ON THE DIOPHANTINE EQUATION $\sigma_2(X_n) = \sigma_n(X_n)$

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Abstract. In this note we investigate the set $S(n)$ of positive integer solutions of the title Diophantine equation. In particular, for a given $n$ we prove boundedness of the number of solutions, give precise upper bound on the common value of $\sigma_2(X_n)$ and $\sigma_n(X_n)$ together with the biggest value of the variable $x_n$ appearing in the solution. Moreover, we enumerate all solutions for $n \leq 16$ and discuss the set of values of $x_n/x_{n-1}$ over elements of $S(n)$.

Dedicated to the memory of professor Andrzej Schinzel.

1. Introduction

Let $\mathbb{N}$ be the set of non-negative integers, $\mathbb{N}_+$ the set of positive integers and for given $k \in \mathbb{N}$ we define $\mathbb{N}_{\geq k}$ as the set of integers $\geq k$. Moreover, for a given set $A$, by $|A|$ we denote the number of elements of the set $A$.

Let $n \in \mathbb{N}_{\geq 3}$ and for given $k \leq n$ consider the $k$-th symmetric polynomial $\sigma_k(x_1, \ldots, x_n) = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1} \cdots x_{i_k}$.

In the sequel, for simplicity of notation, we will write $\mathbf{1}_k$ instead of $(1, \ldots, 1)$, where the number of 1’s is equal to $k$.

The question whether the sum of elements of a given finite set can be equal to the product of these elements is a classical one. Equivalently, we ask for which $n \in \mathbb{N}_{\geq 2}$ the Diophantine equation $\sigma_1(X_n) = \sigma_n(X_n)$ has a solution in positive integers $x_1, \ldots, x_n$. This question was investigated by many authors. For each $n \geq 3$ the equation has a solution $(\mathbf{1}_{n-2}, 2, n)$. In particular, $N(n) \geq 1$, where $N(n)$ is the number of solutions. Schinzel showed that there is no other solution for $n = 6$ or $n = 24$. He also investigated the existence of rational solutions in the case of $n = 3$ and proved that for each $m$ there are at least $m$ triples of integers with the same sum and the same product [8]. Misiurewicz has shown that $n = 2, 3, 4, 6, 24, 114, 174, 444$ are the only values of $n < 10^3$ for which $N(n) = 1$ [6] (recall that the value 114 was misprinted as 144 in [3]). This was later extended by Brown and by Singmaster, Bennett and Dunn to $n \leq 1444000$. All these results were improved by Weingartner. He proved that $N(n) > 1$ for $444 < n < 10^{11}$ [9]. This was possible due to connection with Sophie-Germain primes [1] (see also [7]). The question whether there is a value $n > 444$ such that $N(n) = 1$ is still open. A nice exposition of the basic results concerning this problem can be found in [2].
For more on the history of this problem and related investigations see section D24 in [4].

Motivated by the findings concerning the equation $\sigma_1(\overline{x}_n) = \sigma_n(\overline{x}_n)$ it is natural to ask a question about the existence of positive integer solutions of the Diophantine equation

\begin{equation}
\sigma_2(\overline{x}_n) = \sigma_n(\overline{x}_n).
\end{equation}

Let us note that (1) is equivalent with the Diophantine equation

\begin{equation}
\sigma_{n-2}\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right) = 1
\end{equation}

which need to be solved in positive integers and its solutions give quite special representations of 1 in terms of Egyptian fractions. By a solution of (1) we mean a sequence $\overline{x}_n = (x_1, \ldots, x_n)$ satisfying the condition $x_i \leq x_{i+1}$ for $i = 1, \ldots, n-1$.

In this paper we are interested in the structure of the set $S(n) = \{\overline{x}_n \in \mathbb{N}^n : \overline{x}_n \text{ is a solution of (1)}\}$ and investigate its various properties. Next, for $i \in \{0, \ldots, n\}$, let us put

\begin{equation}
S_i(n) = \{\overline{x}_n \in S(n) : x_1 = \ldots = x_{n-i} = 1, 2 \leq x_{n-i+1} \leq \ldots \leq x_n\},
\end{equation}

i.e., $S_i(n) \subset S(n)$ contain solutions of (1) with exactly $i$ terms different than 1. We clearly have the (disjoint) decomposition

\begin{equation}
S(n) = \bigcup_{i=0}^{n} S_i(n).
\end{equation}

Let us describe the content of the paper in some details. In Section 2 finiteness of the set $S(n)$ is proved together with the maximum value of $x_n$ which can appear in the solution $\overline{x}_n$ of (1) (Theorem 2.7). We also present a bound for $x_{n-2}$ (Corollary 3.4). Using the obtained results we compute all the solutions of (1) for $n \leq 16$.

In Section 3 we investigate the set $S_3(n)$ in details. Our findings allow us to prove that

\begin{equation}
|S(n)| \geq |S_3(n)| \geq \frac{1}{2} \tau\left(\frac{1}{2}(n-2)(3n-1)\right),
\end{equation}

where $\tau(m)$ is the number of divisors of a positive integer $m$. Moreover, we investigate the behaviour of $x_n/x_{n-1}$, where $x_{n-1}, x_n$ are components of $\overline{x}_n \in S(n)$. In particular, we prove that the set of rational values $x_n/x_{n-1}$, where $\overline{x}_n \in S(n)$ and $n \in \mathbb{N}_{\geq 3}$, is dense in the set $[1, +\infty)$.

Finally, in the last section we state several questions and conjectures which, we believe, will motivate further investigations.

2. BOUNDEDNESS OF $x_n$ AND ENUMERATION OF $S(n)$ FOR $n \leq 16$

Before we start, we mention two basic identities involving symmetric polynomials. More precisely, we have

\begin{equation}
\sigma_1(\overline{x}_{k_1}, \overline{y}_{k_2}) = \sigma_1(\overline{x}_{k_1}) + \sigma_1(\overline{y}_{k_2})
\end{equation}

and

\begin{equation}
\sigma_2(\overline{x}_{k_1}, \overline{y}_{k_2}) = \sigma_1(\overline{x}_{k_1})\sigma_1(\overline{y}_{k_2}) + \sigma_2(\overline{x}_{k_1}) + \sigma_2(\overline{y}_{k_2}),
\end{equation}

where $\overline{x}_{k_1}, \overline{y}_{k_2}$ are independent sets of variables. In the sequel we will use these identities several times.
We start with a simple bound for $x_1 \cdot \ldots \cdot x_{n-2}$.

**Lemma 2.1.** If $n \geq 3$ and $\overline{X}_n \in S(n)$ then $x_1 \cdot \ldots \cdot x_{n-2} \leq \binom{n}{2}$. In particular, $x_{n-2} \leq \binom{n}{2}$.

**Proof.** If $\overline{X}_n \in S(n)$ with $x_1 \leq \ldots \leq x_n$ then

$$\sigma_n(\overline{X}_n) = \sigma_2(\overline{X}_n) \leq \binom{n}{2} x_{n-1} x_n,$$

and dividing by $x_{n-1}x_n$ we get the inequality from the statement. \hfill \Box

On the other hand, we have a lower bound $x_1 \cdot \ldots \cdot x_{n-2} \geq 2$ as the following holds.

