A REAL PART THEOREM FOR THE HIGHER DERIVATIVES OF ANALYTIC FUNCTIONS IN THE UNIT DISK

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Abstract. Let \( n \) be a positive integer. Let \( U \) be the unit disk, \( p \geq 1 \) and let \( h^p(U) \) be the Hardy space of harmonic functions. Kresin and Maz'ya in a recent paper found a representation for the function \( H_{n,p}(z) \) in the inequality
\[
|f^{(n)}(z)| \leq H_{n,p}(z) \|\Re(f - P_l)\|_{h^p(U)}, \quad \forall f \in h^p(U), \quad z \in U,
\]
where \( P_l \) is a polynomial of degree \( l \leq n - 1 \). We find or represent the sharp constant \( C_{p,n} \) in the inequality
\[
H_{n,p}(z) \leq C_{p,n} (1 - |z|^2)^{1/p} + n.
\]
This extends a recent result of Kalaj and Marković, where only the case \( n = 1 \) was considered. As a corollary, an inequality for the modulus of \( n \)-th derivative of an analytic function defined in a complex domain with the bounded real part is obtained. This result improves a recent result of Kresin and Maz'ya.

1. Introduction and statement of the results

A harmonic function \( f \) defined in the unit disk \( U \) of the complex plane \( \mathbb{C} \) belongs to the harmonic Hardy class \( h^p = h^p(U), \ 1 \leq p < \infty \) if the following growth condition is satisfied
\[
\|f\|_{h^p} := \left( \sup_{0<r<1} \left| \int_T |f(\rho e^{i\theta})|^p d\theta \right| \right)^{1/p} < \infty
\]
where \( T \) is the unit circle in the complex plane \( \mathbb{C} \). The space \( h^\infty(U) \) consists of all bounded harmonic functions.

If \( f \in h^p(U) \), then there exists the finite radial limit
\[
\lim_{r \to 1^-} f(r\zeta) = f^*(\zeta) \quad \text{(a.e. on } T) \]
and the boundary function \( f^* \) belong to the space \( L^p(T) \) of \( p \)-integrable functions on the circle.

It is well known that a harmonic function \( f \) in the Hardy class \( h^p(U) \) can be represented as the Poisson integral
\[
f(z) = \int_T P(z, \zeta) d\mu(\zeta), \ z \in U
\]
where
\[
P(z, \zeta) = \frac{1 - |z|^2}{|z - \zeta|^2}, \ z \in U, \ \zeta \in T
\]
is the Poisson kernel and \( \mu \) is a complex Borel measure. In the case \( p > 1 \) this measure is absolutely continuous with respect to the Lebesgue measure and

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\[ d\mu(\zeta) = f^*(\zeta)d\sigma(\zeta). \] Here \( d\sigma \) is Lebesgue probability measure in the unit circle. Moreover we have
\[ \|f\|_{h^p} = \|f^*\|_{p}, \quad p > 1 \]
and
\[ \|f\|_{h^1} = \|\mu\| \]
where we denote by \( \|\mu\| \) the total variation of the measure \( \mu \).

For previous facts we refer to the book [1, Chapter 6]. In the sequel for \( p \geq 1 \) and \( m \) a positive integer, as in [10] we use the notation
\[ E_{m,p}(\Re f) := \inf_{p \in \mathbb{P}_m} \|\Re(f - \mathcal{P}_l)\|_{h^p} \]
for the best approximation of \( \Re f \) by the real part of algebraic polynomials in the \( h^p(U) \)-norm, where \( \mathbb{P}_m \) is the set of all algebraic polynomials of degree at most \( m \).

The starting position of this paper is the following proposition of Maz’ya and Kresin [10] Proposition 5.1.

**Proposition 1.1.** Let \( f \) be analytic on \( U \) with \( \Re f \in h^p(U) \), \( 1 \leq p \leq \infty \). Further, let \( n \geq 1 \), and let \( \mathcal{P}_l \) be a polynomial of degree \( l \leq n - 1 \). Then for any fixed point \( z, |z| = r < 1 \), the inequality
\[ |f^{(n)}(z)| \leq H_{n,p}(r)\|\Re(f - \mathcal{P}_l)\|_{h^p} \]
holds with the sharp factor
\[ H_{n,p}(r) = \frac{n!}{\pi} \sup_{\alpha} \left\{ \int_{|\zeta| = 1} \left| \Re \frac{\zeta e^{i\alpha}}{(\zeta - r)^{n+1}} \right|^q |d\zeta| \right\}^{1/q} \]
and \( 1/q + 1/p = 1 \). In particular
\[ |f^{(n)}(z)| \leq H_{n,p}(r)E_{n-1,p}(\Re f). \]

