Gauge-independent Husimi functions of charged quantum particles in the electro-magnetic field

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Abstract

Gauge-independent Husimi function (Q−function) of states of charged quantum particles in the electro-magnetic field is introduced using the gauge-independent Stratonovich-Wigner function, the corresponding dequantizer and quantizer operators transforming the density matrix of state to the Husimi function and vice versa are found explicitly, and the evolution equation for such function is derived.

Also own gauge-independent non-Stratonovich Wigner function is suggested and its Husimi function is obtained. Dequantizers and quantizers for these Wigner and Husimi functions are given.

Keywords: Husimi function, gauge invariance, phase space representation, evolution equation, quantizer, dequantizer.

1 Introduction

Since the very beginning of the creation of quantum mechanics, the question of its formulation in terms of the distribution function on the phase space, like the classical kinetic theory, has attracted the attention of many scientists, despite the fact that the Heisenberg uncertainty relation prohibits the existence of the joint distribution function of the position and momentum in the quantum case.

A great success in this connection was the introduction of the Wigner function [1] and the writing of the dynamic equation for it [1, 2]. For the N−dimensional system the Wigner function is introduced as the Weyl symbol [3] of the density matrix \( \hat{\rho}(t) \) of the state

\[
W(q, P, t) = \frac{1}{(2\pi \hbar)^N} \int \langle q - \frac{u}{2} | \hat{\rho}(t) | q + \frac{u}{2} \rangle \exp \left( \frac{i}{\hbar} uP \right) d^N u,
\]

where \( P \) is the generalized momentum corresponding to the generalized momentum operator \( \hat{P} = -i\hbar \partial/\partial q \). The transformation inverse to (1) has the form:

\[
\langle q | \hat{\rho}(t) | q' \rangle = \int W \left( \frac{q + q'}{2}, P, t \right) \exp \left( \frac{i}{\hbar} P(q - q') \right) d^N P.
\]

Despite the fact that the Wigner function can take negative values, it is successfully applied in many applications since the 1950s (see, e.g., [4, 5, 6, 7, 8]) and to the present. The properties of the Wigner function were considered, e.g., in [9].

At the same time, if we smoothly average the Wigner function on the scales of the hyper-volume \( \hbar^N \) of phase space, then it can be made nonnegative. This way in the article [10] and later in [11] the Husimi function (the so called Q−function) was introduced

\[
Q(q, P, t) = \frac{1}{(2\pi \hbar)^N} \langle \alpha | \hat{\rho}(t) | \alpha \rangle,
\]

representing up to the normalization factor the transition probability of a quantum system from the state \( \hat{\rho}(t) \) into a coherent state \( | \alpha \rangle \) [12, 13], where \( \alpha = (2\hbar)^{-1/2} (\lambda^{1/2} q + i \lambda^{-1/2} P) \), \( \lambda = m\omega \). Up to the inessential phase factor the state \( | \alpha \rangle \) is defined in the position representation as follows:

\[
\langle q' | \alpha \rangle = \left( \frac{\lambda}{\pi \hbar} \right)^{N/4} \exp \left[ -\frac{\lambda}{2\hbar} (q - q')^2 - \frac{i}{\hbar} P(q - q') \right].
\]
As is known, the state $|\alpha\rangle$ minimizes the Heisenberg uncertainty relation, and the remarkable property of the $Q$–function is its connection with the Wigner function by means of the formula:

$$Q(q, P, t) = \frac{1}{(\pi\hbar)^N} \int \exp \left( -\frac{\lambda}{\hbar}(q - q')^2 - \frac{\lambda\hbar}{4}(P - P')^2 \right) W(q', P', t)d^Nq'd^NP',$$

representing the averaging of the function $W(q, P, t)$ in the phase space with respect to a Gaussian distribution centered at the point $(q, P)$. Formula (5) can be rewritten in the operator form,

$$Q(q, P, t) = \exp \left( -\frac{\lambda}{4\hbar}\frac{\partial^2}{\partial_q^2} + \frac{\lambda\hbar}{4}\frac{\partial^2}{\partial_P^2} \right) W(q, P, t),$$

and the inverse transform of (6) is obviously has the shape:

$$W(q, P, t) = \exp \left( -\frac{\lambda}{4\hbar}\frac{\partial^2}{\partial_q^2} - \frac{\lambda\hbar}{4}\frac{\partial^2}{\partial_P^2} \right) Q(q, P, t).$$