**Lemma 2.2.** For $n \geq 3$ we have $S_0(n) = S_1(n) = S_2(n) = \emptyset$.

**Proof.** Suppose that $\overline{X}_n \in S_i(n)$ for some $i \leq 2$. Then

$$\sigma_2(\overline{X}_n) \geq x_{n-1} x_n + x_{n-1} + x_n > x_{n-1} x_n = \sigma_n(\overline{X}_n),$$

which is a contradiction. \hfill \Box

Lemma 2.2 states that if $n \in \mathbb{N}_{\geq 3}$ and $S_i(n) \neq \emptyset$, then $i \geq 3$. However, if $S_i(n) \neq \emptyset$, then $i$ cannot be too big comparing to $n$. This is due to the lemma below.

**Lemma 2.3.** Let $n \in \mathbb{N}_{\geq 3}$. If $S_i(n) \neq \emptyset$, then $i \leq 2 + \log_2 \binom{n}{2}$.

**Proof.** Let $\overline{X}_n \in S_i(n)$. Then

$$2^{i-2} \leq x_{n-i+1} \cdot \ldots \cdot x_{n-2} \leq \binom{n}{2},$$

where the last inequality follows from Lemma 2.1. After taking the logarithm with base 2 from the left hand side and right hand side of the above inequality we get

$$i - 2 \leq \log_2 \binom{n}{2}.$$ The lemma is proved. \hfill \Box

As an immediate consequence we get the following.

**Corollary 2.4.** Let $n \in \mathbb{N}_{\geq 3}$ and $\overline{X}_n \in S(n)$. If $n \geq 6$ then $x_1 = 1$. If $n \geq 8$ then $x_1 = x_2 = 1$.

Writing $\overline{X}_{n-2}$ we mean $(x_1, \ldots, x_{n-2})$. The next result shows how to find $x_{n-1}, x_n \in \mathbb{N}$ such that $\overline{X}_n \in S(n)$, where $\overline{X}_{n-2}$ is fixed.

**Theorem 2.5.** Let $n \in \mathbb{N}_{\geq 3}$ and $\overline{X}_n \in S(n)$. Then

$$x_{n-1} = \frac{\sigma_1(\overline{X}_{n-2}) + d_1}{\sigma_{n-2}(\overline{X}_{n-2}) - 1}, \quad x_n = \frac{\sigma_1(\overline{X}_{n-2}) + d_2}{\sigma_{n-2}(\overline{X}_{n-2}) - 1},$$

where $d_1, d_2 \in \mathbb{N}$ are such that

$$d_1d_2 = \sigma_1(\overline{X}_{n-2})^2 + \sigma_2(\overline{X}_{n-2})(\sigma_{n-2}(\overline{X}_{n-2}) - 1) =: f(n, \overline{X}_{n-2}).$$
Proof. Let us observe that
\[
\begin{align*}
& (\sigma_{n-2}(\mathbf{X}_{n-2}) - 1)(\sigma_n(\mathbf{X}_n) - \sigma_2(\mathbf{X}_n)) \\
& = (\sigma_{n-2}(\mathbf{X}_{n-2}) - 1) \\
& \cdot \left( (\sigma_{n-2}(\mathbf{X}_{n-2}) - 1)x_{n-1}x_n - \sigma_1(\mathbf{X}_{n-2})(x_{n-1} + x_n) - \sigma_2(\mathbf{X}_{n-2}) \right) \\
& = \left( (\sigma_{n-2}(\mathbf{X}_{n-2}) - 1)x_{n-1} - \sigma_1(\mathbf{X}_{n-2}) \right) \left( (\sigma_{n-2}(\mathbf{X}_{n-2}) - 1)x_n - \sigma_1(\mathbf{X}_{n-2}) \right) \\
& - f(n, \mathbf{X}_{n-2}).
\end{align*}
\]
Thus, because \(\sigma_{n-2}(\mathbf{X}_{n-2}) \geq 2\) we see that \(\mathbf{X}_n \in S(n)\) if and only if there are positive integers \(d_1, d_2\) such that \(d_1d_2 = f(n, \mathbf{X}_{n-2})\) and the system of equations
\[
(\sigma_{n-2}(\mathbf{X}_{n-2}) - 1)x_{n-1} - \sigma_1(\mathbf{X}_{n-2}) = d_1, \quad (\sigma_{n-2}(\mathbf{X}_{n-2}) - 1)x_n - \sigma_1(\mathbf{X}_{n-2}) = d_2.
\]
has a solution in integers \(x_{n-1}, x_n\). Solving for \(x_{n-1}, x_n\) we get the expressions from the statement. \(\square\)

Remark 2.6. According to Lemma 2.1 and Theorem 2.5 it suffices to find all the possible values of \(x_{n-1}\) and \(x_n\) for all \(\mathbf{X}_{n-2}\) such that \(\sigma_{n-2}(\mathbf{X}_{n-2}) \leq \binom{n}{2}\). Since there are only finitely many such \((n - 2)\)-tuples \(\mathbf{X}_{n-2}\), the number of solutions of (1) is finite for each \(n \in \mathbb{N}_{\geq 3}\).

Now we state the result that gives us an upper bound for unknown \(x_n\), where \(n \in \mathbb{N}_{\geq 3}\) is fixed.

Theorem 2.7. Let \(n \in \mathbb{N}_{\geq 3}\) and \(m \in \mathbb{N}\) satisfies
\[
m = \sigma_2(\mathbf{X}_n) = \sigma_n(\mathbf{X}_n).
\]
Then \(m \leq n^2(3n - 5)\) and \(x_n \leq \frac{1}{2}n(3n - 5)\).

Before we prove Theorem 2.7 we need some preparation. First, if \(\mathbf{X}_n \in S_i(n)\), then \(i \geq 3\) by Lemma 2.2. Thus, we may put
\[
y_j = x_{n-i+j}, \quad j \in \{1, \ldots, i-2\}
\]
and
\[
\mathbf{Y}_{i-2} = (y_1, \ldots, y_{i-2}).
\]
Of course,
\[
\mathbf{Y}_{i-2} \in \mathbb{N}_{\geq 2}^{i-2}, \quad \mathbf{X}_{n-2} = (\mathbf{Y}_{n-i}, \mathbf{Y}_{i-2})
\]
and
\[
\sigma_{n-2}(\mathbf{X}_{n-2}) = \sigma_{i-2}(\mathbf{Y}_{i-2}).
\]
Let us estimate \(\sigma_1(\mathbf{Y}_{i-2})\) and \(\sigma_2(\mathbf{Y}_{i-2})\) from above in terms of \(\sigma_{i-2}(\mathbf{Y}_{i-2})\).

