Regularity results of the speed of biased random walks on Galton-Watson trees

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Abstract

We prove that the speed of $\lambda$-biased random walks on a supercritical Galton-Watson tree without leaves is differentiable when $\lambda \in (0, 1)$, and give an expression of the derivative using a certain 2-dimensional Gaussian random variable. The proof heavily uses the renewal structure of Galton-Watson trees that was introduced in [LPP2].

1 Introduction

Let $T$ be a Galton-Watson tree with offspring distribution $\{p_k\}_{k \geq 0}$ such that $p_n \neq 1$ for any $n \in \mathbb{N}$. We will denote by $P$ the distribution of the Galton-Watson tree. In this paper, we always assume that $T$ is supercritical (i.e., $m := \sum_{k \geq 1} kp_k > 1$) and has no leaves (i.e., $p_0 = 0$). Fix $\lambda > 0$. For a given infinite rooted tree $T$ without leaves, we consider the $\lambda$-biased random walk $(Z_n)_{n \geq 0}$ defined on a probability space $(\tilde{\Omega}(T), \mathcal{F}, P_T^{\lambda})$ starting at the root whose transition probabilities $\{A_\lambda(x, y)\}_{x, y \in T}$ are given as follows: denote the root of $T$ by $e(T)$, and for $x \in T$, the number of its offspring by $\nu(x)$. From the root $e(T)$, the random walk moves to one of its children equally likely, and from $x \neq e$ which has children $x_1, ..., x_{\nu(x)}$, the random walk moves to one of neighbors of $x$ according to the following formula.

$$ A_\lambda(x, \pi(x)) := \frac{\lambda}{\lambda + \nu(x)}, \quad A_\lambda(x, x_i) := \frac{1}{\lambda + \nu(x)}, \quad \text{for } 1 \leq i \leq \nu(x), $$

where $\pi(x)$ is the parent of $x$. For $x, y \in T$, denote by $d(x, y)$ the distance between $x$ and $y$, and by $d(x)$ the distance between $e(T)$ and $x$. Behaviors of $\lambda$-biased random walks on the Galton-Watson tree $T$ have been extensively studied over decades. For instance, it is proved in [L] that when $\lambda \in (0, m)$, the $\lambda$-biased random walk on the Galton-Watson tree $T$ is transient $\mathbb{P}$-a.s. Later, it is shown in [LPP1, LPP2] that when $\lambda \in (0, m)$ and $p_0 = 0$, the sequence $n^{-1}d(Z_n)$ converges $P_\lambda$-almost surely and in $L^1(P_\lambda)$, where $P_\lambda$ is the so-called annealed measure and defined by

$$ P_\lambda(\cdot) := \int \mathbb{P}(dT) P_T^{\lambda}(\cdot). $$

Moreover, it is proved in [LPP1, LPP2] that the limit of $n^{-1}d(Z_n)$ is a deterministic positive constant, and usually called the speed of the $\lambda$-biased random walk on the Galton-Watson tree, and we will denote it by $v_\lambda > 0$. The aim of this paper is to study how the value of $v_\lambda$ depends on the parameter of bias $\lambda$, and the following claim is our main result.

When $0 < \lambda < 1$, the function $\lambda \mapsto v_\lambda$ is differentiable, and there is an explicit expression for the derivative $v_\lambda'$ using Gaussian random variables. (Theorem 3.5)
In the unpublished note [A2], Ajékon also showed the differentiability of the function $\lambda \mapsto v_\lambda$ for $0 < \lambda < 1$, and gave an expression of the derivative that differs from ours.

The key ingredients of the proof are the renewal structure and the Girsanov-like formula. The renewal structure is a method to decompose paths of a random walk into i.i.d. components, and it is frequently used to analyze random walks in random environments. In [LPP2], Lyons, Pemantle and Peres constructed the renewal structure for supercritical Galton-Watson trees, which we will heavily utilize in this paper. See [BFS, BGN, DGPZ] for applications of this method to analysis of the speed of random walks in random environments. In particular, we refer to the paper [BGN], where the authors study the speed of biased random walks on a random conductance model, since one of their main results and the strategy of their proof is similar to ours.

The renewal structure is a method to decompose paths of a random walk into i.i.d. components. We then prove moment estimates of regeneration times. In Section 3, we will prove the main result of this paper by using the formula (1). We remark here that regeneration times are not stopping times, thus the formula (1) does not apply directly to them. See the proof of Proposition 2.6 for how to overcome this problem.

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2.2 Regeneration times

In this section, we introduce regeneration times and prove their moment estimates, which will turn out to be very important for this study.

Definition 2.1. For an infinite rooted tree $T$ without leaves and $x \in T$, define $P_{\lambda,x}^{T} (\cdot) := P_{\lambda}^{T} (\cdot | Z_{0} = x)$. (Thus, $P_{\lambda}^{T} = P_{\lambda, e(T)}^{T}$.) We will denote the expectation with respect to $P_{\lambda}$ (resp. $P_{\lambda}^{T}$) by $E_{\lambda}$ (resp. $E_{\lambda}^{T}$).

Definition 2.2. Let $(Z_{n})_{n \geq 0}$ be the $\lambda$-biased random walk on an infinite rooted tree $T$ without leaves.

1. A time $n \in \mathbb{N}$ is called a regeneration time if $Z_{n} \neq Z_{k}$ for all $k < n$ and $Z_{k} \neq Z_{n-1}$ for all $k > n$. 
The sequence \(\tau\) is clearly stated in the proof of [LPP2, Proposition 3.4]. The fifth claim immediately follows from [DGPZ, Proposition 3.4]. The second claim is shown in [LPP2, Lemma 3.3]. The first claim and the third claim are shown in [LPP2, Proposition 3.4].

