The $f$–vector of the clique complex of chordal graphs and Betti numbers of edge ideals of uniform hypergraphs

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Abstract

We describe the Betti numbers of the edge ideals $I(G)$ of uniform hypergraphs $G$ such that $I(G)$ has linear graded free resolution.

We give an algebraic equation system and some inequalities for the components of the $f$–vector of the clique complex of an arbitrary chordal graph.

Finally we present an explicit formula for the multiplicity of the Stanley-Reisner ring of the edge ideals of any chordal graph.

1 Introduction

Let $X$ be a finite set and $E := \{E_1, \ldots, E_n\}$ a finite collection of non empty subsets of $X$. The pair $H = (X, E)$ is called a hypergraph. The elements of $X$ are called the vertices and the elements of $E$ are called the edges of the hypergraph.

We say that a hypergraph $H$ is $d$-uniform, if $|E_i| = d$ for every edge $E_i \in E$.

Keywords. Betti number, chordal graph, Hilbert function, Stanley-Reisner ring

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Let $\mathbb{Q}$ denote the rational field. Let $R$ be the graded ring $\mathbb{Q}[x_1,\ldots,x_n]$.

The vector space $R_s = \mathbb{Q}[x_1,\ldots,x_n]_s$ consists of the homogeneous polynomials of total degree $s$, together with 0.

We may think of an edge $E_i$ of a hypergraph as a squarefree monomial $x^{E_i} := \prod_{j \in E_i} x_j$ in $R$.

We can associate an ideal $I(H) \subseteq R$ to a hypergraph $H$. The edge ideal $I(H)$ is the ideal $\langle x^{E_i} : E_i \in E \rangle$, which is generated by the edges of $H$.

The edge ideal was first introduced by R. Villareal in [20]. Later edge ideals have been studied very widely, see for instance [5, 6, 7, 8, 9, 11, 12, 18, 20, 22, 23].

In [9] R. Fröberg characterized the graphs $G$ such that $G$ has a linear free resolution. He proved:

**Theorem 1.1** Let $G$ be a simple graph on $n$ vertices. Then $R/I(G)$ has linear free resolution precisely when $\overline{G}$, the complementary graph of $G$ is chordal.

In [6] E. Emtander generalized Theorem 1.1 for generalized chordal hypergraphs. He proved that the Stanley–Reisner ring of the incidence complex $\Delta(H)$ corresponding to $H$, where $H$ is a generalized chordal hypergraph, has a linear free resolution. In [22] R. Woodroofe extended the definition of chordality from graphs to clutters.

In this article we prove explicit formulas for the Betti numbers of the edge ideals of $m$-uniform hypergraphs $H$ such that $R/I(H)$ has linear free resolution.

Let $\Delta$ be a simplicial complex. A facet $F$ is called a leaf, if either $F$ is the only facet of $\Delta$, or there exists an other facet $G$, $G \neq F$ such that $H \cap F \subseteq G \cap F$ for each facet $H$ with $H \neq F$. A facet $G$ with this property is called a branch of $F$.

Zheng (see [23]) calls the simplicial complex $\Delta$ a quasi–tree if there exists a labeling $F_1,\ldots,F_m$ of the facets such that for all $i$ the facet $F_i$ is a leaf of the subcomplex $\langle F_1,\ldots,F_i \rangle$. We call such a labeling a leaf order.

A graph is called chordal if each cycle of length $> 3$ has a chord.

We recall here for the famous Dirac’s Theorem (see [4]).

**Theorem 1.2** (Dirac) A finite graph $G$ on $[n]$ is a chordal graph iff $G$ is the 1–skeleton of a quasi–tree.
Let $G$ be a finite graph on $[n]$. A **clique** of $G$ is a subset $F$ of $[n]$ such that $\{i, j\} \in E(G)$ for all $i, j \in F$ with $i \neq j$.

We write $\Gamma(G)$ for the simplicial complex on $[n]$ whose faces are the cliques of $G$.

In our article we give an algebraic equation system for the components of the $f$–vector of the clique complex of an arbitrary chordal graph.

