Tian’s properness conjectures: an introduction to Kähler geometry

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Dedicated to Gang Tian on the occasion of his 60th birthday

Abstract

This manuscript served as lecture notes for a minicourse in the 2016 Southern California Geometric Analysis Seminar Winter School. The goal is to give a quick introduction to Kähler geometry by describing the recent resolution of Tian’s three influential properness conjectures in joint work with T. Darvas. These results—inspired by and analogous to work on the Yamabe problem in conformal geometry—give an analytic characterization for the existence of Kähler–Einstein metrics and other important canonical metrics in complex geometry, as well as strong borderline Sobolev type inequalities referred to as the (strong) Moser–Trudinger inequalities.

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Harmonic functions are special. They enjoy a high degree of regularity, and in some vague sense are considered to be more aesthetically pleasing than an arbitrary function. In geometry, one similarly seeks aesthetically pleasing structures on a given space. A typical example is that of an Einstein structure. Among all Riemannian structures Einstein structures are special in many ways; the interested reader is referred to the book by Besse [22].

Harmonic functions can be defined as solutions to the Laplace equation. A fundamental result in analysis is that harmonic functions are also characterized as minimizers of the Dirichlet energy. This result is fundamental in many ways. First, the Dirichlet energy makes sense for functions whose gradient is merely square integrable while Laplace’s equation requires two derivatives to exist pointwise. Second, it gives an approach to actually constructing harmonic functions.

Kähler–Einstein metrics can be defined as solutions to a fully nonlinear analogue of the Laplace equation. A nonlinear analogue of the Dirichlet energy was introduced by Mabuchi [30]...
years ago. A basic aspect of this analogy is that the Euler–Lagrange equation for the Mabuchi functional is precisely the Kähler–Einstein equation. One possible way to view these lectures is as an attempt to explain some aspects of this analogy in more detail. In doing so, we strive to give a quick—and at least partly introductory—course in Kähler geometry.

**A second prologue**

The Kähler–Einstein problem is also strongly motivated by an analogy with the Yamabe problem, that is, of course, itself motivated by the classical Dirichlet problem described above. The following table serves as an overall guidance to the Kähler–Einstein problem, especially for those familiar with the resolution of the Yamabe problem. Our goal in these lectures is to describe the right column of this table, culminating in a complete analytic description of “Tian’s properness conjectures” and “Tian’s Moser–Trudinger conjecture” at the bottom right.

A few remarks are in place. First, this table is highly schematic, and its main purpose is to highlight some possible analogies between the two analytic problems. Second, the infimum in the definition of $\mu[\omega]$ is, of course, solely for the analogy, since $\mu[\omega] = \frac{1}{2} \int_M R_{\omega} \omega^n / \int_M \omega^n$ is a cohomological invariant of the Kähler class. Third, an alternative sufficient and necessary condition that appeared very recently and after these lectures were delivered can now be described in terms of an invariant that is different but related to $\alpha[\omega]$ coming from algebraic geometry and K-stability [33, 34, 93]. Yet, this last characterization is purely algebraic, and so it is less pertinent to the analogy with the Yamabe problem. Finally, one may also discuss the more general problem of constant scalar curvature (csc) Kähler metrics. While this is beyond the scope of the lectures, it is worth mentioning briefly the state-of-the-art on this problem at the time the lectures were given. Indeed, in [44] aside from solving the Kähler–Einstein case we also were able to reduce the general csc problem to the regularity of weak minimizers of the Mabuchi energy. Shortly afterwards, our techniques played a rôle in the resolution of this regularity problem, and hence of the analytic characterization of constant scalar curvature Kähler metrics [17, 32, 33]. Some of these important developments are already described in the survey [11], while others just appeared and seem to involve important new ideas beyond the scope of these lectures.
| **structure** | Yamabe problem | Kähler–Einstein problem |
|---------------|---------------|------------------------|
| **class**     | Riemannian manifold $(M^n, g)$ | Kähler manifold $(M^{2n}, J, \omega)$ |
| **problem**   | (non-)existence of constant scalar curvature metric in a conformal class $(M^n, [g])$ | (non-)existence of a Kähler–Einstein metric in a Kähler class $(M^{2n}, [\omega])$ |
| **equation**  | $u^{N-2}g$ has constant scalar curvature $\mu$ (here $N := \frac{2n}{n-2}$) | $\omega_\varphi := \omega + \sqrt{-1} \partial \bar{\partial} \varphi$ has constant Ricci curvature $\mu$ |

**functional**

Yamabe energy

$$Y(u) := \int_M \left((N+2)|\nabla_g u|^2 + R_g u^2\right) dV_g / \left(\int_M u^N dV_g\right)^{2/N}$$

Mabuchi energy

$$E(\varphi) := \int_M \log \frac{\omega_\varphi^n}{\omega^{n+1}} dV_g$$

**sign invariant**

solution ($\mu \leq 0$) always exists (Aubin, Trudinger, Yamabe)

sufficient criterion ($\mu > 0$) exists if $\mu_g < \mu_{[g_{S^n}]}$ (Aubin)

necessary condition Aubin’s criterion always holds

strong borderline Sobolev inequality on $S^n$
1 Introduction

The main motivation for these lectures are three conjectures: Tian’s properness conjectures and Tian’s Moser–Trudinger conjecture. Consider the space

$$\mathcal{H} = \{ \omega_\varphi := \omega + \sqrt{-1} \partial \bar{\partial} \varphi : \varphi \in C^\infty(M), \omega_\varphi > 0 \}$$  \hspace{1cm} (1)

of all Kähler metrics representing a fixed cohomology class on a compact Kähler manifold \((M,J,\omega)\).

Motivated by results in conformal geometry and the direct method in the calculus of variations, in the 90’s Tian introduced the notion of “properness on \(\mathcal{H}^K\)” \cite[Definition 5.1]{tian91} in terms of the Aubin nonlinear energy functional \(J\) \cite{aubin91} and the Mabuchi K-energy \(E\) \cite{mabuchi85} as follows (both functionals are defined \(\S 3\) below, see (7) and (12)).

**Definition 1.1.** The functional \(E : \mathcal{H} \to \mathbb{R}\) is said to be proper if

$$\forall \omega_j \in \mathcal{H}, \lim_j J(\omega_j) \to \infty \implies \lim_j E(\omega_j) \to \infty.$$  \hspace{1cm} (2)

Tian made the following influential conjecture \cite[Remark 5.2]{tian91}, \cite[Conjecture 7.12]{tian96}. Denote by \(\text{Aut}(M,J,\omega)\) the identity component of the group of automorphisms of \((M,J,\omega)\), and denote by \(\text{aut}(M,J,\omega)\) its Lie algebra, consisting of holomorphic vector fields.

**Conjecture 1.2.** (Tian’s first properness conjecture) Let \((M,J,\omega)\) be a Fano manifold. Let \(K\) be a maximally compact subgroup of \(\text{Aut}(M,J)\). Then \(\mathcal{H}^K\) contains a Kähler–Einstein metric if and only if \(E\) is proper on the subset \(\mathcal{H}^K \subset \mathcal{H}\) consisting of \(K\)-invariant metrics.

Tian’s conjecture is central in Kähler geometry since it predicts an analytic characterization of Kähler–Einstein manifolds. Appropriate analogues of this conjecture in conformal geometry are known and were crucial in the solution of the famous Yamabe problem concerning the existence of constant scalar curvature metrics in conformal classes. We briefly discuss this in Section \(\S 16\).

The conjecture has attracted much attention over the past two and a half decades including motivating much work on equivalence between algebro-geometric notions of stability and existence of canonical metrics, as well as on the interface of pluripotential theory and Monge–Ampère equations. While the algebraic-geometric characterization of Kähler–Einstein manifolds has been finally obtained \cite{bando85, takayama86}, the analytic characterization of Conjecture 1.2 has remained open. We refer to the surveys \cite{tian00, tian01, tian02, tian03}.

Conjecture 1.2 (which we refer to as the Tian’s first properness conjecture) gives a characterization of Kähler–Einstein manifolds in terms of the Mabuchi energy. Thus, it can be seen as the analogue of the properness of the Yamabe energy which led to the resolution of the Yamabe problem. Another central theorem in conformal geometry is Aubin’s strong Moser–Trudinger inequality on spheres. Tian’s Moser–Trudinger conjecture suggests a Kähler geometry analogue of this inequality on any Kähler–Einstein manifold, that we now turn to describe.

Denote by \(\Lambda_1\) the real eigenspace of the smallest positive eigenvalue of \(-\Delta_\omega\), and set

$$\mathcal{H}_1^\perp := \{ \varphi \in \mathcal{H} : \int \varphi \psi \omega^n = 0, \forall \psi \in \Lambda_1 \}.$$  

When \(\omega\) is Kähler–Einstein, it is well-known that \(\Lambda_1\) is in a one-to-one correspondence with holomorphic gradient vector fields \cite{lang87}. Tian made the following conjecture in the 90’s \cite[Conjecture 5.5]{tian90}, \cite[Conjecture 6.23]{tian96}, \cite[Conjecture 2.15]{tian92}. 

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Conjecture 1.3. (Tian’s Moser–Trudinger conjecture) Suppose $(M, J, \omega)$ is Fano Kähler-Einstein. Then for some $C, D > 0$,
\[
E(\varphi) \geq CJ(\varphi) - D, \quad \varphi \in \mathcal{H}^2_{\omega}.
\]

By the end of these lectures we will present results that resolve both Conjectures 1.2 and 1.3. For Conjecture 1.2, the special case when $K$ is trivial has already been known for almost 20 years from the work of Tian and Tian–Zhu [90, 94]. Treating the general case has remained open since. Somewhat surprisingly, Conjecture 1.2 was actually disproved by Darvas and the author recently [44]. More precisely, Theorem 6.5 establishes precisely for which manifolds Conjecture 1.2 holds, giving a converse to a result of Phong et al. [71]. Next, and this is the second main result of [44], an alternative formulation to Conjecture 1.2 is established—which we refer to as Tian’s second properness conjecture—giving the sought after analytic characterization for Kähler–Einstein metrics. This is stated in Theorem 12.4. Finally, the strong Moser–Trudinger inequality for Kähler–Einstein manifolds is established, confirming Conjecture 1.3 [44]. This is stated in Corollary 16.7.

We leave out a few relevant topics due to space and time limitations. Notably, we mostly do not delve into the pluripotential theoretic and Bergman kernel aspects of the proof of Theorem 12.4 for which we refer to Darvas’ survey that has appeared in the meantime [41]. On the other hand, our treatment is rather self-contained and reviews most of the basics, giving an opportunity to the interested reader for a rapid introduction to current research in Kähler geometry.

2 Kähler and Fano manifolds

In these lectures all manifolds will be assumed to be complex. Complex manifolds are just like topological or differentiable manifolds except that the transition functions between the different charts in the atlas are required to be holomorphic in both directions (i.e., biholomorphic). Thus, all our manifolds will be of even dimension. In dimension two all complex manifolds are also Kähler. In higher dimensions, however, the latter condition is rather subtle and reflects the existence of a Riemannian structure highly compatible with the given complex structure— we explain this next.

Let $(M, J, g)$ denote a complex manifold together with a Riemannian metric $g$ on $M$ that is compatible with $J$ in the sense that
\[
g(x, y) = g(Jx, Jy), \quad \forall x, y \in \Gamma(M, TM),
\]
where $\Gamma(M, TM)$ denote smooth vector fields on $M$. Since $J^2 = -I$, the formula $\omega := \omega = g(J \cdot, \cdot)$ defines a 2-form on $M$. We call $(M, J, g)$ a Kähler manifold if the form $\omega$ is a closed 2-form,
\[
d\omega = 0.
\]

Kähler manifolds have many other equivalent characterizations; we refer the reader to [77] §2.1.4.

In these lectures we will be interested in the curvature of Kähler manifolds. In particular, we will be interested in trying to understand when Kähler manifolds admit Einstein metrics. Just as the Riemannian metric can be transformed using the complex structure into a skew-symmetric form, so can the Ricci curvature tensor $\text{Ric}$. We denote the Ricci form by
\[
\text{Ric} \omega = \text{Ric} g(J \cdot, \cdot).
\]
Thus, on a Kähler manifold the Einstein equation
\[ \text{Ric } g = c g, \]
transforms to
\[ \text{Ric } \omega = c \omega. \] (3)
Let \( z_1, \ldots, z_n \) be local holomorphic coordinates on a neighborhood in \( M \). In those coordinates express the form \( \omega \),
\[ \omega = g_{ij} dz^i \wedge \overline{dz^j}. \]
As discovered by Schouten [80], Schouten and van Dantzig [81, 82], and Kähler [63] (see [77, p. 35] for more references) the Ricci form then has the following expression
\[ \text{Ric } \omega = -\sqrt{-1} \partial \overline{\partial} \log \det [g_{ij}]. \] (4)
In fact, the proof is not hard. First, recall some useful formulæ:

**Exercise 2.1.** Let \( D \) be a constant coefficient first-order operator defined on some domain in \( \mathbb{R}^m \) and let \( A \) be a matrix-valued function on the same domain. Then,
\[ D \log \det A = A^{ij} DA_{ij}, \]
and
\[ DA^{ij} = -A^{it} DA_{ts} A^{sj}, \]
where \( A^{ij} \) is the coefficient in the \( i \)-th row and \( j \)-th column of the inverse matrix of \( A \).

Therefore,
\[ \partial \overline{\partial} \log \det [g_{ij}] = -g^{it} g_{ls, k} g^{sj} g_{ij, \overline{l}} + g^{ij} g_{\overline{i}j, k}. \]
Now, \( d\omega = 0 \) implies that both \( \partial \omega = \overline{\partial} \omega = 0 \). Thus, \( g_{ij, k} = g_{kij} \) and \( g_{ij, \overline{k}} = g_{i\overline{k}j} \).

**Exercise 2.2.** Complete the proof of (4).

Thus, if \( \eta \) is any Kähler form such that locally \( \eta = h_{ij} dz^i \wedge \overline{dz^j} \) then
\[ \text{Ric } \omega - \text{Ric } \eta = -\sqrt{-1} \partial \overline{\partial} \log \frac{\det [h_{ij}]}{\det [g_{ij}]} = \sqrt{-1} \partial \overline{\partial} \log \frac{\eta^n}{\omega^n} \] (5)
is an exact two form on \( M \) since \( \log \frac{\eta^n}{\omega^n} \) is a globally defined smooth function on \( M \) as \( \frac{\eta^n}{\omega^n} > 0 \). Therefore, the Ricci form of any Kähler metric is not only a closed two-form (as is evident from (4)), but also lies in a fixed cohomology class. Up to a constant factor, this class is called the first Chern class of \( M \) and is denoted by \( 2\pi c_1(M) \).

The point of this discussion is that Einstein metrics on a Kähler manifold can exist only if the equality of cohomology classes
\[ \mu [\omega] = 2\pi c_1(M) \] (6)
holds. Now, as a rule of thumb, Einstein metrics of negative Ricci curvature exist in abundance on Riemannian manifolds, while Einstein metrics of positive Ricci curvature are quite harder to come by [22]. Somewhat analogously, it is easier to prove existence of Kähler–Einstein metrics of negative Ricci curvature, i.e., when \( \mu < 0 \): a fundamental theorem of Aubin and Yau states that then (6) also implies that there exists a unique Kähler–Einstein metric whose cohomology class is \( [\omega] \). Around the same time, Yau also showed that the same is true when \( \mu = 0 \). However, when \( \mu > 0 \) it was shown by Matsushima already in the 50’s that (6) is not sufficient. Kähler manifolds for which (6) holds with \( \mu > 0 \) are called Fano manifolds. Thus, it is natural to ask:
Question 2.3. When does a Fano manifold admit a Kähler–Einstein metric?

As just explained, if such a Kähler–Einstein metric exists it must have positive Ricci curvature. (Conversely, if a Kähler manifold admits a Kähler metric of positive Ricci curvature it is Fano.) In these lectures we describe an answer to this question. The key player will be the Mabuchi energy, which we now turn to describe.

3 The Mabuchi energy

Before defining the Mabuchi energy we introduce several other basic functionals.

The two most basic functionals, introduced by Aubin [4], are defined by

\[
J(\varphi) = J(\omega_\varphi) := V^{-1} \int_M \varphi \omega^n - \frac{V^{-1}}{n+1} \int_M \varphi \sum_{l=0}^{n} \omega^{n-l} \wedge \omega^l_{\varphi},
\]

\[
I(\varphi) = I(\omega_\varphi) := V^{-1} \int_M \varphi (\omega^n - \omega^n_{\varphi}).
\]

(7)

Here,

\[
V = \int \omega^n_{\varphi},
\]

is a constant independent of \(\omega_\varphi \in \mathcal{H}\).

Exercise 3.1. Show that \(V\) is \(n!\) times the volume of \(M\) with respect to the Riemannian metric \(g\). (See the end of the proof of Proposition 2.1 in [37] for a solution.)

The notation \(J(\varphi) = J(\omega_\varphi)\) (and similarly for \(I\)) is justified by the fact that \(J(\varphi) = J(\varphi + c)\) for any \(c \in \mathbb{R}\). These two functionals, as well as their difference, are mostly equivalent, in the sense that,

\[
\frac{1}{n^2}(I - J) \leq \frac{1}{n(n+1)} I \leq \frac{1}{n} J \leq I - J \leq \frac{n}{n+1} I \leq nJ.
\]

(8)

Remark 3.2. We will be rather sloppy and often say “\(\varphi \in \mathcal{H}\)” when we really mean \(\omega_\varphi \in \mathcal{H}\). However, see Remark 8.17 where we start being more precise.

A closely related functional is the Aubin–Mabuchi functional, introduced by Mabuchi [66, Theorem 2.3],

\[
AM(\varphi) := V^{-1} \int_M \varphi \omega^n - J(\varphi) = \frac{V^{-1}}{n+1} \sum_{j=0}^{n} \int_M \varphi \omega^j \wedge \omega^{n-j}_{\varphi},
\]

(9)

Exercise 3.3. Prove the integration by parts formula

\[
\int g \sqrt{-1} \partial \bar{\partial} f \wedge \alpha^j \wedge \beta^{n-j-1} = \int f \sqrt{-1} \partial \bar{\partial} g \wedge \alpha^j \wedge \beta^{n-j-1},
\]

whenever \(\alpha, \beta\) are smooth closed \((1, 1)\)-forms and \(f, g \in C^2(M)\). Then, show that if \(\delta \mapsto \varphi(\delta)\) denotes a \(C^1\) curve in \(\mathcal{H}\), (in the sense that \(\delta \mapsto \varphi(\delta)(x)\) is \(C^1\) map for each \(x \in M\), and \(\omega_{\varphi(\delta)} \in \mathcal{H}\) for each \(\delta\)),

\[
\frac{d}{d\delta} AM(\varphi(\delta)) = V^{-1} \int \frac{d}{d\delta} \varphi(\delta) \omega^n_{\varphi(\delta)}.
\]

(10)
Denote by
\[ \text{Ent}(\nu, \chi) = \frac{1}{V} \int_M \log \frac{\chi}{\nu} \chi, \]
the entropy of the measure \( \chi \) with respect to the measure \( \nu \) (where here \( V = \int_M \chi = \int_M \nu \)).

The Mabuchi energy (sometimes also called the K-energy as in Mabuchi’s original article)

\[ E : \mathcal{H} \to \mathbb{R}, \]

is defined by \([78, (5.27)]\), \([66]\),

\[ E(\omega, \phi) = E(\phi) := \text{Ent}(e^{f_\omega} \omega^n, \omega^n) - \mu AM(\phi) + \mu V^{-1} \int_M \phi \omega^n. \] (12)

Here, \( f_\omega \) is a smooth function depending on \( \omega \) that we define next.

**Definition 3.4.** The Ricci potential of \( \omega \), \( f_\omega \), satisfies

\[ \sqrt{-1} \partial \bar{\partial} f_\omega = \text{Ric} \omega - \mu \omega, \] (13)

where it is convenient to require the normalization \( \int e^{f_\omega} \omega^n = \int \omega^n \).

**Exercise 3.5.** Show that

\[ AM(\phi) = (I - J)(\phi) + V^{-1} \int \phi \omega^n, \] (14)

and therefore the last two terms in (12) equal \(-\mu(I - J)(\omega, \omega_\phi)\), so

\[ E(\omega_\phi) = \text{Ent}(e^{f_\omega} \omega^n, \omega^n) - \mu(I - J)(\omega, \omega_\phi). \] (15)

From this formula, we see that understanding the K-energy essentially means understanding the interplay between the entropy and the Aubin functional \( I - J \) (or, the equivalent functionals \( I \) or \( J \), recall (8)). This is in some sense the holy grail, the difficult analytical question at the heart of Tian’s conjecture. A first (and fundamental) result in this direction is Theorem 3.8 below, however only after much more work do we obtain a clearer picture of this relationship, culminating in Theorem [12.4].

**Exercise 3.6.** Show that indeed \( E(\omega_\phi) = E(\phi) \), i.e., that \( E(\phi + C) = E(\phi) \) for any \( C \in \mathbb{R} \).

There is another way to write the K-energy:

\[ E(\phi) := \text{Ent}(\omega^n, \omega_\phi^n) + s_0 AM(\phi) - \frac{1}{V} \sum_{j=0}^{n-1} \int_M \phi \text{Ric} \omega \wedge \omega_j \wedge \omega^{n-1-j}, \] (16)

where \( s_0 = V^{-1} \int_M s_\omega \omega^n \) is the average scalar curvature. Of course,

\[ V^{-1} \int_M s_\omega \omega^n = V^{-1} \int_M n \text{Ric} \omega \wedge \omega^{n-1} = n \mu \]

so \( s_0 = n \mu \).

