Abstract

In [15] M. E. Rudin proved (under CH) that for each P-point \( u \) there is a P-point \( v \) such that \( v >_{RK} u \). In [1] A. Blass improved that theorem assuming \( \text{MA}^1 \) in the place of \( \text{CH} \), in that paper he also proved that under \( \text{MA}^1 \) each RK-increasing sequence of P-points is upper bounded by a P-point. We improve Blass results simultaneously in 3 directions - we prove it for each class of index \( \geq 2 \) of P-hierarchy (P-points coincidence with a class \( P_2 \) of P-hierarchy), assuming \( b = \mathfrak{c} \) in the place of \( \text{MA} \) and we show that there are at least \( b \) many Rudin-Kisler incomparable such upper bounds.

1 Introduction

We proved in [17] that a class of P-points is precisely a class \( P_2 \) of P-hierarchy which is a classification of ultrafilters on \( \omega \) into \( \omega_1 \) disjoint classes. It is natural to ask which properties of the class of P-points are (or are not) also properties of other classes of P-hierarchy. We have started this work in earlier papers [17] and [18] where also Rudin-Kisler ordering was examined. Here, inspired by papers of M. E. Rudin and A. Blass we continue our investigation. The P-hierarchy is defined by monotone sequential contours, and since this ideas are not widely known here we recall all necessary informations.

In [5] S. Dolecki and F. Mynard introduced monotone sequential cascades - special kind of trees - as a tool to describe topological sequential spaces. Cascades and their contours appeared to be also an useful tool to investigate certain types of ultrafilters on \( \omega \), namely ordinal ultrafilters and the P-hierarchy (see [17], [18]).

1) The theorem was stated under \( \text{MA} \), but in fact the Blass proof works also under \( p = \mathfrak{c} \), which was mentioned in [1].

Key words: P-hierarchy, P-points, Rudin-Kisler ordering; 2010 MSC: 03E05; 03E17
The cascade is a tree $V$, ordered by "$\subseteq$", without infinite branches and with the minimal element $\emptyset_V$. A cascade is sequential if for each non-maximal element of $V$ ($v \in V \setminus \max V$) the set $v^+V$ of immediate successors of $v$ (in $V$) is countably infinite. We write $v^+$ instead of $v^{+W}$ if it is known in which cascade the successors of $v$ are considered. If $v \in V \setminus \max V$, then the set $v^+$ (if infinite) may be endowed with an order of the type $\omega$, and then by $(v_n)_{n<\omega}$ we denote the sequence of elements of $v^+$, and by $v^{(n)W}$ the $n$-th element of $v^{+W}$.

The rank of $v \in V$ ($r_V(v)$ or $r(v)$) is defined inductively as follows: $r(v) = 0$ if $v \in \max V$, and otherwise $r(v)$ is the least ordinal greater than the ranks of all immediate successors of $v$. The rank $r(V)$ of the cascade $V$ is, by definition, the rank of $\emptyset_V$. If it is possible to order all sets $v^+$ (for $v \in V \setminus \max V$) so that for each $v \in V \setminus \max V$ the sequence $(r(v^{(n)}))_{n<\omega}$ is non-decreasing (in other words if for each $v \in V \setminus \emptyset_V$ the set $\{v \in (w)^+: r(v) < \alpha\}$ is finite for each $\alpha < r(w)$), then the cascade $V$ is monotone, and we fix such an order on $V$ without indication.

For $v \in V$ by $v^\dagger$ we understand $\{w \in V : v \subseteq w\}$ with preserved order, if $V$ is a monotone sequential cascade and $U \# \int V$ then by $V^{\#V}$ we understand the biggest monotone sub-cascade of cascade $V$ such that for each element $w \in V^{\#U}$ we have $\max (w^\dagger) \in U$.

Let $W$ be a cascade, and let $\{V_w : w \in \max W\}$ be a set of pairwise disjoint cascades such that $V_w \cap W = \emptyset$ for all $w \in \max W$. Then, the confluence of cascades $V_w$ with respect to the cascade $W$ (we write $W \leftrightarrow V_w$) is defined as a cascade constructed by an identification of $w \in \max W$ with $\emptyset_{V_w}$ and according to the following rules: $\emptyset_W = \emptyset_{W \leftrightarrow V_w}$; if $w \in W \setminus \max W$, then $w^{+W \leftrightarrow V_w} = w^{+W}$; if $w \in V_{w_0}$ (for a certain $w_0 \in \max W$), then $w^{+W \leftrightarrow V_w} = w^{+V_{w_0}}$; in each case we also assume that the order on the set of successors remains unchanged. By $(n) \leftrightarrow V_n$ we denote $W \leftrightarrow V_w$ if $W$ is a sequential cascade of rank 1.

