Honeycomb Hubbard Model at van Hove Filling II: Lower Bounds of the Self-Energy

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Abstract

In this paper we complete the proof that the honeycomb Hubbard model with renormalized chemical potential \( \mu = 1 \) is not a Fermi liquid in the mathematically precise sense of Salmhofer, by establishing the necessary lower bounds for second derivatives of the self-energy w.r.t. the external momentum.

1 Introduction and main result

This is the second paper which is devoted to the rigorous study of the doped Hubbard model on the two-dimensional honeycomb lattice in which the value of the renormalized chemical potential \( \mu \) is equal to the hopping parameter \( t \) (which is set to be 1). For a general introduction to this model we refer to the paper \([6]\) and the references therein. We assume its results and notations. In \([6]\), using the Fermionic cluster expansions and the renormalization group analysis around the Fermi surface, we prove that the renormalized two-point Schwinger functions and the self-energy are uniformly bounded in the domain \( R_T = \{ \lambda \mid |\lambda| \cdot |\log T| \leq c \} \).

Before proceeding, let us recall the Salmhofer’s criterion on the Fermi liquid \([7]\):

**Definition 1.1** (Salmhofer’s criterion). A 2-dimensional many-fermion system at positive temperature is a Fermi liquid if the thermodynamic limit of the momentum space Green’s functions exists for \( |\lambda| < \lambda_0(T) \) and if there are constants \( C_0, C_1, C_2 > 0 \) independent of \( T \) and \( \lambda \) such that the following holds:

(a) The perturbation expansion for the momentum space self-energy \( \hat{\Sigma}(k, \lambda) \) converges for all \( (\lambda, T) \) with \( |\lambda \log T| < C_0 \).

(b) The self-energy \( \hat{\Sigma}(k, \lambda) \) satisfies the following regularity conditions:

- \( \hat{\Sigma}(k, \lambda) \) is twice differentiable in \( k_0, k_+, k_- \) and

\[
\max_{\beta=2} \| \partial^{\alpha}_{\lambda^\beta} \hat{\Sigma}(k, \lambda) \|_{\infty} \leq C_1, \quad \alpha = 0, \pm.
\]
• The restriction of the self-energy on the Fermi surface is $C^{\beta_0}$ differentiable w.r.t. the momentum, in which $\beta_0 > 2$, and

$$\max_{\beta = \beta_0} \| \partial^3_{k_0} \Sigma(k, \lambda) \|_\infty \leq C_2, \ \alpha = 0, \pm. \quad (2)$$

In this paper we prove that a certain second derivative of the self-energy at a particular value of the external momentum is not uniformly bounded in the domain $\mathcal{R}_T = \{ \lambda : |\lambda| \cdot |\log^2 T| \leq c \}$ where we have established analyticity. This domain being smaller than the Salmhofer’s criterion, together with the non-uniformly boundedness of the second derivative of the self-energy, it suffices to disprove this criterion and to conclude that the two-dimensional half-filled Hubbard model is not a Fermi liquid. The main result of this paper is

**Theorem 1.1.** Let $\Sigma(q, \lambda)$ be the self-energy of the honeycomb Hubbard model (to be defined in Section 2, see also [6] for more details) in the momentum space with external quasi-momentum $q = (k_0, q_+, q_-)$. There exists a positive constant $K$, which depends on the model but is independent of the temperature $T$, such that $\forall \lambda \in \mathcal{R}_T$,

$$\left| \partial^2_{q_+} \Sigma(q, \lambda) \right|_{k_0=\pi T, \ q_+=0, \ q_-=0} \geq \frac{K}{T}. \quad (3)$$

For proving this result it is enough to establish a lower bound for dominant contribution w.r.t. $\lambda$ to the self-energy (cf.[6], Remark 7.2), which is the amplitude of the sunshine graph (see Figure 1). The latter quantity is noted simply by $\partial^2_{q_+} \Sigma^{Sun}$. A complication is that both $\Sigma(q, \lambda)$ and $\Sigma^{Sun}(q, \lambda)$ are matrices. Fortunately for disproving Salmhofer’s criterion, one needs only one element of the matrix $\partial^2_{q_+} \Sigma^{Sun}$, say, in the direction of $\alpha = 1, \alpha' = 1$.

This paper is organized as follows: in section 2 we separate $\partial^2_{q_+} \Sigma^{Sun}(\pi T, 0, 0)$ into a dominant contribution $\partial^2_{q_+} \Sigma^{Sun, dom}(\pi T, 0, 0)$ and a rest $\partial^2_{q_+} \Sigma^{Sun, error}(\pi T, 0, 0)$. In section 3 we provide a lower bound in $\frac{K}{T}$ for the dominant contribution $\Sigma^{Sun, dom}(q, \lambda)$. It relies on a method based on residues for complex integrals. In section 4 we provide an upper bound in $\frac{K}{T} + K_2$ for the rest contribution $\partial^2_{q_+} \Sigma^{Sun, error}(\pi T, 0, 0)$, where $N$, the number of sectors, in related to sector analysis [3, 5, 11]. Since $N$ can be rather large, independent of the temperature, the proof is complete.

Figure 1: The sunshine graph.

2 The model

The honeycomb Hubbard model that we consider has been defined in [6]. Here we briefly recall some basic notions for the reader’s convenience. The infinite honeycomb lattice
is the superposition of two triangle lattices: \( \Lambda = \Lambda_A \cup \Lambda_B \), in which \( \Lambda_A = \{ \mathbf{x} \mid \mathbf{x} = n_1 \mathbf{l}_1 + n_2 \mathbf{l}_2, n_1, n_2 \in \mathbb{Z} \} \subset \mathbb{R}^2 \) is the infinite triangular lattice generated by the basis vectors \( \mathbf{l}_1 = \frac{1}{2}(3, \sqrt{3}) \), \( \mathbf{l}_2 = \frac{1}{2}(3, -\sqrt{3}) \), and \( \Lambda_B = \Lambda_A + \mathbf{d}_1 \) is the shifted of \( \Lambda_A \) by the vector \( \mathbf{d}_1 = (1, 0) \). For a fixed \( L \in \mathbb{N}_+ \), define also the finite lattice of size \( L \), which is the torus \( \Lambda_L = \Lambda/L \). Let \( \beta \) be the inverse temperature, then the set of temperature-space variables \( (x_0, \mathbf{x}) \) is denoted by \( \Lambda_{\beta,L} = [-\beta, \beta] \times \Lambda_L \). Let \( \Lambda_L' \) be the dual lattice of \( \Lambda_L \) with basis vectors \( \mathbf{G}_1 = \frac{2\pi}{3}(1, \sqrt{3}) \), \( \mathbf{G}_2 = \frac{2\pi}{3}(1, -\sqrt{3}) \), the first Brillouin zone is defined as \( \mathcal{D}_L := \mathbb{R}^2/\Lambda_L^* \). The dual variable to \( x_0 \) is denoted by \( k_0 \) (also called the Matsubara frequency), which takes values in the set \( \mathbb{T}_\beta := \pi T + 2\pi T \mathbb{Z} \). The space of frequency-momentum is noted by \( \mathcal{D}_{\beta,L} = \mathbb{T}_\beta \times \mathcal{D}_L \). The non-interacting propagator is

\[
\hat{C}(k) = \frac{1}{-ik_0 I + E(k, \mu)}, \tag{4}
\]

in which \( I \) is the \( 2 \times 2 \) identity matrix, \( E(k, \mu) = \begin{pmatrix} -\mu & -\Omega^*(k) \\ -\Omega(k) & -\mu \end{pmatrix} \) is called the band matrix, in which \( \Omega(k) = 1 + 2e^{-ik_1} \cos(\sqrt{3}k_2) \) is the non-interacting complex dispersion relation and \( \Omega^*(k) \) is the complex conjugate. \( \mu \in \mathbb{R} \) is the chemical potential. Inverting \[4\], we obtain:

\[
\hat{C}(k) = \frac{1}{k_0^2 + e(k, \mu) - 2i\mu k_0} \begin{pmatrix} ik_0 + \mu & -\Omega^*(k) \\ -\Omega(k) & ik_0 + \mu \end{pmatrix}, \tag{5}
\]

in which

\[
e(k, \mu) := -\det[E(k, \mu)] = 4 \cos(3k_1/2) \cos(\sqrt{3}k_2/2) + 4 \cos^2(3k_2/2) + 1 - \mu^2, \tag{6}
\]

which is closely related to the band structure. The propagator in the direct space is defined as:

\[
C_\beta(x - y) := \lim_{L \to \infty} \frac{1}{\beta |\Lambda_L|} \sum_{k = (k_0, k) \in \mathcal{D}_{\beta,L}} e^{ik_0(x_0 - y_0) + i\mathbf{k}(x - y)} \hat{C}(k_0, \mathbf{k}). \tag{7}
\]

The many-body interaction potential is:

\[
\mathcal{V}_L(\psi, \lambda) = \lambda \sum_{a, a' = \uparrow, \downarrow} \int_{\Lambda_{\beta,L}} d^3x \, \psi_{x, a}^{\dagger} \psi_{x, a'} \psi_{x, a'}^{\dagger} \psi_{x, a} + \frac{1}{\beta |\Lambda_L|} \sum_{k \in \mathcal{D}_{\beta,L}} \sum_{\alpha = \uparrow, \downarrow} \sum_{\alpha' = \uparrow, \downarrow} \left[ \delta \mu(\lambda) \delta_{\alpha \alpha'} + \nu(\mathbf{k}, \lambda) \right] \psi_{k, \tau, a}^{\dagger} \psi_{k, \tau, a'} \tag{8}
\]

in which the first term is the Hubbard interaction potential and the two terms in the second line are the counter-terms, which satisfy \( \delta \mu(0) = 0 \) and \( \nu(\mathbf{k}, 0) = 0 \). Remark that only the 4-Fermion Hubbard interaction term is needed in the calculation of the self-energy.

**Remark 2.1.** We are mainly interested in the infinite volume limit of the various quantities, eg. the Schr"{o}dinger's functions, the self-energy, ect., which are taken by removing the infrared cutoff step by step following the renormalization group analysis. In the following we assume that the limits \( L \to \infty \) for the interesting quantities have already been taken. Thus we introduce the following notations: \( \lim_{L \to \infty} \Lambda_{\beta,L} := \Lambda_\beta \) and \( \lim_{L \to \infty} \mathcal{D}_{\beta,L} := \mathcal{D}_\beta \).
The non-interacting Fermi surface (F.S.) is defined as:

\[ \mathcal{F}_0 := \{ \mathbf{k} = (k_1, k_2) \in \mathbb{R}^2 \mid e(\mathbf{k}, \mu) = 0 \}. \tag{9} \]

For \( \mu = 1 \), \( \mathcal{F}_0 \) consists of a set of exact triangles, called the Fermi triangles, among which the following two:

\[ \mathcal{F}_0^+ = \{ k_2 = \sqrt{3} k_1 - 2 \pi \sqrt{3}, k_1 \in \left[ \frac{2 \pi}{3}, \pi \right] \cup \{ k_2 = -\sqrt{3} k_1 + 2 \pi \sqrt{3}, k_1 \in \left[ \frac{2 \pi}{3}, \frac{2 \pi}{3} \right] \} \quad \cup \{ k_2 = \frac{\pi}{\sqrt{3}}, k_1 \in \left[ \frac{\pi}{3}, \pi \right] \}, \tag{10} \]

\[ \mathcal{F}_0^- = \{ k_2 = \sqrt{3} k_1 - 2 \pi \sqrt{3}, k_1 \in \left[ \frac{2 \pi}{3}, \frac{2 \pi}{3} \right] \cup \{ k_2 = -\sqrt{3} k_1 + 2 \pi \sqrt{3}, k_1 \in \left[ \frac{2 \pi}{3}, \pi \right] \} \quad \cup \{ k_2 = -\frac{\pi}{\sqrt{3}}, k_1 \in \left[ \frac{\pi}{3}, \pi \right] \}, \tag{11} \]

are considered as the fundamental Fermi triangles. All the other Fermi triangles are translations of them. The vertices of the Fermi triangles are called the van Hove singularities. Due to the \( \mathbb{Z}^3 \) symmetry of the Fermi triangle (see Figure 2), it is convenient to introduce a new basis \( (e_+, e_-) \):

\[ e_+ = \frac{\pi}{3}(1, \sqrt{3}), \quad e_- = \frac{\pi}{3}(-1, \sqrt{3}), \tag{12} \]

which is neither orthogonal nor normal. Let \( (k_+, k_-) \) be the coordinates of the momentum in the new basis, the transformation law is

\[ k_1 = \frac{\pi}{3}(k_+ - k_-), \quad k_2 = \frac{\pi}{\sqrt{3}}(k_+ + k_-). \tag{13} \]

In the new coordinate system, the first Brillouin zone, denoted by \( \mathcal{D}_1 \), becomes a rescaled rhombus in which \( k_+ \in [0, 2] \) and \( k_- \in [-2, 0] \). See Figure 2 for an illustration. We have

\[ e(k_+, k_-, 1) = 8 \cos \frac{\pi (k_+ + k_-)}{2} \cos \frac{\pi k_+}{2} \cos \frac{\pi k_-}{2}, \tag{14} \]

and the Fermi triangles are defined by the equations \( k_+ = \pm 1 \), \( k_- = \pm 1 \) and \( \frac{k_+ + k_-}{2} = \pm 1 \). Now we consider the dual variables \( (x_+, x_-) \) to the lattice momentum \( (k_+, k_-) \).

