respect to $\lambda$ and let $\lambda = 1$. Then,

$$x \cdot \nabla v(x) - 2v(x) = \int_{R^n} \frac{y \cdot \nabla (u^{q+1}(y)v^p(y))}{|x-y|^{n-2}} dy. \quad (2.11)$$

Similarly, we have

$$x \cdot \nabla u(x) - 2u(x) = \int_{R^n} \frac{y \cdot \nabla (u^q(y)v^{p+1}(y))}{|x-y|^{n-2}} dy. \quad (2.12)$$

Multiplying (2.12) by $u^q v^{p+1}$ and integrating, we obtain

$$\int_{R^n} u^q(x)v^{p+1}(x)[x \cdot \nabla u(x)]dx - 2\int_{R^n} u^{q+1}(x)v^{p+1}(x)dx = \int_{R^n} u^q(x)v^{p+1}(x) \int_{R^n} \frac{y \cdot \nabla (u^q(y)v^{p+1}(y))}{|x-y|^{n-2}} dy dx. \quad (2.13)$$

Using the Fubini theorem and (1.7), we compute that the right hand side of (2.13)

$$\int_{R^n} u^q(x)v^{p+1}(x) \int_{R^n} \frac{y \cdot \nabla (u^q(y)v^{p+1}(y))}{|x-y|^{n-2}} dy dx$$

$$= \int_{R^n} y \cdot \nabla[u^q(y)v^{p+1}(y)] \int_{R^n} u^{q+1}(x)dx dy$$

$$= \int_{R^n} y \cdot \nabla[u^q(y)v^{p+1}(y)]u(y)dy.$$

Integrating by parts, and using (2.9), we deduce that

$$\int_{R^n} u^q(x)v^{p+1}(x) \int_{R^n} \frac{y \cdot \nabla (u^q(y)v^{p+1}(y))}{|x-y|^{n-2}} dy dx$$

$$= -n \int_{R^n} u^{q+1}(x)v^{p+1}(x)dx - \int_{R^n} u^q(x)v^{p+1}(x)[x \cdot \nabla u(x)]dx.$$

Combining with (2.13), we obtain

$$\int_{R^n} [2(q+1)u^q(x)v^{p+1}(x)[x \cdot \nabla u(x)] + (n-2)(q+1)u^{q+1}(x)v^{p+1}(x)]dx = 0. \quad (2.14)$$

Similarly, from (2.11) we also get

$$\int_{R^n} [2(p+1)u^{q+1}(x)v^p(x)[x \cdot \nabla v(x)] + (n-2)(p+1)u^{q+1}(x)v^{p+1}(x)]dx = 0. \quad (2.15)$$

Clearly,

$$\nabla[u^{q+1}v^{p+1}] = (q+1)u^q v^{p+1}\nabla u + (p+1)u^{q+1}v^p \nabla v.$$

Summing (2.14) and (2.15) together implies that

$$2\int_{R^n} x \cdot \nabla (u^{q+1}(x)v^{p+1}(x))dx + (n-2)(p+q+2)\int_{R^n} u^{q+1}(x)v^{p+1}(x)dx = 0.$$

Inserting (2.10) into this result, we see

$$[(n-2)(p+q+2) - 2n] \int_{R^n} u^{q+1}(x)v^{p+1}(x)dx = 0.$$

Therefore, $p + q + 1 = \frac{n+2}{n-2}$ is derived finally. Proposition 1.4 is complete.
Proof of Corollary 1.5. As a corollary of Proposition 1.1 and Theorem 1.2, we know that (1.1) has no positive solution if \( p + q + 1 \leq \frac{n}{n-2} \). In fact, Theorem 1.2 in [21] shows that \( u \equiv v \) and (1.1) is reduced to (1.4) in the case of \( p + q + 1 \leq \frac{n}{n-2} \). By the nonexistence result of the Lane-Emden equation (cf. [18]), we also see this Liouville theorem of (1.1).

When \( \frac{n}{n-2} < p + q + 1 < \frac{n+2}{n-2} \), (1.1) maybe has a pair of solution \((u, v)\) which satisfies that one decays to zero and the other converges to a positive constant. This is different from the Liouville theorem of (1.4).

In fact, Corollary 6.4 in [1] shows that if \((u, v)\) solves (1.1) on an exterior domain, when \(|x| \to +\infty\), either

\[ u, v = O(|x|^{-\frac{2}{p+q}}) \tag{2.16} \]

or \( q > \frac{n}{n-2} \) and

\[ |x|^\theta u(x) \to C_1, \quad \text{and} \quad v \to C_2 \tag{2.17} \]

for some positive constants \( C_1 \) and \( C_2 \), where \( \theta = \frac{2}{q-1} \) or \( \theta = n-2 \). In both cases (2.16) and (2.17), we can always deduce \( u^{q+1}v^{p+1} \in L^1(\mathbb{R}^n) \). By Proposition 1.1 and Theorem 1.3, we can obtain the conclusion in Corollary 1.5.

