Topological phases of spinless $p$-orbital fermions in zigzag optical lattices

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Motivated by the experiment [St-Jean et al., Nature Photon. 11, 651 (2017)] on topological phases with collective photon modes in a zigzag chain of polariton micropillars, we study spinless $p$-orbital fermions with local interorbital hoppings and repulsive interactions between $p_x$ and $p_y$ bands in zigzag optical lattices. We show that spinless $p$-band fermions in zigzag optical lattices can mimic the interacting Su-Schrieffer-Heeger model and the effective transverse field Ising model in the presence of local hoppings. We analytically and numerically discuss the ground-state phases and quantum phase transitions of the model. This work provides a simple scheme to simulate topological phases and the quench dynamics of many-body systems in optical lattices.

I. INTRODUCTION

Topological phases of matter are fascinating quantum states in modern condensed matter physics, which are characterized by some prominent features such as string orders, robust edge states beyond the Landau-Ginzburg symmetry-breaking theory [1]. The Su-Schrieffer-Heeger (SSH) model that exhibits topological nontrivial phases was originally proposed for fermionic particles with staggered hoppings in polyacetylene chains [2, 3]. The SSH model is a simple but very important model in studying the topology of the single-particle band structure in solid-state physics. Thanks to the rapid development of quantum simulations [4–6], the SSH model was recently realized in many platforms, such as coupled semiconductor micropillars with the collective photon modes [7, 8], and optical lattices with ultracold atoms [9–12].

A natural proposal to realize the SSH model in optical lattices is to create a double well superlattice with the same unit cell as the original SSH model [9–12]. Interestingly, an orbital version of the SSH Hamiltonian was implemented by using polariton micropillars in a $p$-band zigzag chain in Ref.[8], where the topological nontrivial phases and topological trivial phases were found to form in the orthogonal $p_x$ and $p_y$ subspaces. However, the impact from the mixing of the $p_x$ and $p_y$ orbitals and the on-site interactions were not investigated in Ref.[8], which we believe are important to engineer rich many-body physics in optical lattices. This is because: (i) In cold atoms, it may be very difficult to prepare orthogonal $p_x$ and $p_y$ orbitals with perfect 90° angles. In fact, it would be interesting to introduce such deformations of the local lattice wells to tune the phase transitions [13] instead of considering only the orthogonal $p_x$ and $p_y$ subspaces. (ii) When placing the bosons or spinful fermions on $p$-band optical lattices, the pair hopping terms due to the Hund effect would cause a mixing of $p_x$ and $p_y$ orbitals of a given lattice well. (iii) In the strong on-site interaction limit, a small mixing of orbitals may lead to a phase transition because the effective coupling strength from the second-order perturbation theory is small.

In this paper, we generalize the work of Ref.[8] that realizes the SSH model with polariton micropillars by considering spinless fermions loaded in a $p$-band zigzag optical lattice with the on-site hopping (band mixing) and on-site interactions, which were discarded in Ref.[8]. We show that the topological phases persists under such local deformations and the phase transition in the strong interacting limit at half-filling is described by the effective transverse field Ising model. We note that the $p$-bands systems in optical lattices have been investigated experimentally [14–18] and theoretically [13, 19–41].

This paper is organized as follows. In Sec.II, we introduce the $p$-band model with spinless fermions in zigzag optical lattices. In Sec.III, we study the quantum phases of spinless $p$-orbital fermions in zigzag optical lattices. This work provides a simple scheme to simulate topological phases and the quench dynamics of many-body systems in optical lattices.

FIG. 1. (Color online) Geometry of the $p$-band model discussed in this work. (a) Zigzag lattice with degenerate $p_x$ and $p_y$ orbitals occupied by spinless fermions, where $t_\parallel$ and $t_\perp$ denote the longitudinal and transverse hopping between the same orbitals in nearest-neighboring lattice sites, $\lambda$ and $U$ refer to the local hopping and interaction between different orbitals in a given site. (b) The equivalent ladder geometry of (a). (c) The representation for the Hamiltonian in Eq. (1) in terms of spinless fermions on a SSH-like chain.

