String Inference from the LCP Array

Juha Kärkkäinen¹, Marcin Piątkowski², and Simon J. Puglisi¹

¹ Helsinki Institute of Information Technology (HIIT) and Department of Computer Science, University of Helsinki, Finland
² Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Toruń, Poland
{juha.karkkainen,simon.puglisi}@cs.helsinki.fi
marcin.piatkowski@mat.umk.pl

Abstract. The suffix array is often augmented with the longest common prefix (LCP) array which is, in essence, a representation of the suffix tree shape. We consider the problem of inferring a string from an LCP array, i.e., determining whether a given array of integers is a valid LCP array, and if it is, reconstructing some text or all texts with that LCP array. We provide two results. (1) We describe a linear time algorithm for inferring a string from an LCP array that contains a single zero (indicating a binary alphabet). For a valid LCP array the algorithm outputs a Burrows-Wheeler transform (BWT), the inversion of which produces a collection of cyclic strings whose generalized LCP array is identical to the input. Furthermore, the algorithm outputs a linear size representation of all such BWTs. (2) We prove that determining whether one of the valid BWTs produced by the above algorithm inverts to a single (cyclic) string rather than a set of strings is NP-hard. This shows that reverse engineering an LCP array is hard if we insist that the result is a single string, but easy over a binary alphabet if the result can be a collection of strings. The latter case for a larger alphabet remains an open problem.

Keywords: LCP array, string inference, BWT, suffix array, suffix tree, NP-hardness

1 Introduction

For a string $X$ of $n$ symbols, the suffix array (SA) [20] contains pointers to the suffixes of $X$, sorted in lexicographical order. The suffix array has a wealth of applications in string processing, and is often augmented with a second array — the longest common prefix (LCP) array — storing the length of the longest common prefix between lexicographically adjacent suffixes; i.e., $LCP[i]$ is the length of the LCP of suffixes $X[SA[i]..n]$ and $X[SA[i-1]..n]$.

In essence, the LCP array corresponds to the shape and string depths of the suffix tree [29] — the compacted trie of all the string’s suffixes. When coupled with the SA, LCP allows top-down and bottom-up traversals of the suffix tree to be efficiently simulated. Such traversals are at the heart of many efficient string processing algorithms (see [12413927]).

These structures can be generalized for a (multi)set of strings instead of a single string.

In this paper, we consider the problem of determining whether a given array of integers is a valid LCP array, and if it is, reconstructing some (set of) strings, or all (sets of) strings, for which it is the LCP array. We call this the String Inference from the LCP Array (SILA) problem. We provide two main results.

1. We describe a linear time algorithm for the SILA problem for LCP arrays containing a single zero (indicating a binary alphabet). For a valid LCP array the algorithm outputs a Burrows-Wheeler transform (BWT), the inversion of which produces a collection of cyclic strings, whose LCP array is identical to the input. Furthermore, the algorithm outputs a linear size representation of all such BWTs.

2. We prove that determining whether one of the valid BWTs produced by the above mentioned algorithm inverts to a single (cyclic) string rather than a set of strings is NP-hard. We call this single string variant problem 1-SILA.
This shows that inferring a string from an LCP array is hard if we insist that the result is a single string, but easy over a binary alphabet if the result can be a collection of strings. The latter case for a larger alphabet remains an open problem.

Our results belong to the growing literature on string inference, and in the remainder of this section we place our results in context, describing prior work on related problems. Then, in Section 2 we lay down basic concepts and notation, before describing our algorithm in Sections 3–4. Sections 5–8 then show that 1-SILA is NP hard.

1.1 Background and Related Work

String inference from a variety of data structures has received a great deal of attention of late, with authors considering border arrays [10,9,8], parameterized border arrays [16], the Lyndon factorization [24], suffix arrays [3,19], KMP failure tables [9,11], prefix tables [5], cover arrays [7], and directed acyclic word graphs [3]. The motivation for studying string inference problems is to gain a better understanding of the combinatorics of these data structures.

To our knowledge we are the first to consider string inference from the LCP array, but several authors have considered the highly-related problem of determining if a given tree is a suffix tree (and, if so, inferring a string from it). Clearly, the suffix array can be extracted from the suffix tree in linear time by writing down the leaf labels in the depth first order. The values of the LCP array are the string depths of nodes in the suffix tree when the nodes are visited in depth-first order. Similarly, the suffix tree can be constructed almost trivially given the text, the suffix array and the LCP array. In fact, given just the LCP array, we get the suffix tree without edge labels; we even get the lengths of the labels for non-leaf edges. Such a labelless suffix tree is known as a Patricia trie [23] or a blind trie [22]. Thus SILA can be considered a variant of string inference from Patricia/blind trie.

The problem of deciding if a given tree corresponds to the suffix tree of any string goes back at least nine years [26] and has recently been considered by three different sets of authors [17,11,28], each deriving a linear time algorithm for a slightly different variant of the problem. Crucially, all three algorithms assume that the suffix tree is augmented with suffix links, which provide a lot of additional information making the task much easier. The LCP array as a suffix tree representation does not provide suffix links. According to Cazaux and Rivals [4], the case without suffix links was considered but not solved in [26]. We also suspect that others have considered the case of no suffix links but unsuccessfully. Indeed, our hardness results show that it is a more difficult variant of the problem.

Apart from the three suffix tree papers discussed above, another (somewhat tangentially) related result is due to He et al. [14], who prove that it is NP hard to infer a string from the longest-previous-factor (LPF) array. It is well known that LPF is a permutation of LCP [6] but otherwise it is a quite different data structure. For example, it is in no way concerned with lexicographical ordering. Like our NP-hardness proof, He et al.’s reduction is from 3-SAT, but the details of each reduction appear to be very different. Moreover, their construction requires an unbounded alphabet while our construction works for a binary alphabet and thus for any alphabet.

To the best of our knowledge, all of the previous string inference problems aim at obtaining a single string from some data structure, and we are the first to consider the generalized case of inferring a set of strings. The fact that SILA is easy but 1-SILA is hard shows that this distinction between a single string and a set of strings makes a crucial difference.