Figure 2: The Brillouin zone \( \mathcal{D}_1 \) in the coordinate system \( (k_+, k_-) \). The vertices of the Fermi triangles are the van Hove singularities.
Without losing generality, we can pick up any lattice point of type $A$ and study the corresponding change of coordinates. Coordinate transforms for points of type $B$ are the same (modulo a shift of the origin), due to the periodic structure on the lattice. Consider the lattice point $x = n_1 l_1 + n_2 l_2$, in which $l_1 = \frac{1}{2}(3, \sqrt{3})$, $l_2 = \frac{1}{2}(3, -\sqrt{3})$ and $n_1, n_2 \in \mathbb{Z}$. We have $x_1 = \frac{3}{2}(n_1 + n_2)$, $x_2 = \frac{\sqrt{3}}{2}(n_1 - n_2)$. The coordinate transformation in the direct space corresponding to (13) is:

$$x_+ = \pi \cdot \left(\frac{x_1}{3} + \frac{x_2}{\sqrt{3}}\right) = \pi n_1, \quad x_- = \pi \cdot \left(-\frac{x_1}{3} + \frac{x_2}{\sqrt{3}}\right) = -\pi n_2, \quad n_1, n_2 \in \mathbb{Z}.$$ 

The free propagator can be written as

$$C(x) = C(x_0, x_+, x_-) = \frac{2\pi^2}{3\sqrt{3}} \int_{\beta} dk_0 \int_{D_{\beta}} dk_+ dk_- \hat{C}(k_0, k_\pm)e^{ik_0 x_0 + i k_\pm x_+ + k_- x_-},$$

(15)

$$\hat{C}(k_0, k_\pm) = \frac{1}{-2ik_0 + e(k_+, k_-) + k_0^2} \begin{pmatrix} ik_0 + 1 & -\tilde{\Omega}^*(k) \\ -\Omega(k) & ik_0 + 1 \end{pmatrix},$$

(16)

in which $\tilde{\Omega}(k) = 1 + 2e^{-i\frac{\pi}{2}(k_+ - k_-)} \cos \frac{\pi}{2}(k_+ + k_-)$ and the prefactor $\frac{2\pi^2}{3\sqrt{3}}$ is the Jacobian of the transform $(k_1, k_2) \to (k_+, k_-)$. The integral $\int dk_0$ is the discrete sum $2\pi T \sum_{n \in \mathbb{Z}} (2n + 1)\pi T$ and the integration $\int dk_+ dk_-$ is constrained in $D_{\beta}$.

Define also the quasi-momentum $q_\pm$ by

$$q_\pm = \begin{cases} k_\pm - 1, & \text{for } k_\pm \geq 0, \\ k_\pm + 1, & \text{for } k_\pm \leq 0. \end{cases}$$

(17)

Obviously we have $|q_\pm| \leq 1$, and (14) can be rewritten as

$$e(q, 1) = -8 \sin \frac{\pi q_+}{2} \sin \frac{\pi q_-}{2} \cos \frac{\pi (q_+ + q_-)}{2}.$$ 

(18)

The first Brillouin zone in the new coordination system $(q_+, q_-)$ is expressed by $\bar{D}$. The fundamental Fermi triangles are defined by: $q_+ = 0$, $q_- = 0$ and $q_+ + q_- = \pm 1$. See Figure 3 for an illustration. Remark that, when the momentums are restricted in the (scaled) first Brillouin zone, we have $k_+ \geq 0$ while $k_- \leq 0$, and the quasi-momentums are given by

$$q_+ = k_+ - 1, \quad q_- = k_- + 1.$$ 

(19)

The free propagator can be rewritten as:

$$C(x_0, x_+, x_-) = \frac{2\pi^2}{3\sqrt{3}} \int_{\beta} dk_0 \int_{\bar{D}} dq_+ dq_- \hat{C}(k_0, q_+, q_-)e^{ik_0 x_0 + q_+ x_+ + q_- x_-},$$

(20)

with

$$\hat{C}(k_0, q_+, q_-) = \frac{1}{-2ik_0 + e(q, 1) + k_0^2} \begin{pmatrix} ik_0 + 1 & -\tilde{\Omega}^*(q) \\ -\Omega(q) & ik_0 + 1 \end{pmatrix},$$

(21)

in which $\tilde{\Omega}(q) = 1 - 2e^{-i\frac{\pi}{2}(q_+ - q_-)} \cos \frac{\pi}{2}(q_+ + q_-)$.

We already said that we consider the second derivative in the $+$ direction of the quantity $\Sigma^{Sun}$, the self-energy contributed by the sunshine graph (see Figure 1), at some
Figure 3: The integration domain \( \tilde{D} \). The vertices of the Fermi triangles are the van Hove singularities.

In particular, say, \( \alpha = 1, \alpha' = 1 \), and at some particular external momentum \( q^e \), which can be chosen as \( k_0 = \pi T, \; q^e_+ = 0 \) and \( q^e_- = 0 \). For this particular set of values, we have:

\[
\partial_+^2 \Sigma^{Sun}(\pi T, 0, 0)|_{\alpha\alpha'} = \int_{\Lambda_\beta} d^3x x_+^2 |C(x)\bar{C}(x)|^2 e^{i\pi T x_0 + i(x_+ + q^e_+ + 2ix_-)}.
\]  

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\]  

Since each matrix element in the numerator of \( \hat{C}(k_0, q_+, q_-) \) (cf. Formula (20)) is bounded both from above and below by a strictly positive constant (cf. [6], Proposition 3.2), we can replace each matrix element of the propagators \( C(x) \) by some positive constant \( K \) times the scalar propagator \( C^I(x) \), which is defined by

\[
C^I(x) = \frac{2\pi^2}{3\sqrt{3}} \int_{\mathbb{T}_\beta} dk_0 \int_{\tilde{D}} dq_+ dq_- e^{ik_0 x_0 + i(q^e_+ + 2ix_-)}.
\]

in which \( \hat{C}^I(k_0, q) = \frac{1}{2i k_0 + e(q_0, 1)} \). In the following we shall forget the matrix index \( \alpha, \alpha' \).

Define \( \tilde{D}_\beta = \mathbb{T}_\beta \times \tilde{D} \), we have:

\[
\partial_+^2 \Sigma^{Sun}(\pi T, 0, 0) = K^3 \int_{\Lambda_\beta} d^3x x_+^2 e^{-i\pi T x_0 + i(x_+ + q^e_+ + 2ix_-)} \int_{\tilde{D}} dq_+ dq_- e^{ik_0 x_0 + i(q^e_+ + 2ix_-)}
\]

\[
\int_{\tilde{D}} dq_+ dq_- e^{ik_0 x_0 + i(q^e_+ + 2ix_-)} \int_{\tilde{D}} dq_3 dq_4 e^{ik_3 o x_0 + i(q^e_+ + 2ix_-)}.
\]

\[
\text{Remark 2.2.} \; \text{Since } x_+ \in \pi \mathbb{Z} \; \text{(cf. Formula (22))}, \; \text{the two factors } e^{2ix_+} \; \text{and } e^{-2ix_-} \; \text{are both equal to one. We shall also omit the inessential constant } K^3.
\]

\[\text{2.1 Sector analysis}\]

In order to establish the upper and lower bounds for \( \partial_+^2 \Sigma^{Sun}(\pi T, 0, 0) \), it is necessary to use the multi-slice representations for \( \hat{C}(x) \). Let \( \chi \in G_0^h(\mathbb{R}) \) be an function, with \( G_0^h \) the Gevrey class of functions of index \( h > 1 \), be an even smooth cutoff function such that \( \chi(t) = 0 \) for \( |t| > 2 \), \( \chi(t) \in (0, 1) \) for \( 1 < |t| \leq 2 \) and \( \chi(t) = 1 \) for \( |t| \leq 1 \). Given any fixed constant \( \gamma \geq 10 \), define the following partition of unity:

\[
1 = \sum_{j=0}^{\infty} \chi_j(t), \quad \forall t \neq 0;
\]

\[
\left\{ \begin{array}{l}
\chi_0(t) = 1 - \chi(t), \\
\chi_j(t) = \chi(\gamma^{2j-2}t) - \chi(\gamma^{2j}t), \; \text{for } j \geq 1.
\end{array} \right.
\]  

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\chi_j(t) = \chi(\gamma^{2j-2}t) - \chi(\gamma^{2j}t), \; \text{for } j \geq 1.
\end{array} \right.
\]
Define $t^{(1)} = \sin^2 \frac{\pi q_+}{2}$, $t^{(2)} = \sin^2 \frac{\pi q_-}{2}$ and $t^{(3)} = \sin^2 \frac{\pi (q_+ + q_-)}{2}$, and to each factor $t^{(a)}$, $a = 1, 2, 3$, we introduce a set of indices, $s^{(a)} = 0, 1, \ldots, j$, called the sector indices, and define the following partition of unity:

$$1 = \sum_{s^{(a)} = 0}^{j} v_{s^{(a)}}(t^{(a)}), \quad \begin{cases} v_0(t^{(a)}) = 1 - \chi(\gamma t^{(a)}), \\ v_{s^{(a)}}(t^{(a)}) = \chi(\gamma^{s+1} t^{(a)}), \\ v_j(t^{(a)}) = \chi(\gamma^{3} t^{(a)}), \end{cases} \text{ for } 1 \leq s^{(a)} \leq j - 1, \ a = 1, 2, 3. \tag{26}$$

The pair of integers $\sigma = (s^{(a)}, s^{(b)})$ are called the sector indices. Define also the generalized sector index by the triple $\bar{\sigma} := (j, \sigma) = (j, s^{(a)}, s^{(b)})$. Define the cutoff function

$$u_{\sigma}(q) = \chi_j[4k_0^2 + e^2(q)]v_{s^{(a)}(\sin^2 \frac{\pi}{2} q_+)}v_{s^{(b)}(\sin^2 \frac{\pi}{2} q_-)}, \ a, b = 1, 2, 3, \ a \neq b, \tag{27}$$

the multi-slice representation of the propagator is defined as $C^I(x) = \sum_{\sigma} C^I_{j,\sigma}(x)$, in which

$$C^I_{j,\sigma}(x) = \frac{2\pi^2}{3\sqrt{3}} \int \frac{dk_0 dq_+ dq_-}{D} e^{i(k_0 x_0 + (q_+ + 1)x_+ + (q_- - 1)x_- - 2ik_0 + e(q, 1)} u_{\sigma}(q). \tag{28}$$

Conservation of momentum at each interaction vertex place the following constraints on the sector indices:

**Lemma 2.1** ([6], Lemma 3.3). Let $t^{(a)}$ and $t^{(b)}$, $a, b = 1, 2, 3, \ a \neq b$, be the factors that are close to zero and $s^{(a)}$, $s^{(b)}$ be the corresponding sector indices. Then $s^{(a)}$ and $s^{(b)}$ must satisfy:

$$s^{(a)} + s^{(b)} \geq j - 1. \tag{29}$$

Before proceeding, let us consider first the upper bound for the sunshine graph.