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without interactions by the single particle spectrum. In Sec.IV, we discuss the quantum phases, phase transitions with interactions and derive the effective transverse field Ising model. Finally, in Sec.V, we summarize this work.

II. MODEL

We consider spinless fermions loaded in a zigzag optical lattice [13, 31, 32] as shown in Fig.1(a), where two degenerate $p_x$ and $p_y$ orbitals are active within the $x$-$y$ plane per lattice site due to a strong confinement along $z$-direction. The Hamiltonian of the system composed of $N$ lattice wells is given by [19, 22],

$$
H = -\sum_{i=1,l=1,2}^N (t_{i,l}^\dagger c_{i+1,l} + \lambda c_{i,p_x}^\dagger c_{i,p_y} + h.c.) + \sum_{i=1}^N U c_{i,p_x}^\dagger c_{i,p_x} c_{i,p_y}^\dagger c_{i,p_y},
$$

with $t = -\frac{1}{2}t_\parallel [1 + (-1)^{i+l}] + \frac{1}{2}t_\perp [1 - (-1)^{i+l}]$, where $t_\parallel$ and $t_\perp$ are the longitudinal and transverse hopping amplitudes along the same orbital $p_x$ (or $p_y$) between two nearest-neighbor lattice sites, and $l = 1, 2$ indicates the $p_x$, $p_y$ orbital in a given lattice site. In Fig.1(a), the longitudinal hopping $t_\parallel$ is much larger than the transverse hopping $t_\perp$ because the overlap integrals of hopping amplitudes are dependent on the orientation of orbitals [19, 22]. The local interorbital hopping $\lambda$ that leads to a mixing of the $p_x$ and $p_y$ orbitals can be tuned by a deformation of the lattice wells such as by an additional weak tilted lattice [13]. Here $c_{i,l}^\dagger, c_{i,l}$ are the creation and annihilation operators at $l$th orbital of the $i$th site, and $U > 0$ is the on-site repulsive interaction between $p_x$ and $p_y$ orbitals in a single given well. It is easy to find that the $p$-band model in zigzag lattices is equivalent to a spinless fermionic model on a two-leg ladder [cf. Fig.1(b)] or a one-dimensional SSH chain [cf. Fig.1(c)], which we will discuss in more detail below.

III. TOPOLOGICAL PHASES IN THE NON-INTERACTING SSH-LIKE MODEL

Let us first consider the noninteracting case ($U = 0$) of Hamiltonian in Eq.(1). In the absence of interorbital hopping ($\lambda = 0$), the $p_x$ orbitals and the $p_y$ orbitals are decoupled into two independent chains (subspaces) with staggered $t_\parallel$ and $t_\perp$ hopping as shown in Fig.1(b). In the chains with open boundary conditions, considering the longitudinal hopping $t_\parallel$ is typically much larger than the transverse hopping $t_\perp$ due to the orientation of orbitals [19, 22], the $p_x$ subspace consequently exhibits a dimerization on the $(2i - 1, 2i)$ bonds without edge states as shown in Fig.2(a), while the $p_y$ subspace forms a dimerization on the $(2i, 2i + 1)$ bonds with topological edge states as demonstrated in Fig.2(b). The odd-bond dimerizations in $p_x$ subspaces correspond to the topological trivial phase while the even-bond dimerization in $p_y$ subspaces exhibit the topological nontrivial phase of the SSH model that was experimentally investigated with polariton micropillars in Ref.[8].

Next, we consider the effect of the orbital deformation that introduces the on-site interorbital hopping ($\lambda \neq 0$) between the $p_x$ and $p_y$ orbitals of the local lattice wells and was neglected in Ref.[8]. In this context, we arrive at the following noninteracting Hamiltonian with periodicity two [see Fig.1(a)] by considering only the leading terms,

$$
H' = -\sum_{i=1,l=1,2}^N (t'c_{i,l}^\dagger c_{i+1,l} + \lambda c_{i,p_x}^\dagger c_{i,p_y} + h.c.),
$$