2 Basic notions

Let \( v \) be a string of length \( n \) and let \( \hat{v} \) be obtained from \( v \) by sorting its characters. The standard permutation \([12,15]\) of \( v \) is the mapping \( \Psi_v : [0..n] \to [0..n] \) such that for every \( i \in [0..n] \) it holds \( \hat{v}[i] = v[\Psi_v(i)] \) and for any \( \hat{v}[i] = \hat{v}[j] \) the relation \( i < j \) implies \( \Psi_v(i) < \Psi_v(j) \). In other words, \( \Psi_v \) corresponds to the stable sorting of the characters. Let \( C = \{c_i\}_{i=1}^n \) be
the disjoint cycle decomposition of $\Psi_v$. We define the inverse Burrows-Wheeler transform IBWT as the mapping from $v$ into a multiset of cyclic strings $W = \{w_i\}_{i=1}^s$ such that for any $i \in [1..s]$ and $j \in [0..|c_i|)$, $w_i[j] = v[w_i(c_i[j])].$

Example 1. For $v = bbaabaa$, we have IBWT($v$) = $\{aab,aab,ab\}$ as illustrated in the following table (showing $\widehat{v}$ and $\Psi_v$) and figure (showing the cycles of $\Psi_v$ as a graph). The character subscripts are provided to make it easier to ensure stability.

| $i$ | $v[i]$ | $\widehat{v}[i]$ | $\Psi_v[i]$ |
|-----|--------|-----------------|------------|
| 0   | b1     | a1             | 2          |
| 1   | b2     | a2             | 3          |
| 2   | a1     | a3             | 4          |
| 3   | a2     | a4             | 5          |
| 4   | b3     | a5             | 6          |
| 5   | a3     | b1             | 7          |
| 6   | b2     | b2             | 2          |
| 7   | a4     | b3             | 1          |
| 8   | b3     | a1             | 3          |

The elements of $W$ are primitive cyclic strings. Cyclic means that all rotations of a string are considered equal. For example, $aab$, $aba$ and $baa$ are all equal. A string is primitive if it is not a concatenation of multiple copies of the same string. For example, $aab$ is primitive but $aabaab$ is not. For any alphabet $\Sigma$, the mapping IBWT is a bijection between the set $\Sigma^*$ of all (non-cyclic) strings and the multisets of primitive cyclic strings over $\Sigma$ \cite{21}.

The set of positions of $W$ is defined as the set of integer pairs $\text{pos}(W) := \{(i,p) : i \in [1..s], p \in [0..|w_i|)\}$. For a position $(i,p) \in \text{pos}(W)$ we define a cyclic suffix $W_{(i,p)}$ as the infinite string that starts at $(i,p)$, i.e., $W_{(i,p)} = w_i[p]w_i[p+1 \mod |w_i|]w_i[p+2 \mod |w_i|], \ldots$. The multiset of all cyclic suffixes of $W$ is defined as $\text{suffix}(W) := \{W_{(i,p)} : (i,p) \in \text{pos}(W)\}$. We say that a string $x$ occurs at position $(i,p)$ in $W$ if $x$ is a prefix of the suffix $W_{(i,p)}$.

The (cyclic) suffix array of a multiset of strings $W$ is defined as an array $\text{SA}_W[j] = (i_j,p_j)$, where $(i_j,p_j) \in \text{pos}(W)$ for all $j \in [0..n)$ and $W_{(i_{j-1},p_{j-1})} \leq W_{(i_j,p_j)}$ for all $j \in [1..n)$. The Burrows-Wheeler transform (BWT) is a mapping from $W$ into the string $v$ defined as $v[j] = w_i[p-1 \mod |w_i|]$, where $(i,p) = \text{SA}_W[j]$, i.e., $v[j]$ is the character preceding the beginning of the suffix $W_{\text{SA}_W[j]}$. The BWT is the inverse of IBWT \cite{21,18}.

The longest-common-prefix array $\text{LCP}_W[1..n]$ is defined as $\text{LCP}_W[j] = \text{lcp}(W_{\text{SA}_W[j-1]}, W_{\text{SA}_W[j]})$ for $0 < j < n$, where $\text{lcp}(x,y)$ is the length of the longest common prefix between the strings $x$ and $y$.

Example 2. For $W = \{ab, aab, aab\}$ we have

$$\text{suffix}(W) = \{\text{aab}\omega, (\text{aab})\omega, (\text{aba})\omega, (\text{aba})\omega, (\text{baa})\omega, (\text{baa})\omega, (\text{baa})\omega\}$$

$$\text{SA}_W = [(2,0), (3,0), (2,1), (3,1), (1,0), (2,2), (3,2), (1,1)]$$

$$\text{LCP}_W = [\omega, 1, \omega, 3, 0, \omega, 2]$$

The suffixes represented by the suffix array entries can also be expressed as follows.

Lemma 1. For $i \in [0..n)$, $W_{\text{SA}_W[i]} = \widehat{v}[i]\widehat{\Psi}_v(i)\widehat{\Psi}_w(i)\widehat{\Psi}_w(i)\widehat{\Psi}_w(i)\ldots$.

3 Basic Properties of Intervals

Many algorithms on suffix arrays and LCP arrays are based on iterating over a specific type of array intervals. In this section, we define these intervals and establish their key properties. For proofs and further details, we refer to \cite{125}.

Let $v \in \{a,b\}^n$ and $W = \text{IBWT}(v)$. Let $\text{SA} = \text{SA}_W$ be the suffix array and $\text{LCP} = \text{LCP}_W$ the LCP array of $W$. Note that from now on, we will assume a binary alphabet.

Definition 1 (x-interval). An interval $[i..j]$, $0 \leq i \leq j \leq n$, is called the x-interval $(x \in \Sigma^*) if and only if

1. $x$ is not a prefix of $W_{\text{SA}_W[i-1]}$ (or $i = 0$)
2. $x$ is a prefix of $W_{\text{SA}_W[k]}$ for all $k \in [i..j)$
3. $x$ is not a prefix of $W_{\text{SA}_W[j]}$ (or $j = n$)
In other words, in the suffix array the x-interval \( \text{SA}[i..j] \) consists of all suffixes of \( W \) with \( x \) as a prefix. Thus the size \( j - i \) of the interval is the number of occurrences of \( x \) in \( W \), which we will denote by \( n_x \).

