On higher derivative corrections of tachyon action

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ABSTRACT

We have examined the momentum expansion of the disk level S-matrix element of two tachyons and two gauge fields to find, up to on-shell ambiguity, the couplings of these fields in the world volume theory of $N$ coincident non-BPS D-branes to all order of $\alpha'$. Using the proposal that the action of D-brane-anti-D-brane is given by the projection of the action of two non-BPS D-branes with $(-1)^F_L$, we find the corresponding couplings in the world volume theory of the brane-anti-brane system. Using these infinite tower of couplings, we then calculate the massless pole of the scattering amplitude of one RR field, two tachyons and one gauge field in the brane-anti-brane theory. We find that the massless pole of the field theory amplitude is exactly equal to the massless pole of the disk level S-matrix element of one RR, two tachyons and one gauge field to all order of $\alpha'$. We have also found the couplings of four tachyons to all order of $\alpha'$ by examining the S-matrix element of four tachyons.
1 Introduction

Brane-anti-brane system has been used to model inflation in string theory\cite{1, 2, 3, 4}. When branes are very far away from each other, the transverse scalar field which describes the motion of one brane in the background of the other brane plays the role of inflaton. The dynamics of the moving brane in this period is very well described at low energy by the DBI action. On the other hand, when branes come within a critical distance from each other, the string stretching between the two branes become tachyonic and inflation ends. The dynamics of the brane-anti-brane in this period is very important for studying the reheating \cite{5, 6, 7}. Brane-anti-brane system has been also used to study spontaneous chiral symmetry breaking in holographic model of QCD \cite{8, 9, 10}. In these studies, flavor branes introduced by placing a set of parallel branes and anti-branes on a background dual to a confining color theory \cite{11}.

It is important then to study the world-volume theory of D$p$-brane-anti-D$p$-brane. The world-volume theory of this system has tachyon, massless and infinite tower of massive fields which can be described by Berkovits’ superstring field theory \cite{12, 13}. The world-volume theory may be rewritten in terms of the tachyon and massless fields and infinite number of derivative terms reflecting the effect of massive fields. We call this field theory “the higher derivative theory”. When the world volume fields vary slowly the higher derivative theory should be reduced to the effective theory. It is known that the vortex solution of the field theory of the D$p$-brane-anti-D$p$-brane pair should describe the stable D$p$-2-brane \cite{14}. An effective action for brane-anti-brane which has this property has been proposed in \cite{15}

$$ S = -T_p \int d^{p+1} \sigma V(T) \left( \sqrt{-\det A^{(1)}} + \sqrt{-\det A^{(2)}} \right), $$  \hspace{1cm} (1)

where

$$ A^{(n)}_{\mu \nu} = \eta_{\mu \nu} + 2\pi \alpha' F^{(n)}_{\mu \nu} + \pi \alpha' (D_\mu T(D_\nu T)^*) + D_\nu T(D_\mu T)^* \right). $$  \hspace{1cm} (2)

where $T_p$ is the D$p$-brane tension and $D_a T = \partial_a T - i(A^{(1)}_a - A^{(2)}_a)T$. The above action is a generalization of the tachyon DBI action \cite{16, 17, 18, 19}. The above action has a vortex solution whose world-volume action is given by the DBI action of stable $D_p$-2-brane \cite{15}. It is difficult to find the higher derivative corrections to this action.

Another proposal for the effective action of the brane-anti-brane pair which is based on the S-matrix elements calculation is given by \cite{20, 21}

$$ S_{DBI} = -T_p \int d^{p+1} \sigma \text{STr} \left( V(T) \sqrt{-\det(\eta_{ab} + 2\pi \alpha' F_{ab} + 2\pi \alpha' D_a T D_b T)} \right), $$  \hspace{1cm} (3)

The trace in the above action should be completely symmetric between all matrices of the form $F_{ab}, D_a T$, and individual $T$ of the tachyon potential. These matrices are

$$ F_{ab} = \begin{pmatrix} F^{(1)}_{ab} & 0 \\ 0 & F^{(2)}_{ab} \end{pmatrix}, \quad D_a T = \begin{pmatrix} 0 & D_a T \\ (D_a T)^* & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix} $$  \hspace{1cm} (4)
To implement the symmetric trace prescription, one must first expand the action then make each term symmetric and finally take the trace. This in particular gives a coupling between $F^{(1)}$ and $F^{(2)}$. There is no such coupling in $F^{(1)}$. The above action has been found from the effective field theory of $N = 2$ non-BPS branes by projecting it with $(-1)^F_L$ where $F_L$ is the spacetime left-handed fermion number. On the other hand, the effective field theory of two non-BPS D-branes has been assumed to be the natural non-abelian extension of the tachyon DBI action, i.e., the action $\mathcal{L}$ without restricting the matrices to $\mathcal{L}$. In this paper, we would like to study the higher derivative corrections to this action.

A method for finding the higher derivative theory is to study the S-matrix elements of this theory and compare them with the S-matrix elements of string theory. If this higher derivative theory is going to be identical with the string theory, the S-matrix elements of the higher derivative theory must be identical to the momentum expansion of the S-matrix elements of string theory. Hence, by calculating the S-matrix elements of string theory and finding their momentum expansions, one can find the appropriate higher derivative couplings in the field theory. For instance, the string theory S-matrix element of one RR field and two tachyons can be reproduced by the higher derivative theory of the brane-anti-brane system if it includes the following couplings $[22]$:

$$\frac{\mu_p}{2} (2\pi\alpha')^2 C_{p-3} \wedge \left( \sum_{n=-1}^{\infty} b_n (\alpha')^{n+1} \partial^{a_1} \cdots \partial^{a_{n+1}} F \wedge \partial a_1 \cdots \partial a_{n+1} F \right)$$

(6)

where $b_n$’s are some known numbers. Or the S-matrix element of one RR and two gauge fields can be reproduced by the higher derivative field theory if it includes

$$\frac{\mu_p}{2} (2\pi\alpha')^2 C_{p-3} \wedge \left( \sum_{n=0}^{\infty} \sum_{p,n,m=0}^{\infty} c_{p,n,m} \left( \frac{\alpha'}{2} \right)^p (\alpha')^{2n+m} (D^a D_a)^n (DT DT^*) \right)$$

(7)

where again $c_{p,n,m}$’s are some known numbers. They can be reproduced by the higher derivative theory if it includes the following couplings for $C_{p-3}$:

$$2i\alpha' (\pi\alpha') \mu_p \sum_{p,n,m=0}^{\infty} c_{p,n,m} \left( \frac{\alpha'}{2} \right)^p (\alpha')^{2n+m} C_{p-3} \wedge \partial^{a_1} \cdots \partial^{a_{2n}} \partial^{b_1} \cdots \partial^{b_m} F$$

$$\wedge (D^a D_a)^p D_{b_1} \cdots D_{b_m} (D_{a_1} \cdots D_{a_2n} DT \wedge D_{a_{n+1}} \cdots D_{a_{2n}} DT^*)$$

(7)

where again $c_{p,n,m}$’s are some known numbers. And the following couplings for $C_{p-1}$:

$$-2\alpha' \mu_p \sum_{n=0}^{\infty} \frac{n}{2} \left( \frac{\alpha'}{2} \right)^n C_{p-1} \wedge (D^a D_a)^n (F|T|^2)$$