**Lemma 2.2.** There exists some positive constant $K_1$ which is independent of $T$, such that

$$|\partial^2_+ \sum_{\text{Sun}}(\pi T, 0, 0)| \leq \frac{K_1}{T}. \tag{30}$$

**Proof.** Using the multi-slice representation for the propagators, we have

$$\partial^2_+ \sum_{\text{Sun}}(\pi T, 0, 0) = -\sum_{j_1, j_2, j_3, \sigma_1, \sigma_2, \sigma_3} \int d^4x \sum_{j_1, j_2, j_3} C^I_{j_1, \sigma_1}(x) C^I_{j_2, \sigma_2}(x) C^I_{j_3, \sigma_3}(x) e^{-i\pi T x_0 - i x_+ + i x_-} \tag{31}$$

Recall the following bound for a sectorized propagator (cf. [6], Lemma 3.6):

$$\|C^I_{j,\sigma}(x)\|_{\alpha} \leq c \gamma^{-s_+ - s_-} e^{-c|\alpha|^2(x)}, \tag{32}$$

in which $c$ is some positive constant, $d_{j,\sigma}(x, y) = \gamma^{-j}|x_0| + \gamma^{-s_+}|x_+| + \gamma^{-s_-}|x_-|$, then we have

$$|\partial^2_+ \sum_{\text{Sun}}(\pi T, 0, 0)| \leq c \gamma^{-s_+ - s_-} \sum_{j_1, j_2, j_3, \sigma_1, \sigma_2, \sigma_3} \gamma^{\inf\{j_1\}} \gamma^{3 \inf\{s_+, i\}} \gamma^{\inf\{s_-, i\}} \sum_{j_1, j_2, j_3} \gamma^{-s_+ - s_-} \gamma^{-\sum_{i=1}^{3} s_{+, i} - \sum_{i=1}^{3} s_{-, i}}, \tag{33}$$

in which $\inf\{j_i\}$ is the smallest scaling index in $\{j_1, j_2, j_3\}$ that we keep when we perform the integration over $x_0$, $\inf\{s_+, i\}$ and $\inf\{s_-, i\}$ are the smallest sector indices in
\( \{s_{+,1}, \cdots, s_{+,3}\} \) and \( \{s_{-,1}, \cdots, s_{-,3}\} \), respectively. Let \( \{a_1, a_2, a_3\} \) be a set of three real numbers, denote by \( \inf_2\{a_i\} \) the next smallest element among \( \{a_1, \cdots, a_3\} \). Then we have
\[
\inf\{a_i\} = \frac{1}{3} \sum_{i=1}^{3} a_i - \frac{1}{3} \left[ \inf\{a_i\} - \inf\{a_i\} \right] - \frac{1}{3} \left[ \sup\{a_i\} - \inf\{a_i\} \right].
\]  
Define \( \delta\{a_i\} = \left[ \inf_2\{a_i\} - \inf\{a_i\} \right] + \left[ \sup\{a_i\} - \inf\{a_i\} \right] \). It is easily seen that \( \delta\{a_i\} \geq 0 \) and \( \inf\{a_i\} = \frac{1}{3} \sum_{i=1}^{3} a_i - \frac{1}{3} \delta\{a_i\} \). Then Formula (33) can be written as:
\[
|\partial^2_k \Sigma^{Sun}(\pi T, 0, 0)| \leq c^3 \sum_{j_1, j_2, j_3} \gamma^{-\sum_{i=1}^{3} s_{+,i} \frac{1}{\sigma_1} \sum_{i=1}^{3} s_{-,i} \frac{1}{\sigma_2} \sum_{i=1}^{3} \delta(j_i)} \gamma_{\inf\{s_{+,i}\}, j_1, j_2, j_3, \sigma_1, \sigma_2, \sigma_3}.
\]  
Now denote by \( \kappa \) the value of the index \( i \) such that \( s_{+,i} = \inf\{s_{+,i}\} \) and write \( \inf\{s_{+,i}\} = j_\kappa - (j_\kappa - s_{+,i}) \), we obtain
\[
|\partial^2_k \Sigma^{Sun}| \leq c^3 \gamma^2 \sum_{s_{+,1}, s_{+,2}, s_{+,3}} \gamma^{-\sum_{i=1}^{3} s_{+,i} \frac{1}{\sigma_1} \sum_{i=1}^{3} \delta(s_{+,i}) - \sum_{i=1}^{3} s_{-,i} \frac{1}{\sigma_2} \sum_{i=1}^{3} \gamma^{-\frac{1}{2} \sum_{i=1}^{3} s_{-,i} \frac{1}{\sigma_2} \sum_{i=1}^{3} \gamma^{-\frac{1}{2} \sum_{i=1}^{3} \frac{1}{\sigma_2}} \sum_{i=1}^{3} \frac{1}{\sigma_2}} \sum_{i=1}^{3} \frac{1}{\sigma_2}} \sum_{i=1}^{3} \frac{1}{\sigma_2}.\]

First of all, summing over the sector indices \( s_{-,1}, \cdots, s_{-,3} \) can be bounded by a factor
\( K = \frac{1}{1 - \gamma - 177} \). For \( i \neq \kappa \), the decaying factor can be used to sum over \( s_{+,i} \), which also costs a constant \( K \). In the same way we can sum over the scaling indices \( j_i \) for \( i \neq \kappa \), each sum costs a constant \( K \). Summation over \( s_{+,\kappa} \) is bounded by
\[
\sum_{0 \leq s_{+,\kappa} \leq j_\kappa} \gamma^{-(j_\kappa - s_{+,\kappa})} \leq \gamma \gamma - 1.
\]  
Finally, summing over \( j_\kappa \), we obtain
\[
|\partial^2_k \Sigma^{Sun}| \leq K \sum_{j_\kappa} \gamma^{j_\kappa} = K \frac{\gamma^{j_{\kappa_{\text{max}}}}}{\gamma - 1},
\]  
in which \( K \) is the constant equal to the multiplication of all the previous constants generated in the estimation. Using the fact that \( \gamma^{j_{\kappa_{\text{max}}}} \sim \frac{1}{T} \), we prove this lemma. 

2.2 The lower bound of the Sunshine graph

Define
\[
C^H(x) := \int_{\mathcal{D}_\beta} dq e^{ik_0 x_0 + (q_+ + 1)x_+ + (q_- - 1)x_-} \hat{C}^H(k_0, q),
\]  
in which \( \hat{C}^H(k_0, q) := \frac{1}{-2k_0 - 2\pi q_+ \sin \pi q_-} \).
Proposition 2.1. We have the following decomposition for $\partial^2_+ \Sigma^{Sun}(\pi T, 0, 0)$:

$$\partial^2_+ \Sigma^{Sun}(\pi T, 0, 0) = \partial^2_+ \Sigma^{Sun,\text{dom}}(\pi T, 0, 0) + \partial^2_+ \Sigma^{Sun,\text{error}}(\pi T, 0, 0),$$

(40)

in which $\partial^2_+ \Sigma^{Sun,\text{dom}}(\pi T, 0, 0)$ is the dominant contribution to $\partial^2_+ \Sigma^{Sun}(\pi T, 0, 0)$ obtained by replacing $C^I(x)$ in [24] by $C^II(x)$, and $\partial^2_+ \Sigma^{Sun,\text{error}}(\pi T, 0, 0)$ is the error term.

Remark 2.3. In order to prove this proposition, it is enough to prove that the upper bound of the error term is much smaller than the lower bound of the dominant term.

The rest of this part is devoted to the construction of $\Sigma^{Sun,\text{error}}(\pi T, 0, 0)$. Let $N < j_{\text{max}}$ be a positive constant which is sufficiently large. Define the unrestricted summation $\sum^N \{ j_1, \{ s_{i, +}, \{ s_{i, -} \} \}$ by:

$$\sum^N \{ j_1, \{ s_{i, +}, \{ s_{i, -} \} \} = \sum_{j_1, \{ s_{i, +} \}} \inf(j_1, j_{\text{max}} - N) \sum_{s_{i, +} = 0} \sum_{s_{i, +} = 0} j_1 \sum_{s_{i, +} = 0} j_{j_2, j_{\text{max}} - N} \sum_{s_{i, +} = 0} s_{i, +} = 0,$$

(41)

which means that at least one sector index $s_{i, +}$ is smaller than $j_{\text{max}} - N$, hence the corresponding coordinate $q_+$ is not sufficiently close to the edge $q_+ = 0$. Define

$$\partial^2_+ \Sigma^{Sun,N}(\pi T, 0, 0) = \sum^N \{ j_1, \{ s_{i, +}, \{ s_{i, -} \} \} \int d^3 x x^2 C_{\vartheta j}(x) \bar{C}_{\vartheta j}(x) e^{-i \pi T x_0 - i x_+ + i x_-},$$

(42)

then we can rewrite $\partial^2_+ \Sigma^{Sun}(\pi T, 0, 0)$ as

$$\partial^2_+ \Sigma^{Sun}(\pi T, 0, 0) = \partial^2_+ \Sigma^{Sun,N}(\pi T, 0, 0) + \partial^2_+ \Sigma^{Sun,\text{error}}(\pi T, 0, 0),$$

(43)

in which each sector index $s_{i, +}$ in $\partial^2_+ \Sigma^{Sun}$ is greater than $j_{\text{max}} - N$. This statement can be reformulated by introducing another smooth cutoff function $u_N(q_+) = \chi(\gamma^{j_{\text{max}} - N} q_+)$.

Define

$$C^{III}(x) = \int d\beta d^3 \pi q e^{i k_0 x_0 + i(q_+ + 1)x_+ + i(q_- - 1)x_-} \tilde{C}^{III}(k_0, q), \quad \tilde{C}^{III}(k_0, q) = \frac{u_N(q_+)}{-2i k_0 + e(q, 1)},$$

(44)

Then we have

$$\partial^2_+ \Sigma^{Sun,N}(\pi T, 0, 0) = \int d^3 x x^2 C^{III}(x) C^{III}(x) e^{-i \pi T x_0 - i x_+ + i x_-}.$$ 

(45)

Define also $\partial^2_+ \Sigma^{Sun}(\pi T, 0, 0)$ by replacing each propagator $\frac{u_N(q_+)}{-2i k_0 + e(q, 1)}$ in (43) by $\frac{u_N(q_+)}{-2i k_0 + 2q_+ \sin \pi q_+}$. The difference $\partial^2_+ \Sigma^{Sun,N}(\pi T, 0, 0) - \partial^2_+ \Sigma^{Sun}(\pi T, 0, 0)$ is an error term. Notice that

$$u_N(q_+) = 1 + [u(q_+ - 1)] + [u_N(q_+) - u(q_+)] : = 1 + 0 + \tilde{u}_N(q_+),$$

(46)

in which the first term 1 means that the cutoff is removed, for which we obtain the dominant contribution $\partial^2_+ \Sigma^{Sun,\text{dom}}$ (cf. [40]). The second term is vanishing identically, since $q_+ \leq 1$ in $\tilde{D}$, and the last term is also an error term. So we obtain

$$\partial^2_+ \Sigma^{Sun,N}(\pi T, 0, 0) = \partial^2_+ \Sigma^{Sun,\text{dom}}(\pi T, 0, 0) + \partial^2_+ \Sigma^{Sun,N,1}(\pi T, 0, 0),$$

(47)
in which
\[
\partial_+^2 \mathcal{S}^\text{Sun}_{N,1}(\pi T, 0, 0) = \int_{A_0} d^3 x x_+^2 e^{-\pi T x_0} \sum_{i=1}^3 \hat{u}_N(q_{i,+}) e^{i k_{1,0} x_0 + i q_{1,+} x_+ + i q_{1,-} x_- - 2i k_0 - 2\pi q_{1,1} \sin \pi q_{1,-}} \times e^{i k_{2,0} x_0 + i q_{2,+} x_+ + i q_{2,-} x_- - 2i k_0 - 2\pi q_{2,1} \sin \pi q_{2,-}} \times e^{i k_{3,0} x_0 + i q_{3,+} x_+ + i q_{3,-} x_- - 2i k_0 - 2\pi q_{3,1} \sin \pi q_{3,-}} \times \hat{u}_N(q_{i,+}) \hat{u}_N(q_{j,+}) \hat{u}_N(q_{k,+}) e^{i k_{1,0} x_0 + i q_{1,+} x_+ + i q_{1,-} x_- - 2i k_0 - 2\pi q_{1,1} \sin \pi q_{1,-}} \times e^{i k_{2,0} x_0 + i q_{2,+} x_+ + i q_{2,-} x_- - 2i k_0 - 2\pi q_{2,1} \sin \pi q_{2,-}} \times e^{i k_{3,0} x_0 + i q_{3,+} x_+ + i q_{3,-} x_- - 2i k_0 - 2\pi q_{3,1} \sin \pi q_{3,-}} \tag{48}
\]

Thus we have constructed the error terms:
\[
\partial_+^2 \mathcal{S}^\text{Sun, error}_{N,1}(\pi T, 0, 0) = \partial_+^2 \mathcal{S}^\text{Sun,N}_{N,1}(\pi T, 0, 0) + \partial_+^2 \mathcal{S}^\text{Sun}_{N,1}(\pi T, 0, 0)
\]
\[
+ [\partial_+^2 \mathcal{S}^\text{Sun}_{N}(\pi T, 0, 0) - \partial_+^2 \mathcal{S}^\text{Sun}_{N}(\pi T, 0, 0)].
\tag{49}
\]

3 The lower bound of \( \partial_+^2 \mathcal{S}^\text{Sun, dom}_{N}(\pi T, 0, 0) \)

Recall that \( \partial_+^2 \mathcal{S}^\text{Sun, dom}_{N} \) is defined by replacing each \( \dot{C}^I(k_0, q) \) in [24] by \( \dot{C}^{II}(k_0, q) \).

