Research Article

Praveen Agarwal* and Juan J. Nieto

Some fractional integral formulas for the Mittag-Leffler type function with four parameters

DOI 10.1515/math-2015-0051
Received June 16, 2015; accepted August 16, 2015.

Abstract: In this paper we present some results from the theory of fractional integration operators (of Marichev-Saigo-Maeda type) involving the Mittag-Leffler type function with four parameters \( \zeta \). Some interesting special cases are given to fractional integration operators involving some Special functions.

Keywords: Marichev-Saigo-Maeda type fractional integral operators, Mittag-Leffler type function with four parameters, Generalized Wright function

MSC: 26A33, 33E12, 33C60, 33E20

1 Introduction and Preliminaries

The fractional calculus and its various applications have become a very popular subject between mathematicians and engineers. New era in the development of this branch of science began 40-50 years ago due to numerous application of fractional-type models and is continued up to now (see [54] and [55]). One can mention a large list of areas of application, in particular, continuum mechanics [8, 39] (including viscoelasticity [27], thermodynamics [17] and anomalous diffusion [38]), astrophysics [30], nuclear physics [53], nanophysics and cosmic physics [57, 58], statistical mechanics [60], fractional order systems and control [7], finance and economics [5], solutions of differential equations [4].

Among the monographs developing the theory of fractional calculus and presenting some applications we have to point out monographs by Diethelm [11], Gorenflo and Mainardi [15], Kiryakova [21], Kilbas, Srivastava and Trujillo [20], Miller and Ross [32], Oldham and Spanier [35], Podlubny [36], and of course the Bible of fractional calculus, monograph by Samko, Kilbas and Marichev [43]. Interested reader can find in these books an extended list of publications on the theory and applications of fractional calculus (see also [56]).

Recently, Mittag-Leffler functions show its close relation to Fractional Calculus and especially to fractional problems which come from applications. This new era of research attract many scientists from different point of view (see [2, 6, 9, 12, 16, 18, 19, 22, 23, 37, 40, 44, 46, 52]). In 1899 G. Mittag-Leffler began the publication of a series of articles under the common title "Sur la representation analytique d’une branche uniforme d’une fonction monogène (On the analytic representation of a single-valued branch of a monogeneity function) published mainly at Acta Mathematica. Nowadays this function and its numerous generalizations are involved in the different fractional
models (see monographs listed above). Motivated by the above works Kiryakova [25, 26] for the first time pointed out the special role of the Mittag-Leffler function and included it into the class of Special Functions for Fractional Calculus. Moreover, based on the role of the Mittag-Leffler function in application, Mainardi called it The Queen of Fractional Calculus (see [27]).

Here, our investigation are based on the so-called Marichev-Saigo-Maeda type generalized fractional operator, i.e., integral transform of the Mellin convolution type with the Appell (or Horn) function $F_3$ developed by Marichev [28] and studied in some recent papers, including the papers by Agarwal et al [2], Choi and Agarwal [10], Saigo and Maeda [42], Saigo and Saxena [45]. The aim of our paper is to present formulas of the Marichev-Saigo-Maeda generalized fractional integration of the generalized Mittag-Leffler type function with four parameters $\xi, \nu, E_{\mu, v}[z]$ which has been recently introduced by Garg et al. [13], and study its various properties, which mainly motivated our present investigation. Throughout this paper, let $\mathbb{C}, \mathbb{R}, \mathbb{R}^+, \mathbb{Z}_0^+$, $\mathbb{N}$ be the sets of complex numbers, real and positive real numbers, nonpositive integers, and positive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2 Definitions and earlier works

For the present investigation, we consider the following definitions and earlier works.

**Definition 2.1.** The Mittag-Leffler type function with four parameters is defined and studied by Garg et al. [13] in the following manner:

$$\xi, \nu, E_{\mu, v}[z] = \sum_{n=0}^{\infty} \frac{(\xi)n}{\Gamma(\mu n + \nu)} z^n \quad (\mu, \nu, \xi, \gamma, z \in \mathbb{C}, \Re(\mu) > \Re(\nu) > 0),$$

(1)

where $(\xi)_n$ is the Pochhammer symbol defined, for $\gamma \in \mathbb{C}$, as follows (see, e.g., [49, p. 2 and p. 5]):

$$(\xi)_n = \begin{cases} 1, & \gamma = 0, \\ \xi(\xi + 1) \cdots (\xi + n - 1), & \gamma \in \mathbb{N}, \\ \frac{\Gamma(\xi + \gamma)}{\Gamma(\xi)} & (\xi \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{cases}$$

and $\Gamma$ being the familiar Gamma function (see, e.g., [48, Section 1.1] and [49, Section 1.1]).

Detail account of several results which included integral representations, recurrence relations, differential formula, fractional derivative and integral, Mellin Barnes integral representation and fractional calculus integral operator involving (1) can be found in the article [13]. Some important special cases of this function are enumerated below:

(i) When $\gamma = 1$ with $(\mu, \nu, \xi, z \in \mathbb{C}, \Re(\mu) > \Re(\nu) > 0)$, the function (1) reduces to the one that has been considered by Garg et al. [13]:

$$\xi, 1, E_{\mu, v}[z] = \sum_{n=0}^{\infty} \frac{(\xi)n}{\Gamma(\mu n + \nu)} z^n.$$  

(2)

(ii) If we set $\gamma = 0$ with $\min\{\Re(\mu), \Re(\nu)\} > 0$, then (1) reduces to the generalized Mittag-Leffler function considered by Wiman [59] (see also [14, p. 39, Eqn. (3.1)]). The case when $\gamma = 0$ and $\nu = 1$ can be found in [33] (see also [14, p. 39]).

