RELATIVE COMPLETED COHOMOLOGIES AND MODULAR SYMBOLS

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ABSTRACT. Generalizing Emerton’s completed cohomologies, we define relative completed cohomologies of arithmetic manifolds. We also define modular symbols for them, and show that the relative completed cohomology spaces interpolate the “nearly ordinary part” of the classical automorphic cohomologies, and the modular symbols defined for them interpolate the classical modular symbols. As applications, we use these modular symbols to construct three families of nearly ordinary p-adic L-functions: (i) Rankin-Selberg p-adic L-functions for $GL_n \times GL_{n-1}$, (ii) Rankin-Selberg p-adic L-functions for $U_n \times U_{n-1}$, and (iii) Standard p-adic L-functions of symplectic type for $GL_{2n}$. We define and calculate explicitly the modifying factors at $\infty$ and at $p$, and determine the exceptional zeros of the $p$-adic L-functions for these examples. The modifying factors at $\infty$ are consistent with the conjectures given by Deligne and Blasius, and the modifying factors at $p$ are consistent with the conjecture given by Coates and Perrin-Riou.

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2020 Mathematics Subject Classification. 11F67, 11F70, 11F33, 22E50.

Key words and phrases. $p$-adic L-function, critical value, completed cohomology, $p$-adic automorphic form, Rankin-Selberg convolution.
1. Introduction

Period integrals of automorphic forms are ubiquitous in the study of complex L-functions. However, due to the lack of $p$-adic Haar measure, there is no obvious way to integrate $p$-adic functions on $p$-adic manifolds. This obstructs the study of $p$-adic L-functions by “$p$-adic period integrals”.

On the other hand, in his seminal work [Em06a], M. Emerton introduces the $p$-adic spaces of completed cohomologies of arithmetic manifolds, which provide $p$-adic interpolations of the classical automorphic cohomologies. One may expect to define “period integrals” on these completed cohomology spaces to get $p$-adic L-functions. Modular symbols, which serve as topological interpretations of period integrals, furnish a fundamental tool for the arithmetic exploration of L-functions. In this paper, generalizing Emerton’s completed cohomologies, we will define relative completed cohomologies and define modular symbols for them. Generalizing the fact that Emerton’s spaces of completed cohomologies interpolate all the classical automorphic cohomologies, we will show that the relative completed cohomology spaces interpolate the “nearly ordinary part” of the classical automorphic cohomology spaces. Moreover, the modular symbols defined for the relative completed cohomologies interpolate the classical modular symbols.

We will give three families of examples to illustrate that our modular symbols produce $p$-adic L-functions that interpolate special values of classical L-functions. We define and calculate explicitly the modifying factors at $\infty$ and at $p$, and determine the exceptional zeros of the $p$-adic L-functions for these examples. The modifying factors at $\infty$ are consistent with the conjectures given by Deligne [Del79] and Blasius [Bl97]. The modifying factors at $p$ are consistent with the conjecture given by Coates and Perrin-Riou [CPR89, Co89].

The construction of relative completed cohomologies and the explicit evaluation of modifying factors at $p$ in this paper have found important arithmetic applications in the recent works [DZ24, Liu23, Liu24, LTX24].

1.1. Some notations. Let us first fix some notation that will be used throughout the paper. Let $p$ be a fixed rational prime. Fix an algebraic closure $\overline{\mathbb{Q}}$ of the field $\mathbb{Q}$ of rational numbers. Let $|\cdot|_p$ denote the multiplicative non-archimedean norm on $\mathbb{C}_p$ that is normalized so that $|p|_p = p^{-1}$. We fix field embeddings

$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p \quad \text{and} \quad \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$$

so that $\overline{\mathbb{Q}}$ is viewed as a subfield of both $\mathbb{C}_p$ and $\mathbb{C}$. Let $E \subset \overline{\mathbb{Q}}$ be a subfield, and $E \subset \mathbb{C}_p$ a closed subfield containing $E$. Write $\mathbb{A}$ for the adele ring of $\mathbb{Q}$. It has the usual decompositions

$$\mathbb{A} = \mathbb{R} \times \mathbb{A}^\infty = \mathbb{R} \times \mathbb{Q}_p \times \mathbb{A}^\infty_p.$$
For every number field $k$, write $k\oplus Q\mathbb{R}$, and for every ring $R$ with identity, write $R^\times$ for the group of its invertible elements. A superscript $\forall$ over a finite-dimensional vector space indicates the dual space.

Given a locally compact Hausdorff topological group $G$, we write $G^\circ$ for the connected component of its identity element, and write $G^\natural := G/G^\circ$. For every locally compact Hausdorff homogeneous space $X$ of $G$, we let $M(X)$ denote the space of all $G$-invariant complex Borel measures on $X$. If $X$ is furthermore totally disconnected, let $D(X) \subset M(X)$ denote the subspace of the measures with respect to which all the open compact subsets have rational volumes. Specifically viewing $G$ as a homogeneous space under the right translation, we have the space $M(G)$, and the space $D(G)$ in the case when $G$ is totally disconnected. Push-forward of measures through the conjugation action yields a representation of $G$ on $M(G)$.

When the homogeneous space $X$ is a topological manifold, we define the $Q$-vector space $O(X) := \{G$-invariant sections of $\mathcal{O}_X\}$, where $\mathcal{O}_X$ denotes the orientation local system on $X$ with coefficients $Q$.

When $G$ is totally disconnected, for every smooth representation $J$ of $G$ over a field of characteristic zero, write $\hat{J} := \varprojlim K J$ for its formal completion, where $K$ runs over all open compact subgroups of $G$, and the implicit homomorphisms defining the inverse limit are the averaging projections. See [Be87, Section 6.3] for more details. This is a (non-necessarily smooth) representation of $G$ which contains $J$ as a subrepresentation.

Let
\begin{equation}
(G, \hat{G}, Z, \iota : \hat{G} \to G, j : \hat{G} \to Z)
\end{equation}
be a 5-tuple where $G$ and $\hat{G}$ are linear algebraic groups over $Q$, $Z$ is an algebraic torus over $Q$, and $\iota$ and $j$ are algebraic homomorphisms over $Q$. Set
\begin{align*}
G := G(Q_p), & & \hat{G} := \hat{G}(Q_p), & & Z := Z(Q_p),
\end{align*}
and write $g$, $\hat{g}$, $z$ respectively for their Lie algebras.

Let $\mathfrak{p}$ be a Lie subalgebra of $\mathfrak{g}$ that is transversal to $\hat{\mathfrak{g}}$ in the sense that
\begin{equation}
\overline{\mathfrak{p}} + \iota(\hat{\mathfrak{g}}) = \mathfrak{g}.
\end{equation}
Here and henceforth, we still use $\iota$ (or $j$) to denote various maps induced by the homomorphism $\iota : \hat{G} \to G$ (or $j : \hat{G} \to Z$). Write
\begin{align*}
\hat{\mathfrak{p}} := \iota^{-1}(\mathfrak{p}) \subset \hat{\mathfrak{g}} & & \text{so that} & & \mathfrak{g}/\mathfrak{p} = \hat{\mathfrak{g}}/\hat{\mathfrak{p}}.
\end{align*}
When no confusion is possible, we will not distinguish a linear algebraic group defined over $Q_p$ with the $p$-adic Lie group of its $Q_p$-points. Let $s \subset \mathfrak{p}$ be a Lie
subalgebra and $Z_0 \subset Z$ an algebraic subtorus such that

$$i(\hat{p}) \subset \mathfrak{s} \quad \text{and} \quad j(\hat{p}) \subset \mathfrak{z}_0,$$

where $\mathfrak{z}_0$ denotes the Lie algebra of $Z_0$.

(The group $Z_0$ is trivial in the three families of examples of this paper.)

Throughout the paper, $\ell = \infty$ or a rational prime. Write

$$D_{Z_0}^{\text{max}} := \prod_{\ell \neq \infty} D_{\ell}^{\text{max}},$$

where $D_{p}^{\text{max}}$ is the maximal compact subgroup of $Z_0$, and $D_{\ell}^{\text{max}}$ is the maximal compact subgroup of $Z(\mathbb{Q}_{\ell})$ for $\ell \neq \infty, p$. Fix a locally constant character

$$(1.3) \quad \varepsilon = \otimes_{\ell \neq \infty} \varepsilon_{\ell} : D_{Z_0}^{\text{max}} \to \mathbb{Q}^\times,$$

to be called the ramification type. We introduce the following sets attached to $Z$, $Z_0$ and $\varepsilon$:

- $\mathcal{X}^{\text{alg}}$ denotes the group of all algebraic characters $w : Z_{\mathbb{Q}} \to \text{GL}(1)/\mathbb{Q}$;
- $\mathcal{X}^{\text{aut}}$ denotes the group of all automorphic characters $\chi : Z(\mathbb{Q}) \backslash Z(\mathbb{A}) \to \mathbb{C}^\times$;
- $\mathcal{X}(\varepsilon)$ denotes the set of all characters $\chi \in \mathcal{X}^{\text{aut}}$ whose restriction to $D_{Z_0}^{\text{max}}$ equals $\varepsilon$;
- $\mathcal{X}_{\ell}$ denotes the group of all continuous characters $\chi_{\ell} : Z(\mathbb{Q}_{\ell}) \to \mathbb{C}^\times$ ($\mathbb{Q}_{\infty} := \mathbb{R}$);
- for $\ell \neq \infty$, $\mathcal{X}(\varepsilon_{\ell})$ denotes the set of all characters $\chi_{\ell} \in \mathcal{X}_{\ell}$ whose restriction to $D_{\ell}^{\text{max}}$ equals $\varepsilon_{\ell}$.

Here $Z_{\mathbb{Q}} := Z \times_{\text{Spec}(\mathbb{Q})} \text{Spec}(\overline{\mathbb{Q}})$, and similar notation for base change will be used without further explanation. Note that for each $\ell \neq \infty$, $\mathcal{X}(\varepsilon_{\ell})$ is a scheme defined over $\mathbb{Q}(\varepsilon_{\ell})$. Here $\mathbb{Q}(\varepsilon_{\ell})$ is the field extension of $\mathbb{Q}$ generated by the values of $\varepsilon_{\ell}$, and similar notations will be used without further explanation.

Let $C(Z, E)_{\text{sm}}$ denote the space of $E$-valued continuous functions on $Z(\mathbb{Q}) \backslash Z(\mathbb{A})$ that are invariant under the translations of some open subgroups of $D_{Z_0}^{\text{max}}$. This is naturally a locally convex topological vector space under the direct limit topology. Moreover, as a locally convex topological vector space over $\mathbb{C}_p$, $C(Z, \mathbb{C}_p)_{\text{sm}} = \bigoplus_\varepsilon C(Z, \mathbb{C}_p)(\varepsilon)$.

Here $\varepsilon$ runs over all characters as in (1.3), and whenever $\mathbb{Q}(\varepsilon) \subset E$, $C(Z, E)(\varepsilon)$ denotes the space of all continuous functions $f : Z(\mathbb{Q}) \backslash Z(\mathbb{A}) \to E$ such that $f(xg) = \varepsilon(g) \cdot f(x)$ for all $x \in Z(\mathbb{A})$, $g \in D_{Z_0}^{\text{max}}$, which is a Banach space under the supremum norm.

For every $w \in \mathcal{X}^{\text{alg}}$ and $\ell = \infty$ or $p$, define a character

$$(1.4) \quad w_\ell : Z(\mathbb{Q}_\ell) \subset Z(\mathbb{C}_\ell) \twoheadrightarrow \mathbb{C}_\ell^\times \quad (\mathbb{C}_\infty := \mathbb{C}).$$
When no confusion is possible, we still use \( w \) to denote the composition of
\[ Z(A) \xrightarrow{\text{projection}} Z(Q_\ell) \twoheadrightarrow \mathbb{C}_\ell^\times. \]

**Definition 1.1.** (a) A character \( \chi_\infty \in \mathcal{X}_\infty \) is said to be algebraic if there is an element \( w \in \mathcal{X}^{\text{alg}} \) such that the character \( w \cdot \chi_\infty \) is locally constant. The algebraic character \( w^{-1} \) is called the weight of \( \chi_\infty \).

(b) A character \( \chi \in \mathcal{X}^{\text{aut}} \) is said to be algebraic if so is its archimedean component \( \chi_\infty \). When this is the case the weight of \( \chi_\infty \) is also called the weight of \( \chi \).

For every automorphic character \( \chi \in \mathcal{X}^{\text{aut}} \) that is algebraic of weight \( w^{-1} \), define a \( p \)-adic automorphic character
\[ \chi^b := (w_\infty \cdot \chi) \cdot w_p^{-1} : Z(Q) \setminus Z(A) \to \mathbb{C}_p^\times. \]

### 1.2. The nearly ordinary part.

Throughout the Introduction we assume that \( G \) is connected and reductive, \( \iota \) is injective so that \( \dot{G} \) is viewed as an algebraic subgroup of \( G \), and \( \dot{G} \cap A \) is zero-dimensional. Here \( A \) denotes the largest central split torus in \( G \). Fix a closed subgroup \( K_\infty \) of \( G(R) \) of the form
\[ K_\infty = A(R)^0 \cdot K'_\infty, \]
where \( K'_\infty \) is a maximal compact subgroup of \( G(R) \). Put
\[ G^0 := K^0_\infty \times G(A^\infty) \]
and
\[ \mathcal{X} := \mathcal{X}_{G,K_\infty} := G(A)/K^\circ_\infty = (G(R)/K^\circ_\infty) \times G(A^\infty). \]

For every open compact subgroup \( K \) of \( G^0 \), define the topological space
\[ S^0_K := G(Q) \setminus \mathcal{X}/K. \]

Let \( V \) be a finite-dimensional \( E \)-vector space carrying a geometrically irreducible algebraic representation of \( G_E \). Here “geometrically irreducible” means that the representation of \( G(Q) \) on \( Q \otimes E V \) is irreducible. Specifically, \( V \) is a representation of \( G(Q) \subset G_E(E) \). Define
\[ H^i_{\Phi}(G,V) := \lim_{\longrightarrow K} H^i_{\Phi}(S^0_K,V_{|K}), \quad (i \in \mathbb{Z}), \]
where \( K \) runs over all open compact subgroups of \( G^0 \), and \( V_{|K} \) is a sheaf of \( E \)-vector spaces over \( S^0_K \) as defined in (1.2). Here the subscript “\( \Phi \)” indicates a support condition for the cohomology group (see Section 3.2). Under the right translations, \( H^i_{\Phi}(G,V) \) is a smooth representation of \( G^0 \) over \( E \).

Set
\[ G^{0,p} := K^0_\infty \times G(A^{\infty p}) \quad \text{so that} \quad G^0 = G^{0,p} \times G. \]

In this Introduction and in the next section we assume that \( p \) is a parabolic subalgebra of \( \mathfrak{g} \). Let \( P \subset G \) denote the normalizer of \( p \), which is a parabolic subgroup of \( G \) whose Lie algebra equals \( p \). Denote by \( N \) the unipotent radical of \( P \) and put \( L := P/N \). The Lie algebras of \( N \) and \( L \) are respectively denoted by \( \mathfrak{n} \) and \( \mathfrak{l} \).
For every representation $J'$ of $\mathcal{G} \times G$, where $\mathcal{G}$ is a certain totally disconnected topological group, write

$$J'_{\text{p-sm}}$$

for the space of all vectors in $J'$ that are fixed by some groups of the form $K \times P$, where $K$ is an open subgroup of $G$ and $P$ is a closed subgroup of $G$ with Lie algebra $\mathfrak{p}$. Similar notation with $\mathfrak{p}$ replaced by other Lie subalgebras will be used without further explanation. For every smooth representation $J$ of $G^\natural$, define

$$B_P(J) := \left( \left( \hat{J} \right)_{\text{p-sm}} \right)^N,$$

which is a smooth representation of $G^\natural \times L$. Here and henceforth, unless otherwise specified, we use a superscript group or Lie algebra to indicate the invariant space. By the second adjointness theorem of Bernstein and Casselman (see [Be87, Theorems 6.4 and 0.1]), $B_P(J)$ is isomorphic to the Jacquet module of $J$ attached to a parabolic subgroup of $G$ opposite to $P$.

Denote by $T$ the largest central torus in $L$, and by $T^\dagger$ the monid consisting of all $t \in T$ such that $|\alpha(t)|_p \geq 1$ for every character $\alpha : T \to \mathbb{C}_p^\times$ that occurs in some irreducible subquotient of the adjoint representation of $P$ on $\mathbb{C}_p \otimes n$ (these irreducible subquotients descend to representations of $L$). Here and henceforth, we often omit the obvious base ring from the notation of a tensor product and hope this causes no confusion. For example, $\mathbb{C}_p \otimes n := \mathbb{C}_p \otimes \mathbb{Q}_p n$.

Set $V := E \otimes V$, which is naturally a representation of $G \subset G_E(E)$. Then $T$ acts on $V^n$ through a character, to be denoted by $\alpha_V : T \to E^\times$.

Now we assume that $\Phi$ satisfies the usual condition (6.17) so that the smooth representation $H^i_{\mathfrak{p}}(G, V)$ (and hence $B_P(H^i_{\mathfrak{p}}(G, V))$) is admissible. The following result is essentially due to Hida (cf. [Hi93, Hi95]), and we will give a proof in Proposition 7.11 in a more general setting.

**Proposition 1.2.** For every character $\chi \in \text{Hom}(T, \mathbb{C}_p^\times)$ that occurs as a subquotient in

$$\mathbb{C}_p \otimes B_P(H^i_{\mathfrak{p}}(G, V)) \otimes D(\mathfrak{g}/\mathfrak{p}),$$

one has that

$$|\chi(t)|_p \leq |\alpha_V(t)|_p \quad \text{for all } t \in T^\dagger.$$

**Definition 1.3.** Let $\chi \in \text{Hom}(T, \mathbb{C}_p^\times)$ be a character satisfying (1.7). It is said to be nearly $V$-ordinary if the equality holds in (1.7) for all $t \in T^\dagger$ (and hence for all $t \in T$).

Let $\mathcal{H}$ be a $G^\natural \times L$-subrepresentation of $B_P(H^i_{\mathfrak{p}}(G, V)) \otimes D(\mathfrak{g}/\mathfrak{p})$ such that all characters of $T$ that occur in $\mathbb{C}_p \otimes \mathcal{H}$ as subquotients are nearly $V$-ordinary. This
is the nearly ordinary part of the classical automorphic cohomology space that has been mentioned before.

1.3. Relative cohomologies and relative completed cohomologies. In what follows we briefly review the notions of relative cohomologies and relative completed cohomologies. See Section 3 for more details. Let $R$ be a commutative ring with identity, and $G$ an open subgroup of $G^\natural$. For each monoid $H^+$, write $R[H^+]$ for the monoid algebra of $H^+$ with coefficients in $R$. As an $R$-module, it is free with basis $H^+$.

For every $R[G(Q) \times G]$-module $M$ and every compact subgroup $C$ of $G$, we define the formal cohomology group

$$H^i_\Phi(C, M) := \lim_{\leftarrow} H^i_\Phi(K, M),$$

where $K$ runs over open compact subgroups of $G$ containing $C$, the transition maps are the push-forward maps, and $H^i_\Phi(K, M)$ is the usual sheaf cohomology group that will be defined in (3.3). Define the relative cohomology group

$$H^i_\Phi(G, M) := \lim_{\leftarrow} H^i_\Phi(D\mathfrak{p}, M),$$

where $D$ runs over open compact subgroups of $G^\natural \cap G$, $\mathfrak{p}$ runs over compact subgroups of $G \cap G$ with Lie algebra $\mathfrak{p}$, and the transition maps are the pull-back maps.

Write $\mathbb{N} := \{0, 1, 2, \ldots\}$ as usual. Define the relative completed cohomology group

($1.8$) $$\tilde{H}^i_\Phi(G, M)(p^k) := \lim_{\leftarrow} \lim_{D\mathfrak{p} \in S} \lim_{k \in \mathbb{N}} H^i_\Phi(D\mathfrak{p}, M/p^k) \quad (M/p^k := M/p^k M),$$

where $D$ runs over open compact subgroups of $G^\natural \cap G$, $\mathfrak{p}$ runs over compact subgroups of $G \cap G$ with Lie algebra $\mathfrak{p}$, and $\mathfrak{s}$ runs over compact subgroups of $G \cap G$ with Lie algebra $\mathfrak{s}$.

**Definition 1.4.** Let $\mathfrak{G}$ be a topological group. An $R[\mathfrak{G}]$-module $M_0$ is said to be $p$-smooth if for every $k \in \mathbb{N}$, some open subgroups of $\mathfrak{G}$ act trivially on $M_0/p^k$.

It is crucial to note that, by using the trivial actions of small open compact subgroups of $G$, the relative completed cohomology group (1.8) is still defined when $M$ is replaced by an $R[G(Q) \times D\mathfrak{G}]$-module that is $p$-smooth as an $R[\mathfrak{G}]$-module, where $D$ is an arbitrary open compact subgroup of $G^\natural \cap G$ and $\mathfrak{G}$ is an arbitrary compact subgroup of $G$ with Lie algebra $\mathfrak{s}$.

View $V$ as an $E[G(Q) \times G]$-module with the given action of $G$ and the trivial action of $G(Q) \times G^\natural$. Write

$$\mathcal{O} := \{x \in E : |x|_p \leq 1\}$$
for the ring of integers of $E$. We define the integral relative cohomology space

$$H_\phi^i(G, V)^{(p, \circ)} := E \otimes H_\phi^i(G, \mathfrak{V})^{(p)},$$

where $\mathfrak{V}$ is an $\mathfrak{O}$-lattice of $V$ (namely a free $\mathfrak{O}$-submodule whose rank equals $\dim V$). This is independent of $\mathfrak{V}$. Similarly, for each $(E \otimes \mathfrak{s})$-submodule $V_0$ of $V$ (which is necessarily stabilized by some compact subgroups of $G$ with Lie algebra $\mathfrak{s}$), we define the integral relative completed cohomology space

$$\widetilde{H}_\phi^i(G, V_0)^{(p \supset \circ), \circ} := E \otimes \widetilde{H}_\phi^i(G, \mathfrak{V}_0)^{(p \supset \circ)},$$

where $\mathfrak{V}_0$ is an $\mathfrak{O}$-lattice of $V_0$. This is independent of $\mathfrak{V}_0$.

We will show in Section 7 that the integral relative completed cohomology space interpolates the nearly ordinary part $\mathcal{H}$. More precisely, we have a commutative diagram

$$
\begin{array}{ccc}
\widetilde{H}_\phi^i(G, V^n)^{(p \supset \circ), \circ} & \xleftarrow{\xi} & \mathcal{H} \\
\downarrow & & \downarrow \\
\widetilde{H}_\phi^i(G, V)^{(p \supset \circ), \circ} & \xleftarrow{H} & H_\phi^i(G, V)^{(p), \circ} \longrightarrow \widetilde{H}_\phi^i(G, V)^{(p)},
\end{array}
$$

(1.9)

where all the arrows are canonically defined. If $P = G$, then $n = \{0\}$ and

$$H_\phi^i(G, V)_p \otimes \mathfrak{D}(\mathfrak{g}/\mathfrak{p}) = H_\phi^i(G, V),$$

and if furthermore $\mathfrak{s} = \{0\}$, then

$$\widetilde{H}_\phi^i(G, V^n)^{(p \supset \circ), \circ} = \widetilde{H}_\phi^i(G, E)^{(g \supset \{0\}), \circ} \otimes V,$$

which agrees with Emerton’s completed cohomology space, and the map $\widetilde{\xi}$ agrees with his interpolation map.

1.4. Modular symbols. We now review some versions of modular symbols that are considered in this paper. See Section 3 for details.

Set $K_{\infty} := K_{\infty} \cap G(\mathbb{R})$. By [Sun13 Proposition 8.1], it equals $K'_{\infty} \cap G(\mathbb{R})$ and is thus compact. With the fixed closed group $K_{\infty}$, we define various cohomology groups for $G$, as we have defined for $G$ in Section 1.3. Similarly these cohomology groups are also defined for $Z$, with the fixed closed subgroup of $Z(\mathbb{R})$ taken to be the group

$$K_{Z, \infty} := A_Z(\mathbb{R}) \cdot (\text{the maximal compact subgroup of } Z(\mathbb{R})),
$$

(1.10)

where $A_Z$ is the maximal split torus in $Z$. In particular, we have an identification

$$\widetilde{H}^0(Z, E)^{(g \supset \mathfrak{g}_0), \circ} = C(Z, E)_{\mathfrak{g}_0, \text{sm}}.$$

For every algebraic character $w \in \mathcal{X}^{\text{alg}}$ that is defined over a subfield $E'$ of $\overline{\mathbb{Q}}$, write $E_{w} := E'$ for the corresponding one-dimensional algebraic representation of
Z_{E'}). Suppose that \( w \in X_{\text{alg}} \) is defined over \( E \). Then the cohomology space \( H^0(Z, E_w) \) equals the space of all locally constant functions \( f : Z(\mathbb{A}) \to E \) such that
\[
f(gx) = w(g) \cdot f(x) \quad \text{for all } g \in Z(\mathbb{Q}), \; x \in Z(\mathbb{A}).
\]
Fix a functional \( \lambda_V \in \text{Hom}_{\mathbb{C}_k}(E_w \otimes V, E) \).

Similar to (1.5), write
\[
\hat{G}^\dagger := \hat{K}^\dagger_{\infty} \times \hat{G}(\mathbb{A}^\infty).
\]
Put
\[
D(\hat{G}) := D(\hat{G}^\dagger) \otimes \mathcal{O}(\hat{G}(\mathbb{R})/\hat{K}^\dagger_{\infty}),
\]
which is a one-dimensional \( \mathbb{Q} \)-vector space. Set
\[
i_0 := \dim(\hat{G}(\mathbb{R})/\hat{K}^\dagger_{\infty}).
\]
Under a natural condition (5.1) on \( \Phi \), the classical modular symbol map is defined to be the composition of
\[
H^0(Z, E_w) \times \left( H^0_{\text{sm}}(G, V) \otimes D(\hat{G}) \right) \xrightarrow{\lambda_V} H_{\text{c}}^0(\hat{G}, E_w \otimes V) \otimes D(\hat{G}) \xrightarrow{\lambda_V} E.
\]
Here the subscript “c” indicates the cohomology with compact support.

Write
\[
D(\hat{G}) = D(\hat{G}, \hat{p}) \otimes D(\mathfrak{g}/\mathfrak{p}),
\]
where
\[
D(\hat{G}, \hat{p}) := D(\hat{G}^{\mathfrak{p}}) \otimes D(\hat{p}) \otimes \mathcal{O}(\hat{G}(\mathbb{R})/\hat{K}^\dagger_{\infty}),
\]
and
\[
\hat{G}^{\mathfrak{p}} := \hat{K}^\dagger_{\infty} \times \hat{G}(\mathbb{A}^{\mathfrak{p}}).
\]
Generalizing the classical modular symbol map, in Section 5 we will define modular symbols for all the spaces occurring in the diagram (1.9). In particular, we have the stable modular symbol map
\[
\mathcal{M}_{\lambda_V} : H^0(Z, E_w) \times \left( H^0_{\mathfrak{p}}(G, V)^p_{\text{sm}} \otimes D(\hat{G}) \right) \to E
\]
which extends the classical modular symbol map (1.12). Recall that \( \mathfrak{s} \supset \iota(\hat{p}) \).

Then we also have the relative completed modular symbol map
\[
\tilde{\mathcal{M}}_{\lambda_0} : \tilde{H}^0(Z, E)_{(\mathfrak{s} \supset \mathfrak{p})_{\text{sm}}} \times \left( \tilde{H}^0_{\mathfrak{p}}(G, V^{\mathfrak{n}})^{\mathfrak{p}_{\text{sm}}} \otimes D(\hat{G}, \hat{p}) \right) \to E,
\]
where $\lambda_0 \in \text{Hom}_{E \otimes \mathbb{F}_p}(V^n, E)$. Both factors in the domain of the bilinear map (1.15) are naturally locally convex topological vector spaces over $E$ under the direct limit topologies. An important fact is that the bilinear map (1.15) is separately continuous. Assume that

$$\text{(1.16)} \quad w_p \text{ is trivial on } Z_0.$$  

Then we have an embedding

$$\text{(1.17)} \quad H^0(L, E_w) \rightarrow \tilde{H}^0(Z, E)^{(\mathfrak{g} \supset \mathfrak{z}_0)^{\circ}} = C(Z, E)_{\mathfrak{z}_0 - \text{sm}}, \quad f \mapsto f \cdot w_p^{-1}.$$  

We have natural inclusions

$$\text{(1.18)} \quad \text{Hom}_{E_{\mathfrak{G}}}(E_w \otimes V, E) \subset \text{Hom}_G(E_w \otimes V, E) \subset \text{Hom}_{E \otimes \mathbb{F}_p}(V, E).$$  

In particular, $\lambda_V$ is also viewed as an element of $\text{Hom}_{E \otimes \mathbb{F}_p}(V, E)$. The following theorem asserts that the relative completed modular symbols interpolate the stable modular symbols on the nearly ordinary part $\mathcal{H}$, for various weights $w$ satisfying (1.16).

**Theorem 1.5** (Theorem 5.13). Suppose that $w \in \mathcal{X}^{\text{alg}}$ is defined over $E$, $w_p$ is trivial on $Z_0$, and $\lambda_V|_{V^u} = \lambda_0$. Then the diagram

$$\text{(1.19)} \quad \begin{array}{ccc}
H^0(L, E_w) \times \left( \mathcal{H} \otimes D(\mathcal{G}, \mathfrak{p}) \right) & \overset{\lambda_0 \otimes \mathfrak{n}}{\longrightarrow} & E \\
\downarrow \mathfrak{m} & & \\
C(Z, E)_{\mathfrak{z}_0 - \text{sm}} \times \left( \tilde{H}^0_{\mathfrak{G}}(\mathcal{G}, V^n)^{(\mathfrak{g} \supset \mathfrak{z}_0)^{\circ}} \otimes D(\mathcal{G}, \mathfrak{p}) \right) & \overset{\text{commutes.}}{\longrightarrow} & E
\end{array}$$

For every $\phi \in \mathcal{H} \otimes D(\mathcal{G}, \mathfrak{p})$, $\mathcal{L}_{\lambda_0}(\cdot, \tilde{\xi}(\phi))$ is a continuous linear functional on $\tilde{H}^0(Z, E)^{(\mathfrak{g} \supset \mathfrak{z}_0)^{\circ}} = C(Z, E)_{\mathfrak{z}_0 - \text{sm}}$, to be denoted by $\mathcal{L}_{\lambda_0 \otimes \mathfrak{p}}$. When $\phi$ is appropriately chosen, this often yields interesting $p$-adic L-functions.

**Remark 1.6.** Suppose that $\mathfrak{s} \supset \mathfrak{n}$ and set $\mathfrak{h} := \mathfrak{s}/\mathfrak{n}$. Let $L_{\mathfrak{h}}$ denote the normalizer of $\mathfrak{s}/\mathfrak{n}$ in $L$. Then the space

$$\text{(1.20)} \quad E \otimes \lim_{\mathcal{H}} \lim_{D \mathfrak{h}} \lim_{k \in \mathbb{N}} H^0_{\mathfrak{h}, P}(D \mathfrak{L}, \mathcal{O}/p^k)$$

is naturally a representation of $G^{\mathfrak{p}} \times L_{\mathfrak{h}}$ over $E$, where $D$ runs over all open compact subgroups of $G^{\mathfrak{p}}$, $\mathfrak{L}$ runs over all compact subgroups of $L$ with Lie algebra $\mathfrak{h}$, and $\mathfrak{L}$ runs over all open compact subgroups of $L$ containing $\mathfrak{h}$, and $H^0_{\mathfrak{h}, P}$ indicates the parabolic cohomology that will be introduced in Definition 6.13. When $G$ is quasi-split and $P$ is a Borel subgroup of $G$, we may use the representation (1.20) to construct the ordinary part of the eigenvariety of $G$. Theorem 1.5 can be applied to this ordinary part to construct multivariable $p$-adic L-functions for Hida.
families. To keep this paper to a reasonable length, we will leave the details of this construction to a future work.

This paper is organized as follows. In Section 2 we outline the general formalism for the rationality of complex L-functions and construction of p-adic L-functions using relative completed cohomologies and modular symbols, and summarize the main results for three families of examples. In Section 3 we develop the basic theory of relative cohomologies and relative completed cohomologies, and in Section 4 we discuss the pull-back and integration of these cohomologies. In Section 5 we compare various modular symbols and prove Theorem 1.5. Sections 6 and 7 are devoted to the theory of parabolic cohomologies and the nearly ordinary part \( H \) of the automorphic cohomology. In Sections 8–10 we apply the main theory of this paper to construct the three families of nearly ordinary p-adic L-functions in Section 2 in details, with explicit modifying factors and exceptional zeros and without any ramification restriction.

2. Complex L-functions and p-adic L-functions

As examples of applications of Theorem 1.5 we will consider three families of automorphic L-functions. In this section, we first develop the general formalism, and then outline the main results for these examples. See Section 2.7 for historical remarks and comparisons with previous results in the literature. The detailed construction will be given in Sections 8–10.

Moreover, for each of the three families, we supply the concrete example of exceptional zeros of p-adic L-functions associated to symmetric powers of elliptic curves over \( \mathbb{Q} \) without complex multiplication.

2.1. Period integrals and complex L-functions. Let \( H \) be an algebraic subgroup of \( G \), together with an automorphic character

\[
\psi_H = \otimes \ell \psi_{H, \ell} : H(\mathbb{Q}) \backslash H(\mathbb{A}) \rightarrow \mathbb{C}^\times.
\]

Set \( \hat{H} := H \cap \hat{G} \). We assume that

\[
\psi_H \text{ is trivial on } \hat{H}(\mathbb{A});
\]

\[
\hat{H} \text{ is contained in the kernel of } j : \hat{G} \rightarrow \mathbb{Z};
\]

\[
M(\hat{H}(\mathbb{A}) \backslash \hat{G}(\mathbb{A})) \neq \{0\}.
\]

Let \( \Pi = \otimes' \Pi_\ell \) be an irreducible subrepresentation of the space of smooth automorphic forms on \( G(\mathbb{Q}) \backslash G(\mathbb{A}) \). For the definition of the restricted tensor product, we realize \( G \) as an algebraic subgroup of a general linear group \( \text{GL}(k)/\mathbb{Q} \) (\( k \in \mathbb{N} \)), and set

\[
H'(\mathbb{Z}_\ell) := \text{GL}_k(\mathbb{Z}_\ell) \cap H'(\mathbb{Q}_\ell)
\]

for every algebraic subgroup \( H' \) of \( G \) and every \( \ell \neq \infty \). For all but finitely many \( \ell \neq \infty \), we fix a nonzero vector \( v^0_\ell \in \Pi_\ell^{G(\mathbb{Z}_\ell)} \), and the restricted tensor products are defined with respect to the family \( \{v^0_\ell\}_\ell \).
Assume that
\[
\dim \text{Hom}_{\mathcal{H}(A)}(\Pi, \psi_H) = 1,
\]
and the integrals
\[
\lambda_H : \Pi \to \mathbb{C}, \quad f \mapsto \int_{\mathcal{H}(Q) \setminus \mathcal{H}(A)} f(x) \psi_H^{-1}(x) \, dx
\]
are absolutely convergent and yield a generator of the one-dimensional space in (2.2). Here \(dx\) is the right invariant Tamagawa measure. Then we have a decomposition
\[
\lambda_H = \bigotimes_\ell \lambda_{H, \ell}, \quad \lambda_{H, \ell} \in \text{Hom}_{\mathcal{H}(Q_\ell)}(\Pi_\ell, \psi_{H, \ell}),
\]
such that \(\lambda_H(v_\ell) = 1\) for all but finitely many \(\ell \neq \infty\).

Fix an \(E^\times\)-valued character \(\varepsilon = \bigotimes_{\ell \neq \infty} \varepsilon_\ell\) as in (1.3). In this subsection and the next one we assume that \(Z_0 = Z\). Suppose that meromorphic continuations and normalizations of the integrals
\[
\int_{\mathcal{H}(Q) \setminus \mathcal{H}(A)} \chi_\ell(j(g)) \cdot (\lambda_{H, \ell} \circ g, f) \, d\mu(g), \quad \chi_\ell \in \mathcal{X}_\ell, \quad f \in \Pi_\ell, \quad \mu \in \text{M}(\mathcal{H}(Q_\ell) \setminus \mathcal{G}(Q_\ell))
\]
yield the “normalized zeta integrals”
\[
\mathcal{P}_\ell^\circ : \mathcal{X}_\ell \times \left( \Pi_\ell \otimes \text{M}(\mathcal{H}(Q_\ell) \setminus \mathcal{G}(Q_\ell)) \right) \to \mathbb{C}
\]
with the following properties:

- if \(\ell = \infty\), then \(\mathcal{P}_\ell^\circ\) is holomorphic in the first variable, linear in the second variable, and continuous;
- if \(\ell \neq \infty\), then \(\mathcal{P}_\ell^\circ\) is algebraic in the first variable and linear in the second variable;
- it holds that
\[
\mathcal{P}_\ell^\circ(\chi_\ell, g, \phi) = \chi_\ell^{-1}(j(g)) \cdot \mathcal{P}_\ell^\circ(\chi_\ell, \phi),
\]
for all \(\chi_\ell \in \mathcal{X}_\ell, \phi \in \Pi_\ell \otimes \text{M}(\mathcal{H}(Q_\ell) \setminus \mathcal{G}(Q_\ell))\), and \(g \in \mathcal{G}(Q_\ell)\);
- there is a family
\[
\{\phi_\ell^\circ \in \Pi_\ell \otimes \text{M}(\mathcal{H}(Q_\ell) \setminus \mathcal{G}(Q_\ell))\}_{\ell \neq \infty}
\]
such that
\[
\mathcal{P}_\ell^\circ(\cdot, \phi_\ell^\circ) \text{ takes a nonzero constant value } (\Omega_{\Pi_\ell}(\varepsilon_\ell))^{-1} \text{ on } \mathcal{X}(\varepsilon_\ell),
\]
where for all but finitely many \(\ell \neq \infty\),
\[
\Omega_{\Pi_\ell}(\varepsilon_\ell) = 1 \quad \text{and} \quad \phi_\ell^\circ = v_\ell^\circ \otimes \mu_\ell^\circ.
\]
Here \(\mu_\ell^\circ \in \text{M}(\mathcal{H}(Q_\ell) \setminus \mathcal{G}(Q_\ell))\) denotes the measure with respect to which \(\mathcal{H}(\mathbb{Z}_\ell) \setminus \mathcal{G}(\mathbb{Z}_\ell)\) has total volume 1.
Let $\chi = \otimes \ell \chi_\ell \in \mathcal{X}(\varepsilon)$. By taking the tensor product we get a homomorphism
\begin{equation}
\mathcal{P}_\chi^\circ := \otimes \mathcal{P}_\ell^\circ(\chi_\ell, \cdot) \in \text{Hom}_{\hat{G}(\mathbb{A})}(\Pi \otimes M(\hat{H}(\mathbb{A}) \backslash \hat{G}(\mathbb{A})), \chi^{-1}).
\end{equation}
Here and henceforth, when no confusion is possible, we do not distinguish a complex character of a group with the vector space $\mathbb{C}$ carrying the corresponding group action. On the other hand, under certain growth assumptions, we get the global period integral map
\[ \mathcal{P}_\chi : \Pi \otimes M(\hat{G}(\mathbb{Q}) \backslash \hat{G}(\mathbb{A})) \to \chi^{-1}, \quad f \otimes \mu \mapsto \int_{\hat{G}(\mathbb{Q}) \backslash \hat{G}(\mathbb{A})} \chi(f(g)) \cdot f(g) \, d\mu(g), \]
which belongs to the hom space in (2.6). By fixing the right invariant Tamagawa measure on $\hat{H}(\mathbb{A})$, and the counting measure on $\hat{G}(\mathbb{Q})$ we have identifications
\begin{equation}
M(\hat{H}(\mathbb{A}) \backslash \hat{G}(\mathbb{A})) = M(\hat{G}(\mathbb{A})) = M(\hat{G}(\mathbb{Q}) \backslash \hat{G}(\mathbb{A})).
\end{equation}
Suppose that certain theory of “unfolding the global period integrals” yields a constant $\mathcal{L}_\Pi(\chi) \in \mathbb{C}$ such that
\[ \mathcal{P}_\chi = \mathcal{L}_\Pi(\chi) \cdot \mathcal{P}_\chi^\circ. \]
The function $\chi \mapsto \mathcal{L}_\Pi(\chi)$ on $\mathcal{X}(\varepsilon)$ is the complex $L$-function that we are interested in.

2.2. Rationality of special values. Now we consider the case when $\Pi$ is cohomological and its basic arithmetic properties have been understood. More precisely, we assume that
\begin{itemize}
  \item the total relative Lie algebra cohomology
    \[ H^\bullet(\mathfrak{g}_C, K^\circ_{\infty}; \mathbb{V} \otimes \Pi_\infty) \quad (\mathfrak{g}_C \text{ denotes the Lie algebra of } G(\mathbb{C})) \]
    is nonzero (in this case we say that $\mathbb{C} \otimes \mathbb{V}'$ is the coefficient system of $\Pi$);
  \item there is a natural injective $\mathbb{G}^\circ$-homomorphism
    \begin{equation}
    H^\bullet(\mathfrak{g}_C, K^\circ_{\infty}; \mathbb{V} \otimes \Pi_\infty) \cong (\otimes_{\ell \neq \infty} \Pi_\ell) \to \mathbb{C} \otimes H^\bullet(\mathbb{G}, \mathbb{V})
    \end{equation}
    whose image is defined over $E$;
  \item for each $\ell \neq \infty$, by certain local theory the functional $\lambda_{H,\ell}$ yields an $E$-form $\Pi_\ell(E)$ of the representation $\Pi_\ell$, such that $\phi^\circ_\ell \in \Pi_\ell(E) \otimes D(\hat{H}(\mathbb{Q}_\ell) \backslash \hat{G}(\mathbb{Q}_\ell))$ and $\Omega_{\Pi_\ell}(\varepsilon_\ell) \cdot \mathcal{P}_\ell^\circ$ is Aut($\mathbb{C}/E(\varepsilon_\ell)$)-equivariant when restricted to $\mathcal{X}(\varepsilon_\ell) \times \left( \Pi_\ell \otimes M(\hat{H}(\mathbb{Q}_\ell) \backslash \hat{G}(\mathbb{Q}_\ell)) \right)$.\end{itemize}
Comparison of the $E$-forms via the homomorphism (2.8) then yields an $E$-form
\[ H^i(\mathfrak{g}_C, K^\circ_{\infty}; \mathbb{V} \otimes \Pi_\infty)(E) \subset H^i(\mathfrak{g}_C, K^\circ_{\infty}; \mathbb{V} \otimes \Pi_\infty), \quad i \in \mathbb{Z}. \]
Write
\begin{equation}
\Pi'_\infty := H^i_\text{top}(\mathfrak{g}_C, K^\circ_{\infty}; \mathbb{V} \otimes \Pi_\infty) \quad \text{and} \quad \Pi'_\infty(E) := H^i(\mathfrak{g}_C, K^\circ_{\infty}; \mathbb{V} \otimes \Pi_\infty)(E).
\end{equation}
Let $X^\text{tor}_\infty \subset X_\infty$ denote the subgroup of the finite order characters. Suppose that we are given a “normalized archimedean modular symbol” map

$$(2.10) \quad \widehat{P}_\infty^\circ : X^\text{tor}_\infty \times \left( \Pi'_\infty \otimes M(\hat{H}(\mathbb{R}))^\vee \otimes O(\hat{G}(\mathbb{R})/\hat{K}_\infty^\circ) \right) \to \mathbb{C},$$

that is linear in the second variable and that

$$\widehat{P}_\infty^\circ(\epsilon, g, \phi) = \epsilon(\cdot(g)) \cdot \widehat{P}_\infty^\circ(\epsilon, \phi),$$

for all $\epsilon, \phi$ in the second factor of the domain, and $g \in \hat{K}_\infty^\circ$. Here $\hat{K}_\infty^\circ$ acts trivially on $M(\hat{H}(\mathbb{R}))^\vee$. We assume that the nonvanishing hypothesis holds, namely, for some vector $\phi$ is the second factor of the domain, $\widehat{P}_\infty^\circ(\cdot, \phi)$ is nowhere vanishing on $X^\text{tor}_\infty$.

We say that a right invariant measure in $M(\hat{H}(\mathbb{R}))$ is rational if its product with every element in $D(\hat{H}(\mathbb{A}_\infty))$ is a rational multiple of the right invariant Tamagawa measure on $\hat{H}(\mathbb{A})$. All such measures constitute a $\mathbb{Q}$-form of $M(\hat{H}(\mathbb{R}))$, to be denoted by $D(\hat{H}(\mathbb{R}))$.

Definition 2.1. The constants

$$\Omega_{\Pi}(\epsilon) := \left( \widehat{P}_\infty^\circ(\epsilon, \phi) \right)^{-1} \in \mathbb{C}^\times, \quad \epsilon \in X^\text{tor}_\infty,$$

are called the periods of $L_{\Pi}$.

The $V$-balanced condition in the following definition is necessary for the arithmetic study of the complex $L$-function $L_{\Pi}$ via modular symbols.

Definition 2.2. (a) A character $w \in X^\text{alg}$ is said to be $V$-balanced if

$$\dim \text{Hom}_{\hat{G}(\mathbb{Q})}(\overline{\mathbb{Q}}_w \otimes V, \overline{\mathbb{Q}}) = 1.$$ 

(b) A character $\chi_\infty \in X_\infty$ or $\chi \in X^\text{aut}$ is $V$-balanced if it is algebraic and the inverse of its weight is $V$-balanced.
composition of
\begin{align}
(2.11) \quad \hat{\mathcal{P}}^\lambda \to & \quad H^0(\mathcal{J}_C, K^0_{Z,\infty}; \mathbb{C}_{w,\infty} \otimes \chi_{\infty}) \\
& \times \left( \Pi'_{\infty} \otimes \mathbb{M}(\mathcal{H}(\mathbb{R}))^\vee \otimes \mathcal{O}(\hat{\mathbb{G}}(\mathbb{R})/\hat{K}^0_{\infty}) \right) \\
\text{pull-back} & \quad H^0(\mathcal{g}_C, \hat{K}^0_{\infty}; \mathbb{C}_{w,\infty} \otimes \chi_{\infty}) \\
& \times \left( H^0(\mathcal{g}_C, \hat{K}^0_{\infty}; V \otimes \Pi_{\infty}) \otimes \mathbb{M}(\mathcal{H}(\mathbb{R}))^\vee \otimes \mathcal{O}(\hat{\mathbb{G}}(\mathbb{R})/\hat{K}^0_{\infty}) \right) \\
\text{cup product} & \quad H^0(\mathcal{g}_C, \hat{K}^0_{\infty}; (\mathbb{C}_{w,\infty} \otimes V) \otimes (\chi_{\infty} \otimes \Pi_{\infty})) \\
& \otimes \mathbb{M}(\mathcal{H}(\mathbb{R}))^\vee \otimes \mathcal{O}(\hat{\mathbb{G}}(\mathbb{R})/\hat{K}^0_{\infty}) \\
\lambda^\otimes \phi_{\infty} & \quad H^0(\mathcal{g}_C, \hat{K}^0_{\infty}; \mathbb{C}) \otimes \mathbb{M}(\hat{\mathbb{G}}(\mathbb{R}))^\vee \otimes \mathcal{O}(\hat{\mathbb{G}}(\mathbb{R})/\hat{K}^0_{\infty}) \\
\text{push-forward of measures} & \quad H^0(\mathcal{g}_C, \hat{K}^0_{\infty}; \mathbb{C}) \otimes \mathbb{M}(\hat{\mathbb{G}}(\mathbb{R})/\hat{K}^0_{\infty})^\vee \otimes \mathcal{O}(\hat{\mathbb{G}}(\mathbb{R})/\hat{K}^0_{\infty}) \\
\end{align}

Here \( \hat{\mathcal{G}}_C \) denotes the Lie algebra of \( \hat{\mathcal{G}}(\mathbb{C}) \). At least in the three families of examples we are concerned, we will prove the following archimedean period relations: when \( \lambda_V \) is nonzero and suitably normalized,
\[
\hat{\mathcal{P}}^\lambda_{\infty}(1, \cdot) = \Upsilon_{\Pi_{\infty}}(\chi_{\infty}) \cdot \hat{\mathcal{P}}^\phi_{\infty}(w_{\infty}, \chi_{\infty}, \cdot)
\]
for some constant \( \Upsilon_{\Pi_{\infty}}(\chi_{\infty}) \in \mathbb{C}^\times \), where
\[
1 \in \mathbb{C} = \mathbb{C}_{w,\infty} \otimes \chi_{\infty} = H^0(\mathcal{J}_C, K^0_{Z,\infty}; \mathbb{C}_{w,\infty} \otimes \chi_{\infty})
\]
and similar notation will be used without explanation.

**Definition 2.3.** The constants
\[
\Upsilon_{\Pi_{\infty}}(\chi_{\infty}) \in \mathbb{C}^\times, \quad \chi_{\infty} \in \mathcal{X}_{\infty} \text{ is } V \text{-balanced},
\]
is called the modifying factors at \( \infty \) for \( \Pi'_{\infty} \).

Similar to (1.17), we have a linear embedding
\begin{align}
(2.12) \quad H^0(\mathcal{J}_C, K^0_{Z,\infty}; \mathbb{C}_{w,\infty} \otimes \chi) & \to \mathbb{C} \otimes H^0(Z, E_w), \quad 1 \mapsto w_{\infty} \cdot \chi.
\end{align}
Under certain growth conditions we have the following commutative diagram, which reflects the fact that modular symbols interpret period integrals:
\begin{align}
(2.13) \quad H^0(\mathcal{J}_C, K^0_{Z,\infty}; \mathbb{C}_{w,\infty} \otimes \chi) \times H^0(g_C, K^0_{\infty}; V \otimes \Pi) \otimes D(\hat{\mathcal{G}}) & \xrightarrow{\hat{\mathcal{P}}^\lambda \otimes \phi_{\infty} \otimes \mathcal{P}^\phi_{\infty}} \mathbb{C} \\
\downarrow & \quad \downarrow \zeta_{\Pi}(\chi)
\end{align}
Here \( \chi \in \mathcal{X}_{\text{aut}} \) is an algebraic automorphic character of weight \( w^{-1} \), and the identification
\[
D(\hat{\mathcal{G}}) = D(\hat{\mathcal{H}}(\mathbb{R}))^\vee \otimes \mathcal{O}(\hat{\mathbb{G}}(\mathbb{R})/\hat{K}^0_{\infty}) \otimes D(\hat{\mathcal{H}}(\mathbb{A}) \setminus \hat{\mathcal{G}}(\mathbb{A}_{\infty}))
\]
is used for the definition of the top horizontal arrow. Implicitly we have used the identification \( D(\hat{H}(\mathbb{R})) \otimes D(\hat{H}(A^\infty)) = \mathbb{Q} \) such that the right invariant Tamagawa measure is identified with \( 1 \in \mathbb{Q} \). In particular if \( \chi \in X(\mathfrak{e}) \), then applying the above commutative diagram to the test vector \( \hat{\phi} \circ \chi \), we obtain the equality

\[
(2.14) \quad \mathcal{M}_{\lambda'}(w_{\infty}\chi, \hat{\phi} \circ \chi \otimes V_{p-sm} \otimes (\otimes_{\ell \neq \infty} D(\mathfrak{g}/p))) = \frac{\Upsilon_{\Pi'_\infty}(\chi) \cdot \mathcal{L}_{\Pi}(\chi)}{\prod_{\ell \neq \infty} \Omega_{\Pi}(\chi_{\ell})}.
\]

In this case the number in (2.14) belongs to \( \mathbb{E} \) as long as \( w_{\infty}\chi \) is \( \mathbb{E} \times \mathbb{E} \)-valued.

2.3. \( p \)-adic \( L \)-functions. The homomorphism (2.8) in degree \( i_0 \) naturally extends to a homomorphism

\[
\iota_{\Pi} : H^0(\mathfrak{g}_C, K_{Z,\infty}; V \otimes \Pi)_{p-sm} = \Pi'_\infty \otimes \left( \hat{\Pi}_p \right)_{p-sm} \otimes (\otimes_{\ell \neq \infty} \rho'_\ell) \quad \rightarrow \quad \mathbb{C} \otimes H^0_\mathfrak{p}(G, V)_{p-sm},
\]

and the commutative diagram (2.13) extends to a commutative diagram (2.15)

\[
\begin{array}{ccc}
H^0(\mathfrak{g}_C, K_{Z,\infty}; C_{w_{\infty}} \otimes \chi) \times H^0(\mathfrak{g}_C, K_{Z,\infty}; V \otimes \Pi)_{p-sm} \otimes D(\hat{G}) & \xrightarrow{\mathcal{M}_{\lambda}} & \mathbb{C} \\
\downarrow{\text{(2.13) and } \iota_{\Pi}} & & \downarrow{\mathcal{L}_{\Pi}(\chi)} \\
(\mathbb{C} \otimes H^0(\mathfrak{Z}, E_{w})) \times (\mathbb{C} \otimes H^0_{\mathfrak{p}}(G, V)_{p-sm} \otimes D(\hat{G})) & \xrightarrow{\mathcal{M}_{\lambda'}} & \mathbb{C}.
\end{array}
\]

The space

\[
\mathcal{B}_P(\Pi_p) := \left( \left( \hat{\Pi}_p \right)_{p-sm} \right)^N
\]

is a smooth representation of \( L \) of finite length, which is defined over \( \mathbb{E} \) by the second adjointness theorem (see [Be87, Theorems 6.4 and 0.1]).

**Definition 2.4.** (a) An irreducible subrepresentation \( \Pi'_p \) of \( \mathcal{B}_P(\Pi_p) \) defined over \( \mathbb{Q} \) is called a refinement of \( \Pi_p \).

