Bypassing the Ambient Dimension:  
Private SGD with Gradient Subspace Identification

Yingxue Zhou, Zhiwei Steven Wu, and Arindam Banerjee

Department of Computer Science & Engineering  
University of Minnesota, Twin Cities  
zhou0877@umn.edu, zsw@umn.edu, banerjee@cs.umn.edu

Abstract

Differentially private SGD (DP-SGD) is one of the most popular methods for solving differentially  
private empirical risk minimization (ERM). Due to its noisy perturbation on each gradient update, the  
error rate of DP-SGD scales with the ambient dimension $p$, the number of parameters in the model. Such  
dependence can be problematic for over-parameterized models where $p \gg n$, the number of training  
samples. Existing lower bounds on private ERM show that such dependence on $p$ is inevitable in the  
worst case. In this paper, we circumvent the dependence on the ambient dimension by leveraging a  
low-dimensional structure of gradient space in deep networks—that is, the stochastic gradients for deep  
ets usually stay in a low dimensional subspace in the training process. We propose Projected DP-SGD  
that performs noise reduction by projecting the noisy gradients to a low-dimensional subspace, which is  
given by the top gradient eigenspace on a small public dataset. We provide a general sample complexity  
analysis on the public dataset for the gradient subspace identification problem and demonstrate that  
under certain low-dimensional assumptions the public sample complexity only grows logarithmically  
in $p$. Finally, we provide a theoretical analysis and empirical evaluations to show that our method can  
substantially improve the accuracy of DP-SGD.

1 Introduction

Many fundamental machine learning tasks involve solving empirical risk minimization (ERM): given a  
loss function $\ell$, find a model $\mathbf{w} \in \mathbb{R}^p$ that minimizes the empirical risk $\hat{L}_n(\mathbf{w}) = \frac{1}{n} \sum_{i=1}^{n} \ell(\mathbf{w},z_i)$, where  
$z_1,\ldots,z_n$ are i.i.d. examples drawn from a distribution $\mathcal{P}$. In many applications, each training example may  
contain highly sensitive information about some individuals. When the models are given by deep neural  
networks, their rich representation can potentially reveal fine details of part of the private data. For example,  
[17] demonstrated that one can extract part of the training images from a facial recognition system.  

Differential privacy (DP) [12] has by now become the standard approach to provide principled and rigorous  
privacy guarantees in machine learning. Roughly speaking, DP is a stability notion that requires that no  
individual example has a significant influence on the trained model. One of the most commonly used algorithm  
for solving private ERM is the differentially-private stochastic gradient descent (DP-SGD) [17, 31]—a private  
variant of SGD that perturbs with each gradient update with random noise vector drawn from an isotropic  
Gaussian distribution $\mathcal{N}(0,\sigma^2 I_p)$, with per-coordinate variance $\sigma^2$ that depends on the chosen privacy  
parameters $\epsilon, \delta$, and sample size $n$. 

1
Due to the gradient perturbation drawn from an isotropic Gaussian distribution, the error rate of DP-SGD has a heavy dependence on the ambient dimension $p$—the number of parameters in the model. In the case of convex loss $\ell$, [7] show that DP-SGD achieves the optimal empirical excess risk of $\tilde{O}\left(\frac{p}{\sqrt{n}}\right)$. For non-convex loss $\ell$, which is more common in neural network training, minimizing $\hat{L}_n(w)$ is in general intractable. However, many (non-private) gradient-based optimization methods are shown to be effective in practice and can provably find approximate stationary points with vanishing gradient norm $\|\nabla \hat{L}_n(w)\|_2$ (see e.g. [13, 27]). Moreover, for a wide family of loss functions $\tilde{L}_n$ under the Polyak-Łojasiewicz condition [30], the minimization of gradient norm implies achieving global optimum. With privacy constraint, [36] recently showed that DP-SGD minimize the empirical gradient norm down to $\tilde{O}\left(\frac{L}{\sqrt{n}e}\right)$ when the loss function $\ell$ is smooth. Furthermore, existing lower bounds results on private ERM [7] show that such dependence on $p$ is inevitable in the worst case. However, many modern machine learning tasks now involve training extremely large models, with the number of parameters substantially larger than the number of training samples. For these large models, the error dependence on $p$ can be a barrier to practical private ERM.

In this paper, we aim to overcome such dependence on the ambient dimension $p$ by leveraging the structure of the gradient space in the training of neural networks. We take inspiration from the empirical observation from [20, 25, 29] that even though the ambient dimension of the gradients is large, the set of sample gradients at most iterations along the optimization trajectory is often contained in a much lower-dimensional subspace. While this observation has been made mostly for non-private SGD algorithm, we also provide our empirical evaluation of this structure (in terms of eigenvalues of the gradient second moments matrix) in Figure [1].

Based on this observation, we provide a modular private ERM optimization framework with two components. At each iteration $t$, the algorithm performs the following two steps:

1) **Gradient dimension reduction.** At each iteration $t$, let $w_t$ be the iterate and $g_t$ be the mini-batch gradient. In general, this subroutines solves the following problem: given any $k < p$, find a linear projection $\hat{V}_k(t) \in \mathbb{R}^{p \times k}$ such that the reconstruction error $\|g_t - \hat{V}_k(t)\hat{V}_k(t)^Tg_t\|$ is small. To implement this subroutine, we follow a long line of work that studies private data analysis with access to an auxiliary public dataset $S_{kh}$ drawn from the same distribution $P$, for which we don’t need to provide formal privacy guarantee [2, 4, 5, 16, 28]. In our case, we compute $\hat{V}_k(t)$ which is given the top $k$ eigenspaces of the gradients evaluated on $S_{kh}$. Alternatively, this subroutine can potentially be implemented through private subspace identification on the private dataset. However, to our best knowledge, all existing methods have reconstruction error scaling with $\sqrt{p}$ [13], which will be propagated to the optimization error.

2) **Projected DP-SGD (PDP-SGD).** Given the projection $\hat{V}_k(t)$, we then perturb gradient in the projected subspace: $\tilde{g}_t = \hat{V}_k(t)\hat{V}_k(t)^T(g_t + b^t)$, where $b^t$ is a $p$-dimensional Gaussian perturbation. The projection mapping provides a large reduction of the noise and enables higher accuracy for PDP-SGD.

We provide both theoretical analyses and empirical evaluations of PDP-SGD:

**Uniform convergence for projections.** A key step in our theoretical analysis is to bound the reconstruction error on the gradients from projection of $\hat{V}_k(t)$. This reduces to bound the deviation $\|\hat{V}_k(t)\hat{V}_k(t)^T - \hat{V}_k\hat{V}_k(t)^T\|_2$, where $V_k(t)$ denotes the top-$k$ eigenspace of the population second moment matrix $\mathbb{E}\nabla^2 \ell(w_t, z)\nabla^2 \ell(w_t, z)^T$.

To handle the adaptivity of the sequence of iterates, we provide a uniform deviation bound for all $w \in W$, where the set $W$ contains all possible iterates. By leveraging generic chaining techniques, we provide a deviation bound that scales linearly with a complexity measure—the $\gamma_2$ function due to Talagrand [33]—of the set $W$. Then, ignoring constants and certain other details, an informal version of the reconstruction error
bound for projection $\hat{V}_k(t)$ is as follows:

$$
\mathbb{E} \left[ \| \hat{V}_k(t)\hat{V}_k(t)^\top - V_k(t)V_k(t)^\top \|_2 \right] \leq O \left( \frac{\sqrt{\ln p} \gamma_2(W, d)}{\sqrt{m}} \right),
$$

(1)

where $\gamma_2(W, d)$ is the complexity measure of the a set $W$ considering metric $d$ (Definition 2). We provide low-complexity examples of $W$ that are supported by empirical observations and show that their $\gamma_2$ function only scales logarithmically with $p$.

**Convergence for convex and non-convex optimization.** Building on the reconstruction error bound, we provide convergence and sample complexity results for our method PDP-SGD in two types of loss functions, including 1) smooth and non-convex, 2) Lipschitz convex. For smooth and non-convex function, ignoring constants and certain other details, an informal version of the convergence rate is as follows:

$$
\mathbb{E} \left[ \| \nabla \hat{L}_n(w_R) \|_2^2 \right] \leq \tilde{O} \left( \frac{k}{n\epsilon} \right) + O \left( \frac{\gamma_2^2(W, d) \ln p}{m} \right),
$$

(2)

where $w_R$ is uniformly sampled from $\{w_1, ..., w_T\}$ and $m$ is the size of public dataset $S_h$. For Lipschitz convex function, ignoring constants and certain other details, an informal version of the convergence rate is as follows:

$$
\mathbb{E} \left[ \hat{L}_n(w) - \hat{L}_n(w^*) \right] \leq O \left( \frac{k}{n\epsilon} \right) + O \left( \frac{\gamma_2(W, d) \ln p}{\sqrt{m}} \right),
$$

(3)

where $w = \sum_{t=1}^{T} w_t$, and $w^*$ is the minima of $\hat{L}_n(w)$. Compared to the error rate of DP-SGD for convex functions [4, 7] and non-convex and smooth functions [36], PDP-SGD demonstrates an improvement over the dependence on $p$ to $k$ in the error rate. PDP-SGD also involves the subspace reconstruction error which depends on $\gamma_2$ functions and size of public dataset. As discussed above, $\gamma_2$ function only scales logarithmically with $p$ supported by empirical observations and our rates only scale logarithmically on $p$.

**Empirical evaluation.** We provide an empirical evaluation of PDP-SGD on two real datasets. In our experiments, we construct the "public" datasets by taking very small random sub-samples of these two datasets (100 and 200 respectively). While these two public datasets are not sufficient for training an accurate predictor, we demonstrate that they provide useful gradient subspace projection and substantial accuracy improvement over DP-SGD.
We analyze our method under two assumptions on the gradients of $\nabla L_n(w)$. We first state the standard definition of differential privacy which requires that no single private example has a significant influence on the algorithm’s output information.

To establish the privacy guarantee of our algorithm, we will combine three standard tools in differential privacy, including 1) the Gaussian mechanism [13] that releases an aggregate statistic (e.g., the empirical average gradient) by Gaussian perturbation, 2) privacy amplification via subsampling [23] that reduces the privacy parameters $\epsilon$ and $\delta$ by running the private computation on a random subsample, and 3) advanced composition theorem [14] that tracks the cumulative privacy loss over the course of the algorithm.

We analyze our method under two assumptions on the gradients of $\ell$.

---

1In this paper, we focus on minimizing the empirical risk. However, by relying on the generalization guarantee of $(\epsilon, \delta)$-differential privacy, one can also derive a population risk bound that matches the empirical risk bound up to a term of order $O(\epsilon + \delta)$ [6][11][22].
Algorithm 1 Projected DP-SGD (PDP-SGD)

1: \textbf{Input:} Training set $S$, public set $S_h$, certain loss $\ell(\cdot)$, initial point $w_0$
2: \textbf{Set:} Noise parameter $\sigma$, iteration time $T$, step size $\eta_t$
3: \textbf{for} $t = 0, ..., T$ \textbf{do}
4: \hspace{0.5cm} Compute top-$k$ eigenspace $\hat{V}_k(t)$ of $M_t = \frac{1}{m} \sum_{i=1}^m \nabla \ell(\mathbf{w}_t, \mathbf{z}_i) \nabla \ell(\mathbf{w}_t, \mathbf{z}_i)^\top$.
5: \hspace{0.5cm} $\mathbf{g}_t = \frac{1}{|B_t|} \sum_{z_i \in B_t} \nabla \ell(\mathbf{w}_t, \mathbf{z}_i)$, with $B_t$ uniformly sampled from $S$ with replacement.
6: \hspace{0.5cm} Project noisy gradient using $\hat{V}_k(t)$: $\tilde{\mathbf{g}}_t = \hat{V}_k(t)^\top (\mathbf{g}_t + \mathbf{b}_t)$, where $\mathbf{b}_t \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_p)$.
7: \hspace{0.5cm} Update parameter using projected noisy gradient: $\mathbf{w}_{t+1} = \mathbf{w}_t - \eta_t \tilde{\mathbf{g}}_t$
8: \textbf{end for}

The PDP-SGD follows the classical noisy gradient descent algorithm DP-SGD, which has been well studied \cite{20, 36, 37}. The typical DP-SGD adds isotropic Gaussian noise $\mathbf{b}_t \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_p)$ to the gradient $\mathbf{g}_t$, i.e., each coordinate of the gradient $\mathbf{g}_t$ is perturbed by the Gaussian noise. Given the dimension of gradient to be $p$, this method ends up in getting a factor of $p$ in the error rate \cite{4, 7}. Our algorithm is inspired by the recent observations that stochastic gradients stay in a low-dimensional space in the training of deep nets \cite{20, 25}. Such observation is also valid for the private training algorithm, i.e., DP-SGD (Figure 1 (b) and (c)). Intuitively, the most information needed for gradient descent is embedded in the top eigenspace of the stochastic gradients. Thus, PDP-SGD performs noise reduction by projecting the noisy gradient $\mathbf{g}_t + \mathbf{b}_t$ to an approximation of such a subspace given by a public dataset $S_h$.