Next, we consider the problem on bounded domain

\[
\begin{align*}
-\Delta u &= (q + 1)u^q v^{p+1}, \quad u > 0 \text{ in } \Omega, \\
-\Delta v &= (p + 1)v^p u^{q+1}, \quad v > 0 \text{ in } \Omega,
\end{align*}
\tag{2.18}
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain. We will verify (1.9) in the case of \( 1 < p+q+1 \leq \frac{n}{n-2} \). In fact, it is a corollary of Corollary 1.5 and the following result.

**Theorem 2.1.** Let \( p + q + 1 < \frac{n+2}{n-2} \). The following two items are equivalent

1. Eqs. (1.1) has no bounded positive classical solution.
2. Positive solution \((u, v)\) of (2.18) satisfies estimates of the boundary blow-up rate

\[ u(x), v(x) \leq Cdist^{-\frac{2}{p+q}}(x, \partial \Omega), \quad \forall x \in \Omega. \tag{2.19} \]

**Proof.** We claim that, if (1.1) does not admit any bounded solution in \( \mathbb{R}^n \). Then there exists \( C = C(n, p, q) > 0 \) (independent of \( \Omega \) and \((u, v)\)) such that any solution \((u, v)\) of (2.18) satisfies (2.19).

Assume that the above conclusion fails. Then, there exists sequences \( \Omega_k, (u_k, v_k), y_k \in \Omega_k \), such that \((u_k, v_k)\) solves (1.1) on \( \Omega_k \), and

\[ M_k := u_k^{\frac{p+1}{q+1}} + v_k^{\frac{p+1}{q+1}}, k = 1, 2, \ldots \]

satisfies \( M_k(y_k) > 2k \text{ dist}^{-1}(y_k, \partial \Omega_k) \). By the Doubling Lemma (Lemma 1.6), it follows that there exists \( x_k \in \Omega_k \) such that

\[ M_k(x_k) > 2k \text{ dist}^{-1}(x_k, \partial \Omega_k), \quad M_k(z) \leq 2M_k(x_k), \quad |z - x_k| \leq k M_k^{-1}(x_k). \]

Write \( \lambda_k = M_k^{-1}(x_k) \) and

\[ \tilde{u}_k(y) := \lambda_k^{\frac{-2}{p+1}} u_k(x_k + \lambda_k y), \quad \tilde{v}_k(y) := \lambda_k^{\frac{-2}{q+1}} v_k(x_k + \lambda_k y), \quad |y| \leq k. \tag{2.20} \]

Clearly, \( u_k, v_k \) are also solutions of system (1.1) for \(|y| \leq k\), and they also satisfy

\[ [\tilde{u}_k^{\frac{p+1}{p-1}} + \tilde{v}_k^{\frac{q+1}{q-1}}](0) = 1, \tag{2.21} \]

\[ [\tilde{u}_k^{\frac{p+1}{p-1}} + \tilde{v}_k^{\frac{q+1}{q-1}}](y) \leq 2, \quad |y| \leq k. \tag{2.22} \]
By using elliptic $L^q$ estimates and standard imbedding theorem, we deduce that some subsequence of $(\tilde{u}_k, \tilde{v}_k)$ converges in $C^1_{\text{loc}}(\mathbb{R}^n)$ to a classical solution $(\tilde{u}, \tilde{v})$ of (1.1) in $\mathbb{R}^n$. Moreover $|\tilde{u}^{\frac{p+2}{2}} + \tilde{v}^{\frac{q+2}{2}}| \equiv 0$ by (2.21), hence $(\tilde{u}, \tilde{v})$ is nontrivial, and moreover, $\tilde{u}, \tilde{v}$ are bounded due to (2.22). This contradicts the assumption of Theorem 2.1.

On the contrary, if the classical positive solution $(u, v)$ of (1.1) satisfies the estimate (2.19), by letting $\Omega \rightarrow \mathbb{R}^n$, we deduce $u = v = 0$ in $\mathbb{R}^n$. This shows the nonexistence of positive solution.

**Remark 2.2.** Theorem 2.1 only works for positive classical solutions. Otherwise, semi-trivial constant solution $(0, 1)$ of (1.1) also satisfies (2.18) and (2.19).

Eq. (2.19) implies an estimate of solutions in the exterior domain. Namely, from (2.19), we can deduce

$$\bar{u}(x), \bar{v}(x) \leq C|x|^{-\frac{2}{p+q}}, \quad |x| \geq 2R.$$  

Here $(\bar{u}, \bar{v})$ solves (2.18) in $\mathbb{R}^n \setminus B_R(0)$.

