Coordinate Dependence of Chern-Simons Theory on Noncommutative $AdS_3$

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ABSTRACT

We investigate the coordinate dependence of noncommutative theory by studying the solutions of noncommutative $U(1,1) \times U(1,1)$ Chern-Simons theory on $AdS_3$ in the polar and rectangular coordinates. We assume that only the space coordinates are noncommuting. The two coordinate systems are equivalent only up to first order in the noncommutativity parameter $\theta$. We investigate the effect of this non-exact equivalence between the two coordinate systems in two cases, a conical solution and a BTZ black hole solution, using the Seiberg-Witten map. In each case, the noncommutative solutions in the two coordinate systems obtained from the corresponding same commutative solution turn out to be different even in the first order in $\theta$.

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1 Introduction

Physics in nonconcommutative spacetime has long been studied [1, 2] since Snyder introduced the notion of quantized spacetime [3]. Among many proposed models, the most common commutation relation (called canonical) between coordinates is

\[
[\hat{x}^\alpha, \hat{x}^\beta] = i\theta^{\alpha\beta},
\]

(1)

where \(\theta^{\alpha\beta} = -\theta^{\beta\alpha}\) are constants. After this canonical noncommutativity was introduced in the string theory context [4, 5], it became the mainly studied commutation relation for physics in noncommutative spacetime.

This commutation relation resembles the fundamental commutation relation of quantum physics. Inspired by Weyl quantization in quantum mechanics [6], a theory on the canonical noncommutative spacetime can be reinterpreted to another theory on the commutative spacetime in which a product of any two functions on the original noncommutative spacetime is replaced with a deformed (\(\star\)) product of the functions on the commutative spacetime, the Moyal product [7]:

\[
(f \star g)(x) \equiv \exp \left[ \frac{i}{2} \theta^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial y^\beta} \right] f(x) g(y) \bigg|_{x=y}.
\]

(2)

Most of the analyses for noncommutative physics are performed by using the Moyal product on the commutative space instead of being treated on noncommutative spaces directly.

What if we use a different coordinate system instead of the canonical coordinate system given by (1)? We expect that the commutation relations for the two coordinate systems would not be exactly equivalent to each other. Would then the physics described in these two coordinates systems be the same? We are used to take general covariance for granted. General covariance in “a noncommutative space” would mean the equivalence among different coordinate systems. However, as we mentioned above different coordinate systems in “a noncommutative space” generally have different commutation relations which are not exactly

\(^1\) In this paper, we only deal with space-space noncommutativity, and time is a commuting coordinate throughout the paper. Thus we use the terms (noncommutative) space and (noncommutative) spacetime interchangeably.

\(^2\) Here, we put the quotation mark since it is not clear at the moment whether we have to treat coordinate systems with different commutation relations of “a given space” as different noncommutative spaces.
equivalent. Therefore we expect that coordinate transformations among different coordinate systems would not yield the same physics contradicting the usual notion of general covariance. Seiberg [8] has already pointed out that general covariance would be broken in theories with emergent spacetime among which model theories on noncommutative spaces are also included. In this paper, we focus on this issue: general covariance on a noncommutative space vs. non-exact equivalence between noncommutative coordinate systems. In order to check this, we compare the solutions of $U(1, 1) \times U(1, 1)$ noncommutative Chern-Simons theory in the rectangular and polar coordinates in 3-dimensional AdS noncommutative spacetime.

Gauge theory on the canonical noncommutative spacetime has been well established using the Seiberg-Witten map [5]. The Seiberg-Witten map is the consistency requirement for a noncommutative gauge transformation of a gauge theory living on a noncommutative spacetime to be equivalent to a gauge transformation of an ordinary gauge theory living on a commutative spacetime. Using this equivalence of the Seiberg-Witten map, one can find the corresponding noncommutative gauge fields in terms of given ordinary gauge fields. The corresponding noncommutative gauge transformation can be found likewise.

For the three dimensional gravity, it has been well known that it is equivalent to a Yang-Mills theory with the Chern-Simons(CS) action in three dimensional spacetime [9] [10]. Thus using the Seiberg-Witten map the noncommutative extension of 3D gravity-CS equivalence was studied in [11][12][13]. Based on these works, Pinzul and Stern [14] obtained noncommutative $AdS_3$ vacuum and conical solution using the Seiberg-Witten map. Rather recently, this method was applied to the rotating BTZ black hole case in [16] with commutation relation of $[\hat{r}, \hat{\phi}] = i\theta$.

For the four dimensional gravity, there is no such known equivalence relation between gravity and gauge theory in four dimensional spacetime. However, using the Poincaré gauge theory approach of Chamseddine [17], a noncommutative Schwarzschild black hole solution was first obtained in [18] using the Seiberg-Witten map. Likewise, the charged black hole solutions in 4D were obtained in [19][20].

In our previous work [21], we studied the rotating BTZ black hole in a noncommutative

\footnote{Before this, the non-rotating BTZ black hole case had been investigated in [15] in a different set-up of geometrical framework.}
polar coordinates with the commutation relation \(^4\)

\[ [\hat{r}, \hat{\phi}] = i\theta \hat{r}^{-1}, \tag{3} \]

which is different from the one used in [16] and is equivalent to the canonical relation [11] up to first order in \(\theta\). In this paper, we study the rotating BTZ black hole case with the canonical commutation relation \([x, y] = i\theta\), and compare it with our previous result [21]. Then we again obtain the conical solution on \(AdS_3\) in the noncommutative polar coordinates with the commutation relation (3) and compare it with the one obtained in [14]. The results exhibit their dependence on a chosen coordinate system.

The paper is organised as follows. In section 2, we consider some aspects related with the Seiberg-Witten map and then investigate the difference between the commutation relations in the polar and rectangular coordinates. In section 3, we obtain the noncommutative BTZ solution with the canonical commutation relation of noncommutative rectangular coordinates, then compare it with the result in the noncommutative polar coordinates obtained in [21]. In section 4, we get the conical solution of noncommutative \(AdS_3\) in the noncommutative polar coordinates, and compare it with the previously obtained solution by Pinzul and Stern [14] in which the canonical commutation relation of the rectangular coordinates was used. We conclude with discussion in section 5.

2 Different noncommutativity and Seiberg-Witten map

Here, we begin with reviewing the Seiberg-Witten map and study related aspects by treating the same map in “a noncommutative spacetime” with different commutation relations. After that we show how these noncommutativities are different in the two following perspectives, coordinates as operators and the Moyal product as a deformed product from twist.

2.1 Seiberg-Witten map in different coordinates

The Sieberg-Witten map matches ordinary gauge fields \(A\) on a commutative spacetime with noncommutative gauge fields \(\hat{A}\) on a noncommutative spacetime such that an ordinary gauge

\(^4\) This is equivalent to \([\hat{r}^2, \hat{\phi}] = 2i\theta\).
transformation of $A$ is equivalent to a noncommutative gauge transformation of $\hat{A}$ [5]:

$$\hat{A}(g \cdot A \cdot g^{-1} - \partial g \cdot g^{-1}) = \hat{g} \ast \hat{A} \ast \hat{g}^{-1} - \partial \hat{g} \ast \hat{g}^{-1},$$

(4)

where $\ast$ denotes the Moyal product, $g, \hat{g}$ are elements of gauge groups for the ordinary and noncommutative gauge theories, respectively. The above equation can be solved to first order in $\theta$ as follows.

$$\hat{A}_\gamma(A) \equiv A_\gamma + A'_\gamma = A_\gamma - i \frac{\theta^{\alpha\beta}}{4} \{A_\alpha, \partial_\beta A_\gamma + F_{\beta\gamma}\},$$

(5)

$$\hat{\lambda}(\lambda, A) \equiv \lambda + \lambda' = \lambda + i \frac{\theta^{\alpha\beta}}{4} \{\partial_\alpha \lambda, A_\beta\} + F_{\beta\gamma} \hat{A}_\gamma,$$

(6)

where $\hat{\lambda}$ and $\lambda$ are noncommutative and ordinary infinitesimal gauge transformation parameters. We note that there are two important factors in the derivation of the solution (5) and (6). One is knowing of the explicit form of the Moyal product up to first order in $\theta$, and the other is the coordinate independence of noncommutativity parameter $\theta$ being used in the Moyal product. One would no longer get the same form of solution for Eq. (4) in the cases of coordinate dependant noncommutativity parameters.

