Newton transformations and motivic invariants at infinity of plane curves

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Abstract
In this article we give an expression of the motivic Milnor fiber at infinity and the motivic nearby cycles at infinity of a polynomial $f$ in two variables with coefficients in an algebraic closed field of characteristic zero. This expression is given in terms of some motives associated to the faces of the Newton polygons appearing in the Newton algorithm at infinity of $f$ without any condition of convenience or non degeneracy. In the complex setting, we compute the Euler characteristic of the generic fiber of $f$ in terms of the area of the surfaces associated to faces of the Newton polygons. Furthermore, if $f$ has isolated singularities, we compute similarly the classical invariants at infinity $\lambda_c(f)$ which measures the non equisingularity at infinity of the fibers of $f$ in $\mathbb{P}^2$, and we prove the equality between the topological and the motivic bifurcation sets and give an algorithm to compute them.

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Introduction

Let \( k \) be an algebraically closed field of characteristic zero. Let \( f \) be a polynomial with coefficients in \( k \). Using the motivic integration theory, introduced by Kontsevich in [21], and more precisely constructions of Denef–Loeser in [12–14] and Guibert–Loeser–Merle in [19], Matsui–Takeuchi in [24,25] and independently the second author in [30,31] and endowed with its monodromy action \( T_\infty \), this ring is a modified Grothendieck ring of varieties over \( k \) endowed with an action of the multiplicative group \( \mathbb{G}_m \) of \( k \). It follows from Denef–Loeser results that the motive \( S_{f,\infty} \) is a “motivic” incarnation of the topological Milnor fiber at infinity of \( f \), denoted by \( F_\infty \) and endowed with its monodromy action \( T_\infty \). For instance, when \( k \) is the field of complex numbers, the motive \( S_{f,\infty} \) realizes on the Euler characteristic of \( F_\infty \) and the monodromy zeta function or the Steenbrink’s spectrum of \( (F_\infty, T_\infty) \).

In the same way, in [30,31] (see also [25]) a notion of **motivic nearby cycles at infinity** \( S_{f,a}^{\infty} \) of \( f \) for a value \( a \) is defined as an element of the Grothendieck ring \( \mathcal{M}_{[a]}^{\mathbb{G}_m} \times \mathbb{G}_m \). This...
motives is constructed using a compactification $X$ of the graph of $f$ and arcs with origin in the closure of the fiber $f^{-1}(a)$ in $X$. It is shown in [30] that this motive does not depend on the chosen compactification. If $f$ has isolated singularities and $k$ is the field of complex numbers, Fantini and the second author proved in [15] that the Euler characteristic of the motive $S_{f,a}$ is equal to $(-1)^{d-1} \lambda_a(f)$, where $d$ is the dimension of the ambient space and $\lambda_a(f)$ is the classical invariant which measures the lack of equisingularity at infinity of the usual compactification of $f$ in $\mathbb{P}_C^d \times \mathbb{P}_C^1$ (see for instance [1,33]). Furthermore, the second author introduced in [30] a motivic bifurcation set $B_{f}^{\text{mot}}$ as the set of values $a$ which belong to the discriminant of $f$ or such that $S_{f,a}^{\infty} \neq 0$. It is shown in [30] that this set is finite, and for instance if $f$ has isolated singularities at infinity, it is proven in [15] that the usual topological bifurcation set $B_{f}^{\text{top}}$ of $f$ is included in $B_{f}^{\text{mot}}$.

In this article we investigate the case of polynomials in $k[x, y]$ in full generality (namely without any assumptions of convenience or non degeneracy w.r.t any Newton polygon) using ideas of Guibert in [17], Guibert, Loeser and Merle in [18], and the works of the first author and Veys in the case of an ideal of $\mathbb{k}$ by introducing in [30] a motivic local monodromy conjecture for the Euler characteristic at infinity or the monodromy around a value of a bifurcation set.

More precisely, the first main results of this article are the expression, in Theorem 3.8 and Theorem 3.23, of the motivic Milnor fiber at infinity $S_{f,\infty}$ and the motivic nearby cycles at infinity $S_{f,a}^{\infty}$, for a value $a$, in terms of some motives associated to faces of the Newton polygons appearing in the Newton algorithm at infinity presented in [5] and recalled in Definition 1.3.2. These formulas use in particular the computation using the Newton algorithm (Theorem 1.8) of some specific motivic nearby cycles $(S_{h^\epsilon,x=0})_{(0,0),0}^{h\epsilon}$, for $h$ any element of $k[x^{-1}, x, y]$ of the form $x^{-M}g(x, y) \epsilon [-1, 1]$. Note that in the statement of these theorems, we give an explicit expression of the rational form of the motivic zeta functions defining $S_{f,a}^{\infty}$ and $S_{f,\infty}$ as the set of values $x$ or such that $S_{f,a}^{\infty} \neq 0$. This is done to study in a future article the motivic monodromy conjecture for $f$ about the monodromy at infinity or the monodromy around a value of a bifurcation set.

Applying the realization of the Euler characteristic on the motives $S_{f,\infty}$ and $S_{f,a}^{\infty}$, we deduce in Corollary 3.12 and Corollary 3.25, a Kouchnirenko type formula for the Euler characteristic of the generic fiber of $f$ and the invariant $\lambda_a(f)$ (in the case of isolated singularities), in terms of the area of surfaces associated to faces of Newton polygons appearing in the Newton algorithm at infinity.

Finally, by area considerations, all these results allow to prove the Fundamental Theorem 3.30 of this article, which states that in the isolated singularity case, we have the equality

$$B_{f}^{\text{top}} = B_{f}^{\text{Newton}} = B_{f}^{\text{mot}}$$

where $B_{f}^{\text{Newton}}$ is the Newton bifurcation set of $f$ which is defined in an algorithmic way (Definition 1.38) using the Newton algorithm at infinity of $f$. Up to factorisation of polynomials, this gives in particular an algorithm to compute the usual topological bifurcation set for curves without assumptions of convenience or non degeneracy toward any Newton polygon, recovering and generalizing the analogous result for curves by Némethi and Zaharia in [26].

1 Newton algorithms

Let $k$ be an algebraically closed field of characteristic zero, with multiplicative group denoted by $\mathbb{G}_m$.

**Definition 1.1** (Coefficients and support) The support of a polynomial $f(x, y) = \sum_{(a,b) \in \mathbb{Z}^2} c_{a,b} x^a y^b$ with coefficients in $k$, is the set defined by

$\sum_{(a,b) \in \mathbb{Z}^2} c_{a,b} x^a y^b$ with coefficients in $k$, is the set defined by
Supp\( (f) = \{(a, b) \in \mathbb{Z}^2|c_{a,b} \neq 0\} \). Sometimes, we will denote by \( c_{a,b}(f) \) the coefficient \( c_{a,b} \) of \( f \).

1.1 Newton algorithm

1.1.1 Newton polygons

**Notation 1.2** Let \( E \) be a subset of \( \mathbb{Z}_{\geq -n} \times \mathbb{Z}_{\geq -m} \) with \((n, m)\) in \( \mathbb{N}^2 \). We denote by \( \Delta(E) \) the smallest convex set containing \( E + \mathbb{R}_+^2 = \{a + b, a \in E, b \in \mathbb{R}_+^2\} \).

**Definition 1.3** (Newton diagram for a set, for a polynomial, vertices) A subset \( \Delta \) of \( \mathbb{R}^2 \) is called Newton diagram if \( \Delta = \Delta(E) \) for some set \( E \) in \( \mathbb{Z}_{\geq -n} \times \mathbb{Z}_{\geq -m} \) with \((n, m)\) in \( \mathbb{N}^2 \). The smallest set \( E_0 \) of \( \mathbb{Z}^2 \) such that \( \Delta = \Delta(E_0) \) is called the set of vertices of \( \Delta \). The Newton diagram \( \Delta(f) \) of a polynomial \( f \) in \( k[x^{-1}, x, y] \) is the diagram \( \Delta(\text{Supp } f) \).

**Remark 1.4** The set of vertices of a Newton diagram is finite.

**Definition 1.5** (Newton polygon, one dimensional faces, zero dimensional faces, horizontal and vertical faces) Let \( \Delta \) be a Newton diagram and \( E_0 = \{v_0, \ldots, v_d\} \) be its set of vertices with \( v_i = (a_i, b_i) \) in \( \mathbb{Z}^2 \) satisfying \( a_{i-1} < a_i \) and \( b_{i-1} > b_i \), for any \( i \) in \( \{1, \ldots, d\} \). For such \( i \), we denote by \( S_i \) the segment \([v_{i-1}, v_i]\) and by \( l_{S_i} \) the line supporting \( S_i \). We define the Newton polygon of \( \Delta \) as the set

\[
\mathcal{N}(\Delta) = \{S_i\}_{i \in \{1, \ldots, d\}} \cup \{v_i\}_{i \in \{0, \ldots, d\}},
\]

the height of \( \Delta \) as the integer \( h(\Delta) = b_0 - b_d \), the one dimensional faces of \( \mathcal{N}(\Delta) \) as the segments \( S_i \), the zero dimensional faces of \( \mathcal{N}(\Delta) \) as the vertices \( v_i \) and among them the vertical face \( \gamma_v \) as \( v_0 \) and the horizontal face \( \gamma_h \) as \( v_d \).

**Definition 1.6** (Newton polygon at the origin, height of a polynomial) The Newton polygon at the origin \( \mathcal{N}(f) \) of a polynomial \( f \) in \( k[x^{-1}, x, y] \) is the Newton polygon \( \mathcal{N}(\Delta(f)) \). The height of \( f \), denoted by \( h(f) \), is the height \( h(\Delta(f)) \).

**Definition 1.7** (Face polynomials, roots and multiplicities, non degenerate case) Let \( f \) be a polynomial in \( k[x^{-1}, x, y] \) and \( \gamma \) be a face of \( \mathcal{N}(f) \). If \( \gamma \) has dimension zero, then \( \gamma \) is a point \((a_0, b_0)\) and we denote by \( f_\gamma \) the monomial \( c_{(a_0,b_0)}(f)x^{a_0}y^{b_0} \). If \( \gamma \) has dimension one, then \( \gamma \) is supported by a line \( l \) and we define

\[
f_\gamma(x, y) := \sum_{(a,b) \in l \cap \mathcal{N}(f)} c_{a,b}(f)x^a y^b.
\]

As the field \( k \) is algebraically closed, there exist \( c \) in \( k \), \((a_\gamma, b_\gamma)\) in \( \mathbb{Z} \times \mathbb{N} \), \((p, q)\) in \( \mathbb{N}^2 \) and coprime, \( r \) in \( \mathbb{N}^* \), \( \mu_i \) in \( k^* \) (all different) and \( v_i \) in \( \mathbb{N}^* \) such that

\[
f_\gamma(x, y) = cx^{a_\gamma}y^{b_\gamma} \prod_{1 \leq i \leq r} (y^p - \mu_i x^q)^{v_i}.
\]

The polynomial \( f_\gamma \) is called the face polynomial of \( f \) associated to the face \( \gamma \). In the one dimensional case, each \( \mu_i \) is called root with multiplicity \( v_i \) of the face polynomial \( f_\gamma \). The set of roots is denoted by \( R_\gamma \). The roots are said to be simple if all the \( v_i \) are equal to 1. Following [22], \( f \) is said non degenerate with respect to its Newton polygon \( \mathcal{N}(f) \), if and only if for each one dimensional face \( \gamma \) in \( \mathcal{N}(f) \), the face polynomial \( f_\gamma \) has no critical points on the torus \( \mathbb{G}_m^2 \), in particular all its roots are simple.
Definition 1.8 (Rational polyhedral convex cone) Let $I$ be a finite set. A rational polyhedral convex cone of $\mathbb{R}^{|I|}\setminus\{0\}$ is a convex part of $\mathbb{R}^{|I|}\setminus\{0\}$ defined by a finite number of linear inequalities with integer coefficients of type $a \leq 0$ and $b > 0$ and stable under multiplication by elements of $\mathbb{R}_{>0}$.

In the well-known following proposition, we introduce notations used throughout this article.

Proposition et notations 1.9 (Function $m$, dual cone $C_\gamma$ and normal vector $\vec{n}_\gamma$) Let $E$ be a subset of some $\mathbb{Z}_{\geq-n} \times \mathbb{Z}_{\geq-m}$ with $(n, m)$ in $\mathbb{N}^2$. Let $(p, q)$ be in $\mathbb{N}^2$ with $\gcd(p, q) = 1$ and

$$l_{(p,q)} : (a, b) \in \mathbb{R}^2 \mapsto ap + bq.$$ 

1. The minimum of the restriction $l_{(p,q)}(\Delta(E))$, denoted by $m(p, q)$, is reached on a face denoted by $\gamma(p, q)$ of $\Delta(E)$. Furthermore, the linear map $l_{(p,q)}$ is constant on the face $\gamma(p, q)$.
2. For any face $\gamma$ of $\Delta(E)$, we denote by $C_\gamma$ the interior in its own generated vector space in $\mathbb{R}^2$, of the positive cone generated by the set $\{(p, q) \in \mathbb{N}^2 \mid \gamma(p, q) = \gamma\}$. This set is called dual cone to the face $\gamma$ and is a relatively open rational polyhedral convex cone of $(\mathbb{R}_{\geq0})^2$.

For a one dimensional face $\gamma$, we denote by $\vec{n}_\gamma$ the normal vector to the face $\gamma$ with integral non negative coordinates and the smallest norm. With these notations we have the following properties.

3. The dual cone $C_\gamma$ of any one dimensional face $\gamma$ of $\Delta(E)$ is the cone $\mathbb{R}_{>0}\vec{n}_\gamma$.
4. Any zero dimensional face $\gamma$ of $\Delta(E)$ is an intersection of two one dimensional faces $\gamma_1$ and $\gamma_2$ of $\Delta(E)$ (may be not compact) and its dual cone $C_\gamma$ is the cone $\mathbb{R}_{>0}\vec{n}_{\gamma_1} + \mathbb{R}_{>0}\vec{n}_{\gamma_2}$.
5. The set of dual cones $(C_\gamma)_{\gamma \in N(\Delta(E))}$ is a fan of $(\mathbb{R}_{>0})^2$, called dual fan of $\Delta(E)$.

1.1.2 Newton algorithm

Definition 1.10 (Newton transformations, Newton transforms and compositions) Let $(p, q)$ be in $\mathbb{N}^2$ with $\gcd(p, q) = 1$. Let $(p', q')$ be in $\mathbb{N}^2$ such that $pp' - qq' = 1$. Let $\mu$ be in $\mathbb{Z}_m$. We define the Newton transformation associated to $(p, q, \mu)$ as the application

$$\sigma_{(p,q,\mu)} : k[x^{-1}, x, y] \to k[x_1^{-1}, x_1, y_1]$$
$$f(x, y) \to f(\mu q' x_1^p, x_1, y_1(y_1 + \mu p')).$$

(1.1)

We call $\sigma_{(p,q,\mu)}(f)$ a Newton transform of $f$ and denote it by $f_{\sigma_{(p,q,\mu)}}$ or simply $f_\sigma$. More generally, let $\Sigma_n = (\sigma_1, \ldots, \sigma_n)$ be a finite sequence of Newton maps $\sigma_i$, we define the composition $f_{\Sigma_n}$ by induction: $f_{\Sigma_1} = f_{\sigma_1}$, $f_{\Sigma_i} = (f_{\Sigma_{i-1}})_{\sigma_i}$ for any $i$.

Remark 1.11 The Newton map $\sigma_{(p,q,\mu)}$ depends on $(p', q')$, nevertheless if $(p' + lq, q' + lp)$ is another pair, then

$$f(\mu q' x_1^p, x_1, y_1(y_1 + \mu p')) = f(\mu q' (x_1 x_1^p p)(y_1 y_1^p lq + \mu p'))$$

for any $f$ in $k[x^{-1}, x, y]$. Furthermore, there is exactly one choice of $(p', q')$ satisfying $pp' - qq' = 1$ and $p' \leq q$ and $q' < p$. In the sequel we will always assume these inequalities. This will make procedures canonical.
Remark 1.12 Sometimes, for instance in Sect. 3.3.7.3, we will work with polynomials in \( \mathbf{k}[x, y, y^{-1}] \). In this case, the Newton polygon \( \mathcal{N}(f) \) is defined as in Definition 1.6 and we go back to the case \( \mathbf{k}[x^{-1}, x, y] \), using a Newton map as

\[
\sigma_{(p, q, \mu)} : \mathbf{k}[x, y, y^{-1}] \rightarrow \mathbf{k}[x_1^{-1}, x_1, y_1] \\
f(x, y) \rightarrow f(x_1^p (y_1 + \mu q), x_1^q \mu^p)
\]

Lemma 1.13 (Newton lemma) Let \( (p, q) \) be in \( \mathbb{N}^2 \) with \( \gcd(p, q) = 1 \). Let \( \mu \) be in \( \mathbb{G}_m \). Let \( f \) be a non zero element in \( \mathbf{k}[x^{-1}, x, y] \) and \( f_1 \) be its Newton transform \( \sigma_{(p, q, \mu)}(f) \) in \( \mathbf{k}[x_1^{-1}, x_1, y_1] \) and \( m \) as above, defined relatively to \( \mathcal{N}(f) \).

1. If there does not exist a one dimensional face \( \gamma \) of \( \mathcal{N}(f) \) whose supporting line has equation \( pa + qb = N \), for some \( N \), then there is a polynomial \( u(x_1, y_1) \) in \( \mathbf{k}[x_1, y_1] \) with \( u(0, 0) \neq 0 \) such that \( f_1(x_1, y_1) = x_1^{m(p, q)} u(x_1, y_1) \).

2. If there exists a one dimensional face \( \gamma \) of \( \mathcal{N}(f) \) whose supporting line has equation \( pa + qb = N \), if \( \mu \) is not a root of \( f_\gamma \), then \( m(p, q) = N \) and there is a polynomial \( u(x_1, y_1) \) in \( \mathbf{k}[x_1, y_1] \) with \( u(0, 0) \neq 0 \) such that \( f_1(x_1, y_1) = x_1^N u(x_1, y_1) \).

3. If there exists a one dimensional face \( \gamma \) of \( \mathcal{N}(f) \) whose supporting line has equation \( pa + qb = N \), if \( \mu \) is a root of \( f_\gamma \) of multiplicity \( v \) then \( m(p, q) = N \) and there is a polynomial \( g_1(x_1, y_1) \) in \( \mathbf{k}[x_1, y_1] \) with \( g_1(0, 0) = 0 \) and \( g_1(0, y_1) \) of valuation \( v \), such that \( f_1(x_1, y_1) = x_1^N g_1(x_1, y_1) \). In that case we have in particular the inequality \( h(f) \geq v \geq h(f_1) \).

Proof This lemma is proved by a simple computation, see [8,9] (or [7, Lemma 2]). \( \square \)

Remark 1.14 If \( f_1(x_1, y_1) \) is equal to \( x_1^{n_1} y_1^{m_1} u(x_1, y_1) \), where \((n_1, m_1)\) belongs to \( \mathbb{Z} \times \mathbb{N}^2 \) and \( u \in \mathbf{k}[x_1, y_1] \) is a unit in \( \mathbf{k}[x_1, y_1] \), we say for short that \( f_1 \) is a monomial times a unit. From this lemma, we see that there is a finite number of triples \((p, q, \mu)\) such that \( \sigma_{(p, q, \mu)}(f) \) is eventually not a monomial times a unit in \( \mathbf{k}[[x_1, y_1]] \). These triples are given by the equations of the faces of the Newton polygon and the roots of the corresponding face polynomials.

We recall the notion of Newton algorithm based on Lemma 1.13 and refer to [2,8,9] for more details.

Definition 1.15 (Newton algorithm) Let \( f \) be a polynomial in \( \mathbf{k}[x^{-1}, x, y] \). The Newton algorithm of \( f \) is defined by induction. It starts by applying Newton transformations given by the equations of the faces of the Newton polygon \( \mathcal{N}(f) \) and the roots of the corresponding face polynomials. Then, this process is applied on each Newton transform until a base case of the form \( u(x, y) x^{-M} y^m \) or \( u(x, y) x^{-M} (y - \mu x^q + g(x, y))^m \) is obtained with \( \mu \in \mathbb{G}_m \), \((M, m)\) in \( \mathbb{Z} \times \mathbb{N} \), \( q \) in \( \mathbb{N} \), \( g(x, y) = \sum_{a+bq>q} c_{a,b} x^a y^b \) in \( \mathbf{k}[x, y] \), and \( u(x, y) \) in \( \mathbf{k}[x, y] \) with \( u(0, 0) \neq 0 \). The output of the algorithm is the tree of the Newton transform polynomials produced.

This definition is a consequence of the following Lemma 1.16 and Theorem 1.17.

Definition-Lemma 1.16 (Stability lemma) Let \( f \) be a polynomial in \( \mathbf{k}[x^{-1}, x, y] \).

1. Let \( \sigma \) be a Newton transformation. If the height of the Newton transform \( f_\sigma \) is equal to the height of \( f \), then the Newton polygon of \( f \) has a unique face \( \gamma \) with face polynomial \( f_\gamma(x, y) = x^k y^l (y - \mu x^q)^v \), with \((k, l, v)\) in \( \mathbb{Z} \times \mathbb{N}^2 \), \( \mu \) in \( \mathbb{G}_m \) and \( q \) in \( \mathbb{N}^* \). If the height in the Newton process remains constant, we say that the Newton algorithm stabilizes.
2. If the Newton algorithm of \( f \) stabilizes with height \( m \) then \( f \) can be written as
\[
  f(x, y) = U(x, y)x^M(y - \mu x^q + g(x, y))^m
\]
with \( \mu \in \mathbb{G}_m, (M, q) \in \mathbb{Z} \times \mathbb{N}, g(x, y) = \sum_{a+bq>q} c_{a,b}x^ay^b \in k[x, y] \) and \( U(x, y) \) in \( k[[x, y]] \) with \( U(0, 0) \neq 0 \).

**Proof**  The proof of point 1 is Lemma 2.11 in [8]. The proof of point 2 is similar to that of [7, Lemma 4]. \( \square \)

**Theorem 1.17**  For all \( f(x, y) \) in \( k[x^{-1}, x, y] \), there exists a natural integer \( n_0 \), such that for any sequence of Newton maps \( \Sigma_n = (\sigma_1, \ldots, \sigma_n) \) with \( n \geq n_0 \), \( f_{\Sigma_n} \) is of the form \( u(x, y)x^{-M}y^n \) or \( u(x, y)x^{-M}(y - \mu x^q + g(x, y))^m \), with \( \mu \in \mathbb{G}_m, (M, m, q) \in \mathbb{Z} \times \mathbb{N}^2, g(x, y) = \sum_{a+bq>q} c_{a,b}x^ay^b \in k[x, y] \), and \( u(x, y) \) belongs to \( k[[x, y]] \) with \( u(0, 0) \neq 0 \).

**Proof**  The proof is similar to [7, Theorem 1]. \( \square \)

### 1.1.3 Local dicritical faces, Newton generic and non generic values and Newton bifurcation set

**Definition 1.18**  \textit{(Local dicritical face, discriminant)}  Let \( f \) be a polynomial in \( k[x^{-1}, x, y] \).
- By definition, if \( f \) belongs to \( k[x, y] \) then it does not have a local dicritical face. Otherwise, a local dicritical face is defined as a one dimensional face of the Newton polygon of \( \Delta(\text{Supp}(f) \cup \{(0, 0)\}) \) which contains \((0, 0)\).
- A local dicritical face is said to be smooth if an equation of its underlying line is \( \alpha + q\beta = 0 \) with \( q \in \mathbb{N} \).
- If \( \gamma \) is a local dicritical face, we define its associated polynomial (relatively to \( f \)) as the polynomial \( \sum_{(a,b) \in \gamma} c_{a,b}(f)x^ay^b \). In particular if \( \gamma \) is a face of \( \mathcal{N}(f) \), then this polynomial is the face polynomial \( f_{\gamma} \). The associated polynomial to \( \gamma \) can be written under the form \( P_{\gamma}(x^{-q}, y^p) \) with \( (p, q) \in (\mathbb{N}^*)^2 \) and coprime, and \( P_{\gamma}(s) \) a polynomial in \( k[s] \). The discriminant of \( P_{\gamma}(s) - c \) with respect to \( s \), element of \( k[c] \), is called discriminant of the face \( \gamma \) (relatively to \( f \)).

**Remark 1.19**  A value \( c_0 \) is a root of the discriminant of \( \gamma \) if and only if the polynomial \( P_{\gamma}(s) - c_0 \) has a multiple root.

**Definition 1.20**  \textit{(Newton generic and non generic values)}  Let \( f \) be a polynomial in \( k[x^{-1}, x, y] \).
- A value \( c_0 \) is Newton non generic for \( f \), if \( f \) admits a local dicritical face \( \gamma \) and \( c_0 \) satisfies one of the two conditions:
  - \( c_0 \neq c_{(0,0)}(f) \) is a root of the discriminant of the face \( \gamma \),
  - \( c_0 = c_{(0,0)}(f) \) is a root of the discriminant of the face \( \gamma \) or \( \gamma \) is not smooth.
- A value \( c_0 \) is Newton generic for \( f \) if it is not Newton non generic.

**Definition 1.21**  \textit{(Local Newton bifurcation set)}  Let \( f \) be in \( k[x^{-1}, x, y] \). The local Newton bifurcation set of \( f \), denoted by \( B_{f,loc}^{\text{Newton}} \), is the set of Newton non generic values of either \( f \) or a Newton transform \( f_\Sigma \) where \( \Sigma \) is a composition of Newton maps in the Newton algorithm of \( f \).
Example 1.22 (Local Newton bifurcation set of the base cases) The local Newton bifurcation set of a base case (Definition 1.15) is contained in \{0\}. It is empty if the base case is assumed with isolated singularities.

Proposition 1.23 (Finiteness of the local Newton bifurcation set) The local Newton bifurcation set of an element of \(k[x^{-1}, x, y]\) is finite.

Proof Indeed a polynomial \(f\) in \(k[x^{-1}, x, y]\) has at most one local dicritical face and its discriminant has finitely many roots. Thus, the number of Newton non generic values of \(f\) is finite. We conclude by Definition 1.15 and Example 1.22.

\[
\begin{align*}
\text{\textbf{1.2 Newton algorithm at infinity}}
\end{align*}
\]

1.2.1 Newton polygon at infinity

Definition 1.24 (Newton polygon at infinity) Let \(E\) be a nonempty finite subset of \(\mathbb{N}^2\). We consider \(\Delta_{\infty,0}(E), \Delta_{\infty,\infty}(E)\) and \(\Delta_{\infty,\infty}(E)\) the smallest convex sets containing respectively \(E + (\mathbb{R}_+ \times \mathbb{R}_-)\), \(E + (\mathbb{R}_- \times \mathbb{R}_-)\) and \(E + (\mathbb{R}_- \times \mathbb{R}_+)\). For any \((i, j)\) equal to \((0, \infty), (\infty, \infty)\) or \((\infty, 0)\), we denote by \(V_{i,j}(E)\) the set of vertices of \(\Delta_{i,j}(E)\). We define the sets

\[
\mathcal{V}_\infty(E) = \mathcal{V}_{\infty,0}(E) \cup \mathcal{V}_{\infty,\infty}(E) \cup \mathcal{V}_{0,\infty}(E) \quad \text{and} \quad \Delta_\infty(E) = \text{convex hull}(\mathcal{V}_\infty(E)).
\]

The set \(\mathcal{V}_\infty(E)\) is called the set of vertices at infinity of \(E\) and we order its elements in the following way:

- on \(\mathcal{V}_{\infty,\infty}(E) \cup \mathcal{V}_{0,\infty}(E)\), we define \((\alpha, \beta) < (\alpha', \beta')\) if and only if \(\alpha < \alpha'\),
- on \(\mathcal{V}_{\infty,\infty}(E) \cup \mathcal{V}_{\infty,0}(E)\), we define \((\alpha, \beta) < (\alpha', \beta')\) if and only if \(\beta > \beta'\).

Write \(\mathcal{V}_\infty(E) = \{v_0, \ldots, v_m\}\), with \(v_i = (\alpha_i, \beta_i)\) and \(v_0 < v_1 < \cdots < v_m\). Let \(S_i\) be the line segment whose endpoints are \(v_{i-1}\) and \(v_i\). We denote by \(S\) the set of these segments. The Newton polygon at infinity of \(E\) is defined as the set

\[
\mathcal{N}_\infty(E) = \{S_1, \ldots, S_m\} \cup \mathcal{V}_\infty(E).
\]

For any \((i, j)\) equal to \((0, \infty), (\infty, \infty)\) or \((\infty, 0)\), we define

\[
\mathcal{N}_{i,j}(E) = \{S \in S \mid S\text{ has both end points in }V_{i,j}(E)\} \cup V_{i,j}(E).
\]

Remark 1.25 The vertical and horizontal faces of \(\mathcal{N}_\infty(E)\) are not contained in the union \(\mathcal{N}(\infty, \infty) \cup \mathcal{N}(0, \infty) \cup \mathcal{N}(\infty, 0)\).

Lemma 1.26 (Function \(m, \text{ dual cone } C_\gamma \text{ and normal vector } \vec{n}_\gamma\)) Let \(E\) be a finite set of \(\mathbb{Z}^2\). Let \((p, q)\) be in \(\mathbb{Z}^2\) with \(\gcd(p, q) = 1\) and \(l_{(p,q)} : (a, b) \in \mathbb{R}^2 \mapsto ap + bq\). Let \(\Delta\) be the convex hull of \(E\) and \(\mathcal{F}(\Delta)\) its set of faces.

1. The maximum of the restriction \(l_{(p,q)}|_{\Delta}\), denoted by \(m(p, q)\), is reached on a facet denoted by \(\gamma(p, q)\) of \(\Delta\). Furthermore, the linear map \(l_{(p,q)}\) is constant on the facet \(\gamma(p, q)\).

2. For any facet \(\gamma\) of \(\Delta\), we denote by \(C_\gamma\) the interior, in its own generated vector space in \(\mathbb{R}^2\), of the positive cone generated by the set \(\{\alpha, \beta\} \in \mathbb{Z}^2 \mid \gamma(\alpha, \beta) = \gamma\}\). This set is called dual cone of the face \(\gamma\) and is a relatively open rational polyhedral convex cone of \(\mathbb{R}^2\).

For a one dimensional face \(\gamma\), we denote by \(\vec{n}_\gamma\) the normal vector to \(\gamma\), exterior to \(\Delta\), with integral coordinates and the smallest norm. With these notations we have:
3. The dual cone $C_\gamma$ of any one dimensional face $\gamma$ of $\Delta$ is the cone $\mathbb{R}_{>0}\tilde{n}_\gamma$.
4. Any zero dimensional face $\gamma$ of $\Delta$ is an intersection of two one dimensional faces $\gamma_1$ and $\gamma_2$ of $\Delta$ and its dual cone $C_\gamma$ is the cone $\mathbb{R}_{>0}\tilde{n}_{\gamma_1} + \mathbb{R}_{>0}\tilde{n}_{\gamma_2}$.
5. The set of dual cones $(C_\gamma)_{\gamma \in \mathcal{F}(\Delta)}$ is a fan of $\mathbb{R}^2$, called dual fan of $\Delta$.

Definition 1.27 (Newton polygon at infinity and global Newton polygon) Let $f$ be a polynomial in $k[x, y]$. We define $N_\infty(f) = N_\infty(\text{Supp} f \cup \{(0, 0)\})$.

\begin{align*}
\Delta_\infty(f) &= \Delta_\infty(\text{Supp} f \cup \{(0, 0)\}) \quad \text{and} \\
\overline{\Delta}(f) &= \text{convex hull}(\text{Supp} f). \\
\text{The set } N_\infty(f) \text{ is called Newton polygon at infinity of } f. \\
\text{Similarly to Definition 1.24, the global Newton polygon } \overline{N}(f) \text{ is defined as the set of vertices and segments of } \overline{\Delta}(f). \\
\text{We define also } N_\infty(f)^{0} \text{ as the set of faces of } N_\infty(f) \text{ which do not contain the origin and we simply denote } N_\infty(\text{Supp} f \cup \{(0, 0)\}) \text{ by } N_\infty(f) \text{ and define similarly } N_{0,\infty}(f) \text{ and } N_{\infty,0}(f).
\end{align*}

We introduce $\tilde{N}(f)$ defined as the set $(\overline{N}(f) \setminus N(f)) \cup \{\gamma_0, \gamma_\infty\}$.

Remark 1.28 If $c \neq c_{(0, 0)}(f)$, we have $N_\infty(f) = N_\infty(\text{Supp}(f - c))$. If $f(0, 0) \neq 0$ then $\overline{N}(f) = N_\infty(f)$.

1.2.2 Newton algorithm at infinity

Definition 1.29 (Newton transformation at infinity) Let $(p, q)$ be a primitive vector of $\mathbb{Z}^2$ and $\mu$ be an element of $k$. We define the Newton transformation at infinity $\sigma(p, q, \mu)$ by

\begin{align*}
\sigma(p, q, \mu) & : k[x, y] \longrightarrow k[v^{-1}, v, w] \\
f(x, y) & \mapsto f(v^{-p}, v^{-q}(w + \mu)) \\
(1.3)
\end{align*}

in the case $p > 0$ and $q > 0$, choosing $(p', q')$ in $\mathbb{Z}^2$ with $qq' - pp' = 1$

\begin{align*}
\sigma(p, q, \mu) & : k[x, y] \longrightarrow k[v^{-1}, v, w] \\
f(x, y) & \mapsto f(v^{-p}, v^{-q}(w + \mu')) \\
(1.4)
\end{align*}

in the case $p > 0$ and $q < 0$, choosing $(p', q')$ in $\mathbb{Z}^2$ with $pp' - qq' = 1$

\begin{align*}
\sigma(p, q, \mu) & : k[x, y] \longrightarrow k[v^{-1}, v, w] \\
f(x, y) & \mapsto f(v^{-p}(w + \mu'), \mu') \\
(1.5)
\end{align*}

in the case $p < 0$ and $q > 0$ choosing $(p', q')$ in $\mathbb{Z}^2$ with $pp' - qq' = 1$

\begin{align*}
\sigma(p, q, \mu) & : k[x, y] \longrightarrow k[v^{-1}, v, w] \\
f(x, y) & \mapsto f((w + \mu), v^{-1}) \\
(1.6)
\end{align*}

in the case $p = 0, q = 1$;

\begin{align*}
\sigma(p, q, \mu) & : k[x, y] \longrightarrow k[v^{-1}, v, w] \\
f(x, y) & \mapsto f(v^{-1}, (w + \mu)) \\
(1.7)
\end{align*}

in the case $p = 1, q = 0$.

Remark 1.30 In the following we will apply these Newton maps, for each face of $N_\infty(f)$ or $\overline{N}(f) \setminus N(f)$ and each root of the face polynomials, and we will get elements in $k[v^{-1}, v, w]$ that we have studied in the previous section.

Notation 1.31 Let $f$ be a polynomial in $k[x, y]$. If $\gamma$ is a one dimensional face of $\overline{N}(f) \setminus N(f)$, then its primitive exterior normal vector $(p, q)$ belongs to $\mathbb{Z}^2 \setminus (\mathbb{Z}_{\leq 0})^2$ and the face $\gamma$ is supported by a line with equation $pa + qb = N$. Furthermore,
- the face $\gamma$ belongs to $\mathcal{N}_{\infty,\infty}(f)$ if and only if $p > 0$ and $q > 0$, and in that case we have the factorisation

$$f_\gamma(x, y) = x^a y^b s \prod_{\mu_i \in R_f} (y^q - \mu_i y^p) v_i$$

and

$$f_\gamma(v^{-p}, v^{-q} w) = v^{-N} w^b s \prod_{\mu_i \in R_f} (1 - \mu_i w^p) v_i$$

- the face $\gamma$ belongs to $\mathcal{N}_{0,\infty}(f)$ if and only if $p < 0$ and $q > 0$, and in that case we have the factorisation

$$f_\gamma(x, y) = x^a y^b s \prod_{\mu_i \in R_f} (\mu_i x^q y^{-p} - 1) v_i$$

and

$$f_\gamma(v^{-p} w, v^{-q}) = v^{-N} w^a s \prod_{\mu_i \in R_f} (\mu_i w^q - 1) v_i$$

- the face $\gamma$ is horizontal if and only if $(p, q) = (0, 1)$, and in that case we have the factorisation

$$f_\gamma(x, y) = x^a y^b s \prod_{\mu_i \in R_f} (x - \mu_i) v_i$$

and

$$f_\gamma(w, v^{-1}) = v^{-N} w^a s \prod_{\mu_i \in R_f} (w - \mu_i) v_i$$

- the face $\gamma$ belongs to $\mathcal{N}_{\infty,0}(f)$, if and only if $p > 0$ and $q < 0$, and in that case we have the factorisation

$$f_\gamma(x, y) = x^a y^b s \prod_{\mu_i \in R_f} (x^q - \mu_i y^p) v_i$$

and

$$f_\gamma(v^{-p} w, v^{-q}) = v^{-N} w^b s \prod_{\mu_i \in R_f} (w^p - \mu_i) v_i$$

- the face $\gamma$ is vertical, if and only if $(p, q) = (1, 0)$, and in that case we have the factorisation

$$f_\gamma(x, y) = x^a y^b s \prod_{\mu_i \in R_f} (y - \mu_i) v_i$$

and

$$f_\gamma(v^{-1}, w) = v^{-N} w^b s \prod_{\mu_i \in R_f} (w - \mu_i) v_i.$$

Each $\mu_i$ is called root of the face polynomial $f_\gamma$ and $R_f$ is the set of these roots.

**Definition 1.32** (Newton algorithm at infinity) The Newton algorithm at infinity of a polynomial $f$ in $K[x, y]$ consists in applying Newton maps at infinity associated to the one dimensional faces of $\overline{\mathcal{N}(f)} \setminus \mathcal{N}(f)$, and then to apply the Newton algorithm to each Newton transform. The output of the algorithm is the tree of the Newton transform polynomials produced.

**Remark 1.33** The Newton algorithm at infinity of $f - c$ with $c$ an element of $K$, depends on $c$. In paragraph 1.2.3, we will define the notions of Newton generic or non generic values and Newton bifurcation set.

**Lemma 1.34** A composition $\Sigma$ of Newton transformations in a Newton algorithm (at infinity) has the form

$$k[x, y] \rightarrow k[v^{-1}, v, w] \quad k[x, y] \rightarrow k[v^{-1}, v, w]$$

with $A$ and $B$ in $\mathbb{Z}$, $a$ and $b$ in $K$, and $Q$ in $k[v, v^{-1}]$.

**Proof** The proof is done by induction on the length of $\Sigma$ using the definition of a Newton transformation in Definitions 1.10 and 1.29. □
Lemma 1.35  Let \( f \) be in \( \mathbb{C}[x, y] \) (resp in \( \mathbb{C}[x^{-1}, x, y] \)) with isolated singularities in \( \mathbb{C}^2 \) (resp in \( \mathbb{C}^* \times \mathbb{C} \)). Then for any composition \( \Sigma \) of Newton transformations, the polynomial \( f_\Sigma \) in \( \mathbb{C}[x_1^{-1}, x_1, y_1] \) has isolated singularities in \( \mathbb{C}^* \times \mathbb{C} \).

Proof  Let \( \Sigma \) be a composition of Newton transformations. It follows from Lemma 1.34 that we can assume

\[
f_\Sigma(x_1, y_1) = f(x_1^A, x_1^B + P(x_1))
\]

with \( A \) and \( B \) in \( \mathbb{Z}\backslash\{0\} \) (the case \( B = 0 \) is similar) and \( P \) in \( \mathbb{C}[x_1, x_1^{-1}] \). We have

\[
\frac{\partial f}{\partial x_1}(x_1, y_1) = \frac{\partial f}{\partial x}(x_1^A, x_1^B + P(x_1)) A x_1^{A-1} + \frac{\partial f}{\partial y}(x_1^A, x_1^B + P(x_1)) (B x_1^{B-1} y_1 + P'(x_1))
\]

\[
\frac{\partial f}{\partial y_1}(x_1, y_1) = \frac{\partial f}{\partial y}(x_1^A, x_1^B + P(x_1)) x_1^B
\]

If \( f_\Sigma \) has non isolated singularities in \( \mathbb{C}^* \times \mathbb{C} \), then there is a point \( M = (a, b) \) in \( \mathbb{C}^* \times \mathbb{C} \) and an infinite sequence of distinct points \( M_n = (a_n, b_n) \) in \( \mathbb{C}^* \times \mathbb{C} \) which converges to \( M \) and such that for any \( n \), we have \( \frac{\partial f}{\partial x}(M_n) = \frac{\partial f}{\partial y}(M_n) = 0 \). As for any \( n \) we have \( a_n \neq 0 \), we conclude that \( \frac{\partial f}{\partial y}(a_n^A, a_n^B b_n + P(a_n)) = \frac{\partial f}{\partial x}(a_n^A, a_n^B b_n + P(a_n)) = 0 \) which proves that all the points \( (a_n^A, a_n^B b_n + P(a_n)) \) are critical points of \( f \). We can assume that all these points are distinct assuming for instance that infinitely many \( a_n \) are distinct (otherwise there are finitely many \( a_n \) but infinitely distinct \( b_n \)). Then we deduce that the point \((a^A, a^B b + P(a))\) is a non isolated critical point of \( f \). Contradiction.

\[\Box\]

1.2.3 Newton generic or non generic values and Newton bifurcation set

Definition 1.36  (Dicritical faces at infinity, discriminant) Let \( f \) be a polynomial in \( \mathbb{k}[x, y] \).

– A dicritical face at infinity of \( f \) is a one dimensional face of the Newton polygon \( N_\infty(f) \) which contains the origin.

– A dicritical face at infinity is said \textit{smooth} if its underlying line has an equation of the form \( px + qy = 0 \) with \( (p, q) \) in \( \mathbb{Z}^2 \) and \( q = 1 \) (resp \( p = 1 \)) if the face belongs to \( N_{0,\infty}(f) \) (resp \( N_{\infty,0}(f) \)).

– If \( \gamma \) is a dicritical face at infinity of \( f \), then the face polynomial \( f_\gamma \) can be written as \( P_\gamma(x^a, y^b) \) with \( P_\gamma \) a polynomial in \( \mathbb{k}[s] \) and \( (a, b) \) coprime integers in \( \mathbb{Z}^2 \). We call \textit{discriminant} of the face \( \gamma \) (relatively to \( f \)), the discriminant of \( P_\gamma(s) - c \) with respect to \( s \), element of \( \mathbb{k}[c] \).

Definition 1.37  (Newton generic and non generic values) Let \( f \) be a polynomial in \( \mathbb{k}[x, y] \). A value \( c_0 \) is said \textit{Newton non generic} for \( f \) if \( f \) has a dicritical face at infinity, denoted by \( \gamma \) and \( c_0 \) satisfies one of the two conditions:

– \( c_0 \neq f(0, 0) \) is a root of the discriminant of \( f_\gamma \),

– \( c_0 = f(0, 0) \) is a root of the discriminant of \( f_\gamma \) or the face \( \gamma \) is not smooth.

Definition 1.38  (Newton bifurcation set) Let \( f \) be a polynomial in \( \mathbb{k}[x, y] \). The \textit{Newton bifurcation set of} \( f \), denoted by \( B_\text{Newton}^f \), is the union of the discriminant of \( f \) (formed by the critical values of \( f \)), the set of Newton non generic values of \( f \) and the set of Newton non generic values of the Newton transforms \( f_\Sigma \), where \( \Sigma \) is a composition of Newton transforms during the Newton algorithm at infinity of \( f \).
Proposition 1.39 (Finiteness of the Newton bifurcation set) The Newton bifurcation set of a polynomial \( f \) in \( k[x, y] \) is finite.

