Braided Quantum Field Theories and Their Symmetries

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Braided quantum field theories, proposed by Oeckl, can provide a framework for quantum field theories that possess Hopf algebra symmetries. In quantum field theories, symmetries lead to non-perturbative relations among correlation functions. We study Hopf algebra symmetries and such relations in the context of braided quantum field theories. We give the four algebraic conditions among Hopf algebra symmetries and braided quantum field theories that are required for the relations to hold. As concrete examples, we apply our analysis to the Poincaré symmetries of two examples of noncommutative field theories. One is the effective quantum field theory of three-dimensional quantum gravity coupled to spinless particles formulated by Freidel and Livine, and the other is noncommutative field theory on the Moyal plane. We also comment on quantum field theory in κ-Minkowski spacetime.

§1. Introduction

Symmetry is one of the most important concepts in quantum field theory. In many examples, it is useful for investigating properties of quantum field theories non-perturbatively, it is a guiding principle in constructing field theories for various purposes, such as grand unification, and it provides powerful methods for finding exact solutions. It also plays important roles in actual renormalization procedures. Therefore, it should be interesting to study symmetries also in the context of noncommutative field theories, which may result from some quantum gravity effects.

A difficulty that arises in such studies is the apparent violation of basic symmetries, such as Poincaré symmetry, in noncommutative spacetimes. For example, the Moyal plane \([x^\mu, x^\nu] = i\theta^{\mu\nu}\) is invariant under translations, but not under Lorentz transformations or rotations. Another example is three-dimensional spacetime with the noncommutativity expressed as \([x^i, x^j] = i\kappa\epsilon^{ijk}x^k\) \((i, j, k = 1, 2, 3)\,\) with the noncommutativity parameter \(\kappa\). This noncommutative spacetime is Lorentz-invariant, but it is not invariant under the translational transformation \(x^i \to x^i + a^i\) with a \(c\)-number \(a^i\). In fact, a naive construction of noncommutative quantum field theory on this spacetime leads to rather disastrous violations of energy-momentum conservation. The violations coming from the non-planar diagrams do not vanish in the commutative limit, \(\kappa \to 0\), as in UV/IR mixing phenomena.

In recent years, however, there has been interesting progress in the understanding of symmetries in noncommutative field theories. In particular, it has been realized that the symmetry transformations in noncommutative spacetimes are not of the usual Lie-algebraic type, but should be generalized to possess Hopf algebraic

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structures. It has been pointed out that the Moyal plane is invariant under the twisted Poincaré transformation in Refs. 12)–14) and under the twisted diffeomorphism in Refs. 15)–18). There have been various proposals to implement the twisted Poincaré invariance in quantum field theories.19)–30) With regard to the noncommutative spacetime with \([x^i, x^j] = i\kappa \varepsilon^{ijk} x_k\), a noncommutative quantum field theory was derived as the effective field theory of three-dimensional quantum gravity with matter.31) The essential difference between it and the naive construction mentioned above is the nontrivial braiding for each crossing in non-planar Feynman diagrams. With this braiding, there exists a kind of conserved energy-momentum in the amplitudes, and the energy-momentum operators possess Hopf algebraic structures.

Our aim in this paper is to systematically understand these Hopf algebraic symmetries and their consequences in noncommutative field theories in the framework of the braided quantum field theories proposed by Oeckl.34) In conventional quantum field theories, symmetries result in non-perturbative relations among correlation functions. In this paper, we find that such relations have natural extensions to the Hopf algebraic symmetries in braided quantum field theories, and we obtain the four necessary conditions for these relations to hold. These conditions should be interpreted as the criteria for the symmetries in braided quantum field theories.

This paper is organized as follows. In the following section, we review braided quantum field theory. This review part faithfully follows the original paper,34) but figures are used more extensively in the proofs and the explanations to make this paper self-contained and intuitively understandable. We start with braided categories and braided Hopf algebras. Then, correlation functions of braided quantum field theory are formulated in terms of them. Finally, braided Feynman rules are given.

In §3, we first review the axioms of the action\(^*\) of an algebra on vector spaces. Then we consider the relations among correlation functions in braided quantum field theory. We find that four algebraic conditions are required for the relations to hold. Then, as concrete examples, we discuss whether the noncommutative field theories mentioned above possess the Poincaré symmetry by checking the four conditions. In the former case, we find that the twisted Poincaré symmetry exists only after the introduction of a non-trivial braiding factor. This result agrees with that of the previous study in Refs. 21) and 35). In the latter case, we find that the theory possesses a kind of translational symmetry that differs from the usual one by multi-field contributions. We also give some examples of such relations among correlation functions and discuss their implications.

The final section is devoted to a summary and comments. Then, we comment on quantum field theory on \(\kappa\)-Minkowski spacetime whose coordinate noncommutativity is expressed as \([x^0, x^j] = \frac{i}{\kappa} x^j\ (j = 1, 2, 3)\).36)

\(^*\) We use italics to distinguish it from the action \(S\).
§2. Review of braided quantum field theory

2.1. Braided categories and braided Hopf algebras

First of all, we review braided categories and braided Hopf algebras.\textsuperscript{34,37} Braided categories are composed of an object $X$, which is a vector space, a dual object $X^\ast$, which is a dual vector space, and morphisms of the form

\begin{align}
\text{ev} & : X^\ast \otimes X \rightarrow \mathbb{k}, \quad \text{(evaluation)} \\
\text{coev} & : \mathbb{k} \rightarrow X \otimes X^\ast, \quad \text{(coevaluation)}
\end{align}

where $\mathbb{k}$ is a $c$-number. The composition of these two morphisms in the obvious way yields the identity. Then, braided categories also possess an invertible morphism,

\begin{equation}
\psi_{V,W} : V \otimes W \rightarrow W \otimes V, \quad \text{(braiding)}
\end{equation}

where $V$ and $W$ are any pair of vector spaces. Generally, the inverse of a braiding is not equal to the braiding itself.

It is required that the braiding is compatible with the tensor product in the sense that the following hold:

\begin{align}
\psi_{U,V \otimes W} &= (\text{id} \otimes \psi_{U,W}) \circ (\psi_{U,V} \otimes \text{id}), \\
\psi_{U \otimes V,W} &= (\psi_{U,W} \otimes \text{id}) \circ (\text{id} \otimes \psi_{V,W}).
\end{align}

Then, the braiding is also required to be \textit{intersectional} under any morphisms in a Hopf algebra. For example, we have

\begin{align}
\psi_{Z,W}(Q \otimes \text{id}) &= (\text{id} \otimes Q)\psi_{V,W} \quad \text{for any } Q : V \rightarrow Z, \\
\psi_{V,Z}(\text{id} \otimes Q) &= (Q \otimes \text{id})\psi_{V,W} \quad \text{for any } Q : W \rightarrow Z,
\end{align}

where $Z$ is a vector space.

We can represent these axioms pictorially.\textsuperscript{38} We depict the morphisms $\text{ev}$, $\text{coev}$, and $\psi$ as in Fig. 1. Thus, the axioms (2.4) are represented as in Fig. 2, and the axioms (2.5) are represented as in Fig. 3.

Next, we consider the polynomials of $X$,

\begin{equation}
\widehat{X} := \bigoplus_{n=0}^{\infty} X^n, \quad \text{with } X^0 := 1 \quad \text{and } X^n := X \otimes \cdots \otimes X, \quad \text{n times,}
\end{equation}

where $1$ is the trivial one-dimensional space. $\widehat{X}$ naturally has the structure of a braided Hopf algebra, as seen from the following:

\begin{align}
\Delta \quad \text{(coproduct)} : \widehat{X} \rightarrow \widehat{X} \otimes \widehat{X}; \quad &\Delta \phi = \phi \otimes 1 + 1 \otimes \phi, \quad \text{and } \Delta(1) = 1 \otimes 1, \\
\epsilon \quad \text{(counit)} : \widehat{X} \rightarrow \mathbb{k}; \quad &\epsilon(\phi) = 0, \quad \text{and } \epsilon(1) = 1, \\
S \quad \text{(antipode)} : \widehat{X} \rightarrow \widehat{X}; \quad &S\phi = -\phi, \quad \text{and } S(1) = 1,
\end{align}

\begin{align}
\eta \quad \text{(unit)} : \mathbb{k} \rightarrow \widehat{X}; \quad &\eta(1) = 1, \\
\cdot \quad \text{(product)} : \widehat{X} \otimes \widehat{X} \rightarrow \widehat{X},
\end{align}

where $\mathbb{k}$ is a $c$-number.
where $\phi \in X$. The tensor product $\otimes$ is the same as the usual product of $X$s, while the new tensor product $\hat{\otimes}$ is the tensor product of $\hat{X}s$. The coproduct $\Delta$, counit $\epsilon$, and antipode $S$ of the products of $X$s are defined inductively by

\[ \Delta \circ \cdot = (\cdot \hat{\otimes} \cdot) \circ (\text{id} \hat{\otimes} \psi \hat{\otimes} \text{id}) \circ (\Delta \hat{\otimes} \Delta), \quad (2.12) \]

\[ \epsilon \circ \cdot = \cdot \circ (\epsilon \hat{\otimes} \epsilon), \quad (2.13) \]

\[ S \circ \cdot = \cdot \circ \psi \circ (S \hat{\otimes} S). \quad (2.14) \]

These axioms are represented diagrammatically in Fig. 4.