**Theorem 1.3** Let $G$ be an arbitrary chordal graph. Let $\Gamma := \Gamma(G)$ be the clique complex of $G$ and $f(\Gamma) := (f_{-1}(\Gamma), \ldots, f_{d-1}(\Gamma))$ be the $f$-vector of the complex $\Gamma$. Here $d = \dim(\Gamma)$. Then

$$-\sum_{i=1}^{p+1} (-1)^i i \binom{f_0}{i+1} + \sum_{j=1}^{p+1} (-1)^{j+p} f_j \binom{f_0 - j - 2}{p - j + 1} = 1 \quad (1)$$

and

$$\sum_{k=1}^{p+1} (-1)^k f_k \left( \sum_{i=k-1}^{p} (-1)^i (2 + i)^i \binom{f_0 - k - 1}{i - k + 1} \right) + \sum_{i=0}^{p} (-1)^i (2 + i)^i (i + 1) \binom{f_0}{i + 2} = 0, \quad (2)$$

for each $j = 1, \ldots, n - d - 1$, where $p := \pdim(R/I(G))$ and $\overline{G}$ is the complement of the graph $G$.

**Remark.** In this Theorem the number of equations depends on the dimension of the complex $\Gamma$. We know from the Auslander–Buchsbaum Theorem that $n - d \leq p$. If $p = n - d$, then the module $M = R/I(\overline{G})$ is Cohen–Macaulay and we know that the complement of the chordal graph $G$ is a $d$–tree (see [21] Theorem 6.7.7, [9]). Consequently we know explicitly the $f$–vector of the clique complexes of $d$–trees.

**Theorem 1.4** Let $G$ be an arbitrary chordal graph. Let $\Gamma := \Gamma(G)$ be the clique complex of $G$ and $f(\Gamma) := (f_{-1}(\Gamma), \ldots, f_{d-1}(\Gamma))$ be the $f$-vector of the complex $\Gamma$. Here $d = \dim(\Gamma)$. Then

$$\sum_{j=1}^{i+1} (-1)^j f_j \binom{f_0 - (j + 1)}{i - j + 1} + (i + 1) \binom{f_0}{i + 2} \geq \binom{p}{i} \quad (3)$$

for each $0 \leq i \leq p$, where $p := \pdim(R/I(\overline{G}))$ and $\overline{G}$ is the complement of the graph $G$. 
In Section 2 we collected some basic results about simplicial complices, free resolutions, Hilbert fuctions and Hilbert series. We present our main results in Section 3. We prove our main results in Section 4.3.

2 Preliminaries

2.1 Simplicial complices and Stanley–Reisner rings

We say that $\Delta \subseteq 2^{[n]}$ is a simplicial complex on the vertex set $[n] = \{1, 2, \ldots, n\}$, if $\Delta$ is a set of subsets of $[n]$ such that $\Delta$ is a down–set, that is, $G \in \Delta$ and $F \subseteq G$ implies that $F \in \Delta$, and $\{i\} \in \Delta$ for all $i$.

The elements of $\Delta$ are called faces and the dimension of a face is one less than its cardinality. An $r$-face is an abbreviation for an $r$-dimensional face. The dimension of $\Delta$ is the dimension of a maximal face. We use the notation $\dim(\Delta)$ for the dimension of $\Delta$.

If $\dim(\Delta) = d - 1$, then the $(d + 1)$–tuple $(f_{-1}(\Delta), \ldots, f_{d-1}(\Delta))$ is called the $f$-vector of $\Delta$, where $f_i(\Delta)$ denotes the number of $i$–dimensional faces of $\Delta$.

Let $\Delta$ be an arbitrary simplicial complex on $[n]$. The Stanley–Reisner ring $R/I_\Delta$ of $\Delta$ is the quotient of the ring $R$ by the Stanley–Reisner ideal $I_\Delta := \langle x^F : F \notin \Delta \rangle$,

generated by the non–faces of $\Delta$.

Let $H = ([n], E(H))$ be a simple hypergraph and consider its edge ideal $I(H) \subseteq R$. It is easy to verify that $R/I(H)$ is precisely the Stanley–Reisner ring of the simplicial complex

$$\Delta(H) := \{ F \subseteq [n] : E \not\subseteq F, \text{ for all } E \in E(H) \}.$$

This complex is called the independence complex of $H$. By definition the edges of $H$ are precisely the minimal non–faces of $\Delta(H)$.

