Non-isomorphic Hopf-Galois structures with isomorphic underlying Hopf algebras.

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Abstract

We give a degree 8 separable extension having two non-isomorphic Hopf-Galois structures with isomorphic underlying Hopf algebras.

1 Introduction

A finite extension of fields $K/k$ is a Hopf Galois extension if there exist a finite cocommutative $k$-Hopf algebra $H$ and a Hopf action of $H$ on $K$, i.e a $k$-linear map $\mu : H \to \operatorname{End}_k(K)$ inducing a bijection $K \otimes_k H \to \operatorname{End}_k(K)$. We shall call such a pair $(H, \mu)$ a Hopf Galois structure on $K/k$. Two Hopf Galois structures $(H_1, \mu_1)$ and $(H_2, \mu_2)$ on $K/k$ are isomorphic if there exists a Hopf algebra isomorphism $f : H_1 \to H_2$ such that $\mu_2 \circ f = \mu_1$.

For a Hopf Galois extension we have the following Galois correspondence theorem:

**Theorem 1.1** ([1], Theorem 7.6). Let $K/k$ be a finite field extension and let $(H, \mu)$ be a Hopf Galois structure on $K/k$. For a $k$-sub-Hopf algebra $H'$ of $H$ we define

$$K^{H'} = \{ x \in K \mid \mu(h)(x) = \varepsilon(h) \cdot x \text{ for all } h \in H' \}$$

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where \( \varepsilon \) is the counit of \( H \). Then, \( K^{H'} \) is a subfield of \( K \), containing \( k \), and
\[
F_H : \{H' \subseteq H \text{ sub-Hopf algebra}\} \to \{\text{Fields } E \mid k \subseteq E \subseteq K\}
\]
\[H' \mapsto K^{H'}\]
is injective and inclusion reversing.

Separable Hopf Galois extensions have been characterized by Greither and Pareigis. Let \( K/k \) be a separable extension of degree \( n \) and let \( \widetilde{K}/k \) be its Galois closure, \( G = \text{Gal}(\widetilde{K}/k) \) and \( G' = \text{Gal}(\widetilde{K}/K) \). The action of \( G \) on the set \( G/G' \) of left cosets induces a group morphism
\[
\lambda : G \to \text{Sym}(G/G') = S_n
\]
\[g \mapsto (\lambda_g : xG' \mapsto gxG') .
\]

**Theorem 1.2** ([5] Theorem 3.1). With the above notations, a finite separable field extension \( K/k \) is a Hopf Galois extension if and only if there exists a regular subgroup \( N \) of \( S_n \) normalized by \( \lambda(G) \). Moreover, there is a one-to-one correspondence between the set of isomorphism classes of Hopf Galois structures on \( K/k \) and the set of regular subgroups \( N \) of \( S_n \) normalized by \( \lambda(G) \).

The underlying Hopf algebra of the Hopf Galois structure corresponding to a subgroup \( N \) is the sub-Hopf algebra \( H = \widetilde{K}[N]^G \) of \( G \)-fixed points in the group algebra \( \widetilde{K}[N] \), where \( G \) acts on \( \widetilde{K} \) by field automorphisms and on \( N \) by conjugation inside \( S_n \) via \( \lambda \).

We shall use the following equivalent condition in our study of Hopf Galois structures.

**Theorem 1.3** ([2] Proposition 1). The group \( \lambda(G) \) normalizes the regular subgroup \( N \) of \( \text{Sym}(G/G') \) if and only if \( \lambda(G) \) is a subgroup of the holomorph \( \text{Hol}(N) = N \rtimes \text{Aut}(N) \) of \( N \).

Within the class \( \mathcal{HG} \) of separable Hopf Galois extensions, Greither and Pareigis defined the subclass \( \mathcal{AC} \) of almost classically Galois extensions and proved that \( \mathcal{AC} \) is included in the class \( \mathcal{B} \) of separable Hopf Galois extensions which may be endowed with a Hopf Galois structure such that the Galois correspondence is bijective ([5] §4). In [3], we proved that the two inclusions \( \mathcal{AC} \subset \mathcal{B} \) and \( \mathcal{B} \subset \mathcal{HG} \) are strict.

