Comment on “Influence Functional for Decoherence of Interacting Electrons in Disordered Conductors” (cond-mat/0510563v1) and on related papers (cond-mat/0510556v1 and cond-mat/0510557v1)

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In recent papers von Delft et al. addressed the problem of low temperature decoherence of electrons in disordered conductors. von Delft has re-derived our influence functional approach for interacting electrons in disordered conductors and observed that it “properly incorporates the Pauli principle” and acknowledged that we “got it completely right” which is “a significant and important achievement”. At the same time, von Delft claimed that – in comparison to our work – he has achieved “a more accurate treatment of recoil effects”. According to von Delft, “with this change” he could reproduce “in a remarkably simple way, the standard, generally accepted results for the decoherence rate”.

Marquardt et al. (MDSA) re-iterated the same conclusions. On one hand, they also admitted that von Delft (JvD) has successfully re-derived our influence functional for interacting electrons JvD et al. and demonstrated that the results of all three papers in the limit $T \to 0$ are inconsistent with simple rules of algebra.

To summarize, JvD et al. make two key statements: (i) our influence functional is correct and exact and (ii) our calculation of the corresponding path integral is not correct as it “unintentionally neglects recoil effects”.

The structure of our Comment is as follows. In Sec. I we inspect JvD’s “influence functional for decoherence” and demonstrate its inconsistency with basic principles of quantum theory. In Sec. II we explicitly analyze the main drawbacks of JvD analysis which have eventually led to the above problems. In Sec. III we briefly discuss insufficient approximations used by MDSA and conclude the discussion by demonstrating that both JvD’s and MDSA’s results for zero temperature electron decoherence rate can be discarded without any calculation just on the basis of simple rules of algebra. This part of our discussion is easily accessible also to non-experts.

I. SOME CONSEQUENCES OF JVD’S ANALYSIS

The central result of our derivations is a general expression for the conductivity of a disordered conductor in terms of the Feynman-Vernon influence functional for interacting electrons. The corresponding expressions are presented in details in our earlier publications and briefly summarized in Appendix A for convenience.
The key role in our analysis is played by the path integral

$$J = \int D^2 p \int D^2 r e^{i(S_0[p_F,r_F] - S_0[p_B,r_B])} F,$$

(1)

which defines the kernel $J$ of the evolution operator on the Keldysh contour. Here $S_0$ is the action for non-interacting electrons (A3) and

$$F = e^{-(1/\hbar)[i(S_0[p_B,r_B]+S_I[r_F,b])]}$$

(2)

is the electron influence functional. It is important that the term $S_R$ (A3) contains the electron density matrix $\rho$ which makes it necessary to keep integrals over both coordinate and momentum variables because the corresponding Hamiltonians turn out to be nonlocal. It is also important to observe that both dissipative term $S_I$ and noise term $S_I$ are purely real functionals.

In order to describe weak localization in the limit $k_Fl \gg 1$ it is sufficient to employ the standard picture of time-reversed paths and evaluate the path integral (A2) within the saddle point approximation. It turns out that the term $S_R$ vanishes for any pair of time-reversed paths while $S_I$ grows with time at all temperatures including $T = 0$ and determines the Cooperon decay at sufficiently long times. One can also evaluate the contribution of fluctuations around time-reversed paths which turns out to be significant only at short times, i.e. in the perturbative (in the interaction) regime. The corresponding analysis was carried out in Ref. 8. Employing an exact transformation of the evolution operators we reduced the result to the form allowing to semiclassically evaluate only those path integrals which do not contain the electron density matrix $\rho$ while the latter was kept in its exact form in the corresponding matrix elements which cannot grow with time. This analysis was performed within the accuracy of the definition of the weak localization correction to conductivity and fully confirmed our earlier results and conclusions.

Unfortunately this latter development was completely ignored by JvD who attempted to treat the path integral yet “somewhat more accurately” than we do. JvD’s key idea is to effectively cut out the momentum variables in Eq. (A2) which would yield the “position-only” representation of the influence functional. Instead of (A2) JvD obtained:

$$\tilde{F}_c = e^{-\frac{1}{\hbar}[iS_0[r_F,r_B] + S_I[r_F,r_B]]},$$

(3)

where $\tilde{S}_R^c$ is defined in Eq. (A10).

Below we will demonstrate that the functional (4) is entirely different from the correct one (2) and it cannot be obtained from the latter by performing momentum integrations or by any other correct means.

