Vortex Structures for an \( SO(5) \) Model of High-\( T_C \) Superconductivity and Antiferromagnetism

Stan Alama ∗ Lia Bronsard †
Tiziana Giorgi ‡

McMaster Univ., Dept. of Math. & Stat., Hamilton Ont., L8S 4K1 Canada

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Abstract

We study the structure of symmetric vortices in a Ginzburg–Landau model based on S. C. Zhang’s \( SO(5) \) theory of high temperature superconductivity and antiferromagnetism. We consider both a full Ginzburg–Landau theory (with Ginzburg–Landau scaling parameter \( \kappa < \infty \)) and a \( \kappa \to \infty \) limiting model. In all cases we find that the usual superconducting vortices (with normal phase in the central core region) become unstable (not energy minimizing) when the chemical potential crosses a threshold level, giving rise to a new vortex profile with antiferromagnetic ordering in the core region. We show that this phase transition in the cores is due to a bifurcation from a simple eigenvalue of the linearized equations. In the limiting large \( \kappa \) model we prove that the antiferromagnetic core solutions are always nondegenerate local energy minimizers and prove an exact multiplicity result for physically relevant solutions.

∗Supported by an NSERC (Canada) Research grant. e-mail: alama@mcmaster.ca
†Supported by an NSERC (Canada) Research grant. e-mail: bronsard@math.mcmaster.ca
‡e-mail: giorgi@math.mcmaster.ca
1 Introduction

In 1986 Bednorz and Müller announced their discovery of high critical-temperature ($T_C$) superconductors, and promptly received the 1987 Nobel Prize for their efforts. This discovery has led to a new flowering of superconductivity theory, since the high temperature phenomenon cannot be explained by the accepted models for conventional superconductors. In particular, many physicists have come to the conclusion that the microscopic BCS theory does not correctly describe the interactions which produce superconductivity at high temperatures. At the present time, there are several competing theories which attempt to explain these interactions. One theory is based on the observation that high-$T_C$ compounds also exhibit an ordered phase called antiferromagnetism when physical parameters (such as temperature, chemical potential or “doping”, and magnetic field) are varied. Antiferromagnetism (abbreviated AF) is an insulating phase of matter in which electron spins orient themselves in the direction opposite to their nearest neighbors. The coexistence of these two phases (AF and SC) in the phase diagram of the high-$T_C$ compounds has led to the speculation that high temperature superconductivity and antiferromagnetism could be explained by the same type of interaction.

Following in this direction, Shou-Cheng Zhang [Z 97] proposed a quantum statistical mechanics model which incorporates AF and high temperature superconductivity (SC). The model is based on a broken $SO(5)$ symmetry tying the complex order parameter of superconductivity to the Néel vector which describes antiferromagnetism. The interactions between the SC and AF order parameters in this model should have some effect on the familiar constructions from conventional superconductivity theory. In a recent paper Arovas, Berlinsky, Kallin, & Zhang [ABKZ 97] introduced a phenomenological Ginzburg–Landau model based on the $SO(5)$ theory, and studied isolated vortex solutions in the plane. Recall that in a conventional superconductor the magnetic field is expelled from the superconducting bulk, and only penetrates in thin tubes (the vortices) where superconductivity is supressed. Hence, in the conventional theory the magnetic field is constrained to a small core of normal (non-SC) phase. Using a simplified model Arovas et al predicted a new kind of vortex structure in the $SO(5)$ model: vortices with antiferromagnetic cores, which should be observed for small values of the chemical potential. They also predicted that (as the chemical potential is gradually decreased) the transition from normal core to AF core vortices occurs in a discontinuous fashion. In other words, AF cores should be produced via a first order phase transition.

In this paper we rigorously analyse vortex cores in the full $SO(5)$ Ginzburg–Landau model and in an “extreme type II” limiting model (also called “high kappa model”) to understand
the nature of the transition between normal core and AF core solutions. For both models we show that the vortex solutions with normal cores become unstable (within the class of radial functions– see (1.1) below,) and vortices with AF cores are produced by bifurcation from the normal core solutions. In the extreme type II model we prove that the transition is continuous (i.e., second order), contrary to the prediction of [ABKZ 97] (see Figure 1.) Furthermore, we show that for each value of the chemical potential there exists a unique stable vortex profile (see Theorem 4.3.)

The full $SO(5)$ Ginzburg–Landau free energy is written in terms of the SC order parameter $\psi \in \mathbb{C}$ and the AF order parameter (Néel vector) $\vec{m} = (m_1, m_2, m_3)$. In non-dimensional form, the free energy is:

$$
\mathcal{F} = \frac{1}{2} \int_{\Omega} \left\{ \frac{\kappa^2}{2} (1 - |\psi|^2 - |\vec{m}|^2)^2 + g\kappa^2|\vec{m}|^2 + |(\frac{1}{i} \nabla - \vec{A})\psi|^2 + |\nabla \vec{m}|^2 + |\nabla \times \vec{A}|^2 \right\} dx.
$$

(We refer to the paper by Alama, Berlinsky, Bronsard & Giorgi [ABBG 98] where the free energy is written in dimensional form.) In these variables, the penetration depth $\lambda = 1$, and the Ginzburg–Landau parameter $\kappa$ is the reciprocal of the correlation length $\xi$. The parameter $g$ measures the strength of doping (chemical potential) of the material. It is this term which breaks the $SO(5)$ symmetry of the potential term. We take $g > 0$: with this assumption superconductivity is preferred in the bulk of the sample.

To study isolated vortex solutions in the plane $\Omega = \mathbb{R}^2$ we seek critical points of $\mathcal{F}$ of the form

$$
\psi = f(r)e^{id\theta}, \quad \vec{A} = S(r) \left( \frac{-\xi}{r^2}, \frac{x}{r^2} \right), \quad \vec{m} = m(r)\vec{m}_0
$$

(1.1)

where $\vec{m}_0$ a fixed unit vector, and $d \in \mathbb{Z} \setminus \{0\}$ represents the degree of the vortex. As for conventional SC vortices, we expect that only the solutions with $d = \pm 1$ will be energy minimizers (see Gustafson [Gu 98], Ovchinnikov & Sigal [OS 97].) Critical points of $\mathcal{F}$ with this ansatz solve the system of equations

$$(GL)_{\kappa, g}
$$

$$
\left\{
\begin{array}{l}
-f'' - \frac{1}{r} f' + \frac{(d-S)^2}{r^2} f = \kappa^2(1 - f^2 - m^2) f, \\
-S'' + \frac{1}{r} S' = (d - S)f^2, \\
-m'' - \frac{1}{r} m' + \kappa^2 gm = \kappa^2(1 - f^2 - m^2) m,
\end{array}
\right.
$$
with \( f(r) \geq 0, f(r), S(r) \to 0 \) as \( r \to 0 \), and \( f(r) \to 1; S(r) \to d \) as \( r \to \infty \); and \( m'(0) = 0, m(r) \to 0 \) as \( r \to \infty \).

In addition, we study the following “extreme Type II” model,

\[
(GL)_{\infty,g} \begin{cases} 
-f'' - \frac{1}{r} f' + \frac{d^2}{r^2} f = (1 - f^2 - m^2) f, \\
-m'' - \frac{1}{r} m' + gm = (1 - f^2 - m^2) m.
\end{cases}
\]

The system \((GL)_{\infty,g}\) is obtained in the limit \( \kappa \to \infty \) after rescaling solutions to \((GL)_{\kappa,g}\) by the correlation length \( \xi = 1/\kappa \). For high \( T_C \) superconductors \( \kappa \) is very large, and hence the vortex cores are very narrow compared to the penetration depth, which measures the length scale for magnetic fields. By rescaling we capture the structure of the vortex cores and decouple the magnetic field, which lives on a much larger length scale. Indeed, the calculations which led Arovas et al [ABKZ 97] to predict AF vortex cores are mostly based on \((GL)_{\infty,g}\) and its associated free energy functional.

We observe that when the AF order parameter \( m = 0 \) the two systems \((GL)_{\kappa,g}\) and \((GL)_{\infty,g}\) reduce to the familiar Ginzburg–Landau vortex equations, well studied in the mathematical literature (see Plohr [P 80], Berger and Chen [BC 89], Chen, Elliot, & Qi [CEQ 94], Brezis, Merle, & Rivière [BMR 94], Ovchinnikov & Sigal [OS 97], for example.) We call these the normal core solutions. In a previous paper [ABG 99] we have proven that when \( \kappa^2 \geq 2d^2 \) there is a unique normal core solution, which is a non-degenerate minimizer of the appropriate free energy functional. This characterization will be essential for our analysis of the normal-to-AF core transition.

We now discuss our results. We define a reduced energy functional defined for functions satisfying the symmetric vortex ansatz (1.1), as well as appropriate function spaces in which that functional is smooth. We find that for every \( \kappa \) (including the extreme type II model) there exists \( g^*_\kappa > 0 \) such that the conventional normal core vortex solutions of \((GL)_{\kappa,g}\) (and \((GL)_{\infty,g}\)) are strict local minimizers of the reduced energy for \( g > g^*_\kappa \), but are not local minimizers when \( 0 < g < g^*_\kappa \). In particular, energy minimizers must have AF order in the vortex core for \( 0 < g < g^*_\kappa \). When \( \kappa^2 \geq 2d^2 \) we show that the AF core solutions bifurcate from the normal core solution at a simple eigenvalue of the linearized system \((GL)_{\kappa,g}\) (or \((GL)_{\infty,g}\).) The bifurcating solutions remain bounded for \( g > 0 \) and lose compactness as \( g \to 0^+ \) with \( f \to 0 \) and \( m \to 1 \).

For the limiting problem \((GL)_{\infty,g}\) we obtain a complete picture of the phase transition to AF cores. This is because all AF core vortex solutions are non-degenerate minima of the
reduced energy. (See Theorem 3.1.) Stable (locally minimizing) solutions with \( m(r) > 0 \) bifurcate from \( m = 0 \) at \( g = g^*_\infty \) to values \( g < g^*_\infty \). Moreover, for each \( g < g^*_\infty \) there exists exactly one solution with \( m(r) > 0 \).

In the language of physics, our results indicate a second order (or continuous) phase transition between normal and AF vortex cores in \((GL)_{\infty,g}\). This information concerning the nature of the transition was not derived in the paper by Arovas et al. [ABKZ 97], and hence the result is new to the physics literature as well. For \((GL)_{\kappa,g}\) Alama, Berlinsky, Bronsard, & Giorgi [ABBG 98] present numerical simulations (based on gradient flow for a finite elements approximation of the free energy) which suggest that the transition is also second order for \( \kappa < \infty \). (See Figure 1.) However we were not able to extend the arguments used in studying the bifurcation curves of \((GL)_{\infty,g}\) to the more complicated system \((GL)_{\kappa,g}\). See Remark 4.3 for further discussion.

Here is an outline of the content of the paper. In the second section we introduce the reduced energy and function spaces, we treat briefly the questions of existence, regularity, and decay of solutions, and we present properties of physically relevant (“admissible”) solutions. We also prove the monotonicity of the solution profiles \((f, S, m)\) under the hypothesis that the solution is a local reduced energy minimizer. This result (Theorem 2.9) is done in the spirit of the weak maximum principle (see Theorem 8.1 of [GT 83].)

Section 3 contains the proof that all solutions of \((GL)_{\infty,g}\) with \( m > 0 \) represent non-degenerate local minima of the reduced energy. This result is the key to understanding the bifurcation diagram for \((GL)_{\infty,g}\). The bifurcation analysis itself occupies Section 4.

The last two sections contain the a priori estimates used in rigorously passing to the limit \( \kappa \to \infty \) and in studying the global behavior of bifurcating continua. In both cases, we require estimates on solutions which are energy-independent. For the limit \( \kappa \to \infty \) this is because the reduced energy of minimizers behaves like \( \log \kappa \), and in studying global bifurcation we require estimates valid for any physically relevant solution (whether it is energy minimizing or not.) The starting point for these estimates is a Pohozaev type identity (see Proposition 5.4.) The proof of convergence to \((GL)_{\infty,g}\) as \( \kappa \to \infty \) is presented in Section 5; other a priori estimates are derived in Section 6.

We wish to thank our colleague John Berlinsky for introducing us to the \( SO(5) \) model, and for his great patience in explaining physics to we mathematicians. We are also obliged to the Brockhouse Institute for Materials Research for supporting a workshop which brought together physicists and mathematicians to discuss issues in superconductivity.
Figure 1: A numerical bifurcation curve, $m(0)$ vs. $g$ for values $\kappa = 20, 40, 120$ and $d = 1$, indicates a second-order transition to AF cores in model (GL)$_{\kappa,g}$. For the high-kappa model (GL)$_{\infty,g}$ we prove that the above image correctly depicts the solution set (see Theorem 4.5). Numerical simulations indicate that the bifurcation occurs at $g^*_\infty \simeq 0.2545$ [ABBG 98].
2 Solutions of the Ginzburg–Landau system

2.1 Preliminaries

Here and in the rest of the paper, we fix the value of \( d \in \mathbb{Z} \setminus \{0\} \). In this section \( \kappa \in \mathbb{R} \) is fixed. Note that without loss of generality we may take \( d > 0 \), since the free energy and the corresponding Euler–Lagrange equations are invariant under the transformation \((\psi, A, m) \rightarrow (\bar{\psi}, -A, m)\).

Following our previous work [ABG 99] on symmetric vortices, we define a function space for which the free energy will be a smooth functional. First we fix some notation: we denote by \( L^p_r \), \( H \) the Lebesgue and Sobolev spaces (respectively) of radially symmetric functions in \( \mathbb{R}^2 \), that is,

\[
L^p_r = \{u(r) : \int_0^{\infty} |u(r)|^p r \, dr < \infty\}, \quad (p < \infty),
\]

\[
H := H^1_r = \{u(r) : \int_0^{\infty} [(u'(r))^2 + (u(r))^2] r \, dr < \infty\},
\]

and analogously for \( L^\infty_r \). We also denote \( \int u(r) r \, dr = \int_0^{\infty} u(r) r \, dr \).

