AN ELEMENTARY APPROACH TO
THE DIMENSION OF MEASURES SATISFYING
A FIRST-ORDER LINEAR PDE CONSTRAINT

ADOLFO ARROYO-RABASA

Abstract. We give a simple criterion on the set of probability tangent measures $\text{Tan}(\mu, x)$ of a positive Radon measure $\mu$, which yields lower bounds on the Hausdorff dimension of $\mu$. As an application, we give an elementary and purely algebraic proof of the sharp Hausdorff dimension lower bounds for first-order linear PDE-constrained measures; bounds for closed (measure) differential forms and normal currents are further discussed. A weak structure theorem in the spirit of [Ann. Math. 184(3) (2016), pp. 1017–1039] is also discussed for such measures.

1. Introduction

The question of determining the dimension of a vector-valued Radon measure satisfying a PDE-constraint is a longstanding one. A good starting point are curl-free measure fields. The seminal work of DE GIORGI [7] on the structure of sets of finite perimeter and the co-area formula [11] from FLEMING & RISHEL yield the estimate $|Du| \ll \mathcal{H}^{d-1}$ for all distributional gradients $Du$ represented by a Radon measure. Later on, FEDERER extended (see [10, Sec. 4.1.21]) this result to the estimate $\|T\| \ll I^m \ll \mathcal{H}^m$ for $m$-dimensional normal currents $T \in \mathcal{N}_m(\mathbb{R}^d)$. Recently, these results have been further extended to deal with more general differential constraints (in the context of $\mathcal{A}$-free measures). Namely, in [6] it is shown that $|\mu| \ll I_{\mathcal{L}^m} \ll \mathcal{H}_{\mathcal{L}^m}$ for measures $\mu$ satisfying a generic constraint of the form $P(D)\mu = 0$, where $P(D)$ is a $k$th-order linear partial differential operator with constant coefficients and $\mathcal{L}^m$ is a positive integer depending only on the principal symbol $P^k$ of $P(D)$. This $\mathcal{L}^m$ dimensional estimate turns out to be sharp for first-order operators; for higher-order operators it is an open question whether it remains an optimal bound (see [6, Conjecture 1.6]).

The compendium of results mentioned above are of stronger structural character than the ones presented on this note, since only bounds on the Hausdorff dimension of such measures will be discussed here. However, they also require a significantly stronger machinery. Our main interest is to give a self-contained and “elementary” proof of the Hausdorff dimension (sharp).
bounds for measures $\mu \in \mathcal{M}(\Omega, E)$ solving, in the sense of the distributions, an equation of the form

$$P(D)\mu := \sum_{i=1}^{d} P_i[\partial_i \mu] + P_0 \mu = 0, \quad P_0, P_i \in F \otimes E,$$

where $E, F$ are finite dimensional euclidean spaces.

The angular stone of our proof rests on a rather simple invariance criterion affecting all normalized blow-ups of a given positive Radon measure $\sigma$, which effortlessly yields a lower bound on the Hausdorff dimension $\dim_{\mathcal{H}}(\sigma)$, where as usual

$$\dim_{\mathcal{H}}(\sigma) := \sup \{ 0 \leq \kappa \leq d : \sigma \ll \mathcal{H}^\kappa \}.$$

This criterion (contained in Lemma 9) links the vector-space dimension, of those directions with respect to which a blow-up of $\sigma$ may be an invariant measure, to a lower bound of the Hausdorff dimension. In particular, this re-directs the study of dimensional estimates for measures satisfying (1), to the study of the structural rigidity of their sets $\text{Tan}(|\mu|, x)$ of probability tangent measures (described in Sec. 3). (A similar method for establishing dimensional estimates has been considered in [4] by AMBROSIO & SONER; see also [13] for the slightly more restrictive context of tangent spaces $T_\sigma(x) \subset \mathbb{R}^d$ introduced by BOUCHITTÉ, BUTTAZZO and SEPPECHER.)

The advantage of this viewpoint lies in the fact that the principal symbol

$$\xi \mapsto P(\xi) := \sum_{i=1}^{d} \xi_i P_i, \quad \xi \in \mathbb{R}^d,$$

being linear as function of $\xi$, precisely characterizes those directions where tangent measures are invariant measures. Thus, allowing one to define a dimension associated to the principal part of the operator:

$$\ell_P := \min_{e \in E \setminus \{0\}} \dim \left( \{P[e] \equiv 0\}^\perp \right).$$

Here, we have used the short-hand notation $\{P[e] \equiv 0\} := \{\xi : P(\xi)[e] = 0\}$. Note that this definition of dimension agrees with the definition given in [6, eq. (1.6)]. It may be worth to mention that, in the context of cocancelling operators (introduced by VAN SCHAFTINGEN [18] and further extended in [6]; see also [16,17]), $P(D)$ is an $(\ell_P - 1)$-cocancelling operator.