In recent years, the Mittag-Leffler function and its various generalizations have become a very popular subject of mathematics and its applications. Among the large number of works regarding the Mittag-Leffler function, for a remarkably clear, insightful, and systematic exposition of the investigations carried out by various authors in the field of mathematical analysis and its applications, the interested reader should refer also to a survey-cum-expository Book by Gorenflo et al., which contains a fairly comprehensive bibliography of as many as 170 further references on the subject.
Definition 2.2. The $H$-function is defined in terms of a Mellin-Barnes integral in the following manner ([29]):

$$H_{p,q}^{m,n}(x) = \frac{1}{2\pi i} \int_{\gamma} \Theta(s) z^{-s} \frac{ds}{s},$$

(3)

where

$$\Theta(s) = \prod_{i=1}^{m} \Gamma(b_i + \beta_i s) \prod_{j=1}^{n} \Gamma(1 - a_i - \alpha_i s) \prod_{j=m+1}^{m+n} \Gamma(1 - b_j - \beta_j s),$$

(4)

where $m$, $n$, $p$, $q$ are integers such that $0 \leq m \leq q$, $0 \leq n \leq p$, and for parameters $a_i$, $b_j \in \mathbb{C}$ and for parameters $\alpha_i$, $\beta_j \in \mathbb{R}^+$, the contour $\gamma$ is suitably chosen, and an empty product, if it occurs, is taken to be unity. The theory of the $H$-function is well explained in the book of Srivastava, Gupta and Goyal ([50], Ch.1) (see also [30]).

Definition 2.3. The generalized Wright's function is defined as follows (see, e.g., [20, p.56, Eqns. (1.11.14) and (1.11.15)):

$$\Psi_q \left[ \left( \frac{a_1}{b_1}, \ldots, \frac{a_p}{b_q} \right) ; \left( \beta_1, \ldots, \beta_p \right) \right] = \sum_{k=0}^{\infty} \prod_{j=1}^{p} \Gamma(\alpha_j + k) \prod_{j=m+1}^{m+n} \Gamma(1 - b_j - \beta_j k) \frac{z^k}{k!},$$

(5)

where the coefficients $A_1, \ldots, A_p \in \mathbb{R}^+$ and $B_1, \ldots, B_q \in \mathbb{R}^+$ with

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0.$$  

(6)

Here, in this paper, our main results are obtained by applying the $\zeta, \gamma E_{\mu, \nu}[z]$ to the fractional integration operators (of Marichev-Saigo-Maeda type) given in (7) and (8), respectively. So we continue to recall the following definitions.

$$(\tau^\alpha_{0+} f)(x) = \frac{x^{-\alpha}}{\Gamma(\alpha)} \int_{0}^{x} (t-x)^{\alpha-1} f(t) \frac{dt}{t}, \quad (\Re(\alpha) > 0),$$

(7)

and

$$(\tau^\alpha_{-} f)(x) = \frac{x^{-\alpha}}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} f(t) \frac{dt}{t}, \quad (\Re(\alpha) > 0).$$

(8)

These operators (integral transforms) were introduced by Marichev [28] as Mellin type convolution operators with a special function $F_3(.)$ in the kernel. These operators were rediscovered and studied by Saigo in [41] as generalization of so-called Saigo fractional integral operators, see [24]. The properties of these operators were studied by Saigo and Maeda [42], in particular, relations of operators with the Mellin transforms, hypergeometric operators (or Saigo fractional integral operators), their decompositions and acting properties in the McBride spaces $F_{p,q}^{\nu}$ (see [31]).

In (7), (8) the symbol $F_3(.)$ denotes so-called 3rd Appell function (known also as Horn function) (see [34, p. 413]):

$$F_3(\alpha, \alpha'; \beta, \beta'; \eta; x, y) = \sum_{m, n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\eta)^{m+n} m! n!} x^m y^n \max \{|x|, |y| \} < 1.$$  

(9)

The properties of this function are discussed in [34, p. 412-415]. In particular, its relation to the Gauss hypergeometric function is presented:

$$F_3(\alpha, \eta - \alpha, \beta, \eta - \beta; \eta; x, y) = 2F_1(\alpha, \beta; \eta; x + y - xy).$$  

(10)

Moreover, it is easily observed that

$$F_3(\alpha, 0, \beta, \beta'; \eta; x, y) = F_3(\alpha, \alpha', \beta; \eta; x, y) = 2F_1(\alpha, \beta; \eta; x),$$  

(11)

and

$$F_3(0, \alpha, \beta, \beta'; \eta; x, y) = F_3(\alpha, \alpha', \beta'; \eta; x, y) = 2F_1(\alpha', \beta'; \eta; x).$$

(12)

It is known that the 3rd Appell function cannot be expressed as a product of two $2F_1$ functions, and satisfy pairs of linear partial differential equations of the second order.
3 Left-sided fractional integration of generalized Mittag-Leffler functions with four parameters

Our results in this Section are based on the preliminary assertions giving composition formula of fractional integral (7) with a power function.

