GENERIC BOHMIAN TRAJECTORIES OF AN ISOLATED PARTICLE

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Abstract

The generic Bohmian trajectories are calculated for an isolated particle in an approximate energy eigenstate, for an arbitrary one-dimensional potential well. It is shown, that the necessary and sufficient condition for there to be a negligible probability of the trajectory deviating significantly from the classical trajectory at any stage in the motion is, that the state be a narrowly localised wave packet. The properties of the Bohmian trajectories are discussed in relationship to the theory of retrodictively optimal simultaneous measurements of position and momentum which was presented in several previous papers. It is shown that the Bohmian velocity at $x$ is the expectation value of the velocity which would be observed at $x$, if one were to make a retrodictively optimal simultaneous measurement of $x$ and $p$, in the limit as the error in the measurement of $x$ tends to zero. This explains the tendency of the Bohmian particle to behave in a highly non-classical manner. It also explains why the trajectories in the interpretation recently proposed by García de Polavieja tend to be much more nearly classical in the limit of large quantum number. The implications for other trajectory interpretations are considered.
1. Introduction

This is the first of two papers in which we investigate the classical limit in the Bohm interpretation of quantum mechanics [1, 2, 3]. It is well known that the Bohmian trajectories can be highly non-classical. We are interested in the question, whether this is true of the macroscopic bodies of our ordinary experience.

Different views have been expressed in the literature. Bohm and Hiley [2] have argued that the Bohm interpretation does successfully account for the existence of an approximately classical level of phenomena due to the effect of the electromagnetic radiation and other particles incident on a macroscopic object such as a planet. On the other hand, Holland [4] has argued that Bohm’s theory may not be rich enough to embrace the full variety of possible classical motions and, in consequence, that it may not be a universal physical theory. Holland [5] has gone on to propose an alternative trajectory interpretation, which he hopes may prove more satisfactory in this respect.

The issues raised by these authors are of some importance. There has been much discussion [6, 7, 8, 9, 10, 11, 12] of the fact that the Bohmian trajectory of a micro-object can, under certain circumstances, be “surreal.” Although this behaviour is highly counter-intuitive, it does not provide the grounds for a clear logical objection since there is no actual conflict with experiment. On the other hand, it would create very serious difficulties for the interpretation if it could be shown that, under the conditions of our ordinary experience, the Bohmian trajectory of a macro-object can be significantly different from the trajectory predicted by classical mechanics. This is because the Bohm interpretation is usually based on what Fine [13] describes as an assumption of “accessibility”. That is, it is assumed that, at least in the case of a macro-object, the position which is (as one would normally say) directly perceived closely corresponds to the position which actually exists (modulo exceptional instances of hallucination etc.). Bell [14] makes the point with his usual vigour and clarity when he says that, in the Bohm interpretation, the positions of macroscopic objects, under the conditions under which we normally experience them, are very far from being “hidden”:

Absurdly, such theories are known as ‘hidden variable’ theories. Absurdly, for there it is not in the wavefunction that one finds an image of the visible world, and the results of experiments, but in the complementary ‘hidden’(!) variables.

It should be noted that it is not simply an instantaneous image that is needed. One also needs to be able to assume that our memory traces, of the way in which a macroscopic body appears to have moved in the past, closely correspond to the way in which it actually moved. In other words, one needs the whole trajectory to be “accessible”, and not just the instantaneous position.

It may be asked whether this assumption is strictly necessary. If one drops the assumption, then one is committed to the view that the actual trajectory of a macroscopic body can be markedly and systematically different from its apparent trajectory. It might, perhaps, be possible to reconstruct the Bohm interpretation along such lines. Indeed, Page [15] has made some definite proposals in this connection. However, as Page points out, one would then be making the interpretation depend on profoundly difficult, and hitherto unresolved questions regarding the nature of human consciousness. Moreover, it is hard to see what would be achieved by postulating the existence of “beables” of such a radically elusive kind. An ontological interpretation such as this has no obvious advantage over the Copenhagen interpretation. In short, although the assumption might, perhaps, not be strictly necessary, dropping the assumption would involve the interpretation in very considerable difficulties. This is why the questions discussed by Bohm and Hiley [2].
and by Holland [4], are important. If it should transpire that the Bohmian trajectories of macroscopic objects are not typically quasi-classical under the conditions of our ordinary experience, then the assumption of accessibility would clearly not be justified.

In the sequel to this paper we will give some additional arguments in support of Bohm and Hiley’s conclusion, that environmental effects cause the Bohmian trajectory of a macroscopic object typically to become quasi-classical. However, before considering the effect of the environment, it is natural to ask what is the typical behaviour when the body is isolated. This is the question addressed in the present paper.

The paper is in two main parts. The purpose of the first part (Sections 2 and 3) is to investigate the sense in which it is true, that the Bohmian trajectories of an isolated body are *generically* non-classical.

Consider a particle moving in an arbitrary one-dimensional potential well. If it is in an energy eigenstate, then the Bohmian velocity is zero. However, it could be argued [16] that the significance of this fact is somewhat unclear, since such states are not typical. One might argue that a macroscopic body is very unlikely to be in an *exact* energy eigenstate. We are therefore led to consider the case when the system is in an approximate energy eigenstate, of the form

\[ |\psi\rangle = \sum_{r=-\Delta n}^{\Delta n} c_r |\bar{n} + r\rangle \]

(1)

where \( |n\rangle \) denotes the \( n \)th energy eigenstate, with energy \( E_n \). Since we are interested in the classical limit we assume that the state is highly excited, \( \bar{n} \gg 1 \). The fact that \( |\psi\rangle \) is an approximate energy eigenstate means that \( \Delta n \ll \bar{n} \).

