Features of ultrasonic non-destructive testing models of rectangular anisotropic elastic waveguides with a membrane coating

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Abstract. A numerical-analytical fuzzy-set method for synthesizing estimates of the influence of uncertainty factors in the form of scatter of the initial physical-mechanical and geometric parameters of a long waveguide of prismatic geometry from anisotropic single-crystal material of a cubic system on the phase velocities of traveling normal elastic waves for applying in ultrasonic nondestructive evaluation are presented. Waveguide has absolutely flexible and inextensible thin coatings of a faces. The technique is based on the description of parameters with scatter errors in the form of fuzzy-intervals quantities and on the transition to fuzzy-set arguments in the analytical representations of wave numbers for various branches of the dispersion spectrum. The alpha-level form of the heuristic generalization principle is used. Examples of a fuzzy-set description of a series of frequency distributions characteristics of phase velocities of traveling normal waves for a waveguide from a silicon single crystal are given.

1. Introduction
In view of the existence of scatter errors in the experimental and technological data on the physical and mechanical constants of materials used in the construction of machines, devices, structures and acoustoelectric devices, as well as scatter errors in their geometric parameters, the very urgent problem of correctly accounting of the influence of uncertainty factors of exogenous parameters on the endogenous characteristics of the corresponding applied calculated mathematical models of their non-destructive ultrasonic diagnostics is arises [1]. The experience of applying the methods of probability theory and mathematical statistics to solving this problem of wave solid mechanics is written, in particular, in [2]. However, in many cases, information on the scatter errors to be taken into account is...
formed on the basis of expert conclusions and experimental estimates, or is based on small experimental samples and, as a result, does not have a correct statistical character. The application in such cases of an approach based on the methods of the theory of fuzzy sets, which creates the possibility of direct operation with uncertain parameters of models without going over to their averaged integral characteristics, is described in publications [3-5].

The study in this paper aims to solve the problem of fuzzy-sets estimation of the phase velocities of traveling normal elastic waves propagating along a rectangular prismatic waveguide of rectangular cross-section made of an anisotropic cubic system material and having thin absolutely flexible inextensible coating the edges, which is important for advanced technologies of non-destructive ultrasonic diagnostics.

2. Methodic of accounting of parametric uncertainty in the model of elastic waves propagation in rectangular anisotropic elastic waveguides with membrane coverings of the faces

The study is based on the use of an applied α-level version of the heuristic principle of generalization [3] to expand of the domains of definition of classical functional mappings on the arguments from the fuzzy sets and on the assumption about effectively describe experimental values of the elastic module with scatter errors for the waveguide material with using normal trapezoidal fuzzy intervals [6].

A long prismatic waveguide of rectangular cross-section made of an anisotropic material of a cubic system with elastic constants $c_{ij}$ and a density $\rho$ is considered. This waveguide of a region $V = \{ |x_1| \leq a, |x_2| \leq b, -\infty < x_3 < \infty \}$ in coordinate space $Ox_1, x_2, x_3$ are occupied. It is assumed that the coordinate directions $Ox_j$ are oriented along the elastically equivalent directions of the waveguide material. Its flat boundary surfaces there are thin flexible inextensible coatings. The representations of modes of normal elastic waves with cyclic frequency $\omega$ and wavenumber $k$ in the considered waveguide, obtained using the classical formulation of the problem in the model of linear dynamic deformation of anisotropic solids, are the result of the analysis of boundary value problems with unknown complex amplitude functions of wave elastic displacements $u_j(x_1, x_2, x_3)$. Boundary value problem contains wave equations

$$L_{\alpha_i}u_{ij} = 0,$$

in which

$$L_{11} = c_1\ddot{\varphi}_1 + c_{14}\ddot{\varphi}_2 + c_{44}\ddot{\varphi}_3 + \Omega^2, \quad L_{22} = c_{44}\ddot{\varphi}_1 + c_{14}\ddot{\varphi}_2 + c_4\ddot{\varphi}_3 + \Omega^2,$$

$$L_{33} = c_4\ddot{\varphi}_1 + c_{44}\ddot{\varphi}_2 + c_4\ddot{\varphi}_3 + \Omega^2, \quad L_{12} = L_{21} = (c_{14} + c_4)^2\ddot{\varphi}_1\ddot{\varphi}_2,$$

$$L_{13} = L_{31} = (c_{14} + c_4)^2\ddot{\varphi}_1\ddot{\varphi}_3, \quad L_{23} = L_{32} = (c_{14} + c_4)^2\ddot{\varphi}_2\ddot{\varphi}_3, \quad \Omega^2 = \rho \omega^2 R^2 c_*^{-1},$$

and also, boundary conditions of the form

$$(u_2)_{x_1 = a} = (u_2)_{x_1 = -a} = (c_{14}\ddot{\varphi}_1 u_1 + c_{44}\ddot{\varphi}_2 u_2 + c_4\ddot{\varphi}_3 u_3)_{x_1 = \pm a} = 0$$

$$(u_1)_{x_1 = b} = (u_2)_{x_1 = -b} = (c_{14}\ddot{\varphi}_1 u_1 + c_{44}\ddot{\varphi}_2 u_2 + c_4\ddot{\varphi}_3 u_3)_{x_1 = \pm b} = 0$$

