An investigation of the relationship between extrapolation parameters and approximate solutions in solving fuzzy linear systems using refinement of Jacobi over relaxation method

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Abstract. Extrapolation parameters in Refinement of Jacobi over Relation method in solving fuzzy linear systems were hypothesized to influence the rate of convergence. This paper aims to investigate the relationship between extrapolation parameters and approximate solutions of fuzzy linear systems when solved using Refinement of Jacobi over Relaxation method. The Algorithms of Refinement of Jacobi over Relaxation method are proposed based on the Jacobi method that specifically used in solving the system. A five by five fuzzy linear system is given to investigate the convergence to exact solution with three different values of extrapolation parameters. The numerical results show that there is a positive correlation between extrapolation parameter and convergence to exact solution. The three extrapolation parameters suggest that convergence to exact solution can be increased in line with the increase in the values of extrapolation parameters.

1. Introduction
One of the most important discussions in applied mathematics is system of linear equations. The systems are germane for modelling and solving in many branches of knowledge areas such as physics, engineering, health sciences, economics, finance and even social sciences. In most of the applications, parameters of the system are represented by crisp. However, there are cases where parameters represented by fuzzy numbers. This situation is true especially when estimation of the system coefficients is imprecise and only some vague knowledge about the actual value of the parameters is available. It has been suggested that the system may be convenient to represent some or all of them with fuzzy numbers. The concept of fuzzy numbers and arithmetic operations with these numbers were first introduced and investigated by Zimmermann [11]. Fuzzy numbers and its arithmetic operations can be used to generalize crisp mathematical concept to fuzzy sets [6].

Fuzzy numbers were further developed in 1990s when Buckley and Qu [3] and Buckley [4] investigated the system and subsequently Buckley [5] proposed a general model for solving a fuzzy n×n linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy n×n linear system by a crisp 2n×2n linear system. Years after that, there were many researchers proposed various methods to solve fuzzy linear system of equations. For example, Friedman et al. [9] developed a method for solving a fuzzy linear system, while Asady et al. [2] solved rectangular fuzzy linear system of equations based on matrix decomposition method. They dealt with Householder process to obtain a QR-decomposition for the coefficient matrix of the extended linear system of the fuzzy linear system equation. Another decomposition method that is Adomian decomposition method was used by Matinfar et al. [10] to solve fuzzy linear equation. Hence it is immensely important to
further develop numerical procedures that would appropriately treat general fuzzy linear systems and solve them.

Another class of methods for solving fuzzy linear systems is iterative methods. The Jacobi method for solving fuzzy linear system was introduced by Allahviranloo [1] for the first time. Dafchahi [7] builds a refinement to the Jacobi method and proposed the Refinement of Jacobi (RJ) method for solution of linear system equations. This method is presented to increase the rate of convergence of Jacobi. Besides the Jacobi method, there are some discussions about others iterative methods. They are the Jacobi over Relaxation (JOR) and the Successive over Relaxation (SOR) methods [8]. What is uniquely in the JOR and the SOR methods is that, there is an extrapolation parameter ($\omega$) in their formulas. The JOR and SOR are the two most popular methods that used an extrapolation parameter. Extrapolation parameter used in both methods is presented by $\omega$ where $0 \leq \omega \leq 1$. Note that the value of extrapolation parameter influences the convergence of the system. Therefore, it motivates to extend the RJ method to be the Refinement of Jacobi over Relaxation (RJOR) method. The extension is done by adding an extrapolation parameter into the formula of RJ method. However, the convergence of the system depends on the extrapolation parameter. It was hypothesized that when smaller value of $\omega$ is used, the system converges very slowly. Therefore, this paper investigates the relationship between extrapolation parameters and convergence to the exact solution when fuzzy linear system is solved using Refinement of Jacobi over Relaxation method.

2. Preliminaries
In this section, some necessary backgrounds and notions of fuzzy linear systems, Jacobi over Relaxation, Successive over Relaxation, and Refinement of Jacobi method are reviewed.

Definition 2.1: Fuzzy linear systems
Friedman et al. [9] have provided definitions of fuzzy linear systems.

\[
\begin{align*}
a_1x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= y_1 \\
a_2x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= y_2 \\
& \quad \vdots \\
a_nx_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= y_n
\end{align*}
\]

The $n \times n$ matrix form of the above equations is $AX = Y$, where the coefficient matrix $A = (a_{ij})$ for $1 \leq i, j \leq n$ is a crisp $n \times n$ matrix and $y_i \in F^1, 1 \leq i \leq n$. This system is called a fuzzy linear system (FLS).

