Some Integral Inequalities For Functions Whose Second Derivatives Are $\varphi-$Convex By Using Fractional Integrals

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ABSTRACT. In this paper, we obtain new estimates on generalization of Hermite-Hadamard, Simpson and Ostrowski type inequalities for functions whose second derivatives is $\varphi-$convex via fractional integrals.

1. INTRODUCTION

The following inequality is called Hermite-Hadamard Inequality;

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2},$$

where $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function and $a, b \in I$ with $a < b$. If $f$ is concave, then both inequalities hold in the reversed direction.

The inequality (1.1) inequality was first discovered by Hermite in 1881 in the Journal Mathesis. This inequality was known as Hermite-Hadamard Inequality, because this inequality was found by Mitrinovic Hermite and Hadamard’s note in Mathesis in 1974.

The inequality (1.1) is studied by many authors, see ([1],[7],[9],[11],[13],[16]–[22]) where further references are listed.

Firstly, we need to recall some concepts of convexity concerning our work.

Definition 1. [6] A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex on $I$ if inequality

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b),$$

holds for all $a, b \in I$ and $t \in [0, 1]$.

Definition 2. [8] Let $s \in (0, 1]$. A function $f : I \subseteq \mathbb{R}_0 = [0, \infty) \rightarrow \mathbb{R}$ is said to be $s-$convex in the second sense if

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b),$$

holds for all $a, b \in I$ and $t \in [0, 1]$.

Tunç and Yıldırım in [22] introduced the following definition as follows:

Definition 3. A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to belong to the class of $MT$($I$) if it is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ satisfies the inequality;

$$f \left( tx + (1-t)y \right) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f (x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f (y).$$

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Dragomir in [3] introduced the following definition as follows:

**Definition 4.** Let \( \varphi : (0, 1) \rightarrow (0, \infty) \) be a measurable function. We say that the function \( f : I \rightarrow [0, \infty) \) is a \( \varphi \)-convex function on the interval \( I \) if \( x, y \in I \) we have

\[
 f(tx + (1-t)y) \leq t \varphi(t) f(x) + (1-t) \varphi(1-t) f(y).
\]

**Remark 1.** According to definition 4 for the special choose of \( \varphi \) we can obtain following

If we take \( \varphi(t) \equiv 1 \), we obtain classical convex.

If we take \( \varphi(t) = t^{s-1} \), we obtain \( s \)-convex.

If we take \( \varphi(t) = \frac{1}{2 \sqrt{t} \sqrt{1-t}} \), we obtain MT-convex.

Now, we will give some definitions and notations of fractional calculus theory which are used later in this paper. Samko et al. in [15] used following definitions as follows:

**Definition 5.** The Riemann-Liouville fractional integrals \( J^\alpha_a f \) and \( J^\alpha_b f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
 J^\alpha_a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a
\]

and

\[
 J^\alpha_b f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b
\]

where \( f \in L_1[a, b] \), respectively. Note that, \( \Gamma(\alpha) \) is the Gamma function and \( J^\alpha_0 f(x) = J^\alpha_b f(x) = f(x). \)

**Definition 6.** The Euler Beta function is defined as follows:

\[
 \beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0.
\]

The incomplete beta function is defined as follows:

\[
 \beta(a, x, y) = \int_0^a t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0, \quad 0 < \alpha < 1.
\]

2. **Main results**

Throughout this paper, we use \( S_f \) as follows:

\[
 S_f(x, \lambda; a, b) \equiv (1-\lambda) \left\{ \frac{(b-x)^{\alpha+1}-(x-a)^{\alpha+1}}{b-a} \right\} f'(x) + (1+\alpha-\lambda) \left\{ \frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a} \right\} f(x) + \lambda \left\{ \frac{(x-a)^{\alpha}(f(a)+(b-x)^{\alpha} f(b))}{b-a} \right\} - \frac{\Gamma(\alpha+2)}{b-a} \left\{ J^\alpha_x f(a) + J^\alpha_x f(b) \right\},
\]
for any \( x \in [a, b], \lambda \in [0, 1] \) and \( \alpha > 0 \).

