Optimal harvesting and stability of predator-prey model with holling type II predation respon function and stage-structure for predator

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Abstract. This article develops mathematical models with structural stages from predators, immature and mature predators. The predation function of mature predators follows the Holling II response function. We assume that the immature predator population has the economic value, therefore the harvesting function is included in this model. In this model an analysis of the equilibrium point and stability of the interior equilibrium point is carried out. Analysis of the stability of the interior equilibrium points is done by linearization method and pay attention to the eigenvalues of the characteristics of the Jacobi matrix obtained. Analysis of equilibrium point stability is carried out before and after harvesting. The result is obtained by each of the three equilibrium points. At the equilibrium point of the interior with stable harvesting a local maximum profit analysis is obtained from the exploitation business. Based on the results of the analysis, it is obtained the value of harvesting business which provides a stable equilibrium point and maximum profit.

1. Introduction

There is a lot of research about modeling predator-prey with the stage-structure. The study was conducted by looking at the interactions of predator-prey in the ecosystem of life. Examined population models through systems of differential equations using stage-structure [1]. Has made and completed a stage-structure modeling of prey predominantly in prey populations [2]. Investigated the interaction between prey populations and stage-structure predator populations in two prey [3]. Discuss the effects of time delay in the model of predator-prey interaction with stage-structure [4]. Investigated the different delay times in a population model, where the population was divided into immature and mature predators [5]. Conducted a study of a predator-prey population model with the same response function Monod-Haldane [6]. Discussed predator-prey models with stage structured in prey [7]. Discussed the stability and maximum profit analysis on the growth model of predator-prey population with structural stages [8].
2. Predator-prey model with stage-structural

\[
\frac{dx}{dt} = x \mu_1 \left(1 - \frac{x}{k_1}\right) - \frac{\beta xy}{\phi + \epsilon x} \\
\frac{dy}{dt} = \frac{c \beta xz}{\phi + \epsilon x} - (\alpha + \delta_1)y \\
\frac{dz}{dt} = (\alpha y - \delta_2)z
\]

with initial values:
\[x(0) > 0; y(0) \geq 0; z(0) \geq 0\]

Mathematical modeling by Subhas Khajanchi, describes three differential equations and consistently only mature predators \((z)\) exploit the prey \((x)\) [9]. Mature predators are assumed to have access/direct contact with prey and have reproductive abilities. While for immature predators not directly related to prey. Immature predators do not have reproductive abilities and their survival is very dependent on the interaction of mature predators and prey. The prediction of mature predators \(\beta\) is assumed to be proportional to the efficiency of change from mature predators \(c\). Besides experiencing a reduction due to the predation function, the prey population grows logistically with \(\alpha\) as the intrinsic growth rate and \(k_1\) as the capacity. It is also assumed that mature predators \(z\) and \(y\) can be harvested separately.

3. Research method

3.1 Literature Study Phase

At this stage identification of problems is done by looking for references that support the research. The understanding of the stability problem is very helpful in solving the model.

3.2 Analysis Model Phase

At this stage, the model is analyzed by searching for the equilibrium point and then checking its stability. Because the model equation is a nonlinear differential equation, the model needs to be linearized first by forming the Jacobian matrix, then the stability will be examined by looking at the eigenvalues or using the Routh-Hurwitz method.

3.3 Model Simulation Phase

At this stage, the simulation is performed to see the action of the solution curve.

3.4 Simulation Analysis Phase

At this stage, the analysis of the results obtained from the simulation is carried out.

3.5 Conclusion Phase

At this stage, the conclusions are determined from the model that has been analyzed for stability and the results of the simulation.
4. Predator-prey model with stage-structural and harvesting

4.1 Model Formulation

We examined the predator-prey model involved in mature predator interactions and prey participation. The following are the assumptions used in the model.

