Tree Quantum Field Theory

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Abstract

We propose a new formalism for quantum field theory (QFT) which is neither based on functional integrals, nor on Feynman graphs, but on marked trees. This formalism is constructive, i.e. it computes correlation functions through convergent rather than divergent expansions. It applies both to Fermionic and Bosonic theories. It is compatible with the renormalization group, and it allows to define non-perturbatively differential renormalization group equations. It accommodates any general stable polynomial Lagrangian. It can equally well treat noncommutative models or matrix models such as the Grosse-Wulkenhaar model. Perhaps most importantly it removes the space-time background from its central place in QFT, paving the way for a nonperturbative definition of field theory in noninteger dimension.

I Introduction

Feynman invented the two pillars of quantum field theory (or QFT): functional integrals and Feynman graphs. However none of them is fully satisfactory. Indeed QFT (and in particular its soul, renormalization theory) requires to compute connected functions. Functional integrals give rise to singular limits such as 0/0 for such connected functions in infinite volume. Feynman graphs apparently solved this problem because connected functions are expressed as the sum of connected graphs. However the price is too heavy: perturbation theory based on Feynman graphs always diverge because there are too many graphs at large order. Since any alternate divergent series can be cut into pieces and rearranged to converge to any number we want, ordinary perturbative QFT does not define anything at a fundamental level. The vast majority of quantum field theorists (with the exception of the small tribe of constructive field theorists) essentially pulls this fundamental problem under the rug. However the tremendous achievements of standard quantum field theory should not be denied either. Functional
integrals can be simulated, for instance for QCD through Monte Carlo numerical experiments. Perturbative computations in QED allow incredibly accurate comparisons of theory and experiments thanks to the smallness of the fine structure constant. But there is no proof (and even no expectation) that these successful computations should converge better and better with more and more computing power. They do not therefore by themselves constitute a true theory in any fundamental sense. We are also well aware about the very interesting axiomatic or algebraic approaches to QFT. However these approaches did not lead up to now to the construction of interacting quantum field theories. Only constructive field theory in the 70’s and 80’s succeeded in defining rigorously some interacting QFT’s but in dimensions less than four. But we must admit that the ugly constructive tools (truncated functional integrals, cluster and Mayer expansions, ”large/small” field expansions) which were neither canonical nor optimal, largely prevented the spread of that approach beyond a small circle of aficionados.

Perhaps axiomatization of QFT might have been premature. Indeed new field theories constantly arise in an extended sense. Condensed matter is clearly better understood in a field theoretic formalism, although that formalism is not relativistic and has finite density. More recently noncommutative QFTs has been shown renormalizable. They show amazing similarities and subtle differences with ordinary QFT.

Ultimately we think that combinatorics is the right approach to QFT and that a QFT should be thought of as the generating functional of a certain weighted species in the sense of [1].

In this paper we perform a step in this direction: we show how to base quantum field theory on trees, which lie at the right middle point between functional integrals and Feynman graphs so that they share the advantages of both, but none of their problems.

The core of our proposal is to distinguish among model independent and model dependent aspects of QFT. There are three model-independent ingredients: a universal vector space algebraically spanned by all marked trees, a universal ”canonical Hamiltonian operator” which essentially glues a new subtree at the mark on the tree, and the canonical forest formula of [2, 3], which is promoted to a central tool of quantum field theory.

A particular (Euclidean) quantum field theory model is a particular positive scalar product on the universal vector space. That scalar product is simply obtained by applying the canonical forest formula to the ordinary perturbative expansion of the considered QFT model. The canonical formula itself is model-independent. What that magic formula does is conceptually not difficult to understand. It just classifies Feynman amplitudes differently, by breaking these amplitudes into pieces and putting these pieces into boxes labeled by trees. The important point is that it does this in a canonical, “democratic”, positivity preserving way.

\footnote{To treat Minkowski signature we need to extend our definition so as to allow nondegenerate but not necessarily positive scalar products. This will not be studied here.}
Model-dependent details such as space-time dimension, interactions and propagators are therefore no longer considered fundamental. They just enter the definition of the matrix elements of this scalar product. These matrix elements are just finite sums of finite dimensional Feynman integrals. It is just the packaging of perturbation theory which is redone in a better way. This is essentially why this formalism accommodates all nice features of perturbative field theory, just curing its single but giant defect, namely divergence.

The most aesthetic and compact formulation of perturbative QFT is the parametric representation, and it is also the one in which space time is no longer at the center of the stage. The associated idea of dimensional interpolation is a beautiful feature of perturbative QFT which was essential in two key milestones in the development of QFT: the proof of renormalizability of non Abelian gauge theories by ’tHooft and Veltmann [4] and the Wilson-Fisher $\epsilon$ expansion [5]. These are certainly milestones to which one would like to give constructive meaning. Parametric representation relies on various types of tree matrix [6] or tree Pfaffian theorems [7, 8]. This again points towards trees as the fundamental structure in QFT.

