Analytic bundle structure on the idempotent manifold

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Abstract
Let $X$ be a (real or complex) Banach space (not necessarily a Hilbert space), and $\mathcal{I}(X)$ be the set of all non-trivial idempotents; i.e., bounded linear operators on $X$ whose squares equal themselves. We show that, when equipped with the Banach submanifold structure induced from $\mathcal{L}(X)$, the subset $\mathcal{I}(X)$ is a locally trivial analytic affine-Banach bundle over the Grassmann manifold $\mathcal{G}(X)$, via the map $\kappa$ that sends $Q \in \mathcal{I}(X)$ to $Q(X)$, such that the affine-Banach space structure on each fiber is the one induced from $\mathcal{L}(X)$. Using this, we show that if $H$ is a real or complex Hilbert space, then the assignment $(E, T) \mapsto T^* \circ P_{E^\perp} + P_E$, where $E \in \mathcal{G}(H)$ and $T \in \mathcal{L}(E, E^\perp)$, induces a real bi-analytic bijection from the total space of the tangent bundle, $T(\mathcal{G}(H))$, of $\mathcal{G}(H)$ onto $\mathcal{I}(H)$ (here, $E^\perp$ is the orthogonal complement of $E$, $P_E \in \mathcal{L}(H)$ is the orthogonal projection onto $E$, and $T^*$ is the adjoint of $T$). Notice that this real bi-analytic bijection is an affine map on each tangent plane. Furthermore, if for every $E \in \mathcal{G}(H)$, we identify $\mathcal{L}(E, E^\perp)$ with a subspace of $\mathcal{L}(H)$ via the embedding $S \mapsto S \circ P_E$, then the inclusion map from $T(\mathcal{G}(H))$ to the trivial Banach bundle $\mathcal{G}(H) \times \mathcal{L}(H)$ is a real analytic immersion. Through this, we give a concrete idempotent in $M_{n^2}(\mathbb{C}(\mathcal{G}(K^n)))$ that represents the $K$-theory class of the tangent bundle $T(\mathcal{G}(K^n))$, when $K$ is either the real field or the complex field.

Keywords Infinite dimensional Grassmannian · Idempotents · Banach bundles · Affine-Banach spaces · Tangent bundles

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1 Introduction

The Grassmannian of a finite dimensional vector space is a very well-studied object. This manifold is important in both pure and applied mathematics (see e.g., [7–9,14,16–19,21,24,28,29,32]). There are two main streams of generalizations of the Grassmannian to the infinite dimensional case. The first one was introduced by Douady in [15] (see also [22,31]). In this case, the Grassmannian of a (either real or complex) Banach space $X$ is the set of closed split subspaces of $X$ equipped with a canonical (respectively, real or complex) analytic Banach manifold structure such that a local chart around a closed split subspace is given by the Banach space of continuous linear operators from that subspace to a complement of it. Another approach was first appeared in the work of Porta and Recht in [30]. In this approach, the Grassmannian of a Banach algebra $D$ is defined to be the set of equivalence classes of idempotents under certain equivalence relation, and is equipped with the quotient topology. In the particular case when $D$ is a $C^*$-algebra, this Grassmannian can be identified, as a topological space, with the set of (self-adjoint) projections, and the latter is a real analytic Banach submanifold of $D$ (see e.g., [2,4,10]; see also [11,13,23] for the generalizations to $JB$-algebras and $JB^*$-triples). A connection between the two approaches was presented in [1], where it was shown that there is a real bi-analytic bijection from the Grassmannian of a complex Hilbert space $H$ (in the sense of Douady) to the set of self-adjoint projections of the $C^*$-algebra $L(H)$ of continuous linear operators on $H$.

Standard results on infinite dimensional Banach manifolds and their relation to operator algebras and to operator theory can be found in the books [3,12,27,31]. In this article, we mainly deal with analytic Banach manifold in the sense of [31].

We will follow [15] for the definition of the Grassmann manifold $G(X)$ of a (either real or complex) Banach space $X$. Moreover, we denote by $I(X)$ the set of all non-zero proper idempotents in the Banach manifold $L(X)$ of all bounded linear operators. Notice that “idempotents” were also called “projections” in some literature, but we prefer the term “idempotents” in order to distinguish them with “self-adjoint projections” on Hilbert spaces (which will also be considered in this article).

We define a map $\kappa: I(X) \to G(X)$ as follows:

$$\kappa(Q) := Q(X) \quad (Q \in I(X)). \quad (1.1)$$

Note that we will exclude the two subspaces $\{0\}$ and $X$ from the definition of $G(X)$ as they are zero dimensional manifolds, and some of the arguments in this paper may not make sense at those points. Furthermore, the fibration structures over $\{0\}$ and $X$ are trivial (actually, they are the two affine subspaces $\{0\}$ and $\{I\}$ respectively), and there is no need to consider them. For this reason, we excluded $0$ and $I$ from the definition of $I(X)$ as well.

We are going to show that under this map, $I(X)$ is a locally trivial analytic affine-Banach bundle over $G(X)$. Notice that since $I(X) \cup \{0, I\} = \{E \in L(X) : E^2 = E\}$, it is likely a known fact that $I(X) \cup \{0, I\}$ is a closed submanifold of $L(X)$. However, in
order to show that $\kappa$ induces a locally trivial analytic affine-Banach bundle structure on $I(X)$, we need a concrete analytic atlas for this submanifold, and it is why the proof of Theorem 3.4 becomes a bit lengthy. Moreover, unlike analytic Banach bundle structure, analytic affine-Banach bundle structure is not well-documented, and we need to say some more words about this in Sect. 2.

The main results of this paper can be summarized in the following (see Theorem 3.4, Propositions 4.1 and 4.3).

**Theorem 1** Let $X$ be a real or complex Banach space, which is not one-dimensional.

(a) Under the Banach submanifold structure on $I(X)$ induced from $L(X)$, one knows that $\kappa : I(X) \to G(X)$ is a locally trivial (respectively, real or complex) analytic affine-Banach bundle, such that the affine-Banach space structures on the fibers of $\kappa$ are the ones induced from $L(X)$.

(b) There exist equivalent Banach space structures on the fiber of $\kappa$, under which $\kappa : I(X) \to G(X)$ becomes a locally trivial continuous Banach bundle.

(c) In the case when $X$ is a complex Banach space, there can never exist Banach space structures on the fiber of $\kappa$ such that $\kappa : I(X) \to G(X)$ becomes a locally trivial complex analytic Banach bundle.

In the case of a real or complex Hilbert space $H$, we also obtain the following:

there is a real analytic immersion from the total space of the tangent bundle of $G(H)$ to $L(H)$ such that the restriction of this immersion on each fiber is affine.

More precisely, an element in the tangent bundle of $G(H)$ can be identified with a pair $(E, T)$, where $E \in G(H)$ and $T$ is a bounded linear operator from $E$ to the orthogonal complement $E^\perp$ of $E$. The following is obtained in Theorem 4.4.

**Theorem 2** Let $H$ be a real or complex Hilbert space, and $T(G(H))$ be the tangent bundle of $G(H)$. The assignment $Q \mapsto (Q(H), P_{Q(H)^\perp} \circ Q^*|_{Q(H)})$, where $P_{Q(H)^\perp}$ is the orthogonal projection onto $Q(H)^\perp$, is a real bi-analytic bijection from $I(H)$ onto $T(G(H))$ such that for each $E \in G(H)$, this bijection is an (respectively, real or complex) affine map from $\{Q \in I(H) : Q(H) = E\}$ onto the tangent plane at $E$.

The well-known Swan’s theorem says that any finite dimensional vector bundle is a subbundle of a trivial bundle. The corresponding result in the infinite dimensional case seems to be open. Nevertheless, one can use the above to show that the tangent bundle of $G(H)$ can be identified with a complemented subbundle of $G(H) \times L(H)$ via the “tautological embedding” (see Corollary 4.5(a)). On the other hand, through the Swan’s theorem, one can define the $K$-theory class of a vector bundle. In the finite dimensional case, we can use the above to write down an explicitly idempotent representing the $K$-theory class of the tangent bundle of the Grassmannian (see Corollary 4.6). Let us state them clearly in the following.

**Corollary 3** (a) Let $H$ be a real or complex Hilbert space. We set

$$L^{E^\perp}(H, E^\perp) := \{T \in L(H) : T(E^\perp) = \{0\}; T(H) \subseteq E^\perp\} \quad (E \in G(H)).$$
The assignment \((E, T) \mapsto (E, T|_E)\) is a fiberwise linear real bi-analytic map from the Banach subbundle \(\{(E, S) : E \in \mathcal{G}(H); S \in \mathcal{L}^{E \perp}(H, E^\perp)\}\) of the trivial bundle \(\mathcal{G}(H) \times \mathcal{L}(H)\) onto \(T(\mathcal{G}(H))\).

(b) Let \(\mathbb{K}\) be the real or the complex field and \(n \geq 2\). The \(\mathbb{K}\)-theory class of the tangent bundle \(T(\mathcal{G}(\mathbb{K}^n))\) is represented by the idempotent \(Q\) in the algebra of continuous maps from \(\mathcal{G}(\mathbb{K}^n)\) to \(\mathcal{L}(M_n(\mathbb{K}))\) given by \(Q(E)(T) := P_{E \perp} \circ T \circ P_E\), for \(E \in \mathcal{G}(\mathbb{K}^n)\) and \(T \in M_n(\mathbb{K})\).

On our way, we also obtain that \(\mathcal{I}(X)\) (under the norm topology) is canonically homeomorphic to the following subspace of the product topological space \(\mathcal{G}(X) \times \mathcal{G}(X)\) (see Corollary 3.6):

\[\{(E, F) \in \mathcal{G}(X) \times \mathcal{G}(X) : E \text{ and } F \text{ are complements to each other}\}\]

Furthermore, similar to the corresponding fact for \(\mathcal{G}(X)\), we will show, in Corollary 5.3, that each orbit in \(\mathcal{I}(X)\) under the conjugate action by the Banach Lie group \(GL(X)\) of continuous invertible operators on \(X\) is a clopen subset, and can be identified bi-analytically with a homogeneous space of \(GL(X)\). We will also verify in Proposition 5.4 that, for any \(n \in \mathbb{N}\), the set \(\{Q \in \mathcal{I}(X) : \dim Q(X) = n\}\) is a connected component of \(\mathcal{I}(X)\).

Using the above, for \(n \in \mathbb{N}\) and \(k \in \{1, \ldots, n\}\), if the map

\[\nu : GL_n(\mathbb{R})/GL_k(\mathbb{R}) \times GL_{n-k}(\mathbb{R}) \to O_n/O_k \times O_{n-k}\]

is given by the Gram–Schmidt process (on column vectors), then \(\nu\) induces a locally trivial real analytic vector bundle structure on the homogeneous space \(GL_n(\mathbb{R})/GL_k(\mathbb{R}) \times GL_{n-k}(\mathbb{R})\), which can be identified with the tangent bundle of \(O_n/O_k \times O_{n-k}\) (see Example 5.5(b)).

2 Notation and preliminaries

Let us begin this paper by giving some notation. Throughout this article, \(\mathbb{K}\) is either the real field \(\mathbb{R}\) or the complex field \(\mathbb{C}\). If \(X\) and \(Y\) are \(\mathbb{K}\)-Banach spaces, we denote by \(\mathcal{L}(X, Y)\) the Banach space of all continuous \(\mathbb{K}\)-linear operators from \(X\) to \(Y\). We will also denote \(\mathcal{L}(X) := \mathcal{L}(X, X)\). Moreover, the identity map in \(\mathcal{L}(X)\) will be denoted by \(I_X\), and sometimes by \(I\) if no confusion arises.

Unless specified otherwise, by \(\mathbb{K}\)-Banach manifolds, we mean \(\mathbb{K}\)-analytic Banach manifolds, in the sense of [12,31].

Throughout this article, \(\mathcal{G}(X)\) is the collection of all non-zero proper closed complemented subspaces (which are also called closed split subspaces) of \(X\). For any \(E \in \mathcal{G}(X)\), we denote \(F \triangleleft E\) if \(F\) is a closed complement of \(E\), and we put

\[\mathcal{C}_E := \{F \in \mathcal{G}(X) : F \triangleleft E\}\]
As in [31, pp. 44–46], we set
\[ p_{E,F} := q_{E,F}^{-1}, \]  
(2.2)
where \( q_{E,F} : \mathcal{L}(E, F) \rightarrow \mathcal{C}_F \) is the bijection given by
\[ q_{E,F}(T) := (I + T)(E) \quad (T \in \mathcal{L}(E, F)). \]  
(2.3)
There is a Hausdorff metrizable topology on \( \mathcal{G}(X) \) such that \( \mathcal{C}_F \) is an open subset of \( \mathcal{G}(X) \) and \( p_{E,F} : \mathcal{C}_F \rightarrow \mathcal{L}(E, F) \) is a homeomorphism. Moreover,
\[ \{(\mathcal{C}_{F_0}, p_{E_0,F_0}, \mathcal{L}(E_0, F_0)) : E_0, F_0 \in \mathcal{G}(X); F_0 \cap E_0 \} \]  
(2.4)
constitutes an analytic atlas for a \( \mathbb{K} \)-Banach manifold structure on \( \mathcal{G}(X) \). When equipped with this structure, \( \mathcal{G}(X) \) is called the Grassmann manifold of \( X \).

A subset \( A \subseteq X \) is called a \( \mathbb{K} \)-affine-Banach subspace if \( A - a_0 \) is a \( \mathbb{K} \)-Banach subspace of \( X \) for an element \( a_0 \in A \). Moreover, a map \( S \) from \( A \) to a \( \mathbb{K} \)-Banach space \( Y \) is said to be \( \mathbb{K} \)-affine if the assignment
\[ S^{a_0} : a - a_0 \mapsto S(a) - S(a_0) \]
is a \( \mathbb{K} \)-linear map on \( A - a_0 \). Note that in this case,
\[ a - b + c \in A \quad \text{and} \quad S(a - b + c) = S(a) - S(b) + S(c) \quad (a, b, c \in A). \]  
(2.5)
We say that \( A \) is isometrically affine isomorphic to \( Y \) if there is a \( \mathbb{K} \)-affine bijection \( T : A \rightarrow Y \) which preserves the metrics, i.e., \( \|T(a) - T(b)\|_Y = \|a - b\|_X \; (a, b \in A) \).

We denote by \( \mathcal{A}(A, B) \) the set of all continuous \( \mathbb{K} \)-affine maps from \( A \) to a \( \mathbb{K} \)-affine-Banach subspace \( B \) of \( Y \). It is clear that \( \mathcal{A}(A, B) \) is a \( \mathbb{K} \)-vector space. The function \( \| \cdot \|_{a_0} \) defined by
\[ \|T\|_{a_0} := \|T^{a_0}\| + \|T(a_0)\| \quad (T \in \mathcal{A}(A, B)) \]
is a complete norm on \( \mathcal{A}(A, B) \).

The following well-known fact ensures a default Banach space structure on \( \mathcal{A}(A, B) \) up to Banach space isomorphism.

**Lemma 2.1** For any \( a_0, a_1 \in A \), the two norms \( \| \cdot \|_{a_0} \) and \( \| \cdot \|_{a_1} \) are equivalent.

