DISTRIBUTIVE LATTICES AND AUSLANDER REGULAR ALGEBRAS

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Abstract. Let $L$ denote a finite lattice with at least two points and let $A$ denote the incidence algebra of $L$. We prove that $L$ is distributive if and only if $A$ is an Auslander regular ring, which gives a homological characterisation of distributive lattices. In this case, $A$ has an explicit minimal injective coresolution, whose $i$-th term is given by the elements of $L$ covered by precisely $i$ elements. We give a combinatorial formula of the Bass numbers of $A$. We apply our results to show that the order dimension of a distributive lattice $L$ coincides with the global dimension of the incidence algebra of $L$. Also we categorify the rowmotion bijection for distributive lattices using higher Auslander-Reiten translates of the simple modules.

CONTENTS

Introduction 1
1. Preliminaries 2
1.1. Preliminaries on representation theory and homological algebra 3
1.2. Preliminaries on lattice theory 4
2. The Auslander condition for lattices 5
3. The Auslander condition for distributive lattices 8
3.1. Main results 9
3.2. Global dimension of the incidence algebra and order dimension 11
3.3. A categorification of the rowmotion bijection for distributive lattice 13
Acknowledgments 14
References 14

INTRODUCTION

The notion of Gorenstein rings is basic in algebra [Ma, BH]. They are commutative Noetherian rings $R$ whose localization at prime ideals have finite injective dimension. One of the characterisations of Gorenstein rings due to Bass [Ba] is that, for a minimal injective coresolution

$$0 \rightarrow R \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \rightarrow I^i \rightarrow \cdots$$

of the $R$-module $R$ and each $i \geq 0$, the term $I^i$ is a direct sum of the injective hull of $R/p$ for all prime ideals $p$ of height $i$. In this case, $I^1$ has flat dimension 1, and this property plays an important role in the study of non-commutative analogue of Gorenstein rings. A noetherian ring $A$, which is not necessarily commutative, satisfies the Auslander condition if there exists an injective coresolution

$$0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \rightarrow I^i \rightarrow \cdots$$

of the right module $A$ such that the flat dimension of $I^i$ is bounded by $i$ for all $i \geq 0$. This condition is left-right symmetric, and there are other equivalent conditions [FGR]. If $A$ has additionally finite injective dimension, $A$ is said to be Auslander–Gorenstein, and if $A$ has finite global dimension, $A$ is said to be Auslander regular. These classes of algebras play important roles in various areas, including homological algebra [AB, AR1, AR2, GN1, GN2, Hu, HI, I1, IwS, MI, Ta], non-commutative algebraic geometry [ATV, L, VO, YZ], analytic $D$-modules [Bj1], combinatorial commutative algebra [Y], Auslander–Reiten
theory [ARS AS CIM IZ 13 I4 IyS] and tilting theory [CS IZ NRTZ PS]. Many well studied classes of rings are Auslander regular, for example enveloping algebras of finite dimensional Lie algebras, Weyl algebras and the ring of \( C \)-linear differential operators on an irreducible smooth subvariety of affine space, see [VO] Chapter 3 for proofs and more examples. For survey articles on rings with the Auslander condition we refer to [B3] and [C].

The incidence algebras of lattices are a central topic in combinatorics, see for example [A] and [Sta]. We refer to [OS] for ring theory of the incidence algebras of posets, to [S] for representation theory of incidence algebras, and to [HI] for commutative ring theory related to distributive lattices. In this article we give a new link between the theory of distributive lattices and the theory of Auslander regular rings:

**Theorem** (=Theorem 3.3, 2.4). Let \( L \) be a finite lattice with incidence algebra \( A \). Then \( L \) is distributive if and only if \( A \) is an Auslander regular algebra.

More strongly, for a distributive lattice \( L \), we construct explicitly a minimal injective coresolution

\[
0 \to A \to I^0 \to I^1 \to \cdots \to I^i \to \cdots
\]

of the right \( A \)-module \( A \) such that \( I^i \) is a direct sum of indecomposable injective \( A \)-modules \( I_x \) corresponding to elements \( x \in L \) with \( |\text{cov}(x)| = i \), and we prove that \( \text{pdim} I(x) = |\text{cov}(x)| \) holds for each \( x \in L \) when \( I(x) \) denotes the indecomposable injective module corresponding to \( x \) (Theorem 3.2). In particular, each non-zero direct summand of \( I^i \) has projective dimension precisely \( i \), and hence \( A \) belongs to a distinguished class of Auslander regular algebras called diagonal Auslander regular (Definition 1.1). Moreover, we give a combinatorial formula of the Bass numbers of \( A \) (Corollary 3.6).

One of the fundamental results in the theory of finite posets is Dilworth's Theorem stating that the order dimension of a distributive lattice \( L \) is equal to the width of the poset of its join-irreducible elements, see [HI] Theorem 1.2 and [HI] Theorem 8.7, Chapter 2). Our next main result gives a homological characterisation of the order dimension for distributive lattices:

**Theorem** (=Theorem 3.11). Let \( L \) be a finite distributive lattice with at least two elements. Then the order dimension of \( L \) equals the global dimension of the incidence algebra of \( L \).

Two important applications of this result are that the global dimension of the incidence algebra of a distributive lattice is independent of the field, a result that is in general not true for arbitrary posets, and the property for a distributive lattice to be planar can be characterised purely homological, namely by having global dimension at most two.

In the last section we will categorify the rowmotion bijection for distributive lattices by using certain functors appearing in the representation theory of the incidence algebras. The rowmotion bijection for a distributive lattice \( L \), given as the set of order ideals of a poset \( P \), is defined as \( \text{row}(x) \) being the order ideal generated by the minimal elements in \( P \setminus x \) for an order ideal \( x \) of \( P \). The rowmotion bijection has appeared under many different names and is a useful tool in the study of combinatorial properties of posets arising for example in Lie theory, see for example the references [Sti] and [TW] that also contain a historical background on the rowmotion map. To categorify the rowmotion map, recall that the \( r \)-Auslander-Reiten translate \( \tau_r \) for \( r \geq 1 \) is the functor \( \tau_r := D \text{Tr} \Omega^{r-1} \) [HI], where \( \text{Tr} \) is the classical Auslander–Bridger transpose [AB]. We summarize the main results on the rowmotion for distributive lattices in the next theorem:

**Theorem** (=Theorem 3.18). Let \( A \) be the incidence algebra of a distributive lattice \( L \). Then we have

\[
\text{top}(\tau_{p_x}(S_x)) = S_{\text{row}(x)} \quad \text{for each} \ x \in L \ \text{and the number} \ p_x \ \text{of covers of} \ x.
\]

Moreover, each simple \( A \)-module \( S \) satisfies \( \text{grade} S = \text{pdim} S \) (i.e. \( S \) is perfect in the sense of [BH]).