(8)
where $a_n$’s are exactly the numbers that appear in (5), and

$$2(\alpha')^2 \mu_p \sum_{p,n,m=0}^{\infty} e_{p,n,m}(\alpha')^{2m+n+1} \left( \frac{\alpha'}{2} \right)^p C_{p-1} \wedge$$

$$2(D_a D^a)^p \left[ -\partial_b \partial_c \partial^{a_1} \cdots \partial^{a_n} \partial_{b_1} \cdots \partial_{b_{2m}} F D_{a_1} \cdots D_{a_n} (D^b D^{b_1} \cdots D^{b_{2m}} T D^c D^{b_{m+1}} \cdots D^{b_{2m}} T^*) \right]$$

$$+ 2D_{a_1} \cdots D_{a_n} (D^b D^{b_1} \cdots D^{b_{2m}} T^* \wedge D_c D^{b_{m+1}} \cdots D^{b_{2m}} T^*) \partial^{a_1} \cdots \partial^{a_n} \partial_{b_1} \cdots \partial_{b_{2m}} F^{bc}$$

$$+ \partial_b \partial^{a_1} \cdots \partial^{a_n} \partial_{b_1} \cdots \partial_{b_{2m}} F_{c} \wedge D_{a_1} \cdots D_{a_n} (D^b D^{b_1} \cdots D^{b_{2m}} T D^c D^{b_{m+1}} \cdots D^{b_{2m}} T^*)$$

where $e_{p,n,m}$ are some other known numbers. It has been argued in [22] that the tachyon couplings in (5), (7) and (8) have no on-shell ambiguity. Having different couplings in the higher derivative theory without on-shell ambiguity, one can then find the effective theory by restricting the fields to be slowly varying fields. It is shown in [22] that the above couplings reduce to the Wess-Zumino effective couplings of brane-anti-brane system [24, 25, 26] for slowly varying fields. In this paper, we would like to extend the above discussion to find the higher derivative tachyon couplings corresponding to the non-abelian tachyon DBI action.

An outline of the rest of paper is as follows. In the next section, we find the momentum expansion of the string theory S-matrix element of two tachyons and two gauge fields in the world volume theory of $N$ non-BPS D-branes. We then write a tower of infinite number of two-tachyon-two-gauge field couplings which reproduce the above momentum expansion. We repeat the same steps to find the couplings of four massless transverse scalar fields and the couplings of four tachyons. In section 3, using the proposal that the action of brane-anti-brane can be found from the action of $N = 2$ non-BPS branes by projecting it with $(-1)^{F_L}$, we find the corresponding couplings in the brane-anti-brane theory. Using the two-tachyon-two-gauge field couplings of brane-anti-brane, we calculate the massless pole of the scattering amplitude of one RR field, two tachyons and one gauge field. We then compare the result with the corresponding massless pole in the string theory S-matrix element [22]. We find exact agreement.

## 2 Higher derivative terms of non-BPS branes

The world volume of $N$ coincident non-BPS D-branes has $N^2$ tachyons and $N^2$ gauge fields. The higher derivative theory of non-BPS branes which includes the kinetic terms and the couplings of these fields may be found by studying the S-matrix elements in the field theory and in the string theory. The S-matrix elements on the world volume of unstable branes may have no clear physical interpretation, however, one expects that the string theory S-matrix elements should be reproduced by the higher derivative theory if the two theories are going to be identical. In this case we would like to find the two-tachyons-two-gauge fields.
couplings and four-tachyons couplings which produce the string theory S-matrix elements to all order of $\alpha'$. We begin with the S-matrix element of two tachyons and two gauge fields.

### 2.1 Two tachyons and two gauge fields couplings

The S-matrix element of two gauge fields and two tachyons in string theory side is given by [27, 28]

$$A = 4i(2\pi\alpha')T_p \left[ \frac{1}{2} \zeta_1 \cdot \zeta_2 \left( -\alpha \frac{\Gamma(-2s)\Gamma(1/2 - 2t)}{\Gamma(-1/2 - 2s - 2t)} - \beta \frac{\Gamma(-2s)\Gamma(1/2 - 2u)}{\Gamma(-1/2 - 2s - 2u)} \right) \right.$$  

$$+ \gamma \frac{\Gamma(1/2 - 2u)\Gamma(1/2 - 2t)}{\Gamma(1 + 2s)} + 2\alpha' \zeta_1 \cdot k_3 \zeta_2 \cdot k_4 \left( \alpha \frac{\Gamma(-2s)\Gamma(1/2 - 2t)}{\Gamma(-1/2 - 2s - 2t)} - \beta \frac{\Gamma(-2s)\Gamma(-1/2 - 2u)}{\Gamma(-1/2 - 2s - 2u)} \right)$$  

$$+ \gamma \frac{\Gamma(-1/2 - 2u)\Gamma(1/2 - 2t)}{\Gamma(-2t - 2u)} \right] .$$  

(10)

where $\zeta_i$ is the polarization of gauge fields. The Mandelstam variables are

$$s = -\alpha'(k_1 + k_2)^2/2 ,$$

$$t = -\alpha'(k_2 + k_3)^2/2 ,$$

$$u = -\alpha'(k_1 + k_3)^2/2 .$$

(11)

The momenta of the gauge fields (tachyons) are $k_1, k_2$ ($k_3, k_4$). The on-shell condition for the tachyons are $k_i^2 = 1/(2\alpha')$, and the Mandelstam variables satisfy the constraint

$$s + t + u = -1/2$$

(12)

The coefficients $\alpha, \beta, \gamma$ are the non-abelian group factors

$$\alpha = \frac{1}{2} \left( \text{Tr}(\lambda_1 \lambda_2 \lambda_3 \lambda_4) + \text{Tr}(\lambda_1 \lambda_3 \lambda_2 \lambda_4) \right) ,$$

$$\beta = \frac{1}{2} \left( \text{Tr}(\lambda_1 \lambda_3 \lambda_4 \lambda_2) + \text{Tr}(\lambda_1 \lambda_2 \lambda_4 \lambda_3) \right) ,$$

$$\gamma = \frac{1}{2} \left( \text{Tr}(\lambda_1 \lambda_4 \lambda_2 \lambda_3) + \text{Tr}(\lambda_1 \lambda_3 \lambda_2 \lambda_4) \right) .$$

(13)

The standard non-abelian kinetic terms of the field theory produce Feynman amplitudes that have massless pole in s-channel and tachyonic poles in $t$- and $u$-channels. It is shown in [29] that the above amplitude reproduce the massless and tachyonic poles of the field theory if one expands the string amplitude around

$$s \to 0, \quad t, u \to -1/4$$

(14)
The other terms of the expansion are speculated in [29, 30, 31] to be related to the higher derivatives of tachyon and gauge fields. In this section we would like to find these higher derivative terms to all order of \( \alpha' \). To this end, we write the amplitude (10) in the following form:

\[
A = 4i(2\pi\alpha')T_p \left( \zeta_1 \cdot \zeta_2 (2t') (2u') - 2\alpha' \zeta_1 \cdot \kappa_3 \zeta_2 \cdot \kappa_4 (2t') - 2\alpha' \zeta_1 \cdot \kappa_4 \zeta_2 \cdot \kappa_4 (2u') \right) 
\times \left( \alpha \frac{\Gamma(2t' + 2u')}{{\Gamma(1 + 2u')}} + \beta \frac{\Gamma(2t' + 2u')}{{\Gamma(1 + 2u')}} + \gamma \frac{\Gamma(-2u')}{\Gamma(1 - 2t' - 2u')} \right)
\]

where \( t' = t + 1/4 = -\alpha' k_2 \cdot k_3 \) and \( u' = u + 1/4 = -\alpha' k_1 \cdot k_3 \). The amplitude must be expanded around \( t', u' \to 0 \)

which is the momentum expansion. Using the Maple, one can expand the amplitude around the above point, i.e.,

\[
A = 4i(2\pi\alpha')T_p \left( \zeta_1 \cdot \zeta_2 (2t') (2u') - 2\alpha' \zeta_1 \cdot \kappa_3 \zeta_2 \cdot \kappa_4 (2t') - 2\alpha' \zeta_1 \cdot \kappa_4 \zeta_2 \cdot \kappa_4 (2u') \right) 
\times \left( \frac{\alpha u' + \beta t' + \gamma s}{4t'u's} + \sum_{n,m=0}^{\infty} \left[ a_{n,m} (\alpha u'^m t'^n + \beta t'^m u'^n) + b_{n,m} \gamma (u'^n t'^m + t'^n u'^m) \right] \right)
\]

where \( b_{n,m} \) is symmetric. Some of the coefficients \( a_{n,m} \) and \( b_{n,m} \) are

\[
a_{0,0} = -\frac{\pi^2}{6}, \quad b_{0,0} = -\frac{\pi^2}{12}, \quad a_{1,0} = 2\zeta(3), \quad a_{0,1} = 0, \quad b_{0,1} = b_{1,0} = -\zeta(3), \\
a_{1,1} = a_{0,2} = -7\pi^4/90, \quad a_{2,0} = -4\pi^4/90, \quad b_{1,1} = -\pi^4/180, \quad b_{0,2} = b_{2,0} = -\pi^4/45, \\
a_{1,2} = a_{2,1} = 8\zeta(5) + 4\pi^2\zeta(3)/3, \quad a_{0,3} = 0, \quad a_{3,0} = 8\zeta(5), \\
b_{0,3} = -4\zeta(5), \quad b_{1,2} = -8\zeta(5) + 2\pi^2\zeta(3)/3
\]

