New non-local SUSY KdV conservation laws from a recursive gradient algorithm

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February 1, 2008

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Abstract

A complete proof of the recursive gradient approach is presented. It gives a construction of all the hierarchy structures of $N = 1$ Super KdV, including the non-local one. A precise definition of the ring of superfields involved in the non-local construction is given. In particular, new non-local conserved quantities of $N = 1$ Super KdV are found.

1 Introduction

KdV equations describe commuting flows in the space of Schrödinger equations,

$$\frac{\partial Q}{\partial t_n} = [M_n, Q],$$

where $Q = \frac{d^2}{dx^2} + U(x, t_n)$ is a Schrödinger operator.

The KdV hierarchy is almost determined by requiring that $[M_n, Q]$ be a zero order differential operator. Besides their relevance as an integrable system, KdV equations are directly related to two-dimensional topological gravity and string theory. It was conjectured by E. Witten [1] that the KdV hierarchy governs the stable intersection theory on the moduli spaces of Riemann surfaces. A generalization of that conjecture considers a Riemann surface $\Sigma$ together with a holomorphic map of $\Sigma$ to a fixed complex manifold [1]. This holomorphic immersion naturally occurs in the formulation of $D = 11$
Supermembranes with central charges \([2, 3, 4]\) which in turn may be formulated as a noncommutative gauge theory \([5]\).

A supersymmetric extension of the KdV equations was introduced in \([6]\) and independently in \([7, 8]\), where a detailed analysis of the system was performed. For a review see \([9]\).

In the same way that the KdV equation is related to the Schrödinger operator of quantum mechanics, the supersymmetric KdV (SKdV) equations are related to supersymmetric quantum mechanics. In \([10]\) it was shown that the Green’s function of the SUSY quantum operator is well defined and that its asymptotic expansion when \(t \to 0^+\) provides all the SKdV hierarchy. In \([8]\) a super Gardner transformation was introduced allowing one to obtain all the known local conserved quantities of the SKdV equations from a single conserved quantity of the Super Gardner equation. This super-transformation generalizes the well-known Gardner transformation for the KdV equation \([12]\). See also \([13, 14, 15]\). An important distinction between the SKdV and KdV hierarchies is that the former presents non-local conserved quantities. The earliest non-local conserved quantities to appear were first presented in \([16]\) and later in \([17]\), where they were obtained from a Lax formulation of the SKdV hierarchy and generated from the super residue of a fractional power of the Lax operator. These non-local conserved quantities are “fermionic” in distinction to the known “bosonic” local ones.

The infinite set of non-local conserved quantities was also obtained from a single fermionic non-local conserved quantity of the Super Gardner equation \([11]\), where the Gardner category was introduced.

In \([10]\) a recursive gradient approach was proposed to analyze the the SKdV (local) hierarchy and its local conserved quantities. The algorithm starting from the gradient of a conserved quantity generates, by application of operators \(P, D^{-2}, K\), a new gradient of an associated new conserved quantity. The algorithm provides all local conserved quantities as well as the SKdV hierarchy of differential equations.

The existence of all such quantities is proven by induction using the exact SUSY sequence introduced in \([10]\).

One crucial step in the proof, which was missing in our previous work, is to show that after applying \(D^{-2}\) one still obtains a local quantity which indeed is the gradient of a conserved quantity.

In the first part of this work we present a complete proof of the recursive gradient approach, for an initial data corresponding to a local conserved quantity of the SKdV equation. In the second part of this work we give a precise definition of the function spaces where the non-local conserved quantities exist. We then apply the recursive gradient approach to an initial data corresponding to a fermionic non-local conserved quantity. It then turns out that one can obtain step by step the complete structure of fermionic non-local conserved quantities. We do not have, however, an inductive proof as in the case of the local initial data. Finally we introduce initial data which give rise to a new set of non-local conserved quantities of the SKdV equations. We find explicitly the first few of
them. The new non-local conserved quantities are bosonic, in distinction to the previously known ones which are fermionic.

2 The Recursive Gradient Algorithm

The Susy KdV equation involves functions $\mathbb{R} \to \Lambda$, with $\Lambda$ a finitely generated exterior algebra. With $\theta \in \Lambda$ one of the generators, the operator $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}$ sends $C^\infty(\mathbb{R}, \Lambda)$ into itself, and interchanges the two direct summands given by the parity of $\Lambda$.

With $\mathcal{C}^\infty(\mathbb{R}, \Lambda)$ the rapidly diminishing functions, the formula $\frac{\partial}{\partial \theta} \int_{-\infty}^{\infty} \Phi(x) dx$ gives a linear functional $\mathcal{C}^\infty(\mathbb{R}, \Lambda) \to \Lambda$ which vanishes on the image of $D$.

From a given $\Phi$ there arises $f(\Phi, D\Phi, D^2\Phi, \ldots)$ where $f$ can be any polynomial in several variables. Then $f$ can give a nonlinear differential equation $\frac{\partial}{\partial t} \Phi(x, t) = f(\Phi, D\Phi, D^2\Phi, \ldots)$, while another polynomial $h$ might give a conserved quantity $\frac{\partial}{\partial \theta} \int h(\Phi, D\Phi, \ldots) dx$.

In the following, an algebraic model is proposed for the study of these questions.

The preceding scenario is replaced by a free derivation algebra on a single fermionic generator, and the $D$ just given is replaced by an algebraically constructed derivation designed to reflect the general properties of the analytical $D$.

Operators, pseudodifferential operators and adjoint involutions are described. The results are then applied to show that the gradients of the local and non-local conserved quantities of the supersymmetric KdV equation are generated by a recursive algorithm formulated in this algebraic context.

3 The derivation algebra

Let $\mathcal{A}$ be the free supersymmetric derivation algebra on a single fermionic generator. It is generated over the real number field by an identity element and elements $a_1, a_2, a_3, \ldots$ subject only to the relations $a_pa_q = a_qa_p(-1)^{pq}$. Anticommutations only occur among $a_1, a_3, a_5, \ldots$, all of whose squares are zero.

The parity involution $u \to \overline{u}$ is the algebra automorphism of $\mathcal{A}$ determined by $\overline{a_p} = a_p(-1)^p$.

Then $\mathcal{A} = \mathcal{A}_{\text{even}} \oplus \mathcal{A}_{\text{odd}}$ by the $\pm 1$ eigenspaces of the parity involution, making $\mathcal{A}$ into a supercommutative superalgebra.

The canonical superderivation $D : \mathcal{A} \to \mathcal{A}$ will give $Da_p = a_{p+1}$ for $p \geq 1$, and satisfy the twisted product rule

$$D(uv) = (Du)v + \overline{u}(Dv)$$

for all $u, v \in \mathcal{A}$. Furthermore $D$ reverses parity, which is to say that $DA_{\text{odd}} \subset \mathcal{A}_{\text{even}}$ and $DA_{\text{even}} \subset \mathcal{A}_{\text{odd}}$.

The Euler operator $E : \mathcal{A} \to \mathcal{A}$ will have values $Ea_p = a_p$ for $p \geq 1$, and satisfy the ordinary product rule

$$E(uv) = (Eu)v + u(Ev),$$
as well as \( E A_{\text{odd}} \subset A_{\text{odd}} \) and \( E A_{\text{even}} \subset A_{\text{even}} \).