Our main result is contained in the following theorem:

**Theorem 1.** Let $\Omega \subset \mathbb{R}^d$ be an open set, let $P(D)$ be a first-order differential operator as in (1), and let $\mu \in \mathcal{M}(\Omega; E)$ be a solution of the equation

$$P(D)\mu = 0 \quad \text{in the sense of distributions on } \Omega.$$

Then,

$$\dim_{\mathcal{H}}(|\mu|) \geq \ell_P.$$

Moreover, this estimate is sharp since the measure

$$\mu = e \mathcal{H}^{\ell_P}, \{P[e] \equiv 0\}^\perp$$

is a solution of (1) on $\mathbb{R}^d$, whenever $e \in E$ is any vector at which the minimum in (2) is attained.
Remark 2. The proof of Theorem 1 does not require, in any way, the structure theorem for PDE-constrained measures [9, Theorem 1.1].

At all points $x$ where $[P(D) \circ \frac{d\mu}{d|\mu|}(x)]$ is elliptic, that is, precisely when the polar $\frac{d\mu}{d|\mu|}(x)$ does not belong to the wave cone set

$$\Lambda_P := \bigcup_{\xi \in \mathbb{R}^d} \ker P(\xi) \subset E,$$

the sets $\text{Tan}(|\mu|, x)$ turn out to be trivial (containing only fully-invariant measures). The invariance criterion then allows us to give the following soft version of [9, Theorem 1.1] (see also [1] in the case of gradients):

Corollary 3 (weak structure theorem). Let $\Omega \subset \mathbb{R}^d$ be an open set, let $P(D)$ be a first-order differential operator as in (1), and let $\mu \in \mathcal{M}(\Omega; E)$ be a solution of the equation

$$P(D)\mu = 0$$

in the sense of distributions on $\Omega$.

Then,

$$|\mu| \cdot \{x \in \Omega : \frac{d\mu}{d|\mu|}(x) \notin \Lambda_P\} \ll \mathcal{H}^{\kappa} \text{ for all } 0 \leq \kappa < d.$$  

Remark 4. The results contained in Theorem 1 and Corollary 3 apply to solutions of the inhomogeneous equation

$$P(D)\mu = \tau \in \mathcal{M}(\Omega; F).$$

To see this, let $\tilde{\mu} = (\mu, \tau)$, $\tilde{E} = E \times F$, and consider the operator $\tilde{P}(D)\tilde{\mu} = P(D)\mu - \tau$.

Further comments. Both Theorem 1 and Lemma 9 do not lead to rectifiability, nor estimates of the form $|\mu| \ll J^\ell$, or even $|\mu| \ll \mathcal{H}^{\ell}$ by the methods presented on this note. This assertion is in line with the following observation. The shortcoming of Corollary 3 — with respect to the (strong) structure theorem — lies in the requirement of $\kappa$ being strictly smaller than $d$. As it has been remarked by De Lellis (see [8, Proposition 3.3]), Preiss’ example [15, Example 5.8(1)] of a purely singular measure with only trivial tangent measures hinders the hope for a traditional blow-up strategy leading to the estimate in the critical case $\kappa = d$.

In a forthcoming paper [5], it will be shown that all functions $u : \Omega \to \mathbb{R}^d$ of bounded deformation satisfy the following rigidity property: every probability tangent measure $\tau \in \text{Tan}(Eu, x)$ can be split as a sum of 1-directional measures (here, $Eu = \frac{1}{2}(Du + Du^t) \in \mathcal{M}(\Omega; \text{sym}(\mathbb{R}^d \otimes \mathbb{R}^d))$ is the distributional symmetric gradient of $u$). Hence, by Lemma 9, one may recover the dimensional estimate $\dim_\mathcal{H}(|Eu|) \geq d - 1$ from [2] through a completely different method. Note however that symmetric gradients satisfy the St. Venant compatibility conditions (see [12, Example 3.10(e)]) which is a 2nd-order differential constraint.

---

2The definition of tangent measure introduced by Preiss in [15] is slightly different than our definition of probability tangent measure. However, the same triviality in the cited example can be inferred for our notion of tangent measure (see [14, Remark 14.4(1)]).
Organization. Applications of our results for several relevant first-order operators are discussed in Section 2; dimension bounds for closed differential forms and normal currents are discussed in Corollaries 6-8. A brief list of definitions (required for the proofs) and the invariance criterion (contained in Lemma 9) are given in Section 3. Section 4 is devoted to the proofs. Lastly, an appendix on multilinear algebra operations has been included, this may be of use for the applications on differential forms and normal currents discussed below.

Acknowledgments. I gratefully thank G. de Philippis and F. Rindler for introducing me to this problem, and to other related questions. I would also like to thank J. Hirsch and P. Gladbach for several fruitful discussions about this subject.

2. Applications

In this section we discuss explicit dimensional bounds for several relevant first-order differential operators.

Here and in what follows $\Omega \subset \mathbb{R}^d$ is an open set.

