AFFINE LIE ALGEBRAS REPRESENTATIONS INDUCED FROM WHITTAKER MODULES

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Abstract. We use induction from parabolic subalgebras with infinite-dimensional Levi factor to construct new families of irreducible representations for arbitrary Affine Kac-Moody algebra. Our first construction defines a functor from the category of Whittaker modules over the Levi factor of a parabolic subalgebra to the category of modules over the Affine Lie algebra. The second functor sends tensor products of a module over the affine part of the Levi factor (in particular any weight module) and of a Whittaker module over the complement Heisenberg subalgebra to the Affine Lie algebra modules. Both functors preserves irreducibility when the central charge is nonzero.

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1. Introduction

Induced representations play very important role in representation theory of Lie algebras as a main tool to construct new irreducible modules and to approach the problem of its classification. Parabolic induction associated with a parabolic subalgebra of an affine Kac-Moody algebra defines a functor from the category of modules over the Levi factor of the parabolic subalgebra to the category of induced modules for the affine Lie algebra.
In this paper we consider parabolic induction when a parabolic subalgebra has infinite-dimensional Levi factor. Study of such induced modules goes back to [JK], [F1], [F2], [BBFK], [FK1], [FK2], [FK18] and references therein. Their structure is understood in the case when the central element acts as a nonzero scalar for some categories of modules over the Levi factor of a parabolic subalgebra. In particular, in a recent paper [GKMOS], it was shown that parabolic induction preserves irreducibility for tensor weight inducing modules with nonzero central charge. These modules are constructed as tensor product of a weight module over the "affine part" of the Levi factor and a weight module over its "Heisenberg part". Previously, particular cases were considered in [BBFK], [FK1], [FK2], [FK18]. Also, free field realizations of induced modules were obtained in [JK], [C], [CF] and [FKS]. In particular, the last paper provided a uniform construction of free field realizations for an arbitrary affine Kac–Moody algebra.

The purpose of current paper is to extend the results of [GKMOS] to the case when the inducing representations are Whittaker modules or mixed tensor modules of weight and Whittaker representations.

Whittaker modules for finite-dimensional semisimple Lie algebras were introduced in [KO] and corresponding parabolic induction (for parabolic subalgebras with finite-dimensional Levi factor) from irreducible Whittaker modules was considered in [Mc] and [MS]. Whittaker modules for Affine Lie algebras were studied in [ALZ] and [GZ]. Unlike in the classical case the structure of Whittaker modules is well understood only for Affine $\mathfrak{sl}(2)$. For example, the universal non-degenerate Whittaker modules with noncritical level and non-degenerate Whittaker character are irreducible as in the classical case, while the Whittaker modules with degenerate Whittaker character and critical level are connected with the vertex theory [ALZ]. The structure problem for Whittaker modules remains open in general.

A different family of imaginary Whittaker modules for non-twisted affine Lie algebras was considered in [Chr] for the simplest case of a parabolic subalgebra with an infinite-dimensional Levi factor which is a sum the Heisenberg and Cartan subalgebras. The inducing modules are Whittaker modules for the Heisenberg subalgebra. When the central charge is nonzero corresponding induced modules are irreducible.

We consider arbitrary parabolic subalgebra with infinite-dimensional Levi factor and consider two induction functors: from the category of Whittaker modules over the Levi factor and from the category of mixed tensor modules. In both cases we show that the functors preserve irreducibility when the central charge is nonzero. This allows to construct new families of irreducible representations for all Affine Lie algebras.

We briefly describe our main results. Let $\widehat{\mathfrak{g}}$ be an affine Kac-Moody algebra with a Cartan subalgebra $\mathfrak{h}$ and $\widehat{\mathfrak{p}}$ a parabolic subalgebra $\widehat{\mathfrak{g}}$ with the Levi decomposition

$$\widehat{\mathfrak{p}} = \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{u}}_+,$$

where $\widehat{\mathfrak{l}}$ is an infinite-dimensional Levi factor. For a $\widehat{\mathfrak{p}}$-module $V$ such that $\widehat{\mathfrak{u}}_+V = 0$ we consider a $\widehat{\mathfrak{p}}$-induced $\widehat{\mathfrak{g}}$-module $M_{\widehat{\mathfrak{p}}}(V)$. In this paper we determine the irreducibility of $M_{\widehat{\mathfrak{p}}}(V)$ for several families of irreducible $\widehat{\mathfrak{p}}$-modules $V$ with nonzero central charge, including imaginary Whittaker modules and generalized imaginary Whittaker modules.
Let $\widehat{I}_1$ be a Lie subalgebra of $\widehat{I}$ generated by the root subspaces of $\widehat{I}$ and by $d$, $H_1$ is its Cartan subalgebra, and $\mathfrak{h}_1$ is the finite dimensional part of $H_1$. Then $\widehat{I} = \widehat{I}_1 \oplus \mathfrak{h}_1^\perp$, where $\mathfrak{h}_1^\perp \subset H$ is the orthogonal complement of $\mathfrak{h}_1$ with respect to the Killing form. Hence $[\widehat{I}_1, \mathfrak{h}_1^\perp] = 0$ and $(H \cap \widehat{I}_1) \cup \mathfrak{h}_1^\perp = H$.

If $\widehat{I}_1$ contains nonzero real root subspaces of $\widehat{I}$ then assume that $\widehat{I}_1$ is the Lie subalgebra of $\widehat{I}$ generated by them and by $d$, or equivalently $\widehat{I}_1$ is the affinization (twisted or untwisted) of a semisimple Lie subalgebra $I_1$. In this case the Levi subalgebra $\widehat{I}$ does not necessarily contain the whole Heisenberg subalgebra.

We summarize our first main result:

**Theorem 1.1.** Let $\widehat{I} = \widehat{I}_1 \oplus \mathfrak{h}_1^\perp$, $a \in \mathbb{C} \setminus \{0\}$, $\lambda \in (\mathfrak{h}_1^\perp)^*$, $\eta : U(\widehat{I}_1^\perp) \to \mathbb{C}$ a nonzero homomorphism and $\widehat{W}_{n,a}$ an irreducible Whittaker $\widehat{I}_1$-module of type $\eta$, central charge $a$ and with the action of $\mathfrak{h}_1^\perp$ defined by $\lambda$. Then the induced generalized imaginary Whittaker $\widehat{g}$-module $M_{\widehat{p}}(\widehat{W}_{n,a})$ is irreducible. In particular, if $\widehat{I} = G + H$, where $G$ is the Heisenberg subalgebra of $\widehat{g}$, $\eta : U(G_+) \rightarrow \mathbb{C}$ an algebra homomorphism such that $\eta|_{\mathfrak{g}_{n,a}} \neq 0$ for infinitely many integer $n > 0$ and $M_{n,a}$ the Whittaker $G \oplus Cd$-module of type $\eta$ and central charge $a$, then the imaginary Whittaker $M_{\widehat{p}}(\widehat{M}_{n,a})$ is irreducible.