Lemma 3.1 ([42, p. 394]). Let \( \alpha, \alpha', \beta, \beta', \eta \in \mathbb{C} \) and

\[
\Re (\eta) > 0, \ Re (\rho) > \max \left\{ 0, \Re (\alpha + \alpha' + \beta - \eta), \Re (\alpha' - \beta') \right\},
\]

then the following relation holds

\[
\left( I_{0+}^{\alpha, \alpha', \beta, \beta', \eta} x^{\rho-1} \right)(x) = \frac{\Gamma(\rho)\Gamma(\rho + \eta - \alpha - \alpha' - \beta)\Gamma(\rho + \beta' - \alpha')}{\Gamma(\rho + \beta')\Gamma(\rho + \eta - \alpha - \alpha')} x^{\rho + \eta - \alpha - \alpha' - 1}.
\]

(13)

The value of the left-sided Marichev-Saigo-Maeda fractional integral (7) for the generalized Mittag-Leffler function (1) is given by the following theorem.

Theorem 3.2. Let the parameters \( \alpha, \alpha', \beta, \beta', \eta, \zeta, \gamma, \rho, \mu, v, x \in \mathbb{C} \) and \( \Re (\mu) > \Re (v) > 0 \) be such that

\[
\Re (\sigma) > 0, \ Re (\eta) > 0, \ Re (\rho) > \max \left\{ 0, \Re (\alpha + \alpha' + \beta - \eta), \Re (\alpha' - \beta') \right\},
\]

then for all \( x > 0 \) the following relation is valid

\[
\left( I_{0+}^{\alpha, \alpha', \beta, \beta', \eta} x^{\rho-1} \right)(x) = \frac{\chi^{\rho + \eta - \alpha - \alpha' - 1}}{\Gamma(\xi)}
\]

\[
\times \Psi_{4} \left[ (\zeta, \gamma, \rho, \eta, \zeta, \gamma, \rho + \eta - \alpha - \alpha' - \beta, \zeta, \gamma, \rho + \beta' - \alpha', \zeta, \gamma, \rho + \eta - \alpha' - \beta', \zeta, \gamma, \rho + \eta - \alpha' - \beta) \right]
\]

(14)

Proof. For convenience, let the left-hand side of the formula (14) be denoted by \( \mathcal{J} \). We apply (1) and use definition of the integral operator (7) and the representation of (1) in terms of generalized Wright function (5). We use then series form definition of the generalized Wright function (5). Finally, we change the order of integration and summation and find

\[
\mathcal{J} = \left( I_{0+}^{(\alpha, \alpha', \beta, \beta', \eta)} x^{\rho-1} \right)(x)
\]

\[
= \sum_{n=1}^{\infty} \frac{\Gamma(\xi, yn)}{\Gamma(\mu n + v)} c^{n} \left( I_{0+}^{(\alpha, \alpha', \beta, \beta', \eta)} x^{\rho + \sigma n - 1} \right)(x).
\]

Due to the convergence conditions of Theorem 3.2, for any \( n \in \mathbb{N}_{0} \), we have \( \Re (\rho + \sigma n) \geq \Re (\rho) > \max \left\{ 0, \Re (\alpha + \alpha' + \beta - \gamma), \Re (\alpha' - \beta') \right\} \)

Therefore we can apply Lemma 3.1 and use (13) with \( \rho \) replaced by \( (\rho + \sigma n) \):

\[
\mathcal{J} = \sum_{n=0}^{\infty} \frac{\Gamma(\xi + yn)}{\Gamma(\xi + yn + \rho n + v)} c^{n} \left( I_{0+}^{(\alpha, \alpha', \beta, \beta', \eta)} x^{\rho + \sigma n + \eta - \alpha - \alpha' - 1} \right)(x)
\]

\[
= \frac{\chi^{\rho + \eta - \alpha - \alpha' - 1}}{\Gamma(\xi)} \Psi_{4} \left[ (\zeta, \gamma, \rho + \sigma n + \eta - \alpha - \alpha' - \beta, \zeta, \gamma, \rho + \sigma n + \eta - \alpha - \alpha' - \beta, \zeta, \gamma, \rho + \sigma n + \eta - \alpha' - \beta', \zeta, \gamma, \rho + \sigma n + \eta - \alpha' - \beta) \right].
\]

(15)

This, in accordance with (5), completes the proof.
For $\gamma = 1$ in (14), Theorem 3.2, yields to the following result:

**Corollary 3.3.** Let the parameters $\alpha, \alpha', \beta, \beta', \eta, \xi, \rho, \mu, v, x \in \mathbb{C}$ and $\Re(\mu) > \Re(v) > 0$ be such that

$$\Re(\sigma) > 0, \Re(\eta) > 0, \Re(\rho) > \max \left\{0, \Re(\alpha + \alpha' - \gamma), \Re(\alpha' - \beta')\right\},$$

then for all $x > 0$ the following result holds:

$$I_{\alpha, \alpha', \beta, \beta', \eta}^{\rho} \left[\mathcal{E}_{\mu, v} \left(c t^{\sigma}\right)\right] (x) = \frac{x^{\rho + \eta - \alpha - \alpha' - 1}}{\Gamma(\xi)} \times$$

$$\times s\Psi_{4} \left(\xi, 1, (\rho, \sigma), (\rho + \eta - \alpha - \alpha' - \beta, \sigma), (\rho + \beta' - \alpha', \sigma), (1, 1)\right)$$

$$\left(v, \mu\right), (\rho + \beta', \sigma), (\rho + \eta - \alpha - \alpha', \sigma), (\rho + \eta - \beta - \alpha', \sigma) \left| c x^{\sigma}\right\right],$$

where $\xi_{\mu, v}[z]$ is another new generalized Mittag–Leffler type function defined as (see [13, p.4, Eqn. (2.2)]):

$$\xi_{\mu, v}[z] = \sum_{n=0}^{\infty} \frac{(\xi)^{n}}{\Gamma(\mu n + \nu)} z^{n},\mu, \nu, z \in \mathbb{C}, \Re(\mu) > \Re(v) > 0.$$

**Remark 3.4.** It is easily seen that setting $\gamma \to 0$ in equation (14) with some suitable parametric replacements in the resulting identities yields the corresponding known integral formulas in Agarwal et al. [11].

### 4 Right-sided fractional integration of generalized Mittag-Leffler functions with four parameters

In this Section, our results are based on the preliminary assertions giving composition formula of fractional integral (8) with a power function.

**Lemma 4.1 ([42, p. 394]).** Let $\alpha, \alpha', \beta, \beta', \eta \in \mathbb{C}$ and

$$\Re(\eta) > 0, \Re(\rho) < 1 + \min \left\{\Re(-\beta), \Re(\alpha + \alpha' - \eta), \Re(\alpha + \beta' - \eta)\right\},$$

then the following relation holds

$$\left(\tau_{\alpha, \alpha', \beta, \beta', \eta}^{\rho - 1}\right) (x) = \frac{\Gamma(1 - \rho - \beta) \Gamma(1 - \rho - \eta + \alpha + \alpha') \Gamma(1 - \rho + \alpha + \beta' - \eta)}{\Gamma(1 - \rho + \alpha + \alpha' + \beta' - \eta) \Gamma(1 - \rho + \alpha - \beta)} x^{\rho + \eta - \alpha - \alpha' - 1}. \quad (17)$$

The value of the right-sided Marichev-Saigo-Maeda fractional integral (8) for the generalized Mittag-Leffler function (1) is given by the following theorem.

**Theorem 4.2.** Let the parameters $\alpha, \alpha', \beta, \beta', \eta, \xi, \gamma, \sigma, \rho, \mu, v, x \in \mathbb{C}$ and $\Re(\mu) > \Re(v) > 0$ be such that

$$\Re(\sigma) > 0, \Re(\eta) > 0, \Re(\rho) < 1 + \min \left\{\Re(-\beta), \Re(\alpha + \alpha' - \eta), \Re(\alpha + \beta' - \eta)\right\},$$

then for all $x > 0$ the following relation is valid

$$\left(\tau_{\alpha, \alpha', \beta, \beta', \eta}^{\rho - 1}\right)_{\xi, \gamma} \left[\mathcal{E}_{\mu, v} \left(\frac{c}{\tau}\right)\right] (x) = \frac{x^{\rho + \eta - \alpha - \alpha' - 1}}{\Gamma(\xi)} \times$$

$$\times s\Psi_{4} \left(\xi, \gamma, (1 - \rho - \beta, \sigma), (1 - \rho - \eta + \alpha + \alpha' - \beta, \sigma), (1 - \rho - \eta + \alpha + \alpha', \sigma), (1, 1)\right)$$

$$\left(v, \mu\right), (1 - \rho - \eta + \alpha + \alpha' + \beta', \sigma), (1 - \rho - \beta + \alpha, \sigma) \left| c x^{\sigma}\right\right].$$

(18)
Proof. For convenience, let the left-hand side of the formula (18) be denoted by $J$. We apply (1) and use definition of the integral operator (8) and the representation of (1) in terms of generalized Wright function (5). We use then series form definition of the generalized Wright function (5). Finally, we change the order of integration and summation and find

$$J = \left( I^{(\alpha, \alpha', \beta, \beta')}_{\xi, \mu, \nu} \left[ x^{\rho-1} \sum_{n=0}^{\infty} \frac{\zeta y_n}{\Gamma(\mu n + \nu)} e^{\sigma n} \right] \right) (x) = \sum_{n=0}^{\infty} \frac{\zeta y_n}{\Gamma(\mu n + \nu)} e^{\sigma n} \left( I^{(\alpha, \alpha', \beta, \beta')}_{\xi, \mu, \nu} \left[ x^{\rho-1} \right] \right) (x).$$

Due to the convergence conditions of Theorem 4.2, for any $n \in \mathbb{N}_0$, we have $\Re (\rho - \sigma n - 1) \leq \Re (\rho - 1) < 1 - \min [\Re (-\beta), \Re (\alpha + \alpha' - \eta), \Re (\alpha + \beta' - \eta)]$