**Exercise 3.7.** Show that (16) coincides with (12) when \( \mu[\omega] = 2\pi c_1(M) \) (= \([\text{Ric} \omega]\)).

The point of (16) is that it actually makes sense on any Kähler manifold. We will however stick to the first formula in these lectures for simplicity.
3.1 The K-energy when $\mu < 0$

Using Exercise 3.5, Conjecture 1.2 is seen to hold in the case $\mu < 0$ as follows. First, convexity of the exponential function implies that

$$\int \log f \, d\nu \leq \log \int f \, d\nu,$$

whenever $d\nu$ is a probability measure (so $\int d\nu = 1$), so

$$\text{Ent}(\nu, \chi) = -\int \log \frac{\nu}{\chi} \chi \geq -\log \int \nu \chi = 0,$$

(17)

i.e., the entropy is always nonnegative. Therefore,

$$E(\phi) = \text{Ent}(e^{f\omega_n}, \omega_n) - \mu (I - J)(\phi) \geq \frac{|\mu|}{n} J(\phi),$$

(18)

as desired.

3.2 The K-energy when $\mu \geq 0$

We now describe a technique, due to Tian [91, §7], to treat some of the cases when $\mu \geq 0$ by showing the entropy itself is always proper. Since

$$\text{Ent}(e^{f\omega_n}, \omega_n) \geq \text{Ent}(\omega_n) - \sup f,$$

(19)

it suffices to estimate $\text{Ent}(\omega_n)$. Since the functionals $I$ and $J$ are interchangeable as far as properness goes (recall (8)), what we would like to show is:

**Theorem 3.8.** There exists a positive $\beta, C$ such that

$$\text{Ent}(\omega_n, \omega_n) \geq \beta I(\varphi) - C = -\beta V^{-1} \int_M \left( \varphi - V^{-1} \int_M \varphi \omega_n \right) \omega_n - C.$$

Rewriting the functional $I$ in this way is useful for the following reason:

$$\beta I(\varphi) - \text{Ent}(\omega_n, \omega_n) \geq \int e^{\log \frac{\omega_n}{\omega_\varphi} - \beta (\varphi - V^{-1} \int_M \varphi \omega_n) \omega_\varphi / V}$$

$$\leq \log \int e^{\log \frac{\omega_n}{\omega_\varphi} e^{-\beta (\varphi - V^{-1} \int_M \varphi \omega_n) \omega_\varphi / V}}$$

$$= \log \int e^{-\beta (\varphi - V^{-1} \int_M \varphi \omega_n) \omega_\varphi / V},$$

(20)

and so the question reduces to whether there exists a positive $\beta$ such that the functional

$$\varphi \mapsto \int e^{-\beta (\varphi - V^{-1} \int_M \varphi \omega_n) \omega_n}$$

is uniformly bounded on $\mathcal{H}$. Observe that we have managed to eliminate the dependence on the measure $\omega_n$. To be more precise, the question is now about integrability properties of functions in $\mathcal{H}$ with respect to a fixed measure. We treat this question in the next subsection. Before doing so, observe that an affirmative answer implies the K-energy $E$ is proper whenever $\mu = 0$. When $\mu > 0$, an affirmative answer implies using (8),

$$E(\varphi) \geq (\beta - n\mu/(n + 1)) I(\varphi).$$

(21)

Thus, if $\beta$ can be taken larger than $n\mu/(n + 1)$ then the K-energy is proper even when $\mu > 0$. 

10
3.3 Tian’s invariant

The preceding question is equivalent to the following:

**Question 3.9.** Is

\[
\alpha(M, [\omega]) = \sup \left\{ \beta : \sup_{\varphi \in H} \int_M e^{-\beta (\varphi - V)^{-1} \varphi \omega^n} \omega^n < C(\beta) \text{ for some constant } C(\beta) > 0 \right\}
\] (22)

positive?

By definition, the number \( \alpha(M, [\omega]) \) is an invariant of the Kähler class \([\omega]\). It was introduced by Tian, who answered Question 3.9 affirmatively [88, Proposition 2.1].

**Theorem 3.10.** \( \alpha(M, [\omega]) > 0 \).

As explained above, Theorem 3.10 implies Theorem 3.8.

Before going into the detailed proof of this theorem we observe that the last statement of the previous subsection can be stated as follows.

**Theorem 3.11.** Suppose (6) holds. The K-energy is proper whenever \( \alpha(M, [\omega]) > n\mu/(n + 1) \).

Thanks to Theorem 3.10 Theorem 3.11 treats in a unified fashion the negative, zero, and some positive cases.

**Remark 3.12.** Using (19) instead of (12) one may generalize Theorem 3.11 to cohomology classes nearby \( c_1(M)/\mu \), as shown recently by Dervan [46, Theorem 1.3].

We now turn to proving Theorem 3.10. The key is an elementary, but by no means trivial, result on subharmonic functions in the plane from Hörmander’s book [61, Theorem 4.4.5]. Denote the ball of radius \( R \) about the origin in \( \mathbb{C} \) by \( B_R(0) \).

**Theorem 3.13.** Let \( R > 0 \) and let \( \psi \) be a smooth subharmonic function defined on \( B_R(0) \subset \mathbb{C} \), satisfying

\[
\begin{align*}
\psi(0) &\geq -1, \\
\psi &\leq 0, \text{ on } B_R(0).
\end{align*}
\] (23)

Then for every \( \rho \in [R/2, e^{-1/2}R] \) there exists a constant \( C \) depending only on \( R, \rho \) such that

\[
\int_{B_\rho(0)} e^{-\psi} \sqrt{-1}dz \wedge d\bar{z} \leq C.
\] (24)

**Proof.** Let \( \tilde{\psi} := \psi + 1 \) (so that \( \tilde{\psi} \leq 1 \) and \( \tilde{\psi}(0) \geq 0 \)). We prove (31) for \( \tilde{\psi} \) which is the same thing as (31) for \( \psi \) up to a factor of \( e \).

The Riesz (or Poisson) representation of a smooth function \( f : B_R(0) \to \mathbb{R} \) takes the form

\[
f(z) = \frac{1}{2\pi} \int_{B_R(0)} \log \left| \frac{Rz - R\zeta}{R^2 - z\zeta} \right| \Delta f(\zeta) \sqrt{-1} d\zeta \wedge d\bar{\zeta} + \int_0^{2\pi} \frac{R^2 - |z|^2}{|z - Re^{\sqrt{-1}\theta}|^2} f(Re^{\sqrt{-1}\theta}) d\theta.
\] (25)

Now we consider (25) for \( f = \tilde{\psi} \) and try to obtain bounds for each of the terms.
Second term: First we show that the second term in (26) for \( f = \tilde{\psi} \) is actually itself uniformly bounded when \( z \in B_{R/2}(0) \). Indeed, putting \( z = 0 \) and \( f = \tilde{\psi} \) in (26),

\[
0 \leq \tilde{\psi}(0) = \frac{1}{2\pi} \int_{B_R(0)} \log \frac{|\zeta|}{R} \Delta \tilde{\psi}(\zeta) \frac{\sqrt{1 - \zeta^2}}{2} d\zeta \land d\bar{\zeta} + \int_{0}^{2\pi} \tilde{\psi}(Re^{\sqrt{-1} \theta}) \frac{d\theta}{2\pi}.
\]

Hence

\[
2\pi \geq 2\pi - 2\pi \tilde{\psi}(0) = \int_{B_R(0)} \log \frac{R}{|\zeta|} \Delta \tilde{\psi}(\zeta) \frac{\sqrt{1 - \zeta^2}}{2} d\zeta \land d\bar{\zeta} + \int_{0}^{2\pi} (1 - \tilde{\psi}(Re^{\sqrt{-1} \theta})) d\theta.
\]  

(26)

Since \( \tilde{\psi} \leq 1 \) the second integrand is nonnegative. So is the first, since \( \Delta \tilde{\psi} \geq 0 \). So each of the terms is nonnegative and hence bounded from above by \( 2\pi \). Therefore,

\[
\int_{0}^{2\pi} |\tilde{\psi}(Re^{\sqrt{-1} \theta})| d\theta \leq \int_{0}^{2\pi} (1 - \tilde{\psi}(Re^{\sqrt{-1} \theta})) d\theta + \int_{0}^{2\pi} 1 \cdot d\theta \leq 2\pi + 2\pi = 4\pi,
\]

hence

\[
\left| \int_{0}^{2\pi} \frac{R^2 - |z|^2}{|z - \zeta|^2} \tilde{\psi}(Re^{\sqrt{-1} \theta}) \frac{d\theta}{2\pi} \right| \leq \sup_{z \in B_r(0), |\zeta| = R} \frac{R^2 - |z|^2}{|z - \zeta|^2} \int_{0}^{2\pi} |\tilde{\psi}(Re^{\sqrt{-1} \theta})| \frac{d\theta}{2\pi} \leq C(r, R) \cdot \frac{1}{2\pi} 4\pi = 6,
\]

(27)

where \( C(r, R) \) is some constant depending only on \( r, R \).

First term: The first term in (25) is not uniformly bounded, however we will show it is uniformly exponentially integrable in the sense of the statement of the theorem. We split this first term into two parts one of which will be actually uniformly bounded (all we really need is a uniform bound from below):

Claim 3.14. For each \( z \) such that \( |z| < r < \rho \) one has

\[
\left| \frac{1}{2\pi} \int_{B_R(0) \setminus B_{R}(0)} \log \left| \frac{Rz - R\zeta}{R^2 - z^2} \right| \frac{\sqrt{1 - \zeta^2}}{2} d\zeta \land d\bar{\zeta} \right| \leq C,
\]

for some constant \( C > 0 \) depending only on \( r, \rho, R \).

Proof. Recall that in (20) each of the terms was bounded by \( 2\pi \), so

\[
\int_{B_R(0)} \log \frac{R}{|\zeta|} \Delta \tilde{\psi}(\zeta) \frac{\sqrt{1 - \zeta^2}}{2} d\zeta \land d\bar{\zeta} \leq 2\pi.
\]

Thus, as \( \log(1 + b) \geq Cb \) for some constant \( C = C(\epsilon) \in (0, 1) \) when \( b \in (0, \epsilon) \),

\[
\int_{B_R(0) \setminus B_{R(1-\epsilon)}(0)} \left( \frac{C|R - |\zeta||}{|\zeta|} \right) \Delta \tilde{\psi}(\zeta) \frac{\sqrt{1 - \zeta^2}}{2} d\zeta \land d\bar{\zeta} \leq 2\pi.
\]

We have,

\[
C \int_{B_R(0)} \frac{|R - |\zeta||}{|\zeta|} \Delta \tilde{\psi}(\zeta) \frac{\sqrt{1 - \zeta^2}}{2} d\zeta \land d\bar{\zeta} \leq 2\pi,
\]

in particular,

\[
C \int_{B_R(0)} |1 - |\zeta||/R| \Delta \tilde{\psi}(\zeta) \frac{\sqrt{1 - \zeta^2}}{2} d\zeta \land d\bar{\zeta} \leq 2\pi.
\]  

(28)
Now,
\[ |R^2 - z\bar{\zeta}| \in \partial B_1(0), \quad \forall \zeta \in \partial B_R(0), \]
as can be checked from the fact that for each \( z \) such that \(|z| < 1\) the map \( \zeta \mapsto \frac{1-z\bar{\zeta}}{z-\zeta} \) is a Möbius map, i.e., sends \( \partial B_1(0) \) to itself and then scaling. Thus, if \( z \in B_r(0) \) with \( r < 1 \), there exists \( C > 0 \) possibly depending on \( r, \rho, R \) such that
\[ \log \left| \frac{Rz - R\zeta}{R^2 - z\bar{\zeta}} \right| \leq \begin{cases} C|1 - |\zeta|/R| & \text{for } \zeta \in (R(1 - \epsilon), R), \\ C & \text{for } \zeta \in (\rho, R(1 - \epsilon)). \end{cases} \]

Then,
\[
\frac{1}{2\pi} \int_{B_R(0)\setminus B_\rho(0)} \log \left| \frac{Rz - R\zeta}{R^2 - z\bar{\zeta}} \right| \Delta \tilde{\psi} \sqrt{-1} \frac{1}{2} d\zeta \wedge d\bar{\zeta} \leq C \frac{1}{2\pi} \int_{B_R(0)\setminus B_{(1-\epsilon)}(0)} |1 - |\zeta|/R| \Delta \tilde{\psi} \sqrt{-1} \frac{1}{2} d\zeta \wedge d\bar{\zeta} + C \frac{1}{2\pi} \int_{B_{(1-\epsilon)}(0)\setminus B_\rho(0)} \Delta \tilde{\psi} \sqrt{-1} \frac{1}{2} d\zeta \wedge d\bar{\zeta}.
\]

The first term on the right hand side is uniformly bounded by \( (28) \). The second term is uniformly bounded by Claim 3.15 below. \( \Box \)

Since \( \rho < e^{-1/2} R \) from now and on we write
\[ \rho = e^{-1/2-\epsilon} R, \]
with \( \epsilon > 0 \) small, say \( \epsilon = 1/500 \), where 500 could be replaced by a generous quantity (cf. [6] Proposition 8.1). In order to estimate \( e^{-\tilde{\psi}} \) we only need to estimate for each \( z \) such that \(|z| < r\) the exponential term (note the minus sign)
\[
\exp \left( - \frac{1}{2\pi} \int_{B_{e^{-1/2-\epsilon} R}(0)} \log \left| \frac{Rz - R\zeta}{R^2 - z\bar{\zeta}} \right| \Delta \tilde{\psi} \sqrt{-1} \frac{1}{2} d\zeta \wedge d\bar{\zeta} \right). \tag{29}
\]

This can be done using Jensen’s inequality, or just the arithmetic mean-geometric mean inequality. For that need to normalize the measure so that it integrates to 1 (i.e., becomes a probability measure). That is need to divide by
\[ a := \frac{1}{2\pi} \int_{B_{e^{-1/2-\epsilon} R}(0)} \Delta \tilde{\psi} \sqrt{-1} \frac{1}{2} d\zeta \wedge d\bar{\zeta}, \]
i.e., the mass of the harmonic measure on this ball. It is well-known that the mass of the harmonic measure on a compact subset of a ball is uniformly bounded by a constant depending on the distance to the boundary of the ball whenever the function is uniformly bounded from above on the whole ball and its value is fixed at one point. Moreover,
\[ a < 2 ! \]
Indeed, this is the reason to choose $\rho = e^{-1/2-\epsilon} R$:

$$a = \frac{1}{2\pi} \int_{B_{\rho/2}(0)} \Delta \tilde{\psi} \frac{\sqrt{-1}}{2} d\zeta \wedge d\bar{\zeta} \leq \frac{1}{2\pi} \int_{B_{\rho/2}(0)} \frac{2 \log \frac{R}{\rho}}{1 + 2\epsilon} \Delta \tilde{\psi} \frac{\sqrt{-1}}{2} d\zeta \wedge d\bar{\zeta} \leq \frac{2}{1 + 2\epsilon} \frac{2\pi}{2\pi} = \frac{2}{1 + 2\epsilon}$$

since earlier we proved the first term of (20) is bounded by 2$\pi$ (note all we did was insert a term between the large parenthesis which is bigger than 1).

For an earlier reference we state the following claim whose proof is identical to the computation of $a$.

**Claim 3.15.** For every $\epsilon \in (0, 1)$ there is a constant $C = C(\epsilon)$ such that

$$\frac{1}{2\pi} \int_{B_{2/(1-\epsilon)}(0)} \Delta \tilde{\psi} \frac{\sqrt{-1}}{2} d\zeta \wedge d\bar{\zeta} \leq C.$$

**Exercise 3.16.** Compute the constant $C(\epsilon)$ in the previous claim.

So we come back to (20), and apply the arithmetic mean-geometric mean inequality:

$$\exp \left( \int_{B_{e^{-1/2-\epsilon} R}(0)} -a \cdot \log \left| \frac{Rz - R\zeta}{R^2 - z\bar{\zeta}} \right| \Delta \tilde{\psi} \frac{\sqrt{-1}}{2} d\zeta \wedge d\bar{\zeta} / (2\pi a) \right) \leq \exp \left( \int_{B_{e^{-1/2-\epsilon} R}(0)} \left| \frac{R^2 - z\bar{\zeta}}{Rz - R\zeta} \right|^a \Delta \tilde{\psi} \frac{\sqrt{-1}}{2} d\zeta \wedge d\bar{\zeta} / (2\pi a) \right) \leq C \int_{B_{e^{-1/2-\epsilon} R}(0)} \left| z - \zeta \right|^a \Delta \tilde{\psi} \frac{\sqrt{-1}}{2} d\zeta \wedge d\bar{\zeta} / (2\pi a).$$

Now this itself may not be bounded, however, it is in $L^1$ in $z$—more precisely in $L^1(B_0(0))$—and this is what we want to show. It is crucial here that $a < 2$ or in other words to chose $\rho = e^{-1/2-\epsilon} R$ earlier. To be precise, we integrate now in $z$ to get

$$\int_{B_{1/2R}(0)} \int_{B_{e^{-1/2-\epsilon} R}(0)} \frac{1}{|z - \zeta|^a} \Delta \tilde{\psi}(\zeta) \frac{\sqrt{-1}}{2} d\zeta \wedge d\bar{\zeta} / (2\pi a) \wedge \frac{\sqrt{-1}}{2} d\zeta \wedge d\bar{\zeta} / 2\pi \leq \frac{2\pi}{2 - a} \int_{B_{1/2+e^{-1/2-\epsilon} R}(0)} \int_{B_{e^{-1/2-\epsilon} R}(0)} \frac{1}{|\xi|^a} \Delta \tilde{\psi}(\xi) \frac{\sqrt{-1}}{2} d\zeta \wedge d\bar{\zeta} / (2\pi a) \wedge \frac{\sqrt{-1}}{2} d\xi \wedge d\bar{\xi} / 2\pi \leq \frac{2\pi}{2 - a} \int_{B_{1/2+e^{-1/2-\epsilon} R}(0)} \int_{B_{e^{-1/2-\epsilon} R}(0)} \Delta \tilde{\psi}(\xi) \frac{\sqrt{-1}}{2} d\zeta \wedge d\bar{\zeta} / (2\pi a) \leq \frac{2\pi}{2 - a} (1/2 + e^{-1/2} - \epsilon) R^{2-a} = C(\epsilon).$$

Note Fubini’s theorem applies thanks to the last estimate, so the change of order of integration is justified, and so the original integral is bounded, concluding the proof of Theorem 3.13. □

**Exercise 3.17.** Show that the fraction in (27) is bounded above by a constant depending only on $r/R$ as claimed and blows up as $r$ approaches 0. When $r = R/2$ show that this constant is equal to 3. (it is even simpler to see it must be $\leq 4$).
Exercise 3.18. Compute the Green kernel of $B_R(0)$ and then derive the Riesz representation formula starting with the identity (cf. [66] [2.4], [93])

$$f(x) = -\int_{B_R(0)} G(x, y) \Delta f(y) dy + \int_{\partial B_R(0)} \partial_r G(x, y) f(y) dy.$$ 

Exercise 3.19. Show that Theorem 3.13 holds for any subharmonic function $f$ by using the fact that the Riesz representation [26] holds with the same expression by interpreting $\Delta f(\zeta) = 2 \zeta$ as the harmonic measure associated to $f$ (a positive measure with respect to which the Green kernel is integrable) [71, Theorem 4.5.1].

Theorem 3.13 can be extended to any dimension as follows [61, Theorem 4.4.5].

Corollary 3.20. The same result holds in $\mathbb{C}^n$ with constants that might additionally depend on $n$. In other words, if $\psi$ is a smooth plurisubharmonic function on $B_R(0) \subset \mathbb{C}^n$, satisfying

$$\psi(0) \geq -1,$$
$$\psi \leq 0, \text{ on } B_R(0),$$

then for every $\rho \in [R/2, e^{-1/2} R]$ there exists a constant $C$ depending only on $R, \rho, n$ such that

$$\int_{B_{\rho}(0)} e^{-\psi} \frac{1}{\sqrt{1-\bar{z}z}} \wedge dz_1 \wedge \cdots \wedge \frac{1}{\sqrt{1-\bar{z}z_n}} \leq C.$$  

(31)

Remark 3.21. In both Theorem 3.13 and Corollary 3.20 one may drop the smoothness assumption on $\psi$: indeed convolve $\psi$ with a smooth mollifier and then observe the integrals on the left had side converge in the limit.

Proof. Write

$$\int_{B_{\rho}(0) \subset \mathbb{C}^n} e^{-\psi} = \int_{\partial B_{\rho}(0) \subset \mathbb{C}^n} dV_{S_{2n-1}}(\lambda) \int_{B_{\rho}(0) \subset \mathbb{C}^n} |w|^{2n-2} e^{-\psi(\lambda w) \sqrt{1-1/dw \wedge dw/2\pi}},$$

(32)

and $|w|^{2n-2} \leq r^{2n-2}$ is uniformly bounded, so we can apply the previous result for $n = 1$. To obtain (32), note that we are integrating over a $2n + 1$ dimensional manifold on the right-hand side and on a $2n$-dimensional one on the left-hand side. We normalize by $2\pi$, the area of $S^1$ since each point is counted “$S^1$ times”, since if wish to write $z = \lambda w$ with $w \in \mathbb{C}$, $\lambda \in S^{2n-1}$, and $|z| = |w|$, $|\lambda| = 1$ then $w$ is only determined in $\mathbb{C}$ up to multiplication by a number in $S^1$! This is not quite precise since we should normalize by an $S^1$ of varying radius! So need to actually divide by $2\pi |w|$, however the change of variables introduces a factor $|w|^{2n-1}$; so we get (32).

