Quantum gravity, minimum length and Keplerian orbits

Zurab Silagadze

Budker Institute of Nuclear Physics and Novosibirsk State University, Novosibirsk 630 090, Russia

We conjecture that the modified commutation relations suggested in the context of quantum gravity (QG) persist also in the classical limit, if the momentum of the classical object is not too large, and calculate the corresponding perihelion precession rate for Keplerian orbits. The main result obtained in this letter is not new. However the derivation is much simpler than the one proposed by Benczik et al. in Phys. Rev. D 66, 026003 (2002) where the corresponding precession rate was calculated for the first time. Our interpretation of the result is also quite different.

PACS numbers: 04.60.-m, 04.60.Bc

Both string theory [2] and some heuristic quantum gravity models [3] imply the existence of minimum length and the corresponding modification of the uncertainty relations near the Planck scale.

When the system has several degrees of freedom, the minimum-length modified commutation relations have the form [4]

\[ [\hat{x}_i, \hat{p}_j] = i\hbar (\delta_{ij} + \beta \hat{p}_i^2 \delta_{ij} + \beta' \hat{p}_i \hat{p}_j) \]  

(1)

The minimum length which follows from these commutation relations is (for more details see [4, 5])

\[ \delta x_{\text{min}} = \sqrt{3 \beta + \beta'} \].

In the particular case \( \beta' = 2\beta \), the realization of this modified Heisenberg algebra to the linear order in \( \beta \) has the form [3, 6]

\[ \hat{x}_i = \hat{x}_0^i, \quad \hat{p}_i = \hat{p}_0^i \left[ 1 + \beta (\hat{p}_0^0)^2 \right] \],

where \( \hat{x}_0^i \) and \( \hat{p}_0^i \) are the standard low-energy position and momentum operators with \( [\hat{x}_0^i, \hat{p}_0^j] = i\hbar \delta_{ij} \). Therefore, up to this accuracy, one gets an universal QG correction to the Hamiltonian [5, 6]

\[ \hat{H} = \frac{\hat{p}}{2m} + V(\hat{x}) = \frac{(\hat{p}_0^0)^2}{2m} + V(\hat{x}_0^0) + \frac{\beta}{m} (\hat{p}_0^0)^4 + O(\beta^2) \].

(2)

To consider the classical limit, the quantum mechanical commutators should be replaced by Poisson brackets according to

\[ \frac{1}{i\hbar} [\hat{A}, \hat{B}] \rightarrow \{A, B\} \]

Therefore we are left with the canonical conjugate variables

\[ \{x_i, p_j\} = \delta_{ij}, \quad \{x_i, x_j\} = 0, \quad \{p_i, p_j\} = 0, \]

and the perturbed Hamiltonian

\[ H = H_0 + \frac{\beta}{m} p^4 \],

(3)

where for Keplerian orbits

\[ H_0 = \frac{p^2}{2m} - \frac{\alpha}{r}. \]

In the above expressions we have dropped the upper index zero in the classical limits of \( \hat{x}_i^0 \) and \( \hat{p}_j^0 \).

Because of the perturbation, the perihelion of the orbit begins to precess. The precession rate is convenient to calculate by using Hamilton vector [7] whose precession rate coincides with the precession rate of perihelion. The Hamilton vector has the form [8]

\[ \vec{u} = \frac{\vec{p}}{m} - \frac{\alpha}{L} \vec{e}_\varphi \]

(4)
and it is conserved in the absence of perturbation:

\[ \dot{\vec{u}} = \{\vec{u}, \mathcal{H}_0\} = 0. \]

Here \( L = mr^2 \dot{\varphi} \) is the orbital momentum and

\[ \vec{e}_\varphi = \frac{\vec{L} \times \vec{r}}{r} \]

is the unit vector in the direction of the polar angle \( \varphi \) in the orbit plane.