Lemma 2.8. We have
\[
\sigma_1(\mathbf{Y}_{i-2}) \leq \frac{i-2}{2i-3} \sigma_{i-2}(\mathbf{Y}_{i-2}) \quad \text{and} \quad \sigma_2(\mathbf{Y}_{i-2}) \leq \frac{(i-2)(i-3)}{2i-3} \sigma_{i-2}(\mathbf{Y}_{i-2}).
\]

Proof. Since \(x_j \geq 2\) for each \(j \in \{1, \ldots, i-2\}\), we have
\[
y_k = \frac{\sigma_{i-2}(\mathbf{Y}_{i-2})}{\prod_{j \neq k} y_j} \geq \frac{\sigma_{i-2}(\mathbf{Y}_{i-2})}{2i-3}
\]
and \[ y_k y_l = \frac{\sigma_{i-2}(Y_{i-2})}{\prod_{j \neq k, l} y_j} \leq \frac{\sigma_{i-2}(Y_{i-2})}{2^{i-4}} \]
for \( k, l \in \{1, \ldots, i - 2\} \). As a result we get
\[ \sigma_1(Y_{i-2}) = \sum_{k=1}^{i-2} y_k \leq (i - 2) \frac{\sigma_{i-2}(Y_{i-2})}{2^{i-3}} \]
and
\[ \sigma_2(Y_{i-2}) = \sum_{1 \leq k < l \leq i-2} y_k y_l \leq \left( \frac{i - 2}{2} \right) \frac{\sigma_{i-2}(Y_{i-2})}{2^{i-4}}. \]
The results follow. \( \square \)

As an immediate consequence of Lemma 2.8, the formulae (2), (3) and the conditions (4) and (5) we get the following.

Corollary 2.9. If \( n \in \mathbb{N}_{\geq 3} \) and \( \overline{X}_n \in S_i(n) \), then
\[ (6) \quad \sigma_1(\overline{X}_{n-2}) \leq n - i + \frac{i - 2}{2^{i-3}} \sigma_n(\overline{X}_{n-2}) \]
and
\[ (7) \quad \sigma_2(\overline{X}_{n-2}) \leq \frac{1}{2}(n - i)(n - i - 1) + \frac{(i - 2)(n - 3)}{2^{i-3}} \sigma_{n-2}(\overline{X}_{n-2}). \]
In particular, for each \( \overline{X}_n \in S(n) \) we have
\[ (8) \quad \sigma_1(\overline{X}_{n-2}) \leq n - 3 + \sigma_{n-2}(\overline{X}_{n-2}) \]
and
\[ (9) \quad \sigma_2(\overline{X}_{n-2}) \leq \frac{1}{2}(n - 3)(n - 4) + (n - 3)\sigma_{n-2}(\overline{X}_{n-2}). \]

Proof. The inequalities (8) and (9) follow as the expressions on the right hand side in (6) and (7) are clearly maximized for \( i = 3 \). \( \square \)

At this moment we are ready to give the proof of Theorem 2.7.

Proof of Theorem 2.7. From Theorem 2.5 we know that \( x_{n-1} = \frac{\sigma_1(\overline{X}_{n-2}) + d_1}{\sigma_{n-2}(\overline{X}_{n-2}) - 1} \) and \( x_n = \frac{\sigma_1(\overline{X}_{n-2}) + d_2}{\sigma_{n-2}(\overline{X}_{n-2}) - 1} \), where \( d_1 d_2 = f(n, \overline{X}_{n-2}) \). Hence,
\[ m = \sigma_{n-2}(\overline{X}_{n-2}) x_{n-1} x_n \]
\[ = \sigma_{n-2}(\overline{X}_{n-2}) \left( f(n, \overline{X}_{n-2}) + (d_1 + d_2) \sigma_1(\overline{X}_{n-2}) + \frac{\sigma_1(\overline{X}_{n-2})^2}{(\sigma_{n-2}(\overline{X}_{n-2}) - 1)^2} \right). \]

We consider two cases.

Case I: \( \sigma_{n-2}(\overline{X}_{n-2}) \leq n \). Then
\[ x_n \leq \frac{\sigma_1(\overline{X}_{n-2}) + f(n, \overline{X}_{n-2})}{\sigma_{n-2}(\overline{X}_{n-2}) - 1} = \frac{\sigma_1(\overline{X}_{n-2})(\sigma_1(\overline{X}_{n-2}) + 1)}{\sigma_{n-2}(\overline{X}_{n-2}) - 1} + \sigma_2(\overline{X}_{n-2}). \]
Using inequalities [8] and [9] we obtain
\[
x_n \leq \frac{(\sigma_{n-2}(X_{n-2}) + n - 3)(\sigma_{n-2}(X_{n-2}) + n - 2)}{\sigma_{n-2}(X_{n-2}) - 1}
+ \frac{1}{2}(n - 3)(n - 4) + (n - 3)\sigma_{n-2}(X_{n-2})
\]
\[
= \sigma_{n-2}(X_{n-2}) + 2n - 4 + \frac{(n - 2)(n - 1)}{\sigma_{n-2}(X_{n-2}) - 1}
+ \frac{1}{2}(n - 3)(n - 4) + (n - 3)\sigma_{n-2}(X_{n-2})
\]
\[
= (n - 2)(\sigma_{n-2}(X_{n-2}) + 2) + \frac{(n - 2)(n - 1)}{\sigma_{n-2}(X_{n-2}) - 1} + \frac{1}{2}(n - 3)(n - 4).
\]

A simple analysis of the last expression treated as a function of \(\sigma_{n-2}(X_{n-2}) \in [2, n]\) shows that this expression is maximized for \(\sigma_{n-2}(X_{n-2}) \in \{2, n\}\) and attains the value
\[
(n - 2)(n + 3) + \frac{1}{2}(n - 3)(n - 4) = \frac{1}{2}n(3n - 5).
\]

Now we estimate the value of \(m\). Since \(d_1 + d_2 \leq 1 + f(n, X_{n-2})\), we have
\[
m \leq \sigma_{n-2}(X_{n-2}) \cdot \frac{f(n, X_{n-2}) + (1 + f(n, X_{n-2}))\sigma_1(X_{n-2}) + \sigma_1(\sigma_{n-2}(X_{n-2}))}{(\sigma_{n-2}(X_{n-2}) - 1)^2}
\]
\[
= \sigma_{n-2}(X_{n-2}) \cdot \frac{\sigma_1(X_{n-2}) + 1}{\sigma_{n-2}(X_{n-2}) - 1} \cdot \frac{\sigma_1(X_{n-2}) + f(n, X_{n-2})}{\sigma_{n-2}(X_{n-2}) - 1}
\]

From the estimation of \(x_n\) we know that \(\frac{\sigma_1(X_{n-2}) + f(n, X_{n-2})}{\sigma_{n-2}(X_{n-2}) - 1} \leq \frac{1}{2}n(3n - 5)\), so it remains to bound \(\sigma_{n-2}(X_{n-2}) \cdot \frac{\sigma_1(X_{n-2}) + 1}{\sigma_{n-2}(X_{n-2}) - 1}\) from above. By [8] we have
\[
\sigma_{n-2}(X_{n-2}) \cdot \frac{\sigma_1(X_{n-2}) + 1}{\sigma_{n-2}(X_{n-2}) - 1} \leq \sigma_{n-2}(X_{n-2}) \cdot \frac{\sigma_1(X_{n-2}) + \sigma_2(\sigma_{n-2}(X_{n-2}) - 1) + \sigma_2(\sigma_{n-2}(X_{n-2}) - 1)}{\sigma_{n-2}(X_{n-2}) - n + 1}.
\]

An analysis of the derivative of the expression on the right hand side as a function of \(\sigma_{n-2}(X_{n-2})\) shows that this expression is maximized for \(\sigma_{n-2}(X_{n-2}) \in \{2, n\}\) and attains the value \(2n\). Summing up,
\[
m \leq n^2(3n - 5).
\]