Proof of Theorem 3.10. Let the injectivity radius of $(M, \omega)$ be $6r$ (so at each point exists a geodesic ball of that radius). Choose an $r$-net of $M$, that is a collection of points $\{x_j\}_{j=1}^N$ such that $M = \cup_j B_r(x_j)$. For each $\varphi \in \mathcal{H}$ one has $n + \Delta_\omega \varphi > 0$. Hence Green’s formula says that [5, Theorem 4.13 (a), p. 108]

$$\varphi(x) = V^{-1} \int_M \varphi \omega^n + V^{-1} \int_M -G(x, y) \Delta_\omega \varphi(y) \omega^n(y) \leq V^{-1} \int_M \varphi \omega^n + nA_\omega,$$

(33)

where $G(x, y) \geq -A_\omega$ and $\int_M G(x, y) \omega^n(y) = 0$ for each $x \in M$. Note that $A_\omega$ is a constant depending only on $(M, \omega)$, in other words, the Green kernel is uniformly bounded from below. 

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Since we are also assuming $\sup \varphi = 0$, we obtain (the RHS of (33) is independent of $x$)
\[ V^{-1} \int_M \varphi \omega^n + nA_\omega \geq 0. \]

Hence, since $\varphi$ is non-positive,
\[ \int_{B_r(x_j)} \varphi \omega^n \geq \int_M \varphi \omega^n \geq -nA_\omega V, \]
that is,
\[ \sup_{B_r(x_j)} \varphi \geq \frac{-nA_\omega V}{\text{vol}(B_r(x_j))}. \] (34)

Choose a local Kähler potential $\psi_j$ satisfying $\sqrt{-1} \partial \bar{\partial} \psi_j = \omega |_{B_{2r}(x_j)}$. Let $C_2 := \sup_j \sup_{B_{2r}(x_j)} \psi_j$. Therefore, since $\varphi \leq 0$,
\[ \psi_j + \varphi \leq C_2, \text{ on } B_{5r}(x_j). \]

Also, let $y_j \in B_r(x_j)$ be such that (using (34))
\[ \varphi(y_j) \geq \frac{-nA_\omega V}{\text{vol}(B_r(x_j))}. \]

We may also assume without loss of generality that $\psi_j(y_j) = 0$, otherwise we add a constant to $\psi_j$ (and then $C_2$ possibly increases).

Now the function $\varphi + \psi_j$ is plurisubharmonic (psh for short) on $B_{5r}(y_j)$ (recall that $\varphi$ is not psh only $\omega$-psh so we add it to it a local potential for $\omega$ in order to be able to apply Hörmander’s result). Also, since $B_{4r}(y_j) \subseteq B_{5r}(x_j)$, we have that on the set $B_{4r}(y_j)$ it holds
\[ \varphi + \psi_j - C_2 \leq 0 \]
\[ (\varphi + \psi_j - C_2)(y_j) \geq -C_2 - \frac{nA_\omega V}{\text{vol}(B_r(x_j))}. \]

Therefore can apply Hörmander’s result to $f_j := (\varphi + \psi_j - C_2)/(C_2 + \frac{nA_\omega V}{\text{vol}(B_r(x_j))})$ on $B_{4r}(y_j)$ (note $C_2 \geq 0$, $A_\omega > 0$, so we are not dividing by zero), namely obtain that
\[ \int_{B_{2r}(y_j)} e^{-f_j} \omega^n |_{B_{2r}(y_j)} < C_j \]

(instead of $2r$ could have taken any number in the range $[2r, e^{-1/2}4r]$). Patching these up, using the fact that $B_{2r}(y_j) \supseteq B_r(x_j)$ and $M$ is covered by the latter, we obtain that regardless of $\varphi$, one has
\[ \int_M e^{-a\varphi} \omega^n < C, \]
where $a := \min_j 1/(C_2 + \frac{nA_\omega V}{\text{vol}(B_r(x_j))})$, and consequently $\alpha(M,[\omega]) \geq a > 0$. \qed
4 The Kähler–Einstein equation

The Kähler–Einstein equation (3) is a fourth order equation in terms of the Kähler potential. The remarkable formula (4) for the Ricci form, however, allows to integrate it to a second order equation. Indeed, subtracting Ric $\omega$ from both sides of the equation and using (5) yields

$$\text{Ric} \, \omega - \text{Ric} \, \omega = \sqrt{-1} \partial \bar{\partial} \log \omega_{\nu} = \mu \omega - \text{Ric} \, \omega = \mu \sqrt{-1} \partial \bar{\partial} \varphi - \sqrt{-1} \partial \bar{\partial} f_{\omega},$$

where $f_{\omega}$ is called the Ricci potential of $\omega$, and satisfies $\sqrt{-1} \partial \bar{\partial} f_{\omega} = \text{Ric} \, \omega - \mu \omega$, where it is convenient to require the normalization

$$\int e^{f_{\omega}} \omega^n = \int \omega^n.$$ We thus obtain the Kähler–Einstein equation,

$$\omega^n = \omega^n e^{f_{\omega} - \mu \varphi}, \quad \text{on } M \quad (35)$$

for a global smooth function $\varphi$ (called the Kähler potential of $\omega$ relative to $\omega$). The function $f_{\omega}$, in turn, is given in terms of the reference geometry and is thus known. Observe that, strictly speaking, the right-hand side of (35) should be

$$\omega^n e^{f_{\omega} - \mu \varphi + C}$$

For some constant $C$; whenever $\mu \neq 0$ we can incorporate the constant $C$ by subtracting $C/\mu$ from $\varphi$ since the left-hand side of (35) is invariant under this. When $\mu = 0$, the constant $C$ must be zero by (13).

Note also that the solution $\varphi$ is only determined up to a constant when $\mu = 0$, while it is uniquely determined when $\mu \neq 0$ by (13).

We close this section by noting a relationship between the K-energy and Kähler–Einstein metrics: the Euler–Lagrange equation of the K-energy is precisely the Kähler–Einstein equation. Indeed,

$$\frac{d}{de} \bigg|_{e=0} E(\varphi(e)) = \frac{d}{de} \bigg|_{e=0} \text{Ent}(e^{f_{\omega}} \omega^n, \omega_{\nu}^n)$$

where we used Exercise 2.1 and the following exercise.

**Exercise 4.1.** Show that the Ricci potential satisfies the following equation

$$f_{\omega_{\nu}} = \log \frac{e^{f_{\omega}} \omega^n_{\nu}}{\omega_{\nu}^n} - \mu \varphi - \log V^{-1} \int_M e^{f_{\omega}} \omega^n_{\nu}.$$ (37)

Thus, the Euler–Lagrange equation of the K-energy is precisely

$$\Delta_{\omega_{\nu}} f_{\omega_{\nu}} = 0,$$
i.e.,
\[ f_{\omega_{\varphi}} = \text{const}, \]
that, recalling Definition 3.4, means that Ric \( \omega_{\varphi} = \mu \omega_{\varphi} \).

## 5 Properness implies existence

In this section we prove the easier part of Conjecture 1.2:

**Theorem 5.1.** If the Mabuchi energy \( E \) is proper on \( \mathcal{H}^K \) then there exists a \( K \)-invariant Kähler–Einstein metric in \( \mathcal{H} \).

This result is due to Tian [90], even though it is in some sense already implicit in Ding–Tian [48]. The proof we give follows the same ideas as in the original proof, with some simplifications in the presentation.

In particular, in combination with Theorem 3.11 we obtain as a corollary a theorem of Tian [88, Theorem 2.1]:

**Corollary 5.2.** Let \( \mu > 0 \) and suppose that (6) holds. Whenever \( \alpha(M, 2\pi c_1(M)/\mu) > n\mu/(n+1) \) there exists Kähler–Einstein metric cohomologous to \( 2\pi c_1(M)/\mu \).

We also obtain as a corollary the theorems of Aubin and Yau:

**Corollary 5.3.** Let \( \mu \leq 0 \) and suppose that (6) holds. Then there exists a unique Kähler–Einstein metric cohomologous to \( \omega \).

**Remark 5.4.** The proof of Corollary 5.3 will we give will not directly use the fact that the K-energy is proper whenever \( \mu \leq 0 \) (which holds according to Theorem 3.11). In fact, the proof of Corollary 5.3 will be more or less a step in the proof of Theorem 5.1.

### 5.1 A two-parameter continuity method

We will give a somewhat nonstandard proof of Theorem 5.1 using a two-parameter continuity method instead of the more standard proofs that use one-parameter continuity methods or the Ricci flow equation. Namely, we consider the two-parameter family of equations,

\[
\omega^n_c = e^{tf\omega_c + ct - s\varphi}\omega^n, \quad c_t := -\log \frac{1}{V} \int_M e^{tf\omega}\omega^n, \quad (s, t) \in A, \tag{38}
\]

where
\[
A := (-\infty, 0] \times [0, 1] \cup [0, \mu] \times \{1\}
\]
is the parameter set—the union of a semi-infinite rectangle and an interval, and show that there exists a unique solution \( \varphi(s, t) \) for each \((s, t) \in A \) once we require that the solution be continuous in the parameters \( s, t \). Of course, we are really looking to show the existence of \( \varphi(\mu, 1) \). Thus, the strategy is to first construct \( \varphi(s, t) \) for other values of \((s, t) \) for which existence is easier to show and then perturbing the equation and eventually arriving at the equation for the values \((\mu, 1) \). Hence, the name ‘continuity method.’

To show existence for the sub-rectangle \((-\infty, 0] \times [0, 1] \) is easier and is precisely what proves Corollary 5.3 Also, one could restrict to the sub-rectangle \((-S, \mu] \times [0, 1] \) for any value \( S > 0 \)
as far as proving the existence theorems is concerned. Working on $A$ requires no more work and is somewhat more natural and canonical, since the value $s = -\infty$ corresponds, in a sense that can be made precise $[93], [62], [9], [12]$, to the initial reference metric $\omega$. Thus, one can view this continuity method as starting with the given reference metric and deforming it to the Kähler–Einstein metric. In fact, the one-parameter continuity method with $t = 1$ fixed and $s$ varying between $-\infty$ and $\mu$ can be viewed as the continuity method analogue of the Kähler–Ricci flow, and is called the Ricci continuity method, introduced in $[76]$ and further developed in $[62]$. One of the reasons we work with the two-parameter family in these lectures is that then the existence of solutions for some parameter values is automatic. Indeed,

$$\varphi(s, 0) = 0, \quad s \in (-\infty, 0].$$

(39)

Working with the one-parameter Ricci continuity method is harder since it is nontrivial to show the existence of solutions for some parameter value $(s, 1)$. The full strength of the Ricci continuity method however goes beyond that of the two-parameter family in that the former can be used to show existence of Kähler–Einstein edge metrics, which are a natural generalization of Kähler–Einstein metrics that allows for a singularity along a complex submanifold of codimension one. In that context the two-parameter continuity method does not always seem to work.

**Exercise 5.5.** Show that for each $(s, t) \in A$,

$$\text{Ric}_\omega \varphi(s, t) = (1 - t)\text{Ric}_\omega + s\omega + (\mu t - s) \omega. \quad (40)$$

Note that this implies that if indeed, as claimed, $\varphi(s, 1) \to 0$ as $s \to -\infty$ then $f_\omega - s \varphi$ should be small, i.e., $\varphi \approx f_\omega / s$ in that regime.

### 5.2 Openness

Let

$$\text{PSH}(M, \omega) = \{ \varphi \in L^1(M, \omega^n) : \varphi \text{ is upper semicontinuous and } \omega_\varphi \geq 0 \}$$

denote the set of $\omega$-plurisubharmonic functions on $M$. Define $M_{s,t} : C^{2,\gamma} \cap \text{PSH}(M, \omega) \to C^{0,\gamma}$ by

$$M_{s,t}(\varphi) := \log \frac{\omega^n}{\omega^n_\varphi} - tf_\omega + s\varphi - ct, \quad (s, t) \in A.$$

If $\varphi(s, t) \in C^{2,\gamma} \cap \text{PSH}(M, \omega)$ is a solution of (38), we claim that its linearization

$$DM_{s,t}|_{\varphi(s,t)} = \Delta_{\varphi(s,t)} + s : C^{2,\gamma} \to C^{0,\gamma}, \quad (s, t) \in A,$$

(41)

is an isomorphism when $s \neq 0$ and $s < \mu$. If $s = 0$, this map is an isomorphism if we restrict on each side to the codimension one subspace of functions with integral equal to 0 with respect to $\omega^n_{\varphi(0,t)}$. Furthermore, we also claim that $C^{2,\gamma} \cap \text{PSH}(M, \omega) \times A \ni (\varphi, s, t) \mapsto M_{s,t}(\varphi) \in C^{0,\gamma}$ is a $C^1$ mapping. Given these claims, the Implicit Function Theorem then guarantees the existence of a solution $\varphi(s, t)$ in $C^{2,\gamma}$ for all $(s, t) \in A$ sufficiently close to $(s, t)$. This solution must necessarily be contained in $\text{PSH}(M, \omega)$ since $M_{s,t}(\varphi(s,t)) = 0$ means that $\omega^n_{\varphi(s,t)} > 0$, and by the continuity of $\omega_\varphi(s,t)$ in the parameters it follows that all the eigenvalues of the metric stay positive along the deformation.

We concentrate on the first claim, since the second claim is easy to check.
Now, $DM_{s,t}$ is an elliptic operator and there is a classical and well-developed theory for those kind of operators acting on Hölder spaces $[56]$. In particular, such an operator has a generalized inverse, or Green kernel. Also, it is Fredholm of index 0. Using the existence of a Green kernel shows that $C^{2,\gamma}$ decomposes as a direct sum $\{Gf : f \in C^{0,\gamma}\} \oplus K_{s,t}$, where $K_{s,t}$ denotes the kernel of the operator $DM_{s,t}$. Thus, whenever $K_{s,t} = \{0\}$ the operator is an isomorphism.

The nullspace of $DM_{s,t}$ is clearly trivial when $s < 0$ since the spectrum of $\Delta_{\varphi(s,t)}$ is contained in $(-\infty, 0]$. When $s = 0$ the nullspace consists of the constants and so is an isomorphism when restricted to functions of zero average. Thus, the claim is verified whenever $s \leq 0$. To deal with the case when $s$ is positive the following lemma is needed.

**Lemma 5.6.** Suppose that $M_{s,1}(\varphi(s,1)) = 0$ and that $s \in (0, \mu)$. Then the spectrum of $\Delta_{\varphi(s,1)}$ is contained in $(-\infty, -s)$.

**Proof.** Let $\psi$ be an eigenfunction of $\Delta_{\omega_{\varphi(s,t)}}$ with eigenvalue $-\lambda_1$. By standard theory, $\psi$ is smooth. The Bochner–Weitzenböck formula states that
\[
\frac{1}{2} \Delta_g |\nabla g f|^2_g = \text{Ric} (\nabla_g f, \nabla_g f) + |\nabla^2 f|^2_g + \nabla f \cdot \nabla (\Delta_g f).
\]
Since $\Delta_g = 2\Delta_\omega$ and $|\nabla^2 f|^2_g = 2|\nabla^{1,0} \nabla^{1,0} f|^2 + 2(\Delta_\omega f)^2$, this becomes
\[
\Delta_\omega |\nabla^{1,0} \psi|^2_g = 2\text{Ric} (\nabla^{1,0} \psi, \nabla^{0,1} \psi) + 2|\nabla^{1,0} \nabla^{1,0} \psi|^2 + 2\lambda_1^2 \psi^2 - 4\lambda_1 |\nabla^{1,0} \psi|^2_\omega.
\]
Integrating (42) and using that $\text{Ric}_\omega(s) > s\omega(s)$ when $s < \mu$ by (40),
\[
\int \left( (2s - 4\lambda_1)|\nabla^{1,0} \psi|^2_\omega + 2\lambda_1^2 \psi^2 \right) \omega^n < 0.
\]
Now,
\[
\int 2\lambda_1^2 \psi^2 \omega^n = - \int 2\lambda_1 \psi \Delta_\omega \psi \omega^n = \int 2\lambda_1 |\nabla^{1,0} \psi|^2_\omega.
\]
Thus,
\[
\int (2s - 2\lambda_1)|\nabla^{1,0} \psi|^2_\omega \omega^n < 0,
\]
so we see that $\lambda_1 > s$. \qed

**Remark 5.7.** Here we see why we cannot use the the rectangle $(-\infty, \mu] \times [0, 1]$ containing $A$: we run into trouble with openness. If we had chosen $\omega$ to have nonnegative Ricci curvature we could have also worked on the larger trapezoid
\[
(-\infty, 0] \times [0, 1] \cup \{(s, t) \in [0, \mu] \times [0, 1] : \mu t \geq s\}.
\]
Producing such an $\omega$ is possible by applying Corollary 5.3 with $\mu = 0$ (whose proof does not require these arguments).
5.3 An $L^\infty$ bound in the sub-rectangle

The following two lemmas will be sufficient for our purposes.

**Lemma 5.8.** Suppose that $\varphi(s,t)$ is a solution of (38). Then whenever $s < 0$,

$$\varphi(s,t) < C(1 + 1/|s|),$$

for some uniform constant $C$ independent of $s$ and $t$.

**Proof.** Let $p$ be a point where the maximum of $\varphi$ is achieved. Then, $\sqrt{-1} \partial \bar{\partial} \varphi(p) \leq 0$. Thus,

$$\frac{\omega^n}{\omega^n(p)} \leq 1,$$

i.e.,

$$tf_\omega(p) + c_t - s\varphi(s,t)(p) \leq 0,$$

so,

$$\max \varphi(s,t) = \varphi(s,t)(p) \leq (-c_t - t \min f_\omega)/|s|.$$  

Similarly, if $q$ is a point where the minimum is achieved,

$$\min \varphi(s,t) = \varphi(s,t)(q) \geq (-c_t - t \max f_\omega)/|s|.$$  

concluding the proof.

**Lemma 5.9.** Suppose that $\varphi(0,t)$ is a solution of (38). Then,

$$\max_M |\varphi(0,t)| < C,$$

for some uniform constant $C$ independent of $t$.

**Proof.** As remarked earlier, the solution in this case is a priori only unique up to a constant. However, we fixed the normalization by requiring that the solution be continuous in the parameters $s,t$. We will eventually show that there are solutions $\varphi(s,t)$ for all $s$ less than 0 and all $t \in [0,1]$. Therefore, $\varphi(s,t)$ converges pointwise to the solution $\varphi(0,t)$ for each fixed $t$ as $s$ tends to 0, and so this solution is actually unique. In particular, since the latter change sign, so must the former. Thus, it is enough to estimate the oscillation of $\varphi(0,t)$ in order to estimate the $L^\infty$ norm of $\varphi(0,t)$, i.e., it suffices to estimate the minimum of

$$\varphi(0,t) - \max \varphi(0,t) - 1.$$  

This bound, due to Yau [99], then follows just as in [91, §5].

5.4 An $L^\infty$ bound in the interval

**Lemma 5.10.** Let $t = 1$. The K-energy is monotonically decreasing in $s$. 


Proof. When $t = 1$, $c_t = 0$. Then,

$$
\frac{d}{ds} E(\varphi(s, 1)) = \frac{d}{ds} \text{Ent}(e^{f_\omega} \omega^n, e^{f_\omega - s\varphi} \omega^n)
- \mu V^{-1} \int_M \dot{\varphi} \omega^n + \mu V^{-1} \int_M \dot{\varphi} \omega^n + \mu V^{-1} \int_M \varphi \Delta \dot{\varphi} \omega^n
= V^{-1} \int_M (-\varphi - s\dot{\varphi} - s\varphi \Delta \dot{\varphi}) \omega^n + \mu V^{-1} \int_M \varphi \Delta \dot{\varphi} \omega^n.
$$

Differentiating (38) yields

$$
\varphi(s, t) = -(\Delta + s)\dot{\varphi}(s, t).
$$

Thus,

$$
\frac{d}{ds} E(\varphi(s, 1)) = V^{-1} \int_M (\Delta \dot{\varphi} - (\mu - s)\Delta \dot{\varphi}(\Delta + s) \dot{\varphi}) \omega^n = -(\mu - s) V^{-1} \int_M \varphi \Delta (\Delta + s) \dot{\varphi} \omega^n < 0,
$$

since $\dot{\varphi}$ is not constant as can be seen from (43) and (38), while $\Delta + s$ is a negative operator for $s < \mu$ thanks to Lemma 5.6.

