A COHOMOLOGICAL INTERPRETATION OF BOGOMOLOV’S INSTABILITY

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Abstract. We give a new proof of Bogomolov’s instability theorem. Furthermore, we prove that it is equivalent to a statement which characterizes when the first cohomology group of a suitable divisor does not vanish.

1. Introduction

In the theory of stable vector bundles on surfaces, the following theorem, known as Bogomolov’s instability theorem, plays a central role:

Theorem 1.1 (Bogomolov). Let \(X\) be a smooth projective surface and \(V\) be a rank 2 vector bundle on \(X\). If \(c_1(V)^2 > 4c_2(V)\), then \(V\) is unstable.

For the original proof we refer to [1]; see also [9]. This theorem was later proved by quite different techniques in [5] and [8]. Furthermore, Reider used Theorem 1.1 to study adjoint linear series on surfaces and to derive his famous theorem, [10].

The first cohomological proof of Reider’s theorem was given by Sakai in [11]. His proof uses ideas of Serrano [12] and generalizes Reider’s theorem to normal surfaces. The key point in Sakai’s proof is the following theorem.

Theorem 1.2 (Sakai). Let \(D\) be a big divisor with \(D^2 > 0\) on a smooth projective surface \(X\). If \(H^1(X, \mathcal{O}_X(K_X + D)) \neq 0\), then there exists an effective divisor \(E\) such that

1. \(D - 2E\) is big;
2. \((D - E) \cdot E \leq 0\).

As shown in [11], Theorem 1.2 can be easily derived from Theorem 1.1. Moreover, Sakai gave an alternative proof based on Miyaoka’s vanishing theorem for the Zariski decomposition of a divisor. Later Ein and Lazarsfeld showed how to apply the Kawamata-Viehweg vanishing theorem to prove a part of Reider’s theorem in [2]. Based on these new techniques Fernández del Busto gave an elegant proof of Bogomolov’s inequality which uses only the Kawamata-Viehweg theorem; see [3]. For a survey on these results we refer to [6].

On the other hand, Mumford showed that we can use Bogomolov’s theorem for rank 2 vector bundles to give a short proof of a generalized Kodaira vanishing for surfaces; see [4]. This vanishing theorem is a little less general than the theorem of Kawamata-Viehweg. These results suggest that there should be a connection between Bogomolov’s instability and some vanishing theorem.

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In this note we prove

**Theorem 1.3.** Bogomolov’s instability theorem is equivalent to Theorem 1.2.

Furthermore, using Sakai’s proof of Theorem 1.2, one gets a new proof of Bogomolov’s instability theorem which is entirely cohomological.

We now outline the proof of Theorem 1.3. After twisting the vector bundle $V$ with a line bundle, we can assume that $V$ has a global section. Using this section we have that the extension class of the vector bundle is nontrivial since $V$ is locally free. The first step of our proof follows Fernández del Busto’s argument [3]. At this point we follow a different strategy. The numerical condition of Bogomolov’s inequality allows us to apply Theorem 1.2, and we show directly that the divisor $E$ gives the destabilizing subsheaf.

2. Preliminaries

For the convenience of the reader we sketch the proof of Theorem 1.2.

**Proof.** Let $D = P + N$ be the Zariski decomposition of $D$ and write $N = \sum \alpha_j E_j$ with each $\alpha_j$ positive and rational. By Sakai’s lemma, Example 9.4.12 in [7], we know that $H^1(X, O_X(K_X + D − \lfloor N \rfloor)) = 0$ so $\lfloor N \rfloor > 0$. Consider the following sequence of divisors:

$$D_0 = D − \lfloor N \rfloor, \ldots, D_k = D_{k-1} + E_{j_k}, \ldots, D_n = D.$$  

If $D_k \cdot E_{j_k} > 0$ for any $k$, we get the vanishing of $H^1(X, O_X(K_X + D))$. Thus we can collect all the $E_{j_k}$’s with positive intersection to construct a sequence $D_0, \ldots, D_k$ such that $(D − D_k) \cdot E_{j} \leq 0$ for all irreducible components $E_{j}$ of $D − D_k$. Now a computation shows that $E := D − D_k$ is the required divisor. □

**Corollary 2.1.** Let $D$ and $E$ be as above. Then

$$H^1(X, O_X(K_X + D − E)) = 0.$$  

**Proof.** By the above construction,

$$H^1(X, K_X + D_0) = H^1(X, K_X + D_k).$$

Since $D_0 = D − \lfloor N \rfloor$ and $D_k = D − E$, the result follows from Sakai’s lemma. □

In conclusion we recall two results which will be used in the proof of the main theorem.

**Lemma 2.2.** Let $f : Y \to X$ be a birational morphism between smooth projective surfaces and $\bar{L}$ a divisor on $Y$. Set $L := f_* \bar{L}$; if $\bar{L}^2 > 0$ and $L$ is big, then $\bar{L}$ is big.