where $t' = -\frac{1}{2}t_\parallel [1 + (-1)^{i+l}]$, and we have discarded the transverse hopping term $t_\perp$ because $t_\perp \ll t_\parallel$. The Bogoliubov-de Gennes (BdG) Hamiltonian of Eq.(2) under periodic boundary conditions can be easily derived as

$$
H' = \begin{pmatrix}
0 & 0 & t_\parallel e^{-ik} & \alpha_k \\
-\lambda & 0 & 0 & b_k \\
0 & t_\parallel & 0 & -\lambda \\
t_\parallel e^{ik} & 0 & -\lambda & 0
\end{pmatrix}
$$

by using the Nambu basis $\psi^T_k = (a_k, b_k, c_k, d_k)$. Here $a_k$, $b_k$, $c_k$, $d_k$ are the annihilation operators in the momen-
term space of $c_{2i−1,p_x}$, $c_{2i−1,p_y}$, $c_{2i,p_x}$, $c_{2i,p_y}$. Diagonalizing the Hamiltonian, we obtain the energy spectrum of the bulk states:

$$ E(k) = \pm \sqrt{\lambda^2 + t_\parallel^2 \pm 2\lambda t_\parallel \cos(k/2)}, $$

with $k = \frac{2\pi}{N}j$ and $j = 1, 2, \ldots, N/2$. Equivalently, four bands in Eq.(5) can be reduced to two bands of the SSH model by simply setting $k' = k/2$ as

$$ E(k') = \pm \sqrt{\lambda^2 + t_\parallel^2 + 2\lambda t_\parallel \cos(k')}, $$

where $k' = \frac{2\pi}{N}j$ and $j = 1, 2, \ldots, N$. Obviously, if the degenerate $p_x$ and $p_y$ orbitals are regarded as two sublattice sites in each unit cell, one can easily arrive at the standard SSH model [see Fig.1(c)]. Consequently, when $\lambda > t_\parallel$, a dimerized state is formed between the $p_x$ and $p_y$ sublattices in each single well as shown in Fig.2(c), which corresponds to the topological trivial phase of SSH model. When $\lambda < t_\parallel$, the system exhibits the dimerization induced interorbital hopping $\lambda$. We note that when the transverse hopping terms $t_\perp$ are finite, the model becomes a SSH model with the third-neighbor hopping [42, 43]. However, it qualitatively would not change the underlying physics with $t_\perp = 0$ because the hopping strength $t_\perp$ is much smaller than $t_\parallel$ due to the orientation of orbitals [19, 22].

IV. EFFECTIVE STRONG-COUPLING MODEL

In the following, we will study the ground-state properties and the associated quantum phase transitions of Hamiltonian in Eq.(1) with on-site repulsive interaction $U \neq 0$. For simplicity but without loss of generality, we still overlook the transverse hopping $t_\perp$ terms in the following discussions. The model is then the usual SSH model with the nearest-neighbor interaction $U$ between the $p_x$ and $p_y$ orbitals within a unit cell. In addition to the topological nontrivial phase and the trivial phase, a density wave phase (or Aoki phase in the Gross-Neveu model) appears [44–46] owing to the presence of local interaction $U$. To understand the nature of the quantum phases and the phase transitions of the $p$-band model in Eq.(1), we derive an effective antiferro-orbital (AF-orbital) Ising model in the strongly interaction limit with $U \gg t_\parallel$, by the second-order perturbation theory at half-filling [13, 19, 22, 31]:

$$ H_{\text{eff}} = \sum_{i=1}^{N} JS_i^z S_{i+1}^z - 2\lambda S_i^x, $$

where $J = 2t_\parallel^2/U$, $S_i^\dagger = c_{i,p_x}^\dagger c_{i,p_y}$ and $S_i^- = (c_{i,p_x}^\dagger c_{i,p_y} - c_{i,p_y}^\dagger c_{i,p_x})/2$. Hence, for $\lambda > t_\parallel^2/2U$, it is a para-orbital phase, while for $\lambda < t_\parallel^2/2U$, it is an antiferro-orbital Ising phase ($p_x, p_y, p_x, p_y, \cdots$), in which one particle is located in the $p_x$ orbital of $i$th well and the other dwells on the $p_y$ orbital of the nearest neighbor $i + 1$th well.

