Unconditional well-posedness for the
Dirac - Klein - Gordon system in two
space dimensions

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Abstract
The solution of the Dirac - Klein - Gordon system in two space dimensions
with Dirac data in $H^s$ and wave data in $H^{s+1/2} \times H^{s-1/2}$ is uniquely determined
in the natural solution space $C^0([0,T], H^s) \times C^0([0,T], H^{s+1/2})$, provided $s > 1/30$. This improves the uniqueness part of the global well-posedness result
by A. Grünrock and the author, where uniqueness was proven in (smaller)
spaces of Bourgain type. Local well-posedness is also proven for Dirac data
in $L^2$ and wave data in $H^{3/5} \times H^{-2/5}$ in the solution space $C^0([0,T], L^2) \times
C^0([0,T], H^{3/5})$ and also for more regular data.

1 Introduction and main results
The Cauchy problem for the Dirac – Klein – Gordon equations in two space di-

$$i(\partial_t + \alpha \cdot \nabla)\psi + M\beta \psi = -\phi \beta \psi$$

$$(-\partial_t^2 + \Delta)\phi + m\phi = -\langle \beta \psi, \psi \rangle$$

with (large) initial data

$$\psi(0) = \psi_0, \phi(0) = \phi_0, \partial_t \phi(0) = \phi_1.$$
We consider Cauchy data in Sobolev spaces: \( \psi_0 \in H^s \), \( \phi_0 \in H^r \), \( \phi_1 \in H^{r-1} \).

Local well-posedness was shown by d’Ancona, Foschi and Selberg \([2]\) in the case \( s > -\frac{1}{4} \) and \( \max(\frac{4}{5} - \frac{3}{5}, \frac{4}{5} + \frac{2}{5}, s) < r < \min(\frac{3}{4} + 2s, \frac{3}{4} + \frac{4}{5}, 1 + s) \). As usually they apply the contraction mapping principle to the system of integral equations belonging to the problem above. The fixed point is constructed in spaces of Bourgain type \( X^{s,b}_X \times X^{r,b}_X \) which are subsets of the space \( C^0([0, T], H^s(\mathbb{R}^2)) \times C^0([0, T], H^r(\mathbb{R}^2)) \).

Thus especially uniqueness is shown also in these spaces of \( X^{s,b}_X \)-type. Thus the question arises whether unconditional uniqueness holds, namely uniqueness in the natural solution space \( C^0([0, T], H^s(\mathbb{R}^2)) \times C^0([0, T], H^r(\mathbb{R}^2)) \) without assuming that the solution belongs to some (smaller) \( X^{s,b}_X \times X^{r,b}_X \)-space.

The question of global well-posedness for the system \([11],[2],[3]\) was recently answered positively for data \( \psi_0 \in H^s \), \( \phi_0 \in H^{s+\frac{1}{2}} \), \( \phi_1 \in H^{s-\frac{1}{2}} \) in the case \( s \geq 0 \) by A. Grünrock and the author \([6]\). They showed existence and uniqueness in Bourgain type spaces \( X^{s,b}_X \) based on certain Besov spaces with respect to time. These solutions were shown to belong automatically to \( C^0([0, T], H^s(\mathbb{R}^2)) \times C^0([0, T], H^r(\mathbb{R}^2)) \). Again the question arises whether unconditional uniqueness holds, namely uniqueness in the natural solution space \( C^0([0, T], H^s(\mathbb{R}^2)) \times C^0([0, T], H^r(\mathbb{R}^2)) \) without assuming that the solution belongs to some (smaller) Bourgain type spaces.

The question of unconditional uniqueness was considered among others by Yi Zhou for the KdV equation \([10]\) and nonlinear wave equations \([11]\), by N. Masmoudi and K. Nakanishi for the Maxwell-Dirac, the Maxwell-Klein-Gordon equations \([7]\), the Klein-Gordon-Zakharov system and the Zakharov system \([8]\), and by F. Planchon \([3]\) for semilinear wave equations.

Our main results read as follows:

**Theorem 1.1** Let \( \psi_0 \in H^s(\mathbb{R}^2) \), \( \phi_0 \in H^r(\mathbb{R}^2) \), \( \phi_1 \in H^{r-1}(\mathbb{R}^2) \), where
\[
\frac{1}{8} > s \geq 0 , \quad \frac{3}{5} - 2s < r < \min(\frac{3}{4} + \frac{3}{2}s, 1 - 2s).
\]

Then the Cauchy problem \([11],[2],[3]\) is unconditionally locally well-posed in
\[
(\psi, \phi, \phi_t) \in C^0([0, T], H^s(\mathbb{R}^2)) \times C^0([0, T], H^r(\mathbb{R}^2)) \times C^0([0, T], H^{r-1}(\mathbb{R}^2)).
\]

Especially we can choose \( s = 0 \) and \( r = \frac{3}{5}+ \).

