Dynamics of A Prey and Two Neutral Predators

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Abstract: We describe a mathematical model for the interaction between three species consists a prey (x1) and two neutral predators (x2, x3), surviving on the common prey (x1). Here all the three species have limited their own natural resources. The system is explained by couple of differential equations. The co-existing state is identified and also characterized the local and global stability analysis at this state. The dynamical behaviour of the system is studied using Numerical simulation. It is shown that the system is globally asymptotically stable.

Keywords: Prey, predator, stability, Numerical simulation

Mathematical classification: 34DXX

1. Introduction:

Ecology deals the interactions among species and their environment. The interactions among the species can be categorized as prey- predator, amensalism, competition etc. Mathematical ecology deals with the stability of eco systems. The research in this area was initiated by Lotka [1] and Volterra [2]. Later on significant contribution is made by the Freedman [5] and Cushing [6]. Mathematical models in ecology, biology and medicine are dealt by Kapur [3, 4]. Prey-predator dynamics is quite interesting in nature. Lakshmi Narayan [7] dealt with prey-predator dynamics with time delay arguments. Paparao [8-10] studied the time delay dynamics in three species ecological models. Shiva Reddy [11] dealt with different prey-predator interactions. In continuation with, we take a logistic growth model of one prey and two predators neutral to each other for investigation. In this model two predators are preying on same prey and two predators are neutral to each other. The model is studied by a couple of differential equations. The co-existing equilibrium point is identified and discussed the dynamics at this point. Numerical simulation is carried out carried out in support of stability analysis. It is shown that the system is globally asymptotically stable.

2. Mathematical model

The mathematical model comprising prey (x1), predator1(x2) and predator2 (x3). Two predators are preying on prey and neutral to each other. The following differential equations represent the system as follows

\[
\begin{align*}
\frac{dx_1}{dt} &= a_1x_1\left[1 - \frac{x_1}{L_1}\right] - \alpha x_1x_2 - \beta x_1x_3 \\
\frac{dx_2}{dt} &= a_2x_2\left[1 - \frac{x_2}{L_2}\right] + \delta x_1x_2 \\
\frac{dx_3}{dt} &= a_3x_3\left[1 - \frac{x_3}{L_3}\right] + \epsilon x_1x_3
\end{align*}
\] (2.1)
2.1: Nomenclature:

\( x_i \): Bio mass of Three populations (prey, predator1 and predator2)

\( a_i \): Natural growth rates of three species (prey, predator1 and predator2)

\( \alpha \): Interaction rate of prey and predator1 species

\( \delta \): Interaction rate of predator1 and prey species

\( \beta \): Interaction rate of prey and predator2 species

\( \epsilon \): Interaction rate of predator2 and prey species

\( L_1, L_2, L_3 \): Carrying capacities of three species

Assume all parameters are positive

3. Equilibrium Points:

Equating the system of equations (2.1) to zero, and derive the co-existing state is given by

\[
E: \text{The Co-existent state or normal steady state} \\
\begin{align*}
\dot{x}_1 &= \frac{L_1 a_1 (a_1 - \alpha L_2 - \beta L_3)}{a_1 a_2 + a_1 \delta a L_2 + a_2 \beta \epsilon L_3} \\
\dot{x}_2 &= \frac{L_2 [a_1 a_2 + a_1 \alpha L_1 + \beta L_2 (\epsilon a_2 - \delta a_1)]}{a_1 a_2 + a_1 \delta a L_2 + a_2 \beta \epsilon L_3} \\
\dot{x}_3 &= \frac{L_3 [a_2 a_3 + a_2 \epsilon L_3 + \alpha L_2 (\delta a_3 - \epsilon a_2)]}{a_2 a_3 + a_2 \delta a L_2 + a_2 \beta \epsilon L_3}
\end{align*}
\]

\( \dot{x}_1 > 0 \) if \( a_1 > \alpha L_2 + \beta L_3 \)

\( \dot{x}_2 > 0 \) if \( \epsilon a_2 > \delta a_3 \)

\( \dot{x}_3 > 0 \) if \( \delta a_3 = \epsilon a_2 \) then system a admit positive equilibrium point and \( a_i > \alpha L_2 + \beta L_3 \)

4. Stability Analysis (Local stability) at Co-existing state:

Theorem 1: The system is locally asymptotically stable at the co-existing state \( E(\bar{x}_1, \bar{x}_2, \bar{x}_3) \)

Proof: The variation matrix for the system (2.1) is given by
\[
J = \begin{bmatrix}
-\frac{a_1 x_1}{L_1} & -\alpha x_1 & -\beta x_1 \\
\delta x_2 & -\frac{a_2 x_2}{L_2} & 0 \\
\varepsilon x_3 & 0 & -\frac{a_3 x_3}{L_3}
\end{bmatrix}
\]

With the characteristic equation 
\[b_0 \lambda^3 + b_1 \lambda^2 + b_2 \lambda + b_3 = 0\]