A quick glance already at the first JvD’s formula (Eq. (1a,b) in Ref. 1) demonstrates that this and subsequent equations violate the energy-time uncertainty relation. Indeed, on p. 5 of Ref. 1 JvD writes that the double path integral entering his Eq. (1b) “gives the amplitude for an electron with energy $\hbar \omega$ to propagate from $r_2$ at time $-\tau/2$ to $r_1$ at $\tau/2$ times the amplitude for it to propagate from $r_1$, at time $\tau/2$ to $r_2$ at $-\tau/2$.” Thus, in JvD’s Eq. (1) and his subsequent analysis both electron energy and time are simultaneously fixed. This is a clear violation of the quantum mechanical energy-time uncertainty relation. No such amplitude can be defined in quantum mechanics.

Let us further examine the “influence functional” defined in Eqs. (2-4) of Ref. 1. It is easy to observe that, while our action $S_R$ (A3) is purely real, JvD’s action $\tilde{S}_R$ already contains an imaginary part. Since this part effectively adds up to the noise term $S_I$ it would imply that the equilibrium Nyquist noise spectrum “felt” by electrons in a disordered conductor would be proportional to the combination $\coth \frac{\omega}{\hbar} + \frac{1}{\hbar}$ which is not to the commonly accepted $\coth \frac{\omega}{\hbar}$ and not to the commonly accepted $\coth \frac{\omega}{2\hbar}$ which also follows from our expression for $S_I$ (A3) and (A6). Already this observation demonstrates that JvD’s “influence functional” is fundamentally different from ours. While the latter takes full account of both classical and quantum noise, JvD’s expressions (A10-A12) would imply “cutting out” the quantum noise for small values of $\epsilon$. As a result, according to JvD in the limit $\epsilon = 0$ and $T \to 0$ electrons would “feel” no noise at all. In this limit JvD’s $\tilde{S}_R$ exactly cancels $S_I$, i.e. the “influence functional” $\tilde{F}_c = 0$ (4) is identically equal to unity for all electron paths. In other words, JvD’s analysis predicts that electron-electron interactions would have no influence at all on the dc conductivity of a disordered conductor at $T = 0$. This is obviously not the case for our influence functional (A3).

Our next observation is that JvD’s “influence functional” violates causality. This fact becomes obvious by inspection of analytic properties of the Fourier-transformed JvD’s function $\tilde{F}_c$ which has poles both in the upper and lower half-planes of the $\omega$-variable, hence, implying that the electron motion should be affected by photons coming both from the past and from the future, an obvious nonsense.

In contrast, any correct expression for the kernel $R(\omega,q)$ should have poles only in the lower half-plane of $\omega$, which is the case for our expression (A6). Applying the least action conditions to our action

$$\delta(S_0[p_F,r_F] - S_0[p_B,r_B]) = 0,$$

(5)

one recovers the standard classical equation of motion for a high energy electron in a dissipative environment

$$m\ddot{r} + \nabla U(r) + e^2 \int_{-\infty}^{t} dt' \nabla R(t-t',r(t) - r(t')) = 0,$$

(6)
where the time integral in a dissipative term extends from $-\infty$ to $t$ in full agreement with the causality principle. In contrast, the same equation derived from JvD’s action \[ \text{A9} \] \[ \text{A10} \] would take the form

$$m \ddot{r} + \nabla U(r) + e^2 \int_{-\infty}^{+\infty} dt' \nabla \hat{R}_t^{\text{Re}}(t-t', r(t)-r(t')) = 0,$$

where the time integral already runs over all times between $-\infty$ and $+\infty$ in a direct conflict with the requirement of causality. In addition, instead of the correct kernel $R(t, r)$ JvD’s Eq. \[ \text{A6} \] contains $\hat{R}_t^{\text{Re}}(t, r)$ \[ \text{A11} \] in a clear contradiction to the well established results. Thus, JvD’s “influence functional” fails already on a classical level being unable to correctly describe particle’s dynamics in a dissipative environment.

In addition to the above problems JvD’s “influence functional” also violates fluctuation-dissipation theorem and detailed balance. The easiest way to observe this violation is to consider the values of $\epsilon$ large compared to temperature $T$. In that case the Pauli principle should not be important and the term $S_R^{\text{Re}}$ should approach the action $S_R$ evaluated without the Pauli principle, e.g. for a muon moving in a metal. This situation was considered, e.g., in Ref. \[ \text{A12} \] and the corresponding expression for $S_R$ obtained there coincides with \[ \text{A11} \] where one should now set $\rho$ equal to zero. In this case FDT is represented by the following relation

$$I(\omega, q) = -\coth \frac{\omega}{2T} \text{Im} R(\omega, q).$$

While FDT \[ \text{A5} \] is manifestly satisfied for our influence functional, JvD’s kernel $\tilde{R}_t(\omega, q)$ \[ \text{A11} \] \[ \text{A12} \] being substituted into Eq. \[ \text{A5} \] instead of $R$, obviously violates this relation. As a direct consequence, JvD’s “influence functional” also violates detailed balance.

