Towards Dynamic-Point Systems on Metric Graphs with Longest Stabilization Time

Leonid W. Dworzanski

National Research University Higher School of Economics, Myasnitskaya ul. 20, Moscow, Russia 101000

Abstract

A dynamical system of points moving along the edges of a graph could be considered as a geometrical discrete dynamical system or as a discrete version of a quantum graph with localized wave packets. We study the set of such systems over metric graphs that can be constructed from a given set of commensurable edges with fixed lengths. It is shown that there always exists a system consisting of a bead graph with vertex degrees not greater than three that demonstrates the longest stabilization time in such a set. The results are extended to graphs with incommensurable edges using the notion of ε-nets and, also, it is shown that dynamical systems of points on linear graphs have the slowest growth of the number of dynamic points.

Keywords: metric graphs, dynamical systems of points, longest stabilization time, saturation time

1. Introduction

A dynamical system of points (DP-system) moving along the edges of a metric graph, yet its dynamics has a discrete nature, could be considered as a simplified discrete model of a quantum graph with narrow localized wave packets [1,2]. A quantum graph is a metric graph equipped with functions on its edges, a differential operator acting on such functions, and matching conditions on its vertices [3]. Quantum graphs occurred as a model or tool in a number of problems in chemistry, physics, engineering, and mathematics since 1930s [4,5]. There exists a correspondence between the statistics of localized solutions on a quantum graph and the dynamics of a DP-system. Points in such a system may represent supports of Gaussian wave packets in a quantum graph and/or projection of wave propagation on medium geodesics [2].

Some results towards the characteristics of the dynamics of such systems were recently obtained in [6–9]. The growth of the number of points moving along edges and its asymptotics are studied for metric trees in [8] reducing the counting problem to counting the number of lattice points in an expanding polyhedra using Barnes multiple Bernoulli polynomials; and, for some special cases, in [2]. Further details and motivation for studying such systems, that emerges from mathematical physics and other fields, can be found in texts [3] and [10].

Recently, in [11], it was shown that, for a subclass of Timed-Arc Petri nets (TaPN-nets), it is possible to overapproximate the number of timers with different values in a TaPN-net by
the number of points in a DP-system; thus, asymptotical estimations obtained for the number of points in a DP-system in [7, 8] could be used to analyse behavioural properties related to timers in TaPN-nets. Additionally, the translation of a DP-system into a TaPN-net was developed and implemented in TAPAAL to permit the analysis of fine TCTL properties of a DP-system.

Additional motivation comes from the problem of upper bounds on the stabilization time for a DP-system on an arbitrary metric graph. While there exists asymptotical estimates for the growth of the number of points in a DP-system on an arbitrary metric graph, estimates for stabilization time of a DP-system are obtained only for two mostly trivial cases — star graphs and graphs with edges of equal length. In this paper, we develop an approach based on graph surgery to show that, in the set of all dynamic-point systems with longest stabilization time (LSTDP-systems) constructed from a fixed set of edges, there exists a DP-system on a metric graph with a specific structure (bead graphs) having a dynamic point at one of its terminal vertices. The results allow to take into account only the set of its edges and LSTDP-systems of a specific structure to study longest stabilization time.

Section 2 contains basic notions and definitions. In section 3, it is shown that longest stabilization time could not always be achieved using only tree metric graphs; also, we introduce notions of point-places and walk classes used in the next section. Section 4 demonstrates the existence of a bead graph with vertex degrees not higher than three among LSTDP-systems. Section 5 extends the result of the previous section to metric graphs with incommensurable edge lengths. Section 6 concludes the paper with some further directions.

2. Preliminaries

A metric graph \( \Gamma \) is a graph consisting of set of vertices \( V \), set of undirected edges \( E \), and length function \( l \) mapping each edge \( e = \{v_1, v_2\} \in E \) to a positive real, i.e., \( l : E \rightarrow \mathbb{R}_+ \). For technical convenience, each undirected edge \( e = \{v_1, v_2\} \in E \) may be considered as a pair of arcs \( \langle v_1, v_2 \rangle \) and \( \langle v_2, v_1 \rangle \), and both arc lengths coincide with the length of \( e \); both notions will be used in the paper interchangeably to shorten some lengthy technical explanations.

The arc opposite to arc \( a = \langle y, x \rangle \) is denoted by \( \bar{a} \), i.e., \( \bar{a} = \langle x, y \rangle \). For two points \( x \) and \( y \) on the graph, metric \( \rho(x, y) \) is the shortest distance between them, where distance is measured along the edges of the graph additively. A walk is a finite or infinite sequence of edges (arcs for directed graphs) which joins a sequence of vertices. A trail (path) is a walk in which all edges (vertices) are distinct.

The set of all walks from vertex \( v \) to vertex \( v' \) is denoted by \( W(v; v') \). For a walk \( w \), the length \( l(w) \) is the sum of lengths of all the arc entries in \( w \). The support of walk \( w \) is the set of all arcs in \( w \) and is denoted by \( S(w) \). The (directed) multisupport of walk \( w \) in graph \( \Gamma \) is a new (directed) graph \( \bar{S}(w) \) obtained from \( \Gamma \) such that it has the same vertices \( V \) as \( \Gamma \) and, for each entry of arc \( \langle v_i, v_j \rangle \) in \( w \), we introduce a new edge \( \{v_i, v_j\} \) (new arc \( \langle v_i, v_j \rangle \)) into \( \bar{S}(w) \), i.e., the number of edges (arcs) in \( \bar{S}(w) \) is equal to the number of arc entries in \( w \). Note that a walk may contain multiple entries of an arc.

