A note on the WKB solutions of difference equations

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Comparison of approximate solutions that were obtained by using different asymptotic methods of solutions of difference equations with the exact solution is presented. Results show that for the studied equation the method of transformation of the difference equation into the Riccati equation gives more correct solutions than the method of direct asymptotic expansion.

1 Introduction

It is common knowledge (see, for example, [1,2,3]) that a second order difference equation
\[ y_{k+2} + f_{1,k} y_{k+1} + f_{0,k} y_k + f_k = 0 \]  
(1)
has approximate solutions if the sequences \( f_{0,k} \) and \( f_{1,k} \) vary sufficiently slowly with \( k \). As in the theory of asymptotic solutions of differential equations, there are two approaches for obtaining approximate solutions of difference equations. In the frame of the first approach, we are finding of approximate solution straightforwardly of the equation (1) [3,4,5]. The second one implies transformation of the equation (1) into the Riccati equation and finding an approximate solution of this equation [6,7]. It can be shown that for differential equations these two methods are equivalent ones and they give the same results. In the case of difference equations the situation is slightly different. The method of finding an approximate solution of the Riccati equation [6] differs from the one that is used for finding an approximate solution straightforwardly of the equation (1) [3,4]. Recently, a new method of transformation of the linear difference equations into a system of the first order equations was proposed [8]. It can also be used for finding an approximate solution of difference equations.

In this note we compare approximate solutions that were obtained by using these methods and compare them with the exact solution.

2 The discrete WKB equations

We represent the solution of the difference equation (1) as the sum of new grid functions [8]
\[ y_k = y_{1,k} + y_{2,k} \]  
(2)
By introducing new unknowns \( y_{n,k} \) instead of the one \( y_k \), we can impose an additional condition. This condition we write in the form
\[ y_{k+1} = g_{1,k} y_{1,k} + g_{2,k} y_{2,k} \]  
(3)
where \( g_{n,k} \) (\( 1 \leq n \leq 2 \)) are the arbitrary sequences.

If
\[ \det \begin{pmatrix} 1 & 1 \\ g_{1,k} & g_{2,k} \end{pmatrix} \neq 0, \]  
(4)
then the representation (2)-(3) is unique. Indeed, from (2) and (3) we can uniquely find \( y_{n,k} \) as a linear combination of \( y_k \). Using (1), (2) and (3) we can write such system of equations.

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2 So called the LG-WKB solutions
\[
\begin{pmatrix}
y_{1,k} \\
y_{2,k}
\end{pmatrix} = T_k \begin{pmatrix}
y_{1,k-1} \\
y_{2,k-1}
\end{pmatrix} + F_k,
\]

(5)

where
\[
T_k = \begin{pmatrix}
-\frac{f_{0,k-1} + g_{1,k-1}(g_{2,k} + f_{1,k-1})}{g_{1,k} - g_{2,k}} & -\frac{f_{0,k-1} + g_{2,k-1}(g_{2,k} + f_{1,k-1})}{g_{1,k} - g_{2,k}} \\
\frac{f_{0,k-1} + g_{1,k-1}(g_{2,k} + f_{1,k-1})}{g_{1,k} - g_{2,k}} & \frac{f_{0,k-1} + g_{2,k-1}(g_{2,k} + f_{1,k-1})}{g_{1,k} - g_{2,k}}
\end{pmatrix},
\]

(6)

\[
F_k = \begin{pmatrix}
-\frac{1}{g_{1,k} - g_{2,k}} f_k \\
\frac{1}{g_{1,k} - g_{2,k}} f_k
\end{pmatrix} = \begin{pmatrix}
-f_k \\
\overline{f}_k
\end{pmatrix}
\]

(7)

In some cases, it is useful to work with S-matrix (see, for example, [8,9,10,11])
\[
\begin{pmatrix}
y_{2,k-1} \\
y_{1,k}
\end{pmatrix} = S_k \begin{pmatrix}
y_{1,k-1} \\
y_{2,k}
\end{pmatrix} + \overline{F}_k,
\]

(8)

where
\[
S_k = \begin{pmatrix}
\frac{T_{k,21}}{T_{k,22}} & 1 \\
T_{k,11}T_{k,22} - T_{k,12}T_{k,21} & \frac{T_{k,12}}{T_{k,22}}
\end{pmatrix},
\]

(9)

\[
\overline{F}_k = \begin{pmatrix}
-\frac{1}{T_{k,22}} \overline{f}_k \\
-\frac{T_{k,12} + T_{k,22}}{T_{k,22}} \overline{f}_k
\end{pmatrix}
\]

(10)

We would like to emphasize that the sequences \(g_{1,k}\) and \(g_{2,k}\) are the arbitrary ones, and we do not impose a condition that the new grid functions \(y_{1,k}, y_{2,k}\) are the solutions of the equation (1).

