On tame strongly simply connected algebras

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Dedicated to José Antonio de la Peña on the occasion of his 60th birthday.

Abstract

In this survey we present the criterion for tameness of strongly simply connected algebras due to Brüstle, de la Peña and Skowroński. We recall relevant concepts of representation theory and discuss some applications and connections to other problems.

1 Introduction

The criterion of tameness of strongly simply connected algebras - Theorem 1 below, due to Thomas Brüstle, José Antonio de la Peña and Andrzej Skowroński [13] can be considered as a culmination of a research program aimed on better understanding of the concept of representation types of algebras. The purpose of this note is to present the result and the on a background of classical concepts and results of representation theory, indicating also some less obvious connections to other mathematical ideas. For more information the reader is referred to [44].

Theorem 1 [13, Main Theorem] Let $A$ be a strongly simply connected algebra. The following statements are equivalent:

1. $A$ is a tame algebra.

2. The Tits form $q_A$ of $A$ is weakly nonnegative.

Throughout this article $A$ is an associative algebra of finite dimension over an algebraically closed field $K$. Usually we assume that $A$ is basic, equivalently, by classical Gabriel’s theorem, $A$ is isomorphic to a bound quiver algebra, that is

$$A \cong KQ/I,$$

for some finite quiver $Q$ and an admissible ideal $I$ in the path algebra $KQ$. We refer to [2] for more information about the basic concepts of representation theory of bound quivers and algebras.
The general question of the representation theory of finite dimensional algebras can be formulated as follows: classify (indecomposable) right $A$-modules of finite $K$-dimension up to isomorphism. Of course, we are far from satisfactory solution of the problem raised this way.

Finite dimensional modules can be represented by sequences of matrices and this can be done in several ways. One way is to consider representations of the bound quiver $(Q, I)$ associated to the algebra $A$, another - based on the idea of analyzing module through its projective presentation - leads to the concept of Roiter’s matrix problems [50] or BOCS introduced by Drozd [19].

Finding isomorphism classes of $A$-modules is equivalent to finding "canonical forms" of matrices up to certain admissible operations. Let us remark that matrix problem algorithms allow to obtain a classification of indecomposable $A$-modules of $K$-dimension bounded by a number fixed à priori. That does not mean, of course, that a complete classification is always possible.

Let us recall some important facts concerning the easiest situation - the case of representation-finite algebras. By the definition, an algebra $A$ is representation-finite (or of finite representation type) if it allows only finitely many isomorphism classes of finite dimensional indecomposable $A$-modules. By famous 1st Brauer-Thrall Conjecture, which is now a theorem by Roiter [49] and Auslander [1], $A$ is representation-finite if and only if there is a common bound $m_A$ for the dimensions of indecomposable $A$-modules. Moreover, if $A$ is not representation-finite (remember that $K$ is algebraically closed), then there exist infinitely many numbers $m \in \mathbb{N}$ such that for each of those $m$ there is infinitely many pairwise non-isomorphic $A$-modules of $K$-dimension $m$. This is the content of the 2nd Brauer-Thrall Conjecture confirmed by several authors, see [1], [8], [12], [21], (also [38]). The reader is referred to [46] for more on Brauer-Thrall conjectures. Validity of the 2nd Brauer-Thrall conjecture allows to prove that given a natural number $d$, the class of $d$-dimensional representation-finite algebras over algebraically closed fields is finitely axiomatizable in a suitable first order language, see [27], [28], [29, Theorem 12.54].

This corresponds to the fact, that given a natural number $d$ there is only finitely many isomorphism classes of representation-finite algebras having the $K$-dimension equal $d$. This is, in turn, a consequence of another deep and difficult result obtained in the 80’s of the 20th century: the proof of the existence of a multiplicative basis of every representation-finite algebra over an algebraically closed field [5].