Therefore we can apply Lemma 4.1 and use (17) with $\rho$ replaced by $(\rho - \sigma n)$:

$$J = \sum_{n=0}^{\infty} \frac{\zeta y_n}{\Gamma(\mu n + \nu)} \frac{\Gamma(1 - \rho - \eta + \sigma n + \alpha + \alpha')} {\Gamma(1 - \rho + \sigma n - \eta + \alpha + \beta')} \frac{\Gamma(1 - \rho + \sigma n - \eta + \alpha + \beta')} {\Gamma(1 - \rho + \sigma n - \eta + \alpha + \beta')} \times \frac{\Gamma(1 - \rho + \sigma n - \beta) \Gamma(1 + n)} {\Gamma(1 - \rho + \sigma n + \alpha - \beta)} \frac{x^{\rho-\sigma n - \eta + \alpha - \alpha' - 1}} {n!}.$$ 

This, in accordance with (5), completes the proof. $\square$

For $\gamma = 1$ in (18), Theorem 4.2, yields the following result:

**Corollary 4.3.** Let the parameters $\alpha, \alpha', \beta, \beta', \eta, \xi, \sigma, \rho, \mu, \nu, x \in \mathbb{C}$ and $\Re (\mu) > \Re (\nu) > 0$ be such that $\Re (\sigma) > 0, \Re (\eta) > 0, \Re (\rho) < 1 + \min [\Re (-\beta), \Re (\alpha + \alpha' - \eta), \Re (\alpha + \beta' - \eta)]$.

then for all $x > 0$ the following relation is valid

$$\left( I^{(\alpha, \alpha', \beta, \beta')}_{1, \mu, \nu} \left[ \frac{\zeta}{\Gamma(\xi)} \right] \right) (x) = \frac{x^{\rho+\eta-\alpha-\alpha'-1}} {\Gamma(\xi)} \times \Psi_{\mu, \nu} \left[ \frac{\zeta}{\Gamma(\xi)}, (1 - \rho - \beta, \sigma), (1 - \rho - \eta + \alpha - \beta', \sigma), (1 - \rho - \eta + \alpha + \alpha', \sigma), (1, 1) \right]$$

$$\left[ \nu, \mu, (1 - \rho, \sigma), (1 - \rho - \eta + \alpha + \alpha' + \beta', \sigma), (1 - \rho - \beta + \alpha, \sigma) \right] c x^{\sigma}.$$ 

**Remark 4.4.** It is easily seen that setting $\gamma \to 0$ in equation (18) with some suitable parametric replacements in the resulting identities yields the corresponding known integral formulas in Agarwal et al. [1].

## 5 Further special cases and concluding remarks

In view of the obvious reduction formula (11), the fractional integration operators (of Marichev-Saigo-Maeda type) given in (7) and (8) reduces to the aforementioned Saigo operators $I_{0+}^{(\alpha, \beta, n)}$ and $I^{(\alpha, \beta, n)}$ defined by (see, for details, [41]; see also [24] and [51] and the references cited therein)

$$\left( I_{0+}^{(\alpha, \beta, n)} f \right) (x) = \frac{x - \alpha - \beta} {\Gamma(\alpha)} \int_{0}^{x} (x - t)^{\alpha - 1} \frac{\zeta}{\Gamma(\xi)} K_{1, \mu, \nu} \left[ \frac{\zeta}{\Gamma(\xi)}, (1 - \rho - \beta, \sigma), (1 - \rho - \eta + \alpha - \beta', \sigma), (1 - \rho - \eta + \alpha + \alpha', \sigma), (1, 1) \right]$$

$$\left[ \nu, \mu, (1 - \rho, \sigma), (1 - \rho - \eta + \alpha + \alpha' + \beta', \sigma), (1 - \rho - \beta + \alpha, \sigma) \right] c x^{\sigma}.$$ 

$$\frac{x^{\sigma}} {\Gamma(\alpha)} \int_{0}^{\infty} f(t) dt, \quad (\Re (\alpha) > 0),$$

(21)
and
$$
\left( T_{\alpha, \beta}^n f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} _2 F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t} \right) f(t) dt, \quad (\Re(\alpha) > 0). \tag{22}
$$

respectively. In the light of above definitions, we have the following relationships (see [47, p.338, Eqns. (2.9) and (2.10)]:

$$
\left( I_{\alpha, \beta}^n f \right)(x) = \left( I_{\alpha, \beta}^n g \right)(x), \quad (\eta \in \mathbb{C}). \tag{23}
$$

$$
\left( I_{\alpha, \beta}^n f \right)(x) = \left( I_{\alpha, \beta}^n g \right)(x), \quad (\eta \in \mathbb{C}). \tag{24}
$$

By setting \(\alpha' = 0\) in Theorems 3.2 and 4.2 and in Corollaries 3.3 and 4.3, if we use the relationships (23) and (24), we can deduce the following interesting corollaries involving the generalized Mittag-Leffler type function with four parameters \(E_{\mu, v}[z]\) defined by (1) and the Saigo fractional integral operators defined by (21) and (22), respectively.