Thus, our algorithm involves two steps at each iteration, i.e., subspace identification and noisy gradient projection. The pseudo-code of PDP-SGD is given in Algorithm 1. At each iteration $t$, in order to obtain an approximated subspace without leaking the information of the private dataset $S$, we evaluate the second moment matrix $M_t$ on the public dataset $S_h$ and compute the top-$k$ eigenvectors $\hat{V}_k(t)$ of $M_t$ (line 4 in Algorithm 1). Then we project the noisy gradient $\mathbf{g}_t + \mathbf{b}_t$ to the top-$k$ eigenspace $\hat{V}_k(t)^\top \hat{V}_k(t)^\top$, i.e., $\tilde{\mathbf{g}}_t = \hat{V}_k(t)^\top \hat{V}_k(t)^\top (\mathbf{g}_t + \mathbf{b}_t)$ (line 6 in Algorithm 1). Later on, PDP-SGD use the projected noisy gradient $\tilde{\mathbf{g}}_t$ to update the parameter $\mathbf{w}_{t+1} = \mathbf{w}_t - \eta_t \tilde{\mathbf{g}}_t$. Let us first state its privacy guarantee.

\footnote{For convex problem, we consider the typical constrained optimization problem such that the optimal solution $\mathbf{w}^*$ is in a set $\mathcal{H}$, where $\mathcal{H} = \{ \mathbf{w} : \| \mathbf{w} \| \leq B \}$ and each step we project $\mathbf{w}_{t+1}$ back to the set $\mathcal{H}$.}

3 Projected Private Gradient Descent

In this section, we first present the Projected DP-SGD (PDP-SGD) algorithm. Then we present the privacy guarantee of the PDP-SGD. Later, we provide the theoretical analysis of PDP-SGD for different types of functions including convex and non-convex functions.
Theorem 1 (Privacy) Under Assumption 7 there exist constants $c_1$ and $c_2$ so that given the number of iterations $T$, for any $\epsilon \leq c_1 q^2 T$, where $q = \frac{B_1}{m}$, PDP-SGD (Algorithm 7) is $(\epsilon, \delta)$-differentially private for any $\delta > 0$, if $\sigma^2 \geq c_2 \frac{G^2 T \ln(\frac{1}{\delta})}{n + \epsilon^2}$.

The privacy proof essentially follows from the same proof of DP-SGD [1, 7]. At each iteration, the update step PDP-SGD is essentially post-processing of Gaussian Mechanism that computes a noisy estimate of the gradient $g_t + b_t$. Then the privacy guarantee of releasing the sequence of $\{g_t + b_t\}_t$ is exactly the same as the privacy proof of Theorem 1 of [1].

3.1 Gradient Subspace Identification

We now analyze the gradient deviation between the approximated subspace $\hat{V}_k(t)\hat{V}_k(t)^\top$ and true (population) subspace $V_k(t)V_k(t)^\top$, i.e., $\|\hat{V}_k(t)\hat{V}_k(t)^\top - V_k(t)V_k(t)^\top\|_2$. To bound $\|\hat{V}_k(t)\hat{V}_k(t)^\top - V_k(t)V_k(t)^\top\|_2$, we first bound the deviation between second moment matrix $\mathbb{E}[\nabla \ell(w_t, \tilde{z}_i(t))]\nabla \ell(w_t, \tilde{z}_i(t))^\top - \mathbb{E} [\nabla \ell(w_t, \tilde{z}_i(t))\nabla \ell(w_t, \tilde{z}_i(t))^\top] \|_2 > u,

\[ \left\| \frac{1}{m} \sum_{i=1}^{m} \nabla \ell(w_t, \tilde{z}_i(t))\nabla \ell(w_t, \tilde{z}_i(t))^\top - \mathbb{E} [\nabla \ell(w_t, \tilde{z}_i(t))\nabla \ell(w_t, \tilde{z}_i(t))^\top] \right\|_2 > u, \]

with probability at most $p\exp(-mu^2/4G)$ and $G$ is as in Assumption 1.

However, this concentration bound does not hold for $w_t$, $\forall t > 0$ in general, since the public dataset $S_h$ is reused over the iterations and the parameter $w_t$ depends on $S_h$. To handle the dependency issue, we bound $\|M_t - \Sigma_t\|_2$ uniformly over all iterations $t \in [T]$ to bound the worst-case counterparts that consider all possible iterates. Our uniform bounding analysis is based on generic chaining (GC) [33], an advanced tool from probability theory. Eventually, the error bound is expressed in terms of a complexity measure called $\gamma_2$ function [33].

Definition 2 ($\gamma_2$ function) For a metric space $\langle A, d \rangle$, an admissible sequence of $A$ is a collection of subsets of $\Gamma = \{A_n : n \geq 0\}$, with $|A_0| = 1$ and $|A_n| \leq 2^{2^n}$ for all $n \geq 1$, the $\gamma_2$ functional is defined by

$$\gamma_2(A, d) = \inf \sup_{A \in \mathcal{A}} \sum_{n \geq 0} 2^n d(A, A_n),$$

where the infimum is over all admissible sequences of $A$.

In Theorem 2 we show that the uniform convergence bound of $\|M_t - \Sigma_t\|_2$ scales with $\gamma_2(W, d)$, where $d$ is the pseudo metric as in Assumption 2 and $W \in \mathbb{R}^p$ is the set that contains all possible iterates in the algorithm, i.e., $w_t \in W$ for all $t \in [T]$. Based on the majorizing measure theorem (e.g., Theorem 2.4.1 in [33]), if the metric $d$ is $\ell_2$-norm, $\gamma_2(W, d)$ can be expressed as Gaussian width of the set $W$, i.e., $w(W) = \mathbb{E}_v[\sup_{w \in W}(w, v)]$ where $v \sim N(0, I_p)$, which only depends on the size of the $W$. In Appendix A.2, we show the complexity measure $\gamma_2(W, d)$ can be expressed as the $\gamma_2$ function measure on the gradient space by mapping the parameter space $W$ to the gradient space, i.e., $f : W \mapsto \mathcal{M}$, where $f$ can be considered as $f(w) = \mathbb{E}_{z \in \mathcal{P}}[\nabla(w, z)]$. To simplify the notation, we use $m = \mathbb{E}_{z \in \mathcal{P}}[\nabla(w, z)]$ to
To measure the value of $\gamma_2(\mathcal{M}, d)$ and have small $\|w\|$ of working with $\mathcal{M}$, we empirically explore the gradient space $\mathcal{M}$ for deep nets. Figure 2 gives an example of the population gradient along the training trajectory of DP-SGD with $\sigma = \{1, 2, 4\}$ for training a 2-layer ReLU on MNIST dataset. Based on the observation in Figure 2 which shows that the each coordinates of the gradient is of small value, it is fair that the gradient space $\mathcal{M}$ is a union of ellipsoids, i.e., there exists $e \in \mathbb{R}^p$ such that $\mathcal{M} = \{ m \in \mathbb{R}^p \mid \sum_{j=1}^p m(j)^2 / e(j)^2 \leq 1, e \in \mathbb{R}^p \}$, where $j$ denotes the $j$-th coordinate. Then we have $\gamma_2(\mathcal{M}, d) \leq c_1 w(\mathcal{M}) \leq c_2 \|e\|_2$ \footnote{In the appendix, we provide more examples $\mathcal{M}$ that are consistent with empirical observations of stochastic gradient distributions and have small $\gamma_2(\mathcal{M}, d)$.}, where $c_1$ and $c_2$ are absolute constants. If the elements of $e$ sorted in decreasing order satisfy $e(j) \leq c_3 \sqrt{j}$ for all $j \in [p]$, then $\gamma_2(\mathcal{M}, d) \leq O\left(\sqrt{\log p}\right)^{\frac{1}{2}}$.

Now we give the uniform convergence bound of $\|M_t - \Sigma_t\|_2$ in Theorem 2.

**Theorem 2 (Second Moment Concentration)** Under Assumption \footnote{Detailed value of constants refer to the proof in the Appendix.} 1, the second moment matrix of the public gradient $M_t = \frac{1}{m} \sum_{i=1}^m \nabla \ell(w_t, z_i)\nabla \ell(w_t, z_i)^\top$ approximates the population second moment matrix $\Sigma_t = \mathbb{E}_{z \sim \mathcal{D}}[\nabla \ell(w_t, z)\nabla \ell(w_t, z)^\top]$ uniformly over all iterations, i.e., for any $u > 0$,

$$\sup_{t \in [T]} \|M_t - \Sigma_t\|_2 \leq O\left(\frac{ugp^{\frac{1}{2}}}{\sqrt{m}}\gamma_2(W, d)\right), \quad (9)$$

with probability at least $1 - c \exp\left(-u^2/4\right)$ where $c$ is an absolute constant $\footnote{Detailed value of constants refer to the proof in the Appendix.} G$ and $p$ are as in Assumption 1 and $\mathcal{W}$ is the set that contains all possible iterates and $d$ is the pseudo-metric as in Assumption 2.

Theorem 2 shows that $M_t$ evaluated on reused public dataset $S_h$, approximates the population second moment matrix $\Sigma_t$ uniformly over all iterations. This uniform bound is derived by the technique GC, which develops sharp upper bounds to suprema of stochastic processes indexed by a set with a metric structure in terms of $\gamma_2$ functions. In our case, $\|M_t - \Sigma_t\|$ is treated as the stochastic process indexed by the set $\mathcal{W} \in \mathbb{R}^p$ such that $w_t \in \mathcal{W}$, which is the set of all possible iterates. The metric $d$ is the pseudo-metric $d: \mathbb{R}^p \times \mathbb{R}^p \mapsto \mathbb{R}$ defined in Assumption 2. If the metric $d$ is $\ell_2$-norm, i.e., $d(w, w') = \|w - w'\|_2$, $\gamma_2(\mathcal{W}, d)$ can be expressed as Gaussian width $\footnote{Detailed value of constants refer to the proof in the Appendix.}$ 34, 35 of the set $\mathcal{W}$. To get a more practical bound, following the above discussion, instead of working with $\mathcal{W}$ over parameters, one can consider working with the set $\mathcal{M}$ of population gradients by. This can be effectively done in practice by applying the technique GC.
defining the pseudo-metric as \( d(w, w') = d(f(w), f(w')) = d(m, m') \), where \( f(w) = E_{z \in P} \left[ \nabla (w, z) \right] \) maps the parameter space to gradient space. Thus, the complexity measure \( \gamma_2(W, d) \) can be expressed as the \( \gamma_2 \) function measure on the population gradient space, i.e., \( \gamma_2(M, d) \). As discussed above, using the \( \ell_2 \)-norm as \( d \), the \( \gamma_2(M, d) \) will be a constant if assuming the gradient space is a union of ellipsoids and uniform bound will only depends on logarithmically on \( p \).

Using the result in Theorem 2 and Davis-Kahan sin-\( \theta \) theorem \cite{Davis1970}, we obtain the subspace construction error \( \| \hat{V}_k(t)V_k(t)^T - V_k(t)V_k(t)^T \|_2 \) in the following theorem.