In [14], Jackeun Park established the following lemma which is necessary to prove our main results:

**Lemma 1.** Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a twice differentiable function on the interior \( I^0 \) of an interval \( I \) such that \( f'' \in L_1 [a, b] \), where \( a, b \in I \) with \( a < b \). Then, for any \( x \in [a, b], \lambda \in [0, 1] \) and \( \alpha > 0 \) we have

\[
S_f (x, \lambda, \alpha; a, b) = \frac{(x-a)^{\alpha+2}}{b-a} \int_0^1 t (\lambda - t^\alpha) f'' (tx + (1-t) a) dt
+ \frac{(b-x)^{\alpha+2}}{b-a} \int_0^1 t (\lambda - t^\alpha) f'' (tx + (1-t) b) dt.
\]

**Theorem 1.** Let \( \varphi : (0, 1) \rightarrow (0, \infty) \) be a measurable function. Assume also that \( f : I \subseteq [0, \infty) \rightarrow \mathbb{R} \) be a twice differentiable function on the interior \( I^0 \) of an interval \( I \) such that \( f'' \in L_1 [a, b] \), where \( a, b \in I^0 \) with \( a < b \). If \( |f''|^{q} \) is \( \varphi \)-convex on \([a, b]\) for some fixed \( q \geq 1 \), then for any \( x = ta + (1-t)b, t \in [0, 1], \lambda \in [0, 1], \) and \( \alpha > 0 \):

\[
|S_f (x, \lambda, t, \varphi; a, b)| \leq A_1^{\frac{1}{q}} (\alpha, \lambda) \left[ \frac{(x-a)^{\alpha+2}}{b-a} \left\{ A_2 (\alpha, \lambda, t, \varphi) |f'' (x)|^q + A_3 (\alpha, \lambda, t, \varphi) |f'' (b)|^q \right\}^{\frac{1}{q}} \right],
\]

the above inequality for fractional integrals holds, where

\[
A_1 (\alpha, \lambda) = \frac{\lambda^{1+\frac{\alpha}{2}} + 1}{\alpha + 2},
A_2 (\alpha, \lambda, t, \varphi) = \int_0^1 |t (\lambda - t^n)| t \varphi (t) dt,
A_3 (\alpha, \lambda, t, \varphi) = \int_0^1 |t (\lambda - t^n)| (1-t) \varphi (1-t) dt.
\]

**Proof.** By using Lemma 1, the power mean inequality, then we get

\[
|S_f (x, \lambda, t, \varphi; a, b)| \\
\leq \frac{(x-a)^{\alpha+2}}{b-a} \left( \int_0^1 |t (\lambda - t^n)| dt \right)^{1-\frac{1}{q}} \left( \int_0^1 |t (\lambda - t^n)| |f'' (tx + (1-t) a)|^q dt \right)^{\frac{1}{q}}
+ \frac{(b-x)^{\alpha+2}}{b-a} \left( \int_0^1 |t (\lambda - t^n)| dt \right)^{1-\frac{1}{q}} \left( \int_0^1 |t (\lambda - t^n)| |f'' (tx + (1-t) b)|^q dt \right)^{\frac{1}{q}}
= A_1^{\frac{1}{q}} (\alpha, \lambda) \left[ \frac{(x-a)^{\alpha+2}}{b-a} \left( \int_0^1 |t (\lambda - t^n)| |f'' (tx + (1-t) a)|^q dt \right)^{\frac{1}{q}} \right.
+ \left. \frac{(b-x)^{\alpha+2}}{b-a} \left( \int_0^1 |t (\lambda - t^n)| |f'' (tx + (1-t) b)|^q dt \right)^{\frac{1}{q}} \right],
\]