In fact, if $(\bar{u}, \bar{v})$ solves (2.18) in $\mathbb{R}^n \setminus B_R(0)$, then its Kelvin transformation $(u, v)$ solves (2.18) in $B_R(0)$. Here $\bar{u}(x) = |x|^p \bar{u}(\frac{x}{|x|^p})$. Let $|x| \geq 2R$, then $\frac{x}{|x|^p} \in B_{\frac{1}{2}}(0)$. From (2.19) with $\Omega = B_R(0)$, it follows that

$$\bar{u}(x) \leq C \frac{1}{|x|^\frac{2}{p+q}} \left( \frac{1}{R} - \frac{1}{|x|} \right)^{-\frac{2}{p+q}} \leq C(2R)^{\frac{2}{p+q}} \frac{1}{|x|^{\frac{2}{p+q}}}.$$  

By the same way of argument above, we can also obtain $\bar{u}(x) \leq C|x|^{-\frac{2}{p+q}}$.

More asymptotic results about the exterior domain problem can be seen in [1].

3. **Critical case: equivalent conditions of classification.** In this section, we will give several necessary and sufficient conditions of the classification of positive solutions of integral system (1.7).

Similar to the derivation of (1.3), we can obtain another corresponding system of (1.7). Without loss of generality, in this section we always omit those constants $C_1, C_2$ in (1.7).

**Lemma 3.1.** If positive functions $u, v \in L^{\infty}_{\text{loc}}(\mathbb{R}^n)$ solve (1.7) with $p + q + 1 = \frac{n+2}{n-2}$, then $u, v$ are given by the radial form (1.5).

**Proof.** It is a direct corollary of Theorem 5 in [6].

**Lemma 3.2.** The radial function (1.5) belongs to $L^\frac{n(p+q)}{2}(\mathbb{R}^n)$ as long as $p+q+1 > \frac{n}{n-2}$ is true.

**Proof.** Clearly, this radial function $w$ is bounded and $w(x) \leq c|x|^{2-n}$ as $|x| > R$ for some $R > 0$. Therefore, by $p + q + 1 > \frac{n}{n-2}$, there holds

$$\int_{\mathbb{R}^n} w^{\frac{n(p+q)}{2}}(x) dx \leq \int_{B_R(0)} w^{\frac{n(p+q)}{2}}(x) dx + \int_{\mathbb{R}^n \setminus B_R(0)} w^{\frac{n(p+q)}{2}}(x) dx < \infty.$$  

Lemma 3.2 is proved.

**Lemma 3.3.** If $u, v \in L^\frac{n(p+q)}{2}(\mathbb{R}^n)$ solve (1.7), then $u, v \in L^t(\mathbb{R}^n)$ for all $t > \frac{n}{n-2}$. On the other hand, $u, v \notin L^t(\mathbb{R}^n)$ for all $t \leq \frac{n}{n-2}$. 
Proof. Write \( w = u + v \). Then \( w \in L^{\frac{n(p+q)}{2}}(R^n) \). From (1.7), it follows that \( w \) satisfies
\[
w(x) = K(x) \int_{R^n} \frac{w^{p+q+1}(y)dy}{|x-y|^{n-2}}.
\]
(3.1)
Here \( K(x) > 0 \) is bounded. Set \( w_A = w \) if \( |x| > A \) or \( w > A; w_A = 0 \) if \( |x| \leq A \) and \( w \leq A \). For \( f \in L^s(R^n) \) with \( s > \frac{n}{n-2} \), define
\[
Tf(x) := K(x) \int_{R^n} \frac{w^{p+q}(y)f(y)dy}{|x-y|^{n-2}},
\]
and
\[
F(x) := K(x) \int_{R^n} \frac{(w-w_A)^{p+q+1}(y)dy}{|x-y|^{n-2}}.
\]
Therefore, \( w \) solves the operator equation \( f = Tf + F \). By the Hardy-Littlewood-Sobolev inequality, we get \( \|Tf\|_s \leq C\|w^{p+q}f\|^{\frac{p}{n-2}} \leq C\|w_A\|^{\frac{p+q}{n-2}} \|f\|_s \) and \( \|F\|_s \leq C\|w-w_A\|^{\frac{p+q+1}{n-2}} \leq C \). Thus, \( T \) is a contraction map from \( L^s(R^n) \) to itself for all \( s > \frac{n}{n-2} \). In view of \( \frac{p+q+1}{2} > \frac{n}{n-2} \) (Otherwise, (1.7) has no solution, which is implied by Theorem 1.2), \( T \) is also a contraction map from \( L^{n(p+q)/2}(R^n) \) to itself. By the regularity lifting lemma of Theorem 3.3.1 in [7], we obtain \( w \in L^s(R^n) \) for \( s > \frac{n}{n-2} \). Thus, \( u, v \in L^s(R^n) \) for \( s > \frac{n}{n-2} \).

On the other hand, \( u, v \in L^s(R^n) \) \( \forall t > \frac{n}{n-2} \).

