Two Cellular Automata for the 3x+1 Map

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Abstract

Two simple Cellular Automata, which mimic the Collatz-Ulam iterated map (3x+1 map), are introduced. These Cellular Automata allow to test efficiently the Collatz conjecture for very large numbers.

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1 Introduction

Cellular Automata (CA) were first introduced in the early fifties by J. Von Neumann [1] in his investigation of "complexity", following an inspired suggestion by S. Ulam. S. Ulam himself was among the first to study intensively CA as well the nice properties of simple iterated maps (among them the $3x+1$ map and the related Collatz conjecture, see below) [2]. Likely, he should been amused to know that there exist simple CA which "compute" this map. Unfortunately, only a very faint light seems to arise from these CA concerning the proof of the Collatz conjecture.

2 The $3x+1$ map and the related conjecture

The $3x+1$ iterated map was introduced in 1937 by L. Collatz, it was investigated by a lot of people and it is also known as Collatz map (sequence), Hasse's algorithm, Syracuse algorithm; the related conjecture is known as $3x+1$ problem, Collatz conjecture, Ulam problem, Kakutani's problem, Syracuse problem, Thwaites conjecture (a large literature on this map, on the related conjecture and on possible generalizations is available, see [3-16]).

The Collatz map (CM):

$$u(t+1) = 3u(t) + 1 \quad \text{if } u(t) \text{ is odd} \; \; \text{(1a)}$$

$$u(t+1) = \frac{u(t)}{2} \quad \text{if } u(t) \text{ is even} \; \; \text{(1b)}$$

$$t \in \mathbb{N}, \; u(0) \text{ positive integer} \; \; \text{(1c)}$$

The Collatz conjecture (CC):

For any initial $u(0)$, the CM attains the final cycle $\ldots 1, 4, 2, 1 \ldots$ \; \; \text{(2)}

Surprisingly enough, the number of steps required to attain the final cycle varies sensibly (and apparently unpredictably) also for contiguous small initial numbers: f.i.

- $u(0) = 6 \rightarrow 3, 10, 5, 16, 8, 4, 2, 1 \ldots$
- $u(0) = 7 \rightarrow 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 \ldots$
- $u(0) = 8 \rightarrow 4, 2, 1 \ldots$

thus respectively 8, 16, 3 steps are needed. Of course numerical computations of CM always confirmed CC, but, in spite of the simplicity of the CM itself, no proof (or disproof) of the CC is yet available. Lagarias (1985) showed that there are no nontrivial cycles with length $<275000$ [3,4]. Conway (1972) proved that Collatz-type problems can be formally undecidable [5]. Thwaites (1996) has offered a £1000 reward for resolving the conjecture [14].
3 Two CA that "compute" the Collatz map

Let us now introduce two simple Cellular Automata that "mimic" the CM. Looking at the CM itself (1) it is clear that computations should be very simple in basis 2 or 3: we take advantage of this. Indeed the first automaton (CA2) computes the map in basis 2, while the second one (CA3) computes it in basis 3.

3.1 CA2

This is an unidimensional CA, the cells are arranged on a line, they can be numbered by \( n \in \mathbb{Z} \) and each cell at the discrete time \( t \) (\( t = 0, 1, 2, \ldots \)) can be in one of two different states so that the state-function \( u(n, t) \) takes values in a finite set: \( u(n, t) \in \mathbb{Z}/2\mathbb{Z} \), say \( \{0, 1\} \). The vacuum-state is 0, i.e.

\[ u(n, t) \rightarrow 0 \quad \text{as} \quad |n| \rightarrow \infty. \tag{3} \]

**Example:**

\[ \ldots 000110100011101001011001000\ldots \]

The evolution law, that allows to construct the state \( u(n, t+1) \) of the CA from the known \( u(n, t) \), consists in the following rules:

**RULE IA:** running from the left to the right (increasing \( n \)) tag the cells from the first 0 followed by a 1 to the first 0 followed by 0 (extrema included);

*applying this rule in the above example, and tagging with a superimposed dot, we get:*

\[ \ldots 00\dot{0}\dot{1}\dot{1}\dot{0}\dot{1}\dot{0}\dot{0}000111010001011001000\ldots \]

**RULE IB:** tag also, if the case, from the next 1 following a 1 to the first 0 followed by 0 (extrema included);

*applying this rule in the above example we get:*

\[ \ldots 00\dot{0}\dot{1}\dot{0}\dot{0}000111010001011001000\ldots \]

**RULE IC:** proceeding to the right, apply RULE IB whenever possible:

*applying this rule in the above example we get:*

\[ \ldots 00\dot{0}\dot{1}\dot{0}\dot{0}000111010001011001000\ldots \]

**RULE II:** the evolved state obtains according to:

\[ u(n-1, t+1) = (u(n, t) + u(n+1, t)) \mod 2, \quad \text{if the cell } n \text{ is not tagged}; \tag{4a} \]

\[ u(n-1, t+1) = (u(n, t) + u(n+1, t) + 1) \mod 2, \quad \text{if the cell } n \text{ is tagged}. \tag{4b} \]
Applying this rule in the above example we get:
...
00100010010100010111001110011110000...