By the previous lemma, the K-energy actually decreases along the interval. By properness this implies that the functional $I - J$ stays uniformly bounded along the interval once we know $\varphi(0, 1)$ exists. This will indeed be the case as we will show (in the first step of the proof) existence in the sub-rectangle for all values $s \leq 0$. Now, the explicit formula (12) for the K-energy hence implies that the entropy is bounded from above along the interval,

$$
\text{Ent}(e^{f_\omega} \omega^n, \omega^n) < C,
$$

thus,

$$
\int (t - 1) f_\omega - s\varphi(s, 1) + c_t) \omega^n < C,
$$

or

$$
\int -\varphi(s, 1) \omega^n < C(1 + 1/s).
$$

Observe that here we may assume that $s > \epsilon > 0$ since by openness about the value $(0, 1)$ we have existence for small positive values of $s$. Going back to (7) now shows that

$$
\int \varphi(s, 1) \omega^n < C(1 + 1/s),
$$

the mean value inequality (33) shows that

$$
\max \varphi(s, 1) < C(1 + 1/s).
$$

It remains to estimate $\min \varphi(s, 1)$. Now, as in the proof of Lemma 5.9 we set

$$
\alpha(s) := \max \varphi(s, 1) - \varphi(s, 1) + 1.
$$

A standard Moser iteration argument now yields an estimate [91, §5]

$$
\|\alpha(s)\|_{L^\infty(M \omega^n_{\varphi(s, 1)})} \leq C \left(\|\alpha(s)\|_{L^1(M \omega^n_{\varphi(s, 1)})}\right),
$$

22
but \[ \|\alpha(s)\|_{L^1(M,\omega^n_{\varphi(s,1)})} = \max \varphi(s,1) + 1 + \int -\varphi(s,1)\omega^n_{\varphi(s,1)} < C(1 + 1/s). \]

Thus, we have proven the following.

**Lemma 5.11.** Suppose that \( \varphi(s,1) \) exists for \( s \in (0,\epsilon) \) for some \( \epsilon > 0 \). Then for every \( s > \epsilon \),

\[
\max_M |\varphi(s,1)| < C(1 + 1/s).
\]

### 5.5 Second order estimates

The reference for this subsection is [78, §7.2–7.4,7.7].

We say that \( \omega,\omega_\varphi \) are uniformly equivalent if

\[
C_1 \omega \leq \omega_\varphi \leq C_2 \omega,
\]

for some constants \( C_2 \geq C_1 > 0 \). We start with a simple result which shows that a Laplacian estimate can be interpreted geometrically. Denote

\[
\text{tr}_\omega \eta := \text{tr}( [g_{ij}]^{-1}[h_{kl}] ),
\]

where \( \omega = g_{ij}dz^i \wedge \overline{dz}^j, \eta = h_{ij}dz^i \wedge \overline{dz}^j \) in local coordinates. Similarly, denote

\[
\text{det}_\omega \eta := \text{det}( [g_{ij}]^{-1}[h_{kl}] ).
\]

**Exercise 5.12.** Show that (49) is implied by either

\[
n + \Delta_\omega \varphi = \text{tr}_\omega \omega_\varphi \leq C_2, \quad \text{and} \quad \text{det}_\omega \omega_\varphi \geq C_1 C_2^{n-1}/(n-1)^{n-1},
\]

or,

\[
n - \Delta_\omega \varphi = \text{tr}_\omega \omega_\varphi \leq 1/C_1, \quad \text{and} \quad \text{det}_\omega \omega_\varphi \leq C_1 C_2^{n-1}(n-1)^{n-1}.
\]

**Exercise 5.13.** Conversely, show that (49) implies

\[
\text{tr}_\omega \omega_\varphi \leq nC_2, \quad \text{and} \quad \text{det}_\omega \omega_\varphi \geq C_1^n,
\]

as well as

\[
\text{tr}_\omega \omega \leq n/C_1, \quad \text{and} \quad \text{det}_\omega \omega_\varphi \leq C_2^n.
\]

(Indeed, \( \sum (1 + \lambda_j) \leq A \), and \( \Pi(1 + \lambda_j) \geq B \) implies \( 1 + \lambda_j \geq (n-1)^{n-1}B/A^{n-1} \); conversely, \( \Pi(1 + \lambda_j) \geq (\frac{1}{n} \sum \frac{1}{1+\lambda_j})^{-n} \geq C_1^n \).

The quantities \( \text{tr}_\omega \omega_\varphi \) and \( \text{det}_\omega \omega_\varphi \) have a nice geometric interpretation. To see that, we will study the geometry of the identity map \( \iota : M \to M! \) Consider \( \partial_\varepsilon^{-1} \) either as a map from \( T^{1,0}M \) to itself, or as a map from \( \Lambda^n T^{1,0} M \) to itself. Alternatively, it is section of \( T^{1,0} \mathcal{M} \otimes T^{1,0} \mathcal{M} \), or of \( \Lambda^n T^{1,0} \mathcal{M} \otimes \Lambda^n T^{1,0} \mathcal{M} \), and we may endow these product bundles with the product metric induced by \( \omega \) on the first factor, and by \( \omega_\varphi \) on the second factor. Then, (50) means that the norm squared of \( \partial_\varepsilon^{-1} \), in its two guises above, is bounded from above by \( C_2 \), respectively bounded from below by \( C_1 C_2^{n-1}/(n-1)^{n-1} \). Similarly, the quantities \( \text{tr}_\omega \omega \) and \( \text{det}_\omega \omega \) and (51) can be interpreted in terms of \( \partial_\varepsilon \).
Now, for us the quantities
\[ \det \omega \varphi(s,t) = e^{t \omega + s \varphi(s,t)} \]
and
\[ \det \omega \varphi(s,t) \omega = e^{-t \omega - s \varphi(s,t)} \]
are already uniformly bounded thanks to the uniform estimate on \( \| \varphi(s,t) \|_{L^{\infty}} \) obtained in \([5.3-5.4]\). Thus, according to Exercise 5.12, it remains to find an upper bound for either \( |\partial \varphi|^{2} \) or \( |\partial t|^{2} \) (from now on we just consider maps on \( T^{1,0}M \)).

The standard way to approach this is by using the maximum principle, and thus involves computing the Laplacian of either one of these two quantities. The classical approach, due to Aubin \([1, 2, 3]\) and Yau \([99]\), is to estimate the first, while a more recent approach is to estimate the second \([76, 62]\), and this builds on using and finessing older work of Lu \([64]\) and Bando–Kobayashi \([7]\). Both of these approaches are explained in a unified manner in \([78, §7]\). The result we need is \([78, Corollary 7.8 (i)]\).

**Lemma 5.14.** Let \( \varphi \in C^{4}(M) \cap PSH(M, \omega) \). Suppose that
\[ \text{Ric} \omega \varphi \geq -C_{1} \omega - C_{2} \omega \varphi, \tag{52} \]
and
\[ \max_{M} \text{Bisection} \omega \leq C_{3}. \tag{53} \]
Then
\[ -n < \Delta \omega \varphi \leq (C_{1} + n(C_{2} + 2C_{3} + 1)) e^{(C_{2} + 2C_{3} + 1) \text{osc} \varphi} - n. \tag{54} \]

To see that this result is applicable, observe first that \( (53) \) holds simply because \( M \) is compact and \( \omega \) is smooth. Second, according to \( (10) \)
\[ \text{Ric} \omega \varphi(s,t) = (1 - t) \text{Ric} \omega + s \omega \varphi(s,t) + (\mu t - s) \omega \]
\[ \geq s \omega \varphi(s,t) + (\mu t + (1 - t)C_{4} - s) \omega, \]
where \( C_{4} \) is a lower bound for the Ricci curvature of \( \omega \), i.e., satisfying
\[ \text{Ric} \omega \geq C_{4} \omega. \]

Therefore, Lemma 5.14 holds with
\[ C_{1} = \max \{0, |\mu t + (1 - t)C_{4} - s|\}, \quad C_{2} = \max \{0, |s|\}, \quad C_{3} = C_{3}(\omega). \]

**5.6 Higher order compactness via Evans–Krylov’s estimate**

To show our solutions along the continuity method are smooth, it suffices to improve the Laplacian estimate to a \( C^{2, \gamma} \) estimate for some \( \gamma > 0 \). Indeed, then it is standard to see that the solutions automatically have uniform \( C^{k, \gamma} \) estimates for each \( k \) (Exercise 5.18). This is obtained via the standard Evans–Krylov estimate, adapted to the complex setting. The standard references for this are the lecture notes of Siu \([84]\) and Blocki \([19, §5]\), as well as the treatment of the real Monge–Ampère equation by Gilbarg and Trudinger \([56]\), with the modification by Wang–Jiang \([96]\), Blocki \([18]\).
Lemma 5.15. Let $\psi \in C^4(M) \cap \text{PSH}(M, \omega)$ be a solution to $\omega^n = \omega^n e^F$. Then
\begin{equation}
||\psi||_{C^2, \gamma} \leq C,
\end{equation}
where $\gamma > 0$ and $C$ depend only on $M, \omega, ||\Delta_\omega \psi||_{C^0}, ||\psi||_{C^0}$, and $||F||_{C^2}$.

Proof. For concreteness, we carry out the proof for out particular $F$ in (38). For each pair $(s, t)$ define $h = h(s, t)$ by
\begin{equation}
\log h := tf_\omega - s\varphi + c_t + \log \det[\psi_{ij}],
\end{equation}
where $\psi$ is a local Kähler potential for $\omega$ on some fixed neighborhood (we will obtain our estimate only on this neighborhood, but then cover $M$ with finitely many such). Set $u := \psi + \varphi$. Each $C^{k, \gamma}$ norm of $\psi$ is bounded by a constant $C = C(k, \gamma, \omega)$, so to get the desired bound on $\varphi$ is tantamount to bounding $u$.

Let $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n$ be a unit vector, and consider $u$ as a function of $(z_1, \ldots, z_n) \in \mathbb{C}^n$. Then,
\begin{equation}
(\log \det[u_{ij}])_{\eta\eta} = -u^{ij} u^{k\ell} u_{\eta\eta k\ell} + u^{ij} u_{\eta\eta j i}.
\end{equation}
(repeated differentiation is justified since by assumption $\varphi$ and hence $u$ belong to $C^4(M)$). Since
\begin{equation}
\log \det[u_{ij}] = \log \det[\psi_{ij} + \varphi_{ij}] = \log h,
\end{equation}
and letting $w := u_{\eta\eta}$,
\begin{equation}
w^{ij} w_{ij} \geq (\log h)_{\eta\eta} = \frac{h_{\eta\eta}}{h} - \frac{|h_{\eta\eta}|^2}{h^2},
\end{equation}
which can be rewritten in divergence form,
\begin{equation}
(hu^{ij} w_i)_j \geq \eta_j (\eta^k h_k)_i - g, \quad g := \frac{|h_{\eta\eta}|^2}{h}.
\end{equation}

Theorem 5.16. [56, Theorem 8.18] Let $\Omega \subset \mathbb{R}^m$, and assume $B_{4\rho} = B_{4\rho}(y) \subset \Omega$. Let $L = D_i(a^{ij} D_j + b^i) + c^i D_i + d$ be strictly elliptic, $\lambda I < [a^{ij}]$, with $a^{ij}, b^i, c^i, d \in L^\infty(\Omega)$, satisfying
\begin{equation}
\sum_{i,j} |a^{ij}|^2 < \Lambda^2, \quad \lambda^{-2} \sum (|b^i|^2 + |c^i|^2) + \lambda^{-1} |d| \leq \nu^2.
\end{equation}

Then if $U \in W^{1,2}(\Omega)$ is nonnegative and satisfies $LU \leq g + D_i f^i$, with $f^i \in L^q, g \in L^{n/2}$ with $q > m$, then for any $p \in [1, \frac{m}{m-2})$, $p = m, m-2, \ldots$, we have
\begin{equation}
\rho^{-m/p}||U||_{L^p(B_{2\rho})} \leq C(\inf_{B_{\rho}} U + \rho^{1-m/q}||f||_{L^q(B_{2\rho})} + \rho^{2-2n/q}||g||_{L^{n/2}(B_{2\rho})})
\end{equation}
\begin{equation}
< C(\inf_{B_{\rho}} U + \rho||f||_{L^\infty(B_{2\rho})} + \rho^2||g||_{L^\infty(B_{2\rho})}),
\end{equation}
with $C = C(n, \Lambda/\lambda, \nu, \rho, q, p)$.

Lemma 5.17. Let $w$ be defined by (55). Suppose that $s > S$. One has
\begin{equation}
\sup_{B_{2\rho}} w - \frac{1}{|B_{\rho}|} \int_{B_{\rho}} w \omega^n \leq C\left(\sup_{B_{2\rho}} w - \sup_{B_{\rho}} w + \rho (\rho + 1)\right),
\end{equation}
with $C = C(M, \omega, S, ||\varphi(s, t)||_{C^0(M)}, ||\Delta_\omega \varphi(s, t)||_{C^0})$.
Proof. By \((\ref{eq:sobolev})\) \(v := \sup_{B_{2\rho}} w - w\) satisfies

\[
(hu^j v_i)_j \leq g - \eta^j (\eta^k h_k)_i,
\]

where \(g := \frac{|h_j|^2}{h_i}\). In general, there are positive bounds on \([u^j_j]\) and \([u^j_i]\), depending only on \(||\Delta \varphi(s, t)||_{C^0}\), and hence similar positive bounds on \([u^j]\) and its inverse, in these coordinates, depending only on \(S, M, \omega, ||\varphi(s, t)||_{C^0}, \omega\), and \(||\varphi(s, t)||_{C^0}\) (since the latter two quantities control \(||\varphi(s, t)||_{C^0,1}\) by interpolation). This, together with\(\ref{eq:uniform-uniform} \) gives the desired inequalities provided that \(v \in W^{1,2}(M, \omega^0)\), which is automatic as \(v\) and \(\omega^0\) are smooth and \(M\) is compact. The lemma follows.

Now, let \(\{V_j\}_{j=1}^n\) be smooth vector fields on \(M\) that span \(T^{1,0} M\) over \(M\) and that on a local chart are given by \(V_k := \frac{\partial}{\partial x_k}, k = 1, \ldots, n\), and denote

\[
M(\rho) := \sup_{|\zeta|, |Z| \in (0, \rho)} \sum_{j=1}^n V_j V_j u, \quad m(\rho) := \inf_{|\zeta|, |Z| \in (0, \rho)} \sum_{j=1}^n V_j V_j u
\]

Our goal is to show that \(\nu(\rho) := M(\rho) - m(\rho)\) is Hölder continuous with respect to \(g_u\), i.e., \(\nu(\rho) \leq C \rho^{\gamma'}\), for some \(\gamma' > 0\), or equivalently that \(\nu(\rho) \leq (1 - \epsilon) \nu(2\rho) + \sigma(\rho)\), for some \(\epsilon \in (0, 1)\) and some non-decreasing function \(\sigma\) \(\cite{friedman1998} \text{ Lemma 8.23}\). Let

\[
M_\eta(\rho) := \sup_{|\zeta|, |Z| \in (0, \rho)} u_{\eta\eta}, \quad m_\eta(\rho) := \inf_{|\zeta|, |Z| \in (0, \rho)} u_{\eta\eta}, \quad \nu_\eta(\rho) := M_\eta(\rho) - m_\eta(\rho).
\]

Equation \((\ref{eq:sobolev})\) implies

\[
\sup_{B_{2\rho}} w - \frac{1}{|B_{\rho}|} \int_{B_{\rho}} w \leq C (\nu_\eta(2\rho) - \nu_\eta(\rho) + \rho(\rho + 1)),
\]

and so it remains to obtain a similar inequality for \(w - \inf_{B_{2\rho}} w\).

Note that \(DF|_{(A - B) \leq F(A) - F(B)}\), by concavity of \(F(A) := \log \det A\) on the space of positive Hermitian matrices. Since \(DF|_{\nabla^1 u} = (\nabla^1 u)^{-1}\), we have

\[
u^j(y)(u^j_j(y) - u^j_j(x)) \leq \log \det u^j_j(y) - \log \det u^j_j(x) \leq |h|_{C^{0,1}|y - x|}.
\]

We now decompose \((u^j_j)\) as a sum of rank one matrices. This will result in the previous equation being the sum of pure second derivatives for which we can apply our estimate from the previous step. By uniform ellipticity this decomposition can be done uniformly in \(y \cite{friedman1998} \text{ p. 103} \text{,} \cite{trudinger1995}\). Namely, we can fix a set \(\{\gamma_k\}_{k=1}^N\) of unit vectors in \(\mathbb{C}^n\) (which we can assume contains \(\gamma_1 = \eta\) as well as a unitary frame of which \(\eta\) is an element) and write

\[
(u^j_j) = \sum_{k=1}^N \beta_k(y) \gamma_k \bar{\gamma}_k,
\]

with \(\beta_k(y)\) uniformly positive depending only on \(n, \lambda\) and \(\Lambda\). Thus \((\ref{eq:sobolev})\) gives

\[
w(y) - w(x) \leq C|y - x| - \sum_{k=2}^N \beta_k(y)(u_{\gamma_k \bar{\gamma}_k}(y) - u_{\gamma_k \bar{\gamma}_k}(x)) \leq C|y - x| + \sum_{k=2}^N \beta_k(y)(\sup_{B_{2\rho}} u_{\gamma_k \bar{\gamma}_k} - u_{\gamma_k \bar{\gamma}_k}(y)).
\]
Now let $w(x) = \inf_{B_{2\rho}} w$, and average over $B_\rho$ to get, using (62),

$$
\frac{1}{|B_\rho|} \int_{B_\rho} w - \inf_{B_{2\rho}} w \leq C \left( \sum_{k=2}^{N} \nu_{2k} (2\rho) - \nu_{2k} (\rho) + \rho (\rho + 1) \right).
$$

Combining this with (62), and summing over $k = 1, \ldots, N$ we thus obtain an estimate on $\nu(\rho)$ of the desired form. Hence $\Delta_\omega \varphi(s) \in C^{0,\gamma'}$ for some $\gamma' > 0$. In fact our proof actually showed that $\varphi_{\eta j} \in C^{0,\gamma'}$ for any $i, j$.

Hence, $|\Delta_\omega \varphi(s, t)|_{C^{0,\gamma'}} \leq C = C(M, \omega, S, ||\Delta_\omega \varphi(s, t)||_{C^0(M)}, ||\varphi(s, t)||_{C^0(M)})$. This concludes the proof of Lemma 5.15. □

**Exercise 5.18.** Suppose that $\varphi \in C^\infty(M)$ satisfies $\omega^n = e^F \omega^n$ and that

$$
||\varphi||_{C^2,\gamma} \leq C.
$$

Show that there exists $C'$ such that

$$
||\varphi||_{C^3,\gamma} \leq C' = C'(M, \omega, ||F||_{C^{1,\alpha}}).
$$

(Hint: Let $D$ be a first order operator with constant coefficients in some holomorphic coordinate chart. Write the Monge–Ampère equation in those coordinates as

$$
\log \det [u_{ij}] = \log \det [\psi_{ij}] + F =: \tilde{F},
$$

as in the proof of Lemma 5.15 and apply $D$ to this equation. By Exercise 2.1 this then gives a Poisson type equation for $Du$,

$$
\tilde{D}u = Du_{ij} = D\tilde{F}.
$$

This is not quite a Poisson equation since the Laplacian on the left hand side depends on $u$ itself! However, since we already have uniform $C^{0,\gamma}$ estimates on $[u^{ij}]$ and $[u_{ij}]$ the usual Schauder estimates [56] give

$$
||Du||_{C^{2,\gamma}} \leq C (||Du||_{C^{0,\gamma}} + ||\tilde{F}||_{C^{0,\gamma}}).
$$

Since this holds for

$$
D \in \left\{ \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial \bar{z}^n} \right\},
$$

we are done.)

Applying the previous exercise repeatedly yields the following improvement of Lemma 5.15.

**Corollary 5.19.** Let $\psi \in C^{k+1}(M) \cap \text{PSH}(M, \omega)$ be a solution to $\omega^n = e^F \omega^n$. Then

$$
||\psi||_{C^k,\gamma} \leq C,
$$

where $\gamma > 0$ and $C$ depend only on $M, \omega$, $||\Delta_\omega \psi||_{C^0}$, $||\psi||_{C^0}$, and $||F||_{C^{k-1}}$, and $C$ depends additionally also on $k$. 27
5.7 Properness implies existence

We now complete the proof of one direction of Conjecture 1.2. Let $K$ be a connected compact subgroup of the automorphism group. Recall that $H^K \subset H$ consists of all $K$-invariant elements of $H$. We denote by $C^k,\gamma_K$ the subset of $C^k,\gamma$ consisting of $K$-invariant functions. Denote by $B \subset A$ the subset of parameter values $(s,t)$ for which there exists a $K$-invariant $C^{2,\gamma}$ solution $\varphi(s,t)$ to (16.4). Note that $(-\infty,0] \times \{0\} \subset B$ since $\varphi(s,0) = 0$ and we can always assume that $\omega$ is $K$-invariant, for instance by taking an arbitrary Kähler metric and averaging it with respect to the Haar measure of $G$ [60, p. 88].

Next, observe that the openness arguments of §5.2 run through unchanged for $K$-invariant solutions. This is because $M_{s,t}: C^{2,\gamma} \cap \text{PSH}(M,\omega) \to C^{0,\gamma}$ defined by

$$M_{s,t}(\varphi) := \log \frac{\omega^n}{\omega'^n} - tf_\omega + s\varphi - c_t, \quad (s,t) \in A,$$

actually maps $C^{2,\gamma}_K \cap \text{PSH}(M,\omega)$ to $C^{0,\gamma}_K$, and therefore

$$DM_{s,t}|_{\varphi(s,t)} = \Delta_\varphi(s,t) + s, \quad (s,t) \in A,$$

maps $C^{2,\gamma}_K$ to $C^{0,\gamma}_K$. In conclusion then, $B$ is a nonempty open subset of $A$. Moreover, if

$$A_S := (-S,-1/S] \times [0,1],$$

we have that $B \cap A_S$ is a nonempty open subset of $A_S$ for any value $S > 1$.