2.2. Braided quantum field theory

Next, we formulate braided quantum field theory\cite{34} in terms of the
braided category and the braided Hopf algebra. We take the vector space \( X \) as the space of a field \( \phi(x) \), where \( x \) is a general index for independent modes of the field. Thus, \( \hat{X} \) is the space of polynomials of the fields, such as \( \phi(x_1)\phi(x_2)\cdots\phi(x_n) \), and \( 1 \) corresponds to the constant field of unit magnitude. We also take the dual vector space \( X^* \) as the space of differentials \( \delta/\delta\phi(x) \). We choose the evaluation and the coevaluation as

\[
ev : \frac{\delta}{\delta\phi(x)} \otimes \phi(x') \rightarrow \delta(x - x'),
\]

\[
\text{coev} : 1 \rightarrow \int x \phi(x) \otimes \frac{\delta}{\delta\phi(x)}.
\]

where the distribution and the integration should be understood symbolically, and the details of their forms, which may contain non-trivial measures, vary from case to case.

The differential on \( \hat{X} \) is defined by

\[
\text{diff} := (\hat{\text{ev}} \otimes \text{id}) \circ (\text{id} \otimes \Delta);
\]

\[
X^* \otimes \hat{X} \rightarrow \hat{X},
\]

where

\[
\hat{\text{ev}}|_{X^* \otimes X^n} = \begin{cases} 
\text{ev} & \text{for } n = 1, \\
0 & \text{for } n \neq 1.
\end{cases}
\]

Diagrammatically, this is given by Fig. 5. To see whether the map \( \text{diff} \) actually does give the differential of products, let us compute the differential of \( \phi(x)\phi(y) \) as a simple example, using the definition (2.17). This becomes

\[
diff\left(\frac{\delta}{\delta\phi(x')} \otimes \phi(x)\phi(y)\right) = (\hat{\text{ev}} \otimes \text{id}) \circ (\text{id} \otimes \Delta)\left(\frac{\delta}{\delta\phi(x')} \otimes \phi(x)\phi(y)\right)
\]

\[
= (\hat{\text{ev}} \otimes \text{id}) \circ \left(\frac{\delta}{\delta\phi(x')} \otimes \Delta(\phi(x)\phi(y))\right)
\]

\[
= (\hat{\text{ev}} \otimes \text{id}) \circ \left(\frac{\delta}{\delta\phi(x')} \otimes (\phi(x)\phi(y) \otimes 1
\]

\[
+ \phi(x)\ot\phi(y) + \psi(\phi(x)\ot\phi(y)) + 1 \ot\phi(x)\phi(y))\right)
\]

\[
= \delta(x' - x) \otimes \phi(y) + (\hat{\text{ev}} \otimes \text{id}) \circ \left(\frac{\delta}{\delta\phi(x')} \otimes \psi(\phi(x)\ot\phi(y))\right),
\]

where we have used the axiom (2.12) in deriving the third line. If the braiding is trivial, we find that the differential (2.17) satisfies the usual Leibniz rule.
Generally, we find the braided Leibniz rule
\[ \partial(\alpha \beta) = \partial(\alpha)\beta + \psi^{-1}(\partial \otimes \alpha)(\beta) \] (2.20)
and
\[ \partial(\alpha) = (\text{ev} \otimes \text{id}^{n-1})(\partial \otimes [n]_{\psi} \alpha), \] (2.21)
where \( \partial \in X^* \) and \( \alpha, \beta \in \hat{X} \), and we have used the simplified notation
\[ \partial(\alpha) := \text{diff}(\partial \otimes \alpha). \] (2.22)
Here, \( n \) is the degree of \( \alpha \), and \([n]_{\psi}\) is called a braided integer, defined by
\[ [n]_{\psi} := \text{id}^n + \psi \otimes \text{id}^{n-2} + \cdots + \psi_{n-2,1} \otimes \text{id} + \psi_{n-1,1}, \] (2.23)
where \( \psi_{n,m} \) is the braiding morphism depicted in Fig. 6.

The proofs of the formulas (2.20) and (2.21) are given in Appendix A.

Now, we define Gaussian integration, which defines the path integral. This definition is given by
\[ \int \partial(\alpha w) := 0 \text{ for } \partial \in X^*, \alpha \in \hat{X}, \] (2.24)
where \( w \in \hat{X} \) is a Gaussian weight. In field theory, \( w \) is the exponential of the free part of the action, \( e^{-S_0} \).

In order to obtain a formula for correlation functions, we define the morphism \( \gamma : X^* \to X \) such that
\[ \partial(w) := -\gamma(\partial)w. \] (2.25)
This morphism is assumed to be commutative with the braiding, as in (2.5). In the case \( w = e^{-S_0} \), we have \( \gamma(\partial) = \partial(S_0) \). In field theory, this is the kinetic part of the action, or the inverse of the propagator.

Starting from (2.24), we can represent correlation functions of a free field theory in terms of the braided category and the braided Hopf algebra. This is the analog of the Wick theorem in braided quantum field theory. The definition of the free \( n \)-point correlation function is given by
\[ Z^{(0)}_n(\alpha) := \frac{\int \alpha w}{\int w}, \] (2.26)
where the degree of \( \alpha \) is \( n \). Algebraically, this is given by
\[ Z^{(0)}_2 = \text{ev} \circ (\gamma^{-1} \otimes \text{id}) \circ \psi, \] (2.27)
\[ Z^{(0)}_{2n} = (Z^{(0)}_2)^n \circ [2n-1]_{\psi}^{!!}, \] (2.28)
\[ Z^{(0)}_{2n-1} = 0, \] (2.29)
where

\[
[2n-1]_\psi'!! := ([1]_\psi' \otimes \text{id}^{2n-1}) \circ ([3]_\psi' \otimes \text{id}^{2n-3}) \circ \cdots \circ ([2n-1]_\psi' \otimes \text{id}),
\]

\[
[n]_\psi':= \text{id}^n + \text{id}^{n-2} \otimes \psi^{-1} + \cdots + \psi^{-1}_{1,n-1}
= \psi_{1,n-1}^{-1} \circ [n]_\psi.
\]

The proofs of (2.27)–(2.29) are in Appendix B.

Next, we consider correlation functions in the case that there exists an interaction. For \( S = S_0 + \lambda S_{\text{int}} \), a correlation function is perturbatively given by

\[
Z_n(\alpha) = \frac{\int \alpha e^{-S}}{\int e^{-S}} = \frac{\int \alpha (1 - \lambda S_{\text{int}} + \cdots) e^{-S_0}}{\int (1 - \lambda S_{\text{int}} + \cdots) e^{-S_0}},
\]

(2.32)

where \( \alpha \in X^n \). Introducing a morphism \( S_{\text{int}}: k \to X^k \), where \( k \) is the degree of \( S_{\text{int}} \), this correlation function is algebraically given by

\[
Z_n = \frac{Z_n^{(0)} - \lambda Z_n^{(0)} \circ (\text{id}^n \otimes S_{\text{int}}) + \frac{1}{2} \lambda^2 Z_{n+2k}^{(0)} \circ (\text{id}^n \otimes S_{\text{int}} \otimes S_{\text{int}}) + \cdots}{1 - \lambda Z_k^{(0)} \circ S_{\text{int}} + \frac{1}{2} \lambda^2 Z_{2k}^{(0)} \circ (S_{\text{int}} \otimes S_{\text{int}}) + \cdots}.
\]

(2.33)

Acting on \( \alpha \in X^n \) with \( Z_n \), we obtain the correlation function (2.32). One can obviously extend \( S_{\text{int}} \) to include various interaction terms.

2.3. Braided Feynman rules

From the results obtained in the preceding subsection, a correlation function can be represented by a sum of diagrams obeying the following rules:

- An \( n \)-point function \( Z_n \) is a morphism \( X^n \to k \). Thus, a Feynman diagram starts with \( n \) strands at the top and must be closed at the bottom.
- The propagator \( Z_2^{(0)}: X \otimes X \to k \) is represented by the left graph in Fig. 7, which is a simplified form of that in Fig. 8.
- The interaction vertex \( S_{\text{int}}: k \to X^k \) is represented by the right graph in Fig. 7. Generally, the order of the strands is important.
- The two kinds of crossings, which are represented in Fig. 9, correspond to the braiding and its inverse.
- Any Feynman diagram is built out of propagators, vertices, and crossings, and is closed at the bottom.

§3. Symmetries in braided quantum field theory

In this section, we study symmetries in braided quantum field theory. In order to represent symmetry transformations of fields, we first review the general description of the action mentioned in §3.1. In §3.2, we study relations among correlation functions. We find four necessary conditions for such relations to follow from the symmetry algebra. In §3.3 and §3.4, we treat two examples of (braided) noncommutative field theories and discuss their Poincaré symmetries.
3.1. General description of an action

Here, we review the action of a general Hopf algebra on vector spaces in mathematical language.\textsuperscript{37,39}

An action $\alpha_V$ is a map $\alpha_V : A \otimes V \rightarrow V$, where $A$ is an arbitrary Hopf algebra and $V$ is a vector space. (In our case, $A$ is a symmetry algebra, and $V = X$ or $X^*$.) We denote the coproduct and the counit of the Hopf algebra by $\Delta'$ and $\epsilon'$, to distinguish them from those of the braided Hopf algebra of fields in §2. We do not present all the axioms of an action, but the important axioms are the following.