This paper mainly concerns with affine-Banach bundles. We need to consider such a general notion (instead of the more well-known notion of Banach bundles; see e.g., [27, p. 41]) because the set of idempotents naturally forms an affine-Banach bundle. Although the concept of affine-Banach bundles should be known to expertise, we will state its precise meaning in the following because we cannot find its explicit definition in the literature.
Definition 2.2 Let $\Omega$ and $\Upsilon$ be Hausdorff spaces. Let $\kappa : \Upsilon \to \Omega$ be a continuous surjection.

(a) Then $(\Upsilon, \Omega, \kappa)$ is called a **locally trivial continuous $K$-affine-Banach bundle** (respectively, **locally trivial continuous $K$-Banach bundle**) if the following conditions are satisfied:

- **(B1)** for each $\omega \in \Omega$, the subset $\Upsilon_\omega := \kappa^{-1}(\omega)$ is homeomorphic to a $K$-affine-Banach subspace of a $K$-Banach space (respectively, homeomorphic to a $K$-Banach space);
- **(B2)** for each $\omega_0 \in \Omega$, there exist an open neighborhood $V_{\omega_0} \subseteq \Omega$ of $\omega_0$ as well as a bi-continuous bijection $\Theta_{\omega_0} : V_{\omega_0} \times \Upsilon_{\omega_0} \to \kappa^{-1}(V_{\omega_0})$ such that $\Theta_{\omega_0}|_{\{\omega\} \times \Upsilon_{\omega_0}}$ is a $K$-affine map (respectively, $K$-linear map) onto $\Upsilon_{\omega_0}$, for every $\omega \in V_{\omega_0}$;
- **(B3)** for $\omega_1, \omega_2 \in \Omega$ with $V_{\omega_1} \cap V_{\omega_2} \neq \emptyset$, the map $\varphi : V_{\omega_1} \cap V_{\omega_2} \to A(\Upsilon_{\omega_1}, \Upsilon_{\omega_2})$ defined by

$$\varphi(\omega)(x) := \Pi_2 \left( \Theta_{\omega_2}^{-1} \circ \Theta_{\omega_1}(\omega, x) \right) \quad (\omega \in V_{\omega_1} \cap V_{\omega_2}, x \in \Upsilon_{\omega_1})$$

is continuous, where $\Pi_2$ is the projection onto the second coordinate.

(b) Suppose that $\Omega$ and $\Upsilon$ are $K$-Banach manifolds. Then $(\Upsilon, \Omega, \kappa)$ is called a **locally trivial $K$-analytic affine-Banach bundle** (respectively, **locally trivial $K$-analytic Banach bundle**), if $\kappa$ is $K$-analytic and the same requirements in part (a) hold with the terms “bi-continuous” and “continuous” in (B2) and (B3) being replaced by “$K$-bi-analytic” and “$K$-analytic”, respectively.

In the case of a Banach bundle, Condition (B3) is equivalent to the corresponding statement when $A(\Upsilon_{\omega_1}, \Upsilon_{\omega_2})$ is replaced by $L(\Upsilon_{\omega_1}, \Upsilon_{\omega_2})$ (because of Lemma 2.1).

We may occasionally use the term “locally trivial continuous affine-Banach bundle” and “locally trivial analytic affine-Banach bundle” etc, if the underlying field $K$ is understood.

A map $\rho$ from an open subset $V_0 \subseteq \Omega$ to $\Upsilon$ is called a **local cross section** if

$$\kappa(\rho(\omega)) = \omega \quad (\omega \in V_0).$$

In the case when $V_0 = \Omega$, we say that $\rho$ is a **global cross section**.

For a continuous (respectively, analytic) Banach bundle, the constant zero map is obviously a continuous (respectively, analytic) global cross section. The following proposition tells us that the only obstruction for an affine-Banach bundle to be a Banach bundle is the existence of global continuous (respectively, analytic) cross sections. This proposition should be considered as well-known. However, for the benefit of the reader, we include its proof here.

**Proposition 2.3** (a) **Let $(\Upsilon, \Omega, \kappa)$ be a locally trivial continuous affine-Banach bundle. If there is a continuous global cross section $\rho : \Omega \to \Upsilon$, then there exist**
Banach space structures on all the fibers of $\kappa$, which are isometrically affine isomorphic to the original affine-Banach space structures on the fibers, such that $(\Upsilon, \Omega, \kappa)$ becomes a locally trivial continuous Banach bundle. (b) Let $(\Upsilon, \Omega, \kappa)$ be a locally trivial analytic affine-Banach bundle. If there is an analytic global cross section $\rho : \Omega \to \Upsilon$, then $(\Upsilon, \Omega, \kappa)$ is a locally trivial analytic Banach bundle, under a Banach space structure on each fiber of $\kappa$ that is isometrically affine isomorphic to the original affine-Banach space structure on the fiber.

Proof (a) Fix $\omega_0 \in \Omega$. The affine-Banach space $\Upsilon_{\omega_0}$ becomes a Banach space when equipped with the following structure:

$$
\|x\|_{\omega_0} := \|x - \rho(\omega_0)\|,
$$

$$
\alpha \cdot_{\omega_0} x := \alpha x + (1 - \alpha)\rho(\omega_0) \quad \text{and} \quad x +_{\omega_0} y := x + y - \rho(\omega_0), \quad (2.6)
$$

where $x, y \in \Upsilon_{\omega_0}$, $\alpha \in \mathbb{K}$ and $\|\cdot\|$ is the norm on the Banach space containing $\Upsilon_{\omega_0}$.

Let $V_{\omega_0}$ and $\Theta_{\omega_0}$ be the maps in Definition 2.2(a). We denote by $\Pi_2 : V_{\omega_0} \times \Upsilon_{\omega_0} \to \Upsilon_{\omega_0}$ the projection onto the second coordinate. Set $\xi_{\omega_0} : V_{\omega_0} \to \Upsilon_{\omega_0}$ to be the continuous map $\Pi_2 \circ \Theta_{\omega_0}^{-1} \circ \rho|_{V_{\omega_0}}$. Then we have

$$
\Theta_{\omega_0}(\omega, \xi_{\omega_0}(\omega)) = \rho(\omega) \quad (\omega \in V_{\omega_0}). \quad (2.7)
$$

Define $\Psi_{\omega_0} : V_{\omega_0} \times \Upsilon_{\omega_0} \to \kappa^{-1}(V_{\omega_0})$ by

$$
\Psi_{\omega_0}(\omega, y) := \Theta_{\omega_0}(\omega, y +_{\omega_0} \xi_{\omega_0}(\omega)) \quad (\omega \in V_{\omega_0}; y \in \Upsilon_{\omega_0}).
$$

Clearly, $\Psi_{\omega_0}$ is continuous. Furthermore, we have

$$
\Psi_{\omega_0}(x) = \Theta_{\omega_0}(x) \oplus (\kappa(x), 2\rho(\omega_0) - \xi_{\omega_0}(\kappa(x))) \quad (x \in \kappa^{-1}(V_{\omega_0})),
$$

where $(\omega, a) \oplus (\omega, b) := (\omega, a +_{\omega_0} b) \quad (\omega \in V_{\omega_0}; a, b \in \Upsilon_{\omega_0})$. From this, and the continuity of $\kappa$, $\Theta_{\omega_0}$ and $\xi_{\omega_0}$, we know that $\Psi_{\omega_0}^{-1}$ is continuous. On the other hand, for a fixed $\omega \in V_{\omega_0}$, since the map $\Theta_{\omega_0}|_{\omega} \times \Upsilon_{\omega_0}$ is affine, it is additive when $\Upsilon_{\omega_0}$ and $\Upsilon$ are equipped with the additions given in (2.6) (see Relation (2.5)). Hence, Relations (2.5) and (2.7) imply that for any $y, z \in \Upsilon_{\omega_0}$ and $\alpha \in \mathbb{K}$, one has

$$
\Psi_{\omega_0}(\omega, y) +_{\omega_0} \Psi_{\omega_0}(\omega, z) = \Theta_{\omega_0}(\omega, y +_{\omega_0} \xi_{\omega_0}(\omega)) + \Theta_{\omega_0}(\omega, z +_{\omega_0} \xi_{\omega_0}(\omega)) - 2\rho(\omega_0)
$$

$$
= \Theta_{\omega_0}(\omega, y + z + \xi_{\omega_0}(\omega) - 2\rho(\omega_0)) = \Psi_{\omega_0}(\omega, y +_{\omega_0} z)
$$

as well as

$$
\alpha \cdot_{\omega_0} \Psi_{\omega_0}(\omega, y) = \alpha \Theta_{\omega_0}(\omega, y + \xi_{\omega_0}(\omega) - \rho(\omega_0)) + (1 - \alpha)\rho(\omega)
$$

$$
= \alpha \Theta_{\omega_0}(\omega, y) + (1 - \alpha)\Theta_{\omega_0}(\omega, \rho(\omega_0)) - \Theta_{\omega_0}(\omega, \rho(\omega_0)) + \Theta_{\omega_0}(\omega, \xi_{\omega_0}(\omega))
$$

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\[ \Theta_{\omega_0}(\omega, \alpha y + (1 - \alpha)\rho(\omega_0) - \rho(\omega_0) + \xi_{\omega_0}(\omega)) = \Psi_{\omega_0}(\omega, \alpha \bullet \omega_0 y). \]

In other words, \( \Psi_{\omega_0} \mid_{\{\omega\} \times \Upsilon_{\omega_0}} \) is linear, and Condition (B2) is verified.

In order to establish Condition (B3) in Definition 2.2(a), we consider \( \omega_1, \omega_2 \in \Omega \) with \( V_{\omega_1} \cap V_{\omega_2} \neq \emptyset \). Let us set two functions \( \xi : V_{\omega_1} \to A(\Upsilon_{\omega_1}, \Upsilon_{\omega_1}) \) and \( \eta : V_{\omega_2} \to A(\Upsilon_{\omega_2}, \Upsilon_{\omega_2}) \) by

\[
\xi(\omega)(x) := x + \omega_1 \xi_{\omega_1}(\omega) \quad \text{and} \quad \eta(\omega)(z) := z + \omega_2 (-1) \bullet_{\omega_2} \xi_{\omega_2}(\omega).
\]

Observe that for \( a \in \Upsilon_{\omega_1} \), if \( \chi_a \in A(\Upsilon_{\omega_1}, \Upsilon_{\omega_1}) \) is the map given by \( \chi_a(x) := x + \omega_1 a \), then \( \chi : a \mapsto \chi_a \) is continuous. Since \( \xi_{\omega_1} \) and \( \xi_{\omega_2} \) are continuous, we know that both \( \xi \) and \( \eta \), being the compositions of \( \xi_{\omega_1} \) with \( \chi \), are continuous. If \( \phi \) is the map in Condition (B3), then for any \( \omega \in V_{\omega_1} \cap V_{\omega_2} \) and \( x \in \Upsilon_{\omega_1} \), one has

\[
\Pi_2 \left( \Psi_{\omega_2}^{-1}(\Psi_{\omega_1}(\omega, x)) \right) = \phi(\omega) \left( x + \omega_1 \xi_{\omega_1}(\omega) \right) + \omega_2 (-1) \bullet_{\omega_2} \xi_{\omega_2}(\omega)
\]

\[
= (\eta(\omega) \circ \phi(\omega) \circ \xi(\omega))(x).
\]

Therefore, Condition (B3) holds when the maps \( \Theta_{\omega_1} \) and \( \Theta_{\omega_2} \) are replaced by \( \Psi_{\omega_1} \) and \( \Psi_{\omega_2} \), respectively.

(b) Note that under the respective assumption, \( \kappa \), \( \Theta_{\omega_0} \), \( \Theta_{\omega_0}^{-1} \), \( \Pi_2 \) as well as \( \rho \) are analytic. Thus, \( \xi_{\omega_0} \), \( \Psi_{\omega_0} \) and \( \Psi_{\omega_0}^{-1} \) are analytic. Observe also that \( a \mapsto \chi_a \) is analytic (as it is continuous and affine). Hence, the maps \( \xi \) and \( \eta \) in part (a) are analytic. The remainder of the proof is the same as that of part (a).

\[ \square \]

Remark 2.4 (a) Suppose that \((\Upsilon, \Omega, \kappa)\) is a locally trivial continuous affine-Banach bundle such that \( \Omega \) is paracompact. Then one will see from the discussion before Proposition 4.1 that there always exists a continuous global cross section. This means that when \( \Omega \) is paracompact, there is no difference between locally trivial continuous affine-Banach bundles over \( \Omega \) and locally trivial continuous Banach bundles over \( \Omega \).

(b) Note that the “\( K \)-analyticity” in the above means the existence of “local power series expansions” (as in Definition 1.6 and p. 36 of [31]). We say that a map from an open subset of a \( K \)-Banach space to another \( K \)-Banach space is \( K \)-differentiable if it is Fréchet differentiable. One can also defined the notion of \( K \)-differentiable affine-Banach bundles if one replaces the terms “bi-continuous” and “continuous” in Conditions (B2) and (B3) of Definition 2.2(a) by “\( K \)-bi-differentiable” and “\( K \)-differentiable”, respectively. The corresponding statement as Proposition 2.3 is also valid for \( K \)-differentiable affine-Banach bundles.

Let us end this section with the following obvious fact for later reference.

Lemma 2.5 If \( R \in \mathcal{L}(X) \) satisfying \( R^2 = 0 \), then \( I + R \) is invertible with inverse \( I - R \).
3 $\mathcal{I}(X)$ is a locally trivial analytic affine-Banach bundle

From now on, $X$ is a $\mathbb{K}$-Banach space with $\dim_\mathbb{K} X > 1$ (could be infinite). Our main concern is the following set

$$\mathcal{I}(X) := \left\{ Q \in \mathcal{L}(X) \setminus \{0, I\} : Q^2 = Q \right\},$$

equipped with the topology induced from $\mathcal{L}(X)$. We will show that it is actually a Banach submanifold of $\mathcal{L}(X)$ is an analytic affine Banach bundle over $\mathcal{G}(X)$ (see Theorem 3.4 below).

For any $Q \in \mathcal{I}(X)$, both $Q(X)$ and $ker Q$ belongs to $\mathcal{G}(X)$, and $Q(X) \cap ker Q$ (i.e., they are complement of each other). Conversely, if $E \in \mathcal{G}(X)$ and $F \in \mathcal{C}_E$ (see (2.1)), there is a unique element $Q_E^F \in \mathcal{I}(X)$ with

$$Q_E^F(X) = E \quad \text{and} \quad ker Q_E^F = F.$$

For every $F_0 \in \mathcal{G}(X)$, let us denote

$$\mathcal{I}(X)^{F_0} := \{ Q \in \mathcal{I}(X) : ker Q = F_0 \}, \quad \mathcal{I}(X)_{F_0} := \{ Q \in \mathcal{I}(X) : Q(X) = F_0 \}$$
as well as

$$\mathcal{L}^{F_0}(X, F_0) := \{ T \in \mathcal{L}(X) : T(X) \subseteq F_0 \text{ and } T(F_0) = \{0\} \}.$$

The starting point of this paper is the following easy observation.

