Order-$\alpha_s^3$ determination of the strange quark mass

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Abstract

We present a QCD sum rule calculation of the strange-quark mass including four-loop QCD corrections to the correlator of scalar currents. We obtain $\bar{m}_s(1\text{ GeV}) = 205.5 \pm 19.1 \text{ MeV}$. 
1. Introduction

A precise determination of the values of the light quark masses is of crucial practical importance for testing in an accurate way the predictions of the Standard Model. In particular, the knowledge of the strange quark mass is relevant for a better understanding of the low-energy phenomenology of QCD and for a precise prediction of the CP-violating parameter $\epsilon'/\epsilon$ in the framework of the Standard Model [12, 13, 14, 15].

The ratios of the light quark masses can be determined in a model-independent way with the help of chiral-perturbation methods [1]. On the other hand, in order to obtain their absolute values, one has to resort either to the method of QCD sum rules [3] or to lattice QCD [16, 30]. While no less fundamental than the first type of predictions, the latter ones have suffered in the past from larger uncertainties.

In [6] a QCD sum rule calculation of the strange quark mass in the scalar channel has been presented, which employed N$^2$LO ($O(\alpha^2_s)$) results for the correlator of two scalar strangeness-changing currents in perturbation theory (see also [7] for a similar calculation) (for earlier calculations, see [4, 5, 9, 10, 11]).

In the meantime the N$^3$LO ($O(\alpha^3_s)$) correction to this correlator has become also known [19]. The perturbative contribution dominates the sum rule [3, 7], so one naturally expects the N$^3$LO correction to alter these results in a significant way. This expectation is also supported by a simple estimate of these corrections [7] which shows that their omission is likely to constitute the main source of errors in the calculations of [6] and [7].

Since the N$^3$LO correction is now known, we are in a position to present a reevaluation of the strange quark mass computation given in [6].

2. Four-loop contributions to the scalar correlator

The QCD sum rule used in this paper is based on the correlator of two scalar currents

$$\psi(Q^2, \alpha_s, m_s, \mu) = i \int dx e^{ix} (0|TJ(x)J^\dagger(0)|0)$$

where $J = \partial_\alpha \bar{s} \gamma^\alpha u = i(m_s - m_u)\bar{s}u$, $Q^2 = -q^2$. It will be more convenient to work with the second derivative of $\psi(Q^2)$, $\psi''(Q^2) = d^2/d(Q^2)^2$, which satisfies a homogeneous renormalization-group equation $\mu \frac{d}{d\mu} \psi''(Q^2) = 0$.

We will write $\psi''(Q)$ as $\psi''_P(Q) + \psi''_{NP}(Q)$, with $\psi''_P(Q)$ is the perturbative part and $\psi''_{NP}(Q)$ contains the vacuum expectation values of the higher dimension operators. For the perturbative part one obtains the following result:

$$\psi''_P(Q) = \frac{6(m_s - m_u)^2}{(4\pi)^2 Q^2} \left\{ 1 + \frac{11\alpha_s}{3\pi} + \frac{\alpha_s^2}{\pi^2} \left\{ \frac{5071}{144} - \frac{35}{2} \zeta(3) \right\} \right\}$$
\[ \alpha_3 \pi^3 \left( -\frac{4781}{9} + \frac{1}{6}a_1 + \frac{475}{4}\zeta(3) \right) \]

\[ + \log \frac{Q^2}{\mu^2} \left[ -2\frac{\alpha_s}{\pi} - \frac{139\alpha_s^2}{6\pi^2} + \frac{\alpha_s^3}{\pi^3} \left( -\frac{2720}{9} + \frac{475}{4}\zeta(3) \right) \right] \]

\[ + \log^2 \frac{Q^2}{\mu^2} \left[ \frac{17\alpha_s^2}{4\pi^2} + \frac{695\alpha_s^3}{8\pi^3} - \frac{221\alpha_s^3 \log Q^2}{24\pi^3} \right] \]