Also the inverse transform of (5) can be expressed as a repeated integral,

$$W(q, P, t) = \int \frac{d^Nq\,d^Nv}{(2\pi\hbar)^{2N}} \int \exp \left[ \frac{\hbar v^2}{4\lambda} + iu(q - q') + \frac{\lambda\hbar v^2}{4} + iv(P - P') \right] Q(q', P', t)d^Nq'd^NP'. \tag{8}$$

Expansion of formula (5) to the classical (non-quantum) case allows to determine the Husimi function $Q_{cl}(q, P, t)$ of the state of the classical system having described by the classical distribution function $W_{cl}(q, P, t)$ as the overlap with a Gaussian distribution.

Consider the motion of a quantum particle having a spin in the electromagnetic field with the vector potential $A(q, t)$ and the scalar potential $\varphi(q, t)$. As it is known, the Hamiltonian of such a system has the form $[14]$:

$$\hat{H} = \frac{1}{2m}(\hat{P} - \frac{e}{c}\hat{A})^2 + e\varphi - \hat{\kappa}B,$$

where $\hat{P} = -i\hbar\partial/\partial q$ is a generalized momentum operator, $m$ and $e$ are mass and charge of the particle, $B = \text{rot}A$ is a magnetic field strength, $\hat{\kappa}$ is an operator of quantum-mechanical magnetic moment

$$\hat{\kappa} = \frac{\kappa}{s}\hat{s},$$

where $s$ is a spin of the particle, $\hat{s}$ is a spin operator, and $\kappa$ is the value of the intrinsic magnetic moment of the particle.

From the classical electrodynamics it is known that potentials of the field are defined only up to the gauge transformation $[15]$:

$$A \to A + \nabla\chi, \quad \varphi \to \varphi - \frac{1}{c}\frac{\partial\chi}{\partial t},$$

where $\chi$ is an arbitrary function of spatial coordinates and time.

Since the electric field intensity $E$ and the magnetic field strength $B$ are defined in terms of the potentials as:

$$E = -\text{grad}\varphi - \frac{1}{c}\frac{\partial}{\partial t}A, \quad B = \text{rot}A,$$

then the gauge transformation $[11]$ does not affect the values of $E$ and $B$. Therefore the part of Hamiltonian $[9]$ responsible for the interaction of the spin with the magnetic field is independent on the gauge transformation, and we can restrict our considerations only to the case of $s = 0$. The generalization to the non-zero spin particles is straightforward (see $[16]$ $[17]$ $[18]$).

The requirement of invariance of the Schrödinger equation under the gauge transformation simultaneously with the gauge-independence of “probability density” $|\Psi|^2$ leads us to the form of the conversion of the wave function $[14]$:

$$\Psi \to \exp \left( \frac{ic}{\hbar\lambda}\chi \right) \Psi.$$

(13)
Accordingly, the conversions of the density matrix of the state of the system under the gauge transformation acquires the form:

\[
\hat{\rho}_c = \exp \left( \frac{i\epsilon}{\hbar c} \chi \right) \hat{\rho} \exp \left( -\frac{i\epsilon}{\hbar c} \chi \right).
\] (14)

In [19] the gauge-independent Wigner function was constructed, and its evolution equation was derived in [20]. The gauge-invariance in the tomographic probability representation of quantum mechanics was considered in [18] (see review articles [21, 22] about the tomographic probability representation).

The aim of this work is introduction of gauge-independent Husimi function (\(Q\)-function) of states of charged quantum particles in the electro-magnetic field, and is derivation of the evolution equation for such function.