First, we show that $A_S \subset B$. Indeed, let $(s,t) \in \partial(B \cap A_S)$, and let $\{(s_j,t_j)\}_j \subset B \cap A_S$ be a subsequence converging to $(s,t)$. According to Lemma 5.8,

$$\sup_j \max_M |\varphi(s_j,t_j)| < C(1 + S).$$

Then, according to Lemma 5.14,

$$\sup_j \max_M |\Delta_\omega \varphi(s_j,t_j)| < C = C(M,\omega,S).$$

Thus, according to Lemma 5.15

$$\sup_j \max_M ||\varphi(s_j,t_j)||_{C^{2,\gamma}} < C = C(M,\omega,S).$$

Therefore, for every $\alpha \in (0,\gamma)$, the functions $\varphi(s_j,t_j)$ converge to $\varphi(s,t)$ in the $C^{2,\alpha}$ topology, and moreover $\varphi(s,t) \in C^{2,\gamma}$. Thus, $(s,t) \in B$. This completes the proof that $A_S \subset B$, So we have shown that

$$\cup_{S>1} A_S = (-\infty,0] \times [0,1] \subset B.$$
Observe that this actually concludes the proof of Corollary 5.3 whenever $\mu < 0$ thanks to the elliptic regularity results mentioned below.

Second, we show that actually $A_\infty \subset B$. Indeed, $(0,0) \in B$, and by openness also $\{(0,t) : 0 < t < \epsilon\} \subset B$, for some $\epsilon > 0$. Applying now Lemma 5.9 instead of Lemma 5.8 we get just as in the previous paragraph

$$
\sup_{t} \max_{M} ||\varphi(0,t)||_{C^{2,\gamma}} < C = C(M,\omega).
$$

Thus, as before it follows that $A_\infty \subset B$. Observe that the solutions we constructed are continuous in the parameters $s,t \in A_\infty$, in particular even up to $s = 0$, since we use the openness argument that relies on the implicit function theorem that necessarily produces solutions that depend continuously on the parameters. This actually concludes the proof of Corollary 5.3 (again, thanks to the elliptic regularity results).

Third, we treat the remaining piece in $A$. First, by openness $\{(s,1) : 0 \leq s < \epsilon\} \subset B$, for some $\epsilon = \epsilon(M,\omega) > 0$. Therefore, by Lemma 5.11 together with the higher-order estimates (as in the preceding paragraphs)

$$
\sup_{s} \max_{M} ||\varphi(s,1)||_{C^{2,\gamma}} < C = C(M,\omega).
$$

Once again, this is enough to conclude that $[0,\mu] \times \{1\} \subset B$. Thus,

$$
B = A,
$$

as desired.

Finally, by Corollary 5.19 (standard elliptic regularity results), the $C^{2,\gamma}$ solutions we constructed are actually smooth. Thus, $\varphi(\mu,1) \in \mathcal{H}^K$, and $\omega_{\varphi(\mu,1)}$ is $K$-invariant Kähler–Einstein metric. This concludes the proof of Theorem 5.1.

6 A counterexample to Tian’s first conjecture and a revised conjecture

Theorem 5.1 shows that properness implies existence. This is one direction of Tian’s first conjecture (Conjecture 1.2). The special case when there are no automorphisms (by which we mean ones homotopic to the identity, i.e., $\text{Aut}(M,J)_0 = \{\text{id}\}$) of the other, harder, direction of Conjecture 1.2 was established by Tian [90] under a technical assumption that was removed by Tian–Zhu [94]. This gave considerable plausibility to the conjecture. We now explain another reason why the general case of the conjecture seems plausible.

6.1 Why Tian’s conjecture is plausible

First we explain why it is natural (in fact, necessary!) for this harder converse direction to only try to establish properness on $\mathcal{H}^K$ and not on all of $\mathcal{H}$. For this, observe first that $E$ is invariant under the action of $\text{Aut}(M,J)_0$ whenever a Kähler–Einstein metric exists:

**Claim 6.1.** Suppose $(M,J,\omega)$ is Fano Kähler–Einstein with $\mu[\omega] = 2\pi c_1(M)$ and $\mu > 0$. Then $E(g^*\omega_{\varphi}) = E(\omega_{\varphi})$ for all $g \in \text{Aut}(M,J)_0$ and $\varphi \in \mathcal{H}$. 



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Proof. By \((36)\),

\[
\left. \frac{d}{dt} \right|_{t=0} E((\exp_t t X)^* \omega) = -V^{-1} \int_M \psi^{X}_{\omega} \Delta_{\omega} f_{\omega} \omega^n
\]

\[
= -V^{-1} \int_M \psi^{X}_{\omega} (s_{\omega} - n\mu) f_{\omega} \omega^n.
\]

(67)

By a theorem of Futaki the functional

\[
\eta \mapsto \int_M \psi^{X}_{\omega} (s_{\omega} - n\mu) f_{\omega} \eta^n
\]

is constant on \(H\) [54, 28, 25]. Since it is zero at \(\omega\) (\(\text{Ric} \omega = \mu \omega\) implies \(s_{\omega} = n\mu\)), it is identically zero. Thus,

\[
\left. \frac{d}{dt} \right|_{t=0} E((\exp_t t X)^* \omega) = 0
\]

(68)

Now, actually

\[
\left. \frac{d}{dt} \right|_{t=s} E((\exp_t t X)^* \omega) = 0
\]

(69)

for every \(s\). Indeed,

\[
\left. \frac{d}{dt} \right|_{t=s} E((\exp_t t X)^* \omega) = \left. \frac{d}{dt} \right|_{t=0} E((\exp_t s t X)^* \omega)
\]

\[
= \left. \frac{d}{dt} \right|_{t=0} E((\exp_t t X)^* ((\exp_t s t X)^* \omega))
\]

\[
= 0,
\]

(70)

by replacing \(\omega\) by \((\exp t t X)^* \omega\) in (68). Since \(\text{Aut}(M, J)_0\) is a covered by its one-parameter subgroup, the statement follows.

On the other hand, the Aubin functional is not invariant under the action of automorphisms. In fact, it might blow up along a one-parameter subgroup. The following lemma is due to Bando–Mabuchi [8, Lemma 6.2].

**Lemma 6.2.** Let \(\omega \in H\) be arbitrary and suppose \(\eta \in H\) is Kähler–Einstein with \(\mu > 0\). The function \(F_\eta : \text{Aut}(M, J)_0 \to \mathbb{R}_+\),

\[
F_\eta : g \mapsto (I - J)(g^* \eta)
\]

is proper (when we identify \(\text{Aut}(M, J)_0\) with its \(\eta\)-orbit in \(H\) and endow this subset of \(H\) with the \(C^2 \gamma(M, \omega)\)-topology).

**Proof.** Indeed, suppose that \(I - J\) is bounded on a sequence \(\omega_j = \omega_{\varphi_j}\) of Kähler–Einstein metrics in \(\text{Aut}(M, J)_0, \eta \in H\). By Claim 6.1 and (15),

\[
C = E(\varphi_1) = E(\varphi_j) = \text{Ent}(e^{f_{\omega} n} \omega^n, \omega^n_{\varphi_j}) - \mu(I - J)(\varphi_j).
\]

Now, normalize \(\varphi_j\) so that (recall (35))

\[
\omega^n_{\varphi_j} = \omega^nf_{\omega} - \mu \varphi_j
\]
(this fixes $\varphi_j$ since $\mu > 0$ and the right hand side must integrate to $V$). Plugging back into the formula for $E(\varphi_j)$ gives

$$C = E(\varphi_j) = -\mu \int \varphi_j \omega^n_{\varphi_j} - \mu (I - J)(\varphi_j),$$

or,

$$-\mu \int \varphi_j \omega^n_{\varphi_j} = C + \mu (I - J)(\varphi_j) \leq C',$$

by assumption that $(I - J)(\varphi_j)$ is uniformly bounded. But now

\begin{align*}
V^{-1} \int \varphi_j \omega^n &= -V^{-1} \int -\varphi_j \omega^n_{\varphi_j} + I(\varphi_j) \\
&\leq -V^{-1} \int -\varphi_j \omega^n_{\varphi_j} + \frac{n + 1}{n} (I - J)(\varphi_j) \\
&\leq -(I - J)(\varphi_j) + \frac{n + 1}{n} (I - J)(\varphi_j) \\
&= \frac{1}{n + 1} (I - J)(\varphi_j) \leq \frac{C}{n + 1}.
\end{align*}

Thus, by (33),

$$\max \varphi_j \leq V^{-1} \int \varphi_j \omega^n + C' < C''.$$

Now a Moser iteration argument just as in §5.4 applies (the Sobolev and Poincaré constants of the Kähler–Einstein metrics of Ricci curvature equal to $\mu > 0$ are all uniform) to give

$$-\min \varphi_j \leq \frac{C}{V} \int -\varphi_j \omega^n_{\varphi_j} + C.$$

Combining the last two equations,

$$\text{osc} \varphi = \max \varphi - \min \varphi \leq C + \frac{C}{V} \int -\varphi_j \omega^n_{\varphi_j} + C'' \leq C''',$$

using the display prior to (71). Since $\varphi_j$ must change signs (from the normalization for $\varphi_j$ inherent in $\omega^n_{\varphi_j} = e^{\ell_\omega - \mu \varphi_j}$ and the one for $f_\omega$ in Definition 3.4), we have showed that

$$\|\varphi_j\|_{L^\infty} < C,$$

and consequently

$$\|\varphi_j\|_{C^k,\gamma} < C(k, \gamma),$$

for all $k, \alpha$, which when $k = 2$ gives

$$C^{-1} \omega \leq \omega_j \leq C \omega.$$

Thus, endowing $\text{Aut}(M, J)_0$ with, say, the $C^{2,\gamma}$-topology we see that the preimage under of $F_\eta$ of compact sets in $\mathbb{R}_+$ are compact in the $C^{2,\gamma}$-topology, i.e., by definition $F_\eta$ (the original $F_\eta$ considered as a map on the group $\text{Aut}(M, J)_0$) is proper. \qed
Corollary 6.3. Suppose \((M, J, \eta)\) is Fano Kähler–Einstein with \(\mu[\omega] = 2\pi c_1(M)\) and \(\mu > 0\) and that \(\text{Aut}(M, J)_0\) is nontrivial. Then \((I - J) : \{g^*\eta : g \in \text{Aut}(M, J)_0\} \to \mathbb{R}_+\) is unbounded from above.

Proof. Indeed, by Corollary 14.7 below \(F_\eta\) descends to a function on \(\text{isom}(M, g)\), still denoted by \(F_\eta\),
\[
F_\eta(X) = (I - J)((\exp_I JX)^*\eta).
\]
Since this function is still proper and \(\text{isom}(M, g)\) is a non-compact vector space, \(F_\eta\) must be unbounded. □

Remark 6.4. There is actually no particular need to look at the orbit of a Kähler–Einstein metric to show unboundedness; the same is true for the orbit of any metric as long as a Kähler–Einstein exists. Indeed, if \(\alpha, \omega, \eta \in H\), with \(\eta\) Kähler–Einstein,
\[
E(g^*\alpha) = E(\omega, g^*\alpha) = E(\omega, \eta) + E(g^*\eta, g^*\alpha) = E(\omega, \eta) + E(\eta, \alpha).
\]
Thus, \(E(g^*\alpha)\) is unbounded if and only if \(E(\eta, \alpha)\) is (as \(E(\eta, \alpha)\) is some fixed constant).

6.2 A counterexample

However, surprisingly, Tian’s first conjecture (which was stated as a theorem in [90, Theorem 4.4]) was recently disproved by Darvas and the author by establishing the following optimal version of Tian’s conjecture.

Theorem 6.5. Suppose \((M, J, \omega)\) is Fano with \(\mu[\omega] = 2\pi c_1(M)\) and \(\mu > 0\), and that \(K\) is a maximal compact subgroup of \(\text{Aut}(M, J)_0\) with \(\omega \in H^K\). The following are equivalent:

(i) There exists a Kähler–Einstein metric in \(H^K\) and \(\text{Aut}(M, J)_0\) has finite center.
(ii) There exists \(C, D > 0\) such that \(E(\eta) \geq CJ(\eta) - D, \eta \in H^K\).

Thus, restricting to the \(K\)-invariant potentials is necessary, but not sufficient, to guarantee properness.

Remark 6.6. The estimate in (ii) gives a concrete version of the properness condition [2]. The direction (i) \(\Rightarrow\) (ii) is due to Phong et al. [71, Theorem 2], building on earlier work of Tian [90] and Tian–Zhu [91] in the case \(\text{Aut}(M, J)_0 = \{\text{id}\}\), who obtained a weaker inequality in (ii) with \(J\) replaced by \(J^\delta\) for some \(\delta \in (0, 1)\) (for more details see also the survey [92, p. 131]).

Example 6.7. [44, Example 2.2] Let \(M\) denote the blow-up of \(\mathbb{P}^2\) at three non colinear points. It is well-known that it admits Kähler–Einstein metrics (see, e.g., [77]). In fact, one way to see this is by showing that Tian’s invariant is equal to 1 for an appropriately chosen group of symmetries [9] and then apply Corollary 5.2 (with \(\mu = 1\)). According to [49, Theorem 8.4.2],
\[
\text{Aut}(M, J)_0 = (\mathbb{C}^*)^2.
\]
We will explain this fact in a moment. Given this, we see that \(\text{Aut}(M, J)_0\) is equal to its center which is clearly not finite. Thus, Conjecture 1.2 fails for \(M\) by Theorem 6.5. Following the appearance of [44], X.-H. Zhu informed the author that using toric methods one can give an alternative proof that Conjecture 1.2 fails in the special case of toric Fano \(n\)-manifolds that satisfy \(\text{Aut}(M, J)_0 = (\mathbb{C}^*)^n\).
To see (72), observe that automorphisms homotopic to the identity map preserve the coho-
mology class of divisors. Thus, they preserve each of the three exceptional divisors. In particular,
they descend to automorphisms of $\mathbb{P}^2$ which preserve the three blowup points. By that we mean
that if $f \in \text{Aut}(M, J)_0$ then $\pi \circ f \circ \pi^{-1} \in \text{Aut}(\mathbb{P}^2)$. Now automorphisms of $\mathbb{P}^2$ are represented
by invertible three-by-three matrices, up to a nonzero complex number. We may assume in this
representation that the three points are then $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ (since they are not collinear!). Thus, each such automorphism is represented by a diagonal matrix. Since the
matrix is invertible, and determined up to a nonzero complex number, that matrix can be taken
to be
\[
\begin{pmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \quad a, b \in \mathbb{C}^\times.
\]
Conversely, the blow-up of $\mathbb{P}^2$ at three non collinear points is a toric manifold so its automorphism
group contains a copy of $(\mathbb{C}^\times)^2$. Thus, (72) is established.

These results motivate a reformulation of Tian’s original conjecture. To present this refor-
mulation we first make an excursion to infinite dimensional metric geometry in the next sections.
In Section 12 we return to state the reformulated conjecture, whose proof is described in Section
15.

7 Infinite dimensional metrics on $\mathcal{H}$

Approaching problems in Kähler geometry through an infinite-dimensional perspective goes back
to Calabi in 1953 [27] and later Mabuchi in 1986 [66]. These works proposed two different weak
Riemannian metrics of $L^2$ type which have been studied extensively since.

The most widely studied such metric is the Mabuchi metric [66],
\[
g_M(\nu, \eta)_{|\varphi} := \int_M \nu \eta \omega^n, \quad \nu, \eta \in T_{\varphi} \mathcal{H}_\omega \cong C^\infty(M),
\]
discovered independently also by Semmes [83] and Donaldson [50] (see, e.g., [77, Chapter 2] for
an exposition and further references).

Calabi’s metric is given by
\[
g_C(\nu, \eta)_{|\varphi} := \int_M \Delta_\varphi \nu \Delta_\varphi \eta \frac{\omega^n}{n!}.
\]
This metric was introduced by Calabi in the 1950s in talks and in a research announcement [27].
It might seem a little less natural at first since it involves more derivatives than the Mabuchi
metric. However, from a Riemannian geometric point of view it is actually more natural, since
it is simply the $L^2$ metric on the level of Riemannian metrics, as the following simple result
shows. To state this result we let

$\mathcal{M}$

denote the infinite-dimensional space of all smooth Riemannian metrics on $M$. The Ebin metric,
also called the $L^2$ metric [51] is defined by
\[
g_E(h, k)_{|g} := \int_M \text{tr}(g^{-1}hg^{-1}k)dV_g,
\]
where \( g \in \mathcal{M} \), \( h, k \in T_g \mathcal{M} \) and \( T_g \mathcal{M} \cong \Gamma(\text{Sym}^2 T^*M) \), the space of smooth, symmetric \((0,2)\)-tensor fields on \( M \).

**Proposition 7.1.** [37, Proposition 2.1] Consider the inclusion \( \iota: \mathcal{H} \hookrightarrow \mathcal{M} \). Then, \( \iota^* g_E = 2g_C \).

In other words, \((\mathcal{H}, 2g_C)\) is isometrically embedded in \((\mathcal{M}, g_E)\), or what is the same, the metric \( g_C \) is induced by the metric \( g_E \).

On the other hand, the Mabuchi metric is more natural from a symplectic or complex geometry point of view. As shown by Semmes and Donaldson, the Mabuchi metric can be considered as an infinite-dimensional analogue of the symmetric space metric structure on spaces of the form \( G^C/G \) where \( G \) is a compact Lie group, but where the group is now infinite-dimensional, more specifically the group of Hamiltonian diffeomorphisms of \((M, \omega)\). We refer the reader to [83, 50, 91, Chapter 4], [87]. In another vein, the Mabuchi metric is also natural from the point of view of semi-classical complex geometry, also referred to as Kähler quantization sometimes. We refer the reader to [77, 72, 52, 79].

### 8 Metric completions of \( \mathcal{H} \)

Historically, Calabi claimed that the completion of his metric “consists of the positive semi-definite Kähler metrics defining the same principal class,” i.e., of

\[
\{\omega_{\varphi} := \omega + \sqrt{-1} \partial \bar{\partial} \varphi : \varphi \in C^\infty(M), \omega_{\varphi} \geq 0\}.
\]

Except from this single line published in his short talk abstract in 1953 [27], there has been no study or even conjectures in the literature concerning metric completions of \( \mathcal{H} \). The first article in this direction is due to Clarke–Rubinstein in 2011 [37], that we now turn to discuss.

#### 8.1 The Calabi metric completion

Denote by \( d_C: \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+ \) the distance function associated to metric \( g_C \). It is defined as follows. A curve \( [0,1] \ni t \mapsto \alpha_t \in \mathcal{H} \) is called smooth if \( \alpha(t,z) \) is smooth in both \( t \) and \( z \). Denote \( \dot{\alpha}_t := \partial \alpha(t)/\partial t \). The length of a smooth curve \( t \to \alpha_t \) is

\[
\ell_C(\alpha) := \int_0^1 \sqrt{g_C(\dot{\alpha}_t, \dot{\alpha}_t)}|_{\alpha_t} dt.
\]  

(76)

**Definition 8.1.** The path length distance of \((\mathcal{H}, g_C)\) is defined by

\[
d_C(\omega, \eta) := \inf\{\ell_C(\alpha) : \alpha : [0,1] \to \mathcal{H} \text{ is a smooth curve with } \alpha(0) = \omega, \alpha(1) = \eta\}.
\]

We refer to the pseudometric \( d_C \) as the Calabi metric.

**Remark 8.2.** As observed already by Calabi, the Calabi–Yau Theorem implies that \((\mathcal{H}, g_C)\) is isometric to a portion of a sphere in \( L^2(M, \omega^n) \), and therefore the Calabi (pseudo)-metric is actually a metric, justifying the above name (see, e.g., [37] pp. 1488–1489] or [29]). Even though we refer to \( d_C \) and to \( g_C \) by the same name, we hope it will be clear below to which one we are referring to from the context.

The Calabi metric completion is given by the following theorem due to Clarke–Rubinstein [37, Theorem 5.6].
Theorem 8.3. The metric completion of \((\mathcal{H}, d_C)\) is given by
\[
\overline{(\mathcal{H}, d_C)} \equiv \{ \varphi \in \mathcal{E}(M, \omega) : \omega^n_\varphi \text{ is absolutely continuous with respect to } \omega^n \text{ and } \omega^n_\varphi/\omega^n \in L^1(M, \omega^n) \},
\]
and is a strict subset of
\[
\mathcal{E}(M, \omega) := \left\{ \varphi \in \text{PSH}(M, \omega) : \lim_{j \to \infty} \int_{\{\varphi \leq -j\}} \left( \omega + \sqrt{-1} \partial \bar{\partial} \max\{\varphi, -j\} \right)^n = 0 \right\}.
\]
Furthermore, convergence with respect to \(d_C\) is characterized as follows. A sequence \(\{\omega^n_\varphi_k\} \subset \mathcal{H}\) converges to \(\omega_\varphi \in \mathcal{H}\) with respect to \(d_C\) if and only if \(\omega^n_\varphi_k \to \omega^n_\varphi\) in the \(L^1\) sense, i.e.,
\[
\int_M \left| \frac{\omega^n_\varphi_k}{\omega^n} - \frac{\omega^n_\varphi}{\omega^n} \right| \omega^n \to 0.
\]

Remark 8.4. Observe that the metric completion turns out to be considerably larger than what Calabi claimed. We also note that Theorem 8.3 was motivated by the computation of the metric completion of the ambient space \((M, g_E)\) obtained in Clarke’s thesis [36]. It is interesting to note that his result does not directly imply Theorem 8.3 as one might suspect from Proposition 7.1.