**Lemma 3.1** Let $X$ be a Banach space and $E_0, F_0 \in \mathcal{G}(X)$ with $E_0 \cap F_0$. Then

$$\mathcal{I}(X)^{F_0} = \mathcal{L}^{F_0}(X, F_0) + Q_{E_0}^{F_0}, \quad \text{and} \quad \mathcal{I}(X)_{F_0} = \mathcal{L}^{F_0}(X, F_0) + Q_{E_0}^{F_0}.$$

**Proof** By considering the bijection $Q \mapsto I - Q$ from $\mathcal{I}(X)^{F_0}$ onto $\mathcal{I}(X)_{F_0}$, one only needs to verify the first equality. In fact, if $E \in \mathcal{C}_{F_0}$ (see (2.1)), then

$$\left( Q_E^{F_0} - Q_{E_0}^{F_0} \right)(x) = Q_E^{F_0}(x) - x = -Q_{E_0}^{F_0}(x) = F_0 \quad (x \in E_0).$$

From this, we obtain $\left( Q_E^{F_0} - Q_{E_0}^{F_0} \right)(x) \subseteq F_0$, and hence $Q_E^{F_0} - Q_{E_0}^{F_0} \in \mathcal{L}^{F_0}(X, F_0)$.

Conversely, consider an element $R \in \mathcal{L}^{F_0}(X, F_0)$. For any $x \in E_0$, as $(I + R)(E_0) \in \mathcal{C}_{F_0}$, one has $Q_{(I + R)(E_0)}^{F_0}(x) = Q_{(I + R)(E_0)}^{F_0}(x + R(x)) = x + R(x)$, which gives

$$R(x) = Q_{(I + R)(E_0)}^{F_0}(x) - Q_{E_0}^{F_0}(x). \quad (3.1)$$

Furthermore, as $R$, $Q_{(I + R)(E_0)}^{F_0}$ and $Q_{E_0}^{F_0}$ all vanish on $F_0$, we know that $R = Q_{(I + R)(E_0)}^{F_0} - Q_{E_0}^{F_0}$. \qed
Consider \( F_0 \in C_{E_0} \) and \( E_0, E_1 \in C_{F_0} \). We have

\[
Q^{F_0}_{E_1}(E_0) = Q^{F_0}_{E_1}(E_0 + F_0) = E_1.
\]  

(3.2)

Lemma 3.1 allows us to define a map \( \pi_{E_0, F_0} \) from \( C_{F_0} \) onto \( \mathcal{L}^{F_0}(X, F_0) \) via

\[
\pi_{E_0, F_0}(E) := Q^{F_0}_E - Q^{F_0}_{E_0} \quad (E \in C_{F_0}).
\]  

(3.3)

Set \( T := p_{E_0, F_0}(E_1) \in \mathcal{L}(E_0, F_0) \) (see (2.2)). Then \( E_1 = (I + T)(E_0) \) (see (2.3)). Thus, applying (3.1) to \( R := T \circ Q^{F_0}_{E_0} \in \mathcal{L}^{F_0}(X, F_0) \), we get \( T(x) = Q^{F_0}_{E_1}(x) - Q^{F_0}_{E_0}(x) \) (\( x \in E_0 \)). In other words,

\[
p_{E_0, F_0} = \Lambda_{E_0, F_0} \circ \pi_{E_0, F_0},
\]  

(3.4)

where

\[
\Lambda_{E_0, F_0} : \mathcal{L}^{F_0}(X, F_0) \to \mathcal{L}(E_0, F_0)
\]  

(3.5)

is the Banach space isomorphism given by restrictions. Notice that the inverse of \( \Lambda_{E_0, F_0} \) is given by compositions of elements in \( \mathcal{L}(E_0, F_0) \) with \( Q^{F_0}_{E_0} \) (considered as an operator from \( X \) onto \( E_0 \)). Furthermore, (3.3) implies

\[
\pi_{E_0, F_0}(E_1) = -\pi_{E_1, F_0}(E_0).
\]  

(3.6)

On the other hand, by Relation (3.4), the analytic atlas of \( \mathcal{G}(X) \) in (2.4) can be rewritten as:

\[
\left\{ \left( C_{F_0}, \pi_{E_0, F_0}, \mathcal{L}^{F_0}(X, F_0) \right) : E_0, F_0 \in \mathcal{G}(X); F_0 \supset E_0 \right\}.
\]  

(3.7)

One good point of this atlas is that elements in \( \mathcal{L}^{F_0}(X, F_0) \) are nilpotent operators of degree two, and Lemma 2.5 applies to them.

Another benefit of this atlas is the following result, which gives a clear picture of the topology on \( \mathcal{G}(X) \). In particular, we know that a sequence \( \{E_k\}_{k \in \mathbb{N}} \) converges to \( E_0 \) is basically the same as \( Q^{F_0}_{E_k}(x) \) converges to \( Q^{F_0}_{E_0}(x) \) in a uniform way on all bounded subsets, for a suitable choice of \( F_0 \). As mentioned in the Introduction, the topology on \( \mathcal{G}(X) \) is metrizable. Therefore, this corollary actually gives an alternative description of the topology on \( \mathcal{G}(X) \).

**Corollary 3.2** Suppose that \( \{E_k\}_{k \in \mathbb{N}} \) is a sequence in \( \mathcal{G}(X) \) and \( E_0 \in \mathcal{G}(X) \). Then \( E_k \to E_0 \) if and only if for every \( F_0 \in C_{E_0} \) (equivalently, there exists \( F_0 \in C_{E_0} \)), there exists \( k_0 \in \mathbb{N} \) such that \( E_k \in C_{F_0} \) when \( k \geq k_0 \) and that \( \| Q^{F_0}_{E_k} - Q^{F_0}_{E_0} \| \to 0 \).

Let \( \kappa : \mathcal{I}(X) \to \mathcal{G}(X) \) be the surjection as given in (1.1). Lemma 3.1 tells us that the fiber \( \kappa^{-1}(E) \) (which coincides with \( \mathcal{I}(X)_E \)) is an affine-Banach subspace of \( \mathcal{L}(X) \), for every \( E \in \mathcal{G}(X) \). The main result of this section is that under these affine-Banach
space structures on the fibers and the Banach submanifold structure on \( \mathcal{I}(X) \) induced from \( \mathcal{L}(X), (\mathcal{I}(X), \mathcal{G}(X), \kappa) \) is a locally trivial analytic affine-Banach bundle.

For the proof of this statement, we need to consider the canonical actions of \( G\mathcal{L}(X) \) on \( \mathcal{I}(X) \) and on \( \mathcal{G}(X) \). Indeed, for any \( W \in G\mathcal{L}(X) \) and \( E, F \in \mathcal{G}(X) \) with \( F^\top E \), one easily sees that
\[
\text{Ad}(W)(Q_E^F) := W \circ Q_E^F \circ W^{-1} = Q_{W(E)},
\]
and this produces a \( \mathbb{K} \)-analytic action of \( G\mathcal{L}(X) \) on \( \mathcal{I}(X) \). On the other hand,
\[
\alpha(W, E) := W(E)
\]
induces a \( \mathbb{K} \)-analytic action of \( G\mathcal{L}(X) \) on \( \mathcal{G}(X) \).

We also need the following easy fact for the proof of the main theorem.

**Lemma 3.3** Let \( Y \) be a \( \mathbb{K} \)-Banach space, and let \( A, B \subseteq Y \) be two \( \mathbb{K} \)-affine-Banach subspaces. Suppose that \( B - b_0 \in \mathcal{G}(Y) \) for an element \( b_0 \in B \). Then there is a continuous affine map \( \Gamma : \mathcal{L}(Y) \to \mathcal{A}(A, B) \) such that whenever \( T \in \mathcal{L}(Y) \) satisfying \( T(A) \subseteq B \), one has \( \Gamma(T) = T|_A \).

In fact, let us pick an element \( D \in C_{B-b_0} \), and define a map \( \tilde{Q} \in \mathcal{A}(Y, B) \) by
\[
\tilde{Q}(y) := Q_{D-b_0}^D(y - b_0) + b_0 \quad (y \in Y).
\]
If we set \( \Gamma(T) := \tilde{Q} \circ T|_A \quad (T \in \mathcal{L}(Y)) \), then clearly, \( \Gamma \) is a continuous affine map satisfying the requirement.

A final piece of well-known information that we need is the following. For \( E, F \in \mathcal{G}(X) \) with \( F^\top E \), one has \( \mathcal{L}^E(X, E) \cap \mathcal{L}^F(X, F) = \{0\} \). We will identify \( \mathcal{L}^E(X, E) \oplus \mathcal{L}^F(X, F) \) with the sum of the two subspaces in \( \mathcal{L}(X) \). If we define a map \( \Delta_{E,F} : \mathcal{L}(X) \to \mathcal{L}(X) \) by
\[
\Delta_{E,F}(T) := Q_E^F \circ T \circ Q_F^E \quad (T \in \mathcal{L}(X)),
\]
then both \( \Delta_{E,F} \) and \( \Delta_{E,F} + \Delta_{F,E} \) are continuous idempotents with
\[
\Delta_{E,F}(\mathcal{L}(X)) = \mathcal{L}^E(X, E) \quad \text{and} \quad (\Delta_{E,F} + \Delta_{F,E})(\mathcal{L}(X)) = \mathcal{L}^E(X, E) \oplus \mathcal{L}^F(X, F).
\]
Consequently, both \( \mathcal{L}^E(X, E) \) and \( \mathcal{L}^E(X, E) \oplus \mathcal{L}^F(X, F) \) are closed complemented subspaces of \( \mathcal{L}(X) \). In the following, we will write elements in \( \mathcal{L}^E(X, E) \oplus \mathcal{L}^F(X, F) \) in either the form \( (R, S) \) or \( R + S \).

Before presenting the main theorem of this section, let us first give an outline of its proof, and give some remarks.

First of all, it is known that \( \kappa \) is continuous (e.g., this can be shown using some results in [5]), but we will give a self-contained argument here. Using this, we will show that \( (\mathcal{I}(X), \mathcal{G}(X), \kappa) \) satisfies Conditions (B1) and (B2) of Definition 2.2(a) (i.e., the continuous case). Next, we construct an analytic atlas for \( \mathcal{I}(X) \) (see (3.19)). We will then verify that \( \mathcal{I}(X) \), under the Banach manifold structure induced from
the atlas in (3.19), is a Banach submanifold of \( \mathcal{L}(X) \), by showing that the canonical inclusion is an analytic immersion.

As said in the Introduction, it is probably well-known that \( \mathcal{I}(X) \) is a submanifold of \( \mathcal{L}(X) \). However, in order to verify that \( \mathcal{I}(X) \) is a locally trivial analytic affine-Banach bundle over \( \mathcal{G}(X) \), we need to use the explicitly atlas for \( \mathcal{I}(X) \) in (3.19). This atlas will also be needed in the later part of this article.

Finally, we will establish Conditions (B2) and (B3) of Definition 2.2(b) (i.e., the analytic case). Observe that if the fiber over each point in \( \mathcal{G}(X) \) were a Banach space, then one might use Proposition 1.2 in Chapter 3 of [27] to simplify the argument. However, since we are in the affine-Banach setting, we give a more direct argument here.

**Theorem 3.4** Let \( X \) be a \( \mathbb{K} \)-Banach space. Then \( \mathcal{I}(X) \) is a \( \mathbb{K} \)-analytic Banach submanifold of \( \mathcal{L}(X) \). Moreover, when equipped with this Banach manifold structure, \((\mathcal{I}(X), \mathcal{G}(X), \kappa)\) is a locally trivial \( \mathbb{K} \)-analytic affine-Banach bundle such that for every \( E \in \mathcal{G}(X) \), the affine-Banach space structure on \( \kappa^{-1}(E) \) is the one induced from \( \mathcal{L}(X) \).

**Proof** Let us first establish the continuity of \( \kappa \). For this, we consider a sequence \( \{Q_{E_n}^{F_n}\}_{n \in \mathbb{N}} \) in \( \mathcal{I}(X) \) converging to \( Q_{E_0}^{F_0} \in \mathcal{I}(X) \). This gives

\[
\| I - (Q_{E_n}^{F_n} + Q_{E_0}^{F_0}) \| \| Q_{E_n}^{F_n} + Q_{E_0}^{F_0} \| - I \| = \| Q_{E_n}^{F_n} - Q_{E_0}^{F_0} \| \to 0.
\]

As \( \mathcal{L}(X) \) is a unital Banach algebra, we know that \( Q_{E_n}^{F_n} + Q_{E_0}^{F_0} \) and \( Q_{E_n}^{F_n} + Q_{E_0}^{F_0} \) are eventually invertible, and we may assume that they are invertible for all \( n \in \mathbb{N} \). The relation \( (Q_{E_n}^{F_n} + Q_{E_0}^{F_0})(X) = X \) and \( \ker (Q_{E_n}^{F_n} + Q_{E_0}^{F_0}) = \{0\} \) will then imply that \( E_n \in \mathcal{C}_{F_0} \).

By Corollary 3.2, we need to show that \( Q_{E_n}^{F_0} \to Q_{E_0}^{F_0} \). Indeed, it is clear that

\[
Q_{E_n}^{F_0} \circ Q_{E_n}^{F_0} = Q_{E_n}^{F_0} \quad \text{and} \quad Q_{E_0}^{F_0} \circ Q_{E_n}^{F_0} = Q_{E_0}^{F_0}.
\]

Moreover, one has \( \| Q_{E_n}^{F_0} - Q_{E_0}^{F_0} \| \leq 1/2 \) when \( n \) is large. In this case,

\[
\| Q_{E_n}^{F_0}(x) \| = \| Q_{E_n}^{F_0}(Q_{E_n}^{F_0}(x)) \| \leq \| Q_{E_0}^{F_0}(Q_{E_n}^{F_0}(x)) \| + \| Q_{E_n}^{F_0}(x) \|/2 \quad (x \in X),
\]

which implies \( \| Q_{E_n}^{F_0} \| \leq 2 \| Q_{E_0}^{F_0} \| \). It follows that

\[
\lambda_0 := \sup_{n \in \mathbb{N}} \| Q_{E_n}^{F_0} \| < \infty.
\]

Now, for any \( \epsilon > 0 \), there is \( n_0 \) such that \( \| Q_{E_n}^{F_0} - Q_{E_0}^{F_0} \| < \epsilon \) whenever \( n \geq n_0 \). Therefore, we obtain from \( Q_{E_0}^{F_0} \circ Q_{E_n}^{F_0} = 0 \) and Relation (3.12) that if \( n \geq n_0 \), then

\[
\| Q_{E_n}^{F_0}(x) - Q_{E_n}^{F_0}(x) \| = \| Q_{E_n}^{F_0}(Q_{E_0}^{F_0}(x)) - Q_{E_0}^{F_0}(Q_{E_n}^{F_0}(x)) \| < \epsilon \lambda_0 \| x \| \quad (x \in X).
\]
Hence, we have \( \| Q_{E_n}^{F_0} - Q_{E_n}^{F_0} \| \to 0 \). This, together with \( \| Q_{E_n}^{F_0} - Q_{E_0}^{F_0} \| \to 0 \), implies the required convergence, and thus \( \kappa \) is continuous.

Secondly, it follows from Lemma 3.1 that \( \kappa^{-1}(E) \) is an affine-Banach subspace of \( \mathcal{L}(X) \), for each \( E \in \mathcal{G}(X) \); in other words, Condition (B1) in Definition 2.2(a) is satisfied.