1. **Preliminaries**

In this article all rings will be finite dimensional algebras over a field \( K \) and all modules will be finitely generated right modules unless otherwise stated.
1.1. Preliminaries on representation theory and homological algebra. We assume that the reader is familiar with the basics on representation theory of finite dimensional algebras and refer for example to the textbooks [ARS] and [SY]. We remark that we adopt the convention of the book [SY] on path algebras to multiply arrows $\alpha_1, \ldots, \alpha_n$ in a path algebra $KQ$ of a quiver $Q$ from left to right, that is $\alpha_1 \cdots \alpha_n$ denotes the path composed of $\alpha_1$ first and $\alpha_n$ last. We denote by $D = \text{Hom}_K(-, K)$ the duality of mod $A$ for a finite dimensional algebra $A$, $J$ the Jacobson radical of $A$ and $A^{\text{op}}$ the opposite algebra of $A$. Being a finite dimensional algebra implies that a finite dimensional module is projective if and only if it is flat and the flat dimension of such a module coincides with the projective dimension of the module, see for example [W, Proposition 4.1.5]. We denote by $\text{pdim}_A$ of the right $A$-module $M$ the projective dimension of a module $M$ and $\text{idim}_A$ the injective dimension of $M$. We denote by $\Omega^1(M)$ the first syzygy of $M$, that is the cokernel of the injective envelope map of $M$. One defines inductively for $n \geq 2$: $\Omega^n(M) := \Omega^1(\Omega^{n-1}(M))$ and $\Omega^{-n}(M) := \Omega^{-1}(\Omega^{-(n-1)}(M))$.

**Definition 1.1.** An algebra $A$ with minimal injective coresolution

\begin{equation}
0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots
\end{equation}

of the right $A$-module $A$ is said to be $n$-Gorenstein for $n \geq 1$ if $\text{pdim} I^i \leq i$ for all $i = 0, \ldots, n - 1$. $A$ is called Auslander regular if it is $n$-Gorenstein for all $n \geq 1$ and additionally has finite global dimension. An Auslander regular algebra $A$ is called diagonal Auslander regular if $\text{pdim}_A X = i$ for all $i \geq 0$ when $X$ is a non-zero direct summand of $I^i$.

Thus an algebra $A$ is 1-Gorenstein if and only if the injective envelope $I^0$ of $A$ is projective. The notion of being $n$-Gorenstein and Auslander regular is left-right symmetric. The notion of diagonal Auslander regular algebras was introduced in [I3].

The dominant dimension $\text{domdim} A$ of an algebra $A$ is a refined notion of $n$-Gorensteinness. It is defined as the infimum of $i \geq 0$ such that $I^i$ in (1.1) is not projective. Thus an algebra $A$ with $\text{domdim} A \geq n$ is $n$-Gorenstein, and the converse is true for $n = 1$. By the Morita-Tachikawa correspondence an algebra $A$ has dominant dimension at least two if and only if $A \cong \text{End}_B(M)$ for another algebra $B$ and a generator-cogenerator $M$ of mod $B$. We refer to [I3] for more on the Morita-Tachikawa correspondence and the dominant dimension of algebras, and to [ARS, AS, I4, IyS] for their importance in representation theory.

We recall homological invariants closely related to $n$-Gorensteinness. The grade of a module $M$ over an algebra $A$ is defined as

$$\text{grade } M := \inf \{ \ell \geq 0 \mid \text{Ext}_A^\ell(M, A) \neq 0 \}$$

and dually the cograde is defined as

$$\text{cograde } M := \inf \{ \ell \geq 0 \mid \text{Ext}_A^\ell(D(A), M) \neq 0 \}.$$ 

If a module $M$ has finite projective dimension, we always have $\text{grade } M \leq \text{pdim } M$, see for example [ARS, Lemma 5.5 (Chapter VI)]. A module $M$ is often called perfect if $\text{pdim } M = \text{grade } M < \infty$. The study of perfect modules is a classical topic in commutative algebra, e.g. [BH, Section 1.4]. The following is immediate from the definitions.

**Proposition 1.2.** Let $A$ be an algebra such that $A$ and $A^{\text{op}}$ are diagonal Auslander regular. Then each simple $A$-module $S$ satisfies $\text{idim } S = \text{pdim } S$ and $\text{cograde } S = \text{idim } S$.

**Proof.** We prove the second equality since the first one is dual. First recall that for any module $M$ of finite injective dimension, $\text{idim } M = \sup \{ i \geq 0 \mid \text{Ext}_A^i(D(A), M) \neq 0 \}$, by the dual of [ARS, Lemma 5.5 (Chapter VI)]. Since we assume that $A$ has finite global dimension as an Auslander regular algebra, any simple module has finite injective dimension and cograde $S \leq \text{idim } S$. Let $(P_i)$ be a minimal projective resolution of $D(A)$ and $S$ a simple $A$-module. Then $\text{Ext}_A^i(D(A), S) \neq 0$ if and only if the projective cover $T$ of $S$ is a direct summand of $P_i$, see for example [Bec, Corollary 2.5.4]. But since we assume that $A^{\text{op}}$ is diagonal Auslander regular, every indecomposable term of $P_i$ has injective dimension equal to $i$. Thus $T$ appears in exactly one of the terms $P_i$ as a direct summand, let us say in term $P_r$. Then $\text{Ext}_A^i(D(A), S) \neq 0$ if and only if $i = r$. Thus $\text{cograde } S = \text{idim } S = r$. \hfill \Box

The Auslander-Bridger transpose $\text{Tr } M$ of a module $M$ over a finite dimensional algebra with minimal projective presentation

$$P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$$
is defined as cokernel of the map \( \text{Hom}_A(f, A) \) so that we obtain the following exact sequence by applying the functor \( \text{Hom}_A(\_, A) \) to the minimal projective presentation of \( M \):

\[
0 \to \text{Hom}_A(M, A) \to \text{Hom}_A(P_0, A) \xrightarrow{\text{Hom}_A(f, A)} \text{Hom}_A(P_1, A) \to \text{Tr} M \to 0.
\]

The \( r \)-Auslander-Reiten translate is defined as \( \tau_r(M) := D \text{Tr} \Omega^{-1}(M) \) for \( r \geq 1 \), where \( \tau = \tau_1 \) is the classical Auslander-Reiten translate. We refer to [ARS] and [I4] for more on the \( r \)-Auslander-Reiten translates. The next lemma follows immediately from the definition of Ext and the Auslander-Bridger transpose:

**Lemma 1.3.** Let \( A \) be a finite dimensional algebra and \( M \) a module of finite projective dimension \( p \geq 1 \). Then \( \text{Ext}^p_A(M, A) \cong \text{Tr} \Omega^{-1}(M) \) as left \( A \)-modules and \( D \text{Ext}^p_A(M, A) \cong \tau_p(M) \) as right \( A \)-modules.

### 1.2. Preliminaries on lattice theory.

Lattices are a central topic in mathematics and there are several textbooks dedicated to the theory of lattices, see for example [Bi] and [G]. For applications of lattices in textbooks, see for example [ARS] and [I4] for more on the Auslander-Reiten translate.