**Lemma 2.** (\( \ell \)-interval). An interval \( [i..j] \), \( 0 \leq i < j \leq n \), is called an \( \ell \)-interval (\( \ell \in \mathbb{N}_0 = \mathbb{N} \cup \{\omega\} \)) if and only if

1. \( \text{LCP}[i] < \ell \) (or \( i = 0 \))
2. \( \min \text{LCP}[i + 1..j] = \ell \) (If \( i + 1 = j \), we define \( \min \text{LCP}[i + 1..j] = \omega \).)
3. \( \text{LCP}[j] < \ell \) (or \( j = n \))

The two types of intervals are closely related.

**Lemma 2.** Every nonempty \( x \)-interval is an \( \ell \)-interval for some (unique) \( \ell \geq |x| \). Every \( \ell \)-interval is an \( x \)-interval for some string \( x \) of length \( \ell \).

**Corollary 1.** If an \( x \)-interval \( [i..j] \) is an \( \ell \)-interval for \( \ell > |x| \), there exists a (unique) string \( y \) of length \( \ell - |x| \) such that \( [i..j] \) is the \( xy \)-interval.

Thus the \( \ell \)-intervals represent the set of all distinct \( x \)-intervals. This and the fact that the total number of \( \ell \)-intervals is \( \mathcal{O}(n) \) are the basis of many efficient algorithms for suffix arrays, see e.g., [125].

## 4 String Inference from LCP Array

We are now ready to describe the algorithm for string inference from an LCP array. Given an LCP array \( \text{LCP}[1..n] \), our goal is to construct a string \( v \in \{a, b\}^n \) such that \( \text{LCP} = \text{LCP}_{\text{IBWT}(v)} \). At first, we assume that such a string \( v \) exists, and consider later what happens if the input is not a valid LCP array (for a binary alphabet).

Let \( \text{RMQ}_{\text{LCP}}[i..j] \) denote the range minimum query over the LCP array that returns the position of the minimum element in \( \text{LCP}[i..j] \), i.e., \( \text{RMQ}_{\text{LCP}}[i..j] = \arg \min_{k \in [i..j]} \text{LCP}[k] \). The LCP array is preprocessed in linear time so that any RMQ can be answered in constant time. Then any \( x \)-interval can be split into two subintervals as shown in the following result.

**Lemma 3.** Let \( [i..j] \) be an \( x \)-interval and an \( \ell \)-interval for \( \ell < \omega \), and let \( k = \text{RMQ}_{\text{LCP}}[i + 1..j] \). Then, for some string \( y \) of length \( \ell - |x| \), \( [i..k] \) is the \( xya \)-interval and \( [k..j] \) is the \( xyb \)-interval.

This approach provides us with an easy way to recursively enumerate all \( \ell \)-intervals. We will also keep track of \( ax \) and \( bx \)-intervals together with any \( x \)-interval, even if we do not know \( x \) precisely. From the intervals we can determine the numbers of occurrences, \( n_{ax} \) and \( n_{bx} \), which are useful in the inference of \( v \).

**Lemma 4.** Let \( [i..j] \) be the \( x \)-interval. Then \( v[i..j] \) contains exactly \( n_{ax} \) a’s and \( n_{bx} \) b’s.

In particular, when either \( n_{ax} \) or \( n_{bx} \) drops to zero, we have fully determined \( v[i..j] \) for the \( x \)-interval \( [i..j] \). In such a case, the LCP array intervals have to satisfy the following property.

**Lemma 5.** Let \( [i_y..j_y] \) be the \( y \)-interval for \( y \in \{x, ax, bx\} \). If \( n_{ax} = j_{ax} - i_{ax} = 0 \), then \( \text{LCP}[i_{bx} + 1..j_{bx}] = 1 + \text{LCP}[i_{x} + 1..j_{x}] \), where \( 1 + A \), for an array \( A \), denotes adding one to all elements of \( A \). Symmetrically, if \( n_{bx} = 0 \), then \( \text{LCP}[i_{ax} + 1..j_{ax}] = 1 + \text{LCP}[i_{x} + 1..j_{x}] \).

The main procedure is given in Algorithm 1. The main work is done in the recursive procedure InferInterval given in Algorithm 2. The procedure gets as input the \( x \)-, \( ax \)- and \( bx \)-intervals for some (unknown) string \( x \), splits the \( x \)-interval into \( xya \)- and \( xyb \)-subintervals based on Lemma 5, and tries to split \( ax \)- and \( bx \)-intervals similarly. If all subintervals are nonempty, the algorithm processes the two subinterval triples recursively (lines 28 and 29).
Algorithm 1: Infer BWT from an LCP array

Input: an array LCP[1..n] of integers and ω's
Output: a string v ∈ {a, b}n such that LCP_BWT(v) = LCP together with a set S of swap intervals, or false if there is no such string v

1. S := ∅;
2. preprocess LCP for RMQs;
3. k := RMQ_LCP[1..n];
4. if LCP[k] ≠ 0 then
   5. if LCP[k] = ω then return a", " otherwise return false
   6. success := InferInterval([0, n], [0, k], [k, n]);
7. if success=false then return false compute W = IBWT(v), SA_W, and LCP_W;
8. if LCP_W ≠ LCP then return return false v, S;

Algorithm 2: InferInterval([i_a..j_a], [i_ax..j_ax], [i_ax..j_bx])

Input: (nonempty) x- ax- and bx-intervals
Output: If successful, set v[i_a..j_a], add the swap intervals within [i_a..j_a] to S and return true. Otherwise return false.