For a metric graph \( \Gamma \), the dynamics of a system of dynamic points \( P_T \) on \( \Gamma \) is defined as following. In the initial state, some vertices of \( \Gamma \) hold a dynamic point. When time starts to flow, each such point \( p \) located in vertex \( v \), for each edge \( e \) incident to \( v \), produces a point \( p' \) on each \( e \), and \( p \) disappears (intuitively, this corresponds to wave packet scattering); each produced point \( p' \) starts moving along corresponding \( e \). Note that if we consider \( e \) as a pair of directed arcs, then \( p' \) is generated on and moving along the arc outgoing from \( v \) respecting the arc direction. All points...
move with the same constant velocity; and, due to new points generation, some arcs may carry more than one point. When a moving point reaches vertex $v'$, again, on each outgoing arc incident to $v'$, a new point is generated. When more than one points reach the vertex simultaneously at $t$, on each outgoing arc, only one point is produced, as if only one point has reached the vertex at $t$; i.e., points met on a vertex fuse, and each coordinate of an arc can carry only one dynamic point. However, points do not collide anywhere on edges except vertices, i.e., if two points met on an edge, they both continue their movement towards their own directions. This becomes clearer if we consider the edge as a pair of arcs; then, the points, converging on an edge, are moving along separate opposite arcs. In Figure 1 the initial set of points consists of two points in vertices $v_1$ and $v_3$. The point in $v_1$ produces a new point on edge $\{v_1, v_2\}$, The point in $v_3$ produces points on edges leading to vertices $v_2, v_4, v_5, v_6, v_7$. After a time unit, there are no points in $v_1$ and $v_3$ (coloured gray), but there are points (coloured black) moving from $v_1$ and $v_3$ to their adjacent vertices.

More examples and further details on DP-systems on metric graphs and some of their extensions can be found in [2, 8, 9].

3. Cutting of a metric graph

The number of dynamic points in $\mathcal{P}_\Gamma$ at time $t$ is denoted by $N_{\mathcal{P}_\Gamma}(t)$. The number of points on edge $e$ at time $t$ is denoted by $N_e(t)$. For dynamical systems of points $\mathcal{P}_\Gamma$ and $\mathcal{P}_\Gamma'$, we say that the growth rate of $\mathcal{P}_\Gamma$ is equal or less than that of $\mathcal{P}_\Gamma'$, if $\forall t \in \mathbb{R}_+ \setminus \text{Coll} : N_{\mathcal{P}_\Gamma}(t) \leq N_{\mathcal{P}_\Gamma'}(t)$, where Coll is the (countable) set of time points when more than one dynamic points meet on a vertex; we exclude such time points as the number of dynamic points decreases for these moments, technically. In what follows, we discuss growth and stabilization implicitly omitting vertices collision time points Coll.

The stabilization time $t_s(\mathcal{P}_\Gamma)$ of DP-system $\mathcal{P}_\Gamma$ on graph $\Gamma$ with edges of commensurable lengths is the value of the period of time from the initial time point to the point in time when the number of dynamic points $N_{\mathcal{P}_\Gamma}(t)$ on $\Gamma$ has been stabilized $\mathcal{P}_\Gamma$, i.e., $\forall t \in \mathbb{R}_+ \setminus ((0, t_s(\mathcal{P}_\Gamma)) \cup \text{Coll}) : N_{\mathcal{P}_\Gamma}(t) = N_{\mathcal{P}_\Gamma}(t_s(\mathcal{P}_\Gamma))$. For a given set of edges $E$, let $\text{LSTD}_{\mathcal{P}_\Gamma}(E)$ be the set of DP-systems constructed from $E$ demonstrating the longest stabilization time ($\text{LSTD}_{\mathcal{P}_\Gamma}(E)$-systems); $\text{LSTD}_{\mathcal{P}_\Gamma}(E)$ is not necessarily a singleton.
Figure 2: Metric graph with a cycle $(v_1, v_4, v_5, v_6)$

Number $N_{P_1}(t)$ increases at time $t_0$ when the number of points simultaneously reached a vertex $v$ at $t_0$ is strictly less than the (out-) degree of $v$.

In this section, it is shown that, while for an arbitrary $DP$-system $P_\Gamma$, there is always a tree $DP$-system which growth rate is equal or less than thus of $P_\Gamma$; set $LSTDP(E)$ does not always contain a $DP$-system on a tree.

To begin with, we suggest to partition walks into equivalence classes under the following equivalence. Let $v$ be a vertex and $e = \{v_a, v_b\}$ be an edge of $\Gamma$. Let set $W(v, e)$ be the union of sets $W(v, v_a)$ and $W(v, v_b)$. Let us introduce an equivalence relation $\approx$ over set of walks $W(v, e)$ defined as follows:

- if walks $w_1$ and $w_2$ both end in $v_a$ or both end in $v_b$, then they are equivalent iff their lengths are congruent modulo $2l(e)$, i.e.,
  $$\forall (w_1, w_2) \in W(v, v_a)^2 \cup W(v, v_b)^2 : w_1 \approx w_2 \iff l(w_1) \equiv l(w_2) \pmod{2l(e)}$$

- if, w.l.o.g., $w_1$ ends in $v_a$, and $w_2$ ends in $v_b$, then they are equivalent iff the difference of their lengths is congruent to $l(e)$ modulo $2l(e)$, i.e.,
  $$\forall (w_1, w_2) \in W(v, v_a) \times W(v, v_b) : w_1 \approx w_2 \iff l(w_1) - l(w_2) \equiv l(e) \pmod{2l(e)}$$

For each edge $e = \{v_1, v_2\}$, set $W(v_0, e)$ is partitioned into equivalence classes under $\approx$. The arrival of point $p$ at vertex $v$ of $e$, moving from $v_0$ to $v$ along walk $w$, induces a new point on $e$ iff $w$ has the minimum length for its class $[w]_\approx$; i.e., only the shortest walks of $[w]_\approx$ induce a new point on $e$. Now, consider two arcs $(v_1, v_2)$ and $(v_2, v_1)$ as a set of point-places forming a loop; the point-places move around the loop along the arc directions with the same velocity as dynamic points. We name them point-places as, when a dynamic point reaches a point-place, it continues its movement bound to the point-place; we denote point-places that will be reached by
dynamic points with empty circles (as in Fig. 2 on edge $e_s$). Thus, for edge $e$, an equivalence class $[w]_e$ of $\mathcal{W}(v_0, e)$ corresponds to a point-place which will be saturated by a dynamic point that came from $v_0$ along the minimal path of $[w]_e$. It is possible to define point-place evolution and corresponding time-dependent equivalence classes in a time-dependent manner; to avoid handling time-dependent coordinates, we just take class representatives for $t = 0$.