It has been shown in [29, 30, 31] that the poles in the above expansion are reproduced by the non-abelian kinetic terms, and the contact terms with coefficients \( a_{0,0} \) and \( b_{0,0} \) are also reproduced by the following terms:

\[
T_p(\pi\alpha')^3 \text{STr} \left( m^2 T^2 F_{\mu\nu} F^{\mu\nu} + D^a T D_a T F_{\mu\nu} F^{\mu\nu} - 4 F^{\mu\alpha} F_{\alpha\beta} D^\beta T D_\mu T \right)
\]

where the covariant derivative is \( D_a T = \partial_a T - i[A_a, T] \), and \( \text{STr} \) is the symmetrised trace prescription. They are the two tachyons and two gauge fields couplings of the non-abelian tachyon DBI action [3]. Writing the symmetric trace in term of ordinary trace, one can write the above couplings as

\[
-T_p(\pi\alpha')^3 (\alpha')^2 (\mathcal{L}^{00}_1 + \mathcal{L}^{00}_2 + \mathcal{L}^{00}_3 + \mathcal{L}^{00}_4)
\]
where

\[ L_{00}^1 = \frac{-\pi^2}{3} m^2 \text{Tr} \left( 2 T T^2 F^\nu_\mu F^\mu_\nu + T F^\nu_\mu T F^\mu_\nu \right) \]

\[ L_{00}^2 = \frac{-\pi^2}{3} \text{Tr} \left( 2 D^a D_\alpha T F^\nu_\mu F^\mu_\nu + D^a T F^\nu_\mu D_\alpha T F^\mu_\nu \right) \]

\[ L_{00}^3 = \frac{2\pi^2}{3} \text{Tr} \left( 2 D^\beta D_\mu T F^\mu_\alpha F^\alpha_\beta + D^\beta T F^\mu_\alpha D_\mu T F^\alpha_\beta \right) \]

\[ L_{00}^4 = \frac{2\pi^2}{3} \text{Tr} \left( 2 D^\beta D_\mu D_\alpha D_\beta F^\mu_\nu F^\nu_\alpha + D^\beta T F^\mu_\alpha D_\mu T F^\alpha_\beta \right) \]

The vertex of two on-shell tachyons and two on-shell gauge fields of the above couplings is in fact the kinematic factor in equation (17) multiplies by \(-\pi^2(\alpha + \beta + \gamma)/6\). At this order the Chan-Paton factors appear in symmetric form, i.e., \(\alpha + \beta + \gamma\). This is the reason that the symmetric trace appears in the field theory couplings (19) at this order. The Chan-Paton factors however does not appear in symmetric form in any other order. So one expects to have no symmetric trace in the higher order terms.

Our strategy for finding the higher derivatives extension of the above couplings is as follows. Since the vertex of the above terms appear as coefficient of all higher order terms in (17), one may find the higher derivative couplings by applying appropriate derivatives on the above couplings. The coefficient of each term in the above couplings is set by \(a_{0,0}\) and \(b_{0,0}\). In the higher derivative extensions one should replace them by \(a_{n,m}\) and \(b_{n,m}\). Let us focus on \(L_{00}^1\) terms which appear in the above couplings as

\[ L_{10}^1 = m^2 \text{Tr} \left( 4 a_{0,0} T^2 F^\nu_\mu F^\mu_\nu + 4 b_{0,0} T F^\nu_\mu T F^\mu_\nu \right) \]  

(21)

Extension of \(a_{0,0}\) to \(a_{1,0}\) is the following:

\[ m^2 (\alpha') \text{Tr} \left( 2 a_{1,0} [D_a T T D^a F^\nu_\mu F^\mu_\nu + T D_a T F^\nu_\mu D^a F^\mu_\nu] \right) \]

\[ = m^2 (\alpha') \text{Tr} \left( 2 a_{1,0} [D_a T T D^a F^\nu_\mu F^\mu_\nu + h.c.] \right) \]  

(22)

where \(D_a F^\nu_\mu = \partial_a F^\nu_\mu - i [A_a, F^\nu_\mu]\). Note that the above Lagrangian is hermitian. Further extension to \(a_{1,1}\) is

\[ m^2 (\alpha')^2 \text{Tr} \left( a_{1,1} [D_a D_b T T D^a F^\nu_\mu D^b F^\nu_\mu + D_a D_b F^\nu_\mu F^\nu_\mu D^a T D^b T + h.c.] \right) \]  

(23)

Now, it is not difficult to extend the above couplings to \(a_{n,m}\) case, i.e.,

\[ m^2 (\alpha')^{n+m} a_{n,m} \text{Tr} \left( D_{nm}(T T F^\nu_\mu F^\mu_\nu) + D_{nm}(F^\nu_\mu F^\nu_\mu T T) + h.c. \right) \]  

(24)

where the higher derivative operator \(D_{nm}\) is defined as

\[ D_{nm}(EFGH) \equiv D_{b_1} \cdots D_{b_n} D_{a_1} \cdots D_{a_n} E F D^a_1 \cdots D^a_n G D^b_1 \cdots D^b_m H \]  

(25)
Similarly, the extension of $b_{0,0}$ to $b_{1,0}$ is the following:

$$m^2 (\alpha') \text{Tr} \left( 2b_{1,0} [D_a T D^a F_{\mu\nu} T F^{\nu\mu} + h.c] \right)$$

Further extension to $b_{1,1}$ is

$$m^2 (\alpha')^2 \text{Tr} \left( b_{1,1} [D_a D_b T D^a F_{\mu\nu} T D^b F^{\nu\mu} + D_a D_b F_{\mu\nu} D^a T F^{\nu\mu} D^b T + h.c.] \right)$$

and extension to $b_{n,m}$ case is

$$m^2 (\alpha')^{n+m} a_{n,m} \text{Tr} \left( \mathcal{D}'_{nm}(T F_{\mu\nu} T F^{\nu\mu}) + \mathcal{D}'_{nm}(F_{\mu\nu} T F^{\nu\mu} T) + h.c. \right)$$

where the higher derivative operator $\mathcal{D}'_{nm}$ is defined as

$$\mathcal{D}'_{nm}(EFGH) \equiv D_b \cdots D_{ba} D_{a_1} \cdots D_a ED^{a_1} \cdots D_{a_n} FGD^{b_1} \cdots D^{b_m} H$$

