Kaon semi-leptonic form factor in lattice QCD

Andreas Jüttner
School of Physics and Astronomy
University of Southampton
Southampton, SO17 1AJ, UK
juettner@soton.ac.uk

Abstract

This talk reviews recent lattice QCD simulations of the $K \to \pi$ semi-leptonic form factor.

1 Introduction

The unitary Cabibbo-Kobayashi-Maskawa matrix \[1, 2\] (CKM) parameterises flavour changing processes in the Standard Model (SM). We have reasons to believe that the SM will have to be extended by new physics contributions. This should also affect the flavour sector which we expect to manifest itself in terms of inconsistencies when overconstraining the parameters of the SM CKM matrix. This talk reports on the status and ongoing improvements of determinations of the $K \to \pi$ semi-leptonic form factors which are a crucial ingredient in the accurate and reliable determination of the CKM matrix element $|V_{us}|$. Due to the non-perturbative nature of the form factor the only systematically improvable way to compute it is a simulation in lattice QCD.

2 Lattice determinations of $f_+^{K\pi}(0)$

The determination of $|V_{us}|$ proceeds as follows: On the one hand, one experimentally measures the rate of a flavour changing process $s \to u$, where $s$ is the strange quark and $u$ the up quark. On the other hand one computes the SM prediction for the same process which comprises contributions from electromagnetic, weak and strong interactions. Since the ultimate goal is a test of the SM any model-dependence should be avoided and this is where progress in lattice simulations is currently being made. The inclusion of electromagnetic and strong isospin breaking corrections in a fully non-perturbative fashion in lattice QCD is still in its infancy (see Tantalo’s talk at

\[1\] Proceedings of CKM 2012, the 7th International Workshop on the CKM Unitarity Triangle, University of Cincinnati, USA, 28 September - 2 October 2012
(this conference) but these effects will likely soon be taken into account at the next
level of sophistication of computations. All results to date have been computed in
pure isospin symmetric \(m_u = m_d\) lattice QCD \cite{1, 5, 6, 7, 8, 9, 10, 11} and the
electromagnetic and isospin corrections are taken into account only at the level of the
analysis of the experimental data using chiral effective theory \cite{12} leading to

\[ |V_{us}| f_{+K\pi}^0(0) = 0.2163(5) . \]  

The remaining bit in the determination of \(|V_{us}|\) is the prediction of \(f_{+K\pi}^0(0)\) and table\textsuperscript{11} summarises current activities which are all based on simulations with either \(N_f = 2, 2 + 1, \) or \(2 + 1 + 1\) flavours of dynamical quarks. With respect to the last edition of
this workshop the number of collaborations working on \(f_{+K\pi}^0(0)\) has increased. This is
a positive development in particular since the various efforts differ in their approach
and therefore systematic effects will likely contribute differently to individual results.

JLQCD, RBC/UKQCD and ETM calculate the form factor in terms of the QCD
matrix element

\[ \langle \pi(p_\pi) | V_\mu | K(p_K) \rangle = f_{+K\pi}^0(q^2)(p_K + p_\pi)_\mu + f_{-K\pi}^0(q^2)(p_K - p_\pi)_\mu , \]  

of the vector current where \(q^2 = (p_K - p_\pi)^2\) is the momentum transfer. MILC instead
compute the form factor from the scalar current matrix element,

\[ \langle \pi(p_\pi) | S | K(p_K) \rangle |_{q^2=0} = f_{0K\pi}^0(0) \frac{m_K^2 - m_\pi^2}{m_s - m_q} . \]  

Note that \(f_{+K\pi}^0(0) = f_{0K\pi}^0(0)\). The form factor as extracted from the vector matrix
element fulfils \(f_{+K\pi}^0(0) = 1\) in the SU(3)-symmetric limit also at finite lattice spacing
which has the benefit of a symmetry-based suppression of cut-off effects. In eq. \textsuperscript{3}
this symmetry is not manifest at finite lattice spacing and one expects larger cut-off
effects. In any case cut-off effects can be parameterised allowing to extrapolate
observables to the continuum limit, provided that results at different lattice spacings
have been generated.