The first quite obvious observation is that, for any dynamical system $\mathcal{P}_\Gamma$, there is a one-point $\text{DP}$-system on a tree constructed from the same edges that has less or equal growth rate. We consider only connected $\text{DP}$-systems with at least one dynamic point. For disconnected graphs, stabilization can be considered for its components independently.

**Lemma 1.** Let $E$ be a given set of edges with a length function $l$ on $E$, and $\mathcal{P}_\Gamma$ be a dynamical system with a graph $\Gamma$ constructed from $E$. Then, there is a dynamical system $\mathcal{P}'_{\Gamma'}$ consisting of a tree graph $\Gamma'$ constructed from $E$ and one dynamic point, such that the growth rate of $\mathcal{P}'_{\Gamma'}$ is equal or slower than thus of $\mathcal{P}_\Gamma$.

**Proof 1.** Let $\mathcal{P}_\Gamma$ have $N_0$ initial points on metric graph $\Gamma$. Obviously, elimination of points in $\mathcal{P}_\Gamma$ can only decrease growth rate. Thus, if $N_0 > 1$, it is safe to eliminate all points in $\mathcal{P}_\Gamma$ except one arbitrary dynamic point $p_0$ in vertex $v_0$.

Let $\Gamma$ be not acyclic. The proof is based on an observation that a cutting of a cycle in $\Gamma$ can only decrease growth rate. Let $(v_{i_1}, v_{i_2}) = a_1, \ldots, a_k = (v_{i_k}, v_{i_1})$ be a cycle in $\Gamma$. Let us split the cycle at arc $a_k$, i.e., a vertex $v'_{i_1}$ is introduced, arcs $a_k$ and $\bar{a}_k$ are replaced with $a_k' = (v_{i_k}, v'_{i_1})$ and $\bar{a}_k'$ in resulting graph $\Gamma'$; in Figure 3, cycle $(v_1, v_3, v_5, v_6)$ is split at vertex $v_1$, i.e., $[v_6, v_1]$ is removed, vertex $v'_{i_1}$ and edge $[v_6, v'_{i_1}]$ are added. For any edge $e$, the operation can only narrow set $\mathcal{W}(v_0, e)$; i.e., all walks of the resultant graph $\Gamma'$ are realizable in $\Gamma$. Thus, no shorter walks are introduced; and, no point-place on $e'$ in $\Gamma'$ is saturated earlier than the corresponding one on $e$ in original $\Gamma$.

By repeating cycle elimination, we obtain $\mathcal{P}'_{\Gamma'}$, on a tree metric graph.

However, we cannot use the same approach to achieve an $\text{LSTDP}$-system on a tree. A cutting may completely eliminate some classes of walks of original $\mathcal{P}_\Gamma$ that stipulate longest stabilization time ($\text{LST}$-classes), i.e., those whose shortest walks are longest ($\text{LST}$-walks) among all the walk classes of $\mathcal{P}_\Gamma$ for all the edges of $\Gamma$. After cutting, the dynamic points corresponding to such $\text{LST}$-classes never occur in $\mathcal{P}_\Gamma$, so stabilization does not depend on (wait for) these dynamic points and may occur earlier. The metric graph $\Gamma$ that demonstrates such phenomenon is depicted in

![Figure 3: Metric graph $\Gamma$ at a) and its possible cuttings $\Gamma'$ at b) and $\Gamma''$ at c)](image-url)
Figure 4: Examples of DP-systems that have stabilization time greater than any acyclic graph with the same set of edges. All edges have length one.

To handle this obstacle, we suggest to fix one of shortest walks in an LST-class, i.e., an LST-walk, and conduct the cutting preserving the walk. In the example above, the ‘missed’ points were generated by the eliminated walks in the class that contains LST-walks $e_1, e_1, e_1, e_2$ and $e_1, e_2, e_2, e_2$. Thus, if we cut graph $\Gamma$ into $\Gamma'$ preserving such walks, then $P_{\Gamma'}$ will have stabilization time not less than $P_\Gamma$ even if some non-LST-classes vanish and overall $N_\Gamma$ could decrease. A possible cutting is depicted in Figure 3c).

If such a cycle cutting preserving LST-walk would always be found, then LSTDP should always contain a tree and, moreover, using a procedure from the next section, a linear graph. Unfortunately, the graphs of DP-systems in Figure 4 cannot be cut into an acyclic graph preserving the stabilization time. All the edges of the graphs in Figure 4 have length equal to 1. For DP-system in 4a), it could be checked manually: the last point is generated on $e_s$ at $t = 4$, while, for a linear graph with 4 edges, stabilization time is 3, for other trees — even less. Figure 4b) shows that an uncuttable cycle could lie not on a path from initial point vertex $v_1$ to stabilization edge $e_s$. Figure 4c) shows that the absolute difference between the stabilization time of LSTDP-system on 4c) ($t_s = 2k + 5$) and a linear graph with the same set of edges ($t_s = k + 5$) could be arbitrary large.

4. Bead DP-systems with longest stabilization time

In this section, by applying LST-path preserving operations, we show that set LSTDP($E$) always contains a DP-system over a graph with a specific structure.

