On the Injectivity of an Integral Operator Connected to Riemann Hypothesis

Dumitru Adam (✉ dumitru_adam@yahoo.ca)

Research Article

**Keywords:** Approximation Subspaces, Integral Operators, Riemann Hypothesis

**Posted Date:** January 25th, 2022

**DOI:** [https://doi.org/10.21203/rs.3.rs-1159792/v4](https://doi.org/10.21203/rs.3.rs-1159792/v4)

**License:** ☑️  This work is licensed under a Creative Commons Attribution 4.0 International License. Read Full License
On the Injectivity of an Integral Operator Connected to Riemann Hypothesis *

Dumitru Adam †
Jan. 13, 2022

Abstract

In 1993, using the Beurling equivalent formulation of the RH ([3]), Alcantara-Bode proved ([2]): the RH holds if and only if the integral operator on the Hilbert space $L^2(0,1)$ having the kernel function defined by the fractional part of $(y/x)$, is injective.

Since then, the injectivity of this integral operator has not been addressed even if the investigation of the injectivity of this operator could be made out of context, by other means than of pure mathematics. In line with this observation is our approach using the operator approximations on a dense family of subspaces on separable Hilbert spaces in order to address its injectivity.

One of the results obtained in this paper, Theorem 2.1 states that given an linear, bounded operator strict positive definite on a dense family of subspaces, having its sequence of injectivity parameters bounded inferior by a strict positive constant, is injective. (The parameters are the inverse condition numbers of the operator restrictions on the family of the subspaces.)

Using a version of this theorem on $L^2(0,1)$ we proved (Theorem 4.1) the injectivity of the integral operator used by Alcantara-Bode in [2] for his equivalent formulation of RH.

subiclass: 65R99, 47G10, 45P05
keywords: Approximation Subspaces, Integral Operators, Riemann Hypothesis

The Problem.

Alcantara-Bode proved ([2], Theorem 1, pg. 152) that, the RH holds if and only if the following Hilbert-Schmidt integral operator $T_\rho$ defined on

---

*Submitted to JNAAT, updated
†No Funding, No Affiliations During This Research. Email: dumitru.adam@yahoo.ca Windsor, On. Ca. & Brasov, Ro
Let $H$ be a separable Hilbert space. The norm considered on $H$ is the norm induced by its inner product. Because a linear bounded operator $T$ and its associated Hermitian and positive definite operator $(T^*T)$ have the same null space $N_T$, we will consider its associated Hermitian operator when we do not have enough information about positivity of $T$ on $H$. Then, we take $T$ or $(T^*T)$ in order to have its positivity $\langle Tu, u \rangle \geq 0$ on $H$ as we need. In the following $T$ is positive definite on $H$ and our aim is to provide methods for investigating its strict positivity on $H$, in other words, in finding if its null space $N_T = \{0\}$.

We will see later that a weaker property than the positivity on $H$ is enough and is mandatory for investigating the injectivity of the linear operators on $H$, that is the strict positivity of the operators on a dense family of approximation subspaces (a-positivity property).

Let $B := \{u \in H; \|u\| = 1\}$ be the unit sphere in $H$. Because a not null element $u \in N_T$ if and only if $\langle u/\|u\| \rangle \in N_T$, we will consider the normalized elements of $N_T$ for convenience.

A family of finite dimension subspaces $\mathcal{F} := \{S_n, n \in N\} \subset H$ is a proper family of approximation subspaces if the following properties hold on the subspaces of the family:
(a) $S_n \subset S_{n+1}, n \in N$

(b) $\beta_n(u) := \|u - u_n\| \to 0$, for $n \to \infty$

for every $u \in B$ where $u_n$ is its orthogonal projection on $S_n$, \( (\forall) n \in N \).

A proper family subspaces is dense in $H$: $\bigcup_{n=1}^{\infty} S_n = H$, property easy to show. The main reason in defining a dense family of subspaces in $H$ being a proper family, is to evidentiate the two properties used extensively in this paper: the inclusion of the approximation subspaces and, the increasing monotone convergence in norm to 1 of the sequence of the projections on approximation subspaces defined for every element from unit sphere $B \subset H$.