2.1. Gradients. The space $\text{BV}(\Omega; \mathbb{R}^m)$ of functions of bounded variation consists of functions $u : \Omega \to \mathbb{R}^m$ whose distributional derivative $Du$ can be represented by a Radon measure $\mu$ in $\mathcal{M}(\Omega; \mathbb{R}^m \otimes \mathbb{R}^d)$. We recall (see [12]) that the gradient $\mu = Du$ is (locally) a curl-free field in the sense that

$$\text{curl}(\mu) := (\partial_i \mu_{kj} - \partial_j \mu_{ki})_{kij} = 0,$$

$1 \leq i, j \leq d$, $1 \leq k \leq m.$

In the case $\text{P}(D\mu) = \text{curl}$ we have $\ker P_{\text{curl}}(\xi)[M] = \{ a \otimes \xi : a \in \mathbb{R}^m \}$, $\xi \in \mathbb{R}^d$, and therefore $\ell_{\text{curl}} = d - 1$. Theorem 1 then recovers the well-known (see [2]) dimensional bound for gradients

$$u \in \text{BV}(\Omega; \mathbb{R}^m) \implies \text{dim}_H(|Du|) \geq d - 1.$$

2.2. Fields of bounded divergence. Consider the divergence operator defined on matrix-fields $\mu \in \mathcal{M}(\Omega; \mathbb{R}^k \otimes \mathbb{R}^d)$ defined as

$$\text{div} \, \mu = \left( \sum_{i=1}^d \partial_i \mu_{ij} \right)_{j}, \quad 1 \leq j \leq k.$$

In this case we get $\mathbb{P}_{\text{div}}(\xi)[M] = M \cdot \xi$ over the space of tensors $M \in \mathbb{R}^k \otimes \mathbb{R}^d$, and $\{ \mathbb{P}_{\text{div}}[M] \equiv 0 \}^\perp = (\ker M)^\perp \cong \text{ran} \, M$. It follows from Riesz' representation theorem $\left( \frac{d\mu}{d|\mu|} (x) \neq 0 \text{ for } |\mu|-\text{a.e. } x \right)$ and Theorem 1 that

$$\text{div} \, \mu \in \mathcal{M}(\Omega; \mathbb{R}^k) \implies \text{dim}_H(|\mu|) \geq 1.$$

In a further refinement, we get the following corollary:

Corollary 5. Let $\mu \in \mathcal{M}(\Omega; \mathbb{R}^k \otimes \mathbb{R}^d)$ satisfy the non-homogeneous equation $\text{div} \, \mu = \tau$ for some $\tau \in \mathcal{M}(\Omega; \mathbb{R}^k)$. Further, assume the set

$$\left\{ x \in \Omega : \text{rank} \left( \frac{d\mu}{d|\mu|} \right) (x) \geq \ell \right\}$$

has full $|\mu^k|$-measure on $\Omega$. Then, $\text{dim}_H(|\mu|) \geq \ell$.  

Proof. In this case \( \dim(\{ \mathcal{P}_{\text{div}}[M] \equiv 0 \}) = \text{rank} \, M \geq \ell \). Then, by (4) and Lemma 9, one gets the desired bound \( \dim_{\mathcal{H}}(|\mu|) \geq \ell \). \qed

2.3. Measure differential forms. Let \( m \in \{0, \ldots, d-1\} \) and let \( \omega \in \mathcal{M}(\Omega; \Lambda^m \mathbb{R}^d) \) be a measure \( m \)-form. The exterior derivative of \( \omega \) is the \((m+1)\)-form distribution

\[
    d\omega := \sum_{i=1}^{m+1} \partial_i \omega_{1\cdots i-1i+1\cdots n} [\, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_i \wedge dx_{i+1} \wedge \cdots \wedge dx_n],
\]

where the \( \omega_{1\cdots i-1i+1\cdots n} = \langle \omega, dx_{i_1} \wedge \cdots \wedge dx_{i_m} \rangle \in \mathcal{M}(\Omega) \) are the coefficients of \( \omega \). The exterior derivative defines a first-order operator of the form (1) with

\[
    V = \Lambda^m \mathbb{R}^d \quad \text{and} \quad F = \Lambda^{m+1} \mathbb{R}^d,
\]

and a principal symbol \( d(\xi) : \Lambda^m \mathbb{R}^d \to \Lambda^{m+1} \mathbb{R}^d \) acting on \( m \)-co-vectors as

\[
    d(\xi)[v^*] = \xi^* \wedge v^*.
\]

Here, \( w^* \in \Lambda^m \mathbb{R}^d \) is the image of \( w \in \Lambda^m \mathbb{R}^d \) under the canonical isomorphism. By Lemma 11 in the Appendix, we get \( \{ d[v^*] \equiv 0 \} = \text{Ann}^1(v^*) \) (see (5) in the Appendix) and therefore \( \ell_d = d - m \).