In [4], Example 2.1, we exhibited a degree 8 Hopf Galois extension of the field \( \mathbb{Q} \) of rational numbers which is not almost classically Galois. In this paper, we study the Hopf Galois structures of this extension and prove that there are two non-isomorphic Hopf Galois structures with isomorphic underlying Hopf algebras. In the last section, we observe that the image of the Galois correspondence of a Hopf Galois structure does not determine the isomorphism class of the underlying Hopf algebra.
2 A Hopf Galois degree 8 extension

Let us consider a Galois extension \( \tilde{K}/k \) with Galois group \( G \) isomorphic to the symmetric group \( S_4 \) and let \( G' \) be an order 3 subgroup of \( G \). Let \( K' \) be the subfield of \( \tilde{K} \) fixed under the action of \( G' \). If \( S_4 \) is realized over \( k \) as the splitting field of a degree four polynomial \( P(X) \in k[X] \), then \( K = k(\alpha, \sqrt{\delta}) \), where \( \alpha \) is a root of \( P(X) \) and \( \delta \) its discriminant. We showed in \([3]\) Proposition 2.1 that such an extension \( K/k \) is Hopf Galois but not almost classically Galois.

2.1 Hopf Galois structures

According to theorem \([1,2]\) in order to determine the Hopf Galois structures of \( K/k \), we look at regular subgroups \( N \) of \( S_8 \). Since the set of isomorphism classes of regular subgroups of \( S_8 \) is in one-to-one correspondence with the set of isomorphism classes of groups of order 8, we have five possible groups \( N \), up to isomorphism, namely the abelian groups \( C_8, C_2 \times C_4 \) and \( C_2 \times C_2 \times C_2 \), the dihedral group \( D_8 \) and the quaternion group \( H_8 \). If we look for holomorphs having order divisible by 24, we are left with \( C_2 \times C_2 \times C_2 \) and \( H_8 \). But the holomorf of \( H_8 \) has no transitive subgroups isomorphic to \( S_4 \). So the only possible \( N \)'s are isomorphic to \( C_2 \times C_2 \times C_2 \).

Let us now determine explicitly the immersion \( \lambda \) of \( G \) into \( Perm(G/G') \). Taking \( G = \langle \tau := (1, 2, 3, 4), \sigma := (1, 2) \rangle \) and \( G' = \langle (2, 3, 4) \rangle \), a left transversal of \( G' \) in \( G \) is \( \{1_G, (1, 2), (1, 3), (1, 4), (2, 3), (1, 2, 3), (1, 3, 4), (1, 4, 2)\} \). By choosing an enumeration of the cosets, we may take \( \lambda(G) = \langle \lambda(\tau) = (1, 2, 3, 4)(5, 6, 7, 8), \lambda(\sigma) = (1, 2)(3, 5)(4, 6)(7, 8) \rangle \) and then \( \lambda(G') = \langle (2, 4, 5)(3, 8, 6) \rangle \).

The conjugation class of regular subgroups of \( S_8 \) isomorphic to \( C_2 \times C_2 \times C_2 \) has length 30. There are exactly four subgroups \( N \) in it satisfying \( \lambda(G) \subset Norm_{S_8}(N) \), namely:

\[
N_1 = \langle r_1 = (1, 3)(2, 4)(5, 7)(6, 8), s_1 = (1, 8)(2, 7)(3, 6)(4, 5), t = (1, 7)(2, 8)(3, 5)(4, 6) \rangle \\
N_2 = \langle r_2 = (1, 3)(2, 6)(4, 8)(5, 7), s_2 = (1, 4)(2, 5)(3, 8)(6, 7), t = (1, 7)(2, 8)(3, 5)(4, 6) \rangle \\
N_3 = \langle r_3 = (1, 6)(2, 4)(3, 8)(5, 7), s_3 = (1, 7)(2, 3)(4, 8)(5, 6), t_3 = (1, 8)(2, 5)(3, 6)(4, 7) \rangle \\
N_4 = \langle r_4 = (1, 3)(2, 5)(4, 7)(6, 8), s_4 = (1, 7)(2, 6)(3, 4)(5, 8), t_4 = (1, 6)(2, 7)(3, 8)(4, 5) \rangle \\
\]

We have then four Hopf Galois structures on \( K/k \), up to isomorphism.
2.2 Galois correspondence

By classical Galois theory, the extension $K/k$ has two strictly intermediate fields, $k(\sqrt{\delta})$ and $k(\alpha)$. We shall now determine the image of the Galois correspondence for each of the Hopf Galois structures. To this end, we compute the subgroups of each of the corresponding $N$’s which are stable under conjugation by $\lambda(G)$ (see the reformulation of the Galois correspondence theorem in terms of groups in [3] Theorem 2.3). The action of $\tau$ and $\sigma$ on the generators of each $N_i$, $1 \leq i \leq 4$, is given in the following table.

