Monochromatic disconnection: Erdős-Gallai-type problems and product graphs

Ping Li, Xueliang Li

1Center for Combinatorics and LPMC
Nankai University, Tianjin 300071, China
qdli_ping@163.com, lxl@nankai.edu.cn

2School of Mathematics and Statistics
Qinghai Normal University, Xining, Qinghai 810008, China

Abstract

For an edge-colored graph \( G \), we call an edge-cut \( M \) of \( G \) monochromatic if the edges of \( M \) are colored with a same color. The graph \( G \) is called monochromatically disconnected if any two distinct vertices of \( G \) are separated by a monochromatic edge-cut. The monochromatic disconnection number, denoted by \( md(G) \), of a connected graph \( G \) is the maximum number of colors that are allowed to make \( G \) monochromatically disconnected. In this paper, we discuss the Erdős-Gallai-type problems for the monochromatic disconnection and the monochromatic disconnection numbers for four graph products, i.e., Cartesian, strong, lexicographic, and tensor products.

Keywords: monochromatic edge-cut, monochromatic disconnection (coloring) number, Erdős-Gallai-type problems, graph products.

AMS subject classification (2010): 05C15, 05C40, 05C35.

1 Introduction

Let \( G \) be a graph and let \( V(G) \), \( E(G) \) denote the vertex set and the edge set of \( G \), respectively. Let \( |G| \) (also \( v(G) \)) denote the number of vertices of \( G \). If there is no confusion, we use \( n \) and \( m \) to denote, respectively, the number of vertices and edges of a graph, throughout this paper. For \( v \in V(G) \), let \( d_G(v) \) denote the degree of \( v \) in \( G \) and let \( N_G(v) \) denote the neighbors of \( v \) in \( G \). We call a vertex \( v \) of \( G \) a \( t \)-degree vertex

---

1Supported by NSFC No.11871034, 11531011 and NSFQH No.2017-ZJ-790.
of $G$ if $d_G(v) = t$. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of $G$, respectively. For all other terminology and notation not defined here we follow Bondy and Murty [1].

For a positive integer $t$, we use $\lbrack t \rbrack$ to denote the set $\{1, 2, \cdots, t\}$ of natural numbers. For a graph $G$, let $\Gamma : E(G) \to \lbrack k \rbrack$ be an edge-coloring of $G$ that allows a same color to be assigned to adjacent edges, and $\Gamma$ is also called a $k$-edge-coloring of $G$ since $k$ colors are used. For an edge $e$ of $G$, we use $\Gamma(e)$ to denote the color of $e$. If $H$ is a subgraph of $G$, we also use $\Gamma(H)$ to denote the set of colors used on all edges of $H$. Let $|\Gamma|$ denote the number of colors in $\Gamma$. An edge-coloring $\Gamma$ of $G$ is trivial if $|\Gamma| = 1$, otherwise, it is non-trivial.

The new concept of monochromatic disconnection of graphs, recently introduced in [10] by us, is actually motivated from the concepts of rainbow disconnection [6] and monochromatic connection [5, 11] of graphs. For an edge-colored graph $G$, we call an edge-cut $M$ a monochromatic edge-cut if the edges of $M$ are colored with a same color. For two vertices $u, v$ of $G$, a monochromatic $uv$-cut is a monochromatic edge-cut that separates $u$ and $v$. An edge-colored graph $G$ is monochromatically disconnected if any two vertices of $G$ has a monochromatic cut separating them. An edge-coloring of $G$ is a monochromatic disconnection coloring (MD-coloring for short) if it makes $G$ monochromatically disconnected. The monochromatic disconnection number, denoted by $md(G)$, of a connected graph $G$ is the maximum number of colors that are allowed to make $G$ monochromatically disconnected. An extremal MD-coloring of $G$ is an MD-coloring that uses $md(G)$ colors. If $H$ is a subgraph of $G$ and $\Gamma$ is an edge-coloring of $G$, we call $\Gamma$ an edge-coloring restricted on $H$.

For a $k$-edge-coloring of $G$ and an integer $j \in \lbrack k \rbrack$, a $j$-induced edge set is the set of edges of $G$ colored with color $j$. We also call a $j$-induced edge set a color-induced edge set. Then an edge-coloring of a graph is an MD-coloring if any two vertices can be separated by a color-induced edge set. We will use this method to verify whether an edge-coloring of a graph is an MD-coloring.

Let $K_n^-$ be a graph obtained from $K_n$ by deleting an arbitrary edge. $K_3$ is also called a triangle. We call a path $P$ a $t$-path if $|E(P)| = t$ and denote it by $P_t$. Analogously, we call a cycle $C$ a $t$-cycle if $|C| = t$ and denote it by $C_t$.

Let $e = uv$ be an edge of $G$. If $d_G(u) = 1$, then we call $u$ a pendent vertex and call $e$ a pendent edge of $G$. A block $B$ of a graph $G$ is trivial if $B = K_2$, otherwise $B$ is non-trivial. The union of two graphs $G$ and $H$ is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

The following results were proved in [10], and they are useful in the sequel.

**Proposition 1.1.** [10] Suppose $G$ is a connected graph that may have parallel edges but
Lemma 1.6. If \( G \) is a connected graph of order \( n \), then \( md(G) = n - 1 \) if and only if \( G \) is a tree.

Proposition 1.7. If \( G \) has \( r \) blocks \( B_1, \ldots, B_r \), then \( md(G) = \sum_{i \in [r]} md(B_i) \).

Furthermore,

1. \( md(G) = n - 1 \) if and only if \( G \) is a tree;
2. if \( G \) is a unique cycle graph, then \( n - 2 \geq md(G) \geq \left\lceil \frac{n}{2} \right\rceil \), with equality when \( G \) is a cycle.

Proposition 1.8. Let \( D \) be a connected subgraph of a graph \( G \). If \( \Gamma \) is an \( MD \)-coloring of \( G \), then \( \Gamma \) is also an \( MD \)-coloring restricted on \( D \).

Lemma 1.9. If \( H \) is a connected spanning subgraph of \( G \), then \( md(H) \geq md(G) \).

From this, one can deduce that \( 1 \leq md(G) \leq n - 1 \) for a connected graph of order \( n \), just by considering a spanning tree of \( G \).

Lemma 1.10. Let \( H \) be the union of some graphs \( H_1, \ldots, H_r \). If \( \bigcap_{i \in [r]} E(H_i) \neq \emptyset \) and \( md(H_i) = 1 \) for each \( i \in [r] \), then \( md(H) = 1 \).

Lemma 1.11. If \( G \) is \( K_n \), \( K_n^{-} \) or \( K_{n,t} \) where \( n \geq 2 \) and \( t \geq 3 \), then \( md(G) = 1 \).

Theorem 1.12. If \( G \) is a 2-connected graph, then \( md(G) \leq \left\lfloor \frac{n}{2} \right\rfloor \).

An edge-cut \( M \) of \( G \) is a matching cut if \( M \) is a matching of \( G \). A graph is called matching immune if it has no matching cut.

Theorem 1.13. If a graph \( G \) is matching immune, then \( e(G) \geq \left\lceil \frac{3}{2}(v(G) - 1) \right\rceil \).

The four main graph products are Cartesian, strong, lexicographic, and tensor products. Let \( G \) and \( H \) be two graphs and \( V(G) \times V(H) = \{(u, v) : u \in V(G) \text{ and } v \in V(H)\} \). The four graph products are defined as follows.

- The Cartesian product of \( G \) and \( H \), written as \( G \Box H \), is the graph with vertex set \( V(G) \times V(H) \), in which two vertices \( (u, v) \) and \( (u', v') \) are adjacent if and only if \( uu' \) is an edge of \( G \) and \( v = v' \), or \( vv' \) is an edge of \( H \) and \( u = u' \).

- The strong product of \( G \) and \( H \), written as \( G \boxtimes H \), is the graph with vertex set \( V(G) \times V(H) \), in which two vertices \( (u, v) \) and \( (u', v') \) are adjacent if and only if \( uu' \) is an edge of \( G \) and \( v = v' \), or \( vv' \) is an edge of \( H \) and \( u = u' \), or \( uu' \) is an edge of \( G \) and \( vv' \) is an edge of \( H \).

- The lexicographic product of \( G \) and \( H \), written as \( G \circ H \), is the graph with vertex set \( V(G) \times V(H) \), in which two vertices \( (u, v) \) and \( (u', v') \) are adjacent if and only if \( uu' \) is an edge of \( G \), or \( u = u' \) and \( vv' \) is an edge of \( H \).
• The tensor product of $G$ and $H$, written as $G \ast H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices $(u, v)$ and $(u', v')$ are adjacent if and only if $uu'$ is an edge of $G$ and $vv'$ is an edge of $H$.

**Proposition 1.9.** For two connected graphs $G$ and $H$, we have

1. $G \boxtimes H$ is a connected spanning subgraph of $G \circ H$.
2. $G \boxtimes H = (G \square H) \cup (G \ast H)$ and $E(G \square H) \cap E(G \ast H) = \emptyset$.