**Corollary 5.1.** Let the parameters \(\alpha, \beta, \eta, \zeta, \gamma, \rho, \mu, v, x \in \mathbb{C}\) and \(\Re(\mu) > \Re(v) > 0\) be such that

$$
\Re(\sigma) > 0, \Re(\eta) > 0, \Re(\rho) > \max \left[ 0, \Re(\eta - \alpha - \beta) \right].
$$

Then each of the following fractional integral formulas holds true for all \(x > 0\)

$$
\left( I_{\alpha, \beta}^n f \right)(x) = \frac{x^{\beta + \eta - \alpha - 1}}{\Gamma(\xi)} \times

\times \Psi_3 \left( \xi, \eta, (\rho + \eta - \alpha - \beta, \sigma), (1, 1) \right)

\left( \eta, \sigma, (\rho + \eta - \alpha - \beta, \sigma) \right) c x^\sigma. \tag{25}
$$

**Corollary 5.2.** Let the parameters \(\alpha, \beta, \eta, \zeta, \gamma, \rho, \mu, v, x \in \mathbb{C}\) and \(\Re(\mu) > \Re(v) > 0\) be such that

$$
\Re(\sigma) > 0, \Re(\eta) > 0, \Re(\rho) > \max \left[ 0, \Re(\eta - \alpha - \beta) \right].
$$

Then each of the following fractional integral formulas holds true for all \(x > 0\)

$$
\left( I_{\alpha, \beta}^n f \right)(x) = \frac{x^{\beta + \eta - \alpha - 1}}{\Gamma(\xi)} \times

\times \Psi_3 \left( \xi, 1, (\rho + \eta - \alpha - \beta, \sigma), (1, 1) \right)

\left( \eta, \sigma, (\rho + \eta - \alpha - \beta, \sigma) \right) c x^\sigma. \tag{26}
$$

**Corollary 5.3.** Let the parameters \(\alpha, \beta, \eta, \zeta, \gamma, \rho, \mu, v, x \in \mathbb{C}\) and \(\Re(\mu) > \Re(v) > 0\) be such that

$$
\Re(\sigma) > 0, \Re(\eta) > 0, \Re(\rho) < 1 + \min \left[ \Re(-\beta), \Re(\alpha - \eta) \right].
$$

Then each of the following fractional integral formulas holds true for all \(x > 0\)

$$
\left( I_{\alpha, \beta}^n f \right)(x) = \frac{x^{\beta + \eta - \alpha - 1}}{\Gamma(\xi)} \times

\times \Psi_3 \left( \xi, \gamma, (1 - \rho - \beta, \sigma), (1 - \rho - \eta + \alpha, \sigma), (1, 1) \right)

\left( \eta, \sigma, (1 - \rho - \beta + \alpha, \sigma) \right) c x^\sigma. \tag{27}
$$
Corollary 5.4. Let the parameters $\alpha, \beta, \eta, \zeta, \sigma, \rho, \mu, v, x \in \mathbb{C}$ and $\Re(\mu) > \Re(v) > 0$ be such that
\[
\Re(\sigma) > 0, \Re(\eta) > 0, \Re(\rho) < 1 + \min[\Re(-\beta), \Re(\alpha - \eta)].
\]
Then each of the following fractional integral formulas holds true for all $x > 0$
\[
\left( I_{-}^{(n,a-n,-\beta)} \left[ t^{\alpha-1} E_{\mu,v} \left( \frac{c}{t^\sigma} \right) \right] \right)(x) = \frac{x^{\alpha+n-\alpha-1}}{\Gamma(\zeta)} \times
\]
\[
\times 4\Psi_3 \left[ \begin{array}{c}
(\zeta, \gamma), (1 - \rho - \beta, \sigma), (1 - \rho - \eta + \alpha, \sigma), (1, 1)
\end{array} \right. (v, \mu), (1 - \rho, \sigma), (1 - \rho - \beta + \alpha, \sigma) \bigg| c, x^\sigma \bigg].
\]
(28)

It is noted that if we set $\beta = -\alpha$ and $\beta = 0$ (21) and (22) yields the Erdélyi-Kober fractional integral operators $E_{0+}^{\alpha,\eta}$ and $K_{0+}^{\alpha,\eta}$, the Riemann-Liouville fractional integral operator $R_{0+}^{\alpha}$, and the Weyl fractional integral operator $W_{0+}^{\alpha}$.

Therefore the results presented here are easily shown to be converted to those corresponding to the above well known fractional operators.

We conclude our present investigation by remarking further that several further consequences of Theorems 3.2 and 3.2 and Corollaries 3.3–5.4 can easily be derived by using some known and new relationships between Mittag-Leffler type function with four parameters $\zeta, \gamma E_{\mu,v}[z]$, which is an elegant unification of various special functions (see [13]), and Fox $H$-function as given in Definition 2.2, after some suitable parametric replacements, which are more simpler fractional integration operators (of Marichev-Saigo-Maeda type), can be deduced from Theorems 3.2 and 3.2, and Corollaries 3.3–5.4 by appropriately applying the following relationships:
\[
\zeta, \gamma E_{\mu,v}[z] = \frac{1}{\Gamma(\gamma)} H_{2,2}^{1,1} \left[ \begin{array}{c}
(1 - \zeta, \eta), (0, 1)
\end{array} \right. (0, 1), (1 - v, \mu) \bigg| z \bigg].
\]
(29)

Acknowledgement: The work of J.J. Nieto has been partially supported by the Ministerio de Economía y Competitividad of Spain under grant MTM2013–43014–P and XUNTA de Galicia under grant R2014/002, and co-financed by the European Community fund FEDER.

