HITTING TIMES OF INTERACTING DRIFTED BROWNIAN MOTIONS
AND THE VERTEX REINFORCED JUMP PROCESS

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Abstract. It is well-known that the first hitting time of 0 by a negatively drifted one
dimensional Brownian motion starting at positive initial position has the inverse Gaussian
law. Moreover, conditionally on this hitting time, the Brownian motion up to that time
has the law of a 3-dimensional Bessel bridge. In this paper, we give a generalization of this
result to a family of Brownian motions with interacting drifts, indexed by the vertices of a
conductance network. The hitting times are equal in law to the inverse of a random potential
that appears in the analysis of a self-interacting process called the Vertex Reinforced Jump
Process [14, 15]. These self-interacting Brownian motions have remarkable properties with
respect to restriction and conditioning, showing hidden Markov properties. This family of
interacting Brownian motions are closely related to the martingale that plays a crucial role
in the analysis of the vertex reinforced jump process and edge reinforced random walk [15]
on infinite graphs.

1. Introduction

We first recall some classic facts about hitting times of standard Brownian motion. Let
\((B_t)_{t \geq 0}\) be a standard Brownian motion and

\[ X(t) = \theta + B(t), \]

be a Brownian motion starting from initial position \(\theta > 0\). It is well known that the first
hitting time of 0

\[ T = \inf\{t \geq 0, \ X(t) = 0\} \]

has the law of the inverse of a Gamma random variable with parameter \((\frac{1}{2}, \frac{\theta^2}{2})\). Moreover,
conditionally on \(T\), \((X_t)_{0 \leq t \leq T}\) has the law of a 3-dimensional Bessel bridge from \(\theta\) to 0 on
time interval \([0, T]\). More generally, if

\[ X(t) = \theta + B(t) - \eta t, \]

is a drifted Brownian motion with negative drift \(-\eta < 0\) starting at \(\theta > 0\), then \(T\) has the
inverse Gaussian distribution with parameters \((\frac{\theta}{\eta}, \theta^2)\), i.e. \(T\) has density on \(\mathbb{R}_+\)

\[ f(t) = \frac{\theta}{\sqrt{2\pi t^3}} \exp \left( -\frac{1}{2} \left( \frac{\theta^2}{t} + \eta^2 t - 2\theta\eta t \right) \right) 1_{t > 0} dt \]

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The 3-dimensional Bessel bridge from \(\theta\) to 0 on time interval \([0, T]\) can be represented by the following
S.D.E.

\[ X(t) = \theta + B(t) + \int_0^t \left( \frac{1}{X(s)} - \frac{X(s)}{T-s} \right) ds, \ 0 \leq t \leq T \]
and conditionally on \(T, (X_t)_{0 \leq t \leq T}\) has the law of a 3-dimensional Bessel bridge from \(\theta\) to 0 on time interval \([0, T]\). (See [17], Theorem 3.1, or [12], p. 317 Corollary 4.6, and [10, 16] for complements)

This paper aims at giving a generalization of these statements on a conductance network, namely a family of interacting drifted Brownian motions indexed by the vertices of the network. The distribution of hitting times of these processes will be given by a multivariate exponential family of distributions introduced by Sabot, Tarres and Zeng [14], which appeared in the context of self-interacting processes and random Schrödinger operators. This family of distributions is also intimately related to the supersymmetric hyperbolic sigma model introduced by Zirnbauer, [18] and investigated by Disertori, Spencer, Zirnbauer, [4, 3], and plays a crucial role in the analysis of the edge reinforced random walk (ERRW) and the vertex reinforced jump process (VRJP), [13].

The generalization of the one dimensional statement presented in this introduction was hinted by the martingales that appear in [13], the latter has played an important role in the analysis of the ERRW and the VRJP on infinite graphs. In Section 2.3, we explain the relations between the stochastic differential equations (S.D.E.s) defined in this paper and the VRJP and in Section 8 we relate the martingales that appear in the study of VRJP to the S.D.E.s.

Note that the computations done in this paper seem to have many similarities with computations done for exponential functional of the Brownian motion in dimension one (see in particular Matsumoto, Yor, [8, 9, 7]). More precisely, it would be possible to write an analogous of the Lamperti transformation that changes the S.D.E. presented below in its exponential functional counterpart with \(\mu = \frac{1}{2}\): the counterpart of the representation of Theorem 1 would correspond to a representation of the S.D.E. with a Brownian motions with opposite drifts as in [7]. We plan to develop these aspects in a further work.

Note also that multidimensional generalizations of the exponential functionals of the Brownian motion, related to some Lie groups, have been described by Chhai* ?bi in [1], even though we do not yet see a clear connection with our work.

2. Statement of the main results

2.1. The multivariate generalization of inverse Gaussian law : the random potential associated with the VRJP. Let \(N\) be a positive integer and \(V = \{1, \ldots, N\}\). Given a symmetric matrix

\[
W = (W_{i,j})_{i,j=1,\ldots,N}
\]

with non negative coefficients \(W_{i,j} = W_{j,i} \geq 0\). We denote by \(\mathcal{G} = (V, E)\) the associated graph with:

\[
V = \{1, \ldots, N\} \text{ and } E = \{\{i, j\}, i \neq j, W_{i,j} > 0\}.
\]

We always assume that the matrix \(W\) is irreducible, i.e. the graph \(\mathcal{G}\) is connected. If \((\beta_i)_{i \in V}\) is a function on the vertices, we set

\[
H_\beta = 2\beta - W,
\]

where \(2\beta\) represents the operator of multiplication by the vector \((2\beta_i)\) (or equivalently the diagonal matrix with diagonal coefficients \((2\beta_i)_{i \in V}\)). We always write \(H_\beta > 0\) to mean that \(H_\beta\) is positive definite. Remark that when \(H_\beta > 0\), its inverse \((H_\beta)^{-1}\) has positive coefficients (since \(\mathcal{G}\) is connected and \(H_\beta\) is an M-matrix, see e.g. [11], Proposition 3).

The following distribution was introduced in [14], and generalized in [6] (unpublished).
Lemma A. Let \((\theta_i)_{i \in V} \in (\mathbb{R}^*_+)^V\) be a positive vector indexed by \(V\). Let \((\eta_i)_{i \in V} \in (\mathbb{R}^*_+)^V\) be a non-negative vector indexed by \(V\). The measure

\[
(2.2) \quad \nu_{V}^{W,\theta,\eta}(d\beta) := \chi_{H_\beta > 0} \left( \frac{2}{\pi} \right)^{|V|/2} \exp \left( -\frac{1}{2} \langle \theta, H_\beta \theta \rangle - \frac{1}{2} \langle \eta, H_\beta^{-1} \eta \rangle + \langle \eta, \theta \rangle \right) \frac{\prod_{i \in V} \theta_i}{\sqrt{\det H_\beta}} d\beta
\]

is a probability distribution on \(\mathbb{R}^V\), where \(\chi_{H_\beta > 0}\) is the indicator function that the operator \(H_\beta\) (defined in (2.1)) is positive definite, \(\langle \cdot, \cdot \rangle\) is the usual inner product on \(\mathbb{R}^V\), and \(d\beta = \prod_{i \in V} d\beta_i\). When \(\eta = 0\), we simply write \(\nu_{V}^{W,\theta}\) for \(\nu_{V}^{W,\theta,0}\).

Moreover, the Laplace transform of (2.2) is explicitly given by

\[
(2.3) \quad \int e^{-\langle \lambda, \beta \rangle} \nu_{V}^{W,\theta,\eta}(d\beta) = e^{-\frac{1}{2} \langle \sqrt{\theta^* + \lambda}, W \sqrt{\theta^* + \lambda} \rangle + \frac{1}{2} \langle \theta, W \theta \rangle + \langle \eta, -\sqrt{\theta^* + \lambda} \rangle \prod_{i \in V} \theta_i} \prod_{i \in V} \theta_i
\]

for all \((\lambda_i)_{i \in V}\) such that \(\lambda_i + \theta_i^2 > 0\) for all \(i \in V\).

Remark 1. The probability distribution \(\nu_{V}^{W,\theta,\eta}\) was initially defined in [14] in the case \(\eta = 0\). In [6] (unpublished), Letac gave a shorter proof of the fact that \(\nu_{V}^{W,\theta}\) is a probability and remarked that the family can be generalized to the family \(\nu_{V}^{W,\theta,\eta}\) above. It appears, see forthcoming Lemma C that the general family \(\nu_{V}^{W,\theta,\eta}\) can be obtained from the family \(\nu_{V}^{W,\theta}\) by taking marginal laws.

Remark 2. The definition of \(\nu_{V}^{W,\theta}\) is not strictly the same as \(\nu_{V}^{W,\theta}\) in [14]. Firstly, compared with the definition of [14], the parameter \(\theta_i\) above corresponds to \(\sqrt{\theta_i}\) in [14]. It is in fact simpler to write the formula as in (2.3) since the quadratic form \(\langle \theta, H_\beta \theta \rangle\) appears naturally in the density and since \(\theta_i\) will play the role of the initial value in the forthcoming S.D.E. Secondly, we do not assume here that the diagonal coefficients of \(W\) are zero. It is obvious that the two definitions are equivalent up to a translation of \(\beta_i\) by \(W_{i,i}\). It will be more convenient here to allow this generality.

Notations 1. To simplify notations, in the sequel, for any function \(\xi : V \to \mathbb{R}\) and any subset \(U \subset V\), we write \(\xi_U\) for the restriction of \(\xi\) to the subset \(U\). We write \(d\beta_U = \prod_{i \in U} d\beta_i\) to denote integration on variables in \(\beta_U\). Similarly, if \(A\) is a matrix and \(U \subset V\), \(U' \subset V\), we write \(A_{U \times U'}\) for its restriction to the block \(U \times U'\). Note also that when \((\xi_i)_{i \in V}\) is in \(\mathbb{R}^V\), we sometimes simply write \(\xi\) for the operator of multiplication by \(\xi\), i.e. the diagonal operator with diagonal coefficients \((\xi_i)_{i \in V}\), as it is done in formula (2.1). Finally, we simply write \(\nu_{U}^{W,\theta,\eta}\) for \(\nu_{U \times U'}^{W_U,\theta_U,\eta_U}\) when \(U \subset V\) is a subset of \(V\) and \(W\) (resp. \(\theta, \eta\)) is a matrix (resp. vectors) in \(\mathbb{R}^V\).

We state the counterpart of Proposition 1 of [14] in the context of the measure \(\nu_{V}^{W,\theta,\eta}\).

Corollary B. \(i\) The random variable

\[
\frac{1}{\sqrt{\theta_i - W_{i,i}}} = \frac{1}{\sqrt{\theta_i - W_{i,i}}} \quad \text{for all } i \in V
\]

follows an inverse Gaussian law with parameters \((\frac{\theta_i}{\sqrt{\theta_i - W_{i,i}}}, \frac{\theta_i^2}{\theta_i - W_{i,i}})\).

\(ii\) The random vector \((\beta_i)\) is 1-dependent, i.e. for any subsets \(V_1 \subset V\), and \(V_2 \subset V\), such that the distance in the graph \(\mathcal{G}\) between \(V_1\) and \(V_2\) is strictly larger than 1, then \(\beta_{V_1}\) and \(\beta_{V_2}\) are independent.

The following lemma was proved in the 3rd arxiv version of [14], in the case of \(\theta = 1\), (p. 18, Lemma 4) but it can be easily extended to the case of general \(\theta\), see Section 3.