**Proposition 1.10.** [14] If $G$ and $H$ are connected graphs, then $G \ast H$ is connected if and only at least one of $G$ and $H$ is not bipartite.

### 2 Preliminaries

Let $e$ and $e'$ be two edges of a graph $G$. We say that $e$ and $e'$ satisfy the relation $\theta$ if there exists a sequence of subgraphs $G_1, \ldots, G_k$ of $G$ where each $G_i$ is either a triangle or a $K_{2,3}$, such that $e \in E(G_1)$ and $e' \in E(G_k)$ and $E(G_i) \cap E(G_{i+1}) \neq \emptyset$ for $i \in [k-1]$. We denote $e \theta e'$ if $e$ and $e'$ satisfy the relation $\theta$. For a graph $G$, if any two edges $e$ and $e'$ of $G$ satisfy $e \theta e'$, then we call the graph $G$ a closure.

**Lemma 2.1.** If $G$ is a closure, then $md(G) = 1$.

*Proof.* Let $\Gamma$ be an extremal $MD$-coloring of $G$ and $e$ be an edge of $G$. For every edge $f$ of $G$, there is a sequence of subgraphs $G_1, \ldots, G_k$ of $G$ such that $e \in E(G_1)$ and $f \in E(G_k)$, and there is an edge $f_i$ of $G$ such that $f_i \in E(G_i) \cap E(G_{i+1})$ for $i \in [k-1]$. Here each $G_i$ is either a $K_3$ or a $K_{2,3}$. Since $md(K_3) = md(K_{2,3}) = 1$, all edges of $G_i$ are colored with a same color. Then $\Gamma(e) = \Gamma(f_1) = \cdots = \Gamma(f)$. Therefore, each edge of $G$ is colored with color $\Gamma(e)$ under $\Gamma$, and hence $md(G) = 1$.

**Lemma 2.2.** Let $G$ be a connected graph and $v \in V(G)$. If $v$ is neither a pendent vertex nor a cut-vertex of $G$, then $md(G) \leq md(G - v)$.

*Proof.* Let $\Gamma$ be an extremal $MD$-coloring of $G$. Then $\Gamma$ is an $MD$-coloring restricted on $G - v$. If $\Gamma(G) - \Gamma(G - v) = \emptyset$, then $md(G) = |\Gamma| = |\Gamma(G - v)| \leq md(G - v)$. Therefore, it is sufficient to show that $\Gamma(G) - \Gamma(G - v) = \emptyset$. Otherwise let $e = vu$ be an edge of $E(G) - E(G - v)$ and $\Gamma(e) \notin \Gamma(G - v)$. Since $d_G(v) \geq 2$, there is another edge incident with $v$, say $f = vw$. Because $v$ is not a cut-vertex, there is a cycle $C$ of $G$ containing $e$ and $f$. Because $\Gamma$ is an $MD$-coloring restricted on $C$, there are at least two edges in the monochromatic $uv$-cut of $C$ and one of them is $e$. Thus $f$ is in the monochromatic $uv$-cut, i.e., $\Gamma(e) = \Gamma(f)$. Then, there is no monochromatic $uw$-cut in $C$, a contradiction. 


Suppose $G$ is a connected graph and $S = \{v_1, \ldots, v_t\}$ is a set of vertices of $G$. Let $G_0 = G$ and $G_i = G - \{v_1, \ldots, v_i\}$ for $i \in [t]$. We call the vertex sequence $\gamma = (v_1, v_2, \ldots, v_t)$ a soft-layer if $d_{G_{i-1}}(v_i) \geq 2$ and $G_i$ is connected for $i \in [t]$. The following result can be derived from Lemma 2.2 directly.

**Lemma 2.3.** Suppose $G$ is a connected graph and $S = \{v_1, \ldots, v_t\}$ is a set of vertices of $G$. If the vertex sequence $\gamma = (v_1, v_2, \ldots, v_t)$ is a soft-layer, then $md(G) \leq md(G_i)$.

**Lemma 2.4.** If $G$ has a matching cut, then $md(G) \geq 2$.

**Proof.** Let $M$ be a matching cut of $G$. Let $\Gamma$ be an edge-coloring of $G$ obtained by coloring $M$ with color 1 and coloring $E(G) - M$ with color 2. Then for any two vertices $u$ and $v$ of $G$, if $uv$ is not an edge of $G$ or $uv \notin M$, then $u, v$ are in different components of $G - (E(G) - M)$; if $uv \in M$, then $u, v$ are in different components of $G - M$. Therefore, $\Gamma$ is an $MD$-coloring of $G$, and hence $md(G) \geq 2$.

**Lemma 2.5.** For a connected graph $G$ and an integer $r$ with $1 \leq r \leq md(G)$, there is an $MD$-coloring $\Gamma$ of $G$ such that $|\Gamma| = r$.

**Proof.** Suppose $\Gamma'$ is an extremal $MD$-coloring of $G$. Then $|\Gamma'| = md(G)$. Let $E_i$ be the $i$-induced edge set for $i \in [md(G)]$. Let $\Gamma$ be an edge-coloring obtained from $\Gamma'$ by recoloring $E' = \bigcup_{i=r}^{md(G)} E_i$ by $r$. Then $|\Gamma| = r$. We now show that $\Gamma$ is an $MD$-coloring of $G$. For two vertices $u, v$ of $G$, since $\Gamma'$ is an extremal $MD$-coloring of $G$, there is an $E_i$ such that $u, v$ are in different components of $G - E_i$. Let $E'' = E_i$ if $i < r$ and $E'' = E'$ if $i \geq r$. Then $u, v$ are in different components of $G - E''$. This implies $\Gamma$ is an $MD$-coloring of $G$.

**Theorem 2.6.** For a connected graph $G$, $md(G) = 1$ if $\delta(G) \geq \lfloor \frac{n}{2} \rfloor + 1$, and the lower bound is sharp.

**Proof.** To prove $md(G) = 1$, it is sufficient to prove $G$ is a closure.

In fact, any two adjacent edges of $G$ are either in a triangle or in a $K_{2,3}$, because for any two adjacent edges $e_1 = ab$ and $e_2 = ac$, $d_G(b) + d_G(c) \geq 2\lfloor \frac{n}{2} \rfloor + 2 \geq n + 1$, and so either $bc$ is an edge of $G$ or $b$ and $c$ have at least three common vertices. For two edges $e_1$ and $e_2$ of $G$, there is a path $P$ of $G$ with pendant edges $e_1$ and $e_2$. Since any two adjacent edges of $P$ are in a $K_3$ or a $K_{2,3}$, $G$ is a closure. Therefore $md(G) = 1$.

Now we show that the bound is sharp, i.e., we need to construct a graph $H$ with $\delta(H) = \lfloor \frac{n}{2} \rfloor$ and $md(H) \geq 2$. Let $A, B$ be two vertex-disjoint complete graphs with $V(A) = \{v_1, \ldots, v_{\lfloor \frac{n}{2} \rfloor}\}$ and $V(B) = \{u_1, \ldots, u_{\lfloor \frac{n}{2} \rfloor}\}$. Let $H$ be a graph obtained from $A$ and $B$ by adding additional edges $e_i = u_iv_i$ for $i \in [\lfloor \frac{n}{2} \rfloor]$. Then $\delta(G) = \lfloor \frac{n}{2} \rfloor$. Because $M = \{e_1, \ldots, e_{\lfloor \frac{n}{2} \rfloor}\}$ is a matching cut of $G$, by Lemma 2.3 $md(G) \geq 2$. ■
3 Erdős-Gallai-type problems

Since for a connected graph $G$, we have $1 \leq md(G) \leq n - 1$, the Erdős-Gallai-type problems for the monochromatic disconnection number are stated as follows.

**Problem A:** Given two positive integers $n$ and $r$ such that $1 \leq r \leq n - 1$, compute the minimum integer $f(n, r)$ such that for any connected graph $G$ of order $n$, if $e(G) \geq f(n, r)$, then $md(G) \leq r$.

**Problem B:** Given two positive integers $n$ and $r$ such that $1 \leq r \leq n - 1$, compute the maximum integer $g(n, r)$ such that for any connected graph $G$ of order $n$, if $e(G) \leq f(n, r)$, then $md(G) \geq r$.

Next we will consider the two problems separately in subsections.

3.1 Solution for Problem A

In order to solve Problem A, we need the following lemmas.

**Lemma 3.1.** Let $G$ be a connected graph with $n$ vertices and $r$ blocks. Then $e(G) \leq \left(\frac{n-r+1}{2}\right) + r - 1$.

**Proof.** Let $H$ be a connected graph with $n$ vertices and $r$ blocks such that $e(H)$ is maximum. We only need to prove $e(H) = \left(\frac{n-r+1}{2}\right) + r - 1$. It is obvious that each block of $H$ is a complete graph. In fact, the graph $H$ has $r-1$ trivial blocks $K_2$ and one block $K_{n-r+1}$, and then $e(H) = \left(\frac{n-r+1}{2}\right) + r - 1$. Otherwise, suppose $H$ has at least two non-trivial blocks $B_1$ and $B_2$ and $|B_1| \geq |B_2|$. Let $H'$ be a graph obtained from $H$ by replacing $B_1$ by $K_{|B_1|+1}$ and replacing $B_2$ by $K_{|B_2|-1}$. Then $H'$ is a graph with $n$ vertices, $r$ blocks and more edges, which contradicts that $e(H)$ is maximum. □

**Lemma 3.2.** Suppose $G$ is a graph with $n \geq 4$ and $e(G) \geq \left(\frac{n-1}{2}\right) + 2$. Then $md(G) = 1$, and the lower bound for $e(G)$ is sharp.

