Policy Learning with Adaptively Collected Data

Ruohan Zhan∗, Zhimei Ren∗, Susan Athey†, and Zhengyuan Zhou‡

1Graduate School of Business, Stanford University
2Department of Statistics, University of Chicago
3Stern School of Business, New York University

First Version: May, 2021;
This Version: November, 2022

Abstract

Learning optimal policies from historical data enables personalization in a wide variety of applications including healthcare, digital recommendations, and online education. The growing policy learning literature focuses on settings where the data collection rule stays fixed throughout the experiment. However, adaptive data collection is becoming more common in practice, from two primary sources: 1) data collected from adaptive experiments that are designed to improve inferential efficiency; 2) data collected from production systems that progressively evolve an operational policy to improve performance over time (e.g. contextual bandits). Yet adaptivity complicates the optimal policy identification ex post, since samples are dependent, and each treatment may not receive enough observations for each type of individual. In this paper, we make initial research inquiries into addressing the challenges of learning the optimal policy with adaptively collected data. We propose an algorithm based on generalized augmented inverse propensity weighted (AIPW) estimators, which non-uniformly reweight the elements of a standard AIPW estimator to control worst-case estimation variance. We establish a finite-sample regret upper bound for our algorithm and complement it with a regret lower bound that quantifies the fundamental difficulty of policy learning with adaptive data. When equipped with the best weighting scheme, our algorithm achieves minimax rate optimal regret guarantees even with diminishing exploration. Finally, we demonstrate our algorithm’s effectiveness using both synthetic data and public benchmark datasets.

Keywords— offline policy learning; adaptive data collection; minimax optimality; personalized decision making; contextual bandits

1 Introduction

The growing availability of user-specific data has welcomed the exciting era of personalized decision making, a paradigm that exploits the heterogeneity in a given population so as to provide individualized service decisions that lead to improved outcomes. This paradigm has found applications in a wide variety of operations management domains. For instance, in healthcare, using electronic medical records, doctors can better prescribe heterogeneous treatments—different types of drugs/therapies or different dosage levels of the same drug—to different patients based on their medical characteristics (Murphy, 2003; Kim et al., 2011; Bertsimas et al., 2017; Fukuoka et al., 2018). In advertising (Charles et al., 2013; Kallus and Udell, 2016; Schnabel et al., 2016; Farias and Li, 2019; Bastani, 2021), using the recorded clientele information, the retailer can send more targeted product promotions—either in mail or online—to different existing and

*Authors contributed equally.
potential customers. In news recommendation (Li et al., 2010, 2011; Zeng et al., 2016; Karimi et al., 2018; Schnabel et al., 2019; Lee et al., 2020; Schnabel et al., 2020), the content provider may stream different news articles and/or media content to users with different digital footprints and perceived interests. In online education (Mandel et al., 2014; Lan and Baraniuk, 2016; Hoiles and Schaar, 2016; Bassen et al., 2020), an institution may want to offer different education plans to different students based on their varied learning styles (visual learner v.s. aural learner v.s. verbal learner, etc.).

A key problem in achieving intelligent personalization through data lies in learning an effective policy (which maps individual characteristics to treatments/actions) in a sample-efficient manner (i.e., making the fullest use of a given dataset so as to learn—to the extent possible—a policy that yields the highest rewards and hence leading to the best outcome for each individual). Particular challenges arise for off-policy evaluation due to missing counterfactual outcomes. Researchers from a variety of fields—including operations research, statistics and machine learning—have devoted extensive efforts to this problem in recent years (Dudık et al., 2011; Zhang et al., 2012; Zhao et al., 2015; Swaminathan and Joachims, 2015a,b,c; Swaminathan et al., 2016; Kitagawa and Tetenov, 2018; Levine et al., 2020; Kallus and Zhou, 2018; Zhou et al., 2022; Joachims et al., 2018; Chernozhukov et al., 2019; Su et al., 2019; Bennett and Kallus, 2020; Sachdeva et al., 2020; Jin et al., 2020; Athey and Wager, 2021) and satisfactorily addressed (discussed in more detail in Section 1.2) various aspects of the policy learning problem when the underlying historical data has been collected with a fixed exploration policy, which results in independent and identically distributed (i.i.d.) data over time. This is an important setting that includes several data-collection mechanisms: A/B testing, randomized control trials, and deploying a fixed operational policy that has built-in randomization.

However, much less is known on this problem when data is collected adaptively, that is, where the policy used to select actions evolves over time in response to observed outcomes rather than stays fixed. The following two broad categories of adaptive data collection are common in practice:

1. **Data from Adaptive Experiments.** Such data are collected from experiments for statistical inference and/or hypothesis testing. Since experiments are costly to conduct, a fixed experiment design (hence a fixed randomization rule) is not efficient. In comparison, adaptive experiment designs can dramatically improve statistical efficiency and are often used instead (Armitage et al., 1960; Simon, 1977; Murphy, 2005; Collins et al., 2007; Cai and He, 2011; Offer-Westort et al., 2019). Experimental data initially collected to answer particular inferential questions can be used for policy learning that falls out of its original design.

2. **Data from Operations Using Bandit Algorithms.** Such data is generated and collected from an operational policy in production systems. Production systems often adaptively choose their operational policies to improve the performance over time. A common family of algorithms are bandit algorithms (particularly contextual bandit algorithms) (Lai and Robbins, 1985; Thompson, 1933; Agrawal and Goyal, 2013; Russo et al., 2017), where treatment assignments are selected to balance exploration and exploitation to maximize the cumulative performance over time, thereby performing better compared to deploying a fixed policy.

Policy learning using adaptively collected data is much more difficult, since there are complex intertemporal dependencies. To see this, suppose we have observational data \( \{(X_t, W_t, Y_t)\}^T_{t=1} \) collected sequentially, where \( X_t \sim \text{i.i.d.} \) \( P_X \) is the context, \( W_t \in \mathcal{W} = \{1, \ldots, K\} \) is the selected action, and \( Y_t = \mu(X_t; W_t) + \epsilon_t \) is the outcome, with \( \{\epsilon_t\}^T_{t=1} \) being i.i.d. zero-mean random variables. Importantly, the samples \( \{(X_t, W_t, Y_t)\}^T_{t=1} \) are not i.i.d. since \( W_t \) is sampled according to probabilities \( \{\epsilon_t(X_t; 1), \ldots, \epsilon_t(X_t; K)\} \)—also known as propensity scores—that are time-varying and dependent on past observations \( \{(X_s, W_s, Y_s)\}^s_{s=1} \). Note that \( \epsilon_t \) is the (randomized) policy used at \( t \) while data was being collected. In practice, these \( \epsilon_t \) functions may be derived from complicated functions with a large number of parameters (e.g., neural networks) and are quickly and constantly updated (e.g., ads serving engines), hence making it cumbersome to record those policies in their entirety. Consequently, in reflecting this constraint, we assume that only \( \epsilon_t(X_t, W_t) \)—the probability of sampling the chosen action \( W_t \) at \( t \)—is recorded for each \( t \), but not the entire function \( \epsilon_t(\cdot; \cdot) \).\(^1\)

With a dataset of the form described above, the goal of policy learning is to select a good policy from a given policy class. A policy \( \pi \)—mapping from contexts to actions—can be evaluated by its policy value \( Q(\pi) = \mathbb{E}[\mu(X, \pi(X))] \), where the expectation is taken with respect to the randomness in \( X \) over the target population. To perform effective policy learning, one needs to select a policy with as large value as possible,\(^1\)

\(^1\)Nevertheless, in our proposal we require the knowledge of \( \epsilon_t(X_t, W_t) \) instead of estimating it from the data, which may be difficult in adaptive experiments since \( \epsilon_t \) can be time-varying.
or equivalently, as small regret as possible, where regret is defined to be the policy value loss relative to
the best value in the policy class. An effective policy learning algorithm should achieve a small regret as
a function of a given finite $T$, the quantitative measure of the algorithm’s sample complexity. However, a
moment of thought reveals that this is a challenging desideratum for the following three reasons.

First, unlike in fixed policy data collection where the propensities $e_t(\cdot; \cdot)$ do not change over time (corre-
sponding to a constant exploration bandwidth), a distinct feature in adaptive collection (when carried out by
popular algorithms used in practice) is that $e_t(\cdot; \cdot)$ shrinks and goes to 0 over time for certain actions (and
contexts). This means that exploration is gradually reduced, thereby resulting in vanishing probabilities of
selecting certain (poorly performing) actions. This crucial benefit of adaptive experiments—that poor arms
can be dropped, while good arms are more likely to be pulled—creates difficulty for offline policy learning, a
purely exploitation task. This is because the data not only have selection bias (as is already present in the
fixed policy setting), but also become increasingly more so overtime, thereby producing skewed data that
makes it difficult to compare the quality (i.e., the outcome) of alternative actions under a given context.

Second, although one might be tempted to think that smaller propensities (particularly towards the end
of data-collection) indicate that certain actions are “clearly bad” as a result of their vanishing probabilities
being selected, it is important to keep in mind that the policy learner does not have access to the propensity
functions $e_t(\cdot; \cdot)$ and hence cannot perform this “action elimination” type of policy learning. Consequently,
when observing that $e_T(X_t; W_T)$ is small on the last timestep $T$ and—assuming that we are confident that
the adaptive data collection process is well-designed and can hence conclude that the action $W_T$ is indeed
the wrong action for $X_T$—we would still not be able to know what other actions are wrong for $X_T$, nor
what other contexts are bad for $W_T$. Further, since we only observe a single data point $e_t(X_t, W_t)$ for
the propensity function $e_t(\cdot; \cdot)$, there is no hope to learn these evolving $e_t(\cdot; \cdot)$’s, a clear distinction from the
fixed policy setting where one can learn the fixed propensity function $e(\cdot; \cdot)$ using the entire training dataset
that it has generated.

Third, an adaptive data collection mechanism may itself be “poorly” designed and hence result in wrong
propensities: $e_t(x, w)$ is small when $w$ is in fact the best action for context $x$. This “poor” design can arise for
different reasons; for instance, it could be that an ineffective adaptive exploration scheme is used, resulting
in over-exploiting certain actions and under-exploring others. Since contextual bandits rely on learning a
complex outcome surface $\mu(\cdot, w)$ for each action $w$, most commonly used algorithms rely on specifying and
estimating a parametric model for $\mu$, making misspecification a real possibility. Alternatively, it could be
that the data were collected to answer a specific inferential question and hence the data-collection process
was steered towards a particular direction. Regardless of the cause, when this occurs, the policy learner
will see very few samples on the good actions but many samples on the bad actions, in effect reducing the
overall useful samples and yielding larger uncertainty about which action is actually good for which context.
This is a risk that the policy learner should address, because it has no control over what the data-collector
does in collecting the data.\(^2\) Hence, policy learning methods should be robust to a wide class of adaptive
data-collection mechanisms, good or bad from the policy learner’s own perspective.

Situated in this challenging and under-explored landscape, we aim to make initial progress into con-
fronting these challenges and focus on developing finite-sample regret bounds that shed light on the design
and implementation of efficient policy learning using adaptively collected data.

1.1 Our Contributions and Related Work

Our contributions are twofold. First, we study the fundamental difficulty of this problem by characterizing
a lower bound for policy learning. In particular, let $\{g_t\}_{t=1}^T$ be any positive lower-bound sequence of propen-
sities $e_t(\cdot; \cdot)$ (i.e., $e_t(\cdot; \cdot)$ is lower bounded by $g_t$ for each $t$).\(^3\) Then, for a worst-case distribution satisfying
the propensity lower bound $\{g_t\}_{t=1}^T$, any policy learning algorithm will incur at least an expected regret
of $\Omega\left(\sqrt{\text{Ndim}(\Pi)} / \sum_{t=1}^T g_t\right)$, where $\text{Ndim}(\Pi)$ refers to the Natarajan dimension of the (multi-action) policy
class $\Pi$—this is a generalization of the VC-dimension for the multi-action policy classes, and in particular
$\text{VC}(\Pi) = \text{Ndim}(\Pi)$ in settings of binary actions (i.e. only two actions are available). Consider an example

\(^2\)If the policy learner also plays the role of collecting data, it should instead focus on adaptive experimental design
or online adaptive learning, rather than offline policy learning, although there are scenarios where the data-collector
has multiple objectives and thus does not optimize for policy learning during the experiment.

\(^3\)The assumption that $g_t > 0$ ensures that each action is sampled with positive probability regardless of the
contexts.
where \( g_t \) decays at a rate with \( g_t = t^{-\alpha} \) and \( \alpha \in [0,1) \), then the worst-case expected regret is lower bounded by 
\[ \Omega\left( \sqrt{\log p \cdot \text{Ndim}(\Pi)} \right) \cdot T^{(\alpha - 1)/2}. \]
This regret lower bound highlights a necessary boundary for the feasibility of policy learning with adaptively collected data: if the adaptive data collection process is overly aggressive in exploitation, leading to an exploration rate that decreases faster than \( \Theta(t^{-1}) \), then the regret will be \( \Omega(1) \)— there is thus no hope of ever learning a near-optimal policy in the worst cases, no matter how large \( T \) is.

On the other hand, when \( \alpha = 0 \), our regret bound recovers the \( \Omega(T^{-1/2}) \) lower bound for the non-adaptive setting where the data is collected by a fixed policy (Kitagawa and Tetenov, 2018).

Second, building on the recent adaptive inference literature, we propose a policy learning algorithm and establish its expected regret bound, which is minimax optimal when the assignment probability lower bound \( g_t \) is known to the policy learner. Our algorithm follows a two-step procedure: 1) construct a policy value estimator \( \hat{Q}(\pi) \) for any fixed \( \pi \in \Pi \) using the collected data; 2) output the policy in \( \Pi \) that maximizes the estimated value: \( \hat{\pi} = \arg\max_{\pi \in \Pi} \hat{Q}(\pi) \).

The specific estimator we use is a variant of the family of generalized augmented inverse propensity weights (AIPW) estimators considered in Luedtke and van der Laan (2016); Hadad et al. (2021); Zhan et al. (2021), which takes the following form:

\[
\hat{Q}_T(\pi) = \frac{\sum_{t=1}^T h_t \hat{\Gamma}_t(\pi)}{\sum_{t=1}^T h_t}, \quad \text{where } \hat{\Gamma}_t(\pi) = \hat{\mu}_t(X_t; \pi(X_t)) + \frac{1}{e_t(X_t; \pi(X_t))} \left( Y_t - \hat{\mu}_t(X_t; \pi(X_t)) \right).
\]

Above, \( \hat{\mu}_t \) is the nuisance estimator of the expected outcome, \( \hat{\Gamma}_t(\pi) \) is the AIPW score (Robins et al., 1994), and \( h_t \) is the pre-specified deterministic weight that remains the same for all \( t \). The purpose of \( h_t \) is to offset the (potentially) large worst-case variance caused by vanishing assignment probabilities \( e_t(X_t; \pi(X_t)) \). Depending on whether \( g_t \) is known or not, \( h_t \) would be chosen differently, which results in different regret bounds (to be elaborated shortly in subsequent paragraphs). These specific choices of \( h_t \) are simple and different from those variants adopted in Luedtke and van der Laan (2016); Hadad et al. (2021); Zhan et al. (2021), which are concerned with devising asymptotic normality for inference, while instead here we aim for controlling worst-case estimation variance and constructing finite-sample regret bounds.

With the weights \( \{h_t\}_{t=1}^T \) plugged into the generalized AIPW estimator, we show that our algorithm has an expected regret upper bound of \( \hat{O}(\kappa(\Pi)) \cdot \sqrt{\frac{\sum_{t=1}^T h_t^2/g_t}{\sum_{t=1}^T h_t^2} + \frac{1}{\sum_{t=1}^T h_t^2/g_t^2}} \cdot \frac{1}{\sqrt{\kappa(\Pi)}} \), and \( \kappa(\Pi) \) is the entropy integral of a policy class \( \Pi \) based on the Hamming distance. Since the data are no longer i.i.d., existing techniques in most policy learning literature no longer apply. Our analysis instead leverages the framework of sequential uniform concentration (Rakhlin et al., 2015). When \( \hat{\mu}_t \) is fitted with observations up to time \( t - 1 \), the AIPW score \( \hat{\Gamma}_t \) is unbiased conditional on the past observations, so we can write \( \hat{Q}_T(\pi) - Q(\pi) \) as the sum of a martingale difference sequence. The supremum of the sum of this martingale difference sequence cannot be bounded with the standard notion of the Rademacher complexity used in existing policy learning literature (e.g., Kallus, 2018; Zhou et al., 2022). We instead consider an analog of the Rademacher process in the adaptive data setting—the tree Rademacher process—and connect the martingale difference sequence with a tree Rademacher process, thereby developing a bound for \( \max_{\pi \in \Pi} (\hat{Q}(\pi) - Q(\pi)) \). However the uniform concentration results in Rakhlin et al. (2015) are not directly applicable here, since here martingale difference terms are not bounded (as required in Rakhlin et al. (2015)) with risk that \( \hat{\Gamma}_t \) may diverge due to vanishing propensities. To address this issue, we introduce a high probability event on which the quadratic variation of the martingale difference sequence is regularized; then we bound the supremum of the tree Rademacher process via a covering of the policy class on the event, by refining and sharpening a chaining technique in Zhou et al. (2022).

The choice of weight \( h_t \) largely decides the regret bound provided by our algorithm. We show that the optimal weight \( h_t^* \) is proportional to the assignment probability lower bound \( g_t \), which yields the expected regret bound \( \hat{O}(\kappa(\Pi)/\sqrt{\sum_{t=1}^T g_t}) \). Note that \( \kappa(\Pi) = O\left( \sqrt{\log p \cdot \text{Ndim}(\Pi)} \right) \), where \( p \) denotes the dimension of the context; in the binary-action case, \( \kappa(\Pi) \leq 2.5 \sqrt{\text{VC}(\Pi)} \) (Jin, 2022). This implies that our upper bound with optimal weight \( h_t^* \) is tight with respect to (abbreviated as w.r.t. hereafter) the sample size and the complexity of the policy class up to logarithmic factors, and is thus minimax optimal. In cases where \( g_t \) is not disclosed to the policy learner, we recommend using uniform weights, that is, estimating policy values with standard AIPW estimator. This choice yields an expected regret bound \( \hat{O}(\kappa(\Pi) \cdot \sqrt{\sum_{t=1}^T g_t^{-1}/q_t^2} + \sum_{t=1}^T \frac{1}{g_t^2}} \),

\[ \sum_{t=1}^T \frac{1}{g_t^2} \]

4We use \( \hat{O}(\cdot) \) to show rates after omitting logarithm factors.
that is in general looser than the minimax regret but performs reasonably well in many cases. In particular, for the same example above in which $g_t = t^{-\alpha}$, using uniform weighting also achieves the minimax optimal regret bound $O(\kappa(\Pi) \cdot T^{(\alpha-1)/2})$, which—by setting $\alpha$ to 0—recovers the minimax optimal regret bound for policy learning under i.i.d. data collection established in Zhou et al. (2022).

After we posted a version of this paper online, Bibaut et al. (2021) made remarkable progress on this policy learning problem with adaptive data. They show that their algorithm (which amounts to a variant of our algorithm that by using uniform weighting $h_t = 1$ and setting nuisance component $\mu_t = 0$) meets our established lower bound in settings where $g_t = t^{-\alpha}$ (though their algorithm does not guarantee minimax optimality beyond those special cases). Encouraged by this positive result, we tighten the upper bound of our algorithm and show that it achieves minimax optimal regret guarantee with optimal weighting $h^*_t = g_t$ in general cases (that is, even when $g_t$ is not of the form $t^{-\alpha}$). Besides, the nuisance component $\mu_t$ in our algorithm (which is set to zero in Bibaut et al. (2021)) would reduce variance in policy value estimation even with misspecification, which in turn improves the value of learned policy. We defer detailed empirical comparison to Appendix C.2.

Finally, leveraging a publicly available piece of software, we consider the policy class of (fixed-depth) decision trees and evaluate the algorithm on both synthetic data and public benchmark datasets. The empirical results show two important strengths of our policy learning algorithm. First, when $g_t$ is unknown, our algorithm narrowly trails and has the same regret decay rate as the setting where $g_t$ is known (and hence the optimal weights $h^*_t$ can be computed exactly). This suggests that although it would be helpful to know $g_t$ and hence leverage it to achieve even better performance, our algorithm would still be functional even when such information is not available. Second, since online learning algorithms (e.g., Thompson sampling) often directly use outcome regression, they are prone to model misspecification, where they allocate small propensities to good actions, incurring large performance gap as the process goes on. Despite this, our offline learning algorithm can still find the $(\epsilon)$-optimal policy, demonstrating its effectiveness and robustness.

### 1.2 Other Related Work

Policy learning with observational data is a growing field that has received increasing attention from different communities. As already mentioned, the existing literature in offline policy learning has primarily focused on data collected by a fixed policy, where they collectively proposed several statistical efficient and/or computationally efficient policy learning algorithms that achieve the minimax optimal regret bounds of $\Theta(T^{-1/2})$ on the expected regret. A sequence of refinements (Zhang et al., 2012; Zhao et al., 2015; Kitagawa and Teteno, 2018; Athey and Wager, 2021) addressed this challenge in the settings of binary actions; and particularly Kitagawa and Teteno (2018) established the tight regret bound with the knowledge of propensities, which was relaxed later in Athey and Wager (2021) that established the optimal dependency with estimated propensities. Extension to multi-action schemes has been successively investigated by Swaminathan and Joachims (2015a); Zhou et al. (2017); Kallus (2018); Zhou et al. (2022); Zhou et al. (2022) established the minimax regret bound using doubly robust AIPW estimator when the propensities are unknown. An important distinction when using AIPW estimator on adaptive data is that one often assumes the knowledge of propensities (Luedtke and van der Laan, 2016; Hadad et al., 2021; Zhan et al., 2021), as is the case in this paper, since they are typically time-varying and are difficult to approximate with limited batch size.

