Shear-Transformation-Zone Theory of Yielding in Athermal Amorphous Materials

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Yielding transitions in athermal amorphous materials resemble critical phenomena. Historically, they have been described by the Herschel-Bulkley rheological formula, which implies singular behaviors at yield points. In this paper, I examine this class of phenomena using an elementary version of the thermodynamic shear-transformation-zone (STZ) theory, focusing on the role of the effective disorder temperature, and paying special attention to scaling and dimensional arguments. I find a wide variety of Herschel-Bulkley-like rheologies but, for fundamental reasons not specific to the STZ theory, conclude that the yielding transition is not truly critical. In particular, there is a correlation length that grows rapidly, but ultimately saturates near the yield point.

I. INTRODUCTION

Numerical simulations and analytic approximations imply that yielding transitions in athermal amorphous materials resemble critical phenomena. These transitions are characterized by fluctuating regions of correlated, irreversible, molecular or granular rearrangements, whose sizes grow as the shear rates decrease and the systems approach their yield points. Recent work on this subject has been influenced by the stochastic model introduced in 1998 by Hébraud and Lequeux. For example, Nicolas, Martens and Barrat have studied a related model that focuses on elastic interactions between localized, stress-driven events. Lemaitre and Caroli have studied yielding in zero-temperature, molecular-dynamics simulations of two-dimensional, binary, Lennard-Jones mixtures. A common outcome of these investigations is a version of the well known Herschel-Bulkley (HB) rheology in which the incremental stress above the yield point is proportional, in these cases, to the square root of the plastic shear rate. This singular behavior is consistent with the idea that some kind of collective motion is occurring at the transition.

From its inception, the theoretical picture that has come to be known as “Soft Glassy Rheology” (SGR) focused explicitly on Herschel-Bulkley and related rheological behaviors. In contrast, the shear-transformation-zone (STZ) theory was never presented clearly in rheological language, which now seems to me to have been a mistake. The STZ theory is an attempt to describe amorphous plasticity in terms that are closer than SGR to the underlying physics and statistical thermodynamics of these processes. It is an intrinsically mean-field theory; it does not explicitly describe correlations between localized shear transformations, at least not in its present form. However, it does describe mean-field collective motion via the dynamics of a thermodynamically defined, effective disorder temperature. The noise generated by driven plastic deformation disorders the system, raises the effective temperature, and thereby creates new flow defects (STZ’s) that, in turn, contribute to the deformation rate. This nonlinear mechanism provides a framework in which to describe a variety of Herschel-Bulkley-like rheological behaviors. A similar mechanism has been shown to account for the basic features of dislocation-induced plasticity in polycrystalline materials, where the generation of defects, i.e. dislocations, is a hardening as well as a softening mechanism.

In fact, HB behavior has appeared implicitly in several earlier STZ-related publications. For example, in (LM), Manning and I analyzed data from glass-dynamics simulations by Haxton and Liu (HL), who saw HB power laws at low temperatures and in a range of relatively high shear rates. Similar results appear (less clearly) in my papers with Egami and Lieou.

In all three of these investigations, our primary interest was in the transitions from yielding to viscous behavior as functions of temperature or packing fraction. We presented our results in the form of log-log plots, focusing primarily on the region of vanishing shear rates, and thus did not pay attention to the rheological significance of the low-temperature data at intermediate and higher shear rates. My purpose here is to look harder at the latter aspects of the theory. I particularly want to emphasize the role of fundamental, dimensional and scaling arguments. It is these general arguments, and not system-specific ones, that lead to the conclusion that yielding in athermal amorphous materials is not a true critical transition.

II. STZ BASICS

A central feature of the STZ theory is that it treats the density and orientations of localized flow defects as dynamical variables. The STZ’s fluctuate into and out of existence in the environment of an elastic solid. When present, they undergo irreversible rearrangements and thus produce plastic deformation in response to stresses. These local transitions have been observed directly in simulations and, most recently, in a detailed experimental study of a colloidal glass by Jensen, Weitz, and Spaepen. The STZ equations of motion that determine these behaviors are subject to constraints imposed by the first and second laws of thermodynamics. This theory has been discussed extensively in the literature along with its applications to shear banding, fracture toughness, oscillatory viscoelastic-
ity \[^{17}\] \], and the like. In what follows, I briefly summarize the main features of the STZ theory without repeating detailed derivations. As part of this summary, however, I raise fundamental issues that I think need special emphasis.