We wish to establish the conditions which the coefficients \( c_r \) must satisfy in order to ensure that there is a negligible probability of non-classical behaviour. We begin, in Section 2, by showing that the necessary and sufficient condition for there to be a negligible probability of the instantaneous Bohmian velocity being significantly different from the classical velocity at any time during the motion is that \( |\psi\rangle \) is a narrowly localised wave-packet.

This result does not entirely settle the question since the instantaneous Bohmian velocity typically undergoes rapid fluctuations. Squires [10] has argued that, for the purposes of a comparison with classical physics, the relevant quantity to consider is, not the instantaneous velocity, but a suitable time-average. In Section 3 we calculate the time-averaged velocity. We show that the necessary and sufficient condition for there to be probability \( \approx 1 \) of the time-averaged velocity always being close to the classical value is again, that \( |\psi\rangle \) is a narrowly localised wave-packet.

These results show that, for an isolated particle in an approximate energy eigenstate, the Bohm interpretation produces quasi-classical trajectories in just those cases where no such interpretation is needed (the conceptual difficulties which originally led Bohm to propose his interpretation arise from the possible occurrence of superpositions of macroscopically distinguishable states, as in the paradigmatic instance of Schrödinger’s cat [17]). They consequently show that the interaction with the environment plays an essential role in the Bohm interpretation, just as it does in other approaches to the problem of interpretation [18, 19, 20, 21, 22].

In the second part of the paper (Sections 4 and 5) we investigate the underlying reasons for the behaviour identified in the preceding Sections. The Bohm interpretation, although it was historically the first, and although it seems to be mathematically the most straightforward, is by no means the only interpretation in which the particles follow well-defined trajectories [4, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41]. The interpretation proposed by García...
de Polavieja is particularly noteworthy from our point of view, since it would appear to produce the correct classical limit even in the case of an isolated system, without any need to take into account the effect of the environment. This suggests that the counter-intuitive behaviour of the Bohmian trajectories is not a general characteristic of every possible trajectory interpretation, but is due instead to the particular manner in which the Bohm interpretation has been constructed. In Section 4 we compare the two interpretations. We argue that the reason the trajectories in García de Polavieja’s interpretation tend to become quasi-classical in the limit of large quantum number is connected with the fact that the phase space distribution for this interpretation is the Husimi function, which plays an important role in the theory of simultaneous measurement processes. We go on to argue that this also gives some insight into the reason why the trajectories in the Bohm interpretation tend to be highly non-classical (if the system is isolated). In Section 5 we briefly consider the implications for other trajectory interpretations.

2. The Instantaneous Velocity

We consider a particle moving in one space dimension under the influence of a potential $V(x)$. For simplicity we will assume that $V(x)$ has a single minimum, and that the spectrum of the Hamiltonian is purely discrete. It would not be difficult to extend the discussion to the case of more general potentials.

Since we are interested in the limit of large quantum number it is appropriate to use the WKB approximation. Let $|n\rangle$ be the $n$th energy eigenstate, and let $E_n$ be the corresponding eigenvalue. Let

$$p_n(x) = \sqrt{2m(E_n - V(x))}$$

be the momentum of a classical particle with mass $m$ and energy $E_n$ located at $x$. Let $a_{n-} < a_{n+}$ be the turning points of the classical motion, and let

$$\tau_n(x) = \int_{a_{n-}}^x dx' \frac{m}{p_n(x')}$$

be the time which the particle would take classically to get from $a_{n-}$ to $x$. Let

$$T_n = 2\tau_n(a_{n+})$$

be the classical period. Define

$$S_n(x) = \int_{a_{n-}}^x dx' p_n(x') + \frac{\hbar}{8}$$

Provided that $x$ is not close to one of the classical turning points we then have

$$\langle x | n \rangle \approx 2 \left( \frac{m}{T_n p_n(x)} \right)^{\frac{1}{2}} \Theta_n(x) \sin \left( \frac{S_n(x)}{\hbar} \right)$$

where

$$\Theta_n(x) = \begin{cases} 
1 & \text{if } a_{n-} < x < a_{n+} \\
0 & \text{if } x < a_{n-} \text{ or } a_{n+} < x
\end{cases}$$

We are interested in the case when the system is in a state of the form defined by Eq. (1). At time $t$ we have (in the Schrödinger picture)

$$|\psi_t\rangle = \sum_{r=\pm \Delta n} c_r \exp \left( -\frac{iE_n r^2 t}{\hbar} \right) |\bar{n} + r\rangle$$
and
\[
\langle x | \psi_t \rangle \approx \sum_{r=-\infty}^{\infty} ic_r \left( \frac{m}{T_{n+r}p_{n+r}(x)} \right) \frac{1}{2} \Theta_{n+r}(x) \times \left( \exp \left[ -\frac{i}{\hbar} (S_{n+r}(x) + E_{n+r}) \right] - \exp \left[ \frac{i}{\hbar} (S_{n+r}(x) - E_{n+r}) \right] \right)
\]
\[
\langle x | \psi_t \rangle \approx \sum_{r=-\infty}^{\infty} ic_r \left( \frac{m}{T_{n+r}p_{n+r}(x)} \right) \frac{1}{2} \Theta_{n+r}(x) \times \left( \exp \left[ -\frac{i}{\hbar} (S_{n+r}(x) + E_{n+r}) t \right] - \exp \left[ \frac{i}{\hbar} (S_{n+r}(x) - E_{n+r}) t \right] \right)
\]
\[\text{(2)}\]