While it is not hard to see that a star metric graph has the stabilization time not more than two times greater than the shortest stabilization time for a fixed set of edges, structure of DP-systems that demonstrate longest stabilization time is not yet understood. Also, DP-systems on a metric
Lemma 2. Let $E$ be a given set of edges with a length function $l$ on $E$. Set $\text{LSTDP}(E)$ contains a bead $\text{LSTDP}$-system $\Gamma$ with a dynamic point at a vertex.

Proof 2. By scaling, i.e., dividing by gcd, w.l.o.g., the lengths of all edges in $E$ can be made integer with gcd$(E)$ equal to 1.

The points generated by dynamic point $p$ reached a vertex are called descendants of $p$, and $p$ is the ancestor of the new points; descendants and ancestor are transitive notions, i.e., descendants of descendants of $p$ are descendants of $p$. The stabilization time for an edge $e$ is denoted by $t_e$, $t_e$ is the earliest time point such that $\forall t > t_e(e) : N_e(t) = N_e(t_e(e))$. $N_e(t)$ stabilizes when edge $e$ receives $N_e(t_e(e))$ points. Assume that $e_p$ is the edge with the longest stabilization time $t_p$ in $\mathcal{P}_\Gamma$. Now consider the last point $p_{s}$ that is generated on $e_p$ at stabilization time $t_p$. Point $p_{s}$ is the descendant of an initial point $p_0$ in $\mathcal{P}_\Gamma$. Elimination of all initial points except $p_0$ in $\mathcal{P}_\Gamma$ does not decrease $t_p(\mathcal{P}_\Gamma)$. Thus, we may consider only DP-systems with one dynamic point.

Let $\mathcal{P}_\Gamma$ be a DP-system on graph $\Gamma$ in set $\text{LSTDP}(E)$ with point $p$ in $\mathcal{P}_\Gamma$. Let $e_p$ be the edge with the longest stabilization time $t_p$ in $\mathcal{P}_\Gamma$, and let $w_s \in \mathcal{W}(0)$ be an LST-walk.

It is clear that, for any walk $w$ from vertex $u$ to $v$, it is possible to obtain a new walk $w'$ with the same length by reordering arc entries in $w$, such that, when any entry of arc $a_p = (v_i, v_j)$, that corresponds to edge $e_p$, is met in $w$ for the first time, walk $w'$ runs back and forth on edge $e_p$, until there are no more entries of arcs $a_p$ or $a_p$ left in $w$, except probably one last entry to preserve ability to reach $u$. Intuitively, consider multisupport $\overline{S}(w)$, which is semi-eulerian by construction, and apply ‘greedy’ modification of Fleury algorithm [12], that always chooses just passed edge again if it may, to find a semi-eulerian walk from $u$ to $v$ on $\overline{S}(w)$; it is possible as Fleury algorithm is path-choice agnostic unless the edge is a bridge. For example, for walk $w = a_2, a_3, a_3, a_2, a_3, a_4$, there is a walk $w' = a_2, a_3, a_2, a_3, a_3, a_4$; we will call such reordered walks greedy.

For a walk $w$, if the number of arcs in $\overline{S}(w)$ corresponding to edge $e$ in $\Gamma$ is odd (even), we will call edge $e$ in $\Gamma$ and the corresponding arcs in $\overline{S}(w)$ odd (even).

Let $w'_s \in \mathcal{W}(0)$ be the greedy version of LST-walk $w_s$, and, obviously, $w'_s$ is an LST-walk in $\mathcal{W}(0)$ itself. Note that if, while traversing $\overline{S}(w'_s)$ according to $w'_s$, we reached arcs corresponding to an edge $e$ of $\Gamma$ which multiplicity is larger than 2, then, by the greedy Fleury procedure, we always move back and forth on these arcs, except the case when the last arc becomes a bridge in $\overline{S}(w'_s)$ during such traversing. Thus, while analysing the structure of a semi-eulerian walk for graph cutting purposes, we may temporary omit all ‘parasite’ pairs of arcs in $\overline{S}(w'_s)$ leaving only one arc for odd and two arcs for even edges; such a multisupport is called reduced and is denoted by $\overline{S}(w'_s)$. To preserve $w'_s$ length, we need to restore such omitted arcs after graph cuttings.

As we will see now, greedy semi-eulerian walks have simpler structure than arbitrary ones. At first, we take greedy walk $w'_s$ and start traversing $\overline{S}(w'_s)$ according to $w'_s$. If walk $w'_s$ reaches vertex $v_i$ with an incident even edge $e = (v_i, v_j)$, and, after traversing back and
forth all the arcs corresponding to e, it ends up in \( v_a \) (i.e., it doesn’t pass beyond \( v_b \)), then we split e from vertex \( v_b \). Such case may happen only when all but last arcs on e are traversed and the last arc is not a bridge, i.e., even if we traverse-remove the last arc \( \langle v_b, v_a \rangle \) corresponding to e in \( \overline{S}(w'_s) \), there are still paths to the rest arcs incident to \( v_b \) in the untraversed fragment of \( \overline{S}(w'_s) \); this implies that when we first reached e before splitting, it lies on a cycle in the untraversed part of \( \overline{S}(w'_s) \). Figure 5 provides an example of such cutting. We have walk \( w_s = v_0, v_1, v_2, v_3, v_4, v_5, v_1, v_2, v_3, v_4, v_1, v_2, v_6 \), its greedy version \( w'_s = v_0, v_1, v_5, v_4, v_1, v_2, v_3, v_4, v_5, v_1, v_2, v_6 \), and a cutting of edge \( e = \langle v_1, v_5 \rangle \) from vertex \( v_5 \) as \( w'_s \) reaches but does not cross (walk beyond) \( v_5 \) from e. After all cuttings, we obtain resultant graph \( \Gamma' \). Obviously, the cutting procedure preserve \( w'_s \); i.e., \( w'_s \) is realizable in \( \Gamma' \). All cycles that contain even edges are cut, i.e., there are not cycles with even edges in \( \Gamma' \). Thus, even edges will correspond to bridges in \( \Gamma' \) after cutting. 