3.1 Integration over \( q_{1,+}, q_{2,+} \) and \( q_{3,+} \) in \( \partial_+^2 \mathcal{S}^\text{Sun, dom}_{N}(\pi T, 0, 0) \)

Let us consider first the integration
\[
\int_{-\infty}^{\infty} dq_{1,+} \frac{e^{i q_{1,+} x_+}}{-2i k_{1,0} - 2\pi q_{1,1} \sin \pi q_{1,-}}, \tag{50}
\]
in which the integrand is a meromorphic function with a pole at \( q_{1,+}^0 = -\frac{i k_{1,0}}{\pi \sin \pi q_{1,-}} \). For \( \sin \pi q_{1,-} \neq 0 \), we can write [50] as:
\[
-\frac{1}{2\pi \sin \pi q_{1,-}} \int_{-\infty}^{\infty} dq_{1,+} \frac{e^{i q_{1,+} x_+}}{q_{1,+} + \frac{i k_{1,0}}{\pi \sin \pi q_{1,-}}}. \tag{51}
\]
If \( x_+ > 0 \), the integration contour is chosen on the the upper half plane (See Figure 4), for which we have:
\[
\int_{-\infty}^{\infty} dq_{1,+} \frac{e^{i q_{1,+} x_+}}{q_{1,+} + \frac{i k_{1,0}}{\pi \sin \pi q_{1,-}}} = \oint dq_{1,+} \frac{e^{i q_{1,+} x_+}}{q_{1,+} + \frac{i k_{1,0}}{\pi \sin \pi q_{1,-}}},
\]
If \( x_+ < 0 \), the integration contour is chosen on the lower half plane, for which we have
Now we consider the integration over $x_0 > 0$. The singularity $q_1^+ = -\frac{i k_0}{\pi \sin \pi q_1}$ is on the upper half plane.

$$\int_{-\infty}^{\infty} dq_1+ \frac{e^{iq_1+x_+}}{q_1+\frac{ik_0}{\pi \sin \pi q_1}} = -\int_{-\infty}^{\infty} dq_1+ \frac{e^{iq_1+x_+}}{q_1+\frac{ik_0}{\pi \sin \pi q_1}}$$

The case of $x_+ = 0$ can be included in either case. So we obtain:

$$\int_{-\infty}^{\infty} dq_1+ \frac{e^{iq_1+x_+}}{-2ik_0 - 2\pi q_1+\sin \pi q_1} = -\frac{i}{\sin \pi q_1} e^{\frac{k_10x_+}{\pi \sin \pi q_1}}$$

In the same way we obtain:

$$\int_{-\infty}^{\infty} dq_i+ \frac{e^{iq_i+x_+}}{2ik_0 - 2\pi q_i+\sin \pi q_i} = -\frac{i}{\sin \pi q_i} e^{\frac{k_i0x_+}{\pi \sin \pi q_i}}$$

for $i = 2, 3$.

Combining these terms we obtain

$$\partial_+^2 \Sigma_{\text{sum, dom}}(\pi T, 0, 0) =$$

$$i \int d^3 x \int dk_{1,0} dq_{1,-} dk_{2,0} dq_{2,-} dk_{3,0} dq_{3,-} x^+ e^{ix_0[k_{1,0}+k_{2,0}+k_{3,0}-\pi T]}$$

$$\times e\left[\chi(x_+ \geq 0)\chi\left(\frac{k_{1,0}}{\pi \sin \pi q_{1,-}} < 0\right)\chi\left(\frac{k_{2,0}}{\pi \sin \pi q_{2,-}} > 0\right)\chi\left(\frac{k_{3,0}}{\pi \sin \pi q_{3,-}} > 0\right)
- \chi(x_+ \leq 0)\chi\left(\frac{k_{1,0}}{\pi \sin \pi q_{1,-}} > 0\right)\chi\left(\frac{k_{2,0}}{\pi \sin \pi q_{2,-}} < 0\right)\chi\left(\frac{k_{3,0}}{\pi \sin \pi q_{3,-}} < 0\right)\right].$$

3.2 Integration over $x_0$ and $k_{3,0}$

Now we consider the integration over $x_0 \in [-\beta, \beta]$. For $T$ fixed, we have $x_0 = a\beta = a\frac{1}{T}$, in which $a \in [-1, 1)$, and $\int_{-\frac{1}{T}}^{\frac{1}{T}} dx_0 = \frac{1}{T} \int_{-1}^{1} da$. So we obtain:

$$\int dx_0 e^{ix_0[k_{1,0}+k_{2,0}+k_{3,0}-\pi T]} = \frac{1}{T} \delta(k_{1,0}+k_{2,0}+k_{3,0} - \pi T).$$

Figure 4: The integral contour for $x_+ > 0$. The singularity $q_1^+ = -\frac{i k_0}{\pi \sin \pi q_1}$ is on the upper half plane.
Then we perform the integration over $k_{3,0}$ against the $\delta$-function. Since $k_{3,0} = (2n + 1)\pi T$, $n \in \mathbb{Z}$, the integration $\int dk_{3,0}$ is indeed the discrete sum $2\pi T \sum_{k_{3,0} \in \pi T + 2\pi T \mathbb{Z}}$, in which the coefficient $T$ cancels the factor $\frac{1}{T}$ in (55). Thus we obtain:

$$
\partial_+^2 \Sigma^{Sun,dom}(\pi T, 0, 0) = i \int dx_+ dx_+ \int \frac{dk_{1,0} dq_{1,0} - dk_{2,0} dq_{2,0} - dq_{3,0}}{\sin \pi q_{1,0} \sin \pi q_{2,0} \sin \pi q_{3,0}} x_+^2
$$

(56)

$$
\times e^{\left(\frac{k_{1,0}}{\pi \sin \pi q_{1,0}} - \frac{k_{2,0}}{\pi \sin \pi q_{2,0}} - \frac{\pi T - k_{1,0} - k_{2,0}}{\pi \sin \pi q_{3,0}}\right)x_+ + i x_+ [q_{1,0} + q_{2,0} + q_{3,0} - q_0]}
$$

$$
\times \left[ \chi(x_+ \geq 0) \chi\left(\frac{k_{1,0}}{\pi \sin \pi q_{1,0}} < 0\right) \chi\left(\frac{k_{2,0}}{\pi \sin \pi q_{2,0}} > 0\right) \chi\left(\frac{\pi T - k_{1,0} - k_{2,0}}{\pi \sin \pi q_{3,0}} > 0\right)
\right] - \left[ \chi(x_+ < 0) \chi\left(\frac{k_{1,0}}{\pi \sin \pi q_{1,0}} > 0\right) \chi\left(\frac{k_{2,0}}{\pi \sin \pi q_{2,0}} < 0\right) \chi\left(\frac{\pi T - k_{1,0} - k_{2,0}}{\pi \sin \pi q_{3,0}} < 0\right) \right].
$$

Consider the following change of variable in the fourth line in the above formula:

$$
\begin{align*}
x'_+ &= -x_+ \\
q'_{1,-} &= -q_{1,-} \\
q'_{2,-} &= -q_{2,-} \\
q'_{3,-} &= -q_{3,-}
\end{align*}
$$

(57)

then these terms are equal to the third line and (56) can be rewritten as:

$$
\partial_+^2 \Sigma^{Sun,dom}(\pi T, 0, 0) = 2i \int dx_+ dx_+ \int \frac{dk_{1,0} dq_{1,0} - dk_{2,0} dq_{2,0} - dq_{3,0}}{\sin \pi q_{1,0} \sin \pi q_{2,0} \sin \pi q_{3,0}} x_+^2
$$

(58)

$$
\times e^{\left(\frac{k_{1,0}}{\pi \sin \pi q_{1,0}} - \frac{k_{2,0}}{\pi \sin \pi q_{2,0}} - \frac{\pi T - k_{1,0} - k_{2,0}}{\pi \sin \pi q_{3,0}}\right)x_+ + i x_+ [q_{1,0} + q_{2,0} + q_{3,0} - q_0]}
$$

$$
\times \chi(x_+ \geq 0) \chi\left(\frac{k_{1,0}}{\pi \sin \pi q_{1,0}} < 0\right) \chi\left(\frac{k_{2,0}}{\pi \sin \pi q_{2,0}} > 0\right) \chi\left(\frac{\pi T - k_{1,0} - k_{2,0}}{\pi \sin \pi q_{3,0}} > 0\right) 
$$

3.3 Integration over $x_-$ and $q_{3,-}$

In this part we consider integration over the coordinates $x_-$, which is indeed the discrete summation $\sum_{x_- \in \pi \mathbb{Z}}$. We obtain:

$$
\sum_{x_- \in \pi \mathbb{Z}} e^{ix_- [q_{1,-} + q_{2,-} + q_{3,-}] - \delta(q_{1,-} + q_{2,-} + q_{3,-} = 0[2])}
$$

(59)

in which $0[2]$ means 0 modulo 2. Integrating over $q_{3,-}$, we obtain:

$$
\partial_+^2 \Sigma^{Sun,dom}(\pi T, 0, 0) = -2i \int dx_+ \int \frac{dk_{1,0} dq_{1,0} - dk_{2,0} dq_{2,0}}{\sin \pi q_{1,0} \sin \pi q_{2,0} \sin \pi (q_{1,-} + q_{2,-})} x_+^2
$$

(60)

$$
\times e^{\left(\frac{k_{1,0}}{\pi \sin \pi q_{1,0}} - \frac{k_{2,0}}{\pi \sin \pi q_{2,0}} + \frac{\pi T - k_{1,0} - k_{2,0}}{\pi \sin \pi (q_{1,-} + q_{2,-})}\right)x_+}
$$

$$
\times \chi(x_+ \geq 0) \chi\left(\frac{k_{1,0}}{\pi \sin \pi q_{1,0}} < 0\right) \chi\left(\frac{k_{2,0}}{\pi \sin \pi q_{2,0}} > 0\right) \chi\left(\frac{\pi T - k_{1,0} - k_{2,0}}{\pi \sin \pi (q_{1,-} + q_{2,-})} < 0\right).
$$

3.4 Integration over $k_{1,0}, k_{2,0}, q_{1,-}, q_{2,-}$

Now we consider the integrations over the variables $k_{1,0}, k_{2,0}, q_{1,-}, q_{2,-}$, which are not independent of each other due to the characteristic functions in (60). Notice that the
integration domain of \( \int dq_1 dq_2 \) is the square \( D_q := [-1,1] \times [-1,1] \) and the integration over \( k_{1,0} \) and \( k_{2,0} \) are indeed the discrete sum over the set \( \pi T + 2 \pi T \mathbb{Z} \). Taking into account the characteristic function \( \chi(x_+ \geq 0) \), we can rewrite (60) as:

\[
\partial^2_+ \Sigma^{Sun, dom}(\pi T, 0, 0) = -2i \sum_{x_+ \in \pi \mathbb{N}} \int dk_{1,0} dk_{2,0} \int_{D_q} \frac{dq_{1,-} dq_{2,-}}{\sin \pi q_{1,-} \sin \pi q_{2,-} \sin (q_{1,-} + q_{2,-})} \\
\times \left[ e^{\frac{k_{1,0} x_+}{\pi \sin \pi q_{1,-}} + e^{-\frac{k_{2,0} x_+}{\pi \sin \pi q_{2,-}}} \right] \\
\times \chi\left( \frac{k_{1,0}}{\pi \sin \pi q_{1,-}} < 0 \right) \chi\left( \frac{k_{2,0}}{\pi \sin \pi q_{2,-}} > 0 \right) \chi\left( \frac{\pi T - k_{1,0} - k_{2,0}}{\pi \sin \pi (q_{1,-} + q_{2,-})} < 0 \right).
\]

(61)

### 3.5 Decomposition of the integration domain

We can divide the integration \( D_q \) into different regions, according to the signs of the functions \( \sin \pi q_{1,-} \), \( \sin \pi q_{2,-} \) and \( \sin (q_{1,-} + q_{2,-}) \) (See Figure 5):

\[
D_q = T^{+++} \cup T^{+-+} \cup T^{++} \cup T^{-+} \cup T^{++} \cup T^{++} \cup \mathcal{T}^{+-+} \cup T^{+-+},
\]