Let us now show how this CA computes the Collatz map. Given a CA2 configuration \( u(n, t) \), f.i.:
\[ u(n, t) = \ldots0010011000\ldots \]
c onsider the configuration \( \tilde{u}(n, t) \) which goes from the leftmost 1 to the rightmost 1:
\[ \tilde{u}(n, t) = 10011 \]

A configuration \( \tilde{u}(n, t) \) is initially finite, due to (3), and stays finite during the evolution, due to rules I and II. Thus we can interpret \( \tilde{u}(n, t) \) as an integer positive (odd) number in basis 2 (proceeding from the left to the right):
\[
\tilde{u}(n, t) = 10011 = 1 \times 2^0 + 0 \times 2^1 + 0 \times 2^2 + 1 \times 2^3 + 1 \times 2^4 = 25
\]

It is straightforward to see that, considering the configurations \( \tilde{u}(n, t) \) as positive (odd) integers, CA2 emulates (1a).

**Proof.** To multiply a number by 3 (namely 11 in basis 2), we have to perform the binary summation of that number and of the same number times 2 (which in our CA2 configuration is the number itself shifted to the right). In other words we have to add any digit to the following one, taking care of the possible amount to be carried: indeed we have to carry 1 after a 1 following a 1 up to the first 0 following a 0. This justifies Rules Ib, Ic, and II. Rule Ia just adds 1 to the result. ■

Now note that the CM rule (3) in our framework is just a trivial shift of the configuration. Thus we can assert that this CA emulate the Collatz map. The Collatz conjecture (2) in the language of this CA reads:
"any initial (non zero) configuration evolves attaining eventually the stable configuration \( \ldots0001000\ldots \)"

Let us give a simple example:
\[
\begin{align*}
u(n, 0) &= \ldots00010011000\ldots \\
u(n, 1) &= \ldots00011010000\ldots \\
u(n, 2) &= \ldots00101110000\ldots \\
u(n, 3) &= \ldots00011101000\ldots \\
u(n, 4) &= \ldots00100010000\ldots \\
u(n, 5) &= \ldots00101100000\ldots \\
u(n, 6) &= \ldots00010100000\ldots \\
u(n, 7) &= \ldots00000100000\ldots
\end{align*}
\]
Interpreting now the configurations as integers, we have the sequence:
25, 19, 29, 11, 17, 13, 5, 1...
which is the odd subsequence of the Collatz sequence:
25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1...

Note the efficiency of CA2 which reaches the final states in only 7 steps (versus the 23 steps needed by the Collatz map). The typical behavior of this automaton is shown in Figure 1.
3.2 CA3

This is an unidimensional CA, the cells are arranged on a line, they can be numbered by \( n \in \mathbb{Z} \) and each cell at the discrete time \( t \) \((t = 0, 1, 2, \ldots)\) can be in one of four different states so that the state-function \( u(n, t) \) takes values in a finite set: \( u(n, t) \in \mathbb{Z}/2\mathbb{Z} \), say \( \{0, 1, 2, 3\} \). Moreover

\[
\begin{align*}
  u(n, t) &\xrightarrow{n \to -\infty} 0 ; \quad (5a) \\
  u(n, t) &\xrightarrow{n \to +\infty} 3; \quad (5b)
\end{align*}
\]

If \( u(n, t) = 3 \) then \( u(n + 1, t) = 3 \). (5c)

Example:

\( \ldots 00012010222102110012333 \ldots \)

The *evolution law*, that allows to construct the state \( u(n, t + 1) \) of the CA from the known \( u(n, t) \), consists in the following two rules:

**RULE III:** tag the cells following a non tagged 1 up to the next 1 or 3 (included);

*applying this rule in the above example, and tagging with a superimposed dot, we get:

\( \ldots 0001\dot{2}\dot{0}\dot{1}02221\dot{0}\dot{2}\dot{1}01\dot{1}001\dot{2}333 \ldots \)

