Injectivity of the Petri map for twisted Brill–Noether loci

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Abstract. Let \( C \) be a generic curve, \( E \) a generic vector bundle on \( C \). Then, for every line bundle on \( C \) the twisted Petri map
\[
P_E : H^0(C, L \otimes E) \otimes H^0(C, K \otimes L^* \otimes E^*) \to H^0(C, K)
\]
is injective.

1. Introduction

Let \( C \) be an algebraic curve defined over an algebraically closed field. Let \( U (r_F, d_F) \) be the moduli space of stable vector bundles of rank \( r \) and degree \( d \). Choose a generic vector bundle \( E \) of rank \( r_E \) and degree \( d_E \) and denote by \( B_{r_F, d_F}^k(E) \) the so called Brill–Noether locus consisting of vector bundles
\[
\{ F \in U (r_F, d_F) \mid \dim H^0(C, F \otimes E) \geq k \}.
\]
These sets play a crucial role in the geometry of moduli spaces of vector bundles. For example, for suitable choices of \( E \) and \( k = 1 \) they give the generalized theta divisor providing generators of the Picard group of the moduli space.

Brill–Noether loci can be given scheme structures as locally defined determinantal varieties (see [7] for an exposition under the additional assumption \( E = O \)). As such their expected dimension is given by the Brill–Noether number
\[
\rho = r_F^2(g - 1) + 1 - k(k - r_F d_E - r_E d_F + r_E r_F(g - 1))
= \dim(U(r_F, d_F) - h^0(F \otimes E) h^1(F \otimes E)).
\]

Here we write \( h^0(F) \) for the dimension of the space of sections \( H^0(C, F) \) and the last equality in the formula above assumes that \( h^0(F \otimes E) = k \). One also expects that when \( \rho \) is negative, these loci are empty.

Consider the so-called Petri map
\[
P_E(F) : H^0(C, F \otimes E) \otimes H^0(C, K \otimes F^* \otimes E^*) \to H^0(C, F \otimes K \otimes F^*)
\]
obtained as the composition of the natural cup-product and tensorization with the identity morphism in $E$. If $h^0(F \otimes E) = k$, the tangent space to $B^k_{r,d}(E)$ at the point $F$ can be identified to the orthogonal to the image of this map. In particular, $F$ is a non-singular point of a component of dimension $\rho$ of $B^k_{r,d}(E)$ if and only if the Petri map is injective. Therefore, proving the injectivity of the Petri map for the generic curve would provide a complete positive answer to the more pressing questions in Brill–Noether Theory. It proves also that when the Brill–Noether number is negative, the locus is actually empty. Moreover, the injectivity of the Petri map helps explain the structure of the tangent cone to the Brill–Noether loci (see [3]).

In the special case in which $r = 1, E = O$, one recovers the classical Brill–Noether loci in the Jacobian. It is well known that in this particular case all our expectations are satisfied: the loci are non-empty on any curve when $\rho > 0$ and the Petri map is injective on the generic curve which implies that these loci have dimension precisely $\rho$ and that the singular locus of $B^k_{r,d}(E)$ is $B^{k+1}_{r,d}(E)$ (see for instance [2]). On the other hand, when $r > 1$, there are counterexamples of particular values of $r, d, k, g$ where the expected results fail, even for the generic curve (see [7]).

In this paper, we want to deal with the case $r_F = 1$ but arbitrary $r_E$. The non-emptiness result under these hypothesis was proved by Ghione in [6]. Therefore, in order to complete the picture, we need to show the injectivity of the Petri map. We will write $L$ instead of $F$. We can identify $L \otimes K \otimes L^*$ with $K$ (as it is done for classical Bril–Noether). We prove the following:

**Theorem 1.1.** Given a generic curve $C$ and a generic vector bundle $E$ on $C$ for any choice of $L \in \operatorname{Pic}^d(C)$ on $C$, the Petri map

$$P_E(L) : H^0(C, L \otimes E) \otimes H^0(C, K \otimes L^* \otimes E^*) \to H^0(C, K)$$

is injective. In particular, the twisted Brill–Noether locus $B^k_{1,d}(E)$ is of the expected dimension $\rho$ and singular only along $B^{k+1}_{1,d}(E)$

In order to prove the injectivity of the Petri map for a generic curve, it suffices to prove it for a special curve. We’ll choose our curve to be reducible with components rational and elliptic. Vector bundles on these curves are easy to describe in terms of the restrictions of the vector bundles to the various components and gluing at the nodes (see [1], [9]). We choose a generic such $E$ and by means of limit linear series prove the result. The tools used are those developed in [8] and [10] which in turn generalizes [5].

2. Preliminaries on reducible curves

Consider a family of curves $\pi : C \to T$. Let $T$ be the spectrum of a discrete valuation ring $\mathcal{O}$ with maximal ideal generated by $t$. Assume that the generic fiber of $\pi$ is a non-singular curve and the special fiber $C$ looks as follows:

Take $g$ elliptic curves $C^i, i = 1, \ldots, g$ and let $P^i, Q^i$ be generic points on $E^i$. Take any number of rational curves $C^0_1, \ldots, C^0_{k_0}, \ldots, C^g_1, \ldots, C^g_{k_g}$ again with points