(b) A refinement \( \Pi'_p \) of \( \Pi_p \) is called nearly ordinary if the (unique) character of \( T \) that occurs in \( C_p \otimes \Pi'_p(\mathbb{Q}) \otimes D(\mathfrak{g}/p) \) is nearly \( V \)-ordinary, where \( \Pi'_p(\mathbb{Q}) \) denotes the \( \mathbb{Q} \)-form of \( \Pi'_p \).

Let \( \Pi'_p \) be a nearly ordinary refinement of \( \Pi_p \) defined over \( \mathbb{E} \), whose \( \mathbb{E} \)-form is denoted by \( \Pi'_p(\mathbb{E}) \). Now we assume that via the homomorphism \( \iota_{\Pi} \), \( \mathcal{H} \) is identified with an \( \mathbb{E} \)-subspace of

\[
H^0(\mathfrak{g}_C, K_{Z,\infty}; V \otimes \Pi)_{p-sm} \otimes D(\mathfrak{g}/p)
\]

of the form

\[
\mathcal{H} = \otimes_{\ell} \mathcal{H}_{\ell},
\]

where
\[ H_\infty = \Pi'_{\infty} (E); \]
\[ H_p = \Pi'_p (E) \otimes D(g/p); \]
\[ H_\ell = \Pi_\ell (E) \text{ for each } \ell \neq \infty, p. \]

In the rest of this section we drop the assumption that \( Z_0 = Z \), and fix an \( E^\times \)-valued character \( \varepsilon = \otimes_{\ell \neq \infty} \varepsilon_\ell \) as in (1.3). Set \( \hat{\mathcal{P}} := \hat{G} \cap \mathcal{P} \) whose Lie algebra equals \( \hat{\mathfrak{g}} \). For the definition of \( p \)-adic \( L \)-functions, suppose that we are given a “normalized refined period” map
\[ \hat{\mathcal{P}}^p : \mathcal{X}_p \times (\Pi'_p \otimes M(\hat{H} \setminus \hat{G})) \to \mathbb{C} \quad (\hat{H} := \hat{H}(\mathbb{Q}_p)) \]
such that
- it is algebraic in the first variable and linear in the second variable;
- it holds that
  \[ \hat{\mathcal{P}}^p (\chi_p, g, \phi) = \chi_p^{-1} (\eta(g)) \cdot \hat{\mathcal{P}}^p (\chi_p, \phi), \]
for all \( \chi_p \in \mathcal{X}_p, \phi \) in the second factor of the domain, and \( g \in \hat{\mathcal{P}}; \)
- there is an element
  \[ \hat{\phi}^p_\varepsilon \in \Pi'_p (E) \otimes D(\hat{H} \setminus \hat{G}) \]
such that \( \hat{\mathcal{P}}^p (\cdot, \hat{\phi}^p_\varepsilon) \) takes a nonzero constant value \((\Omega_{\Pi'_p (\varepsilon_p)})^{-1}\) on \( \mathcal{X}(\varepsilon_p) \), and the map \( \Omega_{\Pi'_p (\varepsilon_p)} \cdot \hat{\mathcal{P}}^p \) is \( \text{Aut}(\mathbb{C}/E) \)-equivariant when restricted to
\[ \mathcal{X}_p (\varepsilon_p) \times (\Pi'_p \otimes M(\hat{H} \setminus \hat{G})). \]

Write
\[
\hat{\phi}^o := \hat{\phi}^\infty \otimes \hat{\phi}^p \otimes (\otimes_{\ell \neq \infty} \phi_\ell)
\in \mathcal{H} \otimes D(\hat{H}(\mathbb{R}))^\vee \otimes O(\hat{G}(\mathbb{R})/\hat{K}_{\infty}) \otimes D(\hat{\mathfrak{g}}/\hat{\mathfrak{p}})^\vee \otimes D(\hat{H}(\mathbb{A}^\infty) \setminus \hat{G}(\mathbb{A}^\infty))
= \mathcal{H} \otimes D(\hat{G}(\mathbb{R}))^\vee \otimes O(\hat{G}(\mathbb{R})/\hat{K}_{\infty}) \otimes D(\hat{\mathfrak{g}}/\hat{\mathfrak{p}})^\vee \otimes D(\hat{H}(\mathbb{A}) \setminus \hat{G}(\mathbb{A}))
= \mathcal{H} \otimes D(\hat{G}, \hat{\mathfrak{g}}).
\]

**Definition 2.5.** The continuous linear functional
\[ \mathcal{L}_{\varepsilon \otimes \mathcal{H}} := (\mathcal{L}_{\lambda_\varepsilon \otimes \mathcal{H}}) \big|_{C(E)(\varepsilon)} \]
is called the \( p \)-adic \( L \)-function attached to \( \varepsilon \otimes \mathcal{H} \).

Note that the \( p \)-adic \( L \)-function \( \mathcal{L}_{\varepsilon \otimes \mathcal{H}} \) defined above uniquely extends to a continuous linear functional \( \mathcal{L}_{\varepsilon \otimes \mathcal{H}} : C(\mathbb{Z}, \mathbb{C}_p)(\varepsilon) \to \mathbb{C}_p. \)
Define a compact \( p \)-adic Lie group
\[ \mathbb{Z}^+_{30} := \lim_3 \mathbb{Z}(\mathbb{Q}) \setminus \mathbb{Z}(\mathbb{A}) / (\mathbb{Z}(\mathbb{R})^0 \cdot D_{\mathbb{Z}_0}^{\max} \cdot 3), \]
where \( \mathfrak{3} \) runs over all open compact subgroups of \( \mathbb{Z}(\mathbb{Q}_p) \). If \( \varepsilon \) is trivial, then as a Banach space \( C(\mathbb{Z}, E)(\varepsilon) \) agrees with \( C(\mathbb{Z}^+_{30}, E) \) (the space of all \( E \)-valued
continuous functions on $\mathbb{Z}^{b, \delta}$) and thus the $p$-adic $L$-function $L_{\varepsilon \otimes \mathcal{H}}$ is an $E$-valued measure on $\mathbb{Z}^{b, \delta}$.

**Proposition 2.6.** If

\begin{equation}
\dim \text{Hom}_p(\chi_p \otimes \mathcal{H}_p \otimes D(\hat{\mathfrak{h}}/\mathfrak{p})^\vee \otimes M(\hat{H}\backslash \hat{G}), \mathbb{C}) \leq 1
\end{equation}

for all $\chi_p \in \mathcal{X}_p(\varepsilon_p)$, then $L_{\varepsilon \otimes \mathcal{H}}$ is independent of $\hat{\phi}_p^\circ$. Similarly, if

\begin{equation}
\dim \text{Hom}_{\mathcal{G}(Q)\ell}(\chi_\ell \otimes \Pi_\ell \otimes D(\hat{H}(\mathbb{Q}_\ell)\backslash \hat{G}(\mathbb{Q}_\ell)), \mathbb{C}) \leq 1
\end{equation}

for all $\ell \neq \infty$, $p$ and all $\chi_\ell \in \mathcal{X}(\varepsilon_\ell)$, then $L_{\varepsilon \otimes \mathcal{H}}$ is independent of the family $\{\phi_\ell^\circ\}_{\ell \neq \infty, p}$.

**Proof.** We only prove the first assertion. The proof of the second one is similar and will be omitted. Without loss of generality we assume that $E = \mathbb{Q}_{\ell}$ so that $E = \mathbb{C}_p$. Suppose that $\chi = \otimes_\ell \chi_\ell \in \mathcal{X}(\varepsilon)$ is a finite order character. Since all such characters are $E^\ell$-valued and span a dense space in $C(\mathbb{Z}, E)(\varepsilon)$, we only need to show that $L_{\varepsilon \otimes \mathcal{H}}(\chi)$ is independent of $\hat{\phi}_p^\circ$.

By definition, $L_{\varepsilon \otimes \mathcal{H}}(\chi) = \overline{\mathcal{M}}_{\lambda_0}(\chi, \tilde{\xi}(\hat{\phi}^\circ))$, and by (2.10), there exists $c_\chi \in E$ such that

\[\overline{\mathcal{M}}_{\lambda_0}(\chi, \tilde{\xi}(\hat{\phi}_\infty^\circ \otimes \varphi_p \otimes (\otimes_{\ell \neq \infty, p} \hat{\phi}_\ell^\circ))) = c_\chi \cdot \hat{\mathcal{P}}_p(\chi_p, \varphi_p)\]

for all $\varphi_p \in \mathcal{H}_p \otimes D(\hat{\mathfrak{h}}/\mathfrak{p})^\vee \otimes D(\hat{H}\backslash \hat{G})$. Thus $L_{\varepsilon \otimes \mathcal{H}}(\chi) = c_\chi \cdot (\Omega_{\Pi_p}(\varepsilon_p))^{-1}$ does not depend on the choice of $\hat{\phi}_p^\circ$ satisfying that $\hat{\mathcal{P}}_p(\cdot, \hat{\phi}_p^\circ) = (\Omega_{\Pi_p}(\varepsilon_p))^{-1}$ on $X_p(\varepsilon_p)$. This proves the proposition.

In conclusion, as least when the conditions of Proposition (2.6) are satisfied, we have defined a $p$-adic $L$-function for $\varepsilon \otimes \mathcal{H}$ which only depends on $\lambda_0$ and $\hat{\phi}_\infty^\circ$.

Since $\hat{G}$ is $P$-spherical (namely (1.2) holds), the map (2.1) for $\ell = p$ naturally extends to a map (the stable normalized zeta integral)

\begin{equation}P_p : \mathcal{X}_p \times \left(\frac{\Pi_p}{\Pi_p^{\text{sm}}} \otimes M(\hat{H}\backslash \hat{G})\right) \rightarrow \mathbb{C},
\end{equation}

\[\chi_p, \phi \otimes \tau \mapsto P_p^0(\chi_p, \phi \otimes \tau),\]

where $\mathfrak{G}$ is a sufficiently small open compact subgroup of $G$ and $\phi_\mathfrak{G}$ denotes the image of $\phi$ under the projection map $\Pi_p \rightarrow \Pi_p^\mathfrak{G}$. The map (2.17) is still algebraic in the first variable, linear in the second variable, $\text{Aut}(\mathbb{C}/E)$-equivariant, and satisfies the analog of (2.5) for $g \in \hat{P}$.

At least when (2.16) holds, for each $\chi_p \in \mathcal{X}_p(\varepsilon_p)$, there is a unique constant $\Upsilon_{\Pi_p}(\chi_p)$ such that

\begin{equation}P_p^0(\chi_p, \cdot)_{|_{\mathcal{H}_p \otimes D(\hat{\mathfrak{h}}/\mathfrak{p})^\vee \otimes M(\hat{H}\backslash \hat{G})} = \Upsilon_{\Pi_p}(\chi_p) \cdot \hat{\mathcal{P}}_p(\chi_p, \cdot).\end{equation}
Definition 2.7. The constants

\[ \Upsilon_{\Pi'_n}(\chi_p) \in \mathbb{C}, \quad \chi_p \in \mathcal{X}_p(\varepsilon_p), \]

are called the modifying factors at \( p \) for \( \Pi'_n \).

Applying the equality (2.18) to the vector \( \hat{\phi}_p \), we know that \( \Upsilon_{\Pi'_n} \) is algebraic as a function on \( \mathcal{X}_p(\varepsilon_p) \). We remark that the \( p \)-adic \( L \)-function and the modifying factors are defined even when no \( w \in \mathcal{X}^{\text{alg}} \) is \( V \)-balanced. However, for all \( V \)-balanced characters \( \chi = \bigotimes \chi_\ell \in \mathcal{X}(\varepsilon) \), by applying the commutative diagram (1.19) to the pair \( (w_\infty \chi, \hat{\phi}_p) \) where \( w \) is the inverse of the weight of \( \chi \), we get the equalities (assuming \( \lambda_V|_{V^n} = \lambda_0 \))

\[
\begin{align*}
L_{\varepsilon \otimes \mathcal{H}}(\chi^\flat) & = \mathcal{M}_0(\chi^\flat, \hat{\xi}(\hat{\phi}^\flat)) \\
& = \mathcal{M} V_n \cdot \chi, \hat{\phi}^\flat \quad \text{by Theorem 1.19} \\
& = \mathcal{L}_{\Pi}(\chi) \cdot \mathcal{P}_p(\chi_p, \hat{\phi}_p^\flat) \cdot \prod_{\ell \neq \infty, p} \mathcal{P}_\ell(\chi_\ell, \hat{\phi}_\ell^\flat) \quad \text{by (2.15)} \\
& = \frac{\Upsilon_{\Pi'_n}(\chi_p) \cdot \Upsilon_{\Pi'_n}(\chi_p) \cdot \mathcal{L}_{\Pi}(\chi)}{(\prod_{\ell \neq \infty, p} \Omega_{\Pi'_n}(\varepsilon_\ell) \cdot \Omega_{\Pi'_n}(\varepsilon_p) \cdot \Omega_{\Pi}(w_\infty, \chi_\infty)}. \tag{2.19}
\end{align*}
\]

Definition 2.8. A character \( \chi' : \mathbb{Z}(\mathbb{Q}) \! \setminus \! \mathbb{Z} \to \mathbb{C}^\times \) is an exceptional zero of the \( p \)-adic \( L \)-function \( L_{\varepsilon \otimes \mathcal{H}} \) if \( \chi' = \chi^\flat \) for some \( V \)-balanced character \( \chi = \bigotimes \chi_\ell \in \mathcal{X}(\varepsilon) \) such that \( \Upsilon_{\Pi'_n}(\chi_p) = 0 \).

It is clear from (2.19) that all exceptional zeros of \( L_{\varepsilon \otimes \mathcal{H}} \) are its zeros.

Lemma 2.9. Suppose that \( P \) is defined over \( \mathbb{Q}_p \cap \mathbb{E} \) and

\[
\dim \text{Hom}_{E \otimes \mathbb{Z}_p}(V^n, E) = 1,
\]

so that \( V^n \) is defined over \( E \) and the one-dimensional space in (2.20) is also defined over \( E \). Assume that \( \lambda_0 \) is a generator of the one-dimensional space in (2.20) and is defined over \( E \). Then for all \( V \)-balanced algebraic character \( w \) of \( \mathbb{Z}_\mathbb{E} \), there exists a unique element \( \lambda_V := \lambda_{V, w} \in \text{Hom}_{\mathbb{E}_w \otimes V}^{\text{c}}(E_w \otimes V, E) \) such that \( \lambda_V|_{V^n} = \lambda_0 \).

Proof. This is implied by the transversal condition. \( \square \)

2.4. Rankin-Selberg \( p \)-adic \( L \)-functions for \( \text{GL}_n \times \text{GL}_{n-1} \). Let \( k \) be a number field with adele ring \( \mathbb{A}_k \). Let \( \Pi_n = \Pi_n \otimes \Pi_{n-1} \) (\( n \geq 2 \)) be an irreducible automorphic representation of \( \text{GL}_n(\mathbb{A}_k) \times \text{GL}_{n-1}(\mathbb{A}_k) \) that is regular algebraic in the sense of [Cl90]. We assume that \( \Pi_n \) is cuspidal, and \( \Pi_{n-1} \) is tamely isobaric in the sense of [LLS24, Section 6.2].
Denote $G_m := \text{Res}_{k/Q} \text{GL}_m$ for every positive integer $m$. Here and henceforth $\text{Res}$ indicates the Weil restriction of scalars. Suppose that the 5-tuple in (1.1) is

\[
\begin{aligned}
g &:= G_n \times G_{n-1}; \\
\hat{G} &:= G_{n-1}; \\
Z &:= G_1;
\end{aligned}
\]

\[i : \hat{G} \rightarrow G, \quad g \mapsto \left( \begin{array}{c} g \\ 0 \\ 0 \end{array} \right), \]

\[j : \hat{G} \rightarrow Z \] is the diagonal embedding;

\[
\begin{aligned}
\hat{\iota} : \hat{G} &\rightarrow G, \\
\hat{\iota} &\mapsto \left( \begin{array}{c} g_0 \\ 0 \\ 1 \end{array} \right), \]

\[
\text{is the determinant homomorphism}.
\]

Then $\Pi$ is an irreducible automorphic representation of $G(\mathbb{A})$. Let $P$ be a Borel subgroup of $G$ as in (8.9), which is transversal to $\hat{G}$ (in the sense that $P \cdot \hat{G}$ is open in $G$). Suppose that $P = P(Q_p)$.

Suppose that $Q(\Pi) \subset E$ and $\Pi_p$ has a nearly ordinary refinement $\Pi'_p \subset \mathcal{B}_p(\Pi_p)$ defined over $E$. Here $Q(\Pi)$ denotes the rationality field of $\Pi$, namely the subfield of $\mathbb{C}$ consisting of the elements that are fixed by all field automorphisms $\sigma$ of $\mathbb{C}$ such that $\Pi_\ell \otimes_{\mathbb{C},\sigma} \mathbb{C} \sim \Pi_\ell$ for all $\ell \neq \infty$ (see [Cl90, 3.1]). Similar notation for rationality fields applies to other automorphic representations. The coefficient system of $\Pi$ is defined over $E$ by [Cl90] (see [JST19, Theorem 2.14]).

The complex $L$-function in Section 2.1 for this example is the Rankin-Selberg $L$-function $L_{\Pi}(\chi) := L(\frac{1}{2}, \Pi \times \chi)$, where $\chi : k^\times \backslash A_k^\times \rightarrow \mathbb{C}^\times$ is a Hecke character. In the previous work [LLS24], under a balanced condition on $V$ we have proved the period relations for all the critical values of $L(s, \Pi \times \chi)$ when $\chi$ is of finite order. The main novelty of the proof is [LLSS23], which gives a comparison between local Rankin-Selberg integrals and the period integrals over the open orbit of the corresponding spherical subgroup acting on the flag variety.

By the same proof, the period relations can be easily extended to the case that $\chi$ is an arbitrary critical algebraic Hecke character (Definition 8.2), and in particular the archimedean period relations give the modifying factors $\Upsilon_{\Pi_\infty}(\chi_\infty)$ at $\infty$ in (2.22). This is the automorphic analog (due to Blasius [Bl97]) of the celebrated conjecture of Deligne [Del79].

Suppose that $Z_0$ is trivial. Then $Z^{\times\{0\}} = \mathcal{C}_k(p^\infty)$ is the idele class group of infinite level at $p$ defined by

\[
\mathcal{C}_k(p^\infty) := \lim_{m \in \mathbb{N} \setminus \{0\}} k^\times \backslash A_k^\times / \left( (k^\times) \circ \prod_{v \mid p} \mathcal{O}_v^\times \cdot (1 + p^m \mathcal{O}_p) \right).
\]

Here and henceforth $k_v$ denotes the local field corresponding to a place $v$ of $k$, with ring of integers $\mathcal{O}_p$ if $v$ is finite, and $\mathcal{O}_p := \mathcal{O}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ where $\mathcal{O}_k$ is the ring of integers of $k$. Denote by $k^{(\infty p)}$ the maximal abelian extension of $k$ that is unramified outside $\infty p$. Then $\mathcal{C}_k(p^\infty) \cong \text{Gal}(k^{(\infty p)}/k)$ by class field theory.

Applying the main theory of this paper as explained earlier, we will construct a $p$-adic $L$-function $\mathcal{L}_{\Pi} := \mathcal{L}_{\varepsilon, \mathbb{C}, R}$, which is a continuous linear functional on $C(Z, E)(\varepsilon)$. Here the ramification type $\varepsilon$ is assumed to be $E^\times$-valued as before. In particular
when \( \varepsilon \) is trivial, \( \mathcal{L}_1 \) is an \( E \)-valued \( p \)-adic measure on \( \mathcal{C}_k(p^{\infty}) \). We emphasize that the construction of \( \mathcal{L}_1 \) does not require the existence of \( V \)-balanced characters.

In Sections 8.3, 8.4 we will explicitly calculate the modifying factor \( \Upsilon_{\Pi_p}(\chi_p) \) at \( p \) as given by (2.22). This is consistent with the conjecture given by Coates and Perrin-Riou in [CPR89, Page 49].

As the first application of Theorem 1.3, we obtain the following main result on the \( p \)-adic L-function for this example, which shows that if there is a \( V \)-balanced character then \( \mathcal{L}_1(\chi) \) interpolates \( L(\frac{1}{2}, \Pi \times \chi) \) for all critical algebraic Hecke characters \( \chi \in \mathcal{X}(\varepsilon) \) with explicit modifying factors at \( p \), and the interpolation is consistent with the Principal Conjecture in [CPR89, Page 49].

**Theorem 2.10** (Theorem 8.8). Let the notations and assumptions be as above. If there is a \( V \)-balanced character, then \( \mathcal{L}_1 \) is the unique continuous linear functional on \( G(Z, E)(\varepsilon) \) such that

\[
\mathcal{L}_1(\chi) = \frac{\Upsilon_{\Pi_p}(\chi) \cdot \Upsilon_{\Pi}(\chi_p) \cdot L(\frac{1}{2}, \Pi \times \chi)}{\mathcal{G}_p(\chi(\pi_p^{\infty})) \cdot \mathcal{G}_p(\chi(\pi_{\Pi_p}^{\infty})) \cdot \Omega_{\Pi_p} \cdot \Omega_{\Pi}(\mathcal{w}_\pi \chi)}
\]

for all critical algebraic Hecke characters \( \chi = \otimes \varepsilon \chi \ell \in \mathcal{X}(\varepsilon) \), where

- \( w \) is the inverse of the weight of \( \chi \);
- \( \Upsilon_{\Pi_p}(\chi) \) is the modifying factor at \( \infty \) in (2.22);
- \( \Upsilon_{\Pi_p}(\chi_p) \) is the modifying factor at \( p \) in (2.23);
- \( \Omega_{\Pi_p} \) is in (2.23);
- \( \mathcal{G}_p(\chi) \) and \( \mathcal{G}_p(\chi_p) \) are the Gauss sums outside \( p \) as in (8.3.6), respectively of \( \chi \) and the central character \( \chi_{\Pi_{n-1}} \) of \( \Pi_{n-1} \);
- \( \{\Omega_{\Pi}(\varepsilon) \in \mathbb{C}^\times \} \) are the Whittaker periods of \( \Pi \) in (8.10).

Here and henceforth, \( \psi \) is the nontrivial additive character of \( k \backslash A_k \) defined as the composition

\[
\psi : k \backslash A_k \xrightarrow{\mathrm{Tr}_{k/Q}} \mathbb{Q} \backslash A \to \mathbb{Q} \backslash A / \prod_{\ell \neq \infty} \mathbb{Z}_\ell = \mathbb{R} / \mathbb{Z} \xrightarrow{\psi_k} \mathbb{C},
\]

where \( \mathrm{Tr}_{k/Q} \) is the trace map, \( \psi_k(x) := e^{2\pi i x} \), \( i := \sqrt{-1} \); and \( \overset{\wedge}{\wedge} \) over an abelian topological group indicates the group of its characters.

Let us explain the modifying factors and \( \Omega_{\Pi_p} \) in Theorem 2.10, which are calculated explicitly based on [LLSS23]. Let \( E_k \) be the set of field embeddings \( \iota : k \to \mathbb{Q} \). Denote the highest weight of \( V^\vee \) by

\[
(\mu, \nu) = (\{\mu_i^\vee\}_{i \in E_k}, \{\nu_i^\vee\}_{i \in E_k}) \in (\mathbb{Z}^n)^{E_k} \times (\mathbb{Z}^{n-1})^{E_k},
\]

where every \( \mu^\vee \) is a sequence of integers \( \mu_1^\vee \geq \mu_2^\vee \geq \cdots \geq \mu_n^\vee \), and \( \nu^\vee \) is similar. Throughout the paper, when \( Z = G_1 \) we write \( w = \prod_{i \in E_k} w_i \), where \( w_i \in \mathbb{Z} \). Then

\[
(2.22) \quad \Upsilon_{\Pi_p}(\chi) = i^{-\sum_{i \in E_k} \sum_{i+1}^{n-1}(n-i)(\mu_i^\vee+\nu_i^\vee-w_i)} \cdot (-1)^{\sum_{i \in E_k} \sum_{i+j \leq n}(\mu_i^\vee+\nu_j^\vee-w_i)}.
\]
The refinement $\Pi'_p$ can be also viewed as a character of the torus $L = P/N$. Assume that conjugation of $P$ to the Borel subgroup of lower triangular matrices in $G$ takes $\Pi'_p$ to a character of the form

$$\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n, \kappa'_1, \kappa'_2, \ldots, \kappa'_{n-1})$$

interpreted in the usual way. Here $\kappa_i = \otimes_{p|l_p} \kappa_{i,p}$ and $\kappa_{i,p}$ is a character of $k_p^\times$ $(i = 1, 2, \ldots, n)$, and similar notations will be used for $\kappa'_j$ $(j = 1, 2, \ldots, n-1)$ and $\chi_p$. Write $\psi = \otimes_v \psi_v$ where $v$ runs over all places of $k$. Then

$$(\Pi'_p \times \chi_p) := \prod_{i,j \geq 1, i+j \leq n} (\kappa_i \kappa'_j \chi_p)(-1) \cdot \prod_{\mathfrak{p} \mid p} \prod_{i+j \leq n} \gamma(n + 1 - i - j, \kappa_{i,p}^{-1} \kappa'_{j,p}^{-1} \chi_p^{-1}).$$

Here and henceforth, for every place $v$ of $k$, $\gamma(s, \omega, \psi_v')$ denotes the Tate $\gamma$-factor for a character $\omega$ of $k_v^\times$ and a nontrivial additive character $\psi_v'$ of $k_v$ (see [T79, Ku03]). Note that $\Upsilon_{\Pi'_p}(\chi_p)$ is an algebraic function on $X_p$ by Proposition 8.7.

For every finite place $v$ of $k$, the conductor of $\psi_v$ is the inverse different of $k_v$ given by

$$d_v^{-1} := \{ x \in k_v : \text{Tr}_{k_v/Q}(x \mathcal{O}_v) \subset \mathbb{Z}_\ell \},$$

where $\ell$ is the residue characteristic of $k_v$. Then the volume of $\mathcal{O}_v$ under the self-dual Haar measure on $k_v$ with respect to $\psi_v$ is equal to

$$(2.24) \quad c_v := (\mathcal{O}_v : \mathfrak{d}_v)^{-1/2}.$$

Write

$$c_p := \prod_{\mathfrak{p} | p} c_{\mathfrak{p}} = \prod_{\mathfrak{p} | p} (\mathcal{O}_\mathfrak{p} : \mathfrak{d}_\mathfrak{p})^{-1/2},$$

and $\omega_{\psi_p}(\kappa) := \prod_{\mathfrak{p} | p} \omega_{\psi_p}(\kappa_{\mathfrak{p}})$, where

$$(2.25) \quad \omega_{\psi_p}(\kappa_{\mathfrak{p}}) := \prod_{i=1}^{n-1} \mathcal{G}_{\psi_p}(\kappa_{i,\mathfrak{p}})^{i-n} \cdot \prod_{j=1}^{n-2} \mathcal{G}_{\psi_p}(\kappa'_{j,\mathfrak{p}})^{-j-n+1}$$

and the Gauss sums are as in [8.3]. Define

$$(2.26) \quad \Omega_{\Pi'_p} := c_p^{(n-1)/2} \cdot \omega_{\psi_p}(\kappa).$$

Using Theorem 2.12 and the formula of $\Upsilon_{\Pi'_p}(\chi_p)$, we determine the exceptional zeros of $L_{\Pi}$ explicitly in Proposition 8.9. The existence of exceptional zeros of $p$-adic $L$-functions is of utmost arithmetic significance. For instance see the discussion in [MTT86] for an elliptic curve $E$ over $\mathbb{Q}$. In this case, assuming that the ramification type $\varepsilon$ in (1.3) is trivial, an exceptional zero occurs if and only if $E$ has split multiplicative reduction at $p$, which corresponds to the Steinberg representation of $\text{GL}_2(\mathbb{Q}_p)$. As a direct application of Proposition 8.9, we have the following generalization.
Table 1. $\chi_p(p)$ for zeros $\chi_p$ of $\Upsilon_{\Pi'_p}$

| $\chi_p(p)$ | $\varepsilon'$ | good | s-mult | ns-mult |
|-------------|----------------|------|--------|--------|
| good        | $\alpha_p^{2i-n-2}\alpha_p^{n+1-2i}$ | $\alpha_p^{2i-n-2}$ | $-\alpha_p^{2i-n-2}$ |
| s-mult      | $\alpha_p^{n+1-2i}$ | 1    | -1     |
| ns-mult     | $-\alpha_p^{n+1-2i}$ | -1   | 1      |

Example 2.11. Let $\varepsilon$ and $\varepsilon'$ be elliptic curves over $\mathbb{Q}$ without complex multiplication. By the modularity of elliptic curves over $\mathbb{Q}$ [W95, BCDT01] and the recent groundbreaking result on symmetric power functoriality in [NT21a, NT21b], $\text{Sym}^n \varepsilon$ and $\text{Sym}^{n-1} \varepsilon'$ ($n \geq 1$) correspond to irreducible regular algebraic cuspidal automorphic representations $\Pi_{n+1}$ and $\Pi_n$ of $\text{GL}_{n+1}(\mathbb{A})$ and $\text{GL}_n(\mathbb{A})$ respectively. Then for $\Pi := \Pi_{n+1} \boxtimes \Pi_n$ we have that

$$L(s, \Pi) = L\left(s + n - \frac{1}{2}, \text{Sym}^n \varepsilon \times \text{Sym}^{n-1} \varepsilon'\right).$$

Assume that $\varepsilon$ and $\varepsilon'$ are $p$-ordinary, so that they have good or multiplicative reductions at $p$. When $\varepsilon$ has good ordinary reduction at $p$, write $\alpha_p$ for one of the two eigenvalues in $\overline{\mathbb{Q}}^\times$ that is a $p$-adic unit, of the $p$-th power Frobenius endomorphism on the Tate module $T_\ell(\tilde{E})$ (see [Sil09, V.2]). Here $\tilde{E}$ is the Néron model of $E$ and $\ell \neq p$ is a prime. Similarly when $\varepsilon'$ has good ordinary reduction at $p$, write $\alpha_p'$ for the corresponding $p$-adic unit.

Given a ramification type $\varepsilon$ in (1.3), which in this case is a finite order character of $\prod_{\ell \neq \infty, p} \mathbb{Z}_\ell^\times$, let $\mathcal{L}_\Pi$ be the corresponding $p$-adic $L$-function. Note that in this example (and Examples 2.13 and 2.15), since the weight $V$ is trivial, all exceptional zeros of $\mathcal{L}_\Pi$ are of finite order.

We first observe that by Proposition 8.9, if $\chi^p$ is an exceptional zero of $\mathcal{L}_\Pi$, where $\chi = \otimes_\ell \chi_\ell \in \mathcal{X}(\varepsilon)$ is a finite order character of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$, then $\chi_p$ is unramified and $\chi_p(p)$ is a root of unity. We list the values $\chi_p(p)$ for all the zeros $\chi_p$ of $\Upsilon_{\Pi'_p}$ given by Proposition 8.9 in Table 1 (depending on different reduction types and ramification types), where “good”, “s-mult” and “ns-mult” indicate good, split multiplicative and nonsplit multiplicative reductions at $p$ respectively, and $1 \leq i \leq n$. However, since $|\alpha_p| = |\alpha_p'| = \sqrt{p}$ by Weil’s conjecture for curves and $\chi_p(p)$ is a root of unity, after ruling out impossible cases we arrive at Table 2, where $d_{\chi_p}$ denotes the order of $\chi_p$ as a zero of $\Upsilon_{\Pi'_p}$.

\[ 2.5. \text{Rankin-Selberg p-adic L-functions for } U_n \times U_{n-1}. \] Thanks to the recent advances [BPLZZ21, BPCZ22] of the global Gan-Gross-Prasad conjecture [GGP12] and Ichino-Ikeda conjecture [II10, Ha14] for unitary groups, our results can be also applied to produce certain anticyclotomic $p$-adic L-functions. The recent work
Table 2. Exceptional zeros for \( \text{Sym}^n E \times \text{Sym}^{n-1} E \)

| \( \chi_p(p) \) | \( E \) | \( \text{good} \) | \( s\text{-mult} \) | \( n\text{s}-\text{mult} \) |
|----------------|----------------|---------|------------|-----------|
| good           | NA             | \( (n \text{ even}, \delta_{\chi_p} = 1) \) | \( (n \text{ even}, \delta_{\chi_p} = 1) \) | \( (n \text{ even}, \delta_{\chi_p} = 1) \) |
| s-mult         | \( (n \text{ odd}, \delta_{\chi_p} = 1) \) | \( (\delta_{\chi_p} = 1) \) | \( (\delta_{\chi_p} = 1) \) | \( (\delta_{\chi_p} = 1) \) |
| ns-mult        | \( (n \text{ odd}, \delta_{\chi_p} = 1) \) | \( (\delta_{\chi_p} = 1) \) | \( (\delta_{\chi_p} = 1) \) | \( \) |

\[ \text{Liu23} \] (see also \[ D23 \]) gives a different approach by using the local Birch lemma as in \[ KMS00, \text{Jan11} \] etc.

Let \( k'/k \) be a quadratic extension of number fields. Let \( \tilde{\pi} = \tilde{\pi}_n \boxtimes \tilde{\pi}_{n-1} \) be an irreducible automorphic representation of \( \text{GL}_n(A_{k'}) \times \text{GL}_{n-1}(A_{k'}) \) that is hermitian isobaric in the sense of \[ BPLZZ21, \text{Definition 1.5} \] such that \( L(1/2, \tilde{\pi}) \neq 0 \). By the Gan-Gross-Prasad conjecture (Theorem \[ 9.1 \]), \( \tilde{\pi} \cong \text{BC}(\pi) \) is the weak base-change of an irreducible cuspidal automorphic representation \( \pi = \pi_n \boxtimes \pi_{n-1} \) of \( G_0(A_k) \) with certain nonvanishing global period. Here \( G_0 := U_n \times U_{n-1} \) is the product of two relevant unitary groups over \( k \), so that \( U_{n-1} \) embeds into \( G_0 \) diagonally.

Assuming that \( \pi \) is everywhere tempered, the Ichino-Ikeda conjecture (Theorem \[ 9.2 \]) makes a further refinement by relating the global and local periods through the following L-function

\[
L(s, \pi \times \chi) := \frac{L(s, \tilde{\pi} \times \tilde{\chi})}{L(s+1/2, \pi, \text{Ad})} \cdot \prod_{i=1}^n L(s+i - \frac{1}{2}, \eta_{k'/k}),
\]

which is always holomorphic at \( s = \frac{1}{2} \), and set

\[
L_\Pi(\chi) := |S_\pi|^{-1} \cdot L(1/2, \pi \times \chi).
\]

Here \( \eta_{k'/k} \) is the quadratic character of \( k^\times \backslash A_k^\times \) associated to \( k'/k \), \( \tilde{\chi} = \text{BC}(\chi) \), \( L(s, \pi, \text{Ad}) \) is the adjoint L-function of \( \pi \), and \( S_\pi \) is a certain component group measuring the Arthur packet of \( \pi \). Here and henceforth, for every finite set \( S \), denote by \( |S| \) its cardinality.

Suppose that the 5-tuple in \( (1.1) \) is

\[
\begin{align*}
  \mathcal{G} & := \text{Res}_{k/Q}(G_0 \times G_0); \\
  \hat{\mathcal{G}} & := \text{Res}_{k/Q}(U_{n-1} \times U_{n-1}); \\
  \iota : \hat{\mathcal{G}} & \to \mathcal{G} \text{ is the product of the diagonal embeddings;} \\
  j : \hat{\mathcal{G}} & \to \mathcal{Z} \text{ is the composition of } \hat{\mathcal{G}} \xrightarrow{\text{det} \times \text{det}^{-1}} \mathcal{Z} \times \mathcal{Z} \xrightarrow{\text{multiplication}} \mathcal{Z}.
\end{align*}
\]

Then \( \Pi \) is an automorphic representation of \( \mathcal{G}(A) \) as before.
For technical reasons, we assume that all places $v \mid \infty p$ of $k$ are split in $k'$. In particular $G$ is quasi-split over $\mathbb{Q}_p$. Suppose that

- $\pi$ is regular algebraic, $\mathbb{Q}(\pi) \subset E$, and the coefficient system of $\pi$ is defined over $E$;
- $G_{E_0}$ is quasi-split, where $E_0 := \mathbb{Q}_p \cap E$;
- $\Pi_p$ has a nearly ordinary refinement $\Pi'_p \subset B_P(\Pi_p)$ defined over $E$, where $P = P_{E_0}(\mathbb{Q}_p)$ and $P_{E_0}$ is a Borel subgroup of $G_{E_0}$ that is transversal to $G_{E_0}$.

Note that $\mathbb{Q}(\pi)$ is a number field by [GL21, 1.4.2].

Suppose that $Z_0$ is trivial. Then

$$Z^\circ \{0\} = C_{k'}'(p^\infty) := \ker(N_{k'/k} : C_{k'}'(p^\infty) \to C_k(p^\infty)),$$

where $N_{k'/k}$ is the norm map. Applying the main theory of this paper when $\varepsilon$ in (1.3) is trivial, we will construct an $E$-valued $p$-adic measure $\mathcal{L}_{\Pi'} := \mathcal{L}_{\varepsilon \otimes \nu'}$ on $C_{k'}'(p^\infty)$. As the second application of Theorem 1.5, we have the following main result for this example, which shows that if there is a $\mathcal{V}$-balanced character then $\mathcal{L}_{\Pi}(\chi)$ interpolates $\mathcal{L}_{\Pi}(\chi)$ for all critical algebraic automorphic characters $\chi$ (Definition 9.7) unramified outside $\infty p$. The explicit modifying factors at $p$ are again consistent with [CPR89, Co89].

**Theorem 2.12** (Theorem 9.13). Let the notations and assumptions be as above. If there is a $\mathcal{V}$-balanced character, then $\mathcal{L}_{\Pi}$ is the unique $E$-valued $p$-adic measure on $C_{k'}'(p^\infty)$ such that

$$\mathcal{L}_{\Pi}(\chi) = \frac{\Upsilon_{\Pi_{\infty}} \cdot \Upsilon_{\Pi_p}(\chi_p) \cdot \mathcal{L}_{\Pi}(\chi)}{\Omega_{\Pi_p} \cdot \Omega_{\Pi}(\omega_{\infty} \chi_{\infty})}$$

for all critical algebraic automorphic characters $\chi = \otimes \varepsilon \chi_{\ell} : U_1(k) \setminus U_1(A_k) \to \mathbb{C}^\times$ unramified outside $\infty p$, where

- $\omega$ is the inverse of the weight of $\chi$;
- $\Upsilon_{\Pi_{\infty}}$ is the modifying factor at $\infty$ in (2.28);
- $\Upsilon_{\Pi_p}(\chi_p)$ is the modifying factor at $p$ in (2.31);
- $\Omega_{\Pi_p}$ is in (2.32);
- $\{\Omega_{\Pi}(\varepsilon) \in \mathbb{C}^\times \}_{\varepsilon \in U_1(A_{k_{\infty}})}$ are the Bessel periods of $\Pi$ in (1.5).

Theorem 2.12 can be easily extended to the case of a general ramification type $\varepsilon$, by twisting $\pi$ with a character $\chi \in X(\varepsilon)$.

In this example, $\Omega_{\Pi_p}(\varepsilon_{\ell}) (\ell \neq \infty, p)$ in (2.19) are all equal to 1. Let $\mu$ explain the modifying factors and $\Omega_{\Pi_p}$ in Theorem 2.12. Suppose that $\mathcal{V} = \mathcal{V}_\mu \otimes \mathcal{V}_\mu'$, where $\mathcal{V}_\mu = F_\mu \boxtimes F_\nu'$ and $(\mu, \nu) \in (\mathbb{Z}/n)_{\delta_k} \times (\mathbb{Z}/n-1)_{\delta_k}$ is as in Section 2.4. Then

$$\Upsilon_{\Pi_{\infty}} := (-1)^{\sum_{i \in S_k} \sum_{j=1}^{n_{i-1}} (n_i + n_{i+1})}.$$

Write $k_p := \mathbb{Q}_p \otimes k$ and fix a $k_p$-algebra isomorphism

$$k_p \otimes_k k' \cong k_p \times k_p.$$
By using the first factor of the product we have an identification
\[ G = \text{GL}_n(k_p) \times \text{GL}_{n-1}(k_p) \times \text{GL}_n(k_p) \times \text{GL}_{n-1}(k_p) \quad (\text{with respect to certain bases}). \]

The refinement \( \Pi'_p \) is also viewed as a character of the torus \( L = P/N \). By MVW involution (see [MVW87] Chapter 4, \( \Pi \)), the conjugation of \( P \) to the Borel subgroup of lower triangular matrices in \( G \) takes \( \Pi'_p \) to a character of the form
\[ (2.30) \quad \kappa = (\kappa_1, \ldots, \kappa_n, \kappa'_{11}, \ldots, \kappa'_{n-1,1}; \kappa_{n,1}^{-1}, \ldots, \kappa_{1,1}^{-1}, \kappa_{1,1}^{j-1}, \ldots, \kappa'_{n-1,n}^{j-1}), \]
where \( \kappa_i = \otimes_{p \mid i} \kappa_{i,p} \) and \( \kappa_{i,p} \) is a character of \( k_p \) \( (i = 1, 2, \ldots, n) \), and similar notation applies for \( \kappa'_j \) \( (j = 1, 2, \ldots, n - 1) \). Then
\[ (2.31) \quad \Upsilon_{\Pi'_p}(\chi_p) := \frac{\omega_{\pi_{n-1,p}}(-1)^n}{\mathcal{L}(\frac{1}{2}, \pi_p \times \chi_p)} \cdot \prod_{p \mid \mathcal{L}} \gamma(n + 1 - i - j, \kappa_{i,p}^{-1} \kappa'_{j,p} \chi_p^{-1}, \psi_p^{-1}) \cdot \prod_{i+j > n} \gamma(i + j - n, \kappa_{i,p} \kappa'_j \chi_p, \psi_p) \]
where \( \pi_{n-1,p} \) is the component of \( \pi_{n-1} \) at \( p \), \( \omega_{\pi_{n-1,p}} \) is the central character of \( \pi_{n-1,p} \), and \( \mathcal{L}(\frac{1}{2}, \pi_p \times \chi_p) \) is the local factor of \( \mathcal{L}(\frac{1}{2}, \pi \times \chi) \) in \( (2.27) \) at \( p \). It is easily verified that \( (2.31) \) is independent of the isomorphism \( (2.29) \). Note that \( \Upsilon_{\Pi'_p} \) is an algebraic function on \( \mathcal{X}_p \) by Proposition 9.12.

Define
\[ (2.32) \quad \Omega_{\Pi'_p} := \prod_{p \mid \mathcal{L}} \omega_{\psi_p}(\kappa_p), \]
where
\[ \omega_{\psi_p}(\kappa_p) := \prod_{i=1}^{n-1} (G_{\psi_p}(\kappa_{i,p}) \cdot G_{\psi_p}(\kappa_{n+1-i,p}))^{i-n} \cdot \prod_{j=1}^{n-2} (G_{\psi_p}(\kappa'_j \chi_p) \cdot G_{\psi_p}(\kappa'_{j-1} \chi_p))^{j-n+1}. \]

The explicit calculation of \( \Upsilon_{\Pi'_p}(\chi_p) \) is again based on [LLSS23], together with a result in [Zh14b] (Lemma 9.3). In the special case that \( \pi_p \) is unramified and \( \chi_p \) is ramified, the value \( \Upsilon_{\Pi'_p}(\chi_p) \) agrees with \( \text{Liu23 Theorem 5.2} \) which is obtained by using the so-called local Birch lemma. Using the formula of \( \Upsilon_{\Pi'_p}(\chi_p) \), we also determine the exceptional zeros of \( \mathcal{L}_\Pi \) (Proposition 9.14).

**Example 2.13.** Let \( E \) and \( E' \) be elliptic curves over \( \mathbb{Q} \) without complex multiplication, and let \( k' \) be a quadratic number field. By [AC89] and [NT21a, NT21b], \( \text{Sym}^n E_{k'} \) and \( \text{Sym}^{n-1} E'_{k'} \) \( (n \geq 1) \) are modular, which correspond to irreducible regular algebraic cuspidal automorphic representations \( \tilde{\pi}_{n+1} \) and \( \tilde{\pi}_n \) of \( \text{GL}_{n+1}(k_{k'}) \) and \( \text{GL}_n(k_{k'}) \) respectively.

Assume that \( E \) and \( E' \) are \( p \)-ordinary, \( k' \) is real quadratic, and \( p \) splits in \( k' \). If \( L(\frac{1}{2}, \tilde{\pi}_{n+1} \times \tilde{\pi}_n) \neq 0 \), then we have a \( p \)-adic measure \( \mathcal{L}_\Pi \) on \( E_{k'}(p) \) associated

\[ \text{with respect to certain bases}. \]
to $\tilde{\pi}_{n+1} \boxtimes \tilde{\pi}_n$ as in Theorem 2.12. By Proposition [9.14] if $\chi^\flat$ is an exceptional zero of $L_\Pi$, where $\chi$ is a finite order character of $U_1(\mathbb{Q}) \backslash U_1(\mathbb{A})$ unramified outside $\infty p$, then $\chi_p$ is unramified and the values of $\chi_p(p)$ (in \{±1\}) are as in Table 2 of Example 2.11 with the order $d_{\chi_p}$ doubled.

2.6. **Standard $p$-adic $L$-functions of symplectic type for $GL_{2n}$**. Let $\pi$ be an irreducible regular algebraic cuspidal automorphic representation of $GL_{2n}(\mathbb{A}_k)$, where $k$ is a number field and $n \geq 1$. We assume that $\pi$ is of symplectic type, which is equivalent to that $\pi$ has a nonzero $(\eta, \psi)$-Shalika integral (see Section 10.2 for the precise statement). Here $\eta : k^\times \backslash \mathbb{A}_k^\times \rightarrow \mathbb{C}^\times$ is a Hecke character, which is necessarily algebraic.

Recall from Section 2.4 that $G_m := \text{Res}_{k/\mathbb{Q}}GL_m$ for every positive integer $m$. Denote by $1_m$ the $m \times m$ identity matrix. Suppose that the 5-tuple in (1.1) is defined over $E$.

Consider the following:

- $G := (G_{2n} \times G)/G_1$, where $G_1 := \{(a \cdot 1_{2n}, a^n) : a \in G_1\}$;
- $\hat{G} := (G_n \times G_n)/G_1$, where $\hat{G}_1 := \{(a \cdot 1_n, a \cdot 1_n) : a \in G_1\}$;
- $Z := G_1$;
- $i : \hat{G} \rightarrow G$, $(g_1, g_2)\hat{G}_1 \mapsto \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$;
- $j : \hat{G} \rightarrow Z$, $(g_1, g_2)\hat{G}_1 \mapsto \frac{\det g_1}{\det g_2}$.

Then $\Pi := \pi \boxtimes \eta^{-1}$ is an automorphic representation of $G(\mathbb{A})$. The complex $L$-function for this example is the standard $L$-function $L_{\Pi}(\chi) := L(\frac{1}{2}, \pi \otimes \chi)$ for $\chi$ a Hecke character of $k^\times \backslash \mathbb{A}_k^\times$, which has been studied through Shalika models and Friedberg-Jacquet integrals in [FJ93].

Let $P$ be a Borel subgroup of $G$ as in (10.3) so that

$$\hat{P} := P \cap \hat{G} = \{(t, t)\hat{G}_1 : t \in T_n\},$$

where $T_n$ is the diagonal maximal torus of $G_n$. Then $j(\hat{P})$ is trivial. Suppose that $P = P(\mathbb{Q}_p)$ and $Z_0$ is trivial. Then $Z^{\epsilon, (0)} = \mathcal{C}_k(p^\infty)$.

Suppose that $Q(\Pi) \subset E$ and $\Pi_p$ has a nearly ordinary refinement $\Pi_p' \subset B_P(\Pi_p)$ defined over $E$. Then the coefficient system of $\Pi$ is defined over $E$. When $\nu$-balanced character exists, the period relations of $L(\frac{1}{2}, \pi \otimes \chi)$ for arbitrary critical algebraic Hecke characters $\chi$ (Definition 10.2) have been proved in full generality in [JST24], which extends the earlier work [JST19].

As before suppose that the ramification type $\varepsilon$ is valued in $E^\times$. Applying the main theory of this paper, we will construct a continuous linear functional $L_{\Pi} := L_{\epsilon \otimes \eta}$ on $C(Z, E)(\varepsilon)$, without assuming the existence of $\nu$-balanced characters. As the third application of Theorem 1.5, we have the following main result for this example, which shows that if there is a $\nu$-balanced character then $L_{\Pi}(\chi^\flat)$ interpolates $L(\frac{1}{2}, \pi \otimes \chi)$ for all critical algebraic Hecke characters $\chi \in X(\varepsilon)$. The explicit modifying factors at $p$ are again consistent with [CPR89, Co89].
Theorem 2.14 (Theorem [10.9]). Let the notations and assumptions be as above. If there is a $V$-balanced character, then $\mathcal{L}_\Pi$ is the unique continuous linear functional on $C(\mathcal{Z}, E)(\varepsilon)$ such that

$$
\mathcal{L}_\Pi(\chi) = \frac{\Upsilon_{\Pi'}(\chi,) \cdot \Upsilon_{\Pi'}(\chi_p) \cdot L(\frac{1}{2}, \pi \otimes \chi)}{\mathcal{G}_\psi(\chi(p))^n \cdot \Omega_{\Pi'} \cdot \Omega_{\Pi}(\omega \chi,)}
$$

for all critical algebraic Hecke characters $\chi = \otimes \chi_\ell \in \mathcal{X}(\varepsilon)$, where

- $w$ is the inverse of the weight of $\chi$;
- $\Upsilon_{\Pi'}(\chi,)$ is the modifying factor at $\infty$ in (2.33);
- $\Upsilon_{\Pi'}(\chi_p)$ is the modifying factor at $p$ in (2.34);
- $\Omega_{\Pi'}$ is in (2.36);
- $\{\Omega_{\Pi}(\varepsilon, ) \in \mathbb{C}^\times\}^{\varepsilon, \in \mathcal{C}^\times}$ are the Shalika periods of $\Pi$ in (10.14).

As mentioned in Theorem 2.10, $\mathcal{G}_\psi(\chi(p))$ denotes the Gauss sum of $\chi$ outside $p$. Let us explain the modifying factors and $\Omega_{\Pi'}$ in Theorem 2.14. Denote the highest weight of $V'$ by $\mu = \{\mu^i\}_{i \in \mathbb{Z}^n} \in (\mathbb{Z}^{2n})^{\mathbb{C}_k}$, where every $\mu^i$ is a sequence of integers $\mu^1 \geq \mu^2 \geq \cdots \geq \mu^2_n$. Then

(2.33) $\Upsilon_{\Pi'}(\chi,) := i^{-\sum_{i \in \mathbb{Z}^n} \sum_{i=1}^n (\mu^i - w_i)}.$

The refinement $\Pi'$ can be also viewed as a character of the torus $L = P/N$. Assume that conjugation of $P$ to the lower triangular Borel subgroup of $G$ takes $\Pi'$ to a character of the form

$$
\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_2n, \eta_{\Pi}^{-1}),
$$

where $\kappa_i = \otimes \psi_{\kappa_i}$ and $\kappa_{i, \psi}$ is a character of $k^{\times} \psi$ (i = 1, 2, ..., 2n). Then

(2.34) $\Upsilon_{\Pi'}(\chi_p) := \frac{1}{L(\frac{1}{2}, \Pi_p \otimes \chi_p)} \prod_{i=1}^n (n + 1 - i, \kappa_{i, \psi}^{-1}, \psi_{\kappa, \psi}^{-1}).$

Note that $\Upsilon_{\Pi'}$ is an algebraic function on $X_p$ by Proposition 10.8.

Recall that $c_p = \prod_{\psi \mid p} c_\psi$, as in Section 2.4. Write

(2.35) $\omega_{\psi}(\kappa) := \prod_{\psi \mid p} \omega_{\psi}(\kappa_{i, \psi}),$ \hspace{1cm} where \hspace{1cm} $\omega_{\psi}(\kappa_{i, \psi}) := \prod_{i=1}^n (i, \psi_{\psi})^{-1}.$

Define

(2.36) $\Omega_{\Pi'} := c_p^\psi \cdot \omega_{\psi}(\kappa).$

Using the explicit formula of $\Upsilon_{\Pi'}(\chi_p)$, we also determine the exceptional zeros of $\mathcal{L}_\Pi$ (Proposition 10.10).
Example 2.15. Let $E$ be an elliptic curve over $\mathbb{Q}$ without complex multiplication. Then $\text{Sym}^{2n-1} E$ $(n \geq 1)$ corresponds to an irreducible regular algebraic cuspidal automorphic representation $\pi$ of $\text{GL}_{2n}(\mathbb{A})$ such that

$$L(s, \pi) = L\left(s + n - \frac{1}{2}, \text{Sym}^{2n-1} E\right).$$

Moreover, $\pi$ has a nonzero $(\eta, \psi)$-Shalika integral with $\eta = 1$.

Assume that $E$ is $p$-ordinary. Given a ramification type $\varepsilon$ in (1.3), we have a $p$-adic $L$-function $L_{\Pi}$ as in Theorem 2.14. By Proposition 10.10, if $\chi^T$ is an exceptional zero of $L_{\Pi}$, where $\chi = \otimes \chi_{\ell} \in \mathcal{X}(\varepsilon)$ is a finite order character of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$, then $\chi_p$ is unramified and $\chi_p(p)$ is a root of unity. There are two cases as in what follows.

If $E$ has good ordinary reduction at $p$ and $\alpha_p$ is the corresponding $p$-adic unit as in Example 2.11, then $\chi_p(p) = \alpha_p^{-1}$, which is impossible since $\alpha_p$ cannot be a root of unity. If $E$ has multiplicative reduction at $p$, then $\chi_p(p) = 1$ or $-1$ according to the reduction is split or nonsplit, in which cases $\chi_p$ is a simple zero of $\Upsilon_{\Pi}$. These results are also consistent with the expectations in terms of Galois representations as in [Gr94, Page 170] (however the construction of $L_{\Pi}$ was not given there), which in particular assert that the trivial character is an exceptional zero of $L_{\Pi}$.