For \( p = 2 \) \((q = 2)\) and \( p = 1 \) \((q = \infty)\) the function \( H_{n,p}(r) \) has been calculated explicitly in [10]. We refer to [10] for the connection of (1.3) and the famous Hadamard-Borel-Carathéodory inequality:
\[ |f(z) - f(0)| \leq \frac{2}{1 - |z|^2} \sup_{|\zeta| < 1} \Re[f(\zeta) - f(0)]. \]

The aim of this paper is to obtain some explicit estimations of \( H_{n,p}(r) \) for general \( p \). The results of this paper are

**Theorem 1.2** (Main theorem). Let \( 1 \leq p \leq \infty \) and let \( q \) be its conjugate. Let \( f \) be analytic on the unit disk \( U \) with \( \Re f \in h^p(U) \), \( 1 \leq p \leq \infty \). Further, let \( n \geq 1 \), and let \( \mathcal{P}_l \) be a polynomial of degree \( l \leq n - 1 \). We have the following sharp inequality
\[ |f^{(n)}(z)| \leq C_{p,n}(1 - r^2)^{-1/p-n}\|\Re(f - \mathcal{P}_l)\|_{h^p}, \]
where
\[ C_{p,n} = \frac{n!}{\pi} 2^{n+1-1/q} \max_{0 \leq \beta \leq \pi/2} F_{q}^{1/q}(\beta) \]
and
\[ F_{q}(\beta) = \int_{0}^{\pi} |\sin((n+1)-2/q) v \cos[v(n+1) + \beta - \frac{\pi}{2}(n-1)]|^{q} dv. \]
In particular
\[ |f^{(n)}(z)| \leq C_{p,n}(1 - r^2)^{-1/p-n}E_{n-1,p}(\Re f). \]
Remark 1.3. In connection with Theorem 1.2 we conjecture that (c.f. Conjecture 4.1)

\[ \max_{0 \leq \beta \leq \pi/2} F_q(\beta) = \max\{F_q(0), F_q(\pi/2)\}. \]

We have the solution for \( q = 1 \) presented in Theorem 1.4. We list some known partial solutions.

- The Hilbert case (see [10, Eq. 5.5.3] or [11, Eq. (6.1.4)]: for \( q = 2 \) and all \( n \), the corresponding function is

\[ F_q(\beta) = \frac{2^n}{\pi^{3/4}} \sqrt{\frac{\Gamma[1/2 + n]}{\Gamma[1 + n]}}. \]

- For \( q = \infty \) and all \( n \) ([10, Eq. 5.4.2] for \( \gamma = 1 \)), \( F_q(\beta) = 2^{n+1} \).

- For \( n = 1 \) and all \( q \), (see [8]) we have

\[ \max_{0 \leq \beta \leq \pi/2} F_q(\beta) = \begin{cases} F_q(0), & \text{if } q > 2, \\ F_q(\pi/2), & \text{if } q \leq 2. \end{cases} \]

We also refer to related sharp inequalities for the derivatives of analytic functions defined in the unit disk [13].

Theorem 1.4. Let \( f \) be analytic on the unit disk \( U \) with bounded real part \( \Re f \) and assume that \( \beta \in [0, \pi] \). Then

\[ |f^{(n)}(z)| \leq C_n (1 - |z|^2)^{-n} O_{n, \Re f}(U), \quad (1.8) \]

where

\[ C_n = \begin{cases} \frac{1}{n!} \frac{(2n)!}{(n!)^2}, & \text{if } n = 2m - 1; \\ \frac{1}{n!} \max\{F(\beta) : 0 \leq \beta \leq \pi\}, & \text{if } n = 2m. \end{cases} \]

\[ F(\beta) = \frac{2^n}{n} \sum_{k=1}^{n+1} \frac{\sin^{n+1} \left( \frac{k\pi - \beta}{n+1} \right)}{k}, \]

and

\[ O_{n, \Re f}(U) = \inf_{P \in P_{n-1}} O_{\Re (f - P)}(U) \]

and \( O_{\Re f}(U) \) is the oscillation of \( \Re f \) on the unit disk \( U \).