We say that the bounded linear operator $T$ is almost strict positive definite on $H$ or a-positive, if there exists a proper family $\mathcal{I}$ on which, for every subspace $S_n \in \mathcal{I}$, there exists $\alpha_n > 0$ such that:

$$\langle T v, v \rangle \geq \alpha_n \|v\|^2, \quad \forall v \in S_n$$

We name the set $\{\alpha_n, n \geq 1\}$, the set of a-positive parameters associated to $T$ on $\mathcal{I}$. In facts, a-positive parameters are the smallest eigenvalues of the operator restrictions to the corresponding subspaces. We will use the term a-positive also associated to the family $\mathcal{I}$ for convenience.

Let $L_\mathcal{I}$ be the set of linear bounded operators on $H$ that are a-positive with respect to the family $\mathcal{I}$.

Taking $\mathcal{I}^0 = \bigcup_{n=1}^{\infty} S_n$ we define $H_{\mathcal{I}^0} := H \setminus \mathcal{I}^0$ and $B_0 := B \cap H_{\mathcal{I}^0}$.

Let $T \in L_\mathcal{I}$. If there exists $u \in B \cap N_T$ then $u \notin S_n$ for every $n \geq 1$, i.e. $u \notin \mathcal{I}^0$. For this reason we will consider in our analysis only $u \in H$ that are eligible, in the sense that these elements from $H$ could be the (normalised) zeros of $T$, $u \in B_0$.

We will refer in this paper to $B_0$ as the set of eligible zeros of the a-positive operators given a proper family $\mathcal{I}$.

**Observations**: we point out two extreme situations that we will exclude from our analysis:

1.) - If $T$ is positive definite on $H$ but is not a-positive on a proper family, then $T$ should have a zero inside the family: there exists $S_n \in \mathcal{I}$ and $v \in S_n$ not null, such that $\langle T v, v \rangle = 0$. So $T$ is not injective and the investigations on its injectivity ends here. If $T$ is not positive definite on $H$ then the observation is valid for its associate Hermitian.

That means, the a-positivity property of an linear bounded operator is mandatory for its injectivity.

2.) - If $T$ is a-positive and the approximation subspaces of $\mathcal{I}$ are invariant subspaces of $T$, then $T$ is injective. For proving it, suppose that $T$ is Hermitian. Then for $v$ in $S_n$ and $w$ in the orthogonal subspace of $S_n$
Supposing \( u \) eligible \( \in N_T \) and taking \( w = (u_n - u) \) and \( v = u_n \) where \( u_n \) is the orthogonal projection of \( u \) on \( S_n \) we obtain that \( T \) violates the a-positivity. Now, if \( T \) is not Hermitian then the arguments made before are valid for its associated Hermitian that, has the same null space and invariant subspaces like \( T \). Thus, \( T \) a-positive having the approximation subspaces as invariant subspaces is injective and no investigations for its injectivity needed.

As a consequence, from now on the linear bounded operators in consideration would be a-positive operators satisfying \( T(S_n) \not\subset S_n \), \( (\forall) n \geq 1 \).

\[ \langle Tw, v \rangle = \langle w, T v \rangle = 0 \] because \( w \) and \( T v \) are in orthonal subspaces. Supposing \( u \) eligible \( \in N_T \) and taking \( w = (u_n - u) \) and \( v = u_n \) where \( u_n \) is the orthogonal projection of \( u \) on \( S_n \) we obtain that \( T \) violates the a-positivity.

§2. An Injectivity Criteria.

Let \( \mathcal{G} \) be a proper family of approximation subspaces on \( H \) and \( T \in L_\mathcal{G} \).

Suppose that there exists \( w \in S_n^\perp \) for which \( Tw \not\in S_n^\perp \). For a such element \( w \) and for every \( v \in S_n \) both not null, the inner product \( \langle Tw, v \rangle \) is well defined in \( H \). Now, because an strict positive operator on a finite dimension subspace has the same eigenvalues as its adjoint, in our case \( T^*|_{S_n} \) and \( T|_{S_n} \) have both the same largest eigenvalue \( \lambda_{\text{max}}^n \) on \( S_n \), we obtain the following estimation for the considered inner product:

\[
| \langle Tw, v \rangle | = | \langle w, T^*v \rangle | \leq \lambda_{\text{max}}^n \| w \| \| v \|
\]