Note that Jensen and Lenzing’s arguments for the finite axiomatizability are not constructive. However, this can be made more explicit by applying a "numerical version" of 2nd Brauer-Thrall Conjecture, see [24, 2.4]. Finally, it is possible to verify in a finite number of steps if given an algebra

\footnote{By [58] it is enough to assume that there exists one such $m$.}
A is representation-finite or not. In practice, for this purpose we use Bon- 
gartz’s criterion for representation-finiteness, which shall be discussed later, 
Auslander-Reiten theory, Galois covering techniques etc. These methods al-
low, in the representation-finite case, to give a list of representatives of the
isomorphism classes of the indecomposable $A$-modules.

If $A$ is not representation-finite, then there are infinitely many iso-
morphism classes of indecomposable modules of dimension $m$ for infinitely many
numbers $m$. It may happen that for every $m$, all but finitely many representa-
tives of those classes can be arranged in finitely many "rational 1-parameter
families". In this case we say that $A$ is of tame representation type or simply
tame. We omit precise formulations of the well-known concepts appear-
ing here, the reader is referred to a reach literature on the subject, see for
example [48], [51], [52].

Otherwise the problem of classification of two non-commuting square ma-
trices up to simultaneous conjugations can be "embedded" into the problem
of classifying the indecomposable $A$-modules, in which case we say that $A$ is
of wild representation type or wild.

This is essentially the content of the famous Tame-Wild Dichotomy:

**Theorem 2** [20], [15] $A$ is either tame or wild.

We can not hope obtain a complete classification of isomorphism classes
of indecomposables in the wild case. Also in the tame case this task can be
unreachable. The level of complication is reflected somehow by the stratifi-
cation of the class of tame algebras into subclasses: domestic, polynomial
growth. Let us recall the notions introduced in [53], see also [51]. See also
[51, Section 14.4], [52, Chapter XIX].

Let $A$ be a tame algebra. Given a natural number $m$ let $\mu_A(m)$ be the
least number of 1-parameter families needed to parameterize (all but finite
number) indecomposables of dimension $m$.

- If $\mu_A(m)$ is bounded, then $A$ is finite growth. By the results of [16] an
  algebra is of finite growth if and only if it is domestic in the sense of
  [17].

- If $\mu_A(m) \leq m^N$ for some $N$ and arbitrary $m$, then $A$ is said to be of
  polynomial growth.

We have remarked that given an algebra $A$ it is possible to verify wether
$A$ is representation-finite or not and it is related with finite axiomatizability
of representation-finiteness. This fact is also connected with Gabriel’s result
that "finite representation type is open" [22], more precisely: representation-
finitesimal algebras of fixed dimension $d$ induce Zariski-open subset in the variety

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2 According to our convention a representation-finite algebra is tame.
of $d$-dimensional associative algebras. Indeed, an analysis of the proof of Geiss’s result on degenerations of wild algebras [23] shows that the openness of the varieties of representation-finite algebras can be derived from the existence of finite number of axioms of representation-finiteness.

Let us collect some known analogies for tameness. First of all, the class of tame algebras (of fixed dimension, over algebraically closed fields) is axiomatizable [30]. This is related with the observation, that if $A$ is wild, then we can verify it in finite number of steps. However we do not know if tame is finitely axiomatizable, that means that there is an algebra $A$ such that we will never be sure if $A$ is tame or not. If we knew that tame is finite axiomatizable, we would prove that "tame is open" [30], by the same kind of arguments that we mentioned above with respect to openness of finite representation type.

Theorem 1 provides us with a criterion for tameness valid for strongly simply connected algebras. The condition equivalent to tameness is expressed in term of Tits quadratic form.

2 Tits quadratic form

Let $A$ be a basic finite dimensional $K$-algebra of the form $KQ/I$, where $Q = (Q_0, Q_1, s, t)$ is a finite acyclic quiver and $I$ is an admissible ideal of the path algebra $KQ$. Recall that by a relation in the path algebra $KQ$ we mean a linear combination of paths having the same sink and the same source. Let $R$ be a minimal set of relations generating the ideal $I$. $R$ decomposes into the disjoint union $\bigcup_{i,j \in Q_0} R_{i,j}$, where $R_{i,j}$ denote the set of the elements starting at the vertex $i$ and ending at $j$. Although $R$ is not uniquely determined by $I$, the numbers $r_{i,j} := |R_{i,j}|$ do not depend on the set $R$ [6].