Define the Hilbert space

\[
X = \{u \in H : \int \frac{u^2}{r^2} r \, dr < \infty\},
\]

with norm

\[
\|u\|_X = \sqrt{\int \left[ (u'(r))^2 + u^2 + \frac{u^2}{r^2} \right] r \, dr}.
\]

The following density and imbedding properties for the space \( X \) are proven in [ABG 99]:

**Lemma 2.1**

i. \( X \) is compactly embedded in \( L^p_r \) for each \( p \in (2, \infty) \).

ii. \( X \) is compactly embedded in \( L^2_{r,loc} \).

iii. For every \( u \in X \),

\[
\|u\|_{\infty}^2 \leq \int \left[ (u')^2 + \frac{u^2}{r^2} \right] r \, dr.
\]
In particular, $X$ embeds continuously into $L^\infty_r$.

iv. $C_0^\infty((0, \infty))$ is dense in $X$.

We note that the compactness of the embedding of $H$ into $L^p_{r,loc}$ ($1 \leq p < \infty$) is just the classical Rellich-Kondrachov Theorem, and the compact embedding of $H$ into $L^p_r$ for $2 < p < \infty$ is due to Strauss [St 77].

2.2 Energy

We now define our energy functionals, using the space $X$ defined above. To keep the appropriate boundary condition at infinity we fix any function $\eta \in C_0^\infty((0, \infty))$ with $\eta(r) = 0$ for $0 \leq r \leq 1$, $\eta(r) = 1$ for all $r \geq 2$, and $0 < \eta < 1$. Then set $f_0 = \eta$, $S_0 = d\eta$, and seek solutions $(f, S, m)$ of (GL)$_{\kappa, g}$ with $f = f_0 + u$, $S = S_0 + rv$, $u, v \in X$, $m \in H$. (Later we will see that this choice poses no restriction on solutions which are physically relevant.) We denote by $Y_0 = X \times X \times H$, and by $Y$ the affine space

$$Y = \{(f, S, m) : f = f_0 + u, \ S = S_0 + rv, \ u, v \in X, \ m \in H \} = Y_0 + (f_0, S_0, 0).$$

For $(f, S, m) \in Y$ we define

$$(2.1) \quad \mathcal{E}_{\kappa, g}(f, S, m) =$$

$$\frac{1}{2} \int \left\{ \left( f' \right)^2 + \left( \frac{S'}{r} \right)^2 + (m')^2 + \kappa^2 gm^2 + \frac{(d - S)^2 f^2}{r^2} + \frac{\kappa^2}{2} (1 - f^2 - m^2)^2 \right\} r \, dr$$

and the functional $I_{k, g} : Y_0 \to \mathbb{R}$ by

$$I_{k, g}(u, v, m) = \mathcal{E}_{\kappa, g}(f_0 + u, S_0 + rv, m) - \mathcal{E}_{\kappa, g}(f_0, S_0, 0).$$

Throughout the paper we will take advantage of these two representations of our spaces and energies, and use the formulation which is more convenient at the given moment.

Defining an energy functional for the limiting problem (GL)$_{\infty, g}$ is trickier, since the naive choice for the energy (namely (2.1) with $S = 0$ and $\kappa = 1$) would be infinite for all $f$ satisfying the desired boundary condition at $r = \infty$. Our solution is to subtract off the offending term from the energy density. Let $f_\infty$ be the (unique) positive solution to the high kappa vortex equation,

$$-f''_\infty - \frac{1}{r} f'_\infty + \frac{d^2}{r^2} f_\infty = (1 - f^2_\infty) f_\infty$$
with $\tilde{f}_\infty(0) = 0$, $\tilde{f}_\infty(r) \to 1$ as $r \to \infty$. The uniqueness of $\tilde{f}_\infty$ was established by Chen, Elliot & Qi [CEQ 94]. The estimates in [CEQ 94] ensure that $\tilde{f}_\infty$ is smooth, $\tilde{f}_\infty(r) \sim r^d$ near $r = 0$, and $(1 - \tilde{f}_\infty) \in H$.

We define the appropriate spaces for the free energy $\mathcal{E}_{\infty, g}$ based on $\tilde{f}_\infty$: let $Z_0 = X \times H$ and

$$Z = \{(f, m) : f = \tilde{f}_\infty + u, \ u \in X, \ m \in H\} = Z_0 + (\tilde{f}_\infty, 0).$$

Then the energy for the high kappa model is:

$$E_{\infty, g}(f, m) =$$

$$\frac{1}{2} \int \left\{ (f')^2 + (m')^2 + gm^2 + \frac{d^2}{r^2} [f^2 - \tilde{f}_\infty^2] + \frac{1}{2} (1 - f^2 - m^2)^2 \right\} r \, dr$$

If we write $f = \tilde{f}_\infty + u$, we reduce to the equivalent functional

$$I_{\infty, g}(u, m) = \mathcal{E}_{\infty, g}(\tilde{f}_\infty + u, m) - \mathcal{E}_{\infty, g}(\tilde{f}_\infty, 0)$$

$$= \frac{1}{2} \int \left\{ (u')^2 + \frac{d^2}{r^2} u^2 + (m')^2 + gm^2$$

$$+ \frac{1}{2} (1 - (\tilde{f}_\infty + u)^2 - m^2)^2 - \frac{1}{2} (1 - \tilde{f}_\infty^2)^2 + 2 (1 - \tilde{f}_\infty^2) \tilde{f}_\infty u \right\} r \, dr.$$ 

By a direct expansion of the energy in powers of $u, v, m$ we see that $I_{k, g} : Y_0 \to IR$ and $I_{\infty, g} : Z_0 \to IR$ are smooth ($C^\infty$) functionals.

When $g > 0$ is fixed, we obtain solutions of $(GL)_{k, g}$ and $(GL)_{\infty, g}$ as global minimizers for $\mathcal{E}_{k, g}$ and $\mathcal{E}_{\infty, g}$ (in the appropriate spaces, $Y$ and $Z$):

**Theorem 2.2** For every fixed $g > 0$, $k \in IR$, $d \in Z - 0$, the functional $I_{k, g}$ admits a minimizer $(u, v, m) \in X \times X \times H$. Moreover, $(f, S, m) = (f_0 + u, S_0 + rv, m)$ is a smooth solution of the system $(GL)_{k, g}$.

**Theorem 2.3** For every fixed $g > 0$ and $d \in Z - 0$, the functional $I_{\infty, g}$ admits a minimizer $(u, m) \in X \times H$. Moreover, $(f, m) = (\tilde{f}_\infty + u, m)$ is a smooth solution of the system $(GL)_{\infty, g}$.

The proofs of Theorems 2.2 and Theorem 2.3 are straightforward but technical, and are deferred to Section 6.
2.3 Admissible solutions

As in [ABG 99], we define a natural class of solutions to the system $(GL)_{\kappa,g}$:

**Definition 2.4** We call $(f^*, S^*, m^*)$ an admissible solution to $(GL)_{\kappa,g}$ if:

i. $(GL)_{\kappa,g}$ holds for all $r \in (0, \infty)$;

ii. $E_{\kappa,g}(f^*, S^*, m^*) < \infty$;

iii. $f^*(r) \geq 0$ and $m^*(r) \geq 0$ for all $r \geq 0$;

iv. $S^*(0) = 0$ and $m^*_0(0) = 0$.

A solution $(f^*, m^*)$ of $(GL)_{\infty,g}$ is called admissible if the above conditions hold, where we replace $\kappa$ by $\infty$ and disregard $S^*$.

A solution to $(GL)_{\kappa,g}$ or $(GL)_{\infty,g}$ with $m^* \equiv 0$ is called a normal core solution.

The admissible solutions are those which are physically relevant in the context of the vortex core problem described in the introduction. We note that the normal core solutions are unique for $\kappa^2 \geq 2d^2$: see [ABG 99] for the case $2d^2 \leq \kappa^2 < \infty$ and [CEQ 94] for $\kappa = \infty$.

We now present some properties of admissible solutions. In the following, we will assume that $\kappa \in \mathbb{R} \cup \{\infty\}$, with the understanding that $S^*_s = 0$ when $\kappa = \infty$.

**Proposition 2.5** Let $(f^*_s, S^*_s, m^*_s)$ be any admissible solution of $(GL)_{\kappa,g}$. Then:

i. For all $r \in (0, \infty)$ it holds $0 < f^*(r) < 1$, $0 \leq m^*(r) < 1$, $f^2_s(r) + m^2_s(r) < 1$, and, if $\kappa \neq \infty$, $0 < S_s(r) < d$.

ii. Either $m^*_s(r) > 0$ for all $r \in [0, \infty)$, or $m^*_s$ vanishes identically.

iii. $f^*_s(r) \to 1$, $m^*_s(r) \to 0$, and, if $\kappa \neq \infty$, $S_s(r) \to d$ as $r \to \infty$. Moreover, there exist constants $\sigma, C_0 > 0$ such that for $\kappa \neq \infty$

\[
0 < 1 - f^*_s(r) \leq C_0 e^{-\sigma r}, \quad 0 < d - S^*_s(r) \leq C_0 e^{-\sigma r}, \quad 0 \leq m^*_s(r) \leq C_0 e^{-\sigma r},
\]

and for $\kappa = \infty$

\[
0 < 1 - f^*_s(r) \leq \frac{d^2}{2r^2} + \frac{8d^2 + d^4}{8r^4} + \mathcal{O}(r^{-6}), \quad 0 \leq m^*_s(r) \leq C_0 e^{-\sigma r},
\]

for all $r > 0$. 

iv. $f_*(r) \sim r^d$, $S_*(r) \sim r^2$ for $r \sim 0$.

v. If $\kappa \neq \infty$, $S'_*(r) > 0$ for all $r > 0$.

**Proof:** The proof is very similar to that of Proposition 2.3 of [ABG 99], so we provide only a sketch. From the finiteness of the free energy we immediately conclude that $m_* \in H$, and hence $m_* \in L_p^p$, for any $p \in [2, \infty]$, and $m_*(r) \to 0$ as $r \to \infty$. Since $f_* \geq 0$, finiteness of energy again implies $1 - f_* \in L^2$ (see (5.10) for details,) and therefore the bound $0 < f_*(r) < 1$ follows exactly as in Proposition 2.3 of [ABG 99]. When $\kappa < \infty$, the bound $0 < S_*(r) < d$ and the proof that $S'_*(r) > 0$ are also unchanged from [ABG 99]. To show $z = f_*^2 + m_*^2 < d$ and the proof that $z$ is a simple consequence of the strong maximum principle.

The exponential decay in (iii) for $m_*$ is consequence of Proposition 7.4 in Jaffe & Taubes [JT 80], and so are the ones for $f_*$ and $S_*$ if $\kappa \neq \infty$. If $\kappa = \infty$, the polynomial decay of $f_*$ can be proven as in Lemma 3.3 in [CEQ 94], since $m_*(r) \leq C(R) r^6$ for any $r > R$ with $C(R)$ a big enough constant.

The behavior at zero given in (iv) can be proven as in [P 80].

We now connect admissible solutions to our space $X$.

**Proposition 2.6** Let $(f_0, S_0, m_0)$, $(f_1, S_1, m_1)$ be admissible solutions to $(GL)_{\kappa,g}$. Then $(f_1 - f_0) \in X$, $[(S_1 - S_0)/r] \in X$, and $m_1, m_0 \in H$.

**Proof:** As already remarked, condition (ii) of the definition of admissible solutions implies $m_1, m_0 \in H$, and $m_1, m_0 \in L_p^p$ for any $p \in [2, \infty]$. Then, the rest of the proposition for $\kappa \neq \infty$ is proven as in Proposition 2.4 of [ABG 99]. When $\kappa = \infty$, we note that $(1 - f_i) \in H$ for $i = 1, 2$, and that by (iv) of Proposition 2.5 we have $(f_1 - f_2)^2 \leq c r^{2d}$ for $r \sim 0$ and again by finiteness of energy we conclude our statement.

**Remark 2.7** In light of Proposition 2.6 we observe that the choice of $f_0, S_0$ in the definition of the space $Y$ may be replaced by any fixed admissible solution of the equations $(GL)_{\kappa,g}$. It will be convenient to choose instead the “basepoint” $(\tilde{f}_\kappa, \tilde{S}_\kappa, 0)$ to be a “normal core” solution to $(GL)_{\kappa,g}$. In other words, an equivalent definition of the space $Y$ is:

$$Y = \{(f, S, m) : f = \tilde{f}_\kappa + u, \ S = \tilde{S}_\kappa + rv, \ u, v \in X, \ m \in H\}$$

We recall that the normal core solutions are uniquely determined for $\kappa^2 \geq 2d^2$. When $\kappa^2 < 2d^2$ we fix any one.
Remark 2.8 Proposition 2.6 also implies that the admissible solutions are exactly those which arise from minimization problems for $\mathcal{E}_{\kappa,g}$ and $\mathcal{E}_{\infty,g}$ in the space $Y$. In particular, as an immediate corollary we obtain the following statement:

$(f_*, S_*, m_*)$ is an admissible solution to $(GL)_{\kappa,g}$ if and only if $f_* \geq 0, m_* \geq 0, (f_*, S_*, m_*) \in Y$ and $\mathcal{E}'_{\kappa,g}(f_*, S_*, m_*)[u, v, w] = 0$ for all $u, v \in X$, and $w \in H$.

An analogous statement holds for the problem $(GL)_{\infty,g}$.