- $\alpha_V$ satisfies the condition
  \[ \alpha_V \circ (\cdot \otimes \text{id}) = \alpha_V \circ (\text{id} \otimes \alpha_V), \] (3.1)
  where the equality acts on $A \otimes A \otimes V$. This means that $\alpha_V((a \cdot b) \otimes V) = \alpha_V(a \otimes (\alpha_V(b \otimes V)))$, where $a, b \in A$. In abbreviated form, we write this as
  \[ (a \cdot b) \triangleright V = a \triangleright (b \triangleright V). \] (3.2)

- An action on 1, which is in a vector space, is defined by
  \[ \alpha_V(a \otimes 1) = \epsilon'(a)1, \] (3.3)
  where $\epsilon'(a)$ is the counit of a Lie algebra $a \in A$.

- An action on a tensor product of vector spaces $V$ and $W$ is defined by
  \[ \alpha_{V \otimes W}(a) := ((\alpha_V \otimes \alpha_W) \circ \Delta')(a) = \sum_i \alpha_V(a_{(1)}^i) \otimes \alpha_W(a_{(2)}^i), \quad a \in A, \] (3.4)
  where $\Delta'(a) = \sum_i a_{(1)}^i \otimes a_{(2)}^i$ is the coproduct of the Hopf algebra $A$. In the case of a usual Lie-algebraic transformation, its coproduct is given by $\Delta'(a) = a \otimes 1 + 1 \otimes a$, where 1 is in $A$. This gives the usual Leibnitz rule.

\textsuperscript{*} We omit the antipode.
Since a Hopf algebra possesses the coassociativity expressed by
\[(\Delta' \otimes \text{id}) \circ \Delta' = (\text{id} \otimes \Delta') \circ \Delta',\]
the action on a tensor product of vector spaces, which is obtained through multiple operations of \(\Delta'\) on \(a\), is actually unique. An important consequence of this fact is that one can divide the action on a tensor product of vector spaces as
\[a \triangleright (V_1 \otimes \cdots \otimes V_{k-1} \otimes V_k \otimes \cdots \otimes V_n) = \sum_i a^{i(1)} \triangleright (V_1 \otimes \cdots \otimes V_{k-1}) \otimes a^{i(2)} \triangleright (V_k \otimes \cdots \otimes V_n)\]
for any \(k\).

### 3.2. Symmetry relations among correlation functions and their algebraic descriptions

The expression for the correlation functions given in (2.33) is perturbative in the interactions, but it is a full-order algebraic description. Therefore we can investigate the symmetry of the theory and the implied relations among correlation functions by using this expression. It may even be the case that the relations hold non-perturbatively.

In usual quantum field theory, if a field theory has a certain symmetry, there is a relation among the correlation functions of the form
\[\sum_{i=1}^{n} \langle \phi(x_1) \cdots \delta_a \phi(x_i) \cdots \phi(x_n) \rangle = 0,\]
where \(\delta_a \phi(x)\) is a variation of a field under a transformation \(a\), with the assumption that the path integral measure and the action are invariant under the transformation.

If the coproduct of a symmetry algebra is not of the usual Lie-algebraic type, and thus the Leibniz rule is deformed, this relation will generally have the form
\[c^{(bi)}_a \langle \phi(x_1) \cdots \delta_b \phi(x_i) \cdots \phi(x_n) \rangle + c^{(bj)}_a \langle \phi(x_1) \cdots \delta_b \phi(x_i) \cdots \delta_c \phi(x_j) \cdots \phi(x_n) \rangle + \cdots = 0,\]
where \(c^{(ij)}_a\) are some coefficients. The essential difference between (3.7) and (3.8) is the multi-field contributions in the latter. In our algebraic language, this relation can be written
\[Z_n(a \triangleright \chi) = \epsilon'(a) Z_n(\chi), \quad \text{for} \ a \in A, \ \chi \in X^n.\]
This is equivalent to Fig. 10 in our diagrammatic representation.

Next, we consider what algebraic structure is required for (3.9) to hold for any \(a\) and \(\chi\), i.e. the structure required for the theory to be invariant under the Hopf algebra transformation \(A\).
Let us write the coproduct of an element \( a \in \mathcal{A} \) as

\[
\Delta'(a) = \sum_s f^s \otimes g^s; \tag{3.10}
\]

where \( f^s, g^s \in \mathcal{A} \). Since this coproduct must satisfy the Hopf algebra axiom,\(^{37}\)

\[
(e' \otimes \text{id}) \Delta'(a) = (\text{id} \otimes e') \Delta'(a) = a, \tag{3.11}
\]

\( f^s \) and \( g^s \) must satisfy

\[
\sum_s e'(f^s) \otimes g^s = \sum_s f^s \otimes e'(g^s) = a. \tag{3.12}
\]

For all the relations among the correlation functions to hold, we have the following four necessary conditions for any action \( a \in \mathcal{A} \).

- (Condition 1) \( S_{\text{int}} \) must satisfy

\[
a \triangleright S_{\text{int}} = e'(a)S_{\text{int}}. \tag{3.13}
\]

- (Condition 2) The braiding \( \psi \) is an intertwining operator; that is

\[
\psi(a \triangleright (V \otimes W)) = a \triangleright \psi(V \otimes W). \tag{3.14}
\]

- (Condition 3) \( \gamma^{-1} \) and \( a \) are commutative;

\[
a \triangleright (\gamma^{-1}(V)) = \gamma^{-1}(a \triangleright V). \tag{3.15}
\]

- (Condition 4) Under an action \( a \), the evaluation map satisfies

\[
ev(a \triangleright (X^* \otimes X)) = e'(a)ev(X^* \otimes X). \tag{3.16}
\]
Conditions 1 through 4 are diagrammatically represented in Fig. 11. It is clear that, when the algebra $\mathcal{A}$ is generated from a finite number of independent elements, it is enough for these generators to satisfy these conditions.

Condition 1 is the requirement of the symmetry at the classical level for the interaction. We can extend this condition to

$$(a \triangleright X^n) \otimes S_{\text{int}}^p = a \triangleright (X^n \otimes S_{\text{int}}^p). \quad (3.17)$$

The proof is the following. From the coproduct (3.10) and its coassociativity, (3.6), the right-hand side of (3.17) is equal to

$$\sum_s (f^s \triangleright (X^n \otimes S_{\text{int}}^{p-1})) \otimes g^s \triangleright S_{\text{int}}. \quad (3.18)$$

Since Condition 1 implies

$$g^s \triangleright S_{\text{int}} = \epsilon'(g^s)S_{\text{int}}, \quad (3.19)$$

(3.18) becomes

$$\sum_s (f^s \triangleright (X^n \otimes S_{\text{int}}^{p-1})) \otimes \epsilon'(g^s)S_{\text{int}}
= a \triangleright (X^n \otimes S_{\text{int}}^{p-1}) \otimes S_{\text{int}}, \quad (3.20)$$

where we have used (3.12). Iterating this procedure, we obtain the left-hand side of (3.17).

Conditions 2–4 can also be extended to

$$[n + kp - 1]_{\psi} \circ (a \triangleright X^{n+kp}) = a \triangleright [n + kp - 1]_{\psi} X^{n+kp}, \quad (3.21)$$

$$(\gamma^{-1} \otimes \text{id}) \frac{n+pk}{2} \circ (a \triangleright X^{n+kp}) = a \triangleright (\gamma^{-1} \otimes \text{id}) \frac{n+pk}{2} X^{n+kp}, \quad (3.22)$$

$$\text{ev}^{-\frac{n+pk}{2}} (a \triangleright (X^* \otimes X) \frac{n+pk}{2}) = \epsilon'(a) \text{ev}^{-\frac{n+pk}{2}} (X^* \otimes X) \frac{n+pk}{2}. \quad (3.23)$$

We can find that the extended conditions (3.17), (3.21), (3.22) and (3.23) can be represented as in Fig. 12. In diagrammatic language, the relations among correlation functions hold if an action can pass downwards through a Feynman diagram and satisfies (3.3).

### 3.3. Symmetries of the effective noncommutative field theory of three-dimensional quantum gravity coupled with scalar particles

In this subsection, we study the Poincaré symmetry of the effective noncommutative field theory of three-dimensional quantum gravity coupled with scalar particles, which was obtained in Ref. 31) by studying the Ponzano-Regge model coupled with spinless particles. The symmetry of this theory is also known as $DSU(2)$, and it is investigated in Refs. 32) and 33). We first review the field theory.\(^{10,31}\)

Let $\phi(x)$ be a scalar field on a three-dimensional space $x = (x^1, x^2, x^3)$. Its Fourier transformation is given by

$$\phi(x) = \int dg \tilde{\phi}(g) e^{\frac{i}{2\kappa} \text{tr}(\mathcal{A}g)}, \quad (3.24)$$
Fig. 11. Conditions 1–4.
where $\kappa$ is a constant, $\mathcal{X} = ix^i \sigma_i$, and $g = P^0 - i\kappa P^i \sigma_i \in SO(3)^*$ with Pauli matrices $\sigma_i$. Here, $\int dg$ is the Haar measure of $SO(3)$ and $P^0 = \pm \sqrt{1 - \kappa^2 P_i P_i}$, by definition. In the following, we will only consider the Euclidean case, but the Lorentzian case can be treated in a similar manner by replacing $SO(3)$ with $SL(2,R)$.