| $\tau$ | $r_1$ | $s_1$ | $t$ | $r_2$ | $s_2$ | $r_3$ | $s_3$ | $t_3$ | $r_4$ | $s_4$ | $t_4$ |
|-------|-------|-------|-----|-------|-------|-------|-------|-------|-------|-------|-------|
| $\sigma$ | $r_1s_1$ | $t$ | $s_1$ | $t_2$ | $s_2t$ | $r_3t_3$ | $r_4s_3$ | $t_3$ | $r_4t_4$ | $r_4s_4$ | $t_4$ |

The $\lambda(G)$-stable subgroups of each $N$ and the corresponding intermediate fields of $K/k$ are given in the following table.

| Groups | Stable subgroups | Fixed subfields |
|--------|-----------------|---------------|
| $N_1$ | $\langle t \rangle, \langle r_1, s_1 \rangle$ | $k(\alpha), k(\sqrt{\delta})$ |
| $N_2$ | $\langle t \rangle$ | $k(\alpha)$ |
| $N_3$ | $\langle r_3, t_3 \rangle$ | $k(\sqrt{\delta})$ |
| $N_4$ | $\langle r_4, t_4 \rangle$ | $k(\sqrt{\delta})$ |

2.3 A Hopf algebra with two different Hopf Galois structures

We shall anlize now if two of the underlying Hopf algebras of the four Hopf Galois structures on $K/k$ may be isomorphic. Two Hopf algebras $H_i = \tilde{K}[N_i]^G$ and $H_j = \tilde{K}[N_j]^G$ are isomorphic if and only if the groups $N_i$ and $N_j$ are $G$-isomorphic. By looking at the stable subgroups of each of the groups $N$, we see that the only possible $G$-isomorphism is between $N_3$ and $N_4$. Now the isomorphism $\Phi : N_3 \to N_4$ defined by $\Phi(r_3) = r_4, \Phi(s_3) = s_4, \Phi(t_3) = t_4$ is clearly a $G$-isomorphism. It may also be seen as induced by conjugation by $s = (1,7)(2,8)(3,5)(4,6)$ and $s$ satisfies $sg^{-1}sg \in Cent_{S_8}(N_3)$ since $s\tau^{-1}s\tau = s\sigma^{-1}s\sigma = Id$. One may check that $N_3$ has no nontrivial $G$-automorphisms, hence $\Phi$ is the unique $G$-isomorphism from $N_3$ onto $N_4$. 

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The isomorphism $\Phi$ induces a Hopf algebra isomorphism between $H_3 = \tilde{K}[N_3]^G$ and $H_4 = \tilde{K}[N_4]^G$ but Theorem 1.2 implies that the corresponding Hopf Galois structures are not isomorphic. We do a direct check of this fact. We determine first the Hopf algebra $H_3 = \tilde{K}[N_3]^G = \{ h \in \tilde{K}[N_3] \mid gh = h, \forall g \in G \}$. From the action of $G$ on $N_3$ determined above we obtain that an element $h = \sum_{n \in N_3} a_n n = a_0Id + a_1r_3 + a_2s_3 + a_3t_3 + a_4r_3s_3 + a_5r_3t_3 + a_6s_3t_3 + a_7r_3s_3t_3 \in \tilde{K}[N_3]$ belongs to $H_3$ if and only if

$$a_0 \in k, a_1 \in \tilde{K}^{(\sigma, \tau^2)}, a_2 \in k(\alpha), a_3 = \sigma \tau(a_1),$$
$$a_5 = \tau(a_1), a_4 = \tau(a_2), a_6 = \tau^2(a_2), a_7 = \tau^3(a_2).$$

If $N$ is a regular subgroup of $Perm(G/G')$ normalized by $G$ and $H = \tilde{K}[N]^G$ is the corresponding Hopf algebra, the Hopf action $\mu : H \to \text{End}_k(K)$ is given by

$$(\sum a_n n) \cdot x = \sum a_n (n^{-1})G(x).$$

To make the Hopf actions $\mu_3$ and $\mu_4$ of $H_3$ and $H_4$ explicit, we first compute the preimage of $\overline{1_G}$ under the elements of $N_3$ and $N_4$.

| $n \in N_3$ | $\overline{1_G}$ | $r_3$ | $s_3$ | $t_3$ | $r_3s_3$ | $r_3t_3$ | $s_3t_3$ | $r_3s_3t_3$ |
|-------------|-------------------|-------|-------|-------|-----------|-----------|-----------|-------------|
| ($n^{-1})G$  | $Id$              | $(1, 4, 2)$ | $(2, 3)$ | $(1, 2, 3)$ | $(1, 3)$ | $(1, 3, 4)$ | $(1, 4)$ | $(1, 2)$   |

| $n \in N_4$ | $\overline{1_G}$ | $r_4$ | $s_4$ | $t_4$ | $r_4s_4$ | $r_4t_4$ | $s_4t_4$ | $r_4s_4t_4$ |
|-------------|-------------------|-------|-------|-------|-----------|-----------|-----------|-------------|
| ($n^{-1})G$  | $Id$              | $(1, 3, 4)$ | $(2, 3)$ | $(1, 4, 2)$ | $(1, 4)$ | $(1, 2, 3)$ | $(1, 2)$ | $(1, 3)$   |