If $U = \{u_s : s \in S\}$ is a family of filters on $X$ and if $P$ is a filter on $S$, then the contour of $\{u_s\}$ along $P$ is defined by

$$\int_P U = \int_P u_s = \bigcup_{P \subseteq P} \bigcap_{s \in P} u_s.$$  

Such a construction has been used by many authors ([3], [5], [9]) and is also known as a sum (or as a limit) of filters.

For the sequential cascade $V$ we define the contour of $V$ (we write $\int V$) inductively: if $r(V) = 1$ then $\int V$ is a co-finite filter on $\max (V)$, if $W = V \leftrightarrow V_w$ then $\int W = \int_V \int V_w$. Similar filters were considered in [10], [11], [3]. Let $V$ be a monotone sequential cascade and let $u = \int V$. Then the
rank \( r(\int V) \) of \( \int V \) is, by definition, the rank of \( V \). It was shown in \([6]\), that for each countable ordinal \( \alpha \geq 1 \), there is a monotone sequential contour of rank \( \alpha \). It was shown in \([6]\) that if \( \int V = \int W \), then \( r(V) = r(W) \). The reader may find more information about monotone sequential cascades and their contours in \([4]\), \([5]\), \([6]\), \([10]\), \([11]\), \([18]\), \([17]\).

We say that an ultrafilter \( u \) belongs to a class \( P_\alpha \) (we write \( u \in P_\alpha \)) if

1) for each \( \beta < \alpha \) there is a monotone sequential contour of rank \( \beta \) contained in \( u \)

2) there is no monotone sequential contour of rank \( \alpha \) contained in \( u \).

Although this paper is self-contained we suggest to look at \([18]\), \([17]\) for more information concerning \( P \)-hierarchy.

IMPORTANT: In the remainder of this paper each filter is considered to be on \( \omega \), unless indicated otherwise, and for \( f, g : \omega \to \omega \) we say that \( f \) dominates \( g \) if \( f(n) > g(n) \) for all(!) \( n < \omega \); this understanding does not change a domination number \( \mathfrak{d} \).

## 2 Results

Let \( V \) be a monotone sequential cascade of rank \( \geq 2 \). If from \( V \) we remove all branches of height 1 obtaining a cascade \( W \) then \( \int V = \int W \). Thus we assume that each cascade of rank \( \geq 2 \) has no branches of height 1.

Let \( V \) be a sequential cascade, we classically identify elements of the cascade with finite sequence of naturals by a function \( f : V \to \omega^{< \omega} \) as follows:

\[
f(\emptyset) = \emptyset; f(w) = f(v)^- n \text{ if } w \text{ is the } n\text{-th element of } v^+.
\]

As a convention, we identify \( v \) with \( f(v) \), and see the cascade as a subset of \( \omega^{< \omega} \).

A sequential cascade \( V \) is absorbing if it fulfills the following condition: if \( (a_1, ..., a_n) \) belongs to \( V \) and \( b_i \geq a_i \) for each \( i \in \{1, ..., n\} \) then \( (b_1, ..., b_n) \) \( \in V \). Note that each absorbing cascade is monotone. A contour of the absorbing cascade is called an absorbing contour.

**Remark 2.1.** For each monotone sequential cascade \( V \) of rank less then, or equal to \( \omega \) there is an absorbing cascade \( W \) such that \( \int V = \int W \). (It was proved in \([2]\) that then \( r(V) = r(W) \)).

**Proof.** For \( V \) of finite rank, it suffices to remove all branches of height less then \( r(V) \) and re-enumerate branches. Take a monotone sequential cascade \( V \) of rank \( \omega \), let \( V = (n) \to V_n \) and in each \( V_n \) remove all branches of height less then \( r(V_n) \) to obtain the cascade we looking for (after re-enumerating of branches). \( \blacksquare \)
We do not know whether we can extend Remark 2.1 to cascades of higher ranks \(^1\), but we have a little weaker Theorem 2.5 for them; first we need a lemma where by \(-1 + \gamma\) we denote \(\gamma\) if \(\gamma\) is infinite, \(\gamma - 1\) if \(\gamma < \omega\).

**Lemma 2.2.** For each countable ordinal \(\gamma\) there is a \((-1 + \gamma + 1)\)-sequence \(((a^n_{\alpha,\gamma})_{n<\omega})_{1<\alpha\leq\gamma}\) of non decreasing \(\omega\)-sequences of ordinal numbers, such that \(\lim_{n<\omega}(a^n_{\alpha,\gamma} + 1) = \alpha\) and \(\alpha < \beta < \gamma\) implies \(a^n_{\alpha,\gamma} \leq a^n_{\beta,\gamma}\) for each natural number \(n\).

*Proof.* In the contrary. Let \(\gamma\) be the first ordinal, such that the claimed sequence does not exist.

If \(\gamma = \delta + 1\) for some \(\delta\), then for each \(n\) it suffices to take: \(a^n_{\alpha,\gamma} = a^n_{\alpha,\delta}\) for each \(\alpha \leq \delta\), and \(a^n_{\gamma,\gamma} = \delta\).