It is worth emphasizing that in contrast to earlier works [AG94, G18, DJR20], our evaluation of $\Upsilon_{\Pi}^p(\chi_p)$ is again by the technique that compares the Friedberg-Jacquet integral with a new integral over the open orbit of the spherical subgroup $\dot{G}$ acting on the flag variety of $G$. This is given by Theorem 10.6 whose proof is based on an application of the Godement-Jacquet integrals [GJ72].

2.7. Historical remarks. Assuming the existence of $V$-balanced characters and rather restrictive ramification conditions, the $p$-adic $L$-functions for $\text{GL}_n \times \text{GL}_{n-1}$ and $\text{GL}_{2n}$ have been constructed in the previous works such as [KMS00, Jan11, Jan15, Jan16, Jan24, Sch93, Sch01] and [AG94, DJR20] among many others, mainly using a different approach called the local Birch lemma. In these works the $p$-adic $L$-functions are evaluated for Hecke characters of the form $\chi_0|\cdot|_k^j$, where $\chi_0$ is of finite order and $j$ is a critical place. In the resulting modifying factors at $p$ one does not see the local $L$-factors as in our modifying factor $\Upsilon_{\Pi}^p(\chi_p)$, but only the local $\varepsilon$-factors which essentially account for the conductor and Gauss sum of $\chi_p$. Thus the connection with the conjecture by Coates and Perrin-Riou is not clear in general. In particular the phenomenon of exceptional zeros does not occur in these works.

For the case of $\text{GL}_{2n}$, in [G18] and more recent works [BDW, BDGJW], under the conditions that the number field $k$ is totally real, $V$-balanced characters exist, and $\Pi_p$ is unramified or parahoric spherical, the $p$-adic $L$-functions are constructed and
the modifying factors are evaluated which are compatible with the general result in Theorem 2.14.

Our construction of $p$-adic L-functions $\mathcal{L}_\Pi$ in Theorems 2.10, 2.12 and 2.14 are different from the previous works in the following aspects.

- The construction of $\mathcal{L}_\Pi$ does not require the existence of $V$-balanced characters. When there is a $V$-balanced character, $\mathcal{L}_\Pi(\chi^b)$ interpolates the classical L-function $\mathcal{L}_\Pi(\chi)$ for all critical algebraic Hecke characters $\chi$ that are not necessarily of the form $\chi_0|_{A_k}^1$ as above.

- Bases on the recent works of the authors and their collaborators, the modifying factors $\Upsilon_{\Pi,\infty}(\chi_\infty)$ at $\infty$ are explicitly calculated and are consistent with the conjectures of Deligne and Blasius, which was not known in previous constructions of $p$-adic L-functions for higher rank groups.

- The modifying factors $\Upsilon_{\Pi,p}(\chi_p)$ at $p$ are defined in a general setting and are explicitly calculated, which are consistent with the conjecture of Coates and Perrin-Riou. The explicit calculation is based on the preparatory work in [LLSS23] of the authors and their collaborators, and the idea of integrations over open orbits in loc. cit. is applicable to other cases (for example, the third family of examples in this paper).

- To determine the exceptional zeros is an important open problem in the theory of $p$-adic L-functions which has been long-standing. Since the modifying factors $\Upsilon_{\Pi,p}(\chi_p)$ at $p$ are explicit, we are able to solve this problem for these cases in Propositions 8.9, 9.14 and 10.10

- Neither ramification conditions on $\Pi_p$ nor assumptions on the number field $k$ are required.

- The method of constructing $p$-adic L-functions is completely general and is potentially applicable to many other cases.

The idea of studying modular symbols of formal vectors as in (1.14) and their $p$-adic interpolations (Theorem 1.5) was first presented by the second named author in the conference on Gan-Gross-Prasad conjecture held in CNRS, Paris, in 2014. This idea is also our guiding principle in the calculations of the modifying factors using principal series representations. However, it took the authors and their collaborators quite a long time to explicitly calculate the modifying factors (both at $\infty$ and at $p$) in the three families of examples considered in this paper. This is the main reason for the delay of the appearance of the current paper.

3. Relative cohomologies and relative completed cohomologies

Recall the notation from Section 1.1. In particular, $G$ is an arbitrary linear algebraic group defined over $\mathbb{Q}$. Fix a closed subgroup of $G(\mathbb{R})$ of the form

$$K_\infty := A(\mathbb{R})^p \cdot K_{\infty}^\circ,$$

where $A$ is a split torus in $G$ defined over $\mathbb{Q}$ that is central in the identity connected component of $G$ modulo its unipotent radical, and $K_{\infty}^\circ$ is a compact subgroup that
normalizes \( A(\mathbb{R})^0 \). As in the Introduction, write
\[
G^\natural := K^\natural \times G(\mathbb{A}^\infty) = G^p \times G,
\]
where \( G^p := K^\natural \times G(\mathbb{A}^{\infty p}) \) and \( G := G(\mathbb{Q}_p) \). Set
\[
\mathcal{X} := G(\mathbb{A})/K^\infty = (G(\mathbb{R})/K^\infty) \times G(\mathbb{A}^\infty),
\]
which carries a left action of \( G(\mathbb{Q}) \) and a right action of \( G^\natural \) that commute with each other. For each open compact subgroup \( \tilde{K} \) of \( G^\natural \), we define a topological space
\[
S^G_\tilde{K} := G(\mathbb{Q}) \backslash \mathcal{X}/\tilde{K},
\]
which is known to be Hausdorff.

Let \( G \) be an open subgroup of \( G^\natural \), \( R \) a commutative ring with identity, and \( M \) an \( R[G(\mathbb{Q}) \times G] \)-module. Recall that \( p \) is a Lie subalgebra of the Lie algebra \( g \) of \( G \). Let \( s \subset p \) be a subalgebra. In the rest of this paper, unless otherwise specified,

(3.1) \[
\begin{align*}
D &\text{ denotes an open compact subgroup of } G^p \cap G; \\
\mathfrak{G} &\text{ denotes an open compact subgroup of } G \cap G; \\
\mathcal{P} &\text{ denotes a compact subgroup of } G \cap G \text{ with Lie algebra } p; \\
\mathfrak{S} &\text{ denotes a compact subgroup of } G \cap G \text{ with Lie algebra } s.
\end{align*}
\]

3.1. Neat elements. Here we review some basic notions concerning neat elements, following \cite{Pi89}. For every field \( k \) of characteristic 0, and every element \( x \in G(k) \), we define a finite subgroup \( \text{Unit}(x) \) of \( \overline{k}^\times \) as in what follows. Fix a faithful finite-dimensional algebraic representation \( V_k \) of \( G_k \) over \( k \), and a field embedding \( \iota_k : \overline{k} \rightarrow k \), where \( k \) is an algebraic closure of \( k \). All the eigenvalues of the linear operator \( x : V_k \otimes \overline{k} \rightarrow V_k \otimes \overline{k} \) generate a subgroup of \( \overline{k}^\times \). We define \( \text{Unit}(x) \) to be the inverse image under \( \iota_k \) of the set of all torsion elements in this subgroup. This is a finite subgroup of \( \overline{k}^\times \), which is independent of the representation \( V_k \) and the embedding \( \iota_k \).

For every \( x^\infty = (x_\ell)_{\ell \neq \infty} \in G(\mathbb{A}^\infty) \)
we define
\[
\text{Unit}(x^\infty) := \bigcap_{\ell \neq \infty} \text{Unit}(x_\ell).
\]
We say that \( x^\infty \) is neat if the group \( \text{Unit}(x^\infty) \) is trivial.

**Definition 3.1.** (a) A compact subgroup of \( G^\natural \) is said to be neat if it is contained in a group of the form \( K^\natural_{\infty} \times K^\infty \), where \( K^\infty \) is an open compact subgroup of \( G(\mathbb{A}^\infty) \) consisting of neat elements. It is said to be completely neat if it is neat and its projection to \( K^\natural_{\infty} \) preserves a \( G(\mathbb{R}) \)-invariant orientation on \( G(\mathbb{R})/K^\infty \).
(b) A compact subgroup of \( C \) of \( G \) is said to be \( D \)-neat if \( DC \) is neat.
3.2. The transfer map of cohomology groups. We will consider cohomologies with support conditions. For this purpose, let $\Phi$ be a family of closed subsets of $G(\mathbb{Q}) \backslash \mathscr{X}$ with the following properties:

- every closed subset of a set in $\Phi$ belongs to $\Phi$;
- the union of two sets in $\Phi$ belongs to $\Phi$;
- for every set $Z \in \Phi$ and every compact subset $X$ of $G^c$, the set $Z.X$ belongs to $\Phi$.

For each open compact subgroup $K$ of $G$, we have a sheaf $M_{[K]}$ of $R$-modules over the topological space $S^G_K$ such that for every open subset $U$ of $\mathscr{X}$ that is left $G(\mathbb{Q})$-invariant and right $K$-invariant, $M_{[K]}(G(\mathbb{Q}) \backslash U/K)$ equals

$$\{G(\mathbb{Q}) \times K\text{-equivariant locally constant maps from } U \text{ to } M\}.$$

When $K$ is neat, $S^G_K$ is a topological manifold and $M_{[K]}$ is a locally constant sheaf.

For simplicity, write

$$H^i_{\Phi}(K, M) := H^i_{\Phi/K}(S^G_K, M_{[K]}), \quad (i \in \mathbb{Z})$$

for the $i$-th cohomology of $S^G_K$ with coefficient $M_{[K]}$ and support $\Phi/K$, where $\Phi/K$ denotes the family of closed subsets $Y$ of $S^G_K$ such that the preimage of $Y$ under the quotient map $G(\mathbb{Q}) \backslash \mathscr{X} \to S^G_K$ belongs to $\Phi$. As usual, we drop the subscript $\Phi$ to indicate the cohomology group with arbitrary support, and replace $\Phi$ by “c” to indicate the cohomology group with compact support. For example, $H^i(K, M) = H^i(S^G_K, M_{[K]})$ and $H^i_c(K, M) = H^i_c(S^G_K, M_{[K]})$.

Let $K'$ be another open compact subgroup of $G$. First assume that $K' \subset K$, so that we have the natural map of topological spaces

$$f_{K',K} : S^G_{K'} \to S^G_K.$$

We will use a subscript $*$ to indicate the push-forward of sheaves. For instance, $f_{K',K,*}$ denotes the push-forward of sheaves by the map $f_{K',K}$. Put

$$M' := \text{Ind}^{G(\mathbb{Q}) \times K'}_{G(\mathbb{Q}) \times K}(M_{|G(\mathbb{Q}) \times K'}) = \text{Ind}^{K'}_{K}(M_{|K'}),$$

regarded as an $R[G(\mathbb{Q}) \times K']$-module.

**Lemma 3.2.** There is a canonical isomorphism of sheaves $M'_{[K]} \cong f_{K',K,*}M_{[K']}$ on $S^G_K$.

**Proof.** We realize $M'$ as the space of maps $f : K \to M$ such that

$$f(k'k_0) = k'.f(k_0) \quad \text{for all } k' \in K', \ k_0 \in K,$$

on which $K$ acts by right translations. Let $U$ be an open subset of $S^G_K$, with preimage $\tilde{U}$ in $\mathscr{X}$. The set of sections $M'_{[K]}(U)$ consists of the $G(\mathbb{Q})$-equivariant maps $s : \tilde{U} \to M'$ such that

$$s(xk^{-1}) = k.s(x) \quad \text{for all } x \in \tilde{U}, \ k \in K,$$
i.e., \( s(xk^{-1})(k_0) = s(x)(k_0k) \) for any \( k_0 \in K \). On the other hand, \((f_{K',K},sM_{[K]})(U)\) consists of the \(G(\mathbb{Q})\)-equivariant maps \( s' : \tilde{U} \to M \) such that
\[
s'(xk^{-1}) = k'.s(x) \quad \text{for all } x \in \tilde{U}, \; k' \in K'.
\]
It is easy to verify that we have mutually inverse maps
\[
M'_{[K]}(U) \to (f_{K',K},sM_{[K']})(U),
\]
\[
s \mapsto (s' : \tilde{U} \to M, \; x \mapsto s(x)(1)),
\]
and
\[
(f_{K',K},sM_{[K']})(U) \to M'_{[K]}(U),
\]
\[
s' \mapsto \left(s : \tilde{U} \to M', \; (x \mapsto (K \to M, \; k_0 \mapsto s'(xk^{-1}_0)))\right).
\]
Thus the sheaves \( M'_{[K]} \) and \( f_{K',K},sM_{[K']} \) are canonically isomorphic. \( \Box \)

Since the fibers of the map \( f_{K',K} \) are finite, by Lemma 3.2 and [Br97, II. Theorem 11.1], we have a canonical isomorphism

(3.4) \[ H^i_\phi(K, M') \cong H^i_\phi(K', M). \]

The \( K \)-homomorphism given by the “orbital map”,

(3.5) \[ \phi : M \to M', \quad v \mapsto (K \to M, \; k \mapsto k.v), \]

induces a homomorphism

(3.6) \[ H^i_\phi(K, M) \to H^i_\phi(K, M'). \]

The composition of (3.6) and (3.4) defines the pull-back homomorphism

\[ \rho_{K,K'} : H^i_\phi(K, M) \to H^i_\phi(K', M). \]

On the other hand, the \( K \)-homomorphism given by the map

(3.7) \[ \phi' : M' \to M, \quad f \mapsto \sum_{kK' \in K/K'} k.f(k^{-1}) \]

induces a homomorphism

(3.8) \[ H^i_\phi(K, M') \to H^i_\phi(K, M). \]

The composition of (3.8) and the inverse of (3.4) defines the push-forward homomorphism

\[ \rho_{K',K} : H^i_\phi(K', M) \to H^i_\phi(K, M). \]

**Lemma 3.3.** Assume that \( K' \subset K \) are open compact subgroups of \( G \). Then

\[ \rho_{K',K} \circ \rho_{K,K'} : H^i_\phi(K, M) \to H^i_\phi(K, M) \]

equals the multiplication by \([K : K']\).
Proof. This follows immediately from the fact that the following composition of (3.5) and (3.7),
\[
\phi' \circ \phi : M \to M
\]
is the multiplication by \([K : K']\).

Now we drop the assumption that \(K' \subset K\) and define the homomorphism
(3.9) \(\rho_{K,K'} : \H^i_{\Phi}(K, M) \to \H^i_{\Phi}(K', M)\)
to be the composition of
\[
\H^i_{\Phi}(K, M) \xrightarrow{\rho_{K,K',K'}} \H^i_{\Phi}(K \cap K', M) \xrightarrow{\rho_{K \cap K',K'}} \H^i_{\Phi}(K', M).
\]
We call the above map the transfer map. When the groups \(K\) and \(K'\) are understood, we also write \(\rho\) for \(\rho_{K,K'}\).

Lemma 3.4. Let \(K_1, K_2, K_3\) be open compact subgroups of \(G\) such that
\[
K_2 = (K_1 \cap K_2)(K_2 \cap K_3).
\]
Then the diagram
\[
\begin{array}{ccc}
H^i_{\Phi}(K_1, M) & \longrightarrow & H^i_{\Phi}(K_2, M) \\
\downarrow & & \downarrow \\
H^i_{\Phi}(K_3, M) & \xrightarrow{[K_1 \cap K_3 : K_1 \cap K_2 \cap K_3]} & H^i_{\Phi}(K_3, M)
\end{array}
\]
commutes, where the bottom horizontal arrow is the multiplication map by \([K_1 \cap K_3 : K_1 \cap K_2 \cap K_3]\), and all the other arrows are the transfer maps.

Proof. This is an exercise in algebraic topology, by using Lemma 3.3. We omit the details.

3.3. Formal cohomologies.

Definition 3.5. For each compact subgroup \(C\) of \(G\), the formal cohomology group is defined to be
(3.10) \(H^i_{\Phi}(C, M) := \lim_{\leftarrow K} H^i_{\Phi}(K, M)\),
where \(K\) runs over all open compact subgroups of \(G\) containing \(C\), and the implicit homomorphisms are the push-forward maps.

When \(C = K\) is open in \(G\), the formal cohomology group coincides with the previously defined module \(H^i_{\Phi}(K, M)\). When \(C\) is neat, in the definition of the formal cohomology (3.10), we may also let \(K\) run over all neat open compact subgroups of \(G\) containing \(C\).
Let $C$ and $C'$ be two compact subgroups of $G$ that are commensurable with each other, namely their intersection is open in both of these two groups. Generalizing the transfer map (3.9), we define the transfer map

$\rho_{C,C'} : H^i_\Phi(C, M) \to H^i_\Phi(C', M)$

as in the following proposition.

**Proposition 3.6.** There exists a unique homomorphism

$\rho_{C,C'} : H^i_\Phi(C, M) \to H^i_\Phi(C', M)$

with the following property: the diagram

$$
\begin{array}{ccc}
H^i_\Phi(C, M) & \xrightarrow{\rho_{C,C'}} & H^i_\Phi(C', M) \\
\downarrow & & \downarrow \\
H^i_\Phi(K, M) & \xrightarrow{\rho_{K,K'}} & H^i_\Phi(K', M)
\end{array}
$$

commutes for all open compact subgroups $K$ and $K'$ of $G$ such that $C \subset K$, $C' \subset K'$ and that the natural map

$C/(C \cap C') \to K/(K \cap K')$

is bijective.

**Proof.** Set

$\mathcal{G}_{C,G} := \{\text{open compact subgroups of } G \text{ containing } C\}$

and define $\mathcal{G}_{C',G}$ similarly. The uniqueness follows from the following assertion: for every $K'' \in \mathcal{G}_{C',G}$, there exist $K \in \mathcal{G}_{C,G}$ and $K' \in \mathcal{G}_{C',G}$ such that $K' \subset K''$ and the natural map

$C/(C \cap C') \to K/(K \cap K')$

is bijective. For the proof of this assertion, take $K' \in \mathcal{G}_{C',G}$ small enough so that

$K' \subset K'' \quad \text{and} \quad K' \cap C = C' \cap C.$

Note that such $K'$ exists because $C$ and $C'$ are commensurable. Let $K_0$ be an open compact subgroup of $G$ that is normalized by $C$ and is small enough so that

$K_0 \subset K' \quad \text{and} \quad ((C \setminus C')K_0) \cap K' = \emptyset.$

Take $K := CK_0$. Then

$K \cap K' = (C \cap C')K_0$

and the map (3.12) is bijective. This proves the assertion.
To prove the existence, it suffices to show that the diagram
\[
\begin{array}{ccc}
H^i_\Phi(K_1, M) & \xrightarrow{\rho} & H^i_\Phi(K_1', M) \\
\downarrow{\rho} & & \downarrow{\rho} \\
H^i_\Phi(K, M) & \xrightarrow{\rho} & H^i_\Phi(K', M)
\end{array}
\]
commutes for all \(K, K_1 \in \mathcal{G}_{\mathbb{C}, G}\) and all \(K', K_1' \in \mathcal{G}_{\mathbb{C}', G}\) such that
\[
K_1 \subset K, \quad K_1' \subset K'
\]
and that the natural maps
\[
C/(C \cap C') \to K/(K \cap K') \quad \text{and} \quad \frac{C}{(C \cap C')} \to \frac{K_1}{(K_1 \cap K_1')}
\]
are bijective. Note that the above conditions imply that the natural map
\[
\frac{K_1}{(K_1 \cap K_1')} \to \frac{K}{(K \cap K')}
\]
is bijective. Thus \(K_1 \cap K \cap K' = K_1 \cap K_1'\), or equivalently
\[
K_1 \cap K' = K_1 \cap K_1' \cap K'.
\]
The proposition then follows from Lemma 3.4. □

We call the map (3.11) a pull-back map or a push-forward map when \(C \supset C'\) or \(C \subset C'\) respectively. Note that the \(R\)-module \(H^i_\Phi(C, M)\) is functorial in \(M\), and the transfer map (3.11) is natural in \(M\), namely the diagram
\[
\begin{array}{ccc}
H^i_\Phi(C, M) & \xrightarrow{\rho_{C, C'}} & H^i_\Phi(C', M) \\
\downarrow{\phi} & & \downarrow{\phi} \\
H^i_\Phi(C, M') & \xrightarrow{\rho_{C, C'}} & H^i_\Phi(C', M')
\end{array}
\]
commutes for all \(R[G(\mathbb{Q}) \times G]\)-modules \(M\) and all \(R[G(\mathbb{Q}) \times G]\)-module homomorphisms \(\phi : M \to M'\).

The following lemma is a generalization as well as a consequence of Lemma 3.4 whose proof is omitted.

**Lemma 3.7.** Let \(C_1, C_2, C_3\) be compact subgroups of \(G\) that are pairwise commensurable with each other. Assume that
\[
C_2 = (C_1 \cap C_2)(C_2 \cap C_3).
\]
Then the diagram
\[
\begin{array}{ccc}
H^i_\Phi(C_1, M) & \longrightarrow & H^i_\Phi(C_2, M) \\
\downarrow & & \downarrow \\
H^i_\Phi(C_3, M) & \xrightarrow{\rho_{C_1 \cap C_3 : C_1 \cap C_2 \cap C_3}} & H^i_\Phi(C_3, M)
\end{array}
\]
commutes, where the bottom horizontal arrow is the multiplication map by \([C_1 \cap C_3 : C_1 \cap C_2 \cap C_3]\), and all the other arrows are the transfer maps.

Recall the convention from (3.1).

**Definition 3.8.** The relative cohomology groups are defined to be

\[
H^i_\varphi(D_\mathcal{S}, M)^{(p)} := \lim_{\varphi \supset \mathcal{S}} H^i_\varphi(D_\mathcal{P}, M)
\]

and

\[
(3.13) \quad H^i_\varphi(G, M)^{(p \supset s)} := \lim_{D_\mathcal{S}} H^i_\varphi(D_\mathcal{S}, M)^{(p)}.
\]

Here the transition maps in (3.13) are the pull-back maps, namely the maps that make all the following diagrams commutative. Here \(D'\) is an open subgroup of \(D\), \(\mathcal{S}'\) is an open subgroup of \(\mathcal{S}\), \(\mathcal{P} \supset \mathcal{S}\), and \(\mathcal{P}'\) is an open subgroup of \(\mathcal{P}\) containing \(\mathcal{S}'\). In this paper, some obvious maps such as the two vertical arrows in the above diagram, will not be named.

Note that the group (3.13) is independent of \(s\) in the sense that the natural homomorphism

\[
(3.14) \quad H^i_\varphi(G, M)^{(p \supset s)} \to H^i_\varphi(G, M)^{(p)} := \lim_{D_\mathcal{P}} H^i_\varphi(D_\mathcal{P}, M)
\]

is an isomorphism. Specializing to the case that \(p = g\), we define

\[
H^i_\varphi(G, M) := H^i_\varphi(G, M)^{(g)} = \lim_{K} H^i_\varphi(K, M),
\]

where \(K\) runs over all open compact subgroups of \(G\).

**Definition 3.9.** The relative completed cohomology groups are defined to be

\[
(3.15) \quad \widetilde{H}^i_\varphi(D_\mathcal{S}, M)^{(p)} := \lim_{k \in \mathbb{N}} H^i_\varphi(D_\mathcal{S}, M/P^k)^{(p)},
\]

and

\[
(3.16) \quad \widetilde{H}^i_\varphi(G, M)^{(p \supset s)} := \lim_{D_\mathcal{S}} \widetilde{H}^i_\varphi(D_\mathcal{S}, M)^{(p)}.
\]
Here the transition maps in (3.16) are the pull-back maps. We have an obvious commutative diagram

\[
\begin{array}{ccc}
\tilde{H}^i_{\phi}(D\mathfrak{S}, M) & \longrightarrow & \tilde{H}^i_{\phi}(D\mathfrak{S}, M) \\
\downarrow & & \downarrow \\
H^i_{\phi}(G, M) & \longrightarrow & \tilde{H}^i_{\phi}(G, M) \\
\end{array}
\]

Relative completed cohomology groups are most interesting when \( M \) is \( p \)-smooth as an \( R[G]\)-module (Definition 1.4). When this is the case, the \( R \)-module \( (3.15) \) (and hence \( (3.16) \)) only depends on the \( R[G] \)-module structure on \( M \).

Remark 3.10. By using the trivial actions of small open compact subgroups of \( G \), the assignments \( \tilde{H}^i_{\phi}(D\mathfrak{S}, \cdot)^{(p)} \) and \( \tilde{H}^i_{\phi}(G, \cdot)^{(p\supset s)} \) are both functors from the category of \( R[G(\mathbb{Q}) \times D\mathfrak{S}] \)-modules that are \( p \)-smooth as \( R[S] \)-modules to the category of \( R \)-modules.

Example 3.11. When \( p = g, s = \{0\} \), and \( M = \mathbb{O} \) is the trivial module, the group \( (3.16) \) agrees with the completed cohomology group introduced by Emerton in [Em06a].

For every compact subgroup \( C \) of \( G \), we define

\[
\tilde{H}^i_{\phi}(C, M) := \lim_{\rightarrow k \in \mathbb{N}} H^i_{\phi}(C, M/p^k).
\]

Example 3.12. We have that

\[
\tilde{H}^i_{\phi}(D\mathfrak{P}, M)^{(p\supset p)} = \tilde{H}^i_{\phi}(D\mathfrak{P}, M)
\]

and so that

\[
\tilde{H}^i_{\phi}(G, M)^{(p\supset p)} = \lim_{\rightarrow D\mathfrak{P}} \tilde{H}^i_{\phi}(D\mathfrak{P}, M).
\]

Example 3.13. The \( \mathbb{O} \)-module \( \tilde{H}^0_{\phi}(G, \mathbb{O})^{(p\supset s)} \) is identified with the \( \mathbb{O} \)-module of all continuous functions \( f : G(\mathbb{Q}) \backslash G \rightarrow \mathbb{O} \) that are invariant under some groups of the form \( D\mathfrak{P} \).

3.4. Hecke maps. Suppose that \( H^+ \) is a submonoid of \( G^\natural \). It is also viewed as a submonoid of \( G(\mathbb{Q}) \times G^\natural \).

Definition 3.14. A compatible action of \( H^+ \) on \( M \) is an \( R \)-linear action

\[
H^+ \times M \rightarrow M, \quad (t, x) \mapsto t \ast x
\]

such that the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{g} & M \\
\downarrow t_* & & \downarrow t_* \\
M & \xrightarrow{tg^{-1}} & M
\end{array}
\]
commutes for all \( t \in H^+ \) and \( g \in G(\mathbb{Q}) \times G \) with \( tgt^{-1} \in G(\mathbb{Q}) \times G \).

Assume that we are given a compatible action (3.18). Let \( t \in H^+ \). Let \( K, K' \) be open compact subgroups of \( G \). Generalizing the definition of \( \rho_{K,K'} \) in (3.9), we shall define in what follows a homomorphism

\[
(3.20) \quad \rho^*_t := \rho^*_{t,K,K'} : H^i_{\Phi}(K, M) \to H^i_{\Phi}(K', M).
\]

We call this map the Hecke map of \( t \).

It follows from the commutative diagram (3.19) that the map

\[
\mathcal{X} \times M \to \mathcal{X} \times M, \quad (x, u) \mapsto (xt^{-1}, t * u)
\]
is equivariant with respect to the isomorphism

\[
G(\mathbb{Q}) \times (G \cap t^{-1}Gt) \to G(\mathbb{Q}) \times (tGt^{-1} \cap G), \quad g \mapsto tgt^{-1}.
\]

Thus it descends to a homomorphism

\[
(3.21) \quad M_{[K \cap t^{-1}K' \cap]} \to M_{[tKt^{-1} \cap K']}
\]
of sheaves over the homeomorphism

\[
S_{K \cap t^{-1}K' \cap}^G \to S_{tKt^{-1} \cap K'}^G,
\]
the class of \( x \in \mathcal{X} \) \( \mapsto \) the class of \( xt^{-1} \).

Write

\[
\rho^*_t : H^i_{\Phi}(K \cap t^{-1}K't, M) \to H^i_{\Phi}(tKt^{-1} \cap K', M),
\]
for the homomorphism induced by (3.21). We define the homomorphism \( \rho^*_t := \rho^*_{t,K,K'} \) in (3.20) to be the composition of

\[
H^i_{\Phi}(K, M) \to H^i_{\Phi}(K \cap t^{-1}K't, M) \stackrel{\rho^*_t}{\to} H^i_{\Phi}(tKt^{-1} \cap K', M) \to H^i_{\Phi}(K', M),
\]
where the first arrow is the pull-back map and the third arrow is the push-forward map.

Hecke maps are natural in the coefficient modules, as in the following lemma.

**Lemma 3.15.** Let \( M' \) be another \( R[G(\mathbb{Q}) \times G] \)-module with a compatible action of \( H^+ \). Let \( \phi : M \to M' \) be an \( H^+ \)-equivariant homomorphism of \( R[G(\mathbb{Q}) \times G] \)-modules. Then for all open compact subgroups \( K, K' \) of \( G \), and all \( t \in H^+ \), the diagram

\[
\begin{array}{ccc}
H^i_{\Phi}(K, M) & \xrightarrow{\rho^*_t} & H^i_{\Phi}(K', M) \\
\phi \downarrow & & \phi \downarrow \\
H^i_{\Phi}(K, M') & \xrightarrow{\rho^*_t} & H^i_{\Phi}(K', M')
\end{array}
\]

commutes.

**Proof.** This is elementary and we omit its proof. \( \square \)
Proposition 3.16. Let $K_1, K_2, K_3$ be open compact subgroups of $G$ and let $s, t \in H^+$. If 

$$K_2 = (sK_1s^{-1} \cap K_2)(K_2 \cap t^{-1}K_3t),$$

then

$$\rho^t_{t,K_2,K_3} \circ \rho^s_{s,K_1,K_2} = [sK_1s^{-1} \cap t^{-1}K_3t : sK_1s^{-1} \cap K_2 \cap t^{-1}K_3t] \rho^s_{s,K_1,K_3}$$

as homomorphisms from $H^i_\Phi(K_1, M)$ to $H^i_\Phi(K_3, M)$.

Proof. For simplicity write $r := ts$ and $K_0 := sK_1s^{-1} \cap K_2 \cap t^{-1}K_3t$.

Then the proposition follows from considering the following commutative diagram:

Here we write $H(K_1)$ for $H^i_\Phi(K_1, M)$ and similarly for other cohomology groups, $(\ast)$ is the multiplication map by $[sK_1s^{-1} \cap t^{-1}K_3t : K_0]$, and all the unnamed arrows are the transfer maps as defined in Section 3.2. □

3.5. The representation $H^i_\Phi(G, M)$ and its formal completion. Take $H^+ := G$. Then the $R[G(Q) \times G]$-module structure on $M$ restricts to an action of $H^+$ on $M$, which is in fact a compatible action. We drop the superscript $*$ to indicate the Hecke maps attached to this compatible action:

$$\rho_t := \rho_{t,K,K'} : H^i_\Phi(K, M) \to H^i_\Phi(K', M).$$

Recall that

$$H^i_\Phi(G, M) = \lim_{\longrightarrow} H^i_\Phi(K, M),$$

where $K$ runs over all open compact subgroups of $G$, and the implicit homomorphisms in the direct limit are the pull-back maps. By using Proposition 3.16 it is easy to see that for every $t \in G$, there is a unique homomorphism

$$\rho_t : H^i_\Phi(G, M) \to H^i_\Phi(G, M), \quad \phi \mapsto t.\phi := \rho_t(\phi)$$
such that the diagram

\[
\begin{array}{ccc}
\mathbb{H}^i_{\Phi}(K, M) & \xrightarrow{\rho_t} & \mathbb{H}^i_{\Phi}(tKt^{-1}, M) \\
\downarrow & & \downarrow \\
\mathbb{H}^i_{\Phi}(G, M) & \xrightarrow{\rho_t} & \mathbb{H}^i_{\Phi}(G, M)
\end{array}
\]

commutes for all open compact subgroups $K$ of $G$. Moreover, the maps in (3.22) for various $t$ yield a representation of $G$ on $\mathbb{H}^i_{\Phi}(G, M)$. This representation is smooth in the sense that every element of $\mathbb{H}^i_{\Phi}(G, M)$ is fixed by some open subgroup of $G$. When $R$ is a $\mathbb{Q}$-algebra and $K$ is a neat open compact subgroup of $G$, [Br97, II. Section 19.1, (47)] implies that

\[
(3.23) \quad \mathbb{H}^i_{\Phi}(G, M)^K = \mathbb{H}^i_{\Phi}(K, M).
\]

In the rest of this section we assume that $R$ is a $\mathbb{Q}$-algebra so that the formal completion

\[
\mathbb{H}^i_{\Phi}(G, M) = \varprojlim_K \mathbb{H}^i_{\Phi}(G, M)^K
\]

is defined, where $K$ runs over all open compact subgroups of $G$, and the transition maps are the averaging projections. For every $\phi \in \mathbb{H}^i_{\Phi}(G, M)$ and every open compact subgroup of $K$ of $G$, write $\phi_K$ for the projection of $\phi$ to $\mathbb{H}^i_{\Phi}(G, M)^K$.

**Lemma 3.17.** For every compact subgroup $C$ of $G$, the natural map

\[
(3.24) \quad \mathbb{H}^i_{\Phi}(G, M)^C \rightarrow \varprojlim_K \mathbb{H}^i_{\Phi}(G, M)^K
\]

is an isomorphism, where $K$ runs over all open compact subgroups of $G$ containing $C$, and the transition maps are the averaging projections.

**Proof.** Note that

\[
(3.25) \quad \mathbb{H}^i_{\Phi}(G, M) = \varprojlim_{K'} \mathbb{H}^i_{\Phi}(K', M),
\]

where $K'$ runs over all open compact subgroups of $G$ normalized by $C$. Let $\{\phi_K\}_{K'}$ be an element in the codomain of the map (3.24). For every open compact subgroup $K'$ of $G$ normalized by $C$, we define

\[
\varphi_{K'} := \phi_{K'C} \in \mathbb{H}^i_{\Phi}(G, M),
\]
Then for every open subgroup $K''$ of $K'$ normalized by $C$, we have that

$$\frac{1}{[K' : K''] K' \cap K'' C:K''} \sum_{g \in K'/K''} g.\varphi_{K''} = \frac{1}{[K' : K' \cap K'' C:K'']} \sum_{g \in K' \cap K'' C:K''} g.\varphi_{K''} = \varphi_{K'C} = \phi_{K'}.$$ 

In view of the identification (3.25), now it is easily checked that the family $\{\varphi_{K'}\}_{K'}$ defines an element of $\widehat{H}_{\Phi}^i(G, M)$, and the map

$$\{\phi_K\}_K \mapsto \{\varphi_{K'}\}_{K'}$$

is an inverse of the map (3.24). □

**Formal cohomologies are related to formal completions as in the following lemma.**

**Lemma 3.18.** Suppose that $C$ is a neat compact subgroup of $G$. Then

$$H^i_{\Phi}(G, M)^C \otimes D(G^\natural) \rightarrow H^i_{\Phi}(C, M), \quad \phi \otimes \mu \mapsto \{\mu(K) \cdot \varphi_K\}_K$$

is a well-defined $R$-module isomorphism, where $K$ runs over all neat open compact subgroups of $G$ containing $C$, and $\varphi_K$ denotes the image of $\phi$ under the projection map $H^i_{\Phi}(G, M) \rightarrow H^i_{\Phi}(G, M)^K = H^i_{\Phi}(K, M)$.

**Proof.** In view of (3.23), this is a direct consequence of Lemma 3.17. □

**Lemma 3.19.** Suppose that $C$ is a neat compact subgroup of $G$ and $C'$ is an open subgroup of $C$. Then the diagram

$$\begin{array}{ccc}
H^i_{\Phi}(G, M)^C \otimes D(G^\natural) \otimes D(C)^\vee & \xrightarrow{\text{evaluation at } \mu_C} & H^i_{\Phi}(C, M) \\
\downarrow \text{inclusion} & & \downarrow \text{pull-back} \\
H^i_{\Phi}(G, M)^{C'} \otimes D(G^\natural) \otimes D(C)^\vee & \xrightarrow{\text{evaluation at } \mu_{C,C'}} & H^i_{\Phi}(C', M)
\end{array}$$

commutes. Here $\mu_C \in D(C)$ denotes the normalized Haar measure, and $\mu_{C,C'} \in D(C)$ denotes the Haar measure on $C$ whose restriction to $C'$ is the normalized Haar measure.
Proof. This is routine to verify, by reducing to the case when $C'$ is a normal subgroup of $C$. □

Recall from (1.6) that

$$\hat{H}^i(\overline{G}, M)_{p-sm} \subset \hat{H}^i(\overline{G}, M)$$

is the submodule consisting of the elements that are fixed by some groups of the form $D\mathfrak{P}$. When $D\mathfrak{P}$ is neat, by Lemma 3.18, we have an isomorphism

$$\hat{H}^i(\overline{G}, M) \otimes D(\mathfrak{p}/p) \cong \hat{H}^i(D\mathfrak{P}, M),$$

where $\mathfrak{P}$ runs over all $D$-neat open compact subgroups of $G \cap G$ containing $\mathfrak{P}$, $\mu_{\mathfrak{P}} \in D(\mathfrak{p}) = D(\mathfrak{P})$ is the measure with $\mu_{\mathfrak{P}}(\mathfrak{P}) = 1$, $\phi_{D\mathfrak{P}}$ denotes the projection of $\phi$ to $\hat{H}^i(\overline{G}, M)_{D\mathfrak{P}} = \hat{H}^i(D\mathfrak{P}, M)$, and as before we use the identification

$$D(\mathfrak{p}/p) = D(\mathfrak{p}) \otimes D(\mathfrak{P}) = D(\mathfrak{G}) \otimes D(\mathfrak{P}).$$

The following lemma is a direct consequence of Lemma 3.19.

**Lemma 3.20.** There is a unique isomorphism

$$\hat{H}^i(\overline{G}, M)_{p-sm} \otimes D(\mathfrak{p}/p) \cong \hat{H}^i(G, M)_{(p)}$$

such that the diagram

$$\begin{array}{ccc}
\hat{H}^i(\overline{G}, M)_{D\mathfrak{P}} & \otimes D(\mathfrak{p}/p) & \cong \hat{H}^i(D\mathfrak{P}, M) \\
& \downarrow & \\
\hat{H}^i(\overline{G}, M)_{p-sm} \otimes D(\mathfrak{p}/p) & \cong \hat{H}^i(G, M)_{(p)}
\end{array}$$

commutes for all $D$ and $\mathfrak{P}$ such that $D\mathfrak{P}$ is neat.

3.6. **Cup product.** Let $M'$ be another $R[\mathbb{G}(\mathbb{Q}) \times \mathbb{G}]$-module so that $M' \otimes M := M' \otimes_R M$ is also an $R[\mathbb{G}(\mathbb{Q}) \times \mathbb{G}]$-module. For the main purpose of this paper, we only consider cup product by degree zero cohomologies.

**Lemma 3.21.** Let $K$ be an open compact subgroup of $\mathbb{G}$, and let $K_1, K_2$ be open subgroups of $K$. Then for each $\eta \in H^0(K, M')$, the diagram

$$\begin{array}{ccc}
H^*_K(K_1, M) & \overset{\eta \cdot (-)}{\longrightarrow} & H^*_K(K_1, M' \otimes M) \\
\rho & & \downarrow \rho \\
H^*_K(K_2, M) & \overset{\eta \cdot (-)}{\longrightarrow} & H^*_K(K_2, M' \otimes_R M)
\end{array}$$

commutes, where $\eta_i := \rho_{K,K_i}(\eta) \in H^i(K_i, M')$ $(i = 1, 2)$ and “$\cdot$” stands for the cup product.

Proof. This is an exercise in algebraic topology. We omit the details. □
By lemma 3.21, we have a unique map
\[(3.30) \quad \sim : H^0(G, M') \times H^i_\phi(G, M) \to H^i_\phi(G, M' \otimes M)\]
such that the diagram
\[
\begin{array}{ccc}
H^0(K, M') \times H^i_\phi(K, M) & \xrightarrow{\text{cup product}} & H^i_\phi(K, M' \otimes M) \\
\downarrow & & \downarrow \\
H^0(G, M') \times H^i_\phi(G, M) & \xrightarrow{\sim} & H^i_\phi(G, M' \otimes M)
\end{array}
\]
commutes for all open compact subgroups \(K\) of \(G\).

Lemma 3.21 also implies that for every compact subgroup \(C\) of \(G\), there is a unique map
\[(3.31) \quad \sim : \left( \lim_{\overset{\delta}{\longrightarrow}} H^0(K, M') \right) \times H^i_\phi(C, M) \to H^i_\phi(C, M' \otimes M),\]
where \(K\) runs over all open compact subgroups of \(G\) containing \(C\), such that for all such \(K\) the diagram
\[
\begin{array}{ccc}
H^0(K, M') \times H^i_\phi(C, M) & \xrightarrow{\sim} & H^i_\phi(C, M' \otimes M) \\
\downarrow & & \downarrow \\
H^0(K, M') \times H^i_\phi(K, M) & \xrightarrow{\text{cup product}} & H^i_\phi(K, M' \otimes M)
\end{array}
\]
commutes. Similarly, for all \(D\) and \(\mathcal{S}\), there is a unique map
\[(3.32) \quad \sim : \left( \lim_{\overset{\delta}{\longrightarrow}} H^0(D\mathcal{S}, M') \right) \times \left( \lim_{\overset{\delta}{\longrightarrow}} H^i_\phi(D\mathcal{P}, M) \right) \to \lim_{\overset{\delta}{\longrightarrow}} H^i_\phi(D\mathcal{P}, M' \otimes M)\]
such that the diagram
\[
\begin{array}{ccc}
H^0(D\mathcal{S}, M') \times H^i_\phi(D\mathcal{P}, M) & \xrightarrow{\text{cup product}} & H^i_\phi(D\mathcal{P}, M' \otimes M) \\
\downarrow & & \downarrow \\
\left( \lim_{\overset{\delta}{\longrightarrow}} H^0(D\mathcal{S}, M') \right) \times \left( \lim_{\overset{\delta}{\longrightarrow}} H^i_\phi(D\mathcal{P}, M) \right) & \xrightarrow{\sim} & \lim_{\overset{\delta}{\longrightarrow}} H^i_\phi(D\mathcal{P}, M' \otimes M)
\end{array}
\]
commutes whenever \(\mathcal{S} \supset \mathcal{P} \supset \mathcal{S}\).

By using Lemma 3.21, it is routine to check that there are three cup product maps as defined in what follows. The cup product map
\[
\sim : H^0(G, M') \times H^i_\phi(G, M) \to H^i_\phi(G, M' \otimes M)
\]
for relative cohomologies is defined by requiring that the diagram

\[
\begin{array}{ccc}
H^0(K, M') \times H^i_{\phi}(D\mathfrak{P}, M) & \xrightarrow{\sim} & H^i_{\phi}(D\mathfrak{P}, M' \otimes M) \\
\downarrow & & \downarrow \\
H^0(G, M') \times H^i_{\phi}(G, M)^{(p)} & \xrightarrow{\sim} & H^i_{\phi}(G, M' \otimes M)^{(p)}
\end{array}
\]

commutes for all \( D, \mathfrak{P} \), and all open compact subgroups \( K \) of \( G \) containing \( D\mathfrak{P} \).

The cup product map

\[
\prec: \tilde{H}^0(G, M')^{(a \geq b)} \times \tilde{H}^i_{\phi}(G, M)^{(p \geq a)} \rightarrow \tilde{H}^i_{\phi}(G, M' \otimes M)^{(p \geq a)}
\]

for relative completed cohomologies is defined by requiring that the diagram

\[
\begin{array}{ccc}
\tilde{H}^0(D\mathfrak{S}, M')^{(a)} \times \tilde{H}^i_{\phi}(D\mathfrak{S}, M)^{(p)} & \rightarrow & \tilde{H}^i_{\phi}(D\mathfrak{S}, M' \otimes M)^{(p)} \\
\downarrow & & \downarrow \\
\tilde{H}^0(G, M')^{(a \geq b)} \times \tilde{H}^i_{\phi}(G, M)^{(p \geq a)} & \rightarrow & \tilde{H}^i_{\phi}(G, M' \otimes M)^{(p \geq a)}
\end{array}
\]

commutes for all \( D \) and \( \mathfrak{S} \), where the top horizontal arrow is the inverse limit over \( k \in \mathbb{N} \) of the maps

\[
\prec: \left( \lim_{\mathfrak{S} \supset \mathfrak{S}} H^0(D\mathfrak{S}, M'/p^k) \right) \times \left( \lim_{\mathfrak{P} \supset \mathfrak{P}} H^i_{\phi}(D\mathfrak{P}, M/p^k) \right) \rightarrow \lim_{\mathfrak{P} \supset \mathfrak{P}} H^i_{\phi}(D\mathfrak{P}, (M' \otimes M)/p^k)
\]

as in (3.32).

When \( R \) is a \( \mathbb{Q} \)-algebra, the cup product map

\[
(3.33) \quad \prec: H^0(G, M') \times H^i_{\phi}(G, M) \rightarrow H^i_{\phi}(G, M' \otimes M)
\]

for the formal completion is defined by requiring that the diagram

\[
\begin{array}{ccc}
H^0(K, M') \times H^i_{\phi}(G, M) & \rightarrow & H^i_{\phi}(G, M' \otimes M) \\
\downarrow & & \downarrow \\
H^0(K, M') \times H^i_{\phi}(G, M)^{K} & \rightarrow & H^i_{\phi}(G, M' \otimes M)^{K}
\end{array}
\]

commutes for all open compact subgroups \( K \) of \( G \). Note that the map \( (3.33) \) restricts to a map

\[
\prec: H^0(G, M') \times H^i_{\phi}(G, M)_{p-sm} \rightarrow H^i_{\phi}(G, M' \otimes M)_{p-sm}.
\]

It is routine to check the following lemma by definitions.
Lemma 3.22. The diagram
\[\begin{array}{ccc}
H^0(G, M') \times H^i_\phi(G, M)^{(p)} & \rightarrow & H^i_\phi(G, M' \otimes M)^{(p)} \\
\downarrow & & \downarrow \\
\tilde{H}^0(G, M')^{(p \otimes \phi)} \times \tilde{H}^i_\phi(G, M)^{(p \otimes \phi)} & \rightarrow & \tilde{H}^i_\phi(G, M' \otimes M)^{(p \otimes \phi)}
\end{array}\]
commutes, and when R is a \(\mathbb{Q}\)-algebra, the diagrams
\[\begin{array}{ccc}
H^0(G, M') \times H^i_\phi(G, M) & \rightarrow & H^i_\phi(G, M' \otimes M) \\
\downarrow & & \downarrow \\
H^0(G, M') \times H^i_\phi(G, M) & \rightarrow & H^i_\phi(G, M' \otimes M)
\end{array}\]
and
\[\begin{array}{ccc}
H^0(G, M') \times \tilde{H}^i_\phi(G, M)_{p-sm} \otimes D(g/p) & \rightarrow & \tilde{H}^i_\phi(G, M' \otimes M)_{p-sm} \otimes D(g/p) \\
\downarrow & & \downarrow \\
H^0(G, M') \times H^i_\phi(G, M)^{(p)} & \rightarrow & H^i_\phi(G, M' \otimes M)^{(p)}
\end{array}\]
commute.

4. Pull-backs and integrations

All unexplained notations in this section will be as in the last section.

Recall the homomorphism \(i : \hat{G} \rightarrow G\) from Section 1.1. Fix a closed subgroup of \(\hat{G}(\mathbb{R})\) of the form \(\hat{K}_\infty = \hat{A}(\mathbb{R})^0 \cdot \hat{K}'_\infty\), where \(\hat{A}\) is a split torus in \(\hat{G}\) defined over \(\mathbb{Q}\) that is central in the identity connected component of \(\hat{G}\) modulo its unipotent radical, and \(\hat{K}'_\infty\) is a compact subgroup that normalizes \(\hat{A}(\mathbb{R})^0\). Assume that \(i(\hat{K}_\infty) \subset K_\infty\). Then we have a map
\[i : \mathcal{X} \rightarrow \mathcal{X}',\]
where
\[\mathcal{X}' := (\hat{G}(\mathbb{R})/\hat{K}_\infty^0) \times \hat{G}(\mathbb{A}^\infty)\]

Put
\[\hat{G} := i^{-1}(G) \subset \hat{G}^2 := \hat{K}^0_\infty \times \hat{G}(\mathbb{A}^\infty)\]

Recall that \(\hat{G} := \hat{G}(\mathbb{Q}_p)\) whose Lie algebra is denoted by \(\hat{g}\). Also recall that \(\hat{p} := i^{-1}(p) \subset \hat{g}\), and let \(\hat{s}\) be a Lie subalgebra of \(i^{-1}(s) \subset \hat{g}\).

Similar to (3.11), in the rest of this paper,
\[
\begin{align*}
\hat{D} & \text{ denotes an open compact subgroup of } \hat{G}^{\times p} \cap \hat{G}; \\
\hat{S} & \text{ denotes an open compact subgroup of } \hat{G} \cap \hat{G}; \\
\hat{Q} & \text{ denotes a compact subgroup of } \hat{G} \cap \hat{G} \text{ with Lie algebra } \hat{p}; \\
\hat{S} & \text{ denotes a compact subgroup of } \hat{G} \cap \hat{G} \text{ with Lie algebra } \hat{s}.
\end{align*}
\]
4.1. Pull-back for formal cohomologies. Let $C$ be a neat compact subgroup of $G$ and $\hat{C}$ a neat compact subgroup of $\hat{G}$ such that

- $\iota(\hat{C}) \subset C$, and
- the map $\iota : \hat{G}/\hat{C} \to G/C$ is a local homeomorphism.

**Lemma 4.1.** For every open compact subgroup $\hat{K}'$ of $\hat{G}$ containing $\hat{C}$, there exist a neat open compact subgroup $\hat{K}$ of $\hat{G}$ containing $\hat{C}$ and a neat open compact subgroup $K$ of $G$ containing $C$ such that

- $\hat{K} \subset \hat{K}'$, $\iota(\hat{K}) \subset K$, and
- the map $\iota : \hat{K}/\hat{C} \to K/C$ is bijective.

**Proof.** Without loss of generality, assume that $\hat{K}'$ is sufficiently small so that the map $\iota : \hat{K}'/\hat{C} \to G/C$ is injective. Take a neat open compact subgroup $K$ of $G$ containing $C$ that is small enough so that $K/C \subset \iota(\hat{K}/\hat{C})$.

Now the lemma follows by taking $\hat{K} := \hat{K}' \cap \iota^{-1}(K)$.

Let $\hat{\Phi}$ be a family of closed subsets of $\hat{G}(\mathbb{Q}) \setminus \mathcal{Z}^*$ that satisfies the analogous conditions for $\Phi$ in Section 3.2. Assume that under the map $\iota : \hat{G}(\mathbb{Q}) \setminus \mathcal{Z}^* \to G(\mathbb{Q}) \setminus \mathcal{Z}^*$, the preimage of every set in $\Phi$ belongs to $\hat{\Phi}$.

We define the pull-back map $\iota^* : H^i_{\hat{\Phi}}(C, M) \to H^i_{\Phi}(\hat{C}, M)$ as in the following lemma.

**Proposition 4.2.** There is a unique map $\iota^* : H^i_{\hat{\Phi}}(C, M) \to H^i_{\Phi}(\hat{C}, M)$ such that the diagram

\[
\begin{array}{ccc}
H^i_{\hat{\Phi}}(C, M) & \xrightarrow{\iota^*} & H^i_{\Phi}(\hat{C}, M) \\
\downarrow & & \downarrow \\
H^i_{\hat{\Phi}}(K, M) & \xrightarrow{\iota^*} & H^i_{\Phi}(\hat{K}, M)
\end{array}
\]

(4.2)

commutes for all neat open compact subgroups $\hat{K}$ of $\hat{G}$ containing $\hat{C}$ and all neat open compact subgroups $K$ of $G$ containing $C$ such that

- $\iota(\hat{K}) \subset K$, and the map $\iota : \hat{K}/\hat{C} \to K/C$ is bijective.

Here the bottom horizontal arrow $\iota^*$ of (4.2) is the usual pull-back map for cohomology groups (similar notation will be used later on).
Proof. The uniqueness follows from Lemma 4.1. To prove the existence, it suffices to show that the diagram

\[
\begin{array}{ccc}
H^i_\Phi(K', M) & \xrightarrow{\iota^*} & H^i_\Phi(\hat{K}', M) \\
\rho \downarrow & & \rho \\
H^i_\Phi(K, M) & \xrightarrow{\iota^*} & H^i_\Phi(\hat{K}, M)
\end{array}
\]

commutes all neat open compact subgroups \(\hat{K}'\) of \(\hat{G}\) containing \(\hat{C}\) and all neat open compact subgroups \(K'\) of \(G\) containing \(C\) such that

\[
K' \subset K, \quad \hat{K}' \subset \hat{K}, \quad \iota(\hat{K}') \subset K'
\]

and that the map

\[
\iota : \hat{K}'/\hat{C} \to K'/C
\]

is bijective.

It follows from (4.3) and (4.5) that the map

\[
\iota : \hat{K}/\hat{K}' \to K/K'
\]

is bijective.

Consider the commutative diagram

\[
\begin{array}{ccc}
S^G_{\hat{K}'} & \xleftarrow{\iota} & S^G_{K'} \\
\downarrow & & \downarrow \\
S^G_{\hat{K}} & \xleftarrow{\iota} & S^G_{K}
\end{array}
\]

The neatness condition implies that the left vertical arrow and the right vertical arrow are finite folds covering maps of topological manifolds, with fibers \(K/K'\) and \(\hat{K}/\hat{K}'\) respectively. Then (4.6) implies that the above commutative diagram is Cartesian. This implies that the diagram (4.4) commutes. \(\square\)

Lemma 4.3. Let \(C' \subset C\) and \(\hat{C}' \subset \hat{C}\) be open subgroups such that \(\iota(\hat{C}') \subset C'\). Then the diagram

\[
\begin{array}{ccc}
H^i_\Phi(C, M) & \xrightarrow{\iota^*} & H^i_\Phi(\hat{C}, M) \\
\downarrow & & \downarrow \\
H^i_\Phi(C', M) & \xrightarrow{\iota^*} & H^i_\Phi(\hat{C}', M)
\end{array}
\]

commutes, where the vertical arrows are the pull-back maps.

