1 Introduction

The connectivity \( \kappa(G) \) of a graph \( G \) is the minimum cardinality of a subset \( V' \) of vertices such that \( G - V' \) is disconnected or trivial. The edge-connectivity \( \lambda(G) \) of a graph \( G \) is the minimum cardinality of a subset \( E' \) of edges such that \( G - E' \) is disconnected. An equivalent definition of connectivity was given in [13]. For each 2-subset \( S = \{u, v\} \) of vertices of \( G \), let \( \kappa(S) \) denote the maximum number of internally vertex-disjoint \((u, v)\)-paths in \( G \). Then \( \kappa(G) = \min \{ \kappa(S) | S \subseteq V \text{ and } |S| = 2 \} \). Similarly, the edge-connectivity also has two equivalent definitions. Let \( \lambda(S) \) denote the maximum number of edge-disjoint \((u, v)\)-paths in \( G \). Then \( \lambda(G) = \min \{ \lambda(S) | S \subseteq V \text{ and } |S| = 2 \} \).
As a means of strengthening the connectivity, the tree connectivity was introduced by Hager [5, 6] (or generalized connectivity by Chartrand et al. [2]) to meet wider applications. Given a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of size at least 2, an $S$-Steiner tree or a Steiner tree connecting $S$ is such a subgraph $T = (V', E')$ of $G$ that is a tree with $S \subseteq V'$. Two $S$-Steiner trees $T$ and $T'$ are said to be internally disjoint if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. Let $\kappa_G(S)$ denote the maximum number of internally disjoint $S$-Steiner trees in $G$. The $k$-tree connectivity (or generalized $k$-connectivity) of $G$, denoted by $\kappa_k(G)$, is then defined as $\kappa_k(G) = \min\{\kappa_G(S) | S \subseteq V(G) \text{ and } |S| = k\}$, where $2 \leq k \leq n$. Clearly, when $k = 2$, $\kappa_2(G)$ is exactly the classical connectivity $\kappa(G)$.

As a natural counterpart of the tree-connectivity, the tree edge-connectivity (or generalized edge-connectivity) was introduced by Li et al. [10]. For $S \subseteq V(G)$ and $|S| \geq 2$, let $\lambda_G(S)$ denote the maximum number of edge-disjoint $S$-Steiner trees in $G$. The $k$-tree edge-connectivity (or generalized $k$-edge-connectivity) of $G$, denoted by $\lambda_k(G)$, is then defined as $\lambda_k(G) = \min\{\lambda_G(S) | S \subseteq V(G) \text{ and } |S| = k\}$, where $2 \leq k \leq n$. It is also clear that when $k = 2$, $\lambda_2(G) = \lambda(G)$.

There have been many results on the $k$-tree (edge-)connectivity, see [3, 7, 9, 10, 12] and a book [8].

The line graph $L(G)$ of $G$ is the graph whose vertex set can be put in one-to-one correspondence with the edge set of $G$ in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of $G$ are adjacent. The connectivity of the line graph of a graph $G$ is closely related to the edge-connectivity of $G$.

**Lemma 1.1** (Chartrand and Stewart [4]). *If $G$ is a connected graph, then $\kappa(L(G)) \geq \lambda(G)$.*

Naturally, one would like to study the relationship between $\kappa_k(L(G))$ and $\lambda_k(G)$, for $k \geq 3$. In [10], Li et al. showed that if $G$ is a connected graph, then $\kappa_3(L(G)) \geq \lambda_3(G)$. In [11], Li et al. showed that if a graph $G$ is connected, then $\kappa_4(L(G)) \geq \lambda_4(G)$. Furthermore, they proved that if a connected graph $G$ has at least $k$ vertices and at least $k$ edges, then $\kappa_k(L(G)) \geq \lceil \frac{3\lambda_k(G)}{4} \rceil - 1$ for any $k \geq 2$. However, they suspect that their result is not sharp and proposed the following conjecture:

**Conjecture 1.1** (Li, Wu, Meng and Ma [11]). *Let $k \geq 2$ be an integer. If a connected graph $G$ has at least $k$ vertices and at least $k$ edges, then $\kappa_k(L(G)) \geq \lambda_k(G)$.*

In this paper, we will confirm this conjecture and prove that the bound is sharp.
2 Main result

Before proving our main result, we first introduce some concepts. A maximal connected subgraph of $G$ is called a component of $G$. A connected acyclic graph is called a tree. The vertices of degree 1 in a tree are called leaves. A connected graph $G$ with $|V(G)| = |E(G)|$ is called a unicyclic graph. A spanning subgraph of a graph $G$ is a subgraph whose vertex set is the entire vertex set of $G$. We refer the reader to [1] for the terminology and notations not defined in this paper.