The Tits quadratic form of the algebra $A$ is the integer quadratic form

$$q_A : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$$

defined by the formula

$$q_A(z) = \sum_{i \in Q_0} z_i^2 - \sum_{i \rightarrow j \in Q_1} z_i z_j + \sum_{i,j \in Q_0} r_{i,j} z_i z_j$$

(1)

for $z = (z_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}$.

3By applying, for example, matrix problems techniques: we classify canonical forms of indecomposable modules of dimension $m$, starting from $m = 1$ and increasing $m$ until we meet 2-parameter family of pairwise non-isomorphic indecomposable $m$-dimensional modules.

4As usual, $Q_0$ is the set of vertices, $Q_1$ is the set of arrow and $s(\alpha)$ and $t(\alpha)$ denote the source and the target of the arrow $\alpha$.

5That means $(KQ_1)^m \subseteq I \subseteq (KQ_1)^2$ for some $m$. 

4
Let us also recall another quadratic form associated with an algebra: the *Euler quadratic form*. For this purpose denote by $S_i$ the simple $A$-module corresponding to the vertex $i \in Q_0$ and assume that $A$ has finite global dimension. The quadratic form

$$
\chi_A : \mathbb{Z}^{Q_0} \to \mathbb{Z},
$$

defined by the formula

$$
\chi_A(z) = \sum_{s=0}^{\infty} \sum_{i,j \in Q_0} (-1)^s \dim \Ext_A^s(S_i, S_j) z_i z_j.
$$

(2)

is called the *Euler quadratic form* of the algebra $A$.

The Euler form is often called a homological form, by obvious reasons, whereas the Tits form - the geometric quadratic form of an algebra. This is because the summands of the right hand side of (1) have interpretations as (bounds of) dimensions of some geometric objects associated with $A$ and $z$. Thanks to this interpretation one can prove that:

1. If $A$ is representation-finite, then $q_A$ is weakly positive, that is, $q_A(z) > 0$ for all non-zero vectors $z$ with all coordinates nonnegative [6].

2. If $A$ is tame, then $q_A$ is weakly nonnegative, that is, $q_A(z) \geq 0$ for all vectors $z$ with all coordinates nonnegative [II].

The assertion 1. is the easier part of the Bongartz’s criterion for representation-finiteness, we shall discuss the converse later. The converse of 2. (valid under suitable conditions on $A$) is the content of Theorem [II]. Let us recall the main steps of the proof of 2, following de la Peña.

It is well-known that the $A$-modules of dimension vector $z = (z_i)_{i \in Q_0}$ can be represented by the tuples $(M_\alpha)_{\alpha \in Q_1} \in \prod M_{z_{\alpha}(a) \times z_{\beta}(a)}(K)$ of matrices satisfying the relations defining the ideal $I$. These tuples form the *variety of modules* of dimension vector $z$ denoted by $	ext{mod}_A(z)$ and, by Krull Theorem, it is easy to observe that

$$
\dim \text{mod}_A(z) \geq \sum_{i \to j \in Q_1} z_i z_j - \sum_{i,j \in Q_0} r_{i,j} z_i z_j.
$$

(3)

The isomorphism classes of modules of dimension vector $z = (z_i)_{i \in Q_0}$ can be identified with the orbits of the natural algebraic group action

$$
Gl(z) \times \text{mod}_A(z) \to \text{mod}_A(z)
$$

of the group $Gl(z) = \prod_{i \in Q_0} Gl_{z_i}(K)$. Clearly

$$
\dim Gl(z) = \sum_{i \in Q_0} z_i^2.
$$

(4)
Now, by (1), (3) and (4),
\[ q_A(z) \geq \dim \text{Gl}(z) - \dim \text{mod}_A(z). \] (5)

