Standard Projective Simplicial Kernels and the Second Abelian Cohomology of Topological Groups

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Abstract

Let \(A\) be an abelian topological \(G\)-module. We give an interpretation for the second cohomology, \(H^2(G, A)\), of \(G\) with coefficients in \(A\). As a result we show that if \(P\) is a projective topological group, then \(H^2(P, A) = 0\) for every abelian topological \(P\)-module \(A\).

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1 Introduction

Let \(G\) be a topological group and \(A\) an abelian topological group. It is said that \(A\) is an abelian topological \(G\)-module, whenever \(G\) acts continuously on \(A\). For all \(g \in G\) and \(a \in A\) we denote the action of \(g\) on \(a\) by \(g a\). Suppose that \(A\) is an abelian topological \(G\)-module. Hu \([4]\) showed that if \(G\) is Hausdorff, then there is a one to one correspondence between the second cohomology, \(H^2(G, A)\), of \(G\) with coefficients in \(A\) and the set, \(\text{Ext}_s(G, A)\), of all equivalence classes of topological extensions with continuous sections. Thus, \(H^2(G, A)\) induces a group structure on \(\text{Ext}_s(G, A)\) and consequently, under this group product, \(H^2(G, A)\) is isomorphic to \(\text{Ext}_s(G, A)\). It is known that \(\text{Ext}_s(G, A)\) with Baer sum is an abelian group and the Baer sum on \(\text{Ext}_s(G, A)\) coincides with the group product induced by \(H^2(G, A)\). In other words, if \(G\) is Hausdorff then, \(H^2(G, A)\) is isomorphic to \(\text{Ext}_s(G, A)\) with Baer sum \([1]\). Similarly, we define \(\text{Opext}_s(G, A)\) and we conclude the same result without Hausdorffness of \(G\), i.e., there is an isomorphism between \(H^2(G, A)\) and \(\text{Opext}_s(G, A)\). By using the concept of projective topological group we introduce the notion of a standard simplicial kernel of a topological group \(G\).

in section 2, we prove that every Markov (Graev) topological group is projective and also we show that projectivity in the category of topological groups is equivalent to \(\mathfrak{g}\)-projectivity in the sense of Hall \([3]\). In addition, we introduce the notion of a standard simplicial kernel of a topological group \(G\). Finally, we

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define $Opext_s(G, A)$ and we conclude that $H^2(G, A) \cong Opext_s(G, A)$, with Baer sum.

In section 3, we give a characterization of $H^2(G, A)$, when $A$ is an abelian topological $G$-module. As a result, we show that if $P$ is a projective topological group then $H^2(P, A) = 0$, for every abelian topological $P$-module.

2 Standard Projective Simplicial Kernels.

In this section, we introduce the notion of a standard projective simplicial kernel of a topological group $G$.

**Definition 2.1.** A topological group $P$ is said to be a projective topological group if, for every continuous epimorphism $\pi : A \to B$, where $\pi$ has a continuous section, and for every continuous homomorphism $f : P \to B$, there exists a continuous homomorphism $g : P \to A$ such that the following diagram is commutative

$$
\begin{array}{c}
P \\
g \downarrow \\
A \xrightarrow{\pi} B \xrightarrow{f} 1
\end{array}
$$

If $X$ is a topological space, then the (Markov) free topological group over $X$ is the pair $(F_X, \sigma_X : X \to F_X)$ in which $F_X$ is a group equipped with the finest group topology such that $\sigma_X$ is continuous. Such a topology always exists [8], and has the following universal property: every continuous map $f$ from $X$ to an arbitrary topological group $G$ lifts to a unique continuous homomorphism $\bar{f}$, i.e., $\bar{f} \circ \sigma_X = f$. Similarly, one can define the Graev free topological group, $(F^*_X, \sigma^*_X)$, over a pointed topological space $(X, e)$. For information on free (abelian) topological groups see [2, 6, 8]. The following facts about Markov (Graev) free topological group are well-known:

1. The Markov (Graev) free topological group over a (pointed) topological space $X$ ($(X, e)$) is the free group with the same canonical map over the (pointed) set $X$ ($(X, e)$);
2. $F_X$ ($F^*_X$) is Hausdorff if and only if $X$ is functionally Hausdorff;
3. $\sigma_X$ ($\sigma^*_X$) is a homeomorphic embedding if and only if $X$ is completely regular;
4. $\sigma_X$ ($\sigma^*_X$) is a closed homeomorphic embedding if and only if $X$ is Tychonov.

The elements of free group, $F_X$, over a set $X$ is denoted by $|x_1|^{\epsilon_1}...|x_n|^{\epsilon_n}$, where $\epsilon_i = \pm 1$. This notation is useful whenever $X$ is a group.