By the definition of the tree edge-connectivity, the following result is obvious.

**Observation 2.1 ([II]).** For any integer $2 \leq s \leq t$, $\lambda_s(G) \geq \lambda_t(G)$.

Now, we give a confirmative solution to Conjecture 1.1.

**Theorem 2.2.** Let $k \geq 2$ be an integer. If a connected graph $G$ has at least $k$ vertices and at least $k$ edges, then $\kappa_k(L(G)) \geq \lambda_k(G)$. Moreover, the bound is sharp.

**Proof.** Let $v_e$ be the vertex of the line graph $L(G)$ corresponding to the edge $e$ of $G$. Assume that $\lambda_k(G) = m$. For any $k$-subset $S_L = \{v_{e_1}, v_{e_2}, \ldots, v_{e_k}\}$ of $V(L(G))$, by the definition of $\kappa_k$, it suffices to show that $\kappa_{L(G)}(S_L) \geq m$.

Now, let $S_G = \{e_1, e_2, \ldots, e_k\}$ and then $S_G \subseteq E(G)$. Denote by $G[S_G]$ the edge-induced subgraph of $G$ whose edge set is $S_G$ and whose vertex set consists of all ends of edges of $S_G$.

We distinguish two cases:

**Case 1:** None of the components of $G[S_G]$ is a tree or unicyclic.

Therefore, for each component $C_t$ of $G[S_G]$, $|E(C_t)| - |V(C_t)| \geq 1$ and so $|E(G[S_G])| = |S_G| = k \geq |V(G[S_G])| + 1$. Let $V(G[S_G]) = Q^*$. It follows that $|Q^*| \leq k - 1$. Since $G$ has at least $k$ vertices, we can take a vertex $v^*$ in $V(G) \setminus Q^*$ and then let $Q = Q^* \cup \{v^*\}$. Now, because $|Q| \leq k$, by Observation 2.1 there are $m$ edge-disjoint $Q$-Steiner trees $T_1, T_2, \ldots, T_m$ in $G$.

Next, in each tree $T_r$ ($1 \leq r \leq m$), we will assign a specific edge to each vertex of $Q^*$. To see this, we let $v^*$ be the root and define the level $l(v)$ of a vertex $v$ in $T_r$ to be the distance from the root $v^*$ to $v$. It is easy to see that, for each vertex $v_i \in Q^* = Q \setminus \{v^*\}$, there is a unique edge $e$ connecting the vertex $v_i$ and a vertex of level $l(v_i) - 1$. Assign the edge $e$ to the vertex $v_i$. We say that the edge $e$ is the corresponding edge of $v_i$ in $T_r$ and denoted by $\hat{e}_i^r$. Note that any two vertices of $Q^*$ in $T_r$ have different corresponding edges. More precisely, $\hat{e}_i^r \neq \hat{e}_j^r$ for any $1 \leq i \neq j \leq |Q^*$.
Now, for each tree \( T_r \) \((1 \leq r \leq m)\) and each edge \( e = v_i v_j \in S_G \), do the following operation. Note that, by Lemma 1.1, \( L(T_r) \) \((1 \leq r \leq m)\) is a connected subgraph of \( L(G) \). Moreover, since \( Q^* = V(G[S_G]) \), both ends of each edge in \( S_G \) belong to \( Q^* \) and so \( v_i, v_j \in V(T_r) \).

**Operation A:** If \( e \in E(T_r) \), it is done; otherwise \( e \notin E(T_r) \), that is \( v_e \notin V(L(T_r)) \). Note that, \( T_1, T_2, \ldots, T_m \) are edge-disjoint and so at most one of them contains the edge \( e \).

If \( e \in E(T_s) \), where \( 1 \leq s \neq r \leq m \), then \( e \) is the corresponding edge of one of its ends in \( T_s \). Without loss of generality, assume that \( e \) is the corresponding edge of \( v_i \) in \( T_s \), that is, \( e^s_i = e \). Now, for \( T_r \), there is an edge \( e^r_j \) corresponding to the vertex \( v_j \), which is the other end of \( e \). Since \( e \) and \( e^r_j \) have the same end \( v_j \), they are adjacent and so \( v_e v^r_j \in E(L(G)) \). Add the vertex \( v_e \) and the edge \( v_e v^r_j \) to \( L(T_r) \).

