MORPHISMS OF VERMA MODULES OVER EXCEPTIONAL LIE SUPeralgebra \(E(5, 10)\).

ALEXEI RUDAKOV

Abstract. In this paper we define the degree of a morphism between (generalized) Verma modules over a graded Lie superalgebra and construct series of morphisms of various degrees between (generalized) Verma modules over the exceptional infinite-dimensional linearly compact simple Lie superalgebra \(E(5, 10)\). We prove that all such morphisms of degree 1 are found.

To Victor Kac for the birthday

0. Introduction.

The paper continues the study of representations of the exceptional infinite-dimensional linearly compact simple Lie superalgebra \(E(5, 10)\) that has begun in [KR3]. Here we deal with morphisms between (generalized) Verma modules. We define a degree of such a morphism and notice that the morphisms described in [KR3] are morphisms of degree 1. In the paper we prove that there are no more morphisms of degree 1. But we find morphisms of degrees 2 and 3 when considering products of morphisms of degree 1.

It takes some efforts to find morphisms of degree 4. We study their properties and use them to construct also some morphisms of degree 5. We show that the picture of complexes of Verma modules over \(E(5, 10)\) given in [KR3] extends naturally with the morphisms of degree 4.

At the moment it is not clear if there are more morphisms of degree \(\geq 3\), but we know that there are no other morphisms of degree 2. We will include the proof of this fact in a subsequent paper.

1. Verma modules.

We keep the notations from [KR2] (see also [KR1], [R]).

Let \(L = \oplus_{j \in \mathbb{Z}} g_j\) be a \(\mathbb{Z}\)-graded Lie superalgebra by finite-dimensional vector spaces. Let

\[
L_- = \oplus_{j < 0} g_j, \quad L_+ = \oplus_{j > 0} g_j, \quad L_0 = g_0 + L_+.
\]

As usual we denote by \(U = U(L)\) the universal enveloping algebra of \(L\) and similarly \(U_0 = U(L_0), U_- = U(L_-)\).

Let us notice that the grading extends to the enveloping algebras, in particular to \(U_-\). It is convenient to change the sign of the degree when considering this grading on \(U_-\). We shall call it the natural grading of \(U_-\).

Given a \(g_0\)-module \(V\), we extend it to a \(L_0\)-module by letting \(L_+\) act trivially, and define the induced \(L\)-module

\[
M(V) = U \otimes_{U_0} V.
\]
We shall use the vector space isomorphism

\[ M(V) = U_- \otimes_C V . \]

**Definition 1.1.** Let \( V \) be a \( g \)-module. The \( L \)-module \( M(V) \) is called a (generalized) Verma module (associated to \( V \)). When we want to emphasize that \( V \) is a finite-dimensional irreducible \( g \)-module, we call the Verma module \( M(V) \) a minimal Verma module.

A minimal Verma module is called non-degenerate if it is irreducible and degenerate if it is not irreducible.

Let \( A \) and \( B \) be two \( g_0 \)-modules and let \( \text{Hom}(A, B) \) be the \( g_0 \)-module of linear maps from \( A \) to \( B \). The following proposition will be used to construct morphisms between the \( L \)-modules \( M(A) \) and \( M(B) \).

**Proposition 1.1.** Let \( \varphi : M(A) \to M(B) \) be a morphism of \( L \)-modules and \( \Phi \in U_- \otimes_C \text{Hom}(A, B) \) be such that

\[ \varphi(1 \otimes a) = \Phi(a) . \]

Then \( \Phi \) has the properties:

\[ v \Phi(a) = \Phi(u a) \text{ for } v \in g_0 , \]
\[ v \Phi(a) = 0 \text{ for } v \in L_+ . \]

and the morphism \( \varphi \) is determined by the rule

\[ \varphi(u \otimes a) = u \Phi(a) . \]

Moreover for any \( \Phi \) with the properties (1.1a), (1.1b) a morphism \( \varphi : M(A) \to M(B) \) is uniquely defined.

**Proof.** If \( \varphi : M(A) \to M(B) \) is a morphism of \( L \)-modules then for \( u \in U(L_0) \),

\[ u \varphi(1 \otimes a) = \varphi(u \otimes a) = \varphi(1 \otimes u a) . \]

Properties (1.1a) and (1.1b) follow.