Lemma 3.4. Assume \( u, v \) are positive solutions of (1.7). If
\[
u, v \in L^t(R^n) \quad \forall t > \frac{n}{n-2},
\]
then \( u, v \) are bounded. Moreover, \( u(x), v(x) = O(|x|^{2-n}) \) as \( |x| \to \infty \).

Proof. Step 1. We claim that \( u, v \) are bounded.

For \( r > 0 \), write
\[
w(x) = K(x) \int_{B_R(x)} \frac{w^{p+q+1}(y)dy}{|x-y|^{n-2}} + K(x) \int_{R^n \setminus B_R(x)} \frac{w^{p+q+1}(y)dy}{|x-y|^{n-2}} \leq K_1 + K_2.
\]
In view of (3.2) and \( p + q + 1 > \frac{n}{n-2} \), we obtain \( w \in L^{p+q+1}(R^n) \). Thus \( K_2 \) is bounded. On the other hand, by the Hölder inequality and Lemma 3.3, we can see that \( K_1 \) is also bounded.

Step 2. \( \lim_{|x| \to \infty} u(x)|x|^{n-2} = \|u^{p+q+1}\|_{L^1(R^n)} < \infty \).

By (3.2) and \( p + q + 1 > \frac{n}{n-2} \), we have \( u^{p+q+1} \in L^1(R^n) \).

For fixed \( R > 0 \), write \( L_1 := \int_{B_R(0)} u^{p+q+1}(y)\left|\frac{|x|^{n-2}}{|x-y|^{n-2}} - 1\right|dy \). Noting \( u^{p+q+1} \in L^1(R^n) \) and using the Lebesgue dominated convergence theorem, we get \( \lim_{|x| \to \infty} L_1 = 0 \).

When \( R \to \infty \), \( L_2 := \int_{R^n \setminus B_R(0)} \int_{B(x, \frac{|x|}{R}) \setminus B(x, \frac{|x|}{R})} u^{p+q+1}(y)\left|\frac{|x|^{n-2}}{|x-y|^{n-2}} - 1\right|dy \). Noting \( u^{p+q+1} \in L^1(R^n) \) and using the Lebesgue dominated convergence theorem, we get \( \lim_{|x| \to \infty} L_2 = 0 \).

To estimate \( L_3 := \int_{B(x, \frac{|x|}{R}) \setminus B(x, \frac{|x|}{R})} u^{p+q+1}(y)\left|\frac{|x|^{n-2}}{|x-y|^{n-2}} - 1\right|dy \), we should observe that \( u, v \) are radially symmetric and decreasing about some \( x_0 \in R^n \). In fact, the argument in section 2 of [24] still works here via the method of moving planes in integral forms established by Chen-Li-Ou [9]. Thus, we can view the maximum point \( x_0 \).
as the origin since $|x|$ is sufficiently large. Write $r = |x|$, and define $\tilde{u}(r) = u(x)$, $\tilde{v}(r) = v(x)$. Thus,

$$L_3 \leq C\tilde{u}'(\frac{|x|}{2})\tilde{u}^{p+1}(\frac{|x|}{2})|x|^n. \quad (3.3)$$

On the other hand, the integrability result (3.2) shows that $u, v \in L^t(R^n)$ with $\frac{1}{t} = \frac{n-2-q}{2}$. Here $\epsilon > 0$ is sufficiently small. This integrability result, together with the decreasing property of $u$, implies

$$\tilde{u}'(\frac{|x|}{2})|x|^n \leq C\int_{B(0,\frac{|x|}{2})\setminus B(0,\frac{|x|}{4})} u'(y)dy \leq C.$$ 

This means $\tilde{u}'(\frac{|x|}{2}) \leq C|x|^{-\frac{n}{t}}$. Similarly, we have $\tilde{v}'(\frac{|x|}{2}) \leq C|x|^{-\frac{n}{t}}$. Combining this consequence with (3.3) yields $L_3 \leq C|x|^{n(1-\frac{q+1}{t}+\frac{t}{p})} \to 0$ as $|x| \to \infty$.

Combining all the estimates for $L_1, L_2$ and $L_3$, we complete the proof. \hfill \square

Clearly, if $u, v$ are bounded and $u(x), v(x) = O(|x|^{2-n})$ as $|x| \to \infty$, then $u, v \in L^{\frac{n}{n-2+q}}(R^n)$ by the same argument of Lemma 3.2. So far, we have obtained the results of

$$(i) \implies (ii) \implies (iii) \implies (iv) \iff (v).$$

From now on, we shall use the Pohozaev identity to prove $(iv) \implies (i)$.

**Lemma 3.5.** Assume $u, v$ solve (1.7). If $u, v \in L^1(R^n)$ for all $t > \frac{n}{n-2}$, then $u, v \in L^\infty(R^n)$ and $p + q + 1 = \frac{n+2}{n-2}$.