Lemma C. Assume that \(\beta\) is a random potential distributed according to \(\nu_{V}^{W,\theta,\eta}\). Let \(U \subset V\).
(i) Then, $\beta_U$ is distributed according to $\nu_W^{\psi, \eta}$ (i.e. $\nu_W^{\psi, \eta}$, of Notations 7) where

\begin{equation}
\hat{\eta} = \eta_U + W_U(U\eta).
\end{equation}

(ii) The law of $\beta_U$, conditionally on $\beta_U$, is $\nu_U^{\psi, \hat{\eta}}$ where

\[ \hat{W} = W_U(U\eta) + W_U((-\beta U\eta)^{-1}) W_U(U\eta), \quad \hat{\eta} = \eta_U + W_U((-\beta U\eta)^{-1}) (\eta_U). \]

\[ \text{2.2. Interacting drifted Brownian motions : main results.} \]

Let $t^0 = (t^0_i)_{i \in V} \in (\mathbb{R}_+)^V$ be a non negative vector. We set

\[ K_0 = \text{Id} - t^0 W, \]

where $t^0$ denotes the operator of multiplication by $t^0$ (or equivalently the diagonal matrix with diagonal coefficients ($t^0_i$)). In the sequel we only consider $t^0_i \geq 0$ for all $i \in V$, note that when $t^0_i > 0$, $\forall i \in V$, we have $K_0 = t^0 (\frac{1}{2t^0_i})$, as in notation (2.1), $\frac{1}{2t^0_i} = \left( \frac{1}{2t^0_i} \right)_{i \in V}$.

For $T = (T_i)_{i \in V} \in (\mathbb{R}_+ \cup \{+\infty\})^V$ and $t \in \mathbb{R}_+$ we write $t \wedge T$ for the vector $(t \wedge T_i)_{i \in V}$.

**Lemma 1.** Let $\theta = (\theta_i)_{i \in V} \in (\mathbb{R}_+)^V$ and $\eta = (\eta_i)_{i \in V} \in (\mathbb{R}_+)^V$ be non-negative vectors. Let $(B_i(t))_{i \in V}$ be a standard $N$-dimensional Brownian motion.

(i) The following stochastic differential equation is well-defined for all $t \geq 0$ and has a unique pathwise solution:

\[ (E_0^{W, \theta, \eta}(Y)) \]

\[ Y_i(t) = \theta_i + \int_0^t 1_{s<T_i} dB_i(s) - \int_0^t 1_{s<T_i} (W \psi(s)_i) ds, \quad \forall i \in V, \]

where $T = (T_i)_{i \in V}$ is the random vector of stopping times defined by

\[ T_i = \inf \{ t \geq 0 ; Y_i(t) - t\eta_i = 0 \}, \quad \forall i \in V. \]

and

\begin{equation}
\psi(t) = K_{t \wedge T}^{-1} Y(t)
\end{equation}

Moreover, $T_i < +\infty$ a.s. for all $i \in V$, and $K_T > 0$ is positive definite.

(ii) Set $X(t) = Y(t) - (t \wedge T) \eta$. The previous S.D.E is equivalent to the following

\[ (E_0^{W, \theta, \eta}(X)) \]

\[ X_i(t) = \theta_i + \int_0^t 1_{s<T_i} dB_i(s) - \int_0^t 1_{s<T_i} ((W \psi(s))_i + \eta_i) ds, \quad \forall i \in V, \]

with

\begin{equation}
\psi(t) = K_{t \wedge T}^{-1} (X(t) + (t \wedge T) \eta)
\end{equation}

and $T_i$ being the first hitting time of 0 by $X_i(t)$.

(iii) The process $\psi(t)$ is a continuous vectorial martingale equal to (recall that $1_{s<T}$ is the operator of multiplication by $1_{s<T_i}$):

\[ (E_0^{W, \theta, \eta}(\psi)) \]

\[ \psi(t) = \theta + \int_0^t K_{s \wedge T}^{-1} (1_{s<T_i} dB_i(s)). \]

Moreover, the quadratic variation of $\psi(t)$ is given by

\[ \langle \psi, \psi \rangle_t = \left( H_{\frac{1}{2tU}} \right)^{-1}. \]

We state our main result below.

**Theorem 1.** Let $(\theta_i)_{i \in V} \in (\mathbb{R}_+)^V$, $(\eta_i)_{i \in V} \in (\mathbb{R}_+)^V$ and $(Y_i(t))_{i \in V}$, $(X_i(t))_{i \in V}$, $(T_i)_{i \in V}$ be as in Lemma 7.
(i) The random vector \( \left( \frac{1}{2T_i} \right)_{i \in V} \) has law \( \nu_{V}^{W,\theta,\eta} \).

(ii) Conditionally on \((T_i)_{i \in V}\), \( (X_i(t))_{0 \leq t \leq T_i} \) are independent 3-dimensional Bessel bridges from \( \theta_i \) to 0 on time interval \([0, T_i]\).

Remark that when \( V = \{1\} \) is a single point and \( W_{1,1} = 0 \), then \( Y(t) \) is a Brownian motion with initial value \( \theta_1 > 0 \), and \( T_1 \) corresponds to the first hitting time of 0 by the drifted Brownian motion \( Y(t) - \theta_1 t \). Hence, it corresponds to the problem presented in (1.1); in particular \( \eta_1 = 0 \) corresponds to (1.2). When \( V = \{1\} \) and \( W_{1,1} > 0 \), \( Y(t)_{t \geq 0} \) is the solution of the S.D.E.

\[
dY_1(t) = 1_{t < T_1} \left( dB_1(t) - \frac{W_{1,1}}{1 - tW_{1,1}} Y(t) dt \right)
\]

with initial condition \( Y_1(0) = \theta_1 \), so that it is a Brownian bridge from \( \theta_1 \) to 0 on time interval \([0, 1/W_{1,1}]\), stopped at the first hitting of 0 by \( Y_1(t) - \theta_1 t \). Consequently, \( Y_1(t) \) has the same law as \( (1 - tW_{1,1}) B_1 \left( \frac{t}{1 - tW_{1,1}} \right) \) up to time \( T_1 \), see e.g. [12] p154, and \( T_1 \) has the same law as \( \frac{1}{1 + \tau W_{1,1}} \) where \( \tau \) is the first hitting time of 0 by a Brownian motion with drift \(-\theta_1\). Therefore, \( \frac{1}{1 + \tau W_{1,1}} \) follows a inverse gaussian law with parameters \( (\sqrt{\frac{\theta_1}{\eta_1}}, \theta_1^2) \), and it is coherent with the expression of marginal law of \( \beta \) in Corollary [3].

We give now two results that show some "abelianity" of the process, in the sense that times on each coordinates can be run somehow independently.

**Theorem 2** (Abelian properties). Let \( (X(t)) \) be the solution of \( E_V^{W,\theta,\eta}(X) \). Denote \( \beta = \frac{1}{2t} \).

(i) (Restriction) Let \( U \subset V \). Then, \( (X_U(t)) \) has the same law as the solution of \( E_U^{W_U,\theta_U,\hat{\eta}}(X) \), where

\[
\hat{\eta} = \eta_U + W_{U,U}(\theta_U).
\]

(ii) (Conditioning on a subset) Let \( U \subset V \). Then, conditionally on \( (X_U(t))_{t \geq 0}, (X_U^c(t))_{t \geq 0} \) has the law of the solutions of the S.D.E. \( \tilde{E}^{W_U,\theta_U,\hat{\eta}}(X) \) where

\[
\tilde{W} = W_{U^c,U^c} + W_{U^c,U} ((H_{\beta})_{U,U})^{-1} W_{U,U^c}, \quad \tilde{\eta} = \eta_U + W_{U,U} ((H_{\beta})_{U,U})^{-1} (\eta_U).
\]

(iii) (Conditioning on the past) Consider \( (t^0_i)_{i \in V} \in (\mathbb{R}_+)^V \). Denote by

\[
\mathcal{F}^X(t^0) = \sigma\{ (X_k(s))_{s \leq t^0_k} : k \in V \},
\]

the filtration generated by the past of the trajectories before time \( (t^0_k)_{k \in V} \). Then, consider for \( t \geq 0 \),

\[
\tilde{X}(t) = X(t^0 + t) = (X_i(t^0_i + t))_{i \in V},
\]

the process shifted by times \( (t^0_i)_{i \in V} \). (Note that the shift in time is not necessarily the same for each coordinate). Conditionally on \( \mathcal{F}^X(t^0) \), the process \( \tilde{X}(t)_{t \geq 0} \) has the same law as the solution of the equation \( \tilde{E}_V^{W,X(t^0),\tilde{\eta}}(X) \) where

\[
\tilde{W} = W(K_{t^0 \wedge T})^{-1}, \quad \tilde{\eta} = \eta + \tilde{W}((t^0 \wedge T)\eta).
\]

In particular, if \( V(t^0) = \{ i \in V, T_i > t^0_i \} \), conditionally on \( \mathcal{F}(t^0) \), \( \left( \frac{1}{T_i - t^0_i} \right)_{i \in V(t^0)} \) has the law \( \nu_{V(t^0)}^{W,X(t^0),\tilde{\eta}} \).
Remark 3. Assertions (ii) and (iii) of the Theorem are the S.D.E. counterpart of the restriction, conditioning properties of Lemma C, and the proof is a simple consequence of Theorem 7 and Lemma C. The assertion (iii) is more involved, see the proofs in Section 7.

Remark 4. In all these statements, the restricted (or conditioned) process that appears is not in general solution of the S.D.E. with the original shifted Brownian motion, but with a different one, which is a priori not a Brownian motion in the original filtration. Nevertheless, when all the \( t_i \) are equal to the same real \( s \), then it is the case : \((X(t+s))_{t \geq 0} \) is solution of the S.D.E. with the shifted Brownian motion \((B(s+t))_{t \geq 0}, \) of forthcoming Proposition 7.

The result in the latter case is much simpler and is a consequence of a plain computation, whereas the general case uses the representation of Theorem 7.

Note that this has consequences on the law of marginals and conditional marginals. Fix some \( i_0 \in V \), if we take \( U = \{i_0\} \), then, by previous considerations, we see that if \( W_{i_0,i_0} = 0 \) (resp. \( W_{i_0,i_0} > 0 \)) the law of the marginal \( X_{i_0}(t) \) is that of a Brownian motion starting at \( \theta_{i_0} \) (resp. a Brownian bridge from \( \theta_{i_0} \) to 0 on time interval \([0,\frac{1}{W_{i_0,i_0}}]) \) and with drift \(-\tilde{\eta}_{i_0} \) and stopped at its first hitting time of 0. Hence, the marginal \((X_{i_0}(t))\) has the same law as the one-dimensional problem (1.2). Moreover, we see that conditionally on \((X_k(t))_{t \geq 0})_{k \neq i_0} \), the process \((X_{i_0}(t))_{t \geq 0} \) has the law of a Brownian bridge from \( \theta_{i_0} \) to 0 on time interval \([0,\frac{1}{W_{i_0,i_0}}]) \) drifted by \(-\tilde{\eta}_{i_0} \) and stopped at its first hitting time of 0, where \( U = V \setminus \{i_0\} \).

