FROM PSEUDO-ROTATIONS TO HOLOMORPHIC CURVES VIA QUANTUM STEENROD SQUARES

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ABSTRACT. In the context of symplectic dynamics, pseudo-rotations are Hamiltonian diffeomorphisms with finite and minimal possible number of periodic orbits. These maps are of interest in both dynamics and symplectic topology. We show that a closed, monotone symplectic manifold, which admits a non-degenerate pseudo-rotation, must have a deformed quantum Steenrod square of the top degree element, and hence non-trivial holomorphic spheres. This result (partially) generalizes a recent work by Shelukhin and complements the results by the authors on non-vanishing Gromov–Witten invariants of manifolds admitting pseudo-rotations.

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1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. In this paper we continue investigating connections between pseudo-rotations and symplectic topology of the underlying manifold. We show that a closed monotone symplectic manifold, which admits a non-degenerate Hamiltonian pseudo-rotation, must have a deformed quantum Steenrod square of the top degree element and, as a consequence, non-trivial holomorphic spheres. This result (partially) generalizes a recent work by Shelukhin, [Sh19b], and complements the results from [CGG].
In the context of symplectic dynamics, pseudo-rotations are Hamiltonian diffeomorphisms with finite and minimal possible number of periodic orbits. Such diffeomorphisms of the disk or the 2-sphere occupy a distinguished place in low-dimensional dynamical systems theory (see, e.g., [AK, FH, FK, FM, LCY] and references therein). In all dimensions, pseudo-rotations can have extremely interesting dynamics. For instance, some manifolds (e.g., \( \mathbb{CP}^n \)) admit pseudo-rotations with finite number of ergodic measures; see [AK, FK, LRS]. Somewhat surprisingly, as has been recently discovered, symplectic topological techniques can be effectively used to study the dynamics of pseudo-rotations in dimension two (see [Br15a, Br15b, BH]) and also in higher dimensions; see [GG18a, GG18b].

Many (in all likelihood, most) closed symplectic manifolds do not admit pseudo-rotations or even Hamiltonian diffeomorphisms with finitely many periodic orbits as the Conley conjecture asserts; see [GG15]. The state of the art result is that a closed symplectic manifold \((M, \omega)\) does not admit such a Hamiltonian diffeomorphism unless there exists a class \(A \in \pi_2(M)\) such that \(\langle [\omega], A \rangle > 0\) and \(\langle c_1(TM), A \rangle > 0\), [Ci, GG17]. For instance, this criterion rules out all cases where the class \([\omega]\) or \(c_1(TM)\) is aspherical and also negative monotone symplectic manifolds. While this theorem is satisfactory on the topological level, one can expect the failure of the Conley conjecture for \(M\) to also have strong consequences on the symplectic topological level. For instance, the so-called Chance–McDuff conjecture, inspired by [McD], claims that a symplectic manifold admitting a Hamiltonian diffeomorphism with finitely many periodic orbits must have non-vanishing Gromov–Witten invariants.

There are a few slightly different working definitions of pseudo-rotations, all reflecting the same condition that the map must have the least possible number of periodic orbits; [CGG, Sh19a, Sh19b]. In this paper, we define a pseudo-rotation as a Hamiltonian diffeomorphism \(\varphi\) such that all iterates \(\varphi^k, k \in \mathbb{N}\), are non-degenerate and the Floer differential (over \(F_2 = \mathbb{Z}_2\)) vanishes for all \(\varphi^k\); see Definition 1.1 and Remark 1.3. (Here and throughout all cohomology groups are with \(F_2\)-coefficients.) As of this writing, all known Hamiltonian diffeomorphisms with finitely many periodic orbits are pseudo-rotations in this sense and these two classes might well coincide; see [Sh19a] for some results in this direction.

Several variants of the Chance–McDuff conjecture have been established recently for pseudo-rotations. In [CGG], it was proved that a weakly monotone symplectic manifold with minimal Chern number \(N > 1\), admitting a pseudo-rotation \(\varphi\), must have deformed quantum product and, in particular, non-vanishing Gromov–Witten invariants, under certain additional index assumptions on \(\varphi\). These extra assumptions appear to be satisfied for most, but certainly not all, pseudo-rotations. Simultaneously and independently, in [Sh19b], along with some other results the quantum Steenrod square of the top degree class was shown to be deformed for monotone symplectic manifolds \(M^{2n}\) with Poincaré duality property (e.g., when \(N \geq n + 1\), admitting pseudo-rotations. (These pseudo-rotations need not be non-degenerate.)

The quantum Steenrod square is a symplectic topological invariant introduced in [Se] and then studied in [Wi18a, Wi18b]; see also [Be, BC, Fu, He, SS, ShZa] for relevant work. It is a cohomology operation \(QS\) on the quantum cohomology \(HQ^*(M)\), which is a deformation of the standard Steenrod square. In other words, \(QS(\alpha) = Sq(\alpha) + O(q)\), where \(q\) is the generator of the Novikov ring and...
$\alpha \in H^*(M)$. Roughly speaking, a deformed quantum Steenrod square, just as a deformed quantum product $\ast$, detects certain holomorphic spheres in $M$, but in general these spheres need not be related to Gromov–Witten invariants. Furthermore, $QS$ can also be viewed as a deformation of the standard quantum product (with respect to a different parameter $h$) in the same sense as $Sq$ can be thought of as a deformation of the cup-product. On the Floer cohomology side, $QS$ is closely related to another quantum cohomology operation also introduced in [Sc], the equivariant pair-of-pants product $\varphi$, which plays a crucial role in our proof. We will briefly discuss both of these operations in Section 3.

Let $\varpi$ be the generator of the top degree cohomology group $H^{2n}(M)$. Our main result (Theorem 1.2) is that $QS(\varpi)$ is different from $Sq(\varpi) = h^{2n}\varpi$ whenever $M$ is monotone and admits a pseudo-rotation. Here we treat the Steenrod square $Sq$ as a degree doubling map $Sq: H^*(M) \rightarrow H^*(M)[[h]]$, $Sq(\alpha) = |\alpha| \sum_{i=0}^{[\alpha]} h^{[\alpha]-i} Sq^i(\alpha)$, where $|h| = 1$ and $Sq^i(\alpha)$ is the $i$-th standard Steenrod square of $\alpha \in H^*(M)$, [Sc, Wi18a, Wi18b]. As a consequence, there is a non-trivial holomorphic sphere through every point of $M$.

