On rooted cluster morphisms and cluster structures in 2-Calabi-Yau triangulated categories

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Abstract

We study rooted cluster algebras and rooted cluster morphisms which were introduced in [ADS14] recently and cluster structures in 2-Calabi-Yau triangulated categories. An example of rooted cluster morphism which is not ideal is given, this clarifies a doubt in [ADS14]. We introduce the notion of frozenization of a seed and show that an injective rooted cluster morphism always arises from a frozenization and a subseed. Moreover, it is a section if and only if it arises from a subseed. This answers the Problem 7.7 in [ADS14]. We prove that an inducible rooted cluster morphism is ideal if and only if it can be decomposed as a surjective rooted cluster morphism and an injective rooted cluster morphism. We also introduce the tensor decompositions of a rooted cluster algebra and of a rooted cluster morphism. For rooted cluster algebras arising from a 2-Calabi-Yau triangulated category C with cluster tilting objects, we give an one-to-one correspondence between certain pairs of their rooted cluster subalgebras which we call complete pairs (see Definition 2.32 for precise meaning) and cotorsion pairs in C.

Key words. Rooted cluster algebra; (Ideal) Rooted cluster morphism; Rooted cluster subalgebra; Cotorsion pair; Cluster structure.

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1 Introduction

Cluster algebras were introduced by Sergey Fomin and Andrei Zelevinsky in [FZ02] with the aim of giving an algebraic frame for the study of the canonical bases in quantum groups and the total positivity in algebraic groups. The further developments were made in a series of papers [FZ03, BFZ05, FZ07]. It has turned out that these algebras have been linked to various areas of mathematics, for example, Poisson geometry, algebraic geometry, discrete dynamical systems, Lie theory and representation theory of finite dimensional algebras.

After a number of studies of the cluster algebras in a combinatorial framework, it seems that a construction of a categorical framework is necessary. In [ADS14], the authors introduced a category $\text{Clus}$ of rooted cluster algebras (see Definition 2.3) of geometry type with non-invertible coefficients. The only difference between a rooted cluster algebra and a cluster algebra is the emphasis of the initial seed in the rooted cluster algebra. The morphisms in this category, which were called the rooted cluster morphisms (see Definition 2.5), are ring homomorphisms which commute with the mutations of rooted cluster algebras. Those bijective morphisms, which are called cluster automorphisms, have been studied in [S10, ASS12, N13, ASS13]. It is proved that the category $\text{Clus}$ has countable coproducts but no products, and the monomorphisms coincide with the injections but not all the epimorphisms are the surjections. In this paper, we study rooted cluster algebras, rooted cluster morphisms furthermore, and rooted cluster subalgebras. In particular we answer some questions appeared in [ADS14] and give a relation between complete pairs (see Definition 2.32) of cluster subalgebras of rooted cluster algebras arising from 2-Calabi-Yau triangulated categories and cotorsion pairs in these 2-Calabi-Yau triangulated categories. We introduce the main results and the structure of the paper as follows:

We study rooted cluster morphisms and ideal rooted cluster morphisms in Section 2. In subsection 2.1 we recall the definitions of rooted cluster algebras and rooted cluster morphisms. In subsection 2.2, we consider the ideal rooted cluster morphism, which is a special rooted cluster morphism such that its image coincides with the rooted cluster algebra of its image seed (see Definition-Proposition 2.6(1)). In general a rooted cluster morphism is not ideal, we give such an example (Example 4), which clarifies a doubt in [ADS14] (compare Problem 2.12 in [ADS14]). Then we prove in Theorem 2.11 that three kinds of important rooted cluster morphisms are ideal, the first one generalizes Corollary 4.5 in [ADS14]. Further, by using some techniques introduced in the following subsections, we show that ideal rooted cluster morphisms have some nice properties. It is showed in Theorem 2.36(2) that an ideal rooted cluster morphism is a composition of a surjective rooted cluster morphism and an injective rooted cluster morphism. In Theorem 2.37 we prove that the inverse statement is also true for an inducible rooted cluster morphism (see Definition 2.8), that is, the rooted cluster morphism which is induced from a ring homomorphism between the corresponding ambient rings.

Recall that a cluster algebra of geometry type is a commutative algebra determined by a seed, more precisely, it is recursively generated by exchangeable variables and frozen variables. Because it is allowed to send an exchangeable variable to an integer and it is also allowed to send a frozen variable to an exchangeable variable or an integer, a rooted cluster morphism is quite complicated. One of the purpose of this paper is to describe these morphisms more precisely. The main idea is to decompose an algebra to several ”simplest” algebras as a tensor product and
consider morphisms on these algebras. Subsection 2.3, 2.4 and 2.5 are devoted to this purpose.

In subsection 2.3, we introduce the decompositions of an ice valued quiver and of an extended skew-symmetrizable matrix. And then we decompose a seed by decomposing the corresponding exchange matrix at the beginning of subsection 2.4. Under these decomposition, the tensor decomposition of a rooted cluster algebra is described in Theorem 2.24. Then we introduce the notion of frozenization of a seed in Definition 2.25 by freezing some initial exchangeable variables. Finally, the tensor decomposition of a rooted cluster morphism is given in Theorem 2.27 by using the frozenization and the decomposition of a rooted cluster algebra. This decomposition allows us to study an explicit rooted cluster morphism by considering some special ideal rooted cluster morphisms which determine the original explicit rooted cluster morphism (see Remark 2.29 for details).

Subsection 2.5 studies the rooted cluster subalgebras of a rooted cluster algebra. Given a seed $\Sigma$ and a frozenization $\Sigma_f$ of $\Sigma$, it is proved in Theorem 2.26 that the rooted cluster algebra $A(\Sigma_f)$ of $\Sigma_f$ is a rooted cluster subalgebra of the rooted cluster algebra $A(\Sigma)$ of $\Sigma$, that is, there is an injective rooted cluster morphism from $A(\Sigma_f)$ to $A(\Sigma)$. Example 2 shows that some special subseeds also give rooted cluster subalgebras. In fact, we prove in Theorem 2.30 that for a given rooted cluster algebra $A(\Sigma)$, each rooted cluster subalgebra $A(\Sigma')$ of $A(\Sigma)$ comes from a frozenization and a subseed, and the corresponding injection from $A(\Sigma')$ to $A(\Sigma)$ is a section if and only if the image seed $f(\Sigma')$ is a subseed of $\Sigma$. The last statement answers the Problem 7.7 in [ADS14]. We also study the rooted cluster morphisms coming from specializations. The simple specialization of a rooted cluster morphism was introduced in [ADS14], we extend it to a general specialization in Definition 2.38 and give some properties in Proposition 2.39, Proposition 2.40 and Corollary 2.41.

By using the results in Section 2, the second purpose of the paper is to establish a relation between the rooted cluster subalgebras of rooted cluster algebras arising from 2-Calabi-Yau triangulated categories with cluster tilting subcategories and the cotorsion pairs in these categories. Recall that cluster categories were firstly introduced in [BMRRT06] (see also [CCS06] for type $A_n$) as a categorification of cluster algebras. The stable module categories of the preprojective algebras of Dynkin type were considered for a similar purpose [GLS06, GLS08]. Both of these categories are 2-Calabi-Yau triangulated categories. An important class of subcategories in a 2-Calabi-Yau triangulated category is cluster tilting subcategories. The cluster structure given by the cluster tilting subcategories is defined in [BIRS09] and also be studied in [FK10]. A 2-Calabi-Yau triangulated category with a cluster structure can be viewed as a categorification of the cluster algebras of the quivers of the cluster tilting subcategories. There are many works on this topic, see survey papers and references cited there [K12] [Re10]. When a triangulated category has a cluster tilting object, one can transform the cluster structure to the cluster algebras by using the cluster map defined in [BIRS09], which is called the cluster character in [Palu08]. A cotorsion pair in a triangulated category was introduced in [IY08], see also in [KR07], and studied in [Ng10, Na11, ZZ11, HJR11, HJR12, ZZ12, ZZZ13, AN12] and many others furthermore. We recall some basic definitions and results on cotorsion pairs in subsection 3.1.

In subsection 3.2, we show an interesting phenomenon that for a functorially finite rigid subcategories $I$ in a 2-Calabi-Yau triangulated category $C$ with a cluster structure, the subfactor
triangulated category \( \perp I[1]/I \) inherits a cluster structure from \( C \).

Subsection 3.3 is devoted to study the cluster substructures in cotorsion pairs. We prove in Theorem 3.10 that in a 2-Calabi-Yau triangulated category \( C \) with a cluster structure given by its cluster tilting subcategories, if the core \( I \) of a cotorsion pair can be extended as a cluster tilting subcategory, then both the torsion subcategory and the torsionfree subcategory in the cotorsion pair have cluster substructures.

Subsection 3.4 is devoted to the correspondence between the complete pairs (see Definition 2.32) of rooted cluster subalgebras of rooted cluster algebras arising from 2-Calabi-Yau triangulated categories and the cotorsion pairs in the triangulated categories. By using the cluster map \( \varphi \) defined in [FK10], we prove in Theorem 3.14(1) that the cluster substructures in a cotorsion pair with core \( I \subseteq \mathcal{T} \) correspond to a complete pair of rooted cluster subalgebras of \( \mathcal{A}(\mathcal{T}) \) with coefficient set \( \varphi(I) \), where \( \mathcal{T} \) is a cluster tilting subcategory in \( C \). Moreover, we can glue these subalgebras together at \( \varphi(I) \) to get a rooted cluster subalgebra \( \mathcal{A}(\mathcal{T}_I) \) of \( \mathcal{A}(\mathcal{T}) \), which is the frozenization of \( \mathcal{A}(\mathcal{T}) \) at \( \varphi(I) \). We prove in Theorem 3.14(2) that any complete pair of rooted cluster subalgebras of \( \mathcal{A}(\mathcal{T}) \) with coefficient set \( \varphi(I) \) such that \( I \) is functorially finite in \( C \) comes from a cotorsion pair in \( C \) with core \( I \). The relation between \( \mathcal{A}(\mathcal{T}_I) \) and the rooted cluster algebra \( \mathcal{A}(\mathcal{T} \setminus I) \) of the cluster structure in the subfactor category \( \mathcal{A}(\mathcal{T}_I)/\mathcal{A}(\mathcal{T} \setminus I) \) is described in Theorem 3.14(4), it shows that specializing \( \mathcal{A}(\mathcal{T}_I) \) at frozen variables in \( \varphi(I) \) gives a surjective rooted cluster morphism from \( \mathcal{A}(\mathcal{T}_I) \) to \( \mathcal{A}(\mathcal{T} \setminus I) \). Finally in Corollary 3.16 by gluing indecomposable components of \( \mathcal{A}(\mathcal{T}_I) \), we classify the cotorsion pairs with core \( I \) in \( C \).

2 Rooted cluster morphisms

2.1 Preliminaries of rooted cluster morphisms

Cluster algebras were introduced in [FZ02]. For the convenience of studying morphisms between cluster algebras, in [ADS14], the authors introduced rooted cluster algebras of geometry type with non-invertible coefficients by fixing an initial seed of cluster algebras. We recall basic definitions and properties on rooted cluster algebras and rooted cluster morphisms in this subsection.

Definition 2.1. [FZ02]

1. A seed is a triple \( \Sigma = (\mathbf{e}x, \mathbf{f}x, B) \) where \( \mathbf{e}x = \{x_1, x_2, \cdots, x_n\} \) is a set with \( n \) elements and \( \mathbf{x} = \mathbf{e}x \cup \mathbf{f}x = \{x_1, x_2, \cdots, x_m\} \) is a set with \( m \) elements. Here \( m \geq n \) be positive integers or countable numbers. \( B = (b_{xy})_{\mathbf{e}x \mathbf{e}x} \in M_{\mathbf{e}x \mathbf{e}x}(\mathbb{Z}) \) is a locally finite integer matrix with a skew-symmetrizable submatrix \( \mathbf{B} \) consisting of the first \( n \) rows.

2. A seed \( \Sigma' = (\mathbf{e}x', \mathbf{f}x', B') \) is called a subseed of a given seed \( \Sigma = (\mathbf{e}x, \mathbf{f}x, B) \), if \( \mathbf{e}x' \subseteq \mathbf{e}x, \mathbf{f}x' \subseteq \mathbf{f}x \) and \( B' = B[\mathbf{e}x' \cup \mathbf{f}x'] \), where \( B[\mathbf{e}x' \cup \mathbf{f}x'] \) is a submatrix of \( B \) corresponding to the subset \( \mathbf{e}x' \cup \mathbf{f}x' \).

3. A seed \( \Sigma'^o = (\mathbf{e}x, \mathbf{f}x, -B) \) is called the opposite seed of a given seed \( \Sigma = (\mathbf{e}x, \mathbf{f}x, B) \).

The set \( \mathbf{x} \) is the cluster of \( \Sigma \). The elements in \( \mathbf{x} \) (\( \mathbf{e}x \) and \( \mathbf{f}x \) respectively) are the cluster variables (the exchangeable variables and the frozen variables respectively) of \( \Sigma \). The matrix \( B \) is called
the exchange matrix of $\Sigma$. It is an extended skew-symmetrizable integer matrix with the principal part $\hat{B}$ skew-symmetrizable. We say a seed $\Sigma = (\mathbf{e}x, \mathbf{f}x, B)$ trivial if the set $\mathbf{e}x$ is empty. We say the rational function field $\mathcal{F}_{\Sigma} = \mathbb{Q}(x_1, x_2, \cdots, x_m)$ the ambient field of $\Sigma$.

Given a seed $\Sigma$ and an exchangeable variable $x$ of $\Sigma$, we can produce a new seed by a mutation defined as follows:

**Definition 2.2.** ([FZ02], Definition 1.4). The seed $\mu_x(\Sigma) = (\mu_x(\mathbf{e}x), \mu_x(\mathbf{f}x), \mu_x(B))$ obtained by the mutation of $\Sigma$ in the direction $x$ is given by:

1. $\mu_x(\mathbf{e}x) = (\mathbf{e}x \setminus \{x\}) \sqcup \{x'\}$ where
   
   $xx' = \prod_{y \in \mathbf{e}x; b_{yx} > 0} y^{b_{yx}} + \prod_{y \in \mathbf{e}x; b_{yx} < 0} y^{-b_{yx}}.$

2. $\mu_x(\mathbf{f}x) = \mathbf{f}x.$

3. $\mu_x(B) = (b'_{yz})_{y \in \mathbf{e}x} \in M_{\mathbf{e}x}(\mathbb{Z})$ is given by
   
   $b'_{yz} = \begin{cases} 
   -b_{yz}, & \text{if } y = z; \\
   b_{yz} + \frac{1}{2}(b_{yx}b_{xz} + b_{yx}b_{xz}) & \text{otherwise}.
   \end{cases}$

Now, we recall the definition of rooted cluster algebras as follows:

**Definition 2.3.** ([ADST14], Definition 1.4). Let $\Sigma = (\mathbf{e}x, \mathbf{f}x, B)$ be a seed.

1. A sequence $(x_1, \cdots, x_l)$ is called $\Sigma$-admissible if $x_1$ is exchangeable in $\Sigma$ and $x_i$ is exchangeable in $\mu_{x_{i-1}} \circ \cdots \circ \mu_{x_1}(\Sigma)$ for every $2 \leq i \leq l$. Denote by $\text{Mut}(\Sigma) = \{\mu_{x_k} \circ \cdots \circ \mu_{x_1}(\Sigma) | n > 0 \text{ and } (x_1, \cdots, x_n) \text{ is } \Sigma}\text{-admissible}\}$ the mutation class of $\Sigma$.

2. A pair $(\Sigma, \mathcal{A})$ is called a rooted cluster algebra with initial seed $\Sigma$, where $\mathcal{A}$ is the $\mathbb{Z}$-subalgebra of $\mathcal{F}_{\Sigma}$ given by:
   
   $\mathcal{A} = \mathbb{Z}\left[ x \mid x \in \bigcup_{(\mathbf{e}x, \mathbf{f}x, B) \in \text{Mut}(\Sigma)} x \right].$

We call the clusters of seeds in $\text{Mut}(\Sigma)$ the clusters of $(\Sigma, \mathcal{A})$. The cluster variables (the exchangeable variables and the frozen variables respectively) arising in these clusters are called cluster variables (the exchangeable variables and the frozen variables respectively) of $(\Sigma, \mathcal{A})$. In particular, the cluster (cluster variable respectively) of $\Sigma$ is called the initial cluster (cluster variable respectively) of $(\Sigma, \mathcal{A})$. The frozen variables of $(\Sigma, \mathcal{A})$ are also called the coefficients of $(\Sigma, \mathcal{A})$. We denote by $\mathcal{X}_{\Sigma}$ the set of cluster variables in $(\Sigma, \mathcal{A})$. We always write $(\Sigma, \mathcal{A})$ as $\mathcal{A}(\Sigma)$ for simplicity. Note that for a trivial seed $\Sigma = (\emptyset, \mathbf{f}x, B)$, the associated rooted cluster algebra is just the polynomial ring $\mathbb{Z}[\mathbf{f}x]$. We refer the readers to [ADST14] for more examples of rooted cluster algebras. It is well known that cluster algebras have many remarkable properties, for example, the Laurent phenomenon. In fact, because the only difference with cluster algebras is the emphasis of the initial seed in rooted cluster algebras, these properties are also true for rooted cluster algebras. Thus we have the following
Theorem 2.4. ([FZ02], Theorem 3.1) ([FZ03], Proposition 11.2). Given a seed \( \Sigma = (ex, fx, B) \) with \( x = ex \sqcup fx = \{x_1, \ldots, x_n\} \sqcup \{x_{n+1}, \ldots, x_m\} \). Let \( \mathcal{L}_{\Sigma, z} := \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, x_{n+1}, \ldots, x_m] \) be the localization of polynomial ring \( \mathbb{Z}[x_1, \ldots, x_n] \) at \( x_1, \ldots, x_n \). Then

\[
\mathcal{A} \subset \bigcup_{\Sigma \in \text{Mut}(\Sigma)} \mathcal{L}_{\Sigma, z} \quad \text{and} \quad \mathcal{A}(\Sigma) \subset \bigcup_{\Sigma' \in \text{Mut}(\Sigma')} \mathcal{L}_{\Sigma', z}.
\]

Definition 2.5. ([ADS14], Definition 2.2). Let \( \Sigma = (ex, fx, B) \) and \( \Sigma' = (ex', fx', B') \) be two seeds. Denote by \( x = ex \sqcup fx \) and \( x' = ex' \sqcup fx' \). A ring homomorphism \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) is called a rooted cluster morphism if

1. (CM1) \( f(ex) \subset ex' \sqcup \mathbb{Z} \);
2. (CM2) \( f(fx) \subset x' \sqcup \mathbb{Z} \);
3. (CM3) For every \((f, \Sigma, \Sigma')\)-biadmissible sequence \((x_1, \ldots, x_i)\), we have \( f(\mu_{x_1} \circ \cdots \circ \mu_{x_i}, \Sigma(f(y))) = \mu_{f(x_1)} \circ \cdots \circ \mu_{f(x_i)}, \Sigma' \) for any \( y \) in \( x \). Here a \((f, \Sigma, \Sigma')\)-biadmissible sequence \((x_1, \ldots, x_i)\) is a \( \Sigma \)-admissible sequence such that \( (f(x_1), \ldots, f(x_i)) \) is \( \Sigma' \)-admissible.

