Virtual element method for the system of time dependent nonlinear convection-diffusion-reaction equation

M. Arrutselvi and E. Natarajan

Department of Mathematics, Indian Institute of Space Science and Technology, Thiruvananthapuram-695547, Kerala, India.

ARTICLE HISTORY
Compiled September 22, 2021

ABSTRACT
In this work, we have discretized a system of time-dependent nonlinear convection-diffusion-reaction equations with the virtual element method over the spatial domain and the Euler method for the temporal interval. For the nonlinear fully-discrete scheme, we prove the existence and uniqueness of the solution with Brouwer’s fixed point theorem. To overcome the complexity of solving a nonlinear discrete system, we define an equivalent linear system of equations. An a priori error estimate showing optimal order of convergence with respect to $H^1$ semi-norm was derived. Further, to solve the discrete system of equations, we propose an iteration method and a two-grid method. In the numerical section, the experimental results validate our theoretical estimates and point out the better performance of the two-grid method over the iteration method.

KEYWORDS
Nonlinear, Time-dependent, Virtual element, System of equations, Two-grid.

1. Introduction

Virtual Element Method (VEM) introduced in [1] is a generalization of finite element method, that can handle meshes with polygonal elements of both convex and non-convex types. Moreover, implementation of higher order VEM is simple, as the computation of integrals arising in the discrete scheme does not require the explicit knowledge of the basis functions. VEM has been successfully applied to linear and nonlinear partial differential equations, ranging from convection-diffusion problems [2, 3], parabolic [4, 5], hyperbolic [6, 7], elliptic [8, 9], and to various problems such as Stokes [11], biharmonic [12], mixed Brinkman [13] and linear elasticity [14], $C^1$ VEM for Cahn-Hilliard problem [15], 3D elasticity problem [16] and so on.

In this paper, we study VEM for a system of time-dependent nonlinear convection-diffusion-reaction equations that arise in several practical applications. We derive the computable VEM discretization with the help of projection operators in section 3 and then prove the existence and uniqueness of the discrete nonlinear system of equations in section 4. In section 5 we have performed the convergence analysis by deriving error estimates showing optimal rate of convergence. In section 6, we have introduced two-grid method in order to reduce the computational cost to solve the system of equations. To establish this fact, we have compared the two-grid approach with the standard iterative procedure in the numerical experiments in section 7.
1.1. Notations

Let $D$ be a subset of $\mathbb{R}^2$ with boundary $\partial D$. The space $L^2(D)$ consists of square integrable functions with inner product $(u, v)_0 := \int_D uv\,dx$ and norm $\| u \|_0 := (\cdot, \cdot)_0^{\frac{1}{2}}$. For index $s \in \mathbb{N}$, $H^s(D)$ is the usual Sobolev space with norm $\| u \|_s := \sum_{\alpha \leq s} \| \partial_\alpha u \|_0$. Let $X$ be a space with inner product $(\cdot, \cdot)_X$ and norm $\| \cdot \|_X$. We define $L^p(0, T; X)$, $1 \leq p \leq \infty$, be space of measurable functions $u : [0, T] \to X$ with norms $\| u \|_{L^p(0, T; X)}$ defined as

$$
\| u \|_{L^p(0, T; X)} := \int_0^T \| u(t) \|_X^p \, dt, \quad 1 \leq p < \infty, \quad \| u \|_{L^\infty(0, T; X)} := \text{ess sup}_{0 \leq t \leq T} \| u(t) \|_X.
$$

For $k \in \mathbb{N}$, we denote $[X]^k := X \times X \times \ldots \times X$ ($k$ - times). The space $[X]_k$ is endowed with the inner product $(u, v) := \sum_{i=1}^k (u_i, v_i)_X$, and the induced norm $\| u \| = (u, u)^{\frac{1}{2}}$.

2. Model Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary, time interval $(0, T) \subset \mathbb{R}$, $m \in \mathbb{N}$ and $D := (0, T) \times \Omega$. We consider the following system of $m$ time-dependent convection-diffusion-reaction equations:

$$
\begin{align*}
\frac{\partial u_i}{\partial t} - \xi_i \Delta u_i + (\omega_1, \omega_2) \cdot \nabla u_i + u_i \sum_{j=1}^m A(i, j) u_j + \sum_{\ell \neq i} Q(i, \ell, j) u_\ell u_j & \quad \text{in } D, \quad i = 1, 2, \ldots, m. \quad (2.1) \\
u_i(t, x) & = 0 \quad \text{on } (0, T) \times \partial \Omega, \quad i = 1, \ldots, m. \quad (2.2) \\
u_i(0, x) & = u_i^0(x) \quad \text{in } \Omega, \quad i = 1, \ldots, m. \quad (2.3)
\end{align*}
$$

Here, the convection parameter $(\omega_1, \omega_2) \in [L^2(0, T; (H^1(\Omega) \cap L^\infty(\Omega))]^2$ along with $(\nabla \cdot (\omega_1, \omega_2))(t, x) = 0$ a.e in $D$ and for $i = 1, \ldots, m$ we have $\xi_i \in L^2(0, T; L^\infty(\Omega))$ are bounded positive diffusion parameters, the source or sink functions $f_i \in L^2(0, T; L^2(\Omega))$. The kinetic coefficients $A, Q, R$ are assumed to be sufficiently regular functions of $x$ and $t$, to ensure an unique solution for (2.1)-(2.3). The homogeneous Dirichlet boundary condition (2.2) was assumed for simplifying the analysis presentation. Our numerical formulation and analysis with slight modification can handle more general boundary conditions arising in practical applications involving combinations of Dirichlet and Neumann boundary conditions.

Let us denote $\mathcal{H} := [L^2(0, T; H_0^1(\Omega)) \cap C(0, T; L^\infty(\Omega))]^m$. Then the variational formulation for the system (2.1)-(2.3) is: Find $u \in \mathcal{H}$ with $\partial u_i/\partial t \in L^2(0, T; H^{-1}(\Omega))$ ($i = 1, \ldots, m$) such that

$$
\begin{align*}
\left( \frac{\partial u_i}{\partial t}, v_i \right)_0 + (\xi_i \nabla u_i, \nabla v_i)_0 + ((\omega_1, \omega_2) \cdot \nabla u_i, v_i)_0 + \left( u_i \sum_{j=1}^m A(i, j) u_j, v_i \right)_0 & \quad \text{for all } v \in \mathcal{H}, \quad i = 1, \ldots, m. \quad (2.4)
\end{align*}
$$

We assume that the exact solution $u$ of (2.1)-(2.3) satisfies the following:

$$
\exists \mu > 0 \quad \text{such that } \| u_i \|_\infty \leq \mu, \quad \forall i \in \{1, 2, \ldots, m\}, \quad \forall t \in [0, T]. \quad (2.5)
$$
3. The Virtual Element Method

In this section we brief the definition of the virtual element space. Let \( \mathbb{P}_p(E) \) denote the space of polynomials with degree less than or equal to \( p \) on \( E \).

We consider \( \{ T_h \}_{h > 0} \) to be a family of partitioning of \( \Omega \) consisting of polygonal elements. For some theoretical estimates to hold, we make the following assumptions on every element in \( T_h \).

For an element \( E \in T_h \), let \( h_E \) be the element diameter and \( h \) represent the maximum diameter over all \( E \in T_h \). In addition, for constants \( \delta > 0, c > 0 \) independent of \( h, E \), we assume an element \( E \in T_h \) fulfil the following (see [11]) : (i) \( E \) is star-shaped with respect to a disc \( D_\delta \) of radius \( \delta h_E \), (ii) for any edge \( e \subset E \), the length \( |e| \geq c h_E \), and (iii) boundary of \( E \) is made up of finite number of edges.

Let us define some necessary operators that projects elements in \( H^1(E) \) onto \( \mathbb{P}_p(E) \). We define the the \( L^2 \) projection operator \( \mathcal{P}_p^0 \) (see [9]) by,

\[
(u - \mathcal{P}_p^0 u, v_p)_{E} = 0 \quad \forall v_p \in \mathbb{P}_p(E).
\]

The gradient projection operator \( \nabla \mathcal{P}_p^0 : H^1(E) \to \mathbb{P}_p(E) \) is such that (see [9]),

\[
(\nabla (u - \mathcal{P}_p^0 u), \nabla v_p)_E = 0 \quad \forall v_p \in \mathbb{P}_p(E) \quad \text{and} \quad \int_{\partial E} (\mathcal{P}_p^0 u - u) \, ds = 0.
\]

For \( u \in H^1(E) \), we shall determine \( \mathcal{P}_{p-1}^0 (\nabla u) \in (\mathbb{P}_{p-1}(E))^2 \) by,

\[
(\nabla u - \mathcal{P}_{p-1}^0 \nabla u, v_{p-1})_E = 0 \quad \forall v_{p-1} \in (\mathbb{P}_{p-1}(E))^2.
\]

Consider the auxiliary space \( W_p^E \) (see [17]) for each \( E \in T_h \),

\[
W_p^E = \left\{ v \in H^1(E) \cap C^0(\partial E) : v|_e \in \mathbb{P}_p(e) \forall e \in \partial E, \Delta v \in \mathbb{P}_p(E) \right\}.
\]

Let \( (\mathbb{P}_p(E)/\mathbb{P}_{p-2}(E)) \) be the set of polynomials of degree exactly equal to \( p - 1 \) and \( p \). Now we define the local virtual element space \( \mathcal{V}_p^E \) as follows,

\[
\mathcal{V}_p^E = \left\{ u \in W_p^E \quad \text{s.t.} \quad (u - \mathcal{P}_p^0 u, q)_E = 0 \quad \forall q \in (\mathbb{P}_p(E)/\mathbb{P}_{p-2}(E)) \right\}.
\]

We consider the following set of degrees of freedom on \( \mathcal{V}_p^E \), \( (V_1) \) the values of \( u \) at the \( n(E) \) vertices of polygon \( E \), \( (V_2) \) the values of \( u \) at \( (p-1) \) internal Gauss-Lobatto quadrature nodes of every edge \( e \in \partial E \), and \( (V_3) \) the moments up to order \( p-2 \) of \( u \) in \( E \), i.e.,

\[
V_3 := \left\{ (u, v_{p-2})_E : v_{p-2} \in \mathbb{P}_{p-2}(E) \right\}.
\]

We note that \( (V_1) - (V_3) \) determine \( u \in \mathcal{V}_p^E \) uniquely on the polygon \( E \), (see [17]). Now we define the global virtual element space \( \mathcal{V}_h^p \) by,

\[
\mathcal{V}_h^p = \left\{ u \in H^1_0(\Omega) : u|_E \in \mathcal{V}_p^E \quad \forall E \in T_h \right\}.
\]