Apart from the non-degeneracy condition, this result generalizes the main theorem from [Sh19b] by eliminating the Poincaré duality requirement; cf. Remark 1.3. On the other hand, it is difficult to compare Theorem 1.2 and the results from [CGG] detecting a deformed quantum product; for these theorems hold under different conditions and provide different symplectic topological information. Note however that the statement that $QS$ is deformed is obviously much weaker than that its 0-th order term in $h$, the quantum square, is deformed, i.e., $\alpha \ast \alpha \neq \alpha \cup \alpha$ for some $\alpha \in H^*(M)$.

The proof Theorem 1.2 hinges on the same idea as the argument in [CGG], although the latter proof is considerably more involved. In both cases, a non-trivial deformation comes roughly speaking from constant (to be more precise, zero energy) pair-of-pants solutions of the Floer equation: equivariant in the present case and standard for the quantum product.

For the standard pair-of-pants product, a zero energy curve is easily seen to be automatically regular provided that the Conley–Zehnder indices allow this. In other words, consider the cohomology pair-of-pants product of iterated capped orbits $\bar{x}^{k_1} \ast \ldots \ast \bar{x}^{k_r}$. Then the least action term in this product is $\bar{x}^k$ with $k = k_1 + \ldots + k_r$, i.e.,

$\bar{x}^{k_1} \ast \ldots \ast \bar{x}^{k_r} = \bar{x}^k + \ldots$,

where the dots stand for higher action terms, if and only if $\bar{x}^k$ has the “right” Conley–Zehnder index. Explicitly, with our conventions, the latter index condition is that

$\mu(\bar{x}^k) = \mu(\bar{x}^{k_1}) + \ldots + \mu(\bar{x}^{k_r}) + (r-1)n$,

and the main difficulty in the proof in [CGG] is to guarantee that this requirement is satisfied for some orbit $\bar{x}$ and that the resulting product is different from the cup product.

On the other hand, for the equivariant pair-of-pants product, $\bar{x}^2$ (or, to be more precise, its product with a suitable power of $h$) is always, without any index requirement, the least action term in $\varphi(\bar{x} \otimes \bar{x})$, although now this is a non-trivial fact proved
in [Se]; see also [ShZa]. This is sufficient to show that the quantum Steenrod square of $\varpi$ is deformed whenever $M$ admits a pseudo-rotation, by using simultaneously the action and $h$-adic filtrations of the equivariant Floer cohomology.

1.2. Main result: from pseudo-rotations to the Steenrod square. While, as has been mentioned above, there are several slightly different ways to define a pseudo-rotation, for our purposes it is convenient to adopt the following definition.

**Definition 1.1.** A Hamiltonian diffeomorphism $\varphi$ of a closed weakly monotone symplectic manifold is called a pseudo-rotation if all iterates $\varphi^k$, $k \in \mathbb{N}$, are non-degenerate and the Floer differential over $\mathbb{F}_2$ vanishes for all $\varphi^k$.

We note that while the Floer differential depends on an auxiliary structure (an almost complex structure), its vanishing is well-defined, i.e., independent of this structure.

We are now in a position to state the main result of this paper. Let $Q\mathcal{S}$ be the quantum Steenrod square; see Section 3 for a brief discussion and further references.

**Theorem 1.2.** Assume that a closed monotone symplectic manifold $(M^{2n}, \omega)$ admits a pseudo-rotation. Then the quantum Steenrod square $Q\mathcal{S}$ of the top degree cohomology class $\varpi \in H^{2n}(M; \mathbb{F}_2)$ is deformed: $Q\mathcal{S}(\varpi) \neq h^{2n}\varpi$.

**Remark 1.3.** Although we deliberately limited our attention to a relatively narrow class of manifolds and pseudo-rotations, we expect the theorem to have several generalizations accessible by the same method with relatively minor modifications. First of all, the theorem should hold for a slightly broader class of pseudo-rotations considered in [Sh19a, Sh19b], allowing for some degenerations.  

Next, one might be able to replace the assumption that $M$ is monotone by the condition that it is weakly monotone; for one can expect the constructions of the equivariant pair-of-pants product $\mathcal{P}$ from [Se] and of the quantum Steenrod square $Q\mathcal{S}$ to extend to this setting with some modifications; see [SW]. Finally, one might also be able to extend Theorem 1.2 to the quantum Steenrod $\mathbb{Z}_p$ cohomology operations (see [ShZa]) although some details of the underlying machinery are yet to be finalized.

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2. Conventions and notation

For the reader’s convenience we set here our conventions and notation and briefly recall some basic definitions. The reader may want to consult this section only as needed.

Throughout this paper, the underlying symplectic manifold $(M, \omega)$ is assumed to be closed and (strictly) monotone, i.e., $[\omega]|_{\pi_2(M)} = \lambda c_1(TM)|_{\pi_2(M)} \neq 0$ for some $\lambda > 0$. The **minimal Chern number** of $M$ is the positive generator $N$ of the subgroup $\langle c_1(TM), \pi_2(M) \rangle \subset \mathbb{Z}$.

A **Hamiltonian diffeomorphism** $\varphi = \varphi_H = \varphi^1_H$ is the time-one map of the time-dependent flow $\varphi^t = \varphi_H^t$ of a 1-periodic in time Hamiltonian $H : S^1 \times M \rightarrow \mathbb{R}$, where $S^1 = \mathbb{R}/\mathbb{Z}$. The Hamiltonian vector field $X_H$ of $H$ is defined by $i_{X_H} \varpi = -dH$. Such time-one maps form the group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms.

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1 In fact, after this work had been completed we have learned of [Sh19c] where a similar theorem has been proved in such a setting by a different method.
of $M$. In what follows, it will be convenient to view Hamiltonian diffeomorphisms as elements of the universal covering $\widetilde{\text{Ham}}(M, \omega)$.

Let $x: S^1 \to M$ be a contractible loop. A capping of $x$ is an equivalence class of maps $A: D^2 \to M$ such that $A|_{S^1} = x$. Two cappings $A$ and $A'$ of $x$ are equivalent if the integral of $\omega$ (or of $c_1(TM)$ since $M$ is strictly monotone) over the sphere obtained by attaching $A$ to $A'$ is equal to zero. A capped closed curve $\bar{x}$ is, by definition, a closed curve $x$ equipped with an equivalence class of cappings, and the presence of capping is indicated by a bar.