**Lemma 2.2.** Every Markov (Graev) free topological group is a projective topological group.
Proof. Assume that $F$ is a Markov free topological group over the space $X$, and let $\pi: A \to B$ be a continuous epimorphism having a continuous section $s$ and $f: F \to B$ a continuous homomorphism. Then, there is a unique continuous homomorphism $\bar{f}: F \to A$ such that the following diagram is commutative

$$
\begin{array}{ccc}
X & \xrightarrow{\sigma_X} & F \\
\downarrow & & \downarrow f \\
F & \xrightarrow{\bar{f}} & A
\end{array}
$$

Hence, $sf = \bar{f}$ on $X$. So, $\pi\bar{f} = \pi(sf) = (\pi s)f = f$ on $X$. Since $X$ generates the group $F$, then it is easy to see that $\pi\bar{f} = f$. Consequently, $F$ is a projective topological group.

Now, let $X$ be a topological space with a fixed point $e \in X$. Assume that $F^*$ is the free topological group in the sense of Graev over $(X, e)$, and let $\pi: A \to B$ be a continuous epimorphism with continuous section $s: B \to A$. We can assume that $s$ is a normal section, i.e., $s(1) = 1$, since it is enough to define the new continuous section $\tilde{s}: B \to A$ by $\tilde{s}(b) = s(b)s(1)^{-1}$. In fact, since $e$ is the neutral element of $F^*$, then $f(e) = 1$. Thus, $sf(e) = 1$. Hence there exists a unique continuous homomorphism $\tilde{f}: F^* \to A$ such that $sf = \tilde{f}$ on $X$. The rest of the proof is the same as the first part.

Note that the free functor $F(-)$ from the category, $\mathcal{T}$, of topological spaces to the category, $\mathcal{T}G$, of topological groups is left adjoint to the forgetful functor, $G: \mathcal{T}G \to \mathcal{T}$. Let $\mathfrak{F}$ be the class of all continuous epimorphisms having a continuous section. Thus, by [3, Theorem 2] one can see the following:

**Remark 2.3.** A topological group $P$ is $\mathfrak{F}$-projective if and only if $P$ is projective.

**Definition 2.4.** Let $M$, $F$ and $G$ be topological groups, and let $i_0, i_1: M \to F$ and $\tau: F \to G$ be continuous homomorphisms. It is said that $(M, i_0, i_1)$ is a simplicial kernel of $\tau$, whenever $\tau i_0 = \tau i_1$ and it has the following universal property:

- if $j_0, j_1: K \to F$ are continuous homomorphisms and $\pi j_0 = \pi j_1$, then there exists a unique continuous homomorphism $h: K \to M$ such that $j_0 = \omega h$ and $j_1 = \iota_1 h$, i.e., $h$ commutes the following diagram.

$$
\begin{array}{ccc}
K & \xrightarrow{j_0} & F \\
\underset{j_1}{\downarrow} & & \downarrow \tau \\
M & \xrightarrow{i_0} & F \\
\underset{i_1}{\downarrow} & & \downarrow \tau \\
& & G
\end{array}
$$

**Definition 2.5.** Let $(M, i_0, i_1)$ be a simplicial kernel of $\tau: P \to G$. The quadruplet $(M, i_0, i_1, \tau)$ is said to be a projective simplicial kernel of $G$, if $P$ is a projective topological group and $\tau$ is a continuous (open) epimorphism which has a continuous section $s: G \to P$. If $P$ is a Markov (Graev) free topological group, then $(M, i_0, i_1, \tau)$ is called a Markov (Graev) simplicial kernel of $G$. 

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Lemma 2.6. If a continuous homomorphism \( f : G \to H \) has a continuous section, then \( f \) is an open epimorphism.

Proof. The proof is a standard argument. \( \square \)

For any topological group \( G \) there exists at least one projective simplicial kernel of \( G \). Since the identity map \( Id_G : G \to G \) lifts to a unique homomorphism \( \tau_G : F_G \to G \). Thus, \( \tau_G \sigma_G = Id_G \). The uniqueness property implies that \( \tau_G \) is a map by the following rule:

\[
\tau_G : F_G \to G, \ |g_1|^{\epsilon_1} \ldots |g_n|^{\epsilon_n} \mapsto g_1^{\epsilon_1} \ldots g_n^{\epsilon_n} \text{ where } \epsilon_i = \pm 1.
\]

The map \( \tau_G \) is called multiplication map. Obviously, \( \sigma_G : G \to F_G, \ g \mapsto |g|, \) is a continuous section for \( \tau_G \). By Lemma 2.6, \( \tau_G \) is an open continuous epimorphism. We take \( M_G = \{(x,y) \mid x,y \in F_G, \tau_G(x) = \tau_G(y)\} \subset F_G \times F_G \). Hence, \( M_G \) has subspace topology induced by the product topology \( F_G \times F_G \). Suppose \( i_0 = i_0^G, i_1 = i_1^G : M_G \to F_G \) are the canonical projection maps, where \( i_0^G(x,y) = x \) and \( i_1^G(x,y) = y \). It is easy to see that \( (M_G, i_0^G, i_1^G) \) is a simplicial kernel of \( \tau_G \). Hence, \( (M_G, i_0^G, i_1^G, \tau_G) \) is a projective simplicial kernel of \( G \) and we call it the standard Markov simplicial kernel of \( G \). By a similar way we can define the standard Graev (projective) simplicial kernel of \( G \).