**Theorem 3 (Subspace Closeness)** Under Assumption \( 1 \) and \( 2 \) with \( V_k(t) \) to be the top-\( k \) eigenvectors of the population second moment matrix \( \Sigma_t \) and \( \alpha_t \) be the eigen-gap at \( t \)-th iterate such that \( \lambda_k(\Sigma_t) - \lambda_{k+1}(\Sigma_t) \geq \alpha_t \), for the \( \hat{V}_k(t) \) in Algorithm 1 if \( m \geq O(\frac{G\rho\sqrt{\ln p\gamma_2(W,d)}}{\min \alpha_t^2}) \), for all \( t \in [T] \), we have

\[
E \left[ \| \hat{V}_k(t)V_k(t)^T - V_k(t)V_k(t)^T \|_2 \right] \leq O \left( \frac{G\rho\sqrt{\ln p\gamma_2(W,d)}}{\alpha_t \sqrt{m}} \right).
\]

Theorem 3 gives the sample complexity of the public sample size and the reconstruction error, i.e., the convergence of the gradient norm if the principal component dominates.

### 3.2 Empirical Risk Convergence Analysis

In this section, we present the error rate of PDP-SGD for convex functions and non-convex (smooth) functions. For non-convex case, we first give the error rate of the \( \ell_2 \)-norm of the principal component of the gradient, i.e., \( \| V_k(t)V_k(t)^T \nabla \hat{L}_n(w_t) \|_2 \). Then we show that if the principal component dominates the residual component of the gradient, the gradient norm also converges. Note that this assumption is suggested by the Figure 1 and recent observations \cite{25, 29} such that only a few top eigenvalues of the gradient space dominate and the rest eigenvalues are of small values (almost zero). For the convex and Lipschitz case, we provide the error rate of the empirical risk under low-rank assumption of the gradient space.

To present our result for non-convex functions, we introduce some new notations here. We use \( \| \nabla \hat{L}_n(w_t) \| \) to denote the principal component of the gradient, i.e., \( \| \nabla \hat{L}_n(w_t) \| = V_k(t)V_k(t)^T \nabla \hat{L}_n(w_t) \). We write \( \| \nabla \hat{L}_n(w_t) \| \perp \) as the residual component, i.e., \( \| \nabla \hat{L}_n(w_t) \| \perp = \nabla \hat{L}_n(w_t) - V_k(t)V_k(t)^T \nabla \hat{L}_n(w_t) \). In the following theorem, we show that the norm of the principal component of the gradient converges implying the convergence of the gradient norm if the principal component dominates.

**Theorem 4 (Smooth and Non-convex)** For \( \rho \)-smooth function \( \hat{L}_n(w) \), under the Assumption \( 1 \) and \( 2 \) let \( \Lambda = \sum_{t=1}^{T} 1/\alpha_t^2 \), for any \( \epsilon, \delta > 0 \), with \( T = O(n^2\epsilon^2) \) iterations, and step size \( \eta_t = \frac{1}{\sqrt{T}} \), PDP-SGD achieves:

\[
\frac{1}{T} \sum_{t=1}^{T} E \| V_k(t)V_k(t)^T \nabla \hat{L}_n(w_t) \|_2^2 \leq \tilde{O} \left( \frac{k\rho G^2}{n\epsilon} \right) + O \left( \frac{\Lambda G^4 \rho^2 \gamma_2^2(W,d) \ln p}{m} \right).
\]

Additionally, assuming the principal component of the gradient dominates, i.e., there exist \( c > 0 \), such that \( \frac{1}{T} \sum_{t=1}^{T} \| \nabla \hat{L}_n(w_t) \| \perp \|_2^2 \leq c \sum_{t=1}^{T} \| \nabla \hat{L}_n(w_t) \| \perp^2 \), we have

\[
E \| \nabla \hat{L}_n(w_T) \|_2^2 \leq \tilde{O} \left( \frac{k\rho G^2}{n\epsilon} \right) + O \left( \frac{\Lambda G^4 \rho^2 \gamma_2^2(W,d) \ln p}{m} \right),
\]
where $w_R$ is uniformly sampled from $\{w_1, ..., w_T\}$.

Theorem 5 shows that PDP-SGD reduces the error rate of a factor of $p$ to $k$ compared to existing results for non-convex and smooth functions [36]. The error rate also includes a term depending on the $\gamma_2$ function and the eigen-gap $\alpha_t$, i.e., $\Lambda = \frac{\sum_{t=1}^T 1/\alpha_t}{T}$. This term comes from the subspace reconstruction error. As discussed in previous section, as the gradient stays in a union of ellipsoids the $\gamma_2$ is a constant. The term $\Lambda$ depends on the eigen-gap $\alpha_t$. Recall that $\lambda_k(\Sigma_t) - \lambda_{k+1}(\Sigma_t) \geq \alpha_t$. As shown by the Figure 4 along the training trajectory, there are a few dominated eigenvalues and the eigen-gap stays significant (even at the last epoch). Then the term $\Lambda$ will be a constant and the bound scales logarithmically with $p$. If one considers the eigen-gap $\alpha_t$ decays as training proceed, e.g., $\alpha_t = \frac{1}{t^{\gamma_2}}$ for $t > 0$, then we have $\Lambda = O(\sqrt{T})$. In this case, with $T = n^2 \epsilon^2$, PDP-SGD requires the public data size $m = O(n \epsilon)$.

For the convex and Lipschitz functions, we consider the low-rank structure of the gradient space, i.e, the population gradient second momment $\Sigma_t$ is of rank-$k$, which is a special case of the principal gradient dominate assumption when $\|\hat{\nabla} L_n(w_t)\|_2 = 0$. We present the error rate of PDP-SGD for such case in the following theorem.

**Theorem 5 (Convex and Lipschitz)**  For $G$-Lipschitz and convex function $\hat{L}_n(w)$, under the Assumption 7 and assuming $\Sigma_t$ is of rank-$k$, let $\Lambda = \frac{\sum_{t=1}^T 1/\alpha_t}{T}$, for any $\epsilon, \delta > 0$, with $T = O(n \epsilon)$, step size $\eta_t = \frac{1}{\sqrt{t}}$, the DPD-SGD achieves

$$
\mathbb{E} \left[ \hat{L}_n(w) - \hat{L}_n(w^*) \right] \leq O \left( \frac{k G^2}{n \epsilon} \right) + O \left( \frac{\Lambda G \rho \gamma_2(W, d) \ln p}{\sqrt{m}} \right),
$$

where $w = \frac{\sum_{t=1}^T w_t}{T}$, and $w^*$ is the minima of $\hat{L}_n(w)$.

Compared to the error rate of DP-SGD for convex functions [4 7], PDP-SGD also demonstrates an improvement from a factor of $p$ to $k$. PDP-SGD also involves the subspace reconstruction error, i.e., $O \left( \frac{\Lambda G \rho \gamma_2(W, d) \ln p}{\sqrt{m}} \right)$ depending on the $\gamma_2$ function and eigen-gap term $\Lambda = \frac{\sum_{t=1}^T 1/\alpha_t}{T}$. Based on the discussion in previous section and a more detailed discussion in Appendix A.2, with suitable assumptions of the gradient structure, e.g., ellipsoids, the $\gamma_2$ is a constant. For the eigen-gap term, if $\alpha_t$ stays as a constant in the training procedure as shown by Figure 1, $\Lambda$ will be a constant and the bound scales logarithmically with $p$. If we assume the eigen-gap $\alpha_t$ decays as training proceed, e.g., $\alpha_t = \frac{1}{t^{\gamma_2}}$ for $t > 0$, then we have $\Lambda = O(\sqrt{T})$. In this case, with $T = n \epsilon$, PDP-SGD requires public data size $m = O(n \epsilon)$.

## 4 Experiments

We empirically evaluate PDP-SGD on training neural networks with two tasks: the MNIST image classification task [24] and Adult income prediction originally from the UCI repository [10]. We compare the performance of PDP-SGD with the baseline DP-SGD for various privacy levels $\epsilon$.

### 4.1 Experimental Setup

**Datasets and Network Structure.** The MNIST dataset contains 60,000 training examples and 10,000 test examples. To construct the private training set, we randomly sample 10,000 samples from the original
Table 1: Neural network and datasets setup.

| Dataset          | Model        | features | classes | Training size | Public size | Test size |
|------------------|--------------|----------|---------|---------------|-------------|-----------|
| MNIST            | 2-Layer ReLU | 784      | 10      | 10,000        | 100         | 10,000    |
| Adult income     | 2-Layer ReLU | 123      | 2       | 29,105        | 200         | 3,256     |

(a) Training accuracy, $\epsilon = 4.13$
(b) Training accuracy, $\epsilon = 0.90$
(c) Training accuracy, $\epsilon = 0.62$
(d) Test accuracy, $\epsilon = 4.13$
(e) Test accuracy, $\epsilon = 0.90$
(f) Test accuracy, $\epsilon = 0.62$

Figure 3: Comparison of DP-SGD and PDP-SGD for MNIST. (a-c) report the training accuracy and (d-f) report the test accuracy for $\epsilon = \{4.13, 0.90, 0.62\}$. The X-axis is the number of epochs, and the Y-axis is the train/test accuracy. The DPD-SGD outperforms DP-SGD for different privacy levels.

training set of MNIST, then we sample 100 samples from the rest to construct the public dataset. The Adult income dataset has been preprocessed into the LIBSVM format [9]. This dataset contains 32,561 examples, each of which has 123 features and a label indicating whether the individual’s annual income is above 50k or not. We first randomly choose 10% of the examples as the test set (3256 examples). From the remaining 29,305 examples, we randomly sample 200 examples as the public set and the remaining 29,105 examples form the private set. We consider a 2-layer ReLU network with 128 nodes each layer for both tasks. The details are given in Table 1.

Training and Hyper-parameter Setting: Cross-entropy is used as our loss function throughout experiments. The mini-batch size is set to be 128 for MNIST and 256 for Adult income. We set learning rate to be 0.01 for both MNIST and Adult income. For MNIST, we run 30, 50, 100 epochs for $\sigma = \{1, 4, 8\}$ respectively and decay learning rate by 0.1 every 30 epochs. For Adult income, we fix 30 epochs for all privacy level. We repeat each experiments 5 times and report the mean and standard deviation of the accuracy on the training and test set. For PDP-SGD, the projection dimension $k$ is a hyper-parameter and it illustrates a trade-off between the reconstruction error and the noise reduction. A small $k$ implies more noise amount will be reduced, and a larger reconstruction error will be introduced. We explored $k = \{5, 10, 20\}$ for both datasets, and we found that $k = 20$ achieves the best performance among the value we searched for both datasets.
Figure 4: Comparison of DP-SGD and PDP-SGD for Adult income. (a-c) report the training accuracy and (d-f) report the test accuracy for $\epsilon = \{1.38, 0.63, 0.31\}$. The X-axis and Y-axis refer to Figure 3. The DPD-SGD outperforms DP-SGD for different privacy levels.

Privacy Parameter Setting: Since gradient norm bound $G$ is unknow for deep learning, we follow the gradient clipping method in [1] to guarantee the privacy. We choose gradient clip size to be 1.0 for MNIST and Adult income. We report results for three choices of the noise scale, i.e., $\sigma = \{1, 4, 8\}$ for MNIST and Adult income. We follow the Moment Accountant (MA) method [8] to calculate the accumulated privacy cost, which depends on the number of epochs, the batch size, $\delta$, and noise variance $\sigma$. With 30, 50, 100 epochs for $\sigma = \{1, 4, 8\}$ respectively for MNIST, fixing $\delta = 10^{-5}$, the $\epsilon$ is $\{4.13, 0.90, 0.62\}$. For Adult income, with 30 epochs for all privacy level, and $\delta = 10^{-5}$, the $\epsilon$ is $\{3.84, 0.63, 0.31\}$ corresponding to $\sigma = \{1, 4, 8\}$.

4.2 Experimental Results

Compare PDP-SGD with DP-SGD. The training accuracy and test accuracy for different levels of privacy, i.e., $\epsilon$, are reported in Figure 3 and Figure 4 corresponding to MNIST and Adult income. For both datasets, PDP-SGD outperforms DP-SGD for all privacy level we have considered. Especially for MNIST, Figure 3 demonstrates that PDP-SGD improve the accuracy substantially over the baseline DP-SGD. The superiority of DP-SGD is obvious when the noise variance is large, i.e., $\sigma = 8$.

