First order devices, hybrid memristors, and the frontiers of nonlinear circuit theory

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Abstract

Several devices exhibiting memory effects have shown up in nonlinear circuit theory in recent years. Among others, these circuit elements include Chua’s memristors, as well as memcapacitors and meminductors. These and other related devices seem to be beyond the, say, classical scope of circuit theory, which is formulated in terms of resistors, capacitors, inductors, and voltage and current sources. We explore in this paper the potential extent of nonlinear circuit theory by classifying such mem-devices in terms of the variables involved in their constitutive relations and the notions of the differential- and the state-order of a device. Within this framework, the frontier of first order circuit theory is defined by so-called hybrid memristors, which are proposed here to accommodate a characteristic relating all four fundamental circuit variables. Devices with differential order two and mem-systems are discussed in less detail. We allow for fully nonlinear characteristics in all circuit elements, arriving at a rather exhaustive taxonomy of $C^1$-devices. Additionally, we extend the notion of a topologically degenerate configuration to circuits with memcapacitors, meminductors and all types of memristors, and characterize the differential-algebraic index of nodal models of such circuits.

Keywords: nonlinear circuit, memristor, memcapacitor, meminductor, nodal analysis, differential-algebraic equation, index.

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1 Introduction

Broadly speaking, nonlinear circuit theory is concerned with the study of constrained ordinary differential equations involving time and four $m$-dimensional variables $q, \varphi, i, v$ (standing for charge, flux, current and voltage, respectively), with the following restrictions:

(a) the $2m$ differential relations $q' = i, \varphi' = v$ always hold;

(b) the vectors $v, i$, satisfy a total amount of $m$ linearly independent relations $Bv = 0$, $Di = 0$ coming from Kirchhoff laws;

(c) the characteristics of devices define $m$ additional relations among the circuit variables.

This means that nonlinear circuit models can be generally written in the form

\[ q' = i \quad (1a) \]
\[ \varphi' = v \quad (1b) \]
\[ 0 = f(q, \varphi, i, v, t), \quad (1c) \]

where $f$ captures Kirchhoff laws but also the constitutive relations of all circuit devices. Aside from the different circuit topologies, reflected in the form of the loop and cutset matrices $B$, $D$ arising in item (b), the differences between circuit families come from the devices’ characteristics referred to in (c).

In classical circuit theory, the devices’ characteristics just involve two out of the four variables mentioned above in one of three ways, relating current and voltage in resistors and controlled sources, charge and voltage in capacitors, and flux and current in inductors. In general, these relations may be nonlinear and/or involve time explicitly; specifically, the current or the voltage in independent sources is an explicit function of time. This provides a setting which models a great variety of circuits arising in electrical and electronic engineering; in particular, many semiconductor devices, including transistors, may be accommodated in this framework by means of equivalent circuits including controlled sources.

Memristors. The memory-resistor or memristor is changing this picture substantially. This device, whose existence was predicted by Leon Chua in 1971 [8] and which actually appeared at the nanometer scale in 2008 [45], is defined by a nonlinear charge-flux relation. The potential applications of this device in the design of non-volatile memories, signal processing, adaptive and learning systems, reconfigurable nanoelectronics, etc., might make the memristor and related devices play a very significant role in electronics in the near future, specially at the nanometer scale. A lot of research is focused on this topic; cf. [2, 7, 19, 20, 22, 28, 30, 31, 32, 33, 34, 35, 36, 40, 41, 43, 44, 49, 50, 51].

Not only regarding applications, but also from a theoretical point of view this scenario poses challenging problems. Several extensions are being developed; some remarkable ones are related to the memristive systems of Chua and Kang [10] and the mem-devices (mem-capacitors and meminductors) recently introduced by Di Ventra et al. [11]. Somehow, the
fundamentals of nonlinear circuit theory are affected, since its concepts and models seem to be moving beyond their classical limits, and several questions arise. One may wonder about the limits to which this framework may extend, but also how to accommodate these new devices in a comprehensive taxonomy, and which aspects should articulate such a taxonomy. In the present paper we will try to answer some of these questions.

In this regard we will find it useful to describe the memristor (say, in a charge-controlled setting) not by means of its flux-charge relation \( \varphi = \phi(q) \) but, instead, via the differentiated relation \( \varphi' = \phi'(q)q' \), that is,

\[
v = M(q)i, \quad (2)
\]

where \( M(q) = \phi'(q) \) is the so-called memristance. The reason to do so is that, in the description of the dynamics of memristive circuits, there is no reason to keep track of both the flux and the charge as dynamic variables since their values are constrained by the relation \( \varphi = \phi(q) \). The memristor has order (or, to avoid terminological ambiguities, state order) one. This means that every memristor introduces one, but not two, degree(s) of dynamic freedom, associated with the charge \( q \); describing the device characteristic via \( (2) \) we get rid of the flux \( \varphi \). A detailed discussion can be found in subsection 2.1.

Now, the key remark in our discussion is to look at \( (2) \) just as a relation involving three out of the four variables \( q, \varphi, i, v \) (all but \( \varphi \)), regardless of its specific form; actually, as in \[11\] we will work with so-called \( q \)-memristors, defined (in a time-invariant setting) by a general, fully nonlinear characteristic of the form \( v = \eta(q, i) \).

The corresponding relation for \( \varphi \)-memristors is \( i = \zeta(\varphi, v) \) and, in particular, the characteristic of Chua’s flux-controlled memristors \[8\] reads as

\[
i = W(\varphi)v, \quad (3)
\]

in terms of the memductance \( W(\varphi) \); the identity \( (3) \) again involves three out of the four variables mentioned above (in this case, all but \( q \)).

**Memcapacitors and meminductors.** In a natural way, this makes us wonder about the remaining relations involving the other two combinations of three variables, namely, \( q, \varphi, v \) and \( q, \varphi, i \). This will lead to the (voltage-controlled) *memcapacitor*, with a characteristic of the form

\[
q = C_m(\varphi)v, \quad (4)
\]

and the (current-controlled) *meminductor*, for which,

\[
\varphi = L_m(q)i. \quad (5)
\]

These devices have been introduced by Di Ventra *et al.* in \[11\] and are discussed in subsection 2.2 using again more general, fully nonlinear characteristics.
Hybrid memristors. Finally, for the sake of completeness one may think about a characteristic actually involving all four variables $q$, $\varphi$, $i$, $v$. For reasons detailed later, we will consider two different settings, defined by the fully nonlinear characteristics $v = \psi(q, \varphi, i)$ and $i = \xi(q, \varphi, v)$. In particular, when these relations are linear in $i$ and $v$, respectively, we get

$$v = M_h(q, \varphi)i$$

and

$$i = W_h(q, \varphi)v.$$  

We will call these circuit elements hybrid memristors, since their memristance and memductance involve both the charge $q$ and the flux $\varphi$. These novel circuit elements, together with their fully nonlinear variants, are introduced here for mathematical completeness, but they also provide a natural framework to accommodate actual physical devices in which different memory effects (namely, memristive, memcapacitive and/or meminductive features) coexist. Find details in subsections 2.3 and 2.4.

These relations comprise all possible combinations of the four variables $q$, $\varphi$, $i$, $v$. This means that the aforementioned devices and their time-varying analogs exhaust what may be called first order circuit theory, within the setting defined by items (a), (b) and (c) on p. 2. Here the term “order” is used to mean differential order (cf. Section 2) and, accordingly, the expression “first order circuit” reflects the fact that only the branch currents and voltages $i$, $v$ and their first integrals $q$, $\varphi$ are involved (compare Definitions 1 and 8 in Sections 2 and 3). Certainly, the reader should not misunderstand our use of this expression with that in elementary circuit theory, referring to a circuit with only one reactive element.