Proof. Pick neat open compact subgroups \(\hat{K}'\) and \(\hat{K}\) of \(\hat{G}\) containing \(\hat{C}'\) and \(\hat{C}\) respectively, such that \(\hat{K}' \subset \hat{K}\) and that \(\hat{K}'/\hat{C}' \to \hat{K}/\hat{C}\) is bijective. It follows from the proof of Proposition 3.6 that such groups \(K'\) form a neighborhood basis of \(\hat{C}'\) in \(G\). Using Lemma 4.1 by shrinking \(K'\) and \(\hat{K}\) if necessary we may assume that there exist neat open compact subgroups \(K'\) and \(K\) of \(G\) such that \(\iota(\hat{K}') \subset K'\),
\( \iota(\hat{K}) \subset K \) and that the maps \( \iota : \hat{K}'/\hat{C}' \to K'/C' \) and \( \iota : \hat{K}/\hat{C} \to K/C \) are bijective. It follows that

\[
K' = \iota(\hat{K}')C' \subset \iota(\hat{K})C = K.
\]

Then we have a commutative diagram

\[
\begin{array}{ccc}
K/C & \xleftarrow{\iota} & \hat{K}/\hat{C} \\
\uparrow & & \uparrow \\
K'/C' & \xleftarrow{\iota} & \hat{K}'/\hat{C}'.
\end{array}
\]

Since all arrows except the left vertical one are bijective, the left vertical arrow is bijective as well. Now the lemma follows by considering the diagram

\[
\begin{array}{ccc}
H_\phi^i(K, M) & \xrightarrow{\iota^*} & H_\phi^i(\hat{K}, M) \\
\downarrow & & \downarrow \\
H_\phi^i(C, M) & \xrightarrow{\iota^*} & H_\phi^i(\hat{C}, M) \\
\downarrow & & \downarrow \\
H_\phi^i(C', M) & \xrightarrow{\iota^*} & H_\phi^i(\hat{C}', M) \\
\downarrow & & \downarrow \\
H_\phi^i(K', M) & \xrightarrow{\iota^*} & H_\phi^i(\hat{K}', M).
\end{array}
\]

Recall that \( \mathfrak{p} \) is transversal to \( \iota(\hat{\mathfrak{g}}) \). The following two lemmas are direct consequences of Lemma 4.3.

**Lemma 4.4.** Assume that \( D\mathfrak{S} \) and \( \hat{D}\hat{\mathfrak{S}} \) are neat, and \( \iota(\hat{D}\hat{\mathfrak{S}}) \subset D\mathfrak{S} \). Then there is a unique homomorphism

\[
\iota^* : H_\phi^i(D\mathfrak{S}, M)^{(\mathfrak{p})} \to H_\phi^i(\hat{D}\hat{\mathfrak{S}}, M)^{(\mathfrak{p})}
\]

such that the diagram

\[
\begin{array}{ccc}
H_\phi^i(D\mathfrak{P}, M) & \xrightarrow{\text{pull-back}} & H_\phi^i(\hat{D}\hat{\mathfrak{P}}, M) \\
\downarrow & & \downarrow \\
H_\phi^i(D\mathfrak{S}, M)^{(\mathfrak{p})} & \xrightarrow{\iota^*} & H_\phi^i(\hat{D}\hat{\mathfrak{S}}, M)^{(\mathfrak{p})}
\end{array}
\]

commutes for all \( \mathfrak{P} \) and \( \hat{\mathfrak{P}} \) such that

\[
\mathfrak{P} \supset \mathfrak{S}, \quad \hat{\mathfrak{P}} \supset \hat{\mathfrak{S}}, \quad D\mathfrak{P} \text{ and } \hat{D}\hat{\mathfrak{P}} \text{ are neat, and } \iota(\hat{\mathfrak{P}}) \subset \mathfrak{P}.
\]
Lemma 4.5. There is a unique homomorphism
\[(4.8) \quad \iota^*: H^i_{\phi}(G, M)^{(p \supset s)} \to H^i_{\hat{\phi}}(\hat{G}, M)^{(\hat{p} \supset \hat{s})}\]
such that the diagram
\[
\begin{array}{ccc}
H^i_\phi(DS, M)^{(p)} & \xrightarrow{\iota^*} & H^i_\phi(D\hat{S}, M)^{(\hat{p})} \\
\downarrow & & \downarrow \\
H^i_\phi(G, M)^{(p \supset s)} & \xrightarrow{\iota^*} & H^i_\phi(\hat{G}, M)^{(\hat{p} \supset \hat{s})}
\end{array}
\]
commutes for all $D, \mathcal{S}, \hat{D}$ and $\hat{\mathcal{S}}$ as in Lemma 4.4.

It is clear that the homomorphism (4.8) is independent of $s$ and $\hat{s}$ (see (3.14)). We also call it the pull-back map and write it as
\[
\iota^*: H^i_\phi(G, M)^{(p)} \to H^i_{\hat{\phi}}(\hat{G}, M)^{(\hat{p})}.
\]
Specializing the above map to the case when $p = g$ and $\hat{p} = \hat{g}$, we obtain the pull-back map
\[
\iota^*: H^i_\phi(G, M) \to H^i_{\hat{\phi}}(\hat{G}, M).
\]

In the setting of Lemma 4.4, the homomorphism (4.7) is natural in the coefficient module $M$. Thus it yields a homomorphism
\[
\iota^*: \tilde{H}^i_\phi(DS, M)^{(p)} \to \tilde{H}^i_\phi(D\hat{S}, M)^{(\hat{p})}.
\]
By taking the direct limits, this further induces a homomorphism
\[
\iota^*: \tilde{H}^i_\phi(G, M)^{(p \supset s)} \to \tilde{H}^i_{\hat{\phi}}(\hat{G}, M)^{(\hat{p} \supset \hat{s})}.
\]
We have obvious commutative diagrams
\[
\begin{array}{ccc}
H^i_\phi(DS, M)^{(p)} & \xrightarrow{\iota^*} & H^i_\phi(D\hat{S}, M)^{(\hat{p})} \\
\downarrow & & \downarrow \\
\tilde{H}^i_\phi(DS, M)^{(p)} & \xrightarrow{\iota^*} & \tilde{H}^i_\phi(D\hat{S}, M)^{(\hat{p})}
\end{array}
\]
and
\[
\begin{array}{ccc}
H^i_\phi(G, M)^{(p)} & \xrightarrow{\iota^*} & H^i_\phi(\hat{G}, M)^{(\hat{p})} \\
\downarrow & & \downarrow \\
\tilde{H}^i_\phi(G, M)^{(p \supset s)} & \xrightarrow{\iota^*} & \tilde{H}^i_{\hat{\phi}}(\hat{G}, M)^{(\hat{p} \supset \hat{s})}.
\end{array}
\]
4.2. Pull-back for formal completions. In this subsection we assume that $R$ is a $\mathbb{Q}$-algebra. The proof of Proposition 4.2 also proves the following proposition.

**Proposition 4.6.** There is a unique map

$$(4.9) \quad i^* : \hat{H}_i^\phi(G,M) \rightarrow \hat{H}_i^\phi(\hat{G},M)$$

such that the diagram

$$\begin{array}{ccc}
\hat{H}_i^\phi(G,M) & \xrightarrow{i^*} & \hat{H}_i^\phi(\hat{G},M) \\
\downarrow \text{the projection map} & & \downarrow \text{the projection map} \\
H_i^\phi(K,M) & \xrightarrow{i^*} & H_i^\phi(\hat{K},M)
\end{array}$$

commutes for all neat open compact subgroups $\hat{K}$ of $\hat{G}$ containing $\hat{C}$ and all neat open compact subgroups $K$ of $G$ containing $C$ such that $i(\hat{K}) \subset K$ and the map $i : \hat{K}/\hat{C} \rightarrow K/C$ is bijective.

Here the identifications $H_i^\phi(K,M) = H_i^\phi(G,M)^K$ and $H_i^\phi(\hat{K},M) = H_i^\phi(\hat{G},M)^{\hat{K}}$ are used.

The proof of Lemma 4.3 also shows the following result.

**Lemma 4.7.** Let $C' \subset C$ and $\hat{C}' \subset \hat{C}$ be open subgroups such that $i(\hat{C}') \subset C'$. Then the diagram

$$\begin{array}{ccc}
H_i^\phi(G,M) & \xrightarrow{i^*} & H_i^\phi(\hat{G},M) \\
\downarrow \text{inclusion} & & \downarrow \text{inclusion} \\
H_i^\phi(G,M) & \xrightarrow{i^*} & H_i^\phi(\hat{G},M)
\end{array}$$

commutes.

By Lemma 4.7 there is a unique homomorphism

$$i^* : \hat{H}_i^\phi(G,M)_{p-sm} \rightarrow \hat{H}_i^\phi(\hat{G},M)_{p-sm},$$

to be called the pull-back map for formal completions, such that the diagram

$$\begin{array}{ccc}
\hat{H}_i^\phi(G,M) & \xrightarrow{D\Psi} & \hat{H}_i^\phi(\hat{G},M) \\
\downarrow & & \downarrow \\
H_i^\phi(G,M)_{p-sm} & \xrightarrow{i^*} & H_i^\phi(\hat{G},M)_{p-sm}
\end{array}$$

commutes for $D$, $\hat{D}$, $\Psi$ and $\hat{\Psi}$ such that $D\Psi$ and $\hat{D}\hat{\Psi}$ are neat, and $i(\hat{D}\hat{\Psi}) \subset D\Psi$. 

The transversality condition implies that $\mathfrak{g}/\mathfrak{p} = \dot{\mathfrak{g}}/\dot{\mathfrak{p}}$.

**Lemma 4.8.** The diagram

\[
\begin{array}{c}
H^i_{\phi}(G, M)_{\text{p-sm}} \otimes D(\mathfrak{g}/\mathfrak{p}) \xrightarrow{\iota^*} H^i_{\phi}(\dot{G}, M)_{\text{p-sm}} \otimes D(\dot{\mathfrak{g}}/\dot{\mathfrak{p}}) \\
\downarrow (3.29) \\
H^i_{\phi}(G, M)^{(p)} \xrightarrow{\iota^*} H^i_{\phi}(\dot{G}, M)^{(\dot{p})}
\end{array}
\]

commutes.

**Proof.** It suffices to show that the diagram

\[
\begin{array}{c}
H^i_{\phi}(G, M)^{D\mathfrak{P}} \otimes D(\mathfrak{g}/\mathfrak{p}) \xrightarrow{\iota^*} H^i_{\phi}(\dot{G}, M)^{D\mathfrak{P}} \otimes D(\dot{\mathfrak{g}}/\dot{\mathfrak{p}}) \\
\downarrow \\
H^i_{\phi}(D\mathfrak{P}, M) \xrightarrow{\iota^*} H^i_{\phi}(\dot{D}\mathfrak{P}, M)
\end{array}
\]

commutes when $D\mathfrak{P}$ and $\dot{D}\mathfrak{P}$ are neat, and $\iota(D\mathfrak{P}) \subset D\mathfrak{P}$. This is routine to check and we omit the details. $\square$

### 4.3. Integrations for formal cohomologies

Fix a $\dot{\mathcal{G}}(\mathbb{R})$-invariant orientation $\omega_{\dot{G}}$ on $\dot{\mathcal{G}}(\mathbb{R})/\dot{K}_\infty$, which always exists and is unique up to sign. It yields a generator of $O(\dot{\mathcal{G}}(\mathbb{R})/\dot{K}_\infty)$, which is still denoted by $\omega_{\dot{G}}$. It also induces an orientation on the manifold

\[
S^G_K := \dot{\mathcal{G}}(\mathbb{Q}) \backslash \mathfrak{g}^\ast / \dot{K},
\]

for every completely neat open compact subgroup $\dot{K}$ of $\dot{\mathcal{G}}$. Using this orientation, the pairing against the fundamental class gives a map

\[
\int_{\omega_{\dot{G}}} : H^i_c(\dot{K}, R) \to R.
\]

Here $R$ carries the trivial action of $\dot{\mathcal{G}}(\mathbb{Q}) \times \dot{\mathcal{G}}$. By convention, the map (4.10) is identically zero unless $i = \dim(\dot{\mathcal{G}}(\mathbb{R})/\dot{K}_\infty)$. Note that for each open compact subgroup $\dot{K}'$ of $\dot{K}$, the diagrams

\[
\begin{array}{c}
H^i_c(\dot{K}', R) \xrightarrow{f_{\omega_{\dot{G}}}} R \\
\downarrow \text{the push-forward map} \\
H^i_c(\dot{K}, R) \xrightarrow{f_{\omega_{\dot{G}}}} R
\end{array}
\]

are commutative.
and
\[
\begin{array}{c}
H^i_c(\hat{K}, R) \xrightarrow{f_{\hat{G}}} R \\
\downarrow \text{the pull-back map} \Downarrow \downarrow \text{multiplication by } [\hat{K} : \hat{K}'] \\
H^i_c(\hat{K}', R) \xrightarrow{f_{\hat{G}}} R
\end{array}
\]
are commutative.

For every completely neat compact subgroup \(\hat{C}\) of \(\hat{G}\), we define the integration map
\[
(4.13) \quad \int_{\hat{G}} : H^i_c(\hat{C}, R) \to R
\]
to be the composition of
\[
H^i_c(\hat{C}, R) \xrightarrow{\text{the projection}} H^i_c(\hat{K}, R) \xrightarrow{f_{\hat{G}}} R,
\]
where \(\hat{K}\) is a completely neat open compact subgroup of \(\hat{G}\) containing \(\hat{C}\). By (4.11), this map is independent of the choice of \(\hat{K}\).

**Lemma 4.9.** Let \(\hat{C}\) and \(\hat{C}'\) be completely neat compact subgroups of \(\hat{G}\) that are commensurable with each other. Then the diagram
\[
\begin{array}{c}
H^i_c(\hat{C}, R) \xrightarrow{f_{\hat{G}}} R \\
\downarrow \text{the transfer map} \Downarrow \downarrow \text{multiplication by } [\hat{C} : \hat{C} \cap \hat{C}'] \\
H^i_c(\hat{C}', R) \xrightarrow{f_{\hat{G}}} R
\end{array}
\]
commutes.

**Proof.** In view of Proposition 3.6, this easily follows from the commutative diagrams (3.11) and (4.12). \(\square\)

As in (1.13), set
\[
D(\hat{G}, \hat{p}) := D(\hat{G}^{\hat{p}}) \otimes O(\hat{G}(\mathbb{R})/\hat{K}_\infty) \otimes D(\hat{p}).
\]
When \(\hat{D}\hat{p}\) is completely neat, we define the integration map
\[
(4.14) \quad \int : H^i_c(\hat{D}\hat{p}, R) \otimes D(\hat{G}, \hat{p}) \to \mathbb{Q} \otimes R,
\]
\[
\phi \otimes \mu \otimes \omega_{\hat{G}} \otimes \mu' \mapsto \mu(\hat{D}) \cdot \mu'(\hat{p}) \otimes \int_{\hat{G}} \phi.
\]
Here we use the obvious identification \(D(\hat{p}) = D(\hat{p})\) via the logarithmic map, and similar identifications will be used without further explanation.
By Lemma 4.9 we have a unique homomorphism

\[(4.15) \int : H^i_c(\hat{G}, R)^{\hat{p}} \otimes D(\hat{G}, \hat{p}) \to \mathbb{Q} \otimes R\]
such that the diagram

\[
\begin{array}{ccc}
H^i_c(\hat{D}\hat{P}, R) \otimes D(\hat{G}, \hat{p}) & \xrightarrow{f} & \mathbb{Q} \otimes R \\
\downarrow & & \downarrow \cong \\
H^i_c(\hat{G}, R)^{\hat{p}} \otimes D(\hat{G}, \hat{p}) & \xrightarrow{f} & \mathbb{Q} \otimes R
\end{array}
\]

commutes for all \(\hat{D}\) and \(\hat{P}\) such that \(\hat{D}\hat{P}\) is completely neat. Specifying the homomorphism (4.15) to the case when \(\hat{p} = \hat{g}\), we get a homomorphism

\[(4.16) \int : H^i_c(\hat{G}, R) \otimes D(\hat{G}) \to R,\]

where

\[D(\hat{G}) := D(\hat{G}^i) \otimes O(\hat{G}(\mathbb{R})/\hat{K}^\infty),\]
as in (1.11).

Set \(\hat{R} := \lim_{\leftarrow k \in \mathbb{N}} R/p^k\). Since the integration map (4.13) is natural in \(R\), it yields a map

\[
\int : \tilde{H}^i_c(\hat{C}, R) \to \hat{R}.
\]

Now we define a map

\[
\int : \tilde{H}^i_c(\hat{G}, R)^{\hat{p} \supset \hat{p}} \otimes D(\hat{G}, \hat{p}) \to \mathbb{Q} \otimes \hat{R},
\]
to be called the integration map for relative completed cohomologies, by the following lemma.

**Lemma 4.10.** There is a unique homomorphism

\[
\int : \tilde{H}^i_c(\hat{G}, R)^{\hat{p} \supset \hat{p}} \otimes D(\hat{G}, \hat{p}) \to \mathbb{Q} \otimes \hat{R}
\]
such that the diagram

\[
\begin{array}{ccc}
\tilde{H}^i_c(\hat{D}\hat{P}, R) \otimes D(\hat{G}, \hat{p}) & \xrightarrow{f} & \mathbb{Q} \otimes \hat{R} \\
\downarrow & & \downarrow \cong \\
\tilde{H}^i_c(\hat{G}, R)^{\hat{p} \supset \hat{p}} \otimes D(\hat{G}, \hat{p}) & \xrightarrow{f} & \mathbb{Q} \otimes \hat{R}
\end{array}
\]
commutes for all \( \mathcal{D} \) and \( \mathcal{P} \) such that \( \mathcal{D}\mathcal{P} \) is completely neat, where the top horizontal arrow is the map
\[
\phi \otimes \mu \otimes \omega_{\mathcal{C}} \otimes \mu' \mapsto \mu(\mathcal{D}) \cdot \mu'(\mathcal{P}) \otimes \int_{\omega_{\mathcal{C}}} \phi.
\]

**Proof.** This is an easy consequence of Lemma 4.9. \( \square \)

The following lemma is easily verified.

**Lemma 4.11.** The diagram
\[
\begin{array}{ccc}
H^i_{c}(G, R)_{\mathcal{P}} \otimes D(G, \mathcal{P}) & \longrightarrow & \hat{H}^i_{c}(\hat{G}, R)_{\mathcal{P}} \otimes D(\hat{G}, \mathcal{P}) \\
\downarrow f & & \downarrow f \\
\mathbb{Q} \otimes R & \longrightarrow & \mathbb{Q} \otimes \hat{R}
\end{array}
\]
commutes.

**4.4. Integrations for formal completions.** In this subsection we assume that \( R \) is a \( \mathbb{Q} \)-algebra. We define the integration map
\[
\int : \hat{H}^i_{c}(\hat{G}, R) \otimes D(\hat{G}) \rightarrow R
\]
as in the following lemma, whose easy proof is omitted.

**Lemma 4.12.** There is a unique \( R \)-homomorphism
\[
(4.17) \quad \int : \hat{H}^i_{c}(\hat{G}, R) \otimes D(\hat{G}) \rightarrow R
\]
such that the diagram
\[
\begin{array}{ccc}
\hat{H}^i_{c}(\hat{G}, R) \otimes D(\hat{G}) & \longrightarrow & R \\
\downarrow \text{projection} & & \downarrow = \\
\hat{H}^i_{c}(\hat{G}, R)^{K} \otimes D(\hat{G}) & \longrightarrow & R
\end{array}
\]
commutes for all completely neat open compact subgroups \( \hat{K} \) of \( \hat{G} \), where the bottom horizontal arrow is the restriction of the map \( 1.16 \). Moreover, the map \( 4.17 \) extends the map \( 1.16 \).

We also have the obvious identification
\[
(4.18) \quad D(\hat{G}) = D(\hat{G}/\mathcal{P}) \otimes D(\hat{G}, \mathcal{P}).
\]

**Lemma 4.13.** The diagram
\[
\begin{array}{ccc}
\hat{H}^i_{c}(\hat{G}, R)_{\mathcal{P} \text{-sm}} \otimes D(\hat{G}) & \longrightarrow & \hat{H}^i_{c}(\hat{G}, R)_{\mathcal{P} \text{-sm}} \otimes D(\hat{G}, \mathcal{P}) \\
\downarrow f & & \downarrow f \\
R & \longrightarrow & R
\end{array}
\]

commutes.

Proof. Suppose that $\tilde{D}\Phi$ is completely neat, and let $\phi \in \hat{H}^i(\tilde{G}, M)$. Write

$$\eta := \mu \otimes \nu \otimes \mu_D \otimes \mu_{\Phi} \otimes \omega \in D(\tilde{G}),$$

where $\mu \in D(\tilde{g}) = D(\tilde{G})$, $\nu \in D(\tilde{p})^\vee = D(\tilde{\Phi})^\vee$ is the functional $\lambda \mapsto \lambda(\tilde{p})$, $\mu_D \in D(\tilde{G})$ is the distribution such that $\mu_D(\tilde{D}) = 1$, and $\mu_{\Phi} \in D(\tilde{\Phi}) = D(\tilde{\Phi})$ is the distribution such that $\mu_{\Phi}(\tilde{\Phi}) = 1$. The image of $\phi \otimes \eta$ under the top horizontal arrow is represented by

$$\{\mu(\tilde{\Phi}) \cdot \phi_{\tilde{D}\Phi}\} \otimes \mu_D \otimes \mu_{\Phi} \otimes \omega \in H^i(\tilde{G}, M) \otimes D(\tilde{G}, \tilde{\Phi}),$$

where $\tilde{\Phi}$ runs over all $\tilde{D}$-neat open compact subgroups of $\tilde{G}$ containing $\tilde{\Phi}$, and $\phi_{\tilde{D}\Phi}$ denotes the projection of $\phi$ to $H^i(\tilde{G}, M)$. We have that

$$\int \{\mu(\tilde{\Phi}) \cdot \phi_{\tilde{D}\Phi}\} \otimes \mu_D \otimes \mu_{\Phi} \otimes \omega \cdot \phi_{\tilde{D}\Phi}$$

$$= \mu(\tilde{\Phi}) \cdot \int \phi_{\tilde{D}\Phi} \text{ (see (4.14))}$$

$$= \int \phi \otimes \eta.$$

This proves the lemma. \hfill \Box

5. Modular symbols

Recall from the Introduction that $E$ is a subfield of $\mathbb{Q}$ that is contained in a closed subfield $E$ of $\mathbb{C}_p$. In this section, let $V$ be a representation of $G(\mathbb{Q})$ over $E$, and $V$ a continuous finite-dimensional representation of $G$ over $E$. Suppose that the representation $V$ is identified with a $G(\mathbb{Q})$-stable $E$-form of $V$. In particular, $V = E \otimes V$ as vector spaces.

Throughout this section we assume that under the map $\iota : \hat{G}(\mathbb{Q}) \setminus \mathcal{X} \rightarrow G(\mathbb{Q}) \setminus \mathcal{X}$,

$$\int \phi \otimes \eta$$

the preimage of every set in $\Phi$ is compact.

5.1. Some cohomology groups. Recall that $p \supset s$ are Lie subalgebras of $g$. Note that every finite-dimensional representation of $s$ over $E$ integrates to continuous representations of some compact subgroups of $G$ with Lie algebra $s$. This establishes an equivalence between the category of all finite-dimensional representations of $s$ over $E$ with the category of all pairs $(\mathcal{G}_1, V_1)$ where $\mathcal{G}_1$ is a compact subgroup of $G$ with Lie algebra $s$, and $V_1$ is a continuous finite-dimensional representation of $\mathcal{G}_1$ over $E$. A morphism $(\mathcal{G}_1, V_1) \rightarrow (\mathcal{G}_2, V_2)$ in the latter category is defined to be a linear map $V_1 \rightarrow V_2$ that is equivariant under some open subgroups of $\mathcal{G}_1 \cap \mathcal{G}_2$. 
Definition 5.1. Let $V_1$ be a finite-dimensional representation of $\mathfrak{s}$ over $E$. Let $\mathfrak{U}_1$ be an $\mathcal{O}$-lattice of $V_1$. Integrate the representation $V_1$ of $\mathfrak{s}$ to a compact subgroup $\mathfrak{S}$ of $G$ with Lie algebra $\mathfrak{s}$ such that $\mathfrak{S}$ stabilizes $\mathfrak{U}_1$. View $\mathfrak{U}_1$ as an $\mathcal{O}[G(\mathbb{Q}) \times G_k^r \times \mathfrak{S}]$-module with the given action of $\mathfrak{S}$ and the trivial action of $G(\mathbb{Q}) \times G_k^r$. Define

$$\tilde{H}^i_{\phi}(G, V_1)^{(\mathfrak{p} \rhd \mathfrak{s}), \circ} := E \otimes \tilde{H}^i_{\phi}(G, \mathfrak{U}_1)^{(\mathfrak{p} \rhd \mathfrak{s})}$$

(see Remark 3.10) to be called the integral relative completed cohomology space.

Note that Definition 5.1 is independent of the choices of $\mathfrak{U}_1$, $\mathfrak{S}$, and the representation of $\mathfrak{S}$ on $V_1$ that integrates the representation of $\mathfrak{s}$. Moreover, $\tilde{H}^i_{\phi}(G, \cdot)^{(\mathfrak{p} \rhd \mathfrak{s}), \circ}$ is a functor from the category of finite-dimensional representation of $\mathfrak{s}$ over $E$ to the category of locally convex topological vector spaces over $E$. Here we write

$$\tilde{H}^i_{\phi}(G, V_1)^{(\mathfrak{p} \rhd \mathfrak{s}), \circ} = \lim_{\mathcal{O}[G(\mathbb{Q}) \times G_k^r \times \mathfrak{S}] \to \mathcal{O}[G(\mathbb{Q}) \times G_{k, \mathfrak{p}}]} E \otimes \tilde{H}^i_{\phi}(G, \mathfrak{U}_1)^{(\mathfrak{p} \rhd \mathfrak{s})},$$

($D$ runs over open compact subgroups of $G_{k, \mathfrak{p}}$ and $\mathfrak{S}_1$ runs over open compact subgroups of $\mathfrak{S}$) and view it as a locally convex topological space under the direct limit topology, where the topology on $E \otimes \tilde{H}^i_{\phi}(G, \mathfrak{U}_1)^{(\mathfrak{p} \rhd \mathfrak{s})}$ is given by the seminorm associated to the image of the natural map

$$\tilde{H}^i_{\phi}(G, \mathfrak{U}_1)^{(\mathfrak{p} \rhd \mathfrak{s})} \to E \otimes \tilde{H}^i_{\phi}(G, \mathfrak{U}_1)^{(\mathfrak{p} \rhd \mathfrak{s})}.$$

Set $V_C := C \otimes_E V$. View $V$ and $V_C$ as $E[G(\mathbb{Q}) \times G_k^r]$-modules with the given action of $G(\mathbb{Q})$ and the trivial action of $G_k^r$. View $V$ as an $E[G(\mathbb{Q}) \times G_k^r]$-module with the given action of $G$ and the trivial action of $G(\mathbb{Q}) \times G_{k, \mathfrak{p}}$.

Definition 5.2. Let $\mathfrak{V}$ be an $\mathcal{O}$-lattice of $V$. Define

$$H^i_{\phi}(G, V)^{(\mathfrak{p} \rhd \mathfrak{s}), \circ} := E \otimes H^i_{\phi}(G, \mathfrak{V})^{(\mathfrak{p} \rhd \mathfrak{s})},$$

to be called the integral relative cohomology space.

Definition 5.2 is independent of $\mathfrak{V}$.

As an example of Definition 5.1, we have the integral relative completed cohomology space

$$\tilde{H}^i_{\phi}(G, V)^{(\mathfrak{p} \rhd \mathfrak{s}), \circ} := E \otimes \tilde{H}^i_{\phi}(G, \mathfrak{V})^{(\mathfrak{p} \rhd \mathfrak{s})},$$

where $\mathfrak{V}$ is an $\mathcal{O}$-lattice of $V$. The natural map

$$H^i_{\phi}(G, \mathfrak{V})^{(\mathfrak{p} \rhd \mathfrak{s})} \to \tilde{H}^i_{\phi}(G, \mathfrak{V})^{(\mathfrak{p} \rhd \mathfrak{s})}$$

(yes see (3.17)) yields a linear map

$$(5.2) \quad H^i_{\phi}(G, V)^{(\mathfrak{p} \rhd \mathfrak{s}), \circ} \to \tilde{H}^i_{\phi}(G, V)^{(\mathfrak{p} \rhd \mathfrak{s}), \circ}.$$

The inclusion map $\mathfrak{V} \to V$ yields a homomorphism

$$H^i_{\phi}(G, \mathfrak{V})^{(\mathfrak{p} \rhd \mathfrak{s})} \to H^i_{\phi}(G, V)^{(\mathfrak{p} \rhd \mathfrak{s})},$$
which further induces a linear map
\[(5.3) \quad H^i_{\Phi}(G, V)^{(p)} \to H^i_{\Phi}(G, V)^{(p)}.\]
Note that both linear maps \((5.2)\) and \((5.3)\) are independent of \(\mathfrak{U}^\ast\).

The map
\[\mathcal{X}^\ast \times V \to \mathcal{X}^\ast \times V, \quad (x, u) \mapsto (x, x^{-1}p^{-1}u)\]
is \(G(\mathbb{Q}) \times G^\ast\)-equivariant, where \(x_p\) denotes the image of \(x\) under the projection map \(\mathcal{X} \to G\). Thus it induces a homomorphism
\[V_{[K]} \to V_{[K]}\]
of sheaves over \(S^\ast_{K^\ast}\), for every open compact subgroup \(K\) of \(G^\ast\). This further induces homomorphisms of various cohomology spaces, to be denoted by
\[(5.4) \quad \begin{cases} 
\iota_V : H^i_{\Phi}(K, V) \to H^i_{\Phi}(K, V); \\
\iota_V : H^i_{\Phi}(G, V) \to H^i_{\Phi}(G, V); \\
\iota_V : \hat{H}^i_{\Phi}(G, V)_{p-sm} \to \hat{H}^i_{\Phi}(G, V)_{p-sm}.
\end{cases}\]

Let \(V_0\) be an \(E \otimes \mathfrak{s}\)-submodule of \(V\). In this section, we will consider modular symbols on the spaces appearing in the following diagram:
\[(5.5) \quad \begin{array}{c}
\hat{H}^i_{\Phi}(G, V_0)_{(p \supset \mathfrak{s}), o} \\
\downarrow \text{the map induced by } V_0 \subset V
\end{array} \quad \begin{array}{c}
\hat{H}^i_{\Phi}(G, V)_{(p \supset \mathfrak{s}), o} \\
\Downarrow \cong
\end{array} \quad \begin{array}{c}
\hat{H}^i_{\Phi}(G, V)_{(p \supset \mathfrak{s}), o} \\
\Downarrow \cong
\end{array} \quad \begin{array}{c}
H^i_{\Phi}(G, V)^{(p), o} \\
\Downarrow \cong
\end{array} \quad \begin{array}{c}
\hat{H}^i_{\Phi}(G, V)_{(p \supset \mathfrak{s}), o} \\
\Downarrow \cong
\end{array} \quad \begin{array}{c}
\hat{H}^i_{\Phi}(G, V)_{(p \supset \mathfrak{s}), o} \\
\Downarrow \cong
\end{array} \quad \begin{array}{c}
\hat{H}^i_{\Phi}(G, V)_{(p \supset \mathfrak{s}), o} \quad \text{(5.2)}
\end{array} \quad \begin{array}{c}
\hat{H}^i_{\Phi}(G, V)_{(p \supset \mathfrak{s}), o} \quad \text{(5.3)}
\end{array} \quad \begin{array}{c}
H^i_{\Phi}(G, V)^{(p)} \quad \text{(5.3)}
\end{array} \quad \text{of sheaves over } S^\ast_{K^\ast}, \text{for every open compact subgroup } K \text{ of } G^\ast.\]

5.2. Modular symbols. Recall that \(Z\) is an algebraic torus over \(\mathbb{Q}\), and \(j : \hat{G} \to Z\)
is an algebraic homomorphism over \(\mathbb{Q}\). With the fixed closed subgroup of \(Z(\mathbb{R})\) as in \((1.10)\), we define various cohomology groups attached to \(Z\) as in the previous sections.

Recall from Section \((1.1)\) that \(Z_0\) is an algebraic subtorus of \(Z := Z(\mathbb{Q}_p)\) whose Lie algebra is denoted by \(\mathfrak{z}_0\). The space \(\hat{H}^0_{\Phi}(Z, E)^{(J \supset \mathfrak{s}_0), o}\) is identified with the space of all continuous \(E\)-valued functions on \(Z(\mathbb{Q}) \backslash Z^2\) that are invariant under some open compact subgroups of \(D^\ast_{Z_0}\).

Recall from the Introduction that
\[\mathfrak{s} \supset j(\mathfrak{p}) \quad \text{and} \quad \mathfrak{z}_0 \supset j(\hat{\mathfrak{p}}).\]
Fix a functional \(\lambda_0 \in \text{Hom}_{E \otimes \mathfrak{g}}(V_0, E)\).
Definition 5.3. The relative completed modular symbol map

\[ \tilde{\mathcal{M}}_{\lambda_0} : \tilde{H}^0(Z, E)^{(q \otimes \mathcal{O}_0)} \times \left( \tilde{H}_q^i(G, \mathcal{V}_0)^{(p \otimes \mathcal{O})} \otimes D(\hat{G}, \hat{\mathcal{P}}) \right) \rightarrow E \]

is the bilinear map whose pull-back to

\[ \tilde{H}^0(Z, \mathcal{O})^{(q \otimes \mathcal{O}_0)} \times \left( \tilde{H}_q^i(G, \mathcal{V}_0)^{(p \otimes \mathcal{O})} \otimes D(\hat{G}, \hat{\mathcal{P}}) \right) \]

equals the composition of

\[ \tilde{H}^0(G, \mathcal{O})^{(q \otimes \mathcal{O}_0)} \times \left( \tilde{H}_q^i(G, \mathcal{V}_0)^{(p \otimes \mathcal{O})} \otimes D(\hat{G}, \hat{\mathcal{P}}) \right) \]

pull-back

\[ \hat{H}^i_c(G, \mathcal{V}_0)^{(p \otimes \mathcal{O})} \otimes D(\hat{G}, \hat{\mathcal{P}}) \]

cup product

\[ \lambda_0 \circ \left( \tilde{H}^0(G, \mathcal{O})^{(q \otimes \mathcal{O}_0)} \times \left( \tilde{H}_q^i(G, \mathcal{V}_0)^{(p \otimes \mathcal{O})} \otimes D(\hat{G}, \hat{\mathcal{P}}) \right) \right) \]

\[ \rightarrow E, \]

where \( \mathcal{V}_0 \) is an \( \mathcal{O} \)-lattice of \( V_0 \) such that \( \lambda_0(\mathcal{V}_0) \subset \mathcal{O} \).

It is clear that the bilinear map (5.6) is well-defined, separately continuous under the natural topologies, and independent of \( \mathcal{V}_0 \).

Let \( w \) be an algebraic character of \( Z_E \) defined over \( E \). As in (1.4) we have a character

\[ w_p : Z \subset Z(E) \rightarrow E^\times. \]

Write \( E_w := E \), to be viewed as an \( E[Z(\mathbb{Q}) \times \mathbb{Z}] \)-module via the action of \( Z \) by \( w_p \) and the trivial action of \( Z(\mathbb{Q}) \times \mathbb{Z}^p \). Fix a functional \( \lambda_V \in \text{Hom}_{E \otimes \hat{\mathcal{P}}}(E_w \otimes V, E) \). Similar to Definition 5.3 (and Definition 5.4 below), we have the relative completed modular symbol map

\[ \tilde{\mathcal{M}}_{\lambda_V} : \tilde{H}^0(Z, E_w)^{(q \otimes \mathcal{O}_0)} \times \left( \tilde{H}_q^i(G, V)^{(p \otimes \mathcal{O})} \otimes D(\hat{G}, \hat{\mathcal{P}}) \right) \rightarrow E. \]

Write \( \mathcal{O}_w := \mathcal{O} \), to be viewed as an \( \mathcal{O}[Z(\mathbb{Q}) \times \mathbb{Z}] \)-submodule of \( E_w \). Similar to Definitions 5.3 we make the following three definitions.

Definition 5.4. The integral relative modular symbol map

\[ \mathcal{M}^{\lambda_V} : H^0(Z, E_w) \times \left( H_q^i(G, V)^{(p)} \otimes D(\hat{G}, \hat{\mathcal{P}}) \right) \rightarrow E \]

is the bilinear map whose pull-back to

\[ H^0(Z, \mathcal{O}_w) \times \left( H_q^i(G, \mathcal{V})^{(p)} \otimes D(\hat{G}, \hat{\mathcal{P}}) \right) \]
equals the composition of

$$
H^0(\mathbb{Z}, \mathcal{O}_w) \times \left( H^r_0(G, \mathcal{M})^{(\mathfrak{p})} \otimes D(\hat{\mathcal{G}}, \mathfrak{p}) \right)
$$

\[\xrightarrow{\text{pull-back}}\]

$$
H^0(\hat{\mathcal{G}}, \mathcal{O}_w) \times \left( H^r_0(\hat{\mathcal{G}}, \mathcal{M})^{(\mathfrak{p})} \otimes D(\hat{\mathcal{G}}, \mathfrak{p}) \right)
$$

\[\xrightarrow{\text{cup product}}\]

$$
H^r_0(\hat{\mathcal{G}}, \mathcal{M})^{(\mathfrak{p})} \otimes D(\hat{\mathcal{G}}, \mathfrak{p})
\xrightarrow{\lambda_V} H^r_0(\hat{\mathcal{G}}, \mathcal{M})^{(\mathfrak{p})} \otimes D(\hat{\mathcal{G}}, \mathfrak{p})
\xrightarrow{\int} E,
$$

where \(\mathcal{M}\) is an \(\mathfrak{O}\)-lattice of \(V\) such that \(\lambda_V(\mathcal{O}_w \otimes \mathcal{M}) \subset \mathfrak{O}\).

The above definition is independent of the choice of \(\mathcal{M}\).

**Definition 5.5.** The relative modular symbol map

(5.8)

$$
\mathcal{M}_V : H^0(\mathbb{Z}, E_w) \times \left( H^r_0(G, V)^{(\mathfrak{p})} \otimes D(\hat{\mathcal{G}}, \mathfrak{p}) \right) \to E
$$

is the composition of

$$
H^0(\mathbb{Z}, E_w) \times \left( H^r_0(G, V)^{(\mathfrak{p})} \otimes D(\hat{\mathcal{G}}, \mathfrak{p}) \right)
$$

\[\xrightarrow{\text{pull-back}}\]

$$
H^0(\hat{\mathcal{G}}, E_w) \times \left( H^r_0(\hat{\mathcal{G}}, V)^{(\mathfrak{p})} \otimes D(\hat{\mathcal{G}}, \mathfrak{p}) \right)
$$

\[\xrightarrow{\text{cup product}}\]

$$
H^r_0(\hat{\mathcal{G}}, E_w \otimes V)^{(\mathfrak{p})} \otimes D(\hat{\mathcal{G}}, \mathfrak{p})
\xrightarrow{\lambda_V} H^r_0(\hat{\mathcal{G}}, E)^{(\mathfrak{p}) \otimes \mathfrak{p}} \otimes D(\hat{\mathcal{G}}, \mathfrak{p})
\xrightarrow{\int} E.
$$

**Definition 5.6.** The stable modular symbol map

$$
\hat{\mathcal{M}}_V : H^0(\mathbb{Z}, E_w) \times \left( H^r_0(G, V)^{(\mathfrak{p}) - \text{sm}} \otimes D(\hat{\mathcal{G}}) \right) \to E
$$

is the composition of

$$
H^0(\mathbb{Z}, E_w) \times \left( H^r_0(G, V)^{(\mathfrak{p}) - \text{sm}} \otimes D(\hat{\mathcal{G}}) \right)
$$

\[\xrightarrow{\text{pull-back}}\]

$$
H^0(\hat{\mathcal{G}}, E_w) \times \left( H^r_0(\hat{\mathcal{G}}, V)^{(\mathfrak{p}) - \text{sm}} \otimes D(\hat{\mathcal{G}}) \right)
$$

\[\xrightarrow{\text{cup product}}\]

$$
H^r_0(\hat{\mathcal{G}}, E_w \otimes V)^{(\mathfrak{p}) - \text{sm}} \otimes D(\hat{\mathcal{G}})
\xrightarrow{\lambda_V} H^r_0(\hat{\mathcal{G}}, E)^{(\mathfrak{p}) - \text{sm}} \otimes D(\hat{\mathcal{G}})
\xrightarrow{\int} E.
$$

Write \(E_w := E\), to be viewed as an \(E[\mathbb{Z}(\mathbb{Q}) \times \mathbb{Z}^2]\)-module via the action of \(\mathbb{Z}(\mathbb{Q}) \subset \mathbb{Z}_E(E)\) by \(w\) and the trivial action of \(\mathbb{Z}^2\). Fix a functional \(\lambda_V \in \text{Hom}_{\hat{G}(\mathbb{Q})}(E_w \otimes V, E)\).
Definition 5.7. The stable modular symbol map

\[ \widetilde{\mathcal{M}}_\lambda : H^0(\mathbb{Z}, E_w) \times \left( \hat{H}^i_{\Phi}(G, V)_{p_{\text{sm}}} \otimes D(\hat{G}) \right) \to E \]

is the composition of

\[ H^0(\mathbb{Z}, E_w) \times \left( \hat{H}^i_{\Phi}(G, V)_{p_{\text{sm}}} \otimes D(\hat{G}) \right) \xrightarrow{\text{pull-back}} H^0(\hat{G}, E_w) \times \left( \hat{H}^i_{\Phi}(\hat{G}, V)_{p_{\text{sm}}} \otimes D(\hat{G}) \right) \]

\[ \xrightarrow{\text{cup product}} H^i_c(\hat{G}, E_w) \otimes V \otimes D(\hat{G}) \]

\[ \xrightarrow{\lambda_V} H^i_c(\hat{G}, E)_{p_{\text{sm}}} \otimes D(\hat{G}) \]

\[ \xrightarrow{f} E. \]

Write \( C_w := C \otimes E_w \), which is a \( \mathbb{C}[\mathbb{Z}(\mathbb{Q}) \times \mathbb{Z}] \)-module. Similar to Definition 5.7, we have the stable modular symbol map

\[ \widetilde{\mathcal{M}}_\lambda : H^0(\mathbb{Z}, C_w) \times \left( \hat{H}^i_{\Phi}(G, V_{C})_{p_{\text{sm}}} \otimes D(\hat{G}) \right) \to \mathbb{C}. \]

5.3. Compatibility of the modular symbols. In this subsection, we show that the seven versions of the modular symbols defined in the last subsection are compatible with respect to the six arrows in the diagram (5.5).

Proposition 5.8. The diagram

\[ H^0(\mathbb{Z}, E_w) \times \left( \hat{H}^i_{\Phi}(G, V)_{p_{\text{sm}}} \otimes D(\hat{G}) \right) \xrightarrow{\widetilde{\mathcal{M}}_\lambda} E \]

\[ \xrightarrow{\otimes f(\cdot)} \]

\[ H^0(\mathbb{Z}, C_w) \times \left( \hat{H}^i_{\Phi}(G, V_{C})_{p_{\text{sm}}} \otimes D(\hat{G}) \right) \xrightarrow{\widetilde{\mathcal{M}}_\lambda} \mathbb{C} \]

commutes.

Proof. This follows from the fact that all the relevant operations are natural in the coefficient modules. \( \square \)

Note that

\[ H^0(\mathbb{Z}, E_w) = H^0(\mathbb{Z}, E_w)^{(\forall), \sigma}, \]

which is identified with the space of all functions \( f : \mathbb{Z}(\mathbb{Q}) \setminus \mathbb{Z}^2 \to E \) such that for some open compact subgroups \( D_Z \) of \( \mathbb{Z}(\mathbb{A}^\infty_p) \) and \( \mathfrak{Z} \) of \( \mathbb{Z} := \mathbb{Z}(\mathbb{Q}_p) \),

\[ f(xg_0g_1^{-1}) = w_p(g_1) \cdot f(x) \quad \text{for all } x \in \mathbb{Z}(\mathbb{Q}) \setminus \mathbb{Z}^2, \ g_0 \in D_Z, \text{ and } g_1 \in \mathfrak{Z}. \]

As an example of the second map in (5.5), we have the map

\[ (5.9) \quad H^0(\mathbb{Z}, E_w) \to H^0(\mathbb{Z}, E_w), \quad f \mapsto f \cdot w_p^{-1}. \]
Proposition 5.9. Suppose that $\lambda_V$ extends $\lambda_V$. Then the diagram

$$
\begin{align*}
H^0(Z, E_w) \times \left( H^i_{\phi}(G, V)_{p-sm} \otimes D(\hat{G}) \right) & \xrightarrow{\cdot \lambda_{\hat{V}}} E \\
\downarrow \text{pull-back} & \downarrow \text{pull-back} \\
H^0(\hat{G}, E_w) \times \left( H^i_{\phi}(\hat{G}, V)_{p-sm} \otimes D(\hat{G}) \right) & \xrightarrow{\cdot \lambda_{\hat{V}}} E
\end{align*}
$$

commutes.

Proof. It is routine to check that the following diagram is commutative:

$$
\begin{align*}
H^0(Z, E_w) \times \left( H^i_{\phi}(G, V)_{p-sm} \otimes D(\hat{G}) \right) & \xrightarrow{\cdot \lambda_{\hat{V}}} H^0(Z, E_w) \times \left( H^i_{\phi}(G, V)_{p-sm} \otimes D(\hat{G}) \right) \\
\downarrow \text{pull-back} & \downarrow \text{pull-back} \\
H^0(\hat{G}, E_w) \times \left( H^i_{c}(\hat{G}, V)_{p-sm} \otimes D(\hat{G}) \right) & \xrightarrow{\cdot \lambda_{\hat{V}}} H^0(\hat{G}, E_w) \times \left( H^i_{c}(\hat{G}, V)_{p-sm} \otimes D(\hat{G}) \right)
\end{align*}
$$

This proves the proposition. \qed

Proposition 5.10. The diagram

$$
\begin{align*}
H^0(Z, E_w) \times \left( H^i_{\phi}(G, V)_{p-sm} \otimes D(\hat{G}) \right) & \xrightarrow{\cdot \lambda_{\hat{V}}} E \\
\downarrow \text{cup product} \quad \text{and} \quad \downarrow \text{cup product} & \downarrow \text{cup product} \\
H^i_{c}(\hat{G}, E_w \otimes V)_{p-sm} \otimes D(\hat{G}) & \xrightarrow{\cdot \lambda_V} H^i_{c}(\hat{G}, E_w \otimes V)_{p-sm} \otimes D(\hat{G}) \\
\downarrow f & \downarrow f \\
E & \xrightarrow{\cdot \lambda_V} E
\end{align*}
$$

commutes.

Proof. This follows from Lemmas 4.13, 3.22 and 4.13. \qed
**Proposition 5.11.** The diagram

\[
\begin{array}{c}
\tilde{H}^0 \left( Z, E_w \right)^{(j \geq 0), \circ} \times \left( \tilde{H}^j_\Phi(G, V)^{(p \geq 0), \circ} \otimes D(\hat{G}, \hat{p}) \right) \xrightarrow{\tilde{\mathcal{H}}_{\lambda V}} E \\
\phantom{=} \downarrow \phantom{=} \\
H^0 \left( Z, E_w \right) \times \left( H^j_\Phi(G, V)^{(p), \circ} \otimes D(\hat{G}, \hat{p}) \right) \xrightarrow{\mathcal{H}_{\lambda V}} E \\
\phantom{=} \downarrow \phantom{=} \\
H^0 \left( Z, E_w \right) \times \left( H^j_\Phi(G, V)^{(p)} \otimes D(\hat{G}, \hat{p}) \right) \xrightarrow{\mathcal{H}_{\lambda V}} E
\end{array}
\]

commutes.

**Proof.** Similar to Proposition 5.8, this follows from the fact that all the relevant operations are natural in the coefficient modules.

Obviously identify \( E_w \otimes V \) with \( V \) as vector spaces. Assume that \( \lambda_0 = (\lambda_V)|_{V_0} \). The following proposition also follows from the fact that all the relevant operations are natural in the coefficient modules.

**Proposition 5.12.** Suppose that \( w_p \) is trivial on \( Z_0 \) so that

\[
\tilde{H}^0 \left( Z, E_w \right)^{(j \geq 0), \circ} = \tilde{H}^0 \left( Z, E \right)^{(j \geq 0), \circ}.
\]

Then the diagram

\[
\begin{array}{c}
\tilde{H}^0 \left( Z, E \right)^{(j \geq 0), \circ} \times \left( \tilde{H}^j_\Phi(G, V_0)^{(p \geq 0), \circ} \otimes D(\hat{G}, \hat{p}) \right) \xrightarrow{\tilde{\mathcal{H}}_{\lambda V}} E \\
\phantom{=} \downarrow \phantom{=} \\
\tilde{H}^0 \left( Z, E_w \right)^{(j \geq 0), \circ} \times \left( \tilde{H}^j_\Phi(G, V)^{(p \geq 0), \circ} \otimes D(\hat{G}, \hat{p}) \right) \xrightarrow{\tilde{\mathcal{H}}_{\lambda V}} E
\end{array}
\]

commutes.

Suppose that we are given an \( E \)-vector space \( \mathcal{H} \) that fits into a commutative diagram

(5.10)

\[
\begin{array}{c}
\tilde{H}^j_\Phi(G, V_0)^{(p \geq 0), \circ} \otimes D(g/p) \xrightarrow{C \otimes (\cdot)} H^j_\Phi(G, V)^{(p \geq 0), \circ} \otimes D(g/p) \\
\tilde{H}^j_\Phi(G, V_0)^{(p \geq 0), \circ} \xrightarrow{\tilde{\xi}} \mathcal{H} \xrightarrow{\xi \circ} H^j_\Phi(G, V)^{(p \geq 0), \circ} \otimes D(g/p) \\
\tilde{H}^j_\Phi(G, V)^{(p \geq 0), \circ} \xrightarrow{\tilde{\xi}} \mathcal{H} \xrightarrow{\xi \circ} H^j_\Phi(G, V)^{(p \geq 0), \circ} \otimes D(g/p)
\end{array}
\]
We will construct such a space $\mathcal{H}$ in the next two sections.

Note that $\mathcal{G}(\mathbb{Q})$ is dense in an open subgroup of $\hat{G}$. Hence as in (1.18) we have natural inclusions

\[
\text{Hom}_{\hat{G}(\mathbb{Q})}(E_w \otimes V, E) \subset \text{Hom}_{E \otimes \hat{p}}(E_w \otimes V, E) \subset \text{Hom}_{E \otimes \hat{p}}(V, E),
\]

where $w$ is an algebraic character of $\mathbb{Z}_E$ defined over $E$ such that $w_p$ is trivial on $Z_0$. In particular, $\lambda_V$ is also viewed as an element of $\text{Hom}_{E \otimes \hat{p}}(V, E)$.

The following theorem asserts that relative completed modular symbols interpolate the stable modular symbols on $\mathcal{H}$, for various weights $w$.

**Theorem 5.13.** Let $\lambda_0 \in \text{Hom}_{E \otimes \hat{p}}(V_0, E)$. Then for all algebraic characters $w$ of $\mathbb{Z}_E$ defined over $E$ such that $w_p$ is trivial on $Z_0$, and all $\lambda_V \in \text{Hom}_{\hat{G}(\mathbb{Q})}(E_w \otimes V, E)$ such that $(\lambda_V)|_{V_0} = \lambda_0$, the diagram

\[
\begin{array}{ccc}
H^0(\mathbb{Z}, C_w) \times \left( \mathbb{H}_\phi(G, V_{\mathbb{C}})_{\text{p-sm}} \otimes D(\hat{G}) \right) & \xrightarrow{\mathcal{M}_{\lambda_V}} & \mathbb{C} \\
\uparrow & & \uparrow \subset \\
H^0(\mathbb{Z}, E_w) \times \left( \mathcal{H} \otimes D(\hat{G}, \hat{p}) \right) & \xrightarrow{\mathcal{M}_{\lambda_V} \circ \tilde{\xi}} & E \\
\text{and } \tilde{\xi} & & \downarrow \subset \\
\tilde{H}^0(\mathbb{Z}, E_{\mathbb{C}}) \times \left( \mathbb{H}_\phi(G, V_0)_{\mathbb{C}} \otimes D(\hat{G}, \hat{p}) \right) & \xrightarrow{\mathcal{M}_{\lambda_0}} & E
\end{array}
\]

commutes. Here the middle horizontal arrow is the composition of

\[
H^0(\mathbb{Z}, E_w) \times \left( \mathcal{H} \otimes D(\hat{G}, \hat{p}) \right) \xrightarrow{\tilde{\xi}} H^0(\mathbb{Z}, E_w) \times \left( \mathbb{H}_\phi(G, V)_{\text{p-sm}} \otimes D(\hat{G}) \right) \xrightarrow{\mathcal{M}_{\lambda_V}} E.
\]

**Proof.** This follows by combining Propositions 5.8–5.12. \qed

### 6. Parabolic cohomologies and the $t$-stable part

In this section, we define parabolic cohomology groups which include the space $\mathcal{B}_P(H_\phi(G, V)) \otimes D(\mathfrak{g}/\mathfrak{p})$ in the Introduction as an example. These can be viewed as variants of the Jacquet modules studied in [Em06b]. We will also define their “$t$-stable parts”, which are related to the nearly ordinary part.

