Homological Stability for Spaces of Subsurfaces with Tangential Structure

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Abstract

Given a closed, simply connected and at least 5-dimensional manifold $M$ together with some $p \in M$ and a two-plane $A$ in $T_p M$, one can consider the space of submanifolds of $M$ that are diffeomorphic to a surface of genus $g$ and that meet $p$ tangential to $A$. We introduce a notion of tangential structure for these subsurfaces and construct a stabilization map for these spaces of subsurfaces which increases the genus by 1. Then we proceed to prove homological stability for these stabilization maps.

As an application of this general result we prove homological stability for spaces of symplectic subsurfaces.

Introduction

The term homological stability refers to a phenomenon in algebraic topology. Consider a sequence of spaces $X_n$ indexed by the natural numbers together with continuous maps $f_n: X_n \to X_{n+1}$, called stabilization maps. We say that such a sequence satisfies **homological stability** if $(f_n)_*: H_* (X_n) \to H_* (X_{n+1})$ is an isomorphism for $* \leq g(n)$, where $g: \mathbb{N} \to \mathbb{N}$ is an increasing function such that $\lim_{n \to \infty} g(n) = \infty$. The most common context for this phenomenon is group homology. For example it was proven in [Nak61] that the classifying spaces of the symmetric groups $BS_n$ together with the maps induced from the inclusion $\{0, \ldots, n\} \to \{0, \ldots, n, n+1\}$ satisfy homological stability.

Harer showed in [Har85] that homological stability also holds true for the mapping class group of closed oriented surfaces of genus $g$ denoted by $\text{Mod}_g$. Even though in this situation we do not have stabilization maps $\text{Mod}_g \to \text{Mod}_{g+1}$, there are geometrically motivated maps $H_* (\text{Mod}_g) \to H_* (\text{Mod}_{g+1})$ that turn out to be isomorphisms in a specific range. Similar results were obtained for the mapping class groups of non-orientable surfaces (see [Wah08]), the spin mapping class groups (see [Har90]) and the mapping class groups of 3-manifolds (see [HW10]).

The example of mapping class groups of a surface is of particular interest for this paper. Let us denote by $\Sigma_g$ a closed, orientable and connected surface of genus $g$, and by $\text{Diff}^+ (\Sigma_g)$ its orientation preserving diffeomorphism
group. We will denote the space of smooth embeddings of $\Sigma_g$ into a possibly infinite-dimensional manifold $X$ by $\text{Emb}(M, X)$. Note that a model for the classifying space $\text{BMod}_g$ is given as follows:

$$\text{BMod}_g := \text{Emb}(\Sigma_g, \mathbb{R}^\infty)/\text{Diff}^+(\Sigma_g)$$

If one replaces $\mathbb{R}^\infty$ in this construction by a finite dimensional manifold $M$, one gets a space, which is not a classifying space for a group, nevertheless it is the space of submanifolds of $M$ which are diffeomorphic to $\Sigma_g$. It is also quite easy to see that this space classifies $\Sigma_g$-bundles together with an embedding in a trivial $M$-bundle. If $M$ is simply-connected, high-dimensional and has a non-empty boundary, and if we replace $\Sigma_g$ by an orientable connected surface of genus $g$ with $b$ boundary components and consider embeddings with a prescribed boundary condition $\delta$, then one can define stabilization maps:

$$\text{Emb}(\Sigma_{g,b}, M, \delta)/\text{Diff}^+(\Sigma_g) \to \text{Emb}(\Sigma_{g',b'}, M, \delta)/\text{Diff}^+(\Sigma_g)$$

These change the number of boundary components and the genus. It was shown in [CRW17] that these stabilization maps induce isomorphism in integral homology in a range that depends on the genus $g$. In this paper we will, among other things, extend this result by introducing a notion of tangential structures into this construction and proving homological stability for these spaces.

**Closed Embedded Subsurfaces with Tangential Structure:** Let us consider the following setup: Let $M$ denote a smooth manifold of dimension at least 5, possibly with boundary, and let again $\Sigma_g$ be a connected and oriented surface of genus $g$. Let $\text{Gr}_2(TM)$ denote the Grassmannian of oriented 2-bundles in $TM$ and note that an embedding $f: \Sigma_g \to M$ lifts to the Grassmannian as a map

$$\text{Gr}(Df): \Sigma_g \to \text{Gr}_2(TM)$$

$$p \mapsto Df(T_p\Sigma_g)$$

Let $B_2(M)$ denote a topological space together with a continuous Hurewicz-fibration $T: B_2(M) \to \text{Gr}_2(TM)$. We will call a lift $T_f$ of $\text{Gr}(Df)$ to $B_2(M)$ a **tangential structure for** $f$ and refer to $(f, T_f)$ as an embedding with tangential structure. We will also refer to $T: B_2(M) \to \text{Gr}_2(TM)$ as a **space of tangential structures of subplanes of** $TM$.

![Diagram](image.png)
Example. Fix a metric on $M$ and let $\gamma^\perp$ denote the orthogonal complement in the pullback of $TM$ of the tautological bundle over $\text{Gr}_2(TM)$. Furthermore let $S(\gamma^\perp)$ denote the sphere bundle of this bundle. Then $\pi: S(\gamma^\perp) \to \text{Gr}_2(TM)$ is a space of tangential structures of subplanes of $TM$. In this case an embedding with tangential structure is an embedding together with a non-vanishing section of its normal bundle. Another example will be introduced in Section 8 in order to prove the corollary mentioned later in this introduction.

Remark. This definition of tangential structures should be thought of as a relative tangential structure in contrast to the usual definition of a tangential structure in terms of a fibration $B \to \text{Gr}_2(\mathbb{R}^\infty)$. Nevertheless it is easy to see that the usual definition is a special case of the present definition by pulling back $B \to \text{Gr}_2(\mathbb{R}^\infty)$ via a classifying map of the tautological bundle over $\text{Gr}_2(TM)$.

Now let us return to the general case and consider the following space of embeddings with tangential structure

$$\text{Emb}^T(\Sigma_g, M) = \left\{ (f, T_f) \mid f: \Sigma_g \to M \text{ is an embedding and } T_f: \Sigma_g \to B_2(M) \text{ makes diagram (1) commute} \right\}$$

The definition of the topology of this space will be given in Section I.

Let $\phi$ denote an orientation preserving diffeomorphism of $\Sigma_g$. Note that $\text{Gr}(D(f \circ \phi)) = \text{Gr}(Df) \circ \phi$. Therefore the group $\text{Diff}^+(\Sigma_g)$ acts freely on $\text{Emb}^T(\Sigma_g, M)$ by precomposition.

In order to stabilize later on we need to restrict ourselves to a smaller set of embeddings (see the remark after Theorem 1 for a justification):

Let us fix $p \in \Sigma_g$, $q \in M$ and $A \in \text{Gr}_2(T_q M)$ together with an $A^T \in T^{-1}(\{A\})$ and define $\text{Emb}^T_{A^T}(\Sigma_g, M)$ as

$$\left\{ (f, T_f) \in \text{Emb}^T(\Sigma_g; M) \mid f(p) = q, \text{ Gr}(Df)(T_p \Sigma_g) = A, T_f(p) = A^T \right\}$$

If we restrict the aforementioned $\text{Diff}^+(\Sigma_g)$ action to the group of orientation preserving diffeomorphisms that fix $p$ denoted by $\text{Diff}^+_p(\Sigma_g)$, then we get a free $\text{Diff}^+_p(\Sigma_g)$ action on $\text{Emb}^T_{A^T}(\Sigma_g, M)$. We will denote the quotient of this action by $\mathcal{E}^T_{A^T}(\Sigma_g, M)$. This is the space of subsurfaces of $M$ we are interested in and we will refer to an element of it as a subsurface with tangential structure of $M$. In Section 2 we will construct a stabilization map

$$\sigma_g: \mathcal{E}^T_{A^T}(\Sigma_g, M) \to \mathcal{E}^T_{A^T}(\Sigma_{g+1}, M)$$

which is heuristically given by flattening the subsurfaces in a neighborhood of $q$ and then taking a connected sum with a torus in this neighborhood. With this map at hand we can formulate our first result:

**Theorem 1.** Let $M$ be a simply connected smooth manifold of dimension at least 5 and let $T: B_2(M) \to \text{Gr}_2(M)$ be a space of tangential structures of
subplanes of $TM$ such that the fiber of $T$ is at least 2-connected. Fix $q \in M$, $A \in \text{Gr}_2(T_qM)$ and $A^T \in T^{-1}(\{A\})$. Then

$$\sigma_g : E^T_A(\Sigma_g, M) \to E^T_A(\Sigma_{g+1}, M)$$

induces an isomorphism in integral homology in degrees less than or equal to $2^{g}$. 

Remark. The role of our fixation of a basepoint, the tangent plane at that point, and the reduction to the case of surfaces with boundary should be compared to the situation of the homological stability phenomenon for the mapping class group. In order to define what homological stability means for the mapping class group of closed surfaces and prove it, one relates it to the mapping class group of surfaces with boundary, where it is quite easy to define stabilization maps. Furthermore there is a stabilization map that decreases the number of boundary components and therefore enables one to return to the case of mapping class groups of closed surfaces.

Symplectic Subsurfaces: In Section 8 we will use Theorem 1 to prove homological stability for spaces of symplectic subsurfaces. In order to do this let $(M, \omega)$ denote a simply connected symplectic manifold of dimension at least 6. We will call an embedding $f: \Sigma_g \to M$ an oriented symplectic embedding if $f^*\omega$ is a symplectic form on $\Sigma_g$ and $\int_{\Sigma_g} f^*\omega > 0$. Fixing $A \in \text{Gr}_2(T_qM)$, which is a symplectic 2-plane, i.e. $\omega|_A$ is non-degenerate, we can consider $\text{SEmb}_A(\Sigma_g, M)$, the space of oriented symplectic embeddings, such that $f(p) = q$ and $Df(T_p\Sigma_g) = A$. Being an oriented symplectic embedding is invariant under the free right action of $\text{Diff}_+^\omega(\Sigma_g)$ and therefore we can form the quotient to obtain $\mathcal{SE}_A(\Sigma_g, M)$, the space of symplectic subsurfaces which are tangential to $A$.

Inspired by a proposition in the context of the h-principle, namely a variation of Theorem 12.1.1 in [EM02], we will construct a tangential structure in Section 8 such that the space of embeddings with this tangential structure models the space of symplectic embeddings very well. Then using Theorem 1 we prove the following corollary:

**Corollary 1.** Let $(M, \omega)$ denote a simply-connected symplectic manifold of dimension at least 6 and fix $q \in M$ and a symplectic 2-plane in $T_qM$. There is a homomorphism of integral homology:

$$f_g : H_*(\mathcal{SE}_A(\Sigma_g, M); \mathbb{Z}) \to H_*(\mathcal{SE}_A(\Sigma_{g+1}, M); \mathbb{Z})$$

And this homomorphism induces an isomorphism for $* \leq 2^{g}$. 

Even though the present stabilization map exists only on the level of homology, the author is fairly certain that with the presented methods it is possible to construct an actual map $\mathcal{SE}_A(\Sigma_g) \to \mathcal{SE}_A(\Sigma_{g+1})$ that realizes $f_g$ (see the end of Section 8 for a more detailed discussion of this).
Embedded Subsurfaces with Boundary and Tangential Structure:

In order to define the stabilization maps in Section 2 and prove Theorem 1 we will result the aforementioned pointed setting to manifolds with boundary. Then we will prove homological stability for spaces of subsurfaces with boundary sitting inside manifolds with boundary. In order to formulate this result we need some more notation:

Let \( \Sigma_{g,b} \) denote an oriented and connected surface of genus \( g \) with \( b \) boundary components and let \( M \) denote a simply-connected manifold of dimension at least 5 with non-empty boundary \( \partial M \) together with a space of tangential structures of subplanes of of its tangent bundle \( T: B_2(M) \to \text{Gr}_2(TM) \). We say that two embeddings \( f, g: (\Sigma_{g,b}, \partial \Sigma_{g,b}) \to (M, \partial M) \) with tangential structures \( T_f \) and \( T_g \) have the same jet along \( \partial M \) if the \( \infty \)-jet of \( f \) and \( g \) agree at \( \partial M \) and \( T_f|_{\Sigma_{g,b}} = T_g|_{\Sigma_{g,b}} \).

Note that the group of diffeomorphisms that fixes the \( \infty \)-jet of the identity at \( \partial \Sigma_{g,b} \), denoted by \( \text{Diff}_0(\Sigma_{g,b}) \), acts freely on the space of embeddings \( (\Sigma_{g,b}, \partial \Sigma_{g,b}) \to (M, \partial M) \). Fix an \( \infty \)-jet at \( \partial M \) of embeddings \( \Sigma_{g,b} \to M \), denoted by \( \delta \), together with a tangential structure for this jet denoted by \( \delta^T \).

Consider the space of embeddings, whose \( \infty \)-jet at \( \partial M \) agrees with \( \delta \) and whose tangential structure at \( \partial M \) agrees with \( \delta^T \). Note that the aforementioned action of \( \text{Diff}_0(\Sigma_{g,b}) \) restricts to an action on this set of embeddings with a specified \( \infty \)-jet and tangential structure at \( \partial M \). We denote the quotient by \( \mathcal{E}^T_{g,b}(M; \delta^T) \) and refer to elements of this space as **subsurfaces with tangential structure in** \( M \). The fixed boundary condition enables us to stabilize these spaces in the following way:

We will write \( I \) for the unit interval \([0,1]\) throughout this text. We can embed \( \partial M \times I \) into \( M \) via a collar that identifies \( \partial M \times \{1\} \) with \( \partial M \) via the identity and we pullback \( T: B_2(M) \to \text{Gr}_2(M) \) to \( \text{Gr}_2(\partial M \times I) \). Let \( P \) denote a subsurface with tangential structure of \( \partial M \times I \) such that every connected component of \( P \) has a non-empty intersection with \( \partial M \times \{0\} \). If we assume that \( P \cap \partial M \times \{0\} \) agrees with the image of the jet \( \delta^T \) and some further reasonable constraints on the "higher jets" of \( P \) as well as the condition that the tangential structure of \( P \) at \( P \cap \partial M \times \{0\} \) agrees with the tangential structure of \( \delta^T \), then the following map is well-defined:

\[
- \cup P: \mathcal{E}^T_{g,b}(M; \delta^T) \to \mathcal{E}^T_{g',b'}(M \cup_{\partial M \times \{0\}} \partial M \times I, \delta'^T)
\]

\[
V \subset M \mapsto V \cup P \subset M \cup_{\partial M \times \{0\}} \partial M \times I
\]

Here \( g', b' \) and \( \delta'^T \) depend on the topology of \( P \) and note that by identifying \( M \cup_{\partial M \times \{0\}} \partial M \times I \) with \( M \), we can justify that this is indeed a stabilization map of the subsurfaces with tangential structure of \( M \).

We will prove that there are three special types of subsurfaces of \( \partial M \times I \) which turn out to be the building blocks of every such \( P \) (Proposition 2.13). These building blocks are of the following form: \( P \) can either be homeomorphic to a collection of cylinders together with a single pair of pants.
or a collection of cylinders together with a single disk. The pair of pants case can be split into two cases: Either the pair of pants meets \( \partial M \times \{0\} \) in two components or in a single component. If \( P \) is of the form described in the first case we will write \( \alpha_{g,b}(M) \) for \(-\cup P\), in the second case we will refer to \(-\cup P\) as \( \beta_{g,b}(M) \). If \( P \) is a collection of cylinders and a disk, note that the requirement for the connected components of \( P \) forces the boundary of the disk to meet \( \partial M \times \{0\} \). In this case we will refer to \(-\cup P\) as \( \gamma_{g,b}(M) \). All of these notations hide some ambiguity given by differently embedded but homeomorphic surfaces, but the statements proven here do not depend on the specific \( P \) and therefore it is dropped from the notation.

With these notations at hand we can formulate our second result:

**Theorem 2.** If \( M \) is a simply connected manifold of dimension at least 5, \( T: B_2(M) \to Gr_2(M) \) is a space of tangential structures of subplanes of \( TM \) such that the fiber of \( T \) is at least 2-connected and \( \delta^T \) is some \( \infty \)-jet, then:

- \( \alpha_{g,b}(M): E^T_{g,b}(M; \delta^T) \to E^T_{g+1,b-1}(M \cup \partial M \times \{0\} \partial M \times I; \delta^T) \) induces an isomorphism in integral homology in degrees less than or equal to \( \frac{2}{3}(g-1) \) and an epimorphism in the next degree.

- \( \beta_{g,b}(M): E^T_{g,b}(M; \delta^T) \to E^T_{g,b+1}(M \cup \partial M \times \{0\} \partial M \times I; \delta^T) \) induces an isomorphism in integral homology in degrees less than or equal to \( \frac{1}{2}(2g-3) \) and an epimorphism in the next degree. If one of the boundaries of the pair of pants intersecting \( \partial M \times \{1\} \) is contractible in \( \partial M \), then \( \beta_{g,b} \) induces a monomorphism in all degrees.

- \( \gamma_{g,b}(M): E^T_{g,b}(M; \delta^T) \to E^T_{g,b-1}(M \cup \partial M \times \{0\} \partial M \times I; \delta^T) \) induces an isomorphism in integral homology in degrees less than or equal to \( \frac{1}{2}g \) and an epimorphism in the next degree. If \( b \geq 2 \), then it is always an epimorphism.

**Overview and Acknowledgments:** This paper can roughly be divided into three parts. The first part consists of the first two sections and contains basic definitions used throughout the paper (Section 1), formulates the stabilization process and by doing this reduces Theorem 1 to Theorem 2 (Section 2).

The second part consists of Sections 3 to 7 and proves Theorem 2. Section 3 contains an overview of the proof of most parts of Theorem 2 concerning the maps of type \( \alpha_{g,b} \) and \( \beta_{g,b} \). Section 4 contains some more definitions and we construct in Section 5 the semi-simplicial resolutions needed for the proof and prove some properties of them. Section 6 finishes the proof of the theorem formulated in Section 3 and finally Section 7 follows a similar but much easier path as the sections before to prove the rest of Theorem 2, namely the stability result for maps of type \( \gamma_{g,b} \) and that maps of type \( \beta_{g,b} \) are monomorphisms if one of the components in \( \partial M \times \{0\} \) is contractible.
Lastly Section 8 applies Theorem 1 to a certain kind of space of tangential structure of subplanes on a symplectic manifold to derive the aforementioned homological stability result for symplectic subsurfaces.

The second part of this paper is inspired by the proof of the second theorem presented in [CRW17]. They prove Theorem 2 without the tangential structure and we were able to adapt their proof to include the tangential structures. While some propositions and lemmas in the second part of this paper could be derived from corresponding statements in [CRW17] using certain pullback diagrams together with Lemma 3.4 some other propositions require a more careful analysis, because of the extra data carried by the tangential structures. Furthermore the results of Section 2 and Section 8 are completely independent of [CRW17].

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1 Basic Definitions

1.1 Spaces of Embeddings and Embedded Submanifolds

The following section will provide almost all necessary definitions for the rest of this paper. Let \((X, \partial X)\) denote a compact smooth manifold with boundary and let \(\partial^0 X\) denote a union of connected components of \(\partial X\). We call an embedding \(c : \partial^0 X \times [0, 1) \to X\) such that \(c(x, 0) = x\) a collar of \(\partial^0 X\).

We say that \((X, \partial X)\) is collared if some \(\partial^0 X\) and a collar of \(\partial^0 X\) are fixed. It is important to note that if \(X\) is collared, then

\[ X_1 := X \cup_{\partial^0 X} \partial^0 X \times [0, 1] \]

has a canonical smooth structure.

**Definition 1.1.** Let \(B\) and \(M\) denote smooth collared manifolds. A **map** \(f : B \to M\) between collared manifolds \(B\) and \(M\) is a map that sends the boundary of \(B\) to the boundary of \(M\) and extends to a map \(F : B_1 \to M_1\) such that for all \(x \in \partial^0 B\) and \(t \in [0, 1]\) \(F(x, t) = (e(x), t)\) for some \(e : \partial^0 B \to \partial^0 M\).

We will use the words manifold and collared manifold synonymously as well as map (or embedding or diffeomorphism) and map between collared
manifolds. Note that as a consequence an embedding from $B$ into $M$ maps the boundary of $B$ to the boundary of $M$ and the map is transverse to the boundary of $M$. If $A \subset B$ and $W \subset M$ are submanifolds we call a map $e: (B, A) \to (M, W)$ an embedding of pairs if $e^{-1}(W) = A$.

**Definition 1.2.** Let $f, g: B \to M$ denote embeddings. We say that $f$ and $g$ have the same jet along $\partial M$ if there exists an open neighborhood $U$ of $\partial B$ such that $f|_U = g|_U$. This defines an equivalence relation on the space of embeddings $\text{Emb}(B, M)$ equipped with the $C^\infty$-topology and we denote the quotient map by

$$J: \text{Emb}(B, M) \to J_\partial(B, M)$$

We write $\text{Emb}(B, M; d) := J^{-1}(d)$.

From here on forth let us specialize to the case where $B$ is a compact connected oriented surface of genus $g$ with $b$ boundary components denoted by $\Sigma_{g,b}$. We denote by $\text{Diff}^+(\Sigma_{g,b})$ the group of orientation-preserving diffeomorphisms of $\Sigma_{g,b}$. This group acts freely on the space of embeddings via precomposition and it also acts on $J_\partial$ via precomposition. We define

$$E^+_g(M) := \text{Emb}(\Sigma_{g,b}, M)/\text{Diff}^+(\Sigma_{g,b})$$

$$J_\partial(\Sigma_{g,b}, M) := J_\partial(\Sigma_{g,b}, M)/\text{Diff}^+(\Sigma_{g,b})$$

The first space will be called the space of subsurfaces. An element $W \in E^+_g(M)$ is an unparametrized embedding or in other words a submanifold $W \subset M$ diffeomorphic to $\Sigma_{g,b}$ and we will call it a subsurface of $M$.

Since the map $J$ is equivariant with respect to the $\text{Diff}^+(\Sigma_{g,b})$-actions we get an induced map $\mathcal{J}$ between the quotient spaces. We define $E^+(\Sigma_{g,b}, M; \delta)$ as $\mathcal{J}^{-1}(\delta)$ for some $\delta \in \mathcal{J}(\Sigma_{g,b}, M)$. Note that if $\delta = [d]$ then

$$E^+_g(M, \delta) \cong \text{Emb}(\Sigma_{g,b}, M; d)/\text{Diff}^+_\partial(\Sigma_{g,b})$$

Here $\text{Diff}^+_\partial(\Sigma_{g,b})$ denotes the group of diffeomorphisms of $\Sigma_{g,b}$ that fixes the jet of the identity along $\partial \Sigma_{g,b}$. As a slight abuse of notation we will also write $\delta$ for the image of some representative of $\delta \in \mathcal{J}(\Sigma_{g,b}, M)$ in $\partial M$. We will call $\delta$ a boundary condition.

These were spaces of subsurfaces with a fixed boundary condition. Now we will speak about pointed subsurfaces. Let $M$ denote a closed simply-connected smooth manifold and abbreviate $\Sigma_{g,0}$ by $\Sigma_g$. Fix $q \in \Sigma_g$, $p \in M$ and an oriented 2-plane $A$ in $T_pM$. We will consider the space

$$\text{Emb}_A(\Sigma_g, M) := \{ f \in \text{Emb}(\Sigma_g, M) \mid f(q) = p \text{ and } Df(T_q \Sigma_g) = A \}$$

equipped with the $C^\infty$-topology. Let $\text{Diff}_q(\Sigma_g)$ denote the subgroup of $\text{Diff}^+(\Sigma_g)$ that fixes $q$. This group acts on $\text{Emb}_A(\Sigma_g, M)$ freely via precomposition and we denote the quotient by $E^+_A(\Sigma_{g,b}, M)$. 

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Remark. In this pointed case we could replace \( M \) by a manifold with boundary and \( \Sigma_g \) by \( \Sigma_{g,b} \) and include some boundary condition \( \delta \) into this notation, but we will not bother with this. If one wants to include a boundary into the pointed case, then all proofs regarding the pointed case in this paper work verbatim and the same results would hold true.

1.2 Tangential structures

Let \( M \) denote a simply-connected manifold possibly with boundary. There is the fiber bundle \( \text{Gr}_2(TM) \to M \), where \( \text{Gr}_2(TM) \) denotes the space of oriented 2-planes in \( TM \). For the purpose of this paper a space of tangential structures of subplanes of \( TM \) will denote a continuous Hurewicz-fibration \( \tau : B_2(M) \to \text{Gr}_2(TM) \). For a submanifold \( V \) of \( M \) we have an induced map \( \text{Gr}_2(TV) \to \text{Gr}_2(TM) \) and we will denote the pullback along this inclusion by \( B_2(V) \). If \( V \) is an oriented surface, then \( \text{Gr}_2(TV) \) is a two-sheeted covering with a distinguished section. In this particular case we will identify the correctly oriented connected component of \( \text{Gr}_2(TV) \) with \( V \) and define \( B_2(V) \) as a fibration over \( V \).