Where
\[b_0 = 1\]
\[b_1 = \left(\frac{a_1 x_1}{L_1} + \frac{a_2 x_2}{L_2} + \frac{a_3 x_3}{L_3}\right)\]
\[b_2 = \left(\frac{a_1 a_2 x_2 x_2}{L_2 L_2} + \frac{a_1 a_2 x_2 x_2}{L_2 L_3} + \frac{a_1 a_2 x_2 x_2}{L_2 L_2} + a_3 \delta x_2 + \beta \varepsilon x_2\right)\]
\[b_3 = x_2 x_2 \left(\frac{a_1 a_2 a_3}{L_2 L_2 L_3} + \frac{a_3 \delta \alpha}{L_3} + \frac{a_2 \beta \varepsilon}{L_2}\right)\]

By using Routh-Hurwitz criteria calculate the following

\[D_1 = b_1, \quad D_2 = \begin{vmatrix} b_1 & b_3 \\ b_0 & b_2 \end{vmatrix}, \quad D_3 = \begin{vmatrix} b_3 & b_1 & b_3 \\ b_0 & b_2 & b_3 \end{vmatrix}\]

\[D_2 = b_2 b_2 - b_1 b_0 D_1 = b_1 = \left(\frac{a_1 x_1}{L_1} + \frac{a_2 x_2}{L_2} + \frac{a_3 x_3}{L_3}\right) > 0\]

\[D_2 = \left(\frac{a_1 x_1}{L_1} + \frac{a_2 x_2}{L_2} + \frac{a_3 x_3}{L_3}\right) \left(\frac{a_1 a_2 x_2 x_2}{L_2 L_2} + \frac{a_1 a_2 x_2 x_2}{L_2 L_3} + \frac{a_1 a_2 x_2 x_2}{L_2 L_2} + a_3 \delta x_2 + \beta \varepsilon x_2\right) - x_2 x_2 \left(\frac{a_1 a_2 a_3}{L_2 L_2 L_3} + \frac{a_3 \delta \alpha}{L_3} + \frac{a_2 \beta \varepsilon}{L_2}\right)\]

\[D_2 > 0 \quad \text{at} \quad (x_1, x_2, x_3)\]
\[
D_3 = (b_2 b_3 - b_3 b_1) b_1
\]
\[
D_3 = \left( \frac{a_1^2 a_2 x_1 x_3}{L_1^3 L_2} + \frac{a_2^2 a_1 x_2 x_3}{L_1^3 L_2} + \frac{a_1 a_2 a_3 x_3}{L_1 L_2} + \frac{a_1 a_2^2 x_3 x_1}{L_1 L_2} + \frac{a_1 a_2 a_3 x_2 x_1}{L_1 L_2} + \frac{a_1 a_2^2 x_2 x_3}{L_1 L_2} \right)
\]
\[
D_3 > 0 \quad \text{at} \quad (\bar{x}_1, \bar{x}_2, \bar{x}_3)
\]

Since all determinants are positive by using Routh–Hurwitz criteria the co-existing state \( E(\bar{x}_1, \bar{x}_2, \bar{x}_3) \) is locally asymptotically stable.

5. Global Stability

Theorem 2: The co-existing state \( E(\bar{x}_1, \bar{x}_2, \bar{x}_3) \) is globally asymptotically stable

Proof: Let the Lyapunov function be
\[
V(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \left[ x_1 - \bar{x}_1 - x_1 \log \left( \frac{\bar{x}_1}{x_1} \right) \right] + m_1 \left[ x_2 - \bar{x}_2 - x_2 \log \left( \frac{\bar{x}_2}{x_2} \right) \right] + m_2 \left[ x_3 - \bar{x}_3 - x_3 \log \left( \frac{\bar{x}_3}{x_3} \right) \right]
\]

(5.1)

Clearly \( V(\bar{x}_1, \bar{x}_2, \bar{x}_3) \) is positive

The time derivative of \( V \) along the solutions of equations (2.1) is
\[
\frac{dV}{dt} = \frac{dx_1}{dt} \left[ 1 - \frac{x_1}{\bar{x}_1} \right] + m_1 \frac{dx_2}{dt} \left[ 1 - \frac{x_2}{\bar{x}_2} \right] + m_2 \frac{dx_3}{dt} \left[ 1 - \frac{x_3}{\bar{x}_3} \right]
\]

(5.2)

\[
= \left[ x_1 - \bar{x}_1 \right] a_1 \left( 1 - \frac{x_1}{L_1} \right) - \alpha x_2 + \beta x_3 + m_1 \left[ x_2 - \bar{x}_2 \right] a_2 \left( 1 - \frac{x_2}{L_2} \right) + \delta x_1 + m_2 \left[ x_3 - \bar{x}_3 \right] a_3 \left( 1 - \frac{x_3}{L_3} \right) + \varepsilon x_1
\]

(5.3)