4.1.1 The prey growth rate uses Logistics Growth.

4.1.2 Predator populations have structural stages, namely immature and mature predators.

4.1.3 In prey on, predation characteristics are assumed to follow the Holling Type II response function.

4.1.4 The population of immature predators and mature predators has an economic value so that they can be harvested.

\[
\frac{dx}{dt} = x\mu_1 \left(1 - \frac{x}{k_1}\right) - \beta \frac{x z}{\phi + x}
\]

\[
\frac{dy}{dt} = \frac{c \beta x z}{\phi + x} - (\alpha + \delta_1) y - q_1 E x
\]

\[
\frac{dz}{dt} = \alpha y - \delta_2 z
\]

In this case, \(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\) are prey population, first predator and second predator. \(\mu_1\) is the logistic growth rate of prey. \(\beta\) and \(c\), each of which is the prediction number and efficiency of change of mature predators. \(q_1\) is the efficiency of changing the immature predators into mature predators. \(\delta_1, \delta_2\), each of which is the mortality rate of immature predators and mature predators. \(q_1\) is the rate of captured immature predators. \(E\) is the harvesting rate of immature predators. Furthermore mature predator predation function follows the Holling II response function \(\frac{\beta x z}{\phi + x}\). In the system model (2) dimensionless variables are used without simplification.

Table 1. The values of some parameters in the system (2) from several sources.

| Parameters | Values | Unit |
|------------|--------|------|
| \(\mu_1\)  | 1.5     | [T]-1|
| \(k_1\)    | 100    | [N]  |
| \(\phi\)   | 8      | -    |
| \(\beta\)  | 0.16   | [N]-1[T]-1|
| \(c\)      | 0.4    | -    |
| \(\alpha\) | 0.02   | [T]-1|
| \(\tau\)   | 0.001  | [N]-1|
| \(\delta_1\)| 0.009 | [T]-1|
| \(\delta_2\)| 0.2   | [T]-1|
| \(q_1\)    | 1      | -    |
| \(q_2\)    | 1      | -    |
4.2 Model Equilibrium Point Analysis

4.2.1 Equilibrium Point without Harvesting

Next we will analyze the equilibrium point of Eq.1. Possible equilibrium points in the dynamics system Eq.1 are $T_1 = (0,0,0)$, $T_2 = (K,0,0)$, and $T_3(x^*,y^*,z^*)$.

with $K = k_1,$

$$x^* = \frac{\phi \delta_2 (\alpha + \delta_1)}{\alpha c + \alpha \delta_2 \alpha - \delta_1 \delta_2 \alpha}$$

$$y^* = \frac{\beta}{c \mu_1 \phi \delta_2 (A)}$$

$$z^* = \frac{\beta}{c \alpha \mu_1 \phi (A)}$$

with $A = \alpha c \beta k_1 - \alpha \delta_2 \alpha - \delta_1 \delta_2 \alpha - \alpha \phi \delta_2 - \phi \delta_2 \delta_1$

4.2.2 Equilibrium Point with Harvesting

The nonnegative equilibrium points obtained from Eq.2 model with $q_i E_i > 0$ are as follows. Possible equilibrium points in system dynamics Eq.1 are $T_1 = (0,0,0)$, $T_2 = (K,0,0)$, and $T_3(x^*,y^*,z^*)$.

with

$$x^* = \frac{\phi \delta_2 (\alpha + \delta_1 + q_1 E)}{\alpha c \beta - \delta_1 \alpha - \delta_2 \alpha - \delta_3 \alpha - \delta_1 q_1 E + B}$$

$$y^* = -\frac{k_1 (ac \beta - \delta_1 \alpha - \delta_2 \alpha - \delta_3 \alpha - \delta_1 q_2 E)}{c \alpha \mu_1 \phi (A)}$$

$$z^* = -\frac{k_1 (ac \beta - \delta_1 \alpha - \delta_2 \alpha - \delta_3 \alpha - \delta_1 q_2 E)}{c \alpha \mu_1 \phi (A)}$$

with $B = \alpha c \beta k_1 - \alpha \delta_2 \alpha - \delta_2 \alpha k_1 - \alpha \phi \delta_2 - \phi \delta_2 \delta_1$

4.2.3 Analysis of stability of equilibrium points

4.2.3.1 Equilibrium Point without Harvesting

Jacobi Matrix from Eq.1 model with $q_i E_i = 0$ is as follows:

$$I = \begin{bmatrix}
I_{11} & 0 & I_{13} \\
I_{21} & I_{22} & I_{23} \\
0 & I_{32} & I_{33}
\end{bmatrix}$$

with:

$$I_{11} = \rho_1 - \frac{2 \rho_1 x^*}{k_1} - \frac{\beta z^*}{\phi + i x^*} + \frac{\beta t z^* x^*}{(\phi + i x^*)^2}$$