Consider the complementary hypergraph $\overline{H}$ of a $d$-uniform hypergraph. This is defined as the hypergraph $(V(H), E(\overline{H}))$ with the edge set

$$E(\overline{H}) := \{ F \subseteq X : |F| = d, \ F \notin E(H) \}.$$

Then the edges of $\overline{H}$ are precisely the $(d - 1)$-dimensional faces of the independence complex $\Delta(H)$.
Specially, let $H = ([n], E(H))$ be a simple graph and consider its edge ideal $I(H) \subseteq R$. Then
\[
\Delta(H) := \{ F \subseteq [n] : \text{ $F$ is an independent set in $H$} \}.
\]
is the independence complex of $H$. Clearly the edges of $H$ are precisely the minimal non–faces of $\Delta(H)$.

Similarly we can define the clique complex of $H$:
\[
\Gamma(H) := \{ F \subseteq [n] : \text{ $F$ is a clique in $H$} \}.
\]

## 2.2 Free resolutions
Recall that for every finitely generated graded module $M$ over $R$ we can associate to $M$ a minimal graded free resolution
\[
0 \longrightarrow \bigoplus_{i=1}^{\beta_p} R(-d_{p,i}) \longrightarrow \bigoplus_{i=1}^{\beta_{p-1}} R(-d_{p-1,i}) \longrightarrow \ldots \longrightarrow \bigoplus_{i=1}^{\beta_0} R(-d_{0,i}) \longrightarrow M \longrightarrow 0,
\]
where $p \leq n$ and $R(-j)$ is the free $R$-module obtained by shifting the degrees of $R$ by $j$.

Here the natural number $\beta_k$ is the $k$’th total Betti number of $M$ and $p$ is the projective dimension of $M$.

The module $M$ has a pure resolution if there are constants $d_0 < \ldots < d_g$ such that
\[
d_{0,i} = d_0, \ldots, d_{g,i} = d_g
\]
for all $i$. If in addition
\[
d_i = d_0 + i,
\]
for all $1 \leq i \leq p$, then we call the minimal free resolution to be $d_0$–linear.

In [19] Theorem 2.7 the following bound for the Betti numbers was proved.

**Theorem 2.1**. Let $M$ be an $R$–module having a pure resolution of type $(d_0, \ldots, d_p)$ and Betti numbers $\beta_0, \ldots, \beta_p$, where $p$ is the projective dimension of $M$. Then
\[
\beta_i \geq \binom{p}{i}
\]
for each $0 \leq i \leq p$.  

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2.3 Hilbert function

Finally let us recall some basic facts about Hilbert functions and Hilbert series.

Let \( M = \bigoplus_{i \geq 0} M_i \) be a finitely generated nonnegatively graded module over the polynomial ring \( R \). Define the Hilbert function \( h_M : \mathbb{Z} \to \mathbb{Z} \) by
\[
h_M(i) := \dim \mathbb{Q} M_i.
\]

If we know the \( f \)-vector of the simplicial complex \( \Delta \), then we can compute easily the Hilbert function \( h_{\mathbb{Q}[\Delta]}(t) \) of the Stanley–Reisner ring \( M := \mathbb{Q}[\Delta] \).

**Lemma 2.2** (Stanley, see Theorem 5.1.7 in [1]) The Hilbert function of the Stanley–Reisner ring \( \mathbb{Q}[\Delta] \) of a \((d-1)\)-dimensional simplicial complex \( \Delta \) is
\[
h_{\mathbb{Q}[\Delta]}(t) = \sum_{j=0}^{d-1} f_j(\Delta) \binom{t - 1}{j}.
\]

In the proof of our main results we use the following Proposition.

**Proposition 2.3** ([3, Chapter 6, Proposition 4.7]) Let \( M \) be a graded \( R \)-module with the graded free resolution
\[
0 \longrightarrow F_n \longrightarrow \ldots \longrightarrow F_1 \longrightarrow M \longrightarrow 0.
\]

If each \( F_j \) is the twisted free graded module \( F_j = \bigoplus_{k=1}^{\beta_{j,k}} R(\delta_{j,k}) \), then
\[
h_M(t) = \sum_{j=1}^{n} (-1)^j \sum_{k=1}^{\beta_{j,k}} \binom{n + \delta_{j,k} + t}{n}.
\]

Let \( \Delta \) be a simplicial complex such that the Stanley-Reisner ring \( R/I_\Delta \) has a linear free resolution. It is known that the generators of \( I_\Delta \) all have the same degree.