After introduced morphisms between rooted cluster algebras in [ADS14], the authors proved that these rooted cluster morphisms consist the set of morphisms of a category \( \text{Clus} \) with the objects given by rooted cluster algebras. This category has countable coproducts but no products. And in this category, the monomorphisms coincide with the injections but not all the epimorphisms are the surjections. They also introduced ideal rooted cluster morphisms to get better understanding of rooted cluster morphisms, which we recall in the following, and will study in the next subsection.

Definition-Proposition 2.6. Given two seeds \( \Sigma = (ex, fx, B) \) and \( \Sigma' = (ex', fx', B') \) with \( x = ex' \sqcup fx \) and \( x' = ex' \sqcup fx' \). Let \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) be a rooted cluster morphism.

1. ([ADS14], Definition 2.8). The image seed of \( \Sigma \) under \( f \) is \( f(\Sigma) = (ex' \cap f(ex), x' \cap f(x) \setminus ex' \cap f(ex), B'[x' \cap f(x)]) \).
2. ([ADS14], Lemma 2.10, Definition 2.11). We have \( \mathcal{A}(f(\Sigma)) \subseteq f(\mathcal{A}(\Sigma)) \). If the inverse inclusion is valid, then \( f \) is called an ideal rooted cluster morphism.

Now we define rooted cluster subalgebras, which is a modified version and a slight generalization of subcluster algebras defined in [BIRS09] (see Remark 2.31 for concrete relation of these two definitions). We will study rooted cluster subalgebras in subsection [2.5] and rooted cluster subalgebras arising from a cotorsion pair in a 2-Calabi-Yau triangulated category in section [3].

Definition 2.7. If there is an injective rooted cluster morphism \( f \) from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma') \), then we say \( \mathcal{A}(\Sigma) \) a rooted cluster subalgebra of \( \mathcal{A}(\Sigma') \).

Example 1. Given two seeds

\[
\Sigma = \begin{pmatrix} x_1, x_2, x_3, & 0 & 1 \\ -1 & 0 & -1 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \Sigma' = \begin{pmatrix} x_1, x_2, x_3, & 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.
\]

Let \( f \) be an identity on \( \mathcal{F}_{\Sigma} = \mathcal{F}_{\Sigma'} = \mathbb{Q}(x_1, x_2, x_3) \). Then \( f \) induces a ring homomorphism from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma') \) which satisfies (CM1) and (CM2). Moreover, it is easy to see that this ring homomorphism is compatible with the mutations in \( \mathcal{A}(\Sigma) \) and \( \mathcal{A}(\Sigma') \) and thus induces a rooted cluster morphism from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma') \). It is injective because \( f \) is injective on \( \mathbb{Q}(x_1, x_2, x_3) \). Therefore \( \mathcal{A}(\Sigma) \) is a rooted cluster subalgebra of \( \mathcal{A}(\Sigma') \).
Example 2. Given a seed $\Sigma = (\text{ex}, \text{fx}, B)$, let $\Sigma' = (\text{ex}', \text{fx}', B')$ be a subseed of $\Sigma$. Then the natural injection from $\text{ex}' \cup \text{fx}'$ to $\text{ex} \cup \text{fx}$ induces an injective ring morphism $j$ from $L_{\Sigma', \Sigma}$ to $L_{\Sigma, \Sigma}$. If $j$ induces a rooted cluster morphism $f$ from $A(\Sigma')$ to $A(\Sigma)$, then $f$ is injective and $A(\Sigma')$ is a rooted cluster subalgebra of $A(\Sigma)$. We will show in Remark 2.9 when the injection $j$ induces a rooted cluster morphism $f$.

2.2 Ideal rooted cluster morphisms

This subsection is devoted to ideal rooted cluster morphisms. For a seed $\Sigma = (\text{ex}, \text{fx}, B)$ with $x = \text{ex} \cup \text{fx} = \{x_1, \ldots, x_n\} \cup \{x_{n+1}, \ldots, x_m\}$, we introduce $L_{\Sigma, Q} := Q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, x_{n+1}, \ldots, x_m]$ as the ambient ring of the rooted cluster algebra $A(\Sigma)$. Then we define the following two kinds of rooted cluster morphisms and consider the relations with ideal rooted cluster morphisms.

Definition 2.8. Given $\Sigma = (\text{ex}, \text{fx}, B)$ and $\Sigma' = (\text{ex}', \text{fx}', B')$ be two seeds with $x = \text{ex} \cup \text{fx}$ and $x' = \text{ex}' \cup \text{fx}'$. Let $f : A(\Sigma) \to A(\Sigma')$ be a rooted cluster morphism.

1. We call $f$ explicit if it is uniquely determined by the images of the initial cluster variables. More precisely, if there is a rooted cluster morphism $f'$ from $A(\Sigma)$ to $A(\Sigma')$ coinciding with $f$ on the initial cluster variables, then $f = f'$.

2. We call $f$ inducible if it can be lifted as a ring homomorphism between the corresponding ambient rings, that is, there is a ring homomorphism from $L_{\Sigma, Q}$ to $L_{\Sigma', Q}$ which induces $f$ from $A(\Sigma)$ to $A(\Sigma')$.

Remark 2.9. It is easy to see that

1. A rooted cluster morphism is inducible if and only if the image of any initial exchangeable cluster variable is not zero;

2. An inducible rooted cluster morphism is explicit.

Thus we state the following problem.

Problem 1: Whether any explicit rooted cluster morphism is inducible or not?

There exists rooted cluster morphism which is not explicit, see the following example.

Example 3. Consider the seed

$$\Sigma = \{(x_1, x_2), (x_3, x_4), \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}\}$$

and the rooted cluster algebra $A(\Sigma) = \mathbb{Z}[x_1, x_2, x_3, x_4, \frac{1+x_2}{x_1}, \frac{x_1+x_3}{x_2}, \frac{x_1+x_3+x_2+x_4}{x_2x_3}]$. Define a map

$$f : \begin{cases} x_1 & \mapsto 0 \\ x_2 & \mapsto -1 \\ x_3 & \mapsto 0 \\ x_4 & \mapsto x_1 \\ \frac{1+x_2}{x_1} & \mapsto y \\ \frac{x_1+x_3}{x_2} & \mapsto 0 \\ \frac{x_1+x_3+x_2+x_4}{x_2x_3} & \mapsto -1 \end{cases}$$
where \( y \) is any given element in \( \mathcal{A}(\Sigma) \). One can easily check that this map induces a ring homomorphism from \( \mathcal{A}(\Sigma) \) to itself. Moreover, there is no \((f, \Sigma, \Sigma')\)-biadmissible sequence, thus \( f \) is a rooted cluster morphism. Because the image of the cluster variable \( \frac{1}{x_i} \) can be chosen as any element in \( \mathcal{A}(\Sigma) \), this rooted cluster morphism is not explicit.

The following example shows that not all the rooted cluster morphisms are ideal. This clarifies a double in [ADS14].

**Example 4.** Consider the rooted cluster morphism \( f \) in Example 3. Let \( y = x_2 \), then we have \( f(\mathcal{A}(\Sigma)) = \mathbb{Z}[x_1, x_2] \) and

\[
  f(\Sigma) = (\emptyset, (x_1), [0]) .
\]

Therefore

\[
  \mathcal{A}(f(\Sigma)) = \mathbb{Z}[x_1]
\]

so that \( \mathcal{A}(f(\Sigma)) \not\subset f(\mathcal{A}(\Sigma)) \) and thus \( f \) is not ideal.

We notice that this counterexample is not an explicit rooted cluster morphism and therefore the degree of freedom in choosing the images of cluster variables is increased significantly. Thus we state the following problem which can be viewed as an improved version of Problem 2.12 in [ADS14].

**Problem 2:** Whether every explicit rooted cluster morphism is ideal or not?

Now assume that we are given a rooted cluster morphism \( f \) from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma') \) with \( \Sigma = (\text{ex}, f\text{ex}, B) \) and \( \Sigma' = (\text{ex}', f'\text{ex}', B') \). Assume that \( S_l = (x_1, \ldots, x_l) \) is a \((f, \Sigma, \Sigma')\)-biadmissible sequence, that is, \( S_l \) is a \( \Sigma \)-admissible sequence and \( f(S_l) = (f(x_1), \ldots, f(x_l)) \) is a \( \Sigma' \)-admissible sequence. Thus we obtain two seeds \( \mu_{S_l}(\Sigma) \) and \( \mu_{f(S_l)}(\Sigma') \). Note that there are isomorphisms of algebras \( \mathcal{A}(\Sigma) \cong \mathcal{A}(\mu_{S_l}(\Sigma)) \) and \( \mathcal{A}(\Sigma') \cong \mathcal{A}(\mu_{f(S_l)}(\Sigma')) \). Thus \( f \) can be viewed as a ring homomorphism from \( \mathcal{A}(\mu_{S_l}(\Sigma)) \) to \( \mathcal{A}(\mu_{f(S_l)}(\Sigma')) \), we denote it by \( f_{S_l} \).

**Lemma 2.10.**

1. \( f_{S_l} \) is a rooted cluster morphism. We call it the mutation of \( f \) along \( S_l \).

2. The seed \( f_{S_l}(\mu_{S_l}(\Sigma)) \) coincides with the seed \( \mu_{f(S_l)}(f(\Sigma)) \).

3. \( f_{S_l} \) is ideal if and only if \( f \) is ideal.

**Proof.** We only prove the case of \( l = 1 \). The general case can be proved inductively on \( l \). We write \( f_{S_{l+1}} \) as \( f' \) for simplicity.

1. For any cluster variable \( \mu_{x_1}(x) \) of \( \mu_{x_1}(\Sigma) \), where \( x \) is a cluster variable of \( \Sigma \), we have \( f(\mu_{x_1}(x)) = \mu_{f(x_1)}(f(x)) \) since \( f \) is a rooted cluster morphism. Thus if \( f(x) \) is a (exchangeable) cluster variable of \( \Sigma \), then \( f(\mu_{x_1}(x)) \) is a (exchangeable) cluster variable of \( \mu_{f(x_1)}(\Sigma') \). If \( f(x) \) is an integer, then \( f(\mu_{x_1}(x)) \) is also an integer. Therefore \( f' \) satisfies (CM1) and (CM2). We now prove (CM3). It is clearly that any \((f'\mu_{s_1}(\Sigma), f'\mu_{s_1}(\Sigma'))\)-biadmissible sequence \((x_2, \ldots, x_l)\) can be extended as a \((f, \Sigma, \Sigma')\)-biadmissible sequence \((x_1, x_2, \ldots, x_l)\). For any cluster variable \( y \) of \( \mu_{s_1}(\Sigma) \), there exists \( x \in \mathcal{X} \) such that \( y = \mu_{s_1}(x) \). Thus \( \mu_{s_1}(\Sigma) = \mu_{s_2, \ldots, s_l}(\Sigma)(y) = \mu_{s_1}(\Sigma)(x) \). On the one hand, because \( f \) is a rooted cluster morphism, we have equalities

\[
  f(\mu_{s_1}(\Sigma)(y)) = f(\mu_{s_2, \ldots, s_l}(\Sigma)(x)) = \mu_{f(s_1)}(\Sigma)(x) \circ \cdots \circ \mu_{s_1}(\Sigma)(x) .
\]

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On the other hand, we have equalities

\[ \mu_{f(x)} \circ \cdots \circ \mu_{f(x_{2})} \circ \mu_{f(x_{1}), \Sigma}(f(x)) \]
\[ = \mu_{f(x)} \circ \cdots \circ \mu_{f(x_{2}), \Sigma}(f(y)) \]
\[ = \mu_{f'(x_{1})} \circ \cdots \circ \mu_{f'(x_{2}), \Sigma}(f'(y)), \]

where the first one due to \( f(y) = f(\mu_{x_{1}}, \Sigma(x)) = \mu_{f(x_{1}), \Sigma}(f(x)) \) and the second one due to \( f = f' \) on the algebra \( \mathcal{A}(\Sigma) \equiv \mathcal{A}(\mu_{x_{1}}(\Sigma)) \). Thus by combining the above equalities, we have

\[ f'(\mu_{x_{1}} \circ \cdots \circ \mu_{x_{2}, \Sigma}(\Sigma)(y)) = f(\mu_{x_{1}} \circ \cdots \circ \mu_{x_{2}, \Sigma}(\Sigma)(y)) = \mu_{f'(x_{1})} \circ \cdots \circ \mu_{f'(x_{2}), \Sigma}(\Sigma)(f'(y)), \]

that is, (CM3) is valid.

2. It is clearly that in these two seeds, the cluster variables coincide with each other, and one can straightforward check that their exchange matrices are also the same. So we are done.

3. This easily follows from the assertion 2.

\[ \square \]

In [ADS14], the authors raised a problem that characterize rooted cluster morphisms which are ideal. We have the following answers in several important cases.

**Theorem 2.11.** Given two seeds \( \Sigma = (\mathbf{ex}, \mathbf{fx}, B) \) and \( \Sigma' = (\mathbf{ex'}, \mathbf{fx'}, B') \) with \( \mathbf{x} = \mathbf{ex} \sqcup \mathbf{fx} \) and \( \mathbf{x'} = \mathbf{ex'} \sqcup \mathbf{fx'} \). Let \( f : \mathcal{A}(\Sigma) \rightarrow \mathcal{A}(\Sigma') \) be a rooted cluster morphism. Then \( f \) is ideal if one of the following conditions is satisfied:

1. \( f(\mathbf{ex}) \subset \mathbf{ex'} \);
2. \( f \) is inducible and \( \Sigma \) is finite acyclic;
3. \( f \) is inducible and surjective.

**Proof.** 1. From the Definition-Proposition 2.6(2), it is sufficient to show that \( f \) maps the cluster variables of \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(f(\Sigma)) \). Because \( f(\mathbf{ex}) \subset \mathbf{ex'} \), each \( \Sigma \)-admissible sequence is \((f, \Sigma, \Sigma')\)-biadmissible. Thus for each cluster variable \( y = \mu_{x_{1}} \circ \cdots \circ \mu_{x_{2}} \circ \mu_{x_{3}, \Sigma}(x) \) in \( \mathcal{A}(\Sigma) \), where the sequence \( (x_{1}, \cdots, x_{i}) \) is \( \Sigma \)-admissible and \( x \in \mathbf{x} \), we have \( f(y) = \mu_{f(x_{1})} \circ \cdots \circ \mu_{f(x_{2})} \circ \mu_{f(x_{3}), \Sigma}(f(x)) \) because of (CM3). It is clearly that \( (x_{1}, \cdots, x_{i}) \) can also be viewed as a \((f, \Sigma, f(\Sigma))\)-biadmissible sequence. Now we inductively prove that

\[ \mu_{f(x_{1})} \circ \cdots \circ \mu_{f(x_{2})} \circ \mu_{f(x_{3}), \Sigma}(f(x)) = \mu_{f(x_{1})} \circ \cdots \circ \mu_{f(x_{2})} \circ \mu_{f(x_{3}), f(\Sigma)}(f(x)) \]

and then \( f(\mathbf{ex}) \) belongs to \( \mathcal{A}(f(\Sigma)) \).

For \( i = 1 \), we have \( \mu_{f(x_{1})}, \Sigma(f(x)) = f(\mu_{x_{1}}, \Sigma(x)) = f(x) \) if \( x \neq x_{1} \). Therefore \( f(x) \neq f(x_{1}) \) and \( \mu_{f(x_{1}), \Sigma}(f(x)) = f(x) = \mu_{f(x_{1}), \Sigma}(f(x)) \).

In fact, if \( f(x) = f(x_{1}) \), then \( \mu_{f(x_{1}), \Sigma}(f(x)) = \frac{m_{1}(x') + m_{2}(x')}{f(x_{1})} \), where \( m_{1}(x') \) and \( m_{2}(x') \) are monomials of \( x' \setminus f(x_{1}) \), and not equal to \( f(x) \).
since the algebraic independence of the initial cluster variables, thus a contradiction. If \( x = x_1 \), then

\[
f(\mu_{x_1}(x_1)) = f \left( \frac{1}{x_1} \left( \prod_{y \in x : b_{y_1} > 0} f(y)^{b_{y_1}} + \prod_{y \in x : b_{y_1} < 0} f(y)^{-b_{y_1}} \right) \right).
\]

Since \( f \) is inducible. On the other hand, we have equalities

\[
\mu_{f(x_1)}(f(x_1)) = \frac{1}{f(x_1)} \left( \prod_{y \in x : f(y) \in \mathbb{R}^+} b_{y_1} f(y)^{b_{y_1}} + \prod_{y \in x : f(y) \in \mathbb{R}^-} -f(y)^{-b_{y_1}} \right)
\]

and

\[
\mu_{f(x_1),f(x)}(f(x_1)) = \frac{1}{f(x_1)} \left( \prod_{y \in x : f(y) \in \mathbb{R}^+} b_{y_1} f(y)^{b_{y_1}} + \prod_{y \in x : f(y) \in \mathbb{R}^-} -f(y)^{-b_{y_1}} \right).
\]

Then from \( f(\mu_{x_1}(x_1)) = \mu_{f(x_1)}(f(x_1)) \) and the algebraic independence of the initial cluster variables, we have

\[
\prod_{y \in x : b_{y_1} > 0} f(y)^{b_{y_1}} = \prod_{z \in \mathbb{R}^+} b_{y_1} f(y)^{b_{y_1}} \quad \text{or} \quad \prod_{y \in x : b_{y_1} < 0} f(y)^{-b_{y_1}} = \prod_{z \in \mathbb{R}^-} b_{y_1} f(y)^{-b_{y_1}}
\]

No matter for which case, the above equalities show that the factors of the right hand monomials in these equalities are of the form \( f(y) \) with \( y \in x \). Thus we have equalities

\[
\prod_{y \in x : f(y) \in \mathbb{R}^+} b_{y_1} f(y)^{b_{y_1}} = \prod_{z \in \mathbb{R}^+} b_{y_1} f(y)^{b_{y_1}} \quad \text{or} \quad \prod_{y \in x : f(y) \in \mathbb{R}^-} -f(y)^{-b_{y_1}} = \prod_{z \in \mathbb{R}^-} b_{y_1} f(y)^{-b_{y_1}}
\]

respectively. Thus \( \mu_{f(x_1)}(f(x_1)) = \mu_{f(x_1),f(x)}(f(x_1)) \). Therefore for each \( x \in x \) we have

\[
\mu_{f(x_1)}(f(x)) = \mu_{f(x_1),f(x)}(f(x)).
\]

(1)
Remark 2.12. In fact, by checking the proof of Lemma 3.1 [ADS14], one can easily notice that it missed a condition that \( f \) should be inducible.

