A new anisotropic solution for ellipsoidal spaces

T. Papakostas
TEI of Crete, Branch of Chania, 3 Romanou Street, Chalepa, Chania 731-33, Crete, Greece
E-mail:taxiar@chania.teicrete.gr

Abstract. We use the notion of ellipsoidal spaces in General Relativity and obtain the corresponding Einstein’s equations in the case of a perfect fluid with heat flux. We present the integration of the resulting equations and two families of solutions.

1. Introduction
The problem of finding an interior solution in the case of stationary and axially symmetric spaces of General Relativity is still open. The importance of this problem is obvious; it gives an alternative model for a rotating star other than the black hole source of the Kerr metric.

There are very few exact solutions describing the above mentioned situation, which all exhibit unsatisfactory features: The Wahlquist solutions [1] have an unphysical equation of state and a bounding surface prolate rather than oblate, in [2] and [3] the energy conditions are not satisfied and in [4] it is not possible to isolate the hydrostatic pressure from the other stresses in order to define the bounding surface of zero pressure.

In this paper we consider a special case of the stationary and axially symmetric spaces, the ellipsoidal spaces. The notion of an ellipsoid of revolution in curved spaces is given first by Krasinski [6] and presented in a geometric way by [7], [8]. In this context we restrict our study in the family [A] of Carter’s spaces and we consider as energy-momentum tensor that of a perfect fluid with heat flux.

In Section 2 we present the Carter’s family [A] of solutions and we introduce the notion of curved spaces admitting a foliation by ellipsoids of revolution. In Section 3 we present the energy-momentum space of a perfect fluid with heat flux and we deduce the corresponding Einstein’s equations. Finally, in Section 4 we integrate the equations and we present the two families of solutions.

2. Carter’s family [A] of solutions and ellipsoidal spaces
We suppose that the space admits an isometry Abelian group of two parameters invertible with non null surface of transitivity [9]. This implies that there is a local coordinate system in which the metric can be written as follows:

\[ ds^2 = (Ldt + Mdz)^2 - (Ndt + Pdz)^2 - S^2 dx^2 - R^2 dy^2, \]  

(2.1)

where \( L, M, N, P, S, R \) are real functions of \( x \) and \( y \). The Killing vectors of the Abelian group are:
The form of the metric for the stationary and axially symmetric spaces has been used by many authors [10-12] and has been proved very effective for the calculations. The Carter’s spaces are characterized by the fact that the Hamilton-Jacobi (HJ) for the geodesics is solvable by separation of variables. Then the metric (2.1) reduces to the following expression:

\[ ds^2 = (x^2 + y^2)^2 \left\{ \frac{E^2(y)}{(x^2 + y^2)^2} (dt - x^2 dz)^2 - \frac{H^2(x)}{(x^2 + y^2)^2} (dt + y^2 dz)^2 - \frac{x^2 dx^2}{F^2(x)} - \frac{y^2 dy^2}{G^2(y)} \right\}. \tag{2.2} \]

In the case of vacuum Einstein’s equations, we obtain the following expressions for the metric (2.2):

\[ G^2 = y^2 E^2, \quad F^2 = x^2 H^2, \quad E^2 = \frac{1}{2} a^2 + by + c, \quad H^2 = -\frac{1}{2} ax^2 + dx + c, \tag{2.3} \]

where \( a, b, c, d \) are constants of integration. The Kerr metric is obtained if we set:

\[ a=2, \quad d=0, \quad b=-2M, \quad c=-a^2, \quad y=r, \quad x=\cos \theta, \tag{2.4} \]

where \( M \) is the mass, \( a \) is the angular momentum per unit mass and \( r, \theta \) are the Boyer-Lindquist coordinates.