Generally one obtains different solutions of the Seiberg-Witten equation for different coordinate systems. To see this let us consider a coordinate transformation $\varphi$ between two coordinate systems $\{x^\alpha\}$ and $\{z^a\}$, say, $\varphi : x^\alpha \rightarrow z^a \equiv z^a(x^\mu)$. Then a Seiberg-Witten solution $\hat{A}_c(z)$ in the coordinate system $\{z^a\}$ can be rewritten in terms of $\hat{A}_c(x)$, the corresponding solution of the Seiberg-Witten equation in the coordinate system $\{x^\alpha\}$:

$$\hat{A}_c(z) = A_c(z) - i \frac{\bar{\theta}^{ab}}{4} \{A_a(z), \partial_b A_c + F_{bc}\}$$

$$\hat{A}_c(z) = \partial^\gamma x^c A_\gamma(x) - i \frac{\bar{\theta}^{ab}}{4} \left\{ \partial^\alpha x^a A_\alpha(x), \partial^\beta \partial^\gamma x^c \left( A_\gamma(x) + F_{\beta\gamma}\right) \right\}$$

$$\hat{A}_c(z) = \left( A_\gamma(x) + i \frac{i}{4} \theta^{\alpha\beta} \{A_\alpha, \partial_\beta A_\gamma + F_{\beta\gamma}\} \right)$$

$$\partial^\gamma x^c (A_\gamma(x), \partial^\alpha x^a A_\alpha + \partial^\beta \partial^\gamma x^c (A_\alpha + F_{\beta\gamma})),$$

where $\bar{\theta}^{ab}$ denote noncommutativity parameters assumed in the coordinate system $\{z^a\}$. This can be reexpressed to show the difference between the two Seiberg-Witten solutions in
\{z^a\} \text{ and } \{x^\alpha\} \text{ coordinate systems,}

\[ \hat{A}_c(z) - \frac{\partial x^\gamma}{\partial z^c} \hat{A}_\gamma(x) = -\frac{i}{4} \tilde{\theta}^{ab} \left( \frac{\partial x^\alpha}{\partial z^a} \right) \left( \frac{\partial^2 x^\beta}{\partial z^b \partial z^c} \right) \{A_\alpha, A_\beta\} + \frac{i}{4} \left( \frac{\partial x^\gamma}{\partial z^c} \right) \left( \theta^{\alpha\beta} - \frac{\partial x^\alpha}{\partial z^a} \frac{\partial x^\beta}{\partial z^b} \tilde{\theta}^{ab} \right) \{A_\alpha(z), \partial_\beta A_\gamma + \mathcal{F}_{\beta\gamma}\}, \] (7)

up to first order in \( \theta \). The first term on the right-hand side vanishes when the transformation \( \varphi \) is linear, i.e., \( \frac{\partial^2 x^\beta}{\partial z^a \partial z^c} \neq 0 \). When \( \frac{\partial^2 x^\beta}{\partial z^a \partial z^c} = 0 \), the solution \( \hat{A}_\mu(\mathcal{A})|_z \) in the coordinate system \( \{z^a\} \) is different from \( \hat{A}_\mu(\mathcal{A})|_x \) obtained in the coordinate system \( \{x^\mu\} \). The second term vanishes when the two noncommutativity parameters, \( \theta^{\alpha\beta} \) and \( \tilde{\theta}^{ab} \), are related as if they are tensors: \( \theta^{\alpha\beta} = \frac{\partial x^\alpha}{\partial z^a} \frac{\partial x^\beta}{\partial z^b} \tilde{\theta}^{ab} \). Although the vanishing condition for the second term does not hold in general, our polar noncommutativity parameter \( \theta/r \) in (3) and the canonical noncommutativity parameter \( \theta \) in (8) satisfy this condition. However, the transformation from the rectangular to the polar coordinates is not linear. Thus, the first term does not vanish and as we shall see this difference will yield different results for the rectangular and polar coordinate systems.

2.2 Coordinates as operators

In the following two subsections, we compare the aspects of noncommutativity in the polar and rectangular coordinate systems especially in using the Seiberg-Witten map. Since we consider only space-space noncommutativity in three dimensional spacetime in this paper, it is sufficient to compare the two sets of coordinate operators \( (\hat{x}, \hat{y}) \) and \( (\hat{r}, \hat{\phi}) \).

In the rectangular coordinate system, the commutation relation is given in the canonical form:

\[ [\hat{x}, \hat{y}] = i\theta. \] (8)

When the two sets of coordinate operators are related by the corresponding classical relation which is not linear, for example \( (x \rightarrow r \cos \phi, \ y \rightarrow r \sin \phi) \), we face the ordering ambiguity if we want to express one set of coordinates in terms of other set of coordinates. Moreover, for

5 The noncommutativity parameter \( \tilde{\theta}^{ab} \) in the polar coordinate system we use in this paper and the canonical one \( \theta^{\alpha\beta} \) satisfy this relation up to first order in \( \theta \). In fact, if the two Moyal products in the two coordinate systems are equal up to first order in \( \theta \), then one can show that this condition holds always regardless of the ordering problem.
the maps between functions of the operators, like a solution $\hat{A}(x, y)$ of the Seiberg-Witten equation on commutative space which corresponds to $A(\hat{x}, \hat{y})$ on noncommutative space, the ambiguity becomes severe.

In [21] it was shown that the commutation relation between polar coordinates is equivalent to the above canonical commutation relation up to first order in $\theta$. The commutation relation chosen there was the relation (3) which is equivalent to

$$[\hat{r}^2, \hat{\phi}] = 2i\theta.$$  

(9)

To see how the above commutation relation and the canonical one (8) is related, we assume that the usual map $(x, y) \to (r, \phi)$ between the rectangular and polar coordinates holds in this noncommutative space,

$$\hat{x} = \hat{r}\cos\hat{\phi}, \quad \hat{y} = \hat{r}\sin\hat{\phi}.$$  

(10)

Using the commutation relation $[\hat{\phi}, \hat{r}^{-1}] = i\theta\hat{r}^{-3}$ deduced from (9) one gets:

$$\hat{x}^2 + \hat{y}^2 := \hat{r}(\hat{r} - \frac{1}{2!}[\hat{\phi}, [\hat{\phi}, \hat{r}]] + \ldots) = \hat{r}^2 - \frac{1}{2!}\theta^2\hat{r}^{-2} + \ldots.$$  

(11)

Then one can readily check how the two commutation relations (8) and (9) are different: Using the commutation relation (8) we have

$$[\hat{x}^2 + \hat{y}^2, \hat{x}] = [\hat{y}^2, \hat{x}] = -2i\theta\hat{y} = -2i\theta\hat{r}\sin\hat{\phi},$$  

(12)

and using (11) this can be rewritten as

$$[\hat{r}^2 + \mathcal{O}(\theta^2), \hat{x}] \cong [\hat{r}^2, \hat{r}\cos\hat{\phi}] = \hat{r}[\hat{r}^2, \cos\hat{\phi}] = -2i\theta\hat{r}\sin\hat{\phi},$$  

(13)

where the relation $[\hat{r}^2, \hat{\phi}] = 2i\theta$ is applied. Therefore, (8) and (9) are equivalent up to first order in $\theta$ and became different from the second order in $\theta$.