Proof. The first claim and the third claim are shown in [LPP2, Lemma 3.3]. The second claim is clearly stated in the proof of [LPP2, Proposition 3.4]. The fifth claim immediately follows from [DGPZ, Lemma 4.2] and the definition of the level regeneration time. Thus, we only need to prove the fourth claim. We first introduce several random times which describe the behavior of regeneration times. Define \(S_1\) and \(U_1\) by

\[ S_1 := \inf\{n \geq 1 : Z_n \notin T(Z_1)\} \]

\[ U_1 := \inf\{n \geq 1 : Z_n \notin T(Z_1)\} \]

We further define \(S_k\) and \(U_k\) recursively by

\[ S_k := \inf\{n \geq U_{k-1} : Z_n \notin \{Z_0, Z_1, \ldots, Z_{U_{k-1}}\}\}, \quad U_k := \inf\{n \geq S_{k-1} : Z_n \notin T(Z_{S_k})\} \]

Then we have \(\tau_1 = S_J\), where \(J\) is the unique integer such that \(S_J < \infty\) and \(U_J = \infty\). Note that conditionally on \(T\), \((S_k)\) and \((U_k)\) are \(\mathcal{F}_n(T)\)-stopping times although \(\tau_1\) is not. Now we get

\[
P_\lambda\left([Z_{\tau_i}Z_{\tau_i+k})k \geq 0 \in A]\right) = \sum_{j \geq 1} P_\lambda\left([Z_{S_j}Z_{S_j+k})k \geq 0 \in A, J = j\right)
\]

\[
= \sum_{j \geq 1} \int P(dT)P_\lambda^{T}\left([Z_{S_j}Z_{S_j+k})k \geq 0 \in A, S_j < \infty, U_j = \infty\right).
\]
In the proof of Proposition 3.4 in [LPP2], it is shown that the pair $< T(Z_{\tau_1}), (Z_{\tau_1+k})_{k \geq 1} >$ is independent of $< T \setminus T(Z_{\tau_1}) \cup \{X_{\tau_1}, (Z_k)_{k \leq \tau_1} >$ under $P_\lambda$. Moreover, Lemma 3.2 in [LPP2] implies that the law of $T(Z_{\tau_1})$ is identical to that of $T$ under $P_\lambda$. Thus, we obtain

$$\sum_{j \geq 1} \int \mathbb{P}(dT) P_{T,j} \left[ (Z_{S_j}^{-1} Z_{S_j+k})_{k \geq 0} \in A, S_j < \infty, U_j = \infty \right]$$

$$= \sum_{j \geq 1} P_{\lambda}(S_j < \infty) P_{\lambda} \left[ \left( P_{\lambda, Z_{S_j}}(T(z_{S_j}))^* \right)^* (Z_{S_j}^{-1} Z_{S_j+k})_{k \geq 0} \in A, \sigma_\lambda(z_{S_j}) = \infty \right]$$

$$= \int \mathbb{P}(dT) P_{\lambda, e(T)}[ (Z_k)_{k \geq 0} \in A, \sigma_\lambda(T) = \infty] \sum_{j \geq 1} P_{\lambda}(S_j < \infty).$$

By substituting the whole space for $A$, we get

$$\sum_{j \geq 1} P_{\lambda}(S_j < \infty) = \left( \int \mathbb{P}(dT) P_{\lambda, e(T)}[ \sigma_\lambda(T) = \infty] \right)^{-1}. $$

This implies the conclusion.

We next study moment estimates of regeneration times.

**Lemma 2.5.** Assume $p_0 = 0$. The random variables $\tau_1$, $\tau_2 - \tau_1$, $r_1$ and $r_2 - r_1$ have finite exponential moment under $P_\lambda$ for any $\lambda \in (0,1)$.

**Proof.** Note that by [DGPZ, Lemma 4.3], the finite exponential moment of $r_1$ implies those of $r_2 - r_1$ and $r_2$. Since $r_2 \geq \tau_2$ almost surely, it suffices to prove the claim for $r_1$. For any $u > 0$ and $\varepsilon > 0$, we have

$$P_\lambda(r_1 > u) \leq P_\lambda(H_{[cu]} \leq r_1) + P_\lambda(H_{[cu]} > u),$$

where $H_k := \inf\{n : d(Z_n) = k\}$ for $k \in \mathbb{N}$. By Proposition 2.3 there exists a constant $c_\lambda > 0$ such that

$$P_\lambda(H_{[cu]} \leq r_1) \leq P_\lambda(d(Z_{r_1}) > [\varepsilon u]) \leq e^{-c_\lambda \varepsilon u}.$$

When $\lambda \in (0,1)$, the exponential decay of $P_\lambda(H_{[cu]} > u)$ immediately follows from the coupling with a biased random walk on $\mathbb{Z}_+$. □

In what follows, we will need uniform moment estimates for regeneration times.

**Proposition 2.6.** Assume $p_0 = 0$. For any $\lambda \in (0,1)$, there exist constants $0 < t_\lambda < \lambda \wedge (1 - \lambda)$, $\eta_\lambda > 0$ and $C_\lambda > 0$ such that

$$P_{\lambda'}(\tau_1 > n) \leq C_\lambda e^{-\eta_\lambda n},$$

$$P_{\lambda'}(\tau_2 - \tau_1 > n) \leq C_\lambda e^{-\eta_\lambda n},$$

for any $\lambda' \in (\lambda - t_\lambda, \lambda + t_\lambda)$ and any $n \geq 1$.

**Proof.** We first prove the claim for $\tau_1$. We will use the following Girsanov formula: let $T$ be a rooted infinite tree without leaves and $(\mathcal{F}_n(T))_{n \geq 0}$ be the filtration on the probability space $(\Omega(T), \mathcal{F}(T), P^\lambda)$ generated by the $\lambda$-biased random walk $(Z_n)$ on $T$. Then for $(\mathcal{F}_n(T))$-stopping time $S$, $\mathcal{F}_S(T)$-measurable function $F : \hat{\Omega}(T) \to \mathbb{R}$ and $h \in \mathbb{R}$, we have

$$E_{\lambda + h}^T \left[ F((Z_k)_{k \geq 0}) \right] = E^T_\lambda \left[ F((Z_k)_{k \geq 0}) \prod_{i=1}^S A_{\lambda + h}(Z_{i-1}, Z_i) \right].$$

(2)
By Lemma 2.5 for fixed $\lambda \in (0,1)$ we can find a constant $\kappa_\lambda > 0$ and $C_\lambda > 0$ such that

$$E_\lambda[\exp(\kappa_\lambda \tau_1)] < C_\lambda.$$  

By using the strong Markov property, we get

$$P_\lambda[S_n < \infty] = \int \mathbb{P}(dT) P_\lambda^T \left[ U_{n-1} < \infty | S_{n-1} < \infty \right] P_\lambda^T \left[ S_{n-1} < \infty \right]$$

$$= \int \mathbb{P}(dT) \mathcal{E} \left[ P_{\lambda,Z_{S_{n-1}}^{T},T} \left[ Z_k \notin T(Z_0) \text{ for some } k \geq 1 \right] \right] P_\lambda^T \left[ S_{n-1} < \infty \right].$$