Now it is time to apply de la Peña’s argument: if \( A \) is tame, representatives of the indecomposables of fixed dimension can be arranged into finite number of "one-parameter families". Therefore, there is a "\(|z|\)-parameter family" of representatives of the modules of dimension vector \( z \), where \( |z| = \sum_{i \in Q_0} z_i \). It follows that if \( A \) is tame, then
\[ \dim \text{Gl}(z) + \sum_{i \in Q_0} z_i \geq \dim \text{mod}_A(z). \] (6)

Combining (5) and (6) we obtain
\[ q_A(z) \geq \dim \text{Gl}(z) - \dim \text{mod}_A(z) \geq -\sum_{i \in Q_0} z_i. \]

As the left hand side of the above inequality depends quadratically on \( z \), whereas the right hand side - linearly, we conclude that \( q_A(z) \geq 0 \) for every \( z \) with nonnegative coordinates and the proof is complete.

As it is mentioned above, the converses of 1. and 2. are not true in general. Let us recall the well known Bongartz’s example showing this.

**Example 1** Let \( Q \) be the quiver

Assume that \( I_1 \) is the ideal generated by all commutativity relations, and \( I_2 \) the ideal is generated by the commutativity of the upper square and two zero-relations \( \alpha\beta \) and \( \gamma\delta \). Then \( A_1 = KQ/I_1 \) is representation-finite, \( A_2 = KQ/I_2 \) is wild, and the Tits quadratic forms of \( A_1 \) and \( A_2 \) coincide and are weakly positive.

A lot of effort has been put into proving the converses of 1. and 2. under various assumptions. The most satisfactory results in this direction are the Bongartz’s criterion for representation-finiteness and Theorem 1 for tameness. The results can be considered as a culmination of a research programm scheduled by Sheila Brenner, who wrote in 1975 about her study on connections between the representation type and definiteness of certain quadratic forms [10]: 

"This paper is written in the spirit of experimental science. It reports some observed regularities and suggests that there should be a theory to explain them."
3 Simply connected and strongly simply connected algebras

Let $A = KQ/I$ be as before, assume that the quiver $Q$ is connected and let $m(I)$ be a set of minimal relations generating $I$. A relation $\sum_{i \in I} \lambda_i u_i \in I$, where $\lambda_i \in K$ and $u_i$ are paths in $Q$, is minimal if $\sum_{i \in J} \lambda_i u_i \notin I$ for any proper subset $J \subseteq I$. We look at $Q$ as a 1-dimensional complex and, whenever two paths $u, v$ are involved into a minimal relation, we attach a 2-cell along the loop $uv^{-1}$:

The fundamental group of the obtained complex is called the fundamental group of the bound quiver $(Q, I)$ and it is denoted by $\pi_1(Q, I)$. Let us remark that $\pi_1(Q, I)$ depends not only on $KQ/I$, it may depend on the particular choice of $I$.

**Definition 1**. The algebra $A$ is simply connected if $\pi_1(Q, I)$ is trivial for every presentation $A \cong KQ/I$.

In the Example, we have $\pi_1(Q, I_1)$ trivial, whereas $\pi_1(Q, I_2) \cong \mathbb{Z} \ast \mathbb{Z}$ is a free non-commutative group with 2 free generators.

Combining the results of [6], [7], [5] and [11] we derive

**Theorem 3** If $A$ is simply connected, then $A$ is representation-finite if and only if $q_A$ is weakly positive.

In fact, the theorem is valid for algebras having simply connected Galois covering. Results on preservation of (locally) representation-finiteness by Galois coverings, [9], are important ingredients in the proof of this general version.

The last sentence of the paper [6] is "There should be a similar result for tame representation type."

Of course, it is expected that weak nonnegativity of the Tits form implies tameness under suitable assumptions. Actually, there are several results of that kind. Such implication has been proved for instance for:

- hereditary algebras [37],
- tilted algebras [35],
- quasitilted algebras [57].

Observe that it is not enough to assume that $A$ is simply connected, as the following example shows:
Example 2 Let $Q$ be the quiver

![Quiver Diagram](https://via.placeholder.com/150)

and let $I$ be the ideal generated by the relations:

$$\alpha \xi, \omega \gamma \sigma \beta, \alpha \eta - \delta \gamma \sigma \beta \eta$$

Then $A = KQ/I$ is simply connected, the Tits quadratic form $q_A$ is weakly nonnegative, but $A$ is wild \[13, Example 1.7\].