3.1. VEM formulation

Consider a partition of \( (0, T) \) into \( N \) disjoint intervals of width \( \delta t (= T/N) \) and denote \( t_n = n \delta t, \ n = 0, 1, \ldots, N \). Let us denote by \( U^n_i \), the function \( U_i \) associated to time \( t_n \), i.e. \( U^n_i := U_i(t_n) \in \mathcal{V}_h^p \). We use backward Euler method to discretize the time derivative term. Hereafter we use the inner product \( (\cdot, \cdot)_0 \) and norms \( \| \cdot \|_k \) split over the mesh elements in \( T_h \), i.e, \( (\cdot, \cdot)_0 := \sum_{E \in T_h} (\cdot, \cdot)_{0,E}, \text{ and } \| \cdot \|_k := \sum_{E \in T_h} \| \cdot \|_{0,E}. \)
As the functions in $V_h^p$ are not explicitly known, using the projection operators described in section 3 we define a computable fully discrete virtual element scheme equivalent to (2.4) as: for each $n = 1, 2, ..., N$, find $(U^n_1, U^n_2, ..., U^n_m) \in V_h^p$ such that

$$U^n_i(x) = u^n_i(x) \quad \text{in } \Omega, \quad i = 1, ..., m,$$

$$\sum_{i=1}^{m} \left\{ (\mathcal{P}(U^n_i - U^{n-1}_i), \mathcal{P}v)_0 + S_1((I - \mathcal{P})(U^n_i - U^{n-1}_i), (I - \mathcal{P})v) \right\} + (\delta t) \left( \xi_i \mathcal{P} \nabla U^n_i, \mathcal{P} \nabla v \right)_0 + (\delta t) S_2((I - \mathcal{P}_p^E)U^n_i, (I - \mathcal{P}_p^E)v) + (\delta t) \left( \omega_1, \omega_2 \right)_0 \cdot \mathcal{P} \nabla U^n_i, \mathcal{P} v \right)_0 + (\delta t) \left( \sum_{j=1}^{m} A(i, j) \mathcal{P} U^n_j, \mathcal{P} v \right)_0 + (\delta t) \left( \sum_{i,j \neq i} Q(i, j) \mathcal{P} U^n_i \mathcal{P} U^n_j, \mathcal{P} v \right)_0 + (\delta t) \left( \sum_{i=1}^{m} R(i, j) \mathcal{P} U^n_j, \mathcal{P} v \right)_0 = (\delta t) \sum_{i=1}^{m} (f^n_i, \mathcal{P} v)_0 \quad \forall \mathcal{V} \in V_h^p,$$

where $S_1$ and $S_2$ are symmetric bilinear functions detailed below.

Consider the local symmetric bilinear maps $S_1^E$ and $S_2^E$ defined on $V_h^E \times V_h^E$ such that there exists non-zero positive constants $\alpha_s, \alpha_s^e, \beta_s$ and $\beta_s^e$, with $\alpha_s \leq \alpha_s^e$ and $\beta_s \leq \beta_s^e$, independent of $h_E$, and $\forall u_h, v_h \in V_h^E \setminus \mathbb{P}_p(E)$,

$$\alpha_s(u_h, v_h)_0,E \leq S_1^E(u_h, v_h) \leq \alpha_s^e(u_h, v_h)_0,E,$$

$$\beta_s(\nabla u_h, \nabla v_h)_0,E \leq S_2^E(u_h, v_h) \leq \beta_s^e(\nabla u_h, \nabla v_h)_0,E.$$  \hspace{1cm} (3.7) \hspace{1cm} (3.8)

A variety of computable choices for $S_1^E$ and $S_2^E$ can be found in the literature. Let us choose $S_1$ and $S_2$, as follows,

$$S_1(u_h, v_h) = \sum_{E \in T_h} \widetilde{S}_1^E(u_h, v_h) \quad \text{and} \quad S_2(u_h, v_h) = \sum_{E \in T_h} (\xi_i^E) \widetilde{S}_2^E(u_h, v_h),$$

where $\xi_i^E := \sup_{x \in E} |\xi_i(x, t_n)|$. In the sequel, we adopt the following notations,

$$m_h(u_h, v_h) := (\mathcal{P} u_h, \mathcal{P} v)_0 + S_1((I - \mathcal{P})u_h, (I - \mathcal{P})v_h),$$

$$a_h(u_h, v_h) := (\xi_i \mathcal{P} \nabla u_h, \mathcal{P} \nabla v)_0 + S_2((I - \mathcal{P}_p^E)u_h, (I - \mathcal{P}_p^E)v).$$

4. Theoretical Analysis

Here, we prove the existence and uniqueness of solution for the fully discrete scheme (3.6). First, we state a variant of Brouwer’s fixed point that will be used to prove the existence of a solution for (3.6).

**Proposition 4.1.** Let $H$ be a finite dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$. Let $L : H \to H$ be a continuous map. If there exists $\rho > 0$ such that, $\langle L(w), w \rangle_H > 0$, $\forall w \in H$ with $\|w\|_H = \rho$, then there exists a $z \in H$ such that $L(z) = 0$ and $\|z\|_H \leq \rho$.

In our analysis, we assume (later proved in sec. 5) that any solution $U^{n-1}_i$ of (3.6) satisfies the following :

$$\exists \mu > 0 \quad \text{s.t} \quad \|U^{n-1}_i\|_\infty \leq (\mu + 1) \quad \forall i \in \{1, ..., m\}.$$  \hspace{1cm} (4.1)

**Remark 1.** On a bounded Lipschitz element $E$, for $\Pi^0_k w \in \mathbb{P}_k(E) \subset H^1(E)$, using the compact
Sobolev embedding $H^1(E) \to L^p(E)$, $2 \leq p < \infty$ and (2.44) in [13], we have
\[
\|\Pi_0^0 w\|_{L^p(E)} \leq C\|\Pi_0^0 w\|_{1,E} \leq C\|w\|_{1,E}.
\]  
\[\text{(4.2)}\]

**Theorem 4.2.** Under the assumption (4.1) and for sufficiently small $(\delta t)$, we assume the existence of unique tuple $U^k := (U^k_1, U^k_2, \ldots, U^k_n) \in \mathcal{V}_h^p$ for $0 \leq k \leq n - 1$ satisfying (3.6). Then for all $1 \leq n \leq N$ there exists a unique solution $U^n := (U^n_1, U^n_2, \ldots, U^n_n) \in \mathcal{V}_h^p$ to the discrete problem (4.2).

**Proof.** We observe that $\mathcal{V}_h^p$ is a finite dimensional Hilbert space with inner product $\langle W, V \rangle = \sum_{i=1}^m (\nabla W_i, \nabla V_i)_{0}$ and the induced norm $\|W\| = \left( \sum_{i=1}^m \|
abla W_i\|_0^2 \right)^{\frac{1}{2}}$.

For $1 \leq n \leq N$, let us define a map $L : \mathcal{V}_h^p \to \mathcal{V}_h^p$ such that
\[
\langle L(W), V \rangle = \sum_{i=1}^m \left\{ m_h(W^n_i, V_i) - m_h(U^{n-1}_i, V_i) + (\delta t) a_h(W^n_i, V_i) + (\delta t) \left( (\omega_1, \omega_2) \cdot \mathcal{P} \nabla W^n_i + \mathcal{P} V_i \right)_{0} + (\delta t) \left( (\omega_1, \omega_2) \cdot \mathcal{P} \nabla W^n_i + \mathcal{P} V_i \right)_{0} \right\}.
\]
\[\sum_{i=1}^m F_i(W, V).
\]
\[\text{(4.3)}\]

We can prove that $L$ is well defined with the help of Riesz representation theorem. Now we shall show that $L$ is a continuous mapping. Let $\epsilon > 0$ be given. Consider arbitrary $W \in \mathcal{V}_h^p$, choose a $\nu > 0$ (depending on $\epsilon$, $W$; for existence of such $\nu$, see Remark 2) and let $Z \in \mathcal{V}_h^p$ be any tuple such that $\|W - Z\| < \nu$. Denote $\psi = W - Z$ and $L^\psi := L(W) - L(Z)$. Then
\[
\|L(W) - L(Z)\|^2 = \langle L^\psi, L^\psi \rangle = \langle L(W), L^\psi \rangle - \langle L(Z), L^\psi \rangle
\]
\[\sum_{i=1}^m \left\{ F_i(W, L^\psi) - F_i(Z, L^\psi) \right\}.
\]
\[\text{(4.4)}\]

Now we estimate the terms in $I := F_i(W, L^\psi) - F_i(Z, L^\psi)$.
\[
I = m_h(\psi^n_i, L^\psi_i) + (\delta t) a_h(\psi^n_i, L^\psi_i) + (\delta t) \left( (\omega_1, \omega_2) \cdot \mathcal{P} \nabla \psi^n_i, \mathcal{P} L^\psi_i \right)_{0} + (\delta t) \left( (\omega_1, \omega_2) \cdot \mathcal{P} \nabla \psi^n_i, \mathcal{P} L^\psi_i \right)_{0} \]
\[+ (\delta t) \left( \sum_{j=1}^n A(i, j) \mathcal{P} Z^n_j - \mathcal{P} Z^n_i \right) \cdot \mathcal{P} L^\psi_i \]
\[+ (\delta t) \left( \sum_{j=1}^n R(i, j) \mathcal{P} \psi^n_j \right) \cdot \mathcal{P} L^\psi_i \]
\[= \sum_{i=1}^m I_r.
\]
\[\text{(4.5)}\]

Using the definition (3.10), property (3.7) and Cauchy-Schwarz inequality, we get
\[
I_1 := m_h(\psi^n_i, L^\psi_i) \leq \max \{1, \alpha^*\} \|\psi^n_i\|_0 \|L^\psi_i\|_0
\]
\[\leq \max \{1, \alpha^*\} \|\psi^n_i\|_0 \|L^\psi_i\|_0.
\]
\[\text{(4.6)}\]

Again, definition (3.11), property (3.8), $\delta t < 1$ and Cauchy-Schwarz inequality implies
\[
I_2 := (\delta t) a_h(\psi^n_i, L^\psi_i) \leq \max \{1, \beta^*\} \|\nabla \psi^n_i\|_0 \|\nabla L^\psi_i\|_0
\]
\[\text{(4.7)}\]
Let $\omega_{\text{max}} := \max\{\|\omega_1\|_{\infty}, \|\omega_2\|_{\infty}\}$, using $\delta t < 1$ and Cauchy-Schwarz inequality we get

$$I_3 \leq \omega_{\text{max}} \|\mathcal{P} \nabla \psi_i^n\|_0 \|\mathcal{P} L_i^\psi\|_0 \leq \omega_{\text{max}} \|\nabla \psi_i^n\|_0 \|L_i^\psi\|_0. \quad (4.8)$$

Adding and subtracting $(\delta t) \left(\mathcal{P} W_i^n (\sum_{j=1}^m A(i,j) \mathcal{P} Z_j^n), \mathcal{P} L_i^\psi\right)_0$ to $I_4$, we get

$$I_4 = (\delta t) \left[\left(\mathcal{P} W_i^n (\sum_{j=1}^m A(i,j) \mathcal{P} (W_j^n - Z_j^n)), \mathcal{P} L_i^\psi\right)_0 + \left(\mathcal{P} (W_i^n - Z_i^n) (\sum_{j=1}^m A(i,j) \mathcal{P} Z_j^n), \mathcal{P} L_i^\psi\right)_0\right].$$