Note that we do cuttings on \( \overline{S}(w'_s) \) and the original graph simultaneously, even we are focused only on \( \overline{S}(w'_s) \) during the procedure. Multisupport \( \overline{S}(w'_s) \) plays auxiliary role to let us see how to cut the original graph preserving the LST-walk \( w'_s \).

Now, we take \( \Gamma' \) after cutting and \( \overline{S}_{\overline{S}}(w'_s) \), and, if there is an edge e in \( \Gamma' \) that corresponds to two or more arcs in \( \overline{S}_{\overline{S}}(w'_s) \), then we remove a pair of arcs that corresponds to e and repeat removing while we can; the result is denoted by \( \overline{S}_{\overline{S}}(w'_s) \). Clearly, when the process ends, all arcs corresponding to even edges will be removed, and, for each odd edge, only one arc is kept. As parity has not changed, all components of odd edges are eulerian, except one that corresponds to a semi eulerian \( v_0v_8 \)-tour. For each eulerian connected component CC in \( \overline{S}_{\overline{S}}(w'_s) \), we fix eulerian walk \( w_{cc} \) within CC. For each vertex \( v_i \) of \( w_{cc} \) with degree \( d(v_i) \) more than two, i.e., \( w_{cc} \)

Figure 5: a) Semi eulerian \( v_0v_8 \)-walk, b) its greedy version, and c) graph \( \Gamma \) after cutting — edge \( \langle v_1, v_5 \rangle \) is split from \( v_5 \)
Figure 6: a) Walk $w_{es}$ in component $CC$ of $\overline{s_{mod2}(w')}$, and b) the resulting component after all cuttings conducted.

crosses vertex $v_i$ exactly $d(v_i)/2$ times, we split $v_i$ into $d(v_i)/2$ vertices and make $w_{cc}$ pass through different copies of $v_i$, thus, not crossing any such vertex $v_i$ more than once. Therefore, by splitting and adding new vertices, we transform $CC$ into one cycle. For the semieulerian component, we do the same process but the result is a linear graph. After cutting, we revive all deleted pairs of arcs on edges, that were removed in $\overline{s_{mod2}(w')}$, and restore path $w'_s$ in the resulting graph; it does not introduce any difficulties as, despite we added new vertices, the set of edges is the same — for example, edge $\langle v_1, v_3 \rangle$ in Figure 6a) corresponds to edge $\langle v_{11}, v_{32} \rangle$ of the resulting graph in Figure 6b). For vertex $v_i$ in $\Gamma'$, there can be several vertices in $\Gamma''$; for example, in Figure 6 for vertex $v_1$ in $\Gamma'$, there are vertices $v_{11}$ and $v_{12}$ in $\Gamma''$. While we reconstructing even edges that are incident to $v_i$ in $\Gamma'$, it is irrelevant which of new vertices corresponding to $v_i$ in $\Gamma''$ we choose as our only focus is to preserve $w'_s$, even less — a path of the same length.

The resulting graph is denoted by $\Gamma''$ and path $w'_s$ reconstructed in $\Gamma''$ is denoted by $w''_s$.

Graph $\Gamma''$ consists of a linear subgraph and a number of cycles corresponding to odd edges, that are connected with bridges corresponding to even edges; to easier imagine it — if each cycle of odd edges is contracted to a vertex, the resulting graph is a tree of even edges. By construction, $\Gamma''$ is a bead graph; all contacting cycles of odd edges are merged into one cycle, and any vertex in $\Gamma''$ has not more than two incident odd edges with exception of $e_s$.

As $w''_s$ has the same length as $w'_s$, $w''_s$ is LST-walk, and $\mathcal{P}_{\Gamma''}$ is an LSTD-system in LSTD($E$).

A graph is called a bead broom graph if it is a bead graph with a fixed connected (linear) subgraph — a handle; all vertices of the handle have at most degree two, except one of its terminal vertices. For example, in Figure 8 the graph is a broom graph with a handle $v_0, v_1, v_2, v_3$.

**Lemma 3.** Let $E$ be a given set of edges with a length function $l$ on $E$. If there is a bead LSTD-system $\Gamma$ in LSTD($E$) with a dynamic point at vertex $v_0$, then LSTD($E$) contains a bead broom LSTD-system $\Gamma'$ with a point at the end of the handle of $\Gamma'$, and the handle contains edge $e_s$ with the longest stabilization time.

**Proof 3.** Let $e_s$ be an edge with longest stabilization time. We apply to $\Gamma$ the procedure described in the proof of Lemma 2 and, after cutting, $\Gamma'$ consists of a linear $v_0 e_s$-subgraph, odd cycles, and even bridges.
Let \( e_s = (v_r, v_q) \), where \( v_r \) is the vertex that is closer to \( v_0 \), i.e., \( v_q \) does not lie on the shortest path \( h \) from \( v_0 \) to \( v_r \). Path \( h \) is unique as \( v_0 \) and \( v_r \) belong to the linear subgraph of odd edges in \( \Gamma \) constructed at the previous phase; \( h \) will become the handle of the resulting bead broom metric graph.

Consider now an arbitrary even edge \( e_b = (v_a, v_b) \) such that \( v_a \) incident to \( h \). Let subgraph \( g \) of \( \Gamma \) be the connected component containing \( v_b \) that appears if we remove bridge \( e_b \) (even \( e_b \) is a bridge as was shown in the the proof of Lemma 2). For example, on the left of Figure 7 such a subgraph containing vertices \( v_4 \) and \( v_5 \) is outlined by a grey wavy line. Vertex \( v_4 \) belongs to \( g \) and is adjacent to \( v_0 \).
New graph $\Gamma'$ is obtained by disconnecting $g$ from vertex $v_q$ in $\Gamma$ and connecting $g$ to $v_q$, i.e., $e_b$ is removed from $\Gamma$, and a new edge $\langle v_b, v_q \rangle$ of length $l(e_b)$ is added. The moved subgraph $g$ with the new edge in $\Gamma'$ will be denoted by $g'$, and each vertex $v$ and each edge $e$ in $g$ will be denoted by $v'$ and $e'$, correspondingly, in $\Gamma'$. In Figure 7 g is moved from $v_0$ to $v_3$. Consider set $W(v_0, e_s)$ in $\Gamma$ and set $W(v_0, e_s)$ in $\Gamma'$. It is needed to ensure that, after such surgery of $\Gamma$, no new walks from $v_0$ to the endpoints of $e_s$ that decrease $t_s(e_s)$ appeared in $\Gamma'$.