For athermal systems undergoing steady-state flow in pure shear, the basic STZ relation between deviatoric stress \(\sigma\) and plastic shear rate \(\dot{\gamma}\) has the form

\[
q \equiv \dot{\gamma}^p l_0 = \epsilon_0 e^{-1/\chi} f(\sigma). \quad (2.1)
\]

Here, \(\chi\) is the effective temperature expressed in units of a characteristic STZ formation energy; and the Boltzmann factor \(e^{-1/\chi}\) is proportional to the density of STZ’s. \(\epsilon_0\) is a dimensionless constant, roughly of the order of unity. It is the product of the prefactor of the Boltzmann exponential, i.e. a site density, and the volume of the deformable core of an STZ. \(f(\sigma)/\tau_0\) is the stress-dependent STZ transition rate, to be specified below. For simplicity, I consider only pure shear in the \(x, y\) plane, so that \(q = q_{xx} = -q_{yy}\) and \(\sigma = \sigma_{xx} = -\sigma_{yy}\).

Note that I have introduced a time constant \(\tau_0\) in Eq.\([2.1]\). Even in athermal systems such as those considered here, the internal dynamics must determine a nonzero time scale on which local configurational fluctuations relax and dissipate energy. By introducing \(\tau_0\), I emphasize that I am not considering the extreme case of athermal, quasi-stationary, numerical models, which exclude any notion of time whatsoever. Nor am I considering models with infinitely long ranged interactions, or even jammed granular materials in which force chains may span the system. \(\tau_0\) must be a well defined, few-body relaxation time. Systems that are driven faster than \(1/\tau_0\) behave differently than those that are driven more slowly; thus \(q \sim 1\) plays a special role in the following analysis.

Derivatives of the Boltzmann factor on the right-hand side of Eq.\([2.1]\) appear in STZ papers such as \[^{7,14}\]. As shown there, this elementary statistical formula is valid, not just for equilibrated systems, but also for systems that are driven persistently away from equilibrium. It follows directly from the requirement that the statistically defined entropy be a non-decreasing function of time. \(\chi\) is the derivative of the configurational energy with respect to the configurational entropy (see Sec. \[III\]): it should not be thought of as a “noise temperature” or used, for example, in a Langevin equation – certainly not without some systematic rationale.

For present purposes, the rate factor in Eq.\([2.1]\) can be written in the form

\[
f(\sigma) = \begin{cases} 
    C(\sigma) \left(1 - \frac{\sigma}{\sigma_y}\right) & \text{if } |\sigma| > \sigma_y; \\
    0 & \text{if } |\sigma| < \sigma_y.
\end{cases} \quad (2.2)
\]

The yield stress \(\sigma_y\) locates an exchange of dynamic stability between non-flowing and flowing states. The fully time dependent STZ equations of motion (see \[^{2}\]) describe the behavior of the density of STZ’s and their average orientations with respect to the stress. These equations have stable fixed points for flowing states only if \(|\sigma| > \sigma_y\), where the rate at which STZ’s are deactivated by making transitions into the forward direction is balanced by the rate at which new STZ’s are formed. In the athermal limit, \(\sigma_y/|\sigma|\) is the fraction of STZ’s that is aligned with the stress; in Eq.\([2.2]\), it plays the role of a back stress. \(\sigma_y\) is a system-specific quantity. It first appears as a factor in the relation between the rate of work done by the external driving force and the frequency of mechanical noise that induces STZ creation and annihilation. (See the discussion of the noise frequency \(\Gamma\) in Sec. \[III\].)