When the perturbation is present, the Hamilton vector is no longer conserved:

\[ \dot{\vec{u}} = \{\vec{u}, \mathcal{H}\} = \frac{\beta}{m} \{\vec{u}, p^4\} = \frac{2\beta}{m} p^2 \{\vec{u}, p^2\}. \]

Substituting (4) and (5), we get

\[ \dot{\vec{u}} = \frac{4\beta \alpha p^2}{mL^2 r^3} \vec{L} \times (\vec{r} \times \vec{L}) = \frac{4\beta \alpha p^2}{m r^3} \vec{r}. \tag{6} \]

The precession rate of the vector \( \vec{u} \) is given by

\[ \vec{\omega} = \frac{\vec{u} \times \dot{\vec{u}}}{u^2}. \]

By using \( (\vec{L} \times \vec{r}) \times \vec{r} = -r^2 \vec{L} \) and \( u = \frac{eL}{r} \), where \( e \) is the eccentricity of the orbit, we get

\[ \omega = \frac{4\beta \alpha p^2}{mr^3 e^2} \left( r - \frac{L^2}{m \alpha} \right). \tag{7} \]

Therefore, under the complete orbital cycle the Hamilton vector and hence the perihelion of the orbit revolves by an angle

\[ \Delta \Theta_p = \int_0^T \omega \, dt = \int_0^{2\pi} \frac{\omega}{\dot{\varphi}} \, d\varphi. \tag{8} \]

While integrating (8), to the first order in the perturbation we can use relations which are valid for the unperturbed orbit

\[ \frac{R}{r} = 1 + e \cos \varphi \]

and

\[ \frac{p^2}{2m} - \frac{\alpha}{r} = -\frac{\alpha}{2a}. \]

Here

\[ R = \frac{L^2}{m \alpha} \]

is the semi-latus rectum of the unperturbed orbit and

\[ a = \frac{R}{1 - e^2} \]

is its semi-major axis.

As a result, we finally get

\[ \Delta \Theta_p = -\frac{4\beta \alpha}{e R} \int_0^{2\pi} \cos \varphi (1 + e^2 + 2e \cos \varphi) \, d\varphi = -\frac{8\pi \beta m \alpha}{R}. \tag{9} \]
In this formula $\alpha = G_N m M$ and $\beta = (\delta x_{\text{min}})^2/5\hbar^2$. Therefore, introducing the Planck mass

$$m_P = \sqrt{\frac{\hbar c}{G_N}},$$

equation (9) can be rewritten in the form

$$\Delta \Theta_p = -\frac{8\pi}{5} \frac{\delta x_{\text{min}}}{R} \left( \frac{m}{m_P} \right)^2 \frac{\delta x_{\text{min}}}{\hbar/Mc}.$$ 

(10)

At first sight the effect can be hugely increased by simply increasing the mass $m$ of the satellite body. However it should be understood that the r.h.s. of the modified commutation relations (1) is, in fact, only a truncation of the true commutation relations to lowest order terms, with higher order contributions neglected (10). This truncation to be valid one needs $\beta p^2 \ll 1$, or $p \ll \hbar/\delta x_{\text{min}}$. Usually it is assumed that $\delta x_{\text{min}}$ is of the order of Planck length

$$l_P = \sqrt{\frac{\hbar G_N}{c^3}} \approx 1.6 \times 10^{-35} \text{ m.}$$

Then $\hbar/l_P \approx 6.6 \text{ kg m/s}$. Of course, for elementary particles or atomic systems this is a huge momentum, but not for macroscopic bodies. For example, for Earth-bound satellites we need the mass of the satellite to be smaller than about one-tenth of a gramme our above analysis to make any sense. However, for the satellites light enough we expect the neglected terms in the true commutation relations to be indeed irrelevant and we get for Earth-bound satellites

$$\Delta \Theta_p \approx -7 \left( \frac{6400 \text{ km}}{R} \right) \left( \frac{m}{0.1 \text{ g}} \right)^2 10^{-2}. $$

(11)

Note that similar investigations in the framework of non-commutative geometry were performed in [11]. Like our case, it was found that the modifications of physics at small scales have rather profound effect on classical physics at large scales, something similar to the UV/IR mixing [12].