Here, we make a short remark about the commutation relation used in [16]. There a noncommutative BTZ solution was worked out in the polar coordinates with the following commutation relation:

$$[\hat{r}, \hat{\phi}] = i\theta.$$  

(14)
If we assume that the usual relationship (10) between the rectangular and polar coordinate systems still holds in the noncommutative case, then we get the following relation by applying the commutation relation (14):

\[
[\hat{x}, \hat{y}] = [\hat{r} \cos \hat{\phi}, \hat{r} \sin \hat{\phi}] = i \theta \hat{r},
\]

which shows that the commutation relations (8) and (14) are not equivalent even by the dimensional count.

### 2.3 Twist perspective

Here, we prefer to use the commutation relation \([\hat{r}^2, \hat{\phi}] = 2i \theta\) in solving the Seiberg-Witten equation for calculational convenience, since the two commutation relations (3) and (9) are exactly equivalent. The reason for this preference can be easily understood if we view the Moyal product from the twist perspective.

It is known that the Moyal product (2) can also be reproduced from the deformed ∗-product (22, 23):

\[
(f ∗ g)(x) ≡ \left[ \mathcal{F}_*^{-1}(f(x) \otimes g(x)) \right],
\]

where the multiplication · is defined as ·[f(x) \otimes g(x)] = f(x)g(x), and the twist element \(\mathcal{F}_*\) is represented with the generators of translation along the \(x^\alpha\) directions, \(P_\alpha\), as follows.

\[
\mathcal{F}_* = e^{\frac{i}{2} \theta_{\alpha\beta} P_\alpha \otimes P_\beta} \rightarrow e^{-\frac{i}{2} \theta_{\alpha\beta} \frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial y^\beta}}. \tag{17}
\]

Using (16) and (17), one can check that \(f ∗ g\) in (16) is indeed equivalent to the Moyal product \(f \cdot g\) given in (2):

\[
(f ∗ g)(x) = \cdot \left[ e^{\frac{i}{2} \theta_{\alpha\beta} \frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial y^\beta}}(f(x) \otimes g(x)) \right] = f(x) \cdot g(x) + \frac{i}{2} \theta_{\alpha\beta} \frac{\partial f(x)}{\partial x^\alpha} \otimes \frac{\partial g(x)}{\partial x^\beta} + \cdots = \exp \left[ \frac{i}{2} \theta_{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial y^\beta} \right] f(x)g(y) \bigg|_{x=y} \equiv (f ∗ g)(x). \tag{18}
\]
Thus knowing the twist element in a given coordinate system helps one to identify the corresponding Moyal product.

The twist element which yields the noncommutativity (8) in the rectangular coordinates, or \([x, y]_* = i\theta\), is given by

\[
F_* = \exp \left[- \frac{i\theta}{2} \left( \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \right) \right].
\]  

(19)

One can rewrite the above exponent up to first order in \(\theta\), as follows:

\[
\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \simeq \frac{1}{r} \frac{\partial}{\partial \phi} \otimes \frac{1}{r} \frac{\partial}{\partial \phi}.
\]  

(20)

We can also define the twist element \(F'_*\) in the polar coordinates which yields the commutation relation \([r, \phi]_* = i\theta/r\) as in (3) and is equivalent to \(F_*\) up to first order in \(\theta\):

\[
F'_* := \exp \left[- \frac{i\theta}{2} \left( \frac{1}{r} \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial \phi} \otimes \frac{\partial}{\partial r} \right) \right]
\]  

(21)

\[
\simeq \exp \left[- \frac{i\theta}{2} \left( \frac{\partial}{\partial r} \otimes \frac{1}{r} \frac{\partial}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial \phi} \otimes \frac{\partial}{\partial r} \right) \right] \simeq F_*.
\]

If we write a twisted product corresponding to \(F'_*\), it would look like the Moyal product (2) except that \(\theta\) becomes coordinate dependent, i.e., \(\theta \rightarrow \theta/r\). To use the solution of the Seiberg-Witten equation without any modification, one should carefully place the factor \(1/r\) in front of the derivative \(\frac{\partial}{\partial r}\) in (21) when one expands the Moyal products in the Seiberg-Witten equation. However, if we rewrite \(F'_*\) in terms of the derivative \(\frac{\partial}{\partial r^2}\), as a new twist element \(F''_*\),

\[
F''_* = \exp \left[- i\theta \left( \frac{\partial}{\partial r^2} \otimes \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \phi} \otimes \frac{\partial}{\partial r^2} \right) \right],
\]  

(22)

this would allow us to use the Seiberg-Witten relation without any modification. The new twist element \(F''_*\) is equivalent to \(F'_*\) and yields the commutation relation \([r^2, \phi]_* = r^2 \star \phi - \phi \star r^2 = 2i\theta\).

### 3 BTZ black hole

Here and in the following section we investigate the effect of non-exact equivalence in noncommutativity using the two known commutative solutions in 3D, the BTZ black hole solution [24, 25] and the conical solution on \(AdS_3\) [14], in two ways.
One way is like the following: To apply the Seiberg-Witten map associated with the noncommutativity in the rectangular coordinates we first transform the commutative solution obtained in the polar coordinates into the one in the rectangular coordinates. Then after getting noncommutative solutions by applying the Seiberg-Witten map with the canonical commutation relation of the rectangular coordinates, we rewrite them back into the polar coordinates. The other way is to use the Seiberg-Witten map directly in the polar coordinates without rewriting the solution back and forth between the polar and the rectangular coordinate systems.

The action of the $\mathbf{(2+1)}$ dimensional noncommutative $U(1,1)\times U(1,1)$ Chern-Simons theory with the negative cosmological constant $\Lambda = -1/l^2$ is given by up to boundary terms [12] [13],

$$\hat{S}(\hat{A}^+, \hat{A}^-) = \hat{S}_+(\hat{A}^+) - \hat{S}_-(\hat{A}^-),$$

$$\hat{S}_\pm(\hat{A}^\pm) = \beta \int \text{Tr}(\hat{A}^\pm \wedge d\hat{A}^\pm + \frac{2}{3} \hat{A}^\pm \wedge \hat{A}^\pm \wedge \hat{A}^\pm),$$

where $\beta = l/16\pi G_N$ and $G_N$ is the three dimensional Newton constant. Here $\hat{A}^\pm = \hat{A}^{A\pm}_A = \hat{A}^a_{\pm} \tau_A + \hat{B}^\pm_{\tau_3}$, with $A = 0, 1, 2, 3$, $a = 0, 1, 2$, $\hat{A}^a_{\pm} = \hat{A}^a_{\pm}$, $\hat{A}^3_{\pm} = \hat{B}^\pm$, and the deformed wedge product $\hat{\wedge}$ denotes that $A \hat{\wedge} B \equiv A_\mu \ast B_\nu \, dx^\mu \wedge dx^\nu$. The noncommutative $SU(1,1)\times SU(1,1)$ gauge fields $\hat{A}$ are expressed in terms of the triad $\hat{e}$ and the spin connection $\hat{\omega}$ as $\hat{A}^a_{\pm} := \hat{\omega}^a_{\pm} \hat{e}^a / l$. In terms of $\hat{e}$ and $\hat{\omega}$ the action becomes [13]