Since we are assuming that \( \Delta n \ll \bar{n} \) we can make some further approximations. Define
\[
p(x) = p_n(x) \quad E = E_n \quad S(x) = S_n(x)
\]
\[
\tau(x) = \tau_n(x) \quad T = T_n \quad a_{\pm} = a_{n\pm}
\]

If \( n - \bar{n} \ll \bar{n} \) we can approximate
\[
p_n(x) - p(x) \approx \frac{m}{\sqrt{2m(E - V(x))}} (E_n - E) = \frac{m}{p(x)} (E_n - E)
\]
We then use the quantisation condition
\[
\int_{a_{n-}}^{a_{n+}} dx p_n(x) = (2n + 1) \frac{\hbar}{4}
\]
to deduce
\[E_n \approx E + (n - \bar{n})\hbar \omega
\]
and
\[S_n(x) \approx S(x) + (n - \bar{n})\hbar \omega \tau(x)
\]
where \( \omega \) is the classical frequency, \( 2\pi/T \). Using these approximations in Eq. (2) we find
\[
\langle x | \psi_t \rangle \approx i \left( \exp \left[ -\frac{i}{\hbar} (S(x) + Et) \right] g_-(x, t) - \exp \left[ \frac{i}{\hbar} (S(x) - Et) \right] g_+(x, t) \right)
\]
\[\text{(3)}\]
(except in the vicinity of the classical turning points). In this expression we have set
\[
g_{\pm}(x, t) = \left( \frac{m}{Tp(x)} \right)^{\frac{1}{2}} \sum_{r=-\infty}^{\infty} c_r \exp \left[ \pm i\hbar (\tau(x) \mp t) \right] \Theta(x)
\]
\[\text{(4)}\]
where
\[
\Theta(x) = \Theta_n(x)
\]
The imaginary exponentials \( \exp \left[ \pm i(S(x) \mp Et)/\hbar \right] \) are rapidly oscillating functions of \( x \) and \( t \), having spatial period \( \hbar/p(x) \) and frequency \( E/\hbar \). The functions \( g_{\pm} \), by contrast, are much more slowly varying, being effectively constant over distances \( \ll (a_{++} - a_{-})/\Delta n \) and times \( \ll T/\Delta n \).

From the form of the expression on the right hand side of Eq. (4) it can be seen that the function \( g_+(x, t) \) propagates to the right at the classical speed \( p(x)/m \) until it reaches the point \( x = a_{++} \), where it is reflected and becomes the function \( g_-(x, t) \). Similarly, \( g_-(x, t) \) propagates to the left at the classical speed until it reaches the point \( x = a_{--} \), where it is reflected and becomes the function \( g_+(x, t) \).

Let us now calculate the instantaneous Bohmian velocity, given by
\[
v_B(x, t) = \frac{\hbar \text{Im} \left( \langle \psi_t | \frac{\partial}{\partial x} (x | \psi_t) \right)}{m \langle x | \psi_t \rangle^2}
\]
\[\text{(5)}\]
We have from Eq. (3)
\[ \frac{\partial}{\partial x} \langle x | \psi_t \rangle \approx \left( \exp \left[ \frac{i}{\hbar} (S(x) + Et) \right] \left( \frac{p(x)}{\hbar} g_-(x, t) + i \frac{\partial}{\partial x} g_-(x, t) \right) \right. \]
\[ + \exp \left[ \frac{i}{\hbar} (S(x) - Et) \right] \left( \frac{p(x)}{\hbar} g_+(x, t) - i \frac{\partial}{\partial x} g_+(x, t) \right) \left( \Theta(x) \right) \]

(except in the vicinity of the classical turning points). The functions \( g_\pm(x, t) \) are effectively constant over distances \( \sim \) the de Broglie wavelength. We may therefore approximate
\[ \frac{p(x)}{\hbar} g_\pm(x, t) \mp i \frac{\partial}{\partial x} g_\pm(x, t) \approx \frac{p(x)}{\hbar} g_\pm(x, t) \] (6)

It is convenient to write \( g_\pm \) in modulus-argument form:
\[ g_\pm(x, t) = \sqrt{\rho_\pm(x, t)} e^{i \phi_\pm(x, t)} \] (7)

In terms of these quantities, and using the approximation of Eq. (3), we have
\[ \text{Im} \left( \langle \psi_t | x \rangle \frac{\partial}{\partial x} \langle x | \psi_t \rangle \right) \approx \frac{p(x)}{\hbar} (\rho_+(x, t) - \rho_-(x, t)) \] (8)

and
\[ |\langle x | \psi_t \rangle|^2 \approx \rho_+(x, t) + \rho_-(x, t) \]
\[ - 2 \sqrt{\rho_+(x, t) \rho_-(x, t)} \cos \left( \frac{2S(x)}{\hbar} + \phi_+(x, t) - \phi_-(x, t) \right) \] (9)

As \( x \) varies the last term on the right hand side of Eq. (3) fluctuates rapidly, with a spatial period \( \sim \) the de Broglie wavelength. The functions \( \rho_\pm \), by contrast, are nearly constant on this scale. It follows that the quantity
\[ \dot{\rho}(x, t) = \rho_+(x, t) + \rho_-(x, t) \] (10)

is the mean \( x \)-space probability density function, averaged over a de Broglie wavelength.