(62)

in which \( T^{+++} \) is the region in which \( \sin \pi q_{1,-} > 0 \), \( \sin \pi q_{2,-} > 0 \) and \( \sin (q_{1,-} + q_{2,-}) > 0 \). The same for the other labeled triangles. It is useful to label these regions by natural numbers: \( T^{(1)} = T^{+++}, T^{(2)} = T^{+-+}, T^{(3)} = T^{++}, T^{(4)} = T^{-+}, T^{(5)} = T^{+-+}, T^{(6)} = T^{++}, T^{(7)} = T^{++} \) and \( T^{(8)} = T^{+-+} \). Let \( A = \partial^2_+ \Sigma^{Sun, dom}(\pi T, 0, 0) \), then we have:

\[
A = \sum_{a=1}^{8} A^{(a)},
\]

(63)

in which

\[
A^{(a)} = -i \sum_{x_+ \in \pi \mathbb{N}} \int dk_{1,0} dk_{2,0} \int_{T^{(a)}} dq_{1,-} dq_{2,-} F(x_+, k_{1,0}, k_{2,0}, q_{1,-}, q_{2,-})
\]

(64)

is the restriction of the integration (61) to the triangle \( T^{(a)} \), \( a = 1, \cdots, 8 \) with integrand \( F(x_+, k_{1,0}, k_{2,0}, q_{1,-}, q_{2,-}) \).
3.6 The vanishing amplitudes

Using the constraints from the characteristic functions, we conclude that:

**Lemma 3.1.**

\[ \mathcal{A}(7) = 0, \quad \mathcal{A}(8) = 0. \]  

*Proof.* We consider first \( \mathcal{A}(7) \), for which the integration domain is \( T^{(++)} \). We have

\[ \sin \pi q_{1,-} > 0, \quad \sin \pi q_{2,-} < 0, \quad \sin \pi (q_{1,-} + q_{2,-}) > 0. \]  

Then the characteristic functions in (61) implies that

\[ k_{1,0} < 0, \quad k_{2,0} < 0, \quad k_{1,0} + k_{2,0} > \pi T. \]  

or

\[ k_{1,0} < 0, \quad k_{2,0} < 0, \quad k_{1,0} + k_{2,0} > -\pi T. \]  

Since \( k_0 \in \pi T + 2\pi T\mathbb{Z} \), the three inequality in (67) or (68) are not compatible. Hence \( \mathcal{A}(7) = 0 \). Repeat this analysis we can prove that \( \mathcal{A}(8) = 0 \).  

3.7 The non-vanishing terms

By Formula (61), the amplitudes \( \mathcal{A}(1), \ldots, \mathcal{A}(6) \) may have different signs and it is important to see if there can be non-trivial cancellations among them. So we consider first the global signs of these terms. We start with the amplitudes \( \mathcal{A}(1) \) and \( \mathcal{A}(2) \). We have:

**Lemma 3.2.**

\[ i[\mathcal{A}(1) + \mathcal{A}(2)] < 0. \]  

*Proof.* Let us consider first \( \mathcal{A}(1) \), which is the restriction of the integral (61) to the domain \( T^{(++)} \), in which we have:

\[ \sin \pi q_{1,-} > 0, \quad \sin \pi q_{2,-} > 0, \quad \sin \pi (q_{1,-} + q_{2,-}) > 0. \]  

The characteristic functions

\[ \chi\left(\frac{k_{1,0}}{\pi \sin \pi q_{1,-}} < 0\right) \chi\left(\frac{k_{2,0}}{\pi \sin \pi q_{2,-}} > 0\right) \chi\left(\frac{\pi T - k_{1,0} - k_{2,0}}{\pi \sin \pi (q_{1,-} + q_{2,-})} < 0\right) \]  

set the following constrains:

\[ k_{1,0} < 0, \quad k_{2,0} > 0, \quad k_{1,0} + k_{2,0} > \pi T. \]  

Since \( k_{1,0} \) and \( k_{2,0} \) are discrete variables, from (72) we can deduce that \( k_{1,0} \in -\pi T - 2\pi TN \), where \( N \) is the set of non-negative integers, and \( k_{2,0} \in \pi T + 2\pi TN \). Define \( s = k_{1,0} + k_{2,0} \), we have \( s \in 2\pi T\mathbb{N}^\ast \), where \( \mathbb{N}^\ast \) is the set of positive integers. Now we consider the summations over \( k_{1,0} \) and \( k_{2,0} \). Since the integration over \( q_{1,-} \) and \( q_{2,-} \)
is absolutely convergent, we can change the order of integration and summation, by
Fubini’s theorem. Let $k_{1,0}$ and $s$ be independent variables, we obtain:

$$
\mathcal{A}(1) = -2i \sum_{x_+ \in \pi N} \sum_{k_{1,0} \in -2\pi TN} \sum_{s \in 2\pi TN^2} x_+^2 e^{-\frac{k_{1,0}x_+}{\pi \sin \pi q_{1,-}} + \frac{k_{1,0}s}{\pi \sin \pi q_{1,-}}} \int_{T^{+++}} dq_{1,-} dq_{2,-} \\
\times e^{-\frac{x_+}{\pi \sin \pi q_{2,-}} + \frac{1}{\pi \sin \pi q_{1,-}} + \frac{1}{\pi \sin \pi (q_{1,-} + q_{2,-})}} e^{\pi s T} \times \chi(k_{1,0} < 0) \chi(k_{2,0} > 0) \chi(k_{1,0} + k_{2,0} > \pi T) \\
= -2i \sum_{x_+ \in \pi N} x_+^2 \int_{T^{+++}} dq_{1,-} dq_{2,-} \\
\times e^{-\frac{x_+}{\pi \sin \pi q_{2,-}} + \frac{1}{\pi \sin \pi q_{1,-}} + \frac{1}{\pi \sin \pi (q_{1,-} + q_{2,-})}} \sum_{n=1}^{\infty} e^{-2(n+1)T} \left[ \frac{1}{\sin \pi q_{1,-}} + \frac{1}{\sin \pi q_{2,-}} \right] x_+ \sum_{p=1}^{\infty} e^{-2pT} \left[ \frac{1}{\sin \pi q_{1,-}} + \frac{1}{\sin \pi (q_{1,-} + q_{2,-})} \right] x_+ \\
= -2i \sum_{x_+ \in \pi N} x_+^2 \int_{T^{+++}} dq_{1,-} dq_{2,-} \\
\times e^{-\frac{x_+}{\pi \sin \pi q_{2,-}} + \frac{x_+}{\pi \sin \pi q_{1,-}} + \frac{x_+}{\pi \sin \pi (q_{1,-} + q_{2,-})}} 1 - e^{-2T} \left[ \frac{1}{\sin \pi q_{1,-}} + \frac{1}{\sin \pi q_{2,-}} \right] x_+ \\
(73)
$$

Now we consider $\mathcal{A}(2)$, for which the integration domain is $T^{---}$, in which we have:

$$
\sin \pi q_{1,-} < 0, \quad \sin \pi q_{2,-} < 0, \quad \sin \pi (q_{1,-} + q_{2,-}) < 0. \\
(74)
$$

The constraints set by the characteristic functions are:

$$
k_{1,0} > 0, \quad k_{2,0} < 0, \quad k_{1,0} + k_{2,0} < \pi T. \\
(75)
$$

Now we perform the following transformation of variables:

$$
\begin{cases}
q_1' &= -q_1, \\
q_2' &= -q_2.
\end{cases} \\
(76)
$$

Then we have

$$
\sin \pi q_{1,-}' > 0, \quad \sin \pi q_{2,-}' > 0, \quad \sin \pi (q_{1,-}' + q_{2,-}') > 0, \\
(77)
$$

which corresponds to the integration in the domain $T^{+++}$. We still call the new variables $q_{1,-}, q_{2,-}$, we obtain:

$$
\mathcal{A}(2) = 2i \sum_{x_+ \in \pi N} \int dk_{1,0} dk_{2,0} \int_{T^{+++}} dq_{1,-} dq_{2,-} \\
\times e^{-\frac{k_{1,0}x_+}{\pi \sin \pi q_{1,-}} + \frac{k_{2,0}x_+}{\pi \sin \pi q_{2,-}} + \frac{k_{1,0}+k_{2,0}}{\pi \sin \pi (q_{1,-} + q_{2,-})}} \sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-}) \\
\times \chi(k_{1,0} > 0) \chi(k_{2,0} < 0) \chi(k_{1,0} + k_{2,0} < \pi T), \\
(78)
$$

$$
(79)
$$
The constrains from the characteristic functions imply that \( k_{1,0} \in \pi T + 2\pi T \mathbb{N}, k_{2,0} \in -\pi T - 2\pi T \mathbb{N} \). Define \( s = k_{1,0} + k_{2,0} \), then we have \( s \in -2\pi T \mathbb{N} \). Consider \( k_{1,0} \) and \( s \) as independent variables and perform the integration over \( k_{1,0} \) and \( s \), we obtain:

\[
\mathcal{A}^{(2)} = 2i \sum_{k_{1,0} \in \pi T + 2\pi T \mathbb{N}} \sum_{s \in -2\pi T \mathbb{N}} x^2_+ \int_{T^{+++}} \frac{dq_{1,-} dq_{2,-}}{\sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-})} e^{-\left( \frac{k_{1,0} x_{1,-} + k_{1,0} x_{2,-}}{\pi \sin \pi q_{1,-} + \pi \sin \pi q_{2,-}} \right)} e^{\left( \frac{T}{\sin \pi q_{1,-} + \sin \pi q_{2,-}} \right)} x_+ 1 - e^{-2T \left( \frac{1}{\sin \pi q_{1,-} + \sin \pi q_{2,-}} \right)} x_+ \]

So we have

\[
\mathcal{A}^{(1)} + \mathcal{A}^{(2)} = 2i \sum_{k_{1,0} \in \pi T + 2\pi T \mathbb{N}} x^2_+ \int_{T^{+++}} \frac{dq_{1,-} dq_{2,-}}{\sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-})} e^{-\left( \frac{T}{\sin \pi q_{1,-} + \sin \pi q_{2,-}} \right)} x_+ 1 - e^{-2T \left( \frac{1}{\sin \pi q_{1,-} + \sin \pi q_{2,-}} \right)} x_+ \]

Let \( i \mathcal{A}_1 = \mathcal{A}^{(1)} + \mathcal{A}^{(2)} \), then we conclude that \( \mathcal{A}_1 \) is positive, since the integrand is positive definite and the measure is positive. So we have

\[
i [\mathcal{A}^{(1)} + \mathcal{A}^{(2)}] = -\mathcal{A}_1 \leq 0. \tag{81}\]

Hence we proved this lemma.

3.8 The amplitudes \( \mathcal{A}^{(3)} \) and \( \mathcal{A}^{(4)} \).

In this part we study the amplitudes \( \mathcal{A}^{(3)} \) and \( \mathcal{A}^{(4)} \). We have

Lemma 3.3.

\[
i [\mathcal{A}^{(3)} + \mathcal{A}^{(4)}] \leq 0. \tag{82}\]

Proof. We use the formalism \([61]\) for the amplitudes \( \mathcal{A}^{(3)} \) and \( \mathcal{A}^{(4)} \). In the domain \( T^{(3)} = T^{+++} \), we have:

\[
\sin \pi q_{1,-} > 0, \quad \sin \pi q_{2,-} > 0, \quad \sin \pi (q_{1,-} + q_{2,-}) < 0. \tag{83}\]

By \([61]\), we have \( k_{1,0} < 0, k_{2,0} > 0, \pi T - k_{1,0} - k_{2,0} > 0 \). Now we consider the integrations over \( k_{1,0} \) and \( k_{2,0} \), which are discrete sums, we have \( k_{1,0} \in -\pi T - 2\mathbb{N} \pi T \), \( \mathbb{N} \) is the set of non-negative integers, and \( k_{2,0} \in \pi T + 2\mathbb{N} \pi T \). Let \( s = \pi T - k_{1,0} - k_{2,0} \). The condition
\( \pi T - k_{1,0} - k_{2,0} > 0 \) implies that \( s \in \pi T + 2N\pi T \). Let \( k_{1,0} \) and \( s \) be independent variables, we obtain:

\[
A^{(3)} = -2i \sum_{x_+ \in \pi N} \int dk_{1,0} dk_{2,0} \int_{\mathcal{T}(+-)} \frac{dq_{1,-} dq_{2,-}}{\sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-})} \\
\times x_+^2 e^{k_{1,0} x_+ \left[ \frac{1}{\sin \pi q_{1,-}} + \frac{1}{\sin \pi q_{2,-}} \right] + \frac{s x_+}{\sin \pi q_{1,-} + \frac{1}{\sin \pi q_{2,-}}} e^{-\frac{\pi T}{\sin \pi q_{2,-}}} \\
\times \chi(k_{1,0} < 0) \chi(k_{2,0} > 0) \chi(k_{1,0} + k_{2,0} > \pi T) \\
= -2i \sum_{x_+ \in \pi N} x_+^2 \int_{\mathcal{T}(+-)} \frac{e^{-\frac{\pi T}{\sin \pi q_{2,-}}} dq_{1,-} dq_{2,-}}{\sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-})} \\
\sum_{s \in \pi T + 2N\pi T} e^{k_{1,0} \left[ \frac{1}{\sin \pi q_{1,-}} + \frac{1}{\sin \pi q_{2,-}} \right] x_+} + \sum_{s \in \pi T + 2N\pi T} e^{s \left[ \frac{1}{\sin \pi q_{1,-} + \frac{1}{\sin \pi q_{2,-}}} \right] x_+}.
\]

Remark that although the sign in the exponential of the last term is not uniform in \( p \), this term is positive definite. Since what we want to determine is the sign of \( i[A^{(3)} + A^{(4)}] \), and by Lemma 2.2 we know that the summation in (84) is not divergent, this subtlety doesn’t cause problem.