Proof The discriminant of \( f \) is a finite set. There is a finite number of dicritical faces of \( f \) and for each face the set of roots of the discriminant of the face polynomial is finite. We conclude by the fact that the Newton algorithm at infinity is finite and the set of Newton non generic values of each Newton transform \( f_{\Sigma} \) occurring in the algorithm is finite. \( \Box \)

Example 1.40 All along this article, we will consider the following example (see also Examples 3.14 and 3.28).

\[
f(x, y) = x^6 y^4 + (4x^5 + 3x^4)y^3 + (6x^4 + 11x^3 + 3x^2)y^2 + (4x^3 + 13x^2 + 2x + 1)y + x^2 + 5x + 1.
\]

A Grobner basis of the Jacobian ideal of \( f \) is

\[
\left(\frac{y^2 + 31024}{455625} y + \frac{92512}{151875}, x - \frac{2581875}{5317088} y - \frac{659}{142422}\right).
\]

Then, the polynomial \( f \) has isolated singularities with in particular only two critical points

\[
\begin{align*}
\gamma_1(0) & : \quad x = -\frac{1}{84} - \frac{17\sqrt{14}}{168}i, \quad y = \frac{15512}{455625} - \frac{94948\sqrt{14}}{455625}i, \\
\gamma_2(0) & : \quad x = \frac{1}{84} + \frac{17\sqrt{14}}{168}i, \quad y = \frac{15512}{455625} + \frac{94948\sqrt{14}}{455625}i.
\end{align*}
\]

The Milnor number of each critical point is equal to one. The discriminant of \( f \) is the set

\[
\text{disc}(f) = \left\{ \frac{86}{135} - i\frac{56\sqrt{14}}{135}, \frac{86}{135} + i\frac{56\sqrt{14}}{135} \right\}.
\]

We apply the Newton algorithm at infinity of \( f \) and compute the Newton bifurcation values of \( f \) given by the algorithm. The polynomial \( f \) does not have any dicritical face at infinity and \( f(0, 0) = 1 \). Let \( c \) be in \( k \). The polygon \( \overline{N}(f - c) \setminus N(f - c) \) only has two one dimensional faces:

- \( \gamma_1(0) \) supported by the line of equation \( -x + 2y = 2 \) and the face polynomial of \( f - c \) is \( y(x^2 y + 1)^3 \),
- \( \gamma_2(0) \) supported by the line of equation \( x - y = 2 \) and the face polynomial of \( f - c \) is \( x^2(xy + 1)^4 \).

We apply the Newton algorithm at infinity:

- For the face \( \gamma_1(0) \), we obtain

\[
\sigma_{\gamma_1(0)} : x = v(w + 1), \quad y = -v^{-2} + v^{-2}(5v + 8w^3 + \ldots).
\]

It does not have a local dicritical face and its Newton polygon has only one one dimensional face, denoted by \( \gamma_1^{(1)} \), with face polynomial \( v^{-2}(5v + 8w^3) \). We continue the algorithm and get

\[
\sigma_{\gamma_1^{(1)}} : v = -8/5v_1^3, \quad w = v_1(w_1 + 1) \text{ and }
\]
\[(f_1)_{\gamma_1^{(1)}}(v_1, w_1) - c = 1500v_1^{-3}(w_1 - 7/30v_1 + \ldots), \quad (1.10)\]

which is a base case (Theorem 1.17). We conclude that the face \(\gamma_1^{(0)}\) does not produce any Newton bifurcation value.

- For the face \(\gamma_2^{(0)}\), we obtain

\[
\sigma_{\gamma_2^{(0)}}: x = v^{-1}, y = v(w - 1) \text{ and }
\]

\[
f_2(v, w) - c := f_2^{\gamma_2^{(0)}}(v, w) - c
\]

\[
= v^{-2}(w^4 + (2 - c)v^2 + 2vw^2 - 4v^2 w - v^3 + \ldots). \quad (1.11)
\]

The polynomial \(f_2\) has a local dicritical face \(\gamma_0^{(2)}\) supported by a line of equation \(2x + y = 0\). The associated polynomial is \(P_{\gamma_0^{(2)}}(s) = s^2 + 2s + 2\). The discriminant of the polynomial \(P_{\gamma_0^{(2)}}(s) - c\) is the polynomial \(4(-1 + c)\) in the variable \(c\).

- Assume \(c = 1\). Thus, \(c\) is the root of the discriminant of \(\gamma_0^{(2)}\) then \(c\) is a Newton bifurcation value of \(f\) as a local dicritical value of \(f_2\). We continue the Newton algorithm, this will be used in Examples 3.14 and 3.28. The Newton polygon \(N(f_2 - 1)\) has only one one dimensional face, denoted by \(\gamma_1^{(2)}\). It is supported by the line of equation \(2x + y = 0\). The face polynomial is \((v^{-4}w^2 + 1)^2\). We get

\[
\sigma_{\gamma_1^{(2)}}: v = -v_1^2, w = v_1(w_1 + 1) \text{ and }
\]

\[
f_3(v_1, w_1) = (f_2)_{\gamma_1^{(2)}}(v_1, w_1) - 1 = 4w_1^2 - 7v_1 + \ldots. \quad (1.12)
\]

The Newton polygon \(N(f_3)\) has only one one dimensional face \(\gamma^{(3)}\), it is supported by the line of equation \(2x + y = 2\). We continue the Newton algorithm and get a base case of Theorem 1.17

\[
\sigma_{\gamma^{(3)}}: v_1 = 4/7v_2^2, w_1 = v_2(w_2 + 1) \text{ and } (f_3)_{\gamma^{(3)}}(w_1, w_2) = 8v_2^2(w_2 + \ldots) \quad (1.13)
\]

- Assume \(c = \gamma_0(0, 0)(f_2) = 2\). Thus, as the dicritical face \(\gamma_0^{(2)}\) is not smooth, \(c\) is a Newton bifurcation value of \(f\). We continue the Newton algorithm. The Newton polygon \(N(f_2 - 2)\) has two one dimensional faces: \(\gamma_1^{(2)}\) supported by the line of equation \(2x + y = 0\), with a face polynomial equal to \(v^{-2}w_1^2(v_2^2 + 2v)\) and \(\gamma_2^{(2)}\) supported by the line of equation \(x + y = 1\), with a face polynomial equal to \(v(2w_2^2v_2^2 - 4v_1^2w_1 - 1)\) which has two simple roots \(r_1\) and \(r_2\).

- For the face \(\gamma_1^{(2)}\) we immediately get a base case of Theorem 1.17

\[
\sigma_{\gamma_1^{(2)}}: v = -1/2v_1^2, w = v_1(w_1 + 1) \text{ and } (f_2)_{\gamma_1^{(2)}}(v_1, w_1) - 2 = 8w_1 - 10v_1 + \ldots \quad (1.14)
\]

- For the face \(\gamma_2^{(2)}\) and one root of the face polynomial, denoted by \(\mu\), we also get a base case

\[
\sigma_{\gamma_2^{(2)}, \mu}: v = v_1, w = v_1(w_1 + \mu) \text{ and } (f_2)_{\gamma_2^{(2)}, \mu}(v_1, w_1) - 2 = v_1(w_1 + \ldots) \quad (1.15)
\]

- We consider \(c \notin \{1, 2\} \cup \text{disc } f\). The Newton polygon \(N(f_2 - c)\) has only one one dimensional face, denoted by \(\gamma_1^{(2)}\). It is supported by the line of equation \(2x + y = 0\).
Its face polynomial is \( P_{y_0}(x^{-1} y^2) - c \) which has two simple roots \( \{ \mu_1, \mu_2 \} \). Let \( \mu \) be one of such roots. Applying the Newton algorithm we obtain

\[
\sigma_{\gamma_1}^{(2), \mu} : v = \mu^{-1} v_1^2, \ w = v_1 (w_1 + 1) \quad \text{and} \quad (f_2)_{\sigma_{\gamma_1}^{(2), \mu}}(v_1, w_1) = w_1 u(v_1, w_1) \quad \text{or} \quad (f_2)_{\sigma_{\gamma_1}^{(2), \mu}}(v_1, w_1) = \ast w_1 + \ast v_1^n + (1.16)
\]

where \( u \) is a unit and “\( \ast \)” are constants. In particular, the Newton transform \( (f_2)_{\sigma_{\gamma_1}^{(2), \mu}}(v_1, w_1) \) is a base case of Theorem 1.17, then \( c \) is not a local Newton bifurcation value of \( f_2 \). Furthermore \( c \) does not belong to \( \text{disc } f \) by assumption, then \( c \) is not a Newton bifurcation value of \( f \).

## 2 Motivic Milnor fibers

We give below an introduction to motivic Milnor fibers and refer to [10,11,13,18,19,23] for further discussion.

### 2.1 Motivic setting

#### 2.1.1 Grothendieck rings

Let \( k \) be a field of characteristic 0 and \( \mathbb{G}_m \) its multiplicative group. We call \( k \)-variety, a separated reduced scheme of finite type over \( k \). We denote by \( \text{Var}_k \) the category of \( k \)-varieties and for any \( k \)-variety \( S \), by \( \text{Var}_S \) the category of \( S \)-varieties, where objects are morphisms \( X \to S \) in \( \text{Var}_k \). We denote by \( \mathcal{M}_S \) the localization of the Grothendieck ring of \( S \)-varieties with respect to the class \([A^1_k \times S \to S]\). We will use also the \( \mathcal{G}_m \)-equivariant variant \( \mathcal{M}^{\mathcal{G}_m}_{S \times \mathbb{G}_m} \) introduced in [19, §2] or [18, §2], which is generated by isomorphism classes of objects, \( Y \to S \times \mathbb{G}_m \) endowed with a monomial \( \mathbb{G}_m \)-action, of the category \( \text{Var}^{\mathcal{G}_m}_{S \times \mathbb{G}_m} \). In this context the class of the projection from \( A^1_k \times (S \times \mathbb{G}_m) \to S \times \mathbb{G}_m \) endowed with the trivial action is denoted by \( \mathbb{L} \). Let \( f : S \to S' \) be a morphism of varieties. The composition by \( f \) induces the direct image group morphism \( f_* \) and the fibred product over \( S' \) induces the inverse image ring morphism \( f^* \)

\[
f_* : \mathcal{M}^{\mathcal{G}_m}_{S \times \mathbb{G}_m} \to \mathcal{M}^{\mathcal{G}_m}_{S' \times \mathbb{G}_m}, \quad f^* : \mathcal{M}^{\mathcal{G}_m}_{S' \times \mathbb{G}_m} \to \mathcal{M}^{\mathcal{G}_m}_{S \times \mathbb{G}_m}.
\]

For a variety \( (X \to S \times \mathbb{G}_m, \sigma) \) where \( \sigma \) is a monomial action of \( \mathbb{G}_m \), we consider its fiber in \( 1 \) denoted by \( X^{(1)} \) and endowed with an induced action \( \sigma^{(1)} \) of the group of roots of unity \( \hat{\mu} \). The corresponding Grothendieck ring to its operation is denoted by \( \mathcal{M}^{\mathcal{G}_m}_S \) and isomorphic to \( \mathcal{M}^{\mathcal{G}_m}_{S \times \mathbb{G}_m} \) (see [19, Proposition 2.6]).

#### 2.1.2 Rational series

Let \( A \) be one of the rings \( \mathbb{Z}[L, L^{-1}], \mathbb{Z}[L^{-1}, (1/(1 - L^{-i})_{i>0})] \) and \( \mathcal{M}^{\mathcal{G}_m}_{S \times \mathbb{G}_m} \). We denote by \( A[[T]]_{sr} \) the \( A \)-submodule of \( A[[T]] \) generated by \( 1 \) and finite products of terms \( p_{e,i}(T) = L^e T^i/(1 - L^e T^i) \) with \( e \in \mathbb{Z} \) and \( i \in \mathbb{N}_{>0} \). There is a unique \( A \)-linear morphism

\[
\lim_{T \to \infty} : A[[T]]_{sr} \to A \quad \text{such that for any subset (} e_i, j_i \}_{i \in I} \text{ of } \mathbb{Z} \times \mathbb{N}_{>0} \text{ with } J \text{ finite or empty, } \lim_{T \to \infty}(\prod_{i \in I} p_{e_i,j_i}(T)) \text{ is equal to } (-1)^{|J|}.
\]
2.1.3 Polyhedral convex cone

We will use the following lemma similar to [17, Lemme 2.1.5] and [19, §2.9].

**Lemma 2.1** (Rational summation on a rational polyhedral convex cone) Let \( \phi \) and \( \eta \) be two \( \mathbb{Z} \)-linear forms defined on \( \mathbb{Z}^2 \). Let \( C \) be a rational polyhedral convex cone of \( \mathbb{R}^2 \setminus \{(0, 0)\} \), such that \( \phi(C) \) and \( \eta(C) \) are subsets of \( \mathbb{N} \). We assume that for any \( n \geq 1 \), the set \( C_n \) defined as \( \phi^{-1}(n) \cap C \cap \mathbb{Z}^2 \) is finite. We consider the formal series in \( \mathbb{Z}\left[\left]\mathbb{L}_i, \mathbb{L}^{-1}\right[\right][(T)] \)

\[
S_{\phi, \eta, C}(T) = \sum_{n \geq 1} \sum_{(k,l) \in C_n} \mathbb{L}^{-\eta(k,l)} T^n.
\]

1. If \( C \) is equal to \( \mathbb{R}_{>0}\omega_1 + \mathbb{R}_{>0}\omega_2 \) where \( \omega_1 \) and \( \omega_2 \) are two non colinear primitive vectors in \( \mathbb{Z}^2 \) with \( \phi(\omega_1) > 0 \) and \( \phi(\omega_2) > 0 \) then, denoting \( P = \{0, 1\omega_1 + 0, 1\omega_2 \} \cap \mathbb{Z}^2 \), we have

\[
S_{\phi, \eta, C}(T) = \sum_{(k_0,l_0) \in P} \frac{\mathbb{L}^{-\eta(k_0,l_0)} T^{\phi(k_0,l_0)}}{(1 - \mathbb{L}^{-\eta(\omega_1)} T^{\phi(\omega_1)})(1 - \mathbb{L}^{-\eta(\omega_2)} T^{\phi(\omega_2)})} \quad \text{(2.1)}
\]

and \( \lim_{T \to \infty} S_{\phi, \eta, C}(T) = 1 = \chi_c(C) \), where \( \chi_c \) is the Euler characteristic with compact support morphism.

2. If \( C \) is equal to \( \mathbb{R}_{>0}\omega \) where \( \omega \) is a primitive vector in \( \mathbb{Z}^2 \) with \( \phi(\omega) > 0 \), then we have

\[
S_{\phi, \eta, C}(T) = \frac{\mathbb{L}^{-\eta(\omega)} T^{\phi(\omega)}}{1 - \mathbb{L}^{-\eta(\omega)} T^{\phi(\omega)}} \quad \text{and} \quad \lim_{T \to \infty} S_{\phi, \eta, C}(T) = -1 = \chi_c(C). \quad \text{(2.2)}
\]

3. If \( C \) is equal to \( \mathbb{R}_{>0}\omega_1 + \mathbb{R}_{>0}\omega_2 \) where \( \omega_1 \) and \( \omega_2 \) are two non colinear primitive vectors in \( \mathbb{N}^2 \) with \( \phi(\omega_1) > 0 \) and \( \phi(\omega_2) > 0 \) then, denoting \( P = \{0, 1\omega_1 + 0, 1\omega_2 \} \cap \mathbb{Z}^2 \), we have

\[
S_{\phi, \eta, C}(T) = \sum_{(k_0,l_0) \in P} \frac{\mathbb{L}^{-\eta(k_0,l_0)} T^{\phi(k_0,l_0)}}{(1 - \mathbb{L}^{-\eta(\omega_1)} T^{\phi(\omega_1)})(1 - \mathbb{L}^{-\eta(\omega_2)} T^{\phi(\omega_2)})} + \frac{\mathbb{L}^{-\eta(\omega_2)} T^{\phi(\omega_2)}}{1 - \mathbb{L}^{-\eta(\omega_2)} T^{\phi(\omega_2)}} \quad \text{(2.3)}
\]

and \( \lim_{T \to \infty} S_{\phi, \eta, C}(T) = 0 = \chi_c(C) \).

**Proof** The points 1 and 2 are similar to [7, Lemma 1]. The point 3 follows from the points 1 and 2. \( \square \)

2.2 Euler characteristic and area

We recall the following definition introduced in [7]. We assume here \( k = \mathbb{C} \).

**Definition 2.2** (Area of a Newton polygon with respect to a polynomial) Let \( \mathcal{N} \) be a Newton polygon or a Newton polygon at infinity. We denote the set of one dimensional faces of \( \mathcal{N} \) by \( (S_i) \) with \( i \in \{1, \ldots, d\} \). Let \( f \) be a polynomial such that \( \mathcal{N}(f) = \mathcal{N} \) or \( \mathcal{N}_\infty(f) = \mathcal{N} \). For each face \( S_i \), we denote by \( r_i \) the number of roots of \( f|_{S_i} \), we denote by \( s_i \) the number of points with integer coordinates on the face \( S_i \) without its vertices and by \( S_i \) its area defined by \( S_i = |\det(v_i, w_i)|/2 \) if \( S_i \) is the segment \([v_i, w_i]\). We define the area of \( \mathcal{N} \) with respect to \( f \) as

\[
S_{\mathcal{N}, f} = \sum_{i=1}^{d} \frac{r_i S_i}{s_i + 1}.
\]
Remark 2.3 Note that if \( f \) is non degenerate with respect to \( \mathcal{N} \), then \( S_{\mathcal{N}, f} = S_{\mathcal{N}} \) where \( S_{\mathcal{N}} \) is the area of \( \mathcal{N} \).

Proposition 2.4 (Quasi-homogeneous case) Let \( f \) be a quasi homogeneous polynomial in \( \mathbb{C}[x^{-1}, x, y] \).
- if \( f(x, y) = x^a y^b \) with \( a \) in \( \mathbb{Z}^* \) and \( b \) in \( \mathbb{N}^* \), then we have
  \[ \chi_c(f^{-1}(1) \cap \mathbb{G}_m^2) = \chi_c(f^{-1}(1)) = 0. \]
- if \( f(x, y) = x^a y^b \prod_{i=1}^{r}(x^q - \mu_i y^p)^{v_i} \) (or \( f(x, y) = x^a y^b \prod_{i=1}^{r}(x^q y^p - \mu_i)^{v_i} \)) with \((p, q)\) a primitive vector of \((\mathbb{N}^*)^2\) and all the \( \mu_i \)'s are different, then we have
  \[ \chi_c(f^{-1}(1) \cap \mathbb{G}_m^2) = -\frac{2rS}{\sum_{i=1}^{r} v_i} = -2S_{\mathcal{N}, f} \]
where \( S \) is the area of the associated triangle with vertices \((0, 0), (a + q \sum_{i=1}^{r} v_i, b), (a, b + p \sum_{i=1}^{r} v_i)\) (respectively of the triangle with vertices \((0, 0), (a, b), (a + q \sum_{i=1}^{r} v_i, b + p \sum_{i=1}^{r} v_i)\)). Furthermore, if \( f \) belongs to \( \mathbb{C}[x, y] \) then we have
  \[ \chi_c(f^{-1}(1) \cap \mathbb{G}_m^2) = -2S_{\mathcal{N}, f}. \]

Proof Assume \( f(x, y) = x^a y^b \) with \( a \neq 0 \) and \( b > 0 \). Let \( d \) be the greatest common divisor of \( a \) and \( b \). We denote \( a' = a/d \) and \( b' = b/d \). Let \( u \) and \( v \) integers such that \( a'u + b'v = 1 \). Considering the change of variables in the torus \( x = x_0^{a'} y_0^{b'}, y = x_0^{-u} y_0^{v} \), we obtain an isomorphism between the algebraic varieties \( f^{-1}(1) \) and \( \mathbb{G}_m \times \mathbb{G}_d \) with \( \mu_d \) the group of \( d \)-roots of unity. In particular, we deduce that \( \chi_c(f^{-1}(1) \cap \mathbb{G}_m^2) \) is zero. Now, we consider the case

\[ f(x, y) = x^a y^b \prod_{i=1}^{r}(x^q - \mu_i y^p)^{v_i} = x^{a+b} \prod_{i=1}^{r}(x^q y^p - \mu_i)^{v_i} \]

with \((p, q)\) a primitive vector of \((\mathbb{N}^*)^2\) and all \( \mu_i \) are different. The case \( f(x, y) = x^a y^b \prod_{i=1}^{r}(x^q y^p - \mu_i)^{v_i} \) is similar. We denote by \( N = ap + bq + pq \sum_{i=1}^{r} v_i \). We consider \( u \) and \( v \) integers such that \( uq - vp = 1 \). On the torus, we use the change of variables \( x = x_1^{a'}, y = y_1^{a''} \) with \( s = 1 \) if \( N \geq 0 \) and \( s = -1 \) if \( N \leq 0 \). Remark that if \( N = 0 \), the two changes of variables are convenient. We obtain the equality

\[ f(x, y) = g(x_1, y_1) = x_1^{au + (b + p \sum_{i=1}^{r} v_i)u} |N| \prod_{i=1}^{r}(x_1 - \mu_i)^{v_i}. \]

In particular the variety \( f^{-1}(1) \cap \mathbb{G}_m^2 \) is isomorphic to the variety \( g^{-1}(1) \cap \mathbb{G}_m^2 \) which Euler characteristic is equal to \(-r|N|\). Indeed if \( N \neq 0 \), the variety is \(|N|\) copies of the graph of the function

\[ x_1 \mapsto \left( x_1^{au + (b + p \sum_{i=1}^{r} v_i)u} \prod_{i=1}^{r}(x_1 - \mu_i)^{v_i} \right)^{-1} \]

defined on \( \mathbb{G}_m \setminus \{ \mu_i \mid 1 \leq i \leq r \} \). If \( N = 0 \), then the variety is isomorphic to \( (g^{-1}(x_1, 1) = 1) \times \mathbb{G}_m \) which has Euler characteristic zero. To conclude, it is enough to use Definition 2.2 and remark that

\[ 2S = \left| \det \left( \left( a + q \sum_{i=1}^{r} v_i, b \right), \left( a, b + p \sum_{i=1}^{r} v_i \right) \right) \right| = |N| \sum_{i=1}^{r} v_i. \]
This formula is correct in the case of \( N = 0 \) because the area of the triangle is zero. \( \square \)

2.3 Arcs

Let \( X \) be a \( k \)-variety. For any integer \( n \), we denote by \( \mathcal{L}_n(X) \) the space of \( n \)-jets of \( X \). This set is a \( k \)-scheme of finite type and its \( K \)-rational points are morphisms \( \text{Spec} \ K[t]/t^{n+1} \to X, \) for any extension \( K \) of \( k \). There are canonical morphisms \( \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X) \). These morphisms are \( A^d_k \)-bundles when \( X \) is smooth with pure dimension \( d \). The arc space of \( X \), denoted by \( \mathcal{L}(X) \), is the projective limit of this system. This set is a \( k \)-scheme and we denote by \( \pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X) \) the canonical morphisms called truncation maps. For more details we refer for instance to [10,13,23].

For a non zero element \( \varphi \) in \( K[[t]] \) or in \( K[t]/t^{n+1} \), we denote by \( \text{ord} (\varphi) \) the valuation of \( \varphi \) and by \( \text{ac}(\varphi) \) the coefficient of \( t^{\text{ord} \varphi} \) in \( \varphi \) called the angular component of \( \varphi \). By convention \( \text{ac}(0) \) is zero. The multiplicative group \( \mathbb{G}_m \) acts canonically on \( \mathcal{L}_n(X) \) by \( \lambda \cdot \varphi(t) := \varphi(\lambda t) \). We consider the application \( \text{origin} \colon \varphi \mapsto (0) = \varphi \mod t \).

Let \( F \) be a closed subscheme of \( X \) and \( \mathcal{I}_F \) be the ideal of regular functions on \( X \) which vanish on \( F \). We denote by \( \text{ord} F \) the function which assigns to each arc \( \varphi \) in \( \mathcal{L}(X) \) the bound \( \min \text{ord} g(\varphi) \) where \( g \) runs on \( \mathcal{I}_{F,\varphi(0)} \).

2.4 The motivic Milnor fiber morphism

2.4.1 Motivic nearby cycles, motivic Milnor fiber

Let \( X \) be a smooth \( k \)-variety of pure dimension \( d \) and \( f : X \to A^1_k \) be a morphism. We set \( X_0(f) \) for the zero locus of \( f \) and for \( n \geq 1 \), we consider the scheme

\[
X_n(f) = \{ \varphi \in \mathcal{L}(X) \mid \text{ord} f(\varphi) = n \}
\]

endowed with the arrow \( (\pi_0, \text{ac} \circ f) \) to \( X_0(f) \times \mathbb{G}_m \). In particular for any \( m \geq n \), the truncation \( \pi_m(X_n(f)) \) denoted by \( X_n^{(m)}(f) \), is a \( (X_0(f) \times \mathbb{G}_m) \)-variety endowed with the standard action of \( \mathbb{G}_m \). By smoothness of \( X \), we have the equality

\[
[X_n^{(m)}(f)]^L_{-md} = [X_n^{(n)}(f)]^L_{-nd} \in \mathcal{M}_{X_0(f) \times \mathbb{G}_m}^{\mathbb{G}_m}
\]

this element is the motivic measure of \( X_n(f) \) denoted by \( \text{mes} (X_n(f)) \) and introduced by Kontsevich in [21]. In [12,14], Denef and Loeser introduce and prove the rationality of the following motivic zeta function

\[
Z_f(T) = \sum_{n \geq 1} \text{mes} (X_n(f)) T^n \in \mathcal{M}_{X_0(f) \times \mathbb{G}_m}^{\mathbb{G}_m}[[T]].
\]

They define the motivic nearby cycles \( S_f \) and the motivic Milnor fiber \( S_{f,x_0} \) at any point \( x_0 \) in \( X_0(f) \) as

\[
S_f = \lim_{T \to \infty} Z_f(T) \in \mathcal{M}_{X_0(f) \times \mathbb{G}_m}^{\mathbb{G}_m} \text{ and } S_{f,x_0} := I_{x_0}^*(S_f) \in \mathcal{M}_{\{x_0\} \times \mathbb{G}_m}^{\mathbb{G}_m}
\]

where \( I_{x_0}^* \) the pull-back morphism induced by the inclusion \( i_{x_0} : \{x_0\} \to X_0(f) \). The motive \( S_{f,x_0} \) realizes on classical invariants of the topological Milnor fiber of \( f \) at \( x_0 \) as the Milnor number (in the isolated singularity case) or the Hodge spectrum ([12, Theorem 4.2.1] and [14, §3.5]).
2.4.2 Motivic nearby cycles morphism

Using the weak factorisation theorem, Bittner [3] extends the motivic nearby cycles as a morphism defined over all the Grothendieck ring \( \mathcal{M}_X \). Guibert, Loeser and Merle [19] give a different construction using the motivic integration theory, we explain it below and we will use it in the following.

**Definition 2.5** Let \( X \) be a smooth \( k \)-variety with pure dimension \( d \), let \( U \) be a Zariski open and dense subset of \( X \), let \( F \) be its complement and let \( f : X \to \mathbb{A}^1_k \) be a morphism. Let \( n \) be in \( \mathbb{N}^* \) and \( \delta > 0 \), we consider the arc space

\[
X_n^\delta(f) := \{ \varphi \in \mathcal{L}(X) \mid \text{ord } f(\varphi) = n, \text{ord } \varphi^* F \leq n\delta \}
\]

endowed with the arrow \((\pi_0, \mathbb{R} \circ f)\) to \( X_0(f) \times \mathbb{G}_m \). Then, we consider the modified motivic zeta function

\[
Z^\delta_{f, U}(T) := \sum_{n \geq 1} \text{mes } (X_n^\delta(f)) T^n \in \mathcal{M}_{X_0(f) \times \mathbb{G}_m}^G[[T]].
\]

**Proposition 2.6** ([19] (§3.8)) Let \( U \) be an open and dense subset of a smooth \( k \)-variety \( X \) with pure dimension. Let \( f : X \to \mathbb{A}^1_k \) be a morphism. There is \( \delta_0 > 0 \) such that for any \( \delta \geq \delta_0 \), the series \( Z_f^\delta(T) \) is rational and its limit is independent of \( \delta \). We will denote by \( S_{f, U} \) the limit \( \lim_{T \to \infty} Z_f^\delta(T) \).

**Remark 2.7** Some remarks:

- With above notations, we deduce immediately from the proof of that proposition, the following equality

\[
f_i(S_{f, U}) = \lim_{T \to \infty} \sum_{n \geq 1} \text{mes } (X_n^\delta(f) \to \{0\} \times \mathbb{G}_m) T^n \in \mathcal{M}^G_{\{0\} \times \mathbb{G}_m}.
\]

Indeed it is enough to compare the computation on a log-resolution of \((X, X \setminus f^{-1}(0))\) of both sides of the equality. The computation of the left hand side term of the equality is done taking track of the origin of arcs above \( f^{-1}(0) \) and then forgetting it after application of the pushforward morphism \( f_i \), whereas the computation of the right hand side term is directly done on the resolution. In the following sections, we will identify \( \mathcal{M}^G_{\{0\} \times \mathbb{G}_m} \) to \( \mathcal{M}^{G_m}_{G_m} \) and we will simply write

\[
f_i(S_{f, U}) = \lim_{T \to \infty} \sum_{n \geq 1} \text{mes } (X_n^\delta(f)) T^n \in \mathcal{M}^{G_m}_{G_m}
\]

considering for any \( n \geq 1, X_n^\delta(f) \) endowed with its structural map to \( \mathbb{G}_m \).

- With the same proof, Proposition 2.6 can be extended to the case of \( X \) not necessary smooth but \( U \) smooth. Indeed, the singular locus of \( X \) is contained in \( F = X \setminus U \) and the first step of the proof is a resolution of \((X, F \cup X_0(f))\).

**Theorem 2.8** ([3,19] (§3.9)) Let \( f : X \to \mathbb{A}^1_k \) be a morphism on a \( k \)-variety not necessary smooth. There is a unique \( \mathbb{M}_k \)-linear group morphism \( S_f : \mathcal{M}_X \to \mathcal{M}^{G_m}_{X_0(f) \times \mathbb{G}_m} \) such that for all proper morphism \( p : Z \to X \) with \( Z \) smooth, and for all open and dense subset \( U \) in \( Z \), \( S_f([p : U \to X]) \) is defined as \( p_i(S_{f \circ p, U}) \).

\[\square\] Springer
2.5 Motivic zeta function and differential form

In the following, we will need to use motivic zeta functions of a function and a differential. These are induced by the change of variables associated to Newton transformations and studied for instance in [2,8,34]. More precisely, we will need to consider a modified version taking account of a closed subset.

Definition 2.9 Let $X$ be a $k$-variety of pure dimension $d$ and $g : X \to \mathbb{A}^1_k$ be a regular map. Let $U$ be a smooth open subvariety of $X$. The singular locus of $X$ is contained in the closed subset $X \setminus U$ denoted by $F$. We assume $U$ to be dense in $X$ and $X$ to be endowed with a differential form $\omega$ of degree $d$ without poles and whose zero locus is a divisor denoted by $D$ and included in $F$. For any $\delta > 0$, $n$ in $\mathbb{N}^*$ and $l$ in $\mathbb{N}^*$, we define

$$X_{n, l}(g, \omega, U) = \{ \varphi \in \mathcal{L}(X) \mid \text{ord}_g(\varphi) = n, \text{ord}_g^*(\mathcal{I}_F) \leq n\delta, \text{ord}_g(\varphi) = l \}$$

endowed with its structural map $(\pi_0, \mathbb{P} \circ g)$ to $(D \cap g^{-1}(0)) \times \mathbb{G}_m$. For any $m \geq n$ the truncation $\pi_m(X_{n, l}(g, \omega, U))$ is endowed with the standard action of $\mathbb{G}_m$. We define the motivic zeta function associated to $(g, \omega, U)$ as

$$Z_{g, \omega, U}^\delta(S, T) = \sum_{n \geq 1} \sum_{l \geq 1} \mes(X_{n, l}(g, \omega, U))^ST^n \in \mathcal{M}_{(D \cap g^{-1}(0)) \times \mathbb{G}_m}[[S, T]].$$

Lemma 2.10 For any $\delta > 0$, the motivic zeta function $Z_{g, \omega, U}^\delta(S, T)$ is rational in variables $S$ and $T$. The evaluation $\mathbb{L}^{-\delta}, T$ is well-defined, and when $T$ goes to infinity this series has a limit independent from $\delta$ large enough.

Proof The differential form $\omega$ defines a divisor on $X$ denoted by $D$. We consider a log-resolution $(h, Y, E)$ of the couple $(X, D \cup g^{-1}(0) \cup F)$ such that $h^{-1}(D), h^{-1}(g^{-1}(0))$ and $h^{-1}(E)$ are normal crossing divisors as union of irreducible components of $E$. We denote by $(E_i)_{i \in A}$ the set of irreducible components of $E$. We consider the following divisors

$$\Jac(h) = \sum_{i \in A} (v_i - 1)E_i, \quad \text{div}(g \circ h) = \sum_{i \in A} N_i(g)E_i,$$

$$\text{div}(h^*\omega) = h^{-1}(D) = \sum_{i \in A} N_i(\omega)E_i \quad \text{and} \quad h^{-1}(F) = \sum_{i \in A} N_i(\mathcal{I}_F)E_i.$$

Following the proof of Denef–Loeser [12, Theorem 2.2.1] and Guibert–Loeser–Merle [19, Proposition 3.8], we have

$$Z_{g, \omega, U}^\delta(S, T) = \sum_{I \in \mathcal{J}} [U_I \to (D \cap g^{-1}(0)) \times \mathbb{G}_m, \sigma]S_I(S, T)$$

where $\mathcal{J} = \{ I \subset A \mid I \cap C_\omega \neq \emptyset \}$. We consider $C_\omega = \{ i \in A \mid \text{ord}_g(\varphi) = 0 \}$, $C_g = \{ i \in A \mid \text{ord}_g(\varphi) = 0 \}$; for any $I$ in $\mathcal{J}, U_I$ is a variety defined in [19, §3.4] endowed with a structural map to the stratum $E_0 = \cap_{i \in I}E_i \cup \cup_{j \notin I}E_j$ composed with $g$ and a structural map to $\mathbb{G}_m$, and

$$S_I(S, T) = \sum_{(k_i) \in C_I} \prod_{i \in I} \left( \mathbb{L}^{-v_i}S^{N_i(\omega)}T^{N_i(g)} \right)^{k_i} \text{ with }$$

$$C_I = \left\{ (k_i) \in \mathbb{N}_{\geq 1}^{|I|} \text{ s.t. } \sum_{i \in I} k_iN_i(\mathcal{I}_F) \leq \delta \sum_{i \in I} k_iN_i(g) \right\}.$$
For instance, if $C^g_f = \mathbb{N}_{\geq 1}^{|I|}$ then we have $S_I(S, T) = \prod_{i \in I} \frac{L_{-v_i}^e S_{N_i(\omega)} T_{N_i(\gamma)}}{1 - L_{-v_i}^e S_{N_i(\omega)} T_{N_i(\gamma)}}$. More generally, the rationality of $S_I$ is shown using a partition of the cone $C^g_f$ in subcones (generated by a basis of primitive vectors of $\mathbb{Z}^{|I|}$) and a toric change of variables. This implies the rationality of the zeta function. Furthermore, we remark that $S_I(\mathbb{L}^{-1}, T)$ and $Z^\delta_{g, \omega, U}(\mathbb{L}^{-1}, T)$ are well defined. Finally, as in the proof of [19, Proposition 3.8] if $I \setminus C_g$ is not empty then $\lim_{T \to \infty} S_I(\mathbb{L}^{-1}, T) = 0$. If $I$ is included in $C_g$ then the limit $\lim_{T \to \infty} S_I(\mathbb{L}^{-1}, T)$ is equal to $(-1)^{|I|}$ if $\delta \geq \sup_{i \in I} \frac{N_i(T_x)}{N_i(g)}$.

Definition 2.11 (Motivic nearby cycles and Milnor fiber relatively to an open set and a differential form) Let $X$ be a $k$-variety of pure dimension $d$ and $g : X \to \mathbb{A}^1_k$ be a regular map. Let $U$ be a smooth open subvariety of $X$ and $F$ be the closed subset $X \setminus U$. Assume $U$ to be dense in $X$. Let $\omega$ be a differential form of degree $d$ without poles and which zero locus is a divisor $D$ included in $F$. We call the zeta function $Z^\delta_{g, \omega, U}(\mathbb{L}^{-1}, T)$, motivic zeta function of $g$ relatively to the open set $U$ and the differential form $\omega$ and we denote it by

$$Z^\delta_{g, \omega, U}(T) = \sum_{n \geq 1} \left( \sum_{l \geq 1} \text{mes}(X^\delta_{n, l}(g, \omega, U)) \mathbb{L}^{-l} \right) T^n \in \mathcal{M}^Gm_{(D \cap g^{-1}(0)) \times \mathbb{G}_m}([[T]]) .$$

We consider also its limit, still called motivic nearby cycles, which does not depend on $\delta \gg 1$,

$$S_{g, \omega, U} = \lim_{T \to \infty} Z^\delta_{g, \omega, U}(T) \in \mathcal{M}^Gm_{(D \cap g^{-1}(0)) \times \mathbb{G}_m} .$$

For any point $x_0$ in $X_0(g)$, we consider the motivic Milnor fiber of $g$ at $x_0$ and relatively to $U$ and $\omega$

$$(S_{g, \omega, U})_{x_0} := i^*_{\{x_0\}} \left( S_{g, \omega, U} \right) \in \mathcal{M}^Gm_{\{x_0\} \times \mathbb{G}_m} .$$

Remark 2.12 Some remarks:

- By Lemma 2.10 the motivic nearby cycles $S_{g, \omega, U}$ depends only on the irreducible components of the divisor of $\omega$ and not on its multiplicities. Thus, if the divisor of $\omega$ is equal to $F$ then $S_{g, \omega, U}$ does not depend on $\omega$.

- The point $x_0$ is the origin of arcs defining $(S_{g, \omega, U})_{x_0}$. As in Remark 2.7 we can identify

$$\mathcal{M}^Gm_{\{x_0\} \times \mathbb{G}_m} \text{ with } \mathcal{M}^Gm_{\mathbb{G}_m} .$$

2.6 The motivic Milnor fiber $(S_{fe, x \neq 0})_{((0, 0), 0)}$ with $f(x, y) = x^{-M} g(x, y)$ and $\varepsilon = \pm$

In Sect. 3.1.2, we will compute the motivic Milnor fiber at infinity (Theorem 3.8) and the motivic nearby cycles at infinity (Theorem 3.23) of a polynomial in $k[x, y]$. For this computation, we will need to compute motivic Milnor fibers of $1/f$ or $f$ along the open set $x \neq 0$ with $f$ an element of $k[x^{-1}, x, y]$. We do it in this section in Theorem 2.22.

2.6.1 Setting

Notation 2.13 In this section we consider an integer $M$ in $\mathbb{Z}$ and a polynomial $f$ in $k[x^{-1}, x, y]$ equal to

$$f(x, y) = x^{-M} g(x, y) = \sum_{(a, b) \in \mathbb{Z} \times \mathbb{N}} c_{a, b} f(x^a y^b)$$
Remark 2.14
along the last coordinate which extends the application
The open set
Remark 2.16
U
open set
singular locus of
We have the following commutative diagram
with value 0. In order to do that, we denote by
F
by Definition 2.9 and Remark 2.16, where for any integers
Notation 2.15
We consider an integer
x
We use notations of Sect. 2.6.1. We fix
2.6.2 The motive
Sfε,ω,x≠0
((0,0),0)
We use notations of Sect. 2.6.1. We fix \( \delta > 0 \). We have
\[
(Z^\delta_{\pi_\epsilon,\omega,U_\varepsilon}(T))_{((0,0),0)} = \sum_{n \geq 1} \left( \sum_{n \delta \geq k \geq 1} \mathbb{L}^{-(v-1)k}\text{mes } (X_\varepsilon,n,k) \right) T^n
\]
\( X_{\varepsilon,n,k} = \{ \varphi \in \mathcal{L}(X) | \varphi(0) = (0,0,0), \ord x(\varphi) = k, \ord \omega(\varphi) = (v-1)k, \ord \pi_{\varepsilon}(\varphi) = \ord z(\varphi) = n \} \)

(2.5)

defined with the structural map \( \mathbb{A} \circ \chi : X_{\varepsilon,n,k} \to \mathbb{G}_{m}, \varphi \mapsto \mathbb{A}(z(\varphi)). \)

**Remark 2.17** The origin of each arc of \( X_{\varepsilon,n,k} \) is \( ((0,0),0) \) and the generic point belongs to \( U_\varepsilon \), so from the definition of \( X_\varepsilon \), for any integers \( n \geq 1 \) and \( k \geq 1 \), there is an isomorphism between \( X_{\varepsilon,n,k} \) and

\[ \{ (x(t), y(t)) \in \mathcal{L}(\mathbb{A}^2) | \ord x(t) = k, \ord \omega(x(t), y(t)) = (v-1)k, \ord y(t) > 0, \ord f_\varepsilon(x(t), y(t)) = n \} \]

defined with the map \( \mathbb{A} \circ f_\varepsilon : (x(t), y(t)) \mapsto \mathbb{A}(f_\varepsilon(x(t), y(t))). \) In this section, we will identify these arc spaces.

**Remark 2.18** We will use the following notation

\[ \left( Z^0_{f_\varepsilon, \omega, x \neq 0}(T) \right)_{((0,0),0)} := \left( Z^0_{\pi_\varepsilon, \omega, U_\varepsilon}(T) \right)_{((0,0),0)} \cdot \]

It follows from the definition of \( \omega \) and Remark 2.12 that the limit

\[ \left( S^0_{f_\varepsilon, \omega, x \neq 0} \right)_{((0,0),0)} := - \lim_{T \to \infty} \left( Z^0_{f_\varepsilon, \omega, x \neq 0}(T) \right)_{((0,0),0)} \]

does not depend on the chosen differential form \( \omega \) with zero locus contained in the divisor \( "x = 0" \) of \( X_\varepsilon \). In the following, we will simply denote it by \( S^0_{f_\varepsilon, x \neq 0} \) \((0,0,0)\). This motive belongs to \( \mathcal{M}^{G_m}_{\mathbb{G}_{m}} \) considered as \( \mathcal{M}^{G_m}_{\mathbb{G}_{m}} \) as in Remark 2.7.

### 2.6.3 Computation of \( \left( S^0_{f_\varepsilon, x \neq 0} \right)_{((0,0),0)} \) in the case \( g(0,0) \neq 0 \)

**Proposition 2.19** Let \( \varepsilon \) be in \( \{ \pm \} \) and \( f(x, y) = x^{-M}g(x, y) \) be in \( \mathbb{k}[x, x^{-1}, y] \) with \( g \) be in \( \mathbb{k}[x, y] \) satisfying \( g(0,0) \neq 0 \).