Choose the proper set of values for \( a_1 = \frac{a_1 x_1}{L_1} + \alpha x_2 + \beta x_3, a_2 = \frac{a_2 x_2}{L_2} - \alpha x_2 - \beta x_3, a_3 = \frac{a_3 x_3}{L_3} - \varepsilon x_1 \)

then (5.3) becomes
\[
= \left[ x_1 - \bar{x}_1 \right] \left[ \frac{a_1 x_1}{L_1} + \alpha x_2 + \beta x_3 - \alpha x_2 - \beta x_3 \right] + m_1 \left[ x_2 - \bar{x}_2 \right] \left[ \frac{a_2 x_2}{L_2} - \alpha x_2 - \beta x_3 + \delta x_1 - \delta x_1 \right] + m_2 \left[ x_3 - \bar{x}_3 \right] \left[ \frac{a_3 x_3}{L_3} - \alpha x_2 + \beta x_3 + \varepsilon x_1 - \varepsilon x_1 \right]
\]

\[
\frac{dV}{dt} = -\frac{a_1}{L_1} (x_1 - \bar{x}_1)^2 - \frac{a_2}{L_2} (x_2 - \bar{x}_2)^2 - \frac{a_3}{L_3} (x_3 - \bar{x}_3)^2 + (x_1 - \bar{x}_1)(x_2 - \bar{x}_2)(\delta m_1 - \alpha) + (x_1 - \bar{x}_1)(x_3 - \bar{x}_3)(\varepsilon m_2 - \beta)
\]

choose \( m_1 = \frac{\alpha}{\delta}, m_2 = \frac{\beta}{\varepsilon} \)
\[
\frac{dV}{dt} = -\frac{a_1}{L_1} (x_1 - \bar{x}_1)^2 - \frac{a_2}{L_2} (x_2 - \bar{x}_2)^2 - \frac{a_3}{L_3} (x_3 - \bar{x}_3)^2 \\
\frac{dV}{dt} = -\left[ \frac{a_1}{L_1} (x_1 - \bar{x}_1)^2 + \frac{a_2}{L_2} (x_2 - \bar{x}_2)^2 + \frac{a_3}{L_3} (x_3 - \bar{x}_3)^2 \right]
\]

\[
\frac{dV}{dt} \leq 0
\]

Hence co-existing state \( E(\bar{x}_1, \bar{x}_2, \bar{x}_3) \) is globally asymptotically stable

4. Numerical example:

Numerical simulation is carried out in support of stability analysis along normal steady state \( E \). In each graph, figure (a) represents time series responses and (b) represents phase portraits.

**Example 1:** let us choose the following values for simulation

\[
a_1=2; \ a_2=3; \ a_3=4; \ \alpha = 0.2, \ \delta=0.3, \ \beta=0.01, \ \varepsilon = 0.01, \ L_1=150, \ L_2=150, \ L_3=150, \ x = 20, \ y =10, \ z = 10
\]

![Fig 1(A)](image1.png) ![Fig 1(B)](image2.png)

The system (2.1) is stable for the initial population strength (20, 10, 10). Since the growth rates of three populations are almost equal, the prey population is extinct. Since two predators are having alternative food sources, they sustain and reach to the fixed equilibrium point.

**Example 2:** let us choose the following values for simulation

\[
a_1=15, \ a_2= 0.5, \ a_3=0.5, \ \alpha = 0.2, \ \delta=0.3, \ \beta=0.01, \ \varepsilon = 0.01, \ L_1=100, \ L_2=100, \ L_3=100, \ x = 2, \ y =10, \ z = 10
\]

![Fig 2(A)](image3.png) ![Fig 2(B)](image4.png)
The system is unstable. Due to the high growth rate of prey population and less growth rates in two predators and availability alternative food, there is a significant growth in three populations is shown in the graphs (2 A &B).

For the same parametric values if we change the initial strengths we can observe the following dynamics

Case (i): $x = 1$, $y=1$, $z = 1$

The system is unstable due to the growth response in prey population is high consequently two predator population are also increased from its initial strength.

Case (ii): $x = 10$, $y=10$, $z = 10$

The system is unstable.

Case (iii): $x = 20$, $y=10$, $z = 10$
Fig 2.3(A)

The system is unstable.

7. Conclusion:

In this work we investigated the stability analysis between one prey and two predators neutral to each other. The mathematical model was described by a couple ordinary differential equations. Co-existing state is identified and the system is locally and globally stable as shown in theorems 1 and 2 respectively, when three growth rates of prey, predator1 and predator2 are identically equal then the prey species would be extinct as shown in the example1(fig 1). If the prey growth rate is high comparatively with two predators, then the prey population is increased from its initial strength and it reaches to 40 as shown in the fig (2). In addition to this two predators population is also increased to 180 as shown in fig (2). Thus system becomes unstable even if initial strengths are changed as shown in the fig (2.1 to 2.3). Hence the system is unstable irrespective of any change in their initial strengths. It is also observed that the sustainability of the prey population is depended on the high growth rate of prey population and low growth rate of predators. If the growth rate (prey) is high, the sustainability is also high and vice versa.

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