\[ - \frac{12(m_s - m_u)^2 m_s^2}{(4\pi)^2 Q^4} \left\{ 1 + \frac{28\alpha_s}{3\pi} + \frac{\alpha_s^2}{\pi^2} \left( \frac{85577}{72} - \frac{77}{3}\zeta(3) \right) \right\} \]

\[ - \log \frac{Q^2}{\mu^2} \left[ \frac{4\alpha_s}{\pi} + \frac{147\alpha_s^2}{2\pi^2} + \frac{25\alpha_s^2 \log Q^2}{2\pi^2} \right] \]

\[ a_1 = \frac{4748953}{864} - \frac{\pi^4}{6} - \frac{91519\zeta(3)}{36} + \frac{715\zeta(5)}{2} \approx 2795.0778. \]

The terms of order \( \alpha_3^3 \) in the \( O(m_q^2) \) part of (2) have been extracted from the recent four-loop calculation of [19]. The terms of order \( \alpha_2^2 \) in the \( O(m_q^4) \) part of (2) can be found in [20].

The exact value of \( a_1 \) agrees well with an estimate [7] of the same quantity based on the assumption of a continued geometric growth of the perturbative series for \( \psi''(Q^2) \), which gave \( a_1 = 2660. \)

We have neglected the light quark mass \( m_u \), except in the overall factors. The renormalized parameters \( \alpha_s \) and \( m_s, m_u \) are taken at the scale \( \mu \). Their \( \mu \)-dependence should cancel against that of the log \( \mu \) factors in (2) so that \( \psi''(Q) \) is \( \mu \)-independent.

As for the nonperturbative contributions, we keep only the dimension-4 operators. These are given, together with their renormalization-group properties and the values of the coefficient functions \( c_i \) to next-to-leading order, in [3, 7] where the references to the original calculations can also be found. We quote here only the final result for \( \psi''_{NP}(Q) \):

\[ \psi''_{NP}(Q) = \frac{(m_s - m_u)^2}{Q^6} \left\{ 2(m_s\bar{u}u)_0 \left( 1 + \frac{\alpha_s}{\pi} \left( \frac{23}{3} - 2 \log \frac{Q^2}{\mu^2} \right) \right) \right. \]

\[ - \frac{1}{9} I_G \left( 1 + \frac{\alpha_s}{\pi} \left( \frac{121}{18} - 2 \log \frac{Q^2}{\mu^2} \right) \right) + I_s \left( 1 + \frac{\alpha_s}{\pi} \left( \frac{64}{9} - 2 \log \frac{Q^2}{\mu^2} \right) \right) \]

\[ - \frac{3}{7\pi^2} m_s^4 \left( \frac{\pi}{\alpha_s} + \frac{155}{24} - \frac{15}{4} \log \frac{Q^2}{\mu^2} \right). \]

\( I_s \) and \( I_G \) are the vacuum expectation values of the two RG-invariant combinations of dimension 4, which are given for \( n_f = 3 \) and to the order we are working, by

\[ I_s = m_s \langle \bar{s}s \rangle_0 + \frac{3}{7\pi^2} m_s^4 \left( \frac{\pi}{\alpha_s} - \frac{53}{24} \right) \]

\[ \dagger \text{The corresponding constant in [3] is called } c_{31} = a_1/6. \]
\[ I_G = -\frac{9}{4} \left( \frac{\alpha_s}{\pi} G^2 \right)_0 \left( 1 + \frac{16 \alpha_s}{9 \pi} \right) + \frac{4 \alpha_s}{\pi} \left( 1 + \frac{91 \alpha_s}{24 \pi} \right) m_s \langle \bar{s}s \rangle_0 + \frac{3}{4 \pi^2} \left( 1 + \frac{4 \alpha_s}{3 \pi} \right) m_s^4. \]  

(5)