We refer to [Bi] and [G] for more on the theory of lattices. A poset is by definition a lattice if any two elements \( x, y \) and \( z \in L \) have a unique supremum, denoted by \( x \vee y \), and a unique infimum, denoted by \( x \wedge y \). Recall that a lattice \( L \) is called distributive if one has \((x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)\) for all elements \( x, y, z \in L \). A sublattice \( T \) of a lattice \( L \) is by definition a nonempty subset of \( L \) such that \( a \vee b \in T \) implies that \( a \wedge b \in T \) and \( a \wedge b \in T \). For elements \( s, t \) in a poset \( P \) we say that \( t \) covers \( s \) or \( s \) is covered by \( t \) if \( s < t \) and no element \( u \in P \) satisfies \( s < u < t \). An element \( p \) of a lattice \( L \) is called join-irreducible if \( p \) is not the minimum \( m \) of \( L \) and for all \( x, y \in L \) one has that \( p = x \vee y \) implies that \( p = x \) or \( p = y \). Meet-irreducible elements are defined dually. The Hasse diagram or Hasse quiver of a poset \( P \) is by definition the quiver with vertices the elements of \( P \) and a directed arrow from \( i \to j \) if \( j \) covers \( i \). For an element \( x \in L \) we denote by \( \text{cov}(x) \) the set of elements that cover \( x \) and by \( \text{cocov}(x) \) the elements that are covered by \( x \). Given two partial orders \( \leq \) and \( \leq^* \) on a finite set \( X \), we say that \( \leq^* \) is a linear extension of \( \leq \) exactly when \( \leq^* \) is a total order and for every \( x \) and \( y \) in \( X \), if \( x \leq y \), then \( x \leq^* y \).

The incidence algebra of a poset \( P \) is by definition the \( K \)-algebra over a field \( K \) with basis given by \( e_i \) for all elements \( x, y \in P \) with \( x \leq y \) and multiplication on basis elements \( e_i e_j \) by \( p_{ij} \) if \( x_j = y_i \) and \( p_{ij} = 0 \) else. In this article we will use the description of incidence algebras by quiver and relations, which is the most suitable for our purposes. That this description is equivalent to the usual definition of incidence algebras is easy to see and explained for example in [S, Example 10 (Chapter 14.1)].

Let \( I \) be an ideal of \( KQ \) generated by all differences \( w_1 - w_2 \) of two parallel paths \( w_1 \) and \( w_2 \) of length at least two that start at the same point point and end at the same point. Then \( A = KQ/I \) is isomorphic to the incidence algebra of \( I \). The quiver \( Q \) of the algebra is acyclic and thus the global dimension of \( A \) is finite. When the points of a quiver algebra are \( 1, \ldots, n \) then we denote by \( e_i \) the primitive idempotents corresponding to \( i \) and by \( S_i \) the simple modules corresponding to \( i \). We denote by \( P(i) = e_i A \) the indecomposable projective modules corresponding to \( i \) and by \( I(i) = D(Ae_i) \) the indecomposable injective modules corresponding to \( i \). We denote by \( p_{ij} \) the arrow from \( x \) to \( y \) in the quiver of \( Q \) when \( x \) is covered by \( y \). For general elements \( a, b \in P \) we denote the unique path from \( a \) to \( b \) by \( p_{ab} \) if \( a \leq b \) with \( e_i = p_{ab} \). We give two examples:

**Example 1.4.** Let \( P = C(n) \) be a chain with \( n \geq 2 \) elements numbered from 0 to \( n - 1 \) such that 0 is the smallest and \( n - 1 \) the largest element. Then the quiver \( Q \) of the incidence algebra \( A \) looks as follows:

\[
0 \xrightarrow{p_{01}} 1 \xrightarrow{p_{12}} 2 \ldots \xrightarrow{p_{n-2,n-1}} n-1
\]

We have \( A = KQ \) since the ideal \( I \) is zero here. It is elementary to check that \( A \) is diagonal Auslander regular with \( \text{gldim} A = 1 \).

Note that \( C(n) \) is actually the unique lattice such that the incidence algebra has global dimension equal to 1, since a quiver algebra of the form \( KQ/I \) with acyclic \( Q \) has global dimension at most equal to 1 if and only if \( I = 0 \) for an admissible ideal \( I \).
Example 1.5. Let $P$ be the Boolean lattice on two elements and $A$ its incidence algebra. Thus $P$ has 4 points, namely $\{1, 2\}$, $\{1\}$, $\{2\}$ and the empty set $\emptyset$. $Q$ looks as follows:

We have $I = (p_{\emptyset}^{(2)}p_{\{1, 2\}}^{\emptyset} - p_{\emptyset}^{\emptyset}p_{\{1\}}^{(1)})$ and $A = KQ/I$. It is elementary to check that $A$ is diagonal Auslander regular with $\text{gldim} A = 2$.

More generally, the incidence algebra $A$ of the Boolean lattice $P$ on $n$ elements is diagonal Auslander regular with $\text{gldim} A = n$.

2. The Auslander condition for lattices

We always assume that a poset $P$ has at least two elements and is connected in this section. Note that the assumption that the poset is connected is no loss of generality as the poset is connected if and only if the incidence algebra of $P$ is a connected algebra. We first determine the dominant dimension of the incidence algebra of posets and thus which of those algebras are 1-Gorenstein.

Proposition 2.1. Let $P$ be a poset with incidence algebra $A$. Then the following are equivalent:
(1) $P$ is bounded.
(2) $A$ is 1-Gorenstein.
(3) There is a non-zero projective-injective module.
(4) The socle of every indecomposable projective $A$-module is simple.

If the poset is bounded, the dominant dimension is exactly one and there is exactly one indecomposable projective-injective module, namely $P(m)$ when $m$ denotes the global minimum of $P$.

Proof. Let $S_i$ denote the simple modules corresponding to the points of $P$. We first show that (3) implies (1). Assume $P$ is not bounded. By duality we can assume there is no global maximum. Let $N = P(a)$ be the indecomposable projective module corresponding to the point $a$. Then the socle of $N$ is isomorphic to the direct sum of simple modules $S_{x_i}$, where $x_i$ are exactly those points in the poset which are larger than $a$ and are maximal, meaning that there are no larger elements in the poset than those $x_i$. Since there is no global maximum, the socle of $N$ is simple if there is only one such $x_i$. Since indecomposable projective modules have a simple socle, in order for $N$ to be injective then, $N$ has to be isomorphic to $I(x_i)$. But $I(x_i)$ has vector space dimension larger than $N$ except when $a$ is a global minimum and in this case the socle is not simple since there is no global maximum. Thus every non-zero indecomposable projective module is not injective. Thus when $A$ has a non-zero projective-injective module, $P$ has to be bounded.