Another strongly related area is offline policy evaluation with adaptively collected data. Estimating policy values on adaptively collected data is much more challenging compared to that on i.i.d. data. For instance, direct methods that fit regression models will be biased (Nie et al., 2018; Shin et al., 2019), while unbiased estimators such as inverse propensity weighted (IPW) estimator can suffer from huge variability (Horvitz and Thompson, 1952; Imbens, 2004). Note that IPW estimators already suffer from large variance with i.i.d. data (where the propensity has a fixed lower bound that does not change), and the problem becomes more acute in the adaptive data collection setting because the propensities are vanishing. The AIPW estimator (Robins et al., 1994; Dudík et al., 2011) combines the outcome modelling and IPW approaches, gaining efficiency and “double robustness” properties with i.i.d. data. In particular, the AIPW estimator is consistent if either the propensity model or the outcome model is consistently estimated. However, AIPW still has challenges when the data is adaptively collected, since vanishing propensities will yield exploding variance and de-stablize the estimator. To deal with this, the literature has seen two approaches for adapting AIPW estimators to offline policy evaluation with adaptively collected data. The first incorporates weight clipping into AIPW (Bembom and van der Laan, 2008; Charles et al., 2013; Wang et al., 2017; Su et al., 2020), where one controls variance by shrinking the weights at the cost of introducing a small bias.
The second approach, described above, is to locally stabilize the elements of the AIPW estimator (Luedtke and van der Laan, 2016; Hadad et al., 2021; Zhan et al., 2021). Our policy learning algorithm uses an estimator that falls into this second approach, where the weights $h_t$ are chosen with the consideration of the worst-case variance in order to be robust.

Finally, there is also an extensive literature on online contextual bandits (Dani et al., 2008; Besbes and Zeevi, 2009; Rigollet and Zeevi, 2010; Abbasi-Yadkori et al., 2011; Chu et al., 2011; Bubeck and Cesa-Bianchi, 2012; Abbasi-Yadkori, 2013; Agrawal and Goyal, 2013; Goldenshluger and Zeevi, 2013; Russo and Van Roy, 2014; Li et al., 2017; Dimakopoulou et al., 2017; Bastani and Bayati, 2020), the online counterpart of policy learning. The literature focuses on analyzing bounds on the regret experienced by alternative algorithms. This problem is distinct from ours and from offline policy learning in general; in the online learning literature, the focus is on the adaptive algorithms and how to assign treatment to units in order to balance the exploration-exploitation trade-off. We close by pointing out that no “online-to-batch” conversion is feasible here: online learning algorithms cannot be directly converted to offline learning algorithms because the offline policy learner does not have the opportunity to choose a particular action. The counterfactual outcomes from alternative actions for a particular unit are unobserved.

2 Offline Policy Learning

Let $T$ be the time horizon, $\mathcal{X}$ the covariate space, and $\mathcal{W}$ the action space with $K$ actions. At time $t \in [T] = \{1, \ldots, T\}$, the experimenter observes a covariate $X_t \in \mathcal{X}$ with $X_t \overset{\text{i.i.d.}}{\sim} P_X$; then the experimenter takes an action $W_t \in \mathcal{W}$ sampled from a multinomial distribution with probability $(e_t(X_t; 1), \ldots, e_t(X_t; K))$; next she receives a response $Y_t$ from the chosen action generated via:

$$Y_t = \mu(X_t; W_t) + \varepsilon_t,$$

where $\{\varepsilon_t\}_{t=1}^T$ is a sequence of i.i.d. zero-mean and $\sigma^2$-variance random variables. We assume that the assignment probability $e_t(x, w)$ is updated for all $(x, w) \in \mathcal{X} \times \mathcal{W}$ using observations up to time $t - 1$. We use $\mathcal{H}_t = \{(X_s, W_s, Y_s)\}_{s=1}^{t-1}$ to denote samples up to time $t$. Throughout, we make the following assumptions on the data generating process.

Assumption 1. The data generating process has:

(a) Bounded outcome: $\exists M \geq 0$ such that $|Y_t| \leq M$ and $|\mu(x; w)| \leq M$ for all $(x, w) \in \mathcal{X} \times \mathcal{W}$.

(b) Known and bounded assignment probability: $e_t(X_t; W_t)$ is known; and $e_t(x; w)$ is lower bounded with probability one, over the sampling of $\{(X_s, W_s, Y_s)\}_{s=1}^{t-1}$, by a positive and nonincreasing deterministic sequence $\{g_t\}$ for all $(x, w) \in \mathcal{X} \times \mathcal{W}$ and all $t \in [T]$.

Assumption 1(b) requires the experimenter to record data $(X_t, W_t, Y_t, e_t(X_t; W_t))$ at each time instead of the standard $(X_t, W_t, Y_t)$, which is feasible in many applications since practitioners often have access to the algorithm used to collect data, and hence have the knowledge of the assignment probability. The requirement of a lower bound on assignment probabilities for all arms is a bit more restrictive—for instance standard Thompson sampling may not fall into this category since the sampling probability on a clearly sub-optimal arm may decay to zero too fast as $T$ goes to infinity—but we note that this is already a generalization of the previous settings with i.i.d. data. In parallel, previous work on online policy learning with i.i.d. data (e.g., Athey and Wager (2021); Zhou et al. (2022)) often assumes the “overlap” condition: the assignment probability $e_t(x; w)$ is lower bounded by a positive constant for all $(x, w) \in \mathcal{X} \times \mathcal{W}$ and all $t \in [T]$. This is in fact a special case of our condition by setting $g_t$ to be a positive constant.

2.1 Offline Policies and Metrics

A policy $\pi$ is a mapping from the covariate space to the action space, i.e., $\pi : \mathcal{X} \mapsto \mathcal{W}$. Given a policy $\pi$, we are interested in its policy value defined as the expected reward incurred by taking actions according to $\pi$:

$$Q(\pi) \triangleq \mathbb{E}_X \left[ \mu(X; \pi(X)) \right].$$
where \( \mathbb{E}_X[\cdot] \) denotes taking expectation w.r.t. \( P_X \). For a class of policies \( \Pi \), we wish to learn the optimal policy \( \pi^* \) within \( \Pi \): 
\[
\pi^* = \arg\max_{\pi \in \Pi} Q(\pi).
\]
The \textit{regret} of a policy \( \pi \in \Pi \) is the policy value difference between the optimal policy and itself:
\[
R(\pi) \triangleq Q(\pi^*) - Q(\pi).
\]
Conceptually, we would like to learn a policy whose regret is as small as possible (so that its policy value is close to that of the optimal policy). To do this, we shall 1) construct a policy value estimator for each \( \pi \in \Pi \), and 2) choose the policy \( \hat{\pi} \) that maximizes the estimated policy value over the class \( \Pi \). Note that here \( \hat{\pi} \) is data-driven, and thus itself and \( R(\hat{\pi}) \) are both random variables. Our strategy of deriving an upper bound for \( R(\hat{\pi}) \) is to establish that our policy value estimator is close to the true value \textit{uniformly} across the policy class. For the rest of the paper, when we say a regret bound for \( \hat{\pi} \), we refer to upper bound of the expected regret.

### 2.2 Policy Class Complexity

Policy class is a central element in the discussion of policy learning. The complexity of a policy class presents a trade-off: a richer policy class means that the corresponding optimal policy has a larger value, while the increasing complexity of the class makes the optimal policy learning intrinsically harder. Ideally we would like to consider a policy class with the appropriate complexity such that the optimal policy has a reasonably high value, and meanwhile it is feasible to learn the optimal policy efficiently. Our first step towards this goal is to characterize the complexity of a policy class based on the Hamming distance.

**Definition 1 (Hamming Distance and the Covering Number).** Consider a covariate space \( \mathcal{X} \) and a policy class \( \Pi \). We define:

(a) the **Hamming distance** between any two policies \( \pi_1, \pi_2 \in \Pi \) given a covariate set \( \{x_{1:n}\} \):
\[
H(\pi_1, \pi_2; \{x_{1:n}\}) \triangleq \frac{1}{n} \sum_{j=1}^{n} 1\{\pi_1(x_j) \neq \pi_2(x_j)\};
\]

(b) \( \epsilon \)-**Hamming covering number** of \( \Pi \) given \( \{x_{1:n}\} \):
\[
\mathcal{N}_H(\epsilon, \Pi; \{x_{1:n}\}) = |\Pi_0|,^5 \text{ where } \Pi_0 \subseteq \Pi \text{ is the smallest policy set such that } \forall \pi \in \Pi \text{, there exists } \pi^* \in \Pi_0 \text{ satisfying } H(\pi, \pi^*; \{x_{1:n}\}) \leq \epsilon;
\]

(c) \( \epsilon \)-**Hamming covering number** of \( \Pi \):
\[
\mathcal{N}_H(\epsilon, \Pi) \triangleq \sup \{\mathcal{N}_H(\epsilon, \Pi; \{x_{1:m}\}) : m \geq 1, x_1, \ldots, x_m \in \mathcal{X}\};
\]

(d) **entropy integral:**
\[
\kappa(\Pi) = \int_0^1 \log \mathcal{N}_H(\epsilon, \Pi) d\epsilon.
\]

The entropy integral is the key quantity characterizing the complexity of a policy class. We will show in Section 4 that the regret of our estimator can be upper bounded by a function of the entropy integral, and in Section 6 we provide a tree-based policy class with finite entropy integral.

### 3 Regret Lower Bound

To characterize the fundamental limit of the offline policy learning problem, we establish a lower bound for the worst-case expected regret in this section. The lower bound is stated in terms of the Natarajan dimension of a multi-action policy class, the definition of which is as follows.

**Definition 2.** Given a \( K \)-action policy class \( \Pi \), we say a set of \( m \) points \( \{x_1, x_2, \ldots, x_m\} \) is shattered by \( \Pi \) if there exist two functions \( f_{-1}, f_1 : \{x_1, x_2, \ldots, x_m\} \mapsto [K] \) such that,

(a) for any \( j \in [m], f_{-1}(x_j) \neq f_1(x_j) \);

---

^5For a set \( A \), we let \( |A| \) denote its cardinality.
(b) for any $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in \{\pm 1\}^m$, there exists a policy $\pi \in \Pi$ such that for any $j \in [m],$

$$\pi(x_j) = \begin{cases} f_{-1}(x_j) & \text{if } \sigma_j = -1; \\ f_1(x_j) & \text{if } \sigma_j = 1. \end{cases}$$

The Natarajan dimension of $\Pi$ is defined to be the size of the largest set of points that can be shattered by $\Pi.$

Returning to our problem, we let $\text{Ndim}(\Pi)$ denote the Natarajan dimension of $\Pi,$ $\mathcal{Z}_T = \{(X_t, W_t, Y_t)\}_{t=1}^T$ represent the collected offline data and $\mathcal{P}$ be the joint distribution of $\mathcal{Z}_T.$ The following theorem shows that, there exists a $\mathcal{P}$ satisfying Assumption 1, such that for any policy $\hat{\pi}$ that is learned from $\mathcal{Z}_T \sim \mathcal{P},$ the expected regret is lower bounded by $\Omega\left(\frac{1}{8}\sqrt{\text{Ndim}(\Pi)}/\sum_{t=1}^T g_t\right).$

**Theorem 1.** Fix a sequence $\bar{g} = \{g_t\}_{t=1}^T,$ and let $\mathcal{P}(\bar{g})$ be the collection of all laws of data generating processes for which Assumption 1 holds with the assignment probability lower bound sequence $\bar{g}.$ For any $\hat{\pi} \in \Pi$ learned from $\mathcal{Z}_T,$ there is

$$\sup_{\mathcal{P} \in \mathcal{P}(\bar{g})} \mathbb{E}_{\mathcal{Z}_T \sim \mathcal{P}} [R(\hat{\pi})] \geq \frac{M}{8} \min \left\{ \frac{1}{3} \sqrt{\text{Ndim}(\Pi)/\sum_{t=1}^T g_t} \right\}.$$

As is self-explained in Theorem 1, when $\text{Ndim}(\Pi) > \frac{\Delta}{2} \sum_{t=1}^T g_t,$ the expected regret in the worst case is lower bounded by a constant, which makes the problem of learning optimal policies infeasible. Hence in this paper, we shall focus on the settings where $\text{Ndim}(\Pi) \leq \frac{\Delta}{2} \sum_{t=1}^T g_t$ and write the lower bound as $O\left(\sqrt{\text{Ndim}(\Pi)/\sum_{t=1}^T g_t}\right)$ elsewhere when no confusion arises.

### 3.1 Proof of Theorem 1

Consider $K$ arms indexed by $[K],$ and write $d = \text{Ndim}(\Pi).$ By the definition of the Natarajan dimension, there exist a set of $d$ points $\{x_1, x_2, \ldots, x_d\} \subset X$ that is shattered by $\Pi,$ i.e., there exist two functions $f_{-1}, f_1 : \{x_1, x_2, \ldots, x_d\} \mapsto [K],$ such that $f_{-1}(x_j) \neq f_1(x_j)$ for any $j \in [d],$ and for any $\sigma \in \{\pm 1\}^d,$ there exists a policy $\pi \in \Pi$ such that $\pi(x_j) = f_{\sigma_j}(x_j).$

To establish the lower bound, we construct $2^d$ distributions for $\mathcal{Z}_T,$ where each $\sigma \in \{\pm 1\}^d$ induces a joint distribution of $\mathcal{Z}_T.$ Fix a $\sigma \in \{\pm 1\}^d.$ For each $t \in [T],$ let $X_t$ be independently and uniformly generated from $\{x_1, x_2, \ldots, x_d\};$ conditional on $X_t = x_j,$ $W_t$ is chosen to be $f_1(x_j)$ with probability (w.p.) $g_t$ and other arms w.p. $(1 - g_t)/(K - 1)$ (thus the distribution of $X_t$ and $W_t$ does not depend on $\sigma$). We now proceed to specify the set of reward distributions for the $K$ arms that depend on $\sigma.$ For any $j \in [d]$ and some (small) $\Delta > 0$ to be specified later, conditional on $X_t = x_j,$

- arm $f_1(x_j): Y_t(f_1(x_j)) \sim M \cdot \text{Bern} \left( \frac{1 + \sigma_j \cdot \Delta}{2} \right),$
- arm $f_{-1}(x_j): Y_t(f_{-1}(x_j)) \sim M \cdot \text{Bern} \left( \frac{1}{2} \right),$
- arm $k: Y_t(k) = 0,$ for any $k \in [K] \setminus \{f_1(x_j), f_{-1}(x_j)\},$

where $\text{Bern}(q)$ denotes the Bernoulli distribution with parameter $q$. By construction, conditional on $X_t = x_j,$ the optimal arm is $f_{\sigma_j}(x_j);$ since $\{x_1, \ldots, x_d\}$ is shattered by $\Pi,$ there exists a policy $\pi_{\sigma} \in \Pi$ that selects the optimal arm for any $x_j \in \{x_1, \ldots, x_d\}.$ It can be easily verified that the distributions constructed satisfy Assumption 1. Let $\mathcal{P}_\sigma(\cdot)$ and $\mathbb{E}_\sigma[\cdot]$ refer to the distribution and expectation taken under the joint distribution induced by $\sigma,$ and let $\mathcal{P}$ be the mixture distribution uniformly drawn from $\{\mathcal{P}_\sigma\}_{\sigma \in \{\pm 1\}^d}.$ For
any policy \( \hat{\pi} \) learned from \( Z_T \sim P, \)

\[
\mathbb{E}_{Z_T \sim P} [R(\hat{\pi})] \geq \frac{1}{2} \sum_{\sigma \in \{\pm 1\}^d} \mathbb{E}_\sigma \left[ Q(\pi^{\sigma,*}) - Q(\hat{\pi}) \right]
\]

\[
\overset{(i)}{=} \frac{1}{2^d} \sum_{\sigma \in \{\pm 1\}^d} \frac{1}{d} \sum_{j=1}^d \mathbb{E}_\sigma \left[ Q(\pi^{\sigma,*}) - Q(\hat{\pi}) \mid X = x_j \right]
\]

\[
\overset{(ii)}{\geq} \frac{M \Delta}{d^{2d+1}} \sum_{\sigma \in \{\pm 1\}^d} \sum_{j=1}^d \mathbb{E}_\sigma \left[ 1 \left\{ \hat{\pi}(x_j) \neq f_{\sigma_j}(x_j) \right\} \right]
\]

\[
= \frac{M \Delta}{d^{2d+1}} \sum_{\sigma \in \{\pm 1\}^d} \sum_{j=1}^d \mathbb{P}_\sigma \left( \hat{\pi}(x_j) \neq f_{\sigma_j}(x_j) \right). \tag{1}
\]

Above, step (i) uses the tower property; in step (ii), the optimal policy is in \( \Pi \) since \( \{x_1, \ldots, x_d\} \) is shattered by \( \Pi \). For a fixed \( \sigma \in \{\pm 1\}^d \), let \( M_j(\sigma) \in \{\pm 1\}^d \) be the vector that differs from \( \sigma \) only in element \( j \): \( [M_j(\sigma)]_j = -\sigma_j \) and \( [M_j(\sigma)]_i = \sigma_i \) for all \( i \neq j \). Equipped with the notation, we have

\[
(1) \overset{(i)}{=} \frac{M \Delta}{d^{2d+1}} \sum_{j=1}^d \sum_{\sigma, \sigma_j = 1} \left( \mathbb{P}_\sigma \left( \hat{\pi}(x_j) \neq f_1(x_j) \right) + \mathbb{P}_{M_j(\sigma)} \left( \hat{\pi}(x_j) \neq f_{-1}(x_j) \right) \right)
\]

\[
\overset{(i)}{\geq} \frac{M \Delta}{d^{2d+1}} \sum_{j=1}^d \sum_{\sigma, \sigma_j = 1} \left( \mathbb{P}_\sigma \left( \hat{\pi}(x_j) = f_{-1}(x_j) \right) + 1 - \mathbb{P}_{M_j(\sigma)} \left( \hat{\pi}(x_j) = f_{-1}(x_j) \right) \right)
\]

\[
\overset{(ii)}{\geq} \frac{M \Delta}{d^{2d+1}} \sum_{j=1}^d \sum_{\sigma, \sigma_j = 1} \left( 1 - \text{TV}(\mathbb{P}_\sigma, \mathbb{P}_{M_j(\sigma)}) \right)
\]

\[
\overset{(iii)}{\geq} \frac{M \Delta}{d^{2d+2}} \sum_{j=1}^d \sum_{\sigma, \sigma_j = 1} \exp \left( - \text{D}_{\text{KL}}(\mathbb{P}_\sigma \| \mathbb{P}_{M_j(\sigma)}) \right), \tag{2}
\]

where \( \text{TV}(P, Q) \) denotes the total variation distance between two distributions \( P \) and \( Q \), and \( \text{D}_{\text{KL}}(P \| Q) \) is the KL-divergence between \( P \) and \( Q \); step (i) is because \( f_1(x_j) \neq f_{-1}(x_j) \), and \( \{\hat{\pi}(x_j) = f_{-1}(x_j)\} \subset \{\hat{\pi}(x_j) \neq f_1(x_j)\} \); step (ii) follows from the definition of the total variation distance and step (iii) is a result of Lemma 1 stated below.

**Lemma 1** (Tsybakov (2008), Lemma 2.6). Let \( P \) and \( Q \) be any two probability measures on the same measurable space. Then

\[
1 - \text{TV}(P, Q) \geq \frac{1}{2} \exp \left( - \text{D}_{\text{KL}}(P \| Q) \right).
\]
The KL-divergence between $\mathbb{P}_\sigma$ and $\mathbb{P}_{M_j(\sigma)}$ can be directly computed as,

$$
D_{\text{KL}}(\mathbb{P}_\sigma \| \mathbb{P}_{M_j(\sigma)}) = \mathbb{E}_\sigma \left[ \log \frac{d\mathbb{P}_\sigma}{d\mathbb{P}_{M_j(\sigma)}}(X_1, W_1, Y_1, \ldots, X_T, W_T, Y_T) \right]
$$

$$
= \mathbb{E}_\sigma \left[ \sum_{t=1}^T \log \frac{d\mathbb{P}_\sigma(Y_t \mid W_t, X_t)}{d\mathbb{P}_{M_j(\sigma)}(Y_t \mid W_t, X_t)} \right]
$$

$$
\leq \sum_{t=1}^T \mathbb{E}_\sigma \left[ \frac{d\mathbb{P}_\sigma(Y_t \mid W_t, X_t)}{d\mathbb{P}_{M_j(\sigma)}(Y_t \mid W_t, X_t)} \right] \left| X_t = x_t \right|
$$

$$
\leq \frac{3 \Delta^2}{d} \sum_{t=1}^T g_t,
$$

(3)

where step (i) uses the linearity and the tower property of expectation; step (ii) is because $\mathbb{P}_\sigma$ differs from $\mathbb{P}_{M_j(\sigma)}$ only when $X = x_i$; step (iii) follows from that $x \log\left(\frac{1 + x}{1 + x} \right) \leq 3x^2$ for $x \in [0, \frac{1}{2}]$; step (iv) is by design of the sampling mechanism of $\{W_t\}_{t=1}^T$. Combining (2) and (3), we have that

$$
\mathbb{E}_{\zeta \sim \mathbb{P}} \left[ Q(\pi^*) - Q(\hat{\pi}) \right] \geq \frac{M \Delta}{8} \exp \left( - \frac{3 \Delta^2}{d} \sum_{t=1}^T g_t \right).
$$

If $d \leq \frac{1}{2} \sum_{t=1}^T g_t$, letting $\Delta = \sqrt{d / \left( \sum_{t=1}^T g_t \right)}$ yields the desired lower bound. Otherwise, choose $\Delta = 1/3$, and we have

$$
\mathbb{E}_{\zeta \sim \mathbb{P}} \left[ Q(\pi^*) - Q(\hat{\pi}) \right] \geq \frac{M}{24} \exp \left( - \frac{1}{3d} \sum_{t=1}^T g_t \right) \geq \frac{M}{24e^3}.
$$

which concludes the proof.