One can repeat similar steps for all other terms in (20). Hence, the proposal for the couplings between two tachyons and two field strengths on the world volume of $N$ non-BPS D-branes, to all order of $\alpha'$, is the following:

$$\mathcal{L} = -T_p (\pi \alpha')(\alpha')^{2+n+m} \sum_{n,m=0}^\infty (\mathcal{L}_{1nm} + \mathcal{L}_{2nm} + \mathcal{L}_{3nm} + \mathcal{L}_{4nm})$$

where

$$\mathcal{L}_{1nm} = m^2 \text{Tr} \left( a_{n,m} [\mathcal{D}_{nm}(T^2 F_{\mu\nu} F^{\nu\mu}) + \mathcal{D}_{nm}(F_{\mu\nu} F^{\nu\mu} T^2)] ight)$$

$$+ b_{n,m} [\mathcal{D}_{nm}(TF_{\mu\nu} T F^{\nu\mu}) + \mathcal{D}'_{nm}(F_{\mu\nu} T F^{\nu\mu} T) + h.c.]$$

$$\mathcal{L}_{2nm} = \text{Tr} \left( a_{n,m} [\mathcal{D}_{nm}(D^a T D_a T F^{\nu\mu}) + \mathcal{D}_{nm}(F_{\mu\nu} F^{\nu\mu} D^a T D_a T)] ight)$$

$$+ b_{n,m} [\mathcal{D}'_{nm}(D^a T F_{\mu\nu} D_a T F^{\nu\mu}) + \mathcal{D}'_{nm}(F_{\mu\nu} D_a T F^{\nu\mu} D^a T) + h.c.]$$

$$\mathcal{L}_{3nm} = -2\text{Tr} \left( a_{n,m} [\mathcal{D}_{nm}(D^a T D_a T F^{\mu\nu} F_{\alpha\beta}) + \mathcal{D}_{nm}(F^{\mu\nu} F_{\alpha\beta} D^a T D_a T)] ight)$$

$$+ b_{n,m} [\mathcal{D}'_{nm}(D^a T F^{\mu\nu} D_a T F_{\alpha\beta}) + \mathcal{D}'_{nm}(F^{\mu\nu} D_a T F_{\alpha\beta} D^a T) + h.c.]$$

$$\mathcal{L}_{4nm} = -2\text{Tr} \left( a_{n,m} [\mathcal{D}_{nm}(D^a T D_a T F_{\alpha\beta} F^{\mu\nu}) + \mathcal{D}_{nm}(F_{\alpha\beta} F^{\mu\nu} D^a T D_a T)] ight)$$

$$+ b_{n,m} [\mathcal{D}'_{nm}(D^a T F_{\alpha\beta} D_a T F^{\mu\nu}) + \mathcal{D}'_{nm}(F_{\alpha\beta} D_a T F^{\mu\nu} D^a T) + h.c.]$$

If one calculates the coupling of two on-shell tachyons and two gauge fields from (29), one will find the contact terms in the amplitude (17).
When the covariant derivative of the field strength and the second covariant derivative of tachyon are zero, the Lagrangian \(\mathcal{L}_{\text{tachyon}}\) reduces to the couplings \(\mathcal{L}_{\text{tachyon}}^{(2)}\) which are the two-tachyons-two-gauge field strengths couplings of the non-abelian tachyon DBI action. This may indicates that the non-abelian tachyon DBI action is the effective action of the non-BPS D-branes when fields vary slowly.

The couplings in \(\mathcal{L}_{\text{tachyon}}^{(2)}\) have on-shell ambiguity, i.e., \(\mathcal{T} \sim 2\alpha' \partial_a \partial^a \mathcal{T}\). This on-shell ambiguity has no effect on the massless and the simple tachyon poles of the S-matrix elements. In the massless pole it is obvious because the tachyons are on-shell. In the tachyon pole \(\mathcal{T}\) appears as \(1/(k^2 - 1/2\alpha')\), whereas \(2\alpha' \partial_a \partial^a \mathcal{T}\) appears as \(2\alpha' k^2/(k^2 - 1/2\alpha')\), however, one can write it as

\[
\frac{2\alpha' k^2}{k^2 - 1/2\alpha'} = \frac{1}{k^2 - 1/2\alpha'} + 2\alpha' \tag{30}
\]

hence in the tachyon pole \(\mathcal{T}\) and \(2\alpha' \partial_a \partial^a \mathcal{T}\) have identical effect. However, the deference is an extra contact term. By studying a S-matrix element in which the couplings \(\mathcal{L}_{\text{tachyon}}^{(2)}\) appear in tachyon poles as well as the contact terms, one way fix the on-shell ambiguity of \(\mathcal{L}_{\text{tachyon}}^{(2)}\).

### 2.2 Four massless scalars couplings

It has been shown in \([30, 31]\) that the S-matrix element of four massless transverse scalars and the S-matrix element of four tachyons can be written in a universal form. So, one may expect that the higher derivative couplings of four tachyons should be similar to the higher derivative couplings of four scalar fields. In fact the tachyon and the scalar fields appear in similar form in the tachyon DBI action. The only difference is that there is a potential for the tachyon, e.g., \(e^{\pi\alpha' m^2 T^2}\) where \(m^2\) is the mass of the tachyon. Therefore, to find the higher derivative couplings of four tachyons, we first find the higher derivative couplings of four scalar fields and then inspired by them we will find the tachyon couplings.

The S-matrix element of four massless transverse scalar vertex operators in the supersting theory is given by \(A = A_s + A_t + A_u\) where

\[
A_s = -4i\mathcal{T}_p \xi_1 \cdot \xi_2 \xi_3 \cdot \xi_4 \left( \alpha \frac{\Gamma(-2s)\Gamma(1 - 2t)}{\Gamma(-2s - 2t)} + \beta \frac{\Gamma(-2s)\Gamma(1 - 2u)}{\Gamma(-2s - 2u)} - \gamma \frac{\Gamma(1 - 2t)\Gamma(1 - 2u)}{\Gamma(1 - 2t - 2u)} \right)
\]
\[
A_u = -4i\mathcal{T}_p \xi_1 \cdot \xi_3 \xi_2 \cdot \xi_4 \left( -\alpha \frac{\Gamma(1 - 2s)\Gamma(1 - 2t)}{\Gamma(1 - 2s - 2t)} + \beta \frac{\Gamma(-2u)\Gamma(1 - 2s)}{\Gamma(-2u - 2s)} + \gamma \frac{\Gamma(-2u)\Gamma(1 - 2t)}{\Gamma(-2u - 2t)} \right)
\]
\[
A_t = -4i\mathcal{T}_p \xi_1 \cdot \xi_4 \xi_2 \cdot \xi_3 \left( \alpha \frac{\Gamma(-2t)\Gamma(1 - 2s)}{\Gamma(-2t - 2s)} - \beta \frac{\Gamma(1 - 2s)\Gamma(1 - 2u)}{\Gamma(1 - 2s - 2u)} + \gamma \frac{\Gamma(-2t)\Gamma(1 - 2u)}{\Gamma(-2t - 2u)} \right)
\]

where \(\xi_i\)'s are the scalars polarization. The on-shell condition for the scalars are \(k_i^2 = 0\), and the Mandelstam variables constrain to the relation

\[
s + t + u = 0 \tag{31}
\]
In this case the massless poles of the Feynman amplitude resulting from the non-abelian kinetic term of the scalars can be produced by the above amplitude expanded at low energy, i.e., \( s, t, u \to 0 \). To find the four scalars couplings to all order of \( \alpha' \), we repeat the steps in the previous section, so rewrite the amplitudes as

\[
A_s = 16i T^p \zeta_1 \zeta_2 \zeta_3 \zeta_4 \times tu \left( \frac{\Gamma(2t + 2u)\Gamma(-2t)}{\Gamma(1 + 2u)} + \frac{\beta(2t + 2u)\Gamma(-2u)}{\Gamma(1 + 2t)} + \frac{\gamma(2t)\Gamma(-2u)}{\Gamma(1 - 2t - 2u)} \right)
\]

\[
A_u = 16i T^p \zeta_1 \zeta_2 \zeta_4 \times ts \left( \frac{\Gamma(-2s)\Gamma(-2t)}{\Gamma(1 - 2s - 2t)} + \frac{\beta(2t + 2s)\Gamma(-2s)}{\Gamma(1 + 2t)} + \frac{\gamma(2t + 2s)\Gamma(-2t)}{\Gamma(1 + 2s)} \right)
\]

\[
A_t = 16i T^p \zeta_1 \zeta_4 \zeta_3 \times us \left( \frac{\Gamma(2s + 2u)\Gamma(-2s)}{\Gamma(1 + 2u)} + \frac{\beta(2u + 2s)\Gamma(-2u)}{\Gamma(1 - 2s - 2u)} + \frac{\gamma(2u + 2s)\Gamma(-2u)}{\Gamma(1 + 2u)} \right)
\]