It follows that \( R/I_\Delta \) is a hypergraph algebra \( R/I(H) \) for some \( k \)-uniform hypergraph \( H \).

2.4 Hilbert–Serre Theorem

Let \( M = \bigoplus_{i \geq 0} M_i \) be a finitely generated nonnegatively graded module over the polynomial ring \( R \). We call the formal power series
\[
H_M(z) := \sum_{i=0}^{\infty} h_M(i) z^i
\]
the Hilbert–series of the module $M$.

The Theorem of Hilbert–Serre states that there exists a (unique) polynomial $P_M(z) \in \mathbb{Q}[z]$, the so-called Hilbert polynomial of $M$, such that $h_M(i) = P_M(i)$ for each $i \gg 0$. Moreover, $P_M$ has degree $\dim M - 1$ and $(\dim M - 1)!$ times the leading coefficient of $P_M$ is the multiplicity of $M$, denoted by $e(M)$.

Thus, there exist integers $m_0, \ldots, m_{d-1}$ such that $h_M(z) = m_0 \cdot \binom{z}{d-1} + m_1 \cdot \binom{z}{d-2} + \ldots + m_{d-1}$, where $\binom{z}{r} = \frac{1}{r!}z(z-1)\ldots(z-r+1)$ and $d := \dim M$. Clearly $m_0 = e(M)$.

We can summarize the Hilbert-Serre theorem as follows:

**Theorem 2.4** (Hilbert–Serre) Let $M$ be a finitely generated nonnegatively graded $R$–module of dimension $d$, then the following statements hold:

(a) There exists a (unique) polynomial $P(z) \in \mathbb{Z}[z]$ such that the Hilbert–series $H_M(z)$ of $M$ may be written as

$$H_M(z) = \frac{P(z)}{(1 - z)^d}$$

(b) $d$ is the least integer for which $(1 - z)^d H_M(z)$ is a polynomial.

## 3 The computation of the Betti–vector from the $f$-vector

### 3.1 Our main result

In our main result we describe explicitly the Betti numbers of the edge ideals $I(G)$ of uniform hypergraphs $G$ such that $I(G)$ has linear free resolution.

**Theorem 3.1** Let $G \subseteq \binom{[n]}{m}$ be an $m$–uniform hypergraph. Suppose that the edge ideal $I(G)$ has an $m$-linear free resolution

$$\mathcal{F}_G : 0 \rightarrow R(-m - g)^{\beta_g} \rightarrow \ldots \rightarrow$$

$$\rightarrow R(-m - 1)^{\beta_1} \rightarrow R(-m)^{\beta_0} \rightarrow I(G) \rightarrow 0. \quad (8)$$

If $\Delta := \Delta(G)$ is the independence complex of $G$ and $f(\Delta) := (f_{-1}(\Delta), \ldots, f_{d-1}(\Delta))$ is the $f$-vector of the complex $\Delta$, then
\[
\beta_i(G) = \sum_{j=1}^{i+1} (-1)^j f_{j+m-2}(\Delta) \binom{f_0(\Delta) - (j + 1)}{i - j + 1} + \binom{i + m - 1}{m - 1} \binom{f_0(\Delta)}{i + m}
\]
(10) for each \(0 \leq i \leq g\).

Remark. J. Herzog and M. Kühl proved similar formulas for the Betti number in [15]. Theorem 1. Here we did not assume that the ideal \(I(G)\) with linear resolution is Cohen–Macaulay.

Proof. Let \(M := R/I(G)\) denote the quotient module of the edge ideal \(I(G)\). Clearly \(R/I(G)\) is the Stanley–Reisner ring of the incidence complex \(\Delta(G)\).

First we compute the Hilbert function \(h_M(t)\) of the quotient module \(M\) from the graded free resolution of \(I(G)\).