#### 6.1. Parabolic pairs

In view of [Be87, Section 5.2] we make the following definition.
Definition 6.1. A parabolic pair in $\mathfrak{g}$ is a pair $\mathfrak{p} \supset \mathfrak{n}$ of its Lie subalgebras such that some $t \in G$ defines $\mathfrak{p} \supset \mathfrak{n}$ in the following sense:

\[
\begin{cases}
  t \text{ normalizes both } \mathfrak{p} \text{ and } \mathfrak{n}; \\
  \text{all eigenvalues (in } \mathbb{C}_p^\times \text{) of } \text{Ad}_t : \mathfrak{n} \to \mathfrak{n} \text{ have } p\text{-adic norms } > 1; \\
  \text{all eigenvalues of } \text{Ad}_t : \mathfrak{p}/\mathfrak{n} \to \mathfrak{p}/\mathfrak{n} \text{ have } p\text{-adic norm } 1; \\
  \text{all eigenvalues of } \text{Ad}_t : \mathfrak{g}/\mathfrak{p} \to \mathfrak{g}/\mathfrak{p} \text{ have } p\text{-adic norms } < 1.
\end{cases}
\]

Such an element $t$ is called a defining element of the parabolic pair.

Here and henceforth $\text{Ad}$ indicates various maps induced by the conjugations. It is clear that every element of $G$ defines a unique parabolic pair in $\mathfrak{g}$.

Definition 6.2. An element of $G$ is said to be split if it is the image of $\mathfrak{p}$ under some (unique) algebraic homomorphism $\mathbb{Q}_p^\times \to G$.

Put

\[\text{Df} (\mathfrak{p}, \mathfrak{n}) := \{ \text{split defining elements } t \in G \text{ of the parabolic pair } \mathfrak{p} \supset \mathfrak{n} \}\].

Lemma 6.3. The set $\text{Df} (\mathfrak{p}, \mathfrak{n})$ is nonempty.

Proof. The lemma follows by the following observation: If we replace $t \in G$ by the semi-simple part of its Jordan decomposition, or its positive power, or its multiplication by an element in a compact subgroup of $G$ that commutes with $t$, then the defined parabolic pair remains unchanged. \hfill \Box

Let $P$ denote the normalizer in $G$ of the pair $\mathfrak{p} \supset \mathfrak{n}$. Note that the Lie algebra of $P$ is equal to $\mathfrak{p}$.

Let $t \in \text{Df} (\mathfrak{p}, \mathfrak{n})$. Note that all eigenvalues of $\text{Ad}_t : \mathfrak{n} \to \mathfrak{n}$ are negative powers of $p$. We have a unique $\text{Ad}_t$-stable decomposition

\[\mathfrak{g} = \tilde{\mathfrak{n}}_t \oplus \mathfrak{l}_t \oplus \mathfrak{n}\]

such that all eigenvalues of $\text{Ad}_t : \tilde{\mathfrak{n}}_t \to \tilde{\mathfrak{n}}_t$ are positive powers of $p$, and $\mathfrak{l}_t$ equals the invariant space of $t$ in $\mathfrak{g}$. Denote by $L_t$ the normalizer of $\mathfrak{l}_t$ in $P$. Note that the Lie algebra of $L_t$ is equal to $\mathfrak{l}_t$.

Lemma 6.4. The subgroups $P$ and $L_t$ of $G$ are both algebraic, and there are unipotent algebraic subgroups $N$ and $\tilde{N}_t$ of $G$ whose Lie algebras are respectively equal to $\mathfrak{n}$ and $\tilde{\mathfrak{n}}_t$.

Proof. By realizing $G$ as an algebraic subgroup of a general linear group, it is easy to see that there are connected algebraic subgroups $P_0$, $L_{t,0}$, $N$ and $\tilde{N}_t$ of $G$ whose Lie algebras are respectively equal to $\mathfrak{p}$, $\mathfrak{l}_t$, $\mathfrak{n}$ and $\tilde{\mathfrak{n}}_t$. Moreover, $P \supset P_0$, $L_t \supset L_{t,0}$, and both $N$ and $\tilde{N}_t$ are unipotent. This implies the lemma. \hfill \Box
Let $N$ and $\bar{N}_t$ be as in Lemma 6.4. It is clear that the multiplication map
\[ \bar{N}_t \times P \to G \]
is an open embedding.

Set $L := P/N$. Denote by $P_0 \subset P$ and $L_0 \subset L$ the identity connected components under the Zariski topologies.

**Lemma 6.5.** For all $t_1, t_2 \in \text{Df}(p, n)$, there exists $g \in N$ such that $g t_1 g^{-1}$ commutes with $t_2$.

*Proof.* In view of Lemma 6.4 we assume without loss of generality that $G = P$ and $P$ is connected as an algebraic group. Then $L_{t_i}$ is connected as an algebraic group, $P = L_{t_i} \ltimes N$, and $t_i$ is a central element of $L_{t_i}$ ($i = 1, 2$). Write

\[ L_{t_i} = L_i \ltimes N_i \]

for a Levi decomposition of $L_{t_i}$ where $N_i$ is the unipotent radical. Then $L_i$ is also a Levi factor of $P$, and $N_i \ltimes N$ equals the unipotent radical of $P$. By uniqueness of Levi factors in characteristic zero, we have elements $g \in N$ and $g' \in \bar{N}_1$ such that

\[ (g g') L_i (g g')^{-1} = L_2. \]

Then we have that

\[ g t_1 g^{-1} = (g g') t_1 (g g')^{-1} \in L_2 \subset L_{t_2}. \]

This proves the lemma.

**Lemma 6.6.** For all $t_1, t_2 \in \text{Df}(p, n)$,

\[ t_1 t_2 = t_2 t_1 \iff \mathfrak{l}_{t_1} = \mathfrak{l}_{t_2} \iff L_{t_1} = L_{t_2}. \]

*Proof.* First assume that $t_1 t_2 = t_2 t_1$. Note that $t_2$ stabilizes $\mathfrak{l}_{t_1}$. Hence it centralizes $\mathfrak{l}_{t_1}$, by considering the eigenvalues of the operator $\text{Ad}_{t_2} : \mathfrak{p} \to \mathfrak{p}$. Thus $\mathfrak{l}_{t_1} \subset \mathfrak{l}_{t_2}$. Similarly $\mathfrak{l}_{t_2} \subset \mathfrak{l}_{t_1}$ and hence $\mathfrak{l}_{t_1} = \mathfrak{l}_{t_2}$. All other implications are obvious.

Note that by Lemmas 6.5 and 6.6 all the groups

\[ \{ L_t : t \in \text{Df}(p, n) \} \]

are conjugate to each other by $N$.

**Lemma 6.7.** The equality $P = L_t \ltimes N$ holds.

*Proof.* Let $g \in P$. By Lemma 6.5 there is an element $g' \in N$ such that $g' g t (g' g)^{-1}$ commutes with $t$. Then by Lemma 6.6 we have that

\[ \mathfrak{l}_t = \mathfrak{l}_{g' g t (g' g)^{-1}} = \text{Ad}_{g' g}(\mathfrak{l}_t) \]

and hence $g' g \in L_t$. This implies the lemma.
6.2. A partial order.

Definition 6.8. For any two open compact subgroups \( G \) and \( G' \) of \( G \), define
\[
G \preceq_P G' \quad \text{if and only if} \quad P \subset P' \quad \text{and} \quad G' \subset GP',
\]
where \( P := G \cap P \) and \( P' := G' \cap P \).

It is easily checked that \( \preceq_P \) is a partial order on the set of all open compact subgroups of \( G \). Note that the conditions in (6.2) imply that
\[
G' = (G \cap G')P' \quad \text{and} \quad P \cap (G G') = P'.
\]
The following lemma will be useful.

Lemma 6.9. Let \( G_1 \preceq_P G_2 \preceq_P G_3 \) be three open compact subgroups of \( G \). Then
\[
G_2 = (G_1 \cap G_2)(G_2 \cap G_3) \quad \text{and} \quad G_1 \cap G_3 \subset G_2.
\]

Proof. The first assertion is obvious in view of the first equality in (6.3). For the second assertion, we have that
\[
G_1 \cap G_3 = G_1 \cap ((G_2 \cap G_3)P_3) \quad \text{and} \quad G_1 \cap (G_2 P) \subset G_2.
\]
Here the last inclusion follows from the second equality in (6.3). \( \square \)

The following lemma is easily checked.

Lemma 6.10. For all open compact subgroups \( G, G' \) of \( G \) with \( G \preceq_P G' \), the natural map
\[
(G' \cap P)/(G \cap G' \cap P) \to G'/(G \cap G')
\]
is bijective.

Let \( L \) be an open compact subgroup of \( L_t \). Let \( L_t \) denote the open compact subgroup of \( L_t \) that corresponds to \( L \) under the obviously isomorphism \( L \cong L_t \). Recall that \( G \) is an open subgroup of \( G^\natural \). Set \( P := P \cap G \),
\[
\mathcal{P}_{\mathfrak{L},P} := \{\text{open compact subgroups} \, \mathfrak{P} \, \text{of} \, P \, \text{such that} \, \mathfrak{P}/(\mathfrak{P} \cap N) = \mathfrak{L}\},
\]
and
\[
\mathcal{G}_{\mathfrak{L},G} := \{\text{open compact subgroups} \, G \, \text{of} \, G \cap G \, \text{such that} \, G \cap P \in \mathcal{P}_{\mathfrak{L},P}\}.
\]
Then
\[
\mathcal{P}_{\mathfrak{L},P} \neq \emptyset \iff \mathcal{G}_{\mathfrak{L},G} \neq \emptyset \iff \mathfrak{L} \subset P/(N \cap P),
\]
in which case \( \mathcal{P}_{\mathfrak{L},P} \) is a directed set under the inclusion relation, and \( \mathcal{G}_{\mathfrak{L},G} \) is a directed set under the partial order \( \preceq_P \).
Lemma 6.11. Let $P$ be an open compact subgroup of $P_0$. Then

$P \subset t^kPt^{-k}$ for some positive integer $k$

$\iff P \subset t^kPt^{-k}$ for all sufficiently large positive integer $k$

$\iff P = (P \cap L_t)(P \cap N)$.

Proof. Suppose that $P \subset t^kPt^{-k}$, where $k$ is a positive integer. Let $g = g_1g_2 \in P$ with $g_1 \in L_t$ and $g_2 \in N$. Then

$g_1 = \lim_{r \to \infty} g_1t^{-rk}g_2t^{rk} = \lim_{r \to \infty} t^{-rk}gt^{rk} \in P$.

This proves the equality $P = (P \cap L_t)(P \cap N)$. The rest of the lemma is obvious. □

Recall that $D$ denotes an open compact subgroup of $G^\bullet \cap G$. We say that $L$ is $D$-neat if so is $L_t$. This is independent of $t \in \text{Def}(p, n)$. Write

$\mathcal{G}^{D,t}_{\mathbb{G},G} := \{D\text{-neat open compact subgroups } \mathcal{G} \text{ of } G \cap G \text{ such that } L_t \subset \mathcal{G} \subset \bar{N}_tL_tN\}$.

The following lemma is obvious.

Lemma 6.12. Suppose that $L \subset L_0$. The set $\mathcal{G}^{D,t}_{\mathbb{G},G}$ is nonempty if and only if $L_t$ is $D$-neat and contained in $P$. When this is the case $\mathcal{G}^{D,t}_{\mathbb{G},G}$ is cofinal in the directed set $\mathcal{G}^{D,t}_{\mathbb{G},G}$.

When $L \subset L_0$, for each $\mathcal{G} \in \mathcal{G}^{D,t}_{\mathbb{G},G}$ we set

$N_{\mathcal{G}} := N_{\mathcal{G},t} := \{k \in \mathbb{N} : \mathcal{G} \preceq t^k\mathcal{G}t^{-k}\}$.

This is a submonoid of the additive monoid $\mathbb{N}$ which contains all but finitely many elements of $\mathbb{N}$.

6.3. Parobolic cohomology.

Lemma 6.13. Let $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ be three open compact subgroups of $G \cap G$ such that $\mathcal{G}_1 \preceq_P \mathcal{G}_2 \preceq_P \mathcal{G}_3$. Then the diagram

$$
\begin{array}{ccc}
H_{\mathcal{G}}^i(D\mathcal{G}_3, M) & \xrightarrow{\rho_{\mathcal{G}_1,\mathcal{G}_2}} & H_{\mathcal{G}}^i(D\mathcal{G}_2, M) \\
\downarrow{\rho_{\mathcal{G}_3,\mathcal{G}_2}} & & \downarrow{\rho_{\mathcal{G}_3,\mathcal{G}_1}} \\
H_{\mathcal{G}}^i(D\mathcal{G}_1, M)
\end{array}
$$

commutes.

Proof. This follows from Lemmas 3.4 and 6.9 □

Suppose that $L$ is contained in $P/(N \cap P)$ so that both $\mathcal{P}_{L,P}$ and $\mathcal{G}_{\mathbb{G},G}$ are nonempty directed sets.
Definition 6.14. In view of Lemma 6.13, we define the parabolic cohomology group

\[ \mathcal{H}_{i \phi, P}(D \mathfrak{L}, M) := \lim_{\psi \in \mathcal{G}_{\mathfrak{L}, \mathfrak{P}}} \mathcal{H}_{i \phi}(D \mathfrak{G}, M) \]

It is clear that the assignment \( \mathcal{H}_{i \phi, P}(D \mathfrak{L}, \cdot) \) is a functor from the category of \( R[G(\mathbb{Q}) \times G] \)-modules to the category of \( R \)-modules. We remark that the parabolic cohomology group only depends on \( P \) in the following sense: if \( G' \) is an open subgroup of \( G \) containing \( D \) such that \( G' \cap P = G \cap P \), then \( \mathcal{G}_{\mathfrak{L}, G} \) is a cofinal subset of \( \mathcal{G}_{\mathfrak{L}, G} \) and hence

\[ \lim_{\psi \in \mathcal{G}_{\mathfrak{L}, G}} \mathcal{H}_{i \phi}(D \mathfrak{G}, M) = \lim_{\psi \in \mathcal{G}_{\mathfrak{L}, G'}} \mathcal{H}_{i \phi}(D \mathfrak{G}, M) \]

We have an identification

\[ (6.4) \mathcal{H}_{i \phi, P}(D \mathfrak{L}, M) = \lim_{\psi \in \mathcal{G}_{\mathfrak{L}, P}} \mathcal{H}_{i \phi}(D \mathfrak{G}, M) \]

as in the following proposition, where the transition maps are the pull-back maps.

Proposition 6.15. The projections yield an isomorphism

\[ (6.5) \mathcal{H}_{i \phi, P}(D \mathfrak{L}, M) \rightarrow \lim_{\psi \in \mathcal{G}_{\mathfrak{L}, P}} \mathcal{H}_{i \phi}(D \mathfrak{G}, M) \]

whose inverse is the unique homomorphism

\[ (6.6) \lim_{\psi \in \mathcal{G}_{\mathfrak{L}, P}} \mathcal{H}_{i \phi}(D \mathfrak{G}, M) \rightarrow \mathcal{H}_{i \phi, P}(D \mathfrak{L}, M) \]

such that the diagram

\[ \lim_{\psi \in \mathcal{G}_{\mathfrak{L}, P}} \mathcal{H}_{i \phi}(D \mathfrak{G}, M) \longrightarrow \mathcal{H}_{i \phi, P}(D \mathfrak{L}, M) \]

\[ \downarrow \quad \downarrow \]

\[ \mathcal{H}_{i \phi}(D(\mathfrak{G} \cap P), M) \longrightarrow \mathcal{H}_{i \phi}(D \mathfrak{G}, M) \]

commutes for all \( \mathfrak{G} \in \mathcal{G}_{\mathfrak{L}, G} \).

Proof. The uniqueness of (6.6) obvious. Lemma 6.10 implies that the diagram

\[ H_{i \phi}(D(\mathfrak{G} \cap P), M) \longrightarrow H_{i \phi}(D \mathfrak{G}', M) \]

\[ \downarrow \quad \downarrow \]

\[ H_{i \phi}(D(\mathfrak{G} \cap P), M) \longrightarrow H_{i \phi}(D \mathfrak{G}, M) \]

commutes, for all \( \mathfrak{G}, \mathfrak{G}' \in \mathcal{G}_{\mathfrak{L}, G} \) with \( \mathfrak{G} \preceq P \mathfrak{G}' \). This proves the existence of the homomorphism (6.6).
Similarly the diagram

\[
\begin{array}{c}
\text{projection} \\
\downarrow \\
\text{pull-back}
\end{array}
\begin{array}{c}
H^i_{\phi, P}(D \mathcal{L}, M) \xrightarrow{\text{projection}} H^i_{\phi}(D \mathcal{L}', M) \\
H^i_{\phi}(D \mathcal{P}, M)
\end{array}
\]

commutes for all $\mathcal{P}, \mathcal{P}' \in \mathcal{P}_{\mathcal{L}, P}$ with $\mathcal{P} \subset \mathcal{P}'$. Hence we have a natural map

\[
H^i_{\phi, P}(D \mathcal{L}, M) \to \lim_{\psi \in \mathcal{P}_{\mathcal{L}, P}} H^i_{\phi}(D \mathcal{P}, M).
\]

It is routine to check that this is the inverse of the map (6.6). □

Let $D'$ be an open subgroup of $D$ and let $\mathcal{L}'$ be an open subgroup of $\mathcal{L}$. We define the pull-back map

\[
\rho_{D \mathcal{L}, D' \mathcal{L}'} : H^i_{\phi, P}(D \mathcal{L}, M) \to H^i_{\phi, P}(D' \mathcal{L}', M)
\]

as in the following lemma.

**Lemma 6.16.** There is a unique homomorphism

\[
\rho_{D \mathcal{L}, D' \mathcal{L}'} : H^i_{\phi, P}(D \mathcal{L}, M) \to H^i_{\phi, P}(D' \mathcal{L}', M)
\]

such that the diagram

\[
\begin{array}{c}
\downarrow \\
\downarrow
\end{array}
\begin{array}{c}
H^i_{\phi, P}(D \mathcal{L}, M) \xrightarrow{\rho_{D \mathcal{L}, D' \mathcal{L}'}} H^i_{\phi, P}(D' \mathcal{L}', M) \\
H^i_{\phi}(D \mathcal{G}, M) \xrightarrow{\rho_{D \mathcal{G}, D' \mathcal{G}'} \rho} H^i_{\phi}(D' \mathcal{G}', M)
\end{array}
\]

commutes for all $\mathcal{G} \in \mathcal{G}_{\mathcal{L}, G}$ and $\mathcal{G}' \in \mathcal{G}_{\mathcal{L}', G}$ with $\mathcal{G}' \leq_P \mathcal{G}$.

**Proof.** It is easy to see that for every $\mathcal{G}' \in \mathcal{G}_{\mathcal{L}', G}$, there is a group $\mathcal{G} \in \mathcal{G}_{\mathcal{L}, G}$ with $\mathcal{G}' \leq_P \mathcal{G}$. This implies the uniqueness assertion of the lemma. Let $\mathcal{G}_1 \in \mathcal{G}_{\mathcal{L}, G}$ and $\mathcal{G}'_1 \in \mathcal{G}_{\mathcal{L}', G}$ with

\[
\mathcal{G} \leq_P \mathcal{G}_1, \quad \mathcal{G}' \leq_P \mathcal{G}'_1, \quad \mathcal{G}_1 \leq_P \mathcal{G}.
\]

Lemma 3.4 and Lemma 6.9 imply that the diagram

\[
\begin{array}{c}
\downarrow \quad \rho \downarrow \\
H^i_{\phi}(D \mathcal{G}, M) \xrightarrow{\rho} H^i_{\phi}(D' \mathcal{G}', M)
\end{array}
\]

commutes. This implies the existence assertion of the lemma. □
Lemma 6.17. Let $D_1 \supset D_2 \supset D_3$ be open compact subgroups of $G^{2,p} \cap G$, and let $\mathcal{L}_1 \supset \mathcal{L}_2 \supset \mathcal{L}_3$ be open compact subgroups of $L \cap G$. Then the diagram
\[
\begin{array}{ccc}
H^i_{\Phi,P}(D_1\mathcal{L}_1, M) & \rightarrow & H^i_{\Phi,P}(D_2\mathcal{L}_2, M) \\
\downarrow & & \downarrow \\
H^i_{\Phi,P}(D_3\mathcal{L}_3, M)
\end{array}
\]
commutes, where the three arrows are the pull-back maps.

Proof. This also follows from Lemma 3.4 and Lemma 6.9. □

It is clear that the pull-back maps are natural in the coefficient modules. The following lemma is routine to check.

Lemma 6.18. The diagram
\[
\begin{array}{ccc}
H^i_{\Phi,P}(D\mathcal{L}, M) & \xrightarrow{\text{pull-back}} & H^i_{\Phi,P}(D'\mathcal{L}', M) \\
\downarrow & & \downarrow \\
H^i_{\Phi}(D\mathfrak{P}, M) & \xrightarrow{\text{pull-back}} & H^i_{\Phi}(D'\mathfrak{P}', M)
\end{array}
\]
commutes for all $\mathfrak{P} \in \mathcal{P}_{\mathcal{L},P}$ and $\mathfrak{P}' \in \mathcal{P}_{\mathcal{L}',P}$ with $\mathfrak{P}' \subset \mathfrak{P}$.

Proof. This is implied by Lemma 6.10. □

In view of Lemma 6.17, we define
\[
(6.7) \quad H^i_{\Phi,P}(G, M) := \lim_{D \mathcal{L}} H^i_{\Phi,P}(D\mathcal{L}, M),
\]
where $D$ runs over open compact subgroups of $G^{2,p} \cap G$ and $\mathcal{L}$ runs over open compact subgroups of $P/(N \cap P)$. Define a homomorphism
\[
(6.8) \quad H^i_{\Phi,P}(D\mathcal{L}, M) \rightarrow H^i_{\Phi}(G, M)^{(p)}
\]
as the composition of
\[
\begin{array}{ccc}
H^i_{\Phi,P}(D\mathcal{L}, M) & \xrightarrow{\text{projection}} & H^i_{\Phi}(D\mathfrak{P}, M) \\
\downarrow & & \downarrow \\
H^i_{\Phi}(G, M)^{(p)}
\end{array}
\]
where $\mathfrak{P} \in \mathcal{P}_{\mathcal{L},G}$. This homomorphism is obviously independent of $\mathfrak{P}$. It follows from Lemma 6.18 that the diagram
\[
\begin{array}{ccc}
H^i_{\Phi,P}(D\mathcal{L}, M) & \xrightarrow{\text{pull-back}} & H^i_{\Phi,P}(D'\mathcal{L}', M) \\
\downarrow & & \downarrow \\
H^i_{\Phi}(G, M)^{(p)}
\end{array}
\]
commutes. Thus the homomorphisms for various $D$ and $\mathcal{L}$ yield a homomorphism
\[
(6.9) \quad H^i_{\Phi,P}(G, M) \rightarrow H^i_{\Phi}(G, M)^{(p)}.
\]
Note that if $L$ is $D$-neat and $L \subset L_0$, then every compact subgroup of $L,N$ is $D$-neat. In the rest of this subsection, assume that $R$ is a $\mathbb{Q}$-algebra, $L$ is $D$-neat, and $L \subset L_0$.

**Proposition 6.19.** The identification (6.4) and the horizontal arrows in (3.27) yield an isomorphism

$$
\hat{H}_i^\Phi(L,M \otimes D(g/p) \cong H_i^\Phi(DL,M).
$$

Moreover, the diagram

$$
\begin{array}{ccc}
\hat{H}_i^\Phi(L,M) & \otimes & D(g/p) \\
\inclusion & \cong & \inclusion \\
\downarrow & & \downarrow \\
\hat{H}_i^\Phi(L',M) & \otimes & D(g/p)
\end{array}
$$

commutes for all open subgroups $D'$ and $L'$ of $D$ and $L$ respectively.

**Proof.** This easily follows from Lemmas 3.19 and 6.18.

Proposition 6.19 implies that

$$
(6.10) \quad H_i^\Phi(P(DL,M) = \left( H_i^\Phi(L,M)_{p-sm}\right)^{N/P} \otimes D(g/p),
$$

and the natural map $H_i^\Phi(DL,M) \rightarrow H_i^\Phi(L,M)$ is injective so that $H_i^\Phi(DL,M)$ is considered as a subspace of $H_i^\Phi(L,M)$. It is clear that both sides of (6.10) carry natural representations of the group $P/(N \cap P)$, and the identification (6.10) respects these representations. Moreover,

$$
(6.11) \quad H_i^\Phi(DL,M) = H_i^\Phi(DL,M)^{DG}.
$$

6.4. The $t$-stable part. In the rest of this section we assume that $L$ is $D$-neat, $L \subset L_0$, and $L_t \subset P$. Then $\mathcal{G}^{DL,G}$ is a nonempty cofinal subset of $\mathcal{G}^{L,G}$. In particular,

$$
H_i^\Phi(DL,M) = \lim_{\Phi \in \mathcal{G}^{DL,G}} H_\Phi^i(D\mathcal{L},M).
$$

Let $\langle t \rangle$ denote the submonoid of $G$ generated by $t$. Suppose that we are given a compatible action of $\langle t \rangle$ on $M$:

$$
\langle t \rangle \times M \rightarrow M, \quad (t^k,u) \mapsto t^k * u.
$$

This action yields Hecke maps as in (3.20), which are homomorphisms to be denoted by

$$
\rho^i_{t^k} := \rho^i_{t^k,K,K'} : H_i^\Phi(K,M) \rightarrow H_i^\Phi(K',M),
$$

where $k \in \mathbb{N}$, and $K$ and $K'$ are open compact subgroups of $G$. 
Let $\mathfrak{G} \in \mathcal{G}_{D,G}^{D,t}$.

**Lemma 6.20.** The map

$$k \mapsto \rho_{k,*}^{t,*}(D\mathfrak{G}, M)$$

defines a representation of the monoid $\mathbb{N}_\mathfrak{G}$ on $H^t_{\Phi}(D\mathfrak{G}, M)$.

**Proof.** This follows from Proposition 3.16 and Lemma 6.9. $\square$

We call the representation in Lemma 6.20 the Hecke representation. Set

$$H^t_{\Phi}(D\mathfrak{G}, M) := \bigcap_{k \in \mathbb{N}_\mathfrak{G}} \rho_{k,*}^{t,*}(H^t_{\Phi}(D\mathfrak{G}, M)).$$

**Lemma 6.21.** Suppose that $\mathfrak{G}, \mathfrak{G}' \in \mathcal{G}_{D,G}^{D,t}$ with $\mathfrak{G} \preceq_P \mathfrak{G}'$. Then the transfer map (6.12)

$$\rho : H^t_{\Phi}(D\mathfrak{G}', M) \rightarrow H^t_{\Phi}(D\mathfrak{G}, M)$$

sends $H^t_{\Phi}(D\mathfrak{G}', M)$ into $H^t_{\Phi}(D\mathfrak{G}, M)$.

**Proof.** It follows from Proposition 3.16 that the transfer map $\rho$ in (6.12) is equivariant under the Hecke representations of the monoid $\{t^k : k \in \mathbb{N}_\mathfrak{G} \cap \mathbb{N}_{\mathfrak{G}'}\}$. Thus

$$\rho \left( H^t_{\Phi}(D\mathfrak{G}', M) \right) = \rho \left( \bigcap_{k \in \mathbb{N}_\mathfrak{G} \cap \mathbb{N}_{\mathfrak{G}'}} \rho_{k,*}(H^t_{\Phi}(D\mathfrak{G}', M)) \right) \subset \bigcap_{k \in \mathbb{N}_\mathfrak{G} \cap \mathbb{N}_{\mathfrak{G}'}} \rho_{k,*}(H^t_{\Phi}(D\mathfrak{G}, M)) = H^t_{\Phi}(D\mathfrak{G}, M).$$

$\square$

**Definition 6.22.** In view of Lemma 6.21, we define the $t$-stable part of the parabolic cohomology group $H^t_{\Phi,P}(D\mathfrak{L}, M)$ to be

$$H^t_{\Phi,P}(D\mathfrak{L}, M) := \lim_{\mathfrak{G} \rightarrow \mathcal{G}_{D,G}^{D,t}} H^t_{\Phi}(D\mathfrak{G}, M) \subset H^t_{\Phi,P}(D\mathfrak{L}, M).$$

Similar to the case of parabolic cohomology groups, the $t$-stable parts also only depend on $P$.

**Lemma 6.23.** Let $D'$ be an open subgroup of $D$, and $\mathfrak{L}'$ an open subgroup of $\mathfrak{L}$. Then the pull-back map (6.13)

$$\rho : H^t_{\Phi,P}(D\mathfrak{L}, M) \rightarrow H^t_{\Phi,P}(D'\mathfrak{L}', M)$$

sends $H^t_{\Phi,P}(D\mathfrak{L}, M)$ into $H^t_{\Phi,P}(D'\mathfrak{L}', M)$.

**Proof.** This is similar to the proof of Lemma 6.21. $\square$
Definition 6.24. In view of Lemma 6.23, we define the $t$-stable part
\begin{equation}
H_{i, P}^t(G, M) := \lim_{D \to \mathcal{D}L} H_{i, P}^t(D \mathcal{L}, M) \subset H_{i, P}^t(G, M),
\end{equation}
where $D$ runs over open compact subgroups of $G^{\mathbb{Z}} \cap G$ and $\mathcal{L}$ runs over $D$-neat open compact subgroups of $(P_0 \cap G)/(N \cap P)$.

6.5. A condition on $\Phi$. Recall the following well-known result, which is more or less a consequence of the Borel-Serre compactification (cf. [BS73], [Pi89], [Bo19, Theorem 17.10] and [Em06a, Page 32]).

Lemma 6.25. Suppose that $G$ is connected and reductive. Then for every neat open compact subgroup $K$ of $G$, $G(Q) \backslash \mathcal{X}/K$ is a topological manifold that is homotopic to a finite simplicial complex.

The above lemma has the following consequence.

Proposition 6.26. Suppose that $G$ is connected and reductive, and $\Phi$ is the family of all closed subsets or the family of all compact subsets of $G(Q) \backslash \mathcal{X}$. Then for all Noetherian commutative rings $R_0$ with identity, all neat open compact subgroups $K$ of $G$, all local systems $\mathcal{L}_0$ over $S_K^G$ of finitely generated $R_0$-modules, and all $i \in \mathbb{Z}$,
\begin{equation}
H_{i}^\Phi(S_K^G, \mathcal{L}_0) \text{ is finitely generated as an } R_0\text{-module.}
\end{equation}

Moreover, the natural map
\begin{equation}
H_{i}^\Phi(S_K^G, \mathcal{L}_0) \otimes_{R_0} M_0 \to H_{i}^\Phi(S_K^G, \mathcal{L}_0 \otimes_{R_0} M_0)
\end{equation}
is an isomorphism for every flat $R_0$-module $M_0$.

Proof. If $\Phi$ is the family of all compact subsets of $G(Q) \backslash \mathcal{X}$, the second assertion of the proposition follows from the universal coefficient theorem for compactly supported cohomology (see [Br97, II. Theorem 15.3] for example). If $\Phi$ is the family of all closed subsets of $G(Q) \backslash \mathcal{X}$, it follows from the same theorem in view of Lemma 6.25.

If $\Phi$ is the family of all closed subsets of $G(Q) \backslash \mathcal{X}$, then the first assertion of the proposition is also implied by Lemma 6.25. If $\Phi$ is the family of all compact subsets of $G(Q) \backslash \mathcal{X}$, then it is still implied by Lemma 6.25 by using Poincaré duality. \qed

From now on, we assume that
\begin{equation}
\text{the support set } \Phi \text{ satisfies the conditions (6.15) and (6.16).}
\end{equation}

6.6. Some properties of the $t$-stable part. In this subsection we further assume that $R$ is Noetherian and $M$ is finitely generated as an $R$-module. We need the following elementary lemma.
Lemma 6.27. Let \( J \) be a finitely generated \( R \)-module and let \( \phi : J \to J \) be an \( R \)-module endomorphism. Then \( \phi \) induces an automorphism on 
\[ \bigcap_{k \in \mathbb{N}} \phi^k(J). \]

Proof. Since \( J \) is a Noetherian \( R \)-module, the sequence
\[ \ker \phi^0 \subset \ker \phi^1 \subset \ker \phi^2 \subset \ldots \]
is stable. In particular, there is a non-negative integer \( r \) such that
\[ (6.18) \quad \ker \phi^r = \ker \phi^{r+1}. \]

Let \( x \in \bigcap_{k \in \mathbb{N}} \phi^k(J) \). Then
\[ x = \phi^{r+1}(y) \]
for some \( y \in J \).

For every \( k \in \mathbb{N} \), there is an element \( z_k \in J \) such that
\[ x = \phi^{r+k+1}(z_k). \]

Then
\[ \phi^{r+1}(y) = \phi^{r+k+1}(z_k), \]
and (6.18) implies that
\[ \phi^r(y) = \phi^{r+k}(z_k). \]

This proves that
\[ \phi^r(y) \in \bigcap_{k \in \mathbb{N}} \phi^{r+k}(J) = \bigcap_{k \in \mathbb{N}} \phi^k(J). \]

In conclusion, we have proved that \( \phi \) restricts to a surjective endomorphism of \( \bigcap_{k \in \mathbb{N}} \phi^k(J) \). It is thus an automorphism since all surjective endomorphisms of Noetherian modules are injective. \( \square \)

The following result is a consequence of Lemma 6.27.

Lemma 6.28. For all \( \mathcal{G} \in \mathcal{G}^{D,t}_{\Sigma, \mathcal{G}} \) and \( k \in \mathbb{N}_\mathcal{G} \), the Hecke map
\[ \rho^t_{\mathcal{G}} : H^i_{\mathcal{G}}(D\mathcal{G}, M) \to H^i_{\mathcal{G}}(D\mathcal{G}, M) \]
is a well-defined automorphism.

Proposition 6.29. For all \( \mathcal{G}, \mathcal{G}' \in \mathcal{G}^{t,D}_{\Sigma, \mathcal{G}} \) such that \( \mathcal{G} \preceq \mathcal{G}' \), the transfer map
\[ \rho : H^i_{\mathcal{G}}(D\mathcal{G}, M) \to H^i_{\mathcal{G}}(D\mathcal{G}, M) \]
induces an isomorphism
\[ \rho : H^i_{\mathcal{G}}(D\mathcal{G}, M) \to H^i_{\mathcal{G}}(D\mathcal{G}, M). \]
Proof. Take a sufficiently large $k \in \mathbb{N}$ so that $k \in N_{\mathcal{G}} \cap N_{\mathcal{G}'}$ and $\mathcal{G}' \leq_P t^k \mathcal{G} t^{-k}$. It follows from Proposition 3.16 that the diagrams
\[
\begin{array}{c}
\xymatrix{
H^i_\Phi(D\mathcal{G}', M) \ar[r]^\rho & H^i_\Phi(D\mathcal{G}, M) \\
H^i_\Phi(D\mathcal{G}', M) \ar[u]^\rho_{^i_k} & H^i_\Phi(D\mathcal{G}', M) \ar[l]_{\rho_{^i_k}}}
\end{array}
\]
and
\[
\begin{array}{c}
\xymatrix{
H^i_\Phi(D\mathcal{G}, M) \ar[r]^\rho & H^i_\Phi(D\mathcal{G}', M) \\
H^i_\Phi(D\mathcal{G}, M) \ar[u]^\rho_{^i_k} & H^i_\Phi(D\mathcal{G}', M) \ar[l]_{\rho_{^i_k}}}
\end{array}
\]
are commutative, and all the six arrows are $\langle t^k \rangle$-equivariant. Thus these two commutative diagrams induce commutative diagrams
\[
\begin{array}{c}
\xymatrix{
H^{i,t}_\Phi(D\mathcal{G}', M) \ar[r]^\rho & H^{i,t}_\Phi(D\mathcal{G}, M) \\
H^{i,t}_\Phi(D\mathcal{G}', M) \ar[u]^\rho_{^{i,t}_k} & H^{i,t}_\Phi(D\mathcal{G}', M) \ar[l]_{\rho_{^{i,t}_k}}}
\end{array}
\]
and
\[
\begin{array}{c}
\xymatrix{
H^{i,t}_\Phi(D\mathcal{G}, M) \ar[r]^\rho & H^{i,t}_\Phi(D\mathcal{G}', M) \\
H^{i,t}_\Phi(D\mathcal{G}, M) \ar[u]^\rho_{^{i,t}_k} & H^{i,t}_\Phi(D\mathcal{G}', M) \ar[l]_{\rho_{^{i,t}_k}}}
\end{array}
\]
Therefore the proposition follows, in view of Lemma 6.28. □

By Proposition 6.29, we have that
\[
H^{i,t}_{\Phi, P}(D\mathcal{L}, M) = H^{i,t}_\Phi(D\mathcal{G}, M)
\]
for all $\mathcal{G} \in \mathcal{G}_{\mathcal{L}, G}^{D,t}$. In particular, $H^{i,t}_{\Phi, P}(D\mathcal{L}, M)$ is independent of $\mathcal{G}$ in the following sense: if $\mathcal{G}'$ is an open subgroup of $\mathcal{G}$ containing $D\mathcal{L}$, then
\[
H^{i,t}_{\Phi, P}(D\mathcal{L}, M) = H^{i,t}_{\Phi, P'}(D\mathcal{L}, M), \quad \text{where } P' := P \cap \mathcal{G}'.
\]

Proposition 6.30. Assume that $P \supset N$ and $\langle t \rangle$ acts on $M$ by automorphisms. Then for all $\mathcal{G} \in \mathcal{G}_{\mathcal{L}, G}^{D,t}$, the projection map
\[
H^i_{\Phi, P}(D\mathcal{L}, M) \to H^i_\Phi(D\mathcal{G}, M)
\]
induces an isomorphism
\[
H^i_{\Phi, P}(D\mathcal{L}, M) \to H^{i,t}_\Phi(D\mathcal{G}, M).
\]
Consequently,
\[ H^i_{\Phi, p}(D\mathcal{L}, M) = H^{i,t}_{\Phi, p}(D\mathcal{L}, M). \]

**Proof.** For all sufficiently large positive integers \( k \) we have that \( t^k \mathfrak{G} t^{-k} \in \mathcal{G}^{D,t}_{\mathcal{L}, G} \) and \( \mathfrak{G} \preceq_p t^k \mathfrak{G} t^{-k} \). It follows from Proposition 3.16 that the diagram
\[
\begin{array}{c}
H^i_{\Phi}(D\mathfrak{G}, M) \\
\Bigg| \\
\downarrow \rho_k^t \\
H^i_{\Phi}(D\mathfrak{G}, M)
\end{array}
\]
\[
\begin{array}{c}
H^i_{\Phi}(D t^k \mathfrak{G} t^{-k}, M) \\
\Bigg| \\
\downarrow \rho \\
H^i_{\Phi}(D\mathfrak{G}, M)
\end{array}
\]
commutes. The top horizontal arrow is an isomorphism since \( (t) \) acts on \( M \) by automorphisms. Thus the image of the projection map
\[ (6.20) \]
\[ H^i_{\Phi, p}(D\mathcal{L}, M) \rightarrow H^i_{\Phi}(D\mathfrak{G}, M) \]
is contained in the image of
\[ \rho_k^t : H^i_{\Phi}(D\mathfrak{G}, M) \rightarrow H^i_{\Phi}(D\mathfrak{G}, M). \]
Therefore the image of (6.20) is contained in \( H^{i,t}_{\Phi}(D\mathfrak{G}, M) \). The proposition then follows from Proposition 6.29. \( \square \)

### 7. The nearly ordinary part

In this section we study the nearly ordinary part of the automorphic cohomology. The nearly ordinary part in the setting of representation theory of \( p \)-adic groups has been studied in \[ \text{Em10}. \]

Let \( V \) be a nonzero continuous finite-dimensional representation of \( G \) over \( E \). Assume that \( V \) is definable over a closed subfield of \( E \) that is a finite extension of \( \mathbb{Q}_p \). In other words, there is a continuous representation of \( G \) on a finite-dimensional vector space \( V' \) over \( E' \) such that \( V = E \otimes V' \) as representations of \( G \), where \( E' \) is a closed subfield of \( E \) that is a finite extension of \( \mathbb{Q}_p \). View \( V \) as an \( E[G(\mathbb{Q}) \times G^1] \)-module with the given action of \( G \) and the trivial action of \( G(\mathbb{Q}) \times G^1 \).

Recall that \( t \in Df(\mathbf{p}, \mathbf{n}) \). Denote by \( V'^t \) the unique \( t \)-stable nonzero subspace of \( V \) with the following properties:

- there is a constant \( r_V \in \mathbb{Q} \) such that \( |a_0|_p = p^{r_V} \) for all eigenvalues \( a_0 \in \mathbb{C}_p^\times \) of the operator \( t : V'^t \rightarrow V'^t \),
- \( |a_1|_p < p^{r_V} \) for all eigenvalues \( a_1 \in \mathbb{C}_p^\times \) of the operator \( t : V/V'^t \rightarrow V/V'^t \).

Here and henceforth, when no confusion is possible, we still use \( t \) to denote various maps attached to \( t \). Note that \( V'^t \) is contained in \( V'^N \), is \( P_0 \)-stable, and is defined over \( E' \) in the following sense:
\[ V'^t = E \otimes V'^t, \quad \text{where } V'^t := V'^t \cap V'. \]
7.1. The nearly ordinary part and Hida’s inequality. As a special case of (6.10), we have an identification (for \( G = G^\# \) and \( M = V \))

\[
(7.1) \quad H_{\Phi, P}^i(G, V) = \left( H_{\Phi}^i(G, V)_{p-sm} \right)^N \otimes D(g/p).
\]

This identification respects the natural actions of \( G^\# \times P \) on the both sides. These actions are still denoted by \( g \mapsto \rho_g \).

Unless otherwise specified, throughout this section \( D \) denotes an open compact subgroup of \( G^\# \) and \( L \) denotes a \( D \)-neat open compact subgroup of \( L_0 \). Note that

\[
H_{\Phi, P}^i(D_L, V) = H_{\Phi, P}^i(G, V) \quad (\text{see (6.11)})
\]

is a \( \rho_t \)-stable subspace and \( H_{\Phi, P}^i(G, V) \) is the union of all such subspaces. Proposition 6.30 implies that \( H_{\Phi, P}^i(D_L, V) \) is finite-dimensional. In conclusion \( H_{\Phi, P}^i(G, V) \) is an admissible smooth representation of \( G^\# \times L \).

Consequently, we have a decomposition

\[
\mathbb{C}_p \otimes H_{\Phi, P}^i(G, V) = \bigoplus_{\nu \in \mathbb{C}_p^\times} \left( \mathbb{C}_p \otimes H_{\Phi, P}^i(G, V) \right)_\nu,
\]

where \( \left( \mathbb{C}_p \otimes H_{\Phi, P}^i(G, V) \right)_\nu \) is the generalized eigenspace of \( \rho_t \) with eigenvalue \( \nu \). In view of Hida’s inequality (Proposition 7.3), we define the nearly ordinary part

\[
H_{\Phi, P}^{i, \text{ord}}(G, V) := H_{\Phi, P}^{i, t-\text{ord}}(G, V) \subset H_{\Phi, P}^i(G, V)
\]

to be the subspace such that

\[
\mathbb{C}_p \otimes H_{\Phi, P}^{i, \text{ord}}(G, V) := \bigoplus_{\nu \in \mathbb{C}_p^\times, |\nu|_p = p^{-v}} \left( \mathbb{C}_p \otimes H_{\Phi, P}^i(G, V) \right)_\nu.
\]

Lemma 7.1. For all \( \mathfrak{G} \in \mathcal{G}_{\mathfrak{G}, G} \) and all \( k \in \mathbb{N} \) such that \( \mathfrak{G} \preceq t^k \mathfrak{G} t^{-k} \), the diagram

\[
\begin{array}{ccc}
H_{\Phi, P}^i(D_\mathfrak{G}, V) & \xrightarrow{\text{projection}} & H_{\Phi}^i(D \mathfrak{G}, V) \\
\rho_t^k \downarrow & & \rho_t^k \\
H_{\Phi, P}^i(D_\mathfrak{G}, V) & \xrightarrow{\text{projection}} & H_{\Phi}^i(D \mathfrak{G}, V)
\end{array}
\]

commutes.
Proof. This follows by considering the commutative diagram

\[
\begin{array}{ccc}
H^i_{\Phi,P}(DL,V) & \xrightarrow{\text{projection}} & H^i_{\Phi}(DG,V) \\
\downarrow{\rho_{t_k}} & & \downarrow{\rho_{t_k}} \\
H^i_{\Phi,P}(DL,V) & \xrightarrow{\text{projection}} & H^i_{\Phi}(D\ell^{t_k}t^{-k},V) \\
\downarrow{\rho_{t_k}} & & \downarrow{\rho} \\
& & H^i_{\Phi}(DG,V).
\end{array}
\]

Here the commutativity of the upper right triangle follows from Proposition 3.16, and the commutativity of the square and the lower left triangle follows by the definitions. \(\square\)

Write \(d_V > 0\) for the denominator of \(r_V\) as an irreducible fraction. Set \(t' := t^{d_V}\).

We define the star action of the monoid \(\langle t' \rangle\) on \(V\) by

\[
\langle t' \rangle \times V \rightarrow V, \quad (t^k, u) \mapsto t^k * u := p^{d_V-r_V} \cdot (t^k, u).
\]

As before, the associated Hecke maps are written as

\[
\rho_{t_k}^* : H^i_{\Phi}(K,V) \rightarrow H^i_{\Phi}(K,V),
\]

where \(K\) is an open compact subgroup of \(G^i\).

The following lemma is obvious.

Lemma 7.2. For all open compact subgroups \(\mathfrak{G}\) of \(G\), the equality

\[
\rho_{t_k}^* = p^{d_V-r_V} \cdot \rho_{t_k} : H^i_{\Phi}(DG,V) \rightarrow H^i_{\Phi}(DG,V)
\]

holds.

Write \(V_t := V/V^t\). Recall that \(\mathcal{O}\) denotes the ring of integers in \(E\). Similarly let \(\mathcal{O}'\) denote the ring of integers in \(E'\). Take a pair \((\mathfrak{W}, \mathfrak{G})\) where \(\mathfrak{W}\) is a lattice of \(V\) and \(\mathfrak{G}\) is an open compact subgroup of \(G\) such that

\[
\begin{cases}
\mathfrak{W} = \mathcal{O} \otimes \mathfrak{W}', & \text{where } \mathfrak{W}' := \mathfrak{W} \cap V'; \\
t' * \mathfrak{W} \subset \mathfrak{W} \text{ and } t' * \mathfrak{W}^t = \mathfrak{W}^t, & \text{where } \mathfrak{W}^t := \mathfrak{W} \cap V^t; \\
\mathfrak{G} \text{ stabilizes } \mathfrak{W}.
\end{cases}
\]

It is easy to see that such a pair exists.

In the rest of this section suppose that \(G = G^{3,p} \times \mathfrak{G}^n\).

The star action \((7.2)\) induces a monoid action

\[
\langle t' \rangle \times \mathfrak{W} \rightarrow \mathfrak{W},
\]

which is still called a star action. The associated Hecke maps are still written as

\[
\rho_{t_k}^* : H^i_{\Phi}(D\mathfrak{G},\mathfrak{W}) \rightarrow H^i_{\Phi}(D\mathfrak{G},\mathfrak{W}),
\]
where $\mathcal{G}$ is an open compact subgroup of $\mathcal{G}_\mathfrak{f}$.

As mentioned in the Introduction, the following result is essentially due to Hida. We will sketch a proof in what follows.

**Proposition 7.3.** For every eigenvalue $a \in \mathbb{C}_p^\times$ of the operator $\rho_t$ on the space $(7.1)$, the inequality

$$|a|_p \leq p^{r_V}$$

holds.

**Proof.** Suppose that $\mathfrak{G}_t \subset \mathcal{G}_\mathfrak{f}$. It suffices to show that every eigenvalue of the operator $\rho_t : H_{\Phi, P}(D\mathfrak{G}, V) \to H_{\Phi, P}(D\mathfrak{G}, V)$ has $p$-adic norm $\leq p^{r_V}$.

Pick a group $\mathfrak{G} \in \mathfrak{G}_{\mathfrak{f}, t}$ such that $\mathfrak{G} \preceq t\mathfrak{G} t^{-1}$. Proposition 6.30 implies that the projection map $H_{\Phi, P}(D\mathfrak{G}, V) \to H_{\Phi}(D\mathfrak{G}, V)$ is injective. In view of Lemma 7.1, it remains to show that all eigenvalues of the operator $\rho_t : H_{\Phi}(D\mathfrak{G}, V) \to H_{\Phi}(D\mathfrak{G}, V)$ have $p$-adic norms $\leq p^{r_V}$. By Lemmas 6.20 and 7.2, this is equivalent to saying that all eigenvalues of the operator $\rho_{t'}^* : H_{\Phi}(D\mathfrak{G}, V) \to H_{\Phi}(D\mathfrak{G}, V)$ have $p$-adic norms $\leq 1$. This is clear by considering the commutative diagram

$$
\begin{array}{ccc}
H_{\Phi}(D\mathfrak{G}, V) & \longrightarrow & H_{\Phi}(D\mathfrak{G}, V) \\
\rho_{t'}^* \Bigg| & & \Bigg| \rho_{t'}^* \\
H_{\Phi}(D\mathfrak{G}, V) & \longrightarrow & H_{\Phi}(D\mathfrak{G}, V).
\end{array}
$$

7.2. The $t'$-stable part and the nearly ordinary part. Every finitely generated $\mathcal{O}$-module is viewed as a topological module under the $p$-adic topology.

**Lemma 7.4.** Let $J$ be an $\mathcal{O}$-module and let $\phi : J \to J$ be an $\mathcal{O}$-module homomorphism. Assume that there is a finitely generated $\mathcal{O}'$-submodule $J'$ of $J$ such that $J = \mathcal{O} \otimes J'$ and $\phi(J') \subset J'$. Then

$$(7.6) \quad J = J_{\text{sta}} \oplus J_{\text{nil}}$$

and $\phi$ induces an isomorphism on $J_{\text{sta}}$, where

$$J_{\text{sta}} := \bigcap_{k \in \mathbb{N}} \phi^k(J) \quad \text{and} \quad J_{\text{nil}} := \{x \in J : \lim_{k \to \infty} \phi^k(x) = 0\}.$$

Moreover, the decomposition $(7.6)$ is defined over $\mathcal{O}'$ in the sense that

$$J_{\text{sta}} = \mathcal{O} \otimes (J_{\text{sta}} \cap J') \quad \text{and} \quad J_{\text{nil}} = \mathcal{O} \otimes (J_{\text{nil}} \cap J').$$
Proof. The assumption of the lemma implies that
\[ J = \lim_{k \in \mathbb{N}} J/p^k. \]
In view of Lemma 6.27, the lemma is easily reduced to the case when \( O = O' \) and \( J \) has finite cardinality. Suppose this is the case. Then it is clear that \( J_{\text{sta}} \cap J_{\text{nil}} = \{0\} \).

Since \( J \) has finite cardinality, there is a positive integer \( k \) such that
\[ \phi^k(J) = \phi^{k+1}(J) = \phi^{k+2}(J) = \cdots. \]
Let \( x \in J \). Then \( \phi^k(x) = \phi^{2k}(y) \) for some \( y \in J \). Write
\[ x = \phi^k(y) + (x - \phi^k(y)). \]
The lemma follows by noting that
\[ \phi^k(y) \in \phi^k(J) = J_{\text{sta}}, \]
and
\[ x - \phi^k(y) \in J_{\text{nil}} \]
since \( \phi^k(x - \phi^k(y)) = 0 \). \( \square \)

In the rest of this section we further assume that \( L_t \subset G \). As before write
\[ P := P \cap G = P \cap \mathfrak{S}_G. \]
Define
\[ H^{i,\text{ord}}_{\Phi,P}(D\mathfrak{L},V) := H^i_{\Phi,P}(D\mathfrak{L},V) \cap H^{i,\text{ord}}_{\Phi,P}(G,V). \]
By using the Hecke maps associated to the star action, we define the \( t' \)-stable part \( H^{i,t'}_{\Phi,P}(D\mathfrak{G}, \mathfrak{W}) \) and so on as in Section 6.4.

**Proposition 7.5.** Let \( \mathfrak{G} \in \mathcal{G}^{D,t}_{\mathfrak{L},G} \). Then the image of \( H^{i,\text{ord}}_{\Phi,P}(D\mathfrak{L},V) \) under the projection map
\[ H^i_{\Phi,P}(D\mathfrak{L},V) \to H^i_{\Phi}(D\mathfrak{G},V) \]
equals \( E \otimes H^i_{\Phi,t'}(D\mathfrak{G}, \mathfrak{W}) \). Consequently, there are identifications
\[ H^{i,\text{ord}}_{\Phi,P}(D\mathfrak{L},V) = E \otimes H^i_{\Phi,P}(D\mathfrak{L}, \mathfrak{W}) \]
and
\[ H^{i,\text{ord}}_{\Phi,P}(G,V) = E \otimes H^i_{\Phi,P}(G, \mathfrak{W}). \]

**Proof.** Pick a positive integer \( k \) such that \( \mathfrak{G} \preceq_P t^k \mathfrak{G} t'^{-k} \). Write \( J := H^i_{\Phi}(D\mathfrak{G}, \mathfrak{W}) \) and
\[ \phi := \rho_{t^k} : J \to J. \]
Then we have a decomposition \( J = J_{\text{sta}} \oplus J_{\text{nil}} \) as in (7.6), with \( J_{\text{sta}} = H^i_{\Phi}(D\mathfrak{G}, \mathfrak{W}) \).
Note that all eigenvalues of
\[ \phi : E \otimes J_{\text{sta}} \to E \otimes J_{\text{sta}} \]
have $p$-adic norm 1, and all eigenvalues of
\[ \phi : E \otimes J_{\text{nil}} \to E \otimes J_{\text{nil}} \]
have $p$-adic norms $< 1$. This implies the first assertion of proposition. The second assertion then follows by Proposition 6.30. \hfill \Box

By the identification \eqref{identification}, we have an inclusion
\[ H^i_{\phi, P}(D L, V) \subset E \otimes H^i_{\phi, P}(D L, \mathcal{V}). \]

7.3. The $t$-stable part and relative completed cohomologies. Put
\[ \mathcal{G}_{L, G, k}^{D, t} := \{ \mathcal{G}_k \in \mathcal{G}_{L, G}^{D, t} : \mathcal{G}_k \text{ stabilizes } \mathcal{V}/p^k \subset \mathcal{V}/p^k \} \quad (k \in \mathbb{N}). \]
Since $N$ acts trivially on $\mathcal{V}/p^k$, this set is cofinal in $\mathcal{G}_{L, G}^{D, t}$.