In particular, it means that the marginal \((X_{i_0}(t))_{t \geq 0} \) is a diffusion process, as well as the (conditional) marginal \((X_{i_0}(t))_{t \geq 0} \) conditioned on \((X_k(t))_{t \geq 0})_{k \neq i_0} \). This Markov property is not obvious in the initial equation \(E_{V}^{\theta_{i_0}}(X)\). Indeed, the process \((X_{i_0}(u))_{u \leq s} \) before time \( s \) affects the drifts of \((X_{k \neq i_0}(u))_{u \leq s} \), and so the values \(X_{k \neq i_0}(s)\), which themselves affect the drift of \( X_{i_0}(s) \).

More generally, there are hidden Markov properties in the restricted process \((X_U(t))_{t \geq 0} \). Indeed, the law of the future path \((X_U(t))_{t \geq s} \) only depends on the past of \((X_U(u))_{u \leq s} \) through the values of \(X_U(s) \) and \((s \wedge T)_{U} \). This is not obvious from the initial equation \(E_{V}^{\theta_{i_0}}(X)\).

The same is true for the process \((X_{U^{-}}(t))_{t \geq 0} \) conditioned on \((X_{U}(t))_{t \geq 0} \).

2.3. Relation with the Vertex Reinforced Jump Process. Let us describe the VRJP in its "exchangeable" time scale introduced in [13]. We consider the VRJP with a general initial local time, as in [13], Section 3.1. The VRJP, with initial local time \((\theta_i)_{i \in V} \), is the self-interacting process \((Z_t)_{t \geq 0} \) that, conditionally on its past at time \( t \), jumps from a vertex \( i \) to \( j \) with rate

\[
W_{i,j} \sqrt{\frac{\theta_j + \ell^Z_j(t)}{\theta_i + \ell^Z_i(t)}},
\]

where \( \ell^Z_j(t) = \int_0^t I_{Z_s=i} ds \) denotes the local time of \( Z \) at site \( i \). In [13], it was proved that this process is a mixture of Markov Jump Processes and that the mixing law can be represented by a marginal of a supersymmetric \( \sigma \)-field investigated by Disertori, Spencer, Zirnbauer in [18, 4, 3]. In [13], it was related to the random potential \( \beta \) of Lemma A.
Theorem D ([13] Theorem 2, [14] Theorem 3). Let $\delta \in V$ where $V$ is finite, and $U = V \setminus \{\delta\}$. Let $(\theta_i)_{i \in V} \in (\mathbb{R}^*_+)^V$ be a positive vector. Consider $\beta = (\beta_j)_{j \in V}$ sampled with distribution $\nu^W_{V,\theta}$. Define $(\psi_j)_{j \in V}$ as the unique solution of

$$\begin{cases}
\psi(\delta) = 1, \\
H_{\beta}(\psi)|_{U} = 0.
\end{cases}$$

Then, the VRJP starting at vertex $\delta$ and initial local times $(\theta_i)_{i \in V}$ is a mixture of Markov jump processes with jumping rates

$$\frac{1}{2} W_{i,j} \frac{\psi_j}{\psi_i}.$$ 

More precisely, it means that

$$\mathbb{P}^{\text{VRJP},\theta}_{\delta}(\cdot) = \int \mathbb{P}^{\psi}_{\delta}(\cdot) \nu^W_{V,\theta}(d\beta),$$

where $\mathbb{P}^{\text{VRJP},\theta}_{\delta}$ is the law of the VRJP starting at vertex $\delta$ and initial local times $(\theta_i)_{i \in V}$ and $\mathbb{P}^{\psi}_{\delta}$ is the law of the Markov jump process with jumping rates (2.7) starting at vertex $\delta$.

Remark that the random variables $(\beta_j)_{j \in U}$ appear as asymptotic holding times of the VRJP. Indeed, let $N_i(t)$ be the number of visits of vertex $i$ by $Z$ before time $t$. Then, by Theorem D, the empirical holding times converge $\mathbb{P}^{\text{VRJP},\theta}$ a.s., i.e. the following limit exists

$$\lim_{t \to \infty} \frac{N_i(t)}{\ell^U_i(t)} = \frac{1}{2} \sum_{j \sim i} W_{i,j} \frac{\psi_j}{\psi_i} = \beta_i, \forall i \in U,$$

and, by Lemma C, $\beta_U$ has law $\nu^W_{U,\theta,\eta}$ where $\eta = W_{U,\delta} \theta_\delta$. Moreover, conditionally on $\beta_U$, the VRJP is a Markov Jump Process with jump rates given by (2.7).

Consider now the S.D.E. $E^{W,U,V,\theta,\eta,U}(Y)$ with same parameters. From Theorem 1, the law $(\frac{1}{t})_{i \in U}$ coincides with that of $\beta_U$. Moreover, if we set

$$\psi_j(\infty) := \lim_{t \to \infty} \psi_j(t), \forall j \in U,$$

then $\psi(\infty) = \left((H_{\frac{1}{2T}})|_{U,U}\right)^{-1} \eta$. Hence, it means that $\psi(\infty)$ coincides with the $\psi$ of Theorem D if we identify $\beta_U$ and $\frac{1}{2T}$. Hence $(\beta_U, \psi)$ of Theorem D has the same law as $(\frac{1}{2T}, \psi)$ arising in the S.D.E. $E^{W,\theta,\eta,Y}(Y)$.

There are remarkable similarities between Theorem 1 and Theorem D. Firstly, $(\beta_i)_{i \in U}$ are homogeneous to the inverse of time, and have same distribution in both cases. Secondly, in both cases, a type of exchangeability appears in the sense that, conditionally on the limit holding times or hitting times, the processes are "trivial": in the case of the VRJP, it becomes Markov; in the case of the S.D.E., the marginals are independent and diffusions processes (in fact Bessel bridges).

In section 8, we push forward this relation, by explaining the martingale property that appears in [13], and the exponential martingale property that extends it in [2], by Theorem 1 and the abelian properties of Theorem 2.

Nevertheless, we do not yet clearly understand the relation between the VRJP and the S.D.E. $E^{W,\theta,\eta}$ beyond these remarks.
2.4. Organization of the paper. In Section 3 we prove the properties related to the distribution $\nu_{V}^{W,\theta,\eta}$, Lemma A, Lemma C and Corollary B. In Section 4 we present some simple key computations that are used several times in the proofs. In Section 5, we prove the results concerning existence and uniqueness of pathwise solution of the S.D.E., Lemma 1, and state and prove Proposition I mentioned in Remark 3 above. Section 6 is devoted to the proof of the main Theorem 1. In Section 7, we prove the abelian properties of Theorem 2. Finally, in Section 8 we explain the relation between the abelian properties of Theorem 2 and the martingale that appears in [15].

3. Proof of the results concerning the distribution $\nu_{V}^{W,\theta,\eta}$: Lemma A, Lemma C and Corollary B

Lemma A and Lemma C are proved in [15] (third arXiv version) in the case $\theta_i = 1$ for all $i \in V$, see Lemma 3 and Lemma 4 therein. The case of general $\theta$ can be deduced from the special case $\theta = 1$ by a change of variables. More precisely, setting $\beta_i' = \theta_i^2 \beta_i$, $W_{i,j}' = \theta_i \theta_j W_{i,j}$, and $\eta_i' = \theta_i \eta_i$, then we have

$$\langle \theta, H_{\beta} \theta \rangle = \langle 1, H_{\beta'} 1 \rangle, \; \langle \eta, H_{\beta}^{-1} \eta \rangle = \langle \eta', (H_{\beta'}^{-1})^{-1} \eta' \rangle, \; \langle \eta, \theta \rangle = \langle \eta', 1 \rangle$$

where $H_{\beta'} = 2\beta' - W'$, so that $\beta \sim \nu_{V}^{W,\theta,\eta}$ if and only if $\beta' \sim \nu_{V}^{W'1,\eta'}$.

Alternatively, one can check that the proofs of [15] work exactly in the same way with a general $\theta$ instead of 1.

Corollary B is a direct consequence of the expression of the Laplace transform. Indeed, under $\nu_{V}^{W,\theta,\eta}$, the Laplace of the marginal $\beta_i - W_{i,i}$ is given for $\zeta \in \mathbb{R}_+$ by

$$\int \exp \left( \zeta (\beta_i - \frac{1}{2} W_{i,i}) \right) \nu_{V}^{W,\theta,\eta}(d\beta) = \frac{\theta_i}{\sqrt{\theta_i^2 + \zeta}} \exp \left( - \left( \sqrt{\theta_i^2 + \zeta} - \theta_i \right) \left( \eta_i + \sum_{j \neq i} W_{i,j} \theta_j \right) \right).$$

It coincides with the Laplace transform of the inverse of the Inverse Gaussian density. More precisely, by changing the parameter of Inverse Gaussian distribution, we have

$$\int_0^\infty \exp \left( - \frac{\zeta}{2x} \right) \left( \frac{\lambda}{2\pi x^3} \right)^{\frac{1}{2}} \exp \left( - \frac{\lambda (x - \mu)^2}{2\mu^2 x} \right) dx = \frac{\sqrt{\lambda}}{\sqrt{\zeta + \lambda}} \exp \left( - \frac{\lambda}{\mu^2} \left( \sqrt{\zeta + \lambda} - \sqrt{\lambda} \right) \right)$$

It means that the law of $2\beta_i - W_{i,i}$ coincides with the law of the inverse of an inverse Gaussian random variable with parameters $(\lambda, \mu)$ such that $\lambda = \theta_i^2$ and $\sqrt{\frac{\lambda}{\mu^2}} = \eta_i + \sum_{j \neq i} W_{i,j} \theta_j$.

4. Simple key formulas

Let us start by a remark. If $(t_i) \in (\mathbb{R}_+)^V$ and $K_t > 0$, then the operator $H_t^{-1}$ is well-defined even when some of the $t_i$’s vanish: indeed, using the identity

$$H_t^{-1} = K_t^{-1} t,$$

the right hand side is perfectly well-defined when $K_t$ is invertible. In all the sequel, we will implicitly consider that $H_t^{-1}$ is defined by this formula when some of the $t_i$’s vanish.

We prove below some simple formulas that will be key tools in forthcoming computations.