4. The sum rule

To enhance the contribution of the low-lying states, one applies a Borel transform to the both sides of the dispersion relation used to define the sum rule [6]. The effect is to transform the power-suppression of the states with a large invariant mass into an exponential one, controlled by the Borel parameter \( M^2 \):

\[ \hat{L}[\psi''(Q^2)] = \frac{1}{M^6} \frac{1}{\pi} \int_0^\infty dt e^{-t/M^2} \text{Im} \psi(t). \]  

(6)

The Borel transform of the l.h.s. can be computed from (2,3) and is given by

\[ \hat{L}[\psi''_P(Q)] = \frac{6(m_s - m_u)^2}{(4\pi)^2 M^2} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{11}{3} - 2\psi(1) \right) + \frac{\alpha_s^2}{\pi^2} \left( \frac{5071}{144} - \frac{35}{2} \zeta(3) + \frac{17}{4} \psi(1) \right) \right. \\
- \frac{139}{6} \psi(1) - \frac{17}{24} \psi^2(1) \right. \\
+ \frac{\alpha_s^3}{\pi^3} \left[ -\frac{4781}{9} + \frac{1}{6} a_1 + \frac{475}{4} \zeta(3) \psi(1) + \frac{823}{6} \zeta(3) \right] \\
- \frac{221}{24} \psi^2(1) + \frac{695}{8} \psi(1) \right. \\
\left. + \frac{221}{48} \psi^2(1) - \frac{2720}{9} \psi(1) - \frac{695}{48} \psi^2(1) \right) + \log \frac{M^2}{\mu^2} \left[ -2 \frac{\alpha_s}{\pi} \right. \\
+ \frac{\alpha_s^2}{\pi} \left. \left( -\frac{139}{6} + \frac{17}{2} \psi(1) \right) \right] \left. + \frac{\alpha_s^3}{\pi^3} \left( -\frac{2720}{9} + \frac{475}{4} \zeta(3) - \frac{221}{8} \psi^2(1) + \frac{695}{48} \psi(1) \right) \right. \\
\left. + \frac{221}{48} \psi^2(1) \right) \right. \\
\left. + \log^2 \frac{M^2}{\mu^2} \left[ \frac{17 \alpha_s^2}{4 \pi^2} + \frac{\alpha_s^3}{\pi^3} \left( \frac{695}{8} - \frac{221}{8} \psi(1) \right) \right] \right. \\
\left. - \log^3 \frac{M^2}{\mu^2} \cdot \frac{221 \alpha_s^3}{24 \pi^3} \right\} \]  

(7)

and respectively

\[ \hat{L}[\psi''_{NP}(Q)] = \frac{(m_s - m_u)^2}{2M^6} \left\{ 2\langle m_s \bar{u}u \rangle_0 \left( 1 + \frac{\alpha_s}{\pi} \left( \frac{14}{3} - 2\psi(1) - 2 \log \frac{M^2}{\mu^2} \right) \right) \right. \\
\right. - \frac{1}{9} I_G \left( 1 + \frac{\alpha_s}{\pi} \left( \frac{67}{18} \right) - 2\psi(1) \right) + \log \frac{M^2}{\mu^2} \right. \\
\left. + I_s \left( 1 + \frac{\alpha_s}{\pi} \left( \frac{37}{9} \right) \right) \right. \\
\left. - \frac{3}{7 \pi^2} m_s^4 \left( \frac{\pi}{\alpha_s} + \frac{5}{4} \psi(1) - \frac{15}{4} \log \frac{M^2}{\mu^2} \right) \right\}. \]  

(8)
The numerical constants entering these expressions have the values \( \psi(1) = -\gamma_E = -0.577 \) and \( \zeta(3) = 1.202 \).