Every walk $w'$ in $W(v_0, v_q)$ of $\Gamma'$ that has no edges of $g'$ is, clearly, in $W(v_0, e_s)$ of $\Gamma$. Let $w'$ be a walk in $W(v_0, e_s)$ of $\Gamma'$ that has edges of $g'$ in its support $S(w')$. For example, consider path $w' = \langle v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_3 \rangle$ in Figure 7 its support is highlighted with gray. For walk $w'$, there is a walk $w$ in $\Gamma$ that is not longer than $w'$. Walk $w$ contains all edges of $w'$ that do not belong to $g'$; in addition, for each edge $\langle v'_i, v'_j \rangle$ of $w'$ in $g'$, $w$ contains corresponding edge $\langle v_i, v_j \rangle$ in $w$. For $\langle v_6, v_7 \rangle$ in $w'$, $w$ contains $e_b$. The lengths of such $w'$ and $w$ are equal; and, thus, $w'$ and $w$ lie in the same class $[w]$. Point $p$ moving along $w$ from $v_0$ to $v_q$ in $\Gamma$ can even induce a new point on $e_s$ earlier than moving along $w'$ in $\Gamma'$ if $w$ does not contain edges of $g'$ beyond endpoint $v_q$ of $e_s$, i.e., $p$ produces a new point on $e_s$ in $l(e_s)$ time units earlier. Thus, for every walk $w'$ in $\Gamma'$, there is a walk in $\Gamma$ that produces a point on $e_s$ not later than $w'$. As the result, $e_s$ will be saturated in $\mathcal{P}_\Gamma$ not earlier than in $\mathcal{P}_\Gamma$, and $\mathcal{P}_\Gamma$ belongs to LSTDP($E$).

![Figure 9: Relocation of subgraph g from v4 to v6.](image)

Eventually, we will show that vertex degrees of LSTDP-system can be reduced to three or less without leaving LSTDP($E$) set.

**Lemma 4.** Let $E$ be a given set of edges with a length function $l$ on $E$. If there is a bead broom DP-system $\mathcal{P}_T$ in LSTDP($E$) with a handle h, an initial point at the end of the handle terminal vertex $v_0$, and $h$ contains an edge $e_s$ with longest stabilization time, then LSTDP($E$) contains a bead DP-system $\mathcal{P}_T'$ with an initial point at one of its terminal vertices and maximum degree of 3.

**Proof 4.** Let $e_s = \langle v_r, v_q \rangle$, where $v_q$ is the vertex that is farther from $v_0$ than $v_r$.

If a vertex $v$ is a leaf or $v$ belongs to a cycle $C$ in $\Gamma$ that has only one incident bridge (even edge) $e_b$, i.e., $C$ is a terminal cycle, and $v$ is not incident to $e_b$, then $v$ is called a bead leaf vertex.
Consider a path $p_1$ from $v_q$ to a bead leaf $v_l$ of $\Gamma$ that does not run through $e_s$, and an arbitrary even edge $e_b$ of $\Gamma$ that is incident to a vertex $v_b$ of $p_1$. Let subgraph $g$ be the connected component that occurs if we remove $e_b$, thus that does not contain $p_1$. Let $v_b$ be another terminal vertex of $e_b$ adjacent to $v_q$. New graph $\Gamma'$ is obtained by disconnecting $g$ from vertex $v_a$ in $\Gamma$ and connecting $g$ to $v_l$, i.e., edge $e_b$ is removed and a new edge $\langle v_b, v_l \rangle$ of length $l(e_b)$ is added. The moved subgraph $g$ with the new edge in $\Gamma'$ will be denoted by $g'$, and each vertex $v$ and each edge $e$ in $g$ will be denoted by $v'$ and $e'$, correspondingly, in $\Gamma'$.

For example, in Figure 5 there is path $w_1 = \langle v_3, v_4, v_5, v_6 \rangle$ from $v_3$ to bead leaf $v_6$. Subgraph $g$ containing vertices $v_7$ and $v_8$ is outlined by a gray wavy line and is incident to vertex $v_4$ of the path. Graph $\Gamma'$ is obtained from $\Gamma$ by moving $g$ from $v_4$ to $v_6$.

The second part of the proof argument mostly resembles thus of the previous theorem. Consider $\mathcal{W}(v_0, e_s)$ in $\Gamma$ and set $\mathcal{W}(v_0, e_s)$ in $\Gamma'$. It is needed to ensure that no new walks from $v_0$ to the endpoints of $e_s$ appear in $\Gamma'$ that may decrease $t_s(e_s)$ in $\Gamma'$.

Each walk $w'$ in $\mathcal{W}(v_0, e_s)$ of $\Gamma'$ that has no edges of $g'$ is in $\mathcal{W}(v_0, e_s)$ of $\Gamma$. Let $w'$ be a walk in $\mathcal{W}(v_0, e_s)$ of $\Gamma'$ that has edges of $g'$ in its support $S(w')$. For walk $w'$, there is a walk $w$ in $\Gamma$ that is not longer than $w'$. Walk $w$ contains all edges of $w'$ that do not belong to $g'$; in addition, for each edge $\langle v', v'' \rangle$ of $w'$ in $t'$, $w$ contains corresponding edge $\langle v_1, v_2 \rangle$ in $w$. The lengths of such $w'$ and $w$ are equal and, thus, $w'$ and $w$ lie in the same class $[w']_t$. For example, in Figure 5 consider walk $w' = \langle v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_10, v_11 \rangle$: its support is highlighted with gray. In $\Gamma$, there is a corresponding walk $w = \langle v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_10, v_11 \rangle$.