Lemma 2. Let $(t_0^i)_{i \in V}$ and $(t^i)_{i \in V}$ be vectors in $\mathbb{R}_+^V$ such that $K_{t_0 + t^i} > 0$. 
(i) We have,
\[ K_{t^0 + t^1} = \tilde{K}_{t^1} K_{t^0}, \]
with
\[ \tilde{K}_{t^1} = \text{Id} - t^1 \bar{W}, \quad \text{where,} \quad \bar{W} = W K_{t^0}^{-1} \]
Hence, we also have, with \[ \tilde{H}_{\frac{t^0}{2(t^0 + t^1)}} = \frac{1}{t^1} - \bar{W}, \quad (\text{where} \; |H| := \det H) \]
\[ \left| H_{\frac{t^0}{2(t^0 + t^1)}} \right| = \left( \prod_{i \in V} \frac{t^1}{t^0_i + t^1_i} \right) \left| K_{\frac{t^0}{2t^0}} \right| \]
(4.2)

(ii) Let
\[ \tilde{\eta} = \eta + \bar{W}(t^0 \eta), \]
then,
\[ \tilde{\eta} = (t^0)^{-1} H_{\frac{t^0}{2t^0}}^{-1} \eta. \]
and,
\[ \left< \tilde{\eta}, (\tilde{H}_{\frac{t^0}{2t^0}})^{-1} \tilde{\eta} \right> = \left< \eta, (H_{\frac{t^0}{2(t^0 + t^1)}})^{-1} \eta \right> - \left< \eta, (H_{\frac{t^0}{2t^0}})^{-1} \eta \right> \]
(4.4)
Proof. (i) We can write
\[ K_{t^0 + t^1} = K_{t^0} - t^1 W = (\text{Id} - t^1 W K_{t^0}^{-1}) K_{t^0} = \tilde{K}_{t^1} K_{t^0}. \]
(ii) Formula (4.3) follows from
\[ \tilde{\eta} = (t^0)^{-1}(\text{Id} + t^0 W K_{t^0}^{-1}) t^0 \eta = (t^0)^{-1} K_{t^0}^{-1} t^0 \eta = (t^0)^{-1} H_{\frac{t^0}{2t^0}}^{-1} \eta, \]
Turning to Formula (4.4), using (4.1), we have
\[ K_{t^0 + t^1} = K_{t^0}^{-1} \tilde{K}_{t^1}^{-1} \]
and
\[ \tilde{H}_{\frac{t^0}{2t^0}}^{-1} = K_{t^0} K_{t^0 + t^1} t^1 \]
\[ = K_{t^0} K_{t^0 + t^1} (t^0 + t^1) \left( \frac{1}{t^0} - \frac{1}{t^0 + t^1} \right) t^0 \]
\[ = t^0 H_{\frac{t^0}{2t^0}}^{-1} H_{\frac{1}{2(t^0 + t^1)}}^{-1} \left( \frac{1}{t^0} - \frac{1}{t^0 + t^1} \right) t^0 \]
\[ = t^0 H_{\frac{t^0}{2t^0}}^{-1} H_{\frac{1}{2(t^0 + t^1)}}^{-1} \left( H_{\frac{1}{2t^0}} - H_{\frac{1}{2(t^0 + t^1)}} \right) t^0 \]
\[ = t^0 H_{\frac{t^0}{2t^0}}^{-1} H_{\frac{1}{2(t^0 + t^1)}}^{-1} H_{\frac{1}{2t^0}} t^0 - t^0 H_{\frac{1}{2t^0}} t^0 \]
(4.5)
Now, (4.3) implies
\[ \tilde{H}_{\frac{t^0}{2t^0}} \tilde{\eta} = t^0 H_{\frac{t^0}{2t^0}}^{-1} H_{\frac{1}{2(t^0 + t^1)}}^{-1} \eta - t^0 \eta, \]
Since \( H_{\frac{1}{2t^0}} \) is symmetric, we get (4.4) by (4.3).
5. Proof of basic properties of the S.D.E. \( E_{V}^{W,\theta,\eta} \): Lemma 1, Proposition 1

Remark that (i) and (iii) of Lemma 1 are equivalent since \( dX(t) = dY(t) - \eta dt \). In order to prove the existence and uniqueness of the pathwise solution of \( E_{V}^{W,\theta,\eta}(Y) \) (or equivalently \( E_{V}^{W,\theta,\eta}(X) \)), we first consider a non stopped version of the S.D.E. \( E_{V}^{W,\theta,\eta}(Y) \), for which the existence and uniqueness is simpler.

**Lemma 3.** Let \( (\theta_i)_{i \in V} \in \mathbb{R}^V_+ \). Let \( h > 0 \) be the smallest positive real such that \( \text{det}(K_h) = 0 \). Then the following S.D.E. is well-defined on time interval \([0, h)\) and has a unique pathwise solution

\[
(5.1) \quad \tilde{Y}_i(t) = \theta_i + B_i(t) - \int_0^t (WK^{-1}(s))_i ds \quad \forall i
\]

Moreover, there exists a time \( \tau < h \) such that \( \tilde{Y}_i(\tau) = \tau \eta_i \) for some vertex \( i \in V \).

**Proof.** As \( WK^{-1} \) is bounded on time interval \([0, h - \epsilon)\) for all \( \epsilon > 0 \), it is a linear S.D.E. with bounded coefficients there is a unique pathwise solution, with continuous simple paths.

To see the existence of \( \tau \), we can define \((Z_t)_{t \geq 0}\) by

\[
(h - t)Z_i\left(\frac{t}{h - t}\right) = \tilde{Y}_i(t), \quad \forall i \in V
\]

and write (5.1) as

\[
(h - t)Z_i\left(\frac{t}{h - t}\right) = \theta_i + B_i(t) - \int_0^t \left[ WK^{-1}(h - s)Z\left(\frac{s}{h - s}\right)\right]_i ds.
\]

By time change \( u = \frac{t}{h - t} \), the S.D.E. is written in the following equivalent form

\[
\frac{1}{u + 1} Z_i(u) = \frac{\theta_i}{h} + B_i\left(\frac{hu}{u + 1}\right) - \int_0^u \left[ WK^{-1}\left(\frac{h}{u}\right)Z(v)\right]_i \frac{1}{(v + 1)^2} dv.
\]

That is

\[
(5.2) \quad dZ_i(u) = \frac{1}{\sqrt{h}} d\tilde{B}_i(u) + \frac{1}{u + 1} \left( \left[ \text{Id} - WK^{-1}\left(\frac{h}{u}\right) \right] Z(u) \right) du.
\]

where \((\tilde{B}_i(t))_{i \in V}\) is a N-dimensional Brownian motion. As \( t \to h \), we have \( u \to \infty \), and there exists \( \tau < h \) such that \( \tilde{Y}_i(\tau) = \tau \eta_i \) if and only if there exists \( \tau' \in \mathbb{R}_+ \) such that \( Z_i(\tau') = \tau' \eta_i \).

Assume by contradiction that none of these \( Z_i \) reach the lines \( y = \eta_i x \), in particular, they are all positive. We use that \( K^{-1}_s \) has positive coefficients and that \( \lim_{s \to h} \min_{i,j}(K^{-1}_s)_{i,j} = +\infty \), which implies that for \( u \) large enough \( \left( \text{Id} - WK^{-1}\left(\frac{h}{u}\right) \right) \) has negative coefficients, hence the drift term in (5.2) is negative. This implies that \( Z_i(u) \) given by (5.2) is stochastically bounded from above by a Brownian motion, at least for \( u \) large enough. Hence, the processes \((Z_i(u))_{u \geq 0}\) reach 0 in finite time, which leads to a contradiction. \( \square \)

**Proof of Lemma 1.** We prove it by recurrence on the size of \( V \). We will gradually define \( Y(t) \), solution to the equation \( E_{V}^{W,\theta,\eta}(Y) \) and \( X(t) = Y(t) - t\eta \). Consider

\[
\tau = \inf\{t \geq 0, \exists i \in V \text{ such that } X_i(t) = 0\}
\]

and denote by \( i_0 \) the vertex in \( V \) such that \( X_{i_0}(\tau) = 0 \). Up to time \( \tau \), the equation \( E_{V}^{W,\theta,\eta}(Y) \) is equivalent to the equation (5.1), hence the equation \( E_{V}^{W,\theta,\eta}(Y) \) is well-defined and has
unique pathwise solution up to time \( \tau \) and \( \tau < \infty \) a.s.. Moreover, \( T_{i_0} = \tau \). Now we set \( \tilde{V} = \{i_0\} \) and

\[
(\tilde{T}_i)_{i \in V} = (T_i - \tau)_{i \in V}
\]

\[
\tilde{W} = WK^{-1}_\tau, \quad \tilde{K}_s = \text{Id} - s\tilde{W}, \quad \tilde{\eta} = \eta + \tilde{W}(\tau \eta).
\]

and use that, by (4.1) applied to \( t^0_i = \tau \) for all \( i \), and \( t^1 = s \wedge \tilde{T} \),

\[
K^{-1}_{(\tau+s)\wedge T} = K^{-1}_{\tau}\tilde{K}^{-1}_{s\wedge \tilde{T}}.
\]

We set

\[
\tilde{X}(s) = X(\tau + s), \quad \tilde{B}(s) = B(\tau + s).
\]

Hence, we have that

\[
(\tau + s) \wedge T = \tau + s \wedge \tilde{T}, \quad WK^{-1}_{(\tau+s)\wedge T} = \tilde{W}\tilde{K}^{-1}_{s\wedge \tilde{T}}
\]

and after time \( \tau \), \( (X_{\tau+t})_{t \geq 0} \) is solution of \( \mathbb{E}^{W,X(\tau),\tilde{\eta}}_{\tilde{V}}(X) \) if and only if \( \tilde{X}(s) \) is solution of

\[
d\tilde{X}(s) = \mathbb{1}_{s < \tilde{T}} dB(s) + \mathbb{1}_{s < \tilde{T}} \left( \tilde{W}\tilde{K}^{-1}_{s\wedge \tilde{T}} \left( \tilde{X}(s) + (s \wedge \tilde{T})\eta + \eta \right) \right) ds.
\]

Using that,

\[
\tilde{W}\tilde{K}^{-1}_{s\wedge \tilde{T}} \left( \tilde{X}(s) + (s \wedge \tilde{T})\eta \right) = \tilde{W}\tilde{K}^{-1}_{s\wedge \tilde{T}} \left( \tilde{X}(s) + \tilde{K}_{s\wedge \tilde{T}}(\tau \eta) + (s \wedge \tilde{T})\tilde{W}(\tau \eta) + (s \wedge \tilde{T})\eta \right) = \tilde{W}\tilde{K}^{-1}_{s\wedge \tilde{T}} \left( \tilde{X}(s) + (s \wedge \tilde{T})\tilde{\eta} \right) + \tilde{W}(\tau \eta)
\]

we see that (5.3) is equivalent to the fact that \( \tilde{X} \) is solution of \( \mathbb{E}^{W,X(\tau),\tilde{\eta}}_{\tilde{V}}(X) \). Since, \( X_{i_0}(\tau) = 0 \) is it equivalent to the fact that \( \tilde{X}_{\tilde{V}} \) is solution of \( \mathbb{E}^{W,X(\tau),\tilde{\eta}}_{\tilde{V}}(X) \). Hence, we conclude by the recurrence hypothesis applied to \( \tilde{V} \), which implies that \( \mathbb{E}^{W,X(\tau),\tilde{\eta}}_{\tilde{V}}(X) \) has a unique pathwise solution.

**Proof of Lemma 4 [33].** Remark first that

\[
\frac{\partial}{\partial t} K^{-1}_{t\wedge T} = K^{-1}_{t\wedge T} \mathbb{1}_{t < T} WK^{-1}_{t\wedge T}.
\]

Differentiating \( \psi(t) = K^{-1}_{t\wedge T}(Y(t)) \), we get,

\[
d\psi_i(t) = (K^{-1}_{t\wedge T}(dY(t)))_i + (K^{-1}_{t\wedge T} \mathbb{1}_{t < T} WK^{-1}_{t\wedge T}(Y(t)))_i dt
\]

\[
= (K^{-1}_{t\wedge T} (\mathbb{1}_{t < T} dB(t)))_i
\]
Moreover, the quadratic variation of \( \psi_i(t) \) and \( \psi_j(t) \) is given by

\[
\langle \psi_i, \psi_j \rangle_t = \sum_{l \in V} \int_0^t (K^{-1}_{s \wedge T_i})_{i,l} \mathbb{1}_{s < T_i} (K^{-1}_{s \wedge T_j})_{j,l} ds
\]

\[
= \sum_{l \in V} \int_0^t (H^{-1}_{\frac{1}{2(s \wedge T_l)}})_{i,l} \left( \frac{1}{s \wedge T_l} \right)^2 \mathbb{1}_{s < T_l} (H^{-1}_{\frac{1}{2(s \wedge T_l)}})_{l,j} ds
\]

\[
= \int_0^t \frac{\partial}{\partial s} (H^{-1}_{\frac{1}{2(s \wedge T_l)}})_{i,j} ds
\]

where in the second equality, we used \( H_\beta \) is a symmetric matrix and \( H^{-1}_{\frac{1}{2(s \wedge T_l)}} = K^{-1}_{t \wedge T_l}(t \wedge T) \), and so that \( H^{-1}_{\frac{1}{2(s \wedge T_l)}} = (t \wedge T)(K^{-1}_{t \wedge T_l})^t \). In the last equality we used that \( H^{-1}_{\frac{1}{2(s \wedge T_l)}} \) is well defined and null for \( t = 0 \). \( \square \)

Similar arguments as above give the following result about stationarity of the equation.