With this choice of representation for our spaces $Y$, $Z$, we now look at the second variation of energy with respect to the variables $(u, v, w) \in X \times X \times H$. We define

$$
\mathcal{E}_{\kappa,g}''(f_*, S_*, m_*)[u, v, w] = \frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{E}_{\kappa,g}(f_* + tu, S_* + tv, m_* + tw)
$$

(2.5) 

$$
= \int \left\{ (u')^2 + (w')^2 + \frac{(d - S_*)^2}{r^2}u^2 + \kappa^2 gw^2 + (v')^2 
\quad + \frac{v^2}{r^2} - \frac{4(d - S_*)}{r}f_*uv + f_*^2 v^2 
\quad - \kappa^2(1 - f_*^2 - m_*^2)(u^2 + w^2) + 2\kappa^2(f_*u + m_*w)^2 \right\} r \, dr.
$$

(2.6) 

$$
\mathcal{E}_{\infty,g}''(f_*, m_*)[u, w] = \frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{E}_{\infty,g}(f_* + tu, m_* + tw)
$$

$$
= \int \left\{ (u')^2 + (w')^2 + \frac{d^2}{r^2}u^2 + gw^2 
\quad - (1 - f_*^2 - m_*^2)(u^2 + w^2) + 2(f_*u + m_*w)^2 \right\} r \, dr.
$$

Note that if we write $f_* = \tilde{f}_* + u_*, S_* = \tilde{S}_* + rv_*$, then

$$
\mathcal{E}_{\kappa,g}''(f_*, S_*, m_*)[u, v, w] = D^2I_{\kappa,g}(u_*, v_*, m_*)[u, v, w],
$$

the usual second Fréchet derivative.

For admissible solutions which are stable, in the sense that the second variation of energy about the solution is a non-negative quadratic form, we have monotonicity of the profiles $f(r), m(r)$.
Theorem 2.9 Suppose \((f, S, m)\) is an admissible solution of \((GL)_{\kappa, g}\), and \(E''_{\kappa, g}(f, S, m) \geq 0\) as a quadratic form acting on \(Y_0\). Then \(f'(r) > 0\) and (if it is not identically zero) \(m'(r) < 0\) for all \(r > 0\).

For the problem \((GL)_{\infty, g}\) the same theorem holds, with exactly the same proof. We will see later that all admissible solutions of \((GL)_{\infty, g}\) with \(m(r) > 0\) are stable (in the above sense), and hence we will obtain the stronger result announced in Corollary 3.2.

Proof: Let \(\tilde{u}(r) = f'(r), \tilde{w}(r) = m'(r)\). Then, differentiating the first and third equations of \((GL)_{\kappa, g}\),

\[-u'' - \frac{1}{r}u' + \frac{(d - S)^2}{r^2}u - \kappa^2(1 - 3f^2 - m^2)u + 2\kappa^2mf \tilde{w} = -\frac{1}{r^2}u' + 2d - S\frac{1}{r}f \left[S'\frac{1}{r} + \frac{d - S}{r^2}\right],\]

\[-w'' - \frac{1}{r}w' + g\kappa^2\tilde{w} - \kappa^2(1 - f^2 - 3m^2)\tilde{w} + 2\kappa^2fm\tilde{u} = -\frac{1}{r^2}\tilde{w}.\]

Suppose there exist intervals \((a, b), (c, d)\) such that

\[\tilde{u}(r) < 0 \quad r \in (a, b), \quad \tilde{u}(a) = 0 = \tilde{u}(b); \text{ or}\]

\[\tilde{w}(r) > 0 \quad r \in (c, d), \quad \tilde{w}(c) = 0 = \tilde{w}(d).\]

Note that by the properties (i), (iii) and (iv) of admissible solutions in Proposition 2.5, \(a \neq 0, b, d < +\infty\). Let

\[u(r) = \begin{cases} \tilde{u}(r), & \text{if } r \in (a, b), \\ 0, & \text{otherwise}, \end{cases} \quad w(r) = \begin{cases} \tilde{w}(r), & \text{if } r \in (c, d), \\ 0, & \text{otherwise}, \end{cases}\]

Then \(u \leq 0, w \geq 0\), and an integration by parts shows that

\[\int (u')^2 r \, dr = -\int_a^b \frac{1}{r} \tilde{u}(r)'r \, dr,\]

and similarly for \(w\). If we now use \((u, 0, w)\) as a test function in the second variation of energy and recall from Proposition 2.3 that \(S(r) < d, S'(r) > 0\) for all \(r > 0\), we obtain

\[0 \leq E''_{\kappa, g}(f, S, m)[u, 0, w]\]
\[ \int \left[ -\frac{1}{r^2} u^2 + 2 \frac{d - S}{r} f \left( \frac{S'}{r} + \frac{d - S}{r^2} \right) u - \frac{1}{r^2} w^2 \right] r \, dr < 0, \]

unless \( u, w \equiv 0 \). Consequently, \( \tilde{u} = f' \geq 0 \) and \( \tilde{w} = m' \leq 0 \). Strict inequality follows from the Strong Maximum Principle, since \( \tilde{u}, \tilde{w} \) satisfy equations of the form

\[-\Delta_r \tilde{u} + c_1(r) \tilde{u} \geq -2\kappa^2 m f \tilde{w} \geq 0, \]
\[-\Delta_r \tilde{w} + c_2(r) \tilde{w} = -2\kappa^2 m f \tilde{u} \leq 0. \]

\[\diamond\]

3 Nondegeneracy of solutions of \((GL)_{\infty,g}\)

**Theorem 3.1** For any admissible solution \((f_*, m_*)\) of \((GL)_{\infty,g}\) with \(m_* > 0\) there exists a constant \(\sigma_* > 0\) such that

\[ \mathcal{E}''_{\infty,g}(f_*, m_*)[u, w] \geq \sigma_*(\|u\|_X^2 + \|w\|_H^2), \]

for all \(u \in X, w \in H\).

**Corollary 3.2** For any admissible solution \((f_*, m_*)\) of \((GL)_{\infty,g}\), \(f'_*(r) > 0\) for all \(r \geq 0\). If \(m_*\) is not identically zero, then \(m'_*(r) < 0\) for all \(r > 0\).

The corollary follows from Theorem 3.1 and the argument of Theorem 2.9 when \(m_* > 0\). Note that when \(m_* \equiv 0\) the system \((GL)_{\infty,g}\) reduces to the single equation studied in [CEQ 94] and the strict monotonicity of \(f_*\) is part of their result. Also, in the case that \(m_* \equiv 0\) the Theorem reduces to \(\mathcal{E}''_{\infty,g}(f_*)[u] \geq \sigma_*\|u\|_X^2\).

The key step in proving Theorem 3.1 is the following identity:

**Theorem 3.3** For any admissible solution \((f_*, m_*)\) of \((GL)_{\infty,g}\) with \(m_* > 0\), and any \(u \in X, w \in H\),

\[ (3.1) \quad \mathcal{E}''_{\infty,g}(f_*, m_*)[u, w] = \int \left\{ f_*^2 \left[ \left( \frac{u}{f_*} \right)' \right]^2 + m_*^2 \left[ \left( \frac{w}{m_*} \right)' \right]^2 + 4(f_* u + m_* w)^2 \right\} r \, dr \]
Proof of Theorem 3.3: First we prove the identity for \( u \in C^\infty_0((0, \infty)) \) and \( w \in C^\infty_0([0, \infty)) \). First, note that using \( f_s > 0 \) and \( m_s > 0 \), we have

\[
(3.2) \quad f_s^2 \left[ \left( \frac{u}{f_s} \right)' \right]^2 = (u')^2 - 2 \frac{uu'f_s'}{f_s} + u^2 \frac{(f_s')^2}{f_s^2},
\]

with a similar identity holding for \( m_s, w \). Hence,

\[
0 = E'_{\infty, g}(f_s, m_s) \left[ \frac{u^2}{f_s}, \frac{w^2}{m_s} \right]
\]

\[
= \int \left\{ (u')^2 + (w')^2 + \frac{d^2}{r^2}u^2 + gw^2 - (1 - f_s^2 - m_s^2)(u^2 + w^2)
\right. \\
\left. - f_s^2 \left[ \left( \frac{u}{f_s} \right)' \right]^2 - m_s^2 \left[ \left( \frac{w}{m_s} \right)' \right]^2 \right\} r dr.
\]

Substituting this in the formula for \( E''_{\infty, g}(f_s, m_s)[u, w] \) we obtain

\[
E''_{\infty, g}(f_s, m_s)[u, w] = \int \left\{ f_s^2 \left[ \left( \frac{u}{f_s} \right)' \right]^2 + m_s^2 \left[ \left( \frac{w}{m_s} \right)' \right]^2 + 2[f_s u + m_s w]^2 \right\} r dr.
\]

To obtain the result for any \((u, w) \in X \times H\), let \( u_n \) be a sequence of \( C^\infty_0((0, \infty)) \) functions converging to \( u \) in \( X \), and \( w_n \) a sequence in \( C^\infty_0([0, \infty)) \) converging to \( w \) in \( H \). By continuity of \( E''_{\infty, g}(f_s, S_s) \), the limit passes in the second variation of \( E_{\infty, g} \). For the right hand side we expand,

\[
\int f_s^2 \left( \left( \frac{u}{f_s} \right)' \right)^2 r dr = \int \left\{ (u')^2 - 2 \frac{f_s u'}{u} uu' + \left( \frac{f_s'}{f_s} \right)^2 u^2 \right\} r dr,
\]

and note that

\[
\left( \frac{f_s'}{f_s} \right)^2 \leq c \left( 1 + \frac{1}{r^2} \right)
\]
since \( f_\ast \sim r^d \) for \( r \sim 0 \). Hence each term is controlled by the \( X \)-norm and can be passed to the limit. A similar argument may be applied for the second term in the right-hand side of (3.1). The quotient is expanded as in (3) above, with \( m_\ast, w \) replacing \( f_\ast, u \). Then we claim that \( m'(r)/m(r) \) is uniformly bounded for \( r \in [0, \infty) \). Indeed, by the basic gradient bound for solutions of the Poisson equation (see section 3.4 of \([GT 83]\)) we have for any \( r_0 > 1 \),

\[
|m'(r_0)| \leq 2 \sup_{|r-r_0| \leq 1} m(r) + \frac{1}{2} \sup_{|r-r_0| \leq 1} |\kappa^2(1-g-f^2-m^2)| \leq C_1 \sup_{|r-r_0| \leq 1} m(r).
\]

Applying the Harnack inequality (Corollary 9.25 of \([GT 83]\)) we then obtain:

\[
\left| \frac{m'(r_0)}{m(r_0)} \right| \leq C_1 \frac{\sup_{|r-r_0| \leq 1} m(r)}{m(r_0)} \leq C_1 \frac{\sup_{|r-r_0| \leq 1} m(r)}{\inf_{|r-r_0| \leq 1} m(r)} \leq C_1',
\]

for all \( r_0 > 1 \). Therefore \( m'/m \) is uniformly bounded, and we may pass to the \( H^1_r \) limit in the second term in (3.1). The last term is clearly continuous in the \( L^2_r \)-norm in both \( u \) and \( w \). In conclusion, we may pass to the limit \( u_n \to u, w_n \to w \) and obtain (3.1) for \( u \in X, w \in H \).

\( \diamond \)

**Proof of Theorem 3.1:** Define

\[
\sigma_* = \inf \{ \mathcal{E}_{\infty,g}'(f_\ast, m_\ast)[u, w] : u \in X, w \in H, \|u\|_X^2 + \|w\|_H^2 = 1 \}.
\]

We must show that \( \sigma_* > 0 \).

By Theorem 3.3, \( \sigma_* \geq 0 \). To obtain a contradiction, assume instead that \( \sigma_* = 0 \). We claim that in this case the infimum is attained at a nontrivial \( (u_\ast, w_\ast) \), with \( \mathcal{E}_{\kappa,g}'(f_\ast, m_\ast)[u_\ast, w_\ast] = \sigma_* = 0 \). But this contradicts Theorem 3.3, and hence \( \sigma_* > 0 \).

We now claim that the infimum \( \sigma_* = 0 \) is attained in \( Z_0 \). Take any minimizing sequence: \( (u_n, w_n) \in X \times H \) with \( \|u_n\|_X^2 + \|w_n\|_H^2 = 1 \) and

\[
\mathcal{E}_{\infty,g}'(f_\ast, m_\ast)[u_n, w_n] \to \sigma_* = 0.
\]

By the Sobolev embedding, there exists a subsequence (still denoted by \( u_n, w_n \)) and \( u_\ast \in X, w_\ast \in H \) so that \( u_n \to u_\ast, w_n \to w_\ast \), weakly in \( X, H \) (respectively), and strongly in \( L^2_{loc} \).
First, we claim that \((u^*, w^*) \neq (0, 0)\). Indeed, if both \(u^*, w^*\) vanish identically then by weak convergence \((u_n, w_n) \rightharpoonup (u^*, w^*) = (0, 0)\) and the compact embeddings,

\[
\int \left( (u'_n)^2 + \frac{d^2}{r^2} u_n^2 + 2\kappa^2 f^2_* u_n^2 + (w'_n)^2 + g\kappa^2 w_n^2 \right) r \, dr
= \mathcal{E}'_{\infty, g}(f_*, m_*)[u_n, w_n]
\]

\[
+ \int \left[ \kappa^2 (1 - f^2_* - m^2_*) (u_n^2 + w_n^2) - 2\kappa^2 m_n^2 w_n^2 - 4\kappa^2 f_* m_* u_n w_n \right] r \, dr
\rightarrow 0.
\]

In particular, \((u_n, w_n) \to (0, 0)\) in the norm on \(X \times H\), which contradicts the fact that \(|u_n|^2_X + |w_n|^2_H = 1\). Thus the claim holds, and \((u^*, w^*) \neq (0, 0)\).