If there exists \( u \in N_T \cap B_0 \), we take \( w := (u_n - u) \in S_n^\perp \) and \( v = u_n \), the projection of \( u \) on \( S_n \). Obviously, \( Tw = Tu_n \not\in S_n^\perp \) (due to the a-positivity of \( T \) \( \langle Tu_n, u_n \rangle \neq 0 \) and, because there exists an index \( n_0(u) \) from which \( u \) is not orthogonal to \( S_n \) from the density of the proper family). Then:

\[
\alpha_n \| u_n \|^2 \leq \langle Tu_n, u_n \rangle = \| \langle Tu_n - u, u_n \rangle \| \text{ and, together with the previous relations we obtain:}
\]

\[
\mu_n \| u_n \| \leq \beta_n(u) \tag{2}
\]

where \( \mu_n := \lambda_{\text{min}}^n / \lambda_{\text{max}}^n \) is the ratio between the smallest and largest eigenvalues of the restriction of \( T \) to the subspace \( S_n \), inequality obtained using the a-positivity property and replacing \( \alpha_n \) with \( \lambda_{\text{min}}^n \). In applied mathematics the ratio between the extreme eigenvalues of a symmetric and positive matrix, \( \lambda_{\text{max}} / \lambda_{\text{min}} \) is called condition number. Then, \( \mu_n \) could be viewed as the inverse of condition number of the operator restriction to \( S_n \).

Pointing out: if there exists \( u \in N_T \cap B_0 \), then (2) holds on every \( S_n \in \mathcal{G} \) \( n \geq n_0(u) \) where \( n_0(u) \) is the finite index from where the orthogonal projections of \( u \) are not null - in virtue of the properties of the proper family given \( u \in B_0 \).

So, the property (2) for an eligible \( u \in N_T \), is valid starting from a finite index \( n_0(u) \); however, it is validated also for indexes \( n < n_0(u) \) because for such indexes \( u_n = 0 \) and \( \beta_n(u) = 1 \).
For $T \in \mathcal{I}$, given $u \in B_0$ we define the expression $	heta_n(u) := \theta^T_n(u)$ on every subspace of the proper family $S_n$, $n \geq n_0(u)$ by:

$$
\theta_n(u) := \beta^2_n - \mu_n^2 \parallel u_n \parallel^2 = 1 - (\mu_n^2 + 1) \parallel u_n \parallel^2
$$

(3)

with $u_n$ the orthogonal projection of $u$ on $S_n$. The last member in the equality is obtained using the relationship for every $u \in B$ and its orthogonal projections on $S_n$ and $S_n^\perp$: $\parallel u_n \parallel^2 + \parallel (u - u_n) \parallel^2 = \parallel u \parallel^2 = 1.$

We will split the set of eligible elements $B_0$ in two disjoint subsets:

1) $B_0^1 := \{u \in B_0; \quad \theta_n(u) \geq 0, n \geq n_0(u)\}$;

for which, $B_0^1 \supset B_0 \cap N_T$

2) $B_0^2 := \{u \in B_0; \quad \theta_n(u) < 0, n \geq n_0(u)\}$;

for which, $B_0^2 \cap N_T = \{\emptyset\}$

**Note.** Because for every normalised $u \in N_T$ the inequality (2) holds globally on the proper family $\mathcal{I}$, this behaviour will be propagated from now on for each statement involving (2); it is the reason we would omit carrying out every time the index specifications referring to all subspaces of the family, when it is evident from context. We would do it until the end of the paragraph.

**Theorem 2.1. (Injectivity Criteria.)** Let $T \in L_3$. If the sequence $\{\mu_n\}, n \geq 1$ is inferior bounded by a strict positive constant independent of $n$, $\mu_n \geq C > 0$ for every $n \geq 1$ then $T$ is injective.

**Proof**

Suppose that the sequence $\{\mu_n\}, n \geq 1$ is inferior bounded by a constant $C$ independent of $n$, strict positive.

Let $u \in B_0$.

1) If there exists an infinite subsequence of subspaces from the family for which $\theta_{n_m}(u) \geq 0$, $n_m \geq n_0(u)$ we obtain:

$$
\parallel u_{n_m} \parallel^2 \leq 1/(1 + \mu_{n_m}^2) < 1/(1 + C^2) < 1
$$

and $\lim_{m \to \infty} \parallel u_{n_m} \parallel^2 < 1/(1 + C^2) < 1.$

Or, $\{u_{n_m}\}, n_m \geq n_0(u)$ should converge in norm to 1 on the approximation subspaces for every $u \in B$. Thus, the inequality $\theta_{n_m}(u) \geq 0$ could not take place on an infinity of subspaces of the family. We should have instead at most only a finite number of subspaces verifying it.