Remark 1.5. If \( w = \Re f \) is a real harmonic function, where \( f \) is an analytic function defined on the unit disk, then the Bloch constant of \( w \) is defined by

\[ \beta_w = \sup_{z \in U} (1 - |z|^2)|\nabla w(z)| = \sup_{z \in U} (1 - |z|^2)|f'(z)| \]

and is less than or equal to \( C_1 = 4/\pi \) provided that the oscillation of \( w \) in the unit disk is \( \leq 1 \). This particular case is well known in the literature see e.g. [3] 6 [7]. In a similar manner we define the Bloch constant of order \( n \) of a harmonic function \( w = \Re f \):

\[ \beta_{n, w} = \sup_{z \in U} (1 - |z|^2)^n |f^{(n)}(z)| \]

and by the previous corollary we find out that \( \beta_{n, w} \leq C_n \) provided \( n \) is an odd integer and the oscillation \( O_{n, \Re f}(U) \) is at most 1.

The following theorem improves one of the main results in [9] (see [9] Corollary 7.1).
Corollary 1.6. Let $\Omega$ be a subdomain of $\mathbb{C}$. Let $z \in \Omega$, and assume that $a_z \in \partial \Omega$ such that $|z - a_z| = d_z = \text{dist}(z, \partial \Omega)$ and that $[\zeta, a_z]$ is the maximal interval containing $z$ with $d_{\zeta} = |\zeta - a_z|$. Let $f$ be a holomorphic function in $\Omega$ with its real part in Lebesgue space $L^\infty(\Omega)$ and let $\|\text{Re} f\|_{L^\infty(\Omega)} \leq 1$. Then the inequality
\[ d_z^n |f^{(n)}(z)| \leq \frac{C_n}{(2 - d_z/d_{\zeta})^n}, \quad z \in \Omega \] (1.9)
holds with $C_n := C_{\infty,n}$ defined in (1.6). In particular,
\[ d_{2^{m-1}} |f(2^{m-1}) (z)| \leq \frac{1}{n!} \left( \frac{(2m)!}{2m!} \right)^2 \frac{1}{(2 - d_z/d_{\zeta})^n}, \quad z \in \Omega. \] (1.10)

Proof. Let $z \in \Omega$ and assume that $\zeta \in \Omega$ satisfies the condition of the theorem. Then $D_{\zeta} := \{ w : |w - \zeta| < d_{\zeta} \} \subset \Omega$. Define $g(w) = f(\zeta + wd_{\zeta})$, $w \in \mathbb{U}$. Then $\text{Re} g \in h^\infty(\mathbb{U})$ and $g^{(n)}(w) = d_z^n f^{(n)}(\zeta + wd_{\zeta})$. By the maximum principle we have
\[ \|\text{Re} f\|_{h^\infty(D_{\zeta})} \leq \|\text{Re} f\|_{L^\infty(\Omega)}. \]
By applying Theorem 1.2 and Theorem 1.4 to $g$ we have
\[ (1 - |w|^2)^n d_z^n |f^{(n)}(\zeta + wd_{\zeta})| \leq C_n. \]
As $z \in [\zeta, a_z]$ it follows that $z = \zeta + s(a_z - \zeta) = \zeta + wd_{\zeta}$, where $w = se^{i\varphi} \in \mathbb{U}$. Since $d_{\zeta} = (1 - s)^{-1}d_z$, and $|w| = |(z - \zeta)/d_z| = s = (d_{\zeta} - d_z)/d_z$ we obtain that
\[ (1 - s^2)^n (1 - s^{-n}) d_z^n |f^{(n)}(z)| \leq C_n, \]
and
\[ d_z^n |f^{(n)}(z)| \leq C_n \frac{1}{(1 + s)^n} = C_n \frac{1}{(2 - d_z/d_{\zeta})^n}. \]

Remark 1.7. In [9] Corollary 7.1 Kresin and Maz'ya proved that
\[ \lim_{\epsilon \to 0^+} \sup_{z \in D_{\epsilon}} d_z^{2^{m-1}} |f(2^{m-1})(z)| \leq 2^{-n} C_n, \] (1.11)
under the condition that $\Omega$ is a planar domain with certain smoothness condition on the boundary, namely assuming that there is $r > 0$ such that for $a \in \partial \Omega$, there is a disk $D_a \subset \Omega$ of radius $r$ with $a \in D_a$. Then $d_{\zeta} \geq r$ for $z \in \Omega$. The inequality (1.11) follows by using (1.6) and letting $\epsilon \to 0$.