Inserting the results of Eqs. (3) and (4) in Eq. (3) we find
\[ v_B(x, t) \approx v_{cl}(x) \]
\[ \times \frac{\rho_+(x, t) - \rho_-(x, t)}{\rho_+(x, t) + \rho_-(x, t) - 2 \sqrt{\rho_+(x, t) \rho_-(x, t)} \cos \left( \frac{2S(x)}{\hbar} + \phi_+(x, t) - \phi_-(x, t) \right)} \] (11)

for \( a_- < x < a_+ \) (except in the immediate vicinity of the turning points). In this expression \( v_{cl}(x) = p(x)/m \), the classical speed at position \( x \).

\( \rho_\pm, \phi_\pm \) are slowly varying functions of \( x \). On the other hand the term \( 2S(x)/\hbar \) is very rapidly varying. It follows that \( v_B(x, t) \) varies rapidly between the extremal values
\[ v_\pm(x, t) = v_{cl}(x) \sqrt{\frac{\rho_+(x, t)}{\rho_+(x, t) + \rho_-(x, t)}} \pm \sqrt{\frac{\rho_-(x, t)}{\rho_+(x, t) + \rho_-(x, t)}} \]
over distances \( \sim \) the de Broglie wavelength.

If \( \rho_+(x, t) \gg \rho_-(x, t) \)
\[ v_-(x, t) \approx v_+(x, t) \approx v_{cl}(x) \]
and the motion is approximately classical. If, on the other hand, \( \rho_+(x, t) \ll \rho_-(x, t) \)
\[ v_-(x, t) \approx v_+(x, t) \approx -v_{cl}(x) \]
The motion is again approximately classical, but in the opposite direction.
Suppose, however, that neither of these conditions is satisfied. In that case \(|v_+(x, t)| \gg v_{cl}(x)|\), and the motion is highly non-classical.

The necessary and sufficient condition for the Bohmian velocity to be close to the one of the two possible values of the classical velocity at position \(x\) is, therefore,

\[
\frac{1}{2} \left( \frac{\rho_+(x, t)}{\rho_+(x, t) + \rho_-(x, t)} \right) \gg 1
\]  

(12)

(except in the vicinity of the points \(x = a_\pm\)).

If we only require the state to be such that there is a high probability of the Bohmian velocity being close to \(\pm v_{cl}(x)\) at all times, then we only need to impose condition (12) at points where the mean probability density \(\bar{\rho} = \rho_+ + \rho_-\) is non-negligible [see the remark following Eq. (11)]. It is, however, important that the inequality always holds true at such points, for every time \(t\). Suppose, for example, that at a particular instant the functions \(\rho_\pm\) are as shown in Fig. 3(a). Then \(v_{cl} \approx v_\pm\) at all values of \(x\) for which \(\bar{\rho}\) is non-negligible. However, at a time \(\sim T/4\) later \(\rho_\pm\) will be as shown in Fig. 3(b), so that there is a significant probability of the particle being in a region where \(\rho_+ \approx \rho_-\), and which is well away from the turning points. From a consideration of this and other examples it can be seen that there will only be a high probability of the velocity being close to \(\pm v_{cl}\) throughout the motion if the state is a highly localised wave packet, so that the peak in \(\bar{\rho}\) is very narrow.

Of course, there will be times when \(\rho_+ \approx \rho_-\) at the turning points, even when the peak in \(\bar{\rho}\) is very narrow. However, this does not invalidate the conclusion. In the first place, the approximations leading to Eq. (11) break down when \(x \approx a_\pm\). In the second place, even if Eq. (11) were valid at the turning points, the fact that \(v_{cl} \approx 0\) at these points means that one can still have \(v_\pm \approx v_{cl}\), even though \(\rho_+ \approx \rho_-\).

3. THE TIME-AVERAGED VELOCITY

We saw in the last section that the instantaneous Bohmian velocity is typically very rapidly fluctuating. However, classical physics is based on observations, not of the instantaneous velocity, but rather of the velocity averaged over a finite time interval. Furthermore, the averaging time may be assumed to be large in comparison with the time-scale of the fluctuations in the instantaneous Bohmian velocity. In order to make the argument complete we consequently need to consider the possibility that the time-averaged Bohmian velocity may be consistent with the predictions of classical physics, even when the instantaneous velocity is not.