The action of a Hamiltonian $H$ on a capped closed curve $\bar{x} = (x, A)$ is

$$A_H(\bar{x}) = -\int_A \omega + \int_{S^1} H_t(x(t)) \, dt.$$  

The space of capped closed curves is a covering space of the space of contractible loops, and the critical points of $A_H$ on this space are exactly the capped 1-periodic orbits of $X_H$.

The $k$-periodic points of $\varphi$ are in one-to-one correspondence with the $k$-periodic orbits of $H$, i.e., of the time-dependent flow $\varphi^t$. Recall also that a $k$-periodic orbit of $H$ is called simple if it is not iterated. A $k$-periodic orbit $x$ of $H$ is said to be non-degenerate if the linearized return map $d\varphi^k: T_{x(0)}M \to T_{x(0)}M$ has no eigenvalues equal to one. We call $x$ strongly non-degenerate if all iterates $x^k$ are non-degenerate.

A Hamiltonian $H$ is non-degenerate if all its 1-periodic orbits are non-degenerate. In what follows we always assume that $H$ is strongly non-degenerate, i.e., all periodic orbits of $H$ (of all periods) are non-degenerate. We denote the collection of capped $k$-periodic orbits of $H$ by $\mathcal{P}_k(H)$ or $\mathcal{P}_k(\varphi)$.

Let $\bar{x}$ be a non-degenerate capped periodic orbit. The Conley–Zehnder index $\hat{\mu}(\bar{x}) \in \mathbb{Z}$ is defined, up to a sign, as in [Sa, SZ]. In this paper, we normalize $\hat{\mu}$ so that $\hat{\mu}(\bar{x}) = n$ when $x$ is a non-degenerate maximum (with trivial capping) of an autonomous Hamiltonian with small Hessian. The mean index $\hat{\mu}(\bar{x}) \in \mathbb{R}$ measures, roughly speaking, the total angle swept by certain (Krein–positive) unit eigenvalues of the linearized flow $d\varphi^k|_{\bar{x}}$ with respect to the trivialization associated with the capping; see [Lo, SZ]. The mean index is defined even when $x$ is degenerate and $|\hat{\mu}(\bar{x}) - \mu(\bar{x})| \leq n$. Moreover, if $x$ is non-degenerate, the inequality is strict:

$$|\hat{\mu}(\bar{x}) - \mu(\bar{x})| < n. \quad (2.1)$$

The mean index is homogeneous with respect to iteration: $\hat{\mu}(\bar{x}^k) = k \hat{\mu}(\bar{x})$. (The capping of $\bar{x}^k$ is obtained from the capping of $\bar{x}$ by taking its $k$-fold cover branched at the origin.)

Fixing an almost complex structure, which will be suppressed in the notation, we denote by $(\text{CF}^*(\varphi), d_F)$ and $\text{HF}^*(\varphi)$ the Floer complex and cohomology of $\varphi$ over $\mathbb{F}_2 = \mathbb{Z}_2$; see, e.g., [MS, Sa]. (As has been mentioned in the introduction, here all complexes and cohomology groups are over $\mathbb{F}_2$.) The complex $\text{CF}^*(\varphi)$ is generated by the capped 1-periodic orbits $\bar{x}$ of $H$, graded by the Conley–Zehnder index, and filtered by the action. The differential $d_F$ is the upward Floer differential: it increases the action and also the index by one. With these conventions, we have the canonical isomorphism

$$\Phi: \text{HQ}^*(M) \cong \text{HF}^*(\varphi)[n], \quad (2.2)$$
where \( HQ^*(M) \) is the quantum homology of \( M \); see, e.g., [Sa, MS] and references therein. Depending on the context, \( \Phi \) is the PSS-isomorphism or the continuation map or a combination of the two.

For instance, assume that \( H \) is \( C^2 \)-small and autonomous (i.e., independent of \( t \)), and has a unique maximum and a unique minimum. Then the top degree cohomology class \( \varpi \in H^{2n}(M) \subset HQ^{2n}(M) \) corresponds to the maximum of \( H \), which has degree \( n \) in \( HF^*(\varphi) \); the unit 1 \( \in HQ^0(M) \) corresponds to the minimum of \( H \) which has degree \( -n \) in \( HF^*(\varphi) \). We denote by \( |\alpha| \) the degree of \( \alpha \in HQ^*(M) \) or \( \alpha \in HF^*(\varphi) \).

The cohomology groups \( HQ^*(M) \) and \( HF^*(\varphi) \) and the complex \( CF^*(\varphi) \) are modules over a Novikov ring \( \Lambda \), and \( HQ^*(M) \cong H^*(M) \otimes \Lambda \) (as a module). There are several choices of \( \Lambda \); see, e.g., [MS]. For our purposes, it is convenient to take the field of Laurent series \( \mathbb{F}_2((q)) \) with \( |q| = 2N \) as \( \Lambda \). Then \( \Lambda \) naturally acts on \( CF^*(\varphi) \) by recapping, and multiplication by \( q \) corresponds to the recapping by \( A \in \pi_2(M) \) with \( \langle c_1(TM), A \rangle = N \).

When \( \varphi \) is a pseudo-rotation (or, more generally, if \( d_{\varphi t} = 0 \)), the isomorphism (2.2) turns into the natural identification
\[
HQ^*(M)[-n] \cong HF^*(\varphi) \cong CF^*(\varphi).
\]
Since any iterate \( \varphi^k \) is also a pseudo-rotation, we have
\[
HQ^*(M)[-n] \cong HF^*(\varphi^k) \cong CF^*(\varphi^k).
\]

The quantum homology \( HQ^*(M) \) carries the quantum product, denoted here by *, which makes it into a graded-commutative algebra over \( \Lambda \) with unit 1. This product is a deformation (in \( q \)) of the cup product: \( \alpha \ast \beta = \alpha \cup \beta + O(q) \). For instance, \( \alpha_1 \ast \alpha_n = q \cdot 1 \) in \( HQ^*(\mathbb{C}P^n) \), where \( \alpha_1 \) stands for the generator (i.e., the only non-zero element) of \( H^{2i}(\mathbb{C}P^n) \). In Floer cohomology, the quantum product corresponds to the so-called pair-of-pants product
\[
HF^*(\varphi) \otimes HF^*(\varphi) \rightarrow HF^*(\varphi^2)[n],
\]
which we also denote by *. We emphasize that with our conventions \( |\alpha \ast \beta| = |\alpha| + |\beta| + n \) in Floer cohomology. When (2.2) is applied to \( \varphi \) and \( \varphi^2 \), the pair-of-pants product turns into the quantum product:
\[
HQ^*(M) \otimes HQ^*(M) \cong HF^*(\varphi) \otimes HF^*(\varphi) \rightarrow HF^*(\varphi^2) \cong HQ^*(M).
\]
Here, for the sake of simplicity, we suppressed the shifts of degree as we often will in what follows.