Beyond this framework, higher order devices involve additional variables such as the first integrals of the charge and the flux (second integrals of the branch current and voltage), to be denoted by $\sigma$ and $\rho$, respectively. With the use of these additional variables we may accommodate devices such as charge-controlled memcapacitors and flux-controlled meminductors [11]. Another extension of the theory is related to the mem-systems introduced by Chua and Kang [10]. Roughly speaking, the restriction on the form of the differential relations considered in (a) is no longer assumed in these systems; by contrast, new variables without a physical meaning in classical circuit-theoretic terms are allowed, providing dynamical states on which the resistance, capacitance and inductance (or better, the memristance, memcapacitance and meminductance) may depend. Higher order devices and mem-systems are briefly discussed in Section 3, a deeper analysis of them is in the scope of future research.

The use of these types of devices in applications, as well as the numerical simulation of the dynamics of nonlinear circuits including them, requires some effort also at the circuit modelling level. We will consider circuit modelling aspects in Section 4. Specifically, we analyze there the differential-algebraic models arising from the nodal analysis of nonlinear circuits including all possible types of first order devices. Our goal in this regard will be the characterization of the index of these models, a notion which defines many key properties
of differential-algebraic systems \cite{14,17,23,37,39}. This way we will arrive at a general index characterization (cf. Theorems 1 and 2 in Section 4) from which the results discussed in \cite{12,39,43,47,48} can be derived as particular cases which apply to circuits including only certain types of devices with differential order one; another particular case of importance concerns circuits with memcapacitors and meminductors, whose index is so far unexplored in the literature. We will also extend the notion of a topologically degenerate configuration to this broader setting.

2 First order devices

Definition 1. A circuit device is said to have differential order one if it is defined by a \(C^1\)-characteristic of the form
\[
h(q, \varphi, i, v, t) = 0, \tag{8}
\]
where at least one of the partial derivatives \(h_q\), \(h_\varphi\) does not vanish identically.

Note that both \(h_q\) and \(h_\varphi\) may of course vanish for specific values of the circuit variables. Definition 1 is supported on the fact that when both \(h_q\) and \(h_\varphi\) vanish identically, the only branch variables involved are the current and/or the voltage. This amounts to what may be called (differential) order zero devices, not involving any dynamics. These are nonlinear resistors, and voltage and current sources. By a nonlinear resistor we mean any device relating current and voltage in an algebraic (i.e., non-differential) manner, such as a diode.

The use of the term “order” is worth a digression. This term has different senses in mathematics, and at least two apply in our context. On the one hand it means the order of derivation of a system of differential equations; in this sense, Newton’s law yields a second order system. We will use the expression \textit{differential order} to mean this. On the other hand, “order” is also used to mean the number of dynamic variables in a system of differential equations; the term is often used in this sense in circuit theory, to refer to the number of state variables associated with a given device. We will describe the latter as the \textit{state order} of a device. Throughout the document and when no label is used, by “order” we refer to the differential order; e.g. a first order device is a device with differential order one.

As detailed in the sequel, the list of devices with differential order one will include

- capacitors, inductors, \(q\)-memristors and \(\varphi\)-memristors, all of them with state order one;
- voltage-controlled memcapacitors, current-controlled meminductors, and hybrid memristors, with state order two.

In practice, at least two of the derivatives \(h_q, h_\varphi, h_i, h_v\) do not vanish identically in first order devices. It is also worth noting that in Definition 1 we implicitly assume the arguments of \(h\) to be one-dimensional; however, allowing \(q, \varphi, i\) and \(v\) to take vector values, this definition easily accommodates coupled devices. In this case the non-vanishing requirement on \(h_q\) and/or \(h_\varphi\) must be replaced by the non-singularity of the corresponding matrices of partial derivatives. This remark will apply throughout the paper, often without explicit mention.
2.1 \( q \)- and \( \varphi \)-memristors

The electromagnetic relations \( q' = i \) and \( \varphi' = v \) link the pairs of fundamental circuit variables \( q - i \) and \( \varphi - v \). First order devices involving the other combinations of two out of the four variables \( q, \varphi, i, v \) are the capacitor, relating \( q \) and \( v \), the inductor, which involves \( \varphi \) and \( i \), and Chua’s memristor, whose characteristic relates \( q \) and \( \varphi \).

The complete dynamical description of the first two devices requires including the differential relation \( q' = i \) for the capacitor (note that \( \varphi \) does not appear in its constitutive relation), and \( \varphi' = v \) for the inductor (\( q \) not being involved). Regarding the memristor, even though both \( q \) and \( \varphi \) arise in its characteristic, these variables are linked together and therefore only one of them introduces a dynamical degree of freedom in the circuit. For this reason, it will be preferred to get rid of either the flux or the charge by means of a relation formulated in terms of the other three variables, as detailed below. This way, not only the capacitor and the inductor but also \( q \)- and \( \varphi \)-memristors will have state order one.

**Definition 2.** A \( q \)-memristor is a device with differential order one, governed by the relations

\[
\begin{align*}
q' & = i \quad (9a) \\
v & = \eta(q, i, t) \quad (9b)
\end{align*}
\]

where \( \eta \) is a \( C^1 \)-map for which neither of the derivatives \( \eta_q, \eta_i \) vanishes identically.

When \( \eta \) is time-invariant and linear in \( i \), we get Chua’s characteristic \( v = M(q)i \), where \( M(q) \) is the memristance \[8\]. The identity \( q(t) = \int_{-\infty}^{t} i(\tau) d\tau \) shows that this voltage-current relation keeps track of the device history; for this reason Chua proposed the name memory-resistor, or memristor for short. In the literature, this device is said to be current-controlled but also charge-controlled, because of the form of the map \( \varphi = \phi(q) \) whose time derivative yields the voltage-current relation \( v = M(q)i \).

For fully nonlinear memristors of the form \[9\] the incremental memristance is defined as the derivative \( \eta_i(q, i, t) \) \[11\]: the device is called strictly locally passive if \( \eta_i(q, i, t) > 0 \) for all \( (q, i, t) \) (or if the matrix \( \eta_i \) is positive definite in coupled cases). The requirement that \( \eta_i \) does not vanish identically distinguishes the device from a nonlinear capacitor. In turn, the non-vanishing condition on \( \eta_q \) makes this device actually different from a nonlinear, current-controlled resistor. Note also that the fully nonlinear form \[9\] makes it possible to accommodate devices displaying memristive effects but whose characteristic does not arise as the time derivative of a \( \varphi \)-\( q \) relation, contrary to Chua’s memristor.

**Definition 3.** A \( \varphi \)-memristor is a device with differential order one, governed by

\[
\begin{align*}
\varphi' & = v \quad (10a) \\
i & = \zeta(\varphi, v, t) \quad (10b)
\end{align*}
\]

where \( \zeta \) is a \( C^1 \)-map such that neither of the derivatives \( \zeta_{\varphi}, \zeta_v \) vanishes identically.
A time-invariant $\varphi$-memristor for which (10b) is linear in $v$ (hence reading as $i = W(\varphi)v$) amounts to Chua’s flux-controlled memristor [8], $W(\varphi)$ being the memductance. In general, the incremental memductance is the derivative $\zeta_v(\varphi, v, t)$, and the device is said to be strictly locally passive if this derivative is always positive. Again, the non-vanishing requirements on $\zeta_\varphi$ and $\zeta_v$ make this device different from a voltage-controlled resistor and an inductor, respectively.

