Vulnerability and Hierarchy of Complex Networks

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We suggest an approach to study hierarchy, especially hidden one, of complex networks based on the analysis of their vulnerability. Two quantities are proposed as a measure of network hierarchy. The first one is the system vulnerability $V$. We show that being quite suitable for regular networks this characteristic does not allow one to estimate the hierarchy of large random networks. The second quantity is a relative variance $h$ of the system vulnerability that allows us to characterize a "natural" hierarchy level of random networks. We find that hierarchical properties of random networks depend crucially on a ratio $\delta$ between the number of nodes and the number of edges. We note that any graph with a transitive isometry group action (i.e. an absolutely symmetric graph) is not hierarchical. Breaking such a symmetry leads to appearance of hierarchy.

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Introduction. It is a traditional feeling that any practical complex system (network) bears, at some degree, a hierarchical property, which means that different parts (elements, clusters, etc.) of the system have different impact on the system performance. The very existence of hierarchy in the system can be both explicit and implicit which do not necessarily coincide; implicit hierarchy can even prevail over the explicit one, as is the case e.g. with "gray cardinal" who can be really much more powerful than the king. Intuitively, it seems that the higher degree (the number of connections to the others) a vertex has, the higher position it occupies in the system hierarchy. Such type of hierarchy we call explicit hierarchy. However, there are situations when vertices with maximal number of edges are not necessarily most vital for the system performance. For instance, all vertices in a binary tree have equal degree, while there is strong hierarchy, i.e. the importance of a particular vertex is dictated by the level it seats on: the vertices which are closer to a root are more important than those lying far from the root. Such type of hierarchy we will refer to as a hidden hierarchy. In this letter we suggest a quantitative way to recover a system hierarchy, especially implicit one, using the system vulnerability properties.

The idea to relate the hierarchy and the vulnerability of the system was inspired by a very simple reason that the more damage can be caused by removal of a particular vertex, the higher position in hidden hierarchy of the system this vertex occupies, and vice versa. As a rough hierarchy measure we suggest a pointwise version of the system vulnerability introduced by Latora and Marchiori, which is quite suitable for the hierarchy characterization of regular systems. We will compare and classify various regular networks with respect to their hierarchy. As it will be demonstrated, the resulting hierarchical properties can be quite different from an intuitive hierarchy. For randomized networks the above described approach turned out to be ineffective, therefore in order to quantify the hierarchy of random networks we use statistical properties of the vulnerability which are quite sensitive to the degree of hierarchy. It will be shown that there exists a "natural" level of hierarchy in randomized networks depending upon the ratio between the number of vertices and the number of edges in the system.

We define a pointwise vulnerability $V(i)$ of the network as relative drop in performance after removal of $i$-th vertex together with all edges connecting it with other vertices, namely

$$V(i) = \frac{E - E(i)}{E}. \quad (1)$$

Here $E = \frac{1}{N(N-1)} \sum_{i \neq j} d_{ij}$ is the global efficiency of the network, $N$ is the total number of vertices in the network, $d_{ij}$ is the minimal distance (either weighted or unweighted) between the $i$-th and $j$-th vertices, and $E(i)$ is the network efficiency after removal of $i$-th vertex and all its edges. Maximal value $V$ of $V(i)$ corresponds to the network vulnerability introduced by Latora and Marchiori. As it was mentioned above, we suggest to classify vertices of the network by the level of their vulnerability $V(i)$. This seems to be a natural way to introduce an hierarchy in any network by relating it to an ordered distribution of vertices with respect to their vulnerability $V(i)$. The most vulnerable vertex occupies the highest position in the system hierarchy.

To illustrate our approach we have calculated the vulnerability of several typical topologically different kinds of networks: a tree-like, torus-like, ring-like, “bush”-like, and “spider”-like networks, which are schematically shown in Fig. We the resulting vulnerability distributions $\rho$ are presented in Fig.2.

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Two types of the considered systems, the torus-like and the ring-like, are not hierarchical, which is caused by existence of transitive action of an isometry group on these networks. In other words, any vertex can be moved to any position by a suitable one-to-one isometrical transformation. As a consequence, all vertices in these networks have the same level of vulnerability, i.e. $V(i) = \text{const}$. We will call any network with a transitive action of isometry group an absolutely symmetric network. The vulnerability distributions in such networks degenerate into a single point as can be seen in Fig.2 for both the ring and the torus-like cases. Obviously, these systems are not hierarchical as all their vertices have equal impact on overall performance.