$$\hat{S} = \frac{1}{8\pi G_N} \int \left( \hat{e}^a \wedge \hat{R}_a + \frac{1}{6l^2} \epsilon_{abc} \hat{e}^a \wedge \hat{e}^b \wedge \hat{e}^c \right)$$

$$- \frac{\beta}{2} \int \left( \hat{B}^+ \wedge d\hat{B}^+ + \frac{i}{3} \hat{B}^+ \wedge \hat{B}^+ \wedge \hat{B}^+ \right) + \frac{\beta}{2} \int \left( \hat{B}^- \wedge d\hat{B}^- + \frac{i}{3} \hat{B}^- \wedge \hat{B}^- \wedge \hat{B}^- \right)$$

$$+ \frac{i\beta}{2} \int (\hat{B}^+ - \hat{B}^-) \wedge \left( \hat{\omega} a \wedge \hat{e}_a + \frac{1}{l^2} \hat{e}^a \wedge \hat{e}_a \right)$$

$$+ \frac{i\beta}{2 l} \int (\hat{B}^+ + \hat{B}^-) \wedge \left( \hat{\omega} a \wedge \hat{e}_a + \hat{e}^a \wedge \hat{\omega}_a \right),$$

up to surface terms, where $\hat{R}^a = d\hat{\omega}^a + \frac{1}{2} \epsilon^{abc} \hat{\omega}_b \wedge \hat{\omega}_c$. The equation of motion can be written as follows.

$$\hat{\mathcal{F}}^\pm \equiv d\hat{A}^\pm + \hat{A}^\pm \wedge \hat{A}^\pm = 0.$$
In the commutative limit this becomes,

\[ F^\pm \equiv dA^\pm + A^\pm \wedge A^\pm = 0, \quad dB^\pm = 0, \quad (26) \]

and the first one can be rewritten as

\[ R^a + \frac{1}{2l^2} e^{abc} e_b \wedge e_c = 0, \quad T^a \equiv de^a + e^{abc} \omega_b \wedge e_c = 0. \quad (27) \]

The solution of the decoupled EOM for \( SU(1,1) \times SU(1,1) \) part was obtained in [25]:

\[ e^0 = m \left( \frac{r^+_d t - r^-_d \phi}{l} \right), \quad e^1 = \frac{l}{n} dm, \quad e^2 = n \left( r^+_d \phi - \frac{r^-_d}{l} dt \right), \]
\[ \omega^0 = -\frac{m}{l} \left( r^+_d \phi - \frac{r^-_d}{l} \right), \quad \omega^1 = 0, \quad \omega^2 = -\frac{n}{l} \left( \frac{r^+_d}{l} dt - r^-_d \phi \right), \quad (28) \]

where \( m^2 = (r^2 - r^2_+) / (r^2_+ - r^2_-) \), \( n^2 = (r^2 - r^2_-) / (r^2_+ - r^2_-) \), and \( r_+, r_- \) are the outer and inner horizons respectively. There it was also shown to be equivalent to the ordinary BTZ black hole solution [24]:

\[ ds^2 = -N^2 dt^2 + N^{-2} dx^2 + r^2 (d\phi + N^\phi dt)^2, \quad (29) \]

where \( N^2 = (r^2 - r^2_+) (r^2 - r^2_-) / l^2 r^2 \) and \( N^\phi = -r_+ r_- / l r^2 \).

### 3.1 Rectangular coordinates

The BTZ solution in the polar coordinates can be rewritten in the rectangular coordinates as follows:

\[ ds^2 = \left[ -N^2 + r^2 (N^\phi)^2 \right] dt^2 - 2y N^\phi dt dx + 2x N^\phi dt dy \\
+ \frac{2xy}{r^2} (N^{-2} - 1) dx dy + \frac{1}{r^2} (N^{-2} x^2 + y^2) dx^2 + \frac{1}{r^2} (N^{-2} y^2 + x^2) dy^2, \quad (30) \]

where \( r^2 = x^2 + y^2, \quad r^2_+ = \frac{M^2}{l^2}, \quad \left\{ 1 + \left[ 1 - \left( \frac{J}{Ml} \right)^2 \right]^{1/2} \right\}, \quad r_- = Jl / 2r_+, \quad N^\phi = -r_+ r_- / l r^2, \) and \( N^2 = (r^2 - r^2_+) (r^2 - r^2_-) / l^2 r^2 \). As in [21], we consider two simple \( U(1) \) fluxes \( B^\pm_\mu = B d\phi = B (xy dy - ydx) / r^2 \) with constant \( B \). Then, the commutative \( U(1,1) \times U(1,1) \) gauge fields \( A^\pm \) can be written as

\[ A^\pm_\mu = A^{\pm A}_\mu \tau_A = A^{a \pm}_\mu \tau_a + B^\pm_\mu \tau_3, \quad (31) \]

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where \( A = 0, 1, 2, 3, a = 0, 1, 2, \ A_{\mu}^{a\pm} = A_{\mu}^{a\pm}, \ A_{\mu}^{3\pm} = B_{\mu}^{\pm} \) and the gauge fields \( A^{a\pm} \) are given by

\[
A^{0\pm} = \pm \frac{m(r_+ \pm r_-)}{l^2} \left[ dt \pm \frac{l}{r^2} (ydx - xdy) \right],
\]

\[
A^{1\pm} = \pm \frac{1}{\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}} (dx + ydy),
\]

\[
A^{2\pm} = -\frac{n(r_+ \pm r_-)}{l^2} \left[ dt \pm \frac{l}{r^2} (ydx - xdy) \right].
\]

(32)

From the commutative \( U(1, 1) \times U(1, 1) \) gauge fields, we get \( \mathcal{A}_{\mu}^{\pm} \) (recall that \( \hat{A}^{\pm} = A_{\mu}^{\pm} + \mathcal{A}_{\mu}^{\pm} \)) via the Seiberg-Witten map (3):

\[
\mathcal{A}_{t}^{\pm} = \frac{i\theta B}{8l^2\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}} \sqrt{\frac{r_+ \pm r_-}{r_+ \mp r_-}} \left( \mp \frac{\sqrt{r^2 - r_+^2}}{\sqrt{r^2 - r_+^2}} \right),
\]

\[
\mathcal{A}_{x}^{\pm} = \frac{i\theta}{8l^2r^4(r^2 - r_+^2)(r^2 - r_-^2)} \left( \begin{array}{c}
-y(U^{\parallel\mp} - V^{\parallel}) \\
\mp Bl(xF^{\parallel\pm} + ilr^2yG)
\end{array} \right),
\]

\[
\mathcal{A}_{y}^{\pm} = \frac{i\theta}{8l^2r^4(r^2 - r_+^2)(r^2 - r_-^2)} \left( \begin{array}{c}
x(U^{\parallel\mp} - V^{\parallel}) \\
\mp Bl(xF^{\parallel\pm} + ilr^2yG)
\end{array} \right),
\]