The star action \eqref{star-action} induces compatible actions
\[ \langle t' \rangle \times \mathcal{V}/p^k \to \mathcal{V}/p^k \]
and
\[ \langle t' \rangle \times \mathcal{V}/p^k \to \mathcal{V}/p^k. \]
As before, associated to these compatible actions we define the Hecke maps and the $t'$-stable parts

\[
\begin{cases}
H^i_{\phi, t'}(D \mathcal{G}, \mathcal{V}/p^k) \subset H^i_{\phi}(D \mathcal{G}, \mathcal{V}/p^k); \\
H^i_{\phi, t'}(D \mathcal{G}_k, \mathcal{V}/p^k) \subset H^i_{\phi}(D \mathcal{G}_k, \mathcal{V}/p^k); \\
H^i_{\phi, P}(D L, \mathcal{V}/p^k) := \varprojlim_{\mathcal{G}_k \in \mathcal{G}_{L, G, k}^{D, t}} H^i_{\phi, t'}(D \mathcal{G}_k, \mathcal{V}/p^k) \subset H^i_{\phi, P}(D L, \mathcal{V}/p^k); \\
H^i_{\phi, P}(D L, \mathcal{V}/p^k) := \varprojlim_{\mathcal{G}_k \in \mathcal{G}_{L, G, k}^{D, t}} H^i_{\phi, t'}(D \mathcal{G}_k, \mathcal{V}/p^k) \subset H^i_{\phi, P}(D L, \mathcal{V}/p^k),
\end{cases}
\]

where $\mathcal{G} \in \mathcal{G}_{L, G}^{D, t}$ and $\mathcal{G}_k \in \mathcal{G}_{L, G, k}^{D, t}$.

**Proposition 7.6.** Let $\mathcal{G}_k \in \mathcal{G}_{L, G, k}^{D, t}$. The inclusion map $\mathcal{V}/p^k \to \mathcal{V}/p^k$ induces an isomorphism
\[ H^i_{\phi, t'}(D \mathcal{G}_k, \mathcal{V}/p^k) \cong H^i_{\phi, t'}(D \mathcal{G}_k, \mathcal{V}/p^k). \]

**Proof.** Let $\mathcal{V}_i \subset \mathcal{V}_i$ denote the image of $\mathcal{V}$ under the quotient map $V \to V_i$. It is clear that the $t'$-stable part
\[ H^i_{\phi, t'}(D \mathcal{G}_k, \mathcal{V}_i/p^k) = \{0\} \quad (j \in \mathbb{Z}). \]
In view of Lemma \ref{coset-lemma}, the proposition then follows by considering the long exact sequence
\[
\cdots \to H^i_{\phi}(-1)(D \mathcal{G}_k, \mathcal{V}_i/p^k) \to H^i_{\phi}(D \mathcal{G}_k, \mathcal{V}/p^k) \\
\to H^i_{\phi}(D \mathcal{G}_k, \mathcal{V}/p^k) \to H^i_{\phi}(D \mathcal{G}_k, \mathcal{V}_i/p^k) \to \cdots.
\]
\hfill \Box
By Proposition 7.6, we have an identification

\[ H^{i,t}_{\phi, P}(DL, \mathfrak{M}/p^k) \cong H^{i,t}_{\phi, P}(DL, \mathfrak{M}/p^k). \]

For each \( \Psi \in \mathcal{D}_{L, P} \), we have a commutative diagram

\[
\begin{array}{ccc}
\lim_{\leftarrow k \in \mathbb{N}} H^{i,t}_{\phi, P}(DL, \mathfrak{M}/p^k) & \cong & \lim_{\leftarrow k \in \mathbb{N}} H^{i,t}_{\phi, P}(DL, \mathfrak{M}/p^k) \\
\downarrow & & \downarrow \\
\lim_{\leftarrow k \in \mathbb{N}} H^i_{\phi}(D\Psi, \mathfrak{M}/p^k) & \cong & \lim_{\leftarrow k \in \mathbb{N}} H^i_{\phi}(D\Psi, \mathfrak{M}/p^k)
\end{array}
\]

where the two vertical arrows are the inverse limits of the projections with respect to the isomorphism (6.5), the down left arrow is the map that makes the diagram commute, and all the other arrows are the coefficient change maps. By compositions in the above diagram we get a commutative diagram

\[
\begin{array}{ccc}
H^{i,t}_{\phi, P}(DL, \mathfrak{M}) & \cong & H^{i,t}_{\phi, P}(DL, \mathfrak{M}) \\
\downarrow & & \downarrow \\
\lim_{\leftarrow k \in \mathbb{N}} H^i_{\phi}(D\Psi, \mathfrak{M}/p^k) & \cong & \lim_{\leftarrow k \in \mathbb{N}} H^i_{\phi}(D\Psi, \mathfrak{M}/p^k)
\end{array}
\]

By using Lemma 6.18 and taking the direct limits, we further get a commutative diagram (see (3.16))

\[
\begin{array}{ccc}
H^{i,t}_{\phi, P}(DL, \mathfrak{M}) & \cong & H^{i,t}_{\phi, P}(DL, \mathfrak{M}) \\
\downarrow & & \downarrow \\
\lim_{\leftarrow k \in \mathbb{N}} H^i_{\phi}(D\Psi, \mathfrak{M}/p^k) & \cong & \lim_{\leftarrow k \in \mathbb{N}} H^i_{\phi}(D\Psi, \mathfrak{M}/p^k)
\end{array}
\]

Finally, taking tensor product with \( E \) we get a commutative diagram

\[
\begin{array}{ccc}
\tilde{H}^{i}_{\phi}(G, \mathfrak{M}) & \cong & H^{i,\text{ord}}(G, V) \\
\downarrow & & \downarrow \\
\tilde{H}^{i}_{\phi}(G, \mathfrak{M}) & \cong & H^{i,\text{ord}}(G, V)
\end{array}
\]
7.4. **A commutative diagram.** We now explain the following diagram of linear maps (see also (1.9)):

\begin{align*}
\tilde{H}^i_\Phi(G, V^t)^{(p \supset p)^{\circ}, \circ} & \xleftarrow{\tilde{\xi}} H^i_\Phi(P) (G, V) \xrightarrow{\xi} H^i_\Phi(G, V)_{p}^{\text{sm}} \otimes D(g/p) \\
\tilde{H}^i_\Phi(G, V)^{(p \supset p)^{\circ}} & \xleftarrow{\tilde{\xi}} H^i_\Phi(G, V)^{(p)^{\circ}} \longrightarrow H^i_\Phi(G, V)^{(p)},
\end{align*}

\[(7.10)\]

- The map \(\tilde{\xi}\) is the inclusion map with respect to the identification (7.1).
- The right vertical arrow is the isomorphism given in Lemma 3.20.
- The map \(\tilde{\xi}\) and the left vertical arrow are given in (7.9).
- Recall that \(H^i_\Phi(G, V)^{(p)^{\circ}} := E \otimes H^i_\Phi(G, V)^{(p)}\). The right bottom horizontal arrow is the linear map induced by the inclusion map \(\mathfrak{W} \to \mathfrak{V}\), and the left bottom horizontal arrow is the linear map induced by the maps \(\mathfrak{W} \to \mathfrak{V}/p^k\) \((k \in \mathbb{N})\).
- The map \(\xi\) is defined by requiring that the diagram

\[
\begin{array}{ccc}
H^i_{\Phi, P}(D\mathfrak{L}, \mathfrak{W}) & \xrightarrow{\text{projection}} & H^i_{\Phi, P}(G, V) \\
\downarrow & & \downarrow \\
H^i_\Phi(D\mathfrak{P}, \mathfrak{W}) & \longrightarrow & H^i_\Phi(G, V)^{(p), \circ}
\end{array}
\]

\[(7.11)\]

commutes for every open compact subgroup \(D\) of \(\mathbb{G}^{k,p}\), every \(D\)-neat open compact subgroup \(\mathfrak{L}\) of \(L_0\) such that \(\mathfrak{L}_t \subset \mathfrak{G}_0\), and every \(\mathfrak{P} \in \mathcal{P}_{\mathfrak{L}, \mathfrak{P}}\). The existence of \(\xi\) is guaranteed by Lemma 6.18.

**Theorem 7.7.** The two squares in (7.10) are commutative.

**Proof.** As before, suppose that \(D\) is an open compact subgroup of \(\mathbb{G}^{k,p}\), \(\mathfrak{L}\) is a \(D\)-neat open compact subgroup of \(L_0\) such that \(\mathfrak{L}_t \subset \mathfrak{G}_0\), and \(\mathfrak{P} \in \mathcal{P}_{\mathfrak{L}, \mathfrak{P}}\). The commutativity of the left square easily follows by considering the commutative diagram

\[
\begin{array}{ccc}
\lim_{k \in \mathbb{N}} H^i_\Phi(D\mathfrak{P}, \mathfrak{W}/p^k) & \xrightarrow{\text{projection}} & \lim_{k \in \mathbb{N}} H^i_{\Phi, P}(D\mathfrak{L}, \mathfrak{W}/p^k) \\
\downarrow & & \downarrow \\
\lim_{k \in \mathbb{N}} H^i_\Phi(D\mathfrak{P}, \mathfrak{W}/p^k) & \xleftarrow{\text{projection}} & \lim_{k \in \mathbb{N}} H^i_{\Phi, P}(D\mathfrak{L}, \mathfrak{W}/p^k).
\end{array}
\]
The commutativity of the right square follows by considering the commutative diagram

\[
\begin{array}{ccc}
H_{i,t}^{i}(D\mathfrak{L},\mathfrak{U}) & \xrightarrow{\Phi} & H_{i}^{i}(G,V) \\
\downarrow \text{projection} & & \downarrow \text{projection} \\
H_{i}^{i}(D\mathfrak{P},\mathfrak{U}) & \xrightarrow{\Phi} & H_{i}^{i}(D\mathfrak{P},V).
\end{array}
\]

Consider the diagram

\[
\begin{array}{ccc}
H_{i}^{i}(G,V) & \xrightarrow{\Phi} & \widehat{H}_{i}^{i}(G,V)_{p-\text{sm}} \otimes D(\mathfrak{g}/\mathfrak{p}) \\
\downarrow & & \downarrow \Phi \circ \mathfrak{g} \\
\widehat{H}_{i}^{i}(G,V)(p\supset p) & \xrightarrow{\Phi \circ \mathfrak{g}} & H_{i}^{i}(G,V)(p),
\end{array}
\]

where the top horizontal arrow is the inclusion map with respect to the identification \([7.1]\), and the other three arrows are as in the diagram \([7.10]\). All the five spaces in the above diagram carry naturally representations of \(G_{\mathfrak{n}} \otimes P\). We still use \(g \mapsto \rho_{g}\) to denote these representations. It is routine to check that the two bottom horizontal arrows are independent of the choice of the lattice \(\mathfrak{U}\), and all the four arrows above are \(G_{\mathfrak{n}} \otimes P\)-equivariant.

### 7.5. Independence of \(t\).

**Lemma 7.8.** If \(V'\) is irreducible as a representation of \(\mathfrak{g}\), then the space \(V^{t}\) is independent of \(t \in \operatorname{Df}(\mathfrak{p}, \mathfrak{n})\).

**Proof.** Assume without loss of generality that \(E = E'\) so that \(V\) is irreducible as an \(E \otimes \mathfrak{g}\)-module.

Note that \(V^{t}\) is contained in \(V^{n}\) and is \(I_{t}\)-stable. Suppose that \(t_{1} \in \operatorname{Df}(\mathfrak{p}, \mathfrak{n})\). First we assume that \(t_{1}\) commutes with \(t\). Then \(I_{t} = I_{t_{1}}\) by Lemma 6.6 and hence

\[
V = U(\mathfrak{g}).V^{t} = U(\mathfrak{g}_{t_{1}}).V^{t}.
\]

Here \(U\) indicates the universal enveloping algebra. This implies that \(V^{t_{1}} \subset V^{t}\). Similarly \(V^{t} \subset V^{t_{1}}\), and hence \(V^{t} = V^{t_{1}}\). The lemma in general then follows by Lemma 6.5 \(\square\)

If \(V^{t}\) is independent of \(t \in \operatorname{Df}(\mathfrak{p}, \mathfrak{n})\), then the space \(\widehat{H}_{i}^{i}(G,V)(p\supset p)\) also carries a representation of \(G_{\mathfrak{n}} \otimes P\), which is still denoted by \(g \mapsto \rho_{g}\). It is easy to see that the left vertical arrow in \([7.10]\) is independent of the choice of the lattice \(\mathfrak{U}\), and is \(G_{\mathfrak{n}} \otimes P\)-equivariant.

The main purpose of this subsection is to prove the following result.
Theorem 7.9. Assume that the subspace $V^t$ of $V$ is independent of $t \in \text{Df}(p, n)$. Then the followings hold true.

(a) The subspace $\text{H}^i_{\Phi, p}(G, V)$ of $\text{H}^i_{\Phi, p}(G, V)$ is $G^{\mathbb{Z}_p} \times L$-stable and is independent of $t \in \text{Df}(p, n)$.

(b) The maps $\xi$ and $\tilde{\xi}$ in (7.10) are both independent of $t \in \text{Df}(p, n)$ and the pair $(\mathfrak{W}, \mathfrak{G}_0)$ in (7.5).

(c) All the arrows in (7.10) are $G^{\mathbb{Z}_p} \times P$-equivariant.

In the rest of this subsection we assume that the subspace $V^t$ of $V$ is independent of $t \in \text{Df}(p, n)$. Then $V^t$ is $P$-stable, and the representation of $P$ on it descends to a representation of $L$.

We say that an element $g_1 \in P$ is a canonical lift of an element $g \in L$ if it belongs to $L_1$ for some $t_1 \in \text{Df}(p, n)$ and the quotient map $L \to L$ sends it to $g$. By (6.1), all canonical lifts of an element $g \in L$ form an $N$-conjugacy class.

Write $T_L$ for the largest central torus in $L_0$. Denote by $T^+_L$ the subset of all $t_1 \in T_L$ such that for some (and hence all) canonical lifts $t_2 \in P$ of $t_1$, all eigenvalues of $\text{Ad}_{t_2} : n \to n$ have $p$-adic norms $\geq 1$, and all eigenvalues of $\text{Ad}_{t_2} : g/p \to g/p$ have $p$-adic norms $\leq 1$.

Lemma 7.10. There is a unique locally constant homomorphism $\beta_V : T_L \to \mathbb{Q}$ such that for every $t_1 \in T_L$, all eigenvalues of $t_1 : V^t \to V^t$ have $p$-adic norm $p^{\beta_V(t_1)}$. Moreover, if $t_1 \in T^+_L$, then all eigenvalues of $t_1 : V/V^t \to V/V^t$ have $p$-adic norms $\leq p^{\beta_V(t_1)}$.

Proof. Let $t_1 \in T_L$. For the first assertion, it suffices to show that all eigenvalues of $t_1 : V^t \to V^t$ have the same $p$-adic norm. Since $t_1$ is the product of a split element with an element in the maximal compact subgroup of $T_L$, without loss of generality we assume that $t_1$ is split. Write $t_1' \in L_1$ for the element corresponding to $t_1$ under the isomorphism $L \cong L_1$. Pick a sufficiently large positive integer $k$ so that $t_1' t^k \in \text{Df}(p, n)$. By assumption, all eigenvalues of $t_1' t^k : V^t \to V^t$ have the same $p$-adic norm, say $p^c$ ($c \in \mathbb{Q}$). Then all eigenvalues of $t_1' : V^t \to V^t$ have the same $p$-adic norm $p^{c-k\cdot \text{v}}$. This proves the first assertion of lemma.

Now suppose that $t_1 \in T^+_L$. As before we assume without loss of generality that $t_1$ is split. Then for all $k \in \mathbb{N}$, all eigenvalues of $t_1^k t : V/V^t \to V/V^t$ have $p$-adic norms $< p^{\beta_V(t_1)}$. Hence all eigenvalues of $t_1 : V/V^t \to V/V^t$ have $p$-adic norms

$$< p^{(k \cdot \beta_V(t_1) - c')/k},$$

where $c' \in \mathbb{Q}$ is independent of $k$. This proves the second assertion. \hfill \square

Let $\beta_V : T_L \to \mathbb{Q}$ be as in Lemma 7.10. Recall that $H^i_{\Phi, p}(G, V)$ is an admissible smooth representation of $G^{\mathbb{Z}_p} \times L$. Hence it is a union of finite-dimensional $T_L$-subrepresentations.

Proposition 7.11. For every character $\chi : T_L \to \mathbb{C}_p^\times$ that occurs in $\mathbb{C}_p \otimes H^i_{\Phi, p}(G, V)$,

$$|\chi(t_+)|_p \leq p^{\beta_V(t_+)} \quad \text{for all } t_+ \in T^+_L.$$
Proof. Without loss of generality assume that $t_+ \in T_L^+$ is split. Write $t'_+ \in L_t$ for the element corresponding to $t_+$ under the isomorphism $L \cong L_t$. Define a character (7.13) $\beta_\chi : T_L \rightarrow \mathbb{R}^\times$, $t_1 \mapsto |\chi(t_1)|_p \cdot p^{-\beta_V(t_1)}$.

Let $k \in \mathbb{N}$. Since $t(t'_+)^k \in Df(p,n)$, by Proposition 7.3 we have that $\beta_\chi([t]_k^t) \leq 1$, where $[t]$ denotes the image of $t$ under the quotient map $P \rightarrow L$. Since $k$ is arbitrary, this implies that $\beta_\chi(t_+) \leq 1$, and hence the proposition follows. □

Lemma 7.12. Let $\chi : T_L \rightarrow \mathbb{C}^\times$ be a character such that (7.12) is satisfied. Then $|\chi(t_1)|_p = p^{-\beta_V(t_1)}$, where $[t]$ denotes the image of $t$ under the quotient map $P \rightarrow L$.

Proof. Since $[t] \in T_L$, we only need to prove the implication "$\Rightarrow$". Suppose that $|\chi([t])|_p = p^{-\beta_V}$. Then $\beta_\chi([t]) = 1$. For every $t_1 \in T_L$, we have that $\beta_\chi(t_1) = \beta_\chi(t_1[k]^k) \leq 1$, where $k$ is a sufficiently large integer so that $t_1[k]^k \in T_L^+$. This proves the lemma.

We prove part (a) of Theorem 7.9 in the following lemma.

Lemma 7.13. The subspace $H^i_{\text{ord},P}(G,V)$ of $H^i_{\phi,P}(G,V)$ is independent of $t \in Df(p,n)$ and is $G^{\flat,p} \times L$-stable.

Proof. It follows from Proposition 7.11 and Lemma 7.12 that the subspace $H^i_{\phi,P}(G,V)$ is independent of $t \in Df(p,n)$. It is $G^{\flat,p} \times L$-stable since $\beta_V$ is $L$-invariant. □

Lemma 7.14. The maps $\xi$ and $\tilde{\xi}$ in (7.10) are both independent of the pair $(\mathfrak{U}, \mathfrak{G}_\mathfrak{U})$ as in (7.3).

Proof. The independence of $\mathfrak{G}$ follows from the observation (6.19). The independence of $\mathfrak{U}$ follows from the fact that all the arrows in (7.8) and the left vertical arrow in (7.11) are natural in the coefficient system $\mathfrak{U}$. □

Lemma 7.15. The images of $\xi$ and $\tilde{\xi}$ are pointwise fixed by $N$.

Proof. Let $\mathfrak{N}$ be an open compact subgroup of $N$. We choose the pair $(\mathfrak{U}, \mathfrak{G}_\mathfrak{U})$ appropriately so that $\mathfrak{G}_\mathfrak{N} \supset \mathfrak{N}$. Then it is easy to see that the images of $\xi$ and $\tilde{\xi}$ are pointwise fixed by $\mathfrak{N}$. This implies the lemma.

Write $\xi_t := \xi$ and $\tilde{\xi}_t := \tilde{\xi}$ to indicate the dependence on $t \in Df(p,n)$.

Lemma 7.16. For every $g \in N$, $\xi_t = \xi_{gtg^{-1}}$ and $\tilde{\xi}_t = \tilde{\xi}_{gtg^{-1}}$. 


Proof. For every $g \in P$, it is routine to check that the diagram
\[
\begin{array}{c}
\tilde{H}_\Phi^i(G, V^t)(p \supseteq p)^o \xrightarrow{\tilde{\xi}_t} H_{\Phi, P}^i(G, V)
\end{array}
\]
\[
\begin{array}{c}
\rho_g \downarrow \\
\tilde{H}_\Phi^i(G, V^t)(p \supseteq p)^o \xrightarrow{\tilde{\xi}_g^{-1}} H_{\Phi, P}^i(G, V)
\end{array}
\]
commutes. If $g \in N$, then the right vertical arrow is the identity map, and it follows from Lemma 7.15 that $\tilde{\xi}_t = \tilde{\xi}_g^{-1}$. The equality $\xi_t = \xi_g^{-1}$ is similarly proved.

Lemma 7.17. For every $t_1 \in Df(p, n)$ that commutes with $t$, $\xi_{t_1} = \xi_t$ and $\tilde{\xi}_{t_1} = \tilde{\xi}_t$.

Proof. Let $T_{L_t}^{+Z}$ denote the monoid of all elements of $T_L^+$ whose image under $\beta_V$ is an integer. Let $T_{L_t}^{+Z}$ denote the submonoid of $L_t$ that corresponds to $T_{L_t}^{+Z}$ under the isomorphism $L \cong L_t$. Define the star action of $T_{L_t}^{+Z}$ on $V$ by
\[
t_+ * u := p^{\beta_V([t_+])} \cdot t_+ \cdot u,
\]
where $[t_+]$ denotes the image of $t_+$ under the isomorphism $L_t \to L$.

By using Lemma 7.10 we choose the pair $(\mathcal{V}, \mathcal{G}_\mathfrak{g})$ appropriately so that besides (7.3) the following condition is also satisfied:
\[
t_+ * \mathcal{V} \subset \mathcal{V} \quad \text{and} \quad t_+ * \mathcal{V}' = \mathcal{V}' \quad \text{for all } t_+ \in T_{L_t}^{+Z}.
\]
Similar to (7.5) we have the Hecke maps
\[
(7.14) \quad \rho^*_{t_+} : H_\Phi^i(D\mathcal{G}, \mathcal{V}) \to H_\Phi^i(D\mathcal{G}, \mathcal{V}), \quad t_+ \in T_{L_t}^{+Z},
\]
where $D$ is an open compact subgroup of $G^{\text{ap}}$ and $\mathcal{G}$ is an open compact subgroup of $\mathcal{G}_\mathfrak{g}$. Similar to Lemma 6.20 the map (7.14) yields an action of the monoid
\[
T_{\mathfrak{g}} := \{ t_+ \in T_{L_t}^{+Z} : \mathcal{G} \preceq_P t_+ \mathcal{G} t_+^{-1} \}
\]
on $H_\Phi^i(D\mathcal{G}, \mathcal{V})$.

Note that $\bar{N}_{t_1} = \bar{N}_t$ for every $t_1 \in Df(p, n)$ that commutes with $t$. Let $\mathcal{L}$ be an open compact subgroup of $L_0$, and recall the corresponding group $\mathcal{L}_t \subset L_t$. Suppose $\mathcal{L}_t \subset \mathcal{G} \subset \bar{N}_t \mathcal{L}_t N$ so that $t_{k} \in T_{\mathfrak{g}}$ for all sufficiently large $k \in \mathbb{N}$. Note that $\langle t' \rangle \cap T_{\mathfrak{g}}$ is cofinal in $T_{\mathfrak{g}}$ in the following sense: for every $t_1 \in T_{\mathfrak{g}}$, there is an element $t_2 \in T_{\mathfrak{g}}$ such that $t_1 t_2 \in \langle t' \rangle \cap T_{\mathfrak{g}}$. Therefore
\[
(7.15) \quad \bigcap_{t_+ \in T_{\mathfrak{g}}} \rho^*_{t_+} \left( H_\Phi^i(D\mathcal{G}, \mathcal{V}) \right) = \bigcup_{k \in \mathbb{N}, \mathcal{G} \preceq_P t^{*k} \mathcal{G} t^{-k}} \rho^*_{t^{*k}} \left( H_\Phi^i(D\mathcal{G}, \mathcal{V}) \right).
\]
This easily implies that $\xi_t$ only depends on $L_t$ and hence $\xi_{t_1} = \xi_t$. The equality $\tilde{\xi}_{t_1} = \tilde{\xi}_t$ is similarly proved.

In view of Lemma 6.31 part (b) of Theorem 7.9 now follows by Lemmas 7.16 and 7.17. To finish the proof of Theorem 7.9 it remains to prove the following result.
Lemma 7.18. The maps $\xi$ and $\tilde{\xi}$ are $G^{\natural,p} \times P$-equivariant.

Proof. It is clear that the maps $\xi$ and $\tilde{\xi}$ are $G^{\natural,p}$-equivariant. By Lemma 7.15 they are $N$-equivariant. It remains to show that they are also $L_t$-equivariant. For every $g \in L_t$, in the notation of the proof of Lemma 7.17, we have a commutative diagram

$$
\lim_{\to} \bigcap_{t, s \in T_G} \rho^*_t \left( H^i_{\phi}(D(G, V)) \right) \xrightarrow{\rho_g} \lim_{\to} \bigcap_{t, s \in T_G} \rho^*_{gtg^{-1}} \left( H^i_{\phi}(D(gGg^{-1}, g, V)) \right)
$$

where $G$ runs over $G_{\Sigma,t}$, $L_t$ is a $D$-neat open compact subgroup of $L_0$, and $P \in \mathcal{P}_{L,P}$. In view of the equality (7.15), this implies that $\xi$ is $L_t$-equivariant. Similar argument shows that $\tilde{\xi}$ is also $L_t$-equivariant. This finishes the proof of the lemma.

7.6. The commutative diagram (5.10). As in Section 5, suppose that $V$ has a $G(Q)$-stable $E$-form $\tilde{V}$. Let $\mathcal{H}$ be an $E$-vector space that fits into a commutative diagram

$$
\begin{array}{ccc}
\mathcal{H} & \rightarrow & H^i_{\phi,P}(G, V) \\
\downarrow & & \downarrow \\
H^i_{\phi}(G, V)_{p-sm} \otimes D(g/p) & \xrightarrow{\iota^r} & H^i_{\phi}(G, V)_{p-sm} \otimes D(g/p).
\end{array}
$$

Then by using the commutative diagrams (7.10) and

$$
\begin{array}{ccc}
\tilde{H}^i_{\phi}(G, V) & \xleftarrow{\text{pull-back}} & \tilde{H}^i_{\phi}(G, V'') \\
\downarrow & & \downarrow \\
\tilde{H}^i_{\phi}(G, V) & \xleftarrow{\text{pull-back}} & \tilde{H}^i_{\phi}(G, V')
\end{array}
$$

$\mathcal{H}$ obviously fits into the commutative diagram (5.10).

8. Application I: Rankin-Selberg $L$-functions for $GL_n \times GL_{n-1}$

In this section we retain the setup in Section 2.4 and construct the $p$-adic $L$-function $L_{\Pi}$ in Theorem 2.10 following the general formalism in Sections 2.1, 2.3. Then we determine the exceptional zeros of $L_{\Pi}$.

8.1. Rankin-Selberg integrals. Let $\Pi = \Pi_n \boxtimes \Pi_{n-1}$ be an irreducible representation of $G(A) = GL_n(A_k) \times GL_{n-1}(A_k)$ ($n \geq 2$) that is realized as a space of smooth automorphic forms on $G(Q) \backslash G(A)$. Assume that $\Pi_n$ is cuspidal. Let
χ : \( k^\times \backslash A_k^\times \rightarrow \mathbb{C}^\times \) be a Hecke character. We have the global Rankin-Selberg integral
\[
P_\chi : \Pi \otimes M(\hat{G}(\mathbb{Q}) \backslash \hat{G}(\mathbb{A})) \rightarrow \chi^{-1}, \quad f \otimes \tau \mapsto \int_{\hat{G}(\mathbb{Q}) \backslash \hat{G}(\mathbb{A})} \chi(\det g) f(g) \, d\tau(g),
\]
which converges absolutely. In this example we take \( H := U \) to be the upper triangular maximal unipotent subgroup of \( G \), and define
\[
\psi_U : U(\mathbb{Q}) \backslash U(\mathbb{A}) \rightarrow \mathbb{C}^\times, \quad ([u_{i,j}], [u'_{k,l}]) \mapsto \psi \left( \sum_{i=1}^{n-1} u_{i,i+1} - \sum_{k=1}^{n-2} u'_{k,k+1} \right),
\]
where \( \psi \) is as in (2.21). Then (2.1) holds. Assume that \( \Pi \) is globally generic, namely the integrals in (2.3) yield a nonzero functional \( \lambda_U \in \text{Hom}_{U(\mathbb{A})}(\Pi, \psi_U) \).

The latter space is known to be at most one-dimensional, and the functional \( \lambda_U \) is called the Whittaker period.

At this point, it will be more familiar to switch the notation and work over the number field \( k \). Let \( U \) be the upper triangular maximal unipotent subgroup of \( G := GL_n \times GL_{n-1} \) (so that \( G = \text{Res}_{k/\mathbb{Q}} G \) and \( U = \text{Res}_{k/\mathbb{Q}} U \)).

We also have \( \hat{G} := GL_n \times GL_{n-1} \) diagonally embedded into \( G \). For every place \( v \) of \( k \), write \( U_v := U(k_v) \), \( G_v := G(k_v) \), \( \hat{G}_v := G(k_v) \), etc. Accordingly we have the decompositions
\[
\Pi = \bigotimes_v \Pi_v := \left( \bigotimes_{v|\infty} \Pi_v \right) \otimes \left( \bigotimes_{v \nmid \infty} \Pi_v \right), \quad \psi_U = \bigotimes_v \psi_{U_v}, \quad \text{and} \quad \lambda_U = \bigotimes_v \lambda_{U_v},
\]
where \( \bigotimes \) stands for the completed projective tensor product. Similar notations will be used without explanation.

Likewise let \( \mathcal{X}_v \) be the group of complex continuous characters of \( k_v^\times \) for every place \( v \) of \( k \), and for a locally constant character \( \varepsilon_v : \mathcal{O}_v^\times \rightarrow \mathbb{C}^\times \) when \( v \) is finite, let \( \mathcal{X}_v(\varepsilon_v) \subset \mathcal{X}_v \) be the subset of characters whose restriction to \( \mathcal{O}_v^\times \) equals \( \varepsilon_v \).

The normalized Rankin-Selberg integral
\[
P^\circ_v : \mathcal{X}_v \times \left( \Pi_v \otimes M(\hat{U}_v \backslash \hat{G}_v) \right) \rightarrow \mathbb{C},
\]
\[
(\chi'_v, f \otimes \tau) \mapsto \frac{1}{L\left( \frac{1}{2}, \Pi_v \times \chi'_v \right)} \int_{\hat{U}_v \backslash \hat{G}_v} \chi'_v(\det g) \langle \lambda_{U_v}, g.f, \tau \rangle \, d\tau(g)
\]
is defined by holomorphic continuation. Write \( \chi = \bigotimes_v \chi_v \). By [JPSS83, J09] and (2.7) we have that
\[
P_\chi = L\left( \frac{1}{2}, \Pi \times \chi \right) \cdot \bigotimes_v P^\circ_v(\chi_v, \cdot).
\]

In the rest of this section, assume that \( \Pi \) is regular algebraic, \( \Pi_{n-1} \) is tamely isobaric as in [LLS24], and \( \chi \) is algebraic. Let \( v \) be a finite place of \( k \). Let \( \ell \) be the
residue characteristic of \( k_v \), and let \( \mu_{\ell^\infty} \subset \mathbb{C}^\times \) be the subgroup of \( \ell \)-power roots of unity. The cyclotomic character at \( \ell \) is given by

\[
\text{Aut}(\mathbb{C}) \xrightarrow{\text{restriction}} \text{Aut}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}) \to \mathbb{Z}_\ell^\times, \quad \sigma \mapsto t_{\sigma,\ell},
\]

such that \( \sigma(\zeta) = \zeta^{t_{\sigma,\ell}} \) for all \( \zeta \in \mu_{\ell^\infty} \). Put

\[
\tau_{\sigma,\ell} := (t_{n,\sigma,\ell}, t_{n-1,\sigma,\ell}) \in G_v,
\]

where \( \tau_{m,\sigma,\ell} := \text{diag}(t_{\sigma,\ell}^{-(m-1)}, \ldots, t_{\sigma,\ell}^{-1}, 1) \in \text{GL}_m(k_v) \quad (m = n, n - 1) \).

Define an action of \( \text{Aut}(\mathbb{C}) \) on \( \text{Ind}^G_v \psi_{U_v} \) (the smooth induction) by

\[
\sigma \varphi(g) := \sigma(\varphi(t_{\sigma,\ell} \cdot g)), \quad \varphi \in \text{Ind}^G_v \psi_{U_v}, \quad g \in G_v.
\]

This action gives the \( \mathbb{Q}(\Pi) \)-form of \( \Pi_v \to \text{Ind}^G_v \psi_{U_v} \) (see [Mah05, Page 594]). The following result is a consequence of [LLS24, Proposition 5.1].

**Proposition 8.1.** For every finite place \( v \) of \( k \), the linear functional

\[
\mathcal{G}_{\psi_v}(\chi_v) \left( \frac{\alpha(n-1)}{2} \right) \cdot \mathcal{G}_{\psi_v}(\chi_{\Pi_{n-1,v}}) \cdot \mathcal{P}_v(\chi_{v}, \cdot) : \Pi_v \otimes M(\hat{U}_v \setminus \hat{G}_v) \to \mathbb{C}
\]

is defined over \( \mathbb{Q}(\Pi_v, \chi_v) \), where \( \chi_{\Pi_{n-1,v}} \) is the central character of \( \Pi_{n-1,v} \), and \( \mathcal{G}_{\psi_v}(\omega_v) \) denotes the Gauss sum of a character \( \omega_v : k_v^\times \to \mathbb{C}^\times \) in (8.3).

Here \( \mathbb{Q}(\Pi_v, \chi_v) \) denotes the composition of the rationality fields \( \mathbb{Q}(\Pi_v) \) and \( \mathbb{Q}(\chi_v) \) which are number fields (see [CI90, 3.1]), and similar notation will be used without explanation. The usual definition of Gauss sum (see [LLS24]) implicitly depends on a generator \( y_v \) of the fractional ideal \( \mathcal{O}_v \cdot c(\omega_v)^{-1} \) of \( \mathcal{O}_v \), where \( c(\omega_v) \) is the conductor of \( \omega_v \). We make a slight modification and put

\[
\mathcal{G}_{\psi_v}(\omega_v) := \int_{\mathcal{O}_v^\times} \omega_v^{-1}(y_v x_v) \cdot \psi_v(y_v x_v) \, dx_v,
\]

where \( dx_v \) is the normalized Haar measure such that \( \mathcal{O}_v^\times \) has total volume 1. Then it is independent of the choice of \( y_v \) and satisfies the property that

\[
\sigma(\mathcal{G}_{\psi_v}(\omega_v)) = \omega_v(t_{\sigma,\ell}) \cdot \mathcal{G}_{\psi_v}(\sigma \omega_v), \quad \sigma \in \text{Aut}(\mathbb{C}).
\]

As before suppose that \( \mathbb{Q}(\Pi) \subset E \). The \( \mathbb{Q}(\Pi_v) \)-form of \( \Pi_v \) induces an \( E \)-form of \( \Pi_v \), to be denoted by \( \Pi_v'(E) \). Put

\[
\Omega_{\Pi_v}(\varepsilon_v) := \mathcal{G}_{\psi_v}(\chi_{\Pi_v'}) \left( \frac{\alpha(n-1)}{2} \right) \cdot \mathcal{G}_{\psi_v}(\chi_{\Pi_{n-1,v}}),
\]

for an arbitrary \( \chi' \in \chi(\varepsilon_v) \), which is clearly well-defined. By Proposition 8.1 and the theory of Rankin-Selberg integrals [IFSS83], there is a family

\[
\{ \phi_v^\varepsilon \in \Pi_v \otimes \text{D}(\hat{U}_v \setminus \hat{G}_v) \}_{v \in \infty}
\]

such that
• for all \( v \uparrow \infty \), \( \phi_v^0 \in \Pi_v(E) \otimes \mathcal{D}(\hat{U}_v \backslash \hat{G}_v) \) and \( \mathcal{P}^0_v(\cdot, \phi_v^0) \) takes the nonzero constant value \((\Omega_{\Pi_v}(\varepsilon_v))^{-1}\) on \( X(\varepsilon_v)\);
• for all but finitely many \( v \uparrow \infty \), \( \Omega_{\Pi_v}(\varepsilon_v) = 1 \) and \( \phi_v^0 \) is the fixed spherical vector used in the restricted tensor product \( \Pi \otimes M(\hat{U}(A_k) \backslash \hat{G}(A_k)) = \bigotimes_v(\Pi_v \otimes M(\hat{U}_v \backslash \hat{G}_v)). \)

For a Hecke character \( \omega = \otimes_v \omega_v : k^\times \backslash A_k^\times \to \mathbb{C}^\times \), define its Gauss sum outside \( p \) by

\[
\mathcal{G}_v(\omega(p)) := \prod_{v \mid \infty p} \mathcal{G}_{\psi_v}(\omega_v).
\]

### 8.2. Archimedean modular symbols

Take \( K_\infty = A(\mathbb{R}) \cdot K'_\infty \subset G(\mathbb{R}) \), where \( A = \text{GL}(1)/\mathbb{Q} \times \text{GL}(1)/\mathbb{Q} \) is the largest central split torus in \( G \) and \( K'_\infty \) is the standard maximal compact subgroup (which is a product of orthogonal groups and unitary groups). Take \( K'_\infty : = K_\infty \cap \hat{G}(\mathbb{R}) \), which is the standard maximal compact subgroup of \( \hat{G}(\mathbb{R}) \). Recall that \( Q(\Pi) \subset E \). Then by \([\text{Cl90}]\) there is a geometrically irreducible algebraic representation \( F_\mu \boxtimes F_\nu \) of \( G_E \) such that the total relative Lie algebra cohomology

\[ H^*(g_C, K'_\infty; (F_\mu^0 \boxtimes F_\nu^0) \otimes \Pi_\infty) \neq \{0\}, \]

where \((\mu, \nu) \in (\mathbb{Z}^n)^{\varepsilon_k} \times (\mathbb{Z}^{n-1})^{\varepsilon_k} \) is as in Section 2.4. See \([\text{LLS24}]\) for more details. Suppose that \( V = F_\mu^0 \boxtimes F_\nu^0 \) (which is also viewed as a representation of \( G(\mathbb{Q}) \)), and that \( V = E \otimes (F_\mu^0 \boxtimes F_\nu^0) \) as a representation of \( G = G(\mathbb{Q}) \subset G_E(E) \).

Recall the \( G^2 \)-homomorphism \((2.8)\), where \( \Phi \) is now the family of closed subsets of \( G(\mathbb{Q}) \backslash \mathcal{X}_{G_n,K_\infty} \), whose obvious projections to \( G_n(\mathbb{Q}) \backslash \mathcal{X}_{G_n,K_n,\infty} \) are relatively compact \((K_n, \infty) \) is the projection of \( K_\infty \) to \( G_n(\mathbb{R}) \). Then it satisfies \((5.1)\) and \((6.17)\) (see Proposition 6.26 and the Künneth formula \([\text{Br97}] \text{ IV. Theorem 7.6}\)). The bottom degree component \( \Pi'_\infty \) of \( H^*(g_C, K'_\infty; V \otimes \Pi_\infty) \) and its \( E \)-form are as in \((2.9)\), with \( i_0 = \dim(\mathcal{G}(\mathbb{R})/K'_\infty) \). As a representation of the component group \( K'_\infty^{2} = k_\infty^{2,2} \), there is a multiplicity-free decomposition

\[ \Pi'_\infty \cong \bigoplus_{e \in k_\infty^{2,2}} e_\infty. \]

**Definition 8.2.** (a) A character \( \chi_\infty \) of \( k_\infty^\times \) is said to be critical for \( \Pi \) if it is algebraic and \( s = \frac{1}{2} \) is a pole of neither \( L(s, \Pi_\infty \times \chi_\infty) \) nor \( L(1-s, \Pi'_\infty \times \chi_\infty^{-1}) \).

(b) A Hecke character \( \chi \) of \( k^\times \backslash A_k^\times \) is said to be critical for \( \Pi \) if so is its archimedean component \( \chi_\infty \).

It is known that if there is a \( V \)-balanced character, then \( \chi_\infty \) is \( V \)-balanced if and only if it is critical for \( \Pi \). Assume that \( \chi_\infty \) is algebraic of weight \( w^{-1} \). By the well-known branching rule for algebraic representations of general linear groups, \( w = \prod_{i \in \mathbb{E}_k} t^{w_i} \) (hence \( \chi_\infty \) is \( V \)-balanced if and only if \((ctf. \text{[KS13]} \text{ [Rag16]}))

\[
\mu_{i+1} + \nu_{n-i} \leq w_i \leq \mu_i + \nu_{n-i} \text{ for all } i = 1, 2, \ldots, n-1 \text{ and } t \in \mathbb{E}_k.
\]
Following [LLSS23] Section 1.3, define a family \( \{z_m \in \text{GL}_m(\mathbb{Z})\}_{m \in \mathbb{N}} \) of matrices inductively by
\[
z_0 := 0 \quad \text{(the unique element of GL}_0(\mathbb{Z})) \quad \text{and} \quad z_1 := [1]
\]
where
\[
z_m := \begin{bmatrix} w_{m-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_{m-2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}, \quad m \geq 2,
\]
where \( w_m := \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \) denotes the \( m \times m \) anti-diagonal permutation matrix, \( e_{m-1} := [0 \cdots 0 \, 1] \in \mathbb{Z}^{1 \times (m-1)} \) is a row vector, and \( t g \) denotes the transpose of a matrix \( g \). Put
\[
(8.8) \quad z := (z_n, z_{n-1}) \in \text{GL}_n(\mathbb{Z}) \times \text{GL}_{n-1}(\mathbb{Z}) \to \text{G}(\mathbb{Q}).
\]
Let \( \mathcal{B} \) be the Borel subgroup of lower triangular matrices in \( \text{G} \), with unipotent radical \( \mathcal{U} \). By [LLSS23] Lemma 1.1], \( \mathcal{B} \, z \, \mathcal{G} \) is Zariski open in \( \mathcal{G} \), and in fact
\[
(8.9) \quad \mathcal{P} \cap \mathcal{G} = \{1\}, \quad \text{where} \quad \mathcal{P} := z^{-1} \mathcal{B} \, z.
\]
Let \( \mathcal{B} \) be the Borel subgroup of upper triangular matrices in \( \text{G} \). By using algebraic induction from \( \mathcal{B} \) as in [LLSS24], we realize \( \mathcal{V} \) as a space of algebraic functions on \( \mathcal{G}_E \). Let \( \mathcal{V} \in \mathcal{V}^\mathcal{U} \) be the unique algebraic function in \( \mathcal{V} \) which equals 1 on \( \mathcal{U} \).

Suppose that \( \mathcal{P} = \mathcal{P}(\mathbb{Q}_p) \) so that \( \mathcal{N} \) is its unipotent radical, and that \( \lambda_0 \in \text{Hom}_E(\mathcal{V}_\mathcal{E}, E) \) is the unique generator such that \( \langle \lambda_0, z^{-1} \mathcal{V} \rangle = 1 \). Recall that \( \mathcal{Z} = \text{G}_1 \).

For every \( \mathcal{V} \)-balanced algebraic character \( \mathcal{W} \) of \( \mathcal{Z}_E \), let \( \lambda_\mathcal{V} \in \text{Hom}_{\mathcal{G}(\mathbb{Q})}(\mathcal{E}_\mathcal{W} \otimes \mathcal{V}, E) \) be as in Lemma 2.9 Then for all algebraic characters \( \chi_\infty \) of \( k_\infty^\circ \) of weight \( w^{-1} \), we have the archimedean modular symbol map
\[
\hat{\mathcal{P}}_{\infty}^\lambda : H^0(\mathcal{J} \mathcal{C}, K^e_\infty; \mathcal{C}_w \otimes \chi_\infty) \times \left( \Pi'_\infty \otimes M(\mathcal{U}(\mathbb{R}))^\vee \otimes O(\mathcal{G}(\mathbb{R})/\hat{K}_\infty^e) \right) \to \mathbb{C}
\]
defined as in (2.11).

Let \( \Pi'_0,\infty \) be the irreducible tempered Casselman-Wallach representation of \( \mathcal{G}(\mathbb{R}) \) whose infinitesimal character equals that of the trivial representation and whose central character equals that of \( (\mathcal{F}_\mathcal{V}^\vee \otimes \mathcal{F}_\mathcal{V}^\vee) \otimes \Pi_\infty \). Define the cohomology group \( \Pi'_0,\infty \) as in (2.9). Following [LLSS24], with the fixed Whittaker functionals we have the translation map
\[
\mathcal{J}_\mu \otimes \mathcal{J}_\nu : \Pi'_0,\infty \to \Pi'_\infty
\]
which is a \( k_\infty^e,\hat{z} \)-equivariant isomorphism. As a specialization of (2.11), we have a map
\[
\hat{\mathcal{P}}_\infty : H^0(\mathcal{J} \mathcal{C}, K^e_\infty; \varepsilon_\infty) \times \left( \Pi'_0,\infty \otimes M(\mathcal{U}(\mathbb{R}))^\vee \otimes O(\mathcal{G}(\mathbb{R})/\hat{K}_\infty^e) \right) \to \mathbb{C}
\]
for every \( \varepsilon_\infty \in \kappa_{\infty}^{\frac{3}{2}} \). Define \( \hat{\mathcal{P}}_\infty^\nu \) in (2.10) to be the map such that for every \( \varepsilon_\infty \in \kappa_{\infty}^{\frac{3}{2}} \), \( \hat{\mathcal{P}}_\infty^\nu(\varepsilon_\infty, \cdot) \) equals the composition of
\[
(1, (\mu \otimes \nu)^{-1} \otimes \id \otimes \id) \rightarrow H^0(\mathfrak{C}, K^\circ_{\infty}; \varepsilon_\infty) \times \left( \Pi'_{\kappa, \infty} \otimes M(\hat{\mathcal{U}}(\mathbb{R}))^\vee \otimes O(\hat{\mathcal{G}}(\mathbb{R})/K^\circ_{\infty}) \right)
\]
\[\hat{\mathcal{P}}_\infty \rightarrow \mathbb{C}.
\]
Here and as usual, \( \id \) denotes the identity map.

We have the following archimedean nonvanishing hypothesis and period relations, which are proved in [Sun17, LLS24] for the essentially tempered case, and extended to all the generic cohomological cases in [JLS24] after a suggestion of Michael Harris.

**Theorem 8.3.** If \( w \) is \( V \)-balanced, then the followings hold.

(a) The map \( \hat{\mathcal{P}}_\infty^\nu \) is nonzero.

(b) \( \hat{\mathcal{P}}_\infty^\nu(1, \cdot) = \Upsilon_{\Pi, \infty}(\chi_\infty) \cdot \hat{\mathcal{P}}_\infty^\nu(w_\infty \chi_\infty, \cdot) \), where \( \Upsilon_{\Pi, \infty}(\chi_\infty) \) is in (2.22).

Following Definition 2.1, fix an element
\[
\hat{\phi}_\infty^\nu \in \Pi'_{\kappa, \infty}(E) \otimes D(\hat{\mathcal{U}}(\mathbb{R}))^\vee \otimes O(\hat{\mathcal{G}}(\mathbb{R})/K^\circ_{\infty})
\]
such that \( \hat{\mathcal{P}}_\infty^\nu(\varepsilon_\infty, \hat{\phi}_\infty^\nu) \neq 0 \) for all \( \varepsilon_\infty \in \kappa_{\infty}^{\frac{3}{2}} \). Define the Whittaker periods
\[
\Omega_{\Pi}(\varepsilon_\infty) := \left( \hat{\mathcal{P}}_\infty^\nu(\varepsilon_\infty, \hat{\phi}_\infty^\nu) \right)^{-1}, \quad \varepsilon_\infty \in \kappa_{\infty}^{\frac{3}{2}}.
\]

### 8.3. Open orbit integrals and normalized refined period

Recall the Borel subgroup \( P \subset G \) from (8.9). Suppose that \( P = P(Q_p) = z^{-1} \mathcal{B} z \), where \( \mathcal{B} = \mathcal{B}(Q_p) \).

Assume that \( \Pi_p' \subset \mathcal{B}_P(\Pi_p) \) is a nearly ordinary refinement of \( \Pi_p \) defined over \( E \), which is also viewed as a character of \( P \) that descends to a character of the torus \( L = P/N \).

For every locally compact Hausdorff topological group \( \mathcal{G} \), write \( \delta_\mathcal{G} : \mathcal{G} \rightarrow \mathbb{C}^\times \) for its modular character. We use \( \Ind \) to denote the normalized smooth induction, and still use a superscript \( \vee \) to indicate the contragredient of an admissible smooth representation of \( G \) (or some other groups when no confusion is possible).

**Lemma 8.4.** One has that \( \dim \text{Hom}_L(\Pi_p', \mathcal{B}_P(\Pi_p)) = 1 \).

**Proof.** We have that
\[
\text{Hom}_L(\Pi_p', \mathcal{B}_P(\Pi_p)) = \text{Hom}_P(\Pi_p' \vee, \Pi_p'^{\vee})
\]
\[
= \text{Hom}_G(\Pi_p' \vee, \Ind_P^G(\Pi_p'^{\vee} \otimes \delta_p^{-1/2}))
\]
\[
= \text{Hom}_G(\Ind_P^G(\Pi_p' \otimes \delta_p^{1/2}), \Pi_p).
\]
This is at most one-dimensional by the uniqueness of the Whittaker functionals on the principal series representations. \( \square \)
By the proof of Lemma 8.4, $\Pi_p$ is isomorphic to a quotient representation of $\text{Ind}^G_P(\Pi'_p \cdot \delta^{1/2})$. For convenience, define a character

$$\kappa := \Pi'_p \circ \text{Ad}(z^{-1}) : B \to \mathbb{E}^\times,$$

so that $\Pi_p$ is also isomorphic to a quotient representation of $I(\tilde{\kappa}) := \text{Ind}^G_B(\tilde{\kappa})$, where $\tilde{\kappa} := \kappa \otimes \delta^{1/2}$.

The most technical input for the evaluation of the modifying factor $s$ at $p$ and $\infty$ is an application of the preparatory result in [LLSS23] to be recalled below. Let $v$ be a finite place of $k$. The result in loc. cit. compares the normalized Rankin-Selberg integral $P^v \circ v$ with the integral over the open $\dot{G}_v$-orbit in the flag variety $B_v \setminus \dot{G}_v$. The same result for the archimedean places has been used in [LLS24, JLS24] to evaluate $\Upsilon_{\Pi_v}(\chi_\infty)$ and prove the archimedean period relations in Theorem 8.3.

Write a continuous character $\varrho : B_v \to \mathbb{C}^\times$ as

$$\varrho := (\varrho_1, \ldots, \varrho_n, \varrho'_1, \ldots, \varrho'_{n-1}) \in \left(\hat{k}_v^\times\right)^{2n-1}.$$

Let $I(\varrho) := \text{Ind}^G_{B_v}\varrho$. By [Wa92] Theorem 15.4.1, there is a unique Whittaker functional $\lambda'_{U_v} \in \text{Hom}_{U_v}(I(\varrho), \psi_{U_v})$ such that

$$\langle \lambda'_{U_v}, f \rangle = \int_{U_v} f(u)\psi_{U_v}^{-1}(u) \, du$$

for all $f \in I(\varrho)$ such that $f|_{U_v} \in S(U_v)$, where we fix the Haar measure $du$ on $U_v$ to be the product of the self-dual Haar measures on $k_v$ with respect to $\psi_v$. Here and henceforth, $S(X)$ denotes the space of compactly supported locally constant complex functions on $X$ when $X$ is a totally disconnected topological space.

We fix the similar Haar measure on $\dot{U}_v$. For $\tau \in M(G_v)$, denote by $\bar{\tau} \in M(\dot{U}_v \setminus \dot{G}_v)$ the quotient of $\tau$ by the fixed measure on $\dot{U}_v$. We have the unnormalized Rankin-Selberg integral map

$$P_v : \mathcal{X}_v \times \left(I(\varrho) \otimes M(\dot{U}_v \setminus \dot{G}_v)\right) \to \mathbb{C} \cup \{\infty\},$$

defined by meromorphic continuation of absolutely convergent integrals.

Following [LLSS23], we also have the open orbit integral map

$$\Lambda_v : \mathcal{X}_v \times \left(I(\varrho) \otimes M(G_v)\right) \to \mathbb{C} \cup \{\infty\},$$

defined by meromorphic continuation (in variables $\varrho$ and $\chi'_v$) of absolutely convergent integrals.

Similar to (2.17), the maps (8.12) and (8.13) naturally extend to maps

$$P_v : \mathcal{X}_v \times \left(I(\varrho)_{p_{v-sm}} \otimes M(\dot{U}_v \setminus \dot{G}_v)\right) \to \mathbb{C} \cup \{\infty\}.$$
and
\[ \Lambda_v : X_v \times \left( \hat{I}(\varrho)_{p_v - \text{sm}} \otimes M(\hat{G}_v) \right) \to \mathbb{C} \cup \{\infty\}, \]
where \( p_v \) denotes the Lie algebra of \( P(k_v) \).