The good news is that our formalism is especially compatible with that parametric representation, to the point that it could be described as a kind of "constructive parametric" formalism. Indeed the canonical forest or tree formula can be adapted so that its corresponding interpolating parameters just coincide with a subset of Feynman-Schwinger parameters, those for the tree considered! In this way tree matrix elements of the scalar products corresponding to QFT models become just finite sums of finite dimensional Feynman integrals in parametric space, with just some funny new condition on the range of integration of the "loop parameters". These conditions are now a really small cost to go from perturbative to constructive QFT!

Remark that Fermionic field theory has undergone quietly this tree revolution almost two decades ago. After a long period of maturation [9, 10, 11], it lead to full constructive results such as the rigorous definition of differential renormalization group equations for Fermions [12] and to a flurry of theorems on condensed matter [13, 14, 15, 16]. However the full power of the idea of basing QFT on trees was still not recognized at that time because Bosonic theories could not be brought into that form.

The stimulation for finding this better formalism came from an unexpected source, namely the discovery of a simple quantum field theory on the four dimensional Moyal-Weyl space, the Grosse-Wulkenhaar model [17]. That model is renormalizable [17, 18, 19, 20] and asymptotically safe [21, 22, 23]. It is therefore an extremely tempting target, as first potential example of a simple and mathematically well-defined non trivial four dimensional QFT.

However the (rather ugly, one must admit) technique of multiscale cluster and Mayer expansions [24, 25, 26, 27], the only available constructive tool for Bosonic quantum field theories, could not be applied to the Grosse-Wulkenhaar model, essentially because the interaction of that model is non-local in $x$-space.

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2Yang-Mills theory is neither simple, nor yet fully well defined in four dimensions.
In matrix base the problem is simply shifted: cluster expansions apply to vector models but not to non-trivial matrix models (they do not provide the right bounds when the size of the matrix increases). Hence something better had to be found.

A first progress occurred one year ago when one of us found that combining the canonical forest formula with the intermediate field method lead to a convergent resummation for matrix models uniformly in the size of the matrix [28]. The resulting loop vertex expansion was devised to treat a renormalization group slice for the Grosse-Wulkenhaar model, and it was quickly realized that this method applied to ordinary quantum field theory on commutative space as well [29]. We recommend the reading of [28, 29] before going further down this paper. However three main drawbacks remained:

• The intermediate field method does not generalize easily to other stable interactions than $\phi^4$, say $\phi^6$ etc... It is probably possible to treat these other cases with more and more intermediate fields but the technique becomes cumbersome.

• A second problem (in fact deeply related to the first) is that the loop vertex expansion of [28, 29] is not easily ordered into a multiscale expansion suited for the renormalization group. This is because for instance a loop made of propagators of two different scales does not factor as two loops, one in each scale, but rather as a sequence of open single-scale resolvents. Although again some ways to circumvent this difficulty do exist, they are not elegant. The conclusion is that some kind of resolvent, rather than loop, should be the right canonical object.

• Finally in the loop vertex expansion, functional integration is still present, although in the reduced, more "model-independent" form of white noise for the intermediate $\sigma$-field. Therefore the formalism does not seem to lead to new insights on QFT in non-integer dimension. Formulating quantum field theory in noninteger dimension is a key benchmark to supersede perturbative field theory while retaining its advantages.

The formalism developed in this paper solves the first two problems at once, and leads to a new way of attacking the third. The vision of QFT that emerges is that of a generating functional for the species of weighted trees that automatically computes connected functions. The vacuum is the trivial tree and the correlation functions are given by "vacuum expectation values" of the resolvent of that combinatoric Hamiltonian operator. The resulting formulation of the theory is given by a convergent rather than divergent expansion. In short and at the most

3Species are roughly speaking structures on finite sets of points, together with generating functionals which allow to extend usual operations on functions, and therefore to formulate rigorously statements such as the logarithm of forests are trees, the derivatives of cycles are chains etc...
naive level, this is because there are much less trees than graphs, but they still capture the vital information about connectedness.

We have worked out our method in the test case of a $\lambda \phi^4$ interaction with a real coupling constant $\lambda$ and in a real dimension $D$. Let us remark that in all known cases where QFT has been built, perturbative theory was Borel summable. We conjecture that this is indeed the case and that our non-perturbative definition of QFT is indeed the Borel sum of ordinary perturbation theory. We formulate a conjecture on positivity of certain matrices which would allow to also extend the theory to real non-integer dimensions, and probably also to complex dimensions with positive real parts. This would be an important step towards making precise the mathematical status of the Wilson-Fisher expansion.

The essential point is that in this reformulation of QFT space-time no longer lies at the center of the stage. Topological notions such as trees now play that central role. Therefore we hope this point of view might ultimately help to answer deep questions, such as: Is quantum gravity a quantum field theory in some “extended sense”? Is it renormalizable?

This paper should be really thought as an introduction to a new line of ideas. The mathematical core of the paper is in section 4 where we give in detail positivity theorems. More precise mathematical details and the exploration of the many conjectures that this work suggests, in particular those of the last sections are devoted to future publications.