If we have an embedding \( e : \Sigma_{g,b} \to M \) we get an induced map

\[
\text{Gr}(De) : B \to \text{Gr}_2(TM)
\]

by sending a point \( p \) in \( B \) to \( De(T_p\Sigma_{g,b}) \) considered as an oriented 2-plane. We will call this map the Grassmannian differential of \( e \).

Definition 1.3. We will call a lift \( T_e \) of \( \text{Gr}(De) \) to \( B_2(M) \) a tangential structure of \( e \).

\[
\begin{array}{ccc}
\Sigma_{g,b} & \xrightarrow{\text{Gr}(De)} & \text{Gr}_2(TM) \\
\downarrow{\uparrow{T_e}} & & \downarrow{T} \\
B_2(M) & & \\
\end{array}
\]

\( \text{Emb}^T(\Sigma_{g,b}, M) \) will denote the space of embeddings with tangential structure. This space is topologized as a subspace of \( \text{Map}(\Sigma_{g,b}, B_2(M)) \times C^\infty(\Sigma_{g,b}, M) \), where we equip the first factor with the compact-open topology and the second factor with the \( C^\infty \) topology.

Even though an element in \( \text{Emb}^T(\Sigma_{g,b}, M) \) consists of a pair of maps, we will most of the time write it as a single map \( f \), which will also represent the underlying embedding. In this case we will refer to the tangential structure of \( f \) as \( T_f \). We will say that two embeddings with tangential structure \( f, g \) have the same jet along \( \partial M \) if the underlying embeddings have the same jet along \( \partial M \) and \( T_f|_{\partial\Sigma_{g,b}} = T_g|_{\partial\Sigma_{g,b}} \).

Let us recall the example of the introduction:
Example. Let $\gamma \perp$ denote the complement of the tautological 2-plane bundle over $\text{Gr}_2(TM)$ i.e. $\pi^*TM/\gamma$, where $\pi: \text{Gr}_2(TM) \to M$ denotes the projection and $\gamma$ denotes the tautological bundle which evidently includes into $\pi^*TN$. Let $S\gamma \perp := \gamma \perp \setminus 0$ denote the complement of the image of the zero section of this bundle. Then $T: S\gamma \perp \to \text{Gr}_2(TM)$ is a Hurewicz-fibration. For an embedding $e: \Sigma_g \to M$ the space $B_2(e(\Sigma_g))$ is the complement of the zero-section of the normal bundle and a tangential structure corresponds to a continuous(!) section of this bundle. Therefore in this case $\text{Emb}^T(\Sigma_g, M)$ is the space of embeddings together with a continuous section of the normal bundle.

Remark. In some situations it might make sense to replace $\text{Map}(\Sigma_{g,b}, B_2(M))$ by $C^\infty(\Sigma_{g,b}, B_2(M))$ and replace the compact-open topology by the $C^\infty$ topology. This makes sense if $B_2(M)$ is some kind of smooth space of tangential structures of subplanes of $TM$ like in the previous example (In the example this would make the sections of the normal bundle a smooth section). To include the smoothness one also has to alter the definition of having the same jet to include the jet of the lifts. Everything proven in this paper works in that case as well.

We will sometimes call an embedding with tangential structure just an embedding, but it will always be clear from the context if it possesses an additional tangential structure.

Since $\text{Gr}(De)(p) = \text{Gr}(D(e \circ \phi^{-1}))(\phi(p))$, we see that $\text{Diff}^+(\Sigma_{g,b})$ acts on $\text{Emb}^T(\Sigma_{g,b}, M)$ via precomposition. We can form the quotient and restrict ourselves to fixed boundary conditions given by equivalence classes of jets to obtain $\text{Emb}^T(\Sigma_{g,b}, M; dT)$ and $\mathcal{E}^T_{g,b}(M; \delta T)$ (note the different definition of having the same jet in this context). We will call the second space the space of subsurfaces with tangential structure and an element in it will be called a subsurface with tangential structure of $M$. If some jet $\delta T$ of an embedding with tangential structure is given, we will sometimes write $\delta$ for the image of the underlying boundary condition without tangential structure. Furthermore if we specify an $A \in \text{Gr}_2(T_pM)$ together with an element $A^T$ in $T^{-1}(A)$ then the definition of $\mathcal{E}^T_{A^T}(\Sigma_{g,b}, M)$ is obvious.

We have forgetful maps $\mathcal{E}^T_{g,b}(M; \delta T) \to \mathcal{E}^+_{g,b}(M; \delta)$ and $\mathcal{E}^T_{A^T}(\Sigma_{g,b}, M) \to \mathcal{E}^+_{A^T}(\Sigma_{g,b}, M)$ and we will prove in Lemma 3.4 that these maps are Hurewicz-fibrations.

Remark. If we have a subsurface with a tangential structure $V$ and some isotopy $(I, 0) \to (\text{Diff}^+(M), Id)$ we can move $V$ along $M$ using this isotopy to obtain $V_t$. Since $T$ is a fibration we can also equip $V_t$ with a tangential structure and if we say nothing about tangential structures in this context, we will assume implicitly that a lift of the aforementioned path in $\mathcal{E}^+(\Sigma_{g,b}, M)$ is chosen. Note that the tangential structure of $V_1$ is not unique i.e. there might be many different tangential structures for $V_1$!
1.3 Retractile Spaces and Fibrations

The following arguments will be used frequently throughout this paper to prove that certain maps are locally trivial fibrations and the ideas date back to [Cer61].

Definition 1.4. Let $G$ denote a topological group and $X$ a $G$-space. We say that $X$ is $G$-locally retractile if every point $x \in X$ possesses an open neighborhood $U_x$ and a continuous map, called the $G$-local retraction around $x$, $\xi : (U_x, x) \to (G, id)$, such that $\xi(y) \cdot x = y$ for all $y \in U_x$.

The following lemma will be used several times throughout this paper.

Lemma 1.5. If $X$ is $G$-locally retractile and $f : Y \to X$ denotes a $G$-equivariant map, then $f$ is a locally trivial fibration.

Proof. For $x \in X$ choose an $U_x$ and a $G$-local retraction around $x \xi : U_x \to G$. Then

$$f^{-1}(\{x\}) \times U_x \to f^{-1}(U_x)$$

$$(z, y) \mapsto \xi(y) \cdot z$$

gives the desired local trivialisation as is easily checked.

Let $\text{Diff}_0(M)$ denote the group of diffeomorphisms of $M$ that fix the $\infty$-jet of the identity of $M$ at $\partial M$. The proof of the following proposition can be found in [Cer61].

Proposition 1.6. Let $d$ denote the jet of an embedding from $B$ into $M$, then $\text{Emb}(B, M; d)$ is $\text{Diff}_0(M)$-locally retractile.

The following proposition can be found in [CRW17] with a proof that relies on a reference to [BF81] or [Mic80]. We will give a sketch of a proof of the proposition, because the ideas are quite nice and similar arguments will occur later on in this work.

Proposition 1.7. The space $\mathcal{E}^+_{g,b}(M; \delta)$ is $\text{Diff}_0(M)$-locally retractile.

Proof. Let $W \subset M$ denote an element in $\mathcal{E}^+_{g,b}(M; \delta)$ and $N$ a tubular neighborhood of $W$ with corresponding projection $\pi : N \to W$. We define

$$U_W = \{V \in \mathcal{E}^+_{g,b}(M; \delta) \mid V \subset N \text{ and } \pi|_V \text{ is a diffeomorphism from } V \text{ to } W\}$$

which is an open subset of $\mathcal{E}^+_{g,b}(M; \delta)$, because the diffeomorphisms form an open subset of the maps from $V$ to $W$. For $V$ in $U_W$ we can define an isotopy of the embedded subsets by pushing $V$ along the fiber of $N$ onto $W$. Furthermore we can use the isotopy extension theorem, to extend these isotopies continuously to isotopies of $M$. The resulting diffeomorphism at time 1 of these isotopies gives us the desired $\xi(V)$ for the $\text{Diff}_0(M)$-local retraction.
The last Proposition implies the following useful observation:

**Corollary 1.8.** The two maps

\[ \text{Emb}(\Sigma_{g,b}, M; d) \to \mathcal{E}^+(\Sigma_{g,b}, M; \delta) \]
\[ \text{Emb}^T(\Sigma_{g,b}, M; d^T) \to \mathcal{E}^T(\Sigma_{g,b}, M; \delta^T) \]

are locally trivial fibrations.

**Proof.** The only thing that might need clarification in this corollary is the second assertion, but to prove this one only has to note that

\[
\begin{array}{ccc}
\text{Emb}^T(\Sigma_{g,b}, M; d^T) & \to & \text{Emb}(\Sigma_{g,b}, M; d) \\
\downarrow & & \downarrow \\
\mathcal{E}^T(\Sigma_{g,b}, M; \delta^T) & \to & \mathcal{E}^+(\Sigma_{g,b}, M; \delta)
\end{array}
\]

is a pullback diagram. \qed

### 1.4 Tubular neighborhoods and Thickened Embeddings

If \( V \subset M \) denotes a submanifold, we will write \( N_M \) for the normal bundle of \( V \) i.e. the quotient of \( TM|_V \) by \( TV \). A tubular neighborhood of \( V \) is an embedding \( f: N_M V \to M \) such that \( f|_V \) is the identity and the composition

\[
TV \oplus N_M V \xrightarrow{\cong} T(N_M V)|_V \xrightarrow{Df} TM|_V \xrightarrow{\text{proj}} N_M V
\]

agrees with the projection onto the second factor. If \( (V, V \cap W) \subset (M, W) \) denotes an embedded pair we define a tubular neighborhood of \( V \) in \( (M, W) \) to be a tubular neighborhood \( f: N_M V \to M \) of \( V \) such that \( f|_{N_W(V \cap W)} \) is a tubular neighborhood of \( V \cap W \) in \( W \). If \( V \) has a boundary we assume that every tubular neighborhood of \( V \) is actually a tubular neighborhood of \( (V, \partial V) \) in \( (M, \partial M) \).

Starting with the vector bundle \( N_M V \) we can obtain a disk bundle by compactifying this fiberwise using a sphere at infinity to obtain \( \overline{N_M V} \). An embedding of \( \overline{N_M V} \to M \) is called a closed tubular neighborhood if its restriction to \( N_M V \) is a tubular neighborhood. We denote by

\[
\text{Tub}(V, M) \subset \text{Emb}((\overline{N_M V}, V), (M, V))
\]

the subspace of closed tubular neighborhoods. Similar notation will occur if we consider tubular neighborhoods of pairs.

A proof of the following lemma is sketched in Section 2.5 of [CRW17].

**Lemma 1.9.** If \( V \) and \( W \) are compact submanifolds of \( M \), then \( \overline{\text{Tub}}(V, M) \) and \( \overline{\text{Tub}}((V, V \cap W), (M, W)) \) are contractible.
Definition 1.10. We call an embedding, possibly equipped with a tangential structure, $f: \Sigma_{g,b} \to M$ together with a closed tubular neighborhood of $f(\Sigma_{g,b})$ a thickened embedding of $\Sigma_{g,b}$. We will write $\overline{T\text{Emb}}(B,M;\delta)$ for the space of thickened embeddings (we could add a $^T$ to indicate that these are thickened embeddings with tangential structure). We could furthermore replace $B$ by $(B,A)$ and $M$ by $(M,W)$ and talk about thickened embeddings of pairs.

Notation. Even though a closed tubular neighborhood is a map we will sometimes use the same notation for its image.

The topology of the space of thickened embeddings is quite tricky and we refer the reader to Section 2.5 in [CRW17] for details. The important thing to note about this space is the following proposition:

Proposition 1.11. $\text{Diff}_{\partial}(M)$ acts via post composition on $\overline{T\text{Emb}}(B,M;d)$ and this space is $\text{Diff}_{\partial}(M)$-locally retractile.

This implies almost immediately the following corollary:

Corollary 1.12. The forgetful map $\overline{T\text{Emb}}(B,M;d) \to \text{Emb}(B,M;d)$ is a locally trivial fibration with fiber over a basepoint $e \in \text{Emb}(B,M;d)$ given by the space of tubular neighborhoods of $e(B)$. Furthermore the forgetful map $\overline{T\text{Emb}}^T(B,M;d^T) \to \text{Emb}^T(B,M;d^T)$ is also a locally trivial fibration with fiber the space of tubular neighborhoods of some $e(B)$.

Proof. The first assertion follows trivially since $\text{Emb}(B,M;d)$ is $\text{Diff}_{\partial}(M)$-locally retractile. The second assertion follows since the following diagram is a pullback diagram:

$$
\begin{array}{ccc}
\overline{T\text{Emb}}^T(B,M;d^T) & \longrightarrow & \overline{T\text{Emb}}(B,M;d) \\
\downarrow & & \downarrow \\
\text{Emb}^T(B,M;d^T) & \longrightarrow & \text{Emb}(B,M;d)
\end{array}
$$

\qed

2 Stabilization Maps

2.1 Pointedly Embedded Subsurfaces

Let $(M,q)$ denote an at least 5-dimensional pointed manifold with a space of tangential structures of subplanes of $TM$: $B_2(M) \to \text{Gr}_2(TM)$ and $(\Sigma_{g},p)$ an oriented closed surface of genus $g$ with basepoint $p$. We want to stabilize $\mathcal{E}_{AT}^T(\Sigma_g,M)$, the space of pointed embedded surfaces in $M$ with a fixed tangent plane at $q$ together with a fixed tangential structure on this tangent plane i.e. construct certain map $\sigma_g: \mathcal{E}_{AT}^T(\Sigma_g,M) \to \mathcal{E}_{AT}^T(\Sigma_{g+1},M)$.
Let $V$ denote a coordinate chart centered around $q$ diffeomorphic to $\mathbb{R}^n$ such that the fixed tangent plane $A$ corresponds to $\mathbb{R}^2$ in $\mathbb{R}^n$, where this inclusion is given by specifying the last $n-2$ coordinates to be zero.

**Definition 2.1.** $\mathcal{E}_{A^T, \text{flat}}^+(\Sigma_g, M)$ is the subspace of $\mathcal{E}_{A^T}^+(\Sigma_g, M)$ of those subsurfaces such that their intersection with the open ball of a fixed radius $R$ in $V$ equals $\mathbb{R}^2 \cap B_R(0)$, where we identify the span of the first two coordinate vectors with $\mathbb{R}^2$, and such that the tangential structure is constantly $A^T$ on $B_{R/2}(0)$ with respect to some fixed trivialization of $B_2(B_R(0))$. Similarly $\mathcal{E}_{A, \text{flat}}^+(\Sigma_g, M)$ is the image of $\mathcal{E}_{A^T, \text{flat}}^+(\Sigma_g, M)$ in $\mathcal{E}_{A}^+(\Sigma_g, M)$ under the forgetful map.

We will need the following two results to define pointed stabilization maps. The first lemma will deal with the case without tangential structure and the subsequent corollary will extend this to the case with tangential structure.

**Proposition 2.2.** There is a coordinate chart centered at $q$ such that the inclusion $i: \mathcal{E}_{A, \text{flat}}^+(\Sigma_g, M) \to \mathcal{E}_{A}^+(\Sigma_g, M)$ is a homotopy equivalence.

**Remark.** It is far easier to prove that the aforementioned inclusion is a weak homotopy equivalence, but we will need a homotopy inverse to the inclusion to construct the stabilization maps and so we have to take this rather technical detour to construct this homotopy inverse.

In order to proof this we will define two subspaces of $\mathcal{E}_{A}^+(\Sigma_g, M)$, which both contain $\mathcal{E}_{A, \text{flat}}^+(\Sigma_g, M)$, and fit in the following diagram

\[
\begin{array}{cccc}
\mathcal{E}_{A}^+(\Sigma_g, M) & \xrightarrow{\Psi_1} & \mathcal{E}_{A^R}^+(\Sigma_g, M) & \xleftarrow{\Psi_2} & \mathcal{E}_{A^T, \text{flat}}^+(\Sigma_g, M) & \xleftarrow{\Psi_3} & \mathcal{E}_{A}^+(\Sigma_g, M)
\end{array}
\]

The hooked arrows are inclusions and the $\Psi_i$ represent their respective homotopy inverses. After the following three lemmas we will prove that all the aforementioned inclusions are indeed homotopy equivalences by constructing the $\Psi_i$ and the necessary homotopies. This will finish the proof of Proposition 2.2. So let us define $\mathcal{E}_{A^R}^+(\Sigma_g, M)$ and $K$:

**Definition 2.3.** Fix $r \in (0, \infty)$ and let $\pi: \mathbb{R}^n \to \mathbb{R}^2$ denote the projection onto the first two coordinates i.e. the map that sends $(x_1, \ldots, x_n)$ to $(x_1, x_2)$. We define

\[
\text{Emb}_{A, r}(\Sigma_g, M) := \left\{ f \in \text{Emb}_A(\Sigma_g, M) \mid \exists U \text{ a closed neighborhood of } q \text{ s.t. } \pi \circ f|_U \text{ is a diffeomorphism with image } \overline{B_r(0)} \subset \mathbb{R}^2 \right\}
\]

The action of $\text{Diff}_+^+(\Sigma_g)$ via precomposition on $\text{Emb}_A(\Sigma_g, M)$ restricts to a free action on $\text{Emb}_{A^R}(\Sigma_g, M)$ and we denote the quotient by $\mathcal{E}_{A^R}^+(\Sigma_g, M)$. For a $V \in \mathcal{E}_{A^R}^+(\Sigma_g, M)$ we will write $W_{r,p}$ for the image of $U$ in $M$. 

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Definition 2.4. Let $K$ denote the subspace of $\mathcal{E}^+_{A,R}(\Sigma_g, M)$ of such subsurfaces $W$ such that $D^2_R := \mathbb{R}^2 \cap B_R(0) \subset W \cap B_R(0)$

Remark. Note that the subtle difference between $K$ and $\mathcal{E}_{A,flat}(\Sigma_g, M)$ is that surfaces in $\mathcal{E}_{A,flat}(\Sigma_g, M)$ intersect $B_R(0)$ exactly in $D^2_R$, while surfaces in $K$ only have to contain $D^2_R$ but there can be more connected components of these surfaces that intersect $B_R(0)$.

In order to prove this proposition we will need the following three easy lemmas:

Lemma 2.5. Let $i: A \to B$ denote an inclusion and let $\Psi: B \to A \subset B$ be a map such that there exists a homotopy $H: B \times I \to B$ such that $H(\cdot, 0) = id_B$ and $H(\cdot, 1) = \Psi$ and $H(A \times I) \subset A$, then $i$ is a homotopy equivalence with homotopy inverse $\Psi$.

Proof. $H$ is already a homotopy between $i \circ f$ and the identity of $B$, therefore we only need to construct a homotopy between $f \circ i$ and the identity on $A$. But $id_B|A = id_A$ therefore the necessary homotopy is again given by $H$. 

This lemma will be used to prove that the $\Psi_i$ are homotopy inverse to the aforementioned inclusions. The next rather technical lemma will be used towards the end of the proof of Proposition 2.2 to show that a map is continuous.

Lemma 2.6. Suppose we have a diagram of the following form:

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{f} & \mathbb{R} \cup \{\infty\} \\
\downarrow{\pi} & & \downarrow{g} \\
X & \xrightarrow{g} & \mathbb{R} \cup \{\infty\}
\end{array}
$$

where $Y$ is compact and $g = \inf_{y \in Y} f(x, y)$. Then $g$ is continuous.

Proof. We will first show that $g^{-1}((M, \infty])$ is open for every $M$. For this one only has to note that $g^{-1}((M, \infty]) = \pi(f^{-1}((\infty, M]))$ which is open since $f$ is continuous and $\pi$ is closed since $Y$ is compact. Analogously for $(M, \infty)$.

Now we will proof that $g^{-1}((\infty, M))$ is open. Observe that

$$
g^{-1}((\infty, M)) = \pi(f^{-1}((\infty, M)))
$$

This is open since $\pi$ is an open map. 

Lastly we need the following lemma before we can start the proof of Proposition 2.2.
Lemma 2.7. $\mathcal{E}_{A,r}^{+}(\Sigma_g, M)$ is an open subset of $\mathcal{E}_{A}^{+}(\Sigma_g, M)$ for every $r \in (0, \infty)$ and if $\epsilon_n$ denotes a sequence of positive real numbers that converges to zero, then

$$\mathcal{E}_{A}^{+}(\Sigma_g, M) = \bigcup_n \mathcal{E}_{A,\epsilon_n}^{+}(\Sigma_g, M)$$

Proof. For every $W \in \mathcal{E}_{A}^{+}(\Sigma_g, M)$ there is an $r > 0$ such that $\pi|_{\pi^{-1}(\overline{B_r}(0)) \cap W}$ restricted to the connected component of $\pi^{-1}(\overline{B_r}(0)) \cap W$ containing $p$ is a diffeomorphism. This follows from the inverse function theorem, because the differential of an embedding representing $W$ is an isomorphism onto $A$, which corresponds to $\mathbb{R}^2 \subset \mathbb{R}^n$. This implies that every $V \in \mathcal{E}_{A}^{+}(\Sigma_g, M)$ is contained in some $\mathcal{E}_{A,r}^{+}(\Sigma_g, M)$ for $r$ small enough.

To finish the proof we have to show that $\mathcal{E}_{A,r}^{+}(\Sigma_g, M)$ is an open subset of $\mathcal{E}_{A}^{+}(\Sigma_g, M)$, which is equivalent to its preimage in $\text{Emb}_{A}(\Sigma_g, M)$ being open. Take some embedding $g$ in this set. There is some closed neighborhood $U$ centered at $q$ in $\Sigma_g$ such that $\pi|_{g(D)}$ is a diffeomorphism onto $\overline{B_r}(0)$. Furthermore this implies that the differential of $\pi \circ g$ is an isomorphism on these points. Since the isomorphisms form an open subset of all linear maps we conclude that there is actually an open neighborhood $U_{\text{big}}$ of $U$ such that on the closure of this neighborhood the differential of $\pi \circ g$ is still an isomorphism. Since $\pi \circ g|_{\overline{U}_{\text{big}}}$ is proper because $\overline{U}_{\text{big}}$ is compact and a local diffeomorphism onto $\mathbb{R}^2$ by the requirement that the differential is an isomorphism, we conclude that it is a covering of its image. Since a closed disk can only cover a closed disk via a diffeomorphism we see that this map is actually a diffeomorphism onto some closed disk in $\mathbb{R}^2$ containing $\overline{B_r}(0)$. Let $2\varepsilon$ denote $\min_{x \in \partial U_{\text{big}}} ||\pi \circ g|| - r > 0$. Consider the following neighborhood of $g$:

$$V_g := \left\{ f \in \text{Emb}_{A}(\Sigma_g, M) \mid \sup_{U_{\text{big}}} ||f - g|| < \varepsilon, D(\pi \circ f)|_{\partial U_{\text{big}}} \text{ is an isomorphism} \right\}$$

With our previous considerations we conclude that $V_g$ is in the preimage of $\mathcal{E}_{A,r}^{+}(\Sigma_g, M)$ in $\text{Emb}_{A}(\Sigma_g, M)$. As it is the prime example of an open neighborhood in the $C^1$ topology and the $C^\infty$ topology is finer than all the $C^k$ topologies we conclude that it is in fact an open neighborhood of $g$. □

Lemma 2.8. If $V$ sits inside another coordinate chart $V'$ and the coordinate change is given by $x = (x_1, \ldots, x_n) \mapsto (\arctan(x_1), \ldots, \arctan(x_n))$, then is the inclusion $i: \mathcal{E}_{A,R}^{+}(\Sigma_g, M) \to \mathcal{E}^{+}(\Sigma_g, M)$ a homotopy equivalence.

Proof. By Theorem 42.3 and the corollary in 27.4 in [KM97] (together with the fact that dividing out a closed subspaces of a nuclear spaces produces a nuclear space) we conclude that $\mathcal{E}_{A}^{+}(\Sigma_g, M)$ is paracompact and Hausdorff.
and by Lemma 27.8 in the same source we conclude that it is actually metrizable. Therefore we can find a continuous function \( \phi: \varepsilon^+_A(\Sigma_g, M) \to (0, R) \), which is \( R \) on \( \varepsilon^+_{A,flat}(\Sigma_g, M) \) and for every \( x \) it is smaller or equal than the maximal \( r \) such that \( \varepsilon^+_{A,r}(\Sigma_g, M) \) contains \( x \) (This uses the fact that \( \varepsilon^+_{A,flat}(\Sigma_g, M) \) is closed in \( \varepsilon^+_A(\Sigma_g, M) \)).