Nonlinear, $C^1$ circuits composed of (current- and voltage-controlled) resistors, voltage and current sources, capacitors, inductors, and $q$- and $\varphi$-memristors display the property that neither the differential nor the state order of their devices exceed one. Several dynamical properties of this type of circuits are discussed in [41]. In the next subsections we discuss the main features of devices with differential order one and state order two.

### 2.2 Memcapacitors and meminductors

The remaining characteristics involving three out of the four variables $q$, $\varphi$, $i$, $v$ naturally lead to the voltage-controlled memcapacitors and current-controlled meminductor discussed below. Both devices will have state order two. We consider fully nonlinear characteristics for memcapacitors and meminductors; when the maps $\omega$ and $\theta$ below are time-independent and linear in $v$ and $i$, respectively, we will get the circuit elements introduced by Di Ventra et al. in [11].

**Definition 4.** A voltage-controlled memcapacitor is a device with differential order one, defined by

\begin{align*}
q' &= i \tag{11a} \\
\varphi' &= v \tag{11b} \\
q &= \omega(\varphi, v, t), \tag{11c}
\end{align*}

where $\omega$ is a $C^1$-map and neither of the derivatives $\omega_\varphi$, $\omega_v$ vanishes identically.

When $\omega$ is time-invariant and linear in $v$, (11c) reads as $q = C_m(\varphi)v$, where $C_m$ is the memcapacitance [11]. The distinctive feature of this device is that the memcapacitance depends on the state variable $\varphi(t) = \int_{-\infty}^t v(\tau)d\tau$, so that the relation $q(t) = C_m(\int_{-\infty}^t v(\tau)d\tau)v(t)$ reflects the device history. Be aware of the circuit-theoretic meaning of this state variable; when the memcapacitance is allowed to depend on abstract state variables we are led to the more general setting of memcapacitive systems (see [11] and Section 3 below). It is also worth mentioning that the relation $q = C_m(\varphi)v$ arises as the derivative of a characteristic $\sigma = \mu(\varphi)$, where $\sigma$ is the time integral of $q$. This yields $\sigma' = q = \mu(\varphi)\varphi' = C_m(\varphi)v$. Notably, the device can be described without recourse to the (second order) variable $\sigma$ (cf. (11)), in contrast to the charge-controlled memcapacitors discussed in Section 3.

For an arbitrary $C^1$-map $\omega$, the incremental memcapacitance $C_m$ is defined as the derivative $\omega_v$ and, in general, depends on $(\varphi, v, t)$. Such a device need not come from a $\sigma - \varphi$ relation, and the requirement that neither $\omega_\varphi$ nor $\omega_v$ vanishes identically makes it actually
different from a capacitor or a memristor, respectively. An instance of a fully nonlinear memcapacitor arising in a Josephson junction model can be found in subsection 2.4.

Definition 5. A current-controlled meminductor is a device with differential order one, governed by

\[ q' = i \]  \hspace{1cm} (12a)  
\[ \varphi' = v \]  \hspace{1cm} (12b)  
\[ \varphi = \theta(q, i, t), \]  \hspace{1cm} (12c)

where \( \theta \) is a \( C^1 \)-map for which neither of the derivatives \( \theta_q, \theta_i \) vanishes identically.

Again, when the map \( \theta \) is linear in \( i \) and does not depend on \( t \), we get the characteristic \( \varphi = L_m(q)i = L_m(\int i(\tau)d\tau)i \) considered by Di Ventra et al. in [11]. Such a characteristic can be obtained as the time derivative of a \( C^1 \)-relation \( \varphi = \kappa(q) \), where \( \varphi \) is the time integral of \( \varphi \); note however that the description (12) does not involve \( \varphi \) (compare with the flux-controlled meminductors in Section 3). Now \( L_m \) is the meminductance. In general, the derivative \( \theta_i(q, i, t) \) is the incremental meminductance. The non-vanishing of this derivative makes the meminductor different from a memristor and, similarly, the assumption that the partial derivative \( \theta_q \) does not vanish identically makes the device different from an inductor.

The appearance of both \( q \) and \( \varphi \) in the characteristics (11c) and (12c) imply that both variables must be present in the dynamical description of memcapacitors and meminductors and, therefore, that these devices have state order two; this is an important difference with capacitors, inductors and \( q \)- and \( \varphi \)-memristors, for which one dynamic variable suffices (together with \( i \) and \( v \)) to describe the device behavior.

2.3 Hybrid memristors

In light of the characteristics (9b), (10b), (11c) and (12c), from a mathematical point of view it is somehow natural to complete the picture by considering a relation which involves all four variables \( q, \varphi, i \) and \( v \). We present below two different settings (dual to each other) in which these four variables might actually arise.

These (so-called hybrid) memristors may account for physical devices in which memory effects of different nature coexist. The simultaneous appearance of memristive, memcapacitive and/or meminductive phenomena has been discussed by different authors. For instance, the coexistence of memristive and memcapacitive effects has been reported to follow from the formation of local dipoles in nanoscale resistors [11], and may also arise in metal-insulator-metal thin films having thickness between the nanometer and the micrometer scales [29]. Capacitive and memristive effects coexist with the nonlinear inductive nature of a Josephson junction in accurate models of this device [21]. From a modelling point of view, hybrid memristors also allow for simplified descriptions of combinations of more basic devices. Find examples in subsection 2.4 below.
**Definition 6.** A current-controlled hybrid memistor is a device with differential order one, defined by the relations

\[
q' = i \\
\varphi' = v \\
v = \psi(q, \varphi, i, t),
\]

where \(\psi\) is a \(C^1\)-map and none of the derivatives \(\psi_q, \psi_{\varphi}, \psi_i\) vanishes identically.

The terminology is motivated by the fact that, in cases in which (13c) is time-independent and linear in \(i\), this relation takes the form \(v = M_h(q, \varphi)i\), providing an analog of Chua’s memristor in which the so-called hybrid memristance \(M_h(q, \varphi)\) now depends on both the charge and the flux; both variables introduce memory effects on the device, because of the relations \(q(t) = \int_{-\infty}^t i(\tau)d\tau, \varphi(t) = \int_{-\infty}^t v(\tau)d\tau\). In general, the incremental hybrid memristance is defined as the derivative \(\psi_i\), and may depend not only on \(q\) and \(\varphi\) but also on the current \(i\) and on \(t\). The hybrid memristor is said to be strictly locally passive when the incremental hybrid memristance is positive (or positive definite in coupled cases).

**Definition 7.** A voltage-controlled hybrid memistor is a device with differential order one, governed by

\[
q' = i \\
\varphi' = v \\
i = \xi(q, \varphi, v, t),
\]

where \(\xi\) is a \(C^1\)-map for which none of the derivatives \(\xi_q, \xi_{\varphi}, \xi_v\) vanishes identically.

Cases in which \(\xi\) is linear in \(v\) and time-invariant yield \(i = W_h(q, \varphi)v\), where \(W_h\) is now the hybrid memductance. In this situation, when \(W_h\) is non-singular for all values of \(q, \varphi\), then obviously the device can be recast in a current-controlled form and vice-versa. Again, in general the incremental hybrid memductance is defined by the derivative \(\xi_v(q, \varphi, v, t)\) and, as before, the device is said to be strictly locally passive when the incremental hybrid memductance is positive (or positive definite).