The operators \( D \) and \( E \) are constructed from the operators \( \frac{\partial}{\partial a_p} : A \to A \), as follows.

Given \( 1 \leq p < \infty \), the complementary subalgebra \( A_p \subset A \) is generated by the identity element and the \( a_q \) for which \( q \neq p \). Then as a vector space direct sum

\[
A = A_p \oplus a_p A_p \oplus a_p^2 A_p + \cdots
\]

and \( \frac{\partial}{\partial a_p} : A \to A \) is defined in the customary fashion.

When \( p \) is even, \( \frac{\partial}{\partial a_p} \) preserves parity and satisfies the ordinary product rule. When \( p \) is odd the direct sum reduces to \( A_p \oplus a_p A_p \), and \( \frac{\partial}{\partial a_p} \) reverses parity and satisfies the twisted product rule. These operators satisfy the commutation rule \( \frac{\partial}{\partial a_p} \frac{\partial}{\partial a_q} = \frac{\partial}{\partial a_q} \frac{\partial}{\partial a_p} (-1)^{pq} \).

The claimed properties of \( D \) and \( E \) then follow from the explicit formulas

\[
D = a_2 \frac{\partial}{\partial a_1} + a_3 \frac{\partial}{\partial a_2} + \cdots
\]

\[
E = a_1 \frac{\partial}{\partial a_1} + a_2 \frac{\partial}{\partial a_2} + \cdots
\]

Furthermore the commutator \( (DE - ED) : A \to A \) satisfies the twisted product rule; the operator identity \( DE = ED \) then follows from its truth on the generating elements \( a_1, a_2, \ldots \). This shows that \( D \) preserves the homogeneous subspaces of \( A \), that is, the eigenspaces of the Euler operator \( E \).

Given \( u \in A \), the possibility \( u \in DA \) is now investigated. Using the congruence notation \( u \equiv v \) when \( u - v \in DA \), the general fact \( (D^2 f)g \equiv -f(D^2 g) \) when applied to

\[
Eu = a_1 \frac{\partial u}{\partial a_1} + a_2 \frac{\partial u}{\partial a_2} + \cdots
\]

gives

\[
Eu \equiv a_1 \left( \frac{\partial u}{\partial a_1} - D^2 \frac{\partial u}{\partial a_3} + D^4 \frac{\partial u}{\partial a_5} - \cdots \right)
+ \, \, a_2 \left( \frac{\partial u}{\partial a_2} - D^2 \frac{\partial u}{\partial a_4} + D^4 \frac{\partial u}{\partial a_6} - \cdots \right).
\]

Another general fact \( a_2 h \equiv a_1 Dh \) then gives

\[
Eu \equiv a_1 Mu
\]

where \( M : A \to A \) is the gradient operator

\[
M = \frac{\partial}{\partial a_1} + D \frac{\partial}{\partial a_2} - D^2 \frac{\partial}{\partial a_3} - D^3 \frac{\partial}{\partial a_4} + \cdots
\]

Evidently the condition \( Mu = 0 \) implies \( Eu \in DA \); if \( u \) has zero constant term its homogeneous components and hence \( u \) itself are in \( DA \). It is also true that \( MD \equiv 0 \) as an operator \( A \to A \).
4 The algebras $\mathcal{O}_p\mathcal{A} \subset \mathcal{P}_{sd}\mathcal{A}$

When the operator $D$ acts on the product of two elements of $\mathcal{A}$, the result is

$$D(uv) = \overline{u}Dv + (Du)v.$$ 

For higher powers of $D$, the supersymmetric binomial coefficients are needed. They are given by the generating functions

$$F_m(x) = \sum_{p=0}^{\infty} \left[ \begin{array}{c} m \\ p \end{array} \right] x^p$$

in which

$$F_m(x) = \begin{cases} (1 + x^2)^n & \text{when } m = 2n \\ (1 + x^2)^n(1 + x) & \text{when } m = 2n + 1. \end{cases}$$

For $\overline{u} = \pm u$ one can use the notation $u = u_M, \overline{u}_M = u_M(-1)^M$. The images of $u_M$ under repeated applications of $D$ can then be written $D^p u_M = u_{M+p}$ with $\overline{u}_{M+p} = u_{M+p}(-1)^{M+p}$.

Then for any integer $m > 0$ the appropriate Leibnitz formula is

$$D^m(u_Mv) = \sum_{p=0}^{m} (-1)^{M(m+p)} \left[ \begin{array}{c} m \\ p \end{array} \right] u_{M+p}D^{m-p}v.$$ 

For $m = 1$ it gives $D(u_Mv) = (-1)^M u_MDv + u_{M+1}v$ as it should. For a proof by induction one passes from $m$ to $m + 1$ by computing

$$D(u_{M+p}D^{m-p}v) = (-1)^{M+p} u_{M+p}D^{m+1-p}v + u_{M+p+1}D^{m-p}v.$$ 

The identity

$$\left[ \begin{array}{c} m \\ p - 1 \end{array} \right] + (-1)^p \left[ \begin{array}{c} m \\ p \end{array} \right] = \left[ \begin{array}{c} m + 1 \\ p \end{array} \right]$$

then gives the desired result: it follows from the recursion

$$xF_m(x) + F_m(-x) = F_{m+1}(x)$$

satisfied by the generating functions.

The algebra $\mathcal{O}_p\mathcal{A}$ consists of the linear transformations $L : \mathcal{A} \rightarrow \mathcal{A}$ which have the form $L = \sum_{n=0}^{N} l_n D^n$ for some $l_n \in \mathcal{A}$ and $0 \leq N < \infty$. The parity involution of $\mathcal{O}_p\mathcal{A}$ sends $L$ to $\overline{L} = \sum_{n=0}^{N} l_n(-D)^n$.

Thus $L$ is “oriented” when $\overline{L} = \pm L$, and

$$\mathcal{O}_p\mathcal{A} = (\mathcal{O}_p\mathcal{A})_{even} \oplus (\mathcal{O}_p\mathcal{A})_{odd}$$
is a superalgebra.

The product of oriented operators is defined by bilinear expansion from the special case
\[
(u_M D^m)(v_N D^n) = \sum_{p \geq 0} (-1)^{N(m+p)} \binom{m}{p} u_M v_{N+p} D^{m+n-p}.
\]

Since the product of operators is defined independently as the composition of linear transformations of a vector space, the associativity of the product would seem to be clear.

But there are no negative powers of $D$ in $O_pA$. For this reason $O_pA$ is enlarged to $P_{sd}A$, whose elements are the formal semi-infinite sums
\[
L = \sum_{-\infty}^{N} l_n D^n
\]
with $l_n \in A$ and $-\infty < N < \infty$. The same parity involution is present, and the product of two oriented elements of $P_{sd}A$ is given by bilinear expansion using the same formula for $(u_M D^m)(v_N D^n)$ as in $O_pA$, but with $0 \leq p < \infty$.

When $m < 0$ the coefficients $\binom{m}{p}$ do not vanish identically for $p >> 0$, and they leave the product as a semi-infinite formal sum.

The associativity equation $A(BC) = (AB)C$ must now be established for any three elements $A, B, C \in P_{sd}A$.