1. k_a := RMQ_LCP[i_a + 1..j_a]; m_a := LCP[k_a];
2. if j_a - i_a = 1 then k_a := i_a; m_a := ω else k_a := RMQ_LCP[i_a + 1..j_a];
   m_a := LCP[k_a] if j_a - i_a = 1 then k_a := i_a; m_a := ω else
   k_a := RMQ_LCP(i_a + 1..j_a); m_a := LCP[k_a];
3. if m_a > m_a + 1 and m_a > m_a + 1 then
   4. if LCP[i_a + 1..j_a] = 1 + LCP[i_a + 1..k_a] then
      5. v[i_a..k_a] = aa...a; v[k_a..j_a] = bb...b;
   6. if LCP[i_a + 1..j_a] = 1 + LCP[i_a + 1..j_a] then add [i_a..j_a] to S return true;
   7. else
      8. v[i_a..k_a] = bb...b; v[k_a..j_a] = aa...a; return true;
9. else if m_a > m_a + 1 then
   10. if k_bx - i_bx = k_a - i_a then
      11. v[i_a..k_a] = bb...b;
      12. return InferInterval(k_a..j_a), [i_ax..j_ax], [k_ax..j_bx]);
   13. else
      14. v[k_a..j_a] = bb...b;
      15. return InferInterval([i_a..k_a], [i_ax..j_ax], [i_bx..j_bx]);
   16. else if m_bx > m_a + 1 then
   17. if k_a - i_a = k_x - i_x then
      18. v[i_a..k_a] = aa...a;
      19. return InferInterval([k_a..j_a], [k_ax..j_ax], [i_bx..j_bx]);
   20. else
      21. v[k_a..j_a] = aa...a;
      22. return InferInterval([i_a..k_a], [i_ax..k_ax], [i_bx..j_bx]);
   23. else
      24. success1 := InferInterval([i_a..k_a], [i_ax..k_ax], [i_bx..k_bx]);
      25. success2 := InferInterval([k_a..j_a], [k_ax..j_ax], [k_bx..j_bx]);
      26. return success1 and success2;

When trying to split the ax-interval, the result may be, for example, that the axya-interval is empty. In this case, we do not need to recurse on the axya-interval since the corresponding part of v must be all b's. The algorithm recognizes the emptiness of axya- or axyb-interval by the fact that m_ax > m_a + 1, but the problem is to decide which is the
empty one. In most cases, this can be determined by comparing the sizes of the different subintervals or even the actual LCP-intervals (see Lemma 5).

There is one case, where algorithm is unable to determine the empty subintervals, which is when $\text{LCP}[i_\alpha + 1..j_\alpha] = \text{LCP}[i_\beta + 1..j_\beta] = 1 + \text{LCP}[i_\alpha + 1..k_\alpha] = 1 + \text{LCP}[k_\alpha + 1..j_\beta]$. In this case, either the $axya$- and $bxyb$-intervals are empty or the $axyb$- and $bxya$-intervals are empty, but there is no way of deciding between the two cases. It turns out that both are valid choices. The algorithm selects $v$ according to one choice (line 8) but records the alternative choice by adding the interval to the set $S$. In such a case, the string $xy$ is called a swap core and the $xy$-interval (equal to the $x$-interval) is called a swap interval.

For each swap interval $[i..j)$, the algorithm sets $v[i..k) = aa\ldots a$ and $v[k..j) = bb\ldots b$, where $k = (i+j)/2$, but swapping the two halves would be an equally good choice. Therefore, if the output of the algorithm contains $s$ swap intervals, it represents a set of $2^s$ distinct strings. The following lemma shows that the swaps indeed do not affect the LCP array.

Lemma 6. Let $v \in \{a,b\}^n$, $W = \text{IBWT}(v)$, $\text{SA} = \text{SA}_W$ and $\text{LCP} = \text{LCP}_W$. Let $x$ be a string that occurs in $W$ and satisfies:

1. $\text{LCP}[i_\alpha + 1..j_\alpha] = \text{LCP}[i_\beta + 1..j_\beta]$, and
2. $v[i_\alpha..j_\alpha] = aa\ldots a$ and $v[i_\beta..j_\beta] = bb\ldots b$,

where $[i_\alpha..j_\alpha)$ is the $z$-interval for $z \in \{xa, xb\}$. Let $v'$ be the same as $v$ except that $v'[i_\alpha..j_\alpha) = bb\ldots b$ and $v'[i_\beta..j_\beta) = aa\ldots a$. Then $\text{LCP}_{\text{IBWT}(v')} = \text{LCP}$.

Proof. Consider first how $\Psi_{v'}$ differs from $\Psi_v$. For any $i \in [0..n)$, if $\Psi_v[i] \not\in [i_\alpha..j_\alpha)$ then $\Psi_{v'}[i] = \Psi_v[i]$. Otherwise $\Psi_{v'}[i] = \Psi_v[i] + n_{xa} \in [i_\alpha..j_\alpha)$ or $\Psi_{v'}[i] = \Psi_v[i] - n_{xa} \in [i_\beta..j_\beta)$, i.e., it is swapped from one side of the interval $[i_\alpha..j_\alpha)$ to the other side.

Now we use Lemma 1 to determine how a suffix at $\text{SA}[i]$ changes with the swap. If $i$ belongs to a cycle that never visits $[i_\alpha..j_\alpha)$, i.e., the suffix does not contain $x$, there is no change. Suppose then that the cycle starting at $i$ first reaches $[i_\alpha..j_\alpha)$ after $k$ steps, and w.l.o.g. assume that it reaches specifically the $xa$-interval, i.e. $\Psi_k[i] \in [i_\alpha..j_\alpha)$. Then for some string $y$ of length $k$, the suffix at $i$ changes from $yxa\ldots$ into $yxb\ldots$. Note also that $yx$ cannot contain $x$ except at the end.

Now consider two adjacent suffixes. If both are of the form $yxa\ldots$, they both change to $yxb\ldots$. The parts after $x$ may change a lot but $\text{LCP}$ of the two suffixes remains the same because $\text{LCP}[i_\alpha + 1..j_\alpha] = \text{LCP}[i_\beta + 1..j_\beta]$. In all other cases (one or both do not contain $x$ or the parts before $x$ differ), the $\text{LCP}$ is determined in the unchanged part of the suffixes. Thus $\text{LCP}_{\text{IBWT}(v')} = \text{LCP}$.