Next, we use lower semicontinuity in the norm and \(L^2_{loc}\) convergence to pass to the limit,

\[
(3.3) \quad \mathcal{E}'_{\infty, g}(f_*, m_*)[u^*, w^*] \leq \liminf_{n \to \infty} \mathcal{E}'_{\infty, g}(f_*, m_*)[u_n, w_n] = 0.
\]

This contradicts Theorem 3.3, since \(\mathcal{E}'_{\infty, g}(f_*, m_*)[u^*, w^*] > 0\). (Note that \(u/f_*\) is non-constant since \(u \in X\) but \(f_* \not\in X\).) We conclude that \(\sigma > 0\), as desired.

\diamond

We note that the same result holds when \(m_* \equiv 0\). Hence following the method of [ABG 99], we obtain another proof of uniqueness for the solution to the high kappa equation for \(f_*\) studied in [CEQ 94].

4 Bifurcation from the normal cores

In this section we show that (when \(\kappa^2 \geq 2d^2\)) AF core solutions are nucleated by means of a bifurcation from the normal core solution family at a simple eigenvalue of the linearized equations. We will also require \textit{a priori} estimates (whose proof we will present in Section 6) to obtain global information about the solutions set for all \(\kappa^2 \geq 2d^2\), and the stronger result of Theorem 3.1 to fully categorize solutions in the extreme type-II model \((GL)_{\infty, g}^\infty\). We present the detailed argument for the problem \((GL)_{\kappa,g}\). The functional analytic framework is entirely similar for the problem \((GL)_{\infty,g}\), and so we omit it and concentrate instead on the more precise global characterization of solutions which we prove for \((GL)_{\infty,g}\).
4.1 Local bifurcation at $g^*_\kappa$

We define a map $\mathcal{F} : Y \times \mathbb{R} \to Y_0^*$ by

$$\langle (u, v, w), \mathcal{F}(f_*, S_*, m_*, g) \rangle_{Y_0, Y_0^*} = \mathcal{E}^\prime_{\kappa, g}(f_*, S_*, m_*)[u, v, w],$$

$(u, v, w) \in Y_0$, $(f_*, S_*, m_*) \in Y$. Its linearization is the operator $\mathcal{F}'(f_*, S_*, m_*, g) \in L(Y_0, Y_0^*)$ defined by

$$\langle (u, v, w), \mathcal{F}'(f_*, S_*, m_*, g)[\varphi, \psi, \xi] \rangle_{Y_0, Y_0^*} = \frac{d}{dt} \bigg|_{t=0} \mathcal{E}^\prime_{\kappa, g}(f_* + t\varphi, S_* + rt\psi, m_* + t\xi)[u, v, w].$$

We remark that the explicit expansion of the energy $I_{\kappa, g}$ in terms of $u_* = f_* - \tilde{f}_\kappa$, $v_* = (S_* - \tilde{S}_\kappa)/r$, $w_*$ ensures that $\mathcal{F}$ is a $C^2$ map in all arguments $u_*, v_*, w_*, g$.

By the natural identification $Y_0 \simeq Y_0^*$ of a Hilbert Space with its dual, we may also represent $\mathcal{F}'$ by $L_g \in L(Y_0, Y_0)$ as

$$\langle (u, v, w), L_g[\varphi, \psi, \xi] \rangle_{Y_0} = \langle (u, v, w), \mathcal{F}'(f_*, S_*, m_*, g)[\varphi, \psi, \xi] \rangle_{Y_0, Y_0^*}.$$

If $i : Z^* \to Z$ is the isomorphism, then $L_g = i \circ \mathcal{F}'(f_*, S_*, m_*, g)$.

**Lemma 4.1** For all $g > 0$, $L_g$ is a Fredholm operator of index zero.

**Proof:** Define an equivalent inner product on $Y_0$,

$$\langle (u, v, w), (\varphi, \psi, \xi) \rangle_{Y_0} = \int \left\{ u'\varphi' + 2\kappa^2 u\varphi + \frac{(d - S_*)^2}{r^2} u\varphi + v'\psi' + v\psi + \frac{1}{r^2} v\psi + w'\xi' + g\kappa^2 w\xi \right\} r \, dr.$$

Then we write

$$\langle (u, v, w), L_g[\varphi, \psi, \xi] \rangle_{Y_0} = \langle (u, v, w), (\varphi, \psi, \xi) \rangle_{Y_0} + \langle (u, v, w), K[\varphi, \psi, \xi] \rangle_{Y_0},$$

where $K$ is defined by

$$\langle (u, v, w), K[\varphi, \psi, \xi] \rangle_{Y_0} = \int \left[ 2\kappa^2 (f_*^2 - 1) w\varphi + 2\kappa^2 f_* m_* (u\xi + w\varphi) + 2\kappa^2 m_*^2 w\xi \right]$$
\[-\kappa^2(1 - f_*^2 - m_*^2)(w\varphi + w\xi) + (f_*^2 - 1)v\psi
\]
\[-2\frac{d - S_*}{r}f_*(w\psi + v\varphi)\]  \[ r \, dr. \]

Recalling the decay properties of \( f_*, S_*, m_* \) and the embedding properties of \( H, X \) we observe that \( K \) is compact, and hence \( L_g = Id_{Y_0} + K \) is Fredholm with index zero.

\[ \Diamond \]

As a direct consequence of Lemma 4.1,
\[ \dim \ker \left( \mathcal{F}' \right) = \dim \ker \left( \mathcal{L}_g \right) = \text{codim} \, \text{Ran} \left( \mathcal{L}_g \right) = \text{codim} \, \text{Ran} \left( \mathcal{F}' \right). \]

Now we may apply the standard bifurcation theory of Crandall & Rabinowitz \([CR\ 71]\) at an eigenvalue \( g^* \) of \( \mathcal{F}'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g^*) \). Indeed, note that when \( m_* = 0 \) the linearization of \( \mathcal{F} \) decouples into two components,

\[ \langle (u, v, w), \mathcal{F}'(f_*, S_*, 0, g)[\varphi, \psi, \xi] \rangle_{Y_0, Y_0^*} = \langle (u, v), \mathcal{F}'_{1,2}(f_*, S_*)[\varphi, \psi] \rangle_{X^2, (X^2)^*} + \langle w, \mathcal{F}'_3(f_*, g)\xi \rangle_{H, H^*}, \]

where

\[ \langle (u, v), \mathcal{F}'_{1,2}(f_*, S_*)[\varphi, \psi] \rangle_{X^2, (X^2)^*} = \langle (u, v, 0), \mathcal{F}'(f_*, S_*, 0, g)[\varphi, \psi, 0] \rangle_{Y_0, Y_0^*} \]

\[ = \int \left[ u'\varphi' + \frac{(d - S_*)}{r^2}u\varphi + v'\psi' + \frac{v\psi}{r^2} \right. \]

\[ + f_*^2w\psi - \frac{2d - S_*}{r}f_*(w\psi + v\varphi) - \kappa^2(1 - 3f_*^2)w\varphi \]  \[ r \, dr, \]

and

\[ \langle w, \mathcal{F}'_3(f_*, g)\xi \rangle_{H, H^*} = \langle (0, 0, w), \mathcal{F}'(f_*, S_*, 0, g)[0, 0, \xi] \rangle_{Y_0, Y_0^*} \]

\[ = \int \left\{ w'\xi' + g\kappa^2 w\xi - \kappa^2(1 - f_*^2)w\xi \right\} r \, dr. \]

By Theorem 3.1 of \([ABG\ 99]\), when \( \kappa^2 \geq 2d^2 \) the operator \( \mathcal{F}'_{1,2} \geq \sigma_* > 0 \) is bounded away from zero (in quadratic form sense.) Hence, if \( (\varphi, \psi, \xi) \in \ker(\mathcal{F}'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g_*^*)) \), we take
(u, v, w) = (ϕ, ψ, 0) and obtain

\[
0 = \langle (\varphi, \psi, 0), F'(f_*, S_*, 0, g^*_n)[\varphi, \psi, \xi]\rangle_{Y_0, Y_0^*} \\
= \langle (\varphi, \psi), F'_{1,2}(f_*, S_*)[\varphi, \psi]\rangle_{X^2, (X^2)^*} \\
\geq \sigma_* (\|\varphi\|^2_X + \|\psi\|^2_X)
\]

In particular, \(\varphi, \psi = 0\).

The operator \(F'_3(f_*, g) = L + g\kappa^2\) where \(L = -\Delta_r - V(r)\) is a Schrödinger operator with potential \(V(r) = \kappa^2(1 - f^2_*(r)) \geq 0\) and \(V(r) \to 0\) as \(r \to \infty\). It is a well-known fact in mathematical physics that in dimension two, such operators have at least one negative eigenvalue:

**Lemma 4.2** Suppose \(V : [0, \infty) \to \mathbb{R}\) is continuous, non-negative, \(V(r) \to 0\) as \(r \to \infty\), and \(V\) is not identically zero, and define \(L = -\Delta - V(r)\) as a self-adjoint operator on the space \(L^2(\mathbb{R}^2)\). Then the ground state energy,

\[
\lambda_0 = \inf \left\{ \frac{\int [(u')^2 - V(r)u^2] r \, dr}{\int u^2 r \, dr} : u \neq 0, \, u \in H \right\} < 0,
\]

and is attained at an eigenfunction \(u_0 \in H\). Moreover, \(\lambda_0\) is an isolated, non-degenerate eigenvalue, \(u_0 \in H\), and \(u_0 > 0\).

The proof follows as an application of the Birman–Schwinger principle in Reed & Simon [RS 78]. We provide an elementary variational proof for the reader’s convenience.

**Proof:** Let

\[
u_n(r) = \begin{cases} 
1, & \text{if } r \leq n, \\
\frac{\ln(r/n^2)}{\ln(1/n)}, & \text{if } n \leq r \leq n^2, \\
0, & \text{if } r \geq n.
\end{cases}
\]

Then,

\[
\int (u'_n)^2 r \, dr = \frac{1}{\ln n} \to 0,
\]

while

\[
\int V(r)u^2_n r \, dr \geq \int_0^n V(r) r \, dr \to \int_0^\infty V(r) r \, dr > 0
\]
(possibly infinite.) Hence, for \( n = N \) large but fixed we have \( \int ((u_N')^2 - V(r)u_N^2) \, r \, dr < 0 \), and hence \( \lambda_0 < 0 \). Since \( L \) is a relatively compact perturbation of \(-\Delta\), \( \lambda_0 \) is a discrete eigenvalue with associated eigenfunction \( u_0 \) contained in the form domain of \( L, H \). By standard arguments, \( u_0 > 0 \) and \( \lambda_0 \) is a simple (non-degenerate) eigenvalue.

By Lemma 4.2

\[-g^*_\kappa = \inf_{w \in H - \{0\}} \frac{\int \left( \frac{1}{\kappa^2}(w')^2 - (1 - \tilde{f}_\kappa^2)w^2 \right) \, r \, dr}{\int w^2 \, r \, dr} < 0,\]

and \( \lambda_0 = -\kappa^2 g^*_\kappa \) is the ground state eigenvalue of the Schrödinger operator \(-\Delta_r - \kappa^2(1 - \tilde{f}_\kappa^2)\).

Since \( \lambda_0 \) is a simple eigenvalue

\[\dim \ker(F'_\tilde{f}_\kappa, g^*_\kappa) = \dim \ker(-\Delta_r - \kappa^2(1 - \tilde{f}_\kappa^2) + \kappa^2 g^*_\kappa) = 1.\]

In conclusion when \( g = g^*_\kappa \) the operator \( F'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g^*_\kappa) \) has a simple eigenvalue and the eigenvector is of the form \((0, 0, w_\kappa)\) with \( w_\kappa \) the (positive) eigenfunction of \( F'_\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g^*_\kappa \).

Finally, we observe that the operator \( (\frac{\partial}{\partial g})F'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g^*_\kappa) \in L(Y_0, Y_0^*)\),

\[\langle (u, v, w), \frac{\partial}{\partial g}F'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g^*_\kappa)[\varphi, \psi, \xi] \rangle_{Y_0, Y_0^*} = \int \kappa^2 w \xi \, r \, dr.\]

At the eigenvalue \( g = g^*_\kappa \) we have

\[\frac{\partial}{\partial g}F'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g^*_\kappa)[0, 0, w_\kappa] = \kappa^2 w_\kappa \notin \text{Ran}(F'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g^*_\kappa)).\]

Therefore Theorem 1.7 of \([CR 71]\) applies, and \( g^*_\kappa \) is a bifurcation point for \( F \) in \( Y \times \mathbb{R} \): there exists a neighborhood \( U \) of \((\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g^*_\kappa)\) in \( Y_0 \times \mathbb{R} \), such that the set of non-trivial solutions of \( F(f, S, m, g) = 0 \) in \( U \) is a unique \( C^1 \) curve parametrized by \( \ker(F'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g^*_\kappa)) \).

**Remark 4.3** Since \( F \) is a smooth \((C^\infty)\) map, we may calculate various derivatives of the bifurcation curve through the normal core solutions at \( g^*_\kappa \). If we parametrize \( g = \gamma(t) \), with \( \gamma(0) = g^*_\kappa \), then we follow Crandall & Rabinowitz \([CR 71]\) or Ambrosetti & Prodi \([AP 93]\) (see...
Remarks 4.3) to calculate derivatives of $\gamma(t)$ and determine the direction of the bifurcation curve locally at $g = g^*_\kappa$. We obtain that $\gamma'(0) = 0$, and

$$\gamma''(0) = -2 \frac{\int \tilde{f}_\kappa u^2 + w^4 \, r \, dr}{\int w^2 r \, dr},$$

where $u^\kappa$ is obtained from the (unique) solution to the linear system

$$F'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g^*_\kappa)[u^*_\kappa, v^*_\kappa, w^*_\kappa] = -\left(2\kappa^2 \tilde{f}_\kappa w^2, 0, 0\right)$$

with $(u^*_\kappa, v^*_\kappa, w^*_\kappa) \perp \ker F'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g^*_\kappa)$. By taking the scalar product of the above system with $(u^*_\kappa, v^*_\kappa, 0)$ (and recalling that $F'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g^*_\kappa)$ is positive definite in the complement of its kernel) we obtain $\int \tilde{f}_\kappa u^2 r \, dr < 0$, and hence the expression for $\gamma''(0)$ is indefinite in sign. In a joint paper with J. Berlinsky [ABBG 98] we present computational evidence that solutions bifurcate to the left, to smaller values $g < g^*_\kappa$. By standard bifurcation theory (see [CR 73], for example) the direction of bifurcation indicates the stability of the solutions, and indeed we observe numerically that the AF core solutions which bifurcate at $g^*_\kappa$ are stable (local energy minimizers.)