PDP-SGD with different $k$. In order to understand the role of projection dimension $k$, in Figure 5, we present the training accuracy and test accuracy for PDP-SGD with $k = 5, 10, 20$ for $\epsilon = 3.84$ (Figure 5(a)) and $\epsilon = 0.63$ (Figure 5(b)) for MNIST dataset. Among the choice of $k$, we can see that PDP-SGD with $k = 20$ performs better that the others in terms of the training and test accuracy. PDP-SGD with $k = 5$ proceeds slower than PDP-SGD with $k = 20$ and $k = 10$. This is due to the larger reconstruction error introduced by projecting the gradient to a much smaller subspace, i.e., $k = 5$. However, compared to the gradient dimension $p = 118, 016$, it is impressive that PDP-SGD with $k = 20$ which projects the gradient to
Figure 5: Training accuracy and test accuracy for PDP-SGD with $k \in \{5, 10, 20\}$ for MNIST. (a) reports the training and test accuracy for $\epsilon = 3.84$; (b) reports the training and test accuracy for $\epsilon = 0.63$. The X-axis and Y-axis refer to Figure 3. PDP-SGD with $k = 20$ performs better than the others in terms of the training and test accuracy.

the a much smaller subspace, can achieve better accuracy than DP-SGD.

5 Conclusion and Future Work

While differentially-private stochastic gradient descent (DP-SGD) algorithms and variants have been well studied for solving differentially private empirical risk minimization (ERM), the error rate of DP-SGD has a dependence on the ambient dimension $p$. In this paper, we aim at bypassing such dependence by leveraging a special structure of gradient space i.e., the stochastic gradients for deep nets usually stay in a low dimensional subspace in the training process. We propose PDP-SGD which projects the noisy gradient to an approximated subspace evaluated on a public dataset. We show that the subspace reconstruction error is small and PDP-SGD reduces the $p$ factor in the error rate to the projection dimension. We evaluate the proposed algorithms on two popular deep learning tasks and demonstrate the empirical advantages of PDP-SGD over DP SGD.

There are several interesting directions for future work. First, it will be interesting to provide a private optimization method that can identify the gradient subspace at each iteration without access to a public dataset. The existing “analyze-Gauss” techniques in [15] will give a bound with reconstruction error scaling with $\sqrt{p}$, which will be propagated to the optimization error. More recently, [32] show that, for a class of generalized linear problems, DP gradient descent (without any projection) achieves an excess empirical risk bound depending on the rank of the feature matrix instead of the ambient dimension. It will also be interesting to explore whether their rank-dependent bounds hold for a more general class of problems. Finally, a question that applies both to private and non-private optimization is to characterize the class of optimization problems with low-dimensional gradient subspaces.

Acknowledgement

The research was supported by NSF grants IIS-1908104, OAC-1934634, IIS-1563950, a Google Faculty Research Award, a J.P. Morgan Faculty Award, and a Mozilla research grant. We would like to thank the Minnesota Super-computing Institute (MSI) for providing computational resources and support.
References

[1] Martin Abadi, Andy Chu, Ian Goodfellow, Brendan McMahan, Ilya Mironov, Kunal Talwar, and Li Zhang. Deep learning with differential privacy. In 23rd ACM Conference on Computer and Communications Security, pages 308–318, 2016.

[2] Brendan Avent, Aleksandra Korolova, David Zeber, Torgeir Hovden, and Benjamin Livshits. BLENDER: enabling local search with a hybrid differential privacy model. In 26th USENIX Security Symposium, pages 747–764, 2017.

[3] Raef Bassily, Albert Cheu, Shay Moran, Aleksandar Nikolov, Jonathan Ullman, and Zhiwei Steven Wu. Private query release assisted by public data. CoRR, abs/2004.10941, 2020.

[4] Raef Bassily, Vitaly Feldman, Kunal Talwar, and Abhradeep Guha Thakurta. Private stochastic convex optimization with optimal rates. In Advances in Neural Information Processing Systems, pages 11282–11291, 2019.

[5] Raef Bassily, Shay Moran, and Noga Alon. Limits of private learning with access to public data. In Advances in Neural Information Processing Systems, pages 10342–10352, 2019.

[6] Raef Bassily, Kobbi Nissim, Adam D. Smith, Thomas Steinke, Uri Stemmer, and Jonathan Ullman. Algorithmic stability for adaptive data analysis. In Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing, pages 1046–1059, 2016.

[7] Raef Bassily, Adam Smith, and Abhradeep Thakurta. Private empirical risk minimization: Efficient algorithms and tight error bounds. In 2014 IEEE 55th Annual Symposium on Foundations of Computer Science, pages 464–473. IEEE, 2014.

[8] Zhiqi Bu, Jinshuo Dong, Qi Long, and Weijie J Su. Deep learning with gaussian differential privacy. arXiv preprint arXiv:1911.11607, 2019.

[9] Chih-Chung Chang and Chih-Jen Lin. LIBSVM: A library for support vector machines. ACM transactions on intelligent systems and technology (TIST), 2(3):1–27, 2011.

[10] Dheeru Dua and Casey Graff. UCI machine learning repository, 2017.

[11] Cynthia Dwork, Vitaly Feldman, Moritz Hardt, Toniann Pitassi, Omer Reingold, and Aaron Leon Roth. Preserving statistical validity in adaptive data analysis. In Proceedings of the 47th Annual ACM on Symposium on Theory of Computing, pages 117–126. ACM, 2015.

[12] Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith. Calibrating noise to sensitivity in private data analysis. In Theory of Cryptography Conference, pages 265–284. Springer, 2006.

[13] Cynthia Dwork, Aaron Roth, et al. The algorithmic foundations of differential privacy. Foundations and Trends® in Theoretical Computer Science, 9(3–4):211–407, 2014.

[14] Cynthia Dwork, Guy N. Rothblum, and Salil P. Vadhan. Boosting and differential privacy. In 51th Annual IEEE Symposium on Foundations of Computer Science, pages 51–60. IEEE Computer Society, 2010.
[15] Cynthia Dwork, Kunal Talwar, Abhradeep Thakurta, and Li Zhang. Analyze gauss: optimal bounds for privacy-preserving principal component analysis. In Symposium on Theory of Computing, pages 11–20. ACM, 2014.

[16] Vitaly Feldman, Ilya Mironov, Kunal Talwar, and Abhradeep Thakurta. Privacy amplification by iteration. In 59th IEEE Annual Symposium on Foundations of Computer Science, pages 521–532, 2018.

[17] Matt Fredrikson, Somesh Jha, and Thomas Ristenpart. Model inversion attacks that exploit confidence information and basic countermeasures. In Proceedings of the 22nd ACM SIGSAC Conference on Computer and Communications Security, page 1322–1333, 2015.

[18] Saeed Ghadimi and Guanghui Lan. Stochastic first- and zeroth-order methods for nonconvex stochastic programming. SIAM Journal on Optimization, 23(4):2341–2368, 2013.

[19] Gene H. Golub and Charles F. Van Loan. Matrix Computations. The Johns Hopkins University Press, third edition, 1996.

[20] Guy Gur-Ari, Daniel A. Roberts, and Ethan Dyer. Gradient descent happens in a tiny subspace. CoRR, abs/1812.04754, 2018.

[21] Roger A Horn and Charles R Johnson. Matrix analysis. Cambridge university press, 2012.

[22] Christopher Jung, Katrina Ligett, Seth Neel, Aaron Roth, Saeed Sharifi-Malvajerdi, and Moshe Shenfeld. A new analysis of differential privacy’s generalization guarantees. volume 151, pages 31:1–31:17, 2020.

[23] S. P. Kasiviswanathan, H. K. Lee, K. Nissim, S. Raskhodnikova, and A. Smith. What can we learn privately? In 2008 49th Annual IEEE Symposium on Foundations of Computer Science, 2008.

[24] Yann LeCun, Léon Bottou, Yoshua Bengio, and Patrick Haffner. Gradient-based learning applied to document recognition. Proceedings of the IEEE, 86(11):2278–2324, 1998.

[25] Xinyan Li, Qilong Gu, Yingxue Zhou, Tiancong Chen, and Arindam Banerjee. Hessian based analysis of SGD for deep nets: Dynamics and generalization. In Proceedings of the 2020 SIAM International Conference on Data Mining, pages 190–198. SIAM, 2020.

[26] Frank McSherry. Spectral methods for data analysis. PhD thesis, University of Washington, 2004.

[27] Yuriii Nesterov. Introductory Lectures on Convex Optimization: A Basic Course. Springer Publishing Company, Incorporated, 1 edition, 2014.

[28] Nicolas Papernot, Martín Abadi, Úlfar Erlingsson, Ian J. Goodfellow, and Kunal Talwar. Semi-supervised knowledge transfer for deep learning from private training data. In 5th International Conference on Learning Representations, 2017.

[29] Vardan Papyan. Measurements of three-level hierarchical structure in the outliers in the spectrum of deepnet Hessians. In International Conference on Machine Learning, pages 5012–5021, 2019.

[30] Boris Polyak. Gradient methods for the minimisation of functionals. Ussr Computational Mathematics and Mathematical Physics, 3:864–878, 12 1963.
[31] Shuang Song, Kamalika Chaudhuri, and Anand D. Sarwate. Stochastic gradient descent with differentially private updates. In *IEEE Global Conference on Signal and Information Processing*, pages 245–248. IEEE, 2013.

[32] Shuang Song, Om Thakkar, and Abhradeep Thakurta. Characterizing private clipped gradient descent on convex generalized linear problems. *arXiv preprint arXiv:2006.06783*, 2020.

[33] M. Talagrand. *Upper and Lower Bounds for Stochastic Processes*. Springer, 2014.

[34] Roman Vershynin. *High-Dimensional Probability: An Introduction with Applications in Data Science*. Cambridge University Press, 2018.

[35] Martin J Wainwright. *High-dimensional statistics: A non-asymptotic viewpoint*, volume 48. Cambridge University Press, 2019.

[36] Di Wang and Jinhui Xu. Differentially private empirical risk minimization with smooth non-convex loss functions: A non-stationary view. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 33, pages 1182–1189, 2019.

[37] Di Wang, Minwei Ye, and Jinhui Xu. Differentially private empirical risk minimization revisited: Faster and more general. In *Advances in Neural Information Processing Systems*, pages 2722–2731, 2017.
A Uniform Convergence for Subspaces: Proofs for Section 3.1

In this section, we provide the proofs for Section 3.1. We first show that the second moment matrix $M_t$ converges to the population second moment matrix $\Sigma_t$ uniform over all iterations $t \in [T]$, i.e., $\sup_{t \in [T]} \|M_t - \Sigma_t\|$. Then we show that the top-$k$ subspace of $M_t$ uniformly converges to the top-$k$ subspace of $\Sigma_t$, i.e., $\|\hat{V}_k(t)V_k(t)^T - V_k(t)V_k(t)^T\|$ for all $t \in [T]$. Our bound depends on the $\gamma_2(\mathcal{W}, d)$ where $\mathcal{W}$ is the set of all possible parameters along the training trajectory. In Section A.2, we show that the bound can be derived by $\gamma_2(\mathcal{M}, d)$ as well, where $\mathcal{M}$ is the set of population gradients along the training trajectory. Then, we provide examples of the set $\mathcal{M}$ and corresponding value of $\gamma_2(\mathcal{M}, d)$.

A.1 Uniform Convergence Bound

Our proofs of Theorem 2 heavily rely on the advanced probability tool, Generic Chaining (GC) [33]. Typically the results in generic chaining are characterized by the so-called $\gamma_2$ function (see Definition 2). [33] shows that for a process $(X_t)_{t \in T}$ and a given metric space $(T, d)$, if $(X_t)_{t \in T}$ satisfies the increment condition

$$\forall u > 0, \mathbb{P}(\|X_s - X_t\| \geq u) \leq 2 \exp \left(-\frac{u^2}{2d(s, t)^2}\right), \quad (14)$$

then the size of the process can be bounded as

$$\mathbb{E} \sup_{t \in T} X_t \leq c\gamma_2(T, d), \quad (15)$$

with $c$ to be an absolute constant.

