SETS NON-THIN AT $\infty$ IN $\mathbb{C}^m$, AND THE GROWTH OF SEQUENCES OF ENTIRE FUNCTIONS OF GENUS ZERO

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Abstract. In this paper we define the notion of non-thin at $\infty$ as follows: Let $E$ be a subset of $\mathbb{C}^m$. For any $R > 0$ define $E_R = E \cap \{ z \in \mathbb{C}^m : |z| \leq R \}$. We say that $E$ is non-thin at $\infty$ if

$$\lim_{R \to \infty} V_{E_R}(z) = 0$$

for all $z \in \mathbb{C}^m$, where $V_E$ is the pluricomplex Green function of $E$.

This definition of non-thinness at $\infty$ has good properties: If $E \subset \mathbb{C}^m$ is non-thin at $\infty$ and $A$ is pluripolar then $E \setminus A$ is non-thin at $\infty$ and $E \cap F \subset \mathbb{C}^m \times \mathbb{C}^m$ is non-thin at $\infty$ (see Lemma 1).

Then we explore the properties of non-thin at $\infty$ sets and apply this to extend the results in [10] and [4].

1. Introduction

Fix $m \in \mathbb{N}$, let $\mathbb{C}^m$ be the usual complex $m$-dimensional complex space. Before going into the main points, we recall some facts about the potential theory in $\mathbb{C}^m$.

Let $U$ be an open subset of $\mathbb{C}^m$. A function $u : U \to [-\infty, \infty)$ (see [5]) is called PSH in $U$ (written $u \in \text{PSH}(U)$) if $u$ is upper-semicontinuous and when restricted to any complex line $L \simeq \mathbb{C}$ then $u$ is subharmonic (see [5]).

A $\text{PSH}(\mathbb{C}^m)$ function $u$ is said to be of minimal growth if

$$u(z) - \log |z| \leq O(1), \text{ as } |z| \to \infty.$$ 

here $|z|$ is the usual Euclidean norm of an element $z \in \mathbb{C}^m$. We denote the set of all such functions by $L$.

Let $E$ be a subset in $\mathbb{C}^m$. Then the pluricomplex Green function of the set $E$ with pole at infinity (see [3], page 184) is $V_E(z) = \sup\{ u(z) : u \in L, \ u|_E \leq 0 \} (z \in \mathbb{C}^m)$.

$V_E$ is also called the Siciak extremal function of the set $E$. By definition we see that $V_E \geq 0$.

For any function $u : \mathbb{C}^m \to (-\infty, \infty)$ we define its upper-semicontinuous regularization $u^*$ (see [3]) by

$$u^*(z) = \lim_{\zeta \to z} \sup u(\zeta).$$
Let $V^*_E(z)$ be the regularization of the Green function $V_E(z)$. It is well-known (see [5]) that there are only three cases:

Case 1: $V^*_E(z) \equiv +\infty$. Then $E$ is pluripolar.

Case 2: $V^*_E(z) \in L$, and $V^*_E \not\equiv 0$.

Case 3: $V^*_E(z) \equiv 0$.

When $m = 1$ then $V^*_E(z) \equiv 0$ iff the set $E$ is non-thin at $\infty$, or equivalently the set $E^* = \{ z : 1/z \in E \}$ is non-thin at 0 (see [10], [3]). Recall that a subset $E$ of $\mathbb{C}^m$ is pluri-thin (or thin for brevity) at a point $a \in \mathbb{C}^m$ if either $a$ is not a limit point of $E$ or there is a neighborhood $U$ of $a$ and a function $u \in \text{PSH}(U)$ such that

$$\lim_{z \to a, \, z \in E \setminus a} u(z) < u(a).$$

If $E$ is not thin then it is called non-thin.

In [10] the authors proved the following result (see also [3], page 270)

**Proposition 1.** Let $E$ be a closed subset of $\mathbb{C}$. Then the following four statements are equivalent

1. $E$ is non-thin at $\infty$.

2. For any $z \in \mathbb{C}$ we have

$$\lim_{R \to \infty} V_{E_R}(z) = 0,$$

where we define $E_R = E \cap \{ z \in \mathbb{C} : |z| \leq R \}$.