**Remark:** Similar results for \( s \geq 1 \) and a suitable range for \( r \) can also be given. If \( 1 > s \geq \frac{1}{6} \) the result remains true if \( \max(\frac{1}{2} - \frac{3}{2}s, s, \frac{3}{2} - \frac{5}{2}s) < r < \min(\frac{1}{2} + \frac{5}{2}s, 6s, 1) \), e.g. if \( s = \frac{1}{6} \) and \( \frac{1}{3} < r < 1 \).

**Theorem 1.2** Let \( \psi_0 \in H^s(\mathbb{R}^2) \), \( \phi_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^2) \), \( \phi_1 \in H^{s-\frac{1}{2}}(\mathbb{R}^2) \) with \( s > \frac{1}{30} \).

Then the Cauchy problem \([11],[2],[3]\) is unconditionally globally well-posed in the space
\[
(\psi, \phi, \phi_t) \in C^0(\mathbb{R}^+, H^s(\mathbb{R}^2)) \times C^0(\mathbb{R}^+, H^{s+\frac{1}{2}}(\mathbb{R}^2)) \times C^0(\mathbb{R}^+, H^{s-\frac{1}{2}}(\mathbb{R}^2)).
\]

This means that existence and uniqueness holds in these spaces.
Remark: The interesting question of unconditional uniqueness in the case of lowest regularity of the data where global existence is known (\(s = 0\) in Theorem 1.2 and \(s = 0, r = \frac{1}{2}\) in Theorem 1.1) unfortunately remains unsolved.

We use the following Bourgain type function spaces. Let \(\tilde{\cdot}\) denote the Fourier transform with respect to space and time. \(X_{s,b}^{+,b}\) is the completion of \(S(R \times R^2)\) with respect to
\[
\|f\|_{X_{s,b}^{+,b}} = \|U_{\pm}(-t)f\|_{H_1^s H_{-1}^s} = \|\langle \xi \rangle^s \langle \tau \rangle^b \hat{f}(\xi, \tau)\|_{L^2},
\]
where \(U_{\pm}(t) := e^{\mp i |\xi| \alpha} |\xi| \alpha \). Then the Cauchy problem (4),(5),(6) is locally well-posed for \(\pm |\xi| \alpha \) belonging to the eigenvalues \(\{\pm |\xi| \alpha\}\). These projections are given by \(\Pi_{\pm}(\xi)\) and splitting the function \(\phi\) into the sum \(\phi = \frac{1}{2}(\phi_+ + \phi_-)\), where \(\phi_{\pm} := \phi \pm iA^{-1/2} \partial_t \phi\), \(A := -\Delta + 1\), the Dirac- Klein - Gordon system can be rewritten as
\[
(i \partial_t + A^{1/2})\phi_{\pm} = \mp A^{-1/2}(\beta(\psi_+ + \psi_-), \psi_+ + \psi_-) \mp A^{-1/2}(m + 1)(\phi_+ + \phi_-). \tag{5}
\]
The initial conditions are transformed into
\[
\psi_{\pm}(0) = \Pi_{\pm}(D)\psi_0, \quad \phi_{\pm}(0) = \phi_0 \pm iA^{-1/2}\phi_1 \tag{6}
\]
We now state again the above mentioned well-posedness results on which our results rely.

**Theorem 2.1 (2)** Let \(\psi_0 \in H^s, \phi_0 \in H^r, \phi_1 \in H^{r-1}\), where
\[
s > -\frac{1}{5}, \quad \max(\frac{1}{4} - \frac{s}{2}, \frac{1}{2}, s) < r < \min(\frac{3}{4}, \frac{3}{2}, \frac{3}{2}, 1 + s)\).
\]
Then the Cauchy problem (4),(5),(6) is locally well-posed for
\[
(\psi_{\pm}, \phi_{\pm}) \in X_{s,b}^{+,b} [0, T] \times X_{s,b}^{+,b} [0, T],
\]
i.e.
\[
(\psi, \phi, \partial_t \phi) \in \times (X_{1+}^{+,b} [0, T] + X_{-1}^{+,b} [0, T]) \times (X_{1+}^{-,b} [0, T] + X_{-1}^{+,b} [0, T])
\]
\[
\times (X_{1+}^{-,1,b} [0, T] + X_{-1+}^{+,b} [0, T]).
\]
This solution belongs to
\[
C^0([0, T], H^s) \times C^0([0, T], H^r) \times C^0([0, T], H^{r-1}).
\]
Remark: The question of uniqueness in the latter (larger) spaces remained open.