With this \( \phi \) at hand we can start constructing the homotopy inverse to the inclusion from \( \varepsilon_A(\Sigma_g, M) \) into \( \varepsilon_{A,R}(\Sigma_g, M) \). Let \( \psi_1 : (0, \infty) \to \text{Diff}_r(M) \) denote a continuous map into the compactly supported diffeomorphisms defined as follows: Fix a smooth monotone function \( f: \mathbb{R} \to [0, 1] \) such that \( f \equiv 0 \) on an open neighborhood of \([-\frac{\pi}{2}, \frac{\pi}{2}]\) and \( f \equiv 1 \) outside of \((-\pi, \pi)\). We define \( \psi \) via the following formula:

\[
\psi_1(t)(x_1, \ldots, x_n) = ((\frac{\pi}{2}f(x_1) + (1 - f(x_1)))x_1, (\frac{\pi}{2}f(x_2) + (1 - f(x_2)))x_2, x_3, \ldots, x_n)
\]

Since this is the identity outside the cube \((-\pi, \pi)^n\) we can consider this as a compactly supported diffeomorphism of \( M \) by extending it via the identity. Let \( \Psi_1 \) denote

\[
\Psi_1: \varepsilon^+_A(\Sigma_g, M) \to \varepsilon^+_A(\Sigma_g, M) \\
W \mapsto \psi(\phi(W))(W)
\]

This maps \( \varepsilon^+_A(\Sigma_g, M) \) into \( \varepsilon^+_{A,R}(\Sigma_{g,b}, M) \). Furthermore this gives us a homotopy equivalence between \( \varepsilon^+_{A,R}(\Sigma_{g,b}, M) \) and \( \varepsilon^+_A(\Sigma_g, M) \) since \( \phi \) can be homotoped to be the map that is constantly \( R \) via a linear interpolation and the induced homotopy maps \( \varepsilon^+_{A,R}(\Sigma_{g,b}, M) \times I \) into \( \varepsilon^+_{A,R}(\Sigma_{g,b}) \). This finishes the construction of \( \Psi_1 \) and the proof that it is indeed a homotopy inverse to the inclusion.

\[\square\]

**Lemma 2.9.** If \( V \) sits inside another coordinate chart \( V' \) and the coordinate change is given by \( x = (x_1, \ldots, x_n) \mapsto (\arctan(x_1), \ldots, \arctan(x_n)) \), then is the inclusion \( i: K \to \varepsilon^+_{A,R}(\Sigma_g, M) \) a homotopy equivalence.

**Proof.** Fix a smooth function \( \xi: \mathbb{R}^2 \to \mathbb{R} \) that is invariant under rotation and \( 1 \) on \( B_R(0) \) and \( 0 \) outside of \( B_{2R}(0) \). We define a map

\[
H: \varepsilon^+_{A,2R}(\Sigma_g, M) \times I \to \varepsilon^+_{A,2R}(\Sigma_g, M)
\]

as follows: We have a map \( h: \varepsilon^+_{A,2R}(\Sigma_g, M) \times I \to \text{Diff}_r(M) \) by sending \( W \) to the diffeomorphism that acts on \( V \) by shifting the fiber of \( \pi \) in which \( x \in W_{2R,p} \) lies by \( \xi(\pi(x))(x - \pi(x)) \) and acts as the identity outside of \( \pi^{-1}(B_{2R}(0)) \). Furthermore without loss of generality we assume \( V \) to lie in a bigger coordinate chart \( V' \) as an open cube and using this we can extend \( h \) to get a diffeomorphisms with compact support on \( V' \) and therefore we can even extend this map to give a diffeomorphisms of \( M \) with compact support.

Note that \( \pi \) extends to \( V' \). For an \( x \in B_{2R}(0) \) we will denote the corresponding point in \( W_{2R,p} \) by \( w_x \). Fix a smooth monotone function
We define the following map
\[ h: E_{A,2R}(\Sigma_g, M) \times I \rightarrow \text{Diff}_c(V') \]
\[ h(W, t)(x) = x + f(||x||)\xi(\pi(x))(\pi(x) - w_{\pi(x)}) \]

Note that \( w_{\pi(x)} \) is only defined when \( ||\pi(x)|| \leq 2R \) but \( \xi(\pi(x)) = 0 \) for \( ||\pi(x)|| \geq 2R \). This map is well-defined, because a map

\[ g: \mathbb{R}^{n-2} \rightarrow \mathbb{R}^{n-2} x \mapsto x + f(||x||)v \]

is a diffeomorphism if \( \max |f'| < ||v|| \) because then \( Dg \) is still an isomorphism which implies that it is locally injective. Furthermore such a \( g \) can be extended to a degree 1 map from \( S^{n-2} \) into itself and therefore it also has to be surjective. Putting these two together we conclude that it is in fact a diffeomorphism, if the derivative of \( f \) is small enough, but we can chose \( f \) retrospectively with a small enough derivative.

Lastly we have to argue that \( h \) is a continuous map as can be seen since it can be written in a neighborhood of \( W \) in terms of the image of \( W \) and the sections of the "normal" bundle of \( W \) (where we require the "normal" bundle to have \( \{0\} \times \mathbb{R}^{n-2} \) as fiber over \( W_{2R,p} \)), which is a neighborhood of \( W \) in \( E_{A,2R}(\Sigma_g, M) \).

We define \( H(W, t) = h(W, t)(W) \). Note that \( E_{A,2R}(\Sigma_g, M) \) includes into \( E_{A,R}(\Sigma_g, M) \) and with a similar construction as in Lemma 2.8 we can see that this inclusion is actually a homotopy equivalence and the homotopy inverse, which is given by postcomposing with a diffeomorphism of \( M \) will be denote by \( \xi \). Note that \( H \) fixes \( \xi(K) \) and so we can apply Lemma 2.5 to see that the inclusion of \( K \) into \( E_{A,R}(\Sigma_g, M) \) followed by \( \xi \) is a homotopy equivalence with homotopy inverse \( H(-, 1) \). We get the following diagram:

\[ \xymatrix{ & E_{A,2R}(\Sigma_g, M) \ar[dl]_{\xi} \ar[dr]^{H(-,1)} & \\
E_{A,R}(\Sigma_g, M) \ar[rr]_{\simeq} & & K } \]

Define \( \Psi_2 \) as \( H(-, 1) \circ \xi \) and it is evident that this is a homotopy inverse to the inclusion of \( K \) into \( E_{A,R}(\Sigma_g, M) \).

Lastly we have the following lemma which finishes the proof Proposition 2.2

**Lemma 2.10.** The inclusion \( i: E_{A,\text{flat}}(\Sigma_g, M) \rightarrow K \) is a homotopy equivalence.
Proof. Consider the function \( \lambda : K \to \mathbb{R} \) that sends \( W \) to \( \inf_{x \in W \setminus D_R^2} ||x|| \). Note that \( \lambda \) is bigger than zero for every surface and bounded by \( R \). Assume for now that this function is continuous. We define \( \tilde{h} : [0, R] \times I \to \text{Diff}(\mathbb{R}^n) \) as follows: \( \tilde{h}(t_1, t_2) \) is a diffeomorphism that is the identity outside the ball of radius \( 2R \) and the identity for \( t_2 = 0 \), and \( h(t_1, t_2) \) sends the ball of radius \( t_1 \) bijectively onto the ball of radius \( Rt_2 + (1 - t_2)t_1 \) and finally only acts on the norm component if an element in \( \mathbb{R}^n \) is represented in polar coordinates.

We then define

\[
\tilde{H} : K \times I \to K
\]

\[
(W, t) \mapsto \tilde{h}(\lambda(W)/2, t)(W)
\]

Then \( \tilde{H}(−, 1) \) is a homotopy inverse to the inclusion of \( \mathcal{E}_{A, flat}(\Sigma_g, M) \) into \( K \) and using Lemma 2.5 both homotopies are given by \( H(−, −) \) as \( H \) fixes \( \mathcal{E}_{A, flat}(\Sigma_g, M) \).

So the only thing left to prove is that \( \lambda \) is continuous. Let \( \tilde{K} \) denote the subspace of those embeddings in \( \text{Emb}_A(\Sigma_g, M) \) whose equivalence class in \( \mathcal{E}_A(\Sigma_g, M) \) lies in \( K \) and we denote by \( pr \) the projection of this space to \( K \). Let \( \text{ev} : \text{Map}(A, B) \times A \to B \) denote the canonical evaluation map. Let \( A = \text{ev}^{-1}(D_R^2)^c \subset \tilde{K} \times \Sigma_g \). \( \text{Diff}_A^+(\Sigma_g) \) acts on \( \tilde{K} \) as well as \( A \) (Here we let \( \text{Diff}_A^+(\Sigma_g) \) from the left on the \( \Sigma_g \) component of \( A \)). We denote the quotient of this action of \( A \) by \( \mathcal{A} \).

The projection onto the first factor from \( A \) to \( \tilde{K} \) descends to a map from \( A \) to \( K \) which we call \( p_A \) and we want this map to be a fiber bundle. Let \( W \in K \) denote an embedded subsurface and \( NW \) a tubular neighborhood of \( W \) with corresponding projection \( \pi_W \). Consider the following set

\[
U_{NW} = \{ W' \in K \mid W' \subset NW \text{ and } \pi_W|_{W'} \text{ is a diffeomorphism} \}
\]

which is open because \( NW \) is open and \( \text{Diff}(W', W) \) is an open subset of \( \text{Maps}(W', W) \) (see Section 1.2 of the second chapter of \cite{Cer61}). We will proof that \( p_A \) is a locally trivial fibration. Note that

\[
p_A^{-1}(U_{NW}) \to p_A^{-1}([W]) \times U_{NW} \quad \text{and} \quad p_A^{-1}([W]) \times U_{NW} \to p_A^{-1}(U_{NW})
\]

\[
[g, x] \mapsto (\pi_W(x), [g]) \quad ([f, x], W') \mapsto [g, \pi_W^{-1}l_W(x)]
\]

are inverses of another. Here \( f \) is any representative of \( W \) and \( [g] \) denotes the equivalence class of a subsurface \( W' \) in \( \mathcal{E}_A(\Sigma_g, M) \). Therefore we conclude that \( p_A \) is a locally trivial fibration, whose fiber is given by \( \Sigma_{g,b} \) with an open disc removed.

So we have the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\text{ev}} & M \\
\downarrow{p_A} & & \downarrow{||-||} \\
K & \xrightarrow{\lambda} & \mathbb{R} \cup \{\infty\}
\end{array}
\]

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where $ev$ denotes again the evaluation map, which descends from $K \times \Sigma_{g,b}$ to $\mathcal{A}$ and $||-||$ denotes the norm mapping in the chart $V$ extended by $\infty$ to the complement of $V$. We give $\mathbb{R} \cup \{\infty\}$ the topology coming from open sets in $\mathbb{R}$ and neighborhoods of $\infty$ have the form $(M, \infty]$. Note that $\lambda(W)$ is given by taking the infimum over the composition of the horizontal maps for all elements in $p_A^{-1}(\{W\})$ and this always lands in $(0, R]$. Since $p_A$ is locally a trivial fibration Lemma 2.7 shows that $\lambda$ is locally continuous and therefore continuous.

The last three lemmas finish the proof of Proposition 2.2.

**Corollary 2.11.** The inclusion $i: \mathcal{E}_{AT,flat}^T(\Sigma_g, M) \to \mathcal{E}_{AT}^T(\Sigma_g, M)$ is a homotopy equivalence.

**Proof.** We define $\mathcal{E}_{AT, flat}^T(\Sigma_g, M)^* = \pi^{-1}(\mathcal{E}_{AT}^+((\Sigma_g, M)))$, where

$$\pi: \mathcal{E}_{AT}^T(\Sigma_g, M) \to \mathcal{E}_{AT}^+(\Sigma_g, M)$$

denotes the forgetful projection. Note that the following diagram is a pullback diagram:

$$\begin{array}{ccc}
\mathcal{E}_{AT, flat}^T(\Sigma_g, M)^* & \longrightarrow & \mathcal{E}_{AT}^T(\Sigma_g, M) \\
\downarrow & & \downarrow \\
\mathcal{E}_{AT, flat}^+(\Sigma_g, M) & \longrightarrow & \mathcal{E}_{AT}^+(\Sigma_g, M)
\end{array}$$

It is a standard fact that a pullback of a homotopy equivalence along a fibration produces an induced map, which is also a homotopy equivalence (see for example Proposition 2.3 in [Fra]). This fact together with Lemma 3.4 (which implies that the right arrow is a fibration) implies that the top map is a homotopy equivalence. Furthermore the homotopy inverse is a map over the homotopy inverse of the base map. Since $\mathcal{E}_{AT}^+(\Sigma_g, M)$ is paracompact (see Proof of Lemma 2.8) and the map $\mathcal{E}_{AT}^T(\Sigma_g, M) \to \mathcal{E}_{AT}^+(\Sigma_g, M)$ is a locally trivial fiber bundle we conclude that it is actually a (Hurewicz-)fibration (see Theorem 13 in Section 2.7 in [Spa66]).

So all that is left to do is trivializing the tangential structure on the disk but this is easily done by using the fixed trivialization of $B_2(D^2_R) \cong D^2 \times F$ and just extending the map by "blowing up" $f$.

**Remark.** For those who are not satisfied with these kind of abstract arguments it is also possible to directly construct the homotopy inverse by noting that all the homotopies in the proof of Proposition 2.2 while not being deformation retracts, are nevertheless fixing the target of the constructed homotopy inverses to the inclusions. Therefore the homotopy from the inclusion followed by its inverse to the identity is just given by "going back" the previously constructed homotopy. Using this and the paracompactness
of \( \mathcal{E}_A^+(\Sigma_{g,b}, M) \) one can write down the homotopy inverses explicitly using a trivialization in interval direction of the correct bundle over \( \mathcal{E}_A^+(\Sigma_{g,b}, M) \times I \) (which can also be done "internally" since all occurring homotopies are given by post composition with diffeomorphisms of \( M \)). After this one just has to add the last part of the previous proof to trivialize the tangential structure on the disk.

To proceed with defining the stabilization maps we need the following corollary of our previous considerations:

**Corollary 2.12.** There is a homotopy equivalence between \( \mathcal{E}_{A,flat}^T(\Sigma_g, M) \) and \( \mathcal{E}^T(\Sigma \setminus B, M \setminus B_{R/2}(q); \delta^T) \), where \( B \) is a small ball centered around \( p \), \( B_{R/2}(q) \) is the preimage of \( B_{R/2}(0) \subset V \) in \( M \) and \( \delta^T \) is the preimage of the intersection of our fixed tangent plane with the sphere of radius \( R/2 \) in \( V \) with the tangential structure stemming from the constant one.

**Proof.** We are interested in the subspace of \( \mathcal{E}_{A,flat}^T(\Sigma_g, M) \) of those subsurfaces that intersect \( V \setminus B_{R/2}(q) \) like \( \partial D_2^2 \cap V \setminus B_{R/2}(q) \), and we will denote it by \( \mathcal{E}_{flat}^T(\Sigma \setminus B, M \setminus B_{R/2}(q); \delta^T) \). By removing the intersection of a subsurface with \( B_{R/2}(0) \) we get a homeomorphism

\[
\mathcal{E}_{A,flat}^T(\Sigma_g, M) \to \mathcal{E}_{flat}^T(\Sigma \setminus B, M \setminus B_{R/2}(q); \delta^T)
\]

Furthermore it is easy to prove that \( \mathcal{E}_{flat}^T(\Sigma \setminus B, M \setminus B_{R/2}(q); \delta^T) \) is homotopy equivalent to \( \mathcal{E}^T(\Sigma \setminus B, M \setminus B_{R/2}(q); \delta^T) \) can be easily done using a collar neighborhood of the boundary of \( M \setminus B_{R/2}(q) \) (Alternatively one can complicate everything a little bit more and produce something in the spirit of the proof of Proposition 2.2).

Now that we have reduced the case of pointed stabilization maps in the inner of a manifold to considerations concerning manifolds with boundary, we will deal with manifolds with boundaries and their stabilization maps for a while in order to define suitable stabilization maps.

### 2.2 Stabilization Maps for Manifolds with Boundary

With the last lemma in mind we will explain the stabilization process for manifolds with boundary to be able to derive the pointed stabilization process afterward.

Let \( M \) denote an at least 5-dimensional collared manifold with boundary \( \partial M \) and a space of tangential structures of subplanes of \( TM \setminus T \partial M \setminus B_2(M) \to Gr_2(M) \). Let \( M_k \) denote \( M \cup_{\partial M \times [0]} \partial M \times [0,k] \). We can identify \( M_k \) with \( M \) using a collar and this enables us to equip \( M_k \) with a space of tangential structures of subplanes of \( TM_k \). Since all collars are isotopic the occurring different spaces of tangential structures of subplanes of \( TM_k \) do not
constitute any differences for the homological properties of the soon defined stabilization maps.

Recall how we defined boundary conditions: We said that two embeddings \( f, g \) have the same jet along \( \partial M \) if there is a neighborhood of \( \partial \Sigma_{g,b} \) on which \( f \) and \( g \) agree. Furthermore we said that all embeddings were maps between collared manifolds i.e. they extend to maps \( f_1: (\Sigma_{g,b})_1 \to M_1 \) and \( g_1: (\Sigma_{g,b})_1 \to M_1 \) such that for all \( x \in \partial \Sigma_{g,b} \) and \( t \in I \) we have \( f_1(x,t) = (f|_{\partial \Sigma_{g,b}}, t) \) and \( g_1(x,t) = (g|_{\partial \Sigma_{g,b}}) \). Since \( f \) and \( g \) have the same jet along \( \partial M \) this implies that \( f_1 \) and \( g_1 \) also agree on \( \partial \Sigma_{g,b} \times I \). Furthermore this implies that by dividing by the group action of \( \text{Diff}^+(\Sigma_{g,b}, M) \) we produce a well-defined subsurface of \( \partial M \times I \) homeomorphic to a collection of cylinders and corresponding to the jet of \( f \) and \( g \). Furthermore this subsurface has a tangential structure at its intersection with \( \partial M \times \{0\} \). We will call this subsurface with the tangential structure at \( \partial M \times \{0\} \) the elongation of the jet \( [f] \) and denote it by \( [f]_1 \).

Fix a boundary condition \( \delta^T \) for elements in \( E^T_{g,b}(M, \delta^T_1) \). For the first kind of stabilization map let \( P \) denote a subsurfaces with tangential structure of \( \partial M \times I \) such that every connected component of the subsurfaces intersects \( \partial M \times \{0\} \) and such that there exists a neighborhood \( U \) of \( \partial P \cap \partial M \times \{0\} \) such that \( U \) is a subset of \( \delta^T_1 \) and a neighborhood of \( \delta^T_1 \cap \partial M \times \{0\} \) and the tangential structure on \( \partial P \cap \partial M \times \{0\} \) agrees with the tangential structure of \( \delta^T_1 \).

Under these considerations the following map is well-defined and continuous:

\[
- \cup P: E^T_{g,b}(M, \delta^T) \to E^T_{g',b'}(M_1, \delta^T_1) \\
W \mapsto W \cup P
\]

Maps of this form are what we call stabilization maps. We will call \( P \cap \partial M \times \{0\} \) the incoming boundary of \( P \) and \( P \cap \partial M \times \{1\} \) the outgoing boundary of \( P \). We have the following lemma, which enables us to cut \( P \) into nice pieces, for which it is easier to prove homological stability.

**Proposition 2.13.** Let \( M \) be a manifold with space of tangential structures of subplanes of \( TM \) as above. If \( P \subset \partial M \times I \) is a subsurface with tangential structure as above then there exists \( P_1, \ldots, P_k \) such that:

(i) \( P_i \) is a 2-dimensional submanifold with tangential structure of \( \partial M \times [i-1,i] \) and all of its connected components but one are homeomorphic to a cylinder.

(ii) \( P_1 \cup \ldots \cup P_k \simeq k \cdot P \subset \partial M \times [0, k] \) (The product means sending \((x,t) \in \partial M \times [0,1] \) to \((x,kt) \in \partial M \times [0, k]) and \( \simeq \) means in this case that the underlying submanifolds are isotopic relative to their boundary and their tangential structures are homotopic.
(iii) the projection onto the second coordinate of \( \partial M \times [i-1, i] \) restricted to \( P_i \) is a Morse function with at most one critical point.

(iv) If this critical points happens to be a minimum or a maximum it has to be a global maximum or minimum of \( P_1 \cup \ldots \cup P_k \).

Remark. Since we required every connected component of \( P \) to meet \( \partial M \times \{0\} \) the last part of the lemma says among other things that there is no minimum.

To prove this lemma we will need to use surgery along a half-disk, a process we will explain now. We define

\[
D^2_x = \{(x_1, x_2) \in D^2 \mid x_2 \geq 0\}
\]
\[
\partial^1 D^2_x = \{x \in D^2_x \mid ||x|| = 1\}
\]
\[
\partial^0 D^2_x = \{(x_1, x_2) \in D^2_x \mid x_2 = 0\}
\]

Let \( W \subset M \) denote a subsurface of \( M \). We fix a circle \( e \subset \partial^1 D^2_x \times I \) which contains \((\partial^1 D^2_x \setminus (B_e(1) \cup B_e(-1))) \times \{0, 1\}\) and connects these two connected arcs via two small circle segments. Fix once and for all an embedding \( a: D^2 \to D^2_x \times I \) such that \( a(\partial D^2) = e \) and \( a(D^2 \setminus \frac{1}{2} D^2) \subset \partial^1 D^2_x \times I \) and such that \( a(\mathbb{R} \cdot (1,0) \cap \frac{1}{2} D^2) \) is very close to \( (\partial^0 D^2_x) \times \{\frac{1}{2}\} \) (see figure 1 to clarify the definition of \( e \) and \( a \)). Note that \( a(D^2) \) cuts \( \partial^1 D^2_x \times I \) into two connected components and let \( X \) denote the "inner one" i.e. the one that does not contain \( \partial^1 D^2_x \times \{0, 1\} \).

**Definition 2.14.** Let \( M \) denote a manifold as above and \( W \) a subsurface of \( M \). Let \( e: (D^2_x \times I, \partial^1 D^2_x \times I) \to (M, W) \) denote an embedding, then we will denote \((W \setminus e(X)) \cup e(\partial D^2)\) by \( W^e \). The above requirements for \( a \) ensure that this is a smooth manifold.

Suppose that the dimension of \( W \) is at least 5 and that \( W \) is a subsurface with tangential structure. In this case \( W \) and \( W^e \) are isotopic and we can use this to give \( W^e \) some tangential structure. Even though this is still not well-defined we will denote any such subsurface with tangential structure by \( W^e \).

Now assume that the dimension of \( M \) is at least 5 and given a map of a half-disk \( e: (D^2_x, \partial^1 D^2_x) \to (M, W) \), which is an embedding at every point except at \((\pm 1, 0)\), where \( De(T_{\pm 1,0} \partial^1 D^2_x) \) agrees with \( De(T_{\pm 1,0} \partial^0 D^2_x) \). We can use the sufficiently high codimension to ensure that \( e \) extends to an embedding \( e': (D^2_x \times I, \partial^1 D^2_x) \to (M, W) \) such that \( e(\partial^0 D^2_x) \) agrees with \( e'(a(\mathbb{R} \cdot (1,0) \cap \frac{1}{2} D^2)) \). In this case we will still write \( W^e \) for the surgery along this extension of \( e \). Armed with this definition we can start proving the previous lemma.

**Proof of Proposition 2.13.** We start by showing that there is an isotopy of \( P \) such that \( \pi: \partial M \times [0, 1] \) restricted to \( P \) is a Morse function with the property
that every critical value corresponds to a unique critical point, which proves the first three parts of the lemma. Then we proceed by explaining how to get rid of local minima. Using the symmetry between minima and maxima given by the flip map \((x, t) \mapsto (x, 1 - t)\) this also explains how to eliminate local maxima.

We have an open inclusion from the set of embeddings of \(P\) into \(\partial M \times I\)

\[
\text{Emb}(P, \partial M \times I) \to C^\infty(P, \partial M \times I) \cong C^\infty(\Sigma, I) \times C^\infty(\Sigma, M)
\]

The equality stems from the identification of maps into products as products of maps and the described map has the following form \(f \mapsto (\pi_{\partial M} \circ f, \pi \circ f)\), where \(\pi_{\partial M}\) denotes the projection onto the first coordinate and \(\pi\) denotes the projection onto the second component. That this map is indeed open is proven in [Cer61] Chapter 2 Section 1.2. Since Morse functions form an open and dense subset of all real functions we get that in \(C^\infty(\Sigma, I) \times C^\infty(\Sigma, \partial M)\) there is an open and dense subset given by \(\text{Morse}(P, I) \times C^\infty(\Sigma, \partial M)\). Since the embeddings are open as well there exists an embedding, such that the
projection onto the second component is a Morse function, which is arbitrary close to the embedding from $P$ into $\partial M \times [0, 1]$ that we started with. Using the local path-connectedness of the space of embeddings (see Theorem 44.1 of [KM97]) we conclude that $P$ is isotopic to an embedding such that the projection restricted to $P$ gives a Morse function.

Furthermore it is easy to see that we can arrange the critical points to have different values by pushing them a small bit up or down.

For the fourth part of the lemma chose a metric $g$ on $\partial M \times I$ such that $\partial_t$ is orthogonal to $T_{(x,t)} \partial M \times \{t\}$ for every $t$ and assume without loss of generality that $\pi|_P$ is a Morse function with distinct critical values.

Before we can continue we have to fix some notation: Consider a critical point $x$ of $\pi|_P$, which is not a maximum. Consider the negative gradient flow of $\pi|_P$ in a neighborhood of $x$. Then there have to be some flow lines $\gamma$ which stem from $x$ meaning that $\lim_{t \to -\infty} \gamma(t) = x$ but since $P$ is compact this gradient flow line has to either meet a boundary component or it is an embedding of $\mathbb{R}$ into $P$ in which case the compactness of $P$ ensures that $\lim_{t \to \infty} \gamma(t)$ exists and has to be a critical value as well. In this case we call $x$ the starting point of $\gamma$ and the other limit point the endpoint of $\gamma$. Furthermore note that if $\gamma$ ends at a point $x$, there also has to be flow line of the gradient flow, which starts at $x$ and goes in the opposite direction as $\gamma$, by which we mean that in a Morse chart centered around $x$ $\gamma$ is given by $e^{1/2t}v$, then the other curve is given by $-e^{1/2t}v$.