3. If sequences $(P_n)$ of polynomials and $k_n \geq \deg(P_n)$ satisfy

$$\limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in E$, then

$$\limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in \mathbb{C}$.

4. $V_E(z) \equiv 0$.

When $m > 1$, as to our knowledge, there was no conception of non-thin at $\infty$. The difficulty here is because we can not reduce the definition of non-thinness of $E$ at $\infty$ to the non-thinness at 0 of some other sets $E^*$, as it was when $m = 1$. However, Proposition 1 suggests a way to define non-thinness at $\infty$ in higher dimensions:

**Definition 1.** Let $E$ be a subset of $\mathbb{C}^m$. For any $R > 0$ define $E_R = E \cap \{ z \in \mathbb{C}^m : |z| \leq R \}$. We say that $E$ is non-thin at $\infty$ if

$$\lim_{R \to \infty} V_{E_R}(z) = 0$$

for all $z \in \mathbb{C}^m$.

If $E$ is not non-thin at $\infty$ then we say $E$ is thin at $\infty$.

This definition of non-thinness at $\infty$ has good properties: If $E \subset \mathbb{C}^m$ is non-thin at $\infty$ and $A$ is pluripolar then $E \setminus A$ is non-thin at $\infty$, if $E \subset \mathbb{C}^m$ and $F \subset \mathbb{C}^n$ are closed sets non-thin at $\infty$ then $E \times F \subset \mathbb{C}^m \times \mathbb{C}^n$ is non-thin at $\infty$ (see Lemma 1).

Our first result in this paper is the following, which is an analog to the case $m = 1$:
Theorem 2. Let $E$ be a closed subset of $\mathbb{C}^m$. Then the following two statements are equivalent:
1. $E$ is non-thin at $\infty$.
2. If sequences $(P_n)$ of polynomials and $k_n \geq \deg(P_n)$ satisfy
   \[
   \limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq 1
   \]
   for all $z \in E$, then
   \[
   \limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq 1
   \]
   for all $z \in \mathbb{C}^m$.

If $E$ is non-thin at $\infty$ then it is easy to see that $V_E \equiv 0$. The converse is not true for $m \geq 2$, by the following example (which essentially is the same as Example 1.1 in [3]).

Example 1. Let $E$ be a subset of $\mathbb{C}^2$ defined by
\[
E = \{(z_1, z_2) \in \mathbb{C}^2 : |z_2| \leq 1\} \cup \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = 0\}.
\]
Then $V_E \equiv 0$ but $V_E$ is thin at $\infty$.

Proof. Use arguments in Example 1.1 in [3], it is easy to see that $V_E \equiv 0$.

If $E$ was non-thin at $\infty$ then since $A = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = 0\}$ is pluripolar, by Lemma 1 we should have $E \setminus A$ is non-thin at $\infty$. In particular we should have $V_{E \setminus A} \equiv 0$. However as computed in [3] we have $V_{E \setminus A}(z) = \log^+ |z|$, which is a contradiction.

Hence $E$ is thin at $\infty$. \hfill \Box

The following result gives a characterization of sets $E$ with $V_E \equiv 0$, which also shows that the non-thin at $\infty$ sets are not rare.

Theorem 3. Let $E$ be a closed subset of $\mathbb{C}^m$. Then $V_E \equiv 0$ iff any open neighborhood of $E$ is non-thin at $\infty$.

In [10], the authors applied the result of Proposition 1 to various problems of convergence of sequences of polynomials in one complex variable. In [4] we extended the results to sequences of entire functions of genus zero of one complex variable. With our definition of non-thinness at $\infty$ for a set $E \subset \mathbb{C}^m$ here, we can extend Theorems 1 and 3 in [4] to the case of entire functions of genus zero of several complex variables, under appropriate assumptions.