Remark 8.5. The space \(\mathcal{E}(M, \omega)\) was introduced by Guedj–Zeriahi [58, Definition 1.1]. The statement of Theorem 8.3 of course assumes that the measure \(\omega^n_\varphi\) can be defined for each \(\varphi \in \mathcal{E}(M, \omega)\). This is indeed the case, but requires considerable background from pluripotential theory. One defines
\[
\omega^n_\varphi := \lim_{j \to -\infty} 1_{\{\varphi > j\}}(\omega + \sqrt{-1} \partial \bar{\partial} \max\{\varphi, j\})^n.
\]

By definition, \(1_{\{\varphi > j\}}(x)\) is equal to 1 if \(\varphi(x) > j\) and zero otherwise, and the measure \((\omega + \sqrt{-1} \partial \bar{\partial} \max\{\varphi, j\})^n\) is defined by the work of Bedford–Taylor [10] since \(\max\{\varphi, j\}\) is bounded. The limit is then well-defined as a Borel measure; for more details we refer to [58, p. 445].

What is perhaps more interesting than computing the metric completion itself, is the fact that this computation yields nontrivial geometric information [37, Theorem 6.3].

Definition 8.6. We say that \((M, J)\) is Calabi–Ricci unstable (or CR-unstable) if there exists a Ricci flow trajectory that diverges in \((\mathcal{H}, d_C)\). Otherwise, we say \((M, J)\) is CR-stable.

Theorem 8.7. A Fano manifold \((M, J)\) is CR-stable if and only if it admits a Kähler–Einstein metric. Moreover, if it is CR-unstable then any Ricci flow trajectory diverges in \((\mathcal{H}, d_C)\).

Theorem 8.7 might seem rather abstract, however it shows that convergence in the metric completion is fundamental geometrically. In addition, it can be stated entirely in terms of an a priori estimates without any reference to the metric completion [37, Corollary 6.9]:

Corollary 8.8. The Ricci flow [78] converges smoothly if and only if
\[
||s - n||_{L^1(R_+, L^2(M, \omega(t)))} < \infty,
\]
where \(s = s(t)\) denotes the scalar curvature of \((M, \omega(t))\).
This improves a result of Phong et al. [71], where (87) is replaced by

\[ |s - n|_{L^1(\mathbb{R}^+,C^0(M))} < \infty, \]

which was proved by completely different methods. The novelty in Corollary 8.8 is that it uses supposedly “soft” infinite-dimensional geometry to prove actual “hard” a priori estimates for a PDE. Of course, the catch is that some analysis does go into computing the metric completion and, aside from that, some PDE techniques are still needed in the proof of Corollary 8.8. But, nevertheless, the idea that some PDE estimates can be explained using infinite-dimensional geometry seems attractive.

**Exercise 8.9.** Show that the length of the curve \( t \mapsto \omega(t) \) with respect to the Calabi metric is equal to

\[ |s - n|_{L^1(\mathbb{R}^+,L^2(M,\omega(t)))} \]

if \( \omega(t) \) satisfies the Ricci flow equation

\[ \frac{\partial \omega(t)}{\partial t} = -\text{Ric}(\omega(t)) + \mu \omega(t), \quad \omega(0) = \omega \in \mathcal{H}. \]  

(78)

Also, show that any solution of (78) that starts in \( \mathcal{H} \) remains in \( \mathcal{H} \) [61]. Thus, it makes sense to write \( \omega(t) = \omega(t) \).

Thus, Corollary 8.8 shows that convergence of the flow is equivalent to having finite distance in the Calabi metric.

**Exercise 8.10.** Rewrite (78) in the form of a complex Monge–Ampère equation

\[ \omega^n = \omega^n e^{f_{-\mu \varphi} + \varphi}, \quad \varphi(0) = \text{const}. \]  

(79)

We remark that, depending on the context, the choice of the constant \( \varphi(0) \) might involve some care (see [35, §10.1], [69, §2]).

**Exercise 8.11.** Assuming the theory of short-time existence for (78) (which replaces the openness arguments for the continuity method) show that for every \( \omega \in \mathcal{H} \) the equation (79) admits a solution for all \( t > 0 \) whenever \( \mu < 0 \). To do this, use Exercise 8.10 as well as the results of [41]. Moreover, show that as \( t \) tends to infinity, the solutions \( \omega(t) \) converge to the Kähler–Einstein metric.

Recently, Darvas generalized Calabi’s metric to a two-parameter family of Finsler metrics, given by

\[ ||\eta||_{C,p,q} := \left( \int_M |\Delta_{\omega} \eta|^p \left( \frac{\omega^n}{\omega^n} \right)^q \frac{\omega^n}{n!} \right)^{1/q}, \]  

(80)

and computed the corresponding metric completions, directly generalizing Theorem 8.3. Denote by \( d_{C,p,q} : \mathcal{H} \times \mathcal{H} \to \mathbb{R^+} \) the path-length distance function associated to (80).

**Theorem 8.12.** [41] Theorem 1.1 Let \( p, q \in (1, \infty) \) and \( q \leq p \). The metric completion of \( (\mathcal{H}, d_{C,p,q}) \) is given by

\[ (\mathcal{H}, d_{C,p,q}) \cong \{ \varphi \in \mathcal{E}(M, \omega) : \omega^n_{\varphi} \text{ is absolutely continuous with respect to } \omega^n \text{ and } \omega^n_{\varphi} / \omega^n \in L^q(M, \omega^n) \}. \]
Furthermore, convergence with respect to $d_{C,p,q}$ is characterized as follows. A sequence $\{\omega_{\varphi_k}\} \subset \mathcal{H}$ converges to $\omega_{\varphi} \in \mathcal{H}$ with respect to $d_{C,p,q}$ if and only if $\omega_{\varphi_k}^n \to \omega_{\varphi}^n$ in the $L^q$ sense, i.e.,

$$\int_M |\omega_{\varphi_k}^n - \omega_{\varphi}^n|^q \omega^n \to 0.$$ 

In particular, the metric completion is independent of $p$. This immediately yields, by the same results of [37] that lead to Corollary 8.8, the following improvement to Corollary 8.8 [40, Theorem 1.1].

**Corollary 8.13.** The Ricci flow (78) converges smoothly if and only if

$$||s - n||_{L^1([\mathbb{R}^+, L^1(M, \omega(t))]} < \infty.$$  

**Exercise 8.14.** Show that the length of the curve $t \mapsto \omega_{\varphi(t)}$ with respect to $d_{C,1,1}$ is equal to

$$||s - n||_{L^1([\mathbb{R}^+, L^1(M, \omega(t))]}$$

if $\omega_{\varphi(t)}$ satisfies the Ricci flow equation (78).

It would be interesting to obtain a proof of Corollary 8.13 using direct flow methods. At the same time, it is remarkable that such metric completion techniques can lead to new estimates on geometric flows. We believe that this circle of ideas should find more applications in other geometric and analytic settings.

### 8.2 The Mabuchi metric completion

As remarked earlier, the Calabi metric is more closely tied with the Riemannian geometry of $M$, and indeed convergence in the Calabi metric is related to convergence of the associated Riemannian volume forms. The Mabuchi metric, on the other hand, is more closely tied with the complex geometry of $M$, and so completely different methods would be needed to compute the Mabuchi metric completion. Using sophisticated techniques from pluripotential theory this was carried through by Darvas. A special case was also obtained around the same time by Guedj [57]. Define,

$$\mathcal{H}_\omega = \{ \varphi : \varphi \in C^\infty(M), \omega_{\varphi} > 0 \},$$

and

$$\mathcal{E}_2 := \{ \varphi \in \mathcal{E}(M, \omega) : \int \varphi^2 \omega_{\varphi}^n < \infty \}.$$ 

A curve $[0, 1] \ni t \mapsto \varphi(t) \in \mathcal{H}_\omega$ is called smooth if $\varphi(t, z) = \varphi(t)(z) \in C^\infty([0, 1] \times M)$. Denote $\dot{\varphi}(t) := \partial \varphi(t)/\partial t$. The length of a smooth curve $t \mapsto \varphi(t)$ is

$$\ell_M(\alpha) := \int_0^1 \sqrt{g_M(\dot{\varphi}(t), \dot{\varphi}(t))}|\varphi(t)| dt.$$  

**Definition 8.15.** The path length distance of $(\mathcal{H}_\omega, d_m)$ is defined by

$$d_m(\varphi_0, \varphi_1) := \inf \{ \ell_M(\varphi) : \varphi : [0, 1] \to \mathcal{H}_\omega \text{ is a smooth curve with } \varphi(0) = \varphi_0, \varphi(1) = \varphi_1 \}.$$ 

We call the pseudometric $d_m$ the *Mabuchi metric*.

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The metric completion of the Mabuchi metric is given by the following theorem of Darvas [38, Theorem 1] which also justifies the name given to $d_M$ above.

**Theorem 8.16.** $(\mathcal{H}_\omega, d_M)$ is a metric space. Moreover, the metric completion of $(\mathcal{H}_\omega, d_M)$ equals $(\mathcal{E}_2, d_{M,2})$, where

$$d_{M,2}(\varphi_0, \varphi_1) := \lim_{k \to \infty} d_M(\varphi_0(k), \varphi_1(k)), \quad (84)$$

for any smooth decreasing sequences $\{\varphi_i(k)\}_{k \in \mathbb{N}} \subset \mathcal{H}$ converging pointwise to $\varphi_i \in \mathcal{E}_2, i = 0, 1$.

Of course, the statement should be understood as also including the claims that: (i) $(84)$ is well-defined independently of the choices of the approximating sequences, (ii) convergence in the metric completion is characterized as follows: $\{\varphi_j\} \subset \mathcal{E}_2$ converges to $\varphi \in \mathcal{E}_2$ if $\lim_j d_{M,2}(\varphi_j, \varphi) = 0$.

**Remark 8.17.** The space $\mathcal{H}$ is the space of Kähler forms, while the space $\mathcal{H}_\omega$ is the space of Kähler potentials. In many instances one can go back and forth between the two carelessly, however in some situations some care is needed. One may also identify the latter as a subspace of the former in several ways, but again some care is needed in doing so. For example,

$$\mathcal{H}_\omega \cap \{\text{AM} = 0\} \quad (85)$$

is a $d_M$-totally geodesic submanifold (hypersurface) of $\mathcal{H}_\omega$ [68, Proposition 2.6.1], [50, §3]. The submanifold $(85)$ can be naturally identified with $\mathcal{H}$. Sometimes, though, we will use identifications different from $(85)$.

In the vein of Remark 8.17, we distinguish between solutions of $(78)$, which we continue to refer to as solutions to the Ricci flow, and solutions of $(79)$, which we refer to as solutions to the Kähler–Ricci flow.

**Exercise 8.18.** Does the map $\omega(t) \mapsto \varphi(t)$ that sends solutions of $(78)$ to solutions of $(79)$, come from the identification of $\mathcal{H}$ with $(85)$?

Theorem 8.16 has already found several geometric applications. The first is the following analogue of Theorem 8.7 for the Mabuchi metric, due to Darvas [38, Theorem 6.1].

**Definition 8.19.** We say that $(M, J)$ is Mabuchi–Ricci unstable (or MR-unstable) if there exists a Kähler–Ricci flow trajectory that diverges in $(\mathcal{H}, d_M)$. Otherwise, we say $(M, J)$ is MR-stable.

**Theorem 8.20.** A Fano manifold $(M, J)$ is MR-stable if and only if it admits a Kähler–Einstein metric. Moreover, if it is MR-unstable then any Ricci flow trajectory diverges in $(\mathcal{H}, d_C)$.

**Exercise 8.21.** Show that the length of the curve $t \mapsto \varphi(t)$ with respect to $d_M$ is equal to

$$\|\int \omega_{\varphi(t)} \|_{L^1(\mathbb{R}_+, L^2(M, \omega_{\varphi(t)}))} \quad (86)$$

if $\varphi(t)$ satisfies $(79)$ (which by Exercise 8.10 implies that $\omega_{\varphi(t)}$ satisfies the Ricci flow equation $(78)$). As observed by Darvas, Theorem 8.20 together with the arguments of [37] imply the following analogue of Corollary 8.13 first obtained by McFeron [68]: the flow $(79)$ converges if and only if $(86)$ is finite.

In fact, the following improvement of the last statement in Exercise 8.21 is due to Darvas. It follows from [38, Theorem 6.1] together with later work of Darvas surveyed in [60].

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Theorem 8.22. The Kähler–Ricci flow (79) converges smoothly if and only if
\[ \|f\|_{L^1(\mathbb{R}_+, L^1(M, \omega_{\varphi(t)}))} < \infty, \]  
(87)
where \( f = f_{\varphi(t)} \) is the Ricci potential along the flow (recall Definition 3.4).

Other applications for Theorem 8.16 include the work of Streets [85], and more recently Berman–Darvas–Lu [16], who show that one gains new insight on the long time behavior of the Calabi flow by placing it in the context of the Mabuchi metric completion; the work of Darvas–He [42], where the asymptotic behavior of the Kähler-Ricci flow in the metric completion is related to destabilizing geodesic rays. We refer the reader to the survey [78] for more references.

9 The Darvas metric and its completion

Perhaps surprisingly, a key observation of Darvas is that not a Riemannian, but rather a Finsler metric, encodes the asymptotic behavior of the Aubin functional \( J \). This is discussed in the Section 10. In this section we introduce the Darvas metric and survey some of its basic properties. In later sections, through considerable more technical work, we survey later work of Darvas–Rubinstein that shows that the Darvas metric also encodes the asymptotic behavior for essentially all energy functionals on \( \mathcal{H} \) whose critical points are precisely various types of canonical metrics in Kähler geometry. In fact, as pointed out in [44, Remark 7.3], the same kind of statement is in general false for the much-studied Riemannian metrics of Calabi and Mabuchi. Thus, the Darvas metric turns out to be fundamental.

The Darvas metric is a weak Finsler metric on \( \mathcal{H}_\omega \) given by [39],
\[ \|\nu\|^D_\varphi := V^{-1} \int_M |\nu|^\omega^n_\varphi, \quad \nu \in T_\varphi \mathcal{H}_\omega = C^\infty(M). \]  
(88)
As in §8.2, define the length of a smooth curve \( t \mapsto \varphi(t) \),
\[ \ell_D(\alpha) := \int_0^1 \int_M |\dot{\varphi}(t)|^\omega^n_{\varphi(t)} \wedge dt. \]  
(89)

Definition 9.1. The path length distance of \( (\mathcal{H}_\omega, d_D) \) is defined by
\[ d_D(\varphi_0, \varphi_1) := \inf \{ \ell_1(\alpha) : \alpha : [0, 1] \to \mathcal{H}_\omega \text{ is a smooth curve with } \alpha(0) = \varphi_0, \alpha(1) = \varphi_1 \}. \]

We call the pseudometric \( d_D \) the Darvas metric.

The following result of Darvas justifies this name. To state the result, consider \([0, 1] \times \mathbb{R} \times M\) as a complex manifold of dimension \( n + 1 \), and denote by \( \pi_2 : [0, 1] \times \mathbb{R} \times M \to M \) the natural projection.

Theorem 9.2. [39, Theorem 3.5] \( (\mathcal{H}_\omega, d_D) \) is a metric space. Moreover,
\[ d_D(\varphi_0, \varphi_1) = \|\varphi_0\|_{\varphi_0} \geq 0, \]  
(90)
with equality iff \( \varphi_0 = \varphi_1 \), where \( \varphi_0 \) is the image of \((\varphi_0, \varphi_1) \in \mathcal{H}_\omega \times \mathcal{H}_\omega \) under the Dirichlet-to-Neumann map for the Monge–Ampère equation,
\[ \varphi \in \text{PSH}(\pi_2^* \omega, [0, 1] \times \mathbb{R} \times M), \quad (\pi_2^* \omega + \sqrt{-1} \partial \bar{\partial} \varphi)^{n+1} = 0, \quad \varphi|_{\{i\} \times \mathbb{R}} = \varphi_i, \quad i = 0, 1. \]  
(91)
Remark 9.3. (i) The Dirichlet-to-Neumann operator simply maps \((\varphi_0, \varphi_1)\) to the initial tangent vector of the curve
\[ t \mapsto \varphi(t) \equiv \varphi_t \]
that solves (91).

(ii) One needs to make sense of the expression \(\dot{\varphi}_0\) in (90) since there is no guarantee that \(\varphi_t\) will be smooth in \(t\). Since \(\varphi\) (considered as a function on \([0, 1] \times \mathbb{R} \times M\)) is \(\pi^*_2\omega\)-psh and independent of the imaginary part of the first variable, it is convex in \(t\). Thus,
\[
\dot{\varphi}_0(x) := \lim_{t \to 0^+} \frac{\varphi(t, x) - \varphi_0(x)}{t}, \tag{92}
\]
with the limit well-defined since the difference quotient is decreasing in \(t\).

The metric completion of the Darvas metric is given by the next result [39, Theorem 2]. The proof is similar in spirit to that of Theorem 8.16, but involves considerable additional technicalities stemming, at least intuitively, from the fact that \(x \mapsto x^2\) is a smooth function while \(x \mapsto |x|\) is only Lipschitz; partly due to this dealing with an \(L^1\) type metric is fundamentally harder in this setting.

**Theorem 9.4.** The metric completion of \((\mathcal{H}_\omega, d_D)\) equals \((E_1, d_D)\), where
\[
d_D(\varphi_0, \varphi_1) := \lim_{k \to \infty} d_D(\varphi_0(k), \varphi_1(k)),
\]
for any smooth decreasing sequences \(\{\varphi_i(k)\}_{k \in \mathbb{N}} \subset \mathcal{H}_\omega\) converging pointwise to \(\varphi_i \in E_1, i = 0, 1\). Moreover, for each \(t \in (0, 1)\), define
\[
\varphi_t := \lim_{k \to \infty} \varphi_t(k), \; t \in (0, 1), \tag{93}
\]
where \(\varphi_t(k)\) is the solution of (91) with endpoints \(\varphi_i(k), i = 0, 1\). Then \(\varphi_t \in E_1\), and the curve \(t \mapsto \varphi_t\) is well-defined independently of the choices of approximating sequences and is a \(d_D\)-geodesic.

10 The Aubin functional and the Darvas distance function

Finally we come to the fact stated at the beginning of the previous section relating the Darvas metric to the Aubin functional.

The subspace
\[
\mathcal{H}_0 := AM^{-1}(0) \cap \mathcal{H}_\omega \tag{94}
\]
is isomorphic to \(\mathcal{H}\) (11), the space of Kähler metrics (recall Remark 8.17). We use this isomorphism to endow \(\mathcal{H}\) with a metric structure, by pulling back the Darvas metric defined on \(\mathcal{H}_\omega\).

**Proposition 10.1.** [39, Remark 6.3] There exists \(C > 1\) such that for all \(\varphi \in \mathcal{H}_0\) (recall 92)),
\[
\frac{1}{C} J(\varphi) - C \leq d_D(0, \varphi) \leq CJ(\varphi) + C.
\]
We refer the reader to [44, Proposition 5.5] for a proof.

Given the equivalence of $J$ and $d_D$ on $H_0$ it is natural to expect that this should extend to the metric completion. This is indeed the case. This amounts to two things: (i) one can extend Aubin’s functional $J$ to the metric completion in a continuous way with respect to the $d_D$-topology, (ii) $H_0$, considered as a submanifold of $H$ endowed with the metric induced by $d_D$, is a totally geodesic metric space whose completion coincides with $E_1 \cap AM^{-1}(0)$, which in turn requires verifying that the Aubin–Mabuchi functional $AM$ can be extended to $E_1$ in a continuous way with respect to the $d_D$-topology. These facts are contained in the following Lemma [44, Lemma 5.2].

**Lemma 10.2.** (i) $AM, J : H_\omega \to \mathbb{R}$ each admit a unique $d_D$-continuous extension to $E_1$ and these extensions still satisfy (9) and (7) (in the sense of pluripotential theory).

(ii) The subspace $(E_1 \cap AM^{-1}(0), d_D)$ is a complete geodesic metric space, coinciding with the metric completion of $(H_0, d_D)$ (recall (94)).

Consequently, from now on we denote by $AM, J$ the unique $d_D$-continuous extensions to $E_1$ given by the previous Lemma.

**Corollary 10.3.** There exists $C > 1$ such that for all $\varphi \in E_1 \cap AM^{-1}(0)$,

$$\frac{1}{C} J(\varphi) - C \leq d_D(0, \varphi) \leq C J(\varphi) + C.$$  

Next, we discuss a concrete formula for the $d_D$ metric relating it to the Aubin–Mabuchi energy and also give a concrete growth estimate for $d_D$. First we need to introduce the following rooftop type envelope for $u, v \in E_1$:

$$P(u, v)(z) := \sup \{ w(z) : w \in PSH(M, \omega), w \leq \min\{u, v\} \}.$$  

Note that $P(u, v) \in E_1$ [38, Theorem 2]. Darvas shows the following beautiful “Pythagorean” formula for $d_D$, as well as a very useful growth estimate [39, Corollary 4.14, Theorem 3].