Theorem 2.2 (\cite{4} or \cite{5}) Let \( s \geq 0 \) and \( \psi_0 \in H^s \), \( \phi_0 \in H^{s+\frac{2}{3}} \), \( \phi_1 \in H^{s-\frac{2}{3}} \). Then the Cauchy problem \((1),(2),(3)\) is globally well-posed for
\[
(\psi_{\pm},\phi_{\pm}) \in X^{s,\frac{2}{3}} \times X^{s+\frac{2}{3}}.
\]
This solution belongs to
\[
(\psi,\phi,\partial_t \phi) \in C^0(\mathbb{R}^+,H^s) \times C^0(\mathbb{R}^+,H^{s+\frac{2}{3}}) \times C^0(\mathbb{R}^+,H^{s-\frac{2}{3}}).
\]
Here the spaces \( X^{s,\frac{2}{3}} \) are certain Bourgain type spaces based on Besov spaces (with respect to time). For a precise definition we refer to \cite{6}.

Remark: Again the question of uniqueness in the latter (larger) spaces remained open.

We recall the following facts about the solution of the inhomogeneous linear problem
\[
\partial_t v - i\phi(D)v = F, \quad v(0) = v_0,
\]
namely
\[
v(t) = U(t)v_0 + \int_0^t U(t-s)F(s)ds,
\]
where
\[
U(t) = e^{it\phi(D)}v_0.
\]

Proposition 2.1 (\cite{4} or \cite{5}) Let \( b' + 1 \geq b \geq 0 \geq b' > -1/2 \). Then the following estimate holds for \( T \leq 1 \):
\[
\|v\|_{X^{s,b}[0,T]} \leq c(T^{\frac{1}{2} - b}\|v_0\|_{H^s} + T^{1 + b'-b}\|F\|_{X^{s',s''}[0,T]}).
\]
Here \( X^{s,b} \) denotes the completion of \( S(\mathbb{R} \times \mathbb{R}^2) \) with respect to the norm \( \|f\|_{X^{s,b}} = \|U(-t)f\|_{H^s_t H^s_x} \) and \( X^{s,b}[0,T] \) the restrictions of these functions to \([0,T]\).

3 Proofs of the theorems

The key result reads as follows:

Theorem 3.1 Let \( \psi_0 \in H^s(\mathbb{R}^2) \), \( \phi_0 \in H^r(\mathbb{R}^2) \), \( \phi_1 \in H^{r-1}(\mathbb{R}^2) \), \( T > 0 \). Assume \( \frac{1}{2} > s \geq 0 \) and \( \frac{2}{3} - 2s < r < 1 - 2s \). Then the Cauchy problem \((1),(2),(3)\) has at most one solution
\[
(\psi,\phi,\partial_t \phi) \in C^0([0,T],H^s(\mathbb{R}^2)) \times C^0([0,T],H^r(\mathbb{R}^2)) \times C^0([0,T],H^{r-1}(\mathbb{R}^2)).
\]
This solution satisfies \( \psi_{\pm} \in X^{s,\frac{1}{2} + \frac{2}{3} + s + \frac{1}{2} + r + 2s} \), \( \phi_{\pm} \in X^{s,\frac{1}{2} + 2s + \frac{1}{2} + r} \).

Proof: We show that any solution
\[
(\psi,\phi,\partial_t \phi) \in C^0([0,T],H^s(\mathbb{R}^2)) \times C^0([0,T],H^r(\mathbb{R}^2)) \times C^0([0,T],H^{r-1}(\mathbb{R}^2))
\]
fulfills $\psi_{\pm} \in X_{\pm}^{-\frac{1}{2}+\frac{s}{2}+\frac{1}{4}+}[0,T]$, $\phi_{\pm} \in X_{\pm}^{-\frac{1}{2}+r+2s+\frac{1}{4}+}[0,T]$. In this space uniqueness holds by the result of d’Ancona, Foschi and Selberg (Theorem 2.1), who had to use the full null structure of the system.

Let $\psi_{\pm} \in C^0([0,T],H^s)$, $\phi_{\pm} \in C^0([0,T],H^r)$ be a solution of (4), (5), (6) in the interval $[0,T]$ for some $T \leq 1$.