**Proof.** The proof proceeds by induction on $n$. If $n = 4$, then $G$ is either a $K_4$ or a $K_{4}$, and so $md(G) = 1$. Let $G$ be a graph with $n > 4$. If $G$ is $K_n$, then $md(G) = 1$. Otherwise there exists a vertex $v$ of $V(G)$ such that $d_G(v) \leq n - 2$. Then $G' = G - v$ satisfies

$$e(G') = e(G) - d_G(v) \geq \left(\frac{n-1}{2}\right) + 2 - (n-2) = \left(\frac{n-2}{2}\right) + 2.$$ 

By induction, $md(G') = 1$.

Because $e(G) \geq \left(\frac{n-1}{2}\right) + 2 = e(K_n) - (n-3)$, $d_G(v) \geq 2$, i.e., $v$ is not a pendent vertex. In fact, $v$ is not a cut-vertex, for otherwise $G$ has at least 2 blocks, and then
Theorem 3.3. Given two positive integers $n$ and $r$ with $1 \leq r \leq n - 1$,

\[
f(n, r) = \begin{cases} 
    \binom{n-r+1}{2} - n + 2r + 1 & 1 \leq r \leq n - 2; \\
    n - 1 & r = n - 1.
\end{cases}
\]

Proof. Although the notation $f(n, r)$ has a special meaning in Problem A, for convenience, we just see it as function on the variables $n$ and $r$ in this proof.

If $n \leq 4$, it is easy to verify that the theorem holds. By Proposition 1.2, $f(n, n-1) = n - 1$ is obvious. By Lemma 3.2, the theorem holds when $r = 1$. Therefore, we only need to show that $f(n, r) = \binom{n-r+1}{2} - n + 2r + 1$ when $n \geq 5$ and $2 \leq r \leq n - 2$.

Let $G_1$ be a graph with $r - 1$ trivial blocks and one non-trivial block $B$, where $|B| = n - r + 1$ and $e(B) = \binom{n-r+1}{2} - n + r + 2$. Then $e(B) = \binom{|B|-1}{2} + 2$, and by Lemma 3.2, $md(B) = 1$. Therefore $md(G_1) = r$ by Proposition 1.2. Let $G_2$ be a graph with $r$ trivial blocks and one non-trivial block $K_{n-r}$. Then $md(G_2) = r + 1$. Because $e(G_1) = f(n, r)$ and $e(G_2) = f(n, r) - 1$, we only need to show that $md(G) \leq r$ when $e(G) = f(n, r)$.

In fact, since every graph with more than $f(n, r)$ edges has a spanning subgraph with exactly $f(n, r)$ edges, by Lemma 1.4 we only need to show that $md(G) \leq r$ when $e(G) = f(n, r)$.

Obviously, the result is true for $n \leq 4$. Suppose the result does not hold for all $n$. Let $n$ be the minimum integer such that there is a positive integer $r$ with $2 \leq r \leq n - 2$, the result is false for some connected graphs $G$ with $|G| = n$ and $e(G) = f(n, r)$. We choose such a graph $G$ with $md(G) \geq r + 1$ such that the number of blocks of $G$ is maximum. Suppose $G$ has $t$ blocks $B_1, \ldots, B_t$. By Lemma 3.1, $t \leq r$. Because $md(G) \geq r + 1$, by Proposition 1.2 there is a block, say $B_1$, with $md(B_1) = k \geq 2$. Let $|B_1| = n_1$. We distinguish the following cases.

Case 1. $t \geq 2$.

Because $|B_1| = n_1 < n$, $e(B_1) \leq f(n_1, k - 1) - 1 = \binom{n_1-k+2}{2} - n_1 + 2(k - 1)$. Let $T^k$ be a graph with $k - 1$ trivial blocks and one block $K_{n_1-k+1}$, then $md(T^k) = k$ and $e(T^k) = \binom{n_1-k+1}{2} + k - 1 = f(n_1, k - 1) - 1 \geq e(B_1)$. Let $G'$ be a graph obtained from $G$ by replacing $B_1$ by $T^k$ and let $G''$ be a connected spanning subgraph of $G'$ with $f(n, r)$ edges. Then $G''$ is a graph with $|G''| = n$, $e(G'') = f(n, r)$ and $md(G'') \geq r + 1$. However, the number of blocks of $G''$ is more than $t$, a contradiction.

Case 2. $t = 1$.


Claim 3.4. For a connected graph $G$ with $e$ edges and let $G/e$ be the two edges incident with $v$. The operation of subdividing the edge $e$ consists of deleting the vertex $v$ and its incident edges $e_1$, $e_2$ and then adding a new edge joining $v_1$ and $v_2$.

Proof. Since $G$ has just one block, $G$ is 2-connected. The average degree of $G$ is

$$\frac{2e(G)}{n} = 2\left(\frac{n-r+1}{2} - n + 2r + 1\right) \leq \frac{n^2 - 2nr + r^2 - n + 3r + 2}{n}.$$ 

Since $G$ is 2 connected, $md(G) = r \leq \left\lceil \frac{n}{2} \right\rceil$ by Theorem 1.7. Because $n \geq 5$ and $r \geq 2$, the difference between the average degree of $G$ and $n - r - 1$ is

$$dif = \frac{2e(G)}{n} - (n - r - 1) = \frac{r^2 + 3r + 2}{n} - r.$$ 

Since $2 \leq r \leq \left\lceil \frac{n}{2} \right\rceil$, if $n \geq 8$, then $dif \leq 0$; if $n = 7$, then $dif < 0$; if $n = 6$, then $dif < 1$; if $n = 5$, then $dif < 1$. This implies that $G$ has a vertex $v$ with $d_G(v) \leq n - r - 1$. Let $G' = G - v$. Then $G'$ is connected and $e(G') \geq e(G) - (n - r - 1) = f(n - 1, r)$. Since $G$ is a minimum counterexample of the theorem and $|G'| = |G| - 1$, $md(G') \leq r$. By Lemma 2.2, $md(G) \leq md(G') \leq r$, which contradicts that $md(G) \geq r + 1$. According to above two cases, such a graph $G$ does not exist, and therefore the theorem holds.

3.2 Results for Problem B

To contract an edge $e$ of a graph $G$ is to delete the edge and then identify its ends, and to contract an edge set $X$ of a graph $G$ is to contract the edges of $X$ one by one. The result graphs are denoted by $G/e$ and $G/X$, respectively. To subdivide an edge of a graph is to insert a new vertex into the edge. Let $v$ be a 2-degree vertex of a graph $G$, and let $e_1 = vv_1$ and $e_2 = vv_2$ be two edges of $G$ incident with $v$. The operation of splitting the edges $e_1$ and $e_2$ from $v$ consists of deleting the vertex $v$ and its incident edges $e_1$, $e_2$ and then adding a new edge joining $v_1$ and $v_2$

Claim 3.4. For a connected graph $G'$, let $c$ be a 2-degree vertex of $G'$ and let $e_1 = ac$ and $e_2 = bc$ be the two edges incident with $c$. Let $G$ be a graph obtained from $G'$ by splitting off the $e_1$ and $e_2$ by a new edge $e$. If $\Gamma'$ and $\Gamma$ are edge-colorings of $G'$ and $G$, respectively, such that $\Gamma'(f) = \Gamma(f)$ when $f \in E(G' - v)$ and $\Gamma'(e_1) = \Gamma'(e_2) = \Gamma(e)$, then $\Gamma'$ is an MD-coloring of $G'$ if and only if $\Gamma$ is an MD-coloring of $G$. Furthermore, $md(G) \leq md(G')$.

Proof. Since $G'$ is a connected graph, $G$ is also connected. Let $E'_i$ and $E_i$ be the $i$-induced edge sets of $G'$ and $G$, respectively. Then $E_i = E'_i$ when $i \neq \Gamma(e)$ and $E_i = E'_i \cup e - (e_1 \cup e_2)$ when $i = \Gamma(e)$. Furthermore, $V(G) = V(G') - c$ and $|\Gamma'(G')| = |\Gamma(G)|$. The relationships between $G - E_i$ and $G' - E'_i$ are shown as follows.

8
1. If \( i \neq \Gamma(e) \), then \( E(G) - E_i \) is a graph obtained from \( G' - E'_i \) by spitting off \( e_1 \) and \( e_2 \) from \( c \);

2. if \( i = \Gamma(e) \), then \( G - E_i = (G' - E'_i) - c \).