$$I_{13} = -\frac{\beta x^*}{\phi + i x^*}$$

$$I_{21} = \frac{c \beta z^*}{\phi + i x^*} - 2 c \beta t x^*$$

$$I_{22} = -\alpha - \delta_1$$
J_{23} = \frac{c\beta x^*}{\phi + i x^*}

J_{32} = \alpha

J_{33} = \delta

by subsidizing the equilibrium value $x^*, y^*, z^*$ a more complete form is obtained. Then it will also be obtained, the characteristic equation of the matrix $\mathbf{J}(T_3)$ is

$$f(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$$

with

$$A_1 = -(J_{11} + J_{22} + J_{33}), A_2 = J_{11}J_{22} + J_{13}J_{31} + J_{23}J_{32} - J_{22}J_{33},
A_3 = J_{11}J_{23}J_{32} - J_{12}J_{31}J_{23} - J_{13}J_{22}J_{31}.$$

To ensure stability, $T_3$ must meet the Routh-Hurwitz criteria [10] that is

$A_1 > 0, A_2 > 0, A_3 > 0$, and $A_1A_2 > A_3$.

4.2.3.1 Equilibrium Point with Harvesting

Jacobi Matrix of Eq.2 model with $q_iE_i > 0$ is as follows

$$J = \begin{bmatrix}
J_{11} & 0 & J_{13} \\
J_{21} & J_{22} & J_{23} \\
0 & J_{32} & J_{33}
\end{bmatrix}$$

with

$$J_{11} = \rho_1 - \frac{2\mu_1 x^*}{\beta x^*} - \frac{\beta z^*}{\phi + i x^*} + \frac{\beta \tau y^* x^*}{(\phi + i x^*)^2},
J_{13} = -\frac{\beta x^*}{\phi + i x^*},
J_{21} = \frac{c\beta z}{\phi + i x^*} - \frac{2c\beta \tau y^* x^*}{(\phi + i x^*)^2},
J_{22} = -\alpha - \delta,
J_{23} = \frac{c\beta x^*}{\phi + i x^*},
J_{33} = \delta.$$

From Jacobi matrix $\mathbf{J}(T_3)$ obtained the following characteristic equation

$$f(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$$

with

$$A_1 = -(J_{11} + J_{22} + J_{33}),
A_2 = J_{11}J_{22} + J_{13}J_{31} + J_{23}J_{32} - J_{22}J_{33},
A_3 = J_{11}J_{23}J_{32} - J_{12}J_{31}J_{23} - J_{13}J_{22}J_{31}.$$

To ensure the stability of $T_3$ must meet the Routh-Hurwitz criteria [10], that is $A_1 > 0, A_2 > 0, A_3 > 0$, and $A_1A_2 > A_3$. So, the $T_3$ equilibrium point is asymptotically stable if it satisfies the condition.
5. Maximum profit in population harvesting

The population assumed to provide economic benefits and value is an immature predator population. So it is exploited by selective harvesting. In the exploitation business costs are needed, and the results of exploitation provide maximum economic benefits.

The total cost function given as

\[ TC = C_1 E \]

The total receipt function given as

\[ TR = p_1 y'E \]

thus the profit function is obtained

\[ \psi = TR - TC \]

\[ \psi(E) = -\frac{p_1 \rho_1 \delta_2 \phi c E (-\frac{q_1 r \epsilon k_1 + \phi q_1 \delta_2)}{k_1 (\alpha \epsilon \beta - \delta_2 r + \epsilon \delta_1 - \delta_2 q_2)} - c_1 E} \]

with

\[ B = \alpha \epsilon \beta k_1 - \alpha r \epsilon k_1 - \delta_2 r k_1 \delta_1 - \alpha \phi \delta_1 - \phi \delta_2 \delta_1 \]

Because the function of profit \( E \) depends on the value of profit and forms a third power equation on \( E \), there will be three profit values, \( E_1, E_2, E_3 \). Of the three values, the profit will be sought which results in the maximum profit for the profit function.