The non-vanishing requirements on the derivatives within the characteristics (13c), (13c) distinguish these devices from the \(q\)- and \(\varphi\)-memristors, memcapacitors and meminductors discussed above. Actually, the non-vanishing of these derivatives implies that, at least locally, both the charge and the flux can be written in terms of the remaining circuit variables. Focusing the attention on the current-controlled case, the fact that \(\psi_q \neq 0\) makes it possible to recast (13c), via the implicit function theorem, as

\[
q = \alpha(\varphi, i, v, t),
\]

and the device exhibits a (generalized) memcapacitance

\[
\alpha_v(\varphi, i, v, t) = \psi_q^{-1}(\alpha(\varphi, i, v, t), \varphi, i, t).
\]
Similarly, the non-zero nature of the partial derivative $\psi_\phi$ yields

$$\varphi = \beta(q, i, v, t),$$

for some locally defined map $\beta$, the (generalized) meminductance being

$$\beta_i(q, i, v, t) = -\psi^{-1}_\phi(q, \beta(q, i, v, t), i, t)\psi_i(q, \beta(q, i, v, t), i, t),$$

as a consequence of the implicit function theorem. Similar remarks apply to voltage-controlled hybrid memristors.

Noteworthy, removing the non-vanishing requirements on the derivatives, the relations (13c), (14c), (15) and (16) account for the constitutive relations of all previous devices. From a dynamical point of view, a remarkable difference between hybrid memristors and both $q$- and $\varphi$-memristors is that hybrid ones have state order two, since both the charge and the flux are necessarily involved in the description of the device. This implies, for instance, that the dynamics of a circuit with just one hybrid memristor and without reactive elements lies on the plane. The actual dynamical properties of circuits with hybrid memristors are in the scope of future research.

### 2.4 Examples

**Example 1.** Our first example attempts to illustrate how the use of fully nonlinear characteristics may allow for simplified descriptions of certain devices. Specifically, we will provide an accurate description of a Josephson junction by means of a fully nonlinear memcapacitor, which accounts for all parasitic effects, connected in parallel to a nonlinear inductor (cf. Figure 1 below).

![Figure 1](image-url)

Figure 1: (a) Josephson junction. (b) Equivalent circuit with a fully nonlinear memcapacitor.

As detailed in [21], realistic models of a Josephson junction should take into not only the usual nonlinear inductive relation

$$i_l = I_0 \sin(k_0 \varphi_l)$$

for certain physical constants $I_0$, $k_0$ (see e.g. [21]), but also the presence of memristive, resistive and capacitive effects; an accurate equivalent circuit of the Josephson junction is defined by the parallel connection of these four elements (cf. [21]), as depicted in Figure 1(a).
The device on the left of Figure 1(a) is a \( \varphi \)-memristor of Chua type, which as reported in [21] captures the presence of a small current component given by

\[ i_w = I_1 \cos(k_1 \varphi_w) v, \]

again for certain constants \( I_1, k_1 \); here \( v \) is the port voltage.

A linear resistor and a linear capacitor in parallel are also present in the description provided in [21], being denoted by \( G, C \), respectively, in Figure 1(a). These elements are defined by the relations \( i_g = G v \) and \( q_c = C v \).

Now, the parallel connection of the memristor and the resistor is obviously governed by the current-voltage relation \( i_{wg} = I_1 \cos(k_1 \varphi_w) v + G v \) or, equivalently, by a charge-flux characteristic of the form

\[ q_{wg} = \frac{I_1}{k_1} \sin(k_1 \varphi_w) + G \varphi_w, \]  

(17)

where we use the fact that the memristor flux is the time-integral of the port voltage \( v \). The expression depicted in (17) shows that the parallel connection of the memristor and the resistor is itself a \( \varphi \)-memristor.

In turn, the parallel connection of the original memristor, the resistor and the capacitor can be described as a single device by setting \( q = q_{wg} + q_c, \varphi = \varphi_w \). Indeed, denoting by \( i \) the sum of the currents through the memristor, the resistor and the capacitor, we get

\[ \begin{align*}
    q' & = i \quad \text{(18a)} \\
    \varphi' & = v \quad \text{(18b)} \\
    q & = \frac{I_1}{k_1} \sin(k_1 \varphi) + G \varphi + C v. \quad \text{(18c)}
\end{align*} \]

This corresponds to a time-invariant, voltage-controlled memcapacitor for which the constitutive relation \( q = \omega(\varphi, v) \) in (11c) takes the specific form depicted in (18c). The corresponding equivalent circuit for the Josephson junction is displayed in Figure 1(b).

**Example 2.** Simple instances of hybrid memristors are defined by the connection of a charge-controlled and a flux-controlled memristor of Chua type; cf. Figure 2, where the charge-controlled memristor and the flux-controlled one are painted in green and yellow, respectively. In spite of their simplicity, these examples are not trivial: the goal is, again, to provide a dynamical description of each connection as a two-terminal device in terms of a single set of variables \( q, \varphi, i, v \).

In both cases, the subscripts 1 and 2 will correspond to variables associated with the charge- and the flux-controlled memristor, respectively. The charge-controlled memristor is assumed to be governed by a relation of the form \( \varphi_1 = \phi(q_1) \), with memristance \( M(q_1) \), and the flux-controlled one is defined by \( q_2 = \gamma(\varphi_2) \), with memductance \( W(\varphi_2) \). Within Figure 2(a) (resp. 2(b)), the memductance \( W(\varphi_2) \) (resp. the memristance \( M(q_1) \)) is assumed not to vanish.

In the series connection of Figure 2(a), elementary circuit theory yields \( i = i_1 = i_2, v = v_1 + v_2 \). In order to arrive at a dynamical description in terms of \( i, v \) and single variables
Figure 2: Examples of (a) a current-controlled and (b) a voltage-controlled hybrid memristor.

$q, \varphi$, we set $q = q_1$ and $\varphi = \varphi_1 + \varphi_2$. Obviously, this yields the relations $q' = i$ and $\varphi' = v$ but, more important, allows for the description of the voltage-current relation in the form

$$v = M(q_1)i + (W(\varphi_2))^{-1}i = [M(q) + (W(\varphi - \varphi_1))^{-1}]i = [M(q) + (W(\varphi - \varphi(q_1)))^{-1}]i = M(q) + (W(\varphi - \phi(q)))^{-1}i.$$

This corresponds to a current-controlled hybrid memristor for which the characteristic is linear in $i$, the hybrid memristance being

$$M_h(q, \varphi) = M(q) + (W(\varphi - \phi(q)))^{-1}.$$

The physical meaning of the variable $q$ is worth some additional remarks. Because of the identities $q'_1 = i_1 = i_2 = q'_2$, the charges $q_1$ and $q_2$ differ in a constant which would be fixed by the initial conditions in Chua’s memristors. In turn, $q$ is defined up to a constant, and therefore can be understood to describe any of both charges except for a fixed quantity. Mathematically, setting $q = q_1 + k$ for any real constant $k$, we would get a dynamical description of the device which, except for an affine change of coordinates, amounts to the one above; the description above assumes $k = 0$ so that $q$ actually equals the charge $q_1$.

It is also worth noting that if $M(q) + (W(\varphi - \phi(q)))^{-1}$ does not vanish for all values of $q, \varphi$, the device also admits a voltage-controlled description. This would be the case, in particular, if the original Chua memristors are strictly locally passive (i.e., if $M > 0, W > 0$ everywhere); in this case the hybrid memristor would itself be strictly locally passive since $M_h$ would be strictly positive.