Definition 2.2: Jacobi over relaxation (JOR) method
Based on Dehgan and Hashemi [8], Jacobi over Relaxation method is an extrapolated of Jacobi method.

\[
\begin{bmatrix}
X \\
\hat{X}
\end{bmatrix} = \begin{bmatrix}
\omega D_1^{-1}Y + \left[I_n - \omega D_1^{-1}B\right]X + \omega D_1^{-1}CX \\
\omega D_1^{-1}\hat{Y} + \left[I_n - \omega D_1^{-1}B\right]\hat{X} + \omega D_1^{-1}C\hat{X}
\end{bmatrix}
\]

So the JOR iterative method will be

\[
\begin{align*}
X^{(m+1)} &= \omega D_1^{-1}Y - \omega D_1^{-1}\left[1 - \frac{1}{\omega}\right]D_1 + L_1 + U_1 \frac{X^{(m)}}{D_1} + \omega D_1^{-1}CX^{(m)} \\
\hat{X}^{(m+1)} &= \omega D_1^{-1}\hat{Y} - \omega D_1^{-1}\left[1 - \frac{1}{\omega}\right]D_1 + L_1 + U_1 \frac{X^{(m)}}{D_1} + \omega D_1^{-1}C\hat{X}^{(m)}
\end{align*}
\]

Hence we have the JOR method in matrix form:
\[ X^{(m+1)} = D^{-1}(D - \omega S)X^{(m)} + \omega D^{-1}Y \] (2.2)

where D is a diagonal matrix, S is a singular matrix and \( \omega \) is an extrapolation parameter.

**Definition 2.3 Successive over relaxation (SOR) method**

There are forward and backward Successive over Relaxation method [8].

The forward SOR iterative method will be

\[
\begin{align*}
X^{(m+1)} &= \omega(D_1 + \omega L_1)^{-1}Y + (D_1 + \omega L_1)^{-1}[(1 - \omega)D_1 - \omega U_1]X^{(m)} - \omega(D_1 + \omega L_1)^{-1}C X^{(m)} \\
\end{align*}
\]

(2.3)

Similarly, the backward SOR iterative method will be

\[
\begin{align*}
X^{(m+1)} &= \omega(D_1 + \omega U_1)^{-1}Y + (D_1 + \omega U_1)^{-1}[(1 - \omega)D_1 - \omega U_1]X^{(m)} - \omega(D_1 + \omega U_1)^{-1}C X^{(m)} \\
\end{align*}
\]

(2.4)

Based on the above equations, D is diagonal for the system, L is strict lower part of the system, U is strict upper part of the systems, C is absolute value of the negative entries of matrix A and X are the variables of the system, where they are lower variables and upper variables.

**Definition 2.4: Refinement of Jacobi method**

Based on the previous study by Buckley [4], stated that Refinement of Jacobi method is an extended of Jacobi method.

\[
AX = b \\
(L + D + U)X = b \\
DX = -(L + U)X + b \\
DX = (D - A)X + b \\
DX = DX + (b - AX) \\
X = X + D^{-1}(b - AX) \\
\]

From above form, the iterative refinement of formula in matrix form is obtained.

\[
X^{(k+1)} - X^{(k)} = D^{-1}(b - AX^{(k+1)}) \\
\]

(2.5)

From Jacobi formula, Refinement of Jacobi method in matrix form is proposed as follows:

\[
X^{(k+1)} = (D^{-1}(L + U))^2 X^{(k)} + (I - D^{-1}(L + U))D^{-1}b \\
\]

(2.6)
Hence, formula (2.6) can be used to solve fuzzy linear systems. Based on the formula, D is diagonal for the system, L is strict lower part of the system, U is strict upper part of the systems, b is fuzzy vector for the system which including fuzzy lower bound and fuzzy upper bound, and X are the variables of the system, where they are also lower variables and upper variables.

### 3. Refinement of Jacobi over relaxation method and extrapolation parameters

In this study, Refinement of Jacobi over Relaxation method is proposed to solve fuzzy linear systems. This method is chosen because Dehgan and Hashemi [8] and Allahviranloo [1] find out that Jacobi over Relaxation and Successive over Relaxation method are iterative methods in fuzzy linear systems. This motivates us to make an extension of Refinement of Jacobi method to Refinement of Jacobi over Relaxation method as an iterative method. Previous studies by Allahviranloo [1] have shown that Successive over Relaxation method is an extension of Gauss Seidel method. Similarly, Jacobi over Relaxation method is an extension of Jacobi method.