6. Numerical simulations

6.1. Non-harvesting numerical simulations

The parameter values used in the simulation are \( p_1 = 1.5, k_1 = 100, \phi = 8, \beta = 0.16, \epsilon = 0.4, \alpha = 0.02, \tau = 0.001, \delta_1 = 0.009, \delta_2 = 0.2, q_1 = 1, \rho_1 = 1.5, k_1 = 100, c = 0.16, \alpha = 0.02, \tau = 0.001, \delta_1 = 0.009, \delta_2 = 0.2, q_1 = 1, \sigma_1 = 1, \sigma_2 = 1 \).

\[ T_3 = (36.41500545, 479.0561912, 479.0561912) \]

Jacobi Matrix from \( T_3 \) is

\[ J(T_3) = \begin{bmatrix} -0.541903290 & 0 & -0.725 \\ 0.3797012500 & -0.02 & 0.29 \\ 0 & 0.02 & -0.2 \end{bmatrix} \]

The characteristic equation formed from \( J(T_3) \) is

\[ f(\lambda) = \lambda^3 + 0.17990325 \lambda^2 + 0.124095853 \lambda + 0.005056826125 \]

The eigen value obtained is \( \lambda_1 = -0.570207, \lambda_2 = -0.08004, \lambda_3 = -0.120648 \)
6.2. Numerical simulation with harvesting

The parameter values used in the simulation are \( \mu_1 = 1.5, \ k_1 = 100, \ \phi = 8, \ \beta = 0.16, \ c = 0.4, \ \alpha = 0.02, \ \gamma = 0.001, \ \delta_1 = 0.009, \ \delta_2 = 0.2, \ q_1 = 1, \ p_1 = 100, \ C_1 = 20 \).

The equilibrium point obtained from the simulation of Eq.1 with the value \( E \) is \( T_3(E) = (x^*, y^*, z^*) \).

with
\[
\begin{align*}
    x^* &= \frac{-80000(1000E + 29)}{1000E - 63/1}, \\
    y^* &= \frac{-4800000(81000E - 4051)}{(1000E - 63/1)^2}, \\
    z^* &= \frac{-4800000(81000E - 4051)}{(1000E - 63/1)^2},
\end{align*}
\]

So that for the profit function value of \( \psi \) is obtained
\[
\psi(E) = \frac{48000000000E (81000E - 4051)}{(1000E - 63/1)^2} - 20E.
\]

The stationary point \( (E) \) the profit function \( \psi(E) \) is \( E_1, E_2, \) and \( E_3 \) which are \( E_1 = 0.02509435131, \ E_2 = 4957.66/921 \) and \( E_3 = -4957.66/921 \). It is clear that for the profit value that meets \( q_1 = E_1 > 0 \) is \( E_1 = 0.02509435131 \) and \( E_2 = 4957.66/921 \). By testing the values of \( E_1 \) and \( E_2 \) in the profit function \( \psi(E) \), then the maximum profit value of \( \psi(E) = 0.02509435131 \) is equal to \( \psi = 603.2077256 \). Also performed is the second numerical test of the partial derivative of the profit function, obtained \( \psi''(E_1) < 0 \) the stationary point \( (E_1) \) is the maximum point.

So the value of \( E_1 = 0.02509435131 \) is used to be a parameter value in the simulation model with harvesting on immature predators so that the equilibrium point is obtained
\[
T_3 = (6819433418, 240.5758995, 24.05758995).
\]

Jacobi matrix from \( T_3 \) that is
\[
J(T_3) = \begin{bmatrix}
-1.018882574 & 0 & -1.352358782 \\
0.1892210197 & -0.05409435131 & 0.5409435131 \\
0 & 0.0 & -0.2
\end{bmatrix}
\]

The characteristic equation formed from \( J(T_3) \) is \( f(\lambda) = \lambda^3 + 1.272976925 \lambda^2 + 0.258823067 \lambda + 0.005117894158 = 0 \).

The eigenvalue obtained is \( \lambda_1 = -1.025354, \lambda_2 = -0.022135, \lambda_3 = -0.225486 \).

Then by using the initial value around the equilibrium point \( T_3 \) namely \( (0) = S_i \), the trajectory curve obtained from the simulation model before and after harvesting, as follows
Figure 1. Plot of prey population growth.

Figure 2. Plot growth of immature predatory populations.

Figure 3. Plot growth of mature predator populations.
In Picture (1), Picture (2), and Picture (3) it is seen that prey, immature and mature predator species will not experience extinction. Over time with the value of harvesting effort \( E_1 \), growth in the prey and mature predator population will continue to grow. Harvesting efforts will result in the growth of the prey population and mature predators will remain sustainable for a long time and provide a profit of \( \psi = 603.207/25 \).

**Conclusion**

The predator-prey population model with stages without harvesting or harvesting has very different population dynamics. In addition to providing maximum benefits \( \psi = 603.207/25 \) after harvesting, the policy of harvesting immature predator populations has an impact on increasing prey population density. Although predator population density is reduced after harvesting, it does not disturb the stability of the existing system.

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