(33)

where

\[
U^{\pm} = (r^2 - r_+^2)(r^2 - r_-^2)[B^2l^2 - (r_+ \pm r_-)]^2 - r^4l^2,
\]

\[
V^{\mp} = Bl(r_+ \mp r_-(r^2 - 2r_+^2)\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)},
\]

\[
F^{\pm} = (r^2 - r - r_+^2)(r^2 - 2r_-^2)(r_+ \pm r_-)\sqrt{r^2 - r_+^2},
\]

\[
G = r^2 \sqrt{\frac{r^2 - r_-^2}{r^2 - r_+^2} - (r^2 - 2r_+^2)} \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}},
\]

(34)
Using the relations between the gauge fields and the triad and spin connection, \( \dot{\theta} / \ell = \dot{A}^+ + \dot{A}^- \) and \( \dot{\omega} = \dot{A}^+ - \dot{A}^- \), we get the following up to first order in \( \theta \).

\[
\begin{align*}
\dot{e}^0 &= \frac{r_+ [r_2 - r_{2+} - \theta B/4]}{l \sqrt{(r_2^2 - r_{2+}^2)(r_{2+}^2 - r_2^2)}} dt + \frac{r_- \sqrt{r_2^2 - r_{2+}^2}}{r_2^2} \left[ 1 + \frac{\theta B}{4r^2} \left( \frac{r_2^2 - 2r_{2+}^2}{r_{2+}^2 - r_2^2} \right) \right] (ydx - xdy), \\
\dot{e}^1 &= - \frac{l (r_2 + r_{2+})}{(r_2^2 - r_{2+}^2)(r_{2+}^2 - r_2^2)} \left[ 1 - \frac{\theta B}{4r^2} \left( \frac{r_{2+}^4 (r_2^2 - 2r_{2+}^2) - r_2^4 (r^2 - 2r_{2+}^2)}{r_{2+}^2 (r_2^2 + r^2) \sqrt{(r_2^2 - r_{2+}^2)(r_{2+}^2 - r_2^2)}} \right) \right] (xdx + ydy), \\
\dot{e}^2 &= \frac{r_+ [r_2 - r_{2+} - \theta B/4]}{l \sqrt{(r_2^2 - r_{2+}^2)(r_{2+}^2 - r_2^2)}} dt - \frac{r_- \sqrt{r_2^2 - r_{2+}^2}}{r_2^2} \left[ 1 + \frac{\theta B}{4r^2} \left( \frac{r_2^2 - 2r_{2+}^2}{r_{2+}^2 - r_2^2} \right) \right] (ydx - xdy), \\
\dot{\omega}^0 &= \frac{r_- [r_2 - r_{2+} - \theta B/4]}{l^2 \sqrt{(r_2^2 - r_{2+}^2)(r_{2+}^2 - r_2^2)}} dt + \frac{r_+ \sqrt{r_2^2 - r_{2+}^2}}{l^2 r^2} \left[ 1 + \frac{\theta B}{4r^2} \left( \frac{r_2^2 - 2r_{2+}^2}{r_{2+}^2 - r_2^2} \right) \right] (ydx - xdy), \\
\dot{\omega}^1 &= 0, \\
\dot{\omega}^2 &= - \frac{r_+ [r_2 - r_{2+} - \theta B/4]}{l^2 \sqrt{(r_2^2 - r_{2+}^2)(r_{2+}^2 - r_2^2)}} dt - \frac{r_- \sqrt{r_2^2 - r_{2+}^2}}{l^2 r^2} \left[ 1 + \frac{\theta B}{4r^2} \left( \frac{r_2^2 - 2r_{2+}^2}{r_{2+}^2 - r_2^2} \right) \right] (ydx - xdy).
\end{align*}
\]

A noncommutative length element can be defined by

\[
ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu \equiv \eta_{ab} \hat{e}^a_\mu \star \hat{e}^b_\nu dx^\mu dx^\nu, \quad (36)
\]

where \( \star \) denotes the Moyal product. Since the length element \( ds^2 \) in (36) has symmetric summation, we end up with a real length element. Thus we define a real noncommutative metric by \( \hat{G}_{\mu\nu} \equiv (\hat{g}_{\mu\nu} + \hat{g}_{\nu\mu})/2 \) as in (14). After transforming it back to the polar coordinates, the length element is given by

\[
ds^2 = \hat{G}_{\mu\nu} dx^\mu dx^\nu = - \mathcal{F}^2 dt^2 + \hat{N}^{-2} dr^2 + 2r^2 N^2 (1 + \frac{\theta B}{2r^2}) dtd\phi + r^2 \left( 1 + \frac{\theta B}{2r^2} \right) d\phi^2, \quad (37)
\]

where

\[
\mathcal{F}^2 = \frac{(r_2^2 - r_{2+}^2 - r_2^2)}{l^2} - \frac{\theta B}{2l^2} = f^2, \tag{38}
\]

\[
\hat{N}^2 = \frac{1}{l^2 r_2^2} \left[ (r_2^2 - r_{2+}^2)(r^2 - r_2^2) - \frac{\theta B}{2r^2} \left( r_2^2 (r_2^2 - r_2^2) + r_{2+}^2 (r^2 - 2r_2^2) \right) \right]. \tag{39}
\]

Now, we investigate the apparent and Killing horizons of the above solution by the following relations:

\[
\hat{G}^{rr} = \hat{G}^{-rr} = \hat{N}^2 = 0, \tag{40}
\]

13
for the apparent horizon (denoted as $\hat{r}$), and
\[ \hat{\chi}^2 = \hat{G}_{tt} - \frac{\hat{G}_{t\phi}^2}{\hat{G}_{\phi\phi}} = 0, \]  
(41)

for the Killing horizon (denoted as $\tilde{r}$). These two equations yield the apparent and Killing horizons up to first order in $\theta$ at
\[ \hat{r}^2_{\pm} = r^2_{\pm} + \frac{\theta B}{2} + \mathcal{O}(\theta^2), \]  
(42)
\[ \tilde{r}^2_{\pm} = r^2_{\pm} + \frac{\theta B}{2} + \mathcal{O}(\theta^2). \]  
(43)

Here the apparent and Killing horizons coincide, and the inner and outer horizons are shifted from the classical (commutative case) value by the same amount $\theta B/2$ due to non-commutative effect of flux. Note that this feature agrees with the result in the commutative (classical) case, in which the apparent and Killing horizons coincide for stationary black holes.

### 3.2 Polar coordinates

Here, we recall the solution in the noncommutative polar coordinates obtained in [21] for comparison. From the consideration in section 2, the Moyal ($\star$) product from $[\hat{R}, \hat{\phi}] = 2i\theta$ is given by
\[ (f \star g)(x) = \exp \left[ i\theta \left( \frac{\partial}{\partial R} \frac{\partial}{\partial \phi'} - \frac{\partial}{\partial \phi} \frac{\partial}{\partial R'} \right) \right] f(x)g(x') \bigg|_{x=x'}, \]  
(44)

where $\hat{R} \equiv \hat{r}^2$. The noncommutative solution $\hat{A}^{\pm}$ is given by
\[ \hat{A}^{\pm}_\mu = \hat{A}^{a\pm}_\mu \tau_a + \hat{B}^{\pm}_\mu \tau_3 = \left( A^{a\pm}_\mu - \frac{\theta}{2} B^\pm_{\phi} \partial_R A^{a\pm}_\mu \right) \tau_a + B^\pm_\mu \tau_3 + \mathcal{O}(\theta^2), \]  
(45)

where we also considered two $U(1)$ fluxes $B^\pm_\mu = B d\phi$ with constant $B$.