In the first equality, we use the fact that $S_k \sim \infty$ a.s. on the event $\{U_k < \infty\}$ for any $k \geq 1$. Since $S_{n-1}$ is what is called a fresh epoch in [LPP2], Lemma 3.2 in [LPP2] implies

$$\int \mathbb{P}(dT) \mathcal{E} \left[ P_{\lambda,Z_{S_{n-1}}^{T},T} \left[ Z_k \notin T(Z_0) \text{ for some } k \geq 1 \right] \right] P_\lambda^T \left[ S_{n-1} < \infty \right]$$

$$= E_\lambda \left[ P_{\lambda,Z_{S_{n-1}}^{T},T} \left[ Z_k \notin T(Z_0) \text{ for some } k \geq 1 \right] \right] P_\lambda[S_{n-1} < \infty].$$

By using the strong Markov property again, we obtain

$$E_\lambda \left[ P_{\lambda,Z_{S_{n-1}}^{T},T} \left[ Z_k \notin T(Z_0) \text{ for some } k \geq 1 \right] \right] = \int \mathbb{P}(dT) \mathcal{E} \left[ P_{\lambda,Z_{S_{n-1}}^{T},T} \left[ \sigma_{\varepsilon}(T(Z_{S_{n-1}})) < \infty \right] \right]$$

$$= \int \mathbb{P}(dT) P_{\lambda,c(T)} \left[ \sigma_{\varepsilon}(T) < \infty \right]$$

$$= E_\lambda \left[ \frac{1}{1 + \lambda R_\lambda(T)} \right],$$

where for a rooted tree $T$, $R_\lambda(T)$ denotes the effective resistance of $\lambda$-biased random walk on $T$ between $c(T)$ and $\infty$. In the second equality, we use Lemma 3.2 in [LPP2] again. The last equality easily follows from the relationship between reversible Markov chains and electric networks. Note that Rayleigh’s monotonicity principle implies that the function

$$\lambda \mapsto E_\lambda \left[ \frac{1}{1 + \lambda R_\lambda(T)} \right]$$

is monotonically decreasing on $(0, m)$. Hence for any $\lambda \in (0, 1)$, there exist constants $\kappa_\lambda > 0, t_\lambda' > 0$ and $C_\lambda' > 0$ such that

$$P_{\lambda'}[S_n < \infty] < C_\lambda' \exp(-\kappa_\lambda')$$

(3)

for any $\lambda' \in (\lambda - t_\lambda', \lambda + t_\lambda')$. Since $S_j \leq \tau_1$ on the event $\{S_j < \infty\}$, we have

$$E_\lambda[\exp(\kappa_\lambda S_j) 1_{\{S_j < \infty\}}] < C_\lambda.$$  

By the definition of transition probabilities $A_\lambda(x, y)$, it is easy to see that for $h > -\lambda - 1$ and $x, y \in T$, we have

$$\left| A_{\lambda+h}(x, y) / A_\lambda(x, y) - 1 \right| \leq \left\{ \begin{array}{ll} \frac{|h|^{\nu(x)}}{\lambda(h) + \nu(x)} & \text{when } y = \pi(x), \\
\frac{|h|}{\lambda(h) + \nu(x)} & \text{when } x = \pi(y). \end{array} \right.$$  

(4)

By the inequality (4) and the Girsanov formula (2), for any $\lambda \in (0, 1)$, there exists a constant $t_\lambda' > 0$ such that for any $\lambda' \in (\lambda - t_\lambda', \lambda + t_\lambda')$ we have

$$E_\lambda \left[ \frac{\kappa_\lambda}{2} S_j \right] 1_{\{S_j < \infty\}} = E_\lambda \left[ \frac{3\kappa_\lambda}{4} S_j \right] 1_{\{S_j < \infty\}} \prod_{i=1}^J \frac{A_{\lambda'}(Z_{i-1}, Z_i)}{A_\lambda(Z_{i-1}, Z_i)}$$

$$\leq E_\lambda \left[ \frac{3\kappa_\lambda}{4} S_j \right] 1_{\{S_j < \infty\}} < C_\lambda.$$  

(5)
We set $t_\lambda := t'_\lambda \land t''_\lambda$. For $l$ sufficiently large, using the Cauchy-Schwarz inequality, Jensen’s inequality and strong Markov property, for any $\lambda' \in (\lambda - t_\lambda, \lambda + t_\lambda)$ we obtain
\[
E_{\lambda'} \left[ \exp \left( \frac{\kappa_{\lambda}}{2l} \tau_1 \right) \right] = \sum_{j=1}^{\infty} \left( \int \mathbb{P}(dT) E_{\lambda'} \left[ \exp \left( \frac{\kappa_{\lambda}}{2l} S_j \right) \mid J = j \right] P_{\lambda'}^T [J = j] \right) \leq \sum_{j=1}^{\infty} \left( \int \mathbb{P}(dT) E_{\lambda'} \left[ \exp \left( \frac{\kappa_{\lambda}}{2l} S_j \right) \mid J = j \right]^2 \right)^{1/2} \left( \int \mathbb{P}(dT) P_{\lambda'}^T [J = j]^2 \right)^{1/2} \leq \sum_{j=1}^{\infty} \left( \int \mathbb{P}(dT) E_{\lambda'} \left[ \exp \left( \frac{\kappa_{\lambda}}{2l} S_j \right) \mid J = j \right] \right)^{1/2} \left( \int \mathbb{P}(dT) P_{\lambda'}^T [S_j < \infty] \right)^{1/2} \leq \sum_{j=1}^{\infty} C_{\lambda}^{jj/2} \left( C_{\lambda} e^{-\kappa_{\lambda} j} \right)^{1/2} = \text{a constant depending only on } \lambda.
\]