After this work was nearly completed, we learned about the paper [1] where the effects of modified commutation relations on the classical Keplerian orbits were also considered by different method. Our final result (9) for perihelion precession perfectly coincides with the result of [1] (Eq. 66) when $\beta' = 2\beta$. However our derivation, being based on the Hamilton vector [7], is much simpler and conclusions are different. Besides, a general case $\beta' \neq 2\beta$, considered in [1], assumes non-commuting spatial coordinates and hence non-commutative geometry. This brings some subtlety in the formalism and we think our independent calculation provides a valuable cross-check of their results.

It was concluded in [1] that the observed precession of the perihelion of Mercury places a severe constraint on the value of the minimum length which should be thirty-three (!) orders of magnitude below the Planck length not to have any observable consequences. We think such a limit looks rather strange. In our opinion the Mercury perihelion precession cannot be used for this goal because the truncated commutation relations [11] are no longer valid for such large momentum.

Nevertheless, if the involved momenta are not very large, we expect the modified commutation relations (11) to be valid and imply an observable modification of classical dynamics of macroscopic/mesoscopic objects (compare with 20 keV = 20 keV insight from [13] that a cryodetector developed for dark matter searches can be used for mass spectroscopy with macromolecules because, being a kind of calorimeter, it doesn’t care whether 20 keV energy comes from an electron or from a huge, slow, 20 keV protein). Does this mean that the quantum gravity effects can be studied in space-based or even in table-top experiments?

In fact, it is not even excluded that such effects were already observed. We mean the notorious Pioneer anomaly [14, 16] and flyby anomalies [12, 16]. However, the momentum scales involved in this phenomena are higher than the critical value 6.6 kg m/s mentioned above which does not allow us to use the classical limit of truncated commutation relations [11] for their analysis. Nevertheless, one cannot a priory exclude that the modification of Newtonian dynamics due to true QG commutation relations, although being much smaller than that follows from the incorrect use of truncated commutation relations, is still big enough to be relevant for these gravitational anomalies.

Anyway, it seems worthwhile to further scrutinize the classical limit of the KMM quantum mechanics [4]. A naive and straightforward approach used in this paper, as well as in [11, 17], implies a deformation of Newtonian dynamics which can be experimentally tested either in experiments aimed to test Newton’s second law [18, 19], or in precision tests of the equivalence principle, because the equivalence principle is expected to be dynamically violated under such a deformation [17].
Another place where such a deformation of Newtonian dynamics can reveal itself is interstellar or interplanetary dust dynamics \cite{17} for which truncated commutation relations should be a good approximation. Note, however, that the dust dynamics is a complicated problem involving, besides gravity, a number of physical effects, such as direct radiation pressure and stellar wind pressure, Poynting-Robertson and pseudo Poynting-Robertson forces, sublimation and mutual collisions of dust grains, dust grain charging and the corresponding influence of magnetic fields due to Lorentz force \cite{20}.

Acknowledgments

The work is supported in part by grants Sci.School-905.2006.2 and RFBR 06-02-16192-a. The author thanks Michael Maziashvili for the useful comments.

\* Electronic address: Z.K.Silagadze@inp.nsk.su

\[1\] S. Benczik, L. N. Chang, D. Minic, N. Okamura, S. Rayyan and T. Takeuchi, Phys. Rev. D \textbf{66}, 026003 (2002) [arXiv:hep-th/0204049].

\[2\] G. Veneziano, Europhys. Lett. \textbf{2}, 199 (1986); D. J. Gross and P. F. Mende, Nucl. Phys. B \textbf{303}, 407 (1988); D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. B \textbf{216}, 41 (1989); K. Konishi, G. Paffuti and P. Provero, Phys. Lett. B \textbf{234}, 276 (1990); R. Guida, K. Konishi and P. Provero, Mod. Phys. Lett. A \textbf{6}, 1487 (1991).

\[3\] M. Maggiore, Phys. Lett. B \textbf{304}, 65 (1993) [arXiv:hep-th/9301067]; F. Scardigli, Phys. Lett. B \textbf{452}, 39 (1999) [arXiv:hep-th/9904025]; R. J. Adler and D. I. Santiago, Mod. Phys. Lett. A \textbf{14}, 1371 (1999) [arXiv:gr-qc/9904026].