We end this subsection with the following lemma, which will be used in the proof of Theorem 2.27 in subsection 2.4.
Lemma 2.13. Let $\Sigma = (\mathbf{ex}, \mathbf{fx}, B)$ and $\Sigma' = (\mathbf{ex}', \mathbf{fx}', B')$ be two seeds, and $f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a rooted cluster morphism with $f(\mathbf{ex}) \subseteq \mathbf{ex}'$. Then $f : \mathbf{ex} \to f(\mathbf{ex})$ induces a rooted cluster isomorphism from $\mathcal{A}(\Sigma_p)$ to $\mathcal{A}(f(\Sigma)_p)$, where $\Sigma_p = (\mathbf{ex}, \emptyset, B[\mathbf{ex}])$ and $f(\Sigma)_p = (f(\mathbf{ex}), \emptyset, B'[f(\mathbf{ex})])$ are principal parts of $\Sigma$ and $f(\Sigma)$ respectively.

Proof. Due to Theorem 2.11(1), $f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ is ideal, therefore it induces a surjective rooted cluster morphism from $\mathcal{A}(\Sigma)$ to $\mathcal{A}(f(\Sigma))$. It is easy to see that if the morphism $f : \mathbf{ex} \to f(\mathbf{ex})$ is injective, then it induces a rooted cluster isomorphism from $\mathcal{A}(\Sigma_p)$ to $\mathcal{A}(f(\Sigma)_p)$. To prove this, let $x \neq y$ be two exchangeable variables of $\Sigma$, then $\mu_{f(x), \Sigma}(f(y)) = f(\mu_{x, \Sigma}(y)) = f(y)$. By the same argument used in the proof of Theorem 2.11(1), we have $f(x) \neq f(y)$. Thus $f : \mathbf{ex} \to f(\mathbf{ex})$ is injective. \hfill $\Box$

2.3 Ice valued quivers

In this subsection, we recall ice valued quivers and their relations with extended skew-symmetrizable matrices (compare [K12]). Although these contents are standard, we write them down here for the convenience of the readers. Then we decompose ice valued quivers and extended skew-symmetrizable matrices, which will be used to decompose rooted cluster algebras and rooted cluster morphisms in the next subsection.

Definition 2.14. A valued quiver is a triple $(Q, v, d)$ where

1. $Q = (Q_0, Q_1)$ is a locally finite, simple-laced quiver without loops. Denote by $Q_0 = \{1, 2, \cdots, n\}$ with $n$ be the cardinality of $Q_0$ which maybe a countable number;
2. $v : Q_1 \to \mathbb{N}^2$ is a function, which maps each arrow $\alpha$ to a non-negative number pair $(v(\alpha)_1, v(\alpha)_2)$;
3. $d : Q_0 \to \mathbb{Z}^*$ is a function such that for each vertex $i$ and each arrow $\alpha : i \to j$, we have $d(i)v(\alpha)_1 = v(\alpha)_2d(j)$.

A valued quiver $(Q, v, d)$ with $v(\alpha)_1 = v(\alpha)_2$ for each arrow $\alpha$ in $Q_1$ is called an equally valued quiver. An equally valued quiver $(Q, v, d)$ correspond to an ordinary quiver $Q'$ in the following way: $Q'$ has the vertex set $Q_0$, and there are $v(\alpha)_1 = v(\alpha)_2$ number of arrows from $i$ to $j$ for any vertices $i, j \in Q_0$ and $\alpha : i \to j$. It is easy to check that the above correspondence gives a bijection between the set of equally valued quivers and the set of ordinary quivers without loops nor 2-cycles. Given a valued quiver $(Q, v, d)$ with $Q_0 = \{1, 2, \cdots, n\}$, we associate a matrix $B = (b_{ij})_{n \times n} \in M_{n \times n}(\mathbb{Z})$ as follows:

$$b_{ij} = \begin{cases} 0 & \text{if there is no arrows between } i \text{ and } j; \\ v(\alpha)_1 & \text{if there is an arrow } \alpha : i \to j; \\ -v(\alpha)_2 & \text{if there is an arrow } \alpha : j \to i. \\ \end{cases}$$

Let $D = (d_{ij})_{n \times n} \in M_{n \times n}(\mathbb{Z})$ be the diagonal matrix with $d_{ii} = d(i), i \in Q_0$, then by Definition 2.14(3), $B$ is a skew-symmetrizable matrix in the sense of $D B$ skew-symmetric. Conversely, one can construct a valued quiver from a skew-symmetrizable matrix, we refer to [K12] for more details, and it is easy to check that these procedures give a bijection between the valued quivers with vertex set $\{1, 2, \cdots, n\}$ and the skew-symmetrizable $n \times n$ integer matrices.

Definition 2.15. An ice valued quiver $Q = (Q_0, Q_1)$ is a quiver without loops nor 2-cycles, where
1. \( Q_0 = Q_0^f \cup Q_0^e = \{1, 2, \ldots, n\} \cup \{n, n+1, \ldots, m\} \) with \( m \geq n \) be positive integers or countable numbers;

2. the full subquiver \( Q^e = (Q_0^e, Q_1^e) \) of \( Q \) is a valued quiver;

3. there are no arrows between vertices in \( Q_0^f \).

The subquiver \( Q^e \) is called the principal part of \( Q \). The vertices in \( Q_0^e \) are called the exchangeable vertices while the vertices in \( Q_0^f \) are called the frozen vertices. For the ice valued quiver \( Q = (Q_0, Q_1) \), the associated matrix \( B = (b_{ij})_{m \times n} \in M_{m \times n}(\mathbb{Z}) \) consists of the following two parts. The principal part of the intersection of the first \( n \) rows and the \( n \) columns is given by the skew-symmetrizable matrix associated with the valued quiver \( Q^e \). Given two vertices \( i \in Q_0^f \) and \( j \in Q_0^e \), the elements in the intersection of the last \( m - n \) rows and the \( n \) columns are as follows:

\[
b_{ij} = \begin{cases} 
    n_{ij} & \text{if there are } n_{ij} \text{ arrows from } i \text{ to } j ; \\
    -n_{ij} & \text{if there are } n_{ij} \text{ arrows from } j \text{ to } i .
\end{cases}
\]

It is easy to see that the above procedure is reversible and gives a bijection between the ice valued quivers with vertex set \( Q_0 = Q_0^f \cup Q_0^e = \{1, 2, \ldots, n\} \cup \{n, n+1, \ldots, m\} \) and the \( m \times n \) extended skew-symmetrizable integer matrices. Using this correspondence, we define the mutation of an ice valued quiver by the matrix mutation defined in Definition 2.2(3). Then the mutation of an ice valued quiver at an exchangeable vertex produces a new ice valued quiver which has the same exchangeable vertices and frozen vertices as the original quiver has. We now introduce the following

**Definition 2.16.** An ice valued quiver \( Q \) is called indecomposable, if it is connected and the principal part \( Q^e \) is also connected.

It is not hard to see that an ice valued quiver is indecomposable if and only if the corresponding extended skew-symmetrizable matrix is indecomposable in the following sense.

**Definition 2.17.** An extended skew-symmetrizable matrix \( B = (b_{ij})_{m \times n} \) is called indecomposable, if it satisfies the following conditions,

1. for any \( 1 \leq s \neq t \leq m \), there exists a sequence \( (i_0 = s, i_1, \ldots, i_l = t) \) with \( 1 \leq i_0, i_1, \ldots, i_l \leq m \) such that \( b_{i_j,i_{j+1}} \neq 0 \) for each \( 0 \leq j \leq l - 1 \);

2. for any \( 1 \leq s \neq t \leq n \), there exists a sequence \( (i_0 = s, i_1, \ldots, i_l = t) \) with \( 1 \leq i_0, i_1, \ldots, i_l \leq n \) such that \( b_{i_j,i_{j+1}} \neq 0 \) for any \( 0 \leq j \leq l - 1 \).

Given an ice valued quiver \( Q \), we denote by \( Q_1^f, Q_2^f, \ldots, Q_t^f \) all the connected components of \( Q^e \), where \( t \) maybe a countable number. For each component \( Q_i^f \), \( 1 \leq i \leq t \), we associate an ice valued quiver \( Q_i \) as a full subquiver of \( Q \), the exchangeable vertices are the vertices of \( Q_i^f \) while the frozen vertices are those frozen vertices of \( Q \) which connected to some vertices in \( Q_i^f \) directly. It is clearly that \( Q_i \) is indecomposable.

The above discussion gives the following proposition:

**Proposition 2.18.** We decompose an ice valued quiver \( Q \) as a collection \( \{Q_1, Q_2, \ldots, Q_t\} \) of indecomposable ice valued quivers in a unique way. These quivers are called indecomposable components of \( Q \).
We also have the following proposition as a matrix version of the above one.

**Proposition 2.19.** We decompose an extended skew-symmetrizable matrix $B$ as a collection \{\(B_1, B_2, \ldots, B_t\)\} of indecomposable extended skew-symmetrizable matrices in a unique way. These matrices are called indecomposable components of $B$.

**Example 5.** Given an ice valued quiver $Q$:

\[
\begin{array}{c}
\begin{array}{c}
4 \\
\downarrow 1,2 \\
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
3 \\
\downarrow 1.2 \\
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
5 \\
\downarrow 6
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
6 \\
\downarrow 7
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
8 \\
\downarrow 5 \\
9
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
3 \\
\downarrow 5
\end{array}
\end{array}
\end{array}
\]

There are following three indecomposable components of $Q$ by decomposing it at frozen vertices 3 and 4.

\[
\begin{array}{c}
\begin{array}{c}
3 \\
\downarrow 5
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
4 \\
\downarrow 1 \\
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1,2 \\
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
5 \\
\downarrow 6 \\
6
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
10 \\
\downarrow 9
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
3 \\
\downarrow 5
\end{array}
\end{array}
\end{array}
\]

The inverse process of the decomposition is the glue of ice valued quivers, which is defined as follows:

Given two ice valued quivers $Q' = (Q'_0, Q'_1)$ and $Q'' = (Q''_0, Q''_1)$, denote by $(Q'^e, v', d')$ and $(Q''^e, v'', d'')$ the principal parts of $Q'$ and $Q''$ respectively. Let $\Delta'$ and $\Delta''$ be two subsets of $Q'_0 \cup Q''_0$ respectively. Assume that there is a bijection $f$ from $\Delta'$ to $\Delta''$ and we identify $\Delta'$ and $\Delta''$ as $\Delta$ under this bijection. Then the glue of $Q'$ and $Q''$ along $f$ (or $\Delta$) is a quiver $Q = (Q_0, Q_1)$, where $Q_0 = (Q'_0 \setminus \Delta') \cup (Q''_0 \setminus \Delta'') \cup \Delta$ and the arrows in $Q_1$ come from $Q'_1$ and $Q''_1$. More precisely, there is an arrow $\alpha : i \to j$ for $i, j \in Q_0$ if and only if there is an arrow $\alpha : i \to j$ in $Q'_1$ or $Q''_1$.

**Proposition 2.20.** The quiver $Q$ constructed above is an ice valued quiver where the principal part is a full subquiver of $Q_0^e \cup Q''_0^e$ and the functions $v$ and $d$ are given by:

\[
v(\alpha) = \begin{cases} 
  v'(\alpha) & \text{if } \alpha \in Q'_1^e \\
  v''(\alpha) & \text{if } \alpha \in Q''_1^e 
\end{cases}
\]

\[
d_i = \begin{cases} 
  d'_i & \text{if } i \in Q'_0^e \\
  d''_i & \text{if } i \in Q''_0^e 
\end{cases}
\]

**Proof.** This can be checked straightforward. \(\square\)

**Remark 2.21.** 1. Given an ice valued quiver, we can glue its indecomposable components in many different ways. However, there exists a natural way to recover the ice valued quiver from its indecomposable components by gluing. For Example 5, one can recover $Q$ by gluing vertices 4 and 8 in $Q_1$ and $Q_2$ respectively, and gluing vertices 3, 9 and 10 in $Q_1$, $Q_2$ and $Q_3$ respectively.
2. Similarly, the glue of extended skew-symmetric matrices can also be defined (compare Definition 7.2 in [ADS14]).

2.4 Tensor decomposition of rooted cluster algebras and of rooted cluster morphisms

Recall that a seed is completely determined by its exchange matrix, which is extended skew-symmetric. Then we decompose a seed as follows by decomposing the corresponding exchange matrix.

**Definition 2.22.** Let $\Sigma$ be a seed with the exchange matrix $B$. Denote by $\{B_1, B_2, \cdots, B_t\}$ the set of indecomposable components of $B$.

1. The seed $\Sigma$ is said indecomposable if $B$ is indecomposable.

2. The decomposition of $\Sigma$ is a set $\text{Ind}(\Sigma) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_t\}$, where each $\Sigma_i$, $1 \leq i \leq t$, is a subseed of $\Sigma$ corresponding to $B_i$. We call each $\Sigma_i$, $1 \leq i \leq t$, an indecomposable component of $\Sigma$.

As an inverse process of decomposition, we glue seeds at frozen variables by gluing the corresponding matrix, or equivalently, gluing the corresponding quiver as defined in above subsection.

**Remark 2.23.**

1. A cutting of a seed along separating families of frozen variables is defined in [ADS14] (Definition 7.2). It is not hard to see that the decomposition of a seed at frozen variables is obtained by cuttings repeatedly.

2. It is also easy to see that an amalgamated sum along glueable seeds, which is defined in [ADS14] (Definition 4.11), is in fact a special case of a glue of seeds defined above. Comparing with the amalgamated sum, in our sense, any two seeds are glueable along each bijection between some frozen variables in these two seeds since we don’t care about the connections between frozen variables.

From now on, we fix the following settings in this subsection. We are given a seed $\Sigma = (\text{ex}, \text{fx}, B)$ with $x = \text{ex} \sqcup \text{fx}$ and $B = (b_{xy})_{x \times \text{ex}}$. We denote by $\text{Ind}(\Sigma) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_t\}$ the set of indecomposable components of $\Sigma$, where for each $1 \leq i \leq t$, the seed $\Sigma_i$ is $(\text{ex}_i, \text{fx}_i, B_i)$ with $x_i = \text{ex}_i \sqcup \text{fx}_i$, and $\mathcal{X}_\Sigma$ is the set of cluster variables of $\Sigma$. Let $\mathcal{A}(\Sigma_1) \otimes_\mathbb{Z} \mathcal{A}(\Sigma_2) \otimes_\mathbb{Z} \cdots \otimes_\mathbb{Z} \mathcal{A}(\Sigma_t)$ be the tensor algebra of the rooted cluster algebras $\mathcal{A}(\Sigma_i), 1 \leq i \leq t$. Then there is a quotient algebra $\mathcal{A}(\Sigma_1) \otimes_\mathbb{Z} \mathcal{A}(\Sigma_2) \otimes_\mathbb{Z} \cdots \otimes_\mathbb{Z} \mathcal{A}(\Sigma_t)/I$ by identifying the elements $1 \otimes 1 \cdots 1 \otimes x^{i^{th}} \otimes 1 \cdots 1$ and $1 \otimes 1 \cdots 1 \otimes x^{j^{th}} \otimes 1 \cdots 1$ for any $x \in \text{fx}$ with $x \in \text{ex}_i \cap \text{ex}_j$ for some $1 \leq i, j \leq t$. Now we state one of the main theorems in this subsection as follows:

**Theorem 2.24.**

1. For each $\Sigma_i \in \text{Ind}(\Sigma)$, the canonical injection $j_i : \text{ex}_i \rightarrow \text{ex}$ induces an injective rooted cluster morphism from $\mathcal{A}(\Sigma_i)$ to $\mathcal{A}(\Sigma)$. Thus $\mathcal{A}(\Sigma_i)$ is a rooted cluster subalgebra of $\mathcal{A}(\Sigma)$.

2. The set $\mathcal{X}_\Sigma$ of cluster variables of $\Sigma$ coincides with the non-joint union $\bigsqcup_{i=1}^t (\mathcal{X}_{\Sigma_i} \setminus \text{fx}_i) \bigsqcup \text{fx}$.

3. The ring homomorphism

$$\tilde{j} : \mathcal{L}_{\Sigma_1, \text{Q}} \otimes_\text{Q} \mathcal{L}_{\Sigma_2, \text{Q}} \otimes_\text{Q} \cdots \otimes_\text{Q} \mathcal{L}_{\Sigma_t, \text{Q}} \rightarrow \mathcal{L}_{\Sigma, \text{Q}}.$$
which is given by

\[ x_1 \otimes x_2 \otimes \cdots \otimes x_t \mapsto j_1(x_1)j_2(x_2)\cdots j_t(x_t) \]

for any \( x_i \in \mathbf{x}_i (1 \leq i \leq t) \), induces a ring isomorphism:

\[ j : \mathcal{A}(\Sigma_1) \otimes_{\mathbb{Z}} \mathcal{A}(\Sigma_2) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathcal{A}(\Sigma_t)/I \to \mathcal{A}(\Sigma). \]

We call \( \mathcal{A}(\Sigma_1) \otimes_{\mathbb{Z}} \mathcal{A}(\Sigma_2) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathcal{A}(\Sigma_t)/I \) the tensor decomposition of \( \mathcal{A}(\Sigma) \).

**Proof.**

1. It is clearly that the map \( j_i : \mathbf{x}_i \to \mathbf{x} \) lifts as a ring homomorphism from \( \mathcal{L}_{\Sigma, \mathbf{Q}} \) to \( \mathcal{L}_{\Sigma, \mathbf{Q}} \) which satisfies (CM1) and (CM2). Note that each \( \Sigma_i \)-admissible sequence \( (x_1, x_2, \cdots, x_t) \) is clearly \( (j_i, \Sigma_i, \Sigma) \)-biadmissible, thus to prove that \( j_i \) induces a homomorphism from \( \mathcal{A}(\Sigma_i) \) to \( \mathcal{A}(\Sigma) \) and satisfies (CM3), it is sufficient to show the equality \( \mu_{x_1} \circ \cdots \circ \mu_{x_i, \Sigma}(y) = \mu_{x_1} \circ \cdots \circ \mu_{x_i, \Sigma}(y) \) for any \( \Sigma_i \)-admissible sequence \( (x_1, x_2, \cdots, x_i) \) and any cluster variable \( y \in \mathbf{x}_i \). Again, we prove the equality by induction on \( l \). The case of \( l = 0 \) is clearly. In the case of \( l = 1 \), if \( y \neq x_1 \), then \( \mu_{x_1, \Sigma}(y) = y = \mu_{x_1, \Sigma}(y) \). If \( y = x_1 \), then we have

\[
\mu_{x_1, \Sigma}(x_1) = \frac{1}{x_1} \left( \prod_{x \in \mathbf{x} ; b_{x_1} > 0} x^{b_{x_1}} + \prod_{x \in \mathbf{x} ; b_{x_1} < 0} x^{-b_{x_1}} \right) = \frac{1}{x_1} \left( \prod_{x \in \mathbf{x} ; b_{x_1} > 0} x^{b_{x_1}} + \prod_{x \in \mathbf{x} ; b_{x_1} < 0} x^{-b_{x_1}} \right) = \mu_{x_1, \Sigma}(x_1)
\]

where the second equality follows from the definition of the indecomposable component. Thus \( \mu_{x_1, \Sigma}(y) = \mu_{x_1, \Sigma}(y) \) for any cluster variable \( y \in \mathbf{x}_i \). In the case of \( l = 2 \), note that \( j_i \) maps each exchangeable variable of \( \Sigma_i \) to an exchangeable variable of \( \Sigma \), and we consider the mutation \( j_{i(x_1)} : \mu_{x_i, \Sigma} \to \mu_{x_1, \Sigma} \) of \( j_i \). It is clearly that \( \mu_{x_1, \Sigma} \) is an indecomposable component of \( \mu_{x_1, \Sigma} \). Then for any cluster variable \( \mu_{x_1, \Sigma}(y) \in \mu_{x_1, \Sigma}(\Sigma_i) \), \( \mu_{x_2, \mu_{x_1, \Sigma}}(\mu_{x_1, \Sigma}(y)) = \mu_{x_2, \mu_{x_1, \Sigma}}(\mu_{x_1, \Sigma}(y)) \) by the case of \( l = 1 \), thus we have \( \mu_{x_2} \circ \mu_{x_1, \Sigma}(y) = \mu_{x_2, \mu_{x_1, \Sigma}}(\mu_{x_1, \Sigma}(y)) = \mu_{x_2, \mu_{x_1, \Sigma}}(\mu_{x_1, \Sigma}(y)) \). Finally, we inductively prove the equality in this way.