**Case II:** \(\sigma_{n-2}(X_{n-2}) \geq n + 1\). Since \(x_{n-1} \geq 2\), we have
\[
d_1 = (\sigma_{n-2}(X_{n-2}) - 1)x_{n-1} - \sigma_1(X_{n-2}) \geq 2(\sigma_{n-2}(X_{n-2}) - 1) - \sigma_1(X_{n-2})
\]
\[
\geq \sigma_{n-2}(X_{n-2}) - n + 1 \geq 2,
\]
where in the first inequality on the last line we used [8]. Hence,
\[
d_2 = \frac{f(n, X_{n-2})}{d_1} \leq \frac{\sigma_1(X_{n-2})^2 + \sigma_2(\sigma_{n-2}(X_{n-2}) - 1)}{\sigma_{n-2}(X_{n-2}) - n + 1}.
\]
Consequently,
\[
x_n \leq \frac{\sigma_1(X_{n-2})(\sigma_{n-2}(X_{n-2}) - n + 1) + \sigma_1(X_{n-2})^2 + \sigma_2(X_{n-2})(\sigma_{n-2}(X_{n-2}) - 1)}{(\sigma_{n-2}(X_{n-2}) - 1)(\sigma_{n-2}(X_{n-2}) - n + 1)}
\]
\[
= \frac{\sigma_1(X_{n-2})(\sigma_{n-2}(X_{n-2}) - n + 1 + \sigma_1(X_{n-2})) + \sigma_2(X_{n-2})(\sigma_{n-2}(X_{n-2}) - 1)}{(\sigma_{n-2}(X_{n-2}) - 1)(\sigma_{n-2}(X_{n-2}) - n + 1)}
\]
\[
\leq \frac{\sigma_1(X_{n-2})(2\sigma_{n-2}(X_{n-2}) - 2) + \sigma_2(X_{n-2})(\sigma_{n-2}(X_{n-2}) - 1)}{(\sigma_{n-2}(X_{n-2}) - 1)(\sigma_{n-2}(X_{n-2}) - n + 1)}
\]
\[
\leq \frac{2\sigma_1(X_{n-2}) + \sigma_2(X_{n-2})}{\sigma_{n-2}(X_{n-2}) - n + 1}
\]
\[
\leq \frac{2(n - 3) + 2\sigma_{n-2}(X_{n-2}) + \frac{1}{2}(n - 3)(n - 4) + (n - 3)\sigma_{n-2}(X_{n-2})}{\sigma_{n-2}(X_{n-2}) - n + 1}
\]
\[
= \frac{\frac{1}{2}(n - 3)n + (n - 1)\sigma_{n-2}(X_{n-2})}{\sigma_{n-2}(X_{n-2}) - n + 1} = \frac{\frac{1}{2}(n - 3)n + (n - 1)^2}{\sigma_{n-2}(X_{n-2}) - n + 1} + n - 1
\]
\[
\leq \frac{\frac{1}{2}(n - 3)n + (n - 1)^2}{2} + n - 1 = \frac{1}{4}(3n^2 - 3n - 2),
\]

where in the third and fifth line we used \([\mathbf{8}]\) and \([\mathbf{9}]\).

Now we estimate the value of \(m\). Since \(d_1 \leq \sigma_{n-2}(X_{n-2}) - n + 1\), the value of \(d_1 + d_2\) is maximized for \(d_1 = \sigma_{n-2}(X_{n-2}) - n + 1 =: d_1(X_{n-2})\) and \(d_2 = \frac{f(n,X_{n-2})}{\sigma_{n-2}(X_{n-2}) - n + 1} =: d_2(X_{n-2})\). Hence, we have
\[
m \leq \sigma_{n-2}(X_{n-2}) \cdot \frac{f(n,X_{n-2}) + (d_1(X_{n-2}) + d_2(X_{n-2}))\sigma_1(X_{n-2}) + \sigma_1(X_{n-2})^2}{(\sigma_{n-2}(X_{n-2}) - 1)^2}
\]
\[
\leq \sigma_{n-2}(X_{n-2}) \cdot \frac{\sigma_1(X_{n-2}) + d_1(X_{n-2})}{\sigma_{n-2}(X_{n-2}) - 1} \cdot \frac{\sigma_1(X_{n-2}) + d_2(X_{n-2})}{\sigma_{n-2}(X_{n-2}) - 1}
\]
From the estimation of \(x_n\) we know that
\[
\frac{\sigma_1(X_{n-2}) + d_2(X_{n-2})}{\sigma_{n-2}(X_{n-2}) - 1} \leq \frac{\frac{1}{2}(n - 3)n + (n - 1)^2}{\sigma_{n-2}(X_{n-2}) - n + 1} + n - 1,
\]
so it remains to bound \(\sigma_{n-2}(X_{n-2})\). From \([\mathbf{3}]\) we have
\[
\sigma_{n-2}(X_{n-2}) \cdot \frac{\sigma_1(X_{n-2}) + d_1(X_{n-2})}{\sigma_{n-2}(X_{n-2}) - 1} \leq \frac{2\sigma_{n-2}(X_{n-2}) - 2}{\sigma_{n-2}(X_{n-2}) - 1}
\]
Hence,
\[
m \leq 2\sigma_{n-2}(X_{n-2}) \left( \frac{\frac{1}{2}(n - 3)n + (n - 1)^2}{\sigma_{n-2}(X_{n-2}) - n + 1} + n - 1 \right).
\]
An analysis of the derivative of the expression on the right hand side as a function of \(\sigma_{n-2}(X_{n-2})\) in \([n + 1, \binom{n}{2}]\) shows that this expression is maximized for \(\sigma_{n-2}(X_{n-2}) = n + 1\) and attains the value \(\frac{1}{2}(n + 1)(3n^2 - 3n - 2)\).

Since \(n \geq 3\), we have \(\frac{1}{4}(3n^2 - 3n - 2) \leq \frac{1}{2}n(3n - 5)\) and \(\frac{1}{4}(n + 1)(3n^2 - 3n - 2) < n^2(3n - 5)\). Thus \(x_n \leq \frac{1}{2}n(3n - 5)\) and \(m \leq n^2(3n - 5)\) in any case. \(\square\)
Remark 2.10. From the proofs of Corollary 2.9 and Theorem 2.7 we see that \( x_n = \frac{1}{n}(3n-5) \) and \( m = n^2(3n-5) \) if and only if \( i = 3 \) and \( \{\sigma_{n-2}(X_{n-2}), x_{n-1}\} = \{2, n\} \), i.e. \( X_n = \{n_{n-3}, 2, n, \frac{1}{n}(3n-5)\} \).

We are also able to estimate \( x_{n-2} \).