A first observation is that any equation $A(BC) = (AB)C$ in $P_{sd}A$ may be multiplied on the left by $h_kI$ and on the right by $D^r$, giving another such equation
\[
(h_k A) (B(CD^r)) = ((h_k A) B) (CD^r).
\]

Then two more special cases are sufficient for the general result. First, when
\[
D^m(u_MIv_NI) = (D^m(u_MI))(v_NI)
\]
is expanded, it is seen to follow from the cancellation identity
\[
\binom{m}{p} \binom{m-p}{q} = \binom{m}{p+q} \binom{p}{q}
\]
which holds for all $p, q \geq 0$ and $-\infty < m < \infty$.

Second, when
\[
D^m(D^n(v_NI)) = D^{n+m}(v_NI)
\]
is worked out, it is seen to follow from the “sum-of-exponents” identities
\[
\binom{n+m}{p} = \sum_{r+s = p} \binom{n}{r} \binom{m}{s} (-1)^{r(m+p+1)}.
\]
These identities in turn follow from the equations connecting the generating function $F_{n+m}(x)$ with $F_n(\pm x)$ and $F_m(x)$.

Multiplying on the left by elements of $A$ and on the right by powers of $D$, we obtain general elements of $P_{sd}A$.

Thus $A(BC) = (AB)C$ is proven when $B = h_kI$ or $B = D^r$.

Finally, to prove associativity for three elements $D^m, u_M D^n,$ and $v_NI$, we compute

\[
D^m((u_M D^n)(v_NI)) = D^m(u_M (D^n (v_NI))) = (D^m(u_M I))(D^n (v_NI))
\]

and

\[
(D^m(u_M D^n))v_NI = ((D^m(u_M I))D^n (v_NI)) = (D^m(u_M I))(D^n (v_NI)).
\]

This equality completes the proof that $P_{sd}A$ is an associative algebra.

A sample formula in $P_{sd}A$ is

\[
D^{-2}(uI) = uD^{-2} - (D^2 u)D^{-4} + (D^4 u)D^{-6} - \cdots \in P_{sd}A;
\]

it will be used in the applications which follow.

5 The Adjoint Involution

The parity-preserving involution $L \to L^*$ of $P_{sd}A$ with itself is determined by the three properties

\begin{enumerate}
  \item $(uI)^* = uI$ for $u \in A$
  \item $D^* = -D$
  \item $(L_1L_2)^* = (-1)^{\lambda_1 \lambda_2}L_2^*L_1^*$,
\end{enumerate}

when $L_1, L_2 \in P_{sd}A$ have parities $(-1)^{\lambda_1}$ and $(-1)^{\lambda_2}$.

The last two properties when applied to powers of $D$ give

\[
(D^n)^* = (-1)^{\frac{n(n+1)}{2}}D^n.
\]

Then, when $u_N \in A$ has parity $(-1)^N$, the adjoint of $L = u_N D^n$ must be defined by

\[
L^* = (-1)^{nN} (D^n)^*(u_NI) = (-1)^{\frac{n(n+1)}{2}} \sum_{p=0}^{\infty} (-1)^{Np} \binom{n}{p} u_{N+p}D^{n-p}
\]
with $u_{N+p} = D^p u_N$ as before. Since every element of $P_{sd}A$ is a formal sum of powers of $D$ multiplied from the left by elements of $A$, this construction gives a well-defined linear transformation $L \rightarrow L^*$ of $P_{sd}A$ into itself.

But to verify (3) when $L_1 = D^m$ and $L_2 = u_N I$, the product $D^m(u_N I)$ must first be expanded as an infinite linear combination of $u_{N+q} D^{m-q}$ with $q \geq 0$, and then the $L \rightarrow L^*$ construction just given must be applied to each term. The result is a double summation over $q \geq 0, p \geq 0$, and property (3) reduces to the identities

$$\sum_{p+q=r>0} (-1)^{pq+m(q-1)} \binom{m}{q} \binom{m-q}{p} = 0.$$

The cancellation identity of the last section puts this into the form

$$\left[ \begin{array}{c} m \\ r \end{array} \right] \sum_{q=0}^r \varepsilon(q) \left[ \begin{array}{c} r \\ q \end{array} \right] = 0$$

with $\varepsilon(q) = (-1)^{q(r-q)+\frac{1}{2}(m-q)(m+1-q)}$.

When $r \geq 2$ the generating function $F_r(x)$ satisfies $F_r(i) = 0$ with $i^2 = -1$, giving

$$0 = \left[ \begin{array}{c} r \\ 0 \end{array} \right] - \left[ \begin{array}{c} r \\ 2 \end{array} \right] + \left[ \begin{array}{c} r \\ 4 \end{array} \right] - \cdots$$

$$0 = \left[ \begin{array}{c} r \\ 1 \end{array} \right] - \left[ \begin{array}{c} r \\ 3 \end{array} \right] + \left[ \begin{array}{c} r \\ 5 \end{array} \right] - \cdots$$

Since $\varepsilon(q+2) = -\varepsilon(q)$ for all integers $q$, the identity is proved in the case $r \geq 2$.

In the remaining case $r = 1$, $\left[ \begin{array}{c} m \\ r \end{array} \right]$ can be nonzero only when $m$ is odd: then $\varepsilon(0) + \varepsilon(1) = 0$ and $\left[ \begin{array}{c} r \\ 0 \end{array} \right] = \left[ \begin{array}{c} r \\ 1 \end{array} \right] = 1$.

Thus (3) is confirmed in all the four cases where $L_1$ and $L_2$ can be $u_N I, v_M I$, or powers of $D$. Using associativity in $v_M u_N D^n$ and $u_N D^n v_M$, these four cases give

$$v_M L \ast = (-1)^{M \lambda} L \ast (v_M I) \ast$$

$$L D^m \ast = (-1)^{\lambda m} (D^m) \ast L^\ast$$

for $L = u_N D^n$ and $\lambda = N + n$.

Finally, the general case $L_1 L_2 = u_N D^n v_M D^m$ can be expanded by applying the preceding special cases. This gives

$$(L_1 L_2)^\ast = \pm (D^m)^\ast (v_M I)^\ast (D^n)^\ast (u_N I)^\ast$$

$$= \pm L_2^\ast L_1^\ast,$$

the $\pm$ sign given by $(-1)^{\lambda_1 \lambda_2}$ with $\lambda_1 = N + n, \lambda_2 = M + m$.

This completes the proof of (3) for all elements of $P_{sd}A$. The involutive property $(L^\ast)^\ast = L$ is a direct consequence.
The Frechet derivative operator

The construction $h \rightarrow L_h$ which takes $h \in \mathcal{A}$ to its Frechet derivative operator $L_h \in \mathcal{O}_p\mathcal{A}$ is now described.

For odd elements $f = -\overline{f}$ in $\mathcal{A}$ the action of $L_h$ is given by

$$h = h(a_1, a_2, a_3, \ldots)$$

$$L_h f = \frac{d}{dt} \bigg|_{t=0} h(a_1 + tf, a_2 + tDf, a_3 + tD^2f, \ldots).$$

For fixed $f$ and varying $h$, the transformation $\mathcal{A} \rightarrow \mathcal{A}$ given by $F h = L_h f$ preserves parity and satisfies the ordinary product rule $F(h_1 h_2) = (F h_1) h_2 + h_1 (F h_2)$. Further, $F a_p = D^{p-1} f$ for all $p \geq 1$.

