Phenomenological description of the vortex density in rotating BEC superfluids

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Abstract

We propose a phenomenological equation for the vortex line density in rotating Bose-Einstein condensates as a function of the angular speed. This equation provides a simple description of the gross features of the increase in vortex number from the appearance of the first vortex to the theoretical rigid-body result for high vortex density, and allows one to compare with analogous situations in superfluid helium, after the suitable changes in the relevant parameters are made.

1 Introduction

The study of quantized vorticity in superfluid helium 4 has been a relevant topic in superfluidity for several decades [1]. In the last decade, much interest has been focused also on the study of quantized vortices in rotating Bose-Einstein condensates of alkali-atomic gases [2]–[10], for several geometries of the confining potential. Here we focus our attention on the vortex density \( L \) per unit transversal area in rotating BEC in terms of the angular velocity \( \Omega \), between the lower critical velocity for the appearance of the first vortex to fast rotations, where vortex discreteness may be neglected, and the system is expected to be characterized by the rigid body result \( L = 2\Omega/\kappa \), with \( \kappa = h/m \) being the vorticity quantum, \( h \) Planck’s constant and \( m \) the mass of the particles.

To compare the behavior of vortices in both systems, helium 4 and BEC, it is interesting to show macroscopic analogies and differences arising from the respective microscopic features of these systems, as for instance the low density, the long coherence length, the size of the vortex radius, and the compressibility of atomic BEC as compared with the high density, the extremely thin vortices, and the incompressibility of liquid helium II. Furthermore, due to very low temperatures, all particles of BEC participate in the superfluid component, whereas in liquid helium only a part of the particles are in the superfluid component and the other ones in the normal viscous component. For the mentioned comparison, it may be useful to explore

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phenomenological frameworks common to both systems in which the macroscopic behavior may be put in a common ground.

The aim of this paper is to adapt an evolution equation for the vortex line density proposed for rotating superfluid helium \[11, 12\] to the vortex density in BEC superfluids, by taking into account the physical differences in the respective situations. This is done in Section 2, whereas Section 3 is devoted to the study of the solutions of the equation and of their stability, from which the vortex density is obtained in terms of the angular velocity and the corresponding vortex number is calculated. In Section 4 we discuss a physical interpretation of the macroscopical equation. In Section 5 we discuss a special situation where the number of vortices seems to have a maximum instead of being a monotonically increasing function of the angular velocity.

2 Evolution equation for vortex density

Recently, two of us proposed a phenomenological equation for the evolution of the vortex line density \[L\] per unit volume in superfluid helium 4 under rotation and counterflow \[11, 12\]. In the case of pure rotation in a cylindrical container, such an equation takes the form \[12\]

\[
\frac{dL}{dt} = -\beta \kappa L^2 + \left[ \alpha_2 \sqrt{\kappa \Omega} - \alpha_3 \frac{\kappa}{d} \right] L^{3/2} - \left[ \beta_1 \Omega - \beta_3 \sqrt{\kappa \Omega} \right] \frac{\kappa}{d} L + \alpha_4 \frac{\kappa}{d^2} L, \tag{2.1}
\]

where \(d\) is the diameter of the container, and \(\alpha_i\) and \(\beta_i\) dimensionless phenomenological coefficients, whose values are obtained by fitting experimental data.

Here, we aim to explore the application of a suitably modified form of this equation to the description of the vortex density in rotating BEC in terms of the rotation frequency. The consequences of equation (2.1) have been studied in detail for superfluid \(^4\)He \[12\], where the situation without vortices \(L = 0\) becomes unstable for values of \(\Omega\) higher than a critical value \(\Omega_1 = (\beta_3/2\beta_2)^2 \kappa/d^2\), and the vortex density \(L\) increases for increasing \(\Omega\) beyond the critical value \(\Omega_1\). For high values of \(\Omega\), it follows from (2.1) that the steady-state solution \(L\) exhibits the well-known behavior \(L = 2\Omega/\kappa\) \[1\]. Thus, equation (2.1) describes the increase from one single vortex to many vortices, and it is more suitable than taking directly the assumption \(L = 2\Omega/\kappa\). Note that for parallel vortices of the same height, \(L\) is simply the number of vortices per unit transversal area, i.e. \(L = N/A\), with \(A\) the total transversal area of the system. Therefore, in this situation the vortex line density per unit volume coincides with the vortex line density per unit transversal area, to which we will pay attention here for rotating BEC.