From this point of view absolute symmetry and hierarchy are mutually excluding properties.

In contrast to absolutely symmetric networks, the spider-like networks have as strong as possible hierarchy, as the removal of a central vertex gives rise to a complete destruction of the network. Quantitatively it means that the vulnerability $V$ of such system equals to $V = 1$, while the second point is related to all other vertices with small value of $V_i$. Notice that the symmetry of a spider-like system is broken only in a single point, the central one. This prompts us to conclude that the level of symmetry is not directly related to the level of hierarchy. Indeed, the spider-like network is both quite symmetric and highly hierarchical, but it is not absolutely symmetric. It seems that interrelation between symmetry and hierarchy is not so simple. The last two examples shown in Fig.2, the tree-like and the bush-like ones, demonstrate another possible type of hierarchy.

Our concept of hierarchy essentially differs from that proposed by Trusina et al \cite{3}. According to their definition the bush-like network (Fig.1d) is maximally hierarchical, while the tree-like one (Fig.1e) is maximally antihierarchical. However as one can see on Fig.2 both these kinds of networks have quite similar hierarchical properties. Moreover, intuitively the tree-like networks represent typically hierarchical systems.

It is interesting to follow the dependence of vulnerability upon the network size (graph order). Such dependencies for regular networks are shown in Fig.3 for the spider, an ideal network (in which all pairs of vertices are connected) and a binary tree. The first two exhibit opposite behavior. The spider-like graph has maximal vulnerability $V = 1$ at any size. On the other hand, the vulnerability of the ideal network tends to zero as the number of vertices grows. The vulnerability of any other network is positioned between the vulnerability values of the spider-like and of the ideal networks of the same size, because any graph is a subgraph of the complete one. This fact is illustrated by the example of binary tree (see Fig.3).

So far we have considered non-random models of networks. Now we will turn to randomized versions of networks and look what happens with the vulnerability $V$ under randomization of the system. To randomize a network we use the standard procedure of rewiring randomly chosen pairs of vertices \cite{4} (note that (i) even a subtle difference in randomization procedure may result in essentially different system behavior, as discussed in \cite{3}; (ii) our procedure is not restricted by quite strong condition of preserving the degree of every individual vertex \cite{5}). At the next step the vulnerability distribution...
The only difference is that now the vulnerability $\rho$ is calculated for every single realization, afterwards a distribution $\langle \rho \rangle$ of the random network is obtained by averaging $\rho$ over all statistical realizations. Now the vulnerability $V$ of the randomized network can be introduced in the same way as it has been done earlier for regular systems. The only difference is that now $V$ is defined from the distribution $\langle \rho \rangle$ rather than from $\rho$. As an example the distribution $\langle \rho \rangle$ for randomized spider-like graph is presented in Fig. 5. This distribution has two essential differences from those for regular networks (compare with Fig. 3): (i) all vulnerability values $V(i)$ in contrast to the regular graphs where very large groups of vertices may have the same value of $V(i)$. The vulnerabilities of totally randomized counterparts of the spider-like, the torus-like and the ideal networks are plotted in Fig. 6 as functions of the system size. Surprisingly, all three curves tend to zero with system size growing to infinity, without any distinction between so different kinds of network. Thus the asymptotic behavior of regular structures is entirely different than that of their totally randomized counterparts. To follow this drastic change we have calculated the vulnerability $V$ of the same networks gradually increasing the degree of randomization $p$, i.e. the fraction of randomly rewired pairs of vertices from zero to one. As one can see from Fig. 6 the impact of randomization on the vulnerability depends crucially on the initial network structure. Being essentially different at no randomization limit $p = 0$ which corresponds to the regular case (see Fig. 3), all three types of networks become indistinguishable at the totally random limit $p = 1$. We conclude that the vulnerability $V(i)$ is not a suitable structural characteristic of random networks because from this point of view large random networks are not hierarchical at all, which seems to be doubtful.