In view of the determination of \(|V_{us}|\) lattice QCD has to provide the form factors
at the kinematical point \(q^2 = 0\). In the finite lattice box hadron momenta assume
 discrete values corresponding to Fourier modes as determined by the choice of boundary
conditions. The kinematical point \(q^2\) is therefore naively (i.e. with periodic bound-
ary conditions) not accessible. Partially twisted boundary conditions however allow
to deal with this problem \cite{18, 19, 20, 4} and by now all groups have adopted this
 technique.

Most current lattice simulations (except for MILC who have first simulation results
with physical valence quark masses for \(f_{+K\pi}^0(0)\)) still simulate unphysically heavy \(u\-
and \(d\)-quarks and results for \(f_{+K\pi}^0(0)\) have to be extrapolated to the physical point
 guided by chiral perturbation theory where the expansion of the form factor is around
Collaboration | Published | $N_f$ | fermion formulation | Lightest pion mass | Technique | $q^2$-dependence | $a$-dependence
---|---|---|---|---|---|---|---
MILC | 2+1+1 | stag. | 135 | S | twbc | ✓ |
MILC | 2+1 | stag. | 270 | S | twbc | ✓ |
JLQCD | 2+1 | ovl. | 290 | V | twbc+interpol. | |
RBC/UKQCD | ✓ | 2+1 | DWF | 170 | V | twbc | ✓ |
ETM | ✓ | 2 | TM | 260 | V | twbc+interpol | ✓ |

Fermion formulations:
- stag. = staggered
- ovl. = overlap
- DWF = Domain Wall Fermions
- TM = twisted mass

Technique:
- $V$ = direct computation of vector current
- $S$ = computation via Ward identity

$q^2$-dependence:
- twbc = twisted boundary conditions
- interpol. = interpolation in momentum transfer to $q^2 = 0$

Table 1: Summary of current efforts for computing $f_{K\pi}^+(0)$ in lattice QCD.

the SU(3)-symmetric limit, $f_{K\pi}^+(0) = 1 + f_2 + f_4 + \ldots$ (also a small mistuning of the $s$-quark mass can in this way be corrected). In this expression $f_2$ is the NLO-term with the decay constant as the only unknown $^{21,5}$ and $f_4$ is the NNLO-term $^{16}$.

The l.h.s. panel in figure 1 shows a typical error budget for $f_{K\pi}^+(0)$ as determined from eq. $^{21}$ which underlines the importance of simulations at the physical point - the chiral extrapolation is by far the dominant source of systematic uncertainty.

### 3 Conclusions

The two largest systematic uncertainties in lattice computations of the $K \to \pi$ semi-leptonic form factor $f_{K\pi}^+(0)$ are due to the interpolation of lattice data in the momentum transfer to $q^2 = 0$ and due to the extrapolation of lattice data in the quark mass. The former has now been removed in all simulations through the use of partially twisted boundary conditions which allow for simulations directly at $q^2 = 0$. The latter is about to be removed by simulating very close to or at the physical point. At the level of precision now reached in these computations it will become important to incorporate also electro-magnetic and isospin effects into the simulation (Nazario Tantalo’s talk at this conference) eventually replacing the dependence on effective theory calculations.

**Acknowledgements:** The research leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement 279757.
| source         | $\delta f_+^{K\pi}(0)$ |
|---------------|----------------------|
| statistical   | 0.3%                 |
| chiral extrapolation | 0.4%             |
| cont. extrapolation  | 0.1%           |
| total         | 0.5%                 |

Figure 1: Error budgets for state-of-the-art lattice computations for $f_+^{K\pi}(0)$ (left, RBC+UKQCD [6, 5]) and summary of current lattice and phenomenological results with the colour coding as in the FLAG report [3].

References

[1] N. Cabibbo, *Unitary Symmetry and Leptonic Decays*, Phys.Rev.Lett. **10** (1963) 531–533.

[2] M. Kobayashi and T. Maskawa, *CP Violation in the Renormalizable Theory of Weak Interaction*, Prog.Theor.Phys. **49** (1973) 652–657.

[3] G. Colangelo, S. Dürr, A. Jüttner, L. Lellouch, H. Leutwyler, et. al., *Review of lattice results concerning low energy particle physics*, Eur.Phys.J. **C71** (2011) 1695, [1011.4408].