At this point the usual procedure is to take advantage of the \( \mu \)-independence of \( \hat{L}[\psi''(Q)] \) and the fact that the operation of Borel transformation does not act on \( \mu \) and choose \( \mu = M \). This “renormalization-group improvement” effectively shifts the logs of \( M^2/\mu^2 \) into the renormalized parameters \( \alpha_s(M) \) and \( m_s(M) \). To the order we are working and for \( n_f = 3 \), these are given by

\[
\frac{\alpha_s(M)}{\pi} = \frac{4}{9} L - \frac{256}{729} LL^2 + \frac{1}{L^3} \left( \frac{6794}{59049} - \frac{16384}{59049} \right) \frac{LL}{L^2} + \frac{16384}{59049} \frac{LL^2}{L^2} \]

\[
m_s(M) = \frac{\hat{m}_s}{(\frac{1}{2}L)^{4/9}} \left[ 1 + \frac{290}{729} \frac{1}{L} - \frac{256}{729} \frac{LL}{L^2} + \left( \frac{550435}{1062882} - \frac{80}{729} \zeta(3) \right) \frac{1}{L^2} \right. \\
\left. + \frac{388736}{531441} \frac{LL}{L^2} + \frac{106496}{531441} \frac{LL^2}{L^2} \right. \\
\left. + \left( \frac{2121723161}{2324522934} + \frac{8}{6561} \pi^4 - \frac{119840}{531441} \zeta(3) - \frac{8000}{59049} \zeta(5) \right) \frac{1}{L^3} \right. \\
\left. + \left( \frac{611418176}{387420489} + \frac{112640}{531441} \zeta(3) \right) \frac{LL}{L^3} + \frac{335011840}{387420489} \frac{LL^2}{L^3} - \frac{149946368}{1162261467} \frac{LL^3}{L^3} \right] .
\]

We have denoted here \( L = \log(M^2/\Lambda_{QCD}^2) \) and \( LL = \log L \). We have used in the recently calculated exact values of the 4-loop beta function \[ \beta_4(n_f = 3) = -\frac{140599}{2304} - \frac{445}{16} \zeta(3) \approx -94.456 \] and of the 4-loop mass anomalous dimension \[ \gamma_4(n_f = 3) = \frac{2977517}{20736} + \frac{3}{16} \pi^4 - \frac{9295}{216} \zeta(3) - \frac{125}{6} \zeta(5) \approx 88.525817 \]

The exact result \[ \gamma_4(n_f = 3) \] is close to an estimate based on the assumption of a geometrical growth for the coefficients of the \( \gamma_m \) anomalous dimension

\[
\gamma_4(n_f = 3) = \frac{\gamma_2^2(n_f = 3)}{\gamma_2(n_f = 3)} \approx 81.368 ,
\]

with \[ \gamma_1(n_f = 3) = 2 , \quad \gamma_2(n_f = 3) = \frac{91}{12} , \quad \gamma_3(n_f = 3) = \frac{8885}{288} - 5 \zeta(3) \].
The usual approach followed in the numerical evaluation of the sum rule \cite{6, 7} has been to expand the Borel transforms (7) and (8) in powers of $1/L$ and truncate the resulting expressions to a given order in this parameter. There are certain errors inherent to this procedure which actually turn out to be important in practice. This can be seen at order $1/L^2$ by examining the structure of the Borel transform of the leading term in (7)

$$
\hat{T}_1(M^2) = \frac{6m_s^2(M^2)}{(4\pi)^2M^2} \left( 1 + \frac{\alpha_s(M^2)}{\pi} c_1 + \left( \frac{\alpha_s(M^2)}{\pi} \right)^2 c_2 + \left( \frac{\alpha_s(M^2)}{\pi} \right)^3 c_3 + \cdots \right), \quad (15)
$$

where we neglected the light quark mass. The first three coefficients $c_i$ have the values $c_1 = 4.8211, c_2 = 21.9765, c_3 = 53.1421$.

We tabulated in Table 1 the values of $m_s^2(M^2)(\alpha_s(M^2)/\pi)^i$ with $\hat{m}_s = 1$ for $i = 0 - 3$ using two different approximations at a typical value of the Borel parameter $M^2 = 3$ GeV$^2$. The first three columns show the values of these parameters computed by expanding in powers of the small parameter $1/L$ up to the shown order. This approximation has been commonly used in the previous literature (for example the results in \cite{6}, \cite{7} have been obtained using a similar expansion to order $1/L^2$).