On the other hand $D : \mathcal{A} \rightarrow \mathcal{A}$ reverses parity and satisfies the twisted product rule.

With $F h = L_h f$, the commutator $(DF - FD) : \mathcal{A} \rightarrow \mathcal{A}$ satisfies the twisted product rule and gives the value zero on all the generators $a_p$. This shows that $[D, F] = 0$ on all elements of $\mathcal{A}$, proving that

$$DL_h f = L_D h f.$$

A second consequence is the explicit formula

$$L_h f = f \frac{\partial h}{\partial a_1} + (D f) \frac{\partial h}{\partial a_2} + (D^2 f) \frac{\partial h}{\partial a_3} + \cdots$$

Indeed, this formula sends $h = a_p$ to $F h = D^{p-1} f$ and satisfies the ordinary product rule when applied to $h_1 h_2$: therefore it must coincide with the $L_h f$ given by the definition not just on the generators $a_p$ but everywhere in $\mathcal{A}$.

The explicit formula gives $L_\overline{a_1} = E h$ and $L_h a_3 = D^2 h$, for example. And when $h$ is oriented with $\overline{h} = h(-1)^\chi$, reorderings and sign changes put $L_h f$ into the standard form $\sum l_n D^n f$.

For oriented elements $h$, the construction $h \rightarrow L_h$ reverses parity, in the sense that when $h$ has parity $(-1)^\chi$, $L_h$ has parity $(-1)^{\chi+1}$. When written out explicitly the Frechet derivative operator is

$$L_h = \sum_{n=1}^{\infty} (-1)^n (\chi+1) \frac{\partial h}{\partial a_n} D^{n-1},$$

giving $L_g = -\left( \frac{\partial g}{\partial a_1} \right) I + \left( \frac{\partial g}{\partial a_2} \right) D - \left( \frac{\partial g}{\partial a_3} \right) D^2 + \cdots$ for example when $\overline{g} = g$ and $\chi$.

When taken together with the construction of adjoint operators, there are two important applications of the Frechet derivative. The first is an analog of the mixed partials criterion in the Poincare lemma:

if $g \in \mathcal{A}$ satisfies $L_g + L_g^* = 0$ then $g$ is the gradient $M h$ of some $h \in \mathcal{A}$, by the exact sequence of calculus of variations.

The second application characterizes those $h \in \mathcal{A}$ which fall into $D^2 \mathcal{A} \subset \mathcal{A}$. 

Indeed, $h = D^2 l$ implies $L_h = D^2 L_l$, which says that $L_h = D^2 Q$ for some $Q \in \mathcal{O}_p \mathcal{A}$. Conversely, applying $L_h = D^2 Q$ to the generating element $a_1$, we obtain $Eh \in D^2 \mathcal{A}$. Since $D$ and $E$ commute, the equation $Eh = D^2 u$ resolves into homogeneous components, giving

$$h = D^2 \left( u_1 + \frac{1}{2} u_2 + \frac{1}{3} u_3 + \cdots \right) \in D^2 \mathcal{A}.$$  

Taking adjoints, $L_h = D^2 Q$ becomes $L_h^* = -Q^* D^2$. This means that $h \in \mathcal{A}$ with zero constant term will fall into $D^2 \mathcal{A}$ if and only if the bottom two coefficients of the adjoint of its Frechet derivative operator are zero, that is,

$$L_h^* = 0 \cdot I + 0 \cdot D + (?D^2 + \cdots)$$

The same reasoning when applied to $D$ instead of $D^2$ suggests that

$$L_h^* = (\pm Mh) I + (?) D + \cdots,$$

this is indeed the case when $h = \pm \overline{h}$.

7 The Recursion

The recursive algorithm for the gradients of conserved quantities claims the existence of even elements $g_2, g_4, g_6, \ldots$ and odd elements $f_3, f_5, f_7, \ldots$ in $\mathcal{A}$ which satisfy

$$P g_n = D^2 f_{n+1}$$

$$K f_{n+1} = D^2 g_{n+2}$$

for the operators

$$P = D^5 + 3a_1 D^2 + a_2 D + 2a_3 I$$

$$K = D^3 + a_1 I.$$  

The additional condition $L_{g_n} + L_{g_n}^* = 0$ is, by the exact sequence, equivalent to the existence of $h_n \in \mathcal{A}$ having $g_n$ as its gradient.

By direct computation one can check that the choice $g_2 = a_2, f_3 = a_5 + 3a_1 a_2, g_4 = a_6 + 3a_2^2 - 2a_1 a_3$ satisfies the recursion, and that the Frechet derivative operators $L_{g_2} = D$ and $L_{g_4} = D^5 - 2a_1 D^2 + 6a_2 D + 2a_3 I$ are antisymmetric. It then remains to be shown that the algorithm continues indefinitely.

The operators $P$ and $K$ appearing in the recursion are both odd, with adjoints $K^* = K$ and $P^* = -P$. When $g = \overline{g}$ and $f = -\overline{f}$ in $\mathcal{A}$, the Frechet derivative operators of $Pg$ and $Kf$ are given by

$$L_{Kf} = KL_f - f I$$

$$L_{Pg} = PL_g + R_g$$

$$R_g = 2g D^2 + (Dg) D + (3D^2 g) I$$
and their adjoints by

\[
L^*_K f = L^*_f K - f I \\
L^*_P g = L^*_g P + R^*_g \\
R^*_g = -2gD^2 - (Dg)D + (2D^2g)I.
\]

Given a satisfactory choice of \(g_2, f_3\) and \(g_4\), it must now be shown that \(f_5\) exists, that \(g_6\) exists, and that \(Lg_6\) is antisymmetric.

The existence of \(f_5 \in A\) with \(D^2f_5 = Pg_4\) is determined by the Frechet derivative operator \(L_{Pg_4}\) whose adjoint \(L^*_{Pg_4}\) must be shown to have bottom two coefficients zero.

Using \(L^*_g = -L_g\) this means that

\[
L_{g_4}P - R^*_{g_4} = 0 \cdot I + 0 \cdot D + \cdots,
\]

which in turn would follow from the general fact

\[
L_g P = (2D^2g)I - (Dg)D + \cdots
\]

for any \(g = g\) in \(A\).

If we compute

\[
P = 2a_3I + a_2D + \cdots \\
DP = 2a_4I - a_3D + \cdots \\
D^2P = 2a_5I + a_4D + \cdots
\]

and recall that

\[
L_g = -\frac{\partial g}{\partial a_1}I + \frac{\partial g}{\partial a_2}D - \frac{\partial g}{\partial a_3}D^2 + \cdots
\]

we obtain

\[
L_g P = l_0I + l_1D + \cdots
\]

with

\[
l_0 = 2 \left( -\frac{\partial g}{\partial a_1}a_3 + \frac{\partial g}{\partial a_2}a_4 - \frac{\partial g}{\partial a_3}a_5 + \cdots \right) \\
l_1 = - \left( \frac{\partial g}{\partial a_1}a_2 + \frac{\partial g}{\partial a_2}a_3 + \frac{\partial g}{\partial a_3}a_4 + \cdots \right).
\]

Because \(g\) is even, \(\frac{\partial g}{\partial a_p}a_q = a_q\frac{\partial g}{\partial a_p}(-1)^{|p|}\).