If \( \varepsilon M \geq 0 \) then we have

\[ \left( Z^0_{f_\varepsilon, x \neq 0}(T) \right)_{((0,0),0)} = 0 \text{ and } \left( S^0_{f_\varepsilon, x \neq 0} \right)_{((0,0),0)} = 0. \]

If \( \varepsilon M < 0 \) then we have

\[ \left( Z^0_{f_\varepsilon, x \neq 0}(T) \right)_{((0,0),0)} = \left[ x^{-\varepsilon M} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m} \right] \frac{L^{-1}T^{-\varepsilon M}}{1 - L^{-1}T^{-\varepsilon M}} \text{ and } \]

\[ \left( S^0_{f_\varepsilon, x \neq 0} \right)_{((0,0),0)} = \left[ x^{-\varepsilon M} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m} \right]. \]

**Proof** Let \( (n, k) \) be in \( \left( \mathbb{N}^* \right)^2 \). If \( \varepsilon M \geq 0 \), then \( \left( Z^0_{f_\varepsilon, x \neq 0}(T) \right)_{((0,0),0)} \) is equal to zero because each arc space \( X_{\varepsilon,n,k} \) is empty. If \( \varepsilon M < 0 \), then by Remark 2.17 the arc space \( X_{\varepsilon,n,k} \) is non-empty if and only if \( n = -\varepsilon M k \). The \( n \)-jet space \( \pi_n(X_{\varepsilon,n,k}) \) endowed with the canonical \( \mathbb{G}_m \)-action on jets is a bundle over \( x^{-\varepsilon M} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m} \) with fiber \( \mathbb{A}^{2n-k} \) and \( \sigma_{\mathbb{G}_m} \) is the action by translation of \( \mathbb{G}_m \) defined by \( \sigma_{\mathbb{G}_m}(\lambda, x) = \lambda x \) for any \( (\lambda, x) \) in \( \mathbb{G}_m^2 \). Then by definition, the motivic measure of \( X_{\varepsilon,n,k} \) is equal to \( [x^{-\varepsilon M} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}]L^{-k} \) and the result follows by summation and limit. 

\( \square \)
2.6.4 Computation of \((S_{f^e,x\neq0})_{((0),0)}\) in the case \(g(0,0) = 0\)

Using notations of Sect. 2.6.1, the Newton algorithm and the strategy of the first author and Vey in [8] (see also [7]), we express the motivic zeta function \(Z^{\mu}_{f^e,x\neq0}(T)_{((0),0)}\) and the motivic Milnor fiber \((S_{f^e,x\neq0})_{((0),0)}\) in terms of Newton polygons of \(f^e\) and its Newton transforms.

**Notation 2.20** We use notations \(R^e\) and \(\sigma(p,q,\mu)\) introduced in Remark 1.7 and Definition 1.10. We define \(\sigma_{G_m}\) and \(\sigma_{G_m^2}\) the actions of \(G_m\) on \(G_m\) and \(G_m^2\) by \(\sigma_{G_m}(\lambda, x) = \lambda x\) and \(\sigma_{G_m^2}(\lambda, (x, y)) = (\lambda x, \lambda y)\). For any \((p, q)\) in \(\mathbb{N}^2\), we consider the differential form \(\omega_{p,q}(v, w) = v^{p+q-1}dv \wedge dw\).

**Theorem 2.22** (Computation of \((S_{f^e,x\neq0})_{((0),0)}\)) Let \(f(x, y) = x^{-M}g(x, y)\) be in \(k[x, x^{-1}, y]\) with \(g\) be in \(k[x, y]\) not divisible by \(x\) and satisfying \(g(0,0) = 0\). Let \(v\) be in \(\mathbb{N}_{\geq1}\) and \(\omega\) be the associated differential form in Notations 2.15. Let \(\varepsilon\) be in \(\{\pm\}\). Then, writing \((a, b)\) the horizontal face \(\gamma_h\) of \(N(f)\) defined in Definition 1.5, we have for any \(\delta > 0\)

\[
(Z^\delta_{f^e,x\neq0}(T))_{((0),0)} = \left[ (x^a y^b)^e : G_m^e \to G_m, \sigma_{G_m^e} \right] R^\delta_{(a,b),\varepsilon,\omega}(T) + \sum_{\gamma \in N(f)} \dim \gamma = 1 \sum_{\mu \in R^\gamma} \left( Z^\delta_{f^e,x\neq0}(T) \right)_{((0),0)}. \tag{2.6}
\]

with \(\sigma(p,q,\mu)\) the Newton transformation defined in Definition 1.10.

- Furthermore, if \(b = 0\) then,
  - in the case \(\varepsilon = +\), the motivic Milnor fiber \((S_{f,x\neq0})_{((0),0)}\) is
    \[
    (S_{f,x\neq0})_{((0),0)} = s^{(+)}[x^a : G_m \to G_m, \sigma_{G_m}] + \sum_{\gamma \in N(f)^+ \setminus \gamma_h} (-1)^{\dim \gamma + 1} [f^e : G_m^2 \setminus (f^e = 0) \to G_m, \sigma_{G_m}] + \sum_{\gamma \in N(f)} \dim \gamma = 1 \sum_{\mu \in R^\gamma} \left( S_{f^e,x\neq0} \right)_{((0),0)} \tag{2.7}
    \]
  - in the case \(\varepsilon = -\), the motivic Milnor fiber \((S_{f^-x\neq0})_{((0),0)}\) is 0 if \(f\) belongs to \(k[x, y]\), otherwise
    \[
    (S_{f^-x\neq0})_{((0),0)} = s^{(-)}[1/x^a : G_m \to G_m, \sigma_{G_m}] + \sum_{\gamma \in N(f)^- \setminus \gamma_h} (-1)^{\dim \gamma + 1} [1/f^e : G_m^2 \setminus (f^e = 0) \to G_m, \sigma_{G_m}] + \sum_{\gamma \in N(f)^-} \dim \gamma = 1 \sum_{\mu \in R^\gamma} \left( S_{f^-x\neq0} \right)_{((0),0)} \tag{2.8}
    \]
- if \(b \neq 0\) then,
  - in the case \(\varepsilon = +\), the motivic Milnor fiber \((S_{f^e,x\neq0})_{((0),0)}\) is
Consider the decomposition of the motivic zeta function (Kouchnirenko type formula for and compute the limit of the zeta function \( Z \))

\[
\begin{align*}
(S_{f,x \neq 0})_{((0,0),0)} &= -s^{(+)\left[ x^a y^b : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{c_m^2} \right]} \\
&+ \sum_{\gamma \in \mathcal{N}(f)^+ \setminus \{ \gamma_h \}} (-1)^{\dim \gamma + 1} \left[ f_{\gamma} : \mathbb{G}_m^2 \setminus \{ f_{\gamma} = 0 \} \to \mathbb{G}_m, \sigma_{\gamma} \right] \\
&+ \sum_{\gamma \in \mathcal{N}(f)^- \setminus \{ \gamma_h \}} \sum_{\mu \in \mathcal{R}_T} (S_{f_{\mu(p,q,\rho)},v \neq 0})_{((0,0),0)} \tag{2.9}
\end{align*}
\]

- in the case \( \varepsilon = - \), the motivic Milnor fiber \((S_{1/f,x \neq 0})_{((0,0),0)} = 0 \) if \( f \) belongs to \( \mathbb{k}[x, y] \), otherwise

\[
(S_{1/f,x \neq 0})_{((0,0),0)} = \sum_{\gamma \in \mathcal{N}(f)^- \setminus \{ \gamma_h \}} (-1)^{\dim \gamma + 1} \left[ f_{\gamma} : \mathbb{G}_m^2 \setminus \{ f_{\gamma} = 0 \} \to \mathbb{G}_m, \sigma_{\gamma} \right] \\
+ \sum_{\gamma \in \mathcal{N}(f)^- \setminus \{ \gamma_h \}} \sum_{\mu \in \mathcal{R}_T} (S_{f_{\mu(p,q,\rho)},v \neq 0})_{((0,0),0)} \tag{2.10}
\]

with \( r = 1 \) if \( b = 0 \) otherwise \( r = 2, s^{(\varepsilon)} = 1 \) if \( \gamma_h \) is in \( \mathcal{N}(f)^{\varepsilon} \) otherwise \( s^{(\varepsilon)} = 0 \), and for any face \( \gamma \) in \( \mathcal{N}(f), R^\delta_{\gamma,y,\omega}(T) \) are rational functions defined in (Propositions 2.31 and 2.32).

**Proof** In the case \( \varepsilon = - \), if \( M \leq 0 \), namely \( f \) belongs to \( \mathbb{k}[x, y] \), then \( 1/f \) does not vanish, so for any integers \( n \) and \( k \), the arc space \( X_{-n,k} \) (defined in formula (2.5)) is empty and the motivic zeta function \((Z_{1/f,0,x \neq 0}(T))_{((0,0),0)} \) is equal to 0. We give the ideas of the general proof and refer to Sect. 2.6.5 for details. In Sect. 2.6.5.1, formula (2.20), for \( \varepsilon \) in \( \{ \pm \} \), we consider the decomposition of the motivic zeta function

\[
(Z_{f^{\varepsilon},\omega,x \neq 0}(T))_{((0,0),0)} = \sum_{\gamma \in \mathcal{N}(f)} Z_{\varepsilon,\gamma,\omega}(T). \tag{2.11}
\]

In Sect. 2.6.5.2, Proposition 2.31, in the case of the horizontal face \( \gamma_h \) defined in Definition 1.5, we show the rationality and compute the limit of \( Z_{\varepsilon,\gamma_h,\omega}(T) \). This limit does not depend on \( \omega \). In Sects. 2.6.5.3 and 2.6.5.4 we consider the case of a non horizontal face \( \gamma \). Depending on the fact that the face polynomial \( f_{\gamma} \) vanishes or not on the angular components of the coordinates of an arc (Remark 2.25), we decompose in formula (2.21) the zeta function \( Z_{\varepsilon,\gamma,\omega}(T) \) as a sum of \( Z_{\varepsilon,\gamma,\omega}(T) \) and \( Z_{\varepsilon,\gamma,\omega}(T) \). In particular if the face \( \gamma \) is zero dimensional then \( Z_{\varepsilon,\gamma,\omega}(T) \) is zero. See also Remark 2.33. In Proposition 2.32 we show the rationality and compute the limit of the zeta function \( Z_{\varepsilon,\gamma,\omega}(T) \). This limit does not depend on \( \omega \). In Sect. 2.6.5.4, Proposition 2.37 we prove the decomposition

\[
Z_{\varepsilon,\gamma,\omega}(T) = \sum_{\mu \in \mathcal{R}_T} \left( Z_{\varepsilon,\gamma,\omega}(T) \right)_{((0,0),0)}.
\]

Applying the Newton algorithm (Lemma 1.13) inductively, using the base cases (Examples 2.38 and 2.39), we recover the rationality of \((Z_{f^{\varepsilon},\omega,x \neq 0}(T))_{((0,0),0)}\), compute \((S_{f^{\varepsilon},\omega,x \neq 0})_{((0,0),0)}\) and check its independence on \( \omega \).

Using Proposition 2.4, we deduce from Theorem 2.22 a Kouchnirenko type formula computing the Euler characteristic of the motivic Milnor fiber \((S_{1/f,x \neq 0})_{((0,0),0)}\). This formula will be used in Corollary 3.25.

**Corollary 2.23** (Kouchnirenko type formula for \( \chi_c((S_{f^{\varepsilon},x \neq 0})_{((0,0),0)}) \) Let \( f(x, y) = x^{-M} g(x, y) \) with \( M \in \mathbb{Z} \) and \( g \) in \( \mathbb{k}[x, y] \) not divisible by \( x \) and satisfying \( g(0, 0) = 0 \). We denote \((a, b) = \gamma_h\) the horizontal face in Definition 1.5.

- If \( b = 0 \) then we have,
• in the case “ε = +”, we have
\[ \tilde{\chi}_c \left( (S_{f,x \neq 0})_{((0,0),0)}^{(1)} \right) = s^{(+)a} - 2 \sum_{\gamma \in N(f)^+} \dim \gamma = 1 \chi_{N(f)} \cdot f_{\gamma} 
+ \sum_{\gamma \in N(f), \dim \gamma = 1} \sum_{\mu \in R_{\mu}} \tilde{\chi}_c \left( (S_{f_{\mu}(p,q,\mu), x \neq 0})_{((0,0),0)}^{(1)} \right) \]

(2.12)

• in the case “ε = −”, \( \tilde{\chi}_c \left( (S_{1/f,x \neq 0})_{((0,0),0)}^{(1)} \right) \) is 0 if \( f \) belongs to \( k[x,y] \), otherwise we have
\[ \tilde{\chi}_c \left( (S_{1/f,x \neq 0})_{((0,0),0)}^{(1)} \right) = s^{(-)a} - 2 \sum_{\gamma \in N(f)^-} \dim \gamma = 1 \chi_{N(f)} \cdot f_{\gamma} 
+ \sum_{\gamma \in N(f)^-, \dim \gamma = 1} \sum_{\mu \in R_{\mu}} \tilde{\chi}_c \left( (S_{1/f_{\mu}(p,q,\mu), x \neq 0})_{((0,0),0)}^{(1)} \right) \]

(2.13)

- If \( b \neq 0 \) then we have,

• in the case “ε = +”, we have
\[ \tilde{\chi}_c \left( (S_{f,x \neq 0})_{((0,0),0)}^{(1)} \right) = -2 \sum_{\gamma \in N(f)^+} \dim \gamma = 1 \chi_{N(f)} \cdot f_{\gamma} 
+ \sum_{\gamma \in N(f), \dim \gamma = 1} \sum_{\mu \in R_{\mu}} \tilde{\chi}_c \left( (S_{f_{\mu}(p,q,\mu), x \neq 0})_{((0,0),0)}^{(1)} \right) \]

(2.14)

• in the case “ε = −”, \( \tilde{\chi}_c \left( (S_{1/f,x \neq 0})_{((0,0),0)}^{(1)} \right) \) is 0 if \( f \) belongs to \( k[x,y] \), otherwise we have
\[ \tilde{\chi}_c \left( (S_{1/f,x \neq 0})_{((0,0),0)}^{(1)} \right) = -2 \sum_{\gamma \in N(f)^-} \dim \gamma = 1 \chi_{N(f)} \cdot f_{\gamma} 
+ \sum_{\gamma \in N(f)^-, \dim \gamma = 1} \sum_{\mu \in R_{\mu}} \tilde{\chi}_c \left( (S_{1/f_{\mu}(p,q,\mu), x \neq 0})_{((0,0),0)}^{(1)} \right) \]

(2.15)

where \( s^{(\epsilon)} = 1 \) if \( \gamma_{\nu} \) is in \( N(f)^{\epsilon} \) otherwise \( s^{(\epsilon)} = 0 \).

### 2.6.5 Proof of Theorem 2.22

In all this subsection we consider the rational function \( f(x,y) \) equal to \( x^{-M}g(x,y) \) with \( M \) in \( \mathbb{Z} \) and \( g \) in \( k[x,y] \) not divisible by \( x \) and satisfying \( g(0,0) = 0 \). We fix also an integer \( \nu \) in \( \mathbb{N}_{\geq 1} \) and denote by \( \omega \) the associated differential form in Notations 2.15. We consider \( \epsilon \) in \( \{\pm\} \) and \( \delta \geq 1 \).

2.6.5.1 Decomposition of the zeta function along \( N(f) \)

**Notation 2.24** For any face \( \gamma \) of \( N(f) \) with dual cone \( C_{\gamma} \), for any integers \( n \geq 1 \) and \( k \geq 1 \), using Remark 2.17 we consider
\[ X_{\ell,n,k}^{\gamma} = \left\{ (x(t), y(t)) \in X_{\ell,n,k} \mid (\text{ord } x(t), \text{ord } y(t)) \in C_{\gamma} \right\} \]
endowed with its structural map to \( (\overline{\mathbb{A}} \circ f^e) \) to \( \mathbb{C}_{m} \) and we decompose
\[ X_{\ell,n,k} = \bigsqcup_{\gamma \in N(f)} X_{\ell,n,k}^{\gamma}. \]
Remark 2.25 For any arc \((x(t), y(t))\) in \(X_{\varepsilon,n,k}^\gamma\), we can write
\[ f(x(t), y(t)) = \tilde{f}(x(t), y(t)) \] with \(\tilde{f}\) in \(k[x, y, u]\) and \(m\) the function defined in Proposition 1.9 relatively to \(\Delta(f)\). As \(\text{ord } f^\varepsilon(x(t), y(t)) = n\) we have
\[ m(\text{ord } x(t), \text{ord } y(t)) \leq n/\varepsilon = n\varepsilon.\]

Two cases occur:
- \(\varepsilon n = m(\text{ord } x, \text{ord } y)\), namely \(n = \varepsilon m(\text{ord } x, \text{ord } y)\), if and only if \(f_y(\overline{\alpha c} x, \overline{\alpha c} y) \neq 0\).
- \(m(\text{ord } x, \text{ord } y) < n\varepsilon\) if and only if \(\dim \gamma = 1\) and \(f_y(\overline{\alpha c} x, \overline{\alpha c} y) = 0\).

Notation 2.26 Let \(\varepsilon\) be in \([\pm]\) and \(\gamma\) be a face of \(\mathcal{N}(f)\). We introduce some notations.
- We consider the cones of \(\mathbb{R}^2\) and \(\mathbb{R}^3\)
  \[ \mathcal{C}_{\varepsilon, \gamma}^{\delta, =} \ni (\alpha, \beta) \in \mathcal{C}_\gamma \mid 0 < \alpha \leq \varepsilon m(\alpha, \beta) \delta, 0 < \beta \] and
  \[ \mathcal{C}_{\varepsilon, \gamma}^{\delta, <} \ni (n, \alpha, \beta) \in \mathbb{R}^3 \times \mathcal{C}_\gamma \mid 0 < \alpha \leq n\delta, 0 < \beta \] (for any \((\alpha, \beta)\) be in \(\mathcal{C}_{\varepsilon, \gamma}^{\delta, =}\) and any \((n, \alpha, \beta)\) be in \(\mathcal{C}_{\varepsilon, \gamma}^{\delta, <}\), we consider the arc spaces \(X_{\varepsilon, \alpha, \beta}^\gamma = \begin{cases} (x(t), y(t)) \in \mathcal{L}(\mathbb{A}^2_k) & \text{ord } x(t) = \alpha, \text{ord } y(t) = \beta, f_y(\overline{\alpha c} x(t), \overline{\alpha c} y(t)) \neq 0, \\ \text{ord } f^\varepsilon(x(t), y(t)) = \varepsilon m(\alpha, \beta) & \end{cases}\) and
  \[ X_{\varepsilon, n, \alpha, \beta}^\gamma = \begin{cases} (x(t), y(t)) \in \mathcal{L}(\mathbb{A}^2_k) & \text{ord } x(t) = \alpha, \text{ord } y(t) = \beta, f_y(\overline{\alpha c} x(t), \overline{\alpha c} y(t)) = 0, \\ \text{ord } f^\varepsilon(x(t), y(t)) = n & \end{cases}\) endowled with \(\overline{\alpha c} f^\delta\), their structural map to \(\mathbb{G}_m\).
- For any \(\delta > 0\) and any face \(\gamma\) in \(\mathcal{N}(f)\), we consider the motivic zeta function
  \[ Z_{\varepsilon, \gamma, \omega}^\delta(T) = \sum_{n \geq 1} \left( \sum_{n\delta \geq k \geq 1} \mathbb{L}^{-(y-1)k} \text{mes } (X_{\varepsilon, n, k}^\gamma) \right) T^n. \tag{2.17} \]
Furthermore, if \(\gamma\) is not horizontal we consider
\[ Z_{\varepsilon, \gamma, \omega}^{\delta, =}(T) = \sum_{n \geq 1} \sum_{(\alpha, \beta) \in \mathcal{C}_{\varepsilon, \gamma}^{\delta, =} \cap \overline{(\mathbb{N})^2}} \mathbb{L}^{-(y-1)\alpha} \text{mes } (X_{\varepsilon, \alpha, \beta}^\gamma) T^n \tag{2.18} \]
and
\[ Z_{\varepsilon, \gamma, \omega}^{\delta, <}(T) = \sum_{n \geq 1} \sum_{(\alpha, \beta) \in \overline{(\mathbb{N})^2} \times \mathcal{C}_{\varepsilon, \gamma}^{\delta, <} \cap \overline{\mathbb{N}^3} \cap \overline{\mathbb{N}^3}} \mathbb{L}^{-(y-1)\alpha} \text{mes } (X_{\varepsilon, n, \alpha, \beta}^\gamma) T^n. \tag{2.19} \]

Remark 2.27 If \(\dim \gamma = 0\) then \(Z_{\varepsilon, \gamma, \omega}^{\delta, <}(T) = 0\).

Remark 2.28 As \(\gamma\) is not the horizontal face, we observe that for any \(n \geq 1\), the following sets are finite
\[ \{(\alpha, \beta) \in \mathcal{C}_{\varepsilon, \gamma}^{\delta, =} \mid n = \varepsilon m(\alpha, \beta)\} \text{ and } \{(\alpha, \beta) \in \mathcal{C}_\gamma \mid (n, \alpha, \beta) \in \mathcal{C}_{\varepsilon, \gamma}^{\delta, <}\}.\]
Proposition 2.29 For any \( \delta > 0 \), for any \( \varepsilon \) in \( \{\pm\} \), we have the decomposition
\[
(Z^\delta_{f^{\varepsilon, \omega}, x \neq 0}(T))_{((0, 0), 0)} = \sum_{\gamma \in \mathcal{N}(f)} Z^\delta_{e, \gamma, \omega}(T)
\] (2.20)
with the following equality for any non horizontal face \( \gamma \) in \( \mathcal{N}(f) \)
\[
Z^\delta_{e, \gamma, \omega}(T) = Z^\delta_{e, \gamma, \omega}(T) + Z^\delta_{e, \gamma, \omega}(T).
\] (2.21)

Proof The proof follows from the additivity of the measure, using equality (2.16) for (2.20) and Remark 2.25 for (2.21).

2.6.5.2 Rationality and limit of \( Z^\delta_{e, \gamma, \omega}(T) \)

In this subsection we study the case of the horizontal face \( \gamma \).

Notation 2.30 If \( x \) is a real number, we will denote by \( \lfloor x \rfloor \) its integral part. We use \( \mathcal{N}(f)^{\varepsilon} \) defined in Notations 2.13.

Proposition 2.31 (Case of the horizontal face) Write \((a, b)\) the horizontal face \( \gamma \) of \( \mathcal{N}(f) \) with \( a \) in \( \mathbb{Z} \) and \( b \) in \( \mathbb{N} \). The dual cone of \( \gamma \) is \( C_\gamma = \mathbb{R}_{>0}(0, 1) + \mathbb{R}_{>0}(p, q) \) with \((p, q)\) a primitive vector and \( p \neq 0 \). Let \( N = ap + bq = (\gamma_1 \mid (p, q)) \).

- If \( b = 0 \), then
  - if \( \varepsilon a > 0 \) then, there is \( \delta_0 > 0 \) such that, for any \( \delta \geq \delta_0 \), the motivic zeta function \( Z^\delta_{e, \gamma, \omega}(T) \) is rational equal to
    \[
    Z^\delta_{e, \gamma, \omega}(T) = [x^{\varepsilon a} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}] R^\delta_{y_h, e, \omega}(T)
    \] (2.22)
    with \( R^\delta_{y_h, e, \omega}(T) \) computed in formula (2.24) and converges to \(-1\), when \( T \to \infty \).
  - if \( \varepsilon a \leq 0 \), namely \( \gamma \notin \mathcal{N}(f)^{\varepsilon} \), then for any \( \delta \geq 1 \), we have \( Z^\delta_{e, \gamma, \omega}(T) = 0 \).

- If \( b \neq 0 \), then we have
  \[
  Z^\delta_{e, \gamma, \omega}(T) = [(x^a y^b)^{\varepsilon} : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}] R^\delta_{y_h, e, \omega}(T)
  \] (2.23)
  - if \( \varepsilon = + \) and \( N > 0 \) then \( R^\delta_{y_h, e, \omega}(T) \) is computed in formula (2.26) and converges to \(1\).
  - if \( \varepsilon = + \) and \( N \leq 0 \), namely \( \gamma \notin \mathcal{N}(f)^{+} \), then \( R^\delta_{y_h, e, \omega}(T) \) is computed in formula (2.27) and converges to \(0\).
  - if \( \varepsilon = - \) and \( N \geq 0 \) then \( R^\delta_{y_h, e, \omega}(T) = 0 \).
  - if \( \varepsilon = - \) and \( N < 0 \) then \( R^\delta_{y_h, e, \omega}(T) \) is computed in formula (2.28) and converges to \(0\).

Proof Assume first \( b = 0 \). Then, by Remark 2.25, for any arc \((x(t), y(t))\) with \(\text{ord } f^\varepsilon(x(t), y(t)) = \varepsilon \text{ord } x(t), \text{ord } y(t)\) in \( C_\gamma \) we have
\[
\text{ord } f^\varepsilon(x(t), y(t)) = \varepsilon \text{ord } x(t).
\]
- Assume \( \varepsilon a \leq 0 \) then for any \((n, k)\) in \( \mathbb{N}^{\varepsilon} \), the set \( X^\delta_{n, k, \gamma} \) is empty and \( Z^\delta_{e, \gamma, \omega}(T) = 0 \).
- Assume \( \varepsilon a > 0 \) and \( \delta > 1 \). As \( a \varepsilon \) is an integer, the condition \( \delta \text{ord } f^\varepsilon(x(t), y(t)) \geq \text{ord } x(t) \) is satisfied. Then applying formula (2.17), we have
  \[
  Z^\delta_{e, \gamma, \omega}(T) = \sum_{k \geq 1} \mathbb{L}^{- (v - 1) k} \text{mes } (X^\gamma_{e, k, \gamma, k}) T^{e a k}.
  \] (2.24)
A couple \((k, l)\) in \((\mathbb{N}^*)^2\) belongs to \(C_{\gamma h}\) if and only if \(pl > qk\). Then, for any \(k \geq 1\) we have

\[
X_{\epsilon, eak, k}^{\gamma h} = \{(x(t), y(t)) \in \mathcal{L}(K_k^2) \mid \text{ord } x(t) = k, \ p.\text{ord } y(t) > qk, \text{ namely ord } y(t) \geq \lfloor qk/p \rfloor + 1\}
\]

and by definition of the motivic measure, we get

\[
\text{mes } (X_{\epsilon, eak, k}^{\gamma h}) = [x^{e\alpha} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}] \mathbb{L}^{-\lfloor qk/p \rfloor - k}
\]

and

\[
Z_{\epsilon, \gamma h, \omega}^k(T) = [x^{e\alpha} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}] \sum_{k \geq 1} \mathbb{L}^{-vk - \lfloor qk/p \rfloor} T^{ke\alpha}.
\]

Let \(k \geq 1\), there is a unique integer \(r \in \{0, \ldots, p - 1\}\) such that \(k = [k/p]p + r\). There is also a unique integer \(\beta\) in \(\{0, \ldots, p - 1\}\) such that \(qr = [q/p]q + [q/p]\) implying \([qk/p] = [k/p]q + [q/p]\). We obtain equality (2.24), writing \(k\) as \(lp + r\) with \(l\) in \(\mathbb{N}\) and \(r\) in \(\{0, \ldots, p - 1\}\), and by decomposition of \(Z_{\epsilon, \gamma h, \omega}^k(T)\) as the sum of \(p\) formal series

\[
Z_{\epsilon, \gamma h, \omega}^k(T) = [x^{e\alpha} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}] \left( -1 + \sum_{r=0}^{p-1} \sum_{l=0}^{p-1} \mathbb{L}^{-v(lp+r)-lq-[q/p]} T^{a\alpha(lp+r)} \right).
\]

Assume \(b \neq 0\), by Remark 2.25, for any arc \((x(t), y(t))\) with \((\text{ord } x(t), \text{ord } y(t))\) in \(C_{\gamma h}\), we have

\[
\text{ord } f^\epsilon(x(t), y(t)) = \epsilon \text{m}(x(t), y(t)) = \epsilon(\text{ord } x(t) + \text{bord } y(t)).
\]

In particular, the motivic zeta function can be written as

\[
Z_{\epsilon, \gamma h, \omega}^k(T) = \sum_{n \geq 1} \sum_{\mathbb{L}_{r>0}^2} \mathbb{L}^{-(\nu-1)k} \text{mes } (X_{\epsilon, k, l}) T^n
\]

with \(X_{\epsilon, k, l} = \{(x(t), y(t)) \in \mathcal{L}(K_k^2) \mid \text{ord } x(t) = k, \text{ord } y(t) = l\}\) with its structural map, \((x(t), y(t)) \mapsto \mathfrak{F}(x(t)^{e\alpha} y(t)^{e\beta})\), and

\[
C_{\epsilon, \gamma h}^k = \{(k, l) \in (\mathbb{R}_{>0})^2 \mid l > qk/p, \epsilon(ak + bl)\delta \geq k\} \subseteq C_{\gamma h}.
\]

For any \((k, l)\) in \(C_{\gamma h} \cap (\mathbb{N}^*)^2\), for any integer \(m \geq m(k, l)\), the \(m\)-jet space \(\pi_m(X_{\epsilon, k, l})\) with the canonical \(\mathbb{G}_m\)-action is a bundle over \(((x^a y^b)^e : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m})\) with fiber \(h^{2m-k-l}\) and for any \((\lambda, x, y)\) in \(\mathbb{G}_m^3, \sigma_{\mathbb{G}_m}(\lambda, (x, y)) = (\lambda k x, \lambda l y)\). By definition of the motivic measure and Remark 2.21, we get mes \((X_{\epsilon, k, l}) = \mathbb{L}^{-k-1}[(x^a y^b)^e : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}]\) and formula (2.23) with

\[
R_{\epsilon, \gamma h, \omega}^k(T) = \sum_{n \geq 1} \sum_{\mathbb{L}_{r>0}^2} \mathbb{L}^{-(\nu-1)k} T^n.
\]

- Assume \(\epsilon = +\) and \(N = ap + bq > 0\), namely \(\frac{a}{p} > -\frac{q}{p}\), then there is \(\delta_0 > 0\) such that for any \(\delta \geq \delta_0\), \(\frac{a}{p} \geq \frac{1}{bb} \frac{a}{\delta}\) then, for any \((k, l)\) in \(C_{\gamma h}\), we have \(\delta(ak + bl) > \delta k(a + bq/p) \geq \delta k(a + (1 - a\delta)/\delta) = k\) inducing the equality
$$C_{\varepsilon, y_\delta} = C_{\gamma_\delta} = \mathbb{R}_{\geq 0}(0, 1) + \mathbb{R}_{\geq 0}(p, q).$$ Then, applying Lemma 2.1 formula (2.1), we obtain
\begin{equation}
R_{\gamma_\delta, \varepsilon, \omega}^\delta(T) = \sum_{(k_0, b_0) \in \mathcal{P}} \frac{L^{-v k_0 - b_0} T^{a k_0 + b l_0}}{(1 - L^{-1} T^b)(1 - L^{-v p - q} T^{b p + q})}
\end{equation}
(2.26)
with \(\mathcal{P} = ([0, 1]([0, 1]+[0, 1](p, q)) \cap \mathbb{N}^2\). We have \(\lim_{T \to \infty} R_{\gamma_\delta, \varepsilon, \omega}^\delta(T) = 1\).

- Assume \(\varepsilon = +\) and \(N = a p + b q \leq 0\), namely \(\frac{a}{p} \geq -\frac{q}{b}\) (this case occurs only for \(a < 0\), then for any \(\delta > 0\), \(1 - \frac{a}{b} \cdot \frac{q}{p} > \frac{q}{p}\) and in that case, we remark that \(C_{\varepsilon, y_\delta} = \{(k, l) \in (\mathbb{R}_{\geq 0})^2 | \frac{1}{k} \geq \frac{L - \delta}{\delta} \geq l > kq/p\}\). We have \(\lim_{T \to \infty} R_{\gamma_\delta, \varepsilon, \omega}^\delta(T) = 0\).

\begin{itemize}
\item If \(N = q b + a p \geq 0\), namely \(\frac{q}{p} \geq -\frac{a}{b}\), then the cone \(C_{\varepsilon, y_\delta}^\delta\) is empty for any \(\delta > 0\), and \(R_{\gamma_\delta, \varepsilon, \omega}^\delta(T) = 0\).
\item If \(N = q b + a p < 0\), namely \(\frac{q}{p} < -\frac{a}{b}\), then there is \(\delta_0 > 0\) such that for any \(\delta \geq \delta_0\), we have \(-(1 + \delta)/(b\delta) > \frac{q}{p}\) and in that case \(C_{\varepsilon, y_\delta}^\delta = \mathbb{R}_{\geq 0}(p, q) + \mathbb{R}_{\geq 0}\omega\delta\) with \(\omega\delta = (b\delta, -(1 + a\delta))\) and by Lemma 2.1 formula (2.3) we have
\begin{equation}
R_{\gamma_\delta, \varepsilon, \omega}^\delta(T) = \sum_{(k_0, b_0) \in \mathcal{P}} \frac{L^{-v k_0 - b_0} T^{a k_0 + b l_0}}{(1 - L^{-((v, 1)(p, q))}(a, b)(\omega\delta)))((1 - L^{-((v, 1)(a)(\omega\delta))})
\end{equation}
(2.28)
with \(\mathcal{P} = ([0, 1](p, q) + [0, 1](p, q)) \cap \mathbb{N}^2\). We have \(\lim_{T \to \infty} R_{\gamma_\delta, \varepsilon, \omega}^\delta(T) = 0\).
\end{itemize}

\[\square\]

2.6.5.3 Rationality and limit of \(Z_{\varepsilon, y_\delta}(T)\) for \(\gamma\) non horizontal
We use the subset \(\mathcal{N}(f)^\varepsilon\) of \(\mathcal{N}(f)\) defined in Notations 2.13 and the actions \(\sigma_\gamma\) defined in Remark 2.21. Similarly to [17,18,30] we have

**Proposition 2.32** Let \(\gamma\) be a non horizontal face of \(\mathcal{N}(f)\). The motivic zeta function \(Z_{\varepsilon, y_\delta}^\delta(T)\) is rational and for \(\delta\) large enough, we have the convergence
\[\lim_{T \to \infty} Z_{\varepsilon, y_\delta}^\delta(T) = s^\varepsilon_T (-1)^{\dim \mathcal{N}(f)(f_\gamma)} = G_m, \sigma_\gamma \in M_{\overline{G}_m}^m\]
with \(s^\varepsilon_T = 1\) if \(\gamma\) is a face in \(\mathcal{N}(f)^\varepsilon\), otherwise \(s^\varepsilon_T = 0\). More precisely, there is a rational function \(R_{\gamma_\delta, \varepsilon, \omega}^\delta(T)\) such that
\[Z_{\varepsilon, y_\delta}^\delta(T) = [f_\gamma^{-\varepsilon} : G_m, \sigma_\gamma] R_{\gamma_\delta, \varepsilon, \omega}^\delta(T).
\] 
(2.29)
If \( \gamma \) is a zero dimensional face \((a, b)\) in \(\mathbb{Z} \times \mathbb{N}\), with \(C_\gamma = \mathbb{R}_{>0} \omega_1 + \mathbb{R}_{>0} \omega_2\) with \(\omega_1\) and \(\omega_2\) primitive vectors in \(\mathbb{N}^n \times \mathbb{N}\),

- if \( \gamma \) belongs to \(\mathcal{N}(f)^\varepsilon\), namely \(\varepsilon((a, b) \mid \omega_1) > 0\) and \(\varepsilon((a, b) \mid \omega_2) > 0\), \(R_{\gamma,\varepsilon,\omega}^\delta(T)\) is a rational function computed in formula \((2.31)\) which does not depend on \(\delta\) large enough and converges to 1,
- if \(\varepsilon((a, b) \mid \omega_1) \leq 0\) and \(\varepsilon((a, b) \mid \omega_2) \leq 0\), then for any \(\delta > 0\), \(R_{\gamma,\varepsilon,\omega}^\delta(T) = 0\),
- otherwise, for any \(\delta > 0\), \(R_{\gamma,\varepsilon,\omega}^\delta(T)\) is a rational function as in formula \((2.32)\) and converges to 0.

If \( \gamma \) is a one dimensional face supported by a line of equation \(ap + bq = N\) with dual cone \(C_\gamma = \mathbb{R}_{>0}(p, q)\) then,

- if \( \gamma \) does not belong to \(\mathcal{N}(f)^\varepsilon\), namely \(\varepsilon N \leq 0\), then \(R_{\gamma,\varepsilon,\omega}^\delta(T) = 0\).
- if \( \gamma \) belongs to \(\mathcal{N}(f)^\varepsilon\), namely \(\varepsilon N > 0\), then for \(\delta \geq \frac{p}{\varepsilon N}\), \(R_{\gamma,\varepsilon,\omega}^\delta(T)\) is computed in formula \((2.33)\), does not depend on \(\delta\) and converges to \(-1\).

**Proof** Let \( \gamma \) be a non horizontal face of \(\mathcal{N}(f)\). We use Notations \(2.26\). For any \((\alpha, \beta)\) in \(C_\gamma \cap (\mathbb{N}^n)^2\), for any integer \(m \geq m(\alpha, \beta)\), the \(m\)-jet space \(\pi_m(X_{\varepsilon,\alpha,\beta})\) with the canonical \(G_m\)-action is a bundle over \((f_\gamma : \mathbb{C}_m^2 \setminus \{f_\gamma = 0\} \to \mathbb{C}_m, \sigma_{\alpha,\beta}\) with fiber \(\mathbb{A}_{2m-\alpha-\beta}\) and for any \((\lambda, x, y)\) in \(\mathbb{C}_m, \sigma_{\alpha,\beta}(\lambda, (x, y)) = (\lambda^\alpha x, \lambda^\beta y)\). Then, by definition of the motivic measure \(\mu((X_{\varepsilon,\alpha,\beta}) = \mathbb{L}^{-\alpha-\beta}[(f_\gamma : \mathbb{C}_m^2 \setminus \{f_\gamma = 0\} \to \mathbb{C}_m, \sigma_{\alpha,\beta}]\in \mathcal{M}_{X_{\varepsilon,\alpha,\beta}}\). Then, using Remark \(2.21\) we have formula \((2.29)\), with

\[
R_{\gamma,\varepsilon,\omega}^\delta(T) = \sum_{n \geq 1} \sum_{\alpha, \beta \in C_{\varepsilon,\gamma}^\delta \cap (\mathbb{N}^n)^2} \mathbb{L}^{-\alpha-\beta}T^n. \tag{2.30}
\]

The proof of Proposition \(2.32\) follows from Lemma \(2.1\) using the following description of the cone \(C_{\varepsilon,\gamma}^\delta\).

- Assume \( \gamma \) is a zero dimensional face equal to \((a, b)\) with \(b > 0\). The associated cone \(C_\gamma\) has dimension 2. It can be described as \(C_\gamma = \mathbb{R}_{>0} \omega_1 + \mathbb{R}_{>0} \omega_2\) where \(\omega_1\) and \(\omega_2\) are the primitive normal vectors of adjacent faces \(\gamma_1\) and \(\gamma_2\) of \(\gamma\), elements of \(\mathbb{N}^n \times \mathbb{N}\), because \(\gamma\) is not horizontal. For any \((\alpha, \beta)\) in \(C_\gamma\), we have \(m(\alpha, \beta) = (\gamma \mid (\alpha, \beta))\).

- Assume \(\varepsilon(\omega_1 \mid \gamma) > 0\) and \(\varepsilon(\omega_2 \mid \gamma) > 0\), namely \(\gamma \in \mathcal{N}(f)^\varepsilon\). Let \(\delta\) be a positive real number satisfying

\[
\delta \geq \max \left( \frac{((1, 0) \mid \omega_1)}{\varepsilon(\gamma \mid (\alpha, \beta))}, \frac{((1, 0) \mid \omega_2)}{\varepsilon(\gamma \mid (\alpha, \beta))} \right).
\]

In that case the cone \(C_{\varepsilon,\gamma}^\delta\) is equal to \(C_\gamma\). Indeed, let \((\alpha, \beta)\) be in \(C_\gamma\). There are real numbers \(\lambda\) and \(\mu\) in \(\mathbb{R}_{>0}\) such that \((\alpha, \beta) = \lambda \omega_1 + \mu \omega_2\). Then, the inequalities defining the cone \(C_{\varepsilon,\gamma}^\delta\) are satisfied:

\[
\varepsilon m(\alpha, \beta) = \varepsilon((\alpha, \beta) \mid \gamma) > 0 \quad \text{and} \quad \delta \varepsilon m(\alpha, \beta) = \varepsilon \delta(\lambda(\gamma \mid \omega_1) + \mu(\gamma \mid \omega_2)) \geq ((1, 0) \mid (\alpha, \beta)) = \alpha.
\]

Then, applying Lemma \(2.1\) we obtain

\[
R_{\gamma,\varepsilon,\omega}^\delta(T) = \sum_{(a_0, b_0) \in \mathcal{P}_\gamma} \frac{\mathbb{L}^{-((v, 1)(a_0, b_0))}T^{\varepsilon((a, b) \mid (a_0, b_0))}}{(1 - \mathbb{L}^{-((v, 1)(\omega_1))}T^{\varepsilon((a, b) \mid \omega_1)})(1 - \mathbb{L}^{-((v, 1)(\omega_2))}T^{\varepsilon((a, b) \mid \omega_2)})}. \tag{2.31}
\]
with \( \mathcal{P}_\gamma = ([0, 1] \omega_1 + [0, 1] \omega_2) \cap \mathbb{N}^2 \), and \( \lim_{T \to \infty} R^5_{\gamma, \varepsilon, \omega}(T) = 1 \).

- Assume \( \varepsilon(\gamma \mid \omega_2) > 0 \) and \( \varepsilon(\gamma \mid \omega_1) = 0 \). The case \( \varepsilon(\gamma \mid \omega_1) > 0 \) and \( \varepsilon(\gamma \mid \omega_2) = 0 \) is symmetrical. By definition we have \( C^{\delta,=}_{\varepsilon, \gamma} = C_\gamma \cap L^{-1}_\delta(\mathbb{R}_{\geq 0}) \) with \( L_\delta : (\alpha, \beta) \mapsto \varepsilon \delta(\gamma \mid (\alpha, \beta)) - \alpha \).

Let \( \omega_\delta = \frac{L_\delta(\omega_1)}{\varepsilon(\gamma \mid \omega_1)} \omega_1 + \omega_2 \) element of \( C_\gamma \). As \( L_\delta(\omega_\delta) = 0 \) and \( L^{-1}_\delta(\mathbb{R}_{\geq 0}) \) is a half-plane, we can conclude \( C^{\delta,=}_{\varepsilon, \gamma} = \mathbb{R}_{\geq 0} \omega_2 + \mathbb{R}_{> 0} \delta \). Then, applying Lemma 2.1 we obtain

\[
R^5_{\gamma, \varepsilon, \omega}(T) = \sum_{(\alpha_0, \beta_0) \in \mathcal{P}^\delta_{\gamma, \varepsilon}} \frac{1}{(1 - L^-((v, 1) | \omega_1) T^\varepsilon((a, b) | (\alpha_0, \beta_0))) (1 - L^-((v, 1) | \omega_2) T^\varepsilon((a, b) | \omega_2))}
\]

with \( \mathcal{P}^\delta_{\gamma, \varepsilon} = ([0, 1] \omega_1 + [0, 1] \omega_2) \cap \mathbb{N}^2 \) and \( \lim_{T \to \infty} R^5_{\gamma, \varepsilon, \omega}(T) = 0 \).