(a) We estimate

$$\|\phi \delta \psi_{\pm}\|_{L^2([0,T),H^{-1+r+s})} \leq c\|\phi \delta \psi_{\pm}\|_{L^2([0,T),L^4)} \leq c\|\phi\|_{L^\infty([0,T),L^4)}\|\psi_{\pm}\|_{L^\infty([0,T),L^4)} \leq c\|\phi\|_{L^\infty([0,T),H^{r+s})}\|\psi_{\pm}\|_{L^\infty([0,T),H^{r+s})} < \infty,$$

where \( \frac{1}{2} = 1 - \frac{s}{2} - \frac{r}{2} + \frac{1}{4}, \frac{1}{2} = -\frac{s}{2} + \frac{s}{2} + \frac{1}{4}. \)

We also have $\psi_{\pm} \in L^2([0,T),H^{-1+r+s})$, because $r < 1$, so that from (4) we get $\psi_{\pm} \in X_{\pm}^{-1+r+s,1}[0,T]$, because

$$\|\psi_{\pm}\|_{X_{\pm}^{1+r+s}[0,T]} \sim \int_0^T \|\psi_{\pm}(t)\|^2_{H^{-1+r+s}} dt + \int_0^T \|(-i\partial_t \pm |D|)\psi_{\pm}(t)\|^2_{H^{-1+r+s}} ds.$$

Interpolation with $\psi_{\pm} \in X_{\pm}^{s,0}[0,T]$ gives $\psi_{\pm} \in X_{\pm}^{1+r+s,1}[0,T]$, where $s_1 = -\frac{1}{2} + \frac{s}{2} + s +$. Remark that $s_1 < 0$ under our assumptions.

(b) In order to show from (5) that $\phi_{\pm} \in X_{\pm}^{-1+r+s,1}[0,T]$ we have to give the following estimates according to Prop. 2.1.

1. \( \|(\beta \Pi_{\pm 1}(D)\psi, \Pi_{\pm 2}\psi')\|_{X_{\pm}^{-1-\frac{1}{4}+}[0,T]} \leq c\|\psi\|_{X_{\pm}^{1-\frac{1}{4}+}[0,T]}\|\psi'\|_{X_{\pm}^{1-\frac{1}{4}+}[0,T]} \)

Here $\pm_1, \pm_2, \pm_3$ denote independent signs. This estimate is proven in [2], Thm. 2 and requires the following conditions: $s_1 > \frac{1}{4} \Leftrightarrow r + 2s > \frac{1}{2}$ and $r_1 < \frac{3}{4} + 2s_1 = -\frac{1}{4} + r + 2s +$. Thus we can choose $r_1 = -\frac{1}{4} + r + 2s +$.

2. \( \|A^{\frac{1}{4}}\phi_{\pm}\|_{X_{\pm}^{-\frac{1}{4}+}[0,T]} \leq \|\phi_{\pm}\|_{L^2([0,T),H^{r-1})} \leq T^{\frac{1}{4}}\|\phi_{\pm}\|_{L^\infty([0,T),H^{r-1})} < \infty \)

3. $\phi_{\pm}(0) \in H^r \subset H^{r_1}$, if $s < \frac{1}{4}$.

Choosing $\psi = \psi_{\pm 1}$ and $\psi' = \psi_{\pm 2}$ in 1. and using 2. and a. we get $\phi_{\pm} \in X_{\pm}^{1-\frac{1}{4}+}[0,T]$.

(c) We have shown that any solution $\psi_{\pm} \in C^0([0,T],H^s)$, $\phi_{\pm} \in C^0([0,T],H^r)$ fulfills $\psi_{\pm} \in X_{\pm}^s[0,T]$, $\phi_{\pm} \in X_{\pm}^r[0,T]$. Now we use the uniqueness part of Theorem 1.2 It requires the following conditions:

$$\max\left(\frac{1}{4} - \frac{s_1}{2} + \frac{s_1}{2}, s_1\right) < r_1 < \min\left(\frac{3}{4} + 2s_1, \frac{3}{4} + \frac{3}{2}s_1, 1 + s_1\right)$$

and $s_1 > -\frac{1}{2}$. An elementary calculation shows that this is equivalent to

$$\frac{3}{5} - 2s < r < 1 - 2s.$$
This gives the claimed result.

**Proof of Theorem 1.1** We combine Theorem 3.1 with the existence part of the local well-posedness result of d’Ancona, Foschi and Selberg (Theorem 2.1). One easily checks that the conditions on $s$ and $r$ reduce to the assumed ranges for these parameters.

**Proof of Theorem 1.2** We use Theorem 3.1 with $s < \frac{1}{8}$, $r = s + \frac{1}{2}$. This requires $\frac{1}{2} - 2s < s + \frac{1}{2} \iff s > \frac{1}{30}$. Combining this with the existence part of the global well-posedness of Grünrock and the author (Theorem 2.2) we get the claimed result.

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