Define a meromorphic function on \( X_v \) by
\[ \Gamma_{\varrho, \psi_v}(\chi'_v) := \prod_{i > j, i + j \leq n} (\varrho_i \cdot \varrho'_j \cdot \chi'_v)(-1) \cdot \prod_{i + j \leq n} \gamma \left( \frac{1}{2}, \varrho_i \cdot \varrho'_j \cdot \chi'_v, \psi_v \right). \]

Here and henceforth
\[ \gamma(s, \omega, \psi_v) := \varepsilon(s, \omega, \psi_v) \cdot \frac{L(1 - s, \omega^{-1})}{L(s, \omega)} \]
denotes the Tate \( \gamma \)-factor of a character \( \omega \in \widehat{k_v}^\times \) defined in \([T79, J79, Ku03]\) using the self-dual Haar measure on \( k_v \) with respect to \( \psi_v \). The following result is implied by \([LLSS23, \text{Theorem 1.6 (b)}]\) and \([LLS24, \text{Corollary 4.3}]\).

**Theorem 8.5.** For every \( \hat{f} \otimes \tau \in \hat{I}(\varrho)_{p_v - \text{sm}} \otimes M(\hat{G}_v) \) the equality
\[ \Lambda_v(\chi'_v, \hat{f} \otimes \tau) = \Gamma_{\varrho, \psi_v}(\chi'_v) \cdot \mathcal{P}_v(\chi'_v, \hat{f} \otimes \tau) \]
holds as meromorphic functions of \( \chi'_v \in X_v \).

Write \( \tilde{k} = \otimes_{q \neq p} \tilde{k}_p \). Then we have that \( I(\tilde{k}) = \otimes_{q \neq p} I(\tilde{k}_p) \). By tensor product over \( q \mid p \), the maps in \((8.12)\) yield a map
\[ \mathcal{P}_p : X_p \times \left( I(\tilde{k}) \otimes M(\hat{U} \setminus \hat{G}) \right) \to \mathbb{C} \cup \{\infty\}, \]
which naturally extends to a map
\[ \mathcal{P}_p : X_p \times \left( I(\tilde{k})_{p - \text{sm}} \otimes M(\hat{U} \setminus \hat{G}) \right) \to \mathbb{C} \cup \{\infty\}. \]
Similarly, the maps in \((8.13)\) yield a map
\[ \Lambda_p : X_p \times \left( I(\tilde{k})_{p - \text{sm}} \otimes M(\hat{G}) \right) \to \mathbb{C} \cup \{\infty\}. \]

Recall the Whittaker functional \( \lambda_{U_p} \) on \( \Pi_p \). Write
\[ (8.14) \quad \xi_p : I(\tilde{k}) \to \Pi_p = \otimes_{q \neq p} \Pi_q \]
for the \( G \)-homomorphism whose composition with \( \otimes_{q \neq p} \lambda_{U_q} \) equals \( \otimes_{q \neq p} \lambda_{U_q} \). It is surjective and naturally extends to a surjective \( G \)-homomorphism
\[ (8.15) \quad \xi_p : I(\tilde{k}) \to \Pi_p. \]

Note that \( I(\tilde{k}) \) is naturally identified with a space of generalized functions on \( G \). Denote by \( I(\tilde{k})_z \) its subspace of the generalized functions supported in \( B_z \), where
where $z$ is as in (8.8). It is easy to see that $\widehat{I}(\tilde{\kappa})_z$ is one-dimensional and the map $\xi_p$ in (8.15) restricts to a $P$-isomorphism
\[ \xi_p : \widehat{I}(\tilde{\kappa})_z \sim \Pi'_p. \]
Moreover, for every generator $\hat{f} \otimes \tau$ of $\widehat{I}(\tilde{\kappa})_z \otimes M(\hat{G})$, $\Lambda_p(\cdot, \hat{f} \otimes \tau)$ is a constant function on $X_p$ with values in $\mathbb{C}^\times$.

Fix the Haar measure on $\hat{U}$ to be the product of the measures on $\hat{U}_p$ for all $\varphi \mid p$, and for $\tau \in M(\hat{G})$ denote by $\bar{\tau} \in M(\hat{U}\backslash \hat{G})$ the quotient of $\tau$ by the fixed measure on $\hat{U}$. Now we define the normalized refined period map to be the composition
\[ \hat{p}_p^\circ : X_p \times \left( \Pi'_p \otimes M(\hat{U}\backslash \hat{G}) \right) \]
\[ \xrightarrow{\xi_p^{-1}} X_p \times \left( \widehat{I}(\tilde{\kappa})_z \otimes M(\hat{G}) \right) \]
\[ \xrightarrow{\bar{\tau} \mapsto \tau} X_p \times \left( \widehat{I}(\tilde{\kappa})_z \otimes M(\hat{G}) \right) \]
\[ \xrightarrow{\Lambda_p} \mathbb{C}. \]

8.4. Rational test vectors and modifying factors at $p$. In what follows we define a rational test vector $\hat{\omega}_p^\circ \in \Pi'_p(E) \otimes D(\hat{U}\backslash \hat{G})$. Recall the $\mathbb{Q}(\Pi_p)$-form of $\Pi_p$ determined by (8.22). Define an Aut($\mathbb{C}/\mathbb{Q}(\kappa_p)$)-action on $I(\tilde{\kappa}_p)$ by
\[ \sigma f(g) := \kappa_p(t_{\sigma,p}) \cdot \sigma(f(g)), \quad \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\kappa_p)), \quad f \in I(\tilde{\kappa}_p), \quad g \in G_p. \]
By (8.4) we have that
\[ \kappa_p(t_{\sigma,p}) = \frac{\sigma(\omega_{\psi_p}(\kappa_p))}{\omega_{\psi_p}(\kappa_p)}, \]
where $\omega_{\psi_p}(\kappa_p)$ is in (2.25). Hence
\[ I(\tilde{\kappa}_p)^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\kappa_p))} = \{ f \in I(\tilde{\kappa}_p) : \omega_{\psi_p}(\kappa_p) \cdot f \text{ is } \mathbb{Q}(\kappa_p)\text{-valued} \}, \]
which is a $\mathbb{Q}(\kappa_p)$-form of $I(\tilde{\kappa}_p)$. By tensor products, we have a $\mathbb{Q}(\Pi_p)$-form of $\Pi_p$ as well as a $\mathbb{Q}(\kappa_p)$-form on $I(\tilde{\kappa})$.

Recall that $c_p = \prod_{\varphi \mid p} c_{\varphi}$, where $c_{\varphi}$ as in (2.24) is the volume of $O_{\varphi}$ under the self-dual Haar measure on $k_{\varphi}$ with respect to $\psi_{\varphi}$.

Proposition 8.6. One has that $\mathbb{Q}(\Pi_p) \subset \mathbb{Q}(\kappa)$ and the surjective homomorphism $c_p^{-(n-1)^2} \xi_p : I(\tilde{\kappa}) \rightarrow \Pi_p$ is $\mathbb{Q}(\kappa)$-rational, where $\xi_p$ is in (8.11).

Proof. From (8.22), it suffices to show that for every $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\kappa_p))$ and $f \in I(\tilde{\kappa}_p)$ it holds that
\[ c_p^{-(n-1)^2} \cdot \langle \lambda'_{U_p}, \sigma f \rangle = \sigma(c_p^{-(n-1)^2} \cdot \langle \lambda'_{U_p}, t_{\sigma,p} f \rangle). \]
We may assume that $f|_{U_p} \in S(U_p)$. Then the above equality can be verified directly using the Jacquet integral (8.11).
Let $\hat{\phi}_p$ denote the generator of $\Pi'_p \otimes M(\hat{U}\backslash \hat{G})$ such that $\hat{P}_p^0(\cdot, \hat{\phi}_p)$ equals the constant function 1 on $X_p$. Write $\hat{\phi}_p = \phi'_p \otimes \bar{\tau}$ such that $\phi'_p \in \Pi'_p$ and $\tau \in D(\hat{G})$. Proposition 8.6 implies that

$$c_p^{-(n-1)^2} \omega_{\psi_p}(\kappa)^{-1} \cdot \phi'_p \in \Pi'_p(E).$$

It is clear that $c_p^{-(n-1)(n-2)/2} \cdot \bar{\tau} \in D(\hat{U}\backslash \hat{G})$. Define

$$\hat{\phi}^\circ_p := (c_p^{-(n-1)^2} \omega_{\psi_p}(\kappa)^{-1} \cdot \phi'_p) \otimes (c_p^{-(n-1)(n-2)/2} \cdot \bar{\tau}) = \Omega_{\Pi'_p}^{-1} \cdot \hat{\phi}_p \in \Pi'_p(E) \otimes D(\hat{U}\backslash \hat{G}),$$

where $\Omega_{\Pi'_p}$ is in (2.23). Then

$$\hat{P}_p^0(\chi'_p, \hat{\phi}^\circ_p) = \Omega_{\Pi'_p}^{-1} \text{ for all } \chi'_p \in X_p.$$

From (2.23) it is clear that

$$\Upsilon_{\Pi'_p}(\chi'_p) = \frac{1}{L(\frac{2}{2}, \Pi_p \times \chi'_p) \cdot \prod_{p \mid \infty} \Gamma_{\hat{\psi}_p}(\chi'_p)}, \quad \chi'_p = \otimes_{v \mid \infty} \chi'_v \in X_p.$$

Then the following is a direct consequence of Theorem 8.5.

**Proposition 8.7.** It holds that

$$\mathcal{P}_p^0(\chi'_p, \hat{\phi}^\circ_p) = \Upsilon_{\Pi'_p}(\chi'_p) \cdot \Omega_{\Pi'_p}^{-1} \text{ for all } \chi'_p \in X_p.$$ 

Consequently $\Upsilon_{\Pi'_p}$ is algebraic on $X_p$.

By Proposition 8.7, $\Upsilon_{\Pi'_p}(\chi_p)$ is the modifying factor at $p$ as in Definition 2.7. It is consistent with the conjecture given by Coates and Perrin-Riou in [CPR89, Co89].

### 8.5. $p$-adic $L$-functions and exceptional zeros

Now we construct Rankin-Selberg $p$-adic $L$-function $\mathcal{L}_{\Pi}$ by combining the previous results. Suppose that $Z_0$ is trivial and recall $\varepsilon = \otimes_{v \mid \infty} \varepsilon_v$ in (1.3).

- For $v \nmid \infty p$, let $\phi_v^\circ \in \Pi_v(E) \otimes D(\hat{U}_v \backslash \hat{G}_v)$ be as in Section 8.1 such that
  $$\mathcal{P}_v^0(\chi'_v, \phi_v^\circ) = \frac{1}{\mathcal{G}_{\psi_v}(\chi_{\Pi_v-1,v}) \cdot \mathcal{G}_{\psi_v}(\chi'_v)^{(n-1)^2}}, \quad \chi'_v \in X(\varepsilon_v).$$

- Let $\lambda_0 \in \text{Hom}_{E}(V^n, E)$ and $\hat{\phi}_\infty^\circ \in \Pi'_\infty(E) \otimes D(\hat{U}(\mathbb{R}))^\vee \otimes O(\hat{G}(\mathbb{R})/K^\circ)$ be as in Section 8.2 such that
  $$\hat{P}_\infty^{\lambda_0}(1, \hat{\phi}_\infty^\circ) = \Upsilon_{\Pi'_\infty}(\chi_\infty) \cdot \hat{P}_\infty^0(w_\infty \chi_\infty, \hat{\phi}_\infty^\circ) = \frac{\Upsilon_{\Pi'_\infty}(\chi_\infty)}{\Omega_{\Pi}(w_\infty \chi_\infty)}$$

for all $V$-balanced characters $w$ and all algebraic characters $\chi_\infty$ of weight $w^{-1}$, where $\lambda_V \in \text{Hom}_{E}(E_w \otimes V, E)$ is the generator such that $\lambda|_{V^n} = \lambda_0$.

- Let $\hat{\phi}_p^\circ \in \Pi'_p(E) \otimes D(\hat{U} \backslash \hat{G})$ be as in Proposition 8.7.
Using all the local test vectors, let
\[ \hat{\phi} := \hat{\phi}_\infty \otimes \hat{\phi}_p \otimes (\otimes_{v|p, \phi_v}) \in H \otimes D(\hat{\mathcal{G}}, \hat{\mathcal{H}}), \]
and following Definition 2.5 let
\[ L_{\Pi} := L_{\chi \otimes \hat{\phi}_\infty} = (L_{\chi \otimes \hat{\phi}_p})|_{C(\mathbb{Z}, E)(\varepsilon)}. \]

We restate Theorem 2.10 below, which is now an immediate consequence of the above results and (2.19).

**Theorem 8.8.** If there is a \( V \)-balanced character, then \( L_{\Pi} \) is the unique continuous linear functional on \( C(\mathbb{Z}, E)(\varepsilon) \) such that
\[ L_{\Pi}(\chi^{\flat}) = \Upsilon_{\Pi}(\chi \otimes \hat{\phi}_\infty) \cdot \Upsilon_{\Pi}(\chi \otimes \hat{\phi}_p) \cdot \Omega_{\Pi}(w_{\infty, \chi}) \]
for all critical algebraic Hecke characters \( \chi = \otimes \chi_\ell \in \mathcal{X}(\varepsilon) \), where \( w \) is the inverse of the weight of \( \chi \).

As mentioned earlier, the existence of exceptional zeros of \( p \)-adic L-functions is of great importance in arithmetic applications. Let us give the application of Theorem 8.8.

**Proposition 8.9.** Under the assumptions in Theorem 8.8, \( \chi^{\flat} \) is an exceptional zero of \( L_{\Pi} \) if and only if there exists a place \( \wp | p \) and \( 1 \leq i < n \) such that
\[ \chi_\wp = \kappa_{i,\wp}^{-1} \cdot \kappa_{n-i,\wp} \cdot |\cdot|_\wp, \]
where \( \kappa_{i,\wp} \)'s and \( \kappa_{j,\wp} \)'s are as in (2.23).

The rest of this section is devoted to the proof of Proposition 8.9. Let us introduce the notion of \( p \)-exponent at a place \( \wp | p \). Denote by \( \mathcal{E}_k \wp \), the set of continuous field embeddings \( k_\wp \hookrightarrow \mathbb{C}_p \), so that \( \mathcal{E}_k \) can be identified with the disjoint union of \( \{\mathcal{E}_k \wp\}_{\wp|p} \). Let \( d_\wp := |\mathcal{E}_k \wp| = [k_\wp : \mathbb{Q}_p] \).

**Definition 8.10.** The \( p \)-exponent of a character \( \omega : k_\wp^\times \to \mathbb{C}_p^\times \) is the number
\[ \exp_p(\omega) \in \mathbb{Q} \] such that
\[ |\omega(a)|_p = |a|_{\wp}^{\exp_p(\omega)} = |a|_{p^{d_\wp \cdot \exp_p(\omega)}} \cdot a \in k_\wp^\times. \]

Here and henceforth, \( |\cdot|_v \) denotes the normalized absolute value on \( k_v \) for every place \( v \) of \( k \).

**Example 8.11.** (a) The normalized absolute value \( |\cdot|_p : k_\wp^\times \to \mathbb{Q}_p^\times \hookrightarrow \mathbb{C}_p^\times \) has \( p \)-exponent \(-1\).
(b) The inclusion \( k_\wp^\times \hookrightarrow \mathbb{C}_p^\times \) has \( p \)-exponent \( d_\wp^{-1} \).
Similarly we define the $p$-exponent of $\tilde{\kappa}_\wp$ as an element of $Q^{2n-1}$. Note that $G_\wp \subset G$ acts diagonally on $V_\wp := \otimes_{i \in E_{k_\wp}}(F_{\mu i} \boxtimes F_{\nu i})$, where $F_{\mu i}$ and $F_{\nu i}$ are of highest weights $\mu_i$ and $\nu_i$ respectively. Let

$$\mu^p := \frac{1}{d_\wp} \sum_{i \in E_{k_\wp}} \mu_i \in Q^n, \quad \nu^p := \frac{1}{d_\wp} \sum_{i \in E_{k_\wp}} \nu_i \in Q^{n-1}.$$ 

Recall that $L$ acts on $V^n$ through a character $\alpha_V : L \to E^\times$. Then the component of $\alpha_{V} \circ \text{Ad}(z^{-1})$ at $\wp$ has $p$-exponent $(-\mu^p, -\nu^p) \in Q^{2n-1}$. The character $\delta_{\Pi_\wp}$ has $p$-exponent $2(\rho_n, \rho_n - 1)$, where $\rho_m := \left( \frac{m-1}{2}, \frac{m-3}{2}, \ldots, \frac{-1}{2} \right) \in Q^m$ ($m = n, n-1$).

Since $\Pi_\wp$ and $\alpha_V \cdot \delta^{-1}_\wp$ have the same $p$-adic norm by Definition 1.13, we have the following result.

**Lemma 8.12.** For $\wp | p$, the character $\tilde{\kappa}_\wp$ has $p$-exponent

$$(-\mu^p - \rho_n, -\nu^p - \rho_{n-1}) \in Q^{2n-1}.$$ 

**Lemma 8.13.** If $\chi = \otimes_v \chi_v$ is $V$-balanced, then for each $\wp | p$ one has that

$$\frac{1}{2} + \exp(\tilde{\kappa}_{i,\wp} \cdot \tilde{\kappa}'_{j,\wp} \cdot \chi_{\wp}) \begin{cases} \leq 0, & \text{if } i + j \leq n, \\ \geq 1, & \text{if } i + j > n. \end{cases}$$

**Proof.** Let $w^{-1}$ be the weight of $\chi$. By Lemma 8.12, we find that

$$\frac{1}{2} + \exp(\tilde{\kappa}_{i,\wp} \cdot \tilde{\kappa}'_{j,\wp}) = w_{\wp} - \mu^p_i - \nu^p_j + i + j - n,$$

where $w_{\wp} := d_{\wp}^{-1} \sum_{i \in E_{k_\wp}} w_i$. The assertion follows easily from (8.7). \qed

Now we prove Proposition 8.9. By the nontrivial estimate of Langlands parameters towards the generalized Ramanujan conjecture in [LRS99, MS04] and the multiplicativity of Rankin-Selberg $L$-functions, $L(\frac{1}{2}, \Pi_p \times \chi_p)$ is finite. Thus $\chi^b$ is an exceptional zero of $L_{\Pi}$ if and only if

$$\prod_{\wp | p, i+j \leq n} \gamma \left( \frac{1}{2}, \tilde{\kappa}_{i,\wp}^{-1} \cdot \tilde{\kappa}'_{j,\wp}^{-1} \cdot \chi_{\wp}^{-1} \cdot \psi_{\wp}^{-1} \right) = 0.$$

The assertion follows easily from Lemma 8.13 and its proof.

9. Application II: Rankin-Selberg $L$-functions for $U_n \times U_{n-1}$

In this section we retain the setup in Section 2.5 and construct the $E$-valued $p$-adic measure $L_{\Pi}$ in Theorem 2.12 following Sections 2.2–2.3, for which we assume that the character $\varepsilon$ in (1.13) is trivial. Then we determine the exceptional zeros of $L_{\Pi}$.
9.1. **Refined Gan-Gross-Prasad conjecture.** We first review the Gan-Gross-Prasad conjecture and its refinement for the Bessel periods of unitary groups. Let \( \hat{\pi}_m, m = n, n-1 \ (n \geq 2) \), be an irreducible isobaric automorphic representation of \( \text{GL}_m(\mathbb{A}_k') \) that is hermitian in the sense of [BPLZZ21, Definition 1.5]. Put \( \hat{\pi} := \hat{\pi}_n \boxplus \hat{\pi}_{n-1} \), which is an automorphic representation of \( \text{GL}_n(\mathbb{A}_k') \times \text{GL}_{n-1}(\mathbb{A}_k') \).

Denote by \( k' \rightarrow k \), \( a \mapsto \bar{a} \) the Galois involution for \( k'/k \), which induces an \( \mathbb{A}_k \)-algebra involution \( \mathbb{A}_k' \rightarrow \mathbb{A}_k' \), \( a \mapsto \bar{a} \).

**Theorem 9.1 (Gan-Gross-Prasad conjecture).** The following two statements are equivalent.

(a) \( L(\frac{1}{2}, \hat{\pi}) \neq 0 \).

(b) There exists a quadruple \( (V_n, V_{n-1}, \pi_n, \pi_{n-1}) \) such that

- \( V_{n-1} \) is a hermitian spaces over \( k' \) of rank \( n-1 \) and \( V_n \) is a Hermitian space over \( k' \) such that \( V_n = V_{n-1} \oplus k' \) is an orthogonal direct sum, where \( k' \) is viewed as a hermitian space under the form \( (a, b) \mapsto ab, a, b \in k' \);
- for \( m = n, n-1 \), \( \pi_m \) is an irreducible cuspidal automorphic representations of \( U_m(\mathbb{A}_k) \) such that \( BC(\pi_m) \cong \hat{\pi}_m \), where \( U_m := U(V_m) \) is the unitary group attached to \( V_m \), and \( BC(\pi_m) \) denotes the weak base-change of \( \pi_m \) to \( \text{GL}_m(\mathbb{A}_k') \);
- there are cusp forms \( \varphi_m \in \pi_m \ (m = n, n-1) \) such that

\[
\int_{U_{n-1}(k) \setminus U_{n-1}(\mathbb{A}_k)} \varphi_n(g) \varphi_{n-1}(g) \, dg \neq 0,
\]

where \( dg \) is the Tamagawa measure.

Suppose that the equivalent statements in Theorem 9.1 hold, and we keep the notation in Theorem 9.1. Then the pair \( (V_n, V_{n-1}) \) is unique up to isomorphism, and the pair \( (\pi_n, \pi_{n-1}) \) is uniquely determined, by the local Gan-Gross-Prasad conjecture [BP16, BP20, Xue23] and Arthur’s multiplicity formula for unitary groups [KMSW14]. Suppose that \( \pi := \pi_n \boxtimes \pi_{n-1} \) is everywhere tempered. Recall the L-function \( L(s, \pi \times \chi) \) in [227], where \( \chi \) is an automorphic character of \( U_1(\mathbb{A}_k) \) with base-change \( \hat{\chi} = BC(\chi) : k' \times \mathbb{A}_{k'}^\times \rightarrow \mathbb{C}^\times, \ a \mapsto \chi(a/\bar{a}) \).

Note that

\[
L(s, \pi, \text{Ad}) = L(s, \hat{\pi}_n, \text{Ad}^{(-1)^n}) L(s, \hat{\pi}_{n-1}, \text{Ad}^{(-1)^{n-1}})
\]

is a product of Asai L-functions.

Recall the embedding

\[
\iota : \hat{\mathbb{G}} = \text{Res}_{k/\mathbb{Q}}(U_{n-1} \times U_{n-1}) \hookrightarrow \mathbb{G} = \text{Res}_{k/\mathbb{Q}}(G_0 \times G_0),
\]

where \( G_0 = U_n \times U_{n-1} \), and the homomorphism \( j : \hat{\mathbb{G}} \rightarrow \mathbb{Z} = \text{Res}_{k/\mathbb{Q}} U_1 \). Let

\[
H := \text{Res}_{k/\mathbb{Q}} G_0 \overset{\text{diag}}{\rightarrow} \mathbb{G}
\]
diagonally embedded as a subgroup of \(G\), and let \(\psi_H\) be the trivial character of \(H(\mathbb{A})\). Then it is clear that (2.7) holds. At this point it is again more familiar to work with \(k\) and introduce
\[
\hat{G} := U_{n-1} \times U_{n-1} \rightarrow G := G_0 \times G_0 \quad \text{and} \quad H := G_0 \rightarrow G.
\]
Fix factorizations \(\pi = \hat{\otimes}_v \pi_v\) and \(\pi^\vee = \hat{\otimes}_v \pi_v^\vee\), where \(v\) runs over all places of \(k\), such that there is a decomposition of \(H(\mathbb{A}_k)\)-invariant map
\[
\langle \cdot, \cdot \rangle = \otimes_v \langle \cdot, \cdot \rangle_v : \Pi = \pi \boxtimes \pi^\vee \rightarrow \mathbb{C},
\]
where \(\langle \cdot, \cdot \rangle_v\) is the pairing between \(\pi_v\) and \(\pi_v^\vee\) given by the Petersson inner product on cusp forms with respect to the Tamagawa measure of \(H(\mathbb{A}_k)\), and \(\langle \cdot, \cdot \rangle_v\) is the natural pairing between \(\pi_v\) and \(\pi_v^\vee\).

Following Theorem 9.1 (b), define the global period integral map
\[
\mathcal{P}_\chi : \Pi \otimes M(\hat{G}(k) \backslash \hat{G}(\mathbb{A}_k)) \rightarrow \chi^{-1},
\]
\[
\varphi \otimes \varphi^\vee \otimes \tau \mapsto \int_{\hat{G}(k) \backslash \hat{G}(\mathbb{A}_k)} \chi(g) \cdot (\varphi \otimes \varphi^\vee)(g) \, d\tau(g).
\]

Let \(X_v\) be the group of complex continuous characters of \(U_1(k_v)\) for every place \(v\) of \(k\), and let \(X_v^\mathrm{un} \subset X_v\) be the subgroup of unramified characters. In particular \(X_v^\mathrm{un}\) consists of the trivial character if \(v\) is nonsplit in \(k\). Define the \(\hat{G}_v\)-invariant normalized zeta integral
\[
\mathcal{P}_v^0 : \mathcal{X}_v \times \left( \Pi_v \otimes M(H_v \backslash \hat{G}_v) \right) \rightarrow \mathbb{C},
\]
\[
(\chi_v', \varphi_v \otimes \varphi_v^\vee \otimes \tau_v) \mapsto \frac{\int_{U_{n-1}(k_v)} \chi_v'(\det g) \langle \pi_v(g) \varphi_v, \varphi_v^\vee \rangle_v \, d\tau_v(g)}{\mathcal{L}(\frac{1}{2}, \pi_v \times \chi'_v)},
\]
where \(U_{n-1}\) embeds into \(G_0\) diagonally as before, \(\varphi_v \otimes \varphi_v^\vee \in \Pi_v = \pi_v \boxtimes \pi_v^\vee\) and \(\tau_v \in M(H_v \backslash \hat{G}_v) = M(U_{n-1}(k_v))\). Here and henceforth \(\mathcal{L}(\frac{1}{2}, \pi_v \times \chi'_v)\) denotes the local factor of \(\mathcal{L}(\frac{1}{2}, \pi \times \chi')\) at \(v\). By the temperedness assumption, the integral converges absolutely and \(\mathcal{L}(\frac{1}{2}, \pi_v \times \chi'_v) \in \mathbb{C}^\times\) when \(\chi'_v\) is unitary. By Lemma 9.5 below, \(\mathcal{P}_v^0\) has a holomorphic continuation in \(\chi'_v \in \mathcal{X}_v\) when \(v\) is split in \(k\). Moreover if \(\chi = \otimes_v \chi_v, \varphi \otimes \varphi^\vee = \otimes_v (\varphi_v \otimes \varphi_v^\vee)\) and \(\tau = \otimes_v \tau_v\) under the identification (2.7), then \(\mathcal{P}_v^0(\chi_v, \varphi_v \otimes \varphi_v^\vee \otimes \tau_v) = 1\) for all but finitely many \(v\) by [Ha14] Theorem 2.12.

The following Ichino-Ikeda conjecture refines the Gan-Gross-Prasad conjecture.

**Theorem 9.2 (Ichino-Ikeda conjecture).** Let the assumptions be as above. Then for \(\chi = \otimes_v \chi_v, \varphi \otimes \varphi^\vee = \otimes_v (\varphi_v \otimes \varphi_v^\vee) \in \Pi\) and \(\tau = \otimes_v \tau_v \in M(\hat{G}(k) \backslash \hat{G}(\mathbb{A}_k))\) it holds that
\[
\mathcal{P}_\chi(\varphi \otimes \varphi^\vee \otimes \tau) = |S_\pi|^{-1} \mathcal{L}(\frac{1}{2}, \pi \times \chi) \prod_v \mathcal{P}_v^0(\chi_v, \varphi_v \otimes \varphi_v^\vee \otimes \tau_v).
\]

In the above, \(S_\pi = S_\pi \times S_{n-1}\) and \(S_m\) is an elementary abelian 2-group whose rank equals the number of cuspidal summands of the isobaric sum \(\pi_m\). Thanks to
the work of [Mok15, KMSW14, AGI+24], it is known that $|S_{\pi_m}|$ is the size of the global Arthur packet of $\pi_m$.

Theorem 9.1 and Theorem 9.2 are proved in [BPLZZ21] for $\tilde{\pi}$ cuspidal (in which case $|S_{\tilde{\pi}}| = 4$), and in [BPCZ22] for $\tilde{\pi}$ hermitian isobaric, improving the previous results in [Zh14a, Zh14b, Xue19, BP21a, BP21b]. They have been further extended to some Eisenstein series very recently in [BPC23].

9.2. Rational forms and integrals of matrix coefficients. In the rest of this section, assume that $\pi$ is regular algebraic, $\mathbb{Q}(\pi) \subset E$ and the coefficient system of $\pi$ is defined over $E$, and that $\chi$ is algebraic.

**Lemma 9.3.** For every finite place $v$ of $k$, there is a unique $\mathbb{Q}(\pi_v)$-form on $\Pi_v$ such that the natural pairing

$$\langle \cdot, \cdot \rangle_v : \Pi_v = \pi_v \boxtimes \pi_v^{\vee} \to \mathbb{C}$$

is defined over $\mathbb{Q}(\pi_v)$.

**Proof.** We sketch the proof, which is similar to that of [JST19 Proposition 3.6]. We have the $G_v$-equivariant embedding

$$\Pi_v \to C^\infty(G_{0,v}), \quad \varphi \otimes \varphi^\vee \mapsto (f_{\varphi,\varphi^\vee}(g) := \langle \pi_v(g)\varphi, \varphi^\vee \rangle_v, \quad g \in G_{0,v}),$$

where $C^\infty(G_{0,v})$ denotes the space of locally constant complex functions on $G_{0,v}$, on which $G_v = G_{0,v} \times G_{0,v}$ acts by translations. Let $\text{Aut}(\mathbb{C})$ act on $C^\infty(G_{0,v})$ by $(\sigma.f)(g) := \sigma(f(g))$, where $\sigma \in \text{Aut}(\mathbb{C})$, $f \in C^\infty(G_{0,v})$. Since $\text{Hom}_{G_v}(\Pi_v, C^\infty(G_{0,v}))$ is one-dimensional, $\text{Aut}(\mathbb{C}/\mathbb{Q}(\pi_v))$ stabilizes the image of $\Pi_v$ in $C^\infty(G_{0,v})$, which induces an action of $\text{Aut}(\mathbb{C}/\mathbb{Q}(\pi_v))$ on $\Pi_v$.

By [GS17] Thm. A.2.4], $\Pi_v$ has a rational form $\Pi_v^\circ$ defined over a finite extension of $\mathbb{Q}(\pi_v)$ such that $\langle \cdot, \cdot \rangle_{\Pi_v^\circ}$ is $\overline{\mathbb{Q}}$-valued. By choosing $\varphi \otimes \varphi^\vee \in \Pi_v^\circ$ such that $\langle \varphi, \varphi^\vee \rangle_v \neq 0$ and using the above multiplicity one result, it is easy to show that some open subgroups of $\text{Aut}(\mathbb{C}/\mathbb{Q}(\pi_v))$ fix $\Pi_v^\circ$ pointwise, where $\text{Aut}(\mathbb{C}/\mathbb{Q}(\pi_v))$ is equipped with the coarsest topology such that for each open $U$, $\text{Aut}(\mathbb{C}/\mathbb{Q}(\pi_v)) \to \text{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}(\pi_v))$ is continuous. It follows that $\Pi_v^{\text{Aut}(\mathbb{C}/\overline{\mathbb{Q}})} = \Pi_v^\circ \otimes \overline{\mathbb{Q}}$.

Hence by [Sp09] Proposition 11.1.6],

$$\Pi_v^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\pi_v))} = \left(\Pi_v^{\text{Aut}(\mathbb{C}/\overline{\mathbb{Q}})}\right)^{\text{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}(\pi_v))},$$

is a $\mathbb{Q}(\pi_v)$-form of $\Pi_v$, which is the unique $\mathbb{Q}(\pi_v)$-form satisfying the lemma. 

For $v \nmid \infty$, define the $\mathbb{Q}(\pi_v)$-form of $\Pi_v$ as in Lemma 9.3 which induces an $E$-form of $\Pi_v$ to be denoted by $\Pi_v(E)$. Recall the normalized zeta integral $\mathcal{P}_v^\circ$.

**Lemma 9.4.** For every finite place $v$ of $k$, the linear functional $\mathcal{P}_v^\circ(\chi_v, \cdot)$ is defined over $\mathbb{Q}(\Pi_v, \chi_v)$.

**Proof.** By the definitions, this follows easily from the $\text{Aut}(\mathbb{C})$-equivariance of the local L-factors in $\mathcal{L}(\frac{1}{2}, \pi_v \times \chi_v)$, which is proved in [Liu23] Lemma 2.5].
We need the following result for the split case. Assume that \( v \) is a place of \( k \) that splits in \( \mathbb{k}' \) and fix an isomorphism \( \mathbb{k}_v \otimes k' \cong \mathbb{k}_v \times k_v \). Via the first factor \( k_v \) we have an identification \( \mathbb{G}_{0,v} = \text{GL}_n(k_v) \times \text{GL}_{n-1}(k_v) \) (with respect to certain bases) and a diagonal embedding of \( \text{GL}_{n-1}(k_v) \) into \( \mathbb{G}_{0,v} \).

Similar to Section \[ \mathbb{S}, \text{1} \] let \( U_v \) be the upper triangular maximal unipotent subgroup of \( \mathbb{G}_{0,v} \), and let \( \psi_{U_v} \) be the generic character of \( U_v \) defined using the character \( \psi_v \) of \( k_v \) which is trivial on \( \hat{U}_v := U_v \cap \text{GL}_{n-1}(k_v) \). By the temperedness assumption, \( \pi_v \) is generic and we fix the Whittaker models

\[
\pi_v \subset \text{Ind}_{U_v}^{\mathbb{G}_{0,v}} \psi_{U_v}, \quad \pi_v' \subset \text{Ind}_{U_v}^{\mathbb{G}_{0,v}} \psi_{U_v}^{-1}
\]

such that the pairing \( \langle \cdot, \cdot \rangle_v \) on \( \pi_v \boxtimes \pi_v' \) is given by

\[
\langle \varphi, \varphi' \rangle_v = \int_{U_v \backslash Q_v} \varphi(g)\varphi'(g) \, dg.
\]

Here \( Q_v \) is a mirabolic subgroup of \( \mathbb{G}_{0,v} \) containing \( U_v \), \( dg \) is a fixed Borel measure on \( U_v \backslash Q_v \) that is rational if \( v \) is finite. Note that the integral \[ \mathbb{9}, \mathbb{1} \] converges absolutely.

Fix the similar Haar measure on \( \hat{U}_v \) as in Section \[ \mathbb{S}, \text{3} \] and for \( \tau \in M(\text{GL}_{n-1}(k_v)) \) denote by \( \tau \in M(U_v \backslash \text{GL}_{n-1}(k_v)) \) the quotient of \( \tau \) by the fixed measure on \( U_v \). The following is \[ \mathbb{Z}, \mathbb{H}, \mathbb{1} \] Proposition 4.10] (stated as \[ \mathbb{L}, \mathbb{I}, \mathbb{2} \] Lemma 2.10])

**Lemma 9.5.** Let \( v \) be a place of \( k \) that splits in \( \mathbb{k}' \), \( \chi_v' \in \mathbb{X}_v \) a unitary character, and \( \tau \in M(\text{GL}_{n-1}(k_v)) \). Then there is a positive constant \( c_\tau \) only depending on \( \tau \) such that

\[
\int_{\text{GL}_{n-1}(k_v)} \chi_v'(\det g) \langle \pi_v(g)\varphi, \varphi' \rangle_v \, d\tau(g)
\]

\[
= c_\tau \int_{\hat{U}_v \backslash \text{GL}_{n-1}(k_v)} \chi_v'(\det g) \varphi(g) \, d\hat{\tau}(g) \int_{\hat{U}_v \backslash \text{GL}_{n-1}(k_v)} \chi_v'^{-1}(\det g) \varphi'(g) \, d\hat{\tau}(g)
\]

for all \( \varphi \otimes \varphi' \in \Pi_v \), where both sides converge absolutely. Moreover if \( v \) is finite and \( \tau_v \in D(\text{GL}_{n-1}(k_v)) \), then \( c_\tau \) is rational.

**Proposition 9.6.** There is a family

\[
\{ \phi_v^\circ \in \Pi_v \otimes D(\hat{\mathbb{H}}_v) \}_{v \in \mathbb{X}}
\]

such that

- for all \( v \not\in \mathbb{X} \), \( \phi_v^\circ \in \Pi_v(\mathbb{E}) \otimes D(\hat{\mathbb{H}}_v) \) and \( \mathbb{P}_v(\cdot, \phi_v^\circ) = 1 \) on \( \mathbb{X}_v \); 
- for all but finitely many \( v \not\in \mathbb{X} \), \( \phi_v^\circ \) is the fixed spherical vector used in the restricted tensor product \( \Pi \otimes M(\hat{\mathbb{H}}(\mathbb{A}_k)) \text{GL}(\mathbb{A}_k)) = \hat{\otimes}_v (\Pi_v \otimes M(\hat{\mathbb{H}}_v)) \).

**Proof.** This follows from \[ \mathbb{H}, \mathbb{A}, \mathbb{1} \] Theorem 2.12], \[ \mathbb{L}, \mathbb{I}, \mathbb{3} \] Proposition 2.11] and Lemmas 9.4 and 9.5. \( \square \)
9.3. **Archimedean modular symbols.** Take $K_\infty$ to be a maximal compact subgroup of $G(\mathbb{R})$ such that the corresponding Cartan involution preserves $G(\mathbb{R})$. Then $\hat{K}_\infty := K_\infty \cap \hat{G}(\mathbb{R})$ is a maximal compact subgroup of $\hat{G}(\mathbb{R})$. There is a geometrically irreducible algebraic representation $V_{\pi} \boxtimes V'_{\pi}$ of $G_E$ such that the total relative Lie algebra cohomology

$$H^\bullet(g_C, K^\circ_\infty; (V_{\pi} \boxtimes V'_{\pi}) \otimes \Pi_\infty) \neq \{0\},$$

where $V_{\pi} = F_{\mu}^\pi \boxtimes F_{\nu}^\pi$, and $(\mu, \nu) \in (\mathbb{Z}^n)^{\mathbb{Z}_+} \times (\mathbb{Z}^{n-1})^{\mathbb{Z}_+}$ is as in Section 2.4. Suppose that $V = V_{\pi} \boxtimes V'_{\pi}$ (which is also viewed as a representation of $G(\mathbb{Q})$), so that $V = E \otimes (V_{\pi} \boxtimes V'_{\pi})$ as a representation of $G = G(\mathbb{Q}_p) \subset G(E)$.

**Definition 9.7.** (a) A character $\chi_\infty$ of $U_1(1_\infty)$ is said to be critical for $\pi$ if it is algebraic and $s = \frac{1}{2}$ is a pole of neither $L(s, \pi_\infty \times \chi_\infty)$ nor $L(1 - s, \pi'_{\infty} \times \chi_\infty^{-1})$. (b) An automorphic character $\chi$ of $U_1(1_\infty) \backslash U_1(\mathbb{A}_k)$ is said to be critical for $\pi$ if so is its archimedean component $\chi_\infty$.

**Lemma 9.8.** If there is a $V$-balanced algebraic character on $Z_E$, then an algebraic character $\chi_\infty$ of $U_1(1_\infty)$ is $V$-balanced if and only if it is critical for $\pi$.

**Proof.** By [Mok15, KMSW14, Ram18] (cf. [BPCZ22, Remark 1.1.4.1]), a weak base-change is automatically a strong base-change, hence $\tilde{\pi}$ is also regular algebraic. It follows easily from the classical branching rule that $\chi$ is $V$-balanced if and only if $\tilde{\chi}$ is $V_{\tilde{\pi}}$-balanced in the sense of Section 3, where $V_{\tilde{\pi}}$ is the rational form of the coefficient system of $\tilde{\pi}$. The assertion follows from this and the fact that $\chi$ is critical for $\pi$ if and only if $\tilde{\chi}$ is critical for $\tilde{\pi}$. \hfill $\square$

Recall the $G^2$-homomorphism (2.8) whose image is defined over $E$ (see [GL21, 1.4.2]), where $\Phi$ is the family of all compact subsets of $G(\mathbb{Q}) \setminus \mathbb{A}$. The bottom degree component of $H^\bullet(g_C, K^\circ_\infty; V \otimes \Pi_\infty)$ is

$$\Pi'_\infty = H^{i_0}(g_C, K^\circ_\infty; V \otimes \Pi_\infty),$$

where $i_0 = \dim(G(\mathbb{R})/K^\circ_\infty)$.

Similarly let $\pi'_{\infty}$ and $\pi'_{\infty}'$ be the bottom degree relative Lie algebra cohomologies of $V_{\pi} \otimes \pi_\infty$ and $V'_{\pi} \otimes \pi'_\infty$, respectively, so that

$$\Pi'_\infty = \pi'_{\infty} \boxtimes \pi'_{\infty}' \cong \bigoplus_{\varepsilon_\infty \otimes \varepsilon'_\infty \in U_1(k_\infty)^2} \varepsilon_\infty \otimes \varepsilon'_\infty,$$

as representations of $\hat{K}^2_\infty = U_1(k_\infty)^2 \times U_1(k_\infty)^2$. In view of Lemma 9.3, we have the $E$-form $\Pi'_\infty(E)$ as in (2.9).

Recall that $P$ is a Borel subgroup of $G$ defined over $\mathbb{Q}_p \cap E$ that is transversal to $G$. Then $N$ is the unipotent radical of $P$. Assume that $\lambda_0 \in \text{Hom}_{\mathbb{P}^1}(V^n, E)$ is a generator defined over $E$. For every $V$-balanced algebraic character $\nu$ of $Z_E$, let $\lambda_V \in \text{Hom}_{\hat{G}(\mathbb{Q})}(E^n \otimes V, E)$ be as in Lemma 2.9. Then for all algebraic characters $\chi_\infty$ of $U_1(1_\infty)$ of weight $\nu^{-1}$, we have the archimedean modular symbol map

$$\hat{P}_{\infty} : H^0(\mathbb{G}_m, K^\circ_\infty; \mathbb{C}_{w_\infty} \otimes \chi_\infty) \times \left(\Pi'_\infty \otimes M(\hat{H}(\mathbb{R}))^V \otimes O(\hat{G}(\mathbb{R})/\hat{K}^\circ_\infty)\right) \to \mathbb{C}$$
defined as in (2.11). The commutative diagram (2.13) is then a consequence of the Ichino-Ikeda conjecture (Theorem 9.2).

Let \( \Pi_{0, \infty} = \pi_{0, \infty} \boxtimes \pi_{0, \infty} \) be the irreducible tempered Casselman-Wallach representation of \( G(\mathbb{R}) \) whose infinitesimal character equals that of the trivial representation and whose central character equals that of \( V \otimes \Pi_{\infty} \). Define the cohomology group \( \Pi'_{0, \infty} \) as in (2.9), so that \( \Pi'_{0, \infty} = \pi'_{0, \infty} \boxtimes \pi'_{0, \infty} \) as in (9.3). Following [LLS24], with the fixed Whittaker functionals we have the translation map

\[
\mathcal{N} = (J_\mu \otimes J_\nu) \otimes (J_\mu^\vee \otimes J_\nu^\vee) : \Pi_{0, \infty} \to \Pi'_{\infty},
\]

where \( \mu^\vee \) and \( \nu^\vee \) denote the highest weights of \( F_\mu^\vee \) and \( F_\nu^\vee \) respectively. As a specialization of (2.11), we have a map

\[
\hat{\mathcal{P}} : H^0(\mathfrak{g}_\mathbb{C}, K^2_{Z, \infty}; \varepsilon_\infty) \times \left( \Pi'_{0, \infty} \otimes M(\mathcal{H}(\mathbb{R}))^{\nu} \otimes O(\hat{G}(\mathbb{R})/K^\circ_{\infty}) \right) \to \mathbb{C}
\]

for every \( \varepsilon_\infty \in U_1(k_{\infty})^2 \).

Define \( \hat{\mathcal{P}}_\infty \) in (10) to be the map such that for every \( \varepsilon_\infty \in U_1(k_{\infty})^2 \), \( \hat{\mathcal{P}}_\infty(\varepsilon_\infty, \cdot) \) equals the composition of

\[
\Pi'_{0, \infty} \otimes M(\mathcal{H}(\mathbb{R}))^{\nu} \otimes O(\hat{G}(\mathbb{R})/K^\circ_{\infty}) \quad \xrightarrow{(1, j_{\hat{k}^{-1}} \otimes \text{id} \otimes \text{id})} \quad H^0(\mathfrak{g}_\mathbb{C}, K^2_{Z, \infty}; \varepsilon_\infty) \times \left( \Pi'_{0, \infty} \otimes M(\mathcal{H}(\mathbb{R}))^{\nu} \otimes O(\hat{G}(\mathbb{R})/K^\circ_{\infty}) \right) \quad \xrightarrow{\hat{\mathcal{P}}_\infty} \quad \mathbb{C}.
\]

Recall the modifying factor \( \Upsilon_{\Pi_{\infty}} \) at \( \infty \) in (2.28). We have the following archimedean nonvanishing hypothesis and period relations.

**Theorem 9.9.** If \( w \) is \( V \)-balanced, then the followings hold.

(a) The map \( \hat{\mathcal{P}}_\infty^\mu \) is nonzero.

(b) \( \hat{\mathcal{P}}_\infty^\mu(1, \cdot) = \Upsilon_{\Pi_{\infty}} \cdot \hat{\mathcal{P}}_\infty^\nu(w_{\infty} \chi_{\infty}, \cdot) \).

**Proof.** Define \( \Upsilon_{\pi_{\infty}^\mu}(\chi_{\infty}) \) and \( \Upsilon_{\pi_{\infty}^\nu}(\chi_{\infty}^{-1}) \) as in (2.22). By Theorem 8.3 and Lemma 9.5, it suffices to show that

\[ \Upsilon_{\pi_{\infty}^\mu}(\chi_{\infty}) \cdot \Upsilon_{\pi_{\infty}^\nu}(\chi_{\infty}^{-1}) = \Upsilon_{\Pi_{\infty}}. \]

It is easy to verify that

\[
\Upsilon_{\pi_{\infty}^\mu}(\chi_{\infty}) \cdot \Upsilon_{\pi_{\infty}^\nu}(\chi_{\infty}^{-1}) = \sum_{i \in \mathcal{E}_k} \sum_{n=-1}^{n-1} (n-i)(\mu_i^\nu - \mu_i^{n+1-i} + \nu_i^\nu - \nu_i^{n-i}) \cdot (-1) \sum_{j \in \mathcal{E}_k} (-1)^{\sum_{i>j, i+j \leq n}(\mu_i^\nu - \mu_i^{n+1-i} + \nu_i^\nu - \nu_i^{n-j})},
\]

where for every \( i \in \mathcal{E}_k, \bar{i} \) denotes the composition of

\[
k \xrightarrow{i} \overline{\mathbb{Q}} \xrightarrow{\text{complex conjugation}} \overline{\mathbb{Q}}.
\]

Note that \( \mu \) and \( \nu \) are pure of weight 0 in the sense of [Cl90], namely \( \mu_i^\nu + \mu_i^{n+1-i} = 0, \ i = 1, 2, \ldots, n \) and \( \nu_j^\nu + \nu_j^{n-j} = 0, \ j = 1, 2, \ldots, n-1 \) for all \( i \in \mathcal{E}_k \). Then (9.4) follows immediately. \( \square \)
Following Definition 2.1, fix an element
\[ \hat{\phi}_\infty^0 \in \Pi'_\infty(E) \otimes D(H(R)) \odot O(G(R)/K^\infty_\infty) \]
such that \( \hat{\phi}_\infty^0(\varepsilon_\infty, \hat{\phi}_\infty^0) \neq 0 \) for all \( \varepsilon_\infty \in \widehat{k}_\infty^{\infty} \). Define the Bessel periods
\[ \Omega_{\Pi}(\varepsilon_\infty) := \left( \hat{\phi}_\infty^0(\varepsilon_\infty, \hat{\phi}_\infty^0) \right)^{-1}, \quad \varepsilon_\infty \in \widehat{U}_1(k_\infty) \]

9.4. Open orbit integrals and normalized refined period. Assume that \( \Pi'_p \subset \mathcal{B}_P(\Pi_p) \) is a nearly ordinary refinement of \( \Pi_p \) defined over \( E \), which is also viewed a character of \( P \) that descends to a character of \( L = P/N \). Without loss of generality assume that \( P = z'^{-1}Bz' \), where \( z' := (z, z) \) with \( z \) in (8.8) and \( B \) is the Borel subgroup of lower triangular matrices in \( G \). Put
\[ \kappa = \Pi'_p \odot \text{Ad}(z'^{-1}) : B \to E \times \hat{\mathbb{C}}, \]
so that \( \Pi_p = \pi_p \boxtimes \pi^\vee_p \) is isomorphic to a quotient representation of
\[ I(\tilde{\kappa}) := \text{Ind}_{\mathbb{B}}^G(\tilde{\kappa}), \quad \text{where} \quad \tilde{\kappa} := \kappa \otimes \delta^{1/2}_B. \]
Recall that by MVW involution, \( \kappa \) is of the form (2.30).

Similar to (8.13), we have the open orbit integral map
\[ \Lambda_p : \mathcal{X}_p \times \left( I(\tilde{\kappa}) \otimes \mathcal{M}(\hat{G}) \right) \to \mathbb{C} \cup \{\infty\}, \]
\[ (\chi'_p, f \otimes \tau') \mapsto \int_G \chi'_p(g) f(z'g) d\tau'(g), \]
defined by meromorphic continuation of absolutely convergent integrals. It naturally extends to a map
\[ \Lambda_p : \mathcal{X}_p \times \left( I(\tilde{\kappa})_{p_{\text{sm}}^\infty} \otimes \mathcal{M}(\hat{G}) \right) \to \mathbb{C} \cup \{\infty\}. \]

For all \( \varphi \mid p \), let \( \lambda'_p \in \text{Hom}_{U_p \times \hat{U}_p}(I(\tilde{\kappa}_p), \psi_{\hat{U}_p} \boxtimes \psi^{-1}_{\hat{U}_p}) \) be given by the Jacquet integrals as in (8.11). Fix the surjective \( G \)-homomorphism
\[ \xi_p : I(\tilde{\kappa}) \to \Pi_p, \quad f \mapsto \langle \otimes_{\varphi \mid p} \lambda'_p, g.f \rangle, \quad g \in G, \]
which naturally extends to a surjective \( G \)-homomorphism
\[ \xi_p : I(\tilde{\kappa})_{z'} \to \Pi'_p. \]
The subspace \( I(\tilde{\kappa})_{z'} \) of the generalized functions supported in \( \mathcal{B}z' \) is one-dimensional and the map (9.7) restricts to a \( P \)-isomorphism
\[ \xi_p : I(\tilde{\kappa})_{z'} \to \Pi'_p. \]

Moreover, for every generator \( \hat{f} \otimes \tau' \) of \( I(\tilde{\kappa})_{z'} \otimes \mathcal{M}(\hat{G}) \), \( \Lambda_p(\cdot, \hat{f} \otimes \tau') \) is a constant function on \( \mathcal{X}_p \) with values in \( \mathbb{C} \times \).
Define the normalized refined period map to be the composition
\[
\hat{\mathcal{P}}_p^\circ : \mathcal{X}_p \times \left( \Pi'_p \otimes M(\hat{H}\setminus \hat{G}) \right) \xrightarrow{\xi_p^{-1}} \mathcal{X}_p \times \left( \hat{I}(\overline{\kappa})_\ell' \otimes M(\hat{H}\setminus \hat{G}) \right) \xrightarrow{\tau \mapsto \sigma_\tau \cdot \sigma_\tau} \mathcal{X}_p \times \left( \hat{I}(\overline{\kappa})_\ell' \otimes M(\hat{G}) \right) \xrightarrow{\Lambda_p} \mathbb{C},
\]
where in the second arrow we use the identifications
\[
M(\hat{H}\setminus \hat{G}) = M(\text{GL}_{n-1}(k_p)) \quad \text{and} \quad M(\hat{G}) = M(\text{GL}_{n-1}(k_p)) \otimes M(\text{GL}_{n-1}(k_p)),
\]
and \(c_\tau\) is given by Lemma 9.5. More precisely, if \(\tau = \otimes_{p \mid \mathfrak{p}} \tau_p\) where \(\tau_p \in M(\hat{H}_p \setminus \hat{G}_p)\), then \(c_\tau = \prod_{p \mid \mathfrak{p}} c_{\tau_p}\) as in Lemma 9.5.

### 9.5. Rational test vectors and modifying factors at \(p\).

In what follows we define a rational test vector \(\hat{\mathcal{P}}_p^\circ \in \Pi'_p(E) \otimes D(\hat{H}\setminus \hat{G})\).

**Lemma 9.10.** For every finite place \(v\) of \(k\) that splits in \(k'\), the pairing (9.1) on \(\pi_v \boxtimes \pi_v^\vee\) is rational with respect to the rational forms of the Whittaker models \(\pi_v \subset \text{Ind}_{U_v}^{G_{0,v}} \psi_{U_v}\) and \(\pi_v^\vee \subset \text{Ind}_{U_v}^{G_{0,v}} \psi_{U_v}^{-1}\) given by (8.2).