This, together with \(D^2 = a_3\frac{\partial}{\partial a_1} + a_4\frac{\partial}{\partial a_2} + \cdots\), proves that \(l_0 = 2D^2g\) and \(l_1 = -Dg\).

With the general fact established, the existence of \(f_5\) is proven.

The existence of \(g_6\) with \(D^2g_6 = Kf_5\) is determined by the Frechet derivative operator \(L_{Kf_5} = KL_{f_5} - f_5I\), whose adjoint \(L^*_f K - f_5I\) must be shown to have bottom two coefficients zero.
Since $D^2(l_0 I + l_1 D^2 + \cdots) = (D^2 l_0) I + (D^2 l_1) D + \cdots$ and $D : A \to A$ is injective, it suffices to prove that

$$D^2 L_{f_5}^* K = (P g_4) I + 0 \cdot D + \cdots$$

But because the adjoint of this operator has bottom two coefficients zero, it is enough to examine

$$G = KL_{f_5} D^2$$

$$G^* = -D^2 L_{f_5}^* K,$$

and to prove that

$$G + G^* = -(P g_4) I + 0 \cdot D + \cdots$$

The equation $D^2 f_5 = P g_4$ gives $D^2 L_{f_5} = PL_{g_4} + R_{g_4}$ and hence

$$G = KD^{-2} R_{g_4} D^2 + KD^{-2} PL_{g_4} D^2.$$

Then, the recursions $D^2 g_4 = K f_3$ and $D^2 f_3 = P g_2$ permit $L_{f_3}$ to be eliminated between the equations $D^2 L_{g_4} = KL_{f_5} - f_3 I$ and $D^2 L_{f_3} = PL_{g_2} + R_{g_2}$, giving

$$D^2 L_{g_4} = -f_3 I + KD^{-2} R_{g_2} + KD^{-2} PL_{g_2}.$$

Taking the adjoint of this equation and remembering the antisymmetry of $L_{g_4}$ and $L_{g_2}$, we get

$$L_{g_4} D^2 = -f_3 I - R_{g_2}^* D^{-2} K + L_{g_2} P D^{-2} K.$$

This equation permits $G$ to be rewritten as

$$G = KD^{-2} R_{g_4} D^2 - KD^{-2} P(f_3 I)$$

$$-KD^{-2} PR_{g_2}^* D^{-2} K + KD^{-2} PL_{g_2} P D^{-2} K.$$

What must then be shown is that $G + G^* = -(P g_4) I + 0 \cdot D + \cdots$; from the original definition $G = KL_{f_3} D^2 \in \mathcal{O}_p A$ it is clear that no negative powers of $D$ will enter. (In fact it turns out that $G + G^* = (-P g_4) I$ exactly.)

Of the four summands in $G$, all but one are elements of $\mathcal{P}_s d A$ of order $\leq 9$. The four summands and their adjoints are

$$A = KD^{-2} R_{g_4} D^2 \quad A^* = D^2 R_{g_4}^* D^{-2} K$$

$$B = -KD^{-2} P(f_3 I) \quad B^* = f_3 P D^{-2} K$$

$$C = -KD^{-2} P R_{g_2}^* D^{-2} K \quad C^* = -KD^{-2} R_{g_2} P D^{-2} K$$

$$F = KD^{-2} PL_{g_2} P D^{-2} K \quad F^* = KD^{-2} PL_{g_2}^* P D^{-2} K.$$

Of these operators only $F$ might have order $> 9$. However of the seven factors appearing in $F$ five are odd, and $\binom{5}{2} = 10$ an even number. Therefore $F^*$ is as stated above without a minus sign, and the induction hypothesis of the antisymmetry of $L_{g_2}$ gives $F + F^* = 0.$
Regarding $C$, it can be proved that $PR_g^* + R_gP = (3P_g)D^2 + (DP_g)D + (2D^2P_g)I$ for any $g = \mathcal{F}$ in $\mathcal{A}$. Since $P_{g_2} = D^2f_3$ we have

$$C + C^* = -KD^{-2}LD^{-2}K$$

with $L = (3D^2f_3)D^2 + (D^3f_3)D + (2D^4f_3)I$.

The coefficients of $A + A^*, B + B^*, C + C^*$ for nonnegative powers of $D$ can be computed. Summing them to get the coefficients of $G + G^*$, we begin with the positive powers and write

| $G + G^*$ | $A + A^*$ | $B + B^*$ | $C + C^*$ |
|------------|------------|------------|------------|
| $1$        | $2D^4g_4$  | $-2a_1D^2f_3 - 2a_3f_3$ | $-2D^5f_3$ |
| $2$        | $2D^3g_4$  | $-3D^4f_3 + 2a_1Df_3 - 2a_2f_3$ | $D^4f_3$ |
| $3$        | $4D^2g_4$  | $-4a_1f_3$ | $-4D^3f_3$ |
| $4$        | $0$        | $-3D^2f_3$ | $3D^2f_3$ |
| $5$        | $0$        | $0$        | $0$        |

The recursion $D^2g_4 = D^3f_3 + a_1f_3$ shows that $(G + G^*)_3 = 0$; likewise $D^3g_4 = D^4f_3 - a_1Df_3 + a_2f_3$ and $D^4g_4 = D^5f_3 + a_1D^2f_3 + a_3f_3$ proving that $(G + G^*)_n = 0$ for all $n \geq 1$. It only remains to compute $(G + G^*)_0$, the coefficient of the identity operator.

We have

$$(G + G^*)_0 = (a_1D^2 - a_2D - 2a_3)g_4 + (-D^6 + 2a_1D^3 - 4a_2D^2 + a_3D - a_4)f_3 + (-5a_1D^3 + 3a_2D^2)f_3.$$  

Adding

$$Pg_4 = (D^5 + 3a_1D^2 + a_2D + 2a_3I)g_4$$

we obtain

$$(G + G^*)_0 + Pg_4 = (D^5 + 4a_1D^2)g_4 + (-D^6 - 3a_1D^3 - a_2D^2 + a_3D - a_4)f_3.$$  

Since $D^2g_4 = D^3f_3 + a_1f_3$, and since the product of operators gives

$$(D^2 + 4a_1I)(D^3 + a_1I) = D^6 + 3a_1D^3 + a_2D^2 - a_3D + a_4I,$$

we have $(G + G^*)_0 + Pg_4 = 0$.

This completes the proof that $G + G^* = -(Pg_4)I$ when $G = KL_{f_5}D^2$, and consequently that $g_6 \in \mathcal{A}_{even}$ exists with $D^2g_6 = Kf_5$.

It only remains to carry out the third and final step, which is to prove $L_{g_6} + L_{g_6}^* = 0$. This will follow from the equations

$$D^2L_{g_6} = KL_{f_5} - f_5I$$
$$L_{g_6}^*D^2 = -L_{f_5}^*K + f_5I$$  

13
which give
\[
D^2(L_{g_6} + L^*_g_6)D^2 = G + G^* - f_3D^2 + D^2(f_5I) = (-Pg_4)I + (D^2f_5)I = 0.
\]

Since \(L_{g_6} + L^*_g_6 \in \mathcal{O}_P \mathcal{A}\) with \(D^2\) an invertible element of \(\mathcal{P}_{sd} \mathcal{A}\), the result follows.