The definition of the star product is given by

$$e^{\frac{i}{2\kappa} \text{tr}(\mathcal{X}g_1)} \star e^{\frac{i}{2\kappa} \text{tr}(\mathcal{X}g_2)} := e^{\frac{i}{2\kappa} \text{tr}(\mathcal{X}g_1g_2)}. \quad (3.25)$$

Differentiating both sides of (3.25) with respect to $P^i_1 := P^i(g_1)$ and $P^i_2 := P^i(g_2)$

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Fig. 12. A relation among correlation functions is satisfied if the four conditions (3.13)–(3.16) are satisfied.

*) The identification $g \sim -g$ is implicitly assumed.
and then taking the limit \( P_1, P_2 \to 0 \), we find the SO(3) Lie-algebraic space-time noncommutativity\(^7\)–\(^9\) \[
[x^i, x^j] = 2i\kappa \epsilon^{ijk} x_k. \tag{3.26}
\]
For example, the action\(^1\) of the \( \phi^3 \) theory is
\[
S = \frac{1}{8\pi\kappa^3} \int d^3x \left[ \frac{1}{2}(\partial_i \phi \star \partial_i \phi)(x) - \frac{1}{2}M^2(\phi \star \phi)(x) + \frac{\lambda}{3!}(\phi \star \phi \star \phi)(x) \right], \tag{3.27}
\]
where \( M^2 = \frac{\sin^2 m\kappa}{\kappa^2} \). Its momentum representation is
\[
S = \frac{1}{2} \int dg (P^2(g) - M^2) \tilde{\phi}(g) \tilde{\phi}(g^{-1}) \]
\[+ \frac{\lambda}{3!} \int dg_1 dg_2 dg_3 \delta(g_1 g_2 g_3) \tilde{\phi}(g_1) \tilde{\phi}(g_2) \tilde{\phi}(g_3), \tag{3.28}\]
from which it is straightforward to read off the Feynman rules.

Some quantum properties of this scalar field theory were analyzed in Ref. 10). As can be seen from (3.26), naive translational symmetry is violated. In fact, its violation is rather disastrous: There exists a kind of conserved energy-momentum in the amplitudes of the tree and the planar loop diagrams, but this energy-momentum is not conserved in the non-planar loop diagrams. Moreover, the violation of the energy-momentum conservation does not vanish in the commutative limit \( \kappa \to 0 \), due to a mechanism similar to the UV/IR phenomena.\(^11\)

In the effective field theory of quantum gravity coupled with spinless particles, however, the Feynman rules also contain a non-trivial braiding rule for each crossing, which comes from the flatness condition in a graph of intersecting particles.\(^31\) This can be incorporated as a braiding between the scalar fields,
\[
\psi(\tilde{\phi}(g_1) \tilde{\phi}(g_2)) = \tilde{\phi}(g_2) \tilde{\phi}(g_2^{-1} g_1 g_2), \tag{3.29}\]
in the braided quantum field theory.

From the direct analysis of the Feynman graphs with this braiding rule, one can easily find that the energy-momentum mentioned above is conserved also in the non-planar diagrams. This suggests the existence of a translational symmetry in quantum field theory. In the following, we discuss the embedding of this field theory into the framework of braided quantum field theory, and check the four conditions for translational and rotational symmetries.

We use the momentum representation, and take \( X \) as the space of \( \tilde{\phi}(g) \) and \( X^* \) as that of \( \frac{\delta}{\delta \phi(g)} \). We choose the braided Hopf algebra of the fields as follows:
\[
\Delta : \tilde{\phi}(g) \to \tilde{\phi}(g) \hat{\otimes} 1 + 1 \hat{\otimes} \tilde{\phi}(g), \tag{3.30}\]
\[
\epsilon : \tilde{\phi}(g) \to 0, \tag{3.31}\]
\[
S : \tilde{\phi}(g) \to -\tilde{\phi}(g), \tag{3.32}\]
\[
\psi : \tilde{\phi}(g_1) \otimes \tilde{\phi}(g_2) \to \tilde{\phi}(g_2) \otimes \tilde{\phi}(g_2^{-1} g_1 g_2). \tag{3.33}\]

\(^{\ast}\) Since in the Ponzano-Regge model, the definition of the weight of the partition function is \( e^{iS} \), despite Euclidean theory, the sign of the mass term is not the usual one.
The evaluation and coevaluation maps are given by
\[
ev : \frac{\delta}{\delta \tilde{\phi}(g)} \otimes \tilde{\phi}(g') \rightarrow \delta(g^{-1}g'),
\]
\[
\text{coev} : 1 \rightarrow \int dg\tilde{\phi}(g) \otimes \frac{\delta}{\delta \tilde{\phi}(g)}.
\]

Also from \(\gamma(\partial) = \partial S_0 = (P^2(g) - m^2)\tilde{\phi}(g^{-1})\), we have
\[
\gamma^{-1}(\tilde{\phi}(g)) = \frac{1}{P^2(g^{-1}) - m^2} \frac{\delta}{\delta \tilde{\phi}(g^{-1})}.
\]

From the algebraic consistencies in Fig. 13, the braidings between \(X\) and \(X^*\) and the braiding between the \(X^*\)s are determined to be
\[
\psi\left(\frac{\delta}{\delta \tilde{\phi}(g_1)} \otimes \tilde{\phi}(g_2)\right) = \tilde{\phi}(g_2) \otimes \frac{\delta}{\delta \tilde{\phi}(g_2^{-1}g_1g_2)},
\]
\[
\psi\left(\tilde{\phi}(g_1) \otimes \frac{\delta}{\delta \tilde{\phi}(g_2)}\right) = \frac{\delta}{\delta \tilde{\phi}(g_2)} \otimes \tilde{\phi}(g_2g_1g_2^{-1}),
\]
\[
\psi\left(\frac{\delta}{\delta \tilde{\phi}(g_1)} \otimes \frac{\delta}{\delta \tilde{\phi}(g_2)}\right) = \frac{\delta}{\delta \tilde{\phi}(g_2)} \otimes \frac{\delta}{\delta \tilde{\phi}(g_2g_1g_2^{-1})}.
\]

In this derivation, we have used the invariance of the Haar measure, i.e. the relation
\[
d(g^{-1}g') = dg'.
\]

Now we consider a translational transformation of the field. If we shift \(x^i\) to \(x^i + \epsilon^i\), a field \(\phi(x)\) becomes
\[
\phi(x) \rightarrow \phi(x + \epsilon)
\]
\[
= \int dg\tilde{\phi}(g)e^{i(x + \epsilon)^i P_i(g)}
\]
\[
\sim \int dg(1 + i\epsilon^i P_i(g))\tilde{\phi}(g)e^{ix^i P_i(g)}.
\]

Thus, in the momentum representation, a translational transformation corresponds to an action
\[
P_i \triangleright \tilde{\phi}(g) = P_i(g)\tilde{\phi}(g), \quad P^0 \triangleright \tilde{\phi}(g) = P^0(g)\tilde{\phi}(g).
\]
From the requirement that the star product (3.25) conserve a kind of momentum, the action on a product of fields should be
\[
P^i \triangleright (\tilde{\phi}(g_1)\tilde{\phi}(g_2)) = P^i(g_1g_2)\tilde{\phi}(g_1)\tilde{\phi}(g_2) = (P^i_1 P^i_2 + P^i_0 P^i + \kappa \epsilon^{ijk} P^i_j P^k_i)\tilde{\phi}(g_1)\tilde{\phi}(g_2),
\]
(3.42)

This determines the coproduct of $P^i$ and $P^0$ as
\[
\Delta'(P^i) = P^0 \otimes P^i + P^i \otimes P^0 + \kappa \epsilon^{ijk} P^j \otimes P^k,
\]
(3.44)
\[
\Delta'(P^0) = P^0 \otimes P^0 - \kappa^2 P^i \otimes P_i.
\]
(3.45)

This coproduct satisfies coassociativity, which essentially comes from the associativity of the group multiplication.

From the axiom (3.11), the counit of $P^i$ and $P^0$ is given by
\[
e'(P^i) = e'(P^0) = 0.
\]
(3.46)

Since the conservation of momentum under the coevaluation map (3.35) requires that the action of $P^i$ on $\int dg(\tilde{\phi}(g) \otimes \frac{\delta}{\delta \tilde{\phi}(g)})$ vanish from (3.3), the action of $P^i$ on $\frac{\delta}{\delta \tilde{\phi}(g)}$ must be
\[
P^i \triangleright \frac{\delta}{\delta \tilde{\phi}(g)} = P^i(g^{-1}) \frac{\delta}{\delta \tilde{\phi}(g)}.
\]
(3.47)

In the following, we see that the momentum algebra satisfies the four conditions (3.13)–(3.16).