$R_*, \ c_*$ – respectively, normalizing parameters for quantities with dimensions of elastic displacements and mechanical stresses. To obtaining the analytical representations for the branches of the analyzed dispersion spectrum the initial representations of the complex amplitude functions of the considered normal waves with the four options for specifying subsets of functions, reflecting the combined symmetry of forms of wave displacement relative to the vertical and horizontal median lines of the cross section are introduced.
\[ u_j(x_1, x_2, x_3) = g_j(x_1, x_3) \exp(ikx_3) \quad (j = 1, 3), \]
\[ g_{1mn}^{(1)}(x_1, x_3) = u_{1mn}^{(1)} \sin \lambda_m x_1 \cos \delta_n x_3, \quad g_{2mn}^{(1)}(x_1, x_3) = u_{2mn}^{(1)} \cos \lambda_m x_1 \cos \delta_n x_3, \]
\[ g_{1mn}^{(2)}(x_1, x_3) = u_{1mn}^{(2)} \sin \lambda_m x_1 \sin \delta_n x_3, \quad g_{2mn}^{(2)}(x_1, x_3) = u_{2mn}^{(2)} \cos \lambda_m x_1 \sin \delta_n x_3, \]
\[ g_{1mn}^{(3)}(x_1, x_3) = u_{1mn}^{(3)} \cos \lambda_m x_1 \cos \delta_n x_3, \quad g_{2mn}^{(3)}(x_1, x_3) = u_{2mn}^{(3)} \sin \lambda_m x_1 \cos \delta_n x_3, \]
\[ g_{1mn}^{(4)}(x_1, x_3) = u_{1mn}^{(4)} \sin \lambda_m x_1 \sin \delta_n x_3, \quad g_{2mn}^{(4)}(x_1, x_3) = u_{2mn}^{(4)} \cos \lambda_m x_1 \sin \delta_n x_3. \]

The parameters \( \lambda_m, \delta_n \) in representations (5) - (8) have the form
\[ \lambda_m = (2m - 1)\pi R, \lambda_n = (2n - 1)\pi R. \]

The introduced complex amplitude functions ensure the fulfillment of boundary conditions (4). The equations for the modes of the investigated waves in the case under consideration are, respectively, described by the equalities to zero of the functional determinants
\[ F_{mn}^{(j)}(k, \Omega) = \det \left| \Delta_{pqmn}^{(j)} \right| = 0 \quad (j, p, q, m, n = 1, 3), \]

in which
\[ \Delta_{1mn}^{(j)} = \Omega^2 - (c_{11}^2 + c_{14}^2 + c_{44}^2), \quad \Delta_{2mn}^{(j)} = \Omega^2 - (c_{14}^2 + c_{44}^2), \quad \Delta_{3mn}^{(j)} = \Omega^2 - (c_{44}^2 + c_{14}^2 + c_{11}^2), \]
\[ \Delta_{12mn}^{(j)} = -\Delta_{12mn}^{(j)}, \quad \Delta_{23mn}^{(j)} = -\Delta_{23mn}^{(j)}, \quad \Delta_{31mn}^{(j)} = -\Delta_{31mn}^{(j)}. \]