The first part of the theorem is proved in Theorem 4.2 while the second part is Theorem 3.3. The second part generalizes the results of [Chr] to all affine Lie algebras. Theorem 1.1 produces new families of irreducible modules for affine Kac-Moody algebras from irreducible Whittaker modules. Moreover, in the case of an non-twisted Affine Lie algebra, this construction can be extended to the inducing modules $(W_{n,a} \otimes E(\mu, a_1, \ldots, a_m)) \otimes \mathbb{C}[d]$, where $W_{n,a}$ is an irreducible Whittaker module of type $\eta$ and central charge $a$ over the central extension of the loop algebra and $E(\mu, a_1, \ldots, a_m)$ is the evaluation module (see Theorem 4.4).

Next we consider induced modules from mixed tensor $\widehat{I}$-modules obtained as a tensor product of irreducible modules for affine Lie subalgebras and Whittaker modules for Heisenberg subalgebras. The case when both components of the tensor module (over the affine subalgebra and the Heisenberg subalgebra) are weight modules was considered in [GKMOS].

Let $\widehat{p} = \widehat{I} \oplus \hat{u}_+$ be a parabolic subalgebra of $\widehat{g}$ containing the whole Heisenberg subalgebra. Then

$$\widehat{I} = \widehat{I}_0 + \widehat{G}(\widehat{I})^\perp \oplus \mathfrak{h}_1^\perp,$$

where $\widehat{I}_0 = \widehat{I}_0 \oplus Cd$ is an Affine subalgebra of $\widehat{I}$ generated by the real root subspaces and extended by the derivation $d$ and $\widehat{G}(\widehat{I})^\perp$ is the orthogonal complement of the Heisenberg subalgebra of $\widehat{I}_0$ extended by the derivation. Consider the natural triangular decomposition

$$\widehat{G}(\widehat{I})^\perp = \widehat{G}(\widehat{I})^\perp_+ \oplus (\mathbb{C}c \oplus Cd) \oplus \widehat{G}(\widehat{I})^\perp_+,$$

where $\mathfrak{g}_{k,a} \cap \widehat{G}(\widehat{I})^\perp \subset \widehat{G}(\widehat{I})^\perp_+$ if and only if $k \in \mathbb{Z}_+$.

Our second main result is the following (see Theorem 4.8).
Theorem 1.2. Let $\lambda \in (\mathfrak{h}_0^+)^*$, $a \in \mathbb{C} \setminus \{0\}$, $M$ an $\hat{\mathfrak{h}}_0$-module with central charge $a$. Let $\eta : U(\hat{G}(\hat{l})^+) \to \mathbb{C}$ be an algebra homomorphism, and $S_{\eta,a}$ a Whittaker $\hat{G}(\hat{l})^+$-module of type $\eta$ and central charge $a$. Consider a mixed tensor $\hat{l}$-module $M \otimes S_{\eta,a}$ on which $\mathfrak{h}_0^+$ acts via $\lambda$ and $d$ has a tensor product action. Then the induced $\hat{\mathfrak{g}}$-module $M(\hat{\mathfrak{p}}(M \otimes S_{\eta,a})) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{p}})} (M \otimes S_{\eta,a})$ is irreducible if and only if $M \otimes S_{\eta,a}$ is irreducible $\hat{l}$-module.

We note that the first item of Theorem 1.2 is a very general statement which a priori does not impose any conditions on $\hat{\mathfrak{h}}_0$-module $M$. This allows to construct a large family of new irreducible representations for all Affine Lie algebras. In particular, Theorem 1.2 extends Theorem 1.1 to arbitrary parabolic subalgebras with infinite dimensional Levi factors.

2. Preliminaries

2.1. Affine Kac-Moody algebras. Let $\hat{\mathfrak{g}}$ be an affine Kac-Moody algebra with a Cartan subalgebra $H$ and the root system $\Delta$ of $\hat{\mathfrak{g}}$. For a root $\alpha$ denote $\hat{\mathfrak{g}}_\alpha = \{x \in \hat{\mathfrak{g}} \mid [h, x] = \alpha(h)x \text{ for every } h \in H\}$. Then $\hat{\mathfrak{g}}$ has the root decomposition

$$\hat{\mathfrak{g}} = H \oplus (\oplus_{\alpha \in \Delta} \hat{\mathfrak{g}}_\alpha).$$

The set of all imaginary roots is $\Delta^{im} = \{k\delta | k \in \mathbb{Z} \setminus \{0\}\}$. Let $\mathfrak{g}$ be an underlined finite-dimensional simple Lie algebra of $\hat{\mathfrak{g}}$ and $\mathfrak{h} \subset H$ a Cartan subalgebra of $\mathfrak{g}$. Then the real roots of $\hat{\mathfrak{g}}$ have the form $\phi + n\delta$ for some $\phi$ spanned by the roots of $\mathfrak{g}$ and some $n \in \mathbb{Z}$. We also have $H = \mathfrak{h} \oplus \mathbb{C}c \oplus C\hat{\delta}$, where $c$ is a central element and $\hat{\delta} \in \mathfrak{h}$ is such that $\delta(d) = 1$ (derivation of $\hat{\mathfrak{g}}$). For any $x \in \hat{\mathfrak{g}}_{\phi+n\delta}$ we have $[d, x] = nx$.

The Heisenberg subalgebra $G \subset \hat{\mathfrak{g}}$ is defined as follows:

$$G = \oplus_{k \in \mathbb{Z} \setminus \{0\}} \hat{\mathfrak{g}}_{k\delta} \oplus \mathbb{C}c.$$ We have $G = G_- \oplus \mathbb{C}c \oplus G_+$, where $G_{\pm} = \oplus_{k>0} \hat{\mathfrak{g}}_{\pm k\delta}$.

Let $\hat{\mathfrak{p}}$ be a parabolic subalgebra $\hat{\mathfrak{g}}$ with the Levi decomposition

$$\hat{\mathfrak{p}} = \hat{\mathfrak{l}} \oplus \hat{\mathfrak{u}}_+,$$

where $\hat{\mathfrak{l}}$ is the Levi factor and $\hat{\mathfrak{u}}_+$ is the radical.