We prove the first result below, that is, \( \Gamma' \) is an \( MD \)-coloring of \( G' \) if and only if \( \Gamma \) is an \( MD \)-coloring of \( G \). Suppose \( \Gamma' \) is an \( MD \)-coloring of \( G' \). Let \( u, v \) be two vertices of \( V(G) \). Since \( u, v \) are also vertices of \( V(G') \), there is an \( E'_i \) such that \( u, v \) are in different components of \( G' - E'_i \). According to the relationship between \( G - E_i \) and \( G' - E'_i \), \( u, v \) are also in different components of \( G - E_i \). Therefore \( \Gamma \) is an \( MD \)-coloring of \( G \). Analogously, suppose \( \Gamma \) is an \( MD \)-coloring of \( G \). Let \( u, v \) be two vertices of \( V(G') \). If \( u \) and \( v \) are in \( V(G') - c = V(G) \), then there is an \( E_i \) such that \( u, v \) are in different components of \( G - E_i \). According to the relationship between \( G - E_i \) and \( G' - E'_i \), \( u, v \) are also in different components of \( G' - E'_i \); if one of the \( u, v \) is \( c \), since \( c \) is an isolate vertex of \( G' - E'_i \), \( u, v \) are in different components of \( G' - E'_i \). Therefore, \( \Gamma' \) is an \( MD \)-coloring of \( G' \).

The second result can be derived from the first result directly. Suppose the edge-coloring \( \Gamma \) is an extremal \( MD \)-coloring of \( G \). Then \( \Gamma' \) is an \( MD \)-coloring of \( G' \). Since \( |\Gamma| = |\Gamma'| \), \( md(G) \leq md(G') \).

**Lemma 3.5.** Let \( M \) be a minimal matching cut of \( G \), and \( G' \) be the underling graph of \( G/M \). Then \( md(G') \leq md(G) - 1 \).

**Proof.** The graph \( G/M \) may have parallel edges but does not have loops. By Proposition 1.1 we only need to prove \( md(G/M) \leq md(G) - 1 \).

Since \( M \) is a minimal matching cut, \( M \) is a bond of \( G \). Then \( G - M \) has two components, say \( D_1 \) and \( D_2 \). We denote \( M = \{ e_1, \ldots, e_t \} \), where \( e_i = a_ib_i \) and \( a_i \) is in \( D_1 \) and \( b_i \) is in \( D_2 \) for every \( i \in [t] \). Suppose the graph \( G/M \) identifies the ends of \( e_i \) into \( c_i \). Let \( A = \bigcup_{i \in [t]} (a_i \cup b_i) \) and let \( f : V(G) \to V(G/M) \) be a mapping such that \( f(u) = u \) when \( u \in V(G) - A \) and \( f(u) = c_i \) when \( u \in \{a_i, b_i\} \).

Let \( \Gamma \) be an extremal \( MD \)-coloring of \( G/M \) with \( \Gamma = [md(G/M)] \) and let \( E_i \) be the \( i \)-induced edge set of \( G/M \). Let \( \Gamma' \) be an edge-coloring of \( G \) such that \( \Gamma(e) = \Gamma'(e) \) when \( e \notin M \) and \( \Gamma'(e) = md(G/M) + 1 \) when \( e \in M \).

For any two vertices \( u, v \) of \( G \), if \( f(u) \) and \( f(v) \) are different vertices of \( G/M \), then there is an \( E_i \) such that \( f(u) \) and \( f(v) \) are in different components of \( G/M - E_i \). Since \( G - E_i \) is a graph obtained from \( G/M - E_i \) by replacing each \( c_i \) by \( e_i \), \( u \) and \( v \) are in different components of \( G - E_i \) also. If \( f(u) = f(v) \), then \( u = a_i \) and \( v = b_i \) for some \( i \in [t] \), \( u \) and \( v \) are in different components of \( G - M \). Therefore, \( \Gamma' \) is an \( MD \)-coloring of \( G \), and so \( md(G/M) = |\Gamma| = |\Gamma'| - 1 \leq md(G) - 1 \).
The following are some definitions.

- A semi-wheel $SW(u; v_1v_2 \cdots v_n)$ is a graph obtained by connecting $u$ to each vertex of the path $P = v_1v_2e_2 \cdots e_{n-1}v_n$.
- For $n \geq 3$, let $D_n$ be a graph obtained from $SW(u; v_1v_2 \cdots v_n)$ by subdividing $uv_2, uv_3, \ldots, uv_{n-1}$. We call $uv_1$ and $uv_n$ the verges of $D_n$.
- For $n \geq 4$, let $F_n$ be a graph obtained from $SW(u; v_1v_2 \cdots v_n)$ by subdividing $uv_2, uv_3, \ldots, uv_{n-2}$.
- We construct a graph $H_n$ as follows:

$$H_n = \begin{cases} 
K_n & n = 1, 2, 3; \\
K_4 & n = 4; \\
D_{\frac{n+1}{2}} & n \text{ is odd and } n \geq 5; \\
F_{\frac{n+2}{2}} & n \text{ is even and } n \geq 6.
\end{cases}$$

- Suppose $v_1$ and $v_2$ are pendent vertices of a path $P$ and $u_1, u_2$ are two different vertices of a graph $G$, and $V(P) \cap V(G) = \emptyset$. We use $I(P, G)$ to denote a graph obtained by identifying $u_i$ of $G$ and $v_i$ of $P$, respectively, for $i \in [2]$.
- Let $n$ and $r$ be two integers with $2 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor$. We construct a graph $H_{n,r}$ below. If $n$ is even and $r < \frac{n}{2}$, then $H_{n,r} = I(P, H_{n-2r+1})$ where $P$ is a $2r$-path; if $n$ is even and $r = \frac{n}{2}$, then $H_{n,r} = C_n$; if $n$ is odd, then $H_{n,r} = I(P, H_{n-2r+2})$ where $P$ is a $(2r-1)$-path.

Remark 1: From the above definitions, we have $e(H_n) = \left\lceil \frac{3}{2}(n-1) \right\rceil$ when $n \geq 3$. For $n \geq 4$, $e(H_{n,r}) = \left\lceil \frac{3}{2}(n-2r) \right\rceil + 2r$ when $n$ is even and $e(H_{n,r}) = \left\lceil \frac{3}{2}(n-2r+1) \right\rceil + 2r - 1$ when $n$ is odd. For convenience of discussion, we denote $\mu_{n,r} = \left\lceil \frac{3}{2}(n-2r) \right\rceil + 2r$ when $n$ is even and $\mu_{n,r} = \left\lceil \frac{3}{2}(n-2r+1) \right\rceil + 2r - 1$ when $n$ is odd, i.e., $e(H_{n,r}) = \mu_{n,r}$.

The following is the proof of $md(H_n) = 1$ for $n \geq 2$. The proof uses an obvious conclusion that any $MD$-coloring of a 4-cycle or a 5-cycle is either trivial or assigning colors 1 and 2 alternately to its edges. Therefore there are two adjacent edges of the 5-cycle receiving a same color when the $MD$-coloring is non-trivial.

Lemma 3.6. $md(H_n) = 1$ for $n \geq 2$.

Proof. Because $H_2 = K_2, H_3 = K_3, H_4 = K_4^-$ and $H_5 = K_{2,3}$, by Lemma 1.6, $md(H_n) = 1$ for $2 \leq n \leq 5$. We proceeds the proof by induction on $n$. The lemma holds when $n \leq 5$. Now we suppose $n \geq 6$.

If $n$ is even, then $H_n = H_{n-1} \cup K_3$ and the intersecting edge of $H_{n-1}$ and $K_3$ is a verge of $H_{n-1}$. Since $md(H_{n-1}) = md(K_3) = 1$, by Lemma 1.5, $md(H_n) = 1$. Therefore, we only need to show that $md(H_n) = 1$ when $n$ is odd. Let $n = 2k - 1$ and $k \geq 3$.

Let $H_n = H_{2k-1}$ be a graph obtained by inserting new vertices $w_2, \cdots, w_{k-1}$ to
uvw, · · ·, uvk−1 of SW(u; v1v2 · · · vk), respectively. Here ei = vi+1v for i ∈ [k − 1] and P = v1e1 · · · ek−1vk is a path.

We proceeds the proof by contradiction. Suppose md(H2k−1) ≥ 2. Then by Lemma 2.5, there exists an MD-coloring Γ of H2k−1 such that |Γ| = 2, i.e., every edge of H2k−1 is either colored by 1 or colored by 2. We distinguish two cases.

Case 1. There exist adjacent edges ei and ei+1 of P such that Γ(ei) = Γ(ei+1).

Let H = H2k−1 − wi+1. Then Γ is an MD-coloring on H. Furthermore, |Γ(H)| = 2. Otherwise suppose all edges of H are colored by 1. Since |Γ| = 2, at least one of ei and e2 is colored by 2 under Γ. Since ei and e2 are in the 5-cycle C = H2k−1[u, w, v, vi, vi+1, wi+1], Γ is not an MD-coloring on C, a contradiction.

Let H′ be a graph obtained from H by splitting off ei and ei+1 from vi+1. By Claim 3.3, there is an MD-coloring Γ′ of H′ such that |Γ′| = 2. However, H′ = H2k−3, and by induction, md(H′) = 1, a contradiction.