The factor \(C(\sigma)\) in Eq.\([2.2]\) is a linear combination of STZ transition rates, and is a symmetric function of \(\sigma\). In LM, Manning and I wrote this term in the form

\[
C(\sigma) = \left[1 + \left(\frac{\sigma}{\sigma_1}\right)^2\right]^{n/2}. \quad (2.3)
\]

Here, I assume that any overall multiplicative constant on the right-hand side of this equation has been incorporated into the rate factor \(1/\tau_0\). For larger stresses, of the order of or greater than some \(\sigma_1\), Manning and I assumed that this rate would grow as the \(n\)’th power of the stress.

There are numerous possible rationales for choosing \(n\). For example, \(n = 1\) would be appropriate for a colloidal suspension in which particle motions are subject to linear viscous drag. A different possibility, for systems in which particles interact via short-ranged repulsive forces, is to invoke Bagnold scaling. As conjectured in LM, if there are no natural stress scales in the system other than the rate-independent \(\sigma_y\) and \(\sigma_1\), then dimensional analysis might require that the dynamic stress \(\sigma\) be proportional to an acceleration, i.e. an inverse time squared, and thus be proportional to the square of a dynamic rate. Conversely, the rate \(C(\sigma)/\tau_0\) would then be proportional to the square root of the stress. This result, by itself, is inconsistent with HB rheology. Nevertheless, the Bagnold assumption with \(n = 1/2\) fits the Haxton-Liu data quite nicely because, as shown in LM, the effective temperature \(\chi\) and the corresponding STZ density are rapidly varying if \(|\sigma| > \sigma_y\), which brings the time scale \(\tau_0\) back into the theory. This Bagnold analysis seems to me to be plausible but not compelling in the present context.

### III. EFFECTIVE DISORDER TEMPERATURE

The equation of motion for the effective temperature \(\chi\) is a statement of the first law of thermodynamics, i.e. energy conservation. Let \(U\) denote the configurational internal energy as a function of the configurational entropy \(S\). Then \(\chi = \partial U/\partial S\), and the first law takes the form

\[
\tau_0 \chi \dot{\bar{S}} \approx \tau_0 V c_{eff} \chi = 2 V q \sigma - Q, \quad (3.1)
\]

where \(\chi \dot{\bar{S}}\) is the rate of change of \(U\), \(c_{eff}\) is an effective specific heat, \(2 q \sigma / \tau_0\) is the rate at which work is done
on the system per unit volume, \( V \) is the volume, and \( Q/\tau_0 \) is the rate at which heat is dissipated. The second law requires that \( Q = V \kappa (\chi - \theta) \), where \( \kappa \) is a non-negative heat transfer coefficient, and \( \theta \) is the ambient temperature expressed in the same units as those used for \( \chi \). For the athermal systems considered here, \( \theta \approx 0 \). Thus, the steady-state equation of motion for \( \chi \) can be written in the form

\[
\tau_0 \dot{\chi} \approx \frac{2q\sigma}{c_{\text{eff}}} - \kappa \chi \equiv \frac{2q\sigma}{c_{\text{eff}}} \left[ 1 - \frac{\chi}{\chi(q)} \right] = 0. \quad (3.2)
\]

With these assumptions, I have defined \( \chi \equiv \dot{\chi}(q) \) in steady state. Note that the underlying physics is thermal transport, which somehow must be encoded in \( \dot{\chi}(q) \).

To evaluate \( \dot{\chi}(q) \), it is useful to think about the analogous relation between ordinary temperature and relaxation rates in glasses. Note first that, if we shear or otherwise "stir" a system more slowly than any of its internal relaxation rates, i.e. set \( q \ll 1 \), then this system must ultimately reach a \( q \)-independent effective temperature, say \( \chi_0 \). It may take longer and longer to equilibrate to \( \chi_0 \) as \( q \to 0 \); but, in this limit, the ultimate state of disorder should depend on the extent – and not the time duration – of the deformation. The glassy analog of this behavior is that the configurational relaxation time \( \tau_\alpha \), like \( 1/q \), diverges (or becomes unmeasurably large) as the temperature approaches an ideal glass transition temperature. In other words, the temperature varies increasingly slowly as \( \tau_\alpha \) goes to infinity. Like the glass transition temperature, \( \chi_0 \) is a system-specific parameter. We will see in Sec. IV that it is important in determining the nature of the yielding transition.