2. Proof of Theorem 1.2

In view of (1.6), we deal with the function
\[ I_\alpha(r) = \int_0^{2\pi} \left| \frac{e^{i(\alpha + t)}}{r e^{i(\alpha + t)}} \right|^q dt, \quad 0 \leq r < 1. \] (2.1)

By making use of the change
\[ e^{it} = \frac{r - e^{is}}{1 - re^{is}}, \]
we obtain
\[ dt = \frac{1 - r^2}{|1 - re^{is}|^2} ds \]
and
\[ r - e^{it} = \frac{(1 - r^2)e^{is}}{1 - re^{is}}. \]
We arrive at the integral
\[
I_\alpha(r) = \int_0^{2\pi} (1 - r^2)^{1 - q - nq} \left( 1 + r^2 - 2r \cos s \right)^{-1 + q} \left| \Re \left[ e^{i(\alpha + s)} (e^{i\alpha} - r)^{-1 + n} \right] \right|^q \, ds
\]
\[
= (1 - r^2)^{1 - q - nq} \int_0^{2\pi} f_\alpha(r, e^{is}) \, ds
\]
where
\[
f_\alpha(z, e^{is}) = \left| \Re \left[ e^{i(\alpha + s)} (z - e^{i\alpha})^{n-1} \right] \right| z - e^{i\alpha} |z - e^{i\alpha}|^{2q-2}.
\]
In order to continue, let’s prove first two lemmas.

**Lemma 2.1.** \( f_\alpha \) is subharmonic in \( z \).

**Proof.** We refer to [12, Chapter 4] and [5, Chapter I § 6] for some basic properties of subharmonic functions. Recall that a continuous function \( g \) defined on a region \( G \subset \mathbb{C} \) is subharmonic if for all \( w_0 \in G \) there exists \( \varepsilon > 0 \) such that
\[
g(w_0) \leq \frac{1}{2\pi} \int_0^{2\pi} g(w_0 + re^{it}) \, dt, \quad 0 < r < \varepsilon.
\]
(2.2)

If \( g(w_0) = 0 \), since \( g \) is non-negative, then (2.2) holds. If \( g(w_0) > 0 \), then there exists a neighborhood \( U \) of \( w_0 \) such that \( g \) is of class \( C^1(U) \) and \( g(w) > 0 \) for \( w \in U \). Thus if \( g \) is \( C^2 \) where it is positive, then it is enough to check that the Laplacian is non-negative there.

Let \( w = e^{i\frac{\alpha + \pi}{n} \pi} z - e^{i\alpha} \) and define
\[
g(w) := f_\alpha(z, e^{i\alpha}) = \left| \Re \left[ w^{n-1} \right] \right|^q |w|^{2q-2}.
\]
Assume that \( \Re(w^{n-1}) > 0 \). Then
\[
g_w = 2^{-q} \bar{w} (\bar{w}w)^{q-4} (\bar{w}^{n-1} + w^{n-1})^{q-2} [(-1 + q)\bar{w}^n w + (-1 + nq)\bar{w}^n w^n].
\]
Further
\[
g_w \bar{w} = \frac{(q - 1)(\bar{w}w)^{q-4} (\bar{w}^{n-1} + w^{n-1})^{q-2}}{2q+2}
\]
\[
\times ((-1 + nq)\bar{w}^2 w^2 + (-1 + nq)\bar{w}^2 w^{2n} + (-2 + q + n^2 q)\bar{w}^{1+n} w^{1+n}).
\]
Observe next that
\[
\frac{(q - 1)(\bar{w}w)^{q-4} (\bar{w}^{n-1} + w^{n-1})^{q-2}}{2q+2} = \frac{(q - 1)|w|^{2q-8} (\Re(w^{n-1}))^{q-2}}{16} \geq 0
\]
and
\[
(-1 + nq)\bar{w}^2 w^2 + (-1 + nq)\bar{w}^2 w^{2n} + (-2 + q + n^2 q)\bar{w}^{1+n} w^{1+n}
\]
\[
= (-2 + q + n^2 q)|w|^{2n+2} - 2(-1 + nq)\Re(\bar{w}^{1+n} w^{1+n})
\]
\[
\geq (n - 1)^2 |w|^{2n+2} \geq 0.
\]
Similarly we treat the case \( \Re(w^{n-1}) < 0 \). Therefore \( \Delta g = 4 g_w \bar{w} \geq 0 \) for \( \Re(w^{n-1}) \neq 0 \). This implies that \( g \) is subharmonic in the whole of \( \mathbb{C} \). Since
\[
f_\alpha(z, e^{i\alpha}) = g(e^{i\frac{\alpha + \pi}{n} \pi} z - e^{i\alpha} e^{-i(\frac{\alpha + \pi}{n} \pi)}),
\]
we have that \( \Delta f_\alpha(z, e^{i\alpha}) = \Delta g(az + b) \) which implies that \( z \to f_\alpha(z, e^{i\alpha}) \) is subharmonic. \( \square \)
Lemma 2.2. For \( \alpha \in [0, \pi] \) we have
\[
I_\alpha(r) \leq (1 - r^2)^{1-q-nq} \max_{0 \leq t \leq 2\pi} \int_0^{2\pi} f_\alpha(e^{it}, e^{is})ds
\]
\[
= (1 - r^2)^{1-q-nq} \int_0^{2\pi} f_\beta(1, e^{is})ds,
\]
for some \( \beta \) possibly different from \( \alpha \).