Case 2. Assigning colors 1 and 2 alternately on P, i.e., Γ(ej) = 1 when j is odd and Γ(ej) = 2 when j is even.

![Figure 1: The graph of Case 2 with k is even.](image)

If Γ(uv1) = Γ(e1) = 1, then Γ is a trivial MD-coloring on the 4-cycle H2k−1[u, v1, v2, w2], and so Γ(uw2) = Γ(w2v2) = 1. Let H be a graph obtained from H2k−1 by splitting off uw2 and w2v2 from w2. Then by Claim 3.3, there is an MD-coloring Γ′ of H′ such that |Γ′| = 2. However, H′ = H2k−2, and by induction, md(H′) = 1, a contradiction.

If Γ(uv1) ≠ Γ(e1), then each 5-cycle Ci = H2k−1[u, wi, vi, vi+1, wi+1] is colored non-trivial under Γ. Furthermore, Γ(wiv) = Γ(ei) for i = 2, · · · , k − 1. This implies that Γ(wk−2v) = Γ(ek−2) = Γ(uw). Since Γ(ek−2) ≠ Γ(ek−1), Γ(uw) ≠ Γ(ek−1), which contradicts that Γ is an MD-coloring restricted on the 4-cycle H2k−1[u, w, v, k−1, v k].

According to that above two cases, one has that md(H2k−1) = 1.

Lemma 3.7. If 2 ≤ r ≤ ⌊n/2⌋ and n ≥ 4, then md(Hn,r) = r.
Proof. Let \( Q_1 = v_1e_1v_2e_2 \cdots v_{2r}e_{2r}v_{2r+1} \) and \( Q_2 = v_1e_1v_2e_2 \cdots v_{2r-1}e_{2r-1}v_{2r} \). Let \( R_1 = H_{n-2r+1} \) and \( R_2 = H_{n-2r+2} \). We will construct \( H_{n,r} \) below. If \( n \) is even and \( r = \frac{n}{2} \), then \( H_{n,r} = C_n \); if \( n \) is even and \( 2 \leq r < \frac{n}{2} \), \( H_{n,r} = I(Q_1, R_1) \); if \( n \) is odd, \( H_{n,r} = I(Q_2, R_3) \).

Case 1. \( n \) is even and \( r = \frac{n}{2} \).

Since \( H_{n,r} = C_n \), by Proposition 1.2, \( md(H_{n,r}) = r \) holds.

Case 2. \( n \) is even and \( 2 \leq r < \frac{n}{2} \).

Color \( e_i \) by \( j \in [r] \) if \( i \equiv j \pmod{r} \) and color the edges of \( R_1 \) by 1. It is easy to verify that the edge-coloring is an MD-coloring of \( H_{n,r} \). Therefore, \( md(H_{n,r}) \geq r \). Since every edge of \( H_{n,r} \) is in some cycles, every color of an extremal MD-coloring of \( H_{n,r} \) colors at least two edges. Furthermore, since \( md(R_1) = 1 \), all edges of \( R_1 \) are colored the same under the extremal MD-coloring. Therefore, there are at most \( r \) colors in the extremal MD-coloring, and so \( md(H_{n,r}) \leq r \). Thus, \( md(H_{n,r}) = r \).

Case 3. \( n \) is odd and \( 2 \leq r \leq \frac{n}{2} \).

Color \( e_i \) by \( j \in [r] \) if \( i \equiv j \pmod{r} \) and color the edges of \( R_2 \) by \( r - j \). It is obvious that the edge-coloring of \( H_{n,r} \) is an MD-coloring. Therefore, \( md(H_{n,r}) \geq r \). As discussed in Case 2, since every color of an extremal MD-coloring of \( H_{n,r} \) colors at least two edges and since \( md(R_2) = 1 \), \( md(H_{n,r}) \leq r \). Thus, \( md(H_{n,r}) = r \).

Since \( md(H_{n,r-1}) = r - 1 \), \( g(n, r) \leq e(H_{n,r-1}) - 1 = \mu_{n,r} \) for \( 3 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor \). Therefore, we have the following result.

**Corollary 3.8.** \( g(n, r) \leq \mu_{n,r} \) for \( 3 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor \).

**Lemma 3.9.** For \( n \geq 4 \), \( g(n, 2) = \left\lceil \frac{3}{2}(n-1) \right\rceil - 1 \). For \( n \geq 6 \), \( g(n, \left\lceil \frac{n}{2} \right\rceil) = \mu_{n,\left\lceil \frac{n}{2} \right\rceil} \).

**Proof.** For \( n \geq 4 \), since \( md(H_n) = 1 \) and \( e(H_n) \leq \left\lceil \frac{3}{2}(n-1) \right\rceil \), \( g(n, 2) \leq \left\lceil \frac{3}{2}(n-1) \right\rceil - 1 \). By Theorem 1.8, \( G \) has a matching cut when \( e(G) \leq \left\lceil \frac{3}{2}(n-1) \right\rceil - 1 \), and by Lemma 2.4, \( md(G) \geq 2 \). Therefore, \( g(n, 2) = \left\lceil \frac{3}{2}(n-1) \right\rceil - 1 \).

If \( n \geq 6 \) and \( n \) is even, \( g(n, \frac{n}{2}) \leq \mu_{n,\frac{n}{2}} = n \) by Corollary 3.8. Since any connected graph \( G \) with \( e(G) \leq n \) is either a tree or a unicycle graph, we have \( md(G) \geq \frac{n}{2} \) by Proposition 1.2. Therefore, \( g(n, \frac{n}{2}) = n \) when \( n \) is even.

If \( n \geq 7 \) and \( e(G) = n + 1 \), we first prove that \( G \) has a minimal matching cut \( M \) such that \(|M| \leq 2 \). If \( G \) has a cut-edge, then \(|M| = 1 \). Otherwise \( G \) has at most two non-trivial blocks. Furthermore, either \( G \) has exactly two 3-degree vertices and the other vertices are 2-degree vertices, or \( G \) has one 4-degree vertex and the other vertices are 2-degree vertices, and both cases imply there are two adjacent 2-degree vertices, say \( u \) and \( v \). Let \( e_1 = xu, e_2 = uv \) and \( e_3 = vy \), where \( x \neq v \) and \( y \neq u \). If \( x \neq y \), \( M = \{e_1, e_3\} \); if \( x = y \), one block of \( G \) is \( K_3 \) and the other block is an \((n - 2)\)-cycle.
Since \( n \geq 7 \), the \((n - 2)\)-cycle has a matching cut \( M \) and \(|M| = 2\). \( M \) is also a matching cut of \( G \).

Now we prove that if \( n \) is odd and \( n \geq 7 \), \( g(n, \lfloor \frac{n}{2} \rfloor) = n + 1 \). By Corollary 3.8, \( g(n, \lfloor \frac{n}{2} \rfloor) \leq \mu_{n, \lfloor \frac{n}{2} \rfloor} = n + 1 \). In order to show \( g(n, \lfloor \frac{n}{2} \rfloor) = \mu_{n, \lfloor \frac{n}{2} \rfloor} = n + 1 \), we need to prove that any graph \( G \) with \(|G| = n \) and \( e(G) \leq n + 1 \) has \( md(G) \geq \lfloor \frac{n}{2} \rfloor \). Let \( G \) be a connected graph with \(|G| \geq 7 \) and \( e(G) \leq n + 1 \). Then \( G \) has a minimal matching cut \( M \) such that \(|M| \leq 2 \). Let \( G' \) be the underlying simple graph of \( G/M \). By Lemma 3.5, \( md(G') \leq md(G) - 1 \). So, we only need to show \( md(G') \geq \lfloor \frac{n}{2} \rfloor - 1 \).

If \(|M| = 1 \), since \(|G'| = e(G') \) is even and \( e(G') = \lfloor G' \rfloor + 1 = \mu_{n-1, \lfloor \frac{n-1}{2} \rfloor - 1} \), \( md(G') \geq \lfloor \frac{n-1}{2} \rfloor - 1 = \lfloor \frac{n}{2} \rfloor - 1 \).

If \(|M| = 2 \), there are two cases to consider.

Case 1. \( n = 7 \).

Then \(|G/M| = 5 \) and \( e(G/M) \leq 6 \). It is easy to verify that \( G/M = H_5 \) is the only such graph with \( md(G/M) = 1 \). If \( G/M \neq H_5 \), then \( md(G/M) = 2 = \lfloor \frac{n}{2} \rfloor - 1 \); if \( G/M = H_5 \), then the graph \( G \) and one of its \( MD \)-colorings are shown as in Figure 2 and so \( md(G) \geq 3 \).

![Figure 2: The graph \( G \) that satisfies \( G/M = H_5 \) and an \( MD \)-coloring of \( G \).](image)

Case 2. \( n \geq 9 \). Since \(|G'| = n - 2 \) is odd and \( e(G') \leq \lfloor G' \rfloor + 1 = \mu_{n-2, \lfloor \frac{n-2}{2} \rfloor - 1} \), by induction, \( md(G') \geq \lfloor \frac{n-2}{2} \rfloor = \lfloor \frac{n}{2} \rfloor - 1 \).