**Proposition 10.4.** Let $u, v \in E_1$. Then,

$$d_D(u, v) = AM(u) + AM(v) - 2AM(P(u, v)).$$  

(95)

Also, there exists $C > 1$ such that for all $u, v \in E_1$,

$$C^{-1} d_D(u, v) \leq \int_M |u - v| \omega^n + \int_M |u - v| \omega^v \leq Cd_D(u, v).$$  

(96)

### 11 Quotienting the metric completion by a group action

We now incorporate automorphisms into the picture. Since automorphisms induce isometries of the various infinite-dimensional metrics we have studied so far it is natural to consider the associated quotient spaces from the metric geometry point of view. In addition, the various functionals we have studied also admit natural descents to the quotient spaces.
11.1 The action of the automorphism group on $\mathcal{H}$

Let $\text{Aut}_0(M, J)$ denote the connected component of the complex Lie group of automorphisms (biholomorphisms, i.e., homeomorphisms that are holomorphic and admit a holomorphic inverse) of $(M, J)$. Denote by $\text{aut}(M, J)$ the Lie algebra of $\text{Aut}_0(M, J)$, consisting of infinitesimal automorphisms, i.e., real vector fields $X$ satisfying $\mathcal{L}_X J = 0$, equivalently,

$$J[X, Y] = [X, JY], \quad \forall X \in \text{aut}(M, J), \forall Y \in \text{diff}(M), \quad (97)$$

where $\text{diff}(M)$ denotes all smooth vector fields on $M$. Thus $\text{aut}(M, J)$ is a complex Lie algebra with complex structure $J$.

The automorphism group $\text{Aut}(M, J)_0$ acts on $\mathcal{H}$ by pullback:

$$f.\eta := f^* \eta, \quad f \in \text{Aut}(M, J)_0, \quad \eta \in \mathcal{H}. \quad (98)$$

Given the one-to-one correspondence between $\mathcal{H}$ and $\mathcal{H}_0$, the group $\text{Aut}(M, J)_0$ also acts on $\mathcal{H}_0$. The action is described in the next lemma.

**Lemma 11.1.** For $\varphi \in \mathcal{H}_0$ and $f \in \text{Aut}(M, J)_0$ let $f.\varphi \in \mathcal{H}_0$ be the unique element such that $f.\omega_\varphi = \omega_{f.\varphi}$. Then,

$$f.\varphi = f.0 + \varphi \circ f, \quad f \in \text{Aut}(M, J)_0, \quad \varphi \in \mathcal{H}_0. \quad (99)$$

**Proof.** Note that (99) is a Kähler potential for $f^* \omega_\varphi$. Indeed, $f \in \text{Aut}(M, J)$ implies that $f^* \sqrt{-1} \partial \bar{\partial} \varphi = \sqrt{-1} \partial \bar{\partial} \varphi \circ f$. That $\text{AM}(f.0 + \varphi \circ f) = 0$ follows from Exercise 11.2 as we have,

$$\text{AM}(f.0 + \varphi \circ f) = \text{AM}(f.0 + \varphi \circ f) - \text{AM}(f.0) = \int_M \varphi \circ f \sum_{j=0}^n f^* \omega_\varphi^{n-j} \wedge f^* \omega_\varphi^j = \text{AM}(\varphi) = 0.$$

(Of course, $\text{AM}(f.0) = 0$ since by definition $f.0 \in \mathcal{H}_0$.)

**Exercise 11.2.** Show that

$$\text{AM}(v) - \text{AM}(u) = \frac{V^{-1}}{n+1} \int_M (v - u) \sum_{k=0}^n \omega_u^{n-k} \wedge \omega_v^k. \quad (100)$$

Among other things, this formula shows that $\text{AM}$ is monotone, i.e.,

$$u \leq v \quad \Rightarrow \quad \text{AM}(u) \leq \text{AM}(v). \quad (101)$$

**Lemma 11.3.** The action of $\text{Aut}(M, J)_0$ on $\mathcal{H}_0$ is a $d_\mathcal{D}$-isometry.

**Proof.** From (99),

$$\frac{d}{dt} f.\varphi_t = \dot{\varphi}_t \circ f,$$

for any smooth path $t \mapsto \varphi_t$ in $\mathcal{H}_0$. Thus, the $d_\mathcal{D}$-length of $t \mapsto f.\varphi_t$ is

$$V^{-1} \int_{[0,1] \times M} |\dot{\varphi}_t \circ f|^2 \omega_\varphi^n \wedge dt = V^{-1} \int_{[0,1] \times M} |\dot{\varphi}_t|^2 \omega_{\varphi_t}^n \wedge dt,$$

equal to the $d_\mathcal{D}$-length of $\varphi_t$. 

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Suppose $G$ is a subgroup of $\text{Aut}(M,J)_0$. By the previous lemma $G$ acts on $\mathcal{H}$ by $d_0$-isometries, hence induces a pseudometric on the orbit space $\mathcal{H}/G$,

$$d_{D,G}(Gu,Gv) := \inf_{f,g \in G} d_D(f.u,g.v).$$

Here, we denote by $Gu$ the orbit of $u$ under the action of $G$. Naturally, $Gu$ is an element of the orbit space $\mathcal{H}/G$. Thus, $d_{D,G}$ measures the distance between orbits.

It is natural to expect that the group action extends to the metric completion. This is indeed the case.

**Lemma 11.4.** Let $(X, \rho)$ and $(Y, \delta)$ be two complete metric spaces, $W$ a dense subset of $X$ and $f : W \to Y$ a $C$-Lipschitz function, i.e.,

$$\delta(f(a), f(b)) \leq C \rho(a,b), \quad \forall a, b \in W. \tag{102}$$

Then $f$ has a unique $C$-Lipschitz continuous extension to a map $\bar{f} : X \to Y$.

**Proof.** Let $w_k \in W$ be a Cauchy sequence converging to some $w \in X$. Lipschitz continuity gives

$$\delta(f(w_k), f(w_l)) \leq C \rho(w_k, w_l),$$

hence $\bar{f}(w) := \lim_k f(w_k) \in Y$ is well defined and independent of the choice of approximating sequence $w_k$. Choose now another Cauchy sequence $z_k \in W$ with limit $z \in X$, plugging in $w_k, z_k$ in (102) and taking the limit gives that $\bar{f} : X \to Y$ is $C$-Lipschitz continuous. \hfill \Box

**Lemma 11.5.** The action of $\text{Aut}(M,J)_0$ on $\mathcal{H}_0$ has a unique $d_0$-isometric extension to the metric completion $(\mathcal{H}_0,d_0) = (\mathcal{E}_1 \cap \text{AM}^{-1}(0), d_0)$.

**Proof.** Because $\text{Aut}(M,J)_0$ acts by $d_0$-isometries, each $f \in \text{Aut}(M,J)_0$ induces a 1-Lipschitz continuous self-map of $\mathcal{H}_0$. By Lemma 11.4, such maps have a unique 1-Lipschitz extension to the completion $\mathcal{E}_1 \cap \text{AM}^{-1}(0)$ and the extension is additionally a $d_0$-isometry. By density, the laws governing a group action have to be preserved as well. \hfill \Box

For any Lie subgroup $K$ of the isometry group of $(M,g_\omega)$ define the subspace

$$\mathcal{H}^K_\omega := \{ \varphi \in \mathcal{H}_\omega : \varphi \text{ is invariant under } K \}, \tag{103}$$

and similarly define $\mathcal{H}^K_0 = \mathcal{H}^K \cap \text{AM}^{-1}(0)$. According to Theorem 9.3, the $d_0$-metric completion of $\mathcal{H}^K_\omega$ is

$$\mathcal{E}_1^K := \{ u \in \mathcal{E}_1 : u \text{ is invariant under } K \}.$$ 

The next result follows using the arguments in the proofs of Lemmas 10.2 and 11.5.

**Lemma 11.6.** The metric completion of $(\mathcal{H}^K_0, d_0)$ is $\mathcal{E}^K_1 \cap \text{AM}^{-1}(0)$.
11.2 The Aubin functional on the quotient space

Let $G \subset \text{Aut}(M,J)_0$ be a subgroup. Following Zhou–Zhu [101, Definition 2.1] and Tian [92, Definition 2.5], define the descent of $J$ to $\mathcal{H}/G$,

$$J_G(Gu) := \inf_{g \in G} J(g.u).$$

By Lemma 11.2 this functional can be extended to a functional $J_G : \mathcal{E}_1 \cap \text{AM}^{-1}(0)/G \to \mathbb{R}$, still satisfying

$$J_G(Gu) = \inf_{g \in G} J(g.u). \quad (104)$$

We now see that the key inequality between the Aubin functional and the Darvas distance function (Proposition 10.1) descends to the metric completion of the quotient space.

**Lemma 11.7.** For $u \in \mathcal{E}_1 \cap \text{AM}^{-1}(0)$ we have

$$\frac{1}{C} J_G(Gu) - C \leq d_{D,G}(G0, Gu) \leq CJ_G(Gu) + C, \quad (105)$$

where $d_{D,G}$ is the pseudometric of the quotient $\mathcal{E}_1 \cap \text{AM}^{-1}(0)/G$.

**Proof.** By Lemma 11.3

$$d_{D,G}(G0, Gu) = \inf_{f \in G} d_D(0, f.u).$$

The result now follows from Proposition 10.1.

12 A modified conjecture: Tian’s second properness conjecture

At last, we return to Conjecture 1.2 and pick up the discussion from where we left it at the end of Section 6. Lemma 11.7 motivates the following modification of Conjecture 1.2.

**Definition 12.1.** Let $F : \mathcal{H} \to \mathbb{R}$ be $G$-invariant.

- We say $F$ is $d_{D,G}$-proper if for some $C, D > 0$,

$$F(u) \geq Cd_{D,G}(G0, Gu) - D.$$  

- We say $F$ is $J_G$-proper if for some $C, D > 0$,

$$F(u) \geq CJ_G(Gu) - D.$$  

**Conjecture 12.2.** (Tian’s second properness conjecture) Let $(M, J, \omega)$ be a Fano manifold. Set $G := \text{Aut}(M,J)_0$. There exists a Kähler–Einstein metric in $\mathcal{H}$ if and only if the descent of the Mabuchi energy $E$ to the quotient space $\mathcal{H}/G$ is $d_{D,G}$-proper (equivalently, $J_G$-proper).

Note that according to Lemma 11.7 both notions of properness are indeed equivalent. Also, the $G$-invariance condition can be considered as a version of the Futaki obstruction [54].

Albeit being a purely analytic criterion, properness should be morally equivalent to properness in a metric geometry sense, namely, that the Mabuchi functional should grow at least linearly relative to some metric on $\mathcal{H}$, and this is precisely the content of Conjecture 12.2.
Remark 12.3. We now come back to the analogy with the Dirichlet energy alluded to in the Prologue. There we seek to minimize the Dirichlet energy, say on the unit ball in $\mathbb{R}^n$,

$$E(f) := \int_{B_1(0)} \sum_{i=1}^n (\partial_{x_i} f)^2 dx^1 \wedge \cdots dx^n.$$ 

The space of competitors $\mathcal{H}$ is now the space of smooth functions with prescribed boundary values $g \in C^\infty(\partial B_1(0))$,

$$\mathcal{H} := \{ f \in C^\infty(B_1(0)) : f|_{\partial B_1(0)} = g \}.$$ 

In some sense, the prescribed boundary values can be morally thought of as the analogue for fixing a Kähler class. What is the analogue of the Aubin functional? In this case it is just $E$ itself, i.e., we put $J = E$, so an analogue of Conjecture 1.2 is trivial here. However, the direct method in the calculus of variations motivates replacing $J$ (which is the $W^{1,2}$ seminorm) with the $W^{1,2}$ norm. Namely, we consider the metric

$$(h,k) := \int \sum_{i=1}^n \partial_{x_i} h \partial_{x_i} k dx^1 \wedge \cdots dx^n + \int h k dx^1 \wedge \cdots dx^n.$$ 

The path-length distance is then just the one coming from the norm $W^{1,2}$, and the properness inequality is a consequence of the Poincaré inequality. This then implies that a minimizer exists in the $W^{1,2}$ completion of $\mathcal{H}$. The Euler–Lagrange equation is precisely the Laplace equation with prescribed boundary data. Elliptic regularity theory then shows the minimizer must be an element of $\mathcal{H}$ itself, hence a smooth harmonic function agreeing with $g$ on the boundary.

In the remainder of these notes, we sketch the resolution of Conjecture 12.2 due to Darvas–Rubinstein [44].

**Theorem 12.4.** Conjecture 12.2 holds.

The proof of this result is completed in Section 15.

Remark 12.5. The easier implication “$J_G$-proper $\Rightarrow$ existence of Kähler–Einstein” is due to Tian [92, Theorem 2.6] and is a modification of the proof of Theorem 5.1. Our proof of Theorem 12.4 also furnishes a new proof of this fact. In the special case of toric Fano manifolds, a variant of the converse direction is due to Zhou–Zhu [101, Theorem 0.2].

### 13 A general existence/properness principle

Motivated by Remark 12.3, we approach Conjecture 12.2 using an abstract metric geometry framework. While seemingly abstract it turns out to be a powerful way of dealing with several different minimization problems in Kähler geometry.

**Notation 13.1.** The data $(\mathcal{R}, d, F, G)$ is defined as follows.

(A1) $(\mathcal{R}, d)$ is a metric space with a distinguished element $0 \in \mathcal{R}$, whose metric completion is denoted $(\overline{\mathcal{R}}, d)$. 

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(A2) $F : \mathcal{R} \to \mathbb{R}$ is lower semicontinuous (lsc). Let $F : \overline{\mathcal{R}} \to \mathbb{R} \cup \{+\infty\}$ be the largest lsc extension of $F : \mathcal{R} \to \mathbb{R}$:

$$F(u) = \sup_{\varepsilon > 0} \left( \inf_{v \in \mathcal{R}} F(v) \right), \quad u \in \overline{\mathcal{R}}.$$ 

For each $u, v \in \mathcal{R}$ define also

$$F(u, v) := F(v) - F(u).$$

(A3) The set of minimizers of $F$ on $\mathcal{R}$ is denoted

$$\mathcal{M} := \{u \in \overline{\mathcal{R}} : F(u) = \inf_{v \in \mathcal{R}} F(v)\}.$$

(A4) Let $G$ be a group acting on $\mathcal{R}$ by $G \times \mathcal{R} \ni (g, u) \to g.u \in \mathcal{R}$. Denote by $\mathcal{R}/G$ the orbit space, by $Gu \in \mathcal{R}/G$ the orbit of $u \in \mathcal{R}$, and define $d_G : \mathcal{R}/G \times \mathcal{R}/G \to \mathbb{R}_+$ by

$$d_G(Gu, Gv) := \inf_{f, g \in G} d(f.u, g.v).$$

**Hypothesis 13.2.** The data $(\mathcal{R}, d, F, G)$ satisfies the following properties.

(P1) For any $\varphi_0, \varphi_1 \in \mathcal{R}$ there exists a $d$-geodesic segment $[0, 1] \ni t \mapsto \varphi_t \in \overline{\mathcal{R}}$ for which $t \mapsto F(\varphi_t)$ is continuous and convex on $[0, 1]$.

(P2) If $\{\varphi_j\}_j \subset \overline{\mathcal{R}}$ satisfies $\lim_{j \to \infty} F(\varphi_j) = \inf_{v \in \mathcal{R}} F$, and for some $C > 0$, $d(0, \varphi_j) \leq C$ for all $j$, then there exists a $u \in \mathcal{M}$ and a subsequence $\{\varphi_{j_k}\}_k$ $d$-converging to $u$.

(P3) $\mathcal{M} \subset \mathcal{R}$.

(P4) $G$ acts on $\mathcal{R}$ by $d$-isometries.

(P5) $G$ acts on $\mathcal{M}$ transitively.

(P6) If $\mathcal{M} \neq \emptyset$, then for any $u, v \in \mathcal{R}$ there exists $g \in G$ such that $d_G(Gu, Gv) = d(u, g.v)$.

(P7) For all $u, v \in \mathcal{R}$ and $g \in G$, $F(g.u, g.v) = F(g.u, g.v)$.

The following result will provide the aforementioned framework for dealing with many minimization problems.

**Theorem 13.3.** Let $(\mathcal{R}, d, F, G)$ be as in Notation 13.1 and satisfying Hypothesis 13.2. Then $\mathcal{M}$ is nonempty if and only if $F : \mathcal{R} \to \mathbb{R}$ is $G$-invariant, and for some $C, D > 0$,

$$F(u) \geq Cd_G(G0, Gu) - D, \quad \text{for all } u \in \mathcal{R}. \quad (106)$$
One direction in this theorem is easy. Namely, if \((106)\) holds, then \(F\) is bounded from below. By \([A2]\)

\[
\inf_{v \in \mathcal{R}} F(v) = \inf_{v \in \mathcal{R}} F(v).
\]  

(107)

This, combined with \((106)\), the \(G\)-invariance of \(F\) and the definition of \(d_G\) implies there exists \(\varphi_j \in \mathcal{R}\) such that \(\lim_j F(\varphi_j) = \inf \mathcal{R} F \) and \(d(0, \varphi_j) \leq d_G(G0, G\varphi_j) + 1 < C\) for \(C\) independent of \(j\). By \([P2]\) \(\mathcal{M}\) is non-empty. For the other direction we refer the reader to \([44, \text{Theorem 3.4}]\).

We have set up things in such a way that the modified properness conjecture, Conjecture \([12.2]\) would become a corollary of Theorem \([13.3]\) applied to the following data

\[
\mathcal{R} = \mathcal{H}_0, \quad d = d_1, \quad F = E, \quad G := \text{Aut}_0(M, J),
\]  

(108)

if this data satisfies the hypothesis of Theorem \([13.3]\). In the next sections we verify that this is indeed the case. Property \([P4]\) has already been verified in Lemma \([11.5]\). In the next few sections we verify the remaining hypothesis of Theorem \([13.3]\).

14 Applying the general existence/properness principle

In this section we briefly motivate—in the context of the Kähler–Einstein problem—some of the key assumptions in the general existence/properness principle. The point is to convince the reader that this principle fits naturally/seamlessly with classical/foundational results in Kähler geometry.

First, a seemingly harmless condition, tucked into the “notation” part of Theorem \([13.3]\) is that the functional we are trying to minimize on the metric completion should be the greatest lower semicontinuous extension (with respect to the path-length metric) of the functional we are trying to study originally on the “regular” objects \(\mathcal{R}\). This turns out to be quite a technical thing to verify. At first, this might cause confusion: indeed any functional admits such an extension by means of the abstract formula

\[
F(u) = \sup_{\epsilon > 0} \left( \inf_{v \in \mathcal{R}} \left( F(v) \right) \right), \quad u \in \overline{\mathcal{R}}.
\]  

(109)

However, the issue is to verify that this abstract formula, say in the case of the Mabuchi energy, coincides with the original defining formula \([12]\) which initially only makes sense on the space of smooth potentials \(\mathcal{R} = \mathcal{H}\). This is because only then can we actually verify that this extended functional satisfies the other hypothesis in Theorem \([13.3]\) (without an explicit formula it is not clear how to proceed). Fortunately, condition \([A2]\) for \([108]\) does hold by the following result \([44, \text{Proposition 5.21}]\).

Proposition 14.1. Formula \([12]\) coincides with formula \([109]\) on \(E_1\). In other words, formula \([12]\) gives the greatest \(d_1\)-lsc extension of \(E : \mathcal{H} \rightarrow \mathbb{R}\) to \(E_1\).

Remark 14.2. The analogue of this result for the Mabuchi metric \(d_{M}\) can be found in \([16]\).

Second, property \([P1]\) holds for the Mabuchi energy due to a result of Berman–Berndtsson \([13, \text{Theorem 1.1}]\). In fact, we remark that it is well-known that the geodesic between smooth endpoints has considerable regularity (as compared to just being in \(\mathcal{R}\)) \([23, 31]\). In \([13]\) it is shown that the Mabuchi energy is convex along such partially regular geodesics.
Third, property \([P2]\) stipulates precompactness of sublevel sets of the Mabuchi energy with respect to the Darvas metric. Pre-compactness with respect to other functionals is a key result in the works \([15, 24]\), and can be adapted to show the aforementioned pre-compactness \([44, \text{Proposition 5.28}]\).

Fourth, property \([P3]\) stipulates regularity of minimizers of the Mabuchi energy in the metric completion. This follows from the regularity result of Berman \([11, \text{Theorem 1.1}]\) combined with the characterization of the metric completion of Darvas (Theorem \([9, \text{Theorem 9.4}]\)).

Fifth, property \([P5]\), modulo property \([P3]\), amounts to the classical Bando–Mabuchi theorem on uniqueness of Kähler–Einstein metrics up to automorphisms.

Sixth, property \([P7]\) says that the Mabuchi functional is exact, or of “Bott–Chern” type, and this is precisely Mabuchi’s original theorem on his functional \([65, \text{Theorem 2.4}]\). For an expository treatment we referred to \([78, \text{§5}]\).

Finally, property \([P6]\) is a new ingredient, and so we go into more detail, sketching property \([P6]\) for \((108)\). It fits nicely into our framework since it shows precisely the role of another classical result in Kähler geometry, namely, Matsushima’s classical theorem about the automorphism group of a Kähler–Einstein manifold. The key result in showing \([P6]\) is the following \([44, \text{Proposition 6.8}]\).

**Proposition 14.3.** Let \((M, J, \omega, g)\) be Kähler–Einstein. Define \((R, d, F, G)\) by \((108)\), and suppose that \([A1], [A4]\) and \([P4]\) hold. Finally, assume the following:

(i) For each \(X \in \text{isom}(M, g)\), \(t \mapsto \exp_t JX.\omega\) is a \(d_\mathcal{D}\)-geodesic whose speed depends continuously on \(X\).