**Proof.** Take \(\varphi_v \in \pi_v\), \(\varphi_v^\vee \in \pi_v^\vee\) and \(\sigma \in \text{Aut}(\mathbb{C})\). Recall the element \(t_{\sigma,\ell} \in G_0(k_v)\) as in (8.4), where \(\ell\) is the residue characteristic of \(k_v\). In fact \(t_{\sigma,\ell} \in Q_v\). From (8.2) we find that
\[
\langle \sigma \varphi_v, \sigma \varphi_v^\vee \rangle_v = \int_{U_v \setminus Q_v} \sigma(\varphi_v(t_{\sigma,\ell} \cdot g_v)\varphi_v^\vee(t_{\sigma,\ell} \cdot g_v)) \, dg_v = \sigma((\varphi_v, \varphi_v^\vee)_v),
\]
which finishes the proof. \(\square\)

For every \(\varphi \mid p\), define an action of \(\text{Aut}(\mathbb{C}/\mathbb{Q}(\kappa_p))\) on \(I(\overline{\kappa}_p)\) by
\[
\sigma f(g) := \kappa_p(t_{\sigma,p}, t_{\sigma,p}) \cdot \sigma(f(g)), \quad \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\kappa_p)), \quad f \in I(\overline{\kappa}_p), \quad g \in G_p.
\]

Then by (8.4) we have that
\[
I(\overline{\kappa}_p)^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\kappa_p))} = \{ f \in I(\overline{\kappa}_p) : \omega_{\psi_{\kappa_p}}(\kappa_p) \cdot f \text{ is } \mathbb{Q}(\kappa_p)-\text{valued} \},
\]
which is a \(\mathbb{Q}(\pi_p)\)-form of \(I(\overline{\kappa}_p)\), where \(\omega_{\psi_{\kappa_p}}(\kappa_p)\) is in (2.32). By tensor products, we have a \(\mathbb{Q}(\pi_p)\)-form of \(\Pi_p\) as well as a \(\mathbb{Q}(\kappa)\)-form of \(I(\overline{\kappa})\).

**Proposition 9.11.** One has that \(\mathbb{Q}(\pi_p) \subset \mathbb{Q}(\kappa)\) and the map \(\xi_p\) in (9.6) is \(\mathbb{Q}(\kappa)\)-rational.

**Proof.** This follows from Lemma 9.10 and the proof of Proposition 8.6. \(\square\)
Let \( \hat{\phi}_p \) denote the generator of \( \Pi'_p \otimes M(\hat{H} \setminus \hat{G}) \) such that \( \hat{P}^o_p(\cdot, \hat{\phi}_p) \) equals the constant function 1 on \( \mathcal{X}_p \). Write \( \hat{\phi}_p = \phi'_p \otimes \tau \) such that \( \phi'_p \in \Pi'_p \) and \( \tau \in D(\hat{H} \setminus \hat{G}) \). Proposition 9.11 implies that

\[
\Omega^{-1}_{\Pi'_p} \cdot \phi'_p = \prod_{\psi \mid p} \omega_{\psi_p}(\kappa_p)^{-1} \cdot \phi'_p \in \Pi'_p(E),
\]

where \( \Omega_{\Pi'_p} \) is in (2.32). Define

\[
\hat{\phi}^o_p := \Omega_{\Pi'_p} \cdot \hat{\phi}_p = \Omega_{\Pi'_p}^{-1} \cdot \phi'_p \otimes \tau \in \Pi'_p(E) \otimes D(\hat{H} \setminus \hat{G}).
\]

Then

\[
\hat{P}^o_p(\chi'_p, \hat{\phi}^o_p) = \Omega_{\Pi'_p}^{-1} \text{ for all } \chi'_p \in \mathcal{X}_p.
\]

Recall the modifying factor \( \Upsilon_{\Pi'_p}(\chi_p) \) at \( p \) in (2.31).

**Proposition 9.12.** It holds that

\[
\hat{P}^o_p(\chi'_p, \hat{\phi}^o_p) = \Upsilon_{\Pi'_p}(\chi'_p) \cdot \Omega_{\Pi'_p}^{-1} \text{ for all } \chi'_p \in \mathcal{X}_p.
\]

Consequently \( \Upsilon_{\Pi'_p} \) is algebraic on \( \mathcal{X}_p \).

**Proof.** By Theorem 8.5 and Lemma 9.5, it is easy to see that the proposition follows from the equality

\[
\prod_{i>j, i+j \leq n} (\kappa_i \kappa'_j \chi_p)(-1) \cdot \prod_{i>j, i+j \leq n} (\kappa'_{n+1-j} \kappa_{n-j} \chi_p)^{-1}(-1) = \omega_n(-1)^n.
\]

By elementary calculations, the left hand side equals \( \prod_{j=1}^{n-1} \kappa'_j(-1)^n \), which is clearly \( \omega_n(-1)^n \). \( \square \)

9.6. \( p \)-adic \( L \)-functions and exceptional zeros. Now we define an \( E \)-valued \( p \)-adic measure \( \mathcal{L}_\Pi \) on \( \mathcal{C}_\mathcal{E}(p^\infty) \). Suppose that \( Z_0 \) and \( \varepsilon = \otimes_{\psi \mid \mathfrak{p}^\infty} \varepsilon_{\psi} \) in (1.3) are both trivial.

- For \( v \mid \infty \), let \( \phi^o_v \in \Pi_v(E) \otimes D(\hat{H}_v \setminus \hat{G}_v) \) be as in Proposition 9.6 such that

\[
\mathcal{P}^o_v(\cdot, \phi^o_v) = 1 \text{ on } \mathcal{X}^{un}_v.
\]

- Let \( \lambda_0 = \text{Hom}_E(V^n, E) \) and \( \hat{\phi}^o_{\infty} \in \Pi'_{\infty}(E) \otimes D(\hat{H}(\mathbb{R}))^\vee \otimes \mathcal{O}(\mathbb{G}(\mathbb{R})/\hat{K}^\infty) \) be as in Section 9.3 such that

\[
\hat{P}^{\omega_{\infty}}(1, \hat{\phi}^o_{\infty}) = \Upsilon_{\Pi_{\infty}} \cdot \hat{P}^{\omega_{\infty}}(\omega_{\infty}, \hat{\phi}^o_{\infty}) = \frac{\mathcal{Y}_{\Omega_{\Pi_{\infty}}(\omega_{\infty}, \chi_{\infty})}}{\Omega_{\Pi_{\infty}}(\omega_{\infty}, \chi_{\infty})}
\]

for all \( V \)-balanced characters \( \omega \) and all algebraic characters \( \chi_{\infty} \) of weight \( \omega^{-1} \), where \( \lambda_V \in \text{Hom}_{\mathbb{C}_p}(E_{\omega} \otimes V, E) \) is the generator such that \( \lambda_V|_{V^*} = \lambda_0 \).

- Let \( \hat{\phi}^o_p \in \Pi'_p(E) \otimes D(\hat{H} \setminus \hat{G}) \) be as in Proposition 9.12.
Using all the local test vectors, let
\[ \tilde{\phi} := \hat{\phi} \otimes \hat{\phi}_p \otimes (\hat{\phi}_v \otimes \phi_v) \in \mathcal{H} \otimes D(\mathcal{G}, \hat{p}), \]
and following Definition 2.5 let
\[ \mathcal{L}_\Pi := \mathcal{L}_{\otimes S} = (\mathcal{L}_{\lambda_0 \otimes \hat{p}}) |_{C(\mathcal{Z}, (a), \mathcal{E})}. \]

We restate Theorem 2.12 below, which is now an immediate consequence of the above results and (2.19).

**Theorem 9.13.** If there is a \( V \)-balanced character, then \( \mathcal{L}_\Pi \) is the unique \( \mathcal{E} \)-valued \( p \)-adic measure on \( \mathcal{C}_{\mathcal{K}}(p^\infty) \) such that
\[ \mathcal{L}_\Pi(\chi^\flat) = \Upsilon_{\Pi} \cdot \Upsilon_{\Pi}'(\chi_p) \cdot \mathcal{L}_\Pi(\chi) \]
for all critical algebraic automorphic characters \( \chi = \otimes \chi_\ell : U_1(k) \setminus U_1(A_k) \to \mathbb{C}^\times \) unramified outside \( \infty p \), where \( \mathcal{W} \) is the inverse of the weight of \( \chi \).

Similar to Proposition 8.9, we can also determine the exceptional zeros of \( \mathcal{L}_\Pi \).

**Proposition 9.14.** Under the assumptions in Theorem 9.13, \( \chi^\flat \) is an exceptional zero of \( \mathcal{L}_\Pi \) if and only if there exists a place \( \wp \mid p \) and \( 1 \leq i < n \) such that
\[ \chi_\wp = \kappa^{-1}_{i,\wp} \cdot \kappa'_{n-i,\wp} \cdot | \cdot |_{\wp} \quad \text{or} \quad \kappa^{-1}_{i+1,\wp} \cdot \kappa'_{n-i,\wp} \cdot | \cdot |_{\wp}^{-1}, \]
where \( \kappa_{i,\wp}'s \) and \( \kappa'_{j,\wp}'s \) are as in (2.31).

**Proof.** Recall that \( \mathcal{L}(1/2, \pi_p \times \chi_p) \in \mathbb{C}^\times \) because \( \pi_p \) is tempered. Thus all the exceptional zeroes arise from the local \( \gamma \)-factors in (2.31). The proposition follows easily in view of Lemma 8.13. \( \square \)

10. **Application III: Standard L-functions of symplectic type for \( GL_{2n} \)**

In this section we retain the setup in Section 2.6 and construct the \( p \)-adic L-function \( \mathcal{L}_\Pi \) in Theorem 2.14 following Sections 2.1–2.3. Then we determine the exceptional zeros of \( \mathcal{L}_\Pi \).

10.1. **Friedberg-Jacquet integrals.** Let \( \pi \) be an irreducible cuspidal automorphic representation of \( GL_{2n}(A_k) \), and let \( \eta : k^\times \setminus A_k^\times \to \mathbb{C}^\times \) be a Hecke character. Combining the works [JS88, AS06, HS09], it is known that the following statements are equivalent:

(a) the twisted exterior square L-function \( L(s, \pi, \lambda^2 \otimes \eta^{-1}) \) has a pole at \( s = 1 \);
(b) \( \pi \) is of symplectic type, namely it is the functorial transfer of an irreducible generic cuspidal automorphic representation of \( GSpin_{2n+1}(A_k) \) with central character \( \eta \);
(c) \( \pi \) has a nonzero \( (\eta, \psi) \)-Shalika integral.
If these equivalent statements hold, then $\pi^\vee \cong \pi \otimes \eta^{-1}$.

Let us explain the statement (c). The Shalika subgroup of $\text{GL}_{2n}$ is defined by

$$S := \left\{ \begin{bmatrix} g & gx \\ 0 & g \end{bmatrix} \middle| g \in \text{GL}_n, x \in M_n \right\},$$

where $M_n$ denotes the space of $n \times n$ matrices. Recall the nontrivial additive character $\psi : k \setminus A_k \to \mathbb{C}^\times$ in (2.21). Let $\psi_S$ be the character

$$\psi_S : S(k) \setminus S(A_k) \to \mathbb{C}^\times, \quad \begin{bmatrix} g & gx \\ 0 & g \end{bmatrix} \mapsto \eta(\det g) \psi(\text{tr } x).$$

Then we say that $\pi$ has a nonzero $(\eta, \psi)$-Shalika integral if $\eta^n$ equals the central character of $\pi$ and there exists $\varphi \in \pi$ such that

$$\langle \lambda_S, \varphi \rangle := \int_{(S(k)A_k) \setminus (S(A_k))} \varphi(h) \cdot \psi_S^{-1}(h) \, dh \neq 0,$$

in which case $\lambda_S \in \text{Hom}_{S(A_k)}(\pi, \psi_S)$ is a nonzero $\psi_S$-Shalika functional on $\pi$.

Assume that the above equivalent statements hold. Recall the algebraic group $G = (G_{2n} \times G_1)/G_1'$ and the automorphic representation $\Pi = \pi \otimes \eta^{-1}$ of $G(A)$. We reformulate the Shalika functional as follows in order to fit the general theory in Section 2.1. Let $A_S \cong \text{GL}_1$ be the center of $S$ and $H := S/A_S$, where $S := \text{Res}_{k/Q} S$ and $A_S := \text{Res}_{k/Q} A_S$. We identify $H$ as a subgroup of $G$ via

$$H \to G, \quad \begin{bmatrix} g & gx \\ 0 & g \end{bmatrix} A_S \mapsto \left( \begin{bmatrix} g & gx \\ 0 & g \end{bmatrix}, \det g \right) G_1'.$$

Define the character

$$\psi_H = \otimes_v \psi_{H_v} : H(Q) \setminus H(A) \to \mathbb{C}^\times, \quad \begin{bmatrix} g & gx \\ 0 & g \end{bmatrix} A_S \mapsto \psi(\text{tr } x),$$

where $v$ runs over places of $k$, $H_v := H(k_v)$, and $H := S/A_S$. Similar notations such as $G_v$ and $G_1'$ will be used without explanation. Then we have the functional $\lambda_H \in \text{Hom}_{H(A)}(\Pi, \psi_H)$ as in Section 2.1. By the uniqueness of twisted Shalika models [CS20, Theorem A] we have a factorization

$$\lambda_H = \otimes_v \lambda_{H_v}, \quad \text{where } \lambda_{H_v} \in \text{Hom}_{H_v}(\Pi_v, \psi_{H_v}).$$

Let $\chi = \otimes_v \chi_v : k^\times \setminus A_k^\times \to \mathbb{C}^\times$ be a Hecke character. Following [FJ93] we have the global Friedberg-Jacquet integral

$$\mathcal{P}_\chi : \Pi \otimes M(\hat{G}(Q) \setminus \hat{G}(A)) \to \chi^{-1}, \quad f \otimes \tau \mapsto \int_{\hat{G}(Q) \setminus \hat{G}(A)} \chi(j(g)) f(g) \, d\tau(g),$$

which converges absolutely.
Let \( \mathcal{X}_v(\varepsilon_v) \subset \mathcal{X}_v \) be as in Section 8.1, and define the normalized Friedberg-Jacquet integral

\[
\mathcal{P}^\circ_v : \mathcal{X}_v \times \left( \Pi_v \otimes M(\hat{\mathcal{H}}_v \backslash \hat{\mathcal{G}}_v) \right) \to \mathbb{C},
\]

\[
(\chi'_v, f \otimes \tau) \mapsto \frac{1}{L(1/2, \pi_v \otimes \chi'_v)} \int_{\hat{\mathcal{H}}_v \backslash \hat{\mathcal{G}}_v} \chi'_v(j(g)) \langle \lambda_{H_v}, g.f \rangle \, d\tau(g)
\]

by holomorphic continuation. By [FJ93, Proposition 2.3],

\[
\mathcal{P}_\chi = L(1/2, \pi_v \otimes \chi'_v) \cdot \mathcal{P}^\circ_v(\chi_v, \cdot).
\]

In the rest of this section, assume that \( \pi \) is regular algebraic and \( \chi \) is algebraic. Let \( v \) be a finite place of \( k \) with residue characteristic \( \ell \). Recall the cyclotomic character \( \text{Aut}(\mathbb{C}) \to \mathbb{Z}_\ell^\times, \sigma \mapsto t_{\sigma, \ell} \). Put

\[
s_{\sigma, \ell} := \left( \begin{bmatrix} t_{\sigma, \ell} \cdot 1_n & 0 \\ 0 & 1_n \end{bmatrix}, 1 \right) \in G_v.
\]

Then we have an action of \( \text{Aut}(\mathbb{C}/\mathbb{Q}(\eta_v)) \) on \( \text{Ind}_{H_v}^{G_v} \psi_{H_v} \) given by

\[
s_{\sigma, \ell} \varphi(g) := \sigma(\varphi(s_{\sigma, \ell} \cdot g)), \quad \varphi \in \text{Ind}_{H_v}^{G_v} \psi_{H_v}, \ g \in G_v.
\]

This defines the \( \mathbb{Q}(\Pi_v) \)-form of \( \Pi_v \to \text{Ind}_{H_v}^{G_v} \psi_{H_v} \) (see [JST19]). The following result is given by [JST19, Theorem 3.1].

**Proposition 10.1.** For every finite place \( v \) of \( k \), the linear functional

\[
\mathcal{G}_v(\chi_v)^n \cdot \mathcal{P}^\circ_v(\chi_v, \cdot)
\]

is defined over \( \mathbb{Q}(\Pi_v, \chi_v) \).

As before suppose that \( \mathbb{Q}(\Pi) \subset E \). The \( \mathbb{Q}(\Pi_v) \)-form of \( \Pi_v \) induces an \( E \)-form of \( \Pi_v \), to be denoted by \( \Pi_v(E) \). By [FJ93] and Proposition 10.1 there is a family

\[
\{ \phi^o_v \in \Pi_v \otimes D(\hat{\mathcal{H}}_v \backslash \hat{\mathcal{G}}_v) \}_{v \mid \infty}
\]

such that

- for all \( v \mid \infty \), \( \phi^o_v \in \Pi_v(E) \otimes D(\hat{\mathcal{H}}_v \backslash \hat{\mathcal{G}}_v) \) and \( \mathcal{P}^\circ_v(\cdot, \phi^o_v) \) takes a nonzero constant value \( (\Omega_{H_v}(\varepsilon_v))^{-1} \) on \( \mathcal{X}(\varepsilon_v) \), where \( \Omega_{H_v}(\varepsilon_v) := \mathcal{G}_v(\chi_v)^n \) for an arbitrary \( \chi'_v \in \mathcal{X}_v(\varepsilon_v) \);
- for all but finitely many \( v \mid \infty \), \( \Omega_{H_v}(\varepsilon_v) = 1 \) and \( \phi^o_v \) is the fixed spherical vector used in the restricted tensor product \( \Pi \otimes M(\hat{\mathcal{H}}(A_k) \backslash \hat{\mathcal{G}}(A_k)) = \hat{\mathcal{O}}_v(\Pi_v \otimes M(\hat{\mathcal{H}}_v \backslash \hat{\mathcal{G}}_v)) \).
10.2. Archimedean modular symbols. Take \( K'_\infty = A(\mathbb{R}) \cdot K'_\infty \subset G(\mathbb{R}) \), where \( A \) is the largest central split torus in \( G \) and \( K'_\infty \) is the standard maximal compact subgroup. Take \( \hat{K}'_\infty := K'_\infty \cap \hat{G}(\mathbb{R}) \), which is the standard maximal compact subgroup of \( \hat{G}(\mathbb{R}) \). Recall that \( \mathbb{Q}(\Pi) \subset E \). Then there is a geometrically irreducible algebraic representation \( F_\mu \boxtimes w_\eta \) of \( G_E \) such that the total relative Lie algebra cohomology

\[
H^*_c(\mathfrak{g}_C, K^0_\infty; (F^\vee_\mu \boxtimes w_\eta^{-1}) \otimes \Pi_\infty) \neq \{0\},
\]

where \( w_\eta \) is the weight of \( \eta^{-1} \) and \( \mu \in (\mathbb{Z}^{2n})^{\mathfrak{c}_k} \) as in Section 2.4. Suppose that \( V = F^\vee_\mu \boxtimes w_\eta^{-1} \) (which is also viewed as a representation of \( G(\mathbb{Q}) \)), so that \( V = E \otimes (F^\vee_\mu \boxtimes w_\eta^{-1}) \) as a representation of \( G = G(\mathbb{Q}_p) \subset G_E(E) \). Recall the \( G^1 \)-homomorphism (2.8), where \( \Phi \) is the family of all compact subsets of \( G(\mathbb{Q}) \setminus \mathcal{Z}^\prime \). We have the degree \( i_0 \) component \( \Pi'_\infty \) of \( H^*_c(\mathfrak{g}_C, K^0_\infty; V \otimes \Pi_\infty) \) and its \( E \)-form \( \Pi'_\infty(E) \) in (2.19), where \( i_0 = \dim(\mathcal{G}(\mathbb{R})/\hat{K}^0_\infty) \).

**Definition 10.2.** (a) A character \( \chi_\infty \) of \( k^\times_\infty \) is said to be critical for \( \pi \) if it is algebraic and \( s = \frac{1}{2} \) is a pole of neither \( L(s, \pi_\infty \otimes \chi_\infty) \) nor \( L(1-s, \pi^\vee_\infty \otimes \chi^{-1}_\infty) \).

(b) A Hecke character \( \chi \) of \( k^\times \backslash G^\times \) is said to be critical for \( \pi \) if so is its archimedean component \( \chi_\infty \).

It is known that (see [JST19, JLST24]) if there is a \( V \)-balanced character, then \( \chi_\infty \) is \( V \)-balanced if and only if it is critical for \( \pi \). Assume that \( \chi_\infty \) is algebraic of weight \( w^{-1} \). Then it is \( V \)-balanced if and only if

\[
(10.2) \quad \mu^\vee_{i+1} \leq w_i \leq \mu^\vee_{i} \quad \text{for all} \ i \in \mathcal{E}_k.
\]

Let \( \mathcal{B} \) be the lower triangular Borel subgroup of \( G \) with unipotent radical \( \mathcal{U} \). Let

\[
(10.3) \quad \mathcal{P} := \gamma^{-1} \mathcal{B} \gamma, \quad \text{where} \quad \gamma := \left( \begin{array}{cc} 1_n & 1_n \\ 0 & w_n \end{array} \right), 1).
\]

Then \( \mathcal{P} \) and \( \hat{\mathcal{G}} \) are transversal, and we have that

\[
\hat{\mathcal{P}} := \mathcal{P} \cap \hat{\mathcal{G}} = \{ (t, t) \hat{G}_1 \mid t \in T_n \},
\]

where \( T_n \) denotes the diagonal maximal torus of \( G_n \).

Let \( \mathcal{B} \) be the upper triangular Borel subgroup of \( G \). Similar to [JST19, JLST24], by using algebraic induction from \( \mathcal{B} \) we realize \( V \) as a space of algebraic functions on \( G_E \). Let \( \nu \in V^\mathcal{P} \) be the unique algebraic function in \( V \) which equals 1 on \( \mathcal{U} \).

Suppose that \( P = \mathcal{P}(\mathbb{Q}_p) \) so that \( N \) is its unipotent radical, and that \( \lambda_0 \in \text{Hom}_E(V^n, E) \) is the unique generator such that \( \langle \lambda_0, \gamma^{-1}, \nu \rangle = 1 \). Recall that \( Z = G_1 \). For every \( V \)-balanced algebraic character \( w \) of \( Z_E \), let \( \lambda_V \in \text{Hom}_{E}(E_\mathcal{W} \otimes V, E) \) be as in Lemma 2.5. Then for all algebraic characters \( \chi_\infty \) of \( k^\times_\infty \) of weight \( w^{-1} \), we have the archimedean modular symbol map

\[
\widehat{\mathcal{P}^\nu}_\infty : H^0(\mathfrak{g}_C, K^0_\infty; \mathbb{C}_{w_\infty} \otimes \chi_\infty) \times \left( \Pi'_\infty \otimes \text{M}(\mathcal{H}(\mathbb{R}))^\vee \otimes \text{O}(\hat{G}(\mathbb{R})/\hat{K}^0_\infty) \right) \to \mathbb{C}
\]

defined as in (2.11).
Let $\Pi_{0,\infty}$ be the irreducible tempered Casselman-Wallach representation of $G(\mathbb{R})$ whose infinitesimal character equals that of the trivial representation and whose central character equals that of $(F_{p}^{\gamma} \boxtimes w_{\eta}^{-1}) \otimes \Pi_{\infty}$. Define the cohomology group $\Pi'_{0,\infty}$ as in (2.9). Following [JLST24], with the fixed Shalika functionals we have the translation map

$$J_{\mu} : \Pi'_{0,\infty} \rightarrow \Pi'_{\infty}$$

which is a $k_{\infty}^{X,\natural}$-equivariant isomorphism. As a specialization of (2.11), we have a map

$$\hat{\mathcal{P}}_{\infty} : H^{0}(\mathfrak{h}_{C}, K_{Z,\infty}^{\natural}; \varepsilon_{\infty}) \times \left( \Pi'_{0,\infty} \otimes M(\mathcal{H}(\mathbb{R}))^{\vee} \otimes O(\hat{G}(\mathbb{R})/\hat{K}_{\infty}^{\natural}) \right) \rightarrow \mathbb{C}$$

for every $\varepsilon_{\infty} \in k_{\infty}^{X,\natural}$. Define $\hat{\mathcal{P}}_{\infty}^{\phi}$ in (2.10) to be the map such that for every $\varepsilon_{\infty} \in k_{\infty}^{X,\natural}$, $\hat{\mathcal{P}}_{\infty}^{\phi}(\varepsilon_{\infty}, \cdot)$ equals the composition of

$$\Pi'_{\infty} \otimes M(\mathcal{H}(\mathbb{R}))^{\vee} \otimes O(\hat{G}(\mathbb{R})/\hat{K}_{\infty}^{\natural})$$

$$\xrightarrow{(1, J_{\mu}^{-1} \otimes \text{id} \otimes \text{id})} H^{0}(\mathfrak{h}_{C}, K_{Z,\infty}^{\natural}; \varepsilon_{\infty}) \times \left( \Pi'_{0,\infty} \otimes M(\mathcal{H}(\mathbb{R}))^{\vee} \otimes O(\hat{G}(\mathbb{R})/\hat{K}_{\infty}^{\natural}) \right)$$

$$\xrightarrow{\hat{\mathcal{P}}_{\infty}} \mathbb{C}.$$

We have the following archimedean nonvanishing hypothesis and period relations, which are proved in [Sun19, JST19, JLST24].

**Theorem 10.3.** If $w$ is $V$-balanced, then the followings hold.

(a) The map $\hat{\mathcal{P}}_{\infty}^{\phi}$ is nonzero.

(b) $\hat{\mathcal{P}}_{\infty}^{\phi}(1, \cdot) = \Upsilon_{\Pi_{\infty}}(\chi_{\infty}) \cdot \hat{\mathcal{P}}_{\infty}^{\phi}(w_{\infty} \chi_{\infty}, \cdot)$, where $\Upsilon_{\Pi_{\infty}}(\chi_{\infty})$ is in (2.33).

Following Definition 2.11 fix an element

$$\hat{\phi}_{\infty}^{\phi} \in \Pi'_{0,\infty}(E) \otimes D(\hat{H}(\mathbb{R}))^{\vee} \otimes O(\hat{G}(\mathbb{R})/\hat{K}_{\infty}^{\natural})$$

such that $\hat{P}_{\infty}^{\phi}(\varepsilon_{\infty}, \hat{\phi}_{\infty}^{\phi}) \neq 0$ for all $\varepsilon_{\infty} \in k_{\infty}^{X,\natural}$. Define the Shalika periods

(10.4) $\Omega_{\Pi}(\varepsilon_{\infty}) := \left( \hat{P}_{\infty}^{\phi}(\varepsilon_{\infty}, \hat{\phi}_{\infty}^{\phi}) \right)^{-1}, \quad \varepsilon_{\infty} \in k_{\infty}^{X,\natural}.$

10.3. Nearlly ordinary refinement. Recall the Borel subgroup $P \subset G$ from (10.3). Suppose that $P = P(Q_{p}) = z^{-1}Bz$, where $B = B(Q_{p})$. Assume that $\Pi'_{p} \subset B_{P}(\Pi_{p})$ is a nearly ordinary refinement of $\Pi_{p}$ defined over $E$, which is also viewed a character of $P$ that descends to a character of $L = P/N$.

As before $\Pi_{p}$ is isomorphic to a quotient representation of $\text{Ind}_{\hat{P}}^{G}(\Pi'_{p} \cdot \delta_{p}^{1/2})$. Define a character

$$\kappa := \Pi'_{p} \circ \text{Ad}(\gamma^{-1}) : \overline{B} \rightarrow E^{\times}.$$

Then $\Pi_{p}$ is also isomorphic to a quotient representation of

$$I(\tilde{\kappa}) := \text{Ind}_{\overline{B}}^{G}(\tilde{\kappa}), \quad \text{where} \quad \tilde{\kappa} := \kappa \otimes \delta_{\overline{B}}^{1/2}.$$
Recall from Section 2.6 that \( \kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{2n}, \eta_p^{-1}) \), where \( \kappa_i = \otimes_{\wp | p} \kappa_{i,\wp} \) \( (i = 1, 2, \ldots, 2n) \). Similarly write
\[
\bar{\kappa} = (\bar{\kappa}_1, \bar{\kappa}_2, \ldots, \bar{\kappa}_{2n}, \eta_p^{-1}),
\]
where \( \bar{\kappa}_i = \otimes_{\wp | p} \bar{\kappa}_{i,\wp} \) \( (i = 1, 2, \ldots, 2n) \) and \( \eta_p = \otimes_{\wp | p} \eta_{\wp} \). Recall that \( \eta_n \) equals the central character of \( \pi \) so that \( \eta_n = \prod_{i=1}^{2n} \kappa_{i,\wp} \). Define \( \mu_{\wp} \) as in Section 8.5 and \( \rho_{2n} := (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, n + \frac{1}{2}) \in \mathbb{Q}^{2n} \), and put
\[
w_{\eta,\wp} := \frac{1}{n} \sum_{i=1}^{2n} \mu_i^p \in \mathbb{Q}.
\]
The following lemma is straightforward.

**Lemma 10.4.** For \( \wp | p \), the character \( \bar{\kappa}_\wp \) has \( p \)-exponent
\[
(-\mu_{\wp} - \rho_{2n}, w_{\eta,\wp}) \in \mathbb{Q}^{2n+1}.
\]

**Proposition 10.5.** For \( i, j = 1, 2, \ldots, 2n \), the equality \( \bar{\kappa}_{i,\wp} \bar{\kappa}_{j,\wp} = \eta_\wp \) holds if and only if \( i + j = 2n + 1 \).

**Proof.** Since \( \pi^\vee_\wp \cong \pi_\wp \otimes \eta_p^{-1} \) and \( \pi^\vee_\wp \) is a subrepresentation of \( I(\bar{\kappa}^{-1}) \), by the uniqueness of cuspidal support (see [Be92, Chapter III, 2.1]) and the MVW involution we have an equality of multi-sets
\[
\{\bar{\kappa}_{1,\wp}^{-1}, \ldots, \bar{\kappa}_{2n,\wp}^{-1}\} = \{\bar{\kappa}_{1,\wp}^{-1}, \ldots, \bar{\kappa}_{2n,\wp}^{-1}\}.
\]
The proposition follows from the comparison of \( p \)-exponents using Lemma 10.4. \( \square \)

To evaluate the modifying factor at \( p \), similar to the method in the Rankin-Selberg case where we have applied Theorem 8.3, the idea is to compare the Friedberg-Jacquet integral with a new integral over the open \( \tilde{G} \)-orbit in the flag variety \( \tilde{B} \backslash G \). We will formulate and prove a general result in the next subsection.

### 10.4 Open orbit integrals and normalized refined period

Let \( v \) be a finite place of \( k \). Let
\[
\varrho = (\varrho_1, \varrho_2, \ldots, \varrho_{2n}, \eta_v) \in (\tilde{\kappa}^\vee_v)^{2n+1}
\]
such that \( \eta_v = \prod_{i=1}^{2n} \varrho_i \). Then \( \varrho \) can be viewed as a character of \( \tilde{B}_v \). Assume that \( \varrho \) is symmetric and good in the sense of [JLST23], namely \( \varrho \) satisfies that
- \( \varrho_1 \varrho_2 \varrho_3 = \cdots = \varrho_{2n} \varrho_{n+1} = \eta_v \);
- \( \varrho_i \varrho_j \neq \eta_v \), for all \( 1 \leq i < j \leq 2n - i \).

By [JLST23, Theorem 1.2], \( I(\varrho) := \text{Ind}_{\tilde{B}_v}^{\tilde{G}_v} \varrho \) has a unique Shalika model, that is,
\[
\dim \text{Hom}_{\tilde{H}_v}(I(\varrho), \psi_{\tilde{H}_v}) = 1.
\]
Let us describe its Shalika model explicitly.

Recall that
\[
P_v = P_v \cap \tilde{G}_v = \{ (t, t) \tilde{G}_{1,v}^t \mid t \in T_{n,v} \},
\]
where $T_{n,v}$ is the diagonal maximal torus of $GL_{n,v} = GL_n(k_v)$. Recall the element $\gamma$ in (10.3), and define

$$\gamma' := \left( \begin{bmatrix} 1_n & 0 \\ 0 & w_n \end{bmatrix}, 1 \right).$$

It is easy to verify that $B_v \gamma' H_v = B_v \gamma H_v \subset G_v$ is open and that $\gamma'^{-1} B_v \gamma' \cap H_v = \hat{P}_v$.

We endow the unipotent radical of $H_v$ with the Haar measure that is the product of self-dual Haar measures on $k_v$ with respect to $\psi_v$. This gives an identification $M(\hat{P}_v \backslash H_v) = M(\hat{P}_v \backslash \hat{H}_v)$. Fix a generator $\nu \in D(\hat{P}_v \backslash \hat{H}_v)$. By [JLST23, Theorem 1.2], there is a unique generator $\lambda'_{H_v} \in \text{Hom}_{H_v}(I(\hat{\sigma}), \psi_{H_v})$ such that

$$\langle \lambda'_{H_v}, f \rangle = \int_{P_v \backslash H_v} f(\gamma' h) \psi_{H_v}^{-1}(h) \, d\nu(h)$$

for all $f \in I(\hat{\sigma})$ with $\text{supp}(f) \subset B_v \gamma' H_v$. Then we have the unnormalized Friedberg-Jacquet integral

$$\Lambda_v : X_v \times \left( I(\hat{\sigma}) \otimes M(\hat{H}_v \backslash \hat{G}_v) \right) \to \mathbb{C} \cup \{\infty\}, \quad (\chi'_v, f \otimes \tau) \mapsto \int_{\hat{P}_v \backslash \hat{G}_v} \chi'_v(j(g)) \langle \chi'_v, f \rangle \, d\tau(g),$$

defined by meromorphic continuation of absolutely convergent integrals. Define

$$\Lambda_v : X_v \times \left( I(\hat{\sigma}) \otimes M(\hat{H}_v \backslash \hat{G}_v) \right) \to \mathbb{C} \cup \{\infty\}, \quad (\chi'_v, f \otimes \tau) \mapsto \prod_{i=1}^n \gamma \left( \frac{1}{2}, \sigma_i \cdot \chi'_v, \psi_v \right) \cdot \text{P}_v(\chi'_v, f \otimes \tau),$$

which is meromorphic in $\chi'_v \in X_v$.

We have the following comparison between Friedberg-Jacquet integral and an open orbit integral, the proof of which uses the functional equation of Godement-Jacquet integrals in [GJ72].

**Theorem 10.6.** Let

$$I(\hat{\sigma})_z := \{ f \in I(\hat{\sigma}) \mid \text{supp}(f) \subset \overline{B}_v \gamma \hat{G}_v \}. \quad \text{Then for every } \chi'_v \in X_v \text{ and } f \otimes \tau \in I(\hat{\sigma})_z \otimes M(\hat{H}_v \backslash \hat{G}_v),$$

$$\Lambda_v(\chi'_v, f \otimes \tau) = \int_{\hat{P}_v \backslash \hat{G}_v} \chi'_v(j(g)) f(\gamma g) \, d\tau'(g),$$

where $\tau' := \nu \otimes \tau$ by using the obvious identification

$$M(\hat{P}_v \backslash \hat{G}_v) = M(\hat{P}_v \backslash \hat{H}_v) \otimes M(\hat{H}_v \backslash \hat{G}_v).$$

Moreover the integral (10.6) is algebraic in $\chi'_v \in X_v$.

**Proof.** Write $\overline{B}_{n,v}$ for the lower triangular Borel subgroup of $GL_{n,v}$. Put

$$\sigma_1 := \text{Ind}_{\overline{B}_{n,v}}^{GL_{n,v}} (\varrho_1 \otimes \cdots \otimes \varrho_n), \quad \sigma_2 := \text{Ind}_{\overline{B}_{n,v}}^{GL_{n,v}} (\varrho_{n+1} \otimes \cdots \otimes \varrho_{2n}).$$
Then the representation $\sigma_1 \otimes \sigma_2 \otimes \eta_v^{-1}$ of $\GL_{n,v} \times \GL_{n,v} \times \GL_{1,v}$ descends to a representation of $\hat{G}_v$. By [JLST23, Theorem 1.5], there is a unique nonzero functional 

$$\lambda_\Delta \in \Hom_{\hat{H}_v}(\sigma_1 \otimes \sigma_2 \otimes \eta_v^{-1}, \mathbb{C})$$

such that 

$$(10.7) \quad \langle \lambda_\Delta, \phi \rangle = \int_{P_v \backslash \hat{H}_v} \phi(\gamma h) \, d\nu(h)$$

for any $\phi \in \sigma_1 \otimes \sigma_2 \otimes \eta_v^{-1}$ such that $\text{supp}(\phi) \subset (\hat{\Gamma}_v \cap \hat{G}_v) \gamma \hat{H}_v$.

Let $\overline{Q}_v$ be the lower triangular parabolic subgroup of $G_v$ of type $(n, n)$, which has Levi subgroup $\hat{G}_v$. Then we have the induction in stages 

$$I(\rho) \cong \Ind_{\overline{Q}_v}^{G_v}(\sigma_1 \otimes \sigma_2 \otimes \eta_v^{-1}), \quad f \mapsto f',$n

where $f'(g') \in \sigma_1 \otimes \sigma_2 \otimes \eta_v^{-1}$ for $g' \in G_v$ is given by 

$$(10.8) \quad f'(g')(g) := f(g g'), \quad g \in \hat{G}_v.$$ 

By [JLST23, Theorem 1.8], if $\text{supp}(f) \subset \overline{Q}_v H_v$, then 

$$\lambda_{\hat{H}_v}(f) = \int_{x \in M_{n,v}} \left\langle \lambda_\Delta, f' \left( \begin{bmatrix} 1_n & x \\ 0 & 1_n \end{bmatrix} \right) \right\rangle \psi_v^{-1}(\text{tr} x) \, dx,$n

where $\lambda_\Delta$ is given by $(10.7)$, and $M_{n,v} := M_n(k_v)$. Here and below, for convenience we use the same notations for elements of $S_v$ and $\GL_{n,v} \times \GL_{n,v}$ and their images in $H_v$ and $\hat{G}_v$ respectively, which should not cause any confusion.

Assume that $f \in I(\rho)_1$ so that $\text{supp}(f) \subset \overline{\Gamma}_v \gamma \hat{G}_v \subset \overline{Q}_v H_v$, and $f'$ is in $(10.8)$. Then $\text{supp}(g f) \subset \overline{Q}_v H_v$ for any $g \in \hat{G}_v$. Thus for $\Re(\chi_v')$ sufficiently large (here $\Re(\chi_v')$ is the real number such that $|\chi_v'(a)| = |a|^{\Re(\chi_v')}$, $a \in k_v^\times$) we have that 

$$\mathcal{P}_v(\chi_v', f \otimes \tau) = \int_{\GL_{n,v}} \int_{M_{n,v}} \left\langle \lambda_\Delta, f' \left( \begin{bmatrix} 1_n & x \\ 0 & 1_n \end{bmatrix} \right) \right\rangle \psi_v^{-1}(\text{tr} x) \, dx \, \chi_v'(\det g) \, d\tau(g) = \int_{\GL_{n,v}} \int_{M_{n,v}} \left\langle \lambda_\Delta, \sigma_1(g) \cdot f' \left( \begin{bmatrix} 1_n & x \\ 0 & 1_n \end{bmatrix} \right) \right\rangle \psi_v^{-1}(\text{tr}(gx)) \, dx \, \chi_v'(\det g) \, \det g |^n \, d\tau(g).$$

Here and below we use the identification $M(\hat{H}_v \backslash \hat{G}_v) = M(\GL_{n,v})$. The support condition on $f$ implies that the function 

$$(10.9) \quad (g, x) \mapsto \left\langle \lambda_\Delta, \sigma_1(g) \cdot f' \left( \begin{bmatrix} 1_n & x \\ 0 & 1_n \end{bmatrix} \right) \right\rangle, \quad (g, x) \in \GL_{n,v} \times M_{n,v},$$

lies in the space $\text{MC}(\sigma_1) \otimes S(M_{n,v})$, where $\text{MC}(\sigma_1)$ denotes the space spanned by the matrix coefficients of $\sigma_1$. More precisely, 

$$\text{MC}(\sigma_1) := \text{Span}\{\varphi_{u,v} : u \in \sigma_1, u^v \in \sigma_1^v\},$$
where \( \varphi_{u,v}(g) := (\sigma_1(g)u,v) \), \( g \in \text{GL}_{n,v} \). Then the above inner integral is the Fourier transform of the function \( \text{(10.9)} \) in the variable \( x \) using \( \psi_v^{-1} \), evaluated at \( (g,g) \in \text{GL}_{n,v} \times M_{n,v} \). Hence the inner integral, as a function in \( g \in \text{GL}_{n,v} \), is the pull-back through the diagonal embedding \( \text{GL}_{n,v} \to \text{GL}_{n,v} \times M_{n,v} \) of a function in \( \text{MC} (\sigma_1) \otimes S(M_{n,v}) \).

Thus \( P_v(\chi_v', f \otimes \tau) \) is a Godement-Jacquet integral \( \text{(GJ72)} \) for the representation \( \sigma_1 \otimes \chi_v' \) of \( \text{GL}_{n,v} \). By the functional equation of Godement-Jacquet integrals and the uniqueness of meromorphic continuation, for \( \text{Re}(\chi_v') \) sufficiently large we have that

\[
\Lambda_v(\chi_v', f \otimes \tau) = \gamma(1/2, \sigma_1 \otimes \chi_v', \psi_v) \cdot P_v(\chi_v', f \otimes \tau)
\]

\[
= \int_{\text{GL}_{n,v}} \langle \lambda_\Delta, \sigma_1(g^{-1}) \cdot f' \begin{bmatrix} 1_n & g \\ 0 & 1_n \end{bmatrix} \rangle \chi_v'^{-1}(\det g) \, d\tau(g)
\]

\[
= \int_{\text{GL}_{n,v}} \langle \lambda_\Delta, f' \begin{bmatrix} g^{-1} & 0 \\ 0 & 1_n \end{bmatrix} \rangle \chi_v'^{-1}(\det g) \, d\tau(g)
\]

Since \( \text{supp}(f) \subset \overline{B_v} \gamma \hat{G}_v \), it is easy to verify that

\[
\int_{\text{GL}_{n,v}} \langle \lambda_\Delta, f' \begin{bmatrix} g & 1_n \\ 0 & 1_n \end{bmatrix} \rangle \chi_v'(\det g) \, d\tau(g)
\]

\[
= \int_{\text{GL}_{n,v}} \int_{P_v \backslash \hat{H}_v} f(\gamma h \begin{bmatrix} g & 1_n \\ 0 & 1_n \end{bmatrix}) \, d\nu(h) \chi_v'(\det g) \, d\tau(g)
\]

\[
= \int_{P_v \backslash \hat{G}_v} f(\gamma g) \chi_v'(\gamma g) \, d\tau'(g).
\]

This proves that \( \text{(10.6)} \) holds for \( \text{Re}(\chi_v') \) sufficiently large, hence for all \( \chi_v' \in X_v \) by the uniqueness of meromorphic continuation.

The maps \( P_v \) and \( \Lambda_v \) naturally extend to maps

\[
X_v \times \left( I(\varphi)_{p - \text{sm}} \otimes M(\hat{H}_v \backslash \hat{G}_v) \right) \to \mathbb{C} \cup \{ \infty \}
\]

where \( p_v \) denotes the Lie algebra of \( P(k_v) \).

By Proposition \( \text{10.5} \), \( \tilde{\kappa}_v \) is symmetric and good. By tensor product over \( \varphi \mid p \), the maps \( P_v \) and \( \Lambda_v \) yield respectively maps

\[
P_v, \Lambda_v : X_v \times \left( I(\tilde{\kappa})_{p - \text{sm}} \otimes M(\hat{H} \backslash \hat{G}) \right) \to \mathbb{C} \cup \{ \infty \}.
\]

Recall the Shalika functional \( \lambda_{H_v} \) on \( \Pi_v \). Write

\[
\xi_p : I(\tilde{\kappa}) \to \Pi_p = \otimes_{v \mid p} \Pi_v
\]
for the $G$-homomorphism whose composition with $\otimes_{\wp} \lambda_{\wp}$ equals $\otimes_{\wp} \lambda_{\wp}'$. It is surjective and naturally extends to a surjective $G$-homomorphism

$$\xi_p : \hat{I}(\hat{k}) \to \hat{\Pi}_p.$$  

Denote by $\hat{I}(\hat{k})_\gamma \subset \hat{I}(\hat{k})$ the subspace of the generalized functions supported in $\overline{B_\gamma}$, where $\gamma$ is as in (10.3). Then $\hat{I}(\hat{k})_\gamma$ is one-dimensional and the map $\xi_p$ in (10.11) restricts to a $P$-isomorphism

$$\xi_p : \hat{I}(\hat{k})_\gamma \sim \Pi'_p.$$  

Moreover, for every generator $\hat{f} \otimes \tau$ of $\hat{I}(\hat{k})_\gamma \otimes M(\hat{H}\backslash \hat{G})$, it follows from Theorem 10.6 that $\Lambda_p(\cdot, \hat{f} \otimes \tau)$ is a constant function on $X_p$ with values in $\mathbb{C}^\times$.

Now we define the normalized refined period map to be the composition

$$\hat{P}^p_\sigma : X_p \times \left( \Pi'_p \otimes M(\hat{H}\backslash \hat{G}) \right) \xrightarrow{\xi_p^{-1}} X_p \times \left( \hat{I}(\hat{k})_\gamma \otimes M(\hat{H}\backslash \hat{G}) \right) \xrightarrow{\Lambda_p} \mathbb{C}.$$  

10.5. **Rational test vectors and modifying factors at $p$.** In what follows we define a rational test vector $\tilde{\phi}_p^\sigma \in \Pi'_p(E) \otimes D(\hat{H}\backslash \hat{G})$. Recall the $\mathbb{Q}(\Pi_p)$-form of $\Pi_p$ given by (10.11). Define an $\text{Aut}(\mathbb{C}/\mathbb{Q}(\kappa_p))$-action on $I(\hat{k}_p)$ by

$$\sigma f(g) = \kappa_p(s_{\sigma, p}) \cdot \sigma(f(g)), \quad \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\kappa_p)), \ f \in I(\hat{k}_p), \ g \in G_p.$$  

Then by (8.4) we have that

$$I(\hat{k}_p)^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\kappa_p))} = \{ f \in I(\hat{k}_p) : \omega_{\psi_p}(\kappa_p) \cdot f \text{ is } \mathbb{Q}(\kappa_p)\text{-valued} \},$$  

which is a $\mathbb{Q}(\kappa_p)$-form of $I(\hat{k}_p)$, where $\omega_{\psi_p}(\kappa_p)$ is in (2.35). By tensor products, we have a $\mathbb{Q}(\Pi_p)$-form of $\Pi_p$ as well as a $\mathbb{Q}(\kappa)$-form of $I(\hat{k})$.

The same proof as that of Proposition 8.6 proves the following result.

**Proposition 10.7.** One has that $\mathbb{Q}(\Pi_p) \subset \mathbb{Q}(\kappa)$ and the surjective homomorphism $c_{\kappa}^{-n^2} \xi_p : I(\hat{k}) \to \Pi_p$ is $\mathbb{Q}(\kappa)$-rational, where $\xi_p$ is in (10.10).

Let $\hat{\phi}_p$ denote the generator of $\Pi'_p \otimes M(\hat{H}\backslash \hat{G})$ such that $\hat{P}^p_\sigma(\cdot, \hat{\phi}_p)$ equals the constant function 1 on $X_p$. Write $\hat{\phi}_p = \phi_p' \otimes \tau$ such that $\phi_p' \in \Pi'_p$ and $\tau \in D(\hat{H}\backslash \hat{G})$.

By Proposition 10.7

$$c_{\kappa}^{-n^2} \omega_{\psi_p}(\kappa)^{-1} \cdot \phi_p' = \Omega_{\Pi'_p}^{-1} \cdot \phi_p' \in \Pi'_p(E),$$  

where $\Omega_{\Pi'_p}$ is in (2.36). Define

$$\hat{\phi}_p^\sigma := \Omega_{\Pi'_p}^{-1} \cdot \hat{\phi}_p \in \Pi'_p(E) \otimes D(\hat{H}\backslash \hat{G}).$$  

Then

$$\hat{P}^p_\sigma(\chi'_p, \hat{\phi}_p^\sigma) = \Omega_{\Pi'_p}^{-1} \text{ for all } \chi'_p \in X_p.$$
From (2.34) it is clear that
\[ \Upsilon_{\Pi'_p}(\lambda'_p) = \frac{1}{L\left(\frac{1}{2}, \pi_p \otimes \chi'_p \right) \cdot \prod_{v|p} \prod_{i=1}^n \gamma_1 \left(\frac{1}{2}, \kappa_i \cdot \chi'_p, \psi'_p \right)} \cdot \chi'_p = \otimes_{v|p} \chi'_p \in \mathcal{X}_p. \]

Then the following is immediate from Theorem [10.6]

**Proposition 10.8.** It holds that
\[ \mathcal{P}^o_p(\chi'_p, \phi^o_p) = \Upsilon_{\Pi'_p}(\lambda'_p) \cdot \Omega^{-1}_{\Pi'_p} \text{ for all } \chi'_p \in \mathcal{X}_p. \]

Consequently \(\Upsilon_{\Pi'_p}\) is algebraic on \(\mathcal{X}_p\).

By Proposition [10.8] \(\Upsilon_{\Pi'_p}(\chi_p)\) is the modifying factor at \(p\) as in Definition [2.7].

It is again consistent with the conjecture given by Coates and Perrin-Riou in [CPR89] [Co89].

10.6. \(p\)-adic L-functions and exceptional zeros. Now we construct standard \(p\)-adic L-function \(\mathcal{L}_\Pi\) by combining the previous results. Suppose that \(Z_0\) is trivial and recall \(\varepsilon = \otimes_{v|\infty} \varepsilon_v\) in [1.3].

- For \(v \nmid \infty\), let \(\phi^o_v \in \Pi_v(E) \otimes D(H_v \backslash \hat{G}_v)\) be as in Section [10.1] such that
  \[ \mathcal{P}^o_v(\chi'_v, \phi^o_v) = \frac{1}{\mathcal{Y}_v(\chi'_v)}, \quad \chi'_v \in \mathcal{X}(\varepsilon_v). \]

- Let \(\lambda_0 \in \text{Hom}_E(V^n, E)\) and \(\hat{\psi}^o_\infty \in \Pi'_\infty(E) \otimes D(H(\mathbb{R}))^\vee \otimes O(\hat{G}(\mathbb{R})/\hat{K}^o)\) be as in Section [10.2] such that
  \[ \hat{\psi}^o_\infty(1, \hat{\phi}^o_\infty) = \Upsilon_{\Pi'_\infty}(\chi_\infty) \cdot \hat{\psi}^o_\infty(w_\infty \chi_\infty, \hat{\phi}^o_\infty) = \frac{\Upsilon_{\Pi'_\infty}(\chi_\infty)}{\Omega_{\Pi'}(w_\infty \chi_\infty)} \]
  for all \(V\)-balanced characters \(w\) and all algebraic characters \(\chi_\infty\) of weight \(w^{-1}\), where \(\lambda_v \in \text{Hom}_{\hat{\phi}_v}(E_w \otimes V, E)\) is the generator such that \(\lambda_v|_{V^n} = \lambda_0\).

- Let \(\hat{\phi}^o_p \in \Pi'_p(E) \otimes D(\mathcal{H} \backslash \hat{G})\) be as in Proposition [10.8].

Using all the local test vectors, let
\[ \hat{\phi}^o := \hat{\phi}^o_\infty \otimes \hat{\phi}^o_p \otimes (\otimes_{v|\infty} \phi^o_v) \in \mathcal{H} \otimes D(\hat{G}, \hat{p}), \]
and following Definition [2.5] let
\[ \mathcal{L}_\Pi := \mathcal{L}_{\varepsilon \otimes \mathcal{H}} = \left(\mathcal{L}_{\lambda_0 \otimes \hat{\phi}^o}\right)|_{C(Z, E)(\varepsilon)}. \]

We restate Theorem [2.14] below, which is now an immediate consequence of the above results and [2.19].

**Theorem 10.9.** If there is a \(V\)-balanced character, then \(\mathcal{L}_\Pi\) is the unique continuous linear functional on \(C(Z, E)(\varepsilon)\) such that
\[ \mathcal{L}_\Pi(\lambda) = \frac{\Upsilon_{\Pi'_\infty}(\chi_\infty) \cdot \Upsilon_{\Pi'_p}(\chi_p) \cdot \left(1/2, \pi \otimes \chi\right)}{\mathcal{Y}_v(\chi_p)^n \cdot \Omega_{\Pi'_p} \cdot \Omega_{\Pi'}(w_\infty \chi_\infty)}. \]
for all critical algebraic Hecke characters $\chi = \otimes \ell \chi_\ell \in X(\varepsilon)$, where $w$ is the inverse of the weight of $\chi$.

We end this section by applying Theorem 10.9 to determine the exceptional zeros of $L_{\Pi}$.

**Proposition 10.10.** Under the assumptions in Theorem 10.9, $\chi^\flat$ is an exceptional zero of $L_{\Pi}$ if and only if there exists a place $\wp \mid p$ such that

$$\chi_\wp = \kappa_{n, \wp}^{-1},$$

where $\kappa_{n, \wp}$ is as in (2.34).

**Proof.** By [LRS99] and [MS04] again, $L(\frac{1}{2}, \pi_p \otimes \chi_\wp)$ is finite. Hence $\chi^\flat$ is an exceptional zero of $L_{\Pi}$ if and only if

$$\prod_{\wp \mid p} \prod_{i=1}^n \gamma \left( \frac{1}{2}, \kappa_{i, \wp}^{-1} \cdot \chi_{\wp}^{-1} \cdot \psi_{\wp}^{-1} \right) = 0.$$

The assertion follows easily from (10.2) and Lemma 10.4. □

**Acknowledgement**

We thank Yifeng Liu for several helpful discussions and for informing us of the papers [Liu23, Liu24, LTX24]. We thank Daniel Disegni and Wei Zhang for reading an earlier version of this paper and sending their paper [DZ24]. We also thank Dihua Jiang and Fangyang Tian for discussions related to Shalika models, and thank Liang Xiao for comments on Emerton’s completed cohomology.

D. Liu was supported in part by National Key R & D Program of China (No. 2022YFA1005300) and Zhejiang Provincial Natural Science Foundation of China (No. LZ22A010006). B. Sun was supported in part by National Key R & D Program of China (No. 2022YFA1005300 and 2020YFA0712600) and New Cornerstone Investigator Program.

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