In contrast with superfluid helium, Bose-Einstein condensates are not confined in material cylindrical containers, but in optical or magnetic potential traps, to which one may add a magnetic rotation in analogy with rotating container experiments for superfluid helium. It is often assumed that the trap has the axially symmetric form \(V(r) = \frac{1}{2}m\Omega_t^2 r^2\) with \(r\) the transversal radius and \(\Omega_t\) the frequency trap, which is a constant characterizing the confining potential, or an ellipsoidal form \(V(x, y, z) = m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2\), where \(\omega_x\), \(\omega_y\) and \(\omega_z\) are the oscillator frequencies characterizing the potential of the trap in the three spatial directions, and that the \(X\) and \(Y\) axes rotate with a frequency \(\Omega\), fixed by the frequency of variation of a magnetic field along the \(X\) and \(Y\) axes \[6\].

Comparing with the rigid cylindrical containers in usual experiments on rotating \(^4\)He superfluids, in Bose-Einstein condensates one has not a definite value for the diameter \(d\) of the
container. Here we will take instead of it the characteristic size of the trap, which controls the spatial extent of the one-particle ground state of the harmonic potential and which is called the oscillator length, $a_{osc} \approx (\kappa/\omega_\perp)^{1/2}$. It is also useful to recall, for future applications, that in axially symmetric traps the stationary number particle density in terms of the radius in the Thomas-Fermi approximation is \[ n(r) = \frac{\hbar}{\pi} \omega_\perp \left( \frac{Na}{l_z} \right)^{1/2} \left[ 1 - \frac{r^2}{R_{TF}^2} \right], \quad (2.2) \]

with $N$ the total number of particles of the condensate, $a$ the s-wave scattering length appearing in the Gross-Pitaevskii equation [7, 10], $l_z$ the average extent of the condensate on the $z$ direction — taken as the direction of the axis of rotation —, $\omega_\perp$ the average transversal frequency of the trap, given by $\omega_\perp^2 = (\omega_x^2 + \omega_y^2)/2$, or simply by $\Omega$ in symmetric traps, and $R_{TF}$ is the Thomas-Fermi radius of the condensate, given by \[ R_{TF} = 2 \left( \frac{Na}{\kappa^2 \pi \omega_\perp} \right)^{1/4}, \quad (2.3) \]

To apply equation (2.1) to rotating BEC, we identify $\delta$ as $a_{osc}$, and (2.1) becomes

\[
\frac{dL}{dt} = -\beta \kappa L^2 + \alpha_2 \sqrt{\kappa \omega_\perp} \left[ \sqrt{\frac{\omega_\perp}{\Omega}} - \frac{\alpha_3}{\alpha_2} \right] L^{3/2} - \beta_1 \omega_\perp \left[ \frac{\beta_3}{\beta_1} \sqrt{\frac{\omega_\perp}{\Omega}} + \frac{\alpha_4}{\beta_1} \right] L, \quad (2.4)
\]

where $\Omega = \Omega/\omega_\perp$ is the dimensionless angular velocity. A further relevant difference with the usual rotating cylinder used in superfluid helium may be the anisotropy of the trap. In this case the coefficients of (2.4) become a function of the eccentricity $\epsilon$, defined as $\epsilon = (\omega_y^2 - \omega_x^2)/(\omega_x^2 + \omega_y^2)$. We do not pretend that equation (2.4) yields a full explanation for the dynamics of the BEC, which would require to include into the description other collective modes, but we aim to identify some of the most salient phenomenological analogies and differences with superfluid helium, which should, in the future, to be understood from a microscopic basis of the Gross-Pitaevskii equation.