Setting $q = q_1 + q_2, \varphi = \varphi_2$, the reader can proceed analogously in order to describe the parallel configuration of Figure 2(b) as a voltage-controlled hybrid memristor, with a characteristic linear in $v$ and hybrid memductance

$$W_h(q, \varphi) = (M(q - \gamma(\varphi)))^{-1} + W(\varphi).$$

These examples show that, even in simple cases, these devices pose interesting problems from the modelling point of view. Needless to say, the scope of hybrid memristors goes however beyond these academic examples. The definition of this circuit element is aimed at modelling devices which are not reducible to a simple connection of $q$- and $\varphi$-memristors, and which may capture the coexistence of different memory effects. In such devices the characteristics need not be linear in $i$ and $v$, and the charge and the flux within the (incremental) memristance or memductance might interact in more intricate ways.
3 Higher order devices and mem-systems

The devices discussed in Section 2 virtually fill the scope of circuit theory within the limits defined by the use of the variables \( q, \varphi, i, v \) and under the restrictions stated in items (a), (b) and (c) on p. 2. The frontier of nonlinear circuit theory in this setting is defined by hybrid memristors, which involve all four fundamental circuit variables. However, other devices recently introduced are located beyond these limits: on the one hand, certain circuit elements involve not only those four variables but also the time-integrals \( \sigma, \rho \) of the charge and the flux, respectively; these include charge-controlled memcapacitors and flux-controlled meminductors [11], but also devices directly relating \( \sigma \) and \( \rho \), as proposed in [3]; all of them would have differential order two. On the other hand, allowing for new state variables, without the restriction imposed by (a), leads to the so-called mem-systems, which in general cannot be accommodated in the framework of Section 2. In this Section we briefly discuss some modelling aspect of these devices, leaving a deeper analysis for future research.

3.1 Second order devices

Definition 8. A circuit device is said to have differential order two if it is defined by a \( C^1 \)-characteristic of the form

\[
\begin{align*}
h(\sigma, \rho, q, \varphi, i, v, t) &= 0, \quad (19)
\end{align*}
\]

where at least one of the partial derivatives \( h_\sigma, h_\rho \) does not vanish identically.

The non-vanishing of \( h_\sigma \) and/or \( h_\rho \) implies that at least one of the differential relations \( \sigma' = q \) (and in turn \( q' = i \)) or \( \rho' = \varphi \) (together with \( \varphi' = v \)) must be included in the dynamical description of the device. The differential order two nature of them stems from the identities \( \sigma'' = i \) and \( \rho'' = v \). Akin to differential order one devices, in practice at least two of the derivatives \( h_\sigma, h_\rho, h_q, h_\varphi, h_i, h_v \) will not vanish identically.

Definition 9. A charge-controlled memcapacitor is a device with differential order two, defined by the relations

\[
\begin{align*}
\sigma' &= q \\
q' &= i \\
v &= \nu(\sigma, q, t),
\end{align*}
\]

where \( \nu \) is a \( C^1 \)-map for which neither of the derivatives \( \nu_\sigma, \nu_q \) vanishes identically.

Definition 10. A flux-controlled meminductor is a device with differential order two, governed by

\[
\begin{align*}
\rho' &= \varphi \\
\varphi' &= v \\
i &= \chi(\rho, \varphi, t)
\end{align*}
\]

where \( \chi \) is a \( C^1 \)-map such that neither of the derivatives \( \chi_\rho, \chi_\varphi \) vanishes identically.
From the descriptions (20) and (21) it follows that charge-controlled memcapacitors and flux-controlled meminductors have state order two; note that neither \( \rho \) nor \( \varphi \) (resp. \( \sigma \), \( q \)) arise in the characteristic (20c) (resp. (21c)).

In particular, when \( \nu \) and \( \chi \) are time-invariant and linear in \( q \) and \( \varphi \), respectively, one gets the devices introduced by Di Ventra et al. in [11], for which the characteristics (20c) and (21c) read as

\[
v = E_m(\sigma)q
\]

and

\[
i = R_m(\rho)\varphi.
\]

Here \( E_m \) and \( R_m \) are the inverse memcapacitance and the inverse meminductance, which introduce memory effects in the circuit because of their dependence on \( \sigma \) and \( \rho \), respectively.

Noteworthy, the relations (22) and (23) arise as the differentiated form of certain mappings \( \varphi = \gamma(\sigma) \) and \( q = \delta(\rho) \), via the relations \( \sigma' = q, \rho' = \varphi \). In a natural way this leads to other second order devices, such as those relating \( \sigma \) and \( \rho \) (cf. [3]). The analysis of the dynamics of nonlinear circuits including these or other higher order devices is an open problem, beyond the scope of the present paper.

### 3.2 Mem-systems

Mem-systems, originally introduced by Chua and Kang in [10], are characterized by the removal of the classical electrical meaning of the state variable which introduces memory into the different devices. This means that new variables arise in the model, and that the specific form of the differential relations within item (a) on p. 2 is no longer assumed. For the sake of brevity, we restrict the discussion to the mem-systems which generalize the devices considered in Definitions 2, 3, 4 and 5, using again fully nonlinear characteristics.

**Memristive systems** are defined either by a system of the form

\[
\begin{align*}
x' &= f(x, i) \quad & (24a) \\
v &= \tilde{\eta}(x, i, t), \quad & (24b)
\end{align*}
\]

or by

\[
\begin{align*}
y' &= g(y, v) \quad & (25a) \\
i &= \tilde{\zeta}(y, v, t). \quad & (25b)
\end{align*}
\]

The distinct feature with respect to \( q \)- and \( \varphi \)-memristors is that now the memristance and the memductance depend on arbitrary state variables \( x, y \), respectively, which do not even need to be one-dimensional; making \( x \equiv q, f(q, i) = i \) and \( y \equiv \varphi, g(\varphi, v) = v \) we get the \( q \)-and \( \varphi \)-memristors introduced in Definitions 2 and 3. Systems in which the characteristics (24b) and (25b) amount to \( v = M(x, i)i \) and \( i = W(y, v)v \), respectively, describe the settings originally discussed by Chua and Kang in [10].
Analogously, a memcapacitive system is governed by the system

\[
q' = i \\
\dot{z} = h(z, v) \\
q = \tilde{\omega}(z, v, t)
\]

whereas the equations defining a meminductive system are

\[
u' = p(u, i) \\
\phi' = v \\
\phi = \tilde{\theta}(u, i, t)
\]

Again, with \(\tilde{\omega}(z, v) = C_m(z)v, \tilde{\theta}(u, i) = L_m(u)i\) we get the systems analyzed by Di Ventra et al. in [11].

Mem-systems have many potential applications, not only in electronics; see e.g. [33, 36] and references therein. These systems are likely to define an active field of research in the near future.

4 Nodal analysis of first order circuits

From a computational point of view, models of the form (1) offer some difficulties for numerical simulation. This is due to the fact that an automatic computation of the loop and cutset matrices \(B, D\) is difficult to perform in practice, specially in high scale integration circuits. For this reason, it is often preferred to introduce the node potentials \(e\) in the model, and describe the circuit equations using nodal analysis. This is the case in most circuit simulation problems, notably in SPICE and its commercial variants, which set up the circuit equations using Modified Nodal Analysis (MNA); cf. [12, 15, 16, 38, 39, 46, 47, 48].