A direct consequence of Theorem 1 is the following corollary.

**Corollary 1.1.** Given $\alpha \in [0, 1)$, let $\mathcal{P}(\{t^{-\alpha}\})$ be the collection of all laws of data generating process for which Assumption 1 holds with the assignment probability lower bound sequence $g_t = t^{-\alpha}$. For any $\hat{\pi} \in \Pi$ learned from $\mathcal{Z}_T$, there is

$$
\sup_{\mathcal{P} \in \mathcal{P}(\{t^{-\alpha}\})} \mathbb{E}_{\zeta \sim \mathbb{P}} \left[ Q(\pi^*) - Q(\hat{\pi}) \right] \geq \frac{M}{8e^3} \cdot \min \left( 1/3, \sqrt{(1 - \alpha) \dim(\Pi)} \cdot T^{\frac{\alpha}{2}} \right).
$$

The proof is completed by noticing that $\sum_{t=1}^T g_t \leq T^{1-\alpha}/(1 - \alpha)$.

**Remark 1.** The overlap condition assumed in literature that studies i.i.d. data (Imbens, 2004; Athey and Wager, 2021; Zhou et al., 2022) is a special case of Corollary 1.1 with $\alpha = 0$, which informs that the regret of any learned policy is at least $\Omega(T^{-1/2})$ under the overlap condition.

## 4 Regret Upper Bound

In this section, we introduce our offline policy learning algorithm based on the generalized AIPW estimator, followed by a regret analysis of the algorithm and a discussion on the choice of weights.
4.1 The Policy Learning Algorithm

Our proposed algorithm consists of two steps. First, it estimates the value of a policy \( \pi \) via reweighting the AIPW scores, where the AIPW score for \( t \in [T] \) is defined as follows

\[
\hat{\Gamma}_t(\pi) = \hat{\mu}_t(X_t; \pi(X_t)) + \frac{1}{e_t(X_t; \pi(X_t))} \cdot \left( Y_t - \hat{\mu}_t(X_t; \pi(X_t)) \right). \tag{4}
\]

Above, \( \hat{\mu}_t(x; w) \) is an estimator of the expectation \( \mu(x; w) \) using \( \mathcal{H}_{t-1} \), and \( e_t(X_t; w) \) is the assignment probability that is also computed based on \( \mathcal{H}_{t-1} \) and is known to the algorithm by Assumption 1. The algorithm then outputs the policy maximizing the estimated policy value.

We start from some properties of the AIPW scores that will be useful later.

**Proposition 1** (Hadad et al. (2021); Zhan et al. (2021)). The AIPW score \( \hat{\Gamma}_t(\pi) \) has the following two properties.

(a) Conditional unbiasedness: \( \mathbb{E} [\hat{\Gamma}_t(\pi) \mid \mathcal{H}_{t-1}, X_t] = Q(X_t, \pi(X_t)) \).

(b) Bounded conditional variance: there exist positive constants \( L, U > 0 \), such that

\[
L \cdot \mathbb{E} [e_t(X_t; \pi(X_t))^{-1} \mid \mathcal{H}_{t-1}] \leq \text{Var}(\hat{\Gamma}_t(\pi) \mid \mathcal{H}_{t-1}) \leq U \cdot \mathbb{E} [e_t(X_t; \pi(X_t))^{-1} \mid \mathcal{H}_{t-1}].
\]

A direct consequence of Proposition 1(a) is that any weighted average of \( \hat{\Gamma}_t(\pi) \) is also unbiased; Proposition 1(b) reveals that the conditional variance of the AIPW score \( \hat{\Gamma}_t(\pi) \) scales with \( \mathbb{E} [e_t(X_t; \pi(X_t))^{-1} \mid \mathcal{H}_{t-1}] \). As is often the case in adaptive experiments, \( e_t(X_t; w) \) goes to zero for some suboptimal action \( w \) as \( t \) increases; consequently, the term \( e_t(X_t; \pi(X_t))^{-1} \) may go to infinity, and the variance of \( \hat{\Gamma}_t(\pi) \) may explode. To offset the potentially large variance of \( \hat{\Gamma}_t \), we further introduce a weight \( h_t \) (the choice of which will be discussed soon) to balance the variance of these AIPW scores. This gives our generalized AIPW estimator:

\[
\hat{Q}_T(\pi) = \frac{\sum_{t=1}^{T} h_t \hat{\Gamma}_t(\pi)}{\sum_{t=1}^{T} h_t}.
\]

Our algorithm then selects the policy that maximizes the above estimator:

\[
\hat{\pi} = \arg\max_{\pi \in \Pi} \hat{Q}_T(\pi). \tag{6}
\]

**Remark 2.** Unlike the choice of weights in Luedtke and van der Laan (2016); Hadad et al. (2021); Zhan et al. (2021), here \( \{h_t\}_{t \in [T]} \) are pre-specified and not adaptive. In fact, the choice of \( h_t \) should take into account the worst-case variance of \( \hat{\Gamma}_t \) over the policy class and all possible data realizations (see more details in Section 4.3).

4.2 Regret Analysis

Below we state the main condition on our weighting scheme.

**Assumption 2.** The weights \( \{h_t\} \) used in (5) satisfy that

\[
L_T(h, g) \triangleq \frac{\sum_{t=1}^{T} h_t^4 / g_t^2}{(\sum_{t=1}^{T} h_t^2 / g_t)^2} \to 0, \quad \text{as} \quad T \to +\infty.
\]

This assumption specifies a regularity condition on the weights, which controls the higher moments of the estimator \( \hat{Q}_T(\pi) \) in (5) in the worst cases and thus enables us to introduce martingale concentration results to prove the uniform convergence of \( \hat{Q}_T(\pi) \). An analogous assumption is also required for weighting schemes used in policy inference with adaptive data (Hadad et al., 2021; Zhan et al., 2021). We now present the bound on the expected regret for the policy obtained via (6).

**Theorem 2.** Suppose Assumptions 1 and 2 hold. For \( T \) such that \( L_T(h, g) < 1/8 \), the expected regret of the policy \( \hat{\pi} \) given by (6) can be upper bounded as,

\[
\mathbb{E} [R(\hat{\pi})] \leq 100M\sqrt{R} \cdot \left( 19\kappa(\Pi) + 7\sqrt{\log \left( \frac{\sum_{t=1}^{T} h_t^4 / g_t}{\sqrt{\sum_{t=1}^{T} h_t^2 / g_t}} + 29 \right)} \cdot \frac{\sqrt{\sum_{t=1}^{T} h_t^2 / g_t}}{\sum_{t=1}^{T} h_t} + 4ML_T(h, g) \right).
\]
**Remark 3.** When \( g_t \) is a positive constant (which corresponds to the overlap condition), the expected regret bound with uniform weights \( (h_t = 1) \) is \( O(\kappa(\Pi) \cdot T^{-1/2}) \); this recovers the rate achieved by Zhou et al. (2022) with i.i.d.
data.

Here, we only discuss the proof of Theorem 2 at a high level and defer the details to Section 5. The main idea is to bound the regret of \( \hat{\pi} \) by the worst-case estimation error of \( \hat{Q}_T \) over \( \Pi \):

\[
R(\hat{\pi}) = Q(\pi^*) - Q(\hat{\pi}) = (Q(\pi^*) - \hat{Q}_T(\pi^*)) + (\hat{Q}_T(\pi) - Q(\pi)) \leq 2 \max_{\pi \in \Pi} |Q(\pi) - \hat{Q}_T(\pi)|,
\]

where the inequality uses the fact that \( \hat{\pi} \) maximizes \( \hat{Q}_T(\pi) \). It thus suffices to bounding the quantity \( \max_{\pi \in \Pi} |Q(\pi) - \hat{Q}_T(\pi)| \). With \( h_t \) being independent of policy \( \pi \) and data realization, we have

\[
\begin{aligned}
\max_{\pi \in \Pi} |Q(\pi) - \hat{Q}_T(\pi)| &\leq \left( \sum_{t=1}^{T} h_t \right)^{-1} \cdot \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} h_t \cdot (\hat{F}_t(\pi) - Q(\pi)) \right| \\
&= (\sum_{t=1}^{T} h_t)^{-1} \cdot \max_{\pi \in \Pi} \left( \sum_{t=1}^{T} h_t \cdot (\hat{F}_t(\pi) - Q(X_t, \pi) + Q(X_t, \pi) - Q(\pi)) \right) \\
&\leq (\sum_{t=1}^{T} h_t)^{-1} \cdot \left( \max_{\pi \in \Pi} \left( \sum_{t=1}^{T} h_t \cdot (\hat{F}_t(\pi) - Q(X_t, \pi)) \right) + \max_{\pi \in \Pi} \left( \sum_{t=1}^{T} h_t \cdot (Q(X_t, \pi) - Q(\pi)) \right) \right).
\end{aligned}
\]

Define the \( \sigma \)-field \( F_t \triangleq \sigma(H_t, X_{t+1}) \). Then by Proposition 1(a) the term \( \sum_{t=1}^{T} h_t(\hat{F}_t(\pi) - Q(X_t, \pi)) \) is a martingale difference sequence w.r.t. the filtration \( \{F_t \}_{t \geq 1} \); the term \( \sum_{t=1}^{T} h_t(Q(X_t, \pi) - Q(\pi)) \) is an empirical process with i.i.d. random variables. We shall establish uniform concentration results for these two terms in Section 5 separately.

### 4.3 Choice of \( h_t \)

We proceed to discuss how to choose weights \( h_t \) to satisfy Assumption 2 and sharpen the regret bound established in Theorem 2. Two scenarios are considered: one with assignment probability lower bound \( g_t \) disclosed and the other without; we summarize the procedure in Algorithm 1.

**Algorithm 1:** Policy Learning via Generalized AIPW Estimator

**Input:** dataset \( \{(X_t, W_t, Y_t)\}_{t=1}^{T} \); policy class \( \Pi \).

for \( t = 1, \ldots, T \) do

1. Fit plug-in estimator \( \hat{\mu}_t(\cdot; w) \) for \( \mu(\cdot; w) \) using data \( \{(X_s, W_s, Y_s)\}_{s=1}^{T-1} \); for all \( w \in W \).

2. Construct the AIPW estimator \( \hat{\Gamma}_t(\pi) = \hat{\mu}_t(X_t; \pi(X_t)) + \frac{1}{c_t(X_t; \pi(X_t))} \cdot (Y_t - \hat{\mu}_t(X_t; \pi(X_t))) \).

if \( g_t \) is known then

- Set \( h_t = g_t \).

else

- Set \( h_t = 1 \).

Construct generalized AIPW estimator \( \hat{Q}_T(\pi) = \sum_{t=1}^{T} h_t \hat{\Gamma}_t(\pi) / \sum_{t=1}^{T} h_t \).

Return: \( \hat{\pi} = \arg\max_{\pi \in \Pi} \hat{Q}_T(\pi) \).

#### 4.3.1 Scenario I: \( g_t \) is known.

With the knowledge of \( g_t \), our goal is to analytically solve the optimal weight \( h^*_t \) that minimizes the regret bound presented in Theorem 2, which we shall soon show to be \( h^*_t \propto g_t \). When \( h_t \propto g_t \), Assumption 2 reduces
to that $\sum_{t=1}^{T} g_t \to +\infty$, which is naturally satisfied when the learning problem is “feasible” (note that when $\liminf_{T \to +\infty} \sum_{t=1}^{T} g_t < +\infty$, by the lower bound no learning algorithm can do better than a constant error in the worst case). Further, the policy $\pi$ obtained with $h^*_t$ is minimax optimal—the expected regret upper bound is $O(\kappa(P) \cdot (\sum_{t=1}^{T} g_t)^{-1/2})$, matching the exact lower bound in Theorem 1 up to logarithmic factors.

To solve for $h^*_t$, we firstly minimize the term $\left( \sum_{t=1}^{T} h^2_t / g_t \right) / \left( \sum_{t=1}^{T} h_t \right)^2$ in (2); as we shall soon see, this minimizer also minimizes the term $L_T(h,g)$ in (2). Let $h_t = h_t / \left( \sum_{s=1}^{T} h_s \right)$ be normalized weights. We rewrite the original problem into the following convex optimization problem:

$$
\min_{(h_t)_{t \in [T]}} \sum_{t=1}^{T} h^2_t / g_t
$$

such that $\sum_{t=1}^{T} h_t = 1; \quad h_t \geq 0, \quad t \in [T].$

The above problem has a unique minimizer $\hat{h}^*_t = g_t / \left( \sum_{s=1}^{T} g_s \right)$ (see Appendix C.1 for details). Plugging $\hat{h}^*$ into $L_T(h,g)$, we have

$$
L_T(\hat{h}^*, g) = \frac{1}{\sum_{t=1}^{T} g_t} \leq \frac{\sum_{t=1}^{T} h^2_t / g_t^2}{\left( \sum_{t=1}^{T} h^2_t / g_t \right)^2} = L_T(h,g), \quad \text{for any positive weights } \{h_t\},
$$

where the inequality is by Cauchy-Schwartz inequality. This choice of $\hat{h}^*$ yields a minimax optimal regret bound summarized in the corollary below.

**Corollary 2.1.** Suppose that Assumption 1 holds for assignment probability lower bound $\{g_t\}_{t \in [T]}$, and that $\sum_{t=1}^{T} g_t \to +\infty$ as $T \to +\infty$. The optimal weights $\{h^*_t\}_{t \in [T]}$ satisfy $h^*_t \propto g_t$ for each $t \in [T]$, with which the policy $\hat{\pi}$ in (6) achieves minimax optimal regret. Specifically, for any $T$ such that $\sum_{t=1}^{T} g_t \geq 8$, the expected regret of the policy $\hat{\pi}$ given by (6) can be upper bounded as,

$$
\mathbb{E}[R(\hat{\pi})] \leq 100M \sqrt{K} \cdot \left( 19\kappa(P) + 7 \sqrt{\log \left( \frac{T}{\sqrt{\sum_{t=1}^{T} g_t}} \right)} + 30 \right) \cdot \frac{1}{\sqrt{T}} + 7 \cdot \frac{1}{\sum_{t=1}^{T} g_t} \cdot \frac{1}{\sqrt{\sum_{t=1}^{T} g_t}}.
$$

The corollary is proven by letting $h_t = g_t$ in Theorem 2 and noting that $L_T(h,g) = \left( \sum_{t=1}^{T} g_t \right)^{-1} \leq \left( \sum_{t=1}^{T} g_t \right)^{-1/2}$.

**4.3.2 Scenario II: $g_t$ is unknown.**

In practice, one may not have access to the assignment probability lower bound $g_t$. We claim in such settings, uniform weighting with $h_t = 1$ can also be effective. In this case, Assumption 2 reduces to that

$$
\frac{\sum_{t=1}^{T} 1/g_t^2}{\left( \sum_{t=1}^{T} 1/g_t \right)^2} \to 0 \quad \text{as} \quad T \to +\infty,
$$

which holds in many cases such as $g_t = t^{-\alpha}$ for some $\alpha \in [0,1)$. The following corollary characterizes the regret incurred by our estimator with uniform weights.

**Corollary 2.2.** Suppose Assumption 1 holds and that

$$
\frac{\sum_{t=1}^{T} 1/g_t^3}{\left( \sum_{t=1}^{T} 1/g_t \right)^2} \to 0 \quad \text{as} \quad T \to +\infty.
$$

The expected regret incurred by policy $\hat{\pi}$ in (6) with uniform weights $h_t = 1$ can be bounded as

$$
\mathbb{E}[R(\hat{\pi})] \leq 100M \sqrt{K} \cdot \left( 19\kappa(P) + 7 \sqrt{\log \left( \sum_{t=1}^{T} 1/g_t \right)} + 29 \right) \cdot \frac{\sqrt{\sum_{t=1}^{T} 1/g_t}}{T} + 4M \cdot \frac{\sum_{t=1}^{T} 1/g_t^3}{\left( \sum_{t=1}^{T} 1/g_t \right)^2}.
$$
In general, the regret bound  \( \tilde{O}(\kappa(T) \cdot \sqrt{\sum_{t=1}^{T} g_t^2} + \sum_{t=1}^{T} 1/(g_t^2)}) \) yielded by uniform weighting is looser than the minimax regret  \( \tilde{O}(\kappa(T) \cdot (\sum_{t=1}^{T} g_t)^{-1/2}) \) yielded by optimal weighting, which can be verified by noticing that  \( \sqrt{\sum_{t=1}^{T} g_t^2} \geq (\sum_{t=1}^{T} g_t)^{-1/2} \) as a result of Cauchy-Schwarz inequality. However, in some cases, these two achieve the same regret decay rate—both are minimax optimal, as illustrated in the following example.

### 4.3.3 A case study.

To provide intuition, consider a special case where the assignment probability lower bound decays polynomially, and in specific we let  \( g_t = t^{-\alpha} \) for some  \( \alpha \in [0, 1) \). We consider weights  \( h_t = t^{-\beta} \) for some nonnegative  \( \beta \) and study how  \( \beta \) affects the regret of  \( \hat{\pi} \) obtained via (6). Theorem 1 shows that the expected regret (in terms of  \( T \)) is lower bounded by  \( \Omega(T^{(\alpha-1)/2}) \). Assumption 2 holds for any  \( \beta < \frac{\alpha+1}{2} \); then with Theorem 2, the expected regret upper bound with  \( h_t = t^{-\beta} \) for  \( \beta \in (0, \frac{\alpha+1}{2}) \) is  \( \tilde{O}(\kappa(T) \cdot T^{(\alpha-1)/2} + T^{4\beta-2\alpha-2}) \). In particular, when  \( \beta \leq \frac{\alpha+1}{4} \), our algorithm achieves expected regret bounds of  \( \tilde{O}(\kappa(T) \cdot T^{(\alpha-1)/2}) \), which matches the exact lower bound and is thus minimax optimal. Notably, uniform weighting (which does not require  \( g_t \) known) achieves the minimax optimal regret upper bound, and this bound inflates the one obtained by optimal weights (which needs  \( g_t \) to be disclosed) by a factor of  \( (1 - \alpha^2)^{-1/2} \). In this regard, when the assignment probability lower bound does not decay too fast in the sense that  \( g_t = t^{-\alpha} \) for some  \( \alpha \in [0, 1) \), the minimax optimality of our algorithm is agnostic to the knowledge of  \( g_t \).

## 5 Proof of the Upper Bound

We now establish the regret bound given in Theorem 2. The problem has been decoupled to showing two uniform concentration results for: (i) the martingale difference sequence  \( \sum_{t=1}^{T} h_t (\hat{\Gamma}_t(\pi) - Q(X_t, \pi)) \), where we condition on an event that the quadratic variation of AIPW scores  \( \hat{\Gamma}_t \) is well regularized across the policy class, which happens with high probability under Assumption 2; and (ii) the sum of independent variable sequence  \( \sum_{t=1}^{T} h_t (Q(X_t, \pi) - Q(\pi)) \), where we apply standard techniques in analyzing empirical processes with i.i.d. data.

### 5.1 Uniform Concentration of Martingale Difference Sequences

Define  \( M_T(\pi) \triangleq \sum_{t=1}^{T} h_t (\hat{\Gamma}_t(\pi) - Q(X_t, \pi)) \). Our goal is to show that with high probability,  \( \max_{\pi \in \Pi} |M_T(\pi)| \) is small. Particularly, we shall restrict our analysis on the event below:

\[
B_T \triangleq \left\{ \sup_{\pi \in \Pi} \sum_{t=1}^{T} h_t^2 \cdot (\hat{\Gamma}_t(\pi) - Q(X_t; \pi))^2 \leq 10KM^2 \cdot \sum_{t=1}^{T} h_t^2 / g_t \right\},
\]

In the following, we write  \( C_T(h, g) \triangleq 10KM^2 \cdot \sum_{t=1}^{T} h_t^2 / g_t \) for notation convenience. On the event  \( B_T \), the quadratic variation of  \( M_T(\pi) \) is controlled, which is critical in showing a fast uniform concentration rate of  \( M_T \). The following lemma quantifies the probability of the event  \( B_T \).

**Lemma 2.** Under Assumption 1,  \( P(B_T) > 1 - 2 \cdot L_T(h, g) \).

The proof is deferred to Appendix B.1. Lemma 2 immediately implies that under Assumption 2, event  \( B_T \) happens with high probability. Moving on, we shall show that  \( M_T \) concentrates uniformly at a fast rate when  \( B_T \) happens.

The traditional symmetrization technique is useful for proving uniform concentration results with i.i.d. data, but is not directly applicable given the adaptive nature of our data. Motivated by Raklin et al. (2015), we leverage a sequential analog of the symmetrization technique to obtain the uniform concentration for martingale empirical process. Note that results in Raklin et al. (2015), which require difference elements to be bounded, cannot be directly invoked in our setting where each element may diverge. To still allow for sequential uniform concentration, we take a different route and condition on event  \( B_T \), such that the sum of quadratic terms is under control.
We present the proof of uniform concentration of $M_T(\pi)$ in two steps. First, we connect the martingale empirical process with a tree Rademacher process (defined shortly afterwards), and next we connect the tree Rademacher process with the entropy integral of the policy class $\kappa(\Pi)$.

**Step 1: Connecting the Martingale Empirical Process with a Tree Rademacher Process.**

We start by defining the notion of a tree, following Rakhlin et al. (2015).

**Definition 3.** A $Z$-valued tree $z$ of depth $T$ is a rooted complete binary tree with nodes labeled by elements of $Z$.