If $P(z)$ is an entire function of genus zero of one complex variable then we have a representation of $P$ by (see [9])
\[
P(z) = az^\alpha \prod_j (1 - \frac{z}{z_j}).
\]
Moreover
\[
\sum_j \frac{1}{|z_j|} < \infty.
\]

In higher dimensions we still have a canonical representation of any entire function of genus zero $P(z_1, \ldots, z_m)$ (see Proposition 3.16 in [7], see also [12]). However in this paper we will use instead the reduction to one variable complex as described below.
Let \( P(z_1, \ldots, z_m) \) be an entire function of genus zero in \( \mathbb{C}^m \). Then for any \( \lambda \in \mathbb{P}^{m-1} \) (here \( \mathbb{P}^{m-1} \) means the complex projective space of dimension \( m - 1 \)) which represents a pluricomplex direction in \( \mathbb{C}^m \), we define a one complex variable function

\[
P_\lambda(w) = P(w\lambda)
\]

where \( w \in \mathbb{C} \). Here we understand the expression \( w\lambda \) as follows: \( w\lambda := w\tilde{\lambda} \in \mathbb{C}^m \) where \( \tilde{\lambda} \in S^{2m-1} = \{z \in \mathbb{C}^m : |z| = 1\} \) is a fixed representative of \( \lambda \). Then \( P_\lambda(w) \) is also an entire function of genus zero of one complex variable hence we can use (1.1) to write

\[
P_\lambda(w) = a_\lambda w^{\alpha_\lambda} \prod_j \left(1 - \frac{w}{w_{\lambda,j}}\right).
\]

We define the counting functions of \( P_\lambda \) and \( P \) as follows: If \( t > 0 \) then

\[
\eta_P(t, \lambda) = \text{the number of elements of the set}\{w \in \mathbb{C} : P_\lambda(w) = 0, |w| \leq t\},
\]

\[
\eta_P(t) = \int_{\mathbb{P}^{m-1}} \eta_P(t, \lambda) d\lambda
\]

where we integrate using the standard metric on \( \mathbb{P}^{m-1} \).

The following result generalizes Theorem 1 in [4]

**Theorem 4.** Let \( E \subset \mathbb{C}^m \) be non-thin at \( \infty \).

Let \( P_n : \mathbb{C}^m \to \mathbb{C} \) be a sequence of entire functions of genus zero, and let \( (k_n) \) be a sequence of positive numbers greater than \( 1 \). For \( \lambda \in \mathbb{P}^{m-1} \) let us define the function \( P_{n,\lambda} : \mathbb{C} \to \mathbb{C} \) by

\[
P_{n,\lambda}(w) = P_n(w\lambda).
\]

Let

\[
P_{n,\lambda}(w) = a_{n,\lambda} w^{\alpha_{n,\lambda}} \prod_j \left(1 - \frac{w}{w_{n,\lambda,j}}\right),
\]

its representation, where

\[
\sum_j \frac{1}{|w_{n,\lambda,j}|} < \infty.
\]

Let \( \eta_{P_n}(t, \lambda) \) and \( \eta_{P_n}(t) \) be counting functions of \( P_n \) as defined above.

Assume that for any compact set \( K \subset \mathbb{C}^m \) there exists \( N_0 \) such that the set

\[
\{|P_n(z)|^{1/k_n} : n \geq N_0, z \in K\}
\]

is bounded from above.

For each \( \lambda \in \mathbb{P}^{m-1} \) assume that the following three conditions are satisfied:

\[
\text{(1.3)} \quad \limsup_{n \to \infty} \frac{\left[\eta_{P_n}(1, \lambda) + \sum_{|w_{n,\lambda,j}| \geq 1} 1/|w_{n,\lambda,j}|\right]}{k_n} < \infty.
\]

\[
\text{(1.4)} \quad \lim_{R \to \infty} \limsup_{n \to \infty} \frac{\left|\sum_{|w_{n,\lambda,j}| \geq R} 1/w_{n,\lambda,j}\right|}{k_n} = 0.
\]

There exists a sequence \( R_{n,\lambda} \to \infty \) such that

\[
\text{(1.5)} \quad \limsup_{n \to \infty} \frac{\eta_{P_n}(R_{n,\lambda}, \lambda)}{k_n} \leq \kappa < \infty,
\]
where $\kappa$ is independent of $\lambda$.

Then the following conclusion is true: If

$$\limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in E$, then we have

$$\limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in \mathbb{C}^m$.