Based on the formula (2.2), there is an extrapolation parameter in formula of Jacobi over Relaxation method. Therefore, Refinement of Jacobi method is extending by adding an extrapolation parameter in the formula (2.6). Hence, we have the Refinement of Jacobi over Relaxation method in matrix form:

$$X^{(k+1)} = (D^{-1}(L + \omega U))X^{(k)} + (I - D^{-1}(L + U))\omega D^{-1}b$$

The above formula also can be written as

$$X^{(k+1)} = (D^{-1}(L + \omega U))X^{(k)} + (I - D^{-1}(L + \omega U))\omega D^{-1}b$$

It is interesting to note that when the both formulas are used to solve fuzzy linear systems, the solution is same even though there are only two extrapolation parameters in formula (3.1) while three extrapolation parameters in formula (3.2). A possible explanation for this might be related to $\omega = 1$. However in this study, three different extrapolation parameters will be used in Refinement of Jacobi over Relaxation method to see how of convergence effects in the systems when $0 \leq \omega \leq 1$. Buckley [5] uses $\omega = 0.87$ for SOR method and $\omega = 0.90$ for JOR method in his numerical examples. Both extrapolation parameters are close to $\omega = 1.0$. So, in order to see how convergence of systems when $0 \leq \omega \leq 1$ are used, we choose a middle point between 0 and 1, and two more points are at two ends between 0 and 1. Therefore, three extrapolation parameters will be used are $\omega = 0.25, 0.5, \text{and} 0.9$.

In order to understand clearly how to get the solution fuzzy linear system by Refinement of Jacobi over Relaxation method, an algorithm is summarized as follows:

**Step 1:** Determine $L$, $U$, $D^{-1}$ and $b$ of the systems.

**Step 2:** Multiply extrapolation parameter $\omega$ with $U$.

**Step 3:** Sum up $L$ and $\omega U$ and multiply them with the inverse of $D$.

**Step 4:** Evaluate $(L + \omega U)D^{-1}$.

**Step 5:** Identify the number of variables $(X)$ of the systems. Make sure they are including lower bound and upper bound variables.

**Step 6:** Multiply the variables with the $(L + \omega U)D^{-1}$.

**Step 7:** Identify an identity matrix for the systems and then subtract it to $(L + U)D^{-1}$.

**Step 8:** Multiply $b$ with the inverse of diagonal matrix of the systems and extrapolation parameter.

**Step 9:** Multiply $\omega D^{-1}b$ with the $I - (L + U)D^{-1}$.

**Step 10:** Substitute all the terms to the formula (3.1).
Step 11: When \( \frac{x_i^{(k+1)}}{x_i^{(k)}} < \varepsilon \), stop the calculation. Note that, the decision to stop the iterations were based on the criterion.

4. Numerical applications

To illustrate the proposed method and demonstrate the relationship, the \( 5 \times 5 \) system retrieved from Dehgan and Hashemi [8] is presented.

**Example**

Consider the \( 5 \times 5 \) fuzzy system

\[
\begin{align*}
8x_1 + 2x_2 + x_3 - 3x_5 &= (r, 2 - r) \\
-2x_1 + 5x_2 + x_3 - x_4 + x_5 &= (4 + r, 7 - 2r) \\
x_1 - x_2 + 5x_3 + x_4 + x_5 &= (1 + 2r, 6 - 3r) \\
x_3 + 4x_4 + 2x_5 &= (1 + r, 3 - r) \\
x_1 - 2x_2 + 3x_5 &= (3r, 6 - 3r)
\end{align*}
\]

The proposed algorithm (refer to Section 3) is implemented to solve the system. Three extrapolation parameters \( \omega = 0.25, 0.5, \) and 0.9 are tested. The solutions for each extrapolation parameters are represented graphically in triangular fuzzy numbers.

The exact solution for the above system is given as

\[
\begin{align*}
x_1 &= (0.729-0.331*r, 0.044+0.354*r) \\
x_2 &= (0.614+0.166*r, 1.077-0.297*r) \\
x_3 &= (0.126+0.291*r, 0.918-0.501*r) \\
x_4 &= (0.242-0.331*r, -0.416+0.326*r) \\
x_5 &= (0.475+0.912*r, 2.395-1.007*r)
\end{align*}
\]

The exact solutions can also be depicted graphically. It is shown in Figure 1.

![Figure 1: Exact solution in triangular fuzzy numbers.](image-url)
The tests are extended to obtain approximation solutions using the algorithm for three different extrapolation parameters. The solutions are summarized in the Table 1-3.