The Einstein’s equations in the presence of the perfect fluid reduce to two equations in the Newman-Penrose formalism:

\[ \Phi_{00} \Phi_{02} = \Phi_{01}^2, \quad \text{and} \quad 2\Phi_{11} = \Phi_{00} + \Phi_{02}. \tag{2.5} \]

These equations imply that the energy-momentum tensor admits one simple eigenvalue (the energy-mass density) and triple eigenvalues (the isotropic hydrostatic pressure). The solution of the first of these equations permits to define \( G^2 \) and \( H^2 \) as follows:

\[ W(y) = \frac{G^2(y)}{E^2(y)} = k_4 y^4 + k_2 y^2 - k_0, \quad Z(x) = \frac{F^2(x)}{H^2(x)} = -k_4 x^4 + k_2 x^2 + k_0, \tag{2.6} \]

where \( k_0, k_2, k_4 \) are constants of integration. The solution of the second of equations (2.5) permits to obtain a generalization of the Wahlquist solution [5].

The Carter’s family [A] of solutions admits a foliation of ellipsoids of revolution if the quotient space of the comoving observers is that of an ellipsoid of revolution in a 3 dimensional curved space [6-8,13]. The vector field of the comoving observers is a linear combination of the Killing fields of Carter’s spaces. The existence of the above-mentioned foliation implies that:

\[ F^2(x) = x^2 (a^2 - x^2), \quad \text{and} \quad G^2(y) = f^2(y) (a^2 + y^2) y^2. \tag{2.7} \]
These expressions of $F^2$ and $G^2$ are the sufficient conditions for the existence of a family of ellipsoids in Carter’s family [A] of solutions.

3. Perfect fluid with heat flux

The energy-momentum tensor of a perfect fluid with heat flux is given by the following expression:

$$T_{ij} = (\varepsilon + p)u_iu_j - pg_{ij} + q_iu_j + q_ju_i,
(3.1)$$

with

$$u_i = \frac{\sqrt{2}}{2} \{\Pi_1(n_i + l_i) - \Pi_2(m_i + \bar{m}_i)\}, \quad q_i = \frac{\sqrt{2}}{2} [q_i n_i + q_2 l_i - q_3 m_i - q_4 \bar{m}_i].
(3.2)$$

The fluid moves along the integral curves of $u_i$; the heat flow is described by the space-like vector field $q_i$ which is orthogonal to $u_i$:

$$u_iu^i = 1, \quad q_iu^i = 0.
(3.3)$$

The energy-mass density is $\varepsilon$ and the hydrostatic pressure $p$. In the (NP) formalism the Einstein’s equations are:

$$2\Phi_{ii} + 6\Lambda = \frac{1}{2} (\varepsilon + p)\Pi_1^2 + \frac{1}{2} (q_1 + q_2)\Pi_1 - p,
$$

$$2\Phi_{ii} - 6\Lambda = \frac{1}{2} (\varepsilon + p)\Pi_2^2 + \frac{1}{2} (q_3 + q_4)\Pi_2 + p,
$$

$$-2\Phi_{0i} = -\frac{1}{2} (\varepsilon + p)\Pi_1\Pi_2 - \frac{1}{2} (q_1\Pi_2 + q_2\Pi_1),
(3.4)$$

$$-2\Phi_{0i} = -\frac{1}{2} (\varepsilon + p)\Pi_1\Pi_2 - \frac{1}{2} (q_3\Pi_2 + q_4\Pi_1),
$$

$$-2\Phi_{12} = -\frac{1}{2} (\varepsilon + p)\Pi_1\Pi_2 - \frac{1}{2} (q_2\Pi_2 + q_4\Pi_1),
$$

$$-2\Phi_{21} = -\frac{1}{2} (\varepsilon + p)\Pi_1\Pi_2 - \frac{1}{2} (q_3\Pi_2 + q_4\Pi_1)$$

$$(q_1 + q_2)\Pi_1 - (q_3 + q_4)\Pi_2 = 0.$$

In Carter’s family [A] of solutions these equations are reduced to the following relations:

$$q_2 = q_1, \quad q_4 = q_3, \quad q_1\Pi_1 = q_3\Pi_2,
(3.5)$$
\[ \varepsilon = 3(\Phi_{00} - \Phi_{02}) + 6\Lambda, \quad (3.6) \]