Let us now show that \( (\mathcal{I}(X), \mathcal{G}(X), \kappa) \) satisfies Condition (B2) of Definition 2.2(a). We will do this via the construction of an analytic (and hence continuous) local right inverse for the evaluation maps from \( \mathcal{G}(X) \) to orbits of the action \( \alpha \) in (3.9). For this, we fix arbitrary elements \( E_0, F_0 \in \mathcal{G}(X) \) with \( E_0 \supset F_0 \). Consider \( E \in \mathcal{C}_{F_0} \). Since \( Q_{E}^{F_0} - Q_{E_0}^{F_0} \in \mathcal{L}^{F_0}(X, F_0) \) (see Lemma 3.1), we know from Lemma 2.5 that if we set

\[
\Xi_{E_0, F_0}(E) := Q_{E}^{F_0} + Q_{E_0}^{F_0},
\]

then \( \Xi_{E_0, F_0}(E) \in G\mathcal{L}(X) \) and

\[
\Xi_{E_0, F_0}(E)^{-1} = Q_{F_0}^{E} + Q_{E_0}^{F_0} = 2I - \Xi_{E_0, F_0}(E).
\]

Recall that \( (\mathcal{C}_{F_0}, \pi_{E_0, F_0}, \mathcal{L}^{F_0}(X, F_0)) \) is a local chart for \( \mathcal{G}(X) \) near \( E_0 \) (see (3.7)). As

\[
\Xi_{E_0, F_0}(\pi_{E_0, F_0}^{-1}(R)) = I + R \quad (R \in \mathcal{L}^{F_0}(X, F_0)),
\]

the map \( \Xi_{E_0, F_0} : \mathcal{C}_{F_0} \to G\mathcal{L}(X) \) is analytic. Moreover, one has

\[
\alpha(\Xi_{E_0, F_0}(E), E_0) = Q_{E}^{F_0}(E_0) = Q_{F_0}^{E_0}(X) = E.
\]

Consequently, \( \Xi_{E_0, F_0} \) is an analytic local right inverse for the evaluation map at \( E_0 \) from \( G\mathcal{L}(X) \) to the orbit \( \alpha(G\mathcal{L}(X), E_0) \).

We now define \( \Theta_{E_0, F_0} : \mathcal{C}_{F_0} \times \mathcal{I}(X)_{E_0} \to \kappa^{-1}(\mathcal{C}_{F_0}) \) by

\[
\Theta_{E_0, F_0}(E, Q) := \text{Ad}(\Xi_{E_0, F_0}(E))(Q) \quad (E \in \mathcal{C}_{F_0}, Q \in \mathcal{I}(X)_{E_0}).
\]

For \( E \in \mathcal{C}_{F_0} \) and \( F \in \mathcal{C}_{E_0} \), we know from (3.8) and (3.16) that

\[
\Theta_{E_0, F_0}(E, Q_{E_0}^{F}) = Q_{E_0, F_0}^{E}(E, Q) = Q_{E}^{Q_{E_0}^{F}(E)}(Q_{E_0}^{F})(E) = Q_{E_0, F_0}^{Q_{E_0}^{F}(E)}(Q_{E_0}^{F})(E).
\]

This implies that \( \Theta_{E_0, F_0} \) is well-defined and injective (since \( \Xi_{E_0, F_0}(E) \) is invertible). It is clear from the definition that \( \Theta_{E_0, F_0} \) is fiberwise affine. Furthermore, for any \( E' \in \mathcal{C}_{F_0} \) and \( F' \in \mathcal{C}_{E_0} \), if we set \( F'' := \Xi_{E_0, F_0}(E')^{-1}(F') \), then \( F'' \in \mathcal{C}_{E_0} \) because \( \text{Ad}(\Xi_{E_0, F_0}(E')^{-1}(Q_{E}^{F}))(Q_{E}^{F}) = Q_{E}^{F''} \), and so, \( \Theta_{E_0, F_0}(E', Q_{E_0}^{F'}) = Q_{E_0}^{F''} \). This means that \( \Theta_{E_0, F_0} \) is surjective.

In the following, we establish the bi-continuity of \( \Theta_{E_0, F_0} \). Let \( \{F_n\}_{n \in \mathbb{N}} \) and \( \{E_n\}_{n \in \mathbb{N}} \) be sequences in \( \mathcal{C}_{E_0} \) and in \( \mathcal{C}_{F_0} \), respectively. If \( \{E_n\}_{n \in \mathbb{N}} \) converges to \( E \in \mathcal{C}_{F_0} \) and
\( \{ Q^F_{E_n} \}_{n \in \mathbb{N}} \) converges to \( Q^F_{E_0} \in \mathcal{I}(X)_{E_0} \), then it follows from Relation (3.14) that
\[
\Theta_{E_0,F_0}(E_n, Q^F_{E_n}) = \Xi_{E_0,F_0}(E_n) Q^F_{E_n} (2I - \Xi_{E_0,F_0}(E_n)) \rightarrow \Theta_{E_0,F_0}(E, Q^F_{E_0}).
\]

Conversely, assume that \( \Theta_{E_0,F_0}(E_n, Q^F_{E_n}) \rightarrow \Theta_{E_0,F_0}(E, Q^F_{E_0}) \). The continuity of \( \kappa \) and Relation (3.18) give \( E_n \rightarrow E \) and \( \Xi_{E_0,F_0}(E_n)(F_n) \rightarrow \Xi_{E_0,F_0}(E)(F) \). Consequently,
\[
F_n = (2I - \Xi_{E_0,F_0}(E_n))(\Xi_{E_0,F_0}(E_n)(F_n)) \rightarrow F.
\]

From this, we know that \( Q^F_{E_0} \rightarrow Q^F_{E_0} \).

We are now ready to construct a Banach manifold structure on \( \mathcal{I}(X) \) that is compatible with the norm topology. For every \( E_0 \in \mathcal{G}(X) \) and \( F_0 \in \mathcal{C}_{E_0} \), we consider the bijection \( \phi_{E_0,F_0} : \mathcal{I}(X)_{E_0} \rightarrow \mathcal{L}^{E_0}(X, E_0) \) induced by Lemma 3.1; namely,
\[
\phi_{E_0,F_0}(Q) := Q - Q^F_{E_0} (Q \in \mathcal{I}(X)_{E_0}).
\]

Set \( \mu_{E_0,F_0} : \kappa^{-1}(\mathcal{C}_{F_0}) \rightarrow \mathcal{L}^{F_0}(X, F_0) \oplus \mathcal{L}^{E_0}(X, E_0) \) to be the map \( (\pi_{E_0,F_0} \times \phi_{E_0,F_0}) \circ \Theta_{E_0,F_0}^{-1} \). We claim that
\[
\left\{ \left( \kappa^{-1}(\mathcal{C}_{F_0}), \mu_{E_0,F_0}, \mathcal{L}^{F_0}(X, F_0) \oplus \mathcal{L}^{E_0}(X, E_0) \right) : E_0, F_0 \in \mathcal{G}(X); F_0 \cap E_0 \right\}
\]
is an analytic atlas for \( \mathcal{I}(X) \).

In fact, \( \mu_{E_0,F_0} \) is a homeomorphism since \( \pi_{E_0,F_0}, \phi_{E_0,F_0} \) and \( \Theta_{E_0,F_0} \) are homeomorphisms. Notice also that if \( E \in \mathcal{C}_{F_0} \) and \( F \in \mathcal{C}_E \), then
\[
\mu_{E_0,F_0}(Q^F_E) = (Q^F_{E_0} - Q^F_{E_0}) \text{Ad}(Q^F_{F_0} + Q^F_{E_0})(Q^F_E) - Q^F_{E_0})
\]
\[
= (Q^F_{E_0} - Q^F_{E_0}) Q^F_{E_0} - Q^F_{E_0} Q^F_{E_0},
\]
by (3.2), (3.18) and the fact that \( Q^F_{E_0} Q^F_E Q^F_{E_0} = Q^F_{E_0} \). Moreover, for \( (R, S) \in \mathcal{L}^{F_0}(X, F_0) \oplus \mathcal{L}^{E_0}(X, E_0) \), one has
\[
\mu_{E_0,F_0}^{-1}(R, S) = (I + R)(S + Q^F_{E_0})(I - R)
\]
\[
= S - SR + Q^F_{E_0} + RS - RSR + R,
\]
because of (3.15), Lemma 2.5 as well as the facts that \( R(X) \subseteq F_0 \) and \( R Q^F_{E_0} = R \).

Assume now that \( E_1, F_1 \in \mathcal{G}(X) \) with \( E_1 \cap F_1 \) such that \( \kappa^{-1}(\mathcal{C}_{F_0}) \cap \kappa^{-1}(\mathcal{C}_{F_1}) \neq \emptyset \). Consider an arbitrary element \( (R, S) \in \mu_{E_0,F_0}(\kappa^{-1}(\mathcal{C}_{F_0} \cap \mathcal{C}_{F_1})) \). By Lemma 3.1, there exists a unique element \( E_R \in \mathcal{C}_{F_0} \) with \( R = Q^F_{E_R} - Q^F_{E_0} \). These produce, via (3.20)
as well as (3.22),
\[
\mu_{E_1,F_1}(\mu_{E_0,F_0}^{-1}(R, S)) = \left( Q_{F_1}^E - Q_{E_1}^E \right) \text{Ad}(Q_{F_1}^{E_1} + Q_{E_1}^{F_1}) \left( \text{Ad}(I + R)(S + Q_{E_0}^{F_0}) - Q_{E_1}^{F_1} \right).
\]

Since \(\{(C_F, \pi_{E,F}, \mathcal{F}(X,F)) : E, F \in \mathcal{G}(X); F \notin E\}\) is an analytic atlas of \(\mathcal{G}(X)\), the map from \(\pi_{E_0,F_0}(C_{F_0} \cap C_{F_1})\) to \(\pi_{E_1,F_1}(C_{F_0} \cap C_{F_1})\) that sends \(Q_{F_0}^{E_0} - Q_{E_0}^{F_0}\) to \(Q_{E_1}^{F_1} - Q_{E_1}^{E_1}\) is analytic. If we set
\[
\Phi(R) := Q_{E_1}^{F_1} - Q_{E_1}^{E_1},
\]
then \((R, S) \mapsto \Phi(R)\) is an analytic map from \(\pi_{E_0,F_0}(C_{F_0} \cap C_{F_1}) \times \mathcal{L}^{E_0}(X, E_0)\) to \(\pi_{E_1,F_1}(C_{F_0} \cap C_{F_1})\). Thus, the assignment
\[
(R, S) \mapsto (I - \Phi(R))(I + R)(S + Q_{E_0}^{F_0})(I - R)(I + \Phi(R)) - Q_{E_1}^{F_1}
\]
is also analytic. Consequently, \(\mu_{E_1,F_1} \circ \mu_{E_0,F_0}^{-1}\) is analytic on \(\mu_{E_0,F_0}(C_{F_0} \cap C_{F_1})\), and (3.19) is an analytic atlas for \(\mathcal{I}(X)\).

Next, we will show that, when equipped with the above manifold structure, \(\mathcal{I}(X)\) is a Banach submanifold of \(\mathcal{L}(X)\), by verifying that the inclusion map \(\iota: \mathcal{I}(X) \to \mathcal{L}(X)\) is an analytic immersion. Let us fix \(E_0 \in \mathcal{G}(X)\) and \(F_0 \in \mathcal{E}_0\). Set \(\theta_{E_0,F_0}\) to be the map \(\iota \circ \mu_{E_0,F_0}^{-1}: \mathcal{L}^{F_0}(X,F_0) \oplus \mathcal{L}^{E_0}(X,E_0) \to \mathcal{L}(X)\). By (3.23), one has
\[
\theta_{E_0,F_0}(R, S) = S - SR + Q_{E_0}^{F_0} + RS - RSR + R.
\]
Hence, \(\iota\) is analytic. Consider \(T(\iota): T(\mathcal{I}(X)) \to T(\mathcal{L}(X))\) to be the tangent map associated to \(\iota\). We need to show that the map
\[
T_{Q_{E_0}^{F_0}}(\iota) = \theta'_{E_0,F_0}(0, 0)
\]
(here, \(\theta'_{E_0,F_0}\) is the derivative of \(\theta_{E_0,F_0}\)) will send \(\mathcal{L}^{F_0}(X,F_0) \oplus \mathcal{L}^{E_0}(X,E_0)\) bijectively onto a closed complemented subspace of \(\mathcal{L}(X)\).

As \(\mathcal{L}^{F_0}(X,F_0) \oplus \mathcal{L}^{E_0}(X,E_0)\) is already a closed complemented subspace of \(\mathcal{L}(X)\), this claim is established if one can show that \(\theta'_{E_0,F_0}(0, 0)\) is the inclusion map. To see this, we observe that for every \(\epsilon \in (0, 1)\) and \((R, S) \in \mathcal{L}^{F_0}(X,F_0) \oplus \mathcal{L}^{E_0}(X,E_0)\), with \(\|R\| + \|S\| < \epsilon\), one has, via (3.24),
\[
\|\theta_{E_0,F_0}(R, S) - \theta_{E_0,F_0}(0, 0) - (R + S)\| = \|RS - SR - RSR\| < \epsilon^2(2 + \epsilon).
\]
This gives \(\theta'_{E_0,F_0}(0, 0)(R, S) = R + S\), as required.

Finally, we will establish that \((\mathcal{I}(X), \mathcal{G}(X), \kappa)\) is a locally trivial \(\mathbb{K}\)-analytic affine-Banach bundle. Indeed, as \(\pi_{E_0,F_0} \circ \kappa \circ \mu_{E_0,F_0}^{-1}(R, S) = R\), we know that \(\kappa: \mathcal{I}(X) \to \mathcal{L}(X)\)
\( \mathcal{G}(X) \) is analytic. Moreover, since the analytic atlas in (3.19) is defined through the map \( \Theta_{E_0,F_0} \) as well as the bi-analytic maps \( \pi_{E_0,F_0} \) and \( \phi_{E_0,F_0} \), it is a tautology that \( \Theta_{E_0,F_0} \) is bi-analytic; i.e., Condition (B2) of Definition 2.2(b) holds.