We mention here some remarks about the result of Theorem 4:

1. Theorem 4 is not a direct consequence of the one-dimensional result in [4]. This is because although $E$ is non-thin at $\infty$ in $\mathbb{C}^m$, we may have that $E \cap L$ is thin at $\infty$ in $L$ (even $E \cap L$ may be the empty set) where $L$ is a complex line in $\mathbb{C}^m$.

2. The way of considering the growth of the sequence $(P_n)$ in $\mathbb{C}^m$ by considering the growth of the restricted sequence to any line $L_\lambda$ is a natural approach.

3. Condition (1.2) is natural. By Theorem 2 in [4] conditions (1.4) and (1.5) (with $\kappa$ may depend on $\lambda$) in Theorem 4 are satisfied in case we have condition (1.3),

$$\limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq 1,$$

for all $z \in \mathbb{C}^m$, and

$$\liminf_{n \to \infty} |P_n(0)|^{1/k_n} > 0.$$

Similarly, we can extend Theorems 3 in [4] by the same manner as that of Theorem 4. Because the statement of the result is rather long, we defer the statement of this result to Section 3 (see Theorem $\mathbf{5}$). Here we outline the notations and results we will use in the statement and the proof of Theorem $\mathbf{5}$.

Let $K \subset \mathbb{C}^m$ be compact. Then the Robin constant of $K$ is defined as (see $\mathbf{8}$)

$$\gamma(K) = \limsup_{z \to \infty} [V^*_E(z) - \log |z|],$$

and the $\mathbb{C}^m$-capacity of $K$ is

$$C(K) = e^{-\gamma(K)}.$$  \hspace{1cm} (1.6)

Let $K \subset \{z \in \mathbb{C}^m : |z| \leq s\}$ be compact and non-pluripolar, where $s > 0$. By Theorem 2 in [4] and Poisson integration formula for harmonic functions (see Theorem 2.2.3 in [5]), there exists a constant $C_m > 0$ depending only on the dimension $m$ such that

$$\sup_{|z| \leq s} V^*_E(z) \leq C_m (\log s + \log 2 - \log C(K)).$$ \hspace{1cm} (1.7)

The following corollary is an analog of Remark 1 in $\mathbf{10}$

**Corollary 1.** Let $E \subset \mathbb{C}^m$ be closed. Let $0 < C_m < +\infty$ be the constant in (1.7). Then $E$ is non-thin at $\infty$ if it satisfies the following condition: There exists $\beta > 0$ such that

$$\limsup_{R \to \infty} \frac{\log C(E_R)}{\log R} \geq \beta > \frac{C_m - 1}{C_m},$$

where $C(E_R)$ is determined from formula (1.6).
Interesting questions may arise such as whether we can in fact get rid of the condition (1.2) or get rid of the constant $\kappa$ in (1.5) so we have complete analogs with the results in [4], or whether we have a Wiener criterion for non-thin at $\infty$ in $\mathbb{C}^m$. We hope to return to these issues in a future paper.

This paper consists of three sections. In Section 2 we prove Theorems 2 and 3 as well as some other properties of sets non-thin at $\infty$. In Section 3 we prove Theorem 4 as well as the analog of Theorem 3 in [4].

2. Non-thin at $\infty$ sets in $\mathbb{C}^m$

In this section we explore some properties of sets which are non-thin at $\infty$ in $\mathbb{C}^m$.

**Lemma 1.**

a) Let $E$ be a subset of $\mathbb{C}^m$. If $E$ is non-thin at $\infty$ and $A$ is pluripolar then $E \setminus A$ is non-thin at $\infty$.

b) Let $E \subset \mathbb{C}^m$ and $F \subset \mathbb{C}^n$. Then $E \times F \subset \mathbb{C}^m \times \mathbb{C}^n$ is non-thin at $\infty$ iff $E \subset \mathbb{C}^m$ and $F \subset \mathbb{C}^n$ are non-thin at $\infty$.

**Proof.**

a) Define $F = E \setminus A$. For any $R > 0$ the set $E_R = E \cap \{z \in \mathbb{C}^m : |z| \leq R\}$ is bounded. Hence by Corollary 5.2.5 in [5] we have $V^*_{E_R(z)} = V^*_{F_R(z)}$.