## 2 Gauge-independent Husimi function

For the construction of quantum Husimi representations, in which the evolution equation would be gauge-independent, we need to introduce gauge-independent quantum Husimi functions. This can be done with the help of a gauge-independent Wigner function obtained in [19],

\[
W_g(q, p, t) = \frac{1}{(2\pi\hbar)^3} \int \exp \left( \frac{i}{\hbar} u \left( p + \frac{e}{c} \int_{-1/2}^{1/2} \text{d}\tau A(q + \tau u, t) \right) \right) \rho \left( q - \frac{u}{2}, q + \frac{u}{2}, t \right) \text{d}^3u,
\] (15)

where \(p\) is a kinetic momentum.

The gauge-independent Husimi function \(Q_g(q, p, t)\) should be introduced using a formula similar to [5]

\[
Q_g(q, p, t) = \frac{1}{(\pi\hbar)^3} \int \exp \left( -\frac{\lambda}{\hbar}(q - q')^2 - \frac{1}{\lambda\hbar}(p - p')^2 \right) W_g(q', p', t) \text{d}^3q' \text{d}^3p'.
\] (16)

With this definition the formulas [6, 7, 8] of the relations between \(Q_g(q, p, t)\) and \(W_g(q', p', t)\) remain valid if we replace the generalized momentum \(P\) by the kinetic momentum \(p\) in them.

Combining formulas (15) and (16) we can write

\[
Q_g(q, p, t) = \int \langle q_1|\hat{\rho}(t)|q_2 \rangle \langle q_2|\hat{U}_{Q_g}(q, p)|q_1 \rangle d^3q_1 d^3q_2,
\] (17)

where we introduce the matrix element for the corresponding dequantizer operator

\[
\langle q_2|\hat{U}_{Q_g}(q, p)|q_1 \rangle = \frac{1}{(2\pi\hbar)^3} \left( \frac{\lambda}{\pi\hbar} \right)^{3/2} \exp \left\{ -\frac{\lambda}{2\hbar}(q - q_2)^2 - \frac{\lambda}{2\hbar}(q - q_1)^2 \right. \\
+ \left. \frac{i}{\hbar}(q_2 - q_1) \left[ p + \frac{e}{c} \int_{-1/2}^{1/2} A(q + \tau q_2, t) \right] d\tau \right\}.
\] (18)

From (18) we can see that \(\hat{U}_{Q_g}(q, p)\) is Hermitian operator, so the Husimi function \(Q_g(q, p, t)\) is real.

The explicit form of the operator \(\hat{U}_{Q_g}(q, p)\) can be written in the integral form

\[
\hat{U}_{Q_g}(q, p) = \int \frac{d^3u d^3v}{(2\pi\hbar)^6} \exp \left[ -\frac{\lambda u^2}{4\hbar} - \frac{v^2}{4\hbar\lambda} - \frac{i}{\hbar}(u p + v q) \right] \exp \left\{ \frac{i}{\hbar} \left[ u \left( \hat{p} - \frac{e}{c} A(q, t) \right) + v \hat{q} \right] \right\}.
\] (19)

The transformation inverse to (17) can be expressed using the matrix element of quantizer operator \(\hat{D}_{Q_g}(q, p)\)

\[
\langle q_1|\hat{\rho}(t)|q_2 \rangle = \int \langle q_1|\hat{D}_{Q_g}(q, p)|q_2 \rangle Q_g(q, p, t) d^3q d^3p,
\] (20)
where

\[
\langle q_1 | \hat{D}_{Q_s}(q, p) | q_2 \rangle = \int \frac{d^3v}{(2\pi \hbar)^3} \exp \left[ \frac{\lambda(q_2 - q_1)^2}{4\hbar} + \frac{v^2}{4\hbar\lambda} - \frac{i\nu}{2\hbar}(2q - q_1 - q_2) - \frac{i\nu}{\hbar}(q_2 - q_1) \right]
\]  

- \frac{ie}{\hbar c}(q_2 - q_1) \int_{-1/2}^{1/2} d\tau A \left( \frac{q_2 + q_1}{2} + \tau(q_2 - q_1), t \right).

