Cosmological solutions of $F(R, T)$ gravity model with $k$ -essence

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Abstract. Now exist several alternative cosmological models that describe observable properties of our universe. In particular, it is such models as $F(R)$ and $F(T)$ gravity. We consider properties of their generalization as $F(R, T)$ model of gravity with k-essence. We obtained some exact solutions of particular cases of scale factor $a$ for general form of the $F(R, T)$ functions with scalar field. These solutions describe the accelerated/decelerated periods of the universe.

1. Introduction
The discovery of the accelerated expansion of the universe requires the modernization of cosmology. As one of the way for the description, this cosmic acceleration is assumed to appear due to matter usually called Dark Energy with negative pressure (DE). But this DE has not yet been discovered. To explain this phenomenon, many theoretical models have been proposed, such as k-essence [1], [2], f-essence [3], [4], [5], [6], [7], g-essence [8], [9], [10] etc. In addition, Modified gravitational models are interesting like $F(R)$ gravity, $F(G)$ gravity, $F(T)$ gravity etc [11], [12], [13], [14], [15] [16]. For investigation of quantum and general gravity theories is interesting to investigate generalization of gravity as $F(R, T)$ gravity with k-essence, where $R$ is the scalar of curvature and $T$ is scalar torsion.

2. $F(R, T)$ gravity
The action of modified $F(R, T)$ gravity has the following form [11]:

$$S_{13} = \int \sqrt{-g} d^4x [F(R, T) + L_m],$$

(1)

where

$$R = \epsilon_1 g^{\mu\nu} R_{\mu\nu} + u,$$

$$T = \epsilon_2 S^\rho_{\mu\nu} T^\rho_{\mu\nu} + v.$$ (2)

Here $L_m$ is the matter Lagrangian, that for k-essence we can rewrite as $L_m = K = X - V(\varphi)$, $\epsilon_i = \pm 1$ is signature, $R$ is the curvature scalar, $T$ is the torsion scalar and $X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$, where $\varphi$- scalar field.
3. $f(R,T,X,\varphi)$ gravity

For action of modified gravity in the most general form we consider here $F(R,T)$ as $f(R,T,X,\varphi) = F(R,T) + C_X X + C(\varphi)$ gravity within the framework of Friedmann-Robertson-Walker (FRW) metric, with line element $ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)$ as

$$S = 2\pi^2 \int dt \, a^3 \left\{ f(R,T,X,\varphi) - \lambda_1 \left[ R - u + 6 \left( \frac{\dot{a}}{a} + \frac{\ddot{a}}{a^2} \right) \right] - \lambda_2 \left[ T - v + 6 \left( \frac{\dot{a}}{a^2} \right) \right] - \lambda_3 \left[ X - \frac{1}{2} \dot{\varphi}^2 \right] \right\}.$$  \hfill (3)

Here $u, v$ - some function of $a, \dot{a}$.

For this signature we have

\begin{align*}
R &= u - 6 \left( \dot{H} + 2H^2 \right), \hfill (4) \\
T &= v - 6H^2, \hfill (5) \\
X &= \frac{1}{2} \dot{\varphi}^2. \hfill (6)
\end{align*}

Hereafter denoted $f(R,T,X,\varphi)$ as $F$. By varying the action with respect to $R$, $T$ and $X$, one obtains

$$\lambda_1 = F_R, \quad \lambda_2 = F_T, \quad \lambda_3 = F_X.$$  \hfill (7)

Here $F_R$ is derivation of $F$ function by $R$, $F_T$ is derivation of $F$ function by $T$ and $F_X$ is derivation of $F$ function by $X$. After an integration by parts, the point-like Lagrangian have the following form

$$L = a^3 \left[ F - (R - u)F_R - (T - v)F_T \right] + 6a\dot{a}^2 \left[ F_R - F_T \right] + 6a^2 \dot{a} \left[ \dot{R}F_{RR} + \dot{T}F_{RT} + \dot{X}F_{RX} + \dot{\varphi}F_{R\varphi} \right] - a^3 F_X \left[ X - \frac{1}{2} \dot{\varphi}^2 \right].$$  \hfill (8)

4. The Noether Symmetries

For this Lagrangian Noether symmetry condition we will write as

$$XL = 0,$$  \hfill (9)

here

$$X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + \gamma \frac{\partial}{\partial T} + \delta \frac{\partial}{\partial X} + \epsilon \frac{\partial}{\partial \varphi} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{R}} + \dot{\gamma} \frac{\partial}{\partial \dot{T}} + \dot{\delta} \frac{\partial}{\partial \dot{X}} + \dot{\epsilon} \frac{\partial}{\partial \dot{\varphi}}.$$  \hfill (10)