Now given \( \Phi \in U_- \otimes_C \text{Hom}(A, B) \) with the properties (1.1a), (1.1b) we may wish to use (1.2) to define \( \varphi \), but in order to conclude that \( \varphi \) is well-defined we have to check for any \( u \in U \), \( v \in U_0 \) the equality

\[ \varphi(u v \otimes a) = \varphi(v \otimes u a) . \]

Clearly it is equivalent to an equality

\[ v \Phi(a) = \Phi(v a) , \]

and it is sufficient to consider cases: \( v \in g_0 \) and \( v \in L_+ \). For the first case the equality is exactly the same as the property (1.1a). In the second case we have \( v a = 0 \) because \( L_+ \) act trivially and we come to the property (1.1b). \( \square \)

Let us denote the morphism defined in the proposition as \( \Phi = \varphi|_A \) and call it the restriction of \( \varphi \).

**Definition 1.2.** We say that a morphism \( \varphi : M(A) \to M(B) \) has degree \( k \) when

\[ \Phi = \varphi|_A = \sum_i u_i \otimes \ell_i , \text{ where } u_i \in U_- , \ell_i \in \text{Hom}(A, B) \]

and \( \text{deg} u_i = k \) for all \( i \).

We shall permit ourselves to denote the morphism of Verma modules and its restriction by the same letter when it does not lead to confusion.
Remark 1.3. If $L_0$ is generated by $\mathfrak{g}_0$ and a subset of $T \subset L_+$, then conditions (1.1) are equivalent to

\begin{align}
&g_0 \cdot \Phi = 0 \\
&t \Phi(a) = 0 \text{ for all } t \in T, \ a \in A.
\end{align}

The "dot" in (1.3a) denotes the action of $g_0$ on the tensor product of $g_0$-modules $U_-$ and $\text{Hom}(A, B)$. Usually it gives a hint to a possible choice of $\Phi$ and may be checked by general invariant-theoretical considerations. The condition (1.3b) is often more difficult to check.

Remark 1.4. We can view $M(V)$ also as the induced $(L_- \oplus \mathfrak{g}_0)$-module: $U(L_- \oplus \mathfrak{g}_0) \otimes \text{Hom}(A, B) V$. Then condition (1.3a) on $\Phi = \sum_i u_i \otimes \ell_i$, where $u_i \in U(L_- \oplus \mathfrak{g}_0)$, $\ell_i \in \text{Hom}(A, B)$, suffices in order (1.2) to give a well-defined morphism of $(L_- \oplus \mathfrak{g}_0)$-modules. One can also replace $\mathfrak{g}_0$ by any of its subalgebras.

2. Lie superalgebra $E(5, 10)$.

Recall some standard notation:

$$W_n = \{ \sum_{j=1}^n P_j(x) \partial_i \mid P_j \in \mathbb{C}[x_1, \ldots, x_n], \ \partial_i \equiv \partial / \partial x_i \}$$

denotes the Lie algebra of formal vector fields in $n$ indeterminates:

$$S_n = \{ D = \sum P_i \partial_i \mid \partial x_i D = 0 \}$$

denotes the Lie subalgebra of divergenceless formal vector fields; $\Omega^k(n)$ denotes the associative algebra of formal differential forms of degree $k$ in $n$ indeterminates, $\Omega^k_\text{cl}(n)$ denoted the subspace of closed forms.

The Lie algebra $W_n$ acts on $\Omega^k(n)$ via Lie derivative $D \rightarrow L_D$. Given $\lambda \in \mathbb{C}$ one can define the twisted action:

$$D \omega = L_D \omega + \lambda (\text{div} D) \omega .$$

The $W_n$-module thus obtained is denoted by $\Omega^k(n)^\lambda$. Recall the following isomorphism of $W_n$-modules

\begin{equation}
W_n \simeq \Omega^{n-1}(n)^{-1} .
\end{equation}

It is obtained by mapping a vector field $D \in W_n$ to the $(n - 1)$-form $\iota_D(dx_1 \wedge \ldots \wedge dx_n)$. Note that (2.1) induces an isomorphism of $S_n$-modules:

\begin{equation}
S_n \simeq \Omega^{n-1}_\text{cl}(n) .
\end{equation}

Recall that the Lie superalgebra $E(5, 10) = E(5, 10)_0 + E(5, 10)_1$ is constructed as follows

$[K], [CK]$: $E(5, 10)_0 = S_5, \ E(5, 10)_1 = \Omega^2_\text{cl}(5)$.