2. It follows from the proof of the above assertion that the set \( \mathcal{P}_\Sigma \) contains the set \( \bigcup_{l=1}^{s} (\mathcal{P}_\Sigma \setminus \mathbf{f}_l) \bigcup \mathbf{f}_l \). Note that any cluster variable of \( \mathcal{A}(\Sigma) \) is of the form \( \mu_{x_1} \circ \cdots \circ \mu_{x_s, \Sigma}(y) \), where \( (x_1, x_2, \cdots, x_s) \) is a \( \Sigma \)-admissible sequence and \( y \in \mathbf{x} \) is a cluster variable of \( \Sigma \). Assume that \( y \in \mathbf{x} \) and \( \{x_kl \}_{l=1}^{s} \) be the maximal subset of \( \{x_1l \}_{l=1}^{s} \) with \( x_i \in \mathbf{x}_i \) for each \( 1 \leq t \leq s \), and \( i_1 < i_2 \) for each \( 1 \leq t_1 < t_2 \leq s \). Then we have \( \mu_{x_1} \circ \cdots \circ \mu_{x_s, \Sigma}(y) = \mu_{x_1} \circ \cdots \circ \mu_{x_s, \Sigma}(y) \) since \( \mu_{x_s, \Sigma} \) is an indecomposable component of \( \Sigma \). Thus the inverse inclusion is valid and \( \mathcal{P}_\Sigma = \bigcup_{l=1}^{s} (\mathcal{P}_\Sigma \setminus \mathbf{f}_l) \bigcup \mathbf{f}_l \).

3. Firstly, it is easy to see that \( \tilde{j} \) is a ring homomorphism and induces a surjective ring homomorphism \( \tilde{j} \) from \( \mathcal{A}(\Sigma_1) \otimes_{\mathbb{Z}} \mathcal{A}(\Sigma_2) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathcal{A}(\Sigma_t)/I \) to \( \mathcal{A}(\Sigma) \) due to the assertion 2. Secondly, the kernel \( \text{Ker}(\tilde{j}) \) of \( \tilde{j} \) is \( I \) and thus we get the conclusion. In fact, \( \text{Ker}(\tilde{j}) = I \) is because \( \text{Ker}(\tilde{j}) = I \).

\( \square \)

**Example 6.** Consider the ice valued quiver \( Q \) and its indecomposable components \( Q_i \) \((1 \leq i \leq 3)\) in Example 5. Denote by \( \mathcal{A} \) and \( \mathcal{A}_t \) \((1 \leq i \leq 3)\) the rooted cluster algebras corresponding to \( Q \) and \( Q_i \) \((1 \leq i \leq 3)\) respectively. The ambient rings of \( \mathcal{A} \) and \( \mathcal{A}_t \) \((1 \leq i \leq 3)\) are respectively as follows:

\[
\mathcal{L} = \mathbb{Q}[x_1^{\pm 1}, x_2 x_2^{\pm 1}, x_3^{\pm 1}, x_4 x_4^{\pm 1}, x_5]
\]

\[
\mathcal{L}_1 = \mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}, x_3, x_4]
\]
$L_2 = \mathbb{Q}[x_{21}^1, x_8, x_9]$, \\
$L_3 = \mathbb{Q}[x_{61}^1, x_{71}^1, x_{10}]$.

Let $j : L_1 \otimes_{\mathbb{Q}} L_2 \otimes_{\mathbb{Q}} L_3 \to L$ be a ring homomorphism with
\[
j(x_i) = \begin{cases} x_i & i = 1, 2, 5, 6, 7 \\
x_3 & i = 3, 9, 10 \\
x_4 & i = 4, 8. \end{cases}
\]

Then from above theorem, it induces a ring isomorphism $j : A_1 \otimes \mathbb{Z} A_2 \otimes \mathbb{Z} A_3 / I \to A$, where $I = \langle x_3 \otimes 1 \otimes 1 \otimes 1, x_9 \otimes 1, x_3 \otimes 1 \otimes 1 - 1 \otimes 1, x_{10} - 1 \otimes 1 \otimes 1, x_8 \otimes 1 \rangle$.

**Definition 2.25.** Let $\Sigma = (\mathbf{ex}, \mathbf{fx}, B)$ be a seed and $\mathbf{ex}_0$ be a subset of $\mathbf{ex}$. The seed $\Sigma_f = (\mathbf{ex} \setminus \mathbf{ex}_0, \mathbf{fx} \cup \mathbf{ex}_0, B_f)$ is called the frozenization of $\Sigma$ at $\mathbf{ex}_0$, where $B_f$ is obtained from $B$ by deleting the columns corresponding to the elements in $\mathbf{ex}_0$.

**Example 7.** Consider the seed
\[
\Sigma = (x_1, x_2, x_3), \emptyset, \begin{bmatrix} 0 & -2 & 6 \\
1 & 0 & -3 \\
-2 & 2 & 0 \end{bmatrix} \] and its ice valued quiver

$\leq \begin{array}{c} 2 \\
1, 2 \\
\downarrow \\
2, 3 \end{array}$

Let $\Sigma_f = (x_1, x_2), (x_3), \begin{bmatrix} 0 & -2 \\
1 & 0 \\
-2 & 2 \end{bmatrix}$

be the frozenization of $\Sigma$ at the exchangeable cluster variable $x_3$. Then the corresponding ice valued quiver is

$\leq \begin{array}{c} 2 \\
1, 2 \\
\downarrow \\
2 \end{array}$

where we have framed the frozen vertex.

**Proposition 2.26.** The natural bijection $j$ from the cluster variables of $\Sigma_f$ to the cluster variables of $\Sigma$ induces an injective rooted cluster morphism from $A(\Sigma_f)$ to $A(\Sigma)$. Thus $A(\Sigma_f)$ is a rooted cluster subalgebra of $A(\Sigma)$. We call $A(\Sigma_f)$ the frozenization of $A(\Sigma)$ at $\mathbf{ex}_0$.

**Proof.** Firstly, it is clearly that $j$ lifts as an injective ring homomorphism from $L_{\Sigma_f, \mathbb{Q}}$ to $L_{\Sigma, \mathbb{Q}}$ which satisfies (CM1) and (CM2). Secondly, each $\Sigma_f$-admissible sequence $(x_1, x_2, \ldots, x_i)$ is clearly $(j, \Sigma_f, \Sigma)$-biadmissible. Then similar to the proof of Theorem 2.24(1), (CM3) is also valid. So we are done. \[\square\]
Now, we consider the decomposition of a rooted cluster morphism by using the frozenization and the decomposition of a rooted cluster algebra.

Given two seeds $\Sigma = (\text{ex}, \text{fx}, B)$ and $\Sigma' = (\text{ex}', \text{fx}', B')$ with $\text{x} = \text{ex} \sqcup \text{fx}$ and $\text{x}' = \text{ex}' \sqcup \text{fx}'$. Let $g : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a rooted cluster morphism. Denote by $\text{ex}_0 = \{ x \in \text{ex} \mid g(x) \in \mathbb{Z} \}$ and $\Sigma_f$ the frozenization of $\Sigma$ at the set $\text{ex}_0$. By $\text{Ind}(\Sigma_f) = \{ \Sigma^1_f, \Sigma^2_f, \ldots , \Sigma^t_f \}$ we denote the set of indecomposable components of $\Sigma_f$. Let $\mathcal{A}(\Sigma^1_f) \otimes_{\mathbb{Z}} \mathcal{A}(\Sigma^2_f) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathcal{A}(\Sigma^t_f)/I$ be the tensor decomposition of $\mathcal{A}(\Sigma_f)$. We have the following theorem which is the last main result in this subsection.

**Theorem 2.27.** Under the above notations, we have:

1. For each $1 \leq i \leq t$, the natural injection $j_i$ from the set of cluster variables of $\Sigma^i_f$ to the set of cluster variables of $\Sigma$ induces an injective rooted cluster morphism $\tilde{j}_i$ from $\mathcal{A}(\Sigma^i_f)$ to $\mathcal{A}(\Sigma)$;

2. For each $1 \leq i \leq t$, the rooted cluster morphism $g_i = g \circ \tilde{j}_i$ is ideal and we call it an indecomposable component of $g$;

3. There is a commutative diagram of rooted cluster morphisms

\[
\begin{array}{ccc}
\mathcal{A}(\Sigma) & \xrightarrow{g} & \mathcal{A}(\Sigma') \\
\downarrow{j_1 \otimes j_2 \otimes \cdots \otimes j_t} & & \downarrow{g_1 \otimes g_2 \otimes \cdots \otimes g_t} \\
\mathcal{A}(\Sigma^1_f) \otimes_{\mathbb{Z}} \mathcal{A}(\Sigma^2_f) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathcal{A}(\Sigma^t_f)/I,
\end{array}
\]

where $\tilde{j}_1 \otimes \tilde{j}_2 \otimes \cdots \otimes \tilde{j}_t(y_1 \otimes y_2 \otimes \cdots \otimes y_t) = \tilde{j}_1(y_1) \tilde{j}_2(y_2) \cdots \tilde{j}_t(y_t)$ and $g_1 \otimes g_2 \otimes \cdots \otimes g_t(y_1 \otimes y_2 \otimes \cdots \otimes y_t) = g_1(y_1)g_2(y_2) \cdots g_t(y_t)$ for any $y_i \in \mathcal{A}(\Sigma_i)$, $1 \leq i \leq t$; moreover, $g_1 \otimes g_2 \otimes \cdots \otimes g_t$ is ideal and we call it the tensor decomposition of $g$.

**Proof.**

1. For each $1 \leq i \leq t$, the map $j_i$ can be viewed as a composition of the natural injection from the set of cluster variables of $\Sigma^i_f$ to the set of cluster variables of $\Sigma$ and the natural injection from the set of cluster variables of $\Sigma_f$ to the set of cluster variables of $\Sigma$. These two injections both induce injective rooted cluster morphisms between the corresponding rooted cluster algebras due to Theorem 2.24(1) and Proposition 2.26 respectively. Thus $j_i$ induces an injective rooted cluster morphism $\tilde{j}_i$ from $\mathcal{A}(\Sigma^i_f)$ to $\mathcal{A}(\Sigma)$.

2. For each $1 \leq i \leq t$, the morphism $g_i$ is a rooted cluster morphism which maps an exchangeable variable of $\Sigma^i_f$ to an exchangeable one of $\Sigma'$. Thus the assertion follows from Theorem 2.11(1).

3. The commutative diagram is straightforward. The morphism $g_1 \otimes g_2 \otimes \cdots \otimes g_t$ is ideal due to Theorem 2.11(1).

**Proposition 2.28.** An explicit rooted cluster morphism $g$ is determined by its indecomposable components. More precisely, if there is another rooted cluster morphism $g'$ from $\mathcal{A}(\Sigma)$ to $\mathcal{A}(\Sigma')$ with the same indecomposable components as $g$ has, then $g = g'$.
variables, the statement follows from the fact that \( g(x) = g_i(x) \) for any indecomposable components \( g_i \) of \( g \) and any \( x \in x_i \).

\[ \square \]

**Remark 2.29.** Essentially speaking, the commutative diagram in Theorem 2.27 gives a tensor decomposition of the rooted cluster morphism \( g \circ j : \mathcal{A}(\Sigma_f) \rightarrow \mathcal{A}(\Sigma') \), where \( j \) is the natural injection from \( \mathcal{A}(\Sigma_f) \) to \( \mathcal{A}(\Sigma') \). On the other hand, from above proposition an explicit rooted cluster morphism \( g \) is determined by \( g \circ j \). Thus the theorem also reveals a tensor decomposition of the explicit morphism \( g \) in some sense.

## 2.5 Injections and specializations

In this subsection, we completely characterize injective rooted cluster morphisms. Then we define the general specialization and give some properties.

Given two seeds \( \Sigma_0 \) and \( \Sigma \). Let \( g : \mathcal{A}(\Sigma_0) \rightarrow \mathcal{A}(\Sigma) \) be an injective rooted cluster morphism. Then from the injectivity, \( g \) induces an injection from the set \( x_{\Sigma_0} \) of initial cluster variables in \( \mathcal{A}(\Sigma_0) \) to the set \( x_{\Sigma} \) of initial cluster variables in \( \mathcal{A}(\Sigma) \). In fact, for any \( x \in x_{\Sigma_0} \), if \( g(x) = n \in \mathbb{Z} \), then we have \( g(n) = g(x) = n \) and a contradiction to the injectivity of \( g \). Thus there is no harm to assume that \( \Sigma_0 = (\text{ex}_0, \text{fx}_0 \sqcup \text{ex}_2, B_0) \) and \( \Sigma = (\text{ex}_0 \sqcup \text{ex}_1 \sqcup \text{ex}_2, \text{fx}_0 \sqcup \text{fx}_1, B_f) \) where \( g \) is an identity on \( x_{\Sigma_0} = \text{ex}_0 \sqcup \text{fx}_0 \sqcup \text{ex}_2 \). We define \( \Sigma_f = (\text{ex}_0 \sqcup \text{ex}_1, \text{fx}_0 \sqcup \text{fx}_1 \sqcup \text{ex}_2, B_f) \) as the frozenization of \( \Sigma \) at \( \text{ex}_2 \) and \( \Sigma_1 = (\text{ex}_1, \text{fx}_1 \sqcup \text{ex}_2, B_1) \) as a subseed of \( \Sigma_f \). Denote by \( \text{Ind}(\Sigma_0) \) (\( \text{Ind}(\Sigma_1) \) and \( \text{Ind}(\Sigma) \) respectively) be the set of indecomposable components of \( \Sigma_0 \) (\( \Sigma_1 \) and \( \Sigma \) respectively).

We have the following characterization of an injective rooted cluster morphism.

**Theorem 2.30.** Let \( \mathcal{A}(\Sigma_0) \) be a rooted cluster subalgebra of \( \mathcal{A}(\Sigma) \) under an injective rooted cluster morphism \( g : \mathcal{A}(\Sigma_0) \rightarrow \mathcal{A}(\Sigma) \) with notations as above.

1. If the seed \( \Sigma_0 \) is indecomposable, then \( \Sigma_0 \) (or \( \text{Ind}(\Sigma_0) \)) is an indecomposable component of \( \Sigma_f \), and the seed \( \Sigma_f \) is obtained by gluing \( \Sigma_0 \) (or \( \text{Ind}(\Sigma_0) \)) and \( \Sigma_1 \) along \( \text{ex}_2 \).

2. There is a bijection \( h \) from \( \text{Ind}(\Sigma_0) \sqcup \text{Ind}(\Sigma_1) \to \text{Ind}(\Sigma_f) \) such that \( h(\Sigma') = \Sigma' \) or \( h(\Sigma') = \Sigma'_{\text{op}} \) for each \( \Sigma' \in \text{Ind}(\Sigma_0) \) and \( h(\Sigma') = \Sigma' \) for each \( \Sigma' \in \text{Ind}(\Sigma_1) \).

3. There is a rooted cluster isomorphism \( \mathcal{A}(\Sigma_0) \cong \mathcal{A}(g(\Sigma_0)) \).

4. The injection \( g \) is a section if and only if \( \text{ex}_2 \) is an empty set.

**Proof.**

1. For each exchangeable cluster variable \( x \in \text{ex}_0 \), we have

\[
\mu_x(x) = \frac{1}{x} \left( \prod_{y \in \text{ex}_0} y^{b_{yx}^0} + \prod_{y \in \text{ex}_2} y^{-b_{yx}^0} \right)
\]

in \( \mathcal{A}(\Sigma_0) \), where \( b_{yx}^0 \) entries in \( B_0 \). Then by viewing \( g \) as an identity on the set \( x_{\Sigma_0} \), due to (CM3), we have the equalities:

\[
\frac{1}{x} \left( \prod_{y \in \text{ex}_0} y^{b_{yx}^0} + \prod_{y \in \text{ex}_2} y^{-b_{yx}^0} \right) = \mu_x(g(x)) = \mu_{g(x)}(g(x)) = \frac{1}{x} \left( \prod_{y \in \text{ex}_0} y^{b_{yx}^0} + \prod_{y \in \text{ex}_2} y^{-b_{yx}} \right),
\]
Remark 2.31.  

1. Given a seed \( \Sigma \) and a subseed \( \Sigma_0 \) of \( \Sigma \), from Theorem 2.24 and the above theorem, the natural injection from \( \mathcal{A}(\Sigma') \) to \( \mathcal{A}(\Sigma_0) \) induces an injective rooted cluster morphology from \( \mathcal{A}(\Sigma') \) to \( \mathcal{A}(\Sigma) \). Thus \( \Sigma' \) or \( \Sigma^{op} \) can be viewed as an indecomposable component of \( \Sigma_f \) from the first statement. It is clearly that \( \text{Ind}(\Sigma_f) \) is a subset of \( \text{Ind}(\Sigma) \), thus we have an injection \( h : \text{Ind}(\Sigma_0) \cup \text{Ind}(\Sigma_1) \rightarrow \text{Ind}(\Sigma) \) satisfies the properties we want. For the bijectivity, it is only need to notice that the set of exchangeable cluster variables arising from the seeds in \( \text{Ind}(\Sigma_0) \cup \text{Ind}(\Sigma_1) \) covers the set of exchangeable cluster variables of \( \Sigma_f \).

3. This assertion is clearly.

4. If \( g \) is a section, that is, there exists a rooted cluster morphism \( f \) from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma_0) \) such that \( fg = id_{\mathcal{A}(\Sigma_0)} \). Then \( f \) induces an identity on \( \text{ex}_2 \) whose elements are exchangeable in \( \Sigma \) and frozen in \( \Sigma_0 \). Thus the set \( \text{ex}_2 \) must be empty because it is forbidden in a rooted cluster morphology from mapping an initial exchangeable variable to a frozen one. Conversely, if \( \text{ex}_2 \) is empty, let \( f \) be the unique ring homomorphism from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma_0) \) which is an identity on \( \text{ex}_0 \cup \text{fx}_0 \) and maps each element in \( \text{ex}_1 \cup \text{fx}_1 \) to 1. Then \( f \) is a rooted cluster morphology and moreover clearly a retraction of \( g \). In fact it is easy to see that \( f \) satisfies the conditions (CM1) and (CM2). For (CM3), one need only notice that in each \((f, \Sigma, \Sigma_0)\)–admissible sequence \((x_1, x_2, \ldots, x_n)\), any variable belongs to \( \mathcal{A}(\Sigma_0) \).

\[ \square \]

Remark 2.31.  

