The Partial Ricci Flow on $g$-foliations

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Abstract

We introduce and study new structures, which generalize the 3-(quasi-)Sasakian structure, an $f$-structure with parallelizable kernel, and an almost para-$\phi$-structure with complemented frames (having constant partial Ricci curvature) and are of particular interest in the study of the partial Ricci flow of metrics on a totally geodesic foliation. We show convergence of the partial Ricci flow on a $g$-foliation with any of our novel structures.

Keywords: $g$-foliation, flow of metrics, Riemannian curvature, partial Ricci curvature.

Mathematics Subject Classifications (2010) Primary 53C12; Secondary 53C21

Introduction

The flow of (pseudo)Riemannian metrics $g_t$ on a smooth manifold $M$ is an evolution of a geometric structure

$$\partial_t g = B(g), \tag{1}$$

where $B(g)$ is a symmetric $(0,2)$-tensor field. The Ricci flow appears when $B(g) = -2 \text{Ric}(g)$, where $\text{Ric} = \text{Trace}_2 R$ is the Ricci curvature of the curvature tensor $R$.

The Levi-Civita connection for (1) evolves as, see e.g. [2],

$$2 g_t((\partial_t \nabla^t)(X, Y), Z) = (\nabla^t_X B_t)(Y, Z) + (\nabla^t_Y B_t)(X, Z) - (\nabla^t_Z B_t)(X, Y) \tag{2}$$

for all $X, Y, Z \in \Gamma(T(M))$. Furthermore, the Riemannian curvature $(0,4)$-tensor evolves as

$$2 \partial_t R(X, Y, Z, V) = \nabla^2_{X,V} B(Y, Z) + \nabla^2_{Y,Z} B(X, V) - \nabla^2_{X,Z} B(Y, V) - \nabla^2_{Y,V} B(X, Z) + B(R(X, Y)Z, V) - B(R(X, Y)V, Z). \tag{3}$$

Some authors consider flows of metrics on a foliated manifold with the metric varying along transverse (to the leaves) distribution, e.g., second-order quasilinear transversally parabolic flows [2], which can be applied to other flows like the transverse Ricci flow and Sasaki-Ricci flow. In [19]–[24], they study flows of metrics on a foliation called extrinsic geometric flows: although the metric varies along normal to the leaves distribution, such flows are parabolic along the leaves.

Let $\tilde{D} = T\mathcal{F}$ be the tangent distribution to an $n$-dimensional foliation $\mathcal{F}$ and $D$ the normal (i.e., $g$-orthogonal to $\tilde{D}$) distribution of dimension $p$. Denote $\updownarrow$ and $\perp$ orthogonal projections onto $\tilde{D}$ and $D$, respectively. A local adapted orthonormal frame $\{E_a, \mathcal{E}_i\}$, where $\{E_a\} \subset \tilde{D}$ and $\{\mathcal{E}_i\} \subset D$, always exists on $M$. The partial Ricci curvature is defined by, see [20] [21],

$$r(X, Y) = \sum_a R(X, E_a, Y, E_a), \quad (X, Y \in TM),$$

and

$$\text{Ric}^\perp(X) = \sum_a (R(X, E_a)E_a) \perp$$

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is the $(1,1)$-tensor adjoint to $r$. Observe that $\text{Ric}^+ = \text{Trace}_2 R^+$, where

$$R^+(X, U, Y, V) := R(X^\perp, U^\perp, Y^\perp, V^\perp),$$

is related to the extrinsic geometry of the distributions. Certain analogue in a sense of the Ricci flow for foliations, is the partial Ricci flow; see \[20\] \[21\],

$$\partial_t g = -2r(g).$$

(4)

Given a leaf-wise constant $\Phi : M \to \mathbb{R}$, the normalized partial Ricci flow is defined by

$$\partial_t g = -2r(g) + 2\Phi g^\perp,$$

(5)

where $g = g^\top + g^\perp$. The flow \(5\), preserves metric on the leaves of $\mathcal{F}$ and the orthogonality of vectors to $T\mathcal{F}$; if $\mathcal{F}$ is either totally umbilical, totally geodesic or harmonic foliation for $t = 0$ then it has the same property for all $t > 0$. It was proposed as the main tool to prescribe the partial Ricci and the mixed sectional curvature of a foliation.

The principal difference of the partial Ricci flow from other known flows with metric varying along $\mathcal{D}$ is that the PDE’s under consideration are parabolic along the leaves and not along $\mathcal{D}$. For a general foliation, the topology of the leaf through a point can change crucially with the point, this gives difficulties in studying such PDE’s. Thus, we assume the following condition:

$$\text{a closed manifold } M \text{ is fibered} \quad \text{instead of being just foliated}. \quad \text{(6)}$$

Observe that if \(6\) holds than all the leaves (fibers) are compact. We will use the following short-time existence and uniqueness result about the partial Ricci flow, see \[20\] \[21\].

**Theorem 1.** The linearization of \(5\) at $g_0$ on a foliation is a leaf-wise parabolic PDE, hence, \(5\) under assumption \(6\) has a unique smooth solution $g_t$ defined on a positive time interval $[0, t_0]$.

In the work, we show that the partial Ricci flow preserves metric of a $g$-foliation. We are interested in applications of \(5\) to $g$-foliations. In this regard, we introduce and study new structures, which generalize the 3-(quasi-)Sasakian structure, an $f$-structure with parallelizable kernel, and an almost para-$\phi$-structure with complemented frames and are of particular interest in the study of the partial Ricci flow of metrics on a totally geodesic foliation. We show convergence of the partial Ricci flow with any of our novel structures.

1 Main results

1.1 Preliminaries: totally geodesic foliations

We assume that the foliations under consideration are totally geodesic. With that in mind we recall some notions and results of \[20\]. The second fundamental tensor and the integrability tensor of a totally geodesic foliation $\mathcal{F}$ vanish. The second fundamental tensor $h : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ and the integrability tensor $T : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ of normal distribution $\mathcal{D}$ are

$$h(X, Y) = (1/2)(\nabla_X Y + \nabla_Y X)^\top, \quad T(X, Y) = (1/2)(\nabla_X Y - \nabla_Y X)^\top, \quad X, Y \in \mathcal{D}.$$  \quad \text{(7)}

The shape operator $A$ and the skew-symmetric operator $T^\sharp$ are given, respectively, by:

$$\langle A\xi, X \rangle = \langle h(X, Y), \xi \rangle, \quad \langle T^\sharp\xi, X \rangle = \langle T(X, Y), \xi \rangle, \quad X, Y \in \mathcal{D}, \quad \xi \in \tilde{\mathcal{D}}.$$  \quad \text{(8)}

Since $T(X, Y) = 1/2 [X, Y]^\top$, the distribution $\mathcal{D}$ is tangent to a foliation $\mathcal{F}^\perp$ if and only if $T = 0$. If $h = 0$ then $\mathcal{D}$ is a totally geodesic distribution. The trace of $h$, called the mean curvature vector of $\mathcal{D}$, is given by $H = \sum_i h(E_i, E_i)$. Define the co-nullity tensor $C : \tilde{\mathcal{D}} \times \mathcal{D} \to \mathcal{D}$ by

$$C(\xi, X) = -(\nabla_X \xi)^\perp, \quad X \in \mathcal{D}, \quad \xi \in \tilde{\mathcal{D}},$$

and set $C_\xi = C(\xi, \cdot)$. We have the identities

$$A_\xi = (C_\xi + C_\xi^*)/2, \quad T^\sharp_\xi = (C_\xi - C_\xi^*)/2, \quad C_\xi = A_\xi + T^\sharp_\xi.$$  \quad \text{(8)}

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The leafwise divergence of a vector field $X \in \mathcal{D}$ is given by
\[ \text{div}_\mathcal{F} X = \sum_i \langle \nabla_{\xi_i} X, \xi_i \rangle, \]
More generally, the divergence of a rank $(1, k)$ tensor $S$ is a rank $(0, k)$ tensor
\[ (\text{div}_\mathcal{F} S)(X_1, \ldots, X_k) = \sum_i \langle (\nabla_{\xi_i} S)(X_1, \ldots, X_k), \xi_i \rangle, \]

**Lemma 1.** For a totally geodesic foliation $\mathcal{F}$ on $(M, g)$ we have
\[ R(\xi_1, X)\xi_2 = (\nabla_{\xi_1} C)(\xi_2, X) - C(\xi_2, C(\xi_1, X)), \quad \xi_1, \xi_2 \in \mathcal{D}, \quad X \in \mathcal{D}, \]
\[ \text{Ric}^\perp = \text{div}_\mathcal{F} h - \sum_i (A^0_{\xi_i} + (T^\perp_{\xi_i})^2), \]
where $(\xi_i)$ is a local orthonormal frame of $\mathcal{D}$.