Lemma 2.11. Let \( n \in \mathbb{N}_{\geq 3} \) and \( X_n \in S(n) \). Assume that \( x_{n-2} \geq 1 + C(n-2)^{2/3} \) for some real number \( C > 0 \). Then

\[
(10) \quad C \leq \frac{1}{C}(n-2)^{-1/3} + \frac{3}{2C}(n-2)^{-1} + \frac{1}{C^2}(n-2)^{-2/3} + (n-2)^{-1/3} + \frac{1}{2C}.
\]

Proof. By Theorem 2.5 and the fact that \( x_n \geq x_{n-1} \geq x_{n-2} \geq 1 + C(n-2)^{2/3} \) we obtain the following chain of inequalities:

\[
1 + C(n-2)^{2/3} \leq x_{n-2} \leq x_{n-1} \leq \frac{\sigma_1(X_{n-2}) + \sqrt{f(n, X_{n-2})}}{\sigma_{n-2}(X_{n-2}) - 1}
\]

\[
= \frac{\sigma_{n-2}(X_{n-2}) - 1 + n - 2}{\sigma_{n-2}(X_{n-2}) - 1} + \frac{\sqrt{\sigma_{n-2}(X_{n-2}) - 1 + n - 2 + \left(\frac{n-2}{n} + (n-3)(\sigma_{n-2}(X_{n-2}) - 1)\right)(\sigma_{n-2}(X_{n-2}) - 1)}}{\sigma_{n-2}(X_{n-2}) - 1}
\]

\[
= 1 + \frac{n - 2}{\sigma_{n-2}(X_{n-2}) - 1}
\]

\[
+ \left(1 + \frac{n - 2}{\sigma_{n-2}(X_{n-2}) - 1}\right)^2 + \frac{n - 3}{2} \frac{n - 2}{\sigma_{n-2}(X_{n-2}) - 1} + n - 3
\]

\[
\leq 1 + \frac{n - 2}{C(n-2)^{2/3}} + \left(1 + \frac{n - 2}{C(n-2)^{2/3}}\right)^2 + \frac{n - 3}{2} \frac{n - 2}{C(n-2)^{2/3}} + n - 3
\]

\[
\leq 1 + \frac{1}{C(n-2)^{1/3}} + \frac{1}{C^2}(n-2)^{1/3} + \frac{1}{2C}(n-2)^{2/3} + \frac{1}{2C}(n-2)^{4/3} - \frac{1}{2C}(n-2)^{1/3} + n - 3
\]

\[
= 1 + \frac{1}{C(n-2)^{1/3}} + \frac{3}{2C}(n-2)^{1/3} + \frac{1}{C^2}(n-2)^{2/3} + n - 2 + \frac{1}{2C}(n-2)^{4/3}.
\]

After reduction the 1’s and dividing by \((n-2)^{2/3}\) we get the the statement. \( \square \)

Corollary 2.12. Let \( n \in \mathbb{N}_{\geq 3} \) and \( X_n \in S(n) \). Then we have

\[
(11) \quad x_{n-2} \leq 1 + [2(n-2)^{2/3}],
\]

where the equality holds only if \( n = 3 \). Moreover, for each \( C > 2^{-1/3} \) there are only finitely many values of \( n \in \mathbb{N}_{\geq 3} \) such that there exists \( X_n \in S(n) \) with \( x_{n-2} > [C(n-2)^{2/3}] \).
Proof. If \( x_{n-2} = 1 + C(n-2)^{2/3} \) for some \( C > 0 \) then by Lemma 2.11 we have

\[
C \leq \frac{1}{C} + \sqrt{\frac{3}{2C} + \frac{1}{C^2} + 1 + \frac{1}{2C}}
\]

since \( n \geq 3 \). The left hand side of (12) is an increasing function of \( C \) while the right hand side is a decreasing one. Moreover, the equality holds for \( C = 2 \). Hence, (12) holds if and only if \( C \leq 2 \). Thus \( x_{n-2} \leq 1 + 2(n-2)^{2/3} \). As \( x \in \mathbb{Z} \), we have

\[
x \leq 1 + [2(n-2)^{2/3}].
\]

Moreover, if \( n > 3 \) then \( C = 2 \) does not satisfy inequality (10). Therefore the equality in (11) holds only if \( n = 3 \).

We are left with the proof of the "moreover" part. Let \( C > 2^{-1/3} \). Assume by contrary that there are infinitely many values of \( n \in \mathbb{N} \) such that there exists \( \overline{x}_n \in S(n) \) with \( x_{n-2} \geq 1 + C(n-2)^{2/3} \). Tending with \( n \) to \( +\infty \) in inequality (10) we obtain

\[
C \leq \frac{1}{\sqrt{2}C},
\]

which means that \( C \leq \frac{1}{\sqrt{2}} \). This is a contradiction with our assumption that \( C > 2^{-1/3} \). \( \square \)

Remark 2.13. Let \( \overline{x}_n \in S(n) \). If the rate of the magnitude of \( x_{n-2} \) is \( 2^{-1/3}n^{2/3} + o(n^{2/3}) \) then it is also the rate of the magnitude of \( x_{n-1} \) and \( x_n \). Indeed, from the proof of Lemma 2.11 we know that

\[
2^{-1/3}(n-2)^{2/3} + o((n-2)^{2/3}) \leq x_{n-2} \leq x_{n-1} = \frac{\sigma_1(\overline{x}_{n-2}) + d_1}{\sigma_{n-2}(\overline{x}_{n-2}) - 1}
\]

where \( \sigma_{n-2}(\overline{x}_{n-2}) = x_{n-2} \) (i.e. \( \overline{x}_{n-2} = (\overline{t}_{n-3}, x_{n-2}) \)) and \( d_1 \sim \sqrt{f(n, \overline{x}_{n-2})} = \sqrt{f(n, x_{n-2})} \to +\infty \). This means that also \( d_2 \sim \sqrt{f(n, x_{n-2})} \) and analogously we compute that

\[
x_n = \frac{\sigma_1(\overline{x}_{n-2}) + d_1}{\sigma_{n-2}(\overline{x}_{n-2}) - 1} = 2^{-1/3}n^{2/3} + o(n^{2/3}).
\]

In fact there is an infinite family of quadruples \((n, x, y, z)\) of positive integers such that \((\overline{t}_{n-3}, x, y, z) \in S_3(n)\) and \( x \sim y \sim z \sim 2^{-1/3}n^{2/3}, n \to +\infty \). Namely,

\[
n = 4(4k^3 + 2k^2 + 2k - 2)^3 + 2,
\]

\[
x = 2(4k^3 + 2k^2 + 2k - 2)
\]

\[
+ 32k^6 + 32k^5 + 32k^4 - 16k^3 - 8k^2 - 10k + 8,
\]

\[
y = 2(4k^3 + 2k^2 + 2k - 2)^2 + 1,
\]

\[
z = (4k^3 + 2k^2 + 2k - 2)(8k^3 + 4k^2 + 6k - 1) + 1,
\]
where $k \in \mathbb{N}$ is sufficiently large. In the above family we have $y = 1 + 2^{-1/3}(n - 2)^{2/3}$. However, it is possible to give a family of quadruples $(n, x, y, z)$ such that $(T_{n-3}, x, y, z) \in S_d(n)$ and $x = 1 + 2^{-1/3}(n - 2)^{2/3}$. Such a family is

\[
\begin{align*}
  n &= 4(4k^3 - 10k^2 + 10k - 6)^3 + 2, \\
  x &= 2(4k^3 - 10k^2 + 10k - 6)^2 + 1, \\
  y &= (4k^3 - 10k^2 + 10k - 6)(8k^3 - 20k^2 + 22k - 11) + 1, \\
  z &= 2(4k^3 - 10k^2 + 10k - 6) \\
  &\quad + 32k^6 - 150k^5 + 352k^4 - 464k^3 + 392k^2 - 202k + 58,
\end{align*}
\]

where $k \in \mathbb{N}$ is sufficiently large.

Gathering all the results above we are able to enumerate all elements of $S(n)$ for $n \leq 16$. More precisely, we have the following.