3. Preliminaries

Drawing heavily from [Se] and also [ShZa, Wi18a, Wi18b], we now recall the construction of the equivariant pair-of-pants product and its relation to the quantum Steenrod square in the case where the ambient manifold \( M \) is closed and monotone. Then we will take a closer look at the effect of the additional condition that the Hamiltonian diffeomorphism \( \varphi \) is a pseudo-rotation.

3.1. Equivariant pair-of-pants product. Let us start by briefly outlining the definition of the equivariant pair-of-pants product, closely following [Se]. In this (sub)section we make little or no use of the fact that the underlying manifold \( M \) is compact and that we are interested in the global, rather than filtered, Floer cohomology although most of the discussion carries over to the filtered setting...
verbatim. These extra restrictions afford some immediate simplifications which we will spell out in Section 3.2 based on [Wi18a, Wi18b]. It is worth pointing out that the underlying assumptions in [Se] are somewhat different from ours: there the ambient manifold $M$ is assumed to be exact rather than closed and monotone, but the definitions and results translate readily to our framework; see [ShZa, Wi18a, Wi18b].

The target space of the equivariant pair-of-pants product is the $\mathbb{Z}_2$-equivariant cohomology $HF^*_{eq}(\varphi^2)$. This is the homology of the complex $CF^*_{eq}(\varphi^2)$, which is the graded $\mathbb{F}_2$-vector space $CF^*_{eq}(\varphi^2)[[h]] = CF^*(\varphi^2) \otimes \mathbb{F}_2[[h]]$ with $[h] = 1$ equipped with the differential
\[
d_{eq} = d_{Fl} + \sum_{j=1}^{\infty} h^j d_j
\]
emulating the construction of the $\mathbb{Z}_2$-equivariant cohomology via Morse theory on the Borel quotient; see [Se, Sect. 4.2] for details. (Here $\mathbb{F}_2[[h]] \cong H^*(\mathbb{RP}^\infty; \mathbb{F}_2).$) This complex and hence the homology carry two increasing filtrations: the action filtration and the $h$-adic filtration.

**Remark 3.1.** In this connection we point out that there are two slightly different constructions of the equivariant Floer cohomology for Hamiltonians with symmetry, e.g., the $S^1$-symmetry for autonomous Hamiltonians and the $\mathbb{Z}_k$-symmetry for $k$-iterated Hamiltonian diffeomorphisms. The first construction uses a parametrized perturbation of the original Hamiltonian and the action functional; see [BO, Vi] and also [GG19]. This is a Floer theoretic analogue of taking a Morse perturbation of the pull-back (which is Morse–Bott) of the original Morse function to the Borel quotient. In the second construction one keeps the Hamiltonian and the action functional unchanged, but uses a parametrized almost complex structure and continuation maps along the gradient lines of an auxiliary Morse function on the classifying space to define the differential; see [Hu, Se, SS]. This approach results in a complex and cohomology a priori better behaving with respect to the action filtration. This is the complex considered here. (The difference becomes apparent in the context of the filtered Leray spectral sequence converging to the equivariant cohomology and associated with the $h$-adic filtration: it is not even clear how to define the $h$-adic filtration in the framework of the first construction without additional assumptions on the perturbation; see [BO].)

The domain of the equivariant pair-of-pants product map is the ordinary group cohomology of $\mathbb{Z}_2$ with coefficients in the complex $CF^*(\varphi) \otimes CF^*(\varphi)$ with the $\mathbb{Z}_2$-action given by the involution $\iota$ interchanging the two factors. In other words, this is the cohomology of the complex
\[
C^* \left( \mathbb{Z}_2; CF^*(\varphi) \otimes CF^*(\varphi) \right) := CF^*(\varphi) \otimes CF^*(\varphi)[[h]]
\]
equipped with the differential $d_{z_2} = d_{Fl} + h(id + \iota)$, where the first term stands for the differential induced by $d_{Fl}$ on the tensor product. This complex also carries two increasing filtrations: the action filtration and the $h$-adic filtration. We note that the complex is “unaware” of simple 2-periodic orbits of $\varphi$.

The **equivariant pair-of-pants product** is the $\mathbb{F}_2[[h]]$-linear map
\[
\varphi: H^* \left( \mathbb{Z}_2; CF^*(\varphi) \otimes CF^*(\varphi) \right) \to HF^*_{eq}(\varphi^2)
\]
induced by the chain map

\[ C^* \left( \mathbb{Z}_2; CF^*(\varphi) \otimes CF^*(\varphi) \right) \to CF^*_{eq}(\varphi^2) \]

constructed in [Se]. This product is a deformation in \( \hbar \) of the pair-of-pants product, i.e., on the chain level the equivariant pair-of-pants product \( \varphi(c_1 \otimes c_2) \) of \( c_1 \) and \( c_2 \) in \( CF^*_{eq}(\varphi) \) has the form \( c_1 * c_2 + O(\hbar) \).

To get a better understanding of how the map \( \varphi \) works, note first that the graded space \( CF^*(\varphi^2) \) has a canonical involution \( \iota' \) given by the shift of time \( x(t) \mapsto x(t+1) \); for the generators of this space are the 2-periodic orbits of \( \varphi \).

In general, this linear map, extended to \( CF^*(\varphi^2)[[h]] \), does not commute with \( d_{P1} \) unless there is a regular 1-periodic (rather than 2-periodic) almost complex structure. However, when this is the case, one can replace the complex \( CF^*_{eq}(\varphi^2) \) by the former complex with the differential \( d_{P1} + \hbar(id + \iota') \). Then \( \varphi \) is a deformation of the map induced by the pair-of-pants product map \( CF(\varphi) \otimes CF(\varphi) \to CF(\varphi^2) \) of the “coefficient” complexes.

**Remark 3.2.** Recall that even when \( \iota' \) does not commute with \( d_{P1} \), it becomes an isomorphism of complexes when the target is equipped with the Floer differential associated with the time-shifted almost complex structure \( J_{t_1} = J_{t_1+1} \). Then, once composed with the continuation map, \( \iota' \) induces an involution of \( HF^*(\varphi^2) \). In our setting, the global Floer cohomology \( HF^*(\varphi^2) \) and the involution are independent of \( \varphi \), and hence this involution is the identity map.