**Proof.** By Lemma 3.4, $u, v \in L^\infty(R^n)$. In the following, we only verify $p + q + 1 = \frac{n+2}{n-2}$.

By an analogous argument of sections 5 and 6 in [16], we can obtain that the positive solutions $u, v$ of (1.7) belong to $C^1(R^n)$ as long as $u, v \in L^\infty(R^n)$.

By Lemma 3.4, we can deduce that $u^{q+1}v^{p+1} \in L^1(R^n)$. Then we obtain $p + q + 1 = \frac{n+2}{n-2}$ from Proposition 1.4 immediately. \hfill \square

4. Supercritical case: stable solutions and Joseph-Lundgren condition.

This section is devoted to the proofs of Lemma 1.9, Theorem 1.10 and Theorem 1.11. For convenience, we denote $p + q + 1$ by $\mu$.

**Proof of Lemma 1.9.** Here we only discuss the function $\pi(x)$. The corresponding result about $\pi(x)$ can be obtained analogously.

(i) From $-\Delta u = (q+1)u^q v^{p+1}$, it follows that

$$-\Delta \pi = (q+1)\overline{u^q v^{p+1}} > 0,$$

and hence

$$-\pi'' - r^{-1} \pi' = -r^{1-n}(r^{n-1} \pi')' > 0.$$ 

Integrating from 0 to $r$, we obtain $\pi'(r) < 0$. It means that $\pi$ is a monotone decreasing function, and hence $\lim_{r \to \infty} \pi(r)$ exists.

(ii) Assume $w = \min\{u, v\} \geq \min\{a, b\} > 0$, then we have $-\Delta \pi \geq \overline{(\pi)^\mu} > C_0$ for some positive constant $C_0$. Similar to the argument in (i), we get $r^{1-n}(r^{n-1} \pi')' < -C_0$. Integrating twice, we have

$$\pi(r) < \pi(0) - \frac{C_0^2}{2}r^2.$$
Thus, there must be a suitable \( r > 0 \), such that the right side of the above inequality equals 0. Therefore, \( \pi(r) < 0 \). This is a contradiction. So there must be \( a = 0 \) or \( b = 0 \).

Hereafter, we only discuss existence/nonexistence of the u-stable solutions and the local u-stable solutions. Thus, we just need \( a = 0 \) and \( q > 1 \) in the proofs of Theorems 1.10 and 1.11. The argument of the v-stable solutions and the local v-stable solutions is analogous.

Before verifying Theorem 1.10, we need the following result.

**Proposition 4.1.** If \( a = 0 \), let \((u, v)\) be a u-stable solution of (1.1) with \( p > 0, q > 1 \). Then for any \( \gamma \in [1, 2q + 2\sqrt{q(q - 1)} - 1) \) and any integer \( m \geq \max\left(\frac{p + 1}{\gamma - 1}, 2\right) \), there exists a constant \( C > 0 \) depending only on \( p, q, m \) and \( \gamma \), such that

\[
\int_{R^n} u^{\gamma + \gamma} \psi^{2m} dx \leq C \int_{R^n} (|\nabla \psi|^2 + |\psi| |\nabla \psi|)^{\frac{\gamma + 1}{\gamma - 1}} dx \tag{4.1}
\]

for all test functions \( \psi \in C^0_0(R^n) \) satisfying \( |\psi| \leq 1 \) in \( R^n \).

**Proof.** According to Lemma 3.1 in [21], if \( a = 0 \), i.e. \( \lim_{r \to \infty}\inf r = 0 \), then \( w_1 \leq w_2 \), where \((w_1, w_2)\) solves (1.3). Thus,

\[
 u \leq \left(\frac{q + 1}{p + 1}\right)^{\frac{1}{q}} v. \tag{4.2}
\]

Next, we split the proof into three steps.

**Step 1.** We claim that for any \( \varphi \in C^\infty_0(R^n) \), there holds

\[
\int_{R^n} |\nabla (u^{\gamma + \gamma})|^2 \varphi^2 dx = \frac{(\gamma + 1)^2}{4\gamma} (q + 1) \int_{R^n} u^{\gamma + \gamma} u^{p+1}\varphi^2 dx + \frac{\gamma + 1}{4\gamma} \int_{R^n} u^{\gamma + 1}\Delta(\varphi^2) dx. \tag{4.3}
\]

In fact, multiplying both sides of the first equation in (1.1) by \( u^\gamma \varphi^2 \) and integrating by parts, we get

\[
\int_{R^n} \gamma |\nabla u|^2 u^{-1} \varphi^2 dx + \int_{R^n} \nabla u \nabla (\varphi^2) u^\gamma dx = (q + 1) \int_{R^n} u^{\gamma + 1} u^{p+1} \varphi^2 dx.
\]

The left hand side is equal to

\[
\frac{\gamma}{(\frac{\gamma + 1}{2})^2} \int_{R^n} |\nabla (u^{\gamma + \gamma})|^2 \varphi^2 dx + \int_{R^n} \nabla \left( \frac{u^{\gamma + 1}}{\gamma + 1} \right) \nabla (\varphi^2) dx
\]

\[
= \frac{\gamma}{(\frac{\gamma + 1}{2})^2} \int_{R^n} |\nabla (u^{\gamma + \gamma})|^2 \varphi^2 dx - \int_{R^n} \frac{u^{\gamma + 1}}{\gamma + 1} \Delta(\varphi^2) dx.
\]

Identity (4.3) then follows by combining two results above.