Suppose that \( (M, i_0, i_1, \tau) \) is a projective simplicial kernel of \( G \). We sometimes denoted it by the following

\[
M \xrightarrow{i_0} P \xrightarrow{\tau} G \xrightarrow{i_1} 1. \tag{*}
\]

If \( (M_G, i_0, i_1, \tau_G) \) is a standard projective simplicial kernel of \( G \), then put \( \Delta_P = \{(x,x) \mid x \in P\} \subset M \). Now, let \( A \) be an abelian topological \( G \)-module. Obviously, \( A \) is a topological \( P \)-module via \( \tau \) and a topological \( M \)-module via \( \tau i_0 \) (or \( \tau i_1 \)). Define \( Z^1(M, A) = \{ [\alpha] \mid \alpha \in \text{Der}_c(M, A), \alpha(\Delta_P) = 0 \} \). It is clear that \( Z^1(M, A) \) is a subgroup of abelian group \( \text{Der}_c(M, A) \).

Consider

\[
\eta : \text{Der}_c(P, A) \to Z^1(M, A)
\]

\[
\alpha \mapsto \alpha i_0 \alpha i_1^{-1}
\]

Obviously, \( \eta \) is well-defined and since \( A \) is an abelian group, then

\[
\eta(\alpha, \beta) = (\alpha, \beta i_0)(\alpha, \beta i_1^{-1}) = (\alpha i_0, \beta i_0) (\alpha i_1^{-1}, \beta i_1^{-1}) = (\alpha i_0 \alpha i_1^{-1})(\beta i_0 \beta i_1^{-1}) = \eta(\alpha) \eta(\beta).
\]

Therefore, \( \eta \) is a homomorphism. Clearly, \( \eta \) induces a congruence equivalence relation \( \sim \) in \( Z^1(M, A) \). In other words, for \( \alpha, \beta \in Z^1(M, A) \), the equivalence relation \( \sim \) is defined as follows:

\[
\alpha \sim \beta \text{ whenever there is } \gamma \in \text{Der}_c(P, A) \text{ and } \beta = \gamma i_0 \gamma i_1^{-1} \alpha.
\]
**Definition 2.7.** It is said that a short exact sequence

\[ e : 1 \to A \xrightarrow{\chi} E \xrightarrow{\pi} G \to 1 \]

is a proper extension of \( G \) by \( A \) whenever \( \chi \) is a homeomorphic embedding and \( \pi \) is an open continuous homomorphism. By a section for \( e \) we mean a continuous map \( s : G \to E \) such that \( \pi s = \text{Id}_G \).

Let \( e \) be a proper extension of \( G \) by \( A \) which has a section \( s : G \to E \). In addition, assume that \( A \) is abelian. Then the proper extension \( e \) gives rise to a topological \( G \)-module structure on \( A \), which is well-defined by

\[ g a = \chi^{-1}(s(g)\chi(a)s(g)^{-1}) \], where \( g \in G \), \( a \in A \).

One can easily see that for each \( g \in G \) and \( a \in A \), the element \( ga \) does not depend on the choice of continuous sections.

Now, let \( A \) be an abelian topological \( G \)-module. We denote by \( \text{opexts}(G, A) \) the set of all proper extensions of \( G \) by \( A \) having continuous sections and corresponding to the given way in which \( G \) acts on \( A \). We define an equivalence relation \( \equiv \) on \( \text{opexts}(G, A) \) as follows:

Let \( e_i : 1 \to A \xrightarrow{\chi_i} E \xrightarrow{\pi_i} G \to 1 \in \text{opexts}(G, A) \) for \( i = 0, 1 \). \( e_0 \equiv e_1 \) whenever there is a continuous (open) homomorphism \( \theta : E_0 \to E_1 \) so that the following diagram is commutative.

\[
\begin{array}{ccc}
1 & \xrightarrow{\chi_0} & A \xrightarrow{\chi_1} E_0 & \xrightarrow{\pi_0} & G & \xrightarrow{\theta} & E_1 & \xrightarrow{\pi_1} & G & \xrightarrow{\pi} & 1 \\
\end{array}
\]

We denote the set of all equivalence classes by \( \text{Opexts}(G, A) \). Note that if \( G \) is Hausdorff, then \( \text{Opexts}(G, A) = \text{Ext}_s(G, A) \). One can verify by a similar argument as in [1, 4] that we have the following.

**Theorem 2.8.** Let \( A \) be an abelian topological \( G \)-module, then \( H^2(G, A) \) in the sense of Hu, is isomorphic to \( \text{Opexts}(G, A) \) with Baer sum.

---

3 the Main Theorem

In this section we will prove the main theorem. Sometimes we denote the inclusion maps and epics by the arrows \( \hookrightarrow \) and \( \twoheadrightarrow \), respectively. Let \( A \) be a topological \( G \)-module. Thus, under the action \( (g, a) \xrightarrow{\phi} g a \) we may consider the topological semidirect product \( A \rtimes_{\phi} G \) (see [7, Section 6]).