- Assume \( \varepsilon(\gamma \mid \omega_2) > 0 \) and \( \varepsilon(\gamma \mid \omega_1) < 0 \). The case \( \varepsilon(\gamma \mid \omega_1) > 0 \) and \( \varepsilon(\gamma \mid \omega_2) < 0 \) is symmetrical. By convexity, annulling the linear form \( (\gamma \mid .) \), the vertex \( \omega = (b, -a) \), is an element of the cone \( C_\gamma \). We observe that any element \( u \in \mathbb{R}_{> 0} \omega + \mathbb{R}_{> 0} \omega \subset C_\gamma \) does not belong to \( C^{\delta,=}_{\varepsilon, \gamma} \), because \( \varepsilon m(u) < 0 \). Then, we conclude that

\[
C^{\delta,=}_{\varepsilon, \gamma} = \{ (\alpha, \beta) \in \mathbb{R}_{> 0} \omega_2 + \mathbb{R}_{> 0} \omega \mid 0 < \alpha < \varepsilon m(\alpha, \beta) \delta, 0 < \beta \}.
\]

As \( \varepsilon(\omega_2 \mid \gamma) > 0 \) and \( \varepsilon(\omega \mid \gamma) = 0 \) we can apply the previous point.

- Assume \( \varepsilon(\gamma \mid \omega_2) \leq 0 \) and \( \varepsilon(\gamma \mid \omega_1) \leq 0 \). Then \( C^{\delta,=}_{\varepsilon, \gamma} \) is empty, because for any \((\alpha, \beta) \in C_\gamma \), \( \varepsilon m(\alpha, \beta) \leq 0 \).

Assume \( \gamma \) is a one dimensional compact face supported by a line with equation \( mp + nq = N \) with \((p, q)\) non negative integers and \( \gcd(p, q) = 1 \). In particular in that case we have \( C_\gamma = \mathbb{R}_{> 0}(p, q) \). Let \((a, b)\) be a point in \( \gamma \) with integral coordinates. Then, for any \((kp, kq)\) in \( C_\gamma \) we have \( m(kp, kq) = ((a, b) \mid (kp, kq)) = kN \) and

\[
C^{\delta,=}_{\varepsilon, \gamma} = \{ (kp, kq) \in C_\gamma \mid 1 \leq kp \leq \varepsilon kN \delta \}.
\]

In particular, under the condition \( \varepsilon N \leq 0 \) the set \( C^{\delta,=}_{\varepsilon, \gamma} \) is empty, otherwise for any \( \delta \geq \frac{p}{\varepsilon N} \), the set \( C^{\delta,=}_{\varepsilon, \gamma} \) is equal to \( C_\gamma \). Applying Lemma 2.1 we obtain the following expression of \( R^5_{\gamma, \varepsilon, \omega}(T) \) implying its convergence to \(-1\)

\[
R^5_{\gamma, \varepsilon, \omega}(T) = \frac{1}{1 - L^-((v, p) + q) T^{\varepsilon N}}.
\]
is empty and $Z_{-\gamma,\epsilon,\omega}^\delta(T) = 0$. This is the reason why the second summation in Eqs. (2.8) and (2.10) is only on $\mathcal{N}(f)^-$. 

**Remark 2.34** By additivity of the measure, by Eq. (2.19) and Definition 1.7 and Remark 2.28, for any one dimensional face $\gamma$ in $\mathcal{N}(f)$, the motivic zeta function $Z_{e,\gamma}^{\delta,<}(T)$ has the following decomposition

$$Z_{e,\gamma}^{\delta,<}(T) = \sum_{\mu \in R_{\gamma}} \sum_{(n,\alpha,\beta) \in C_{e,\gamma}^{\delta,<} \cap \mathbb{N}^3} \mathbb{L}^{-(\nu-1)\alpha} \text{mes} \left( X_{(n,\alpha,\beta),\mu} \right) T^n$$

where $X_{(n,\alpha,\beta),\mu} = \left\{ (x(t), y(t)) \in \mathcal{L}(A_{k}^2) \left| \frac{\bar{a}c}{(y(t))^p} = \mu \bar{a}c (x(t))^q, \text{ord} x(t) = \alpha, \text{ord} y(t) = \beta, f_{\sigma}(x(t), y(t)) = n \right. \right\}$. 

**Proposition 2.35** Let $\gamma$ be a one dimensional face of $\mathcal{N}(f)$, let $\mu$ be a root of $f_{\gamma}$ and $\sigma_{(p,q,\mu)}$ the induced Newton transform (Definition 1.10). For any $k > 0$, denoting $\alpha = pk$ and $\beta = qk$, we have

$$\text{mes} \left( X_{(n,\alpha,\beta),\mu} \right) = \mathbb{L}^{-(p+q-1)k} \text{mes} \left( Y_{(n,k)}^{\sigma_{(p,q,\mu)}} \right)$$

with $Y_{(n,k)}^{\sigma_{(p,q,\mu)}} = \left\{ (v(t), w(t)) \in \mathcal{L}(A_{k}^2) \left| \text{ord} v(t) = k, \text{ord} w(t) > 0, \text{ord} f_{\sigma}(v(t), w(t)) = n \right. \right\}$. 

**Proof** The proof is similar to that of [8, Lemma 3.3] (see also [7, Proposition 6] or the proof of Proposition 3.53 below). \hfill \square

**Remark 2.36** Let $\gamma$ be a one dimensional face of $\mathcal{N}(f)$ supported by a line of equation $ap + bq = N$. Let $\mu$ be a root of $f_{\gamma}$ and $\sigma_{(p,q,\mu)}$ the induced Newton transform. By Lemma 1.13, there is a polynomial $f_{\sigma}(p,q,\mu)$ in $k[v, w]$ such that

$$f_{\sigma}(p,q,\mu)(v, w) = v^N \tilde{f}_{\sigma}(p,q,\mu)(v, w).$$

In particular the Newton transform $f_{\sigma}(p,q,\mu)$ satisfies the same type of conditions on $f$, defined in Sect. 2.6.1, and the motive $(S_{f_{\sigma}(p,q,\mu), v \neq 0})_{((0,0),0)}$ is well defined.

**Proposition 2.37** Let $\gamma$ be a one dimensional face of $\mathcal{N}(f)$ supported by a line of equation $ap + bq = N$ with $p$ and $q$ in $(\mathbb{N}^*)^2$ and coprime. With Notations 2.20, for $\delta$ large enough, the motivic zeta function $Z_{e,\gamma,\omega}^{\delta,<}$ can be decomposed as

$$Z_{e,\gamma,\omega}^{\delta,<}(T) = \sum_{\mu \in R_{\gamma}} \left( Z_{f_{\sigma}(p,q,\mu), v \neq 0}^{\delta/p} T \right)_{((0,0),0)}.$$

In particular we have,

$$- \lim_{T \to \infty} \left( Z_{f_{\sigma}(p,q,\mu), v \neq 0}^{\delta/p} T \right)_{((0,0),0)} = \left( S_{f_{\sigma}(p,q,\mu), v \neq 0}^{\delta/p} T \right)_{((0,0),0)} \in \mathcal{M}_{G_{m}}^{G_{m}}.$$

Furthermore, by Remark 2.33, if “$\epsilon = -$” then $Z_{\gamma,\epsilon,\omega}^{\delta,<} = 0$ for all face $\gamma$ not in $\mathcal{N}(f)^-$. 

**Proof** Let $\gamma$ be a one dimensional face of $\mathcal{N}(f)$. For any element $(\alpha, \beta) \in C_{\gamma}$ there is $k > 0$ such that $\alpha = pk$ and $\beta = qk$. Note that $m(\alpha, \beta) = m(pk, qk) = kN$. By Notations 2.26, we remark that the cone

$$C_{e,\gamma}^{\delta,\epsilon} \cap \mathbb{N}^3 = \left\{ (n, \alpha, \beta) \in \mathbb{R}_{>0} \times C_{\gamma} \left| m(\alpha, \beta) < \epsilon n, 1 \leq \alpha \leq n\delta \right. \right\} \cap \mathbb{N}^3$$
Using Proposition 2.35 we have
\[
\lim_{T \to \infty} \left( Z^{\delta/p}_{f^\varepsilon \gamma \mu, v \neq 0}(T) \right)_{(0,0)} = \left( S^{\delta/p}_{f^\varepsilon \gamma \mu, v \neq 0}(T) \right)_{(0,0),0} \in \mathcal{M}_{\mathbb{G}_m}.
\]

2.6.5.5 Base case \( f(x, y) = U(x, y)x^{-M}y^m \)

**Example 2.38** Let \( f(x, y) = U(x, y)x^{-M}y^m \) with \( M \) in \( \mathbb{Z} \), \( m \) in \( \mathbb{N} \) and \( U \) in \( \mathbb{k}[[x, y]] \) with \( U(0, 0) \neq 0 \). The Newton polygon \( \mathcal{N}(f) \) has only one face \((-M, m)\) denoted by \( \gamma_h \).

- If \( m = 0 \), then
  - if \( -\varepsilon M \leq 0 \), then for any \( \delta \geq 1 \), \( Z^{\delta}_{f^{\varepsilon} \gamma_h, \omega}(T) = 0 \) and \( (S_{f^{\varepsilon} \gamma_h, v = 0})_{(0,0),0} = 0 \)
  - if \( -\varepsilon M > 0 \), then there is \( \delta_0 > 0 \) such that for any \( \delta \geq \delta_0 \), \( Z^{\delta}_{f^{\varepsilon} \gamma_h, \omega}(T) \) is computed in formula (2.24) and
    \[
    (S_{f^{\varepsilon} \gamma_h, v = 0})_{(0,0),0} = [x^{-\varepsilon M} : \mathbb{G}_m \to \mathbb{G}_m, \sigma \mathbb{G}_m] \quad (2.35)
    \]
    where \( \sigma \mathbb{G}_m \) is the action by translation of \( \mathbb{G}_m \) on \( \mathbb{G}_m \).

- If \( m \neq 0 \) then
  - if \( \varepsilon = + \) and \(-M > 0 \) then \( R^{\delta}_{f^{\varepsilon} \gamma_h, \omega}(T) \) is computed in formula (2.26) and
    \[
    (S_{f^{\varepsilon} \gamma_h, v = 0})_{(0,0),0} = -[x^{-\varepsilon M} : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma \mathbb{G}_m]
    \]
  - if \( \varepsilon = + \) and \(-M \leq 0 \) then \( R^{\delta}_{f^{\varepsilon} \gamma_h, \omega}(T) \) is computed in formula (2.27) and
    \[
    (S_{f^{\varepsilon} \gamma_h, v = 0})_{(0,0),0} = 0
    \]
• if “ε = −” and −M ≥ 0 then $R^\delta_{\gamma, \varepsilon, \omega}(T) = 0$ and $(S_{1/f, x \neq 0})_{(0, 0), 0} = 0$
• if “ε = −” and −M < 0 then $R^\delta_{\gamma, \varepsilon, \omega}(T)$ is computed in formula (2.28) and $(S_{1/f, x \neq 0})_{(0, 0), 0} = 0$.

**Proof** As $U$ is a unit, as all the arcs $(x(t), y(t))$ used in the computation of the motivic Milnor fiber at the origin satisfy $(x(0), y(0)) = (0, 0)$, we can assume $U(x, y) = 1$. As $f$ is a monomial its Newton polygon $\mathcal{N}(f)$ has only one face, the horizontal face $\gamma_h = (−M, m)$ and the proof follows immediately from Proposition 2.31. □

2.6.5.6 Base case $f(x, y) = U(x, y)x^{-M}(y - \mu x^q + g(x, y))^m$

**Example 2.39** Let $f(x, y) = U(x, y)x^{-M}(y - \mu x^q + g(x, y))^m$ with $\mu \in \mathbb{G}_m$, $M \in \mathbb{Z}$, $q \in \mathbb{N}$, $m \in \mathbb{N}_*$, $U \in \mathbb{k}[x, y]$ with $U(0, 0) \neq 0$ and $g(x, y) = \sum_{a+bq > q} c_{a,b} x^a y^b$ in $\mathbb{k}[x, y]$. We denote by $γ$ the one dimensional compact face. Let $v \in \mathbb{N}_{\geq 1}$ and $ω$ the associated differential form in Notations 2.15. Then, there is $δ_0 > 0$, such that for any $δ \leq δ_0$, we have

\[
(Z^\delta_{f, \varepsilon, \omega, x \neq 0}(T))_{(0, 0), 0} = \left[ x^\varepsilon(-M + mq) : \mathbb{G}_m \rightarrow \mathbb{G}_m, \sigma_{\mathbb{G}_m} \right] R^\delta_{(-M + mq, 0), \varepsilon, \omega}(T) + \left[ x^{-\varepsilon M}(y - \mu x^q)^{\varepsilon m} : \mathbb{G}_m \rightarrow \mathbb{G}_m, \sigma_{\mathbb{G}_m} \right] R^\delta_{\gamma, \varepsilon, \omega}(T) + \left[ x^{-\varepsilon M} y^{\varepsilon m} : (y = \mu x^q) \cap \mathbb{G}_m \times \mathbb{G}_m \rightarrow \mathbb{G}_m, \sigma_{\mathbb{G}_m} \right] S^\delta_{\omega}(T)
\]

where $\sigma_{\mathbb{G}_m}$ is the action $\sigma_{\mathbb{G}_m}((x, y, \xi)) = (\lambda x, \lambda y, \lambda \xi)$ and $R^\delta_{(-M + mq, 0), \varepsilon, \omega}$, $R^\delta_{\gamma, \varepsilon, \omega}$, and $S^\delta_{\omega}(T)$ are rational functions defined in Propositions 2.31 and 2.32 and depending on the context, formulas (2.38), (2.39), (2.40) and (2.41).

Furthermore,

- If $−M > 0$ then $(S_{f, x \neq 0})_{(0, 0), 0} = 0$(2.36)
- If $−M \leq 0$ then $(S_{f, x \neq 0})_{(0, 0), 0} = 0$ (2.37)

**Proof** The proof is similar to [7, Example 3]. We start the proof by some preliminary remarks.

- The dual cone to the face $x^{-M} y^m$ is $\mathbb{R}_{>0}(1, 0) + \mathbb{R}_{>0}(1, q)$. Thus, the face $x^{-M} y^m$ belongs to $\mathcal{N}(f)^+$ if and only if $−M > 0$. Furthermore, if $M > 0$ then the face $x^{-M} y^m$ belongs to $\mathcal{N}(f)^-$ if and only if $−M + mq < 0$.
- As $U$ is a unit, as all the arcs $(x(t), y(t))$ used in the computation of the motivic Milnor fiber at the origin satisfy $(x(0), y(0)) = (0, 0)$, we can assume in the following $U(x, y) = 1$. We denote by $h(x, y)$ the polynomial $y - \mu x^q + g(x, y)$. We denote by $γ$ the compact one-dimensional face of the Newton polygon of $f$ with face polynomial $x^{-M}(y - \mu x^q)^m$.
- The Newton polygon of $f$ has three face polynomials $x^{-M + mq}$, $x^{-M} y^m$ and $f_T(x, y) = x^{-M}(y - \mu x^q)^m$. Applying the decomposition formula (2.20), we get for any $δ \geq 1$:

\[
Z^\delta_{f, \varepsilon, \omega, x \neq 0}(T) = Z^\delta_{\varepsilon, x^{q-M}, \omega}(T) + Z^\delta_{\varepsilon, x^{-M} y^m, \omega}(T) + Z^\delta_{\varepsilon, y, \omega}(T) + Z^\delta_{\varepsilon, y^2, \omega}(T).
\]
The rationality and the limit of $Z_{\varepsilon,x^q m - M, \omega}^\delta(T)$, $Z_{\varepsilon,x^q m - M y^m, \omega}^\delta(T)$ and $Z_{\varepsilon,y, \omega}^\delta(T)$ are given in Propositions 2.31 and 2.32.

As $C_y = \mathbb{R}_{>0}(1, q)$, the set $C_{\varepsilon,y}^{\delta, \omega} \cap \mathbb{N}^3$ (Notations 2.26) is in bijection with $C_\delta = \left\{ (n, k) \in (\mathbb{N}^*)^2 \mid -Mk + qkm < \varepsilon n, 0 < k \leq n\delta \right\}$. Then, by its definition in formula (2.19), we have

$$Z_{\varepsilon,y, \omega}^{\delta, \omega}(T) = \sum_{(n,k) \in C_\delta} \mathbb{L}^{-(v-1)k} \text{mes } (X_{n,(k,q)}) T^n$$

with for any $(n,k)$ in $\mathbb{N}^2$

$$X_{n,(k,q)} = \left\{ \varphi = (x(t), y(t)) \in \mathcal{L}(\mathbb{A}_k^2) \mid \begin{array}{l}
\text{ord } x(t) = k, \text{ord } y(t) = qk, \ \text{ord } h(\varphi(t)) = n + Mkm
\end{array} \right\}.$$

We introduce, for any $(k,l)$ in $(\mathbb{N}^*)^2$

$$X_{l,k}^{\varepsilon}(h) = \left\{ \varphi = (x(t), y(t)) \in \mathcal{L}(\mathbb{A}_k^2) \mid \begin{array}{l}
\text{ord } x(t) = k, \text{ord } y(t) = qk, \ \text{ord } h(\varphi(t)) = l
\end{array} \right\}$$

endowed with the map to $\mathbb{G}_m : (x(t), y(t)) \mapsto \left( (\text{ord } x)^{-M} (\text{ord } h(\varphi))^m \right)^\varepsilon$. Remark that, if $X_{l,k}^{\varepsilon}$ is not empty, then $l > qk$.

We introduce $\tilde{C}_\delta = \{(k,l) \in (\mathbb{R}_{>0})^2 \mid l > kq, \varepsilon(-Mk + ml) > 0, k \leq \varepsilon(-Mk + ml)\delta\}$. Writing $n = \varepsilon(-Mk + ml)$, we have

$$Z_{\varepsilon,y, \omega}^{\delta, \omega}(T) = \sum_{(k,l) \in \tilde{C}_\delta \cap (\mathbb{N}^*)^2} \mathbb{L}^{-(v-1)k} \text{mes } (X_{l,k}^{\varepsilon}(h)) T^{\varepsilon(-Mk+ml)}$$

As $h$ is a polynomial Newton non-degenerate, we have (see for instance [17,19, Lemme 2.1.1] or [30, Lemme 3.17])

$$\text{mes } (X_{l,k}^{\varepsilon}(h)) = [(x^{-M} \xi^m)^\varepsilon : (y = \mu x^q) \cap \mathbb{G}_m^2 \times \mathbb{G}_m \rightarrow \mathbb{G}_m, \sigma_{k,l}] \mathbb{L}^{-k-l}$$

with $\sigma_{k,l}(\lambda, (x, y, \xi)) = (\lambda^k x, \lambda^{kq} y, \lambda^l \xi)$. Using the construction of the Grothendieck ring $\mathcal{M}_{\mathbb{G}_m}^{\mathbb{G}_m}$, we obtain the equality

$$[(x^{-M} \xi^m)^\varepsilon : (y = \mu x^q) \cap \mathbb{G}_m^2 \times \mathbb{G}_m \rightarrow \mathbb{G}_m, \sigma_{k,l}] = [(x^{-M} \xi^m)^\varepsilon : (y = \mu x^q) \cap \mathbb{G}_m^2 \times \mathbb{G}_m \rightarrow \mathbb{G}_m, \sigma_{1,1}]$$

(see [7, Example 3] for details). Then we have,

$$Z_{\varepsilon,y, \omega}^{\delta, \omega}(T) = [(x^{-M} \xi^m)^\varepsilon : (y = \mu x^q) \cap \mathbb{G}_m^2 \times \mathbb{G}_m \rightarrow \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}^{\delta, \omega}] S_{\omega}^{\delta}(T)$$

with

$$S_{\omega}^{\delta}(T) = \sum_{n \geq 1} \sum_{(k,l) \in \tilde{C}_\delta} \mathbb{L}^{-vk-l} T^n.$$

The rationality result is a consequence of Lemma 2.1.

- If $M \leq 0$ and "$\varepsilon = +$" then the assumption $-Mk + ml > 0$ is always satisfied, the condition $k \leq (-Mk + ml)\delta$ is also satisfied for any $\delta \geq 1$. Then we have
\[ \tilde{C}^\delta = \{(k, l) \in (\mathbb{R}_+)^2 \mid kq < l \} = \mathbb{R}_+0(0, 1) + \mathbb{R}_+0(1, q) \] and by Lemma 2.1 denoting \( \mathcal{P} = ([0, 1][0, 1]+[0, 1][1, q]) \cap \mathbb{N}^2 \) we have

\[ S_{\omega}^\delta(T) = \sum_{(k_0, l_0) \in \mathcal{P}} \frac{\mathbb{I}_{-v^{k_0-l_0}} T^\varepsilon(-Mk_0+ml_0)}{(1 - L^{-1}T^{eM})(1 - L^{-v-q} T^\varepsilon(-M+mq))} \xrightarrow{T \to \infty} 1. \]  

(2.38)

If \( M > 0 \) and \( \varepsilon = + \) remark that for any \( \delta > 0 \), \( \frac{m\delta}{1+M\delta} < \frac{m}{M} \) and

\[ \tilde{C}^\delta = \{ (k, l) \in (\mathbb{R}_+)^2 \mid kq < l, \ k \leq \frac{im\delta}{1+M\delta} \}. \]

Furthermore, \( \frac{m\delta}{1+M\delta} \to \frac{m}{M} \) when \( \delta \to +\infty \). Thus,

- if \( mq - M \leq 0 \), namely \( \gamma \notin \mathcal{N}(f)^+ \) then we have the inequalities \( m\delta/(1 + M\delta) < m/M \leq 1/q \) implying

\[ \tilde{C}^\delta = \{ (k, l) \in (\mathbb{R}_+)^2 \mid k \leq \frac{m\delta}{1+M\delta} l \} = \mathbb{R}_+0(0, 1) + \mathbb{R}_+0\omega^\delta \]

with \( \omega^\delta = (1, (1+M\delta)/(m\delta)) \). By Lemma 2.1 denoting \( \mathcal{P} = ([0, 1][0, 1]+[0, 1][1, q]) \cap \mathbb{N}^2 \) we have

\[ S_{\omega}^\delta(T) = \sum_{(k_0, l_0) \in \mathcal{P}} \frac{\mathbb{I}_{-v^{k_0-l_0}} T^\varepsilon(-Mk_0+ml_0)}{(1 - L^{-1}T^{eM})(1 - L^{-v-q} T^\varepsilon(-M+mq))} \xrightarrow{T \to \infty} 0. \]  

(2.39)

- if \( mq - M > 0 \), namely \( \gamma \notin \mathcal{N}(f)^- \), then for \( \delta \) large enough we have, the inequalities \( 1/q < m\delta/(1 + M\delta) < m/M \) inducing

\[ \tilde{C}^\delta = \{ (k, l) \in (\mathbb{R}_+)^2 \mid k < l/q \} = \mathbb{R}_+0(0, 1) + \mathbb{R}_+0(1, q). \]

By Lemma 2.1 denoting \( \mathcal{P} = ([0, 1][0, 1]+[0, 1][1, q]) \cap \mathbb{N}^2 \) we have

\[ S_{\omega}^\delta(T) = \sum_{(k_0, l_0) \in \mathcal{P}} \frac{\mathbb{I}_{-v^{k_0-l_0}} T^\varepsilon(-Mk_0+ml_0)}{(1 - L^{-1}T^{eM})(1 - L^{-v-q} T^\varepsilon(-M+mq))} \xrightarrow{T \to \infty} 1. \]  

(2.40)

- if \( M \leq 0 \) and \( \varepsilon = - \), namely \( \gamma \notin \mathcal{N}(f)^- \), then the cone \( \tilde{C}^\delta \) is empty and \( S_{\omega}^\delta(T) = 0 \).

- if \( M > 0 \) and \( \varepsilon = - \), namely \( \gamma \in \mathcal{N}(f)^- \), then we have

\[ \tilde{C}^\delta = \{(k, l) \in \mathbb{R}_+^2 \mid l/q > k > ml/M, \ k \leq (Mk - ml)\delta \}. \]

- If \( M - mq \leq 0 \), then the cone \( \tilde{C}^\delta \) is empty and \( S_{\omega}^\delta(T) = 0 \).

- If \( M - mq > 0 \), then there is \( \delta_0 > 0 \) such that for any \( \delta > \delta_0 \), \( \frac{m}{M} < \frac{m\delta}{M\delta-1} < \frac{1}{q} \) and we conclude that

\[ \tilde{C}^\delta = \{ (k, l) \in \mathbb{R}_+^2 \mid \frac{m\delta}{M\delta-1} \leq k < \frac{l}{q} \} = \mathbb{R}_+0(1, q) + \mathbb{R}_+0\omega^\delta \]

with here \( \omega^\delta = (1, (M\delta - 1)/(m\delta)) \) and by Lemma 2.1 denoting \( \mathcal{P} = ([0, 1][1, q]+[0, 1][1, q]) \cap \mathbb{N}^2 \) we have \( \lim_{T \to \infty} S_{\omega}^\delta(T) = 0 \) with

\[ S_{\omega}^\delta(T) = \sum_{(k_0, l_0) \in \mathcal{P}} \frac{\mathbb{I}_{-v^{k_0-l_0}} T^\varepsilon(-Mk_0+ml_0)}{(1 - L^{-v-q} T^\varepsilon(-M+mq))} \xrightarrow{T \to \infty} 0. \]  

(2.41)
Finally applying Propositions 2.31 and 2.32 we obtain

- if $M < 0$ then
  
  
  $$(S_{f,x \neq 0})_{((0,0),0)} = \left[x^{-M + mq} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}\right] + \left[x^{-M} (y - \mu x^q)^m : \mathbb{G}_m^2 \setminus (y = \mu x^q) \to \mathbb{G}_m, \sigma\right] - \left[x^{-M} \xi^m : (y = \mu x^q) \cap \mathbb{G}_m^2 \times \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}\right]$$

- if $M = 0$ then
  
  $$(S_{f,x \neq 0})_{((0,0),0)} = \left[x^{mq} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}\right] + \left[(y - \mu x^q)^m : \mathbb{G}_m^2 \setminus (y = \mu x^q) \to \mathbb{G}_m, \sigma\right] - \left[\xi^m : (y = \mu x^q) \cap \mathbb{G}_m^2 \times \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}\right]$$

- if $M > 0$ then
  
  $$(S_{f,x \neq 0})_{((0,0),0)} = s^{(+) left}\left[x^{-M + mq} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}\right] + s^{(+)}\left[x^{-M} (y - \mu x^q)^m : \mathbb{G}_m^2 \setminus (y = \mu x^q) \to \mathbb{G}_m, \sigma\right] - s^{(+)}\left[x^{-M} \xi^m : (y = \mu x^q) \cap \mathbb{G}_m^2 \times \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}\right]$$

the motivic Milnor fiber $(S_{1/f,x \neq 0})_{((0,0),0)}$ is 0 if $M \leq 0$ otherwise, if $M > 0$

$$(S_{1/f,x \neq 0})_{((0,0),0)} = s^{(-)}\left[x^{M- mq} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}\right] + s^{(-)}\left[x^M (y - \mu x^q)^m - m : \mathbb{G}_m^2 \setminus (y = \mu x^q) \to \mathbb{G}_m, \sigma\right] - s^{(-)}\left[x^M \xi^m : (y = \mu x^q) \cap \mathbb{G}_m^2 \times \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}\right]$$

with for any $\varepsilon \in \{-, +\}$, $s^{(\varepsilon)} = 1$ if $-M + mq > 0$ and otherwise $s^{(\varepsilon)} = 0$. Then for any $M$ in $\mathbb{Z}$ and $m \geq 1$, equalities (2.36) and (2.37) are induced by the following equalities

$$[x^{-M + mq} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}] + [x^{-M} (y - \mu x^q)^m : \mathbb{G}_m^2 \setminus (y = \mu x^q) \to \mathbb{G}_m, \sigma] = [x^{-M} (y - \mu x^q)^m : \mathbb{G}_m \times \mathbb{A}_k^1 \setminus (y = \mu x^q) \to \mathbb{G}_m, \sigma]$$

$$[x^{-M} (y - \mu x^q)^m : \mathbb{G}_m \times \mathbb{A}_k^1 \setminus (y = \mu x^q) \to \mathbb{G}_m, \sigma] = [x^{-M} \xi^m : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{1,1}]$$

$$[x^{-M} \xi^m : (y = \mu x^q) \cap \mathbb{G}_m^2 \times \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}] = [x^{-M} \xi^m : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{1,1}]$$

These equalities follow from the construction of the Grothendieck ring $\mathcal{M}_{\mathbb{G}_m}$ and the isomorphisms in the category $Var_{\mathbb{G}_m}$

$$\mathbb{G}_m \times \mathbb{A}_k^1 \setminus (y = \mu x^q) \to \mathbb{G}_m^2$$

$(x, y) \mapsto (x, z = y - \mu x^q)$ and $(y = \mu x^q) \times \mathbb{G}_m^2 \times \mathbb{G}_m \to \mathbb{G}_m^2$

$(x, y, \xi) \mapsto (x, \xi)$. 

\[\square\]
3 Motivic invariants at infinity and Newton transformations

Definition 3.1 A compactification of a polynomial \( f \) in \( \mathbb{k}[x, y] \) is a data \( (X, i, \hat{f}) \) with \( X \) an algebraic \( \mathbb{k} \)-variety, \( \hat{f} \) a proper map and \( i \) an open dominant immersion, such that the following diagram is commutative

\[
\begin{array}{ccc}
\mathbb{A}^2_\mathbb{k} & \xrightarrow{i} & X \\
\downarrow f & & \downarrow \hat{f} \\
\mathbb{A}^1_\mathbb{k} & \xrightarrow{j} & \mathbb{P}^1_\mathbb{k}
\end{array}
\]

where \( j \) is the open dominant immersion from \( \mathbb{A}^1_\mathbb{k} \) to \( \mathbb{P}^1_\mathbb{k} \) which maps a point \( a \) to \( [1 : a] \). With these notations, we denote by \( X_\infty \) the closed subset \( X \setminus i(\mathbb{A}^2_\mathbb{k}) \) and by \( \infty \) the point \([0 : 1]\). We identify \( a \) with the point \([1 : a]\) and we denote \( 1/\hat{f} \) the extension of \( 1/f \) on \( X \setminus \hat{f}^{-1}(0) \) and for any value \( a \), we consider \( \hat{f} - a \) the extension of \( f - a \) on \( X \setminus \hat{f}^{-1}(\infty) \). In the following, we will compute rational forms of motivic zeta function using a specific compactification defined in Sect. 3.3.1.

3.1 Motivic Milnor fiber at infinity

In this subsection, we recall the notion of Milnor fiber at infinity and motivic Milnor fibers at infinity of a polynomial \( f \) in \( \mathbb{C}[x, y] \), which is a consequence of the studies of Bittner [3], Guibert–Loeser–Merle [20] on motivic Milnor fibers and developed for instance in [16,24,30].

3.1.1 Milnor fibration at infinity

The following result is a consequence of ideas of Thom, see for instance [27].

Theorem 3.2 Let \( f \) be a polynomial in \( \mathbb{C}[x, y] \). There is \( R > 0 \) such that the restriction

\[
f : \mathbb{C}^2 \setminus f^{-1}(D(0, R)) \to \mathbb{C} \setminus D(0, R)
\]

is a \( C^\infty \) – locally trivial fibration called the Milnor fibration at infinity of \( f \). The Milnor fiber at infinity is up to an homeomorphism the fiber \( f^{-1}(a) \) for \( a > R \). The monodromy at infinity is induced by the action of \( \pi_1(\mathbb{C} \setminus D(0, R)) \) on \( f^{-1}(a) \).

3.1.2 Motivic Milnor fiber at infinity

Definition 3.3 Let \( (X, i, \hat{f}) \) be a compactification of a polynomial \( f \) in \( \mathbb{k}[x, y] \). For any \( \delta > 0 \) and \( n \) in \( \mathbb{N}^* \), we consider

\[
X_n^\delta(1/\hat{f}) = \{ \varphi(t) \in \mathcal{L}(X) \mid \text{ord } \varphi^*: (\mathcal{I}_{X_\infty}) \leq n\delta, \text{ ord } 1/\hat{f}(\varphi(t)) = n \}
\]

with its structural map to \( \hat{f}^{-1}(\infty) \times \mathbb{G}_m, \varphi \mapsto (\varphi(0), \varphi(1/\hat{f}(\varphi(t)))) \).

Remark 3.4 The motivic measure of \( X_n^\delta(1/\hat{f}) \) belongs to \( \mathcal{M}_{\hat{f}^{-1}(\infty) \times \mathbb{G}_m} \). Indeed, even if \( X \) is singular, the singular locus is contained in \( X_\infty \), and it follows from [13, Lemma 4.1] that the condition \( \text{ord } \varphi^*: (\mathcal{I}_{X_\infty}) \leq n\delta \) implies that it is not necessary to complete the Grothendieck ring to compute the measure of \( X_n^\delta(1/\hat{f}) \) for any \( n \) and \( \delta \).
Applying [20, §3.9], with notations of Sect. 2.5, Remark 2.7 and Theorem 2.8, we get [28, Theorem 3.4]:

**Theorem 3.5** (Motivic Milnor fiber at infinity) Let \((X, i, \hat{f})\) be a compactification of a polynomial \(f\) in \(k[x, y]\) and \(\delta > 0\). The modified zeta function

\[ Z^\delta_{1/\hat{f}, i}(k\hat{A}^2) (T) = \sum_{n \geq 1} \mes \left( X_n^\delta(1/\hat{f}) \right) T^n \]

is rational for \(\delta\) large enough and has a limit when \(T\) goes to infinity independent from the parameter \(\delta\). We denote

\[ S_{1/\hat{f}}(i : k\hat{A}^2 \to X) = -\lim_{T \to \infty} Z^\delta_{1/\hat{f}, i}(k\hat{A}^2) (T) \in \mathcal{M}_{\mathcal{G}_m}^{G_m \times G_m} and \]

\[ S_{f, \infty} = \hat{f}_* S_{1/\hat{f}}(i : k\hat{A}^2 \to X) \in \mathcal{M}_{\mathcal{G}_m}^{G_m \times G_m}. \]

The motive \(S_{f, \infty}\) does not depend on the chosen compactification and is called motivic Milnor fiber at infinity of \(f\).

**Remark 3.6** Some remarks:

- In the following, similarly to Remark 2.7, we will identify \(\mathcal{M}_{\mathcal{G}_m}^{G_m \times G_m} with \mathcal{M}_{\mathcal{G}_m}^{G_m}.\)

- For any constant \(c\), for any arc \(\varphi\) in \(\mathcal{L}(X)\), for any positive integer \(n\), \(\text{ord} (\hat{f} - c)(\varphi) = -n\) if and only if \(\text{ord} \hat{f}(\varphi) = -n\) which implies the equality \(S_{f, \infty} = S_{\hat{f} - c, \infty}\). So to compute \(S_{f, \infty}\), we will always assume that \((0, 0)\) is a point of the support, namely \(f(0, 0) \neq 0\). In that case, by Remark 1.28, \(\mathcal{N}(f)\) is equal to \(\mathcal{N}_{\infty}(f)\).

**Notation 3.7** For a one-dimensional face \(\gamma\) in \(\mathcal{N}_{\infty}(f)\) (or \(\mathcal{N}(f)\)) with primitive exterior normal vector \((p, q)\), we define

\[ c(p, q) = \begin{cases} 
  p + q, & \text{if } p > 0 \text{ and } q > 0, \\
  p, & \text{if } p > 0 \text{ and } q < 0, \\
  q, & \text{if } p < 0 \text{ and } q > 0, \\
  1, & \text{if } (p, q) = (1, 0) \text{ or } (p, q) = (0, 1)
\end{cases} \quad \text{and} \quad \omega_{p, q}(v, w) = v^{(p + q - 1)} dv \wedge dw. \]

\[ (3.1) \]

**Theorem 3.8** (Computation of \(S_{f, \infty}\)) Let \(f\) in \(k[x, y]\) and not in \(k[x] or k[y]\). Using the compactification \((X, i, \hat{f})\) of Sect. 3.3.1,

- if \(f(x, y) = P(x^a y^b) with P in k[s]\) of degree \(d\) with \((a, b) \in \mathbb{N}^2\), then if \(\delta > \max(\frac{1}{da}, \frac{1}{db})\) we have

\[ Z^\delta_{1/\hat{f}, i}(k\hat{A}^2) (T) = [1/(x^a y^b)^d : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}] R^\delta_{\mathbb{G}_m} (T) \text{ and} \]

\[ S_{f, \infty} = [1/(x^a y^b)^d : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}] \]

\[ (3.2) \]

- otherwise, there is \(\delta' > 0\) such that for any \(\delta \geq \delta'\),

\[ Z^\delta_{1/\hat{f}, i}(k\hat{A}^2) (T) = e_{(a_0, 0)}[1/x^{a_0} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}] R^\delta_{(a_0, 0)} (T) \]

\[ + e_{(b_0, 0)}[1/y^{b_0} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}] R^\delta_{(0, b_0)} (T) \]

\[ + \sum_{\gamma \in N_{\infty}(f)^\mu} [1/f_\gamma : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_\gamma] R^\delta_{\mathbb{G}_m} (T) \]

\[ + \sum_{\gamma \in N_{\infty}(f)^\mu} \sum_{\mu \in R^\gamma} Z^\delta_{c(p, q)} \omega_{p, q, u \neq 0} (T) \]
and the motivic Milnor fiber at infinity is
\[
S_{f, \infty} = \varepsilon_{(a_0, 0)}[1/x^{a_0} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}] + \varepsilon_{(0, b_0)}[1/y^{b_0} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}]
\]
\[
+ \sum_{\gamma \in \mathcal{N}_\infty(f)^\circ} \varepsilon_\gamma [1/f_\gamma : \mathbb{G}_m^2 \setminus f_\gamma^{-1}(0) \to \mathbb{G}_m, \sigma_\gamma]
\]
\[
+ \sum_{\gamma \in \mathcal{N}_\infty(f)^\circ, \dim \gamma = 1} \sum_{\mu \in R_\gamma} \left( S_{1/f_\gamma o(p,q,\mu), v^{\neq 0}} \right) \quad (0, 0, 0).
\] (3.4)

In particular, we have
\[
S_{f, \infty} = \sum_{\gamma \in \mathcal{N}_\infty(f)^\circ} \varepsilon_\gamma S_{f, \infty} + \sum_{\gamma \in \mathcal{N}_\infty(f)^\circ, \dim \gamma = 1} \sum_{\mu \in R_\gamma} \left( S_{1/f_\gamma o(p,q,\mu), v^{\neq 0}} \right) \quad (0, 0, 0)
\]
\[
- \left( S_{1/(f_\gamma o(p,q,\mu), v^{\neq 0})} \right) \quad (0, 0, 0). \] (3.5)

All these formulas use the notation \( N_\infty(f)^\circ \) of Definition 1.27, Notations 1.31 and 3.7, and the following:

- \( \varepsilon_{(a_0, 0)} \) and \( \varepsilon_{(0, b_0)} \) are respectively equal to 1 and otherwise 0, if and only if \( (a_0, 0) \) and \( (0, b_0) \) are respectively faces of \( N_\infty(f)^\circ \). \( R^\delta_{(a_0, 0)}(T) \) and \( R^\delta_{(0, b_0)}(T) \) are respectively defined in Eqs. (3.41) and (3.42) with limit equal to −1.

- The expression of \( R_\gamma^\delta(T) \) is given for any zero dimensional face \( \gamma \in N_\infty(f)^\circ \) by Eqs. (3.51), (3.53), (3.54), (3.56), (3.58), (3.60), (3.62), (3.64) and (3.66) for the case \( f = P(x^a, y^b) \), and Eq. (3.65) for one-dimensional faces of \( N_\infty(f)^\circ \). In particular, in the general case (formula (3.3)), we have \( -\lim_{T \to \infty} R_\gamma^\delta(T) = \varepsilon_\gamma \) where \( \varepsilon_\gamma \) is \( (-1)^{\dim \gamma + 1} \) (and otherwise 0) if \( \gamma \) is not contained in a face which contains the origin.

**Proof** We give the ideas of the general proof using above notations and refer to Sect. 3.3 for details. It is similar to the proof of Theorem 2.22. We consider first a polynomial \( f \) in \( k[x, y] \) which is not of the form \( f(x, y) = P(x^a, y^b) \) with \( P \) in \( k[s] \) with \( (a, b) \) in \( \mathbb{N}^2 \). We work with the Newton polygon at infinity \( N_\infty(f) \). Note that by Remark 3.6, we can assume \( f(0, 0) \neq 0 \) and in that case we have \( \overline{N}(f) = N_\infty(f) \). In Sect. 3.3.1 we consider the compactification \( (X, i, \hat{f}) \) of the graph of \( f \) in \( (\mathbb{P}^1_k)^3 \). In Proposition 3.44, we decompose the motivic zeta function of \( 1/\hat{f} \) along \( \overline{N}(f) \)

\[
Z^\delta_{1/\hat{f}, i(A^3_k)}(T) = \sum_{\gamma \in \overline{N}(f)} Z^\delta_{\gamma, -}(T). \] (3.6)

By Remark 3.42, it is enough to consider faces \( \gamma \) in \( N_\infty(f)^\circ \) namely faces of \( N_\infty(f) \) which do not contain the origin. In Proposition 3.46, we show the rationality and compute the limit of \( Z^\delta_{\gamma, -}(T) \) in the case of a zero dimensional face \( \gamma \) contained in a coordinate axis. In Proposition 3.48 and Sect. 3.3.7, we consider the case of a face \( \gamma \) not contained in a coordinate axis. Depending on the fact that the face polynomial \( f_\gamma \) vanishes or not on the angular components of the coordinates of an arc (Remark 3.41), we decompose in formula (3.40) the zeta function \( Z^\delta_{\gamma, -}(T) \) as a sum of \( Z^\delta_{\gamma, =}(T) \) and \( Z^\delta_{\gamma, <}(T) \). In particular if the face \( \gamma \) is zero dimensional then \( Z^\delta_{\gamma, <}(T) \) is zero. In Proposition 3.48 we show the rationality and compute the limit of the zeta function \( Z^\delta_{\gamma, =}(T) \). In Propositions 3.54 and 3.56, we prove the decomposition

\[
Z^\delta_{\gamma, -}(T) = \sum_{\mu \in R_\gamma} \left( Z^\delta_{1/f_\gamma o(p,q,\mu), \omega^{\neq 0}} \right) \quad (0, 0, 0).
\]
We use Sect. 2.22 to obtain the rationality and an expression of the limit of 
$(Z^h_{1/f,j(\mathbb{k})}^\delta,\gamma)(\mathbb{k})$ for $q \neq 0$, $v \neq 0, (0,0,0)$. Similarly in Sects. 3.3.7.3 and 3.3.7.4 we consider the case of the horizontal and vertical faces.

We assume now that $f(x, y) = P(x^a y^b)$ with $f(0, 0) \neq 0$, $(a, b)$ in $(\mathbb{N}^*)^2$ and $P$ in $k[y]$ of degree $d$. We denote by $\gamma$ the face $(a d, b d)$. Remark that by assumption $N_{\infty}(f)$ is the segment $[0, 0), (da, db)]$ not contained in a coordinate axis. Then, by Proposition 3.44, Remarks 3.42 and 3.50, we have the equality

$$Z^h_{1/f,j(\mathbb{k})}^\delta(T) = Z^h_{\gamma,-}(T) = Z^h_{\gamma,-}(T).$$

Then formulas (3.2) follow from formula (3.66) of Proposition 3.48. The proof of equation (3.5) follows from Theorem 3.8 applied to $f$ and each face polynomials $f_\gamma$ for $\gamma$ in $N_{\infty}(f)^o$ applying Remark 3.6.