Thus, for every walk $w'$ in $\Gamma'$, there is a walk in $\Gamma$ that produces a point on $e_s$ not later than $w'$. As the result, $e_s$ will be saturated in $\mathcal{P}_\Gamma$ not earlier than in $\mathcal{P}_\Gamma'$; therefore, $\mathcal{P}_\Gamma'$ belongs to $\text{LSTD}(E)$.

For any vertex $v$ in $\Gamma$ with four or more incident edges, $v$ may have two odd incident edges only if $v$ lies on a cycle, $v$ may have three odd incident edges if $v$ is a terminal vertex of $e_s$; all other incident edges of $v$ are even. One of even incident edges may belong to the $e_s$-path; for each other even incident edge we move it with its connected subgraph to a bead leaf. After the procedure is applied repeated, we obtain a bead graph with no vertices of degree four or more.

As the set of metric graphs that can be constructed from the given finite set of edges $E$ is finite, there is a metric graph with the longest stabilization time built from $E$. Lemmas 2,3,4 combined give us the following theorem.

**Theorem 5.** Let $E$ be a given set of edges with a length function $l$ on $E$. There is a DP-system on a bead metric graph with vertices of degree three or less and with an initial point in one of its terminal vertices that belongs to $\text{LSTD}(E)$.

As the resulting tree DP-system $\mathcal{P}_\Gamma$ in Lemma 1 does not contain cycles at all, if we fix stabilization edge $e_s$ and apply the transformations of a graph suggested in the proofs of Lemmas 2,3,4 to $\mathcal{P}_\Gamma$, then we obtain a linear graph $\mathcal{P}'_\Gamma$. As such transformations does not add paths of new lengths, $\mathcal{P}'_\Gamma$ will demonstrate less or equal growth rate than $\mathcal{P}_\Gamma$. As this can be done for any $\mathcal{P}_\Gamma$, we can improve Lemma 1.

**Corollary 6.** Let $E$ be a given set of edges with a length function $l$ on $E$, and $\mathcal{P}_\Gamma$ be a dynamical system with a graph $\Gamma$ constructed from $E$. Then, there is a dynamical system $\mathcal{P}'_\Gamma$, consisting of a linear graph $\Gamma'$ constructed from $E$ and one dynamic point at its vertex, such that the growth rate of $\mathcal{P}'_\Gamma$ is equal or less than thus of $\mathcal{P}_\Gamma$. 

12
It is important to note that the suggested procedures are quite local, i.e., they are bound to a vertex and stabilization edge. This means, for example, that we cannot use it, at least straightforwardly, to construct or prove that there exists a linear metric graph with many dynamic points that demonstrates the slowest growth rate at all edges, as moving subgraphs may reduce the growth rate of one edge but intensify that of another.

5. Saturation time for bead DP-systems with incommensurable edge lengths

DP-system on a metric graph with incommensurable edges (i.e., the rank of its edges over $\mathbb{Q}$ is greater than one) never stabilizes and, moreover, ergodic [2]. However, it is possible to extend the notion of stabilization time to systems with incommensurable edges using the notion of $\varepsilon$-net [13].

For a point $x$ on $\Gamma$, a closed 1-ball $B_\varepsilon(x)$ with radius $\varepsilon$ around $x$ is the set of all points in $\Gamma$ whose distance from $x$ in the metric $\rho$ is less than $\varepsilon$, i.e., $\{p \in \Gamma | \rho(p,x) \leq \varepsilon\}$. A finite subset $N_\varepsilon$ of points of metric graph $\Gamma$ is an $\varepsilon$-net if, for each point $p \in \Gamma$, distance $\rho(p,N_\varepsilon)$ is less than $\varepsilon$. Thus, the closed 1-balls with centres in $N_\varepsilon$ cover the whole $\Gamma$. Now we consider $\varepsilon$-net over point-places, and 1-ball $B_\varepsilon(x)$ are set of points at $\varepsilon$-distance or less from moving point-place $x$. In Figure 10 edge $(v_1,v_2)$ holds only one dynamic point $p$, which saturates 1-ball (i.e., line segment bounded by the circle) $B_\varepsilon(x_1)$; note that, points of $\varepsilon$-net, coloured gray, are not necessarily regularly distributed. For DP-system $P_\Gamma$ and $\varepsilon$-net $N_\varepsilon$ on point-places of $\Gamma$, the saturation time $t_s(N_\varepsilon)$ of $P_\Gamma$ is the earliest point in time when, for each point $x$ in $N_\varepsilon$, there is a point $q$ in $P_\Gamma$ with $\rho(x,q) \leq \varepsilon$. For given $\varepsilon$, the saturation time $t_s(\varepsilon)$ of $P_\Gamma$ is the supremum of the set of saturation times for all possible $\varepsilon$-nets on $\Gamma$, i.e., $t_s(\varepsilon) = \sup\{t_s(N_\varepsilon)\}$. We call 1-ball $B_\varepsilon(x)$ on $\Gamma$ with centre $x$ and radius $\varepsilon$ saturated when point $p$ of $P_\Gamma$ is bound to a place-point in $B_\varepsilon(x)$.

**Theorem 7.** Let $E$ be a given set of edges with a length function $l$ on $E$, and the rank of $E$ over $\mathbb{Q}$ is greater than one. There is a DP-system on a bead metric graph with vertices of degree three or less and with an initial point in one of its terminal vertices that has the longest saturation time among systems in DP($E$).