Thus a strengthening of the concept of simply connectedness is needed.

Definition 2 \[55\] The algebra $A$ is strongly simply connected if every convex subcategory $C$ of $A$ is simply connected.

It is proved in \[55\] that the condition in the definition above is equivalent to the vanishing of the first Hochschild cohomology groups $H^1(C)$ (with coefficients in the bimodule $CC_C$) for every convex subcategory $C$ of $A$. By results of Bretscher - Gabriel \[11\] and Martinez - de la Peña \[36\] a triangular representation-finite algebra is simply connected if and only if it is strongly simply connected.

4 Strongly simply connected of polynomial growth

The proof of Theorem 1 depends essentially on some older result on polynomial growth algebras.

Theorem 4 Let $A$ be a strongly simply connected algebra. The following statements are equivalent:

1. $A$ is tame of polynomial growth.
2. The Tits form $q_A$ of $A$ is weakly nonnegative and $A$ does not contain a convex subcategory, which is pg-critical.

The list of pg-critical algebras is given in \[39\].

Representation theoretical properties of tame polynomial growth algebras differ substantially from those of remaining tame algebras. In particular, we observe characteristic behavior of the Tits and Euler quadratic forms in case of polynomial growth strongly simply connected algebras. Let us recall two theorems:

Theorem 5 \[42\] Let $A$ be a strongly simply connected algebra. The following conditions are equivalent:
1. A is tame of polynomial growth.

2. For every indecomposable module $X$ of dimension vector $z$

\[
\chi_A(z) = \dim G(z) - \dim \text{mod}_A(z) \geq 0.
\]

In this case every indecomposable module is a smooth point of the module scheme.$^6$

**Theorem 6** $^4$ Let $A$ be a strongly simply connected algebra and $X$ an indecomposable $A$-module of dimension vector $z$. Then:

1. $0 \leq q_A(z) \leq 18|Q_0|.$
2. $0 \leq \chi_A(z) \leq 2 + \#\{\text{projective-injective indecomposable } A\text{-modules}\}.$
3. If $X$ is a faithful module, then $q_A(z) \leq 2$ and $\chi_A(z) \leq 2.$

**Remark 1** There are also interesting connections between the growth of tame algebra and certain phenomena on the level of infinite-dimensional modules.

Let us recall that a module is **superdecomposable** if it has no indecomposable direct summands. Here we assume that the field $K$ is countable.

1. A wild algebra possesses a superdecomposable pure-injective module $^{25}$.
2. A non-domestic string algebra possesses a superdecomposable pure-injective module $^{45}$.
3. An algebra admitting a strongly simply connected Galois covering of non-polynomial growth, in particular, a strongly simply connected algebra of non-polynomial growth, possesses a superdecomposable pure-injective module, under the additional assumption that $\text{char}(K) \neq 2$ $^{32}, 33$.
4. Tubular algebras possess a superdecomposable pure-injective module $^{26}$.
5. A strongly simply connected algebra is domestic if and only if does not possess a superdecomposable pure-injective module, under the additional assumption that $\text{char}(K) \neq 2$ $^{40}$.

$^6$Let us remark, that here we mean the module scheme, see $^{42}$ for the details, not the module variety $\text{mod}_A(z)$
5 On the proof of the main theorem

We know that the Tits quadratic form of a tame algebra is weakly nonnegative, so we concentrate on the converse implication for strongly simply connected algebras. Very briefly the proof can be sketched as follows. By Theorem 4 a strongly simply connected algebra $A$ is tame of polynomial growth if and only if the Tits quadratic form $q_A$ is weakly nonnegative and $A$ does not contain a $pg$-critical convex subcategory. Thus we can restrict our attention to $A = KQ/I$ with $q_A$ weakly nonnegative and containing a $pg$-critical convex subcategory. Without loss of generality we can assume that additionally $A$ has an indecomposable module whose support contains all sources and all sinks of $Q$. The structure of such algebras can be described quite precisely; they are so called $D$-algebras in the terminology of [13]. Moreover, to $A$ there is associated a mild and smooth algebra $A^*$, which is tame if and only if $A$ is tame. The reader is again referred to [13] for the details. So we need to prove that $A^*$ is tame. This final step is done by Geiss’s degeneration theorem [23], since $A^*$ degenerates to a special biserial algebra, which is tame by [59], [18], [14].