**Remark 3.9** This theorem extends in the case of curves (without non degeneracy or convenient conditions), the computation in the non degenerate case of $S_{f,\infty}$ done in [25,28].

**3.1.3 Realization results**

In this section we assume $k = \mathbb{C}$.

**3.1.3.1 Generalized Kouchnirenko formula for the generic fiber**. Using Denef-Loeser results (see for instance [19,3.17]) we have ([28, §2.4], [24,25]).

**Theorem 3.10** Let $f$ be a polynomial in $\mathbb{C}[x,y]$. We have the equality $\tilde{\chi}_c(S^{(1)}_{f,\infty}) = \chi_c(F_{\infty})$ where $F_{\infty}$ is the Milnor fiber at infinity of $f$ and $\tilde{\chi}_c : M^a_{h} \rightarrow \mathbb{Z}$ is the Euler characteristic realization.

**Remark 3.11** We recall that the Milnor fiber at infinity $F_{\infty}$ of $f$ is the fiber $f^{-1}(R)$ for $R$ large enough, then it is homeomorphic to the generic fiber $f^{-1}(a_{\text{gen}})$ of $f$, then we have $\chi_c(F_{\infty}) = \chi_c(f^{-1}(a_{\text{gen}}))$.

Using that result, Proposition 2.4 and Theorem 3.8 we have

**Corollary 3.12** (Generalized Kouchnirenko formula for the generic fiber) Let $f$ be a polynomial in $\mathbb{C}[x,y]$, not in $\mathbb{C}[x]$ or $\mathbb{C}[y]$. With notations of Theorem 3.8 and Sect. 2.2, we have

- if $f(x, y) = P(x^a y^b)$ with $P$ in $k[y]$ with $(a, b)$ in $(\mathbb{N}^*)^2$ then we have $\chi_c(F_{\infty}) = \chi_c(f^{-1}(a_{\text{gen}})) = 0$,
- otherwise in the general case we have

$$\chi_c(F_{\infty}) = \chi_c(f^{-1}(a_{\text{gen}})) = \varepsilon(a_0, 0)a_0 + \varepsilon(0,b_0)b_0 - 2 \sum_\gamma N_{\infty}(f)^{\gamma} \dim \gamma = 1 \sum_{\mu \in R_{\gamma}} \tilde{\chi}_c \left(S^{(1)}(1/f)_{\infty,\mu}, \varepsilon_{\gamma} \right)$$

$$\left((S^{(1)}(1/f)_{\infty,\mu}, \varepsilon_{\gamma})=(0,0)\right)$$

(3.7)

with $\varepsilon(a_0, 0)$ (resp. $\varepsilon(0,b_0)$) is equal to 1 and otherwise 0, if and only if $(a_0, 0)$ (resp. $(0, b_0)$) is a face of $N_{\infty}(f)$.

**Remark 3.13** Using as usual other realizations, we can obtain formula of the monodromy zeta function at infinity of $f$ or the spectrum at infinity of $f$ in terms of the iterated Newton polygons of $f$ in the Newton algorithm at infinity.
Example 3.14 We extend Example 1.40. We observe that \( \overline{N}(f) = N_\infty(f) \) and by Theorem 3.8 we have

\[
S_{f, \infty} = \left\{ \frac{1}{y} : \mathbb{C}_m \to \mathbb{C}_m, \sigma_{G_m} \right\} + \left\{ \frac{1}{x^2} : \mathbb{C}_m \to \mathbb{C}_m, \sigma_{G_m} \right\} + \left( (y^2 + 1)^3 : \mathbb{C}_m \setminus (y^2 + 1 = 0) \to \mathbb{C}_m, \sigma_{G_m} \right) + \left( (x^2 + 1)^4 : \mathbb{C}_m \setminus (x^2 + 1 = 0) \to \mathbb{C}_m, \sigma_{G_m} \right) + \left( \frac{1}{x^6 y^4} : \mathbb{C}_m \to \mathbb{C}_m, \sigma_{G_m} \right) + \left( S_{1/f_1, v \neq 0} \right)_{(0,0),0} + \left( S_{1/f_2, v \neq 0} \right)_{(0,0),0} \tag{3.8}
\]

with \( f_1 \) and \( f_2 \) defined in formulas (1.9) and (1.11).

– Applying Theorem 2.22, we have

\[
\left( S_{1/f_1, v \neq 0} \right)_{(0,0),0} = \left\{ v : \mathbb{C}_m \to \mathbb{C}_m, \sigma_{G_m} \right\} + \left( (8v^{-2}w^3 + 5v^{-1})^{-1} : \mathbb{C}_m \setminus (8v^{-2}w^3 + 5v^{-1} = 0) \to \mathbb{C}_m, \sigma_{G_m} \right) + \left( S_{1/f_1, v \neq 0} \right)_{(0,0),0}.
\]

As \( (f_1)_{\sigma_{G_m}} \) is a base case of Theorem 1.17, with \( M = 3 \) and \( m = 1 \), by Example 2.39 we have \( (S_{1/(f_1 f_1), v \neq 0})_{(0,0),0} = 0 \).

– As the set \( \mathcal{N}(f_2) \) is empty, applying Theorem 2.22, we have \( (S_{1/f_2, v \neq 0})_{(0,0),0} = 0 \).

Assume now \( k = \mathbb{C} \). By Corollary 3.12, we compute the Euler characteristic of the generic fiber of \( f \). We have

\[
\chi_c((y^2 + 1)^3 = 1) \cap \mathbb{C}_m^2 = -2, \quad \chi_c((x^2 + 1)^4 = 1) \cap \mathbb{C}_m^2 = -2, \quad \chi_c((8v^{-2}w^3 + 5v^{-1} = 1) \cap \mathbb{C}_m^2 = -3.
\]

by Corollary 2.23 and Proposition 2.4. We conclude by formulas (3.7) and (2.12) that

\[
\chi_c(F_\infty) = 1 + 2 - 2 - 2 - 0 + (1 - 3 - 0) + 0 = -3.
\]

3.2 Topological bifurcation set, motivic bifurcation set, motivic nearby cycles at infinity

3.2.1 Topological bifurcation set

The following result is a consequence of ideas of Thom, see for instance [27].

Theorem 3.15 Let \( f \) be a polynomial in \( \mathbb{C}[x, y] \). There is a finite set \( B \) such that the restriction

\[
f : \mathbb{C}^2 \setminus f^{-1}(B) \to \mathbb{C} \setminus B
\]

is a \( C^\infty \)-locally trivial fibration. The smallest convenient set \( B \), denoted by \( B_f^{\top} \), is called topological bifurcation set of \( f \).

Hà and Lê gave the following description of the topological bifurcation set

Theorem 3.16 (Hà-Lê [35]) Let \( f \) be a polynomial in \( \mathbb{C}[x, y] \). The topological bifurcation set is

\[
B_f^{\top} = \{ a \in \mathbb{C} \mid \chi_c(f^{-1}(a)) \neq \chi_c(f^{-1}(a_{gen})) \}.
\]
3.2.2 \( \lambda \)-Invariant

Let \( f \) be a non constant polynomial in \( \mathbb{C}[x, y] \). We denote by \( d \) its degree and by \( f_0, \ldots, f_d \), its homogeneous components. In this paragraph, we recall for any value \( a \) the definition of the invariant \( \lambda_a(f) \), which roughly speaking measures the non equisingularity at infinity of the fibers of \( f \) in \( \mathbb{P}^2_{\mathbb{C}} \), see for instance formula (3.10). We consider first, the algebraic variety \( \mathcal{V} = \{(x : y : z), \ a) \in \mathbb{P}^2_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}} \mid G(x, y, z, a) = 0 \} \) with \( G(x, y, z, a) = \tilde{f}(x, y, z) - az^d \) where \( \tilde{f} \) is the homogeneous polynomial associated to \( f \). We denote by \( i_V : \mathbb{A}^1_{\mathbb{C}} \to \mathcal{V} \) the open dominant immersion which maps \( \{x : y : z\} \) to \( (x : y : 1), \ f(x, y) \). We denote by \( \tilde{f}_V : \mathcal{V} \to \mathbb{A}^1_{\mathbb{C}} \) the application which maps \( \{x : y : z\} \) to \( a \). The triple \( (\mathcal{V}, i_V, \tilde{f}_V) \) is a compactification of \( f \). The singular locus of \( \mathcal{V} \) is the closed subset, denoted by \( \mathcal{V}_{sing} \), and given by the equations

\[
\frac{\partial f_d}{\partial x} = \frac{\partial f_d}{\partial y} = f_{d-1} = z = 0. \tag{3.9}
\]

We observe that \( \mathcal{V}_{sing} \) is equal to the product \( \mathcal{P} \times \mathbb{A}^1_{\mathbb{C}} \) where \( \mathcal{P} \) is the closed subset of \( \mathbb{P}^2_{\mathbb{C}} \) defined by (3.9). As \( \frac{\partial f_d}{\partial x}, \frac{\partial f_d}{\partial y} \) and \( f_{d-1} \) are homogeneous polynomials in two variables, their zero locus in \( \mathbb{P}^1_{\mathbb{C}} \) is a finite set, thus \( \mathcal{P} \) is a finite set.

We assume \( f \) has isolated singularities. This is equivalent here to have reduced fibers. Let \( p_0 = [x_0 : y_0 : 0] \) be an element of \( \mathcal{P} \). We can assume for instance \( x_0 \neq 0 \). Then we work in the chart \( x \neq 0 \) with the coordinates \( u = y/x \) and \( v = z/x \). We define \( u_0 = y_0/x_0, v_0 = 0 \) and for any \( a \) in \( \mathbb{C} \), we consider the polynomial \( H_a(u, v) = G(1, u, v, a) \). The point \( (u_0, v_0) \) is an isolated critical point of \( H_a \), and we denote by \( \mu_{p_0}(a) \) the Milnor number \( \mu_{(u_0, v_0)}(H_a) \), it does not depend on the choice of the chart. It follows from Thom–Mather theorem that the set of values \( \mu_{p_0}(a) \) parametrised by \( a \) is finite. We finally define for any value \( a \), the classical invariants

\[
\lambda_{p_0, a}(f) = \mu_{p_0}(a) - \mu_{p_0}(a_{gen}) \quad \text{and} \quad \lambda_a(f) = \sum_{p_0 \in \mathcal{P}} \lambda_{p_0, a}(f). \tag{3.10}
\]

By upper semi-continuity of the function \( a \mapsto \mu_{p_0}(a) \) (see for instance [4, Prop 2.3]), we observe that \( \lambda_a(f) \geq 0 \) and equal to zero for almost every value \( a \). This invariant was studied by Suzuki in [31], Hà and Lê in [35] or the first author in [6]. We refer to [1] or [33] for generalization in dimension \( \geq 3 \). It follows from these references and Theorem 3.16 that

**Theorem 3.17** Let \( f \) be a polynomial in \( \mathbb{C}[x, y] \) with isolated singularities. We denote by \( \chi_c(f^{-1}(a_{gen})) \) the Euler characteristic of the generic fiber of \( f \). For any value \( a \) in \( \mathbb{C} \), we have

\[
\chi_c(f^{-1}(a)) = \chi_c(f^{-1}(a_{gen})) + \mu_a(f) + \lambda_a(f) \quad \text{and} \quad \chi_c(f^{-1}(a_{gen})) = 1 - (\mu(f) + \lambda(f)) \tag{3.11}
\]

with \( \mu_a(f) \) equal to the sum of Milnor numbers of critical points of \( f^{-1}(a) \), \( \mu(f) \) is the sum of Milnor numbers and \( \lambda(f) \) is the sum of all \( \lambda_a(f) \). In particular, we have the equality \( B^\text{top}_f = \text{disc}(f) \cup \{a \in \mathbb{C} \mid \lambda_a(f) \neq 0\} \).

3.2.3 Motivic nearby cycles at infinity

**Definition 3.18** (The morphism \( S^\infty_{f-a} \)) Let \( f \) be a polynomial in \( \mathbb{k}[x, y] \) and \( (X, i, \hat{f}) \) be a compactification of \( f \), with \( X_\infty = X \setminus i(\mathbb{A}^2_{\hat{k}}) \). Let \( a \) be a value in \( \mathbb{A}^1_{\hat{k}} \). We denote by \( S^\infty_{f-a} : \)}
\( \mathcal{M}_X \to \mathcal{M}_{f-a}^{G_m} \) the composition of the morphism \( S_{f-a} : \mathcal{M}_X \to \mathcal{M}_{f-1(a)}^{G_m} \) (Theorem 2.8) with the morphism \( i^* : \mathcal{M}_{f-1(a)}^{G_m} \to \mathcal{M}_{f(a)}^{G_m} \) (Sect. 2.1.1) and induced by the canonical injection \( i : (X_{f-1(a)}) \times G_m \to f^{-1}(a) \times G_m \).

We recall some notions of [30, §4] (see also constructions of Matsui, Takeuchi and Tăbără in [24,25,32]).

**Theorem 3.19** (Motivic nearby cycles at infinity) For any value \( a \) in \( \mathbb{A}^1_k \), the motive \( S_{f,a}^{\infty} \) defined as

\[
S_{f,a}^{\infty} = \hat{f}_! S_{f-a}^\infty ([i : \mathbb{A}^2_k \to X]) \in \mathcal{M}_{[a]}^{G_m} \times G_m
\]

does not depend on the chosen compactification \((X, i, \hat{f})\) and is called motivic nearby cycles at infinity of \( f \) for the value \( a \).

**Remark 3.20** Some remarks:

- The motive \( S_{f,a}^{\infty} ([i : \mathbb{A}^2_k \to X]) \) is the limit of the motivic zeta function

\[
Z_{f-a, i(\mathbb{A}^2_k)}^{\delta, \infty} (T) = \sum_{n \geq 1} \text{mes} (X_n^{\delta, \infty} (f - a)) T^n \in \mathcal{M}_{(X_{\infty} \cap f^{-1}(a)) \times G_m} [[T]]. \tag{3.12}
\]

with \( \delta \) an integer large enough (Proposition 2.6) and for any \( n \geq 1 \)

\[
X_n^{\delta, \infty} (f - a) = \{ \varphi(t) \in \mathcal{L}(X) \mid \varphi(0) \in X_{\infty}, \text{ord} \varphi^* \mathcal{I}_{X_{\infty}} \leq n \delta, \text{ord} (f - a)(\varphi(t)) = n \}.
\]

endowed with the structural application to \((f^{-1}(a) \cap X_{\infty}) \times G_m, \varphi \mapsto (\varphi(0), \overline{\varphi}((f - a)(\varphi)))\).

- Similarly to Remark 2.7, we will identify \( \mathcal{M}_{[a]}^{G_m} \times G_m \) with \( \mathcal{M}_{G_m}^{G_m} \).

In [15], Fantini and the second author proved the following result stated in dimension 2 here:

**Theorem 3.21** Let \( f \) be a polynomial in \( \mathbb{C}[x, y] \) with isolated singularities. Then, for any value \( a \), we have the equality

\[
\bar{\lambda}_e (S_{f,a}^{\infty, (1)}) = -\lambda_a (f)
\]

with \( S_{f,a}^{\infty, (1)} \) in \( \hat{M}_k \) the fiber in \( I \) of \( S_{f,a}^{\infty} \) and \( \bar{\lambda}_e : \hat{M}_k \to \mathbb{Z} \) the Euler characteristic realization.

**Remark 3.22** Let \( f \) be a polynomial in \( \mathbb{R}[x, y] \). In the following we compute \( S_{f,a}^{\infty} \) in terms of the combinatorics of iterated Newton polygons. The general case \( a \neq 0 \) can be deduced by translation. We use Notations 1.27 of Newton polygons.

**Theorem 3.23** (Computation of \( S_{f,a}^{\infty} \)) Let \( f \) be a polynomial in \( \mathbb{K}[x, y] \), not in \( \mathbb{K}[x] \) or \( \mathbb{K}[y] \). Using the compactification \((X, i, \hat{f})\) of Sect. 3.3.1, there is \( \delta' > 0 \) such that for any \( \delta \geq \delta' \):

- If \( \overline{\mathcal{N}}(f) \) is not a segment, then we have
\[ Z_{f,i(\mathbb{A}^2_k)}^{\delta,\infty}(T) = \sum_{\gamma \in \mathcal{N}(f)} [f_{\gamma} : \mathbb{G}_m^2 \setminus f_{\gamma}^{-1}(0) \to \mathbb{G}_m, \sigma_{\gamma}] R_{\gamma}^{\delta,=}(T) + \sum_{\gamma \in \mathcal{N}(f), \dim \gamma = 1} \sum_{\mu \in R_{\gamma}} (Z_{f_{\mu(p,q)}, \gamma}^{\delta/c(p,q)})(0,0,0) \] (3.13)

\[ S_{f,0}^{\infty} = \sum_{\gamma \in \mathcal{N}(f), \dim \gamma = 1} \sum_{\mu \in R_{\gamma}} (S_{f_{\mu(p,q), \gamma}, \gamma}^{\delta/c(p,q)})(0,0,0) \] (3.14)

- If \( f \) is the monomial \( x^a y^b \), then we have

\[ Z_{f,i(\mathbb{A}^2_k)}^{\delta,\infty}(T) = [x^a y^b : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}] R_{(a,b)}^{\delta,=}(T) \] and \( S_{f,0}^{\infty} = 0 \). (3.15)

- If \( \mathcal{N}(f) \) is a segment, we denote by \( \gamma \) the one dimensional face of \( \mathcal{N}(f) \). It is a segment with vertices \( (a_0, b_0) \) and \( (a_1, b_1) \) such that \( a_0 \leq a_1 \) and if \( a_0 = a_1 \) then \( b_0 < b_1 \). We define \( \delta_{(a_0, b_0)} \) as 1 if \( (a_0, b_0) \neq (0,0) \) otherwise 0. We choose an equation of the underlying line of \( \gamma \) as \( ap + bq = N \) with \( (p, q) \) in \( \mathbb{Z}^2(\mathbb{Z}_{\leq 0})^2 \) and coprime.

- If \( pq < 0 \), we have

\[ Z_{f,i(\mathbb{A}^2_k)}^{\delta,\infty}(T) = \delta_{(a_0, b_0)} [x^{a_0} y^{b_0} : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}] R_{(a_0, b_0)}^{\delta,=}(T) + [x^{a_1} y^{b_1} : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}] R_{(a_1, b_1)}^{\delta,=}(T) + \sum_{\mu \in R_{\gamma}} (Z_{f_{\mu(p,q), \gamma}, \gamma}^{\delta/c(p,q)})(0,0,0) \] (3.16)

\[ S_{f,0}^{\infty} = \delta_{(a_0, b_0)} \epsilon_{(a_0, b_0)} [x^{a_0} y^{b_0} : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}] + \epsilon_{(a_1, b_1)} [x^{a_1} y^{b_1} : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}] + \delta_N \sum_{\mu \in R_{\gamma}} [x^{[N]} y^{(\mu)} : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}] \] (3.17)

with \( \delta_N = -1 \) if \( N \neq 0 \) otherwise 0, and for any root \( \mu, v(\mu) \) is the multiplicity of \( \mu \).

- If \( pq \geq 0 \) we have

\[ Z_{f,i(\mathbb{A}^2_k)}^{\delta,\infty}(T) = [x^{a_0} y^{b_0} : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}] R_{(a_0, b_0)}^{\delta,=}(T) + [x^{a_1} y^{b_1} : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2}] R_{(a_1, b_1)}^{\delta,=}(T) + \sum_{\mu \in R_{\gamma}} (Z_{f_{\mu(p,q), \gamma}, \gamma}^{\delta/c(p,q)})(0,0,0) \] (3.18)

\[ S_{f,0}^{\infty} = 0. \] (3.19)

All these formulas use Notations 3.7 and for any face \( \gamma \), the expression of \( R_{\gamma}^{\delta,=}(T) \) is given in formula (3.48) of Proposition 3.47, and formulas (3.67), (3.69), (3.71) and (3.72) of Springer.
Proposition 3.49 and \( \varepsilon_{\gamma} \) belongs to \([-2, -1, 0]\) if \( \gamma \) is zero-dimensional and to \([0, 1]\) if \( \gamma \) is one-dimensional.

**Proof** We give the ideas of the general proof using above notations and refer to Sect. 2.6.5 for details. It is similar to the proof of Theorem 2.22 or Theorem 3.8. We consider a polynomial \( f \in k[x, y] \) not in \( k[x] \) or \( k[y] \). In Sect. 3.3.1 we consider the compactification \((X, i, \hat{f})\) of the graph of \( f \) in \((\mathbb{P}^1_k)^3\). In Proposition 3.44, we consider the decomposition

\[
Z^\delta_{f, i(k^2)}(T) = \sum_{\gamma \in \overline{N}(f)} Z^\delta_{\gamma, +}(T)
\]  

(3.20)

Assume \( \overline{N}(f) \) is not a segment. We only consider faces \( \gamma \), such that the dual cone \( C_{\gamma} \) is not contained in \( \mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\leq 0} \), then we only consider faces in \( \overline{N}(f) \). Depending on the fact that the face polynomial \( f_{\gamma} \) vanishes or not on the angular components of the coordinates of an arc (Remark 3.41), we decompose in formula (3.40), the zeta function \( Z^\delta_{\gamma, +}(T) \) as a sum of \( Z^\delta_{\gamma, =}(T) \) and \( Z^\delta_{\gamma, <}(T) \). In particular if the face \( \gamma \) is zero dimensional then \( Z^\delta_{\gamma, <}(T) \) is zero and the case of faces contained in coordinate axes is done in Proposition 3.47. In Proposition 3.49 we show the rationality and compute the limit of the zeta function \( Z^\delta_{\gamma, +}(T) \).

In Propositions 3.54 and 3.56, we prove the decomposition

\[
Z^\delta_{\gamma, +}(T) = \sum_{\mu \in R_{\gamma}} \left( \frac{Z^\delta_{C(p, q)\setminus{\varepsilon_{\gamma}}}(T)}{Z_{f_{\sigma(p, q, a)}, \omega_{p, q}, b \neq 0}(0, 0, 0)} \right)
\]

Similarly in Sects. 3.3.7.3 and 3.3.7.4 we consider the case of the horizontal and vertical faces.

Assume \( \overline{N}(f) \) is a segment. We denote by \( \gamma \) the one dimensional face of \( \overline{N}(f) \). It is a segment with vertices \((a_0, b_0)\) and \((a_1, b_1)\) such that \( a_0 \leq a_1 \) and if \( a_0 = a_1 \) then \( b_0 < b_1 \). We choose an equation of the underlying line of \( \gamma \) as \( ap + bq = N \) with \((p, q)\) in \( \mathbb{Z}^2 \setminus \{(0, 0)\}^2 \) and coprime.

- Assume \( pq < 0 \). In that case we have \( C_{\gamma} \cap \Omega = \mathbb{R}_{>0}(p, q) + \mathbb{R}_{<0}(-p, -q) \), then using similar ideas as in the previous case, we obtain formula (3.16). We remark also that for any Newton transformation at infinity \( \sigma \) associated to a root \( \mu \) of \( f \) with multiplicity \( \nu \), the Newton transforms \( f_{\sigma(p, q, a)} \) and \( f_{\sigma(-p, -q, \mu)} \) have the form \( u(v, w)v^{-N}w^v \) and \( u(v, w)v^Nw^v \) with \( u \) a unit. Applying Example 2.38, formula (3.17) is satisfied.

- Assume \( pq \geq 0 \). In that case we have \( C_{\gamma} \cap \Omega = \mathbb{R}_{>0}(p, q) \) and then, using similar ideas as in two previous cases, we obtain formula (3.18). We prove now formula (3.19). First of all, as \( pq \geq 0 \) we necessarily have \( N > 0 \), then \( \varepsilon_{\gamma} = 0 \) and as above \( (S_{f_{\sigma(p, q, a)}, p \neq 0})(0, 0, 0) \) is zero for any roots \( \mu \) of \( f \). Furthermore,

  - if \( a_0 = 0 \) (resp \( b_1 = 0 \)) then the intersection \( C_{(a_0, b_0)} \cap H_{(a_0, b_0)} \cap \Omega \) is empty and \( \varepsilon_{(0, b_0)} = 0 \) (similarly \( \varepsilon_{(a_1, 0)} = 0 \)).
  - If \( a_0 \neq 0 \) (resp \( b_1 \neq 0 \)) then we have

\[
\Omega \cap C_{(a_0, b_0)} \cap H_{(a_0, b_0)} = \mathbb{R}_{>0}(0, -1) + \mathbb{R}_{>0}(b_0, -a_0) \subset \mathbb{R}_{>0} \times \mathbb{R}_{<0}
\]

and we obtain \( \varepsilon_{(a_0, b_0)} = 0 \) (similarly \( \varepsilon_{(a_1, b_1)} = 0 \)) by Proposition 3.49.

The monomial case follows from Proposition 3.49. □
Remark 3.24 Let \( f \) be a polynomial in \( \mathbb{k}[x, y] \) with \( \overline{N}(f) \) not a segment. We use notations of Theorem 3.23.

We refer to Proposition 3.47 for the value of \( \varepsilon_\gamma \) for \( \gamma \) a face equal to \((a, 0)\) with \( a > 0 \) or \((0, b)\) with \( b > 0 \). We precise two particular cases

- if the dual cone of the face \((a, 0)\) is \( C_{(a,0)} = \mathbb{R}_{>0}(0, -1) + \mathbb{R}_{>0}\eta \) with \((\eta \mid (1, 0)) > 0\). Then we have \( \varepsilon_{(a,0)} = 0 \). Indeed, the half-space \( H_{(a,0)} = \{ (\alpha, \beta) \mid a\alpha < 0 \} \) does not intersect \( C_{(a,0)} \), and we conclude by Proposition 3.47.
- if the dual cone of the face \((0, b)\) is \( C_{(0,b)} = \mathbb{R}_{>0}(-1, 0) + \mathbb{R}_{>0}\eta \) with \((\eta \mid (0, 1)) > 0\). Then we have \( \varepsilon_{(0,b)} = 0 \). Indeed, the half-space \( H_{(0,b)} = \{ (\alpha, \beta) \mid b\beta < 0 \} \) does not intersect \( C_{(0,b)} \), and we conclude by Proposition 3.47.

By Proposition 3.49 and its notations, we have \( \varepsilon_\gamma = 0 \) for any zero-dimensional face \( \gamma \) intersection of two one-dimensional faces \( \gamma_1 \) and \( \gamma_2 \), with exterior normal vectors \( \omega_1 \) and \( \omega_2 \) such that \((\gamma \mid \omega_1) \geq 0\) and \((\gamma \mid \omega_2) \geq 0\). Indeed, under this assumption, the intersection \( C_\gamma \cap H_\gamma \) is empty.

If a face \( \gamma \) belongs to \( \overline{N}(f) \) but is not the horizontal or vertical face \( \gamma_h \) or \( \gamma_v \) of \( f \) in Definition 1.5, then its dual cone \( C_\gamma \) does not intersect \( \Omega \), and \( \varepsilon_\gamma = 0 \).

From Theorems 3.21 and 3.23, we deduce the following Kouchnirenko type formula for the invariant \( \lambda \).

Corollary 3.25 (Kouchnirenko type formula for the invariant \( \lambda \)) Assume \( \mathbb{k} = \mathbb{C} \). Let \( f \) be a polynomial in \( \mathbb{k}[x, y] \) not in \( \mathbb{k}[x] \) or \( \mathbb{k}[y] \), with isolated singularities.

Assume \( \overline{N}(f) \) is not a segment. Then we have,

\[
\lambda_0(f) = 2 \sum_{\gamma \in \overline{N}(f), \text{dim } \gamma = 1} \varepsilon_\gamma S_N(f_\gamma, f_\gamma) - \sum_{\gamma \in \overline{N}(f), \text{dim } \gamma = 1} \sum_{\mu \in R_\gamma} \check{\varepsilon}(S_{f_\gamma, N, \mu}, \mu \neq 0)(0,0,0) \tag{3.21}
\]

where \( \varepsilon_\gamma \) belongs to \( \{0, 1\} \) for faces of dimension 1.

Assume \( f \) to be quasi homogeneous polynomial with simple roots and which is not monomial. Let \( \gamma \) be its one-dimensional face with primitive exterior normal vector \((p, q)\) and underlying line of equation \( ap + bq = N \). Then, we have

\[
\lambda_0(f) = 2 \varepsilon_\gamma S_N(f_\gamma, f_\gamma) \tag{3.22}
\]

where \( \varepsilon_\gamma \) belongs to \( \{0, 1\} \). More precisely we have, if \( pq \geq 0 \) then \( \lambda_0(f) = 0 \), if \( pq < 0 \) and \( N = 0 \) then \( \lambda_0(f) = 0 \) and if \( pq < 0 \) and \( N \neq 0 \) then \( \lambda_0(f) = 2S_N(f_\gamma, f_\gamma) = 2S_\gamma \) where \( S_\gamma \) is the area associated to the face \( \gamma \) (Proposition 2.4).

Proof This corollary follows from Theorems 3.21, 3.23, Remark 3.24 and Proposition 2.4.

Example 3.26 (Broughton’s example) We study here Broughton example \( f(x, y) = x(xy-1) \).

The global Newton polygon of \( f \) is a segment, then applying Theorem 3.23 we have

\[
S_{f,0}^\infty = \varepsilon_x \left[ x : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2} \right] + \varepsilon_{x^2 y} \left[ x^2 y : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2} \right] + \varepsilon_{x^2 y-x} \left[ x^2 y - x : \mathbb{G}_m^2 \setminus (xy = 1) \to \mathbb{G}_m, \sigma_y \right] + (S_{f_\sigma(-1,1), x_1 \neq 0}(0,0,0), 0) + (S_{f_\sigma(1,-1), x_1 \neq 0}(0,0,0), 0). \tag{3.23}
\]

We compute now the coefficients \( \varepsilon_x, \varepsilon_{x^2 y}, \varepsilon_{x^2 y-x} \) and the motives \( (S_{f_\sigma(-1,1), x_1 \neq 0}(0,0,0), 0), (S_{f_\sigma(1,-1), x_1 \neq 0}(0,0,0), 0) \).
We have $C_x = \{(\alpha, \beta) \mid \alpha + \beta < 0\}$ and $H_x = \{(\alpha, \beta) \mid \alpha < 0\}$. In particular we have
\[ C_x \cap H_x \cap (\mathbb{R}_{>0} \times \mathbb{R}_{<0}) = \emptyset \text{ and } C_x \cap H_x \cap (\mathbb{R}_{<0} \times \mathbb{R}_{>0}) = \mathbb{R}_{>0}(-1, 0) + \mathbb{R}_{>0}(-1, 1). \]

Then, by Proposition 3.47 we conclude that $\varepsilon_x = -1$.

We have $C_{x^2} = \{(\alpha, \beta) \mid \alpha + \beta > 0\}$ and $H_{x^2} = \{(\alpha, \beta) \mid 2\alpha + \beta < 0\}$. In particular we have
\[ C_{x^2} \cap H_{x^2} \cap (\mathbb{R}_{>0} \times \mathbb{R}_{<0}) = \emptyset \text{ and } C_{x^2} \cap H_{x^2} \cap (\mathbb{R}_{<0} \times \mathbb{R}_{>0}) = \mathbb{R}_{>0}(-1, 1) + \mathbb{R}_{>0}(-1, 2). \]

Then, by point 2(b)ii of Proposition 3.49 we conclude that $\varepsilon_{x^2} = 0$.

By point 4b of Proposition 3.49, we have $\varepsilon_{x^2 - x} = 1$.

We have $f_{\sigma(-1,1,1)}(x_1, y_1) = x_1 1 (y_1 + 1)$ and by Example 2.38
\[ (S_{f\sigma(-1,1,1),x_1\neq 0})(0,0) = -(x_1 1 : G_m^2 \to G_m, \sigma_{G_m^2}). \]

We have $f_{\sigma(1,-1,1)}(x_1, y_1) = x_1^{-1} 1$ and by Example 2.38
\[ (S_{f\sigma(1,-1,1),x_1\neq 0})(0,0) = 0. \]

From equality 3.23, we obtain
\[ S_{f,0}^\infty = -[x : G_m^2 \to G_m, \sigma_{G_m^2}] + [x^2 y - x : G_m^2 \setminus (xy = 1) \to G_m, \sigma_y] \]
\[ -[xy : G_m^2 \to G_m, \sigma_{G_m^2}] \]
and we have
\[ S_{f,0}^{\infty,(1)} = -L \text{ and } \lambda_0(f) = 1 \]
using the equalities $[(xy = 1) \cap G_m^2, \sigma_\mu] = [(z = 1) \cap G_m^2, \sigma_\mu] = L - 1$, by the isomorphism $(x, y) \to (z = xy, y)$ of $G_m^2$ and
\[ ((xy - 1)x = 1) \cap G_m^2, \sigma_{G_m^2} = [G_m \setminus \{1\}, \sigma_{G_m^2}] = L - 2 \]
writing $y = (1 + x)x^{-2}$ with the condition $y \neq 0$.}

Let $c \neq 0$ be in $\mathbb{C}$. Applying Theorem 3.23 we have
\[ S_{f,c}^\infty = \varepsilon_x[x : G_m^2 \to G_m, \sigma_{G_m^2}] + \varepsilon_{x^2}[x^2 y : G_m^2 \to G_m, \sigma_{G_m^2}] \]
\[ + \varepsilon_{x-c}[x - c : G_m^2 \setminus (x = c) \to G_m, \sigma_{G_m^2}] \]
\[ + \varepsilon_{x^2 - c}[x^2 y - c : G_m^2 \setminus (xy = c) \to G_m, \sigma_{G_m^2}] \]
\[ + \varepsilon_{x^2 - x}[x^2 y - x : G_m^2 \setminus (xy - 1) \to G_m, \sigma_{G_m^2}] \]
\[ + (S_{f\sigma(-1,2,c),x_1\neq 0})(0,0,0) + (S_{f\sigma(1,-1,1),x_1\neq 0})(0,0,0). \]

We compute now the coefficients $\varepsilon_x$, $\varepsilon_{x^2}$, $\varepsilon_{x-c}$, $\varepsilon_{x^2 y-c}$, $\varepsilon_{x^2 y-x}$ and the motives $(S_{f\sigma(-1,2,c),x_1\neq 0})(0,0,0)$, $(S_{f\sigma(1,-1,1),x_1\neq 0})(0,0,0)$.

- We have $C_x = \mathbb{R}_{>0}(0, -1) + \mathbb{R}_{>0}(1, -1)$ and $H_x = \{(\alpha, \beta) \mid \alpha < 0\}$. In particular, $C_x \cap H_x$ is empty, so by Proposition 3.47, $\varepsilon_x = 0$.

- We have $C_{x^2} = \mathbb{R}_{>0}(-1, 2) + \mathbb{R}_{>0}(1, -1)$ and $H_{x^2} = \{(\alpha, \beta) \mid 2\alpha + \beta < 0\}$. In particular, $C_{x^2} \cap H_{x^2}$ is empty, so by point 2a of Proposition 3.49 we have $\varepsilon_{x^2} = 0$.

- We have $C_{x-c} = \mathbb{R}_{>0}(0, -1)$, so $C_{x-c} \cap \Omega$ is empty, so by point 3a of Proposition 3.49, $\varepsilon_{x-c} = 0$.

- By point 3a of Proposition 3.49, we have $\varepsilon_{x^2 y-c} = 0$ and $\varepsilon_{x^2 y-x} = 0$. 


\[ \text{Springer} \]
Theorem 3.23, we have proved that for any convenient and non-degenerate polynomial \( f \), Example 3.27

Let \( \gamma \) be a one dimensional face in \( \mathcal{N}^{\infty}_{\infty}(f) \), applying the Newton transformation \( \sigma_{(p,q,\mu)} \), we obtain a Newton transform of the form \( f_{\sigma_{(p,q,\mu)}}(x_1, y_1) = x_1^{-N} y_1 u(x_1, y_1) \) with \( u \) a unit or \( f_{\sigma_{(p,q,\mu)}}(x_1, y_1) = 0 \). Then, applying Examples 2.38 and 2.39, we have \( (S_{f_{\sigma_{(p,q,\mu)}}})_{(x_1 \neq 0, y_1 \neq 0)}(0,0,0) = 0 \).

For the vertical face (not contained in the coordinate axis), the face polynomial is \( f_y(x, y) = x^M T(y) \), with \( M > 0 \). We remark that for any root \( \mu \) of \( T \), the polynomial \( f(1/x, y + \mu) \) as the form \( x^{-M} (c_y y + c_x x^m + g(x, y)) \) with \( c_y \) and \( c_x \) in \( \mathbf{k} \), \( m > 0 \) and \( g(x, y) = \sum_{a+b m > 0} c_{a,b} x^a y^b \). Then, applying Example 2.39, we have \( (S_{f_{(0,m,u)}})_{x \neq 0}(0,0,0) = 0 \). The result and the proof are the same for the horizontal case.

Then, by formula (3.14), \( S^{\infty}_{f,c} = 0 \).

Example 3.28 We consider the polynomial \( f \) defined in Example 1.40. We remark that \( f(0,0) = 1 \). Let \( c \) be in \( \mathbf{k} \). We use all the notations of Example 1.40. By formula (3.14) of Theorem 23, we have

\[
S^{\infty}_{f,c} = \epsilon_{y}[y : \mathbb{G}_m^2 \rightarrow \mathbb{G}_m] + \epsilon_{x^2}[x^2 : \mathbb{G}_m^2 \rightarrow \mathbb{G}_m] + \epsilon_{y}(y^2 y + 1)^3 : \mathbb{G}_m^2 \rightarrow \mathbb{G}_m, \sigma_{y_1}(0) + \epsilon_{x^2}(x^2 y + 1)^4 : \mathbb{G}_m^2 \rightarrow \mathbb{G}_m, \sigma_{y_2}(0) + \epsilon_{x^4 y} x^4 y^6 : \mathbb{G}_m^2 \rightarrow \mathbb{G}_m, \sigma_{y_3}(0) + (S_{f_1-c,v \neq 0})_{(0,0,0)} + (S_{f_2-c,v \neq 0})_{(0,0,0)}.
\]
By Remark 3.24 and Propositions 3.47 and 3.49, we have \(\varepsilon_y = 0, \varepsilon_{x^2} = 0, \varepsilon_{y^1(0)} = 0, \varepsilon_{x^3, y^6} = 0\). Furthermore, \(\mathcal{N}(f_1 - c)^+\) is empty and by Theorem 2.22, we have \((S_{f_1 - c, v \neq 0})_{(0,0),0} = (S_{(f_1)}_{\sigma_{y^1}} - c, v \neq 0)_{(0,0),0} = 0\) because \((f_1)_{\sigma_{y^1}} - c\) is an Example 2.39 with \(M = 3\) and \(m = 1\) by formula (1.9), then we obtain the equality

\[
S_{f, c}^\infty = (S_{f_2 - c, v \neq 0})_{(0,0),0}.
\]

- We assume \(c \notin \{1, 2\}\). The set \(\mathcal{N}(f_2 - c)^+\) is empty and by Theorem 2.22, we have

\[
(S_{f_2 - c, v \neq 0})_{(0,0),0} = \left( (S_{(f_2)}_{\sigma_{y^1}, v_1} - c, v \neq 0)_{(0,0),0} + (S_{(f_2)}_{\sigma_{y^1}, v_2} - c, v \neq 0)_{(0,0),0} \right) = 0
\]

because \((f_2)_{\sigma_{y^1}} - c\) for \(i \in \{1, 2\}\) is an Example 2.38 or Example 2.39 with \(M = 0\) and \(m = 1\) by formula (1.11). Assume \(k = \mathbb{C}\), as \(S_{f, c}^\infty = 0\), it follows from Corollary 3.25 or Theorem 3.21 that \(\lambda_c(f) = 0\).

- We assume \(c = 2\). We observe that \(\mathcal{N}(f_2)^+ = \{\gamma_v, \gamma_v^2(2)\}\). Then, applying Theorem 2.22, we have

\[
(S_{f_2 - v, v \neq 0})_{(0,0),0} = [v : \mathbb{G}_m \to \mathbb{G}_m]
\]

\[
+ \left[ 2v^{-1}w^2 - 4w - v : \mathbb{G}_m^2 \setminus (2v^{-1}w^2 - 4w - v = 0) \to \mathbb{G}_m, \sigma_{\gamma_v^2(2)} \right]
\]

\[
+ \left( (S_{(f_2)}_{\sigma_{y^1}} - 2, v_1 \neq 0)_{(0,0),0} + (S_{(f_2)}_{\sigma_{y^1}, v_1} - 2, v_1 \neq 0)_{(0,0),0} \right)
\]

\[
+ \left( (S_{(f_2)}_{\sigma_{y^1}, v_2} - 2, v_1 \neq 0)_{(0,0),0} \right).
\]

As the Newton transform \((f_2)_{\sigma_{y^1}} - 2\) is an Example 2.39 with \(M = 0\) and \(m = 1\), we have \((S_{(f_2)}_{\sigma_{y^1}} - 2, v_1 \neq 0)_{(0,0),0} = 0\). As well, the Newton transforms \((f_2)_{\sigma_{y^1}, v_1} - 2\) and \((f_2)_{\sigma_{y^1}, v_2} - 2\) are Examples 2.39 with \(M = -1\) and \(m = 1\), then we have

\[
(S_{(f_2)}_{\sigma_{y^1}, v_1} - 2, v_1 \neq 0)_{(0,0),0} = (S_{(f_2)}_{\sigma_{y^1}, v_2} - 2, v_1 \neq 0)_{(0,0),0} = - \left[ xy : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2} \right].
\]

We conclude that

\[
S_{f,2}^\infty = [v : \mathbb{G}_m \to \mathbb{G}_m]
\]

\[
+ \left[ 2v^{-1}w^2 - 4w - v : \mathbb{G}_m^2 \setminus (2v^{-1}w^2 - 4w - v = 0) \to \mathbb{G}_m, \sigma_{\gamma_v^2(2)} \right]
\]

\[
-2 \left[ xy : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_{\mathbb{G}_m^2} \right].
\]

Assume \(k = \mathbb{C}\). By Proposition 2.4 we have \(\chi_v((2v^{-1}w^2 - 4w - v = 1) \cap \mathbb{G}_m^2) = -2\). It follows from Corollary 3.25 or Theorem 3.21 that

\[
\lambda_2(f) = -\chi_v(S_{f,2}^\infty(1)) = -(1 - 2 + 0) = 1.
\]

- We assume \(c = 1\). Applying Theorem 2.22, we have

\[
(S_{f_2 - v, v \neq 0})_{(0,0),0} = (S_{f_3, v_1 \neq 0})_{(0,0),0},
\]

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with
\[
(S_{f_3}, v_1 \neq 0)(0,0), 0 = [v_1 : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}] + [4 w_1^2 - 7 v_1 : \mathbb{G}_m \setminus (4 w_1^2 = 7 v_1) \\
\to \mathbb{G}_m, \sigma_{\mathbb{G}_m}(2)] + (S_{f_3}, v_2 \neq 0)(0,0), 0,
\]

with
\[
(S_{f_3}(\sigma_{\mathbb{G}_m}))^2, 0 \neq 0) = -[v_2^2 w_2 : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}].
\]

because \( f_3 \) is an Example 2.39 with \( M = -2 \) and \( m = 1 \). Assume \( k = \mathbb{C} \). By Proposition 2.4 we have \( \chi_c((4 w_1^2 - 7 v_1 = 1) \cap \mathbb{G}_m^2) = -2 \). It follows from Corollary 3.25 or Theorem 3.21 that \( \lambda_1(f) = -\chi_c(S_\infty(1)) = -(1 - 2 + 0) = 1 \).