Proof. Since \( z \to f_\alpha(z, e^{it}) \) is subharmonic in \(|z| < 1\), and \((t, z) \to f_\alpha(z, e^{it})\) is continuous in \([0, \pi] \times U\), then the integral mean \( I(z) = \int_0^{2\pi} f_\alpha(z, e^{it})ds \) is a subharmonic function in \(|z| < 1\). Therefore
\[
\int_0^{2\pi} f_\alpha(z, e^{is})ds \leq \max_t \int_0^{2\pi} f_\alpha(e^{it}, e^{is})ds.
\]
Since
\[
f_\alpha(e^{it}, e^{is}) = |\Re[e^{i(\alpha+s)(e^{it} - e^{is})n^{-1})]}| e^{it} - e^{is}|^{2q-2}
\]
\[
= |\Re[e^{i(\beta+u)}(1 - e^{iu})n^{-1})]| |1 - e^{iu}|^{2q-2},
\]
for \( u = s - t \) and \( \beta = \alpha + nt \) we obtain the second statement of the lemma. \( \square \)

Proof of Theorem 1.2. As above, we have
\[
|\Re[e^{i(\alpha+s)(e^{it} - e^{is})n^{-1})]}|^{q}
\]
\[
= 2^{(n-1)q/2} \left| \cos[\beta - \frac{\pi}{2}(n-1) + \frac{n+1}{2}2u] \right|^q (1 - \cos u)^{(n-1)q/2}
\]
and
\[
(1 - e^{iu})^{2q-2} = 2^{q-1} (1 - \cos u)^{q-1}.
\]
In view of Lemma 2.2 we have
\[
I_\alpha(r) = 2^{(n+1)q/2-1} (1 - r^2)^{1-(1+n)q} F_q(\beta),
\]
where
\[
F_q(\beta) = \int_0^{2\pi} (1 - \cos u)^{(n+1)q/2-1} \left| \cos[\frac{n+1}{2} + \beta - \frac{\pi}{2}(n-1)] \right|^q du. \tag{2.3}
\]
Moreover
\[
F_q(\beta) = 2^{(n+1)q/2} \int_0^{\pi} |\phi_\beta(v)|^q dv
\]
where
\[
\phi_\beta(v) = \sin^{(n+1)-2/q} v \cos[v(n+1) + \beta - \frac{\pi}{2}(n-1)]. \tag{2.4}
\]
It can be proved easily that \( F_q(\beta) = F_q(\pi - \beta) \). The last fact implies that it is enough to find the maximum in \([0, \pi/2]\). \( \square \)

3. The case \( q = 1 \) and the proof of Theorem 1.4

We divide the proof into two cases and use the notation \( F = F_q \).
3.1. The odd \( n \).

For \( n = 2m - 1 \) and \( q = 1 \) we have

\[
F(\beta) = 2^m \int_0^\pi \sin^{n-1} v \cos[(n+1)v + \beta]|dv.
\]  

(3.1)

Then

\[
\frac{d}{dx} \cos(\beta + nx) \sin^nx_n = (-1)^{m-1} \phi_\beta(x) = \sin^{a-1} x \cos[\beta + (n+1)x].
\]

Since \( F \) is \( \pi \)-periodic we can assume that \(-\pi/2 \leq \beta \leq \pi/2\). Assume that \( 0 \leq \beta < \pi/2 \) (the second case can be treated similarly). Then \( \cos[\beta + 2mx] \geq 0 \) and \( 0 \leq x \leq \pi \) if and only if one of the following relations hold

- \( \beta \leq \beta + 2mx < \pi/2 \)
- \( -\pi/2 + 2k\pi < \beta + 2mx < \pi/2 + 2k\pi, \) for \( 1 \leq k \leq m - 1 \) or
- \( -\pi/2 + 2m\pi < \beta + 2mx < \beta + 2m\pi \)

or, what is the same, if:

- \( a_0 = 0 < x < b_0 = \frac{\pi - 2\beta}{4m} \)
- \( a_k := \frac{\pi + 4k\pi - 2\beta}{4m} < x < b_k := \frac{\pi + 4k\pi - 2\beta}{4m}, \) for \( 1 \leq k \leq m - 1 \) or
- \( a_m := \frac{-\pi + 4m\pi - 2\beta}{4m} < x < b_m := \pi. \)