**Step 2.** We claim that for any \( \varphi \in C^\infty_0(R^n) \), there holds

\[
[(p + 1) \frac{\gamma + 1}{\gamma - 1} (q + 1) \frac{1}{\gamma} \left( q - \frac{1}{4\gamma} \right) \frac{(\gamma + 1)^2}{4\gamma}] \int_{R^n} u^{\gamma + \gamma} \varphi^2 dx
\]

\[
\leq \int_{R^n} u^{\gamma + 1} |\nabla \varphi|^2 dx + \left( \frac{\gamma + 1}{4\gamma} - \frac{1}{2} \right) \int_{R^n} u^{\gamma + 1} \Delta(\varphi^2) dx. \tag{4.4}
\]
In fact, take \( \psi = u^{\frac{q+1}{2}} \) in (1.10). Thus,

\[
q(q + 1) \int_{\mathbb{R}^n} u^{q+\gamma} v^{p+1} \varphi^2 dx \\
\leq \int_{\mathbb{R}^n} |\nabla (u^{\frac{q+1}{2}})|^2 \varphi^2 dx + \int_{\mathbb{R}^n} u^{\gamma+1} |\nabla \varphi|^2 dx - \int_{\mathbb{R}^n} \frac{1}{2} u^{\gamma+1} \Delta (\varphi^2) dx.
\]

Inserting (4.3) into this result, we have

\[
q(q + 1) \int_{\mathbb{R}^n} u^{q+\gamma} v^{p+1} \varphi^2 dx \\
\leq \left( \frac{(\gamma + 1)^2}{4q} \right) (q + 1) \int_{\mathbb{R}^n} u^{q+\gamma} v^{p+1} \varphi^2 dx + \frac{\gamma + 1}{4\gamma} \int_{\mathbb{R}^n} u^{\gamma+1} \Delta (\varphi^2) dx \\
+ \int_{\mathbb{R}^n} u^{\gamma+1} |\nabla \varphi|^2 dx - \frac{1}{2} \int_{\mathbb{R}^n} u^{\gamma+1} \Delta (\varphi^2) dx.
\]

Combining with (4.2), we immediately obtain identity (4.4).

Step 3. We claim that for any \( \gamma \in [1,2q + 2\sqrt{q(q-1)} - 1] \) and any integer \( m \geq \max\{\frac{q+1}{p}, 2\} \), there exists a constant \( C > 0 \), depending only on \( p, q, m \) and \( \gamma \), such that

\[
\int_{\mathbb{R}^n} u^{\mu+\gamma} \psi^{2m} dx \leq C \int_{\mathbb{R}^n} (|\nabla \psi|^2 + |\psi||\Delta \psi|)^{\frac{q+1}{q+\gamma}} dx
\]

for all test functions \( \psi \in C_0^\infty(\mathbb{R}^n) \) satisfying \( |\psi| \leq 1 \) in \( \mathbb{R}^n \).

In fact, from (4.4), it follows that for any \( \varphi \in C_0^\infty(\mathbb{R}^n) \),

\[
\alpha \int_{\mathbb{R}^n} u^{\mu+\gamma} \varphi^{2m} dx \leq \int_{\mathbb{R}^n} u^{\gamma+1} |\nabla \varphi|^2 dx + \beta \int_{\mathbb{R}^n} u^{\gamma+1} \varphi \Delta \varphi dx,
\]

where \( \alpha = (p + 1) \frac{\gamma + 1}{q + 1} (q + 1) \frac{\gamma + 1}{\gamma + 2} (q - \frac{\gamma + 1}{\gamma + 2}) \), \( \beta = \frac{\gamma + 1}{\gamma + 2} \). In view of \( p, q > 1 \) and \( \gamma \in [1, 2q + 2\sqrt{q(q-1)} - 1] \), we have \( \alpha > 0 \) and \( \beta \leq 0 \).

For any \( \psi \in C_0^\infty(\mathbb{R}^n) \) satisfying \( |\psi| \leq 1 \) in \( \mathbb{R}^n \), we take \( \varphi = \psi^m \) in (4.6). A direct computation implies

\[
\int_{\mathbb{R}^n} u^{\mu+\gamma} \psi^{2m} dx \leq C_1 \int_{\mathbb{R}^n} u^{\gamma+1} |\nabla \psi|^{2m-2} |\nabla \psi|^2 + |\psi \Delta \psi| dx,
\]

where \( C_1 = \frac{m^2 + \beta m (m-1)}{\alpha} > -\frac{\beta m}{\alpha} \geq 0 \).