By Examples 1.40 and 3.14, we have \( \chi_c(f^{-1}(a_{\text{gen}})) = -3 \), \( \mu(f) = 2 \) and \( \lambda(f) = 2 \). The second formula 3.11 is satisfied.

### 3.2.4 Motivic bifurcation set

We recall in dimension 2, the notion of motivic bifurcation set defined in [15,30].

**Definition 3.29** (Motivic bifurcation set) For any polynomial \( f \) in \( k[x, y] \), the motivic bifurcation set defined as
\[
B_{f}^{\text{mot}} = \{a \in \mathbb{A}^1_k | S_{f,a}^\infty \neq 0 \} \cup \text{disc}(f)
\]
is a finite set.

### 3.2.5 Equalities of bifurcation sets

In this section we prove the following theorem.

**Theorem 3.30** Let \( f \) be a polynomial in \( \mathbb{C}[x, y] \) not in \( \mathbb{C}[x] \) or \( \mathbb{C}[y] \). If \( f \) has isolated singularities then
\[
B_{f}^{\text{top}} = B_{f}^{\text{Newton}} = B_{f}^{\text{mot}}.
\]

**Proof** Indeed, we deduce from Theorems 3.17 and 3.21 that \( B_{f}^{\text{top}} \) is included in \( B_{f}^{\text{mot}} \) (see [15]). We deduce from Proposition 3.34 that \( B_{f}^{\text{Newton}} \) is included in \( B_{f}^{\text{top}} \). We deduce from Proposition 3.36 the inclusion of \( B_{f}^{\text{mot}} \) in \( B_{f}^{\text{Newton}} \).

**Remark 3.31** This result generalizes the result in the non-degenerate case of Némethi and Zaharia in [26]. We recover also from Proposition 1.39 that \( B_{f}^{\text{mot}} \) is finite. From Theorem 3.30 and [15], we deduce the equality
\[
B_{f}^{\text{top}} = B_{f}^{\text{Newton}} = B_{f}^{\text{Serre}} = B_{f}^{\text{mot}}
\]
where \( B_{f}^{\text{Serre}} \) is the Serre bifurcation set defined in [15] using analytic geometry in the Berkovich sense.

To prove Proposition 3.34, we need the following Lemmas 3.32 and 3.33.
Lemma 3.32 \textit{Let} $h$ \textit{be polynomial in} $\mathbb{k}[x^{-1}, x, y]$ \textit{which is not of the form} $x^{-M}u(x, y)$ \textit{with} $u$ \textit{a unit. Then, we have}
\[ \tilde{\chi}( (S_{h,x\neq0})^{(1)}_{(0,0), 0}) \leq 0. \]

\textbf{Proof} \textit{To compute} $\tilde{\chi}( (S_{h,x\neq0})^{(1)}_{(0,0), 0})$, \textit{we use formula (2.12) and (2.14) of Corollary 2.23.}
\textit{We prove the result by induction on the Newton process.}

- \textit{For the base case Example 2.38 with} $m \neq 0$ \textit{and base case Example 2.39, we have}
\[ \tilde{\chi}( (S_{h,x\neq0})^{(1)}_{(0,0), 0}) = 0. \]

- \textit{By Newton algorithm, at each step the considered polynomial is not of the type of Example 2.38 with} $m = 0$. \textit{Indeed, by assumption} $h$ \textit{is not of the form} $x^{-M}u(x, y)$ \textit{with} $u$ \textit{a unit. If} $h$ \textit{is an Example 2.38 necessarily} $m \neq 0$. \textit{Otherwise,} $h$ \textit{has at least one compact one dimensional face, and for each of them, for each root} $\mu$ \textit{of the face polynomial the corresponding multiplicity} $\nu$ \textit{is larger than 1 and the Newton transform is}
\[ h_{\sigma(p,q,\mu)}(x, y) = x^N \left( y^\nu u(y) + \sum_{(k,l) \neq \gamma} x^{(k,l)(p,q)-(N+\nu)u_{k,l}(y)} \right) \]
\textit{where} $u$ \textit{and} $u_{k,l}$ \textit{are units in} $y$. \textit{Then,} $h_{\sigma(p,q,\mu)}$ \textit{is not of the type of Example 2.38 with} $m = 0$. \textit{Now assume the induction hypothesis:} $\tilde{\chi}( (S_{h^\sigma(p,q,\mu)})^{(1)}_{(0,0), 0}) \leq 0$, \textit{for each face of the Newton polygon of} $h$ \textit{and for each root of the corresponding face polynomial. Then, by formulas (2.12) and (2.14) of Corollary 2.23 we get the result. Indeed,}

- \textit{for each one dimensional face} $\gamma$ \textit{in} $\mathcal{N}(h)^+$ \textit{we have} $-2S_{N(h_\gamma), f_\gamma} \leq 0$.
- \textit{if we assume that the Newton polygon} $\mathcal{N}(h)$ \textit{has a face} $\gamma$ \textit{with vertices} $(-M, d)$ \textit{and} $(a, 0)$, \textit{with} $-M$ \textit{and} $a$ \textit{in} $\mathbb{Z}$, \textit{d in} $\mathbb{N}$ \textit{and} $a > -M$. \textit{We denote by} $h_\gamma$ \textit{the face polynomial. It follows from Definition 2.2 and its notations that}
\[ a - 2S_{N(h_\gamma), h_\gamma} = a \left( 1 - \frac{rd}{s+1} \right) \leq 0 \] (3.24)
\textit{because} $d \geq s + 1$. \textit{In particular this is equal to zero if and only if} $d = s + 1$ \textit{and} $r = 1$. \textit{Then, we conclude by induction using the Newton algorithm and get the result.} \hfill \Box

Lemma 3.33 \textit{Let} $h$ \textit{be in} $\mathbb{C}[x^{-1}, x, y]$ \textit{with isolated singularities and} $c_0$ \textit{be in} $\mathbb{C}$. \textit{If} $\tilde{\chi}( (S_{h-c_0, x\neq0})^{(1)}_{(0,0), 0}) = 0$ \textit{then} $c_0$ \textit{does not belong to} $\mathcal{B}_h^{\text{Newton}}$.

\textbf{Proof} \textit{As} $h$ \textit{has isolated singularities in} $\mathbb{C}^* \times \mathbb{C}$, \textit{then by Lemma 1.35, for any composition} $\Sigma$ \textit{of Newton transformations, the polynomial} $h_\Sigma$ \textit{has isolated singularities in} $\mathbb{C}^* \times \mathbb{C}$. \textit{Let} $c_0$ \textit{be in} $\mathbb{C}$. \textit{We assume} $\tilde{\chi}( (S_{h-c_0, x\neq0})^{(1)}_{(0,0), 0}) = 0$. \textit{We remark that using Corollary 2.23 and Lemma 3.32 (and formula (3.24) in its proof) we have}
\[ \tilde{\chi}( (S_{h-c_0})^{(1)}_{(0,0), 0}) = 0 \] (3.25)
\textit{for any composition} $\Sigma$ \textit{of Newton transformations. We denote by} $c_{(0,0)}(h)$ \textit{the constant term of} $h$. \hfill \$\copyright$ Springer
Proposition 3.34 \[ \text{Let } f \text{ be a polynomial in } \mathbb{C}\setminus\text{disc}(f) \text{ with isolated singularities. Let } c_0 \text{ be in } \mathbb{C} \text{ and } \lambda_{c_0}(f) = 0 \text{ then } c_0 \text{ does not belong to } B^\text{Newton}_f. \text{ Then, we have the inclusion } B^\text{Newton}_f \subset B^\text{top}_f. \]

Proof As \( f \) has isolated singularities in \( \mathbb{C}^2 \), it follows from Lemma 1.35 that for any composition \( \sum \) of Newton transformations, \( f_\sum \) has isolated singularities in \( \mathbb{C}^* \times \mathbb{C} \). Let \( c_0 \) be in \( \mathbb{C} \setminus \text{disc}(f) \) with \( \lambda_{c_0}(f) = 0 \). We consider the computations of \( \lambda_{c_0}(f) \) in Corollary 3.25 for \( f - c_0 \).
Assume $f - c_0$ is not a quasi homogeneous polynomial. As $\lambda_{c_0}(f) = 0$, using Lemma 3.32 we observe that in formula (3.21) we have $\varepsilon_\gamma = 0$ for any one dimensional face $\gamma$ in $\overline{N}(f - c_0)$ and for any Newton transformation in the first step of the Newton algorithm at infinity of $f - c_0$, we have

$$\bar{\lambda}_{\gamma} \left((S(f-c_0)_{\sigma(p,q,\mu)},v\neq0)((0,0),0)\right) = 0. \quad (3.27)$$

As $f_{\sigma(p,q,\mu)} \in \mathbb{C}[x^{-1}, x, y]$ and $f_{\sigma(p,q,\mu)} - c_0 = (f - c_0)_{\sigma(p,q,\mu)}$, we deduce by Lemma 3.33 that $c_0$ does not belong to $B_{\text{Newton}}^{\Sigma} f_{\sigma(p,q,\mu)}$ and then neither, to any $B_{\text{Newton}}^{\Sigma}$ for any composition $\Sigma$ of Newton transformations. Thus, to prove the result it is enough to show that $c_0$ is not a non Newton generic value at infinity. We proceed by contradiction. Assume $c_0$ is a non Newton generic value at infinity. So, there is a dicritical face at infinity $\gamma_0$ of $f$ with face polynomial $P_{\gamma_0}(x^m y^n)$ with $P_{\gamma_0}$ in $\mathbb{C}[s]$.

1. Assume $c_0 \neq f(0,0)$. The polynomial $P_{\gamma_0}(s) - c_0$ has a root $\mu$ of multiplicity $v \geq 2$.

We denote by $(p, q)$ the primitive exterior normal vector to the face $\gamma_0$. Applying the corresponding Newton transformation $\sigma(p,q,\mu)$, we get

$$(f - c_0)_{\sigma(p,q,\mu)} = u(w)w^v + \sum_{(k,l) \in \text{supp}(f-c_0)\setminus\{\gamma_0\}} u_1^{((k,l))((p,q))} u_{(k,l)}(w) \quad (3.28)$$

where $u$ and $u_{k,l}$ are units. We remark that $\mathcal{N}^+(f - c_0)$ is not empty and contains a face of dimension 1, then it follows from Corollary 2.23 that $\bar{\lambda}_{\epsilon} \left((S(f-c_0)_{\sigma(p,q,\mu)},v\neq0)((0,0),0)\right) \neq 0$ which is a contradiction with formula (3.27).

2. Assume $c_0 = f(0,0)$. Let $\gamma'$ be the face of maximal dimension of $\overline{\mathcal{N}}(f - c_0)$ such that $\gamma \cap \gamma_0 \neq \emptyset$. It has only one vertex $(a_0, b_0)$ if its dimension is zero, otherwise it has two vertices $(a_0, b_0)$ and $(a_1, b_1)$ with $a_0 < a_1$.

- Assume that the polynomial $P_{\gamma_0}(s) - c_0$ has a root $\mu$ with a multiplicity $v \geq 2$.
  (a) Assume the root $\mu \neq 0$. The situation is similar to above case 1 and gives a contradiction.
  (b) Assume the root $\mu = 0$. As $v \geq 2$ we have $a_0 \geq 2$. If $x^{a_0}$ divides $f - c_0$ then $f - c_0$ does not have isolated singularities in $\mathbb{C}^2$. Then, $x^{a_0}$ does not divide $f - c_0$, so there is a one dimensional face $\gamma'$ in $\overline{\mathcal{N}}(f - c_0)\setminus\mathcal{N}(f - c_0)$ with vertex $(a_0, b_0)$. As $\gamma_0$ is a dicritical face, the face $\gamma'$ is supported by a line of equation $ap + bq = N$ with $(p, q)$ the normal vector exterior to $\overline{\mathcal{N}}(f - c_0)$ and $N < 0$. This one dimensional face $\gamma'$ induces a non zero area with $\varepsilon'_\gamma = 1$ (by Proposition 3.49), then by Lemma 3.32 and formula (3.21), we have $\lambda_{c_0}(f) \neq 0$. Contradiction.

- Assume that the dicritical face at infinity $\gamma_0$ is non smooth. As $c_0$ is a non critical value of $f$ and $c_0 = f(0,0)$, the point $(0,0)$ is not critical and $x$ or $y$ is in the support of $f - c_0$. Assume for instance $x$ is in the support (the case of $y$ in the support is similar). The dicritical face $\gamma_0$ is supported by an equation $p\alpha x + q\beta y = 0$ with $p < 0$ and $q > 0$. By assumption $\gamma_0$ is non smooth, then $q$ is different from 1. In particular, the face $\gamma$ does not have a point of coordinate $(1, |p|)$ and we necessarily have $a_0 > 1$. Then, we are in the similar case as 2b and obtain a contradiction.

- Assume $f - c_0$ is a quasi homogeneous polynomial. As $\lambda_{c_0}(f) = 0$, by Corollary 3.25, we consider the cases $pq < 0$ with $N = 0$ and $pq \geq 0$.
  - Assume $pq < 0$ and $N = 0$. If $f - c_0$ has a root $\mu$ with multiplicity $v \geq 2$, then $f$ does not have isolated singularities. If $f(0,0) = c_0$ then as $c_0$ is not a critical value, $(0,0)$ is
not a critical point then \( x \) or \( y \) belongs to the support of \( f - c_0 \) which is a contradiction with the assumption \( N = 0 \). Then \( c_0 \) does not belong to \( B_f^{\text{Newton}} \).

- Assume \( pq \geq 0 \). As \( f - c_0 \) is quasi homogeneous, we have \( f(0, 0) = c_0 \). As \( c_0 \) is not critical then as above \( (0, 0) \) is not a critical point and we can assume that \( x \) is in the support, then up to a constant, there is \( p \) in \( \mathbb{N}^* \) such that \( f(x, y) - c_0 = y^p - \mu x \). Then \( c_0 \) does not belong to \( B_f^{\text{Newton}} \).

**Proposition 3.35** Let \( h \) be in \( k[x^{-1}, x, y] \) and not in \( k[x, y] \). Let \( c \) be in \( k \setminus B_f^{\text{Newton}} \). Then we have \( (S_{h-c,x \neq 0})_{(0,0),0} = 0 \).

**Proof** (The base cases). We prove here Proposition 3.35 for the base cases Examples 2.38 or 2.39 with each case \( M > 0 \). The general case is proved below by induction using Newton algorithm.

- Consider \( h(x, y) = U(x, y)x^{-M_0}y^{m_0} \) with \( U \) a unit and \( M_0 > 0 \). We assume \( U(0, 0) = 1 \). Let \( c \) be in \( k \setminus B_f^{\text{Newton}} \).
  - Assume \( c \neq 0 \). The polynomial \( h \) has a dicritical face \( \gamma \) which is also a face of \( h - c \). The face polynomial of \( h - c \) is \( x^{-M_0}y^{m_0} - c = \prod_{i=1}^{d}(x^{-q}y^{p} - \mu_i) \), where \( d = \gcd(M_0, m_0), (p, q) = (m_0, M_0)/d \) and \( \mu_i \) are the \( d \)-roots of \( c \). In particular the set \( N(h - c)^+ \) is empty. Then, applying Theorem 2.22 we have
    \[
    (S_{h-c,x \neq 0})_{(0,0),0} = \sum_{i=1}^{d} (S_{(h-c)^{\sigma(p,q,\mu_i)},v \neq 0})_{(0,0),0} = 0
    \]
    because for each root \( \mu_i \) the Newton transform \( (h - c)^{\sigma(p,q,\mu_i)} \) is an Example 2.38 or 2.39 with \( M = 0 \) and \( m = 0 \).
  - Assume \( c = 0 \), by Example 2.38, we have \( (S_{h,v \neq 0})_{(0,0),0} = 0 \) because \( M_0 > 0 \).

- We consider \( h(x, y) = U(x, y)x^{-M_0}(y - \mu_0x^q + g(x, y))^{m_0} \) with \( U \) a unit, \( M_0 > 0 \) and \( g(x, y) = \sum_{a+bq > q} c_{a,b}x^a y^b \). We assume \( U(0, 0) = 1 \). The polynomial \( h \) has a one dimensional face with vertices \( (-M_0, m_0) \) and \( (-M_0 + m_0q, 0) \). We denote the constant term of \( h \) by \( c_{(0,0)}(h) \).
  - Assume \( -M_0 + m_0q < 0 \). Then, the set \( N(h - c)^+ \) is empty for any value \( c \). The polynomial \( h - c \) has only one one dimensional face denoted by \( \gamma \) and its face polynomial \( (h - c)^{\gamma} = x^{-M_0}(y - \mu_0x^q)\) is an Example 2.38 or 2.39 with \( M = -M_0 + m_0q < 0 \) and \( m = m_0 \), so we have \( (S_{(h-c)^{\gamma},v \neq 0})_{(0,0),0} = 0 \). Applying Theorem 2.22 we have
    \[
    (S_{h-c,x \neq 0})_{(0,0),0} = 0
    \]
  - Assume \( -M_0 + m_0q = 0 \). The polynomial \( h \) has a dicritical face \( \gamma \). The polynomial face \( h^{\gamma} \) is equal to \( P(x^{-q}y) \) with \( P(s) = (s - \mu_0)^{m_0} \) in \( k[s] \). Let \( c \notin B_f^{\text{Newton}} \).
  - If \( c \neq c_{(0,0)}(h) \) then \( \gamma \) is a face of \( h - c \) and the set \( N(h - c)^+ \) is empty. The polynomial \( h - c \) has only one one dimensional face, which is the dicritical face \( \gamma \). As \( c \) does not belong to \( B_f^{\text{Newton}} \) the face polynomial \( (h - c)^{\gamma} \) has simple roots. Then, for any root \( \mu \) of \( (h - c)^{\gamma} \), the Newton transform \( (h - c)^{\gamma^{\sigma(1,q,\mu)}} \) is an Examples 2.38 or 2.39 with \( M = 0 \) and \( m = 1 \). Then, we have \( (S_{(h-c)^{\gamma^{\sigma(1,q,\mu)}},v \neq 0})_{(0,0),0} = 0 \) and as above applying Theorem 2.22 we have
    \[
    (S_{h-c,x \neq 0})_{(0,0),0} = 0
    \]
  - If \( c = c_{(0,0)}(h) \), then as \( c \notin B_f^{\text{Newton}} \) the face \( \gamma \) is smooth and the polynomial \( P(s) - c \) has simple roots. In particular, the polynomial \( h \) has a monomial \( x^{-M'}y \) with \( M' > 0 \).
Assume the horizontal face of $h - c$ is $(a, 0)$. Necessarily we have $a > 0$. The polynomial $h - c$ has at most two one dimensional faces: the associated face to the dicritical face of $h$ still denoted by $\gamma$, and a one dimensional with vertices $(a, 0)$ and $(-M', 1)$ denoted by $\gamma'$. We can assume the face polynomial $(h - c)_{\gamma'}$ to be equal to $x^{-M'}(y - \lambda x^{a + M'})$. We have $\mathcal{N}(h - c)^+ = \{(a, 0), \gamma'\}$. The polynomial $(h - c)_{\sigma_{(1, a + M', \lambda)}}$ is an Examples 2.38 or 2.39 with $M = a > 0$ and $m = 1$ then we have

$$(S_{(h - c)_{\sigma_{(1, a + M', \lambda)}}}, y \neq 0)(0, 0), 0 = -[x^a y : G_m^2 \to G_m, \sigma_{G_m}].$$

By Theorem 2.22 we have

$$(S_{h - c, x \neq 0})_{(0, 0), 0} = [x^a : G_m \to G_m, \sigma_{G_m}]$$

$$+ [x^{-M'}(y - \lambda x^{a + M'}) : G_m^2 \setminus (y = \lambda x^{a + M'}) \to G_m, \sigma_{G_m}]$$

$$- [x^a y : G_m^2 \to G_m] + \sum_{\mu \in R_y} (S_{(h - c)_{\sigma_{(1, M', \mu)}}}, y \neq 0)(0, 0), 0.$$

As in the proof of Example 2.39 we have

$$[x^a : G_m \to G_m, \sigma_{G_m}] + [x^{-M'}(y - \lambda x^{a + M'}) : G_m^2 \setminus (y = \lambda x^{a + M'}) \to G_m, \sigma_{G_m}]$$

$$= [x^{-M'} y : G_m^2 \to G_m, \sigma_{G_m}] = [x^a y : G_m^2 \to G_m].$$

Furthermore, as $c$ does not belong to $B_{h_{\text{Newton}}}^+$, any roots $\mu$ in $R_y$ has multiplicity one, then the Newton transform $(h - c)_{\sigma_{(1, M', \mu)}}$ is an Examples 2.38 or 2.39 with $M = 0$ and $m = 1$, then we have $(S_{(h - c)_{\sigma_{(1, M', \mu)}}, y \neq 0})(0, 0), 0 = 0$, and, we conclude that $(S_{h - c, x \neq 0})(0, 0), 0 = 0$.

If the horizontal face is not of the form $(a, 0)$ then, it is the face $(-M', 1)$. Then, the set $\mathcal{N}(h - c)^+$ is empty, and as above as $c$ does not belong to $B_{h_{\text{Newton}}}^+$ we have $(S_{h - c, x \neq 0})(0, 0), 0 = 0$.

Assume $-M_0 + m_0q > 0$.

If $c$ is different from 0, then $\mathcal{N}(h - c)^+$ is empty. The polynomial $h - c$ has only one face of dimension one with face polynomial $x^{-M_0}y^{m_0} - c$, the Newton transform $(h - c)_{\sigma_{(m_0, M_0, c)}}$ is an Examples 2.38 or 2.39 with $M = 0$ and $m = 1$, then we conclude as above that $(S_{h - c, x \neq 0})(0, 0), 0 = 0$.

If $c$ is equal to zero, and does not belong to $B_{h_{\text{Newton}}}^+$, then with the same proof as in the case $-M_0 + m_0q = 0$ and $c = c_0(0)(h)$, we obtain $(S_{h - c, x \neq 0})(0, 0), 0 = 0$.

$\square$

**Proof** (The general case) We consider the general case of $h$ in $k[x^{-1}, x, y] \setminus k[x, y]$ and argue by induction using the Newton algorithm. The base cases in the induction are Examples 2.38 or 2.39 with $M > 0$; they are treated above. Let $c$ be a value not in $B_{h_{\text{Newton}}}^+$.

- Assume $c$ to be different from the constant term $c_0(0)(h)$ of $h$. Thus, $(0, 0) \in \text{Supp}(h - c)$ and $\mathcal{N}(h - c)^+$ is empty.

- Assume $h$ has a dicritical face $\gamma$. Then, $\gamma$ is a face of $\mathcal{N}(h - c)$. Furthermore, as $c \notin B_{h_{\text{Newton}}}^+$, all the roots $\mu$ of the face polynomial $(h - c)_\gamma$ have multiplicity one, and for each associated Newton transformation $\sigma_{(p, q, \mu)}$, the polynomial $(h - c)_\sigma$ is a base.
case Example 2.38 or 2.39 with $M = 0, m = 1$, then $(S_{h-c})_{\alpha(p,q),\mu}, v \neq 0)(0,0),0 = 0$. Thus, applying Theorem 2.22 we have

$$(S_{h-c,x \neq 0})(0,0),0 = \sum_{\gamma' \in \mathcal{N}(h-c) \setminus \{\gamma\}} \sum_{\mu \in R_{\gamma'}} \left( S_{h-c})_{\sigma(p',q',\mu),v \neq 0} \right)(0,0),0$$

Furthermore, for any face $\gamma' \in \mathcal{N}(h-c) \setminus \{\gamma\}$, $\gamma'$ is supported by a line of equation $\alpha p' + \beta q' = N_{\gamma'}$ with $(p', q')$ the normal of the face $\gamma'$ in $\mathcal{N}(h-c)$ and $N_{\gamma'} < 0$. Then, for each Newton transformation $\sigma_{(p',q',\mu)}$ associated to a root $\mu$ of the face polynomial $(h - c)_{\gamma'}$, the Newton transformation $(h - c)_{\sigma_{(p',q',\mu)}} = h_{\sigma_{(p',q',\mu)}} - c$ belongs to $k[x^{-1}, x, y]\backslash k[x, y]$. Finally, as $c$ does not belong to $B_h^\text{Newton}$, by definition $c$ does not belong to $B_h^\text{Newton}_{\sigma_{(p',q',\mu)}}$. Then, applying the induction we obtain

$$(S_{h-c})_{\sigma_{(p',q',\mu)},v \neq 0}(0,0),0 = 0$$

and we conclude that $(S_{h-c,x \neq 0})(0,0),0 = 0$.

- Assume $h$ does not have a dicritical face. Then as $h$ belongs to $k[x^{-1}, x, y]\backslash k[x, y]$, necessarily the horizontal face of $h$ is $(a, 0)$ with $a < 0$. Then, the set $\mathcal{N}(h-c)^+$ is empty and as above applying Theorem 2.22 we have

$$(S_{h-c,x \neq 0})(0,0),0 = \sum_{\gamma' \in \mathcal{N}(h-c)} \sum_{\mu \in R_{\gamma'}} \left( S_{h-c})_{\sigma_{(p',q',\mu)},v \neq 0} \right)((0,0),0)$$

where for each face $\gamma'$, for each associated Newton transformation $\sigma_{(p',q',\mu)}$, the polynomial $(h - c)_{\sigma_{(p',q',\mu)}}$ belongs to $k[x^{-1}, x, y]\backslash k[x, y]$ and $c$ does not belong to $B_h^\text{Newton}_{\sigma_{(p',q',\mu)}}$. Then, applying the induction we obtain

$$(S_{h-c})_{\sigma_{(p',q',\mu)},v \neq 0}(0,0),0 = 0$$

and we conclude that $(S_{h-c,x \neq 0})(0,0),0 = 0$.

- Assume $c$ is equal to the constant term $c((0,0)(h)$ of $h$. In particular $(0, 0)$ does not belong to the support of $h - c$.

- If $h$ does not have a dicritical face, then we argue as above and get the result.
- Assume that $h$ has a dicritical face $\gamma$. As $c$ does not belong to $B_h^\text{Newton}$, the dicritical face $\gamma$ is smooth and supported by a line of equation $\alpha \beta q = 1$. Furthermore the face polynomial of $h_\gamma$ is $P_\gamma(x^{-q} y)$ and the polynomial $P_\gamma(s) - c$ has simple roots. Then, we deduce that $h$ has a monomial $x^{-q} y$ and for each non zero root $\mu$ of $P_\gamma(s) - c$ we have as above $(S_{h-c})_{\alpha(p,q,\mu),v \neq 0}(0,0),0 = 0$.
- Assume the horizontal face of $h - c$ is $(-q, 1)$ then $\mathcal{N}(h-c)^+$ is empty and again by Theorem 2.22 we have

$$(S_{h-c,x \neq 0})(0,0),0 = \sum_{\gamma' \in \mathcal{N}(h-c) \setminus \{\gamma\}} \sum_{\mu \in R_{\gamma'}} \left( S_{h-c})_{\sigma_{(p',q',\mu)},v \neq 0} \right)((0,0),0)$$

where for each face $\gamma'$, for each associated Newton transformation $\sigma_{(p',q',\mu)}$, the polynomial $(h - c)_{\sigma_{(p',q',\mu)}}$ belongs to $k[x^{-1}, x, y]\backslash k[x, y]$ and $c$ does not belong to $B_h^\text{Newton}_{\sigma_{(p',q',\mu)}}$. Then, applying the induction we obtain

$$(S_{h-c})_{\sigma_{(p',q',\mu)},v \neq 0}(0,0),0 = 0$$

and we conclude that $(S_{h-c,x \neq 0})(0,0),0 = 0$.
- Assume the horizontal face $h - c$ is $(a, 0)$. Necessarily we have $a > 0$. We denote by $\gamma'$ the face with vertices $(a, 0)$ and $(-q, 1)$. In that case we have $\mathcal{N}(h-c)^+ = \{(a, 0), \gamma'\}$. The face polynomial $(h - c)_{\gamma'}$ is equal to $\alpha x^{-q} (y - \mu x^{a+q})$. The Newton transformation $(h - c)_{\sigma_{(a+q,\mu)}}$ is a base
case 2.38 or 2.39 with $M = -a < 0$ and $m = 1$. Then, we have
\[
(S_{h-c}(y), q', q', \mu))v \neq 0) (0, 0, 0) = -[x^a y : \mathbb{G}^2_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}].
\]
By Theorem 2.6 we have
\[
(S_{h-c,x} - (0) (0, 0) = [x^a : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}]
\]
\[
+ \left[ x^m \frac{y}{\mu} x^a + q : \mathbb{G}^2_m \setminus (y = \mu x^a + q) \to \mathbb{G}_m, \sigma_{\mathbb{G}_m} \right]
\]
\[
- \left[ x^a y : \mathbb{G}^2_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m} \right]
\]
\[
+ \sum_{\gamma' \in N(h-c) \setminus \{\gamma, \gamma\}} \sum_{\mu \in R_{\gamma'}} \left( S_{h-c}(y), q', q', \mu \right) v \neq 0 \right) (0, 0, 0)
\]
Using similar ideas of the end of the proof of Example 2.39, we have
\[
[x^a : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}] + [x^m \frac{y}{\mu} x^a + q : \mathbb{G}^2_m \setminus (y = \mu x^a + q) \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}]
\]
\[-[x^a y : \mathbb{G}^2_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}] = 0
\]
Furthermore, for any face $\gamma''$ in $N(h-c) \setminus \{\gamma, \gamma\}$, for each associated Newton transformation $\sigma_{\gamma''}(p', q', \mu)$, the polynomial $(h-c)_{\gamma''}(p', q', \mu)$ belongs to $k[x, y] \setminus k[x, y]$ and $c$ does not belong to $B_{h-c}^{\text{Newton}}$. Then, applying the induction we obtain $(S_{h-c}(y), q', q', \mu) v \neq 0) (0, 0, 0) = 0$ and we conclude that $(S_{h-c,x} - (0) (0, 0) = 0$.

\[\square\]

**Proposition 3.36** For any polynomial $f \in k[x, y]$, we have the inclusion $B_{f,1}^{\text{not}} \subset B_{f,1}^{\text{Newton}}$ and $B_{f,1}^{\text{not}}$ is finite.

**Proof** We assume $c \notin B_{f,1}^{\text{Newton}}$ and prove that $S_{f,c} = 0$ using formulas (3.14), (3.17) and (3.19).

- Assume $c \neq f(0, 0)$. We remark first that the point $(0, 0)$ belongs to the support of $f - c$, then we have $\varepsilon_{\gamma} = 0$ for any face $\gamma$ in $N_h(f - c) = S_{f,c}$. Indeed:
  - each one-dimensional face $\gamma$ not contained in a coordinate axis is supported by a line of equation $ap + bq = N$ with $N > 0$, and $(p, q)$ the primitive exterior normal vector to $N_{\infty}(f)$. Then applying Proposition 3.49, we obtain that $\varepsilon_{\gamma} = 0$.
  - each one dimensional face $\gamma$ contained in a coordinate axis, has a dual cone $C_{\gamma}$ which does not intersect $\Omega$, then by Proposition 3.49 $\varepsilon_{\gamma} = 0$.
  - for any face $\gamma$ which is not horizontal, not vertical, and not in $N_{0,0}(f)$, $N_{0,0}(f)$, and $N_{\infty,\infty}(f)$, the dual cone $C_{\gamma}$ does not intersect $\Omega$, then by Proposition 3.49 $\varepsilon_{\gamma} = 0$.
  - As $(0, 0)$ belongs to the convex polygon $N_{\infty}(f - c)$, using Remark 3.24 we have $\varepsilon_{\gamma} = 0$ for any zero dimensional face of $N_{\infty}(f - c)$.

Then, we have
\[
S_{f,c} = \sum_{\gamma \in N_h(f - c)} \sum_{\mu \in R_{\gamma}} (S_{h,c}(y), q', q', \mu) v \neq 0 \right) (0, 0, 0).
\]
• If \( \gamma \) is a dicritical face at infinity, then the face polynomial \((f - c)_{c}\) is equal to \(P_{c}(-y^{d}x^{d}y^{p}) - c \) (or \(P_{c}(-y^{d}x^{d}y^{p}) - c \) with \(P_{c}\) in \(k[s]\). As \(c\) is a Newton generic value, all the roots of the polynomial \(P_{c}(s) - c\) have multiplicity one, then for any root \(\mu\), the Newton transform \((f - c)_{c}\sigma(p,q,\mu)\) belongs to \(k[x, y]\), and is a base case of type 2.38 or 2.39 with \(M = 0\) and \(m = 1\). Then \((S_{f(\cdots)}(c, y) \neq 0)\) for each root \(\mu\).

• If \(\gamma\) is not a critical face, then the face polynomial \((f - c)_{c}\) is equal to the face polynomial \(f_{c}\). Furthermore, it is supported by a line of equation \(\alpha p + \beta q = N_{\gamma}\) with \((p, q)\) the exterior normal to the Newton polygon \(N/(f - c)\) equal to \(N_{\gamma}(f)\) and \(N_{\gamma} > 0\). Then, for each root \(\mu\) of the face polynomial \((f - c)_{c}\) = \(f_{c}\), the Newton transform we have \((f - c)_{c}\sigma(p,q,\mu)\) = \(f_{c}\) and \(f_{c}(p,q,\mu)\) belongs to \(k[x, y]\). Furthermore, as \(c\) does not belong to \(B_{f_{c}}\), \(c\) does not belong to \(B_{f_{c}}\) then applying Proposition 3.35 we get \((S_{f(\cdots)}(c, y) \neq 0)(0,0)) = 0\).

Then we conclude that \(S_{f(\cdots)}(c, y) = 0\).

- Assume \(c = f(0,0)\). In that case, \(c\) does not belong to the discriminant of \(f\), the point \((0,0)\) is not a critical point of \(f - c\), then \(f - c\) has a monomial \(x\) or \(y\). Assume for instance \(x\) is a monomial of \(f - c\) (the case \(y\) is a monomial is similar). We denote by \(\gamma\) the dicritical face of \(f = c\) not contained in the coordinate axes \(R_{>0}(1, 0)\).

• Assume \(\gamma\) not contained in the coordinate axes \(R_{>0}(0,1)\). As \(c\) is a generic Newton value for \(f\), this face is smooth, the face polynomial \((f - c)_{c}\gamma(x, y)\) is equal to \(P_{c}(x^{d}y) - c\) with \(P_{c}(s) - c\) a polynomial with simple roots. Then, \(c\) has a monomial \(x^{d}y\). Considering, the monomials \(x\) and \(x^{d}y\) we observe that each one dimensional face \(\gamma' \neq \gamma\) is supported by a line of equation \(\alpha p' + \beta q' = N_{\gamma'}\) with \((p', q')\) the exterior normal vector and \(N_{\gamma'} > 0\). Then, for each root \(\mu\) of the face polynomial \((f - c)_{c}\gamma',\) for each Newton transformation \(\sigma(p',q',\mu)\), the Newton transform \((f - c)_{c}\sigma(p',q',\mu)\) belongs to \(k[x^{-1}, x, y]k[x, y]\) and \(c\) does not belong to \(B_{f_{c}}\) because \(c\) does not belong to \(B_{f_{c}}\).

Then applying Proposition 3.35 we get \((S_{f(\cdots)}(c, y) \neq 0)\) for each root \(\mu\). Then, we conclude that \(S_{f(\cdots)}(c, y) = 0\).

• Assume \(\gamma\) contained in the coordinate axes \(R_{>0}(0,1)\). The polynomial \(f\) has a face \((0, b)\). Using the monomial \(y^{b}\) and \(x\) we observe that each one dimensional face \(\gamma'\) of \(N/(f - c)\) \(\gamma/(f - c)\) is supported by a line of equation \(\alpha p' + \beta q' = N_{\gamma'}\) with \((p', q')\) the exterior normal vector and \(N_{\gamma'} > 0\). Using the same ideas as in the previous point, we obtain \(S_{f(\cdots)}(c, y) = 0\).

\[\Box\]

### 3.3 Complement of the proof of theorems 3.8 and 3.23

Let \(f\) be a polynomial in \(k[x, y]\) not in \(k[x]\) or \(k[y]\), we denote by \(d_{x}\) and \(d_{y}\) the degrees of \(f\) in variables \(x\) and \(y\).

#### 3.3.1 Compactification

In the following, we consider the compactification \(X, i, \hat{f}\) of \(f\) with \(X\) the algebraic variety

\[
X = \{(x_{0} : x_{1}), (y_{0} : y_{1}), (z_{0} : z_{1}) \in (\mathbb{P}^{1}k)^{3} | z_{0}x_{0}^{d_{x}}y_{0}^{d_{y}}f(x_{0}^{1}, y_{0}^{1}) = z_{1}x_{0}^{d_{x}}y_{0}^{d_{y}} \}
\]

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i and j are the following open dominant immersions and \( \hat{f} \) is the following projection which is proper

\[
i : \mathbb{A}_k \to X \quad \text{and} \quad j : \mathbb{A}_k \to \mathbb{P}^1_k \quad \text{such that} \quad f : \mathbb{P}^1_k \to \mathbb{P}^1_k \quad \text{and} \quad a \mapsto [1 : a], \quad (x_0 : x_1), (y_0 : y_1), (z_0 : z_1) \mapsto [z_0 : z_1].
\]

**Remark 3.37** We use notations of Definition 3.1 of the closed subset at infinity \( X_\infty \) and values \( \infty \) and \( a \). The fiber

\[
\hat{f}^{-1}(\infty) = \{(0 : 1), (y_0 : y_1), (0 : 1)\} \in X \cup \{(y_0 : y_1) \in \mathbb{P}^1 \}
\]

is covered by three charts:

\[
\hat{f}^{-1}(\infty) = (\hat{f}^{-1}(\infty) \cap \{x_1 \neq 0, y_1 \neq 0\}) \cup (\hat{f}^{-1}(\infty) \cap \{x_0 \neq 0, y_1 \neq 0\}) \cup (\hat{f}^{-1}(\infty) \cap \{x_1 \neq 0, y_0 \neq 0\}).
\]

### 3.3.2 The motivic zeta function \( Z_\delta^{\hat{f}^\varepsilon, j(\mathbb{A}^2_k)}(T) \)

Let \( \varepsilon \) be in \([\pm]\). If “\( \varepsilon = +\)” we simply write \( \hat{f}^+ \), this function extends \( f \) and is equal to \( z_1/z_0 \). If “\( \varepsilon = -\)” we simply write \( 1/\hat{f}^- \) for \( \hat{f}^- \), this function extends \( 1/f \) and is equal to \( z_0/z_1 \). For any \( n \geq 1 \), for any \( \delta \geq 1 \), we consider

\[
X_n^\delta(\hat{f}^\varepsilon) = \{ \varphi(t) \in \mathcal{L}(X) \mid \varphi(0) \in X_\infty, \text{ ord } \varphi(I_{X_\infty}) \leq n\delta, \text{ ord } \hat{f}^\varepsilon(\varphi(t)) = n \}
\]

endowed with the map \( \hat{f}^\varepsilon \) to \( \mathbb{G}_m \). Following Definition 2.5 we denote

\[
Z_\delta^{\hat{f}^\varepsilon, j(\mathbb{A}^2_k)}(T) = \sum_{n \geq 1} \text{mes}(X_n^\delta(\hat{f}^\varepsilon))T^n \in \mathcal{M}_{\mathbb{G}_m}[[T]]
\]

and by Sects. 3.1.2 and 3.2.3, we have \( S_{f, \infty} = -\lim_{T \to \infty} Z_\delta^{1/\hat{f}^\varepsilon, j(\mathbb{A}^2_k)}(T) \) and \( S_{f, 0} = -\lim_{T \to \infty} Z_\delta^{\hat{f}^\varepsilon, j(\mathbb{A}^2_k)}(T) \).

### 3.3.3 Description of arcs of \( X_n^\delta(\hat{f}^\varepsilon) \)

**Notation 3.38** An arc \( \varphi \) in \( \mathcal{L}(X) \) has the form

\[
\varphi(t) = ([x_0(t) : x_1(t)], [y_0(t) : y_1(t)], [z_0(t) : z_1(t)])
\]

where coordinates are formal series in \( k[[t]] \) satisfying the equation

\[
z_0(t)x_0(t)^{d_2}y_0(t)^{d_2}f \left( \frac{x_1(t)}{x_0(t)}, \frac{y_1(t)}{y_0(t)} \right) = z_1(t)x_0(t)^{d_2}y_0(t)^{d_2}
\]

and such that none of the couples \( (x_0(0), x_1(0)), (y_0(0), y_1(0)) \) and \( (z_0(0), z_1(0)) \) is equal to \( (0, 0) \). In the following, we only work with arcs not contained in the closed subset \( X \setminus i(\mathbb{G}_m^2) \), namely such that the orders of \( x_0(t), x_1(t), y_0(t), y_1(t) \) are finite. The set of arcs contained in \( X \setminus i(\mathbb{G}_m^2) \) has motivic measure zero. For an arc \( \varphi \) as above, we will denote

\[
x(\varphi) = x_1(t)/x_0(t) \quad \text{and} \quad y(\varphi) = y_1(t)/y_0(t).
\]

With these notations, we have \( \text{ord } \hat{f}^\varepsilon(\varphi) = \text{ord } f^\varepsilon(x(\varphi), y(\varphi)) = \varepsilon \text{ord } (z_1(t)/z_0(t)) \).

**Remark 3.39** Let \( \delta > 0 \) and \( n \geq 1 \). Let \( \varphi \) be an arc in \( X_n^\delta(\hat{f}^\varepsilon) \). Its origin \( \varphi(0) \) belongs to \( X_\infty \), then the product \( x_0(0)y_0(0) \) is equal to 0 and we consider the following three cases:
1. if $x_0(0) = y_0(0) = 0$ then the arc can be written as
   \[
   \varphi(t) = ([A(t) : 1], [B(t) : 1], [z_0(t) : z_1(t)])
   \]  
   with $\text{ord } A(t) > 0, \text{ord } B(t) > 0$ and $\text{ord } f^e(1/A(t), 1/B(t)) = n$. Remark that $\text{ord } x(\varphi) = -\text{ord } A(t)$, $\text{ord } y(\varphi) = -\text{ord } B(t)$. The origin $\varphi(0) = ([0 : 1], [0 : 1], [z_0(0) : z_1(0)])$ belongs to the chart $x_1y_1 \neq 0$. As the equation of $X_\infty$ around $\varphi(0)$ is $x_0y_0 = 0$, we have the equality $\text{ord } \varphi^* \left( \mathcal{I}_{X_\infty} \right) = \text{ord } A(t) + \text{ord } B(t)$. 

2. if $x_0(0) \neq 0$ and $y_0(0) = 0$, then the arc can be written as
   \[
   \varphi(t) = ([1 : A(t)], [B(t) : 1], [z_0(t) : z_1(t)])
   \]  
   with $\text{ord } A(t) > 0, \text{ord } B(t) > 0$ and $\text{ord } f^e(A(t), 1/B(t)) = n$. Remark that $\text{ord } x(\varphi) = -\text{ord } A(t)$, $\text{ord } y(\varphi) = -\text{ord } B(t)$. The origin $\varphi(0) = ([1 : A(0)], [0 : 1], [z_0(0) : z_1(0)])$ belongs to the chart $x_0y_1 \neq 0$. As the equation of $X_\infty$ around $\varphi(0)$ is $y_0 = 0$, we have the equality $\text{ord } \varphi^* \left( \mathcal{I}_{X_\infty} \right) = \text{ord } B(t)$.