We will consider the homogeneous difference equations. It is means that \(f_k = 0\).

Now we can describe several properties of the normal system of difference equations
\[
\begin{pmatrix}
y_{1,k} \\
y_{2,k}
\end{pmatrix} = T_k \begin{pmatrix}
y_{1,k-1} \\
y_{2,k-1}
\end{pmatrix},
\]

(11)

From (6) it follows that we can choose the sequences \(g_{1,k}\) and \(g_{2,k}\) in a such way that matrix \(T_k\) will be triangular or even diagonal one. It is realized by setting \(T_{k,12} = 0\) and \(T_{k,21} = 0\).

These conditions give the non-linear second-order rational difference equation (Riccaty type [8,12,13,14]) for the sequences \(g_{1,k}\) and \(g_{2,k}\)
\[
f_{0,k} + g_{1(2),k}(g_{1(2),k+1} + f_{1,k}) = 0.
\]

(12)

In this case the matrix \(T_k\) is a diagonal one and the system (11) takes the form:
\[
y_{1,k} = g_{1,k-1}y_{1,k-1},
\]
\[
y_{2,k} = g_{2,k-1}y_{2,k-1}.
\]

(13)
Solutions $y_{1,k}, y_{2,k}$ are the linearly independent ones.

The characteristic equation of the difference equation (1) is

$$\rho_k^2 + f_{1,k} \rho_k + f_{0,k} = 0.$$

(14)

Let $g_{1,k} = \rho_k^{(1)}$, $g_{2,k} = \rho_k^{(2)}$, where $\rho_k^{(1)}$, $\rho_k^{(2)}$ are the solutions of the characteristic equation (14)

$$\rho_k^{(1,2)} = \frac{f_{1,k}}{2} \pm \frac{1}{2} \sqrt{f_{1,k}^2 - 4f_{0,k}}.$$

(15)

The matrix $T_k$ takes the form

$$T_k = \begin{pmatrix}
\rho_k^{(1)} & \rho_k^{(1)} - \rho_k^{(2)} & \rho_k^{(2)} - \rho_k^{(1)} \\
\rho_k^{(1)} - \rho_k^{(2)} & \rho_k^{(2)} & \rho_k^{(1)} - \rho_k^{(2)} \\
\rho_k^{(1)} - \rho_k^{(2)} & \rho_k^{(1)} - \rho_k^{(2)} & \rho_k^{(1)} - \rho_k^{(2)}
\end{pmatrix}. $$

(16)

If the sequences $f_{0,k}$ and $f_{1,k}$ vary sufficiently slowly with $k$

$(f_{0,k} = f_0(\varepsilon k), f_{1,k} = f_1(\varepsilon k), 0 \leq \varepsilon \ll 1)$, then the differences $(\rho_k^{(1,2)} - \rho_k^{(1,2)})$ are the small values and we can neglect the non-diagonal terms in the matrix $T_k$. This gives

$$y_{1,k} = \rho_k^{(1)} \left(1 - \frac{\rho_k^{(1)} - \rho_k^{(2)}}{\rho_k^{(1)} - \rho_k^{(2)}}\right) y_{1,k-1},$$

$$y_{2,k} = \rho_k^{(2)} \left(1 + \frac{\rho_k^{(2)} - \rho_k^{(1)}}{\rho_k^{(1)} - \rho_k^{(2)}}\right) y_{2,k-1}.$$ 

(17)