Denote by $\hat{\mathfrak{u}}_-$ the opposite radical, that is if $\hat{\mathfrak{g}}_\alpha \subset \hat{\mathfrak{u}}_+$ then $\hat{\mathfrak{g}}_{-\alpha} \subset \hat{\mathfrak{u}}_-$. Then we have $\hat{\mathfrak{g}} = \hat{\mathfrak{p}} \oplus \hat{\mathfrak{u}}_-$.
Let $V$ an $\hat{\mathfrak{g}}$-module. Then we have a $\hat{\mathfrak{p}}$-module structure on $V$ by setting $\hat{\mathfrak{u}}_+ V = 0$. Define an induced $\hat{\mathfrak{g}}$-module

$$M_{\hat{\mathfrak{p}}}(V) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{p}})} V,$$

which is isomorphic to $U(\hat{\mathfrak{u}}_-) \otimes V$ as a vector space, where $\hat{\mathfrak{u}}_-$ is the opposite radical of $\hat{\mathfrak{p}}$.

If $V$ is an irreducible $\hat{\mathfrak{g}}$-module then the action of $c$ on $V$ is scalar and its value is called the central charge of $V$.

A $\hat{\mathfrak{g}}$-module $\hat{\mathfrak{g}}$-module $V$ is called weight (with respect to the Cartan subalgebra $H$) if

$$M = \bigoplus_{\lambda \in H^*} M_\lambda,$$

where $M_\lambda = \{ v \in M \mid hv = \lambda(h)v, \forall h \in H^* \}$.

If $\mathfrak{a} \subset \hat{\mathfrak{g}}$ is a Lie subalgebra then weight structure on $\mathfrak{a}$-modules is determined with respect to $\mathfrak{a} \cap H$.

2.2. Whittaker modules for Affine Lie algebras. Consider the triangular decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus H \oplus \hat{\mathfrak{g}}_+$, where $\hat{\mathfrak{g}}_\pm = \sum_{\alpha \in \Delta_{\pm}} \widehat{\mathfrak{g}}_\alpha$ and $\Delta_{\pm}$ are positive and negative roots.

Let $\eta : U(\hat{\mathfrak{g}}_+) \rightarrow \mathbb{C}$ be an algebra homomorphism with $\eta(\hat{\mathfrak{g}}_+) \neq 0$. Then $\eta(\hat{\mathfrak{g}}_+ \hat{\mathfrak{g}}_+) = 0$. In particular, $\eta(\hat{\mathfrak{g}}_{kd}) = 0$ for all $k > 0$.

A $\hat{\mathfrak{g}}$-module $V$ is a Whittaker module of type $\eta$ if $V$ is generated by a Whittaker element $v$ such that $xv = \eta(x)v$ for any $x \in U(\hat{\mathfrak{g}}_+)$. For $a \in \mathbb{C}$ consider a 1-dimensional $\hat{\mathfrak{g}}_+ \oplus \mathbb{C}c$-module $\mathbb{C}v$ such that $v$ is a Whittaker element of type $\eta$ and $cv = av$. Define the following universal Whittaker module of type $\eta$ and central charge $a$:

$$V_{\eta,a} = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+ \oplus \mathbb{C}c)} \mathbb{C}v.$$

Any quotient of $V_{\eta,a}$ is a Whittaker module of type $\eta$ [GZ] Lemma 5.2].

Denote by $\mathfrak{g}$ the Affine Lie subalgebra of $\hat{\mathfrak{g}}$ obtained by removing the derivation $d$: $\mathfrak{g} = \hat{\mathfrak{g}}_- \oplus \mathfrak{h} \oplus \mathbb{C}c \oplus \hat{\mathfrak{g}}_+$. Whittaker $\mathfrak{g}$-modules of type $\eta : U(\hat{\mathfrak{g}}_+) \rightarrow \mathbb{C}$ are defined similarly as above. If $W_{\eta,a}$ is a Whittaker $\mathfrak{g}$-modules of type $\eta$ and central charge $a$ then we get the induced Whittaker $\hat{\mathfrak{g}}$-module of type $\eta$:

$$\hat{W}_{\eta,a} = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g})} W_{\eta,a} \simeq \mathbb{C}[d] \otimes W_{\eta,a}.$$

Note that the action of $d$ is free on $\hat{W}_{\eta,a}$. On the other hand, for the same Whittaker function there could exist irreducible Whittaker modules with free and with diagonalizable action of $d$ [ALZ].

3. Imaginary Whittaker modules for Affine Lie algebra

3.1. Whittaker modules for Heisenberg subalgebra. Let $\eta : U(G_+) \rightarrow \mathbb{C}$ be an algebra homomorphism with $\eta(G_+) \neq 0$. A $G$-module $V$ is a Whittaker module of type $\eta$ if $V$ is generated by a Whittaker element $v$ such that $xv = \eta(x)v$ for any $x \in U(G_+)$ [Ch].
For $a \in \mathbb{C}$ consider a 1-dimensional $G_+ \oplus \mathbb{C}c$-module $\mathbb{C}v$ such that $v$ is a Whittaker element of type $\eta$ and $cv = av$. Define a Whittaker module of type $\eta$ as follows:

$$M_{\eta,a} = U(G) \otimes_{U(G_+ \oplus \mathbb{C}c)} \mathbb{C}v.$$ 

Note that module $M_{\eta,a}$ is not $\mathbb{Z}$-graded. The following properties of $M_{\eta,a}$ were shown in [Chr, Proposition 5, Proposition 6, Corollary 8]:

**Proposition 3.1.**
- The module $M_{\eta,a}$ is $U(G_-)$-free;
- The module $M_{\eta,a}$ is irreducible if $a \neq 0$; In this case $M_{\eta,a}$ is the unique (up to isomorphism) irreducible Whittaker module of type $\eta$ with central charge $a$.

Next we extend the Heisenberg subalgebra by the derivation $d$ and set $\tilde{G} = G \oplus \mathbb{C}d$.

A Whittaker $\tilde{G}$-module of type $\eta$ and central charge $a \in \mathbb{C}$ is defined as follows:

$$\tilde{M}_{\eta,a} = U(\tilde{G}) \otimes_{U(G_+ \oplus \mathbb{C}c)} \mathbb{C}v.$$ 

**Proposition 3.2.**
- The module $\tilde{M}_{\eta,a}$ is $U(G_- \oplus \mathbb{C}d)$-free;
- The module $\tilde{M}_{\eta,a}$ is a $G$-submodule of $\tilde{M}_{\eta,a}$;
- If $\eta|_{\mathfrak{b}_n} \neq 0$ for infinitely many integer $n > 0$ then the module $\tilde{M}_{\eta,a}$ is irreducible. In this case $\tilde{M}_{\eta,a}$ is the unique (up to isomorphism) irreducible Whittaker module of type $\eta$ with central charge $a$.

**Proof.** Follows from Propositions 20, 21, 25 and Corollary 28 of [Chr].

### 3.2. Imaginary Whittaker modules

Consider a parabolic subalgebra $\widehat{\mathfrak{p}} = \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{u}}_+$, where $\widehat{\mathfrak{l}} = G + H = (G + \mathbb{C}d) \oplus \mathfrak{h}$ and $[G, \mathfrak{h}] = 0$.