**Theorem 2.14.** We have the following equalities of sets:

- $S(3) = \{(2, 3, 6), (2, 4, 4), (3, 3, 3)\}$;
- $S(4) = \{(1, 2, 4, 14), (2, 2, 2, 6)\}$;
- $S(5) = \{(1, 1, 2, 5, 25), (1, 1, 2, 7, 11), (1, 1, 3, 3, 22), (1, 1, 3, 4, 9), (1, 2, 2, 2, 18), (1, 2, 4, 4), (2, 2, 2, 2, 3)\}$;
- $S(6) = \{(1, 1, 1, 2, 6, 39), (1, 1, 1, 2, 7, 22), (1, 1, 1, 3, 4, 18), (1, 1, 1, 3, 6, 8)\}$;
- $S(7) = \{(1, 1, 1, 1, 2, 7, 56), (1, 1, 1, 1, 2, 8, 31), (1, 1, 1, 2, 11, 16), (1, 1, 1, 1, 3, 4, 46), (1, 1, 1, 1, 3, 6, 12), (1, 1, 1, 1, 4, 6, 7), (1, 1, 1, 2, 2, 3, 20)\}$;
- $S(8) = \{\langle T_5, 2, 8, 76\rangle, \langle T_5, 2, 10, 30\rangle, \langle T_5, 4, 4, 22\rangle, \langle T_4, 2, 2, 3, 50\rangle, \langle T_3, 2, 2, 3, 5\rangle\}$;
- $S(9) = \{\langle T_6, 2, 9, 99\rangle, \langle T_6, 2, 15, 21\rangle, \langle T_6, 3, 5, 78, 78\rangle, \langle T_6, 3, 6, 29\rangle, \langle T_6, 3, 8, 15\rangle\}$;
- $S(10) = \{\langle T_7, 2, 10, 125\rangle, \langle T_7, 2, 11, 67\rangle, \langle T_7, 2, 13, 38\rangle, \langle T_7, 3, 6, 51\rangle, \langle T_7, 3, 7, 28\rangle, \langle T_7, 4, 4, 93\rangle, \langle T_7, 4, 5, 26\rangle, \langle T_7, 6, 7, 77\rangle, \langle T_6, 2, 3, 3, 21\rangle, \langle T_6, 3, 3, 8\rangle\}$;
- $S(11) = \{\langle T_8, 2, 11, 154\rangle, \langle T_8, 2, 12, 82\rangle, \langle T_8, 2, 13, 58\rangle, \langle T_8, 2, 14, 46\rangle, \langle T_8, 2, 16, 34\rangle, \langle T_8, 2, 18, 28\rangle, \langle T_8, 2, 19, 26\rangle, \langle T_8, 2, 22, 22\rangle, \langle T_8, 3, 6, 118\rangle, \langle T_8, 3, 7, 43\rangle, \langle T_8, 3, 8, 28\rangle, \langle T_8, 3, 10, 18\rangle, \langle T_8, 3, 13, 13\rangle, \langle T_8, 4, 5, 40\rangle, \langle T_8, 4, 6, 22\rangle, \langle T_8, 4, 7, 16\rangle, \langle T_8, 4, 8, 13\rangle, \langle T_8, 4, 10, 10\rangle, \langle T_8, 5, 5, 19\rangle, \langle T_8, 6, 6, 10\rangle, \langle T_7, 7, 7, 7\rangle, \langle T_7, 7, 2, 4, 97\rangle, \langle T_7, 7, 2, 5, 27\rangle, \langle T_7, 2, 2, 6, 17\rangle, \langle T_7, 2, 2, 7, 13\rangle, \langle T_6, 2, 2, 2, 3, 11\rangle\}$;
- $S(12) = \{\langle T_9, 2, 12, 186\rangle, \langle T_9, 2, 16, 46\rangle, \langle T_9, 2, 18, 36\rangle, \langle T_9, 4, 6, 30\rangle, \langle T_9, 4, 8, 16\rangle, \langle T_9, 6, 6, 12\rangle, \langle T_8, 2, 3, 4, 18\rangle, \langle T_8, 2, 4, 4, 10\rangle, \langle T_8, 2, 4, 6, 6\rangle, \langle T_7, 2, 2, 2, 101\rangle\}$;
- $S(13) = \{\langle T_{10}, 2, 13, 221\rangle, \langle T_{10}, 2, 23, 31\rangle, \langle T_{10}, 3, 7, 166\rangle, \langle T_{10}, 3, 12, 21\rangle, \langle T_{10}, 4, 5, 155\rangle, \langle T_{10}, 5, 5, 34\rangle, \langle T_9, 2, 3, 3, 129\rangle, \langle T_9, 3, 3, 16\rangle, \langle T_8, 2, 2, 4, 4, 4\rangle\}$;
- $S(14) = \{\langle T_{11}, 2, 14, 259\rangle, \langle T_{11}, 2, 15, 136\rangle, \langle T_{11}, 2, 16, 95\rangle, \langle T_{11}, 2, 19, 54\rangle, \langle T_{11}, 3, 8, 100\rangle, \langle T_{11}, 3, 10, 38\rangle, \langle T_{11}, 4, 6, 63\rangle, \langle T_{11}, 4, 7, 34\rangle, \langle T_{10}, 2, 2, 5, 159\rangle, \langle T_9, 2, 2, 3, 5, 5\rangle\}$;
- $S(15) = \{\langle T_{12}, 2, 15, 300\rangle, \langle T_{12}, 2, 16, 157\rangle, \langle T_{12}, 2, 25, 40\rangle, \langle T_{12}, 2, 27, 36\rangle, \langle T_{12}, 3, 8, 222\rangle, \langle T_{12}, 3, 9, 79\rangle, \langle T_{12}, 3, 13, 27\rangle, \langle T_{12}, 3, 14, 24\rangle, \langle T_{12}, 4, 6, 105\rangle, \langle T_{12}, 4, 13, 14\rangle, \langle T_{12}, 2, 3, 4, 45\rangle, \langle T_{11}, 2, 3, 7, 12\rangle, \langle T_{10}, 2, 2, 3, 33\rangle\}$;
- $S(16) = \{\langle T_{13}, 2, 16, 344\rangle, \langle T_{13}, 2, 22, 62\rangle, \langle T_{13}, 4, 6, 232\rangle, \langle T_{13}, 4, 8, 38\rangle, \langle T_{13}, 8, 8, 10\rangle, \langle T_{12}, 2, 2, 6, 107\rangle, \langle T_{12}, 2, 2, 7, 46\rangle, \langle T_{12}, 2, 3, 6, 18\rangle, \langle T_{11}, 2, 2, 3, 46\rangle\}$.
In particular, we have the following values of $|S(n)|$.

| $n$ | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $|S(n)|$ | 3   | 2   | 7   | 4   | 7   | 5   | 5   | 10  | 26  | 10  | 9   | 10  | 13  | 9   |

Table 1. The number of elements of $S(n)$ for $n \leq 16$.

Moreover, for each $n \in \mathbb{N}_{\geq 3}$ we have $S(n) \neq \emptyset$.