\[ p = (\Phi_{00} - \Phi_{02}) - 6\Lambda, \quad (3.7) \]

\[ q_1 = \frac{4}{(\Pi_1^2 + \Pi_2^2)} \left[ \Phi_{00}\Pi_2 - (\Phi_{00} - \Phi_{02})\Pi_1\Pi_2^2 \right], \quad (3.8) \]

\[ q_3 = \frac{4}{(\Pi_1^2 + \Pi_2^2)} \left[ \Phi_{00}\Pi_1 - (\Phi_{00} - \Phi_{02})\Pi_1^2\Pi_2 \right], \quad (3.9) \]

\[ \Phi_{00}\Pi_2^2 - 2\Phi_{01}\Pi_1\Pi_2 + \Phi_{02}\Pi_1^2 = 0, \quad (3.10) \]

\[ \Pi_1^2 - \Pi_2^2 = 1, \quad (3.11) \]

\[ 2\Phi_{11} - \Phi_{00} - \Phi_{02} = 0. \quad (3.12) \]

The relation (3.12) is a differential equation for the unknown functions \( E \) and \( H \).

4. The integration of equation \( 2\Phi_{11} - \Phi_{00} - \Phi_{02} = 0 \)

We write explicitly this equation as:

\[
2\left[ x^3y(x^2 + y^2)^2 \right]WE_{yy}^2 + (x^2 + y^2)\left[ -2x^3(5y^2 + x^2) \right]WE_y^2 + \\
+ 8x^3y^3(W + Z)E^2 \\
-2\left[ x^3y(x^2 + y^2)^2 \right]ZH_{xx}^2 - (x^2 + y^2)\left[ -2y^3(5x^2 + y^2) \right]ZH_y^2 + x^3y^2(x^2 + y^2)Z_x \]

\[ -8x^3y^3(W + Z)H^2 = 0, \quad (4.1) \]

where \( W \) and \( Z \) are defined as follows:

\[ W(y) = \frac{G^2(y)}{E^2(y)}, \quad \text{and} \quad Z(x) = \frac{F^2(x)}{H^2(x)}. \quad (4.2) \]

If we divide Equation (4.1) by \( x^3y^3 \) and differentiate five times with respect \( x \) and \( y \) we can easily show that:

\[ W = kE^2 + k_1y^4 + k_3y^3 + k_2y^2 + k_4y + k_5, \quad (4.3) \]

\[ Z = kH^2 + l_4x^4 + l_3x^3 + l_2x^2 + l_1x + l_0. \quad (4.4) \]
If we impose now relations (2.7) we have an ellipsoidal space and the solution of (4.1) is split into two families:

I. $k \neq 0$

$$H^2(x) = \frac{a^2x^2}{l_0},$$

$$E^2(y) = \frac{l_0\sqrt{(l_0^2 + a^4k)y^4 + (4a^2k_0l_0 + 4a^6k)y^2 + 4a^4k_0^2}}{2a^2kl_0} - \frac{(l_0^2 + a^4k)y^2 + 2a^2k_0l_0}{2a^2kl_0},$$

II. $k = 0$

$$H^2(x) = \frac{a^4x^2(a^2 - x^2)}{l_0(a^2 - 2x^2)},$$

$$E^2(y) = \frac{1}{4l_0} \cdot b(16l_0y^4 - a^4k_2y^2 - a^4k_0^2) \frac{\{32l_0y^2 - a^2\sqrt{64k_0l_0 + a^4k_2^2 - a^4k_2}\}^{1/2}}{32l_0y^2 + a^2\sqrt{64k_0l_0 + a^4k_2^2 - a^4k_2}} - \frac{a^4}{4l_0}. $$

The resulting spaces are of type I in the Petrov classification and we intend to make a complete study of them. Relations (3.5) - (3.11) will permit to calculate the heat flux vector, the mass-energy density, the hydrostatic pressure and try to define the surface of zero pressure in order to check if these spaces can be matched to the Kerr metric.

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