Using the Maple, one can expand the amplitude around \( s, t, u \to 0 \), i.e.,

\[
A_s = 16i T^p \zeta_1 \zeta_2 \zeta_3 \zeta_4 t u \times \left( \frac{\alpha u + \beta t + \gamma s}{4 tu s} + \sum_{n,m=0}^\infty \left[ a_{n,m}(\alpha u^n t^m + \beta t^n u^m) + b_{n,m}\gamma(u^n t^m + t^n u^m) \right] \right)
\]

\[
A_u = 16i T^p \zeta_1 \zeta_2 \zeta_4 \times ts \left( \frac{\gamma s + \beta t + \alpha u}{4 tu s} + \sum_{n,m=0}^\infty \left[ a_{n,m}(\gamma s^n t^m + \beta t^n s^m) + b_{n,m}\alpha(s^n t^m + t^n s^m) \right] \right)
\]

\[
A_t = 16i T^p \zeta_1 \zeta_4 \zeta_3 \times su \left( \frac{\alpha u + \gamma s + \beta t}{4 su t} + \sum_{n,m=0}^\infty \left[ a_{n,m}(\alpha u^n s^m + \gamma s^n u^m) + b_{n,m}\beta(u^n s^m + s^n u^m) \right] \right)
\]

On can also write the last term in each amplitude in another form, e.g., the last term in the first line can be written also as \( \sum_{n,m=0}^\infty (tu)^m (s)^n \). They produce different four scalars couplings. Up to total derivative terms, the differences are in the couplings which involve \( \partial_a \partial^a \phi^i \). They have no effect on the simple massless poles of S-matrix elements because canceling \( k^2 \) with the massless propagator one finds a contact term.

The massless poles in (32) are reproduced by the non-abelian kinetic terms of the scalar field, and the contact terms with coefficients \( a_{0,0} \) and \( b_{0,0} \) are also reproduced by the following terms:

\[
-T_p \text{Str} \left( -\frac{1}{4} D_a \phi^j D_b \phi_i D^b \phi^j D^a \phi_j + \frac{1}{8} (D_a \phi^i D^a \phi_i)^2 \right) \quad (33)
\]

Writing the symmetric trace in term of ordinary trace, one can write it as

\[
T_p (L_{5}^{00} + L_{6}^{00} + L_{7}^{00}) \quad (34)
\]
where
\[
\begin{align*}
\mathcal{L}_5^{(0)} &= -\frac{1}{8\pi^2} \text{Tr} \left( 4a_{0,0} D_\alpha \phi^i D_\beta \phi_i D^\beta \phi_j D^\alpha \phi_j + 4b_{0,0} D_\alpha \phi^i D^\beta \phi_j D^\alpha \phi_i \right) \\
\mathcal{L}_6^{(0)} &= -\frac{1}{8\pi^2} \text{Tr} \left( 4a_{0,0} D_\alpha \phi^i D_\beta \phi_i D^\alpha \phi_j D^\beta \phi_j + 4b_{0,0} D_\alpha \phi^i D^\alpha \phi_i D_\beta \phi_j \right) \\
\mathcal{L}_7^{(0)} &= \frac{1}{8\pi^2} \text{Tr} \left( 4a_{0,0} D_\alpha \phi^i D^\alpha \phi_i D_\beta \phi_j D^\beta \phi_j + 4b_{0,0} D_\alpha \phi^i D^\beta \phi_j D^\alpha \phi_i \right)
\end{align*}
\]

To check the consistency of the above couplings with the \((a_{0,0}, b_{0,0})\) order contact terms of the string theory amplitude \((32)\), one uses the relation \(2k_1 \cdot k_2 (k_1 \cdot k_3) = -(k_1 \cdot k_2)(k_3 \cdot k_4) - (k_1 \cdot k_3)(k_2 \cdot k_4) + (k_2 \cdot k_3)(k_1 \cdot k_4)\). Now one can extend it easily to the higher derivative terms as
\[
\frac{1}{4\pi^2} T_p (\alpha')^{n+m} \sum_{m,n=0}^{\infty} (\mathcal{L}_5^{nm} + \mathcal{L}_6^{nm} + \mathcal{L}_7^{nm})
\]

where
\[
\begin{align*}
\mathcal{L}_5^{nm} &= -\text{Tr} \left( a_{n,m} \mathcal{D}_{nm} [D_\alpha \phi^i D_\beta \phi_i D^\beta \phi_j D^\alpha \phi_j] + b_{n,m} \mathcal{D}'_{nm} [D_\alpha \phi^i D^\beta \phi_j D_\beta \phi_i D^\alpha \phi_j] + h.c. \right) \\
\mathcal{L}_6^{nm} &= -\text{Tr} \left( a_{n,m} \mathcal{D}_{nm} [D_\alpha \phi^i D_\beta \phi_i D^\alpha \phi_j D^\beta \phi_j] + b_{n,m} \mathcal{D}'_{nm} [D_\beta \phi_i D^\beta \phi_j D_\alpha \phi_i D^\alpha \phi_j] + h.c. \right) \\
\mathcal{L}_7^{nm} &= \text{Tr} \left( a_{n,m} \mathcal{D}_{nm} [D_\alpha \phi^i D^\alpha \phi_i D_\beta \phi_j D^\beta \phi_j] + b_{n,m} \mathcal{D}'_{nm} [D_\alpha \phi^i D^\beta \phi_j D^\alpha \phi_i D^\beta \phi_j] + h.c. \right)
\end{align*}
\]

It is not difficult to check that the infinite tower of four scalar couplings \((35)\) produce the string theory S-matrix element \((32)\). The above couplings can be extended to the coupling of four gauge fields by using T-duality. Up to total derivative terms and the terms like \(FFDFDF\), which is zero on-shell, one can show that the \(FFDFDF\) terms are those appear in the literature. We now turn to the couplings of four tachyons.

### 2.3 Four tachyons couplings

The S-matrix element of four open string tachyon vertex operators in the superstring theory is given by \([27] [28]\)
\[
A = -12i T_p \left( \frac{\alpha \Gamma(-2t)\Gamma(-2s)}{\Gamma(-1 - 2t - 2s)} + \beta \frac{\Gamma(-2s)\Gamma(-2u)}{\Gamma(-1 - 2s - 2u)} + \gamma \frac{\Gamma(-2t)\Gamma(-2u)}{\Gamma(-1 - 2t - 2u)} \right)
\]

where the Mandelstam variables are those defined in \((11)\) and satisfy the constraint
\[
s + t + u = -1.
\]

The standard non-abelian kinetic term in field theory produces massless poles in \(s\), \(t\), \(u\)-channels. However, the constraint \((37)\) does not allow us to sent all \(s, t, u\) to zero at the
same time to produce massless poles. It is shown in [29, 30, 31] that in order to produce the massless poles, one should first arrange the amplitude in a specific form, i.e., one should write $A = A_s + A_t + A_u$ where

$$
A_s = -4iT_p \left( \frac{\Gamma(2s)\Gamma(2t)}{\Gamma(-1-2s-2t)} + \frac{\Gamma(-2s)\Gamma(2u)}{\Gamma(-1-2u-2s)} + \frac{\Gamma(-2t)\Gamma(-2u)}{\Gamma(-1-2t-2u)} \right)
$$

$$
A_u = -4iT_p \left( \frac{\Gamma(-2s)\Gamma(-2t)}{\Gamma(-1-2s-2t)} + \frac{\Gamma(-2u)\Gamma(-2s)}{\Gamma(-1-2u-2s)} + \frac{\Gamma(-2u)\Gamma(-2t)}{\Gamma(-1-2u-2t)} \right)
$$

$$
A_t = -4iT_p \left( \frac{\Gamma(-2t)\Gamma(-2s)}{\Gamma(-1-2s-2t)} - \frac{\Gamma(-2s)\Gamma(-2u)}{\Gamma(-1-2u-2s)} + \frac{\Gamma(-2u)\Gamma(-2t)}{\Gamma(-1-2t-2u)} \right)
$$

Then one should send

$$
\begin{align*}
s - \text{channel} : & \lim_{s \to 0, t, u \to -1/2} A_s \\
t - \text{channel} : & \lim_{t \to 0, s, u \to -1/2} A_t \\
u - \text{channel} : & \lim_{u \to 0, s, t \to -1/2} A_u
\end{align*}
$$

These limits are consistent with the constraint (37).