3. if $x_0(0) = 0$ and $y_0(0) \neq 0$, then the arc can be written as
   \[
   \varphi(t) = ([A(t) : 1], [1 : B(t)], [z_0(t) : z_1(t)])
   \]  
   with $\text{ord } A(t) > 0, \text{ord } B(t) > 0$ and $\text{ord } f^e(1/A(t), B(t)) = n$. Remark that $\text{ord } x(\varphi) = -\text{ord } A(t)$, $\text{ord } y(\varphi) = -\text{ord } B(t)$. The origin $\varphi(0) = ([0 : 1], [1 : B(0)], [z_0(0) : z_1(0)])$ belongs to the chart $x_1y_0 \neq 0$. As the equation of $X_\infty$ around $\varphi(0)$ is $x_0 = 0$, we have the equality $\text{ord } \varphi^* \left( \mathcal{I}_{X_\infty} \right) = \text{ord } A(t)$.

**Notation 3.40** We denote by $\Omega$ the set $\mathbb{Z}^2 \setminus (\mathbb{Z}_{\leq 0})^2$.

### 3.3.4 Decomposition of the motivic zeta function $Z^\mathbf{G}_{\mathbf{f}, \mathbf{i}(\beta^2)}(T)$

**Remark 3.41** We write $f(x, y) = \sum_{(a, b) \in \mathbb{N}^2} c_{a, b}x^a y^b$. Let $(\alpha, \beta)$ be in $\Omega$. By Lemma 1.26, the restriction to $\overline{\mathcal{A}}(f)$ (Definition 1.27) of the linear form $l(\alpha, \beta)$ equal to $((\alpha, \beta), \cdot)$ has a maximum $m(\alpha, \beta)$ along a face denoted by $\gamma(\alpha, \beta)$ in $\overline{\mathcal{N}}(f)$. Then, we denote
   \[
   \tilde{f}_{\gamma(\alpha, \beta)}(x, y, u) := f_{\gamma(\alpha, \beta)}(x, y) + \sum_{(a, b) \notin \gamma(\alpha, \beta)} c_{a, b}u^{m(\alpha, \beta) - (\alpha a + \beta b)} x^a y^b
   \]  
   where
   \[
   f_{\gamma(\alpha, \beta)}(x, y) = \sum_{(a, b) \in \gamma(\alpha, \beta)} c_{a, b}x^a y^b.
   \]

Let $P(t)$ and $Q(t)$ be invertible formal series. Then, we obtain the equalities
   \[
   f \left( \frac{P(t)}{t^\alpha}, \frac{Q(t)}{t^\beta} \right) = t^{-m(\alpha, \beta)} \tilde{f}_{\gamma(\alpha, \beta)}(P(t), Q(t), t)
   \]  
   and
   \[
   \varepsilon \text{ord } f^e \left( \frac{P(t)}{t^\alpha}, \frac{Q(t)}{t^\beta} \right) = -m(\alpha, \beta) + \text{ord } \tilde{f}_{\gamma(\alpha, \beta)}(P(t), Q(t), t).
   \]  
   In particular, by equation (3.31), for an arc $\varphi$ in $\mathcal{L}(X)$, writing $x(\varphi) = P(t)/t^\alpha$ and $y(\varphi) = Q(t)/t^\beta$, we have:
   - if $f_{\gamma(\alpha, \beta)}(x(\varphi), y(\varphi)) \neq 0$ then $\varepsilon \text{ord } \tilde{f}_{\gamma(\alpha, \beta)}(\varphi(t)) = -m(-\text{ord } x(\varphi), -\text{ord } y(\varphi))$,
   - if $f_{\gamma(\alpha, \beta)}(x(\varphi), y(\varphi)) = 0$ then $\varepsilon \text{ord } \tilde{f}_{\gamma(\alpha, \beta)}(\varphi(t)) > -m(-\text{ord } x(\varphi), -\text{ord } y(\varphi))$. 

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and by Remark 3.39, we have ord $\varphi^*(I_{F_{\infty}}) = c(\alpha, \beta)$ with

$$c(\alpha, \beta) = \begin{cases} 
\alpha & \text{if } \alpha > 0 \text{ and } \beta \leq 0 \\
\beta & \text{if } \beta > 0 \text{ and } \alpha \leq 0 \\
\alpha + \beta & \text{if } \alpha > 0 \text{ and } \beta > 0 
\end{cases} \quad (3.36)$$

**Remark 3.42** In this remark we assume “$\varepsilon = -$”. By Remark 3.6 we can assume that $(0, 0)$ belongs to the support of $f$ and in particular $N_{\infty}(f) = \overline{N}(f)$. Then, for any $(\alpha, \beta)$ in $\Omega = \mathbb{Z}^2 \setminus (\mathbb{Z} \leq 0)^2$, $m(\alpha, \beta) \geq 0$. Furthermore, if $m(\alpha, \beta) > 0$ then the face $\gamma(\alpha, \beta)$ belongs to $N_{\infty}(f)^{\circ}$, the set of faces of $N_{\infty}(f)$ which does not contain 0. For instance, for any arc $\varphi$ in $\mathcal{L}(X)$, with $-(\text{ord } x(\varphi), \text{ord } y(\varphi))$ in $\Omega$, if $\text{ord } 1/f(\varphi) > 0$ then the corresponding face $\gamma(-\text{(ord } x(\varphi), \text{ord } y(\varphi)))$ belongs to $N_{\infty}(f)^{\circ}$.

**Notation 3.43** Let $\varepsilon$ be in $\{\pm\}$ and $\gamma$ be a face of $\overline{N}(f)$. By Remarks 3.39 and 3.41, we introduce

- Let $C_{\gamma}$ be the dual cone of $\gamma$. We consider the following polyhedral rational cones of $\mathbb{R}^2$ and $\mathbb{R}^3$

$$C_{\gamma,=}^{\delta,=} := \{ (\alpha, \beta) \in C_{\gamma} \mid c(\alpha, \beta) \leq -\varepsilon m(\alpha, \beta) \delta \} \quad (3.37)$$

$$C_{\gamma,=}^{\delta,=} := \{ (n, (\alpha, \beta)) \in \mathbb{R}_0 \times C_{\gamma} \mid c(\alpha, \beta) \leq n\delta, -m(\alpha, \beta) < \varepsilon n \} \quad (3.38)$$

- For any integer $n$, we consider

$$X_{n,\gamma}^{\delta,=}(\hat{f}^{=}) = \{ \varphi \in X_n^{\delta}(\hat{f}^{=}) \mid -\text{(ord } x(\varphi), \text{ord } y(\varphi)) \in C_{\gamma} \} \quad (3.39)$$

endowed with its structural map $\overline{\alpha e} \hat{f}^{=}$ to $\mathbb{G}_m$. We consider the motivic zeta function in $\mathcal{M}_{\mathbb{G}_m}[[T]]$

$$Z_{\gamma,=}^{\delta,=}(T) = \sum_{n \geq 1} \text{mes} (X_{n,\gamma}^{\delta}(\hat{f}^{=}))T^n.$$ 

For any face $\gamma$ in $N(f) \setminus \{\gamma_{H, H}\}$, the dual cone $C_{\gamma}$ does not intersect $\alpha$, then $Z_{\gamma,=}^{\delta,=}(T) = 0$.

- We assume that the face $\gamma$ is not contained in a coordinate axis. For any integer $n \geq 1$ and $\delta > 0$, for any $(\alpha, \beta)$ in $C_{\gamma} \cap \Omega$, we consider the arc spaces endowed with their structural map $\overline{\alpha e} \hat{f}^{=}$ to $\mathbb{G}_m$

$$X_{n,(\alpha,\beta)}^{=}:= \{ \varphi \in X_n^{=} \mid -(\text{ord } x(\varphi), \text{ord } y(\varphi)) = (\alpha, \beta) \},$$

$$X_{n,(\alpha,\beta)}^{=}:= \{ \varphi \in X_n, (\alpha, \beta)(\hat{f}^{=}) \mid f_{\gamma}(\overline{\alpha e} x(\varphi), \overline{\alpha e} y(\varphi)) \neq 0 \},$$

and

$$X_{n,(\alpha,\beta)}^{\leq}:= \{ \varphi \in X_{n,(\alpha, \beta)}(\hat{f}^{=}) \mid f_{\gamma}(\overline{\alpha e} x(\varphi), \overline{\alpha e} y(\varphi)) = 0 \}.$$ 

The sets $\Omega \cap C_{\gamma,=}^{\delta,=}$ and $C_{\gamma,=}^{\delta,=} \cap (\mathbb{N}) \times \Omega$ are finite for any integer $n$, and we introduce

$$Z_{\gamma,=}^{\delta,=}(T) = \sum_{n \geq 1} \sum_{(\alpha, \beta) \in C_{\gamma,=}^{\delta,=} \cap \Omega} \text{mes} (X_{m(\alpha, \beta), (\alpha, \beta)}^{=}^{\delta,=}(\hat{f}^{=}))T^n$$

and

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\[ Z_{\gamma,\varepsilon}^{\delta,\prec}(T) = \sum_{n \geq 1} \sum_{(\alpha, \beta) \in \Omega} \text{mes} (X_{n,(\alpha,\beta)}(f_\varepsilon)) T^n. \]

We deduce from Remark 3.41 and from the additivity of the measure the following proposition:

**Proposition 3.44** For any \( \delta > 0 \), for any \( \varepsilon \) in \( \{\pm\} \), we have the decomposition

\[ Z_{f_\varepsilon, i(\Delta_\delta^\varepsilon)}^{\delta}(T) = \sum_{\gamma \in N(f)} Z_{\gamma,\varepsilon}^{\delta}(T). \]

For any face \( \gamma \) which is not contained in a coordinate axes we have

\[ Z_{\gamma,\varepsilon}^{\delta}(T) = Z_{\gamma,\varepsilon}^{\delta,=}(T) + Z_{\gamma,\varepsilon}^{\delta,\prec}(T). \quad (3.40) \]

**Remark 3.45** If \( \gamma \) is the origin, then \( Z_{\gamma,\varepsilon}^{\delta}(T) = 0 \) because for any \( n \), \( X_{n,\gamma}^{\delta,\varepsilon} \) is empty.

### 3.3.5 The formal series \( Z_{\gamma,\varepsilon}^{\delta} \) for \( \gamma \) contained in a coordinate axis

**Proposition 3.46** (Case “\( \varepsilon = - \)”) Let \( \gamma \) be a face \((0, b_0)\) in \( N_\infty(f) \). There is \( \delta_0 \) such that for any \( \delta \geq \delta_0 \), the motivic zeta function \( Z_{\gamma,-}^{\delta}(T) \) is rational and

\[- \lim Z_{\gamma,-}^{\delta}(T) = [y^{-b_0} : \mathbb{G}_m \to \mathbb{G}_m]. \]

More precisely, writing \( C_\gamma = \mathbb{R}_{>0}(-1, 0) + \mathbb{R}_{>0} \eta \) with \( \eta \) in \( \mathbb{Z} \times \mathbb{N}^2 \), the dual cone of \( \gamma \), we have

- if \( \eta = (p, q) \) with \( p \leq 0 \) and \( q > 0 \), then we have

\[ Z_{\gamma,-}^{\delta}(T) = [1/y^{b_0} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}] R_\gamma^{\delta}(T), \]

with

\[ R_\gamma^{\delta}(T) = -1 + \frac{1}{1 - \mathbb{L}^{p-q}T^{b_0}} \sum_{r=0}^{q-1} \mathbb{L}^{-r-\lfloor -p/r \rfloor} T^{rb_0}. \quad (3.41) \]

- if \( \eta = (p, q) \) with \( p > 0 \) and \( q > 0 \), then we have

\[ Z_{\gamma,-}^{\delta}(T) = [1/y^{b_0} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}] R_\gamma^{\delta}(T), \]

with

\[ R_\gamma^{\delta}(T) = \frac{\mathbb{L}^{-1} T^{b_0}}{1 - \mathbb{L}^{-1} T^{b_0}} \]

\[ + (\mathbb{L} - 1) \left( \sum_{(a_0, b_0) \in P_{\gamma,\prec,\succ}} \frac{\mathbb{L}^{-a_0-b_0} T^{b_0} T^{b_0}}{(1 - \mathbb{L}^{-1} T^{b_0})(1 - \mathbb{L}^{-p-q} T^{b_0})} + \frac{T^{b_0} \mathbb{L}^{-1}}{1 - \mathbb{L}^{-1} T^{b_0}} \right) \quad (3.42) \]

with \( P_{\gamma,\prec,\succ} = \{(0, 1)(0, 1)+0, 1\} \cap \mathbb{Z}^2 \).

Similarly, let \( \gamma \) be a face \((a_0, 0)\) in \( N_\infty(f) \). There is \( \delta_0 \) such that for any \( \delta \geq \delta_0 \), the motivic zeta function \( Z_{\gamma,-}^{\delta}(T) \) is rational and

\[- \lim Z_{\gamma,-}^{\delta}(T) = [x^{-a_0} : \mathbb{G}_m \to \mathbb{G}_m]. \]

**Proof** The dual cone \( C_\gamma \) of \( \gamma \) is equal to \( \mathbb{R}_{>0}(-1, 0) + \mathbb{R}_{>0} \eta \) with \( \eta \) a primitive vector in \( \mathbb{Z} \times \mathbb{Z}_{>0} \).
– Assume that \( \eta \) belongs to \( \mathbb{Z}_{\leq 0} \times \mathbb{Z}_{> 0} \). Writing \( \eta = (p, q) \) with \( p \leq 0 \) and \( q > 0 \). We observe that
\[
C_\gamma = \{(\alpha, \beta) \in \mathbb{R}^2 \mid \beta > 0, \alpha < \beta(p/q) = -\beta |p/q| \} = C_\gamma \cap (\mathbb{R}_{< 0} \times \mathbb{R}_{> 0})
\]
and we conclude that for any \( n \geq 1 \),
\[
X_{n, \gamma}^\delta(1/\hat{f}) = \{\varphi \in X_{n}^\delta(1/\hat{f}) \mid \text{ord } x(\varphi) \geq [\text{ord } y(\varphi) |p/q|] + 1\}.
\]
By Remark 3.41, for any arc \( \varphi \) in \( X_{n, \gamma}^\delta(1/\hat{f}) \) we have \( \text{ord } 1/\hat{f}(\varphi) = -\text{ord } y(\varphi)b_0 \). Assume \( \delta > 1/b_0 \), then we have the inequality \( \text{ord } \varphi^*(X_{\infty}) = -\text{ord } y(\varphi) \leq -\delta b_0 \text{ord } y(\varphi) \) and \( Z_{\gamma, \varphi}(T) = \sum_{k \geq 1} \text{mes } (X_k^\delta(1/\hat{f}))T^{b_0k} \) with for any \( k \geq 1 \),
\[
X_k^\delta(1/\hat{f}) = \{\varphi \in \Lambda(X) \mid -\text{ord } y(\varphi) = k, \text{ ord } x(\varphi) \geq [k |p/q|] + 1\}.
\]
As in the proof of formula (2.24) in Proposition 2.31, we get \( Z_{\gamma, \varphi}(T) = [1/y^{b_0} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}]R_{\gamma, (\leq, >)}(T) \), with
\[
R_{\gamma, (\leq, >)}(T) = -1 + \frac{1}{1 - L^{1-|p/q|}T^{b_0}} \sum_{r=0}^{q-1} L^{-r-1/|p/q|}T^{r+b_0} \xrightarrow{T \to \infty} -1 \tag{3.43}
\]
– Assume that \( \eta \) belongs to \( \mathbb{Z}_{> 0} \times \mathbb{Z}_{> 0} \). Writing \( \eta = (p, q) \) with \( p > 0 \) and \( q > 0 \). We observe that
\[
C_\gamma = (\mathbb{R}_{< 0} \times \mathbb{R}_{> 0}) \cup (\mathbb{R}_{> 0}(0, 1) + \mathbb{R}_{\geq 0}\eta)
\]
In particular we denote
\[
C_{\gamma, (\leq, >)} := C_\gamma \cap (\mathbb{R}_{< 0} \times \mathbb{R}_{> 0}) = \mathbb{R}_{< 0} \times \mathbb{R}_{> 0} \quad \text{and}
\]
\[
C_{\gamma, (\geq, >)} := C_\gamma \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}_{> 0}) = \mathbb{R}_{> 0}(0, 1) + \mathbb{R}_{\geq 0}\eta. \tag{3.44}
\]
We decompose the zeta function following these quadrants \( Z_{\gamma, \varphi}(T) = Z_{\gamma, (\leq, >)}(T) + Z_{\gamma, (\geq, >)}(T) \) with
\[
Z_{\gamma, (\leq, >)}(T) = \sum_{n \geq 1} \text{mes } (X_{n, (\leq, >)}^\delta(1/\hat{f}))T^n \quad \text{and}
\]
\[
Z_{\gamma, (\geq, >)}(T) = \sum_{n \geq 1} \text{mes } (X_{n, (\geq, >)}^\delta(1/\hat{f}))T^n
\]
with for any \( n \geq 1 \), \( X_{n, (\leq, >)}(1/\hat{f}) = \{\varphi \in X_{n, \gamma}^\delta(1/\hat{f}) \mid -(\text{ord } x(\varphi), \text{ ord } y(\varphi)) \in C_{\gamma, (\leq, >)}\} \), and
\[
X_{n, (\geq, >)}(1/\hat{f}) = \{\varphi \in X_{n, \gamma}^\delta(1/\hat{f}) \mid -(\text{ord } x(\varphi), \text{ ord } y(\varphi)) \in C_{\gamma, (\geq, >)}\}.
\]
• It follows from formula (3.43) that for \( \delta > 1/b_0 \), we have \( Z_{\gamma, (\leq, >)}(T) = [1/y^{b_0} : \mathbb{G}_m \to \mathbb{G}_m, \sigma_{\mathbb{G}_m}]R_{\gamma, (\leq, >)}(T) \) with
\[
R_{\gamma, (\leq, >)}(T) = \frac{L^{1} - 1}{1 - L^{1}T^{b_0}} \xrightarrow{T \to \infty} -1. \tag{3.45}
\]
• By Remark 3.39, for any arc \( \varphi \) in \( X_{n, (\geq, >)}^\delta(1/\hat{f}) \) we have the equality
\[
\text{ord } \varphi^*(I_{\infty}) = -\text{ord } x(\varphi) - \text{ord } y(\varphi).
\]
For any \( \delta \) satisfying \( \delta > \delta_3 = \max\left(\frac{1}{b_0}, \frac{p+q}{b_0q}\right) = \frac{p+q}{b_0q} \), we have the inequality \( \text{ord } \varphi^*(I_{\infty}) \leq \delta \text{ord } (1/\hat{f})(\varphi) = -\delta b_0 \text{ ord } y(\varphi) \), for any arc \( \varphi \) in \( \Lambda(X) \) with
By symmetry, we obtain a similar result for a zero-dimensional face \( \gamma \).

Let \( \varepsilon = \) \( \frac{1}{2} \). Then, we have the equality \( X_{(\alpha, \beta)} = \{ \varphi \in \mathcal{L}(X) \mid -\text{ord } x(\varphi), \text{ord } y(\varphi) = (\alpha, \beta) \}. \)

It follows from standard computations of motivic invariants at infinity of plane curves that \( \text{mes } (X_{(\alpha, \beta)}) = \mathbb{L}^{-\alpha-\beta}[y^{-b_0} : \mathbb{G}_2^2 \to \mathbb{G}_m] = \mathbb{L}^{-\alpha-\beta}(\mathbb{L}^{-1}[y^{-b_0} : \mathbb{G}_m \to \mathbb{G}_m]). \)

Then, we have the equality \( Z^{\delta}_{\gamma, (+)}(T) \) as the formal series \( \sum_{n \geq 1} \sum_{\beta \in C_{\gamma, (+)}} \mathbb{L}^{-\alpha-\beta}T^{b_0} \).

In particular applying Lemma 2.1 and using \( \mathcal{P}_\gamma = \{(0, 1)(0, 1), 0 \} \cap \mathbb{Z}^2 \) we obtain that
\[
R_{\gamma, (+)}(T) = \sum_{(\alpha, \beta) \in \mathcal{P}_\gamma} \frac{\mathbb{L}^{-\alpha-\beta}T^{b_0}}{(1 - \mathbb{L}^{-1}T^{b_0})} + \frac{T^{b_0} \mathbb{L}^{-1}}{1 - \mathbb{L}^{-1}T^{b_0}} \quad \text{as } \gamma \to \infty
\] (3.46).

By symmetry, we obtain a similar result for a zero-dimensional face \( (a_0, 0) \).

**Proposition 3.47** (Case “\( \varepsilon = + \)”)

Let \( \gamma \) be a face \((a, 0)\) or \((0, b)\) of \( \mathcal{N}(f) \). There is \( \delta_0 \) such that for any \( \delta \geq \delta_0 \), the formal series \( Z^{\delta, +}_{\gamma, +}(T) \) is rational and equal to
\[
Z^{\delta, +}_{\gamma, +}(T) = [x^a : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_y] \quad \text{or} \quad Z^{\delta, +}_{\gamma, +}(T) = [y^b : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_y]
\]
with \( R^{\delta, +}_{\gamma}(T) \) expressed in (3.48). It admits a limit \( \lim Z^{\delta, +}_{\gamma, +}(T) \) in \( \mathcal{M}_{\mathbb{G}_m^2}^\gamma \) with
\[
- \text{ lim } Z^{\delta, +}_{\gamma, +}(T) = \varepsilon_Y[x^a : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_y] \quad \text{or} \quad \text{ lim } Z^{\delta, +}_{\gamma, +}(T) = \varepsilon_Y[y^b : \mathbb{G}_m^2 \to \mathbb{G}_m, \sigma_y]
\]
with \( \varepsilon_Y = -\chi_c(C^{\delta, +}_\gamma \cap \Omega) \) belongs to \( \{0, -1\} \). More precisely, in the case \( \gamma = (a, 0) \) (the case \( \gamma = (0, b) \) is similar), let \( H_Y = \{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha < 0 \} \) and \( C_\gamma \) be the dual cone of \( \gamma \).

The cone \( C^{\delta, +}_\gamma \), defined in formula (3.37), is included in \( C_\gamma \cap H_Y \), and

- if \( C_\gamma \cap H_Y \cap \Omega = \emptyset \) then \( C^{\delta, +}_\gamma \cap \Omega \) is empty and \( \varepsilon_Y = 0 \),
- otherwise, there is \( \eta = (p, q) \) with \( p < 0 \) and \( q > 0 \) such that \( C_\gamma \cap H_Y \cap \Omega = \mathbb{R}_{>0}(-1, 0) + \mathbb{R}_{>0}\eta \).

There is \( \delta_1 > 0 \) such that for any \( \delta > \delta_1 \), we have \( C^{\delta, +}_\gamma \cap \Omega = \mathbb{R}_{>0}(-1, 0) + \mathbb{R}_{>0}\eta \) and \( \varepsilon_Y = -1 \).

**Proof** Let \( \gamma \) be a face \((a, 0)\) in \( \mathcal{N}(f) \) with \( a > 0 \). The dual cone of \( \gamma \) has dimension 2. Remark that, for any integer \( n \geq 1 \), for any arc \( \varphi \in X^{h, \gamma}_n(f) \), by Remark 3.41 we have \( \text{ord } \hat{f}(\varphi) = \text{ord } X(\varphi) a = n > 0 \) which implies that \( \text{ord } X(\varphi) > 0 \), namely \( -\text{ord } x(\varphi), \text{ord } y(\varphi) \) belongs to \( C_\gamma \cap H_Y \cap \Omega \). If \( C_\gamma \cap H_Y \cap \Omega \) is empty then \( Z^{\delta, +}_{\gamma, +}(T) = 0 \).
Assume $C_{\gamma} \cap H_{\gamma} \cap \Omega = \mathbb{R}_{>0}(-1, 0) + \mathbb{R}_{>0}(p, q)$ with $p < 0$ and $q > 0$. We observe that

$$C_{\gamma} \cap H_{\gamma} \cap \Omega = \{(\alpha, \beta) \in \mathbb{Z}^2 \mid \beta > 0, \alpha < 0, \beta p/q > \alpha\}$$

and we conclude that for any $n \geq 1$,

$$X_{n, \gamma}^{\delta}(\hat{f}) = \{\varphi \in X_{n}^{\delta}(\hat{f}) \mid \text{ord } y(\varphi) < 0, \text{ord } x(\varphi) > 0, |q/p| \text{ord } x(\varphi) > -\text{ord } y(\varphi)\}.$$

For any arc $\varphi$ in $X_{n, \gamma}^{\delta}(\hat{f})$, we have $\varphi(0) = ([1:0], [0:1], [1:0])$ and by formula (3.33) in Remark 3.39, we have

$$\text{ord } \varphi^*(I_{X_{\infty}}) = -\text{ord } y(\varphi) \leq \delta \text{ord } x(\varphi). \quad (3.47)$$

For any $\delta \geq |q/p|/a$, we have

$$X_{n, \gamma}^{\delta}(\hat{f}) = \{\varphi \in C(X) \mid \text{ord } \hat{f}(\varphi) = n, (-\text{ord } x(\varphi), -\text{ord } y(\varphi)) \in C_{\gamma} \cap H_{\gamma} \cap \Omega\}$$

and

$$Z_{\gamma, +}^{\delta}(T) = \sum_{n \geq 1} \text{mes } (X_{n, \gamma}^{\delta}) T^n = \sum_{k \geq 1} \sum_{(-k, l) \in C_{\gamma} \cap H_{\gamma} \cap \Omega} \text{mes } (X_{k, l}) T^k$$

with for any

$$(-k, l) \in C_{\gamma} \cap H_{\gamma} \cap \Omega, X_{k, l} = \{\varphi \in X_{n, \gamma}^{\delta}(\hat{f}) \mid (-\text{ord } x(\varphi), -\text{ord } y(\varphi)) = (-k, l)\}.$$

By construction of the motivic measure we have $\text{mes } (X_{k, l}) = \mathbb{L}^{-k-l}[x^a : \mathbb{G}_m^2 \rightarrow \mathbb{G}_m, \sigma_{\mathbb{G}_m}]$. Then, by application of Lemma 2.1 we obtain the expression

$$Z_{\gamma, +}^{\delta}(T) = [x^a : \mathbb{G}_m^2 \rightarrow \mathbb{G}_m, \sigma_{\mathbb{G}_m}] R_{\gamma}^{\delta = -}(T),$$

where denoting

$$P = ([0, 1](-1, 0)+[0, 1](p, q)) \cap \mathbb{Z}^2,$$

we have

$$R_{\gamma}^{\delta = -}(T) = \sum_{(k_0, l_0) \in P} \frac{\mathbb{L}^{k_0-l_0} T^{-ak_0}}{(1 - \mathbb{L}^{-1} T^a)(1 - \mathbb{L}^{-q} T^{-ap})} \rightarrow 1 = \chi_\mathcal{E}(C_{\gamma} \cap H_{\gamma} \cap \Omega). \quad (3.48)$$

3.3.6 The formal series $Z_{\gamma, \mathcal{E}}^{\delta, \mathcal{E}}(T)$ for $\gamma$ not contained in a coordinate axes

Proposition 3.48 (Case $\mathcal{E} = -$) We assume $f(0, 0) \neq 0$.

- If $f$ can be written as $f(x, y) = P(x^a y^b)$ with $P$ in $k[s]$ of degree $d$ with $(a, b)$ in $(\mathbb{N}^*)^2$. Let $\gamma$ be the face $(ad, bd)$ of $\mathcal{N}_\infty(f)$. If $\delta > \max(1/(da), 1/(db))$ then we have

$$Z_{\gamma, -}^{\delta, \mathcal{E}}(T) = \left[1/(x^a y^b) : \mathbb{G}_m^2 \rightarrow \mathbb{G}_m, \sigma_{\mathbb{G}_m}\right] R_{\gamma}^{\delta}(T) \quad \text{and}$$

$$-\lim Z_{\gamma, -}^{\delta, \mathcal{E}}(T) = \left[1/(x^a y^b) : \mathbb{G}_m^2 \rightarrow \mathbb{G}_m, \sigma_{\mathbb{G}_m}\right]. \quad (3.49)$$

where $R_{\gamma, -}^{\delta}(T)$ is rational and given in formula (3.66).
Assume \( f \) is not of the previous form. Let \( \gamma \) be a face in \( N_\infty(f)^\circ \) and not contained in a coordinate axes. There is \( \delta_0 > 0 \) such that for any \( \delta > \delta_0 \), we have

\[
Z_{\gamma,-}^\delta(T) = \left[ 1/f_\gamma : \mathbb{G}_m^2 \setminus f_\gamma^{-1}(0) \to \mathbb{G}_m, \sigma_\gamma \right] R_{\gamma,-}^\delta(T) \quad \text{and} \quad -\lim Z_{\gamma,-}^\delta(T) = \varepsilon_\gamma \left[ 1/f_\gamma : \mathbb{G}_m^2 \setminus f_\gamma^{-1}(0) \to \mathbb{G}_m, \sigma_\gamma \right]
\]

with \( \varepsilon_\gamma = -\chi_c(C_{\gamma,-}^\delta \cap \Omega) \) equal to \((-1)^{\dim \gamma + 1}\) if \( \gamma \) is not contained in a face which contains 0 and otherwise is equal to 0, and \( R_{\gamma,-}^\delta(T) \) is rational and given in formula (3.51) (with (3.53), (3.54), (3.56), (3.58), (3.60), (3.62), (3.64)) for zero-dimensional faces and in formula (3.65) for one-dimensional faces.

**Proof** We assume first that \( f(0, 0) \neq 0 \) and \( f \) is not of the form \( P(x^ay^b) \) with \( P \in k[s] \) with \((a, b) \) in \((\mathbb{N}^*)^2 \). Let \( \gamma \) be a face in \( N_\infty(f)^\circ \) not contained in a coordinate axes and \( \delta > 0 \). By assumption \((0, 0) \) does not belong to the face \( \gamma \), then for any \((\alpha, \beta) \) in the dual cone \( C_\gamma \), \( m(\alpha, \beta) > 0 \). Then, by homogeneity of the functions \( m \) and \( c \) defined in formula (3.36), there is \( \delta' \) such that for any \( \delta > \delta' \) the cone \( C_{\gamma,-}^\delta \) defined in formula (3.37) is non empty. Indeed, let \((\alpha, \beta) \) be in the dual cone \( C_\gamma \), we have \( m(\alpha, \beta) > 0 \), there is \( \delta' > 0 \) such that \( c(\alpha, \beta) \leq \delta'm(\alpha, \beta) \), then \((\alpha, \beta) \) belongs to \( C_{\gamma,-}^\delta \). Then, for any \( \delta > \delta' \), \( \delta'/\delta(\alpha, \beta) \) belongs to \( C_{\gamma,-}^\delta \). Let \( \delta > \delta' \) and \((\alpha, \beta) \) be an element of \( C_{\gamma,-}^\delta \). By standard arguments on the definition of the motivic measure, it follows from [30, Lemma 3.16] that the motivic measure of \( X_{m(\alpha, \beta), (\alpha, \beta)}(1/f) \) is equal to \( \mathbb{L}^{-|\alpha|-|\beta|} \left[ 1/f : \mathbb{G}_m^2 \setminus f^{-1}(0) \to \mathbb{G}_m, \sigma_{\alpha, \beta} \right] \). By Remark 2.21 (see also [30, Proposition 3.13] for details) this measure does not depend on \((\alpha, \beta) \) in \( C_\gamma \) and we replace \( \sigma_{\alpha, \beta} \) by \( \sigma_\gamma \). Note that, as the face \( \gamma \) is not contained in a coordinate axes, for any integer \( n \), the set of \((\alpha, \beta) \) in \( C_{\gamma,-}^\delta \cap \Omega \) with \( n = m(\alpha, \beta) \) is finite. Indeed, as the face \( \gamma \) is not contained in a coordinate axes, this face has a point \((a_0, b_0) \) in \((\mathbb{N}^*)^2 \), then for any \((\alpha, \beta) \) in \( C_{\gamma,-}^\delta \) we have \( m(\alpha, \beta) = a_0\alpha + b_0\beta \), then the equation and inequation \( n = m(\alpha, \beta) = a_0\alpha + b_0\beta \), \( c(\alpha, \beta) \leq n\delta \) have finitely many solutions in \( C_{\gamma,-}^\delta \cap \Omega \). Then, we have the equality

\[
R_{\gamma,-}^\delta(T) = \sum_{n \geq 1} \sum_{(\alpha, \beta) \in C_{\gamma,-}^\delta \cap \Omega \atop n = m(\alpha, \beta)} \mathbb{L}^{-|\alpha|-|\beta|} T^n.
\]

As \( \gamma \) belongs to \( N_\infty(f) \), we remark that \( C_\gamma \) is included in \( \Omega \). The application \( c \) is linear on each cone \( \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \) with symbols “\( \cdot \)” and “\( \cdot \)” in \( \{>, <, =\} \) and \( \mathbb{R}_{=0} = \{0\} \). Then, we decompose the cone \( C_\gamma \) along the quadrants of \( \mathbb{R}^2 \), as the disjoint union

\[
C_\gamma = \bigcup_{\{?,!\} \in \{<,=,>\}^2 \setminus \{<,=\}^2} C_\gamma \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})
\]

and we have

\[
R_{\gamma,-}^\delta(T) = \sum_{(?,!) \in \{<,=,>\}^2 \setminus \{<,=\}^2} \sum_{(\alpha, \beta) \in C_{\gamma,-}^\delta \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \cap \Omega \atop n = m(\alpha, \beta)} \mathbb{L}^{-|\alpha|-|\beta|} T^n.
\]

(3.51)
By Lemma 2.1, each $R^\varepsilon_{\gamma, (?)}(T)$ is rational and its limit when $T$ goes to infinity is $\chi_e(C_{\gamma, -}^{\delta, \varepsilon} \cap (R_{>0} \times R_{>0}))$. By additivity, we obtain the rational form of $R^\delta_{\gamma, (?)}(T)$ and its limit is $\chi_e(C_{\gamma}^{\delta, \varepsilon})$. In the following, we study the cones $C_{\gamma}^{\delta, \varepsilon}$ and $C_{\gamma, -}^{\delta, \varepsilon}$ in $(R_{>0} \times R_{>0})$.

Assume $\gamma$ is a zero dimensional face equal to $(a_0, b_0)$ in $(N^*)^2$ namely not contained in a coordinate axes. As $f$ is not quasi homogeneous, the face $\gamma$ is the intersection of two one-dimensional faces $\gamma_1$ and $\gamma_2$ with primitive exterior normal vectors $\eta_1$ and $\eta_2$, such that we have the inequality of measure of oriented angles

$$\text{mes}((1, 0), \eta_1) > \text{mes}((1, 0), \eta_2). \quad (3.52)$$

The dual cone of $\gamma$ is $C_\gamma = R_{>0}\eta_1 + R_{>0}\eta_2$ and for any $\delta > 0$, by definition $C_{\gamma, -}^{\delta, \varepsilon} = \{(\alpha, \beta) \in C_\gamma \mid c(\alpha, \beta) \leq m(\alpha, \beta)\delta\}$.

Assume the faces $\gamma_1$ and $\gamma_2$ do not contain the origin. Hence, the vectors $\eta_1$ and $\eta_2$ belong to $\Omega$ and by convexity of the Newton polygon ($\gamma \mid \eta_1 > 0$ and $\gamma \mid \eta_2 > 0$, these vectors belong to the half plane ($\gamma \mid \gamma$ > 0). We remark that for any $\delta > 0$

$$\delta_0 = \max \left( \frac{|(\eta_1 | (1, 0)] + |(\eta_1 | (0, 1)]|}{|\eta_1 | \gamma \rangle}, \frac{|(\eta_2 | (1, 0)] + |(\eta_2 | (0, 1)]|}{|\eta_2 | \gamma \rangle} \right),$$

we have for any $(\alpha, \beta)$ in $C_\gamma$, the inequalities $c(\alpha, \beta) \leq |\alpha| + |\beta| < \delta m(\alpha, \beta)$ inducing the equality $C_\gamma = C_{\gamma, -}^{\delta, \varepsilon}$ and $\chi_e(C_{\gamma, -}^{\delta, \varepsilon}) = 1$. Then, by Lemma 2.1 we obtain for any “?” and “!” in $\{<, >\}$

- if $C_\gamma \cap (R_{>0} \times R_{>0}) = \emptyset$ then $R^\delta_{\gamma, (?)}(T) = 0$,
- if $C_\gamma \cap (R_{>0} \times R_{>0})$ is generated by two vectors $\omega_1$ and $\omega_2$ then

$$R^\delta_{\gamma, (?)}(T) = \sum_{(\alpha_0, \beta_0) \in P_{\gamma, (?)}} \frac{\mathbb{L}^{-((e_1, e_2)(\alpha_0, \beta_0) T((\alpha_0, \beta_0) \gamma))}}{(1 - \mathbb{L}^{-((e_1, e_2) \omega_1) T(\omega_1 \gamma))}(1 - \mathbb{L}^{-((e_1, e_2) \omega_2) T(\omega_2 \gamma)})},$$

with $P_{\gamma, (?)} = \{(0, 1) | \omega_1 + (1, 0) | \omega_2) \cap Z^2$ and

$$\varepsilon_1 = \begin{cases} 1 & \text{if “?” is “>”} \\ -1 & \text{if “?” is “<”} \end{cases} \quad \text{and} \quad \varepsilon_2 = \begin{cases} 1 & \text{if “!” is “>”} \\ -1 & \text{if “!” is “<”} \end{cases}.$$

If “?” is “=” and “!” is “>” (the case “?” is “>” and “!” is “=” similar), we obtain also

- if $C_\gamma \cap (0 \times R_{>0}) = \emptyset$ then $R^\delta_{\gamma, (?)}(T) = 0$,
- if $C_\gamma \cap (0 \times R_{>0}) = R_{>0}(0, 1)$ then

$$R^\delta_{\gamma, (?)}(T) = \frac{\mathbb{L}^{-T((0, 1) \gamma))}}{1 - \mathbb{L}^{-T((0, 1) \gamma))}}. \quad (3.54)$$

Assume the origin $(0, 0)$ is contained in $\gamma_1$ and not in $\gamma_2$ (the case where the origin is contained in $\gamma_2$ and not in $\gamma_1$ is similar). Then, as $b_0 > 0$, by convention (3.52) on $\eta_1$ and $\eta_2$ and the fact that these normal vectors are exterior to $N_{\infty}(f)$, $\eta_1$ belongs to $R_{>0}(-b_0, a_0)$ and by convexity of the Newton polygon, we have $\eta_1 | \gamma > 0$. For any $(\alpha, \beta)$ in $C_\gamma$, there is $x > 0$ and $y > 0$ such that $(\alpha, \beta) = x \eta_1 + y \eta_2$ and $m(\alpha, \beta) = (\gamma \mid (\alpha, \beta)) = y(\gamma \mid \eta_2)$. In the following, we show that for any $\delta$ large enough, we have the equality $C_{\gamma, -}^{\delta, \varepsilon} = R_{>0}\eta_1 + R_{>0}\eta_2$, with $\eta_1 = (1 - \delta b_0, \delta a_0)$ and $\chi_e(C_{\gamma, -}^{\delta, \varepsilon}) = 0$. In order to prove this description, we study below each intersection $C_{\gamma, -}^{\delta, \varepsilon} \cap (R_{>0} \times R_{>0})$ for any “?” and “!” in $\{>, =, <\}$ and take their union.
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Indeed, for any $|\eta| > 0$, and in the non empty case we have by Lemma 2.1 and recall that $(\alpha, \beta)$

$$L_\delta(\eta) = (|\eta| (0, 1)) - \delta(\gamma | \eta) < 0$$

for any $\delta > \delta_1 = \frac{\eta |0(1)|}{(\gamma | \eta)}$.

And by Lemma 2.1, using $P_{\gamma, (\gamma, <)} = ([0, 1]|\eta + [0, 1]|\eta) \cap \mathbb{Z}^2$, we have

$$R_{\gamma, (\gamma, <)}(T) = \sum_{(\alpha, \beta) \in P_{\gamma, (\gamma, <)}} \left[ \frac{L_{-1}(T((\alpha, \beta) | \gamma))}{(1 - L^{-1}((0, 1)|\eta))) \eta(T(\eta | \eta))} \right]$$

$$+ \frac{L_{-1}((0, 1)|\eta))}{(1 - L^{-1}((0, 1)|\eta))) \eta(T(\eta | \eta))}.$$  

The cone $C_\gamma \cap \{0\} \times \mathbb{R}_{\geq 0}$ is empty or equal to $\mathbb{R}_{\geq 0}(0, 1)$. As $b_0 > 0$, for any $\delta > \delta_2 = 1/b_0$ we have

$$C_{\gamma, <}^\delta \cap \{0\} \times \mathbb{R}_{\geq 0} = C_\gamma \cap \{0\} \times \mathbb{R}_{\geq 0}$$

and in the non empty case we have by Lemma 2.1

$$R_{\gamma, (\gamma, <)}(T) = \frac{L_{-1}(T((0, 1)|\gamma))}{(1 - L^{-1}(0, 1)|\gamma))}.$$  

The cone $C_\gamma \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})$ is empty or equal to $\mathbb{R}_{\geq 0}(0, 1) + \mathbb{R}_{\geq 0}|\eta$ with $\eta$ equal to $(1, 0)$ or $\eta_2$. In particular in the non empty case $(|\eta| (1, 0)) > 0$ and $(|\eta| (0, 1)) > 0$. For $\delta > \delta_3 = \max\left(\frac{1}{(\gamma | \eta))}, \eta(0(1)) \frac{1}{(\gamma | \eta)}\right)$ we have

$$C_{\gamma, <}^\delta \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) = C_\gamma \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) = \mathbb{R}_{\geq 0}(0, 1) + \mathbb{R}_{\geq 0}|\eta),$$

Indeed, for any $(\alpha, \beta)$ in $C_\gamma \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})$, we have $C(\alpha, \beta) = \alpha + \beta$. Let $\delta > \delta_3$ and $(\alpha, \beta)$ in $C_\gamma$. There is $\lambda > 0$ and $\mu > 0$ such that $(\alpha, \beta) = \lambda(0, 1) + \mu|\eta$ and we conclude that $(\alpha, \beta)$ belongs to $C_{\gamma, <}$ by

$$C(\alpha, \beta) = \alpha + \beta = \lambda + \mu(|\eta | (1, 0)) \leq \delta[\lambda(\gamma | (0, 1)) + \mu(\gamma | \eta)].$$

Then, in the non empty case, we have by Lemma 2.1 with $P_{\gamma, (\gamma, >)} = ([0, 1](0, 1) + [0, 1]|\eta) \cap \mathbb{Z}^2$,

$$R_{\gamma, (\gamma, >)}(T) = \sum_{(\alpha, \beta) \in P_{\gamma, (\gamma, >)}} \left[ \frac{L_{-1}(T((\alpha, \beta) | \gamma))}{(1 - L^{-1}(0(1)|\eta))) \eta(T(\eta | \eta))} \right]$$

The cone $C_\gamma \cap (\mathbb{R}_{\geq 0} \times \{0\})$ is empty or equal to $\mathbb{R}_{\geq 0}(1, 0)$. If $a_0 = 0$ then the cone is empty, otherwise $a_0 > 0$ and for $\delta > \delta_4 = 1/a_0$, we have

$$C_{\gamma, <}^\delta \cap (\mathbb{R}_{\geq 0} \times \{0\}) = C_\gamma \cap (\mathbb{R}_{\geq 0} \times \{0\}).$$
We assume now that there is $(a, b)$ in $(\mathbb{N}^*)^2$ and a polynomial $P$ in $k[x]$ such that $f(x, y) = P(x^ay^b)$. We denote by $\gamma$ the face $(ad, bd)$. Then the rationality of $Z^\delta_{\gamma, -}(T)$ and its rational form follows from above ideas, using the fact that

$$C_{\gamma, -}^\delta \cap (\mathbb{R}_{>0} \times \mathbb{R}_{<0}) = \mathbb{R}_{>0} \eta_1 + \mathbb{R}_{<0} \eta_2.$$
with \( \eta_\delta = (1 - bd\delta, ad\delta) \) and \( \eta_\delta' = (bd\delta, 1 - ad\delta) \). More precisely, we have for any \( \delta > \max(1/(da), 1/(bd)) \)

\[
R_{\gamma, +}^\delta(T) = \sum_{(a_0, b_0) \in \mathcal{P}_{\gamma, (<, >)}(\mathbb{G}^2_m \setminus f_\gamma^{-1}(0) \to \mathbb{G}_m, \sigma_\gamma)} \eta_{\gamma, +}(T)
\]

with \( \mathcal{P}_{\gamma, (<, >)} = (0, 1)(0, 1) + [0, 1] f_\gamma \) and \( \mathcal{P}_{\gamma, (>, <)} = (0, 1)(1, 0) + [0, 1] f_\gamma \) in \( \mathbb{Z}^2 \). In particular \(- \lim R_{\gamma, +}^\delta(T) = 1 \).

