Most Vertex Superalgebras Associated to an Odd Unimodular Lattice of Rank 24 Have an $N=4$ Superconformal Structure

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ABSTRACT

The classification of odd positive-definite unimodular lattices $M$ of rank 24 was completed by Borcherds. There are 273 isometry classes of such lattices. Associated to them are vertex superalgebras $V_M$ of central charge $c=24$. We show that at least 267 (but not all) of these vertex superalgebras contain an $N=4$ superconformal subalgebra of central charge $c=6$. This is achieved by studying embeddings $L^+ \subseteq M$ of a certain rank 6 lattice $L^+$. 

1. Introduction

The results discussed in this article concern positive-definite, integral, unimodular lattices of rank 24 and the vertex algebras that they define. These lattices, which have been classified by the work of Witt, Kneser, Niemeier, Conway and Sloane, and Borcherds (for further detail and additional references cf. [3]) can be odd or even and the minimal weight can be 1, 2, 3, or 4. For the sake of brevity, then, it is convenient to call any positive-definite, unimodular lattice of rank 24 a Niemeier lattice, adding modifying adjectives when appropriate. Thus what is called a Niemeier lattice in the literature is what we here call an even Niemeier lattice.

Quite often, the basic axiom for a vertex algebra, i.e., the Jacobi identity, is supplemented by additional conditions such as the existence of a Virasoro field, or a superconformal structure. In the case of vertex superalgebras, there is a hierarchy of such structures, often denoted by $N=1, N=2, N=4, \ldots$ becoming more intricate with increasing $N$. For general background we refer the reader to [5], [10], [12], and for an early comparison of these super structures, see Green et al. [8].

Vertex algebras containing an $N=4$ superconformal structure of central charge 6, in particular, have come into recent prominence because of their intervention in Mathieu Moonshine [6]. Experience suggests that moonshine-like phenomena may be related to Niemeier lattices as we have defined them and thus one is lead to consider odd Niemeier lattices $M$ of rank 24 and the vertex superalgebras $V_M$ that they define. A natural question presents itself: does $V_M$ have an $N=4$ structure? In this paper we will show that most of the vertex superalgebras $V_M$ do have an $N=4$ superconformal structure.

This question was considered in [13] for general lattice vertex superalgebras $V_L$. However, the precise definition of the (small) $N=4$ algebra is awkward to handle (cf. [13], Section 2) and makes the question hard to deal with. A way around this difficulty was proposed in [13]. Namely, it was shown (loc. cit., Proposition 26) that a certain integral rank 6 lattice, denoted by $L^+$, has the property that $V_{L^+}$ contains an $N=4$ superconformal algebra $A$ and that $V_L^+$ and $A$ share the same Virasoro element (the canonical one defined by the Siegel-Sugawara construction). The lattice $L^+$ is spanned by elements $\alpha_1, \ldots, \alpha_6$ and $h = \frac{1}{4}(\alpha_1 + \cdots + \alpha_6)$ and equipped with a bilinear form satisfying $(\alpha_i, \alpha_j) = 3 \delta_{ij}$ for $i, j = 1, \ldots, 6$, so that $(h, h) = 2$.

In this way one can establish the existence of an $N=4$ superconformal subalgebra of central charge 6 in $V_M$ by proving the existence of a subalgebra $L^+ \subseteq M$, for we then have the tower $A \subseteq V_{L^+} \subseteq V_M$. A similar question for the vertex superalgebra associated to the odd unimodular lattice $D_{12}^+$ of rank 12 has been investigated in Section 5.4 of Creutzig et al. [4]. Our main results, obtained by a computer search, identify exactly which odd Niemeier lattices admit such an embedding.