The leading term of this expansion produces the massless poles of the higher derivative theory, and the other terms are speculated to be related to the higher derivatives of the tachyon [29, 30, 31]. In this section we would like to find these higher derivative terms up to on-shell ambiguity. So we write the amplitude in the following form:

$$
\begin{align*}
A_s &= 16iT_p t' u' \left( \frac{\Gamma(2t' + 2u')\Gamma(-2t')}{\Gamma(1+2u')} + \frac{\Gamma(2t' + 2u')\Gamma(-2u')}{\Gamma(1+2t')} + \frac{\Gamma(-2t')\Gamma(-2u')}{\Gamma(-1-2t'-2u')} \right) \\
A_u &= 16iT_p t' s' \left( \frac{\Gamma(-2s')\Gamma(-2t')}{\Gamma(1-2s'-2t')} + \frac{\Gamma(2t' + 2s')\Gamma(-2s')}{\Gamma(1+2t')} + \frac{\Gamma(2t' + 2s')\Gamma(-2t')}{\Gamma(-1-2t'-2u')} \right) \\
A_t &= 16iT_p u' s' \left( \frac{\Gamma(2s' + 2u')\Gamma(-2s')}{\Gamma(1+2u')} + \frac{\Gamma(-2u')\Gamma(-2s')}{\Gamma(-1-2s'-2u')} + \frac{\Gamma(2u' + 2s')\Gamma(-2u')}{\Gamma(1+2s')} \right)
\end{align*}
$$

(40)

where $s' = s + 1/2 = -\alpha' k_1 \cdot k_2$, $t' = t + 1/2 = -\alpha' k_2 \cdot k_3$ and $u' = u + 1/2 = -\alpha' k_1 \cdot k_3$. Now the field theory corresponds to expanding the above amplitude around

$$
s', t', u' \to 0
$$

(41)

which is the momentum expansion. Note that $s + t' + u' = 0$, $u + s' + t' = 0$ and $t + s' + u' = 0$. Like the scalar case, one should expand the amplitude around the above point, i.e.,

$$
A_s = 16iT_p t' u' \times
$$

$$
\left( \frac{\alpha u' + \beta t' + \gamma s}{4t'u's} + \sum_{n,m=0}^{\infty} \left[ a_{n,m}(\alpha u^m t^m + \beta t^m u^m) + b_{n,m} \gamma (u^m t^m + t^m u^m) \right] \right)
$$

11
\[ A_u = 16i T_p t' s' \times \left( \frac{\gamma s' + \beta t' + \alpha u}{4t's'u} + \sum_{n,m=0}^{\infty} \left[ a_{n,m}(\gamma s' t'^m + \beta t's'^m) + b_{n,m}(s's'^m + t'^m s^m) \right] \right) \]

\[ A_t = 16i T_p s' u' \times \left( \frac{\alpha u' + \gamma s' + \beta t}{4s'u't} + \sum_{n,m=0}^{\infty} \left[ a_{n,m}(\alpha u^n s'^m + \gamma s^n u'^m) + b_{n,m}(u^n s'^m + s'^n u'^m) \right] \right) \]

The poles in the above expansion are reproduced by the non-abelian kinetic terms, and the contact terms with coefficients \( a_{0,0} \) and \( b_{0,0} \) are reproduced by the following terms \([29, 30, 31]\):

\[-(2\pi \alpha')^2 T_p S T r \left( \frac{m^4}{8} T^4 + \frac{m^2}{4} T^2 D_a T D^a T - \frac{1}{8} (D_a T D^a T)^2 \right)\]

which are the four tachyons coupling of the non-abelian tachyon DBI action. To check this explicitly, one needs the on-shell relation \( 2k_1 \cdot k_2 \cdot k_1 - k_2 \cdot k_1) = m^4 - m^2(k_2 \cdot k_3 + k_1 \cdot k_4) - (k_1 \cdot k_2)(k_3 \cdot k_4) - (k_1 \cdot k_3)(k_2 \cdot k_4) + (k_2 \cdot k_3)(k_1 \cdot k_4) \). Writing the symmetric trace in term of ordinary trace, one can write \([42]\) as

\[ T_p(\alpha')^2 (L_{00}^0 + L_{90}^0 + L_{10}^0 + L_{01}^0 + L_{12}^0) \]

where

\[ L_{00}^0 = 2m^4 \left( a_{0,0} T^4 + b_{0,0} T^4 \right) \]
\[ L_{90}^0 = 4m^2 \left( a_{0,0} T^2 D^a T D_a T + b_{0,0} T D^a T D_a T \right) \]
\[ L_{10}^0 = -2 \left( a_{0,0} D_a T D_b T D^a T D_b T + b_{0,0} D_a T D^a T D_b T D_b T \right) \]
\[ L_{01}^0 = -2 \left( a_{0,0} D_a T D_b T D^a T D_b T + b_{0,0} D_a T D^a T D_b T D_b T \right) \]
\[ L_{12}^0 = 2 \left( a_{0,0} D_a T D^a T D_b T D_b T + b_{0,0} D_a T D^a T D_b T D_b T \right) \]

Note that the last three lines add up to

\[ \frac{\pi^2}{6} Tr \left( 2D^a T D_a T D^b T D_b T + D^a T D^b T D_a T D_b T \right) \]

However, we have written them in the form \([44]\) to extend them easily to the higher derivative terms using the fact that the higher derivative extension of \([34]\) is \([35]\). The higher derivative extension of \( L_{00}^0 \) is like the higher derivative extension of \( L_{20}^0 \) which appears in \([29]\). Therefore, the higher derivative extension of \([33]\) is the following:

\[ T_p(\alpha')^{2+n+m} \sum_{m,n=0}^{\infty} \left( L_{8}^{nm} + L_{9}^{nm} + L_{10}^{nm} + L_{11}^{nm} + L_{12}^{nm} \right) \]

\[ 12 \]
where

\[ L_{8}^{nm} = m^{4} \text{Tr} \left( a_{n,m} D_{nm}[TTTT] + b_{n,m} D'_{nm}[TTTT] + h.c. \right) \]
\[ L_{9}^{nm} = m^{2} \text{Tr} \left( a_{n,m} [D_{nm}(T D^{\alpha} T D_{\alpha} T)] + D_{nm}(D^{\alpha} T D_{\alpha} TTT) \right) + h.c. \]
\[ L_{10}^{nm} = -\text{Tr} \left( a_{n,m} D_{nm}[D_{\alpha} T D_{\beta} T D^{\alpha} T D^{\beta} T] + b_{n,m} D'_{nm}[D_{\alpha} T D^{\alpha} T D_{\beta} T D^{\beta} T] + h.c. \right) \]
\[ L_{11}^{nm} = -\text{Tr} \left( a_{n,m} D_{nm}[D_{\alpha} T D_{\beta} T D^{\alpha} T D^{\beta} T] + b_{n,m} D'_{nm}[D_{\beta} T D^{\beta} T D_{\alpha} T D^{\alpha} T] + h.c. \right) \]
\[ L_{12}^{nm} = \text{Tr} \left( a_{n,m} D_{nm}[D_{\alpha} T D^{\alpha} T D_{\beta} T D^{\beta} T] + b_{n,m} D'_{nm}[D_{\alpha} T D_{\beta} T D^{\alpha} T D^{\beta} T] + h.c. \right) \]

It is not difficult to check that the infinite tower of four tachyons couplings (45) produce the string theory S-matrix element. It is interesting to note that when the second derivative of tachyon is zero, the Lagrangian (45) reduces to DBI couplings (42) for abelian tachyon. For nonabelian case, it does not reduces to the tachyon couplings of the non-abelian DBI action. However, as we mentioned before the couplings (45) have on-shell ambiguity, i.e., \( T \sim 2 \alpha' \partial_{\alpha} \partial^{\alpha} T \). As long as the on-shell ambiguity is not fixed, one cannot use the tachyon couplings (45) to find the effective tachyon couplings. It has been check in [32] that the couplings (42) have no on-shell ambiguity, i.e., these couplings appear in the tachyon pole and contact terms of the S-matrix element of four tachyons and one gauge field at \( \zeta(2) \) order. It would be interesting to compare the infinite tachyon poles and contact terms of the S-matrix element found in [32] with the tachyon pole and the contact terms of the scattering amplitude of four tachyons and one gauge field using the infinite tower of tachyon couplings in (45). This calculation may fix the on-shell ambiguity of all couplings in (45).