To describe brackets consider an algebra $\tilde{E}(5, 10) = \tilde{E}(5, 10)_0 + \tilde{E}(5, 10)_1 = W_5 + \Omega^2(5)$ where the subalgebra $W_5$ acts on $\tilde{E}(5, 10)_1$ via the Lie derivative, but for $\omega_2, \omega'_2 \in \tilde{E}(5, 10)_1$ the brackets are

$$[\omega_2, \omega'_2] = \omega_2 \wedge \omega'_2 \in \Omega^2(5) \rightarrow W_5 \quad \text{(see (2.1))} .$$

Now $E(5, 10)$ is a subalgebra of $\tilde{E}(5, 10)$. Let as note that $\tilde{E}(5, 10)$ is not a Lie superalgebra, the Jacobi identity is no longer true in this larger algebra, but true in $E(5, 10)$. 

As in [KR2] we use for the odd elements of $E(5,10)$ the notation $d_{ij} = dx_i \wedge dx_j$ $(i,j = 1,2,\ldots,5)$; recall that we have the following commutation relation \( (f,g \in \mathbb{C}[x_1,\ldots,x_5]) \):

\[
[f d_{jk}, g d_{lm}] = \epsilon_{ijklm} f g \partial_i,
\]

where $\epsilon_{ijklm}$ is the sign of the permutation $ijk\ell m$ if all indices are distinct and 0 otherwise.

Recall that the Lie superalgebra $L = E(5,10)$ carries a unique consistent irreducible $\mathbb{Z}$-gradation $L = \oplus_{j \geq -2} g_j$. It is defined by:

\[
\deg x_i = 2 = -\deg \partial_i, \quad \deg d_{ij} = -1.
\]

One has: $g_0 \simeq \mathfrak{sl}_5(\mathbb{C})$ and the $g_0$-modules occurring in the $L_-$ part are:

\[
g_{-1} = \langle d_{ij} | i,j = 1,\ldots,5 \rangle \simeq \Lambda^2 \mathbb{C}^5,
\]

\[
f g_{-2} = \langle \partial_i | i = 1,\ldots,5 \rangle \simeq \mathbb{C}^5^*.
\]

Recall also that $g_1$ consist of closed 2-forms with linear coefficients, that $g_1$ is an irreducible $g_0$-module and $g_j = [g_1 \ldots] = g_j^I$ for $j \geq 1$.

3. Degenerate Verma modules and morphisms of degree 1.

We take for the Borel subalgebra of $g_0 \simeq \mathfrak{sl}_5$ the subalgebra of the vector fields

\[
\{ \sum_{i \leq j} a_{ij}(x_i \partial_j) \mid a_{ij} \in \mathbb{C}, \, \text{tr}(a_{ij}) = 0 \} = \langle x_i \partial_j, \, i \leq j, \, h_i = x_i \partial_i - x_{i+1} \partial_{i+1} \rangle.
\]

We denote by $F(n_1,n_2,n_3,n_4)$ the finite-dimensional irreducible $g_0$-module with highest weight $(n_1,n_2,n_3,n_4)$. Let

\[
M(n_1,n_2,n_3,n_4) = M(F(n_1,n_2,n_3,n_4))
\]

denote the corresponding generalized Verma module over $E(5,10)$.

Let us repeat a conjecture from [KR3] (where it contains a misprint)

**Conjecture 3.1.** (The following is a complete list of degenerate Verma modules over $E(5,10)$):

\[
M(m,n,0,0), \, M(m,0,0,n), \, \text{and} \, M(0,0,m,n) \quad \text{(for any } m,n \in \mathbb{Z}_+).\]

In this section we construct three series of morphisms of degree one of Verma modules which shows, in particular, that all modules from the list given by Conjecture 3.1 are indeed degenerate.

We let (see [KR3], but our notations differ slightly):

\[
S_A = S(\mathbb{C}^5 + \Lambda^2 \mathbb{C}^5), \quad S_B = S(\mathbb{C}^5 + \mathbb{C}^5^*), \quad S_C = S(\mathbb{C}^5 + \Lambda^2 \mathbb{C}^5^*).
\]

Denote by $z_i$ $(i = 1,\ldots,5)$ the standard basis of $\mathbb{C}^5$ and by $x_{ij} = -x_{ji}$ $(i,j = 1,\ldots,5)$ the standard basis of $\Lambda^2 \mathbb{C}^5$. Let $z_i^*$ and $x_{ij}^*$ be the dual bases of $\mathbb{C}^5^*$ and $\Lambda^2 \mathbb{C}^5^*$, respectively. Then $S_A$ is the polynomial algebra in 15 indeterminates $x_i$ and $x_{ij}$, $S_B$ is the polynomial algebra in 15 indeterminates $z_i^*$ and $x_{ij}^*$ and $S_C$ is the polynomial algebra in 10 indeterminates $z_i$ and $z_i^*$.