Now let us proceed with the proof of the theorem. Let $p$ denote a minimum for $\pi|_P$. Since every connected component of $P$ meets $\partial M \times \{0\}$ there has to be a 1-handle, which cancels the 0-handle given by $p$. In other words there has to exist a gradient flow line $\gamma_1$ for $\pi|_P$ starting at $p$, which ends at an index 1 critical point $q$. Following the negative gradient flow along the opposite direction defined by $\gamma_1$ at $q$ gives us a gradient flow line $\gamma_2$ that ends at some critical point $p'$ which is lower with respect to its $t$-component than $q$. By reparametrizing and including their endpoint and starting point we can consider $\gamma_1$ and $\gamma_2$ as arcs starting and ending at critical points. We denote by $\gamma$ the concatenation of $\gamma_1$ and $\gamma_2$. By changing $P$ a little bit but fixing $\pi|_P$ we can arrange that there is no open segment of $\gamma$, where $\pi_{\partial M} \circ \gamma$ is constant. Fix a strictly increasing or strictly decreasing function $f : [0, 1] \to [0, 1]$ such that $f(0) = \pi(p')$ and $f(1) = \pi(p)$. Then define

$$H : I \times I \to \partial M \times I$$

$$(t', s) \mapsto (\pi_M(\gamma(s)), (1-t)'\pi(\gamma(s)) + t'(f(s))$$

Using the main theorem of [Whi36], which says that embeddings are dense in the space of mappings, we can alter $H$ a little bit such that $H$ turns into an embedding of a half disc $D^2_+ = \{(x, y) \in D^2 \mid y \geq 0\}$ such that $D\pi(TH(D^2_+)) = < \partial_t >$, $D\pi(H|_{\partial_1 D^2_+}) = \lambda \partial_t$ for $\lambda < 0$ or $\lambda > 0$ depending on the height of $p$ and $p'$ and that $H|_{\partial_1 D^2_+} = \gamma$. Furthermore we alter $H$
a bit more, using the denseness of transversal maps, that $H$ still fulfills the
conditions above and that $H(D_2^+ \cap P = \gamma$.

Now the idea is to do surgery along $H$ to cancel pairs of critical points
to get rid of minima (In fact we have to change $H$ a little bit for technical
reasons, but this doesn’t change the idea). This strategy works since $P$ and
$P_2H$ are isotopic by using a deformation of $H$ to $\gamma$.

We have to consider three and a half cases: First consider the case, where $p'$
is also a minimum. Without loss of generality we assume that $p'$ is lower
than $p$. In this case we replace $\gamma$ by the curve that ends a little bit before
$p'$ with respect to the flow time and alter $H|_{\partial D_2^+}$ a little bit such that it is
tangential to $\gamma$ at $H((\pm 1,0))$. We alter $\gamma$ a bit further such that it starts
a little bit before $p$ in the sense that we go a short time in the opposite
direction of $\gamma$ at $p$ and proceed in the same fashion as before to produce an
$H'$ with the aforementioned properties. Then $P_2H'$ is isotopic to $P$, $\pi|_{P_2H'}$
is still a Morse function with the same critical points as $\pi|_P$ except for $p$
and $q$ (this is possible by choosing a very small extension of $H$). In this case
we have eliminated a minimum and start this process anew with another
minimum.

The next case is the case where $p'$ is an index 1 critical point which is
lower than $p$. In this case we alter $\gamma$ exactly as in the previous case and then
$\pi|_{P_2H}$ is still a Morse function with the same critical points as $\pi|_P$ except for $p$
and $q$. In this case we have again eliminated a minimum without
producing any new critical points. The last full case, where $p'$ is an index 1
critical point, which is higher than $p$, is a little bit different compared to the
previous two quite similar cases. In this case we let $\gamma$ start not at $p$ but at
$\gamma(\epsilon)$ for some small epsilon and choose $H|_{\partial D_2^+}$ to be tangential to $\gamma$ at its
starting point. Furthermore we alter $\gamma$ at its endpoint by stopping a short
time before arriving at $p'$ and then going down with respect to $\pi$ avoiding $p'$
and points that lie on the gradient flow line which ends at $p'$ and stopping at
a point which is lower than $p'$. As before $\pi|_{P_2H}$ is a Morse function but the
only critical point that could have vanished on $P_2H$ is $q$ by the restrictions
on the derivative of $H$. Since deleting only one critical point changes the
Euler characteristic of $P$ we know that this procedure has to create an index
1 critical point somewhere, but the only place where this can occur is in a
small neighborhood of $H(-1,0)$ because at the other boundary point of $\gamma$
we can arrange $H|_{\partial D_2^+}$ to be tangential to $\gamma$. Another way to see this is to
note that $H|_{\partial D_2^+}$ becomes part of a new gradient flow line of $\pi|_{P_2H}$, which
has to flow down to $p$ after passing a neighborhood of $H(-1,0)$, where it has
to flow up. Because every point close to $H(-1,0)$ is a point that can
flow down along $\text{grad}(\pi|_{P_2H})$ to a critical point that is not $p$, we conclude
that the gradient flow line $H|_{\partial D_2^+}$ left the open unstable manifold of $p$.
This procedure moves the peak of $\gamma$ below $p'$ so that $p'$ does not play a role
anymore in the canceling of $p$.
All in all these considerations imply that we can proceed again as before but we will not run into \( p' \) again essentially reducing the number of critical points that come into question for this construction.

The last "half" case is the case, where \( \gamma \) doesn’t run into a critical point after passing the index 1-critical point, but rather runs into the boundary \( P \cap \partial M \times \{0\} \). This case is handled completely analogous to the first two cases.

This lemma allows us to split every stabilization map into a composition of maps which are given by taking the union with some \( P_i \). But the fourth condition limits the topology of \( P_i \) to be either a pair of pants with incoming boundary consisting of one or two circles (a single index 1 critical point) and corresponding outgoing boundary or a disk with incoming boundary a circle (a single index 2 critical point). Accordingly we will write

\[
\begin{align*}
\alpha_{g,b}(P) & : \mathcal{E}^T_{g,b}(M, \delta^T) \to \mathcal{E}^T_{g+1,b-1}(M_1, \delta^T) \\
\beta_{g,b}(P) & : \mathcal{E}^T_{g,b}(M, \delta) \to \mathcal{E}^T_{g,b+1}(M_1, \delta^T) \\
\gamma_{g,b}(P) & : \mathcal{E}^T_{g,b}(M, \delta) \to \mathcal{E}^T_{g,b-1}(M_1, \delta^T)
\end{align*}
\]

for the corresponding stabilization maps i.e. \( \alpha_{g,b}(P) \) corresponds to a map which is given by stabilization with a pair of pants with incoming boundary two circles, \( \beta_{g,b}(P) \) is a map which is given by stabilization with a pair of pants with incoming boundary a single circle and \( \gamma_{g,b}(P) \) is given by taking the union with a disk.

**Notation.** Even though different choices of \( P \) produce non-homotopic stabilization maps we will usually suppress the \( P \) from the notation for example write \( \alpha_{g,b}(P) \) instead of \( \alpha_{g,b}(P) \) , because the mentioned stability results are independent of any particular choice of \( P \). If multiple maps of this type occur we will usually include the manifold and the boundary conditions into the notation. We will write \( \alpha_{g,b}(M; \delta^T, \delta^T) \) for example.

**Construction of \( \sigma_g \):** So with this notation at hand let us return to the pointed stabilization process: We define our pointed stabilization map \( \sigma_g \) as follows:

\[
\begin{array}{c}
\mathcal{E}^T_{A^T}(\Sigma_g, M) \xrightarrow{\sigma_g} \mathcal{E}^T_{A^T, flat}(\Sigma_g, M) \xrightarrow{\beta_g,1} \mathcal{E}^T(\Sigma_{g,1}, M \setminus B_R(q), \delta^T_1) \\
\downarrow \quad \downarrow \beta_g,2 \\
\mathcal{E}^T_{A^T}(\Sigma_{g+1}, M) & \xleftarrow{\psi} \mathcal{E}^T_{A^T, flat}(\Sigma_{g+1}, M) \xleftarrow{\alpha_g,2} \mathcal{E}^T(\Sigma_{g+1,1}, M \setminus B_R(q), \delta^T_2) \\
\end{array}
\]
The top vertical arrows stem from Proposition 2.2 and Corollary 2.11. The maps on the right are the stabilization maps we have just defined and we restrict \( \delta_T^3 \) to be the same as \( \delta_T^1 \). Now \( \Psi \) is defined in a similar way as the map from \( \mathcal{E}_{T, flat}(\Sigma_g, M) \) to \( \mathcal{E}_T(\Sigma_{g,1}, M \setminus B_R(q), \delta_T^1) \). Using a diffeomorphism from \( M \setminus B_R(q) \setminus B_R(q) \) enables us to define \( \Psi \) by gluing in a disk isomorphic to \( \mathbb{R}^2 \cap B_R(0) \) in \( V \). It is now easy to see that \( \psi \) is a homotopy equivalence from \( \mathcal{E}_T(\Sigma_{g+1}, 1, M \setminus B_R(q), \delta_T^3) \) to \( \mathcal{E}_{T, flat}(\Sigma_{g+1}, M) \). Indeed it is a homotopy inverse to the map occurring in Corollary 2.12.

Since we defined \( \sigma_g \) in terms of maps of type \( \alpha \) and \( \beta \) and homotopy equivalences that stem from Proposition 2.2 and Corollary 2.11, it is now clear that Theorem 1 follows directly from Theorem 2.

3 Formulating the Proof Strategy and Initial Results

The goal of the next four sections is to prove the following theorem which implies the first two parts of the Theorem 2.

**Theorem 3.** Let \( M \) be a simply-connected manifold with a space of tangential structures of subplanes of \( TM \): \( B_2(M) \to \text{Gr}_2(TM) \) of dimension at least 5 such that the fiber of \( T \) is at least 2-connected. Then

(i) \( H_k(C(\alpha_{g,b}(M))) = 0 \) for \( k \leq \frac{1}{3}(2g + 1) \)

(ii) \( H_k(C(\beta_{g,b}(M))) = 0 \) for \( k \leq \frac{2}{3}g \).

Here \( C(f) \) denotes the pair consisting of the mapping cone of \( f \) relative to the top of the cone. We will prove this theorem by induction using the following assertions:

**A_g:** \( H_k(C(\alpha_{h,b}(M))) = 0 \) for all manifolds fulfilling the aforementioned prerequisites as long as \( h \leq g \) and all \( k \leq \frac{1}{3}(2h + 1) \)

**B_g:** \( H_k(C(\beta_{h,b}(M))) = 0 \) for all manifolds fulfilling the aforementioned prerequisites as long as \( h \leq g \) and all \( k \leq \frac{2}{3}h \)

In addition to these two assertions we will need four auxiliary assertions. All the occurring objects and maps will be defined in Section 5. The assertions are that for any element \( u \in A^T(M, \delta, \ell)_0 \) (A space that will be defined in Section 5):

**SA_g:** \( H_k(C(\beta_{h,b-1}(M(a))) \to C(\alpha_{h,b}(M))) \) is surjective for all manifolds as above, all \( h \leq g \) and all \( k \leq \frac{1}{3}(2h + 1) \)

**0A_g:** \( H_k(C(\beta_{h,b-1}(M(a))) \to C(\alpha_{h,b}(M))) \) is the zero morphism for all manifolds as above, all \( h \leq g \) and all \( k \leq \frac{1}{3}(2h + 2) \)

**SB_g:** \( H_k(C(\alpha_{h-1,b+1}(M(a))) \to C(\beta_{h,b}(M))) \) is surjective for all manifolds as above, all \( h \leq g \) and all \( k \leq \frac{2}{3}h \)

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$0B_g: H_k(C(\alpha_{h-1,b+1}(M(a))) \to C(\beta_{h,b}(M)))$ is the zero morphism for all manifolds as above, all $h \leq g$ and all $k \leq \frac{1}{3}(2h + 1)$

The maps mentioned in the assertions above will be called approximate augmentations (see Definition 5.11). The following lemma summarizes the implications, we will prove, to finish the proof of Theorem 3

Lemma 3.1. If $M$ satisfies the properties of Theorem 3 then

(i) $SA_g, OA_g \implies A_g$  (iii) $B_g \implies SA_g$  (v) $B_g, SA_{g-1} \implies 0A_g$

(ii) $SB_g, OB_g \implies B_g$  (iv) $A_{g-1} \implies SB_g$  (vi) $A_{g-1}, SB_{g-1} \implies 0B_g$

The proof of this lemma will be distributed throughout the next sections. We will start by proving (i) and (ii), which is possible without even knowing what the maps in question are:

Proof of (i)+(ii) of Lemma 3.1. (i) The morphism induced by the approximate augmentation

$$H_k(C(\beta_{g,b-1}(M(a)))) \to H_k(C(\alpha_{g,b}(M)))$$

is both zero and an epimorphism in the relevant degrees, which implies $A_g$. Similarly for (ii) □

The rest of this section will contain a short overview of the proof and establish the initial step for the aforementioned induction process. The next section will establish techniques, namely semi-simplicial resolutions and some related useful statements, so that we can properly formulate the assertions for the induction process in Section 5 and prove (iii) and (iv) of Lemma 3.1. Once we have establish the necessary lemmas in Section 5 we will only need to plug in the correct spaces into the statement of Lemma 4.3 to conclude the implications (iii) and (iv).

The proof of parts (v) and (vi) of Lemma 3.1 will occupy all of Section 6 and will essentially be technically independent of Section 4. After we have proven Theorem 3 we will prove homological stability for maps of type $\gamma_{g,b}$ in Section 7 using the homological stability of maps of type $\beta$.

3.1 The Initial Step

To establish the initial step we will first show that $\pi_0(\mathcal{E}^+_{g,b}(M; \delta))$ is an affine $\pi_2(M)$-space and that the stabilization maps are equivariant with respect to this action. Then we will relate this to $\pi_0(\mathcal{E}^T_{g,b}(M; \delta))$ using a nice fibration.

Lemma 3.2. Suppose $M$ denotes a simply-connected smooth manifold of dimension at least 5. Then $H_2(M; \mathbb{Z})$ acts freely and transitively on the set
\( \pi_0(\mathcal{E}_{g,b}^+(M; \delta)) \). If \( \partial : H_2(M, \delta; \mathbb{Z}) \to H_1(\delta; \mathbb{Z}) \) denotes the boundary homomorphism and \([\delta]\) denotes the fundamental class of \( \delta \), then the map

\[
\pi_0(\mathcal{E}_{g,b}^+(M; \delta)) \to \partial^{-1}([\delta])
\]

\([f] \to f_*([\Sigma_{g,b}, \partial \Sigma_{g,b}])\)

is an isomorphism of \( H_2(M; \mathbb{Z}) \)-sets.

**Proof.** Since \( M \) is at least 5-dimensional and simply connected the main result of [Hae62] says that the \( \text{Diff}(\Sigma_{g,b}) \)-equivariant inclusion

\[
\text{Emb}(\Sigma_{g,b}, M; \delta) \to \text{Map}(\Sigma_{g,b}, M; \delta)
\]

induces a bijection on \( \pi_0 \).

Let \( \Sigma_{g,b}^1 \) denote the 1-skeleton of \( \Sigma_{g,b} \), to which only a single 2-cell needs to be attached to obtain \( \Sigma_{g,b} \). The cofibration \( \Sigma_{g,b}^1 \to \Sigma_{g,b} \) induces a Serre-fibration

\[
\text{Map}(\Sigma_{g,b}, M; \delta) \to \text{Map}(\Sigma_{g,b}^1, M; \delta)
\]

Since \( \Sigma_{g,b}^1 \) is homotopy equivalent to \( \bigvee S^1 \) and \( M \) is simply-connected, we conclude that \( \text{Map}(\Sigma_{g,b}^1, M; \delta) \) is path-connected. Let \( \Phi \) denote the restriction of an embedding of \( \Sigma_{g,b} \) to its 1-skeleton.

The fiber over \( \Phi \) of the aforementioned fibration is given by the space \( \text{Map}(D^2, M; \Phi \circ \sigma) \), the space of maps from \( D^2 \) to \( M \) which agree with \( \Phi \circ \sigma \) on the boundary, where \( \sigma \) describes the boundary map of the two-cell in our surface. By considering the long exact sequence for the pair \( (M, \Phi(\Sigma_{g,b}^1)) \)

\[
\pi_2(\Phi(\Sigma_{g,b}^1)) = 0 \longrightarrow \pi_2(M) \longrightarrow \pi_2(M, \Phi(\Sigma_{g,b}^1)) \xrightarrow{\partial} \pi_1(\Phi(\Sigma_{g,b}^1))
\]

we conclude that \( \pi_2(M) \) acts transitively and freely on \( \pi_0(\text{Map}(D^2, M; \Phi)) = \partial^{-1}(\Phi \circ \sigma) \). Furthermore since \( \pi_0(\text{Map}(\Sigma_{g,b}^1, M; \delta)) = \{\ast\} \) we get from the long exact sequence of the aforementioned fibration that the first map in the following diagram is a surjection.

\[
\pi_0(\text{Map}(D^2, M; \Phi)) \longrightarrow \pi_0(\text{Map}(\Sigma_{g,b}, M; \delta)) \longrightarrow \partial^{-1}([\delta])
\]

But the long exact sequence in homology of \( (M, \delta) \) tells us that \( \partial^{-1}([\delta]) \) has a free and transitive \( H_2(M) \) action and we see that the composition of the maps in the diagram is equivariant with respect to the Hurewicz homomorphism. Therefore we conclude that the composition is an isomorphism, which furthermore implies that \( \pi_0(\text{Map}(\Sigma_{g,b}, M; \delta)) \cong \partial^{-1}([\delta]) \). Since the action of \( \text{Diff}^+(\Sigma_{g,b}) \) does not change the image of the fundamental class in \( H^2(M, \delta) \) we conclude that this isomorphism descends to an equivariant isomorphism \( \pi_0(\mathcal{E}_{g,b}^+(M; \delta)) \to \partial^{-1}(\delta) \). \( \square \)
This lemma implies almost directly the following corollary.

**Corollary 3.3.** If \( M \) is simply-connected and of dimension at least 5, then every stabilization map induces an isomorphism between \( \pi_0(E_{g,b}^+(M;\delta)) \) and \( \pi_0(E_{g+1,b-1}^+(M;\delta')) \) or \( \pi_0(E_{g,b+1}^+(M;\delta')) \) depending on the stabilization map in question.

**Proof.** The stabilization maps are given by gluing in surfaces, which induce \( H_2(M) \)-equivariant maps between \( \partial^{-1}([\delta]) \) and \( \partial^{-1}([\delta']) \). Since both of these spaces are affine \( H_2(M) \)-spaces, the map is actually a bijection.

To relate all of this to \( E_{T,g,b}(M;\delta) \) we want to show that the forgetful map from \( E_{g,b}(M;\delta) \to E_{g,b}^+(M;\delta) \) is a fibration with a path-connected fiber.

**Lemma 3.4.** The forgetful map \( \pi: E_{g,b}^+(M;\delta) \to E_{g,b}(M;\delta) \) is a Hurewicz fibration. The fiber over a surface \( W \) is given by \( \Gamma(B_2(W)) \), the space of sections of \( B_2(W) \).

**Proof.** Let Homeo\(_M(B_2(M)) \) denote the group of homeomorphisms of \( B_2(M) \) which cover a diffeomorphism of \( Gr_2(TM) \) induced from a diffeomorphism of \( M \) i.e. the following diagram commutes and the lower map is a diffeomorphism

\[
\begin{array}{ccc}
B_2(M) & \longrightarrow & B_2(M) \\
\downarrow & & \downarrow \\
Gr_2(TM) & \xrightarrow{Gr(Df)} & M \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & M
\end{array}
\]

This group acts on \( E_{g,b}^+(M;\delta^T) \) and \( E_{g,b}^+(M;\delta) \) by post-composition. It was the content of Lemma 3.1 that \( E_{g,b}^+(M;\delta) \) is Diff(M)-locally retractible, but is it also Homeo\(_M(B_2(M))\)-locally retractible? Let \( \xi: U \to Diff(M) \) denote a Diff(M)-local retraction around some \( V \). Note that Diff(M) is locally contractible. Shrinik \( U \) such that \( \xi(U) \) lies in a contractible neighborhood of the identity \( V \). Note that Diff(M) is paracompact (see Lemma 41.11 of [KM97] to conclude that Diff(M) with the \( C^\infty \) topology is metrizable) and the neighborhood on which Diff(M) is contractible can be chosen as a closed neighborhood.

We get a map \( Gr(ev): Gr_2(TM) \times V \to Gr_2(TM) \) and we can use the contractibility of \( V \) together with the paracompactness of \( V \) and the compactness of \( M \) to conclude that \( Gr(ev)^*B_2(M) \) is isomorphic to \( \pi^*B_2(M) \), where \( \pi: Gr_2(TM) \times V \to Gr_2(TM) \) denotes the projection. Using this isomorphism it is easy to construct a lift of \( \xi \) to conclude that \( E_{g,b}^+(M;\delta) \) is Homeo\(_M(B_2(M))\)-locally retractible.
By Lemma 1.5 we conclude that this map is a locally trivial fiber bundle and using the paracompactness of $E_{g,b}^+(M;\delta)$ (see the proof of Lemma 2.8) we conclude that it is actually a Hurewicz fibration by Theorem 13 in section 2.7 of [Spa66].

The specification of the fiber is evident.

The same proof works almost verbatim in the pointed case one only has to note that $E_A^+(\Sigma_{g,b},M) = \text{Diff}_p(M)$ locally retractile, where $\text{Diff}_p(M)$ is the group of diffeomorphisms of $M$ that fix $T_pM$ pointwise.

With this lemma at hand we can prove the following lemma, which provides the start of the induction.

**Lemma 3.5.** The stabilization maps induce bijections on the $\pi_0$-level of $E_{g,b}^+(M;\delta)$.

**Proof.** We will do the proof for maps of type $\alpha_{g,b}$, which are defined using a bordism $P$. The other case is completely analogous. By the previous lemma we get the following long exact sequences and since the stabilization maps are defined for $E_{g,b}(M;\delta)$ as well as $E_{g,b}^+(M;\delta)$ we get the described maps between them.

$$
\cdots \longrightarrow \pi_0(\Gamma(B_2(W))) \longrightarrow \pi_0(E_{g,b}^+(M;\delta)) \longrightarrow \pi_0(E_{g,b}(M;\delta)) \longrightarrow \{\ast\} \bigg\downarrow \alpha_{g,b}^* \bigg\downarrow \alpha_{g,b}^* \\
\cdots \longrightarrow \pi_0(\Gamma(B_2(W \cup P))) \longrightarrow \pi_0(E_{g+1,b-1}^+(M;\delta)) \longrightarrow \pi_0(E_{g+1,b-1}(M;\delta)) \longrightarrow \{\ast\}
$$

To finish the proof we will show that $\pi_0(\Gamma(B_2(W)))$ consists of a single point for every $W$ and therefore the claim follows from Lemma 3.3. Using obstruction theory (see for example Chapter 1 of [MT68]) we see that the obstruction to a homotopy between any two sections of this bundle lies in $H^2(W;\pi_2(F))$, where $F$ denotes the fiber of $T$, which is 0. Therefore we conclude that $\pi_0(\Gamma(B_2(W)))$ consists of a single point as claimed.

**Remark.** The previous lemma is also true if we only require the fiber of $T: B_2(M) \to \text{Gr}_2(TM)$ to be simply-connected, but this complicates the proof and is of no use for us, as we need the 2-connectedness of the fiber later on.

### 4 Resolutions via Semi-Simplicial Spaces

The following section is based entirely on the fourth section of [CRW17]. Let $\Delta_{inj}^{op}$ denote the category, whose objects are non-empty finite ordinals and whose morphisms are injective order preserving maps. A **semi-simplicial space** is a contravariant functor $X_\bullet : \Delta_{inj}^{op} \to \text{Top}$. We denote the image of $[n] := \{0, \ldots, n\}$ by $X_n$ and we denote by $\partial_j$ the face maps that stem from the inclusion from $\{0, \ldots, n-1\} \to \{0, \ldots n\}$ that misses $j \in \{0, \ldots n\}$. 

...
A semi-simplicial space $X_\bullet$ together with a continuous map $\epsilon : X_0 \to X$ is called a semi-simplicial space augmented over $X$ if $\epsilon \partial_0 = \epsilon \partial_1 : X_1 \to X$. We call $\epsilon$ the augmentation. If $X_\bullet$ and $Y_\bullet$ denote (augmented) semi-simplicial spaces we call a natural transformation between the functors a semi-simplicial map. If the semi-simplicial spaces are augmented we furthermore require the maps given by the natural transformation to commute with the augmentations.