## 3 Higher derivative terms of brane-anti-brane

Having found the couplings in the world-volume theory of \( N \) non-BPS D-brane, we can now find the corresponding couplings in the world-volume theory of brane-anti-brane. It has been proposed in [20] that the action of D-brane-anti-D-brane may be given by the projection of the action of two non-BPS D-brane with \((-1)^{F_{L}}\) where \( F_{L} \) is the spacetime left hand fermion number. According to this proposal, the couplings in the brane-anti-brane theory can be read from the non-BPS branes couplings by using the matrices (4) for the field strength and for the tachyon. If one replaces them into (29) and performing the trace, one finds the following couplings between two tachyons and two gauge fields:

\[ \mathcal{L}_{DD} = -T_{\mu} (\pi \alpha')(\alpha')^{2+n+m} \sum_{n,m = 0}^{\infty} (\mathcal{L}^{nm}_{1DD} + \mathcal{L}^{nm}_{2DD} + \mathcal{L}^{nm}_{3DD} + \mathcal{L}^{nm}_{4DD}) \]  

(46)
where

\[
\mathcal{L}^{nm}_{1DD} = m^2 \left( a_{nm}[\mathcal{D}_{nm}(TT^* F^{(1)\mu\nu}) + \mathcal{D}_{nm}(F^{(1)\mu\nu} TT^*) + c.c] + b_{nm}[\mathcal{D}'_{nm}(TF^{(2)\mu\nu}) + \mathcal{D}'_{nm}(F^{(1)\mu\nu} TT^*) + c.c.}\right)
\]

\[
\mathcal{L}^{nm}_{2DD} = \left( a_{nm}[\mathcal{D}_{nm}(D^\alpha T D_\alpha T^* F^{(1)\mu\nu}) + \mathcal{D}_{nm}(F^{(1)\mu\nu} D^\alpha T D_\alpha T^*) + c.c] + b_{nm}[\mathcal{D}'_{nm}(D^\alpha TF^{(2)\mu\nu}) + \mathcal{D}'_{nm}(F^{(1)\mu\nu} D^\alpha T D_\alpha T^*) + c.c.]\right)
\]

\[
\mathcal{L}^{nm}_{3DD} = -2 \left( a_{nm}[\mathcal{D}_{nm}(D^\beta T D_\beta T^* F^{(1)\mu\nu}) + \mathcal{D}_{nm}(F^{(1)\mu\nu} D^\beta T D_\beta T^*) + c.c] + b_{nm}[\mathcal{D}'_{nm}(D^\beta TF^{(2)\mu\nu}) + \mathcal{D}'_{nm}(F^{(1)\mu\nu} D^\beta T D_\beta T^*) + c.c.]\right)
\]

\[
\mathcal{L}^{nm}_{4DD} = -2 \left( a_{nm}[\mathcal{D}_{nm}(D^\beta T D_\beta T^* F^{(1)\mu\nu}) + \mathcal{D}_{nm}(F^{(1)\mu\nu} D^\beta T D_\beta T^*) + c.c] + b_{nm}[\mathcal{D}'_{nm}(D^\beta TF^{(2)\mu\nu}) + \mathcal{D}'_{nm}(F^{(1)\mu\nu} D^\beta T D_\beta T^*) + c.c.]\right)
\]

plus \( F^{(1)} \rightarrow F^{(2)} \) for \( a_{nm} \) terms and \( F^{(1)} \leftrightarrow F^{(2)} \) for \( b_{nm} \) terms. In above equation, the covariant derivative of field strength is ordinary derivative and \( D_\alpha \cdots D_\alpha T = \partial_\alpha D_\alpha \cdots D_\alpha T - i(A^{(1)}_{\alpha_1} - A^{(2)}_{\alpha_1} ) D_\alpha \cdots D_\alpha T \). Note that there are couplings between \( F^{(1)} \) and \( F^{(2)} \). For the case \( T = T^* \) and \( F^{(1)} = F^{(2)} \), the Lagrangian [16] is the same as the Lagrangian [23] for one non-BPS D-brane as expected. One can do similar steps to find the four tachyon couplings in the brane-anti-brane world-volume theory.

We have found the tachyon couplings from the contact terms of the momentum expansion of the string theory S-matrix elements. A nontrivial consistency check of the higher derivative theory with string theory is that the tachyon couplings reproduce also the infinite massless or tachyonic poles of the string theory S-matrix elements. As an example we consider the momentum expansion of the S-matrix element of one RR field \( C_{p-1} \), two tachyons and one gauge field which is nonzero in the world volume theory of brane-anti-brane. This expansion has infinite contact terms and also infinite massless and tachyonic poles [22]. The contact terms and the tachyonic poles are reproduced by appropriate couplings of one RR field, two tachyons and one gauge field in field theory which has been found in [22], i.e., the higher derivative couplings in the Introduction section. The massless poles are the following:

\[
i^\mu_p(\alpha')^2 \frac{e^{a_{0}\cdots a_{p-1}}H_{a_{0},a_{p-1}}}{p!(s+t+u+1/2)} \left[ k_{2a}(t+1/4)(\alpha' k_3) - \frac{1}{2} \xi_a(s+1/4)(t+1/4) + (3 \leftrightarrow 2) \right]
\]

\[
\times \sum_{n,m=0}^{\infty} d_{n,m}(s+t+1/2)^{n}(t+1/4)(s+1/4)^{m}
\]

where some of the coefficients \( d_{n,m} \) are

\[
d_{0,0} = -\pi^2/3, \quad d_{1,0} = 8\zeta(3)
\]

\[
d_{2,0} = -7\pi^4/45, \quad d_{0,1} = \pi^4/45, \quad d_{3,0} = 32\zeta(5), \quad d_{1,1} = -32\zeta(5) + 8\zeta(3)\pi^2/3
\]
The Mandelstam variables in (47) are

\[
s = -\frac{\alpha'}{2}(k_1 + k_3)^2, \quad t = -\frac{\alpha'}{2}(k_1 + k_2), \quad u = -\frac{\alpha'}{2}(k_2 + k_3)^2
\] (48)

In above, \(k_1\) is momentum of the gauge field and \(k_2, k_3\) are the tachyon momenta. It has been speculated in [22] that the above massless poles should be reproduced by the higher derivative theory once one knows the two-tachyon-two-gauge field couplings.

Now using the two-tachyon-two-gauge field couplings (46), one can reproduce the string theory massless poles (47). To this end, consider the amplitude for decaying one RR field to two tachyons and one gauge field in the world-volume theory of brane-anti-brane which has the following Feynman diagrams:

![Feynman diagrams](image)

Figure 3: The Feynman diagrams corresponding to the amplitude in (49).