**Theorem 3.3.** Let $a \in \mathbb{C} \setminus \{0\}$, $\lambda \in (\mathfrak{h})^*$ and $\eta : U(G_+) \to \mathbb{C}$ an algebra homomorphism such that $\eta|_{\mathfrak{b}_n} \neq 0$ for infinitely many integer $n > 0$. Then the induced $\widehat{\mathfrak{g}}$-module

$$M_{\widehat{\mathfrak{p}}}(\widetilde{M}_{\eta,a}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{p}})} \widetilde{M}_{\eta,a}$$

is irreducible, where the action of $\mathfrak{h}$ on $\widetilde{M}_{\eta,a}$ is defined via $\lambda$.

**Proof.** For non-twisted Affine Lie algebras the statement is [Chr, Theorem 50]. For an arbitrary Affine Lie algebra the result is a particular case of Theorem 3.2 shown below.

### 4. Generalized Imaginary Whittaker modules

Now we consider a general parabolic subalgebra $\widehat{\mathfrak{p}} = \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{u}}_+$ with infinite dimensional Levi factor $\widehat{\mathfrak{l}}$. 
4.1. Inducing from Whittaker modules. Let \( \hat{I}_1 \) be a Lie subalgebra of \( \hat{I} \) generated by the subspaces \( \hat{g}_\alpha \cap \hat{I}, \alpha \in \Delta, \) and \( d. \) Set \( G(\hat{I}) = \hat{I} \cap G \) for the Lie subalgebra of \( \hat{I} \) spanned by its imaginary root subspaces. Denote by \( \hat{h}_l = H_l \cap \hat{h} \). Let \( \hat{h}_l^+ \) be the orthogonal complement to \( \hat{h}_l \) with respect to the Killing form. Then we have \( \hat{I} = \hat{I}_1 \oplus \hat{h}_l \), \( H = H_l \oplus \hat{h}_l^+ \) and \( [\hat{I}_1, \hat{h}_l^+] = 0. \) For simplicity we assume now that if \( \hat{I}_1 \) contains nonzero real root subspaces of \( \hat{I} \) then \( \hat{I}_1 \) is generated by them and by \( d. \) This means that \( \hat{I}_1 \) is the affinization (twisted or untwisted) of a semisimple or abelian Lie subalgebra \( I_1. \) Under this assumption we have \( G(\hat{I}) \subset \hat{I}_1 \). Consider the standard triangular decomposition of \( \hat{I}_1: \hat{I}_1 = \hat{I}_1^+ \oplus H_1 \oplus \hat{I}_1^- \), where \( \hat{I}_1^+ = \sum_{\alpha \in \Delta_+} \hat{g}_\alpha \cap \hat{I}_1 \), and set \( G(\hat{I})^\perp = G(\hat{I}) \cap \hat{I}_1^\perp. \) Let \( G(\hat{I})^\perp \) be the orthogonal complement of \( G(\hat{I}) \) (with respect to the Killing form), that is \( G = G(\hat{I}) + G(\hat{I})^\perp, [G(\hat{I})^\perp, \hat{I}_1] = 0 \) and \( G(\hat{I}) \cap G(\hat{I})^\perp = \mathbb{C}c. \)

We have a natural triangular decomposition

\[
G(\hat{I})^\perp = G(\hat{I})_+^\perp \oplus \mathbb{C}c \oplus G(\hat{I})_+^\perp,
\]

where \( \hat{g}_k \cap G(\hat{I})^\perp \subset \hat{G}(\hat{I})_+^\perp \) if and only if \( k \in \mathbb{Z}_{\pm} \). Under our assumption we have \( G(\hat{I})^\perp_+ \subset \hat{u}_+. \)

The general case of an arbitrary reductive \( I_1 \) will be discussed in the next section. If \( \hat{I}_1 \) has no nonzero real root subspaces then without loss of generality we assume that it contains the whole Heisenberg subalgebra, that is \( G(\hat{I}) = G. \)

For a nonzero homomorphism \( \eta: U(\hat{I}_1) \to \mathbb{C} \) and \( a \in \mathbb{C} \setminus \{0\} \) consider an irreducible Whittaker \( \hat{I}_1 \)-module \( \hat{W}_{\eta,a} \) of type \( \eta \) and central charge \( a. \)

**Lemma 4.1.** If \( a \neq 0 \) then \( G(\hat{I})_\cdot \) is torsion free on \( \hat{W}_{\eta,a}. \)

**Proof.** Let \( v_\eta \) be a Whittaker vector in \( \hat{W}_{\eta,a}. \) Set \( \hat{I}_k = \hat{g}_k \cap \hat{I} \). Then \( \hat{g}_k v_\eta = 0 \) for any \( k > 0 \) as \( \hat{I}_k \subset [\hat{I}_1^+, \hat{I}_1^-] \) and \( \eta([\hat{I}_1^+, \hat{I}_1^-]) = 0. \) Suppose that for some nonzero \( x \in \hat{I}_{-k} \) and \( u \in U(\hat{I}_1^-) \) we have \( xu v_\eta = 0 \) while \( uv_\eta \neq 0. \) Assume for simplicity that \( \hat{I} \) is non-twisted and take any nonzero \( \bar{x} \in \hat{I}_k \) such that \( \bar{x} x = c. \) If \( \bar{x}uv_\eta = 0 \) then

\[
0 = \bar{x}xuv_\eta = [\bar{x}, x]uv_\eta = auv_\eta,
\]

which is a contradiction.

Assume \( \bar{x}uv_\eta \neq 0. \) We have

\[
\bar{x}uv_\eta = [\bar{x}, u]v_\eta + u\bar{x}v_\eta = [\bar{x}, u]v_\eta,
\]

since \( \eta(\bar{x}) = 0. \) Applying \( \bar{x} \) sufficiently many times we get that for some \( m > 0, \)

\[
ad(\bar{x})^m(u) \in U(\hat{I}_1^-) \quad \text{and} \quad \bar{x}^muv_\eta = \bar{x}ad(\bar{x})^m(u)v_\eta = [\bar{x}, ad(\bar{x})^m(u)]v_\eta = \eta([\bar{x}, ad(\bar{x})^m(u)])v_\eta = 0.
\]

Choose the smallest \( k > 0 \) such that \( \bar{x}^kuv_\eta = 0 \) but \( \bar{x}^kuv_\eta \neq 0. \) Now we have

\[
0 = \bar{x}^{k+1}xuv_\eta = [\bar{x}^{k+1}, x]uv_\eta + x\bar{x}^{k+1}uv_\eta = (k + 1)x^kuv_\eta,
\]

giving a contradiction.
Similarly, one can extend the arguments above to an arbitrary nonzero element \( y \in U(G(\hat{\mathfrak{l}})_-) \) and show that \( yuv_\eta \neq 0 \) if \( u\nu_\eta \neq 0 \) completing the proof. The twisted case is treated analogously.