Noticing that \( m \geq \max\{\frac{q+1}{p}, 2\} \) implies \( (2m - 2) \frac{q+1}{\gamma+1} \geq 2m \), and \( |\psi| \leq 1 \), we can use the Hölder’s inequality to deduce that

\[
\int_{\mathbb{R}^n} u^{\mu+\gamma} \psi^{2m} dx \leq C_1 \left( \int_{\mathbb{R}^n} u^{\gamma+1} |\psi|^{2m-2} \right)^{\frac{q+1}{\gamma+1}} \left( \int_{\mathbb{R}^n} |\nabla \psi|^2 + |\psi \Delta \psi| \right)^{\frac{\gamma + 1}{\gamma + 2}}.
\]

Therefore, the desired conclusion follows immediately

\[
\int_{\mathbb{R}^n} u^{\mu+\gamma} \psi^{2m} dx \leq C_1 \int_{\mathbb{R}^n} (|\nabla \psi|^2 + |\psi||\Delta \psi|)^{\frac{\gamma + 1}{\gamma + 2}} dx.
\]
Proof of Theorem 1.10. For every \( R > 0 \), we take a function \( \psi_R(x) = \varphi\left(\frac{|x|}{R}\right) \), where \( \varphi \in C_0^\infty(R) \), \( 0 \leq \varphi \leq 1 \), and

\[
\varphi(t) = \begin{cases} 
1 & \text{if } |t| \leq 1, \\
0 & \text{if } |t| \geq 2.
\end{cases}
\] (4.10)

Let us fix \( p > 0 \) and \( q > 1 \). We first observe that for any \( \gamma \in [1, 2q + 2\sqrt{q(q-1)} - 1) \) and any integer \( m \geq \max\left\{ \frac{2q+\gamma}{p+q}, 2 \right\} \), Proposition 4.1 shows that

\[
\int_{B(0,R)} u^{\mu+\gamma} dx \leq C(p, q, m, \gamma, n) R^{n-2\frac{\mu+\gamma}{p+q}}, \quad \forall R > 0,
\] (4.11)

where \( C(p, q, m, \gamma, n) \) is a positive constant independent of \( R \).

Next, under the assumption \( n - 2 < \frac{4q+4\sqrt{q(q-1)}}{p+q} \), we can always choose \( \gamma \in [1, 2q + 2\sqrt{q(q-1)} - 1) \) such that

\[
n - 2\frac{\mu+\gamma}{\mu-1} < 0.
\] (4.12)

Letting \( R \to +\infty \) in (4.11), we see

\[
\int_{\mathbb{R}^n} u^{\mu+\gamma} dx = 0,
\]

which yields \( u \equiv 0 \). This is a contradiction and Theorem 1.10 is proved.

\( \square \)

Proof of Theorem 1.11. 1) Let \( n - 2 \geq \frac{4q+4\sqrt{q(q-1)}}{p+q} \). Namely, \( \mu \geq 1 + \frac{4}{n-2}(q + \sqrt{q(q-1)}) \).

Write

\[
u_s(r) = Lr^{-\frac{n-\mu}{\mu-1}}, \quad v_s = L\sqrt{\frac{p+1}{q+1}} r^{-\frac{2}{p-1}}, \quad r > 0
\]

with

\[
L^{\mu-1} = \frac{2}{\mu-1} [n - 2 - \frac{2}{\mu-1}(p+1)]^{\frac{n+1}{p+1}} (q+1)^{\frac{n+1}{p+1}}.
\]

It is easy to see that for \( r > 0 \),

\[
-\triangle u_s = -u_s'' - \frac{n-1}{r} u_s' = (q+1)\mu u_s^q v_s^{p+1},
\]

\[
-\triangle v_s = -v_s'' - \frac{n-1}{r} v_s' = (p+1)\mu u_s^{q+1} v_s^p.
\] (4.13)

This result shows that \((u_s, v_s)\) is a pair of radial singular solution of (1.1) on \( \mathbb{R}^n \setminus \{0\} \).

In view of \( \mu > \frac{n+2}{n-2} \), by Proposition 3.4 in [23], the nontrivial radial regular solutions of (1.3) are included in a family \( \{u_\alpha\}_{\alpha > 0} \) with \( u_\alpha \) decreasing in \( r \),

\[
r^{\frac{n-\mu}{\mu-1}} u_\alpha(r) \to L, \quad \text{as } r \to +\infty.
\] (4.14)

In addition, \( u_\alpha(0) = \alpha^{\frac{2}{n-\mu}}, \) and \( u_\alpha(r) = \alpha^{\frac{2}{n-\mu}} u_1(\alpha r) \) where \( u_1 \) solves (1.6).