This completes the proof of the indefinite continuation of the recursive algorithm for the gradients of the local conserved quantities of the Susy KdV equation.

8 Rings of Superfields

A superfield is an infinitely differentiable function \(\Phi : \mathbb{R} \to \Lambda\). The ring \(C^\infty_{NL}\) consists of the nonlocal superfields, those that diminish rapidly at \(x = -\infty\) and increase slowly at \(x = +\infty\). This means that \(\lim_{x \to -\infty} |x|^N \Phi(x) = 0\) for all \(-\infty < N < \infty\) when \(x \to -\infty\) and for some \(N\) as \(x \to +\infty\). The same condition is assumed to hold for all \((\frac{\partial}{\partial x})^p \Phi(x)\).

Two more rings of superfields satisfy the inclusions
\[
C^\infty_I \subset C^\infty_I \subset C^\infty_{NL}.
\]

The ideal \(C^\infty_I\) is the Schwartz space of superfields which diminish rapidly at \(\pm \infty\) together with all \(x\)-derivatives while \(C^\infty_I\) is defined by the condition \(\frac{\partial}{\partial \theta} \Phi \in C^\infty_I\).

The rings \(C^\infty_I\) and \(C^\infty_{NL}\) are invariant under the action of the operator \(D\). In the smaller ring \(D\) is not invertible, \(DC^\infty_I\) being only a proper subspace. But in the larger ring \(C^\infty_{NL}\) the formulas
\[
\Phi(x) = \xi(x) + \theta u(x)
\]
\[
(D^{-1} \Phi)(x) = \int_{-\infty}^{x} u(s) ds + \theta \xi(x)
\]
give \(D^{-1} : C^\infty_{NL} \to C^\infty_{NL}\), the inverse to the bijection of \(C^\infty_{NL}\) with itself which is given by the operator \(D\).

9 Integration

The integration linear functional \(C^\infty_I \to \Lambda\) is given by
\[
\Phi(x) = \xi(x) + \theta u(x)
\]
\[
\int \Phi = \int_{-\infty}^{\infty} u(x) dx.
\]
Evidently \(\int D\Phi = 0\) when \(\Phi \in C^\infty_I\).
Because $C^\infty_1$ is an ideal in $C^\infty_{NL}$, one has integration-by-parts formulas

$$\int (D\Phi)\Psi = \pm \int \Phi(D\Psi)$$

when $\Psi \in C^\infty_{NL}$ and $\Phi \in C^\infty_1$ is oriented.

10 Gradients

A function $H : C^\infty_1 \rightarrow \Lambda$ may be said to have another function $\Gamma : C^\infty_1 \rightarrow C^\infty_{NL}$ as its gradient if, for any $\Phi, \dot{\Phi} \in C^\infty_1$,

$$\frac{d}{dt}|_{t=0} H(\Phi + t\dot{\Phi}) = \int \dot{\Phi} \Gamma(\Phi).$$

In what follows $H$ will have the form $H(\Phi) = \int h(\Phi)$ for some $h : C^\infty_1 \rightarrow C^\infty_1$.

To know that $H$ is a conserved quantity for a differential equation, the preceding equation need only hold when $\dot{\Phi}$ is given in terms of $\Phi, D\Phi, \ldots$ by the differential equation, provided that $\int \dot{\Phi} \Gamma(\Phi) = 0$. Then $\Gamma$ can be called a “restricted” gradient of $H$.

11 The Recursive Algorithm

Given an odd $\Phi \in C^\infty_1$, two operators acting on superfields are given by

$$P = D^5 + 3\Phi D^2 + (D\Phi)D + (2D^2\Phi)I$$

$$K = D^3 + \Phi I.$$

Then five superfields $\Gamma_0, \Omega_1, \Gamma_2, \Omega_3, \Gamma_4$ satisfy the recursion if

$$P\Gamma_0 = \Omega_1 = D^2\Gamma_2$$

$$K\Gamma_2 = \Omega_3 = D^2\Gamma_4.$$

An infinite sequence of superfields $\{\ldots, \Omega_{-1}, \Gamma_0, \Omega_1, \Gamma_2, \ldots\}$ satisfies the recursion if $\Gamma_m, \Omega_{m+1}, \ldots, \Gamma_{m+4}$ are connected by the same equations when $m = 0, \pm 4, \pm 8, \ldots$

Supposing $\Omega_n = \Gamma_m = 0$ for negative integers, the choice of initial value $\Gamma_0 = \frac{1}{2}$ has been shown in the preceding sections to produce $\Omega_1, \Gamma_2, \ldots$ that stay within $C^\infty_1$, despite the apparent presence of $D^{-2}$ in the recursion.

Moreover, $\Gamma_0 = \frac{1}{2}$ gives

$$\Omega_5 = D^6\Phi + 3\Phi D^3\Phi + 3(D\Phi)(D^2\Phi),$$

which defines the SUSY K-dV equation

$$\frac{\partial}{\partial t} \Phi(x, t) = \Omega_5(\Phi, D\Phi, \ldots)$$

for time-dependent odd superfields in $C^\infty_1$. 15
12 Local Conserved Quantities

It was also proved that the superfields $\Gamma_4, \Gamma_8, \ldots \in C_1^\infty$ produced by $\Gamma_0 = \frac{1}{2}$ are all gradients of functions $H_m(\Phi) = \int h_m : C_1^\infty \to \Lambda$ where the $h_m$ are again polynomials in $\Phi, D\Phi, \ldots$, according to the SUSY exact sequence.

The proof that the $H_m$ are conserved quantities follows from the operators that appear in the recursion:

If $L$ is any one of the three operators $P, K$ and $D^2$ one has

$$\int (L f) g = \pm \int f (L g)$$

when $f$ and $g$ are oriented elements of $C_1^\infty$, and at least one of them is in $C_1^\infty$.

Since $\Omega_n \in C_1^\infty$ for all odd $n$, this gives

$$\int \Omega_n \Gamma_m = \int \Omega_{n-4} \Gamma_{m+4} = \cdots 0$$

for all odd $n$ and even $m$, in consequence of the recursion relations.

This shows that the $H_m$ are conserved quantities for the differential equations given by the $\Omega_n$.

13 Nonlocal Conserved Quantities

In general the successive application of the operators $P, K$ and $D^2$ can only be expected to produce superfields in $C_{NL}^\infty$.

Nonetheless, other choices of initial values such as $\Gamma_0 = \theta, \Gamma_2 = 1, \Gamma_2 = \theta$ will produce infinite sequences of superfields satisfying the recursion, because $D^2 \theta = D^2 1 = 0$.

The choice $\Gamma_0 = \frac{1}{2}$ produces $\Omega_1, \Gamma_2, \ldots$ that stay within $C_1^\infty$. The other three choices produce superfields in $C_{NL}^\infty$. The initial values $\Gamma_0 = \theta, \Gamma_2 = 1$ produce the infinite sequence of gradients and non-local fermionic conserved quantities already known in the literature [17, 15]. The initial value $\Gamma_2 = \theta$ give rise to new non-local bosonic conserved quantities.

