DEFINABILITY OF A VARIETY
GENERATED BY A COMMUTATIVE MONOID
IN THE LATTICE OF COMMUTATIVE
SEMIGROUP VARIETIES

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Abstract. Let $M$ be a commutative monoid. We construct a first-order formula that defines the variety generated by $M$ in the lattice of all commutative semigroup varieties.

A subset $A$ of a lattice $\langle L; \lor, \land \rangle$ is called definable in $L$ if there exists a first-order formula $\Phi(x)$ with one free variable $x$ in the language of lattice operations $\lor$ and $\land$ which defines $A$ in $L$. This means that, for an element $a \in L$, the sentence $\Phi(a)$ is true if and only if $a \in A$. If $A$ consists of a single element, we speak about definability of this element.

We denote the lattice of all commutative semigroup varieties by $\text{Com}$. A set of commutative semigroup varieties $X$ (or a single commutative semigroup variety $X$) is said to be definable if it is definable in $\text{Com}$. In this situation we will say that the corresponding first-order formula defines the set $X$ or the variety $X$.

Let $M$ be a commutative monoid. In [10, Corollary 4.8], we provide an explicit first-order formula that defines the variety generated by $M$ in the lattice of all semigroup varieties. The objective of this note is to modify the arguments from [10] in order to present an explicit formula that defines the variety generated by $M$ in the lattice $\text{Com}$.

We will denote the conjunction by $\&$ rather than $\land$ because the latter symbol stands for the meet in a lattice. Since the disjunction and the join in a lattice are denoted usually by the same symbol $\lor$, we use this symbol for the join and denote the disjunction by $\lor$. Evidently, the relations $\leq$, $\geq$, $<$ and $>$ in a lattice $L$ can be expressed in terms of, say, meet operation $\land$ in $L$. So, we will freely use these four relations in formulas. Let $\Phi(x)$ be a first-order formula. For the sake of brevity, we put

$$\min_x \{ \Phi(x) \} \equiv \Phi(x) \& (\forall y) \left( y < x \rightarrow \neg \Phi(y) \right).$$

Clearly, the formula $\min_x \{ \Phi(x) \}$ defines the set of all minimal elements of the set defined by the formula $\Phi(x)$.

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Many important sets of semigroup varieties admit a characterization in the language of atoms of the lattice $\text{Com}$. The set of all atoms of a lattice $L$ with $0$ is defined by the formula

$$A(x) \equiv (\exists y) \left( (\forall z) (y \leq z) \& \min_x \{x \neq y\}\right).$$

A description of all atoms of the lattice $\text{Com}$ directly follows from the well-known description of atoms of the lattice of all semigroup varieties (see [2, 8], for instance). To list these varieties, we need some notation.

By $\text{var } \Sigma$ we denote the semigroup variety given by the identity system $\Sigma$. A pair of identities $wx = xw = w$ where the letter $x$ does not occur in the word $w$ is usually written as the symbolic identity $w = 0_1$. Let us fix notation for several semigroup varieties:

- $A_n = \text{var } \{x^n y = y, xy = yx\}$ — the variety of Abelian groups whose exponent divides $n$,
- $SL = \text{var } \{x^2 = x, xy = yx\}$ — the variety of semilattices,
- $ZM = \text{var } \{xy = 0\}$ — the variety of null semigroups.

**Lemma 1.** The varieties $A_p$ (where $p$ is a prime number), $SL$, $ZM$ and only they are atoms of the lattice $\text{Com}$. □

Put

$$\text{Neut}(x) \equiv (\forall y, z) \left( (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)\right).$$

An element $x$ of a lattice $L$ such that the sentence $\text{Neut}(x)$ is true is called neutral. We denote by $T$ the trivial semigroup variety.

**Lemma 2** ([6, Theorem 1.2]). A commutative semigroup variety $V$ is a neutral element of the lattice $\text{Com}$ if and only if either $V = COM$ or $V = M \vee N$ where $M$ is one of the varieties $T$ or $SL$, while the variety $N$ satisfies the identity $x^2 y = 0$. □

For convenience of references, we formulate the following immediate consequence of Lemmas 1 and 2.

**Corollary 3.** An atom of the lattice $\text{Com}$ is a neutral element of this lattice if and only if it coincides with one of the varieties $SL$ or $ZM$. □

A semigroup variety $V$ is called chain if the subvariety lattice of $V$ is a chain. Clearly, each atom of $\text{Com}$ is a chain variety. The set of all chain varieties is definable by the formula

$$\text{Ch}(x) \equiv (\forall y, z) (y \leq x \& z \leq x \rightarrow y \leq z \lor z \leq y).$$

We adopt the usual agreement that an adjective indicating a property shared by all semigroups of a given variety is applied to the variety itself; the expressions like “group variety”, “periodic variety”, “nil-variety” etc. are understood in this sense.

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1This notation is justified because a semigroup with such identities has a zero element and all values of the word $w$ in this semigroup are equal to zero.
Put
\[ N_k = \text{var} \{ x^2 = x_1 x_2 \cdots x_k = 0, x y = y x \} \] (k is a natural number),
\[ N_\omega = \text{var} \{ x^2 = 0, x y = y x \}, \]
\[ N_3^c = \text{var} \{ x y z = 0, x y = y x \} \]
(in particular \( N_1 = \mathcal{T} \) and \( N_2 = \mathcal{ZM} \)). The lattice of all Abelian periodic group varieties is evidently isomorphic to the lattice of natural numbers ordered by divisibility. This readily implies that non-trivial chain Abelian group varieties are varieties \( A_{p^k} \) with prime \( p \) and natural \( k \), and only they. Combining this observation with results of [9], we have the following

**Lemma 4.** The varieties \( A_{p^k} \) with prime \( p \) and natural \( k \), \( SL, N_k, N_\omega, N_3^c \) and only they are chain varieties of commutative semigroups.

Fig. 1 shows the relative location of chain varieties in the lattice \( \text{Com} \).

**Figure 1.** Chain varieties of commutative semigroups

Combining above observations, it is easy to verify the following

**Proposition 5.** The set of varieties \( \{ A_p \mid p \text{ is a prime number} \} \) and the varieties \( SL \) and \( ZM \) are definable.

