On $\psi$- basic Bernoulli-Ward polynomials

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Abstract

The Ward solution of $\psi$- difference calculus nonhomogeneous equation

$$\Delta_\psi f = \varphi \quad \varphi = ?$$

is found in the form of

$$f(x) = \sum_{n \geq 1} \frac{B_n}{n!} \varphi^{(n-1)}(x) + \int_\psi \varphi(x) + p(x)$$

(where $B_n$ denote $\psi$-Bernoulli-Ward numbers [1]) - in the framework of the $\psi$-Finite Operator Calculus [2] - [5]. Specifications to $q$-calculus case and the new Fibonomial calculus case [5, 6] are made explicit.

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1 Remark on Notation and References

At first let us anticipate with $\psi$- remark. $\psi$ denotes an extension of

$$\left\{ \frac{1}{n!} \right\}_{n \geq 0}$$
sequence to quite arbitrary one ("admissible") and the specific choices are
for example: Fibonomially - extended \((F_n, \ n \geq 0 - \text{Fibonacci sequence})\)
Gauss \(q\)-extended

\[
\{\psi_n\}_{n \geq 0} = \left\{ \frac{1}{F_n!} \right\}_{n \geq 0}, \quad \{\psi_n\}_{n \geq 0} = \left\{ \frac{1}{n_q!} \right\}_{n \geq 0}
\]
admissible sequences of extended umbral operator calculus - see more below.
With such an extension we may \(\psi\) - mnemonic repeat with exactly the same
simplicity and beauty much of what was done by Rota years ago. Thus due
to efficient usage we get used to write down these extensions in mnemonic
upside down notation \([2, 5]\)

\[
n_\psi \equiv \psi_n, \quad x_\psi \equiv \psi(x) \equiv \psi_x, \quad n_\psi! = n_\psi(n-1)_\psi!, \quad 0_\psi = 1
\]

\[
x_\psi^k = x_\psi(x-1)_\psi \ldots (x-k+1)_\psi \equiv \psi(x)\psi(x-1)\ldots\psi(x-k+1)
\]

You may consult for further development and use of this notation \([1, 5]\)
and references therein. Summing up - we say it again. The papers of main
reference are \([1, 2, 3]\). For not only mnemonic reasons we follow here the
notation from \([2, 3]\) and we shall take the results from \([1]\) as well as from
\([2, 3]\) - for granted. (Note the access also via ArXiv to \([3, 5]\)). For other
respective references see: \([2, 3, 5]\). Note that we use \(\psi\)-extension symbols - as
in popular \(q\)-calculi (for example \(n_q\)) - in upside down notation; for example:

\[
\psi_n \equiv n_q, \quad x_q = \psi(x) = \frac{1 - qx}{1 - q}.
\]

\(B_n\) denote here \(\psi\)-Bernoulli-Ward numbers.

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Let the \(\psi\)-difference delta operator be defined as \(\Delta_\psi = E^\psi(\partial_\psi) - id\). Then \(\psi\)-
basic Bernoulli-Ward polynomials \(\{B_n(x)\}_{n \geq 0}\) might be defined equivalently
by

\[
\sum_{s=0}^{n-1} \binom{n}{s}_\psi B_s(x) = nx^{n-1}; \ n \geq 2; \ B_0(x) = 1,
\]

\((1)\)
where $B_n = B_n(0)$ denote $\psi$-Bernoulli-Ward numbers: $\{B_s\}_{s \geq 0}$ or via

$$B_n(x) = \sum_{s=0}^{n} \binom{n}{s} B_s x^{n-s} \equiv (x + \psi B)^n \quad n \geq 0. \tag{2}$$

$\psi$-basic Bernoulli-Ward polynomials $\{B_n(x)\}_{n \geq 0}$ are generalized Appell polynomials i.e.