Concluding this section, we have demonstrated that there exists no environment in Nature which could be described by the “influence functional” defined in Eqs. (2-4) of Ref. \[ \text{A1} \]. Hence, this “influence functional” is unsuitable for calculations of any physical quantity including, of course, the electron decoherence rate.

II. MAIN PROBLEMS OF JVD’S ANALYSIS

We now turn to the main problems of JvD’s calculation which have eventually led to alarming consequences outlined above.

As it was already emphasized, the path integral over momentum variables (Eqs. \[ \text{A2} \], \[ \text{A3} \], \[ \text{A4} \]) is strongly non-Gaussian, and, hence, cannot be exactly evaluated. JvD does not even attempt to directly perform the momentum integrals. Instead, he carries out a set of manipulations and formulates “a rule of thumb”, Eq. (B.91). Unfortunately this central part for the whole analysis (pp. 54-55 of the paper) is by far less detailed than the rest of the paper and, in fact, contains almost no equal signs (substituted in eqs. (B.91-B.93) by the sign “→”).

What was actually done by JvD with our influence functional? JvD uses it only at the very first step expanding $F$ in Eqs. \[ \text{A12} \] in powers of $iS_R + S_I$. In this way JvD reproduces all Keldysh diagrams for the problem in question. This step is correct and – in full agreement with our statement \[ \text{A5-8} \] – demonstrates that no diagrams are missing within our influence functional approach. From this point on JvD performs his own analysis of the diagrammatic series. This analysis has no direct connection to any path integral calculation, since all Keldysh diagrams can, of course, be also recovered without path integrals. JvD’s key steps are as follows:

1. Rewriting the Keldysh Green function for electrons as

$$G^{K}(E - \omega) = \left[G^{R}(E - \omega) - G^{A}(E - \omega)\right] \tanh \frac{E - \omega}{2T},$$

JvD neglects $G^{A}$ on the forward branch of the Keldysh contour, i.e. he makes the replacement

$$G^{K}(E - \omega) \rightarrow G^{R}(E - \omega) \tanh \frac{E - \omega}{2T},$$

see his Eq. (B.93). Similarly, on the backward branch of the Keldysh contour JvD replaces

$$G^{K}(E - \omega) \rightarrow -G^{A}(E - \omega) \tanh \frac{E - \omega}{2T}.$$

This approximation is equivalent to simply dropping certain classes of diagrams. Already at this point JvD violates causality for electrons.

2. JvD splits the Pauli factor $\tanh \frac{E - \omega}{2T}$ from $G^{K}$ (or, better to say, from the remaining part of $G^{K}$ containing $G^{R}$ or $G^{A}$ only) and transfers it to the photon propagators $L^{R}$ on the forward or $L^{A}$ on the backward branches of the Keldysh contour, i.e. he makes the following replacement

$$G^{K}(E - \omega) L^{A,B}(\omega) \rightarrow \pm G^{R,A}(E - \omega) \tanh \frac{\epsilon - \omega}{2T} L^{R,A}(\omega).$$

Note, that the energy $\epsilon$ under tanh is now different from $E$. According to JvD, $\epsilon$ is set to be constant which is not sensitive to the pre-history. At this stage JvD violates the energy-time uncertainty relation as well as causality for photons. This step in combination with the previous one is equivalent to JvD’s “rule of thumb”, Eq. (B.91), applied to the first order diagrams.

3. JvD spreads his “rule of thumb” to all orders of the perturbation theory. For that purpose in addition to steps (1) and (2) JvD substitutes

$$\tanh \frac{E - \omega_1 - \omega_2 - \ldots - \omega_n}{2T} \rightarrow \tanh \frac{\epsilon - \omega_1}{2T}$$

in all the diagrams whenever more than one photon frequency under tanh is encountered. As a result, all the complicated Pauli factors are reduced.
JvD’s “rule of thumb” is also illustrated in Fig. 1.

FIG. 1: JvD’s “rule of thumb” illustrated for one of the third order diagrams. All three steps (1) to (3) are evident. Notice that the Golden rule combination coth $\omega_\Omega + \tanh \omega_\Omega$ appears only after JvD’s “rule of thumb” is applied, while the initial exact expression for this diagram is by far more complicated and it does not contain this simple combination of coth and tanh.