The key property, [Se, Thm. 1.3], of the equivariant pair-of-pants product map \( \varphi \) is that when, for example, \((M, \omega)\) is symplectically aspherical it becomes an isomorphism once \( \hbar^{-1} \) is attached to the ground ring, i.e., after taking tensor product with the ring of Laurent series \( \mathbb{F}_2((\hbar)) \). This yields the Floer theoretic analogue of Borel’s localization relating the filtered cohomology \( HF^*(\varphi) \) and \( HF^*_{eq}(\varphi^2) \) and, as a consequence, a variant of Smith’s inequality, cf. [ÇG, He, Sh19b, ShiZa]. Although in this paper we do not directly use any of these results, we will briefly revisit them in Remark 3.5.

Next, consider the map \( S: CF^*(\varphi) \to CF^*(\varphi) \otimes CF^*(\varphi) \) given by \( c \mapsto c \otimes c \) for all \( c \in CF^*(\varphi) \). When needed, we extend this map to \( CF^*(\varphi)[[h]] \) by setting \( S(hc) = hS(c) \). In general, \( S \) is neither linear nor, when linear, is it a chain map. However, \( S \) is well-defined on the level of cohomology and becomes linear when multiplied by \( \hbar \), i.e., as a map

\[ hS: HF^*(\varphi)[[h]] \to H^* \left( \mathbb{Z}_2; CF^*(\varphi) \otimes CF^*(\varphi) \right). \tag{3.1} \]

For instance, to see the linearity it suffices to observe that

\[ h((c_1 + c_2)^2 - c_1^2 - c_2^2) = dz_2(c_1 \otimes c_2), \]

when \( d_{P1}(c_1) = 0 = d_{P1}(c_2) \). As a consequence, \( S \) itself is linear on the level of cohomology when the target has no \( h \)-torsion.

Composing the cohomology map \( S \) with the equivariant pair-of-pants map \( \varphi \), we obtain a map

\[ \mathcal{P}_S: HF^*(\varphi)[[h]] \to HF^*_{eq}(\varphi^2); \]

the notation is borrowed from [Wi18a, Wi18b]. This map is linear whenever the target has no \( h \)-torsion and, as we will soon see, this condition is automatically satisfied in the case we are interested in.
Remark 3.3. Following [Se, Sect. 2.1] note that for purely algebraic reasons there is a canonical isomorphism

$$H^* \left( \mathbb{Z}_2; \mathrm{CF}^*(\varphi) \otimes \mathrm{CF}^*(\varphi) \right) \cong H^* \left( \mathbb{Z}_2; \mathrm{HF}^*(\varphi) \otimes \mathrm{HF}^*(\varphi) \right)$$

Hence the cohomology group on the right can also be thought of as the domain of the equivariant pair-of-pants product $\varphi$ and the target of the map $S$. Furthermore, the map

$$S: \mathrm{HF}^*(\varphi)((h)) \to H^* \left( \mathbb{Z}_2; \mathrm{CF}^*(\varphi) \otimes \mathrm{CF}^*(\varphi) \right) \otimes F_2((h)),$$

which is linear regardless of whether or not the target of $S$ has $h$-torsion, is also an isomorphism and again for purely algebraic reasons.

In general, we still have $\mathcal{PS}$ defined on the chain level as a map

$$\mathrm{CF}^*(\varphi)[[h]] \to \mathrm{CF}^*_{eq}(\varphi^2)$$

such that $\mathcal{PS}(c) = c \ast c + O(h)$ for $c \in \mathrm{CF}^*(\varphi)$, but it is neither linear nor a chain homomorphism. (Here we have once again ignored the shift of degree: by construction, $|\mathcal{PS}(c)| = 2|c| + n$.)

By construction the equivariant pair-of-pants map $\mathcal{P}$ and the map $\mathcal{PS}$ preserve the action filtration; cf. Remark 3.6.

One of the key ingredients in the proof of [Se, Thm. 1.3] is the following result, which also plays a central role in our argument and which, slightly deviating from [Se], we state for the map $\mathcal{PS}$ rather than for $\varphi$.

**Proposition 3.4** ([Se], Prop. 6.7). Consider a collection of orbits $\bar{x}_i \in \bar{\mathcal{P}}_i(\varphi)$, $i = 1, \ldots, \ell$, such that $A_H(\bar{x}_i) = a$ for $i = 1, \ldots, \ell_0 \leq \ell$ and $A_H(\bar{x}) > a$ for the remaining orbits. Then

$$\mathcal{PS}: \sum_{i=1}^{\ell} \bar{x}_i \mapsto \sum_{i=1}^{\ell_0} h^{m_i} \bar{x}_i^2 + \ldots,$$

where $\bar{x}_i^2 \in \bar{\mathcal{P}}_2(\varphi)$ is the second iterate of $\bar{x}_i$ and

$$m_i = 2\mu(\bar{x}_i) - \mu(\bar{x}_i^2) + n$$

(3.2)

and the dots stand for a sum of capped orbits with action strictly greater than $2a$.

For instance, $\mathcal{PS}(\bar{x}) = h^m \bar{x}^2 + \ldots$, where $m = 2\mu(\bar{x}) - \mu(\bar{x}^2) + n$ and the remaining terms have action higher than that of $\bar{x}^2$. In particular, $\bar{x}^2$ with some power of $h$ is necessarily present in $\mathcal{PS}(\bar{x})$.

Proposition 3.4 is an equivariant analogue of the standard fact that a constant solution of the Floer or Cauchy–Riemann equation is automatically regular whenever the relative index of the solution is zero, which in turn is a consequence of that the kernel of the linearized operator at the constant solution with suitable boundary conditions is trivial; see, e.g., [CGG, Lemma 3.1], [MS, Lemma 6.7.6], [Se, p. 971] and [Sa, Sect. 2.7]. However, the step from a non-equivariant to equivariant setting is non-trivial. We refer the reader to [Se] for the proof; see also [ShZa] for an alternative approach and generalizations.

**Remark 3.5** (Borel’s localization theorem according to [Se]). As has been mentioned above, one consequence of Proposition 3.4 is [Se, Thm. 1.3] asserting, in particular, that for a symplectically aspherical manifold the equivariant pair-of-pants product $\varphi$ in the filtered Floer homology becomes an isomorphism after tensoring with
\( F_2((h)). \) (The theorem follows from the proposition via applying the comparison theorem to the action filtration spectral sequence.) Since \( S \) is an isomorphism modulo h-torsion for purely algebraic reasons (see Remark 3.3), this yields that the map \( \mathcal{PS} \) is also an isomorphism, as well as the variants of Borel’s localization and Smith’s inequality in the filtered Floer homology.