Theorem 1. Algorithm 1 returns a representation of the set of all strings $v$ such that $\text{LCP}_{\text{IBWT}(v)}$ is the input array, or false if no such string exists.

Proof. Since the algorithm verifies its result (lines 9 and 10), it will always return false if the input is not a valid LCP array. Given a valid LCP array, Algorithm 2 sets all elements of $v$ since it recurses on any subinterval that it doesn’t set. All the choices made by the algorithm are forced by the lemmas in this and the previous section. The swap intervals record all alternatives in the cases where the content of $v$ could not be fully determined, and Lemma 6 shows that all of those alternatives have the same LCP array.

It is also easy to see that the algorithm runs in linear time.

Example 3. Let us consider an integer array $\mathcal{L}[1..7] = [1, 4, 0, 2, 1, 3]$. Using the above algorithms we will try reconstruct a string $v$, such that $\text{LCP}_{\text{IBWT}(v)} = \mathcal{L}$. Since $\mathcal{L}[3] = 0$ $w$ contains 3 occurrences of $a$ and 4 occurrences of $b$, and the initial call to Algorithm 2 is $\text{InferInterval}(0[0..7], [0..3], [3..7])$ (see Figure 3 (1)). We then have $m_x = \mathcal{L}[3] = 0$, $m_{ax} = \mathcal{L}[1] = 1$ and $m_{bx} = \mathcal{L}[5] = 1$, which leads to the recursive calls $\text{InferInterval}(0..3, [0..1], [3..5])$ and $\text{InferInterval}(3..7, [1..3], [5..7])$. 


When processing InferInterval([0..3], [0..1], [3..5]) (see Figure 3(2)), we find that \( m_{bx} = m_x + 1 = 2 \) but \( m_{ax} = \omega \) because the \( ax \)-interval has size 1. Thus we set \( v[0..1] = b \) (line 18) and make the recursive call InferInterval([1..3], [0..1], [4..5]).

When processing InferInterval([1..3], [0..1], [4..5]) (see Figure 3(3)), we find that both the \( ax \)- and the \( bx \)-interval have size 1. In such a case, we always have a swap interval. Here we set \( v[1..3] = ab \) and add \([1..3]\) into \( S \).

When processing InferInterval([3..7], [1..3], [5..7]) (see Figure 3(4)), we have \( m_x = L[5] = 1 \) but \( m_{ax} = 4 > m_x + 1 \) and \( m_{bx} = 3 > m_x + 1 \). Comparing \( L[2..3] = [4] \) and \( L[4..5] = [4] \) (line 10), we find that they do not match. Thus we set \( v[3..5] = bb \) and \( v[5..7] = aa \).

The final result is \( v = b[ab]bbaa \), where the only swap interval is marked with brackets. The main algorithm then computes \( W = IBWT(v) = \{aabb, abb\} \), verifies that \( LCP_W = L \) and outputs \( b[ab]bbaa \). It is easy to verify that \( LCP_{IBWT(babbbabba)} = L \) too.

5 Coupling Constrained Eulerian Cycle

We will now set out to prove the NP-completeness of the Single String Inference from LCP Array (1-SILA) problem where, given an array \( LCP \), we want to determine if there exists a single cyclic string \( W \) such that \( LCP_W = LCP \). The algorithm of the previous section produces a representation of a potentially exponential size set \( V \) and the NP-completeness of 1-SILA shows that determining whether \( IBWT(v) \) is a single string rather than a (multi)set for any \( v \in V \) is NP-complete too.

The proof is done by a reduction from 3-SAT according to the following outline:

- In this section, we define another problem called Coupling Constrained Eulerian Cycle (CCEC), which plays the role of an intermediary, and prove its NP-completeness by a reduction from 3-SAT.
- In Section 6 we describe a reduction from 1-SILA to CCEC to establish a connection between the two problems.
- In Section 7 we describe an incomplete (one-sided) reduction from CCEC to 1-SILA.
- In Section 8 we show that the reduction from the previous section can be made complete in the special case where the CCEC instance was derived from a 3-SAT instance using the construction described in the present section.

Consider a directed graph \( G \) of degree two, i.e., every vertex in \( G \) has exactly two incoming and two outgoing edges. If \( G \) is connected, it is Eulerian. An Eulerian cycle can pass through each vertex in two possible ways as illustrated here:

To distinguish between the two ways, we will call them the straight state and the crossing state of the vertex. We consider each vertex to be a switch that can be flipped between these two states. The combination of vertex states is called the graph state. For an arbitrary graph state, the paths in the graph form, in general, a collection of cycles. The Eulerian cycle problem can then be stated as finding a graph state such that there is only a single cycle; we call such a graph state Eulerian.

In the Coupling Constrained Eulerian Cycle (CCEC) problem, we are given a graph as described above, an initial graph state, and a partitioning of the set of vertices. If we flip a vertex state, we must simultaneously flip the states of all the vertices in the same partition, i.e., the vertices in a partition are coupled (see Figure 2 for an example). A graph state that is achievable from the initial state by a set of such partition flips is called a feasible state. The CCEC problem is to determine if there exists a feasible graph state that is Eulerian.

Theorem 2. The coupling constrained Eulerian cycle problem is NP-complete.
The proof is by reduction from 3-SAT. To obtain a CCEC graph from a 3-CNF formula, a gadget of five vertices is constructed from each clause and these gadgets are connected by a cycle. In each gadget, three of the vertices are labeled by the literals of the corresponding clause; the other two are called free vertices. See Fig. 1 for an illustration.

Each labeled vertex is in a straight state if the labeling literal is false and in a crossing state if the literal is true; their initial state corresponds to some arbitrary truth assignment to the variables. For each variable \( x_i \), there is a vertex partition consisting of all vertices labeled by \( x_i \) or \( \neg x_i \), so that flipping this partition corresponds to changing the truth value of \( x_i \). Each free vertex forms a singleton partition and has an arbitrary initial state. Thus a graph state is feasible iff the labeled vertex states correspond to some truth assignment.