**RULE IV:** the evolved state of CA3 obtains according to the following table:

| tagged cells | not tagged cells |
|--------------|------------------|
| 0 → 1        | 0 → 0            |
| 1 → 2        | 1 → 0            |
| 2 → 2        | 2 → 1            |
| 3 → 2        | 3 → 3            |

(6)

*applying this rule in the above example we get:

\( \ldots 000210110122002002233 \ldots \)

Let us now show how this CA computes the Collatz map. Given a \( u(n, t) \), f.i.:

\( u(n, t) = \ldots 000101121333 \ldots \)

consider the configuration \( \tilde{u}(n, t) \) which goes from the leftmost not-zero number (included) to the first 3 (excluded):

\( \tilde{u}(n, t) = 101121. \)

A configuration \( \tilde{u}(n, t) \) is initially finite, due to (6), and stays finite during the evolution, due to rules III and IV. Thus we can interpret \( \tilde{u}(n, t) \) as an integer positive number in basis 3 (proceeding from the right to the left):

\( \tilde{u}(n, t) = 101121 \) (basis 3) = \( 1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 + 1 \cdot 3^3 + 0 \cdot 3^4 + 1 \cdot 3^5 = 286 \) (basis 10).
Suppose now that $\tilde{u}(n, t)$ is even (as in the above case); then in $\tilde{u}(n, t)$ there is an even number of 1 and therefore, according to Rule III, in $u(n, t)$ the first 3 cannot be tagged:

$$u(n, t) = \ldots 000101121333 \ldots.$$ 

In this case Rules III and IV perform just the division of the number $\tilde{u}(n, t)$ by 2, according to the CM rule $\text{(1b)}$. The proof of this statement is straightforward and it is based mainly on the trivial fact that

$$3^{k+1} = (2 + 1) \cdot 3^k. \quad (7)$$

Let us explicit this using the above example:

$$\tilde{u}(n, t) = 101121 = 1 \cdot 3^5 + 0 \cdot 3^4 + 1 \cdot 3^3 + 1 \cdot 3^2 + 2 \cdot 3^1 + 1 \cdot 3^0 =$$

$$(2 + 1) \cdot 3^4 + 0 \cdot 3^4 + 1 \cdot 3^3 + 1 \cdot 3^2 + 2 \cdot 3^1 + 1 \cdot 3^0 =$$

$$2 \cdot 3^4 + (2 + 1) \cdot 3^3 + 1 \cdot 3^2 + 2 \cdot 3^1 + 1 \cdot 3^0 =$$

$$2 \cdot 3^4 + 4 \cdot 3^3 + (2 + 1) \cdot 3^3 + 2 \cdot 3^1 + 1 \cdot 3^0 =$$

$$2 \cdot 3^4 + 4 \cdot 3^3 + 0 \cdot 3^2 + 4 \cdot 3^1 + (2 + 1) \cdot 3^0 + 1 \cdot 3^0 =$$

$$2 \cdot 3^4 + 4 \cdot 3^3 + 0 \cdot 3^2 + 4 \cdot 3^1 + 4 \cdot 3^0.$$

Thus clearly:

$$\tilde{u}(n, t)/2 = 101121 \quad \text{(basis 3)} = \text{12022 (basis 3)} = 143 \quad \text{(basis 10).}$$

Indeed the tagging Rule III and the Rule IV perform this task on the CA configuration:

$$u(n, t) = \ldots 000101121333 \ldots; \quad \text{apply Rule III:}$$

$$\ldots 000101121333 \ldots; \quad \text{apply Rule IV:}$$

$$u(n, t + 1) = \ldots 00012022333 \ldots \Rightarrow \tilde{u}(n, t + 1) = 12022.$$ 

Let us now consider the case of odd $\tilde{u}(n, t)$, f.i. $u(n, t) = \ldots 000101121333 \ldots \Rightarrow \tilde{u}(n, t) = 101120 \quad \text{(basis 3)} = 285 \quad \text{(basis 10)}.$

CM rule $\text{(1b)}$ yields

$$285 \cdot 3 + 1 \quad \text{(basis 10)} = 856 \quad \text{(basis 10)} = 1011201 \quad \text{(basis 3).}$$

It is plain that in terms of the CA3 configuration this could be accomplished just by changing the leftmost 3 into a 1. But then the new number is even and thus one should again apply CM rule $\text{(1b)}$ yielding as a result:

$$285 \Rightarrow 856 \Rightarrow 428 \quad \text{(basis 10)} = 120212 \quad \text{(basis 3).}$$

It is straightforward to see that Rules III and IV perform these two consecutive tasks in just one step.