Summarizing, the above set of steps is equivalent to (a) dropping certain classes of diagrams and (b) replacing infinitely many remaining diagrams by completely different ones. JvD’s “rule of thumb” is also illustrated in Fig. 1.

Unfortunately no serious justification for these manipulations was offered by JvD. For instance, according to JvD’s Eq. (1) some contributions to the Cooperon self-energy corresponding to the so-called Hijami box should vanish after impurity averaging in the first order in the limit of zero frequency and wave vectors. For some unclear reasons JvD believes that this observation should be sufficient in order to perform his step (1), i.e. to disregard terms with $G^A$ and $G^R$, already before impurity averaging and in all orders of the perturbation theory.

As for the step (2), according to JvD it is possible to neglect all energy fluctuations “if one so chooses”. This approximation “is expected to work well if the relevant physics is dominated by low frequencies”. This argument is logically inconsistent, since in the very beginning it already assumes what one needs to prove in the end. In practice, as it is argued by JvD in Sec. B6.2, one should disregard “accumulation of energy changes” in all vertex diagrams in all orders and also in the self-energy diagrams in the second and all higher orders in the interaction. Our analysis, in contrast, demonstrates that this “energy accumulation” is important and cannot be disregarded.

The combination of JvD’s steps (1) and (2) makes it impossible to fully reconstruct all contributions to the Cooperon already in the first order in the interaction. For instance, the first order non-Golden-rule terms defined, e.g., in Eqs. (70) of Ref. [5] are completely missing in JvD’s Eqs. (1-4) and his subsequent analysis. The time dependence encoded in these non-Golden-rule terms is indeed slower than linear. Nevertheless, these first order terms are important at $T \to 0$ since in 1d and 2d cases they actually diverge in the long time limit and, hence, also contribute to decoherence, see Sec. IVB of Ref. [5].

JvD’s step (3) introduces yet one more uncontrolled approximation by neglecting all but one photon frequencies under each tanh. To support this step JvD argues that all these frequencies are smaller than $T$. It is worth stressing that in the exact diagrammatic expansion the photon frequencies are not restricted by temperature. The contribution of high frequencies is important and may even lead to ultra-violet divergencies. Only after the steps (1) and (2) such divergencies could disappear. However, even if one adopts these steps, it would still be inconsistent to disregard all but one photon frequency. Since all these frequencies can be of the same order, it is simply a matter of consistency of any approximation to either keep or disregard them all. In either case it would then be impossible to recover the desired combination $\coth \omega + \tanh \omega$. Perhaps, this observation might suggest a clue why only one photon frequency was kept by JvD under each tanh.

The net result of all these manipulations is nothing but simple exponentiation of the first order self-energy diagrams evaluated within the approximation which effectively ignores all contributions not containing the combination $\coth \omega + \tanh \omega$. For the latter reason already the first order terms of the perturbation series cannot be fully recovered from JvD’s Eqs. (1-4). However, even if all the first order terms would be correctly reproduced by JvD, it would still be completely useless to simply exponentiate the first order result. The problem in question...
is essentially non-perturbative and, hence, contributions of all orders should be fully included.\textsuperscript{14,5,6,8}

After all these manipulations with diagrams JvD goes back to path integrals and constructs the functional \( \Pi \) which, being expanded in the interaction, should generate JvD’s diagrams obtained after his steps (1) to (3). Obviously, this functional has nothing to do with our influence functional \( \mathcal{I} \) no matter whether one carries out momentum integrations or not. In fact, JvD himself acknowledges that “an accurate treatment of effects occurring in second or higher order is beyond the accuracy of” his “influence functional approach”. In contrast to \( \mathcal{I} \), our influence functional \( \Pi \) – as was also confirmed by JvD – includes all RPA diagrams in all orders and, hence, is suitable for non-perturbative calculations. Thus, the crucial difference between \( \Pi \) and \( \mathcal{I} \) becomes evident already by careful reading of last paragraphs of Sec. B6.2 of Ref. 1.

Unfortunately in other parts of that paper the wording is sometimes not so clear. For instance, in Sec. 3 on p. 9 JvD first acknowledges that “we got it completely right in the position-time representation” and then continues “unfortunately, however, it did not occur to them to use the frequency representation (4e)”. Having read these statements, the reader could easily conclude that JvD only passes to a different representation of the same influence functional. In reality, however, at this point JvD replaces our correct influence functional \( \Pi \) by a very different object \( \Pi \) which properties have already been discussed in Sec. I of this Comment.