Condition 1 follows from the relation
\[
P^i \triangleright S_{\text{int}}
\]
\[
= \int dg_1 dg_2 dg_3 \delta(g_1 g_2 g_3) P^i \triangleright (\tilde{\phi}(g_1)\tilde{\phi}(g_2)\tilde{\phi}(g_3))
\]
\[
= \int dg_1 dg_2 dg_3 \delta(g_1 g_2 g_3) P^i(g_1 g_2 g_3)(\tilde{\phi}(g_1)\tilde{\phi}(g_2)\tilde{\phi}(g_3))
\]
\[
= 0.
\]
(3.48)

Condition 2 follows from the relation
\[
\psi(P^i \triangleright (\tilde{\phi}(g_1)\tilde{\phi}(g_2))) = P^i(g_1 g_2)(\tilde{\phi}(g_2)\tilde{\phi}(g^{-1}_2 g_1 g_2)),
\]
\[
P^i \triangleright \psi(\tilde{\phi}(g_1)\tilde{\phi}(g_2)) = P^i(g_2 g^{-1}_2 g_1 g_2)(\tilde{\phi}(g_2)\tilde{\phi}(g^{-1}_2 g_1 g_2)).
\]

Condition 3 follows from the relation
\[
P^i \triangleright \gamma^{-1}(\tilde{\phi}(g)) = \frac{1}{P^2(g^{-1}) - m^2} P^i(g) \frac{\delta}{\delta \tilde{\phi}(g^{-1})},
\]
\[
\gamma^{-1}(P^i \triangleright \tilde{\phi}(g)) = \frac{1}{P^2(g^{-1}) - m^2} P^i(g) \frac{\delta}{\delta \tilde{\phi}(g^{-1})}.
\]
Condition 4 follows from the relation
\[ \text{ev}(P^i \triangleright \left( \frac{\delta}{\delta \phi(g_1)} \otimes \tilde{\phi}(g_2) \right)) = P^i(g^{-1}_1 g_2) \text{ ev}(\left( \frac{\delta}{\delta \phi(g_1)} \otimes \tilde{\phi}(g_2) \right)) = 0. \] (3.49)

Thus we find that the effective braided noncommutative field theory of three-dimensional quantum gravity coupled with spinless particles possesses translational symmetry.

Next we consider rotational symmetry. Rotational symmetry corresponds to the action
\[ \Lambda \triangleright \tilde{\phi}(g) = \tilde{\phi}(h^{-1}g h), \] (3.50)

which is the usual Lie-group action. The action on the tensor product is
\[ \Lambda \triangleright (\tilde{\phi}(g_1) \otimes \tilde{\phi}(g_2)) = \tilde{\phi}(h^{-1}g_1 h) \otimes \tilde{\phi}(h^{-1}g_2 h). \] (3.51)

Thus, the coproduct of the rotational symmetry is given by
\[ \Delta'(\Lambda) = \Lambda \otimes \Lambda. \] (3.52)

From the axiom (3.11), the counit of \( \Lambda \) is given by
\[ \epsilon'(\Lambda) = 1. \] (3.53)

Condition 1 follows from the relation
\[ \Lambda \triangleright S_{\text{int}} = \int dg_1 dg_2 dg_3 \delta(g_1 g_2 g_3) \Lambda \triangleright (\tilde{\phi}(g_1) \tilde{\phi}(g_2) \tilde{\phi}(g_3)) \]
\[ = \int dg_1 dg_2 dg_3 \delta(g_1 g_2 g_3) (\tilde{\phi}(h^{-1}g_1 h) \tilde{\phi}(h^{-1}g_2 h) \tilde{\phi}(h^{-1}g_3 h)) \]
\[ = \epsilon'(\Lambda) S_{\text{int}}. \] (3.54)

Condition 2 follows from the relation
\[ \psi(\Lambda \triangleright (\tilde{\phi}(g_1) \otimes \tilde{\phi}(g_2))) = \tilde{\phi}(h^{-1}g_2 h) \otimes \tilde{\phi}(h^{-1}g_2^{-1}g_1 g_2 h), \]
\[ \Lambda \triangleright \psi(\tilde{\phi}(g_1) \otimes \tilde{\phi}(g_2)) = \tilde{\phi}(h^{-1}g_2 h) \otimes \tilde{\phi}(h^{-1}g_2^{-1}g_1 g_2 h). \] (3.55)

Condition 3 follows from the relation
\[ \Lambda \triangleright \gamma^{-1}(\tilde{\phi}(g)) = \frac{1}{P^2(g^{-1}) - m^2 \delta \tilde{\phi}(h^{-1}g^{-1} h)}, \]
\[ \gamma^{-1}(\Lambda \triangleright \tilde{\phi}(g)) = \frac{1}{P^2(h^{-1}g^{-1} h) - m^2 \delta \tilde{\phi}(h^{-1}g^{-1} h)} \]
\[ = \frac{1}{P^2(g^{-1}) - m^2 \delta \tilde{\phi}(h^{-1}g^{-1} h)}. \] (3.56)
Condition 4 follows from the relation
\[
\text{ev} \left( A \triangleright \left( \frac{\delta}{\delta \tilde{\phi}(g_1)} \otimes \tilde{\phi}(g_2) \right) \right) = \text{ev} \left( \frac{\delta}{\delta \tilde{\phi}(h^{-1}g_1h)} \otimes \tilde{\phi}(h^{-1}g_2h) \right) \\
= \delta(g_1^{-1}g_2) \\
= \epsilon'(A) \text{ev} \left( \frac{\delta}{\delta \tilde{\phi}(g_1)} \otimes \tilde{\phi}(g_2) \right).
\]
(3.57)

Thus we find that this braided noncommutative field theory also possesses rotational symmetry.

### 3.4. Twisted Poincaré symmetry of noncommutative field theory on the Moyal plane

In this subsection, we discuss the twisted Poincaré symmetry of noncommutative field theory on the Moyal plane \([x^\mu, x^\nu] = i\theta^{\mu\nu}\).

For example, the action of the \(\phi^3\) theory is given by
\[
S = \int d^Dx \left[ \frac{1}{2} (\partial_\mu \phi \ast \partial^\mu \phi)(x) - \frac{1}{2} m^2 (\phi \ast \phi)(x) + \frac{\lambda}{3!} (\phi \ast \phi \ast \phi)(x) \right],
\]
(3.58)

where the star product is given by
\[
\phi(x) \ast \phi(y) = e^{\frac{i}{2} \theta^{\mu\nu} \partial_\mu \phi(x) \partial^\nu \phi(y)} \bigg|_{x=y}.
\]
(3.59)

In the momentum representation, the action is
\[
S = \int d^Dp \left[ \frac{1}{2} (p^2 - m^2) \tilde{\phi}(p) \tilde{\phi}(-p) \\
+ \frac{\lambda}{3!} \int d^Dp_1 d^Dp_2 d^Dp_3 e^{-\frac{i}{2} \theta^{\mu\nu} \partial_\mu \phi_1 \partial^\nu \phi_2} \delta(p_1 + p_2 + p_3) \tilde{\phi}(p_1) \tilde{\phi}(p_2) \tilde{\phi}(p_3) \right].
\]
(3.60)

We take \(X\) as the space of \(\tilde{\phi}(p)\) and \(X^*\) as that of \(\frac{\delta}{\delta \phi(p)}\). Then, we choose the braided Hopf algebra as follows:
\[
\Delta : \tilde{\phi}(p) \rightarrow \tilde{\phi}(p) \otimes 1 + 1 \otimes \tilde{\phi}(p),
\]
(3.61)
\[
\epsilon : \tilde{\phi}(p) \rightarrow 0,
\]
(3.62)
\[
S : \tilde{\phi}(p) \rightarrow -\tilde{\phi}(p).
\]
(3.63)

From the relation \(\gamma(\partial) = \partial S_0 = (p^2 - m^2) \tilde{\phi}(-p)\), we have
\[
\gamma^{-1}(\tilde{\phi}(p)) = \frac{1}{p^2 - m^2} \frac{\delta}{\delta \tilde{\phi}(-p)}.
\]
(3.64)

Let us consider the twisted Poincaré symmetry.\(^{12)}\)–\(^{14)}\) The coproduct and the counit of the twisted Poincaré algebra are given by
\[
\Delta'(P^\mu) = P^\mu \otimes 1 + 1 \otimes P^\mu,
\]
\[ \epsilon'(P^\mu) = 0, \]
\[ \Delta'(M^{\mu\nu}) = M^{\mu\nu} \otimes 1 + \epsilon' \otimes M^{\mu\nu} \]
\[ - \frac{1}{2} \theta^{\alpha\beta} [(\delta_\alpha^\mu P^\nu - \delta_\nu^\mu P^\alpha) \otimes P_\beta + P_\alpha \otimes (\delta_\beta^\nu P^\mu - \delta_\mu^\nu P^\beta)], \]
\[ \epsilon'(M^{\mu\nu}) = 0. \]  
(3.65)