3 Stationary solutions of the evolution equation

Now, we study the stationary solutions of equation (2.4) and their corresponding domain of stability. The stationary solutions are

\[ L = 0, \quad (3.5) \]

\[ L_{\pm}^{1/2} = \frac{\sqrt{\omega_\perp}}{2\beta \sqrt{\kappa}} \left( \alpha_2 \sqrt{\frac{\omega_\perp}{\Omega}} - \alpha_3 \right) \pm \frac{\omega_\perp}{4\kappa \beta^2} \left[ (\alpha_2 \sqrt{\frac{\omega_\perp}{\Omega}} - \alpha_3)^2 - 4\kappa \beta_1 \left( \frac{\beta_3}{\beta_1} \sqrt{\frac{\omega_\perp}{\Omega}} + \frac{\alpha_4}{\beta_1} \right) \right]. \quad (3.6) \]

The first solution ($L = 0$) corresponds to the absence of vortices, and the second one describes an increase in the vortex density for increasing $\Omega$. Regarding the stability of solution $L = 0$ (no vortices), and according to equation (2.4), we consider the evolution equation for the perturbation $\delta L$ around $L = 0$, which is

\[ \frac{\delta L}{dt} = -\beta_1 \omega_\perp \left[ \frac{\beta_3}{\beta_1} \sqrt{\frac{\omega_\perp}{\Omega}} + \frac{\alpha_4}{\beta_1} \right] \delta L. \quad (3.7) \]
From this equation we can establish that the solution $L = 0$ is stable if
\[
\beta_1 \Omega - \beta_3 \sqrt{\Omega} + \alpha_4 \geq 0.
\] (3.8)

In superfluid helium \[12\], a satisfactory choice of the parameters is to assume $\beta_3^2 = 4\alpha_4\beta_1$, in which case the critical value of $\Omega$ is
\[
\Omega_c = \left( \frac{\beta_3}{2\beta_1} \right)^2 = \frac{\alpha_4}{\beta_1},
\] (3.9)
and $L = 0$ is stable for $\Omega < \Omega_c$. For symmetric traps, the critical value in BEC is $\Omega_c = 1/\sqrt{2}$, corresponding to half the frequency of the quadrupole mode \[5\]–\[9\] which is a collective surface mode with angular momentum $l = 2$; when resonantly excited, quantized vortices formed on the surface come into the condensate. In eccentric traps, the value of $\Omega_c$ depends on the eccentricity $\epsilon$, and it is close to $\Omega(\epsilon) \approx \frac{1}{\sqrt{2}} - \epsilon$ (some authors \[7\] give this value whereas other ones \[6\] indicate $\Omega(\epsilon) \approx \frac{1}{\sqrt{2}} - 0.91\epsilon$). In any case, the critical value of $\Omega$ for the formation of the first vortices is lower in eccentric traps than in symmetric traps, which reflects the fact that the finite energy barrier that vortices must overcome to move into the condensate is lowered if the condensate is elongated.

For $\Omega \geq \Omega_c$, the behavior of $L$ in terms of $\Omega$ is described by (3.6), which tends to $L = 2\Omega_{\perp}/\kappa$ for high values of $\Omega$.

![Figure 1: Vortex density $L$ in a BEC in terms of the dimensionless angular velocity for two values of the eccentricity $\epsilon = 0$ (continuous line) and $\epsilon = 0.037$ (dashed line). The graphics are plotted using the value $\omega_{\perp}/2\pi = 219$ Hz for the the average transversal frequency of the trap from Ref. \[4\].](image)