In Sec. 4 of Ref. 1 JvD incorrectly ascribes to us certain manipulations with his functional \( \Pi \) which we have never performed and could never perform simply because we do not have such a functional at all in any of our papers. The procedure discussed in Sec. 4 of Ref. 1 has, therefore, nothing to do with our saddle point analysis\textsuperscript{15,6,8} and JvD’s claim that this analysis “neglects recoil effects” is highly misleading. Obviously, all recoil effects are fully included in our path integral \( \Pi \). Evaluation of this non-Gaussian path integral within the saddle point approximation is a legitimate and standard mathematical procedure. It is certainly correct within its applicability range and by no means it implies neglecting recoil effects, as it was incorrectly alleged by JvD.

### III. FURTHER REMARKS AND ALGEBRAIC CONSIDERATIONS

For completeness, let us briefly address MDSA papers\textsuperscript{12,3}. The analysis in Sec. VII of Ref. 2 is essentially identical to that of Ref. 1 and, hence, all our critique of the latter paper equally applies to MDSA’s work\textsuperscript{2}. In particular, the action defined in Eqs. (93)-(94) of Ref. 2 as well as MDSA’s “plausibility arguments” also violate fundamental principles of quantum theory as already discussed in Sec. I of this Comment. MDSA’s statement that our effective action \( iS + S_I \) “is essentially the same as” that defined in their Eqs. (93)-(94) is explicitly incorrect, as it was already argued above. These two actions are so different, that they even yield different classical equations of motion, respectively Eqs. (6) and (7).

The MDSA’s discussion of both recoil effects and the Pauli principle is again highly misleading as it is based exclusively on the Golden rule approximation and clearly contradicts the energy-time uncertainty relation. Since MDSA use the Pauli principle only as the energy constraint and argue that electrons lose coherence “by spontaneous emission”, their line of reasoning is not at all specific to fermions and can equally be applied, e.g., to a quantum particle interacting with Caldeira-Leggett environment of harmonic oscillators. Unfortunately MDSA avoid even mentioning about the exact solution for the latter model which can be considered as a primer on zero temperature quantum decoherence by interactions as well as a demonstration of insufficiency of any Golden-rule-type approximation for the problem in question.

As for Ref. 3 its main drawback lies in the assumption of purely exponential decay of the Cooperon at all times adopted by the authors. With this assumption any time dependence slower than exponential would simply be excluded from the analysis from the very beginning. The only contribution which could be captured under this assumption is again the Golden rule combination coth \( \frac{\omega}{2T} \) + tanh \( \frac{\omega}{2T} \). In essence, all three approaches\textsuperscript{1,2,3} do not go beyond exponentiating the first order Golden rule terms of the perturbation theory. Therefore, it is not at all surprising that all three approaches yield the same incorrect results for the electron decoherence rate at \( T = 0 \).

Despite all our critique the works\textsuperscript{1,2,3} have at least one important merit. Namely, these authors have re-derived our influence functional and acknowledged that our expressions \( \mathcal{I}^{R}, \mathcal{A}^{R} \) are exact. This observation is crucial as it leaves practically no room for further discussions. Moreover, it makes it easy also for non-experts to judge which conclusion is correct without even looking into complicated diagrams and path integrals. It is actually sufficient to observe – as JvD and MDSA do – that both \( S_R \) and \( S_I \) are purely real.

Consider two sums over \( N \) realizations:

\[
A = \frac{1}{N} \sum_{n=1}^{N} e^{-i\alpha_n - b_n}, \quad B = \frac{1}{N} \sum_{n=1}^{N} e^{i\alpha_n - i\alpha_n - b_n},
\]

where \( \alpha_n, b_n \) and \( \alpha_n \) are all real numbers. Obviously, \( B \) is transformed into \( A \) by a trivial shift \( \alpha_n \rightarrow \alpha_n + \alpha_n \). Provided all \( b_n \) are much larger than one, \( b_n \gg 1 \), both \( A \) and \( B \) are exponentially small,

\[
A \sim B \sim \exp(-b), \quad b = \min_n b_n,
\]

no matter what the values \( \alpha_n \) and \( \alpha_n \) are. The same is, of course, true for our path integral, one should only replace realizations by trajectories and numbers by functionals \( \alpha_n \rightarrow S_R, b_n \rightarrow S_I \).
Regrettably, these trivial algebraic considerations were not respected by JvD. On p. 50 (very end of Sec. B5.8) he argues that “this general argument would work if the measure used in the path integral were real, however, it does not apply to the present case”... “where the measure $e^{i\alpha n S}$ is complex”. In other words, JvD agrees that in the sum $A$ an imaginary part $i\alpha n$ has no chance to cancel a real one $b_n$, but according to him — this cancellation can happen in the sum $B$ where a complex measure $e^{i\alpha n}$ is added. JvD concludes “Indeed, it is shown in the main text”... “that contributions from $iS_R$ and $S_I$ do partially cancel each other”.