**Proposition 1** (Stationarity). (i) Consider \((t^0_i)_{i \in V} \in (\mathbb{R}_+)^V\) a non-negative vector such that \( K_{t^0} \) is positive definite. We denote by \( E_{V,t^0}^{W,\theta,\eta} \) the equation \( E_{V,t^0}^{W,\theta,\eta} \) (respectively for \( Y \) or \( X \)) where \( K_{t \wedge T} \) is replaced by \( K_{t^0 + t \wedge T} \) in (respectively) (2.5) or (2.6). Then, the equation \( E_{V,t^0}^{W,\theta,\eta} \) is equivalent to the equation \( E_{V,t^0}^{W,\theta,\eta} \) where

\[
\tilde{W} = WK_{t^0}^{-1}. 
\]

(ii) If \((X(t))_{t \geq 0} \) is the solution of \( E_{V,t^0}^{W,\theta,\eta}(X) \) and \( s \geq 0 \), then \((X(t+s))_{t \geq 0} \) is solution of the S.D.E. \( E_{V,t^0}^{W,X(s),\tilde{\eta}}(X) \) with the shifted brownian motion \((B(t+s))_{t \geq 0} \), and with

\[
\tilde{W} = WK_{s \wedge T}^{-1}, \quad \tilde{\eta} = \eta + \tilde{W}(s \wedge T)\eta, \quad \tilde{\eta}(t) = \eta_{V(t)}. 
\]

**Remark 5.** Comparing \( \textbf{iii} \) with Theorem \( \textbf{iii} \) \( \textbf{iii} \), see when \( \eta = 0 \), conditionally on \( F^X(t^0) \), the process \((X(t^0+T))_{t \geq 0} \) has the same law as the solution of \( E_{V,t^0}^{W,\theta,\eta} \) defined in the proposition above, which means that the process \((X(t^0+T))_{t \geq 0} \) has the same law as if the time on each coordinate was started at \((t^0_i)_{i \in V} \).

**Remark 6.** The statement \( \textbf{iii} \) must be compared with Theorem \( \textbf{iii} \), see Remark \( \textbf{iii} \) above.

**Proof of Proposition 1.** \( \textbf{iii} \) The equation \( E_{V,t^0}^{W,\theta,\eta}(Y) \) corresponds to the equation \( E_{V,t^0}^{W,\theta,\eta}(X) \) where \( K_{t \wedge T} \) is replaced by \( K_{t^0 + t \wedge T} \). By Lemma \( \textbf{iii} \) we have that

\[
WK_{t^0 + t \wedge T}^{-1} = \tilde{W}K_{t \wedge T}^{-1},
\]

where \( \tilde{W} = WK_{t^0}^{-1} \) as defined in Proposition \( \textbf{iii} \), and \( K_{t \wedge T} = \text{Id} - (t \wedge T)\tilde{W} \). The result follows.

\( \textbf{iii} \) Set \((\tilde{X}(t))_{t \geq 0} := (X(t+s))_{t \geq 0}, (\tilde{B}(t))_{t \geq 0} := (B(t+s))_{t \geq 0}, \) and \( \tilde{T} = T - s \wedge T \). Remark that by Lemma \( \textbf{iii} \)

\[
(s + t) \wedge T = s \wedge T + t \wedge \tilde{T}, \quad WK_{(s+t) \wedge T} = \tilde{W}K_{t \wedge T}.
\]
with $\tilde{W}$ defined in \eqref{eq:tildeW} and as usual $\tilde{K}_{t\wedge T} = \text{Id} - (t \wedge T)\tilde{W}$. The S.D.E. $E_{V}^{W,\theta,\eta}(X)$ after time $s$ is thus equivalent to

$$d\tilde{X}(t) = 1_{t < \tilde{T}}d\tilde{B}_i(t) - 1_{t < \tilde{T}_i} \left( \tilde{W}\tilde{K}_{t\wedge T}^{-1}\left(\tilde{X}(t) + (s \wedge T)\eta + (t \wedge \tilde{T})\eta\right) + \eta \right)_i dt, \quad \forall i \in V,$$

By Lemma 2, we have that

$$\tilde{W}\tilde{K}_{t\wedge T}^{-1}\left(\tilde{X}(t) + (s \wedge T)\eta + (t \wedge \tilde{T})\eta\right) = \tilde{W}\tilde{K}_{t\wedge T}^{-1}\left(\tilde{X}(t) + \tilde{K}_{t\wedge T}( (s \wedge T)\eta + (t \wedge \tilde{T})\eta\right) = \tilde{W}\tilde{K}_{t\wedge T}^{-1}\left(\tilde{X}(t) + (t \wedge \tilde{T})\eta\right) + \tilde{W}(s \wedge T)\eta$$

Hence, $\tilde{X}(t)$ is solution of

$$d\tilde{X}(t) = 1_{t < \tilde{T}_i}d\tilde{B}_i(t) - 1_{t < \tilde{T}_i} \left( \tilde{W}\tilde{K}_{t\wedge T}^{-1}\left(\tilde{X}(t) + (t \wedge \tilde{T})\eta\right) + \eta \right)_i dt, \quad \forall i \in V,$$

Since, $\tilde{X}(0) = X(s)$, we have the result. \hfill \Box

6. PROOF OF THEOREM \ref{thm:main}

We provide below a convincing but incomplete argument for the proof of Theorem \ref{thm:main}. We do not know yet how to turn this argument into a rigorous alternative proof, even though we think that it should be possible. The rigorous proof is given in Section 6.2.

6.1. A convincing but incomplete argument for Theorem \ref{thm:main}. Let $\lambda \in \mathbb{R}^V_+$ be a non negative vector on $V$. As

$$\exp\left(-\langle \eta, H^{-1}_\beta \lambda \rangle - \frac{1}{2} \langle \lambda, H^{-1}_\beta \lambda \rangle \right) \nu_{V}^{W,\theta,\eta} = \exp\left(-\langle \lambda, \theta \rangle \right) \nu_{V}^{W,\theta,\eta + \lambda},$$

we have,

$$\int \exp\left(-\langle \eta, H^{-1}_\beta \lambda \rangle - \frac{1}{2} \langle \lambda, H^{-1}_\beta \lambda \rangle \right) \nu_{V}^{W,\theta,\eta}(d\beta) = \exp\left(-\langle \lambda, \theta \rangle \right).$$

On the other hand, consider $Y(t)$, solution of $E_{V}^{W,\theta,\eta}(Y)$ and the associated processes $(X(t))$, $(\psi(t))$. By Lemma \ref{lem:main} and \ref{lem:main'} proposition 3.4 p 148, we know that

$$\exp\left(-\langle \lambda, \psi(t) \rangle - \frac{1}{2} \langle \lambda, H^{-1}_{\frac{1}{2}(T - T)} \lambda \rangle \right),$$

is a continuous martingale, dominated by 1. Moreover, we have that $X(t) \to 0$, a.s., when $t \to \infty$, hence, a.s.,

$$\lim_{t \to \infty} \psi(t) = K_T^{-1}(T\eta) = H^{-1}_{\frac{1}{2T}}\eta.$$

By dominated convergence, it implies that

$$\mathbb{E}\left( \exp\left(-\langle \lambda, H^{-1}_{\frac{1}{2T}}\eta \rangle - \frac{1}{2} \langle \lambda, H^{-1}_{\frac{1}{2T}}\lambda \rangle \right) \right) = \exp\left(-\langle \lambda, \psi(0) \rangle \right) = \exp\left(-\langle \lambda, \theta \rangle \right).$$

Hence, it implies that both $\beta$ under $\nu_{V}^{W,\theta,\eta}$ and $\frac{1}{2T}$ obtained from $E_{V}^{W,\theta,\eta}$ satisfy the same functional identity \eqref{eq:func_id}. Note that the dimension of the space of variables $(\lambda_i)_{i \in V}$ and of the random variables $(\beta_i)_{i \in V}$ is the same. Nevertheless, it is not clear if the functional identity \eqref{eq:func_id} characterizes the distribution $\nu_{V}^{W,\theta,\eta}$, at least we have no proof of this fact.
If such an argument were available, it would imply Theorem 1. Indeed, using the stationarity of the equation, Proposition 1, it would be possible to deduce Theorem 1 by enlargement of filtration (see [5]). We do not give the detail of the argument here since the first part of the proof is missing.

6.2. Proof. The strategy of the proof of Theorem 1 is in the spirit of the proof of Theorem 2, ii) of [13]: we start from the mixture of Bessel processes and we prove that this mixture has the same law as the solutions of the S.D.E. $E^{W,\theta,\eta}(X)$. We use in a crucial way the fact that the law $\nu_{V}^{W,\theta,\eta}$ is a probability density with explicit normalizing constant.

6.2.1. The classical statement for $N = 1$. We denote by $W = C(\mathbb{R}^+, \mathbb{R})$ the Wiener space. For $\theta > 0$, denote by $P_{\theta}$ the law of $X_{\theta,t}$ where $X_{\theta} = \theta + B_{t}$ and $B_{t}$ is a standard Brownian motion and $T = \inf\{t \geq 0, X_{\theta} = 0\}$ is the first hitting time of $0$. We denote by $Q^{3,T}_{\theta,0}$ the law of the 3-dimensional Bessel Bridge from $\theta > 0$ to $0$ on time interval $[0, T]$, as defined in [12], section XI-3. We always consider that the Bessel bridge is extended to time interval $[0,\infty)$, with constant value equal to $0$ after time $T$. As mentioned in the introduction it is known (see [17], [12], p317), that, under $P_{\theta}$, $\frac{\beta_{t}}{\sqrt{t}}$ has the law Gamma($\frac{1}{2}, \frac{\theta^{2}}{4}$) and that, conditionally on $T$, $(X_{\theta})_{i \geq 0}$ has law $Q^{3,T}_{\theta,0}$. Otherwise stated it means that the following equality of probabilities holds

\[ P_{\theta}(\cdot) = \int_{0}^{\infty} Q^{3,T}_{\theta,0}(\cdot) d\theta e^{-\frac{\theta^{2}}{4T}} dT \]

6.2.2. Proof of Theorem 1 (i) and (ii). We use the formulation of Lemma 1 (ii), and we will prove that if $(X_{\theta}(t))_{i \in V}$ satisfies $E^{W,\theta,\eta}(X)$, then $\beta := \frac{1}{\sqrt{T}}$ is distributed as $\nu_{V}^{W,\theta,\eta}$ and conditionally on $T$, the coordinates $(X_{\theta}(t))_{i \geq 0}$ are independent 3-dimensional Bessel bridges from $\theta_{i}$ to $0$ on time interval $[0, T]$.