Recall that $J$ is the unique integer such that $S_J < \infty$ and $U_J = \infty$. The above estimate implies the claim for $\tau_1$. (In the last inequality, we use \[\[\text{ and } \[\text{. The desired estimate of } \tau_2 - \tau_1 \text{ can be shown by using that of } \tau_1 \text{ as follows: by the third claim in Proposition 2.4, we have}
\[
E_{\lambda'} \left[ \exp \left\{ \frac{\kappa_{\lambda}}{2l} (\tau_2 - \tau_1) \right\} \right] \leq E_{\lambda'} \left[ \exp \left( \frac{\kappa_{\lambda}}{2l} \tau_1 \right) \right] \left( \int \mathbb{P}(dT) P_{\lambda',\sigma(T)}^T [\sigma_T = \infty] \right)^{-1} = E_{\lambda'} \left[ \exp \left( \frac{\kappa_{\lambda}}{2l} \tau_1 \right) \right] \left( E_{\lambda} \left[ \frac{1}{1 + \lambda R_{\lambda}(T)} \right] \right)^{-1}.
\]

By Rayleigh’s monotonicity principle, the function
\[
\lambda \mapsto \left( E_{\lambda} \left[ \frac{1}{1 + \lambda R_{\lambda}(T)} \right] \right)^{-1}
\]
is monotonically increasing on $\lambda \in (0, m)$. This implies that $\left( E_{\lambda} \left[ \frac{1}{1 + \lambda R_{\lambda}(T)} \right] \right)^{-1}$ is locally uniform in $\lambda$. Hence we finish the proof of the second claim. The first claim can be proved similarly. \hfill \square

### 2.3 Wald’s identities

In what follows, we will use the following elementary formulas, called Wald’s identities.

**Proposition 2.7.** Let $X_1, \ldots, X_n, \ldots$ be i.i.d. random variables and $N$ be a stopping time. Then, we have the following.

**Wald’s identity** If $E[X_1] < \infty$ and $E[N] < \infty$, then $E[\sum_{i=1}^{N} X_i] = E[X_1]E[N]$.

**Wald’s second identity** If $E[X_1^2] < \infty$ and $E[N] < \infty$, then $E[(\sum_{i=1}^{N} X_i - E[X_1]N)^2] = E[X_1^2]E[N]$.

**Wald’s third identity** If $E[e^{\theta X_1}] < \infty$ for some $\theta \in \mathbb{R}$ and $N$ is bounded, then
\[
E \left[ \frac{e^{\theta \sum_{i=1}^{N} X_i}}{E[e^{\theta X_1}]} \right] = 1.
\]

### 3 Expressions of derivatives of the speed

The following result gives the finite approximation of the derivative.
Proposition 3.1. Assume $\lambda \in (0, m)$. Let $h$ tend to 0 and $n$ tend to $\infty$ in such a way that $h^2 n$ tends to 1. (i.e., $hn \sim n^{1/2}$.) Then
\[
\frac{v_{\lambda+h} - v_\lambda}{h} - \frac{E_{\lambda+h}[d(Z_n)] - E_\lambda[d(Z_n)]}{hn}
\]
tends to 0.

Proof. Define $\eta_n := \inf\{k : \tau_k \geq n\}$ for $k \in \mathbb{N}$, then $\eta_n$ is a stopping time with respect to the filtration generated by random variables $\tau_1$, and $\{\tau_{i+1} - \tau_i\}_{i \geq 1}$. By the definition of $\eta_n$, we have
\[
n \leq \tau_{\eta_n} \leq n + \max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i).
\]
This implies that
\[
|E_\lambda[d(Z_n)] - E_\lambda[d(Z_{\tau_{\eta_n}})]| \leq E_\lambda \left[ \max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i) \right].
\]
By Wald’s identity, we have
\[
n \leq E_\lambda[\tau_{\eta_n}] = E_\lambda[\tau_1] + E_\lambda[\eta_n]E_\lambda[\tau_2 - \tau_1] \leq n + E_\lambda \left[ \max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i) \right],
\]
and
\[
E_\lambda[d(Z_{\tau_{\eta_n}})] = E_\lambda[d(Z_{\tau_1})] + E_\lambda \left[ \sum_{i=1}^{\eta_n-1} (d(Z_{\tau_{i+1}}) - d(Z_{\tau_i})) \right]
\]
\[
= E_\lambda[d(Z_{\tau_1})] + (E_\lambda[\eta_n] - 1) \cdot E_\lambda[d(Z_{\tau_2}) - d(Z_{\tau_1})].
\]
Hence, we get
\[
|E_\lambda[d(Z_n)] - n v_\lambda| \leq |E_\lambda[d(Z_n)] - E_\lambda[d(Z_{\tau_{\eta_n}})]| + |E_\lambda[d(Z_{\tau_{\eta_n}})] - n v_\lambda|
\]
\[
\leq E_\lambda \left[ \max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i) \right] + E_\lambda[d(Z_{\tau_2})] + |E_\lambda[\eta_n] \cdot E_\lambda[d(Z_{\tau_2}) - d(Z_{\tau_1})] - n v_\lambda|.
\]
By (6) and the equality $v_\lambda = E_\lambda[d(Z_{\tau_2}) - d(Z_{\tau_1})]/E_\lambda[\tau_2 - \tau_1]$, we have
\[
|E_\lambda[\eta_n] \cdot E_\lambda[d(Z_{\tau_2}) - d(Z_{\tau_1})] - n v_\lambda| \leq E_\lambda[\tau_1] + E_\lambda \left[ \max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i) \right].
\]
By combining (7) and (8) with Proposition 2.6 we get
\[
|E_\lambda[d(Z_n)] - n v_\lambda| \leq 2 E_\lambda \left[ \max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i) \right] + E_\lambda[\tau_1] + E_\lambda[d(Z_{\tau_2})].
\]
It is obvious that Proposition 2.6 implies that there exists a constant $C_\lambda > 0$ such that
\[
E_\lambda[\tau_1] + E_\lambda[d(Z_{\tau_2})] \leq C_\lambda,
\]
for any $\lambda' \in (\lambda - t_\lambda, \lambda + t_\lambda)$, where $t_\lambda$ is a positive constant given in the statement of Proposition 2.6. In order to complete the proof of Proposition 3.1 it suffices to show that for any $\varepsilon > 0$, there exist constants $c_\lambda > 0$ and $t_\lambda > 0$ such that
\[
E_{\lambda'} \left[ \max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i) \right] \leq c_\lambda n^\varepsilon,
\]
for any $\lambda' \in (\lambda - t_\lambda, \lambda + t_\lambda)$. The estimate (9) can be proved as follows: for $\rho > 0$, we have

$$E_{\lambda'} \left[ \max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i) \right] \leq n^\rho + \sum_{k \geq [n^\rho]} P_{\lambda'} \left( \max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i) \geq k \right).$$