Unfortunately it did not occur to JvD to make a shift $a_n \to a_n + \alpha n$ in order to observe the full equivalence of $A$ and $B$. Instead, he expands the exponent in the sum in $B$ to the first order in $i\alpha n$ and $b_n$, observes that both contributions $i \sum_n e^{i\alpha n} a_n$ and $\sum_n e^{i\alpha n} b_n$ may have real parts which can (partially or exactly) cancel each other and then re-exponentiates the result, i.e. writes

\[ \hat{B} \sim \exp \left( -\frac{1}{N} \sum_{n=1}^{N} e^{i\alpha n} (ia_n + b_n) \right). \] (10)

In this way JvD erroneously arrives at the “cancellation” in the exponent. There is, of course, no need to analyze infinite series of complicated diagrams in order to see that the quantity $\hat{B}$ has in general nothing to do with $B$, except in the perturbative limit $\alpha n \ll 1$ and $b_n \ll 1$.

Finally, let us emphasize yet another point which we have already discussed in Ref. 4.5.6. Very generally, evaluating a non-Gaussian path integral around certain saddle point paths, at $T \to 0$ one arrives at an effective action

\[ S_{\text{eff}}(t) = S^{(1)}(t) - \hbar \ln[A(h, t)], \] (11)

where $S^{(1)}(t)$ is the classical ($h$-independent) action on the relevant saddle point paths and $\hbar \ln[A(h)]$ represents the quantum correction ($A$ being the pre-exponent). Obviously, this quantum correction can only be important if $S^{(1)}(t) \lesssim \hbar$, i.e. in the perturbative (short time) limit $t \ll \tau_p$, in which case partial cancellation of the term $S^{(1)}(t)$ by the term $\hbar \ln[A(h)]$ is, of course, possible. However, for $S^{(1)}(t) \gg \hbar$ (i.e. for $t \gg \tau_p$) there is no way to cancel the classical action $S^{(1)}(t)$ by the quantum correction which formally tends to zero for $\hbar \to 0$. Hence, in order to determine the scale $\tau_p$, it is absolutely sufficient to evaluate the action $-iS_R - S_I$ on pairs of time-reversed saddle point paths, find the classical ($h$-independent) action $S^{(1)}(t)$ and obtain the decoherence time $\tau_p$ from the condition $S^{(1)}(\tau_p) \sim \hbar$. This is exactly what was done in our papers 4,5,6.

In conclusion, we have demonstrated that JvD’s and MDSA’s analysis 2,4,6 fails to correctly evaluate the low temperature decoherence rate for electrons in disordered conductors.

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APPENDIX A

For reference purposes we briefly recapitulate our path integral representation for the conductance $\sigma$ of a disordered conductor in the presence of electron-electron interactions. One finds

\[ \sigma = \frac{e^2}{3m} \int_{-\infty}^{t} dt' \int dF_{t} dr_{t} (\nabla_{r_{t}'} - \nabla_{r_{t}}) \bigg|_{r_{t}' = r_{t}} \]

\[ J(t, t'; r_{t}', r_{t}; r_{t}', r_{t}') (r_{t}' - r_{t}) \rho (r_{t}', r_{t}) \] (A1)

Here $\rho (r_{t}', r_{t})$ is the equilibrium single-electron density matrix, the function $J$ is given by the path integral over the coordinates and momenta on the forward $(r_{t}, p_{t})$ and backward $(r_{t}', p_{t})$ branches of Keldysh contour,

\[ J = \int J_{r_{t}'} = r_{t} dr_{t} \int \mathcal{D}r_{t} \int \mathcal{D}r_{t'} \int \mathcal{D}p_{t} \int \mathcal{D}p_{t'} \]

\[ \times \exp \{ iS_{0} (p_{t}, r_{t}) - iS_{0} (p_{t}', r_{t}') \} \]

\[ \times \exp \{ -iS_{R} (p_{t}, r_{t}; p_{t}', r_{t}') - S_{I} (r_{t}, r_{t}) \}. \] (A2)

