Note on the energy of regular graphs *

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Abstract

For a simple graph $G$, the energy $E(G)$ is defined as the sum of the absolute values of all the eigenvalues of its adjacency matrix $A(G)$. Let $n, m$, respectively, be the number of vertices and edges of $G$. One well-known inequality is that $E(G) \leq \lambda_1 + \sqrt{(n - 1)(2m - \lambda_1)}$, where $\lambda_1$ is the spectral radius. If $G$ is $k$-regular, we have $E(G) \leq k + \sqrt{k(n - 1)(n - k)}$. Denote $E_0 = k + \sqrt{k(n - 1)(n - k)}$. Balakrishnan [Linear Algebra Appl. 387 (2004) 287–295] proved that for each $\epsilon > 0$, there exist infinitely many $n$ for each of which there exists a $k$-regular graph $G$ of order $n$ with $k < n - 1$ and $E(G) < E_0 < \epsilon$, and proposed an open problem that, given a positive integer $n \geq 3$, and $\epsilon > 0$, does there exist a $k$-regular graph $G$ of order $n$ such that $\frac{E(G)}{E_0} > 1 - \epsilon$. In this paper, we show that for each $\epsilon > 0$, there exist infinitely many such $n$ that $\frac{E(G)}{E_0} > 1 - \epsilon$. Moreover, we construct another class of simpler graphs which also supports the first assertion that $\frac{E(G)}{E_0} < \epsilon$.

Keywords: graph energy; regular graph; Paley graph; open problem

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1 Introduction

Let $G$ be a simple graph with $n$ vertices and $m$ edges. Denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ the eigenvalues of $G$. Note that $\lambda_1$ is called the spectral radius. The energy of $G$ is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. For more information on graph energy we refer to [5, 6], and for terminology and notations not defined here, we refer to Bondy and Murty [2].

One well-known inequality for the energy of a graph $G$ is that $E(G) \leq \lambda_1 + \sqrt{(n-1)(2m-\lambda_1)}$. If $G$ is $k$-regular, we have $E(G) \leq k + \sqrt{k(n-1)(n-k)}$. Denote $E_0 = k + \sqrt{k(n-1)(n-k)}$. In [1], Balakrishnan investigated the energy of regular graphs and proved that for each $\epsilon > 0$, there exist infinitely many $n$ for each of which there exists a $k$-regular graph $G$ of order $n$ with $k \leq n - 1$ and $\frac{E(G)}{E_0} < \epsilon$. In this paper, we construct another class of simpler graphs which also support the above assertion. Furthermore, we show that for each $\epsilon > 0$, there exist infinitely many $n$ satisfying that there exists a $k$-regular graph $G$ of order $n$ with $k < n - 1$ and $\frac{E(G)}{E_0} > 1 - \epsilon$, which answers the following open problem proposed by Balakrishnan in [1]:

Open problem. Given a positive integer $n \geq 3$ and $\epsilon > 0$, does there exist a $k$-regular graph $G$ of order $n$ such that $\frac{E(G)}{E_0} > 1 - \epsilon$ for some $k < n - 1$?

2 Main results

Throughout this paper, we denote $V(G)$ the vertex set of $G$ and $E(G)$ the edge set of $G$. Firstly, we will introduce the following useful result given by So et al. [3].

Lemma 1 Let $G - e$ be the subgraph obtained by deleting an edge $e$ of $E(G)$. Then

$$E(G) \leq E(G - e) + 2.$$

We then formulate the following theorem by employing the above lemma.

Theorem 1 ([1]) For any $\epsilon > 0$, there exist infinitely many $n$ for each of which there exists a $k$-regular graph $G$ of order $n$ with $k < n - 1$ and $E(G)/E_0 < \epsilon$. 