Referring to Eq. (11) we see that the equation of motion is, approximately,

\[
\frac{dx}{dt} \approx \frac{v_{cl}(x)(\rho_+(x, t) - \rho_-(x, t))}{\rho_+(x, t) + \rho_-(x, t) - 2\sqrt{\rho_+(x, t)\rho_-(x, t)} \cos \left[\frac{\chi_0}{2} S(x) + \phi_+(x, t) - \phi_-(x, t)\right]}
\]

We wish to solve this equation subject to the initial condition \(x = x_0\) when \(t = t_0\). We may assume \((t - t_0) \ll T/\Delta n\) and \(|x - x_0| \ll (a_+ - a_-)/\Delta n\). We can then further approximate

\[
\frac{dx}{dt} \approx \frac{v_{cl}(x_0) \sinh \chi_0}{\cosh \chi_0 - \cos \left[\phi_0 + \frac{\chi_0}{x_0}(x - x_0)\right]}
\]

where

\[
\chi_0 = \frac{1}{2} \log \left( \frac{\rho_+(x_0, t_0)}{\rho_-(x_0, t_0)} \right)
\]

\[
\phi_0 = \frac{2}{\hbar} S(x_0) + \phi_+(x_0, t_0) - \phi_-(x_0, t_0)
\]
Figure 1. The graphs in (a) show the functions $\rho_{\pm}$ at a time when $\rho_+$ has a single, rather broad peak, centred in the middle of the interval $(a_-, a_+)$, and when $\rho_-$ is everywhere negligible. In this situation $v_B$ is everywhere close to $v_{cl}$. The graphs in (b) show the situation at a time $\sim \frac{T}{\lambda}$ later. There is then a significant probability of the particle being in a region where $v_B$ fluctuates violently, up to a maximum which is much greater than the speed which would be expected classically. The arrows show the direction of propagation of the functions. See the discussion in the paragraph following Eq. (12)

and where $\lambda_0 = \hbar / p(x_0)$ is the de Broglie wavelength at position $x_0$. The solution to this equation is

$$
\left(1 - \frac{\lambda_0 \sech \chi_0}{4\pi(x-x_0)} \left[ \sin \left( \phi_0 + \frac{4\pi}{\lambda_0} (x-x_0) \right) - \sin \phi_0 \right] \right) (x-x_0) \\
\approx (v_{cl}(x_0) \tanh \chi_0) (t-t_0) \quad (13)
$$

If $x-x_0 \gg \lambda_0$ we may write

$$
x \approx x_0 + \bar{v}_v (t-t_0)
$$
where $\bar{v}_T$ is the time-averaged velocity

$$\bar{v}_T = \frac{1}{\tau} \int_{t_0}^{t_0+\tau} dt \frac{dx}{dt} \approx v_{cl}(x_0) \tanh \chi_0$$

$\tau$ being the time to move one de Broglie wavelength, $\lambda_0/(v_{cl}(x_0) \tanh \chi_0)$.

In Fig. 2 we illustrate this result by plotting $x-x_0$ as a function of $t-t_0$ for the case $\phi_0 = 0, \chi_0 = 0.01$. See Eq. (13). The broken line shows the time-averaged trajectory $x = x_0 + \bar{v}_T(t-t_0)$. Units have been chosen so that $\lambda_0 = v_{cl}(x_0) = 1$.

The graphs in Fig. 3 illustrate the fact that if $\chi_0 \approx 0$ (so that $\rho_+ \approx \rho_-$) the speed peaks at a value which is very much greater than $v_{cl}$. However, they also show that this phenomenon is of very short duration. An equally significant feature of the motion is the fact that the speed is, for the most part, very much smaller than $v_{cl}$. As a result $|\bar{v}_T| \ll v_{cl}$.

The condition for $|\bar{v}_T|$ to be close to $\pm v_{cl}$ is that $|\chi_0| \gg 0$. This is the same as the condition derived in the last section [c.f. Eq. (12)]. Consequently, the conclusion still stands, that there is only a high probability of the motion being quasi-classical in the case of a narrowly localised wave-packet.

4. Comparison with García de Polavieja’s Interpretation

We now consider the underlying reasons for the behaviour discussed in the last two sections. It might be thought that the counter-intuitive behaviour of the Bohmian trajectories (of an isolated particle) is not very surprising. According to the Copenhagen school of thought the concept of a precisely defined, completely determinate trajectory is illegitimate. The Copenhagen interpretation is not so
Figure 3. In graph (a) the solid line shows the dependence of velocity on time for the case $\phi_0 = 0$, $\chi_0 = 0.01$, with units chosen as in Fig. 2, so that $\lambda_0 = v_{cl}(x_0) = 1$. The broken line shows the time-averaged velocity $\bar{v}_T (= 0.01 \times$ the classical velocity). It can be seen that during the greater part of the motion the particle is travelling more slowly than this. In (b) the graph is reproduced with a different choice of scale on the $v$ axis, so as to include the maxima at $200 \times$ the classical velocity. It can be seen that the peaks are extremely narrow.
widely accepted as once was the case. Nevertheless, there continues to be a wide-
spread feeling that the concept of a determinate trajectory, though not excluded in
point of strict logic, represents an artificial construction which is imposed on the
theory by arbitrary fiat. Someone who takes this point of view may feel that it is
only to be expected that the trajectories will tend to be strikingly counter-intuitive.

In fact, however, it would appear from the work of García de Polavieja [33] that
there is at least one interpretation of this kind in which the trajectories are much
better behaved, and in which one already obtains the correct classical limit even in
the case of an isolated system, without having to take into account the effect of the
environment. This suggests that the counter-intuitive behaviour of the Bohmian
trajectories may actually be due, not to constraints inherent in the very idea of a
trajectory interpretation, but rather to features specific to the particular manner in
which the Bohm interpretation realises this idea. We will now try to identify these
features, by making a comparison between the Bohm interpretation and García de
Polavieja’s interpretation.