Thus the action of the twisted Lorentz algebra on the tensor product is

\[ M^{\mu\nu} \triangleright (\tilde{\phi}(p_1) \otimes \tilde{\phi}(p_2)) = M^{\mu\nu} \triangleright \tilde{\phi}(p_1) \otimes \tilde{\phi}(p_2) + \tilde{\phi}(p_1) \otimes M^{\mu\nu} \triangleright \tilde{\phi}(p_2) \]
\[ - \frac{1}{2} \theta^{\alpha\beta} [(\delta_\alpha^\mu P^\nu - \delta_\nu^\mu P^\alpha) \triangleright \tilde{\phi}(p_1) \otimes P_\beta \triangleright \tilde{\phi}(p_2) \]
\[ + P_\alpha \triangleright \tilde{\phi}(p_1) \otimes (\delta_\beta^\nu P^\mu - \delta_\mu^\nu P^\beta) \triangleright \tilde{\phi}(p_2)], \]  
(3.66)

where \( M^{\mu\nu} \triangleright \tilde{\phi}(p) = i(p^\mu \partial/p_{\nu} - p^\nu \partial/p_{\mu}) \tilde{\phi}(p) \) and \( P^{\mu} \triangleright \tilde{\phi}(p) = p^\mu \tilde{\phi}(p) \). The actions of \( M^{\mu\nu} \) and \( P^{\mu} \) on \( \frac{\delta}{\delta \phi(p)} \) are

\[ M^{\mu\nu} \triangleright \frac{\delta}{\delta \phi(p)} = i(p^\mu \partial/p_{\nu} - p^\nu \partial/p_{\mu}) \frac{\delta}{\delta \phi(p)}, \]  
(3.67)

\[ P^{\mu} \triangleright \frac{\delta}{\delta \phi(p)} = -p^\mu \frac{\delta}{\delta \phi(p)}. \]  
(3.68)

We easily find that the three conditions (3.13), (3.15) and (3.16) are satisfied, but (3.14) is not if the braiding is trivial. In order to keep the invariance, the braiding must be chosen as

\[ \psi(\tilde{\phi}(p_1) \otimes \tilde{\phi}(p_2)) = e^{i\theta^{\alpha\beta} p_2^\alpha \otimes p_1^\beta} (\tilde{\phi}(p_2) \otimes \tilde{\phi}(p_1)). \]  
(3.69)

This agrees with the result given in the previous works.\(^{21,35}\)

We can easily check that translational symmetry holds, since the coproduct \( \Delta'(P^\mu) \) follows the usual Leibniz rule.

3.5. Relations among correlation functions: Examples

Now we have confirmed, to all orders of perturbation, that the two theories considered in the preceding sections possess symmetry relations among the correlation functions implied by the Hopf algebra symmetries. In §3.3, we determined how the translational generator acts on a product of fields in (3.42) and (3.43) in the momentum representation. Because the physical meaning of this Hopf algebra transformation is not clear, it would be interesting to explicitly elucidate the symmetry relations among correlation functions. The same is true in the case of the twisted Lorentz symmetry studied in §3.4. In this subsection, we explicitly work out some relations among correlation functions in the two theories.

In the effective quantum field theory of quantum gravity, the action of the translational generators on a correlation function is given by

\[ \langle \tilde{\phi}(g_1) \cdots \tilde{\phi}(g_n) \rangle \rightarrow i\epsilon P_i (g_1 \cdots g_n) \langle \tilde{\phi}(g_1) \cdots \tilde{\phi}(g_n) \rangle \]  
(3.70)
in the momentum representation, where \( \epsilon^i \) is an infinitesimal parameter. Thus we obtain the relation
\[
P_i(g_1 \cdots g_n) \langle \hat{\phi}(g_1) \cdots \hat{\phi}(g_n) \rangle = 0. \tag{3.71}
\]
This is a (modified) momentum conservation law; the correlation function has support only on the vanishing momentum subspace, \( P_i(g_1 \cdots g_n) = 0 \). This all-order relation in quantum field theory is a simple but important implication of the Hopf algebraic translational symmetry. This provides a good example of the physical importance of a Hopf algebraic symmetry: a Hopf algebraic symmetry leads to a (modified) conservation law.

It would also be interesting to determine the relations in the coordinate representations, where the fields are defined by \( \phi(x) = \int_p e^{ip \cdot x} \hat{\phi}(p) \). As explicitly noted in the preceding subsections, we stress that the basis of the spaces \( X \) of the field variables in the path integrals are parameterized in terms of momenta, and that the fields \( \phi(x) \) are defined as some \( c \)-number linear combinations of them. Therefore, an action \( a \in A \) of a symmetry transformation acts as
\[
a \triangleright \phi(x) = \int_p e^{ip \cdot x} (a \triangleright \hat{\phi}(p)), \tag{3.72}
\]
and the symmetry relations of the correlation functions can be obtained through inverse Fourier transformations (with possible non-trivial measures) of those in the momentum representations.

For example, in the case of the two-point function, after the inverse Fourier transformation, the relation among correlation functions is given by
\[
\langle \partial^i \phi(x_1) \phi(x_2) + \phi(x_1) \partial^i \phi(x_2) \rangle = 0, \tag{3.73}
\]
where we have used the relation (3.71). Interestingly, this is the usual relation in a translationally invariant quantum field theory. In the case of the three-point function, however, the relation is given by
\[
\langle \partial^i \phi(x_1) \sqrt{1 + \kappa^2 \partial^2} \phi(x_2) \sqrt{1 + \kappa^2 \partial^2} \phi(x_3) + \sqrt{1 + \kappa^2 \partial^2} \phi(x_1) \partial^i \phi(x_2) \sqrt{1 + \kappa^2 \partial^2} \phi(x_3) + \sqrt{1 + \kappa^2 \partial^2} \phi(x_1) \partial^i \phi(x_2) \sqrt{1 + \kappa^2 \partial^2} \phi(x_3) \\
+ \sqrt{1 + \kappa^2 \partial^2} \phi(x_1) \partial^i \phi(x_2) \partial^k \phi(x_3) - \kappa \epsilon^{ijk} \partial_j \phi(x_1) \partial_k \phi(x_2) \partial_k \phi(x_3) + \kappa^2 \partial^i \phi(x_1) \partial^j \phi(x_2) \partial^k \phi(x_3) - \kappa^2 \partial^i \phi(x_1) \partial^j \phi(x_2) \partial^k \phi(x_3) - \kappa^2 \partial^i \phi(x_1) \partial^j \phi(x_2) \partial^k \phi(x_3) \rangle = 0. \tag{3.74}
\]
This is quite a non-trivial relation among correlation functions, and it would be difficult to derive if the Hopf algebra symmetry in the quantum field theory were not first known. This is another interesting example implying the physical importance of a Hopf algebra symmetry. In general, the relation has the form
\[
\left( \sum_{l=1}^{n} \partial_{x_{1l}} - i \sum_{l<m} \kappa \epsilon^{ijk} \partial_{x_{1j}} \partial_{x_{m}} \right) \langle \phi(x_1) \cdots \phi(x_n) \rangle = 0. \tag{3.75}
\]
In the \( \kappa \to 0 \) limit, this relation approaches the usual relation. Thus, the Hopf algebra symmetry is a kind of translational symmetry modified by adding \( \kappa \)-dependent higher-order derivative multi-field contributions.

We can proceed in a similar manner for the twisted Lorentz symmetry. We have the following general form of such a symmetry relation:

\[
M_{\mu\nu} > \langle \tilde{\phi}(p_1) \cdots \tilde{\phi}(p_n) \rangle = 0. \tag{3.76}
\]

In the case of the two-point function, this relation is given by

\[
\langle (x_1^\mu \partial^\nu - x_2^\nu \partial^\mu) \phi(x_1) \phi(x_2) + \phi(x_1) \phi(x_2) (x_2^\mu \partial^\nu - x_2^\nu \partial^\mu) \phi(x_2) \rangle = 0, \tag{3.77}
\]

where we have used momentum conservation. This is the same as the relation in a Lorentz invariant quantum field theory. In the case of the three-point function, this relation is given by

\[
\langle (x_1^\mu \partial^\nu - x_3^\nu \partial^\mu) \phi(x_1) \phi(x_2) \phi(x_3) \\
+ \phi(x_1)(x_2^\nu \partial^\mu - x_3^\nu \partial^\mu) \phi(x_2) \phi(x_3) + \phi(x_1) \phi(x_2)(x_3^\nu \partial^\mu - x_3^\nu \partial^\mu) \phi(x_3) \\
+ \frac{1}{2} i \theta^{\alpha \mu} (\partial_\alpha \phi(x_1) \partial^\nu \phi(x_2) \phi(x_3) + \partial_\alpha \phi(x_1) \phi(x_2) \partial^\nu \phi(x_3) + \phi(x_1) \partial_\alpha \phi(x_2) \partial^\nu \phi(x_3) \\
- \partial^\nu \phi(x_1) \partial_\alpha \phi(x_2) \phi(x_3) - \partial^\nu \phi(x_1) \phi(x_2) \partial_\alpha \phi(x_3) - \phi(x_1) \partial_\alpha \phi(x_2) \partial^\nu \phi(x_3) \\
- \frac{1}{2} i \theta^{\alpha \mu} (\partial_\alpha \phi(x_1) \partial^\nu \phi(x_2) \phi(x_3) + \partial_\alpha \phi(x_1) \phi(x_2) \partial^\nu \phi(x_3) + \phi(x_1) \partial_\alpha \phi(x_2) \partial^\nu \phi(x_3) \\
- \partial^\nu \phi(x_1) \partial_\alpha \phi(x_2) \phi(x_3) - \partial^\nu \phi(x_1) \phi(x_2) \partial_\alpha \phi(x_3) - \phi(x_1) \partial_\alpha \phi(x_2) \partial^\nu \phi(x_3)) \rangle = 0. \tag{3.78}
\]

In general, the relation among correlation functions has

\[
\langle (x_{1\mu} \partial_{1\nu} - x_{1\nu} \partial_{1\mu}) + \cdots + (x_{n\mu} \partial_{n\nu} - x_{n\nu} \partial_{n\mu}) + O(\theta) \rangle \langle \phi(x_1) \cdots \phi(x_n) \rangle = 0 \tag{3.79}
\]

in the coordinate representation. The leading terms correspond to the usual Lorentz transformation, \( x^\mu \to x^\mu + \epsilon^{\mu \nu} x^\nu \).