### 4.2 Global bifurcation for $(GL)_{\infty,g}$

We obtain the same abstract bifurcation result for the extreme type-II model, $(GL)_{\infty,g}$. Namely, the value

$$g^*_{\infty} = -\inf_{w \in H-\{0\}} \frac{\int [(w')^2 - (1 - \tilde{f}_{\infty}^2)w^2] \, r \, dr}{\int w^2 r \, dr} > 0$$

is a bifurcation point for nontrivial $(m > 0)$ solutions from the (trivial) curve of normal core solutions $(\tilde{f}_{\infty}, 0, g)$. But in this case we can make a much more precise statement:

**Proposition 4.4** Let

$$\Sigma = \{(f, m, g) : (f, m) \text{ is an admissible solution to } (GL)_{\infty,g} \text{ with } m > 0\}.$$ 

Then $\mathcal{C} = \Sigma \cup \{(\tilde{f}_{\infty}, 0, g^*_\infty)\}$ is a connected $C^1$ curve, parametrized by $g$. Moreover for any $g_0 > 0$, $\mathcal{C} \cap \{g \geq g_0\}$ is compact.
As a consequence we have the following exact solvability theorem for \((GL)_{\infty, g}\).

**Theorem 4.5** For \(g \geq g^*_\infty\), the normal core solutions \((\tilde{f}_\infty, 0)\) are the only admissible solutions of \((GL)_{\infty, g}\).

For \(0 < g < g^*_\infty\) there is a unique solution with \(m > 0\). This solution is the global minimizer of \(E_{\infty, g}\).

The proofs of these two results hinge on the powerful Theorem 3.1 and the following compactness theorem, which will be proven in Section 6:

**Theorem 4.6** Let \(0 < a < b\). Then the set of all admissible solutions of \((GL)_{\infty, g}\) with \(g \in [a, b]\) is compact in \(Z\).

**Proof of Theorem 4.4:** Let \(C'\) be a maximally connected component of \(C\), and suppose \((f_0, m_0, g_0) \in C'\) but \((f_0, m_0, g_0) \neq (\tilde{f}_\infty, 0, g^*_\infty)\). Since \(m = 0\) only when \(g = g^*_\infty\), we must have \(m_0 > 0\). By Theorem 3.1, \((f_0, m_0, g_0)\) is a nondegenerate zero of \(F\) in \(Z \times \mathbb{R}\), so by the Implicit Function Theorem there exists a neighborhood \(U\) of \((f_0, m_0, g_0)\) in \(Z \times \mathbb{R}\), an interval \(J = (g_0 - \delta, g_0 + \delta)\), and a \(C^1\) function \(\Phi : J \to Z\) so that all solutions of \(F = 0\) in \(U\) are of the form \((\Phi(g), g)\) with \(g \in J\).

Let
\[
\hat{g} = \sup\{g : \text{there exists a solution } (f, m, g) \in C'\} > g_0.
\]

Note first that any solution must satisfy
\[
0 \leq \int (m')^2 r \, dr \leq \int (1 - g)m^2 r \, dr,
\]
and hence \(g < 1\) for any solution with \(m \neq 0\). Since by Proposition 4.6, \(C' \cap \{g \geq g_0\} = C' \cap \{g_0 \leq g \leq 1\}\) is compact, there exists a solution at \(g = \hat{g}\), \((\hat{f}, \hat{m}, \hat{g}) \in C'\). First, we claim that \(\hat{m} = 0\). If not, then by Proposition 2.5, \(\hat{m}(r) > 0\) for all \(r > 0\), so by Theorem 3.1, \((\hat{f}, \hat{m})\) is a nondegenerate minimum of \((GL)_{\hat{g}, \infty}\). By the Implicit Function Theorem argument above there exist a \(C^1\) curve of nontrivial solutions through \((\hat{f}, \hat{m}, \hat{g})\), parametrized by \(g\). In particular, we contradict the definition of \(\hat{g}\) is the supremum of all \(g\) for solutions in the connected component \(C'\). Hence \(\hat{m} = 0\), as desired.

Now we show that \(\hat{g} = g^*_\infty\). Take a sequence \((f_n, m_n, g_n) \in C'\) with \(g_n \to \hat{g}\), so the above arguments imply that \(f_n - \tilde{f}_\infty \to 0\) in \(X\) and \(m_n \to 0\) in \(H\). Let
\[
t_n = \int (1 - f_n^2)m_n^2 r \, dr \to 0.
\]
Then \( w_n = m_n/t_n \) solves

\[
(4.1) \quad -w_n'' - \frac{1}{r}w_n' + g_n w_n = (1 - f_n^2 - m_n^2)w_n.
\]

Since

\[
(4.2) \quad \int \left((w_n')^2 + gw_n^2\right) r \, dr = \int (1 - f_n^2 - m_n^2)w_n^2 r \, dr \leq \int (1 - f_n^2)w_n^2 r \, dr = 1,
\]

(by the choice of \( t_n \)) we have \( \|w_n\|_H \leq 1/g \) and we may extract a subsequence (which we continue to call \( w_n \)) which converges \( w_n \to w_\infty \) weakly in \( H \) and strongly in \( L^2_{\text{loc}} \). By the strong convergence of \( f_n \to \tilde{f}_\infty \) we have \( \int (1 - \tilde{f}_\infty^2)w_\infty^2 r \, dr = 1 \), so \( w_\infty \neq 0 \), and \( w_\infty \geq 0 \). Passing to the limit in \((4.2)\) we have

\[
(4.3) \quad \int \left((w_\infty')^2 + \hat{g}w_\infty^2 - (1 - \tilde{f}_\infty^2)w_\infty^2\right) r \, dr = 0,
\]

for all \( \varphi \in H \). This can only occur when \( \hat{g} = g_\infty^* \), the ground state eigenvalue of the above Schrödinger operator.

We have just shown that the point \((\tilde{f}_\infty, 0, g_\infty^*)\) belongs to every connected component of \( C \), and hence \( C \) is connected. The solution set \( C \) is everywhere a \( C^1 \) curve: for \( g > g_\infty^* \) this results from the Implicit Function Theorem argument in the first paragraph, and at \( g_\infty^* \) it is a consequence of bifurcation from a simple eigenvalue \([CR\,71]\). We now claim that there exists exactly one solution in \( C \) for every \( g \leq g_\infty^* \). Suppose not, and consider

\[
D = \{ g \in (0, g_\infty^*) : \text{there exist two distinct solutions } (f_{g,1,1}, m_{g,1}), (f_{g,2,2}, m_{g,2}) \text{ in } C \text{ at } g \},
\]

and \( g_0 = \sup D \).

First, we note that \( g_0 < g_\infty^* \). To see this we note that the only solution in \( C \) with \( g = g_\infty^* \) is the normal core solution, and the bifurcation theorem ensures that the solution set in a neighborhood of the bifurcation point \((\tilde{f}_\infty, 0, g_\infty^*)\) is a single smooth curve.

Next, we claim that \( g_0 \notin D \). Indeed, if \( g_0 \in D \) there exist two distinct solutions \((f_{g_0,1,1}, m_{g_0,1})\) and \((f_{g_0,2,2}, m_{g_0,2})\) for \( g = g_0 \). By the Implicit Function Theorem argument of the first paragraph there exist neighborhoods \( U_1 \) (of \((f_{g_0,1,1}, m_{g_0,1}, g_0))\) and \( U_2 \) (of \((f_{g_0,2,2}, m_{g_0,2}, g_0))\) in \( Z \times \mathbb{R} \) such that all solutions of \( \mathcal{F} = 0 \) in \( U_1, U_2 \) are given by smooth curves parametrized by \( g \). In particular, \( C \) contains two distinct solutions for \( g \) in an interval to the right of \( g_0 \), contradicting the definition of \( g_0 \) as the supremum.

Hence \( g_\infty^* > g_0 \notin D \), and there exists a sequence \( g_k \to g_0 \) for which \( C \) contains two distinct solutions, \((f_{g_k,1,1}, m_{g_k,1}), (f_{g_k,2,2}, m_{g_k,2})\). By Theorem \([4.6]\), along some subsequence these
solutions converge, and since \( g_0 \notin D \), they both converge to a single solution, \((f_{g_0}, m_{g_0})\). But this contradicts the Implicit Function Theorem argument, which implies that the solution set near \((f_{g_0}, m_{g_0}, g_0)\) is a single curve parametrized by \(g\). We conclude that the AF core solutions are unique for each \( g \in (0, g_\infty^*) \).

\[ \diamond \]

4.3 Behavior for \( g \to 0, \kappa < \infty \)

For the problem \((GL)_{\kappa,g}\) we do not have the strong information provided by Theorem 3.1 which determines the global structure of the solution set, and hence we cannot make the same elegant conclusion about the uniqueness of AF core solutions. However we may still say something about the global structure of the continuum bifurcating from the normal cores at \( g = g_\kappa^* \). When \( \kappa^2 \geq 2d^2 \) we may apply the Global Bifurcation Theorem of Rabinowitz [Ra 71] to conclude that the continuum \( \Sigma_\kappa \) of zeros of \( \mathcal{F}(f,S,m,g) = 0 \) with \( m > 0 \) is unbounded in the space \( Y \times \mathbb{R} \). (Note that \( \Sigma_\kappa \) cannot contain any other eigenvalues of the linearization about the normal core solutions, as is easily seen from the calculations (4.1)–(4.3) above.) In the next section we will prove the following a priori estimate, which has as a direct consequence the fact that \( \Sigma_\kappa \) can only become unbounded as \( g \to 0^+ \):

**Theorem 4.7** Let \( d, \kappa \) be fixed. For any compact interval \( J \in (0, \infty) \) there exists \( C_0 = C_0(\kappa, d, J) > 0 \) such that every admissible solution \((f,S,m)\) of \((GL)_{\kappa,g}\) with \( g \in J \) satisfies \( \|(f,S,m)\|_Y \leq C \).

Let us now concentrate on this loss of compactness in the continuum \( \Sigma_\kappa \) as \( g \to 0^+ \). We prove:

**Theorem 4.8** For any sequence of (absolute) minimizers \((f_g, S_g, m_g)\) \( Y \) with \( g \to 0^+ \) we have \( f_g \to 0 \) in \( X_{loc} \), \( S_g \to 0 \) locally uniformly, and \( m_g \to 1 \) in \( H_{loc} \).

Fix \( \kappa \in \mathbb{R} \), and for any \( g > 0 \) consider a minimizer \((f_g, S_g, m_g)\) \( Y \) of \( \mathcal{E}_{\kappa,g} \).

**Lemma 4.9**

\[ \mathcal{E}_{\kappa,g}(f_g, S_g, m_g) \to 0, \quad \text{as} \quad g \to 0. \]

**Proof:** We will show that for any \( \varepsilon > 0 \) there exist \( g_\varepsilon > 0 \) and \( H \) radial functions \((f_\varepsilon, S_\varepsilon, m_\varepsilon)\) \( Y \) such that \( 0 < \mathcal{E}_{\kappa,g}(f_\varepsilon, S_\varepsilon, m_\varepsilon) < \varepsilon \), for any \( g < g_\varepsilon \).
For a fixed $\rho > 0$, we define

$$u_\rho(r) = \begin{cases} 1, & \text{if } r \leq \rho, \\ \ln(r/\rho^2)/\ln(1/\rho), & \text{if } \rho \leq r \leq \rho^2, \\ 0, & \text{if } r \geq \rho^2, \end{cases}$$

we consider

$$f_\rho(r) = \cos(u_\rho(r)\pi/2), \quad m_\rho(r) = \sin(u_\rho(r)\pi/2),$$

and

$$S_\rho(r) = \begin{cases} 0, & \text{if } r \in (0, \rho^2), \\ d, & \text{if } r \in (\rho, \infty), \end{cases}$$

A direct computation shows that

$$\mathcal{E}_{\kappa,g}(f_\rho, S_\rho, m_\rho) \leq C_{\rho^2} + \frac{\pi^2}{4\ln \rho} + \frac{\kappa^2}{2}g\rho^4$$

for any $g > 0$.