2. Results

In order to state our main results, we need to review the theory of Niemeier lattices [1, 3]. The important idea of neighboring lattices is due to Kneser [11]. Suppose that $A$ and $B$ are a pair of equal rank lattices. We say that $A$ and $B$ are neighbors in case their intersection $D$ (in a common Euclidean space $E$ containing both) has index 2 in each of them. If $A$ and $B$ are neighboring even Niemeier lattices, then there is a unique odd Niemeier lattice $M$ that neighbors both $A$ and $B$. ($A$, $B$, and $M$ are the lattices in $E$ containing $D$ with index 2). Furthermore, every odd Niemeier lattice $M$ arises in this way from an unordered pair ($A$, $B$) of even Niemeier lattices that is unique up to isometry. We say that $M$ has type $(A, B)$. In general there will be several isometry classes of $M$ of the same type. An even Niemeier lattice is characterized by its (semisimple) root system (which by convention is empty in the case of the Leech lattice.
Thus, the type of an odd Niemeier lattice \( M \) is also determined by the corresponding pair \((\Phi_A, \Phi_B)\) of root systems, which we also refer to as the type of \( M \).

We are also interested in the minimum norm \( \mu \) of a Niemeier lattice \( M \), i.e., the least value of \( (\alpha, \alpha) \) for \( \alpha \neq 0 \in M \). If \( M \) is even, then \( \mu = 2 \) with the single exception of the Leech lattice \( \Lambda \) which has \( \mu = 4 \). For odd \( M \) there are three possibilities as in Table 1.

The unique Niemeier lattice of minimum norm 3 is called the odd Leech lattice \( \Lambda^{\text{odd}} \), it has type \((\emptyset, A_1^{24})\).

If \( M \) is an odd Niemeier lattice of minimum norm 1, then its two even Niemeier neighbors are necessarily isometric. This is because the Weyl reflection in the dual lattice \( M^{\ast} \) determined by a norm 1 vector exchanges the two even neighbors. Thus, the type of such a lattice always takes the form \((\Phi, \Phi)\) where \( \Phi \) is the root system common to the even neighbors.

This situation does not necessarily pertain in the case of odd Niemeier lattices with \( \mu = 2 \). The neighboring graph, with nodes given by even Niemeier lattices and edges between \( A \) and \( B \) indexed by the odd lattices of type \((A, B)\) and with \( \mu = 2 \) was first constructed by Borcherds [1]. It is Figure 17.1 in [3].

We can now state our main results.

**Theorem 1.** Suppose that \( M \) is an odd Niemeier lattice. Then the following hold:

1. If \( \mu = 1 \), then either \( M \) is of type \((\emptyset, \emptyset)\), or else there is a sublattice \( L^+ \subseteq M \).
2. If \( \mu = 2 \), then exactly one of the following holds:
   a. \( M \) has type \((E_8^3, D_8^3)\), \((D_{16}E_8, A_{15}D_9)\), \((D_{12}^2, A_{12}^2)\) or \((D_{24}, A_{24})\).
   b. there exists a sublattice \( L^+ \subseteq M \).
3. If \( \mu = 3 \), then the odd Leech lattice \( \Lambda^{\text{odd}} \) does not contain \( L^+ \).

**Remark 2.** (i) There is a unique odd Niemeier lattice for each of the types listed in Theorem 1(2)(a). Thus there are exactly 152 isometry classes of odd Niemeier lattices of minimum norm 2 that contain the lattice \( L^+ \).

(ii) The odd Leech lattice \( \Lambda^{\text{odd}} \) cannot possibly contain \( L^+ \) because the even sublattice of \( \Lambda^{\text{odd}} \) is contained in \( \Lambda \) and hence has minimum norm \( \geq 4 \), whereas \( L^+ \) contains vectors of norm 2 (namely \( \pm h \)).

(iii) For similar reasons, the unique lattice of type \((\emptyset, \emptyset)\) in Theorem 1(1) cannot contain \( L^+ \). Thus we can paraphrase Theorem 1(1) as follows: every one of the 116 odd Niemeier lattices with \( \mu = 1 \) that could contain \( L^+ \) (i.e., they contain a root vector, which is a necessary condition), does contain \( L^+ \).

The proof of Theorem 1 is computational. We used the neighborhood method as implemented in MAGMA [2] together with some optimizations to create the list of 273 odd Niemeier lattices together with their two even neighbors. We also computed the automorphism group and checked the result by a mass formula ([3], Theorem 1 of Chapter 16).