Considering $A_{\text{max}} := \max_{i,j} |A(i,j)|$, using generalized Hölder’s inequality, $\delta t < 1$ and triangle inequality we obtain

$$I_4 \leq A_{\text{max}} \left[\|\mathcal{P} W_i^n\|_{L^4(\Omega)} \sum_{j=1}^m \|\psi_j^n\|_{L^4(\Omega)} + \|\mathcal{P} \psi_i^n\|_{L^4(\Omega)} \sum_{j=1}^m \|\mathcal{P} Z_j^n\|_{L^4(\Omega)} \right] \|\mathcal{P} L_i^\psi\|_0.$$

Using $[4.2]$, property of $\mathcal{P}$, Poincaré inequality (with constant $C_P$), Hölder’s inequality and noting $\|Z\| < \nu + \|W\|$, we get

$$I_4 \leq C A_{\text{max}} \left[2\|W_i^n\| + \nu\right] \|\psi\| \|L_i^\psi\|. \quad (4.9)$$

Note that, by adding and subtracting $\mathcal{P} W_j^n \mathcal{P} Z_j^n$ we have

$$[\mathcal{P} W_i^n \mathcal{P} W_j^n - \mathcal{P} Z_i^n \mathcal{P} Z_j^n] = \mathcal{P} \psi_i^n \mathcal{P} W_j^n + \mathcal{P} Z_i^n \mathcal{P} \psi_j^n. \quad (4.10)$$

Next, let $Q_{\text{max}} := \max_{i,j} Q(i,j)$, using $\delta t < 1$, (4.10) and similar to (4.9), we have

$$I_5 \leq Q_{\text{max}} \left[\sum_{j \neq i} \mathcal{P} \psi_j^n \sum_{j \neq i} \mathcal{P} W_j^n + \sum_{j \neq i} \mathcal{P} Z_j^n \sum_{j \neq i} \mathcal{P} \psi_j^n\right]_0 \leq C Q_{\text{max}} \left[2\|W\| + \nu\right] \|\psi\| \|L_i^\psi\|. \quad (4.11)$$

Denoting $R_{\text{max}} := \max_{i,j} R(i,j)$, using $\delta t < 1$, Cauchy-Schwarz inequality, triangle inequality and Poincaré inequality, we obtain

$$I_6 \leq R_{\text{max}} \left(\sum_{j=1}^m \|\mathcal{P} \psi_j^n\|_0\right) \|\mathcal{P} L_i^\psi\|_0 \leq C_P^2 R_{\text{max}} \|\psi\| \|L_i^\psi\|. \quad (4.12)$$

Substituting the results (4.6)-(4.8), (4.9), (4.11)-(4.12) into (4.5), and using Poincaré inequality, we obtain

$$I \leq \left[C_P^2 \max\{1, \alpha^*\} + (\max_{\xi} \xi^E) \max\{1, \beta^*\} + C_P \omega_{\text{max}}\right] \|\nabla \psi_i^n\|_0 \|\nabla L_i^\psi\|_0$$

$$+ \left[C (A_{\text{max}} + Q_{\text{max}}) \left(2\|W\| + \nu\right) + C_P^2 R_{\text{max}}\right] \|\psi\| \|L_i^\psi\|. \quad (4.13)$$

Summing (4.13) over $i = 1$ to $m$, and using Hölder’s inequality, we get

$$\|L(W) - L(Z)\| \leq (C_1 \nu + C_2)\|W - Z\|, \quad (4.14)$$
where, \( C_1 = m C (A_{\text{max}} + Q_{\text{max}}) \).

\[
C_2 = \left[ C P^2 \max\{1, \alpha^*\} + (\max_{E} \xi_E^r) \max\{1, \beta^*\} + C P \omega_{\text{max}} \right] + m \left[ C (A_{\text{max}} + Q_{\text{max}}) 2\|W\| + C P^2 \right].
\]

(4.15) (4.16)

Under suitable choice for \( \nu \) (see Remark 2), the estimate (4.14) implies \( \|L(W) - L(Z)\| < \epsilon \). Hence \( L \) is continuous.

Consider the fixed constant,

\[
K = \frac{2 (\max\{1, \alpha^*\} \|U^n-1\| + 1)}{C \min\{1, \alpha\} + (\delta t) \min\{1, \beta, \xi_i, 0\}}.
\]

(4.17)

We will show \( \langle L(W), W \rangle > 0 \), for all \( W \in \mathcal{V}_h^p \) with \( \|W\| = K \). We have, \( \langle L(W), W \rangle = \sum_{i=1}^{m} \mathcal{F}_i(W, W) \). Consider

\[
\mathcal{F}_i(W, W) = m_h(W^n_i, W^n_i) - m_h(U^n_{i-1}, W^n_i) + (\delta t) a_h(W^n_i, W^n_i)
\]

\[
+ (\delta t) \left( (\omega_1, \omega_2) \cdot \mathcal{P} \nabla W^n_i, \mathcal{P} W^n_i \right)_0 + (\delta t) (\mathcal{P} J_{W^n_i} (\sum_{j=1}^{m} A(i, j) \mathcal{P} W^n_j), \mathcal{P} W^n_i)_0
\]

\[
+ (\delta t) \left( \sum_{j \neq i} Q(i, j) \mathcal{P} W^n_j \mathcal{P} W^n_i, \mathcal{P} W^n_i \right)_0 + (\delta t) \left( \sum_{j=1}^{m} R(i, j) \mathcal{P} W^n_j, \mathcal{P} W^n_i \right)_0 - (\delta t) (f^n_i, \mathcal{P} W_i)_0 := \sum_{r=1}^{8} II_r.
\]

(4.18)

Using Cauchy-Schwarz inequality, (3.7), (3.8) and Poincaré inequality, we have

\[
\sum_{r=1}^{3} II_r \geq \min\{1, \alpha\} \|W^n_i\|_0^2 - \max\{1, \alpha^*\} C P^2 \|\nabla W^n_i\|_0 \|\nabla U^n_{i-1}\|_0 + (\delta t) \min\{1, \beta\} \|\nabla W^n_i\|_0^2
\]

\[
= \xi_{i,0} := \min_{x \in \Omega} \|\xi_i(t_n, x)\|. \quad \text{Again using Cauchy-Schwarz inequality and Poincaré inequality, we get}
\]

\[
|II_4| \leq (\delta t) \omega_{\text{max}} C P \|\nabla W^n_i\|_0^2
\]

\[
\text{Similar to (4.9), (4.11) and (4.12), we obtain},
\]

\[
\sum_{r=3}^{7} II_r \leq (\delta t) C \left( A_{\text{max}} + Q_{\text{max}} \right) \|W\|_0^3 + (\delta t) R_{\text{max}} C P^2 \|W\|_0^2.
\]

\[
\text{Using Cauchy-Schwarz inequality and Poincaré inequality, we get}
\]

\[
II_8 \geq -(\delta t) C P \|f^n_i\|_0 \|\nabla W^n_i\|_0.
\]

\[
\text{From the estimates (4.19)-(4.22), we obtain,}
\]

\[
\mathcal{F}_i(W, W) \geq \min\{1, \alpha\} \|W^n_i\|_0^2 - \max\{1, \alpha^*\} \|W^n_i\|_0 \|U^n_{i-1}\|_0 + (\delta t) \min\{1, \beta\} \|\nabla W^n_i\|_0^2 - (\delta t) \omega_{\text{max}} C P \|\nabla W^n_i\|_0^2 - (\delta t) C \left( A_{\text{max}} + Q_{\text{max}} \right) \|W\|_0^3 - (\delta t) R_{\text{max}} C P \|W\|_0^2.
\]

\[
\text{Summing } \mathcal{F}_i(W, W) \text{ over } i = 1 \text{ upto } m, \text{ equivalence of } \|\cdot\|_0, \|\nabla \cdot\|_0 \text{ in } V_h^p \text{ and using Hölder’s}
\]
implies ∥scheme (3.6) and an equivalent linear formulation which is stated as follows: systems are time-dependent, is cumbersome and time-consuming. Hence we modify the nonlinear ν > a

Remark 2. Let any inequality, we get

Let any

For sufficiently small (δt), we can have the estimate

Substituting (4.24) into (4.23) and using (4.17), we obtain

Thus by Proposition 4.1 existence of a solution $U^m ∈ V_h^p$ to the discrete problem (3.6) is proved. Next we show uniqueness of the solution $U^m ∈ V_h^p$ to (3.6) by the method of contradiction. Let $U^m, ˜U^m ∈ V_h^p$ be two distinct solutions of (3.6). Denote $E^m := U^m − ˜U^m ∈ V_h^p$. Then we have

From previous estimates we have

Choose sufficiently small (δt) such that

implies $∥E^m∥^2 ≤ 0$. Therefore $U^m = ˜U^m$.

Remark 2. Let any $ε > 0$ be given and note that $C_1 > 0$, $C_2 > 0$. Choose $0 < ε_ε < ε$. Then note that the quadratic equation $C_1 ν^2 + C_2 ν − ε_ε = 0$ has real positive solution. This guarantees a $ν > 0$ satisfying $ν ≤ \frac{ε}{C_1 ν + C_2}$.

5. Linearized scheme

Considering the implementation aspects, solving highly nonlinear systems, especially when the systems are time-dependent, is cumbersome and time-consuming. Hence we modify the nonlinear scheme (3.6) and an equivalent linear formulation which is stated as follows:
for each $n = 1, 2, \ldots, N$. Find $(U^n_1, U^n_2, \ldots, U^n_m) \in \mathbf{V}^P_h$ such that $\forall \mathbf{V} \in \mathbf{V}^P_h$:

\[
U^n_i(x) = u^n_i(x) \quad \text{in } \Omega, \quad i = 1, \ldots, m,
\]

\[
\sum_{i=1}^{m} \left\{ m_h(U^n_i, V_i) - m_h(U^{n-1}_i, V_i) + (\delta t) a_h(U^n_i, V_i) + (\delta t) l_{1,h}(U^n_i, V_i) + (\delta t) l_{2,h}(U^n_i, V_i) \right\} = (\delta t) \sum_{i=1}^{m} \left\{ -l_{4,h}(U^n_i, V_i) - l_{5,h}(U^n_i, V_i) + (f^n_i, \mathbf{V}_i)_0 \right\}, \quad (5.1)
\]

where,

\[
l_{1,h}(U^n_i, V_i) = \left( \omega_1, \omega_2 \right)_i (\mathbf{P}^2 \mathbf{V}_i, \mathbf{P} \mathbf{V}_i)_0, \quad l_{2,h}(U^n_i, V_i) = \left( \mathbf{P}^2 \mathbf{U}_i^n, \left( \sum_{j=1}^{m} A(i,j) \mathbf{P}^2 \mathbf{U}^{n-1}_j \right), \mathbf{P}_i \mathbf{V}_i \right)_0, \quad l_{3,h}(U^n_i, V_i) = \left( \mathbf{R}(i,i) \mathbf{P}^2 \mathbf{U}_i^n, \mathbf{P} \mathbf{V}_i \right)_0, \quad l_{4,h}(U^n_i, V_i) = \left( \left( \sum_{j \neq i} Q(i,j) \mathbf{P}^2 \mathbf{U}^{n-1}_j \right), \mathbf{P} \mathbf{V}_i \right)_0, \quad l_{5,h}(U^n_i, V_i) = \left( \left( \sum_{j \neq i} \mathbf{R}(i,j) \mathbf{P}^2 \mathbf{U}^{n-1}_j \right), \mathbf{P} \mathbf{V}_i \right)_0.
\]

Consider a finite sequence $\{\phi^n_i\}_{i=1}^{N}$ of functions in $\mathbf{V}^P_h$ associated to different time levels. In our analysis we use the norm $[\phi]_{0,k}$ that is defined as,

\[
[\phi]_{0,k} := \left( (\delta t) \sum_{i=1}^{N} \left( \sum_{j=1}^{m} \|\phi^n_i\|^2_{k} \right) \right)^{\frac{1}{2}}. \tag{5.2}
\]

In this section, we shall first discuss the well-posedness of the linear virtual element formulation \[5.1\]. Then we derive a priori error estimates involving the numerical solution of \[5.1\] with respect to the norm $[\cdot]_{0,1}$.