Corollary 6. Let \( \omega \in \mathcal{M}(\Omega; \Lambda^m) \) be a measure \( m \)-form satisfying \( d\omega = \eta \) for some \( \eta \in \mathcal{M}(\Omega; \Lambda^{m+1} \mathbb{R}^d) \). Then, \( \omega \) satisfies the dimensional estimate

\[
    \dim_{\mathcal{H}}(|\omega|) \geq d - m.
\]

2.4. Normal currents. Let \( 1 \leq m \leq d \) be an integer. The space of \( m \)-currents consists of all distributions \( T \in \mathcal{D}'(\Omega; \Lambda^m \mathbb{R}^d) \). In duality with the space of smooth differential forms and the exterior derivative, one defines the boundary of a current \( T \) as the \((m-1)\)-current acting on \( C_c^\infty(\Omega; \Lambda^{m-1} \mathbb{R}^d) \) as \( \partial T[\omega] = T(\partial \omega) \). The space \( \mathcal{N}_m(\Omega) \) of \( m \)-dimensional normal currents is defined as the space of \( m \)-currents \( T \), such that both \( T \) and \( \partial T \) can be represented by a measure, that is,

\[
    \mathcal{N}_m(\Omega) \equiv \{ T \in \mathcal{M}(\Omega; \Lambda^m \mathbb{R}^d) : \partial T \in \mathcal{M}(\Omega; \Lambda^{m-1} \mathbb{R}^d) \}.
\]

The total variation of a normal current \( T \) is denoted by \( ||T|| \); and we write \( T = T' + T'' \) to denote its polar decomposition. The boundary operator on \( \mathcal{N}_m(\Omega) \) defines a first-order operator of the form (1), with a principal symbol \( d^*(\xi) : \Lambda_m \mathbb{R}^d \to \Lambda_{m-1} \mathbb{R}^d \) acting on \( m \)-vectors as the interior multiplication

\[
    d^*(\xi)[v] = v^* \cdot \xi \quad \text{where} \quad \langle v^*, \xi \rangle = \langle v, \xi^* \wedge z^* \rangle.
\]

Using the notation contained in the appendix, we readily check that \( \{ d^*[v] \equiv 0 \} = \text{Ann}_1(v) \). By means of Lemma 12 and definition (2), we conclude \( \ell_{d^*} = m \). Theorem 1 gives an alternative proof of the known dimensional estimates for normal currents:

Corollary 7. Let \( T = T' + T'' \in \mathcal{N}_m(\Omega) \) be an \( m \)-dimensional normal current on \( \Omega \). Then, \( ||T|| \) satisfies the dimensional estimate

\[
    \dim_{\mathcal{H}}(||T||) \geq m.
\]

Moreover, by the natural association between fields with bounded divergence and one-dimensional normal currents, Corollary 3 and Proposition 5 yield a simple proof of the following soft version of [9, Corollary 1.12]:
Corollary 8. Let $T_1 = \overline{T}_1\|T_1\|, \ldots, T_d = \overline{T}_d\|T_d\| \in \mathbb{N}(\Omega)$ be one-dimensional normal currents and assume there exists a positive Radon measure $\sigma \in \mathcal{M}(\Omega)$ satisfying the following properties:

(i) $\sigma \ll \|T_i\|$ for all $i = 1, \ldots, d$, 
(ii) $\text{span} \{\overline{T}_1(x), \ldots, \overline{T}_d(x)\} = \mathbb{R}^d$ for $\sigma$-almost every $x \in \mathbb{R}^d$.

Then, $\sigma \ll \mathcal{H}^\kappa$ for all $0 \leq \kappa < d$.

3. Preliminaries

Let $E$ be a finite dimensional euclidean space. We denote by $\mathcal{M}(\Omega; E) \cong C_c(\Omega; E)^*$ the space of $E$-valued Radon measures over $\Omega$. For a vector-valued measure $\mu \in \mathcal{M}(\Omega; E)$, we write the Radon–Nykodým–Lebesgue decomposition of $\mu$ as

$$\mu = \mu^{ac} \mathcal{L}^d + g_\mu |\mu|,$$

where $\mu^{ac} \in L^1(\Omega; E)$, $|\mu^a| \perp \mathcal{L}^d, \Omega$, and $g_\mu \in L^1(\Omega, |\mu^a|; E)$.

The map $T_{r,x}^{\tau}(y) = (y - x)/r$, which maps the open ball $B_r(x) \subset \mathbb{R}^d$ into the open unit ball $B_1 \subset \mathbb{R}^d$, induces a (isometry) push-forward $T_{\#}^{\tau} : \mathcal{M}(\mathbb{R}^d; E) \to \mathcal{M}(\mathbb{R}^d; E)$. A (normalized) sequence of the form

$$\gamma_j = \frac{1}{|\mu|((B_r(x))} (T_{\#}^{r_j,x_0}\mu)_1B_1, \quad r_j \downarrow 0, \quad j \in \mathbb{N},$$

is called a bounded blow-up sequence of $\mu$ at $x_0$. If $\tau = \text{w}^*-\lim \gamma_j$ on $\mathcal{M}(\overline{B_1})$, we say that $\sigma$ is a probability tangent measure of $\mu$ at $x_0$, symbolically we denote this by

$$\tau \in \text{Tan}(\mu, x_0).$$

Observe that $|\tau|(\overline{B_1}) = 1$ and, at a $|\mu|$-Lebesgue point $x_0 \in \Omega$, it holds

$$\tau \in \text{Tan}(\mu, x_0) \iff \tau = \frac{d\mu}{d|\mu|}(x_0)|\tau| \quad \text{and} \quad |\sigma| \in \text{Tan}(|\mu|, x_0).$$