**Theorem 3.1.** Let \( A \) be an abelian topological \( G \)-module and let \((M, \iota_0, \iota_1, \tau)\) be a standard projective simplicial kernel of \( G \). Then, \( H^2(G, A) \) is canonically isomorphic to \( Z^2(M, A) / \sim \).
Proof. By Theorem 2.8, it is enough to show that $\text{Opext}_s(G, A)$ is isomorphic to $Z^1(M, A)/\sim$.

Let $\alpha \in Z^1(M, A)$. Since $A$ is a $G$-module, then $A$ is a $P$-module via $\tau$. Thus, we may consider topological semidirect product $A \rtimes P$. Now, we define a relation $\sim$ in the semidirect product $A \rtimes P$, as follows

$$(a, x) \sim (b, y) \text{ if and only if } \tau(x) = \tau(y) \text{ and } a\alpha(x, y) = b.$$ 

Obviously, $\sim$ is reflexive. Let $x, y$ and $z$ be arbitrary elements of $P$ such that $\tau(x) = \tau(y) = \tau(z)$. Then

$$\alpha(x, y)\alpha(y, z) = \alpha(x, z) \tag{3.1}$$

since,

$$\alpha(x, z) = \alpha((x, y)(1, y^{-1}z)) = \alpha(x, y).\alpha(1, y^{-1}z) = \alpha(x, y).\alpha(y^{-1}y, y^{-1}z)$$

$$= \alpha(x, y).\alpha((y^{-1}, y^{-1})(y, z)) = \alpha(x, y).\alpha(y^{-1}, y^{-1}).\alpha(y, z))$$

$$= \alpha(x, y).\alpha(1, y^{-1}\alpha(y, z)) = \alpha(x, y)\alpha(y, z).$$

Now if $(a, x) \sim (b, y)$ and $(b, y) \sim (c, z)$, then

$$\tau(x) = \tau(y) = \tau(z), \text{ a}\alpha(x, y) = b \text{ and } b\alpha(y, z) = c.$$ 

Hence, by (3.1), $a\alpha(x, z) = a\alpha(x, y)\alpha(y, z) = b\alpha(y, z) = c$. This implies that $\sim$ is transitive. By (3.1), we have:

$$\alpha^{-1}(x, y) = \alpha(y, x). \tag{3.2}$$

If $(a, x) \sim (b, y)$, then $\tau(x) = \tau(y)$ and $a\alpha(x, y) = b$. Thus, $a = b\alpha^{-1}(x, y) = b\alpha(y, x)$. Consequently, $\sim$ is an equivalence relation in $A \rtimes P$.

Now, Define $N_\alpha = \{(a, x)|(a, x) \in A \rtimes P, (a, x) \sim (1, 1)\}$. Then, by the definition of $\sim$ and the identity (3.2), we have

$$N_\alpha = \{(a, x)|(a, x) \in A \rtimes P, \tau(x) = 1, a\alpha(x, 1) = 1\}$$

$$= \{(a, x)|(a, x) \in A \rtimes P, \tau(x) = 1, a = \alpha(1, x)\}.$$ 

Suppose that $(a, x), (b, x) \in N_\alpha$. Since $\tau(x) = 1$, then $(a, x)(b, y) = (a^b, xy) = (ab, xy)$. On the other hand, $\alpha(1, xy) = \alpha(1, x)\alpha(1, y) = ab$. Thus, $(a, x)(b, y) \in N_\alpha$. Also $(a, x)^{-1} = (\tau^{-1}a, x^{-1}) = (a^{-1}, x^{-1})$ and $\alpha(1, x^{-1}) = \alpha(xx^{-1}, x^{-1}) = \alpha(x, 1)^\tau\alpha(x^{-1}, x^{-1}) = \alpha(x, 1) = \alpha^{-1}(1, x) = a^{-1}$, hence $(a, x)^{-1} \in N_\alpha$. Therefore $N_\alpha$ is a subgroup of $A \rtimes P$.

Denote by $E_\alpha$ the quotient space $(A \rtimes P)/\sim$.

We show that $\sim$ is a congruence (i.e., $\sim$ is compatible with the group product) and therefore $E_\alpha$ is a group.

Let $(a, x) \sim (a', x')$ and $(b, y) = (b', y')$. Then $\tau(x) = \tau(x'), \tau(y) = \tau(y')$, $a\alpha(x, x') = a'$ and $b\alpha(y, y') = b'$. 

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Lemma 1.6, \( \phi \). Hence, \( (\psi \alpha, (\psi \alpha)) = (1, x)N_\alpha \). This implies that \( N_\alpha \) is a normal subgroup of \( A \times P \). Therefore \( E_\alpha \) is a topological group.

There is a diagram as follows:

\[
\begin{array}{ccc}
M & \longrightarrow & P \\
\downarrow{\alpha} & & \downarrow{\psi} \\
A & \longrightarrow & E_\alpha \\
\end{array}
\]

where \( \sigma(a) = (a, 1)N_\alpha \), \( \psi(a, x)N_\alpha = \tau(x) \) and \( \beta(x) = (1, x)N_\alpha \).