There is a geometric realization functor (compare [ER17])

$$||\cdot|| : \text{Semi-simplicial spaces} \to \text{Top}$$

and we call a semi-simplicial space augmented over $X$ a resolution if the induced map between the geometric realization of the semi-simplicial space and $X$ is a weak equivalence. We call it an $n$-resolution if the induced map is $n$-connected.

Furthermore we call an augmented semi-simplicial space $\epsilon : X_\bullet \to X$ an augmented topological flag complex if

i) the product map $X_i \to X_0 \times_X \ldots \times_X X_0$ is an open embedding

ii) a tuple $(x_0, \ldots, x_i)$ is in $X_i$ if and only if for each $0 \leq j < k \leq i$ the pair $(x_j, x_k) \in X_0 \times_X X_0$ is in $X_1$.

The following three lemmas will be crucial for the proofs of the following sections and can be found in Section 4 of [CRW17].

**Lemma 4.1.** Let $\epsilon : X_\bullet \to X$ be an augmented topological flag complex. Suppose that

i) $X_0 \to X$ has local sections that is $\epsilon$ is surjective and for each $x_0 \in X_0$ such that $\epsilon(x_0) = x$ there is a neighborhood $U$ of $x$ and a map $s : U \to X_0$ such that $\epsilon(s(y)) = y$ and $s(x) = x_0$

ii) given any finite collection $\{x_1, \ldots, x_n\} \subset X_0$ in a single fiber of $\epsilon$ over some $x \in X$, there is an $x_\infty$ in that fiber such that each $(x_i, x_\infty)$ is an 1-simplex

Then $||\epsilon|| : ||X_\bullet|| \to X$ is a weak homotopy equivalence

**Lemma 4.2.** Let $\epsilon : X_\bullet \to X$ be an augmented semi-simplicial space. If each $\epsilon_i$ is a fibration and $\text{Fib}_x(\epsilon_i)$ denotes its fiber at $x$, then the realization of the semi-simplicial space $\text{Fib}_x(\epsilon_\bullet)$ is weakly homotopy equivalent to the homotopy fiber of $||\epsilon||$ at $x$.

If $f : X \to Y$ denotes a map of topological spaces, we will write $C(f)$ for the pair $(M_f, X)$, where $M_f$ denotes the mapping cylinder of $f$. 

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Lemma 4.3. Let $f_\bullet: X_\bullet \to Y_\bullet$ be a map of augmented semi-simplicial spaces such that $\|f_\bullet\|: \|X_\bullet\| \to X$ is $(l - 1)$-connected and $\|f_\bullet\|: \|Y_\bullet\| \to Y$ is $l$-connected. Suppose there is a sequence of path connected based spaces $(B_i, b_i)$ and maps $p_i: Y_i \to B_i$, and form the map

$$g_i: \text{HoFib}_{b_i}(p_i \circ f_i) \to \text{HoFib}_{b_i}(p_i)$$

induced by the composition with $f_i$.

\[
\begin{array}{ccc}
\text{HoFib}_{b_i}(p_i \circ f_i) & \xrightarrow{g_i} & \text{HoFib}_{b_i}(p_i) \\
\downarrow & & \downarrow \\
X_i & \xrightarrow{f_i} & Y_i \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

Suppose that there is a $k \leq l + 1$ such that

$$H_q(C(g_i)) = 0 \text{ when } q + i \leq k, \text{ except if } (q, i) = (k, 0)$$

Then the map induced in homology by the composition of the inclusion of the fiber and the augmentation map

$$H_q(C(g_0)) \longrightarrow H_q(C(f_0)) \longrightarrow H_q(C(f))$$

is an epimorphism in degrees $q \leq k$.

If in addition $H_k(C(g_0)) = 0$, then $H_q(C(f)) = 0$ in degrees $q \leq k$.

5 Resolutions of the Space of Subsurfaces and the Proof of (iii) and (iv) of Lemma 3.1

The goal of this section is to establish a resolution of $E_{g,b}^T(M; \delta^T)$ and to understand how this resolution behaves with respect to stabilization maps to properly formulate the four assertions in Section 3 and establish the spaces and maps necessary for the application of Lemma 1.3. We will first establish $O_{g,b}(M; \delta^T; \ell)$ a $(g - 1)$-resolution of $E_{g,b}^T(M; \delta^T)$, which will play the role of $X_\bullet$ and $Y_\bullet$ in Lemma 1.3. Then we will construct lifts $(\alpha_{g,b})_\bullet$ and $(\beta_{g,b})_\bullet$ of $\alpha_{g,b}$ and $\beta_{g,b}$, which will play the role of $f_\bullet$ in Lemma 4.3 (depending on whether we prove (iii) or (iv)). Finally we will define $A_T^T(M; \delta^T, \ell)_i$, which will play the role of $B_i$ in Lemma 4.3 and then calculate the homotopy fibers and the map between them mentioned in Lemma 1.3. These maps between the fibers will be the maps mentioned in the four assertions of Section 3.
Constructing $X_\bullet$ and $Y_\bullet$ for the Application of Lemma 4.3} Recall the following notation introduced in Section 2:

(i) $D^2_+ = \{(x,y) \in \mathbb{R}^2 : y \geq 0, \| (x,y) \| \leq 1 \}$

(ii) $\partial^0 D^2_+ = \{(x,y) \in \partial D^2_+ : y = 0 \}$

(iii) $\partial^1 D^2_+ = \{(x,y) \in \partial D^2_+ : \| (x,y) \| = 1 \}$

From here on forth let $(M, \partial M, \partial^0 M)$ denote an at least 5-dimensional simply-connected manifold together with a union of connected components of its boundary called $\partial^0 M$ and a fibration $B_2(M) \to \text{Gr}_2(TM)$ with a fiber that is at least 2-connected. Furthermore let $\delta^T$ denote some fixed boundary condition for embeddings with tangential structure of a surface $\Sigma_{g,b}$ into $M$. Let $\ell$ denote a codimension 0 ball in $\partial^0 M$ that intersects $\delta$ in two intervals $\ell_0$ and $\ell_1$, which we label and orient once and for all.

To define the aforementioned resolution we will need the following definition:

**Definition 5.1.** Let $W \in \mathcal{E}^+_{g,b}(M; \delta)$. We will call an embedding

$$a: (D^2_+, \partial^1 D^2_+) \to (M, W)$$

which maps $\partial^0 D^2_+$ to $\ell$ and $(1,0)$ to $\ell_0$ and $(-1,0)$ to $\ell_1$ an **arc in $W$ with embedded boundary isotopy**. We will call a thickened embedding $(a,a)$ (i.e. a pair consisting of an embedding and a closed tubular neighborhood of the image), where $a$ is an arc together with an embedded boundary isotopy and $a = (a_M, a_W)$ is a tubular neighborhood of $(e(D^2_+), e(\partial^1 D^2_+))$ in $(M, W)$ (i.e. $a_M$ is a closed tubular neighborhood of $e(D^2_+)$ and the restriction to the normal bundle of $a(\partial^1 D^2_+)$ in $W$ is given by $a_W$), a **thickened arc in $W$ with embedded boundary isotopy** if the image of $a_M$ restricted to the normal bundle of $\partial^0 D^2_+$ in $\partial M$ lies in $\ell$. In this case we will call the image of $a|_{\partial^1 D^2_+}$ resp. the image of $a_W$ the **underlying arc** resp. **thickened underlying arc**. For notational reasons we will usually write $a$ for the thickened arc in $W$ with embedded boundary isotopy even tough the correct notation would include $a_M$ and $a_W$ as well.

If we only consider embeddings $a: D^2_+ \to M$ such that $e(\partial^1 D^2_+) \subset M \setminus \partial M$, $e(\partial^0 D^2_+) \subset \ell$ and $e((-1)^k,0)) \subset \ell_k$, we will drop the $W$ from the notation i.e. we will call such an $a$ an **arc with embedded boundary isotopy**. Similarly for a **thickened arc with embedded boundary isotopy**.

With this notation at hand we can proceed to define the resolution of $\mathcal{E}^T_{g,b}(M; \delta^T)$.

**Definition 5.2.** Let $\mathcal{O}^T_{g,b}(M; \delta^T, \ell)_\bullet$ be the following semi-simplicial space: The space of $i$-simplices consists of tuples $(W, (a^0, a^0), \ldots, (a^i, a^i))$ such that:
(i) \( W \in \mathcal{E}_{g,b}^T(\Sigma, M; \delta^T) \) is a surface with tangential structure in \( M \).

(ii) All the \((a^k, a^k)\) are thickened arcs in \( W \) with embedded boundary isotopy.

(iii) The images of all \( a^k_M \) are disjoint.

(iv) \( W \) without all underlying thickened arcs is connected, i.e. the arc system consisting of the underlying arcs is coconnected.

(v) The starting and endpoints of the underlying arcs are ordered from 0 to \( i \) in \( \ell_0 \) and ordered from \( i \) to 0 in \( \ell_1 \) (Note that this makes sense as \( \ell_i \) is oriented). We will say that the arc system consisting of the underlying arcs is ordered.

The \( j \)-th face map forgets the \( j \)-th embedding and we topologize the set of \( i \)-simplices as a subspace of

\[
\mathcal{E}_{g,b}^T(M; \delta) \times \mathcal{T}\text{Emb}(D^2_+ \times [i])
\]

This semi-simplicial space possesses an augmentation map \( \epsilon \) to \( \mathcal{E}_{g,b}^T(M; \delta^T) \) given by forgetting the thickened arcs with embedded boundary isotopy.

**Notation.** If there is no chance of misunderstanding we will write \( O_{T}^T_{g,b}(M; \delta^T) \) as a shorthand notation for \( O_{T}^{g,b}(M; \delta^T, \ell) \). Sometimes we want to distinguish between the cases, where the intersection of \( \delta \) and \( \ell \) meets a single connected component of \( \delta \) or two different components. To emphasize this we will sometimes write \( O_{T}^T_{g,b}(M; \delta^T) \) for the single component case and \( O_{T}^T_{g,b}(M; \delta^T) \) for the different components case.

The following proposition shows that this is indeed a nice resolution. The crucial parts of the following proof were communicated to me by Frederico Cantero as a proposed fix to some issues that arose in the proof of Proposition 5.3 in [CRW17].

**Proposition 5.3.** \( O_{T}^T_{g,b}(M; \delta^T) \) is a \((g-1)\)-resolution of \( \mathcal{E}_{g,b}^T(M; \delta^T) \) i.e. \( \epsilon: \left\| O_{g,b}^T(M; \delta^T) \right\| \to \mathcal{E}_{g,b}^T(M; \delta^T) \) is \((g-1)\)-connected.

**Proof.** We will denote by \( O_{g,b}(M; \delta) \) the semi-simplicial space defined just like \( \mathcal{E}_{g,b}^T(M; \delta^T) \), but with \( \mathcal{E}_{g,b}^T(M; \delta^T) \) replaced by \( \mathcal{E}_{g,b}^+ (M; \delta^T) \).

Since \( \mathcal{E}_{g,b}^+ (M; \delta^T) \) is Diff\(\partial (M)\)-locally retractile and the augmentation maps from \( O_{g,b}(M; \delta) \) to \( E_{g,b}(M; \delta) \) are equivariant with respect to the natural action of Diff\(\partial (M)\) via post composition, Lemma 1.5 implies that the augmentation maps are actually locally trivial fibrations. Using Lemma 4.2 we get that the homotopy fiber \( \text{HoFib}_W(\|e_i\|) \) is weakly homotopy equivalent to \( \|\text{Fib}_W(e_i)\| \), where this is the geometric realization of the level-wise homotopy fiber. The \( i \)-simplices of \( \text{Fib}_W(e_i) \) are certain thickened arcs in \( W \) with embedded boundary isotopy.
Fix \( x_k \in \ell_k \) and charts \( U_k \subset W \) centered at \( x_k \). We say that an embedding \( u: I \to W \) meets \( x_k \) in a nice way if its image in \( U_k \) is a straight ray meeting 0.

Let \( X(W; x_0, x_1)_\bullet \) denote the following semi-simplicial set: Its set of zero simplices consists of embeddings of an arc into \( W \) that meets \( x_0 \in \ell_0 \) and \( x_1 \in \ell_1 \) in a nice way together with a tubular neighborhood of said arc and we furthermore require these arcs to be non-isotopic to a part of the boundary. The set of \( i \)-simplices is given by collections of 0-simplices such that the isotopy class of the arc system is coconnected and such that the tubular neighborhoods are disjoint except for their intersection with \( U_k \).

Furthermore we require the arcs to intersect only in \( x_k \) and the ordering of the arcs at \( x_k \) with respect to the angle to be order-preserving at \( x_0 \) and order-reversing at \( x_1 \).

There is a map \( f: \text{Fib}_W(\epsilon_\bullet) \to X(W; x_0, x_1) \) given by sending an element in the fiber to the underlying thickened arcs and then adding a collar of \( W \) to \( W \) and then joining the arcs to \( x_k \) in a controlled way to produce an element in \( X \).

Let \( \text{Fib}_W(\epsilon_\bullet)\delta \) denote the semi-simplicial set \( \text{Fib}_W(\epsilon_\bullet) \) i.e. the set of \( i \)-simplices is given by the underlying set of \( \text{Fib}_W(\epsilon_i) \) . Now we want to apply Theorem A.7 of \cite{Kup13}, which tells us that if we have a map

\[
f: \text{Fib}_W(\epsilon_\bullet)\delta \to X(W; x_0, x_1)
\]

such that

(i) \( X(W; x_0, x_1) \) is weakly Cohen-Macauley of dimension \( (g - 1) \) i.e. it is \( (g - 2) \)-connected and the link of every \( p \)-simplex is \( (g - 2 - p - 2) \)-connected.

(ii) \( \text{Fib}_W(\epsilon_\bullet) \) is a Hausdorff ordered flag space

(iii) \( \|f\|: \|\text{Fib}_W(\epsilon_\bullet)\delta\| \to \|X(W; x_0, x_1)\| \) is simplexwise injective i.e. for every \( p \)-simplex \( \sigma = \{y_0, \ldots, y_p\} \) with \( p \geq 1 \) and \( y_i \neq y_j \) for \( i \neq j \) we have \( f(y_i) \neq f(y_j) \).

(iv) For all finite collections \( \{y_1, \ldots, y_k\} \subset \text{Fib}_W(\epsilon_0) \) and \( p_0 \in X(W; x_0, x_1) \) such that \( (p_0, f(y_i)) \) is an 1-simplex in \( X(W; x_0, x_1) \) then there exists an element \( y_0 \in \text{Fib}_W(\epsilon_0) \) such that \( f(y_0) = p_0 \) and such that \( (y_0, y_i) \) is a 1-simplex in \( \text{Fib}_W(\epsilon_\bullet) \).

All these conditions are fulfilled the aforementioned theorem implies that \( \|\text{Fib}_W(\epsilon_\bullet)\| \) is \( (g - 2) \)-connected. The second and third properties are easy observations. For the first property note that Theorem 2.9 in \cite{Nar15} proves that \( \|X\| \) is \( (g - 2) \)-connected but if one looks at the proof carefully one notes that it is actually proven that \( X \) is weakly Cohen-Macauley of dimension \( (g - 1) \).
For the fourth property note that we can find a thickened embedded arc $\gamma$ in $W$ such that $f(\gamma) = p_0$ (this makes sense, since $f$ only takes the underlying thickened arcs of the elements in Fib$_W(\epsilon_\ast)$) and such that the tuple consisting of $\gamma$ and the thickened underlying arcs of all $y_i$ form an ordered and coconnected arc system. Now note that by simply-connectedness of $M$ and the main result of [Hae62] we can find an extension of $\gamma$ to an arc in $M$ with embedded boundary isotopy. Since the dimension of $M$ is at least 5 we conclude that a small perturbation of this embedding yields an embedding disjoint from all the other $y_k$. Adding a sufficiently small tubular neighborhood finishes the proof of the fourth assumption and therefore gives us the desired claim about the connectivity of Fib$_W(\epsilon_\ast)$.

To finish the proof, just observe that the following diagram is a pullback diagram, where the lower map denotes the map forgetting the tangential structure.

\[
\begin{array}{ccc}
\|O^T_{g,b}(M; \delta^T)\| & \rightarrow & \|O_{g,b}(M; \delta)\| \\
\downarrow & & \downarrow \\
E^T_{g,b}(M; \delta^T) & \rightarrow & E^+_{g,b}(M; \delta)
\end{array}
\]

This observation concludes the proof. 

\[\square\]

**Constructing $B_i$ for the Application of Lemma 4.3** We will need to establish some more notation before we can define the spaces representing $B_i$.

**Definition 5.4.** Let $e: D^2_+ \rightarrow M$ denote a thickened arc with an embedded boundary isotopy. If we fix a subbundle $L$ of $N_M e(D^2_+)\big|_{\partial^1 D^2_+}$ of dimension 1, we will denote the restriction of $e_M$ to $L$ by $e_L$. If $e(L(\pm 1, 0) \subset \ell_0 \cup \ell_1$, we will call a tuple $(e, e_M, e_L)$ a **thickened strip with embedded boundary isotopy**. If we add a tangential structure for the image of $e_L$, which agrees with the one specified by $\delta^T$, wherever this makes sense, we will call this a **thickened strip with tangential structure and embedded boundary isotopy**. We will call the image of $e_L$ the **strip** of $(e, e_M, e_L)$.

**Notation.** Similar as before we will usually suppress $e_M$ and $e_L$ from this notation and only write $e$ for the tuple $(e, e_M, e_L)$.

With this definition at hand we can define the spaces, which will represent $B_i$ later on.

**Definition 5.5.** Let $A^T(M; \delta^T, \ell)$ denote the set of tuples $(a^0, \ldots, a^i)$ such that all the $a^k$ are thickened strips with tangential structures and embedded boundary isotopies such that all the images of $a^k_M$ are disjoint and the starting and endpoints of the underlying arcs are ordered from 0 to $i$ in $\ell_0$ and ordered from $i$ to 0 in $\ell_1$.
We topologize this as a subset of $\overline{\text{Emb}}(D^2 \times [i], M) \times \text{Emb}^T(I \times I \times [i], M)$, where the thickened arcs with embedded boundary isotopies correspond to elements the components in the first factor and the strips and their tangential structure to the second factor.

There is a continuous map from $O_{g,b}^T(M; \delta^T, \ell)_i \to A^T(M; \delta^T, \ell)_i$ which forgets the surface but keeps the tubular neighborhood in the surface as $a^k_L$ and equips the image of $a^k_L$ with the restriction of the tangential structure on $W$.

The following lemma will compute the homotopy fibers that occur in the later use of Lemma 4.3.

Lemma 5.6. The restriction map $O_{g,b}^T(M; \delta^T, \ell)_i \to A(M; \delta^T, \ell)_i$ is a Serre fibration and the fiber over a point $u = (a^0, \ldots, a^i)$ can be identified with $E_{g-i-b+i+1}^T(M(a); \delta^T(a), \ell)$ if we are considering $O_{g,b}^T(M; \delta^T, \ell)_i$ or given by $E_{g-i-b+i-1}^T(M(a); \delta^T(a), \ell)$ if we are considering $O_{g,b}^T(M; \delta^T, \ell)_i$.

Here we define $M(a) = M \cup \bigcup_k a^k_M(\partial^1 D^2_2)$ and $\delta^T(a)$ is given by the boundary of the image of $L$ under all $a^k_M$ together with $\delta \cup \bigcup_k a^k_L(L|_{(\pm 1,0)})$ and we equip this with a tangential structure by restricting the one on the images of $a^k_L$ and the tangential structure on $\delta^T$. Note that $M(a)$ is a manifold with corners and $\delta^T(a)$ is a boundary condition for a manifold with corners, but we circumvent this by fixing a homeomorphism from $M(a)$ to $M$ that is a diffeomorphism at all points except the corner points i.e. a homeomorphisms that pushes the dent which came from removing $a$ to the outside.

Note that the boundary condition $\delta^T(a)$ maps the corners of $\Sigma_{g,b}(a)$ i.e. the surface with the corresponding arcs removed to the corners of $M(a)$. Therefore postcomposing a subsurface with the aforementioned homeomorphism gives an embedded subsurface without corners in a manifold without corners. Therefore we can identify $E_{g-i-1,b+i+1}^T(M(a); \delta^T(a), \ell)$ and $E_{g-i,b+i-1}^T(M(a); \delta^T(a), \ell)$ via homeomorphisms with spaces of subsurfaces without corners with tangential structure of $M$. This allows us to resolve the issues with the occurrence of manifolds with corners silently.

Proof. To prove that these maps are indeed Serre fibrations we have to consider the following lifting problem:

$$
\begin{array}{ccc}
D^n \times \{0\} & \longrightarrow & D^n \times I \\
\downarrow f_0 & & \downarrow f \\
O_{g,b}^T(M; \delta^T, \ell)_i & \longrightarrow & A^T(M; \delta^T, \ell)_i \\
\downarrow & & \downarrow \\
O_{g,b}(M; \delta, \ell)_i & \stackrel{f_{\text{br}}}{\longrightarrow} & A(M; \delta, \ell)_i
\end{array}
$$
Here $A(M; \delta, \ell)_i$ is defined just like $A^T(M; \delta^T, \ell)_i$ but without the tangential structures on the strips. Since $A(M; \delta, \ell)_i$ is $\text{Diff}_{\partial}(M)$-locally retractile (see Proposition 2.16 in [CRW17]), we conclude that the map labeled fibr is actually a fibration. Now proceed by lifting $f$ to $O_{g,b}(M; \delta, \ell)$. Such a map corresponds to a submanifold of $D^n \times I \times M$ such that the preimage with respect to the projection $D^n \times I \times M \rightarrow D^n \times I$ of $(x, t)$ for some point $(x, t) \in D^n \times I$ is a subsurface of $M$ of the form $\Sigma_{g,b}$.

Note that the map from $\text{Emb}_{g,b}(M) \rightarrow \mathcal{E}_{g,b}^+(M; \delta)$ is a locally trivial fibration and therefore we can produce a section of this map over the image of $D^n \times I$. So we can lift the map from $D^n \times I$ to $O_{g,b}(M; \delta, \ell)$ to an actual embedding denoted by $F$ of $D^n \times I \times \Sigma_{g,b}$ into $D^n \times I \times M$.

Furthermore note that by taking preimages of the images of $a^L_k$ corresponding to $f(x, t)$ we obtain a map from $D^n \times I$ to $\text{Emb}^{\text{ord}}((|t| \times I, |t| \times \{0\}, |t| \times \{1\}), (\Sigma_{g,b}, (\ell_0, \ell_1)))$, the space of ordered thickened embeddings of arcs, where ordered means that the ordering at the endpoints is as in the definition of $O(\Sigma_{g,b}; \ell_0, \ell_1)$. $\text{Diff}_{\partial}(\Sigma_{g,b})$, the group of diffeomorphisms fixing $\ell \cap \delta$ set wise, acts on $\text{Emb}^{\text{ord}}((|t| \times I, |t| \times \{0\}, |t| \times \{1\}), (\Sigma_{g,b}, (\ell_0, \ell_1)))$ and it is $\text{Diff}_{\partial}(\Sigma_{g,b})$-locally retractile by Lemma 1.6. Therefore the map $\text{Diff}_{\partial}(\Sigma_{g,b}) \rightarrow \text{Emb}^{\text{ord}}((|t| \times I, |t| \times \{0\}, |t| \times \{1\}), (\Sigma_{g,b}, (\ell_0, \ell_1)))$ given by taking a fixed system of thickened arcs and then acting on it via the action of $\text{Diff}_{\partial}(\Sigma_{g,b})$ is equivariant with respect to this action and the action via left multiplication on the diffeomorphism group. From this we conclude that the map is actually a locally trivial fibration. Using the contractibility of $D^n \times I$ we can construct a lift of the map from $D^n \times I$ to $\text{Emb}^{\text{ord}}((|t| \times I, |t| \times \{0\}, |t| \times \{1\}), (\Sigma_{g,b}, (\ell_0, \ell_1)))$ to the diffeomorphism group.

Using this we can reparametrize $F$ such that $\{x\} \times \{t\} \times \sigma$ gets mapped to the images of $a^L_k$ corresponding to $f(x, t)$. Here $\sigma$ denotes the image of some element in $\text{Emb}^{\text{ord}}((|t| \times I, |t| \times \{0\}, |t| \times \{1\}), (\Sigma_{g,b}, (\ell_0, \ell_1)))$. Now all that is left to do is to equip $F(D^n \times I \times \Sigma_{g,b})$ with a tangential structure that agrees with the one specified on $F(D^n \times I \times \sigma)$ via $f$ and the one on $D^n \times \{0\} \times \Sigma_{g,b}$ specified via $f_0$. Let $\text{Gr}(F)$ denote the map from $D^n \times I \times \Sigma_{g,b}$ to $\text{Gr}_2(TM)$ that sends a point to the tangent space in $TM$ of the corresponding surface. Now define $B_2(D^n \times I \times M) := \text{Gr}(F)^*(B_2(TM))$.