Set $R_\xi : X \to R(\xi, X)\xi$. The symmetric and antisymmetric parts of $R_\xi$ with $\xi_i = \xi$ are
\[ R_\xi = \nabla_\xi A_\xi - A^0_\xi - (T^\perp_\xi)^2, \quad \nabla_\xi T^\perp_\xi = A_\xi T^\perp_\xi + T^\perp_\xi A_\xi, \]
respectively. The notion of the $\mathcal{D}$-truncated $(r, 2)$-tensor $B^\perp$ for $r = 0, 1$ will be helpful:
\[ B^\perp(X, Y) := B(X^+, Y^\perp). \]
For example, the $\mathcal{D}$-truncated metric tensor is $g^\perp(X, Y) := (X^+, Y^\perp)$, $X, Y \in TM$. Using that definition we can formulate and prove the following result, cf. [20].

**Proposition 1.** The geometric quantities of a totally geodesic foliation related to $\mathcal{D}$ evolve by (with a $\mathcal{D}$-truncated symmetric $(0, 2)$-tensor $B(g)$) according to
\[ 2 \partial_t A_\xi = -\nabla_\xi B^\perp + [A_\xi - T^\perp_\xi, B^\perp], \quad \partial_t T^\perp_\xi = -B^\perp T^\perp_\xi, \quad \xi \in \mathcal{D}. \]

**Proof.** Note that $\partial_t T = 0$. For all $X, Y \in \mathcal{D}$, using (2) and (7), we have
\[ 2 g(\partial_t (\nabla_X Y), \xi) = (\nabla_X B)(Y, \xi) + (\nabla^\perp_X B)(X, \xi) - (\nabla^\perp_Y B)(X, Y) = -\nabla^\perp_Y B)(X, Y) - B(Y, \nabla_X \xi) - B(X, \nabla_Y \xi). \]
From this and symmetry of $\partial_t \nabla^\perp$, we have
\[ 2 g(\partial_t h(X, Y), \xi) = -\nabla_\xi B(X, Y) + B(Y, C_\xi(X)) + B(X, C_\xi(Y)). \]
Using
\[ g(A_\xi(X, Y), \xi) = g(h(X, Y), \xi), \quad g(T^\perp_\xi(X, Y), \xi) = g(T(X, Y), \xi) \quad (X, Y \in \mathcal{D}), \]
we then find
\[ g(\partial_t A_\xi(X, Y), \xi) = \partial_t g(h(X, Y), \xi) - (\partial_t g)(A_\xi(X, Y)), \quad g(\partial_t T^\perp_\xi(X, Y), \xi) = - (\partial_t g)(T^\perp_\xi(X, Y)). \]
The above, (8) and (12) yield (11). \qed

**Remark 1.** Finally, let us consider the behaviour of tensor fields associated to Riemannian metric with respect to diffeomorphisms. Let $(M, g)$ be a Riemannian metric and let $f : M \to M'$ be a diffeomorphism. Then $f$ is an isometry between $(M, g)$ and $(M', g' = f_* g)$. The Levi-Civita connections $\nabla^g$ and $\nabla^{g'}$ are $f$-related, i.e.,
\[ df(\nabla^g_X Y) = \nabla^{g'}_{df(X)} df(Y) \]
for any vector fields $X, Y$ on $M$. Therefore, cf. [13] Propositions VI.1.2 and VI.1.4], the torsion tensor $T^g$ is $f$-related to $T^{g'}$ and the curvature tensor field $R^g$ is $f$-related to $R^{g'}$. If we just consider $f$-related orthonormal bases then any tensor field $B$ obtained via contractions, traces etc., are $f$-related, i.e.,
\[ df(B(g))_x = B(g')_{f(x)}. \]
If we consider truncated or “partial” tensor fields, we have to consider diffeomorphisms $f$, which preserve the foliation, i.e.,
\[ df(T \mathcal{F}) \subset T \mathcal{F}. \]
Then $df(T \mathcal{F}^{\perp g}) \subset T \mathcal{F}^{\perp g'}$; thus, truncated (partial) tensors are also related.
1.2 $\mathfrak{g}$-manifolds and $\mathfrak{g}$-foliations

If there is a Lie algebra homomorphism from a Lie algebra $\mathfrak{g}$ into the Lie algebra $\mathcal{X}(M)$ of all vector fields on the manifold $M$ we say that $M$ is a $\mathfrak{g}$-manifold or that $\mathfrak{g}$ acts on $M$.

In their paper [1], D. Alekseevsky and P. Michor have began the study of smooth manifolds $M$ with an action of constant rank of a Lie algebra $\mathfrak{g}$, or a bit more restrictively, a locally free action. In such a case one can define on the manifold $M$ a $\mathfrak{g}$-valued 1-form, which they call a principal connection. In the paper they investigate the properties of this form and the topology of $\mathfrak{g}$-manifolds. They complete their study introducing characteristic classes associated to this structure with values in the basic cohomology of the foliation defined by the action of $\mathfrak{g}$.

Several classes of $\mathfrak{g}$-manifolds with some additional geometrical structures have been studied in great detail. If we take a 1-dimensional Lie algebra, a $\mathfrak{g}$-manifold is then just a smooth manifold with a nonvanishing vector field. In this case we have almost contact, strict contact, (almost) contact metric structures as well as $K$-contact and Sasakian ones. If we take a higher dimensional abelian Lie algebra, then as examples of such $\mathfrak{g}$-manifolds can serve $K, S$ or $C$ manifolds, e.g., cf. [10, 9]. 3-Sasakian manifolds, cf. [7], form a special class of $so(3)$-manifolds. G. Cairns demonstrated that the study of geometrical and topological properties of totally geodesic foliations can be reduced to the study of these properties of the derived $\mathfrak{g}$-foliation, cf. [8]. As the author is interested more in foliations themselves, these structures are called tangentially Lie foliations. Obviously, a manifolds with a tangentially Lie foliation is a $\mathfrak{g}$-manifold, and a $\mathfrak{g}$-manifold with a locally free $\mathfrak{g}$-action admits a tangentially $\mathfrak{g}$-foliation. If the action of the Lie algebra is not locally free then the situation is a bit more complicated. Another context in which $\mathfrak{g}$-manifolds appear is the geometrical study of ODE and PDE on smooth manifolds. If an ODE or PDE admits a nontrivial symmetry group, or more precisely nontrivial infinitesimal symmetries and these infinitesimal symmetries define a regular foliation, the ambient manifold is a $\mathfrak{g}$-manifold. The properties of this foliated $\mathfrak{g}$-manifold influence the properties of the solutions of our system of differential equations, and in some cases permit to solve the system effectively, cf. [18, 26].

A foliation $\mathcal{F}$ on a Riemannian manifold $(M, g)$ is called a tangentially Lie foliation if there is a complete Lie parallelism $\{\xi_1, \ldots, \xi_n\}$, along its leaves, that preserves the horizontal subbundle $\mathcal{H} = \mathcal{F}^\perp$. Another term used is a tangentially $\mathfrak{g}$, where in this case $\mathfrak{g}$ is the Lie subalgebra of $\mathcal{X}(M)$ spanned by the tangential vector fields $X_i$. For convenience, we can suppose that the vector fields $X_i$ are orthonormal.