**Proof.** Lemma 3 in [11]. □

**Proposition 2.3.** Let $f : \bar{X} \to X$ be a birational morphism between smooth projective surfaces. Let $\bar{D}$ be a divisor on $\bar{X}$ such that $\bar{D}^2 > 0$. Suppose there is a divisor $\bar{E}$ which satisfies the conclusions of Theorem 1.2 and let $D := f_* \bar{D}$, $E := f_* \bar{E}$ and $\alpha := D^2 − \bar{D}^2$. If $D$ is nef and $E$ is effective, we have

$$0 \leq D \cdot E < \alpha / 2.$$  

**Proof.** See Proposition 2 in [11]. □
3. Main theorem

We can now prove the main result of the paper.

Proof of Theorem 1.3 As mentioned before, Theorem 1.2 can be easily proved using Bogomolov’s instability; see [11], p. 307.

We now want to show that Theorem 1.2 implies Bogomolov’s theorem. Since the inequality in Theorem 1.1 is invariant under twisting with a line bundle, we can assume that \( V \) is globally generated, \( \det(V) \) is ample and \( c_2(V) > 0 \). Taking a general section \( s \) of \( V \), we get the following exact sequence:

\[
0 \to O_X \to V \to L \otimes I_Z \to 0,
\]

where \( L := \det(V) \) and \( Z \) is the zero locus of \( s \). Then we have \( c_2(V) = |Z| \), the length of \( Z \).

Since \( V \) is locally free, the above extension is nontrivial and then

\[
H^1(X, O_X(K_X + L) \otimes I_Z) \neq 0.
\]

Let \( \pi : Y \to X \) be the blow up of \( X \) at all points in \( Z \). Let \( E_j \) be the exceptional curve over \( x_j \in Z \). Then

\[
H^1(Y, O_Y(K_Y + \pi^*L - 2 \sum_j E_j)) = H^1(X, O_X(K_X + L) \otimes I_Z) \neq 0.
\]

Define \( \widetilde{L} := \pi^*L - 2 \sum_j E_j \). Thus, we have

\[
\widetilde{L}^2 = (\pi^*L)^2 + 4 \sum_j E_j^2 = c_1^2(V) - 4c_2(V) > 0,
\]

so \( \widetilde{L} \) is big by Lemma 2.2.

By applying Theorem 1.2 we get an effective divisor \( \widetilde{E}_s \) such that

1. \( \widetilde{L} - 2 \widetilde{E}_s \) is big;
2. \( (\widetilde{L} - \widetilde{E}_s) \cdot \widetilde{E}_s \leq 0 \).

Note that \( \widetilde{E}_s \) depends on the section \( s \) that we chose at the beginning. Let \( E_s := \pi_*\widetilde{E}_s \). We want to show that, for any \( s, E_s \) passes through at least one point of \( Z \). Let \( \widetilde{E}_s := \pi^*E_s + \sum a_i E_i \), where \( E_i \) are the exceptional divisors. It suffices to show that there exists an index \( i \) such that \( a_i < 0 \). Write \( \widetilde{L} - \widetilde{E}_s = \pi^*W_s - \sum(a_i + 2)E_i \), where \( W_s := L - E_s \). Thus by (2) we have

\[
E_s \cdot W_s + \sum_i a_i(a_i + 2) \leq 0.
\]

Then if we show that \( E_s \cdot W_s > 0 \), we must have a negative \( a_i \), and then \( x_i \in \text{Supp}(E_s) \). By construction \( L = E_s + W_s \), \( L \cdot E_s > 0 \) and

\[
L \cdot W_s = (L - 2E_s) \cdot L + L \cdot E_s = (\widetilde{L} - 2\widetilde{E}_s) \cdot \pi^*L + L \cdot E_s > 0
\]

by (1). From the Hodge index theorem we get \( E_s \cdot W_s > 0 \).

Now we need a result in [8], called the uniform multiplicity property. See also [6].

Lemma 3.1. Choosing \( s \) and \( E_s \) generally we can assume that the multiplicity of \( E_s \) at every point of \( Z \) is the same.
Since for any \( s \) there exists \( x \in Z \) such that \( x \in \text{Supp}(E_s) \), by the uniform multiplicity property, we can choose \( s \) and \( E_s \) generally such that \( Z \subset \text{Supp}(E_s) \).

By the uniform multiplicity property, we can choose \( s \) and \( E_s \) generally such that \( Z \subset \text{Supp}(E_s) \). Thus, \( O_X(L - E) \) is a subsheaf of \( V \).

It remains to prove that \( V \) is unstable. This is equivalent to showing that

\[
(L - 2E)^2 > 0, \quad (L - 2E) \cdot L > 0.
\]

For the first inequality we consider the following exact sequence:

\[
0 \to O_X(L - E) \to V \to O_X(E) \otimes I_{Z'} \to 0,
\]

for some zero dimensional scheme \( Z' \). Then \( c_1(V) = L \) and \( c_2(V) = (L - E) \cdot E + |Z'| \), and by hypothesis we get

\[
(L - 2E)^2 > 4|Z'| > 0.
\]

For the second one we note that

\[
\alpha = c_2^1(V) - c_1^2(V) + 4c_2(V) = 4c_2(V),
\]

and Proposition \ref{prop:2.3} gives the following:

\[
L \cdot E < 2c_2(V).
\]

Then

\[
L^2 > 4c_2(V) > 2L \cdot E.
\]

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