Given two irreducible $g_0$-modules $E$ and $F$, we denote by $(E \otimes F)_{\text{high}}$ the highest irreducible component of the $g_0$-module $E \otimes F$. If $E = \oplus_i E_i$ and $F = \oplus_j F_j$ are direct sums of irreducible $g_0$-modules, we let $(E \otimes F)_{\text{high}} = \oplus_{i,j} (E_i \otimes F_j)_{\text{high}}$. If $E$ and $F$ are again irreducible $g_0$-modules, then $S(E \otimes F) = \oplus_{m,n \in \mathbb{Z}_+} S^m E \otimes S^n F$, and we let $S_{\text{high}}(E \otimes F) = \oplus_{m,n \in \mathbb{Z}_+} S^m E \otimes S^n F$ for $E \otimes F$. We also denote by $S_{\text{low}}(E \otimes F)$ the $g_0$-invariant complement to $S_{\text{high}}(E \otimes F)$.
It is easy to see that we have as \( g_0 \)-modules:
\[
S_{A,\text{high}} \simeq \oplus_{m,n \in \mathbb{Z}_+} F(m, n, 0, 0),
\]
\[
S_{B,\text{high}} \simeq \oplus_{m,n \in \mathbb{Z}_+} F(m, 0, n, 0),
\]
\[
S_{C,\text{high}} \simeq \oplus_{m,n \in \mathbb{Z}_+} F(0, 0, m, n).
\]

We are going to construct morphisms \( \nabla_X \in \text{End}_L(M(S_{X,\text{high}})) \) of degree 1 for \( X = A, B \) or \( C \). Let us start with the following operators:
\[
(3.1) \quad \nabla_X = \sum_{i,j=1}^5 d_{ij} \otimes \theta_{ij}^X,
\]
where \( \theta_{ij}^X \in \text{Hom}(S_X, S_X) \), \( X = A, B \) or \( C \). Namely we take
\[
\theta_{ij}^A = \frac{d}{dx_{ij}}, \quad \theta_{ij}^B = z_i^* \frac{d}{dz_j} - z_j \frac{d}{dz_i}, \quad \theta_{ij}^C = x_{ij}^*.
\]

In order to get the morphisms of \( S_{X,\text{high}} \) we need following more specific realizations of these spaces.

**Proposition 3.2.** The \( g_0 \)-module \( S_{C,\text{high}} \) is equal to a factor of the polynomial ring \( S_C \) by the ideal \( S_{C,\text{low}} \) generated by the relations:
\[
(3.2a) \quad x_{ab}^* x_{cd}^* - x_{ac}^* x_{bd}^* + x_{ad}^* x_{bc}^* = 0 \quad \text{for } a,b,c,d = 1, \ldots, 5,
\]
\[
(3.2b) \quad x_{ab}^* z_i^* - x_{ac}^* z_b^* + x_{ad}^* z_c^* = 0 \quad \text{for } a,b,c = 1, \ldots, 5.
\]

This follows from the fact that the orbit of the sum of highest weight vectors in \( F(0,0,0,0) \oplus F(0,0,1,0) \) is a spherical variety.

**Proposition 3.3.** The \( g_0 \)-module \( S_{B,\text{high}} \) is equal to a factor of the polynomial ring \( S_B \) by the ideal \( S_{B,\text{low}} \) generated by the relation:
\[
(3.3) \quad \sum_{i=1}^5 z_i z_j^* = 0.
\]

Similarly we use the fact that the orbit of the sum of highest weight vectors in \( F(1,0,0,0) \oplus F(0,0,0,1) \) is a spherical variety.

**Proposition 3.4.** The \( g_0 \)-module \( S_{A,\text{high}} \) is equal to the subspace of all polynomials \( f \in S_A \) that satisfy the following equations:
\[
(3.4a) \quad \left( \frac{d}{dx_{ab}} \frac{d}{dx_{cd}} - \frac{d}{dx_{ac}} \frac{d}{dx_{bd}} + \frac{d}{dx_{ad}} \frac{d}{dx_{bc}} \right) f = 0 \quad \text{for } a,b,c,d = 1, \ldots, 5,
\]
\[
(3.4b) \quad \left( \frac{d}{dx_{ab}} \frac{d}{dz_c} - \frac{d}{dx_{ac}} \frac{d}{dz_b} + \frac{d}{dx_{ac}} \frac{d}{dx_{bc}} \right) f = 0 \quad \text{for } a,b,c = 1, \ldots, 5.
\]

We notice that the realization of \( S_C \) as differential operators on \( S_A \) makes the perfect duality. The proposition means that \( S_{A,\text{high}} \) coincides with the orthogonal complement to \( S_{C,\text{low}} \), which is clear because \( S_{A,\text{high}} \) is dual to \( S_{C,\text{high}} \).