At the other end of the range of shear rates, the glass analogy implies Arrhenius behavior. Sufficiently far above the glass temperature, the configurational relaxation rate \( \tau_\alpha^{-1} \) (or the inverse viscosity, or the diffusion constant, or the like) usually appears experimentally to be controlled by a thermally activated process with a temperature-independent energy barrier. Similarly, for values of \( q \) not too small, the Haxton-Liu measurements of the effective temperature are well described by a relation of the form \( q \approx q_0 e^{-A/\chi} \), where \( A \) is an activation energy expressed here, like \( \chi_0 \), in units of the STZ formation energy. To interpolate between these two limiting behaviors in \( \dot{\chi} \), my coauthors and I have postulated a modified Vogel-Fulcher-Tamann (VFT) formula:

\[
q(\dot{\chi}) = q_0 \exp \left[ -\frac{A}{\chi} - \alpha_{\text{eff}}(\dot{\chi}) \right], \quad (3.3)
\]

where

\[
\alpha_{\text{eff}}(\dot{\chi}) = \frac{\chi_0}{\dot{\chi} - \chi_0} \exp \left[ -b \frac{\dot{\chi} - \chi_0}{\chi_0 - \chi_0} \right]. \quad (3.4)
\]

The first term on the right-hand side of Eq. (3.3) is the conventional VFT divergence at \( \dot{\chi} = \chi_0 \). Here it is cut off at \( \dot{\chi} \) by an exponential factor so that the behavior of \( q(\dot{\chi}) \) in Eq. (3.3) is dominated by the Arrhenius term for \( \dot{\chi} \sim 0 \). This interpolation formula has no special physical significance that I am aware of and, so far, its details have not seemed to be absolutely crucial for interpreting experimental or computational data. On the other hand, the parameter \( A \) plays a special role in what follows.

In interpreting Eq. (3.3), note that the STZ theory has an absolute upper limit of validity where \( \chi \to \infty \). At such large values of \( \chi \), the density of STZ’s becomes large, and the theory is no longer consistent with a model of a solid containing a dilute population of flow defects. A natural way to formulate the theory, then, is to choose \( q_0 = 1 \) so that the maximum theoretical shear rate is \( \dot{\chi}_{\text{max}} = \frac{1}{\tau_0} \). Above this rate, the system has insufficient time to relax between irreversible rearrangements; i.e. it behaves like a fluid. Adopting this convention, however, means that we need to use a physically realistic estimate of \( \tau_0 \).

### IV. HERSCHEL-BULKLEY BEHAVIOR

At large stresses and shear rates, the preceding set of equations immediately produces a generalized Herschel-Bulkley relation. In this limit,

\[
q \approx \epsilon_0 e^{-1/\chi} \left( \frac{\sigma}{\sigma_1} \right)^n \approx q_0 e^{-A/\chi}. \quad (4.1)
\]

Eliminating \( \chi \), we find

\[
\sigma \approx \sigma_1 \left( \frac{q_0}{\epsilon_0} \right)^{1/n} \left( \frac{q}{q_0} \right)^{\beta}; \quad \beta = \frac{A - 1}{nA}. \quad (4.2)
\]

Then, for example, the Bagnold choice \( n = 1/2 \), and \( A = 4/3 \), produces the conventional HB power law, \( \beta = 1/2 \). However, with different choices of parameters, this STZ-based theory of athermal yielding produces a wide range of rheological behaviors, well beyond the simple square-root law.

