A conditional greedy algorithm for edge-coloring

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Abstract

We present a novel algorithm for edge-coloring of multigraphs. The correctness of this algorithm for multigraphs with $\chi' > \Delta + 1$ ($\chi'$ is the chromatic edge number and $\Delta$ is the maximum vertex degree) would prove a long standing conjecture in edge-coloring of multigraphs.

1 Introduction.

The **chromatic index** $\chi'(G)$ of a multigraph $G(V, E)$ is the minimal number of colors that can be assigned to the edges of $G$ so that no two adjacent edges receive the same color. Clearly, $\Delta(G) \leq \chi'(G)$, where $\Delta(G)$ is the maximal vertex degree in $G$. The famous result by Vizing ([13]) establishes $\chi' = \chi'(G) \leq \Delta(G) + p(G)$, where $p(G)$ is the maximal number of parallel edges in $G$. In particular, for graphs, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. A polynomial algorithm to determine the exact value of the chromatic index is unlikely to exist, as the problem was proved by Holyer ([6]) to be NP-hard even for cubic graphs. It is suspected that for every multigraph with $\chi' > \Delta + 1$, its chromatic index is determined by the parameter $\omega(G)$ called the multigraph density:

$$\omega(G) = \max_{H \subseteq G} \left[ \frac{e(H)}{v(H)/2} \right],$$

where $H$ is a sub-multigraph of $G$, and $v(H)$ (resp. $e(H)$) denotes the number of vertices (resp. edges) in $H$. It is easy to see that $\omega(G) \leq \chi'(G)$ for every multigraph $G$. A conjecture connecting $\chi', \omega$, and $\Delta$ was independently proposed by Goldberg ([2]), Gupta([4]), and Seymour ([10]) more than 40 years ago (see [11]): if $\chi' > \Delta + 1$, then $\chi' = \omega$. In ([3]), it was additionally conjectured that if $\Delta > \omega$, then $\chi' = \Delta$. The value $\max\{\Delta, \omega\}$ is the polynomially computable fractional chromatic index of $G$ ([1, 9, 10]). See [1, 3, 5, 7, 8, 11, 12] for some partial results towards the proof. All these results are obtained by using algorithms that are based on the recoloring of maximal two-colored chains and the operation of fan-recoloring discovered by Vizing.
and by Gupta. In this paper, we present a different type of an algorithm which colors the edges in a pre-determined sequence so that every intermediate partial edge-coloring is admissible. Proving that the algorithm colors all edges would imply the conjecture above.

**Notations.** Given set $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced on $S$. For $S, T \subseteq V(G)$, $\deg(S, T)$ denotes the number of edges $xy$ such that $x \in S$ and $y \in T$. The degree of $x$ is $\deg(x) = \ deg(\{x\}, V - x)$; if $x \in S$, the degree of $x$ in $G[S]$ is denoted $\deg_S(x)$; $\delta_S(x) = \deg(x) - \deg_S(x)$; $\delta(x) = \deg(S, V - S) = \sum_{x \in S} \delta_S(x)$. We say that edge $e = xy$ crosses set $S \subseteq V$, if $|\{x, y\} \cap S| = 1$.

The multigraphs considered in this paper are connected and have more than one vertex; thus, the degree of every vertex is positive. The colors used for edge-colorings are always integers from the interval $[1, k]$, for some $k \geq 1$; such colorings are called $k$-edge-colorings. Obviously, every $k$-edge-coloring is also a $(k + 1)$-edge-coloring.

A coloring for which all edges are colored is called complete. A partial edge-coloring may have edges that are not colored; it is still required that any two adjacent colored edges have distinct colors. If $\phi$ is a partial edge-coloring and edge $e$ is not colored by $\phi$, then we write $\phi(e) = \emptyset$.

Given two partial $k$-colorings $\phi_1$ and $\phi_2$, we say that $\phi_2$ is an extension of $\phi_1$, denoted $\phi_1 \preceq \phi_2$, if for every edge $xy$ which is $\phi_1$-colored, $\phi_2(xy) = \phi_1(xy)$. If coloring $\phi_2$ is obtained from $\phi_1$ by coloring an edge $e$ using color $i$, we write $\phi_2 = \text{ext}(\phi_1, e, i)$.

Let $\phi$ be a partial $k$-edge-coloring of $G$ and $S \subseteq V(G)$. Then $E_{\phi}^{(un)}(S)$ denotes the set of all edges in $E(S)$ that are not colored. For $i \in [1, k]$, an edge is called $i$-free, if it is not colored and it is not adjacent to any edge colored $i$. A vertex $x$ is called $i$-free, if it is incident to an $i$-free edge. The set of all $i$-free vertices in $S$ is denoted $S_{\phi}^{(i)}$. The cover value of $S$ by $\phi$ is defined as

$$\text{cov}_{\phi}(S) = \sum_{i=1}^{k} \left\lfloor \frac{|S_{\phi}^{(i)}|}{2} \right\rfloor.$$ 

A partial coloring $\phi$ is said to cover set $S \subseteq V(G)$, if $\text{cov}_{\phi}(S) \geq |E_{\phi}^{(un)}(S)|$. A partial coloring is called admissible, if it covers every subset $S \subseteq V$.

Further in the paper, $k = \omega(G)$.

## 2 Edge-coloring algorithm

The algorithm ConditionalGreedy presented below colors the edges one-by-one using the smallest available colors so that the resulting partial colorings are admissible. The edges are colored in the order developed by procedure Reorder, which first sorts out the vertices and then the edges.

**Procedure Reorder** $G$;

**Input:** A connected graph $G$;
Output: An ordered sequence \( \{e_1, \ldots, e_m\} \) of the edges in \( G \).

- let \( |V(G)| = n \);
  select \( x_1 \) as a vertex of the maximal degree in \( G \);
- for every \( i \in [2, n] \), select \( x_i \) as a vertex \( z \in V - \{x_1, \ldots, x_{i-1}\} \) which maximizes \( \deg(\{z\}, \{x_1, \ldots, x_{i-1}\}) \);
- the edges are sorted lexicographically: for any two edges \( e_a = x_i x_j \) and \( e_b = x_k x_l \), where \( i < j \) and \( k < l \), \( e_a \) precedes \( e_b \) if either \( i < k \), or \( i = k \) and \( j < l \) (the equality for parallel edges is broken arbitrary).

Procedure Conditional_Greedy;
Input: A connected multigraph \( G \); colors 1, 2, \ldots, \( \omega \); Output: An edge-coloring \( \phi \).

1. \text{Reorder}(G); let \( e_1, e_2, \ldots, e_m \) be the output order of the edges;
2. For \( i = 1, \ldots, m \), set \( \phi(e_i) = \emptyset \);
3. for \( i = 1, \ldots, m \),
   \{find the smallest \( c \in [1, \omega] \), for which \( \text{ext}(\phi, e_i, c) \) is admissible;
   if such a color does not exist,
   halt and output \( \phi \);
   else \( \phi = \text{ext}(\phi, e_i, c) \);\}
4. output \( \phi \).

Conjecture 1 If \( \chi'(G) > \Delta(G) + 1 \), then Conditional_Greedy outputs a complete \( \omega \)-edge-coloring.

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