**Theorem 5.1.** Under the assumption \[4.1\] and for sufficiently small $(\delta t)$, we assume there exists a unique $U^k \in \mathbf{V}^P_h$ for $0 \leq k \leq n - 1$ satisfying \[5.1\]. Then for all $1 \leq n \leq N$ there exists a unique solution $U^n \in \mathbf{V}^P_h$ to the linear discrete problem \[5.1\].

**Proof.** The proof is analogous to the proof of Theorem \[4.2\]. \hfill $\Box$

The following are some valid results that will help prove our error estimate. Consider the local polynomial interpolation estimates (see Lemma 5.1 in \[9\]): for all $E \in \mathcal{T}_h$ and any $w \in H^{s+1}(E)$,

\[
\|[w - \mathbf{P}w]_{m,E}\|_{m,E} \leq C h^{s+1-m} |w|_{s+1,E}, \quad m, s \in \mathbb{N} \cup \{0\}, \quad m \leq s + 1 \leq k + 1. \tag{5.3}
\]

\[
\|[w - \mathbf{P}^k w]_{m,E}\|_{m,E} \leq C h^{s+1-m} |w|_{s+1,E}, \quad m, s \in \mathbb{N}, \quad m \leq s + 1 \leq k + 1, \quad s \geq 1. \tag{5.4}
\]

The virtual interpolation estimate given below can be found in \[19\]. For $1 \leq s \leq k$, $\forall E \in \mathcal{T}_h$ and for every $w \in H^{1+s}(E)$, there exists $\mathbf{W} \in V^k_h$ satisfying

\[
\|[w - \mathbf{W}]_{E} + h |w - \mathbf{W}|_{1,E} + h \|w - \mathbf{W}\|_{\infty,E} \leq C h^{1+s} |w|_{1+s,\Omega}. \tag{5.5}
\]

**Remark 3.** On a bounded domain $E$ with area of $E \approx h^2$, and also if $\|v\|_{\infty} < \infty$, $\|v\|_0 < \infty$, then using inverse inequality on polynomials, property of $\mathbf{P}$ and standard estimation, we have

\[
\|\mathbf{P} v\|_{\infty} \leq h_E^{-1} \|\mathbf{P} v\|_0 \leq h_E^{-1} \|v\|_0 \leq h_E^{-1} h_E \|v\|_{\infty} \leq \|v\|_{\infty}. \tag{5.6}
\]

**Theorem 5.2.** Let $u^n \in H^{s+1}(\Omega)^m$, $s \geq 1$ be the smooth exact solution satisfying \[2.1\]-\[2.3\] and $U^n \in \mathbf{V}^P_h$ be the solution of discrete form \[5.1\] at $n$th time step. Under the assumptions
(2.5) and (4.1), the numerical solution \( U^n \) converges to the exact solution \( u^n \), as \( h \to 0 \) and we obtain the estimates,

\[
[U^n - u^n]_{0,1} \leq A_1 h^s + A_2 (\delta t).
\]

where \( A_1 \) and \( A_2 \) are independent of \( h \) and \( \delta t \).

**Proof.** Let \( U^n \in \mathcal{V}_h^p \) be the interpolant of \( u^n \) onto \( \mathcal{V}_h^p \). We denote,

\[
\varphi^n := u^n - U^n = (u^n - u^n) + (U^n - U^n) = \eta^n + \chi^n.
\]

We need to estimate \( \chi^n \in \mathcal{V}_h^p \). Since \( u^n \) also satisfies the variational form (2.4), \( U^n \) satisfies (5.1) and rewriting, we have

\[
\sum_{i=1}^{m} \left\{ \left( \frac{\partial u_i^n}{\partial t}, V_i \right)_0 - \frac{1}{\delta t} m_h(u_i^n - u_i^{n-1}, V_i) + \left( \xi_i \nabla u_i^n, \nabla V_i \right)_0 - a_h(u_i^n, V_i) \right\} := I
\]

\[
\left( \omega_1, \omega_2 \right) \cdot \nabla u_i^n, V_i \right)_0 - l_{1,h}(U_i^n, V_i) + \left( u_i^n \left( \sum_{j=1}^{m} A(i, j) u_j^n \right), V_i \right)_0 - l_{2,h}(U_i^n, V_i)
\]

\[
+ \left( R(i, i) u_i^n, V_i \right)_0 - l_{3,h}(U_i^n, V_i) = - \left( \sum_{i,j \neq i} Q(i, j, i) u_i^n u_j^n, V_i \right)_0 - l_{4,h}(U_i^n, V_i)
\]

\[
- \left( \sum_{i,j \neq i} R(i, j) u_i^n u_j^n, V_i \right)_0 - l_{5,h}(U_i^n, V_i) + (f_i^n, (I - \mathcal{P}) V_i)_0 \right\}. \tag{5.9}
\]

Adding and subtracting the term \( \frac{1}{\delta t} m_h(u_i^n - u_i^{n-1}, V_i) \) to \( I \) and using (5.8), we get,

\[
I = \left( \frac{\partial u_i^n}{\partial t}, V_i \right)_0 - \frac{1}{\delta t} m_h(u_i^n - u_i^{n-1}, V_i) + \frac{1}{\delta t} m_h(\eta_i^n - \eta_i^{n-1}, V_i) + \frac{1}{\delta t} m_h(\chi_i^n - \chi_i^{n-1}, V_i). \tag{5.10}
\]

Next to \( II \), adding and subtracting \( a_h(u_i^n, V_i) \) and using (5.8), we obtain

\[
II = \left( \xi_i \nabla u_i^n, \nabla V_i \right)_0 - a_h(u_i^n, V_i) + a_h(\eta_i^n, V_i) + a_h(\chi_i^n, V_i). \tag{5.11}
\]

Substituting (5.10) and (5.11) into (5.9), letting \( V_i := \chi_i^n \) and rearranging, we have

\[
\sum_{i=1}^{m} \left\{ m_h(\chi_i^n - \chi_i^{n-1}, \chi_i^n) + \delta t a_h(\chi_i^n, \chi_i^n) \right\}
\]

\[
= -\delta t \left( \frac{\partial u_i^n}{\partial t}, \chi_i^n \right)_0 + m_h(u_i^n - u_i^{n-1}, \chi_i^n) - m_h(\eta_i^n - \eta_i^{n-1}, \chi_i^n) - \delta t a_h(\eta_i^n, \chi_i^n)
\]

\[
+ \delta t a_h(u_i^n, \chi_i^n) - \delta t \left[ \xi_i \nabla u_i^n, \nabla \chi_i^n \right]_0 + \delta t \left[ l_{1,h}(U_i^n, \chi_i^n) - (\omega_1, \omega_2) \cdot \nabla u_i^n, \chi_i^n \right]_0
\]

\[
- \delta t \left[ \left( u_i^n \left( \sum_{j=1}^{m} A(i, j) u_j^n \right), \chi_i^n \right)_0 - l_{2,h}(U_i^n, \chi_i^n) \right] - \delta t \left[ \sum_{i,j \neq i} Q(i, j, i) u_i^n u_j^n, \chi_i^n \right]_0 - l_{4,h}(U_i^n, \chi_i^n)
\]

\[
- \delta t \left[ \sum_{i,j \neq i} R(i, j) u_i^n u_j^n, \chi_i^n \right)_0 - l_{5,h}(U_i^n, \chi_i^n) \right\} + \delta t (f_i^n, (I - \mathcal{P}) \chi_i^n)_0 \right\}. \tag{5.12}
\]

Using (3.7), property of \( \mathcal{P} \) and \( mn \leq \frac{m^2}{\theta} + n^2 \theta \) (with \( \theta := \min\{1, \alpha_s\}/2 \)) we have
\( m_h(\chi^n_i, \chi^n_i) \geq \min\{1, \alpha_s\} \|\chi^n_i\|^2. \)  \hspace{2cm} (5.13)

\( m_h(\chi^{n-1}_i, \chi^n_i) \leq (1 + \alpha^*) \|\chi^{n-1}_i\|_0 \|\chi^n_i\|_0 \leq \frac{2(1 + \alpha^*)^2}{\min\{1, \alpha_s\}} \|\chi^{n-1}_i\|_0^2 + \frac{\min\{1, \alpha_s\}}{2} \|\chi^n_i\|_0^2. \)  \hspace{2cm} (5.14)

Then (5.13) and (5.14) implies

\[ m_h(\chi^n_i - \chi^{n-1}_i, \chi^n_i) \geq \min\{1, \alpha_s\} \|\chi^n_i\|^2 - \frac{2(1 + \alpha^*)^2}{\min\{1, \alpha_s\}} \|\chi^{n-1}_i\|_0^2. \]  \hspace{2cm} (5.15)

Using definition (3.11) and (3.8), we obtain

\[ \delta t \, a_h(\chi^n_i, \chi^n_i) \geq \delta t \, \min\{1, \beta_s\} \xi_{i,0} \|\nabla\chi^n_i\|^2. \]  \hspace{2cm} (5.16)

Using (3.10), add and subtract \((u^n_i - u^{n-1}_i, \mathcal{P}\chi^n_i + \chi^n_i)\), and property of \(\mathcal{P}\), we get

\[ I_1 := m_h(u^n_i - u^{n-1}_i, \chi^n_i) - \delta t \left( \frac{\partial u^n_i}{\partial t}, \chi^n_i \right)_0 \]
\[ = S_1((I - \mathcal{P})(u^n_i - u^{n-1}_i), (I - \mathcal{P})\chi^n_i) + (u^n_i - u^{n-1}_i, (I - \mathcal{P})\chi^n_i)_0 \]
\[ + \left[ (u^n_i - u^{n-1}_i, \chi^n_i) - \delta t \left( \frac{\partial u^n_i}{\partial t}, \chi^n_i \right)_0 \right]. \]