It is obvious that \( \sigma \) and \( \beta \) are continuous homomorphisms. Consider the continuous map \( \phi : A \times P \to P, (a, x) \mapsto \tau(x) \).

We know that \( \tau \) has a continuous section \( s : G \to P \). Hence, \( \beta \circ s \) is a continuous section for \( \psi \), since \( \psi \beta \circ s = (\psi \beta) \circ s = \tau \circ s = \text{Id}_G \). Thus, by Lemma 1.6, \( \phi \) is an open epimorphism.

If \( \sigma(a) \in N_\alpha \), then \( (a, 1) \in N_\alpha \). Thus, by definition of \( N_\alpha \), we get \( a = \alpha(1, 1) = 1 \). Therefore \( \sigma \) is one to one.

\( \psi \circ \sigma(a) = \psi((a, 1)N_\alpha) = \tau(1) = 1 \), thus \( \text{Im} \sigma \subset \text{Ker} \psi \). On the other hand, if \( \psi(a, x)N_\alpha = 1 \), then \( \tau(x) = 1 \) and \( \sigma(\alpha a(x, 1)) = (a, x)N_\alpha \), i.e., \( \text{Ker} \psi \subset \text{Im} \sigma \). Therefore, \( \text{Im} \sigma = \text{Ker} \psi \).

Now, we prove that \( \sigma \) is a homeomorphic embedding.

Define the map \( \chi : A \times P \to A \) via \( \chi(a, x) = a\alpha(x, s\tau(x)s(1)^{-1}) \). Obviously, \( \chi \) is continuous. It is clear that \( (A \times \{1\})N_\alpha \) is a topological subgroup of \( A \times P \) and \( (A \times \{1\})N_\alpha = \{(a, x)\mid a \in A, x \in \text{Ker} \tau\} \). Take \( \tilde{\chi} = \chi |_{(A \times \{1\})N_\alpha} \). Thus, \( \tilde{\chi} \) is continuous and \( \tilde{\chi}(a, x) = a\alpha(x, 1) \). Note that \( \tilde{\chi} \) is a homomorphism, because

\[
\tilde{\chi}((a, x)(b, y)) = \tilde{\chi}(ab, xy) = ab\alpha(xy, 1) = ab\alpha(x, 1)\alpha(xy, x)
\]

If \( (a, x) \in N_\alpha \), then \( \tilde{\chi}(a, x) = a\alpha(x, 1) = \alpha(1, 1)\alpha(x, 1) = \alpha(1, 1) = 1 \). Hence \( \tilde{\chi}(N_\alpha) = 1 \). So, \( \chi \) induces the continuous homomorphism

\[
\xi : ((A \times \{1\})N_\alpha)/N_\alpha \to A, \xi((a, x)N_\alpha) = \tilde{\chi}(a, x).
\]

It is clear that \( \text{Im} \sigma = ((A \times \{1\})N_\alpha)/N_\alpha \). We have \( \xi \circ \sigma(a) = a \) and

\[
\sigma \circ \xi((a, x)N_\alpha) = \sigma(\alpha a(x, 1)) = (a\alpha(x, 1), 1)N_\alpha = (a, x)(\alpha(x, 1), x^{-1})N_\alpha
\]

Thus, \( \sigma^{\text{Im} \sigma : A \to \text{Im} \sigma, \sigma^{\text{Im} \sigma}(a) = (a, 1)N_\alpha, \) is a topological isomorphism. Therefore, \( \sigma \) is a homeomorphic embedding.
Denote by $e_\alpha$ the extension $0 \to A \overset{\sigma}{\to} E_\alpha \overset{\psi}{\to} G \to 1$. We have $[e_\alpha] \in Opext_s(G, A)$. Since $s : G \to P$ is a continuous section for $\tau$, then $\beta \circ s$ is a continuous section for $\psi$ and

$$
\beta \circ s(g)\sigma(a)\beta \circ s(g)^{-1} = (1, s(g))N_\alpha(a, 1)N_\alpha(1, s(g)^{-1})N_\alpha
$$

$$
= (1, s(g))(a, 1)(1, s(g)^{-1})N_\alpha = (s(g), a, s(g))(1, s(g)^{-1})N_\alpha
$$

$$
= (s(g), a, 1)N_\alpha = (\eta g, a)N_\alpha = \sigma(\eta g).
$$

In addition, $\sigma$ is one to one, thus the extension $e_\alpha$ induces the given action of $G$ on $A$. Hence $[e_\alpha] \in Opext_s(G, A)$.

Consider the map $\zeta : Z^1(M, A) \to Opext_s(G, A)$ via $\alpha \mapsto [e_\alpha]$. Suppose that $\alpha, \bar{\alpha} \in Z^1(M, A)$ and $\bar{\alpha} = \eta(\gamma)\alpha$, for some $\gamma \in \text{Der}_c(P, A)$. Define the map $\varepsilon : A \times P \to A \times P$ by $(a, x) \mapsto (a\gamma^{-1}(x), x)$. Obviously, $\varepsilon$ is continuous, and

$$
\varepsilon((a, x)(b, y)) = \varepsilon(a^2b, xy) = (a^2b\gamma^{-1}(xy), xy) = (a^2b\gamma^{-1}(y)\gamma^{-1}(x), xy)
$$

$$
= (a\gamma^{-1}(x)^2(b\gamma^{-1}(y), xy) = (a\gamma^{-1}(x), x)(b\gamma^{-1}(y), y);
$$

that is, $\varepsilon$ is a homomorphism.