For this an other facts about $\text{Tan}(\mu, x)$, we refer the interested reader to the monograph [3, Sec. 2.7].

For a finite dimensional euclidean vector space $W$, we write $\text{Gr}(W)$ to denote the Grassmanian of all linear subspaces of $W$, and $\text{Gr}(\ell, W)$ to denote the set of $\ell$-dimensional subspaces of $W$; when $W = \mathbb{R}^d$ we shall simply write $\text{Gr}(d)$ and $\text{Gr}(\ell, d)$ respectively. For given $V \in \text{Gr}(d)$, a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is called $V$-invariant if

$$\tau_{\#} \mu = \mu \quad \text{for all translations} \quad \tau : \mathbb{R}^d \to \mathbb{R}^d \quad \text{satisfying} \quad \tau(V) = V.$$

The subspace of $V$-invariant measures is denoted $\mathcal{M}^V(\mathbb{R}^d)$. Note that this space is sequentially weak-* closed in $\mathcal{M}(\mathbb{R}^d)$.

The dimension criterion is contained in the next result:

Lemma 9 (invariance criterion). Let $0 \leq \ell \leq d$ be a positive integer and let $\sigma \in \mathcal{M}(\Omega)$ be a positive measure. Assume that at, $\sigma^\ell$-almost every $x \in \Omega$, every bounded tangent measure $\tau \in \text{Tan}(\sigma^\ell, x)$ can be split on $B_1$ as a finite sum

$$\tau = (\tau_1 + \cdots + \tau_k)_1B_1, \quad k = k(\sigma) \in \mathbb{N},$$

(C)
where, for each $1 \leq h \leq k$, $\tau_h$ is a $V_h$-invariant measure for some $V_h \in \text{Gr}(\ell_h, d)$ with $\ell_h \geq \ell$. Then, $\sigma$ satisfies the dimensional estimate
\[
\dim_H(\sigma) \geq \ell.
\]

4. Proofs

We begin by proving an estimate on the upper Hausdorff densities.

**Lemma 10.** Let $0 \leq \ell \leq d$ be a positive integer and let $\sigma \in \mathcal{M}_{\text{loc}}(\Omega)$ be a positive measure. Let $x \in \Omega$ be a $[\mu^s]$-Lebesgue point and assume that every bounded tangent measure $\tau \in \text{Tan}(\mu^s, x)$ can be split on $B_1$ as a finite sum
\[
\tau = \tau_1 + \cdots + \tau_k, \quad k = k(\sigma) \in \mathbb{N},
\]
where each $\tau_h$ is a $V_h$-invariant measure for some $V_h \in \text{Gr}(\ell_h, d)$ with $\ell \leq \ell_h$.

Then, the upper $\kappa$-density of $\mu$ at $x$ is equal to zero for all $\kappa \in [0, \ell)$, that is,
\[
\theta^{*\kappa}(\mu^s, x) := \limsup_{r \downarrow 0} \frac{\mu(Q_r(x))}{r^\kappa} = 0 \quad \forall \kappa \in [0, \ell).
\]

**Proof.** It suffices to show that $\theta^{*\kappa}(\mu^s, x)$ is finite for all $\kappa \in [0, \ell)$. The fact that $\theta^{*\kappa}(\mu^s, x)$ is equally zero will then follow from the next simple observation: if $\theta^{*\kappa_1}(\mu^s, x) > 0$, then $\theta^{*\kappa}(\mu^s, x) = \infty$ for all $\kappa \in (\kappa_1, \ell)$.

We argue by contradiction. Assume that $\theta^{*\kappa}(\mu^s, x) = \infty$ for some $\kappa \in [0, \ell)$ and let $t \in (0, 4^{-\frac{d-\ell}{\kappa}})$. Then, by [3, Proposition 2.42], there exists a bounded tangent measure $\tau \in \text{Tan}(\mu^s, x)$ with $t^\kappa \leq \tau(B_t) \leq \tau(B_1) \leq 1$. On the other hand, by assumption, we may find a positive integer $k = k(\tau)$ such that
\[
\tau = \tau_1 + \cdots + \tau_k,
\]
where each $\tau_h$ is a positive $V_h$-directional measure, for all $1 \leq h \leq k$. Let us denote by $p_h : \mathbb{R}^d \to V_h^\perp$ the canonical projection so that
\[
\tau_h(F) \leq \mathcal{L}^{\ell_h}(1 - p_h)F \cdot \tilde{\tau}_h(p_h F) \quad F \subset B_1,
\]
where up to a linear isometry transformation we have $\tau_h = \mathcal{L}^{\ell_h} \otimes \tilde{\tau}_h$.