If \(\gamma\) is a limit, take an increasing sequence \((\gamma_n)\) of ordinals, such that \(\gamma_1 = 1\) and \(\lim_{n<\omega}(\gamma_n + 1) = \gamma\). We define:

\[
a^2_{\alpha,\gamma} = 1
\]

for \(\alpha\) such that \(\gamma_n < \alpha \leq \gamma_{n+1}\), for \(m \leq n\) let \(a^m_{\alpha,\gamma} = \gamma_m\)

for \(\alpha\) such that \(\gamma_n < \alpha \leq \gamma_{n+1}\), for \(m > n\) let \(a^m_{\alpha,\gamma} = \max\{\gamma_n, a^m_{\alpha,\gamma_{n+1}}\}\)

Standard check shows that the defined sequence fulfills the claim. \(\blacksquare\)

In forthcoming paper [18] we showed the following Remark 2.3 with standard proof by induction with respect to rank.

**Remark 2.3.** [18] For each monotone sequential cascade \(V\), for each ordinal \(\alpha < r(V)\) there exists a monotone sequential cascade \(W\) of rank \(\alpha\) such that \(\int W \subset \int V\).

**Lemma 2.4** (Folklore). Let \((a_n)\) and \((b_n)\) be nondecreasing sequences of ordinals such that \(a_1 = b_1\) and \(\lim_{n<\omega} a_n = \lim_{n<\omega} b_n\). Then there is a nondecreasing, finite-to-one surjection \(f: \omega \to \omega\) such that \(b_n \geq a_{f(n)}\).

*Proof.* : Put \(f(1) = 1\), now let \(f(n) = f(n-1) + 1\) if \(a_{f(n-1)+1} \leq b_n\), and \(f(n) = f(n-1)\) otherwise. \(\blacksquare\)

**Theorem 2.5.** For each monotone sequential cascade \(V\) there is an absorbing cascade \(W\) such that \(r(V) = r(W)\) and \(\int W \subset \int V\).

*Proof.* Fix \(\gamma\) and a \((-1 + \gamma + 1)\)-sequence \(((a^n_{\alpha,\gamma})_{n<\omega})_{1<\alpha\leq\gamma}\) from the Lemma 2.2. We will show a little more, i.e. that for each monotone sequential cascade \(V\) of rank \(\gamma\) there is an absorbing cascade \(W\) such that: \(r(w^+_n) = a^n_{r(V),\gamma}\) for each \(w \in W \setminus \max(W)\).

Since a set \(\{v : v \in \emptyset^+_V, r(v) < a^1_{r(V),\gamma}\}\) is finite, so without loss of generality, we can assume that \(r(V) \geq a^1_{r(V),\gamma}\) for all \(v \in \emptyset^+_V\).

\(^1\) We suppose that there is a counter-example for each \(\alpha > \omega\)
Put \( b_n = r(\emptyset^{+}_V) \) and \( a_n = a^{\gamma, \gamma}_n \) and fix a function \( f \) from the Lemma 2.4. By Remark 2.3 for each \( n < \omega \) there is a monotone sequential cascade \( T_n \) such that \( r(T_n) = a^{\gamma, \gamma}_n \) and \( \int T_n \subset \int V_n \).

Let \( K_n \) be a cascade obtained from cascades \( T_m \) for \( m \in f^{-1}(n) \) by identifying all such \( \emptyset_{T_m} \), i.e. \( \emptyset^+_{K_n} = \bigcup_{m \in f^{-1}(n)} \emptyset^+_{T_m} \), and if \( k \in K \), \( k \neq \emptyset_{K} \) then \( k \in T_m \) for some \( m \in f^{-1}(n) \) and \( k^{+} = k^{+T_m} \). Now, for each \( K_n \) we use an inductive assumption, obtaining \( W_n \), and the confluence \( (k) \mapsto K_k \) is the cascade we are looking for.

**Corollary 2.6.** An ultrafilter \( u \) contains a contour of a monotone sequential cascade of rank \( \alpha \) if and only if \( u \) contains a contour of an absorbing cascade of the rank \( \alpha \).

Let \( V \) be a sequential cascade, and let \( f, g : V \setminus \max(V) \rightarrow \omega \). We say that a function \( f \) \( V \)-dominates \( g \) (in symbols \( f \geq_V g \)) if there is \( U \in \int V \) such that \( f(v) \geq g(v) \) for each \( v \in U \). In this meaning we define the \( V \)-dominating family and analogically \( V \)-dominating number \( \emptyset_V \). Define also the set \( V(f) \) inductively: \( \emptyset_V \in V(f) \) and if \( v \in V(f) \) then \( v \setminus k \in V(f) \) if \( k \geq f(v) \). For \( U \in \int V \) we define in the analogous way \( f_U : \text{dom}(f_U) \rightarrow \omega : \emptyset_V \in \text{dom}(f_U) \), if \( v \in \text{dom}(f_U) \) then \( f(v) = \min \{ n < \omega : \forall m \geq n, f(v \setminus n) \# U \} \).