In field theory, this amplitude is given by

\[
\mathcal{A} = V_a(C_{p-1}, A)G_{ab}(A)V_b(A, T_1, T_1, A^{(1)})
\] (49)

where \(A\) should be \(A^{(1)}\) and \(A^{(2)}\). In above \(T_1\) is the real component of the complex tachyon, \(i.e., T = (T_1 + iT_2)/\sqrt{2}\). It has been argued in [22] that the coupling between one RR and one gauge field which is given by the Wess-Zumino terms, and the kinetic term of the gauge field have no higher derivative correction. Hence, they are given by [23]

\[
G_{ab}(A) = \frac{i\delta_{ab}}{(2\pi\alpha')^2 T_p (\alpha' k^2 / 2)}
\]

\[
V_a(C_{p-1}, A^{(1)}) = i\mu_p (2\pi\alpha') \frac{1}{p!} \epsilon_{a_0...a_{p-1}a} H^{a_0...a_{p-1}}
\] (50)

\[
V_a(C_{p-1}, A^{(2)}) = -i\mu_p (2\pi\alpha') \frac{1}{p!} \epsilon_{a_0...a_{p-1}a} H^{a_0...a_{p-1}}
\]

The vertexes \(V_b(A, A^{(1)}, T_1, T_1)\) which have higher derivative corrections, can be derived from [40]. One finds that the vertex \(V_b(A^{(1)}, A^{(1)}, T_1, T_1)\) is given by

\[
-i T_p (\pi\alpha') (\alpha')^2 (-\alpha')^{n+m} a_{n,m} (k_b [(s + 1/4)(2k_2 \cdot \xi) + (t + 1/4)(2k_3 \cdot \xi)]
\] (51)
\[ k = k_2 + k_3 \]

and the vertex \( V(A(2), A(1), T_1, T_1) \) by

\[
-\frac{1}{2} \text{Tr}(\pi \alpha\,')(\alpha')^2 (-\alpha')^{n+m} b_{n,m}(k_0 [(s + 1/4)(2k_2 \cdot \xi) + (t + 1/4)(2k_3 \cdot \xi)])
\]

\[
+ (k_2 \cdot k)^n (k_3 \cdot k_1)^n + (k_2 \cdot k)(k_1 \cdot k_3)^n + (k_1 \cdot k)(k_2 \cdot k_3)^n
\]

where \( k^a \) is the momentum of the off-shell gauge field. Now one can write \( k_1 \cdot k = -(k_1 \cdot k_2 + k_1 \cdot k_3), \) \( k_2 \cdot k = k_1 \cdot k_3 - k^2 \) and \( k_3 \cdot k = k_1 \cdot k_2 - k^2 \). The \( k^2 \) in the above vertex will be canceled in the denominator of the gauge field propagator resulting a bunch of contact terms of one RR, two tachyons and one gauge field, \( i.e., \) the diagram (b). They should be subtracted from the contact terms that have been extracted from the S-matrix element of one RR, two tachyons and one gauge field, \( i.e., \) the couplings in (9). Let us at the moment ignore the contact terms and consider only the massless poles of the amplitude (49), \( i.e., \) diagram (a). Replacing (5.2), (5.1) and (5.0) in (49), one finds the following massless pole:

\[
2\pi \mu_p(\alpha')^2 \varepsilon^a_{\alpha' p_{-1} a_{-1}} H_{a_{-1} p_{-1} a_{-1}} [k_2(t')(\alpha' \xi \cdot k_3) - \frac{1}{2} \xi a(s')(t') + (3 \leftrightarrow 2)]
\]

\[
\times \sum_{n,m=0}^{\infty} ((a_{n,m} - b_{n,m})(t')^n (s')^m) + (t')^n (s')^m)
\]

where \( t' = t + 1/4 = -\alpha' k_1 \cdot k_2 \) and \( s' = s + 1/4 = -\alpha' k_1 \cdot k_3 \).

The above amplitude should be compared with the massless poles in (47). Let us compare them for some values of \( n, m \). For \( n = m = 0 \), the amplitude (53) has the following factor:

\[ 4(a_{0,0} - b_{0,0}) = -\frac{\pi^2}{3} \]

which is equal to \( d_{0,0} \). At the order of \( \alpha' \), the amplitude (53) has the following factor:

\[ 2(a_{1,0} + a_{0,1} - b_{0,1})(s' + t') = 8\xi(3)(s + t + 1/2) \]

which is equal to \( d_{1,0}(s + t + 1/2) \) in (47). At the order of \( (\alpha')^2 \), the amplitude (53) has the following factor:

\[
4(a_{1,1} - b_{1,1})(s')(t') + 2(a_{0,2} + a_{2,0} - b_{0,2} - b_{2,0})[(s')^2 + (t')^2]
\]

\[
= -\frac{7\pi^4}{45}(s' + t')^2 + \frac{\pi^4}{45}(s')(t') = d_{2,0}(s + t + 1/2)^2 + d_{0,1}(s + 1/4)(t + 1/4)
\]
At the order of \((\alpha')^3\), the amplitude (53) has the following factor:

\[
2(a_{3,0} + a_{0,3} - b_{0,3} - b_{3,0})(s')^3 + 2(a_{1,2} + a_{2,1} - b_{1,2} - b_{2,1})(s')(t')(s' + t')
\]

\[
= 32\zeta(5)(s' + t')^3 + (-32\zeta(5) + 8\pi^2\zeta(3)/3)(s')(t')(s' + t')
\]

which is again exactly equal to the corresponding terms in (47). Similar comparison can be done for all order of \(\alpha'\). Hence, the field theory amplitude (53) reproduces exactly the infinite tower of the massless pole of string theory S-matrix element (47). In particular, this consistency requires to have couplings between \(F^{(1)}\) and \(F^{(2)}\) which is in fact the case, as the coefficients \(b_{n,m}\) in (46) are non-zero.

Finally, let us now return to the contact terms that the field theory amplitude (49) produces. Using the Binomial formula, one can write the contact terms as the following:

\[
-\frac{i\mu_p(\alpha')^2^{a_{a_0\cdots a_{p-1}}H_{a_{0\cdots a_p-1}}}}{p!} k_{2a}(t')(\alpha'\xi,k_3) - \frac{1}{2} \xi_a(s')(t') + (3 \leftrightarrow 2) \sum_{n,m=0}^\infty (a_{n,m} - b_{n,m})
\]

\[
\left[ 2m \sum_{\ell=1}^m \binom{m}{\ell} (t^{m-\ell}s^m + s^{m-\ell}t^m) + 2n \sum_{\ell=1}^n \binom{n}{\ell} (t^{n-\ell}s^n + s^{n-\ell}t^n) \right] (\alpha'k^2)^{\ell-1}
\]

\[
+ \sum_{\ell=1,j=1}^{n,m} \binom{n}{\ell} \binom{m}{j} (t^{n-\ell}s^{m-j} + s^{n-\ell}t^{m-j})(\alpha'k^2)^{\ell+j-1}
\]

(54)

Note that the above couplings have at least four momenta. They can be rewritten in the following form:

\[
\frac{i(\alpha')^2\mu_p}{p!} \frac{\epsilon_{a_{a_0\cdots a_{p-1}}H_{a_{0\cdots a_p-1}}}}{p!} k_{2a}(t + 1/4)(\alpha'\xi,k_3) - \frac{1}{2} \xi_a(s + 1/4)(t + 1/4) + (3 \leftrightarrow 2)
\]

\[
\times \sum_{p,n,m=0}^\infty \epsilon'_{p,n,m}(s + t + u + 1/2)^p(s + t + 1/2)^n((t + 1/4)(s + 1/4))^m
\]

(55)

where \(\epsilon'_{p,n,m}\) can be written in term of \(a_{n,m}\) and \(b_{n,m}\). The contact terms of one RR, two tachyons and one gauge field that have been extracted from string theory S-matrix element in (29) have the above structure. Hence, the coefficients \(\epsilon_{p,n,m}\) in the couplings (3) should be replaced by

\[
\epsilon_{p,n,m} \rightarrow \epsilon'_{p,n,m}
\]

(56)

This makes the higher derivative theory to produce the string theory S-matrix element. Since the couplings (46) have on-shell ambiguity, the contact terms in (54) have also on-shell ambiguity. The on-shell ambiguity of the couplings in (29) or in (46), however, may be fixed by studying the S-matrix element of three tachyons and two gauge fields because the couplings (29) appear in the tachyon poles and in the contact terms of this S-matrix element. It would be interesting to perform this calculation.

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