**Corollary 3.5.** (a) The operator \( \nabla_A \) preserves the subspace \( S_{A,\text{high}} \) of \( S_A \) and therefore defines a map \( \nabla_A \in \text{End}(S_{A,\text{high}}) \).
(b) The operator \( \nabla_B \) preserves the ideal \( S_{B,\text{low}} \) of \( S_B \) and thus determines a map of the factor rings \( \nabla_B \in \text{End}(S_{B,\text{high}}) \).
(c) The operator \( \nabla_C \) preserves the ideal \( S_{C,\text{low}} \) of \( S_C \) and determines a map of the factor rings \( \nabla_C \in \text{End}(S_{C,\text{high}}) \).
The statements (a) and (c) are evident. For (b) we notice that $\nabla_B$ acts as a derivative, thus it is enough to check that it annihilates the generator of the ideal $S_{B,\text{low}}$ given by (3.3), which is straightforward.

To keep with the notations in [KR3] we let:

$$V_A = S_{A,\text{high}}, \quad V_B(\text{resp. } C) = S_{B(\text{resp. } C)}/S_{B(\text{resp. } C),\text{low}} \simeq S_{B(\text{resp. } C),\text{high}}.$$  

**Theorem 3.6.** (a) The operators $\nabla_X$ define $E(5,10)$-morphisms $M(V_X) \to M(V_X)$ ($X = A, B$ or $C$).

(b) Morphism $\nabla_X$ restricted to $M = M(n_1, n_2, n_3, n_4)$ is trivial if $X = A$ and $M = M(m, 0, 0, 0)$, or $X = B$ and $M = M(0, 0, 0, n)$.

(c) The non-zero morphisms $\nabla_X$ are morphisms of degree 1.

**Proof.** It is immediate to see that $g_0 : \nabla_X = 0$. By Remark 1.4 we conclude that there exist the corresponding $(U_\cdot \oplus g_0)$-morphisms $M(S_X) \to M(S_X)$ that we permit ourselves to denote by the same symbols $\nabla_X$. These are evidently morphisms of degree 1.

In order to apply Proposition 1.1 and get $E(5,10)$-morphisms of modules $S_{X,\text{high}}$ we need to check that

$$g_1 : \nabla_X(s) = 0 \text{ for } s \in S_{X,\text{high}}.$$  

Now in order to check (3.5) we may use Remark 1.3 with $t = x_5d_{45}$. Namely

$$(x_5d_{45}) \sum_{i,j=1}^5 d_{ij} \otimes \theta_{ij}^X(s) =$$

$$= [x_5d_{45}, d_{12}] \otimes \theta_{12}^X(s) + [x_5d_{45}, d_{13}] \otimes \theta_{13}^X(s) + [x_5d_{45}, d_{23}] \otimes \theta_{23}^X(s) + 0$$

$$= (x_5\partial_3) \otimes \theta_{12}^X(s) - (x_5\partial_2) \otimes \theta_{13}^X(s) + (x_5\partial_1) \otimes \theta_{23}^X(s).$$

It is not difficult to check that for $X = A$ the right hand side is equal to zero modulo relations (3.4).

Now let us consider $X = B$. Here for $s \in S_B$

$$(x_5\partial_c)\theta_{ab}^B(s) = \left( z_5z_5^e \frac{d}{dz_5} \left( z_5^e - z_5 \right) + \frac{d}{dz_5} \left( z_5 - z_5^e \right) \right) \theta_{ab}^X(s).$$

This immediately gives us zero at the right hand side of (3.6).

When $X = C$ we have

$$(x_5\partial_c)\theta_{ab}^C(s) = -z_c^a \theta_{ab}^X(s),$$

thus the right hand side of (3.6) is zero modulo the relations (3.2b).  

The non-zero maps $\nabla_X$ are illustrated in Figure 2.

The nodes in the quadrants $A, B$ and $C$ represent generalized Verma modules $M(m, 0, 0, 0)$, $M(0, 0, m, n)$ and $M(m, 0, 0, n)$, respectively. The arrows represent the $E(5,10)$-morphisms $\nabla_X$, $X = A, B$ or $C$ in the respective quadrants.

**Proposition 3.7.** $(\nabla_X)^2 = 0$ ($X = A, B$ or $C$).

It is not difficult to check the following lemma.