If a clause is false for a given truth assignment, the labeled vertices in the corresponding gadget are all in a straight state. This separates a part of the gadget from the main cycle and thus the graph state is not Eulerian. If a clause is true, at least one of the labeled vertices in the gadget is in a crossing state. Then we can always choose the state of the free vertices so that the full gadget is connected to the main cycle. Thus there exists a feasible Eulerian graph state iff there exists a truth assignment to the variables that satisfies all clauses.

For purposes that will become clear later, we modify the above construction by adding some extra components to the graph without changing the validity of the reduction. Specifically, for each variable \( x_i \) in the 3-CNF formula, we add the following gadget to the main cycle:

The vertices in the gadget are treated similarly to the other vertices in the graph: they belong to the partition with the other vertices labeled by \( x_i \) or \( \neg x_i \), and the initial state is determined by the truth value of the labeling literal. It is easy to see that the gadget will be fully connected to main cycle whether \( x_i \) is true or false. Thus the extra gadgets have no effect on the existence of an Eulerian cycle.

### 6 1-SILA to CCEC

The next step is to establish a connection between the 1-SILA and CCEC problems by showing a reduction from 1-SILA to CCEC. Although the direction of the reduction is opposite to what we want, this construction plays a key role in the analysis of the main construction described in the following sections.

Given a 1-SILA instance (an array), we use Algorithm 1 to produce a representation of a set \( V \) of strings. We will write \( V \) as a string with brackets marking the swaps. For example, \( V = b[ab][ab]a = \{ bababa, bababaab, bbaabaa, bbaabaab \} \). In Example 1 we saw that the inverse BWT of a string \( v \in V \) can be represented as a graph \( G_v \) where the vertices are labeled by positions in \( v \) and there is an edge between vertices \( i \) and \( j \) if, for some character \( c \in \{a, b\} \),
and some integer \( k \), \( \hat{v}[i] = c \) is the \( k \)th occurrence of \( c \) in \( \hat{v} \) and \( v[j] = c \) is the \( k \)th occurrence of \( c \) in \( v \). Such an edge \( (i, j) \) is labeled by \( c_k \). Note that \( \forall v \in V, \hat{v} \) is the same; we will denote it by \( \hat{V} \). We form a generalized graph \( G_V \) as a union of the graphs \( G_v, v \in V \). See Fig. 2.

Fig. 2. The graphs \( G_V \) (a) and \( \tilde{G}_V \) (b) for \( V = b|ab|aabbbaa|ab|aa \), which is the BWT with swaps produced from the LCP array \( \text{LCP} = [2, 5, 1, 4, 3, 4, 2, 0, 3, 2, 5, 3, 1] \). The solid edges in \( G_V \) are the edges of \( G_v \) for \( v = babaabbbaaabbaa \). The cycles in \( \tilde{G}_V \) with all vertices in the straight state corresponds to the cycles in \( G_v \). The only non-singleton partition in \( G_v \) is \( \{3, 5, 6, 4\} \).

The graph \( G_V \) can be constructed as follows. Consider \( a_k \) (the \( k \)th \( a \)) in \( \hat{V} \), say at position \( i \). If \( a_k \) is outside any swap region in \( V \), say at position \( j \), there is a single edge \( (i, j) \) in \( G_V \) labeled by \( a_k \). If \( a_k \) is within a swap region in \( V \), it has two possible positions in the strings \( v \in V \), say \( j \) and \( j' \). That same pair of positions are also the possible positions of some \( b \), say \( b_k = \hat{V}[i'] \). Then \( g_v \) has two edges, \( (i, j) \) and \( (i, j') \), labeled with \( a_k \) and two edges, \( (i', j) \) and \( (i', j') \), labeled with \( b_k \). The positions/vertices \( j \) and \( j' \) are called a swap pair.

To obtain a CCEC graph \( \tilde{G}_V \), we make two modifications to \( G_V \). First, we merge each swap pair into a single vertex. Each merged vertex now has two incoming and two outgoing edges and all other vertices have one incoming and one outgoing edge. Second, we remove all vertices with degree one by concatenating their incoming and outgoing edges. See Fig. 2.

The initial state of the vertices in \( \tilde{G}_V \) is set so that the cycles in \( G_V \) correspond to the cycles in \( G_v \) for some \( v \in V \). Two vertices in \( \tilde{G}_V \) belong to the same partition if their labels belong to the same swap interval in \( V \). Then we have a one-to-one correspondence between swaps in \( V \) and partition flips in \( \tilde{G}_V \). If this CCEC instance has a solution, the Eulerian cycle spells a single string realizing the input LCP array. If the CCEC instance has no solution, the original 1-SILA problem has no solution either.

### 7 CCEC to 1-SILA

In this section, we will make a (not fully successful) attempt at a reduction from CCEC to 1-SILA. The above 1-SILA to CCEC reduction transforms each pair of swapped positions into a vertex and each swap interval into a vertex partition. Our construction creates a 1-SILA instance such that the resulting BWT has the necessary swaps to produce the CCEC instance vertices and partitions. However, the BWT also has some unwanted swaps producing spurious vertices that break the reduction. In the next section, we will show how the effect of the spurious vertices can be neutralized in a special case.

Starting from a CCEC instance, the transformation constructs a set of cyclic strings, and the 1-SILA instance is the LCP array of that string set. The construction associates two strings to each vertex and the cyclic strings are formed by concatenating the vertex strings according to the cycles in the graph in its initial state. The two passes of the cycles through a vertex must use different strings but it does not matter which pass uses which string.