The lattice \( L^+ \) has a basis consisting of the vector \( h \) of norm 2 and the five pairwise orthogonal norm 3 vectors \( \alpha_1, \ldots, \alpha_5 \) with scalar product \((\alpha_i, h) = 1\). For each odd Niemeier lattice \( M \), this permits use of the following search method for the sublattice \( L^+ \): do a backtracking search by selecting all possible norm 2 vectors \( h \in M \), then for given vector \( h \) take all norm 3 vectors \( \alpha_1 \) with \((h, \alpha_1) = 1\), then for such pairs \((h, \alpha_1)\) select all norm 3 vectors \( \alpha_2 \) with \((h, \alpha_2) = 1\) and \((\alpha_1, \alpha_2) = 0\), then, in similar way, all possible vectors \( \alpha_3, \alpha_4, \text{and} \alpha_5 \). If we find a tuple \((h, \alpha_1, \ldots, \alpha_5)\), then we stop our search for the lattice \( L^+ \). Otherwise we continue until all possibilities are exhausted.

To speed up the computation, we used the automorphism group \( O(M) \) of \( M \). First we compute the orbits of \( O(M) \) on the roots of norm 2. Then it is enough to choose a random vector \( h \) in \( \phi \). Then we compute the pointwise stabilizer \( H_1 \) of \( h \) in \( O(M) \) and decompose the set of norm 3 vectors \( \alpha \) with \((h, \alpha) = 1\) into \( H_1 \)-orbits. Then it is enough to choose a random vector \( \alpha_1 \) in each such orbit. Then we compute the pointwise stabilizer \( H_2 \) of the two vectors \( h \) and \( \alpha_1 \) in \( O(M) \) and select a random vector among the possible \( \alpha_2 \) in each \( H_2 \)-orbit.

**Remark 3.** (i) The computation took less than 24 hours on a single processor machine.

(ii) We did not classify all orbits of sublattices \( L^+ \subseteq M \) under the action of \( O(M) \), although this can easily be achieved with our method.

(iii) We also tried a different computational approach. We constructed a single embedding of \( L^+ \) into an odd Niemeier lattice \( M \) and determined the orthogonal complement lattice \( K = (L^+ \otimes \mathbb{R})^\perp \cap M \). Then we tried the neighborhood method to compute all lattices \( K' \) in the genus of \( K \). By computing all gluings of all such lattices \( K' \) with \( L^+ \), we obtain odd Niemeier lattices \( M' \) having \( L^+ \) as a sublattice and all such lattices can be constructed that way.

| \( \mu \) | 1 | 2 | 3 |
|---|---|---|---|
| # of classes | 116 | 156 | 1 |

Table 1. Minimum norms for odd Niemeier lattices.
We found more than 10,000 rank 18 lattices $K'$ covering most of the genus of $K$. However, the estimated run time for completing the calculation was more than a month. Apart from a few missing cases, the list of lattices $M'$ obtained in this way was identical with the one from our theorem.

**Remark 4.** The two lattice vertex superalgebras $V_M$ corresponding to the lattice $M$ in Theorem 1 with $\mu = 1$ and type $(\emptyset, \emptyset)$ and to the odd Leech lattice $\Lambda^{\text{odd}}$ with $\mu = 3$ do not have an $N=4$ superconformal structure for any central charge since the Lie algebra $(V_M)_1$ is Abelian, i.e., does not contain an $\mathfrak{sl}_2$ subalgebra. It is known, however, that the vertex superalgebra $V_{\Lambda^{\text{odd}}}$ has at least two inequivalent $N=1$ superconformal structures with $c = 24$ [7].

The following two questions remain open:

(i) Do the four exceptional lattice vertex superalgebras $V_M$ corresponding to part (2) of Theorem 1 have an $N=4$ superconformal structure not coming from a lattice embedding $L^+ \subseteq M$?

(ii) Do most of the self-dual vertex superalgebras of central charge 24 (cf. [9]), which are not of lattice type have an $N=4$ superconformal structure?

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