For a given $\varepsilon > 0$, we choose a $\rho_\varepsilon$ such that $\frac{C}{\rho_\varepsilon^2} + \frac{\pi^2}{4\ln \rho_\varepsilon} < \frac{\varepsilon}{2}$, and a $g_\varepsilon = g_\varepsilon(\rho_\varepsilon)$ for which

$$\frac{\kappa^2}{2}g_\varepsilon\rho_\varepsilon^4 < \frac{\varepsilon}{2},$$

i.e. $g_\varepsilon < \frac{\varepsilon}{2\kappa^2\rho_\varepsilon^4}$. Then, $\mathcal{E}_{\kappa,g}(f_g, S_g, m_g) \leq \mathcal{E}_{\kappa,g}(f_{\rho_\varepsilon}, S_{\rho_\varepsilon}, m_{\rho_\varepsilon}) < \varepsilon$, for any $g < g_\varepsilon$.

\[\Diamond\]

**Proof of Theorem 4.8** By Lemma 4.9 each term in the energy tends to zero as $g \to 0$. First, note that $\int (S_g'/r)^2 r \, dr \to 0$ combined with (1.4) in [BC 89] implies that

$$S_g(r)/r \to 0 \text{ uniformly.}$$

For any $R_0 > 0$, we then have

$$o(1) = \int \frac{(d - S_g)^2}{r^2} f_g^2 r \, dr \geq \int_0^{R_0} \frac{(d - S_g)^2}{r^2} f_g^2 r \, dr$$

$$= \int_0^{R_0} \frac{d^2}{r^2} f_g^2 r \, dr + o(1).$$
In particular, \( f_g \to 0 \) in \( L^2_{loc}, X_{loc} \). Finally, by the reverse triangle inequality,

\[
o(1) = \sqrt{\frac{K^2}{2} \int_0^{R_0} (1 - f^2 - m^2_g)^2 r \, dr} = \frac{K}{\sqrt{2}} \|1 - f^2 - m^2_g\|_{L^2([0,R_0])}
\]

\[
\geq \frac{K}{\sqrt{2}} \left[ \|1 - m^2_g\|_{L^2([0,R_0])} - \|f^2_g\|_{L^2([0,R_0])} \right]
\]

\[
\geq \frac{K}{\sqrt{2}} \|1 - m_g\|_{L^2([0,R_0])} + o(1),
\]

where we have also used \( 0 \leq f_g < 1, 0 < m_g < 1 \), and \( f_g \to 0 \) in \( L^2_{loc} \). In conclusion \( m_g \to 1 \) in \( L^2_{loc} \) and in fact in \( H^1_{loc} \), since \( \int (m_g')^2 r \, dr \to 0 \) by the energy estimate.

\[\Box\]

5 The limit \( \kappa \to \infty \)

In this section we show that the problem \((GL)_{\infty,g}\) arises as a limiting case of \((GL)_{\kappa,g}\) as \( \kappa \to \infty \). For any solution \((f_\kappa, S_\kappa, m_\kappa)\) of \((GL)_{\kappa,g}\), define

\[
\hat{f}_\kappa(r) = f_\kappa \left( \frac{r}{\kappa} \right), \quad \hat{S}_\kappa(r) = S_\kappa \left( \frac{r}{\kappa} \right), \quad \hat{m}_\kappa(r) = m_\kappa \left( \frac{r}{\kappa} \right).
\]

We prove:

**Theorem 5.1** Let \((f_\kappa, S_\kappa, m_\kappa)\) be any family of solutions of \((GL)_{\kappa,g}\) for \( \kappa > 0 \), and \((\hat{f}_\kappa, \hat{S}_\kappa, \hat{m}_\kappa)\) defined as in (5.1). For any sequence \( \kappa_n \to \infty \), there exists a subsequence and a solution \((f_\infty, m_\infty)\) of \((GL)_{\infty,g}\) so that (as \( \kappa_n \to \infty \)), \( \hat{f}_{\kappa_n} - f_\infty \to 0 \) in \( X \), \( \hat{m}_{\kappa_n} - m_\infty \to 0 \) in \( H \), and \( \hat{S}_{\kappa_n} \to 0 \) locally uniformly. Moreover:

i. If \( g \geq g^*_\infty \), then \( m_\kappa \to 0 \);

ii. If \( m_\kappa \not\equiv 0 \) for all large \( \kappa \) and \( g \neq g^*_\infty \), then \( \lim_{\kappa \to \infty} \hat{m}_\kappa = m_\infty > 0 \).

As a simple consequence of the uniform convergence of \( \hat{f}_\kappa \to \hat{f}_\infty \) we have the following

**Corollary 5.2**

\[
g^*_\infty = \lim_{\kappa \to \infty} g^*_\kappa.
\]
Remark 5.3 This implies that the bifurcation diagram for $(GL)_{\kappa, g}$ with $\kappa$ very large should strongly resemble the very precise image given for $(GL)_{\infty, g}$ by Theorem 4.5. In particular, for any fixed $g > g^*_\infty$ $(GL)_{\kappa, g}$ cannot have solutions $(f_{\kappa, g}, S_{\kappa, g}, m_{\kappa, g})$ with $m_{\kappa, g} > 0$ for $\kappa$ large.

Simple calculations using the energy $E_{\kappa, g}$ show that $\inf_Y E_{\kappa, g} \sim \ln \kappa$, and hence we require energy-independent estimates for our solutions $(\hat{f}_\kappa, \hat{S}_\kappa, \hat{m}_\kappa)$. To obtain these estimates we begin with a simple version of the celebrated Pohozaev identity. This identity will also be essential for proving the a priori estimates used in the bifurcation analysis in the previous section.

Proposition 5.4 For any finite energy solution $(f, S, m)$ of $(GL)_{\kappa, g}$ we have

$$g\kappa^2 \int m^2 r \, dr + \frac{\kappa^2}{2} \int (1 - f^2 - m^2)^2 r \, dr = \int \left[ \frac{S'}{r} \right]^2 \, r \, dr.$$ 

For any finite energy solution $(f, m)$ of $(GL)_{\infty, g}$ we have

$$g \int m^2 r \, dr + \frac{1}{2} \int (1 - f^2 - m^2)^2 r \, dr = \frac{d^2}{2}.$$ 

Proof: We multiply the first equation in $(GL)_{\kappa, g}$ by $f'(r)r$ and integrate $r \, dr$ to obtain:

$$\frac{\kappa^2}{2} \int (1 - f^2 - m^2)(f^2)' r^2 \, dr = \int (d - S)^2 \left( \frac{1}{2} f^2 \right)' \, dr$$

$$= \int (d - S) S' f^2 \, dr = - \int \left( \frac{S'}{r} \right)' S' r \, dr$$

$$= \int \frac{S'}{r} (S' r)' \, dr = \int \left( \frac{S'}{r} \right)^2 r \, dr,$$

using the equation for $S(r)$, and integrating by parts whenever necessary. We also multiply the third equation in $(GL)_{\kappa, g}$ by $m'(r)r$ and integrate $r \, dr$ to obtain:

$$\frac{\kappa^2}{2} \int (1 - f^2 - m^2)(m^2)' r^2 \, dr = \frac{g\kappa^2}{2} \int (m^2)' r^2 \, dr = -g\kappa^2 \int m^2 r \, dr.$$
Together,

\[
\int \left( \frac{S'}{r} \right)^2 r \, dr = g \kappa^2 \int m^2 r \, dr + \frac{\kappa^2}{2} \int (1 - f^2 - m^2)(m^2 + f^2)' r^2 dr \\
= g \kappa^2 \int m^2 r \, dr + \frac{\kappa^2}{2} \int (1 - f^2 - m^2)^2 r \, dr.
\]

For the case \( \kappa = \infty \) we proceed in the same way, except the equation for \( f \) yields

\[
\int (1 - f^2 - m^2) \left( \frac{f^2}{2} \right)' r^2 dr = \frac{d^2}{2}.
\]

The calculation then continues as above.

\[\diamondsuit\]

**Proof: Step 1:** Bounding the sequence.

From the Pohozaev identity (Proposition 5.4) and Lemma 4.2 of [BC 89] after rescaling we have:

\[
d^2 \geq \int \left( \frac{S'}{r} \right)^2 r \, dr
\]

(5.2) \hspace{1cm} = \kappa^2 \int \left( \frac{\hat{S}}{r} \right)^2 r \, dr

(5.3) \hspace{1cm} = g \int \hat{m}^2 r \, dr + \frac{1}{2} \int (1 - \hat{f}^2 - \hat{m}^2)^2 r \, dr.

Using (5.2) and Lemma 1.2 (ii) in [BC 89] we have

\[
\sup_{r \in [0, \infty)} \left| \frac{\hat{S}(r)}{r} \right| \to 0,
\]

(5.4)

and hence \( \hat{S} \to 0 \) locally uniformly. From (5.3) we obtain the uniform bound \( \| \hat{m} \|_2 \leq C \) (depending on \( g \), which we assume is fixed.) From the equation for \( m \), after a change of
scale, we obtain:

\[
\int \left[ (\hat{m}'_\kappa)^2 + g\hat{m}_\kappa^2 \right] r \, dr = \int (1 - \hat{f}'_\kappa - \hat{m}'_\kappa)^2 r \, dr \leq \int \hat{m}_\kappa^2 r \, dr \leq C,
\]

and therefore \( \|\hat{m}_\kappa\|_H \leq C \) uniformly in \( \kappa \).

Recalling Proposition 2.5, any solution satisfies \( 0 < \hat{m}_\kappa(r) < 1 \), and we may conclude that \( \|\hat{m}_\kappa\|_q \leq \|\hat{m}_\kappa\|_2 \leq C \) for all \( p \in [2, \infty] \). By the triangle inequality,

\[
\|1 - \hat{f}'_\kappa\|_2 \leq \|1 - \hat{f}'_\kappa - \hat{m}'_\kappa\|_2 + \|\hat{m}'_\kappa\|_2 \\
\leq |d| + C,
\]

and hence we obtain

\[
\int (1 - \hat{f}'_\kappa)^2 r \, dr \leq \int (1 - \hat{f}'_\kappa)^2 r \, dr \leq C
\]

(since \( \hat{f}'_\kappa \geq 0 \).)

Choose a function \( \eta \in C^\infty(\mathbb{R}) \) with

\[
\eta(r) = \begin{cases} 
1, & \text{if } r \leq 2, \\
0, & \text{if } r \geq 3,
\end{cases}
\]

and \( 0 \leq \eta(r) \leq 1 \) for all \( r \). Using \( \eta^2 \hat{f}_\kappa \) as a test function in the weak form of the rescaled equation for \( \hat{f}_\kappa \),

\[
\int \eta^2 \left[ (\hat{f}'_\kappa)^2 + \frac{d^2}{r^2} \hat{f}_\kappa \right] r \, dr = \int \left[ (1 - \hat{f}'_\kappa - \hat{m}'_\kappa) \eta^2 - \eta \hat{f}_\kappa \hat{f}'_\kappa \right] r \, dr \\
\leq \int \left[ \frac{1}{2} (1 - \hat{f}'_\kappa)^2 + \frac{1}{2} \eta^4 + \frac{1}{2} \eta^2 (\hat{f}'_\kappa)^2 + 2 \hat{f}_\kappa^2 (\eta')^2 \right] r \, dr \\
\leq C + \frac{1}{2} \int \eta^2 (\hat{f}'_\kappa)^2 r \, dr.
\]

Absorbing the last term back to the left hand side,

\[
(5.5) \quad \int \eta^2 \left[ (\hat{f}'_\kappa)^2 + \frac{d^2}{r^2} \hat{f}_\kappa \right] r \, dr \leq C.
\]
Now choose another smooth function \( f_0 \) with
\[
f_0(r) = \begin{cases} 0, & \text{if } r \geq 1, \\ 1, & \text{if } r \geq 2, \end{cases}
\]
and \( 0 \leq f_0(r) \leq 1 \). Note that with this choice \( f_0^2 + \eta^2 \geq 1 \). We use \((\hat{f}_\kappa - 1)f_0^2\) as a test function in the equation for \( \hat{f}_\kappa \) to obtain
\[
\int \left[ (\hat{f}_\kappa')^2 + \frac{d^2}{r^2}(\hat{f}_\kappa^2 - 1)^2 \right] r \, dr = \int \left[ (1 - \hat{\Phi}_\kappa^2 - \hat{m}_\kappa^2)\hat{f}_\kappa(\hat{f}_\kappa - 1)f_0^2 - (\hat{f}_\kappa - 1)\hat{f}_\kappa'f_0 \right] r \, dr \\
+ \int \left[ -\frac{d^2}{r^2}(\hat{f}_\kappa - 1)f_0^2 + \frac{\hat{S}_\kappa(2d - \hat{S}_\kappa)}{r^2}(\hat{f}_\kappa - 1)^2 f_0^2 \right] r \, dr \\
\leq C \int_1^2 (\hat{f}_\kappa - 1)|\hat{f}_\kappa'|f_0 \, r \, dr \\
+ \int_1^\infty \left[ \frac{d^2}{r^2}(1 - \hat{f}_\kappa) + \frac{Ad}{r^2}(\hat{f}_\kappa - 1)^2 \right] r \, dr \\
\leq \frac{1}{2} \int (\hat{f}_\kappa')^2 f_0^2 r \, dr + C\|\hat{f}_\kappa - 1\|_2^2 + \int \left[ \frac{C}{r^4} + (\hat{f}_\kappa^2 - 1)^4 \right] r \, dr \\
\leq C + \frac{1}{2} \int (\hat{f}_\kappa')^2 f_0^2 r \, dr.
\]
(Note that in the first line, the first integrand is non-positive.) In conclusion,
\[
(5.6) \quad \int \left[ (\hat{f}_\kappa')^2 + \frac{d^2}{r^2}(\hat{f}_\kappa^2 - 1)^2 \right] r \, dr \leq C.
\]

Now define \( u_\kappa = \hat{f}_\kappa - f_0 \in X \). Then from (5.3), (5.6) we obtain:
\[
\int \left[ (u_\kappa')^2 + u_\kappa^2 + \frac{d^2}{r^2}u_\kappa^2 \right] r \, dr \leq 2 \int \left[ (\hat{f}_\kappa')^2 + (f_0)^2 \right] r \, dr + \int_0^2 \frac{2d^2}{r^2}\hat{f}_\kappa^2 r \, dr
\]
\[ + \int_{2}^{\infty} (d^2 + 1)(\hat{f}_\kappa - 1)^2 \]
\[ \leq 2 \int \left[ (\eta^2 + f_0^2)(\hat{f}_\kappa')^2 + \frac{d^2}{r^2} \hat{f}_\kappa^2 \eta^2 \right] r \, dr + C \]
\[ \leq C. \]

In other words, \( u_\kappa \) is uniformly bounded in \( X \), and we may extract weakly convergent subsequences \( u_n = u_{\kappa_n} \rightharpoonup u_\ast \) (in \( X \)), \( m_n = m_{\kappa_n} \rightharpoonup m_\ast \) (in \( H \)).