By Proposition 2.6 for $\lambda' \in (\lambda - t_\lambda, \lambda + t_\lambda)$ and sufficiently large $k$, we have

$$P_{\lambda'} \left( \max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i) \geq k \right) = 1 - \{1 - P_{\lambda'} (\tau_2 - \tau_1 \geq k) \}^n \leq 1 - (1 - C_\lambda k^{-l})^n \leq 2C_\lambda nk^{-l}.$$

Thus, for sufficiently large $n$, we get

$$E_{\lambda'} \left[ \max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i) \right] \leq n^\rho + 4C_\lambda n^{-\varepsilon(l-1) + 1}.$$

By taking $l > 0$ sufficiently large, we obtain the estimate (9). Therefore, we have shown that

$$\frac{E_{\lambda+h}[d(Z_n)] - n\nu_{\lambda+h}] - E_{\lambda}[d(Z_n) - n\nu_{\lambda}]}{hn}$$

tends to 0 when $h$ tends to 0 and $n$ tends to $\infty$ in such a way that $h^2n$ tends to 1.

By Proposition 3.1 in order to show the differentiability of the function $\lambda \mapsto \nu_{\lambda}$, it suffices to prove the existence of the limit

$$\lim_{h,n} \frac{E_{\lambda+h}[d(Z_n)] - E_{\lambda}[d(Z_n)]}{hn},$$

where $h$ tends to 0 and $n$ tends to $\infty$ in such a way that $h^2n$ tends to 1. We recall here that the annealed CLT for $\{n^{-1/2}(d(Z_n) - n\nu_{\lambda}) \}_{n \geq 1}$ is shown in [PZ, Corollary 4]. Besides, we will need the following estimate.

Lemma 3.2. For $\lambda \in (0, m)$, we have

$$\sup_n \frac{1}{n} E_{\lambda}[(d(Z_n) - n\nu_{\lambda})^2] < \infty.$$

Proof. We already know that

$$|d(Z_n) - n\nu_{\lambda}| \leq |d(Z_{\tau_m}) - n\nu_{\lambda}| + \max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i),$$

and

$$d(Z_{\tau_m}) - d(Z_{\tau_1}) - n\nu_{\lambda}
= \sum_{i=1}^{\tau_m - 1} (d(Z_{\tau_{i+1}}) - d(Z_{\tau_i}) - E_{\lambda}[d(Z_{\tau_2}) - d(Z_{\tau_1})]) + ((\eta - 1)E_{\lambda}[d(Z_{\tau_2}) - d(Z_{\tau_1})] - n\nu_{\lambda}).$$

Hence, we get

$$E_{\lambda}[(d(Z_n) - n\nu_{\lambda})^2]$$
$$\leq 16 \left\{ E_{\lambda} \left[ \max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i) \right]^2 + E_{\lambda}[d(Z_{\tau_1})^2] + E_{\lambda} \left[ ((\eta - 1)E_{\lambda}[d(Z_{\tau_2}) - d(Z_{\tau_1})] - n\nu_{\lambda})^2 \right] 
+ E_{\lambda} \left\{ \sum_{i=1}^{\tau_m - 1} (d(Z_{\tau_{i+1}}) - d(Z_{\tau_i}) - E_{\lambda}[d(Z_{\tau_2}) - d(Z_{\tau_1})])^2 \right\} \right\}.$$
Wald’s second identity implies
\[
E_\lambda \left( \sum_{i=1}^{n-1} (d(Z_{\tau_{i+1}}) - d(Z_{\tau_i})) \right)^2
\]
\[= E_\lambda \left( d(Z_{\tau_n}) - d(Z_{\tau_1}) - E_\lambda [d(Z_{\tau_2}) - d(Z_{\tau_1})]^2 \right) (E_\lambda[\eta_n] - 1),
\]
thus, by the estimate (6), we have
\[
\sup_n \frac{1}{n} E_\lambda \left( \sum_{i=1}^{n-1} (d(Z_{\tau_{i+1}}) - d(Z_{\tau_i})) \right)^2 < \infty.
\]
It is obvious that \( n^{-1} E_\lambda [d(Z_{\tau_1})^2] \) is bounded in \( n \), and it is not difficult to see that Proposition 2.3 implies \( n^{-1} E_\lambda [(\max_{1 \leq i \leq n}(\tau_{i+1} - \tau_i))^2] \) is also bounded in \( n \). Hence, we get the conclusion if we show
\[
\sup_{n \geq 1} \frac{1}{n} E_\lambda \left( (\eta_n - 1) E_\lambda [d(Z_{\tau_2}) - d(Z_{\tau_1})] - n \nu_\lambda \right)^2 < \infty.
\]
It is shown in Chapter 4 that
\[
E_\lambda[\eta_n^2] = E_\lambda[\eta_n]^2 + O(n) = \frac{n^2}{E_\lambda[\tau_2 - \tau_1]^2} + O(n).
\]
By using the formula \( \nu_\lambda = E_\lambda [d(Z_{\tau_2}) - d(Z_{\tau_1})] / E_\lambda[\tau_2 - \tau_1] \) and the estimate (6), we get
\[
E_\lambda \left( (\eta_n - 1) E_\lambda [d(Z_{\tau_2}) - d(Z_{\tau_1})] - n \nu_\lambda \right)^2
\]
\[= E_\lambda[\eta_n]^2 E_\lambda [d(Z_{\tau_2}) - d(Z_{\tau_1})]^2 - \frac{E_\lambda[d(Z_{\tau_2}) - d(Z_{\tau_1})]^2 n^2}{E_\lambda[\tau_2 - \tau_1]^2} + O(n) = O(n),
\]
which implies the conclusion. \(\square\)

For any \( \lambda \in (0, 1) \), Lemma 3.2 implies the uniform integrability of the sequence \( \{d(Z_n) - n \nu_\lambda \} \) under the annealed measure \( P_\lambda \). On the other hand, by Lemma 2.5 and a standard argument in the renewal theory, we get that for \( \lambda \in (0, 1) \) the sequence \( \{d(Z_n) - n \nu_\lambda \} \) weakly converges to a Gaussian random variable as \( n \) tends to \( \infty \) under the annealed measure \( P_\lambda \). Hence, we have
\[
\lim_{n \to \infty} \frac{E_\lambda[d(Z_n) - n \nu_\lambda]}{\sqrt{n}} = 0,
\]
for any \( \lambda \in (0, 1) \). Thus, in order to prove the differentiability of the speed, we only need to show the existence of the limit
\[
\frac{1}{hn} E_{\lambda+h}[d(Z_n) - n \nu_\lambda], \tag{10}
\]
for any sequence \( h \) and \( n \) such that \( h \to 0 \) and \( n \to \infty \) in such a way that \( h^2 n \to 1 \).