If $\{\ldots, \tilde{\Gamma}_m, \tilde{\Omega}_{m+1}, \ldots\} \subset C_{NL}^\infty$ is such a sequence then $\Omega_n \tilde{\Gamma}_m = 0$ continues to hold for even $m$ and odd $n$ if $\tilde{\Gamma}_m = \tilde{\Omega}_{m+1} = 0$ when $m < 0$. This suggests that some $\tilde{H}_m : C_1^\infty \to \Lambda$ may exist having $\tilde{\Gamma}_m$ as its gradient.

If so, $\tilde{\Gamma}_m$ would be a nonlocal conserved quantity for the SUSY KdV equation.

This possibility is checked for the initial value $\tilde{\Gamma}_2 = \theta$, and is seen to hold at least for the first two gradients. The computations follow.
14 New Non-local Conserved Quantities

Writing \( a_1 = \Phi \), any odd element of \( C^\infty_1 \), the images under applications of \( D \) and \( D^{-1} \) are written \( a_n = D^n \Phi \in C^\infty_{NL} \) when \( n \leq 0 \), \( a_n \in C^\infty_1 \) when \( n \geq 1 \).

Multiplication from the left by \( a_n \) gives a linear operator \( C^\infty_{NL} \to C^\infty_{NL} \), as do \( D \) and \( D^{-1} \).

The formulas
\[
Da_1 + a_1 D = a_2 \\
Da_2 - a_2 D = a_3 \\
Da_3 + a_3 D = a_4, \\
\vdots
\]
are identities in the ring of linear operators \( C^\infty_{NL} \to C^\infty_{NL} \).

The KdV element corresponding to \( a_1 = \Phi \) is \( b_3 = D^2 b_1 \) with \( b_1 = a_5 + 3a_1 a_2 \).

Since the integration functional \( C^\infty_I \to \Lambda \) is identically zero on \( DC^\infty_1 \subset C^\infty_I \), it suffices to do computations in the quotient space \( C^\infty_I / DC^\infty_1 \). Thus for example \( b_1 D^2 \Phi = b_2 D \Phi = -b_3 \Phi \) for any \( \Phi \in C^\infty_{NL} \), because \( D(b_1 D \Phi), D^2(b_1 \Phi) \in DC^\infty_1 \).

Further, a function \( h : C^\infty_I \to C^\infty_I \) gives a conserved quantity for Super KdV if \( \delta h = \frac{d}{dt} |_{t=0} h(a_1 + tb_3) = 0 \), as an element of the quotient space.

15 The first gradient

The operators in the recursion are written as before as
\[
K = D^3 + a_1 I \\
P = D^5 + 3a_1 D^2 + a_2 D + 2a_3 I.
\]
With \( 0 = \cdots = \tilde{\Gamma}_0 = \tilde{\Omega}_1 \) and \( \tilde{\Gamma}_2 = \theta \), the next step in the recursion is \( \tilde{\Gamma}_4 = D^{-2}K\tilde{\Gamma}_2 = D^{-2}a_1 \theta \).

Then \( b_3 \tilde{\Gamma}_4 = -b_1 a_1 \theta \).

The function \( h = a_1 a_{-1} \theta \) sends \( C^\infty_1 \) into itself, and its gradient is computed by
\[
\delta h = b_3 a_{-1} \theta + a_1 b_1 \theta \\
= -b_1 D^2 a_{-1} \theta - b_1 a_1 \theta \\
= -2b_1 a_1 \theta.
\]
The equality
\[
\frac{d}{dt} |_{t=0} h(a_1 + tb_3) = 2b_3 \tilde{\Gamma}_4(a_1) = 0
\]
in the quotient space \( C^\infty_I / DC^\infty_1 \) proves that \( \int h \) is a conserved quantity for the KdV equation.
16 The second gradient

In general the recursion operator taking gradient to gradient can be written as

\[ D^{-2}K D^{-2}P = D^4 + D^{-2}L_2 + D^{-2}L_3 \]

\[ L_2 = -2a_1D^3 + 4a_2D^2 - a_3D + 2a_4I \]

\[ L_3 = 2a_1D^{-2}a_1D^2 + a_1D^{-1}a_1D, \]

after the operator identities \( Da_1 = a_2 - a_1D \) and \( D^2a_1 = a_3 + a_1D^2 \) are taken into account.

From \( \tilde{\Gamma}_4 = D^{-2}a_1\theta \), the recursion gives the second gradient \( \tilde{\Gamma}_8 \) as the sum of three terms. An antigradient of \( \tilde{\Gamma}_8 \) would satisfy

\[ \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \tilde{\h}(a_2 + \varepsilon b_3) = b_3\tilde{\Gamma}_8 = -b_1D^2\tilde{\Gamma}_8. \]

Since \( \tilde{\Gamma}_8 = (D^4 + D^{-2}L_2 + D^{-2}L_3)\tilde{\Gamma}_4 \), we should examine

\[ b_1D^2\tilde{\Gamma}_8 = b_1D^6\tilde{\Gamma}_4 + b_1L_2\tilde{\Gamma}_4 + b_1L_3\tilde{\Gamma}_4. \]

The first term is

\[ b_1D^6\tilde{\Gamma}_4 = b_1D^4a_1\theta = b_1a_5\theta. \]

An antigradient is given by

\[ h = \frac{1}{2}a_1a_3\theta \]

because

\[ \delta h = \frac{1}{2}b_3a_3\theta + \frac{1}{2}a_1b_5\theta \]

\[ = \frac{1}{2}b_1D^2a_3\theta - \frac{1}{2}b_3a_1\theta \]

\[ = -\frac{1}{2}b_1a_5\theta - \frac{1}{2}b_1D^4a_1\theta \]

\[ = -b_1a_5\theta. \]

Therefore the function \( C_1^\infty \to C_1^\infty \) given by \(-h\) has the first term of \( \tilde{\Gamma}_8 \) as its gradient.

When the operator \( L_2 \) is applied to \( \tilde{\Gamma}_4 = D^{-2}a_1\theta \), the result is

\[ (2a_2a_4 - a_6a_3 + 2a_1a_2)\theta - a_2a_3, \]

the second term in \( D^2\tilde{\Gamma}_8 \).
Working with $h = a_{-1}a_1a_2\theta$, we find that $\delta h = x + y + z$, with

\[
\begin{align*}
    x &= b_1a_1a_2\theta \\
    y &= a_{-1}b_3a_2\theta \\
    z &= a_{-1}a_1b_4\theta.
\end{align*}
\]

Then

\[
\begin{align*}
    y &= -b_3a_{-1}a_2\theta \\
    &= b_1D^2a_{-1}a_2\theta \\
    &= b_1a_1a_2\theta + b_1a_{-1}a_4\theta,
\end{align*}
\]

while

\[
\begin{align*}
    z &= -b_2D^2a_{-1}a_1\theta \\
    &= -b_2a_{-1}a_3\theta \\
    &= -b_1Da_{-1}a_3\theta \\
    &= -b_1a_0a_3\theta + b_1a_{-1}a_4\theta - b_1a_{-1}a_3.
\end{align*}
\]

This gives

\[
\delta h = b_1(2a_1a_2\theta - a_0a_3\theta + 2a_{-1}a_4\theta) - b_1a_{-1}a_3 = b_1L_2\tilde{\Gamma}_4,
\]

proving that $h = a_{-1}a_1a_2\theta$ has for its gradient the second term in $\tilde{\Gamma}_8$.