It can be shown that these equations coincide with the equations of the discrete WKB approach (see, for example, [6,7]). Indeed, from (13) it follows that the discrete WKB equations can be obtained with using an approximate solutions of the Riccati equation (12) under assumption that $f_{0,k}$ and $f_{1,k}$ vary sufficiently slowly with $k$. The Riccati equations can be transformed with an iteration procedure into the quadratic equations

$$\left(g_{(1,2),k-1}\right)^2 + g_{(1,2),k-1} f_{1,k-1} + f_{0,k-1} + \rho_k^{(1,2)} \left(\rho_k^{(1,2)} - \rho_k^{(1,2)}\right) = 0.$$ 

(18)

It is one of the possible forms of the quadratic equation (compare with [6,7]) that can be obtained at the second iteration. Its solutions differ from the ones that were obtained in [6,7] by an amount of order $\varepsilon^2$.

The approximate solutions of this equations with error of $O(\varepsilon^2)$ are

$$g_{(1,2),k-1} \approx \rho_k^{(1,2)} \left(1 \mp \frac{\rho_k^{(1,2)} - \rho_k^{(1,2)}}{\rho_k^{(1)} - \rho_k^{(2)}}\right). $$

(19)

Comparison (17) and (19) shows that these two different approaches give the same result, and the equations (17) and (13) coincide.

The solutions of the equations (17) at $k > k_0$ can be write as
\[ y_k^{(1,2)}(r) = \prod_{s=k+1}^{k} T_{s,1,2} y_0^{(i)}(r) = y_0^{(i)}(r) \prod_{s=1}^{k} \rho_s^{(i)} \left( 1 \mp \frac{\rho_s^{(i,2)} - \rho_s^{(1,2)}}{\rho_s^{(i)} - \rho_s^{(2)}} \right) = \]
\[ = y_0^{(i)}(r) \exp \left( \sum_{s=k+1}^{k} \ln \rho_s^{(i,2)} + \sum_{s=k+1}^{k} \ln \left( 1 \mp \frac{\rho_s^{(i,2)} - \rho_s^{(1,2)}}{\rho_s^{(i)} - \rho_s^{(2)}} \right) \right) \approx y_0^{(i)}(r) \exp \left( \sum_{s=k+1}^{k} \ln \rho_s^{(i,2)} \mp \sum_{s=k+1}^{k} \frac{\rho_s^{(i,2)} - \rho_s^{(1,2)}}{\rho_s^{(i)} - \rho_s^{(2)}} \right) \]
\[ \approx y_0^{(i)}(r) \exp \left( \sum_{s=k+1}^{k} \ln \rho_s^{(i,2)} - \sum_{s=k+1}^{k} \frac{\sqrt{f_{1,s}^2 - 4f_{0,s}^2} - \sqrt{f_{1,s-1}^2 - 4f_{0,s-1}^2}}{2\sqrt{f_{1,s}^2 - 4f_{0,s}^2}} \pm \sum_{s=k+1}^{k} \frac{f_{1,s} - f_{1,s-1}}{2\sqrt{f_{1,s}^2 - 4f_{0,s}^2}} \right) \]

We can transform the second term in the exponent into the multiplier
\[
\frac{1}{(f_{1,k}^2 - 4f_{0,k}^2)^{1/4}} \exp \left( \sum_{s=k+1}^{k} \ln \rho_s^{(i,2)} \pm \sum_{s=k+1}^{k} \frac{f_{1,s} - f_{1,s-1}}{2\sqrt{f_{1,s}^2 - 4f_{0,s}^2}} \right) .
\]

The comparison this formula with the one that was obtained by finding an approximate solution straightforwardly of the equation (1) [3] gives some difference. The formula (21) contains additional sum in the exponent (the second sum).

3 Numerical model
In the frame of the Coupling Cavity Model (CCM) electromagnetic field in each cavity of the chain of resonators are represented as the expansion with the short-circuit resonant cavity modes [15,16,17,18,19,20]
\[
\bar{E}^{(q)} = \sum_{q=0, m, n} e_q^{(k)} \bar{E}_q^{(k)}(\bar{r}) ,
\]
where \( q = \{0, m, n\} \) and such coupling equations for \( e_{010}^{(n)} \) can be obtained [21,22,23,24]
\[
Z_k e_{010}^{(n)} = \sum_{j=-N}^{N} e_{010}^{(j)} \alpha_{010}^{(k,j)} .
\]
Here \( e_{010}^{(k)} \) - amplitudes of \( E_{010} \) modes, \( Z_k = 1 - \frac{\omega_0^{(k,2)}}{\omega_0^{(k,1)}} - \alpha_{010}^{(k,1)} \), \( \omega_0^{(k,1)} \) - eigen frequencies of these modes, \( \alpha_{010}^{(k,j)} \) - real coefficients that depend on both the frequency \( \omega \) and geometrical sizes of all volumes. Sums in the right side can be truncated
\[
Z_k^{(N)} e_{010}^{(N,k)} = \sum_{j=-N}^{N} e_{010}^{(j)} \alpha_{010}^{(k,j)} .
\]