Under the previous assumption $\beta_3^2 = 4\alpha_4\beta_1$, equation (2.4) may be rewritten as
\[
\frac{dL}{dt} = -\beta_1 L \Omega + \sqrt{\kappa \Omega_{\perp}} \left[ \sqrt{\Omega} - \sqrt{\Omega_c} \right] L^{3/2} - \beta_1 \omega_{\perp} \left[ \sqrt{\Omega} - \sqrt{\Omega_c} \right]^2 L,
\] (3.10)
whose stable solution for $\Omega > \Omega_c$ is simply

$$L^{1/2} = \frac{\alpha_2 \sqrt{\omega_\perp}}{2\beta} \left[ 1 + \sqrt{1 - \frac{4\beta \beta_1}{\alpha_2^2}} \right] \frac{\sqrt{\Omega} - \sqrt{\Omega_c}}{\sqrt{\kappa}} = \sqrt{2\omega_\perp} \frac{\sqrt{\Omega} - \sqrt{\Omega_c}}{\sqrt{\kappa}}. \quad (3.11)$$

The value of the combination of coefficients appearing in the prefactor of (3.11) is dictated by the fact that for $\Omega \gg \Omega_c$, $L$ tends to the rigid body result $L \approx 2\Omega \omega_\perp / \kappa$. In Fig. 1 we plot $L$ in terms of $\Omega$ for a given $\omega_\perp$ and for two values of eccentricity of the potential, $\epsilon = 0$ (continuous line) and $\epsilon = 0.037$ (dashed line) [4]. As expected, graphics of $L$ in figure 1 confirm the fact that when the eccentricity increases the first vortex in BEC appears for a smaller value of rotation. Expression (3.11) generalizes the well-known expression $L = 2\Omega / \kappa$ and, from a practical perspective, it is more useful than it, because in BEC, in contrast to helium 4, the angular velocity cannot be increased indefinitely, because of the effects of the centrifugal force. Note, of course, that these lines are not completely realistic, and only discontinuous values of them, corresponding to integer numbers of vortices, are actually meaningful.

![Figure 2](image-url): The number of vortex $N_V$ in a BEC in terms of the dimensionless angular velocity for two values of the eccentricity $\epsilon = 0$ (continuous line) and $\epsilon = 0.037$ (dashed line). The graphics are plotted using the values $\omega_\perp / 2\pi = 219$ Hz, $N = 10^5$, $a = 8$ $\mu$m and $l_z = 49$ $\mu$m from Ref. [4]. The horizontal lines indicate the ranges of $\Omega$ to which the number of vortices corresponds: the first range corresponds to 1 vortex while the second one corresponds to 2/4 vortices.

In many analyses, one measures the number $N_V$ of the vortices rather than their areal density. To go from $L$ to $N_V$ one must multiply $L$ times the transversal area of the condensate, taking into account that the value of its radius depends on the rotation rate, because of the centrifugal force [10]. One has

$$N_V = L \pi R^2 = \frac{L \pi R_{TF}^2}{1 - (\Omega / \omega_\perp)^2} = \frac{L \pi R_{TF}^2}{1 - \Omega^2}, \quad (3.12)$$

with $R_{TF}$ the Thomas-Fermi radius [23]. Note, then, that for a given value of the angular frequency the total number of vortices increases when the number of particles increases, pro-
portionally to \((N a/l_z)^{1/2}\); thus, it depends not directly on \(N\), but rather on the number of particles per unit transverse length. A second aspect is that \(\Omega\) cannot increase indefinitely, because at \(\Omega = \omega_\perp\) the centrifugal force is so high that the condensate is flattened out and it may disperse, unless a suitable quartic potential is added to the harmonic trap. However, since both \(L\) and \(R\) increase with \(\Omega\), it follows that as far as the condensate exists, the total number of vortices given in (3.12) increase as function of the angular velocity, as shown in Fig. 2, where \(N_V\) is plotted for two values of the eccentricity \(\epsilon = 0\) (continuous line) and \(\epsilon = 0.037\) (dashed line). There a direct comparison between the graphics of our model and the experimental data of Ref. [4] is carried out. Note, incidentally, that the relation between \(L\) and \(N_V\) is very different in BEC than in confined superfluid helium, where the number of vortex is

\[
N = L \pi R^2 \approx \frac{2\pi R^2 \Omega}{\kappa} \left[1 - \sqrt{\frac{\Omega_c}{\Omega}}\right],
\]

with \(R\) the radius of the cylinder, \(\kappa\) the vorticity quantum — which is different with respect to that of BEC — and \(\Omega_c\) the critical rotation velocity for the appearing of the first vortex in rotating helium II. The difference between two relations (3.12) and (3.13)) is caused by the fact that in the latter the radius of the sample always coincides with the radius of the container and it does not depend on the angular velocity, whereas in BEC the change in the radius with \(\Omega\) is very relevant.