\[4\] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D \textbf{52}, 1108 (1995) [arXiv:hep-th/9412167].

\[5\] F. Brau, J. Phys. A \textbf{32}, 7691 (1999) [arXiv:quant-ph/9905033].

\[6\] S. Das and E. C. Vagenas, Phys. Rev. Lett. \textbf{101}, 221301 (2008) [arXiv:0810.5333 [hep-th]].

\[7\] O. I. Chashchina and Z. K. Silagadze, Phys. Rev. D \textbf{77}, 107502 (2008) [arXiv:0802.2431 [gr-qc]]; B. Davies, Am. J. Phys. \textbf{51}, 909 (1983); D. Ebner, Am. J. Phys. \textbf{53}, 374 (1985).

\[8\] R. P. Martínez-y-Romero, H. N. Núñez-Yépez and A. L. Salas-Brito, Eur. J. Phys. \textbf{14}, 71 (1993); H. N. Núñez-Yépez and A. L. Salas-Brito, Eur. J. Phys. \textbf{21}, L39 (2000); G. Munoz, Am. J. Phys. \textbf{71}, 1292 (2003) [arXiv:physics/0303106]; J. T. Wheeler, Can. J. Phys. \textbf{83}, 91 (2005) [arXiv:physics/0511054].

\[9\] M. G. Stewart, Am. J. Phys. \textbf{73}, 730 (2005).

\[10\] J. Y. Bang and M. S. Berger, Phys. Rev. D \textbf{74}, 125012 (2006) [arXiv:gr-qc/0610056].

\[11\] B. Mirza and M. Dehghani, Commun. Theor. Phys. \textbf{42}, 183 (2004) [arXiv:hep-th/0211190]; A. E. F. Djemai, Int. J. Theor. Phys. \textbf{43}, 299 (2004) [arXiv:hep-th/0309034]; J. M. Romero and J. D. Vergara, Mod. Phys. Lett. A \textbf{18}, 1673 (2003) [arXiv:hep-th/0303064].

\[12\] A. G. Cohen, D. B. Kaplan and A. E. Nelson, Phys. Rev. Lett. \textbf{82}, 4971 (1999) [arXiv:hep-th/9803132].

\[13\] L. Stodolsky, [arXiv:0810.4446 [physics.ins-det]].

\[14\] J. D. Anderson, P. A. Laing, E. L. Lau, A. S. Liu, M. M. Nieto and S. G. Turshev, Phys. Rev. D \textbf{65}, 082004 (2002) [arXiv:gr-qc/0104064]; M. M. Nieto and J. D. Anderson, Contemp. Phys. \textbf{48}, 41 (2007) [arXiv:0709.3866 [gr-qc]].

\[15\] J. D. Anderson, J. K. Campbell, J. E. EkELund, J. Ellis and J. F. Jordan, Phys. Rev. Lett. \textbf{100}, 091102 (2008); J. D. Anderson, J. K. Campbell and M. M. Nieto, New Astron. \textbf{12}, 383 (2007) [arXiv:astro-ph/0608087].

\[16\] C. Lämmerzahl, O. Preuss and H. Dittus, [arXiv:gr-qc/0604052].

\[17\] S. Benczik, L. N. Chang, D. Minic, N. Okamura, S. Rayyan and T. Takeuchi, [arXiv:hep-th/0209119].

\[18\] J. H. Gundlach, S. Schlamminger, C. D. Spitzer, K. Y. Choi, B. A. Woodahl, J. J. Coy and E. Fischbach, Phys. Rev. Lett. \textbf{98}, 150801 (2007).

\[19\] C. Lämmerzahl, Eur. Phys. J. ST \textbf{163}, 255 (2008).

\[20\] A. Krivov, H. Kimura and I. Mann, Icarus \textbf{134}, 311 (1998); M. Horányi, Ann. Rev. Astron. Astrophys. \textbf{34}, 383 (1996).