Now we consider \( A^{(4)} \), which is equal to the integration (61) constrained in the domain \( \mathcal{T}(-+) \). So we have

\[
\sin \pi q_{1,-} < 0, \quad \sin \pi q_{2,-} < 0, \quad \sin \pi (q_{1,-} + q_{2,-}) > 0,
\]

and

\[
k_{1,0} > 0, \quad k_{2,0} < 0, \quad k_{1,0} + k_{2,0} > \pi T.
\]

Now we perform the following change of variables:

\[
\begin{cases}
q_{1,-}' = -q_{2,-}, \\
q_{2,-}' = -q_{1,-}, \\
k_{1,0}' = k_{2,0}, \\
k_{2,0}' = k_{1,0}.
\end{cases}
\]

Then we have

\[
\sin \pi q_{1,-}' > 0, \quad \sin \pi q_{2,-}' > 0, \quad \sin \pi (q_{1,-} + q_{2,-}) < 0,
\]

hence the integration domain for \( q_{1,-}' \) and \( q_{2,-}' \) is \( \mathcal{T}(+-) \). Let us denote the variables \( q_{1,-}' \), \( q_{2,-}' \) by \( q_{1,-}, q_{2,-} \) and the variables \( k_{1,0}', k_{2,0}' \) by \( k_{1,0}, k_{2,0} \). Define the new variable \( s = k_{1,0} + k_{2,0} - \pi T \), which takes values in the set \( \pi T + 2N\pi T \), and consider \( s \) and \( k_{1,0} \) as independent variables, we obtain:

\[
A^{(4)} = 2i \sum_{x_+ \in \pi N} x_+^2 \int_{\mathcal{T}(+-)} \frac{e^{-\frac{\pi T}{\sin \pi q_{2,-}}} dq_{1,-} dq_{2,-}}{\sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-})} \\
\sum_{s \in \pi T + 2N\pi T} e^{k_{1,0} \left[ \frac{1}{\sin \pi q_{1,-}} + \frac{1}{\sin \pi q_{2,-}} \right] x_+} + \sum_{s \in \pi T + 2N\pi T} e^{s \left[ \frac{1}{\sin \pi q_{1,-} + \frac{1}{\sin \pi q_{2,-}}} \right] x_+}.
\]

17
Finally we have

\[ A^{(3)} + A^{(4)} = 2i \sum_{x \in \pi N} x^2 \int_{T^{++-}} e^{-\frac{\pi}{\sin \pi q_1} dq_{1^-} dq_{2^-}} \sin \pi q_1^- \sin \pi q_2^- \sin \pi (q_{1^-} + q_{2^-}) \]  

(90)

\[ \sum_{k_1,0 \in -\pi T - 2\pi T} e^{k_1,0 \left( \frac{1}{\sin \pi q_1^-} + \frac{1}{\sin \pi q_2^-} \right) x^+} \sum_{s \in \pi T + 2\pi T} e^{\frac{\pi}{\sin \pi q_1 - \pi q_2} T^+} \left[ e^{-\frac{\pi}{\sin \pi q_2} x^+} - e^{\frac{\pi}{\sin \pi q_2} T^+} \right]. \]

Let \( A^{(3)} + A^{(4)} = iA_2. \) Since \( \sin \pi (q_1^- + q_{2^-}) < 0 \) in the domain \( T^{++-} \), we conclude that \( A_2 \geq 0 \) and

\[ i[A^{(3)} + A^{(4)}] = -A_2 \leq 0. \]

(91)

Hence we proved this lemma.

3.9 The amplitudes \( A^{(5)} \) and \( A^{(6)} \)

In this part we consider \( A^{(5)} \) and \( A^{(6)} \). Like in the previous two cases, there is a difference of the global sign between \( A^{(5)} \) and \( A^{(6)} \), and we have to consider the sum of the two terms. We have:

Lemma 3.4.

\[ i[A^{(5)} + A^{(6)}] \leq 0. \]

(92)

Proof. Recall that

\[ A^{(5)} = -2i \sum_{x \in \pi N} \int dk_{1,0} dk_{2,0} \int_{T^{++-}} dq_{1^-} dq_{2^-} \sin \pi q_1^- \sin \pi q_2^- \sin \pi (q_{1^-} + q_{2^-}) \]

\[ \times x^2 e^{\frac{k_{1,0}^2}{\sin \pi q_1^-} - \frac{k_{2,0}^2}{\sin \pi q_2^-} + \frac{\pi}{\sin \pi q_1^- \pi q_2^-} T^- k_{1,0} - k_{2,0}} \]

\[ \times \chi(k_{1,0} < 0) \chi(k_{2,0} > 0) \chi(\frac{\pi}{\sin \pi q_1^-} > 0) \chi(\frac{\pi}{\sin \pi q_2^-} < 0). \]

(93)

In \( T^{++-} \), we have:

\[ \sin \pi q_1^- > 0, \quad \sin \pi q_2^- < 0, \quad \sin \pi (q_{1^-} + q_{2^-}) < 0. \]

(94)

Using the characteristic functions we have:

\[ k_{1,0} < 0, \quad k_{2,0} < 0, \quad \pi T - k_{1,0} - k_{2,0} > 0, \]

(95)

which implies that \( k_{1,0} \in -\pi T - 2\pi T, k_{2,0} \in -\pi T - 2\pi T \), where \( \mathbb{N} \) is the set of non-negative functions. Define \( s = \pi T - k_{1,0} - k_{2,0} \), then we have: \( s \in 3\pi T + 2\pi T. \)

18
Let $s$ and $k_{1,0}$ be the independent variables in the integral, we obtain:

\[
A^{(5)} = -2i \sum_{x_+ \in \pi \mathbb{N}} \int dk_{1,0} dk_{2,0} \int_{T^{(-+)}} \frac{dq_{1,-} dq_{2,-}}{\sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-})} \\
\times x_+^2 e^{\frac{1}{\pi \sin \pi q_{1,-}^{+}} + \frac{1}{\pi \sin \pi q_{2,-}^{+}} + s x_+ + \frac{1}{\pi \sin \pi q_{3,-}^{+}}} e^{-\frac{T}{\sin \pi q_{2,-}}} \\
\times \chi(k_{1,0} < 0) \chi(k_{2,0} < 0) \chi(k_{1,0} + k_{2,0} < \pi T) \\
= -2i \sum_{x_+ \in \pi \mathbb{N}} x_+^2 \int_{T^{(-+)}} \frac{dq_{1,-} dq_{2,-}}{\sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-})} \\
\sum_{k_{1,0} \in -\pi T - 2\pi \mathbb{N} T} e^{k_{1,0} \left( \frac{1}{\sin \pi q_{1,-}} + \frac{1}{\sin \pi q_{2,-}} \right) x_+} \\
\sum_{s \in \pi T + 2\pi \mathbb{N} T} e^{s \left( \frac{1}{\sin \pi (q_{1,-} + q_{2,-})} \right) x_+}.
\]

Now we consider $A^{(6)}$. Recall that

\[
A^{(6)} = -2i \sum_{x_+ \in \pi \mathbb{N}} \int dk_{1,0} dk_{2,0} \int_{T^{++}} \frac{dq_{1,-} dq_{2,-}}{\sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-})} \\
\times x_+^2 e^{\frac{k_{1,0}}{\sin \pi q_{1,-}} + \frac{k_{2,0}}{\sin \pi q_{2,-}} + \frac{s}{\pi \sin \pi (q_{1,-} + q_{2,-})} x_+} \\
\times \chi(\frac{k_{1,0}}{\pi \sin \pi q_{1,-}} < 0) \chi(\frac{k_{2,0}}{\pi \sin \pi q_{2,-}} > 0) \chi(\frac{\pi T - k_{1,0} - k_{2,0}}{\pi \sin \pi (q_{1,-} + q_{2,-})} < 0).
\]

we have:

\[
\sin \pi q_{1,-} < 0, \quad \sin \pi q_{2,-} > 0, \quad \sin \pi (q_{1,-} + q_{2,-}) > 0,
\]

and the constraints are:

\[
k_{1,0} < 0, \quad k_{2,0} > 0, \quad \pi T - k_{1,0} - k_{2,0} < 0.
\]

Like in the previous subsection, we perform the following change of variables:

\[
\begin{align*}
q_{1,-}' &= -q_{2,-}, \\
q_{2,-}' &= -q_{1,-}, \\
k_{1,0}' &= k_{2,0}, \\
k_{2,0}' &= k_{1,0}.
\end{align*}
\]

Then we have

\[
\sin \pi q_{1,-}' > 0, \quad \sin \pi q_{2,-}' < 0, \quad \sin \pi (q_{1,-} + q_{2,-}) < 0,
\]

hence the integration domain for $q_{1,-}'$ and $q_{2,-}'$ is $T^{(++)}$. Again, we denote the variables $q_{1,-}', q_{2,-}'$ by $q_{1,-}, q_{2,-}$ and the variables $k_{1,0}', k_{2,0}'$ by $k_{1,0}, k_{2,0}$. Let $s = \pi T - k_{1,0} - k_{2,0}$, which takes values in the set $\pi T + 2\pi \mathbb{N} T$, and consider $s$ and $k_{1,0}$ as independent variables, we have:

\[
A^{(6)} = 2i \sum_{x_+ \in \pi \mathbb{N}} x_+^2 \int_{T^{(++)}} \frac{dq_{1,-} dq_{2,-}}{\sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-})} \\
\sum_{k_{1,0} \in -\pi T - 2\pi \mathbb{N} T} e^{k_{1,0} \left( \frac{1}{\sin \pi q_{1,-}} + \frac{1}{\sin \pi q_{2,-}} \right) x_+} \\
\sum_{s \in \pi T + 2\pi \mathbb{N} T} e^{s \left( \frac{1}{\sin \pi (q_{1,-} + q_{2,-})} \right) x_+}.
\]
We have (cf. Formula (81)):

\[ A^{(5)} + A^{(6)} = 2i(2\pi T)^2 \sum_{x_+ \in \pi N} x_+^2 \int_{T(-+)} \frac{e^{-\frac{T}{\sin \pi q_1_- \sin \pi q_2_- \sin \pi (q_1_- + q_2_-)}}}{dq_1_- dq_2_-} \]

\[ \sum_{k_{1,0} \in -\pi T - 2\pi N} e^{k_{1,0} \left[ \sum_{s \in \pi T + 2\pi N} e^{\frac{s}{\sin \pi q_{2}^- + \sin \pi (q_{1}^- + q_{2}^-)}} \right] x_+} \]

\[ = 2i \sum_{x_+ \in \pi N} x_+^2 \int_{T(-+)} \frac{e^{-\frac{T}{\sin \pi q_1_- \sin \pi q_2_- \sin \pi (q_1_- + q_2_-)}}}{dq_1_- dq_2_-} \]

3.10 The lower bound of the non vanishing terms

There exists a positive constant \( K \), which may depends on the model but is independent of the temperature \( T \), such that:

\[ |A^{(1)} + A^{(2)} + \cdots + A^{(6)}| = |A_1 + A_2 + A_3| \geq \frac{K}{T}. \] (105)

Since the amplitudes \( A_1, \cdots, A_3 \) have the same signs, we first prove that:

**Lemma 3.5.** There exists a positive constant \( K_1 \leq K \) such that

\[ |A_1| \geq \frac{K_1}{T}. \] (106)

**Proof.** We have (cf. Formula (81)):

\[ |A_1| = 2(2\pi T)^2 \sum_{x_+ \in \pi N} x_+^2 \int_{T(++)} \frac{dq_1_- dq_2_-}{\sin \pi q_1_- \sin \pi q_2_- \sin \pi (q_1_- + q_2_-)} \]

\[ e^{-\left( \frac{T}{\sin \pi q_1_-} + \frac{T}{\sin \pi q_2_-} + \frac{T}{\sin \pi (q_1_- + q_2_-)} \right)x_+} \]

\[ 1 - e^{-2T \left( \frac{1}{\sin \pi q_{1}^-} + \frac{1}{\sin \pi q_{2}^-} \right)x_+} \] (107)