Let \( n \) be the number of vertices in the CCEC graph and let \( m \) be the number of vertex partitions. We number the vertices from 1 to \( n \) and the partitions from 1 to \( m \). Small
occurrences of $x$ is a free vertex. The extra vertex is connected to the graph by making two existing edges $ba$ form any string not mentioned above either does not occur at all or contains a substring of the form $x$. If $i..j$ is an $x$-interval for a string $x$ such that either $occ(axa) = occ(bxb) = occ(x)/2$ or $occ(axb) = occ(bxa) = occ(x)/2$, where $occ(y)$ is the number of occurrences of $y$ in $W$. Let us now analyze the transformation by changing the 1-SILA instance back to a CCEC instance using the construction of the previous section. Specifically, we will analyze the swaps in the BWT produced from the LCP array. Let $W$ be the set of cyclic strings constructed from the CCEC instance, and let $W$ be the BWT with swaps constructed from $LCP_W$. An interval $[i..j]$ in $W$ is a swap interval if and only if the following conditions hold:

1. $[i..j]$ is an $x$-interval for a string $x$ such that either $occ(axa) = occ(bxb) = occ(x)/2$ or $occ(axb) = occ(bxa) = occ(x)/2$, where $occ(y)$ is the number of occurrences of $y$ in $W$. Let us first consider strings of the form $x = ba^k$. If $k > m$, $occ(x) \leq 1$ and $x$ cannot be a swap core. For $k \in [1..m]$, $x$ is always a swap core and corresponds to the CCEC partition numbered $k$. Let $v = \text{BWT}(W)$ and let $V'$ be $v$ together with the swaps with cores of the form $x = ba^k$. It is easy to verify that a CCEC instance constructed from $V'$ as described in the previous section is identical to the original CCEC instance. Thus, if there were no other swap cores, we would have a perfect reduction. In the rest of this section, our goal is to identify all other potential swap cores that may break the reduction.

We will systematically examine all strings, starting with unary strings. First, $b$, $bb$ and $a^k$ for $k < m + 2n$ are not swap cores because they are preceded and succeeded by a more often than by $b$. We can also eliminate all other unary strings since they occur at most once. We also note that any string beginning (ending) with $bb$ cannot be a swap core because it is always preceded (succeeded) by $a$. Let us then consider strings $x$ of the following forms:

- $x = ba^k$. If $k < m + 2n - 1$, $occ(xa) > occ(xb)$, and if $k \geq m + 2n$, $occ(x) \leq 1$. In either case, $x$ is not a swap core. On the other hand, $x = ba^{m+2n-1}$ is always a swap core with two occurrences.
- $x = a^k b$. This case is symmetric to the one above except we cannot be certain whether $x = a^{m+2n-1} b$ is swap core or not since the characters following the two occurrences of $x$ are not fully determined. However, we count $x$ as a potential swap core.
- $x = a^k ba^k$ and $x = a^k bba^k$. If $k > m$, we have $occ(x) = 0$, and if $k = h < m$, we have $occ(ax) > occ(bx)$. Also, if $k = m$, we have $occ(ax) > occ(bx)$ but now this relies on the fact that the partition numbered $m$ is non-singleton. In all cases, $x$ is not a swap core.
- $x = a^k ba^k$ and $x = a^k bba^k$ for $k < h$. If $k > m$, we have $occ(x) = 0$ and if $h \leq m$, we have $occ(ax) > occ(bx)$. If $k \leq m < h$, $x$ is obviously not a swap core if $occ(x) < 2$ but also not if $occ(x) > 2$ because then we must have $occ(axa) > occ(xab)$. On the other hand, if $k \leq m < h$ and $occ(x) = 2$, then $x$ might be a swap core.
- $x = a^k b^h$ and $x = a^k bba^h$ for $k > h$. This is symmetric to the case above.
- $x = ba^k b^h, x = ba^k bba^h, x = a^k ba^h b$ and $x = a^k bba^k b$. If $k > m$, $occ(x) \leq 1$. If $k \leq m$, every occurrence of $x$ is either preceded (the first two cases) or succeeded (the latter two cases) by the same character. Thus $x$ is never a swap core.
- $x = a^k ba^ka^h$ and $x = a^k bba^i bba^h$ for $i \in [1..m]$. Obviously, $x$ is not a swap core if $occ(x) < 2$ but also not if $occ(x) > 2$ because then $occ(xa) > occ(xb)$. If $occ(x) = 2$ then $x$ may or may not be a swap core.

Any string not mentioned above either does not occur at all or contains a substring of the form $ba^k b$ for $k > m$ and occurs once.

Note that each extra swap core has exactly two occurrences and thus corresponds to a free vertex. The extra vertex is connected to the graph by making two existing edges to pass through the new vertex. Which two edges are affected depends on where the two occurrences of $x$ are in $W$, specifically, on what are the nearest preceding and succeeding
swap core occurrences. Note that the extra vertices can never be isolated. Thus the addition of the extra vertices can never eliminate an Eulerian cycle.

Let us summarize the above analysis. We want to compare the 1-SILA instance to the original CCEC instance from which it was derived. To do this, we construct a CCEC instance equivalent to the 1-SILA instance using the reduction of Section 6. The latter CCEC instance is created in two phases. First we construct the instance from the BWT \( V' \) with only the desired swaps with the cores of the form \( ba^kb \). This instance is identical to the original. Then we add the missing swaps, each of which adds a free vertex to the graph. If the original CCEC instance has a solution, so does the derived CCEC instance and thus also the 1-SILA instance. However, the derived CCEC instance might have a solution even if the original CCEC instance does not because of the extra connections created by the extra vertices. Thus this CCEC to 1-SILA reduction is one-sided only.

8 1-SILA is NP-Complete

We are now ready to show that 1-SILA is NP-complete using the reduction chain 3-SAT \( \rightarrow \) CCEC \( \rightarrow \) 1-SILA. The first step was described in Section 5 and the second step uses the construction of Section 7 with two differences. First, we don’t assume an arbitrary CCEC instance but one derived from a 3-SAT instance including the extra gadgets. Second, we use more specific rules for assigning the partition and vertex numbers:

- The biggest partition number is assigned to the partition corresponding to the variable \( x_1 \), the second biggest to variable \( x_2 \), and so on.
- The three biggest vertex numbers are assigned to the vertices labeled \( x_1 \) in the extra gadget for the variable \( x_1 \), the next three biggest to the vertices labeled \( x_2 \) and so on. Within each extra gadget, the biggest number is assigned to the middle one of the three vertices.

Otherwise the construction follows previous sections.

Our goal is now to show that the spurious vertices derived from the undesirable swap cores can create new connections within the extra gadgets but do not affect the clause gadgets. Thus the existence of a feasible Eulerian state is not affected by the spurious vertices.