From Table 1 one can see that truncating $m_s^2(\alpha_s/\pi)^2$ to order $1/L^2$ results in an error of the order of 100%, as the $1/L^3$ correction to its value is comparable with the $1/L^2$ term itself. The result of truncating to order $1/L^2$ is to overestimate the $\mathcal{O}(\alpha_s^2)$ correction to the sum rule by a factor of 2. Neglecting this fact could

| $m_s^2$ | $1/L$ | $1/L^2$ | $1/L^3$ | $m_s^{(2)}$, $\alpha_s^{(1)}$ | $m_s^{(3)}$, $\alpha_s^{(2)}$ | $m_s^{(4)}$, $\alpha_s^{(3)}$ |
|-------|-------|--------|--------|----------------|----------------|----------------|
| 0.6942 | 0.6673 | 0.6638 | 0.6942 | 0.6675 | 0.6641 |
| 0.1011 | 0.0725 | 0.0720 | 0.1017 | 0.0695 | 0.0727 |
| — | 0.0148 | 0.0063 | — | 0.0072 | 0.0080 |
| — | — | 0.0022 | — | — | 0.0009 |

Table 1. Values of $m_s^2(M^2)(\alpha_s(M^2)/\pi)^i$ at $M^2 = 3$ GeV$^2$ ($\Lambda_{\overline{MS}}^{\nu_f=3} = 380$ MeV) used for the discussion of the validity of the truncation approximation in the text.
result in the paradoxical consequence that adding the $O(\alpha_s^3)$ correction decreases the perturbative contribution $\hat{T}_1(M^2)$ to the sum rule, although each separate term in the $\alpha_s$ expansion is positive!

The last three columns of Table 1 show the untruncated values for these parameters (e.g. $m_s(q), \alpha_s(q)$ means that the full $i$-loop expression for the running mass and the $j$-loop one for the running coupling have been used). One can see that these quantities are more stable when going from one order in perturbation theory to another than the truncated quantities. They can be therefore expected to give a closer estimate of the true size of each correction term and will be used in our numerical analysis below.

Another difference from the treatment followed in [6, 7] will be the use of the 4-loop formulas for the running parameters (9,10) in all our expressions (7), (8) and (19). We recall that the truncation approach (working to a finite order in $1/L$) employs running parameters at lower orders in the loop expansion in the power-suppressed terms. The numerical differences between these two approaches are not significant. However, the former one is physically preferable as it uses as small expansion parameter $\alpha_s(M)$ which is what is directly measured in practice (as opposed to $1/L$).

The condensates entering the renormalization-group invariant condensates (4) and (5) are taken, as in [6], at the reference scale $\mu_0 = 1$ GeV and have the values $\langle \bar{u}u \rangle |_{\mu_0} = -(0.225)^3$ GeV and $\langle \bar{c}cG^2 \rangle = 0.02 - 0.06$ GeV$^4$. The amount of SU(3)-breaking in the scalar condensate $\langle \bar{s}s \rangle/\langle \bar{u}u \rangle$ will be varied between 0.7 and 1. The up quark mass has been taken [8] as $\bar{m}_u(1$ GeV$)=5$ MeV.

The hadronic contribution to the sum rule is expressed below in terms of the scalar form-factor $d(s)$ in $K_{33}$ decays. One has [3]

$$\frac{1}{\pi} \text{Im } \psi(s) = \frac{3}{32\pi^2} |d(s)|^2 \sqrt{\left(1 - \frac{(m_K + m_{\pi})^2}{s}\right) \left(1 - \frac{(m_K - m_{\pi})^2}{s}\right)}.$$  \hspace{1cm} (16)