The above symmetry relations on the Moyal plane can be represented in manners similar to those in the usual commutative cases if we use star products. In the papers,\textsuperscript{24–30} it is pointed out that in the coordinate representation, correlation functions on the Moyal plane should be defined with star products extended to noncoincident points (see also Ref. 43), instead of usual products, because in that case, the usual commutation relation \([x_i^\mu, x_j^\nu] = 0 \ (i, j = 1, \ldots, n)\) is not invariant under the twisted Poincaré transformation. Carrying out the Fourier transformation of the symmetry relation (3.76) for the momentum representation in such a noncommutative coordinate representation, we obtain the symmetry relations in the form of star tensor products. Explicitly, (3.77) becomes

\[
\langle ((x_1^\mu \partial^\nu - x_2^\nu \partial^\mu) \phi(x_1)) * \phi(x_2) + \phi(x_1) * ((x_2^\mu \partial^\nu - x_2^\nu \partial^\mu) \phi(x_2)) \rangle = 0, \tag{3.80}
\]

and (3.78) becomes

\[
\langle ((x_1^\mu \partial^\nu - x_1^\nu \partial^\mu) \phi(x_1)) * \phi(x_2) * \phi(x_3) \rangle.
\]
\[+ \phi(x_1) \ast ((x_2^\mu \partial^\nu - x_2^\nu \partial^\mu) \phi(x_2)) \ast \phi(x_3) + \phi(x_1) \ast \phi(x_2) \ast ((x_3^\nu \partial^\rho - x_3^\rho \partial^\nu) \phi(x_3))\]

\[+ \frac{1}{2} i \theta^{\alpha \mu} (\partial_\alpha \phi(x_1) \ast \partial^\nu \phi(x_2) \ast \phi(x_3) + \partial_\alpha \phi(x_1) \ast \phi(x_2) \ast \partial^\nu \phi(x_3))\]

\[+ \phi(x_1) \ast \partial_\alpha \phi(x_2) \ast \partial^\nu \phi(x_3) - \partial^\nu \phi(x_1) \ast \partial_\alpha \phi(x_2) \ast \phi(x_3)\]

\[- \partial^\nu \phi(x_1) \ast \phi(x_2) \partial_\alpha \ast \phi(x_3) - \phi(x_1) \ast \partial^\nu \phi(x_2) \ast \partial_\alpha \phi(x_3))\]

\[- \frac{1}{2} i \theta^{\alpha \nu} (\partial_\alpha \phi(x_1) \ast \partial^\nu \phi(x_2) \ast \phi(x_3) + \partial_\alpha \phi(x_1) \ast \phi(x_2) \partial^\nu \ast \phi(x_3))\]

\[- \phi(x_1) \ast \partial_\alpha \phi(x_2) \ast \partial^\nu \phi(x_3) - \partial^\nu \phi(x_1) \ast \partial_\alpha \phi(x_2) \ast \phi(x_3)\]

\[- \partial^\nu \phi(x_1) \ast \phi(x_2) \partial_\alpha \ast \phi(x_3) - \phi(x_1) \ast \partial^\nu \phi(x_2) \ast \partial_\alpha \phi(x_3))\] = 0. \hspace{1cm} (3.81)

More generally, we can derive the symmetry relations of the correlation functions for the tensor fields \(\phi_{\alpha_1 \cdots \alpha_n}(x) \equiv \partial_{\alpha_1} \cdots \partial_{\alpha_n} \phi(x)\). For example, in the case of the three-point function of the tensor fields, the symmetry relation becomes

\[\langle \langle (M^{1 \mu\nu})_{\alpha_1 \cdots \alpha_l} \delta_{\beta_1 \cdots \beta_l} (x_1) \rangle \ast \phi_{\beta_1 \cdots \beta_m} (x_2) \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\]

\[+ \phi_{\alpha_1 \cdots \alpha_l}(x_1) \ast \langle \langle (M^{2 \mu\nu})_{\beta_1 \cdots \beta_m} \delta_{\delta_1 \cdots \delta_m} (x_2) \rangle \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\]

\[+ \phi_{\alpha_1 \cdots \alpha_l}(x_1) \ast \phi_{\beta_1 \cdots \beta_m} (x_2) \ast \langle \langle (M^{3 \mu\nu})_{\gamma_1 \cdots \gamma_n} \delta_{\delta_1 \cdots \delta_n} (x_3) \rangle \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\]

\[- \frac{1}{2} \theta^{\alpha \mu} [\partial_\alpha \phi_{\alpha_1 \cdots \alpha_l}(x_1) \ast \partial^\nu \phi_{\beta_1 \cdots \beta_m} (x_2) \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\]

\[+ \partial_\alpha \phi_{\alpha_1 \cdots \alpha_l}(x_1) \ast \phi_{\beta_1 \cdots \beta_m} (x_2) \partial^\nu \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\]

\[+ \phi_{\alpha_1 \cdots \alpha_l}(x_1) \ast \partial_\alpha \phi_{\beta_1 \cdots \beta_m} (x_2) \partial^\nu \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\]

\[- \partial^\nu \phi_{\alpha_1 \cdots \alpha_l}(x_1) \ast \partial_\alpha \phi_{\beta_1 \cdots \beta_m} (x_2) \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\]

\[- \partial^\nu \phi_{\alpha_1 \cdots \alpha_l}(x_1) \ast \phi_{\beta_1 \cdots \beta_m} (x_2) \partial_\alpha \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\]

\[- \phi_{\alpha_1 \cdots \alpha_l}(x_1) \ast \partial^\nu \phi_{\beta_1 \cdots \beta_m} (x_2) \partial_\alpha \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\]

\[+ \frac{1}{2} \theta^{\alpha \nu} [\partial_\alpha \phi_{\alpha_1 \cdots \alpha_l}(x_1) \ast \partial^\mu \phi_{\beta_1 \cdots \beta_m} (x_2) \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\]

\[+ \partial_\alpha \phi_{\alpha_1 \cdots \alpha_l}(x_1) \ast \phi_{\beta_1 \cdots \beta_m} (x_2) \partial^\mu \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\]

\[+ \phi_{\alpha_1 \cdots \alpha_l}(x_1) \ast \partial_\alpha \phi_{\beta_1 \cdots \beta_m} (x_2) \partial^\mu \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\]

\[- \partial^\mu \phi_{\alpha_1 \cdots \alpha_l}(x_1) \ast \partial_\alpha \phi_{\beta_1 \cdots \beta_m} (x_2) \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\]

\[- \partial^\mu \phi_{\alpha_1 \cdots \alpha_l}(x_1) \ast \phi_{\beta_1 \cdots \beta_m} (x_2) \partial_\alpha \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\]

\[- \phi_{\alpha_1 \cdots \alpha_l}(x_1) \ast \partial^\nu \phi_{\beta_1 \cdots \beta_m} (x_2) \partial_\alpha \ast \phi_{\gamma_1 \cdots \gamma_n} (x_3)\] = 0. \hspace{1cm} (3.82)

where

\[(M^{\mu \nu})_{\alpha_1 \cdots \alpha_n} = (L^{\mu \nu})_{\alpha_1 \cdots \alpha_n} \beta_1 \cdots \beta_n + (S^{\mu \nu})_{\alpha_1 \cdots \alpha_n} \beta_1 \cdots \beta_n,\]

\[(L^{\mu \nu})_{\alpha_1 \cdots \alpha_n} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \delta_{\alpha_1 \beta_1} \cdots \delta_{\alpha_n \beta_n},\]

\[(S^{\mu \nu})_{\alpha_1 \cdots \alpha_n} = i(\eta^{\nu \beta_1} \delta_{\alpha_1} \delta_{\alpha_2} \beta_2 \cdots \delta_{\alpha_n} \beta_n - \eta^{\mu \beta_1} \delta_{\alpha_1} \beta_1 \alpha_2 \beta_2 \cdots \delta_{\alpha_n} \beta_n). \hspace{1cm} (3.83)\]

If we take the operators \((M^{i \mu \nu})_{\alpha_1 \cdots \alpha_n} \beta_1 \cdots \beta_n (i = 1, 2, 3)\) out of the star products, the \(\theta^{i \mu \nu}\) dependent terms are canceled. The final expressions are simply the usual Lorentz rotations of the coordinates and the tensorial indices in the correlation functions. This is fully consistent with the results given in Ref. 29).
3.6. Origin of Hopf algebra symmetries

To further study the meanings of these additional terms, let us more closely consider the transformation properties of the star products. In the latter case, it is known that the $\theta^{\mu\nu}$ dependence of the twisted Lorentz transformation (3.66) comes from the Lorentz transformation of $\theta^{\mu\nu}$ itself. To see this, let us consider an infinitesimal Lorentz transformation $\Lambda^\mu_\nu = \delta^\mu_\nu + \epsilon^\mu_\nu$. Under this operation, the transformation of $\theta^{\mu\nu}$ is given by

$$
\theta^{\mu\nu} \rightarrow \theta^{\mu\nu} + \epsilon^\mu_\rho \theta^{\rho\nu} + \epsilon^\nu_\rho \theta^{\mu\rho} := \theta^{\mu\nu} + \delta \theta^{\mu\nu}.
$$

If we consider not only the transformation of the coordinates, $x' = x + \epsilon^{\mu\nu} x_\nu$, but also (3.84), and assume that $\phi(x) \ast_{\theta} \phi(x)$ and $\phi'(x') \ast_{\theta + \delta \theta} \phi'(x')$ are equal, we obtain, after the Fourier transformation,

$$
\tilde{\phi}'(p_1) \otimes \tilde{\phi}'(p_2)
= (1 - \frac{i}{2} \epsilon^{\mu\nu} M_{\mu\nu} \otimes 1 + 1 \otimes \epsilon^{\mu\nu} M_{\mu\nu} + \delta \theta^{\mu\nu} P_\mu \otimes P_\nu)) \tilde{\phi}(p_1) \otimes \tilde{\phi}(p_2)
= (1 - \frac{i}{2} \epsilon^{\mu\nu} \Delta' M_{\mu\nu}) \tilde{\phi}(p_1) \otimes \tilde{\phi}(p_2),
$$

which agrees with (3.66). This shows that the additional part of the coproduct of $M_{\mu\nu}$ takes into account the transformation of the non-dynamical background parameter, $\theta^{\mu\nu}$.