In conclusion, we proved in this subsection that the three combined terms \([A^{(1)} + A^{(2)}], [A^{(3)} + A^{(4)}]\) and \([A^{(5)} + A^{(6)}]\) have the same signs, hence there is no further cancellation among them.
Lemma 3.7. There exists some positive constants $T$ such that we obtain:
\[
1 - e^{-2T \left( \frac{1}{\sin \pi q_1} + \frac{1}{\sin \pi q_2} \right) x^+} \leq 1, \quad 1 - e^{-2T \left( \frac{1}{\sin \pi q_2} + \frac{1}{\sin \pi (q_1 + q_2)} \right) x^+} \leq 1.
\] (108)

we obtain:
\[
|A_1| \geq 2(2\pi T)^2 \sum_{x_+ \in \mathbb{N}} x_+^2 \int_{T(++)} dq_1 dq_2 \frac{dq_1 dq_2}{\sin \pi q_1 \sin \pi q_2 \sin \pi (q_1 + q_2)} \quad (109)
\]
\[
x e^{-\left( \frac{T}{\sin \pi q_1} + \frac{T}{\sin \pi q_2} + \frac{T}{\sin \pi (q_1 + q_2)} \right) x^+} \left[ 1 - e^{-2T x^+} \right].
\]

For any fixed $\varepsilon > 0$ (say, $\varepsilon = \frac{1}{100}$), define $T_\varepsilon(++) \subset T(++)$ as the measurable subset of $T(++)$ such that $\forall(q_1, q_2) \in T_\varepsilon(++)$, we have:
\[
\sin \pi q_1 \geq \varepsilon, \quad \sin \pi q_2 \geq \varepsilon \quad \text{and} \quad \sin (q_1 + q_2) \geq \varepsilon.
\] (110)

**Lemma 3.8.** Consider the integration in (109) constrained in the domain $T_\varepsilon(++)$. Then $\forall(q_1, q_2) \in T_\varepsilon(++)$, we can exchange the summation and the integration in (109).

**Proof.** Since the domain $T_\varepsilon(++)$ has a finite, positive measure, by Lebesgue domination integration theorem, we conclude that the integral and the summation on the r.h.s. of (109) is absolute convergent. By Fubini’s theorem, we can change the order of the multiple integrations, hence exchange summation $\sum_{x_+} \text{ and the integration } \int dq_1 dq_2$.

Then we can write (109) as:
\[
|A_1| \geq 2(2\pi T)^2 \int_{T_\varepsilon(++)} dq_1 dq_2 \sum_{x_+ \in \mathbb{N}} x_+^2 e^{-\frac{2T x^+}{\sin \pi q_2}} \left[ 1 - e^{-2T x^+} \right].
\] (113)
Let $K_3$ be the area of $T_{e^{(+++)}}$, for which we have $0 < K_3 < \frac{1}{2}$, and perform the summation over $x_+$, we have

$$|A_1| \geq 8\pi^4 T^2 K_3 \sum_{n=0}^{\infty} n^2 e^{-\frac{3\pi T + n}{e}} \left[ 1 - e^{-2\pi T n} \right].$$  (114)

Using the formula:

$$\sum_{n=0}^{\infty} n^2 e^{-an} = \frac{e^{-a} + e^{-2a}}{(1 - e^{-a})^3}, \quad \text{for } a > 0,$$  (115)

we obtain:

$$|A_1| \geq \frac{8\pi^4 K_3 T^2}{T} \left[ \frac{e^{-\frac{3\pi T}{e}} + e^{-\frac{4\pi T}{e}}}{(1 - e^{-\frac{3\pi T}{e}})^3} - \frac{e^{-\left(\frac{3\pi T + 2\pi T}{e}\right)} + e^{-\left(\frac{6\pi T + 4\pi T}{e}\right)}}{(1 - e^{-\left(\frac{3\pi T + 2\pi T}{e}\right)})^3} \right] \geq \frac{8\pi T}{T} K_3 \left[ \left(\frac{2}{(\frac{2}{e})^3} - \frac{2}{\left(\frac{2}{(\frac{2}{e})^3} + 2\right)} \right) + O(T) \right].$$  (116)

Choosing $K_1 = 8\pi K_3 \left[ \left(\frac{2}{(\frac{2}{e})^3} - \frac{2}{\left(\frac{2}{(\frac{2}{e})^3} + 2\right)} \right) \right]$, we conclude that

$$|A_1| \geq \frac{K_1}{T},$$  (117)

hence we prove this lemma.

\textbf{Proof of lemma 3.5.} Since $A_1$, $A_2$ and $A_3$ have the same sign, we have

$$|A_1 + A_2 + A_3| \geq |A_1| \geq \frac{K_1}{T}.$$  (118)

Thus we proved this lemma by choosing $K = K_1$.

\textbf{4 Estimation of the error terms}

In order to prove the upper bound for the error term $\partial^2_+ \Sigma_{\text{Sun, error}}(\pi T, 0, 0)$ (cf. Formula (49)), it is enough to prove the upper bound for each term in the sum. First of all, we have:

\textbf{Lemma 4.1.}

$$|\partial^2_+ \Sigma_{\text{Sun}, N}(\pi T, 0, 0)| \leq \frac{K_1}{\gamma N T}, \quad |\partial^2_+ \Sigma_{\text{Sun}, 1}(\pi T, 0, 0)| \leq \frac{K_1}{\gamma N T},$$  (119)

where $K_1$ is the constant in Lemma 2.2.

\textbf{Proof.} Due to the restricted sum in $\partial^2_+ \Sigma_{\text{Sun}, N}(\pi T, 0, 0)$ and the presence of the cutoff function $u_N(q_+) - u(q_+)$ in $\partial^2_+ \Sigma_{\text{Sun}}(\pi T, 0, 0)$, the coordinate $q_+$ in these terms are away from the axis $q_+ = 0$ and the propagator $C^{II}$ has the same scaling property as $C^{I}$. 

22
Following the proof of Lemma 2.2, we can obtain the similar bounds as in Formula (121) and obtain:
\[
\left| \partial_+^2 \sum_{N}^{\text{Sun}} (\pi T, 0, 0) \right| \leq K' \sum_{j_n} \gamma^{j_n} \sum_{s_{+n}} \gamma^{-(j_n-s_{+n})}, \tag{120}
\]
\[
\left| \partial_+^2 \sum_{N,1}^{\text{Sun}} (\pi T, 0, 0) \right| \leq K' \sum_{j_n} \gamma^{j_n} \sum_{s_{+n}} \gamma^{-(j_n-s_{+n})}, \tag{121}
\]
for some positive constant $K'$. Using the decomposition $\sum_{j_n=0}^{j_{\max}} = \sum_{j_n=0}^{j_{\max}-N} + \sum_{j_n=j_{\max}-N}^{j_{\max}}$, the l.h.s. of (120) can be bounded by
\[
K'' \sum_{j_n=0}^{j_{\max}-N} \gamma^{j_n} \lesssim 2K'' \gamma^{j_{\max}} \frac{\gamma^{j_{\max}}}{\gamma^{N/2}},
\]
for some positive constant $K'' > K'$. The l.h.s. of (121) can be bounded by
\[
K' \sum_{j_n=j_{\max}-N}^{j_{\max}} \gamma^{j_n} \sum_{0 \leq s_{+n} \leq j_{\max}-N} \gamma^{-(j_n-s_{+n})} \lesssim \tilde{K}'' \gamma^{-N} \gamma^{j_{\max}} \sim \frac{\tilde{K}''}{\gamma^{N/2}},
\]
for some positive constants $K''$ and $\tilde{K}''$. Choosing $K_1 = \max\{2K'', \tilde{K}''\}$, we conclude this lemma.

**Lemma 4.2.** There exists a constant $K_2 > 0$ such that
\[
\left| \partial_+^2 \sum_{N}^{\text{Sun}} (\pi T, 0, 0) - \partial_+^2 \sum_{N}^{\text{Sun}} (\pi T, 0, 0) \right| \leq K_2.
\]

**Proof.** We have
\[
\begin{align*}
\partial_+^2 \sum_{N}^{\text{Sun}} (\pi T, 0, 0) - \partial_+^2 \sum_{N}^{\text{Sun}} (\pi T, 0, 0) &= \sum_{\left\{ \sigma, \bar{\sigma} \in \mathbb{N}, i=1,\ldots,3 \right\}} d^3 x \int \frac{d^3 p}{(2\pi)^3} e^{-i p_{+0} + \imath q_{+1} x_{+1} + \imath q_{+2} x_{+2} + \imath q_{+3} x_{+3}} \times \left( C_{\sigma_1}(x) \bar{C}_{\sigma_2}(x) \bar{C}_{\sigma_3}(x) - C_{\bar{\sigma}_1}(x) \bar{C}_{\bar{\sigma}_2}(x) \bar{C}_{\bar{\sigma}_3}(x) \right),
\end{align*}
\]
in which
\[
C_{\sigma_1}(x) = \int_{D_\beta} dq_1 e^{i k_{10} x_{0} + i (q_{+1} + 1) x_{+1} + i (q_{+2} - 1) x_{+2}} - i k_0 - 2\pi q_{+1} \sin \pi q_{+2} u_{\sigma_1}(q_1). \tag{126}
\]
The square brackets of (125) can be written as
\[
\begin{align*}
C_{\sigma_1}(x) \bar{C}_{\sigma_2}(x) \bar{C}_{\sigma_3}(x) - C_{\bar{\sigma}_1}(x) \bar{C}_{\bar{\sigma}_2}(x) \bar{C}_{\bar{\sigma}_3}(x) &= |C_{\sigma_1}(x) - C_{\bar{\sigma}_1}(x)| C_{\sigma_2}(x) \bar{C}_{\sigma_3}(x) + C_{\bar{\sigma}_1}(x) \bar{C}_{\sigma_2}(x) \bar{C}_{\sigma_3}(x) + C_{\sigma_1}(x) \bar{C}_{\bar{\sigma}_2}(x) \bar{C}_{\bar{\sigma}_3}(x) - C_{\bar{\sigma}_1}(x) \bar{C}_{\bar{\sigma}_2}(x) \bar{C}_{\bar{\sigma}_3}(x).
\end{align*}
\]
Consider first the difference:
\[
C_{\sigma_1}(x) - C_{\bar{\sigma}_1}(x) = \int_{D_\beta} dq_1 e^{i k_{10} x_{0} + i (q_{+1} + 1) x_{+1} + i (q_{+2} - 1) x_{+2}} - i k_0 - 2\pi q_{+1} \sin \pi q_{+2} u_{\sigma_1}(q_1) \left( \hat{C}(k_{10}, q_{+1}, \pm) - \hat{C}_{\bar{\sigma}_1}(k_{10}, q_{+1}, \pm) \right). \tag{128}
\]
By inserting and subtracting the term

\[
\frac{1}{-2ik_0 - 4\pi q_1,+ \sin \frac{n}{2} q_1,- \cos \frac{n}{2} (q_1,+ + q_1,-)},
\]

we obtain

\[
\hat{C}(k_0, q_1, \pm) - \hat{C}'(k_0, q_1, \pm) = \hat{C}_a(q) + \hat{C}_b(q),
\]

in which

\[
\hat{C}_a(q) = \frac{\left[ \sin \frac{n}{2} q_1,+ - \frac{n}{2} q_1,+ \right]}{-2ik_0 - 8\pi \sin \frac{n}{2} q_1,+ \sin \frac{n}{2} q_1,- \cos \frac{n}{2} (q_1,+ + q_1,-)}
\times \sin \frac{n}{2} q_1,- \cos \frac{n}{2} (q_1,+ + q_1,-)
\frac{\left[ -2ik_0 - 4\pi q_1,+ \sin \frac{n}{2} q_1,- \cos \frac{n}{2} (q_1,+ + q_1,-) \right]}{\left[ -2ik_0 - 4\pi q_1,+ \sin \frac{n}{2} q_1,- \cos \frac{n}{2} (q_1,+ + q_1,-) \right]}.
\]

and

\[
\hat{C}_b(q) = \frac{4\pi q_1,+ \sin \frac{n}{2} q_1,- \left[ \cos \frac{n}{2} (q_1,+ + q_1,-) - \cos \frac{n}{2} q_1,- \right]}{-2ik_0 - 4\pi q_1,+ \sin \frac{n}{2} q_1,- \cos \frac{n}{2} (q_1,+ + q_1,-)} \left[ -2ik_0 - 2\pi q_1,+ \sin \pi q_1,- \right].
\]

**Lemma 4.3.** There exist some positive constant $O_1$ and $O_2$ such that

\[
|\sin \frac{n}{2} q_1,+ - \frac{n}{2} q_1,+| \leq O_1 q_1,+,
\]

\[
||\cos \frac{n}{2} (q_1,+ + q_1,-) - \cos \frac{n}{2} q_1,-|| \leq O_2 q_1,+ |\sin \frac{n}{2} q_1,+ - \frac{n}{2} q_1,+|.
\]

**Proof.** The first inequality can be proved using the well known fact: For $x < 1$, there exists some positive constant $O_1$ such that $|\sin x - x| \leq O_1 x^2$. The second inequality can be proved by expanding $\cos \frac{n}{2} (q_1,+ + q_1,-)$ and using the fact that $|\cos \frac{n}{2} q_1,+ - 1| \leq O_2 q_1,+$, for some positive constant $O_2$. 