The functions $\alpha, \beta, \gamma, \delta, \epsilon$ depend on the variables $a, R, T, X, \varphi$ and then

\begin{align*}
\dot{\alpha} &= \alpha_a \dot{a} + \alpha_R \dot{R} + \alpha_T \dot{T} + \alpha_X \dot{X} + \alpha_\varphi \dot{\varphi}, \hfill (11) \\
\dot{\beta} &= \beta_a \dot{a} + \beta_R \dot{R} + \beta_T \dot{T} + \beta_X \dot{X} + \beta_\varphi \dot{\varphi}, \hfill (12) \\
\dot{\gamma} &= \gamma_a \dot{a} + \gamma_R \dot{R} + \gamma_T \dot{T} + \gamma_X \dot{X} + \gamma_\varphi \dot{\varphi}, \hfill (13) \\
\dot{\delta} &= \delta_a \dot{a} + \delta_R \dot{R} + \delta_T \dot{T} + \delta_X \dot{X} + \delta_\varphi \dot{\varphi}, \hfill (14) \\
\dot{\epsilon} &= \epsilon_a \dot{a} + \epsilon_R \dot{R} + \epsilon_T \dot{T} + \epsilon_X \dot{X} + \epsilon_\varphi \dot{\varphi}. \hfill (15)
\end{align*}

Using this we can receive:
\[ 0 = \begin{aligned} & 6(\alpha F_R - F_T) + \beta a [F_{RR} - F_{TR}] + \gamma a [F_{RT} - F_{TT}] + \delta a [F_{RX} - F_{TX}] + \\
& + 2 a [F_{ST} - F_{SR}] + a a [F_{RS}] + \alpha_2 [F - F_T] + \beta_a a^2 F_{RR} + \gamma a a^2 F_{TR} + \delta a^2 F_{RX} + \epsilon a^2 F_{R\varphi} a^2 + \\
& + R^2 6a^2 F_{RR} + T^2 6a^2 F_{RT} + X^2 6a^2 F_{RX} + \dot{\varphi}^2 (\epsilon a^3 F_X + 6 a^2 F_{R\varphi} + \\
& + 3 a^2 F_X + \frac{1}{2} a^3 F_{XX} + \frac{1}{2} a^3 F_{XT} + \frac{1}{2} a^3 F_{XX} + \epsilon a^3 F_{\varphi} \varphi) + \\
& + \dot{a} R^2 6a (\beta a F_{RRR} + \gamma a F_{RTT} + \delta a F_{RXX} + \epsilon a F_{RX\varphi} + 2 a R [F - F_T] + \\
& + (\alpha a + \beta a + 2 a) F_{RR} + \gamma a F_{TR} + \delta a F_{RX} + \epsilon a F_{R\varphi} + \\
& + \dot{a} T^2 6a (\beta a F_{TRR} + \gamma a F_{RTT} + \delta a F_{TRX} + \epsilon a F_{TR\varphi} + 2 a T [F - F_T] + (\alpha a + \beta a + \gamma T a) F_T + \\
& + \beta \alpha a F_{RR} + \delta a F_{RX} + \epsilon a F_{R\varphi} + \dot{a} \dot{X} 6a (2 a X [F - F_T] + a a F_{RR} + \\
& + \alpha \gamma X F_{TT} + \beta a F_{RXX} + \gamma a F_{RTX} + \epsilon a F_{RXX} + (2 a + \alpha a + \beta a + \delta X a) F_{RX} + \\
& + \epsilon X a F_{R\varphi} + \dot{a} \dot{\varphi} 6a (2 a [F - F_T] + a a F_{RR} + \gamma a F_{RT} + \epsilon a F_{R\varphi} + a \frac{a^2}{6} F_X + \\
& + 2 \alpha a F_{R\varphi} + \beta a F_{R\varphi} + \gamma a F_{RT\varphi} + \delta a F_{RX\varphi} + \epsilon a F_{R\varphi\varphi} + \\
& + \dot{a} a F_{R\varphi} + \epsilon a F_{R\varphi\varphi} + \dot{R} T^2 6a (\alpha a F_{TR} + \alpha a F_{RR}) + \\
& + \dot{R} X 6a^2 (\alpha X F_{RR} + \alpha R F_{RX}) + 6 T \dot{X} 6a^2 (\alpha X F_{TR} + \alpha T F_{RX}) + \\
& + \dot{R} \dot{\varphi} 6a (\alpha X F_{RR} + \epsilon a F_{X} + 6 a F_{R\varphi}) + \dot{\varphi} (6 a\alpha^2 F_{R\varphi} + \\
& + \epsilon F_{X} a^3 F_X + 6 a^2 F_{RX} + \dot{X} \varphi (\epsilon a X^3 F_X + 6 a X a^2 F_{RX} + 6 a F_{R\varphi} X^2 + \\
& + 3 a^2 F_X + \beta a F_{RX} + \gamma a F_{XT} + \delta a F_{XX} + \epsilon a F_{X\varphi} X a^2). \end{aligned} \] (16)