**Proof.** By Lemma 1, all varieties mentioned in the proposition are atoms of \( \text{Com} \). By Corollary 3, the varieties \( SL \) and \( ZM \) are neutral elements in \( \text{Com} \), while \( A_p \) is not. Fig. 1 shows that the varieties \( ZM \) and \( A_p \) are proper subvarieties of some chain varieties, while \( SL \) is not. Therefore the formulas
\[
SL(x) \iff A(x) \& \text{Neut}(x) \& (\forall y) \left( \text{Ch}(y) \& x \leq y \rightarrow x = y \right),
\]
\[
ZM(x) \iff A(x) \& \text{Neut}(x) \& (\exists y) \left( \text{Ch}(y) \& x < y \right)
\]
define the varieties \( SL \) and \( ZM \) respectively, while the the formula
\[
\text{GrA}(x) \iff A(x) \& \neg \text{Neut}(x) \& (\exists y) \left( \text{Ch}(y) \& x < y \right)
\]
define the set \( \{ A_p \mid p \text{ is a prime number} \} \).
Note that in fact each of the group atoms $A_p$ is individually definable (see Proposition 15 below). The definability of the varieties $SL$ and $ZM$ is mentioned in [4, Proposition 3.1] without any explicitly written formulas.

Recall that a semigroup variety is called *combinatorial* if all its groups are trivial.

**Proposition 6.** The sets of all Abelian periodic group varieties, all combinatorial commutative varieties and of all commutative nil-varieties of semigroups are definable.

*Proof.* It is well known that a commutative semigroup variety is an Abelian periodic group variety [a combinatorial variety, a nil-variety] if and only if it does not contain the varieties $SL$ and $ZM$ [respectively, the varieties $A_p$ for all prime $p$, any atoms except $ZM$]. Therefore, the sets of all Abelian periodic group varieties, all combinatorial commutative varieties and of all commutative nil-varieties are definable by the formulas

$$\text{Gr}(x) \equiv (\forall y) (A(y) \& y \leq x \rightarrow \text{GrA}(y));$$
$$\text{Comb}(x) \equiv (\forall y) (A(y) \& y \leq x \rightarrow \neg \text{GrA}(y));$$
$$\text{Nil}(x) \equiv (\forall y) (A(y) \& y \leq x \rightarrow \text{ZM}(y))$$

respectively.

The claim that the set of all Abelian periodic group varieties is definable in $\text{Com}$ is proved in [4] without any explicitly written formula defining this class.

Identities of the form $w = 0$ are called 0-reduced. We denote by $\text{COM}$ the variety of all commutative semigroups. A commutative semigroup variety is called 0-reduced in $\text{Com}$ if it is given within $\text{COM}$ by 0-reduced identities only.

**Proposition 7.** The set of all 0-reduced in $\text{Com}$ commutative semigroup varieties is definable.

*Proof.* Put

$$\text{LMod}(x) \equiv (\forall y, z) (x \leq y \rightarrow x \vee (y \wedge z) = y \wedge (x \vee z)).$$

An element $x$ of a lattice $L$ such that the sentence $\text{LMod}(x)$ is true is called lower-modular. Lower-modular elements of the lattice $\text{Com}$ are completely determined in [7, Theorem 1.6]. This result immediately implies that a commutative nil-variety is lower-modular in $\text{Com}$ if and only if it is 0-reduced in $\text{Com}$. Therefore the formula

$$0\text{-red}(x) \equiv \text{Nil}(x) \& \text{LMod}(x)$$

defines the set of all 0-reduced in $\text{Com}$ varieties.

The following general fact will be used in what follows.

**Lemma 8.** If a countably infinite subset $S$ of a lattice $L$ is definable in $L$ and forms a chain isomorphic to the chain of natural numbers under the order relation in $L$ then every member of this set is definable in $L$. 

\[ \square \]
Proof. Let \( S = \{ s_n \mid n \in \mathbb{N} \} \), \( s_1 < s_2 < \cdots < s_n < \cdots \) and let \( \Phi(x) \) be the formula defining \( S \) in \( L \). We are going to prove the definability of the element \( s_n \) for each \( n \) by induction on \( n \). The induction base is evident because the element \( s_1 \) is definable by the formula \( \min_x \{ \Phi(x) \} \). Assume now that \( n > 1 \) and the element \( s_{n-1} \) is definable by some formula \( \Psi(x) \). Then the formula

\[
\min_x \{ \Phi(x) \& (\exists y) (\Psi(y) \& y < x) \}
\]

defines the element \( s_n \). □

The following lemma is a part of the semigroup folklore. It is known at least since earlier 1980’s (see [5], for instance). In any case, it immediately follows from Lemma 2 of [11] and the proof of Proposition 1 of the same article.

**Lemma 9.** If \( V \) is a commutative semigroup variety with \( V \neq \mathsf{COM} \) then \( V = \mathcal{K} \sqcup \mathcal{N} \) where \( \mathcal{K} \) is a variety generated by a monoid, while \( \mathcal{N} \) is a nil-variety. □

Let \( C_{m, 1} \) denote the cyclic monoid \( \langle a \mid a^m = a^{m+1} \rangle \) and let \( C_m \) be the variety generated by \( C_{m, 1} \). It is clear that

\[
C_m = \text{var} \{ x^m = x^{m+1}, xy = yx \}.
\]

In particular, \( C_{1, 1} \) is the 2-element semilattice and \( C_1 = \mathcal{S} \mathcal{L} \). For notation convenience we put also \( C_0 = T \). The following lemma can be easily extracted from the results of [3].

**Lemma 10.** If a periodic semigroup variety \( V \) is generated by a commutative monoid then \( V = \mathcal{G} \sqcup C_m \) for some Abelian periodic group variety \( \mathcal{G} \) and some \( m \geq 0 \). □

Lemmas 9 and 10 immediately imply

**Corollary 11.** If \( V \) is a commutative combinatorial semigroup variety then \( V = C_m \sqcup \mathcal{N} \) for some \( m \geq 0 \) and some nil-variety \( \mathcal{N} \). □

Let now \( V \) be a commutative semigroup variety with \( V \neq \mathsf{COM} \). Lemmas 9 and 10 imply that \( V = \mathcal{G} \sqcup C_m \sqcup \mathcal{N} \) for some Abelian periodic group variety \( \mathcal{G} \), some \( m \geq 0 \) and some commutative nil-variety \( \mathcal{N} \). Our aim now is to provide formulas defining the varieties \( \mathcal{G} \) and \( C_m \).