Otherwise, none of the trees \( T_1, T_2, \ldots, T_m \) contains the edge \( e = v_i v_j \). In this case, we can add the vertex \( v_e \) and either the edge \( v_e v^r_i \) or the edge \( v_e v^r_j \) to \( L(T_r) \), where \( e^r_i \) and \( e^r_j \) are the corresponding edges of \( v_i \) and \( v_j \) in \( T_r \), respectively. \( \Box \)

Now, \( L(T_1), L(T_2), \ldots, L(T_m) \) are transformed into \( m \) connected subgraphs of \( L(G) \), each of which contains the vertex set \( S_L \). Next, for each of the obtained subgraphs of \( L(G) \), take a spanning tree \( T^*_r \) \((1 \leq r \leq m)\). Because \( V(T^*_r) \supseteq S_L \) \((1 \leq r \leq m)\), it remains to show that \( T^*_1, T^*_2, \ldots, T^*_m \) are internally disjoint. Note that if \( v_e \notin V(L(T_r)) \), for some \( e \in S_G \) and \( r \in \{1, 2, \ldots, m\} \), \( v_e \) must be a leaf of \( T^*_r \).

Since \( T_1, T_2, \ldots, T_m \) are edge-disjoint in \( G \), \( L(T_1), L(T_2), \ldots, L(T_m) \) are vertex-disjoint in \( L(G) \). Moreover, the vertices added to \( L(T_r) \) by Operation A are all from \( S_L \). Therefore, \( T^*_1, T^*_2, \ldots, T^*_m \) are vertex-disjoint except \( S_L \), that is, \( V(T^*_r) \cap V(T^*_s) = S_L \), for any \( 1 \leq r < s \leq m \).

Now, assume that there are two trees \( T^*_r \) and \( T^*_s \) such that \( E(T^*_r) \cap E(T^*_s) \neq \emptyset \) \((1 \leq r < s \leq m)\). Let \( e^* \in E(T^*_r) \cap E(T^*_s) \). Since \( V(T^*_r) \cap V(T^*_s) = S_L \), both ends of \( e^* \) belong to \( S_L \). Thus, without loss of generality, let \( e^* = v_{e_1} v_{e_2} \). If \( L(T_r) \) contains neither \( v_{e_1} \) nor \( v_{e_2} \), by Operation A, both \( v_{e_1} \) and \( v_{e_2} \) are leaves of \( T^*_r \) and hence it is impossible that \( v_{e_1} v_{e_2} \in E(T^*_s) \). So is \( L(T_s) \). And \( L(T_r) \) and \( L(T_s) \) are vertex-disjoint \((T_r \text{ and } T_s \text{ are edge-disjoint})\). Thus, without loss of generality, suppose that \( v_{e_1} \in L(T_r) \) and \( v_{e_1} \in L(T_s) \), and so \( v_{e_1} \notin L(T_r) \) and \( v_{e_2} \notin L(T_s) \). Since \( v_{e_1} \) and \( v_{e_2} \) are adjacent in \( L(G) \), \( e_1 \) and \( e_2 \) are adjacent in \( G \). Assume that \( v_1 \) is the common end of \( e_1 \) and \( e_2 \) in \( G \) and let \( e_1 = v_i v_j \). Since \( v_{e_2} v_{e_1} \in E(T^*_s) \), we added the vertex \( v_{e_2} \) and the edge \( v_{e_2} v_{e_1} \) to \( L(T_s) \). So by Operation A, we know that \( e_1 \) is exactly the corresponding edge of \( v_i \) in \( T_s \), that is, \( e_1 = e^r_i \). Again by Operation A, since \( e_1 \notin E(T_r) \), we added the vertex \( v_{e_1} \) and the
edge $v_{e_1}v_{e_2}$ to $L(T_r)$, where $e_1$ and $e_2$ have the same end $v_j$. Since $e_1 \neq e_2$ and $e_1$ and $e_2$ have the same end $v_i$, it is impossible that $\tilde{e}_j = e_2$. Therefore, by Operation A, it is impossible that $v_{e_1}v_{e_2} = e^* \in E(T^*_r)$, a contradiction. It follows that $T^*_1, T^*_2, \ldots, T^*_m$ are edge-disjoint.

Thus, in this case, $T^*_1, T^*_2, \ldots, T^*_m$ are $m$ internally disjoint trees connecting $S_L$ in $L(G)$.

**Case 2:** There is a component of $G[S_G]$ which is either a tree or unicyclic.

For each component $C_t$ of $G[S_G]$ which is neither a tree nor unicyclic, add all vertices of $C_t$ to the vertex set $Q_1$ and add all edges of $C_t$ to the edge set $S^1_G$. Clearly, if $Q_1 \neq \emptyset$, $|S^1_G| > |Q_1|$.