From (3.1) for

\[
g_n(x) = \frac{\cos(\beta + nx) \sin^n x}{n},
\]

because \( g_n(\pi) - g_n(0) = 0 \), we have

\[
F(\beta) = 2^m \int_0^\pi |\phi_\beta(v)|dv = 2 \cdot 2^m \int_{0 \leq v \leq \pi: \phi_\beta(v) \geq 0} \phi_\beta(v)dv - 2^m \int_0^\pi \phi_\beta(v)dv = 2^{m+1} \int_{0 \leq v \leq \pi: \phi_\beta(v) \geq 0} \phi_\beta(v)dv.
\]

Therefore

\[
\frac{F(\beta)}{2^{m+1}} = \sum_{k=0}^m [g_n(b_k) - g_n(a_k)] = g_n(b_0) - g_n(a_m) + \sum_{k=1}^{m-1} [g_n(b_k) - g_n(a_k)].
\]

But for \( 1 \leq k \leq m \)

\[
g_n(b_k) = \frac{\sin^{2m} b_k}{n} \quad \text{and} \quad g_n(a_k) = -\frac{\sin^{2m} a_k}{n}.
\]

Therefore

\[
F(\beta) = \frac{2^{m+1}}{n} \sum_{k=1}^{2m} \sin^{2m} \left[ \frac{-2\beta + (2k - 1)\pi}{4m} \right] = \frac{2^{m+1}}{n} \sum_{k=1}^{2m} \sin^{2m} \left[ \frac{\gamma + k\pi}{2m} \right],
\]

where \( \gamma = -\pi/2 - \beta \). Now by invoking [2, Lemma 3.5], we have

\[
f(\beta) := \sum_{k=1}^{2m} \sin^{2m} \left[ \frac{\gamma + k\pi}{2m} \right] = \frac{2}{B(\frac{1}{2}, m)},
\]  

(3.2)

and therefore

\[
F(\beta) = \frac{4m}{n^{2m}} \binom{2m}{m}.
\]
3.2. The even \( n \). For \( n = 2m \) and \( q = 1 \)

\[
F(\beta) = 2^{(2m+1)/2} \int_0^{\pi} \sin^{n-1} v |\sin[v(n+1)+\beta]| dv. \tag{3.3}
\]

Let

\[
g(x) = \frac{\sin(\beta + nx) \sin x}{n}.
\]

Since

\[
d\frac{dx}{dx} g(x) = (-1)^m \phi_\beta(x) = \sin^{n-1} x \sin[\beta + (n+1)x],
\]

from (3.3) and

\[
F(\beta) = 2^{(n+1)/2} \int_0^{\pi} |\phi_\beta(v)| dv
\]

we obtain

\[
F(\beta) = 2^{(n+1)/2} \int_0^{\pi} \phi_\beta(v) dv - 2^{(n+1)/2} \int_0^{\pi} \phi_\beta(v) dv
\]

we obtain

\[
F(\beta) = 2^{(n+1)/2} \int_0^{\pi} \phi_\beta(v) dv - 2^{(n+1)/2} \int_0^{\pi} \phi_\beta(v) dv.
\]

After some elementary transformations we obtain

\[
2^{-(n+1)/2} m F(\beta) = \sum_{k=0}^{n+1} \sin^{1+n} \left[ \frac{-\beta + k\pi}{1 + n} \right]. \tag{3.4}
\]

This fraction the proof of Theorem 1.4.

4. APPENDIX

In this section we include a possible strategy how to determine the maximum of the function \( F \) in \([0, \pi/2]\) provided that \( n = 2m \) is an even integer. First of all

\[
2^{-(2m+1)/2} m F'(\beta) = - \sum_{k=1}^{1+2m} \cos \left[ \frac{-\beta + k\pi}{1 + 2m} \right] \sin^{2m} \left[ \frac{-\beta + k\pi}{1 + 2m} \right].
\]

Let

\[
h_k(\beta) = \cos \left[ \frac{-\beta + k\pi}{1 + 2m} \right] \sin^{2m} \left[ \frac{-\beta + k\pi}{1 + 2m} \right].
\]

Then for \( 1 \leq k \leq 2m \), \( h_k(0) + h_{2m+1-k}(0) = 0 \), \( h_{2m+1}(0) = 0 \) and \( h_{m+1}(\pi/2) = 0 \) and \( h_{m+1}(\pi/2) = 0 \). It follows that

\[
F'(0) = F'(\frac{\pi}{2}) = 0.
\]

Thus 0 and \( \pi/2 \) are stationary points of \( F \).