Now we perform the integration over $(k_0, q)$, we obtain

\[
|C_{\gamma}(x) - C_{\gamma}'(x)| \leq |O_1' \gamma^{-s_1,+ - s_1,-} + O_2' \gamma^{-3s_1,+ - s_1,-} + O_3' \gamma^{-2s_1,+ - 2s_1,-}| e^{-c_{\gamma}'(x)}
\]

\[
\leq 2O_1' \gamma^{-2s_1,+ - s_1,-} e^{-c_{\gamma}'(x)},
\]

for some positive constants $O_1', \cdots, O_3'$. Repeating the same analysis we can obtain:

\[
|\hat{C}_{\gamma}(x) - \hat{C}'_{\gamma}(x)| \leq 2O_1' \gamma^{-s_1,+ - s_1,-} e^{-c_{\gamma}'(x)}, \text{ for } i = 2, 3.
\]

So we gained the convergent factor $2O_1' \gamma^{-s_1,+}$ from each difference. Since $(125)$ is linear in the difference, we gain the convergent factor $\gamma^{-s_1,+}$ in $(125)$ w.r.t. $|\partial^2_t \Sigma_{\text{sun}}^N(\pi T, 0, 0)|$. Since $\gamma^{-s_1,+} \leq \gamma^{-j_{\text{max}} N} \sim T^{-N}$, $|\partial^2_t \Sigma_{\text{sun}}^N(\pi T, 0, 0)| \leq \frac{K_1}{T^{-N}}$, we obtain:

\[
|\partial^2_t \Sigma_{\text{sun}}^N(\pi T, 0, 0) - \partial^2_t \Sigma_{\text{sun}}^N(\pi T, 0, 0)| \leq 2O_1' \frac{K_1}{T^{-N}} \cdot T^{-N} \gamma^{-N} \leq K_2,
\]

for some positive constant $K_2$. 

24
Proof of Proposition 2.1. By Lemma 4.1, Lemma 4.2 and Lemma 3.5 we conclude that the upper bounds for the error terms is bounded by $\frac{K_1}{N^2} + K_2$, for some positive constant $K_1$ and $K_2$ that are independent of $T$, which is much smaller than the lower bound of the dominant term, which is $\frac{K}{N}$, for $N$ large enough and $T$ small enough. Since the cutoff function $N$ is independent of $T$, this concludes this proposition.

Since the dominant contribution to the sunshine amplitude is also the dominant contribution to the self-energy, this concludes also Theorem 1.1.

5 Appendix: Another way of estimating $A^{(1)}$ and $A^{(2)}$

In this appendix we provide another way of estimating the amplitudes $A^{(1)}$ and $A^{(2)}$. Let us go back to Formula (56) and perform the following change of variable:

$$
\begin{align*}
\begin{cases}
x_+ &= -x_+ \\
k_1' &= -k_1 \\
k_2' &= -k_2
\end{cases}
\end{align*}
$$

(137)

then (56) can be further simplified as:

$$
\partial^2_+ \Sigma^{\text{Sun,dom}}(\pi T, 0, 0) = i \int dx_+ dx_- \int \frac{dk_{1,0} dq_{1,-} dq_{2,-} dq_{3,-}}{\sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi q_{3,-}}
$$

(138)

$$
\times x_+^2 e^{-k_{1,0} \frac{x_+}{\sin \pi q_{1,-}} - k_{2,0} \frac{x_+}{\sin \pi q_{2,-}} + k_{1,0} + k_{2,0} \frac{x_+}{\sin \pi q_{3,-}}} e^{i \pi T} [\chi(x_+ < 0) \chi(\frac{k_{1,0}}{\sin \pi q_{1,-}} < 0) \chi(\frac{k_{2,0}}{\sin \pi q_{2,-}} > 0) \chi(\frac{\pi T - k_{1,0} - k_{2,0}}{\sin \pi q_{3,-}} > 0) - e^{-2ix_+} \chi(\frac{\pi T + k_{1,0} + k_{2,0}}{\sin \pi q_{3,-}} < 0)]
$$

Performing the integrations over $x_-$ and $q_{3,-}$ and taking into account that $x_+ \in \pi T \mathbb{Z}$, we obtain:

$$
\partial^2_+ \Sigma^{\text{Sun,dom}}(\pi T, 0, 0) = -i \sum_{x_+ \in \pi \mathbb{N}} \int dk_{1,0} dk_{2,0} \int_{D_q} \frac{dq_{1,-} dq_{2,-}}{\sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-})}
$$

(139)

$$
\times x_+^2 e^{-k_{1,0} \frac{x_+}{\sin \pi q_{1,-}} - k_{2,0} \frac{x_+}{\sin \pi q_{2,-}} + k_{1,0} + k_{2,0} \frac{x_+}{\sin \pi q_{3,-}}} e^{i \pi T - k_{1,0} - k_{2,0} \frac{x_+}{\sin \pi (q_{1,-} + q_{2,-})}} \chi(\frac{\pi T - k_{1,0} - k_{2,0}}{\sin \pi (q_{1,-} + q_{2,-})} < 0) \chi(\frac{k_{1,0}}{\sin \pi q_{1,-}} < 0) \chi(\frac{k_{2,0}}{\sin \pi q_{2,-}} > 0) \chi(\frac{\pi T + k_{1,0} + k_{2,0}}{\sin \pi (q_{1,-} + q_{2,-})} > 0).
$$

Lemma 5.1. We have

$$
iA^{(1)} \leq 0.
$$

(140)
Proof. By definition, $A^{(1)}$ is equal to the integral $\text{[139]}$, constrained in the domain $T^{(++)}$, in which we have: $\sin \pi q_{1,-} > 0$, $\sin \pi q_{2,-} > 0$ and $\sin (q_{1,-} + q_{2,-}) > 0$. The characteristic functions

$$
\chi\left(\frac{k_{1,0}}{\pi \sin \pi q_{1,-}} - 0\right)\chi\left(\frac{k_{2,0}}{\pi \sin \pi q_{2,-}} > 0\right)\chi\left(\frac{\pi T - k_{1,0} - k_{2,0}}{\pi \sin \pi (q_{1,-} + q_{2,-})} < 0\right)
$$

(141)

set the following constraints: $k_{1,0} < 0$, $k_{2,0} > 0$, and $k_{1,0} + k_{2,0} > \pi T$, and the characteristic functions

$$
\chi\left(\frac{k_{1,0}}{\pi \sin \pi q_{1,-}} - 0\right)\chi\left(\frac{k_{2,0}}{\pi \sin \pi q_{2,-}} > 0\right)\chi\left(\frac{\pi T + k_{1,0} + k_{2,0}}{\pi \sin \pi (q_{1,-} + q_{2,-})} > 0\right)
$$

(142)

set the following constraints: $k_{1,0} < 0$, $k_{2,0} > 0$, $k_{1,0} + k_{2,0} > -\pi T$. We have

$$
A^{(1)} = -i \sum_{x_{+} \in \pi N} \sum_{k_{1,0} \in \pi T + 2\pi T \mathbb{Z}} \sum_{s \in 2\pi T \mathbb{Z}} x_{+}^2
$$

$$
\times \int_{T^{(++)}} \frac{dk_{1,0}dk_{2,0}}{\sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-})} \chi(k_{1,0} < 0)\chi(k_{2,0} > 0)
$$

$$
\times \int_{T^{(++)}} \frac{dq_{1,-}dq_{2,-}}{\pi \sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-})} \chi(k_{1,0} + k_{2,0} > \pi T)
$$

$$
\times \left[ e^{\pi (q_{1,-} + q_{2,-})} + \chi(k_{1,0} + k_{2,0} > -\pi T) \right].
$$

(143)

Now we consider the integration over $k_{1,0}$ and $k_{2,0}$, which are summations over the variables $k_{1,0}, k_{2,0} \in \pi T + 2\pi T \mathbb{Z}$. Define $s = k_{1,0} + k_{2,0}$, we have $s \in 2\pi T \mathbb{Z}$ and

$$
A^{(1)} = -i \sum_{x_{+} \in \pi N} \sum_{k_{1,0} \in \pi T + 2\pi T \mathbb{Z}} \sum_{s \in 2\pi T \mathbb{Z}} x_{+}^2
$$

$$
\times \int_{T^{(++)}} \frac{dk_{1,0}dk_{2,0}}{\sin \pi q_{1,-} \sin \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-})} e^{\frac{k_{1,0}x_{+}}{\sin \frac{\pi}{\pi q_{1,-}}} + \frac{1}{\sin \frac{\pi}{\pi q_{2,-}}}}
$$

$$
\times \left[ e^{s x_{+}} \left( \frac{1}{\sin \pi q_{1,-}} + \frac{1}{\sin \pi q_{2,-}} \right) \chi(k_{1,0} < 0)\chi(s > k_{1,0})
$$

$$
\times \chi(s > \pi T) - e^{-\frac{s}{\sin \pi q_{1,-} + q_{2,-}}} + \chi(k_{1,0} + k_{2,0} > -\pi T) \right].
$$

(144)

The condition $k_{1,0} < 0$ implies that $k_{1,0} \in -\pi T - 2\pi T \mathbb{N}$, the condition $s > -\pi T$ implies that $s \in 2\pi T \mathbb{Z}$, while $s > \pi T$ implies that $s \in 2\pi T + 2\pi T \mathbb{N}$. These conditions imply already $s > k_{1,0}$ so that this constraint can be omitted. The summation $\sum_{k_{1,0}} \sum_{s} \cdots$ is

$$
(2\pi T)^2 \sum_{n=0}^{\infty} e^{-\left(2n+1\right)Tx_{+}} \left( \frac{1}{\sin \pi q_{1,-}} + \frac{1}{\sin \pi q_{2,-}} \right)
$$

$$
\sum_{p=1}^{\infty} e^{-2pT x_{+}} \left( \frac{1}{\sin \pi q_{2,-}} + \frac{1}{\sin \pi (q_{1,-} + q_{2,-})} \right) e^{\frac{T}{\sin \pi (q_{1,-} + q_{2,-})} x_{+}}
$$

$$
- \sum_{p=0}^{\infty} e^{-2pT x_{+}} \left( \frac{1}{\sin \pi q_{2,-}} + \frac{1}{\sin \pi (q_{1,-} + q_{2,-})} \right) e^{-\frac{T}{\sin \pi (q_{1,-} + q_{2,-})} x_{+}}
$$

$$
= -e^{-Tx_{+}} \left( \frac{1}{\sin \pi q_{1,-}} + \frac{1}{\sin \pi q_{2,-}} \right) x_{+} + \left( 1 - e^{-2T x_{+}} \right) \left( \frac{1}{\sin \pi q_{2,-}} + \frac{1}{\sin \pi (q_{1,-} + q_{2,-})} \right),
$$

26
which is negative definite. Let the above expression be $-\bar{A}_1$, in which $\bar{A}_1$ is positive definite. So we obtain:

$$A^{(1)} = i \sum_{x_+ \in \mathbb{Z}} x_+^2 \int_{\mathbb{T}} dq_{1,-} dq_{2,-} \frac{dq_{1,-} dq_{2,-}}{\pi q_{1,-} \pi q_{2,-} \sin \pi (q_{1,-} + q_{2,-})} \bar{A}_1,$$

which is equal to half of (73). Since the r.h.s. of (146) is positive, we have $iA^{(1)} \leq 0$.

Carrying out exactly the same analysis we can prove:

**Lemma 5.2.** $A^{(2)} = A^{(1)}$, and $iA^{(2)} \leq 0$.

**Proof.** That $A^{(2)} = A^{(1)}$ can be easily proven by using the symmetry properties of the two amplitudes. This proof is left to the reader as an exercise.

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