Proposition 3.49 [Case \( \varepsilon = + \)] Let \( \gamma \) be a face of \( N(f) \) not contained in the coordinate axes. There is \( \delta_0 \) such that for any \( \delta > \delta_0 \), the formal series \( Z_{\gamma, +}^\delta(T) \) is rational and equal to

\[
Z_{\gamma, +}^\delta(T) = [f_\gamma : \mathbb{G}^2_m \setminus f_\gamma^{-1}(0) \to \mathbb{G}_m, \sigma_\gamma]R_{\gamma, +}^\delta(T)
\]

with \( R_{\gamma, +}^\delta(T) \) expressed in 3.67, 3.69, 3.71 and 3.72. It admits a limit \(- \lim Z_{\gamma, +}^\delta(T) \) in \( \mathcal{M}_{\mathbb{G}^2_m} \) with

\[
- \lim Z_{\gamma, +}^\delta(T) = \varepsilon_\gamma [f_\gamma : \mathbb{G}^2_m \setminus f_\gamma^{-1}(0) \to \mathbb{G}_m, \sigma_\gamma]
\]

with \( \varepsilon_\gamma = -\chi_c(C_{\gamma, +}^\delta \cap \Omega) \) belongs to \( [0, -1, -2] \) if \( \gamma \) is zero-dimensional and to \( \{0, 1\} \) if \( \gamma \) is one-dimensional. More precisely:

1. Assume \( \gamma \) is the origin then for any \( \delta > 0 \), \( Z_{\gamma, +}^\delta(T) = 0 \) and \( \varepsilon_\gamma = 0 \).
2. Assume \( \gamma \) is zero dimensional equal to \((a_0, b_0)\). Let

\[
H_\gamma = \{ (\alpha, \beta) \in \mathbb{R}^2 \mid (\alpha, \beta) \mid (\gamma) < 0 \}
\]

and \( C_\gamma \) be the dual cone of \( \gamma \). We have \( C_{\gamma, +}^\delta \subset H_\gamma \) and

\[
C_{\gamma, +}^\delta \cap \Omega = C_{\gamma, +}^\delta \cap ((\mathbb{R}_{>0} \times \mathbb{R}_{<0}) \cup (\mathbb{R}_{<0} \times \mathbb{R}_{>0}))
\]

For any \((?, !)\) in \((\langle, >\rangle, (>, <))\)

(a) if \( C_\gamma \cap H_\gamma \cap (\mathbb{R}_{>0} \times \mathbb{R}_{>0}) = \emptyset \) then \( C_{\gamma, +}^\delta \cap (\mathbb{R}_{>0} \times \mathbb{R}_{>0}) \) is empty and its Euler characteristic is zero

(b) if \( C_\gamma \cap H_\gamma \cap (\mathbb{R}_{>0} \times \mathbb{R}_{<0}) = \mathbb{R}_{>0}\omega_1 + \mathbb{R}_{>0}\omega_2 \), with \( \omega_1 \) and \( \omega_2 \) in the closure \( \overline{H_\gamma} \) and not colinear then:

(i) if \( (\gamma \mid \omega_1) < 0 \) and \( (\gamma \mid \omega_2) < 0 \) then, there is \( \delta_1 > 0 \) such that for any \( \delta > \delta_1 \), the cone \( C_{\gamma, +}^\delta \cap (\mathbb{R}_{>0} \times \mathbb{R}_{>0}) \) is equal to \( \mathbb{R}_{>0}\omega_1 + \mathbb{R}_{>0}\omega_2 \), with Euler characteristic equal to 1.

(ii) if \( (\gamma \mid \omega_1) = 0 \) and \( (\gamma \mid \omega_2) < 0 \) then there is \( \delta_1 > 0 \) such that for any \( \delta > \delta_1 \), considering the vector \( \omega_1 = (-b_0 - 1/\delta, a_0 - 1/\delta) \), the cone \( C_{\gamma, +}^\delta \cap H_\gamma \cap (\mathbb{R}_{>0} \times \mathbb{R}_{>0}) \) is equal to \( \mathbb{R}_{>0}\omega_1 + \mathbb{R}_{>0}\omega_2 \), with Euler characteristic equal to 0.

3. Assume \( \gamma \) is a one-dimensional face supported by a line \( ap + bq = N \) with \( \eta_\gamma = (p, q) \) the primitive exterior normal vector of the face \( \gamma \) in \( N(f) \), and \( C_\gamma = \mathbb{R}_{>0}\eta_\gamma \), we have

(a) if \( N \geq 0 \) then the cone \( C_{\gamma, +}^\delta \cap \Omega \) is empty and \( \varepsilon_\gamma = 0 \).
(b) if \( N < 0 \) then there is \( \delta_2 \) such that for any \( \delta > \delta_2 \) we have \( C_{\gamma, +}^{\delta, \infty} \cap \Omega = C_{\gamma} \cap \Omega \), in particular
(i) if \( \eta_\gamma \) belongs to \( \Omega \) then, \( C_{\gamma, +}^{\delta, \infty} \cap \Omega = \mathbb{R}_{> 0} \eta_\gamma \) with Euler characteristic \(-1\), then 
\( \epsilon_\gamma = 1 \),
(ii) otherwise, \( C_{\gamma, +}^{\delta, \infty} \cap \Omega \) is empty with Euler characteristic 0 and \( \epsilon_\gamma = 0 \).

4. Assume \( \gamma \) is a one-dimensional face supported by a line \( ap + bq = N \) with \((p, q)\) the primitive normal vector of the face \( \gamma \) in \( \mathcal{N}(f) \), and \( C_{\gamma} = \mathbb{R}_{> 0}(p, q) + \mathbb{R}_{> 0}(-p, -q) \) with \( pq < 0 \) (this case occurs if and only if \( \mathcal{N}(f) \) is a segment), we have

(a) if \( N = 0 \) then the cone \( C_{\gamma, +}^{\delta, \infty} \cap \Omega \) is empty, and \( \epsilon_\gamma = 0 \).
(b) if \( N \neq 0 \) then there is \( \delta_2 \) such that for any \( \delta > \delta_2 \) we have \( C_{\gamma, +}^{\delta, \infty} \cap \Omega = \mathbb{R}_{> 0}(p, q) \) or \( \mathbb{R}_{> 0}(-p, -q) \) and its Euler characteristic is \(-1\) and \( \epsilon_\gamma = 1 \).

**Proof** Let \( \gamma \) be a face in \( \mathcal{N}(f) \) and \( \delta > 0 \). If the cone \( C_{\gamma}^{\delta, \infty} \) is empty then the result is immediate. If there is \( \delta' > 0 \) such that the cone \( C_{\gamma}^{\delta', \infty} \cap \Omega \) is non empty then, for any \( \delta > \delta' \) the cone \( C_{\gamma}^{\delta, \infty} \cap \Omega \) is non empty. In the following of the proof, we work with this assumption. Let \( \delta > 0 \) and \((\alpha, \beta)\) be an element of \( C_{\gamma}^{\delta} \cap \Omega \), then similarly to the proof of Proposition 3.48, the motivic measure of \( X_m(\alpha, \beta)(f) \) is equal to \( \mathbb{L}^{-|\alpha| - |\beta|} [f_{\gamma} : \mathcal{G}_m^2 \setminus f_{\gamma}^{-1}(0) \rightarrow \mathbb{G}_m, \sigma_{\gamma}] \). Then, we have the equality \( Z_{\gamma, +}^{\delta, \infty}(T) = \{ f_{\gamma} : \mathcal{G}_m^2 \setminus f_{\gamma}^{-1}(0) \rightarrow \mathbb{G}_m, \sigma_{\gamma} \} R_{\gamma}^{\delta, \infty}(T) \), where \( R_{\gamma}^{\delta, \infty}(T) = \sum n \geq 1 \sum (\alpha, \beta) \in C_{\gamma, +}^{\delta, \infty} \cap \Omega \mathbb{L}^{-|\alpha| - |\beta|} T^n \). The application \( c \), defined in formula

\[
(n) = -m(\alpha, \beta)
\]

(3.6), is linear on each cone \( \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \) with “\( \gamma’ \)” and “\( \eta’ \)” in \([> , < , =]\) and \( \mathbb{R}_{= 0} = \{ 0 \} \). Then, we consider the cone \( C_{\gamma} \) as the disjoint union \( C_{\gamma, +}^{\delta, \infty} \cap \Omega = \bigsqcup_{(?, ?) \in \{ < , = , > \} \setminus \{ < , = \}^2 \} C_{\gamma, +}^{\delta, \infty} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \)

and we have

\[
R_{\gamma}^{\delta, \infty}(T) = \sum_{(?, ?) \in \{ < , = , > \} \setminus \{ < , = \}^2} R_{\gamma,(?, ?)}^{\delta, \infty}(T) \quad \text{with}
\]

\[
R_{\gamma,(?, ?)}^{\delta, \infty}(T) = \sum_{n \geq 1} \sum (\alpha, \beta) \in C_{\gamma, +}^{\delta, \infty} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \cap \Omega \mathbb{L}^{-|\alpha| - |\beta|} T^n. \tag{3.67}
\]

By Lemma 2.1, each \( R_{\gamma,(?, ?)}^{\delta, \infty}(T) \) is rational and its limit, when \( T \) goes to infinity is \( \chi_{c}(C_{\gamma, +}^{\delta, \infty} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})) \). By additivity, we obtain the rational form of \( R_{\gamma}^{\delta, \infty}(T) \) and its limit is \( \chi_{c}(C_{\gamma, +}^{\delta, \infty} \cap \Omega) \). In the following, we study the cones \( C_{\gamma, +}^{\delta, \infty} \cap \Omega \) and \( C_{\gamma, +}^{\delta, \infty} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \) for any \((?, ?) \in \{ < , = , > \} \setminus \{ < , = \}^2 \).

- Assume \( \gamma \) is a zero dimensional face, written as \( \gamma = (a_0, b_0) \). Recall that \( C_{\gamma, +}^{\delta, \infty} = \{(\alpha, \beta) \in C_{\gamma} \mid 0 < c(\alpha, \beta) \leq -m(\alpha, \beta) \delta \} \).

We consider \( H_{\gamma} = \{(\alpha, \beta) \in \mathbb{R}^2 \mid m(\alpha, \beta) = ((\alpha, \beta) \mid \gamma) < 0 \} \).

By definition, \( C_{\gamma, +}^{\delta, \infty} \) is a subset of \( H_{\gamma} \) then we have \( C_{\gamma, +}^{\delta, \infty} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) = \emptyset \) for “\( ? \)” and “\( \eta \)” in \([= , > \)”. Then, we only study the cases \( C_{\gamma, +}^{\delta, \infty} \cap (\mathbb{R}_{> 0} \times \mathbb{R}_{< 0}) \) and \( C_{\gamma, +}^{\delta, \infty} \cap (\mathbb{R}_{< 0} \times \mathbb{R}_{> 0}) \)
which are similar. If $C_\gamma \cap H_\gamma \cap (\mathbb{R}_{<0} \times \mathbb{R}_{>0})$ is non empty then, there are two non colinear vectors $\omega_1$ and $\omega_2$, such that

$$C_\gamma \cap H_\gamma \cap (\mathbb{R}_{<0} \times \mathbb{R}_{>0}) = \mathbb{R}_{>0} \omega_1 + \mathbb{R}_{>0} \omega_2$$

with $(\gamma \mid \omega_1) \leq 0$ and $(\gamma \mid \omega_2) \leq 0$.

- If $(\gamma \mid \omega_1) < 0$ and $(\gamma \mid \omega_2) < 0$ then, for $\delta \geq 2\delta_1$ with

$$\delta_1 = \max \left( \left( \frac{|(\omega_1 \mid (0,1))|}{|(\gamma \mid \omega_1)|}, \frac{|(\omega_2 \mid (0,1))|}{|(\gamma \mid \omega_2)|} \right) \right)$$

we have

$$C_\gamma^{\delta,+} \cap (\mathbb{R}_{<0} \times \mathbb{R}_{>0}) = C_\gamma \cap H_\gamma \cap (\mathbb{R}_{<0} \times \mathbb{R}_{>0}). \quad (3.68)$$

Indeed, for $\delta \geq \delta_1$ and for any $(\alpha, \beta)$ in $C_\gamma \cap H_\gamma \cap (\mathbb{R}_{<0} \times \mathbb{R}_{>0})$, there are $\lambda > 0$ and $\mu > 0$ such that, $(\alpha, \beta) = \lambda \omega_1 + \mu \omega_2$. As $(\omega_1 \mid \gamma) < 0$ and $(\omega_2 \mid \gamma) < 0$, $(\alpha, \beta)$ belongs to $C_\gamma^{\delta,+}$ thanks to

$$c(\alpha, \beta) = \beta = \lambda(\omega_1 \mid (0,1)) + \mu(\omega_2 \mid (0,1))$$

$$\leq \delta_1 (\mu (\omega_2 \mid \gamma)) + \mu (\omega_2 \mid \gamma)) = -\delta_1 (\gamma \mid \alpha, \beta) \mid \gamma) \leq -\delta m(\alpha, \beta).$$

Then, by equality $(3.68)$, and Lemma 2.1 we conclude that $R_{\gamma, \langle \cdot, \cdot \rangle}^{\delta,+}(T)$ is rational and equal to

$$R_{\gamma, \langle \cdot, \cdot \rangle}^{\delta,+}(T) = \sum_{(\omega_0, \omega_0) \in \mathcal{P}_{\gamma, \langle \cdot, \cdot \rangle}} \frac{L^{\omega_0 - \beta_0}T^{-(\omega_0, \beta_0)\gamma}}{(1 - L^{-((1,1)|\omega_0)}T^{-(\omega_1|\gamma)}(1 - L^{-((1,1)|\omega_2)}T^{-(\omega_2|\gamma)}))}$$

with $\mathcal{P}_{\gamma, \langle \cdot, \cdot \rangle} = \{0, 1] \omega_0 + \{0, 1] \omega_2\} \cap \mathbb{R}_{>0}$. The limit of $R_{\gamma, \langle \cdot, \cdot \rangle}^{\delta,+}(T)$ is $\chi_c(C_\gamma^{\delta,+}) = 1$.

- Assume $(\gamma \mid \omega_1) = 0$ and $(\gamma \mid \omega_1) < 0$ (the case $(\gamma \mid \omega_1) = 0$ and $(\gamma \mid \omega_1) < 0$ is similar). Then, for any $\lambda > 0$ and $\mu > 0$, we have $m(\lambda \omega_1 + \mu \omega_2) = \mu(\omega_2 \mid \gamma)$. Assume $\delta > \delta_1 = -\frac{1}{\gamma \mid \omega_1}$, and denote $\omega_\delta = (b_0 - 1/\delta, a_0)$. We recall that $\gamma = (\omega_0, b_0)$. Denote $L_\delta$ the function $(\alpha, \beta) \mapsto \beta + \delta(\gamma \mid (\alpha, \beta))$ on $C_\gamma^{\delta,+} \cap (\mathbb{R}_{<0} \times \mathbb{R}_{>0})$. Then, $L_\delta(\omega_0) < 0$ and $L_\delta(\omega_0) = 0$ and we have

$$C_\gamma^{\delta,+} \cap (\mathbb{R}_{<0} \times \mathbb{R}_{>0}) = \mathbb{R}_{>0} \omega_\delta + \mathbb{R}_{>0} \omega_2. \quad (3.70)$$

Then, by equality $(3.70)$, and Lemma 2.1 we conclude that the zeta function $R_{\gamma, \langle \cdot, \cdot \rangle}^{\delta,+}(T)$ is rational with

$$R_{\gamma, \langle \cdot, \cdot \rangle}^{\delta,+}(T) = \sum_{(\omega_0, \omega_0) \in \mathcal{P}_{\gamma, \langle \cdot, \cdot \rangle}} \frac{L^{\omega_0 - \beta_0}T^{-(\omega_0, \beta_0)\gamma}}{(1 - L^{-((1,1)|\omega_0)}T^{-(\omega_1|\gamma)}(1 - L^{-((1,1)|\omega_2)}T^{-(\omega_2|\gamma)}))}$$

with $\mathcal{P}_{\gamma, \langle \cdot, \cdot \rangle} = \{0, 1] \omega_0 + \{0, 1] \omega_2\} \cap \mathbb{R}_{>0}$. The limit of $R_{\gamma, \langle \cdot, \cdot \rangle}^{\delta,+}(T)$ is $\chi_c(C_\gamma^{\delta,+}) = 0$.

- We consider the case of a one dimensional face $\gamma$. The face $\gamma$ is supported by a line of equation $ap + bq = N$ with $(p, q)$ the primitive normal vector to the face $\gamma$ exterior to the Newton polygon $\mathcal{N}(f)$.

Assume $C_\gamma = \mathbb{R}_{>0}(p, q)$. We have for any $k$ in $\mathbb{R}^+$, $m(pk, qk) = kN$, and

$$C_\gamma^{\delta,+} = \{ pk, qk \in C_\gamma \mid k \in \mathbb{R}^+, 0 < c(pk, qk) \leq -Nk\delta \}.$$
• If \( N \geq 0 \) then the cone \( C_{\gamma,+}^\delta \cap \Omega \) is empty.
• If \( N < 0 \) then there is \( \delta_2 = -c(p, q)/N \) such that for any \( \delta > \delta_2 \) we have \( C_{\gamma,+}^\delta \cap \Omega = C_\gamma \cap \Omega \), in particular
  • if \( (p, q) \) belongs to \( \Omega \) then, \( C_{\gamma,+}^\delta \cap \Omega = \mathbb{R}_{>0}(p, q) \) with Euler characteristic \(-1\), with
    \[
    R_{\gamma}^\delta(T) = \frac{\mathbb{L}|p|-|q|T^{-N}}{1 - \mathbb{L}|p|-|q|T^{-N}} \tag{3.72}
    \]
  • otherwise, \( C_{\gamma,+}^\delta \cap \Omega \) is empty with Euler characteristic 0 and \( R_{\gamma}^\delta(T) = 0 \).
Assume \( C_\gamma = \mathbb{R}_{>0}(p, q) + \mathbb{R}_{>0}(-p, -q) \) with \( pq < 0 \) (this case only occurs in the case where \( \mathcal{N}(f) \) is a segment).
• If \( N = 0 \) then the cone \( C_{\gamma,+}^\delta \cap \Omega \) is empty and \( R_{\gamma}^\delta(T) = 0 \).
• If \( N \neq 0 \) then there is \( \delta_2 \) such that for any \( \delta > \delta_2 \) we have \( C_{\gamma,+}^\delta \cap \Omega = \mathbb{R}_{>0}(p, q) \) or \( \mathbb{R}_{>0}(-p, -q) \) and its Euler characteristic is \(-1\) and \( R_{\gamma}^\delta(T) \) is given by formula (3.72).
The bound \( \delta_0 \) in the statement can be chosen larger then the maximum of the bounds \( \delta_1 \) and \( \delta_2 \) above. \( \square \)

### 3.3.7 The formal series \( Z_{\gamma,\nu}^\psi \) for a face \( \gamma \) not contained in a coordinate axes

**Remark 3.50** (Vanishing \( f_\gamma \)) Let \( \gamma \) be a face in \( \mathcal{N}(f) \). Let \( (\alpha, \beta) \) be in \( C_\gamma \) and \( \varphi \) be in an arc in \( \mathcal{L}(X) \) with ord \( x(\varphi) = -\alpha \) and ord \( y(\varphi) = -\beta \). Then, by Remark 3.41 \( \epsilon \text{ord } \hat{f}_{\psi}(\varphi) > -m(\alpha, \beta) \) if and only if \( f_\gamma(\infty x(\varphi), \infty y(\varphi)) = 0 \). In particular in that case \( \gamma \) is a one-dimensional face of \( \mathcal{N}(f) \).

**Remark 3.51** We only consider one dimensional faces with dual cone \( C_\gamma \) in \( \Omega \), then there are five cases to study: the face \( \gamma \) belongs to \( \mathcal{N}_{\infty,\infty}(f), \mathcal{N}_0,\infty(f), \mathcal{N}_{\infty,0}(f) \) or is horizontal or vertical.

3.3.7.1 The face \( \gamma \) belongs to \( \mathcal{N}_{\infty,\infty}(f) \). The dual cone of the one dimensional face \( \gamma \) is the cone \( C_\gamma = \mathbb{R}_{>0}(p, q) \) with \( (p, q) \) the primitive normal vector to \( \gamma \) exterior to \( \mathcal{N}(f) \) and \( \gamma \) is supported by a line of equation \( pa + q\beta = N \). As \( \gamma \) belongs to \( \mathcal{N}_{\infty,\infty}(f) \), we have \( p > 0 \) and \( q > 0 \). We write \( f_\gamma(x, y) = x^{\psi_\gamma}y^{\nu_\gamma} \prod_{\mu \in \mathcal{R}_\gamma}(x^\mu - \mu y^p)^{\nu_\mu} \). Let \( \mu \) be a root in \( \mathcal{R}_\gamma \). Using Notations 3.38 and Eq. (3.37), for any \( (n, (\alpha, \beta)) \) in \( C_{\gamma,\nu}^\psi \), we denote \( X_{n,(\alpha,\beta),\mu}(\hat{f}_\psi) = \{ \varphi \in X_{n,(\alpha,\beta)}(\hat{f}_\psi) \mid -(\text{ord } x(\varphi), \text{ord } y(\varphi)) = (\alpha, \beta), \infty x(\varphi)^q = \mu \infty y(\varphi)^p \} \) endowed with the induced structural map to \( \mathbb{G}_m \).

**Remark 3.52** The origin of any arc of \( X_{n,(\alpha,\beta),\mu}(\hat{f}_\psi) \) is the point \((0 : 1), (0 : 1), (0 : 1)\) in \( X_\infty \) and by Remark 3.39 (formula 3.32) the arc space \( X_{n,(\alpha,\beta),\mu}(\hat{f}_\psi) \) is isomorphic to the arc space

\[
\left\{ (A(t), B(t)) \in \mathcal{L}(\mathbb{A}_k^2) \mid \text{ord } A(t) = \alpha, \text{ord } B(t) = \beta, \text{ord } f(1/A(t), 1/B(t)) = n, \infty A(t)^{-q} - \mu \infty B(t)^{-p} = 0 \text{ namely } \infty B(t)^p = \mu \infty A(t)^q \right\}.
\]

**Proposition 3.53** Let \( \mu \) be a root in \( \mathcal{R}_\gamma \) and \( \sigma_{(p,q,\mu)} \) the induced Newton transform

\[
\sigma_{(p,q,\mu)} : k[x, y] \longrightarrow k[u^{-1}, v, w] \\
g(x, y) \mapsto g_{(p,q,\mu)}(v, w) = g(\mu^q v^{-p}, v^{-q}(w + \mu^p))
\]

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defined in Definition 1.29 with \( qq' - pp' = 1 \). For any \( k > 0 \), for any \((n, (\alpha, \beta))\) in \( C^k_\nu \) with \( \alpha = pk \) and \( \beta = qk \), we have

\[
\text{mes} \left( X_{n,(\alpha,\beta),\mu}(\hat{f}^x) \right) = \mathbb{L}^{-p+q-1}k \text{mes} \left( Y_{n,k}(f_{\sigma(p,q,\mu)}) \right) \in M^{\mathbb{G}_m}_{\mathbb{G}_m}
\]

with \( Y_{n,k}(f_{\sigma(p,q,\mu)}) = \left\{(v(t), w(t)) \in \mathbb{C}(A^2_k) \mid \begin{array}{l} \text{ord } v(t) = k, \text{ord } w(t) > 0, \\
\text{ord } (f_{\sigma(p,q,\mu)})^x(v(t), w(t)) = n \end{array} \right\} \) endowed with the structural map \( \overline{\text{ac}}(f_{\sigma(p,q,\mu)}) \) to \( \mathbb{G}_m \).

**Proof** The proof is inspired from that of [8, Lemma 3.3]. Let \( L \) be an integer bigger than \( \alpha \), \( \beta \) and \( n \).

\[
X_{(n,a,b),\mu}^{(L)} = \left\{ (A(t), B(t)) \in \left( \mathbb{k}[[t]]/(t^{L+1}) \right)^2 \mid \begin{array}{l} \text{ord } A(t) = \alpha, \text{ord } B(t) = \beta, \\
\overline{\text{ac}} B(t)^p = \mu \overline{\text{ac}} A(t)^q \end{array} \right\}
\]

which is isomorphic to the jet-spaces \( \pi_L(X_{n,(\alpha,\beta),\mu}(\hat{f}^x)) \) by Remark 3.52.

\[
\overline{X}_{(n,a,b),\mu}^{(L)} = \left\{ (\psi_1(t), \psi_2(t)) \in \left( \mathbb{k}[[t]]/(t^{L+1}) \right)^2 \mid \begin{array}{l} \psi_1(0) = 0, \psi_2(0) = 0, \\
\psi_1^p = \mu \psi_1^q \end{array} \right\} = n.
\]

\[
Y_{(n,k)}^{(L)} = \left\{ (v'(t), w'(t)) \in \left( \mathbb{k}[[t]]/(t^{L+1}) \right)^2 \mid \begin{array}{l} \text{ord } (f_{\sigma(p,q,\mu)})^x(v'(t), w'(t)) = n, \text{ord } v'(t) = k, \text{ord } w'(t) \geq 1 \end{array} \right\}
\]

which is \( \pi_L(Y_{n,k}(f_{\sigma(p,\mu,q)})) \). The application

\[
(\varphi_1, \varphi_2) \mapsto (t^\alpha \varphi_1 \mod t^{L+1}, t^\beta \varphi_2 \mod t^{L+1})
\]

induces a structure of bundle on \( \overline{X}_{(n,a,b),\mu}^{(L)} \) over \( X_{(n,a,b),\mu}^{(L)} \) with fiber \( A^\alpha + \beta \). Also, the application

\[
(\varphi_1, \varphi_2) \mapsto (t^k \varphi_1 \mod t^{L+1}, \varphi_2 \mod t^{L+1})
\]

induces a structure of bundle on \( Y_{(n,k)}^{(L)} \) over \( Y_{(n,k)}^{(L)} \) with fiber \( A^k \). We deduce the equalities

\[
[X_{(n,a,b),\mu}^{(L)}] = \mathbb{L}^{-\alpha - \beta}[\overline{X}_{(n,a,b),\mu}^{(L)}] \text{ and } [Y_{(n,k)}^{(L)}] = \mathbb{L}^k[Y_{(n,k)}^{(L)}].
\]

We consider the application

\[
\Phi_{\sigma(p,q,\mu)} : Y_{(n,k)}^{(L)} \to X_{(n,a,b),\mu}^{(L)}
\]

\[
(\varphi_1, \varphi_2) \mapsto (t^{\alpha - \beta} \varphi_1^p, \varphi_1^q(\psi_2 + \mu \varphi_2)^{-1}) =: (\varphi_1, \varphi_2).
\]

Using the relation \( qq' - pp' = 1 \), we can check that \( \Phi_{\sigma(p,q,\mu)}(Y_{(n,k)}^{(L)}) \subset X_{(n,a,b),\mu}^{(L)} \).

Indeed, if \((\varphi_1(t), \varphi_2(t))\) is equal to \( \Phi_{\sigma(p,q,\mu)}(\psi_1(t), \psi_2(t)) \) then

\[
\varphi_2(0)^p - \mu \varphi_1(0)^q = (\psi_1(0)^q \mu^{-p'})^p - \mu (\mu^{-q} \psi_1(0)^p)^q = 0
\]

and using the relations \( \alpha = pk \), \( \beta = qk \), and the definitions we deduce the equality

\[
f^x(1/t^\alpha \varphi_1(t), 1/t^\beta \psi_2(t)) = ((f_{\sigma(p,q,\mu)}(t^k \varphi_1(t)), \psi_2(t)))^x.
\]

We prove that \( \Phi_{\sigma(p,q,\mu)} \) is an isomorphism building the inverse application. Consider \( \psi(t) = (\varphi_1(t), \varphi_2(t)) \) in \( X_{(n,a,b),\mu}^{(L)} \). Remark that if there is \( \psi(t) = (\varphi_1(t), \varphi_2(t)) \) in \( Y_{(n,k)}^{(L)} \) such that \( \varphi(t) = \Phi_{\sigma(p,q,\mu)}(\psi(t), \psi_2(t)) \) then we have the equality \( \varphi_2(0)^{q'} / \varphi_1(0)^{p'} = \psi_1(0) \).

Furthermore, denoting \( \varphi_1(t) = \varphi_1(0) \varphi_1(t) \), by Hensel lemma, there is a unique formal series \( a(t) \) such that \( a(0) = 1 \) and \( a(t)^p = \varphi_1(t) \). The formal series \( a(t) \) is denoted by \( \varphi_1(t)^{1/p} \).

Hence, the inverse map is given by

\[
(\varphi_1(t), \varphi_2(t)) \mapsto \left( \frac{\varphi_2(0)^{q'}}{\varphi_1(0)^p} \varphi_1(t)^{1/p} \mod t^{L+1}, -\mu^{-p} \left( \frac{\varphi_2(0)^{q'}}{\varphi_1(0)^p} \varphi_1(t)^{1/p} \right)^q \varphi_2(t)^{-1} \mod t^{L+1} \right).
\]
Thus $\Phi_{\sigma(p,q,\mu)}$ is an isomorphism, we have $[X^{(L)}_{(n,\alpha,\beta),\mu}] = \mathbb{L}^{-\alpha-\beta+k}[Y^{(L)}_{(n,k)}]$, and conclude by definition of the motivic measure.

\textbf{Proposition 3.54} For $\delta$ large enough, the motivic zeta function $Z_{\gamma,e}^{\delta}\langle$ is rational and can be decomposed as

$$Z_{\gamma,e}^{\delta} (T) = \sum_{\mu \in R_Y} \left( Z_{\delta(p,q)}^{\delta(p,q)} \left( f_{\sigma(p,q,\mu)}, \omega_{p,q, \epsilon}, v \neq 0 \right) \right) (0,0),$$

and

$$- \lim_{T \to \infty} Z_{\gamma,e}^{\delta} (T) = \sum_{\mu \in R_Y} \left( S \left( f_{\sigma(p,q,\mu)} \right), \epsilon \right) (0,0),$$

(3.73)

with the differential form $\omega_{p,q}(v, w) = v^{(p+q-1)}dv \wedge dw$.

\textbf{Remark 3.55} In the statement of the proposition, we can also use the differential $\omega_{p,q}(v, w) = v^{(p+q-1)} (w + \mu^{p'})^{-2} dv \wedge dw$.

because we work locally at $(0,0)$, in particular $ord v(t) > 0$, then we have equality of orders $ord \omega_{p,q}(v(t), w(t)) = ord \omega_{p,q}(v(t), w(t)) = (p+q-1) ord v(t)$.

\textbf{Proof} For any element $(\alpha, \beta)$ in $C_\gamma$ there is $k > 0$ such that $\alpha = pk$ and $\beta = qk$. In particular, as $\gamma$ belongs to $N_{\infty, \infty}(f), p > 0$ and $q > 0$, and we have $\alpha > 0$ and $\beta > 0$. The set of integer points of the cone $C_\gamma^{\delta,\epsilon}$ defined in equation (3.37) is,

$$C_\gamma^{\delta,\epsilon} \cap \mathbb{N}^3 = \{ (n, (\alpha, \beta)) \in \mathbb{R}_{>0} \times C_\gamma \mid -m(\alpha, \beta) < \epsilon n, 1 \leq \alpha + \beta \leq n\delta \} \cap \mathbb{N}^3,$$

is in bijection with the cone $\overline{C}_{\gamma}^{\delta,\epsilon} = \{ (n,k) \in \mathbb{N}^{k+1} \mid -m(pk, qk) < \epsilon n, 1 \leq (p+q)k \leq n\delta \}$, by $(n, k) \mapsto (n, pk, qk)$. Using this notation we prove the equality 3.73. Indeed, using Proposition 3.53 we have

$$Z_{\gamma}^{\delta,\epsilon} (T) = \sum_{\mu \in R_Y} \sum_{n \geq 1} \sum_{(n, (\alpha, \beta)) \in C_\gamma^{\delta,\epsilon} \cap \mathbb{N}^3} mes \left( X_{(n, (\alpha, \beta)),\mu} \right) T^n$$

$$= \sum_{\mu \in R_Y} \sum_{n \geq 1} \sum_{(n,k) \in \overline{C}_{\gamma}^{\delta,\epsilon}} \mathbb{L}^{-(p+q+1)k} mes \left( Y_{(n,k)} \left( \left( f_{\sigma(p,q,\mu)} \right)^{\epsilon} \right) \right) T^n,$$

but using the definition of the zeta function in Sect. 2.5 (see also Eq. (2.4)) we have

$$\left( Z_{\delta(p,q)}^{\delta(p,q)} \left( f_{\sigma(p,q,\mu)}, \omega_{p,q, \epsilon}, v \neq 0 \right) \right) (0,0),$$

$$= \sum_{n \geq 1} \left[ \sum_{k \geq 1} \mathbb{L}^{-(p+q+1)k} \right] mes \left( Y_{(n,k)} \left( \left( f_{\sigma(p,q,\mu)} \right)^{\epsilon} \right) \right) T^n$$

with

$$Y_{(n,k)} \left( \left( f_{\sigma(p,q,\mu)} \right)^{\epsilon} \right) = \begin{cases} (v(t), w(t)) \in \mathcal{L}(\alpha_k^\delta) & (v(0), w(0)) = 0, ord v(t) = k \leq n\delta/(p+q) \\ ord \left( f_{\sigma(p,q,\mu)} \right)^{\epsilon} (v(t), w(t)) = n \end{cases}.$$

In particular we can conclude thanks to Sect. 2.5, for $\delta$ large enough, that the motivic zeta function is rational and has a limit independent on $\delta$ when $T$ goes to infinity

$$- \lim_{T \to \infty} \left( Z_{\delta(p,q)}^{\delta(p,q)} \left( f_{\sigma(p,q,\mu)}, \omega_{p,q, \epsilon}, v \neq 0 \right) \right) (0,0),$$

$$= \left( S \left( f_{\sigma(p,q,\mu)} \right)^{\epsilon}, \omega_{p,q, \epsilon}, v \neq 0 \right) (0,0),$$

$\in \mathcal{M}_{Gm}^{\mathbb{G}_m}$. □
3.3.7.2 The face $\gamma$ belongs to $N_{0,\infty}(f)$ or $N_{0,\infty}(f)$. Let $\gamma$ be a face in $N_{i(\gamma),j(\gamma)}$ with $(i(\gamma), j(\gamma))$ equal to $(\infty, 0)$ or $(0, \infty)$ with $C_{\gamma} = \mathbb{R}_{>0}(p, q)$ with $p > 0$ and $q < 0$, or, $p < 0$ and $q > 0$. We denote by $R_{\gamma}$ the roots of $f_{\gamma}$. (see Notations 1.31).

**Proposition 3.56** For $\delta$ large enough, the motivic zeta function $Z_{\gamma, <}^{\delta}$ is rational with

$$Z_{\gamma, <}^{\delta}(T) = \sum_{\mu \in R_{\gamma}} \left( Z_{f_{\mu}}^{\delta/c(p,q)}(v, w) \right)_{(0,0,0)}$$

and

$$- \lim_{T \to \infty} Z_{\gamma, <}^{\delta}(T) = \sum_{\mu \in R_{\gamma}} \left( S_{f_{\mu}}(v, w) \right)_{(0,0,0)}$$

with $\omega_{p,q}(v, w) = v^{(|p|+|q|-1)}dv \wedge dw$ and the convenient Newton transformations defined in Definition 1.29 in (1.4) and (1.5).

**Proof** The proof is similar to the proof of Proposition 3.54. \(\square\)

**Remark 3.57** Assume $\varepsilon = +$. In the case where $C_{\gamma} = \mathbb{R}_{>0}(p, q) + \mathbb{R}_{>0}(-p, -q)$ with $pq < 0$ then applying twice the previous proposition we obtain, that for $\delta$ large enough, the motivic zeta function $Z_{\gamma, <}^{\delta}$ is rational with

$$Z_{\gamma, <}^{\delta}(T) = \sum_{\mu \in R_{\gamma}} \left( Z_{f_{\mu}}^{\delta/c(p,q)}(v, w) \right)_{(0,0,0)}$$

It has a limit (independent from $\delta$)

$$- \lim_{T \to \infty} Z_{\gamma, <}^{\delta}(T) = \sum_{\mu \in R_{\gamma}} \left( S_{f_{\mu}}(v, w) \right)_{(0,0,0)}$$

3.3.7.3 The face $\gamma$ is horizontal. There are at most two one-dimensional horizontal faces. There is only one with exterior normal vector in $\Omega$, we denote it $\gamma_H$. The face polynomial $f_{\gamma_H}$ has the form $\gamma^M T(x)$ where $M \geq 1$ and $T$ is a polynomial in $\mathbf{k}[x]$. We denote by $R_{\gamma_H}$ the set of roots of $T$ in $\mathbb{C}_{\gamma_H}$ and call it set of roots of $f_{\gamma_H}$. In that case, we have $C_{\gamma_H} = \mathbb{R}_{>0}(0, 1)$.

**Remark 3.58** Let $(n, (\alpha, \beta))$ in $C_{\gamma_H, <}^{\delta}$ necessarily $\alpha = 0$ and $\beta > 0$. By Remark 3.39, any arc $\varphi$ in $X_{n, (\alpha, \beta)}$ can be written as $\varphi(i) = (\{1 : A(t)\}, \{B(t) : 1\}, \{z_0(t) : z_1(t)\})$ with ord $A(t) > 0$ and ord $B(t) > 0$ and ord $f_{\gamma_H}^{\varepsilon}(\varphi(i)) = ord f_{\gamma_H}^\varepsilon(A(t), 1/B(t)) = n$. Furthermore, as $n < -\varepsilon(\alpha, \beta)$, $A(0)$ is a root of $f_{\gamma_H}$.

**Proposition 3.59** For $\delta$ large enough, the motivic zeta function $Z_{\gamma_H, <}^{\delta}$ is rational with

$$Z_{\gamma_H, <}^{\delta}(T) = \sum_{\mu \in R_{\gamma_H}} \left( Z_{f_{\mu}}^{\delta/c(p,q)}(v, w) \right)_{(0,0,0)}$$

and

$$- \lim_{T \to \infty} Z_{\gamma_H, <}^{\delta}(T) = \sum_{\mu \in R_{\gamma_H}} \left( S_{f_{\mu}}(v, w) \right)_{(0,0,0)}$$

where for any root $\mu$ of $f_{\gamma_H}$, we consider $f_{\mu}(x, y) = f(x + \mu, 1/y)$.

**Remark 3.60** The motives $(S_{f_{\mu}}^{\delta/c(p,q)}(v, w))_{(0,0,0)}$ are computed in Sect. 2.6.

**Proof** For any $n \geq 1$, $\delta \geq 1$ and $\mu \in R_{\gamma_H}$ we consider

$$X_{n, (\alpha, \beta, \gamma_H, \mu)} \{ (A(t), B(t)) | A(0) = \mu, B(0) \neq 0, 0 < ord B(t) \leq n \delta, ord f_{\gamma_H}^{\varepsilon}(A(t), 1/B(t)) = n \}$$

and using Remark 3.58 we obtain the decomposition of the zeta function $Z_{\gamma_H, <}^{\delta}(T) = \sum_{\mu \in R_{\gamma_H}} Z_{f_{\mu}}^{\delta}(T)$ with for any root $\mu$ in $R_{\gamma_H}$,

$$Z_{f_{\mu}}^{\delta}(T) = \sum_{n \geq 1} \left( X_{n, (\alpha, \beta, \gamma_H, \mu)}^{\delta} \right) T^n.$$
We have the isomorphism

\[ X^\delta_{n, y \neq 0, (0, 0)} (f_{0, \infty, \mu}) \rightarrow X^\delta_{n, \gamma^V, \mu} (w(t), B(t)) \quad \rightarrow (w(t) + \mu, B(t)) \]

where

\[ X^\delta_{n, y \neq 0, (0, 0)} (f_{0, \infty, \mu}) = \{(w(t), t^k B(t)) \mid \text{ord } w(t) > 0, 0 < \text{ord } B(t) \leq n\delta, \text{ ord } f_{0, \infty, \mu}(w(t), B(t)) = n \}. \]

Then, we conclude that we have the equality \( Z^\delta_{\Gamma f^V, \gamma^V, \mu} (T) = (Z^\delta_{f_{0, \infty, \mu}, y \neq 0} (T))((0, 0), 0) \), which induces the result. \( \square \)

The face \( \gamma \) is vertical. There are at most two one-dimensional vertical faces. There is only one with exterior normal vector in \( \Omega \), we denote it \( \gamma^V \). The face polynomial \( f^V \) has the form \( x^M T(y) \) where \( M \geq 1 \) and \( T \) is a polynomial in \( \mathbb{k}[y] \). We denote by \( R_{\gamma^V} \) the set of roots of \( T \) in \( \mathbb{C}^m \). We call this set, set of roots of \( f_{\gamma^V} \). In that case, we have \( C_{\gamma^V} = \mathbb{R}_{>0}(1, 0) \).

**Proposition 3.61** For \( \delta \) large enough, the motivic zeta function \( Z_{\Gamma f^V, x}^{\delta, <} \) is rational with

\[ Z_{\Gamma f^V, x}^{\delta, <} = \sum_{\mu \in R_{f_{\gamma^V}}} \left( Z_{f_{\infty, 0}, \mu, x \neq 0}^{\delta, \gamma^V, \mu} ((0, 0), 0) \right) \quad \text{and} \quad \lim_{T \to \infty} Z_{\Gamma f^V, x}^{\delta, <} (T) = \sum_{\mu \in R_{f_{\gamma^V}}} \left( S_{f_{\infty, 0}, \mu, x \neq 0} \right) ((0, 0), 0), \]

where for any root \( \mu \) of \( f_{\gamma^V} \) we consider \( f_{\infty, 0, \mu}(x, y) = f(1/x, y + \mu) \).

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