It is well known that each periodic semigroup variety \( \mathcal{X} \) contains its greatest nil-subvariety. We denote this subvariety by \( \text{Nil}(\mathcal{X}) \). Put

\[
\mathcal{D}_m = \text{Nil}(C_m) = \text{var} \{ x^m = 0, xy = yx \}
\]

for every natural \( m \). In particular, \( \mathcal{D}_1 = T \) and \( \mathcal{D}_2 = \mathcal{N}_\omega \).

**Proposition 12.** For each \( m \geq 0 \), the variety \( C_m \) is definable.

**Proof.** First, we are going to verify that the formula

\[
\text{All-C}_m(x) = \text{Comb}(x) \& (\forall y, z) (\text{Nil}(y) \& x = y \lor z \rightarrow x = z)
\]

defines the set of varieties \( \{ C_m \mid m \geq 0 \} \) in \( \mathsf{Com} \). Let \( V \) be a commutative semigroup variety such that the sentence \( \text{All-C}_m(V) \) is true. Then \( V \) is combinatorial. Now Corollary 11 successfully applies with the conclusion that \( M = C_m \sqcup \mathcal{N} \) for
some \( m \geq 0 \) and some commutative nil-variety \( N \). The fact that the sentence \( \text{A11-C}_m(N) \) is true shows that \( M = C_m \).

Let now \( m \geq 0 \). We aim to verify that the sentence \( \text{A11-C}_m(C_m) \) is true. It is evident that the variety \( C_m \) is combinatorial. Suppose that \( C_m = M \lor N \) where \( N \) is a nil-variety. It remains to check that \( N \subseteq M \). We may assume without any loss that \( N = \text{Nil}(C_m) = D_m \). It is clear that \( M \) is a commutative and combinatorial variety. Corollary 11 implies that \( M = C_r \lor N' \) for some \( r \geq 0 \) and some nil-variety \( N' \). Then \( N' \subseteq \text{Nil}(C_m) = N \), whence

\[
C_m = M \lor N = C_r \lor N' \lor N = C_r \lor N.
\]

It suffices to prove that \( N \subseteq C_r \) because \( N \subseteq C_r \lor N' = M \) in this case. The equality \( C_m = C_r \lor N \) implies that \( C_r \subseteq C_m \), whence \( r \leq m \). If \( r = m \) then \( N \subseteq C_r \), and we are done. Let now \( r < m \). Then the variety \( C_m = C_r \lor N \) satisfies the identity \( x^r y^m = x^{r+1} y^m \). Recall that the variety \( C_m \) is generated by a monoid. Substituting 1 for \( y \) in this identity, we obtain that \( C_m \) satisfies the identity \( x^r = x^{r+1} \). Therefore \( C_m \subseteq C_r \) contradicting the inequality \( r < m \).

Thus we have proved that the set of varieties \( \{C_m \mid m \geq 0\} \) is definable by the formula \( \text{A11-C}_m(x) \). Now Lemma 8 successfully applies with the conclusion that the variety \( C_m \) is definable for each \( m \).

**Proposition 13.** For every natural number \( m \), the variety \( D_m \) is definable.

**Proof.** Every commutative semigroup variety either coincides with \( \text{COM} \) or is periodic. Thus the formula

\[
\text{Per}(x) \iff (\exists y)(x < y)
\]

defines the set of all periodic commutative varieties. In particular, if \( X \) is a commutative variety such that the sentence \( \text{Per}(X) \) is true then the variety \( \text{Nil}(X) \) there exists. Put

\[
\text{Nil-part}(x, y) \iff \text{Per}(x) \land y \leq x \land \text{Nil}(y) \land (\forall z)(z \leq x \land \text{Nil}(z) \rightarrow z \leq y).
\]

Clearly, if \( X \) and \( Y \) are commutative semigroup varieties then the sentence \( \text{Nil-part}(X, Y) \) is true if and only if \( X \) is periodic and \( Y = \text{Nil}(X) \). Let \( C_m \) be the formula defining the variety \( C_m \). The variety \( D_m \) is defined by the formula

\[
\text{D}_m(x) \iff (\exists y)(C_m(y) \land \text{Nil-part}(y, x))
\]

because \( D_m = \text{Nil}(C_m) \).

If \( X \) is a commutative nil-variety of semigroups then we denote by \( \text{ZR}(X) \) the least 0-reduced in \( \text{Com} \) variety that contains \( X \). Clearly, the variety \( \text{ZR}(X) \) is given within \( \text{COM} \) by all 0-reduced identities that hold in \( X \). If \( u \) is a word and \( x \) is a letter then \( c(u) \) denotes the set of all letters occurring in \( u \), while \( \ell_x(u) \) stands for the number of occurrences of \( x \) in \( u \).

**Lemma 14.** Let \( m \) and \( n \) be natural numbers with \( m > 2 \) and \( n > 1 \). The following are equivalent:

(i) \( \text{Nil}(A_n \lor X) = \text{ZR}(X) \) for any variety \( X \subseteq D_m \);

(ii) \( n \geq m - 1 \).
Proof. (i)→(ii) Suppose that \( n < m - 1 \). Let \( \mathcal{X} \) be the subvariety of \( \mathcal{D}_m \) given within \( \mathcal{D}_m \) by the identity
\[
x^{n+1}y = xy^{n+1}.
\]
Since \( n+1 < m \), the variety \( \mathcal{X} \) is not 0-reduced in \( \textbf{Com} \). Note that \( \mathcal{X} \subseteq \text{Nil}(\mathcal{A}_n \lor \mathcal{X}) \) because \( \mathcal{X} \) is a nil-variety. The identity (1) holds in the variety \( \mathcal{A}_n \lor \mathcal{X} \), and therefore in the variety \( \text{Nil}(\mathcal{A}_n \lor \mathcal{X}) \). But the latter variety does not satisfy the identity \( x^{n+1}y = 0 \) because this identity fails in \( \mathcal{X} \). We see that the variety \( \text{Nil}(\mathcal{A}_n \lor \mathcal{X}) \) is not 0-reduced in \( \textbf{Com} \). Since the variety \( \text{ZR}(\mathcal{X}) \) is 0-reduced in \( \textbf{Com} \), we are done.