Suppose that \( \varphi \) is the map in Condition (B3) for \( \Theta_{E_1,F_1}^{-1} \circ \Theta_{E_0,F_0} \). It follows from (3.15) and Lemma 2.5 that for any \( E \in \mathcal{C}_{F_0} \cap \mathcal{C}_{F_1} \) and \( Q \in \mathcal{I}(X)_{E_0} \), one has

\[
\varphi(E)(Q) = \text{Ad}(\Xi_{E_1,F_1}(E)^{-1}) \circ \text{Ad}(\Xi_{E_0,F_0}(E))(Q)
= (I - \pi_{E_1,F_1}(E))(I + \pi_{E_0,F_0}(E))Q(I - \pi_{E_0,F_0}(E))(I + \pi_{E_1,F_1}(E)).
\]

Let us define \( \psi : \mathcal{C}_{F_0} \cap \mathcal{C}_{F_1} \rightarrow \mathcal{L}(\mathcal{L}(X)) \) by

\[
\psi(E)(T) = \text{Ad}(I - \pi_{E_1,F_1}(E)) \circ \text{Ad}(I + \pi_{E_0,F_0}(E))(T) \quad (T \in \mathcal{L}(X)).
\]

Since both \( \pi_{E_0,F_0} \) and \( \pi_{E_1,F_1} \) are analytic, we know that the map \( \psi \) is analytic (notice that when \( A \) is a Banach algebra, \( a \mapsto L_a \) and \( a \mapsto R_a \) are bounded linear maps from \( A \) to \( \mathcal{L}(A) \) and hence are analytic, where \( L_a(x) := ax \) and \( R_a(x) := xa \)). On the other hand, as \( \mathcal{L}(X) \in \mathcal{G}(\mathcal{L}(X)) \), we obtain a continuous affine map \( \Gamma : \mathcal{L}(\mathcal{L}(X)) \rightarrow \mathcal{A}(\mathcal{I}(X)_{E_0}, \mathcal{I}(X)_{E_1}) \) satisfying the condition in Lemma 3.3. It follows that

\[
\Gamma(\psi(E)) = \psi(E)|_{\mathcal{I}(X)_{E_0}} = \varphi(E) \quad (E \in \mathcal{C}_{F_0} \cap \mathcal{C}_{F_1}).
\]

Since \( \Gamma \circ \psi \) is an analytic map from \( \mathcal{C}_{F_0} \cap \mathcal{C}_{F_1} \) to \( \mathcal{A}(\mathcal{I}(X)_{E_0}, \mathcal{I}(X)_{E_1}) \), Condition (B3) of Definition 2.2(b) is satisfied.

Example 3.5 We equip \( \mathbb{C}^2 \) with the usual Euclidean norm. For \( \lambda \in \mathbb{C} \), we set \( E_\lambda := \left\{ \begin{bmatrix} a \\ \lambda a \end{bmatrix} : a \in \mathbb{C} \right\} \) and \( E_\infty := \left\{ \begin{bmatrix} 0 \\ b \end{bmatrix} : b \in \mathbb{C} \right\} \). Then \( \mathcal{G}(\mathbb{C}^2) = \left\{ E_\lambda : \lambda \in \mathbb{C} \cup \{ \infty \} \right\} \) and

\[
\mathcal{I}(\mathbb{C}^2) = \left\{ \begin{bmatrix} 1 & \gamma \\ 0 & 0 \end{bmatrix} : \gamma \in \mathbb{C} \right\} \cup \left\{ \begin{bmatrix} 1 - \alpha \\ \alpha/\lambda(1-\alpha) \\ \alpha \end{bmatrix} : \alpha, \lambda \in \mathbb{C}; \lambda \neq 0 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ \delta & 1 \end{bmatrix} : \delta \in \mathbb{C} \right\}.
\]

Notice that \( E_0 \uparrow E_\infty, Q_{E_0}^{E_\infty} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( Q_{E_\infty}^{E_0} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \). Moreover, for each \( \lambda \in \mathbb{C} \setminus \{0\} \), we have \( E_\lambda \uparrow E_\infty, E_\lambda \uparrow E_0, Q_{E_\lambda}^{E_\infty} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \) as well as \( Q_{E_\lambda}^{E_0} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \).

On the other hand, \( \mathcal{I}(\mathbb{C}^2)_{E_0} = \left\{ \begin{bmatrix} 1 & \gamma \\ 0 & 0 \end{bmatrix} : \gamma \in \mathbb{C} \right\}, \mathcal{I}(\mathbb{C}^2)_{E_\infty} = \left\{ \begin{bmatrix} 0 & 0 \\ \delta & 1 \end{bmatrix} : \delta \in \mathbb{C} \right\}, \mathcal{L}_{E_\infty}(\mathbb{C}^2, E_\infty) = \left\{ \begin{bmatrix} 0 & 0 \\ \delta & 0 \end{bmatrix} : \delta \in \mathbb{C} \right\} \) and \( \mathcal{L}_{E_0}(\mathbb{C}^2, E_0) = \left\{ \begin{bmatrix} 0 & \gamma \\ 0 & 0 \end{bmatrix} : \gamma \in \mathbb{C} \right\}. \]
Furthermore, for \( \lambda \in \mathbb{C} \setminus \{0\} \), one has \( \mathcal{I}(\mathbb{C}^2)_{E_{\lambda}} = \left\{ \begin{bmatrix} 1 - \alpha & \alpha / \lambda \\ \lambda(1 - \alpha) & \alpha \end{bmatrix} : \alpha \in \mathbb{C} \right\} \) and

\[
\mathcal{L}^{E_{\lambda}}(\mathbb{C}, E_{\lambda}) = \left\{ \alpha \begin{bmatrix} -1 & 1 / \lambda \\ -\lambda & 1 \end{bmatrix} : \alpha \in \mathbb{C} \right\}.
\]

The map \( \pi_{E_0, E_{\infty}} : \mathcal{C}_{E_{\infty}} \to \mathcal{L}^{E_{\infty}}(\mathbb{C}^2, E_{\infty}) \) is given by

\[
\pi_{E_0, E_{\infty}}(E_{\lambda}) = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix} \quad (\lambda \in \mathbb{C}).
\]

We also know that the map \( \pi_{E_{\infty}, E_0} : \mathcal{C}_{E_0} \to \mathcal{L}^{E_0}(\mathbb{C}^2, E_0) \) is given by

\[
\pi_{E_{\infty}, E_0}(E_{\infty}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \pi_{E_{\infty}, E_0}(E_{\lambda}) = \begin{bmatrix} 0 & 1 / \lambda \\ 0 & 0 \end{bmatrix} \quad (\lambda \in \mathbb{C} \setminus \{0\}).
\]

Consider \( \lambda \in \mathbb{C} \). If \( \lambda \neq 0 \), then \( \mu_{E_0, E_{\infty}} : \kappa^{-1}(E_{\lambda}) \to \mathcal{L}^{E_{\infty}}(\mathbb{C}^2, E_{\infty}) \oplus \mathcal{L}^{E_0}(\mathbb{C}^2, E_0) \) is given by

\[
\mu_{E_0, E_{\infty}} \left( \begin{bmatrix} 1 - \alpha & \alpha / \lambda \\ \lambda(1 - \alpha) & \alpha \end{bmatrix} \right) = \left( \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix}, \begin{bmatrix} 0 & \alpha / \lambda \\ 0 & 0 \end{bmatrix} \right).
\]

In the case when \( \lambda = 0 \), the map \( \mu_{E_0, E_{\infty}} : \kappa^{-1}(E_0) \to \mathcal{L}^{E_{\infty}}(\mathbb{C}^2, E_{\infty}) \oplus \mathcal{L}^{E_0}(\mathbb{C}^2, E_0) \) is given by \( \mu_{E_0, E_{\infty}} \left( \begin{bmatrix} 1 & \gamma \\ 0 & 0 \end{bmatrix} \right) = \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \gamma \\ 0 & 0 \end{bmatrix} \right) \).

On the other hand, the map \( \mu_{E_{\infty}, E_0} : \kappa^{-1}(\mathcal{C}_{E_0}) \to \mathcal{L}^{E_0}(\mathbb{C}^2, E_0) \oplus \mathcal{L}^{E_{\infty}}(\mathbb{C}^2, E_{\infty}) \) is given by

\[
\mu_{E_{\infty}, E_0} \left( \begin{bmatrix} 1 - \alpha & \alpha / \lambda \\ \lambda(1 - \alpha) & \alpha \end{bmatrix} \right) = \left( \begin{bmatrix} 0 & 1 / \lambda \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \lambda(1 - \alpha) & 0 \end{bmatrix} \right)
\]

and \( \mu_{E_{\infty}, E_0} \left( \begin{bmatrix} 0 & 0 \\ \delta & 1 \end{bmatrix} \right) = \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \delta & 0 \end{bmatrix} \right) \).

The two charts \( \left( \kappa^{-1}(\mathcal{C}_{E_{\infty}}), \mu_{E_0, E_{\infty}}, \mathcal{L}^{E_{\infty}}(\mathbb{C}^2, E_{\infty}) \oplus \mathcal{L}^{E_0}(\mathbb{C}^2, E_0) \right) \) and

\[
\left( \kappa^{-1}(\mathcal{C}_{E_0}), \mu_{E_{\infty}, E_0}, \mathcal{L}^{E_0}(\mathbb{C}^2, E_0) \oplus \mathcal{L}^{E_{\infty}}(\mathbb{C}^2, E_{\infty}) \right)
\]

form an analytic atlas for \( \mathcal{I}(\mathbb{C}^2) \), which produces the Banach submanifold structure induced from \( M_2(\mathbb{C}) \).

Let us denote by \( \mathcal{V}(X) \) the set of “self-inverse mappings”; i.e.

\[
\mathcal{V}(X) := \{ V \in GL(X) \setminus \{I, -I\} : V^2 = I \}.
\]

For each \( V \in \mathcal{V}(X) \), let us put \( X^V := \{ x \in X : V(x) = x \} \), and set \( \tilde{\kappa} : \mathcal{V}(X) \to \mathcal{G}(X) \) to be the map given by \( \tilde{\kappa}(V) := X^V \). Since the continuous affine bijection \( T \mapsto 2T - I \)
sends \( \mathcal{I}(X) \) onto \( \mathcal{V}(X) \), Lemma 3.1 tells us that \( \tilde{k}^{-1}(E) \) is an affine-Banach subspace of \( \mathcal{L}(X) \), and Theorem 3.4 tells us that \( (\mathcal{V}(X), \mathcal{G}(X), \tilde{k}) \) is a locally trivial analytic affine-Banach bundle over \( \mathcal{G}(X) \). If we define \( \tilde{\mu}_{E,F} : \tilde{k}^{-1}(\mathcal{C}_F) \to \mathcal{L}_E^F(X, F) \oplus \mathcal{L}_E^E(X, E) \) by

\[
\tilde{\mu}_{E,F}(V) := (Q_{E(F)}^F_{(I_X+V)(X)} - Q_{E}^F, Q_{E}^E V Q_{F}^E / 2) \quad (V \in \mathcal{V}(X)_E)
\]

(see (3.21)), then \( \{(\tilde{k}^{-1}(\mathcal{C}_F), \tilde{\mu}_{E,F}, \mathcal{L}_E^F(X, F) \oplus \mathcal{L}_E^E(X, E)) : E, F \in \mathcal{G}(X); F \cap E\} \) is an analytic atlas for the Banach submanifold structure on \( \mathcal{V}(X) \) induced from \( \mathcal{L}(X) \).

Another disguised form of \( \mathcal{I}(X) \) is the subspace

\[
\mathcal{G}(X) \times_C \mathcal{G}(X) := \{(E, F) \in \mathcal{G}(X) \times \mathcal{G}(X) : F \cap E\}
\]

of \( \mathcal{G}(X) \times \mathcal{G}(X) \). We will say some words about this subspace in the following.

**Corollary 3.6** Suppose that \( \mathcal{T} \) is the topology on \( \mathcal{G}(X) \times_C \mathcal{G}(X) \) induced from the product topology on \( \mathcal{G}(X) \times \mathcal{G}(X) \). There is a \( \mathbb{K} \)-Banach manifold structure on \( \mathcal{G}(X) \times_C \mathcal{G}(X) \) compatible with \( \mathcal{T} \) such that under the projection \( \kappa_1 : \mathcal{G}(X) \times_C \mathcal{G}(X) \to \mathcal{G}(X) \) onto the first coordinate, one obtains a locally trivial analytic affine-Banach bundle structure on \( \mathcal{G}(X) \times_C \mathcal{G}(X) \).

**Proof** By Theorem 3.4, it suffices to show that \( \Psi : (E, F) \mapsto Q_E^F \) is a homeomorphism from \( \mathcal{G}(X) \times_C \mathcal{G}(X) \) onto \( \mathcal{I}(X) \). However, thanks to the continuity of \( \kappa \), it remains to establish the continuity of \( \Psi \). For this, let us consider a sequence \( \{(E_n, F_n)\}_{n \in \mathbb{N}} \) in \( \mathcal{G}(X) \times_C \mathcal{G}(X) \) converging to \( (E_0, F_0) \in \mathcal{G}(X) \times_C \mathcal{G}(X) \). By Corollary 3.2, we may assume that \( E_n \in \mathcal{C}_{F_0} \) and \( F_n \in \mathcal{C}_{E_0} \) for all \( n \in \mathbb{N} \). Relation (3.14) and Corollary 3.2 then produce \( Q_{E_0}^{E_0, F_0}(E_n^{-1}(F_n)) \to Q_{E_0}^{F_0} \). Now, the continuity of \( \Theta_{E_0, F_0} \) gives the required convergence: \( Q_{E_0}^{E_n, F_0} = \Theta_{E_0, F_0} \left( E_n, Q_{E_0}^{E_0, F_0}(E_n^{-1}(F_n)) \right) \to Q_{E_0}^{F_0} \). \( \square \)

The above Banach manifold structure will be state explicitly in the following. For \( (E_0, F_0) \in \mathcal{G}(X) \times_C \mathcal{G}(X) \), we define \( \tilde{\mu}_{E_0, F_0} : \kappa_1^{-1}(\mathcal{C}_{F_0}) \to \mathcal{L}_{E_0}^{F_0}(X, F_0) \oplus \mathcal{L}_{E_0}^{E_0}(X, E_0) \) to be the map

\[
\tilde{\mu}_{E_0, F_0}(E, F) := (Q_{E_0}^{E_0, F_0} - Q_{E_0}^{F_0}, Q_{E_0}^{F_0} Q_{E_0}^{F_0} Q_{F_0}^{E_0}) \quad (E \in \mathcal{C}_{F_0}; F \in \mathcal{C}_{E}).
\]

Then \( \{(\tilde{k}^{-1}(\mathcal{C}_F), \tilde{\mu}_{E,F}, \mathcal{L}_E^F(X, F) \oplus \mathcal{L}_E^E(X, E)) : (E, F) \in \mathcal{G}(X) \times_C \mathcal{G}(X) \} \) is an analytic atlas for \( \mathcal{G}(X) \times_C \mathcal{G}(X) \).

### 4 \( \mathcal{I}(X) \) as a Banach bundle

It is natural to ask whether \( (\mathcal{I}(X), \mathcal{G}(X), \kappa) \) is actually a Banach bundle, instead of an affine-Banach bundle. The first proposition in this section states that one can really regard \( (\mathcal{I}(X), \mathcal{G}(X), \kappa) \) as a *continuous* Banach bundle.

In fact, Proposition 2.3 tells us that it suffices to show the existence of a continuous global cross section for \( (\mathcal{I}(X), \mathcal{G}(X), \kappa) \). In order to construct such a global cross
section, let us fix an element $F_E \in C_E$ for every $E \in \mathcal{G}(X)$. Since $\mathcal{G}(X)$ is metrizable (see e.g., [15, Sect. 2.1]), there is a partition of unity $\{\psi_E\}_{E \in \mathcal{G}(X)}$, consisting of continuous functions, dominated by the open covering $\{C_{F_E}\}_{E \in \mathcal{G}(X)}$ of $\mathcal{G}(X)$. On the other hand, as $\kappa^{-1}(C_{F_E})$ is homeomorphic to a trivial affine-Banach bundle, there exists a local cross section on it. Now, a standard “scaled-sum construction” will produce the required continuous global cross section (observe that as $\mathcal{T}(X)_E$ is an affine subspace of $\mathcal{L}(X)$, it is closed under convex combinations).