**Step 2:** Strong convergence.
We next show that the sequences \( u_n, m_n \) converge in norm. Let \( f_n = f_0 + u_n \) and \( S_n = \hat{S}_{\kappa_n} \).

First note that
\[
(1 - f_n^2 - m_n^2)m_n - (1 - f_p^2 - m_p^2)m_p = (1 - f_n^2)w - (m_n^2 + m_p^2)w + (f_n^2 - f_p^2)m_p.
\]

Hence, using compact embeddings of \( X, H \) into \( L^q \) for \( 2 < q < \infty \),
\[
\int \left[ ((m_n - m_p)^2) + g(m_n - m_p)^2 \right] r \, dr
\]
\[
= \int \left[ [(1 - f_n^2) - (m_n^2 + m_p^2)(m_n - m_p)] (m_n - m_p)^2 \right] r \, dr
\]
\[
+ \int (f_n + f_p)(u_n - u_p)m_p(m_n - m_p) r \, dr
\]
\[
= o(1).
\]

Therefore, \( m_n \to m_\ast \) in norm.

We proceed in the same way with \( u_n \):
\[
\int \left\{ (u_n' - u_p')^2 + \frac{(d - S_n)^2}{r^2} f_n - \frac{(d - S_p)^2}{r^2} f_p \right\} (u_n - u_p) \, dr
\]
\[
= \int \left\{ (1 - f_n^2)(f_n - (1 - f_p^2 - m_p^2)f_p) \right\} (u_n - u_p) \, dr
\]
\[
(5.7)
\]

Now we expand,
\[
\frac{(d - S_n)^2}{r^2} f_n - \frac{(d - S_p)^2}{r^2} f_p = \left[ \frac{d^2}{r^2} - \frac{S_n}{r^2} (2d - S_n) \right] (u_n - u_p)
\]
+ f_0 \left[ \frac{S_p}{r^2} (2d - S_p) - \frac{S_n}{r^2} (2d - S_n) \right].

Now we take each term separately:

\[
\int_0^1 \frac{S_n}{r^2} (2d - S_n) (u_n - u_p)^2 \, r \, dr \leq \sup_{r \in [0,1]} |S_n| \int_0^1 \frac{2d(u_n - u_p)^2}{r^2} \, r \, dr \to 0,
\]

since $S_n \to 0$ locally uniformly, and $u_n$ are uniformly bounded.

\[
\int_1^{\infty} \frac{S_n}{r^2} (2d - S_n) (u_n - u_p)^2 \, r \, dr \leq \sup |S_n| \int_1^{\infty} 2d(u_n - u_p)^2 \, r \, dr \to 0,
\]

by (5.4). Choose $r_0 > 0$ so that \( \int_{r_0}^{\infty} r^{-2} \, dr < \varepsilon^2/4 \), and \( \kappa \) sufficiently large so that

\[
dr_0^2 \sup \frac{|S_p|}{r} \| u_n - u_p \|_X < \varepsilon^2/2.
\]

Then,

\[
\int_0^{r_0} f_0 \frac{S_p}{r^2} (2d - S_p) (u_n - u_p) \, r \, dr \leq dr_0^2 \sup_{0 \leq r \leq r_0} \frac{|S_p|}{r} \sqrt{\int_0^{r_0} \frac{(u_n - u_p)^2}{r^2} \, r \, dr} \leq \varepsilon/2,
\]

and

\[
\int_{r_0}^{\infty} f_0 \frac{S_p}{r^2} (2d - S_p) (u_n - u_p) \, r \, dr \leq 2d^2 \left[ \int_{r_0}^{\infty} \frac{dr}{r^3} \right]^{1/2} \left[ \int_0^{\infty} (u_n - u_p)^2 \, r \, dr \right]^{1/2} < \varepsilon/2.
\]

We return to (5.7), and substitute the above estimates:

\[
\int \left\{ (u_n' - u_p')^2 + \frac{d^2}{r^2} (u_n - u_p)^2 \right\} \, r \, dr + o(1)
\]
\[
= \int \left\{ (1 - f_n^2 - m_n^2) f_n - (1 - f_p^2 - m_p^2) f_p \right\} (u_n - u_p) \, r \, dr
\]
\[
= \int \left\{ (1 - 3f_0^2)(u_n - u_p)^2 - 3f_0(u_n^2 - u_p^2)(u_n - u_p) \right. \\
\left. - (u_n^3 - u_p^3)(u_n - u_p) - f_0(m_n^2 - m_p^2)(u_n - u_p) \right\} \, r \, dr
\]
\[
= -2 \int (u_n - u_p)^2 \, r \, dr + o(1),
\]
where we use the facts that \( m_n \to m_* \) strongly in \( H \), \( u_n \) is bounded in \( X \), and \( u_n \to u_* \) in \( L^2_{\text{loc}} \). In conclusion, the subsequence \( u_n \to u_* \) strongly in \( X \).

**Step 3:** Determining when \( m_\infty = 0 \).

Since all solutions of (GL)\(_{\infty,g}\) with \( g \geq g_*\) have \( m_\infty = 0 \), we have \( \hat{m}_\kappa \to 0 \) when \( g \geq g_*\).

On the other hand, suppose \( \hat{m}_\kappa > 0 \) for all sufficiently large \( \kappa \), but \( \hat{m}_\kappa \to 0 \). By uniqueness of the normal core solution, \( \hat{f}_\kappa \to \tilde{f}_\infty \), the unique solution of
\[
-\Delta_r \tilde{f}_\infty + \frac{d^2}{r^2} \tilde{f}_\infty = (1 - \tilde{f}_\infty^2) \tilde{f}_\infty.
\]

Let
\[
t_\kappa = \int (1 - \hat{f}_\kappa^2) \hat{m}_\kappa^2 \, r \, dr \to 0,
\]
and set \( w_\kappa = \hat{m}_\kappa / t_\kappa \). Then
\[
-\Delta_r w_\kappa + gw_\kappa = (1 - \hat{f}_\kappa^2 - \hat{m}_\kappa^2) w_\kappa.
\]
Since
\[
\int \left[ (w_\kappa')^2 + gw_\kappa^2 \right] \, r \, dr = \int (1 - \hat{f}_\kappa^2 - \hat{m}_\kappa^2) w_\kappa^2 \, r \, dr \leq 1
\]
(by the choice of \( t_\kappa \)), the bound \( \|w_\kappa\|_H \leq 1/g \) results. We extract a subsequence (which we still denote by \( w_\kappa \)) with \( w_\kappa \rightharpoonup w_\infty \) weakly in \( H \). Note that \( w_\infty \geq 0 \). By the choice of \( t_\kappa \), the uniform convergence \( \hat{f}_\kappa \to \tilde{f}_\infty \), and the \( L^2_{\text{loc}} \) convergence of \( w_\kappa \to w_\infty \) we have:
\[
\int (1 - \tilde{f}_\infty^2)w_\infty^2 \, r \, dr = \int \left[ (1 - \tilde{f}_\infty^2)(w_\infty^2 - w_\kappa^2) + (\hat{f}_\kappa^2 - \tilde{f}_\infty^2)w_\kappa^2 + (1 - \hat{f}_\kappa^2)w_\kappa^2 \right] \, r \, dr
\]
\[
= 1 + o(1).
\]
In particular $w_\infty \neq 0$. By weak convergence we may pass to the limit in the equation for $w_\kappa$, and hence $w_\infty$ is a nontrivial non-negative solution of

$$ -\Delta_r w_\infty + gw_\infty = (1 - f_\infty^2)w_\infty. $$

This can only occur when $g = g_\infty^*$. 

This completes the proof of Theorem 5.1.

6 Estimates and existence

In this section we derive the technical estimates which were needed in our analysis of the bifurcation problem in Section 4. We also provide the details of the proof of existence of minimizers of the energies $E_{\kappa,g}$ and $E_{\infty,g}$.

6.1 A priori estimates

We may now prove a priori estimates for the solutions of our system $(\text{GL})_{\kappa,g}$, Theorem 4.7, as well as the compactness result for solutions of $(\text{GL})_{\infty,g}$ (both theorems as stated in the previous section.) Note that both theorems are stated for all solutions, not only energy minimizers, and hence we will use our Pohozaev identity (Proposition 5.4) to obtain energy independent estimates. As before, we denote by $\tilde{f}_\kappa$, $\tilde{S}_\kappa$ a normal core solution at $\kappa$, and $u = f - \tilde{f}_\kappa$, $v = (S - \tilde{S}_\kappa)/r$.

By the Pohozaev identity and Lemma 4.2 of [BC 89] we have

$$ \kappa^2 \int \left[ gm^2 + \frac{1}{2}(1 - f^2 - m^2)^2 \right] r \, dr = \int \left( \frac{S'}{r} \right)^2 r \, dr \leq \frac{d^2}{2}. \tag{6.1} $$

In particular, we obtain

$$ \int (1 - f)^2 r \, dr \leq C + C/g, \quad \int m^2 r \, dr \leq C/g $$

with constant $C$ depending on $\kappa, d$. From the first estimate we obtain

$$ \|u\|_2 \leq \|\tilde{f}_\kappa + u - 1\|_2 + \|\tilde{f}_\kappa - 1\|_2 \leq C + C/g. $$

The equation for $m$ together with the second estimate gives:

$$ \int \left[ (m')^2 + \kappa^2 gm^2 \right] r \, dr = \kappa^2 \int (1 - f^2 - m^2)m^2 r \, dr \leq \kappa^2 \int m^2 r \, dr \leq C/g. $$
In particular, \( \|m\|_H \leq C \), \( \|u\|_2 \leq C \), and the constant depending on \( \kappa, d \) may be chosen uniformly for \( g \in J \).

Using the right half of (6.1) we have

\[
\frac{d^2}{2} \geq \int \left( \frac{S'}{r} \right)^2 r \, dr = \int \left[ \left( \frac{\tilde{S}_{\kappa}'}{r} \right)^2 + 2 \frac{\tilde{S}_{\kappa}' (rv)'}{r} + \left( \frac{(rv)'}{r} \right)^2 \right] r \, dr.
\]

Since

\[
\left| 2 \int \frac{\tilde{S}_{\kappa}' (rv)'}{r} r \, dr \right| \leq 2 \int \left( \frac{\tilde{S}_{\kappa}'}{r} \right)^2 + \frac{1}{2} \int \left( \frac{(rv)'}{r} \right)^2 r \, dr,
\]

and

\[
\int \left( \frac{(rv)'}{r} \right)^2 r \, dr = \int \left( (v')^2 + \frac{v^2}{r^2} \right) r \, dr,
\]

we may conclude from (6.2) that

\[
\int \left( (v')^2 + \frac{v^2}{r^2} \right) r \, dr \leq C,
\]

with constant depending only on \( d \). From the embedding properties of \( X \), Lemma 2.1, we conclude that \( \|v\|_\infty \leq C \).

We now use \( v \) as a test function in the weak form of the equation for \( S \) to obtain an estimate:

\[
\left| \int \frac{d - S}{r} f^2 v r \, dr \right| = \left| \int \frac{S'}{r^2} (rv)' r \, dr \right| \leq \frac{1}{2} \int \left( \frac{S'}{r^2} \right)^2 r \, dr + \frac{1}{2} \int \left[ (v')^2 + \frac{v^2}{r^2} \right] r \, dr \leq C.
\]
On the other hand, expanding the left-hand side of (6.4),

\[(6.5) \quad \int \left( \frac{d - S}{r} \right) f^2 v r \, dr = \int \left[ \frac{d - \tilde{S}_\kappa}{r} - v \right] (\tilde{f}_\kappa + u)^2 v r \, dr \]

\[= \int \left( \frac{d - \tilde{S}_\kappa}{r} \right) [\tilde{f}_\kappa^2 + 2 \tilde{f}_\kappa u + u^2] v r \, dr - \int 2 \tilde{f}_\kappa u v^2 r \, dr \]

\[- \int v^2 \tilde{f}_\kappa^2 r \, dr - \int v^2 u^2 r \, dr. \]

To bound the term \(\int v^2 \tilde{f}_\kappa^2 r \, dr\), we need to evaluate the other terms:

\[\left| \int \frac{d - \tilde{S}_\kappa}{r} \tilde{f}_\kappa^2 v r \, dr \right| \leq 2 \int \left[ \frac{d - \tilde{S}_\kappa}{r} \right]^2 \tilde{f}_\kappa^2 r \, dr + \frac{1}{8} \int \tilde{f}_\kappa^2 v^2 r \, dr \leq C + \frac{1}{8} \int \tilde{f}_\kappa^2 v^2 r \, dr, \]

\[2 \left| \int \frac{d - \tilde{S}_\kappa}{r} \tilde{f}_\kappa u v r \, dr \right| \leq \int \left[ \frac{d - \tilde{S}_\kappa}{r} \right]^2 \tilde{f}_\kappa^2 r \, dr + \|v\|_\infty^2 \|u\|_2^2 \leq C, \]

\[\left| \int \frac{d - \tilde{S}_\kappa}{r} u^2 v r \, dr \right| \leq \frac{1}{2} \|u\|_2^2 \|v\|_\infty^2 + \frac{1}{2} \int \left[ \frac{d - \tilde{S}_\kappa}{r} \right]^2 u^2 r \, dr, \]

\[2 \left| \int \tilde{f}_\kappa u v^2 r \, dr \right| \leq 8 \|u\|_2^2 \|v\|_\infty^2 + \frac{1}{8} \int \tilde{f}_\kappa^2 v^2 r \, dr \leq C + \frac{1}{8} \int \tilde{f}_\kappa^2 v^2 r \, dr, \]

\[\int v^2 u^2 r \, dr \leq \|v\|_\infty^2 \|u\|_2^2 \leq C. \]

Hence,

\[(6.6) \quad \frac{3}{4} \int v^2 \tilde{f}_\kappa^2 r \, dr \leq C + \frac{1}{2} \int \left[ \frac{d - \tilde{S}_\kappa}{r} \right]^2 u^2 r \, dr \]