It will be proven later in Lemma 5.7 that $D^n \times \{0\} \times \Sigma_{g,b} \cup D^n \times I \times \sigma$ is a strong deformation retract of $D^n \times I \times \Sigma_{g,b}$. This enables us to identify $B_2(D^n \times I \times \Sigma_{g,b})$ with $r^*(B_2(D^n \times \{0\} \times \Sigma_{g,b} \cup D^n \times I \times \sigma))$ for some strong deformation retract $r$, where $B_2(D^n \times \{0\} \times \Sigma_{g,b} \cup D^n \times I \times \sigma)$ denotes the restriction of $B_2(D^n \times I \times \Sigma_{g,b})$ to the space in brackets. This enables us to give the rest of the points of $D^n \times I \times \Sigma_{g,b}$ a tangential structure. Furthermore this lift is continuous as it stems from a map to the space of embeddings with tangential structure, which is continuous because it is defined using the correspondence between maps into mapping spaces and
Lastly we have to determine the fiber over \( a \). Note that removing \( i + 1 \) strips \( s_k \) from a surface increases its Euler characteristic by \( i + 1 \) since it corresponds to taking out \( i + 1 \) one-cells. So to calculate the genus and the number of boundary components of the fiber it is enough to specify its number of boundary components. We will take out the \( i + 1 \) strips consecutively and there are two cases we have to distinguish. Either all boundary points lie in the same connected component of the boundary of the surface (Case 1) or they all lie in different connected components (Case 2). But note that by removing \( s_0 \) in Case 2 we reduce the calculation to Case 1 for \( \langle s_k \rangle \geq 0 \). But in the first case we see that taking out an arc increases the number of boundary components by one and the requirement for the ordering of the arcs ensures that the consecutive arcs all connect the same connected component of the boundary.

All in all we conclude that in Case 2 the number of boundary components changes to \( b + i - 1 \) and in the first case it changes to \( b + i + 1 \). Using the formula 
\[ g = \frac{1}{2}(2 - \chi - b) \] we can compute the genus to get the above specifications of the fiber.

**Lemma 5.7.** Let \( M \) denote a metric space and \( N \subset M \) a closed subset, with a closed neighborhood \( V \) such that \( N \) is a strong deformation retract of \( V \) via a strong deformation \( \Phi(x,t) \). Then \( M \times \{0\} \cup N \times I \) is a strong deformation retract of \( M \times I \).

The assumptions about \( M \) are quite restricting and the lemma should be true in a more general setting, but for the present context it certainly suffices.

**Proof.** The trick is to write down the correct formulas for a deformation retract. Define \( d: V \rightarrow \mathbb{R} \) as 
\[ d(x) = \frac{d(x,N)}{d(x,N) + d(x,\partial V)} \]
As assumed above let \( \Phi(-,t) \) denote a homotopy between the identity on \( V \) and a retraction of the neighborhood. Then the deformation retract in the product space and the corresponding homotopy is given by the following:

\[
\Psi((x,s),t) = \begin{cases} 
(\Phi(x,0),st) & \text{for } d(x) - s \geq 0 \\
(\Phi(x,(1-t)(s-d(x)_s)),s - (1-t)d(x)) & \text{for } d(x) - s \leq 0 \\
(x,st) & \text{for } x \notin V 
\end{cases}
\]

**Remark.** It is evident that if needed and in the correct context this proof can be altered to produce a smooth deformation retract.
**Definition 5.8.** For some $a \in A(M; \delta^T, \ell)_0$ we call the composition of the inclusion of the fiber in Lemma 5.6 with the augmentation of $O^T_{g,b}(M; \delta^T, \ell)$ the **approximate augmentation** of the resolution $O^T_{g,b}(M; \delta^T)_\bullet$ over the 0-simplex $u$.

**Constructing $f_\bullet$ in the application of Lemma 4.3** We want to extend the maps $\alpha_{g,b}$ and $\beta_{g,b}$ to the aforementioned resolution as shown below:

\[
\begin{align*}
O^T_{g,b}(M; \delta^T, \ell) &\longrightarrow O^T_{g+1,b-1}(M_1; \delta^T, \ell) \\
\downarrow_{\epsilon_i} &\quad \downarrow_{\epsilon_i} \\
E^T_{g,b}(M, \delta^T) &\xrightarrow{\alpha_{g,b}(M; \delta^T, \delta^T)} E^T_{g+1,b-1}(M_1, \delta^T)
\end{align*}
\]

\[
\begin{align*}
O^T_{g,b}(M; \delta^T, \ell) &\longrightarrow O^T_{g,b+1}(M_1; \delta^T, \ell) \\
\downarrow_{\epsilon_i} &\quad \downarrow_{\epsilon_i} \\
E^T_{g,b}(M, \delta^T) &\xrightarrow{\beta_{g,b}(M; \delta^T, \delta^T)} E^T_{g,b+1}(M_1, \delta^T)
\end{align*}
\]

Let $P$ denote the subsurface in $\partial^0 M \times I$ used in the definition of the stabilization maps. We define $\tilde{\ell}_i = \ell_i \times \{1\}$ for $i \in \{0, 1\}$ and assume without loss of generality that $P \cap (\ell \times I) = (\ell \cap \delta) \times I$ in particular $\tilde{\ell} \cap \delta = (\ell \cap \delta) \times \{1\}$, where $\delta^T$ denotes the boundary condition of the image of the stabilization map. (Here we isotope $P$ not relative the boundary to get that $\ell \times I$ is contained in $P$).

Define $\bar{a}$ for $a \in A(M; \delta^T, \ell)_0$ as $a(\partial^0 D^2) \times I$. This allows us to extend the stabilization maps to $O^T_{g,b}(M; \delta^T, \ell)_1$ as follows

\[
(W, a) \rightarrow (W \cup P, a \cup \bar{a} = (a^0 \cup \bar{a}^0, \ldots, a^i \cup \bar{a}^i))
\]

and we write $\bar{a}$ for $a \cup \bar{a}$. This yields the dashed lifts of $\alpha_{g,b}(M; \delta^T, \delta^T)$. Since they commute with our face maps and with the augmentation maps we get a map of semi-simplicial spaces

\[
\alpha_{g,b}(M; \delta^T, \delta^T)_\bullet : C^T_{g,b}(M; \delta^T, \ell)_\bullet \rightarrow C^T_{g+1,b-1}(M_1; \delta^T, \ell)_\bullet
\]

which is augmented over $\alpha_{g,b}(M; \delta^T, \delta^T)$. We can do the same to get a map

\[
\beta_{g,b}(M; \delta^T, \delta^T)_\bullet : C^T_{g,b+1}(M; \delta^T, \ell)_\bullet \rightarrow C^T_{g,b}(M_1; \delta^T, \ell)_\bullet
\]

augmented over $\beta_{g,b}(M; \delta^T, \delta^T)$.

All of these considerations imply the following corollary.

**Corollary 5.9.** The semi-simplicial pair $C(\alpha_{g,b}(M; \delta^T, \delta^T)_\bullet)$ together with the natural augmentation map to $C(\alpha_{g,b}(M; \delta^T, \delta^T))$ is a $g$-resolution i.e. the map between pairs is $g$-connected.
The semi-simplicial pair \( C(\beta_{g,b}(M; \delta^T, \delta^T)\) together with the natural augmentation map to \( C(\beta_{g,b}(M; \delta^T, \delta^T)) \) is a \((g-1)\)-resolution.

There is a commutative square

\[
\begin{array}{ccc}
\mathcal{O}_{g,b}^T(M; \delta^T, \ell)_i & \xrightarrow{(\alpha_{g,b})_i} & \mathcal{O}_{g+1,b-1}^T(M_1, \delta^T, \ell)_i \\
A^T(M; \delta^T, \ell)_i & \xrightarrow{u \to \bar{u}} & A^T(M_1; \delta^T, \ell)_i \\
\end{array}
\]

where the lower map is given by joining elements with \( \bar{a}_j \), which is evidently a homotopy equivalence. By commutativity of the above square we get a map between the fibers over the points \( a \) and \( \bar{a} \)

\[
\mathcal{E}^T_{g-i,b+i+1}(M(u); \delta^T(u)) \to \mathcal{E}^T_{g-i,b+i}(M_1(\bar{u}); \delta^T(\bar{u}))
\]

If \( P \) denotes the bordism defining the map \( \alpha_{g,b} \) in question, then this map is given by taking the union with \( P(\bar{a}) := P \cup \bigcup_k \bar{a}_k \). Consider the following diagram, where the vertical maps are the homeomorphisms introduced in the discussion after Lemma 5.6

\[
\begin{array}{ccc}
\mathcal{E}^T_{g-i,b+i-1}(M(a), \delta^T(a)) & \xrightarrow{-\cup P(a)} & \mathcal{E}^T_{g-i,b+i}(M_1(\bar{a}), \delta^T(\bar{a})) \\
\mathcal{E}^T_{g-i,b+i-1}(M, \delta^{\eta T}) & \xrightarrow{f} & \mathcal{E}^T_{g-i,b+i-1}(M_1, \delta^{\eta T}) \\
\mathcal{E}^T_{g-i,b+i-1}(M, \delta^{\eta T}) & \xrightarrow{f} & \mathcal{E}^T_{g-i,b+i-1}(M_1, \delta^{\eta T}) \\
\end{array}
\]

We want to understand the map labeled \( f \). By picking sections of \( N_{\partial M \times I} \bar{a} \) extending the ones given by \( \partial \bar{a} \) we get embeddings \( b_k : I \times I \to \partial M \times I \) for \( k \in \{0, \ldots, 2i + 1\} \), which connect pairs of arcs in the boundary of \( P(a) \) one of which lies in \( \ell_0 \times I \) and one in \( \ell_1 \times I \) and we order these embeddings according to their points in \( \ell_0 \times \{0\} \). Then the map labeled \( f \) is given by \(- \cup (P \cup b_k(I \times I)) \). Note that \( P(a) \) has multiple connected components and the union with \( b_{2k+1} \) and \( b_{2k+2} \) whenever these indexes make sense produces a cylinder so one only has to consider the connected component, which is connected via \( b_0 \) and \( b_{2i+1} \) and this connected component is easily observed to be a pair of pants glued at the waist so that \( f \) corresponds to a map of type \( \beta_{g-i,b+i-1} \) and so we will denote the aforementioned map \(- \cup P(a) \) by \( \beta_{g-i,b+i-1}(M(a); \delta^T(a), \delta^T(\bar{a})) \) if we want to be more precise.

As the map \( A^T(M; \delta^T, \ell)_i \to A^T(M_1(a); \delta^T(\bar{a})) \) is a homotopy equivalence we conclude that the space \( \mathcal{E}^T_{g-i,b+i-1}(M(a); \delta^T(a)) \) is homotopy equivalent to the homotopy fiber of the composition of the restriction map \( \rho \) of \( O_{g,b}^T(M; \delta^T, \ell)_i \to A^T(M; \delta^T, \ell)_i \) with the aforementioned homotopy equivalence (the importance of this observation is rather subtle, but it is necessary to apply Lemma 4.3). Moreover we have shown that the map between the fibers is a stabilization map of the form \( \beta_{g-i,b+i-1}(M(a); \delta^T(a), \delta^T(\bar{a})) \).
As a consequence we get the following diagram:

\[
\begin{array}{c}
\mathcal{E}^T_{g-i,b+i-1}(M(a);\delta^T(a)) \xrightarrow{\beta_{g-i,b+i-1}(M(a);\delta^T(a),\bar{\delta}^T(\bar{a}))} \mathcal{E}^T_{g-i,b+i}(M_1(\bar{a});\bar{\delta}^T(\bar{a})) \\
\cong \\
\text{HoFib}_{\bar{a}}(\rho) \\
\cong \\
\mathcal{O}^T_{g,b}(M;\delta^T,\ell) \xrightarrow{\alpha_{g,b}(M,\delta^T,\bar{\delta}^T)} \mathcal{O}^T_{g+1,b-1}(M_1,\bar{\delta}^T,\bar{\ell}) \\
\rho \\
\Rightarrow \\
\mathcal{A}^T(M_1,\bar{\delta}^T,\bar{\ell}) \xrightarrow{\rho'} \\
\mathcal{A}^T(M_1,\bar{\delta}^T,\bar{\ell})
\end{array}
\]

This gives that the pair \((\text{HoFib}_{\bar{a}}(\rho'),\text{HoFib}_{\bar{a}}(\rho))\) is homotopy equivalent to the pair of the stabilization map \((\beta_{g-i,b+i-1}(M(a);\delta^T(a),\bar{\delta}^T(\bar{a})))\). Following the same procedure with maps of the form \(\beta_{g,b}(M;\delta^T,\bar{\delta}^T)\) we obtain the following corollary:

**Corollary 5.10.** The induced map between the homotopy fibers of

\[(\alpha_{g,b}(M;\delta^T)) \rightarrow \mathcal{A}^T(M_1,\bar{\delta}^T,\bar{\ell}),\]

is given by \(\beta_{g-i,b+i-1}(M(a);\delta^T(a),\bar{\delta}^T(\bar{a}))\) and analogously the induced map between the homotopy fibers of

\[(\beta_{g,b}(M;\delta^T)) \rightarrow \mathcal{A}^T(M_1,\bar{\delta}^T,\bar{\ell}),\]

is given by \(\alpha_{g-i-1,b+i+1}(M(a);\delta^T(a),\bar{\delta}^T(\bar{a}))\).

**Definition 5.11.** We call the map from \(C(\beta_{g-i,b+i-1}(M(a);\delta^T(a),\bar{\delta}^T(\bar{a})))\) to \(C(\alpha_{g,b}(M;\delta^T))\) given by the composition of the inclusion of the fiber into \(C(\alpha_{g,b}(M;\delta^T))\) with the projection onto \(C(\alpha_{g,b}(M;\delta^T))\) the approximate augmentation of the resolution \((\alpha_{g,b}(M;\delta^T))\), and analogously for \((\beta_{g,b}(M;\delta^T))\).

These are the maps mentioned in the four auxiliary assertions in Section 3.

**Proof of (iii) and (iv) of Lemma 3.1.** By corollary 5.10 we get the following diagram:
Furthermore the augmentation $O'_{g,b}(M,\delta) \to E_{g,b}(M,\delta)$ is $(g-2)$-connected by Proposition 5.3. Since $M(a)$ is homeomorphic to $M_B^g$ implies that $H_q(\beta_{g-i,b+i-1}) = 0$ for $q \leq \frac{1}{3}(g-2)$. This gives us all the ingredients to apply Lemma 4.3 with $k = \frac{1}{3}(2g+1)$, which implies that the induced map $H_q(\beta_{g,b}) \to H_q(\alpha_{g,b})$ is an epimorphism for $q \leq k$, which is exactly the statement of $SA_g$. Similarly for iv).

6 Proving Part (v) and (vi) of Lemma 3.1

Fix an element $a = (a, a_M, a_L) \in A^0_\ell(M;\ell)$. We will have to take a closer look at the approximate augmentations of the resolutions $O'_{g,b}(M;\delta,\ell)$ denoted by $b_{g,b-1}(a)$ and the approximate augmentation of the resolution $O_{g,b}(M;\delta,\ell)$ denoted by $a_{g,b}(a)$. Using Corollary 5.10 we get the following diagram:

\[
\begin{array}{ccc}
E_{g-1,b+i-1}^T(M(a),\delta(a)) & \xrightarrow{\beta_{g-1,b+i-1}^T} & E_{g,b}^T(M_1(\bar{a},\delta(\bar{a}))) \\
\downarrow & & \downarrow \\
O'_{g,b}(M,\delta)_i & \xrightarrow{(a_{g,b})_i} & O'_{g+1,b-1}^T(M_1,\delta)_i \\
\downarrow & & \downarrow \\
E_{g,b}^T(M,\delta) & \xrightarrow{a_{g,b}} & E_{g+1,b-1}^T(M_1,\bar{\delta})
\end{array}
\]

We are interested in constructing the dashed map, such that both triangles commute up to homotopy. This will give us some algebraic insight, which will be crucial to the proof of parts (v) and (vi) of Lemma 3.1. Constructing this map and the corresponding homotopies will occupy the first part of this section.

The constructions will be a little bit easier if we extend the collar a bit. For this we will again denote by $M_\ell = M \cup_{\partial M \times \{0\}} \partial M \times [0,\ell]$ and for some set $X \subset \partial^0 M \times [0,1]$ we will denote by $X + i \subset \partial^0 M \times [i,i+1]$ the translated set. Furthermore we will write $P$ for the bordism defining $\alpha_{g,b}$ and we will also write $a_L$ and $\bar{a}_L$ for their images and we will do the same for $a_M$ and $\bar{a}_M$. By enlarging the above diagram we get the following diagram, where the lower vertical maps are obviously homotopy equivalences and it is easy to
see using collars that the lower rectangle commutes up to homotopy.

\[
\begin{array}{c}
\mathcal{E}_{g,b-1}(M(a); \delta(a)) \\
\mathcal{E}_{g,b}(M; \delta) \\
\mathcal{E}_{g,b}(M_2; \delta + 2)
\end{array}
\begin{array}{c}
\xrightarrow{\beta_{g,b-1}(M(a); \delta(a)) - \cup P(\bar{a})}
\xrightarrow{\alpha_{g,b}(M; \delta) - \cup P}
\xrightarrow{\alpha_{g,b}(M_2) - \cup (P + 2)}
\end{array}
\begin{array}{c}
\mathcal{E}_{g,b}(M_1(\bar{a}); \bar{\delta}(\bar{a})) \\
\mathcal{E}_{g+1,b-1}(M_1; \bar{\delta}) \\
\mathcal{E}_{g+1,b-1}(M_3; \bar{\delta} + 2)
\end{array}
\]

Let \( N \) denote \( \bar{a}_M \cup \partial^0 M \times [1, 2] \). Since \( M_1(\bar{a}) \cup N = M_2 \) any trivial cobordism with tangential structure (i.e. a cobordism which increases neither the number of boundary components nor the genus) \( Q \subset N \) satisfying the boundary condition \( \xi = (\delta \times \{2\}) \cup \delta(\bar{a}) \) defines a map

\[
- \cup Q : \mathcal{E}_{g,b}^T(M_1(\bar{a}), \bar{\delta}(\bar{a})) \to \mathcal{E}_{g,b}^T(M_2; \delta + 2)
\]

Such a map gives us some dashed arrow in the above diagram, but to ensure that the triangles commute we need some further assumptions for \( Q \). The following lemma says that a nice enough \( Q \) exists that only depends on \( P \) such that the aforementioned dashed arrow exists and such that the diagram is homotopy commutative.

**Lemma 6.1.** There exists an \( \ell \) that depends only on \( P \), for which there exists:

(i) A trivial cobordism \( Q \in \mathcal{E}^T(N; \xi) \)

(ii) Isotopies

\[
P(\bar{a}) \cup Q \simeq a_L \cup (\delta^T \times [0, 2]) \subset a_M \cup \partial^0 M \times [0, 2] \\
Q \cup (P + 2) \simeq \bar{a}_L \cup (\delta^T \times [1, 3]) \subset \bar{a}_M \cup \partial^0 M \times [1, 3]
\]

relative to their respective boundaries.

To prove this lemma we need some definitions in the context of surgery theory in an ambient manifold and in the presence of tangential structures.

Fix an embedding \( a : D^2 \times \partial D^1 \to D^2 \times D^1 \) such that \( a|_{[\partial D^2 \times \partial D^1]} \) is the identity and the image of \( \{ x \in D^2 \mid \|x\| \geq \frac{1}{2}\} \times \partial D^1 \) is contained in \( \partial D^2 \times D^1 \).

**Definition 6.2.** Let \( W \subset M \) be an oriented subsurface with tangential structure and let \( \gamma \subset W \) be an oriented embedded circle. This orientation induces an orientation of \( N_W \gamma \). Since the normal bundle is 1-dimensional
this is the same as a trivialisation. Let $s_0$ denote a non-zero section given by this trivialisation. Let $(D, s)$ be a pair consisting of an embedded oriented 2-disc $D \subset M$ bounding $\gamma$ transverse to $W$ and a section $s$ of $S(N_M D)$ extending $s_0$.

A **surgery datum with tangential structure** for $(W, (D, s))$ contains first of all an embedding of pairs $e: (D^2 \times D^1, \partial D^2 \times D^1) \to (M, W)$ such that:

(i) the restriction of $e$ to $D^2 \times \{0\}$ is an orientation preserving diffeomorphism onto $D$

(ii) the canonical section of $S(N_{D^2 \times D^1} D^2 \times \{0\})$ is mapped to the section $s$. 

Secondly it contains a section $\phi$ of $B_2(e(a(D^2 \times \partial D^1)))$ which equals the tangential structure of $W$ on $e(a(\{x \in D^2 \mid ||x|| \geq \frac{1}{2}\} \times \partial D^1))$

The **ambient surgery on $W$ along $\gamma$ by means of the pair $(D, s)$ and the surgery datum $(e, \phi)$**, denoted by $W^\# D$ (even tough this notation is a priori not unambiguous, see the following Lemma for clarification), is the submanifold of $M$ obtained by removing $e((\partial D^2 \times D^1)$ from $W$ and then gluing in $e(a(D^2 \times \partial D^1)$. The new tangential structure on this submanifold is given by the union of the original tangential structure and $\phi$.

**Lemma 6.3.** The isotopy class of $W^\# D$ is determined by $W$ and $(D, s)$.

**Proof.** It is briefly mentioned in [CRW17] how to prove that the aforementioned surgery without the tangential structure is well defined up to isotopy and we can always lift these isotopies to the tangential structure level to reduce the question to whether the tangential structure is uniquely determined up to homotopy.

If we have two different tangential structures on a connected component of $e(a(D^2 \times \partial D^1) used for a surgery then they agree on the boundary and therefore the only obstruction to a homotopy between these two lies in $H^3(D^2 \times I, S^1 \times I \cup D^2 \times \{\pm 1\}; \pi_2(F))$, which vanishes by assumption. Therefore the two tangential structures are actually homotopic.

**Lemma 6.4.** Let $M$ be a compact manifold of dimension at least 5 possibly with boundary and with a space of tangential structures of subplanes of $TM$ $T: B_2(M) \to Gr_2(TM)$ and let $B$ in $M$ be a ball. Let $W$ be an embedded oriented surface in $B$ and let $(\gamma, \gamma')$ be either a pair of collared arcs in $W$ with the same boundary, which are not isotopic to an arc that lies completely in the boundary, or a pair of curves in $W$ such that their respective complements are connected. Then there is an isotopy $f_t: M \to M$ supported on $B$ i.e. it is the identity outside of $B$ and constant on the boundary of $B$ such that

(i) $f_1(W) = W$
(ii) $f_1(\gamma) = \gamma'$

Furthermore there is a continuous choice of tangential structures of $f_1(W)$ such that the tangential structures of $W$ and $f_1(W)$ agree. By a slight abuse of notation we will write $f_1(W)$ for $f_1(W)$ with this tangential structure.

Proof. Note that $\text{Diff}_+(W)$ acts transitively on the space of non-separating arcs (curves) in $W$. Fix a parametrization of $W$ i.e a preimage of $W \in E^T(W; B; \delta^T)$ in $\text{Emb}^T(W; B; \delta)$ called $\phi$. Then the proof of Lemma 3.2 showed that $\phi$ and $\phi \circ \psi$ are isotopic relative their boundaries for any element in $\text{Diff}_+(W)$ as they induce the same image in homology. Using the isotopy extension theorem we get the desired isotopy of $M$. \hfill $\square$

With all these tools at hand we are able to prove Lemma 6.1. The proof follows the proof of Lemma 7.1 in [CRW17] almost verbatim, but special care is needed because we have to consider the tangential structure which complicates the necessary surgeries a little bit.

Proof of Lemma 6.1. Let us assume without loss of generality (see Proposition 2.13) that $\pi|_P$ is a Morse function. Let $\phi$ denote the underlying arc of $\bar{a}$. Let $D_\phi$ be the image of $D^2_+$ under $\bar{a}$ and let $\sigma$ be the image of $\partial^0 D^2_+$, so that $D_\phi$ is a half-disk in $\bar{a}_M$ bounding $\phi \cup \sigma$. Observe that the isotopy class of $\sigma$ is determined by $\ell$. Chose a section $s_\phi$ of $S(N_M D_\phi)$ that agrees with the section on $\phi$ given by the orientation of $\bar{a}_L$, and $\phi$ and is collared i.e. as $\bar{a}$ is collared, there is an $\epsilon$ for which $D_\phi = \sigma \times (1-\epsilon, 1]$ and we have a canonical identification $N_{\partial M \times (1-\epsilon, 1]} D_\phi \cap \partial M \times (1-\epsilon, 1] \cong (N_{\sigma} \partial M) \times (1-\epsilon, 1]$; say that $s$ is collared if $s(x, t) = s(s(x, 0), t)$ under this identification. Note that since the fiber of $T$: $B_2(M) \rightarrow \text{Gr}_2(M)$ is simply-connected we can give $D_\phi$ some tangential structure such that $(D_\phi, s_\phi)$ is a valid surgery datum.

Let $p_0$ and $p_1$ be the initial and final points of the path $\phi$. By definition these are points that lie in the pair of pants component of $P$. Let $\beta$ be a path embedded in $P$ from $p_0$ to $p_1$ with a unique maximum at the unique critical point $p$ of $\pi$: $P \subset \partial^0 M \times [0, 1] \rightarrow [0, 1]$ and such that $T_p \beta \subset T_p P$ is the unstable subspace. Now chose $\ell$ so that $\sigma$ is isotopic to $\beta$ in $\partial M \times [0, 1]$.