\[ \square \]

**Theorem 4.2.** Let \( \hat{\mathfrak{l}} = \hat{\mathfrak{l}}_1 \oplus \mathfrak{h}_1^+ \) with semisimple or abelian \( \mathfrak{l}_1 \), \( \lambda \in (\mathfrak{h}_1^+)^* \), \( a \in \mathbb{C} \setminus \{0\} \), \( \eta \neq 0 \) and \( \hat{\mathcal{W}}_{\eta,a} \) an irreducible Whittaker \( \hat{\mathfrak{l}}_1 \)-module of type \( \eta \) and central charge \( a \). Define the action of \( \mathfrak{h}_1^+ \) on \( \hat{\mathcal{W}}_{\eta,a} \) by \( \lambda \) making it a \( \hat{\mathfrak{g}} \)-module with \( \hat{u}_+ \cdot \hat{\mathcal{W}}_{\eta,a} = 0 \). Then the induced \( \hat{\mathfrak{g}} \)-module

\[ M_{\hat{\mathfrak{g}}}(\hat{\mathcal{W}}_{\eta,a}) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{p}})} \hat{\mathcal{W}}_{\eta,a} \]

is irreducible.

**Proof.** Take any nonzero element of \( v \in M_{\hat{\mathfrak{g}}}(\hat{\mathcal{W}}_{\eta,a}) \). By the PBW theorem the element \( v \) can be written in the following form:

\[ v = \sum_i u_i w_i, \]

where \( u_i \in U(\hat{\mathfrak{u}}_-) \), \( w_i \in \hat{\mathcal{W}}_{\eta,a} \). We assume that \( u_i \) are linearly independent and \( w_i \neq 0 \) for all \( i \). We can also assume that \( v \) is \( \mathfrak{h}_1^+ \)-weight element. Then for each \( i \), \( u_i \in U(\hat{\mathfrak{g}})_{-\phi_i + k_i \delta} \) for some \( \phi_i \) generated by the roots of \( \mathfrak{g} \) and some integer \( k_i \). Decompose each \( \phi_i \) into a sum of simple roots. Note that the total number of simple roots of \( \mathfrak{g} \) in \( \phi_i \) which are not roots of \( \mathfrak{l} \) is the same for all \( i \) due to the fact that \( v \) is an \( \mathfrak{h}_1^+ \)-weight element. Denote this number by \( \tau(v) \). Also, denote by \( \tau_i(v) \) the total number of simple roots of \( \mathfrak{l} \) in \( \phi_i \). Note that \( \tau_i(v) \) can be different from \( \tau_j(v) \) if \( i \neq j \). We proceed by induction on \( \tau(v) \). Suppose that \( \tau(v) = 1 \) and \( \beta \) is a unique root of \( \mathfrak{g} \) in all \( \phi_i \). Then for each \( i \), \( u_i \in \hat{\mathfrak{g}}_{-\phi_i + k_i \delta} \) and \( \phi_i = \beta + \alpha_i \), where \( \alpha_i \) decomposes into a sum of simple roots of \( \mathfrak{l} \). Consider such indices \( i \) for which \( \tau_i(v) \) is the least possible and among those the ones for which \( k_i \) is the least possible. Fix an index \( j \) satisfying all these conditions. Without loss of generality we may assume that \( j = 1 \). For a sufficiently large \( N \) \((N \gg k_i \text{ for all } i)\) choose a nonzero \( u_N \in \hat{\mathfrak{g}}_{\beta + \alpha_1 + (-N-k_1) \delta} \). Then we have

\[ u_N \cdot v = \sum_i u_N u_i w_i = \sum_i [u_N, u_i] w_i = x_{-N} w_1 + \sum_{j \geq 2} y_j^i w_j, \]

where \( x_{-N} \in \hat{\mathfrak{g}}_{-N \delta} \) is nonzero and \( y_j^i \in U(\hat{\mathfrak{l}}) \) for all \( j \). Rewriting the sum if necessary we can assume that all summands are linearly independent. In the twisted case we restrict ourselves to those \( N \) for which \( \beta + \alpha_1 + (-N-k_1) \delta \) is a root. Note \( x_{-N} w_i = 0 \) for all \( i \) if \( N \) is sufficiently large. Moreover, for such \( N \), we have \( \hat{x}_{-N} w_1 \neq 0 \). Indeed, write \( \hat{x}_{-N} = x_{-N}^1 + x_{-N}^u \), where \( x_{-N}^1 \in \hat{\mathfrak{l}}_{-N \delta} \) and \( x_{-N}^u \in G(\hat{\mathfrak{l}})_-^\perp \). Then \( x_{-N}^1 w_1 \neq 0 \) by Lemma 4.1 and \( x_{-N}^u w_1 \neq 0 \) as the central charge is nonzero. Choose \( x_{N}^u \in G(\hat{\mathfrak{l}})_+^\perp \) such that \([x_{N}^u, x_{-N}] = c \). Since \([G(\hat{\mathfrak{l}})_+^\perp, x_{-N}^1] = 0 \) we have

\[ x_{N}^u x_{-N} w_1 = x_{N}^u (x_{-N}^1 + x_{-N}^u) w_1 = a w_1. \]
Hence, $x_{-N}w_1 \neq 0$. We claim now that $u_Nv \neq 0$. Indeed, suppose $u_Nv = 0$ for all admissible $N$. Then choose a nonzero $x_N \in \mathfrak{g}_{N\delta}$ such that $[x_N, x_{-N}] = c$. Suppose that $[x_N, y_N^j]w_j = 0$ for all $j \geq 2$. Then $0 = x_Nu_Nv = x_Nx_{-N}w_1 = aw_1$, which is a contradiction.

Let $[x_N, y_N^j]w_j \neq 0$ for some $j \geq 2$ and assume $j = 2$ for simplicity. Then

$$0 = x_Nx_{-2N}w_1 + \sum_{j\geq 2} x_Ny_{2N}^j w_j = \sum_{j\geq 2} [x_N, y_{2N}^j] w_j = \sum_{j\geq 2} b_j y_{N}^j w_j,$$

for some $b_j \in \mathbb{C}$ and $b_2 \neq 0$. Hence $y_{N}^j w_j, j \geq 2$ are linearly dependent, which is a contradiction.

Suppose now $\tau(v) > 1$. Then the same argument as in the proof of [BBFK, Lemma 5.3] shows that there exists $u \in U(\mathfrak{u}_+)$ such that $u \cdot v = 0$ and $\tau(u \cdot v) < \tau(v)$. Then the proof is completed by induction. □

**Remark 4.3.** One can extend Lemma 4.1 and Theorem 4.2 to the case when the whole Heisenberg subalgebra $G$ is contained in $\mathfrak{l}$, that is $G(\mathfrak{l})^+ \subset \mathfrak{l}$. This will follow from Theorem 4.8 as a particular case.