Let us show on examples ideas how to perform degenerations mentioned above. There are two basic, rather known, tricks. First is to degenerate a commutative square to a square with a zero relation:

$$\alpha \gamma = \beta \delta \quad \alpha \gamma = 0$$

The degeneration is performed by the following "passage to the limit":

$$\alpha \gamma - t \beta \delta \to \alpha \gamma$$ as $t \to 0$.

Another trick is the following:

$$\alpha \gamma = \beta \delta \quad \alpha \gamma = 0, \epsilon^2 = 0$$

This is based on the observation that the path algebra of the two point quiver without arrows (○ ○) is isomorphic to $K \times K \cong K[X]/(X(X-t))$ for $t \neq 0$ and $K[X]/(X(X-t)) \to K(\epsilon \bigcirc)/\epsilon^2$ as $t \to 0$. 

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Combining this tricks we can degenerate the pg-critical algebra defined by the bound quiver
to the special biserial algebra defined by the quiver

with zero-relations $\alpha \beta = \gamma \delta = \eta \xi = \varepsilon^2 = 0$.

6 Some consequences and comments

The main theorem yields a possibility of checking in a finite number of steps whether given a strongly simply connected algebra is tame or not. It is not difficult to observe that also it is possible to check algorithmically whether $A = KQ/I$ is strongly simply connected - here the Hochschild cohomology approach is useful. Thanks to this we obtain

**Theorem 7** \[31\] Given a number $d$, the class of $d$-dimensional tame strongly simply connected algebras over algebraically closed fields is finitely axiomatizable.

A further consequence is

**Theorem 8** \[31\] Given a number $d$ and an algebraically closed field $K$, the class of $d$-dimensional tame strongly simply connected $K$-algebras induces a Zariski-open subset in the variety of associative $d$-dimensional $K$-algebras.

Let us finish with presenting some general idea of connecting certain properties which seem to be of different origin at the first sight. In the above presentation we tried to highlight connections between some general properties. Namely, we know that:
1. Finite representation type is open [22].
2. Finite representation type is finitely axiomatizable [28].
3. There is a criterion of representation-finiteness (expressed in terms of the Tits quadratic form) valid for simply connected algebras.
4. Galois coverings behave nice with respect to representation-finiteness.

Some time ago Jose Antonio de la Peña asked what remains of that in the class of tame algebras? Let us write down that:

1. We can formulate the conjecture that tame is open.
2. This conjecture is equivalent to the conjecture that tame is finitely axiomatizable [30].
3. We have quadratic form criterion for tameness of strongly simply connected algebras - Theorem [1].
4. We can formulate the "Galois Covering Preserve Tameness (GCPT)" Conjecture: Assume that a torsion-free group $G$ acts freely on the objects of a locally bounded category $\tilde{A}$. Then $A = \tilde{A}/G$ is tame if and only if $\tilde{A}$ is tame.

Let us remark that the "only if" part of GCPT is known by [17]. The "if" has been announced by Drozd and Ovsienko. But we are still - excluding Theorem [1] - on a conjectural level. However there is an unexpected connection between some of the conjectures.

It is known that we can express the theory of Galois coverings in the language of graded algebras [24]. This language is more suitable to our purposes.

Let us consider graded algebras over fields of characteristic $p$ (0 or a prime). We have the following

**Theorem 9** [24] If tame graded algebras of fixed dimension form a finitely axiomatizable class, then GCPT holds for $p'$-residually finite group $G$.

We say that a group $G$ is $p'$-residually finite if for any $a \in G$, $a \neq e$, there exists $H < G$ of finite index not divisible by $p$ such that $a \notin H$.

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