To apply the GC result to establish Theorem 2, we treat $\|M_t - \Sigma_t\|_2$ as the process $X_t$ over the iterations. In detail, since $M_t = \frac{1}{m} \sum_{i=1}^m \nabla \ell (w_t, \tilde{z}_i) \nabla \ell (w_t, \tilde{z}_i)^T$, and $\Sigma_t = \mathbb{E}_{z \sim P} \left[ \nabla \ell (w_t, z) \nabla \ell (w_t, z)^T \right]$, the $\|M_t - \Sigma_t\|_2$ is a random process indexed by $w_t \in \mathcal{W}$, with $\mathcal{W}$ to be the set of all possible iterates obtained by the algorithm.

We first show that the variable $\|M_t - \Sigma_t\|_2$ satisfies the increment condition as stated in (14) in the Lemma 1. Before we present the proof of Lemma 1, we introduce the Ahlswede-Winter Inequality [21, 35], which will be used in the proof of Lemma 1. Ahlswede-Winter Inequality shows that positive semi-definite random matrix with bounded spectral norm concentrates to its expectation with high probability.

**Theorem 6 (Ahlswede-Winter Inequality)** Let $Y$ be a random, symmetric, positive semi-definite $p \times p$ matrix, such that $\|\mathbb{E}[Y]\| \leq 1$. Suppose $\|Y\| \leq R$ for some fixed scalar $R \geq 1$. Let $\{Y_1, \ldots, Y_m\}$ be independent copies of $Y$ (i.e., independently sampled matrices with the same distribution as $Y$). For any $u \in [0, 1]$, we have

$$\mathbb{P} \left( \left\| \frac{1}{m} \sum_{i=1}^m Y_i - \mathbb{E}[Y_i] \right\|_2 > u \right) \leq 2p \cdot \exp \left(-mu^2/4R\right) . \quad (16)$$

To make the argument clear, we use a more informative notation for $M_t$ and $\Sigma_t$. Recall the notation of $M_t$ and $\Sigma_t$ such that

$$M_t = \frac{1}{m} \sum_{i=1}^m \nabla \ell (w_t, \tilde{z}_i) \nabla \ell (w_t, \tilde{z}_i)^T, \quad (17)$$
where

\[ \Sigma_t = \mathbb{E}_{z \sim \mathcal{P}} \left[ \nabla \ell (w_t, z) \nabla \ell (w_t, z)^\top \right], \quad (18) \]

given the dataset \( S_h = \{ \tilde{z}_1, \ldots, \tilde{z}_m \} \) and distribution \( \mathcal{P} \) where \( \tilde{z}_i \sim \mathcal{P} \) for \( i \in [m] \), the \( M_t \) and \( \Sigma_t \) are functions of parameter \( w_t \), so we use \( M(w_t) \) and \( \Sigma(w_t) \) for \( M_t \) and \( \Sigma_t \) interchangeably in the rest of this section, i.e,

\[ M(w) = \frac{1}{m} \sum_{i=1}^{m} \nabla \ell (w, \tilde{z}_i) \nabla \ell (w, \tilde{z}_i)^\top \quad (19) \]

and

\[ \Sigma(w) = \mathbb{E}_{z \sim \mathcal{P}} \left[ \nabla \ell (w, z) \nabla \ell (w, z)^\top \right]. \quad (20) \]

**Lemma 1** With Assumption 2 and 7 hold, for any \( w, w' \in \mathcal{W} \) and \( \forall u > 0 \), we have

\[ \mathbb{P} \left( \| M(w) - \Sigma(w) \|_2 - \| M(w') - \Sigma(w') \|_2 \geq \frac{u}{\sqrt{m}} \cdot 4G\rho d(w, w') \right) \leq 2p \cdot \exp \left( -u^2/4 \right), \quad (21) \]

where \( d : \mathcal{W} \times \mathcal{W} \mapsto \mathbb{R} \) is the pseudo-metric in Assumption 2.

**Proof:** We consider random variable

\[ X_i = \nabla \ell (w, \tilde{z}_i) \nabla \ell (w, \tilde{z}_i)^\top - \nabla \ell (w', \tilde{z}_i) \nabla \ell (w', \tilde{z}_i)^\top + 2G\rho d(w, w') \mathbb{I}_p, \quad (22) \]

where \( \mathbb{I}_p \in \mathbb{R}^{p \times p} \) is the identity matrix.

Note that \( 2G\rho d(w, w') \) is deterministic and the randomness of \( X_i \) comes from \( \nabla \ell (w, \tilde{z}_i) \) and \( \nabla \ell (w', \tilde{z}_i) \).

By triangle inequality and the construction of \( X_i \), we have

\[ \| M(w) - \Sigma(w) \|_2 - \| M(w') - \Sigma(w') \|_2 \\
= \| M(w) - \mathbb{E}[M(w)] \|_2 - \| M(w') - \mathbb{E}[M(w')] \|_2 \\
\leq \| M(w) - \mathbb{E}[M(w)] - (M(w') - \mathbb{E}[M(w')]) \|_2 \\
= \| M(w) - M(w') - \mathbb{E}[(M(w) - M(w'))] \|_2 \\
= \left\| \frac{1}{m} \sum_{i=1}^{m} \left( \nabla \ell (w, \tilde{z}_i) \nabla \ell (w, \tilde{z}_i)^\top - \nabla \ell (w', \tilde{z}_i) \nabla \ell (w', \tilde{z}_i)^\top \right) - \mathbb{E} \left[ \nabla \ell (w, \tilde{z}_i) \nabla \ell (w, \tilde{z}_i)^\top - \nabla \ell (w', \tilde{z}_i) \nabla \ell (w', \tilde{z}_i)^\top \right] \right\|_2 \\
\leq \left\| \frac{1}{m} \sum_{i=1}^{m} \left( \nabla \ell (w, \tilde{z}_i) \nabla \ell (w, \tilde{z}_i)^\top - \nabla \ell (w', \tilde{z}_i) \nabla \ell (w', \tilde{z}_i)^\top + 2G\rho d(w, w') \mathbb{I}_p \right) \\
- \mathbb{E} \left[ \nabla \ell (w, \tilde{z}_i) \nabla \ell (w, \tilde{z}_i)^\top - \nabla \ell (w', \tilde{z}_i) \nabla \ell (w', \tilde{z}_i)^\top + 2G\rho d(w, w') \mathbb{I}_p \right] \right\|_2 \\
= \left\| \frac{1}{m} \sum_{i=1}^{m} X_i - \mathbb{E}[X_i] \right\|_2 \quad (23) \]

To apply Theorem 6 for \( \left\| \frac{1}{m} \sum_{i=1}^{m} X_i - \mathbb{E}[X_i] \right\|_2 \), we first show that the random symmetric matrix \( X_i \) is positive semi-definite.
By Assumption 1 and Assumption 2 and definition

\[
\begin{align*}
\| \nabla \ell(w, \tilde{z}_i) &\nabla \ell(w, \tilde{z}_i)^T - \nabla \ell(w', \tilde{z}_i)\nabla \ell(w', \tilde{z}_i)^T \|_2 \\
= &\sup_{x: \|x\|=1} x^T (\nabla \ell(w, \tilde{z}_i)\nabla \ell(w, \tilde{z}_i)^T - \nabla \ell(w', \tilde{z}_i)\nabla \ell(w', \tilde{z}_i)^T) x \\
= &\sup_{x: \|x\|=1} x^T \nabla \ell(w, \tilde{z}_i)\nabla \ell(w, \tilde{z}_i)^T x - x^T \nabla \ell(w', \tilde{z}_i)\nabla \ell(w', \tilde{z}_i)^T x \\
= &\sup_{x: \|x\|=1} (x, \nabla \ell(w, \tilde{z}_i))^2 - (x, \nabla \ell(w', \tilde{z}_i))^2 \\
= &\sup_{x: \|x\|=1} (x, \nabla \ell(w, \tilde{z}_i) + \nabla \ell(w', \tilde{z}_i)) (x, \nabla \ell(w, \tilde{z}_i)) - (x, \nabla \ell(w', \tilde{z}_i)) \\
\leq &\sup_{x: \|x\|=1} 2G (x, \nabla \ell(w, \tilde{z}_i)) - (x, \nabla \ell(w', \tilde{z}_i)) \\
= &2G \| \nabla \ell(w, \tilde{z}_i) - \nabla \ell(w', \tilde{z}_i) \|_2 \\
\leq & 2G \rho \|w - w'\|
\end{align*}
\]

(24)

For any non-zero vector \(x \in \mathbb{R}^p\), we have

\[
\begin{align*}
x^T X_i x &= x^T (\nabla \ell(w, \tilde{z}_i)\nabla \ell(w, \tilde{z}_i)^T - \nabla \ell(w', \tilde{z}_i)\nabla \ell(w', \tilde{z}_i)^T + 2G \rho \|w - w'\|_p) x \\
&= x^T (\nabla \ell(w, \tilde{z}_i)\nabla \ell(w, \tilde{z}_i)^T - \nabla \ell(w', \tilde{z}_i)\nabla \ell(w', \tilde{z}_i)^T) x + 2G \rho \|w - w'\|_p \|x\|_2^2 \\
&\geq -\|\nabla \ell(w, \tilde{z}_i)\nabla \ell(w, \tilde{z}_i)^T - \nabla \ell(w', \tilde{z}_i)\nabla \ell(w', \tilde{z}_i)^T\|_2 \|x\|_2^2 + 2G \rho \|w - w'\|_p \|x\|_2^2 \\
&= (2G \rho \|w - w'\| - \|\nabla \ell(w, \tilde{z}_i)\nabla \ell(w, \tilde{z}_i)^T - \nabla \ell(w', \tilde{z}_i)\nabla \ell(w', \tilde{z}_i)^T\|_2) \|x\|_2^2 \\
&\geq 0,
\end{align*}
\]

(25)

where (a) is true because \(\|\nabla \ell(w, \tilde{z}_i)\nabla \ell(w, \tilde{z}_i)^T - \nabla \ell(w', \tilde{z}_i)\nabla \ell(w', \tilde{z}_i)^T\|_2 \leq 2G \rho \|w - w'\|\) as shown in (24).

Let \(Y_i = \frac{X_i}{4G \rho \|w - w'\|}\), with (24), we have

\[
\begin{align*}
\|Y_i\|_2 &= \frac{\|\nabla \ell(w, \tilde{z}_i)\nabla \ell(w, \tilde{z}_i)^T - \nabla \ell(w', \tilde{z}_i)\nabla \ell(w', \tilde{z}_i)^T + 2G \rho \|w - w'\|_p\|_2}{4G \rho \|w - w'\|} \\
&\leq \frac{\|\nabla \ell(w, \tilde{z}_i)\nabla \ell(w, \tilde{z}_i)^T - \nabla \ell(w', \tilde{z}_i)\nabla \ell(w', \tilde{z}_i)^T\|_2 + \|2G \rho \|w - w'\|_p\|_2}{4G \rho \|w - w'\|} \\
&\leq \frac{2G \rho \|w - w'\| + 2G \rho \|w - w'\|}{4G \rho \|w - w'\|} \\
&= 1.
\end{align*}
\]

(26)

So that \(\|Y_i\|_2 \leq 1\) and \(\|\mathbb{E}[Y_i]\|_2 \leq 1\). Then, from Theorem 3 with \(R = 1\), we have for any \(u \in [0, 1]\)

\[
P \left( \left\| \frac{1}{m} \sum_{i=1}^{m} Y_i - \mathbb{E}[Y_i] \right\| > u \right) \leq 2p \cdot \exp \left( -mu^2/4 \right).
\]

(27)

Note that \(\frac{1}{m} \sum_{i=1}^{m} Y_i - \mathbb{E}[Y_i]\) is always bounded by 1 since \(\|Y_i\|_2 \leq 1\) and \(\|\mathbb{E}[Y_i]\|_2 \leq 1\). So the above inequality holds for any \(u > 1\) with probability 0 which is bounded by \(2p \cdot \exp \left( -mu^2/4 \right)\). So that we have
for any \( u > 0 \),
\[
P\left( \left\| \frac{1}{m} \sum_{i=1}^{m} Y_i - \mathbb{E}[Y_i] \right\| > u \right) \leq 2p \cdot \exp \left( -mu^2/4 \right). \tag{28}
\]

So that for any \( u > 0 \),
\[
P\left( \left\| \frac{1}{m} \sum_{i=1}^{m} X_i - \mathbb{E}[X_i] \right\| > \frac{u}{\sqrt{m}} \cdot 4G\rho d(w, w') \right) \leq 2p \cdot \exp \left( -u^2/4 \right). \tag{29}
\]

Combine (29) and (23), we have
\[
P\left( \left\| M(w) - \Sigma(w) \right\|_2 - \left\| M(w') - \Sigma(w') \right\|_2 \geq \frac{u}{\sqrt{m}} \cdot 4G\rho d(w, w') \right)
\[
= P\left( \left\| M(w) - E[M(w)] \right\|_2 - \left\| M(w') - E[M(w')] \right\|_2 > \frac{u}{\sqrt{m}} \cdot 4G\rho d(w, w') \right)
\[
\leq P\left( \left\| \frac{1}{m} \sum_{i=1}^{m} X_i - \mathbb{E}[X_i] \right\| > \frac{u}{\sqrt{m}} \cdot 4G\rho d(w, w') \right)
\]
\[
\leq 2p \cdot \exp \left( -u^2/4 \right) \tag{30}
\]

That completes the proof.