To compare the behavior of vortices in rotating Bose-Einstein condensates and in rotating superfluid helium 4, it is also worth to stress that the characteristic values of the angular velocities and of the vortex densities are rather different in both cases; in typical experiments with helium the angular frequency is less than 10 rad/s, whereas in BEC may be of the order of 300 rad/s; and the vortex line densities are of the order of \(10^4\) vortex lines/cm\(^2\) for superfluid helium and of \(10^7\) vortex lines/cm\(^2\) in BEC. However, we should not compare the actual values of \(\Omega\) but characteristic dimensionless rotation velocity. In BEC, it is defined as \(\Omega \equiv \Omega/\omega_\perp\), whereas for superfluid helium it is given by \(\Omega \equiv \Omega d^2/\kappa\); for \(^4\)He, the value of the quantum of vorticity \(\kappa\) is \(\kappa \approx 9.97 \times 10^{-4}\) cm\(^2\)s\(^{-1}\); thus, the value of the dimensionless velocity corresponding to 10 rad/s for helium in a container of diameter 1cm, would be of the order of \(10^5\), much higher than those explored in Bose-Einstein condensates. This difference can also be related to the fact that the quanta of vorticity of helium II and BEC are very different, as the quantum of vorticity in rubidium BEC is about 20 times lower than in helium II because of the lower atomic mass of helium with respect to that of rubidium.

The equation (2.4) has been applied to superfluid helium in rotating cylinders. It would be interesting to have information for rotating containers with ellipsoidal cross section, in order to explore the influence of the eccentricity on the vortex line density, but we are not aware of experimental results in this problem. A parallel possibility would be to compare rotating containers with rectangular cross section [13].

4 A physical interpretation of the evolution equation

To have a more explicit understanding of equation (2.4) or (3.10) we recall that in rotating Bose-Einstein condensates and Helium II, the vortices are produced at the surface of the condensate and penetrate into it pulled by the rotating drive. The repulsive interaction amongst them tends to push them apart because vortices rotating in the same direction experience
an effective repulsive interaction \([4, 5]\). This competition between centripetal driving and repulsive force yields eventually to a regular vortex lattice. This idea suggests us to identify the terms appearing in equation (2.4) in terms of these features. In this way, we propose to interpret (2.4) as

\[
\frac{dL}{dt} = [v_{\text{in}} - v_{\text{out}}]L^{3/2},
\]

with \(v_{\text{in}}, v_{\text{out}}\) being the respective ingoing and outgoing vortex drift velocities, proportional to the respective driving forces. The exponent \(3/2\) for \(L\) in (4.14) is chosen on dimensional grounds, since \(L\) has dimensions of (length\(^{-2}\)). There is not an univocal way to decide to which term, either \(v_{\text{in}}\) or \(v_{\text{out}}\), must be assigned each term in (2.4) or in or (3.10). Here we propose to relate the terms linear in \(L^{3/2}\) to the inflowing flux, because the corresponding velocity would not depend on the vortex density, but only on the rotation. Following this tentative criterion, the respective values of \(v_{\text{in}}\) and \(v_{\text{out}}\) would be

\[
v_{\text{out}} = \beta \kappa L^{1/2} + \beta_1 \omega_\perp \left[ \sqrt{\Omega - \sqrt{\Omega_c}} \right]^2 L^{-1/2},
\]

\[
v_{\text{in}} = \alpha_2 \sqrt{\kappa \omega_\perp} \left[ \sqrt{\Omega - \sqrt{\Omega_c}} \right].
\]