where

\[
A_1 (\alpha, \lambda) = \int_0^1 |t (\lambda - t^n)| dt = \left( \frac{\alpha^{\frac{1}{\alpha}} + 1}{\alpha + 2} - \frac{\lambda}{2} \right).
\]
By substituting (2.3) and (2.4) in (2.2), we get
\begin{equation}
\begin{aligned}
&I_1 = \int_0^1 |t (\lambda - t^\alpha)| \| f'' (tx + (1-t) a) \|^q dt \\
&\le \int_0^1 |t (\lambda - t^\alpha)| \left\{ t \varphi (t) \| f'' (x) \|^q + (1-t) \varphi (1-t) \| f'' (a) \|^q \right\} dt \\
&= A_2 (\alpha, \lambda, t, \varphi) \| f'' (x) \|^q + A_3 (\alpha, \lambda, t, \varphi) \| f'' (a) \|^q,
\end{aligned}
\end{equation}
and similarly we can obtain
\begin{equation}
\begin{aligned}
&I_2 = \int_0^1 |t (\lambda - t^\alpha)| \| f'' (tx + (1-t) b) \|^q dt \\
&\le \int_0^1 |t (\lambda - t^\alpha)| \left\{ t \varphi (t) \| f'' (x) \|^q + (1-t) \varphi (1-t) \| f'' (b) \|^q \right\} dt \\
&= A_2 (\alpha, \lambda, t, \varphi) \| f'' (x) \|^q + A_3 (\alpha, \lambda, t, \varphi) \| f'' (b) \|^q.
\end{aligned}
\end{equation}
By substituting (2.3) and (2.4) in (2.2), we get
\begin{equation}
\begin{aligned}
&|S_f (x, \lambda, \alpha, t, \varphi; a, b)| \\
&\le \left( \frac{\alpha \lambda^1 + \frac{\alpha + 1}{2}}{\alpha + 2} - \frac{1}{2} \right) \frac{1}{\varphi^q} \left\{ \int_0^1 |f'' (x)\|^q \int_0^1 |t (\lambda - t^\alpha) t \varphi (t) dt \right\} \\
&+ |f'' (a)|^q \int_0^1 |t (\lambda - t^\alpha) (1-t) \varphi (1-t) dt \right\} \frac{1}{\varphi} \\
&+ \frac{(b-x)^{\alpha + 2}}{b-a} \left\{ \int_0^1 |f'' (x)\|^q \int_0^1 |t (\lambda - t^\alpha) t \varphi (t) dt \right\} \frac{1}{\varphi} \\
&+ |f'' (b)|^q \int_0^1 |t (\lambda - t^\alpha) (1-t) \varphi (1-t) dt \right\} \frac{1}{\varphi}.
\end{aligned}
\end{equation}
Thus the proof is completed. \(\square\)

**Corollary 1.** Let \( \varphi (t) = 1 \) in Theorem 1, then we get the following inequality:
\begin{equation}
\begin{aligned}
&|S_f (x, \lambda, \alpha; a, b)| \\
&\le \left( \frac{\alpha \lambda^1 + \frac{\alpha + 1}{2}}{\alpha + 2} - \frac{1}{2} \right) \frac{1}{\varphi^q} \left\{ \int_0^1 |f'' (x)|^q \int_0^1 |t (\lambda - t^\alpha) t \varphi (t) dt \right\} \\
&+ \frac{(b-x)^{\alpha + 2}}{b-a} \left\{ \int_0^1 |f'' (x)|^q \int_0^1 |t (\lambda - t^\alpha) t \varphi (t) dt \right\} \frac{1}{\varphi} \\
&+ \frac{(b-x)^{\alpha + 2}}{b-a} \left\{ \int_0^1 |f'' (x)|^q \int_0^1 |t (\lambda - t^\alpha) t \varphi (t) dt \right\} \frac{1}{\varphi}.
\end{aligned}
\end{equation}
Where
\begin{equation}
\begin{aligned}
A_2 (\alpha, \lambda) &= \int_0^1 |t (\lambda - t^\alpha)| t \varphi (t) dt = \frac{3 - (\alpha + 3) \lambda + 2 \alpha \lambda^1 + \frac{\alpha}{2}}{3 (\alpha + 3)} \\
A_3 (\alpha, \lambda) &= \int_0^1 |t (\lambda - t^\alpha) (1-t) \varphi (1-t) dt \frac{\alpha \lambda^1 + \frac{\alpha}{2}}{\alpha + 2} - \frac{2 \alpha \lambda^1 + \frac{\alpha}{2}}{3 (\alpha + 3)} + \frac{\alpha \lambda}{6} - \frac{\alpha}{(\alpha + 2) (\alpha + 3)}.
\end{aligned}
\end{equation}
Corollary 3. Let \( \varphi(t) = t^{s-1} \) in Theorem 1, then we have

\[
|S_f(x, \lambda, \alpha, t, \varphi; a, b)| \leq \left( \frac{\lambda^{s+2}+1}{\alpha s+2} - \frac{2}{\alpha s+2} \right) \frac{1}{\lambda^s+1} \left[ \frac{(x-a)^{n+2}}{b-a} \left\{ |f''(x)|^q A_4(\alpha, \lambda, s) + |f''(a)|^q A_5(\alpha, \lambda, t, \varphi) \right\} \right]^{\frac{1}{q}}
\]

\[+ \frac{(b-x)^{n+2}}{b-a} \left\{ |f''(x)|^q A_4(\alpha, \lambda, s) + |f''(b)|^q A_5(\alpha, \lambda, t, \varphi) \right\} \]

Where

\[
A_4(\alpha, \lambda, s) = \frac{2\lambda^{s+2}+1}{\alpha s+2} - 2 \frac{\lambda^{s+2}+1}{\alpha s+2} + \frac{1}{\alpha s+2}
\]

\[
A_5(\alpha, \lambda, t, \varphi) = \lambda \beta \left( \lambda^s+1, 2, s+1 \right) - \beta \left( \lambda^s+1, \alpha+2, s+1 \right)
\]

+ \beta \left( 1 - \lambda^s+1, \alpha+2, s+1 \right) - \lambda \beta \left( 1 - \lambda^s+1, 2, s+1 \right).