In the case of \( N = 1 \), the system of coupled equations (24) is very similar to the one that can be constructed on the basis of equivalent circuits approach (see, for example [25,26,27,28]). But in the frame of the CCM the coefficients \( \alpha_{010}^{(k,j)} \) are electrodynamically strictly defined for arbitrary \( N \) and can be calculated with necessary accuracy. In the theory of RF filters the coupling matrix circuit model is used intensively (see, for example, [29] and cited there literature). The main problem is how to calculate the matrix elements.

Amplitudes of other modes (\( (m,n) \neq (1,0) \)) can be found by summing the relevant series
\[
e_{010}^{(k)} = \frac{\omega_0^{(k,2)}}{\omega_0^{(k,1)}} - \frac{\omega_0^{(k,2)}}{\omega_0^{(k,1)}} - \alpha^{(k,1)} \sum_{j=-N}^{N} e_{010}^{(j)} \alpha_{010}^{(k,j)} .
\]

For the chain of cylindrical resonators longitudinal component of electric field at \( r = 0 \) (on the system longitudinal axis) is:
\[ E_z^{(k)} = \sum_{m,n} e_{0mn}^{(i)} \cos \left( \frac{\pi}{d} nz \right). \]  

(26)

If we can ignore “long coupling” interaction, the set of coupling equations (24) takes the form

\[ Z_k e_{010}^{(k)} = e_{010}^{(k-1)} a_{010}^{(k,k-1)} + e_{010}^{(k+1)} a_{010}^{(k,k+1)}, \]

(27)

where

\[ Z_k = \left( 1 - \frac{\omega^2}{\omega_0^{(k+1)^2}} - a_{010}^{(k,k)} - i \frac{\omega}{\omega_0^{(k)}} \right). \]

The set of coupling equations (27) can be considered as the second-order difference equation. This difference equation, which defines the amplitudes of the basic modes \( e_{010}^{(k)} \), is the main equation of the CCM. It is reasonable to note that the amplitudes of the basic modes \( e_{010}^{(k)} \) are non-measured values. Indeed, we can measure the components of electric field in any point, for example, by the nonresonant perturbation method, but we cannot measure \( e_{0mn} \) and have to use numerical methods for finding these amplitudes by using the expansion (22). This circumference creates difficulties in studying the properties of the real slow-wave waveguides, including their tuning [30,31]. The similar situation arises also in other electrodynamic models. For example, the space harmonics in homogeneous periodic waveguides are non-measured values, too.

Below we will consider the chain of cylindrical resonators that are connected via circular central openings in the walls – the disk loaded waveguides (DLW)\(^3\). It was shown that the DLWs, that are usually used in linacs, with disk spacing large enough \((d \geq \lambda / 3)\) can be describe with sufficient accuracy by the difference equation (27) [32]. Appropriate values of the coupling coefficients \( a_{010}^{(k,k)}, \ a_{010}^{(k,k+1)} \) at fixed frequency can be approximated by some functions of geometrical sizes. Calculations on the base of the CCM show that for the most often used in linacs DLWs such approximations can be used

\[
\begin{align*}
\alpha_{010}^{(k,k)} &= -\frac{u_k \bar{P}_k + u_{k+1} \bar{P}_{k+1}}{b_k^2 d_k}, \\
\alpha_{010}^{(k,k-1)} &= \frac{u_k}{b_k^2 d_k}, \\
\alpha_{010}^{(k,k+1)} &= \frac{u_{k+1}}{b_k^2 d_k},
\end{align*}
\]