(21)

In these formulas it is assumed that \( Q_g(q, p, t) \in S(\mathbb{R}^6) \), where \( S(\mathbb{R}^6) \) is a Schwartz space, and first we take the integral over \( d^3q \), and after that the integrals over \( d^3p \) and \( d^3v \) are taken.

Assuming a special order of integration, the explicit form of the operator \( \hat{D}_{Q_s}(q, p) \) can be written as:

\[
\hat{D}_{Q_s}(q, p) = \int \frac{d^3ud^3v}{(2\pi \hbar)^3} \exp \left[ \frac{\lambda u^2}{4\hbar} + \frac{v^2}{4\hbar\lambda} - \frac{i}{\hbar}(up + vq) \right] \exp \left\{ \frac{i}{\hbar} \left[ u \left( \hat{P} - \frac{e}{c} \hat{A}(q, t) \right) + vq \right] \right\}.
\]

(22)

3 Evolution equation for the gauge-independent Husimi function

To begin with, let us recall the Liouville equation in the electro-magnetic field for the classical distribution function. For the classical ensemble of non-interacting particles with mass \( m \) and charge \( e \) this equation in the phase space has the form:

\[
\left\{ \partial_t + \frac{p}{m} \partial_q + e \left( E(q, t) + \frac{1}{mc} [p \times B(q, t)] \right) \partial_p \right\} W_{cl}(q, p, t) = 0,
\]

(23)

where \( p \) is a kinetic momentum, \( E(q, t) \) and \( B(q, t) \) are electric and magnetic fields, defined by formulas (12), \( W_{cl}(q, p, t) \) is a distribution function of non-interacting particles.

The distribution function \( W_{cl}(q, p, t) \) is independent on the gauge transformation [15] because the Liouville equation (23) includes only gauge-independent intensities of the electro-magnetic field.

Gauge-independent Moyal equation for the Wigner function \( W_g(q, p, t) \) has the form [20]:

\[
\left\{ \partial_t + \frac{1}{m} (p + \Delta \hat{p}) \partial_q + e \left( \hat{E} + \frac{1}{mc} [(p + \Delta \hat{p}) \times \hat{B}] \right) \partial_p \right\} W_g(q, p, t) = 0,
\]

(24)

where

\[
\Delta \hat{p} = -\frac{e}{c} \frac{\hbar}{i} \left[ \frac{\partial}{\partial p} \times \int_{-1/2}^{1/2} d\tau B \left( q + i\hbar \tau \frac{\partial}{\partial p}, t \right) \right],
\]

\[
\hat{E} = \int_{-1/2}^{1/2} d\tau E \left( q + i\hbar \tau \frac{\partial}{\partial p}, t \right), \quad \hat{B} = \int_{-1/2}^{1/2} d\tau B \left( q + i\hbar \tau \frac{\partial}{\partial p}, t \right).
\]

This equation in the classical limit \( \hbar \to 0 \) is converted into Liouville equation (23).

Since the relations between the functions \( W_g(q, p, t) \) and \( Q_g(q, p, t) \) are the same as between \( W(q, p, t) \) and \( Q(q, p, t) \), then the correspondence rules between operators acting on the Wigner function and the Husimi function do not change. Consequently, we can write:

\[
q W_g(q, p) \leftrightarrow (q + \frac{\hbar}{2} \partial_q) Q(q, p, t),
\]

\[
p W_g(q, p) \leftrightarrow (p + \frac{\hbar}{2} \partial_p) Q(q, p, t),
\]

\[
\partial_q W_g(q, p) \leftrightarrow \partial_q Q(q, p, t),
\]

\[
\partial_p W_g(q, p) \leftrightarrow \partial_p Q(q, p, t).
\]