To explore some of these possibilities, I start by reanalyzing the low temperature HL data. I have made two changes from the earlier LM analysis. First, instead of using the microscopic time scale adopted by HL, I have deduced a presumably more realistic \( \tau_0 \) by setting \( q_0 = 1 \). The resulting \( \tau_0 \) is about a factor 12 larger than the original HL value, but is still a microscopic time scale. Second, in our previous attempt to fit all the HL data, including at high temperatures, with as few parameters as possible, Manning and I used what I now think was an unrealistically small value of \( \sigma_1 \), and thus needed to use a correspondingly small value of \( \epsilon_0 \). In simple athermal systems, however, there is really only one physically meaningful stress scale, which is set by the shear modulus \( \mu \). Both \( \sigma_0 \) and \( \sigma_1 \) should be very roughly of the order of \( \mu \). Therefore, I have set \( \sigma_1 = \sigma_y \), and have measured the dynamic stress \( \sigma \) in units of \( \sigma_y \). Then I have adjusted \( \epsilon_0 \)
to fit the data, finding \( c_0 \approx 0.26 \) in accord with my expectation that this parameter should be roughly equal to unity. Other parameters, as given in LM, are: \( A = 1.5 \) (so that \( \beta = 2/3 \) as observed), \( \chi_0 = 0.2 \) (directly observed as shown in Fig. 2), \( \chi_1 = 0.26 \), \( \chi_A = 0.3 \), and \( b = 3 \).

Figures 1 and 2 show the HL data in their original form as a log-log plot of \( \sigma/\sigma_y \) versus shear rate \( q \), and a semilog plot of the effective temperature \( \chi(q) \), along with the theoretical curves evaluated using the full equations given above. In Fig. 2 note that the upper limit of the data at \( q \approx 0.25 \) occurs at \( \chi \approx 0.7 \), which should be about at the limit of validity of the STZ theory, as intended. Figure 3 is a direct plot of the data in which the HB form is easily visible, with the crossover to the \( q \ll 1 \) behavior squeezed into a small part of the graph near \( q = 0 \). For comparison, the dashed curve shows the approximation in Eq. (4.2) displaced upward by one unit of the yield stress. As can just be seen in both the data and the theory, the stress rises linearly above the yield point:

\[
\sigma - \sigma_y \approx q \frac{\sigma_y}{c_0} e^{1/\chi_0}. \tag{4.3}
\]

For the chosen parameters, this formula means that the initial slope of the graph in Fig. 3 is large but finite, approximately 570. The rapid growth of \( \chi \) and the corresponding growth of the STZ density cause the material to soften, with the result that the curve bends over into power-law behavior at small values of \( q \).

To illustrate the variety of rheological behaviors that emerge from this theory, I show in Fig. 4 a set of five different curves of stress versus shear rate, using all but one of the same parameters that were used to fit the HL data shown in Figs. 1 - 3. The exception is that I have chosen a sequence of different values of the dimensionless activation energy \( A \) that controls the dissipation rate in Eq. (3.2). From bottom to top in the figure, these values are \( A = 2.0, 1.33, 1.1, 1.0, \) and \( 0.9 \). The corresponding values of the HB exponent are \( \beta = 1.0, 0.5, 0.18, 0, \) and \( -0.22 \). The second curve, with \( \beta = 0.5 \), is the square-root law that appears in the fluctuation-theory literature, e.g. [1–3]; and the third and fourth curves, with small values of \( \beta \), are included just to show the transition between stable and unstable rheologies.

The more interesting cases shown in Fig. 4 are the first and the fifth. The first, at the bottom with \( \beta = 1 \), illustrates one of the ways in which this theory can produce a conventional, linear, Bingham rheology. By far the most realistic way for the linear law to be observed, however, is when \( \tau_0 \) is a microscopic time, perhaps a few molecular vibration periods, so that shear rates with \( q \sim 1 \) are out of the range of most experiments. Then, Eq. (4.3), with a constant \( \chi_0 \), is accurate throughout an observable range of shear rates, \( q \ll 1 \). This is what happened, for example, in earlier analyses of plasticity in metallic glasses (e.g. [18]), where \( \chi_0 \) was more nearly of the order of 0.1 instead of 0.2. Yet another possibility, still within the scope of this theory, is when materials are intrinsically disordered and soft, i.e., when \( \chi_0 \) is large, so that the initial slope of \( \sigma(q) \) in Eq. (4.3) is small and remains

FIG. 1: Log-Log plot of stress \( \sigma \) in units of the yield stress \( \sigma_y \) as a function of dimensionless shear rate \( q \). The data points are from HL, with rescaled values of \( q \) as described in the text.