On the other hand, while for a closed symplectic manifold \( M \) the analogues of Borel’s localization and Smith’s inequality hold trivially in the global Floer cohomology, the filtered versions (in the most naive form) fail without the assumption that \( (M, \omega) \) is symplectically aspherical; see, however, \([Sh19a]\). Moreover, \( \mathcal{PS} \) need not be an isomorphism even globally without this assumption. In fact, as Example 3.7 shows, \( \mathcal{PS} \) is not an epimorphism already for \( M = S^2 \).

Remark 3.6 (Regularity). To ensure that the regularity condition is satisfied for the equivariant pair-of-pants product, a certain arbitrarily small “inhomogeneous perturbation”, i.e., an \( s \)-dependent perturbation of the Hamiltonian, is introduced to the Floer equation in \([Sc]\). This perturbation is compactly supported in \( s \) and thus does not affect the initial and terminal Hamiltonians and the actions. However, it does effect the relation between the energy of a pair-of-pants curve and the action difference. As a consequence, the equivariant pair-of-pants product \( \varphi \) is now action increasing only up to an \( \epsilon \)-error, which goes to zero with the size of the perturbation. Therefore, \( \mathcal{PS} \) also preserves the action filtration only up to an \( \epsilon \)-error. In particular, if all fixed points of \( \varphi \) have distinct actions and \( \epsilon > 0 \) is small, \( \mathcal{PS} \) literally preserves the action filtration. This would already be sufficient for our purposes; see Remark 4.1. However, since pair-of-pants curves connecting different orbits must have energy \emph{a priori} bounded away from zero (cf. \([GG17, \text{ Prop. 2.2}]\)), the map \( \mathcal{PS} \) always preserves the action filtration when the inhomogeneous perturbation is small enough, and Proposition 3.4 holds as stated.

3.2. Quantum Steenrod square. The counterpart of the map \( \mathcal{PS} \) on the side of the quantum cohomology is the quantum Steenrod square \( QS \). This quantum cohomology operation is studied in detail in \([Wi18a, Wi18b]\), but the first Morse/Floer theoretic descriptions of the Steenrod squares go back to \([Be, BC, Fu]\). Throughout this section it is essential that the manifold \( M \) is closed.

Following \([Sc, Wi18a, Wi18b]\) and slightly changing the usual notation, let us define the Steenrod square as the degree doubling linear map

\[
\text{Sq}: H^*(M) \to H^*(M)[[h]], \quad \text{Sq}(\alpha) = \sum_{i=0}^{[\alpha]} h^{[\alpha]-i} \text{Sq}^i(\alpha),
\]

(3.3)

where \( \text{Sq}^i \) are the standard Steenrod squares. In particular, \( |\text{Sq}^i(\alpha)| = |\alpha| + i \), and \( \text{Sq}^0(\alpha) = \alpha \) and \( \text{Sq}^{[\alpha]}(\alpha) = \alpha \cup \alpha \). For instance,

\[
\text{Sq}(\varpi) = h^{2n} \varpi.
\]

(3.4)

The quantum Steenrod square is a degree doubling map

\[
Q\text{S}: H^*(M) \to HQ^*(M)[[h]],
\]

which is a certain deformation of \( \text{Sq} \) in \( q \):

\[
Q\text{S}(\alpha) = \text{Sq}(\alpha) + O(q)
\]

(3.5)

for \( \alpha \in H^*(M) \). For instance,

\[
Q\text{S}(\varpi) = h^{2n} \varpi + O(q),
\]
and $QS$ is undeformed at $\varpi \in H^{2n}(M)$ if and only if the higher order terms in $q$ vanish.

It is convenient to formally extend $Sq$ and $QS$ to the maps

$$Sq : HQ^*(M)[[h]] \to HQ^*(M)[[h]]$$

and

$$QS : HQ^*(M)[[h]] \to HQ^*(M)[[h]],$$

which are linear over $\mathbb{F}_2[[h]]$ and homogeneous of degree two in $q$ and no longer degree doubling. Then $QS$ is still a deformation of $Sq$ in $q$ in the sense of (3.5).

The next ingredient we need is the equivariant continuation/PSS map introduced in [Wi18b]. This is the map $\Phi_{eq}$ from the $\mathbb{Z}_2$-equivariant cohomology of $HQ^*(M)$ with trivial action to the equivariant Floer cohomology of $\varphi^2$:

$$\Phi_{eq} : HQ^*_{eq}(M) \to HF^*_{eq}(\varphi^2)[n].$$

Since the $\mathbb{Z}_2$-action in the cohomology is trivial, we have $HQ^*_{eq}(M) = HQ^*(M)[[h]]$ and one can also think of the cohomology on the left hand side as the target space of $QS$.

Just as an ordinary continuation/PSS map $\Phi$, its equivariant counterpart $\Phi_{eq}$ is an $\mathbb{F}_2[[h]]$-linear isomorphism. As a consequence, $HF^*_{eq}(\varphi^2)$ has no $h$-torsion and the map $\mathcal{P}S$ is linear.

The spaces and maps we have introduced fit together into the following commutative diagram, where we again suppressed the shifts of degree by the continuation/PSS maps:

$$
\begin{array}{ccc}
HQ^*(M)[[h]] & \xrightarrow{\Phi} & HF^*(\varphi)[[h]] \\
\downarrow QS & & \downarrow \mathcal{P}S \\
HQ^*_{eq}(M) & \xrightarrow{\Phi_{eq}} & HF^*_{eq}(\varphi^2) \\
\cong & & \cong \\
HQ^*(M)[[h]] & \xrightarrow{\Phi} & HF^*(\varphi^2)[[h]]
\end{array}
$$

(3.6)

We emphasize that here the continuation/PSS maps $\Phi$ for $\varphi$ and $\varphi^2$, and $\Phi_{eq}$ (i.e., the horizontal arrows) are $\mathbb{F}_2[[h]]$-linear isomorphisms. The requirement that the top square is commutative can be viewed as the definition of $QS$, [Wi18b]. Likewise, the condition that the bottom square is commutative is the definition of $F$, i.e.,

$$F = \Phi_{eq}^{-1}.$$

It is worth pointing out that in general the map $F$ need not preserve the action filtration.