In interpretations of the kind we are considering one has to make a choice as
to the intrinsic probability distribution describing the “beables” of the theory. In
the Bohm interpretation one starts with an intrinsic configuration space probability
distribution, which is taken to be $| \langle x | \psi \rangle | ^2$. In García de Polavieja’s interpretation,
by contrast, one starts with an intrinsic phase space probability distribution, which
is taken to be the Husimi, or $Q$-function defined by [42, 43, 44, 48, 51]

$$Q_\lambda(x,p) = \frac{1}{\hbar} | \langle (x,p) | \lambda \rangle \rangle |^2$$

(15)

where $|(x,p)\rangle$ is the coherent state with $x$-representation wave function

$$\langle x' | (x,p)\rangle = \left(\frac{1}{\pi \lambda^2}\right)^{\frac{1}{4}} \exp \left[ -\frac{1}{2\lambda^2} (x' - x)^2 + \frac{i}{\hbar} px' - \frac{i}{2\hbar} px \right]$$

(16)

The significance of the parameter $\lambda$ can be understood by referring to the role
that $Q_\lambda(x,p)$ plays in the theory of measurement. The function $| \langle x | \psi \rangle | ^2$ gives
the distribution of results for ideal, perfectly accurate measurements of position
only. $Q_\lambda(x,p)$, by contrast, gives the distribution when one makes simultaneous,
imperfectly accurate measurements of both position and momentum [45, 46, 50]
(for reviews of the theory of simultaneous measurement processes, and additional
references, see Busch [17] and Leonhardt [18]). The significance of the parameter
$\lambda$ is that it specifies the relative accuracy of the measurements of $x$ and $p$. That
is, $Q_\lambda(x,p)$ describes the outcome when $x$ is measured to retrodictive accuracy
$\Delta_{ei}x = \pm \lambda/\sqrt{2}$ and $p$ is measured to retrodictive accuracy $\Delta_{ei}p = \pm \hbar/(\sqrt{2}\lambda)$, and
when, in addition, the measurements of $x$ and $p$ are retrodictively unbiased [50].
Such measurements are optimal, in the sense that the lower bound set by the
inequality $\Delta_{ei}x\Delta_{ei}p \geq \hbar/2$ is actually achieved [19].

It can be seen that García de Polavieja’s interpretation is in fact, not a single
interpretation, but rather an infinite family of interpretations, parameterised by $\lambda$.
In order to obtain the correct classical limit for an isolated system the value of $\lambda$
must be appropriately chosen. To see this, let us calculate $Q_\lambda$ for states of the type
defined by Eq. (1).

\footnote{As explained in ref. [19] this inequality is not the same as the uncertainty principle usually
so called. The quantities $\Delta_{ei}x$, $\Delta_{ei}p$ are errors, not uncertainties.}
Eqs. (3) and (16) imply
\[ \langle (x, p) | \psi_i \rangle \]
\[ \approx i \left( \frac{1}{\pi \lambda^2} \right)^{\frac{1}{2}} \exp \left[ -\frac{i}{\hbar} \left( E_t - \frac{1}{2} p x \right) \right] \int dx' \exp \left[ -\frac{1}{2\lambda^2} (x' - x)^2 - \frac{i}{\hbar} p x' \right] \]
\[ \times \left( \exp \left[ -\frac{i}{\hbar} S(x') \right] g_-(x') - \exp \left[ \frac{i}{\hbar} S(x') \right] g_+(x') \right) \] (17)
away from the classical turning points. Suppose that
\[ \lambda \ll \lambda_+(x) = \min \left( \frac{a_+ - a_-}{\Delta n}, \left( \frac{\hbar}{|p'(x)|} \right)^{\frac{1}{2}} \right) \] (18)
We can then approximate
\[ g_\pm (x') \approx g_\pm (x) \quad \text{and} \quad S(x') \approx S(x) + p(x)(x' - x) \]
Making these approximations in Eq. (17), carrying out the Gaussian integration, and substituting the result in (15) gives
\[ Q_\lambda(x, p) \approx \frac{\lambda}{\sqrt{\pi h}} \left\{ \exp \left[ -\frac{\lambda^2}{h^2} (p + p(x))^2 \right] \rho_-(x) + \exp \left[ -\frac{\lambda^2}{h^2} (p - p(x))^2 \right] \rho_+ (x) \right. \]
\[ -2 \exp \left[ \frac{\lambda^2}{h^2} (p^2 + (p(x))^2) \right] \cos \left[ \frac{2}{h} S(x) + \phi_+(x) - \phi_-(x) \right] S(x) \rho_+(x) \rho_-(x) \right\} \] (19)
where \( \rho_\pm, \phi_\pm \) are the quantities defined by Eq. (7). Suppose that we also have
\[ \lambda \gg \lambda_-(x) = \frac{\hbar}{p(x)} \] (20)
In that case the third, oscillatory term in parentheses on the right hand side of Eq. (19) is negligible. Also, the Gaussian peaks in the first and second terms are very narrow in comparison with the classical momentum \( p(x) \), and may therefore be regarded as approximate \( \delta \)-functions. We conclude
\[ Q_\lambda(x, p) \approx \delta \left( p + p(x) \right) \rho_-(x) + \delta \left( p - p(x) \right) \rho_+ (x) \] (21)
provided that \( \lambda_-(x) \ll \lambda \ll \lambda_+(x) \). This is a possible classical distribution for a particle of energy \( E_n \)—which suggests that the trajectories in García de Polavieja’s interpretation will also be approximately classical when \( \lambda \) lies within the stated range.
Suppose, on the other hand, that \( \lambda \ll \lambda_-(x) \). In that case the widths of the Gaussian peaks on the right hand side of Eq. (19) are much larger than \( p(x) \), so that we have, approximately,
\[ Q_\lambda(x, p) \approx \frac{\lambda}{\sqrt{\pi h}} \exp \left[ -\frac{\lambda^2}{h^2} p^2 \right] \]
\[ \times \left\{ \rho_-(x) + \rho_+ (x) - 2 \sqrt{\rho_-(x) \rho_+(x)} \cos \left[ \frac{2}{h} S(x) + \phi_+(x) - \phi_-(x) \right] \right\} \]
Comparing this expression with Eq. (11) we see that
\[ Q_\lambda(x, p) \approx \frac{\lambda}{\sqrt{\pi h}} \left| \langle x | \psi_i \rangle \right|^2 \] (22)
If, on the other hand, \( \lambda \to \infty \), then it is not difficult to show [12]
\[
Q_\lambda(x, p) \approx \frac{1}{\sqrt{\pi} \lambda} \exp \left[ -\frac{1}{\lambda^2} x^2 \right] |\langle \psi_t | \psi_\lambda \rangle|^2
\]