Based on the above result, now we come to the proof of Theorem 2. The proof follows the Generic Chaining argument, i.e., Chapter 2 of [33].

**Theorem 2 (Second Moment Concentration)** Under Assumption 1, 2, the second moment matrix of the public gradient \( M_t = \frac{1}{m} \sum_{i=1}^{m} \nabla \ell(w_t, \tilde{z}_i) \nabla \ell(w_t, \tilde{z}_i)^\top \approx \) approximates the population second moment matrix \( \Sigma_t = E_{z \sim P}[\nabla \ell(w_t, z) \nabla \ell(w_t, z)^\top] \) uniformly over all iterations, i.e., for any \( u > 0 \),
\[
\sup_{t \in [T]} \left\| M_t - \Sigma_t \right\|_2 \leq O \left( \frac{uG\rho \sqrt{\ln p\gamma_2(W, d)}}{\sqrt{m}} \right), \tag{9}
\]
with probability at least \( 1 - c \exp\left( -u^2/4 \right) \) where \( c \) is an absolute constant, \( G \) and \( \rho \) are as in Assumption 1 and 2, \( W \) is the set that contains all possible iterates and \( d \) is the pseudo metric as in Assumption 2.

**Proof:** Note that equation (9) is a uniform bound over iteration \( t \in [T] \). To bound \( \sup_{t \in [T]} \left\| M_t - \Sigma_t \right\|_2 \), it is sufficient to bound
\[
\sup_{w \in W} \left\| M(w) - \Sigma(w) \right\|_2, \tag{31}
\]
where \( W \) contains all the possible trajectories of \( w_1, ..., w_T \).
We consider a sequence of subsets \( W_n \) of \( W \), and
\[
\text{card } W_n \leq N_n, \tag{32}
\]

\[\text{Detailed value of constants refer to the proof in the Appendix}\]
where \( N_0 = 1; N_n = 2^{2^n} \) if \( n \geq 1 \).

Let \( \pi_n(w) \in \mathcal{W}_n \) be the approximation of any \( w \in \mathcal{W} \). We decompose the \( \|M(w) - \Sigma(w)\|_2 \) as

\[
\|M(w) - \Sigma(w)\|_2 - \|M(\pi_0(w)) - \Sigma(\pi_0(w))\|_2 = \sum_{n \geq 1} (\|M(\pi_n(w)) - \Sigma(\pi_n(w))\|_2 - \|M(\pi_{n-1}(w)) - \Sigma(\pi_{n-1}(w))\|_2),
\]

which holds since \( \pi_n(w) = w \) for \( n \) large enough.

Based on Lemma[1] for any \( u > 0 \), we have

\[
\|M(\pi_n(w)) - \Sigma(\pi_n(w))\|_2 - \|M(\pi_{n-1}(w)) - \Sigma(\pi_{n-1}(w))\|_2 \geq \frac{u}{\sqrt{m}} \cdot 4G\rho d(\pi_n(w), \pi_{n-1}(w)),
\]

with probability at most \( 2p \exp(-\frac{u^2}{4}) \).

For any \( n > 0 \) and \( w \in \mathcal{W} \), the number of possible pairs \( (\pi_n(w), \pi_{n-1}(w)) \) is

\[
\text{card } \mathcal{W}_n \cdot \text{card } \mathcal{W}_{n-1} \leq N_n N_{n-1} \leq N_{n+1} = 2^{2^{n+1}}.
\]

Apply union bound over all the possible pairs of \( (\pi_n(w), \pi_{n-1}(w)) \), we have any for \( n > 0 \) and \( w \in \mathcal{W} \)

\[
\|M(\pi_n(w)) - \Sigma(\pi_n(w))\|_2 - \|M(\pi_{n-1}(w)) - \Sigma(\pi_{n-1}(w))\|_2 \geq u 2^{n/2} d(\pi_n(w), \pi_{n-1}(w)) \cdot \frac{4G\rho}{\sqrt{m}}
\]

with probability

\[
\sum_{n \geq 1} 2p \cdot 2^{2^{n+1}} \exp(-u^2 2^{n-2}) \leq c' p \exp\left(-\frac{u^2}{4}\right),
\]

where \( c' < 1 \) is a positive constant.

Then we have

\[
\sum_{n \geq 1} (\|M(\pi_n(w)) - \Sigma(\pi_n(w))\|_2 - \|M(\pi_{n-1}(w)) - \Sigma(\pi_{n-1}(w))\|_2)
\]

\[
\geq \sum_{n \geq 1} u 2^{n/2} d(\pi_n(w), \pi_{n-1}(w)) \cdot \frac{4G\rho}{\sqrt{m}}
\]

\[
\geq \sum_{n \geq 0} u 2^{n/2} d(w, \mathcal{W}_n) \cdot \frac{4G\rho}{\sqrt{m}}
\]

with probability at most \( c' p \exp\left(-\frac{u^2}{4}\right) \).

From Theorem[6] let \( Y_i = \nabla \ell(\pi_0(w), \bar{z}) \nabla \ell(\pi_0(w), \bar{z})^T \), so that \( \|Y_i\| \leq 1 \) and \( \|E[Y_i]\| \leq 1 \). Then we have

\[
\|M(\pi_0(w)) - \Sigma(\pi_0(w))\|_2 \geq \frac{u G}{\sqrt{m}},
\]

with probability at most \( 2p \exp\left(-\frac{u^2}{4}\right) \).
Combine (33), (38) and (39), we have

\[
\sup_{w \in \mathcal{W}} \| M(w) - \Sigma(w) \|_2 \geq \sup_{w \in \mathcal{W}} \sum_{n \geq 0} u 2^{n/2} d(w, W_n) \cdot \frac{4G \rho}{\sqrt{m}} + u \frac{G}{\sqrt{m}}
\]

\[
= u \left( G \frac{4\rho \gamma_2(W, d) + 1}{\sqrt{m}} \right)
\]

with probability at most \((c' + 2 \exp(-u^2 \frac{\rho}{m}))\).

That completes the proof.

Now we provide the proof of Theorem 3.

**Theorem 3 (Subspace Closeness)** Under Assumption 1 and 2 with \(V_k(t)\) to be the top-\(k\) eigenvectors of the population second moment matrix \(\Sigma_t\) and \(\alpha_t\) be the eigen-gap at \(t\)-th iterate such that \(\lambda_k(\Sigma_t) - \lambda_{k+1}(\Sigma_t) \geq \alpha_t\), for the \(\hat{V}_k(t)\) in Algorithm 1 if \(m \geq \frac{O(G \rho \sqrt{\ln \rho \gamma_2(W, d)})^2}{\min_t \alpha_t^2}\), for all \(t \in [T]\), we have

\[
\mathbb{E} \left[ \| \hat{V}_k(t) \hat{V}_k(t)^T - V_k(t) V_k(t)^T \|_2 \right] \leq O \left( \frac{G \rho \sqrt{\ln \rho \gamma_2(W, d)}}{\alpha_t \sqrt{m}} \right),
\]

**Proof:** Recall that \(\hat{V}_k(t)\) is the top-\(k\) eigenspace of \(M_t\). Let \(V_k(t)\) be the top-\(k\) eigenspace of \(\Sigma_t\).

\[
\hat{V}_k(t) \hat{V}_k(t)^T - V_k(t) V_k(t)^T = \Pi_{M_t}^{(k)} (\mathbb{I} - \Pi_{\Sigma_t}^{(k)}) + (\mathbb{I} - \Pi_{M_t}^{(k)}) \Pi_{\Sigma_t}^{(k)}
\]

\[
\Rightarrow \| \hat{V}_k(t) \hat{V}_k(t)^T - V_k(t) V_k(t)^T \|_2 \leq \| \Pi_{M_t}^{(k)} (\mathbb{I} - \Pi_{\Sigma_t}^{(k)}) \|_2 + \| (\mathbb{I} - \Pi_{M_t}^{(k)}) \Pi_{\Sigma_t}^{(k)} \|_2.
\]

where \(\Pi_{M_t}^{(k)} = \hat{V}_k(t) \hat{V}_k(t)^T\) denotes the projection to the top-\(k\) subspace of the symmetric PSD \(M_t\) and \(\Pi_{\Sigma_t}^{(k)} = V_k(t) V_k(t)^T\) denotes the projection to the top-\(k\) subspace of the symmetric PSD \(\Sigma_t\). Then, from Davis-Kahan (Corollary 8 in [26]) and using the fact for symmetric PSD matrices eigen-values and singular values are the same, we have

\[
\| \Pi_{M_t}^{(k)} (\mathbb{I} - \Pi_{\Sigma_t}^{(k)}) \|_2 \leq \frac{\| M_t - \Sigma_t \|_2}{\lambda_k(M_t) - \lambda_{k+1}(\Sigma_t)}
\]

\[
\| (\mathbb{I} - \Pi_{M_t}^{(k)}) \Pi_{\Sigma_t}^{(k)} \|_2 \leq \frac{\| M_t - \Sigma_t \|_2}{\lambda_k(\Sigma_t) - \lambda_{k+1}(M_t)}.
\]

Recall, from [21] (Section 4.3) and [19] (Section 8.1.2), e.g., Corollary 8.1.6, we have

\[
|\lambda_k(M_t) - \lambda_k(\Sigma_t)| \leq \| M_t - \Sigma_t \|_2.
\]

From Theorem 2 and Lemma 2 with \(Y = \| M_t - \Sigma_t \|_2, A = c, B = \frac{4G \rho \sqrt{\ln \rho \gamma_2(W, d)}}{\sqrt{m}}\), we have

\[
\mathbb{E} \left[ \| M_t - \Sigma_t \|_2 \right] \leq O \left( \frac{G \rho \sqrt{\ln \rho \gamma_2(W, d)}}{\sqrt{m}} \right),
\]

Let \(c_0 = O \left( G \rho \sqrt{\ln \rho \gamma_2(W, d)} \right)\). For \(m \geq \frac{c_0^2}{\alpha_t^2}\), we have

\[
\mathbb{E} \left[ \| M_t - \Sigma_t \|_2 \right] \leq \frac{uc_0}{\sqrt{m}} \leq \frac{\alpha_t}{2}.
\]
Then, for (43), we have
\[
\| \Pi_{M_t}^{(k)} (\mathbb{I} - \Pi_{\Sigma_t}^{(k)}) \|_2 \leq \frac{\| M_t - \Sigma_t \|_2}{\lambda_k(M_t) - \lambda_{k+1}(\Sigma_t)}
\]
\[
= \frac{\| M_t - \Sigma_t \|_2}{\lambda_k(\Sigma_t) - \lambda_{k+1}(\Sigma_t) - (\lambda_k(\Sigma_t) - \lambda_k(M_t))}
\]
\[
\leq \frac{\| M_t - \Sigma_t \|_2}{\alpha_t - \| M_t - \Sigma_t \|_2}.
\]
(48)

Then, for (44), we have
\[
\| (\mathbb{I} - \Pi_{M_t}^{(k)}) \Pi_{\Sigma_t}^{(k)} \|_2 \leq \frac{\| M_t - \Sigma_t \|_2}{\lambda_k(\Sigma_t) - \lambda_{k+1}(M_t)}
\]
\[
= \frac{\| M_t - \Sigma_t \|_2}{(\lambda_k(\Sigma_t) - \lambda_{k+1}(\Sigma_t)) + (\lambda_{k+1}(\Sigma_t) - \lambda_{k+1}(M_t))}
\]
\[
\leq \frac{\| M_t - \Sigma_t \|_2}{\alpha_t - \| M_t - \Sigma_t \|_2}.
\]
(49)

Combining these two bounds and (46), with \( c_0 = O\left(G\rho \sqrt{\ln p / \gamma^2 d}ight) \), we have
\[
E \left[ \| \hat{V}_k(t)^T \hat{V}_k(t) - V_k(t)^T V_k(t) \|_2 \right] \leq \frac{2\alpha_t}{\alpha_t - E[\| M_t - \Sigma_t \|_2]} - 2 \leq \frac{2c_0}{\alpha_t - \frac{c_0}{\sqrt{m}}}
\]
(50)

Using (47) such that \( \frac{c_0}{\sqrt{m}} \leq \frac{\alpha_t}{2} \), we have
\[
E \left[ \| \hat{V}_k(t)^T \hat{V}_k(t) - V_k(t)^T V_k(t) \|_2 \right] \leq \frac{G\rho \sqrt{\ln p / \gamma^2 (W, d)}}{\alpha_t \sqrt{m}}.
\]
(51)

That completes the proof.