Each undesirable swap core is of one of the following forms: \( ba^{m+2n-1}b, a^{m+2n-1}b, a^kb^h, a^kbba^h, a^kbab^a, a^kbba^h \). Furthermore, we know that each such swap core has exactly two occurrences, which means that the values \( k \) and/or \( h \) have to be sufficiently large. Because we chose to assign the biggest vertex numbers to the vertices in the extra gadgets, all the additional connections are within the extra gadgets.

This completes the proof and we have the following result.

**Theorem 3.** 1-SILA is NP-complete.

**References**

1. Mohamed Ibrahim Abouelhoda, Stefan Kurtz, and Enno Ohlebusch. Replacing suffix trees with enhanced suffix arrays. *Journal of Discrete Algorithms*, 2(1):53–86, 2004.
2. Alberto Apostolico. The myriad virtues of subword trees. In Alberto Apostolico and Zvi Galil, editors, *Combinatorial Algorithms on Words*, NATO ASI Series F12, pages 85–96. Springer-Verlag, Berlin, Germany, 1985.
3. Hideo Bannai, Shunsuke Inenaga, Ayumi Shinohara, and Masayuki Takeda. Inferring strings from graphs and arrays. In Branislav Rovan and Peter Vojtás, editors, *Mathematical Foundations of Computer Science 2003, 28th International Symposium, MFCS 2003, Bratislava, Slovakia, August 25-29, 2003, Proceedings*, volume 2747 of *Lecture Notes in Computer Science*, pages 208–217. Springer, 2003.
4. Bastien Cazaux and Eric Rivals. Reverse engineering of compact suffix trees and links: A novel algorithm. *J. Discrete Algorithms*, 28:9–22, 2014.
5. Julien Clément, Maxime Crochemore, and Giuseppina Rindone. Reverse engineering prefix tables. In Susanne Albers and Jean-Yves Marion, editors, *26th International Symposium on Theoretical Aspects of Computer Science*, STACS 2009, February 26-28, 2009, Freiburg, Germany, Proceedings, volume 3 of LIPIcs, pages 289–300. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, Germany, 2009.

6. Maxime Crochemore and Lucian Ilie. Computing longest previous factor in linear time and applications. *Inf. Process. Lett.*, 106(2):75–80, 2008.

7. Maxime Crochemore, Costas S. Iliopoulos, Solon P. Pissis, and German Tischler. Cover array string reconstruction. In Amirhood Amir and Laxmi Parida, editors, *Combinatorial Pattern Matching*, 21st Annual Symposium, CPM 2010, New York, NY, USA, June 21-23, 2010. Proceedings, volume 6129 of Lecture Notes in Computer Science, pages 251–259. Springer, 2010.

8. Jean-Pierre Duval, Thierry Lecroq, and Arnaud Lefèvre. Border array on bounded alphabet. *Journal of Automata, Languages and Combinatorics*, 10(1):51–60, 2005.

9. Jean-Pierre Duval, Thierry Lecroq, and Arnaud Lefèvre. Efficient validation and construction of border arrays and validation of string matching automata. *RAIRO-Theor. Inf. Appl.*, 43(2):281–297, 2009.

10. František Franěk, S. Gao, WeiLin Lu, Patrick J. Ryan, William F. Smyth, Yu Sun, and Lu Yang. Verifying a border array in linear time. *Journal on Combinatorial Mathematics and Combinatorial Computing*, 42:223â€“236, 2002.

11. Pawel Gawrychowski, Artur Jez, and Lukasz Jez. Validating the knuth-morris-pratt failure function, fast and online. *Theory Comput. Syst.*, 54(2):337–372, 2014.

12. Ira M. Gessel and Christophe Reutenauer. Counting permutations with given cycle structure and descent set. *Journal of Combinatorial Theory, Series A*, 64(2):189–215, 1993.

13. Dan Gusfield. *Algorithms on Strings, Trees, and Sequences: Computer Science and Computational Biology*. Cambridge University Press, Cambridge, United Kingdom, 1997.

14. Jing He, Hongyu Liang, and Guang Yang. Reversing longest previous factor tables is hard. In Frank Dehne, John Iacono, and Jürg-Rüdiger Suck, editors, *Algorithms and Data Structures - 12th International Symposium, WADS 2011, New York, NY, USA, August 15-17, 2011. Proceedings*, volume 6844 of Lecture Notes in Computer Science, pages 488–499. Springer, 2011.

15. Peter M. Higgins. Burrows-Wheeler transformations and de Bruijn words. *Journal of Computational Biology*, 4:579–597, 1997.

16. Tomohiro I, Shunsuke Inenaga, Hideo Bannai, and Masayuki Takeda. Inferring strings from Lyndon factorization. In Erzsébet Csuhaj-Varjú, Martin Dietzfelbinger, and Zoltán Ésik, editors, *Mathematical Foundations of Computer Science 2014 - 39th International Symposium, MFCS 2014, Budapest, Hungary, August 25-29, 2014. Proceedings, Part II*, volume 8635 of Lecture Notes in Computer Science, pages 251–259. Springer, 2014.

17. Giovanni Manzini and Paolo Ferragina. Engineering a lightweight suffix array construction algorithm. *Algorithmica*, 40(1):33–50, 2004.

18. Donald R Morrison. Patriciaâ€™s practical algorithm to retrieve information coded in alphanumeric. *Journal of the ACM (JACM)*, 15(4):514–534, 1968.

19. Tomohiro I, Shunsuke Inenaga, Hideo Bannai, and Masayuki Takeda. Inferring strings from Lyndon factorization. In Erzsébet Csurj-Kuszmics, Zoltán Ésik, and Dominik Kempa, editors, *Proceedings - 12th International Symposium, WADS 2011, New York, NY, USA, August 15-17, 2011. Proceedings*, volume 6844 of Lecture Notes in Computer Science, pages 488–499. Springer, 2011.

20. Peter Weiner. Linear pattern matching algorithms. In *Proceedings of the 14th Annual IEEE Symposium on Switching and Automata Theory*, page 1â€“11, 1973.
A  String inference example

Fig. 3. Graphical illustration of Example 3.