Fix a sequence $R_n \to \infty$, using the same argument as in the proof of Theorem 2 there exists a pluripolar set $B$ such that

$$\lim_{n \to \infty} V_{F_{R_n}}(z) = \lim_{n \to \infty} V^*_{F_{R_n}}(z) = \lim_{n \to \infty} V^*_{E_{R_n}}(z) = 0.$$  

for all $z \in \mathbb{C}^m \setminus B$. Then a little more argument completes the proof of this part.

b) Let $R > 0$. Then one can check easily that

$$\max\{V_{E_R}(z), V_{F_R}(w)\} \leq V^*_{E_R \times F_R}(z, w) \leq V_{E_R}(z) + V_{F_R}(w),$$

where $z \in \mathbb{C}^m$, $w \in \mathbb{C}^n$. This completes the proof of case b). \(\square\)

Now we proceed to proving Theorem 2.

**Proof.** In this proof fix a sequence $R_n \to \infty$.

(2 $\Rightarrow$ 1) Since $E$ is closed, for any $R > 0$ we have $E_R$ is compact.

Fixed $z_0 \in \mathbb{C}^m$. By Siciak’s theorem (see Theorem 5.1.7 in [5]) there exists a sequence of polynomials $(P_n)$ of degree $(k_n)$ such that

$$\|P_n\|_{E_{R_n}}^{1/k_n} \leq 1,$$

for all $n = 1, 2, \ldots$, and

(2.1) $$\lim_{n \to \infty} \log |P_n(z_0)|^{1/k_n} = \lim_{n \to \infty} V_{E_{R_n}}(z_0).$$

Then for any $z \in E$ we have

$$\limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq 1.$$

Hence by assumption that Statement 2 is true, we have

$$\limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in \mathbb{C}^m$. In particular, with $z = z_0$ we get from (2.1) that

$$\lim_{n \to \infty} V_{E_{R_n}}(z_0) \leq 0.$$
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Since \( z_0 \in \mathbb{C}^m \) is arbitrary and obviously \( V_E(z) \geq 0 \) for all \( z \), we get Statement 1.

\((1 \Rightarrow 2)\) We use the ideas in [10]. Assume that Statement 1 is true. Consider any sequence \((P_n)\) of polynomials and \( k_n \geq \deg(P_n) \) such that

\[ \limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq 1 \]

for all \( z \in E \). For each \( n \) define

\[ v_n(z) = \log |P_n(z)|^{1/k_n}. \]

Note that Statement 1 implies that for \( R \) large enough then \( E_R \) is non-pluripolar.

Fix \( R > 0 \) such that \( E_R \) is non-pluripolar.

For any \( h, l \in \mathbb{N} \) define

\[ E_R^{h,l} = \bigcap_{n=h}^{\infty} \{ z \in E_R : v_n(z) < \frac{1}{l} \}. \]

Then \( E_R^{h,l} \subset E_R^{h+1,l} \) and from (2.2)

\[ \bigcap_{h=1}^{\infty} E_R^{h,l} = E_R \]

for any \( l \in \mathbb{N} \).

By definition of the pluricomplex Green function, for any \( h, l \in \mathbb{N}, n \geq h \) and \( z \in \mathbb{C}^m \)

\[ v_n(z) \leq V_{E_R^{h,l}}(z) + \frac{1}{l} \leq V_{E_R^{h,l}}^*(z) + \frac{1}{l}. \]

Hence take the limsup of the above inequality as \( n \to \infty \) we get

\[ v(z) = \limsup_{n \to \infty} v_n(z) \leq V_{E_R^{h,l}}^*(z) + \frac{1}{l}, \]

for all \( h, l \in \mathbb{N}, z \in \mathbb{C}^m \). Take the limit of this inequality as \( h \to \infty \), using (2.2), we see from Corollary 5.2.6 in [5] that

\[ v(z) \leq \lim_{h \to \infty} V_{E_R^{h,l}}^*(z) + \frac{1}{l} = V_{E_R}^*(z) + \frac{1}{l}, \]

for all \( l \in \mathbb{N}, z \in \mathbb{C}^m, R > 0 \). Since \( l \) is arbitrary, we get

(2.4)

\[ v(z) \leq V_{E_R}^*(z), \]

for all \( z \in \mathbb{C}^m, R > 0 \).