Then from the Sieberg-Witten map we obtain the noncommutative triad and spin con-
\[ e^0 = \left( m - \frac{\theta B}{2} m' \right) \left( \frac{r_+}{l} dt - r_- d\phi \right) + \mathcal{O}(\theta^2), \]

\[ e^1 = l \left[ m' \left( \frac{m'}{n} - \frac{\theta B}{2} \left( \frac{m'}{n} \right) \right) \right] dR + \mathcal{O}(\theta^2), \]

\[ e^2 = \left( n - \frac{\theta B}{2} n' \right) \left( r_+ d\phi - \frac{r_-}{l} dt \right) + \mathcal{O}(\theta^2), \]

\[ \omega^0 = -\frac{1}{l} \left( m - \frac{\theta B}{2} m' \right) \left( r_+ d\phi - \frac{r_-}{l} \right) + \mathcal{O}(\theta^2), \]

\[ \omega^1 = \mathcal{O}(\theta^2), \]

\[ \omega^2 = -\frac{1}{l} \left( n - \frac{\theta B}{2} n' \right) \left( \frac{r_+}{l} dt - r_- d\phi \right) + \mathcal{O}(\theta^2), \]

where \( \cdot \) denotes the differentiation with respect to \( R = r^2 \). It should be noted that in the polar coordinates we get a real metric, \( \hat{e}_\mu \hat{e}_\nu = \hat{e}_\mu \hat{e}_\nu \). Rewriting \( R \) back to \( r^2 \), we get

\[ ds^2 = -f^2 dt^2 + \hat{N}^{-2} dr^2 + 2r^2 N^\phi dt d\phi + \left( r^2 + \frac{\theta B}{2} \right) d\phi^2 + \mathcal{O}(\theta^2), \]

where

\[ N^\phi = -r_+ r_- / l r^2, \]

\[ f^2 = \frac{(r^2 - r_+^2 - r_-^2)}{l^2} - \frac{\theta B}{2l^2}, \]

\[ \hat{N}^2 = \frac{1}{l^2 r^2} \left[ (r^2 - r_+^2)(r^2 - r_-^2) - \frac{\theta B}{2} (2r^2 - r_+^2 - r_-^2) \right]. \]

In this solution, the apparent and Killing horizons denoted as \( \hat{r} \) and \( \tilde{r} \), respectively, are given by:

\[ \hat{r}_\pm^2 = r_\pm^2 + \frac{\theta B}{2} + \mathcal{O}(\theta^2), \]

\[ \tilde{r}_\pm^2 = r_\pm^2 \pm \frac{\theta B}{2} \left( \frac{r_+^2 + r_-^2}{r_+^2 - r_-^2} \right) + \mathcal{O}(\theta^2). \]

Unlike the rectangular case, the apparent and the Killing horizons in this case do not coincide. Note that the outer horizons coincide only in the non-rotating limit in which the inner horizon of the commutative solution vanishes (\( r_- = 0 \)).
4 Conical solution on AdS$_3$

In this section we first reobtain the noncommutative conical solution in the rectangular coordinates and check it with the previously obtained one in [14]. Then, we repeat the analysis in the polar coordinates and compare the two results.

4.1 Rectangular coordinates

We begin with a nonsingular conical metric on AdS$_3$ in the polar coordinates ($t, r, \phi$) [14],

$$ds^2 = H^{-2} \left[ -(2 - H)^2 (dt + Jd\phi)^2 + (1 - M)^2 r^2 d\phi^2 + dr^2 \right],$$

(53)

where $M, J$ are mass and angular momentum of the source respectively, and $H = (1 - r^2/4l^2)$. The above metric can be transformed to the rectangular coordinates and the corresponding triad and spin connection in the rectangular coordinates are given by

$$e^0 = \frac{2 - H}{H} [dt - \frac{J}{r^2} (ydx - xdy)],$$

$$e^1 = \frac{1}{H} \left[ (1 - \frac{My^2}{r^2}) dx + \frac{Mxy}{r^2} dy \right],$$

$$e^2 = \frac{1}{H} \left[ \frac{Mxy}{r^2} dx + (1 - \frac{Mx^2}{r^2}) dy \right],$$

$$\omega^0 = \frac{(2 - M)H - 2(1 - M)}{Hr^2} (xdy - ydx),$$

$$\omega^1 = \frac{y}{l^2 H} \left[ dt - \frac{J}{r^2} (ydx - xdy) \right],$$

$$\omega^2 = -\frac{x}{l^2 H} \left[ dt - \frac{J}{r^2} (ydx - xdy) \right].$$

(54)

As in the previous subsection we consider the same commutative $U(1,1) \times U(1,1)$ gauge fields. After applying the Seiberg-Witten map we get $A^\pm_{i\mu}$ as follows.

$$\mathcal{A}^\pm_{i} = \frac{i \theta}{8l^3 H^2} \left( \begin{array}{cc} \mp (B + 2) & -Bl(2 - H)e^{-i\phi}/r \\ Bl(2 - H)e^{-i\phi}/r & \pm (B - 2) \end{array} \right),$$

$$\mathcal{A}^\pm_{x} = \frac{i \theta}{8l^2 r^3 H^2} \left( \begin{array}{cc} -ryu_B^\pm & \pm iBv^\pm \\ \mp iBv^\pm & -ryu_B^{\pm} \end{array} \right),$$

$$\mathcal{A}^\pm_{y} = \frac{i \theta}{8l^2 r^3 H^2} \left( \begin{array}{cc} rux_B^\pm & \pm iBh^\pm \\ \mp iBh^\pm & -rux_B^{\pm} \end{array} \right),$$

(55)
where

\[ u_B^\pm = [(M + B)l \pm J]^2 - 2(1 - H)[J^2 \pm 2(M + B + 1)Jl + (B^2 + 2MB + M(M + 2)l^2)] + (1 - H)^2[(M - B - 2)l \pm J]^2, \]

\[ v^\pm = lx(2 - H) + iy(Ml - l \pm J)(3H - 2), \]

\[ h^\pm = ly(2 - H) - ix(Ml - l \pm J)(3H - 2). \]  

(56)

Using the same relations between the gauge fields and the triad and spin connection given in the previous section, we obtain the noncommutative triad and spin connection up to first order in \( \theta \) as follows.

\[
\hat{e}_0 = \frac{2 - H}{H} \left[ \left( 1 - \frac{\theta B}{4l^2H(2 - H)} \right) dt - \frac{J}{r^2} \left( 1 - \frac{\theta B}{2r^2} \right) (ydx - xdy) \right],
\]

\[
\hat{e}_1 = \frac{1}{r^2H} \left[ Mxy - \frac{\theta B}{16l^2H} \left( 3(M - 1)y^2 + \frac{4(M - 2)l^2y^2}{r^2} + x^2 + 4l^2 \right) \right] dx + \frac{1}{H} \left[ (1 - \frac{Mx^2}{r^2}) + \frac{\theta Bxy}{16l^2r^4H} ((2 - 3M)r^2 + 4(M - 2)l^2) \right] dy,
\]

\[
\hat{e}_2 = \frac{1}{H} \left[ (1 - \frac{Mx^2}{r^2}) + \frac{\theta Bxy}{16l^2r^4H} ((2 - 3M)r^2 + 4(M - 2)l^2) \right] dx + \frac{1}{r^2H} \left[ Mxy + \frac{\theta B}{16l^2r^4H} \left( 3(M - 1)x^2 - \frac{4(M - 2)l^2x^2}{r^2} - y^2 + 4l^2 \right) \right] dy,
\]