**Theorem 3.10.** For \( n \geq 2 \) and \( 1 \leq r \leq n - 1 \),

\[
g(n, r) = \begin{cases} 
\binom{n}{2} & r = 1; \\
\lfloor \frac{n}{2} \rfloor(n - 1) - 1 & r = 2; \\
\mu_{n,r} & 3 \leq r \leq \lfloor \frac{n}{2} \rfloor - 1; \\
\mu_{n,\lfloor \frac{n}{2} \rfloor} & n \geq 6 \text{ and } r = \lfloor \frac{n}{2} \rfloor; \\
n - 1 & \lfloor \frac{n}{2} \rfloor + 1 \leq r \leq n - 1.
\end{cases}
\]

**Proof.** It is easy to verify that \( g(n, 1) = \binom{n}{2} \) and \( g(n, r) = n - 1 \) when \( n - 1 \geq r \geq \lfloor \frac{n}{2} \rfloor \). By Corollary 3.8, \( g(n, r) \leq \mu_{n,r} \) when \( 3 \leq r \leq \lfloor \frac{n}{2} \rfloor - 1 \). By Lemma 3.3, \( g(n, \lfloor \frac{n}{2} \rfloor) = \mu_{n,\lfloor \frac{n}{2} \rfloor} \) when \( n \geq 6 \) and \( g(n, 2) = \lfloor \frac{n}{2} \rfloor(n - 1) - 1 \) when \( n \geq 4 \).
Remark 2: Given two integers \( r \) and \( n \) satisfying that \( n \geq 7 \) and \( 3 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \). If \( g(n, r) = \mu_{n,r} \) holds when \( n \) is odd, then \( g(n, r) = \mu_{n,r} \) holds when \( n \geq 8 \) is even.

Proof. Suppose \( G \) is a graph with \( e(G) \leq \mu_{n,r} \) where \( n \geq 8 \) is even and \( 3 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \). Since \( \frac{2e(G)}{|G|} < 3 \), there is a vertex \( v \) with degree two or one. If \( d_G(v) = 1 \), let \( G' = G - v \), and then \( md(G') = md(G) - 1 \); if \( d_G(v) = 2 \), then let \( G' \) be a graph obtained from \( G \) by splitting off the two edges incident with \( v \). By Claim 3.4 \( md(G') \leq md(G) \). Therefore, \( md(G') \leq md(G) \) in both cases. Since \( r \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \), \( r \leq \left\lfloor \frac{n+1}{2} \right\rfloor \) also. Since \( G' = n - 1 \) is odd and \( e(G') = e(G) - 1 = \mu_{n-1,r} \), \( md(G') \geq r \). Therefore, \( md(G) \geq r \).

Remark 3: For \( n \geq 8 \), since \( g(n, 2) = \left\lfloor \frac{3}{2}(n - 1) \right\rfloor - 1 \) and \( g(n, \left\lfloor \frac{n}{2} \right\rfloor) = \mu(n, \left\lfloor \frac{n}{2} \right\rfloor) \), we have \( \frac{g(n, 2) - g(n, \left\lfloor \frac{n}{2} \right\rfloor)}{\left\lfloor \frac{n}{2} \right\rfloor - 2} = 1 \) when \( n \) is even and \( \frac{g(n, 2) - g(n, \left\lfloor \frac{n}{2} \right\rfloor)}{\left\lfloor \frac{n}{2} \right\rfloor - 2} < 1 \) when \( n \) is odd. This implies that the average value of \( g(n, r - 1) - g(n, r) \) is less than or equal to 1. Furthermore, if \( n \) is odd, there exists an integer \( r \) such that \( 4 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor \) and \( g(n, r - 1) = g(n, r) \).

4 Results for graph products

Since an \( MD \)-coloring of a 4-cycle is either trivial or assigning 1 and 2 alternately to its edges, the opposite edges of a 4-cycle are colored the same under its every \( MD \)-coloring.

Theorem 4.1. For two connected graphs \( G \) and \( H \), \( md(G\Box H) = md(G) + md(H) \).

Proof. Let \( |G| = n_1 \) and \( |H| = n_2 \). Let \( V(G) = \{u_1, \ldots, u_{n_1}\} \) and \( V(H) = \{v_1, \ldots, v_{n_2}\} \). For an edge \( e = u_iu_j \) of \( G \) and an edge \( f = v_sv_t \) of \( H \), let

\[
S_e = \{((u_i, v_r), (u_j, v_r)) : r \in [n_2]\} \quad \text{and} \quad S_f = \{((u_r, v_s), (u_r, v_t)) : r \in [n_1]\}.
\]

It is obvious that every edge of \( G\Box H \) is in a unique \( S_e \), where \( e \) is either in \( E(G) \) or in \( E(H) \). Therefore, \( \bigcup_{e \in E(G)} S_e = E(G\Box H) \).

Let \( \Gamma \) be an extremal \( MD \)-coloring of \( G\Box H \). Then we have the following result.

Claim 4.2. \( |\Gamma(S_e)| = 1 \) for every \( e \in E(G) \cup E(H) \).

Proof. Without loss of generality, let \( e = u_1u_2 \) be an edge of \( G \). For any two edges \( h_1 = ((u_1, v_r), (u_2, v_r)) \) and \( h_2 = ((u_1, v_j), (u_2, v_j)) \) of \( S_e \), there is a \( v_1v_j \)-path \( P \) of \( H \). W.l.o.g., let \( v_i = v_1 \) and \( P = v_1f_1v_2f_2\cdots v_{j-1}f_{j-1}v_j \). Then \( L = e\Box P \) is a subgraph of \( G\Box H \).

Because \( e\Box f_r \) is a 4-cycle for \( r \in [j - 1] \), and \( ((u_1, v_r), (u_2, v_r)) \) and \( ((u_1, v_{r+1}), (u_2, v_{r+1})) \) are opposite edges of \( e\Box f_r \), \( ((u_1, v_r), (u_2, v_r)) \) and \( ((u_1, v_{r+1}), (u_2, v_{r+1})) \) are colored the same under \( \Gamma \). Therefore, \( h_1 \) and \( h_2 \) are colored the same under \( \Gamma \).

\[ \blacksquare \]
Because $u_1 \Box H$ and $G \Box v_1$ are subgraphs of $G \Box H$, by Proposition 1.3, $\Gamma$ is an $MD$-coloring restricted on $G \Box v_1$ and $u_1 \Box H$. Since $G \cong G \Box v_1$ and $H \cong u_1 \Box H$, $|\Gamma(G \Box v_1)| \leq md(G)$ and $|\Gamma(u_1 \Box H)| \leq md(H)$. Now we choose an edge $h$ of $G \Box H$ arbitrarily. Without loss of generality, suppose $h = ((u_i, v_i), (u_j, v_j))$ (or $h = ((u_r, v_s), (u_r, v_t))$). Then by Claim 3.2, there is an edge $e = ((u_i, v_i), (u_j, v_j))$ of $G \Box v_1$ (or an edge $e = ((u_j, v_s), (u_i, v_t))$ of $u_1 \Box H$), such that $\Gamma(h) = \Gamma(e)$. This implies that $\Gamma(G \Box v_1) \cup \Gamma(u_1 \Box v_1) = \Gamma$. Since $\Gamma$ is an extremal $MD$-coloring of $G \Box H$, $md(G \Box H) = |\Gamma| \leq md(G) + md(H)$.

We need to prove $md(G \Box H) \geq md(G) + md(H)$ below. Let $\Gamma_1$ be an extremal $MD$-coloring of $G$ and $\Gamma_2$ be an extremal $MD$-coloring of $H$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Since every edge $h$ of $G \Box H$ is in a unique $S_h$, where $e$ is either in $E(G)$ or $E(H)$, we construct an edge-coloring $\Gamma$ of $G \Box H$ such that $\Gamma(h) = \Gamma_1(e)$ when $e \in E(G)$ and $\Gamma(h) = \Gamma_2(e)$ when $e \in E(H)$. Since $|\Gamma| = |\Gamma_1| + |\Gamma_2| = md(G) + md(H)$, in order to prove $md(G \Box H) \geq md(G) + md(H)$, we only need to prove that $\Gamma$ is an $MD$-coloring of $G \Box H$.

We need to prove that there is a monochromatic cut between any two different vertices of $G \Box H$. We set the two different vertices and denote them by $w_0 = (u_i, v_s)$ and $w_r = (u_j, v_t)$, here either $u_i \neq u_j$ or $v_s \neq v_t$, say $v_s \neq v_t$. Since $\Gamma_2$ is an extremal $MD$-coloring of $H$, there is a monochromatic $u_i \cdot v_t$-cut of $H$, and we suppose that the color of the monochromatic $u_i \cdot v_t$-cut is $c$. If any $w_0w_r$-path of $G \Box H$ has an edge that is colored by $c$ under $\Gamma$, then the set of these edges is a monochromatic $w_0w_r$-cut of $G \Box H$ under $\Gamma$. We will show the existence below.