Recall that $V = \{1, \ldots, N\}$, and denote by $W_{V} = C(\mathbb{R}^+, \mathbb{R}^{V})$ the $N$-dimensional Wiener space and $(X(t))_{t \geq 0}$ the canonical process. For $\theta = (\theta_{i})_{i \in V} \in \mathbb{R}_{+}^{V}$, we set

\[ P_{V,\theta} = \otimes_{i \in V} P_{\theta_{i}}, \]

the probability on $W_{V}$ such that $(X_{\theta}(t))_{i \in V}$ are $N$ independent Brownian motions starting at positions $(\theta_{i})$ and stopped at their first hitting times of $0$. The assertions of Theorem 1 (i) and (ii) are equivalent to the fact that the law of the solution of the S.D.E. $E^{W,\theta,\eta}(X)$ is a mixture of independent Bessel bridges $Q^{3,T}_{\beta_{i},0}$, where $\beta$ is a random vector with distribution $\nu_{V}^{W,\theta,\eta}$. Otherwise stated, it means that the probability distribution $P_{V}^{W,\theta,\eta}$ defined by

\[ P_{V}^{W,\theta,\eta}(\cdot) := \int \left( \otimes_{i \in V} Q^{3,T}_{\beta_{i},0} \right) (\cdot) \nu_{V}^{W,\theta,\eta}(d\beta), \]

is the law of the solution of the S.D.E. $E^{W,\theta,\eta}(X)$. The strategy is now to write the Radon-Nikodym derivative of $P_{V}^{W,\theta,\eta}$ with respect to $P_{V,\theta}$ as an exponential martingale, and then to apply Girsanov’s theorem.

In the sequel, we adopt the following notations:

\[ T := \frac{1}{2\beta}, \quad \text{so that} \quad H_{\beta} = \frac{1}{T}K_{T}. \]
From (6.2), it is clear that $P_{V,\theta,\eta}$ is absolutely continuous with respect to $P_{V,\theta}$, and changing from variables $\beta$ to $T$ in $\nu_{V,\theta,\eta}(d\beta)$, we get that

$$d\frac{P_{V,\theta,\eta}}{P_{V,\theta}} = 1_{H_{T} > 0} \cdot e^{-\frac{1}{2} \left< \theta, H_{T} \theta \right> + \frac{1}{2} \left< \eta, H_{T}^{-1} \eta \right>} \frac{1}{\sqrt{|H_{T}|}}.$$

(6.3)

Let $t > 0$, define

$$\begin{align*}
\tilde{V}(t) &:= \{ i \in V, T_i > t \}, \\
\beta(t) &:= \frac{1}{2(t \wedge T)}, \\
\tilde{W}(t) &:= WK_{t\wedge T}^{-1} = W + WK_{t\wedge T}^{-1}(t \wedge T)W
\end{align*}$$

where the last equality comes from the fact that $K_{t\wedge T}^{-1} = Id + (t \wedge T)WK_{t\wedge T}^{-1}$ by classical first order expansion. Note that $\tilde{W}(t)$ is symmetric since $K_{t\wedge T}^{-1}(t \wedge T) = H_{t\wedge T}^{-1}$. We also set,

$$\begin{align*}
\tilde{T}(t) &:= T - t \wedge T, \\
\tilde{\beta}(t) &:= \frac{1}{2T(t)}, \\
\tilde{K}_{\tilde{T}}(t) &:= Id - \tilde{T}(t)\tilde{W}(t), \\
\tilde{H}_{\tilde{\beta}}(t) &:= 2\tilde{\beta}(t) - \tilde{W}(t) = \frac{1}{T(t)}\tilde{K}_{\tilde{T}}(t), \\
\tilde{\eta}(t) &:= \eta + \tilde{W}(t)(t \wedge T)\eta
\end{align*}$$

Note that $(\tilde{H}_{\tilde{\beta}(t)})^{-1}$ is well defined for all $t$ using $(\tilde{H}_{\tilde{\beta}(t)})^{-1} = (\tilde{K}_{\tilde{T}}(t))^{-1}\tilde{T}(t)$, see beginning of section 4. By lemma 4.3 applied to $t^0 = t \wedge T$ and $t^1 = \tilde{T}$, we get that

$$\tilde{\eta}(t) = (t \wedge T)^{-1}H_{\tilde{\beta}(t)}^{-1}\eta.$$

We first prove the following lemma.

**Lemma 4.** Let

$$M_t = \exp \left( -\frac{1}{2} \left< X(t), \tilde{W}(t)X(t) \right> + \frac{1}{2} \left< \tilde{\eta}(t), (\tilde{H}_{\tilde{\beta}(t)})^{-1}\tilde{\eta}(t) \right> - \left< \tilde{\eta}(t), X(t) \right> \right) \sqrt{|\tilde{K}_{\tilde{T}}(t)|}$$

Under $P_{V,\theta}$, we have

$$\frac{M_t}{M_0} = \exp \left( -\int_0^t \left< W\psi(s) + \eta, dX_s \right> - \frac{1}{2} \int_0^t \left< W\psi(s) + \eta, \mathbb{1}_{s < T}(W\psi(s) + \eta) \right> ds \right)$$

with

$$\psi(t) = K_{t\wedge T}^{-1}(X(t) + (t \wedge T)\eta).$$

**Proof of Lemma 4** We will compute the Itô derivative of $\ln M_t$, the following formulae will be used several times

$$\frac{\partial}{\partial t} K_{t\wedge T} = -1_{t < T}W, \quad \frac{\partial}{\partial t} K_{t\wedge T}^{-1} = K_{t\wedge T}^{-1}1_{t < T}WK_{t\wedge T}^{-1}, \quad \frac{\partial}{\partial t} \tilde{W}(t) = \tilde{W}(t)1_{t < T}\tilde{W}(t).$$

(6.6)

$$\frac{\partial}{\partial t} H_{\tilde{\beta}(t)}^{-1} = H_{\tilde{\beta}(t)}^{-1}1_{t < T} \left( \frac{1}{t \wedge T} \right)^2 H_{\tilde{\beta}(t)}^{-1}$$

(6.7)
By (6.6) and Itô formula, we have
\[ d \left\langle X(t), \tilde{W}(t)X(t) \right\rangle = 2 \left\langle dX(t), \tilde{W}(t)X(t) \right\rangle + \left\langle \tilde{W}(t)X(t), 1_{t<T} \tilde{W}(t)X(t) \right\rangle dt + \text{Trace}(\tilde{W}(t)1_{t<T})dt \]
(6.8)

where in the second term we used that the operator \( \tilde{W}(t) \) is symmetric.

By (4.4) of Lemma 2 applied to \( t^0 = t \wedge T \) and \( t^1 = \tilde{T} \), we get
\[ \left\langle \tilde{\eta}(t), (\tilde{H}_{\beta}(t))^{-1}\tilde{\eta}(t) \right\rangle = \left\langle \eta, (H_{\beta})^{-1}\eta \right\rangle - \left\langle \eta, (H_{\beta}(t))^{-1}\eta \right\rangle \]
(6.9)

Using (6.7) and (6.4), it implies,
\[ d \left\langle \tilde{\eta}(t), (\tilde{H}_{\beta}(t))^{-1}\tilde{\eta}(t) \right\rangle = - \left\langle \tilde{\eta}(t), 1_{t<T} \tilde{\eta}(t) \right\rangle dt. \]
(6.10)

We have also
\[ \frac{\partial}{\partial t} \tilde{\eta}(t) = \tilde{W}(t)1_{t<T}\eta + \tilde{W}(t)1_{t<T} \tilde{W}(t)(t \wedge T)\eta = \tilde{W}(t)1_{t<T} \tilde{\eta}(t). \]
Hence,
\[ d \left\langle \tilde{\eta}(t), X(t) \right\rangle = \left\langle \tilde{\eta}(t), dX(t) \right\rangle + \left\langle \tilde{\eta}(t), 1_{t<T} \tilde{W}(t)X(t) \right\rangle dt. \]
(6.11)

Finally, using (4.4) of Lemma 4.1 applied to \( t^0 = t \wedge T \) and \( t^1 = \tilde{T} \), we get
\[ K_T^{-1} = K_{t \wedge T}^{-1}(\tilde{K}_T(t))^{-1} \]
which implies by (6.6),
\[ \frac{\partial}{\partial t} \ln |\tilde{K}_T(t)| = - \frac{\partial}{\partial t} \ln |K_{t \wedge T}| = - \text{Trace}(1_{t<T}WK_{t \wedge T}^{-1}) = - \text{Trace}(1_{t<T} \tilde{W}(t)). \]
(6.12)

Combining (6.8), (6.9), (6.10), and (6.12), we get using that \( \tilde{W}(t)X(t) + \tilde{\eta}(t), \]
\[ d \ln M_t = - \left\langle dX(t), \tilde{W}(t)X(t) + \tilde{\eta}(t) \right\rangle - \frac{1}{2} \left\langle \tilde{W}(t)X(t), 1_{t<T} \tilde{W}(t)X(t) \right\rangle dt \]
\[ - \frac{1}{2} \left\langle \tilde{\eta}(t), 1_{t<T} \tilde{\eta}(t) \right\rangle dt - \left\langle \tilde{\eta}(t), 1_{t<T} \tilde{W}(t)X(t) \right\rangle dt \]
\[ = - \left\langle W\psi(t) + \eta, dX(t) \right\rangle - \frac{1}{2} \left\langle W\psi(t) + \eta, 1_{t<T} (W\psi(t) + \eta) \right\rangle dt \]

Consider now a positive measurable test function \( \phi((X_s)_{s \leq t}) \). Denote by \( \mathbb{E}_V^{W, \theta, \eta} \), (resp. \( \mathbb{E}_{V, \theta} \)), the expectation with respect to \( \mathbb{P}_V^{W, \theta, \eta} \), (resp. \( \mathbb{P}_{V, \theta} \)). We have, by (6.3),
\[ \mathbb{E}_V^{W, \theta, \eta}(\phi((X_s)_{s \leq t})) \]
\[ = \mathbb{E}_{V, \theta}\left( \phi((X_s)_{s \leq t})1_{H_{\theta} > 0} \cdot e^{\frac{1}{2}(W_{\theta} - \eta_{(K_T)^{-1}T\eta} + \langle \eta, \theta \rangle) \frac{1}{\sqrt{|K_T|}}} \right) \]
\[ = \mathbb{E}_{V, \theta}\left( \frac{M_t}{M_0} \phi((X_s)_{s \leq t})1_{H_{\theta} > 0} \cdot e^{\frac{1}{2}(X(t), \tilde{W}(t)X(t)) - \frac{1}{2} \left\langle \tilde{\eta}(t), (\tilde{H}_{\beta}(t))^{-1}\tilde{\eta}(t) \right\rangle + \langle \tilde{\eta}(t), X(t) \rangle} \frac{1}{\sqrt{|\tilde{K}_T(t)|}} \right) \]
Let us denote by $(\cdot,\cdot)_V$ the usual scalar product on $\mathbb{R}^V$ (we keep denoting by $(\cdot,\cdot)$ the usual scalar product on $\mathbb{R}^t$). As $X(t)$ vanishes on $V \setminus \tilde{V}(t)$, we have

$$\langle X(t), \tilde{W}(t)X(t) \rangle = \langle X(t), \tilde{W}(t)X(t) \rangle_{\tilde{V}(t)}, \quad \langle \tilde{\eta}(t), X(t) \rangle = \langle \tilde{\eta}(t), X(t) \rangle_{\tilde{V}(t)},$$