Using (3.7), Cauchy-Schwarz inequality and Taylor’s theorem w.r.t \(t\) (neglecting \((\delta t)^2\) term), we get

\[ I_1 \leq (\alpha^* + 1) \|u^n_i - u^{n-1}_i\|_0 \|\chi^n_i\|_0 + \left[ (u^n_i - u^{n-1}_i, \chi^n_i) - \delta t \left( \frac{\partial u^n_i}{\partial t}, \chi^n_i \right)_0 \right] \]
\[ \leq (\alpha^* + 1) \delta t \|u^n_i\|_0 \|\chi^n_i\|_0 + \delta t \left( \frac{u^n_i - u^{n-1}_i}{\delta t} - \frac{\partial u^n_i}{\partial t}, \chi^n_i \right)_0. \]  \hspace{2cm} (5.17)

By applying Taylor series for variable \(t\) with the remainder in terms of integral and using Cauchy-Schwarz inequality for the second term of (5.17), we have

\[ I_1 \leq (\alpha^* + 1) \delta t \|u^n_i\|_0 \|\chi^n_i\|_0 + (\delta t)^2 \|\chi^n_i\|^2. \]  \hspace{2cm} (5.18)

Using the inequality \(mn \leq \frac{m^2}{\theta} + n^2\theta\) with \(\theta := \min\{1, \alpha_s\}/8(\alpha^* + 1)\) for the first term and \(\theta := 1\) for the second term of (5.18), we get,

\[ I_1 \leq (\delta t)^2 \frac{8(\alpha^* + 1)^2}{\min\{\alpha^*, 1\}} \|u^n_i\|^2_0 + \min\{\alpha^*, 1\} \frac{1}{8} \|\chi^n_i\|^2_0 + (\delta t)^2 \|u^n_i\|^2_0 + \delta t \|\chi^n_i\|^2_0. \]  \hspace{2cm} (5.19)

Using (3.7), (3.8), Cauchy-Schwarz inequality and the inequality \(mn \leq \frac{m^2}{\theta} + n^2\theta\) with \(\theta := \min\{1, \alpha_s\}/8(\alpha^* + 1)\) for the first term and \(\theta := (\xi_{i,0} \min\{1, \beta_s\})/8(\beta^* + 1)\) for the second term, we get,
\[ I_2 := |m_h(\eta_i^n - \eta_i^{n-1}, \chi_i^n)| + |\delta t a_h(\eta_i^n, \chi_i^n)| \]
\[ \leq (1 + \alpha^2_\eta) \eta_i^n - \eta_i^{n-1} \| \chi_i^n \|_0 + (\delta t) (1 + \beta^2_\eta) \| \nabla \eta_i^n \|_0 \| \nabla \chi_i^n \|_0 \]
\[ \leq 8(\alpha^2 + 1)^2 \min\{\alpha, 1\} \frac{2}{8} \left( \| \eta_i^n \|_0^2 + \| \eta_i^{n-1} \|_0^2 \right) + \frac{\min\{\alpha, 1\}}{8} \| \chi_i^n \|_0^2 + \delta t \frac{8(1 + \beta^2_\eta)}{8} \| \nabla \eta_i^n \|_0^2 \]
\[ + \delta t \xi_i,0 \min\{1, \beta_\eta\} \| \nabla \chi_i^n \|_0^2 \]
\[ \leq 8(\alpha^2 + 1)^2 \min\{\alpha, 1\} \frac{2}{8} C h^{2s+2} \left( \| u_i^n \|_{s+1}^2 + \| u_i^{n-1} \|_{s+1}^2 \right) + \frac{\min\{\alpha, 1\}}{8} \| \chi_i^n \|_0^2 \quad \text{\textcolor{red}{use (5.5)}} \]
\[ + \delta t \frac{8(1 + \beta^2_\eta)}{8} C h^{2s} \| u_i^n \|_{s+1}^2 + \delta t \xi_i,0 \min\{1, \beta_\eta\} \| \nabla \chi_i^n \|_0^2. \] 

Adding and subtracting \( \delta t (\xi_i \nabla u_i^n, \mathcal{P} \nabla \chi_i^n) \)
we get
\[ I_3 := \delta t a_h(u_i^n, \chi_i^n) - \delta t (\xi_i \nabla u_i^n, \nabla \chi_i^n) \]
\[ = \delta t (\xi_i \mathcal{P} \nabla u_i^n - \xi_i \nabla u_i^n, \mathcal{P} \nabla \chi_i^n) + \delta t (\mathcal{P}(\xi_i \nabla u_i^n) - \xi_i \nabla u_i^n, \nabla \chi_i^n) \]
\[ + \delta t S_2((I - I_{\Lambda}^\chi) u_i^n, (I - I_{\Lambda}^\chi) \chi_i^n) \]

Using Cauchy-Schwarz inequality, (3.8), (5.3), (5.4) and the inequality \( mn \leq \frac{m^2}{\theta} + n^2\theta \) with \( \theta := \xi_i,0 \min\{1, \beta_\eta\} \), we get

\[ I_3 \leq \delta t \max_E \xi_i^E \| (\mathcal{P} - I) \nabla u_i^n \|_0 \| \nabla \chi_i^n \|_0 + \delta t \| \mathcal{P}(\xi_i \nabla u_i^n) - \xi_i \nabla u_i^n \|_0 \| \nabla \chi_i^n \|_0 \]
\[ + \delta t \beta_\eta \| (I - I_{\Lambda}^\chi) u_i^n \|_0 \| \nabla (I - I_{\Lambda}^\chi) \chi_i^n \|_0 \]
\[ \leq \delta t \left\{ \max_E \xi_i^E \| \nabla (\mathcal{P} - I) u_i^n \|_0 + C h^s \| \xi_i \nabla u_i^n \|_s + \beta_\eta \| \nabla (I - I_{\Lambda}^\chi) u_i^n \|_0 \right\} \| \nabla \chi_i^n \|_0 \]
\[ \leq \delta t \left( \max_E \xi_i^E + \| \nabla (\mathcal{P} - I) u_i^n \|_0 + \beta_\eta \right) C h^s \| u_i^n \|_{s+1} \| \nabla \chi_i^n \|_0 \]
\[ \leq \left( \max_E \xi_i^E + \| \nabla (\mathcal{P} - I) u_i^n \|_0 + \beta_\eta \right)^2 \delta t \frac{8}{\xi_i,0 \min\{1, \beta_\eta\}} C h^{2s} \| u_i^n \|_{s+1}^2 + \delta t \frac{8}{\xi_i,0 \min\{1, \beta_\eta\}} \| \nabla \chi_i^n \|_0^2. \] 

Next adding and subtracting the term \( \delta t ((\omega_1, \omega_2) \cdot \mathcal{P} \nabla u_i^n, \mathcal{P} \chi_i^n) \), using Lemma 1 of [20, Cauchy-Schwarz inequality and triangle inequality, we obtain

\[ I_4 := \delta t ((\omega_1, \omega_2) \cdot \mathcal{P} \nabla u_i^n, \mathcal{P} \chi_i^n) - \delta t ((\omega_1, \omega_2) \cdot \nabla u_i^n, \chi_i^n) \]
\[ = \delta t ((\omega_1, \omega_2) \cdot \mathcal{P} \nabla u_i^n, \mathcal{P} \chi_i^n) - \delta t ((\omega_1, \omega_2) \cdot \nabla u_i^n, \chi_i^n) + \delta t ((\omega_1, \omega_2) \cdot \mathcal{P} \nabla \varphi_i^n, \mathcal{P} \chi_i^n) \]
\[ \leq C \delta t ((\omega_1, \omega_2))_{\infty, s} h^{s+1} \| u_i^n \|_{s+1} \| \nabla \chi_i^n \|_0 + \max \delta t \| \nabla \chi_i^n \|_0 \| \chi_i^n \|_0 + \max \delta t \| \nabla \chi_i^n \|_0 \| \chi_i^n \|_0. \]

Using the inequality \( mn \leq \frac{m^2}{\theta} + n^2\theta \) with \( \theta := \xi_i,0 \min\{1, \beta_\eta\} \) and from (5.5), we get

\[ I_4 \leq (C ((\omega_1, \omega_2))_{\infty, s})^2 \frac{8}{\xi_i,0 \min\{1, \beta_\eta\}} \delta t h^{2s+2} \| u_i^n \|_{s+1}^2 + \frac{8}{\xi_i,0 \min\{1, \beta_\eta\}} \delta t \| \nabla \chi_i^n \|_0^2 \]
\[ + \omega_{\max}^2 \delta t C h^{2s} \| u_i^n \|_{s+1}^2 + \delta t \| \chi_i^n \|_0^2 \]
\[ + \delta t \frac{8}{\xi_i,0 \min\{1, \beta_\eta\}} \| \nabla \chi_i^n \|_0^2 + \delta t \frac{8}{\xi_i,0 \min\{1, \beta_\eta\}} \| \chi_i^n \|_0^2. \]
Adding and subtracting the terms \( \delta t \left( \sum_{j=1}^{m} A(i,j) \partial u_{j}^{n-1} \right) \), we get

\[
I_5 := \delta t \left( \sum_{j=1}^{m} A(i,j) \partial u_{j}^{n-1} \right) - \delta t \left( \sum_{j=1}^{m} A(i,j) \partial u_{j}^{n-1} \right) \]

\[
= \delta t \left( \sum_{j=1}^{m} A(i,j) \partial u_{j}^{n-1} \right) + \delta t \left( \sum_{j=1}^{m} A(i,j) \partial u_{j}^{n-1} \right) \]

Using generalised Hölder’s inequality and property of \( \mathcal{P} \), we obtain,

\[
I_5 \leq \delta t A_{\text{max}} \left[ \|u_{i}^{n}\|_{\infty} \sum_{j=1}^{m} \|u_{j}^{n-1} - u_{j}^{n-1}\|_{0} + \|\partial u_{i}^{n}\|_{\infty} \sum_{j=1}^{m} \|\partial u_{j}^{n-1}\|_{\infty} + \|\partial u_{i}^{n}\|_{\infty} \sum_{j=1}^{m} \|\partial u_{j}^{n-1}\|_{0} \right] \|\partial u_{i}^{n}\|_{0}
\]

Using Taylor’s theorem w.r.t \( t \) (neglecting \( (\delta t)^2 \) term), the estimates in (5.3), (5.5), (2.5), (4.1) and then Young’s inequality, we have,

\[
I_5 \leq \delta t A_{\text{max}} \left[ \mu \sum_{j=1}^{m} C \delta t \|u_{j}^{n}\|_{0} + m (\mu + 1) \left( \|\partial u_{i}^{n}\|_{0} + \|\partial \chi_{i}^{n}\|_{0} \right) + \mu \sum_{j=1}^{m} C H^{s+1} \|u_{j}^{n}\|_{s+1} + \right.
\]

\[
\left. + m (\mu + 1) C H^{s+1} \|u_{i}^{n}\|_{s+1} + \mu \sum_{j=1}^{m} \left( H^{s+1} \|u_{j}^{n-1}\|_{s+1} + \|\partial \chi_{j}^{n-1}\|_{0} \right) \right] \|\partial \chi_{i}^{n}\|_{0}
\]