The values of $s$ appearing in this relation are not accessible in $K_{33}$ decays, but extend above the $K\pi$-production threshold $s_{th} = (m_K + m_{\pi})^2$. We will parametrize the scalar form-factor $d(s)$ in this region by a sum of Breit-Wigner resonances corresponding to the two bound states with the quantum numbers of the scalar current $K_0^*(1430)$ and $K_0^*(1950)$ [10, 7]

$$|d(s)|^2 = |d(s_{th})|^2 \frac{\Gamma_1}{(M_1^2 - s)^2 + M_1^2 \Gamma_1^2(s)} + \frac{\gamma^2 \Gamma_2}{(M_2^2 - s)^2 + M_2^2 \Gamma_2^2(s)}.$$

The threshold value of the scalar form-factor $d(s_{th})$ has been computed to order $p^4$ in chiral perturbation theory [4] with the result $|d(s_{th})|^2 = 0.35$ GeV$^2$. The same
quantity has been recently extracted \cite{26} from data on s-wave phase shifts for Kπ scattering \cite{27} with a similar result \[ d(s_{th})^2 = 0.33 \pm 0.02 \text{ GeV}^2 \]. The values of the other parameters in (17) are \[ M_1 = 1423 \pm 10 \text{ MeV}, \Gamma_1 = 268 \pm 25 \text{ MeV}, M_2 = 1945 \pm 22 \text{ MeV}, \Gamma_2 = 201 \pm 86 \text{ MeV}, \gamma = 0.5 \pm 0.3 \). The energy-dependent widths \( \Gamma_i(s) \) are given by

\[
\Gamma_i(s) = \Gamma_i \left( \frac{1 - \frac{(m_K + m_\pi)^2}{s}}{1 - \frac{(m_K + m_\pi)^2}{M_i^2}} \right) \left( \frac{1 - \frac{(m_K - m_\pi)^2}{s}}{1 - \frac{(m_K - m_\pi)^2}{M_i^2}} \right). \tag{18}
\]

For the region of large invariant mass of the hadronic states \((s > s_0)\), parton-hadronic duality can be expected to hold to a high degree of precision. This allows us to take the spectral density \( \text{Im} \psi(s) \) to be equal to the imaginary part of the QCD expression (2). This is given by

\[
\frac{1}{\pi} \text{Im} \psi(s) = \frac{3}{8\pi^2} (m_s - m_u)^2 s \left\{ \frac{1}{3} - 2 \ln \frac{s}{\mu^2} \right\} + \frac{\alpha_s}{\pi} \left\{ -\frac{35\zeta(3)}{2} + \frac{9631}{144} - \frac{17\pi^2}{12} - \frac{95}{3} \ln \frac{s}{\mu^2} + \frac{17}{4} \ln^2 \frac{s}{\mu^2} \right\} + \frac{\alpha_s^2}{\pi^2} \left\{ \frac{4748953}{5184} - \frac{\pi^4}{36} - \frac{91519\zeta(3)}{216} + \frac{715\zeta(5)}{12} - \frac{229\pi^2}{6} \right\} \cdot
\]

\[
+ \left( -\frac{4781}{9} + \frac{221\pi^2}{24} + \frac{475\zeta(3)}{4} \right) \ln \frac{s}{\mu^2} - \frac{s}{\mu^2} \frac{229}{2} \ln^2 \frac{s}{\mu^2} - \frac{221}{24} \ln^3 \frac{s}{\mu^2} \right\} \bigg) \bigg] - \frac{3}{4\pi^2} (m_s - m_u)^2 \left\{ \frac{1}{3} - 4 \ln \frac{s}{\mu^2} \right\} + \frac{\alpha_s}{\pi} \left\{ \frac{77\zeta(3)}{3} + \frac{5065}{72} - \frac{25\pi^2}{6} - \frac{97}{2} \ln \frac{s}{\mu^2} + \frac{25}{2} \ln^2 \frac{s}{\mu^2} \right\} + \frac{m_s^2(s)}{s} \left\{ \frac{45}{56\pi^2} m_s^4(s) - \frac{2\alpha_s(s)}{\pi} \langle m_s \bar{u} u \rangle_0 + \frac{\alpha_s(s)}{9\pi} I_G - \frac{\alpha_s(s)}{\pi} I_s \right\}. \]
5. Results and discussion