Now we show that (1) implies (2). Let $P$ be bounded with global minimum $m$ and global maximum $M$. First note that the socle of $P(m)$ is equal to $S_M$ since the unique longest path that starts at $m$ ends at $M$. Thus $P(m)$ embeds into the indecomposable injective module $I(M)$ and since both modules have the same vector space dimension, they must be isomorphic. Then there is the projective-injective indecomposable module $P(m) \cong I(M)$ and injections $f_i : P(i) \to P(m)$ for any $i \neq m$ given by left multiplication by the unique path from $m$ to $i$. This shows that $A$ has dominant dimension at least one and that $P(m)$ is the unique indecomposable projective-injective module in that case. That (2) implies (3) is trivial since every algebra with positive dominant dimension must contain a projective-injective module by definition.

Now assume (4) so that every indecomposable projective $A$-module has a simple socle. Then $A$ is also 1-Gorenstein, see [Th, Theorem 4]. Thus (4) implies (2).

Now assume (2) (which we showed before is equivalent to (3)). Then we saw that every indecomposable projective module embeds into the unique indecomposable projective-injective module $P(m)$. But $P(m)$ has a simple socle and thus every indecomposable projective module has a simple socle as a submodule of $P(m)$, which shows that (2) implies (4).

Now assume that the incidence algebra $A$ has dominant dimension at least two. By the Morita-Tachikawa correspondence we have $A \cong \text{End}_B(W)$, where $B \cong eAe$ when $e$ is an idempotent such that $eA$ is the minimal faithful projective-injective module of $A$ and the $B$-module $W$ is a generator-cogenerator of mod $B$. But then the algebra $eAe$ is isomorphic to the field $K$ because the quiver of $A$ is acyclic and $e$ is a primitive idempotent (since $A$ contains exactly one indecomposable projective-injective module) and thus with $B$ also $A$ would be semisimple as the endomorphism ring of a module over a semisimple algebra. But since we assume that $P$ is connected and has at least two elements, $A$ is not semisimple.

This is a contradiction and thus the dominant dimension is equal to one. \hfill $\Box$

Remark 2.2. In [BS] Proposition 1.1], a proof that a connected poset is a 1-Gorenstein algebra if and only if it is bounded can be found, while our result [2.1] is more precise since it gives the exact value of the dominant dimension.

Corollary 2.3. Let $P$ be a lattice. Then the incidence algebra of $P$ is 1-Gorenstein with dominant dimension precisely equal to one.

Since all incidence algebras of lattices are 1-Gorenstein algebras, this class of algebras is a natural class to check for the Auslander condition. The next theorem shows that a lattice whose incidence algebra is Auslander regular has to be distributive.

Theorem 2.4. Let $P$ be a lattice with incidence algebra $A$. If $P$ is not distributive, then $A$ is not 2-Gorenstein.

Proof. Assume $P$ is not distributive and has a global maximum $M$ and a global minimum $m$. For the proof note that an algebra $A$ satisfies the 2-Gorenstein condition if and only if for every indecomposable
projective module $P(i)$ we have $\text{pdim}(J^i) \leq l$ for $l = 0$ and $l = 1$ when $(J^i)$ denotes a minimal injective coresolution of $P(i)$. Also recall that in 2.1 we saw that each injective envelope of an indecomposable projective module is equal to $P(m)$ and dually the projective cover of each indecomposable injective module is equal to $P(m) \cong I(M)$. By 1.6 $P$ not being distributive means that $P$ has a sublattice isomorphic to a diamond or a pentagon. We look at both cases separately.

Case 1: Assume $P$ has a sublattice isomorphic to the diamond. We can picture the situation as follows when looking at the Hasse quiver $Q$ of $P$:

```
     m
d1   \phantom{1}   d2   \phantom{1}   d3
   \downarrow         \downarrow         \downarrow
b1     \phantom{1}       b2     \phantom{1}       b3
   \downarrow         \downarrow         \downarrow
c1   \phantom{1}       c2   \phantom{1}       c3
   \downarrow         \downarrow         \downarrow
a2     \phantom{1}       a1
   \downarrow         \downarrow         \downarrow
M
```

Here $a_1$ is the infimum of $b_1, b_2$ and $b_1$ and $a_2$ is the supremum of $b_1, b_2$ and $b_3$. Note that we possibly can have that $a_1 = m$ and $a_2 = M$. For $i = 1, 2, 3, d_i$ is the point between $a_1$ and $b_1$ that covers $a_1$ and $c_i$ is the point between $b_i$ and $a_2$ that is covered by $a_2$. Let

$$0 \rightarrow P(b_1) \xrightarrow{g} P(m) \rightarrow U \rightarrow 0$$

be the short exact sequence such that the left map $g$ is the injective envelope of the projective non-injective module $P(b_1)$. The map $P(b_1) \rightarrow P(m)$ is given by left multiplication by the unique path $p_{b_1}^m$ from $m$ to $b_1$ and thus the cokernel $U$ of this map is isomorphic to $e_m A / p_{b_1}^m A$. Now note that the path $p_{c_2}^m$ from $m$ to $c_2$ is non-zero in $e_m A / p_{b_1}^m A$ but $p_{c_2}^m x = 0$ in $e_m A / p_{b_1}^m A$ for any element $x$ in the radical of $A$. Thus the socle of the module $e_m A / p_{b_1}^m A$ contains the simple module $S_{c_2}$ as a direct summand. This implies that the injective envelope of the module $e_m A / p_{b_1}^m A$ has the indecomposable injective module $I(c_2)$ corresponding to the point $c_2$ as a direct summand. We will show that $I(c_2)$ has projective dimension at least two in order to finish the proof in case 1. In the short exact sequence

$$0 \rightarrow K \rightarrow P(m) \xrightarrow{h} I(c_2) \rightarrow 0,$$

the right map $h$ is the projective cover of $I(c_2)$ and given by left multiplication by the element $(p_{c_2}^m)^*$ when $(p_{c_2}^m)^*$ denotes the vector space dual of the path $p_{c_2}^m$. This follows from the fact that $\text{top}(I(c_2)) = \text{top}(D(Ae_{c_2})) = D(\text{soc}(Ae_{c_2})) = D(p_{c_2}^m))$. Now the kernel $K$ of $h$ contains the path from $m$ to $d_1$, $u_1 = p_{d_1}^m$, and the path from $m$ to $d_3$ $u_2 = p_{d_3}^m$. This is because for $a, a' \in A$ we have: $h(e_m a)(a' e_{c_2}) = (p_{c_2}^m)^* (e_m a a' e_{c_2}) = 0$ for all $a' \in A$ if $a = p_{d_1}^m$, since $p_{c_2}^m$ does not factor as a path over $u_1 = p_{d_1}^m$ and similarly for $u_2$. Note that since the socle of $P(m)$ is simple, every submodule of $P(m)$ also has simple socle and thus is indecomposable. That the kernel $K$ contains $u_1$ and $u_2$ gives us that $K$ is not a cyclic module generated by a path and thus $K$ can not be projective, since every indecomposable projective submodule of $e_m A$ is of the form $pA$ for some path $p$ and thus especially a cyclic module. As the kernel $K$ of the projective cover $h$ of $I(c_2)$ is not projective, $I(c_2)$ has projective dimension at least two. This finishes the proof for case 1.