Let us return to general $\mathfrak{g}$-manifolds. There is a Lie algebra homomorphism $^\ast : \mathfrak{g} \to \mathcal{X}(M)$ from $\mathfrak{g}$ into the Lie algebra $\mathcal{X}(M)$ of all vector fields on $M$. Let $\{\xi_i\}$ denote a basis of $\mathfrak{g}$ and let denote $^\ast(\xi_i) = \xi_i^\ast$ and we also write $\xi^\ast$ for any $^\ast(\xi) \in \mathfrak{g}$. This somewhat cumbersome notation is meant to accommodate for that our main motivation and examples come from actions of Lie group on a manifold (and it is then customary to mark the transported invariant fields with $^\ast$). Since $^\ast$ is a homomorphism of Lie algebras, the distribution spanned by its image is involutive and defines a foliation $F_{\mathfrak{g}}$, called the characteristic foliation of the $\mathfrak{g}$-manifold $M$. We are interested only in those $\mathfrak{g}$-manifolds for which the characteristic foliation is regular and its dimension is equal to the dimension of the Lie algebra $\mathfrak{g}$. Let $G$ be a simply connected Lie group, whose Lie algebra is isomorphic to $\mathfrak{g}$. Then there is a natural locally free right action of $G$ on the manifold $M$, whose orbits are the leaves of of the foliation $\mathcal{F}$, cf. [17].

The $\mathfrak{g}$-valued 1-form $\eta$ on $M$ given by the formula

$$\eta(X) = \Sigma_i g(X, \xi_i^\ast)\xi_i, \quad X \in T_xM,$$

is called a transverse connection. The above $\mathfrak{g}$-valued one-forms are considered in [1] and called connections. Such a one-form with values in $\mathfrak{g}$ is determined by its kernel, i.e., a subbundle supplementary to the characteristic foliation. P. Molino investigated $\mathfrak{g}$-foliations in [14], where such subbundles are called presque-connexions. A Lie algebra $\mathfrak{g}$-valued one-form $\eta$ is called a principal connection form if the following conditions are satisfied:

1. $\eta$ is $\mathfrak{g}$-equivariant, i.e., for all $v \in \mathfrak{g}$ we have $L_v\eta = -\text{ad}(v)\eta$.

2. For any $x \in M$ and $v \in \mathfrak{g}$, we have $v^\ast(x) = \eta(v^\ast(x))^\ast(x)$.
Any connection form $\eta$ defines a connection understood as the projection from the tangent bundle onto its subbundle of vectors tangent to the leaves of the characteristic foliation.

On $\mathfrak{g}$-manifolds with regular characteristic foliation principal connections exist in several important cases, cf. Proposition 3.2 of [1]:

1. The action is locally trivial.
2. There exists a $\mathfrak{g}$-invariant Riemannian metric on $M$.
3. The $\mathfrak{g}$-action is induced by a proper action of a Lie group $G$ with Lie algebra $\mathfrak{g}$.

The $\mathfrak{g}$-valued 2-form $\Omega$ defined as $\Omega = d\eta + (1/2)[\eta, \eta]$ is called the curvature form of the transverse connection $\eta$. The form $\Omega$ vanishes on vertical vector fields, and it is identically zero if and only if the horizontal bundle $\mathcal{H}$ is integrable.

As in the case of classical connections we have the following property, cf. [1]:

**Proposition 2.** For all elements $g \in G$ we have $g^*\eta = \text{Ad}(g^{-1})\eta$ and $g^*\Omega = \text{Ad}(g^{-1})\Omega$.

For more properties of principal connections on $\mathfrak{g}$-manifolds as well the conditions ensuring the existence, see [1].

### 1.3 The partial Ricci flow on $\mathfrak{g}$-foliations

Here, we study tangentially Lie foliations, whose characteristic foliation is regular and its dimension is equal to the dimension of the Lie algebra $\mathfrak{g}$. We restrict our attention to compatible metrics of $\mathfrak{g}$. For compatible metrics, the characteristic foliation is Riemannian, and the mixed sectional curvature is nonnegative, thus, $\text{Ric}_g^\perp \geq 0$.

**Lemma 2.** The class of compatible metrics of $\mathfrak{g}$-foliations is preserved by the partial Ricci flow.

**Proof.** Consider solution of (1) with the initial condition $g = g_0$ - a compatible metric, where $B(g) = -2r(g)$ is a symmetric $(0,2)$-tensor depending on the metric $g$. If a smooth diffeomorphism $\phi: M \to M$ is an isometry of $g$, then $\phi^*g = g$. Therefore, $\phi^*r(g) = r(\phi^*g) = r(g)$ for any isometry $\phi$ of $g$ preserving $\mathcal{D}$, see Remark [1]. Then

$$\partial_t(\phi^*g_t) = \phi^*(\partial_t g_t) = \phi^*(r(g_t)) = r(\phi^*g_t).$$

Hence, the family $\phi^*g_t$ is the evolution of $\phi^*g_0 = g_0$. From the uniqueness of solution, see Theorem [1] we get $\phi^*g_t = g_t$ for any $t$. So the isometry $\phi$ of $g$ is also an isometry of any metric of its evolution. \qed

By Lemma [2] assumption $h = 0$, and Lemma [1] we obtain the following.

**Proposition 3.** For any $\mathfrak{g}$-foliation we have

$$R(\xi, X)\xi_j = \nabla_{\xi_j} T^2_{\xi_j} (X) - T^2_{\xi_j} T^2_{\xi_j} (X), \quad X \in \mathcal{D}, \; \xi, \xi_j \in \widetilde{\mathcal{D}},$$

$$\text{Ric}_g^\perp = -\sum_i (T^2_{\xi_i})^2.$$  \hspace{1cm} (14) \hspace{1cm} (15)

Moreover, (14) yields $R_{\xi} = -(T^2_{\xi})^2 \geq 0$ and $\nabla_{\xi} T^2_{\xi} = 0$ for $\xi \in \widetilde{\mathcal{D}}$.

**Lemma 3.** For compatible metrics of $\mathfrak{g}$-foliations, $\|T\|$ is leafwise constant and

$$\nabla_{\xi} \text{Ric}_g^\perp = [\text{Ric}_g^\perp, T^2_{\xi}], \quad \xi \in \widetilde{\mathcal{D}}.$$  \hspace{1cm} (16)

**Proof.** By Lemma [2] one may consider the partial Ricci flow family $g_t$ of compatible metrics. By (14) with $B = -2r(g) + 2\Phi g^{-1}$, using $A_{\xi} = 0$ and $\nabla_{\xi} \text{id}^\perp = 0$ (for Riemannian foliations), we get (16). Taking trace of (16) and using (15), we find $\text{Ric}_g^\perp = \|T\|^2$ and $\xi(\|T\|^2) = 0$. \qed

**Proposition 4.** The partial Ricci flow (5) for compatible metrics on $\mathfrak{g}$-foliations obeys the following ODE’s in variable $t$:

$$\partial_t T^2_{\xi} = 2(\text{Ric}_g^\perp - \Phi \text{id}^\perp) T^2_{\xi},$$

$$\partial_t \text{Ric}_g^\perp = 2 \text{Ric}_g^\perp (\text{Ric}_g^\perp - 2\Phi \text{id}^\perp) - 2 \sum_i T^2_{\xi_i} \text{Ric}_g^\perp T^2_{\xi_i}.$$  \hspace{1cm} (17) \hspace{1cm} (18)
Proof. By (11) with \( B = -2r(g) + 2\Phi g^\perp \), using \( A_\xi = 0 \) and \( \nabla_\xi g^\perp = 0 \) (for Riemannian foliations), we get (17). Derivation of (15) in \( t \) and using (15) and (17) yield (18). \( \Box \)

**Remark 2.** The leafwise Laplacian of a \( g \)-foliation \( \mathcal{F} \) acts on tensors by \( \Delta_\mathcal{F} = \text{Trace}_\mathcal{F}(\nabla^2) = \sum_i \nabla_{\xi_i,\xi_i} \). Applying \( \nabla_\xi \) to (16) with \( \xi = \xi_i \) and summing by \( i \), we obtain

\[
\Delta_\mathcal{F} \text{Ric}^\perp = -2(\text{Ric}^\perp)^2 - 2 \sum_i T_{\xi_i}^\perp \text{Ric}^\perp T_{\xi_i}^\perp.
\]

The above and (18) provide the following PDE: \( \partial_t \text{Ric}^\perp = \Delta_\mathcal{F} \text{Ric}^\perp + 4 \text{Ric}^\perp(\text{Ric}^\perp - \Phi \text{id}^\perp) \).