A tree $z$ is identified with a sequence of labeling functions $(z_1, \ldots, z_T)$, where $z_i : \{\pm 1\}^{i-1} \mapsto Z$ labels the nodes on the $i$-th level. To be more specific, $z_1$ refers to the root node; $z_i$ for $i > 1$ refers to the node on the $i$-th level of the tree with the following rule: for a $\{\pm 1\}$-valued sequence of length $i - 1$, $z_i$ maps the sequence to a node by following a path on the tree, with $-1$ referring to “left” and $+1$ to “right”. As an example, Figure 1 plots a tree of depth 3, where $z_3(-1, -1)$ corresponds to the blue node, and $z_3(1, -1)$ is the red node.

![Figure 1: Illustration of a tree of depth 3. The blue node corresponds to $z_3(-1, -1)$, and the red node corresponds to $z_3(1, -1)$.

We proceed to state the definition of a tree Rademacher process:

**Definition 4** (Definition 2 of Rakhlin et al. (2015)). Suppose $Z$ is the sample space. Let $\{f_{\pi} : Z \rightarrow \mathbb{R} \mid \pi \in \Pi\}$ be a class of functions indexed by $\Pi$ and $\epsilon_1, \ldots, \epsilon_T$ be independent Rademacher random variables such that $P(\epsilon_1 = 1) = P(\epsilon_t = -1) = 1/2$. Given a $Z$-valued tree $z$ of depth $T$, we define the following stochastic process as a tree Rademacher process (indexed by $\Pi$):

$$f_{\pi} \mapsto \sum_{t=1}^{T} \epsilon_t \cdot f_{\pi}(z_t(\epsilon_1, \ldots, \epsilon_{t-1}))$$

Defining $Z_t \overset{\Delta}{=} (W_t, Y_t, \{\epsilon_t(X_t; w)\}_{w \in W}, \{\mu_t(X_t; w)\}_{w \in W})$, we write $f(h_t, X_t, Z_t; \pi) = h_t \cdot \hat{\Gamma}(\pi)$. Let $Z$ be the space $Z_t$ lives in and $z$ be a $Z$-valued binary tree. For notational simplicity, we write $x = \{x_1, \ldots, x_T\}$ to denote realized values of the covariates and $z_t(\epsilon) = z_t(\epsilon_1, \ldots, \epsilon_{t-1})$ to denote a node at depth $t$. Following Rakhlin et al. (2015), we introduce a decoupled tangent sequence (which is similar to the symmetrized sequence in the i.i.d. case and is defined in Definition 6) with respect to the data sequence $(Z_1, \ldots, Z_T)$. Similar to analyzing the martingale empirical process on event $\mathcal{B}_T$, we also restrict our analysis of the tree Rademacher process on the event $\mathcal{B}_T(x, z)$, which is defined with respect to a realization of covariates $x$ and a tree $z$ as follows:

$$\mathcal{B}_T(x, z) \overset{\Delta}{=} \left\{ \epsilon \in \{\pm 1\}^T : \sup_{\pi' \in \Pi} \sum_{t=1}^{T} \left( f(h_t, X_t, z_t(\epsilon); \pi') - h_t Q(X_t; \pi) \right)^2 \leq 2C_T(h, g) \right\}.$$ 

Lemma 3 shows that the tail of $\sup_{\pi \in \Pi} |M_T(\pi)|$ on the event $\mathcal{B}_T$ is bounded by the tail of the supremum of tree Rademacher process on the event $\mathcal{B}_T(x, z)$.
Lemma 3. Suppose that Assumption 1 holds, and that $L_T(h, g) \leq 1/8$. Then for any $\eta > \sqrt{8C_T(h, g)/KT^2}$, it holds that

$$
\mathbb{P}\left( \sup_{\pi \in \Pi} \mathbb{E}_T(\pi) \geq \eta T, B_T \mid x \right) \leq 4 \cdot \mathbb{P}_\epsilon \left( \sup_{\pi \in \Pi} \sum_{t=1}^T \epsilon_t f(h_t, X_t, z_t(\epsilon); \pi) \geq \eta T, B_T(x, z) \mid x, z \right),
$$

where $z$ is a $\mathbb{Z}$-valued tree of depth $T$, and $\mathbb{P}_\epsilon$ denotes the distribution of the Rademacher random variables $\{\epsilon_t\}_{t=1}^T$.

The proof is deferred to Appendix B.2.

Step 2: Connecting the Tree Rademacher Process with Policy Class Complexity. We now proceed to connect the tail bound of the supremum of tree Rademacher process with the complexity of the policy class. Note that this Rademacher process is defined on a tree, and the Hamming distance in Definition 1 cannot be directly applied here to characterize Rademacher complexity. Alternatively we adopt the notion of distance between policies on the tree process defined in Rakhlin et al. (2015) and modify it slightly for notational convenience in the proof.

Definition 5. Given covariates $x$ and a tree $z$ of depth $T$:

(a) The $\ell_2$ distance between two policies $\pi_1, \pi_2 \in \Pi$ w.r.t. $x, z$ and a sequence of $\{\pm 1\}$-valued random variables $\epsilon_{1:T}$ is defined as

$$
\ell_2(\pi_1, \pi_2; z, x, \epsilon_{1:T}) \triangleq \sqrt{\frac{\sum_{t=1}^T \left( f(X_t, z_t(\epsilon); \pi_1) - f(X_t, z_t(\epsilon); \pi_2) \right)^2}{16C_T(h, g)}}.
$$

(b) A set $S \subset \Pi$ is a (sequential) $\eta$-cover of a policy class $\Pi$ under the $\ell_2$-distance w.r.t. $x$ and $z$, if for any $\pi \in \Pi$ and any $\epsilon_{1:T} \in B_T(x, z)$, there exists some $s \in S$ such that $\ell_2(s, \pi; z, x, \epsilon_{1:T}) \leq \eta$.

(c) The $\eta$-covering number of a policy class $\Pi$ w.r.t. $x$ and $z$ is defined as

$$
N_2(\eta, \Pi; z, x) = \min \left\{ |S| : S \text{ is an } \eta \text{-cover of } \Pi \text{ under the } \ell_2 \text{ w.r.t. } z \text{ and } x \right\}.
$$

The Hamming distance is connected with the $\ell_2$ distance as follows.

Lemma 4. Under Assumption 1, for any realization of covariates $x$, any tree $z$ of depth $T$ and for any $\eta > 0$, we have $N_2(\eta, \Pi; z, x) \leq N_1(\eta^2, \Pi)$.

The proof of Lemma 4 is provided in Appendix B.3. We are now ready to bound the tail of supremum of tree Rademacher process using the entropy integral defined under Hamming distance.

Lemma 5. Under Assumptions 1, consider a realization of covariates $x$ and a tree $z$ of depth $T$. Given any $\delta \in (0, 1)$,

$$
\mathbb{P}_\epsilon \left( \max_{\pi \in \Pi} \sum_{t=1}^T \epsilon_t f(h_t, X_t, z_t(\epsilon); \pi) \geq \zeta, B_T(x, z) \mid x, z \right) \leq \delta,
$$

where $\zeta = 24\sqrt{C_T(h, g)} \left( 2\sqrt{2} + 2\sqrt{2\kappa(\Pi)} + \sqrt{\log(5/(3\delta))} + 1/\sqrt{T} \right)$.

We refer readers to Appendix B.4 for the complete proof of Lemma 5. We here present the three key parts in our proof to provide some intuition.

• **Part I: Policy decomposition.** Let $J = \lfloor \log_2(T) \rfloor$. Given $\epsilon = \epsilon_{1:T}$, we define a sequence of projection operators $A_0, A_1, \ldots, A_J$, where each $A_j$ maps a policy $\pi$ to its $j$-th approximation $A_j(\pi; \epsilon)$. As $j$ increases from 0 to $J$, the approximation becomes finer: $\{A_0(\pi; \epsilon)\}$ is a singleton indicating the coarsest
approximation, and \( A_J \) refers to the finest approximation. The construction of such a sequence will be discussed in detail in Appendix B.4. We decompose the tree Rademacher process as follows:

\[
\sum_{t=1}^{T} \epsilon_t f(x_t, z_t(\epsilon); \pi) = \sum_{t=1}^{T} \epsilon_t \left( \sum_{j=1}^{J} f(x_t, z_t(\epsilon); A_j(\pi; \epsilon)) - f(x_t, z_t(\epsilon); A_{j-1}(\pi; \epsilon)) + f(x_t, z_t(\epsilon); A_0(\pi; \epsilon)) \right) \\
+ \sum_{t=1}^{T} \epsilon_t \left( f(x_t, z_t(\epsilon); \pi) - f(x_t, z_t(\epsilon); A_J(\pi; \epsilon)) \right)
\]

Above, term (i) is of the order \( \sqrt{C_T(h, g)} \) with high probability, and terms (ii) is negligible w.r.t. the first term, which are shown separately in the following steps.

- **Part II: The Effective Term.** Given a realization of the covariates \( x \) and a tree \( z \) of depth \( T \), for any \( \delta > 0 \), with probability at least \( 1 - \delta \), either \( \epsilon \notin B_T(x, z) \), or there is

\[
\max_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t \left( \sum_{j=1}^{J} f(x_t, z_t(\epsilon); A_j(\pi; \epsilon)) - f(x_t, z_t(\epsilon); A_{j-1}(\pi; \epsilon)) + f(x_t, z_t(\epsilon); A_0(\pi; \epsilon)) \right) \right| < 24\sqrt{C_T(h, g)} \left( 2\sqrt{2} + 2\sqrt{2}\kappa(\Pi) + \sqrt{\log \left( \frac{5}{3\delta} \right)} \right).
\]

where the probability is w.r.t. the distribution of the Rademacher sequence \( \epsilon \).

- **Part III: The Negligible Terms.** Term (ii) in (7) is negligible with respect to \( \sqrt{C_T(h, g)} \) on the event \( B_T(x, z) \), and specifically,

\[
\sup_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t \left( f(x_t, z_t(\epsilon); \pi) - f(x_t, z_t(\epsilon); A_J(\pi; \epsilon)) \right) \right| \leq 4\sqrt{C_T(h, g)/T}.
\]

### 5.2 Uniform Concentration of Independent Difference Sequences

We now establish concentration results for \( \sum_{t=1}^{T} h_t(Q(X_t, \pi) - Q(\pi)) \). We note that covariates \( X_{1:T} \) are exogenous; thus \( \{Q(X_t, \pi)\}_{t=1}^{T} \) are bounded i.i.d. random variables. We shall apply the standard toolkit for the uniform concentration of i.i.d. data.

**Lemma 6.** Under Assumption 1, with probability at least \( 1 - \delta \),

\[
\max_{\pi \in \Pi} \left| \sum_{t=1}^{T} h_t(Q(X_t, \pi) - Q(\pi)) \right| \leq 8\sqrt{2M} \sqrt{\sum_{t=1}^{T} h_t^2 \left( 13 + 4\kappa(\Pi) + \sqrt{\log(1/\delta)} + 1/\sqrt{T} \right)}.
\]

The proof is deferred to Appendix B.5 for details. At a high level, we shall decompose \( \sum_{t=1}^{T} h_t(Q(X_t, \pi) - Q(\pi)) \) via a sequence of policy approximation operators, and bound each difference with standard uniform concentration techniques for i.i.d. data.

### 5.3 Expected Regret Upper Bound

We are now ready to put the pieces together and prove Theorem 2. Suppose that Assumptions 1 and 2 hold. Consider a \( T \) such that \( L_T(h, g) \leq 1/8 \).

\[
E[R(\tilde{\pi})] = E[R(\tilde{\pi}) \cdot 1\{B_T\}] + E[R(\tilde{\pi}) \cdot 1\{B_T^c\}] \leq E[R(\tilde{\pi}) \cdot 1\{B_T\}] + 2M \mathbb{P}(B_T^c) \\
\leq E[R(\tilde{\pi}) \cdot 1\{B_T\}] + 4ML_T(h, g),
\]

17
where the last inequality is a result of Lemma 2. Next, using the decomposition introduced earlier,

\[
E[R(\tilde{\pi}) \cdot 1\{B_T\}] \leq 2 \left( \sum_{t=1}^{T} h_t \right)^{-1} \left\{ E \left[ \sup_{\pi \in \Pi} \sum_{t=1}^{T} h_t (\tilde{\Gamma}_t(\pi) - Q(X_t; \pi)) \right] \cdot 1\{B_T\} 
+ E \left[ \sup_{\pi \in \Pi} \sum_{t=1}^{T} h_t (Q(X_t; \pi) - Q(\pi)) \right] \right\}.
\]

(8)

For the first expectation, letting \( \zeta = 24 \sqrt{C_T(h, g)} \cdot (2 + 2\sqrt{2} + 2\kappa(\Pi)) + \sqrt{\log \left( \frac{2M \sum_{t=1}^{T} h_t / g_t}{\sqrt{C_T(h, g)}} \right) + 1/\sqrt{T}} \), we have

\[
E \left[ \sup_{\pi \in \Pi} |\mathcal{M}_T(\pi)| \cdot 1\{B_T\} \right] \leq 4\zeta \ + \ E \left[ \sup_{\pi \in \Pi} |\mathcal{M}_T(\pi)| \cdot 1 \left\{ B_T, \sup_{\pi \in \Pi} |\mathcal{M}_T| \geq 4\zeta \right\} \right] 
\leq 4\zeta + 4M \left( \sum_{t=1}^{T} h_t / g_t \right) \cdot P \left( B_T, \sup_{\pi \in \Pi} |\mathcal{M}_T| \geq 4\zeta \right) 
\leq 4\zeta + 16 \sqrt{C_T(h, g)},
\]

where in the last step we apply Lemma 3 and Lemma 5 with \( \eta = 4\zeta / T \) and \( \delta = \frac{1}{M} \sqrt{\frac{C_T(h, g)}{\sum_{t=1}^{T} h_t / g_t}} \) (it can be checked that \( \eta > \sqrt{SC_T(h, g)/(KT^2)} \)).

For the second expectation, we take \( \delta = \sqrt{C_T(h, g)/(M \sum_{t=1}^{T} h_t)} \) in Lemma 6 and get

\[
E \left[ \sup_{\pi \in \Pi} \sum_{t=1}^{T} h_t (Q(X_t; \pi) - Q(\pi)) \right] 
\leq 8\sqrt{2}M \left( \sum_{t=1}^{T} h_t^2 \right) \left( 13 + 4\kappa(\Pi) + \sqrt{\log \left( \frac{M \sum_{t=1}^{T} h_t / g_t}{\sqrt{C_T(h, g)}} \right) + 1/\sqrt{T}} \right) + 2\sqrt{C_T(h, g)}.
\]

Summing the two terms up, we have

\[
(8) \leq M \sqrt{\frac{K \sum_{t=1}^{T} h_t^2 / g_t}{\sum_{t=1}^{T} h_t}} \cdot \left( 2200 + 1900\kappa(\Pi) + 630\sqrt{\log \left( \frac{\sum_{t=1}^{T} h_t / g_t}{\sqrt{C_T(h, g)}} \right) + 630/\sqrt{T}} \right).
\]

The proof is hence completed.

6 Simulations: Policy Learning with Decision Trees

In this section, we provide experimental evidence on the effectiveness of Algorithm 1, using both synthetic datasets and classification datasets from OpenML (Vanschoren et al., 2013). We investigate 1) how offline learning compares with its online counterpart when there is model misspecification; and 2) how different choices of weights \( h_t \) influence the regret of offline-learned policy. Throughout the experiments, we use linear models to fit the nuisance estimator \( \mu_t \) on data \( \mathcal{H}_{t-1} \).

Policy Class. Exact policy learning via maximizing policy value estimation generally leads to a nonconvex optimization problem and can be infeasible for arbitrary policy classes. We hereby focus on a policy class of decision trees with fixed depth, which has a finite entropy integral (Zhou et al., 2022). To learn the policy that maximizes generalized AIPW estimator, we apply a publicly available solver PolicyTree that finds the global optimum in polynomial runtime via an exhaustive and unconstrained tree search (Sverdrup et al., 2020). Algorithm 2 adapts the software to our problem setting with customized inputs, so that weights \( h_t \) are incorporated in the value estimator.

\[ ^6 \text{Reproduction code can be found at https://github.com/gsbDBI/PolicyLearning.} \]
aligned with our recommendation of weights in Algorithm 1—when the assignment probability lower bound does not satisfy Assumption 2 and thus does not have guaranteed regret decay in Theorem 2. These findings are consistent with our analysis in Section 4.3.3, where the optimal weight \( h_t^* = t^{-\alpha} \) (which minimizes the theoretical regret bound in Theorem 2) achieves the smallest regret. Besides, weights \( h_t = t^{-\beta} \) with \( \beta < \frac{2\alpha+3}{\alpha} \) yield the same regret decay rate as \( h_t^* \); this rate is faster than the regret obtained by setting \( \beta = 2\alpha \), which choice of weights does not satisfy Assumption 2 and thus does not have guaranteed regret decay in Theorem 2. These findings are aligned with our recommendation of weights in Algorithm 1—when the assignment probability lower bound \( g_t \) is unknown, one can choose \( h_t = 1 \) to achieve a reasonable regret decay rate, which is minimax optimal when \( g_t \) decays slower than \( \Theta(t^{-1}) \).

Algorithm 2: Policy Learning via Generalized AIPW Estimator: Invoking PolicyTree

**Input:** dataset \( \{(X_t, W_t, Y_t)\}_{t=1}^T \); weights \( \{h_t\}_{t=1}^T \); decision tree depth \( L \).

for \( t = 1, \ldots, T \) do

1. Fit plug-in estimator \( \hat{\mu}_t(\cdot; w) \) for \( \mu(\cdot; w) \) using data \( \{(X_s, W_s, Y_s)\}_{s=1}^{t-1} \) for all \( w \in W \).
2. Construct the AIPW estimator \( \hat{\Gamma}_t(\pi) = \hat{\mu}_t(X_t; \pi(X_t)) + \frac{1}{c \pi(X_t)} \cdot (Y_t - \hat{\mu}_t(X_t; \pi(X_t))) \).
3. Construct reweighted AIPW estimator \( \bar{\Gamma}_t = \frac{h_t}{\sum_{s=1}^{t} h_s} \hat{\Gamma}_t \).

Call PolicyTree with input \( \{(X_t)_{t=1}^T, \{\bar{\Gamma}_t\}_{t=1}^T, L\} \), and obtain the learned policy \( \hat{\pi} \).

Data-Collection Agent. At each time, the experimenter first computes each arm’s preliminary assignment probabilities \( \{e_t(X_i; w)\}_{w \in W} \) based on past observations via a Linear Thompson sampling agent (Agrawal and Goyal, 2013); then a lower bound \( g_t = t^{-\alpha}/K \) (with \( \alpha = 0.5 \)) is imposed: arms with \( e_t(X_i; w) < g_t \) have assignment probability \( e_t(X_i; w) = g_t \); others will be shrunk by setting \( e_t(X_i; w) = g_t + c(e_t(X_i; w) - g_t) \), where \( c \) ensures \( \sum_{w \in W} e_t(X_i; w) = 1 \). This type of flooring scheme is a generalization of commonly-enforced overlap practice in randomized controlled trials and has been increasingly used in adaptive experimentation. The floor allows for diminishing exploration on suboptimal arms, but imposes a positive probability of sampling each arm to facilitate post-experiment analyses (which often require non-zero assignment probabilities everywhere when using methods based on inverse probability weighting) (Offer-Westort et al., 2021).

6.1 Synthetic Data

We consider a contextual bandit problem with two arms. At each time, the experimenter observes a covariate \( X_t \in \mathbb{R}^3 \) that is i.i.d. sampled from Uniform\([-2, 2]^3\]. The outcome model only depends on the first coordinate of the covariate: given \( x = (x_1, x_2, x_3) \), arm 1 has conditional mean \( \mu_1(x) = x_1^2 \), and arm 2 has conditional mean \( \mu_2(x) = 2 - x_1^2 \); see the left panel in Figure 2 for illustration. The observed response is perturbed by i.i.d. standard Gaussian noise.

To evaluate learned policies, we in addition sample 100,000 observations (covariates and potential outcomes) from the same underlying distribution as test data to calculate the regret. The right panel in Figure 2 demonstrates that given each sample size, the out-of-sample regrets (on the test data) of the current data-collection agent (note that this agent updates its policy with growing sample size) and of the policies obtained via Algorithm 2 with different choices of \( h_t \).

We first compare online learning with offline learning when there is model misspecification. Both the data-collection agent and the nuisance component \( \hat{\mu}_t \) in the AIPW scores assume a linear outcome model, while the true \( \mu \) is quadratic. The data-collection agent thus fails to learn the optimal policy; in fact, as shown in Figure 2, its out-of-sample regret decays much slower than the regret of any policy learned via Algorithm 2.

Figure 2 also demonstrates that performances of different choices of weights are consistent with our analysis in Section 4.3.3, where the optimal weight \( h_t^* = t^{-\alpha} \) (which minimizes the theoretical regret bound in Theorem 2) achieves the smallest regret. Besides, weights \( h_t = t^{-\beta} \) with \( \beta < \frac{2\alpha+3}{\alpha} \) yield the same regret decay rate as \( h_t^* \); this rate is faster than the regret obtained by setting \( \beta = 2\alpha \), which choice of weights does not satisfy Assumption 2 and thus does not have guaranteed regret decay in Theorem 2. These findings are aligned with our recommendation of weights in Algorithm 1—when the assignment probability lower bound \( g_t \) is unknown, one can choose \( h_t = 1 \) to achieve a reasonable regret decay rate, which is minimax optimal when \( g_t \) decays slower than \( \Theta(t^{-1}) \).
Figure 2: A synthetic example. A Thompson sampling agent collects data with assignment probability lower bound $g_t = t^{-\alpha}/2$ ($\alpha = 0.5$). The left panel demonstrates the arm outcome models, which only depend on the first coordinate. The right panel shows the out-of-sample regrets of policy learned with different choices of weights $h_t$ and the data-collection agent. With model misspecification, the data-collection agent fails to learn the optimal policy and has the largest regret. Among different weights used for offline learning, the policy learned with optimal weights $h_t = t^{-\alpha}$ has the smallest regret. Error bars are 95% confidence intervals averages across 1000 replications.