(28)

where

\[ u_k = \frac{a a_0^3}{b_k^2 d_e} p_k^{(c)}, \quad \bar{b}_k = \frac{b_k}{b_*}, \quad \bar{d}_k = \frac{d_k}{d_*}, \quad a_k \quad \text{- the hole radius between } k-1 \text{ and } k \text{ resonators,}
\]

\[ b_k \quad \text{- the radius of } k \text{ cylindrical resonator,} \quad d_k \quad \text{- the resonator length,} \quad b_* = \frac{\lambda_0}{\omega}, \quad d_* \quad \text{- normalizing parameters,} \quad \omega_0^{(k)} = \frac{\lambda_0}{b_k^2}, \quad J_0(\lambda_0) = 0, \quad \alpha = \frac{2}{3\pi J_1^2(\lambda_0)}, \quad \bar{P}_k = \frac{p_k^{(c)}}{p_k^{(c)}}.
\]

Analysis shows that we can consider parameters \( p_k^{(c)}, p_k^{(c)} \) as the functions of the geometric sizes of the diaphragms only (the opening radius \(a_k\), the thickness \( t_k \) of the diaphragm between \( k-1 \) and \( k \) resonators and the radius of the rounding of the disk hole edges).

For \( t_k = 0.4 \text{ cm}, \quad d_k = 3.0989 \text{ cm}, \) parameters \( p_k^{(c)}, p_k^{(c)} \) can be represented\(^4\) as

\(3\) DLW structures are the most often used in linacs and represent the chain of cavities in which the phase varies smoothly from cell to cell in such way, that an accelerated particle constantly locates in accelerating field.

\(4\) For simplicity, we will consider the case without of the rounding of the disk hole edges. For taking into account the rounding of the disk hole edges.
The parameter $p_k^{(c)}$ determines the deviation of the dependence of the coupling coefficient $\alpha_{010}^{(k,k-1)}$ on $a_k$ from the law $a_k^3$, $p_k^{(c)}$ - the deviation of the dependence of the resonator frequency shift due to the hole in the $k$-disk on $a_k$ from the law $a_k^3$ (see (28)).

For simplicity we will suppose that $p_k^{(c)} = p_k^{(c)} = 1$. This assumption corresponds the case when the cavities are connected through the small round openings in the thin diaphragms.

We will consider the chain of the identical resonators $(1, 1, \ldots, 1)$ that consist of two semi-infinite homogeneous parts (I and II) that are connected via the inhomogeneous transient cells. The variable parameter is the radius of the opening that connects two resonators. The difference equation (27) can be written as $(y_k = e_k)$

$$y_{k+1} + \left[ \frac{2(1 - \cos \varphi_I) - (\overline{u}_k + \overline{u}_{k+1})}{\overline{u}_{k+1}} \right] y_k + \frac{\overline{u}_k}{\overline{u}_{k+1}} y_{k-1} = 0, \quad (30)$$

where $\varphi_I$ - the phase shift between cells in the first homogeneous part,

$$\overline{u}_k = \frac{u_k}{u_I} = \frac{(1 - \cos \varphi_I)}{(1 - \cos \varphi_k)} \quad (31).$$

If we set $\varphi_k = \varphi_I$ ($\overline{u}_k = 1$) the equation (30) has such solutions of the characteristic equation (14)

$$\rho_k^{(1,2)} = -\frac{f_{1,k}}{2} \pm \frac{1}{2} \sqrt{f_{1,k}^2 - 4f_{R,k}} = \cos \varphi_I \pm i \sin \varphi_I = \exp(\pm i\varphi_I). \quad (32)$$

Using (31), we can transform the difference equation (30) as

$$y_{k+1} + \left[ -2 \cos \varphi_{k+1} + \frac{\cos \varphi_{k+1} - \cos \varphi_k}{(1 - \cos \varphi_k)} \right] y_k + \left[ 1 - \frac{\cos \varphi_{k+1} - \cos \varphi_k}{(1 - \cos \varphi_k)} \right] y_{k-1} = 0. \quad (33)$$

If the differences $(\cos \varphi_{k+1} - \cos \varphi_k)$ are small, then the solutions of the characteristic equation can be estimated as

$$\rho_k^{(1,2)} \approx \exp(\pm i\varphi_{k+1}). \quad (34)$$

Let consider the task of diffraction of the incident wave on the gradual transition between two homogeneous chains of resonators. In this case in the first homogeneous part we have two waves: the incident wave $(\rho_k^{(1)} = \exp(i\varphi_I))$ with the amplitude equals to 1 and the reflected wave $(\rho_k^{(2)} = \exp(-i\varphi_I))$ with the amplitude $R$ (the reflection coefficient). In the second homogeneous part we have only one transmitted wave $(\rho_k^{(2)} = \exp(i\varphi_R))$ with the amplitude $T$ (the transmission coefficient).