The frequency dependences for the phase velocities of traveling normal waves with various defined parameters \((j, m, n = 1, 3)\) by the wave elastic displacement forms are described by the relations
\[ \Phi_{1j}(\Omega, c_{11}, c_{12}, c_{44}, a, b, m, n) = [\varphi_{1jmn} - 2^{1/3} \varphi_{3jmn} / (3\varphi_{2jmn}^{(1)}) + \varphi_{2jmn} / (3 \cdot 2^{1/3} \varphi_{1jmn}^{(1)})]^{1/2} \]
\[ \Phi_{2j}(\Omega, c_{11}, c_{12}, c_{44}, a, b, m, n) = [\varphi_{1jmn} - (1 + 3^{1/3}i) \varphi_{3jmn} / (3 \cdot 2^{1/3} \varphi_{2jmn}^{(1)}) - (1 - 3^{1/3}i) \varphi_{2jmn} / (3 \cdot 2^{1/3} \varphi_{1jmn}^{(1)})]^{1/2} \]
\[ \Phi_{3j}(\Omega, c_{11}, c_{12}, c_{44}, a, b, m, n) = [\varphi_{1jmn} - (1 - 3^{1/3}i) \varphi_{3jmn} / (3 \cdot 2^{1/3} \varphi_{2jmn}^{(1)}) - (1 + 3^{1/3}i) \varphi_{2jmn} / (3 \cdot 2^{1/3} \varphi_{1jmn}^{(1)})]^{1/2}, \]
\[ \varphi_{1jmn} = \varphi_{1jmn}(\Omega, c_{11}, c_{12}, c_{44}, a, b, m, n) = -r_{2mn}^{(j)} / (3 r_{1mn}^{(j)}) \]
\[ \varphi_{2jmn} = \varphi_{2jmn}(\Omega, c_{11}, c_{12}, c_{44}, a, b, m, n) = -r_{2mn}^{(j)} / (3 r_{1mn}^{(j)}) \]
\[ \varphi_{3jmn} = \varphi_{3jmn}(\Omega, c_{11}, c_{12}, c_{44}, a, b, m, n) = -r_{2mn}^{(j)} + 3 r_{1mn}^{(j)} (r_{1mn}^{(j)})^2 / 3. \]
\[
\tau_{1mn}^{(j)} = -c_1^2 + c_1^2 \mu_{44}^{(j)} + \nu_{11}^{(j)} + \nu_{22}^{(j)} + c_3^2 \rho_{23}^{(j)} + c_4^2 (\eta_{12}^{(j)} + \eta_{23}^{(j)})^2,
\]

\[
\tau_{2mn}^{(j)} = -c_4^2 \mu_{44}^{(j)} + c_3^2 \rho_{23}^{(j)} - c_2^2 \rho_{23}^{(j)} + c_3^2 \rho_{23}^{(j)} + c_4^2 (\eta_{12}^{(j)} + \eta_{23}^{(j)})^2 + 2 \chi_{12mn}^{(j)} + \chi_{13mn}^{(j)} + \chi_{23mn}^{(j)} - (\chi_{12mn}^{(j)} - (\chi_{13mn}^{(j)} + \chi_{23mn}^{(j)})^2 + \chi_{11mn}^{(j)} + \eta_{33}^{(j)} + c_1(\chi_{13mn}^{(j)})^2),
\]

\[
\eta_{1mn}^{(j)} = \Omega_m^2 - (c_1 \lambda_m + c_4 \delta_n^2), \quad \eta_{22mn}^{(j)} = \Omega_m^2 - (c_2 \lambda_m + c_4 \delta_n^2), \quad \eta_{33}^{(j)} = \Omega_m^2 - (c_4 \lambda_m + c_1 \delta_n^2),
\]

Obtaining fuzzy-set estimates for the frequency parametric distributions of the phase velocities of normal waves is based on the use of representations (12) and the transition in them to arguments that are fuzzy quantities. According to the assumptions about the existence of scatter errors in the values of the initial physical-mechanical and geometric parameters of the waveguide, a hypothesis about the possibility of an effective description of the uncertain exogenous parameters \(c_{11}, c_{12}, c_{44}, \rho, a, b\) of the considered model by normal trapezoidal fuzzy intervals \(\bar{c}_{11}, \bar{c}_{12}, \bar{c}_{44}, \bar{\rho}, \bar{a}, \bar{b}\) is specified as

\[
\bar{c}_{11} = (c_{111}, c_{112}, c_{113}, c_{114}), \quad \bar{c}_{12} = (c_{121}, c_{122}, c_{123}, c_{124}), \quad \bar{c}_{44} = (c_{441}, c_{442}, c_{443}, c_{444}),
\]

and are represented by expansions in sets of \(\alpha\)-level

\[
\bar{c}_{11} = \bigcup_{\alpha \in [0,1]} \{c_{111}, c_{112}, c_{113}, c_{114}\}, \quad \bar{c}_{12} = \bigcup_{\alpha \in [0,1]} \{c_{121}, c_{122}, c_{123}, c_{124}\}, \quad \bar{c}_{44} = \bigcup_{\alpha \in [0,1]} \{c_{441}, c_{442}, c_{443}, c_{444}\},
\]