(57)

\[
\hat{\omega}_0 = \frac{1}{Hr^2} \left[ (2 - M)H - 2(1 - M) - \frac{\theta B}{2r^2}[2(M - 1) - (M + 2)H] \right] (ydx - xdy),
\]

\[
\hat{\omega}_1 = \frac{y}{l^2H} \left[ 1 - \frac{\theta B(2 - H)}{4r^2H} \right] dt - \frac{Jy}{l^2r^2H} \left[ 1 - \frac{\theta B(2 - 3H)}{4r^2H} \right] (ydx - xdy),
\]

\[
\hat{\omega}_2 = \frac{-x}{l^2H} \left[ 1 - \frac{\theta B(2 - H)}{4r^2H} \right] dt + \frac{Jx}{l^2r^2H} \left[ 1 - \frac{\theta B(2 - 3H)}{4r^2H} \right] (ydx - xdy).
\]
Now, the length element of this solution becomes:

\[
\begin{align*}
\frac{ds^2}{d\xi^2} &= -\left(\frac{2 - H}{H}\right)^2 \left[1 - \frac{\theta B}{2l^2 H(2 - H)}\right] dt^2 + H^{-2} \left[1 - \frac{\theta B}{2r^2} \left(\frac{2 - H}{H}\right)\right] dr^2 \\
&\quad - 2J \left(\frac{2 - H}{H}\right)^2 \left[1 + \frac{\theta B}{2r^2} \left(1 - \frac{r^2}{l^2 H(2 - H)}\right)\right] dt d\phi \\
&\quad + \frac{1}{H^2} \left[\left((M - 1)^2 r^2 - J^2 (2 - H)^2\right)\right] \\
&\quad - \frac{\theta B}{2r^2 H} \left[2J^2 (H^3 - 6H^2 + 10H - 4) - (M - 1)^2 r^2 (3H - 2)\right] d\phi^2.
\end{align*}
\]

The above solution is not a black hole solution. However, in order to compare the effect of noncommutativity in different coordinate systems, we again consider the same quantities used to evaluate the two horizons, apparent and Killing horizons in the BTZ black hole case, now denoted as \(\hat{r}_A\) and \(\tilde{r}_K\). From the same determining relations, \(\hat{G}^{rr} = \hat{G}^{-1}_{rr} = 0\) and \(\hat{\chi}^2 = \hat{G}_{tt} - \hat{G}_{t\phi}/\hat{G}_{\phi\phi} = 0\) for \(\hat{r}_A\) and \(\tilde{r}_K\) respectively, we get

\[
\begin{align*}
\hat{r}_A^2 &= 4l^2, \\
\tilde{r}_K^2 &= 0,
\end{align*}
\]

up to first order in \(\theta\). The values obtained above coincide with the values in the commutative case. We consider that this matches with the feature appeared in the BTZ solution of the rectangular coordinates given in section 3.1. There the apparent and Killing horizons coincide in the noncommutative case just as in the commutative case.

### 4.2 Polar coordinates

Now we do the same analysis in the polar coordinates using \(R \equiv r^2\). The length element can be written in the \((t, R, \phi)\) coordinates as

\[
\begin{align*}
\frac{ds^2}{d\xi^2} &= H^{-2} \left[-(2 - H)^2 (dt + J d\phi)^2 + (1 - M)^2 Rd\phi^2 + \frac{dR^2}{4R}\right].
\end{align*}
\]

\(^6\) Our conical solution differs from the result obtained in [14] in one respect, in the use of gauge parameter: We use \(\hat{g} = \hat{g}(g, A)_{B \neq 0}\) with nonzero flux while in [14] they used \(\hat{g} = \hat{g}(g, A)_{B = 0}\) with zero flux.
Then the triad and spin connection are given by

\[
e^0 = \frac{2 - H}{H} (dt + Jd\phi),
\]

\[
e^1 = \frac{1}{H} \left[ \frac{\cos \phi}{2\sqrt{R}} dR - (1 - M)\sqrt{R} \sin \phi d\phi \right],
\]

\[
e^2 = \frac{1}{H} \left[ \frac{\sin \phi}{2\sqrt{R}} dR + (1 - M)\sqrt{R} \cos \phi d\phi \right],
\]

(62)

\[
\omega^0 = \frac{1}{H} [(2 - M)H - 2(1 - M)]d\phi,
\]

\[
\omega^1 = \sqrt{R} \sin \phi \frac{1}{l^2 H} (dt + Jd\phi),
\]

\[
\omega^2 = -\sqrt{R} \cos \phi \frac{1}{l^2 H} (dt + Jd\phi).
\]

We consider the same \( U(1) \) fluxes \( B^\pm_\mu = B d\phi \) with constant \( B \). Then, the noncommutative solution \((\hat{A}^\pm_\mu = A^\pm_\mu + A'^\pm_\mu)\) is given by

\[
A'^\pm_\mu = \pm \frac{i}{8l^3 H^2} \left( \begin{array}{cc} 2 + B & \pm B l (2 - H) e^{-i\phi}/\sqrt{R} \\ \mp B l (2 - H) e^{i\phi}/\sqrt{R} & 2 - B \end{array} \right),
\]

\[
A'^\pm_\mu = \pm \frac{\theta B (3H - 2)}{16l^2 H^2 \sqrt{R}} \left( \begin{array}{cc} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{array} \right),
\]

\[
A'^\pm_\phi = \frac{i \theta (l - M l \mp J)}{8l^3 H^2} \left( \begin{array}{cc} 2 + B & \pm B l (2 - H) e^{-i\phi}/\sqrt{R} \\ \mp B l (2 - H) e^{i\phi}/\sqrt{R} & 2 - B \end{array} \right).
\]

(63)

Then using the same relations between the gauge fields and the triad and spin connection given in the previous section, the noncommutative triad and spin connection are given by

\[
\hat{e}^0 = \frac{H (2 - H) - \theta B/4l^2}{H^2} (dt + Jd\phi),
\]

\[
\hat{e}^1 = \frac{\cos \phi}{2\sqrt{R} H} \left[ 1 + \frac{\theta B}{4R} \left( \frac{3H - 2}{H} \right) \right] dR - \frac{(1 - M)\sqrt{R} \sin \phi}{H} \left[ 1 - \frac{\theta B}{4R} \left( \frac{2 - H}{H} \right) \right] d\phi,
\]

\[
\hat{e}^2 = \frac{\sin \phi}{2\sqrt{R} H} \left[ 1 + \frac{\theta B}{4R} \left( \frac{3H - 2}{H} \right) \right] dR + \frac{(1 - M)\sqrt{R} \cos \phi}{H} \left[ 1 - \frac{\theta B}{4R} \left( \frac{2 - H}{H} \right) \right] d\phi,
\]

\[
\hat{\omega}^0 = \frac{1}{H} \left[ (2 - M)H - 2(1 - M) + \frac{\theta B (1 - M)}{4l^2 H} \right] d\phi,
\]

\[
\hat{\omega}^1 = \frac{\sqrt{R} \sin \phi}{l^2 H} \left[ 1 - \frac{\theta B}{4R} \left( \frac{2 - H}{H} \right) \right] (dt + Jd\phi),
\]

\[
\hat{\omega}^2 = -\frac{\sqrt{R} \cos \phi}{l^2 H} \left[ 1 - \frac{\theta B}{4R} \left( \frac{2 - H}{H} \right) \right] (dt + Jd\phi).
\]