**Remark 2.7.** For an absorbing cascade \( V \) a family \( F \) of functions \( V \setminus \max V \rightarrow \omega \) is \( V \)-dominating if and only if \( F(f) : f \in F \) is a base of \( \int V \).

For an absorbing cascade \( V \) and for a function \( f : V \setminus \max V \rightarrow \omega \), we define inductively a partial function \( f_{\text{Shift}} : V \supset \text{dom}(f_{\text{Shift}}) \rightarrow V \) as follows:

\[
\emptyset_V \in \text{dom}(f_{\text{Shift}}) \text{ and } f_{\text{Shift}}(\emptyset_V) = \emptyset_V;
\]

if \( v \in \text{dom}(f_{\text{Shift}}) \) and \( f_{\text{Shift}}(v) \notin \max V \) then \( (v \setminus k) \in \text{dom}(f_{\text{Shift}}) \) for \( k \geq f(v) \) and \( f_{\text{Shift}}(v \setminus k) = f_{\text{Shift}}(v) \setminus (k - f(v) + 1) \).

Note that \( f_{\text{Shift}} \) is a bijection and \( \text{rng}(f_{\text{Shift}}) = V \).

If \( g \) is also a function \( V \setminus \max(V) \rightarrow \omega \), then \( f_{\text{Shift}}(g)(v) : V \rightarrow \omega \) is defined by: \( f_{\text{Shift}}(g)(v) = g(f_{\text{Shift}})^{-1}(v) \).

**Theorem 2.8.** Let \( V \) be an absorbing cascade, a family \( F \subset \omega^{V \setminus \max(V)} \) is \( V \)-dominating if and only if \( F^* = \{ f_{\text{Shift}}(f) : f \in F \} \) is a dominating family on \( V \setminus \max(V) \).

**Proof.** Let \( F \subset \omega^{V \setminus \max(V)} \) be a \( V \)-dominating family, and let \( f : V \setminus \max V \rightarrow \omega \). Take \( g : V \setminus \max(V) \rightarrow \omega \) defined as follows: \( g(a_1, \ldots, a_n) = \max \{ f((b_1, \ldots, b_n)) : b_i \leq a_i, (b_1, \ldots, b_n) \in V \} \). Since \( F \) is \( V \)-dominating, there is \( h \in F \) that \( V \)-dominates \( g \). Clearly \( h_{\text{Shift}}(h) \geq f \).
Now let $f$ be a witness that $F^*$ is not dominating on $V$, define $g$ as above and observe that $g$ is not $V$-dominating by any $f \in F$. ■

By Corollary 2.6 Remark 2.7 and Theorem 2.8 we have:

**Corollary 2.9.** For each absorbing cascade $V$ there is $d_V = d$.

**Remark 2.10** (Folklore). The minimum of cardinalities of families of non-dominating families such that the sum of all that families is a dominating family is $b$

**Proof.** For each $\alpha$, $\alpha < \lambda < b$, let $F_\alpha$ be a non-dominating family of functions $\omega \to \omega$. Let $f_\alpha$ be a function non dominated by $F_\alpha$, let $f$ be a function that dominates all $f_\alpha$ (there is some since $\lambda < b$). Clearly $f$ can not be dominated by any element of $\bigcup_{\alpha < \lambda} F$.

Let $(f_\alpha)_{\alpha < b}$ be e non-limited sequence of functions. Define $F_\alpha$ as a family of all functions $f$ such that $f$ does not dominate $\{f_\beta : \beta \leq \alpha\}$. Clearly $F_\alpha$ is non-dominating, and $\bigcup_{\alpha < b} F_\alpha = \omega^\omega$ and so is dominating. ■

Let $A \subset X$, a supersets closure of $A$ is a family $\text{SC}(A) = \bigcup A \in A \langle A \rangle$. Let $u$ be a filter on $X$, we say that a family $P \subset 2^X$ is a $\pi$-base of $u$ if $P$ has finite intersection property, and if $u \subset \text{SC}(P)$.

**Corollary 2.11.** A sum of less then $b$ families $A_\alpha$ that do not contain a $\pi$-base of absorbing contour of rank $\alpha > 1$ does not contain any $\pi$-base of monotone sequential contour of rank $\alpha$