In Fig.1 are presented plots of the invariant mass $\hat{m}_s$ and of the running mass at the scale 1 GeV as a function of the Borel parameter $M^2$ for different values of the continuum threshold $s_0$ and the central value of the QCD scale $\Lambda_{MS}^{n_f=3} = 380$ MeV [28]. We extract our results from the region in $M^2$ corresponding to the stability interval $M^2 = 2 - 9$ GeV$^2$, obtaining in this way $\hat{m}_s = 172 - 191$ MeV respectively $m_s(1 \text{ GeV})=191-213$ MeV. The error arises mainly from the $s_0$ and $M^2$ dependence, the errors due to the condensates being negligible, under 1-2%.

The effect on $\hat{m}_s$ of changing $\Lambda_{MS}^{n_f=3}$ between the limits 280-480 MeV is shown in Fig.2. The continuum threshold has been chosen such that optimal stability is obtained for each value of $\Lambda_{MS}^{n_f=3}$. For $\Lambda_{MS}^{n_f=3} = 280, 380, 480$ MeV we find $s_0 = 5.0, 6.0$ and 6.9 GeV$^2$. The corresponding values for the invariant mass are $\hat{m}_s = 231 - 232$ MeV, 181-182 MeV and 140-147 MeV. This rather large spread of values is considerably reduced for the running mass at the scale 1 GeV, for which we obtain $m_s(1 \text{ GeV}) = 209-210$ MeV, 201-202 MeV and 211-221 MeV.

As one can see from Fig.2 the larger values of $m_s(1 \text{ GeV})$ arises from including the large value of the QCD scale $\Lambda_{MS}^{n_f=3} = 480$ MeV. If this curve is eliminated the following results are obtained:

$$\hat{m}_s = 181 - 232 \text{ MeV}, \quad m_s(1 \text{ GeV}) = 201 - 210 \text{ MeV}. \quad (20)$$

A similar observation has been made in [29] in the context of the QCD sum rule for the $\rho$ meson width, where even lower values for $\Lambda_{MS}^{n_f=3}$ are advocated, of the order of 220 MeV. We will adopt therefore in the following (20) as our result incorporating the theoretical errors arising from varying $\Lambda_{MS}^{n_f=3} = 280 - 380$ MeV.

These results are significantly higher than the $O(\alpha_s^2)$ results of [6, 7], so that an explanation for this difference is necessary. As mentioned already (see the discussion surrounding Eq.(16)) the approach used in this paper differs from that of [6, 7] in that the Borel transform is not expanded in powers of $1/\ln(\Lambda^2/M^2)$ but all orders in this parameter are kept. As an effect the leading term in the perturbative contribution to the sum rule (of $O(m_s^2/M^2)$) is smaller than in [6, 7], even after including the 4-loop contribution, by about 8%. This results in an increase in the invariant mass by 4%, respectively 7-9 MeV. Another effect which pushes the result to the high side is the increase of the $O(m_s^4/M^4)$ term, when adding the $O(\alpha_s^2)$ contribution. Since this term contributes with a negative sign to the theoretical side of the sum rule, it results also in a small increase of 1-2 MeV in the final result.