**Example 3.7.** Let $M = S^2$. Then $QS(1) = 1$ and $QS(\varpi) = h^2 \varpi + q \cdot 1$, where $1$ is the generator of $H^0(S^2)$. Continuing the discussion from Remark 3.5 and making $h$ invertible, it is easy to see now that neither $\varpi$ nor $q \cdot 1$ nor $q \varpi$ is in the image of $QS$:

$$QS : HQ^*(S^2)((h)) \to HQ^*_{eq}(S^2) \otimes \mathbb{F}_2((h)) \cong HQ^*(M)((h)).$$

As a consequence of the diagram (3.6), $\mathcal{P}S : HF^*(\varphi)((h)) \to HF^*_{eq}(\varphi^2) \otimes \mathbb{F}_2((h))$ is not onto for any Hamiltonian diffeomorphism $\varphi : S^2 \to S^2$. Note that at the same
time the classical Steenrod square map $\text{Sq}: H^*(M)((h)) \to H^*(M)((h))$ induced by (3.3) is a linear isomorphism over $\mathbb{F}_2((h))$ for any closed manifold $M$.

Finally, as has been mentioned above, the complex $\text{CF}_*^\psi (\varphi^2)$ carries the h-adic filtration. The resulting spectral sequence (the Leray spectral sequence in the equivariant cohomology) converges to the graded space associated with the h-adic filtration of $\text{HF}_*^\psi (\varphi^2)$. It readily follows from the fact that $\Phi_{eq}$ is an isomorphism that this spectral sequence collapses on the $E_1$-page: $E_1 = \text{HF}^* (\varphi^2)[[h]] = E_\infty$. With this in mind we can view $\text{HF}^* (\varphi^2)[[h]]$ as the graded space associated with the h-adic filtration on $\text{HF}_*^\psi (\varphi^2)$. We will need the following simple observation:

**Lemma 3.8.** For any Hamiltonian diffeomorphism $\varphi$, the equivariant continuation map $\Phi_{eq}$ induces the ordinary continuation map $\Phi$ on the graded vector space $E_\infty = \text{HF}^* (\varphi^2)[[h]]$, i.e.,

$$\Phi_{eq} = \Phi + O(h). \quad (3.7)$$

**Proof.** Let $f$ be a $C^2$-small Morse function on $(M, \omega)$, unrelated to the pseudo-rotation $\varphi$. Set $QC^*(M) = CM^*(f) \otimes \Lambda$ where $CM^*(f)$ is the Morse complex of $f$. The complex $QC^*(M)[[h]]$ has a natural h-adic filtration and the resulting spectral sequence collapses in the $E_1$-term; for $h$ is not involved in the differential. Furthermore, recall that $\text{HF}^* (\varphi)[[h]]$ is the $E_1$-term associated with the h-adic filtration of $\text{CF}_*^\psi (\varphi^2)$. We claim that, in the obvious notation,

$$E_1(\Phi_{eq}) = \Phi. \quad (3.8)$$

Then, since both spectral sequences collapse on the $E_1$-page, (3.7) readily follows from (3.8).

It remains to establish (3.8). Let $\psi_f$ be the Hamiltonian diffeomorphism generated by $f$. Following [Wi18b], we write the chain level definitions of $\Phi_{eq}$ and $\Phi$ as the compositions

$$QC^*(M)[[h]] \xrightarrow{\Psi_{eq}} \text{CF}^* (\psi_f^2)[[h]] \xrightarrow{C}_{eq} \text{CF}^* (\varphi^2)[[h]] = \text{CF}_*^\psi (\varphi^2)$$

and

$$QC^*(M)[[h]] \xrightarrow{\Psi} \text{CF}^* (\psi_f^2)[[h]] \xrightarrow{C} \text{CF}^* (\varphi^2)[[h]].$$

Here the map $\Psi$ is the PSS map for $f$ or, to be more precise, for $2f$. Furthermore, $C$ is the continuation from the Floer complex of $\varphi^2$ to the Floer complex of $\psi_f$ for a fixed (e.g., linear) homotopy between $f$ and the Hamiltonian $H$ generating $\varphi$. The maps $\Psi_{eq}$ and $C_{eq}$ are the equivariant counterparts of $\Psi$ and, respectively, $C$. (Note that the differential in the Morse complex of $f$, and hence in the Floer complex of $\psi_f^2$, might be non-trivial; for $M$ need not admit a perfect Morse function.) By definition (see [Wi18b]), $\Psi_{eq} = \Psi + O(h)$ and $C_{eq} = C + O(h)$, and (3.8) follows. □

### 3.3. Enter pseudo-rotations

Assume now that $\varphi$ is a pseudo-rotation or more generally that every 2-periodic point is a fixed point and that $d_{\varphi^2} = 0$ for $\varphi$ and hence for $\varphi^2$. Then $\text{HF}^* (\varphi) = \text{CF}^* (\varphi)$ and $\text{HF}^* (\varphi^2) = \text{CF}^* (\varphi^2)$.

Furthermore, from the collapse of the Leray spectral sequence it then follows inductively that $d_{\varphi^2} = 0$ and we have the identifications

$$F_0: \text{HF}_*^\psi (\varphi^2) = \text{CF}_*^\psi (\varphi^2) = \text{CF}^* (\varphi^2)[[h]], \quad (3.9)$$

which, in contrast with the natural map $F$, are specific to the case of pseudo-rotations, but might exist under somewhat less restrictive conditions. Of course,
$F_0 = \text{id}$, but we prefer to use a different notation at this point to emphasize the fact that $F_0$ is defined only under some additional assumptions on $\varphi$ and $\varphi^2$.

The next important (but simple) ingredient of our proof is the following:

**Lemma 3.9.** We have $F = F_0 + O(h)$.

This lemma is an immediate consequence of Lemma 3.8 and the identifications (3.9).

4. Proof of the main theorem

With all preparations in place, we are now ready to prove the main result of the paper, Theorem 1.2. For the sake of simplicity, we will assume that all capped periodic orbits have distinct action: the argument extends to the general case in a straightforward way.

We will argue by contradiction: throughout the proof we assume that $\mathcal{QS}(\varpi)$ is undeformed, i.e., $\mathcal{QS}(\varpi) = h^{2n}\varpi$; see (3.4) and (3.5). The proof comprises two steps, and this assumption, which we aim to disprove, is used in both steps.