### 4.2. Inducing from Whittaker and evaluation modules.

In this section we assume that $\mathfrak{g}$ is a non-twisted affine Lie algebra. We will extend the results of the previous section to the family of smooth inducing $\mathfrak{l}$-modules which are the tensor products of Whittaker and evaluation modules [GZ].

Let $\mathfrak{l}_1 \subset \mathfrak{g}$ be a semisimple Lie subalgebra, $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{h}_1^+, \mathfrak{l}_1 = \mathfrak{l}_1 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathfrak{C}c \oplus \mathfrak{C}d$ and $\mathfrak{l}_2 = \mathfrak{l}_1 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathfrak{C}c$. For a nonzero homomorphism $\eta : U(\mathfrak{l}_1^+) \to \mathbb{C}$ and $a \in \mathbb{C} \setminus \{0\}$ let $W_{\eta, a}$ be an irreducible Whittaker $\mathfrak{l}_2$-module of type $\eta$ and central charge $a$.

For every positive integer $m$ consider $\mu = (\mu_1, \ldots, \mu_m) \in (\mathfrak{h}_1^*)^m \setminus \{0\}$ with all dominant integral $\mu_i$ and a sequence $a_1, \ldots, a_m$ with all nonzero and pairwise distinct complex entries. Consider the evaluation $\mathfrak{l}_1$-module $E(\mu, a_1, \ldots, a_m)$ which is a tensor product of finite-dimensional irreducible $\mathfrak{l}_2$-modules with highest weights $\mu_1, \ldots, \mu_m$ and the action is defined as follows:

$$(x \otimes t^n)(v_1 \otimes \ldots \otimes v_m) = \sum_{i=1}^m a_i^n(v_1 \otimes \ldots \otimes xv_i \otimes \ldots \otimes v_m).$$

Taking the tensor product $W_{\eta, a} \otimes E(\mu, a_1, \ldots, a_m)$ we get an $\mathfrak{l}_2$-module which induces the following $\mathfrak{l}_1$-module [GZ]:

$$(W_{\eta, a} \otimes E(\mu, a_1, \ldots, a_m))[d] = U(\mathfrak{l}_1) \otimes_{U(\mathfrak{l}_2)} (W_{\eta, a} \otimes E(\mu, a_1, \ldots, a_m)).$$

Theorem 4.2 can be easily extended to the inducing modules $(W_{\eta, a} \otimes E(\mu, a_1, \ldots, a_m))[d]$.

**Theorem 4.4.** Let $\lambda \in (\mathfrak{h}_1^*)^r$, $a \in \mathbb{C} \setminus \{0\}$, $\eta \neq 0$, $\mu = (\mu_1, \ldots, \mu_m) \in (\mathfrak{h}_1^*)^m \setminus \{0\}$ with all dominant integral $\mu_i$ and $a_1, \ldots, a_m$ pairwise distinct nonzero numbers. Let $W_{\eta, a}$ be an irreducible Whittaker $\mathfrak{l}_2$-module of type $\eta$ and central charge $a$. Define the action of $\mathfrak{h}_1^+$ on
(\(W_{\eta,a} \otimes E(\mu, a_1, \ldots, a_m)[d]\) by \(\lambda\) making it a \(\hat{\mathfrak{p}}\)-module with trivial action of \(\hat{u}_+\). Then the induced \(\hat{\mathfrak{g}}\)-module

\[
M_{\hat{\mathfrak{g}}}((W_{\eta,a} \otimes E(\mu, a_1, \ldots, a_m))[d]) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{p}})} (W_{\eta,a} \otimes E(\mu, a_1, \ldots, a_m))[d]
\]

is irreducible.

Proof. Note that \((W_{\eta,a} \otimes E(\mu, a_1, \ldots, a_m))[d]\) is irreducible \(\hat{\mathfrak{t}}_1\)-module by [GZ] Corollary 3.5. The proof is analogous to the proof of Theorem 4.2. We leave the details to the reader.

\[\square\]

4.3. Inducing from mixed tensor modules. From now on we assume that the Levi factor \(\hat{\mathfrak{l}} = \hat{\mathfrak{l}}_1 + \mathfrak{h}_1^+)\) of the parabolic subalgebra \(\hat{\mathfrak{p}} = \hat{\mathfrak{l}} \oplus \hat{\mathfrak{u}}_+\) contains the whole Heisenberg subalgebra \(G\).

Denote by \(\hat{\mathfrak{l}}_0\) the Lie subalgebra of \(\hat{\mathfrak{l}}\) generated by all its real root subspaces. Let \(G(\hat{\mathfrak{l}})\) be the Lie subalgebra of \(\hat{\mathfrak{l}}_0\) spanned by its imaginary root subspaces, \(G(\hat{\mathfrak{l}})^\perp\) the orthogonal complement of \(G(\hat{\mathfrak{l}})\) in \(G\): \(G = G(\hat{\mathfrak{l}}) + G(\hat{\mathfrak{l}})^\perp\), \([G(\hat{\mathfrak{l}})^\perp, \hat{\mathfrak{l}}_0] = 0\) and \(\hat{\mathfrak{l}}_0 \cap G(\hat{\mathfrak{l}})^\perp = \mathbb{C}c\).

We have

\[
\hat{\mathfrak{l}} = \hat{\mathfrak{l}}_0 + G(\hat{\mathfrak{l}})^\perp + h_1^+ + \mathbb{C}d = (\hat{\mathfrak{l}}_0 + G(\hat{\mathfrak{l}})^\perp) \oplus \mathfrak{h}_1^+, \tag{4.3.1}
\]

where \(\hat{\mathfrak{l}}_0 = \hat{\mathfrak{l}}_0 \oplus \mathbb{C}d\) and \(\tilde{G}(\hat{\mathfrak{l}})^\perp = G(\hat{\mathfrak{l}})^\perp \oplus \mathbb{C}d\). The Lie algebras \(G(\hat{\mathfrak{l}})^\perp\) and \(\tilde{G}(\hat{\mathfrak{l}})^\perp\) inherit the following triangular decompositions

\[
G(\hat{\mathfrak{l}})^\perp = G(\hat{\mathfrak{l}})^\perp_\perp \oplus \mathbb{C}c \oplus G(\hat{\mathfrak{l}})^\perp_+, \quad \tilde{G}(\hat{\mathfrak{l}})^\perp = \tilde{G}(\hat{\mathfrak{l}})^\perp_\perp \oplus (\mathbb{C}c \oplus \mathbb{C}d) \oplus \tilde{G}(\hat{\mathfrak{l}})^\perp_+
\]

from the corresponding decomposition of \(G\). Here \(G(\hat{\mathfrak{l}})^\perp_\perp = \tilde{G}(\hat{\mathfrak{l}})^\perp = G(\hat{\mathfrak{l}})^\perp \cap G_\pm\).