**Proof.** Suppose on the contrary that there is a pair of witnesses - a sequence $(A_\alpha)_{\alpha < \lambda < b}$ and a monotone sequential cascade $W$, and fix a classical - defined by functions $V \setminus \max(V) \to \omega$ - base $B$ of $V$. By Theorem 2.5 or by Remark 2.1 (depending on the rank of the cascade) there is an absorbing cascade $V$ of the rank $r(V) = r(W)$ such that $\int V \subset \int W$, clearly each $\pi$-base of $\int W$ is also a $\pi$-base of $V$, so it suffices to prove this corollary for absorbing cascades. Since a family $A_\alpha$ does not contain a $\pi$-base of $\int V$ and since $A_\alpha$ is a $\pi$-base of $B_\alpha = \{B \in B : B \supset A, A \in A_\alpha\}$, thus $B_\alpha$ does not contain $\int V$, and therefore by Remark 2.7 a family $\{f_B; B \in B_\alpha\}$ is not $V$-dominating for each $\alpha < \lambda$. By Theorem 2.8 $\{f_{\text{Shift}}(f_B); B \in B_\alpha\}$ is not dominating so by Remark 2.10 a family $\bigcup_{\alpha < \lambda}\{f_{\text{Shift}}(f_B); B \in A_\alpha\}$ is not dominating, and by Theorem 2.8 $\bigcup_{\alpha < \lambda}\{f_B; B \in B_\alpha\}$ is not $V$-dominating. Therefore by Remark 2.7 a family $\bigcup_{\alpha < \lambda} B_\alpha$ does not contain a base of $\int V$, but since $\bigcup_{\alpha < \lambda} B_\alpha$ contains all supersets of elements of $\bigcup_{\alpha < \lambda} A_\alpha$ which belong to $B$ thus $\bigcup_{\alpha < \lambda} B_\alpha$ does not contain a $\pi$-base of $\int V$, and since $\bigcup_{\alpha < \lambda} A_\alpha \subset \bigcup_{\alpha < \lambda} B_\alpha$ thus $\bigcup_{\alpha < \lambda} A_\alpha$ does not contain a $\pi$-base of $\int V$. ■
Corollary 2.12. An increasing (⊂) sequence of length less than \( b \) of filters that do not contain a \( \pi \)-base of an absorbing sequential contour of rank \( \alpha > 1 \) does not contain any monotone sequential contour of rank \( \alpha \).

Remark 2.13. If \( v \) is a filter and \( T \) is a set such that \( T \# v \) then if \( v \mid T \) does not contain any monotone sequential contour of rank \( \alpha \) than \( v \) does not contain any \( \pi \)-base of any monotone sequential contour of rank \( \alpha \).

Remark 2.14. Let \( u \) be a filter which does not contain any \( \pi \)-base of any monotone sequential cascade of rank \( \alpha \), and let \( f : \omega \to \omega \), then a filter \( \{f^{-1}(U) : U \in u\} \) does not contain any \( \pi \)-base of any monotone sequential cascade of rank \( \alpha \).

Proof. For each \( n \) such that \( f^{-1}(n) \) is nonempty put \( x_n = \min(f^{-1}(n)) \) and let \( X \) be a set of all such \( x_n \)'s. It suffices to consider \( \{\{f[U] : U \in u\} \mid X \) which is a copy of \( u \).

Theorem 2.15. \( \square \) Theorem 2.5] Let \( (\alpha_n)_{n<\omega} \) be a non-decreasing sequence of ordinals less than \( \omega_1 \), let \( \alpha = \lim_{n<\omega}(\alpha_n) \), let \( 1 < \beta < \omega_1 \). If \( u_n \in \mathcal{P}_{\alpha_n} \) is a discrete sequence of ultrafilters and \( u \in \mathcal{P}_\beta \) then \( \bigcap_u u_n \in \mathcal{P}_{\alpha+(1+\beta)} \).

Remark 2.16. Let \( u \) be such a filter that there is a map \( \omega \to \omega \) that \( f(u) = \mathfrak{t} \). If \( \langle u \cup \mathcal{A} \rangle \) is an ultrafilter for some family (of sets) \( \mathcal{A} \) then card \( (\mathcal{A}) \geq u \).

Proof. Let \( f : \omega \to \omega \) be a function, such that \( f(u) = \mathfrak{t} \), if \( u \cup \mathcal{A} \) is an ultrafilter thus \( f(\langle u \cup \mathcal{A} \rangle) \) is a free ultrafilter and so \( \{f[A] : A \in \mathcal{A}\} \) is a base of ultrafilter i.e. it has a cardinality of at least \( u \).

Theorem 2.17. \( \{b = c\} \) Let \( 1 < \xi \leq \omega_1 \) and let \( p \in \mathcal{P}_\xi \), then there is \( \mathfrak{U} \subset \mathcal{P}_\alpha \) of cardinality \( b \) that \( u \succ_{\text{RK}} p \) for each \( u \in \mathfrak{U} \), and that elements of \( \mathfrak{U} \) are Rudin-Kisler incomparable. \(^1\)

Proof.

For \( \xi = \omega_1 \) the claim is obvious. Fix \( 1 < \xi < \omega_1 \). Let \( f : \omega \to \omega \) be a finite-to-one function such that \( \sup\{n : n \in U\} = \omega \) for each \( U \in p \).