If there exists a finite number of subspaces on which $\theta_{n_m}(u) \geq 0$, $n_m \geq n_0(u)$ then we will shrink the family including these subspaces inside of
the subspaces already verifying the opposite relationship, obtaining $u \notin B_0^1$ when the sequence $\{\mu_n\}, n \geq 1$ is inferior bounded by a not null constant.

Then, we are in the situation $\theta_n(u) < 0$ for every $n \geq n_0(u)$ on the $\mathcal{X}$, i.e. $u \in B_0^2$.

2) If $\theta_n(u) < 0$ holds for $n \geq n_0(u)$, then $\beta_n(u) < \mu_n \parallel u_n \parallel, n \geq n_0(u)$. But, $u$ satisfying such relationship could not be in $N_T$ because a reverse inequality given in (2) holds for eligible $u$ in $N_T$. The only restriction we put on $u \in H$, has been its eligibility, $u \in B_0$. Thus, if $\{\mu_n\}, n \geq 1$ is inferior bounded, does not exists $u \in B_0$ that is in $N_T$. Or, $B_0$ contains all normalized zeros of $T$, so $N_T = \{0\}$.

§3. On Approximations of the Integral Operators. On the separable Hilbert space $L^2(0,1)$, we will take in the following paragraphs the proper family $\mathcal{X} := \{S_h, nh = 1, n \geq 2\}$, where $S_h := \text{span}\{\chi_{n,k}, k = 1, n, nh = 1\}$ built on simple functions that are linear combinations of indicator functions of disjoint intervals

$$
\chi_{n,k} := \chi_{n,k}(t) = \begin{cases} 
1 & t \in \triangle_k := ((k - 1)h, kh] \\
0 & \text{otherwise}
\end{cases}
$$

(4)

is dense in $L^2(0,1)$ and meets the requests imposed by the definition of a proper family of approximation subspaces introduced in the previous paragraph.

Citing (Buescu & Paixão, 2007), the family of functions $\{\chi_{n,k}\}, k = 1, n, nh = 1,$ is defining a trace class integral operator with the kernel

$$
r_h(x, y) = h^{-1} \sum_{k=1}^{n} \chi_{n,k}(x)\chi_{n,k}(y)
$$

that is an orthogonal projection in $L^2(0,1)$ on $S_h$ having the orthogonal eigenfunctions $\{\chi_{n,k}\}, k = 1, n$. In fact, $S_h \in \mathcal{X}$ are generated by the families of the orthogonal eigenfunctions of the projection operators $\{P_h := P_{S_h}\}, nh = 1, n \geq 2$. Moreover, because the orthogonality of eigenfunctions is dictated by the disjoint interval support, $\chi_{n,k}(x)\chi_{n,j}(y) = \delta_{k,j}$, the entries in the matrix representation (5) bellow of the operator restrictions on approximation subspaces will be zero outside the diagonal and, it explain the form of the discrete approximations of the kernel functions on approximation subspaces defined bellow.

Thus, the linear integral operator (like in (1))

$$
T_\rho u := (T_\rho u)(y) = \int_0^1 \rho(y, x)u(x)dx
$$
has the discrete approximations of the kernel function $\rho$ like in [5],[6]:

$$\rho_h(y, x) = h^{-1} \sum_{k=1}^{n} \chi_{n,k}(x) \rho(y, x) \chi_{n,k}(y)$$

As an observation, the authors of [5], [6] used this family of approximations in dealing with decay rate of integral operators eigenvalues when their kernel functions are Mercer like kernels ([9]), i.e. Hermitian and at least continue (see also [10]).