The actions are defined as follows

\[ S_{0} [p, r] = \int_{t'}^{t} dt'' \left( p^{2} - \frac{p'^{2}}{2m} - U(r) \right), \] (A3)

\[ S_{R} = \frac{e^{2}}{2} \int_{t'}^{t} dt_{1} \int_{t'}^{t} dt_{2} \]

\[ \times \left\{ R(t_{1} - t_{2}, r_{t}(t_{1}) - r_{t}(t_{2}))(1 - 2\rho (p_{t}, r_{t})) \right. \]

\[ - R(t_{1} - t_{2}, r_{t}(t_{1}) - r_{t}(t_{2}))(1 - 2\rho (p_{t}', r_{t}')) \]

\[ + R(t_{1} - t_{2}, r_{t}(t_{1}) - r_{t}(t_{2}))(1 - 2\rho (p_{t}', r_{t}')) \]

\[ - R(t_{1} - t_{2}, r_{t}(t_{1}) - r_{t}(t_{2}))(1 - 2\rho (p_{t}, r_{t}')) \} \right\}, \] (A4)

\[ S_{I} = \frac{e^{2}}{2} \int_{t'}^{t} dt_{1} \int_{t'}^{t} dt_{2} \left\{ I(t_{1} - t_{2}, r_{t}(t_{1}) - r_{t}(t_{2})) \right. \]

\[ + I(t_{1} - t_{2}, r_{t}(t_{1}) - r_{t}(t_{2})) \]

\[ - I(t_{1} - t_{2}, r_{t}(t_{1}) - r_{t}(t_{2})) \]

\[ - I(t_{1} - t_{2}, r_{t}(t_{1}) - r_{t}(t_{2})) \} \right\}. \] (A5)

Here $U(r)$ is the impurity potential and $\rho (p, r)$ is the electron density matrix. With sufficient accuracy one can set $1 - 2\rho (p, r) \approx \tanh \left( \frac{[p'^{2}/2m + U(r) - \mu]}{2T} \right)$. The functions $R(t, r)$ and $I(t, r)$ read

\[ R(t, r) = \int \frac{d\omega dq}{(2\pi)^{4}} \frac{4\pi}{q^{2} \epsilon (\omega, q)} e^{-i\omega t + iq r}, \] (A6)

\[ I(t, r) = \int \frac{d\omega dq}{(2\pi)^{4}} \Im \left( \frac{-4\pi}{q^{2} \epsilon (\omega, q)} \right) \coth \left( \frac{\omega}{2T} \right) e^{-i\omega t + iq r}, \]
where $\epsilon(\omega, q)$ is the dielectric function

$$
\epsilon(\omega, q) = 1 + \frac{4\pi \sigma_D}{i\omega + Dq^2}, \quad (A7)
$$

$\sigma_D$ is the Drude conductivity and $D$ is the diffusion coefficient.

For comparison, we also present the results obtained by JvD, see Eqs. (1–4) of that paper. According to JvD the conductivity should read

$$
\sigma = \frac{2e^2}{3m^2} \int dx \left( \nabla r_1 - \nabla r_{1'} \right) \bigg|_{r_1=r_{1'}} \times \left( \nabla r_2 - \nabla r_{2'} \right) \bigg|_{r_2=r_{2'}} \frac{d\epsilon}{4T \cosh^2 \epsilon/2T} \times \int_0^\infty dt \dot{p}^\epsilon(t; r_1, r_2; r_{1'}, r_{2'}), \quad (A8)
$$

where

$$
P^\epsilon = \int_{r_2(-\tau/2)=r_{2'}} d\mathbf{r}_F \int_{r_2(\tau/2)=r_{1'}} d\mathbf{r}_B \times \exp \left\{ i\tilde{S}_0[\mathbf{r}_F] - i\tilde{S}_0[\mathbf{r}_B] - i\tilde{S}_R[\mathbf{r}_F, \mathbf{r}_B] - S_1[\mathbf{r}_F, \mathbf{r}_B] \right\}, \quad (A9)
$$

$$
\tilde{S}_0[\mathbf{r}] = \int_{-\tau/2}^{\tau/2} dt'' \left( m\mathbf{r}^2/2 - U(\mathbf{r}) \right) \text{ and }
\tilde{S}_R[\mathbf{r}] = \frac{e^2}{2} \int_0^\tau dt \int dt' \left\{ \dot{R}_e(t_1 - t_2, \mathbf{r}_F(t_1) - \mathbf{r}_F(t_2)) \right.
+ \dot{R}_e(t_1 - t_2, \mathbf{r}_B(t_1) - \mathbf{r}_B(t_2))
- \dot{R}_e(t_1 - t_2, \mathbf{r}_F(t_1) - \mathbf{r}_B(t_2))
- \dot{R}_e(t_1 - t_2, \mathbf{r}_B(t_1) - \mathbf{r}_F(t_2)) \right\}, \quad (A10)
$$

where $\dot{R}_e(t, \mathbf{r}) = \dot{R}_e^B(t, \mathbf{r}) + i \dot{R}_e^I(t, \mathbf{r})$,