If $(a, x) \in N_\alpha$, then

$$
\bar{\alpha}(1, x) = (\gamma_0, 1) = \gamma_0(1, x)\gamma^{-1}(1, x)\alpha(1, x) = \gamma^{-1}(x)\alpha(1, x)
$$

$$
= a\gamma^{-1}(x).
$$

Hence, $(a\gamma^{-1}(x), x) \in N_{\bar{\alpha}}$, i.e., $\varepsilon(N_\alpha) \subset N_{\bar{\alpha}}$. Thus, $\varepsilon$ induces the continuous homomorphism $\bar{\varepsilon} : E_\alpha \to E_{\bar{\alpha}}, (a, x)N_\alpha \mapsto (a\gamma^{-1}(x), x)N_{\bar{\alpha}}$.

Suppose that the extensions $e_\alpha : 0 \to A \overset{\sigma}{\to} E_\alpha \overset{\psi}{\to} G \to 1$ and $e_{\bar{\alpha}} : 0 \to A \overset{\sigma}{\to} E_{\bar{\alpha}} \overset{\bar{\psi}}{\to} G \to 1$ are corresponding to $\alpha$ and $\bar{\alpha}$, respectively. We show that the following diagram commutes.

$$
e_\alpha : 0 \to A \overset{\sigma}{\to} E_\alpha \overset{\psi}{\to} G \to 1
$$

$$
e_{\bar{\alpha}} : 0 \to A \overset{\sigma}{\to} E_{\bar{\alpha}} \overset{\bar{\psi}}{\to} G \to 1
$$

Because, $\bar{\psi}(a, x)N_{\bar{\alpha}} = \bar{\psi}(a\gamma^{-1}(x), x)N_{\bar{\alpha}} = \tau(x) = \psi(a, x)N_\alpha$, and $\bar{\varepsilon}\sigma(a) = \bar{\varepsilon}(a, 1)N_{\bar{\alpha}} = (a\gamma^{-1}(1, 1))N_{\bar{\alpha}} = (a, 1)N_{\bar{\alpha}} = \bar{\sigma}(a)$. i.e., $[e_{\bar{\alpha}}] = [e_\alpha]$. Thus, $\zeta$ induces the map $\Theta : \text{Coker}\eta \to Opext_s(G, A)$

$$
[\alpha] \mapsto [e_\alpha].
$$

We will prove that $\Theta$ is a bijective map.

Let $e : 0 \to A \overset{\sigma}{\to} C \overset{\psi}{\to} G \to 1$ be an extension with a continuous section $s : G \to A$ and corresponding to the given way in which $G$ acts
We have $\bar{\tau}$. Consider the commutative diagram.

\[
P \xrightarrow{\tau} G \xrightarrow{1} \\
\downarrow{\beta} \quad \downarrow{\psi} \\
C \xrightarrow{\psi} G \xrightarrow{1}
\]

commutes. Now, define $\alpha_{\beta} : M \to A$ by $\alpha_{\beta}(m) = (\beta \psi(m))^{-1}(m)$. Note that $\alpha_{\beta}$ is well-defined, because $\psi(\beta \psi(m)) = \psi \beta \psi(m) = \tau \tau^{-1} = 1$. Thus, $\text{Im} \alpha_{\beta} \subset \text{Ker} \psi = \text{Im} \sigma = A$. For simplicity, we denote $\alpha_{\beta}$ by $\beta \psi^{-1}$. It is clear that $\alpha_{\beta}$ is continuous and in addition, $\alpha_{\beta}$ is a crossed homomorphism, since

\[
\alpha_{\beta}(x_1, y_1, x_2, y_2) = \alpha_{\beta}(x_1 x_2, y_1 y_2) = \beta(x_1 x_2) \beta^{-1}(y_1 y_2) = \beta(x_1) \beta^{-1}(y_1) \beta(x_2) \beta^{-1}(y_2) = \alpha_{\beta}(x_1, x_2) \alpha_{\beta}(y_1, y_2)
\]

and $\alpha_{\beta}(x, x) = \beta(x) \beta^{-1}(x) = 1$. i.e., $\alpha_{\beta}(\Delta_P) = 1$, hence $\alpha_{\beta} \in Z^1(M, A)$. Consequently, $\beta$ and $\alpha_{\beta}$ make the following diagram commutative.

\[
\begin{array}{ccc}
M & \xrightarrow{\tau} & G \\
\downarrow{\alpha_{\beta}} & & \downarrow{\psi} \\
A & \xrightarrow{\sigma} & C
\end{array}
\]