From Proposition 2.3 we conclude that the Hilbert function \(h_M(t)\) of \(M\) is
\[
h_M(t) = \binom{t + n}{n} + \sum_{i=0}^{g} (-1)^{i+1} \beta_i(G) \binom{t + n - m - i}{n}.
\]
(11)

From the Vandermonde identities (see e.g. [10], 169–170)
\[
\binom{t + n}{n} = \sum_{j=0}^{n} \binom{n}{j} \binom{t}{j}
\]
and
\[
\binom{t + n - m - i}{n} = \sum_{j=0}^{n} \binom{t}{j} \binom{n - m - i}{n - j}
\]
for each \(i \geq 0\), we infer that
\[
h_M(t) = \sum_{j=0}^{n} \binom{n}{j} \binom{t}{j} + \sum_{i=0}^{g} (-1)^{i+1} \beta_i(G) \left( \sum_{j=0}^{n} \binom{t}{j} \binom{n - m - i}{n - j} \right) =
\]
\[
= \sum_{j=0}^{n} \binom{n}{j} \binom{t}{j} + \sum_{j=0}^{n} \binom{t}{j} \left( \sum_{i=0}^{g} (-1)^{i+1} \beta_i(G) \binom{n - m - i}{n - j} \right)
\]

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\[ = \sum_{j=0}^{n} \binom{t}{j} \binom{n}{j} + \sum_{i=0}^{g} (-1)^{i+1} \binom{n-m-i}{n-j} \beta_i(G) \]  

(12)

On the other hand we can apply Lemma 2.2 for the simplicial complex \( \Delta \). We get

\[ h_M(t) = \sum_{j=0}^{n} f_{j-1}(\Delta) \binom{t}{j}. \]

(13)

But the polynomials \( \{ \binom{t}{j} : j \in \mathbb{N} \} \) constitute a basis of the polynomial ring \( \mathbb{Q}[t] \).

Hence equations (12) and (13) imply that

\[ f_{j-1}(\Delta) = \binom{n}{j} + \sum_{i=0}^{j-m} (-1)^{i+1} \binom{n-m-i}{n-j} \beta_i(G) \]

(14)

for each \( 0 \leq j \leq n \).

Now we can prove equation (10) by induction.

It is clear that

\[ \beta_0(G) + f_{m-1}(\Delta) = \binom{n}{m}. \]

Hence we settled the case \( i = 0 \).

Suppose that equation (10) is true for each \( 0 \leq i \leq j-m-1 \). Now we prove equation (10) for \( j-m \).

It follows from equation (14) that

\[ (-1)^{j-m} \beta_{j-m}(G) = \sum_{i=0}^{j-m-1} (-1)^{i+1} \binom{n-m-i}{n-j} \beta_i(G) + \binom{n}{j} - f_{j-1}(\Delta). \]

(15)

Hence substituting equation (10) for \( \beta_i \), where \( 0 \leq i \leq j-m-1 \), and rearranging the terms yields to equation (10) for \( j-m \). \( \square \)

In the proof of Theorem 1.3 we need for the following Corollary.

**Corollary 3.2** Let \( G \subseteq [n]_2 \) be a 2–uniform hypergraph. Suppose that the edge ideal \( I(G) \) has an 2-linear free resolution

\[ \mathcal{F}_G : 0 \rightarrow S(-2-g)^{\beta_g} \rightarrow \ldots \rightarrow S(-3)^{\beta_3} \rightarrow S(2)^{\beta_0} \rightarrow I(G) \rightarrow 0. \]

(16)
If $\Delta := \Delta(G)$ is the independence complex of $G$ and $f(\Delta) := (f_{-1}(\Delta), \ldots, f_{d-1}(\Delta))$ is the $f$-vector of the complex $\Delta$, then

$$
\beta_i(G) = \sum_{j=1}^{i+1} (-1)^j f_j(\Delta) \binom{f_0(\Delta) - (j + 1)}{i - j + 1} + (i + 1) \binom{f_0(\Delta)}{i + 2} \quad (17)
$$

for each $0 \leq i \leq g$.

### 3.2 Examples

We give here two applications of Corollary 3.2.