(ii)→(i) Let \( n \geq m - 1 \) and \( \mathcal{X} \subseteq \mathcal{D}_m \). One can verify that \( \mathcal{A}_n \lor \mathcal{X} = \mathcal{A}_n \lor \text{ZR}(\mathcal{X}) \). Note that this equality immediately follows from [6, Lemma 2.5] whenever \( n \geq m \). We reproduce here the corresponding arguments for the sake of completeness. It suffices to check that \( \mathcal{A}_n \lor \text{ZR}(\mathcal{X}) \subseteq \mathcal{A}_n \lor \mathcal{X} \) because the opposite inclusion is evident. Suppose that the variety \( \mathcal{A}_n \lor \mathcal{X} \) satisfies an identity \( u = v \). We need to prove that this identity holds in \( \mathcal{A}_n \lor \text{ZR}(\mathcal{X}) \). Since \( u = v \) holds in \( \mathcal{A}_n \), we have \( \ell_x(u) = \ell_x(v)(\text{mod } n) \) for any letter \( x \). If \( \ell_x(u) = \ell_x(v) \) for all letters \( x \) then \( u = v \) holds in \( \mathcal{A}_n \lor \text{ZR}(\mathcal{X}) \) because this variety is commutative. Therefore we may assume that \( \ell_x(u) \neq \ell_x(v) \) for some letter \( x \). Then either \( \ell_x(u) \geq n \) or \( \ell_x(v) \geq n \). We may assume without any loss that \( \ell_x(u) \geq n \).

Suppose that \( n \geq m \). Then the identity \( u = 0 \) holds in the variety \( \mathcal{D}_m \), whence it holds in \( \mathcal{X} \). This implies that \( v = 0 \) holds in \( \mathcal{X} \) too. Therefore the variety \( \text{ZR}(\mathcal{X}) \) satisfies the identities \( u = 0 = v \). Since the identity \( u = v \) holds in \( \mathcal{A}_n \), it holds in \( \mathcal{A}_n \lor \text{ZR}(\mathcal{X}) \), and we are done.

It remains to consider the case \( n = m - 1 \). Let \( x \) be a letter with \( x \in c(u) \cup c(v) \) and \( \ell_x(u) \neq \ell_x(v) \). If either \( \ell_x(u) \geq m \) or \( \ell_x(v) \geq m \), we go to the situation considered in the previous paragraph. Let now \( \ell_x(u), \ell_x(v) < m \). Since \( \ell_x(u) \geq n = m - 1 \), \( \ell_x(u) = \ell_x(v)(\text{mod } n) \) and \( \ell_x(u) \neq \ell_x(v) \), we have \( \ell_x(u) = n = m - 1 \) and \( \ell_x(v) = 0 \). The latter equality means that \( x \notin c(v) \). Substituting \( 0 \) for \( x \) in \( u = v \), we obtain that the variety \( \mathcal{X} \) satisfies the identity \( v = 0 \). We go to the situation considered in the previous paragraph again.

We have proved that \( \mathcal{A}_n \lor \mathcal{X} = \mathcal{A}_n \lor \text{ZR}(\mathcal{X}) \). Therefore \( \text{ZR}(\mathcal{X}) \subseteq \text{Nil}(\mathcal{A}_n \lor \mathcal{X}) \). If the variety \( \mathcal{X} \) satisfies an identity \( u = 0 \) then \( u^{n+1} = u \) holds in \( \mathcal{A}_n \lor \mathcal{X} \). This readily implies that \( u = 0 \) in \( \text{Nil}(\mathcal{A}_n \lor \mathcal{X}) \). Hence \( \text{Nil}(\mathcal{A}_n \lor \mathcal{X}) \subseteq \text{ZR}(\mathcal{X}) \). Thus \( \text{Nil}(\mathcal{A}_n \lor \mathcal{X}) = \text{ZR}(\mathcal{X}) \). \(\square\)

Now we are well prepared to prove the following

**Proposition 15.** An arbitrary Abelian periodic group variety is definable.

**Proof.** Abelian periodic group varieties are exhausted by the trivial variety and the varieties \( \mathcal{A}_n \) with \( n > 1 \). The trivial variety is obviously definable. For brevity, put
\[
\text{ZR}(x, y) \equiv \text{0-red}(y) \land x \leq y \land (\forall z)(\text{0-red}(z) \land x \leq z \rightarrow y \leq z).
\]
The sentence \( \text{ZR}(\mathcal{X}, \mathcal{Y}) \) is true if and only if \( \mathcal{Y} = \text{ZR}(\mathcal{X}) \). Let \( m \) be a natural number with \( m > 2 \). In view of Lemma 14, the formula
\[
\mathcal{A}_{2^m-1}(x) \equiv \text{Gr}(x) \land (\forall y, z, t)(\text{D}_m(y) \land z \leq y \land \text{Nil-part}(x \lor z, t) \rightarrow \text{ZR}(z, t))
\]
defines the set of varieties \( \{ A_n \mid n \geq m - 1 \} \). Therefore the formula

\[
A_n(x) \iff A_{\geq n}(x) \& \neg A_{\geq n+1}(x)
\]

defines the variety \( A_n \).

It was proved in [4] that each Abelian group variety is definable in the lattice \( \text{Com} \). However this paper contain no explicit first-order formula defining any given Abelian periodic group variety.

Now we are ready to achieve the goal of this note.

**Theorem 16.** A semigroup variety generated by a commutative monoid is definable.

**Proof.** Let \( \mathcal{V} \) be a variety generated by some commutative monoid. According to Lemma 10, \( \mathcal{V} = A_n \vee C_m \) for some \( n \geq 1 \) and \( m \geq 0 \). It is easy to check that the parameters \( n \) and \( m \) in this decomposition are defined uniquely. Therefore the formula

\[
(\exists y, z) \left( A_n(y) \& C_m(z) \& x = y \lor z \right)
\]

defines the variety \( \mathcal{V} \) (we assume here that \( A_1 \) is the evident formula defining the variety \( A_1 = T \)).