**Table 1:** Approximation solutions and triangular fuzzy numbers for $\omega = 0.25$

| Extrapolation parameter | Solutions |
|-------------------------|-----------|
| $\omega = 0.25$         | $x_1 = (0.153 - 0.095^* r, -0.074 + 0.121^* r)$  
                        | $x_2 = (0.260 - 0.039^* r, 0.200 - 0.030^* r)$  
                        | $x_3 = (0.105 + 0.013^* r, 0.163 - 0.081^* r)$  
                        | $x_4 = (0.142 - 0.105^* r, -0.085 + 0.100^* r)$  
                        | $x_5 = (0.231 + 0.172^* r, 0.571 - 0.223^* r)$ |

**Table 2:** Approximation solutions and triangular fuzzy numbers for $\omega = 0.5$

| Extrapolation parameter | Solutions |
|-------------------------|-----------|
| $\omega = 0.5$         | $x_1 = (0.355 - 0.203^* r, -0.128 + 0.245^* r)$  
                        | $x_2 = (0.509 - 0.057^* r, 0.403 - 0.027^* r)$  
                        | $x_3 = (0.199 + 0.037^* r, 0.324 - 0.165^* r)$  
                        | $x_4 = (0.276 - 0.219^* r, -0.221 + 0.211^* r)$  
                        | $x_5 = (0.440 + 0.350^* r, 1.119 - 0.440^* r)$ |
Table 3: Approximation solutions and triangular fuzzy numbers for $\omega = 0.9$

| Extrapolation parameter | Solutions |
|-------------------------|-----------|
| $\omega = 0.9$          | $x_1 = (0.719 - 0.354^* r, -0.079 + 0.384^* r)$  |
|                        | $x_2 = (0.717 + 0.044^* r, 0.842 - 0.175^* r)$  |
|                        | $x_3 = (0.232 + 0.175^* r, 0.707 - 0.377^* r)$  |
|                        | $x_4 = (0.346 - 0.362^* r, -0.447 + 0.353^* r)$ |
|                        | $x_5 = (0.588 + 0.729^* r, 2.050 - 0.835^* r)$  |

From these results, it can be seen that there are relationships between an exact solution and the approximation solutions when three different extrapolation parameters are used. When extrapolation parameter used is small, the rate of convergence is very low. On the other hand, when bigger extrapolation parameter is used, the rate of convergence is increased. The system seems converged properly to the exact solution as the values of extrapolation parameters increases. The approximate solutions indicate the importance of extrapolation parameters in solving the system using Refinement of Jacobi over Relaxation. The exact solution possible is achieved when extrapolation parameter is 1. This investigation concludes that the rate of convergence is influenced by the values of extrapolation parameters in solving fuzzy linear systems with Refinement of Jacobi over Relaxation method.

5. Conclusions
This paper was presented to investigate the effect of extrapolation parameters to convergence rate when the fuzzy linear system is solved using Refinement of Jacobi over Relaxation method. Using the numerical applications, it can be concluded that there was a relationship between extrapolation parameter and approximate solutions. Convergence to the exact solution seems to be improved by increasing the values of extrapolation parameters. It would suggest that there is inefficiently to choose other extrapolation parameters other than $\omega = 1$. The most probable explanation is the Refinement of Jacobi over Relaxation method is reduced to Refinement of Jacobi method when $\omega = 1$. Therefore, fuzzy linear systems may be solved by this proposed method and converge to the exact solution when the appropriate value of extrapolation parameter is selected.
6. References

[1] Allahviranloo T 2005 The Adomian decomposition method for fuzzy system of linear equations Appl. Math. Comput. 163, pp. 553–63
[2] Asady B, Abbassbandy S and, Alavi M 2005 Fuzzy general linear systems Appl. Math. and Comput. 169 34-40
[3] Buckley J J and Qu Y 1991 Solving systems of linear fuzzy equations Fuzzy Sets and Systems 43 33–43
[4] Buckley J J 1992 Solving fuzzy equations in economics and finance Fuzzy Sets and Systems 48 289–96
[5] Buckley J J 1992 Solving fuzzy equations Fuzzy Sets and Systems 50 1–14
[6] Chang S L and Zadeh L A 1972 On fuzzy mapping and control IEEE Trans., Syst. Man Cyb. 2
[7] Dafchahi F N 2008 A new refinement of Jacobi method for solution of linear system equations AX = b Int.J.Contemp.Math.Sciences 3 17 819-27
[8] Dehgan M. and Hashemi B 2006 Iterative solution of fuzzy linear systems, Applied Mathematics and Computation 175, 645-74
[9] Friedman M, Ming M and, Kandel A 1998 Fuzzy linear systems Fuzzy Sets and Systems 96, 201-09
[10] Matinfar M, Nasseri S H and Sohrabi M 2008 Solving fuzzy linear system of equations by using householder decomposition method Appl. Math. Sci. 2 52 2569-75
[11] Zimmermann H J 1996 The Fuzzy Set Theory and its Applications (Kluwer, Dordrecht)