S. Jacques proved in [16] that the total $i$'th Betti numbers of the complete graph $K_n$ with $n$ vertices are

$$
\beta_i = (i + 1) \binom{n}{i + 2}
$$

for each $0 \leq i \leq n - 2$. This is clear from Corollary 3.2 because then $\tilde{G} = ([n], \emptyset)$ and the graph $\tilde{G}$ is chordal.

Now consider the computation of the total Betti numbers of the complete bipartite graphs $K_{n,m}$. Clearly $\overline{K_{n,m}}$ is a chordal graph, hence it follows from Theorem 1.1 that the edge ideal $I$ has a linear free resolution.

Define the ideal

$$I := I(K_{n,m}) = \langle x_i y_j : 1 \leq i \leq n, 1 \leq j \leq m \rangle.$$

It is easy to verify that the incidence complex $\Delta(K_{n,m})$ is the disjoint union of two simplices, one of dimension $n - 1$, the other of dimension $m - 1$.

Hence we get that

$$f_i(\Delta(K_{n,m})) = \binom{n}{i + 1} + \binom{m}{i + 1}$$

for each $i \geq 0$.

Finally it follows from [16, Corollary 5.2.5] and Corollary 3.2 that

$$\beta_i(K_{n,m}) = \sum_{j+l=i+2, j,l \geq 1} \binom{n}{j} \binom{m}{l} = \sum_{j=1}^{i+1} (-1)^j \left( \binom{n}{j + 1} + \binom{m}{j + 1} \right) \binom{n + m - j - 1}{i - j + 1} + (i + 1) \binom{n + m}{i + 2}.\]
4 The proof of our main result

4.1 A generalization of Herzog–Kühl Theorem

We need for the following easy Lemma:

Lemma 4.1 Let $K(z) = \sum_{i=0}^{p} c_i z^{d_i} \in \mathbb{Q}[z]$ be an arbitrary polynomial over $\mathbb{Q}$. Then $K$ is divisible by $(1 - z)^m$ iff $K^{(j)}(1) = 0$ for each $j = 0, \ldots, m - 1$.

We can prove Theorem 1.3 with the following generalization of the famous Herzog–Kühl Theorem (Theorem 1 in [15]). We can prove this Theorem using the same method as in [15], but for the reader’s convenience we include here the proof.

Theorem 4.2 Let $M$ be an $R$–module having a pure resolution of type $(d_0, \ldots, d_p)$ and Betti numbers $\beta_0, \ldots, \beta_p$, where $p$ is the projective dimension of $M$. Let $d$ denote the dimension of the module $M$. Suppose that $d + 1 \leq n$. Then

$$\sum_{i=0}^{p} (-1)^i \beta_i = 0 \quad (18)$$

and

$$\sum_{i=0}^{p} (-1)^i \beta_i d_i (d_i - 1) \cdots (d_i - j + 1) = 0 \quad (19)$$

for each $j = 1, \ldots, n - d - 1$.

Proof.

Since the Hilbert–series is additive on short exact sequences, and since

$$H_R(z) = \frac{1}{(1 - z)^n},$$

and consequently

$$H_{R(-d)}(z) = \frac{z^d}{(1 - z)^n},$$

the pure resolution

$$0 \rightarrow \bigoplus_{k=1}^{\beta_p} R(-d_p) \rightarrow \bigoplus_{k=1}^{\beta_{p-1}} R(-d_{p-1}) \rightarrow \ldots \rightarrow \bigoplus_{k=1}^{\beta_0} R(-d_0) \rightarrow M \rightarrow 0,$$
yields
\[ H_M(z) = \sum_{i=0}^{p} (-1)^i \beta_i \frac{z^{d_i}}{(1 - z)^n}, \]  
(20)
where \( p = \text{pdim}(M) \).

Write \( d := \text{dim}M \), and let \( m := \text{codim}(M) = n - d \). It follows from the Auslander–Buchbaum formula that \( m \leq p \). We infer from the Theorem of Hilbert–Serre that we can write
\[ H_M(z) = \frac{P(z)}{(1 - z)^d}. \]  
(21)
Comparing the two expressions (20) and (21) for \( H_M \), we find
\[ (1 - z)^m P(z) = \sum_{i=0}^{p} (-1)^i \beta_i z^{d_i}. \]  
(22)
This formula shows that \((1 - z)^m\) divides \( \sum_{i=0}^{p} (-1)^i \beta_i z^{d_i} \) (in the ring \( \mathbb{Z}[x] \)). It follows from Lemma 4.1 that \((\beta_0, \ldots, \beta_p)\) solves the equation system (18), (19).

\[ \Box \]

4.2 The multiplicity of Stanley-Reisner ideals of chordal graphs

We can derive easily the following Corollary.