As detailed below, the models arising from nodal analysis naturally take the form of a differential-algebraic equation (DAE) [4, 14, 23, 37, 39]. The main problem in the analysis of such differential-algebraic models is the characterization of the index, a concept which, roughly speaking, extends the notion of the Kronecker-Weierstrass index of a matrix pencil [13] and measures the numerical difficulties faced in simulation [4, 17]. Index one and index two systems require specific numerical techniques, and because of this it is important to characterize the circuit configurations which lead to models with these indices. Therefore, we undertake in this Section the index analysis of nodal models of general first order circuits, using the tractability index framework [14, 24, 25, 48] and extending the results discussed in [12, 39, 43, 47] to circuits with (voltage-controlled) memcapacitors, (current-controlled) meminductors, and hybrid memristors. Nonlinear circuits with Chua’s memristors, memcapacitors and meminductors define a particular case of great interest in current applications [19, 20, 22, 32, 33, 34, 35, 36].

Also from an analytical point of view it is important to characterize index one circuit configurations. In index one systems, all the dynamic variables of the different devices contribute to the state dimension of the problem; more precisely, in index one cases the state
dimension (also called the \textit{order of complexity}) of a first order circuit equals the sum of the state orders of the devices with differential order one. By contrast, in higher index problems which arise from so-called \textit{topologically degenerate} configurations the feasible values for these dynamic variables are restricted by algebraic (non-differential) constraints. In a classical setting the topologically degenerate configurations are VC-loops (loops just defined by capacitors and –possibly– voltage sources) and IL-cutsets (cutsets just including inductors and –possibly– current sources); note that V-loops and I-cutset are excluded in well-posed problems. As a byproduct of our analysis we introduce the notion of a topologically degenerate configuration for circuits including memcapacitors and meminductors.

4.1 The nodal model

Assuming a time-invariant setting for simplicity, the nodal model of a general first order circuit reads as

\begin{align}
q'_c &= i_c \\
q'_{mc} &= i_{mc} \\
\varphi'_l &= A_l^T e \\
\varphi'_{ml} &= A_{ml}^T e \\
q'_m &= i_m \\
q'_{ml} &= i_{ml} \\
q'_{hm} &= i_{hm} \\
q'_{hw} &= i_{hw} \\
\varphi'_w &= A_w^T e \\
\varphi'_{mc} &= A_{mc}^T e \\
\varphi'_{hm} &= A_{hm}^T e \\
\varphi'_{hw} &= A_{hw}^T e \\
0 &= A_c i_c + A_{mc} i_{mc} + A_u i_u + A_{il} i_l + A_{ml} i_{ml} + A_g \gamma_g (A_g^T e) + A_w \zeta (\varphi_w, A_w^T e) + \\
&+ A_r i_r + A_m i_m + A_{hm} i_{hm} + A_{hw} i_{hw} + A_j i_s (t) \\
0 &= q_c - \gamma_c (A_c^T e) \\
0 &= q_{mc} - \omega (\varphi_{mc}, A_{mc}^T e) \\
0 &= v_s (t) - A_w^T e \\
0 &= \varphi_l - \gamma_l (i_l) \\
0 &= \varphi_{ml} - \theta (q_{ml}, i_{ml}) \\
0 &= \gamma_r (i_r) - A_r^T e \\
0 &= \eta (q_m, i_m) - A_m^T e \\
0 &= \psi (q_{hm}, \varphi_{hm}, i_{hm}) - A_{hm}^T e \\
0 &= i_{hw} - \xi (q_{hw}, \varphi_{hw}, A_{hw}^T e).
\end{align}
The two distinct features of nodal models are the use of node potentials \( e \) and the description of Kirchhoff laws as \( Ai = 0, v = A^T e \), in terms of the reduced incidence matrix \( A \). Provided that the circuit is connected and that a reference node has been chosen, this matrix is defined as \( A = (a_{ij}) \) with

\[
a_{ij} = \begin{cases} 
1 & \text{if branch } j \text{ leaves node } i \\
-1 & \text{if branch } j \text{ enters node } i \\
0 & \text{if branch } j \text{ is not incident with node } i,
\end{cases}
\]

for all nodes except for the reference one. In (28) this matrix is partitioned by columns according to the electrical nature of the corresponding branches.

Several additional remarks are in order. The size of the model (28) owes to its very general nature, which accommodates a great variety of devices, namely, capacitors, inductors, current-controlled and voltage-controlled resistors, \( q \)-memristors, \( \varphi \)-memristors, memcapacitors, meminductors, current-controlled and voltage-controlled hybrid memristors, and voltage and current sources. The subscripts corresponding to these devices are \( c, l, r, g, m, w, mc, ml, hm, hw, u \) and \( j \), respectively. For the sake of simplicity capacitors and inductors are assumed to be voltage- and current-controlled, respectively (cf. the maps \( \gamma_c \) and \( \gamma_l \) in (28n) and (28q)), and we eliminate the branch currents \( i_g, i_w \) of voltage-controlled resistors and \( \varphi \)-memristors by means of the maps \( \gamma_g \) and \( \zeta \), respectively. The map \( \gamma_r \) in (28s) defines the characteristic of current-controlled resistors, whereas \( i_s(t) \) and \( v_s(t) \) are the excitations in the current and voltage sources, respectively. The maps \( \eta, \omega, \theta, \psi, \xi \) are those arising in the characteristics (9b), (11c), (12c), (13c) and (14c), except for the fact that they account for the whole sets of \( q \)-memristors, memcapacitors, meminductors and hybrid memristors, and hence need not be scalar; the same holds, of course, for \( \varphi \)-memristors and the map \( \zeta \).

For later use, denote by \( R, G, C, L, M, W, C_m, L_m, M_h \) and \( W_h \) the incremental resistance, conductance, capacitance, inductance, memristance, memductance, memcapacitance, meminductance, hybrid memristance and hybrid memductance matrices, defined by the derivatives \( \gamma'_c, \gamma'_g, \gamma'_r, \gamma'_l, \eta_{im}, \zeta_{vw}, \omega_{vm_e}, \theta_{im_l}, \psi_{ihm}, \xi_{vw} \), respectively. These matrices need not be diagonal, meaning that full coupling is allowed within each of these sets of devices.

### 4.2 Topologically nondegenerate configurations and index one models

Semiexplicit differential-algebraic equations are defined by a system of the form

\[
\begin{align*}
x' &= f(x, y, t) \quad (29a) \\
0 &= g(x, y, t), \quad (29b)
\end{align*}
\]

where \( x \in \mathbb{R}^r \) denotes the differential or dynamic variables, \( y \in \mathbb{R}^p \) stands for the algebraic ones, \( f \in C^1(\mathbb{R}^{r+p+1}, \mathbb{R}^r) \), and \( g \in C^1(\mathbb{R}^{r+p+1}, \mathbb{R}^p) \). The DAE (29) is said to be index one around a given \((x^*, y^*, t^*)\) satisfying (29b) if the matrix of partial derivatives \( g_y(x^*, y^*, t^*) \) is invertible. Often, this non-singularity requirement holds everywhere. This definition applies
in the context of the geometric, differentiation and tractability indices, and for both analytical and numerical purposes it is important to characterize index one systems in practice; detailed discussions about the different index notions can be found in [4, 14, 17, 23, 37, 39].