Let $P = w_0h_0w_1h_1 \cdots w_{r-1}h_{r-1}w_r$ be a $w_0w_r$-path of $G \Box H$. Here $h_i = w_iw_{i+1}$ is an edge of $G \Box H$. For convenience, we denote $w_k$ by $(u_k, v_k)$ for $0 \leq k \leq r$, and then $i = s = 0$ and $j = t = r$. Because $h_k = w_kw_{k+1} = ((u_k, v_k), (u_{k+1}, v_{k+1}))$ is an edge of $G \Box H$, either $v_kv_{k+1}$ is an edge of $H$ or $v_k = v_{k+1}$. Therefore, $L = v_sv_1 \cdots v_{r-1}v_t$ is a $v_sv_t$-walk of $H$ (it may have $v_k = v_{k+1}$ for some $0 \leq k \leq r - 1$). Then $L$ contains a $v_sv_t$-path $L'$ of $H$. This implies that there is an edge of $L'$, which is also an edge of $L$, is colored by $c$. Suppose the edge is $e = vLv_{l+1}$. Then $h_l = ((u_l, v_l), (u_{l+1}, v_{l+1}))$ is an edge of $P$ colored by $c$. This implies that any $w_0w_r$-path of $G \Box H$ has an edge that is colored by $c$ under $\Gamma$.

Since the $w_0w_r$-path $P$ is chosen arbitrarily, there is a monochromatic $w_0w_r$-cut of $G \Box H$ under $\Gamma$, and since the vertices $w_0$ and $w_r$ are chosen arbitrarily, $\Gamma$ is an $MD$-coloring of $G \Box H$.

Because any three graphs $G_1, G_2$ and $G_3$ satisfy $G_1 \Box G_2 \Box G_3 = (G_1 \Box G_2) \Box G_3$, the following result is obvious.

**Corollary 4.3.** For $k$ connected graphs $G_1, \cdots, G_k$, $md(G_1 \Box \cdots \Box G_k) = \sum_{i \in [k]} md(G_i)$. 

15
Lemma 4.4. If $m \geq 1$ and $n \geq 1$, then $P_m \boxtimes P_n$ is a closure.

Proof. The proof is by induction on $m + n$. It is easy to verify that $P_1 \boxtimes P_1 = K_4$, and so the result holds for $m + n = 2$. Suppose $m + n > 2$ and $m \geq 2$. Let $P_m = u_0e_1u_1e_2\cdots u_{m-1}e_mu_m$ and $P_n = v_0f_1v_1f_2\cdots v_{n-1}f_nv_n$. Let $P' = P_m - e_m$, and by induction, both $P' \boxtimes P_n$ and $e_m \boxtimes P_n$ are closures. Since $h = ((u_{m-1}, v_0), (u_{m-1}, v_1))$ is a common edge of $P' \boxtimes P_n$ and $e_m \boxtimes P_n$, $P_m \boxtimes P_n$ is a closure. \qed

Theorem 4.5. For two connected graphs $G$ and $H$ with $|G| \geq 2$ and $|H| \geq 2$, $\text{md}(G \boxtimes H) = 1$.

Proof. By Lemma 2.1, if we prove $G \boxtimes H$ is a closure, then we are done. Let $h_1 = ((x_1, y_1), (x_2, y_2))$ and $h_2 = ((a_1, b_1), (a_2, b_2))$ be two distinct edges of $G \boxtimes H$. Let $e_1 = x_1x_2$, $e_2 = a_1a_2$, $f_1 = y_1y_2$ and $f_2 = b_1b_2$. Then $e_i$ (or $f_i$) is either an edge or a vertex of $G$ (or $H$) for $i = 1, 2$. Therefore, there is a path $P'$ of $G$ connects $e_1$ and $e_2$, that is, $e_1$ is either a pendent edge of $P'$ if $e_1$ is an edge, or a pendent vertex of $P'$ if $e_1$ is a vertex, and so is $e_2$. Analogously, there is a path $P''$ of $H$ connects $f_1$ and $f_2$. Furthermore, at least one of $e_1$ and $f_1$ is an edge, and at least one of $e_2$ and $f_2$ is an edge.

Case 1. None of $P'$ and $P''$ is a single vertex.

Since at least one of $e_1$ and $f_1$ is an edge, and at least one of $e_2$ and $f_2$ is an edge, without loss of generality, we assume $e_1$ and $f_1$ are edges. Then $h_1 \in E(e_1 \boxtimes f_1)$ and $h_2 \in E(e_2 \boxtimes f_2)$. Since both $e_1 \boxtimes f_1$ and $e_2 \boxtimes f_2$ are subgraphs of $P' \boxtimes P''$, both $h_1$ and $h_2$ are in $P' \boxtimes P''$. By Lemma 1.4 $P' \boxtimes P''$ is a closure, and then $h_1\theta h_2$ is in $P' \boxtimes P''$. Therefore, $h_1\theta h_2$ is also in $G \boxtimes H$.

Case 2. One of $P'$ and $P''$ is a single vertex, say $P'$.

Since at least one of $e_1$ and $f_1$ is an edge, and at least one of $e_2$ and $f_2$ is an edge, and since $e_1 = e_2$ is a vertex of $G$, both $f_1$ and $f_2$ are edges of $H$. Since $|G| \geq 2$, there is an edge of $G$, say $e$, incident with $e_1$. It is easy to verify that both $h_1$ and $h_2$ are in $e \boxtimes P''$. Since $e \boxtimes P''$ is a closure by Lemma 1.4, $h_1\theta h_2$ in $e \boxtimes P''$. Therefore, $h_1\theta h_2$ is also in $G \boxtimes H$. \qed

Because $G \boxtimes H$ is a connected spanning subgraph of $G \circ H$ by Proposition 1.9, by Lemma 1.4, the following result is obvious.

Theorem 4.6. If $G$ and $H$ are connected graphs with $|G| \geq 2$ and $|H| \geq 2$, then $\text{md}(G \circ H) = 1$.

Lemma 4.7. $\text{md}(K_2 \ast K_n) = \text{md}(P_3 \ast K_3) = 1$ where $n \geq 5$. 

16
Proof. We first show that \(md(K_2 \ast K_n) = 1\) for \(n \geq 5\). Let \(V(K_2) = \{x_1, x_2\}\) and \(V(K_n) = \{y_1, \ldots, y_n\}\). We construct a bipartite graph \(G_{2,n}\) with bipartition \(S_1 = \{v_1, v_2, \ldots, v_n\}\) and \(S_2 = \{v_1^2, v_2^2, \ldots, v_n^2\}\), and \(v_i^1\) connects \(v_i^1\) if and only if \(i \neq j\) and \(s \neq t\). Then \(K_2 \ast K_n \cong G_{2,n}\); this is because there is a bijection \(f\) between \(V(K_2) \times V(K_n)\) and \(V(G_{2,n})\), such that \(f(x_i, y_j) = v_i^j\), and then \(((x_i, y_j), (x_s, y_t))\) is an edge of \(K_2 \ast K_n\) if and only if \(v_i^j v_s^t\) is an edge of \(G_{2,n}\). Therefore, by Lemma 2.1, we only need to prove that \(G_{2,n}\) is a closure when \(n \geq 5\).

Let \(e = v_i^1 v_j^2\) and \(f = v_i^1 v_t^2\) be two edges of \(G_{2,n}\). Then \(i \neq j\) and \(s \neq t\). Let \(A = \{i, j, s, t\}\).

If \(|A| = 4\), since \(n \geq 5\), there is an integer \(w \in [n]\) such that \(w \not \in A\). Then \(i, j, s, t, w\) are pairwise different, and so \(G_1 = G_{2,n}[v_i^1, v_j^2, v_s^1, v_t^2, v_w^2] \cong K_{2,3}\). Therefore, \(\theta f\).

If \(|A| = 3\), then if \(e\) and \(f\) have no common vertex, for convenience, let \(i = t = 1, j = 2\) and \(s = 3\). Then \(G_1 = G_{2,n}[v_i^1, v_j^2, v_1^1, v_2^2] \cong K_{2,3}\) and \(G_2 = G_{2,n}[v_1^1, v_2^2, v_3^2, v_4^3] \cong K_{2,3}\). Since \(e \in E(G_1), f \in E(G_2)\) and \(v_1^2 v_2^1 \in E(G_1) \cap E(G_2), \theta f\). If \(e\) and \(f\) have a common vertex, for convenience, let \(i = s = 1, j = 2\) and \(t = 3\). Then \(G'_1 = G_{2,n}[v_i^1 = v_s^1 = v_1^1, v_j^2 = v_2^2, v_4^3, v_3^2] \cong K_{2,3}\) and both \(e\) and \(f\) are in \(G'_1, \theta f\).

If \(|A| = 2\), then \(e\) and \(f\) are two non-adjacent edges. Let \(i = t = 1\) and \(j, s = 2\) for convenience. Then \(G_1 = G_{2,n}[v_i^1, v_j^2, v_i^1, v_2^2] \cong K_{2,3}\) and \(G_2 = G_{2,n}[v_1^1, v_2^2, v_3^2, v_3^3] \cong K_{2,3}\). Since \(e \in E(G_1), f \in E(G_2)\) and \(v_4^3 v_3^2 \in E(G_1) \cap E(G_2), \theta f\).