Case 2: Now assume that $P$ has a sublattice isomorphic to a pentagon. We can picture the situation as
follows when looking at the Hasse quiver $Q$ of $P$:

Here $a_1$ is the infimum of $b_1$ and $b_2$ and $a_2$ is the supremum of $b_1$ and $b_3$. Note that we possibly can have $m = a_1$ and $M = a_2$. $c_2$ is covered by $b_3$ and is between $b_2$ and $b_3$ and $c_1$ covers $b_1$ and is between $b_1$ and $a_2$. Let

$$0 \to P(b_3) \xrightarrow{g} P(m) \to U \to 0$$

be the short exact sequence such that $g$ is the injective envelope of $P(b_3)$. Then $g$ is given by left multiplication with the unique path $p^m_{b_3}$ from $m$ to $b_3$ and thus $U = e_mA/p^m_{b_3}A$. Now the path $p^m_{c_2}$ from $m$ to $c_2$ is nonzero in $e_mA/p^m_{b_3}A$ and $p^m_{c_2}x = 0$ in $e_mA/p^m_{b_3}A$ for every $x$ in the radical of $A$ and thus the socle of $U$ contains $S_{c_2}$. This shows that the injective envelope of $e_mA/p^m_{b_3}A$ contains $I(c_2)$ as a direct summand. We show that the projective dimension of the indecomposable injective module $I(c_2)$ corresponding to the point $c_2$ is at least two. Let

$$0 \to K \to P(m) \xrightarrow{h} I(c_2) \to 0$$

be the exact sequence such that $h$ is the projective cover of $I(c_2)$. Then $h$ is given by left multiplication with the vector space dual of the the path $p^m_{c_2}$. We again show that $K$ is not a cyclic module to finish the proof. In order to see this just note that the kernel $K$ of $h$ contains the paths $p^m_{b_3}$ from $m$ to $b_3$ and $p^m_{b_1}$ from $m$ to $b_1$ and as in case 1 one concludes that $K$ can not be cyclic and thus not be projective. This finishes the proof of case 2.

We give an example showing that there are posets that contain a pentagon as a subposet but whose incidence algebra is Auslander regular. This shows that the proof of 2.4 can not be extented to general bounded posets.

**Example 2.5.** The following poset $P$ with Hasse quiver $Q$ is a bounded poset, which contains a pentagon as a subposet and is not a lattice. Its incidence algebra $A$ is Auslander regular and has global dimension equal to 3.

![Diagram](https://via.placeholder.com/150)

3. **The Auslander condition for distributive lattices**

In the previous section we saw that non-distributive lattices can not be Auslander regular. In this section we prove that the incidence algebras of distributive lattices are Auslander regular algebras and give applications.
3.1. Main results. We assume in the following that all lattices have at least two elements. For a subset $X$ of a lattice $L$ we denote by $\text{min}(X)$ the set of minimal elements in $X$ and dually $\text{max}(X)$ denotes the set of maximal elements in $X$. When $\text{min}(X)$ consists of a unique element, we identify the set $\text{min}(X)$ with this element and dually we identify $\text{max}(X)$ with its unique element if there is a unique element in $\text{max}(X)$. We denote by $\text{M-irr}(L)$ the set of meet-irreducible elements of a lattice $L$ and by $\text{J-irr}(L)$ the set of join-irreducible elements of $L$.

**Proposition 3.1.** Let $L$ be a distributive lattice.

1. (i) For $x \in L$, we have $\text{min}([m, x]^c) \subseteq \text{J-irr}(L)$.
   (ii) There is a bijection $\alpha_x : \text{min}([m, x]^c) \to \text{cov}(x)$ given by $\alpha_x(z) = z \lor x$.
   (iii) For each subset $S \subseteq \text{min}([m, x]^c)$, let $y := \bigvee S$. Then $|\text{max}([y, M]^c)| = |S|$.

2. (i) For $y \in L$, we have $\text{max}([y, M]^c) \subseteq \text{M-irr}(L)$.
   (ii) There is a bijection $\beta_y : \text{max}([y, M]^c) \to \text{cocov}(y)$ given by $\beta_y(z) = z \land y$.
   (iii) For each subset $S \subseteq \text{max}([y, M]^c)$, let $x := \bigwedge S$. Then $|\text{min}([m, x]^c)| = |S|$.

**Proof.** We prove (1), the proof of (2) is dual. We use $\mathbb{I}$ to represent $L = \mathbb{O}_P$ as the distributive lattice of order ideals of a poset $P$. For $p \in P$, we denote by

$I(p) := \{q \in P \mid q \leq p\} \subseteq P$ the order ideal of $P$ generated by $p$ and $\mathcal{J}(p) := \{q \in P \mid p \not\leq q\} \subseteq P$.

(i) We regard $x \in L$ as an order ideal $x \subseteq P$. For each $z \in \text{min}([m, x]^c)$, take a minimal element $p$ in $z \setminus x$. Then $z = I(p)$ holds since $[m, x]^c \ni I(p) \subseteq z$ and $z$ is minimal in $[m, x]^c$. In particular, $z = I(p)$ is join-irreducible since it covers a unique element $I(p) \setminus \{p\}$.

(ii) By the above argument, there is a bijection $\text{min}(P \setminus x) \to \text{min}([m, x]^c)$ given by $p \to I(p)$. On the other hand, elements in $\text{cov}(x)$ are the order ideals $z$ of $P$ such that $z \supseteq x$ and $|z \setminus x| = 1$. Thus there is a bijection $\text{min}(P \setminus x) \to \text{cov}(x)$ given by $p \to x \cup I(p)$, the assertion follows.

(iii) Let $S = \{P_i \mid 1 \leq i \leq r\}$. Then

$y = \bigvee S = \bigcup_{i=1}^{r} I(p_i)$.

We prove $\text{max}([y, M]^c) = \mathcal{J}(p_i) \mid 1 \leq i \leq r$. Then $|\text{max}([y, M]^c)| = r = |S|$ holds. Clearly $\mathcal{J}(p_i) \subseteq \text{max}([y, M]^c)$ holds. Conversely, for each $z \in \text{max}([y, M]^c)$, there exists $1 \leq i \leq r$ such that $p_i \not\in z$. Then $z = \mathcal{J}(p_i)$ holds since $[y, M]^c \ni \mathcal{J}(p_i) \supseteq z$ and $z$ is maximal in $[y, M]^c$. $\square$

Now we state our main result.