(64)
The noncommutative length element defined in the same way as in the previous section is given by in terms of \( r \) as follows.

\[
\begin{align*}
\text{d}s^2 &= -\hat{\mathcal{F}}^2 \text{d}t^2 + \hat{\mathcal{N}}^{-2} \text{d}r^2 - 2J \hat{\mathcal{F}}^2 \text{d}t \text{d}\phi \\
& + \frac{(1 - M)^2 r^2 - J^2(2 - H)^2}{H^2} \left[ 1 - \frac{\theta B}{2l^2} \left( \frac{2 - H}{H} \right) \right] \left[ \frac{(1 - M)^2 r^2 - J^2}{(1 - M)^2 r^2 - J^2(2 - H)^2} \right] \text{d}\phi^2 + \mathcal{O}(\theta^2),
\end{align*}
\]

where

\[
\begin{align*}
\hat{\mathcal{F}}^2 &= \left( \frac{2 - H}{H} \right)^2 \left[ 1 - \frac{\theta B}{2l^2} \frac{1}{H(2 - H)} \right], \\
\hat{\mathcal{N}}^2 &= H^2 \left[ 1 - \frac{\theta B}{2r^2} \left( \frac{3H - 2}{H} \right) \right].
\end{align*}
\]

Here we again consider the same quantities \( \hat{r}_A \) and \( \tilde{r}_K \) defined in the previous subsection to investigate the effect of noncommutativity in different coordinate systems. Now they are given by

\[
\begin{align*}
\hat{r}_A^2 &= 4l^2 + \mathcal{O}(\theta^2), \\
\tilde{r}_K^2 &= \frac{\theta B}{4} + \mathcal{O}(\theta^2).
\end{align*}
\]

Unlike the rectangular case in the previous subsection in which both \( \hat{r}_A \) and \( \tilde{r}_K \) coincide with the classical values, here only \( \hat{r}_A \) coincides with the classical value \( r_A = 2l \). For \( \tilde{r}_K \), which would correspond to the Killing horizon of a black hole, does not coincide with the classical value \( r_K = 0 \). However, in the non-rotating limit (\( J = 0 \)), the solution for \( \tilde{r}_K \) does not exist, and this feature agrees with that of the commutative case in which the solution for \( r_K \) does not exist either in the non-rotating limit. Thus we see that the same pattern holds in the polar coordinates as in the BTZ case, namely in the non-rotating limit the same feature appears in both commutative and noncommutative cases.

5 Disccussion

In this paper, in order to investigate the non-exact equivalence between noncommutative coordinate systems we obtain a noncommutative BTZ black hole solution in the canonical
rectangular coordinates via Seiberg-Witten map, and compare it with the previously obtained result in the noncommutative polar coordinates [21]. We repeat the same analysis for the conical solution in noncommutative AdS$_3$ using the same action to see whether there exists any similarity between the two cases.

What we have learned can be illustrated as follows:

$$
\begin{array}{c}
\mathcal{A}(r, \phi) \xrightarrow{\text{II}} \mathcal{B}(x, y) \\
[\hat{r}, \hat{\phi}] = i\tilde{\theta} \\
\downarrow \\
\hat{\mathcal{A}}(r, \phi) \xleftarrow{\text{IV}} \hat{\mathcal{B}}(x, y) \\
\end{array}
$$

where $\mathcal{B}(x, y) \equiv \mathcal{A}[r(x, y), \phi(x, y)]$ and the maps II, IV are the coordinate transformations $(x, y) \leftrightarrow (r, \phi)$ in a commutative space, and the maps I, III denote corresponding Seiberg-Witten maps. For a function $\mathcal{A}(r, \phi)$, for example, the Carlip et. al.’s BTZ black hole solution in the polar coordinates [25], we have two different routes of getting Seiberg-Witten solutions $\hat{\mathcal{A}}(r, \phi)$, via I or via II $\rightarrow$ III $\rightarrow$ IV. From the observation of Eq. (7) in section 2, we know that the two solutions via the different routes would be different, i.e. $\hat{\mathcal{A}}(r, \phi) \neq \hat{\mathcal{B}}[x(r, \phi), y(r, \phi)]$, since the transformation $(x, y) \leftrightarrow (r, \phi)$ is not linear. The results in sections 3 and 4 just support this observation.

Another lesson we get is from the following observations. 1) In the rectangular coordinates, the feature appeared in the solution of the commutative case remains intact in the noncommutative case: In the BTZ case, both apparent and Killing horizons coincide. In the conical solution, the commutative and the noncommutative results are the same. 2) In the polar coordinates, the feature appeared in the commutative case is not maintained in the noncommutative case: In the BTZ case, apparent and Killing horizons do not coincide. In the conical solution, the commutative and the noncommutative results do not agree. However, in the non-rotating limit the feature appeared in the commutative case is maintained in the noncommutative case: In the BTZ case, apparent and Killing horizons do coincide. In the conical solution case, the commutative and noncommutative results agree.

Thus we are left with a task of understanding the differed behaviors in the polar coordinates. Our understanding is as follows. In the BTZ case, the Killing vector which determines
the Killing horizon is dependent on the translation generator along the \( \hat{\phi} \) direction, while the apparent horizon is determined by the null vector given by the translation generator along the radial \( \hat{r} \) direction. Hence in the rotating case the relation between the two horizons is affected by the noncommutativity between the two coordinates (\( \hat{r}, \hat{\phi} \)), and will differ from the commutative case. The two horizons will not coincide. In the non-rotating case, the Killing vector does not depend on the translation generator along the \( \hat{\phi} \) direction and thus no effect of noncommutativity among (\( \hat{r}, \hat{\phi} \)) enters, resulting the same relation as in the commutative case. In the conical solution case, since we used the same defining relations for \( \hat{r} \) and \( \tilde{r} \) as in the BTZ case, we expect the same.

In the rectangular coordinates, the above noncommutative effect does not enter since we are applying the above operation (getting a solution for \( \hat{r} \) and \( \tilde{r} \)) to the result obtained by commutative coordinate transformation after the Seiberg-Witten map, thus wiping out the noncommutative characteristics. Note that the result obtained in the rectangular coordinates for the BTZ case differs from the commutative result. However, the feature that the apparent and Killing horizons coincide remains the same as in the commutative case. Namely, we simply obtained a differed geometry from the commutative case due to noncommutative effect by the Seiberg-Witten map. However, the noncommutative effect in getting the solution of \( \hat{r} \) and \( \tilde{r} \) was lost.

Thus as it was pointed out in [26] that the conventional sense of diffeomorphism is not invariant in noncommutative theory, we better use the same coordinate system throughout the process of solution finding, matching the coordinate system such that the operational meaning of noncommutativity can be kept. For instance, the commutation relation \([\hat{x}, \hat{y}] = i\theta\) has translational symmetry, while the commutation relation \([\hat{r}^2, \hat{\phi}] = 2i\theta\) has rotational symmetry. So if we use \([\hat{r}^2, \hat{\phi}] = 2i\theta\) instead of \([\hat{x}, \hat{y}] = i\theta\), this means that we choose the rotational symmetry (translational symmetry along \( \phi \) direction) at the cost of the translational symmetry along the \( x \) and \( y \) directions. We consider this as the underlying reason for the differences in the results obtained in the paper.
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