As it was already mentioned, the value $\langle V \rangle$ of the mean vulnerability may also be used to characterize the network hierarchy level. However this quantity also tends to zero with decrease of the system size, as well as the vulnerability $V$. Generally, both the mean and the maximal parameter values are very rough characteristics of large random systems. For example, the mean ocean level is constant. From this point of view the ocean is rather homogeneous system without any hierarchy. However this is not correct as there are always highly hierarchical storm regions. Another example is the mean temperature of patients in a hospital that tells nothing about the local situations. To obtain more detailed description one has to use other parameters which are more sensitive to deviations from the mean value, as it is usual for statistics. As such additional parameter we introduce a relative variance $h$ of the pointwise vulnerability $V_i$ as a measure of a random network hierarchy:

$$h = \frac{\langle \Delta V^2 \rangle}{\langle V \rangle^2},$$

where $V$ stands for the vulnerability of a single statistical realization of the network, $\langle \Delta V^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} (V(i) - \langle V \rangle)^2$, and $\langle V \rangle = \frac{1}{N} \sum_{i=1}^{N} V_i$ is the mean pointwise vulnerability. The parameter $h$ is a measure of the fluctuation level and, as will be seen, can be used to describe the hierarchical properties of both regular and random networks. This means that in the case of ocean mentioned above wave intensities should be analyzed instead of wave amplitudes in order to find the hierarchical structure.
The dependence of $h$ upon the system size for the randomized versions of the spider-like, the torus-like, and the ideal-like types of network is presented in Fig. 4. In contrast to asymptotic behavior of the vulnerability $V$ that tends to zero at large size limit for all three networks, the asymptotic behavior of the relative variance $h$ exhibits a crucial difference for different types of networks. For the ideal network $h \to 0$ which means that the level of fluctuations remains extremely low despite the total randomization of the network. The situation changes drastically for the spider-like system where the relative variance $h$ grows with the system size and becomes even bigger than one for large enough systems, i.e. the fluctuations are huge in spider-like systems. Again we have two extreme situations, namely $h \to 0$ for the ideal and $h \sim 1$ or bigger for spider-like random networks (compare with Fig. 3). The value of $h$ for the randomized torus is between these two extreme values.

What makes totally random networks of the same size $N$ so different? What is their “memory” about their non-randomized parents? We believe that the role of such a relict factor is played by the relative number of connections (edges), therefore to classify different types of random networks according to asymptotic behavior of their fluctuations $h$ we introduce the ratio $\delta$ of the number of vertices to the number of edges. This quantity equals to $N/(N-1)$ for the tree and the spider-like graphs and $\delta = 2/(N-1)$ for the ideal network. It should be noted that the randomization procedure used here strictly preserves $\delta$. As it can be seen in Fig. 7 the level of fluctuations characterized by the value of $h$ increases with $\delta$, which is clear since the larger $\delta$ the lesser the number of edges and so there are more options for randomization. Thus, the relative variance $h(\delta)$ introduced above can be used as the hierarchy measure for random networks.

Summary and discussion. We have introduced the vulnerability distributions for complex networks and used them for studying the network hierarchical structure. Such distributions are essentially different for different types of networks and can be quite complicated for rigorous analysis. To estimate the hierarchy level of networks, both regular and random, we have used the following global characteristics: the network vulnerability $V$ and the relative variance $h$ of the pointwise vulnerability $V(i)$. It has been demonstrated that the relative variance is a more delicate hierarchy characteristic than the vulnerability, especially for random networks in which the hierarchy level $h$ depends crucially upon the ratio $\delta$ of the number of vertices to the number of edges. Any random network has its natural hierarchy level $h(\delta)$ which is minimal for the ideal random network with $\delta \to 0$ and is maximal for the spider-like random network with $\delta \sim 1$ (strictly speaking, one can imagine networks with the number of edges lesser than that in the spider-like graph, i.e. $\delta > 1$, but such networks are disconnected, and this situation is beyond the scope of this letter). This prompts us to make a more general conclusion that stochasticity itself does not exclude the presence of hierarchy, and any randomized system can have some local islands with quite hierarchical structures. Complexity of such hierarchy islands is higher for systems with restricted number of connections.

Our approach is fully applicable when a link hierarchy is of interest rather than hierarchy of vertices. Then the quantity $E(i)$ in eq. (1) would mean the network efficiency after removal of $i$-th link, hence eq. (1) will define a linkwise vulnerability $h$. Hierarchical structures based on pointwise and linkwise vulnerabilities of the same network can look entirely different. The obvious example is the spider-like network where the pointwise based approach results in strongest possible hierarchy ($h = 1$), whereas the linkwise vulnerability tends to zero in large size limit, meaning that the there is no hierarchy of links whatsoever.

Although all systems considered here are idealized mathematical models, both regular and random ones, our approach is applicable for analysis of the vulnerability $V$ and the degree of hierarchy $h$ of any real network. Finally, apart from $V$ some other hierarchy measures based upon vulnerability distributions may be used, such as mean vulnerability, the number of vulnerability levels, etc.

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