Theorem 2. Let \( \varphi : (0, 1) \rightarrow (0, \infty) \) be a measurable function. For \( f : I \subset [0, \infty) \rightarrow \mathbb{R} \) be a twice differentiable function on the interior \( I^0 \) assume also that \( f'' \in L_1[a, b] \) where \( a, b \in I^0 \) with \( a < b \). If \( |f''|^q \) is \( \varphi \)-convex on \( [a, b] \) for some fixed \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then for any \( x \in [a, b], \lambda \in [0, 1] \) and \( \alpha > 0 \) the following inequality holds

\[
|S_f(x, \lambda, \alpha, t, \varphi; a, b)| \leq B^+ (\alpha, \lambda, p) \left[ \frac{(x-a)^{n+2}}{b-a} \left\{ |f''(x)|^q + |f''(a)|^q \right\} t^\varphi(t) dt \right]^{\frac{1}{q}}
\]

\[+ \frac{(b-x)^{n+2}}{b-a} \left\{ |f''(x)|^q + |f''(b)|^q \right\} \frac{1}{t^\varphi(t) dt} \]

where

\[
B(\alpha, \lambda, p) = \frac{\lambda^{1+p+\alpha}}{\alpha} \left\{ \Gamma(1+p) \Gamma \left( \frac{1+p+\alpha}{\alpha} \right) \left( {}_2F_1 \left( 1, 1+p, 2+p + \frac{1+p}{\alpha}, 1 \right) \right) \right\}
\]

+ \beta \left( 1 + p, -\frac{1+p+\alpha}{\alpha} \right) - \beta \left( \lambda, 1 + p, -\frac{1+p+\alpha}{\alpha} \right),

also, for \( 0 < b < c \) and \( |z| < 1 \), \( {}_2F_1 \) is hypergeometric function defined by

\[
{}_2F_1(a, b, c, z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt.
\]
Proof. By using Lemma 1 and the Hölder inequality, we have the below inequality

\[
|S_f(x, \lambda, \alpha, t, \varphi; a, b)| \\
\leq \frac{(t-a)^{\gamma+2}}{b-a} \left( \int_0^1 |t (\lambda - t^\alpha)|^p \, dt \right)^{\frac{1}{p}} \left( \int_0^1 |f''(tx + (1-t)a)|^q \, dt \right)^{\frac{1}{q}} \\
+ \frac{(b-x)^{\gamma+2}}{b-a} \left( \int_0^1 |t (\lambda - t^\alpha)|^p \, dt \right)^{\frac{1}{p}} \left( \int_0^1 |f''(tx + (1-t)b)|^q \, dt \right)^{\frac{1}{q}} \\
= \left( \frac{1}{\lambda} \int_0^1 |t (\lambda - t^\alpha)|^p \, dt \right)^{\frac{1}{p}} \left[ \frac{(x-a)^{\gamma+2}}{b-a} \left( \int_0^1 |f''(tx + (1-t)a)|^q \, dt \right)^{\frac{1}{q}} \\
+ \frac{(b-x)^{\gamma+2}}{b-a} \left( \int_0^1 |f''(tx + (1-t)b)|^q \, dt \right)^{\frac{1}{q}} \right].
\]

(2.6)

Since \(|f''|\) is \(\varphi\)-convex on \([a, b]\), we have

\[
\int_0^1 |f''(tx + (1-t)a)|^q \, dt \leq \int_0^1 t \varphi(t) |f''(x)|^q \, dt \\
+ \int_0^1 (1-t) \varphi(1-t) |f''(a)|^q \, dt \\
= \left( |f''(x)|^q + |f''(a)|^q \right) \int_0^1 t \varphi(t) \, dt,
\]

(2.7)

and using same technique, we get

\[
\int_0^1 |f''(tx + (1-t)b)|^q \, dt \leq \int_0^1 t \varphi(t) |f''(x)|^q \, dt \\
+ \int_0^1 (1-t) \varphi(1-t) |f''(b)|^q \, dt \\
= \left( |f''(x)|^q + |f''(b)|^q \right) \int_0^1 t \varphi(t) \, dt.
\]