Next, we use that $4t < 1$ and that $\ell \leq \ell_h$ (for all $1 \leq h \leq k$) to obtain the estimate
\[
t^\kappa \leq \tau(B_t) \leq \tau_1(B_t) + \cdots + \tau_k(B_t) \leq (2t)^{\ell_1} \tilde{\tau}_1(p_1 B_{4t}) + \cdots + (2t)^{\ell_k} \tilde{\tau}_k(p_k B_{4t}) \leq (2t)^{\ell_1} 2^{-(d-\ell_1)} \tau_1(B_1) + \cdots + (2t)^{\ell_k} 2^{-(d-\ell_k)} \tau_k(B_1) \leq 2^{d\ell t} \tau(B_1) \leq 2^{d\ell t}.
\]

This chain of inequalities implies $2^{-\frac{d}{\kappa}} t \leq t$, which directly contradicts our choice of $t$. This shows $\theta^{*\kappa}(\mu, x) < \infty$, as desired. \hfill \Box

**Proof of Lemma 9.** Fix an arbitrary $\kappa \in [0, \ell)$. By the previous lemma and the assumption we know that the set $\Theta^\kappa_0 := \{ x \in \Omega : \theta^{*\kappa}(\sigma, x) = 0 \}$ has full $[\sigma^s]$-measure on $\Omega$. Hence, $\sigma^s \cup \Theta^\kappa_0 = \sigma^s$. Moreover, for every $\varepsilon > 0$, it holds $\theta^{*\kappa}(\sigma^s, x) \leq \varepsilon$ for all $x \in \Theta^\kappa_0$. Then, the upper-density criterion contained in [3, Theorem 2.56] holds and therefore
\[
\sigma^s \cup \Theta^\kappa_0 \leq 2^\kappa \varepsilon \mathcal{H}^{\kappa^s} \Theta^\kappa_0 \quad \text{for all } \varepsilon > 0.
\]
Letting $\varepsilon \downarrow 0$ we deduce that $\sigma^s(F) = 0$ whenever $\mathcal{H}^s(F \cap \Theta^c_0) < \infty$ for a Borel set $F \subset \Omega$. By the definition of Hausdorff dimension, this implies $\dim_H(\sigma^s) \geq \kappa$. Since $\kappa \in [0, \ell)$ was chosen arbitrarily and $\dim_H(\sigma) = \dim(\sigma^s)$, we conclude that $\dim_H(\sigma) \geq \ell$.

Proof of Theorem 1. Let $x \in \Omega$ be a $|\mu|^s$-Lebesgue point so that every probability tangent measure $\sigma \in \Tan(\mu^s, x)$ can be written as $\sigma = e|\sigma|$ with $e = \frac{\partial \mu}{\partial \mu^s}(x) \in E$ and $|\sigma| \in \Tan(|\mu|^s, x)$.

Fix $\sigma \in \Tan(\mu^s, x)$. Note that $P^1(D)\sigma = 0$ in the sense of distributions on $B_1$, where $P^1(D)$ is the principal part of $P(D)$. This follows from the scaling rule

$$P^1(D)[T^d_{\#}x|\sigma] = -r_j \cdot P_0[T^d_{\#}x|\mu],$$

where the term in the right-hand side converges strongly to zero (in the sense of distributions) as $j \to \infty$. We now use the fact that $B_1$ is a star-shaped domain to define smooth approximations of $\sigma$ on $B_1$ as follows. Fix $\delta > 0$ to be a small parameter and define $\sigma_\delta := (T^d_{\#}1_{\delta})\sigma * \rho_\delta \in C^\infty(B_1; E)$, where $\rho_\delta$ is a standard mollifier at scale $\delta$. In this way $\sigma_\delta \mathcal{L}^d, B_1 \stackrel{\ast}{\to} \sigma$ and $|\sigma_\delta| \mathcal{L}^d, B_1 \stackrel{\ast}{\to} |\sigma|$ as $\delta \downarrow 0$ on $B_1$. Observe that, for each $\delta > 0$, the measure $\sigma_\delta$ (which satisfies $\sigma_\delta = e|\sigma_\delta|$) solves (in the classical sense) the homogeneous equation

$$P^1(D)\sigma_\delta = \sum_{i=1}^n P_i[e](\partial_i|\sigma_\delta|) = 0$$

on $B_1$. In symbolic language this reads $\mathbb{P}(\nabla|\sigma_\delta|)[e] = 0$, or equivalently, in terms of the differential inclusion,

$$\nabla(|\sigma_\delta|) \in \{\mathbb{P}[e] \equiv 0\}$$

on $B_1$. We deduce that $\nabla(|\sigma_\delta|)(x)[\xi] = 0$ for all $\xi \in \{\mathbb{P}[e] \equiv 0\}^\perp$ and all $x \in B_1$. In particular, for every $\delta > 0$, the measure $|\sigma_\delta| \mathcal{L}^d, B_1$ is $\{\mathbb{P}[e] \equiv 0\}^\perp$-invariant. Since the space of $\{\mathbb{P}[e] \equiv 0\}^\perp$-invariant measures is sequentially weak-* closed, we infer that

$$|\sigma| \in \Tan(|\mu|^s, x)$$

is a $\{\mathbb{P}[e] \equiv 0\}^\perp$-invariant measure on $B_1$.