Table 1: Characteristics of 82 public OpenML datasets used for sequential classification in Section 6.2.
A linear Thompson sampling agent with assignment probability lower bound $g_t = t^{-\alpha}/K$ ($\alpha = 0.5$ and $K$ is number of classes/arms) conducts sequential classification for each dataset. The $x, y$-coordinates are the normalized regret of policies learned by setting $h_t = 1$ and $h_t = t^{-\beta}$ with $\beta \in \{\frac{1}{2}\alpha, \alpha, 2\alpha\}$. Orange and blue points are those with $x < y$ and $x > y$ respectively. Policy learned with optimal weights $h_t = t^{-\alpha}$ performs best.

Table 2: Summary statistics of regret of policy learned with different choices of weights $h_t$ on 82 OpenML datasets. The assignment probability lower bound is $g_t = t^{-\alpha}/K$, where $K$ is the number of arms (classes). Optimal weighting with $h_t = t^{-\alpha}$ performs the best in terms of the average and median of regrets over all datasets; it also achieves the smallest regret in most datasets (64 out of 82.)

| Weighting                  | $h_t = 1$ | $h_t = t^{-\alpha/2}$ | $h_t = t^{-\alpha}$ | $h_t = t^{-2\alpha}$ |
|---------------------------|-----------|------------------------|---------------------|----------------------|
| Averaged regret over 82 datasets | 0.313     | 0.307                  | 0.301               | 0.353                |
| Median regret among 82 datasets | 0.225     | 0.219                  | 0.215               | 0.281                |
| Number of datasets on which such weighting achieves smallest regret | 5         | 10                     | 64                  | 3                    |

6.2 Multi-class Classification Data

We adapt 82 multi-class classification datasets from OpenML (Vanschoren et al., 2013) into contextual bandit problems of sequential classification, following literature Dudík et al. (2011); Dimakopoulou et al. (2017); Su et al. (2020). Specifically, each class represents an arm, and each feature vector denotes a covariate that is sampled uniformly from the data; the counterfactual outcomes of different arms correspond to the one-hot encoding of the associated label. That is, we set the expected potential outcome for a given “arm” (a possible label) to one if the arm is the same as the label, and zero otherwise. The observed outcome is perturbed with a standard Gaussian noise. We can simulate an online learning algorithm with this artificial definition of arms and outcomes. In the experiment, we again use a floored Thompson sampling agent to collect data. See Appendix C.3 for the list of datasets. Table 1 summarizes the statistics of the datasets.

Figure 3 demonstrates regret of policies learned with different choices of weights $h_t$ on various datasets. Particularly, we use uniform weights $h_t = 1$ as a baseline, and compare it with other choices of weights $h_t = t^{-\beta}$ with $\beta = \{\alpha/2, \alpha, 2\alpha\}$. Each point represents a dataset, and its $x, y$ coordinates are normalized regrets using the uniform weights $h_t = 1$ and the weights $h_t = t^{-\beta}$, with the normalization term being the largest regret among all datasets. One can see that the best performance is achieved at optimal weights $h_t = t^{-\alpha}$, while the worst one is given by weights $h_t = t^{-2\alpha}$. We also observe that in the first two panels of Figure 3, setting $h_t = \{t^{-\alpha/2}, t^{-\alpha}\}$ yields smaller regret than uniform weighting $h_t = 1$ in most datasets.
(most points are in orange in these two panels), but the improvement is mild. This again agrees with our analysis in Section 4.3.3, which shows these three choices of weighting $h_t = \{1, t^{-\alpha/2}, t^{-\alpha}\}$ have the same minimax regret decay rate $O(T^{(\alpha-1)/2})$. As such, we again recommend using uniform weighting when the assignment probability lower bound $g_t$ is unknown, as presented in Algorithm 1.

We further provide auxiliary summary statistics in Table 2, where for each weighting scheme, we list the average and median of regrets over all datasets; we also compute the number of datasets on which each weighting scheme achieves the smallest regret. Again, we find that optimal weighting with $h_t = t^{-\alpha}$ has the best performance.

7 Discussion

In this paper, we propose an approach to policy learning with adaptively collected data. Our main result is that, in the regime where assignment probabilities are bounded below by a nonincreasing positive sequence, one can leverage generalized AIPW estimators to approximate policy value, and the policy that maximizes the estimates within a pre-specified class is asymptotically optimal. Our approach is built upon the semi-parametric literature and does not require knowledge of outcome model. Particularly, when equipped with the knowledge of lower bound on assignment probabilities, our algorithm achieves rate-optimal guarantees for minimax regret.

A number of interesting research directions remain open. A natural extension is to adapt weights $h_t$ to a smaller subset of policy classes, or further, the policy currently being evaluated. Here we choose $h_t$ to offset the worst-case variance in the value estimation (and thus constant over the policy class and for any data realization); as a result, the effective sample size may not grow as fast as the actual sample size. However, if one can adaptively choose $h_t$ with respect to the evaluated policy and the data collected, as proposed in Hadad et al. (2021), it may result in better empirical performance with improved effective sample size. Another potential line of research is to accommodate our offline learning framework to batch learning in the online setting. Most of the online learning literature that has established regret bounds is built upon functional form assumptions about the outcome model, while the generalized AIPW estimator does not require such knowledge, suggesting its potential to innovate the design of adaptive experiments.

Acknowledgements

The authors would like to thank Vitor Hadad, David A. Hirshberg, Stefan Wager, and Ruoxuan Xiong for helpful discussions. The authors are also grateful for the generous support provided by Golub Capital Social Impact Lab. S.A. acknowledges generous support from the Office of Naval Research grant N00014-19-1-2468. R.Z. acknowledges generous support from the PayPal Innovation Fellowship.

References

Abbasi-Yadkori, Y. (2013). Online learning for linearly parametrized control problems. PhD thesis, University of Alberta.

Abbasi-Yadkori, Y., Pál, D., and Szepesvári, C. (2011). Improved algorithms for linear stochastic bandits. Advances in neural information processing systems, 24.

Agrawal, S. and Goyal, N. (2013). Thompson sampling for contextual bandits with linear payoffs. In International Conference on Machine Learning, pages 127–135. PMLR.

Armitage, P. et al. (1960). Sequential medical trials. In Sequential Medical Trials, Blakewell Scientific Publication.

Athey, S. and Wager, S. (2021). Policy learning with observational data. Econometrica, 89(1):133–161.

Bassen, J., Balaji, B., Schaarschmidt, M., Thille, C., Painter, J., Zimmero, D., Games, A., Fast, E., and Mitchell, J. C. (2020). Reinforcement learning for the adaptive scheduling of educational activities. In Proceedings of the 2020 CHI Conference on Human Factors in Computing Systems, pages 1–12.
Bastani, H. (2021). Predicting with proxies: Transfer learning in high dimension. *Management Science, 67*(5):2964–2984.

Bastani, H. and Bayati, M. (2020). Online decision making with high-dimensional covariates. *Operations Research, 68*(1):276–294.

Bembom, O. and van der Laan, M. J. (2008). Data-adaptive selection of the truncation level for inverse-probability-of-treatment-weighted estimators. In *U.C. Berkeley Division of Biostatistics Working Paper Series*. bepress.

Bennett, A. and Kallus, N. (2020). Efficient policy learning from surrogate-loss classification reductions. In *International Conference on Machine Learning*, pages 788–798. PMLR.

Bercu, B., Delyon, B., and Rio, E. (2015). *Concentration inequalities for sums and martingales*. Springer.

Bertsimas, D., Kallus, N., Weinstein, A. M., and Zhuo, Y. D. (2017). Personalized diabetes management using electronic medical records. *Diabetes care, 40*(2):210–217.

Besbes, O. and Zeevi, A. (2009). Dynamic pricing without knowing the demand function: Risk bounds and near-optimal algorithms. *Operations Research, 57*(6):1407–1420.

Bibaut, A., Kallus, N., Dimakopoulou, M., Chambaz, A., and van der Laan, M. (2021). Risk minimization from adaptively collected data: Guarantees for supervised and policy learning. *Advances in Neural Information Processing Systems, 34*.

Bubeck, S. and Cesa-Bianchi, N. (2012). Regret analysis of stochastic and nonstochastic multi-armed bandit problems. In *arXiv:1204.5721*. arXiv preprint.

Cai, D. and He, X. (2011). Manifold adaptive experimental design for text categorization. *IEEE Transactions on Knowledge and Data Engineering, 24*(4):707–719.

Charles, D., Chickering, M., and Simard, P. (2013). Counterfactual reasoning and learning systems: The example of computational advertising. *Journal of Machine Learning Research, 14*.

Chernozhukov, V., Demirer, M., Lewis, G., and Syrgkanis, V. (2019). Semi-parametric efficient policy learning with continuous actions. *Advances in Neural Information Processing Systems, 32*.

Chu, W., Li, L., Reyzin, L., and Schapire, R. (2011). Contextual bandits with linear payoff functions. In *Proceedings of the Fourteenth International Conference on Artificial Intelligence and Statistics*, pages 208–214. JMLR Workshop and Conference Proceedings.

Collins, L. M., Murphy, S. A., and Strecher, V. (2007). The multiphase optimization strategy (most) and the sequential multiple assignment randomized trial (smart): new methods for more potent ehealth interventions. *American journal of preventive medicine, 32*(5):S112–S118.

Dani, V., Hayes, T. P., and Kakade, S. M. (2008). Stochastic linear optimization under bandit feedback. In *21st Annual Conference on Learning Theory*. PMLR.

Dimakopoulou, M., Zhou, Z., Athey, S., and Imbens, G. (2017). Estimation considerations in contextual bandits. In *arXiv:1711.07077*. arXiv preprint.

Duchi, J. (2016). Lecture notes for statistics 311/electrical engineering 377. *URL: https://stanford.edu/class/stats311/Lectures/full_notes.pdf*. Last visited on, 2:23.

Dudík, M., Langford, J., and Li, L. (2011). Doubly robust policy evaluation and learning. In *arXiv:1103.4601*. arXiv preprint.

Farias, V. F. and Li, A. A. (2019). Learning preferences with side information. *Management Science, 65*(7):3131–3149.
Fukuoka, Y., Zhou, M., Vittinghoff, E., Haskell, W., Goldberg, K., and Aswani, A. (2018). Objectively measured baseline physical activity patterns in women in the mped trial: Cluster analysis. *JMIR public health and surveillance*, 4(1):e10.

Goldenshluger, A. and Zeevi, A. (2013). A linear response bandit problem. *Stochastic Systems*, 3(1):230–261.

Hadad, V., Hirshberg, D. A., Zhan, R., Wager, S., and Athey, S. (2021). Confidence intervals for policy evaluation in adaptive experiments. *Proceedings of the National Academy of Sciences*, 118(15):e2014602118.

Hoiles, W. and Schaar, M. (2016). Bounded off-policy evaluation with missing data for course recommendation and curriculum design. In *International conference on machine learning*, pages 1596–1604. PMLR.

Horvitz, D. G. and Thompson, D. J. (1952). A generalization of sampling without replacement from a finite universe. *Journal of the American statistical Association*, 47(260):663–685.

Imbens, G. W. (2004). Nonparametric estimation of average treatment effects under exogeneity: A review. *Review of Economics and statistics*, 86(1):4–29.

Jin, Y. (2022). Upper bounds on the natarajan dimensions of some function classes. *arXiv preprint arXiv:2209.07015*.

Jin, Y., Yang, Z., and Wang, Z. (2020). Is pessimism provably efficient for offline rl? In *arXiv:2012.15085*. arXiv preprint.

Joachims, T., Swaminathan, A., and de Rijke, M. (2018). Deep learning with logged bandit feedback. In *International Conference on Learning Representations*.

Kallus, N. (2018). Balanced policy evaluation and learning. *Advances in neural information processing systems*, 31.

Kallus, N. and Udell, M. (2016). Dynamic assortment personalization in high dimensions. In *arXiv:1610.05604*. arXiv preprint.

Kallus, N. and Zhou, A. (2018). Confounding-robust policy improvement. In *arXiv:1805.08593*. arXiv preprint.

Karimi, M., Jannach, D., and Jugovac, M. (2018). News recommender systems–survey and road ahead. *Information Processing & Management*, 54(6):1203–1227.

Kim, E. S., Herbst, R. S., Wistuba, I. I., Lee, J. J., Blumenschein, G. R., Tsao, A., Stewart, D. J., Hicks, M. E., Erasmus, J., Gupta, S., et al. (2011). The battle trial: personalizing therapy for lung cancer. *Cancer discovery*, 1(1):44–53.

Kitagawa, T. and Tettenov, A. (2018). Who should be treated? empirical welfare maximization methods for treatment choice. *Econometrica*, 86(2):591–616.

Lai, T. L. and Robbins, H. (1985). Asymptotically efficient adaptive allocation rules. *Advances in applied mathematics*, 6(1):4–22.

Lan, A. S. and Baraniuk, R. G. (2016). A contextual bandits framework for personalized learning action selection. In *EDM*, pages 424–429.

Lee, D., Oh, B., Seo, S., and Lee, K.-H. (2020). News recommendation with topic-enriched knowledge graphs. In *Proceedings of the 29th ACM International Conference on Information & Knowledge Management*, pages 695–704.

Levine, S., Kumar, A., Tucker, G., and Fu, J. (2020). Offline reinforcement learning: Tutorial, review, and perspectives on open problems. In *arXiv:2005.01643*. arXiv preprint.

Li, L., Chu, W., Langford, J., and Schapire, R. E. (2010). A contextual-bandit approach to personalized news article recommendation. In *Proceedings of the 19th international conference on World wide web*, pages 661–670.
Li, L., Chu, W., Langford, J., and Wang, X. (2011). Unbiased offline evaluation of contextual-bandit-based news article recommendation algorithms. In Proceedings of the fourth ACM international conference on Web search and data mining, pages 297–306.

Li, L., Lu, Y., and Zhou, D. (2017). Provably optimal algorithms for generalized linear contextual bandits. In International Conference on Machine Learning, pages 2071–2080. PMLR.

Luedtke, A. R. and van der Laan, M. J. (2016). Statistical inference for the mean outcome under a possibly non-unique optimal treatment strategy. Annals of statistics, 44(2):713.

Mandel, T., Liu, Y.-E., Levine, S., Brunskill, E., and Popovic, Z. (2014). Offline policy evaluation across representations with applications to educational games. In AAMAS, pages 1077–1084.

Murphy, S. A. (2003). Optimal dynamic treatment regimes. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 65(2):331–355.

Murphy, S. A. (2005). An experimental design for the development of adaptive treatment strategies. Statistics in medicine, 24(10):1455–1481.

Nie, X., Tian, X., Taylor, J., and Zou, J. (2018). Why adaptively collected data have negative bias and how to correct for it. In International Conference on Artificial Intelligence and Statistics, pages 1261–1269. PMLR.

Offer-Westort, M., Coppock, A., and Green, D. P. (2019). Adaptive experimental design: Prospects and applications in political science. In SSRN 3364402. SSRN.

Offer-Westort, M., Rosenzweig, L. R., and Athey, S. (2021). Optimal policies to battle the coronavirus “infodemic” among social media users in sub-saharan africa. In OSF Registered Study. Oen Science.

Rakhlin, A., Sridharan, K., and Tewari, A. (2015). Sequential complexities and uniform martingale laws of large numbers. Probability Theory and Related Fields, 161(1-2):111–153.

Rigollet, P. and Zeevi, A. (2010). Nonparametric bandits with covariates. In arXiv:1003.1630. arXiv preprint.

Robins, J. M., Rotnitzky, A., and Zhao, L. P. (1994). Estimation of regression coefficients when some regressors are not always observed. Journal of the American statistical Association, 89(427):846–866.

Russo, D. and Van Roy, B. (2014). Learning to optimize via posterior sampling. Mathematics of Operations Research, 39(4):1221–1243.

Russo, D., Van Roy, B., Kazerouni, A., Osband, I., and Wen, Z. (2017). A tutorial on thompson sampling. In 1707.02038. arXiv preprint.

Sachdeva, N., Su, Y., and Joachims, T. (2020). Off-policy bandits with deficient support. In Proceedings of the 26th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining, pages 965–975.

Schnabel, T., Amershi, S., Bennett, P. N., Bailey, P., and Joachims, T. (2020). The impact of more transparent interfaces on behavior in personalized recommendation. In Proceedings of the 43rd International ACM SIGIR Conference on Research and Development in Information Retrieval, pages 991–1000.

Schnabel, T., Bennett, P. N., and Joachims, T. (2019). Shaping feedback data in recommender systems with interventions based on information foraging theory. In Proceedings of the Twelfth ACM International Conference on Web Search and Data Mining, pages 546–554.

Schnabel, T., Swaminathan, A., Singh, A., Chandak, N., and Joachims, T. (2016). Recommendations as treatments: Debiasing learning and evaluation. In International conference on machine learning, pages 1670–1679. PMLR.

Shin, J., Ramdas, A., and Rinaldo, A. (2019). Are sample means in multi-armed bandits positively or negatively biased? In arXiv:1905.11397. arXiv preprint.
Simon, R. (1977). Adaptive treatment assignment methods and clinical trials. *Biometrics*, pages 743–749.

Su, Y., Dimakopoulou, M., Krishnamurthy, A., and Dudík, M. (2020). Doubly robust off-policy evaluation with shrinkage. In *International Conference on Machine Learning*, pages 9167–9176. PMLR.

Su, Y., Wang, L., Santacatterina, M., and Joachims, T. (2019). Cab: Continuous adaptive blending for policy evaluation and learning. In *International Conference on Machine Learning*, pages 6005–6014. PMLR.

Sverdrup, E., Kanodia, A., Zhou, Z., Athey, S., and Wager, S. (2020). Policypred: Policy learning via doubly robust empirical welfare maximization over trees. *Journal of Open Source Software*, 5(50):2232.

Swaminathan, A. and Joachims, T. (2015a). Batch learning from logged bandit feedback through counterfactual risk minimization. *The Journal of Machine Learning Research*, 16(1):1731–1755.

Swaminathan, A. and Joachims, T. (2015b). Counterfactual risk minimization: Learning from logged bandit feedback. In *International Conference on Machine Learning*, pages 814–823. PMLR.

Swaminathan, A. and Joachims, T. (2015c). The self-normalized estimator for counterfactual learning. In *advances in neural information processing systems*, pages 3231–3239. Citeseer.

Swaminathan, A., Krishnamurthy, A., Agarwal, A., Dudík, M., Langford, J., Jose, D., and Zitouni, I. (2016). Off-policy evaluation for slate recommendation. In *arXiv:1605.04812*. arXiv preprint.

Thompson, W. R. (1933). On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. *Biometrika*, 25(3/4):285–294.

Tsybakov, A. B. (2008). *Introduction to nonparametric estimation*. Springer Science & Business Media.

Vanschoren, J., van Rijn, J. N., Bischl, B., and Torgo, L. (2013). Openml: Networked science in machine learning. *SIGKDD Explorations*, 15(2):49–60.

Victor, H. and Giné, E. (1999). *Decoupling: From Dependence to Independence*. Springer New York.

Wang, Y.-X., Agarwal, A., and Dudík, M. (2017). Optimal and adaptive off-policy evaluation in contextual bandits. In *International Conference on Machine Learning*, pages 3589–3597. PMLR.

Zeng, C., Wang, Q., Mokhtari, S., and Li, T. (2016). Online context-aware recommendation with time varying multi-armed bandit. In *Proceedings of the 22nd ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 2025–2034.

Zhan, R., Hadad, V., Hirshberg, D. A., and Athey, S. (2021). Off-policy evaluation via adaptive weighting with data from contextual bandits. In *Proceedings of the 27th ACM SIGKDD Conference on Knowledge Discovery & Data Mining*, pages 2125–2135.

Zhang, B., Tsiatis, A. A., Davidian, M., Zhang, M., and Laber, E. (2012). Estimating optimal treatment regimes from a classification perspective. *Stat*, 1(1):103–114.

Zhao, Y.-Q., Zeng, D., Laber, E. B., Song, R., Yuan, M., and Kosorok, M. R. (2015). Doubly robust learning for estimating individualized treatment with censored data. *Biometrika*, 102(1):151–168.

Zhou, X., Mayer-Hamblett, N., Khan, U., and Kosorok, M. R. (2017). Residual weighted learning for estimating individualized treatment rules. *Journal of the American Statistical Association*, 112(517):169–187.

Zhou, Z., Athey, S., and Wager, S. (2022). Offline multi-action policy learning: Generalization and optimization. In *Operations Research*. INFORMS.
A Auxiliary Definitions and Lemmas

Definition 6 (Decoupled tangent sequence). Fix a sequence of random variables \( \{Z_t\}_{t \geq 1} \) adapted to the filtration \( \{\mathcal{F}_t\}_{t \geq 1} \). A sequence of random variables \( \{Z'_t\}_{t \geq 1} \) is said to be a decoupled sequence tangent to \( \{Z_t\}_{t \geq 1} \) if for each \( t \), conditioned on \( Z_1, \ldots, Z_{t-1} \), the random variables \( Z_t \) and \( Z'_t \) are independent and identically distributed.