$$\partial_{\psi} B_n(x) = n_{\psi} B_{n-1}(x) \tag{3}$$

and being $\psi$-Sheffer they naturally do satisfy the $\psi$-Sheffer-Appell identity \cite{3, 2}

$$B_n(x + \psi y) = \sum_{s=0}^{n} \binom{n}{s} B_s(y) x^{n-s}. \tag{4}$$

$\psi$-basic Bernoulli-Ward polynomials $\{B_n(x)\}_{n \geq 0}$ are also equivalently characterized via their $\psi$-exponential generating function

$$\sum_{n \geq 0} z^n B_n(x) \frac{1}{n_{\psi}!} = \frac{z}{\exp_{\psi} \{z\} - 1} \exp_{\psi} \{xz\} \tag{5}$$

while the $\psi$-exponential generating function of $\psi$-Bernoulli-Ward numbers $B_n = B_n(0)$ is

$$\sum_{n \geq 0} z^n B_n = \frac{z}{\exp_{\psi} \{z\} - 1} , B_n = B_n(0) \quad n \geq 0. \tag{6}$$

Compare the above with the theorem 16.2 in \cite{1}. There one also shows that

$$\frac{B_{r+1}(\pi) - B_r}{(r+1)_{\psi}} = \sum_{k=0}^{n-1} \overline{k^r} ; r \geq 1. \tag{7}$$

where

$$\overline{k^r} = (1 + \psi 1 + \psi 2 + \ldots + \psi 1)^r \leftarrow k \text{ summands} \tag{8}$$

and $\psi$-multinomial formula reads (see \cite{1})

$$(x_1 + \psi x_2 + \psi \ldots + \psi x_k)^n = \sum_{s_1, \ldots, s_k = 0}^{n} \binom{n}{s_1, \ldots, s_k} x_1^{s_1} \ldots x_k^{s_k}$$

$$s_1 + s_2 + \ldots + s_k = n \tag{9}$$
where
\[
\binom{n}{s_1, \ldots, s_k}_\psi = \frac{n_\psi!}{(s_1)_\psi! \cdots (s_k)_\psi!}.
\]

Naturally \( \psi \)-basic Bernoulli-Ward polynomials \( \{B_n(x)\}_{n \geq 0} \) satisfy the \( \psi \)-difference equation
\[
\Delta \psi B_n(x) = n_\psi x^{n-1}; \quad n \geq 0
\]

hence they play the same role in \( \psi \)-difference calculus as Bernoulli polynomials do in standard difference calculus (see: Theorem 16.1 in [1]) due to the following: The central problem of the \( Q(\partial_\psi) \)-difference calculus is:
\[
Q(\partial_\psi)f = \varphi \quad \varphi = ?
\]

where \( f, \varphi \) - are for example formal series of polynomials.

The idea of finding solutions is then following. As we know [2, 3] any \( \psi \)-delta operator \( Q \) is of the form \( Q(\partial_\psi) = \partial_\psi \hat{B} \) where \( \hat{B} \in \Sigma_\psi \). Consider then
\[
\Delta \psi = \partial_\psi \hat{B} \equiv \partial_\psi \sum_{k \geq 0} \frac{1}{(k+1)_\psi} \partial_\psi^k \equiv E(\partial_\psi) - \text{id},
\]
hence \( \hat{B} \in \Sigma_\psi \) we have for \( \hat{B} \)-recognized as \( \psi \)-Bernoulli operator - the obvious expression
\[
\hat{B} = \frac{\partial_\psi}{\Delta \psi} = \sum_{n \geq 0} \frac{B_n}{n_\psi!} \partial_\psi^n.
\]

Now multiply by \( \hat{B} \equiv \frac{\partial_\psi}{e^{\psi-1}} \equiv \sum_{n \geq 0} \frac{B_n}{n_\psi!} \partial_\psi^n \) the equation \( \Delta \psi f = \varphi \) in order to get
\[
\partial_\psi f = \sum_{n \geq 0} \frac{B_n}{n_\psi!} \varphi^{(n)}, \quad \varphi^{(n)} = \partial_\psi \varphi^{(n-1)}.
\]