References

[1] Agarwal P, Trujillo J J, Rogosin S V, Certain fractional integral operators and the generalized multiindex Mittag–Leffler functions, Proc. Indian Acad. Sci. Math. Sci. (In press)
[2] Agarwal P, Chnad M and Jain S, Certain integrals involving generalized Mittag-Leffler functions, Proc. Nat. Acad. Sci. India Sect. A, (2015); doi:10.1007/s40010-015-0209-1
[3] Agarwal P, Jain S, Chnad M, Dwivedi S K, Kumar S, Bessel functions Associated with Saigo-Maeda fractional derivative operators, J. Fract. Calc. 5(2) (2014) 102–112
[4] Al-Bassam M A, Luchko Y F, On generalized fractional calculus and it application to the solution of integro-differential equations. J. Fract. Calc. 7 (1995) 69–88
[5] Baleanu D, Diethelm K, Scalas E, Trujillo J J, Fractional Calculus: Models and Numerical Methods (2012) (N. Jersey, London, Singapore: World Scientific Publishers)
[6] Capelas de Oliveira E, Mainardi F, VAZ J Jr, Models based on Mittag-Leffler functions for anomalous relaxation in dielectrics, Eur. Phys. J. Special Topics 193 (2011) 161–171;http://dx.doi.org/10.1140/epjst/e2011-01388-0
[7] Caponetto R, Dongola G, Fortuna L, and Petráš I, Fractional Order Systems: Modeling and Control Applications (2010) (Singapore: World Scientific Pub Co Inc)
[8] Caputo M, Mainardi F, Linear models of dissipation in anelastic solids. Riv. Nuovo Cimento (Ser. II), 1 (1971) 161–198
[9] Choi J, Agarwal P, A note on fractional integral operator associate with multiindex Mittag-Leffler functions, Filomat (In press)
[10] Choi J, Agarwal P, Certain integral transform and fractional integral formulas for the generalized Gauss hypergeometric functions, Abstr. Appl. Anal. 2014 (2014) 735946, 7 pages, available online at http://dx.doi.org/10.1155/2014/735946
[11] Diethelm K, The Analysis of Fractional Differential Equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type (2010) (Berlin: Springer) Springer Lecture Notes in Mathematics No 2004
Some fractional integral operator for M-L functions