The nodal model (28) has a semiexplicit form, the set of dynamic variables being

\[ q_c, q_{mc}, \varphi_l, \varphi_{ml}, q_m, q_{ml}, q_{hm}, q_{hw}, \varphi_w, \varphi_{mc}, \varphi_{hm}, \varphi_{hw}, \]

whereas the algebraic ones are

\[ e, i_c, i_{mc}, i_u, i_l, i_{ml}, i_r, i_m, i_{hm}, i_{hw}. \]

We address below the characterization of index one configurations for (28), under the assumption that certain circuit matrices are positive definite; recall that a given matrix \( K \) is positive definite if \( u^T K u > 0 \) for any non-vanishing real vector \( u \), and that this notion expresses mathematically a strict passivity requirement on the corresponding devices. The proof proceeds by showing how the non-singularity of the matrix defining index one configurations can be reduced to a form already analyzed in the context of nonlinear circuits without memristive devices in [39]. The result stated in Theorem 1 actually motivates the following definition.

**Definition 11.** A first order circuit is said to be topologically nondegenerate if it does not display either loops defined by voltage sources, capacitors and/or memcapacitors, or cutsets composed of current sources, inductors and/or meminductors.

This extends a well-known notion for RLC circuits, for which topologically nondegenerate configurations preclude loops defined by voltage sources and/or capacitors, and cutsets composed of current sources and/or inductors. Stemming from the work of Bashkow [1] in the classical RLC setting, this provides a way to formulate a state space model of the circuit dynamics by means on the notion of a proper tree. Note that in our present setting neither \( q \)- or \( \phi \)-memristors, nor hybrid ones, introduce topological degeneracies.

**Theorem 1.** Assume that the capacitance \( C \), the memcapacitance \( C_m \), the inductance \( L \) and the meminductance \( L_m \) are non-singular matrices, and that the resistance \( R \), the conductance \( G \), the memristances \( M \), \( M_h \) and the memductances \( W \), \( W_h \) are positive definite.

Then the model (28) is index one if and only if the circuit is topologically nondegenerate.

**Proof.** The matrix of partial derivatives of the right-hand side of (28) w.r.t. all variables but time, to be denoted by \( F \), has the form

\[ F = \begin{pmatrix} 0 & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \]

where the block \( F_{22} \) stands for the partial derivatives of the restrictions (28m)–(28v) with respect to the algebraic variables [31]. The non-singularity of \( F_{22} \) characterizes index one
configurations, and this matrix reads as

\[
F_{22} = \begin{pmatrix}
A_g A_g^T + A_w W A_w^T & A_c & A_{mc} & A_u & A_l & A_{ml} & A_r & A_m & A_{hm} & A_{hw} \\
-CA_c^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-C_m A_{mc}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-A_u^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -L & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_m & 0 & 0 & 0 & 0 & 0 \\
-A_r^T & 0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 \\
-A_m^T & 0 & 0 & 0 & 0 & 0 & 0 & M & 0 & 0 \\
-A_{hm}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_h & 0 \\
-W_h A_{hw}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{hw}
\end{pmatrix},
\]

the block \( I_{hw} \) being an identity matrix whose size is defined by the number of voltage-controlled hybrid memristors. The non-singularity of \( C, C_m, L, L_m \) obviously reduces the problem to the characterization of the non-singularity of

\[
\begin{pmatrix}
A_g A_g^T + A_w W A_w^T & A_c & A_{mc} & A_u & A_l & A_{ml} & A_r & A_m & A_{hm} & A_{hw} \\
-CA_c^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-C_m A_{mc}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-A_u^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -L & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_m & 0 & 0 & 0 & 0 & 0 \\
-A_r^T & 0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 \\
-A_m^T & 0 & 0 & 0 & 0 & 0 & 0 & M & 0 & 0 \\
-A_{hm}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_h & 0 \\
-W_h A_{hw}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{hw}
\end{pmatrix},
\]

and, by means of a Schur reduction \cite{18,39}, the non-singularity of this matrix amounts to that of

\[
\begin{pmatrix}
A_g A_g^T + A_w W A_w^T & A_c & A_{mc} & A_u & A_l & A_{ml} & A_r & A_m & A_{hm} & A_{hw} \\
-CA_c^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-C_m A_{mc}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-A_u^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -L & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_m & 0 & 0 & 0 & 0 & 0 \\
-A_r^T & 0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 \\
-A_m^T & 0 & 0 & 0 & 0 & 0 & 0 & M & 0 & 0 \\
-A_{hm}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_h & 0 \\
-W_h A_{hw}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{hw}
\end{pmatrix}
\]

This matrix has the structure arising in \cite{39} Theorem 5.1(1)] (cf. eq. (5.43) there); from this result it follows that, in the present setting, the non-singularity of this matrix relies on the absence of loops composed of voltage sources, capacitors and/or memcapacitors and cutsets defined by current sources, inductors and/or meminductors, as we aimed to show.

\[\square\]

### 4.3 Topological degeneracies: Index two

In presence of the topologically degenerate configurations discussed above (i.e. loops defined by voltage sources, capacitors and/or memcapacitors, or cutsets composed of current sources,
inductors and/or meminductors), the index one condition for the nodal system (28) fails. In this situation, for both analytical and numerical purposes it is important to characterize whether the model is index two or not. The index two notion for a DAE is more intricate than the index one concept introduced above. Again, the reader is referred to [4, 14, 17, 23, 37, 39] for different approaches to the index notion.

In particular, the *tractability index* notion, together with the projector-based framework supported on it [14, 24, 25, 26, 39, 42, 48], has been proved to be a valuable tool in circuit simulation [12, 15, 16, 27, 39, 47, 48]. In order to introduce this notion, we look at (28) as a semilinear problem of the form

\[ E z' = f(z, t), \]  

(33)

where \( E \) is a block-diagonal matrix \( \text{block-diag}\{I, 0\} \), and \( z \) joins together the variables denoted by \( x \) and \( y \) in (29). Consider the *matrix pencil* \( \lambda E - F [13] \), \( F \) being the matrix of partial derivatives \( f_z \). As detailed in the references above, the pencil is said to have tractability index one if \( E_1 = E - FQ \) is a non-singular matrix. In turn, if \( E_1 \) is singular, we let \( Q_1 \) be any projector onto ker \( E_1 \), and the pencil is said to have tractability index two if \( E_2 = E_1 - F_1Q_1 \) is non-singular, where \( F_1 = F(I - Q) \).

Iteratively, this approach provides a general index notion which can be shown to equal the Kronecker-Weierstrass (or nilpotency) index of the pencil, being well-suited for computational purposes. Moreover, this concept can be extended to nonlinear and/or time-varying settings under suitable assumptions on the system operators; restricting the attention to DAEs of the form (33) in an index two context, these assumptions amount to requiring that \( Q_1 \) be a continuous projector onto the kernel of \( E_1(z) \). Supported on these ideas, we show below that the nodal model (28) is indeed index two in the presence of degenerate configurations. Note that in the index two context the normal tree method of Bryant [5, 6] applies in order to derive a state space equation.

**Theorem 2.** Assume that the capacitance \( C \), the memcapacitance \( C_m \), the inductance \( L \), the meminductance \( L_m \), the resistance \( R \), the conductance \( G \), the memristances \( M, M_h \) and the memductances \( W, W_h \) are positive definite.

Then the nodal model (28) has tractability index two in the presence of topologically degenerate configurations.

**Proof.** As indicated above, the model (28) has the form depicted in (33) with \( E = \text{block-diag}\{I, 0\} \) and \( F \) being the matrix of partial derivatives displayed in (32). Letting \( Q \) be a projector onto the kernel of \( E \) with the structure block-diag\{0, I\}, we arrive at

\[ E_1 = \begin{pmatrix} I & -F_{12} \\ 0 & -F_{22} \end{pmatrix}. \]

Note, incidentally, that this makes it clear why the index one condition arising in Theorem 1 relies on the non-singularity of \( F_{22} \). As indicated in the proof of Theorem 1 the presence of loops defined by voltage sources, capacitors and/or memcapacitors, and/or cutsets composed
of current sources, inductors and/or meminductors makes $F_{22}$ (and hence $E_1$) a singular matrix.