Next, for each component $C_t$ of $G[S_G]$ which is either a tree or unicyclic, if $C_t$ is unicyclic, choose an edge $e_t$ from $C_t$ such that $C_t - e_t$ is a tree and let one end of $e_t$ as the root $r_t$; otherwise, select an arbitrary vertex as the root $r_t$. For $C_t$ (if $C_t$ is a tree) or $C_t - e_t$ (if $C_t$ is unicyclic), define the level $l(v)$ of a vertex $v$ to be the distance from the root $r_t$ to $v$. Notice that each edge in the tree $C_t$ (or $C_t - e_t$ if $C_t$ is unicyclic) joins vertices on consecutive levels. Then, for each edge $e = uv$, where $l(u) + 1 = l(v)$, we assign the vertex $v$ which has higher level to the edge $e$ and say that the vertex $v$ is the corresponding vertex of the edge $e$. If $C_t$ is unicyclic, let the root $r_t$ be the corresponding vertex of the remaining edge $e_t$. Now, each edge of $C_t$ has a corresponding vertex. By the definition, it is obvious that any two edges of $C_t$ have different corresponding vertices. Add the corresponding vertices of all edges of $C_t$ to the vertex set $Q_2$ and add all edges of $C_t$ to the edge set $S^2_G$. Clearly, $|S^2_G| = |Q_2|$.

Moreover, it is clear that $Q_1 \cap Q_2 = \emptyset$, $S^1_G \cap S^2_G = \emptyset$ and $S_G = S^1_G \cup S^2_G$. Let $S^1_L = \{v_e \in S^1_G\}$ and $S^2_L = \{v_e \in S^2_G\}$. Then $S_L = S^1_L \cup S^2_L$. Let $Q = Q_1 \cup Q_2$. We have $|Q| = |Q_1| + |Q_2| \leq |S^1_G| + |S^2_G| = |S_G| = k$. Since $Q \subseteq V(G)$, by Observation 2.1, there are $m$ edge-disjoint $Q$-Steiner trees $T_1, T_2, \ldots, T_m$ in $G$. Note that both ends of each edge in $S^1_G$ belong to $Q_1$, but there may be an edge in $S^2_G$, only one end of which belongs to $Q_2$. Thus, we use different methods to deal with the edges in $S^1_G$ and $S^2_G$.

For every edge of $S^1_G$, we take the same approach as Case 1. In each tree $T_r$ ($1 \leq r \leq m$), since $Q_2 \neq \emptyset$ (it is possible that $Q_1 = \emptyset$), take an arbitrary vertex $v^*$ in $Q_2$ as the root and define the level $l(v)$ of a vertex $v$ in $T_r$ to be the distance from the root $v^*$ to $v$. For each vertex $v_i \in Q_1$ (if $Q_1 \neq \emptyset$), there is a unique edge $e$ connecting the vertex $v_i$ and a vertex of level $l(v_i) - 1$. Let the edge $e$ be the corresponding edge of $v_i$ in $T_r$, denoted by $\tilde{e}^r_i$. Any two vertices of $Q_1$ in $T_r$ have different corresponding edges. Now, apply Operation A to each tree $T_r$ ($1 \leq r \leq m$) and each edge $e = v_iv_j \in S^1_G$. Then, each
vertex of $S^1_L$ is added to $L(T_r)$ ($1 \leq r \leq m$).

Next, for each edge $e_i$ of $S^2_G$ and each tree $T_r$ ($1 \leq r \leq m$), do the following operation.

**Operation B:** If $e_i \in E(T_r)$, it is done; otherwise $e_i \notin E(T_r)$, that is $v_{e_i} \notin V(L(T_r))$. By the definitions of $S^2_G$ and $Q_2$, there is a corresponding vertex $v_i$ of $e_i$, and $v_i \in Q_2 \subseteq Q$ and so $v_i \in V(T_r)$. Thus, there exists an edge $\tilde{e}_i \neq e_i$ incident with $v_i$ in the tree $T_r$. Since $e_i$ and $\tilde{e}_i$ have the same end $v_i$, they are adjacent and so $v_{e_i}v_{\tilde{e}_i} \in E(L(G))$. Add the vertex $v_{e_i}$ and the edge $v_{e_i}v_{\tilde{e}_i}$ to $L(T_r)$.