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A subset $A$ of a lattice $\langle L; \lor, \land \rangle$ is called definable in $L$ if there exists a first-order formula $\Phi(x)$ with one free variable $x$ in the language of lattice operations $\lor$ and $\land$ which defines $A$ in $L$. This means that, for an element $a \in L$, the sentence $\Phi(a)$ is true if and only if $a \in A$. If $A$ consists of a single element, we speak about definability of this element.

We denote the lattice of all commutative semigroup varieties by $\text{Com}$. A set of commutative semigroup varieties $X$ (or a single commutative semigroup variety $\mathcal{X}$) is said to be definable if it is definable in $\text{Com}$. In this situation we will say that the corresponding first-order formula defines the set $X$ or the variety $\mathcal{X}$.

Let $M$ be a commutative monoid. In [10, Corollary 4.8], we provide an explicit first-order formula that defines the variety generated by $M$ in the lattice of all semigroup varieties. The objective of this note is to modify the arguments from [10] in order to present an explicit formula that defines the variety generated by $M$ in the lattice $\text{Com}$.

We will denote the conjunction by $\&$ rather than $\land$ because the latter symbol stands for the meet in a lattice. Since the disjunction and the join in a lattice are denoted usually by the same symbol $\lor$, we use this symbol for the join and denote the disjunction by $\lor$. Evidently, the relations $\leq$, $\geq$, $<$ and $>$ in a lattice $L$ can be expressed in terms of, say, meet operation $\land$ in $L$. So, we will freely use these four relations in formulas. Let $\Phi(x)$ be a first-order formula. For the sake of brevity, we put

$$\min_x \{\Phi(x)\} \equiv \Phi(x) & (\forall y)(y < x \rightarrow \neg \Phi(y)).$$

Clearly, the formula $\min_x \{\Phi(x)\}$ defines the set of all minimal elements of the set defined by the formula $\Phi(x)$.

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Many important sets of semigroup varieties admit a characterization in the language of atoms of the lattice $\text{Com}$. The set of all atoms of a lattice $L$ with $0$ is defined by the formula

$$A(x) \equiv (\exists y) \left( (\forall z) (y \leq z) \& \min_x \{x \neq y\} \right).$$

A description of all atoms of the lattice $\text{Com}$ directly follows from the well-known description of atoms of the lattice of all semigroup varieties (see [2, 8], for instance). To list these varieties, we need some notation.

By $\text{var} \Sigma$ we denote the semigroup variety given by the identity system $\Sigma$. A pair of identities $wx = xw = w$ where the letter $x$ does not occur in the word $w$ is usually written as the symbolic identity $w = 0^1$. Let us fix notation for several semigroup varieties:

$$A_n = \text{var} \{ x^n y = y, xy = yx \}$$ — the variety of Abelian groups
whose exponent divides $n$,

$$SL = \text{var} \{ x^2 = x, xy = yx \}$$ — the variety of semilattices,

$$ZM = \text{var} \{ xy = 0 \}$$ — the variety of null semigroups.

**Lemma 1.** The varieties $A_p$ (where $p$ is a prime number), $SL$, $ZM$ and only they are atoms of the lattice $\text{Com}$. \hfill $\Box$

Put

$$\text{Neut}(x) \equiv (\forall y, z) \left( (x \lor y) \land (y \lor z) \land (z \lor x) = (x \land y) \lor (y \land z) \lor (z \land x) \right).$$

An element $x$ of a lattice $L$ such that the sentence $\text{Neut}(x)$ is true is called neutral. We denote by $T$ the trivial semigroup variety.

**Lemma 2** ([6, Theorem 1.2]). A commutative semigroup variety $V$ is a neutral element of the lattice $\text{Com}$ if and only if either $V = \text{COM}$ or $V = M \lor N$ where $M$ is one of the varieties $T$ or $SL$, while the variety $N$ satisfies the identity $x^2 y = 0$. \hfill $\Box$

For convenience of references, we formulate the following immediate consequence of Lemmas 1 and 2.

**Corollary 3.** An atom of the lattice $\text{Com}$ is a neutral element of this lattice if and only if it coincides with one of the varieties $SL$ or $ZM$. \hfill $\Box$

A semigroup variety $V$ is called chain if the subvariety lattice of $V$ is a chain. Clearly, each atom of $\text{Com}$ is a chain variety. The set of all chain varieties is definable by the formula

$$\text{Ch}(x) \equiv (\forall y, z) (y \leq x \& z \leq x \rightarrow y \leq z \text{ OR } z \leq y).$$

We adopt the usual agreement that an adjective indicating a property shared by all semigroups of a given variety is applied to the variety itself; the expressions like “group variety”, “periodic variety”, “nil-variety” etc. are understood in this sense.

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1. This notation is justified because a semigroup with such identities has a zero element and all values of the word $w$ in this semigroup are equal to zero.
Put
\[ N^k = \text{var}\{x^2 = x_1 x_2 \cdots x_k = 0, xy = yx\} \quad (k \text{ is a natural number}), \]
\[ N^\omega = \text{var}\{x^2 = 0, xy = yx\}, \]
\[ N^3_3 = \text{var}\{xyz = 0, xy = yx\} \]
(in particular \( N^1 = T \) and \( N^2 = ZM \)). The lattice of all Abelian periodic group varieties is evidently isomorphic to the lattice of natural numbers ordered by divisibility. This readily implies that non-trivial chain Abelian group varieties are varieties \( A^p_k \) with prime \( p \) and natural \( k \), and only they. Combining this observation with results of [9], we have the following

**Lemma 4.** The varieties \( A^p_k \) with prime \( p \) and natural \( k \), \( SL \), \( N_k \), \( N^\omega \), \( N^3_3 \) and only they are chain varieties of commutative semigroups. \( \square \)

Fig. 1 shows the relative location of chain varieties in the lattice \( \text{Com} \).

![Figure 1. Chain varieties of commutative semigroups](image)

Combining above observations, it is easy to verify the following

**Proposition 5.** The set of varieties \( \{A_p | p \text{ is a prime number}\} \) and the varieties \( SL \) and \( ZM \) are definable.