These distributions are both highly non-classical. It follows that the trajectories in García de Polavieja’s interpretation will also be highly non-classical in the limit as \( \lambda \) becomes very small, or very large.

One can understand the reason why \( Q_\lambda \) behaves in this way if one considers its interpretation as the probability distribution describing the outcome of a retrodictively optimal joint measurement of \( x \) and \( p \) [42]. Eq. (23) describes a situation in which the error in \( x \) is very small, and the error in \( p \) is correspondingly large. The fact that the error in \( p \) is large means that the measurement is too insensitive to pick up any correlation between the values of \( x \) and \( p \). On the other hand the fact that the error in \( x \) is small means that the measurement is able pick up the very rapid variation in the probability distribution which occurs in the direction parallel to the \( x \) axis, over distances \( \sim \) the de Broglie wavelength. Analogous statements apply to Eq. (22), except that now it is the measurement of \( p \) which is very accurate, and the measurement of \( x \) which is correspondingly inaccurate. Eq. (21), by contrast, describes a situation in which both \( x \) and \( p \) are measured to an intermediate degree of accuracy. The errors are both small enough to ensure that the measured values of \( x \) and \( p \) are highly correlated. At the same time, they are both large enough to ensure that the measurement is insensitive to the very rapid variations in the functions \( |\langle x | \psi_t \rangle|^2 \) and \( |\langle p | \psi_t \rangle|^2 \).

Classical physics is based on situations in which \( x \) and \( p \) have both been determined to an intermediate degree of accuracy— which is the why the distribution of Eq. (21) is of classical form.

Let us now relate this discussion to the behaviour of the Bohmian trajectories. Let \( \bar{v}_\lambda(x) \) be the mean velocity at \( x \) in García de Polavieja’s interpretation:
\[
\bar{v}_\lambda(x) = \frac{\int dp \, p \, Q_\lambda(x, p)}{m \int dp \, Q_\lambda(x, p)}
\]

\( \bar{v}_\lambda(x) \) is also the mean velocity which would be observed at \( x \) if one were to make retrodictively optimal joint measurements of \( x \) and \( p \) to accuracies \( \pm \lambda/\sqrt{2} \) and \( \pm \hbar/\sqrt{2\lambda} \) respectively. Substituting the expression given by Eq. (19) in this equation and taking the limit as \( \lambda \to 0 \) gives
\[
\lim_{\lambda \to 0} (\bar{v}_\lambda(x)) = v_{cl}(x) \frac{\rho_+(x) - \rho_-(x)}{\rho_+(x) + \rho_-(x) - 2\sqrt{|\rho_+(x)|\rho_-(x)} \cos \left[ \frac{\lambda}{2} S(x) + \phi_+(x) - \phi_-(x) \right]}
\]

where \( v_{cl}(x) = p(x)/m \). Comparing with Eq. (11) we see that
\[
\lim_{\lambda \to 0} (\bar{v}_\lambda(x)) = v_B(x)
\]

In Section 3 we saw that the instantaneous Bohmian speed \( |v_B(x)| \) tends to take values which are much larger than the classical value, while in Section 3 we saw that \( |\bar{v}_\lambda(x)| \) exhibits the opposite behaviour, often taking values which are much less than the classical speed. The result just derived explains both these features.

The reason for the velocity spikes illustrated in Fig. 3(b) is that \( v_B(x) \) is the mean observed velocity at \( x \) in the limit as the measurement of position becomes almost perfectly accurate. In order to carry out such a measurement it would be necessary to use a probe whose momentum was large in comparison with the momentum of the particle. Under such conditions violent fluctuations in the observed velocity are not unexpected.
The reason that $|\bar{\nu}_T(x)|$ is often much less than the classical speed is that the Bohmian velocity is related specifically to the mean observed velocity at $x$. Suppose, for example, that the particle was in an exact energy eigenstate. In that case $ho_-(x) = \rho_+(x)$, and $\bar{\nu}_\lambda(x) = 0$. In García de Polavieja’s interpretation (for intermediate values of $\lambda$) this implies the classical picture of an ensemble of particles, one half of which are moving at the classical speed to the right, while the other half are moving at the classical speed to the left, with only the mean velocity being zero. In the Bohm interpretation, by contrast, it implies the highly non-classical picture of an ensemble in which each individual particle has velocity zero. The picture is non-classical because it takes a quantity having the observational significance of a mean, and interprets it as a property of individual particles.