**Theorem 3.2.** Let $L$ be a distributive lattice with incidence algebra $A$ and $x, y \in L$.

1. $\text{pdim} I(x) = |\text{min}([m, x]^c)| = |\text{cov}(x)| := \ell$ holds, and $I(x)$ has a minimal projective resolution

$0 \to P_{\ell} \to \cdots \to P_0 \to I_x \to 0$ with $P_0 = P(m)$ and $P_r = \bigoplus_{S \subseteq \text{min}([m, x]^c), |S| = r} P(\bigvee S)$ for $1 \leq r \leq \ell$.

2. $\text{idim} P(y) = |\text{max}([y, M]^c)| = |\text{cocov}(y)| := \ell$ holds, and $P(y)$ has a minimal injective coreolution

$0 \to P(y) \to I^0 \to \cdots \to I^\ell \to 0$ with $I^0 = I(M)$ and $I^r = \bigoplus_{S \subseteq \text{max}([y, M]^c), |S| = r} I(\bigwedge S)$ for $1 \leq r \leq \ell$.

**Proof.** We show (1), the proof of (2) is dual. The elements $\bigvee S \in L$ for $S \subseteq \text{min}([m, x]^c)$ are pairwise different since elements in a distributive lattice have a unique irredundant join-irreducible representation. Thus the terms $P(\bigvee S)$ are pairwise non-isomorphic. Let $f_0 : P_0 = P(m) \to I_x$ be a natural surjection. For two elements $y \geq z$ in $L$, we denote by $\epsilon(y, z) : P(y) \to P(z)$ the natural inclusion. Moreover, we fix an arbitrary total ordering of elements in $\text{min}([m, x]^c)$, and for subsets $T \subseteq S \subseteq [m, x]^c$ with $T = S \setminus \{y\}$, let $\epsilon(S, T) := (-1)^{|\{z \in S \mid z \geq y\}|}$. We construct a map $f_r : P_r \to P_{r-1}$ by

$f_r := (f_{S, T})_{S, T} : P_r = \bigoplus_{S \subseteq \text{max}([m, x]^c), |S| = r} P(\bigvee S) \to P_{r-1} = \bigoplus_{T \subseteq \text{max}([m, x]^c), |T| = r-1} P(\bigvee T),$

where $f_{S, T} := \begin{cases} \epsilon(S, T) \epsilon(\bigvee S, \bigvee T) & \text{if } S \supseteq T, \\ 0 & \text{else.} \end{cases}$
Now we check \( f_{r-1}f_r = 0 \). This is clear for \( r = 1 \) since \( \text{Hom}_A(P(y), I(x)) = 0 \) holds for any \( y \in [m, x]^c \).

Assume \( r \geq 2 \). Take a direct summand \( P(\bigvee S) \) of \( P_r \) and \( P(\bigvee U) \) of \( P^{r-2} \). Clearly, the \((S, U)\)-entry of \( f_{r-1}f_r \) is non-zero only when there exist \( y \neq z \in S \) such that \( U = S \setminus \{y, z\} \). If there exist such \( y, z \), then the \((S, U)\)-entry of \( f_{r-1}f_r \) is

\[
\left( P(\bigvee S) \xrightarrow{f_{r-1}(y, u)} P(\bigvee S \setminus \{y\}) \xrightarrow{f_{r-1}(y, u)} P(\bigvee U) \right) + \left( P(\bigvee S) \xrightarrow{f_{r-1}(z, t)} P(\bigvee S \setminus \{z\}) \xrightarrow{f_{r-1}(z, t)} P(\bigvee U) \right),
\]

which is easily shown to be zero because of sign \( \epsilon \).

To check exactness of the sequence, it suffices to show that the sequence

\[
0 \to P_r e_w \xrightarrow{f_r e_w} \cdots \xrightarrow{f_1 e_w} P_0 e_w \xrightarrow{f_0 e_w} I(x) e_w \to 0
\]

is exact for each \( w \in L \). Let \( R_w := \{ y \in \min([m, x]^c) \mid y \leq w \} \). Then we have

\[
I(x) e_w = \begin{cases} \{k \mid w \leq x\} & \text{if } 0 \leq r, \\ \{0\} & \text{else} \end{cases}
\]

and \( P_r e_w = \bigoplus_{S \subseteq R_w, |S| = r} k \) for \( 0 \leq r \).

Thus, if \( R_w = \emptyset \), then \((3.1)\) is exactly clear. In the rest, assume \( R_w \neq \emptyset \). For subsets \( T \subseteq S \subseteq R_w \) with \(|S| = |T| = |T| + 1\), the \((S, T)\)-entry of \( f_{r-1}f_r : P_r e_w \to P_{r-1} e_w \) is \( \epsilon(S, T) \iota_1 \). Thus the sequence \((3.1)\) is isomorphic to the tensor product of \(|R_w|\) copies of \( k \to k \) (i.e. the Koszul complex), and hence exact. Since the terms \( P(\bigvee S) \) are pairwise non-isomorphic, the projective resolution is minimal. Therefore, the projective dimension of \( I(x) \) is equal to \( \min([m, x]^c) \) and \( |\min([m, x]^c)| = |\text{cov}(x)| \) by \(3.1\) \( \square \)

Immediately, we obtain the following result.

**Theorem 3.3.** The incidence algebra \( A \) of a distributive lattice \( L \) is diagonal Auslander regular. More precisely, \( \text{pdim} I(x) = |\text{cov}(x)| \) holds for each \( x \in L \), and \( A \) has a minimal injective resolution

\[
0 \to A \to I^0 \to I^1 \to \cdots \to I^i \to \cdots
\]

such that \( I^i \) is a direct sum of copies of \( I(x) \) for elements \( x \in L \) with \( |\text{cov}(x)| = i \).

**Proof.** By Theorem 3.2(2), each indecomposable direct summand of \( I^r \) can be written as \( I(x) \) for \( x := \bigwedge S \), where \( S \subseteq \max([y, M]^c) \) with \(|S| = r \). By Theorem 3.2(1) and Proposition 3.1(2)(iii), we have

\[
\text{pdim} I(x) = |\text{cov}(x)| = |\min([m, x]^c)| = |S| = r.
\]

We refer to [Y] for a similar type of results for the incidence algebras of the posets associated with simplicial complexes.

Our previous results imply the following corollary, which makes Proposition 1.2 more explicit.

**Corollary 3.4.** Let \( A \) be the incidence algebra of a distributive lattice \( L \). Then the simple \( A \)-module \( S_x \) corresponding to \( x \in L \) satisfies

\[
\text{grade} S_x = \text{pdim} S_x = |\text{cov}(x)| \quad \text{and} \quad \text{cograde} S_x = \text{idim} S_x = |\text{cove}(x)|.
\]

**Proof.** We prove the first equalities. By Theorem 3.3 and the same argument as in the proof of Proposition 1.2 if an integer \( i \geq 0 \) satisfies \( \text{Ext}^i_A(S_x, A) \neq 0 \), we have \( i = |\text{cov}(x)| \). Since \( \text{gldim} A \) is finite, \( \text{Ext}^i_A(S_x, A) \neq 0 \) for \( i = \text{pdim} S_x \). Thus we obtain \( \text{grade} S_x = \text{pdim} S_x = |\text{cov}(x)| \). \( \square \)

As in the case of commutative rings [BH] (see also [GN1]), we introduce the following notion.