1. Given a seed \( \Sigma \) and a subseed \( \Sigma_0 \) of \( \Sigma \), from Theorem 2.24 and the above theorem, the natural injection from \( \mathcal{A}(\Sigma') \) to \( \mathcal{A}(\Sigma_0) \) induces an injective rooted cluster morphology from \( \mathcal{A}(\Sigma') \) to \( \mathcal{A}(\Sigma) \) if and only if \( \Sigma_0 \) is a glue of some indecomposable components of \( \Sigma \) in a natural way.

2. The third part of the above theorem shows that the rooted cluster subalgebra in the sense of Definition 2.7 coincides with the subcluster algebra defined in section IV.1 [BIRS09] up to a rooted cluster isomorphism.

3. The fourth part of the above theorem answers the Problem 7.7 in [ADS14]; more precisely, an injective rooted cluster morphology \( g \) is a section if and only if \( g(\Sigma_0) \) is a subseed of \( \Sigma' \).

4. The above theorem allows us to find out all the rooted cluster subalgebras of a given rooted cluster algebra up to isomorphism. Let \( \Sigma = (\text{ex}, \text{fx}, B) \) be a seed and \( \text{ex}' \) be a subset of \( \text{ex} \). We have a seed \( \Sigma_f \) by freezing variables in \( \text{ex}' \). Then the inverse process of the proof of Theorem 2.30(1) shows that a glue \( \Sigma' \) of some indecomposable components of \( \Sigma_f \) determines a rooted cluster subalgebra of \( \mathcal{A}(\Sigma) \), that is, \( \mathcal{A}(\Sigma') \) is a rooted cluster subalgebra of \( \mathcal{A}(\Sigma) \). The above theorem shows that up to rooted cluster isomorphism, each rooted cluster subalgebra of \( \mathcal{A}(\Sigma) \) arises from this way.

For a rooted cluster algebra, we introduce the following notion of complete pairs of subalgebras with given coefficients.
Definition 2.32. Let $\Sigma = (\mathbf{ex}, B)$ be a seed and $\Sigma_f$ be the frozenization of $\Sigma$ at a subset $\mathbf{ex}'$ of $\mathbf{ex}$. Let $\Sigma_1 = (\mathbf{ex}_1, f_1, B_1)$ and $\Sigma_2 = (\mathbf{ex}_2, f_2, B_2)$ be two subseeds of $\Sigma_f$. We call the pair $(\mathcal{A}(\Sigma_1), \mathcal{A}(\Sigma_2))$ a complete pair of subalgebras of $\mathcal{A}(\Sigma)$ with coefficient set $\mathbf{ex} \sqcup \mathbf{ex}'$ if $\mathcal{A}(\Sigma_1)$ and $\mathcal{A}(\Sigma_2)$ are both rooted cluster subalgebras of $\mathcal{A}(\Sigma)$ and $\mathbf{ex} \sqcup \mathbf{ex} = \mathbf{ex}_1 \sqcup f_1 \sqcup \mathbf{ex}_2 \sqcup f_2$.

Proposition 2.33. Let $\Sigma = (\mathbf{ex}, B)$ be a seed and $\Sigma_f$ be the frozenization of $\Sigma$ at a subset $\mathbf{ex}'$ of $\mathbf{ex}$. Let $\Sigma_1$ be a subseed of $\Sigma_f$ such that $\mathcal{A}(\Sigma_1)$ is a rooted cluster subalgebra of $\mathcal{A}(\Sigma)$. Then there exists a unique subseed $\Sigma_2$ of $\Sigma_f$ such that the pair $(\mathcal{A}(\Sigma_1), \mathcal{A}(\Sigma_2))$ is a complete pair of subalgebras of $\mathcal{A}(\Sigma)$ with coefficient set $\mathbf{ex} \sqcup \mathbf{ex}'$.

Proof. It follows from Remark 2.31(1) that $\Sigma_1$ is a glue of some indecomposable components of $\Sigma_f$. Let $\Sigma_2$ be the subseed of $\Sigma_f$ which is a glue of the rest indecomposable components of $\Sigma_f$. Then the conclusion follows from Theorem 2.30(2) and Theorem 2.24(2).

Remark 2.34. It is not hard to see that the above proposition shows that in a complete pair of subalgebras, these two subalgebras determine with each other, and the pair is symmetric, that is, $(\mathcal{A}(\Sigma_1), \mathcal{A}(\Sigma_2))$ is a complete pair of subalgebras if and only if $(\mathcal{A}(\Sigma_2), \mathcal{A}(\Sigma_1))$ is a complete pair of subalgebras. For rooted cluster algebras arising from 2-CY triangulated categories, the correspondences between these pairs and cotorsion pairs in the 2-CY triangulated categories will be characterized in Theorem 3.14(3). In particular, the above properties of determining of each other and symmetry of complete pairs correspond to the relative properties (see in subsection 3.1) of the corresponding cotorsion pairs respectively.

Lemma 2.35. Let $\Sigma = (\mathbf{ex}, B)$ be a seed with $x = \mathbf{ex} \sqcup \mathbf{fx}$. Let $\Sigma' = (\mathbf{ex}', B')$ be a subseed of a frozenization of $\Sigma$ such that $\mathcal{A}(\Sigma')$ is a rooted cluster subalgebra of $\mathcal{A}(\Sigma)$ under the natural injection from $x'$ to $x$, where $x' = \mathbf{ex}' \sqcup \mathbf{fx}'$.

1. Let $\Sigma'' = (\mathbf{ex}'', B'')$ be a frozenization of $\Sigma$ with $x'' = \mathbf{ex}'' \sqcup \mathbf{fx}''$. Then under the natural injection from $x'$ to $x''$, $\mathcal{A}(\Sigma')$ is a rooted cluster subalgebra of $\mathcal{A}(\Sigma'')$ if and only if $\mathbf{ex}' \subseteq \mathbf{ex}''$.

2. Let $\Sigma'' = (\mathbf{ex}'', B'')$ be a subseed of a frozenization of $\Sigma$ with $x' \subseteq x'' = \mathbf{ex}'' \sqcup \mathbf{fx}''$. Then under the natural injection from $x'$ to $x''$, $\mathcal{A}(\Sigma')$ is a rooted cluster subalgebra of $\mathcal{A}(\Sigma'')$ if and only if $\mathbf{ex}' \subseteq \mathbf{ex}''$.

3. The set $(S, \leq)$ of frozenizations of $\Sigma$ is a partial order set, where $\Sigma_1 \leq \Sigma_2$ if $\mathbf{ex}_1 \subseteq \mathbf{ex}_2$ for any two seeds $\Sigma_1 = (\mathbf{ex}_1, f_1, B_1)$ and $\Sigma_2 = (\mathbf{ex}_2, f_2, B_2)$ in $S$. For any $\Sigma_1 \leq \Sigma_2$ in $S$, $\mathcal{A}(\Sigma_1)$ is a rooted cluster subalgebra of $\mathcal{A}(\Sigma_2)$.

Proof. 1. If $\mathcal{A}(\Sigma')$ is a rooted cluster subalgebra of $\mathcal{A}(\Sigma'')$ under the natural injection from $x'$ to $x''$, then we have $\mathbf{ex}' \subseteq \mathbf{ex}''$ due to (CM2). Conversely, it follows from $\mathbf{ex}' \subseteq \mathbf{ex}''$ that $\Sigma'$ is a subseed of the frozenization $\Sigma''$ of $\Sigma''$ at $\mathbf{ex}'' \setminus \mathbf{ex}'$. Let $x \in \mathbf{ex}'$ and $y \in \mathbf{ex}''$. If $x$ and $y$ are connected directly in $\Sigma''$, then they are connected directly in $\Sigma$. Thus $y \in x'$ because $\mathcal{A}(\Sigma')$ is a rooted cluster subalgebra of $\mathcal{A}(\Sigma)$. Therefore $\Sigma'$ is a glue of some indecomposable components of $\Sigma''$. Then by using the above theorem again, $\mathcal{A}(\Sigma')$ is a rooted cluster subalgebra of $\mathcal{A}(\Sigma'')$, and thus of $\mathcal{A}(\Sigma'')$ by Proposition 2.26.

2. Similar to the above proof.
3. It is clearly that the relation $\preceq$ gives a partial order on the set $\mathcal{S}$. Note that if $\Sigma^1_f \preceq \Sigma^2_f$ for $\Sigma^1_f, \Sigma^2_f \in \mathcal{S}$, then $\Sigma^1_f$ is a frozenization of $\Sigma^2_f$ at $\text{ex}_2 \setminus \text{ex}_1$. Thus $\mathcal{A}(\Sigma^1_f)$ is a rooted cluster subalgebra of $\mathcal{A}(\Sigma^2_f)$ by Proposition 2.26.

The following two theorems give the relations between ideal rooted cluster morphisms and injective rooted cluster morphisms.

**Theorem 2.36.** Given two seeds $\Sigma = (\text{ex}, \text{fx}, B)$ and $\Sigma' = (\text{ex}', \text{fx}', B')$ with $\text{ex} = \text{ex} \sqcup \text{fx}$ and $\text{ex}' = \text{ex}' \sqcup \text{fx}'$. Let $f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a rooted cluster morphism.

1. The rooted cluster algebra $\mathcal{A}(f(\Sigma))$ is a rooted cluster subalgebra of $\mathcal{A}(\Sigma')$ under the natural injection on the initial cluster variables.

2. If $f$ is ideal then it is a composition of a surjective rooted cluster morphism and an injective rooted cluster morphism.

**Proof.** 1. We consider the image seed $f(\Sigma) = (\text{ex}' \cap f(\text{ex}), \text{ex}' \cap f(\text{ex}) \setminus \text{ex}' \cap f(\text{ex}'), B' \setminus f(\text{ex}'))$ and the frozenization $\Sigma'_f = (\text{ex}' \setminus \text{ex}'_0, f(\text{ex})' \cap \text{ex}'_0, f(\text{ex}')')$ of $\Sigma'_f$ at $\text{ex}'_0$, where $\text{ex}'_0 = \{f(x) \in \text{ex}' \mid \forall x \in \text{fx}\}$. Then by a same argument used in the proof of Theorem 2.11(1), there is no $y \in \text{ex}$ such that $f(x) = f(y)$. Thus $f(\text{ex}) \cap \text{ex}'_0 = \emptyset$ and $\text{ex}' \cap f(\text{ex}) \subseteq \text{ex}' \setminus \text{ex}'_0$, $\text{ex}' \cap f(\text{ex}) \subseteq \text{ex}' \setminus \text{ex}'_0$. Therefore, the seed $f(\Sigma)$ is a subseed of $\Sigma'_f$. Given an exchangeable variable $x_1$ of $f(\Sigma)$, we can assume that $x_1 = f(x)$ for some $x \in \text{ex}$. Then we have the following equalities:

$$f(\mu_{x,\Sigma}(x)) = f\left(\prod_{y \in \text{ex} \setminus \text{ex}_1} y^b_{xy} + \prod_{y \in \text{ex}_1} y^{-b_{xy}} \right) = \frac{1}{x_{1}} \prod_{y \in \text{ex} \setminus \text{ex}_1} f(y)^{b_{xy}} + \prod_{y \in \text{ex}_1} f(y)^{-b_{xy}}$$

$$\mu_{x_1,\Sigma}(x_1) = \prod_{y_1 \in \text{ex}' \setminus \text{ex}_1} y_{1}^{b_{1y1}} + \prod_{y_1 \in \text{ex}'_1} y_{1}^{-b_{1y1}}$$

and

$$\mu_{x_1,\Sigma}(x_1) = \prod_{y_1 \in \text{ex}' \setminus \text{ex}_1} y_{1}^{b_{1y1}} + \prod_{y_1 \in \text{ex}'_1} y_{1}^{-b_{1y1}}$$

Because $f$ is a rooted cluster morphism, we have $f(\mu_{x,\Sigma}(x)) = \mu_{x_1,\Sigma}(x_1)$. Then by a similar trick used in the proof of Theorem 2.11(1) we get $\mu_{x_1,\Sigma}(x_1) = \mu_{x_1,\Sigma}(x_1)$. Thus for each $y_1 \in \text{ex}'$ with $b_{y_1x_1} \neq 0$, we have $y_1 \in \text{ex}' \cap f(\text{ex})$. Therefore, $f(\Sigma)$ is a glue of some indecomposable components of $\Sigma'_f$. Finally, $\mathcal{A}(f(\Sigma))$ is a rooted cluster subalgebra of $\mathcal{A}(\Sigma'_f)$ and thus a rooted cluster subalgebra of $\mathcal{A}(\Sigma')$.

2. Because $f$ is ideal, it induces a surjective rooted cluster morphism $f' : \mathcal{A}(\Sigma) \to \mathcal{A}(f(\Sigma))$. It follows from the above assertion that $j : \mathcal{A}(f(\Sigma)) \to \mathcal{A}(\Sigma')$ is an injective rooted cluster morphism. Thus $f$ is a composition of rooted cluster morphisms $f'$ and $j$. 

\qed
Theorem 2.37. Let \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) be an inducible rooted cluster morphism. Assume that it is a composition of rooted cluster morphisms \( f' : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma'') \) and \( f'' : \mathcal{A}(\Sigma'') \to \mathcal{A}(\Sigma') \) where \( f' \) is surjective and \( f'' \) is injective. Then \( \mathcal{A}(\Sigma') \cong \mathcal{A}(f(\Sigma)) \) and \( f \) is ideal.

Proof. Note that \( f' \) is a surjective inducible rooted cluster morphism because \( f \) is inducible. Then from Theorem 2.11 \( f' \) is ideal and thus \( f'(\mathcal{A}(\Sigma)) = \mathcal{A}(f'(\Sigma)) \). On the one hand, \( f'(\Sigma) = \Sigma'' \) from Lemma 3.1 ([ADS14]). On the other hand, we have \( f''(\mathcal{A}(\Sigma'')) = \mathcal{A}(f''(\Sigma'')) \) because \( f'' \) is injective and thus ideal due to Corollary 4.5 in [ADS14]. Therefore there are equalities \( f(\mathcal{A}(\Sigma)) = f''(\mathcal{A}(\Sigma)) = f''(f'(\mathcal{A}(\Sigma))) = f''(\mathcal{A}(\Sigma'')) = \mathcal{A}(f''(\Sigma'')). \) Thus to prove that \( f \) is ideal, that is, \( f(\mathcal{A}(\Sigma)) = \mathcal{A}(f(\Sigma)), \) it is only need to show that \( \mathcal{A}(f''(\Sigma'')) \subseteq \mathcal{A}(f(\Sigma)) \). Let \( x \in x_{f''(\Sigma'')} \) be a cluster variable of \( f''(\Sigma'') \) and assume that \( y \in x_{\Sigma''} \) is a cluster variable such that \( f''(y) = x \). By Lemma 3.1 ([ADS14]), there is a cluster variable \( y' \in x_{\Sigma} \) such that \( f'(y') = y \). Then \( x = f(y') \in x_{f(\Sigma)} \) and thus \( x_{f''(\Sigma'')} \subseteq x_{f(\Sigma)}. \) Similarly, we have \( e_{x_{f''(\Sigma'')}} \subseteq e_{x_{f(\Sigma)}} \). Finally, from Lemma 2.35, we have \( f''(\mathcal{A}(\Sigma'')) \subseteq \mathcal{A}(f(\Sigma)) \) and thus \( f \) is ideal. The rooted cluster isomorphism \( \mathcal{A}(\Sigma') \cong \mathcal{A}(f(\Sigma)) \) is clearly.

In the rest of this subsection, we consider the specialization of a seed at some cluster variables. Let \( \Sigma = (e_{x}, f_{x}, B) \) be a seed with \( x = e_{x} \cup f_{x} \). Let \( x' = e_{x'} \cup f_{x'} \) be a subset of \( x \) with \( e_{x'} \subseteq e_{x} \) and \( f_{x'} \subseteq f_{x} \). Denote by \( \Sigma' = (e_{x'}, f_{x'}, B') \) the subseed of \( \Sigma \).

Definition 2.38. The map \( f : x \to x' \cup \mathbb{Z} \) is called a specialization of \( \Sigma \) at \( x \setminus x' \), which is given by:

\[
 f(x) = \begin{cases} 
 x & \text{if } x \in x' \\
 n & \text{if } x \not\in x \setminus x'.
\end{cases}
\]

If the map \( f \) induces a rooted cluster morphism \( \tilde{f} \) from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma') \), then we call \( \tilde{f} \) the specialization of rooted cluster algebra \( \mathcal{A}(\Sigma) \) at \( x \setminus x' \). If \( x \setminus x' = \{x\} \) for some \( x \), then we call \( f \) a simple specialization.

The simple specialization is also defined in [ADS14] and it is needed that \( f(x) \neq 0 \) for \( x \in x \setminus x' \). It is well-known that if \( x \) is frozen, then specializing \( x \) to 1 induces a rooted cluster morphism from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma') \), see for instance [PZ07]. Generally we have the following analogue:

Proposition 2.39. Let \( f \) be a specialization defined as above. Assume that \( e_{x'} = e_{x} \) and \( f(x) = 1 \) for any cluster variables \( x \in x \setminus x' \), then

1. The map \( f \) induces a rooted cluster morphism \( \tilde{f} \) from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma') \):

2. The morphism \( \tilde{f} \) is a composition of several simple specializations of rooted cluster algebras at frozen variables.

Proof. 1. It is clearly that \( f \) induces a ring homomorphism \( f : \mathcal{L}_{\Sigma, Q} \to \mathcal{L}_{\Sigma', Q} \). Now we inductively show that \( f \) maps each cluster variable in \( \mathcal{A}(\Sigma) \) to a cluster variable in \( \mathcal{A}(\Sigma') \) and thus induces a morphism \( \tilde{f} \) from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma') \). For a cluster variable \( \mu_{x, \Sigma}(y) \) in \( \mathcal{A}(\Sigma) \), where \( x \in e_{x} \) and \( y \in x \), we have \( f(\mu_{x, \Sigma}(y)) = f(y) = \mu_{\tilde{f}(x), \Sigma'}(f(y)) \) if \( x \neq y \). If \( x = y \), then

\[
 f(\mu_{x, \Sigma}(x)) = f \left( \frac{1}{x} \sum_{\substack{z_{h_{\alpha}} \in x_{\Sigma} \cap \mathbb{Z}, h_{\alpha} > 0}} z^{h_{\alpha}} + \sum_{\substack{z_{h_{\alpha}} \in x_{\Sigma} \cap \mathbb{Z}, h_{\alpha} < 0}} z^{-h_{\alpha}} \right) = \frac{1}{x} \sum_{\substack{z_{h_{\alpha}} \in x_{\Sigma'} \cap \mathbb{Z}, h_{\alpha} > 0}} z^{h_{\alpha}} + \sum_{\substack{z_{h_{\alpha}} \in x_{\Sigma'} \cap \mathbb{Z}, h_{\alpha} < 0}} z^{-h_{\alpha}} = \mu_{x, \Sigma'}(x),
\]

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As defined in Definition 2.38, let \( f(x) = 1 \) for any \( x \in \mathbf{x} \setminus \mathbf{x}' \). Note that \( x_{\mu,\Sigma}(\mathbf{x}) = \mathbf{x} \setminus \{x\} \cup \{\mu,\Sigma(x)\} \), \( x_{\mu,\Sigma}(\mathbf{x}') = \mathbf{x}' \setminus \{x\} \cup \{\mu,\Sigma(x)\} \) and \( f : \mathcal{L}_{\Sigma,\mathcal{Q}} \to \mathbb{L}_{\Sigma,\mathcal{Q}} \) induces a specialization of \( \mu,\Sigma(\mathbf{x}) \) at \( x_{\mu,\Sigma}(\mathbf{x}) \setminus \mathbf{x}' \). Then we continue the induction by considering the specialization \( \tilde{f}_i : x_{\mu,\Sigma}(\mathbf{x}) \to x_{\mu,\Sigma}(\mathbf{x}') \). In fact, from the above discussion, the ring homomorphism \( \tilde{f} : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) satisfies conditions (CM1), (CM2) and (CM3). Therefore it is a rooted cluster morphism.