By (4.5), since $\tilde{H}_\beta(t) = \tilde{K}_t^{-1}(t)\tilde{T}(t)$ and since $\tilde{T}(t)$ vanishes on the subset $V \setminus \tilde{V}(t)$ and $\tilde{H}_\beta(t)$ is symmetric, we get

$$\langle \tilde{\eta}(t), (\tilde{H}_\beta(t))^{-1}\tilde{\eta}(t) \rangle = \langle \tilde{\eta}(t), (\tilde{H}_\beta(t))^{-1}\tilde{\eta}(t) \rangle_{\tilde{V}(t)},$$

Moreover,

$$|\tilde{K}_t(t)| = |\text{Id} - \tilde{T}(t)\tilde{W}(t)| = |(\text{Id} - \tilde{T}(t)\tilde{W}(t))_{\tilde{V}(t)}\tilde{V}(t)|$$

and

$$\mathbb{1}_{H_\beta(t)}>0 = \mathbb{1}_{H_\beta(t)}>0\mathbb{1}_{\tilde{H}_\beta(t)}>0$$

thus

$$\mathbb{1}_{H_\beta(t)}>0\mathbb{1}_{\tilde{H}_\beta(t)}>0\mathbb{e}^{\int_0^t (X(s) + \eta, dX_s)} - \frac{1}{2} \int_0^t (W\psi(s) + \eta, 1_{s \leq T}(W\psi(s) + \eta)) ds$$

where we used Lemma 4 in the second equality. It implies that

$$\mathbb{P}_{V,\theta}^{W,\eta}(\phi(\tilde{(X_s)_{s \leq t}})) = \mathbb{E}_{V,\theta}\left(\mathbb{1}_{H_\beta(t)}>0\frac{M_t}{M_0}\phi((X_s)_{s \leq t})\mathbb{E}_{\tilde{V}}(\tilde{X}(t), \tilde{\eta}(t))\right),$$

Finally, by Girsanov’s theorem, we know that under the law

$$e^{\int_0^t (W\psi(s) + \eta, dX_s)} - \frac{1}{2} \int_0^t (W\psi(s) + \eta, 1_{s \leq T}(W\psi(s) + \eta)) ds\mathbb{P}_{V,\theta},$$

the process

$$(\tilde{B}(t))_{t \geq 0} := (X_t + \int_0^t 1_{s \leq T}(W\psi(s) + \eta) ds)_{t \geq 0}$$

is a Brownian motion stopped at time $T$, the first hitting time of 0 by $(X(t))$. (Indeed, recall that $\mathbb{P}_{V,\theta}$ is the law of independent Brownian motions starting at $\theta$ and stopped at their first hitting time of 0). Hence,

$$dX(t) = 1_{t < T}d\tilde{B}(t) + 1_{t < T}(W\psi(t) + \eta) dt,$$

and under the law (6.13), $X$ is solution of the S.D.E $E_{V}^{W,\theta,\eta}(X)$ with driving Brownian motion $\tilde{B}$. By Lemma 1 we know that a.s. under the law (6.13), we have $H_\beta(t) > 0$, thus $\mathbb{P}_{V,\theta}^{W,\eta}$ and (6.13) are equal. Hence, under $\mathbb{P}_{V,\theta}^{W,\eta}$, $(X(t))$ has the law of the solutions of the S.D.E $E_{V}^{W,\theta,\eta}(X)$.
7. Proof of the abelian properties : Theorem 2

Proof of Theorem 2 (ii). Consider first the restriction property (i). By Theorem 1 conditionally on $T$, $(X_i(t))_{i \in V}$ are independent Bessel bridges from $\theta_i$ to 0 in time $T_i$. By Theorem 1 and Lemma $\Box$ $\frac{1}{2T_i}$ is $\nu_{W_{U,V},\theta_i,\eta}$ distributed. By Theorem 1 applied to the set $U$ and parameters $W_{U,V}, \theta_V, \eta$, it implies that $X_U$ has the law of the solutions of $E_{U}^{W,\theta,\eta}(X)$.

For (iii), the same argument applies, using that $\beta_U$, conditionally on $\beta_V$, is $\nu_{U,V,\theta_i,\eta}$ distributed. □

Proof of Theorem 2 (iii). Recall that we denote by $X(t)$ (resp. $(X_i(t))_{i \in V}$) the canonical process on Wiener space $W = C(\mathbb{R}_+, \mathbb{R})$ (resp. $W_V = C(\mathbb{R}_+, \mathbb{R}^V)$). Recall that $E_{\theta,0}^{3,T}$ and $E_{\theta,0}^{3,T}$ denotes the law (resp. the expectation) on $W$ of a Bessel bridge from $\theta$ to 0 on time interval $[0, T]$ (and extended by 0 for $t \geq T$). Recall also that $E_{\theta,0}^{W,\theta,\eta}(\cdot)$ denotes the expectation with respect to the law on $W_V$ of the solution of the S.D.E. $E_{U}^{W,\theta,\eta}(X)$.

Following [12] p.463, under $E_{\theta,0}^{3,T}$, the law of $X(t)$ for some $0 < t < T$ is given by $p_{\theta,0}^{3,t} (y) dy$ on $\mathbb{R}_+$, with

$$
p_{\theta,0}^{3,t} (y) = \frac{1}{\sqrt{2\pi t}} e^{\frac{y^2}{2t}} \frac{y}{\sqrt{2}} e^{\frac{\beta t}{2}} \left( e^{-\frac{(y-\theta)^2}{2t}} - e^{-\frac{(y+\theta)^2}{2t}} \right), \quad \forall y \geq 0.
$$

Moreover, the Markov property of the Bessel bridge implies that under $E_{\theta,0}^{3,T}$ and conditionally on $X(t) = x$, $0 < t < T$, the law of $((X(u))_{0 \leq u \leq t}, (X(t+u))_{0 \leq u \leq T-t})$ is given by

$$(7.1) \quad E_{\theta,0}^{3,T} \otimes E_{x,0}^{3,T-t}.
$$

Let us denote by $\nu_{V}^{W,\theta,\eta}(dT)$ the law of $T = \frac{1}{\beta}$ when $\beta$ follows the law $\nu_{V}^{W,\theta,\eta}(d\beta)$, so that

$$
\nu_{V}^{W,\theta,\eta}(dT) = 1_{H \geq 0} (2/\pi)^{\frac{|V|}{2}} e^{-\frac{\beta}{2}} (\frac{\beta}{\pi}) e^{-\frac{\beta}{2} (\theta + \beta)} e^{-\frac{\beta}{2} (\theta - \beta)} e^{\frac{\beta}{2} (\eta T)} e^{\frac{\beta}{2} (\eta - \beta)} \prod_{i \in V} \theta_i \prod_{i \in V} \frac{1}{2T_i^2} dT_i.
$$

Let $(t^0_i)_{i \in V} \in \mathbb{R}_+^V$ be as in the statement of the theorem. Let $\tilde{V} \subset V$, and denote by $V(t^0) = \{i \in V, T_i > t^0_i\}$. We denote by $A(t^0, T)$, the event

$$
A(t^0, T) = \{V(t^0) = \tilde{V}\} = \{T > t^0_i, i \in \tilde{V}\} \cap \{T \leq t^0_i, i \in \tilde{V}\}.
$$

Let $h, g$ be test functions. By Theorem 1 we have

$$
E_{V}^{W,\theta,\eta} \left[ 1_{V(t^0) = \tilde{V}} h((X_i([0, t^0_i]))_{i \in V}) g((X_i([t^0_i, T_i]))_{i \in \tilde{V}}) \right] = \int 1_{A(t^0, T)} \otimes \prod_{i \in V} E_{\theta_i,0}^{3,T_i} [h((X_i([0, t^0_i]))_{i \in V}) g((X_i([t^0_i, T_i]))_{i \in \tilde{V}})] d\nu_{V}^{W,\theta,\eta}(T).
$$

By the Markov property (7.1), we have on the event $A(t^0, T)$, that

$$
\otimes \prod_{i \in V} E_{\theta_i,0}^{3,T_i} [h((X_i([0, t^0_i]))_{i \in V}) g((X_i([t^0_i, T_i]))_{i \in \tilde{V}})] = \int_{\mathbb{R}_+} H(x_{\tilde{V}}, t^0_{\tilde{V}}, T_{\tilde{V}}) \prod_{i \in \tilde{V}} E_{x_i,0}^{3,T_i-t^0_i} [g((X_i([0, T_i - t^0_i]))_{i \in \tilde{V}})] \prod_{i \in \tilde{V}} p_{\theta_i,0}^{3,t_i} (x_i) dx_i.
$$
where

\begin{equation}
H(x, t^0, T_V) = \left( \bigotimes_{i \in \check{V}} \mathbb{E}^{3, t^0_{i, 0}}_{\theta, x_i} \bigotimes_{i \in \check{V}} \mathbb{E}^{3, T_{i, 0}}_{\theta, 0} \right) [h((X_i[0, t^0_i]))_{i \in \check{V}}]
\end{equation}

is a function that only depends on \((x_i, t^0_i)_{i \in \check{V}}, (T_i)_{i \in \check{V}}\). We thus get,

\[
\mathbb{E}^{W, \theta, \eta}_{\check{V}} \left[ \mathbf{1}_{V(t) = \check{V}} h((X_i[0, t^0_i])_{i \in \check{V}})g((X_i([t^0_i, T_i]))_{i \in \check{V}}) \right] = \int 1_{A(t^0, T)} H(x, t^0, T_{\check{V}, c}) \bigotimes_{i \in \check{V}} \mathbb{E}^{3, T_{i, 0} - t^0_i} \left[ g((X_i([0, T_i - t^0_i]))_{i \in \check{V}}) \right] \left( \prod_{i \in \check{V}} p_{\theta, 0}^{3, t^0_{i, 0}}(x_i) dx_i \right) d\nu^{W, \theta, \eta}_{\check{V}}(T)
\]

In the sequel, on the event \(A(t^0, T)\), we set

\[
(\check{T}_i)_{i \in \check{V}} = (T_i - t^0_i)_{i \in \check{V}}.
\]

The strategy is now to show that we can combine the terms \(\prod_{i \in \check{V}} p_{\theta, 0}^{3, t^0_{i, 0}}(x_i)\) and the measure \(d\nu^{W, \theta, \eta}_{\check{V}}(T)\) in such a way that on the event \(A(t^0, T)\), changing from variables \((T_i)_{i \in \check{V}}\) to variables \((\check{T}_i)_{i \in \check{V}}\), we end up with a function of \((x, t^0, T_{\check{V}, c})\) and the measure \(\nu^{W, x, \check{\eta}}_{\check{V}}(d\check{T})\), see forthcoming formula (7.3).