It is useful to use the S-matrix method for finding solution of the considered task. If we find the S-matrix of the part of the infinite chain that contains the transition cells and several homogeneous cells on both sides of the inhomogeneous part (cell numbers are $k = 1 \div N$) and the set of the additional S-matricies, we get the solution of the considered task. Indeed, from the definition of the S-matrix we have

$$y_k = y_{1,k} + y_{2,k}, \quad (35)$$

$$y_{1,1} = S_{11}^{(N)} y_{1,1} + S_{12}^{(N)} y_{2,N}, \quad (36)$$

$$y_{1,1} = S_{21}^{(N)} y_{1,1} + S_{22}^{(N)} y_{2,N},$$

Suppose $y_{1,1} = 1, y_{2,N} = 0$, then
For the additional $S^{(k)}$-matrices
\[ y_{2,k} = S_{11}^{(k)} y_{1,1} + S_{12}^{(k)} y_{2,k}, \quad 2 \leq k \leq N, \]
\[ y_{1,k} = S_{21}^{(k)} y_{1,1} + S_{22}^{(k)} y_{2,k}. \]
From these equations it follows that
\[ y_{2,k} = \frac{R - S_{11}^{(k)}}{S_{12}^{(k)}}, \]
\[ y_{1,k} = S_{21}^{(k)} y_{1,1} + S_{22}^{(k)} \frac{R - S_{11}^{(k)}}{S_{12}^{(k)}}. \]
\[ y_k = y_{1,k} + y_{2,k} = S_{21}^{(k)} + (S_{22}^{(k)} + 1) \frac{R - S_{11}^{(k)}}{S_{12}^{(k)}}. \]
If we know the transfer matrix $T_k$, we can find the S-matrix by using the formula (9). We will use three type of the transfer matrix $T_k$. The first matrix is the strict one (16)
\[ T_k^{(0)} = \begin{pmatrix} \rho_{k-1}^{(1)} & \rho_{k-1}^{(2)} \\ \rho_{k}^{(1)} - \rho_{k-1}^{(2)} & \rho_{k}^{(1)} - \rho_{k}^{(2)} \end{pmatrix}. \]
The second matrix
\[ T_k^{(1)} = \begin{pmatrix} \rho_{k-1}^{(1)} \left( 1 - \frac{\rho_{k-1}^{(1)} - \rho_{k}^{(1)}}{\rho_{k}^{(1)} - \rho_{k}^{(2)}} \right) & 0 \\ 0 & \rho_{k}^{(2)} \left( 1 - \frac{\rho_{k}^{(1)} - \rho_{k}^{(2)}}{\rho_{k}^{(1)} - \rho_{k}^{(2)}} \right) \end{pmatrix}. \]
gives such solution in the WKB approach
\[ y_k^{(1,2)} \sim \frac{1}{(f_{1,k}^2 - 4f_{0,k}^2)^{1/4}} \exp \left( \sum_{s=k_0+1}^k \ln \rho_{r,s}^{(1,2)} \pm \sum_{s=k_0+1}^k \frac{f_{1,s} - f_{1,s-1}}{2 \sqrt{f_{1,s}^2 - 4f_{0,s}}} \right). \]
The third matrix
\[ T_k^{(2)} = \begin{pmatrix} \rho_{k-1}^{(1)} \left( 1 - \frac{\rho_{k-1}^{(1)} - \rho_{k}^{(1)}}{\rho_{k}^{(1)} - \rho_{k}^{(2)}} \right) & 0 \\ 0 & \rho_{k}^{(2)} \left( 1 + \frac{\rho_{k}^{(1)} - \rho_{k}^{(2)}}{\rho_{k}^{(1)} - \rho_{k}^{(2)}} \right) \end{pmatrix}, \]
where $\rho_k^{(1,2)} = \pm \frac{1}{2} \sqrt{f_{1,k}^2 - 4f_{0,k}}$, gives such solution in the WKB approach
\[ y_k^{(1,2)} \sim \frac{1}{(f_{1,k}^2 - 4f_{0,k}^2)^{1/4}} \exp \left( \sum_{s=k_0+1}^k \ln \rho_{r,s}^{(1,2)} \right). \]
This case corresponds of approximate solution that was obtained without using the Riccati equation [3,4].
For numerical calculation we used the linear phase transition between homogeneous parts.
\[
\varphi_k = \begin{cases} 
\varphi_l, & 1 \leq k \leq N_h \\
\varphi_l + \frac{\varphi_h - \varphi_l}{N - 2N_h}(k - N_h), & N_h + 1 \leq k \leq N - N_h \\
\varphi_h, & N - N_h + 1 \leq k \leq N 
\end{cases}
\] (46)