Let us state this clearly as follows.

**Proposition 4.1** If $X$ is a $\mathbb{K}$-Banach space, then $(\mathcal{I}(X), \mathcal{G}(X), \kappa)$ is a locally trivial continuous $\mathbb{K}$-Banach bundle, under equivalent $\mathbb{K}$-Banach space structures on all the fibers of $\kappa$.

By Proposition 2.3(b), the only obstruction for $\mathcal{I}(X)$ to be a locally trivial analytic Banach bundle is the existence of an analytic global cross section. However, a complex analytic global cross section does not exist even in the case when $X = \mathbb{C}^2$.

**Example 4.2** Let $E_\lambda$ be the subspace in Example 3.5. Suppose that there is a complex analytic global cross section $\rho : \mathcal{G}(\mathbb{C}^2) \to \mathcal{I}(\mathbb{C}^2)$. Then one can find a function $\alpha : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ satisfying

$$\rho \circ \pi^{-1}_{E_0, E_{\infty}} \left( \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 - \alpha(\lambda) & \alpha(\lambda)/\lambda \\ \lambda(1 - \alpha(\lambda)) & \alpha(\lambda) \end{bmatrix} \quad (\lambda \in \mathbb{C}\setminus\{0\}).$$

As $\rho \circ \pi^{-1}_{E_0, E_{\infty}}$ is complex analytic, we know that $\alpha$ is holomorphic. Moreover, the compactness of $\mathcal{G}(\mathbb{C}^2)$ tells us that the image of $\rho$ is norm-bounded. From this, we deduce that the three functions

$$\lambda \mapsto \alpha(\lambda), \quad \lambda \mapsto \alpha(\lambda)/\lambda \quad \text{and} \quad \lambda \mapsto \lambda(1 - \alpha(\lambda))$$

are bounded on $\mathbb{C}\setminus\{0\}$. As $\alpha$ is bounded, it has a removable singularity at 0. Thus, $\alpha$ extends to a bounded entire function on $\mathbb{C}$, which can only be a constant function. On the other hand, since $\lambda \mapsto \alpha(\lambda)/\lambda$ is bounded as well, we know that $\alpha$ is the constant zero function. However, this will contradict with the boundedness of $\lambda \mapsto \lambda(1 - \alpha(\lambda))$. Consequently, there does not exist a complex analytic global cross section on $\mathcal{I}(\mathbb{C}^2)$. In other words, $(\mathcal{I}(\mathbb{C}^2), \mathcal{G}(\mathbb{C}^2), \kappa)$ is not a locally trivial complex analytic Banach bundle.

More generally, there is no complex differentiable global cross section on $(\mathcal{I}(X), \mathcal{G}(X), \kappa)$ for any complex Banach space $X$.

**Proposition 4.3** Let $X$ be a complex Banach space. We denote by $\mathcal{G}(X)_\text{fin}$ the subset of $\mathcal{G}(X)$ consisting of finite dimensional subspaces, and set

$$\mathcal{I}(X)_\text{fin} := \kappa^{-1}(\mathcal{G}(X)_\text{fin}).$$

Then the subbundle $(\mathcal{I}(X)_\text{fin}, \mathcal{G}(X)_\text{fin}, \kappa)$ of $(\mathcal{I}(X), \mathcal{G}(X), \kappa)$ is not a locally trivial complex differentiable Banach bundle (see Remark 2.4(b)).
Proof Suppose on the contrary that $(I(X)_{\text{fin}}, \mathcal{G}(X)_{\text{fin}}, \kappa)$ is a locally trivial complex differentiable Banach bundle. Then there is a complex differentiable global cross section $\rho: \mathcal{G}(X)_{\text{fin}} \to I(X)_{\text{fin}}$. Consider $E_0 \in \mathcal{G}(X)_{\text{fin}}$ and $F_0 \in \mathcal{C}_{E_0}$.

Fix an operator $T \in \mathcal{L}(E_0, F_0)$ and put $Y := E_0 + T(E_0)$. Then $Y$ is a finite dimensional subspace. For any $\lambda \in \mathbb{C}$, one has

$$p_{E_0, F_0}^{-1}(\lambda T) = (I + \lambda T)(E_0) \subseteq Y,$$

where $p_{E_0, F_0}$ is given by (2.2). Hence, $\{p_{E_0, F_0}^{-1}(\lambda T) : \lambda \in \mathbb{C}\} \subseteq \mathcal{G}(Y) \subseteq \mathcal{G}(X)_{\text{fin}}$.

Choose any $x \in X$ and any $f$ in the dual space, $X^*$, of $X$. As the inclusion map $\iota: I(X) \to \mathcal{L}(X)$ is complex analytic, we know that the function $\chi$ on $\mathbb{C}$ defined by

$$\chi(\lambda) := f(\rho(p_{E_0, F_0}^{-1}(\lambda T))(x))$$

is holomorphic (since (2.4) is an analytic atlas). On the other hand, as $\mathcal{G}(Y)$ is compact and $\rho|_{\mathcal{G}(Y)}$ is continuous, we know that

$$\sup_{E \in \mathcal{G}(Y)} \|\rho(E)\| < \infty.$$

This means that $\chi$ is bounded and hence it should be a constant function.

Since $x$ and $f$ are arbitrarily chosen, we know that $\lambda \mapsto \rho(p_{E_0, F_0}^{-1}(\lambda T))$ is a constant map, for each fixed $T \in \mathcal{L}(E_0, F_0)$. Consequently, the map $\rho \circ p_{E_0, F_0}^{-1} : \mathcal{L}(E_0, F_0) \to I(X)$ is constant (since all the rays pass through 0), but this contradicts with the fact that $\rho$ is a cross section on $\mathcal{C}_{F_0}$. □

In the following, we consider the case of Hilbert spaces. Although the above proposition tells us that for a complex Hilbert space, the set of idempotents can never be a locally trivial complex analytic Banach bundle, it is nevertheless a real analytic Banach bundle.

Moreover, we have the stronger conclusion that if $H$ is a $\mathbb{K}$-Hilbert space, then $(I(H), \mathcal{G}(H), \kappa)$ can be identified, as real analytic Banach bundles, with the tangent bundle of $\mathcal{G}(H)$. In fact, the total space $T(\mathcal{G}(H))$ of the tangent bundle of $\mathcal{G}(H)$ is the disjoint union $\biguplus_{E \in \mathcal{G}(H)} \mathcal{L}(E, E^{\perp})$ of Banach spaces, equipped with an appropriate Banach manifold structure. By Lemma 3.1, for every $E \in \mathcal{G}(H)$ and $T \in \mathcal{L}(E, E^{\perp})$, one knows that $T^*P_{E^{\perp}} + P_E$ is in $I(X)$, where $T^*$ is the adjoint of $T$ and

$$P_E := Q_E^{E^{\perp}}. \quad (4.1)$$

Theorem 4.4 Let $H$ be a real or complex Hilbert space. The assignment $(E, T) \mapsto T^*P_{E^{\perp}} + P_E$ is a fiberwise (respectively, real or complex) affine real bi-analytic bijection from $T(\mathcal{G}(H))$ onto the Banach submanifold $I(H)$ of $\mathcal{L}(H)$.

Proof For any $E \in \mathcal{G}(H)$ and $F \in \mathcal{C}_E$, we can identify, via the Banach space isomorphism $\Lambda_{F,E}$ in (3.5),

$$\mathcal{L}(F, E) \cong \mathcal{L}^E(H, E) \quad (4.2)$$

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(recall that in this case, \( T \in \mathcal{L}(F, E) \) is identified with \( T Q_F^E \)).

Let us set \( \hat{I}(H) := \bigoplus_{E \in \mathcal{G}(H)} \mathcal{L}(E^\perp, E) \) and denote by \( \hat{k} : \hat{I}(H) \to \mathcal{G}(H) \) the map that sends \( T \in \mathcal{L}(E^\perp, E) \) to \( E \). Through the identification in (4.2), together with the equality
\[
\mathcal{L}^E(H, E) = \mathcal{I}(H)_E - P_E \quad (E \in \mathcal{G}(H)),
\]

one may see \((\mathcal{I}(H), \mathcal{G}(H), \kappa)\) as \((\hat{I}(H), \mathcal{G}(H), \hat{k})\), and obtains a (respectively, real or complex) Banach manifold structure on \( \hat{I}(H) \). In this case, the following is an analytic atlas for this structure:
\[
\{ (\hat{k}^{-1}(C_{E_0}^\perp), \hat{\mu}_{E_0}, \mathcal{L}^{E_0^\perp}(H, E_0^\perp) \oplus \mathcal{L}^{E_0^\perp}(H, E_0)) : E_0 \in \mathcal{G}(H) \},
\]

where
\[
\hat{\mu}_{E_0}(T) := \left( Q_E^{E_0^\perp} - P_{E_0}, P_{E_0}(T \circ P_{E_0} + P_E)P_{E_0^\perp} \right) \quad (E \in C_{E_0}^\perp, T \in \mathcal{L}(E^\perp, E))
\]
(4.3)

(c.f. (3.21)). Furthermore, for \( R \in \mathcal{L}^{E_0^\perp}(H, E_0^\perp) \) and \( S \in \mathcal{L}^{E_0^\perp}(H, E_0) \), we have, via (3.22),
\[
\hat{\mu}_{E_0}^{-1}(R, S) = (I + R)(S + P_{E_0})(I - R)|_{(I + R)(E_0)^\perp}.
\]
(4.4)

In order to verify this theorem, it suffices to show that the fiberwise (respectively, real or complex) linear map
\[
\Phi : \hat{I}(H) \to \mathbf{T}(\mathcal{G}(H))
\]
that sends an operator to its adjoint is real bi-analytic.

To do this, let us fix \( E_0 \in \mathcal{G}(H) \). Consider \( E_1 \in C_{E_0}^\perp \) and \( T \in \mathcal{L}(E_1, E_1^\perp) \). We define \( \gamma(s) := p_{E_1, E_1^\perp}^{-1}(sT) \) \((s \in (-1, 1))\). We also define
\[
\nu_{E_0}(T) := \left( \pi_{E_0, E_0^\perp}(E_1), \Psi(T)P_{E_0} \right),
\]
(4.5)
where \( \Psi(T) := (P_{E_0, E_0^\perp} \circ \gamma)'(0) \). As in [12, p. 69 and p. 73],
\[
\left( \bigoplus_{E \in C_{E_0}^\perp} \mathcal{L}(E, E^\perp), \nu_{E_0}, \mathcal{L}^{E_0^\perp}(H, E_0^\perp) \times \mathcal{L}^{E_0^\perp}(H, E_0^\perp) \right)
\]
is a local chart of \( \mathbf{T}(\mathcal{G}(H)) \) around \( E_0 \).

Now, we are required to identify the map \( p_{E_0, E_0^\perp} \circ p_{E_1, E_1^\perp}^{-1} \), in order to express \( \nu_{E_0} \). For this, we need to express the two maps \( p_{E_0, E_0^\perp} \circ p_{E_1, E_1^\perp}^{-1} \) and \( p_{E_1, E_1^\perp} \circ p_{E_1, E_1^\perp}^{-1} \). Let
us put $B := p_{E_1, E_0}^\perp(E_0)$ and $C := p_{E_1^\perp, E_0}^\perp(E_0)$. Denote

$$D := I_{E_1} + B \quad \text{and} \quad A := I_{E_1^\perp} + C,$$

where $I_{E_1}$ and $I_{E_1^\perp}$ are the identity maps on $E_1$ and $E_1^\perp$, respectively. Then $D = (I + \pi_{E_1, E_0}^\perp(E_0))|_{E_1}$ is an operator from $E_1$ to $E_0$ and $A = (I + \pi_{E_1^\perp, E_0}^\perp(E_0))|_{E_1^\perp}$ is an operator from $E_1^\perp$ to $E_0^\perp$. As in the proof of [31, Lemma 3.12], for any $S \in \mathcal{L}(E_1, E_1^\perp)$, one has

$$p_{E_0, E_0}^\perp \circ p_{E_1, E_0}^{-1} \circ p_{E_1, E_0}^{-1} \circ p_{E_1, E_1}^{-1}(S) = (AS(I_{E_1} - CS)^{-1} - B)D^{-1}.$$ 

Hence, for $s \in (-1, 1)$, we have

$$(p_{E_0, E_0}^\perp \circ \gamma)'(s) = AT(I_{E_1} - sCT)^{-1}D^{-1} + sAT(I_{E_1} - sCT)^{-2}CTD^{-1},$$

which gives

$$\Psi(T) = AD^{-1} = (I + CP_{E_1^\perp})TD^{-1}. \quad (4.6)$$

Set $R := \pi_{E_0, E_0}^\perp(E_1)$. It follows from the equality $BQ_{E_1}^\perp = \pi_{E_1, E_0}^\perp(E_0)$ as well as Relation (3.6) that $R = -BQ_{E_1}^\perp$. This means that

$$D = (I - R)|_{E_1}, \quad (4.7)$$

and Lemma 2.5 gives $D^{-1} = (I + R)|_{E_0}$. On the other hand, as $R = Q_{E_1}^\perp - P_{E_0}$, we have

$$CP_{E_1^\perp} = \pi_{E_1^\perp, E_1}^\perp(E_0^\perp) = Q_{E_0}^\perp - P_{E_1^\perp} = P_{E_1} - R - P_{E_0}. \quad (4.8)$$

From these, we conclude that

$$\nu_{E_0}(T) = \left(\pi_{E_0, E_0}^\perp(E_1), (P_{E_1} - \pi_{E_0, E_0}^\perp(E_1) + P_{E_0}^\perp)TP_{E_1}(I + \pi_{E_0, E_0}^\perp(E_1))P_{E_0}\right). \quad (4.9)$$

Now, the adjoint map $\Phi$ restricts to a fiberwise (respectively, real or complex) linear bijection

$$\Phi_{E_0} : \bigcup_{E \in C_{E_0^\perp}} \mathcal{L}(E^\perp, E) \to \bigcup_{E \in C_{E_0^\perp}} \mathcal{L}(E, E^\perp).$$
Under the corresponding local charts of $\hat{T}(H)$ and $T(\mathcal{H}(H))$ about $C_{E_0}^\perp$, the map $\Phi_{E_0}$ is transformed into a map that sends $(R, S) \in \mathcal{L}^{E_0\perp}(H, E_{0\perp}) \oplus \mathcal{L}^{E_0}(H, E_0)$ to

$$\left( R, (P_{E_0\perp} + P_{(I+R)(E_0)}) - R)(I - R^*)(S^* + P_{E_0})(I + R^*)(I + R)P_{E_0} \right)$$

(see Relations (4.4) and (4.9)). This shows that $\Phi_{E_0}$ is real analytic, since $(I + R)(E_0) = \pi_{E_0, E_0\perp}(R)$ and the assignment $E \mapsto P_E$ is a real analytic map because of [1, Proposition 4(4)] (note that [1, Proposition 4(4)] is also valid in the case of a real Hilbert space using the same argument).