Now, we will show that $[\alpha_{\beta}]$ is independent of the choice of $\beta$. Consider the commutative diagram.

\[
\begin{array}{ccc}
M & \xrightarrow{\tau} & G \\
\downarrow{\alpha_{\beta}} & & \downarrow{\psi} \\
A & \xrightarrow{\sigma} & C
\end{array}
\]

Define $\gamma(x) = (\tilde{\beta} \beta^{-1})(x)$, for all $x \in P$. Since $\psi(\gamma(x) = \psi \tilde{\beta}(x) \psi^{-1}(x) = \tau(x) \tau^{-1}(x) = 1$, then $\gamma(x) \in \text{Ker} \psi = A$. Hence, we can define the continuous map $\gamma : P \to A$, $x \mapsto \gamma(x)$. Now $\gamma$ is a crossed homomorphism, since

\[
\gamma(xy) = (\tilde{\beta} \beta^{-1})(xy) = \tilde{\beta}(xy) \beta^{-1}(xy) = \tilde{\beta}(x) \tilde{\beta}(y) \beta^{-1}(y) \beta^{-1}(x) = \gamma(xy)
\]

We have $\tilde{\beta} = \gamma \beta$. Thus,
\[ \alpha_\beta = \beta_{t_0^2}^{-1} = \gamma_{t_0} \gamma_{t_0}^{-1} \gamma_{t_0}^{-1} = \gamma_{t_0} (\beta_{t_0^2} \gamma_{t_0}^{-1}) \gamma_{t_0}^{-1} = \gamma_{t_0} \alpha_\beta \gamma_{t_0}^{-1} = \alpha_\beta (\gamma_{t_0}^{-1}) . \]

i.e., \( \alpha_\beta \sim \alpha_\beta \).

Now, we will show that \( \Theta([\alpha_\beta]) = [e] \). It is sufficient to prove that \( e_{\alpha_\beta} \sim e \). Define the map \( \nu : A \times P \to C \) via \( (a, x) \mapsto a \beta(x) \). Obviously, \( \nu \) is continuous, and since

\[ \nu(1(a, x)(b, y)) = \nu(a^x b, xy) = a^x b \beta(xy) = a \beta(x) \beta(x)^{-1} (\tau b) \beta(y) = \nu(a, x)^{\beta(\psi)}(\tau b) \beta(y) = \nu(a, x)(\tau b)(y) = \nu(a, x) \nu(b, y), \]

then \( \nu \) is a homomorphism. On the other hand, if \( (a, x) \in N_{\alpha_\beta} \), then

\[ \nu(a, x) = a \beta(x) = \alpha_\beta(1, x) \beta(x) = \beta^{-1}(x) \beta(x) = 1. \]

Thus, \( \nu : A \times P \to C \) induces the continuous homomorphism \( \hat{\nu} : E_{\alpha_\beta} \to C \), \( (a, x)N_{\alpha_\beta} \mapsto a \beta(x) \). Note that \( \hat{\nu} \) commutes the following diagram

\[
\begin{array}{ccc}
\varepsilon_{\alpha_\beta} : 0 & \longrightarrow & A \\
\downarrow & & \downarrow \beta \phi \nu \psi \text{G} \downarrow\gamma \\
e : 0 & \longrightarrow & A \times C \rightarrow C \\
\text{a} & \phi \rightarrow & C \rightarrow G \\
\text{a} & \phi \rightarrow & G \\
\end{array}
\]

Since \( \hat{\nu} \hat{\sigma}(a) = \hat{\nu}(a, 1)N_{\alpha_\beta} = a \beta(1) = a = \sigma(a) \), and \( \psi \hat{\nu}(a, x)N_{\alpha_\beta} = \psi(a \beta(x)) = \psi(x) = \tau(x) = \psi((a, x)N_{\alpha_\beta}) \). Therefore, \( e_{\alpha_\beta} \sim e \), i.e., \( \Theta \) is onto.

We show that \( \Theta \) is one to one.

Let \( e_i : 0 \longrightarrow A^{\alpha_i} \longrightarrow C_i \longrightarrow G \longrightarrow 1 \), \( i = 0, 1 \), be the extensions with continuous sections and corresponding to the given way in which \( G \) acts on \( A \). Suppose that \( e_1 \sim e_2 \) are equivalent, i.e., there is continuous homomorphism \( \omega : C_1 \to C_2 \) such that \( \omega(a) = a, \forall a \in A \), and \( \psi_2 \omega = \psi_1 \). There is a continuous homomorphism \( \beta : P \to C_1 \) such that \( \psi_1 \beta = \tau \). Take \( \bar{\beta} = \omega \circ \beta \). Consider the following diagram.

\[
\begin{array}{ccc}
M & \text{t} & \text{P} \longrightarrow & G \\
\downarrow \alpha & \text{t} & \text{P} \longrightarrow & G \longrightarrow 1 \\
0 & A^{\alpha_1} & \longrightarrow & C_1 \longrightarrow G \longrightarrow 1 \\
\text{a} & \phi \rightarrow & C_1 \rightarrow G \longrightarrow 1 \\
\end{array}
\]