Let \( \mathcal{A}_n \) be a family of such subsets of \( \omega \) that there is \( P \in p \) that card \( (f^{-1}(m)) = \text{card} (f^{-1}(m) \cap A) + n \) for each \( m \in P \).

\(^1\) We can obtain an easier version of the Theorem in a much shorter way: (P-points exist) Let \( \alpha \) be infinite countable ordinal, than for each \( u \in \mathcal{P}_\alpha \) there is \( v \in \mathcal{P}_\alpha \) such that \( v \succ_{\text{RK}} u \).

Proof. Let \( \alpha \geq \omega_1 \), and take any \( u \in \mathcal{P}_\alpha \). Consider a partition \( (\mathcal{A}_n) \) of \( \omega \) into \( \omega_1 \) infinite sets. For each \( n \) let \( u_n \) be a P-point such that \( \mathcal{A}_n \in u_n \). Put \( v = \bigcap_u u_n \), by Theorem 2.15 \( v \in \mathcal{P}_\alpha \). on the other hand for a function \( f(m) = n \) for each \( m \in \mathcal{A}_n \) we have \( f(v) = u \) and there is no set \( V \in v \) such that \( f \mid V \) is one-to-one, so it can not be Rudin-Kisler equivalent.

For \( \alpha = \omega_1 \) the claim is obvious.
Proposition 2.18. For each natural number $i$ a family $\mathbb{B}_i = \{f^{-1}[P] \cap A^c : P \in p, A \in \mathcal{A}_i\}$ does not contain a $\pi$-base of any monotone sequential contour of rank $\xi$.

On the contrary. Let $i = \min \{j < \omega : \text{there is a } \pi\text{-base monotone sequential cascade of rank } \alpha \text{ contained in } \mathbb{B}_j\}$, let $\mathbb{P}$ be a $\pi$-base of absorbing sequential contour $\int V$ of rank $\xi$, such that $\mathbb{P} \subset \mathbb{B}_j$. For each $U \in \int V$, for each $k \leq i$ let $W_k(U) = \{n \in \omega : \text{card} (f^{-1}(m)) = \text{card} (f^{-1}(m) \cap U) + k\}$, since $p$ is an ultrafilter, then for each $U \in \int V$ there is $k(U) \leq i$ such that $W_k(U) \in p$. Define also $A_k(U) = \{n < \omega : n \in f^{-1}(m) \text{ for such } m \text{ that card} (f^{-1}(m)) = \text{card} (f^{-1}(m) \cap U) + k \text{ and } n \notin U\}$. By minimality of $i$ there is $U_0 \in V$ such that $k(U_0) = i$, moreover if $U_1 \subset U_0$ then $k(U_1) = i$ and $A_i(U_1) \subset A_i(U_0)$. Thus $\int V \subset \text{SC} (\{f^{-1}[P] \cap A_i(U_0)^c : P \in p\})$ what is impossible since $\text{SC} (\{f^{-1}[P] \cap A_i(U_0)^c : P \in p\}) = \text{SC} (\{(f |_{A_i(U_0)^c})^{-1}[P] : P \in p\})$ so by Remark 2.14 it does not contain any monotone sequential cascade of rank $\xi$. □

Proposition 2.19. $\langle f^{-1}[P] \cap A^c : P \in P, A \in \mathcal{A} \rangle$ do not contain any monotone sequential contour of rank $\xi$.

By Proposition 2.18 each $\mathbb{B}_i = \{f^{-1}[P] \cap A^c : P \in p, A \in \mathcal{A}_i\}$ do not contain a $\pi$-base of any monotone sequential contour of rank $\xi$, thus by Corollary 2.11 $\bigcup_{n<\omega} \mathbb{B}_n$ does not contain a $\pi$-base of any monotone sequential contour of rank $\xi$, but $\bigcup_{n<\omega} \mathbb{B}_n = \langle f^{-1}[P] \cap A^c : P \in P, A \in \mathcal{A} \rangle$, i.e., is a filter and so since it does not contain a $\pi$-base of any monotone sequential contour of rank $\xi$, it also does not contain any monotone sequential contour of rank $\xi$. □

Clearly there is no set $\bar{A}$ that $\langle f^{-1}[P] \cap A^c \cap \bar{A} : P \in P, A \in \mathcal{A} \rangle$ is an ultrafilter thus there is a sequence of pairwise disjoint sets $(C_n)_{n<\omega}$ such that $C_n \# \langle f^{-1}[P] \cap A^c : P \in P, A \in \mathcal{A} \rangle$ and $\bigcup_{n<\omega} C_n = \omega$. Let $\mathcal{T} = \langle \{f^{-1}[P] \cap A^c : P \in P, A \in \mathcal{A} \} \cup \{\bigcup_{n>m} C_n : m < \omega\}\rangle$.