Lemma 2 (Lemma 2.2.3 in [33]) Consider a r.v. \( Y \geq 0 \) which satisfies
\[
\forall u > 0, \mathbb{P}(Y \geq u) \leq A \exp \left( - \frac{u^2}{B^2} \right)
\]
(52)

for certain numbers \( A \geq 2 \) and \( B > 0 \). Then
\[
EY \leq CB \sqrt{\log A},
\]
(53)

where \( C \) denotes a universal constant.
A.2 Geometry of Gradients and $\gamma_2$ functions.

In this section, we justify our assumptions about the gradient space and provide more examples of the gradient space structure and the corresponding $\gamma_2$ functions.

To utilize the structure of gradients as example shown in Figure 2 instead of focusing on the $\gamma_2(W,d)$, we can also derive the uniform convergence bound using the measurement $d$ on the set of population gradient $m = E_{z \in P} [\nabla (w,z)]$. We consider a mapping $f : W \mapsto M$ from the parameter space $W$ to the gradient space $M$, where $f$ can be considered as $f(w) = E_{z \in P} [\nabla (w,z)]$. With $m_t = E_{z \in P} [\nabla (w,z)]$ and $M$ to be the space of the population gradient $m_t \in M$, the pseudo metric $d(w, w')$ can be written as $d(w, w') = d(f(w), f(w')) = d(m, m')$ with $m = E_{z \in P} [\nabla (w,z)]$ and $m' = E_{z \in P} [\nabla (w',z)]$.

With such a mapping $f$, the admissible sequence $\Gamma_W = \{W_n : n \geq 0\}$ of $W$ in the proof of Theorem 2 corresponds to the admissible sequence $\Gamma_M = \{M_n : n \geq 0\}$ of $M$. The $\gamma_2(W,d)$ is defined as $\gamma_2(W,d) = \inf_{\Gamma_W} \sup_{w \in W} \sum_{n \geq 0} 2^{n/2} d(w, W_n)$, where $\Gamma_M = \inf_{\Gamma_M} \sup_{m \in M} \sum_{n \geq 0} 2^{n/2} d(m, M_n) = \gamma_2(M,d)$. Considering $d(m, m') = \|m - m'\|_2$, the $\gamma_2(M,d)$ will be the same order as the Gaussian width of $M$, i.e., $w(M) = \inf_{\Gamma_M} \sup_{m \in M} \|m - \mu\|_2$.

**Ellipsoid.** The Gaussian width $w(M)$ depends on the structure of the gradient $m$. In Figure 2 we observe that, for each coordinates of the gradient is of small value along the training trajectory and thus $M$ includes all gradients living in an ellipsoid, i.e., $M = \{m_t \in \mathbb{R}^p | \sum_{j=1}^p m_t(j)^2/e(j)^2 \leq 1, e \in \mathbb{R}^p\}$. Then we have $\gamma_2(M,d) \leq c_1 w(M) \leq c_2 \|e\|_2$, where $c_1$ and $c_2$ are absolute constants. If the elements of $e$ sorted in decreasing order satisfy $e(j) \leq c_3/\sqrt{j}$ for all $j \in [p]$, then $\gamma_2(M,d) \leq O(\sqrt{\log p})$.

**Composition.** Based on the composition properties of $\gamma_2$ functions, one can construct additional examples of the gradient spaces. If $M = M_1 + M_2 = \{m_1 + m_2, m_1 \in M_1, m_2 \in M_2\}$, the Minkowski sum, then $\gamma_2(M,d) \leq c(\gamma_2(M_1,d) + \gamma_2(M_2,d))$ (Theorem 2.4.15 in [33]), where $c$ is an absolute constant. If $M$ is a union of several subset, i.e., $M = \cup_{i=1}^p M_i$, then by using an union bound on Theorem 2 we have $\gamma_2(M,d) \leq \sqrt{\log D} \max M_i \gamma_2(M_i,d)$. Thus, if $M$ is an union of $D = p^a$ ellipsoids, i.e., polynomial in $p$, then $\gamma_2(M, \|\cdot\|_2) \leq O(\sqrt{\log p})$.

### B Proofs for Section 3.2

In this section, we present the proofs for Section 3.2.

**Theorem 4 (Smooth and Non-convex)** For $\rho$-smooth function $\hat{L}_n(w)$, under the Assumption 1 and 2, let $\Lambda = \sum_{i=1}^p 1/\alpha_i^2$, for any $\epsilon, \delta > 0$, with $T = O(n^2 \epsilon^2)$ iterations, and step size $\eta_t = \frac{1}{\sqrt{T}}$, PDP-SGD achieves:

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\| \nabla \hat{L}_n(w_t) \|^2_2] \leq O\left( \frac{\kappa \rho G^2}{n \epsilon} \right) + O\left( \frac{\Lambda G^4 \rho^2 \gamma_2^2(W,d) \ln p}{m} \right). \quad (11)$$

Additionally, assuming the principal component of the gradient dominates, i.e., there exist $c > 0$, such that $\frac{1}{T} \sum_{t=1}^T \| \nabla \hat{L}_n(w_t) \|^2_2 \leq c \frac{1}{T} \sum_{t=1}^T \| \nabla \hat{L}_n(w_t) \|^2_2$, we have

$$\mathbb{E}[\| \nabla \hat{L}_n(w_R) \|^2_2] \leq O\left( \frac{\kappa \rho G^2}{n \epsilon} \right) + O\left( \frac{\Lambda G^4 \rho^2 \gamma_2^2(W,d) \ln p}{m} \right), \quad (12)$$

where $w_R$ is uniformly sampled from $\{w_1, ..., w_T\}$.
Proof: Recall that $g_t = \frac{1}{|B_t|} \sum_{z_i \in B_t} \nabla \ell(w_t, z_i)$. With $B_t$ uniformly sampled from $S$, we have
\begin{equation}
E_t[g_t] = \nabla \hat{L}_n(w_t). \tag{54}
\end{equation}
Recall that the update of Algorithm 1 is
\begin{equation}
\bar{g}_t = \hat{V}_k \hat{V}_k^T (g_t + b_t), \quad \text{and} \quad w_{t+1} = w_t - \eta \bar{g}_t. \tag{55}
\end{equation}
Let $\bar{g}_t = g_t + \hat{V}_k \hat{V}_k^T b_t$, and $\Delta_t = \hat{V}_k \hat{V}_k^T g_t - g_t$. Then we have
\begin{equation}
\bar{g}_t = g_t + \Delta_t. \tag{56}
\end{equation}
Since $b_t$ is a zero mean Gaussian vector, we have
\begin{equation}
E_t[\bar{g}_t] = g_t. \tag{57}
\end{equation}
For $\rho$-smooth\footnote{The Assumption 2 suggests that the $\hat{L}_n(w)$ is $\rho$-smooth.} function $\hat{L}_n(w)$, conditioned on $w_t$, we have
\begin{align*}
E_t \left[ \hat{L}_n(w_{t+1}) \right] & \leq \hat{L}_n(w_t) + E_t \left[ \nabla \hat{L}_n(w_t), w_{t+1} - w_t \right] + \frac{\rho}{2} \eta_t^2 E_t \left[ \| \bar{g}_t \|_2^2 \right] \\
& = \hat{L}_n(w_t) - \eta_t E_t \left[ \nabla \hat{L}_n(w_t), \bar{g}_t + \Delta_t \right] + \frac{\rho}{2} \eta_t^2 E_t \left[ \| \bar{g}_t + \Delta_t \|_2^2 \right] \\
& = \hat{L}_n(w_t) - \eta_t \left\langle \nabla \hat{L}_n(w_t), \nabla \hat{L}_n(w_t), E_t[\Delta_t] \right\rangle + \rho \eta_t^2 E_t \left[ \| \bar{g}_t \|_2^2 + \| \Delta_t \|_2^2 \right] \\
& \leq \hat{L}_n(w_t) - \eta_t \left\| \nabla \hat{L}_n(w_t) \right\|_2^2 - \eta_t \left\langle \nabla \hat{L}_n(w_t), E_t[\Delta_t] \right\rangle + \rho \eta_t^2 E_t \left[ \| \bar{g}_t \|_2^2 + \| \Delta_t \|_2^2 \right]. \tag{58}
\end{align*}
Rearrange the above inequality, we have
\begin{equation}
\eta_t \left\| \nabla \hat{L}_n(w_t) \right\|_2^2 + \eta_t E_t \left\langle \nabla \hat{L}_n(w_t), \Delta_t \right\rangle \leq \hat{L}_n(w_t) - E_t \left[ \hat{L}_n(w_{t+1}) \right] + \rho \eta_t^2 E_t \left[ \| \bar{g}_t \|_2^2 + \| \Delta_t \|_2^2 \right]. \tag{59}
\end{equation}
For $D_{t,1}$, let
\begin{equation}
\nabla \hat{L}_n(w_t) = \Pi_{Q_t} \nabla \hat{L}_n(w_t) + \Pi_{Q_t^\perp} \nabla \hat{L}_n(w_t) \tag{60}
\end{equation}
where $Q_t$ is $\hat{V}_k(t) \hat{V}_k(t)^T$ and $\Pi_{Q_t}$ is the projection. $Q_t^\perp$ is the null space of $Q$.
\begin{equation}
\Pi_{Q_t} \nabla \hat{L}_n(w_t) = \hat{V}_k(t) \hat{V}_k(t)^T \nabla \hat{L}_n(w_t) \tag{61}
\end{equation}
and
\begin{equation}
\Pi_{Q_t^\perp} \nabla \hat{L}_n(w_t) = \left[ I - \hat{V}_k(t) \hat{V}_k(t)^T \right] \nabla \hat{L}_n(w_t). \tag{62}
\end{equation}
We have
\begin{align*}
\Delta_t &= \Pi_{Q_t} [\hat{V}_k(t) \hat{V}_k(t)^T g_t - g_t] + \Pi_{Q_t^\perp} [\hat{V}_k(t) \hat{V}_k(t)^T g_t - g_t] \\
&= -\Pi_{Q_t^\perp} [g_t]. \tag{63}
\end{align*}
So that we have

\[
D_{t,1} = \mathbb{E}_t \left\langle \nabla \hat{L}_n(w_t), \Delta_t \right\rangle \\
= -\mathbb{E}_t \left\langle \Pi_{Q_t^i}[\nabla \hat{L}_n(w_t)] + \Pi_{Q_t^i}[\nabla \hat{L}_n(w_t)]^T, \Pi_{Q_t^i}[g_t] \right\rangle \\
= -\mathbb{E}_t \left\langle \Pi_{Q_t^i}[\nabla \hat{L}_n(w_t)], \left[ I - \hat{V}_k(t) \hat{V}_k(t)^T \right] g_t \right\rangle \\
= -\left\langle \left[ I - \hat{V}_k(t) \hat{V}_k(t)^T \right] \nabla \hat{L}_n(w_t), \left[ I - \hat{V}_k(t) \hat{V}_k(t)^T \right] \nabla \hat{L}_n(w_t) \right\rangle \\
\leq -\nabla \hat{L}_n(w_t)^T \left[ I - \hat{V}_k(t) \hat{V}_k(t)^T \right] \nabla \hat{L}_n(w_t)
\]  