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**Proof.** Let $q > 2$ be a positive integer. We take $q$ copies of the complete graph $K_q$. Denote by $v_1, \ldots, v_q$ the vertices of $K_q$ and the corresponding vertices in each copy by $v_1[i], \ldots, v_q[i]$, for $1 \leq i \leq q$. Let $G_{q^2}$ be a graph consisting of $q$ copies of $K_q$ and $q^2$ edges by joining vertices $v_j[i]$ and $v_j[i + 1]$ (for $1 \leq i < q$), $v_j[q]$ and $v_j[1]$ where $1 \leq j \leq q$. Obviously, the graph $G_{q^2}$ is $q + 1$ regular. Employing Lemma 1, deleting all the $q^2$ edges joining two copies of $K_q$, we have $E(G_{q^2}) \leq E(qK_q) + 2q^2$. Thus, $E(G_{q^2}) \leq 2q(q - 1) + 2q^2$. Then, it follows that

$$\frac{E(G_{q^2})}{E_0} \leq \frac{4q^2 - 2q}{q + 1 + \sqrt{(q + 1)(q^2 - 1)(q^2 - q - 1)}} \leq \frac{4q^2 - 2q}{(q^2 - q - 1)\sqrt{q} + 1} \to 0 \text{ as } q \to \infty.$$ 

Thus, for any $\varepsilon > 0$, when $q$ is large enough, the graph $G_{q^2}$ satisfies the required condition. The proof is thus complete.

**Theorem 2** For any $\varepsilon > 0$, there exist infinitely many $n$ satisfying that there exists a $k$-regular graph of order $n$ with $k < n - 1$ and $E(G)/E_0 > 1 - \varepsilon$.

**Proof.** It suffices to verify an infinite sequence of graphs satisfying the condition. To this end, we focus on the Paley graph (for details see [4]). Let $p \geq 11$ be a prime and $p \equiv 1 (mod 4)$. The Paley graph $G_p$ of order $p$ has the elements of the finite field $GF(q)$ as vertex set and two vertices are adjacent if and only if their difference is a nonzero square in $GF(q)$. It is well known that the Paley graph $G_p$ is a $(p - 1)/2$-regular graph. And the eigenvalues are $\frac{p - 1}{2}$ (with multiplicity 1) and $-\frac{1 \pm \sqrt{p}}{2}$ (both with multiplicity $\frac{p - 1}{2}$). Consequently, we have

$$E(G_p) = \frac{p - 1}{2} + \frac{-1 + \sqrt{p}}{2} \cdot \frac{p - 1}{2} + \frac{1 + \sqrt{p}}{2} \cdot \frac{p - 1}{2} = (p - 1)\frac{1 + \sqrt{p}}{2} > \frac{p^{3/2}}{2}.$$ 

Moreover, $E_0 = \frac{p - 1}{2} + \frac{p - 1}{2}(p - 1)(p - \frac{1}{2})$, we can deduce that

$$\frac{E(G_p)}{E_0} > \frac{\frac{p^{3/2}}{2}}{\frac{p - 1}{2} + \sqrt{p} + 1} > \frac{\frac{p^{3/2}}{2}}{\frac{p}{2} + 2} \to 1 \text{ as } p \to \infty.$$ 

Therefore, for any $\varepsilon > 0$ and some integer $N$, if $p > N$, it follows that $E(G_p)/E_0 > 1 - \varepsilon$. The theorem is thus proved.
References

[1] R. Balakrishnan, The energy of a graph, *Lin. Algebra Appl.* **387** (2004) 287–295.

[2] J.A. Bondy, U.S. R. Murty, *Graph Theory*, Springer–Verlag, Berlin, 2008.

[3] W. So, M. Robbiano, N.M.M. de Abreu, I. Gutman, Applications of a theorem by Ky Fan in the theory of graph energy, *Lin. Algebra Appl.*, doi:10.1016/j.laa.2009.01.006.

[4] C. Godsil, G. Royle, *Algebraic Graph Theory*, Springer–Verlag, New York, 2001.

[5] I. Gutman, The energy of a graph: old and new results, in: Betten, A., Kohnert, A., Laue, R., Wassermann, A. (Eds.), *Algebraic Combinatorics and Applications*, Springer–Verlag, Berlin, (2001) 196–211.

[6] I. Gutman, X. Li, J. Zhang, *Graph Energy*, in: M. Dehmer, F. Emmert-Streib (Eds.), *Analysis of Complex Networks: From Biology to Linguistics*, Wiley-VCH Verlag, Weinheim, (2009) 145–174.