where \( \varphi_l = \frac{\pi}{3}, \varphi_h = \frac{2\pi}{3}, N_h = 100, N = 250 \).

Calculation results of are presented in Figure 1-Figure 6.

Calculations on the basis of the matrix \( T_k^{(0)} \) give the following values of the reflection coefficient and the transmission coefficient: \( |R_0| = 3.05E-008, |T_0| = 1.7316 \). Approximate models give \( |R_1| = 0, |T_1| = 1.7395, |R_2| = 0, |T_2| = 1.7330 \).

We see that the amplitude distributions are nearly the same for three types of the transition matrix. We can also see that the phase distribution for the third matrix \( T_k^{(2)} \) differs from the phase distributions for the matrices \( T_k^{(0)} \) and \( T_k^{(1)} \). This difference is not small.

Let estimate the difference between the distributions that are based on using the matrices \( T_k^{(1)} \) and \( T_k^{(2)} \). From (43) and (45) it follows that the difference is in the additional sum that stands in the exponent

\[
\Delta P = \sum_{x=0}^{x=N} \frac{f_{k,x} - f_{k,x-1}}{2\sqrt{f_{k,x}^2 - 4f_{k,x}}}. 
\] (47)

Using the estimation (34), we have

\[
\Delta P \approx \sum_{x=0}^{x=N} \frac{\cos \varphi_x - \cos \varphi_{x-1}}{2i \sin \varphi_x} = \sum_{x=0}^{x=N} \frac{\Delta x}{2i \sqrt{1 - x^2}} \approx \\
\approx \int_{x_1}^{x_2} \frac{dx}{2i \sqrt{1 - x^2}} = \frac{1}{2i} \arcsin x \bigg|_{x_1}^{x_2} = -i \frac{1}{2} (\varphi_h - \varphi_l) = -i \frac{\pi}{6}. 
\] (48)

Results of the numerical calculations are in good agreement with this estimation (see Figure 6, cavity numbers >150).

**Figure 1** The amplitude distribution of the exact solution (calculation with the matrix \( T_k^{(0)} \))
Figure 2: The phase distribution of the exact solution

Figure 3: The phase shift \((\Phi_{0,k} = \arg(y_{0,k+1}) - \arg(y_{0,k}))\) distribution of the exact solution

Figure 4: Deviation of the absolute values of the approximate solutions from the exact one \((|y_k| - |y_{0,k}|)\), 1 - calculation with the matrix \(T_k^{(1)}\), 2 - with the matrix \(T_k^{(2)}\)
Figure 5 Deviation of the phase shift of the approximate solutions from the exact one: 1- calculation with the matrix $T^{(1)}_k$, 2 - with the matrix $T^{(2)}_k$

Figure 6 The phase distribution of the different solutions: 0 - calculation with the matrix $T^{(0)}_k$, 1- with the matrix $T^{(1)}_k$, 2 - with the matrix $T^{(2)}_k$

Conclusions
We compared approximate solutions that were obtained by using different asymptotic methods of solutions of difference equations with the exact solution. Results show that for the studied equation the method of transformation of the difference equation into the Riccati equation gives more correct solutions than the method of direct asymptotic expansion.

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