Obviously, ($\forall$) $u \in H$, its projection on $S_h$ take the form of a simple function:

$$P_{|S_h} u = \sum_{k=1}^{n} \langle u, \chi_{n,k} \rangle \chi_{n,k}.$$  

Accordingly, taking any $v_h \in S_h$ not null of the form $v_h(t) = \sum_{k=1}^{n} c_k \chi_{n,k}(t)$, the discrete approximation of the integral operator verifies the following equality:

$$\langle T^h \rho v_h, v_h \rangle = \int_{0}^{1} \int_{0}^{1} \rho_h(y, x)v_h(x)v_h(y)dxdy$$

$$= h^{-1} c_h^T M_h c_h$$  

(5)

Here the $(n \times n)$ diagonal matrix $M_h = [d^h_{k}]_{k=1,n}$ is the matrix representation of the restriction $T^h \rho$ of the integral operator $T \rho$ to the subspace $S_h$ and the vector $c_h$ is the vector of coordinates $\{c_k, k=1,n\}$ of $v_h$ in $C^n$ verifying $\|v_h\|^2 = h\|c_h\|^2_{C^n}$.

Taking a look at (5) and at the matrix representation $M_h$ of the $T^h \rho$ on $S_h$, if the matrix is strict positive definite then: $\lambda_{min}(M_h) = \min_{k=1,n} d^h_{k}$, from where:

$$\langle T^h \rho v_h, u_h \rangle \geq h^{-2} \lambda_{min}(M_h) \|v_h\|^2$$

So, the a-positivity parameters associated to $T \rho$ should have the form like $\alpha_n = h^{-2} \lambda_{min}(M_h)$, nh = 1, n $\geq$ 1.

**Remark 3.1)** The linear bounded integral operator $T \rho$ is a-positive on $\mathfrak{Z}_\chi$ if and only if on approximation subspaces its kernel function is verifying

$$d^h_{k} := \int_{\Delta_k} \int_{\Delta_k} \rho(y, x)dxdy > 0$$  

(6)

for $k = 1,n$, nh = 1, i.e. the diagonal entries in the diagonal matrices $M_h$ are strict positive.

Here is the main theorem (Theorem 2.1) reformulated for integral operators on $L^2(0,1)$ taking the proper family $\mathfrak{Z}_\chi$. 

7
Lemma 3.1. (An Injectivity Criteria for Integral Operators.) Let $T_\rho \in L_{\mathcal{I}}\chi$. If the sequence $\mu_h(T_\rho), nh = 1, n \geq 1$

$$\mu_h(T_\rho) = \frac{\min_{k=1,n} d_k^n}{\max_{k=1,n} d_k^n}$$  \hspace{1cm} (7)

is inferior bounded, $\mu_h(T_\rho) > C$ by a strict positive constant $C$ independent of $n, nh = 1$, then $T_\rho$ is injective.

Proof.

In the view of the previous paragraphs, if $T \in L_{\mathcal{I}}\chi$ is a linear bounded integral operator its a-positivity parameters are given by $\alpha_n = h^{-2} \min_{k=1,n} d_k^n, nh = 1, n \geq 2$. Thus, the injectivity criteria (Theorem 2.1) could be used in determining the injectivity of $T_\rho$ observing that in this case $\{\mu_n, n \geq 1\}$ are given as in (7).

§4. On the integral operator used in RH Equivalence.

We will use the proper family of approximation subspaces introduced in previous paragraph, $\mathcal{I}_\chi$ on which, for our integral operator of interest we will apply the Remark 3.1 for cheking its a-positivity and the Lemma 3.1 for its injectivity.

Let the linear bounded integral operator $T_\rho$ defined by

$$(T_\rho u)(y) := \int_0^1 \{\frac{y}{x}\} u(x) dx$$  \hspace{1cm} (8)

on $L^2(0,1)$ having the kernel function $\rho(y, x) := \{\frac{y}{x}\}$. This kernel function is the fractional part function of the quantity between brackets having jumps on the lines like $y = k \cdot x, k \geq 1$ resulting in being a discontinue function.

Checking for the invariant subspaces of the considered integral operator (see Observation 2, Par. 1) we observe that $(T_\rho \chi_{n,k})(y) \notin \text{span}(\{\chi_{n,k}\}, 1 \leq k \leq n, nh = 1)$ easy to show for $k \geq 2$: by integrating on $\triangle_k$ we obtain a function of the first degree in the variable $y$, that is not a linear combination of indicator intervals $\{\chi_{n,k}\}, 1 \leq k \leq n, nh = 1$. Thus, the approximation subspaces are not invariant subspaces of $T_\rho$ and we could proceed with the steps of checking the a-positivity and the injectivity of the integral operator.