[4] JLQCD Collaboration Collaboration, [JLQCD 11], T. Kaneko, et. al., *Kaon semileptonic form factors in QCD with exact chiral symmetry*, PoS LATTICE2011 (2011) 284, [1112.5259].

[5] [RBC/UKQCD 10], P. A. Boyle, et. al., *$K \rightarrow \pi$ form factors with reduced model dependence*, Eur. Phys. J. C**69** (2010) 159–167, [1004.0886].

[6] [RBC/UKQCD 07], P. A. Boyle, et. al., *$K_{i3}$ semileptonic form factor from 2+1 flavour lattice QCD*, Phys. Rev. Lett. **100** (2008) 141601, [0710.5136].

[7] [ETM 10D], V. Lubicz, F. Meschia, L. Orifici, S. Simula, and C. Tarantino, *Improved analysis of the scalar and vector form factors of kaon semileptonic decays with $N_f = 2$ twisted-mass fermions*, PoS LAT2010 (2010) 316, [1012.3573].
[8] [ETM 09A], V. Lubicz, F. Mescia, S. Simula, and C. Tarantino, $K \to \pi \ell \nu$ semileptonic form factors from two-flavor lattice QCD, Phys. Rev. D80 (2009) 111502, [0906.4728].

[9] [QCDSF 07], D. Brömmel, et. al., Kaon semileptonic decay form factors from $N_f = 2$ non-perturbatively $O(a)$-improved Wilson fermions, PoS LAT2007 (2007) 364, [0710.2100].

[10] [RBC 06], C. Dawson, T. Izubuchi, T. Kaneko, S. Sasaki, and A. Soni, Vector form factor in $K_{l3}$ semileptonic decay with two flavors of dynamical domain-wall quarks, Phys. Rev. D74 (2006) 114502, [hep-ph/0607162].

[11] [JLQCD 05], N. Tsutsui, et. al., Kaon semileptonic decay form factors in two-flavor QCD, PoS LAT2005 (2006) 357, [hep-lat/0510068].

[12] M. Antonelli, V. Cirigliano, G. Isidori, F. Mescia, M. Moulson, et. al., An Evaluation of $|V_{us}|$ and precise tests of the Standard Model from world data on leptonic and semileptonic kaon decays, Eur.Phys.J. C69 (2010) 399–424, [1005.2323].

[13] [Kastner 08], A. Kastner, and H. Neufeld, The $K_{l3}$ scalar form factors in the Standard Model, Eur. Phys. J. C57 (2008) 541–556, [0805.2222].

[14] [Cirigliano 05], V. Cirigliano, et. al., The Green function and SU(3) breaking in $K_{l3}$ decays, JHEP 04 (2005) 006, [hep-ph/0503108].

[15] [Jamin 04], M. Jamin, J. A. Oller, and A. Pich, Order $p^6$ chiral couplings from the scalar $K\pi$ form factor, JHEP 02 (2004) 047, [hep-ph/0401080].

[16] [Bijnens 03], J. Bijnens, and P. Talavera, $K_{l3}$ decays in chiral perturbation theory, Nucl. Phys. B669 (2003) 341–362, [hep-ph/0303103].

[17] [LR 84], H. Leutwyler, and M. Roos, Determination of the elements $V_{us}$ and $V_{ud}$ of the Kobayashi-Maskawa matrix, Z. Phys. C25 (1984) 91.

[18] P. F. Bedaque and J.-W. Chen, Twisted valence quarks and hadron interactions on the lattice, Phys.Lett. B616 (2005) 208–214, [hep-lat/0412023].

[19] C. Sachrajda and G. Villadoro, Twisted boundary conditions in lattice simulations, Phys.Lett. B609 (2005) 73–85, [hep-lat/0411033].

[20] P. Boyle, J. Flynn, A. Jüttner, C. Sachrajda, and J. Zanotti, Hadronic form factors in Lattice QCD at small and vanishing momentum transfer, JHEP 0705 (2007) 016, [hep-lat/0703005].

[21] J. Gasser and H. Leutwyler, Low-Energy Expansion of Meson Form-Factors, Nucl.Phys. B250 (1985) 517–538.