**Example 1.** Let \( \mathcal{F} \) be a one-dimensional foliation by geodesics spanned by a unit vector field \( \xi \) (e.g., \( \xi \) is a unit Killing vector). Then \( \text{Ric}^\perp = R_\xi \) (the Jacobi operator in the \( \xi \)-direction). Recall that \( C_\xi \) is the conullity tensor, \( T_{\xi_i}^\perp \) the integrability tensor, and \( A_\xi \) the shape operator of the normal distribution \( \mathcal{D} \). Then

\[
\partial_t R_\xi = \nabla_\xi^2 R_\xi - (\nabla_\xi R_\xi) C_\xi - C_\xi^* \nabla_\xi R_\xi - R_\xi C_\xi^2 -(C_\xi^*)^2 R_\xi + 2 C_\xi^* R_\xi C_\xi, \\
\partial_t \text{Ric}_{\xi,\xi} = \xi(\xi(\text{Ric}_{\xi,\xi})) - 2 \text{Trace}(A_\xi \nabla_\xi R_\xi) - 4 \text{Trace}((T_{\xi_i}^\perp)^2 R_\xi).
\]

For (5), we also have

\[
\partial_t T_{\xi}^\perp = 2(\nabla_\xi A_\xi)T_{\xi}^\perp - 2 A_\xi^2 T_{\xi}^\perp - 2 (T_{\xi}^\perp)^3 - 2 \Phi T_{\xi}^\perp, \\
\partial_t A_\xi = \nabla_\xi(\nabla_\xi A_\xi) - 2 A_\xi \nabla_\xi A_\xi + [A_\xi^2, T_{\xi}^\perp] - 2 (T_{\xi_i}^\perp)^2 A_\xi - 2 T_{\xi_i}^\perp A_\xi T_{\xi_i}^\perp.
\]

For a \( g \)-foliation \( \mathcal{F} \), we have \( \nabla_\xi R_\xi = 0 \), and (18) reads as

\[
\partial_t \text{Ric}^\perp = -2 R_\xi(\text{Ric}^\perp + \Phi \text{id}^\perp).
\]

The following theorem shows that metrics of \( g \)-foliations with certain conditions can be deformed to metrics of the same type but with leafwise constant partial Ricci curvature.

**Theorem 2.** Let \( \mathcal{F} \) be a \( g \)-foliation of \( (M, g_0) \) spanned by \( p \geq 1 \) orthonormal vector fields \( \{\xi_i\} \). If \( r(g_0) > 0 \) on the normal distribution \( \mathcal{D} \) then (5) with \( \Phi > 0 \) has a unique solution \( g_t \) (\( t \in \mathbb{R} \)).

(i) If the following recurrent relation holds for some \( \lambda : M \to \mathbb{R} : \)

\[
\nabla_\xi \text{Ric}^\perp = \xi(\lambda)(\text{Ric}^\perp - (1/p) \|T\|^2 \text{id}^\perp), \quad \xi \in \tilde{\mathcal{D}},
\]

then there exists \( \lim_{t \to -\infty} \text{Ric}^\perp(g_t) = (\Phi - (\Delta_\mathcal{F} \lambda + |\nabla^\mathcal{F} \lambda|^2)/4) \text{id}^\perp. \)

(ii) If \( \text{Ric}^\perp = \mu \text{id}^\perp \) for some leafwise constant positive function \( \mu : M \to \mathbb{R} \), then there exists metric \( \tilde{g} = \lim_{t \to -\infty} g_t \) and we have \( \lim_{t \to -\infty} r(g_t) = \Phi \tilde{g}^\perp. \)

Proof. For a bundle-like metric, \( \mathcal{F} \) is Riemannian, thus \( A_i = 0 \) and \( C_i = T_{\xi_i}^\perp \). By conditions, \( 0 < \mu_{\min} \text{id}^\perp \leq \text{Ric}^\perp \leq \mu_{\max} \text{id}^\perp \), where \( \mu_{\min} \) (\( \mu_{\max} \)) is minimal (maximal) eigenvalue of \( \text{Ric}^\perp \). We then have

\[
0 < \mu_{\min} \text{Ric}^\perp \leq - \sum_i T_{\xi_i}^\perp \text{Ric}^\perp T_{\xi_i}^\perp \leq \mu_{\max} \text{Ric}^\perp.
\]

Thus, (18) yields the following differential inequalities:

\[
2 \text{Ric}^\perp(\text{Ric}^\perp + (\mu_{\min} - 2\Phi) \text{id}^\perp) \leq \partial_t \text{Ric}^\perp \leq 2 \text{Ric}^\perp(\text{Ric}^\perp + (\mu_{\max} - 2\Phi) \text{id}^\perp).
\]

Consider the comparison matrix ODE with \( 2\Phi > \alpha \in \mathbb{R}, \)

\[
\partial_t Z = 2Z(\alpha + (2\Phi) \text{id}^\perp).
\]

Let \( \mu_i(t) \) be the eigenvalue and \( e_i(t) \) the \( g_t \)-unit eigenvector of the solution of (21). Observe that (21) preserves the directions of \( \{e_i\} \) and yields the system

\[
\dot{\mu}_i = 2\mu_i(\mu_i + \alpha - 2\Phi), \quad 1 \leq i \leq n.
\]
It has global solution $\mu_i(t) = \frac{\mu_0(0)(2\Phi-\alpha)}{\mu_0(0)+\exp(4\Phi t)(2\Phi-\alpha-\mu_0(0))}$ with $\mu_i(0) > 0$; moreover, $\lim_{t \to -\infty} \mu_i(t) = 2\Phi - \alpha$. By the above, (13) has a global solution $\text{Ric}^+(t)$. Thus, (5) has solution $g_t \ (t \in \mathbb{R})$.

(i) Set $\Psi_1 := \frac{1}{4}(\Delta g \lambda + \|\nabla g \lambda\|)^2 - \Phi$ and $\Psi_2 := \frac{1}{4} \|\nabla g \lambda\|^2 \|\tau\|^2 \geq 0$. Notice that (20) is compatible with (10): RHS of equations have zero traces. If (20) holds then

$$\Delta g \text{Ric}^+ = \left(4\Psi_1 + \Phi\right) \text{Ric}^+ - \Psi_2 \text{id}^+.$$ 

Hence, each eigenvalue $\mu_j$ of $\text{Ric}^+$ satisfies ODE $\dot{\mu}_j = 4\mu_j(\mu_j + \Psi_1) - \Psi_2$, which has two stationary solutions $\mu_{\pm} = \frac{1}{4}(-\Psi_1 \pm \sqrt{\Psi_1^2 + \Psi_2})$ (functions on $M$). Here, $\mu_+ > 0$ is attractor for $t \to -\infty$. Since we assume $\text{Ric}^+ > 0$, by the above, $\lim_{t \to -\infty} \text{Ric}^+(g_t) = \mu_+ \text{id}^+$.

(ii) If $\text{Ric}^+ = \mu \text{id}^+$ then $\text{Ric}^+(t) = \mu(t) \text{id}^+$ for all $t$, where $\mu = 4\mu(\mu - \Phi)$. Hence, $\mu(t) = \frac{\mu(0)\Phi}{\mu_0(0)+\exp(4\Phi t)(\Phi-\mu(0))}$ with $\mu(0) > 0$ and $\lim_{t \to -\infty} \mu(t) = \Phi$. Let $\{e_i(t)\}$ be a $g_t$-orthonormal frame of $\mathcal{D}$. We then have $\partial_t e_i = (\mu - \Phi)e_i$. Since $e_i(t) = z(t)e_i(0)$ with $z(0) = 1$, then $\partial_t \log z(t) = \mu(t) - \Phi$. By the above, $z(t) = (\mu(t)/\mu(0))^{1/4}$. Hence,

$$g_t(e_i(0), e_j(0)) = z^{-2}(t)g_0(e_i(0), e_j(0)) = \delta_{ij}(\mu^2(t)/(\mu_i(0)\mu_j(0)))^{-1/4}.$$ 

As $t \to -\infty$, $g_t$ converges to the metric $\hat{g}$ determined by $\hat{g}(e_i(0), e_j(0)) = \delta_{ij}\sqrt{\mu(0)/\Phi}$. □

A nonsingular Killing vector clearly defines a Riemannian flow; moreover, a Killing vector of unit length generates a geodesic Riemannian flow. Recall [5] that a $K$-contact structure is a contact metric structure, for which the characteristic (Reeb) vector field is Killing.

The next corollary of Theorem [2] generalizes [5] Proposition 7.4.