(25)
With the help of (25), equation (24) is transformed to the evolution equation for the gauge-independent Husimi function $Q_g(q, p, t)$

$$\left\{ \partial_t + \frac{1}{m} \left( p + \frac{\hbar}{2} \partial_p + [\Delta \hat{P}]_Q \right) \partial_q 
+ e \left( [\hat{E}]_Q + \frac{1}{mc} \left( p + \frac{\hbar}{2} \partial_p + [\Delta \hat{P}]_Q \right) \times [\hat{B}]_Q \right) \partial_p \right\} Q_g(q, p, t) = 0, \quad (26)$$

where

$$[\Delta \hat{P}]_Q = -\frac{e \hbar}{c} \left[ \partial_p \times \int_{-1/2}^{1/2} B \left( q + \frac{\hbar}{2\lambda} \partial_q + i\hbar \tau \partial_p, t \right) \tau d\tau \right],$$

$$[\hat{E}]_Q = \int_{-1/2}^{1/2} E \left( q + \frac{\hbar}{2\lambda} \partial_q + i\hbar \tau \partial_p, t \right) d\tau,$n

$$[\hat{B}]_Q = \int_{-1/2}^{1/2} B \left( q + \frac{\hbar}{2\lambda} \partial_q + i\hbar \tau \partial_p, t \right) d\tau. \quad (27)$$

As it should be, equation (26) in the classical limit $\hbar \to 0$ is converted into the Liouville equation (23).

### 4 Non-Stratonovich type of gauge-independent Wigner and Husimi functions

In the previous sections we considered the Wigner and Husimi functions on the basis of the definition of Stratonovich [19]. However, it is possible to introduce a gauge-independent Wigner function and its corresponding Husimi function according to (5) by other ways.

Let us define the gauge-invariant dequantizer for the new Wigner function $W_g(q, p)$ as follows:

$$\hat{U}_{W_g}(q, p) = \exp \left[ \frac{ie}{\hbar c} \int_0^1 d\tau A(\tau \hat{q}, t) \right]$$

$$\times \int \frac{d^3u d^3v}{(2\pi \hbar)^6} \exp \left\{ \frac{i}{\hbar} \left[ u(\hat{P} - p) + v(\hat{q} - q) \right] \right\} \exp \left[ -\frac{ie}{\hbar c} \int_0^1 d\tau A(\tau \hat{q}, t) \right]. \quad (28)$$

Since for the Wigner function the dequantizer-quantizer scheme is self-dual, then the following equality takes place for the corresponding quantizer: $D_{W_g}(q, p) = (2\pi \hbar)^3 \hat{U}_{W_g}(q, p)$. Calculation of the matrix element of (28) yields

$$(q_2 | \hat{U}_{W_g}(q, p) | q_1) = (2\pi \hbar)^{-3} \delta \left( q - \frac{q_2 + q_1}{2} \right)$$

$$\times \exp \left[ \frac{ip}{\hbar} (q_2 - q_1) + \frac{ie}{\hbar c} q_2 \int_0^1 d\tau A(\tau q_2, t) - \frac{ie}{\hbar c} q_1 \int_0^1 d\tau A(\tau q_1, t) \right]. \quad (29)$$

Taking into account (5), we find the dequantizer for the corresponding Husimi function $\Omega_g(q, p)$

$$\hat{U}_{\Omega_g}(q, p) = \exp \left[ \frac{ie}{\hbar c} \int_0^1 d\tau A(\tau \hat{q}, t) \right] \int \frac{d^3u d^3v}{(2\pi \hbar)^6} \exp \left[ -\frac{\lambda u^2}{4\hbar} - \frac{v^2}{4\hbar \lambda} - \frac{i}{\hbar} (up + vq) \right]$$

$$\times \exp \left[ \frac{i}{\hbar} (u\hat{P} + v\hat{q}) \right] \exp \left[ -\frac{ie}{\hbar c} \int_0^1 d\tau A(\tau \hat{q}, t) \right]. \quad (30)$$
The matrix element \( \langle q_2 | \hat{U}_{\Omega_{g}}(q, p) | q_1 \rangle \) is obviously equal to the following:

\[
\langle q_2 | \hat{U}_{\Omega_{g}}(q, p) | q_1 \rangle = \frac{1}{(2\pi\hbar)^3} \left( \frac{\lambda}{\pi\hbar} \right)^{3/2} \exp \left[ -\frac{\lambda}{2\hbar} (q - q_2)^2 - \frac{\lambda}{2\hbar} (q - q_1)^2 \right] + \frac{i\hbar}{\lambda} (q_2 - q_1)p + \frac{ie}{\hbar c} q_2 \int_0^1 d\tau A(\tau q_2, t) - \frac{ie}{\hbar c} q_1 \int_0^1 d\tau A(\tau q_1, t) \right].
\] (31)