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II General Formalism

II.1 The Universal Vector Space

The basic quantities of field theories, the $N$-point correlation functions, are described by sums of connected graphs and are functions of a certain set of external invariants. In the parametric representation we know that the dependence in terms of the Euclidean invariants is associated to a sum over two-trees of the graph, which are similar to spanning trees but with one special deleted or cut line. The invariant associated to such a two tree is the square of the sum over all incoming momenta on any of the two pieces defined by the cut. It can be computed on any of them because by momentum conservation the corresponding invariants are equal. Note also that cuts which don’t have any incoming line on one piece don’t contribute as their associated invariant is 0.

Motivated by this observation we introduce the family of labeled marked

\footnote{We use labeled trees because they are the most standard ones. Labeled means that vertices are labeled; the total number of labeled (unmarked) trees with $n$ vertices and $n-1$ lines is $n^{n-2}$ (Cayley’s theorem).}
trees with one external point or source and one mark, both of which are leaves. The order \( n \) of such a tree is defined as the number of vertices (excluding the mark and the external point). To the source vertex will be associated an external variable, eg a spatial position \( x \). The line with the mark at one end is special and should be thought as a half-line, waiting to be glued to another one of the same type.

![Figure 1: A marked tree of order \( n=6 \). The mark is the white box; the black box is the source at position \( x \).](image)

The universal vector space for QFT is an (infinite dimensional) vector space \( \mathcal{E} \) which is the algebraic vector space spanned by the countable basis \( e_T \) for each such marked tree \( T \). It foliates as

\[
\mathcal{E} = \bigoplus_{N \geq 1} \mathcal{E}_N
\]

where \( \mathcal{E}_N \) is spanned by marked trees with \( N \) sources. It contains a natural exhausting sequence of finite dimensional spaces if we fix eg the total number of lines of the tree. Finally we recall that each element of \( \mathcal{E} \) is a linear combination

\[
\sum_T \lambda_T e_T
\]

where the sum over \( T \) is finite.

This universal space will be decorated by a (model dependent) scalar product and will then give rise under completion to various (model-dependent) Hilbert spaces \( \bar{\mathcal{E}} \).

### II.2 The Universal Hamiltonian

Interesting operators on \( \mathcal{E} \) may be obtained by operations such as gluing or contracting lines of trees. We focus on one particular operator which plays the key role in what follows and which we call the "universal Hamiltonian". It is in fact only really defined in \( \bar{\mathcal{E}} \), but it is an inductive limit of a family of operators \( H_n \) defined in \( \mathcal{E} \) as follows.

We need first to introduce a new category of trees called elementary 2-marked trees. They have no sources, two special marked leaves, and the property that
the (unique) path from one gluing point to the other one contains exactly one vertex.

![Figure 2: An elementary 2-marked tree of order n=5.](image)

There is a natural gluing operation of a marked tree (with $N$ sources) $T$ and an elementary two-marked tree $S$. It creates a larger marked tree (with $N$ sources) $S \star T$. It glues the marked point of $T$ to one of the marked point of $S$ fusing their (half)-lines into a single line.

We now define the $n$-th order universal or ”abstract” Hamiltonian $H_n$ by its action on basis vectors $H_n e_T = - \sum_{S, n(S) \leq n} e_{S \star T}$ where $n(S)$ is the order of $S$, in which marks do not count.

**Remarks**

- The definition of the gluing does not depend on which mark we chose in $S$ for the gluing.
- The sum over $S$ being finite, the operators $H_n$ are well-defined on $\mathcal{E}$.
- $\lim_n H_n$ does not exist on $\bar{\mathcal{E}}$, but $\lim_n (1 + H_n)^{-1}$ will exist.

![Figure 3: The gluing operation.](image)

This completes the list of universal model-independent structure. Of course when really changing QFT, eg to NCQFT the categories have to change, eg fermions imply oriented trees, NCQFT imply ribbon trees and so on.

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For ribbon graphs, the $\star$ operation is not symmetric but $H_n$ remains symmetric.
II.3 The Forest Formula

Consider \( n \) points; the set of pairs \( P_n \) of such points which has \( n(n-1)/2 \) elements \( \ell = (i, j) \) for \( 1 \leq i < j \leq n \), and a smooth function \( f \) of \( n(n-1)/2 \) variables \( x_\ell \), \( \ell \in P_n \). Noting \( \partial_\ell \) for \( \frac{\partial}{\partial x_\ell} \), the standard canonical forest formula is

\[
f(1, \ldots, 1) = \sum_{F} \left[ \prod_{\ell \in F} \int_{0}^{1} dw_\ell \left( \prod_{\ell \in F} \partial_\ell \right) [x^{F}_\ell (\{w_\ell\})] \right] \quad (\text{II-3})
\]

where

- the sum over \( F \) is over forests over the \( n \) vertices, including the empty one
- \( x^{F}_\ell (\{w_\ell\}) \) is the infimum of the \( w_\ell \) for \( \ell' \) in the unique path from \( i \) to \( j \) in \( F \), where \( \ell = (i, j) \). If there is no such path, \( x^{F}_\ell (\{w_\ell\}) = 0 \) by definition.
- The symmetric \( n \) by \( n \) matrix \( X^{F}(\{w\}) \) defined by \( X^{F}_{ii} = 1 \) and \( X^{F}_{ij} = x^{F}_{ij}(\{w_\ell\}) \) for \( 1 \leq i < j \leq n \) is positive.