Conversely, consider again $E_0 \in \mathcal{H}(H)$ and $E_1 \in C_{E_0}$. Suppose that $R, \tilde{S} \in \mathcal{L}^{E_0\perp}(H, E_{0\perp})$ such that $E_1 = \pi_{E_0, E_0\perp}(R)$. Let $A, B, C$ and $D$ be the operators defined in the above. If $T \in \mathcal{L}(E_1, E_{1\perp})$ such that $\Psi(T)P_{E_0} = \tilde{S}$, then it follows from (4.6) that

$$T = A^{-1}\tilde{S}|_{E_0 D}.$$ 

Hence, we have $v_{E_0}^{-1}(R, \tilde{S}) = A^{-1}\tilde{S}|_{E_0 D}$ (see (4.5)).

Since $A = (I + \pi_{E_1, E_1}(E_0\perp))|_{E_1\perp}$, it follows from Lemma 2.5 that $A^{-1} = (I - \pi_{E_1, E_1}(E_0\perp))|_{E_1\perp}$. Consequently, Relations (4.7) and (4.8) imply

$$v_{E_0}^{-1}(R, \tilde{S}) = (I + R + P_{E_0} - P_{E_1})\tilde{S}(I - R)|_{E_1}.$$

This, together with Relation (4.3), tells us that $\Phi_{E_0}^{-1}$ is transformed (under the corresponding local charts of $\hat{T}(H)$ and $T(\mathcal{H}(H))$ about $C_{E_0\perp}$) into a map sending $(R, \tilde{S}) \in \mathcal{L}^{E_0\perp}(H, E_{0\perp}) \oplus \mathcal{L}^{E_0\perp}(H, E_{0\perp})$ to

$$\left( R, P_{E_0}(P_{E_1}(I - R^*)\tilde{S}^*)(I + R^* + P_{E_0} - P_{E_1})P_{E_1} + P_{E_1})P_{E_0\perp} \right).$$

Thus, $\Phi_{E_0}^{-1}$ is also real analytic. This completes the proof.

Suppose that $K$ is a real Hilbert space. Under the identification of $\mathcal{H}(K)$ with

$$\mathcal{P}(K) := \{ P_E : E \in \mathcal{H}(K) \} \subseteq \mathcal{L}(K)$$

as Banach manifolds (see [1]), the fiber of $\hat{T}(K)$ over the base points $P_E$ will contain $P_E$. Therefore, the presentation of $\hat{T}(K)$ as the tangent bundle of $\mathcal{P}(K)$ is nice one, in the sense that the tangent plane at every point contains that point.

We may also identify the tangent bundle of $\mathcal{H}(K)$ with either $\mathcal{V}(K)$ or $\mathcal{Y}(K) \times_C \mathcal{H}(K)$ as real analytic Banach bundles (see the discussions following Example 3.5). On the other hand, it is possible to identify $\mathcal{H}(K)$ with the real Banach submanifold

$$\mathcal{V}_{sa}(K) := \{ V \in G\mathcal{L}(K) \setminus \{I, -I\} : V^* = V = V^2 \}$$
(via the continuous affine bijection sending \( P \in \mathcal{P}(K) \) to \( 2P - I \in \mathcal{V}_{sa}(K) \)). Therefore, the Banach submanifold \( \mathcal{V}(K) \) can be identified with the tangent bundle of the Banach submanifold \( \mathcal{V}_{sa}(K) \), under a suitable fibration map with each fiber being an affine-Banach subspace of \( \mathcal{L}(K) \) containing the base point.

Note that the corresponding statement of Theorem 4.4 for general Banach spaces is in general false because \( \mathcal{L}(E, F) \) may not be isomorphic to \( \mathcal{L}(F, E) \) for \( E, F \in \mathcal{G}(X) \) with \( E \tau F \).

Let us define a map \( \tau : \tilde{\mathcal{I}}(H) \to \mathcal{G}(H) \times \mathcal{L}(H) \) by

\[
\tau(Q) := (\kappa(Q), \iota(Q)) \quad (Q \in \tilde{\mathcal{I}}(H)),
\]

where \( \iota : \tilde{\mathcal{I}}(H) \to \mathcal{L}(H) \) is the inclusion map. Clearly, \( \tau \) is a homeomorphism onto its image. Moreover, as in the proof of Theorem 3.4, \( \tau \) is an analytic immersion.

We set

\[
\tilde{\mathcal{I}}(H) := \bigcup_{E \in \mathcal{G}(H)} \mathcal{L}^{E}(H, E),
\]

and define the bundle map \( \tilde{\kappa} : \tilde{\mathcal{I}}(H) \to \mathcal{G}(H) \) canonically. One can see from the proof of Theorem 4.4 that \((\tilde{\mathcal{I}}(H), \mathcal{G}(H), \tilde{\kappa})\) is a locally trivial real analytic Banach bundle.

Consider a “\( \mathcal{L}(H) \)-valued metric” on \( \tilde{\mathcal{I}}(H) \) given by

\[
\langle S, T \rangle_{\mathcal{L}(H)} := ST^*, \quad \text{for any } S, T \in \tilde{\mathcal{I}}(H) \text{ with } \tilde{\kappa}(S) = \tilde{\kappa}(T). \quad (4.10)
\]

Notice that \( \langle \cdot, \cdot \rangle_{\mathcal{L}(H)} \) satisfies all the requirements of a metric (i.e., fiberwise inner product) except that it takes values in \( \mathcal{L}(H) \) instead of the scalar field. Note also that

\[
\langle S, T \rangle_{\mathcal{L}(H)} \in P_{\tilde{\kappa}(S)}\mathcal{L}(H)P_{\tilde{\kappa}(T)}. \quad (4.11)
\]

For each \( x \in H \), we can also define

\[
\langle S, T \rangle_{x} := \langle ST^*x, x \rangle_{H}.
\]

Then \( \{\langle \cdot, \cdot \rangle_{x}\}_{x \in H} \) is a family of pseudo-metric on \( (\tilde{\mathcal{I}}(H), \mathcal{G}(X), \tilde{\kappa}) \) which is separating; in the sense that \( S = 0 \) whenever we have \( \langle S, S \rangle_{x} = 0 \) for all \( x \in H \). It is easy to see, via the fiberwise (respectively, real or complex) linear analytic immersion induced by \( \tau \) (when \( \tilde{\mathcal{I}}(H) \) is identified with \( \tilde{\mathcal{I}}(H) \)), that \( \langle \cdot, \cdot \rangle_{x} \) is real analytic. On the other hand, if \( H \) is separable and \( \{x_n\}_{n\in\mathbb{N}} \) is a countable dense subset of \( H \), then the sequence of pseudo-metric \( \{\langle \cdot, \cdot \rangle_{x_n}\}_{n\in\mathbb{N}} \) is also separating.

Let \( \mathcal{G}(H)_{\text{fin}} \) be the subset of \( \mathcal{G}(H) \) consisting of finite dimensional subspaces, and we put

\[
\tilde{\mathcal{I}}(H)_{\text{fin}} := \tilde{\kappa}^{-1}(\mathcal{G}(H)_{\text{fin}}).
\]

Then \( (\tilde{\mathcal{I}}(H)_{\text{fin}}, \mathcal{G}(H)_{\text{fin}}, \tilde{\kappa}) \) is a locally trivial real analytic Banach bundle. For any \( S, T \in \tilde{\mathcal{I}}(H)_{\text{fin}} \) with \( \tilde{\kappa}(S) = \tilde{\kappa}(T) \), the operator \( \langle S, T \rangle_{\mathcal{L}(H)} \) is of finite rank (see...
(4.11)). Therefore, we can define a metric $\langle \cdot, \cdot \rangle_{\text{fin}}$ on $\tilde{T}(H)_{\text{fin}}$ by

$$\langle S, T \rangle_{\text{fin}} := \text{Tr}(ST^*)$$

for any $S, T \in \tilde{T}(H)_{\text{fin}}$ with $\tilde{\kappa}(S) = \tilde{\kappa}(T)$, (4.12)

where Tr is the canonical densely defined trace on $\mathcal{L}(H)$.

One can see from the identification in (4.2) that $\tilde{T}(H)$ is the same as $\hat{T}(H)$. Hence, the above discussion, together with Theorem 4.4, gives the following (notice that both $(F, S) \mapsto (F, S + P_F)$ and $(F, R) \mapsto (F, R^*)$ are real bi-analytic maps from $\mathcal{G}(H) \times \mathcal{L}(H)$ to itself).

**Corollary 4.5** Let $H$ be a real or complex Hilbert space.

(a) The assignment $(E, T) \mapsto (E, TP_E)$ is a real analytic immersion from $\mathbf{T}([\mathcal{G}(H)])$ to the trivial Banach bundle $(\mathcal{G}(H) \times \mathcal{L}(H), \mathcal{G}(H), \kappa_0)$ which is fiberwise linear. This immersion is a homeomorphism from $\mathbf{T}([\mathcal{G}(H)])$ onto the Banach subbundle $\{(E, S) : E \in \mathcal{G}(H); S \in \mathcal{L}^E(H, E^\perp)\}$.

(b) The $\mathcal{L}(H)$-valued metric $\langle \cdot, \cdot \rangle_{\mathcal{L}(H)}$ in (4.10) induces a separating family of real analytic pseudo-metrics on $\mathbf{T}([\mathcal{G}(H)])$. If $H$ is separable, then one can find a countable separating family of real analytic pseudo-metrics on $\mathbf{T}([\mathcal{G}(H)])$.

(c) The maps $\langle \cdot, \cdot \rangle_{\text{fin}}$ in (4.12) is a real analytic metric on $\mathbf{T}([\mathcal{G}(H)]_{\text{fin}})$.

Note that since all the tangent spaces of $\mathcal{G}(H)_{\text{fin}}$ are isomorphic to Hilbert spaces, it is already known that a real analytic metric exists on $\mathbf{T}([\mathcal{G}(H)]_{\text{fin}})$. The above gives an explicit construction of such a metric.

Part (a) above tells us that $\mathbf{T}([\mathcal{G}(H)])$ can be identified, as locally trivial continuous Banach bundles, with the Banach subbundle $\{(E, S) : E \in \mathcal{G}(H); S \in \mathcal{L}^E(H, E^\perp)\}$ of the trivial bundle $\mathcal{G}(H) \times \mathcal{L}(H)$. From this, as well as the continuity of $E \mapsto P_E$, we obtain the following (see (3.11)). Note that $C(X; A)$ in this corollary denotes the algebra of continuous maps from a topological space $X$ to a Banach algebra $A$.

**Corollary 4.6** For any integer $n \geq 2$, the $K$-theory class of the finite dimensional bundle $\mathbf{T}([\mathcal{G}(\mathbb{K}^n)])$ is represented by the idempotent $E \mapsto \Delta_{E^\perp,E}$ in $C([\mathcal{G}(\mathbb{K}^n); \mathcal{L}(\mathcal{L}(\mathbb{K}^n))]) = M_n^2(C(\mathcal{G}(\mathbb{K}^n); \mathbb{K}))$, where $\Delta_{E^\perp,E} \in \mathcal{L}(\mathcal{L}(\mathbb{K}^n))$ is the map given by (3.10).

On the other hand, notice that the fiber of the Banach bundle $\hat{T}(H)$ over $E \in \mathcal{G}(H)$ is a Hilbert $\mathcal{L}(E^\perp)$-module with inner product $\langle S, T \rangle_0 := S^*T$ (in the usual convention, a Hilbert $C^*$-module is a right module and the operator-valued inner product is conjugate linear in the first variable; see e.g., [26]). One may use it to define another family of pseudo-metrics on $\mathbf{T}([\mathcal{G}(H)])$.

### 5 $\mathcal{I}(X)$ as a disjoint union of homogeneous spaces

In the case when $X$ is finite dimensional, it is well-known that $\mathcal{G}(X)$ can be identified with a disjoint union of quotient spaces of $G\mathcal{L}(X)$ by closed Lie subgroups. The corresponding fact for $\mathcal{I}(X)$ may also be known. We are going to look at the infinite dimensional situation.
In the following, we will consider the analytic actions $\alpha$ of $\mathcal{GL}(X)$ on $\mathcal{I}(X)$ and $\mathcal{G}(X)$, respectively (see (3.8) and (3.9)). We have already constructed in the proof of Theorem 3.4 an analytic local right inverse $\tilde{\Sigma}_{E_0,F_0}$ (see (3.15)) for the evaluation map at $E_0$ from $\mathcal{GL}(X)$ to the orbit $\alpha(G\mathcal{L}(X),E_0)$, for every $E_0 \in \mathcal{G}(X)$. The following lemma gives an analytic local right inverse for $\alpha$.

**Lemma 5.1** For $(E_0,F_0) \in \mathcal{G}(X) \times \mathcal{C}\mathcal{G}(X)$, the map from $\kappa^{-1}(C_{F_0})$ to $\mathcal{GL}(X)$ defined by

$$\tilde{\Sigma}_{E_0,F_0}(Q_E^F) := (Q_E^F + Q_{E_0}^F)(Q_E^F + Q_{F_0}^E)(Q_E^F \in \kappa^{-1}(C_{F_0}))$$

is an analytic local right inverse for the evaluation map at $Q_{E_0}^F$ from $\mathcal{GL}(X)$ onto the orbit of $Q_{E_0}^F$ under the action $\alpha$.

**Proof** Consider $Q_E^F \in \kappa^{-1}(C_{F_0})$. We know that both $Q_{E_0}^F + Q_{E_0}^F$ and $Q_{E_0}^F + Q_{E_0}^F$ are invertible (see Lemmas 2.5 and 3.1). Moreover, one easily check that

$$\tilde{\Sigma}_{E_0,F_0}(Q_E^F)(E_0) = E \quad \text{and} \quad \tilde{\Sigma}_{E_0,F_0}(Q_E^F)(F_0) = F.$$

This implies that $\tilde{\Sigma}_{E_0,F_0} : \kappa^{-1}(C_{F_0}) \rightarrow \mathcal{GL}(X)$ is a local right inverse for the evaluation map.

Suppose that $(R,S) = \mu_{E_0,F_0}(Q_E^F)$. It follows from (3.21) and (3.22) that

$$Q_E^F = (I + R)(S + Q_{E_0}^F)(I - R)$$

and $R = Q_{E_0}^F + Q_{E_0}^F - I$. Consequently,

$$\tilde{\Sigma}_{E_0,F_0}(\mu_{E_0,F_0}^{-1}(R,S)) = (I - (I + R)(S + Q_{E_0}^F)(I - R) + R + Q_{E_0}^F)(R + I),$$

which is analytic as required. \hfill \Box

The existences of local analytic right inverses for the evaluation maps from $\mathcal{GL}(X)$ to the orbits of $\alpha$ and $\alpha$, respectively, imply that these two actions are locally transitive in the sense of [31, Definition 8.20]. Hence, if we set

$$\mathcal{GL}(X)_Q^Q := \{ W \in \mathcal{GL}(X) : WQW^{-1} = Q \}$$

and $\mathcal{GL}(X)_E^E := \{ W \in \mathcal{GL}(X) : WE = E \}$,

for any $Q \in \mathcal{I}(X)$ and $E \in \mathcal{G}(X)$, then [31, Proposition 8.21] produces the following result.