The following proposition shows how to construct irreducible \(\hat{\mathfrak{l}}\)-modules.

**Proposition 4.5.**

- Let \(\lambda \in (\mathfrak{h}_1^+)^*\), \(M\) and \(S\) are irreducible modules over \(\hat{\mathfrak{l}}_0\) and \(G(\hat{\mathfrak{l}})^\perp\) respectively with the same central charge \(a \in \mathbb{C}\). Then \(M \otimes S\) is an irreducible \((\hat{\mathfrak{l}}_0 + G(\hat{\mathfrak{l}})^\perp + \mathfrak{h}_1^+)-\)module with central charge \(a\) and with the action of \(\mathfrak{h}_1^+\) defined by \(\lambda\): \(h(v \otimes w) = \lambda(h)(v \otimes w)\) for all \(h \in \mathfrak{h}_1^+, v \in M, w \in S\).

- Let \(V\) be an irreducible \((\hat{\mathfrak{l}}_0 + G(\hat{\mathfrak{l}})^\perp + \mathfrak{h}_1^+)-\)module with central charge \(a\) and with the action of \(\mathfrak{h}_1^+\) defined by some \(\lambda \in (\mathfrak{h}_1^+)^*\). Suppose \(S\) is an irreducible \(G(\hat{\mathfrak{l}})^\perp\)-submodule of \(V\). Then \(V \simeq M \otimes S\) for some irreducible \(\hat{\mathfrak{l}}_0\)-module \(M\) with central charge \(a\).

- Let \(\lambda \in (\mathfrak{h}_1^+)^*\), \(M\) a weight \(\hat{\mathfrak{l}}_0\)-module which irreducible as \(\hat{\mathfrak{l}}_0\)-module with central charge \(a \in \mathbb{C} \setminus \{0\}\), \(S\) an irreducible Whittaker \(G(\hat{\mathfrak{l}})^\perp\)-module of type \(\eta\) with \(\eta|_{G(\hat{\mathfrak{l}})^\perp} \neq 0\) for infinitely many integer \(n > 0\) and the same central charge. Then \(M \otimes S\) is an irreducible \(\hat{\mathfrak{l}}\)-module with central charge \(a\), with the action of \(\mathfrak{h}_1^+\) defined by \(\lambda\) and with the tensor product action of \(d\).
Proof. If $M$ and $S$ have central charge $a$ then they are irreducible modules over $U(\hat{\mathfrak{g}}_{0})/(c-a)$ and $U(G(\hat{\mathfrak{g}}_{0}))/(c-a)$ respectively. Hence, $M \otimes S$ is an irreducible $U(\hat{\mathfrak{g}}_{0} + G(\hat{\mathfrak{g}}_{0}))/\mathfrak{g}$-module and the first statement is clear. The second statement follows from [LX, Lemma 2.2].

For the third statement, suppose $M \otimes S$ contains a nonzero $\hat{\mathfrak{g}}$-submodule $D$ and consider any nonzero element $v \in D$. Then
\[ v = \sum_{i \in I} v_{i} \otimes w_{i}, \]
for some finite set $I$, where $v_{i} \in M$ and $w_{i} \in S$. We can assume that $w_{i} = d^{k_{i}}z_{i}$, $i \in I$, where $z_{i} \in U(G(\hat{\mathfrak{g}}_{0}))$, $k_{i} \geq 0$, and the elements $v \otimes w_{i}$ are linearly independent. If $i \in I$ and $n > 0$ then for any nonzero $x \in G(\hat{\mathfrak{g}}_{0} \cap \mathfrak{g})$ such that $[x, z_{i}] = 0$, we have
\[ (x - \eta(x))(d^{k_{i}}z_{i}) = \sum_{j=1}^{k_{i}-1} \eta(x)\lambda_{j}d^{j}z_{i} \]
for some integers $\lambda_{j}$. Note that the degree of $d$ in the right hand side is smaller than in the original element. Since the restrictions of $\eta$ on $G(\hat{\mathfrak{g}}_{0} \cap \mathfrak{g})$ are nonzero for infinitely many $n > 0$, there exists $u \in U(G(\hat{\mathfrak{g}}_{0}))$ such that
\[ v' = u \cdot v = \sum_{i \in I'} v'_{i} \otimes w'_{i}, \]
for some new set of indices $I'$, $v'_{i} \in M$ and $w'_{i} \in M_{\eta,a}$. Again we assume that all $v'_{i} \otimes w'_{i}$ are linearly independent. As the Whittaker $G(\hat{\mathfrak{g}}_{0})$-module $M_{\eta,a}$ is irreducible, we find $u' \in U(G(\hat{\mathfrak{g}}_{0}))$ such that
\[ u' \cdot v' = \sum_{i \in I''} v''_{i} \otimes 1, \]
where $1$ is the generator of $S$, for some set of indices $I''$ and some linearly independent elements $v''_{i} \in M$. Then $u' \otimes v' \in D$ is nonzero. We see that $D$ contains a nonzero element $m \otimes 1$ with $m \in M$. Then $U(\hat{\mathfrak{g}}_{0})(m) \otimes 1 = M \otimes 1 \subset D$ (recall that $M$ is irreducible as $\hat{\mathfrak{g}}_{0}$-module). Finally we have
\[ U(G(\hat{\mathfrak{g}}_{0}))(M \otimes 1) = M \otimes S \subset D, \]
implying that $D = M \otimes S$. Hence $M \otimes S$ is irreducible. \hfill $\square$

Remark 4.6. Examples of weight $\tilde{\mathfrak{g}}_{0}$-module which are irreducible as $\hat{\mathfrak{g}}_{0}$-module include irreducible Verma modules, imaginary Verma modules with nonzero central charge, and modules induced from irreducible diagonal $G$-modules with nonzero central charge, among the others.

Remark 4.7. If $M$ is a weight $\tilde{\mathfrak{g}}_{0}$-module and $S$ is a weight $G(\hat{\mathfrak{g}}_{0})$-module with the same central charge then $M \otimes S$ is a tensor $\tilde{\mathfrak{g}}$-module [FK18], [GKMOS]. Modules induced from
weight tensor modules were studied in [GKMOS]. Inspired by [GKMOS] we will consider a different family of inducing modules which we call mixed tensor modules. These are modules of the form $M \otimes S$ where $M$ is an $\hat{\mathfrak{l}}_0$-module and $S$ is a Whittaker $G(\hat{\mathfrak{l}})\perp$-module with the same central charge.