**Proof 5.** At time $t_s(\varepsilon)$, points of $P_\Gamma$ form an $\varepsilon$-net themselves; if they do not, it is possible to construct $N_\varepsilon$ that is not saturated at $t_s(\varepsilon)$ by taking the centre of a non-covered ball into $N_\varepsilon$, which contradicts the definition of $t_s(\varepsilon)$. Thus, $t_s(\varepsilon)$ is the time point when all the non-saturated balls vanish. Fix ball $B_\varepsilon(x)$ that is saturated at $t_s(\varepsilon)$ by dynamic point $q$ and build $\varepsilon$-net $N_\varepsilon$ such
that, within $B_\varepsilon(x)$, only the centre of $B_\varepsilon(x)$ belongs to $N_\varepsilon$. Let $q_0$ in vertex $v_0$ be the ancestor of $q$ in the initial state of $P\gamma$, $e$ be an edge containing centre $x$, and $p$ be the path of $q$ from $v_0$ to $B_\varepsilon(x)$. By the graph surgery suggested in Lemma 2–Theorem 5, we build a bead graph with vertex degrees not greater than three that preserves $v_0e$-path $p$.

6. Conclusions and future work

Neither explicit formula nor asymptotic estimates are known for the longest stabilization time of DP-systems constructed from set of edges $E$ or for the stabilization time of an arbitrary DP-system in the general case [2]. Theorem 5 allows us to obtain the longest stabilization time of $DP(E)$ by studying only bead graphs of degree not higher than 3. It also allows to narrow down the search state if we want to compute the longest stabilization time algorithmically.

The proofs are mostly agnostic to the numerical properties of edge lengths and specific structure of a metric graph. Thus, it allowed us to extend the results to incommensurable metric graphs by adapting the notion of stabilization time using $\varepsilon$-nets. It could be interesting to combine the suggested graph surgery with [9].

Considering the support set of an $LST$-walk, one could hypothesizes that, even the absolute difference between longest stabilization time of $DP(E)$ and set of linear graphs built from $E$ could be arbitrary large (Fig. 4), there is always a linear graph with stabilization time not more than two times shorter than the longest stabilization time in $DP(E)$.

Acknowledgements

The reported study was funded by RFBR, project number 20-07-01103a. Author expresses gratitude to V. Chernyshev for providing the state-of-the-art view of the field and pointing to some recent results, and to A. G. Khovanskii for explaining the useful notion of $\varepsilon$-net.

References

[1] V. L. Chernyshev, Time-dependent schrödinger equation: statistics of the distribution of gaussian packets on a metric graph, Proceedings of the Steklov Institute of Mathematics 270 (1) (2010) 246–262.
[2] V. L. Chernyshev, A. A. Tolchennikov, A. I. Shafarevich, Behavior of quasi-particles on hybrid spaces. relations to the geometry of geodesics and to the problems of analytic number theory, Regular and Chaotic Dynamics 21 (5) (2016) 531–537. doi:https://doi.org/10.1134/S156035471605004X
[3] G. Berkolaiko, P. Kuchment, Introduction to quantum graphs, Vol. 186, American Mathematical Soc., 2013. doi:https://doi.org/10.1090/surv/186
[4] L. Pauling, The diamagnetic anisotropy of aromatic molecules, The Journal of Chemical Physics 4 (10) (1936) 673–677. doi:https://doi.org/10.1063/1.1749766
[5] P. Exner, J. Lipovský, Topological bulk-edge effects in quantum graph transport, Physics Letters A (2020) 126–39. doi:https://doi.org/10.1016/j.physleta.2020.126390
[6] A. A. Tolchennikov, V. L. Chernyshev, A. I. Shafarevich, Asymptotic properties and classical dynamical systems in quantum problems on singular spaces, Nelineinaya Dinamika [Russian Journal of Nonlinear Dynamics] 6 (3) (2010) 623–638. doi:https://doi.org/10.1134/S008154381003020X
[7] V. Chernyshev, A. Tolchennikov, Asymptotic estimate for the counting problems corresponding to the dynamical system on some decorated graphs, Ergodic Theory and Dynamical Systems 38 (5) (2018) 1697–1708. doi:https://doi.org/10.1017/etds.2016.102
[8] V. Chernyshev, A. Tolchennikov, Correction to the leading term of asymptotics in the problem of counting the number of points moving on a metric tree, Russian Journal of Mathematical Physics 24 (3) (2017) 290–298. doi:https://doi.org/10.1134/s1061920817030029
[9] V. L. Chernyshev, A. A. Tolchennikov, The second term in the asymptotics for the number of points moving along a metric graph, Regular and Chaotic Dynamics 22 (8) (2017) 937–948. doi:https://doi.org/10.1134/ S1560354717080032
[10] G. Berkolaiko, R. Carlson, S. A. Fulling, P. Kuchment, et al., Quantum Graphs and Their Applications: Proceedings of an AMS-IMS-SIAM Joint Summer Research Conference on Quantum Graphs and Their Applications, June 19-23, 2005, Snowbird, Utah, Vol. 415, American Mathematical Soc., 2006. doi:https://doi.org/10.1090/conm/415

[11] L. W. Dworzanski, A. A. Izmaylov, Automated analysis of dp-systems using timed-arc petri nets via tapaal tool, in: Proceedings of ISP RAS, Russian Academy of Science, Vol. 32, 2020, pp. 155–166. doi:https://doi.org/10.15514/ISPRAS-2020–32(6)-12

[12] H. Fleischner, Eulerian Graphs and Related Topics, Vol. 50 of Annals of Discrete Mathematics, Elsevier, 1991, Ch. X. Algorithms for Eulerian Trails and Cycle Decompositions, Maze Search Algorithms, pp. X.1 – X.34. doi: https://doi.org/10.1016/S0167-5060(08)70158-4

[13] F. Hausdorff, Set theory, American Mathematical Society, Providence, R.I, 2005.