**Definition 3.5.** Let \( L \) be a poset and \( A \) the incidence algebra of \( L \). For \( M \in \text{mod} A \), let

\[
0 \to M \to I^0 \to I^1 \to \cdots
\]

be a minimal injective resolution. For \( x \in L \) and \( i \geq 0 \), we define the \( i \)-th Bass number \( \mu^i(x, M) \) of \( M \) as the multiplicity of \( I(x) \) in \( I^i \).

We have the following description of the Bass number of \( A \).
Corollary 3.6. Let $L$ be the distributive lattice of the order ideals of a poset $P$, and $A$ the incidence algebra of $L$. For $x, y \in L$ and $i \geq 0$, we have

$$\mu^i(x, P(y)) = \begin{cases} 1 & \text{if } i = |\text{cov}(x)| \text{ and } \min(P \setminus x) \subseteq \max(y), \\ 0 & \text{else.} \end{cases}$$

$$\mu^i(x, A) = \begin{cases} |\{y \in L \mid \min(P \setminus x) \subseteq \max(y)\}| & \text{if } i = |\text{cov}(x)|, \\ 0 & \text{else.} \end{cases}$$

Proof. It suffices to prove the first equality. By Theorem 3.2, the following conditions are equivalent.

(i) $\mu^i(x, P(y)) \neq 0$.
(ii) $\mu^i(x, P(y)) = 1$.
(iii) There exists $S \subseteq \max([y, M]^c)$ such that $|S| = i$ and $\bigwedge S = x$.

The equality $\bigwedge S = x$ in (iii) is equivalent to $S = \{J(p) \mid p \in \min(P \setminus x)\}$. In this case, $|S| = |\text{cov}(x)|$ holds. Moreover, there is a bijection $\max(y) \to \max([y, M]^c)$ given by $p \mapsto J(p) := \{q \in P \mid p \not\leq q\}$. Thus (iii) is equivalent to the following.

(iv) $i = |\text{cov}(x)|$ and $\min(P \setminus x) \subseteq \max(y)$.

Thus the first equality holds.

We give an explicit example:

Example 3.7. Let $L$ be a distributive lattice with incidence algebra $A$ and let $x \in L$ such that $\min([m, x]^c) = \{y_1, y_2, y_3\}$. Then the minimal projective resolution of $I(x)$ is as follows:

$$0 \to P(y_1 \lor y_2 \lor y_3) \to P(y_1 \lor y_2) \oplus P(y_1 \lor y_3) \oplus P(y_2 \lor y_3) \to P(y_1) \oplus P(y_2) \oplus P(y_3) \to P(m) \to I(x) \to 0.$$

As an explicit example we can take $L$ to be the Boolean lattice of subsets of $\{1, 2, 3, 4\}$ and $x = \{4\}$. Then $\min([\emptyset, \{4\}]^c) = \{\{1\}, \{2\}, \{3\}\}$ and the minimal projective resolution of $I(\{4\})$ is:

$$0 \to P(\{1, 2, 3\}) \to P(\{1, 2\}) \oplus P(\{1, 3\}) \oplus P(\{2, 3\}) \to P(\{1\}) \oplus P(\{2\}) \oplus P(\{3\}) \to P(\emptyset) \to I(\{4\}) \to 0.$$

The next example shows that for a non-distributive lattice $L$ it is in general not true that $\text{pdim} I(m) = |\min([m, x]^c)|$.

Example 3.8. Let $L$ be the diamond lattice with minimum $x = m$. Then $\text{pdim} I(m) = 2$ but $|\min([m, x]^c)| = 3$. Note that $I(m) \cong S_m$ and $S_m$ has projective dimension two, while $m$ has three covers.

3.2. Global dimension of the incidence algebra and order dimension. Now we use our results to calculate the global dimension of the incidence algebra of a distributive lattice. First we start with the general definition of order dimension for posets.

Definition 3.9. The order dimension of a poset $P$ is by definition the minimal number $t$ such that there exist $t$ linear extensions of $P$ whose intersection is equal to $P$.

We refer for example to [CLM] Chapter 6 or the book [Tr] for more characterisations and properties of the order dimension. The next result is the classical theorem of Dilworth.

Theorem 3.10. [D] theorem 1.2] Let $L$ be a distributive lattice. Then the order dimension of $L$ coincides with $\max\{|\text{cov}(x)| \mid x \in L\}$.

The theorem of Dilworth can be used to calculate the global dimension for incidence algebras of distributive lattices as the next theorem shows.

Theorem 3.11. Let $L$ be a distributive lattice. Then the global dimension of the incidence algebra of $L$ is equal to the order dimension of $L$.

Proof. By the order dimension of $L$ coincides with $\max\{|\text{cov}(x)| \mid x \in L\}$. But by (1):

$$\max\{|\text{cov}(x)| \mid x \in L\} = \text{pdim} I(x) \mid x \in L \} = \text{pdim} D(A) = \text{gldim} A,$$

where in the last step we used $\text{pdim} D(A) = \text{gldim} A$ which holds for any finite dimensional algebra $A$ with finite global dimension, see for example [ARS] Lemma 5.5 (Chapter VI).
We give an explicit description of minimal projective resolutions of simple $A$-modules, which give another proof of the equality $\text{pdim} \ S_x = |\text{cov}(x)|$ in Corollary 3.4.

**Proposition 3.12.** Let $A$ be the incidence algebra of a distributive lattice $L$ and $x \in L$ and corresponding simple module $S_x$. Then $\text{pdim} \ S_x = |\text{cov}(x)| =: \ell$ and $S_x$ has a minimal projective resolution

$$0 \to P_\ell \to \cdots \to P_0 \to S_x \to 0$$

with $P_0 = P_x$ and $P_r = \bigoplus_{C \subseteq \text{cov}(x), |C| = r} P(\bigvee C)$ for $1 \leq r \leq \ell$.

**Proof.** It is clear that $S_M$ is projective. Let $x \neq M$ and let $L' := [x, M]$ be the interval from $x$ to $M$ in $L$ and note that $L'$ is also a distributive lattice. Let $A'$ denote the incidence algebra of $L'$ and let $I^L(i)$ denote the indecomposable injective $A'$-modules corresponding to the points $i \in L'$. Note that the $A'$-modules $I^L(i)$ are also $A$-modules and we have $I^L(x) \cong S_x$ as $A$-modules. The minimal projective resolution of $I^L(x)$ as an $A'$-module coincides with the minimal projective resolution as an $A$-module and thus we immediately obtain the result from 3.2 since in $L'$ the element $x$ is the minimum of $L'$ and thus $\min([x, x]_{L'})$ (here $[a, b]_{L'}$ denotes the interval in the distributive lattice $L'$) is exactly the set of covers $\text{cov}(x)$ of $x$ in $L$. □

Note that in example 2.5, we saw a poset $P$ that is Auslander regular but the order dimension of $P$ is 2 and thus not equal to the global dimension, which is equal to 3, of the incidence algebra of $P$ so that the previous theorem does not extend to posets with Auslander regular incidence algebra.