A particular variant of this formula (II-3) is in fact better suited to direct application to the parametric representation of Feynman amplitudes. It consists in changing variables \( x \to 1 - x \) and rescaling to \([0, 1] \to [0, \infty]\) of the range of the variables. One gets that if \( f \) is smooth with well defined limits for any combination of \( x_\ell \) tending to \( \infty \),

\[
f(0, \ldots, 0) = \sum_{F} \left[ \prod_{\ell \in F} \int_{0}^{\infty} ds_\ell \left( \prod_{\ell \in F} \partial_\ell \right) [-\partial_\ell f] [x^{F}_\ell (\{s_\ell\})] \right] \quad (\text{II-4})
\]

where

- the sum over \( F \) is like above,
- \( x^{F}_\ell (\{s_\ell\}) \) is the supremum of the \( s_\ell \) for \( \ell' \) in the unique path from \( i \) to \( j \) in \( F \), where \( \ell = (i, j) \). If there is no such path, \( x^{F}_\ell (\{s_\ell\}) = \infty \) by definition. This is because the change of variables exchanged inf and sup.

To distinguish these two formulas we call \( w \) the parameters of the first one (like "weakening") since the formula involves infima, and \( s \) the parameters of the second one (like "strengthening" or "supremum") since the formula involves suprema.

III QFT Models as scalar products

A QFT model is defined perturbatively by gluing propagators and vertices and computing corresponding amplitudes according to certain Feynman rules.

Model dependent features imply the types of lines (propagators for various particles, bosons or fermions), the type of vertices, the space time dimension and its metric (Euclidean/Minkowsky, curved...)
We now reorganize the Feynman rules by breaking Feynman graphs into pieces and putting them into boxes labeled by trees according to the canonical formula. The content of each tree box is then considered the matrix element of a certain scalar product.

Equivalently we can obtain our formalism by applying the canonical forest formula to the $n$-th order of perturbation on a functional integral. Of course we arrive at the same point. However one can use either formula (II-3), introducing new weakening parameters $w,$ or formula (II-4) in which the $s$ parameters are directly the Feynman parameters of the parametric representation of the tree lines. This second point of view is much better suited to multiscale analysis, but for pedagogical reasons we give both formulas. We illustrate our formalism with the example of the $\lambda \phi^4$ theory in real space time dimension $D$. Other stable polynomial interactions could be treated in the same way.

### III.1 Propagator

For Bosonic scalar field theory in integer dimension the usual massive propagator is, up to inessential constants, which we forget from now on

$$C(k) = \frac{1}{k^2 + m^2} = \int_0^\infty e^{-\alpha(k^2 + m^2)} d\alpha$$ (III-1)

$$C(x,y) = \int_0^\infty \frac{e^{-\alpha m^2 - |x-x'|^2/4\alpha}}{\alpha^{D/2}} d\alpha$$ (III-2)

We also note $D$ the propagator at fixed value of the Feynman parameter

$$D(s; x, x') = e^{-sm^2 - |x-x'|^2/4s} s^{D/2}$$ (III-3)

### III.2 Tree Amplitudes

We consider now ordinary trees with $N$ external points, $N \geq 1$. To any such tree we shall associate an amplitude by applying formulas (II-3) or (II-4) to the functional integral defining the theory, which is only at this stage a heuristic tool. The goal is to obtain a forest formula for unnormalized functions, from which a tree formula for normalized connected functions follows.

We show now two ways to apply these formulas. As a pedagogical exercise in subsection III.2.1 we use (II-3) to decouple vertices in the most naive way. This does not optimize multiscale analysis.

Then we show in subsection III.2.2 how to apply the second formula (II-4) directly on the Feynman parameters in the parametric representation, This has two main advantages. First Feynman parameters precisely provide scale analysis, so that we obtain a much better formalism for future applications in which renormalization will enter the picture. Second, in the parametric representation space time is a parameter and we get in this way a program to define QFT constructively at non integer dimension, which we sketch in section VII.
III.2.1 Scale independent amplitudes

The first (naive interpolation) computes an amplitude (in $x$ space representation)

$$A(T, z_1, \ldots, z_N) = \frac{(-\lambda)^n}{(4!)^n n!} \int_{v \in T} dx_v D_v \prod_{\ell \in T} \int_0^1 dw_\ell \prod_{\ell \in T} C(x_\ell, x'_\ell) \quad (\text{III-4})$$

$$\int d\mu_{CT(w)} \left( \prod_{\ell \in T} \delta \phi(x_\ell) \delta \phi(x'_\ell) \prod_{i=1}^N \phi(z_i) \prod_v \phi^4(x_v) \right)$$

where $C^T(x, x', w) = C(x, x') \inf_{\ell \in P^T(x, x')} w_\ell$, and $P^T(x, x')$ is the path in $T$ from $x$ to $x'$. The functional derivations $\prod_{\ell \in T} \delta \phi(x_\ell) \delta \phi(x'_\ell)$ of course are constrained to apply in such a way as to create the lines of the tree $T$. This formula comes by applying the interpolation formula (II-3) where a distinct field copy is associated to each vertex with a degenerate copy-blind covariance. This copy-blind covariance is weakened on off diagonal terms by the $w$ factors. This is exactly the same method that was used in [28] for the $\sigma$ field.