Now, after applying Operations A and B, $L(T_1), L(T_2), \ldots, L(T_m)$ are transformed into $m$ connected subgraphs of $L(G)$, each of which contains the vertex set $S_L = S^1_L \cup S^2_L$. For each of the obtained subgraphs of $L(G)$, take a spanning tree $T^*_r$ ($1 \leq r \leq m$). Note that, if $v_e \notin V(L(T_r))$, for some $e \in S_G$ and $r \in \{1, 2, \ldots, m\}$, whether $e \in S^1_G$ or $S^2_G$, that is, whether Operation A or Operation B is applied, $v_e$ must be a leaf of $T^*_r$. Because $V(T^*_r) \supseteq S_L$ for any $1 \leq r \leq m$, it remains to show that $T^*_1, T^*_2, \ldots, T^*_m$ are internally disjoint.

Since $L(T_1), L(T_2), \ldots, L(T_m)$ are vertex-disjoint in $L(G)$ and the vertices added to $L(T_r)$ by Operations A and B are all from $S_L$, $T^*_1, T^*_2, \ldots, T^*_m$ are vertex-disjoint except $S_L$, that is, $V(T^*_r) \cap V(T^*_s) = S_L$, for any $1 \leq r < s \leq m$.

To complete the proof, it remains to show that $T^*_1, T^*_2, \ldots, T^*_m$ are edge-disjoint. By contradiction, assume that there are two trees $T^*_r$ and $T^*_s$ such that $E(T^*_r) \cap E(T^*_s) \neq \emptyset$ ($1 \leq r < s \leq m$). Let $e^* \in E(T^*_r) \cap E(T^*_s)$. Since $V(T^*_r) \cap V(T^*_s) = S_L$, both ends of $e^*$ belong to $S_L$. Thus, without loss of generality, let $e^* = v_{e_1}v_{e_2}$. If $L(T_r)$ contains neither $v_{e_1}$ nor $v_{e_2}$, then whether apply Operation A or Operation B, both $v_{e_1}$ and $v_{e_2}$ are leaves of $T^*_r$, which is impossible. So is $L(T_s)$. And $L(T_r)$ and $L(T_s)$ are vertex-disjoint ($T_r$ and $T_s$ are edge-disjoint). Thus, without loss of generality, suppose that $v_{e_2} \in L(T_r)$ and $v_{e_1} \in L(T_s)$, and so $v_{e_1} \notin L(T_r)$ and $v_{e_2} \notin L(T_s)$. Since $v_{e_1}$ and $v_{e_2}$ are adjacent in $L(G)$, $e_1$ and $e_2$ are adjacent in $G$. Therefore, $e_1$ and $e_2$ belong to the same component of $G[S_G]$. Hence, by the definitions of $S^1_G$ and $S^2_G$, both $e_1$ and $e_2$ belong to $S^1_G$. If both $e_1$ and $e_2$ belong to $S^2_G$, by Operation A, it is impossible that $v_{e_1}v_{e_2} = e^* \in E(T^*_r) \cap E(T^*_s)$. The proof is the same as that of Case 1.

If both $e_1$ and $e_2$ belong to $S^2_G$, since $e_1 \notin E(T_r)$, by Operation B, we added the vertex $v_{e_1}$ and the edge $v_{e_1}v_{\tilde{e}_1}$ to $L(T_r)$, where the common end $v_1$ of $e_1$ and $\tilde{e}_1$ in $G$ is the corresponding vertex of $e_1$. Similarly, since $e_2 \notin E(T_s)$, by Operation B, we added the vertex $v_{e_2}$ and the edge $v_{e_2}v_{\tilde{e}_2}$ to $L(T_s)$, where the common end $v_2$ of $e_2$ and $\tilde{e}_2$ in $G$ is the corresponding vertex of $e_2$. Since $v_1 \neq v_2$ by the definition of $Q_2$, at least one
of the equations $e_1^* = e_2$ and $e_2^* = e_1$ is not true. So $v_{e_1}v_{e_1^*} \neq v_{e_1}v_{e_2}$ or $v_{e_2}v_{e_2^*} \neq v_{e_1}v_{e_2}$. It is impossible that $e^* = v_{e_1}v_{e_2} \in E(T^*_r) \cap E(T^*_s)$, a contradiction. It follows that $T^*_1, T^*_2, \ldots, T^*_m$ are edge-disjoint.

Thus, in both cases, there always exist $m$ internally disjoint trees connecting $S_L$ in $L(G)$ and so $\kappa_{L(G)}(S_L) \geq m$. By the arbitrariness of $S_L$, we conclude that $\kappa_k(L(G)) \geq m$.

For a cycle $C_n$ with $n \geq k$, since $L(C_n) = C_n$, $\kappa_k(L(C_n)) = \lambda_k(C_n) = 1$ for $k \geq 3$ and $\kappa_2(L(C_n)) = \lambda_2(C_n) = 2$. Thus, the bound is sharp. The proof is complete.

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