Finally, since $x$ was chosen to be an arbitrary $|\mu|^s$-Lebesgue point, $|\mu|$ satisfies (C) with $\ell = \ell_\mathbb{P}$. We conclude, by Lemma 9, that $\dim_H(|\mu|) \geq \ell_\mathbb{P}$.

Proof of Corollary 3. By the very definition of $\Lambda_\mathbb{P}$, it follows that $\{\mathbb{P}[e] \equiv 0\} = \{0\}$ for all $e \notin \Lambda_\mathbb{P}$. Let us write $S_{\mathbb{P}, \mu} := \{x \in \Omega : \frac{\partial \mu}{\partial \mu^s}(x) \notin \Lambda_\mathbb{P}\}$. From (4), it follows that $|\mu| | S_{\mathbb{P}, \mu}$ satisfies the assumptions of Lemma 9 with $\ell = d$. Therefore $\dim_H(|\mu| | S_{\mathbb{P}, \mu}) = d$. The sought estimate is then an immediate consequence of the definition of Hausdorff dimension.

Appendix A. Multilinear algebra

Let $V$ be a finite dimensional euclidean space. The exterior algebra $\wedge^* V$ is a graded algebra with the “$\wedge$” product. Specifically $\wedge : \wedge^p V \times \wedge^q V \to \wedge^{p+q} V : (\xi^*, \omega^*) \mapsto \xi^* \wedge \omega^*$. 
Lemma 11. Let $V$ be an euclidean space of dimension $d$, let $m \in \{0, \ldots, d\}$ be a positive integer, and let $v^* \in \bigwedge^m V$ be a non-zero $m$-covector. Then

$$\text{Ann}^1(v^*) \subseteq \text{Gr}(\ell, V) \quad \text{for some } 0 \leq \ell \leq m.$$  
Moreover, if $v^*$ is a simple $m$-covector, then $\ell = m$.

Proof. The assertion that $\text{Ann}^1(v)$ is in fact a linear space follows immediately from the bi-linearity of the wedge product. Notice also that, on simple vectors $v^* = v_1^* \wedge \cdots \wedge v_m^*$, the result is immediate since then $\text{Ann}^1(v^*) = \text{span}\{v_1, \ldots, v_m\}$ (so that $\ell = m$ in this case). Any automorphism $\varphi$ of $V$ lifts to an automorphism $\Phi$ on $\bigwedge^* V$ satisfying

$$\Phi(v_1^* \wedge \cdots \wedge v_m^*) = \varphi(v_1^*) \wedge \cdots \wedge \varphi(v_m^*).$$

Hence, once $v^* \in \bigwedge^m V$ is fixed, we may assume without loss of generality that $\text{Ann}^1(v^*) = \text{span}\{e_1, \ldots, e_\ell\}$ for some $0 \leq \ell \leq d$, where $\{e_1, \ldots, e_d\}$ is an orthonormal basis of $V$. Indeed, let $\{\xi_1, \ldots, \xi_\ell\}$ be a normal basis of $\text{Ann}^1(v)$ and let $\varphi$ be the automorphism of $V$ satisfying $\varphi(\xi_i^*) = e_i^*$ for all $1 \leq i \leq \ell$ and $\varphi(w^*) = w^*$ for all $w^* \in \text{Ann}^1(v^*) \perp$. Then,

$$\Phi(\xi_i^* \wedge v^*) = e_i^* \wedge \varphi(v^*).$$

Let us fix $i_0 \in \{1, \ldots, \ell\}$ and observe that

$$e_{i_0}^* \wedge v^* = \sum_{1 \leq i_1 < \cdots < i_m \leq d \atop i_1, \ldots, i_m \neq i_0} v_{i_1 \ldots i_m} (e_{i_0}^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_m}^*),$$

where $v_{i_1 \ldots i_m} = \langle v^*, e_{i_1} \wedge \cdots \wedge e_{i_m} \rangle$. On the other hand, the set

$$\{ e_{i_0}^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_m}^* : 1 \leq i_1 < \cdots < i_m \leq d, i_1, \ldots, i_m \neq i_0 \}$$

forms a set of linearly independent $m$-covectors in $\bigwedge^{m+1} V$. Therefore, $e_{i_0}^* \wedge v^* = 0$ if and only if $e_{i_0}^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_m}^* = 0$ for all $1 \leq i_1 < \cdots < i_m \leq d$ such that $v_{i_1 \ldots i_m} \neq 0$. Since $1 \leq i_0 \leq \ell$ was chosen arbitrarily, this yields the set contention

$$\text{Ann}^1(v^*) \subseteq \bigcap_{i_1, \ldots, i_m \neq i_0 \atop v_{i_1 \ldots i_m} \neq 0} \text{Ann}^1(e_{i_1}^* \wedge \cdots \wedge e_{i_m}^*).$$

By the first observation, on the dimension of annihilators of simple vectors, we conclude that $\dim[\text{Ann}^1(v^*)] = \ell \leq m$. 