Since the back, the bottom, the front squares, and the middle triangle are commutative, so is the left triangle, i.e., \( \alpha_{\bar{\beta}} = \alpha_{\beta} \). This shows that \( \Theta \) is one to one. Consequently, \( \Theta \) is bijective.
Finally, we prove that $\Theta$ is a homomorphism. Suppose that $\Theta([\alpha_i]) = [e_i], i = 0, 1$. In another words, the following diagram is commutative.

\[
\begin{array}{c}
M \xrightarrow{i_0} P \xrightarrow{\tau} G \rightarrow 1 \\
\downarrow \alpha_i \downarrow \beta_i \\
e_i : 0 \xrightarrow{} A \xrightarrow{\sigma_i} E_i \xrightarrow{\psi_i} G \rightarrow 1
\end{array}
\]

for $i = 0, 1$. Hence, we have the commutative diagram

\[
\begin{array}{c}
M \xrightarrow{i_0} P \xrightarrow{\tau} G \rightarrow 1 \\
\downarrow \alpha_1 \times \alpha_2 \downarrow \beta_1 \times \beta_2 \\
e_1 \times e_2 : 0 \xrightarrow{} A \times A \xrightarrow{\sigma_1 \times \sigma_2} E_1 \times E_2 \xrightarrow{\psi_1 \times \psi_2} G \times G \rightarrow 1
\end{array}
\]

where $\Delta_G(g) = (g, g), \beta_1 \times \beta_2(p) = (\beta_1(p), \beta_2(p)), \alpha_1 \times \alpha_2(m) = (\alpha_1(m), \alpha_2(m)), \sigma_1 \times \sigma_2(a, b) = (\sigma_1(a), \sigma_2(b))$ and $\psi_1 \times \psi_2(e_1, e_2) = (\psi_1(e_1), \psi_2(e_2))$.

Then by definition of pushout and pullback extensions [Al], we get the following diagram

\[
\begin{array}{c}
M \xrightarrow{i_0} P \xrightarrow{\tau} G \rightarrow 1 \\
\downarrow \alpha_1 \times \alpha_2 \downarrow \beta_1 \times \beta_2 \\
\Delta_G \\
e_3 : A \xrightarrow{} E_3 \xrightarrow{\nabla_A} A \times A \xrightarrow{\sigma_1 \times \sigma_2} E_1 \times E_2 \xrightarrow{\psi_1 \times \psi_2} G \times G \rightarrow 1 \\
e_4 : A \xrightarrow{} E_4 \xrightarrow{} G \times G
\end{array}
\]

in which, extension $e_3 = \nabla_A(e_1 \times e_2) : 0 \xrightarrow{} A \xrightarrow{} E_3 \xrightarrow{} G \times G \rightarrow 1$ is the pushout extension of $e_1 \times e_2$, and extension $e_4 = (\nabla_A(e_1 \times e_2))\Delta_G : 0 \xrightarrow{} A \xrightarrow{} E_3 \xrightarrow{} G \rightarrow 1$ is the pullback extension of $\nabla_A(e_1 \times e_2)$.

Also, $\nabla_A$ is the codiagonal map, i.e., $\nabla_A(a, b) = ab$, and $\alpha_1 \alpha_2(m) = \alpha_1(m)\alpha_2(m)$. Obviously, the left and the right squares are commutative. Since $P$ is a projective topological group, then the existence of $\gamma : P \rightarrow A$ is guaranteed and commutes the right-up square. Therefore in the right cube of diagram (***) all the up, down, front, back and right squares are commutative. So $\gamma$ commutes the middle square. On the other hand, in the left cube of diagram (***) all the right, left, back, front and down squares are commutative. Thus, $\gamma$ commutes the left-up square of diagram (***) Therefore, the whole top of diagram (***) is commutative. This means that $\Theta([\alpha_1 \alpha_2]) = [e_4]$. On the other hand by
definition of Baer sum in the $\text{Opext}_s(G, A)$, $[\nabla_A(e_1 \times_2) \Delta_G] = [e_1] + [e_2]$. Consequently, $\Theta(\alpha_1, \alpha_2) = [e_1] + [e_2] = \Theta(\alpha_1) + \Theta(\alpha_2)$, i.e., $\Theta$ is an isomorphism and thereby $\Theta$ is an isomorphism. The proof is completed.

Note that Theorem 3.1 is similar to [5, Proposition 6] in a topological context.

**Corollary 3.2.** Let $P$ be a projective topological group, then $H^2(P, A) = 0$, for every abelian topological $P$-module $A$. In particular, for every Markov (Graev) free topological groups $F$, $H^2(F, A) = 0$, for every abelian topological $F$-module $A$.

**Proof.** It is clear that $(\Delta_P, \iota_0, \iota_G, Id_P)$ is a standard projective simplicial kernel of $P$. Thus, every continuous crossed homomorphism $\alpha : \Delta_P \to A$ is the zero map. Consequently, $H^2(P, A) = 0$.

Thus, we have discovered a general result of [4, (5.6)].

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