**Corollary 1.** Let a Riemannian manifold $(M^{2n+1}, g_0)$ admit a unit Killing vector field $\xi$. If the Jacobi operator $R_\xi$ is positive definite on the orthogonal to $\xi$ distribution $\mathcal{D}$ at $t = 0$, then the flow (15) with $\Phi > 0$ has a unique solution $g_t \ (t \in \mathbb{R})$ and there exists metric $\hat{g} = \lim_{t \to -\infty} g_t$. Moreover, if $\Phi = 1$ then $M$ is a $K$-contact manifold for the metric $\hat{g}$. □

**Proof.** By the proof of Theorem [2] for $p = 1$, applied to (19), (15) has a unique solution $g_t \ (t \leq 0)$ and there exists $\hat{g} = \lim_{t \to -\infty} g_t$. If $\Phi = 1$ then $\lim_{t \to -\infty} R_\xi(g_t) = \text{id}^+$. Then we have

$$\partial_t e_i = (\mu_i - \Phi)e_i, \quad \partial_t (\log \mu_i) = 4(\mu_i - \Phi),$$

where $g_t(e_i(t), e_j(t)) = \delta_{ij}$. Thus, the partial Ricci flow preserves the directions of eigenvectors of $\text{Ric}^+$ and $\mu_i(t) = \frac{\mu_i(0)\Phi}{\mu_i(0)+\exp(4\Phi t)(\Phi-\mu_i(0))}$ with $\mu_i(0) > 0$. Since $e_i(t) = z_i(t)e_i(0)$ with $z_i(t) = 1$, then $\partial_t \log z_i = \mu_i - 1$. The above provide $z_i = (\mu_i(t)/\mu_i(0))^{1/4}$. Hence,

$$g_t(e_i(0), e_j(0)) = z_i^{-1}z_j^{-1}(t)g_0(e_i(0), e_j(0)) = \delta_{ij}(\mu_i(t)\mu_j(t)/(\mu_i(0)\mu_j(0)))^{-1/4}.$$ 

As $t \to -\infty$, $g_t$ converges to a metric $\hat{g}$ determined by $\hat{g}(e_i(0), e_j(0)) = \delta_{ij}(\mu_i(0)\mu_j(0))^{1/4}$. If $\xi$ is Killing for $g$ then $\xi$ is Killing for $\hat{g}$; by [5] Proposition 7.4, $M$ is $K$-contact for metric $\hat{g}$. □

2 Applications

In the study of flows there are two important problems to consider: the limit sets and the stationary/fixed points. We have investigated the first problem in the previous section. Presently we will describe three examples of structures with constant partial Ricci curvature, which provide examples of fixed points (stable in some classes of metrics) of the flow (15) with certain values of $\Phi$. In the final part of the section we introduce structures, generalizing known ones, and show convergence of the partial Ricci flow on new structures.
2.1 Structures with constant partial Ricci curvature

In the last 60 years many very special Riemannian structures have been studied. In many cases totally geodesic foliations appear quite naturally. Some of them provide examples very interesting from the point of view of our study.

**Example 2.** Recall that an almost contact manifold \( (M, \phi, \xi, \eta) \) is a \((2n + 1)\)-dimensional manifold \( M \), which carries a \((1, 1)\)-tensor field \( \phi \), a vector field \( \xi \), called characteristic or Reeb vector field, and a dual 1-form \( \eta \), satisfying

\[
\phi^2 = -\text{id} + \eta \otimes \xi, \quad \eta(\xi) = 1,
\]

where \( \text{id} : TM \to TM \) is the identity mapping. Then \( \phi \xi = 0, \eta \circ \phi = 0 \) and \( \phi \) has rank \( 2n \), see [5] Theorem 4.1. There exists a metric \( g \) (in general, not unique) such that, see e.g. [5],

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

and in this case we get an almost contact metric structure. Setting \( Y = \xi \), we get \( \eta(X) = g(X, \xi) \). For a contact metric structure we have, in addition,

\[
g(X, \phi Y) = d\eta(X, Y).
\]

An almost contact metric manifold \( (M, \phi, \xi, \eta, g) \) is normal if \( N_\phi := N_\phi + 2d\eta \otimes \xi = 0 \), where

\[
N_\phi(X, Y) := \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]
\]

is the Nijenhuis torsion of \( \phi \). A quasi-Sasakian structure is a normal almost contact metric structure, whose fundamental 2-form \( F \) defined by \( F(X, Y) = g(X, \phi Y) \) is closed. A Sasakian structure is a normal contact metric structure. They are characterized by the following property:

\[
R(X, \xi)Y = (\xi, Y)X - (X, Y)\xi.
\]  

Observe that (22) is equivalent to the following two conditions:

\[
\begin{align*}
R(X, \xi)X &= X, \quad X \in \mathcal{D}, \quad (23) \\
(R(X, Y)Z)^\top &= 0, \quad X, Y, Z \in \mathcal{D} \quad \text{(the curvature-invariance of \( \mathcal{D} \)).} \quad (24)
\end{align*}
\]

When the foliation under consideration is of higher dimension, i.e. the dimension of the characteristic foliation is of dimension greater than 1, there are several possible options. In the case of dimension 3, an (almost) contact metric structure can be replaced by an (almost) contact 3-structure, defined as a set of three (almost) contact structures, \((M, \phi_i, \xi_i, \eta^i)\),

\[
\phi_i^2 = -\text{id} + \eta^i \otimes \xi_i, \quad \eta^i(\xi_j) = \delta^i_j
\]

with the same compatible metric \( g \), i.e., \( g(\phi_i X, \phi_i Y) = g(X, Y) - \eta^i(X)\eta^i(Y) \), obeying

\[
\phi_k = \phi_i \circ \phi_j - \eta^i \otimes \xi_j = -\phi_j \circ \phi_i + \eta^j \otimes \xi_i
\]

for any cyclic permutation \((i, j, k)\) of \((1, 2, 3)\), see [5]. Further generalization to almost \( k \)-contact (metric) manifolds was given in [4]. The dimension of \( M \) with an almost contact 3-structure is \( 4n + 3 \). We get a 3-(quasi-)Sasakian structure if each of \((\phi_i, \xi_i, \eta^i)\) is a (quasi-)Sasakian structure,

\[
g(X, \phi_i Y) = d\eta^i(X, Y), \quad i = 1, 2, 3.
\]

Note that every contact metric 3-structure is 3-Sasakian. Let a distribution \( \tilde{\mathcal{D}} \) be spanned by 3 characteristic vector fields \( (\xi_i) \), and \( \mathcal{D} \) its orthogonal complement. For a 3-quasi-Sasakian structure we have \([\xi_i, \xi_j] = c\xi_k\) for some \( c \in \mathbb{R} \) and any cyclic permutation \((i, j, k)\) of \((1, 2, 3)\); thus, \( \tilde{\mathcal{D}} \) is integrable; moreover, it defines a totally geodesic Riemannian foliation with the property \( T^\perp_{\xi_i} = \phi_i \) along \( \mathcal{D} \). Hence, \( \text{Ric}^\perp = -\sum_i(T^i)^2 = 3\text{id}^\perp \), i.e., the partial Ricci curvature is \( 3 \), and the metric \( g \) is a fixed point of the flow \( \Phi \) with \( \Phi = 3 \).
Example 3. Recall that an \textit{f-structure} on a manifold $M^{2n+p}$ is a non-null (1, 1)-tensor field $f$ on $M$ of constant rank $2n$ such that $f^3 + f = 0$. A manifold $M$, provided with an $f$-structure, is said to be an \textit{$f$-manifold}, and it is known that $TM$ splits into sum of subbundles $D = f(TM)$ and $\overline{D} = \ker f$, and that the restriction of $f$ to $D$ determines a complex structure on it.