Then $\beta$ is isotopic to $\phi$. We can find an embedded disk that bounds the curve $\beta \cup \sigma$ and therefore we conclude that there is a ball $B \subset a_M \cup \partial^0 M \times [0, 1]$ containing this disk and $D_\phi$. In addition $\beta$ and $\phi$ are both non-separating in $a_L \cup P$ as they both meet only one critical point and the stable submanifold of the other critical point provides a path between the two sides of the paths. Using Lemma 6.4 we conclude that there exists an isotopy $f_1$ of $a_M \cup \partial^0 M \times [0, 1]$ supported on $B$ and constant on the boundary such that

(i) $f_1(w \cup P) = w \cup P$

(ii) $f_1(\phi) = \beta$

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We define $D_\beta$ as $f_1(D_\phi)$ and $s_\beta = f_1(s_\phi)$. To define a tangential structure on $D_\beta$ and an isotopy of disks with tangential structures note that we already have a fixed tangential structure on $D^2 \times \{0\} \cup \partial D^2 \times I$ and that this space is a deformation retract of $D^2 \times I$, which enables us to extend the tangential structure to the whole cylinder.

We can alter $D_\phi$ a bit such that $D_\beta$ has no critical points with respect to $\pi$. Define $\text{refl}(P), \text{refl}(\beta), \text{refl}(D_\beta) \subset \partial M \times [1, 2]$ to be the reflections of $P, \beta$ and $D_\beta$ along $\partial M \times \{1\}$. We give $\text{refl}(P)$ some arbitrary tangential structure, extending the one given at the boundary, which is possible since the corresponding obstruction classes vanish since the fiber of $T: B_2(M) \to \text{Gr}_2(TM)$ is 1-connected. Similarly we can equip $\text{refl}(D_\beta)$ with a tangential structure, which agrees with the one of $D_\beta$ on its intersection with $\partial M \times \{1\}$.

The union $\bar{a}_L \cup \text{refl}(P)$ is a submanifold of $N$ satisfying the boundary condition $\xi$, but it is not a trivial cobordism. The circle $\phi \cup \text{refl}(\beta) \subset \bar{a}_L \cup \text{refl}(P)$ is bounded by the disc $D_\phi \cup \text{refl}(D_\beta)$ on which we have the section $s_\phi \cup \text{refl}(s_\beta)$. We define $Q := (\bar{a}_L \cup \text{refl}(P)) \sharp(D_\phi \cup \text{refl}(D_\beta))$. We will proceed by showing that $P(a) \cup Q$ is isotopic to $a_L \cup \partial^0 \times [1, 2]$, which proves the first part of the lemma. We can extend $f_t$ to $a_M \cup \partial^0 M \times [0, 2]$ by the identity on $\partial^0 M \times [1, 2]$. Using $f_t$ we get the following isotopy of submanifolds with tangential structure:

$$P(a) \cup Q = P(a) \cup ((\bar{a}_L \cup \text{refl}(P)) \sharp(D_\phi \cup \text{refl}(D_\beta))$$

$$\simeq f_1([P(a) \cup \bar{a}_L \cup \text{refl}(P)] \sharp(D_\phi \cup \text{refl}(D_\beta)))]$$

$$= f_1(P \cup a_L \cup \text{refl}(P)) \sharp(f_1(D_\phi) \cup \text{refl}(D_\beta))$$

$$= (a_L \cup P \cup \text{refl}(P)) \sharp(D_\beta \cup \text{refl}(D_\beta))$$

$$= a_L \cup (P \cup \text{refl}(P)) \sharp(D_\beta \cup \text{refl}(D_\beta))$$

But by construction $\pi: (P \cup \text{refl}(P)) \sharp(D_\beta \cup \text{refl}(D_\beta)) \to [0, 2]$ has no critical points, since both critical points of $P \cup \text{refl}(P)$ are canceled by the surgery along $D_\beta \cup \text{refl}(D_\beta)$, and it is therefore a disjoint union of cylinders. Furthermore note that the part in $\partial^0 M \times [1, 2]$ is a reflection of the part in $\partial^0 \times [0, 1]$ and therefore it is homotopic to $\delta \times [0, 2]$ and again using the main result of [Hae62] we get that they are actually isotopic. Lifting this isotopy gives us that $\delta \times [0, 2]$ has some tangential structure, which agrees with the one on $\delta$ at the boundary. To conclude that the desired isotopy can actually be extended to be an isotopy of subsurfaces with tangential structures, we would need a homotopy between two tangential structures on a cylinder. The obstruction classes for the existence of such a homotopy all lie in homology groups which are zero by the assumption on the connectedness of the fiber of $T: B_2(M) \to \text{Gr}_2(TM)$.

Now note that $f_1(P(a) \cup Q)$ is isotopic to $a_L \cup \delta \times [0, 2]$. Gluing of the Morse function $\pi$ on $f_1(P(a) \cup Q) \cap \delta \times [0, 2]$ and some Morse function on $w$ with a single index 1 critical point gives a Morse function with a single
critical point on $f_1((P(a) \cup Q)$. This gives us a Morse function with a single critical point on $P(a) \cup Q$, but note that $P(a)$ has a single critical point from, which we conclude that $Q$ has none. Therefore we conclude that $Q$ is indeed a trivial cobordism (Alternatively this can be shown using Euler-characteristic calculations).

To conclude the second part of Lemma 6.1 we want that $Q \cup (P + 2)$ is isotopic to $\overline{a_L} \cup \delta \times [1, 3]$. Let $q_0$ and $q_1$ denote the initial resp. final point of the underlying arc of $a$ and let $\alpha$ denote an embedded path in $P$ from $q_0$ to $q_1$ with a unique maximum with respect to $\pi$ at the unique critical point of $P$ and such that $\alpha$ represents the stable submanifold in a neighborhood of this critical point. Now let $D_\alpha$ denote some collared half-disk transverse to $P$ bounding $\alpha \cup \rho$, for some path $\rho$ in $\partial^0 M \times \{0\}$, which does not meet $\partial^0 M \times \{0\} \cap P$ except for $q_0$ and $q_1$. This half disk exists using again the main result of [Hae62]. Furthermore assume that $D_\alpha$ meets $D_\beta$ in the unique critical point of $P$ and we equip $D_\alpha$ with some tangential structure compatible with the one on $P$.

This implies that the intersection of $D_\phi \cup \text{refl}(D_\beta)$ and $\text{refl}(D_\alpha) \cup (D_\alpha + 2)$ consists of a single point and therefore we can find a ball $B$ that contains both these disks. Moreover $\phi \cup \text{refl}(\beta)$ and $\text{refl}(\alpha) \cup (\alpha + 2)$ meet transversely in a single point from which we conclude that both curves are non-separating. Using Lemma 6.4 we get an isotopy $g_t$ of $\overline{a_M} \cup \partial^0 M \times [1, 3]$ supported on $B$ such that:

1. $g_1(\overline{a_L} \cup \text{refl}(P) \cup (P + 2)) = \overline{a_L} \cup \text{refl}(P) \cup (P + 2)$
2. $g_1(\phi \cup \text{refl}(\beta)) = \text{refl}(\alpha) \cup (\alpha + 2)$

$g_1(D_\phi \cup \text{refl}(D_\beta))$ and $\text{refl}(D_\alpha) \cup (D_\alpha + 2)$ are both contained in $B$ and agree on their boundary. Using the main result of [Hae62] again, we conclude that there exists an isotopy $h_t$ of $\overline{a_M} \cup \partial^0 M \times [1, 3]$ supported on $B$ such that:

1. $h_t(\overline{a_L} \cup \text{refl}(P) \cup (P + 2)) = \overline{a_L} \cup \text{refl}(P) \cup (P + 2)$
2. $h_1(g_1(D_\phi \cup \text{refl}(D_\beta))) = \text{refl}(D_\alpha) \cup (D_\alpha + 2)$

Furthermore we endow $\text{refl}(D_\alpha) \cup (D_\alpha + 2)$ with the section $h_1(g_1(s_\phi \cup \text{refl}(s_\beta))$ and we equip $\text{refl}(D_\alpha)$ with some tangential structure compatible with the one at $(D_\alpha + 2)$ and $(P + 2)$. All in all we get:

$$Q \cup (P + 2) = (\overline{a_L} \cup \text{refl}(P)) \zeta(D_\phi \cup \text{refl}(D_\beta)) \cup (P + 2)$$

$$\simeq h_1 \circ g_1(\overline{a_L} \cup \text{refl}(P)) \zeta(D_\phi \cup \text{refl}(D_\beta)) \cup (P + 2)$$

$$= (\overline{a_L} \cup \text{refl}(P) \cup (P + 2)) \zeta(h_1(g_1(D_\phi \cup \text{refl}(D_\beta))))$$

$$= (\overline{a_L} \cup \text{refl}(P) \cup (P + 2)) \zeta(\text{refl}(D_\alpha) \cup (D_\alpha + 2))$$

Just as before we conclude that $(\text{refl}(P) \cup (P + 2)) \zeta(\text{refl}(D_\alpha) \cup (D_\alpha + 2))$ has no critical points and is therefore a disjoint union of cylinders and we
conclude that these cylinders are isotopic to \( \bar{\delta} \times [1, 3] \) and by the uniqueness of tangential structures on the cylinder we conclude that we can concatenate this isotopy with some homotopy of tangential structures to get an isotopy of subsurfaces with tangential structures.

Remark. The crucial part in this proof is that the cylinder possesses only one tangential structure up to homotopy. This is the main obstruction to extending this whole proof to the case where only \( \pi_1(F) \) has to vanish, but the author is not aware of any way to extend this.

6.1 Finishing the Proof of (v) and (vi) of Lemma 3.1

If we have an inclusion \( X \to A \), we will denote by \( C(X,A) \) its mapping cone. \( \Sigma X \) will denote the unreduced suspension of \( X \) and \( CX \) will denote the unreduced cone over \( X \).

The last piece of argument for the proof of Lemma 3 will again follow almost verbatim in Section 7.2 in [CRW17]. The following technical lemma can also be found in [CRW17] as Lemma 7.5 and it is the technical foundation of the argument.

Lemma 6.5. Suppose we have a map of pairs

\[
\begin{array}{ccc}
A & \overset{i}{\longrightarrow} & X \\
\downarrow g & & \downarrow f \\
A' & \overset{j}{\longrightarrow} & X'
\end{array}
\]

such that there exists a map \( t: X \to A' \) making the bottom triangle commute up to homotopy \( H \). Then the induced maps between mapping cones \( (f, g): C(X,A) \to C(X',A') \) factors as \( C(X,A) \xrightarrow{p} CA \cup_i CX \xrightarrow{h} C(X',A') \), where \( p \) comes from the Puppe sequence of the pair \( (X,A) \). If there is in addition a homotopy \( G \), which makes the bottom triangle commute, then the composite \( CA \cup_i CX \xrightarrow{h} C(X',A') \xrightarrow{j} CA' \cup_j CX' \) is nullhomotopic.

We return to the situation of the beginning of this section, where we had chosen a 0-simplex \( a_0 \in \mathcal{O}_{g,b}^{T}(M; \delta^T) \). Suppose we have another 0-simplex \( a_1 \) in \( \mathcal{O}_{g,b}^{T}(M(a_0); \delta^T(a_0)) \). Notice that we can consider \( a_1 \) as an element in the resolutions of \( (M; \delta^T) \) as well. We will consider the following enormous diagram, in which we leave out the boundary conditions for notational reasons:
The same holds for the homotopies.

Proof of (v) and (vi) of Lemma 3.1.

If we apply Lemma 6.1 to the square (3) we get a map defined via $Q \subset N = (\overline{a}_M)_0 \cup \partial^0 M(a_1) \times [1, 2]$. The cobordism $Q' = Q \cup (\overline{a}_L)_1 \subset N' = (\overline{a}_M)_1 \cup \partial^0 M \times [1, 2]$ provides a diagonal map $\sim Q'$ for the square (1). Furthermore as the isotopies constructed in the aforementioned Lemma for $Q$ were the identity on the boundary of $N$ we can extend them via the identity to get isotopies for $Q'$. All in all this gives:

**Corollary 6.6.** The composition of the diagonal map of (3) with $b_{g-1,b}(a_1)$ is the same as the composition of $b_{g,b-1}(\overline{a}_1)$ with the diagonal map of (1). The same holds for the homotopies.

In a sense this means that the above diagram is "cylindrical" if we identify the first and the last line. All these considerations make it fairly easy to finish the proof of Lemma 3.1.

**Proof of (v) and (vi) of Lemma 3.1.** Using the commutativity in the above diagram and the previous corollary we get homotopies

$$h \circ \Sigma a_{g-1,b}(a_1) \simeq (a_{g,b}(a_1), b_{g,b-1}(a_1)) \circ h'' \simeq h' \circ p' \circ h'' \simeq *$$

Here $(a_{g,b}(a_1), b_{g,b-1}(a_1))$ denotes the induced map from $C(\beta_{g,b-1}(M(a_1)))$ to $C(\alpha_{g,b}(M))$ in the aforementioned diagram. Now note that $\Sigma a_{g-1,b}$ is equivalent to a map of type $\alpha_{g-1,b}$ in the following sense: There is a diagram of the following form, where the vertical maps are homeomorphisms given by identifying $M(a)$ with $M$ just like it was discussed in Section 5.
Using these considerations we conclude that $\Sigma_{a_{g-1},b}$ induces an epimorphism in degrees $\leq \frac{1}{3}(2(g - 1) + 1) + 1$. This implies that $h$ is the zero morphism in these degrees. Therefore we conclude that $h \circ p$, which is homotopic to $C(\beta_{g,b-1}(M(a_0))) \to C(\alpha_{g,b}(M))$, induces the zero morphism in these degrees as well.

The second part of this lemma is proven in exactly the same way but replacing $\alpha_{g,b}$ in the third line of the diagram above with $\beta_{g,b}$ and then replacing the resolutions and maps correspondingly.

7 Homological Stability for Capping off Boundary Components

All that is left to do to finish the proof of Theorem 2 is to prove the homological stability statement for maps of type $\gamma$ and prove that $\beta_{g,b}$ induces a monomorphism in integral homology provided that one of the newly created boundary components is contractible. To do this we will first observe that homological stability for maps of type $\gamma_{g,b}$ where $b > 1$ is an easy corollary of homological stability of maps of type $\beta_{g,b}$ and then we will establish everything needed to use Lemma 4.3 again to relate the homological stability for maps of type $\gamma_{g,1}$ to maps of type $\gamma_{g,b}$ for $b > 1$.

So consider a subsurface $P \subset \partial^0 M \times [0,1]$, which defines a map of type $\beta_{g,b}$. Suppose that one of the boundary components of the outgoing boundary of the pair of pants component of $P$ is contractible in $\partial M \times [0,1]$. Fix a contraction and, using the main result of [Hac62], realize it as an embedding $\Phi: (D^2, \partial D^2) \to (\partial M \times [0,1], \partial M \times \{1\})$.

Then $-\cup (\text{refl}(\Phi(D^2)) + 1)$ together with a tangential structure extending the one on the boundary given by $P$ is a map of type $\gamma_{g,b+1}$ and by construction $\gamma_{g,b+1} \circ \beta_{g,b}$ is a union with a cylinder $Q$. Let $\text{refl}(Q)$ denote the reflected cylinder, equipped with some tangential structure extending the one on the outgoing boundary component of $Q$. Then $-\cup Q \cup \text{refl}(Q)$ is homotopic to the identity as was shown in the proof of Lemma 6.1. Therefore $\gamma_{g,b+1} \circ \beta_{g,b}$ is a homotopy equivalence and induces an isomorphism in homology. This implies furthermore that $\beta_{g,b}$ induces a monomorphism in integral homology in all degrees.

Similarly given a disk defining a map of type $\gamma_{g,b}$ such that there exists another component of $\delta$ in the same connected component of $\partial M$, we can find a pair of pants defining a map $\beta_{g,b-1}$ such that $\gamma_{g,b} \circ \beta_{g,b-1}$ is a homotopy equivalence. But note that we have already shown that $\beta_{g,b-1}$ induces an isomorphism in homology in degrees $\leq \frac{2}{3}g - 1$ and an epimorphism in the next degree. Moreover in this case $\beta_{g,b-1}$ induces a monomorphism in all degrees from which we conclude that it actually induces an isomorphism in all degrees $\leq \frac{2}{3}g$. Together with the fact that $\gamma_{g,b}$ is a left inverse for $\beta_{g,b-1}$ this implies that $\gamma_{g,b}$ induces an isomorphism in these degrees as well. All in all this gives:
Lemma 7.1. If one of the new boundary components of the bordism that defines a map of type $\beta_{g,b}$ is contractible in $\partial^0 M$, then $\beta_{g,b}$ induces a monomorphism in homology in all degrees.

Similarly if there exists another boundary component of $\delta$ in the same connected component of $\partial M$, where the disk component of the bordism defining $\gamma_{g,b}$ lies, then $\gamma_{g,b}$ induces an epimorphism in homology in all degrees and furthermore an isomorphism in degrees $\leq 2g$.

Therefore we only have to concern ourselves with the case where there is no more connected component of $\delta$ in the same connected component as the disk component of the bordism defining $\gamma_{g,b}$. We will tackle this case again using a certain resolution, which lets us relate $C(\gamma_{g,b})$ to $C(\gamma_{g,b+i})$. All in all this section is very similar to Section 5 albeit it is much easier and does not require any inductive arguments.

Let $\ell$ denote a ball that is in the same connected component of $\partial M$ as the disk component of the bordism defining some fixed map of type $\gamma_{g,b}$ and that is disjoint from $\delta$.

Definition 7.2. Fix an embedded subsurface $W \in \mathcal{E}^T_{g,b}(M;\delta_T)$. We call an embedding $e: (I, \frac{1}{2}) \to (M, W)$ such that $e(0) \in \ell$ and $e(1) \in M \setminus \partial M$ a boundary path of $W$. If we add a tubular neighborhood of $e$ in the pair $(M, W)$ denoted by $e = (e_M, e_W)$ such that $e \cap \partial M \subset \ell$, then we will call the pair $(e, e)$ a thickened boundary path of $W$.

If we only consider embeddings as above $e: (I, 0) \to (M, \partial M)$ without a particular subsurface, we will drop the $W$ from the notation i.e. we will call these just boundary paths. Similarly for thickened boundary paths.

With this notation at hand we can proceed to define another resolution of $\mathcal{E}^T_{g,b}(M;\delta_T)$.

Definition 7.3. Let $Q^T_{g,b}(M;\delta, \ell)_*$ denote the semi-simplicial space, whose $i$-simplices are tuples $(W, (q^0, q^0), \ldots, (q^i, q^i))$, such that:

(i) $W \in \mathcal{E}^T_{g,b}(M;\delta_T)$

(ii) $(q^k, q^k)$ is a thickened boundary path of $W$.

(iii) the neighborhoods $q^0, \ldots, q^i$ are pairwise disjoint.

The $j$-th face map forgets the $j$-th boundary path and we topologize this as a subspace of $\mathcal{E}^T_{g,b}(M;\delta) \times \text{TEmb}(I \times [i], M)$. There is an augmentation map $\epsilon_*$ to $\mathcal{E}^T_{g,b}(M;\delta)$, which forgets the boundary paths. If the notation is not misleading we will write $Q^T_{g,b}(M;\delta)$.

These spaces will play the role of $X_*$ and $Y_*$ in the application of Lemma 4.3.
Lemma 7.4. If $M$ is connected and has dimension at least 3, then the semi-simplicial space $Q^T_{g,b}(M;\delta)$ is a resolution of $\mathcal{E}^T_{g,b}(M;\delta)$.

Proof. First we will start by proving that the augmentation maps are Serre fibrations. For this we denote by $Q_{g,b}(M;\delta,\ell)$ a semi-simplicial space augmented over $\mathcal{E}^+_g(M;\delta)$, which is defined just like $Q^T_{g,b}(M;\delta)$ but replacing $\mathcal{E}^T_{g,b}(M;\delta)$ by $\mathcal{E}^+_g(M;\delta)$. Then there is the following commutative square:

$$
\begin{array}{ccc}
Q^T_{g,b}(M;\delta) & \longrightarrow & \mathcal{E}^T_{g,b}(M;\delta) \\
\downarrow & & \downarrow \\
Q_{g,b}(M;\delta) & \longrightarrow & \mathcal{E}^+_g(M;\delta)
\end{array}
$$

By Lemma 1.7, $\mathcal{E}^+_g(M;\delta)$ is $\text{Diff}(M)$-locally retractile furthermore $\text{Diff}(\partial M)$ acts on $Q_{g,b}(M;\delta)$, therefore we conclude that the bottom map is a Hurewicz-fibration. Furthermore it is obvious that the square is a pullback square, from which we conclude that the top map is also a Hurewicz-fibration.

By Lemma 4.2 this yields that $\text{Fib}(\epsilon_{\bullet})$ is homotopy equivalent to $|\epsilon_{\bullet}|$. The space of $i$-simplices of the aforementioned simplicial space is given by $\text{TEmb}((I \times [i], \{\frac{1}{2}\} \times [i]), (M, W); q)$, where $q$ encapsulates the condition that $f(0) \in \ell$ and $f(1) \in M \setminus \partial M$.

It is easy to see that $\text{Fib}(\epsilon_{\bullet})$ is a topological flag complex. We will consider $\text{Fib}(\epsilon_{\bullet})$ as a semi-simplicial space augmented over a point and use Lemma 4.1 to show that it is contractible. All that is left to show to apply the lemma is that for $\{x_1, \ldots x_n\} \subset \text{Fib}(\epsilon_{\bullet})$, there exists an $x_{\infty}$ such that $(x_i, x_{\infty})$ is a 1-simplex for every $i$. Since $M$ is at least 3-dimensional we conclude that $M \setminus (x_0' \cup \ldots \cup x_n')$ is still path-connected. Therefore we can find an embedded path in this complement starting at $\ell$ and ending at some point in the inner meeting $W$ at $\frac{1}{2}$. This embedded path together with some sufficiently small neighborhood serves as our $x_{\infty}$.

Definition 7.5. Let $(e, e)$ denote a thickened boundary path. If we fix a two plane $A$ in $N_{Me}(I)|_{\frac{1}{2}}$, we will denote the restriction of $e_M$ to $A$ by $e_A$. We will call a tuple $(e, e_M, e_A)$ a thickened boundary path with a disk. If we furthermore add a tangential structure for the image of $e_A$ we will call this a thickened boundary path with a disk with tangential structure.

The following spaces will play the role of the $B_i$ in the application of Lemma 4.3.

Definition 7.6. Let $R^T_i(M; \ell)$ denote the set of tuples $((q^0, q^0), \ldots, (q^i, q^i))$, where $(q^k, q^{k'})$ are thickened boundary paths with a disk with tangential structure such that all $q^i$ are disjoint. We topologize this as a subspace of.
\(\text{TEmb}(I \times [i]) \times \text{Emb}^T(D^2 \times [i]),\) where the \((q^0, q_M^1)\) correspond to the first factor and \((q_M^1)\) and its tangential structure to the second factor.

There is a map from \(Q_T^{g,b}(M; \delta, \ell)\) to \(R^T_i(M; \ell)\), which forgets the surface, but remembers the tubular neighborhood in the surface and the corresponding tangential structure on it.

**Lemma 7.7.** For a point \(q = ((q^0, q^0), \ldots, (q^i, q^i)) \in R^T_i(M; \ell)\) there is a homotopy fiber sequence:

\[
E_{g,b+i+1}(M(q); \delta^T(q)) \longrightarrow Q_{g,b}(M; \delta^T, \ell) \longrightarrow R_i(M; \ell)
\]

where \(M(q)\) denotes \(M \setminus (\bigcup_j q^j_M)\) and \(\delta^T(q)\) denotes \(\delta \cup (\bigcup_j \partial q^j_L)\).

The issues here that arise from the occurring manifolds with corners are dealt with in the exact same fashion as in the discussion after Lemma 5.6.

**Proof.** Similarly as before we define \(R_i(M; \ell)\) as \(R^T_i(M; \ell)\) without the tangential structure and we have an augmentation map from \(Q_{g,b}(M; \delta)\) to it. Since \(R_i(M; \ell)\) is \(\text{Diff}_\partial(M)\)-locally retractile and the augmentation map is equivariant we conclude that the augmentation map \(Q_{g,b}(M; \delta) \rightarrow R_i(M; \ell)\) is a locally trivial fibration.

Consider a map \(f : D^n \times I \rightarrow R^T_i(M, \ell)\) that we want to lift along \(f_0 : D^n \times \{0\} \rightarrow Q_{g,b}^T(M, \delta)\). We have the following diagram:

\[
\begin{array}{ccc}
D^n \times \{0\} & \xrightarrow{f_0} & Q_{g,b}^T(M, \delta) \\
\downarrow & & \downarrow \\
D^n \times I & \xrightarrow{T} & R^T_i(M, \ell) \rightleftharpoons_{\text{fibr}} R_i(M, \ell)
\end{array}
\]

The dashed arrow was constructed using the fact that the right map is a fibration. Note that a map \(X \rightarrow E_{g,b}^T(M, \delta)\) can be thought of as a submanifold of \(X \times M\) isomorphic to \(X \times \Sigma_g\) and similarly for \(Q_{g,b}^T(M, \delta)\).