(ii) \(\text{Aut}(M, J)_0 \times \text{Aut}(M, J)_0 \ni (f, g) \mapsto d(f.u, g.v)\) is a continuous map for every \(u, v \in \mathcal{H}\).

Then property \([P6]\) holds.

Condition (i) is essentially a corollary of \((90)\), while (ii) follows from \((96)\). Property \([P6]\) stipulates that a certain infimum over the group \(G\) is attained. Thus, for the proof of Proposition \(14.3\) we decompose the group \(G\) into a compact part and a non-compact part in such a way that the compact part acts by \(d\)-isometries while on the non-compact part (but finite-dimensional!) we have \(d\)-properness. Then, together with conditions (i) and (ii), the existence of a minimizer is guaranteed.

The aforementioned decomposition of the group into a compact and a non-compact part is stated in Corollary \(14.7\) below. It should be well-known and relies on classical results that we now recall. First, we recall Matsushima’s classical theorem \([67, \text{Théorème 1}]\). We refer to Gauduchon \([55]\) for more details. Let \(g(\cdot, \cdot) = \omega(\cdot, J\cdot)\) denote the Riemannian metric associated to \((M, J, \omega)\). Denote by \(\text{Isom}(M, g)_0\) the identity component of the isometry group of \((M, g)\). Since \(M\) is compact so is \(\text{Isom}(M, g)_0\) \([73, \text{Proposition 29.4}]\). Denote by \(\text{isom}(M, g)\) the Lie algebra of \(\text{Isom}(M, g)_0\).

**Theorem 14.4.** Let \((M, J, \omega, g)\) be a Fano Kähler manifold. Suppose \(g\) is a Kähler–Einstein metric. Then,

\[ \text{aut}(M, J) = \text{isom}(M, g) \oplus J\text{isom}(M, g). \]

The following result is classical, and we only state its Kähler–Einstein version, whose proof we sketch.

**Theorem 14.5.** Let \((M, J, \omega, g)\) be Kähler–Einstein. Then any maximally compact subgroup of \(\text{Aut}(M, J)_0\) is conjugate to \(\text{Isom}(M, g)_0\).
Proof. By a Theorem of Iwasawa–Malcev [85, Theorem 3.5], if $G$ is a connected Lie group then its maximal compact subgroup must be connected and any two maximal compact subgroups are conjugate. But then by Theorem [14.4] Isom($M, g_0$) has to be a maximal compact subgroup of Aut($M, J_0$).

Next, we need a version of the classical Cartan decomposition [26, Proposition 32.1, Remark 31.1].

**Theorem 14.6.** Let $S$ be a compact connected semisimple Lie group. Denote by $(S^C, J)$ the complexification of $S$, namely the unique connected complex Lie group whose Lie algebra is the complexification of that of $s$, the Lie algebra of $S$. Then the map $C$ from $S \times s$ to $S^C$ given by

$$(s, X) \mapsto C(s, X) := s \exp_I JX$$

is a diffeomorphism.

Combining Theorems [14.4] [14.5] and [14.6] we obtain the decomposition of Aut($M, J_0$) into a compact and a non-compact part that is needed for the proof of Proposition [14.3]. For details on how the following result yields Proposition [14.3] we refer to [44, §6], where a more general result is proven in the constant scalar curvature setting (when the Cartan type decomposition is not given by classical results and we construct instead a “partial Cartan decomposition” that may only be surjective).

**Corollary 14.7.** Let $(M, J, \omega, g)$ be Kähler–Einstein. Then the map $C$ from Isom($M, g_0$) to Aut($M, J_0$) given by

$$(s, X) \mapsto C(s, X) := s \exp_I JX$$

is a diffeomorphism.

15 A proof of Tian’s second properness conjecture

As already explained at the end of Section 13, and as we started to elaborate in the previous section, we prove Theorem [12.4] by applying Theorem [13.3] to data (108). Thus, it only remains to verify that this data satisfies the hypothesis of Theorem [13.3].

First, we go over Notation [13.1]. First, in (A1), $R = E_1 \cap AM^{-1}(0)$ by Theorem [9.4] and Lemma [10.2]. Observe that (A2) holds by Proposition [14.1]. In (A3), the minimizers of $F$ are denoted by $M$. Finally, (A4) holds since $G \subset$ Aut($M, J_0$) implies that if $g \in G$ and $\eta \in H$ then $g.\eta$ is both Kähler and cohomologous to $\eta$, i.e., $g.\eta \in H$. Thus, it remains to verify Hypothesis [13.2].

Properties (P1)–(P7) were all verified in [14] with the exception of property (P4), that itself follows from Lemma [11.3].

Finally, we need to justify why we did not state $E$ must be Aut($M, J_0$)-invariant in Theorem [12.4], while it is needed to apply Theorem [13.3]. This follows from Futaki’s theorem [54, p. 437]. Indeed, as in the proof of Claim [6.1]

$$\frac{d}{dt} E((\exp_I tX)^* \omega_\tau) = C_X,$$
for some $R \ni C_X$ depending on $X$ but not on $\omega_\varphi \in \mathcal{H}$. Also,
\[
\frac{d}{dt}E((\exp_I tX)^* \omega_\varphi) = -C_X.
\]
Thus, unless this derivative, i.e., $C_X$, is zero for every $X \in \text{aut}(M, J)$ and $\omega_\varphi \in \mathcal{H}$, the functional $E$ cannot be bounded from below. Now, properness of $E$ with respect to any nonnegative functional implies $E$ is bounded from below. Thus, $J_G$-properness of $E$ implies it is $\text{Aut}(M, J)_0$-invariant.

16 A proof of Tian’s third conjecture: the strong Moser–Trudinger inequality

We now explain the proof of Tian’s third conjecture, namely the strong Moser–Trudinger inequality for Kähler–Einstein manifolds. First, let us recall the statement.

Denote by $\Lambda_1$ the real eigenspace of the smallest positive eigenvalue of $-\Delta_\omega$, and set
\[
\mathcal{H}_\omega^\bot := \{ \varphi \in \mathcal{H} : \int \varphi \psi \omega^n = 0, \forall \psi \in \Lambda_1 \}.
\]

**Conjecture 16.1.** Suppose $(M, J, \omega)$ is Fano Kähler–Einstein. Then for some $C, D > 0$,
\[
E(\varphi) \geq CJ(\varphi) - D, \quad \varphi \in \mathcal{H}_\omega^\bot.
\] (113)

Observe that no invariance properties are assumed, and the functionals are not taken on the quotient space. Instead, an orthogonality assumption is made.

Conjecture 1.3 was originally motivated by results in conformal geometry related to the determination of the best constants in the borderline case of the Sobolev inequality. By restricting to functions orthogonal to the first eigenspace of the Laplacian, Aubin was able to improve the constant in the aforementioned inequality on spheres [5, p. 235]. This can be seen as the sort of coercivity of the Yamabe energy occurring in the Yamabe problem, and it clearly fails without the orthogonality assumption due to the presence of conformal maps. Conjecture 1.3 stands in clear analogy with the picture in conformal geometry, by stipulating that coercivity of the K-energy holds in ‘directions perpendicular to holomorphic maps’ (when $\omega$ is Kähler–Einstein, it is well-known that $\Lambda_1$ is in a one-to-one correspondence with holomorphic gradient vector fields, in fact this is how Matsushima’s Theorem 14.4 is proven [55, 30]). It can be thought of as a higher-dimensional fully nonlinear generalization of the classical Moser–Trudinger inequality.

It is a rather simple consequence of the work of Bando–Mabuchi [8] that when a Kähler–Einstein metric exists, $J_G$-properness implies $J$-properness on $\mathcal{H}_\omega^\bot$ [90, Corollary 5.4], [101, Lemma A.2], [92, Theorem 2.6]. We now explain how to carry this through. The key is to study the Aubin functional restricted to orbits of $\text{Aut}(M, J)_0$ and identify the minimizers and relate them to the first eigenspace.

Fix $\eta \in \mathcal{H}$. Let $F_\eta : \text{Aut}(M, J)_0 \to \mathbb{R}_+$ be given by
\[
F_\eta(g) := (I - J)(g^* \eta) = V^{-1} \frac{1}{n+1} \int_M \sqrt{-1} \partial \bar{\partial} \varphi_g \wedge \bar{\partial} \varphi_g \wedge \sum_{l=0}^{n-1} (n-l) \omega^{n-l-1} \wedge (g^* \eta)^l,
\]
where $\varphi_g \in \mathcal{H}_\omega$ is such that $g^* \eta = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_g$ (i.e., where the $I - J$ energy of $g^* \eta$ with respect to the reference form $\omega$).
Lemma 16.2. Suppose \((M,J,\eta=\omega_\psi)\) is Fano Kähler–Einstein. Then \(h \in \text{Aut}(M,J)_0\) is a critical point of \(F_\eta\) precisely if \(-\varphi_h \in H^\perp_{h^*\eta}\).

Proof. Using (114) and (10),

\[
\frac{d}{d\delta}(I-J)(\varphi(\delta)) = \frac{d}{d\delta} \text{AM}(\varphi) - \frac{d}{d\delta} V^{-1} \int \varphi(\delta) \omega^n_{\varphi(\delta)} \\
= V^{-1} \int \frac{d}{d\delta} \varphi(\delta) \omega^n_{\varphi(\delta)} - V^{-1} \int \left( \frac{d}{d\delta} \varphi(\delta) + \varphi(\delta) \Delta_{\omega_{\varphi(\delta)}} \frac{d}{d\delta} \varphi(\delta) \right) \omega^n_{\varphi(\delta)}
\]

Writing \(g_t = h \exp_t tX\) with \(X \in \text{aut}(M,J)\), observe that

\[
\sqrt{-1} \partial \bar{\partial} \varphi_h = \sqrt{-1} \partial \bar{\partial} \varphi_{g_0}
\]

Writing \(g_t = h \exp_t tX\) with \(X \in \text{aut}(M,J)\), observe that

\[
\sqrt{-1} \partial \bar{\partial} \varphi_h = \sqrt{-1} \partial \bar{\partial} \varphi_{g_0}
\]

Therefore,

\[
\frac{d}{dt} \bigg|_{t=0} F_\eta(g_t) = -V^{-1} \int \varphi_h \Delta_{h^*\eta} \varphi_h (h^*\eta)^n
\]

\[
= -V^{-1} \int \varphi_h \Delta_{h^*\eta} \psi^X_{h^*\eta} (h^*\eta)^n
\]

\[
= V^{-1} \int \varphi_h \psi^X_{h^*\eta} (h^*\eta)^n.
\]

Since this holds for all \(X \in \text{aut}(M,J)\), it follows that \(\varphi_h \in H^\perp_{h^*\eta}\).

Lemma 16.3. Suppose \((M,J,\eta=\omega_\psi)\) is Fano Kähler–Einstein. By Theorem 14.5 then \(\text{Aut}(M,J)_0 = K^C\) for a maximally compact subgroup \(K\). Suppose \(\omega \in H^K\). Then \(F_\eta\) has a unique critical point which is a global minimum.

Proof. We start with the following observation.

Exercise 16.4. If \(g \in \text{Aut}(M,J)_0\) preserves \(\omega\) then

\[
(I-J)(g^*\eta) = (I-J)(\eta).
\]

Thus, using the Cartan decomposition (Corollary 14.7), \(F_\eta\) descends to a function on \(\text{isom}(M,g)\), still denoted by \(F_\eta\),

\[
F_\eta(X) = (I-J)((\exp_t tX)^*\eta).
\]

Now, we show that the function \((\exp_t tX)^*\eta\) satisfies a useful equation.

The Hodge decomposition implies that every \(X \in \text{aut}(M,J)\) can be uniquely written as

\[
X = X_H + \nabla \psi^X_\omega - J\nabla \psi^1_X\omega,
\]

where \(\nabla\) is the gradient with respect to the Riemannian metric associated to \(J\) and \(\omega\), and \(X_H\) is the \(g_\omega\)-Riemannian dual of a \(g_\omega\)-harmonic 1-form.
By (116) and the fact that $X \in \text{isom}(M, g_\eta)$ (here $g_\eta$ denotes the Riemannian metric associated to $J$ and $\eta$) it follows that
\[ JX = \nabla \psi^J_X \] (117)
is a gradient (with respect to $g_\eta$) vector field [66, Theorem 3.5]. We set
\[ \omega_{\varphi(t)} := \omega(t) = \exp_t tJX.\eta. \]
Thus,
\[ \dot{\omega}(t) = \frac{d}{dt} \exp_t tJX.\eta = \mathcal{L}_{JX} \eta \circ \exp_t tJX = \sqrt{-1} \partial \bar{\partial} \psi^J_X \circ \exp_t tJX, \] (118)
and
\[ \ddot{\omega}(t) = \sqrt{-1} \partial \bar{\partial} ((JX)(\psi^J_X)) \circ \exp_t tJX \\
= \sqrt{-1} \partial \bar{\partial} (d\psi^J_X (JX)) \circ \exp_t tJX \\
= \sqrt{-1} \partial \bar{\partial} |\nabla \psi^J_X| \circ \exp_t tJX, \]
since the $\eta$-Riemannian dual of $d\psi^J_X$ is $\nabla \psi^J_X$. Thus,
\[ \ddot{\varphi}(t) - |\nabla \dot{\varphi}(t)|^2 \omega_{\varphi(t)} = 0. \] (121)

Next, we can generalize this computation slightly to obtain an equation for the function \( \exp_t J((1 - t)Y + tZ)^*\eta \). By (116) and the fact that $X \in \text{isom}(M, g_\eta)$ (here $g_\eta$ denotes the Riemannian metric associated to $J$ and $\eta$) it follows that
\[ J(Z - Y) = \nabla \psi^J_{(Z - Y)} \] (119)
is a gradient (with respect to $g_\eta$) vector field [66, Theorem 3.5]. We set
\[ \omega_{\varphi(t)} := \omega(t) = (\exp_t J((1 - t)Y + tZ))^*\eta. \]
Thus,
\[ \dot{\omega}(t) = \frac{d}{dt} (\exp_t J((1 - t)Y + tZ))^*\eta \\
= \mathcal{L}_{J(Z - Y)} \eta \circ \exp_t J((1 - t)Y + tZ) = \sqrt{-1} \partial \bar{\partial} \psi^J_{(Z - Y)} \circ \exp_t J((1 - t)Y + tZ), \] (120)
and
\[ \ddot{\omega}(t) = \sqrt{-1} \partial \bar{\partial} ((J(Z - Y))(\psi^J_{(Z - Y)})) \circ \exp_t J(Z - Y) \\
= \sqrt{-1} \partial \bar{\partial} |\nabla \psi^J_{(Z - Y)}|^2 \circ \exp_t J((1 - t)Y + tZ), \]
Thus, again,
\[ \ddot{\varphi}(t) - |\nabla \dot{\varphi}(t)|^2 \omega_{\varphi(t)} = 0. \] (121)

Observe that
\[ F_\eta((1 - t)Y + tZ) = (I - J)((\exp_t J((1 - t)Y + tZ))^*\eta). \]
Therefore,
\[
\frac{d}{dt} \bigg|_0 F_{\eta}(1 - t)Y + tZ = -V^{-1} \int \varphi(t)\Delta_{g_t^*\eta}\dot{\varphi}(t)(g_t^*\eta)^n \\
= -V^{-1} \int \varphi(t)\Delta_{\omega(t)}\dot{\varphi}(t)\omega(t)^n \\
= -V^{-1} \int \dot{\varphi}(t)\Delta_{\omega(t)}\varphi(t)\omega(t)^n \\
= V^{-1} \int \dot{\varphi}(t)n(\omega - \omega(t)) \wedge \omega(t)^{n-1},
\]
(122)
where \( g_t := \exp_t J((1 - t)Y + tZ) \), since \( g_t^*\eta = \omega(t) \). Also, using \((121)\),
\[
\frac{d^2}{dt^2} \bigg|_0 F_{\eta}(1 - t)Y + tZ = V^{-1} \int \ddot{\varphi}(t)n(\omega - \omega(t)) \wedge \omega(t)^{n-1} \\
- V^{-1} \int \dot{\varphi}(t)n\sqrt{-1}\partial\bar{\partial}\dot{\varphi}(t) \wedge \omega(t)^{n-1} \\
+ V^{-1} \int \dot{\varphi}(t)n(n - 1)(\omega - \omega(t)) \wedge \sqrt{-1}\partial\bar{\partial}\dot{\varphi}(t) \wedge \omega(t)^{n-2} \\
= nV^{-1} \int |\nabla\dot{\varphi}|^2\omega \wedge \omega(t)^{n-1} - nV^{-1} \int |\nabla\dot{\varphi}|^2\omega(t)^n \\
+ nV^{-1} \int \sqrt{-1}\partial\dot{\varphi}(t) \wedge \sqrt{-1}\partial\bar{\partial}\dot{\varphi}(t) \wedge \omega(t)^{n-1} \\
+ V^{-1} \int \dot{\varphi}(t)n(n - 1)\omega \wedge \sqrt{-1}\partial\bar{\partial}\dot{\varphi}(t) \wedge \omega(t)^{n-2} \\
- V^{-1} \int \dot{\varphi}(t)n(n - 1)\sqrt{-1}\partial\bar{\partial}\dot{\varphi}(t) \wedge \omega(t)^{n-1} \\
= nV^{-1} \int |\nabla\dot{\varphi}|^2\omega \wedge \omega(t)^{n-1} \\
- n(n - 1)V^{-1} \int \sqrt{-1}\partial\dot{\varphi}(t) \wedge \sqrt{-1}\partial\bar{\partial}\dot{\varphi}(t) \wedge \omega(t)^{n-2} \\
= \frac{n}{V} \int \left( |\nabla\dot{\varphi}|^2\omega(t) - (n - 1)\sqrt{-1}\partial\dot{\varphi}(t) \wedge \sqrt{-1}\partial\bar{\partial}\dot{\varphi}(t) \right) \wedge \omega \wedge \omega(t)^{n-2} \\
\geq \frac{n}{V} \int \sqrt{-1}\partial\dot{\varphi}(t) \wedge \sqrt{-1}\partial\bar{\partial}\dot{\varphi}(t) \wedge \omega \wedge \omega(t)^{n-2} > 0,
\]
(123)
since if \( \alpha, \beta \) are two positive (1,1)-forms then \((\text{tr}_\omega \alpha - \beta) \geq 0\), in general, so have \( |\nabla\dot{\varphi}|^2\omega(t) - n\sqrt{-1}\partial\dot{\varphi}(t) \wedge \sqrt{-1}\partial\bar{\partial}\dot{\varphi}(t) \geq 0\). Thus, \( F_{\eta} \) is strictly convex on the vector space isom\((M, g)\). Now, observe that it is a proper function by Lemma \(6.2\). Since a proper strictly convex function
attains a unique minimum, the proof is complete. \hfill \Box

**Exercise 16.5.** Prove the formula (see, e.g., [77, p. 140])

\[(I - J)(\omega, \eta) = J(\eta, \omega),\]

where \((I - J)(\omega, \eta)\) is just \((I - J)(\varphi)\) for any \(\varphi\) such that \(\eta = \omega_{\varphi}\), while \(J(\eta, \omega)\) is just \(J\) (recall (7) “of” \(\omega\) “with respect to” the reference \(\eta\), in the sense that

\[J(\eta, \omega) = V^{-1} \int_M \varphi \eta^n - \frac{V^{-1}}{n+1} \int_M \psi \sum_{l=0}^n \eta^{n-l} \wedge \omega^l,\]

where \(\psi\) satisfies \(\omega = \eta_{\psi}\).

**Proposition 16.6.** Suppose \((M, J, \eta)\) is Fano Kähler–Einstein. If \(E\) is \(J_G\)-proper then (113) holds.

**Proof.** According to Lemma 16.2, the functional

\[g \mapsto (I - J)(\omega, g^* \eta)\]

has a critical point at the identity \(g = \text{id}\) if \(\eta = \omega - \sqrt{-1} \partial \bar{\partial} \varphi\) when \(\varphi \in H^\perp_\eta\). Now, by Exercise 16.5, this is tantamount to the functional

\[g \mapsto J(g^* \eta, \omega)\]

having a critical point at the identity \(g = \text{id}\) if \(\eta = \omega - \sqrt{-1} \partial \bar{\partial} \varphi\) when \(\varphi \in H^\perp_\eta\).

Suppose now that indeed \(\varphi \in H^\perp_\eta\). Then the functional (124) has a critical point at \(g = \text{id}\). By Lemma 16.3, this is the unique minimum of this functional. Thus, using also \(\text{Aut}(M, J)_\eta\)-invariance of \(J\) yields

\[J(\varphi) =: J(\eta, \eta_{\varphi}) = J(\eta, \omega) = \inf_{g \in G} J(g^* \eta, \eta_{\varphi}) = \inf_{\varphi \in \mathcal{H}^\perp} J(\eta, g^* \eta_{\varphi}).\]

The last expression is precisely \(J_G(\varphi)\) (with respect to the reference metric \(\eta\) (not \(\omega\)!)). By assumption \(E\) is \(J_G\)-proper, so, say, for concreteness,

\[E(\varphi) \geq C J_G(\eta) - D = C J(\varphi) - D,\]

as desired. (Observe that the proof also gives the converse, namely that if (113) holds then \(E\) is \(J_G\)-proper.) \hfill \Box

Therefore, Theorem 12.4 and Proposition 16.6 confirm Tian’s conjecture.

**Corollary 16.7.** Conjecture 1.3 holds.
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