We here provide more detail in constructing the decoupled tangent sequence. First note that \( e_t \) and \( \hat{\mu}_t \) are measurable w.r.t. \( \mathcal{F}_{t-1} \), since they are fit with historical data \( \mathcal{H}_{t-1} \). We then construct \( Z'_t = (W'_t, Y'_t, \{e_t(X_i; w)\}_{w \in W}, \{\hat{\mu}_t(X_i; w)\}_{w \in W}) \), where \( W'_t \in W \) is sampled from a categorical distribution specified by the assignment probabilities \( \{e_t(X_i; w)\}_{w \in W} \); and \( Y'_t = \mu(X_i; W'_t) + e'_t \) with \( e'_t \) independently and identically distributed as \( e_t \). By definition, the random variables \( \{Z'_t\}_{t \geq 1} \) are conditionally independent given the \( \{\{e_t(X_i; w)\}_{w \in W}, \{\hat{\mu}_t(X_i; w)\}_{w \in W}\} \), which can be used to induce the master \( \sigma \)-field in Proposition 6.1.5. of Victor and Giné (1999). Particularly, we note that \( \{Z'_t\}_{t \geq 1} \) are conditionally independent given \( (Z'_t)_{t \geq 1} \).

Definition 7 (\( \hat{\ell}_2 \) distance). Given a realization of the covariates \( x_{1:T} \), we define
(a) the inner product distance between two policies \( \pi_1, \pi_2 \in \Pi \) w.r.t. \( x_{1:T} \) as
\[
\hat{\ell}_2(\pi_1, \pi_2; x_{1:T}) \triangleq \sqrt{\sum_{t=1}^T [h_t Q(x_t; \pi_1) - h_t Q(x_t; \pi_2)]^2 / 4M^2 \sum_{t=1}^T h_t^2}.
\]
(b) a set \( S \) is a (sequential) \( \eta \)-cover of a policy class \( \Pi \) under the \( \hat{\ell}_2 \)-distance w.r.t. \( x_{1:T} \), if for any \( \pi \in \Pi \), there exists some \( s \in S \) such that
\[
\hat{\ell}_2(s, \pi; x_{1:T}) \leq \eta.
\]
(c) the covering number of a policy class \( \Pi \) w.r.t. \( x_{1:T} \) as
\[
N_{\hat{\ell}_2}(\eta, \Pi; x_{1:T}) = \min \{|S| : S \text{ is an } \eta \text{-cover of } \Pi \text{ under the } \hat{\ell}_2 \text{ w.r.t. } x_{1:T}\}.
\]

Lemma 7 (Theorem 3.26 of Bercu et al. (2015)). Let \( \{M_n\}_{n \geq 1} \) be a square integrable martingale w.r.t. the filtration \( \mathcal{F}_{n \geq 1} \) such that \( M_0 = 0 \). Define
\[
[M]_n \triangleq \sum_{k=1}^n (M_k - M_{k-1})^2, \quad \langle M \rangle_n \triangleq \sum_{k=1}^n \mathbb{E}[(M_k - M_{k-1})^2 | \mathcal{F}_{k-1}].
\]
Then for any positive \( x \) and \( y \),
\[
\mathbb{P}(M_n \geq x, [M]_n + \langle M \rangle_n \leq y) \leq \exp \left(-\frac{x^2}{2y}\right).
\]

B Proof of Main Lemmas

B.1 Proof of Lemma 2

For notational simplicity, we define
\[
\tilde{\Gamma}_t(X_i; w) = \hat{\mu}(X_i; w) + \frac{1}{e_t(X_i; w)} \cdot (Y_t - \hat{\mu}(X_i; w)).
\]
Then for any \( \pi \in \Pi \),
\[
\mathbb{P} \left( \sup_{\pi \in \Pi} \sum_{t=1}^T h_t^2 \cdot (\tilde{\Gamma}_t(\pi) - Q(X_i; \pi))^2 \geq C_T(h, g) \right) \leq \mathbb{P} \left( \sum_{t=1}^T \sum_{w \in W} h_t^2 \cdot (\tilde{\Gamma}_t(X_i; w) - \mu(X_i; w))^2 \geq C_T(h, g) \right) \tag{9}
\]
Next, we note that
\[
\mathbb{E} \left[ (\hat{\Gamma}_t(X_t; w) - \mu(X_t; w))^2 \mid \mathcal{H}_{t-1} \right] = \mathbb{E} \left[ \frac{(1 - \varepsilon_t(X_t; w)) \cdot (\hat{\mu}(X_t; w) - \mu(X_t; w))^2 + \sigma^2}{\varepsilon_t(X_t; w)} \mid \mathcal{H}_{t-1} \right] \leq \frac{5M^2}{g_t},
\]
and consequently,
\[
\sum_{t=1}^{\tau} \sum_{w \in W} h_t^2 \mathbb{E} \left[ (\hat{\Gamma}_t(X_t; w) - \mu(X_t; w))^2 \mid \mathcal{H}_{t-1} \right] \leq \frac{C_T(h, g)}{2}.
\]
Letting \( D_t(X_t) \triangleq \sum_{w \in W} (\hat{\Gamma}_t(X_t; w) - \mu(X_t; w))^2 \), we have
\[
(9) \leq \mathbb{P} \left( \sum_{t=1}^{\tau} h_t^2 \cdot \left( D_t(X_t) - \mathbb{E}[D_t(X_t) \mid \mathcal{H}_{t-1}] \right) \geq \frac{C_T(h, g)}{2} \right)
\leq \frac{4}{C_T(h, g)^2} \mathbb{E} \left[ \sum_{t=1}^{\tau} h_t^2 \cdot \left( D_t(X_t) - \mathbb{E}[D_t(X_t) \mid \mathcal{H}_{t-1}] \right)^2 \right]
= \frac{4}{C_T(h, g)^2} \cdot \sum_{t=1}^{\tau} h_t^4 \cdot \mathbb{E} \left[ \text{Var}(D_t(X_t) \mid \mathcal{H}_{t-1}) \right],
\]
where in the last inequality we use the property of a martingale. Above, the variance term can be further bounded as
\[
\mathbb{E} \left[ (D_t(X_t) - \mathbb{E}[D_t(X_t) \mid \mathcal{H}_{t-1}])^2 \right] \leq 2 \cdot \left( \mathbb{E}[D_t(X_t)^2] + \mathbb{E}[\mathbb{D}_t(X_t) \mid \mathcal{H}_{t-1}]^2 \right)
\leq 4 \mathbb{E}[D_t(X_t)^2]
= 4 \mathbb{E} \left[ \left( \sum_{w \in W} (\hat{\Gamma}_t(X_t; w) - \mu(X_t; w))^2 \right)^2 \right]
\leq 4K \sum_{w \in W} \mathbb{E} \left[ (\hat{\Gamma}_t(X_t; w) - \mu(X_t; w))^4 \right]
\leq \frac{48K^2 M^4}{g_t^2},
\]
where step (i) is because \((a + b)^2 \leq 2(a^2 + b^2)\); step (ii) is due to Jensen’s inequality and step (iii) follows from the Cauchy-Schwarz inequality. Combining everything, we have
\[
(10) \leq 192K^2 M^4 \cdot \frac{\sum_{t=1}^{\tau} h_t^4 / g_t^3}{C_T(h, g)^2} \leq 2 \cdot \frac{\sum_{t=1}^{\tau} h_t^4 / g_t^3}{\left( \sum_{t=1}^{\tau} h_t^2 / g_t^3 \right)^2}.
\]

**B.2 Proof of Lemma 3**

Recall that \( Z_t = (W_t, Y_t, \{e_t(X_t; w)\})_{w \in W}, \{\tilde{\mu}_t(X_t; w)\})_{w \in W} \). Let \( Z'_1, Z'_2, \ldots, Z'_T \) be a decoupled sequence tangent to \( Z_1, Z_2, \ldots, Z_T \) conditional on \( X_{1:T} \) as defined in Definition 6, where \( Z'_t = (W'_t, Y'_t, \{e'_t(X_t; w)\})_{w \in W}, \{\tilde{\mu}_t(X_t; w)\})_{w \in W} \). By Chebychev’s inequality, for any \( \eta > 0 \)
\[
\mathbb{P} \left( \left( \sum_{t=1}^{\tau} f(h_t, X_t, Z'_t; \pi) - h_t \cdot Q(X_t; \pi) \right) \geq \frac{T\eta}{2} \mid Z_{1:T}, X_{1:T} \right) \leq \mathbb{E} \left[ \left( \sum_{t=1}^{\tau} f(h_t, X_t, Z'_t; \pi) - h_t \cdot Q(X_t; \pi) \right)^2 \mid Z_{1:T}, X_{1:T} \right].
\]
Conditional on \( Z_{1:T} \) and \( X_{1:T} \), \( Z'_t \) is independent of \( Z'_s \) for any \( t \neq s \). Consequently,
\[
\mathbb{E} \left[ \left( \sum_{t=1}^{\tau} f(h_t, X_t, Z'_t; \pi) - h_t \cdot Q(X_t; \pi) \right)^2 \mid Z_{1:T}, X_{1:T} \right] = \sum_{t=1}^{\tau} \mathbb{E} \left[ \left( f(h_t, X_t, Z'_t; \pi) - h_t Q(X_t; \pi) \right)^2 \mid Z_{1:T}, X_{1:T} \right].
\]

28
For a fixed $t \in [T]$, we expand the summand as

\[
E \left( (f(h_t, X_t, Z_t'; \pi) - h_t \cdot Q(X_t; \pi))^2 \mid Z_{1:T}, X_{1:T} \right)
= h_t^2 \cdot E \left( (\tilde{f}'_t(X_t; \pi(X_t)) - \mu(X_t, \pi(X_t)))^2 \mid Z_{1:T}, X_{1:T} \right)
= h_t^2 \cdot E \left( \left( \frac{1}{c_t(X_t; \pi(X_t))} \cdot (h'_t - \mu(X_t, \pi(X_t))) \right)^2 \mid Z_{1:T}, X_{1:T} \right)
+ \left( 1 - \frac{1}{c_t(X_t; \pi(X_t))} \right) \cdot \left( \tilde{\mu}_t(X_t, \pi(X_t)) - \mu(X_t, \pi(X_t)) \right)^2
\]

Combining the above calculation, we have that

\[
\frac{h_t^2 \sigma_t^2}{g_t^2} + \frac{h_t^2 (1 - c_t(X_t; \pi(X_t)))(\tilde{\mu}_t(X_t, \pi(X_t)) - \mu(X_t, \pi(X_t)))^2}{c_t(X_t; \pi(X_t))} \leq \frac{5M^2h_t^2}{g_t^2}
\]

Similar to the proof of Lemma 2, we can bound the quadratic variation.

\[
P \left( \sup_{t \in [T]} \sum_{t=1}^{T} \left( f(h_t, X_t, Z_t'; \pi) - h_t \cdot Q(X_t; \pi) \right)^2 \geq \frac{T \mu_t}{2} \mid Z_{1:T}, X_{1:T} \right) \leq \frac{20M^2 \cdot \sum_{t=1}^{T} h_t^2 / g_t}{K \eta T^2} \leq \frac{2C_T(h, g)}{K \eta T^2}
\]

Next, since the conditional expectation of $\sum_{w \in \mathcal{W}} (\hat{f}'(X_t; w) - \mu(X_t; w))^2$ can be bounded deterministically as below:

\[
\sum_{t=1}^{T} \sum_{w \in \mathcal{W}} h_t^2 \cdot E \left[ (\hat{f}'(X_t; w) - \mu(X_t; w))^2 \mid X_{1:T}, Z_{1:T} \right] \leq \sum_{t=1}^{T} \sum_{w \in \mathcal{W}} \frac{5M^2h_t^2}{c_t(X_t, w)} \leq \frac{C_T(h, g)}{2}
\]

we subsequently have

\[
(12) \leq \mathbb{P} \left\{ \sum_{t=1}^{T} \sum_{w \in \mathcal{W}} h_t^2 \left( \left( \hat{f}'^2(X_t; w) - \mu(X_t; w) \right)^2 - E \left[ (\hat{f}'(X_t; w) - \mu(X_t; w))^2 \mid X_{1:T}, Z_{1:T} \right] \right)^2 \right\} \leq \frac{C_T(h, g)}{2} \mid X_{1:T}, Z_{1:T} \right\}
\]

\[
\leq \frac{4}{C_T(h, g)^2} \cdot E \left( \sum_{t=1}^{T} \sum_{w \in \mathcal{W}} h_t^2 \left( \left( \hat{f}'^2(X_t; w) - \mu(X_t; w) \right)^2 - E \left[ (\hat{f}'(X_t; w) - \mu(X_t; w))^2 \mid X_{1:T}, Z_{1:T} \right] \right)^2 \right) \mid X_{1:T}, Z_{1:T} \right]\]

\[
\leq \frac{4K}{C_T(h, g)^2} \cdot \sum_{t=1}^{T} \sum_{w \in \mathcal{W}} h_t^2 \left( \left( \hat{f}'^2(X_t; w) - \mu(X_t; w) \right)^2 - E \left[ (\hat{f}'(X_t; w) - \mu(X_t; w))^2 \mid X_{1:T}, Z_{1:T} \right] \right)^2 \mid X_{1:T}, Z_{1:T} \right]
\]

where the equality is due to the independence between $Z_{1:T}$ conditional on $X_{1:T}$ and $Z_{1:T}$. For a given $t \in [T]$ and $w \in \mathcal{W}$,

\[
E \left[ (\hat{f}'^2(X_t; w) - \mu(X_t; w))^2 \mid X_{1:T}, Z_{1:T} \right] \leq E \left[ (\hat{f}'(X_t; w) - \mu(X_t; \mu))^2 \mid X_{1:T}, Z_{1:T} \right] \leq \frac{48M^4}{c_t(X_t, w)}.
\]

\[
\text{Var} \left( \hat{f}'(X_t; w) - \mu(X_t; w) \right)^2 \mid X_{1:T}, Z_{1:T} \right] \leq \frac{48M^4}{c_t(X_t, w)}.
\]
Consequently, we have
\[
(13) \leq \frac{192K^4M^4}{C_T(h,g)^2} \sum_{t=1}^{T} \sum_{w \in W} \frac{h_t^4}{c_t^4(X_t;w)} \leq \frac{192K^4M^4}{C_T(h,g)^2} \sum_{t=1}^{T} \frac{h_t^4}{g_t^2}.
\]  
(14)

With condition that \((\sum_{t=1}^{T} h_t^4/g_t^3)/(\sum_{t=1}^{T} h_t^2/g_t) \leq 1/8\), the right-hand side of (14) is bounded by 1/4. Choose \(\eta > \sqrt{8C_T(h,g)/KT^2}\), then the right-hand side of (11) is bounded by 1/4. Collectively, we have for any \(\pi \in \Pi\),
\[
P \left( \left| \sum_{t=1}^{T} f(h_t, X_t, Z_t^i; \pi) - h_t \cdot Q(X_t; \pi) \right| < \frac{T \eta}{2}, \sup_{\pi \in \Pi} \sum_{t=1}^{T} \left( f(h_t, X_t, Z_t^i; \tilde{\pi}) - h_t \cdot Q(X_t; \tilde{\pi}) \right)^2 < C_T(h, g) \mid Z_{1:T}, X_{1:T} \right) \geq \frac{1}{2},
\]

Given \(Z_1, \ldots, Z_T\) and \(X_1, \ldots, X_T\), let \(\pi^*\) denote the policy that maximizes \(\sum_{t=1}^{T} f(h_t, X_t, Z_t^i; \pi) - h_t \cdot Q(X_t, \pi)\), from the above we have,
\[
P \left( \left| \sum_{t=1}^{T} f(h_t, X_t, Z_t^i; \pi^*) - h_t Q(X_t, \pi^*) \right| < \frac{T \eta}{2}, \sup_{\pi \in \Pi} \sum_{t=1}^{T} \left( f(h_t, X_t, Z_t^i; \pi) - h_t \cdot Q(X_t; \pi) \right)^2 < C_T(h, g) \mid Z_{1:T}, X_{1:T} \right) \geq \frac{1}{2}.
\]

Define the events
\[
A = \left\{ \sup_{\pi \in \Pi} \left| \sum_{t=1}^{T} f(h_t, X_t, Z_t^i; \pi) - h_t \cdot Q(X_t; \pi) \right| \geq \eta T \right\},
\]
\[
B = \left\{ \sup_{\pi \in \Pi} \sum_{t=1}^{T} \left( f(h_t, X_t, Z_t^i; \pi) - h_t \cdot Q(X_t; \pi) \right)^2 \leq C_T(h, g) \right\}.
\]

By the tower property, we have that
\[
\frac{1}{2} \leq P \left( \left| \sum_{t=1}^{T} f(h_t, X_t, Z_t^i; \pi^*) - h_t \cdot Q(X_t, \pi^*) \right| < \frac{T \eta}{2}, \sup_{\pi \in \Pi} \sum_{t=1}^{T} \left( f(h_t, X_t, Z_t^i; \pi) - h_t \cdot Q(X_t; \pi) \right)^2 < C_T(h, g) \mid A \cap B, X_{1:T} \right).
\]

This implies that
\[
\frac{1}{2} P \left( \left| \sum_{t=1}^{T} f(h_t, X_t, Z_t^i; \pi^*) - Q(X_t, \pi^*) \right| \geq \eta T, \sup_{\pi \in \Pi} \sum_{t=1}^{T} h_t^2 \cdot \left( f(h_t, X_t, Z_t^i; \pi) - Q(X_t; \pi) \right)^2 \leq C_T(h, g) \mid X_{1:T} \right) \leq P \left( \left| \sum_{t=1}^{T} f(h_t, X_t, Z_t^i; \pi^*) - h_t \cdot Q(X_t, \pi^*) \right| < \frac{T \eta}{2}, \sup_{\pi \in \Pi} \sum_{t=1}^{T} \left( f(h_t, X_t, Z_t^i; \pi) - h_t \cdot Q(X_t; \pi) \right)^2 \leq C_T(h, g) \mid A \cap B, X_{1:T} \right) \times P \left( \left| \sum_{t=1}^{T} f(h_t, X_t, Z_t^i; \pi^*) - h_t \cdot Q(X_t; \pi^*) \right| \geq \eta T, \sup_{\pi \in \Pi} \sum_{t=1}^{T} \left( f(h_t, X_t, Z_t^i; \pi) - h_t \cdot Q(X_t; \pi) \right)^2 \leq C_T(h, g) \mid X_{1:T} \right) \leq P \left( \sum_{t=1}^{T} f(h_t, X_t, Z_t^i; \pi^*) - h_t \cdot Q(X_t, \pi^*) \right| < \frac{T \eta}{2}, \sup_{\pi \in \Pi} \sum_{t=1}^{T} \left( f(h_t, X_t, Z_t^i; \pi) - h_t \cdot Q(X_t; \pi) \right)^2 \leq C_T(h, g), \right. \]
\[
\left| \sum_{t=1}^{T} f(h_t, X_t, Z_t^i; \pi^*) - h_t \cdot Q(X_t, \pi^*) \right| \geq \eta T, \sup_{\pi \in \Pi} \sum_{t=1}^{T} \left( f(h_t, X_t, Z_t^i; \pi) - h_t \cdot Q(X_t; \pi) \right)^2 \leq C_T(h, g) \mid X_{1:T} \right).
\]
The above can be further bounded by

\[
\mathbb{P} \left( \left| \sum_{t=1}^{T} f(h_t, X_t, Z_t; \pi) - f(h_t, X_t, Z_t; \pi^*) \right| \geq \frac{\eta T}{2}, \sup_{\pi \in \Pi} \sum_{t=1}^{T} (f(h_t, X_t, Z_t; \pi) - h_t Q(X_t; \pi))^2 \leq C_T(h, g) \right) \\
\sup_{\pi \in \Pi} \sum_{t=1}^{T} (f(h_t, X_t, Z_t; \pi) - h_t Q(X_t; \pi))^2 \leq C_T(h, g) \right) \\
\leq \mathbb{P} \left( \sup_{\pi \in \Pi} \sum_{t=1}^{T} f(h_t, X_t, Z_t; \pi) - f(h_t, X_t, Z_t; \pi) \right) \leq \frac{\eta T}{2}, \sup_{\pi \in \Pi} \sum_{t=1}^{T} (f(h_t, X_t, Z_t; \pi) - h_t Q(X_t; \pi))^2 \leq C_T(h, g) \right) \\
\leq \mathbb{P} \left( \sup_{\pi \in \Pi} \sum_{t=1}^{T} f(h_t, X_t, Z_t; \pi) - f(h_t, X_t, Z_t; \pi) \right) \geq \frac{\eta T}{2}, \sup_{\pi \in \Pi} \sum_{t=1}^{T} (f(h_t, X_t, Z_t; \pi) - h_t Q(X_t; \pi))^2 + (f(h_t, X_t, Z_t; \pi) - h_t Q(X_t; \pi))^2 \leq 2C_T(h, g) \right) \\
\left(15\right)
\]

Since conditional on \(Z_{t, T-1}\) and \(X_{1:T}\), \(Z_T\) is independent of and identically distributed as \(Z_T^t\),

\[
\left(15\right) \leq \mathbb{E}_{\pi_T} \left[ \mathbb{P}_{Z_T, Z_T^t} \left( \sup_{\pi \in \Pi} \sum_{t=1}^{T-1} f(h_t, X_t, Z_t; \pi) - f(h_t, X_t, Z_t; \pi) + \epsilon_T \cdot \left( f(h_t, X_t, Z_T^t; \pi) - f(h_t, X_t, Z_T^t; \pi) \right) \right) > \frac{T \eta}{2} \right], \\
\sup_{\pi \in \Pi} \sum_{t=1}^{T-1} (f(h_t, X_t, Z_t; \pi) - h_t \cdot Q(X_t; \pi))^2 + (f(h_t, X_t, Z_T^t; \pi) - h_t \cdot Q(X_t; \pi))^2 \leq 2C_T(h, g) \right] \\
\leq \sup_{z_T, z_T^t} \mathbb{E}_\pi \left[ \left\{ \sup_{\pi \in \Pi} \sum_{t=1}^{T-1} f(h_t, X_t, Z_t; \pi) - f(h_t, X_t, Z_t; \pi) + \epsilon_T \left( f(h_t, X_t, z_T^t; \pi) - f(h_t, X_t, z_T^t; \pi) \right) \right] > \frac{T \eta}{2} \right] \\
\times \left\{ \sup_{\pi \in \Pi} \sum_{t=1}^{T} (f(h_t, X_t, Z_t; \pi) - h_t \cdot Q(X_t; \pi))^2 + (f(h_t, X_t, Z_t; \pi) - h_t Q(X_t; \pi))^2 \leq 2C_T(h, g) \right\},
\]