**Remark 3.3.** Though we proved Proposition 3.1 and Lemma 3.2 for \( \lambda \in (0, m) \), we only use them for \( \lambda \in (0, 1) \) in what follows.

### 3.1 The Girsanov formula

In this subsection, we will relate the quantities \( E_\lambda[d(Z_n) - n \nu_\lambda] \) and \( E_{\lambda+h}[d(Z_n) - n \nu_\lambda] \) by using the Girsanov formula (2). By the Taylor expansion, there exists \( s = s(x, y) \in [0, 1] \) such that
\[
\log \frac{A_{\lambda+h}(x,y)}{A_\lambda(x,y)} = h B_\lambda(x,y) + \frac{h^2}{2} C_\lambda(x,y) + \frac{h^3}{6} D_{\lambda+h}(x,y),
\]
where

\[
B_\lambda(x, y) = \frac{d}{d\lambda} \log A_\lambda(x, y) = \begin{cases} 
0 & \text{when } x = e, \\
\frac{1}{\lambda} - \frac{1}{\lambda + \nu(x)} & \text{when } y = \pi(x), \\
\frac{1}{\lambda + \nu(x)} & \text{when } x = \pi(y),
\end{cases}
\]

and

\[
C_\lambda(x, y) = \frac{d}{d\lambda} B_\lambda(x, y) = \begin{cases} 
0 & \text{when } x = e, \\
-\frac{1}{\lambda^2} + \frac{1}{(\lambda + \nu(x))^2} & \text{when } y = \pi(x), \\
\frac{1}{(\lambda + \nu(x))^2} & \text{when } x = \pi(y),
\end{cases}
\]

Since \(|B_\lambda(x, y)| \leq \frac{1}{\lambda} + 1, |C_\lambda(x, y)| \leq \frac{1}{\lambda^2} + 1, |D_\lambda(x, y)| \leq \frac{1}{\lambda^3} + 2\), we get

\[
1 = \sum_y A_{\lambda + h}(x, y) = \sum_y A_\lambda(x, y) \exp \left( hB_\lambda(x, y) + \frac{h^2}{2} C_\lambda(x, y) + \frac{h^3}{6} D_{\lambda + sh}(x, y) \right),
\]

\[
= \sum_y A_\lambda(x, y) \left( 1 + hB_\lambda(x, y) + \frac{h^2}{2} B_\lambda^2(x, y) + \frac{h^3}{2} C_\lambda(x, y) + O(h^3) \right).
\]

This implies that for any \(x \in \mathcal{T}\),

\[
\sum_y A_\lambda(x, y) B_\lambda(x, y) = 0, \quad P_\lambda \text{-a.s.},
\]

(11)

\[
\sum_y A_\lambda(x, y) \left( B_\lambda^2(x, y) + C_\lambda(x, y) \right) = 0, \quad P_\lambda \text{-a.s.}
\]

(12)

By using the equality (11), we obtain

\[
E_\lambda^T [B_\lambda(Z_j, Z_{j+1}) | Z_j] = \sum_y B(Z_j, y) A_\lambda(Z_j, y) = 0, \quad P_\lambda^{\mathcal{T}} \text{-a.s.}
\]

This implies

\[
E_\lambda^T [B_\lambda(Z_j, Z_{j+1})] = E_\lambda[ B_\lambda(Z_j, Z_{j+1})] = 0, \quad P_\lambda \text{-a.s.}
\]

(13)

Similarly, we have

\[
E_\lambda[ B_\lambda^2(Z_j, Z_{j+1}) + C_\lambda(Z_j, Z_{j+1})] = 0.
\]

(14)

By using these expressions, we can rewrite (10) as follows.

\[
\frac{1}{hn} E_\lambda [d(Z_n) - n\lambda] = \frac{1}{hn} E_\lambda \left[ (d(Z_n) - n\lambda) \cdot \exp(hP_n - h^2Q_n + R_n) \right],
\]

(15)

where

\[
P_n := \sum_{j=0}^{n-1} B_\lambda(Z_j, Z_{j+1}),
\]

\[
Q_n := \sum_{j=0}^{n-1} \frac{1}{2} B_\lambda^2(Z_j, Z_{j+1}),
\]

\[
R_n := \sum_{j=0}^{n-1} \left\{ h^2 \left( \frac{1}{2} B_\lambda^2(Z_j, Z_{j+1}) + \frac{1}{2} C_\lambda(Z_j, Z_{j+1}) \right) + \frac{h^3}{6} D_{\lambda + sh}(Z_j, Z_{j+1}) \right\}.
\]
We now let \( h \) tend to 0 and \( n \) tend to \( \infty \) in such a way that \( h^2 n \) tends to 1. Hence, \( h \sim n^{-1/2} \) as \( n \) tends to \( \infty \). So the limits of \( hP_n \) and \( h^2 Q_n \) should be described by the CLT and the LLN respectively, and the limit of \( R_n \) must be negligible.

**1. The CLT for \( P_n \)** We have already seen that \( E_\lambda[P_n] = 0 \). Besides, by the renewal structure of Galton-Watson trees, we know that \( \{\sum_{j=\tau_i}^{\tau_{i+1}} B_\lambda(Z_j, Z_{j+1})\}_{i \geq 1} \) are i.i.d. random variables. By Wald’s identity, we get

\[
E_\lambda \left[ \sum_{j=0}^{\tau_{n-1}} B_\lambda(Z_j, Z_{j+1}) \right] = E_\lambda \left[ \sum_{j=0}^{\tau_{n-1}} B_\lambda(Z_j, Z_{j+1}) \right] + E_\lambda \left[ \sum_{j=\tau}^{\tau_{n-1}} B_\lambda(Z_j, Z_{j+1}) \right],
\]