From (31) it is obvious that up to the normalization factor the dequantizer \( \hat{U}_{\Omega_{g}}(q, p) \) is the projector of the considered state \( \hat{\rho}(t) \) onto the pure state \( |\Psi_{q,p}\rangle \), i.e. \( \hat{U}_{\Omega_{g}}(q, p) = (2\pi\hbar)^{-3} |\Psi_{q,p}\rangle \langle \Psi_{q,p}| \), where we have up to the phase factor in the position representation:

\[
\langle q'|\Psi_{q,p}\rangle = \left( \frac{\lambda}{\pi\hbar} \right)^{3/4} \exp \left[ -\frac{\lambda}{2\hbar} (q - q')^2 - \frac{i}{\hbar} p(q - q') + \frac{ie}{\hbar c} q' \int_0^1 d\tau A(\tau q', t) \right].
\] (32)

Calculations of the matrix element of the quantizer for the function \( \Omega_{g}(q, p) \) give rise to the expression

\[
\langle q_1 | \hat{D}_{\Omega_{g}}(q, p) | q_2 \rangle = \int \frac{d^3v}{(2\pi\hbar)^3} \exp \left[ \frac{\lambda(q_2 - q_1)^2}{4\hbar} + \frac{v^2}{4\hbar\lambda} - \frac{iv}{2\hbar} (2q - q_1 - q_2) - \frac{ip}{\hbar} (q_2 - q_1) \right]
\]

\[
- \frac{ie}{\hbar c} q_2 \int_0^1 d\tau A(\tau q_2, t) + \frac{ie}{\hbar c} q_1 \int_0^1 d\tau A(\tau q_1, t) \right].
\] (33)

When using quantizer (33), it is assumed that \( \Omega_{g}(q, p, t) \in S(\mathbb{R}^6) \), where \( S(\mathbb{R}^6) \) is a Schwartz space, and first we take the integral over \( d^3q \), and after that the integrals over \( d^3p \) and \( d^3v \) are taken. With the same stipulation, the explicit form of the operator \( \hat{D}_{\Omega_{g}}(q, p) \) can be written as:

\[
\hat{D}_{\Omega_{g}}(q, p) = \exp \left[ \frac{ie}{\hbar c} \hat{q} \right] \int \frac{d^3u d^3v}{(2\pi\hbar)^3} \exp \left[ \frac{\lambda u^2}{4\hbar} + \frac{v^2}{4\hbar\lambda} - \frac{i}{\hbar} (up + vq) \right]
\]

\[
\times \exp \left[ \frac{i}{\hbar} \left[ u\hat{P} + v\hat{q} \right] \right] \exp \left[ -\frac{ie}{\hbar c} \hat{q} \right] \int_0^1 d\tau A(\tau \hat{q}, t) \right].
\] (34)

Knowing the quantizers and dequantizers for functions \( M_{g}(q, p) \) and \( \Omega_{g}(q, p) \) one can find the evolution equations for them. Since \( M_{g}(q, p) \) and \( \Omega_{g}(q, p) \) do not depend on the gauge, then their evolution equations must also be gauge-independent.

### 5 Conclusion

In conclusion, I point out the main results of the paper. The gauge-independent Husimi function \( (Q-\text{function}) \) of states of charged quantum particles in the electro-magnetic field was introduced using the gauge-independent Stratonovich-Wigner function, the corresponding dequantizer and quantizer operators transforming the density matrix of state to the such Husimi function and vice versa were found explicitly, and the evolution equation for such function was derived. Also own non-Stratonovich gauge-independent Wigner function and its Husimi function were suggested and their dequantizers and quantizers were obtained.

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