Proof. To find all elements of $S(n)$ we used a simple computer search. More precisely, we solved the equation (1) for $x_n$ and get

$$x_n = \frac{\sigma_2(X_{n-1})}{\sigma_{n-1}(X_{n-1}) - \sigma_1(X_{n-1})}.$$

Next, for given $n$, we computed, via Lemma 2.3, the number of 1’s in the solution of (1) and used the bounds

$$x_{n-1} \leq 1 + \frac{1}{2}n(3n - 5), \quad x_{n-2} \leq 1 + \left\lfloor \frac{2(n - 2)^{2/3}}{2} \right\rfloor$$

to check all possibilities $x_1 \leq x_2 \leq \ldots \leq x_{n-2} \leq x_{n-1}$ for which $x_n$ computed above is a positive integer.

To get the last statement it is enough to note that for $n \geq 3$ we have $(1, \ldots, 1, 2, n, n(3n - 5)) \in S(n)$. All these computations took less than 2 hours on a laptop with 32 GB of RAM and i7 type processor. □

Corollary 2.15. We have

$$\limsup_{n \to +\infty} \max_{X_n \in S(n)} \frac{x_n}{x_{n-1}} = +\infty.$$

Proof. Because for each $n \in \mathbb{N}_{\geq 3}$ we have $(1, \ldots, 1, 2, n, n(3n - 5)/2) \in S(n)$ we get

$$\limsup_{n \to +\infty} \max_{X_n \in S(n)} \frac{x_n}{x_{n-1}} \geq \limsup_{n \to +\infty} \frac{n(3n - 5)}{2n} = +\infty,$$

and hence the result. □

3. ANALYSIS OF $S_3(n)$ AND APPLICATIONS

From Theorem 2.3 we know that $(T_{n-3}, x, y, z) \in S_3(n)$ if and only if

$$y = \frac{n + d_1 - 2}{x - 1} + 1, \quad z = \frac{n + d_2 - 2}{x - 1} + 1,$$

where

$$d_1d_2 = \frac{1}{2}(n - 2)((x + 1)n + 2x^2 - 3x - 3) = f(n, (T_{n-3}, x)) =: f(n, x).$$

It is clear that $|S_3(n)|$ is finite because from Corollary 2.12 we know that $x \leq 1 + \left\lfloor 2(n - 2)^{2/3} \right\rfloor$. Thus, we can even present a crude upper bound

$$|S_3(n)| \leq \sum_{2 \leq x \leq 1 + \left\lfloor 2(n - 2)^{2/3} \right\rfloor} \tau(f(n, x)),$$

where $\tau(m)$ is the number of positive integer divisors of $m$. 

Theorem 3.2. More precisely, the following is true.
\[
\begin{align*}
d &\leq d_k
\text{ and } \limsup n = 299.
\end{align*}
\]

In the considered range the value of \(|S_3(n)|\) attains maximum equal to 213 for \(n = 299\).

From Theorem 2.14 we know that \(S(n)\) is nonempty. We prove that \(|S(n)| \geq 3\).

More precisely, the following is true.

Theorem 3.2. (1) For \(n = 3\) and each \(n \in \mathbb{N}_{\geq 5}\) we have \(|S_3(n)| \geq 3\).

(2) We have
\[
|S_3(n)| \geq \frac{1}{2} \tau \left( \frac{1}{2} (n-2)(3n-1) \right).
\]

In particular \(\limsup_{n \to +\infty} |S_3(n)| = \limsup_{n \to +\infty} |S(n)| + \infty\).

Proof. To get the first statement we note that \(|S_3(3)| = 3\), \(|S_3(5)| = 3\), \(|S_3(6)| = 4\), \(|S_3(7)| = 6\), \(|S_3(8)| = 3\), i.e., we can assume that \(n \geq 9\). We observe that for each \(n \geq 5\), the \(n\)-tuple \((T_{n-3}, 2, n, \frac{1}{2} n(3n - 5)) \in S_3(n)\). If \(n \equiv 1 \pmod{2}\) we have two additional \(n\)-tuples
\[
\left( T_{n-3}, 2, 2n - 3, \frac{1}{2} (5n - 3) \right), \quad \left( T_{n-3}, 3, n - 1, \frac{3}{2} (n + 1) \right).
\]

Similarly, if \(n \equiv 0 \pmod{2}\) we have the following additional solutions
\[
\left( T_{n-3}, 2, \frac{1}{2} (3n - 4), 2(2n - 1) \right), \quad \left( T_{n-3}, 4, \frac{1}{2} n, 2(n + 3) \right).
\]

The second statement is a consequence of the following reasoning. Take \(x = 2\) in (13). Then the corresponding values of \(y, z\) are
\[
y = n + d_1 - 1, \quad z = n + \frac{f(n, 2)}{d_1} - 1
\]
and are integers. Thus, the number of different pairs satisfying \(y \leq z\) is equal to \(\left\lceil \frac{1}{2} \tau (f(n, 2)) \right\rceil\) and we get the required inequality. Moreover, if we take \(n \equiv 2 \pmod{2q_1 \ldots q_k}\), where \(k \in \mathbb{N}_+\) and \(q_1 < \ldots < q_k\) are primes, then \(\frac{1}{2} \tau (f(n, 2)) \geq k\) and \(\limsup_{n \to +\infty} |S_3(n)| = +\infty\). Thus \(\limsup_{n \to +\infty} |S(n)| = +\infty\).

\(\square\)

Remark 3.3. Let us note that if we take \(x = 3\) in (13) then the necessary and sufficient condition for \(n\) in order to get \((T_{n-3}, 3, y, z) \in S(n)\) is \(n \not\equiv 0 \pmod{4}\).

We are in position to compute the value of \(\liminf_{x_n \in S(n)} x_n / x_{n-1}\). More precisely, the following is true.

Theorem 3.4. There are infinitely many pairs \((n, X)\) of positive integers such that \((T_{n-3}, 2, X, X) \in S_3(n) \subset S(n)\). In particular, we have
\[
\liminf_{n \to +\infty} \min_{x_n \in S(n)} x_n / x_{n-1} = 1.
\]
ON THE DIOPHANTINE EQUATION $\sigma_2(\overline{X}_n) = \sigma_n(\overline{X}_n)$

Proof. To get the result we show that there are infinitely many values of $n$ such that there is an integer $X$ such that $(\bar{T}_{n-3}, 2, X, X) \in S_3(n) \subset S(n)$. We know that this is enough to show that $d^2 = f(n, 2)$ has infinitely many solutions in positive integers. Then $X = n + d - 1$ is a solution we are looking for. The equation $d^2 = f(n, 2)$ is equivalent with Pell type equation $Y^2 - 24d^2 = 25$, where $Y = 6m - 7$. Using standard method, one can check that for each $m \in \mathbb{N}$ the pair $(d_m, Y_m)$, where $d_m = f_m(1, 10), Y_m = f_m(5, 49)$ for $m = 0, 1, \ldots$ and $f_m(a, b) = 10f_{m-1}(a, b) - f_{m-2}(a, b)$ with $f_0(a, b) = a, f_1(a, b) = b$, satisfies $Y_m^2 - 24d_m^2 = 1$ and hence the pair $(5d_m, 5Y_m)$ solves the equation $Y^2 - 24d^2 = 25$. To get integer values of $n$ we need to take $m \equiv 1 \pmod{2}$. With $n$ chosen in this way the equation $\sigma_2(\overline{X}_n) = \sigma_n(\overline{X}_n)$ has the solution

$$\overline{X}_{n(m)} = (\bar{T}_{n(m)-3}, 2, n(m) + d_{2m+1} - 1, n(m) + d_{2m+1} - 1), m = 0, 1, \ldots,$$

satisfying $x_{n-1} = x_n$ and hence the result. \hfill \Box

As an example, we computed the values of $n = n(m)$ and the corresponding solution $x_n$ for $m = 0, 1, \ldots, 5$.

| $m$ | 0  | 1  | 2  | 3  | 4  | 5  |
|-----|----|----|----|----|----|----|
| $n$ | 42 | 402 | 392042 | 38416002 | 3764376042 | 368870436002 |
| $x_n$ | 91 | 8901 | 872191 | 85465801 | 8374776291 | 820642610701 |

Table 2. Some values of $n$ such that there is an $x_n$ satisfying $(\bar{T}_{n-3}, 2, x_n, x_n) \in S(n)$.