On the other hand, we have

\[
E_\lambda \left[ \sum_{j=0}^{\tau_{n-1}} B_\lambda(Z_j, Z_{j+1}) \right] = E_\lambda \left[ \sum_{j=\tau}^{\tau_{n-1}} B_\lambda(Z_j, Z_{j+1}) \right] - E_\lambda [P_n], \quad (\because \ E_\lambda[P_n] = 0)
\]

\[
= E_\lambda \left[ \sum_{j=\tau}^{\tau_{n-1}} B_\lambda(Z_j, Z_{j+1}) \right] - E_\lambda \left[ \sum_{j=0}^{n} B_\lambda(Z_j, Z_{j+1}) \right], \quad (\because \ \text{the definition of } P_n)
\]

\[
= E_\lambda \left[ \sum_{j=\tau}^{\tau_{n-1}} B_\lambda(Z_j, Z_{j+1}) \right] \leq \left( \frac{1}{\lambda} + 1 \right) E_\lambda [\max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i)], \quad (\because \ |B_\lambda(x, y)| \leq \frac{1}{\lambda} + 1)
\]

Since it is shown in the estimate 2 that \( E_\lambda[\eta_n] \) grows linearly in \( n \), we obtain by the above estimate that \( E_\lambda \left[ \sum_{j=\tau_1}^{\tau_2} B_\lambda(Z_j, Z_{j+1}) \right] = 0 \). This equality together with the moment estimate for \( \tau_2 - \tau_1 \) implies the annealed CLT for \( P_n \).

**2. The LLN for \( Q_n \)** We know that

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{\tau_{n-1}} B_\lambda^2(Z_j, Z_{j+1}) = E_\lambda \left[ \sum_{j=\tau_1}^{\tau_{n-1}} B_\lambda^2(Z_j, Z_{j+1}) \right], \quad P_\lambda\text{-a.s.}
\]

Hence,

\[
\lim_{n \to \infty} \frac{1}{\eta_n} \sum_{j=0}^{\tau_{n-1}} B_\lambda^2(Z_j, Z_{j+1}) = E_\lambda \left[ \sum_{j=\tau_1}^{\tau_{n-1}} B_\lambda^2(Z_j, Z_{j+1}) \right], \quad P_\lambda\text{-a.s.}
\]

Combining the above with formulas

\[
\eta_n \leq n + \max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i) \quad \text{and} \quad \lim_{n \to \infty} \eta_n/n = 1/E_\lambda[\tau_2 - \tau_1], \quad P_\lambda\text{-a.s.,}
\]

we get

\[
\lim_{n \to \infty} \frac{1}{n} Q_n = \frac{1}{2E_\lambda[\tau_2 - \tau_1]} E_\lambda \left[ \sum_{j=\tau_1}^{\tau_{n-1}} B_\lambda^2(Z_j, Z_{j+1}) \right], \quad P_\lambda\text{-a.s.}
\]
3. The estimate for $R_n$  

Note that we have

$$\left| \sum_{j=0}^{n-1} \frac{h^3}{6} D_{\lambda+h} (Z_j, Z_{j+1}) \right| \leq \frac{h}{3} \left( \frac{1}{\lambda^3} + 1 \right),$$

and it is already shown that

$$E_{\lambda} \left[ \sum_{j=0}^{n-1} \left( B_{\lambda}^2(Z_j, Z_{j+1}) + C_{\lambda}(Z_j, Z_{j+1}) \right) \right] = 0.$$

Hence, by using the similar argument to the above one, we see that

$$\lim_{n \to \infty} R_n = 0, \ P_\lambda\text{-a.s.}$$

Note also that $R_n$ satisfies the following uniform estimate.

$$|R_n| \leq h^2 n \left( \frac{1}{\lambda} + 1 + \frac{1}{2} \left( \frac{1}{\lambda^2} + 1 \right) \right) + \frac{h}{3} \left( \frac{1}{\lambda^3} + 1 \right),$$

$$\leq 2 \left( \frac{1}{\lambda} + 1 + \frac{1}{2} \left( \frac{1}{\lambda^2} + 1 \right) \right) + \frac{1}{\lambda^3} + 1.$$ 

4. The joint CLT for $\left( n^{-1/2} (d(Z_n) - n v_{\lambda}), n^{-1/2} P_n \right)_{n \geq 1}$  

We have already given a proof of the annealed CLT for sequences of random variables $\left\{ n^{-1/2} (d(Z_n) - n v_{\lambda}) \right\}_{n \geq 1}$ and $\left\{ n^{-1/2} P_n \right\}_{n \geq 1}$, but in what follows, we need the joint CLT for the sequence of random vectors $\left( n^{-1/2} (d(Z_n) - n v_{\lambda}), n^{-1/2} P_n \right)$.

By Proposition 3.4 for any $\lambda \in (0, m)$, under $P_\lambda$

$$\left( d(Z_{\tau_i}) - d(Z_{\tau_i}), \sum_{j=\tau_i}^{\tau_{i+1}-1} B_{\lambda}(Z_j, Z_{j+1}) \right)_{i \geq 1}$$

are i.i.d. $\mathbb{R}^2$-valued random variables. This fact together with the moment estimate of regeneration times immediately implies the following result.

**Proposition 3.4.** For any $\lambda \in (0, m)$, the sequence $\left\{ (n^{-1/2} (d(Z_n) - n v_{\lambda}), n^{-1/2} P_n) \right\}_{n \geq 1}$ under $P_\lambda$ weakly converges to a centered Gaussian random vector $(X, Y)$ with the covariance matrix $\Sigma_{\lambda} := (\sigma_{ij}(\lambda))_{0 \leq i,j \leq 1}$ that is given by

$$\sigma_{00}(\lambda) := \frac{1}{E_{\lambda}[\tau_2 - \tau_1]} \left( d(Z_{\tau_2}) - d(Z_{\tau_1}) \right),$$

$$\sigma_{11}(\lambda) := \frac{1}{E_{\lambda}[\tau_2 - \tau_1]} E_{\lambda}\left[ \sum_{j=\tau_1}^{\tau_2-1} B_{\lambda}^2(Z_j, Z_{j+1}) \right],$$

$$\sigma_{10}(\lambda) = \sigma_{01}(\lambda) := \frac{1}{E_{\lambda}[\tau_2 - \tau_1]} E_{\lambda}\left[ \left( d(Z_{\tau_2}) - d(Z_{\tau_1}) \right) \cdot \left( \sum_{j=\tau_1}^{\tau_2-1} B_{\lambda}(Z_j, Z_{j+1}) \right) \right].$$

Note that in the last formula, we use $E_{\lambda}\left[ \sum_{j=\tau_1}^{\tau_2-1} B_{\lambda}(Z_j, Z_{j+1}) \right] = 0$.