In order to construct a projector $Q_1$ onto $\ker E_1$, we need to detail the structure of $F_{12}$ and $F_{21}$ in (32). These blocks are defined by the derivatives of (28a)–(28l) w.r.t. the algebraic variables (31) and the derivatives of the restrictions (28m)–(28v) w.r.t. the dynamic variables (30), respectively, and they read as

\[
F_{12} = \\
\begin{pmatrix}
0 & I_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{mc} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A^T_l & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A^T_{ml} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{ml} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{hm} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{hw} & 0 & 0 & 0 \\
A^T_w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A^T_{mc} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A^T_{hm} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A^T_{hw} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

and

\[
F_{21} = \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{mc} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_l & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{ml} & 0 & K_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & K_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & K_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & K_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_7 & 0 & 0 \\
\end{pmatrix}
\]

with $K_0 = A_w \partial \zeta / \partial \varphi_w$, $K_1 = -\partial \omega / \partial \varphi_{mc}$, $K_2 = -\partial \theta / \partial q_{ml}$, $K_3 = \partial \eta / \partial q_m$, $K_4 = \partial \psi / \partial q_{hm}$, $K_5 = \partial \psi / \partial \varphi_{hm}$, $K_6 = -\partial \xi / \partial q_{hw}$, $K_7 = -\partial \xi / \partial \varphi_{hw}$.

Now, the projector $Q_1$ may be chosen to have the structure

\[
Q_1 = \begin{pmatrix}
0 & Q_a \\
0 & Q_b \\
\end{pmatrix}
\]
where $Q_a$ and $Q_b$ are

$$
\begin{pmatrix}
0 & \hat{Q}_{11} & \hat{Q}_{12} & \hat{Q}_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \hat{Q}_{21} & \hat{Q}_{22} & \hat{Q}_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
A_I^T \hat{Q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{ml}^T \hat{Q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \hat{Q}_{11} & \hat{Q}_{12} & \hat{Q}_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \hat{Q}_{21} & \hat{Q}_{22} & \hat{Q}_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \hat{Q}_{31} & \hat{Q}_{32} & \hat{Q}_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\end{pmatrix}
$$

Here $\hat{Q}$ is a projector onto $\ker(A_c\ A_{mc}\ A_u\ A_g\ A_w\ A_r\ A_m\ A_{hm}\ A_{hw})^T$, being non-trivial in the presence of cutsets defined by inductors, meminductors and/or current sources, whereas

$$
\hat{Q} = \begin{pmatrix} \hat{Q}_{11} & \hat{Q}_{12} & \hat{Q}_{13} \\ \hat{Q}_{21} & \hat{Q}_{22} & \hat{Q}_{23} \\ \hat{Q}_{31} & \hat{Q}_{32} & \hat{Q}_{33} \end{pmatrix}
$$

is a projector onto $\ker(A_c\ A_{mc}\ A_u)$, which does not vanish in the presence of loops composed of capacitors, memcapacitors and/or voltage sources.

Additionally, the matrix $F_1 = F(I - Q)$ has the expression

$$
\begin{pmatrix} 0 & 0 \\ F_{21} & 0 \end{pmatrix}.
$$

This gives $E_2 = E_1 - F_1 Q_1$ the form

$$
E_2 = \begin{pmatrix} I & -F_{12} \\ 0 & -F_{22} - F_{21} Q_a \end{pmatrix}
$$

and, except for the $-$ sign, the lower-right block (which characterizes the non-singularity of $E_2$) reads

$$
F_{22} + F_{21} Q_a =
\begin{pmatrix}
A_g G A_g^T + A_w W A_w^T - C A_c^T & A_c & A_{mc} & A_u & A_l & A_{ml} & A_r & A_m & A_{hm} & A_{hw} \\
-C_m A_{mc}^T & \hat{Q}_{11} & \hat{Q}_{12} & \hat{Q}_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
-A_{r}^T & \hat{Q}_{21} & \hat{Q}_{22} & \hat{Q}_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{ml}^T \hat{Q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{ml}^T \hat{Q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-A_{r}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-A_{r}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-A_{r}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-A_{h}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-A_{h}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-W_h A_{hw}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
$$
Now, by means of a Schur reduction the non-singularity of this matrix is easily proved equivalent to that of

\[
\begin{pmatrix}
A_1 M_1 A_1^T + A_2 M_2 A_2^T \tilde{Q} & A_3 & A_4 \\
-M_3 A_3^T & \tilde{Q} & 0 \\
-A_4^T & 0 & 0
\end{pmatrix}
\]

with \(A_1 = (A_g \ A_w \ A_r \ A_h \ A_{hw})\), \(M_1 = \text{block-diag}(G, W, R^{-1}, M^{-1}, M_h^{-1}, W_h)\), \(A_2 = (A_l \ A_{ml})\), \(M_2 = \text{block-diag}(L^{-1}, L_m^{-1})\), \(A_3 = (A_c \ A_{mc})\), \(M_3 = \text{block-diag}(C, C_m)\), \(A_4 = A_u\), and

\[
\tilde{Q} = \begin{pmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{21} & \hat{Q}_{22} \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} \hat{Q}_{13} \\ \hat{Q}_{23} \end{pmatrix}.
\]

The form of this matrix amounts to that arising in the index-two analysis of Augmented Nodal Analysis models of nonlinear circuits without memristive devices, which is proved in [39, Theorem 5.1(2)] to be non-singular provided that \(M_1, M_2, M_3\) are positive definite. In this case, the definiteness of these matrices follows from the assumption that the circuit matrices are positive definite and the proof is complete.

In problems without memristive devices, Theorems 1 and 2 particularize to the results obtained in [12, 39, 47]. Our results also extend the characterization derived in [43] for cases including Chua’s memristors. The scope of the general index characterization here presented covers all types of devices with differential order one, and therefore applies also to circuits with \(q\)- and \(\varphi\)-memristors, voltage-controlled memcapacitors, current-controlled meminductors, and hybrid memristors. These results should be useful in future analytical or numerical studies of general first order circuits.

5 Concluding remarks

We have presented in this paper a comprehensive taxonomy of a variety of devices which have arisen in nonlinear circuit theory in the last few years. This taxonomy is organized around the notions of the differential and the state order of a device. An exhaustive list of possibly nonlinear devices with differential order one has been discussed: besides capacitors and inductors, these include \(q\)- and \(\varphi\)-memristors, memcapacitors, meminductors, and also the hybrid memristors here introduced. Hybrid memristors display a characteristic relating all four fundamental circuit variables, and account for devices in which memory effects of resistive, capacitive and inductive nature coexist. All these devices are discussed using fully nonlinear characteristics, and particularize to Chua’s memristors and to the memcapacitors and meminductors of Di Ventra et al., respectively, when the corresponding constitutive relations are linear in certain variables. A detailed analysis of the differential-algebraic index of circuits including all possible types of first order devices is also included.
Many aspects remain open and define lines for future investigation. These include numerical issues, modelling aspects involving e.g. branch-oriented systems and hybrid analysis, or dynamical properties related to the nature of these circuits’ operating points, their stability, bifurcations, as well as the eventual existence and characterization of periodic solutions, oscillations or chaotic effects. Higher order devices and mem-systems are also in the scope of future research.

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