**Remark 1**

We recall that such amplitudes are distributions in the external arguments $z_1, \ldots, z_N$ which may be singular at coinciding points. We can smear them against test functions $f_1, \ldots, f_N$. It is well known that bare Feynman amplitudes may diverge for $D$ large enough. This is tackled through renormalization theory, which has of course to enter the picture when necessary. Here and in what follows the reader may assume $\Re D < 2$ to separate the issues.

**Remark 2** Working with the $\phi^4$ means that we cannot produce any tree with degree more than 4 at any vertex. Therefore we could restrict the space $\mathcal{E}$ with that condition. We prefer to consider that all amplitudes for trees which violates that condition are zero. The amplitude for a tree such as the one of Figure 4 should be thought as obtained by first completing all vertices of degree less than 4 to degree 4 by adding the necessary fields, i.e. half lines, then summing over all their contraction schemes with the correct weakening parameters.

III.2.2 Parametric amplitudes

This method was introduced for Fermions in [12].

$$A(T, z_1, \ldots, z_N) = \frac{(-\lambda)^n}{(4!)^n n!} \int_{v \in T} dx_v D_v \prod_{\ell \in T} \int_0^\infty ds_\ell \prod_{\ell \in T} [D(s_\ell; x_\ell, x'_\ell)]$$

$$\int d\mu_{CT(s)} \left( \prod_{\ell \in T} \delta \phi(x_\ell) \delta \phi(x'_\ell) \prod_{i=1}^N \phi(z_i) \prod_v \phi^4(x_v) \right)$$

(III-5)
where
\[ C^T(s; x, x') = \int_{\sup l \in \mathcal{P}_T(x, x')}^{\infty} \frac{e^{-\alpha m^2 - |x-x'|^2/4\alpha}}{\alpha^{D/2}} d\alpha \] (III-6)
is the propagator with a restricted integration range in parametric space. This is obtained by applying formula (II-4) directly to the Feynman parameters.

### III.3 The scalar product

There is a natural gluing operation \( \star \) on marked trees with sources. To two such marked trees \( T \) and \( T' \) with \( p \) and \( q \) sources it associates an ordinary tree \( T \star T' \) with \( p + q \) sources, by gluing the two marked (half lines) into an ordinary line (always called \( \ell_0 \) in what follows).

![Figure 4: The gluing of two marked trees into an ordinary tree](image)

We now consider the infinite matrix \( <e_T, e_{T'}> \). This matrix is obviously symmetric because \( T \star T' = T' \star T \).

**Theorem III.1.** This matrix is positive, hence defines a scalar product on \( \mathcal{E} \)

The theorem means that \( \sum_T \lambda_T e_T, \sum_T \lambda_T e_T \geq 0, \forall \lambda_T \), the sum being over finitely many marked trees.

The operator \( H_n \) is symmetric with respect to that scalar product, because

\[ <e_T, H_n e_{T'}> = <e_T, \sum_{S,n(S) \leq n} e_{S \star T'}> = - \sum_{S,n(S) \leq n} A(T \star (S \star T')) = <H_n e_T, e_{T'}> \] (III-7)
because

\[ T \star (S \star T') = (S \star T) \star T' \] (III-8)

Remark that for ribbon graphs, equation (III-8) is not true but (III-7) still holds because of the summation.

**Theorem III.2.** The operator \( H_n \) is positive, i.e. \( \sum_T \lambda_T e_T, H_n(\sum_T \lambda_T e_T) \geq 0, \forall \lambda_T, \forall n \), the sum being over finitely many marked trees.

The proofs of these two main theorems is given in the following section.

This positivity is a kind of abstract tree version of the well known OS Euclidean positivity axiom. Developing this analogy should lead to an axiomatic formulation of Tree QFT that we leave for the future.
IV The positivity theorems

IV.1 Positivity of the scale independent formula

Let us call $z_T$ the collection of fixed external positions $z_1, \ldots, z_N$ of a marked tree $T$ with $N$ sources, and $f_T(z_1, \ldots, z_N)$ a test function for these sources. We want to prove that

$$I = \sum_{T, T'} \lambda_T \lambda_{T'} \int dz_T dz_{T'} A(T \ast T', z_T, z_{T'}) f_T(z_T) f_{T'}(z_{T'}) \geq 0 \quad \text{(IV-1)}$$

hence that

$$I = \sum_{T, T'} \lambda_T \lambda_{T'} \int dz_T dz_{T'} f_T(z_T) f_{T'}(z_{T'}) \left( \frac{(-\lambda)^{n(T \ast T')}}{(4!)^{n(T \ast T')} n(T \ast T')!} \right)$$

$$\left( \prod_{\ell \in T \ast T'} \frac{\delta}{\delta \phi(x_{\ell})} \frac{\delta}{\delta \phi(x'_{\ell})} \right) \prod_{i=1}^{N(T \ast T')} \phi(z_i) \prod_{v \in T \ast T'} \phi^4(x_v) \geq 0 \quad \text{(IV-2)}$$

We reorganize this sum by fixing the number $p \geq 0$ of lines which cross from $T$ to $T'$ in the functional integration above, and we prove that for any fixed $p$ the sum above is positive.