2. This can be easily proved by induction.

As defined in Definition 2.38, let \( f \) be a specialization of a seed \( \Sigma \). Denote by \( \Sigma_f \) the frozenization of \( \Sigma \) at \( \mathbf{x} \setminus \mathbf{x}' \). By \( \text{Ind}(\Sigma_f) = \{\Sigma_1^f, \Sigma_2^f, \cdots, \Sigma_t^f\} \), we denote the set of indecomposable components of \( \Sigma_f \). For each \( 1 \leq i \leq t \), let \( f_i : \mathcal{A}(\Sigma_f) \to \mathcal{A}(\Sigma') \) be the indecomposable component of \( f \) (see Theorem 2.27)).

**Proposition 2.40.** Under the above notations, assume that \( f \) induces a morphism \( \tilde{f} \) from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma') \) and \( f(x) = 1 \) for any cluster variables \( x \in \mathbf{x} \setminus \mathbf{x}' \), then

1. The morphism \( \tilde{f} \) is an ideal surjective rooted cluster morphism;
2. Each indecomposable component \( f_i \) induces a rooted cluster morphism \( \tilde{f}_i : \mathcal{A}(\Sigma_f^i) \to \mathcal{A}(f_i(\Sigma_f^i)) \) which is a composition of several simple specializations at frozen variables.

**Proof.**

1. Similar to the proof of Proposition 6.9 [ADS14].

2. For each \( 1 \leq i \leq t \), from Theorem 2.27, \( f_i : \mathcal{A}(\Sigma_f^i) \to \mathcal{A}(\Sigma') \) is ideal and thus induces a rooted cluster morphism from \( \mathcal{A}(\Sigma_f^i) \to \mathcal{A}(f_i(\Sigma_f^i)) \). Moreover, note that \( \tilde{f}_i : \mathcal{A}(\Sigma_f^i) \to \mathcal{A}(f_i(\Sigma_f^i)) \) is a specialization satisfies the conditions in Proposition 2.39 thus it is a composition of several simple specializations at frozen variables.

Note that the first statement in the above proposition is a generalized version of Proposition 6.9 [ADS14]. In [ADS14], the authors showed that in some special cases, specializing an exchangeable variable to 1 also induces a rooted cluster morphism from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma') \), for example, when the seed \( \Sigma \) arising from a marked Riemann surface or the seed \( \Sigma \) is a finite acyclic seed. For the case of \( \Sigma \) be finite acyclic, we have the following conclusion which can be viewed as a generalization of Corollary 6.14 [ADS14].

**Corollary 2.41.** Let \( f \) be a specialization of seed \( \Sigma \) with notations in Definition 2.38 Assume that \( f(x) = 1 \) for each \( x \in \mathbf{x} \setminus \mathbf{x}' \). If \( \Sigma \) is finite acyclic, then \( f \) induces a rooted cluster morphism from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma') \).

**Proof.** From the Proposition 2.40 it is sufficient to show that \( f \) induces a morphism from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma') \). Because \( \Sigma \) is finite acyclic and thus finite generated, using the assumption that \( f(x) = 1 \) for each \( x \in \mathbf{x} \setminus \mathbf{x}' \), we can prove this similar to the proof of Theorem 2.11(2). □
3 Relations with cluster structures in 2-Calabi-Yau triangulated categories

In this section, we consider the cluster substructure of a functorial finite extension-closed subcategory in a 2-Calabi-Yau triangulated category which has a cluster structure given by cluster tilting subcategories. We establish a relation between functorial finite extension-closed subcategories, cluster substructures and rooted cluster subalgebras.

3.1 Preliminaries of 2-Calabi-Yau triangulated categories

Let \( k \) be an algebraically closed field and \( C \) a triangulated category. We always assume that it is \( k \)-linear, Hom-finite and Krull-Schmidt. Denote by \([1]\) the shift functor in \( C \). \( C \) is called 2-Calabi-Yau if there is a bifunctorial isomorphism \( \text{Ext}^1_C(X, Y) \cong \text{Ext}^1_C(Y, X) \) for all objects \( X \) and \( Y \) in \( C \), where \( D = \text{Hom}_k(\cdot, k) \) is the \( k \)-duality. For a subcategory \( \mathcal{B} \) of \( C \), we always mean that \( \mathcal{B} \) is a full additive category which closed under direct sums, direct summands and isomorphisms. Denote by \( \text{ind}\mathcal{B} \) the set of isomorphism classes of indecomposable objects in \( \mathcal{B} \). For an object \( X \) in \( C \), we denote by \( \text{add}X \) the subcategory additive generated by \( X \). For a set \( \mathcal{X} \) of isomorphism classes of indecomposable objects in \( C \), we denote by \( \text{add}\mathcal{X} \) the subcategory additive generated by \( \mathcal{X} \). In this way, we have for a subcategory \( \mathcal{B}, \mathcal{B} = \text{add}(\text{ind}\mathcal{B}) \). For subcategories \( \mathcal{B} \) and \( \mathcal{D} \), we denote by \( \mathcal{B} \cap \mathcal{D} \) the subcategory \( \text{add}((\text{ind}\mathcal{B}) \cap \text{ind}\mathcal{D}) \). A subcategory \( \mathcal{B} \) of \( C \) is called contravariantly finite if for each object \( C \) in \( C \), there exist an object \( B \in \mathcal{B} \) and a morphism \( f \in \text{Hom}_C(B, C) \) such that the morphism \( f : \text{Hom}_C(\cdot, B) \rightarrow \text{Hom}_C(\cdot, C) \) is surjective on \( \mathcal{B} \), where \( f \) maps any morphism \( g \in \text{Hom}_C(B', B) \) to \( fg \) for each \( B' \in \mathcal{B} \). Here, \( f \) is called a right \( \mathcal{B} \)-approximation of the object \( C \). Moreover, if \( f : B \rightarrow C \) has no direct summand of the form \( D \rightarrow 0 \) as complex, we say \( f \) right minimal. A contravariantly finite subcategory, a left approximation and a left minimal map are defined in a dual way. If a subcategory is both contravariantly finite and contravariantly finite, then we call it functorially finite. A functorially finite subcategory \( \mathcal{T} \) of \( C \) is called a cluster tilting subcategory if the following conditions are satisfied:

1. \( \mathcal{T} \) is rigid, that is, \( \text{Ext}^1_C(T, T') = 0 \) for all \( T, T' \in \mathcal{T} \);
2. For an object \( X \in C, X \in \mathcal{T} \) if and only if \( \text{Ext}^1_C(X, T) = 0 \) for any \( T \in \mathcal{T} \).

An object \( T \) in \( C \) is called a cluster tilting object if \( \mathcal{T} = \text{add}T \) is a cluster tilting subcategory. The cluster tilting subcategories in a 2-Calabi-Yau triangulated category have many remarkable properties. For example the property given in the following

Lemma 3.1. (Theorem 5.3 [IY08]) Let \( \mathcal{T} \) be a cluster tilting subcategory in a 2-Calabi-Yau triangulated category \( C \) and \( T \) be an indecomposable object in \( \mathcal{T} \). Then there is a unique (up to isomorphism) indecomposable object \( T' \neq T \) in \( C \) such that \( \mu_T(\mathcal{T}) := \text{add}(\text{ind}\mathcal{T} \setminus \{T\} \cup \{T'\}) \) is a cluster tilting subcategory in \( C \), where \( T' \) belongs to the following two triangles:

\[
T \xrightarrow{f} E \xrightarrow{g} T' \rightarrow T[1]
\]  \hspace{1cm} \text{and} \hspace{1cm}

\[
T' \xrightarrow{s} E' \xrightarrow{t} T \rightarrow T'[1]
\]

with \( g \) and \( t \) are minimal right \( \text{add}(\text{ind} \mathcal{T} \setminus \{T\}) \)-approximation and \( f \) and \( s \) are minimal left \( \text{add}(\text{ind} \mathcal{T} \setminus \{T\}) \)-approximation.

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We recall the definitions of cotorsion pairs, relative cluster tilting subcategories of cotorsion pairs and t-structures from [IY08, Na11, ZZ12, BBD81, BR07].

Definition 3.3.

The cluster structure of a triangulated category is defined in [BIRS09] and also be studied in [FK10]. For the general definition of a cluster structure, we refer to [BIRS09]. In our settings, we use Theorem II 1.6 in [BIRS09] to define the cluster structure as follows. For the general definition of a cluster structure, we refer to [BIRS09]. In our settings, we use Theorem II 1.6 in [BIRS09] to define the cluster structure as follows.

Remark 3.4.

1. The conditions in the definition of cotorsion pairs provide that for any cotorsion pairs $(\mathcal{X}, \mathcal{Y})$ in the usual sense [KR08] [IY08], $\mathcal{X}$ and $\mathcal{Y}$ are functorially finite.

2. It is easy to see that $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair if and only if $\mathcal{X}$ is co-vanishingly finite and $(\mathcal{X}, \mathcal{Y}[1])$ is a torsion pair in the sense of Definition 2.2 in [IY08].

3. It is proved in [IY08] (Proposition 2.3) that the functorially finite extension-closed subcategories of $C$ one-to-one correspond to the cotorsion pairs in $C$. The existence of the exchange triangles for each indecomposable object in $T$ guarantees that $Q(T)$ is a locally finite quiver.

The cluster structure of a triangulated category is defined in [BIRS09] and also be studied in [FK10]. For the general definition of a cluster structure, we refer to [BIRS09]. In our settings, we use Theorem II 1.6 in [BIRS09] to define the cluster structure as follows.

Definition 3.2. We say that $C$ has a cluster structure given by its cluster tilting subcategories if the quiver $Q(T)$ has no loops nor 2-cycles for each cluster tilting subcategory $T$ of $C$.

We recall the definitions of cotorsion pairs, relative cluster tilting subcategories of cotorsion pairs and t-structures from [IY08, Na11, ZZ12, BBD81, BR07].

Definition 3.3.

1. A pair $(\mathcal{X}, \mathcal{Y})$ of functorially finite subcategories of $C$ is called a cotorsion pair if $\text{Ext}_1^C(\mathcal{X}, \mathcal{Y}) = 0$ and

\[
C = \mathcal{X} \ast \mathcal{Y}[1] := \{ Z \in C \mid \exists \text{ a triangle } X \to Z \to Y[1] \to X[1] \text{ in } C \text{ with } X \in \mathcal{X}, Y \in \mathcal{Y} \}.
\]

We call the subcategory $\mathcal{X}$ ($\mathcal{Y}$ respectively) the torsion subcategory (torsion-free subcategory respectively) of $(\mathcal{X}, \mathcal{Y})$ and $I = \mathcal{X} \cap \mathcal{Y}$ the core of $(\mathcal{X}, \mathcal{Y})$.

2. [BBD81, BR07] A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in $C$ is called a t-structure if $\mathcal{X}$ is closed under $[-1]$ (equivalently $\mathcal{Y}$ is closed under $[1]$).

3. [ZZ12] Let $\mathcal{X}$ be a subcategory of $C$. A functorially finite subcategory $\mathcal{D}$ of $\mathcal{X}$ is called a $\mathcal{X}$-cluster tilting subcategory provided that for an object $D$ of $\mathcal{X}$, $D \in \mathcal{D}$ if and only if $\text{Ext}_1^C(X, D) = 0$ (thus $\text{Ext}_1^C(D, X) = 0$) for any object $X \in \mathcal{X}$. From now on to the end of the paper, we always assume that a 2-Calabi-Yau triangulated category $C$ has a cluster tilting subcategory. For a cluster tilting subcategory $T$, one can associate it a quiver $Q(T)$ as follows: The vertices of $Q(T)$ are the isomorphism classes of indecomposable objects in $T$ and the number of arrows from $T_i$ to $T_j$ is given by the dimension of the space $\text{irr}(T_i, T_j) = \text{rad}(T_i, T_j)/\text{rad}^2(T_i, T_j)$ of irreducible morphisms, where $\text{rad}(,)$ is the radical in $T$. The existence of the exchange triangles for each indecomposable object in $T$ guarantees that $Q(T)$ is a locally finite quiver.

The subcategory $\mu_T(T)$ is the mutation of $T$ at $T$. The first triangle is the right exchange triangle of $T$ in $T$ and the second one is the left exchange triangle of $T$ in $T$. We say a cluster tilting subcategory $T'$ is reachable from $T$, if it can be obtained by finite number of mutations from $T$. A rigid object in $C$ is called reachable from $T$ if it belongs to a cluster tilting subcategory which is reachable from $T$. Denote by $\mathcal{R}(T)$ the subcategory of $C$ additive generated by rigid objects in $C$ which are reachable from $T$.
4. It can be easily derived from the 2-Calabi-Yau property of \( C \) that if \((\mathcal{X}, \mathcal{Y})\) is a cotorsion pair of \( C \), then \((\mathcal{Y}, \mathcal{X})\) is also a cotorsion pair. Due to the symmetry, \( \mathcal{X} \) and \( \mathcal{Y} \) possess similar properties. Thus for the convenience, we only state definitions and properties about torsion subcategory \( \mathcal{X} \) in the following of the paper.

3.2 Cluster structures in subfactor triangulated categories

Let \( I \) be a functorially finite rigid subcategory in \( C \). Then the subfactor category \( C' = \perp I[1]/I \) is a 2-Calabi-Yau triangulated category with triangles and cluster tilting subcategories induced from triangles and cluster tilting subcategories of \( C \) respectively \([1Y08]\). For an object \( X \in \perp I[1], \) we denote also by \( X \) the object in the quotient category \( C' \). For a morphism \( f \in \text{Hom}_C(X_1, X_2) \) with \( X_1, X_2 \in \perp I[1], \) we denote by \( f \) the residue class of \( f \) in the quotient category \( C' \). Now assume that \( C \) has a cluster structure, we show in this subsection that the subfactor category \( \perp I[1]/I \) inherits the cluster structure from \( C \).

**Lemma 3.5.** Let \( T \) be a cluster tilting subcategory in \( C' \) and \( T \) be an indecomposable object in \( T \). Then \( T \oplus I \) is a cluster tilting subcategory of \( C \).

1. Let triangles

\[
T \rightarrow E \rightarrow T' \rightarrow T(1) \quad \text{and} \quad T' \rightarrow E' \rightarrow T \rightarrow T'(1)
\]

be the exchange triangles of \( T \) in \( T \), where \( (1) \) is the shift functor in \( C' \). Then in the triangulated category \( C \) the exchange triangles of \( T \) in \( T \oplus I \) are of the forms

\[
\ast \quad T \rightarrow E \oplus I \rightarrow T' \rightarrow T[1] \quad \text{and} \quad T' \rightarrow E' \oplus I' \rightarrow T \rightarrow T'[1],
\]

where \( I \) and \( I' \) are both in \( I \).

2. Let triangles

\[
T \rightarrow E \oplus I \rightarrow T' \rightarrow T[1] \quad \text{and} \quad T' \rightarrow E' \oplus I' \rightarrow T \rightarrow T'[1]
\]

be the exchange triangles of \( T \) in \( T \oplus I \), where \( I \) and \( I' \) are both in \( I \), and \( E \) and \( E' \) have no direct summands in \( I \). Then in the triangulated category \( C' \), they respectively induce

\[
T \rightarrow E \rightarrow T' \rightarrow T(1) \quad \text{and} \quad T' \rightarrow E' \rightarrow T \rightarrow T'(1)
\]

as the exchange triangles of \( T \) in \( T \).

**Proof.** 1. It follows from \([1Y08]\)(Theorem 4.9) that \( T \oplus I \) is a cluster tilting subcategory of \( C \). Because \( T \rightarrow E \rightarrow T' \rightarrow T(1) \) is a triangle in \( C' \), from the triangulated structure
of $C'$ (see section 4 in [IY08]), we have the following commutative diagram of morphisms between triangles in $C$:

\[
\begin{array}{ccccccc}
\star & T & \rightarrow & E \oplus I_1 & \rightarrow & T' \oplus I_2 & \rightarrow & T[1] \\
\| & & \downarrow^{(b_1)} & & \downarrow & & \| \\
T & \alpha & \rightarrow & I_0 & \rightarrow & T(1) & \rightarrow & T[1],
\end{array}
\]

where $\alpha$ is a left $I$-approximation in $C$, and $I_1$ and $I_2$ are both in $I$. We claim that $(a_1 \ a_2)$ is a left $\text{add}(\text{ind}(T \oplus I) \setminus \{T\})$-approximation in $C$. Let $T_0$ be an indecomposable object in $\text{add}(\text{ind}(T \oplus I) \setminus \{T\})$ and $f$ be a morphism in $\text{Hom}_C(T, T_0)$. If $T_0 \in I$, then there exists $g \in \text{Hom}_C(I_0, T_0)$ such that $f = ga$ because $\alpha$ is a left $I$-approximation. Thus we have $f = ga_1b_1 + ga_2b_2$, that is, $f$ factor through $(a_1, a_2)$. If $T_0 \in \text{add}(\text{ind}T \setminus \{T\})$, then there exists $h \in \text{Hom}_C(E, T_0)$ such that $f = ha_1$ since $a_1$ is a left $\text{add}(\text{ind}T \setminus \{T\})$-approximation in $C$. Therefore the right exchange triangle of $T$ in $\mathcal{T} \oplus I$ is a direct summand of the triangle $\star$ as a complex. Note that each indecomposable direct summand of $E$ is not in $\text{add}I_2$, then it belongs to the middle term of the right exchange triangle of $T$ in $\mathcal{T} \oplus I$. Therefore the right exchange triangle of $T$ is of the form of triangle $\star$. Similarly we can prove the case of the left exchange triangle of $T$ in $\mathcal{T} \oplus I$.