An interesting case of $f$-structure, introduced in [13], occurs when $\overline{D}$ is parallelizable, see [12], i.e., there exist vector fields $\xi_i, i \in \{1, \ldots, p\}$, with their dual 1-forms $\eta^i$, satisfying
\[
f^2 = -\text{id} + \sum_i \eta^i \otimes \xi_i, \quad \eta^i(\xi_j) = \delta^i_j.
\] (25)

From (25) and $f^3 + f = 0$ it follows that $f \xi_i = 0$ and $\eta^i \circ f = 0$ for any $i \in \{1, \ldots, p\}$. Such an $f$-structure is called a \textit{globally framed $f$-structure}, briefly denoted $f$-pk-structure, also see, e.g., [10]. Moreover, a manifold $M$ endowed with a $f$-pk-structure is called a $f$-pk-manifold, and it is denoted with $(M, f, \xi, \eta^i)$; the vector fields $\xi_i, i \in \{1, \ldots, p\}$, are called the characteristic vector fields. It is also known that an $f$-structure, on a manifold $M$, is called \textit{normal} if the tensor field $N_f = N_f + 2 \sum \eta_i \otimes \xi_i$ vanishes. A pseudo-Riemannian metric $g$ is compatible (then we obtain a metric $f$-pk-structure), if
\[
g(f(X), f(Y)) = g(X, Y) - \sum_i \varepsilon_i \eta^i(X) \eta^i(Y),
\] (26)
where $\varepsilon_i = g(\xi_i, \xi_i) = \pm 1$. From (26) we get
\[
g(X, \xi_i) = \varepsilon_i \eta^i(X), \quad g(X, f(Y)) = -g(f(X), Y), \quad X, Y \in TM.
\]

Define the 2-form $F$ putting
\[
F(X, Y) = g(X, f(Y)), \quad X, Y \in TM.
\] (27)
A metric $f$-pk-manifold is called \textit{indefinite K-manifold} if it is normal and $dF = 0$. Two subclasses of $K$-manifolds are defined as follows: a metric $f$-pk-manifold is called \textit{indefinite almost C-manifold} if $\eta_i \circ f = 0$ for any $i$, a metric $f$-pk-manifold is called \textit{indefinite almost S-manifold} if $\eta^i = F$ for any $i$. On an indefinite $K$-manifold there exists a $p$-parameter group of isometries generated by the set of Killing vector fields $\{\xi_i\}$, see [6]. Since the sectional curvature is $K(X, \xi_i) = \varepsilon_i$ for unit $X$, then the \textit{partial Ricci curvature} is $\sum_i \varepsilon_i$, and the metric $g$ is a fixed point of (3) with certain $\Phi$.

Example 4. Recall that a smooth manifold $M^{2n+p}$ is called an \textit{almost para-$\phi$-manifold} if it admits a $(1, 1)$-tensor field $\phi$ of rank $2n$, satisfying $\phi^2 - \phi = 0$. The kernel distribution $\overline{D} = \ker \phi$ has dimension $p$ and $\pm 1$ eigen-distributions of $\phi$, denoted by $D^+$ and $D^-$, respectively, have the same dimension equal to $n$. Set $D := \phi(TM) = D^+ \oplus D^-$. Also if there exist vector fields $\xi_i (i \leq p)$, and their dual 1-forms $\eta^i$, satisfying the compatibility conditions
\[
\phi^2 = \text{id} - \sum_i \eta^i \otimes \xi_i, \quad \eta^i(\xi_j) = \delta^i_j,
\] (28)
then $M$ is an \textit{almost para-$\phi$-manifold with complemented frames}. From (28) and $\phi^3 - \phi = 0$ it follows that $\phi \xi_i = 0$ and $\eta^i \circ \phi = 0$ for any $i \in \{1, \ldots, p\}$. If an almost para-$\phi$-manifold $M$ admits a (compatible) pseudo-Riemannian metric $g$ such that
\[
g(\phi X, \phi Y) = -g(X, Y) + \sum_i \varepsilon_i \eta^i(X) \eta^i(Y),
\]
then it is called a \textit{metric almost para-$\phi$-manifold}. A compatible metric $g$ is always of signature $(n + p, n)$, see, e.g., [11],
\[
g(X, Y) = (1/2) \left( \overline{g}(X, Y) - \overline{g}(\phi X, \phi Y) + \sum_i \eta^i(X) \eta^i(Y) \right),
\]
where $\overline{g}(X, Y) = G(\phi^2 X, \phi^2 Y) + \sum_i \eta^i(X) \eta^i(Y)$ and $G$ is any given metric on $M$. An almost para-$\phi$-manifold is \textit{normal} if the tensor $N_{\phi}^{(1)} := N_{\phi} - 2 \sum \eta_i \otimes \xi_i$ vanishes.

On a metric almost para-$\phi$-manifold, we define a 2-form by
\[
F(X, Y) = g(X, \phi(Y)), \quad X, Y \in TM.
\] (29)
If $F = dh^i$ for any $i$, then $M$ is called a para-$\phi$-manifold. A para-$S$-manifold is a normal para-$\phi$-manifold. In this case, see [11, 16], $\nabla X \xi_i = -\phi(X)$ for all $i$ and

$$R(X,Y)\xi_i = \sum_j [\eta^j(X)\phi^2(Y) - \eta^j(Y)\phi^2(X)].$$

For $X = \xi_i, Y \in D$ we get $R(\xi_i, Y)\xi_i = Y$. Hence, $Ric^\perp = p id^\perp$, i.e., the partial Ricci curvature is $p$, and the metric $g$ is a fixed point of the flow $\Phi$ with $\Phi = p$.

In the next three subsections we introduce and study structures, which generalize the ones presented above and are of particular interest in the study of the partial Ricci flow.

### 2.2 Weak almost contact structure

Here, we introduce and study structures, which generalize contact metric structures (see Example [21]), and show convergence of the partial Ricci flow for these structures.

**Definition 1.** A weak almost contact structure $(\phi, \xi, \eta)$ is an odd-dimensional manifold $M$, endowed with a $(1,1)$-tensor field $\phi$, a vector field $\xi$, called characteristic, and a dual 1-form $\eta$, satisfying

$$\phi^2 = -Q + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

where $Q$ is a nonsingular $(1,1)$-tensor field such that $Q\xi = \xi$.

The following proposition generalizes [5, Theorem 4.1].

**Proposition 5.** Suppose that $M^{2n+1}$ has a weak almost contact structure $(\phi, \xi, \eta)$. Then $\phi$ has rank $2n$ and

$$\phi \xi = 0, \quad \eta \circ \phi = 0, \quad [Q, \phi] = 0.$$

**Proof.** By (30), $\phi^2 \xi = 0$, hence, either $\phi \xi = 0$ or $\phi \xi$ is a nontrivial vector of $\ker \phi$. Applying (30) to $\phi \xi$, we get $Q(\phi \xi) = \eta(\phi \xi)\xi$. If $\phi \xi = \mu \xi$ for some nonzero $\mu : M \to \mathbb{R}$ then $0 = \phi^2 \xi = \mu \cdot \phi \xi = \mu^2 \xi$ or $\phi \xi = \mu^2 \xi$ is a contradiction. Assuming $\phi \xi = \mu \xi + X$ for some $\mu : M \to \mathbb{R}$ and nonzero $X \in \ker \eta$, again by (30) we get $QX = 0$ – a contradiction. Hence, $\phi \xi = 0$.

Next, since $\phi \xi = 0$ everywhere, rank $\phi < 2n + 1$. If a vector field $\xi$ satisfies $\phi \xi = 0$, then (30) gives $Q\xi = \eta(\xi)\xi$. One may write $\xi = \mu \xi + X$ for some $\mu : M \to \mathbb{R}$ and $X \in \ker \eta$. This yields $QX = 0$, hence, $\xi$ is collinear with $\xi$, and so rank $\phi = 2n$.

To show $\eta \circ \phi = 0$, observe that from (30) and $Q\xi = \xi$ we get $[Q, \phi] = 0$. Since $\phi \xi = 0$, we also have, applying (30),

$$\eta(\phi X) = \phi^3 X + Q(\phi X) = Q(\phi X) = [Q, \phi](X) = 0$$

for any $X$, that proves the claim. \[\square\]

**Definition 2.** If for a weak almost contact structure $(\phi, \xi, \eta)$ there exists a metric $g$ such that

$$g(\phi X, \phi Y) = g(X, QY) - \eta(X)\eta(Y),$$

then we get a weak almost contact metric structure. If, in addition,

$$g(X, \phi Y) = dh(X, Y),$$

then we obtain a weak quasi-Sasakian structure, and in this case, $dh(\xi, \cdot) = 0$.