FIG. 2: Semi-log plot of the steady-state effective temperature \( \chi \) as a function of dimensionless shear rate \( q \). The data points are from HL, with rescaled values of \( q \) as described in the text.
linear at observably large shear rates.

The fifth interesting case in Fig. 3 at the top with $\beta = -0.22$, exhibits shear-rate weakening above $q \approx 0.07$, and therefore must be dynamically unstable. Daub and Carlson [19] have examined a model of this kind in detail, and have shown that it produces both shear-banding and stick-slip instabilities. More recently, Lieou et al. [20] have applied this kind of theory – but with $\beta > 0$ – to granular materials of the kind that occur in fault gouge. They have added terms to Eq.(3.2) to account for shear-rate dependent frictional dissipation due to collisions between angular grains, and also to account for dissipation induced by tapping or other vibrational perturbations. In this generalization of the theory, the effective temperature can become a non-monotonic function of $q$, and the volume may also vary non-monotonically. But these modifications are beyond the scope of the present discussion, because they implicitly introduce new intrinsic time scales analogous to $\tau_0$.

V. SPATIAL HETEROGENEITIES

As a first step in studying spatial variations in this theory, I have linearized the equation of motion for $\chi$, Eq.(5.2), in the space and time dependent variable $\chi' = \chi - \hat{\chi}(q)$, and have added a diffusion term:

$$\tau_0 \dot{\chi} = -\frac{2\sigma q}{c_{eff}} \frac{\chi'}{\hat{\chi}(q)} + D \nabla^2 \frac{\chi'}{\hat{\chi}(q)}.$$  (5.1)

Because $\hat{\chi}(q)$ diverges when $q \rightarrow q_0$, it is best to linearize in $\chi'/\hat{\chi}(q)$, and to define the diffusion coefficient $D$ accordingly. Diffusion terms of this kind have appeared previously in the STZ literature, e.g. in [13, 19]; but it seems to me that $D$ has never been given the attention that it deserves. Here, I argue that $D$ must have the form $\Gamma \ell^2$, where $\Gamma/\tau_0$ is the noise-generated attempt frequency for activation and annihilation of STZ’s, and $\ell$ is the elementary diffusion length, which must be the average spacing between STZ’s. These results follow from dimensional analysis. The only scalar quantity in this system with dimensions of rate is the power per unit volume, $\sigma q/\tau_0$. To convert this quantity into a frequency, divide by a stress times a volume (an energy), say, $\sigma_0 v_0$, and multiply by the only relevant volume, i.e. the volume per STZ, $v_0 e^{1/\chi}$. The factors $v_0$ cancel, and subsequent athermal STZ analysis reveals that $\sigma_0$ is equal to the yield stress $\sigma_y$. Thus, $\Gamma = 2q(\sigma/\sigma_y)e^{1/\chi}$. Similarly, for dimension $d$, $\ell^d = v_0 e^{1/\chi}$. The result is

$$D = 2a^2 q \frac{\sigma}{\sigma_y} e^{(1+2/d)/\chi},$$  (5.2)

where $v_0 = a^d$ defines a characteristic interparticle length scale $a$. 

FIG. 3: Stress $\sigma$ in units of the yield stress $\sigma_y$ as a function of dimensionless shear rate $q$. The data points are from HL, with rescaled values of $q$ as described in the text. The dashed line is the approximation given in Eq.(4.2) displaced upward by one unit of the yield stress.

FIG. 4: Stress $\sigma$ in units of the yield stress $\sigma_y$ as a function of dimensionless shear rate $q$, for five different values of the dimensionless activation energy $A$. From bottom to top, $A = 2.0, 1.33, 1.1, 1.0$, and $0.9$. The corresponding values of the HB exponent are $\beta = 1.0, 0.5, 0.18, 0$, and $-0.22$. 