For each \( n \in \mathbb{N} \) define

\[ A_n = \{ z \in \mathbb{C}^m : V_{E_{R_n}}(z) < V_{E_{R_n}}^*(z) \} \]

then \( A_n \) is pluripolar thus has Lebesgue measure zero, and \( V_{E_{R_n}}(z) = V_{E_{R_n}}^*(z) \) for \( z \in \mathbb{C}^m \backslash A_n \). Hence

\[ A = \bigcup_{n=1}^{\infty} A_n \]

is also pluripolar and of Lebesgue measure zero, and we have \( V_{E_{R_n}}(z) = V_{E_{R_n}}^*(z) \) for \( z \in \mathbb{C}^m \backslash A \) and \( n \in \mathbb{N} \). Hence for \( z \in \mathbb{C}^m \backslash A \), apply (2.4) we get

\[ v(z) \leq \lim_{n \to \infty} V_{E_{R_n}}^*(z) = \lim_{n \to \infty} V_{E_{R_n}}(z) = 0 \]
for all \( z \in \mathbb{C}^m \setminus A \). Since \( A \) is of Lebesgue measure zero, by definition of \( v(z) \), we see that \( v(z) \leq 0 \) for all \( z \in \mathbb{C}^m \). This completes the proof of Theorem 2. \( \square \)

Now we prove Theorem 3.

Proof. (\( \Rightarrow \)) Let \( E \) be a closed subset of \( \mathbb{C}^m \). Assume that \( V_E \equiv 0 \). We will show that any open neighborhood of \( E \) is non-thin at \( \infty \). Let \( F \) be any open neighborhood of \( E \). Then for any \( R > 0 \) since \( E_R \) is compact, \( F \) is open and contains \( E_R \), we can find \( 1 > \epsilon = \epsilon_R > 0 \) such that \( E_R, \epsilon \subset F \).

By Corollary 5.1.5 in [5] we have \( V_{E_R, \epsilon}^* (z) = 0 \) for \( z \in E_R, \epsilon \supset E_R \). Now it is obvious that \( E_R, \epsilon \subset \cap F_{R+1} \) hence for \( z \in F_R \)

\[
(2.5) \quad V_{F_R+1}^*(z) = 0.
\]

Define \( v(z) = \lim_{R \to \infty} V_{F_R}^*(z) \), then \( v \) is the limit of a decreasing sequence of PSH functions hence \( v \) is itself PSH.

By (2.5) for \( z \in E \) we have \( v(z) \equiv 0 \). Thus by definition of the pluricomplex Green function \( v(z) \leq V_E(z) = 0 \) for all \( z \in \mathbb{C}^m \). This shows that \( F \) is non-thin at \( \infty \).

(\( \Leftarrow \)) Assume that \( E \) is an arbitrary subset of \( \mathbb{C}^m \) with \( V_E \not\equiv 0 \). Then \( V_E(z_0) > 0 \) for some \( z_0 \in \mathbb{C}^m \), hence by definition of the pluricomplex Green function, there exist a function \( u \in L \) such that \( u(z) \leq 0 \) for \( z \in E \), and \( u(z_0) > 0 \). Define \( F = \{ z \in \mathbb{C}^m : u(z) < u(z_0)/2 \} \).

\( F \) is open because \( u \) is upper-semicontinuous, and \( E \subset F \) because \( u|_E \leq 0 \) and \( u(z_0) > 0 \). Now \( u(z) < u(z_0)/2 \) for \( z \in F \), hence we have

\[
u(z) \leq V_F(z) + u(z_0)/2
\]

for all \( z \in \mathbb{C}^m \). In particular choose \( z = z_0 \) we have \( V_F(z_0) \geq u(z_0)/2 > 0 \), hence \( F \) is thin at \( \infty \). \( \square \)

Theorem 3 applied to Example 1 shows that any nonempty open cone in \( \mathbb{C}^m \) is non-thin at \( \infty \). The following result provide other sources of sets which are non-thin at \( \infty \).