3.2 The proof of the differentiability of the speed

In this subsection, we will prove that the function $\lambda \mapsto v_{\lambda}$ is a differentiable function on $(0, 1)$, and the derivative is expressed as covariance of some 2-dimensional Gaussian random vector.
Theorem 3.5. The function $\lambda \mapsto v_\lambda$ is differentiable on $(0, 1)$. Moreover, the derivative of the speed $v'_\lambda$ can be expressed as the covariance of a 2-dimensional Gaussian random variables, namely there exists a centered 2-dimensional Gaussian random vector $(X, Y)$ with the covariance matrix $\Sigma_\lambda$ such that $v'_\lambda = E_\lambda[XY]$.

Proof. Recall that for the identity [15], we let $h$ tend to 0 and $n$ tend to $\infty$ in such a way that $h^2n$ tends to 1. Then, once we justify that we can pass to the limit in (15), by using Proposition 3.4 we will get

$$\frac{1}{hn}E_\lambda\left[\left(d(Z_n) - nv_\lambda\right) \cdot \exp(hP_n - h^2Q_n + R_n)\right]$$

$$\rightarrow E_\lambda\left[X \exp\left(Y - \frac{1}{2} Var(Y)\right)\right],$$

where $(X, Y)$ is a centered Gaussian vector with the covariance matrix $\Sigma_\lambda$. Since it is shown in [10] that

$$\text{Var}(Y) = \frac{1}{E_\lambda[\tau_2 - \tau_1]} E_\lambda\left[\sum_{j=\tau_1}^{\tau_2-1} B^2_\lambda(Z_j, Z_{j+1})\right],$$

the above convergence implies

$$\frac{v_{\lambda+h} - v_\lambda}{h} \rightarrow E_\lambda\left[X \exp\left(Y - \frac{1}{2} Var(Y)\right)\right] = E_\lambda[XY] = \sigma_{10}(\lambda).$$

Note that the integration by parts formula for Gaussian laws is used in the last step. In order to justify the step (17), it suffices to show the uniform integrability of

$$\left\{\frac{1}{hn}\left(d(Z_n) - nv_\lambda\right) \cdot \exp(hP_n - h^2Q_n + R_n)\right\}_{n \geq 1}.$$  

By Hölder’s inequality, we have

$$E_\lambda\left[\left(\frac{1}{hn}\left(d(Z_n) - nv_\lambda\right) \cdot \exp(hP_n - h^2Q_n + R_n)\right)^{6/5}\right]$$

$$\leq E_\lambda\left[\frac{1}{(hn)^2}\left(d(Z_n) - nv_\lambda\right)^2\right]^{3/5} E_\lambda[\exp(3hP_n - 3h^2Q_n + 3R_n)]^{2/5}$$

In Proposition 3.2 we have already seen that $E_\lambda\left[\frac{1}{(hn)^2}\left(d(Z_n) - nv_\lambda\right)^2\right]$ is bounded in $n$. We will prove that $E_\lambda[\exp(3hP_n - 3h^2Q_n + 3R_n)]$ is also bounded in $n$. Recall that $Q_n \geq 0$ and there exists a constant $C_\lambda > 0$ such that $|R_n| \leq C_\lambda$ for any $n \geq 1$. Thus, we get

$$E_\lambda[\exp(3hP_n - 3h^2Q_n + 3R_n)] \leq \exp(3C_\lambda)E_\lambda[\exp(3hP_n)].$$

Hence, it suffices to show that $E_\lambda[\exp(3hP_n)]$ is bounded in $n$ when $h$ tends to 0 and $n$ tends to $\infty$ in such a way that $h^2n$ is bounded. Since $\left\{\sum_{j=\tau_1}^{\tau_i-1} B_\lambda(Z_j, Z_{j+1})\right\}_{i \geq 1}$ are i.i.d. random variables with finite exponential moment when $p_0 = 0$ and $\lambda \in (0, 1)$, with a slight modification of the proof of Lemma 3.7 in [M], we know that $E_\lambda[\exp(ahP_{\tau_1-1})]$ is bounded in $n$ for any $a > 0$, and

$$E_\lambda[\exp(3hP_{\tau_1-1})] = E_\lambda[\exp(ahP_{\tau_1-1})]E_\lambda[\exp(ah \sum_{j=\tau_1}^{\tau_2-1} B_\lambda(Z_j, Z_{j+1}))]^{n-1}.$$  

Wald’s third identity implies that

$$E_\lambda\left[\frac{\exp(ahP_{\tau_1})}{E_\lambda[\exp(ah \sum_{j=\tau_1}^{\tau_2-1} B_\lambda(Z_j, Z_{j+1}))]^{n}}\right] = 1, \text{ for any } n \geq 1.$$
Since the denominator of the integrand in the above equality is bounded in $n$, $E_\lambda[\exp(\alpha h P_{\tau_n})]$ is also bounded in $n$ for any $\alpha > 0$. By Hölder’s inequality, we get

$$E_\lambda[\exp(3h P_n)] = E_\lambda[\exp(3h P_{\tau_n}) \exp(-3h \sum_{j=n+1}^{\tau_n} B_\lambda(Z_j, Z_{j+1}))],$$

$$\leq E_\lambda[\exp(6h P_{\tau_n})]^{1/2} E_\lambda[\exp(-6h \sum_{j=n+1}^{\tau_n} B_\lambda(Z_j, Z_{j+1}))]^{1/2},$$

$$\leq E_\lambda[\exp(6h P_{\tau_n})]^{1/2} E_\lambda[\exp(-6h(\lambda^{-1} + 1)(\tau_{\tau_n} - n))]^{1/2},$$

$$\leq E_\lambda[\exp(6h P_{\tau_n})]^{1/2} E_\lambda[\exp(-6h(\lambda^{-1} + 1)(\max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i)))]^{1/2},$$

and it immediately follows from Proposition 2.5 that

$$E_\lambda[\exp(-6h(\lambda^{-1} + 1)(\max_{1 \leq i \leq n} (\tau_{i+1} - \tau_i)))]$$

is bounded in $n$. (Note that $h \sim n^{-1/2}$.)

\[ \square \]

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