**Lemma 3.8.** Equation $(\nabla_X)^2 = 0$ for $\nabla_X \in \text{End}_L(M(V_X))$ is equivalent to the system of equations:

$$\theta_{ab}^X \theta_{ca}^X = \theta_{ac}^X \theta_{bd}^X + \theta_{ad}^X \theta_{bc}^X = 0 \text{ for } a, b, c, d = 1, \ldots, 5.$$
For $X = A$ the equations follow from Proposition 3.4, for $X = C$ the equations follow immediately from Proposition 3.2. In the case $X = B$ denote

$$[ab/cd] = x_a^* x_b^* \frac{d}{dx_c} \frac{d}{dx_d}.$$

Then we have

$$\theta^B_{ab} \theta^B_{cd} = [ac/bd] - [ad/bc] - [bc/ad] + [bd/ac],$$

and similarly

$$\theta^B_{ac} \theta^B_{bd} = [ab/cd] - [ad/bc] - [bc/ad] + [cd/ab],$$

$$\theta^B_{ab} \theta^B_{cd} = [ac/bd] - [bd/ac] + [cd/ab].$$

The equation follows.

**Theorem 3.9.** All the morphisms of degree 1 are the following:

(a) $\nabla_A : M(n_1, n_2, 0, 0) \to M(n_1, n_2 - 1, 0, 0)$ for $n_1 \geq 0, n_2 > 0$;

(b) $\nabla_B : M(n_1, 0, 0, n_4) \to M(n_1 - 1, 0, 0, n_4 + 1)$ for $n_1 > 0, n_4 \geq 0$;

Figure 1. Morphisms of degree 1.
(c) $\nabla_C : M(0, 0, n_3, n_4) \to M(0, 0, n_3, n_4 + 1)$ for $n_3 \geq 0$, $n_4 \geq 0$.

Remark 3.10. In short these are the morphisms represented in Figure 2, i.e. all the non-zero morphisms $\nabla_X$.

Proof. A morphism $\varphi : M(\bar{n}') \to M(\bar{n}'')$ of degree 1 is defined according to Proposition 1.1 by $\Phi$ of the form

$$\Phi = \sum_{i,j=1}^{5} d_{ij} \otimes \theta_{ij},$$

Because of (1.3a) the basis $\{\theta_{ij}\}$ is dual to $\{d_{ij}\}$ and that means

$$\Phi = (x_q \partial_b) \theta_{ij} = -\delta_{ai} \theta_{bj} - \delta_{aj} \theta_{ib},$$

The equation (1.3b) where we consider $t = x_q d_{pq}$ gives (the calculations are similar to those in (3.6)):

$$(x_q d_{pq}) \sum_{i,j=1}^{5} d_{ij} \otimes \theta_{ij}(s) =$$

$$= (x_q \partial_c) \otimes \theta_{ab}(s) - (x_q \partial_b) \otimes \theta_{ac}(s) + (x_q \partial_a) \otimes \theta_{bc}(s) = 0,$$

or

$$(x_q \partial_c)\theta_{ab}(s) + (x_q \partial_b)\theta_{ca}(s) + (x_q \partial_a)\theta_{bc}(s) = 0,$$

for different $a, b, c, q$.

Choose $s$ to be the highest weight vector in $F(\bar{n}')$.

Suppose first that $\theta_{45}(s) \neq 0$ and $a, q \leq 3$, $b = 4$, $c = 5$. Then (3.9) and (3.8) imply

$$(x_q \partial_a) \otimes \theta_{45}(s) = -\theta_{a4} (x_{q} \partial_{1})s - \theta_{5a} (x_{q} \partial_{4})s = 0.$$

Therefore the weight $\text{wt}(\theta_{45}(s))$ has the form $(0, 0, m, n)$ for some $m, n \geq 0$. We conclude that

$$\bar{n}' = \text{wt}(s) = (0, 0, m + 1, n).$$

This clearly implies $\varphi = \nabla_C$.

We assume $\theta_{45}(s) = 0$ for the following.

If we apply $x_c \partial_q$ to (3.9) we get

$$(3.10) \quad \theta_{ab}(h_{cq}s) + \theta_{aq}(x_c \partial_q)(s) - \theta_{aq}(x_q \partial_b)(s) + \theta_{bc}(x_c \partial_q)(s) - \theta_{bq}(x_q \partial_c)(s) = 0.$$

For $a = 3, b = 5, c = 2, q = 4$ it becomes

$$h_{24} \theta_{35}(s) = -\theta_{23}(x_2 \partial_3)(s) + \theta_{43}(x_4 \partial_3)(s) - \theta_{52}(x_2 \partial_5)(s) + \theta_{54}(x_4 \partial_5)(s) = (x_3 \partial_3)\theta_{54}(s) = 0.$$

Suppose that $\theta_{35}(s) \neq 0$. Then $\theta_{35}(s)$ is a highest weight vector and its weight has the form $\text{wt}(\theta_{35}(s)) = (m, 0, 0, n)$ for some $m, n \neq 0$. But when $\text{wt}(s) = (m, 0, 0, n) - (0, 1, -1, 1)$ is not dominant.