Now adding and subtracting \( \delta t \left( R(i,i) (\mathcal{P}u_{i}^{n} + \partial u_{i}^{n}) \right) \), using Cauchy-Schwarz inequality and property of \( \mathcal{P} \), we get,
\[ I_6 := \delta t (R(i, i)u^n_i, \chi^n_i)_0 - \delta t (R(i, i)PU^n_i, \mathcal{P}\chi^n_i)_0 \]
\[ = \delta t (R(i, i)(u^n_i - \mathcal{P}u^n_i), \chi^n_i)_0 + \delta t (R(i, i)\mathcal{P}\varphi^n_i), \chi^n_i)_0 + \delta t (R(i, i)PU^n_i, \chi^n_i - \mathcal{P}\chi^n_i)_0 \]
\[ \leq \delta t R_{\max} \|u^n_i - \mathcal{P}u^n_i\|_0 \|\chi^n_i\|_0 + \delta t R_{\max} \|\varphi^n_i\|_0 \|\chi^n_i\|_0 + \delta t R_{\max} (\mathcal{P}U^n_i, \chi^n_i - \mathcal{P}\chi^n_i)_0 \]
\[ \leq \delta t R_{\max} \left[ 7 h^{s+1} \|u^n_i\|_{s+1} + \|\eta^n_i\|_0 \|\chi^n_i\|_0 \right] \|\chi^n_i\|_0 \quad \text{(use (5.5), (5.8))} \]
\[ \leq \delta t R_{\max} \left[ 2 C h^{s+1} \|u^n_i\|_{s+1} + \|\chi^n_i\|_0 \right] \|\chi^n_i\|_0 \quad \text{(use (5.5))} \]

Using Young’s inequality, we obtain
\[ I_6 \leq R_{\max} \left[ \delta t C 4 h^{2s+2} \|u^n_i\|^2_{s+1} + 2 \delta t \|\chi^n_i\|_0^2 \right]. \quad (5.24) \]

Adding and subtracting \( \left( \sum_{\ell,j \neq i} Q(i, \ell, j)PU^{n-1}_{\ell}PU^{n-1}_j, \chi^n_i \right)_0 \), we get
\[ I_7 = \delta t \left( \left( \sum_{\ell,j \neq i} Q(i, \ell, j)u^n_\ell u^n_j, \chi^n_i \right)_0 - \delta t \left( \left( \sum_{\ell,j \neq i} Q(i, \ell, j)PU^{n-1}_{\ell}PU^{n-1}_j, \mathcal{P}\chi^n_i \right)_0 \right. \right. \]
\[ = \delta t \left( \left. \left( \sum_{\ell,j \neq i} Q(i, \ell, j)(u^n_\ell u^n_j - \mathcal{P}U^{n-1}_{\ell} \mathcal{P}U^{n-1}_j), \chi^n_i \right)_0 \right. \right. \]
\[ + \delta t \left( \left. \left. \left( \sum_{\ell,j \neq i} Q(i, \ell, j)PU^{n-1}_{\ell}PU^{n-1}_j, \chi^n_i - \mathcal{P}\chi^n_i \right)_0 \right. \right) \right) \quad \text{(5.25)} \]

We estimate
\[ u^n_\ell u^n_j - \mathcal{P}U^{n-1}_{\ell} \mathcal{P}U^{n-1}_j = u^n_\ell (u^n_\ell - \mathcal{P}U^{n-1}_\ell) + \mathcal{P}U^{n-1}_\ell (u^n_\ell - \mathcal{P}U^{n-1}_\ell) \]
\[ = u^n_\ell \left[ (u^n_\ell - u^n_j) + (I - \mathcal{P})u^{n-1}_\ell + \mathcal{P}\varphi^{n-1}_\ell \right] \]
\[ + \mathcal{P}U^{n-1}_\ell \left[ (u^n_\ell - u^n_j) + (I - \mathcal{P})u^{n-1}_\ell + \mathcal{P}\varphi^{n-1}_\ell \right]. \quad (5.26) \]

Combining (5.25) and (5.26), and using generalised Hölder’s inequality we obtain
\[ I_7 \leq \delta t Q_{\max} \sum_{\ell \neq i} \|u^n_\ell\|_\infty \sum_{j \neq i} \left[ \|u^n_\ell - u^n_j\|_0 + \|\mathcal{P}\varphi^{n-1}_\ell\|_0 \right] \|\chi^n_i\|_0 \]
\[ + \delta t Q_{\max} \sum_{j \neq i} \|\mathcal{P}U^{n-1}_{\ell}\|_\infty \sum_{\ell \neq i} \left[ \|u^n_\ell - u^n_j\|_0 + \|\mathcal{P}\varphi^{n-1}_\ell\|_0 \right] \|\chi^n_i\|_0 \]
\[ + \delta t Q_{\max} \sum_{\ell \neq i} \|\mathcal{P}U^{n-1}_{\ell}\|_\infty \sum_{j \neq i} \left( \mathcal{P}U^{n-1}_j, \chi^n_i - \mathcal{P}\chi^n_i \right)_0. \]

Using Taylor’s theorem w.r.t t (neglecting \((\delta t)^2\) term), the estimates in (5.6), (5.3), (5.8), (5.5), (2.5), (4.1) and Young’s inequality, we have,
\[ I_7 \leq \delta t Q_{\max} \left[ \mu \sum_{j \neq i} \left[ \|u^n_j\|_0 + 2 h^{s+1} \|u^n_j\|_{s+1} + \|\chi^{n-1}_j\|_0 \right] \|\chi^n_i\|_0 \right. \]
\[ + (\mu + 1) \sum_{\ell \neq i} \left[ \|u^n_\ell\|_0 + 2 h^{s+1} \|u^n_\ell\|_{s+1} + \|\chi^{n-1}_\ell\|_0 \right] \|\chi^n_i\|_0 \right) + 0 \]
\[ \leq 2 (\delta t) Q_{\max} \left[ \mu (\mu + 1) \left\{ \|u^n_j\|_0 + 2 h^{s+1} \|u^{n-1}_j\|_{s+1} + \|\chi^{n-1}_j\|_0 \right\} \|\chi^n_i\|_0 \right. \]
\[ \leq 2 Q_{\max} \left[ (\delta t)^2 \|u^n_\ell\|_0^2 + \delta t 4 h^{2s+2} \|u^{n-1}\|_{s+1}^2 + \delta t \|\chi^{n-1}\|_0^2 + 3 \delta t \|\chi^n_i\|_0^2 \right]. \quad (5.27) \]

Adding and subtracting \( \left( \sum_{j \neq i} R(i, j)PU^{n-1}_j, \chi^n_i \right)_0 \) using Cauchy-Schwarz inequality and tri-
Using Taylor’s theorem w.r.t \((\delta t)^2\) term, (5.3), property of \(\mathcal{P}\), (5.8), (5.5) and Young’s inequality, we obtain,

\[
I_8 \leq \delta t \delta t_{\text{max}} \sum_{j \neq i} \left[ (\delta t) \|u_j^n\|^2_0 + 2 h^{s+1} \|u_j^{n-1}\|_s + C_m \delta t \|u_j^{n-1}\|_0 \right] \|\nabla \chi^n\|^2_0.
\]

Note that \(I_9 := (\delta t) (f^n, (I - \mathcal{P})\chi^n)_0 = (\delta t) ((I - \mathcal{P})f^n, \chi^n)_0\). Then using Cauchy-Schwarz inequality and Young’s inequality, we have,

\[
I_9 \leq \delta t \| (I - \mathcal{P})f^n \|_x \| \chi^n \|_0 \leq \delta t C h^{s+1} \| f^n \|_{s+1} \| \chi^n \|_0 \leq \delta t C h^{2s+2} \| f^n \|^2_{s+1} + \delta t \| \chi^n \|^2_0.
\]

Substituting the estimates (5.15), (5.16), (5.19)-(5.24) and (5.27)-(5.28) into the equation (5.12), and simplifying, we obtain

\[
\begin{align*}
\frac{\min\{1, \alpha_s\}}{4} \|\chi^n\|^2_0 + \frac{\xi_{\min} \min\{1, \beta_s\}}{2} (\delta t) \|\nabla \chi^n\|^2_0 \leq \\
2 \|\chi^{n-1}\|^2_0 + C_1 \delta t \|\chi^n\|^2_0 + C_2 \delta t \|\chi^{n-1}\|^2_0 + C_3 \delta t h^{2s} \|u^n\|^2_{s+1} \\
+ C_4 \delta t h^{2s} \|u^{n-1}\|^2_{s+1} + C_5 (\delta t)^2 \|\nabla u^n\|^2_0 + (\delta t)^2 \|\nabla f^n\|^2_0 + \delta t h^{2s} \|f^n\|^2_{s+1},
\end{align*}
\]

where,

\[
\begin{align*}
C_1 & := \frac{8 \omega^2_{\text{max}}}{\xi_{\min} \min\{1, \beta_s\}} + A_{\text{max}} (4 \mu + 3m(\mu + 1)) + R_{\text{max}} (3 + 2C) + Q_{\text{max}} Cm(\mu + 1), \\
C_2 & := m\mu A_{\text{max}} + CQ_{\text{max}} (\mu + 1)m^2 + mR_{\text{max}}, \\
C_3 & := \frac{C_{16}(\alpha^* + 1)}{\min\{1, \alpha_s\}} + C\omega^2_{\text{max}} + CA_{\text{max}} (2m + 3m\mu) + C4R_{\text{max}} \\
& \quad + \frac{8C}{\xi_{\min} \min\{1, \beta_s\}} \left[ (1 + \beta_s) + \max_i \left\{ (\max E \xi_i^k + \|\frac{\partial^s \xi_i}{\partial x^s}\|_x + \beta_s)^2 \right\} \right] + C \|(\omega_1, \omega_2)\|^2_{3c, s}, \\
C_4 & := \frac{C_{16}(\alpha^* + 1)}{\min\{1, \alpha_s\}} + C\mu A_{\text{max}} + CQ_{\text{max}} m^2(\mu + 1) + 4CmQ_{\text{max}}, \\
C_4 & := \frac{8(\alpha^* + 1)}{\min\{1, \alpha_s\}} + CA_{\text{max}} m\mu + CQ_{\text{max}} m^2(\mu + 1) + CmR_{\text{max}}.
\end{align*}
\]
Summing from \( n = 1 \) to \( k \), and noting \( \| \chi^0 \|_0 = 0 \) (choose \( U^0 = U^0 \)) we get,

\[
\frac{\min\{1, \alpha_s\}}{4} \| x^k \|_0^2 + \frac{\xi_{\min} \min\{1, \beta_s\}}{2} \sum_{n=1}^{k} (\delta t) \| \nabla x^n \|_0^2
\]

\[
\leq C \frac{(1 + \alpha_s)^2}{\min\{1, \alpha_s\}} \| x^0 \|_0^2 + C (C_1 + C_2) \sum_{n=1}^{k} \delta t \| x^n \|_0^2 + (C_3 + C_4) h^{2s} \sum_{n=1}^{k} \delta t \| u^n \|_{s+1}^2
\]

\[
+ (\delta t)^2 \sum_{n=1}^{k} (C_5 \| u^n \|_0^2 + \| u_{tt} \|_0^2) + h^{2s} \sum_{n=1}^{k} \delta t \| f^n \|_{s+1}^2
\]

\[
\leq \tilde{C}_1 \sum_{n=1}^{k} \delta t \| x^n \|_0^2 + \tilde{C}_2 h^{2s} \left( [u]_{0,s+1}^2 + [f]_{0,s+1} \right) + \tilde{C}_3 (\delta t)^2 \left( [u]_{\infty,0}^2 + [u_{tt}]_{\infty,0}^2 \right),
\]

where \( \tilde{C}_i \)’s are dependent on \( C_1 - C_5 \) and are independent of \( h \) and \( \delta t \). From (5.31), note that,

\[
\frac{\min\{1, \alpha_s\}}{4} \| x^k \|_0^2 \leq \tilde{C}_1 \sum_{n=1}^{k} \delta t \| x^n \|_0^2 + \tilde{C}_2 h^{2s} \left( [u]_{0,s+1}^2 + [f]_{0,s+1} \right) + \tilde{C}_3 (\delta t)^2 \left( [u]_{\infty,0}^2 + [u_{tt}]_{\infty,0}^2 \right).
\]