**Corollary 3.13.** The global dimension of the incidence algebra of a distributive lattice over a field $K$ is independent of the field.

**Proof.** This follows immediately from 3.11, since the order dimension of the lattice does not depend on the field. □

Note that the previous corollary might be surprising since in general the global dimension of the incidence algebra of a poset can depend on the field, see for example [IZ, Proposition 2.3].

Recall that a poset $P$ is called planar if its Hasse quiver can be drawn in the plane without intersections of arrows. Our next application reveals the surprising fact that being planar can be characterised purely homological for distributive lattices.

**Corollary 3.14.** Let $L$ be a distributive lattice with incidence algebra $A$. Then $L$ is planar if and only if the global dimension of $A$ is at most two.

**Proof.** The equivalence of (1) and (2) follows immediately by 3.11 combined with the fact that a lattice is planar if and only if its order dimension is at most 2, see for example [Tr, theorem 5.1 (Chapter 3)]. □

We give an example that shows that in general the global dimension of the incidence algebra of a (non-distributive) lattice does not coincide with the order dimension of this lattice.

**Example 3.15.** Let $L$ be the lattice with the following Hasse diagram:

![Hasse diagram](image)

Then $L$ has order dimension 3 but the global dimension of the incidence algebra of $L$ is equal to two.
3.3. A categorification of the rowmotion bijection for distributive lattice. We saw in the previous section that the incidence algebra of distributive lattices are diagonal Auslander regular algebras. In this section we apply this result to categorify the rowmotion bijection for distributive lattices. Let \( L \) be a distributive lattice given as the set of order ideals of a poset \( P \). Then the rowmotion bijection \( \text{row} \) of \( L \) is given by
\[
\text{row}(x) = \bigcup_{p \in \min(P \setminus x)} I(p)
\]
for an order ideal \( x \) of \( P \). This is a bijection on \( L \) and this bijection has several applications for the combinatorial study of posets in Lie theory, we refer for example to [TW] and [S] for more on rowmotion for lattices.

Our key observation is the next theorem which shows that any Auslander regular algebra \( A \) has a canonical bijection between simple \( A \)-modules and simple \( A^{op} \)-modules.

**Theorem 3.16.** [II, Theorem 2.10] Let \( A \) be an Auslander regular algebra. Then the map \( S \mapsto \text{soc} \text{Ext}^p_A(S,A) \) for \( g = \text{grade} S \) gives a bijection between the simple \( A \)-modules and the simple \( A^{op} \)-modules. Moreover it preserves the grade.

We are ready to introduce the following notion.

**Definition 3.17.** Let \( A \) be an Auslander regular algebra. Using (3.3) and Theorem 3.16, we obtain a permutation \( g_A \) of the simple \( A \)-modules, which we call the grade bijection, given by
\[
g_A(S) := \text{top}(D \text{Ext}^p_A(S,A)) \quad \text{where} \quad g = \text{grade} S.
\]

Now we assume that \( A \) and \( A^{op} \) are diagonal Auslander regular. Then by Proposition 1.2, we have grade \( S = \text{pdim} S \) for each simple \( A \)-module \( S \), and by Lemma 1.3, we have
\[
g_A(S) = \text{top}(D \text{Ext}^p_A(S,A)) \cong \text{top}(\tau_p(S)) \quad \text{for} \quad p = \text{pdim} S,
\]
where \( \tau_p \) is the \( p \)-Auslander-Reiten translate.

Our main result shows that the grade bijection \( g_A \) gives a homological realization of the rowmotion bijection row.

**Theorem 3.18.** Let \( A \) be the incidence algebra of a distributive lattice \( L \). Then we have
\[
g_A(S_x) = S_{\text{row}(x)} \quad \text{for each} \quad x \in L.
\]

**Proof.** Fix \( x \in L \) and let \( g := \text{grade} S_x = |\text{cov}(x)| \). Since \( g_A(S_x) \) is simple, it suffices to show that \( S_{\text{row}(x)} \) is a factor \( A \)-module of \( D \text{Ext}^3_A(S_x,A) \), or equivalently, \( S_{\text{row}(x)}^{op} \) is a sub \( A^{op} \)-module of \( \text{Ext}^3_A(S_x,A) \). To prove this, it suffices to prove the following two claims.

1. \( \text{Ext}^3_A(S_x, P(\text{row}(x))) \neq 0 \).
2. \( \text{Ext}^3_A(S_x, P(y)) = 0 \) for each \( y \in L \) with \( y < \text{row}(x) \).

On the other hand, the following conditions are equivalent for \( y \in L \) by Corollary 3.6

(i) \( \text{Ext}^3_A(S_x, P(y)) \neq 0 \).
(ii) \( \min(P \setminus x) \leq \text{max}(y) \).

By (3.2), \( \min(P \setminus x) = \text{max}(|\text{row}(x)|) \) holds, and any \( y \) satisfying (ii) satisfies \( \text{row}(x) \leq y \). Thus the claims (1) and (2) hold. \( \square \)

Combining our results, we obtain an equality \( |\text{cov}(\text{row}(x))| = |\text{cov}(x)| \) for each \( x \in L \):
\[
|\text{cov}(\text{row}(x))| = |\text{cogr}(S_{\text{row}(x)})| = |\text{cogr}(g_A(S_x))| = |\text{cogr}(g(S_x))| = |\text{cov}(x)|.
\]

This gives a homological proof of a classical result of Dilworth in the special case of distributive lattices, see for example [KRY, Theorem 3.5.1].

The result of this section motivates to classify all posets whose incidence algebra is a Auslander regular algebra since by 3.16 we obtain a bijection for such posets that generalises the rowmotion bijection for distributive lattices.

We give one example of a poset that is not a lattice but whose incidence algebra is diagonal Auslander regular and display the bijection obtained from 3.16.
Example 3.19. Let $P$ be the poset with the following Hasse diagram:

```
| 1 |
|---|
| 2 |
|---|
| 3 |
|---|
| 4 |
| 5 |
| 6 |
| 7 |
```

The incidence algebra $A$ of $P$ has global dimension three and is a diagonal Auslander regular algebra, but $P$ is not a lattice. The map $g_A$ sends $S_1$ to $S_8$, $S_2$ to $S_5$, $S_3$ to $S_4$, $S_4$ to $S_7$, $S_5$ to $S_6$, $S_6$ to $S_2$, $S_7$ to $S_3$ and $S_8$ to $S_1$.

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