**Corollary 4.3** Let \( M \) be an \( R \)-module having a pure resolution of type \((d_0, \ldots, d_p)\) and Betti numbers \( \beta_0, \ldots, \beta_p \), where \( p \) is the projective dimension of \( M \). Let \( d \) denote the dimension of the module \( M \). Suppose that \( d + 1 \leq n \). Then
\[ \sum_{i=0}^{p} (-1)^i \beta_i d_i^j = 0 \]  
(23)
for each \( j = 0, \ldots, n - d - 1 \).

**Remark.** It follows easily that these equations are linearly independent.
Corollary 4.4 Let $M$ be an $R$–module having a pure resolution of type $(d_0, \ldots, d_p)$ and Betti numbers $\beta_0, \ldots, \beta_p$, where $p$ is the projective dimension of $M$. Let $d$ denote the dimension of the module $M$ and $m := \text{codim}(M) = n - d$. Suppose that $d + 1 \leq n$. Then

$$e(M) = (-1)^m \frac{p!}{m!} \sum_{i=0}^{p} (-1)^i \beta_i \left( \frac{d_i}{p} \right).$$

Proof. It comes out from the definition that

$$e(M) = ((1 - z)^d \cdot H_M(z)) |_{z=1} = P(1).$$

Hence we infer from equation (22) that

$$e(M) = P(1) = \left( \frac{(-1)^m m!}{m!} ((1 - z)^m P)^{(m)} |_{z=1} = \right)$$

$$= \left( \frac{(-1)^m m!}{m!} \sum_{i=0}^{p} (-1)^i \beta_i p! \left( \frac{d_i}{p} \right) = \right)$$

$$= (-1)^m \frac{p!}{m!} \sum_{i=0}^{p} (-1)^i \beta_i \left( \frac{d_i}{p} \right).$$

\[ \square \]

Now we can describe easily the multiplicity of the Stanley–Reisner ideals of chordal graphs.

Corollary 4.5 Let $G$ be an arbitrary chordal graph and $H := \overline{G}$ denote the complement of the graph $G$. Let $\Gamma := \Gamma(G)$ be the clique complex of $G$ and $f(\Gamma) := (f_{-1}(\Gamma), \ldots, f_{d-1}(\Gamma))$ be the $f$-vector of the complex $\Gamma$. Let $p$ be the projective dimension of $R/I(H)$. Let $d$ denote the dimension of the module $R/I(H)$ and $m := \text{codim}(R/I(H)) = n - d$. Then

$$e(R/I(H)) = (-1)^m \frac{p!}{m!} \sum_{i=0}^{p} (-1)^i \left( \sum_{j=1}^{i+1} (-1)^j f_j \left( \frac{f_0 - (j + 1)}{i - j + 1} \right) + (i + 1) \left( \frac{f_0}{i + 2} \right) \right) \left( \frac{i + 2}{p} \right)$$

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Proof. It follows from Theorem 1.1 that the module $M := R/I(H)$ has a 2-linear resolution:
\[
\mathcal{F}_H : 0 \rightarrow S(-2-p)^{\beta_p} \rightarrow \ldots \rightarrow S(-3)^{\beta_1} \rightarrow S(-2)^{\beta_0} \rightarrow R \rightarrow M \rightarrow 0. \tag{24}
\]
where $p$ is the projective dimension of $M$.

If we apply Theorem 4.4 for the module $M$, we get that
\[
e(R/I(H)) = (-1)^m \frac{p!}{m!} \sum_{i=0}^{p} (-1)^i \beta_i \left( \binom{i+2}{p} \right), \tag{25}
\]
Now using Theorem 3.2 and substituting
\[
\beta_i(H) = \sum_{j=1}^{i+1} (-1)^j f_j \left( \binom{f_0 - (j + 1)}{i - j + 1} \right) + (i + 1) \left( \binom{f_0}{i + 2} \right) \tag{27}
\]
into (25), we get our result. \hfill \Box

4.3 The proofs

Proof of Theorem 1.3 Let $H := \overline{G}$ denote the complement of the graph $G$.