Using representations (21)-(23) obtaining of parametric fuzzy estimates \(\bar{v}_{s}^{(j,m,n)}(\omega)\) \((s = 1, 3)\) is realized by passing in functional dependencies (12) to fuzzy-interval arguments using the \(\alpha\)-level form of the heuristic principle of generalization [5-8]. As a result, for fuzzy-set characteristics \(\bar{v}_{s}^{(j,m,n)}(\omega)\) \((s = 1, 3)\) are obtained representations of the form

\[
\bar{v}_{s}^{(j,m,n)}(\omega) = \bigcup_{\alpha \in [0,1]} [v_{s}^{(j,m,n)}(\omega), \bar{v}_{s}^{(j,m,n)}(\omega)]
\]

\[
\bar{v}_{s}^{(j,m,n)}(\omega) = \inf_{c_{11} \in [c_{111}, c_{112}], c_{12} \in [c_{121}, c_{122}], c_{44} \in [c_{441}, c_{442}], a \in [a_{1}, a_{2}], b \in [b_{1}, b_{2}], \rho \in [\rho_{1}, \rho_{2}], m \in [m_{1}, m_{2}], n \in [n_{1}, n_{2}]} \{\omega/\Phi_{s}^{(j,m,n)}(\Omega, c_{11}, c_{12}, c_{44}, a, b, m, n)\}
\]
\[-(j,m,n)\]

\[
\Omega_{\text{int}}(\omega) = \sup_{c_{11} \in [c_{11}, c_{11}]} \left\{ \frac{\omega}{\Phi_{s}(\Omega, c_{11}, c_{12}, c_{44}, a, b, m, n)} \right\}
\]

3. Numerical results

The developed technique is applied for the particular case of a waveguide made of a silicon single crystal when the following fuzzy interval parameters are specified:

\[
\tilde{c}_{11} = (164c., 166c., 167c., 169c.), \quad \tilde{c}_{12} = (63c., 65c., 66c., 67c.), \quad \tilde{c}_{44} = (77c., 79c., 80c., 82c.) \quad (26)
\]

\[
\tilde{\rho} = (2.30\rho, 2.32\rho, 2.33\rho, 2.34\rho), \quad c_{s} = 10^{9}[Pa], \quad \rho = 10^{3}[kg/m^{3}].
\]

The calculation results are presented by parametrical frequency distributions for the characteristics of fuzzy-set descriptions of the phase velocities of normal waves with different types of symmetry and various indicators of the variability of wave displacement forms in the waveguide section, as well as descriptions of the of membership functions for fuzzy-set quantities \(\tilde{v}_{s}^{(j,m,n)}(\omega)\) \((s = 1, 3)\) at some fixed values of the cyclical frequency of \([\text{rad} / c]\) parameter of normal waves. The parameters of the section dimensions \(a, b\) in the calculations were considered as clear values without scatter of values and were taken equal to \(a = 2L, \quad b = 3L, \quad L = 1[m]\).

Figures 1-4 characterize the results of calculations for the case of waves of a mode \((s=1)\) with the type of symmetry \((j=1)\) and indicators of the variability for the displacement field in the section \((m=1, n=1)\). The outer lines in figure 1 correspond to the levels \(\mu = 0\) of belonging of the corresponding characteristic to the fuzzy set \(v_{s}^{(j,m,n)}(\omega)\) of its expected values \(\tilde{v}_{s}^{(j,m,n)}(\omega)\) and the inner lines correspond to the levels \(\mu = 1\) and limit the ranges of the most reliable values \(v_{s}^{(j,m,n)}(\omega)\) for the considered scatter errors of the initial parameters.

**Figure 1.** Distributions of values \(v_{1}^{(j,m,n)}(\omega) \in \tilde{v}_{1}^{(j,m,n)}(\omega)\).

**Figure 2.** Membership function \(\mu_{v_{1}^{(j,m,n)}(\omega)}(v_{1}^{(1,1,1)}(\omega))\) for \(v_{1}^{(j,m,n)}(300)\).
The form of the membership functions in figures 2-4 allows us to draw conclusions about the degree of confidence in the achievement of the corresponding values by the endogenous speed parameters at various values of the cyclic frequency parameter. It also follows from these figures that the spreads of endogenous parameters are of the same order as the specified spreads of exogenous parameters and, in the case under consideration, are about 3%.

4. Conclusion
A description and an example of the application of a theoretical technique for obtaining fuzzy-set estimates of the spread in the values of phase velocities for ultrasonic normal waves used in nondestructive ultrasonic testing technologies for deformable structures with undefined physical and mechanical parameters are presented. The technique allows using uncertain initial information of an expert nature and obtaining adequate estimates of the scatter of endogenous parameters, having a similar order with the scatter of values of exogenous parameters of the model.

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