Furthermore, [23] also shows that when \( \mu > p_{ij} \),

\[
u_\alpha(r) < u_s(r), \quad \text{for } r > 0.
\] (4.15)

Write \( v_\alpha = (\frac{p+1}{q+1})^{1/2} u_\alpha \). Then \((u_\alpha(r), v_\alpha(r))\) solves (1.1). We will prove that \((u_\alpha(r), v_\alpha(r))\) is also a pair of local u-stable solution of (1.1).
First we notice that
\[
q\left(\frac{2}{\mu-1}\right)(n - 2 - \frac{2}{\mu-1}) \leq \frac{(n-2)^2}{4}
\]
\[
\Longleftrightarrow (\mu - 1)^2(n - 2)^2 - 8q(\mu - 1)(n - 2) + 16q \geq 0 \quad (4.16)
\]
\[
\Longleftrightarrow n - 2 \geq \frac{4q + 4\sqrt{q(q-1)}}{\mu - 1}.
\]

By (4.14), for any \(\varepsilon > 0\), there exists \(R > 0\) such that \(u_\alpha(r) \leq (L + \varepsilon)r^{-\frac{n-2}{\mu-2}}\) for all \(r > R\). Hence, for every \(\psi \in C_0^\infty (R^n \setminus B_R(0))\), we have
\[
\int_{R^n \setminus B_R(0)} |\nabla \psi|^2 \, dx - q(q + 1) \int_{R^n \setminus B_R(0)} u_\alpha^{q-1}v_\alpha^{p+1} \psi^2 \, dx
\]
\[
\geq \int_{R^n \setminus B_R(0)} |\nabla \psi|^2 \, dx - \int_{R^n \setminus B_R(0)} q\left(\frac{2}{\mu-1}\right)(n - 2 - \frac{2}{\mu-1})|x|^{-2} \psi^2 \, dx \quad (4.17)
\]
\[-C(q,n)\varepsilon^{\mu-1} \int_{R^n \setminus B_R(0)} |x|^{-2} \psi^2 \, dx.
\]

For the above function \(\psi\), \(\int_{R^n \setminus B_R(0)} |x|^{-2} \psi^2 \, dx\) is a convergent improper integral. Thus, using (4.16), we can deduce from (4.17) that
\[
\int_{R^n \setminus B_R(0)} |\nabla \psi|^2 \, dx - q(q + 1) \int_{R^n \setminus B_R(0)} u_\alpha^{q-1}v_\alpha^{p+1} \psi^2 \, dx
\]
\[
\geq \int_{R^n \setminus B_R(0)} |\nabla \psi|^2 - \frac{(n - 2)^2}{4}|x|^2 \psi^2 \, dx - C_2 \varepsilon^{\mu-1}.
\]

The Hardy’s inequality is strict for any \(C_0^\infty (R^n)\)-function (cf. [10]). Therefore, for the above function \(\psi\), there exists a positive constant \(\eta\) such that
\[
\int_{R^n \setminus B_R(0)} |\nabla \psi|^2 - \frac{(n - 2)^2}{4}|x|^2 \psi^2 \, dx \geq \eta > 0.
\]

Choosing \(\varepsilon^{\mu - 1} = \frac{\eta}{R^2}\), we have
\[
\int_{R^n \setminus B_R(0)} |\nabla \psi|^2 \, dx - q(q + 1) \int_{R^n \setminus B_R(0)} u_\alpha^{q-1}v_\alpha^{p+1} \psi^2 \, dx \geq 0.
\]

This means that \((u_\alpha(r), v_\alpha(r))\) is a local \(u\)-stable solution of (1.1).

2) Let \(\mu = \frac{n+2}{n-2}\). Assume that \(u\) can be written as (1.5) and \(v = (\frac{p+1}{q+1})^{1/2}u\).

Then, \((u, v)\) solves (1.1). Thus, there exists \(R > 0\) such that as \(|x| > R\),
\[
q(q + 1) \int_{R^n} u_\alpha^{q-1}v_\alpha^{p+1} \psi^2 \, dx \leq \frac{(n - 2)^2}{4} \int_{R^n} |x|^{-2} \psi^2 \, dx, \quad \forall \psi \in C_0^\infty (R^n \setminus B_R(0)).
\]

Using Hardy’s inequality, we see that for any \(\psi \in C_0^\infty (R^n \setminus B_R(0))\),
\[
\int_{R^n} |\nabla \psi|^2 \, dx - q(q + 1) \int_{R^n} u_\alpha^{q-1}v_\alpha^{p+1} \psi^2 \, dx \geq 0.
\]

Namely, \((u, v)\) is a local \(u\)-stable solution. Theorem 1.11 is proved. \(\square\)

**Remark 4.2.** When \(p + q + 1 > B_j\), by the analogous argument above, we see that \((u_\alpha, v_\alpha)\) is also a \(u\)-stable solution by using (4.15).

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E-mail address: liyayun.njnu@qq.com
E-mail address: leiyutian@njnu.edu.cn