For purposes of comparison we give also the results obtained if both sides of the sum rule had been truncated to order $1/L^3$ (in the leading terms of $O(m_s^2/M^2)$). For $\Lambda_{MS}^{n_f=3} = 380$ MeV the best stability is obtained for $s_0 = 5.7$ GeV$^2$ and the results for the strange quark mass are $\hat{m}_s = 175 - 176$ MeV, respectively.
Changing the continuum threshold \(s_0\) by \(\pm 0.5 \text{ GeV}^2\) about this value gives the broader range of values \(\hat{m}_s = 167 - 184 \text{ MeV}, \ m_s(1 \text{ GeV}) = 183 - 202 \text{ MeV}\). Choosing \(\Lambda_{n_f=3}^{MS} = 280,480 \text{ MeV}\) gives (for \(s_0 = 4.8, 6.7 \text{ GeV}^2\)) the results \(\hat{m}_s = 225 - 227, 135 - 139 \text{ MeV}\), respectively \(m_s(1 \text{ GeV}) = 203 - 205, 204 \text{ MeV}\). These results are somewhat smaller than the ones obtained in the nontruncation approach \((20)\) but still larger than the ones obtained in \([3,4]\). The reason for this is that, as explained in Sec.4, the leading term on the theoretical side of the sum rule decreases by about 6% when going from \(1/L^2\) to \(1/L^3\). This is partly compensated by a similar decrease in the contribution of the perturbative continuum when truncated to the same order in \(1/L\), such that the final result for \(m_s\) is smaller than in the untruncated approach.

Finally we should include the errors induced by the variation of the parameters (masses and widths) of the resonances and of the normalization factor \(d(s_{th})\). The former give an additional error of about \(\pm 14 \text{ MeV}\) on the value of \(\hat{m}_s\) and of \(\pm 15 \text{ MeV}\) on \(m_s(1 \text{ GeV})\). The latter induces an error of about \(\pm (11-12) \text{ MeV}\) on both mass parameters.

Adding all these errors in quadrature we obtain from \((20)\) our final result

\[
\bar{m}_s(1 \text{ GeV}) = 205.5 \pm 19.1 \text{ MeV}.
\]  

This value lies on the high side of the existing QCD sum rule calculations of \(m_s\) \([3,4]\), coming closest to the recent result obtained to three-loop order in \([5]\), of \(196.7 \pm 29.1 \text{ MeV}\). The comparatively low results obtained in \([3,4]\) were in good agreement with the lattice \([10]\) results for the strange quark mass. With our new value the disagreement between the two is back in place. In \([10]\) the value \(\bar{m}_s(2 \text{ GeV}) = 128 \pm 18 \text{ MeV}\) was obtained which gives \(\bar{m}_s(1 \text{ GeV}) = 172 \pm 24 \text{ MeV}\). Recently, a new lattice calculation has appeared \([30]\) with lower results: in the quenched approximation \(\bar{m}_s(2 \text{ GeV}) = 90 \pm 20 \text{ MeV}\), corresponding to \(\bar{m}_s(1 \text{ GeV}) = 121 \pm 27 \text{ MeV}\), and for \(n_f = 2\) an even lower value \(\bar{m}_s(2 \text{ GeV}) = 70 \pm 15 \text{ MeV}\), respectively \(\bar{m}_s(1 \text{ GeV}) = 94 \pm 20 \text{ MeV}\). Conceivable explanations for this discrepancy are a) significant systematic errors in the parametrization of the hadronic density and b) large contributions of direct instantons to the correlator of scalar currents \([31,32,33]\). In our case, we consider b) to be little probable given the large scales \(M^2 = 2 - 9 \text{ GeV}^2\) at which our determination is performed (for an explicit calculation in the pseudoscalar current case see \([32]\)). Thus further progress in improving the accuracy of the strange quark mass determination using the methods of the present paper can only come from a better knowledge of the hadronic density function. With the advent of a \(\tau\)-charm factory it should be possible to directly measure it in the future in semileptonic \(\tau\) decays.
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Figure captions

**Fig.1** Dependence of the invariant strange quark mass $\hat{m}_s$ on the Borel parameter $M^2$ and on the continuum threshold $s_0$ (the three lower curves). The upper three curves show the running mass at the scale 1 GeV for the same values of the parameters. ($\Lambda_{QCD} = 380$ MeV)

**Fig.2** Dependence of the results on the value of the QCD scale $\Lambda_{QCD}$. The continuous lines are the results for the running mass at the scale 1 GeV and the dotted lines show the invariant mass $\hat{m}_s$. 
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