12. Dzrbashyan M M, On the integral transforms generated by the generalized Mittag-Leffler function, Izv. AN Arm. SSR 13(3) (1960) 21–63
13. Garg M, A. Sharma and P. Manohar, A Generalized Mittag-Leffler Type Function with Four Parameters, Thai J. Math., (In press)
14. Gorenflo R, Kilbas A A., Mainardi F., Rogosin S V., Mittag-Leffler Functions, Related Topics and Applications (Springer 2014) 454 pages.
15. Gorenflo R, Mainardi F, Fractional calculus: integral and differential equations of fractional order, in: A. Carpinteri and F. Mainardi (Editors) Fractals and Fractional Calculus in Continuum Mechanics 223–276 (1997) (Springer Verlag, Wien)
16. Haubold H J, Mathai A M, and Saxena R K, Mittag-Leffler functions and their applications, J. Appl. Math. 2011 (2011) 298628, 51 pages; available online at http://dx.doi.org/10.1155/2011/298628
17. Hiller R (Edt), Applications of Fractional Calculus in Physics (2000) (New Jersey, London, Hong Kong:Word Scientific Publishing Co.)
18. Kilbas A A, Koroleva A A, Rogosin S V, Multi-parametric Mittag-Leffler functions and their extension, Fract. Calc. Appl. Anal. 16(2) (2013) 378–404
19. Kilbas A A, Saigo M and Saxena R K, Solution of Volterra integro-differential equations with generalized Mittag-Leffler function in the kernels, J. Integral Equations Appl. 14(4) (2002) 377–386
20. Kilbas A A, Srivastava H M, Trujillo J J, Theory and Applications of Fractional Differential Equations (2006) North-Holland Mathematics Studies, 204 (Elsevier, Amsterdam, etc)
21. Kiryakova V, Generalized Fractional Calculus and Applications (1994) (Harlow, Longman)
22. Kiryakova V, Multiindex Mittag-Leffler functions, related Gelfond-Leontiev operators and Laplace type integral transforms, Fract. Calc. Appl. Anal. 2(4) (1999) 445–462
23. Kiryakova V, Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus, J. Comput. Appl. Math. 118 (2000) 241–259
24. Kiryakova V, On two Saigo’s fractional integral operators in the class of univalent functions, Fract. Calc. Appl. Anal. 9(2) (2006) 160–176
25. Kiryakova V S, The special functions of fractional calculus as generalized fractional calculus operators od some basic functions, Comp. Math. Appl. 59(3) (2010) 1128–1141
26. Kiryakova V S, The multi-index Mittag-Leffler function as an important class of special functions of fractional calculus, Comp. Math. Appl. 59(5) (2010) 1885–1895
27. Mainardi F, Fractional Calculus and Waves in Linear Viscoelasticity (2010) (London: Imperial College Press)
28. Marichev O I, Volterra equation of Mellin convolution type with a Horn function in the kernel, Izv. AN BSSR Ser. Fiz.-Mat. Nauk 1 (1974) 128–129
29. Mathai A M, Saxena R K, The H -function with Applications in Statistics and Other Disciplines, Halsted Press [John Wiley & Sons], New York, London, Sydney, 1978
30. Mathai A M, Saxena R K, Haubold H J, The H -function. Theory and Applications (2010) (Dordrecht: Springer)
31. McBride A C, Fractional Calculus and Integral Transforms of Generalized Functions (1979) (Research Notes in Math. 31) (Pitman, London)
32. Miller K S, Ross B, An Introduction to the Fractional Calculus and Fractional Differential Equations (1993) (New York: John Wiley and Sons)
33. Mittag-Leffler G M, Sur la nouvelle fonction Eγ(x), C. R. Acad. Sci. Paris 137 (1903) 554–558
34. NIST Handbook of Mathematical Functions. Edited by Frank W.J. Olver (editor-in-chief), D.W. Lozier, R.F. Boisvert, and C.W. Clark. Gaithersburg, Maryland, National Institute of Standards and Technology, and New York, Cambridge University Press, 951 + xv pages and a CD, (2010)
35. Oldham K B, Spanier J, The Fractional Calculus (1974) (New York: Academic Press)
36. Podlubny I, Fractional Differential Equations (1999) (New York: Academic Press)
37. Prabhakar T R, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J. 19 (1971) 7–15
38. Prakasa Rao B L S, Statistical inference for fractional diffusion processes (2010) (Chichester: John Wiley & Sons Ltd.)
39. Rabotnov Yu N, Elements of Hereditary Solid Mechanics (1980) (Moscow:MIR)
40. Rogosin S V, ”The Role of the Mittag-Leffler Function in Fractional Modeling” Mathematics 2015, 3, 368-381; doi:10.3390/math3020368
41. Saigo M, On generalized fractional calculus operators. In: Recent Advances in Applied Mathematics (Proc. Internat. Workshop held at Kuwait Univ.). Kuwait Univ., Kuwait, (1996) 441–450
42. Saigo M, Maeda N, More generalization of fractional calculus, In: Transform Methods and Special Functions, Varna 1996 (Proc. 2nd Intern. Workshop, P. Rusev, I. Dimovski, V. Kiryakova Eds.), IMI-BAS, Sofia, (1998) 396–400
43. Samko S G, Kilbas A A, Marichev O I, Fractional Integrals and Derivatives: Theory and Applications (1993) (New York and London: Gordon and Breach Science Publishers)
44. Saxena R K and Nishimoto K, N-Fractional Calculus of Generalized Mittag-Leffler functions, J. Fract. Calc. 37(2010) 43–52
45. Saxena R K, Saigo M, Generalized fractional calculus of the H -function associated with the Appell function, J. Fract. Calc. 19 (2001) 89–104
[46] Shukla A K and Prajapati J C, On a generalization of Mittag-Leffler function and its properties, J. Math. Anal. Appl. 336 (2007) 797–811
[47] Srivastava H M and Agarwal P, Certain fractional integral operators and the generalized incomplete hypergeometric functions, Appl. Appl. Math. 8(2) (2013) 333–345
[48] Srivastava H M, Choi J, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001
[49] Srivastava H M, Choi J, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London and New York, 2012
[50] Srivastava H M, Gupta K C, Goyal S P, The H-functions of One and Two Variables with Applications, South Asian Publishers, New Delhi, Madras, 1982
[51] Srivastava H M and Saigo M, Multiplication of fractional calculus operators and boundary value problems involving the euler-darboux equation, J. Math. Anal. Appl. 121 (1987) 325–369
[52] Srivastava H M, Tomovski Ž, Fractional claculus with an integral operator containing generalized Mittag-Leffler function in the kernel, Appl. Math. Comput. 211(1) (2009) 198–210
[53] Tarasov V E, Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media (2010) (Beijing: Springer, Heidelberg; Higher Education Press)
[54] Tenreiro Machado J A, Kiryakova V, Mainardi F, A poster about the old history of fractional calculus, Fract. Calc. Appl. Anal. 13 (4)(2010) 447–454
[55] Tenreiro Machado J A, Kiryakova V, Mainardi F, A poster about the recent history of fractional calculus, Fract. Calc. Appl. Anal. 13(3)(2010) 329–334
[56] Tenreiro Machado J A, Kiryakova V, Mainardi F, Recent history of fractional calculus, Commun. Nonlinear Sci. Numer. Simulat. 16(3) (2011) 1140–1153; available online at http://dx.doi.org/10.1016/j.cnsns.2010.05.027
[57] Uchaikin V V, Fractional derivatives for physicists and engineers. Vol. I. Background and theory(2013) (Beijing: Springer, Berlin - Higher Education Press)
[58] Uchaikin V V, Fractional derivatives for physicists and engineers, Vol. II Applications(2013) (Beijing: Springer, Berlin - Higher Education Press)
[59] Wiman A, Über den Fundamentalsatz in der Theorie der Funktionen $E_{p} (x)$. Acta Math. 29(1905) 191–201
[60] Zaslavsky G M, Hamiltonian Chaos and Fractional Dynamics (2005) (Oxford: Oxford University Press)