Then consider the module $M := R/I(H)$. It follows from Theorem 1.1 that the module $M$ has a 2-linear resolution:
\[
\mathcal{F}_H : 0 \rightarrow S(-2-p)^{\beta_p} \rightarrow \ldots \rightarrow S(-3)^{\beta_1} \rightarrow S(-2)^{\beta_0} \rightarrow R \rightarrow M \rightarrow 0. \tag{26}
\]
where $p$ is the projective dimension of $M$.

If we apply Theorem 3.2 for the graph $F := H$, then we get that
\[
\beta_i(H) = \sum_{j=1}^{i+1} (-1)^j f_j(\Delta) \left( \binom{f_0(\Delta) - (j + 1)}{i - j + 1} \right) + (i + 1) \left( \binom{f_0(\Delta)}{i + 2} \right) \tag{27}
\]
for each $0 \leq i \leq p$.

Now we can apply Theorem 4.2. If we substitute the expressions (27) for $\beta_i(H)$ into the equation system (18), (19) and rearrange the obtained equations, we get our result.
Namely
\[
\sum_{i=0}^{p} (-1)^i \beta_i = \sum_{i=0}^{p} (-1)^i \left( \sum_{j=1}^{i+1} (-1)^j f_j \left( f_0 - j - 1 \atop i - j + 1 \right) + (i + 1) \left( f_0 \atop i + 2 \right) \right)
\]
\[
= \sum_{i=0}^{p} (-1)^i (i + 1) \left( f_0 \atop i + 2 \right) + \sum_{i=0}^{p} (-1)^i \left( \sum_{j=1}^{i+1} (-1)^j f_j \left( f_0 - (j + 1) \atop i - j + 1 \right) \right)
\]
\[
= \sum_{i=0}^{p} (-1)^i (i + 1) \left( f_0 \atop i + 2 \right) + \sum_{j=1}^{p+1} (-1)^j f_j(\Gamma) \left( \sum_{i=j}^{p} (-1)^i \left( f_0 - (j + 1) \atop i - j + 1 \right) \right)
\]
\[
= - \sum_{i=1}^{p+1} (-1)^i i f_0 \left( f_0 \atop i + 1 \right) + \sum_{j=1}^{p+1} (-1)^j + p f_j(\Gamma) \left( f_0(\Gamma) - j - 2 \right) = -1,
\]
because
\[
\sum_{i=j-1}^{p} (-1)^i \left( f_0 - (j + 1) \atop i - j + 1 \right) = (-1)^p \left( f_0 - j - 2 \atop p - j + 1 \right).
\]

Similarly
\[
\sum_{i=0}^{p} (-1)^i d_i^j \beta_i = \sum_{i=0}^{p} (-1)^i (2+i)^j \left( \sum_{j=1}^{i+1} (-1)^j f_j \left( f_0 - j - 1 \atop i - j + 1 \right) + (i + 1) \left( f_0 \atop i + 2 \right) \right)
\]
\[
= \sum_{k=1}^{p+1} (-1)^k f_k(\Gamma) \left( \sum_{i=k-1}^{p} (-1)^i (2+i)^j \left( f_0 - k - 1 \atop i - k + 1 \right) \right) +
\]
\[
+ \sum_{i=0}^{p} (-1)^i (2+i)^j (i + 1) \left( f_0 \atop i + 2 \right) = 0. \tag{28}
\]

**Proof of Theorem 1.4** Let \( H := \overline{G} \) denote the complement of the graph \( G \). Applying Theorem 1.1 and Theorem 3.2 for the graph \( F := H \), we get again (27). Hence we infer from Theorem 4 that
\[
\sum_{j=1}^{i+1} (-1)^j f_j \left( f_0 - (j + 1) \atop i - j + 1 \right) + (i + 1) \left( f_0 \atop i + 2 \right) \geq \binom{p}{i}. \tag{29}
\]
for each $0 \leq i \leq p$. 

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