So we have $\theta_{35}(s) = \theta_{45}(s) = 0$.

If $\theta_{25}(s) \neq 0$, then it is a highest weight vector and similarly we may use (3.10) for $a = 2, b = 5, c = 1, q = 4$. We come to $h_{14} \theta_{25}(s) = 0$, which gives an impossible weight for $\theta_{25}(s)$.

Thus $\theta_{25}(s) = 0$.

Substitute $a = 1, b = 5, 1 < c, q < 5$ in (3.10)

$$h_{cq} \theta_{15}(s) = -\theta_{1c}(x_c \partial_q)(s) + \theta_{1q}(x_q \partial_c)(s) + (x_c \partial_q)\theta_{5q}(s) - (x_q \partial_c)\theta_{5q}(s) = 0.$$

We see that if $\theta_{15}(s) \neq 0$, then it is the highest weight vector with the weight $\text{wt}(\theta_{15}(s)) = (m, 0, 0, n)$. It is easy to conclude that $\varphi = \nabla_B$ in this case.
We have all $\theta_{ij} = 0$. Let $1 < a, b < 5$. Again (3.10) gives
$$h_{12} \theta_{ab}(s) = 0.$$ 
It follows that non of the vectors $\theta_{ab}(s)$ can be the high weight vector, they all are to be zero. Only $\theta_{1j}(s)$ for $j = 2, 3, 4$ could be non-zero.

For $b = 3, 4$ we have
$$h_{25} \theta_{1b}(s) = \theta_{12}(x_2 \partial_5)(s) - (x_5 \partial_2)\theta_{15}(s) - (x_2 \partial_1)\theta_{2b}(s) + (x_5 \partial_4)\theta_{55}(s) = 0.$$ 
So $\text{wt}(\theta_{1b}(s))$ should be $(m, 0, 0, 0)$ and this is not possible. The only choice is $\theta_{1b}(s) = 0$ for $b = 3, 4$.

We are left with the case when all $\theta_{ab}(s) = 0$ except $\theta_{12}(s)$, which is the highest weight vector. Then similarly we get $h_{35} \theta_{12}(s) = 0$ and therefore $\text{wt}(\theta_{12}(s)) = (m, n, 0, 0)$. Then $\varphi = \nabla A$. □

4. Morphisms of degree 4.

Our goal is to construct a morphism of degree 4.

Let us start with the formula
$$t = \sum u_a \otimes \bar{z}^a,$$
where $\{\bar{z}^a\}$ is the monomial basis of $S^4(C^5)$ and $\{u_a\}$ is the dual basis of the irreducible $\mathfrak{sl}_5$-submodule $S^4(C^5)$ in $U_-$ with the highest vector $d_{12}d_{13}d_{14}d_{15}$. This implies
$$u_{(30000)} = d_{12}d_{13}d_{14}d_{15}.$$ 
It is more convenient to write the multi-index in the "multiplicative" form, $[1^3]$ instead of (30000) and so on. Then
$$u_{(122)} = u_{(21000)} = d_{12}d_{23}d_{14}d_{15} + d_{12}d_{13}d_{24}d_{15} + d_{12}d_{13}d_{14}d_{25}.$$ 
The expressions of this type are not unique, for example
$$u_{(22)} = d_{21}d_{23}d_{24}d_{25} = d_{12}d_{23}d_{24}d_{25},$$ 
$$u_{(33)} = d_{31}d_{32}d_{34}d_{35} = d_{13}d_{23}d_{34}d_{35},$$ 
$$u_{(44)} = d_{41}d_{42}d_{43}d_{45} = d_{14}d_{24}d_{34}d_{45},$$ 
but each $u_a$ can be written as a sum of monomials in $d_{ij}$ of degree 4.