The gluing line $\ell_0$ in $T \ast T'$ has by convention index 0 and associated weakening parameter $w_0$. It has propagator $C(x_0, x'_0)$. The other crossing lines have indices $i = 1, \ldots, p$. We use the identity $\inf(w) = \int_0^{\inf w} du$ to express as integrals over new parameters $u_i$ the weakening parameters of the $p$ crossing lines. We use the multinomial identity plus relabeling to attribute vertices either to the right or to the left of the star operation. This replaces the $1/n(T \ast T')!$ symmetry factor by "factorized" factors $1/n(T)!n(T')!$. Finally for $i = 0, \ldots, p$ we cut the crossing lines $C(x_i, x'_i)$ in the middle with respect to new variables $y_i$, as $C(x_i, x'_i) = \int d^D y_i C^{1/2}(y_i, y_i) C^{1/2}(y_i, y_i)$. This rewrites $I$ in formula (IV-2) as

$$I = \sum_{p \geq 0} \frac{1}{p!} \int_0^1 \prod_{i=1}^p dw_0 \prod_{i=1}^p \int_0^{w_0} du_i \prod_{i=0}^p \int d^D y_i \left( K_p(\{y\}, \{u\}) \right)^2 \quad \text{(IV-3)}$$

$$K_p(\{y\}, \{u\}) = \sum_T \frac{\lambda_T (-\lambda)^{n(T)}}{(4!)^{n(T)} n(T)!} \int dz_T f_T(z_T) \int \prod_{v \in T} dw_v \prod_{\ell \in T} \int_0^1 dw_\ell \quad \text{(IV-4)}$$

$$C^{1/2}(y_0, x_0) \prod_{\ell \in T} C(x_\ell, x'_\ell) \int d\mu_{CT(w)} \left( \prod_{i=1}^p \int d^D x_i C^{1/2}(y_i, x_i) \frac{\delta}{\delta \phi(x_i)} \right)$$

$$\left( \frac{\delta}{\delta \phi(x_0)} \prod_{\ell \in T} \frac{\delta}{\delta \phi(x_\ell)} \frac{\delta}{\delta \phi(x'_\ell)} \right) \prod_{j=1}^{N(T)} \prod_{\ell \in T} \phi(z_j) \prod_{v \in T} \phi^4(x_v) \prod_{i=1}^p \chi(w_i \geq u_i)$$
where $w_i^T$ is the infimum over the parameters in the unique path in $T$ going from $x_i$ to $x_0$. Note that the non trivial function $\prod_{i=1}^p \chi(\inf w_i^T \geq u_i)$ can be computed only after the action of the functional derivatives, as it depends on this action. The important point is that the condition that the crossing lines are multiplied by the infimum of the $w$ over the path in $T \ast T'$ can be factorized in these non trivial functions, thanks to the $u$ parameters. This trick is a multiparameter generalization of the identity

$$\int_0^1 \int_0^1 dx dy \inf(x,y) f(x)f(y) = \int_0^1 ds \int_s^1 \int_0^1 dx dy f(x)f(y) \geq 0 \quad \text{(IV-5)}$$

From (IV-3) follows immediately that $I \geq 0$.

Finally the positivity for $H$ can be proved exactly in a similar manner, but we have to split the central vertex in two halves. In short the role of the $C_0$ line is replaced by a propagator which is in fact a delta function and there is no $w_0$ parameter. Apart from these details the factorization is identical. Remark that this split of $H$ as a square can be performed for any even polynomial, not only $\phi^4$. This is definitely an advantage of this method over the intermediate field method and loop vertex expansion of $[28]-[29]$.

Figure 5: Positivity of the $H$ operator. The dotted line represents a delta function.

**IV.2 Positivity of the parametric formula**

We use now formula (II-4). The outcome is only slightly different, and reads for an integer dimension $D$:

$$I = \sum_{p \geq 0} \frac{1}{p!} \int_0^\infty ds_0 \prod_{i=1}^p \int_{s_0}^\infty dt_i \prod_{i=0}^p \int d^D y_i \left( K_p(s_0, \{y_i\}, \{t_i\}) \right)^2 \quad \text{(IV-6)}$$
where $s_T$ is the supremum over the parameters in the unique path in $T$ going from $x_i$ to $x_0$.

The proof that $H$ is positive is similar.