Write

$$\Phi(\varpi) = \bar{x} + \ldots,$$

where the dots stand for the terms with action strictly greater than the action of $\bar{x}$. Thus $\mu(\bar{x}) = n$. In the first step we prove the theorem under the additional condition that the index of $\bar{x}$ jumps from $\bar{x}$ to $\bar{x}^2$, i.e.,

$$\mu(\bar{x}^2) > \mu(\bar{x}) = n.$$

(4.2)

This condition might or might not hold for $\varphi$ and in the second step we show that (4.2) is necessarily satisfied for a sufficiently high iterate of $\varphi$. This will complete the proof of the theorem.

**Step 1.** Thus let us prove the theorem under the additional condition (4.2), where $\bar{x}$ is given by (4.1). From the top square in the diagram (3.6) and Proposition 3.4, we obtain the following commutative square:

$$
\begin{array}{ccc}
\varpi & \xrightarrow{\Phi} & \bar{x} + \ldots \\
\downarrow & & \downarrow \\
\mathcal{QS} & \xrightarrow{F_0} & \mathcal{PS} \\
\downarrow & & \downarrow \\
h^{2n}\varpi & \xrightarrow{\Phi_{eq}} & h^m\bar{x}^2 + \ldots
\end{array}
$$

In the bottom right corner the dots again stand for higher action terms and, by (3.2) and (4.2),

$$m = 3n - \mu(\bar{x}^2) < 2n.$$

This contradicts the fact that $\Phi_{eq}$ is $F_2[[h]]$-linear.

**Step 2.** To finish the proof it remains to make sure that (4.2) is satisfied after, if necessary, replacing $\varphi$ by its iterate.

The condition that $\mu(\bar{x}) = n$ guarantees that, by (2.1), $\tilde{\mu}(\bar{x}) > 0$ and hence $\mu(\bar{x}^k) \to \infty$, and furthermore that the sequence $\mu(\bar{x}^k)$ is increasing (but not necessarily strictly increasing):

$$\mu(\bar{x}^k) \nearrow \infty.$$

(4.3)

There are several ways to show this. For instance, let us adopt the argument from [GG18b, Sect. 4]; see also [GG18b, Formula (6.1)]. Namely, let $P \in \text{Sp}(2n)$ be the linearized flow along $\bar{x}$. Since the index sequence $\mu(P^k)$ is invariant under iso-spectral deformations, we can assume without loss of generality that $P(1)$ is
semi-simple. Then $P$ can be expressed as the product of a loop $\phi$ and $P \in \widetilde{Sp}(2n)$ which decomposes as a direct sum of elements of $\widetilde{Sp}(2)$ or $\widetilde{Sp}(4)$ of the following three types: short rotations of $\mathbb{R}^2$ (by an angle $\theta \in (-\pi, \pi)$), positive and complex hyperbolic transformations of $\mathbb{R}^2$ or $\mathbb{R}^4$ with zero index, and negative hyperbolic transformations of $\mathbb{R}^2$. (A negative hyperbolic transformation is the counterclockwise rotation in $\pi$ composed with a positive hyperbolic transformation with zero index.) Then, using the condition that $\mu(P) = n$, we can redistribute the loop part $\phi$ among individual terms and write $P$ as $\bigoplus P_i$, where $P_i$ is either a counterclockwise rotation by $\theta \in (0, 2\pi)$ or a counterclockwise negative hyperbolic transformation.

Clearly, each of the sequences $\mu(P_k)$ is increasing, and hence so is $\mu(\bar{x}_k)$.

As a consequence of (4.3), there exists $r = 2\ell_0 \geq 1$ such that $\mu(\bar{x}_k) = n$ for $k \leq r$ but $\mu(\bar{x}_{2r}) > n$.

We claim that

$$\Phi(\varpi) = \bar{x}^2 + \ldots$$

as long as $\ell \leq \ell_0$, where the dots stand again for the terms of higher action. In particular, we can replace $\varphi$ by $\varphi^\circ$ to guarantee that (4.2) is satisfied.

To prove (4.4), arguing by induction, it is enough to show that (4.1) still holds for $\varphi^2$, i.e.,

$$\Phi(\varpi) = \bar{x}^2 + \ldots,$$

provided, of course, that it holds for $\varphi$ and $\mu(\bar{x}^2) = n = \mu(\bar{x})$.

To establish (4.5), let us trace the image of $\varpi$ through the diagram (3.6). We have

$$\varpi \xrightarrow{\Phi} \bar{x} + \ldots \xrightarrow{\Phi_{eq}} \bar{x}^2 + \ldots \xrightarrow{\Phi_{eq}} \bar{x}^2 + \bar{x} + \ldots \xrightarrow{\Phi_{eq}} \bar{x}^2 + \bar{x} + \ldots$$

We emphasize that in the left column of the diagram, we have used, as in Step 1, the background assumption that $QS(\varpi)$ is undeformed. Next, let us take a closer look at what $R$ and $R'$ are.

The remainder $R$ in $\mathcal{PS}(\bar{x} + \ldots)$ is a sum of capped 2-periodic orbits of $\varphi$ with action strictly greater than the action of $\bar{x}^2$ and possibly $h$-dependent coefficients. The condition that

$$\mathcal{PS}(\bar{x} + \ldots) = \Phi_{eq}(h^{2n}\varpi) = h^{2n}\Phi_{eq}(\varpi)$$

guarantees that $\mathcal{PS}(\bar{x} + \ldots)$ is divisible by $h^{2n}$. Let us write

$$R = \sum \bar{z}_j + O(h),$$

where $\bar{z}_j$ are some capped 2-periodic orbits of $\varphi$ with action strictly greater than the action of $\bar{x}^2$.

By Lemma 3.9, $F(\bar{x}^2) = \bar{x}^2 + O(h)$ and

$$R' = \sum \bar{z}_j + O(h).$$
However, $\Phi$ is a non-equivariant continuation/PSS map and thus $\Phi(\varpi) \in CF^*(\varphi^2)$. Therefore, $R' \in CF^*(\varphi^2)$ since
\[ \hbar^{2n}(x^2 + R') = \hbar^{2n}\Phi(\varpi). \]
This proves (4.5) and completes the proof of the theorem. ☐

Remark 4.1. Returning to the regularity question (see Remark 3.6), note that, although this is not really necessary, we could have assumed throughout the proof that all periodic orbits of $\varphi$ have distinct actions. Indeed, it is clear from the proof that it suffices to assume that $\varphi$ is a pseudo-rotation up to a certain iteration order $r$, which is completely determined by the indices and the mean indices of 1-periodic orbits. Then the actions can be made distinct by an arbitrarily small perturbation of $\varphi$ keeping it a pseudo-rotation up to arbitrarily large iteration order.

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