The outgoing velocity depends on the vortex line density: it depends on the inverse of the average vortex separation (which is given by \(L^{-1/2}\)). Thus, (4.15) could be considered as the first terms in a kind of virial expansion of a repulsive force between vortices in terms of powers of \(L^{1/2}\). In vortex arrays in rotating containers, the free energy has a term which may be interpreted as a repulsion force. The ingoing velocity is related to the angular velocity: it is positive at values of \(\Omega\) higher than \(\Omega_c\). This could probably be related to an energy barrier which the vortices must overcome in order to move into the condensate. Then, the interpretation of a dynamical equation for \(L\) for rotating superfluids — BEC or liquid helium — is rather different than for counterflow systems, where it relies on the dynamics of vortex breaking and reconnections, as proposed Schwarz \([14]\), whose microscopic view is not useful for parallel vortex lines. Furthermore, in Schwarz microscopical model for the dynamics of \(L\) under counterflow in superfluid helium, all the volume contributes to the production term of \(L\), because of the elongation of vortex loops, whereas in rotating systems the vortices are produced on the surface. A relevant difference between BEC and superfluid helium may be the extent of vortex interaction, because of the considerable width of vortices in BEC as compared with the atomic width of vortex lines in superfluid helium. This effect should be reflected in the value of coefficient \(\beta\).

5 An anomalous non-monotonic density variation

The usually expected behavior of \(L\) in terms of \(\Omega\) is a monotonical increase as mentioned in Section 3. However, in an experiment by Hodby et al. \([6]\) a different behavior was observed, which we briefly report here for the sake of completeness, although it seems rather exceptional, as it has not been reproduced, to our knowledge. In \([6]\) Hodby et al. studied experimentally the evolution of the number of vortices for a condensate of \(2 \cdot 10^4\) atoms of \(^{87}\)Rb for an oblate geometry \((\omega_\perp < \omega_z)\) within a rotating magnetic elliptical trap. They observed that when the eccentricity is adiabatically ramped from \(\epsilon = 0\) to a given final value \(\epsilon\), there were both a lower
\( \overline{\Omega}_1(\epsilon) \) and an upper \( \overline{\Omega}_2(\epsilon) \) values of rotation rates for which vortices were nucleated; these values depend on the eccentricity, and were given respectively by [6]

\[
\overline{\Omega}_1(\epsilon) \approx 0.71 - 0.91 \epsilon, \quad (5.17)
\]

\[
\Omega_1 = \frac{2}{\Omega_2} \left( \frac{2\Omega_2^2 - 1}{3} \right)^{2/3} = \epsilon. \quad (5.18)
\]

Hodby et al. [6] measured the number of vortices as a function of \( \overline{\Omega} \) for two values of the eccentricity, and found that this has a maximum \((L_{\text{max}}, \overline{\Omega}_{\text{max}})\) within the interval \( \overline{\Omega}_1 \leq \overline{\Omega} \leq \overline{\Omega}_2 \). This information may also be synthesized into (2.4), by identifying its coefficients in terms of \( \overline{\Omega}_1(\epsilon), \overline{\Omega}_2(\epsilon), \overline{\Omega}_{\text{max}}(\epsilon) \) and \( L_{\text{max}}(\epsilon) \).

To describe Hodby’s results one should assume \( \beta_3^2 > 4 \alpha_4 \beta_1 \) instead of the equality of both terms assumed in (3.9). In this case, there will be an instability range for \( L = 0 \) for \( \overline{\Omega} \) between \( \overline{\Omega}_1^{1/2} \) and \( \overline{\Omega}_2^{1/2} \) given by

\[
\frac{\Omega_1}{2} = \frac{\beta_3^2}{2 \beta_1} - \frac{\sqrt{\beta_3^2 - 4 \alpha_4 \beta_1}}{2 \beta_1}, \quad (5.19)
\]

\[
\frac{\Omega_2}{2} = \frac{\beta_3^2}{2 \beta_1} + \frac{\sqrt{\beta_3^2 - 4 \alpha_4 \beta_1}}{2 \beta_1}, \quad (5.20)
\]

from which we deduce immediately that \( \beta_3 / \beta_1 = \sqrt{\overline{\Omega}_1} + \sqrt{\overline{\Omega}_2} \) and \( \alpha_4 / \beta_1 = \sqrt{\overline{\Omega}_1 \overline{\Omega}_2} \).