**Proof.** By Lemma 1, all varieties mentioned in the proposition are atoms of \( \text{Com} \). By Corollary 3, the varieties \( SL \) and \( ZM \) are neutral elements in \( \text{Com} \), while \( A_p \) is not. Fig. 1 shows that the varieties \( ZM \) and \( A_p \) are proper subvarieties of some chain varieties, while \( SL \) is not. Therefore the formulas
\[
SL(x) = A(x) \& \text{Neut}(x) \& (\forall y) \left( \text{Ch}(y) \& x \leq y \rightarrow x = y \right),
\]
\[
ZM(x) = A(x) \& \text{Neut}(x) \& (\exists y) \left( \text{Ch}(y) \& x < y \right)
\]
define the varieties \( SL \) and \( ZM \) respectively, while the the formula
\[
\text{Gr}(x) = A(x) \& \neg \text{Neut}(x) \& (\exists y) \left( \text{Ch}(y) \& x < y \right)
\]
define the set \( \{A_p | p \text{ is a prime number}\} \). \( \square \)
Note that in fact each of the group atoms $A_p$ is individually definable (see Proposition 15 below). The definability of the varieties $SL$ and $ZM$ is mentioned in [4, Proposition 3.1] without any explicitly written formulas.

Recall that a semigroup variety is called combinatorial if all its groups are trivial.

**Proposition 6.** The sets of all Abelian periodic group varieties, all combinatorial commutative varieties and of all commutative nil-varieties of semigroups are definable.

**Proof.** It is well known that a commutative semigroup variety is an Abelian periodic group variety [a combinatorial variety, a nil-variety] if and only if it does not contain the varieties $SL$ and $ZM$ [respectively, the varieties $A_p$ for all prime $p$, any atoms except $ZM$]. Therefore, the sets of all Abelian periodic group varieties, all combinatorial commutative varieties and of all commutative nil-varieties are definable by the formulas

$$\text{Gr}(x) \equiv (\forall y) \left( A(y) & y \leq x \rightarrow \text{Gr}A(y) \right);$$
$$\text{Comb}(x) \equiv (\forall y) \left( A(y) & y \leq x \rightarrow -\text{Gr}A(y) \right);$$
$$\text{Nil}(x) \equiv (\forall y) \left( A(y) & y \leq x \rightarrow ZM(y) \right)$$

respectively. \hfill \Box

The claim that the set of all Abelian periodic group varieties is definable in $\text{Com}$ is proved in [4] without any explicitly written formula defining this class.

Identities of the form $w = 0$ are called 0-reduced. We denote by $\text{COM}$ the variety of all commutative semigroups. A commutative semigroup variety is called 0-reduced in $\text{Com}$ if it is given within $\text{COM}$ by 0-reduced identities only.

**Proposition 7.** The set of all 0-reduced in $\text{Com}$ commutative semigroup varieties is definable.

**Proof.** Put

$$\text{LMod}(x) \equiv (\forall y, z) \left( x \leq y \rightarrow x \lor (y \land z) = y \land (x \lor z) \right).$$

An element $x$ of a lattice $L$ such that the sentence $\text{LMod}(x)$ is true is called lower-modular. Lower-modular elements of the lattice $\text{Com}$ are completely determined in [7, Theorem 1.6]. This result immediately implies that a commutative nil-variety is lower-modular in $\text{Com}$ if and only if it is 0-reduced in $\text{Com}$. Therefore the formula

$$\text{0-red}(x) \equiv \text{Nil}(x) & \text{LMod}(x)$$

defines the set of all 0-reduced in $\text{Com}$ varieties. \hfill \Box

The following general fact will be used in what follows.

**Lemma 8.** If a countably infinite subset $S$ of a lattice $L$ is definable in $L$ and forms a chain isomorphic to the chain of natural numbers under the order relation in $L$ then every member of this set is definable in $L$. 
Proof. Let \( S = \{ s_n \mid n \in \mathbb{N} \} \), \( s_1 < s_2 < \cdots < s_n < \cdots \) and let \( \Phi(x) \) be the formula defining \( S \) in \( L \). We are going to prove the definability of the element \( s_n \) for each \( n \) by induction on \( n \). The induction base is evident because the element \( s_1 \) is definable by the formula \( \min_x \{ \Phi(x) \} \). Assume now that \( n > 1 \) and the element \( s_{n-1} \) is definable by some formula \( \Psi(x) \). Then the formula
\[
\min_x \{ \Phi(x) & (\exists y) (\Psi(y) & y < x) \}
\]
defines the element \( s_n \).

The following lemma is a part of the semigroup folklore. It is known at least since earlier 1980’s (see [5], for instance). In any case, it immediately follows from Lemma 2 of [11] and the proof of Proposition 1 of the same article.

**Lemma 9.** If \( V \) is a commutative semigroup variety with \( V \neq \text{COM} \) then \( V = K \lor N \) where \( K \) is a variety generated by a monoid, while \( N \) is a nil-variety.

Let \( C_{m,1} \) denote the cyclic monoid \( \langle a \mid a^m = a^{m+1} \rangle \) and let \( C_m \) be the variety generated by \( C_{m,1} \). It is clear that
\[
C_m = \text{var}\{x^m = x^{m+1}, xy = yx\}.
\]
In particular, \( C_{1,1} \) is the 2-element semilattice and \( C_1 = \mathcal{S} \mathcal{L} \). For notation convenience we put also \( C_0 = \mathcal{T} \). The following lemma can be easily extracted from the results of [3].

**Lemma 10.** If a periodic semigroup variety \( V \) is generated by a commutative monoid then \( V = G \lor C_m \) for some Abelian periodic group variety \( G \) and some \( m \geq 0 \).

Lemmas 9 and 10 immediately imply

**Corollary 11.** If \( V \) is a commutative combinatorial semigroup variety then \( V = C_m \lor N \) for some \( m \geq 0 \) and some nil-variety \( N \).

Let now \( V \) be a commutative semigroup variety with \( V \neq \text{COM} \). Lemmas 9 and 10 imply that \( V = G \lor C_m \lor N \) for some Abelian periodic group variety \( G \), some \( m \geq 0 \) and some commutative nil-variety \( N \). Our aim now is to provide formulas defining the varieties \( G \) and \( C_m \).