**Theorem 4.1.** (a) For each $n_1 \geq 3$ there exist a morphism of degree 4
$$t_{AB} : M(n_1, 0, 0, 0) \rightarrow M(n_1 - 3, 0, 0, 0).$$
(b) For any $n_4 \geq 0$ there exist a morphism of degree 4
$$t_{BC} : M(0, 0, 0, n_4) \rightarrow M(0, 0, 0, n_4 + 3).$$

**Remark 4.2.** We may also write $t_{AB}$ as a morphism of the combined Verma module
$$M(C[z_i]) = \oplus M(n, 0, 0, 0) \text{ in itself, } \quad t_{AB} : M(C[z_i]) \rightarrow M(C[z_i])$$
in itself given by (4.1) with $\bar{z}_i = \partial_i$. Similarly
$$t_{BC} : M(C[z_i]) \rightarrow M(C[z_i])$$
is a morphism given by (4.1) with $\bar{z}_i$ equal to the multiplication by $z_i^*$. 
Proof of the theorem. We see immediately that Remark 4.2 determines a \((L, \oplus g_0)\)-morphism of Verma modules. One has to check the condition (1.3b) where we may suppose that \(a\) is the highest weight vector of \(A\). For \(t_{AB}\) this means
\[
x_{5d45} \cdot t_{AB} \otimes z^n_1 = n(n-1)(n-2)x_{5d45} \cdot d_{12}d_{13}d_{14}d_{15} \otimes z^{n-3}_1 = n(n-1)(n-2)(x_5 \partial_1d_{13}d_{14}d_{15} + d_{12}(x_5 \partial_2)d_{14}d_{15}) \otimes z^{n-3}_1 = 0.
\]

The case \((b)\) amounts to a much more complicated computation. We shall write only few hints. Evidently the symmetric group \(S_5\) acts on the indices. The following simple observation helps in the computations.

Lemma 4.3. For \(\pi \in S_5\) we have
\[
\pi \cdot u_\alpha = \text{sign}(\pi) u_{\pi(\alpha)}.
\]

Clearly we have the equalities
\[
u_{[12]} = (x_2 \partial_1) u_{[13]} \quad \text{and} \quad u_{[123]} = (x_3 \partial_1) u_{[12]}.
\]
that help us to write the elements \(u_\alpha\) explicitly. We leave it to the reader to check the rest of the computation. \(\square\)

Proposition 4.4. Whenever defined, the following compositions are zero
\[
t_{AB} \cdot \n_A = 0, \quad \n_B \cdot t_{AB} = 0, \\
t_{BC} \cdot \n_B = 0, \quad \n_C \cdot t_{BC} = 0, \\
t_{BC} \cdot t_{AB} = 0.
\]
The proof is more or less immediate.

This means that we have got complexes that are shown in the following picture.

5. Morphisms of other degrees.

There morphisms of degrees 2, 3 and 5 that can be constructed as compositions of the morphisms of degree 1 and 4 described above.

Proposition 5.1. There are morphisms of degree 2
\[
\n_{AB} = \n_B \cdot \n_A : M(m, 1, 0, 0) \to M(m-1, 0, 0, 1) \quad \text{for } m > 0, \\
\n_{BC} = \n_C \cdot \n_B : M(1, 0, 0, n) \to M(0, 0, 1, n+1) \quad \text{for } n \geq 0, \\
\n_{AC} = \n_C \cdot \n_A : M(1, 0, 0, 0) \to M(0, 0, 0, 1).
\]

It is not difficult to see that the morphisms are non-zero and they are evidently morphisms of degree 2.

Proposition 5.2. There is a morphism of degree 3
\[
\n_{ABC} = \n_C \cdot \n_B \cdot \n_A : M(1, 1, 0, 0) \to M(0, 0, 1, 1).
\]

It is a simple calculation to check that the morphism in question is non-zero.

Proposition 5.3. There are two morphisms of degree 5
\[
t' = \n_C \cdot t_{AB} : M(3, 0, 0, 0) \to M(0, 0, 1, 0) \quad \text{and} \\
t'' = t_{BC} \cdot \n_A : M(0, 1, 0, 0) \to M(0, 0, 0, 3).
\]
Again we only need to notice that the morphisms are non-zero which amounts to some calculation. It is tempting to believe that we have found already all morphisms between degenerate Verma modules.

**Conjecture 5.4.** The morphisms listed in Propositions 5.1, 5.2, 5.3 and Theorems 3.9, 4.1 are all morphisms between degenerate (minimal) Verma modules for \( E(5,10) \), in particular there are no morphisms of degrees larger than 5.

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**Author addresses:**

NIISI RAN, section Mathematics,  
Nahimovskij prosp.36, kor.1,  
Moscow 117218 RUSSIA

Dept of Mathematics, GU-VSE,  
Moscow, RUSSIA

email: rrudakov@gmail.com