Remark 2.20. A filter $\langle \mathcal{T} \cup \bar{A}\rangle$ does not contain any monotone sequential contour of rank $\xi$ for any $\bar{A} \# \mathcal{T}$, and there is a function $f : \omega \rightarrow \omega$ that $f(\mathcal{T}) = \mathfrak{F}_\mathcal{T}$.

First part follows from Proposition 2.19, and a number of generators added to the family $\langle f^{-1}[P] \cap A^c \cap \bar{A} : P \in P, A \in \mathcal{A} \rangle$ in the virtue of Corollary 2.11. To see the second part of Remark 2.20 it suffices to consider a function $f : \omega \rightarrow \omega$ such that $f(n) = m$ if $n \in C_m$. □

We enlist all absorbing cascades of rank $\xi$ in a sequence $(V_\alpha)_{\alpha<\mathfrak{b}}$ and all functions $\omega \rightarrow \omega$ in a sequence $(f_\beta)_{\beta<\mathfrak{b}}$. We will build a family $\{(f_\alpha)^3\beta<\mathfrak{b}\}$ of increasing $\mathfrak{b}$-sequences $(\mathcal{F}_\alpha)_{\alpha<\mathfrak{b}}$ of filters such that:
1) Each $\mathcal{F}_\beta^\alpha$ is generated by $\mathcal{T}$ together with some family of cardinality $< b$ of sets;
2) $\mathcal{F}_0^\beta = \mathcal{T}$ for each $\beta < b$;
3) For each $\alpha, \beta < b$, there is $F \in \mathcal{F}_\alpha^{\beta+1}$ such that $F^c \in \bigcap V_\alpha$;
4) For limit $\alpha$ for each $\beta$, $\mathcal{F}_\alpha^\beta = \bigcup_{\gamma < \alpha} \mathcal{F}_\gamma^\beta$;
5) For each $\alpha$, for each $\gamma < \alpha$, for each $\beta_1, \beta_2 < \alpha$ there is a set $F \in \mathcal{F}_\alpha^{\beta_1+1}$, such that $\left( f_{\gamma} [F] \right)^c \in \mathcal{F}_\alpha^{\beta_2}$.

Existence of such families is a standard work by induction with respect to $\alpha$ with sub-induction with respect to $\gamma < \alpha$, with sub-sub induction with respect to $\beta_1 < \gamma$ and with sub-sub-sub-induction with respect to $\beta_2 < \beta_1$, using Remark 2.16, Remark 2.20 and Remark 2.10.

Now it suffice for each $\beta < b$ take any ultrafilter extending $\bigcup_{\beta < b} \mathcal{F}_\beta^\alpha$.

Theorem 2.21. ($b = c$) Let $1 < \xi \leq \omega_1$, and let $(p_n)_{n < \omega}$ be a RK-increasing sequence of elements of $P_\xi$, then there exists $u \in P_\xi$ such that $u >_{RK} p_n$ for each $n < \omega$.

Proof. Let $f_n$ be a function $\omega \to \omega$ - witness that $p_{n+1} >_{RK} p_n$. For each natural number $m$ consider on $\omega \times \omega$ a family of sets $B_m$ such that $B_m | (\omega \times \{n\}) = \langle \{f_{n-1} \circ f_{n-1} \circ \ldots \circ f_m^{-1}(P) : P \in p_n\} \rangle$ for $n \geq m$. Let $B = \bigcup_{n < \omega} B_m$.

Clearly $B$ is a filter, and each ultrafilter which extends $B$ is RK greater then each $p_n$. Also, by Remark 2.14, each $B_n$ does not contain any monotone sequential contour of rank $\xi$ so by Corollary 2.12, $B$ does not contain any monotone sequential contour of rank $\xi$.

We enlist all absorbing cascades of rank $\xi$ in a sequence $(V_\alpha)_{\alpha < b}$ and we will build an increasing $b$-sequence of filters $\mathcal{F}_\alpha$ such that:
1) $\mathcal{F}_0 = B$.
2) For each $\alpha$, there is such $F \in \mathcal{F}_{\alpha+1}$ that $F^c \in \bigcap V_\alpha$;
3) For a limit $\alpha$, $\mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \mathcal{F}_\beta$.

The rest of the proof is an easier version of the final part of the proof of Theorem 2.17.

Corollary 2.22. ($b = c$) Let $1 < \xi \leq \omega_1$, and let $(p_n)_{n < \omega}$ be a RK-increasing sequence of elements of $P_\xi$, then there exists a family $\mathcal{U} \subset P_\xi$ of cardinality $b$ such that $u >_{RK} p_n$ for each $u \in \mathcal{U}$, and that elements of $\mathcal{U}$ are Rudin-Kisler incomparable.

Proof. Just combine Theorem 2.21 with Theorem 2.17.