Let $\alpha_1 = \alpha, \alpha_2, \alpha_3, \alpha_4$ be the four roots of the polynomial $P(X)$ in $\tilde{K}$. We consider the element $h = \alpha_1^2s_3 + \alpha_2^2r_3s_3 + \alpha_3^2s_3t_3 + \alpha_4^2r_3s_3t_3 \in H_3$. We have

$$\mu_3(h)(\alpha_1) = \alpha_1^3 + \alpha_2^2\alpha_3 + \alpha_3^2\alpha_4 + \alpha_4^2\alpha_2,$$
$$\mu_4(h)(\alpha_1) = \mu_4(\alpha_1^2s_4 + \alpha_2^2r_4s_4 + \alpha_3^2s_4t_4 + \alpha_4^2r_4s_4t_4)(\alpha_1)$$
$$= \alpha_1^3 + \alpha_2^2\alpha_4 + \alpha_3^2\alpha_2 + \alpha_4^2\alpha_3.$$

In order to see $\alpha_2^2\alpha_3 + \alpha_3^2\alpha_4 + \alpha_4^2\alpha_2 \neq \alpha_2^2\alpha_4 + \alpha_3^2\alpha_2 + \alpha_4^2\alpha_3$, we write both elements in the $k$-basis $(\alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2)_{0 \leq i_1 \leq 3, 0 \leq i_2 \leq 2, 0 \leq i_3 \leq 2}$ of $\tilde{K}$. Writing $P(X) = X^4 - b_1X^3 + b_2X^2 - b_3X + b_4$, we obtain
\[
\alpha_2^2 \alpha_3 + \alpha_2^2 \alpha_4 + \alpha_2^2 \alpha_2 = \alpha_4^3 + (\alpha_2 - b_1) \alpha_3^3 + (-\alpha_2 \alpha_3 + b_2 + \alpha_3^2) \alpha_4 + (-\alpha_3^2 + b_1 \alpha_3) \alpha_2 - b_3,
\]
\[
\alpha_2^2 \alpha_4 + \alpha_2^2 \alpha_2 + \alpha_2^2 \alpha_3 = -\alpha_4^3 + (-\alpha_2 + b_1) \alpha_3^2 + ((b_1 - \alpha_3) \alpha_2 - b_2 + b_1 \alpha_3 - \alpha_3^2) \alpha_4 + \alpha_2 \alpha_3^2.
\]

3 Final remarks

Let \( K/k \) be a separable extension of degree \( n \) and let \( G \) be the Galois group of the Galois closure of \( K/k \). Clearly, the fact that two regular subgroups of \( S_n \) are \( G \)-isomorphic implies that the corresponding Hopf Galois structures have the same Galois correspondence image. The converse is not true. In [3] Theorem 3.4, we give a family of Hopf Galois extensions having two Hopf Galois structures of cyclic and Frobenius type, respectively, with the same Galois correspondence image.

Let us now consider a Galois extension \( K/k \) with Galois group a Hamiltonian group \( G \), i.e. \( G \) is a non-abelian group such that all its subgroups are normal subgroups. All Hamiltonian groups are of the form \( H \times B \times D \), where \( H \) is the quaternion group, \( B \) is the direct sum of some number of copies of the cyclic group \( C_2 \), and \( D \) is a periodic abelian group with all elements of odd order (see [6] §12.5). Let \( n \) be the order of \( G \). We may consider two Hopf Galois structures on \( K/k \) associated to two subgroups of \( S_n \) isomorphic to \( G \), the group \( \rho(G) \), where \( \rho \) is induced by the action of \( G \) on itself by right translation, and the group \( \lambda(G) \). For the first one, the Hopf algebra is the group algebra \( k[G] \) and the Hopf action is the linear extension of the action of the Galois group \( G \) by \( k \)-automorphisms of \( K \). For the second one, the image of the Galois correspondence is the set of intermediate fields \( E \) such that \( E/k \) is Galois ([3], Theorem 5.3), which in the case of Hamiltonian groups is the whole sublattice. We obtain then two Hopf Galois structures with same Galois correspondence image associated to two isomorphic but not \( G \)-isomorphic regular subgroups of \( S_n \).

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