Now, we rewrite the non zero stationary solution (3.5) as

\[
\sqrt{L} = \frac{\alpha_2 \sqrt{\omega_\perp}}{2 \sqrt{\kappa}} \left( \sqrt{\overline{\Omega}_1} - \sqrt{\overline{\Omega}_2} \right) \left[ 1 \pm \frac{\sqrt{4 \beta_1 \sqrt{\overline{\Omega}_1} - \sqrt{\overline{\Omega}_1}}}{\alpha_2 \sqrt{\overline{\Omega}_2} - \sqrt{\overline{\Omega}_1}} \right], \quad (5.21)
\]

with \( \alpha_2 = \alpha_2 / \beta \) and \( \beta_1 = \beta_1 / \beta \). We have taken \( \alpha_3 / \alpha_2 = \sqrt{\overline{\Omega}_2} \) in order to ensure that the solution of \( L \) will vanish at \( \overline{\Omega} = \overline{\Omega}_2 \). This condition implies a restriction on the coefficients in (2.4), namely \( \alpha_3 / \alpha_2 = \beta_3 / \beta_1 - (\alpha_4 \alpha_2) / (\beta_3 \alpha_3) \).

The branch \( L_+ \) is not considered here because inside the interval \([\overline{\Omega}_1, \overline{\Omega}_2]\) it is negative; whereas \( L_- \) vanishes at \( \overline{\Omega}_1 \) and \( \overline{\Omega}_2 \) and it has a maximum for a value of the frequency \( \sqrt{\overline{\Omega}_{\text{max}}} \), given by the condition

\[
\frac{\alpha_2}{\sqrt{\beta_1}} = \frac{\sqrt{\overline{\Omega}_{\text{max}}} - \sqrt{\overline{\Omega}_1}}{\sqrt{\overline{\Omega}_2} - \sqrt{\overline{\Omega}_{\text{max}}}} - 1. \quad (5.22)
\]

The corresponding maximum value \( L_{\text{max}} \) is

\[
\beta_1 = \frac{\kappa L_{\text{max}}}{\omega_\perp [\sqrt{\overline{\Omega}_2} - \sqrt{\overline{\Omega}_{\text{max}}}]^2}. \quad (5.23)
\]

Thus, the coefficients in (2.4) have been identified in terms of \( \overline{\Omega}_1(\epsilon), \overline{\Omega}_2(\epsilon), \overline{\Omega}_{\text{max}}(\epsilon) \) and \( L_{\text{max}}(\epsilon) \), from which their values could be found, except for \( \beta \), related to the dynamics of the vortex line density. The coefficients \( \alpha_i, \beta_i \) depend on the eccentricity \( \epsilon \), but not on \( \omega_\perp \) neither on the mass of the particles, which appears in \( \kappa \).
To compare with the experimental values of Hodby et al. [6] one must go from the values of \( L \) (number of vortices per unit area) to the average number of vortices in the condensate, \( N \), which are related through \( L = N/A \), with \( A \) the transversal area of the condensate. This will be estimated as \( A = (\pi/4)d^2 = (\pi/4)(\kappa/\omega_\perp) \). In fact, it must be recalled that the transverse radius depends on the angular velocity \( \Omega \) as commented in (3.12). In the trap used by Hodby et al. [6], \( \omega_\perp \approx 124\pi s^{-1} \), and for \( ^{87}\text{Rb} \), the vorticity quantum \( \kappa = h/m \) is \( 4.54 \times 10^{-5} \text{cm}^2\text{s}^{-1} \). Thus, we take for the average area of the condensate \( A \approx 9.15 \times 10^{-8} \text{cm}^2 \). In Fig. 3 the curves of our model and the experimental results by Hodby et al. are shown [6]. The experimental data are disperse but the general qualitative features are well exhibited. We have taken for \( \Omega_1 \), \( \Omega_2 \), \( \Omega_{\text{max}} \) and \( L_{\text{max}} \) their experimental values — in fact, the experimental value for \( \Omega_2 \) found by Hodby et al. does not coincide exactly with the theoretical value coming from (5.18).