Theorem 2.23. [17, Theorem 2.8] The following statements are equivalent:
1) $P$-points exist;
2) Classes $\mathcal{P}_\alpha$ are nonempty for each countable successor $\alpha$;
3) There exists a countable successor $\alpha > 1$ such that the class $\mathcal{P}_\alpha$ is nonempty.
Theorem 2.24. \( \mathfrak{d} = \mathfrak{c} \) if, and only if, every filter generated by less than \( \mathfrak{c} \) elements can be extended to a P-point.

Theorem 2.25. \((b = c)\) Each class of \( \mathcal{P} \)-hierarchy is non-empty.

Proof. For classes of index \( \xi \in \{1, \omega_1\} \) we are done (in ZFC) by [18, Corollary 6.4]

For successor \( 1 < \xi < \omega_1 \), since \( b \leq d \), it suffice to combine Theorem 2.23 and Theorem 2.24.

For limit \( \xi < \omega_1 \) a proof is essentially the same as final part of the proof of Theorem 2.17:

We enlist all absorbing cascades in a sequence \( (V_\alpha)_{\alpha < b} \). Let \( (\int V_n) \) be an increasing \( (\subseteq) \) sequence of monotone sequential contours such that \( \lim_{n<\omega}(r(V_n) + 1) = \xi \), such sequences exist in ZFC - and were constructed in the proof of [3, Theorem 4.6]. Thus by Corollary 2.11 \( \bigcup_{n<\omega} \int V_n \) does not contain a \( \pi \)-base of monotone sequential cascade of rank \( \xi \).

We enlist all absorbing cascades of rank \( \xi \) in a sequence \( (V_\alpha)_{\alpha < b} \) and we will build an increasing \( b \)-sequence of filters \( F_\alpha \) such that:

1) \( F_0 = \bigcap_{n<\omega} V_n \).
2) For each \( \alpha \), there is \( F \in F_{\alpha+1} \) such that \( F^c \subseteq \int V_\alpha \);
3) For a limit \( \alpha \), \( F_\alpha = \bigcup_{\beta < \alpha} F_\beta \).

The rest of the proof is an easier version of the final part of the proof of Theorem 2.17. \( \blacksquare \)

Corollary 2.26. \((b = c)\) Each class of \( \mathcal{P} \)-hierarchy of index \( \geq 1 \) contains a family of cardinality \( b \) of pairwise Rudin-Kisler incomparable ultrafilters.

Proof. Just combine Theorem 2.25 with Theorem 2.17. \( \blacksquare \)

References

[1] A. Blass, Rudin - Kisler ordering on P-points, Trans. Amer. Math. Soc. 179 (1973) 145-166.
[2] A. Blass, Combinatorial Cardinal Characteristics of the Continuum; in Handbook of Set Theory; Mathew Foreman and Akihiro Kanamori editors, Springer (2010)
[3] M. Daguene, Emploi des filtres sur N dans l’étude descriptive des fonctions, Fund. Math., 95 (1977), 11-33.
[4] S. Dolecki, Multisequences, Quaestiones Mathematicae, 29 (2006), 239-277.
[5] S. Dolecki, F. Mynard, Cascades and multifilters, Topology Appl., 104 (2002), 53-65.
[6] S. Dolecki, A. Starosolski, S. Watson, Extension of multisequences and countable uniradial class of topologies, Comment. Math. Univ. Carolin., 44, 1 (2003), 165-181.
[7] Z. Frolík, Sums of ultrafilters, Bull. Amer. Math. Soc., 73 (1967), 87-91.
[8] G. Grimeisen, Gefilterte Summation von Filtern und iterierte Grenzprozesse, I, Math. Annalen, 141 (1960), 318-342.
[9] G. Grimeisen, Gefilterte Summation von Filtern und iterierte Grenzprozesse, II, Math. Annalen, 144 (1961), 386-417.
[10] M. Katětov, On descriptive classes of functions, Theory of Sets and Topology - a collection of papers in honour of Felix Hausdorff, D. V. W. (1972).
[11] M. Katětov, On descriptive classification of functions, General Topology and its Relations to Modern Analysis and Algebra II, Proc. Sympos. Prague, 1971.
[12] J. Ketonen, On the existence of P-points in the Stone-Čech compactification of integers, Fund. Math. 92 (1976), 91-94.
[13] C. Laflamme, A few special ordinal ultrafilters, J. Symb. Log. 61, 3 (1996), 920-927.
[14] M. Machura, A. Starosolski, How high can Baumgartner’s I-ultrafilters lie in the P-hierarchy ?, preprint. [http://arxiv.org/abs/1108.1818](http://arxiv.org/abs/1108.1818)
[15] M. E. Rudin, Partial orders on the types of $\beta N$, Trans. Amer. Math. Soc., 155 (1971), 353-362.
[16] A. Starosolski, Fractalness of supercontours, Top. Proc. 30, 1 (2006), 389-402.
[17] A. Starosolski, P-hierarchy on $\beta \omega$, J. Symb. Log. 73, 4 (2008), 1202-1214.
[18] A. Starosolski, Ordinal ultrafilters versus P-hierarchy, preprint

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