**Proposition 5.2** Let $X$ be a $\mathbb{K}$-Banach space. Suppose that $Q \in \mathcal{I}(X)$ and $E \in \mathcal{G}(X)$. Then both $\mathcal{GL}(X)_Q^Q$ and $\mathcal{GL}(X)_E^E$ are analytic $\mathbb{K}$-Banach Lie subgroups of $\mathcal{GL}(X)$. The orbits $\text{Ad}(\mathcal{GL}(X), Q)$ and $\alpha(\mathcal{GL}(X), E)$ are clopen in $\mathcal{I}(X)$ and $\mathcal{G}(X)$, respectively. Moreover, the canonical bijections from, respectively, $\mathcal{GL}(X)/\mathcal{GL}(X)_Q^Q$ and $\mathcal{GL}(X)/\mathcal{GL}(X)_E^E$ onto $\text{Ad}(\mathcal{GL}(X), Q)$ and $\alpha(\mathcal{GL}(X), E)$ are $\mathbb{K}$-bi-analytic.
For every $W \in G\mathcal{L}(X)$, we denote by $[W]_Q$ and $[W]_E$ the images of $W$ in $G\mathcal{L}(X)/G\mathcal{L}(X)^Q$ and $G\mathcal{L}(X)/G\mathcal{L}(X)^E$, respectively.

**Corollary 5.3** (a) The $\mathbb{K}$-Banach submanifold $\mathcal{I}(X)$ of $\mathcal{L}(X)$ can be identified with a disjoint union of homogeneous spaces of the form $G\mathcal{L}(X)/G\mathcal{L}(X)^Q$ for some $Q \in \mathcal{I}(X)$, via the map $\Sigma_Q : G\mathcal{L}(X)/G\mathcal{L}(X)^Q \to \mathcal{I}(X)$ given by $\Sigma_Q([W]_Q) := WQW^{-1}$.

(b) The $\mathbb{K}$-Banach manifold $\mathcal{G}(X)$ can be identified with a disjoint union of homogeneous spaces of the form $G\mathcal{L}(X)/G\mathcal{L}(X)^E$ for some $E \in \mathcal{G}(X)$, via the map $\Sigma^E : G\mathcal{L}(X)/G\mathcal{L}(X)^E \to \mathcal{G}(X)$ given by $\Sigma^E([W]_E) := WE$.

(c) Let $Q \in \mathcal{I}(X)$. The assignment $\nu_0 : [W]_Q \mapsto [W]_{Q(X)}$ is a well-defined $\mathbb{K}$-analytic map from $G\mathcal{L}(X)/G\mathcal{L}(X)^Q$ to $G\mathcal{L}(X)/G\mathcal{L}(X)^{Q(X)}$ such that $\kappa \circ \Sigma_Q = \Sigma^{Q(X)} \circ \nu_0$.

Our next question concerns with connected components of $\mathcal{I}(X)$. Note that when $G\mathcal{L}(X)$ is connected, Proposition 5.2 tells us that subsets of the form $\text{Ad}(G\mathcal{L}(X), Q)$ and $\alpha(G\mathcal{L}(X), E)$ are all the components of $\mathcal{I}(X)$ and $\mathcal{G}(X)$, respectively. In this case, if we define an equivalence relation $\sim$ on $\mathcal{G}(X) \times C\mathcal{G}(X)$ such that $(E_1, F_1) \sim (E_2, F_2)$ if and only if $E_1$ and $F_1$ are Banach space isomorphic to $E_2$ and $F_2$, respectively, then all the disjoint components of $\mathcal{I}(X)$ are of the form

$$\{Q^F_E : (E, F) \in \mathcal{G}(X) \times C\mathcal{G}(X) \text{ with } (E, F) \sim (E_0, F_0)\}$$

for some $(E_0, F_0) \in \mathcal{G}(X) \times C\mathcal{G}(X)$.

In the case when $H$ is an infinite Hilbert space, Kuiper’s theorem tells us that $G\mathcal{L}(H)$ is connected (see [25]; see also [20] for the case when the Hilbert space is non-separable). Therefore, one can determine connected components of both $\mathcal{I}(H)$ and $\mathcal{G}(H)$ through the dimensions of subspaces and those of their orthogonal complements (because every idempotent in $\mathcal{L}(H)$ is similar to a self-adjoint projection; see e.g., [6, Proposition 4.6.2]).

However, $G\mathcal{L}(X)$ is in general not connected. Nonetheless, the above still holds for finite dimensional subspaces. We will establish this fact in our next result.

**Proposition 5.4** Suppose that $X$ is an infinite dimensional $\mathbb{K}$-Banach space and $n \in \mathbb{N}$. Then $\mathcal{I}_n(X) := \{Q \in \mathcal{I}(X) : \dim Q(X) = n\}$ and $\mathcal{G}_n(X) := \{E \in \mathcal{G}(X) : \dim E = n\}$ are connected components of $\mathcal{I}(X)$ and $\mathcal{G}(X)$, respectively.

**Proof** In the following, for any $F, F_1, F_2 \in \mathcal{G}(X)$, we will write $F = F_1 \oplus F_2$ if $F_1 \cap F_2 = (0)$ and $F = F_1 + F_2$.

By Proposition 5.2, it suffices to show that $\mathcal{I}_n(X)$ and $\mathcal{G}_n(X)$ are connected. Moreover, since $\kappa$ is a continuous surjection from $\mathcal{I}_n(X)$ onto $\mathcal{G}_n(X)$, we only need to establish the path connectedness of $\mathcal{I}_n(X)$. Fix $E_1, E_2 \in \mathcal{G}_n(X)$ as well as $F_1 \in \mathcal{C}_{E_1}$ and $F_2 \in \mathcal{C}_{E_2}$. Notice that $E_1 + E_2$ is finite dimensional. We fix a subspace $F \in \mathcal{C}_{E_1 + E_2}$. There exist finite dimensional subspaces $\tilde{E}_1$ and $\tilde{E}_2$ such that

$$E_1 \oplus \tilde{E}_2 = E_1 + E_2 = \tilde{E}_1 \oplus E_2.$$

We know from Lemma 3.1 that $Q^F_{E_1}$ is joined to $Q^F_{\tilde{E}_1}$ via a continuous path in $\mathcal{I}(X)_{E_1}$. Similarly, $Q^F_{E_2}$ is joined by a continuous path to $Q^F_{\tilde{E}_2}$. Therefore,
in order to show that $Q_{E_1}^{F_1}$ and $Q_{E_2}^{F_2}$ are in the same connected component, we only need to construct a continuous path joining $Q_1$ and $Q_2$.

Pick a subspace $E_3$ of $F$ with dimension $n$ and choose $\tilde{F} \in \mathcal{G}(F)$ with $F = E_3 \oplus \tilde{F}$. Set $Q_0 := Q_{E_3}^{\tilde{F} \oplus (E_1 + E_2)}$. Under the decomposition $X = E_3 \oplus (\tilde{F} \oplus \tilde{E}_2) \oplus E_1$, one can write

$$Q_0 = \begin{bmatrix} I_{E_3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{E_1} \end{bmatrix}. $$

Since $\dim E_1 = \dim E_3 = n$, there is an isomorphism $\Phi : E_1 \to E_3$. For $t \in [0, \pi/2]$, we have

$$W_t = \begin{bmatrix} \cos t \cdot I_{E_3} & 0 & \sin t \cdot \Phi \\ 0 & I_{\tilde{F} \oplus \tilde{E}_2} & 0 \\ -\sin t \cdot \Phi^{-1} & 0 & \cos t \cdot I_{E_1} \end{bmatrix} \in \mathcal{L}(X).$$

It is not hard to check that $W_t \in GL(X)$. Moreover, as $W_0Q_0W_0^{-1} = Q_0$ and $W_{\pi/2}Q_0W_{\pi/2}^{-1} = Q_1$, we see that $Q_0$ and $Q_1$ are joined by a continuous path of idempotents.

In the same way, $Q_0$ and $Q_2$ are joined by a continuous path of idempotents, via the presentations of $Q_0$ and $Q_2$ under the decomposition $X = E_3 \oplus (\tilde{F} \oplus \tilde{E}_1) \oplus E_2$. This completes the proof of the connectedness of $\mathcal{I}_n(X)$. \hfill $\square$

In the remainder of this section, we will again look at the case of a $\mathbb{K}$-Hilbert space $H$. For any $Q \in \mathcal{I}(H)$, one can find $P \in \mathcal{I}(H)$ and $W \in GL(H)$ such that

$$P^* = P \quad \text{and} \quad Q = WPW^{-1}$$

(see e.g., [6, Proposition 4.6.2]). Hence, $\text{Ad}(GL(H), P) = \text{Ad}(GL(H), Q)$. This means that in order to study components of $\mathcal{I}(H)$ it suffices to consider components generated by self-adjoint projections.

Let us denote $\mathcal{U}(H) := \{V \in \mathcal{L}(H) :VV^* = I = V^*V\}$, and

$$\mathcal{U}(H)^P := \mathcal{U}(H) \cap GL(H)^P,$$

which coincides with $\mathcal{U}(H) \cap GL(H)^P$. It is well-known that $\mathcal{U}(H)$ is a real analytic Banach manifold, and the assignment $V \mapsto VP(H)$ induces a real bi-analytic map from $\mathcal{U}(H)/\mathcal{U}(H)^P$ onto a clopen subset of $\mathcal{G}(H)$ (see e.g., [1, Proposition 3(3)]).

On the other hand, by Corollary 5.3(a), the assignment $W \mapsto WPW^{-1}$ induces a $\mathbb{K}$-bi-analytic map from $GL(H)/\mathcal{L}(H)^P$ onto a clopen subset of $\mathcal{I}(H)$. Consequently, using Theorem 4.4, we have the following.

For $P \in \mathcal{I}(H)$ with $P^* = P$, there is a map $\psi^P : GL(H)/GL(H)^P \to \mathcal{U}(H)/\mathcal{U}(H)^P$ such that $(GL(H)/GL(H)^P, \mathcal{U}(H)/\mathcal{U}(H)^P, \psi^P)$ can be identified, as locally trivial real analytic $\mathbb{K}$-Banach bundle, with the tangent bundle of $\mathcal{U}(H)/\mathcal{U}(H)^P$.\hfill $\Box$
It is natural to ask if there is a direct and explicit way to express this map $v^P$ (which is defined through several identifications). Note that by Corollary 5.3(a), if $W \in GL(H)$ and $V \in \mathcal{U}(H)$, then

$$v^P([W]_P) = [V]_P \quad \text{if and only if} \quad WP(H) = VP(H),$$

and this relation determines $v^P$.

In the following, we will describe $v^P$ via the Gram-Schmidt process, when $k := \dim_{\mathbb{K}} P(H) < \infty$. In this case, the image of $\mathcal{U}(H)/\mathcal{U}(H)^P$ in $\mathcal{G}(H)$ is precisely the subset $\mathcal{G}_k(H)$ of $k$-dimensional subspaces. Consider $W \in GL(H)$. We are required to give a canonical way to express the element $V \in \mathcal{U}(H)$ that satisfies $WP(H) = VP(H)$.

Pick an orthonormal basis $\{\xi_1, \ldots, \xi_k\}$ for the Hilbert space $P(H)$, and extend it to an orthonormal basis $B$ of $H$. By applying the Gram–Schmidt process to $\{W\xi_1, \ldots, W\xi_k\}$, one obtains a collection $\{\xi_1, \ldots, \xi_k\}$ of orthogonal unit vectors, and we extend it to an orthonormal basis $D$ of $H$. Now, consider $V \in \mathcal{U}(H)$ satisfying $V(B) = D$ and $V\xi_i = \xi_i$ for $i \in \{1, \ldots, k\}$. Then one clearly has $WP(H) = VP(H)$.

In the case when $\dim H < \infty$, we can actually apply the Gram–Schmidt process to the finite subset $\{W\xi : \xi \in B\}$ and obtain an element $V \in \mathcal{U}(H)$ satisfying our requirement. In particular, this gives the following.

**Example 5.5** (a) Consider an integer $n \geq 2$. Suppose that $Q \in \mathcal{I}(\mathbb{K}^n)$ with $k := \dim_{\mathbb{K}} Q(\mathbb{K}^n)$. There exists $W \in GL(\mathbb{K}^n)$ such that $WQW^{-1}$ is the diagonal matrix $P_k$ with the first $k$ entries in the diagonal being $1$ and all the other entries being $0$. In this case, $GL(\mathbb{K}^n)^P_k = GL(\mathbb{K}^k) \times GL(\mathbb{K}^{n-k})$ and $\mathcal{U}(\mathbb{K}^n)^P_k = \mathcal{U}(\mathbb{K}^k) \times \mathcal{U}(\mathbb{K}^{n-k})$. As in the above, the map $v$ from $GL(\mathbb{K}^n)$ to $\mathcal{U}(\mathbb{K}^n)$ given by the Gram–Schmidt process on the column vectors of matrices produces a map

$$v : GL(\mathbb{K}^n)/GL(\mathbb{K}^k) \times GL(\mathbb{K}^{n-k}) \to \mathcal{U}(\mathbb{K}^n)/\mathcal{U}(\mathbb{K}^k) \times \mathcal{U}(\mathbb{K}^{n-k})$$

and this induces a locally trivial real analytic $\mathbb{K}$-Banach bundle structure on $GL(\mathbb{K}^n)/GL(\mathbb{K}^k) \times GL(\mathbb{K}^{n-k})$.

(b) Let $n \geq 2$ and $k \in \{1, \ldots, n-1\}$. Denote by $GL_n$ and $O_n$ the sets of all $n \times n$ real invertible matrices and orthogonal matrices, respectively. Suppose that

$$v : GL_n/GL_k \times GL_{n-k} \to O_n/O_k \times O_{n-k}$$

is the map given by the Gram–Schmidt process on the column vectors of matrices. Then there is a bi-analytic bijection from the tangent bundle of $O_n/O_k \times O_{n-k}$ onto $GL_n/GL_k \times GL_{n-k}$ that respects the corresponding fiber maps. Consequently, the Gram–Schmidt process on the column vectors of elements in $GL_n$ produces the tangent bundle structure on $GL_n/GL_k \times GL_{n-k}$.

The above example tells us that one can use the Gram–Schmidt process to obtain the tangent bundle of $O_n/O_k \times O_{n-k}$.
The referee was so kind to give another explicit construction for $\nu^P$ when $P(H)$ is not necessarily finite dimensional. Let us present the construction of the referee in the following remark.

**Remark 5.6** Set $K := WP(H)$, $L := P(H)$ and $Q := WPW^{-1}$. Consider $P_K \in \mathcal{L}(H)$ to be the self-adjoint projection as in (4.1). Define

$$S := QP_K + (I - Q)(I - P_K).$$

As $P_K, Q \in \mathcal{I}(H)$ and $Q(H) = K = P_K(H)$, one obtains from Relation (3.12) that

$$QS = P_K = SP_K \quad (5.1)$$

Furthermore, we know that $S|_K = I|_K$ and that $S|_{K^\perp} = (I - Q)|_{K^\perp}$ is a Banach space isomorphism from $K^\perp$ onto ker $Q$. From these, we know that $S$ is invertible. Now, we put

$$T := S^{-1}W \quad \text{and} \quad V := T|T|^{-1}.$$ 

Then $TP = P_K T$ (see Relation (5.1)), and hence $PT^* = T^*P_K$. This implies that $P$ commutes with $T^*T$ and hence with $|T|$. Therefore, $VPV^{-1} = P_K$ and we have $VP(H) = K = WP(H)$ as required.

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