It can be shown that for \( \gamma = \beta + \pi/2 \) and for \( \pi/2 \leq \gamma \leq \pi \)

\[
\frac{m F(\beta)}{2^{(n+1)/2}} = \sum_{j=0}^{m} (-1)^j \frac{(1+n) \cos \left[ \frac{(1+2j)\gamma}{1+n} \right]}{2^n (m-j)} + 2 \sin^{1+n} \left[ \frac{\gamma - \pi/2}{1 + n} \right]. \tag{4.1}
\]
We expect that the formula (4.1) can be more useful than (3.4) in finding the maximum of the function $F(\beta)$, however it seems that the corresponding problem is hard. By using the software "Mathematica 8" we can see that $F(0) < F(\beta) < F(\pi/2)$ provided that $n = 4k$ and $0 < \beta < \pi/2$ and $F(\pi/2) < F(\beta) < F(0)$ provided that $n = 4k + 2$ and $0 < \beta < \pi/2$ (cf. Conjecture 4.1). We do not have a proof of the previous fact but we include in this paper the following special cases.

4.1. The case $m = 1$ ($n = 2$) and $q = 1$. We have

$$F(\beta) = \frac{\sqrt{2}}{2} \left( 3\sqrt{3} \cos \frac{\beta}{3} + 4 \sin \frac{3\beta}{3} \right)$$

and

$$F'(\beta) = -\sin \frac{\beta}{2} \left( \sqrt{3} - 2 \sin \frac{2\beta}{3} \right).$$

Thus $F'(\beta) = 0$ if and only if $\beta = 0$ or $\beta = \pi/2$. The minimum of $F(\beta)$ is $F(\pi/2) = \frac{5\sqrt{2}}{2}$ and the maximum is $F(0) = \frac{3\sqrt{2}}{2}$.

4.2. The case $m = 2$ ($n = 4$) and $q = 1$. In this case

$$F(\beta) = \frac{\sqrt{2}}{8} \left( 10\sqrt{5} + 2\sqrt{5} \cos[\beta/5] - 5\sqrt{5} - 2\sqrt{5} \cos[3\beta/5] + 16 \sin[\beta/5]^3 \right).$$

Then it can be proved that $F$ is increasing in $[0, \pi/2]$ and

$$F(0) = 5/4 \sqrt{12.5 + \sqrt{5}} \approx 4.79845 < F(\pi/2) = \sqrt{381/32 + 5\sqrt{5}} \approx 4.80485.$$

By differentiating the subintegral expression (1.7) w.r.t $\beta$ we can easily conclude that $\beta = 0$ and $\beta = \pi/2$ are stationary points of $F$ provided that $q \geq 1$ and $n \in \mathbb{N}$. This and some experiments with the software "Mathematica 8" leads to the following conjecture

**Conjecture 4.1.** Denote by $[a]$ the integer part of $a$. We conjecture that:

- $F_q$ is decreasing on $[0, \frac{\pi}{2}]$ for $q > 2$
- $F_q$ is nondecreasing (nonincreasing) on $[0, \frac{\pi}{2}]$ for $q \leq 2$ and $\frac{(n+1)q}{2}$ is an even (odd) integer.

5. Appendix B

In this appendix we offer some numerical estimation that confirm that our conjecture is true, at least for $q = 1$. Whereas as is observed in (3.2) for $0 \leq \beta \leq \pi$

$$f(\beta) = f_n(\beta) = \sum_{k=1}^{s} \sin^s \left( -\beta + k\pi \right)$$

(5.1)

is the constant $2/B(\frac{1}{s}, \frac{1}{s})$ for even $s$, if $s = n + 1$ is odd the maximum exceeds $2/B(\frac{1}{s}, \frac{1}{s})$ by a tiny amount that is very nearly

$$\frac{4}{\pi} \cdot \frac{1}{s+2} \cdot \frac{2}{s+4} \cdot \frac{3}{s+6} \cdots \frac{s}{3s} = \frac{4}{\pi} \frac{s!!}{(3s)!!} = (27 + o(1))^{-s/2}$$

for large $s$. Here and later we use "$u!!" only for positive odd $u$ to mean the product of all odd integers in $[1, u]$; that is, $u!! := u!/(2^uv)$ where $u = 2v + 1$. In order to outline the proof of the last statement we do as follows.
For \( s = 2m + 1 \) we define the function \( g \) as follows
\[
g(x) := f(x + \frac{\pi}{2}) = g(-x) = -g(x + s\pi),
\]
which has a finite Fourier expansion in cosines of odd multiples of \( X := x/s \), namely
\[
f(x) = (-1)^{m} 2^{-s} \sum_{j=0}^{s} (-1)^{j} \frac{s^{j}}{j} \frac{\cos tX}{\sin \frac{\pi}{2}}
\]
where \( t = s - 2j \). We deduce from (5.1) that
\[
f(\beta) - f(\beta + \pi) = 2 \sin s \left( \frac{\beta}{s} \right),
\]
from which it follows that \( g(x) \) is maximized somewhere in \( |x| \leq \pi/2 \), but that changing the optimal \( x \) by a small integral multiple of \( \pi \) reduces \( g \) by a tiny amount; this explains the near-maxima we observed at \( x = \pm \pi \) for \( 2|m \), and indeed the further oscillations for both odd and even \( m \) that we later noticed as \( s \) grows further.