1. On the a-positivity of the integral operator $T_\rho$.

The entries of the diagonal matrices $M_h$, (see (6)) defined by the kernel
approximations on the subspaces \( \{ S_h \}, nh = 1, n \geq 2 \), are:

\[
d_h^1 = \frac{h^2}{4}(3 - 2\gamma); \quad d_h^k = \frac{h^2}{2}\left(-1 + \frac{2k - 1}{k - 1}\ln\left(\frac{k}{k-1}\right)^{k-1}\right)
\]

for \( k = 2, n, nh = 1 \) where \( \gamma \approx 0.5772156... \) is the Euler-Mascheroni constant rounded here to 7 digits. These entries are strict positive valued and bounded by \( d_h^1 \) and \( d_h^2 \). Moreover, the sequence \((-1 + \frac{2k-1}{k-1}\ln\left(\frac{k}{k-1}\right)^{k-1})\) is monotone and converges to 0.5 for \( k \to \infty \). Approximated values modulo the factor \( h^2 \), of the entries are:

\[
d_h^1 \approx 0.461392, d_h^2 \approx 0.636625, d_h^3 \approx 0.506887, ...
\]

Then, by Remark 3.1 \( T_\rho \) is a-positive on the approximation family having the \( a \)-positivity parameters given by \( \alpha_n := h^{-2}d_h^1 = h^{-2}\frac{1}{4}(3 - 2\gamma) \approx 0.461392 \) \( h^{-2}, nh = 1 \).

2. On the injectivity of the integral operator \( T_\rho \).

The injectivity parameters \( \{ \mu_n \}, n \geq 2 \) defined in (7) are \( \mu_n = \frac{d_h^k}{d_h^2}, n \geq 2, nh = 1 \) and will be all independent of \( h \) because all entries \( d_h^k, 1 \leq k \leq n \) are on the same order in \( h \). The evaluations of \( \{ \mu_n \}, n \geq 1 \) are:

\[
\mu_n(T_\rho) = \frac{3 - 2\gamma}{-2 + 6\ln(4/3)} \geq 0.724746, n \geq 2, nh = 1
\]

Hence the sequence \( \mu_n(T_\rho), n \geq 1 \), is bounded by a constant strict positive on every approximation subspace of the family.

By the Lemma 3.1 follows that the integral operator \( T_\rho \) is injective, equivalently, \( N_{T_\rho} = \{0\} \).

We just proved the following theorem:

**Theorem 4.1** The linear bounded integral operator \( T_\rho \) defined by

\[
(T_\rho u)(y) := \int_0^1 \rho(y, x)u(x)dx
\]

on \( L^2(0,1) \) having the kernel function \( \rho(y, x) := \{ \frac{y}{x} \} \) is injective.■

References

[1] Adam, D. (1989) "On Mesh Independence by Preconditioning", *Int. J. Of Comp. Math.*, 1989.

[2] Alcantara-Bode, J. (1993) "An Integral Equation Formulation of the Riemann Hypothesis", *Integr Equat Oper Th*, Vol. 17, 1993.
[3] Balazard M., Saias E. (2000) "The Nyman–Beurling equivalent form for the Riemann hypothesis", Expo. Math. 18, 131–138 (2000)

[4] Beurling, A. (1955) "A closure problem related to the Riemann zeta function", Proc. Nat. Acad. Sci. 41 pg. 312-314, 1955.

[5] Buescu, J., Paixão A. C. (2007) "Eigenvalue distribution of Mercer-like kernels", Math. Nachr. 280, No. 9–10, pg. 984 – 995, 2007.

[6] Chang, C.H., Ha, C.W. (1999) "On eigenvalues of differentiable positive definite kernels", Integr. Equ. Oper. Theory 33 pg. 1-7, 1999.

[7] Fan, K. (1954) "Inequalities for eigenvalues of Hermitian matrices", Nat. Bur. Standards Appl. Math. Ser. 39 pg. 131-139, 1954.

[8] Kolotilina, L. Yu. (1995) "Interrelations between eigenvalues and diagonal entries of Hermitian matrices implying their block diagonality", Zap. Nauchn. Sem. POMI, 1995, Volume 229, 153–158

[9] Mercer, J., (1909) "Functions of positive and negative type and their connection with the theory of integral equations", Philosophical Transactions of the Royal Society A 209, 1909

[10] Reade, J.P., (1983) "Eigenvalues Of Positive Definite Kernels", SIAM J. Math. Anal. Vol. 14, No. 1” January 1983.