$$
\dot{R}_e^B(t, \mathbf{r}) = \frac{1}{2} \int \frac{d\omega d\mathbf{q}}{(2\pi)^4} \text{ Re } \left[ R(\omega, \mathbf{q}) \right] e^{-i\omega t + i\mathbf{q}\mathbf{r}} \quad (A11)
\times \left[ \tanh \frac{\epsilon - \omega}{2T} + \tanh \frac{\epsilon + \omega}{2T} \right],
\dot{R}_e^I(t, \mathbf{r}) = \frac{1}{2} \int \frac{d\omega d\mathbf{q}}{(2\pi)^4} \text{ Im } \left[ R(\omega, \mathbf{q}) \right] e^{-i\omega t + i\mathbf{q}\mathbf{r}} \quad (A12)
\times \left[ \tanh \frac{\epsilon - \omega}{2T} - \tanh \frac{\epsilon + \omega}{2T} \right],
$$

and $R(\omega, \mathbf{q}) = \int d\mathbf{r} R(\mathbf{r}, t) e^{i\omega t - i\mathbf{q}\mathbf{r}}$.

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1. J. von Delft, cond-mat/0510563
2. F. Marquardt, J. von Delft, R.A. Smith, V. Ambegaokar, cond-mat/0510566
3. J. von Delft, F. Marquardt, R.A. Smith, V. Ambegaokar, cond-mat/0510557
4. D.S. Golubev and A.D. Zaikin, Phys. Rev. B 59, 9195 (1999)
5. D.S. Golubev and A.D. Zaikin, Phys. Rev. B 62, 14061 (2000).
6. D.S. Golubev, A.D. Zaikin, and G. Schön, J. Low. Temp. Phys. 126, 1355 (2002).
7. Our derivation is indeed a formally exact procedure, as correctly stated by MDSA. At the same time the final expression for the influence functional is tractable only within RPA.
8. D.S. Golubev and A.D. Zaikin, J. Low. Temp. Phys. 132, 11 (2003).
9. K.A. Eriksen, P. Hedegard, and H. Bruus, Phys. Rev. B 64, 195327 (2001).
10. See, e.g., R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals (McGraw Hill, NY, 1965).
11. This property is general for the models in question and has nothing to do with the presence or absence of $\rho$ in $S_R$.
12. See, e.g., L.D. Landau and E.M. Lifshits, Electrodynamics of continuous media. (Pergamon Press, Oxford, 1960).
13. D.S. Golubev, G. Schön, and A.D. Zaikin, J. Phys. Soc. Jap. 72, Suppl. A, 30 (2003).
14. Let us remind the reader that the importance of Hikami boxes for the problem in question was repeatedly emphasized by other authors who made this point central for their critique of our calculation. Now JvD, in contrast, argues that Hikami boxes can be safely neglected in the first order. Here we do not need to elaborate more on this controversy between JvD and the authors simply because the first order terms, even if fully and correctly recovered, are still insufficient to draw any conclusion about the electron decoherence time $\tau_\epsilon$ which can only be evaluated non-perturbatively, see, e.g. Refs. 15-17 for further discussion.
15. I.L. Aleiner, B.L. Altshuler, and M.E. Gershenson, Waves Random Media 9, 201 (1999); I.L. Aleiner, B.L. Altshuler, and M.G. Vavilov, J. Low. Temp. Phys. 126, 1377 (2002). JvD is trying to defend the validity of the Golden rule approximation at $T = 0$ and to counter our critique in his Appendices A (pp. 17, 19) and C3 (p. 63) arguing that by making a better approximation for the initial electron density matrix “all time integrals”... “can be extended to range over the entire real axis”. As a result, according to JvD, only “simple convolution integrals” are encountered and “oscillating cos-terms” should be absent. Unfortunately, these JvD’s arguments are in error. In Ref. 18 we have retained the electron density matrix in its exact form and have demonstrated that in the relevant physical limit $k_F l \gg 1$ it can be factorized from the evolution operators describing the electron time dynamics. Hence, the non-Golden-rule terms survive already in the first order no matter whether one neglects the effect of interactions on the initial density matrix or not.
16. JvD seems to fully agree with this statement. On p. 56 (Sec. B6.2) of Ref. 16 he himself points out that his approach “is equivalent to reexponentiating the first order term in the expansion of the Cooperon in powers of the interaction propagator”.
17. JvD seems to fully disagree with this statement. On p. 56 (Sec. B6.2) of Ref. 16 he himself points out that his approach “is equivalent to reexponentiating the first order term in the expansion of the Cooperon in powers of the interaction propagator”. 