By duality, the exterior product induces then interior multiplication on the algebra of vectors $\bigwedge^*_d V$. This is a bilinear map $\iota: \bigwedge_q^p V \times \bigwedge^p V \mapsto \bigwedge^p_q V : (v, w^*) \mapsto v \iota w^*$, where $v \iota w$ acts on $(q-p)$-co-vectors $z^*$ as

$$\langle v \iota w^*, z^* \rangle = \langle v, w^* \wedge z^* \rangle.$$  

Similarly as before, when $p = 1$, we may consider its corresponding annihilator

$$\text{Ann}_1(v) := \{ \xi \in \mathbb{R}^d : v \iota \xi^* = 0 \}.$$
A similar (dual) proof to the one of Lemma 11 yields the following result:

**Lemma 12.** Let $V$ be an euclidean space of dimension $d$, let $m \in \{0, \ldots, d\}$, and let $v \in \bigwedge^m V$ be a non-zero $m$-vector. Then

$$\text{Ann}_1(v) \in \text{Gr}(d - \ell, V)$$

for some $0 \leq \ell \leq m$.

Furthermore, if $v$ is a simple $m$-vector, then $\ell = d - m$.

**References**

[1] G. Alberti, *Rank one property for derivatives of functions with bounded variation*, Proc. Roy. Soc. Edinburgh Sect. A 123 (1993), no. 2, 239–274. MR1215412

[2] L. Ambrosio, A. Coscia, and G. Dal Maso, *Fine properties of functions with bounded deformation*, Arch. Rational Mech. Anal. 139 (1997), no. 3, 201–238. MR1480240

[3] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000. MR1857292

[4] L. Ambrosio and H. M. Soner, *A measure theoretic approach to higher codimension mean curvature flows*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze Ser. 4, 25 (1997), no. 1-2, 27–49 (en). MR1655508

[5] A. Arroyo-Rabasa, *Rigidity of tangent functions of bounded variation and its applications in the calculus of variations*. In preparation.

[6] A. Arroyo-Rabasa, G. De Philippis, J. Hirsch, and F. Rindler, *Dimensional estimates and rectifiability for measures satisfying linear pde constraints*, ArXiv e-prints: 1811.01847 (2018), available at 1811.01847.

[7] E. De Giorgi, *Frontiere orientate di misura minima*, Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960-61, Editrice Tecnico Scientifica, Pisa, 1961. MR0179651

[8] C. De Lellis, *A note on Alberti’s rank-one theorem*, Transport equations and multi-D hyperbolic conservation laws, 2008, pp. 61–74. MR2504174

[9] G. De Philippis and F. Rindler, *On the structure of $A$-free measures and applications*, Ann. Math. 184 (2016), no. 3, 1017–1039. MR3549629

[10] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR0257325

[11] W. H. Fleming and R. Rishel, *An integral formula for total gradient variation*, Arch. Math. (Basel) 11 (1960), 218–222. MR0114892

[12] I. Fonseca and S. Müller, *$A$-quasiconvexity, lower semicontinuity, and Young measures*, SIAM J. Math. Anal. 30 (1999), no. 6, 1355–1390. MR1718306

[13] I. Fragalà and C. Mantegazza, *On some notions of tangent space to a measure*, Proc. Roy. Soc. Edinburgh Sect. A 129 (1999), no. 2, 331–342. MR1686704

[14] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995. MR1333890

[15] D. Preiss, *Geometry of measures in $\mathbb{R}^n$*: distribution, rectifiability, and densities, Ann. of Math. (2) 125 (1987), no. 3, 537–643. MR890162

[16] B. Raita, $L^1$-estimates and $A$-weakly differentiable functions, University of Oxford, Technical Report OxPDE-18/01, 2018.

[17] D. Spector and J. V. Schaftingen, *Optimal embeddings into lorentz spaces for some vector differential operators via gagliardo’s lemma*, ArXiv e-prints: 1811.02691 (2018), available at 1811.02691.

[18] J. Van Schaftingen, *Limiting Sobolev inequalities for vector fields and canceling linear differential operators*, J. Eur. Math. Soc. 15 (2013), no. 3, 877–921. MR3085095

Mathematics Institute, The University of Warwick

E-mail address: adolfo.arroyo-rabasa@warwick.ac.uk