Setting $Y = \xi$ in (31) and using the property $Q\xi = \xi$, we get $\eta(X) = g(X, \xi)$. By (31), the tensor field $Q$ is self-adjoint,

$$g(QX, Y) = g(X, QY),$$

**Proposition 6.** For a weak almost contact metric structure, the tensor $\phi$ is skew-symmetric,

$$g(\phi X, Y) = -g(X, \phi Y).$$
Proof. For any $Y \in \mathcal{D}$ there is $\tilde{Y} \in \mathcal{D}$ such that $\phi Y = \tilde{Y}$. Thus, (33) follows from (31) and (32) for $X \in \mathcal{D}$ and $\tilde{Y}$.

Notice that (33) characterizes in a sense compatible metrics $g$.

**Definition 3.** We say that an endomorphism $F$ of $\mathcal{D}$ has a skew-symmetric representation if for any $x \in M$ there exist a neighborhood $U_x \subset M$ and a local frame $\{e_i\}$ on $U_x$, for which $F$ has a skew-symmetric matrix.

Next, we generalize the fact that any almost contact structure admits a compatible metric.

**Proposition 7.** If $\phi$ in (30) has a skew-symmetric representation then the weak almost contact structure admits a compatible metric.

Proof. Let (30) holds and $\phi$ has a skew-symmetric matrix in a local frame $\{e_i\}$ on a domain $U \subset M$. There exists metric $g_U$ on $U$ such that $\{e_i\}$ is orthonormal. Thus, (33) holds for $g_U$, in particular, $g_U(\phi(X), X) = 0$ for all $X \in TU$. The last property is preserved under summation of metrics. Hence, applying partition of unity, we obtain metric $g$ on $M$ with the same property $g(\phi(X), X) = 0$ for all $X \in TM$, i.e., $\phi$ is skew-symmetric for $g$. By the above, (31) holds.

**Example 5.** Let $(M, g, \phi, \xi, \eta)$ be an almost contact metric manifold. For arbitrary $(1, 1)$-tensor $\phi'$ commuting with $\phi$ on $M$, define $\tilde{\phi} = \phi + \phi'$ and $Q = \text{id} - (\phi\phi' + \phi'\phi) - (\phi')^2$ on $\mathcal{D}$. Then $(M, \tilde{\phi}, \xi, \eta)$ is a weak almost contact manifold when $|\phi'|$ is sufficiently small.

**Definition 4.** A set of $p$ weak almost contact structures $(\phi_i, \xi_i, \eta^i)$ with the same tensor $Q$ on an $(n + p + np)$-dimensional manifold $M$, satisfying the following condition:

$$
\phi_i \circ \phi_j = -\delta_{ij}Q + \eta^j \otimes \xi_i + \sum_k \varepsilon_{ijk} \phi_k, \quad i, j, k \in \{1, \ldots, p\},
$$

(34)

where $\varepsilon_{ijk}$ is the totally antisymmetric symbol, will be called a weak almost $p$-contact structure. If there exists a metric $g$ compatible with each of our weak almost contact structures,

$$
g(\phi_i X, \phi_j Y) = g(X, QY) - \eta^i(X)\eta^j(Y),
$$

(35)

then we obtain a weak almost $p$-contact metric manifold.

For $Q = \text{id}$ we get the structure considered in [4], and for $p = 3$, it generalizes an almost 3-contact (metric) structure on $M^{4n+3}$. From (33) it follows that, see also [4] for $Q = \text{id}$,

$$
\phi_i \xi_j = \sum_k \varepsilon_{ijk} \xi_k, \quad \eta^i \circ \phi_j = \sum_k \varepsilon_{ijk} \phi_k.
$$

Define smooth distributions $\tilde{\mathcal{D}} = \text{span}(\xi_1, \ldots, \xi_p)$ and $\mathcal{D} = \bigcap \ker \eta_i$. Then $Q|_{\tilde{\mathcal{D}}} = \text{id}^\top$. By $Q\xi_i = \xi_i$ and (34) the tensor $Q$ is nonsingular, and by (35) $Q$ is self-adjoint. Observe that for a weak almost $p$-contact metric structure, the tensors $\phi_i$ are skew-symmetric,

$$
g(\phi_i X, Y) = -g(X, \phi_i Y), \quad i \in \{1, \ldots, p\},
$$

(36)

and the Reeb vector fields $\{\xi_i\}$ are orthonormal with respect to $g$.

The following statement is similar to Proposition 7.

**Proposition 8.** If the tensor fields $\phi_i$ in (34) have a skew-symmetric representation with the same local frames, then the weak almost $p$-contact structure admits a compatible metric.

We get a weak quasi-$p$-Sasakian structure if each of $(\phi_i, \xi_i, \eta^i)$ satisfies, in addition,

$$
g(X, \phi_i Y) = d\eta^i(X, Y).
$$

(37)

Observe that (37) yields $T^2_{\xi_i} = \phi_i$ on $\mathcal{D}$. Hence, we get the following.
Theorem 3. Let \((\phi_i, \xi, \eta_i')\) be a weak quasi-p-Sasakian structure on \((M, g)\) such that \(\tilde{D} = \text{span}(\xi_1, \ldots, \xi_p)\) determines a \(g\)-foliation. Consider the partial Ricci flow (39) of metrics \(g_t\) on \(M\), and redefine on the normal distribution \(D\),

\[
Q(t) = (1/p) \text{Ric}^\perp_t, \quad \phi_i(t) = T^\perp_{\xi_i}(t).
\]

If \(r(g) > 0\) on \(D\) then (39) with \(\Phi > 0\) has a unique global solution \(g_t\) \((t \in \mathbb{R})\); and there exists metric \(\tilde{g} = \lim_{t \to -\infty} g_t\) for which \(\lim_{t \to -\infty} r(g_t) = \Phi \tilde{g} \perp\), and certainly, \(\lim_{t \to -\infty} Q(t)|_D = (\Phi/p) \text{id} \perp\). In particular, for \(\Phi = p\) the limit yields an almost \(p\)-contact metric structure.

Proof. Observe that \((\phi_i(t), \xi, \eta_i')\) is a weak quasi-p-Sasakian structure on \((M, g_t)\) with \(Q(t)\). In our case, by (14) we have \(\text{Ric}^\perp = -\sum_i \phi_i^2 = pQ\) on \(D\), thus

\[
\sum_i T^\perp_{\xi_i} \text{Ric}^\perp T^\perp_{\xi_i} = -(\text{Ric}^\perp)^2.
\]

By the above and (18), we obtain the ODE

\[
\partial_t \text{Ric}^\perp = 4 \text{Ric}^\perp (\text{Ric}^\perp - \Phi \text{id} \perp).
\]

Each eigenvalue \(\mu_j\) of \(\text{Ric}^\perp\) satisfies ODE \(\mu_j = 4\mu_j (\mu_j - \Phi)\), which has solution \(\mu_j(t) = \frac{\mu_j(0) \Phi}{\mu_j(0) + \exp(4\Phi t)}\) (functions on \(M\)) with \(\mu_j(0) > 0\) and \(\lim_{t \to -\infty} \mu_j(t) = \Phi\). As in the proof of Theorem 2 we complete the proof. \(\square\)

Corollary 2. Let \((\phi_i, \xi, \eta_i')\) be a weak 3-quasi-Sasakian structure on \((M^{4n+3}, g)\) such that \(\tilde{D} = \text{span}(\xi_1, \xi_2, \xi_3)\) determines a \(g\)-foliation. If \(r(g_0) > 0\) on the normal distribution \(D\) then (39) with \(\Phi = 3\) has a unique global solution \(g_t\) \((t \in \mathbb{R})\); moreover, there exists metric \(\tilde{g} = \lim_{t \to -\infty} g_t\), for which we get a 3-\(q\)-Sasakian structure.

Proof. By Theorem 3 with \(p = 3\), the limit metric \(\tilde{g}\) determines a 3-\(q\)-Sasakian structure. \(\square\)

Corollary 3. Let \((\phi, \xi, \eta)\) be a weak quasi-Sasakian structure on \((M^{2n+1}, g)\) such that \(\xi\) is a unit Killing vector. If \(r(g_0)|_D > 0\) then (12) with \(\Phi = 1\) has a unique global solution \(g_t\) \((t \in \mathbb{R})\); moreover, there exists metric \(\tilde{g} = \lim_{t \to -\infty} g_t\), for which we get a quasi-Sasakian structure.

Proof. By Theorem 3 with \(p = 1\), the limit metric \(\tilde{g}\) determines a quasi-Sasakian structure with \(R\xi = \text{id}\), see (22). \(\square\)

2.3 Weak almost \(f\)-structure

In this subsection we introduce and study a structure, which generalizes an almost \(f\)-structure (see Example 3), and show the convergence of the partial Ricci flow for this structure.