Thus by discrete Gronwall’s lemma, we obtain,

\[
\| x^n \|_0^2 \leq C \left[ \tilde{C}_2 h^{2s} \left( [u]_{0,s+1}^2 + [f]_{0,s+1} \right) + \tilde{C}_3 (\delta t)^2 \left( [u]_{\infty,0}^2 + [u_{tt}]_{\infty,0}^2 \right) \right].
\]

Now substituting (5.32) into (5.31), we get

\[
\sum_{n=1}^{k} (\delta t) \| \nabla x^n \|_0^2 \leq C \left[ \tilde{C}_2 h^{2s} \left( [u]_{0,s+1}^2 + [f]_{0,s+1} \right) + \tilde{C}_3 (\delta t)^2 \left( [u]_{\infty,0}^2 + [u_{tt}]_{\infty,0}^2 \right) \right].
\]

Using (5.8), (5.33) and (5.5), we obtain

\[
[u^n - U^n]_{0,1}^2 \leq C \left( h^{2s} + (\delta t)^2 \right).
\]

Next we will prove that the induction assumption \( \| U^n \|_\infty \leq (\mu + 1) \) holds, using the following proposition (see sec.5.6 in [21]):

**Proposition 5.3.** Let \( w \in V^h \) and using the assumptions on the elements \( E \in \mathcal{T}_h \), we have the norm inequality,

\[
\| w \|_{\infty,E} \leq h^{-1} \| w \|_{0,E}. \tag{5.34}
\]

**Lemma 5.4.** Let us consider the discrete solution \( U^n \) of (3.6) at the \( n \)th time step along with estimate (5.32). Then

\[
\| U^n \|_\infty \leq (\mu + 1). \tag{5.35}
\]

**Proof.** From (5.8), (2.5), (5.34), (5.5) and (5.32), we have

\[
\| U^n \|_\infty \leq \| U^n - u^n \|_\infty + \| u^n \|_\infty \leq \| \eta^n \|_\infty + h^{-1} \| \chi^n \|_0 + \mu
\]

\[
\leq C \left( h^s + h^{s-1} + h^{-1} \delta t \right) [u]_{\infty,s+1} + \mu.
\]

For sufficiently small \( h \) and \( \delta t \), we obtain (5.35). \( \square \)
6. Implementation details

We outline two techniques to solve the system of equations obtained from the discrete formulation (5.1).

1. Iteration method

At each time step, to enhance the accuracy of our numerical solution, we solve the discrete linear system (5.1) by using an iteration method. In Algorithm 1, the iteration procedure is detailed in steps (2.2)-(2.3), and the advantages in its implementation is described in [22].

Algorithm 1 :
(1) Let \( U^0 := u^0 \) on the virtual element space \( V_h \) and fix \( tol > 0 \).
(2) START FOR \( n=1,2,...,N \) DO
  (2.1) Set \( N_0 := U_{n-1} \) and let \( r = 0 \).
  (2.2) Set \( r = r + 1 \). Find \( \mathcal{N}_r \in V_h \) satisfying \( \forall V \in V_h : \)
  \[
  \sum_{i=1}^{m} \left\{ m_h(\mathcal{N}_r, V_i) - m_h(U_{n-1}^{i-1}, V_i) + (\delta t) a_h(\mathcal{N}_r, V_i) + (\delta t) l_{1,h}(\mathcal{N}_r, V_i) + (\delta t) l_{2,h}(\mathcal{N}_r, V_i) + (\delta t) l_{3,h}(\mathcal{N}_r, V_i) \right\} = (\delta t) \sum_{i=1}^{m} \left\{ -l_{4,h}(\mathcal{N}_r, V_i) - l_{5,h}(\mathcal{N}_r, V_i) + (f^n, PV_i) \right\}.
  
(2.3) Repeat steps (2.2) UNTIL \( \text{norm} (\mathcal{N}_r - \mathcal{N}_{r-1}) < tol \).
(2.4) Set \( U^n := \mathcal{N}_r \). END FOR ○

2. Two-grid method

Usually two-grid method involves two meshes \( \mathcal{T}_H, \mathcal{T}_h \) of different mesh diameters \( H, h \) ( with \( H < h \) ) along with the corresponding VEM spaces \( \mathcal{V}_H \) and \( \mathcal{V}_h \), known as coarse space and fine space, respectively. To our knowledge, this is the first instance where we use a two-grid concept to solve the discrete scheme obtained by VEM discretization of a system of equations. Now we present our two-grid method to solve the virtual element formulation (5.1).

Algorithm 2 :
(a) Fix \( ctol > 0 \) and an integer \( fiter > 0 \).
(b) START FOR \( n=1,2,...,N \) DO
  (b.1) On coarse mesh \( \mathcal{T}_H \), find \( \tilde{U}^n \in \mathcal{V}_H \) solving (5.1) using the steps (2.1)-(2.4) in Algorithm 1 by fixing \( tol := ctol \).
  (b.2) Interpolate \( \tilde{U}^n \in \mathcal{V}_H \) to \( \mathcal{V}_h \) and denote as \( \mathcal{T}U^n \).
  (b.3) Set \( \mathcal{N}_0 := \mathcal{T}U^n \) and \( r = 0 \). On fine mesh \( \mathcal{T}_h \), execute \( fiter \) times the step (2.2) of Algorithm 1, and obtain the solution \( U^n \in \mathcal{V}_h \). END FOR ○

In Algorithm 2, we can choose \( ctol \) relatively larger, say \( 10^{-3} \) and a small \( fiter \), say 1,2 or 3.

In the section we shall see that for the optimal performance of our two-grid technique, the choice of \( H \) wrt \( h \) and the number of iterations \( fiter \) in fine space, are problem dependent.

7. Numerical examples

In this section we consider two problems whose exact solutions are known. For the numerical experiment, we use three types of meshes, namely, distorted square mesh, nonconvex mesh and regular voronoi mesh. The choice of meshes include convex and concave elements. A sample of each mesh is shown in figure 1.
To obtain the error, we evaluate norm of the difference of exact and numerical solution, at the final time step $T$. Let $u$ and $u_h$ denote the exact and discrete solution, respectively, at time $T$. Then the $L^2$ norm and $H^1$ semi-norm of the error at $T$, denoted by $e_{h,0}$ and $e_{h,1}$ respectively, are calculated using the expressions,

$$
e_{h,0}^2 = \sum_{E \in \mathcal{T}_h} \| u - \Pi^0_p u_h \|^2_E, \quad e_{h,1}^2 = \sum_{E \in \mathcal{T}_h} \| \nabla (u - \Pi^1_p u_h) \|^2_E.$$

The first example consists of a system of two equations, for which we show the rate of convergence of $H^1$ semi-norm with respect to $h$, for the VEM orders 1, 2 and 3. The second example is a system of four equations and in this we show rate of convergence wrt $t$ for second-order VEM. On both the examples, for VEM order $k=2$, we compare the performances of Iteration method (Algorithm 1) and two-grid method (Algorithm 2), based on accuracy in terms of error values $e_{h,0}$, $e_{h,1}$ and on CPU time.

### 7.1. Example 1

Let $m = 2$, $\Omega = [0,1] \times [0,1]$ and $T = 1$. The choice for parameters in (2.1) are $\xi_1 = 1$, $\xi_2 = 2$, $\omega = (1, 2)$, $Q = [0]$, $A = [1, 1.5; 1.1, 2]$, $R = [-1, 0; 2, 0]$ and the functions $f_1, f_2$ are such that the exact solution $u_1, u_2$ are defined as:

$$u_1(x, y, t) := e^t x y (x - 1)^2 (y - 1)^2 \quad \text{and} \quad u_2(x, y, t) := e^{-t} x y (x - 1) (y - 1).$$

Using Algorithm 1, we solve the system of equations arising from the scheme (5.1) for VEM orders $k = 1, 2$ and 3, over the distorted square mesh. We choose $\delta t \approx O(h^{k+1})$ and tolerance value $tol := 10^{-7}$. The error $e_{h,1}$ for the solutions $u_1, u_2$ along with rate of convergence (roc) are tabulated in table 1. We see optimal rate wrt $H^1$ seminorm which is in accordance with our theoretical estimate.

Now for VEM order $k=2$, we present a comparison of iteration method and two-grid method. For this sake, we fix $\delta t = 10^{-3}$. In Algorithm 1, $tol$ is set to $10^{-6}$. In the two-grid scheme, we choose $H = 2h$, set the coarse mesh solve tolerance $ctol$ as $10^{-3}$ and perform one fine grid iteration, i.e. set $filter = 1$. The results obtained for distorted square mesh and nonconvex mesh can be found in table 2 and table 3, respectively.

We note that on both the meshes, the two-grid method performs efficiently by consuming considerably less CPU time than the standard scheme, without compromising on the accuracy.