Remark 1. Let \( \Delta \) be a collection of complex lines \( L \in \mathbb{C}^m \) such that

\[
\bigcup_{L \in \Delta} L = \mathbb{C}^m.
\]

Let \( E \) be a closed subset of \( \mathbb{C}^m \) such that for each \( L \in \Delta \) the set \( E \cap L \) considered as a subset in the one-dimensional complex line \( L \) is non-thin at \( \infty \). Then \( E \) is non-thin at \( \infty \) as a subset in \( \mathbb{C}^m \).

Proof. By Theorem 2 we need only to show that: If \( (P_n) \) is a sequence of polynomials, and \( k_n \geq \deg(P_n) \) such that

\[
\lim_{n \to \infty} |P_n(z)|^{1/k_n} \leq 1
\]
for \( z \in E \), then
\[
\limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq 1
\]
for all \( z \in \mathbb{C}^m \).

Let \( w \in \mathbb{C} \) be a coordinate for \( L \) such that the coordinates \( z_1, \ldots, z_n \) of \( \mathbb{C}^m \) are linear functions of \( w \) when restricted on \( L \), and denote by \( P_{n,L}(w) \) the restriction of \( P_n \) to \( L \). Then \( P_{n,L}(w) \) is a polynomial in one complex variable \( w \) of degree
\[
\deg(P_{n,L}(w)) \leq \deg(P_n) \leq k_n.
\]
Then since
\[
\limsup_{n \to \infty} |P_{n,L}(w)|^{1/k_n} \leq 1
\]
for \( w \in E \cap L \), and \( E \cap L \) is non-thin at \( \infty \) in the complex line \( L \), hence
\[
\limsup_{n \to \infty} |P_n(w)|^{1/k_n} \leq 1
\]
for all \( w \in L \). Since this is true for any complex line \( L \in \Delta \), it is also true for their union, which is equal to \( \mathbb{C}^m \).

3. The growth of sequences of entire functions of genus zero

In this section we apply the results in previous section to sequences of entire functions of genus zero in \( \mathbb{C}^m \).

First we give a proof of Theorem 4.

**Proof.** Define functions from \( \mathbb{C}^m \) to \([-\infty, +\infty]\) by
\[
u(z) = \limsup_{n \to \infty} u_n(z),
\]
where
\[
u_n(z) = \sup_{j \geq n} \frac{1}{k_j} \log |P_j(z)|.
\]
From the assumption (1.2), by Theorem 4.6.3 in [5], \( u^* \) is PSH. Moreover there is a pluripolar set \( A \subset \mathbb{C}^m \) such that \( u(z) = u^*(z) \) for all \( z \in \mathbb{C}^m \setminus A \).

Fix \( \lambda \in \mathbb{P}^{m-1} \). Choose \( R_n = R_{n,\lambda} \) as the sequence in condition (1.5). Then from the proof of Theorem 1 in [4] we see that for all \( w \in \mathbb{C} \)
\[
\limsup_{n \to \infty} u_n(w,\lambda) = \limsup_{n \to \infty} \frac{1}{k_n} \log |Q_n(\lambda)(w)|
\]
where \( Q_{n,\lambda} : \mathbb{C} \to \mathbb{C} \) is defined by
\[
Q_{n,\lambda}(w) = a_n,\lambda w^{\alpha,\lambda} \prod_{|w,\lambda,j| \leq R_n} \left( 1 - \frac{w}{w,\lambda,j} \right).
\]
Condition (1.12) shows that
\[
\limsup_{n \to \infty} \frac{1}{k_n} \log |Q_{n,\lambda}(w)| \leq \kappa \log C_0
\]
for all \( w \in \mathbb{C} \) with \( |w| \leq \epsilon \). This, together with condition (1.5) and potential theory in one complex variable (Berstein’s lemma for polynomials, see also the proof of Theorem 2), show that there exists \( C > 0 \) such that
\[
\limsup_{n \to \infty} \frac{1}{k_n} \log |Q_{n,\lambda}(w)| \leq \kappa \log^+ |w| + C
\]
for all $w \in \mathbb{C}$. Moreover $C$ is independent of $\lambda$. Hence $u^*/\kappa$ is in $L$. Let $F = E \setminus A$, then by Lemma 1 $F$ is non-thin at $\infty$. In particular $V_F \equiv 0$. Now for all $z \in F$ we have by assumption

$$u^*(z) \leq 0,$$

thus

$$u^*(z) \leq \kappa V_F(z) = 0$$

for all $z \in \mathbb{C}^m$. □

Now we consider the analog of Theorem 3 in [4]. We have the following result.