Let us denote by \(\langle \cdot, \cdot \rangle_{\check{V}}\) the usual scalar product on \(\mathbb{R}^{\check{V}}\) (recall that we keep denoting by \(\langle \cdot, \cdot \rangle\) the usual scalar product on \(\mathbb{R}^{\check{V}}\)). Note that \(\check{\eta}\) and \(\check{W}\) defined in Theorem 2 (iii) correspond to \(\check{\eta}\) and \(\check{W}\) of Lemma 2 for \(t^0 \wedge T\) and \(\check{T}\). Hence, by (4.4) of Lemma 2 we get that

\[
\langle \check{\eta}, (H_{\frac{1}{2T}})^{-1} \check{\eta} \rangle_{\check{V}} - \langle \eta, (H_{\frac{1}{2T}})^{-1} \eta \rangle = - \langle \eta, (H_{\frac{1}{2t^0_i}})^{-1} \eta \rangle
\]

and by (1.2) of Lemma 2

\[
\left| \left( \frac{H_{\frac{1}{2T}}}{H_{\frac{1}{2t^0_i}}} \right)_{\check{V}} \right| = \left| K_{t^0 \wedge T} \prod_{i \in \check{V}} \left( \frac{T_i}{\check{T}_i} \right) \right|
\]

Note that we have

\[
\prod_{i \in \check{V}} p_{\theta, 0}^{3, t^0_{i, 0}}(x_i) = e^{-\frac{1}{2} \langle x, \frac{1}{2T} \check{W} \rangle_{\check{V}}} e^{\frac{1}{2} \langle \theta, \frac{1}{2T} \check{W} \rangle_{\check{V}}} \prod_{i \in \check{V}} \left( e^{-\frac{\langle x_i - \theta_i \rangle^2}{2t^0_i}} - e^{-\frac{\langle x_i + \theta_i \rangle^2}{2t^0_i}} \right) \frac{1}{\sqrt{2\pi t^0_i}} \frac{x_i}{\theta_i} \left( \frac{T_i}{\check{T}_i} \right)^{3/2}
\]

Changing from variables \((T_i)_{i \in \check{V}}\) to \((\check{T}_i)_{i \in \check{V}}\), we get

\begin{equation}
1_{A(t^0, T)} \left( \prod_{i \in \check{V}} p_{\theta, 0}^{3, T_{i, 0}}(x_i) \right) d\nu^{W, \theta, \eta}_{\check{V}}(d\check{T}) = 1_{T_i < t^0_i, i \in \check{V}} \Xi(x, t^0_{\check{V}}, T_{\check{V}}, c) d\nu^{W, x, \check{\eta}}_{\check{V}}(d\check{T}) \prod_{i \in \check{V}} dT_i
\end{equation}

for some explicit function \(\Xi(x, t^0_{\check{V}}, T_{\check{V}, c})\) that only depends on \((x_i, t^0_i)_{i \in \check{V}}, (T_i)_{i \in \check{V}}\).
Continuing our computation, we have
\[
E_V^{W,\theta,\eta} \left[ \mathbb{1}_{V(\omega')=\hat{V}} h((X_i[0, t_i^0])_{i \in \hat{V}}) g((X_i([t_i^0, T_i]))_{i \in \hat{V}}) \right]
\]
(7.4)
\[
= \int \mathbb{1}_{T_i < t_i^0, i \in \hat{V}, \nu H(x_i, t_i^0, T_i)} \Xi(x_i, t_i^0, T_i) E_V^{W,\theta,\eta} \left[ g((X_i([0, T_i]))_{i \in \hat{V}}) \right] \prod_{i \in \hat{V}} dx_i \prod_{i \in \hat{V}^c} dT_i
\]
Let us apply the last equality to the case where \( h \) and \( g \) are replaced by
\[
\tilde{h}((X_i[0, t_i^0])_{i \in \hat{V}}) := h((X_i[0, t_i^0])_{i \in \hat{V}}) E_V^{W,\theta,\eta} \left[ g((X_i([0, T_i]))_{i \in \hat{V}}) \right],
\]
\( \tilde{g} := 1 \)
The identity (7.2) gives in this case
\[
E_V^{W,\theta,\eta} \left[ \mathbb{1}_{V(\omega')=\hat{V}} \tilde{h}((X_i[0, t_i^0])_{i \in \hat{V}}) E_V^{W,\theta,\eta} \left[ g((X_i([0, T_i]))_{i \in \hat{V}}) \right] \right]
\]
(7.5)
\[
= \int \mathbb{1}_{T_i < t_i^0, i \in \hat{V}, \nu H(x_i, t_i^0, T_i)} \Xi(x_i, t_i^0, T_i) E_V^{W,\theta,\eta} \left[ g((X([0, T]))_{i \in \hat{V}}) \right] \prod_{i \in \hat{V}} dx_i \prod_{i \in \hat{V}^c} dT_i
\]
where, using (7.2) applied to \( \tilde{h} \) instead,
\[
\tilde{H}(x_i, t_i^0, T_i) = H(x_i, t_i^0, T_i) E_V^{W,\theta,\eta} \left[ g((X([0, T]))_{i \in \hat{V}}) \right].
\]
Remark that the right-hand sides of (7.4) et (7.5) are thus the same. Hence, we conclude that
\[
E_V^{W,\theta,\eta} \left[ \mathbb{1}_{V(\omega')=\hat{V}} h((X_i[0, t_i^0])_{i \in \hat{V}}) g((X_i([t_i^0, T_i]))_{i \in \hat{V}}) \right]
\]
\[
= \mathbb{E}_V^{W,\theta,\eta} \left[ \mathbb{1}_{V(\omega')=\hat{V}} h((X_i[0, t_i^0])_{i \in \hat{V}}) E_V^{W,\theta,\eta} \left[ g((X_i([t_i^0, T_i]))_{i \in \hat{V}}) \right] \right].
\]
Summing on all possible choices of \( \hat{V} \), we exactly get that the law of \( (X_i([t_i^0, T_i])) \), conditionally on \( F^X(t_i^0) \), is the law of the solutions of the S.D.E. \( E_{V_n}^{W,\theta,\eta}(X) \).

\[\square\]

8. Relation with the martingales associated with the VRJP

Consider in this section that \( V \) is infinite and that \( W \) is such that the associated graph \( G \) has finite degree at each vertex and is connected. Following [13], we extend the definition of the distribution \( \nu_V^{W,\theta} \) to the case of this infinite graph. We assume to be coherent with [13] that \( W \) is zero on the diagonal. Note that we slightly generalize the definition of [13] since we consider a general vector \((\theta_i)_{i \in V} \in (\mathbb{R}^+)^V \), which is equal to 1 in [13]. (But as noted at the beginning of section 3 it is in fact not more general since we can always take \( \theta \) to 1 by a change of variables on \( \beta \) and \( W \).)

Let us recall the construction of the distribution \( \nu_V^{W,\theta} \) obtained by Kolmogorov’s extension Theorem. The approach is slightly different from that of [13] and make use of Lemma C [i]. Let \( V_n \) be an increasing sequence of subsets such that \( \cup_{n \geq 1} V_n = V \). Consider the vector \( \eta^{(n)} \in (\mathbb{R}^+)^{V_n} \) defined by
\[
\eta^{(n)} = W_{V_n, V_n}^\vee (\theta_{V_n}^\vee).
\]
(8.1)
By lemma \ref{lem:compatibility} \ref{lem:kolmogorov}, the sequence of distribution $\nu_{V_n}^{W,\eta^{(n)}}$ is compatible, hence by Kolmogorov theorem it can be extended to a measure $\nu_{V}^{W,\theta}$ on $(\mathbb{R}_+)^V$. We define the Schrödinger operator

$$H_\beta := 2\beta - W,$$

on $\mathbb{R}^V$ associated with the potential $\beta \sim \nu_{V}^{W,\theta}$. Note that $H_\beta \geq 0$ as the limit of $(H_\beta)_{V_n,V_n}$ which is positive definite since $\beta_{V_n}$ has law $\nu_{V_n}^{W,\theta,\eta^{(n)}}$.

In \cite{mm} we considered the sequence of functions $(\psi^{(n)}_{j})_{j \in V} \in (\mathbb{R}_+)^V$ defined by

$$\begin{cases} (H_\beta \psi^{(n)}_{j})_{V_n} = 0 \\
\psi^{(n)}_{V_n^{i}} = \theta_{V_n^{i}} \end{cases}$$

and the operators $(\hat{G}^{(n)}(i,j))_{i,j \in V_n}$ by

$$\begin{cases} \hat{G}^{(n)}_{V_n,V_n} = ((H_\beta)_{V_n,V_n})^{-1}, \\
\hat{G}^{(n)}_{i,j} = 0, \text{ if } i \text{ or } j \text{ in not in } V_n \end{cases}$$

Let $\mathcal{F}_n = \sigma(\beta_i, \ i \in V_n)$, the sigma field generated by $\beta_{V_n}$. In \cite{mm}, Proposition 9, it was proved that $\psi^{(n)}(i)$ is a vectorial $\mathcal{F}_n$-martingale, with quadratic variation given by $\hat{G}^{(n)}(i,j)$, i.e. that for all $i,j$ in $V$ and all $n$

$$\mathbb{E} \left( \psi^{(n+1)}_{i,j} \psi^{(n+1)\ (j)} - \hat{G}^{(n+1)}(i,j) \mathcal{F}_n \right) = \psi^{(n)}_{i} \psi^{(n)}_{j} - \hat{G}^{(n)}(i,j).$$

It was extended in \cite{gmm} to an exponential martingale property, namely it was proved that for any compactly supported function $\lambda \in (\mathbb{R}_+)^V$,

$$e^{-\langle \lambda, \psi^{(n)} \rangle} - \frac{1}{2} \langle \lambda, \hat{G}^{(n)} \lambda \rangle,$$

is a $\mathcal{F}_n$-martingale.

We can interpret the functions $\psi^{(n)}$ that appear above in terms of the S.D.E.s. Consider $X^{(n)}$ the solution of the S.D.E. $\dddot{\hat{F}}_{V_n}^{W,\eta^{(n)}}$ where $\eta^{(n)}$ is defined in (8.1). Denote by $T^{(n)}$ the associated stopping times and $\beta^{(n)} = \frac{1}{2T^{(n)}}$ and

$$K^{(n)}_{t\wedge T^{(n)}} = \text{Id}_{V_n,V_n} - (t \wedge T^{(n)})W_{V_n,V_n}, \quad \psi^{(n)}(t) = \left(K^{(n)}_{t\wedge T^{(n)}} \right)^{-1} X^{(n)}(t),$$

the associated operator and martingale that appear in Lemma \ref{lem:kolmogorov}. We always consider that $\psi^{(n)}(t)$ is extended to the full set $V$ by $\psi^{(n)}_{V_n^{i}} = \theta_{V_n^{i}}$. Considering (8.2), we have that

$$\lim_{t \to \infty} \psi^{(n)}_{j}(t) = \psi^{(n)}_{j}.$$

Hence the function $\psi^{(n)}$ appears as the limit of the continuous martingale $\psi^{(n)}(t)$.

It is possible to interpret the exponential martingale property (8.3) in terms of the abelian properties, see Theorem \ref{thm:abelian}. More precisely, conditionally on $\sigma(\beta_{V_n})$, it is possible to construct a continuous martingale that interpolates between $\psi^{(n)}$ and $\psi^{(n+1)}$ and with total quadratic variation given by $\hat{G}^{(n+1)} - \hat{G}^{(n)}$, which explains the exponential martingale property as a consequence the standard exponential martingale property for continuous martingales. We do not gives details of this computation which requires heavy notations (but the authors will provide details under request).
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