### 7.2. Example 2

Consider the example given in [22]. We take $T = 1$, $\Omega = [0,1] \times [0,1]$ and $m = 4$. Let $\xi_1 = 1$, $\xi_2 = 2$, $\xi_3 = 1.5$, $\xi_4 = 3$, $\omega = (1, 2)$, $Q = [0]$.
The optimal rate of one for Euler discretization was obtained as seen in table 4. In the iteration method, set \( \delta t \) to \( 10^{-6} \). For the two-grid method, we choose \( H = \frac{1}{4} \) for \( h = \frac{1}{10} \) and \( H = \frac{1}{2} \) for \( h = \frac{1}{20} \). The optimal rate of one for Euler discretization was obtained as seen in table 4. Next, let us fix \( \delta t = 10^{-3} \). In the iteration method, set \( tol \) to \( 10^{-6} \). For the two-grid method, we choose \( H = \frac{1}{4} \) for \( h = \frac{1}{10} \) and \( H = \frac{1}{2} \) for \( h = \frac{1}{20} \). Set \( c_{tol} \) as \( 10^{-3} \) and perform three fine grid iteration, i.e. set \( f\text{ilter} = 3 \). We compute the results for second order VEM on regular Voronoi mesh (tables 5-6) and nonconvex mesh (tables 7-8).
Table 3.: CPU time and error values $e_{h,0}$, $e_{h,1}$ for nonconvex mesh.

| $\delta t$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
|------------|-------|-------|-------|-------|
| $\frac{1}{16}$ | 5.67e-3 | 1.62e-3 | 2.79e-3 | 1.59e-3 |
| $\frac{1}{32}$ | 2.71e-3 | 8.47e-4 | 1.33e-3 | 8.06e-3 |
| $\frac{1}{128}$ | 1.32e-3 | 4.33e-4 | 6.51e-4 | 4.69e-3 |
| $\frac{1}{256}$ | 6.56e-4 | 2.22e-4 | 3.22e-4 | 3.44e-4 |

Table 4.: Rate of convergence wrt $\delta t$ for Voronoi mesh.

| $h$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
|-----|-------|-------|-------|-------|
| $\frac{1}{16}$ | 4.25658e-4 | 4.642544e-2 | 3.562504e-4 | 3.825499e-2 |
| $\frac{1}{32}$ | 5.554100e-5 | 1.091555e-2 | 4.261762e-5 | 9.094519e-3 |
| $\frac{1}{64}$ | 2.650741e-5 | 2.670695e-3 | 1.014168e-5 | 2.238159e-3 |
| $\frac{1}{128}$ | 2.138414e-5 | 3.755053e-3 | 2.928331e-5 | 6.665600e-2 |
| $\frac{1}{256}$ | 1.287428e-5 | 9.125372e-4 | 3.569207e-5 | 1.602872e-2 |

Table 5.: Error values $e_{h,0}$, $e_{h,1}$ for Voronoi mesh.

| $h$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
|-----|-------|-------|-------|-------|
| $\frac{1}{16}$ | 4.249176e-4 | 4.642545e-2 | 3.561006e-4 | 3.825466e-2 |
| $\frac{1}{32}$ | 5.021513e-5 | 1.091543e-2 | 4.472873e-5 | 9.094471e-3 |
| $\frac{1}{64}$ | 2.244500e-5 | 2.670014e-3 | 6.332434e-6 | 2.238086e-3 |
| $\frac{1}{128}$ | 1.493283e-5 | 1.586688e-2 | 2.454744e-5 | 2.765073e-1 |
| $\frac{1}{256}$ | 1.876699e-5 | 3.755002e-3 | 3.033806e-4 | 6.665607e-2 |
| $\frac{1}{512}$ | 1.165073e-5 | 9.124825e-4 | 4.246686e-5 | 1.602876e-2 |

Table 6.: Error values $e_{h,0}$, $e_{h,1}$ for Voronoi mesh.
Two-grid method
Iteration method

| $h$   | $e_{h,0}$ | $e_{h,1}$ | $u_1$ | $e_{h,0}$ | $e_{h,1}$ | $u_2$ |
|-------|----------|----------|-------|----------|----------|-------|
| $1/16$ | 5.839672e-4 | 6.380904e-2 | 4.549473e-3 | 5.060191e-2 |
| $1/32$ | 7.722887e-5 | 1.598509e-2 | 6.083321e-5 | 1.264378e-2 |
| $1/64$ | 2.739111e-5 | 4.000714e-3 | 1.014168e-5 | 3.159645e-3 |

| $h$   | $e_{h,0}$ | $e_{h,1}$ | $u_3$ | $e_{h,0}$ | $e_{h,1}$ | $u_4$ |
|-------|----------|----------|-------|----------|----------|-------|
| $1/16$ | 1.982376e-4 | 2.062463e-2 | 3.430639e-4 | 3.807783e-1 |
| $1/32$ | 2.786719e-5 | 5.174224e-3 | 4.306165e-4 | 9.573265e-2 |
| $1/64$ | 1.308594e-5 | 1.206721e-3 | 5.438818e-5 | 2.398772e-2 |

Table 7: Error values $e_{h,0}$, $e_{h,1}$ for nonconvex mesh.

Two-grid method

| $h$   | $e_{h,0}$ | $e_{h,1}$ | $u_1$ | $e_{h,0}$ | $e_{h,1}$ | $u_2$ |
|-------|----------|----------|-------|----------|----------|-------|
| $1/16$ | 5.835024e-4 | 6.380999e-2 | 4.842840e-3 | 5.060195e-2 |
| $1/32$ | 7.370203e-5 | 1.598590e-2 | 6.132023e-5 | 1.264375e-2 |
| $1/64$ | 2.687501e-5 | 4.000648e-3 | 1.103731e-5 | 3.159606e-3 |

| $h$   | $e_{h,0}$ | $e_{h,1}$ | $u_3$ | $e_{h,0}$ | $e_{h,1}$ | $u_4$ |
|-------|----------|----------|-------|----------|----------|-------|
| $1/16$ | 1.984016e-4 | 2.062461e-2 | 3.430072e-4 | 3.807783e-1 |
| $1/32$ | 2.619106e-5 | 5.174197e-3 | 4.353126e-4 | 9.573265e-2 |
| $1/64$ | 1.293289e-5 | 1.206694e-3 | 5.470843e-5 | 2.398773e-2 |

Table 8: Error values $e_{h,0}$, $e_{h,1}$ for nonconvex mesh.

| $h$   | $H$ | Voronoi mesh | Nonconvex mesh |
|-------|-----|--------------|---------------|
|       |     | Iteration method | Two-grid method | Iteration method | Two-grid method |
|       | Time (sec) | Time (sec) | Time (sec) | Time (sec) |
| $1/16$ | $1/1$ | 387 | 306 | 441 | 355 |
| $1/32$ | $1/1$ | 2842 | 1846 | 3249 | 2131 |
| $1/64$ | $1/2$ | 55142 | 29915 | 57954 | 30776 |

Table 9: CPU time for Voronoi and nonconvex mesh.

A glance of the tables shows that for both the meshes, the accuracy of $e_{h,0}$, $e_{h,1}$ on standard scheme and two-grid schemes remains the same. Table clearly shows the computational efficiency of two-grid method. In particular, the effectiveness of two-grid method is more for smaller $h$ (ie. for large degrees of freedom).

8. Conclusion

In this work we have studied the virtual element method for a system of nonlinear advection-diffusion-reaction equation. To circumvent the difficulty of solving a nonlinear system, we have modified it to a linear discrete formulation whose solution also satisfies the model problem. For solving the arising system of equations, we propose iteration method and two-grid method. From the two examples, we clearly see the computational efficiency of two-grid method over iteration method. We also note that the choice of parameters $H$ and $fit_{er}$ in Algorithm 2 are problem dependent.
References

[1] L. Beirao Da Veiga, F Brezzi, A Cangiani, G Manzini, L. D Marini, and A Russo. Basic principles of virtual element methods. *Math Models Methods Appl Sci*, 23:199–214, 2013.

[2] M Arrutselvi and E Natarajan. Virtual element method for nonlinear convection-diffusion-reaction equation on polygonal meshes. *Int. J. Comput. Math.*, 98(9):1852–1876, 2021.

[3] M Arrutselvi and E Natarajan. Virtual element stabilization of convection-diffusion equation with shock capturing. *Journal of Physics: Conf. Ser.*, 1850:1–12, 2021.

[4] Dibyendu Adak, E Natarajan, and Sarvesh Kumar. Convergence analysis of virtual element methods for semilinear parabolic problems on polygonal meshes. *Numer Meth Part D E*, 35(1):222–245, 2019.

[5] G Vacca and L. Beirao Da Veiga. Virtual element methods for parabolic problems on polygonal meshes. *Numer Meth Part D E*, 31(6):2110–2134, 2015.

[6] Dibyendu Adak, E Natarajan, and Sarvesh Kumar. Virtual element method for semilinear hyperbolic problems on polygonal meshes. *Int J Comput Math*, 96(5):971–991, 2019.

[7] G Vacca. Virtual element methods for hyperbolic problems on polygonal meshes. *Comput Math Appl*, 74(5):882–898, 2017.

[8] Dibyendu Adak, S Natarajan, and E Natarajan. Virtual element method for semilinear elliptic problems on polygonal meshes. *Appl Numer Math*, 145:175–187, 2019.

[9] L. Beirao Da Veiga, F Brezzi, L. D Marini, and A Russo. Virtual element method for general second-order elliptic problems. *Math Models Methods Appl Sci*, 26(4):729–750, 2016.

[10] Andrea Cangiani, Gianmarco Manzini, and Oliver J. Sutton. Conforming and nonconforming virtual element methods for elliptic problems. *IMA Journal of Numerical Analysis*, 37(3):1317–1354, 2017.

[11] P. F Antonietti, L. Beirao Da Veiga, D Mora, and M Verani. A stream virtual element formulation of the stokes problem on polygonal meshes. *SIAM J Numer Anal*, 52(1):386–404, 2014.

[12] P. F Antonietti, G Manzini, and M Verani. The fully nonconforming virtual element method for biharmonic problems. *Math Models Methods Appl Sci*, 28(2):387–407, 2018.

[13] Ernesto Caceres, Gabriel N. Gatica, and Filander A. Sequeira. A mixed virtual element method for the brinkman problem. *Math Models Methods Appl Sci*, 27(4):707–743, 2017.

[14] Ernesto CÁceres, Gabriel N. Gatica, and Filander A. Sequeira. A mixed virtual element method for a pseudostress-based formulation of linear elasticity. *Appl Numer Math*, 135:423–442, 2019.

[15] P. F Antonietti, L. Beirao Da Veiga, S Scacchi, and M Verani. A c1 virtual element method for the cahn-hilliard equation with polygonal meshes. *SIAM J Numer Anal*, 54(1):34–56, 2016.

[16] A. L Gain, C Talischi, and G. H Paulino. On the virtual element method for three-dimensional linear elasticity problems on arbitrary polyhedral meshes. *Comput Methods Appl Mech Eng*, 282:132–160, 2014.

[17] B Ahmad, A Alsaedi, F Brezzi, D Marini, L, and A Russo. Equivalent projectors for virtual element methods. *Comput. Math. with Appl.*, 66:376–391, 2013.

[18] Susanne C. Brenner, Qingguang Guan, and Li-Yeng Sung. Some estimates for virtual element methods. *Comput. Methods Appl Math.*, 17(4):553–574, 2017.

[19] Susanne C. Brenner and Li-Yeng Sung. Virtual element methods on meshes with small edges or faces. *Math. Models Methods Appl. Sci.*, 28(1):1291–1336, 2018.

[20] M. F Benedetto, S Berrone, A Borio, S Pieraccini, and S Scialo. Order preserving supg stabilization for the virtual element formulation of advection-diffusion problems. *Comput Methods Appl Mech Eng*, 311:18–40, 2016.

[21] Wen-Ming He and Hailong Gou. Optimal maximum norm estimates for virtual element methods. https://arXiv:2105.11621[math.NA], 2021.

[22] Biyue Liu. An error analysis of a finite element method for a system of nonlinear advection-diffusion-reaction equations. *Applied Numer. Math.*, 59:1947–1959, 2009.