It is well known that each periodic semigroup variety \( X \) contains its greatest nil-subvariety. We denote this subvariety by \( \text{Nil}(X) \). Put
\[
D_m = \text{Nil}(C_m) = \text{var}\{x^m = 0, xy = yx\}
\]
for every natural \( m \). In particular, \( D_1 = \mathcal{T} \) and \( D_2 = N_\omega \).

**Proposition 12.** For each \( m \geq 0 \), the variety \( C_m \) is definable.

**Proof.** First, we are going to verify that the formula
\[
\text{All-C}_m(x) \Rightarrow \text{Comb}(x) \land (\forall y, z) (\text{Nil}(y) \land x = y \lor z \rightarrow x = z)
\]
defines the set of varieties \( \{ C_m \mid m \geq 0 \} \) in \( \text{Com} \). Let \( V \) be a commutative semigroup variety such that the sentence \( \text{All-C}_m(V) \) is true. Then \( V \) is combinatorial. Now Corollary 11 successfully applies with the conclusion that \( M = C_m \lor N \) for
some \(m \geq 0\) and some commutative nil-variety \(N\). The fact that the sentence \(\text{All-}C_m(\mathcal{V})\) is true shows that \(\mathcal{M} = C_m\).

Let now \(m \geq 0\). We aim to verify that the sentence \(\text{All-}C_m(C_m)\) is true. It is evident that the variety \(C_m\) is combinatorial. Suppose that \(C_m = \mathcal{M} \vee \mathcal{N}\) where \(\mathcal{N}\) is a nil-variety. It remains to check that \(\mathcal{N} \subseteq \mathcal{M}\). We may assume without any loss that \(\mathcal{N} = \text{Nil}(C_m) = D_m\). It is clear that \(\mathcal{M}\) is a commutative and combinatorial variety. Corollary 11 implies that \(\mathcal{M} = C_r \vee N'\) for some \(r \geq 0\) and some nil-variety \(N'\). Then \(N' \subseteq \text{Nil}(C_m) = N\), whence

\[
C_m = \mathcal{M} \vee \mathcal{N} = C_r \vee N' \vee N = C_r \vee N.
\]

It suffices to prove that \(\mathcal{N} \subseteq C_r\) because \(\mathcal{N} \subseteq C_r \vee N' = \mathcal{M}\) in this case. The equality \(C_m = C_r \vee N\) implies that \(C_r \subseteq C_m\), whence \(r \leq m\). If \(r = m\) then \(\mathcal{N} \subseteq C_r\), and we are done. Let now \(r < m\). Then the variety \(C_m = C_r \vee N\) satisfies the identity \(x^r y^m = x^{r+1} y^m\). Recall that the variety \(C_m\) is generated by a monoid. Substituting 1 for \(y\) in this identity, we obtain that \(C_m\) satisfies the identity \(x^r = x^{r+1}\). Therefore \(C_m \subseteq C_r\) contradicting the inequality \(r < m\).

Thus we have proved that the set of varieties \(\{C_m \mid m \geq 0\}\) is definable by the formula \(\text{All-}C_m(x)\). Now Lemma 8 successfully applies with the conclusion that the variety \(C_m\) is definable for each \(m\).

\[\square\]

**Proposition 13.** For every natural number \(m\), the variety \(D_m\) is definable.

**Proof.** Every commutative semigroup variety either coincides with \(\text{COM}\) or is periodic. Thus the formula

\[
\text{Per}(x) \equiv (\exists y)(x < y)
\]

defines the set of all periodic commutative varieties. In particular, if \(\mathcal{X}\) is a commutative variety such that the sentence \(\text{Per}(\mathcal{X})\) is true then the variety \(\text{Nil}(\mathcal{X})\) there exists. Put

\[
\text{Nil-part}(x, y) \equiv \text{Per}(x) \& y \leq x \& \text{Nil}(y) \& (\forall z)(z \leq x \& \text{Nil}(z) \rightarrow z \leq y).
\]

Clearly, if \(\mathcal{X}\) and \(\mathcal{Y}\) are commutative semigroup varieties then the sentence \(\text{Nil-part}(\mathcal{X}, \mathcal{Y})\) is true if and only if \(\mathcal{X}\) is periodic and \(\mathcal{Y} = \text{Nil}(\mathcal{X})\). Let \(C_m\) be the formula defining the variety \(C_m\). The variety \(D_m\) is defined by the formula

\[
D_m(x) \equiv (\exists y)(C_m(y) \& \text{Nil-part}(y, x))
\]

because \(D_m = \text{Nil}(C_m)\).

\[\square\]

If \(\mathcal{X}\) is a commutative nil-variety of semigroups then we denote by \(\text{ZR}(\mathcal{X})\) the least 0-reduced in \(\text{Com}\) variety that contains \(\mathcal{X}\). Clearly, the variety \(\text{ZR}(\mathcal{X})\) is given within \(\text{COM}\) by all 0-reduced identities that hold in \(\mathcal{X}\). If \(u\) is a word and \(x\) is a letter then \(c(u)\) denotes the set of all letters occurring in \(u\), while \(\ell_x(u)\) stands for the number of occurrences of \(x\) in \(u\).

**Lemma 14.** Let \(m\) and \(n\) be natural numbers with \(m > 2\) and \(n > 1\). The following are equivalent:

(i) \(\text{Nil}(A_n \vee \mathcal{X}) = \text{ZR}(\mathcal{X})\) for any variety \(\mathcal{X} \subseteq D_m\),

(ii) \(n \geq m - 1\).
Proof. (i) — (ii) Suppose that $n < m − 1$. Let $\mathcal{X}$ be the subvariety of $\mathcal{D}_m$ given within $\mathcal{D}_m$ by the identity
\[(1) \quad x^{n+1}y = xy^{n+1}.\]
Since $n+1 < m$, the variety $\mathcal{X}$ is not 0-reduced in $\text{Com}$. Note that $\mathcal{X} \subseteq \text{Nil}(\mathcal{A}_n \lor \mathcal{X})$ because $\mathcal{X}$ is a nil-variety. The identity (1) holds in the variety $\mathcal{A}_n \lor \mathcal{X}$, and therefore in the variety $\text{Nil}(\mathcal{A}_n \lor \mathcal{X})$. But the latter variety does not satisfy the identity $x^{n+1}y = 0$ because this identity fails in $\mathcal{X}$. We see that the variety $\text{Nil}(\mathcal{A}_n \lor \mathcal{X})$ is not 0-reduced in $\text{Com}$. Since the variety $\text{ZR}(\mathcal{X})$ is 0-reduced in $\text{Com}$, we are done.