2. The right exchange triangle $T \xrightarrow{(a_1 \ a_2)} E \oplus I \rightarrow T' \rightarrow T[1]$ in $C$ induces an triangle $T \xrightarrow{a_1} E \rightarrow T' \rightarrow T(1)$ in $C'$ [IY08]. It is easy to see that $a_1$ is a minimal left $\text{add}(\text{ind}T \setminus \{T\})$-approximation in $C'$ due to $(a_1, a_2)$ is a minimal left $\text{add}(\text{ind}(T \oplus I) \setminus \{T\})$-approximation in $C$. Therefore by the uniqueness of the minimal left approximation, $T \xrightarrow{a_1} E \rightarrow T' \rightarrow T(1)$ is the right exchange triangle of $T$ in $\mathcal{T}$. Similarly we can prove the case of the left exchange triangle.

\[\square\]

**Proposition 3.6.** The subfactor category $\mathcal{T}[1]/I$ has a cluster structure.

**Proof.** Let $\mathcal{T}$ be a cluster tilting subcategory of $C'$, we claim that its quiver $Q(\mathcal{T})$ is a full subquiver of $Q(\mathcal{T} \oplus I)$. It is only need to show that for any indecomposable objects $T$ and $T_0$ in $\mathcal{T}$, by viewing $T$ and $T_0$ as the vertices in the quiver, the number of arrows between these two vertices in the quiver $Q(\mathcal{T})$ is equal to the number of arrows between these vertices in the quiver $Q(\mathcal{T} \oplus I)$. In fact, because in the right exchange triangle $T \xrightarrow{f} E \rightarrow T' \rightarrow T(1)$ of $T$ in $\mathcal{T}$, $f$ is a minimal left $\text{add}(\text{ind}T \setminus \{T\})$ approximation of $T$. Thus $T_0$ is a direct summand of $E$ with degree $n$ if and only if there are irreducible morphism from $T$ to $T_0$ and the dimension of the irreducible morphism space $\text{irr}(T, T_0)$ is equal to $n$. Therefore there are $n$ arrows from $T$ to $T_0$ in the quiver $Q(\mathcal{T})$. By considering the left exchange triangle $T' \xrightarrow{\tilde{g}} E' \rightarrow T \rightarrow T'(1)$ of $T$ in $\mathcal{T}$, a similar argument shows that the degree of $T_0$ in $E'$ is equal to the number of arrows from $T_0$ to $T$ in the quiver $Q(\mathcal{T})$. Similarly, the numbers of arrows between vertices in $Q(\mathcal{T} \oplus I)$ are also determined by the exchange triangles in $\mathcal{T} \oplus I$. Then it follows from the above lemma that the quiver $Q(\mathcal{T})$ is a full subquiver of $Q(\mathcal{T} \oplus I)$ by deleting vertices given by isomorphism.

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classes of indecomposable objects in \( I \). Thus \( Q(T) \) has no 2-cycles since \( T \oplus I \) is a cluster tilting subcategory of \( C \) which has a cluster structure; moreover, it is clearly that \( Q(T) \) has no loops. So we are done. \( \square \)

### 3.3 Cluster substructures in cotorsion pairs

In [ZZ12], the authors studied the cotorsion pairs in a 2-Calabi-Yau triangulated category with cluster tilting objects and give a classification of cotorsion pairs in the category. For 2-Calabi-Yau triangulated category \( C \) with a cluster tilting subcategory, we have the following analogy of Proposition 5.3 in [ZZ12]. Before state the proposition, we recall a result in [BIRS09], which is a key point in the following study of cluster substructures in cotorsion pairs.

**Lemma 3.7.** (Proposition II.2.3 [BIRS09]) Let \( (\mathcal{X}, \mathcal{Y}) \) be a cotorsion pair of \( C \). Then the core \( I = \mathcal{X} \cap \mathcal{Y} \) is a functorially finite rigid subcategory and there is a decomposition of triangulated category \( I[1]/I = \mathcal{X}/I \oplus \mathcal{Y}/I \).

**Proposition 3.8.** Let \( (\mathcal{X}, \mathcal{Y}) \) be a cotorsion pair of \( C \) with core \( I = \mathcal{X} \cap \mathcal{Y} \). Assume that there is a cluster tilting subcategory \( T \) contains \( I \) as a subcategory. Then

1. Any cluster tilting subcategory \( T' \) containing \( I \) as a subcategory can be uniquely written as \( T' = \mathcal{T}_X \oplus I \oplus \mathcal{T}_Y \), such that \( \mathcal{T}_X \oplus I \) is a \( \mathcal{X} \)-cluster tilting subcategory and \( T_Y \oplus I \) is a \( \mathcal{Y} \)-cluster tilting subcategory.

2. Any \( \mathcal{X} \)-cluster tilting subcategory is of the form \( T_{X} \oplus I \oplus T_{Y} \), where \( T' = \mathcal{T}_X \oplus I \oplus T_{Y} \) is a cluster tilting subcategory of \( C \) and \( T_Y \oplus I \) is a \( \mathcal{Y} \)-cluster tilting subcategory.

3. The correspondence \( T' \rightarrow T_{X} \oplus I \oplus T_{Y} \) gives a bijection between the cluster tilting subcategories in \( C \) containing \( I \) as a subcategory and the pairs of the \( \mathcal{X} \)-cluster tilting subcategories and of the \( \mathcal{Y} \)-cluster tilting subcategories.

**Proof.** Because \( I \subseteq T \) and \( T \) is rigid, we have \( T' \subseteq \mathcal{T}_X[I] \); moreover due to the decomposition \( \mathcal{T}_X[I] = \mathcal{T}_X[I] / \mathcal{T}_Y[I] \), \( T \) can be decomposed as \( \mathcal{T}_X[I] \oplus I \oplus \mathcal{T}_Y[I] \) in the quotient category \( \mathcal{X}/I \), where \( \mathcal{T}_X[I] \subseteq \mathcal{X} \) and \( T_Y[I] \subseteq \mathcal{Y} \). Thus \( T = \mathcal{T}_X[I] \oplus I \oplus \mathcal{T}_Y[I] \) in \( C \). By using the decomposition \( \mathcal{T}_X[I] = \mathcal{T}_X[I] / \mathcal{T}_Y[I] \), the proof of (1) is similar to the proof of Proposition 5.3(1) in [ZZ12]. For the statement (2), note that the existence of the cluster tilting subcategory \( T \) guarantees that any \( \mathcal{X} \)-cluster tilting subcategory can be extended as a cluster tilting subcategory in \( C \).

In fact, it is clearly that any \( \mathcal{X} \)-cluster tilting subcategory contains \( I \) as a subcategory, and let \( T_{X} \oplus I \) be a \( \mathcal{X} \)-cluster tilting subcategory. Then we claim that \( T_{X} \) is a cluster tilting subcategory in \( \mathcal{X}/I \). Let \( X \) be an object in \( \mathcal{X}/I \), because \( T_{X} \oplus I \) is contravariantly finite in \( \mathcal{X} \), there exist an object \( T \in T_{X} \oplus I \) and a morphism \( f \in \text{Hom}_C(T, X) \) such that \( f : \text{Hom}_C(-, T) \rightarrow \text{Hom}_C(-, X) \) is surjective on \( T_{X} \oplus I \). Then it is easy to check that \( f : \text{Hom}_{\mathcal{X}/I}(-, T) \rightarrow \text{Hom}_{\mathcal{X}/I}(-, X) \) is surjective on \( T_{X} \). Therefore \( T_{X} \) is contravariantly finite in \( \mathcal{X}/I \). Similarly, \( T_{Y} \) is contravariantly finite and thus functorially finite in \( \mathcal{X}/I \). Let \( X \) be an object in \( \mathcal{X}/I \) such that \( \text{Ext}^{1}_{\mathcal{X}/I}(T, X) = 0 \) for any \( T \in T_{X} \). We now prove that \( X \in T_{Y} \). In fact, from \( \text{Ext}^{1}_{\mathcal{X}/I}(T, X) = \text{Ext}^{1}_{\mathcal{X}/I}(T, X) = 0 \), we have \( \text{Ext}^{1}_{C}(I, X) = 0 \) due to Lemma 4.8 in [IY08]. Then we have \( \text{Ext}^{1}_{C}(T, X) = 0 \) for any \( T \in T_{X} \oplus I \) since \( \text{Ext}^{1}_{C}(I, X) = 0 \) for any \( I \in I \). Then \( X \in T_{X} \oplus I \) since \( T_{X} \oplus I \) is a \( \mathcal{X} \)-cluster tilting subcategory in \( \mathcal{X}/I \). Therefore \( X \) belongs to \( T_{X} \).

We have proved the claim. Note that \( T_Y \oplus I \) is a \( \mathcal{Y} \)-cluster tilting subcategory in \( \mathcal{Y} \) from the
statement (1), then by a similar argument, \( \mathcal{C}_\mathcal{X} \) is a cluster tilting subcategory in \( \mathcal{Y}/\mathcal{I} \). Then from Lemma 3.7, it is clearly that \( \mathcal{C}_{\mathcal{X}}' \oplus \mathcal{C}_\mathcal{Y} \) is a cluster tilting subcategory in \( \mathcal{Y}/\mathcal{I} \). Therefore \( \mathcal{C}_{\mathcal{X}}' \oplus \mathcal{I} \oplus \mathcal{C}_\mathcal{Y} \) is a cluster tilting subcategory in \( \mathcal{C} \) \([\text{[Y08]}\text{Theorem 4.9}]\). Finally the statement (3) follows from (1) and (2).

**Definition 3.9.** For a \( \mathcal{X} \)-cluster tilting subcategory \( \mathcal{C}_{\mathcal{X}}' \oplus \mathcal{I} \) in \( \mathcal{X} \), we define \( Q(\mathcal{C}_{\mathcal{X}}' \oplus \mathcal{I}) \) as an ice quiver with the exchangeable vertices given by the isomorphism classes of indecomposable objects in \( \mathcal{C}_{\mathcal{X}}' \) and the frozen vertices given by the isomorphism classes of indecomposable objects in \( \mathcal{I} \). For two vertices \( T_i \) and \( T_j \) (not both the frozen variables), the number of arrows from \( T_i \) to \( T_j \) is given by the dimension of irreducible morphism space \( \text{irr}(T_i, T_j) \) in \( \mathcal{C}_{\mathcal{X}}' \oplus \mathcal{I} \).

Now we state the main result in this subsection.

**Theorem 3.10.** Let \( \mathcal{C} \) be a 2-Calabi-Yau triangulated category with a cluster tilting subcategory and \((\mathcal{X}, \mathcal{Y})\) be a cotorsion pair of \( \mathcal{C} \) with core \( \mathcal{I} = \mathcal{X} \cap \mathcal{Y} \). Assume that there is a cluster tilting subcategory in \( \mathcal{C} \) contains \( \mathcal{I} \) as a subcategory. If the cluster tilting subcategories in \( \mathcal{C} \) forms a cluster structure, then the \( \mathcal{X} \)-cluster tilting subcategories form a cluster structure of \( \mathcal{X} \) with coefficient subcategory \( \mathcal{I} \): it is a cluster substructure of \( \mathcal{C} \) in the sense of section II.2\([\text{BIRS09]}\], more precisely, the following conditions are satisfied:

1. For each \( \mathcal{X} \)-cluster tilting subcategory \( \mathcal{C}_{\mathcal{X}} \oplus \mathcal{I} \) in \( \mathcal{X} \) and an indecomposable object \( T_0 \) in \( \mathcal{C}_{\mathcal{X}} \), there is a unique (up to isomorphism) indecomposable object \( T_0' \not\cong T_0 \) in \( \mathcal{X} \) such that \( T_0' \oplus I := \text{add}(\text{ind} \mathcal{C}_{\mathcal{X}} \setminus \{T_0\} \cup \{T_0'\}) \oplus \mathcal{I} \) is a \( \mathcal{X} \)-cluster tilting subcategory in \( \mathcal{X} \).

2. In the situation of (1), there are triangles

\[
T_0 \xrightarrow{f} E \oplus I \xrightarrow{g} T_0' \xrightarrow{g'} T_0[1]
\]

in \( \mathcal{X} \) with \( g \) and \( t \) are minimal right \((\mathcal{C}_{\mathcal{X}} \cap \mathcal{C}_{\mathcal{Y}}') \oplus \mathcal{I}\)-approximation and \( f \) and \( s \) are minimal left \((\mathcal{C}_{\mathcal{X}} \cap \mathcal{C}_{\mathcal{Y}}') \oplus \mathcal{I}\)-approximation. The subcategory \( \mu_{T_0}(\mathcal{C}_{\mathcal{X}} \oplus \mathcal{I}) := \mathcal{C}_{\mathcal{X}}' \oplus \mathcal{I} \) is called the mutation of \( \mathcal{C}_{\mathcal{X}} \oplus \mathcal{I} \) at \( T_0 \). These two triangles are called the right exchange triangle and the left exchange triangle of \( T_0 \) in \( \mathcal{C}_{\mathcal{X}} \oplus \mathcal{I} \) respectively.

3. For each \( \mathcal{X} \)-cluster tilting subcategory \( \mathcal{C}_{\mathcal{X}} \oplus \mathcal{I} \) in \( \mathcal{X} \), there are no loops nor 2-cycles in the ice quiver \( Q(\mathcal{C}_{\mathcal{X}} \oplus \mathcal{I}) \).

4. In the situation of (1), passing from \( Q(\mathcal{C}_{\mathcal{X}} \oplus \mathcal{I}) \) to \( Q(\mathcal{C}_{\mathcal{X}}' \oplus \mathcal{I}) \) is given by the Fomin-Zelevinsky mutation at the vertex of \( Q(\mathcal{C}_{\mathcal{X}} \oplus \mathcal{I}) \) corresponding to \( T_0 \).

5. There is a subcategory \( \mathcal{A} \) of \( \mathcal{C} \) such that \( \mu_{T_0} \circ \circ \circ \mu_{T_0}(\mathcal{C}_{\mathcal{X}} \oplus \mathcal{I}) \oplus \mathcal{A} \) is a cluster tilting subcategory in \( \mathcal{C} \) for any finite mutation \( \mu_{T_0} \circ \circ \circ \mu_{T_0}(\mathcal{C}_{\mathcal{X}} \oplus \mathcal{I}) \) of \( \mathcal{C}_{\mathcal{X}} \oplus \mathcal{I} \).

**Proof.** 1. From Proposition 3.8 we can assume that each \( \mathcal{X} \)-cluster tilting subcategory is of the form \( \mathcal{C}_{\mathcal{X}} \oplus \mathcal{I} \), where \( \mathcal{C} = \mathcal{C}_{\mathcal{X}} \oplus \mathcal{I} \oplus \mathcal{C}_{\mathcal{Y}} \) is a cluster tilting subcategory in \( \mathcal{C} \) with \( \mathcal{C}_{\mathcal{Y}} \in \mathcal{Y} \). Let

\[
T_0 \xrightarrow{f} E \oplus I \xrightarrow{g} T_0' \xrightarrow{g'} T_0[1] \quad \text{and} \quad (5)
\]

\[
T_0' \xrightarrow{s} E' \oplus I' \xrightarrow{t} T_0 \xrightarrow{t'} T_0'[1] \quad (6)
\]
be the exchange triangles of $\mathcal{T}$ in $C$ with $I \in \mathcal{I}$ and $E$ and $E'$ have no direct summands in $I$. Then from Lemma 3.5 they induce exchange triangles

$$T_0 \xrightarrow{f} E \xrightarrow{g} T_0' \rightarrow T_0(1) \quad \text{and}$$  

$$T_0 \xrightarrow{s} E' \xrightarrow{t} T_0 \rightarrow T_0'(1)$$  

in the subfactor category $\mathcal{T}^\perp I[1]/I$ respectively. Moreover we have $E \in \mathcal{T}_X \oplus I$ in the triangle (5), this is because $\mathcal{T}^\perp I[1]/I = \mathcal{T}_X/I \oplus \mathcal{T}_Y/I$ and thus any morphism from $T_0$ to $\mathcal{T}_Y$ factor through $I$. Similarly, $E' \in \mathcal{T}_X \oplus I$ in the triangle (6). Since $\mathcal{T}_X/I$ is a triangulated subcategory of $\mathcal{T}^\perp I[1]/I$, the triangles (7) and (8) are both in $\mathcal{T}_X/I$ and thus $T_0'$ belongs to $\mathcal{T}_X$. Because $\mathcal{T}_X' \oplus I \oplus \mathcal{T}_Y$ is a cluster tilting subcategory in $C$, $\mathcal{T}_X' \oplus I$ is a $\mathcal{X}$-cluster tilting subcategory due to Proposition 3.3. The uniqueness of $\mathcal{T}_X' \oplus I$ comes from the uniqueness of $\mathcal{T}_X' \oplus I \oplus \mathcal{T}_Y$. We have proved the statement.

2. It is only need to show that in the triangles (5) and (6), $g$ and $t$ are minimal right $(\mathcal{T}_X \cap \mathcal{T}_Y') \oplus I$-approximation and $f$ and $s$ are minimal left $(\mathcal{T}_X \cap \mathcal{T}_Y') \oplus I$-approximation. This easily follows from the fact that (5) and (6) are the exchange triangles.

3. We claim that the arrows between any two vertices (not both frozen) in the ice quiver $Q(\mathcal{T}_X \oplus I)$ are coincide with the arrows between these vertices in the quiver $Q(\mathcal{T})$. In fact, given vertices $T_0$ and $T_1$ in the ice quiver $Q(\mathcal{T}_X \oplus I)$ with $T_0$ exchangeable, by unifying the vertices in the quiver and the isomorphism classes of indecomposable objects in subcategory, the proof of the statement (2) shows that $\mathcal{T}_X \oplus I$ and $\mathcal{T}$ have the same exchange triangles at $T_0$, thus the number of arrows from $T_0$ to $T_1$ and the number of arrows from $T_1$ to $T_0$ in the quivers $Q(\mathcal{T}_X \oplus I)$ and $Q(\mathcal{T})$ are all determined by degree of $T_1$ in $E \oplus I$ and $E' \oplus I'$ respectively. Thus the conclusion follows from the assumption that $C$ has a cluster structure.

4. We can prove this similar to the proof of Theorem II.1.6 in [BIRS09].

5. Let $\mathcal{A} = \mathcal{T}_Y$ be the subcategory of $\mathcal{Y}$ in the proof of statement (1). Note that any finite mutation $\mu_{T_0} \circ \cdots \mu_{T_0}(\mathcal{T}_X \oplus I)$ is a cluster tilting subcategory in $\mathcal{T}_X/I$, then from the proof of Proposition 3.3, $\mu_{T_0} \circ \cdots \mu_{T_0}(\mathcal{T}_X \oplus I) \oplus \mathcal{T}_Y$ is a cluster tilting subcategory in $C$.

\[ \square \]

Remark 3.11. 1. If $C$ has a cluster tilting object (for example, cluster categories [BMRRT06], Amiot’s generalized cluster categories [A09], stable categories related to preprojective algebras [GLS06], [BIRS09]), then each cluster tilting subcategory of $C$ has finite number of isomorphism classes of indecomposable objects and each rigid subcategory can be completed as a cluster tilting subcategory [Y08, D08, ZZ]. Thus there always exists a cluster tilting subcategory containing rigid subcategory $I$ as a subcategory, which is assumed in above theorem.