Definition 5. A weak \(f\)-structure on a manifold \(M^{2n+p}\) is defined by a \((1,1)\)-tensor field \(f\) satisfying

\[
f^3 + fQ = 0, \quad \text{rank } f = 2n,
\]

where \(Q\) is a nonsingular \((1,1)\)-tensor field such that \(Q\xi = \xi\) for \(\xi \in \text{ker } f\).

Notice that \(TM\) splits into two complementary subbundles \(D = f(TM)\) and \(\tilde{D} = \text{ker } f\). Hence, \(Q\) is a nonsingular \((1,1)\)-tensor field such that \(Q|_{\tilde{D}} = \text{id}|_{\tilde{D}}\). A weak globally framed \(f\)-structure occurs when there exist vector fields \(\xi_i, i \in \{1, \ldots, p\}\), with their dual 1-forms \(\eta_i\), satisfying

\[
f^2 = -Q + \sum_i \eta_i \otimes \xi_i, \quad \eta_i(\xi_j) = \delta^i_j.
\]

Similarly to Example 5 we can build an example of a weak almost \(f\)-manifold.

Proposition 9. Suppose that \(M^{2n+p}\) has a weak \(f\)-structure \((f, \xi_i, \eta_i')\), see (39). Then \(f \xi_i = 0\) and \(\eta_i' \circ f = 0\) for any \(i \in \{1, \ldots, p\}\).
Proof. By (39), $f^2 \xi_i = 0$. Applying (38) to $\xi_i$, we get $f \xi_i = 0$. To show $\eta^i \circ f = 0$, observe that from (39) and $Q \xi_i = \xi_i$, we get

$$[Q, f] = 0.$$ 

Since $f \xi_i = 0$, we also have, applying (39),

$$\eta^i(fX) = f(f^2X) + Q(fX) = Q(fX) - f(QX) = [Q, f](X) = 0$$

for any $X$, that proves the claim. 

A pseudo-Riemannian metric $g$ is compatible (and then we get a metric weak $f$-structure), if

$$g(fX, fY) = g(X, QY) - \sum_i \varepsilon_i \eta^i(X) \eta^i(Y),$$

where $\varepsilon_i = g(\xi_i, \xi_i) = \pm 1$. From (40) we get

$$g(X, \xi_i) = \varepsilon_i \eta^i(X).$$

By (40), a weak metric almost $f$-manifold has self-adjoint tensor $Q$, i.e., $g(QX, Y) = g(X, QY)$.

Similarly to Proposition 6, for a metric weak almost $f$-structure, the tensor $f$ is skew-symmetric,

$$g(fX, Y) = -g(X, fY),$$

and the Reeb vector fields $\{\xi_i\}$ are orthonormal with respect to $g$.

Next we generalize the fact that any almost $f$-structure admits a compatible metric.

Proposition 10. If the tensor field $f$ in (39) has a skew-symmetric representation then the weak $f$-structure admits a compatible metric.

Proof. This is similar to proof of Proposition 7. 

Theorem 4. Let $(f, \xi_i, \eta^i)$ be a metric weak $f$-structure on $(M, g)$ such that $\tilde{D} = \text{span}(\xi_1, \ldots, \xi_p)$ determines a $g$-foliation. Consider the partial Ricci flow (5) of metrics $g_t$ on $M$, and redefine

$$Q(t) = (1/p) \text{Ric}_f^\perp, \quad f(t) = T_\xi^\perp(t)$$

on $D$. Then $(f(t), \xi_i, \eta^i)$ is a weak almost $f$-structure on $(M, g_t)$ with $Q(t)$. If $r(g) > 0$ on normal distribution $D$ then (5) with $\Phi > 0$ has a unique global solution $g_t$ $(t \in \mathbb{R})$; and there exists metric $\tilde{g} = \lim_{t \to -\infty} g_t$ for which $\lim_{t \to -\infty} r(g_t) = \Phi \tilde{g}^\perp$, and certainly, $\lim_{t \to -\infty} Q(t)|_D = (\Phi/p) \text{id}^\perp$. In particular, for $\Phi = p$ the limit yields a metric almost $f$-structure.

Proof. This is similar to the proof of Theorem 3.

2.4 Weak almost para-$\phi$-structure

Here, we introduce and study structure, which generalizes an almost para-$\phi$-structure (see Example 5), and show convergence of the partial Ricci flow for this structure.

Definition 6. A weak para-$\phi$-structure on a manifold $M^{2n+p}$ is defined by a $(1,1)$-tensor field $\phi$ of rank $2n$ satisfying

$$\phi^3 - \phi Q = 0, \quad \text{rank} \phi = 2n,$$

where $Q$ is a nonsingular $(1,1)$-tensor field such that $Q \xi = \xi$ for $\xi \in \text{ker} \phi$.

The kernel distribution $\tilde{D} = \text{ker} \phi$ has dimension $p$. Set $D = \phi(TM)$. Hence, $Q$ is a nonsingular $(1,1)$-tensor field such that $Q|_D = \text{id}_D$.

A weak para-$\phi$-structure with complemented frames occurs when there exist vector fields $\xi_i$, $1 \leq i \leq p$, and dual 1-forms $\{\eta^i\}$, satisfying the following compatibility conditions:

$$\phi^2 = Q - \sum_j \eta^j \otimes \xi_j, \quad \eta^i(\xi_j) = \delta^i_j,$$

then $M$ is a weak almost para-$\phi$-manifold with complemented frames. Similarly to Example 5, we can build an example of a weak almost para-$\phi$-manifold.

The following statement is similar to Proposition 10.
Proposition 11. Suppose that $M^{2n+p}$ has a weak $\phi$-structure $(\phi, \xi_i, \eta^i)$, see \[43\]. Then $\phi \xi_i = 0$ and $\eta^i \circ \phi = 0$ for any $i \in \{1, \ldots, p\}$.

Proof. This is similar to the proof of Proposition \[9\].

A weak almost para-$\phi$-manifold $M$ with complemented frames and a compatible pseudo-Riemannian metric $g$ such that

$$g(\phi X, \phi Y) = -g(X, QY) + \sum_i \xi_i \eta^i(X) \eta^i(Y),$$

is called a metric weak para-$\phi$-manifold. By (32), a weak metric almost para-$\phi$-manifold has self-adjoint $Q$, i.e., $g(QX, Y) = g(X, QY)$.

For a metric weak almost para-$\phi$-manifold, the tensor $\phi$ is skew-symmetric,

$$g(\phi X, Y) = -g(X, \phi Y), \quad (44)$$

and the Reeb vector fields $\{\xi_i\}$ are orthonormal with respect to $g$. Similarly to Proposition \[7\] we generalize the fact that any almost para-$\phi$-structure admits a compatible metric.

Proposition 12. If $\phi$ has a skew-symmetric representation, then the weak para-$\phi$-structure \[43\] admits a compatible metric.

Theorem 5. Let $(\phi, \xi_i, \eta^i)$ be a metric weak para-$\phi$-structure on $(M, g)$ and $\overline{\mathcal{D}} = \text{span}(\xi_1, \ldots, \xi_p)$ determines a $g$-foliation. Consider the partial Ricci flow \[15\] of metrics $g_t$ on $M$, and redefine

$$Q(t) = (1/p) \text{Ric}^\perp_t, \quad \phi(t) = T^\phi_t(\xi)$$

on $\mathcal{D}$. Then $(\phi(t), \xi_i, \eta^i)$ is a metric weak para-$\phi$-structure on $(M, g_t)$ with $Q(t)$. If $r(g) > 0$ on normal distribution $\mathcal{D}$ then \[15\] with $\Phi > 0$ has a unique global solution $g_t$ ($t \in \mathbb{R}$); and there exists metric $\tilde{g} = \lim_{t \to -\infty} g_t$ for which $\lim_{t \to -\infty} r(g_t) = \Phi \tilde{g}^\perp$, and certainly, $\lim_{t \to -\infty} Q(t)|_{\overline{\mathcal{D}}} = (\Phi/p) \text{id}^\perp$. In particular, for $\Phi = p$ the limit yields a metric almost para-$\phi$-structure.

Proof. This is similar to the proof of Theorem \[8\].

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