V The Hilbert space and the two point function

The Hilbert space is defined as the completion $\mathcal{E}$ of the canonical space for the chosen scalar product. It is therefore model dependent. We already know from Haag’s theorem that this should be the case. Our scalar product typically (for instance for the $\phi^4$ theory) has matrix elements of order $n!$ so that the Hilbert space is roughly made of infinite sums of trees with coefficients decaying as $1/n!$.

The full abstract Hamiltonian operator $H$ is the inductive limit $\lim_{n \to \infty} H_n$. It is not well defined on $\mathcal{E}$ nor on the whole of $\overline{\mathcal{E}}$, because it is an unbounded operator. However the interacting propagator $1 + H$ can be defined as $\lim_{n \to \infty} 1 + H_n$, and should be a bounded operator on $\overline{\mathcal{E}}$.

An analogy that may help to grasp the situation is that of the ordinary Laplacian; although $\Delta$ is not a bounded operator on $L^2$, we can perfectly define $(1 - \Delta)^{-1}$ on that space.

**Definition V.1.** The interacting propagator or two point function is then defined non-perturbatively by

$$S_2(x, y) = <e_0(x), \frac{1}{1 + H} e_0(y)> \quad (V-1)$$

where $e_0(x)$ is the trivial tree whose black box position is at $x$.

We do not give details here about existence of the limit $\frac{1}{1 + H} = \lim_{n \to \infty} \frac{1}{1 + H_n}$ but it should follow easily from the positivity of all $H_n$’s, and lead to the norm bound $\| \frac{1}{1 + H} \| \leq 1$. To establish the decay properties of $S_2(x, y)$ as $\|x - y\| \to \infty$ however is expected to require expansion steps followed by inequalities similar to those of [29].
VI  

\textbf{N point functions}

It is also possible to define $N$ point functions, but it requires to enlarge slightly the formalism. Formally the $N$ point functions can be obtained by gluing two marked trees with $p$ and $q$ sources with $p + q = N$. However the crucial non-perturbative ingredient is hidden in the positivity of $H$ and the resolvent $(1 + H)^{-1}$ in (V-1). To make use also of this resolvent, we can define for any $N$ point function its skeleton, which is made of at most $N-2$ particular ”crossroad” vertices $V$ and of thick lines.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{tree.png}
\caption{The skeleton of a tree with $N = 6$ sources and three ”crossroads”.}
\end{figure}

The thick lines correspond to resolvents $(1 + H)^{-1}$. A crossroad vertex of degree $d$ can be though as having $d$ gluing marks or white boxes, and an associated coupling constant $-\lambda$; in a $\phi^4$ theory we must again have $d \leq 4$. In the universal space $cE$ such a vertex is not an operator but an operad, that it is tensor of degree $d$. The non perturbative definition of $N$ point Schwinger functions is then

\textbf{Definition VI.1.} The interacting $N$ point functions are defined non-perturbatively by

$$S_N(x_1, \ldots, x_N) = \sum_{N\text{-skeleton } S \in \mathcal{S}} \prod_{v \in S} V_v \prod_{\ell \in S} \left( \frac{1}{1 + H} \right) \prod_{j=1}^N e_0(x_j) \quad (\text{VI-1})$$

where the gluing are made according to the skeleton $S$ as in Figure 6, with hopefully transparent notations. The important point is that the sum over $S$ is finite, hence this is a convergent definition, as all divergences of ordinary perturbation theory have been hidden in the $(1 + H)_{\ell}$ resolvents.

A few additional remarks are in order.

Like the loop-vertex expansion this formalism applies equally well to non-commutative QFT’s or to matrix models. The key non-perturbative bound is indeed a norm bound: $H \geq 0$ implies $\left\| \frac{1}{1 + H} \right\| \leq 1$. This is the exact analog of the loop-vertex bound, in which an operator $\sigma$ Hermitian implied $\left\| \frac{1}{1 + i\sigma} \right\| \leq 1$
These bounds extend exactly in the same way to matrix models. But the advantage of this formalism is that in the parametric version, ordering the tree prepares exactly the theory for (continuous) multiscale analysis and renormalization. Ultraviolet divergences in this formalism simply occur when $D \geq 2$ in the form of divergence of certain matrix elements of the scalar product which define the quantum field theory. Hence this scalar product itself should be defined inductively over scales using running constants to absorb the divergences as usual [25].

VII The case of noninteger $D > 0$

We would like to define QFT non perturbatively in non integer (positive) dimension.

The key should be given by the parametric representation. It was remarked very early [30] that Feynman amplitudes in parametric space only involve Gaussian integrations and the result is therefore given in terms of determinants ("Symanzik polynomials") to the power $D$ times quadratic forms in the external invariants which are rotation invariant so involve only scalar external invariants.