This feature of the Bohm interpretation is related to the fact that the equation of motion is first-order in time, so that the velocity is a simple function of position. The Bohm interpretation is consequently unable to describe a situation in which both signs of the velocity occur with non-negligible probability.

5. Other Trajectory Interpretations

It should be noted that García de Polavieja’s interpretation has certain drawbacks. In the first place, the equations of motion involve an infinite series, whose individual terms are often only defined in a distributional sense, and whose convergence properties are unclear. In fact, as we will show in a subsequent article, the analytic properties of the Husimi function can be used to re-write the equations of motion in a different form, which involves an absolutely convergent series of holomorphic functions. Nevertheless, it does not seem to be possible to avoid the use of an infinite series—which is clearly undesirable from a calculational point of view. Another possible difficulty stems from the fact that the range of admissible values of $\lambda$ depends on the potential. It is not entirely clear that there exists a single value of $\lambda$ which would be admissible for all physically reasonable choices of potential. It would be interesting to know if there exists some other trajectory interpretation, which also produces the correct classical limit for an isolated system, but which does not have the same disadvantages as García de Polavieja’s interpretation. The discussion in the last section provides some indications as to the direction one might take in such an enquiry.

We would particularly stress the significance of the intrinsic phase space probability distribution, describing the “beables” of the theory. This function is usually chosen so as to have, as one of its marginal distributions, either the function $|\langle x | \psi \rangle|^2$, or the function $|\langle p | \psi \rangle|^2$. Roy and Singh, in a very interesting series of papers [30, 31, 32], have proposed an interpretation in which the intrinsic phase space distribution has both these functions as its marginals. The distribution $Q_\lambda(x,p)$, by contrast, has neither function as a marginal. At first sight it may appear that this makes it an unnatural choice [53]. However, the discussion in the last section shows that it actually has some important advantages. As we saw, it is just because the Bohm interpretation does have $|\langle x | \psi \rangle|^2$ as the intrinsic $x$-space distribution that it tends to produce the velocity spikes illustrated in Fig. 3.

As we remarked in Section 4, the reason that the distribution in Eq. (21) is of classical form is that it describes the kind of measurement on which classical physics is based, in which $x$ and $p$ are both determined to an intermediate degree of accuracy. The function $|\langle x | \psi \rangle|^2$, by contrast, describes a measurement in which $x$ is determined with perfect accuracy, and $p$ is not determined at all. Such measurements require experimental conditions which are very unlike the conditions of our ordinary experience. In particular, they involve a very significant perturbation of the momentum of the particle whose position is being measured. Consequently, it is
perhaps not surprising that an interpretation based on the function $Q_\lambda(x,p)$ (with $\lambda$ appropriately chosen) gives trajectories which are much more nearly classical than one which is based on the function $|\langle x | \psi \rangle|^2$.

6. Conclusion

In this paper we have only considered the behaviour of an isolated system. Of course, the macroscopic bodies of our ordinary experience never are isolated (by definition since, if they were isolated, we would not be able to experience them). Bohm and Hiley argue that the effect of the interaction with the environment is to cause the Bohmian trajectory to become quasi-classical. In the sequel to this paper we will give some further arguments in support of their conclusion.

The Bohm interpretation is sometimes seen as being in opposition to the Copenhagen interpretation, so that one has to take up a position either for or against. This does not appear to have been the view of Bohm himself. On p.5 of The Undivided Universe Bohm and Hiley argue that “there should be a kind of dialogue between different interpretations rather than a struggle to establish the primacy of any one of them”. One of the most interesting features of the Bohm interpretation is the way in which it serves to illuminate some of Bohr’s key concepts from a somewhat unexpected direction (indeed, the title The Undivided Universe does itself contain an allusion to one of Bohr’s concepts). The results we have been discussing provide some further illustrations of the connection between the Bohm interpretation and other approaches to the problem of interpretation: for they show that the interaction with the environment plays a crucial role in the Bohm interpretation just as it does in the decoherent histories [18, 19] and existential [21] interpretations.

Our discussion also casts some light on the concept of a “hidden variable”. It would be reasonable to say that the trajectory is hidden if the system is isolated, but not hidden if the system is open, so that information about the trajectory is recorded in the environment. Since the Bohm interpretation makes statements about the trajectory of an isolated particle, it is therefore reasonable to describe it as a “hidden-variables theory”.

There are many different trajectory interpretations, which all make different predictions regarding the motion of an isolated particle. However, this has no bearing on their empirical acceptability since the motion of an isolated particle is “hidden”, and so it cannot be empirically determined (by definition: the particle cannot be observed by something external to it if it is not interacting with something external to itself). In order to be empirically acceptable it is only necessary that the various interpretations all make the same predictions regarding the motion of a particle which is not isolated.

This point is somewhat reminiscent of Copenhagen doctrines regarding the role of the external observer. There is, however, an important difference since the proponents of the Copenhagen interpretation appeared to make physical processes depend on the actual presence of such an observer. They thereby introduced an unacceptable element of subjectivity into physical theory. No such subjectivity is present here. It is indeed the case that, if the system has interacted with the rest of the universe, then there is the possibility of an external observer using the interaction to acquire information about the system. However, the interaction is not dependent on this happening. On the contrary, it is a completely objective process which would occur even if there were no observers.

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