(64)

\[-\eta_t \| \nabla \hat{L}_n(w_t) \|_2^2 - \eta_t D_{t,1} = -\eta_t \nabla \hat{L}_n(w_t)^T \left[ I \| \nabla \hat{L}_n(w_t) \|_2 + \eta_t \nabla \hat{L}_n(w_t)^T \left[ I - \hat{V}_k(t) \hat{V}_k(t)^T \right] \nabla \hat{L}_n(w_t) \right] \nabla \hat{L}_n(w_t) \]

\[= -\eta_t \nabla \hat{L}_n(w_t)^T \left[ \hat{V}_k(t) \hat{V}_k(t)^T \right] \nabla \hat{L}_n(w_t) \]

\[= -\eta_t \| \hat{V}_k(t) \hat{V}_k(t)^T \nabla \hat{L}_n(w_t) \|_2^2
\]  

(65)

So that we have

\[\eta_t \| \hat{V}_k(t) \hat{V}_k(t)^T \nabla \hat{L}_n(w_t) \|_2^2 \leq \hat{L}_n(w_t) - \mathbb{E}_t \left[ \hat{L}_n(w_{t+1}) \right] + \rho \eta_t^2 \mathbb{E}_t \left[ \| g_t \|_2^2 + \| \Delta_t \|_2^2 \right]_{D_{t,2}}
\]  

(66)

Note that

\[\| \Delta_t \|_2^2 = \| \hat{V}_k \hat{V}_k^T g_t - g_t \|_2^2 = \| \Pi_{Q_t^i}[g_t] \|_2^2 \leq G^2.
\]  

(67)

We also have

\[\mathbb{E}_t \| g_t \|_2^2 \leq \mathbb{E}_t \left[ \left\| 2 \| g_t \|_2^2 + 2 \left\| \hat{V}_k \hat{V}_k^T b_t \right\|_2^2 \right\|_2^2 \right] \leq 2 (G^2 + k \sigma^2).
\]  

(68)

So that

\[D_{t,2} = \mathbb{E}_t \left[ \| g_t \|_2^2 + \| \Delta_t \|_2^2 \right] \leq 3G^2 + k \sigma^2.
\]  

(69)

Bringing the upper bound of \(D_{t,2}\) to (66), setting \(\eta_t = \frac{1}{\sqrt{T}}\), using telescoping sum and taking the expectation over all iterations, we have

\[\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_t \| \hat{V}_k(t) \hat{V}_k(t)^T \nabla \hat{L}_n(w_t) \|_2^2 \leq \frac{\hat{L}_n(w_1) - \hat{L}_n^*}{\sqrt{T}} + \frac{\rho (3G^2 + k \sigma^2)}{\sqrt{T}}
\]  

(70)

With triangle inequality and Theorem [3], we have

\[\| \hat{V}_k(t) \hat{V}_k(t)^T \nabla \hat{L}_n(w_t) \|_2^2 \leq 2 \| \hat{V}_k(t) \hat{V}_k(t)^T \nabla \hat{L}_n(w_t) \|_2^2 + 2 \left\| \left( \hat{V}_k(t) \hat{V}_k(t)^T - \hat{V}_k(t) \hat{V}_k(t)^T \right) \nabla \hat{L}_n(w_t) \right\|_2^2 \]

\[\leq 2 \| \hat{V}_k(t) \hat{V}_k(t)^T \nabla \hat{L}_n(w_t) \|_2^2 + O \left( \frac{G^4 \rho^2 \ln \rho \gamma^2_2(W, d)}{\alpha^2 t m} \right).
\]  

(71)
With (70), we have
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left\| V_k(t) V_k(t)^T \nabla \hat{L}_n(w_t) \right\|_2^2 \leq \frac{\hat{L}_n(w_1) - \hat{L}_n^*}{\sqrt{T}/2} + \frac{\rho(3G^2 + k\sigma^2)}{\sqrt{T}/2} + O \left( \frac{\Lambda G^4 \rho^2 \gamma_2^2(W, d) \ln p}{m} \right).
\]
(72)

where \( \Lambda = \sum_{t=1}^{T} \frac{1}{\alpha_t} \).

Let \( \nabla \hat{L}_n(w_t) \) and \( \nabla \hat{L}_n(w_t) \).

Assuming there exist \( c > 0 \), we have
\[
\sum_{t=1}^{T} \mathbb{E} \left\| \nabla \hat{L}_n(w_t) \right\|_2^2 \leq c \sum_{t=1}^{T} \mathbb{E} \left\| \nabla \hat{L}_n(w_t) \right\|_2^2.
\]
(73)

Then we have
\[
\frac{1}{T} \mathbb{E} \left\| \nabla \hat{L}_n(w_t) \right\|_2^2 \leq (1 + c) \left( \frac{\hat{L}_n(w_1) - \hat{L}_n^*}{\sqrt{T}/2} + \frac{\rho(3G^2 + k\sigma^2)}{\sqrt{T}/2} + O \left( \frac{\Lambda G^4 \rho^2 \gamma_2^2(W, d) \ln p}{m} \right) \right).
\]
(74)

Take \( T = n^2 \varepsilon^2 \), with \( \mathbb{E} \left[ \| \nabla \hat{L}_n(w_R) \|_2^2 \right] = \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \| \nabla \hat{L}_n(w_t) \|_2^2 \), we have
\[
\mathbb{E} \| \nabla \hat{L}_n(w_R) \|_2^2 \leq \tilde{O} \left( \frac{kG^2}{n\varepsilon} \right) + O \left( \frac{\Lambda G^4 \rho^2 \gamma_2^2(W, d) \ln p}{m} \right),
\]
(75)

where \( w_R \) is uniformly sampled from \( \{w_1, \ldots, w_T\} \).

**Theorem 5 (Convex and Lipschitz)** For \( G \)-Lipschitz and convex function \( \hat{L}_n(w) \), under the Assumption [72], and assuming \( \Sigma_t \) is of rank-\( k \), let \( \Lambda = \sum_{t=1}^{T} \frac{1}{\alpha_t} \), for any \( c, \delta > 0 \), with \( T = O(n\varepsilon) \), step size \( \eta_t = \frac{1}{\sqrt{T}} \), the DPD-SGD achieves
\[
\mathbb{E} \left[ \hat{L}_n(w) \right] - \hat{L}_n(w^*) \leq O \left( \frac{kG^2}{n\varepsilon} \right) + O \left( \frac{\Lambda G^4 \rho^2 \gamma_2^2(W, d) \ln p}{m} \right),
\]
(13)

where \( w = \sum_{t=1}^{T} w_t \), and \( w^* \) is the minimizer of \( \hat{L}_n(w) \).

**Proof:** By convexity of \( \hat{L}_n(w) \), we have
\[
\hat{L}_n(w_t) - \hat{L}_n(w^*) \leq \langle w_t - w^*, \nabla \hat{L}_n(w_t) \rangle.
\]
(76)

At iteration \( t \), we have \( w_{t+1} = w_t - \eta_t \bar{g}_t \), where \( \bar{g}_t = \bar{V}_k \bar{V}_k^T g_t + \bar{V}_k \bar{V}_k^T b_t \).

Let \( g_t = g_t + \bar{V}_k \bar{V}_k^T b_t \), and \( \Delta_t = \bar{V}_k \bar{V}_k^T g_t - g_t \). Then we have
\[
\bar{g}_t = g_t + \Delta_t.
\]

Recall that \( g_t = \frac{1}{|B_t|} \sum_{z_i \in B_t} \nabla \ell(w_t, z_i) \). With \( B_t \) uniformly sampled from \( S \), we have
\[
\mathbb{E}_t[g_t] = \nabla \hat{L}_n(w_t).
\]
(78)
Since $b_t$ is a zero mean Gaussian vector, we have $E_t[g_t] = g_t$.

By convexity, conditioned at $w_t$, we have
\[
E_t[\tilde{L}_n(w_t) - \tilde{L}_n(w^*)] \leq E_t(w_t - w^*, g_t) = E_t(w_t - w^*, \tilde{g}_t) \\
= \frac{1}{\eta_t} E_t(w_t - w^*, w_t - w_{t+1} - \eta_t \Delta_t) \\
= \frac{1}{2\eta_t} E_t(\|w_t - w^*\|^2 + \eta_t^2 \|\tilde{g}_t\|^2 - \|w_{t+1} - w^* - \eta_t \Delta_t\|^2) \\
= \frac{1}{2\eta_t} E_t(\|w_t - w^*\|^2 + \eta_t^2 \|\tilde{g}_t\|^2 - \|w_{t+1} - w^*\|^2 - \|w_{t+1} - w^* - \eta_t \Delta_t\|^2 + 2\langle w_{t+1} - w^*, \eta_t \Delta_t \rangle) \\
= \frac{1}{2\eta_t} E_t(\|w_t - w^*\|^2 - \|w_{t+1} - w^*\|^2) + \frac{\eta_t}{2} E_t(\|\tilde{g}_t\|^2) - \frac{\eta_t}{2} E_t(\|\Delta_t\|^2) + E_t(w_{t+1} - w^*, \Delta_t) \\
\leq \left(\frac{1}{2\eta_t} - \frac{\eta_t}{2}\right) E_t(\|w_t - w^*\|^2 - \|w_{t+1} - w^*\|^2) + \eta_t (G^2 + k\sigma^2) + B E_t \|\Delta_t\|_2, \tag{79}
\]
where (a) is true since
\[
E_t(\|\tilde{g}_t\|^2) \leq E_t[2\|g_t\|^2 + 2\|\hat{V}_k V_k^T b_t\|^2] \\
\leq 2(G^2 + k\sigma^2), \tag{80}
\]
and $\|w_{t+1} - w^*\|^2 \leq B^2$.

Let $\eta_t = \frac{1}{\sqrt{T}}$, taking the expectation over all iterations and sum over $t = 1, ..., T$, we have
\[
\frac{1}{T} E \left[ \sum_{t=1}^T \tilde{L}_n(w_t) - \tilde{L}_n(w^*) \right] \leq \frac{\|w_0 - w^*\|^2 / 2 + G^2 + k\sigma^2}{\sqrt{T}} + \frac{B \sum_{t=1}^T E_t \|\Delta_t\|_2}{T}. \tag{81}
\]

From Theorem\[\] we have
\[
E_t(\|\Delta_t\|_2) = E \left[ \left\| \hat{V}_k(t) \hat{V}_k(t)^T g_t - V(t) V(t)^T g_t \right\|_2 \right] \\
\leq E \left[ \left\| \hat{V}_k(t) \hat{V}_k(t)^T - V(t) V(t)^T \right\|_2 \right] G \\
\leq G E \left[ \left\| \hat{V}_k(t) \hat{V}_k(t)^T - V_k(t) V_k(t)^T \right\|_2 + \|V_k(t) V_k(t)^T - V(t) V(t)^T\|_2 \right] \\
\leq O \left( \frac{G \rho \ln p_{\gamma_2}(W, d)}{\alpha_t \sqrt{m}} \right), \tag{82}
\]
where the last inequality holds because the $\Sigma_t$ is of rank $k$ and $V(t) = V_k(t)$.

Bring this to (81), use the fact that $\|w_0 - w^*\| < B$, with Jensen’s inequality we have
\[
E \left[ \tilde{L}_n(w) - \tilde{L}_n(w^*) \right] \leq \frac{B^2 + G^2 + k\sigma^2}{\sqrt{T}} + O \left( \frac{e G \rho \ln p_{\gamma_2}(W, d)}{\sqrt{m}} \right). \tag{83}
\]

\[\]For convex problem, we consider $w \in \mathcal{H}$, where $\mathcal{H} = \{w : \|w\| \leq B\}$. 

27
where $\Lambda = \sum_{t=1}^{T} \frac{1}{\alpha_t}$.

With $\sigma^2 = \frac{G^2 T}{n \epsilon^2}$, let $T = n \epsilon$, we have

$$E \left[ \hat{L}_n(\bar{w}) - \hat{L}_n(w^*) \right] \leq O \left( \frac{kG^2B^2}{n \epsilon} \right) + O \left( \frac{\Lambda G \rho \ln \gamma_2(W,d)}{\sqrt{m}} \right).$$

(84)

That completes the proof.