Hodby et al. give the value of \( \Omega_{\text{max}} \) for two values of \( \epsilon \), namely, \( \epsilon = 0.041 \), for which \( \Omega_{\text{max}} \approx 0.74 \), and \( \epsilon = 0.084 \) for which \( \Omega_{\text{max}} \approx 0.785 \). In view of the values reported by these authors, we suggest for \( \Omega_{\text{max}}(\epsilon) \) the ansatz

\[
\Omega_{\text{max}}(\epsilon) \approx \frac{\Omega_c + \Omega_2(\epsilon)}{2} = \frac{\sqrt{2}}{4} + \frac{1}{2} \Omega_2(\epsilon). 
\]

(5.24)

In fact, the values of \( \Omega_2(\epsilon) \) observed in Fig. 1 of [6] were \( \Omega_2(\epsilon = 0.041) \approx 0.77 \pm 0.02 \) and \( \Omega_2(\epsilon = 0.084) \approx 0.86 \pm 0.02 \), in such a way that (5.24) yields indeed a reasonable estimate for \( \Omega_{\text{max}}(\epsilon) \) in these cases.

6 Conclusions

We adapted a previous dynamical equation proposed for the vortex density in superfluid helium to BEC condensates. The adaptation is not obvious, because it requires to consider the form of the trap potential, the radius of the BEC cloud in terms of the number of particles and the angular velocity, and information on the critical value of \( \Omega \) related to the resonant excitation of the quadrupolar mode.

An interesting aspect of our proposal is the simplicity of the expression (3.11) for the vortex density in term of the angular rotation. The lines shown in the figure are indicative, as in fact the number of vortices only take discontinuous values, whereas the actual values are related to the discrete number of vortices. Relation (3.11) shows the considerable difference between the vortex density starting from \( L = 0 \) at \( \Omega = \frac{\sqrt{2}}{2} \omega_\perp \) and tending to \( L = \frac{2\Omega}{\kappa} \) for an asymptotic limit of \( \Omega \gg \omega_\perp \) which is never reached, as the centrifugal force limits \( \Omega \) to values less or equal than \( \omega_\perp \), with \( \omega_\perp \) the transversal frequencies of the confining trap. Thus, (3.11) albeit phenomenological, may be useful for simple estimates of the density, and also of the number of vortices, as given by (3.12).

We also included in Section 5 a seemingly anomalous situation where the number of vortices is not an increasing function of \( \Omega \), but is reported to have a maximum. To our knowledge, this observation has not been replicated by other groups, and it could be related to some metastable situation. In any case, we have reported it here to have a better appreciation of the role of some restrictions on the values of the numerical coefficients, as that reported in (3.9), whose breakdown could lead to a situation as that reported in Section 5 instead to the much more generic results of Section 3.

A physical interpretation of the meaning of the several terms in (2.4) is still open, and is related to the entrance into the condensate of vortices formed in the walls, and their subse-
quent mutual interactions inside the condensate.

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Figure 3: Representation of the number of vortices $N$ in terms of the dimensionless rotational velocity $\bar{\Omega}$ according to equation (5.21). The points correspond to the experimental data of Hodby et al. [6]. The continuous lines correspond to our model, taking for $\bar{\Omega}_1$, $\bar{\Omega}_2$, $\bar{\Omega}_{\text{max}}$ and $L_{\text{max}}$ the experimental values. The upper line corresponds to the eccentricity $\epsilon_1 = 0.084$ and the lower one to eccentricity $\epsilon_2 = 0.041$. 