Using the fact that \(\text{Emb}(\Sigma_{g,b}, M; \delta) \rightarrow E_{g,b}(M; \delta)\) is a fibration and that \(D^n \times \{0\}\) is contractible we can choose a parametrization of the submanifold in \(D^n \times \{0\} \times M\) obtained via \(f_0\). Furthermore note that by taking preimages of \(q_L^j\) corresponding to \(f_0(x, 0)\) we obtain a map from \(D^n \times \{0\}\) to \(\text{TEmb}([i], \Sigma_g)\). \(\text{Diff}_{\partial}(\Sigma_{g,b})\) acts on \(\text{TEmb}([i], \Sigma_{g,b})\) and it is \(\text{Diff}(M)\)-locally retractile by Lemma 1.5. Therefore the map from \(\text{Diff}_{\partial}(\Sigma_{g,b}) \rightarrow \text{TEmb}([i], \Sigma_{g,b})\) given by taking a fixed system of points and then acting on it via the action of \(\text{Diff}_{\partial}(\Sigma_{g,b})\) is equivariant with respect to this action and the action via left multiplication on the diffeomorphism group. From this we conclude that this map is actually a locally trivial fibration. Using the contractibility of \(D^n \times \{0\}\) we can construct a lift of the map from \(D^n \times \{0\} \rightarrow \text{TEmb}([i], \Sigma_{g,b})\) to the diffeomorphism group.
Using this we can reparametrize the embedding
\[ D^n \times \{0\} \times \Sigma_{g,b} \to D^n \times \{0\} \times M \]
such that, if \( \sigma \) denotes the image of some element in \( \text{TEmb}([i], \Sigma_{g,b}) \), then 
\( D^n \times \{0\} \times \sigma \) gets mapped to the \( q_j \) corresponding to \( f(x,0) \).

Using the dashed arrow in the diagram above we can extend this embedding to an embedding of 
\( D^n \times I \times \Sigma_{g,b} \) and with the same argument we can arrange this embedding to send 
\( D^n \times I \times \sigma \) to the \( q_j \) corresponding to \( f(x,t) \). All that is left to do is to equip 
\( D^n \times I \times \Sigma_{g,b} \) with a tangential structure compatible with the one given at 
\( D^n \times \sigma \cup D^n \times \{0\} \times \Sigma_{g,b} \) via \( f \) and \( f_0 \). Let us denote the pullback of \( B_2(M) \) via the Grassmannian differential in \( \Sigma_{g,b} \) direction 
\( (x,t,p) \mapsto T_p f(x,t) \) by \( B_2(D^n \times I \times \Sigma_{g,b}) \). Note that by Lemma \[5.7\] this is a pullback of 
\( B_2(D^n \times I \times \sigma \cup D^n \times \{0\} \times \Sigma_{g,b}) \), the restriction of the aforementioned \( B_2(D^n \times I \times \Sigma_{g,b}) \) to the space in brackets, which enables us to extend the given tangential structure to a tangential structure on the whole submanifold.

We still have to identify the fiber. The fiber over a point \( q \in R_i(M; \ell) \) is given by all the surfaces \( W \) such that 
\( W \cap q_j M = q_j L \) and the tangential structure at \( q_j \) agrees with the one specified by \( q \). This space can easily be identified with the aforementioned fiber.

Similar to our previous considerations we want to understand how this resolution behaves under stabilization maps or in other words construct \( f_\bullet \) in Lemma \[4.3\]. Let us take a look at the following diagram, in which we want to constructed the dashed arrow:

\[
\begin{align*}
Q^T_{g,b}(M; \delta, \ell) & \quad \text{----} \quad Q^T_{g,b-1}(M; \bar{\delta}, \bar{\ell}) \text{.} \\
E^T_{g,b}(M; \delta) & \quad \gamma_{g,b}(M; \delta, \bar{\delta}) \quad E^T_{g,b}(M; \bar{\delta})
\end{align*}
\]

Let us denote the surface defining \( \gamma_{g,b}(M; \delta, \bar{\delta}) \) by \( P \). Replace \( P \) by a different but isotopic bordism if necessary to ensure that \( (\ell \times I) \cap P = \emptyset \). We will write \( \bar{\ell} \) for \( \ell \times I \). Similarly to our previous notations for a \( q \in R_i^T(M; \ell) \) we denote by \( \bar{q}_j = q_j(0) \times I \) and by \( \bar{q}_j = q_j \cup \bar{q}_j \). Now the dashed arrow is given by sending \( W \) to \( W \cup P \) and \( q_j \) to \( \bar{q}_j \). These maps commute with the face and augmentation maps and so they define a map of semi-simplicial spaces, which we denote by \( \gamma_{g,b}(M; \delta, \bar{\delta}) \).

Furthermore there is a map from \( R_i^T(M; \ell) \) to \( R_i^T(M_1, \bar{\ell}) \) that sends \( q \mapsto \bar{q} \). This map is obviously a homotopy equivalence. We have the following
diagram:

\[ \begin{array}{ccc}
\mathcal{E}^T_{g,b+i+1}(M(q); \delta(q)) & \longrightarrow & (\mathcal{Q}^T_{g,b}(M; \delta, \ell))_i \\
\downarrow \gamma_{g,b+i+1} & & \downarrow \gamma_{g,b}(M; \delta, \bar{\delta})_i \\
\mathcal{E}^T_{g,b+i}(M_1(\bar{q}); \delta(\bar{q})) & \longrightarrow & (\mathcal{Q}^T_{g,b-1}(M_1; \bar{\delta}, \bar{\ell}))_i \\
\end{array} \]

The commutativity of the right square gives us a map between the fibers and it is easy to see that the left vertical arrow is given by \( \gamma_{g,b+i+1} \). Putting this together yields:

**Corollary 7.8.** The induced map between the homotopy fibers of

\( (\gamma_{g,b}(M; \delta^T))_i \rightarrow R^T(M_1, \bar{\ell})_i \)

is given by \( \gamma_{g,b+i+1}(M(q); \delta^T(q), \delta^{T}(\bar{q})) \).

Finally with all these tools at hand we are able to conclude:

**Proposition 7.9.** Let \( M \) be a simply-connected manifold of dimension at least 5 with non-empty boundary equipped with a space of tangential structures of subplanes of its tangent bundle \( T: B_2(M) \rightarrow Gr_2(TM) \) such that the fiber of \( T \) is at least 2-connected. Let \( \delta \) denote a boundary condition. Then

\[ H_k(C(\gamma_{g,b}(M; \delta, \bar{\delta}))) = 0 \]

for \( k \leq \frac{2}{3}g + 1 \).

Of course this proposition finishes the proof of Theorem 2.

**Proof.** We want to apply Lemma 4.3. \( R^T(M; \ell) \) will be our \( B_i \) and the resolution \( \mathcal{Q}^T_{g,b}(M; \delta, \ell) \) will be \( X_\bullet \), while \( \mathcal{Q}^T_{g,b-1}(M_1; \delta, \bar{\ell}) \) is \( Y_\bullet \). \( f_\bullet \) is given by \( \gamma_{g,b}(M; \delta, \bar{\delta})_\bullet \). Corollary 7.7 specified the occurring homotopy fibers.

We chose \( \ell = \infty \) and \( k = \frac{4}{3}g + 1 \). By Corollary 7.8 together with Lemma 7.1 we conclude that the homology of the mapping cone of the map between the fibers equals zero in the desired range i.e.

\[ H_k(C(\gamma_{g,b+i+1}(M(p); \delta(p), \bar{\delta}(\bar{p})))) = 0 \]

for \( k \leq \frac{2}{3}g + 1 \), which finishes the proof of the proposition.

8 Homological Stability for Symplectic Subsurfaces

In this last section we will explain how to use Theorem 1 to prove homological stability for symplectic subsurfaces of a given closed simply-connected symplectic manifold \((M, \omega)\) of dimension \(2n \geq 6\). The proof will be based on the h-principle.
Let $\pi \colon \text{Gr}_2(TM) \to M$ denote the Grassmannian of oriented two-planes in $TM$ and $\Sigma_g$ an oriented, connected, closed surface of genus $g$. Fix $p \in \Sigma_g$, $q \in M$ and $A \in \text{Gr}_2(T_q M)$ such that $\omega|_A$ is non-degenerate and the orientation of $A$ agrees with the orientation induced by $\omega|_A$.

**Definition 8.1.** We say an embedding $f \colon \Sigma_g \to M$ is an oriented symplectic embedding if

(i) $f(p) = q$ and $Df(T_p \Sigma_g) = A$

(ii) $\omega|_{Df(T_x \Sigma_g)}$ is non-degenerate for every point $x \in \Sigma_g$

(iii) $\int_{\Sigma_g} f^* \omega > 0$

We equip the set of symplectic embeddings with the $C^\infty$-topology and we denote this space by $\text{SEmb}_A(\Sigma_g, M)$.

Similarly we say that a smooth map $F \colon \Sigma_g \to \text{Gr}_2(TM)$ is a formal solution to the oriented symplectic embedding problem if

(i) $\pi \circ F$ is an embedding such that $(\pi \circ F)(p) = q$ and $D(\pi \circ F)(T_p \Sigma_g) = A$

(ii) $\omega|_{F(x)}$ is non-degenerate for every point $x \in \Sigma_g$

(iii) $\int_{\Sigma_g} (\pi \circ F)^* \omega > 0$

Note that the last condition for an oriented symplectic embedding ensures that the orientations of $\Sigma_g$ induced by $\omega$ and by the orientation of $\Sigma_g$ agree.

**Remark.** For an embedding $f : \Sigma_g \to M$ we call the map

$$x \mapsto Df(T_x \Sigma_g)$$

the Grassmannian differential and denote it by $\text{Gr}(Df)$. Then it is obvious, that for an oriented symplectic embedding $f : \Sigma_g \to M$ the Grassmannian differential $\text{Gr}(Df) : \Sigma_g \to \text{Gr}_2(TM)$ is a formal solution to the oriented symplectic embedding problem.

**Definition 8.2.** We call a continuous map $H : \Sigma_g \times I \to \text{Gr}_2(TM)$ a solution homotopy if

(i) There exists an embedding $f : \Sigma_g \to M$ such that $H(-, 0)$ agrees with $\text{Gr}(Df)$, $f(p) = q$ and $Df(T_p \Sigma_g) = A$.

(ii) The following diagram commutes:

$$\begin{array}{ccc}
\Sigma_g \times I & \xrightarrow{H} & \text{Gr}_2(TM) \\
\downarrow^{pr} & & \downarrow^{\pi} \\
\Sigma_g & \xrightarrow{f} & M
\end{array}$$
(iii) $H(-, 1)$ is a formal solution to the oriented symplectic embedding problem

We topologize the set of solution homotopies as a subspace of $\text{Emb}_A(\Sigma_g, M) \times \text{Map}(\Sigma_g, \text{Map}(I, \text{Gr}_2(TM)))$, where the first factor is equipped with the $C^\infty$ topology and the second one with the compact-open topology.

Note that there is an inclusion from the space of symplectic embeddings into the space of solutions homotopies by sending an embedding to the constant solution homotopy over that embedding. The proof of the following theorem can be found in Section 12 of [EM02].

**Theorem 8.1.** If $(M, \omega)$ is an at least six-dimensional symplectic manifold, then the inclusion of the symplectic embeddings into the space of solution homotopies is a weak equivalence.

**Understanding Solution Homotopies as Tangential Structures:** Inspired by Theorem 8.1 we want to construct a space of tangential structures of subplanes of $TM$ that reformulates the space of solution homotopies in terms of embeddings with this tangential structure.

We call an element $W \in \text{Gr}_2(T_p M)$ a symplectic two-plane if $\omega|_W$ is non-degenerate and we denote by $(A_2(M))_p$ the open subspace of symplectic two-planes of $T_p M$ and by $A_2(TM)$ the space of all symplectic two-planes of $TM$. For a $V \in \text{Gr}_2(T_p M)$ we will denote the homotopy fiber of the inclusion $(A_2(M))_p \to \text{Gr}_2(T_p M)$ by $\mathcal{PA}_2(M)_V$. More explicitly this is the space of paths in $\text{Gr}_2(T_p M)$ starting at $V$ and ending in $(A_2(M))_p$. If we allow $V$ to vary, we get the mapping path space $\mathcal{PA}_2(M)_p$ of the inclusion $(A_2(M))_p \to \text{Gr}_2(T_p M)$, i.e.

$$\mathcal{PA}_2(M)_p := \{ \gamma : I \to \text{Gr}_2(T_p M) \mid \gamma(1) \in (A_2(M))_p \}$$

This space sits inside a bigger space, which is equipped with the compact open topology

$$\mathcal{PA}_2(M) := \{ \gamma : I \to \text{Gr}_2(TM) \mid \gamma(1) \in A_2(M) \text{ and } \pi \circ \gamma = \text{const} \}$$

If we trivialize $\text{Gr}_2(TM)$ via a Darboux chart defined on $U \subset M$, we see that the inclusion of $A_2(M) \to \text{Gr}_2(TM)$ is locally equivalent to $(A_2(M))_p \times U \to (\text{Gr}_2(TM))_p \times U$. This implies that

$$\mathcal{PA}_2(M) \to \text{Gr}_2(TM)$$

$$\gamma \mapsto \gamma(0)$$

is locally equivalent to $\mathcal{PA}_2(M)_p \times U \to \text{Gr}_2(T_p M) \times U$, but this map is a Hurewicz-fibration as it is the product of two Hurewicz-fibrations, one being the projection of the mapping path space $\mathcal{PA}_2(M)_p \to \text{Gr}_2(T_p M)$.
and the other one being the identity. Using Theorem 13 in Chapter 2.7 of [Spa66] we conclude that it is indeed a Hurewicz-fibration as it is locally a Hurewicz-fibration and the basespace $\text{Gr}_2(TM)$ is paracompact. All in all we conclude:

**Lemma 8.3.**

$$ T: \mathcal{P}A_2(M) \rightarrow \text{Gr}_2(TM) $$

$$ \gamma \mapsto \gamma(0) $$

is a space of tangential structures of subplanes of $TM$, such that the fiber is 2-connected.

**Proof.** The claim that the $T$ is indeed a fibration was shown before. So the only thing left is the computation of the connectivity of the fiber:

The fiber over an oriented 2-plane $V$ in $T_pM$ is given by the set of paths in $\text{Gr}_2(T_pM)$ from $V$ to $(A_2(M))_p$. As we have mentioned before this is the homotopy fiber of the inclusion of $(A_2(M))_p$ into $\text{Gr}_2(T_pM)$. Let us abbreviate these spaces by $A$ and $G$ respectively. Choose some almost complex structure $J$ on $TM$ compatible with the symplectic form $\omega$. Let $V_2^R(T_pM)$ denote the compact 2-Stiefel manifold i.e. the manifold of real 2-frames of $T_pM$ and let $V_2^{sym}(T_pM)$ denote the space of 2-frames $(v_1, v_2)$ such that $\omega(v_1, v_2) = 1$. Furthermore define $V_1^C(T_pM)$ as the compact Stiefel manifold of complex 1-frames. Then we have an inclusion

$$ V_1^C(T_pM) \rightarrow V_2^{sym}(T_pM) $$

$$ v \mapsto (v, J(v)) $$

and we claim that this is a homotopy equivalence. Indeed the homotopy is given by

$$ ((v_1, v_2), t) \mapsto (v_1, (1 - t)v_2 + tJ(v_1)) $$

which is easily seen to be well-defined. Therefore we get the following maps between fiber sequences:

$$ S^1 \simeq Sp(2, \mathbb{R}) \xrightarrow{\pi} SO(2) \cong S^1 $$

$$ V_1^C(T_pM) \simeq V_2^{sym}(T_pM) \xrightarrow{\pi} V_2^R(T_pM) $$

$$ A \xrightarrow{\pi} G $$

Note that $V_1^C(T_pM) \cong S^{2n-1}$ and $V_2^R(T_pM) \cong STS^{2n-1}$ (the unit sphere bundle of the tangent bundle of $S^{2n-1}$). Since both of these spaces are $2n - 3$-connected and the top horizontal map is a homotopy equivalence, we conclude that the map from $A \rightarrow G$ is an isomorphism on $\pi_i$ for $i \leq 3$ or in other words $\pi_i(G, A) = 0$ for $i \leq 3$, which implies that the homotopy fiber is two-connected, since $\pi_{i+1}(X, B) \cong \pi_i(\text{HoFib}(B \rightarrow X))$. □
Spaces of Symplectic Subsurfaces and Homological Stability: The group $\text{Diff}_{+}^+(\Sigma)$ acts freely on $\text{SEmb}_A(\Sigma, M)$ via precomposition as being a symplectic embedding is independent of the parametrization. We denote the quotient by this group action by $\mathcal{SE}_V(\Sigma, M)$. Elements of this space are referred to as symplectic subsurfaces. By a slight abuse of notation let $A$ also denote the path in $\mathcal{P}_A(M)$ that is constantly $A$. The inclusion of the space of symplectic embeddings into the space of solution homotopies descends to an inclusion $\mathcal{SE}_A(\Sigma, M) \to \mathcal{ET}_A(\Sigma, M)$.

This inclusion is evidently not $\pi_0$-surjective and in order to use this inclusion later on we have to alter its image. We define:

$$\text{Emb}_{T}^+(\Sigma, M)^\omega := \{ f \in \text{Emb}_{T}^+(\Sigma, M) | \int_{\Sigma} f^* \omega > 0 \}$$

$$\mathcal{E}_A^i(\Sigma, M)^\omega := \{ W \in \mathcal{E}_A^i(\Sigma, M) | \int_{W} \omega > 0 \}$$

Then the following corollary is obvious:

**Corollary 8.4.** $\text{Emb}_{T}^+(\Sigma, M)^\omega$ is exactly the space of solution homotopies.

Observe that the stabilization map of $\mathcal{E}_A^i(\Sigma, M)$ defined in Section 2 restricts to $\mathcal{E}_A^i(\Sigma, M)^\omega$ namely

$$\sigma_g : \mathcal{E}_A^i(\Sigma, M)^\omega \to \mathcal{E}_A^i(\Sigma+1, M)^\omega$$

since for every $W \in \mathcal{E}_A^i(\Sigma, M)$ we have $[W] = [\sigma_g(W)] \in H_2(M)$ and if $[\omega]$ denotes the real cohomology class corresponding to $\omega$, then $\int_{W} \omega > 0$ is equivalent to $[\omega]([W]) > 0$. Furthermore the proof of Lemma 3.2 and Lemma 3.5 showed that $\mathcal{E}_A^i(\Sigma, M)^\omega$ is a union of connected components of $\mathcal{E}_A^i(\Sigma, M)$ and that $(\sigma_g)_* : \pi_0(\mathcal{E}_A^i(\Sigma, M)^\omega) \to \pi_0(\mathcal{E}_A^i(\Sigma+1, M)^\omega)$ is an isomorphism. Therefore Theorem 1 implies that homological stability holds for $\mathcal{E}_A^i(\Sigma, M)^\omega$ as well i.e. eventually the homology of $\mathcal{E}_A^i(\Sigma, M)^\omega$ becomes independent of the genus $g$.

**Remark.** The same arguments show that even in the general case, homological stability still holds for all kinds of homological constraints we could put on $\mathcal{E}_V^i(\Sigma, M)$ as long as these constraints depend only on $H_2(M)$.

The following proposition allows us to relate this homological stability to $\mathcal{SE}_A(\Sigma, M)$.

**Proposition 8.5.** $\mathcal{SE}_A(\Sigma, M) \to \mathcal{ET}_A(\Sigma, M)$ is a weak equivalence.
Proof. We have the following map between fibrations:

\[
\begin{array}{cccc}
\text{Diff}_p^+(\Sigma) & \xrightarrow{id} & \text{Diff}_p^+(\Sigma) \\
\downarrow & & \downarrow \\
\text{SEmb}_A(\Sigma, M) & \rightarrow & \text{Emb}_A^T(\Sigma, M)^\omega \\
\downarrow & & \downarrow \\
\mathcal{S}\mathcal{E}_A(\Sigma, M) & \rightarrow & \mathcal{E}_A^T(\Sigma, M)^\omega
\end{array}
\]

Note that the lower map on the left side is a fibration as a restriction of a fibration and that the horizontal arrow in the middle is a weak equivalence by Theorem 8.1. Therefore we conclude from the 5-Lemma and the long exact sequence of homotopy groups for a fibration that the inclusion is in fact a weak equivalence.

This proposition together with Theorem 1 implies Corollary 1 immediately:

**Corollary 1.** Let \((M, \omega)\) denote a simply-connected symplectic manifold of dimension at least 6. For \(q \in M\) and a symplectic 2-plane in \(T_q M\). There is a homomorphism of integral homology:

\[
f_g : H_*(\mathcal{S}\mathcal{E}_A(\Sigma_g, M); \mathbb{Z}) \rightarrow H_*(\mathcal{S}\mathcal{E}_A(\Sigma_{g+1}, M); \mathbb{Z})
\]

And this homomorphism induces an isomorphism for \(* \leq \frac{2}{3}g\).

The main problem here is that the present text does not provide an actual stabilization map for symplectic subsurfaces in the spirit of the pointed stabilization maps \(\sigma_g\), but only an abstract one that uses the inverses of the aforementioned weak equivalence and therefore only exists on the level of homology.

Nevertheless the author is fairly confident that the proof of Proposition 2.2 and its corollaries can easily be adapted to work in the symplectic setting. One only has to be a little bit more careful in choosing the deformations so that the embedded subsurfaces stay symplectic. All in all this should be doable if one alters the definition of \(\mathcal{E}_A^T(M)\) a little bit to include something like the angle between the tangent planes and \(A\).

Furthermore there are nice symplectic embeddings of tori with a hole into \(\mathbb{R}^{2n}\), which should provide the necessary tori in \(B_R(0)\) such that it should be possible to produce an actual stabilization map in the symplectic case, which is defined like \(\sigma_g\) i.e. it is defined by taking the connected sum with one of these symplectic tori and therefore maps symplectic subsurfaces to symplectic subsurfaces.
References

[BF81] E. Binz and H. R. Fischer, *The manifold of embeddings of a closed manifold*, Differential Geometric Methods in Mathematical Physics, Lecture Notes in Physics, Berlin Springer Verlag, vol. 139, 1981, pp. 310–325.

[Cer61] J. Cerf, *Topologie de certains espaces de plongements*, Thèses présentées à la Faculté des Sciences de l’Université de Paris. Série A, Gauthier-Villars, 1961.

[CRW17] Federico Cantero and Oscar Randal-Williams, *Homological stability for spaces of embedded surfaces*, Geom. Topol. 21 (2017), no. 3, 1387–1467.

[EM02] Y. Eliashberg and N.M. Mishachev, *Introduction to the h-principle*, Graduate studies in mathematics, American Mathematical Society, 2002.

[ER17] J. Ebert and O. Randal-Williams, *Semi-simplicial spaces*, ArXiv: 1705.03774 (2017).

[Fra] Martin Frankland, *Math 527 - homotopy theory homotopy pullbacks*, [http://www.home.uni-osnabrueck.de/mfrankland/Math527/Math527_0308_20170508.pdf](http://www.home.uni-osnabrueck.de/mfrankland/Math527/Math527_0308_20170508.pdf), Accessed: 27.5.2018.

[Hae62] André Haefliger, *Plongements différentiables de variétés dans variétés.*, Commentarii mathematici Helvetici, vol. 36, Pages 47-82, 1961/62.

[Har85] John L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Annals of Mathematics 121 (1985), no. 2, 215–249.

[Har90] _____, *Stability of the homology of the moduli spaces of riemann surfaces with spin structure*, Mathematische Annalen 287 (1990), 323–334.

[HW10] Allen Hatcher and Nathalie Wahl, *Stabilization for mapping class groups of 3-manifolds*, Duke Math. J. 155 (2010), no. 2, 205–269.

[KM97] A. Kriegl and P.W. Michor, *The convenient setting of global analysis*, Mathematical Surveys, American Mathematical Society, 1997.

[Kup13] A. Kupers, *Homological stability for unlinked circles in a 3-manifold*, ArXiv: 1310.8580v2 (2013).

[Mic80] P.W. Michor, *Manifolds of differentiable mappings*, Shiva mathematics series, Shiva Pub., 1980.
[MT68] R.E. Mosher and M.C. Tangora, *Cohomology operations and applications in homotopy theory*, Dover Books on Mathematics Series, Dover Publications, 1968.

[Nak61] Minoru Nakaoka, *Homology of the infinite symmetric group*, Annals of Mathematics **73** (1961), no. 2, 229–257.

[Nar15] S. Nariman, *Stable homology of surface diffeomorphism groups made discrete*, ArXiv: 1506.00033v1 (2015).

[Spa66] E.H. Spanier, *Algebraic topology*, Springer New York, 1966.

[Wah08] Nathalie Wahl, *Homological stability for the mapping class groups of non-orientable surfaces*, Inventiones Mathematicae **171** (2008), no. 2, 389–424.

[Whi36] Hassler Whitney, *Differentiable manifolds*, Annals of Mathematics **37** (1936), no. 3, 645–680.