From a Noether symmetry we have:

\[ \dot{a}^2 : \begin{aligned} & (\alpha + 2 a \alpha a) [F_R - F_T] + a [\beta a F_{RR} + \gamma a F_{TR}] a - \\
& - \beta a F_{TR} - \gamma a F_{TT} = 0, \end{aligned} \] (17)

\[ \dot{R}^2 : \begin{aligned} & 6 a R^2 F_{RR} = 0, \end{aligned} \] (18)

\[ \dot{T}^2 : \begin{aligned} & 6 a T^2 F_{RT} = 0, \end{aligned} \] (19)

\[ \dot{X}^2 : \begin{aligned} & 6 a X a^2 F_{RX} = 0, \end{aligned} \] (20)

\[ \dot{\varphi}^2 : \begin{aligned} & \epsilon a a + \frac{3 a}{2} = 0, \end{aligned} \] (21)

\[ \dot{a} R : \begin{aligned} & 2 a F_{RR} + (\beta a F_{RR} + \gamma a F_{RT}) R + a a a F_{RR} = 0, \end{aligned} \] (22)

\[ \dot{a} T : \begin{aligned} & 2 a F_{RT} + (\beta a F_{RR} + \gamma a F_{RT}) T + a a a F_{RT} = 0, \end{aligned} \] (23)

\[ \dot{a} X : \begin{aligned} & (\beta a F_{RR} + \gamma a F_{RT}) X = 0, \end{aligned} \] (24)

\[ \dot{a} \varphi : \begin{aligned} & 2 a [F_R - F_T] + (a \beta a F_{RR} + a \gamma F_{TR}) \varphi + a \frac{a^2}{6} F_X = 0, \end{aligned} \] (25)

\[ \dot{R} T : \begin{aligned} & \alpha R a^2 F_{TR} + \alpha T a^2 F_{RR} = 0, \end{aligned} \] (26)

\[ \dot{R} X : \begin{aligned} & \alpha X a^2 F_{RR} + \alpha R a^2 F_{RX} = 0, \end{aligned} \] (27)

\[ \dot{T} X : \begin{aligned} & \alpha X a^2 F_{TR} + \alpha T a^2 F_{RX} = 0, \end{aligned} \] (28)
\begin{align}
\dot{R}\phi & : \quad 6\alpha\varphi a^2 F_{RR} + \epsilon R a^3 F_X = 0, \\
\dot{T}\phi & : \quad 6\alpha\varphi a^2 F_{TR} + \epsilon T a^3 F_X = 0, \\
\dot{X}\phi & : \quad \epsilon_X a^3 F_X = 0,
\end{align}

(29) (30) (31)

\begin{align}
3\alpha \left[ F - (R - u) F_R - (T - v) F_T + \frac{1}{3} a \left( u a F_R + v a F_T \right) \right] + \nonumber \\
+ \alpha \left[ -(R - u) F_{RR} - (T - v) F_{TR} \right] + \gamma a \left[ -(R - u) F_{RT} - (T - v) F_{TT} \right] + \\
+ \dot{\alpha} a [u a F_R + v a F_T] + \dot{\varphi} a [u a F_R + v a F_T] + \\
+ \dot{R} a [u a F_R + v a F_T] + \dot{T} a [u a F_R + v a F_T] + \\
+ \dot{X} \alpha a [u a F_R + v a F_T] + \epsilon a [F_{\varphi}] - (a 3 F_X) X = 0.
\end{align}

(32)

5. The Noether Symmetries Solution

First variant for solution we have find for \( F_{RR} = F_{RT} = F_{RX} = 0 \), and solution of this is a linear equation \( F = s_1(\varphi) R + s_2(\varphi) T + s_3(\varphi) X + s_4(\varphi) \).

Second variant for solution we have find for \( \alpha_R = \alpha_T = \alpha_X = 0 \). In general the solution here will have this form:

\[
\frac{f(R, T, X, \varphi)}{F} = F'(C_1(\varphi) R + C_2(\varphi) T) + C_X X + C(\varphi).
\]

(33)

Than we can rewrite solution for action in most simple form as

\[
S = \int \sqrt{-g} d^4 x [\alpha R + \beta T + \alpha u + \beta v + L_m] = \int \sqrt{-g} d^4 x [\alpha R + \bar{L}_m],
\]

(34)

where \( \bar{L}_m = \alpha u + \beta v + L_m \).

Here \( L_m : \rho, p \) rewritten as \( \bar{L}_m : \bar{\rho}, \bar{p} \).

Then the action we rewrite as

\[
S = \int \sqrt{-g} d^4 x [\alpha R + \beta T + L_m].
\]

(35)

For Friedman - Robertson - Walker (FRW) metric we can write density \( \rho \) and pressure \( p \) as

\[
\rho = 3 H^2, \quad -p = 2 \dot{H} + 3 H^2.
\]

(36)

The solution we look far as \( a = a_0 t^n \), where \( n \) is constant. Than, since we solve the system for k-essences, we has

\[
\bar{\rho} = \bar{K} = \frac{(2 - 3 n) n}{t^2},
\]

(37)

\[
\bar{p} = 2 X \bar{K} - \bar{K} = \frac{3 n^2}{t^2}.
\]

Solving these equations we obtain

\[
\bar{K} = \frac{(3 n - 2) n}{C_1^2 X^{\frac{2(2 - 3 n)}{2 - 3 n}}},
\]

\[
X = C_1 \frac{2(2 - 3 n)}{t^{2(2 - 3 n) - 1}}.
\]

(38)
6. Conclusions
We considered the generalization of $F[R, T]$ gravity with k-essence and found the exact analytical solution for this model.

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8. References
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