Finally, the operator $L_3$ is applied to $\tilde{\Gamma}_4 = D^{-2}a_1\theta$, giving just

\[
a_1D^{-1}a_1D^{-1}a_1\theta.
\]

Using $D^{-1}a_1\theta = a_0\theta - a_{-1}$, $Da_{-1}a_0 = a_0^2 - a_{-1}a_1$, and $Da_0^2\theta = 2a_0a_1\theta + a_0^2$, one can show that

\[
2a_1D^{-1}a_1D^{-1}a_1\theta = a_0^2a_1\theta + 2a_{-1}a_0a_1 + a_1D^{-1}a_0^2,
\]

this being a constant multiple of the third term of $D^2\tilde{\Gamma}_8$.

An antigradient exists, and is a constant multiple of

\[
h = a_0^4\theta - 4a_{-1}a_0^3 + 3a_0^2D^{-1}a_0^2,
\]

a function $C^\infty_\iota \to C^\infty_{NL}$. In order for it to be integrable we need to show that $\frac{\partial h}{\partial \theta} \in C^\infty_\iota$, that is, $h \in C^\infty_\iota$.

It is easy to see that $\frac{\partial h}{\partial \theta}a_0^N \in C^\infty_\iota$ for all $N > 0$. Remembering that $C^\infty_\iota$ is an ideal in $C^\infty_{NL}$ we have

\[
\frac{\partial h}{\partial \theta} = a_0^4 - 4a_0^3\frac{\partial}{\partial \theta}a_{-1} + 3a_0^2\frac{\partial}{\partial \theta}D^{-1}a_0^2,
\]

except for a term in $C^\infty_\iota$. 

19
Since \( \left( \frac{\partial}{\partial \theta} D^{-1} + D^{-1} \frac{\partial}{\partial \theta} \right) \Phi = \Phi \) for all \( \Phi \in C^\infty_{NL} \), we obtain

\[
\frac{\partial}{\partial \theta} a_{-1} = \frac{\partial}{\partial \theta} D^{-1} a_0 = a_0 - D^{-1} \frac{\partial a_0}{\partial \theta} \equiv a_0 \text{ mod } C^\infty_1
\]

while

\[
\frac{\partial}{\partial \theta} D^{-1} a_0^2 = a_0^2 - D^{-1} \frac{\partial a_0^2}{\partial \theta} \equiv a_0^2 \text{ mod } C^\infty_1,
\]

because \( D^{-1} \frac{\partial}{\partial \theta} C^\infty_1 \subset C^\infty_1 \).

This proves that \( h \) takes its values in \( C^\infty_I \), as claimed, because the powers of \( a_0 \) all cancel.

The gradients of the three terms of \( h \) are now computed.

\[
\delta(a_0^4 \theta) = 4a_0^3 b_2 \theta = 4b_1 Da_0^3 \theta = b_1 (12a_0^2 a_1 \theta + 4a_0^3).
\]

\[
\delta(a_{-1} a_0^3) = b_1 a_0^3 + 3a_{-1} a_0^2 b_2 = b_1 a_0^3 + 3b_1 Da_{-1} a_0^2 = b_1 a_0^3 + 3b_1 (a_0^3 - 2a_{-1} a_0 a_1) = b_1 (4a_0^3 - 6a_{-1} a_0 a_1).
\]

\[
\delta_{\frac{1}{2}}(a_0^2 D^{-1} a_0^2) = x + y \text{ in which}
\]

\[
x = a_0 b_2 D^{-1} a_0^2 = b_1 D a_0 D^{-1} a_0^2 = b_1 (a_1 D^{-1} a_0^2 + a_0^3),
\]

while \( y = a_0^2 D^{-1} a_0 b_2 \).

Using \( D^{-1} a_0 b_2 = a_0 b_1 - D^{-1} a_1 b_1 \) and \( a_1 b_1 = a_1(a_5 + 3a_1 a_2) = D^2 a_1 a_3 \), we obtain

\[
y = b_1 a_0^3 - a_0^2 Da_1 a_3.
\]

But since \( (D^{-1} a_0^2)(Da_1 a_3) \in C^\infty_1 \), we can apply \( D \), obtaining

\[
a_0^2 Da_1 a_3 = (D^{-1} a_0^2) D^2 a_1 a_3 = (D^{-1} a_0^2)a_1 b_1,
\]

mod \( DC^\infty_1 \).
This gives
\[ y = b_1 a_0^3 - (D^{-1}a_0^2)a_1 b_1 \]
\[ = b_1 (a_0^3 + a_1 D^{-1}a_0^2), \]
and therefore
\[ \delta(a_0^2 D^{-1}a_0^2) = b_1 (4a_0^3 + 4a_1 D^{-1}a_0^2). \]

Taken in combination with
\[ \delta(a_0^4 \theta) = b_1 (4a_0^3 + 12a_0^2 a_1 \theta) \]
\[ -4\delta(a_{-1} a_0^3) = b_1 (-16a_0^3 + 24a_{-1} a_0^2 a_1), \]
\[ 3\delta(a_0^2 D^{-1}a_0^2) = b_1 (12a_0^3 + 12a_1 D^{-1}a_0^2) \]
this completes the proof that
\[ \delta h = b_1 (12a_0^2 a_1 \theta + 24a_{-1} a_0 a_1 + 12a_1 D^{-1}a_0^2) \]
\[ = b_1 (24a_1 D^{-1}a_1 D^{-1} a_1 \theta) = 24b_1 L_3 \bar{\Gamma}_4. \]

To sum up: the superfield \( \bar{\Gamma}_8(a_1) \) produced from the initial value \( \bar{\Gamma}_2 = \theta \) and the subsequent constructions given by the recursive algorithm has been shown to appear in an equation
\[ \frac{d}{dt} \bigg|_{t=0} \int \tilde{h}_8(a_1 + tb_3) = \int b_3 \tilde{\Gamma}_8(a_1) \]
for a certain \( \tilde{h}_8 : C^\infty_1 \rightarrow C^\infty_1 \).

This shows that \( a_1 \rightarrow \int \tilde{h}_8(a_1) \), a function \( C^\infty_1 \rightarrow \Lambda \), is a nonlocal conserved quantity for the SUSY KdV equation.

## 17 Conclusions

We presented a complete proof of the gradient recursion algorithm for the \( N = 1 \) SKdV system. We introduced the precise ring of superfields where the non-local gradients and conserved quantities appear. All the local and non-local hierarchy of the \( N = 1 \) SKdV is obtained from the gradient recursion algorithm. In particular we found new non-local conserved quantities of the \( N = 1 \) SKdV equation. These new conserved quantities are bosonic in contrast to the already known fermionic non-local conserved quantities. They were constructed step by step using the recursive gradient algorithm. That suggests that there might exist a new non-local conserved quantity of the Super Gardner equation (S. Andrea, A. Restuccia and A. Sotomayor, work in progress).

The recursive gradient approach may also be extended for \( N = 2 \) SKdV equations [18], we expect to report on this shortly.
Acknowledgments The work of A.R. was supported by PROSUL under contract CNPq 490134/2006-8 and Decanato de Investigación y Desarrollo(DID USB), Proyecto G11.

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