**Theorem 5.** Let $E \subset \mathbb{C}^m$ be closed and satisfy the following condition: there exists $\beta > 0$ such that

$$\limsup_{R \to \infty} \frac{\log C(E_R)}{\log R} \geq \beta > \frac{C_m - 1}{C_m},$$

where $C(E_R)$ is determined from formula (1.6), and $0 < C_m < \infty$ is the constant in (1.7).

Let $(P_n)$ and $(k_n)$ be sequences satisfying conditions (1.2), (1.3), (1.4) and (1.5) of Theorem 4. Then for all $\lambda \in \mathbb{P}^{m-1}$ we have

$$\exp\{\limsup_{n \to \infty} \frac{1}{2\pi k_n} \int_{0}^{2\pi} \log |P_n(e^{i\theta}\lambda)| d\theta\} = C_\lambda < \infty.$$

Assume that for all $z \in E$ we have

$$\limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq h(|z|)$$

where

$$\limsup_{R \to \infty} \frac{\log h(R)}{\log R} \leq \tau < \infty.$$

Then for any $w \in \mathbb{C}$ and $\lambda \in \mathbb{P}^{m-1}$ we have

$$\limsup_{n \to \infty} |P_n(w\lambda)|^{1/k_n} \leq C_\lambda (1 + |w|)^{\tau/[1 - C_m(1 - \beta)]}.$$

**Proof.** Define functions $u$ and $Q_{n,\lambda}$ as in the proof of Theorem 4. Without loss of generality we may assume that $u \geq 0$. By part (i) of Lemma 1 in [4] and the proof of Theorem 4 there exists a constant $C > 0$ such that

$$u^*(z) \leq \kappa \log^+ |z| + C,$$

for all $z \in \mathbb{C}^m$. Now we define

$$\kappa_0 = \limsup_{z \in \mathbb{C}^m, z \to \infty} \frac{u^*(z)}{\log |z|}.$$

For any $R > 0$, by the definition of the pluricomplex Green function

$$u^*(z) \leq \kappa_0 V_{E_R}(z) + h(R).$$

For any $\lambda \in \mathbb{P}^{m-1}$ define

$$\kappa(\lambda) = \limsup_{w \in \mathbb{C}, w \to \infty} \frac{u(w\lambda)}{\log |w|} \leq \kappa_0.$$
Fixed $\lambda \in \mathbb{P}^{m-1}$. By definition
\[ u(w\lambda) = \limsup_{n \to \infty} \frac{1}{k_n} \log |Q_n(w\lambda)| \]
for all $w \in \mathbb{C}$. Hence using (3.6), for any $s > 0$ we have
\[
\limsup_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{k_n} \log |Q_n(se^{i\theta}\lambda)| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} u(se^{i\theta}\lambda) d\theta \leq \kappa_0 \frac{1}{2\pi} \int_0^{2\pi} V_{E_s}(se^{i\theta}\lambda) d\theta + h(s).
\]
By the previous inequality, (1.7) and assumption (3.2) we get
\[ \liminf_{s \to \infty} \limsup_{n \to \infty} \frac{1}{\log s} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{k_n} \log |Q_n(se^{i\theta}\lambda)| d\theta \leq \kappa_0 C_m (1 - \beta) + \tau. \]
Then by Lemma 2 in [4] we have
\[ \kappa(\lambda) \leq \kappa_0 C_m (1 - \beta) + \tau. \]
Since (3.8) is true for any $\lambda \in \mathbb{P}^{m-1}$, by definition (3.5) and Lemma 2 in [4] we have
\[ \kappa_0 \leq \kappa_0 C_m (1 - \beta) + \tau, \]
or equivalently
\[ \kappa_0 \leq \frac{\tau}{1 - C_m (1 - \beta)}. \]
From the above inequality, use Lemma 2 in [4] we get the conclusion of Theorem 5. □

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