Let $\eta : U(\hat{\mathfrak{g}}(\hat{\mathfrak{l}})^\perp) \rightarrow \mathbb{C}$ be an algebra homomorphism, $a \in \mathbb{C} \setminus \{0\}$ and $S_{\eta,a}$ a Whittaker $\hat{\mathfrak{g}}(\hat{\mathfrak{l}})^\perp$-module of type $\eta$ and central charge $a$.

The following theorem generalizes Theorem 3.3 for arbitrary parabolic subalgebra and for mixed tensor modules. Note that we are not imposing any conditions on $\hat{\mathfrak{l}}_0$-module $M$.

**Theorem 4.8.** Let $\lambda \in (\mathfrak{h}_1^\perp)^*$, $a \in \mathbb{C} \setminus \{0\}$, $M$ an $\hat{\mathfrak{l}}_0$-module with central charge $a$. Consider a mixed tensor $\hat{\mathfrak{l}}$-module $M \otimes S_{\eta,a}$ on which $\mathfrak{h}_1^\perp$ acts via $\lambda$ and $d$ has a tensor product action. Then the induced $\hat{\mathfrak{g}}$-module

$$M_{\hat{\mathfrak{g}}}(M \otimes S_{\eta,a}) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}})} (M \otimes S_{\eta,a})$$

is irreducible if and only if $M \otimes S_{\eta,a}$ is irreducible $\hat{\mathfrak{l}}$-module.

**Proof.** If $M \otimes S_{\eta,a}$ is not irreducible $\hat{\mathfrak{l}}$-module then clearly $M_{\hat{\mathfrak{g}}}(M \otimes S_{\eta,a})$ is not irreducible by the basic properties of induced modules. Conversely, assume that a mixed tensor $\hat{\mathfrak{l}}$-module $M \otimes S_{\eta,a}$ is irreducible.

Let $v \in M_{\hat{\mathfrak{g}}}(M \otimes S_{\eta,a})$ be a nonzero element. Then $v$ can be written as follows:

$$v = \sum_i u_i (v_i \otimes w_i),$$

for some $u_i \in U(\hat{\mathfrak{u}}_\pm)$, $v_i \in M$, $w_i \in S_{\eta,a}$. We assume that the elements $u_i(v_i \otimes w_i)$ in the decomposition of $v$ are linearly independent. We can also assume that $v$ is $\mathfrak{h}_1^\perp$-weight and hence $u_i \in U(\hat{\mathfrak{g}})_{-\varphi_i + k_i}$ or each $i$, for some $\varphi_i$ in the root lattice of $\mathfrak{g}$ and some integer $k_i$. Each $\varphi_i$ can be written as a sum of simple roots of $\mathfrak{g}$. Denote by $\tau(\varphi_i)$ the total number of simple roots of $\mathfrak{g}$ in $\varphi_i$ which are not the roots of $\hat{\mathfrak{I}}$. Clearly $\tau(\varphi_i) = \tau(\varphi_j)$ for all $i, j$ since $v$ is a weight element. We denote this number by $\tau(v)$. Suppose that $\tau(v) = 1$, that is $\varphi_i$ contains a unique simple root which is not a root of $\hat{\mathfrak{I}}$, and for all $i$ this root is the same. As with the weight modules (see [FK18], [GKMOS]) this is the most difficult case which requires a special treatment.

Choose any index $i_0$ for which $k_{i_0}$ is the largest (there could be several indices like that, in this case choose any of them), and choose sufficiently large positive $N$. Let $u \in \hat{\mathfrak{g}}_{\varphi_{i_0} - (N + k_{i_0})}$ be a nonzero element. We have

$$u \cdot v = \sum_{t \in T} y_{-N}(v_t \otimes w_t) + \sum_{j \in J} y_{-N-s_j}(v_j \otimes w_j) + \sum_{k \in K} z_k(v_k \otimes w_k),$$

where $T, J, K$ some sets of indices ($i_0 \in T$), $s_j > 0$, $y_{-N-s_j} \in \hat{\mathfrak{g}}_{-(N-s_j)}$, $z_k \in U(\hat{\mathfrak{l}}_0)$. Moreover, each $y_{N-s_j}$ can be written as $y_{N-s_j}^{(1)} + y_{N-s_j}^{(2)}$, where $y_{N-s_j}^{(1)} \in \hat{\mathfrak{l}}_0$ and $y_{N-s_j}^{(2)} \in$
Then irreducibility follows. Note that $u + v$ for all $s$.

Affine Lie algebras remain valid in this case and hence we obtain more families of irreducible modules for sional Levi factors that contain a proper subalgebra of the Heisenberg algebra. All proofs hold for more general parabolic subalgebras with infinite dimensional Levi factors that contain a proper subalgebra of the Heisenberg algebra. All proofs complete the proof. Then we complete the proof by induction on \[\tau(v)\] one can easily find $\hat{M}$ is irreducible.

Suppose that $u \cdot v = 0$. Then $\sum_{t \in T} v_t \otimes w_t = 0$. But $u_i = u_j$ for all $i, j \in T$ implying that \(\{u_i(v_i \otimes w_i), i \in T\}\) are linearly dependent, which is a contradiction. Hence $u \cdot v \neq 0$. This completes the proof in the case $\tau(v) = 1$. If $\tau(v) > 1$ then by the argument in the proof of [BBFK] Lemma 5.3 one can easily find $u \in U(\tilde{\mu}_+)$ such that $u \cdot v \neq 0$ and $\tau(u \cdot v) < \tau(v)$. Then we complete the proof by induction on $\tau(v)$.

In particular Theorem 4.8 is applied in the case of mixed tensor modules with weight $\tilde{\eta}$-module $M$.

**Corollary 4.9.** Let $\lambda \in (\mathfrak{h}_0^+)^*$, $a \in \mathbb{C} \setminus \{0\}$ and $M$ a weight $\tilde{\eta}_0$-module which is irreducible as $\tilde{\eta}_0$-module with central charge $a$. If $\eta|_{\tilde{G}(0)_{\tau}^\circ \tilde{g}_0a} \neq 0$ for infinitely many integers $n > 0$ then the induced $\tilde{\mathfrak{g}}$-module

$$M_{\tilde{g}}(M \otimes S_{\eta,a}) = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}})} (M \otimes S_{\eta,a})$$

is irreducible.

**Proof.** The module $S_{\eta,a}$ is irreducible by the assumption on $\eta$. Then $M \otimes S_{\eta,a}$ is irreducible by Proposition 4.5. It remains to apply Theorem 4.8.

**Remark 4.10.** All results hold for more general parabolic subalgebras with infinite dimensional Levi factors that contain a proper subalgebra of the Heisenberg algebra. All proofs remain valid in this case and hence we obtain more families of irreducible modules for Affine Lie algebras.
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