In the above example:

$$\ldots 000101120333 \ldots \Rightarrow \text{Rule III} \ldots 000101120333 \ldots \Rightarrow \text{Rule IV} \ldots 00010212333 \ldots$$

Note that, even if all ”computations” are in basis 3, we need four states for CA3. Another oddity, in terms of usual unidimensional CA, is the presence of two different vacuum states (0 on the left, 3 on the right).

Note also that the final cycle, eventually attained according to the Collatz conjecture, is visualized by CA3 in the following way:

$$\ldots 0001333 \ldots \Rightarrow \text{Rule III} \ldots 0001333 \ldots \Rightarrow \text{Rule IV} \ldots 0000233 \ldots \Rightarrow \text{Rule III} \ldots 0000233 \ldots \Rightarrow \text{Rule IV} \ldots 0000133 \ldots$$

A typical behavior of this CA is shown in Figure 2.
4 Final remarks

Looking at Figures 1 and 2, we can see that one edge of the configuration moves with constant average velocity while the other exhibit erratic behavior: this somehow justify the conjecture (obviously prove not!). However if the introduction of these two CA doesn’t seem to give a great contribution to the aim of proving the Collatz conjecture, on the other hand it allows to explore intensively the $3x + 1$ map also for very large initial integers. Thus our CA offer a good example of how CA themselves can be used as fast computing machines even if implemented on sequential computers (much more if running on dedicated parallel machines). Moreover this approach allows convenient investigation of the statistical behavior of iterates. Indeed in [17], the authors proposed a Quasi Cellular Automaton for the $3x + 1$ map: they did not introduce a real CA but merely displayed the iterated of CM in basis 2, simulating in fact our CA2. In the same work generic behaviors associated with classes of seed values and periodic and chaotic structures within iterate patterns were investigated. The same investigation could be pursued for our CA3.

Finally some further remarks:

- our CA, according to the Wolfram heuristic classification [18], should be Class 1 CA (thus they should be not interesting ones...);
- our CA, due to the tagging rules I and III, are ”non local” CA (namely the evolved state of a cell depends in principle on the whole previous configuration). Indeed the evolution laws for these CA exhibit some analogies with the so called ”fast rule” introduced in [19] for the parity-rule ”filter” CA [20]: this suggests that it should be possible (may be useful) to find a ”local”, possibly ”filter”, formulation of these CA;
- empirically, computer investigation shows that ”small deformations” of the evolution law of CA2 and, to a lesser extent, of CA3 give rise mostly to new CA which themselves end up with final cycles (see Figures 3,4,5: of course this is not equivalent to change ”slightly” the rules of the Collatz map, thus we have non conventional generalizations of the map itself);
- the scheme here introduced could be easily and profitably extended to emulate and explore other different iterated maps.

Figure captions

Figures 1-5 are space-time diagrams of the evolution of the involved CA, space on the vertical axis, time flows left-right on the horizontal axis, the states of the cells are shown in different shades of grey.

- Figure 1: the typical behavior of CA2, starting from a random chosen initial configuration of 300 cells (corresponding to an integer of order $2^{300}$)
and ending with the final cycle predicted by the Collatz conjecture, is shown.

- Figure 2: the typical behavior of CA3, starting from a random chosen initial configuration of 100 cells (corresponding to an integer of order $3^{100}$) and ending with the final cycle predicted by the Collatz conjecture, is shown. Note that, in order to get a final stable configuration, the average drift velocity was corrected, adding a downward shift at any 2 time-steps.

- Figure 3: the typical behavior of a variant of CA2, starting from a random chosen initial configuration of 450 cells (corresponding to an integer of order $2^{450}$) and ending with the same final cycle of CA2, is shown. The evolution law of this CA is the same of CA2 except for the Rule Ib that now reads: tag also, if the case, from the next 1 following a 1 or followed by a 1 to the first 0 followed by 0 (extrema included).

- Figure 4: the typical behavior of a variant of CA2, starting from a random chosen initial configuration of 400 cells (corresponding to an integer of order $2^{400}$) and ending with a final cycle (more complex than that of CA2), is shown. The evolution law of this CA is the same of CA2 except for the ending condition of tagging in Rule Ia, Ib that now reads: to the first 0 followed by 0 and following a 0 (extrema included).

- Figure 5: the typical behavior of a variant of CA3, starting from a random chosen initial configuration of 200 cells (corresponding to an integer of order $3^{200}$) and ending with a simple final cycle, is shown. The evolution law of this CA is the same of CA3 except for Rule IV, that now reads: tag the cells following a non tagged 1 up to the cell that is 1 or 3 or is followed by a 2.
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