We start again from formula (IV-7), but we want to perform all momentum and spatial integrations to obtain determinants raised to the power $D$. Let us rewrite $K$ in this form. For each value of the functional derivatives

$$
\prod_{i=1}^{p} \frac{\delta}{\delta \phi(x_i)} \left( \prod_{\ell \in T} \frac{\delta}{\delta \phi(x_\ell)} \frac{\delta}{\delta \phi(x_\ell')} \right)
$$

(VII-1)

we obtain a piece of a Feynman amplitude for a particular graph. Let us generically call $G$ a label for all these pieces of graphs. In any such $G$ the $x_i$ are now identified with some particular $x_v$’s, and we get a particular function $\chi_G(s, t) = \prod_{v} \chi(s^T_v \leq t_v)$. It is then not difficult to perform for each such $G$ all internal spatial integrations $\prod_v d^D x_v$ in $K$. The Feynman-Symanzik parametric representation then states that $K$ is a sum of quadratic forms on the invariants built on the external variables which are now $z_T$, and the $\{y\}$, divided by a polynomial in the Feynman parameters, called $U_G(s, t)$ to the power $D/2$.

$$
K_p(s_0, \{y\}, \{t\}) = \sum_G c_G \prod_{\ell \in T} \int_0^\infty ds_{\ell} \chi_G(s, t)e^{Q_G(y), z_T, s_0, \{t\}} / U_G^{D/2}(s, t)
$$

(VII-2)

The last remaining difficulty is that we still have to perform the integrals $d^D y_i$ in

$$
I = \sum_{p \geq 0} \frac{1}{p!} \int_0^\infty ds_0 \prod_{i=1}^{p} \int_{s_0}^\infty dt_i \prod_{i=0}^{p} d^D y_i \left( K_p(s_0, \{y\}, \{t\}) \right)^2
$$

(VII-3)
to get a formula in which the dimension purely enters as a parameter. We can put \( y_0 \) to the origin to break translation invariance. Substituting the form \(\text{(VII-2)}\) into \(\text{(VII-3)}\) we can put the result into the form

\[
\sum_{a,b} \lambda_a \lambda_b \det(Q_a + Q_b)^{-D/2} \tag{VII-4}
\]

where \( Q_a \) is a \( p \) by \( p \) positive quadratic form on the invariants built on the \( y_i \) variables, and \( a \) is a simpler label for the piece of graph \( G \) under consideration.

We conjecture that such quadratic forms are always positive:

**Conjecture 1** Let \( Q_a, a = 1, ..., q \) be a family of \( q \) \( p \) by \( p \) positive quadratic forms with positive coefficients Then the matrix \( M_{ab} = \det(Q_a + Q_b)^{-D/2} \) is positive for any \( D > 0 \).

The conjecture is obviously true for integer \( D \), and for \( q = 2 \) and any \( D \) and \( p \), since a two by two symmetric matrix with positive coefficients

\[
M = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \tag{VII-5}
\]

is positive if and only any of its Hadamard positive power

\[
M_d = \begin{pmatrix} a^d & c^d \\ c^d & b^d \end{pmatrix} \tag{VII-6}
\]

is positive; indeed positivity in that case reduces to the condition \( ab \geq c^2 \). However this is no longer true for \( q \geq 3 \). It is easy to check eg that for positive \( r \)

\[
M(r) = \begin{pmatrix} 1 & r & 0 \\ r & 1 & r \\ 0 & r & 1 \end{pmatrix} \tag{VII-7}
\]

is positive if and only if \( r \leq 1/\sqrt{2} \); hence

\[
M(4/5) = \begin{pmatrix} 1 & .8 & 0 \\ .8 & 1 & .8 \\ 0 & .8 & 1 \end{pmatrix} \tag{VII-8}
\]

is not positive but its Hadamard square

\[
M_2 = M(16/25) = \begin{pmatrix} 1 & .64 & 0 \\ .64 & 1 & .64 \\ 0 & .64 & 1 \end{pmatrix} \tag{VII-9}
\]

is positive. We conjecture however that matrices obtained from determinants of sums of positive quadratic forms as in \(\text{(VII-4)}\) are never of this kind.

The conjecture if true would lead to the first non perturbative definition of QFT’s in non-integer dimension.
VIII  Complex parameters, Borel Summability

It is natural to expect that for complex coupling constant sufficiently small with positive real part, the formula $\sqrt{V-1}$ still makes sense, and that for $0 < \Re D < 2$ our non-perturbative definition is in fact the Borel sum of the ordinary perturbative series.

**Conjecture 2** The two point function $S_2$ is well defined by $\sqrt{V-1}$ for $\lambda \phi^4_D$ for $0 < \Re D < 2$, and for in a Nevanlinna-Sokal disk $\Re^{-1} \geq R^{-1}$. It is the Borel sum of its perturbative series.

We do not expect the proof of this conjecture to be very difficult, but the resolvent $H$ is not linear in $\lambda$ so the problem looks more like analyticity of a continued fraction rather than of a simple resolvent as was the case in [28]-[29].

If Conjecture 1 is true, we expect this Conjecture 2 to extend at least to the region $0 \leq D \leq 4$, if we add the ultraviolet subtractions corresponding to the mass renormalizations for $2 \leq \Re D < 2$. The local "Borel germ" of perturbation theory was shown to exist in that region $0 \leq D \leq 4$ in [31].

The proof of Conjecture 2 involves presumably to define complex suitable extensions of the real vector space $E$ and the real symmetric $H$ operator. This and many other applications of our formalism are devoted to future publications.

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