31
where we use Fubini’s theorem in the inequality. Continue doing this for \( T - 1 \), we have that

\[
P \left( \sup_{\pi \in \Pi} \left| \sum_{t=1}^{T} f(h_t, X_t, Z_t; \pi) - f(h_t, X_t, Z_t; \tilde{\pi}) \right| > \frac{T \eta}{2} \right) 
\]

\[
= \mathbb{E} \left[ P \left( \sup_{\pi \in \Pi} \left| \sum_{t=1}^{T} f(h_t, X_t, Z_t; \pi) - f(h_t, X_t, Z_t; \pi) \right| > \frac{T \eta}{2} \right) \right] 
\]

\[
\leq \mathbb{E} \left[ \sup_{z_T, z_{T-1}, \ldots, z_1} \mathbb{E}_{\pi_T} \left[ 1 \left\{ \sup_{z_T, z_{T-1}, \ldots, z_1} \sup_{\pi \in \Pi} \left| \sum_{t=1}^{T} f(h_t, X_t, Z_t; \pi) - f(h_t, X_t, Z_t; \pi) \right| > \frac{T \eta}{2} \right) \right] \right].
\]

Repeating the above steps for \( T - 2, \ldots, 1 \), we have that

\[
(15) \leq \sup_{z_1, z_1'} \sup_{z_2, z_2'} \cdots \sup_{z_T, z_T'} \mathbb{E}_{\pi_1} \mathbb{E}_{\pi_2} \cdots \mathbb{E}_{\pi_T} \left[ \sup_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t (\sum_{t=1}^{T} f(h_t, X_t, Z_t; \pi) - f(h_t, X_t, z_t; \pi)) \right| > \frac{T \eta}{4} \right] \right].
\]

Note here the t-th supremum is taken over \( z_t \in Z \). Assuming the above supremum can all be attained—so that at \( t \) the maximum \( z_t^* \) depends on \( \epsilon_j, \ldots, \epsilon_{t-1} \)—the above probability can essentially be written as

\[
2P \left( \sup_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t \cdot f(h_t, X_t, z_t^* (\epsilon_1, \ldots, \epsilon_{t-1}, X_{t}; \pi)) \right| \geq \frac{T \eta}{4} \right)
\]

\[
\leq 2 \sup_{\pi} \mathbb{P} \left( \sup_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t \cdot f(X_t, z_{t+1} (\epsilon_1, \ldots, \epsilon_{t-1}; X_{t+1}; \pi)) \right| \geq \frac{T \eta}{4} \right).
\]

where \( z \) is a \( Z \)-valued tree of depth \( T \). If the supremum cannot be obtained, a limiting argument can be
applied to show the same conclusion. Finally, we conclude that
\[
\mathbb{P} \left( \sup_{\pi \in \Pi} \left| \sum_{t=1}^{T} h_t \left( \hat{\Gamma}_t(X_t; \pi) - Q(\pi) \right) \right| \geq \eta T, \sup_{\pi \in \Pi} \sum_{t=1}^{T} \left( f(h_t, X_t, Z_t; \pi) - h_t Q(X_t; \pi) \right)^2 \leq C_T(h, g) \mid X_{1:T} \right) \leq \frac{T \eta}{4},
\]

\[
\sup_{\pi \in \Pi} \sum_{t=1}^{T} \left( f(h_t, X_t, z_t(\epsilon_1, \ldots, \epsilon_{t-1}; \pi)) - h_t Q(X_t; \pi) \right)^2 \leq 2C_T(h, g).
\]

**B.3 Proof of Lemma 4**

For \( \eta > 0 \), let \( N_0 = N_0(\eta^2, \Pi) \). Without loss of generality we assume \( N_0 < \infty \), otherwise the argument trivially holds. Fix a realization of the covariates \( x \) and a tree \( \pi \). When \( \mathcal{B}_T(x, z) \) is empty, the result trivially holds. We assume in the following that \( \mathcal{B}_T(x, z) \) is non-empty. Next, consider the following optimization problem:

\[
\sup_{p_{a,t}, p_{b,t}, \pi \in \mathcal{B}_T(x, z)} | f(x_t, z_t(\epsilon_t); p_{a,t}) - f(x_t, z_t(\epsilon_t); p_{b,t}) |.
\]

Let \( (p^*_{a,t}, p^*_{b,t}, \epsilon_t) \) denote the policies and the Rademacher sequence that attains the supremum (we assume without loss of generality that the supremum is attainable; otherwise we can simply apply a limiting argument). For a positive integer \( m \), define

\[
n_t = \left\lfloor \frac{m \left( f(h_t, x_t, z_t(\epsilon_t); p^*_{a,t}) - f(h_t, x_t, z_t(\epsilon_t); p^*_{b,t}) \right)^2}{16C_T(h, g)} \right\rfloor,
\]

and

\[
\{ \tilde{x}_1, \ldots, \tilde{x}_n \} = \{ x_1, x_2, \ldots, x_T \}
\]

where \( x_t \) appears \( n_t \) times. Let \( S = \{ \pi_1, \ldots, \pi_{N_0} \} \) be the set of \( N_0 \) policies that \( \eta^2 \)-covers \( \Pi \) w.r.t. the Hamming distance defined with \( \{ \tilde{x}_1, \ldots, \tilde{x}_n \} \). Consider now an arbitrary policy \( \pi \in \Pi \). By the definition of a covering set, there exists \( \pi' \in S \) such that

\[
H(\pi, \pi'; \tilde{x}_{1:n}) \leq \eta^2.
\]

Further fix an arbitrary Rademacher sequence \( \epsilon \in \mathcal{B}_T(x, z) \), and by the definition of \( n \), we have

\[
n = \sum_{t=1}^{T} \frac{m \left( f(h_t, x_t, z_t(\epsilon_t'; \pi^*_{a,t}) - f(h_t, x_t, z_t(\epsilon_t'; \pi^*_{b,t}) \right)^2}{16C_T(h, g)} \right\rfloor \leq \sum_{t=1}^{T} 1 + \frac{3m}{16C_T(h, g)} \left( (f(z_t(\epsilon_t'; \pi^*_{a,t}) - h_t \cdot Q(x_t; \pi^*_{a,t}))^2 + h_t^2 \cdot (Q(x_t; \pi^*_{a,t}) - Q(x_t; \pi^*_{b,t}))^2 \right.
\]

\[
\left. + (h_t \cdot Q(x_t; \pi^*_{b,t}) - f(z_t(\epsilon_t'; \pi^*_{b,t}) )^2 \right)
\]

\[
\leq T + \frac{3m}{16C_T(h, g)} \sum_{t=1}^{T} \left( (f(z_t(\epsilon_t'; \pi^*_{a,t}) - h_t \cdot Q(x_t; \pi^*_{a,t}))^2 + (f(z_t(\epsilon_t'; \pi^*_{b,t}) - h_t \cdot Q(x_t; \pi^*_{b,t}))^2 + 4M^2 h_t^2 \right)
\]

\[
\leq m + T.
\]
On the other hand, by the definition of the Hamming distance,

\[
H(\pi, \pi' \mid \bar{x}_{1:n}) = \frac{1}{n} \sum_{i=1}^{n} 1\{\pi(\bar{x}_i) \neq \pi'(\bar{x}_i)\} \\
= \frac{1}{n} \sum_{i=1}^{T} \left[ \frac{m}{16C_T(h, g)} \left( f(h_t, x_t, z_t(\epsilon_i^t); \pi_{u,t}^*) - f(h_t, x_t, z_t(\epsilon_i^t); \pi_{u,t}) \right)^2 \right] 1\{\pi(x_t) \neq \pi'(x_t)\} \\
\geq \frac{m}{m + T} \sum_{i=1}^{T} \left[ \frac{m}{16C_T(h, g)} \left( f(h_t, x_t, z_t(\epsilon); \pi) - f(h_t, x_t, z_t(\epsilon); \pi') \right)^2 \right] 1\{\pi(x_t) \neq \pi'(x_t)\} \\
\geq \frac{m}{m + T} \sum_{i=1}^{T} \left[ \frac{m}{16C_T(h, g)} \left( f(h_t, x_t, z_t(\epsilon); \pi) - f(h_t, x_t, z_t(\epsilon); \pi') \right)^2 \right] 1\{\pi(x_t) \neq \pi'(x_t)\} \\
= \frac{m}{m + T} \ell_2^2(\pi, \pi'; \mathbf{z}, \mathbf{x}, \epsilon),
\]

where step (i) is because \( f(h_t, x_t, z_t(\epsilon); \pi) = f(h_t, x_t, z_t(\epsilon); \pi') \) if \( \pi(x_t) = \pi'(x_t) \). Additionally, with the choice of \( \pi' \), we have that

\[
\eta^2 \geq H(\pi, \pi' \mid \bar{x}_{1:n}) \geq \frac{m}{m + T} \ell_2^2(\pi, \pi'; \mathbf{z}, \mathbf{x}, x_t, 1),
\]

which yields \( \ell_2(\pi, \pi'; \mathbf{z}, x_t, 1, \epsilon) \leq \sqrt{1 + T/m} \cdot \eta \). In other words, for any \( \epsilon \in \mathcal{B}_T(\mathbf{x}, \mathbf{z}) \) and any \( \pi \in \Pi \), there exists \( \pi' \in S \) such that \( \ell_2(\pi, \pi'; \mathbf{z}, \mathbf{x}, \epsilon) \leq \sqrt{1 + T/m} \cdot \eta \). This says,

\[
N_2\left(\sqrt{1 + T/m} \cdot \eta, \Pi; \mathbf{x}, \mathbf{z}\right) \leq N_H(\eta^2, \Pi; \mathbf{x}, \mathbf{z}) \leq N_H(\eta^2, \Pi).
\]

Letting \( m \) go to infinity, we arrive at

\[
N_2(\eta, \Pi; \mathbf{z}, \mathbf{x}) \leq N_H(\eta^2, \Pi).
\]

### B.4 Proof of Lemma 5

#### B.4.1 Policy Decomposition.

For notational brevity, we write \( \epsilon_{j,T} \) as \( \epsilon \) if no confusion can arise. With \( J = \lceil \log_2(T) \rceil \), we construct a sequence of projection operators \( A_0, A_1, \ldots, A_J \), where conditioning on realizations of \( \mathbf{x}, \mathbf{z}, \mathbf{e} \), each \( A_j \) maps a policy \( \pi \) to its \( j \)-th approximation \( A_j(\pi; \epsilon) \); as \( j \) increases from 0 to \( J \), the approximation becomes finer.

Let \( \eta_j = \epsilon^{1/2-j} \), and \( S_j \) be a set of policies that \( \eta_j \)-covers \( \Pi \) under the \( \ell_2 \) distance for \( j = 0, 1, \ldots, J \). Now for any sequence \( \epsilon \in \mathcal{B}_T(\mathbf{x}, \mathbf{z}) \) and any policy \( \pi \in \Pi \), there exists \( \pi' \in S_j \) such that \( \ell_2(\pi, \pi'; \mathbf{z}, \mathbf{x}, \epsilon) \leq \eta_j \). By definition, we can choose \( S_j \) such that \( |S_j| = N_2(\eta_j, \Pi; \mathbf{z}, \mathbf{x}) \), for any \( j \in [J] \). We now proceed to construct the sequential approximation operators via a backward selection scheme. For any policy \( \pi \in \Pi \), fix a sequence \( \epsilon \in \mathcal{B}_T(\mathbf{x}, \mathbf{z}) \). For \( j \in [J] \), we define the \( j \)-th approximation mapping \( A_j(\pi; \epsilon) \) to be:

\[
A_j(\pi; \epsilon) = \arg\min_{\pi' \in S_j} \ell_2(\pi, \pi'; \mathbf{z}, \mathbf{x}, \epsilon).
\]

By the definition of \( S_j \), \( \ell_2(\pi, A_j(\pi); \mathbf{z}, \mathbf{x}, \epsilon) \leq \eta_j \). In particular, we define \( A_0(\pi) = (0, 0, \ldots, 0) \). Note that \( A_0(\pi) \) is not exactly an element in \( \Pi \)—-it is not a policy—it however serves as a 1-cover of \( \Pi \): for any \( \pi \in \Pi \), any \( \mathbf{z} \) and any \( \mathbf{x} \),

\[
\ell_2(\pi, A_0(\pi); \mathbf{z}, \mathbf{x}, \epsilon) = \sqrt{\sum_{i=1}^{T} \left( f(x_t, z_t(\epsilon); \pi) - 0 \right)^2} \\
\leq \sqrt{\sum_{i=1}^{T} \left( f(h_t, x_t, z_t(\epsilon); \pi) - h_t Q(X_t; \pi) \right)^2 + \sum_{i=1}^{T} h_t^2 \cdot Q(x_t; \pi)^2} \\
\leq 1,
\]

34
where step (i) is due to the triangular inequality. Given the approximation operators, we decompose the tree Rademacher process as follows:

\[
\sum_{t=1}^{T} \epsilon_t f(x_t, z_t(\epsilon); \pi) = \sum_{t=1}^{T} \epsilon_t \left( \sum_{j=1}^{J} f(x_t, z_t(\epsilon); A_j(\pi; \epsilon)) - f(x_t, z_t(\epsilon); A_{j-1}(\pi; \epsilon)) \right)
\]

\[
+ \sum_{t=1}^{T} \epsilon_t \left( f(x_t, z_t(\epsilon); \pi) - f(x_t, z_t(\epsilon); A_j(\pi; \epsilon)) \right).
\]

(16)

**B.4.2 The Effective Term.**

We now focus on term (i) in (16). For \( j \in \{1, \ldots, J\} \), any \( \pi_s \in S_j \) and \( \pi_r \in S_{j-1} \), we define a new tree \( w^{j, s, r}_t \), where for a sequence \( \epsilon \in B_T(x, z) \) and \( t \in [T] \):

- If there exists \( \pi \in \Pi \) and \( \epsilon' \in B_T(x, z) \) that matches with \( \epsilon \) up to time \( t \) such that \( A_j(\pi; \epsilon') = \pi_s \) and \( A_{j-1}(\pi; \epsilon') = \pi_r \), then

\[
w^{j, s, r}_t(\epsilon) = f(x_t, z_t(\epsilon); \pi_s) - f(x_t, z_t(\epsilon); \pi_r).
\]

- Otherwise \( w^{j, s, r}_t(\epsilon) = 0 \).

We can check that \( w^{j, s, r}_t \) is well-defined. Write the set of all such trees as \( W_j \triangleq \{ w^{j, s, r}_t : 1 \leq s \leq |S_j|, 1 \leq r \leq |S_{j-1}| \} \) (where we enumerate the elements \( S_j \) and \( S_{j-1} \) in an arbitrary order); by definition, \( |W_j| \leq |S_j||S_{j-1}| \). With the new tree process, the sum of differences in term (i) can be bounded as

\[
\sum_{t=1}^{T} \epsilon_t \left( \sum_{j=1}^{J} f(x_t, z_t(\epsilon); A_j(\pi; \epsilon)) - f(x_t, z_t(\epsilon); A_{j-1}(\pi; \epsilon)) \right)
\]

\[
= \sum_{j=1}^{J} \sum_{t=1}^{T} \epsilon_t \left( f(x_t, z_t(\epsilon); A_j(\pi; \epsilon)) - f(x_t, z_t; A_{j-1}(\pi; \epsilon)) \right)
\]

\[
\leq \sum_{j=1}^{J} \sup_{w \in W_j} \sum_{t=1}^{T} \epsilon_t \ w_t(\epsilon).
\]

Next, for any \( j \in \{1, \ldots, J\} \), any sequence \( \epsilon \in B_T(x, z) \), and any \( w \in W_j \), if there exists \( \pi \in \Pi \) such that \( A_j(\pi; \epsilon) = \pi_s \) and \( A_{j-1}(\pi; \epsilon) = \pi_r \), then

\[
\left( \sum_{t=1}^{T} w_t(\epsilon)^2 \right)^{\frac{1}{2}} \leq \sum_{t=1}^{T} \left( f(x_t, z_t(\epsilon); A_j(\pi; \epsilon)) - f(x_t, z_t(\epsilon); A_{j-1}(\pi; \epsilon)) \right)^2
\]

\[
\leq \sum_{t=1}^{T} \left( f(x_t, z_t(\epsilon); A_j(\pi; \epsilon)) - f(x_t, z_t(\epsilon); \pi) \right)^2 + \sum_{t=1}^{T} \left( f(x_t, z_t(\epsilon); \pi) - f(x_t, z_t(\epsilon); A_{j-1}(\pi; \epsilon)) \right)^2
\]

\[
\leq 4 \sqrt{C_T(h, g) \cdot (\eta_j + \eta_j - 1)} = 12 \sqrt{C_T(h, g) \cdot \eta_j}.
\]

Above, step (i) is due to the triangle inequality and step (ii) is due to the choice of \( A_j(\pi) \) and \( A_{j-1}(\pi) \). If no such \( \pi \) exists, then there exists a time \( t_0 \leq T \) such that \( w_{t_0-1}(\epsilon) \neq 0 \) and \( w_t(\epsilon) = 0 \) for all \( t \geq t_0 \). When \( t_0 = 1 \), \( w(\epsilon) \) is trivially zero. When \( t_0 > 1 \), by definition of \( w \), there exists a policy \( \pi \in \Pi \) and a sequence
\[ \epsilon' \in B_T(x, z) \] that agrees with \( \epsilon \) up to time \( t_0 - 1 \) such that \( A_j(\pi; \epsilon') = \pi_s \) and \( A_{j-1}(\pi; \epsilon') = \pi_r \). Using the previous argument, we arrive at

\[ \sum_{t=1}^T w_t(\epsilon')^2 \leq 12 \sqrt{C_T(h, g) \cdot \eta_j}. \]

Combining the above, we have for any \( j \in \{1, \ldots, J\} \) and any (constant to be specified) \( t_j > 0 \),

\[
\mathbb{P}_\epsilon \left( \max_{\pi \in \Pi} \left| \sum_{t=1}^T \epsilon_t \left( f(x_t, z_t(\epsilon); A_j(\pi; \epsilon)) - f(x_t, z_t(\epsilon); A_{j-1}(\pi; \epsilon)) \right) \right| \geq t_j, \epsilon \in B_T(x, z) \right) \\
\leq \mathbb{P}_\epsilon \left( \max_{\pi \in \Pi} \left| \sum_{t=1}^T \epsilon_t \right| \geq t_j, \max_{\pi \in \Pi} \sum_{t=1}^T \epsilon_t(\pi) \right) \leq 44C_T(h, g) \cdot \eta_j^2 \\
\leq N_2^{\pi}((\eta_j^2, \Pi; x, z), \epsilon) \cdot \mathbb{P} \left( \sum_{t=1}^T \epsilon_t(\pi) \geq t_j \right) \leq 144C_T(h, g) \cdot \eta_j^2 \\
\leq N_2^{\pi}((\eta_j^2, \Pi; x, z), \epsilon) \cdot \exp \left( - \frac{t_j^2}{576C_T(h, g) \cdot \eta_j^2} \right) \\
\leq N_3^{\pi}((\eta_j^2, \Pi) \cdot \exp \left( - \frac{t_j^2}{576C_T(h, g) \cdot \eta_j^2} \right) \\
\]

Above, step (i) is by the union bound, step (ii) follows from Lemma 7 and step (iii) is a result of Lemma 4.

For \( \delta > 0 \), setting \( t_j = 24 \sqrt{C_T(h, g) \cdot \eta_j \sqrt{\log (5j^2N_2^{\pi}(\eta_j^2, \Pi)/3\delta)}} \), we have

\[ \mathbb{P} \left( \max_{\pi \in \Pi} \left| \sum_{t=1}^T \epsilon_t \left( f(x_t, z_t(\epsilon); A_j(\pi; \epsilon)) - f(x_t, z_t(\epsilon); A_{j-1}(\pi; \epsilon)) \right) \right| \geq t_j, \epsilon \in B_T(x, z) \right) \leq \frac{3\delta}{5j^2}. \]

Taking the union bound over \( j \in \{1, \ldots, J\} \), we have

\[ \mathbb{P} \left( \max_{\pi \in \Pi} \left| \sum_{t=1}^T \epsilon_t \left( f(x_t, z_t(\epsilon); A_j(\pi; \epsilon)) - f(x_t, z_t(\epsilon); A_{j-1}(\pi; \epsilon)) \right) \right| \geq \sum_{j=1}^J t_j, \epsilon \in B_T(x, z) \right) \leq \frac{3\delta \sum_{j=1}^J \frac{1}{j^2}}{5} = \delta. \]

Consequently, we conclude that with probability at least \( 1 - \delta \), on the event \( \{ \epsilon \notin B_T(x, z) \} \)

\[ \max_{\pi \in \Pi} \left| \sum_{t=1}^T \epsilon_t \left( f(x_t, z_t(\epsilon); A_j(\pi; \epsilon)) - f(x_t, z_t(\epsilon); A_{j-1}(\pi; \epsilon)) \right) \right| \]

\[ \leq \sum_{j=1}^J t_j = \sum_{j=1}^J 24 \sqrt{C_T(h, g) \cdot \eta_j \sqrt{2 \log(j) + 2 \log (N_2(\eta_j^2, \Pi)) + \log (5/(3\delta))}} \]

\[ \leq 24 \sqrt{C_T(h, g) \cdot \eta_j \left( \sqrt{2 \log(j)} + \sqrt{2 \log (N_2(\eta_j^2, \Pi)) + \log (5/(3\delta))} \right)} \]

\[ \leq 24 \sqrt{C_T(h, g) \cdot \left( 2 \sqrt{2} + 2 \sqrt{2} \log(\Pi) + \log (5/(3\delta)) \right)} \]
Above, step (a) is because $\sqrt{x + y + z} \leq \sqrt{x} + \sqrt{y} + \sqrt{z}$ for $x, y, z \geq 0$; step (b) is due to the fact that $\sum \eta_j \log j \leq \sum \eta_j j \leq 2$ and that $\sum \eta_j \sqrt{\log N_H(\eta_j^2, \Pi)} = 2(\eta_j - \eta_{j+1}) \sqrt{\log N_H(\eta_j^2, \Pi)} \leq 2 \int_0^1 \sqrt{\log N_H(\eta^2, \Pi)}\;d\epsilon = 2\epsilon(\Pi)$.