This also suggests that in and near the interval \( |x| \leq \pi/2 \) our function \( g \) should be very nearly approximated for large \( s \) by an even periodic function \( \tilde{g}(x) \) of period \( \pi \). We next outline the derivation of such an approximation, with \( \tilde{g}(x) \) having an explicit cosine-Fourier expansion
\[
\tilde{g}(x) = g_0 + \frac{1}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l s^{l-1}}{(2l)!} \cos \left( \frac{(2l+1)x}{s} \right)
\]
where \( g_0 = 2/B(\frac{1}{2}, \frac{s}{2}) \) and, for \( l > 0 \),
\[
g_l = (-1)^{m+l-1} \frac{4}{\pi} \frac{s^{l+1}}{2l+1} \frac{(2l-1)!!}{((2l+1)!!)}
\]
with the double-factorial notation defined as above. Thus
\[
\tilde{g}(x) = g_0 + \frac{4s}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l (2l-1)!!}{((2l+1)!!)} \left( \frac{s^{l}}{(3s)!!} \cos 2x - \frac{1}{3} \frac{(3s)!!}{(5s)!!} \cos 4x + \frac{1}{5} \frac{(5s)!!}{(7s)!!} \cos 6x \pm \cdots \right).
\]
For large \( s \), this is maximized at \( x = 0 \) or \( x = \pm \pi/2 \) according as \( m \) is even or odd. Since we already know by symmetry arguments that \( g'(0) = g'(|\pi/2|) = 0 \), this point or points will also be where \( g \) is maximized, once it is checked that \( g - \tilde{g} \) and its first two derivatives are even tinier there.

The key to all this is the partial-fraction expansion of the factor \( 1/\sin(\pi t/2s) \) in the Fourier series of \( g \), obtained by substituting \( \theta = \pi t/2s \) into
\[
\frac{1}{\sin \pi \theta} = \frac{1}{\pi} \sum_{l=-\infty}^{\infty} \frac{(-1)^l}{\theta - l}
\]
with the conditionally convergent sum interpreted as a principal value or Cesàro limit etc. On the other hand the main term, for \( l = 0 \), yields the convolution of \( \cos^s(x/s) \) with a symmetrical square wave, which is thus maximized at \( x = 0 \) and almost constant near \( x = 0 \); we identify the constant with \( 2/B(\frac{1}{2}, \frac{s}{2}) \) using the known product formula for
\[
\int_{-\pi/2}^{\pi/2} \cos X dX.
\]
The new observation is that each of the error terms \( (-1)^l / (\theta - l) \) likewise yields the convolution with a square wave of
\[
(-1)^l \cos(2lx) \cos^s(x/s).
\]
If we approximate this square wave with a constant, we get the formula for $g_1$ displayed above, via the formula for the $s$-th finite difference of a function $1/(j_0 - j)$.

The error in this approximation is still tiny (albeit not necessarily negative) because $\cos^s(x/s)$ is minuscule when $x$ is within $\pi/2$ of the square wave’s jump at $\pm \pi s/2$.

We’ve checked these approximations numerically to high precision (modern computers and gp make this easy) for $s$ as large as 100 or so, in both of the odd congruence classes mod 4, and it all works as expected; for example, when $s = 99$ we have $f(0) - g_0 = 2.57990478176660 \ldots 10^{-70}$, which almost exactly matches the main term $g_1 = (4/\pi) 99! 99!!/297!!$ but exceeds it by $5.9110495 \ldots 10^{-102}$, which is almost exactly $g_2 = (4/\pi) 99! 297!!/(3 \cdot 495!!)$ but too large by $7.92129 \ldots 10^{-120}$, which is almost exactly $g_3 = (4/\pi) 99! 495!!/(5 \cdot 693!!)$, etc.; and likewise for $s = 101$ except that the maximum occurs at $\beta = \pi/2$ and is approximated by an alternating sum $g_1 - g_2 + g_3 \ldots$ (actually here this approximation is exact because $x = 0$).

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