Finally, we use \(u\) as a test function in the weak form of the equation for \(f\). Recalling the definition of \(\tilde{f}_\kappa\) as a normal core solution, we expand and cancel terms to arrive at:

\[(6.7) \quad \int \left[ (u')^2 + \left( \frac{d - \tilde{S}_\kappa}{r} \right)^2 \right] r \, dr = \int \left[ 2 \left( \frac{d - \tilde{S}_\kappa}{r} \right) (\tilde{f}_\kappa + u)uv - (\tilde{f}_\kappa + u)uv^2 \right] r \, dr \]
$$+\kappa^2 \int \left[ (1 - 3\tilde{f}_\kappa^2)u^2 - 3\tilde{f}_\kappa u^3 - u^4 - m^2fu \right] r\,dr.$$ 

Each term on the right hand side may be controlled as follows:

$$\left| \int m^2fu r\,dr \right| \leq \frac{1}{2} \int m^4 + \frac{1}{2} \int u^2 r\,dr \leq C,$$

$$\left| \int (1 - 3\tilde{f}_\kappa^2)u^2 r\,dr \right| \leq 3\|u\|^2_2 \leq C,$$

$$2 \left| \int \left( \frac{d - \tilde{S}_\kappa}{r} \right) \tilde{f}_\kappa u v r\,dr \right| \leq \|u\|^2_2\|v\|^2_\infty + \int \left( \frac{d - \tilde{S}_\kappa}{r} \right)^2 \tilde{f}_\kappa^2 r\,dr \leq C,$$

$$\left| \int \tilde{f}_\kappa u^3 r\,dr \right| \leq \frac{3}{2} \int \tilde{f}_\kappa^2 u^2 r\,dr + \frac{1}{2} \int u^4 r\,dr,$$

$$2 \left| \int \left( \frac{d - \tilde{S}_\kappa}{r} \right) u^2 v r\,dr \right| \leq 6\|u\|^2_2\|v\|^2_\infty + \frac{1}{6} \int \left( \frac{d - \tilde{S}_\kappa}{r} \right)^2 u^2 r\,dr$$

$$\leq C + \frac{1}{6} \int \left( \frac{d - \tilde{S}_\kappa}{r} \right)^2 u^2 r\,dr,$$

$$\left| \int \tilde{f}_\kappa uv^2 r\,dr \right| \leq \frac{1}{2} \|u\|^2_2\|v\|^2_\infty + \frac{1}{2} \int \tilde{f}_\kappa^2 v^2 r\,dr \leq C + \frac{1}{3} \int \left( \frac{d - \tilde{S}_\kappa}{r} \right)^2 u^2 r\,dr,$$

where in the last estimate we apply (5.6). Using (5.7) we have

$$\int \left[ (u')^2 + \frac{1}{2} \left( \frac{d - \tilde{S}_\kappa}{r} \right)^2 u^2 \right] r\,dr \leq C.$$ 

Consequently, $\|u\|_X \leq C$. Returning to (6.4), it follows that

$$\int \tilde{f}_\kappa^2 v^2 r\,dr \leq C + \frac{2}{3} \int \left( \frac{d - \tilde{S}_\kappa}{r} \right)^2 u^2 r\,dr \leq C,$$

and hence (6.3) yields $\|v\|_X \leq C$. This concludes the proof of Theorem 4.7.
An analogous result may be proven for solutions of \((GL)_{\infty,g}\):

**Theorem 6.1** Let \(d\) be fixed. For any compact interval \(J \in (0, \infty)\) there exists \(C_0 = C_0(d, J) > 0\) such that every admissible solution \((f,m)\) of \((GL)_{\infty,g}\) with \(g \in J\) satisfies \(||(f,m)||_Z^0 \leq C\).

The proof of Theorem 6.1 is similar to (and simpler than) the previous one, and is left to the reader.

### 6.2 Compactness

Here we prove Theorem 4.6, which asserts that the family of solutions to \((GL)_{\infty,g}\) with \(g\) bounded away from zero is a compact set. The same result holds for \((GL)_{\kappa,g}\), although the proof is more complicated due to the additional terms involving \(S(r)\).

**Proof of Theorem 4.6:** Suppose \(f_n = \tilde{f}_\infty + u_n, m_n, g_n\) are a sequence of solutions of \((GL)_{\infty,g_n}\) with \(g_n \in [a, b]\). By the Theorem we have \(\|u_n\|_X, \|m_n\|_H \leq C\), and hence we may extract a subsequence with \(u_n \to \tilde{u}, m_n \to \tilde{m}, \) and \(g_n \to \tilde{g} \in [a, b]\). Then we have

\[
\int [(u'_n - u'_k)^2 + \frac{d^2}{r^2}(u_n - u_k)^2] \, r \, dr =
\]

\[
\int [(1 - f^2_n - m^2_n)f_n - (1 - f^2_k - m^2_k)f_k] (u_n - u_k) \, r \, dr
\]

(6.8)

\[
\int [(m'_n - m'_k)^2 + \tilde{g}(m_n - m_k)^2] \, r \, dr =
\]

\[
\int [(1 - f^2_n - m^2_n)m_n - (1 - f^2_k - m^2_k)m_k] (m_n - m_k) \, r \, dr + o(1).
\]

(6.9)

We now expand the two right-hand side terms. First, we use the embedding properties of \(X, H\) and the fact that \(0 \leq f_n < 1\) for any solution to show:

\[
[1 - f^2_n - m^2_n]f_n - (1 - f^2_k - m^2_k)f_k (u_n - u_k) =
\]

\[
- \int 2\tilde{f}^2(u_n - u_k)^2 \, r \, dr + \int (1 - \tilde{f}^2)x(u_n - u_k)^2 \, r \, dr
\]

\[
-2 \int [\tilde{f}_\infty(u_n + u_k)(u_n - u_k)^2 - f_n(u_n + u_k)(u_n - u_k)^2] \, r \, dr
\]
\[-\int [f_n(m_n + m_k)(m_n - m_k)(u_n - u_k) + (u^2_n + m^2_k)(u_n - u_k)^2] \, r \, dr\]

\[= -\int 2\tilde{f}_\infty(u_n - u_k)^2 \, r \, dr + o(1).\]

Applying the above estimate to (6.8) we have

\[
\int \left[(u'_n - u'_k)^2 + \left(\frac{d^2}{r^2} + 2\tilde{f}_\infty\right)(u_n - u_k)^2\right] \, r \, dr \to 0
\]
as \(n, k \to \infty\), so \(u_n \to \tilde{u}\) in norm on the space \(X\).

Similarly, we estimate

\[
\int \left[(1 - f^2_n - m^2_n)m_n - (1 - f^2_k - m^2_k)m_k\right] (m_n - m_k) \, r \, dr =
\]

\[
\int (1 - \tilde{f}_\infty)^2(m_n - m_k)^2 \, r \, dr - \int [2\tilde{f}_\infty u_n(m_n - m_k)^2
\]

\[+ 2\tilde{f}_\infty m_k(m_n - m_k)(u_n - u_k) + u^2_n(m_n - m_k)^2] \, r \, dr\]

\[= -\int [m_k(u_n^2 - u_k^2)(m_n - m_k) + (m^3_n - m^3_k)(m_n - m_k)] \, r \, dr\]

\[= o(1).\]

Therefore, (6.9) implies that \(m_n \to \tilde{m}\) in \(H\). By passing to the limit in the weak formulation of \((GL)_{g_n,\infty}\) we easily obtain that \((\tilde{f}, \tilde{m})\) solve \((GL)_{\tilde{g},\infty}\), and hence the specified solution set is compact.

\(\diamondsuit\)

### 6.3 Existence

Let \((u_n, v_n, m_n)\) be a minimizing sequence for \(I_{\kappa, g}\), so \((f_n, S_n, m_n) = (f_0 + u_n, S_0 + rv_n, m_n)\) is a minimizing sequence for \(\mathcal{E}_{\kappa, g}\). To prove Theorem 2.2 we first observe that the energy \(\mathcal{E}_{\kappa, g}\) is a sum of positive terms, and hence each is individually bounded. In particular, \(m_n\) is uniformly bounded in \(H\).
Now we must estimate \( u_n \). First, note that \( E_{\kappa,g}(|f_n|, S_n, m_n) = E_{\kappa,g}(f_n, S_n, m_n) \), and so we may assume that our minimizing sequence satisfies \( f_n(r) \geq 0 \) for all \( r \). Next, we observe

\[
\|1 - f_n^2\|_2 \leq \|1 - f_n^2 - m_n^2\|_2 + \|m_n^2\|_2 \leq \|1 - f_n^2 - m_n^2\|_2 + C.
\]

Hence we conclude that

\[
C \geq E_{\kappa,g}(f_n, S_n, m_n) \geq \int \left[ (f_n')^2 + \left( \frac{S_n'}{r} \right)^2 - \frac{(d - S_n)}{r^2} f_n^2 + \frac{\kappa^2}{2} (1 - f_n^2)^2 \right] r \, dr.
\]

The right-hand side of the above inequality is the free energy of conventional Ginzburg–Landau vortices studied in [ABG 99]. The boundedness of \( \|u_n\|_X, \|v_n\|_X \) then follows from the argument of Proposition 4.2 of [ABG 99]. We may then pass to the limit in \( E_{\kappa,g} \) via lower semicontinuity of the norms and Fatou’s Lemma.

To prove Theorem 2.3 let \((u_n, m_n)\) be a minimizing sequence for \( I_\infty \) in \( X \times H \), so \((f_n, m_n) = (\tilde{f}_\infty + u_n, m_n)\) is a minimizing sequence for \( E_{\kappa,g} \). Choose \( r_\delta \geq 1 \) so that \( \frac{d^2}{r_\delta^2} \leq \frac{g}{2} \). Then

\[
E_{\kappa,g}(f_n, m_n) \geq \int_0^{r_\delta} \left[ (m_n')^2 + \left( \frac{S_n'}{r} \right)^2 + \frac{(d - S_n)^2 f_n^2}{r^2} + \frac{\kappa^2}{2} (1 - f_n^2)^2 \right] r \, dr + \int_{r_\delta}^{\infty} \left[ (m_n')^2 + (g - d^2/r^2) m_n^2 + \frac{d^2}{r^2} (f_n^2 + m_n^2 - 1) \right. \\
+ \left. \frac{1}{2} (f_n^2 + m_n^2 - 1)^2 + \frac{d^2}{r^2} (1 - \tilde{f}_\infty^2) \right] r \, dr \\
\geq \int_0^{r_\delta} \left[ (m_n')^2 + \frac{g}{2} m_n^2 \right] r \, dr - \int_0^{r_\delta} \frac{d^2}{r^2} \tilde{f}_\infty^2 r \, dr + \int_{r_\delta}^{\infty} \left[ \frac{d^2}{r^2} (1 - \tilde{f}_\infty^2) - \frac{d^4}{2r^4} \right] r \, dr,
\]

where we have used the elementary bound \( ax + x^2/2 \geq -a^2/2 \). In particular, \( E_{\kappa,g} \) is bounded below and the minimizing sequence has \( \|m_n\|_H \leq C \) uniformly in \( n \). By the Sobolev embedding, we also conclude that \( \|m_n\|_p \leq C_p \) for all \( p \in [2, \infty) \).
Now we must estimate $u_n$. As above we note that $\mathcal{E}_{\infty,g}(|f_n|, m_n) = \mathcal{E}_{\infty,g}(f_n, m_n)$, and so we may assume that our minimizing sequence satisfies $f_n(r) \geq 0$ for all $r$, and the bound (6.10) holds. Note that we also have:

$$\|u_n\|_2 \leq \|\tilde{f}_\infty - 1\|_2 + \|1 - f_n\|_2 \leq C + \|1 - f_n\|_2. \tag{6.11}$$

By the estimate on $m_n$, (6.10), and (6.11) we now have

$$C \geq \int \left[ \left( f'_n \right)^2 + \frac{d^2}{r^2} (f_n^2 - \tilde{f}_\infty^2) + \frac{1}{2} (1 - f_n^2)^2 \right] r \, dr$$

$$= \int \left[ \left( u'_n \right)^2 + 2\tilde{f}_\infty u'_n + \left( \tilde{f}'_\infty \right)^2 + \frac{d^2}{r^2} (2\tilde{f}_\infty u_n + u_n^2) + \frac{1}{2} (1 - f_n^2)^2 \right] r \, dr$$

$$= \int \left[ \left( u'_n \right)^2 + \left( \tilde{f}'_\infty \right)^2 + \frac{d^2}{r^2} u_n^2 + \frac{1}{2} (1 - f_n^2)^2 + 2(1 - \tilde{f}_\infty^2)\tilde{f}_\infty u_n \right] r \, dr$$

$$\geq \int \left[ \left( u'_n \right)^2 + \frac{d^2}{r^2} u_n^2 + \frac{1}{4} u_n^2 - 8(1 - \tilde{f}_\infty^2)^2 - \frac{1}{8} \tilde{f}_\infty^2 u_n^2 \right] r \, dr - C$$

$$\geq \int \left[ \left( u'_n \right)^2 + \frac{d^2}{r^2} u_n^2 + \frac{1}{8} u_n^2 \right] r \, dr - C.$$

In conclusion $\|u_n\|_X \leq C$. We extract a subsequence for which both $u_n \rightharpoonup u_0$ and $m_n \rightharpoonup m_0$ weakly in $X, H$ respectively, and pointwise almost everywhere.

By semicontinuity of the norm, Fatou's Lemma (for the positive terms) and the $L^2_{r,loc}$ convergence of $u_n \rightarrow u_0$ we can pass to the limit in (2.3):

$$I_{\infty,g}(u_0, m_0) \leq \liminf_{n \rightarrow \infty} I_{\infty}(u_n, m_n) = \inf_{X \times H} I_{\infty}.$$

So the infimum of $I_{\infty}$ is attained.

$\Diamond$
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