The solution then reads:
\[
f(x) = \sum_{n \geq 1} \frac{B_n}{n_\psi!} \varphi^{(n-1)}(x) + \int_\psi \varphi(x) + p(x),
\]

where \( p \) is "\( + \psi \)-periodic" i.e. \( p(x+\psi 1) = p(x) \) i.e. \( \Delta \psi p = 0 \). Here the \( \psi \)-integration \( \int_\psi \varphi(x) \) is defined as in [2]. We recall it in brief. Let us introduce the following representation for \( \partial_\psi \) "difference-ization"
\[
\partial_\psi = \hat{n}_\psi \partial_0; \quad \hat{n}_\psi x^{n-1} = n_\psi x^{n-1}; \quad n \geq 1,
\]
where $\partial_{\psi} x^n = x^{n-1}$ i.e. $q = 0$ "Jackson derivative" $\partial_{\psi}$ is identical with divided difference operator. Then we define the linear mapping $\int_{\psi}$ accordingly:

$$\int_{\psi} x^n = \left(\hat{x} \frac{1}{n_{\psi}}\right) x^n = \frac{1}{(n + 1)_{\psi}} x^{n+1}; \quad n \geq 0$$

where of course $\partial_{\psi} \circ \int_{\psi} = id$.

3 Two Illustrative Specifications

3.1 $q$-umbral case [1]-[5]

The following choice [2, 3, 4, 5] of the admissible sequence $\psi_n(q) = [R(q^n)]^{-1}$ and then $R(x) = \frac{1-x}{1-q}$ results in the well known $q$-factorial $n_q! = n_q(n - 1)_q!$, $(n_{\psi} = n_q)$ while the $\psi$-derivative $\partial_{\psi}$ becomes the Jackson’s derivative $\partial_q$:

$$(\partial_q \varphi)(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)x} [1].$$

The $\psi$- integration [2, 5] becomes the well known $q$- integration and we arrive at the $q$- Bernoulli numbers and $q$- Bernoulli polynomials (for further references to Cigler, Roman and others see[2, 3, 4, 5]).

3.2 FFOC - case [5]

In straightforward analogy - (see FFOC-Fibonomial Finite Operator Calculus, Example 2.1 in [5]) - consider now the Fibonomial coefficients ($F_n$ - Fibonacci numbers)

$$\binom{n}{k}_F = \frac{F_n!}{F_k! F_{n-k}!} = \binom{n}{n-k}_F,$$

where $n_F \equiv F_n \neq 0$, $n_F! = n_F(n - 1)_F(n - 2)_F(n - 3)_F \ldots 2_F 1_F; \quad 0_F! = 1$;

$$n^k_F = n_F(n - 1)_F \ldots (n - k + 1)_F; \quad \binom{n}{k}_F \equiv \frac{n^k_F}{k_F!}$$

and difference operator $\partial_F$ linearly extended from $\partial_F x^n = n_F x^{n-1}; \quad n \geq 0$ - we shall call the $F$-derivative. Then in conformity with [1] and with notation as in [2, 3] one writes:
(1) \((x + F a)^n \equiv \sum_{k \geq 0} \binom{n}{k}_F a^k x^{n-k}\) \(\text{where } \binom{n}{k}_F = \frac{n!}{k!F!}\)
and \(n^k_F = n_F(n-1)_F \ldots (n-k+1)_F;\)

(2) \((x + F a)^n \equiv E^n(a)(\partial_F x^n); \quad E^n(\partial_F) = \sum_{n \geq 0} \frac{a^n}{n!F!}\partial_F^n;\)
\(E^n(\partial_F)f(x) = f(x+Fa), E^n(\partial_F)\) is the corresponding generalized translation operator.

The \(\psi\)-integration becomes now still not explored \(F\)-integration and we arrive at the \(F\)-Bernoulli numbers and \(F\)-Bernoulli polynomials - all to be investigated soon.

Note: recently a combinatorial interpretation of Fibonomial coefficient has been found \([6, 7]\).

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