2. It follows from the proof of the above theorem that for each $\mathcal{X}$-cluster tilting subcategory, its exchange triangles are coincide with the relative exchange triangles of any cluster tilting subcategory of $C$ which contains it.
3. It is proved in [BIRS09] (Theorem II.3.8 and Theorem III.5.1) that some functorially finite extension closed subcategories arising from the module category of preprojective algebras have cluster substructures of the module category.

4. The mutation of $\mathcal{X}$-cluster tilting objects and the weak cluster structure of $\mathcal{X}$ were also defined in [ZZ11], see Remark 5.11[ZZ11].

3.4 Cluster structures and rooted cluster algebras

In this subsection, we fix the following settings. We always assume that $\mathcal{C}$ is a 2-Calabi-Yau triangulated category with a cluster structure given by its cluster tilting subcategories. Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair of $\mathcal{C}$ with core $I = \mathcal{X} \cap \mathcal{Y}$, where we assume that $I$ is a subcategory of a cluster tilting subcategory $\mathcal{T}$ in $\mathcal{C}$. Then Proposition 3.8 guarantees that we can write $\mathcal{T} = \mathcal{T}_X \oplus I \oplus \mathcal{T}_Y$ with $\mathcal{T}_X \oplus I$ being $\mathcal{X}$-cluster tilting and $\mathcal{T}_Y \oplus I$ being $\mathcal{Y}$-cluster tilting. We collect the following notations which we fix in this subsection.

- $Q(\mathcal{T})$: The quiver of the cluster tilting subcategory $\mathcal{T}$.
- $Q(\mathcal{T}_X \oplus I)$: The ice quiver of the $\mathcal{X}$-cluster tilting subcategory $\mathcal{T}_X \oplus I$.
- $Q(\mathcal{T}_Y \oplus I)$: The ice quiver of the $\mathcal{Y}$-cluster tilting subcategory $\mathcal{T}_Y \oplus I$.
- $Q(\mathcal{T}_I)$: The ice quiver by freezing the vertices in $Q(\mathcal{T})$ which are determined by the isomorphism classes of the indecomposable objects in $I$.
- $Q(\mathcal{T} \setminus I)$: The quiver of the cluster tilting subcategory $\mathcal{T} \setminus I$ in the subfactor triangulated category $\mathcal{C}[1]/\mathcal{I}$.

The rooted cluster algebras corresponding to the above quivers are denoted by $\mathcal{A}(\mathcal{T})$, $\mathcal{A}(\mathcal{T}_X \oplus I)$, $\mathcal{A}(\mathcal{T}_Y \oplus I)$, $\mathcal{A}(\mathcal{T}_I)$ and $\mathcal{A}(\mathcal{T} \setminus I)$ respectively.

- $R(\mathcal{T})$: The subcategory of $\mathcal{C}$ additive generated by rigid objects which are reachable from $\mathcal{T}$ by finite number of mutations, where the mutation is defined in Subsection 3.1.
- $R(\mathcal{T}_X \oplus I)$: The subcategory of $\mathcal{C}$ additive generated by rigid objects which are reachable from $\mathcal{T}_X \oplus I$ by finite number of mutations, where the mutation is defined in Theorem 3.10(2).
- $R(\mathcal{T}_Y \oplus I)$: The subcategory of $\mathcal{C}$ additive generated by rigid objects which are reachable from $\mathcal{T}_Y \oplus I$ by finite number of mutations.
- $R(\mathcal{T}_I)$: The subcategory of $\mathcal{C}$ additive generated by rigid objects which are reachable from $\mathcal{T}_I$ by finite number of mutations not at the indecomposable objects in the subcategory $\mathcal{I}$.
- $R(\mathcal{T} \setminus I)$: The subcategory of $\mathcal{C}$ additive generated by rigid objects which are reachable from $\mathcal{T} \setminus I$ in the subfactor triangulated category $\mathcal{C}[1]/\mathcal{I}$, where we view these reachable rigid objects as objects in $\mathcal{C}$.

Then from Lemma 3.5, it is not hard to see that $\text{ind} R(\mathcal{T}_I) = \text{ind} R(\mathcal{T} \setminus I) \sqcup \text{ind} I$. We have a sequence $R(\mathcal{T}_X \oplus I) \subseteq R(\mathcal{T}_I) \subseteq R(\mathcal{T})$ of inclusions, where the first one follows from Theorem 3.10(2) and the second one is clearly. Denote by $i_1 : R(\mathcal{T}_X \oplus I) \to R(\mathcal{T}_I)$ and $i_2 : R(\mathcal{T}_I) \to R(\mathcal{T})$ the natural embedding functors under the above inclusions. We define a canonical functor
\( p : R(T) \to R(T \setminus I) \), which is an identity on \( R(T \setminus I) \) and maps an object in \( I \) to zero object.

The cluster map, which is called the cluster character in Palu08, is defined in BIRS09. One can use it to transform a cluster structure in \( C \) to a cluster algebra. We recall the following definition of cluster map from BIRS09 and FK10 in our settings. For more details, we refer to BIRS09 and FK10.

**Definition 3.12.** Denote by \( \mathbb{Q}(X_T) \) the rational function field of \( X_T \), where \( X_T \) is the indeterminate set which is indexed by the isomorphism classes of indecomposable objects in \( T \). A map \( \varphi \) from \( R(T) \) to \( \mathbb{Q}(X_T) \) is called a cluster map if the following conditions are satisfied:

1. For the object \( M \cong M' \), we have \( \varphi(M) = \varphi(M') \).
2. For any indecomposable object \( T_i \) in \( T \), we have \( \varphi(T_i) = x_i \) where \( x_i \in X_T \) is the element indexed by \( T_i \).
3. For any \( M \) and \( N \) in \( R(T) \) with \( \dim \text{Ext}^1_C(M, N) = 1 \) (thus \( \dim \text{Ext}^1_C(N, M) = 1 \)), we have \( \varphi(M)\varphi(N) = \varphi(V) + \varphi(V') \) where \( V \) and \( V' \) are in the non-split triangles

\[
M \to V \to N \to M[1] \quad \text{and} \quad N \to V' \to M \to N[1].
\]
4. For any \( M \) and \( N \) in \( R(T) \), we have \( \varphi(M \oplus N) = \varphi(M)\varphi(N) \). In particular, \( \varphi(0) = 1 \).

Then the cluster map \( \varphi \) constructs a connection between a cluster structure (cluster substructure respectively) of \( C \) and the rooted cluster algebras (the rooted cluster subalgebras respectively). More precisely, we have the following proposition, where the first statement can be proved similar to the proof of Proposition 2.3 in FK10 and the second one can be easily derived from the first one and Theorem 3.10. The last one is clearly follows from the first one.

**Proposition 3.13.** 1. The map \( \varphi \) induces a surjection from the set of isomorphism classes of indecomposable objects in \( R(T) \) onto the set of cluster variables in \( A(T) \), and also induces a surjection from the set of cluster tilting subcategories reachable from \( T \) onto the set of clusters of \( A(T) \).

2. The map \( \varphi \) induces a surjection \( \varphi_1 \) from the set of isomorphism classes of indecomposable objects in \( R(T \setminus X \oplus I) \) onto the set of cluster variables in \( A(T \setminus X \oplus I) \), and also induces a surjection from the set of \( X \)-cluster tilting subcategories reachable from \( T \setminus X \oplus I \) onto the set of clusters of \( A(T \setminus X \oplus I) \).

3. The map \( \varphi \) induces a surjection \( \varphi_2 \) from the set of isomorphism classes of indecomposable objects in \( R(T_I) \) onto the set of cluster variables in \( A(T_I) \), and also induces a surjection from the set of cluster tilting subcategories reachable from \( T \) by finite number of mutations not at indecomposable objects in \( I \) onto the set of clusters of \( A(T_I) \).

From Theorem 3.6 the subfactor category \( \perp - I[1] / I \) inherits a cluster structure. By view \( R(T \setminus I) \) as a subcategory in \( \perp - I[1] / I \), we denote by \( \varphi' \) the cluster map from \( R(T \setminus I) \) to \( A(T \setminus I) \). Now we state our main result in this section.

**Theorem 3.14.** Under the above settings,
1. The rooted cluster algebras $A(T_X \oplus I)$ and $A(T_Y \oplus I)$ are both rooted cluster subalgebras of $A(T_I)$ and thus rooted cluster subalgebras of $A(T)$. Moreover, $A(T_I)$ is the glue of $A(T_X \oplus I)$ and $A(T_Y \oplus I)$ at $\varphi(I)$. 

2. Any rooted cluster subalgebra of $A(T)$ with coefficient set $\varphi(I)$ such that $I$ is functorially finite in $C$ is of the form $A(T_X' \oplus I)$, where $T_X' \oplus I$ is a $X'$-cluster tilting in a cotorsion pair $(X', Y')$ with core $I$. 

3. The correspondences $(X', Y') \mapsto (A(T_X' \oplus I), A(T_Y' \oplus I))$ give a bijection between the following two sets:

\[ \{ \text{Cotorsion pairs in } C \text{ with core } I \} \]
\[ \downarrow \]
\[ \{ \text{Complete pairs of rooted cluster subalgebras of } A(T) \text{ with coefficient set } \varphi(I) \text{ such that } I \text{ is functorially finite in } C \} \]

This bijection induces the following bijection:

\[ \{ \text{T-structures in } C \} \]
\[ \downarrow \]
\[ \{ \text{Complete pairs of rooted cluster subalgebras of } A(T) \text{ without coefficients} \} \]

4. The specialization at $\varphi(I)$ induces a rooted cluster surjection $\pi$ from $A(T_I)$ to $A(T \setminus I)$. 

5. We have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{R}(T \setminus I) & \xrightarrow{\varphi} & \mathcal{R}(T_I) \\
\downarrow i_1 & & \downarrow i_2 \\
\mathcal{R}(T_X \oplus I) & \xrightarrow{\varphi_1} & \mathcal{A}(T \setminus I) \\
\downarrow \varphi & & \downarrow \varphi_2 \\
\mathcal{A}(T_X \oplus I) & \xrightarrow{j_1} & \mathcal{A}(T_I) \\
\downarrow & & \downarrow j_2 \\
\mathcal{A}(T_X \oplus I) & & \mathcal{A}(T_X \oplus I) \\
\end{array}
\]

where $j_1$ and $j_2$ are injections arising from subalgebras in the first statement.

**Proof.**

1. On the one hand, from Theorem 5.10 the quiver $Q(T_X \oplus I)$ is a full subquiver of $Q(T_I)$. On the other hand, from the decomposition of the triangulated category $\perp I[1]/I = X'/I \oplus Y'/I$, the morphisms between objects in $T_X$ and $T_Y$ factor through $I$ and thus in the quiver $Q(T_I)$ there are no arrows between the vertices in $indT_X$ and the vertices in $indT_Y$. Therefore $Q(T_X \oplus I)$ is a glue of some connected components of $Q(T_I)$. Thus from the Remark 2.31(1), $A(T_X \oplus I)$ is a rooted cluster subalgebra of $A(T_I)$. Similarly,
Remark 3.15.  

1. If \( \mathcal{A}(T \varnothing \oplus I) \) is a rooted cluster subalgebra of \( \mathcal{A}(T) \). Moreover, it is clearly that \( \mathcal{A}(T) \) is a glue of \( \mathcal{A}(T \varnothing \oplus 1) \) and \( \mathcal{A}(T \varnothing \oplus I) \) along \( \varphi(I) \). Because \( \mathcal{A}(T) \) is a rooted subalgebra of \( \mathcal{A}(T) \) by Proposition 3.26. \( \mathcal{A}(T \varnothing \oplus I) \) and \( \mathcal{A}(T \varnothing \oplus I) \) are rooted cluster subalgebras of \( \mathcal{A}(T) \).

2. From Theorem 2.30, any rooted cluster subalgebra of \( \mathcal{A}(T) \) with coefficient set \( \varphi(I) \) is of the form \( \mathcal{A}(T' \oplus I) \) where \( T' \oplus I \subseteq T, \) and in the quiver \( Q(T) \), there are no arrows between vertices in \( \text{ind}T' \) and vertices in \( \text{ind}(T' \oplus I) \). Let \( T'' = \text{add}([\text{ind}T \setminus \text{ind}(T' \oplus I)]) \) be a subcategory of \( C \). Now we consider the pair \( (T', T'') \) in the subfactor category \( \mathcal{A}(T'/I) \), which is a 2-Calabi-Yau triangulated category since \( I \) is functorially finite in \( C \). Note that \( T' \oplus T'' \) is a cluster tilting subcategory in \( \mathcal{A}(T'/I) \) and \( \text{Hom}_{\mathcal{A}(T'/I)}(T', T'') = 0 \). Thus we have \( \mathcal{A}(T'/I) = (T' \oplus T'') \oplus (T'(1) \oplus T''(1)) = T' \oplus T'(1) \oplus T'' \oplus T''(1) = C_1 \oplus C_2 \) as a decomposition of triangulated category by Proposition 3.5 [ZZ11]. Let \( \pi : I[1] \rightarrow I[1]/I \) be the natural projection. Then because \((C_1, C_2)\) is a cotorsion pair in \( \mathcal{A}(T'/I) \) with core \( 0, (X', Y') = (\pi^{-1}(C_1), \pi^{-1}(C_2)) \) is a cotorsion pair in \( C \) with core \( I \) by Theorem 3.5 [ZZ11]. It is clearly that \( T' \oplus I \) is \( T' \)-cluster tilting.

3. The first assertion follows from above statements (1) and (2). The second assertion follows from the fact that \( I \) is a t-structure if and only if \( I = 0 \) (Proposition 2.39 [DK08, ZZ]).

4. It is easily follows from Lemma 3.5 that \( Q(T \setminus I) \) is a full subquiver of \( Q(T) \) by deleting all the frozen variables. Therefore the conclusion follows due to Proposition 2.39.

5. Note that the maps \( \varphi_1, \varphi_2 \) and \( \varphi' \) are all induced by \( \varphi \), and the injections \( i_1, i_2, j_1, \) and \( j_2 \), the surjection \( p \) and \( \pi \) are all canonical, thus the commutative diagram is natural valid.

\[ \square \]

Remark 3.16.  

1. If \( C \) has a cluster tilting object, then any rigid subcategory \( I \) is additive generated by an object in \( \mathcal{C}(\mathcal{A}(T)) \). Thus \( I \) is functorially finite in \( C \). Therefore there is a bijection between the following two sets:

\[ \{ \text{Cotorsion pairs in } C \text{ with core } I \} \]

\[ \Downarrow \]

\[ \{ \text{Complete pairs of rooted cluster subalgebras of } \mathcal{A}(T) \text{ with coefficient set } \varphi(I) \}. \]

In this case, each rooted cluster subalgebra \( \mathcal{A}(T \varnothing \oplus I) \) of \( \mathcal{A}(T) \) has a 2-Calabi-Yau categorification by the stably 2-Calabi-Yau category \( \mathcal{A} \) in the sense of [BIRS09, FK10].

2. It follows from above statement that if a rooted cluster algebra \( \mathcal{A}(\Sigma) \) has a 2-Calabi-Yau categorification by a 2-Calabi-Yau triangulated category with a cluster tilting object, then any rooted cluster subalgebra of \( \mathcal{A}(\Sigma) \) has a 2-Calabi-Yau categorification by a stably 2-Calabi-Yau category.

Corollary 3.16. Under the settings assumed at the beginning of this subsection, we unify the following six kinds of decompositions:

1. The decomposition of the triangulated category \( \mathcal{A}(T) \) in the sense of [BIRS09, ZZ12].
2. The decomposition of the cluster tilting subcategory $\mathcal{T}$ in $\frac{\varphi(I)}{I}$ in the sense of [ZZ12].

3. The decomposition of the ice quiver $Q(\mathcal{T})$ in the sense of Proposition 2.18.

4. The decomposition of the exchange matrix $B(\mathcal{T})$ in the sense of Proposition 2.19.

5. The decomposition of the seed $\Sigma(\mathcal{T})$ in the sense of Definition 2.22.

6. The decomposition of the rooted cluster algebra $\mathcal{A}(\mathcal{T})$ in the sense of Theorem 2.24.

Thus from the above theorem, we can classify the cotorsion pairs in $\mathcal{A}$ with core $I$ by gluing indecomposable components of $\mathcal{A}(\mathcal{T})$ (or equivalently of $Q(\mathcal{T})$, of $B(\mathcal{T})$ and of $\Sigma(\mathcal{T})$ respectively). In fact, each way of gluing all the indecomposable components of $\mathcal{A}(\mathcal{T})$ to a complete pair of rooted cluster subalgebras of $\mathcal{A}(\mathcal{T})$ with coefficient set $\varphi(I)$ gives a unique cotorsion pair in $\mathcal{A}$ with core $I$.

**Proof.** It is proved in Theorem 3.10 [ZZ12] that the first two decompositions are unified. We have proved in subsections 2.3 and 2.4 that the last four decompositions are unified. Now we prove the decomposition of the cluster tilting subcategory $\mathcal{T}$ in $\frac{\varphi(I)}{I}$ and the decomposition of the quiver $Q(\mathcal{T})$ is unified. Let $\mathcal{T} = \bigoplus_{i=1}^n T_i$ be the decomposition of $\mathcal{T}$ in $\frac{\varphi(I)}{I}$, this means that on the one hand $\mathcal{T}$ is a direct sum of categories $T_i$, $1 \leq i \leq n$, that is, $\text{Hom}_{\frac{\varphi(I)}{I}}(T_j, T_k) = 0$ for any $T_j \in \mathcal{T}_j, T_k \in \mathcal{T}_k$ with $j \neq k$, and on the other hand each $T_i$, $1 \leq i \leq n$, is a direct sum of smaller subcategories. Then from $\text{Hom}_{\frac{\varphi(I)}{I}}(T_j, T_k) = 0$ for any $T_j \in \mathcal{T}_j, T_k \in \mathcal{T}_k$ with $j \neq k$, any morphism from $T_j$ to $T_k$ factor through an object in $I$. Therefore there are no arrows in $Q(\mathcal{T})$ from the vertex of $T_j$ to the vertex of $T_k$. For any $1 \leq i \leq n$, denote by $Q(T_i)$ the ice quiver with exchangeable vertices given by isomorphisms of indecomposable objects $T_i$ in $\mathcal{T}_i$ and frozen vertices given by isomorphisms of indecomposable objects $I_i$ in $I$ such that there are irreducible morphism between $T_i$ and $I_i$. Then $Q(\mathcal{T})$ is indecomposable since $\mathcal{T}_i$ is indecomposable; moreover, $Q(\mathcal{T}_i)$ is a indecomposable component of $Q(\mathcal{T}_i)$. Therefore the decomposition $\mathcal{T} = \bigoplus_{i=1}^n T_i$ determines the decomposition $\{Q(T_i)\}_{i=1}^n \uplus \{Q(0)\}_{i=1}^n$ of $Q(\mathcal{T})$, where $\{Q(0)\}_{i=1}^n$ is the set of isolated points in $Q(\mathcal{T})$. An inverse argument shows that the decomposition of $\mathcal{T}$ is also determined by the decomposition of the quiver $Q(\mathcal{T})$. Hence decomposition of $\mathcal{T}$ and the decomposition of $Q(\mathcal{T})$ are unified. Thus all these decompositions are unified. \qed

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