The following graph shows the stress ratio $\Sigma/\Sigma_1$ as a function of shear rate $q$ for different values of $q_2$. The stress ratio is defined as $\Sigma/\Sigma_1$, where $\Sigma$ is the stress at a given shear rate $q$, and $\Sigma_1$ is the yield stress at the reference shear rate $q_1$. The graph demonstrates how the stress ratio changes with shear rate and how it is affected by the different values of $q_2$.
I emphasize that I have not attempted here to construct a complete theory of the experimental data, even within the assumptions that led to Eq. (5.3). To do that, I would have had to couple the equation of motion for $\chi$, Eq. (5.1), to a position-dependent and fully tensorial version of the STZ theory. For example, we know by symmetry that, in plug flow of this kind, both the shear rate and the shear stress vanish at the center of the gap, near where the values of $\xi$ in Fig. 5 are shown to saturate. Such an analysis might not make qualitative changes in the results shown here, which relate to measurements away from the center where shear rates are nonzero. However, the full analysis would be absolutely essential for computing the strain-rate profile across the flow, as is attempted by Jop et al. in their estimate of a different, more slowly varying, strain-rate dependent length scale. Much of that kind of analysis has been carried out by Hennan and Kamrin in their recent theory of nonlocal rheology in dense granular flows. It should be useful to compare their theory with this one.

VI. REMARKS AND QUESTIONS

The equations of motion presented here, when generalized for variations in space and time and coupled to the equations of motion for elastic fields, constitute a self-contained theory of athermal amorphous rheology. We know from recent experience that this theory does a good job of describing dynamic phenomena such as shear banding instabilities, brittle and ductile failure in the neighborhoods of crack tips, stick-slip behavior of granular materials in earthquake faults, and the like. In analogy to the well known equations of motion for fluids, it seems reasonable to expect that these rheological equations of motion, when solved for large systems coupled to external forces and boundary conditions, will predict chaotic behaviors with heterogeneous deformations and local failures on many different length and time scales.

This theory contains system-specific, physical ingredients such as the yield stress $\sigma_y$ and the parameters $A$ and $\chi_0$ that determine the effective disorder temperature; but it is not phenomenological in the same sense as are conventional rheological theories such as those described in Sec. III. For example, the deviation from criticality at the yielding transition in this theory is not an artifact of any purely phenomenological assumption. As argued in Sec. III, the state of disorder at small shear rate $q$ must be independent of $q$, i.e., $\chi \rightarrow \chi_0 > 0$. A nonzero degree of disorder means that, in steady-state deformation, there must be a constant density of some kind of flow defects and, therefore, a linear relation between $q$ and the stress increment just above the yield point. This theoretical conclusion is supported by the numerical simulations shown in Fig. 4 and the experimental data in Fig. 5.

Evidence in favor of criticality at yielding transitions comes from stochastic models of the kind orig-
ominated by Hébraux and Lequeux \[1\] and extended in \[2\]. More generally, the yielding mechanism in solids looks as if it should be in the same universality class as the depinning mechanism that produces, for example, broad distributions of slipping events on earthquake faults. Renormalization-group analyses of depinning models have produced scaling relations similar to the Herschel-Buckley law. (For example, see Fisher et al. \[26\].) The Hébraux and Lequeux model produces a Herschel-Bulkley exponent of exactly $1/2$ in the limit of zero strain rate. Even if we assume that some limiting approximation is being made that eliminates the saturation effect (perhaps by implicitly setting $\chi_0 = 0$), the question remains: why is $\beta = 1/2$ in this model? We know that other HB exponents do occur, e.g. in the Haxton-Liu results discussed here, and in a variety of experimental rheological situations.

A related question is whether the deterministic nature of the rheological theory presented here might be inadequate for describing broad distributions of event sizes. The scaling analysis in \[26\] starts with the assumption of a quenched random pinning force. Hébraux and Lequeux couch their analysis in terms of probability distributions over the magnitudes of local stresses. Is that kind of analysis essential? Or might the observed behaviors be results of deterministic chaos, as was the case in our earlier slider-block studies of earthquake dynamics? \[27\]

Large-scale numerical solutions of the present rheological equations of motion might help answer such questions.

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