The former case can be treated in a similar manner. The definition of the star product is given by

$$
e^{ix^i P_i(g_1)} \ast x e^{ix^i P_i(g_2)} = e^{ix^i P_i(g_1 g_2)},
$$

where we have explicitly indicated the coordinate in which the star product is taken. Then we recognize that $e^{i(x+\epsilon)^i P_i(g_1)} \ast_{x+\epsilon} e^{i(x+\epsilon)^i P_i(g_2)}$ and $e^{i(x+\epsilon)^i P_i(g_1)} \ast x e^{i(x+\epsilon)^i P_i(g_2)}$ give distinct values. Specifically, if the coordinate of the star product is also shifted, we have

$$
e^{i(x+\epsilon)^i P_i(g_1)} \ast_{x+\epsilon} e^{i(x+\epsilon)^i P_i(g_2)} = e^{i(x+\epsilon)^i P_i(g_1 g_2)},
$$

but, if not, we have

$$
e^{i(x+\epsilon)^i P_i(g_1)} \ast x e^{i(x+\epsilon)^i P_i(g_2)} = e^{i\epsilon^i P_i(g_1)} e^{i\epsilon^i P_i(g_2)} e^{ix^i P_i(g_1 g_2)}.
$$

Therefore, if we take the translationally transformation as (3.87), and carry out the same procedure in deriving (3.26), we always obtain a translational invariant commutation relation,*

$$
[(x + \epsilon)^i, (x + \epsilon)^j]_{x+\epsilon} = 2i \kappa e^{ijk} (x + \epsilon)_k.
$$

Now, assuming that $\phi(x) \ast_x \phi(x)$ and $\phi'(x') \ast_x \phi'(x')$ are equal under the translation $x^i \rightarrow x'^i = x^i + \epsilon^i$, we obtain, after the Fourier transformation,

$$
\tilde{\phi}'(g_1) \tilde{\phi}'(g_2) = (1 - i \epsilon^i P_i(g_1 g_2)) \tilde{\phi}(g_1) \tilde{\phi}(g_2),
$$

* There is a similar treatment in Ref. 42.
which is the same as (3.42).

From these two examples, we conjecture that the multi-field contributions in (3.8) come from the transformation properties of the star products.

§4. Summary and comments

We have investigated symmetries in noncommutative field theories in the framework of braided quantum field theory. We have obtained the algebraic conditions for a Hopf algebra to be a symmetry of a braided quantum field theory, by studying the necessary conditions for the relations among correlation functions generated by the transformation algebra to hold. Then we applied our study to the Poincaré symmetries in the effective noncommutative field theory of three-dimensional quantum gravity coupled with spinless particles and in the noncommutative field theory on the Moyal plane. In the former case, we can understand the braiding between fields, which was derived from a three-dimensional quantum gravity computation, from the viewpoint of the translational symmetry of noncommutative field theory in a Lie-algebraic noncommutative spacetime. In the latter case, we found that the twisted Lorentz symmetry on the Moyal plane is a symmetry of the quantum field theory only after the inclusion of the nontrivial braiding factor. This finding is in agreement with that given in the previous works.\textsuperscript{28,35} Then, we discussed the meaning of the Hopf algebra symmetries from the viewpoint of the coordinate representation.

In a recent study, a noncommutative field theory on $\kappa$-Minkowski spacetime was investigated.\textsuperscript{36} Because this noncommutativity of the coordinates is given by $[x^0, x^j] = \frac{i}{\kappa} x^j$, this noncommutative field theory does not possess naive translational symmetry. In that case, we may introduce a non-trivial braiding between fields, as in the effective field theory discussed in §3.3, to maintain the momentum conservation. However, while the effective field theory possesses the braided category structure because of the invariance of the Haar measure $d(g^{-1}g'g) = dg'$, the measure of the momentum space of the field theory on $\kappa$-Minkowski spacetime is only left-invariant.\textsuperscript{36} Therefore, it is yet unclear whether we can embed this field theory on $\kappa$-Minkowski spacetime into the framework of braided quantum field theory.

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Appendix A

Proofs of the Formulas (2.20) and (2.21)

Here, we give proofs of the formulas (2.20) and (2.21) using diagrams. First, we use the formula
\[
\hat{ev}(\partial \otimes \alpha \beta) = \hat{ev}(\partial \otimes \alpha)\epsilon(\beta) + \hat{ev}(\partial \otimes \beta)\epsilon(\alpha),
\]
(A.1)
where \(\alpha, \beta \in \hat{X}\). This is clear from the definition of \(\hat{ev}\).

Figure 14 gives the proof of (2.20). In the first line, we use the axiom (2.12), and in the second line we use the lemma (A.1). We find the last line from a property of counits.

Next, we prove (2.21). By using the braided Leibniz rule (2.20) as \(\alpha \in X \otimes \hat{X}\), the left-hand side of (2.21) takes the form expressed in Fig. 15. The first term in Fig. 15 becomes \((ev \otimes id^n - 1)(\partial \otimes id^n \alpha)\) if we use the definition of the coproduct given in (2.9).

In the second term of Fig. 15, we divide \(\hat{X}\) into \(X \otimes \hat{X}\) and carry out the same iteration as above. For example, if the degree of \(\hat{X}\) is 3, the second term in Fig. 15 can be reduced as in Fig. 16. We have used \(\Delta X = X \otimes 1 + 1 \otimes X\) in the second line of Fig. 16. The result agrees with (2.21).

In the same way, we can obtain the formula (2.21) in general.

Appendix B

Proofs of (2.27)–(2.29)

From the definition of \(\gamma\) (2.25), we find
\[
\alpha aw = -\alpha \text{diff}(\gamma^{-1}(a) \otimes w)
\]
(B.1)
for \(a \in X\) and \(\alpha \in \hat{X}\). On the other hand, adding \(\gamma^{-1}\) and \(\psi\) to the braided Leibniz rule (2.20), as in Fig. 17, we find
\[
\alpha \text{diff}(\gamma^{-1}(a) \otimes w) = \text{diff}(\psi(\alpha \otimes \gamma^{-1}(a))w) - (\text{diff} \circ \psi(\alpha \otimes \gamma^{-1}(a)))w.
\]
(B.2)
Combining (B.1) and (B.2), we obtain
\[
\alpha aw = -\text{diff}(\psi(\alpha \otimes \gamma^{-1}(a))w) + (\text{diff} \circ \psi(\alpha \otimes \gamma^{-1}(a)))w.
\]
(B.3)
Integrating both sides of (B.3) and using (2.24), we find
\[
Z^{(0)}(\alpha a) = Z^{(0)}(\text{diff} \circ \psi(\alpha \otimes \gamma^{-1}(a))).
\]
(B.4)
If \(\alpha\) is \(b \in X\), then we have
\[
Z^{(0)}(ba) = Z^{(0)}(\text{diff} \circ \psi(b \otimes \gamma^{-1}(a)))
= ev \circ \psi(b \otimes \gamma^{-1}(a))
= ev \circ (\gamma^{-1} \otimes id) \circ \psi(b \otimes a).
\]
(B.5)
Fig. 14. The proof of (2.20).

Fig. 15. The left-hand side of (2.21).
Thus we obtain (2.27).

By setting $\alpha = 1$, it is clear that we have

$$Z_{(0)}^1(a) = 0. \quad (B.6)$$

Next, we rewrite (B.4) for $\alpha \in X^{n-1}$ using the formula (2.21). This is expressed diagrammatically in Fig. 18. The second equality is due to (2.21). Thus, we obtain

$$Z_{(0)}^n = (Z_{n-2}^{(0)} \otimes Z_{2}^{(0)}) \circ ([n-1]_0 \otimes \text{id}) \quad (B.7)$$

Iterating this, we find (2.28) for even $n$ and (2.29) for odd $n.$
Fig. 17. The diagram obtained by adding $\gamma^{-1}$ and $\psi$ over the braided Leibniz rule.

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Fig. 18. Diagrammatic proof of (B.7).

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