We note that in contrast to SU(2) symmetric Heisenberg interactions in spin models, the orbital exchange Hamiltonian evokes Ising-type interactions without quantum fluctuations, similar to the systems with $t_\parallel$ orbital degeneracy [47, 48]. Especially, the interorbital hopping $\lambda$ herein is responsible for substantial quantum fluctuations and plays a role of an external transverse field, which is hardly experimentally controlled in the orbital-only models of Mott insulators [47, 48]. To verify our theoretical analysis, we compute the correlation function,

$$ C_{1,j} = \langle S_i^z S_j^z \rangle, $$

and the fidelity susceptibility per orbital [49–52],

$$ \chi_{\text{L}}(\lambda) = \frac{1}{2N} \lim_{\delta \lambda \to 0} \frac{-2 \ln F(\lambda, \lambda + \delta \lambda)}{(\delta \lambda)^2}, $$

with periodic boundary conditions. Where $F(\lambda, \lambda + \delta \lambda) = |\langle \psi_0(\lambda) | \psi_0(\lambda + \delta \lambda) \rangle|$ evaluates the overlap of two infinitesimally close states. The numerical results are

![Graph](image-url)
obtained and presented in Fig. 3 by performing the exact diagonalization with periodic boundary conditions up to $N = 11$ wells [equivalent to $N = 22$ orbitals of Fig. 1(c)]. As is shown in Fig. 3(b), one can see clearly that a quantum phase transition occurs between the antiferro-orbital Ising phase ($\lambda < \lambda_c$) and the para-orbital phase ($\lambda > \lambda_c$). The phase transition is also detected by the peak of the fidelity susceptibility as shown in Fig. 3(c). The dependence of the critical values $\lambda_c$ on $U$ is presented in Fig. 3(a), which agrees well with the analytical results $\lambda_c = t_j^2/2U$ from the perturbation theory. Regarding the finite-size scaling of the peak of the fidelity susceptibility for a continuous phase transitions in one-dimensional system [50–54],

$$\chi_N^m \propto N^{2/\nu-1},$$

we obtain the critical exponent of the correlation length $\nu \approx 0.98$ consistent with Ising transition $\nu = 1$ from maximal values of the fidelity susceptibility as shown in Fig. 3(c). Consequently, one can simulate the Ising phase transition or dynamical quantum phase transitions [55, 56] with spinless fermions in zigzag lattices.

V. CONCLUSION

In summary, we have shown that spinless fermions loaded in a $p$-band zigzag optical lattice can engineer the interacting SSH model, which shows a topological phase transition from the trivial phase to the topological nontrivial phase, where the edge states appear in open boundary conditions. In the strong interaction limit, the transverse field Ising model can be mimicked owing to the on-site band mixing and repulsion. We show the spinless fermions in $p$-band zigzag lattice can host rich quantum phases and the associated phase transitions due to the interplay between the lattice geometry, the deformation of the lattice wells and the interactions.

In addition, when the dipolar particles are loaded into the lattices, one may simulate the long-range interacting SSH and long-range Ising models [5]. Consequently, our proposal opens a simple way to study quantum phase transitions and the quench dynamics, such as dynamical quantum phase transitions with broken symmetries [57] of many-body systems. We note that it may also be possible to simulate a non-Hermitian SSH model or a non-Hermitian Ising model if the gain and loss are introduced into the systems [58, 59]. Moreover, it would be very interesting to investigate the bosons placed in the zigzag optical lattices to understand the Hund effects in the future.

Finally, we would like to emphasize that the orbital symmetry in $p$-band zigzag lattice leads to $z$-component Ising interactions along any direction in the $xy$ plane in the regime of large $U$, i.e., for $U \gg t_j$, in stark contrast to the ferromagnetic Kitaev interactions with nonequivalent components of Ising superexchange along different axes [60–62]. We note that the hole propagation described by the $tJ_z$ model in antiferro-orbital and para-orbital background may lead to a nontrivial many-body problem [47, 63].

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