Now we prove \(md(P_3 \ast K_3) = 1\). The graphs \(P_3, K_3\) and \(P_3 \ast K_3\) are shown as on the left-hand-side of Figure 3 and we write the vertex \((y_i, x_j)\) of \(P_3 \ast K_3\) as \(v_i^j\). The planar embedding of \(G = P_3 \ast K_3\) is shown as on the right-hand-side of Figure 3. We will complete the proof by checking all the possible edge-colorings of \(P_3 \ast K_3\).

![Figure 3: The graph P_3 * K_3.](image)

The central cycle \(C = G[v_1^1, v_2^2, v_1^3, v_2^1, v_3^2]\) of \(G\) is crucial for our discussion. Since the opposite edges of \(C_4\) are colored the same under any \(MD\)-coloring, \(\Gamma(G) = \Gamma(C)\) for any \(MD\)-coloring of \(G\). If \(md(G) \geq 2\), by Lemma 2.1, there is an \(MD\)-coloring.
Γ' of G such that |Γ'| = 2. All possible edge-colorings of C under Γ' are shown as in Figure 4A, B, C and D, and the colors of the other edges are also labeled. If Γ' is an edge-coloring shown as in Figure 4A, then Γ' is not an MD-coloring restricted on the cycle \( C_1 = G[v_3^1, v_3^2, v_1^1, v_2^3, v_3^1] \); if Γ' is an edge-coloring shown as in Figure 4B, C or D, then Γ' is not an MD-coloring restricted on the cycle \( C_2 = G[v_3^2, v_3^1, v_4^1, v_3^2, v_3^1] \). All the four cases contradict that Γ' is an MD-coloring of G, and so \( md(G) = 1 \).

Figure 4: All possible 2-edge-coloring of \( P_3 \times K_3 \).

**Lemma 4.8.** Let G and H be two connected graphs and let G' be a connected subgraph of G. If at least one of G' and H is non-bipartite graph and \( \delta(H) \geq 2 \), then \( md(G \ast H) \leq md(G' \ast H) \).

**Proof.** We proceed the proof by induction on \(|G| - |G'||\). If \(|G| - |G'| = 0\), then G' is a spanning subgraph of G. This implies that G' \ast H is a spanning subgraph of G \ast H. Since at least one of G' and H is not bipartite, by Proposition 1.10 both of G \ast H and G' \ast H are connected graphs. Then by Lemma 1.4 \( md(G \ast H) \leq md(G' \ast H) \), and the result thus holds.

Now we suppose \(|G| - |G'|| \geq 1\). Since G' is a connected subgraph of G, there is a spanning tree of G such that one of its leaves, say u, is not in \( V(G') \). Let \( G^* = G - u \). Then G* is a connected subgraph of G containing G' as its subgraph. Furthermore, both of G \ast H and G* \ast H are connected by Proposition 1.10. Since \(|G^*| - |G'| < |G| - |G'||\), by induction, \( md(G^* \ast H) \leq md(G' \ast H) \).

18
Let $V(H) = \{w_1, w_2, \cdots, w_n\}$ and let $S = \{(u, w_i) : i \in [n]\}$. Then $S$ is an independent set of $G * H$. Furthermore, $G * H - S = G^* * H$. For an element $(u, w)$ of $S$, since $\delta(H) \geq 2$, there are two neighbors of $w$ in $H$, say $w_1$ and $w_2$. Let $v$ be a neighbor of $u$ in $G$. Then $((u, w), (v, w_1))$ and $((u, w), (v, w_2))$ are edges of $G * H$ incident with $(u, w)$. Therefore, each vertex of $S$ has a degree at least two in $G * H$. Let $\gamma = ((u, w_1), \cdots, (u, w_n))$ be a vertex sequence of $G * H$. Then $\gamma$ is a soft-layer. By Lemma 2.3 $md(G * H) \leq md(G^* * H)$. Since $md(G^* * H) \leq md(G' * H)$, $md(G * H) \leq md(G' * H)$.

**Theorem 4.9.** Let $G'$ and $H'$ be connected subgraphs of the connected graphs $G$ and $H$, respectively, and all the four graphs do not have pendent edges. If at least one of $G'$ and $H'$ is non-bipartite, then $md(G * H) \leq md(G' * H')$.

**Proof.** Since at least one of $G'$ and $H$ is non-bipartite and $\delta(H) \geq 2$, by Lemma 4.8 $md(G * H) \leq md(G' * H')$. Analogously, since at least one of $G'$ and $H'$ is non-bipartite and $\delta(G') \geq 2$, $md(H * G') = md(H' * G') = md(G' * H')$. Therefore, $md(G * H) \leq md(G' * H')$.

The odd girth of a non-bipartite graph $G$ is the length of a minimum odd cycle of $G$, and we denote it by $g_o(G)$. If $G$ is a bipartite graph, we define $g_o(G) = +\infty$, this is because a bipartite graph has no odd cycle.

**Corollary 4.10.** Let $G$ and $H$ be two connected non-trivial graphs both without pendent edges and at least one of them is non-bipartite. Then $md(G * H) \leq \min\{g_o(G), g_o(H)\}$.

**Proof.** Without loss of generality, suppose $G$ contains an odd cycle $O$ such that $|O| = \min\{g_o(G), g_o(H)\}$. Since $H$ has no pendent edge, $H$ has a cycle $O'$ by Lemma 4.9 $md(G * H) \leq md(O * O')$. By Lemma 4.8 $md(O * O') \leq md(O * K_2)$. Since $O * K_2$ is a $(2|O|)$-cycle, $md(O * K_2) = |O| = \min\{g_o(G), g_o(H)\}$. Therefore, $md(G * H) \leq md(O * K_2) = \min\{g_o(G), g_o(H)\}$.

**Corollary 4.11.** Let $G$ and $H$ be two connected graphs. Then

1. if $G$ is neither a tree nor a unicycle graph with the cycle $K_3$, and $H$ contains a triangle but does not have pendent edges, then $md(G * H) = 1$;

2. if $|G| \geq 2$ and $H = K_n$ where $n \geq 5$, then $md(G * H) = 1$.

**Proof.** We prove the first result. Let $G'$ be a graph obtained from $G$ by deleting pendent edges one by one. Since $G$ is neither a tree nor a unicycle graph with the cycle $K_3$, $G'$ has no pendent edges and is not a $K_3$. Therefore, $G'$ contains a 3-path, say $P$. By Theorem
4.9  \( md(G*H) \leq md(G'*K_3) \). By Lemma 4.8 and 4.7, \( md(G'*K_3) \leq md(P*K_3) = 1 \). So, \( md(G*H) = 1 \).

Since \( md(G*K_n) \leq md(K_2*K_n) \) and \( md(K_2*K_n) = 1 \) for \( n \geq 5 \) by Lemma 4.8 and 4.7 respectively, the second result can be derived directly.

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM 244, Springer, 2008.

[2] P. Bonsma, A.M. Farley, A. Proskurowski, Extremal graphs having no matching cuts, J. Graph Theory 69(2012), 206–222.

[3] Q. Cai, X. Li, D. Wu, Erdős-Gallai-type results for colorful monochromatic connectivity of a graph, J. Comb. Optim. 33(1)(2017), 123–131.

[4] Q. Cai, X. Li, D. Wu, Some extremal results on the colorful monochromatic vertex-connectivity of a graph, J. Comb. Optim. 35(2018), 1300–1311.

[5] Y. Caro, R. Yuster, Colorful monochromatic connectivity, Discrete Math. 311(2011), 1786–1792.

[6] G. Chartrand, S. Devereaux, T.W. Haynes, S.T. Hedetniemi, P. Zhang, Rainbow disconnection in graphs, Discuss. Math. Graph Theory 38(4)(2018), 1007–1021.

[7] R. Gu, X. Li, Z. Qin, Y. Zhao, More on the colorful monochromatic connectivity, Bull. Malays. Math. Sci. Soc. 40(4)(2017), 1769–1779.

[8] H. Jiang, X. Li, Y. Zhang, Erdős-Gallai-type results for total monochromatic connection of graphs, Discuss. Math. Graph Theory, in press.

[9] Z. Jin, X. Li, K. Wang, The monochromatic connectivity of some graphs, submitted, 2016.

[10] P. Li, X. Li, Monochromatic disconnection of graphs, arXiv:1901.01372 [math.CO].

[11] X. Li D. Wu, A survey on monochromatic connections of graphs, Theory & Appl. Graphs 0(1)(2018), Art.4.

[12] Y. Mao, Z. Wang, F. Yanling, C. Ye, Monochromatic connectivity and graph products, Discrete Math, Algorithm. Appl. 8(01)(2016), 1650011.
[13] D. Gonzlez-Moreno, M. Guevara, J.J. Montellano-Ballesteros, Monochromatic connecting colorings in strongly connected oriented graphs, Discrete Math. 340(4)(2017), 578–584.

[14] P.M. Weichsel, The Kronecker product of graphs. Proc. Amer. Math. Soc. 13(1963), 47–52.