### B.4.3 The Negligible Term.

We proceed to show term (ii) in (16) is negligible. On the event $\{\epsilon \in B_T(x, z)\}$,

$$
\sup_{\pi \in \Pi} \left| \sum_{t=1}^T \epsilon_t \left( f(x_t, z_t(\epsilon); \pi) - f(x_t, z_t(\epsilon); A_J(\pi, \epsilon)) \right) \right| \leq \left( \sup_{\pi \in \Pi} \sqrt{T} \cdot \sum_{t=1}^T \left( f(x_t, z_t(\epsilon); \pi) - f(x_t, z_t(\epsilon); A_J(\pi, \epsilon)) \right)^2 \right)^{\frac{1}{2}} 
$$

$$
\leq 4 \cdot 2^{-J} \cdot \sqrt{C_T(h, g)T} \leq 4 \sqrt{C_T(h, g)}/\sqrt{T},
$$

where in step (a) we use the Cauchy-Schwarz inequality; step (b) is due to the choice of $A_J(\pi, \epsilon)$.

### B.5 Proof of Lemma 6

Let $\epsilon_1, \ldots, \epsilon_T$ denote a sequence of i.i.d. Rademacher random variables with $\mathbb{P}(\epsilon_t = 1) = \mathbb{P}(\epsilon_t = -1) = 1/2$ for $t \in [T]$. Let $X'_1, \ldots, X'_T$ be an independent copy of $X_1, \ldots, X_T$. Then for any $\eta > 0$, we have

$$
\mathbb{E} \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^T h_t Q(X_t; \pi) - h_t Q(\pi) \right| \right] = \mathbb{E} \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^T h_t Q(X_t; \pi) - h_t \mathbb{E}[Q(X_t; \pi)] \right| \right] 
$$

$$
\leq \mathbb{E} \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^T h_t Q(X_t; \pi) - h_t \mathbb{E}[Q'(X_t; \pi)] \right| \right] 
$$

$$
\leq \max_{\pi \in \Pi} \left| \sum_{t=1}^T \epsilon_t h_t (Q(X_t; \pi) - Q(X'_t; \pi)) \right| 
$$

$$
\leq 2 \mathbb{E} \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^T \epsilon_t h_t Q(X_t, \pi) \right| \right].
$$

By the tower property,

$$
\mathbb{E} \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^T \epsilon_t h_t Q(X_t, \pi) \right| \right] = \mathbb{E} \left[ \mathbb{E}_{\epsilon_{1:T}} \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^T \epsilon_t h_t Q(X_t, \pi) \right| \right] \right].
$$

We now focus on bounding the inner expectation, and for brevity we write $\mathbb{E}_\epsilon = \mathbb{E}_{\epsilon_{1:T}}$. To do this, we shall use a distance between two policies (defined in Definition 7) as in the non i.i.d. case. Given the $\ell_2$ distance, we construct the policy approximation operators $A_j : \Pi \rightarrow \Pi$ via a backward selection scheme as before. Let $J = \lceil \log_2 T \rceil$ and $\eta_j = 2^{-j}$ for $j = 0, 1, \ldots, J$. For any covariate realization, let $S_j$ denote the smallest $\eta_j$-covering set of $\Pi$ w.r.t. the $\ell_2$ distance. Then for any $\pi \in \Pi$,

- define $A_j(\pi) = \arg\min_{\pi' \in S_j} \hat{\ell}_2(\pi, \pi'; x_{1:T})$;
- for each $j = J - 1, \ldots, 1$, define $A_j(\pi) = \arg\min_{\pi' \in S_j} \hat{\ell}_2(A_{j+1}(\pi), \pi'; x_{1:T})$;
- define $A_0(\pi) = (0, 0, \ldots, 0)$.

Similar to the non i.i.d. case, $A_0(\pi)$ is not exactly in $\Pi$. But $A_0(\pi)$ can serve as a 1-cover of $\Pi$ since for any $x_{1:T}$ and any $\pi \in \Pi$,

$$
\hat{\ell}_2(\pi, A_0(\pi); x_{1:T}) = \frac{\sqrt{\sum_{t=1}^T h_t^2 Q^2(X_t; \pi)}}{2M \sqrt{\sum_{t=1}^T h_t^2}} \leq 1.
$$
With the sequence of policy approximation operators, we can decompose the inner expectation as follows:

\[
\mathbb{E}_t \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t h_t \left( Q(X_t, \pi) - Q(X_t, A_j(\pi)) \right) \right| \right] \leq \mathbb{E}_t \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t \cdot \left( h_t Q(X_t, \pi) - h_t Q(X_t, A_j(\pi)) \right) \right| \right]
\]

\[
+ \mathbb{E}_t \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t \cdot \left( \sum_{j=1}^{J} h_t Q(X_t, A_j(\pi)) - h_t Q(X_t, A_{j-1}(\pi)) \right) \right| \right].
\]

We proceed to bound the above two terms separately.

**The Negligible Term.** Conditional on \(X_{1:T}\), we have,

\[
\max_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t h_t \left( Q(X_t, \pi) - Q(X_t, A_j(\pi)) \right) \right| \leq \max_{\pi \in \Pi} \sqrt{T} \sqrt{\sum_{t=1}^{T} h_t^2 \cdot \left( Q(X_t, \pi) - Q(X_t, A_j(\pi)) \right)^2} = 2M \sqrt{T} \sum_{t=1}^{T} h_t^2 \cdot \max_{\pi \in \Pi} \tilde{\ell}_2(\pi, A_j(\pi); X_{1:T}) \leq 2M \sqrt{T} \sum_{t=1}^{T} h_t^2 / T,
\]

where step (i) follows from Cauchy-Schwarz inequality and step (ii) is due to the choice of \(A_j(\pi)\). Consequently,

\[
\mathbb{E}_t \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t h_t \left( Q(X_t, \pi) - Q(X_t, A_j(\pi)) \right) \right| \right] \leq 2M \sqrt{T} \sum_{t=1}^{T} h_t^2 / T.
\]

**The Effective Term.** For \(j \in [J]\), a positive integer \(k\), and a positive constant \(t_{j,k}\) to be specified later,

\[
P_t \left( \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t h_t \left( Q(X_t, A_j(\pi)) - Q(X_t, A_{j-1}(\pi)) \right) \right| \geq t_{j,k} \right) \leq 2 \exp \left( - \frac{t_{j,k}^2}{8u_{j-1}^2 M^2 \sum_{t=1}^{T} h_t^2} \right)
\]

\[
\leq 2 N_{\ell_2}(\eta_j, \Pi; x_{1:T},) \cdot \exp \left( - \frac{t_{j,k}^2}{8u_{j-1}^2 M^2 \sum_{t=1}^{T} h_t^2} \right),
\]

where the inequality (i) is due to Hoeffding’s inequality and the definition of \(A_j(\pi)\) and \(A_{j-1}(\pi)\). Let \(t_{j,k} = 2\sqrt{2} u_{j-1} M \sqrt{\sum_{t=1}^{T} h_t^2} \cdot \sqrt{\log \left( 2^{k+2} j^2 N_{\ell_2}(\eta_j, \Pi; x_{1:T}) \right)} \) and applying a union bound over \(j \in [J]\), we obtain

\[
P_t \left( \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t h_t \left( Q(X_t, A_j(\pi)) - Q(X_t, A_{j-1}(\pi)) \right) \right| \geq \sum_{j=1}^{J} t_{j,k} \right) \leq \sum_{j=1}^{J} P_t \left( \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t h_t \left( Q(X_t, A_j(\pi)) - Q(X_t, A_{j-1}(\pi)) \right) \right| \geq t_{j,k} \right)
\]

\[
\leq 2 \sum_{j=1}^{J} N_{\ell_2}(\eta_j, \Pi; x_{1:T},) \cdot \exp \left( - \frac{t_{j,k}^2}{8u_{j-1}^2 M^2 \sum_{t=1}^{T} h_t^2} \right) \leq \frac{1}{2^{k+1}} \sum_{j=1}^{J} \frac{1}{2^{j}} \leq \frac{1}{2^k}.
\]

where in the last inequality we use \(\frac{2^2}{4} \leq 4\). As before, we connect the covering number under \(\ell_2\)-distance with that under the Hamming distance. The connection is characterized by Lemma 8.

**Lemma 8.** Under Assumption 1, for any realization of covariates \(x_{1:T}\) and for any \(\eta > 0\), we have \(N_2(\eta, \Pi; x_{1:T}) \leq N_H(\eta^2, \Pi)\).
We defer the proof of Lemma 8 to Appendix B.6. Using the connection, we have with probability at least $1 - 1/2^k$,

$$\max_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t \left( \sum_{j=1}^{J} h_t Q(X_t, A_j(\pi)) - h_t Q(X_t, A_{j-1}(\pi)) \right) \right|$$

$$\leq \sum_{j=1}^{J} 4\sqrt{2} \eta_j M \sqrt{\sum_{t=1}^{T} h_t^2 \cdot \left( \sqrt{(k+2) \log 2 + \sqrt{2} \log j + \sqrt{\log N_{\Pi}(\eta_j, \Pi; x_1:T)} \right)}$$

$$\leq \sum_{j=1}^{J} 4\sqrt{2} \eta_j M \sqrt{\sum_{t=1}^{T} h_t^2 \cdot \left( \sqrt{(k+2) \log 2 + \sqrt{2} \log j + \sqrt{\log N_{\Pi}(\eta_j^2, \Pi; x_1:T)} \right)}$$

$$\leq 4\sqrt{2} M \sqrt{\sum_{t=1}^{T} h_t^2 \cdot \left( \sqrt{(k+2) \log 2 + 2\sqrt{2} + 2\kappa(\Pi)} \right)}.$$

where in step (i) we use that $\sqrt{a + b + c} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$ for $a, b, c \geq 0$. Step (ii) is due to the fact that $\sum \eta_j \sqrt{\log j} \leq \sum \eta_j \leq 2$ and that $\sum \eta_j \sqrt{\log N_{\Pi}(\eta_j^2, \Pi)} = \sum 2(\eta_j - \eta_{j+1}) \sqrt{\log N_{\Pi}(\eta_j^2, \Pi)} \leq 2 \int_0^1 \sqrt{\log N_{\Pi}(\eta_j^2, \Pi)} = 2\kappa(\Pi)$.

As a result, we can bound the expectation as:

$$\mathbb{E}_\epsilon \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t \left( \sum_{j=1}^{J} h_t Q(X_t, A_j(\pi)) - h_t Q(X_t, A_{j-1}(\pi)) \right) \right| \right]$$

$$= \int_0^\infty \mathbb{P}_\epsilon \left( \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t \left( \sum_{j=1}^{J} h_t Q(X_t, A_j(\pi)) - h_t Q(X_t, A_{j-1}(\pi)) \right) \right| > s \right) \, ds$$

$$\leq 4\sqrt{2} M \sqrt{\sum_{t=1}^{T} h_t^2 \left( 8 \sqrt{\log 2 + 4\sqrt{2} + 4\kappa(\Pi)} \right)}$$

$$\leq 4\sqrt{2} M \sqrt{\sum_{t=1}^{T} h_t^2 \left( 13 + 4\kappa(\Pi) \right)}.$$

where in the last equality we use $8 \sqrt{\log 2 + 4\sqrt{2}} < 13$. Combining the bounds for the negligible term and the effective bound, we have

$$\mathbb{E}_\epsilon \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t h_t Q(X_t, \pi) \right| \right] \leq 2M \sqrt{\sum_{t=1}^{T} h_t^2/T + 4\sqrt{2} M \sqrt{\sum_{t=1}^{T} h_t^2 \left( 13 + 4\kappa(\Pi) \right)}}.$$

Therefore,

$$\mathbb{E} \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} h_t Q(X_t; \pi) - h_t Q(\pi) \right| \right] \leq 2\mathbb{E}_\epsilon \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} \epsilon_t h_t Q(X_t, \pi) \right| \right]$$

$$\leq 4M \sqrt{\sum_{t=1}^{T} h_t^2/T + 8\sqrt{2} M \sqrt{\sum_{t=1}^{T} h_t^2 \left( 13 + 4\kappa(\Pi) \right)}}.$$

39
Finally for any \( \eta > 0 \),
\[
\mathbb{P} \left( \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} h_t(Q(X_t, \pi) - Q(\pi)) \right| \geq 8M \sqrt{\frac{T}{\sum_{t=1}^{T} h_t^2}} + 8\sqrt{2M} \sqrt{\sum_{t=1}^{T} h_t^2 \left( 13 + 4\kappa(\Pi) \right) + \eta} \right) \\
\leq \mathbb{P} \left( \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} h_t(Q(X_t, \pi) - Q(\pi)) \right| - \mathbb{E} \left[ \max_{\pi \in \Pi} \left| \sum_{t=1}^{T} h_t(Q(X_t, \pi) - Q(\pi)) \right| \right] \geq \eta \right) \\
\leq \exp \left( - \frac{\eta^2}{2M^2 \sum_{t=1}^{T} h_t^2} \right),
\]
where the last inequality is due to bounded difference inequality (Duchi, 2016). Choosing \( \eta = \sqrt{2 \sum_{t=1}^{T} h_t^2 M \log(1/\delta)} \) yields the desired result.

### B.6 Proof of Lemma 8

Consider \( \eta > 0 \), and let \( N_0 = N_0(\eta^2, \Pi) \). We assume without loss of generality that \( N_0 < \infty \). Fix a realization of covariates \( x_{1:T} \), and consider the following optimization problem
\[
\max_{\pi_{a,t}, \pi_{b,t}} |h_t(Q(x_t, \pi_{a,t}) - h_t(Q(x_t, \pi_{b,t}))|.
\]
Let \( \pi_{a,t}^* \) and \( \pi_{b,t}^* \) denote the policies that achieve the maximum (we assume without loss of generality that the maximum is attainable; otherwise we can simply apply a limiting argument). We have
\[
|h_t(Q(x_t, \pi_{a,t}^*)) - h_t(Q(x_t, \pi_{b,t}^*))| \leq 2M h_t.
\]
For a positive integer \( m \), we define for any \( t \in [T] \) that
\[
n_t = \left\lfloor \frac{m|h_t(Q(x_t, \pi_{a,t}^*)) - h_t(Q(x_t, \pi_{b,t}^*))|^2}{4M^2 \sum_{t=1}^{T} h_t^2} \right\rfloor,
\]
and
\[
\{\tilde{x}_1, \ldots, \tilde{x}_n\} = \{x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_T, \ldots, x_T\},
\]
where \( x_t \) appears for \( n_t \) times. Let \( \mathcal{S} = \{\pi_1, \ldots, \pi_{N_0}\} \) denote the set of \( N_0 \) policies that \( \eta^2 \)-cover \( \Pi \) w.r.t. the Hamming distance defined under \( \tilde{x}_1, \ldots, \tilde{x}_n \). Consider an arbitrary \( \pi \in \Pi \). By the definition of a covering set, there exists \( \pi' \in \mathcal{S} \), such that
\[
H(\pi, \pi'; \tilde{x}_{1:n}) \leq \eta^2.
\]
By the definition of \( n \), we have
\[
n = \sum_{t=1}^{T} \left\lfloor \frac{m|h_t(Q(x_t, \pi_{a,t}^*)) - h_t(Q(x_t, \pi_{b,t}^*))|^2}{4M^2 \sum_{t=1}^{T} h_t^2} + 1 \right\rfloor \leq m + T.
\]
On the other hand,
\[
H(\pi, \pi'; \tilde{x}_{1:n}) = \frac{1}{n} \sum_{i=1}^{n} 1\{\pi(\tilde{x}_i) \neq \pi'(\tilde{x}_i)\}
\]
\[
\geq \frac{1}{m + T} \sum_{t=1}^{T} \frac{m|h_t(Q(x_t, \pi_{a,t}^*)) - h_t(Q(x_t, \pi_{b,t}^*))|^2}{4M^2 \sum_{t=1}^{T} h_t^2} \cdot 1\{\pi(x_t) \neq \pi'(x_t)\}
\]
\[
\geq \frac{1}{m + T} \sum_{t=1}^{T} \frac{m|h_t(Q(x_t, \pi) - h_t(Q(x_t, \pi'))|^2}{4M^2 \sum_{t=1}^{T} h_t^2} \cdot 1\{\pi(x_t) \neq \pi'(x_t)\}
\]
\[
\geq \frac{m}{m + T} \sum_{t=1}^{T} \frac{|h_t(Q(x_t, \pi) - h_t(Q(x_t, \pi'))|^2}{4M^2 \sum_{t=1}^{T} h_t^2}
\]
\[
= \frac{m}{m + T} \hat{H}(\pi, \pi'; x_{1:T}).
\]
Equivalent, for any \( \pi \in \Pi \), there exists \( \pi' \in S \) such that \( \hat{\ell}_2(\pi, \pi'; x_{1:T}) \leq \sqrt{1 + T/m} \cdot \eta \). Consequently,

\[
N_{\hat{\ell}_2}(\sqrt{\frac{m + T}{m}} \cdot \eta, \Pi; x_{1:T}) \leq N_0 = N_{H}(\eta^2, \Pi).
\]

Letting \( m \to \infty \) completes the proof.

C Additional Details

C.1 Solving for the Optimal Weights

The optimization problem in (4.3.1) is a convex one, and we can use the method of Lagrange multipliers to find the global minimum. We construct the Lagrangian as follows:

\[
\mathcal{L}(\tilde{h}_1, \ldots, \tilde{h}_T; \lambda) = \sum_{t=1}^{T} \frac{\tilde{h}_t^2}{g_t} + \lambda \left( \sum_{t=1}^{T} \tilde{h}_t - 1 \right),
\]

with the constraints \( \tilde{h}_t \geq 0 \) for \( t \in [T] \). The corresponding KKT condition can be written as:

\[
\begin{align*}
\tilde{h}_t &\geq 0, \text{ for } t \in [T], \\
\nabla_{\tilde{h}_t} \mathcal{L} &= 2\tilde{h}_t/g_t + \lambda = 0, \\
\nabla_{\lambda} \mathcal{L} &= 1 - \sum_{t=1}^{T} \tilde{h}_t = 0.
\end{align*}
\]

Solving the above equations, we arrive at the solution \( \tilde{h}_t^* = g_t/\left(\sum_{s=1}^{T} g_s\right) \) for all \( t \in [T] \).

C.2 Comparing with Bibaut et al. (2021)

We hereby provide more detailed comparison with Bibaut et al. (2021). Their approach is a variant of our algorithm, where they use uniform weights with \( h_t = 1 \) and IPW estimator with \( \hat{\mu}_t = 0 \) to estimate policy values. In general, uniform weighting with \( h_t = 1 \) does not achieve minimax regret guarantee. When weights \( \{h_t\} \) are fixed, different choices of nuisance component \( \hat{\mu}_t \)—whether setting it to zero (IPW estimator) as in Bibaut et al. (2021) or not as in our proposal—would not affect regret rate. Yet this nuisance component choice can largely influence the variance of policy value estimation, which in turn affects the value of learned policy. In particular, suppose that \( \hat{\mu}_t \overset{a.s.}{\longrightarrow} \mu_\infty \) for some \( \mu_\infty(\cdot) \), our algorithm achieves a regret bound of \( \hat{\mathcal{O}}(\kappa(\Pi)) \sqrt{MT^2 + \sup_{x,w} \|\mu_\infty(x, w) - \mu(x, w)\|^2} \cdot \frac{\sum_{t=1}^{T} h_t^2/g_t^2 + \sum_{t=1}^{T} h_t^4/g_t^3}{\sum_{t=1}^{T} h_t^2/g_t} \). Thus a good estimation of \( \mu \) would reduce the regret bound up to a constant factor. One possible fix to using IPW estimator would be de-meaning the outcomes beforehand, but the effect of this data-preprocessing approach may be weakened when there exists a fair amount of variation in outcomes across contexts and actions.

To further see the benefits of using nuisance components in the AIPW estimator, we compare with Bibaut et al. (2021) empirically. Consider the same data generating process and the same policy class we provide in Section 6.1. Fixing weights \( h_t = 1 \), we compare the policies learned with the IPW estimator (Bibaut et al., 2021), the AIPW estimator with correctly specified \( \hat{\mu}_t \) (which has the same quadratic functional form in context as the underlying outcome model), and the AIPW estimator with mis-specified \( \hat{\mu}_t \) (which wrongly assumes the underlying outcome model to be linear). Figure 4 shows that, as compared to using IPW estimator, policies learned with both AIPW estimators—even the one with mis-specified nuisance component—effectively reduce the estimation variance and improve the value of learned policies noticeably.

C.3 Multi-class Classification Datasets

The list of OpenML datasets used in Section 6.2 is

'GAMETES_Epistasis_2-Way_20atts_0_4H_EDM-1_1',
'BNG_sick_nominal_1000000', 'waveform-5000', 'BNG_credit-a',

41
Figure 4: Comparing with Bibaut et al. (2021). Policies learned with AIPW estimators used in our algorithm effectively reduce the variance and improve the value of learned policy.