(ii) — (i) Let $n \geq m − 1$ and $\mathcal{X} \subseteq \mathcal{D}_m$. One can verify that $\mathcal{A}_n \lor \mathcal{X} = \mathcal{A}_n \lor \text{ZR}(\mathcal{X})$. Note that this equality immediately follows from [6, Lemma 2.5] whenever $n \geq m$. We reproduce here the corresponding arguments for the sake of completeness. It suffices to check that $\mathcal{A}_n \lor \text{ZR}(\mathcal{X}) \subseteq \mathcal{A}_n \lor \mathcal{X}$ because the opposite inclusion is evident. Suppose that the variety $\mathcal{A}_n \lor \mathcal{X}$ satisfies an identity $u = v$. We need to prove that this identity holds in $\mathcal{A}_n \lor \text{ZR}(\mathcal{X})$. Since $u = v$ holds in $\mathcal{A}_n$, we have $\ell_x(u) \equiv \ell_x(v) \pmod{n}$ for any letter $x$. If $\ell_x(u) = \ell_x(v)$ for all letters $x$ then $u = v$ holds in $\mathcal{A}_n \lor \text{ZR}(\mathcal{X})$ because this variety is commutative. Therefore we may assume that $\ell_x(u) \neq \ell_x(v)$ for some letter $x$. Then either $\ell_x(u) \geq n$ or $\ell_x(v) \geq n$. We may assume without any loss that $\ell_x(u) \geq n$.

Suppose that $n \geq m$. Then the identity $u = 0$ holds in the variety $\mathcal{D}_m$, whence it holds in $\mathcal{X}$. This implies that $v = 0$ holds in $\mathcal{X}$ too. Therefore the variety $\text{ZR}(\mathcal{X})$ satisfies the identities $u = 0 = v$. Since the identity $u = v$ holds in $\mathcal{A}_n$, it holds in $\mathcal{A}_n \lor \text{ZR}(\mathcal{X})$, and we are done.

It remains to consider the case $n = m − 1$. Let $x$ be a letter with $x \in c(u) \cup c(v)$ and $\ell_x(u) \neq \ell_x(v)$. If either $\ell_x(u) \geq m$ or $\ell_x(v) \geq m$, we go to the situation considered in the previous paragraph. Let now $\ell_x(u), \ell_x(v) < m$. Since $\ell_x(u) \geq n = m − 1$, $\ell_x(u) \equiv \ell_x(v) \pmod{n}$ and $\ell_x(u) \neq \ell_x(v)$, we have $\ell_x(u) = n = m − 1$ and $\ell_x(v) = 0$. The latter equality means that $x \not\in c(v)$. Substituting 0 for $x$ in $u = v$, we obtain that the variety $\mathcal{X}$ satisfies the identity $v = 0$. We go to the situation considered in the previous paragraph again.

We have proved that $\mathcal{A}_n \lor \mathcal{X} = \mathcal{A}_n \lor \text{ZR}(\mathcal{X})$. Therefore $\text{ZR}(\mathcal{X}) \subseteq \text{Nil}(\mathcal{A}_n \lor \mathcal{X})$. If the variety $\mathcal{X}$ satisfies an identity $u = 0$ then $u^{n+1} = u$ holds in $\mathcal{A}_n \lor \mathcal{X}$. This readily implies that $u = 0$ in $\text{Nil}(\mathcal{A}_n \lor \mathcal{X})$. Hence $\text{Nil}(\mathcal{A}_n \lor \mathcal{X}) \subseteq \text{ZR}(\mathcal{X})$. Thus $\text{Nil}(\mathcal{A}_n \lor \mathcal{X}) = \text{ZR}(\mathcal{X})$.

Now we are well prepared to prove the following

**Proposition 15.** An arbitrary Abelian periodic group variety is definable.

**Proof.** Abelian periodic group varieties are exhausted by the trivial variety and the varieties $\mathcal{A}_n$ with $n > 1$. The trivial variety is obviously definable. For brevity, put
\[
\text{ZR}(x, y) \equiv 0\text{-red}(y) \land x \leq y \land (\forall z)\left(0\text{-red}(z) \land x \leq z \rightarrow y \leq z\right).
\]
The sentence $\text{ZR}(\mathcal{X}, \mathcal{Y})$ is true if and only if $\mathcal{Y} = \text{ZR}(\mathcal{X})$. Let $m$ be a natural number with $m > 2$. In view of Lemma 14, the formula
\[
A_{\geq m−1}(x) \equiv \text{Gr}(x) \land (\forall y, z, t)\left(\text{D}_m(y) \land z \leq y \land \text{Nil-part}(x \lor z, t) \rightarrow \text{ZR}(z, t)\right)
\]
defines the set of varieties \( \{ A_n \mid n \geq m - 1 \} \). Therefore the formula

\[
A_n(x) \iff A_{\geq n}(x) \land \neg A_{\geq n+1}(x)
\]

defines the variety \( A_n \). □

It was proved in [4] that each Abelian group variety is definable in the lattice \( \text{Com} \). However this paper contain no explicit first-order formula defining any given Abelian periodic group variety.

Now we are ready to achieve the goal of this note.

**Theorem 16.** A semigroup variety generated by a commutative monoid is definable.

**Proof.** Let \( \mathcal{V} \) be a variety generated by some commutative monoid. According to Lemma 10, \( \mathcal{V} = A_n \lor C_m \) for some \( n \geq 1 \) and \( m \geq 0 \). It is easy to check that the parameters \( n \) and \( m \) in this decomposition are defined uniquely. Therefore the formula

\[
(\exists y, z) \left( A_n(y) \land C_m(z) \land x = y \lor z \right)
\]

defines the variety \( \mathcal{V} \) (we assume here that \( A_1 \) is the evident formula defining the variety \( A_1 = T \)). □

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