(2.8)

On the other hand we can obtain the following equality:

\[
B(\alpha, \lambda, p) = \int_0^1 |t (\lambda - t^\alpha)|^p \, dt \\
= \int_0^{\lambda^\frac{1}{\alpha}} \{t(\lambda - t^\alpha)\}^p \, dt + \int_{\lambda^\frac{1}{\alpha}}^1 \{t(t^\alpha - \lambda)\}^p \, dt \\
= C_1(\alpha, \lambda, p) + C_2(\alpha, \lambda, p).
\]

(2.9)
By letting \( \lambda - t^\alpha = u \) and \( t^\alpha = u \), respectively, we have

\begin{equation}
C_1(\alpha, \lambda, p) = \int_0^{\lambda \frac{p}{\alpha}} \{ t (\lambda - t^\alpha) \}^p \, dt
= \frac{1}{\alpha} \int_0^{\lambda} u^p (\lambda - u)^{\frac{1+p}{\alpha} - \alpha} \, du
= \frac{1}{\alpha} \int_0^{1} \lambda^{\frac{p}{\alpha} + 1/p} u^p \left( 1 - \frac{1}{\alpha} \right)^{- \frac{1}{\alpha}} \lambda \, dy
= \frac{\lambda^{\frac{p}{\alpha} + 1/p}}{\alpha} \int_0^{1} y^p \left( 1 - \frac{1}{\alpha} \right)^{- \frac{1}{\alpha}} \, dy
= \frac{\lambda^{\frac{p}{\alpha} + 1/p}}{\alpha} \Gamma \left( 1 + p \right) \Gamma \left( \frac{1+p+\alpha}{\alpha} \right) _2 F_1 \left( 1, 1+p, 2+p + \frac{1+p}{\alpha}, 1 \right),
\end{equation}

and

\begin{equation}
C_2(\alpha, \lambda, p) = \int_{\lambda}^{1} \frac{1}{\lambda + \frac{p}{\alpha}} \{ t (t^\alpha - \lambda) \}^p \, dt
= \frac{1}{\alpha} \int_{\lambda - \frac{1+p}{\alpha}}^{1} (u - \lambda)^p \, du
= \frac{\lambda^{\frac{p}{\alpha} + 1/p}}{\alpha} \left\{ \beta \left( 1 + p, -\frac{1+p+\alpha}{\alpha} \right) - \beta \left( \lambda, 1 + p, -\frac{1+p+\alpha}{\alpha} \right) \right\}.
\end{equation}

Thus, we get the desired result. \( \square \)

**Corollary 4.** Let \( \varphi(t) = 1 \) in Theorem 2, then we get the following inequality for any \( x \in [a, b], \lambda \in [0, 1] \) and \( \alpha > 0 \);

\[
|S_f(x, \lambda, \alpha, t, \varphi; a, b)| \leq \left( \int_0^1 |t (\lambda - t^\alpha)|^p \, dt \right)^{\frac{q}{p}} \left[ \frac{(x-a)^{\alpha+2}}{b-a} \left\{ \frac{|f''(x)|^q + |f'(a)|^q}{2} \right\} \right]^{\frac{1}{q}}
+ \frac{(b-x)^{\alpha+2}}{b-a} \left\{ \frac{|f''(x)|^q + |f'(b)|^q}{2} \right\}^{\frac{1}{q}}.
\]

**Corollary 5.** If we choose \( \varphi(t) = 1 \) and \( x = \frac{a+b}{2} \) in Theorem 2, we can obtain the corollary 2.6, 2.7, 2.8 in (14), respectively for \( \lambda = \frac{1}{3}, \lambda = 0, \lambda = 1 \).

**Corollary 6.** Let in \( \varphi(t) = t^\alpha \) Theorem 2, then we obtain

\[
|S_f(x, \lambda, \alpha, t, \varphi; a, b)| \leq \left( \int_0^1 |t (\lambda - t^\alpha)|^p \, dt \right)^{\frac{q}{p}} \left[ \frac{(x-a)^{\alpha+2}}{b-a} \left\{ \frac{|f''(x)|^q + |f'(a)|^q}{s+1} \right\} \right]^{\frac{1}{q}}
+ \frac{(b-x)^{\alpha+2}}{b-a} \left\{ \frac{|f''(x)|^q + |f'(b)|^q}{s+1} \right\}^{\frac{1}{q}}.
\]
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