In Theorem 3.4 we proved that $\lim \inf \{x_n/x_{n-1} : \overline{X}_n \in S(n)\}$ is equal to 1. In this context it is natural to investigate the set

$$D := \left\{ \frac{x_n}{x_{n-1}} : \overline{X}_n \in S(n), n \in \mathbb{N}_{\geq 3} \right\} \subset \mathbb{Q}$$

and the set

$$D_i := \left\{ \frac{x_n}{x_{n-1}} : \overline{X}_n \in S_i(n) \right\} \subset \mathbb{Q}$$

where $i \in \mathbb{N}_{\geq 3}$, and ask whether $D$ and $D_i$ are dense in the set $[1, +\infty)$.

We prove the following.

**Theorem 3.5.** The sets $D_3$ and $D$ are dense in $[1, +\infty)$.

Proof. It suffices to show that $D_3$ is dense in $[1, +\infty)$ as $D \supset D_3$.

One can check that the set of limit points of $D_3$ contains the set

$$\left\{ \frac{\sqrt{24ab} + b}{\sqrt{24ab} + a} : a, b \in \mathbb{N}_+, \ 24ab \text{ is not a square} \right\}$$

Indeed, if $(\bar{T}_{n-3}, x, x_{n-1}, x_n) \in S_3(n)$ and we take $x = 2$ in (13) then $x_{n-1} = n + d_1 - 1, x_n = n + d_2 - 1$ with $d_1d_2 = f(n, 2)$. If we put $d_1 = at$ and $d_2 = bt$ then we deal with Pell type equation $abt^2 = f(n, 2)$ or equivalently

$$u^2 - 24abt^2 = 25,$$

where $u = 6n - 7$. From the theory of Pell equations we know that this equation has infinitely many solutions in positive integers provided that $24ab$ is not a square of an integer. Moreover, since $(u, t) = (5, 0)$ is a solution of (14), there are infinitely
many solutions of (14) with \( u \equiv 5 \pmod{6} \), which ensures the integrality of \( n \). Hence, \((d_1, d_2, n)\) satisfies
\[
d_1 \sim a \frac{6n - 7}{\sqrt{24ab}}, \quad d_2 \sim b \frac{6n - 7}{\sqrt{24ab}}, \quad n \to +\infty.
\]
In consequence
\[
\lim_{n \to +\infty} \frac{x_n}{x_{n-1}} = \lim_{n \to +\infty} \frac{n + d_2 - 1}{n + d_1 - 1} = \lim_{n \to +\infty} \frac{n + b \frac{6n - 7}{\sqrt{24ab}} - 1}{n + a \frac{6n - 7}{\sqrt{24ab}} - 1} = \frac{\sqrt{24ab} + 6b}{\sqrt{24ab} + 3a}.
\]
Since the function \([1, +\infty) \ni x \mapsto \sqrt{\frac{6x + 1}{6x+2}} \in [1, +\infty)\) is a continuous surjection, we infer that the set
\[
\left\{ \frac{\sqrt{24ab} + b}{\sqrt{24ab} + a} : a, b \in \mathbb{N}_+, \ 24ab \text{ is not a square} \right\}
\]
is dense in \([1, +\infty)\). Hence, the closure of \(D_3\) is the whole interval \([1, +\infty)\). \(\square\)

4. Questions and conjectures

In this section we collect some questions and conjectures which appeared during various stages of our investigations.

The most important (and probably the most difficult) is the following

**Question 4.1.** What is the order of growth of the number \(|S(n)|\)? Is it true that there is \( \varepsilon > 0 \) such that \(|S(n)| > n^\varepsilon \)? In particular, is the equality
\[
\liminf_{n \to +\infty} |S(n)| = +\infty
\]
true?

This is a difficult question. We believe that the inequality \(|S(n)| > \log n\) is true. In this context one can also ask the following questions.

**Question 4.2.** What is the average order of the function \(|S(n)|\)?

**Question 4.3.** Is the function \(|S(n)|\) eventually increasing?

For given \( n \in \mathbb{N}_{\geq 3} \) and \( \overline{X}_n \in S(n) \) let us put \( X_n = \{x_1, \ldots, x_n\} \) and
\[
M(n) : = \min \{|X_n| : \overline{X}_n \in S(n)\}.
\]
From emptiness of \( S_i(n) \) for \( i = 0, 1, 2 \) and our investigations concerning the structure of the set \( S_3(n) \), or to be more precise the proof of Theorem 3.4, we know that
\[
\liminf_{n \to +\infty} M(n) \leq 3.
\]
Let \( n \geq 6 \). Then \( M(n) = 1 \) is impossible by Corollary 2.4. Moreover, \( M(n) = 2 \) if and only if there is \( k \in \{1, \ldots, n-1\} \) and \( x \in \mathbb{N}_{\geq 2} \) such that
\[
\sigma_2(\overline{T_{n-k}, x_k}) = \sigma_n(\overline{T_{n-k}, \overline{x_k}}),
\]
where \( \overline{x_k} = (x, \ldots, x) \), where we have \( k \)-occurrences of \( x \).
Equivalently, we deal with the equation
\[ y^2 = 8x^k + 4kx^2 - 4kx + 1 =: P_k(x), \]
where \( y = 2n + 2k(x - 1) - 1 \). Let \( C_k \) denote the curve defined by the equation above. One can check, using [3] Theorem 3.1], that the discriminant of \( P_k(x) \) is non-zero, and thus the polynomial \( P_k \) has no multiple roots (moreover, since the Newton polygon of \( P_k \) with respect to 2-adic valuation has only one slope, equal to \( 3/k \), we see that for each \( k \in \mathbb{N}_{\geq 3} \) not divisible by 3 the polynomial \( P_k \) is irreducible over the field \( \mathbb{Q}_2 \) of 2-adic numbers and thus irreducible over \( \mathbb{Q} \). In consequence, the genus of the curve \( C_k \) is equal to \( \lceil (k - 1)/2 \rceil \). For \( k = 3 \) the curve \( C_k \) is an elliptic curve and we have \( C_k(\mathbb{Q}) \simeq \mathbb{Z} \times \mathbb{Z}_3 \), where the infinite part is generated by the point \( P = (8, 24) \), and the torsion part is generated by \( T = (0, 1) \). Using standard methods one can check that the only integer points on \( C_3 \) which lead to solutions of the equation (15) are: \( (8, 24), (172, 83679) \).

The point \( (7, 141) \) gives the solution \( (k, n, x) = (4, 47, 7) \) and the point \( (172, 83679) \) gives the solution \( (k, n, x) = (4, 41156, 172) \) of the equation (15). If \( k \geq 5 \) then the curve \( C_k \) is of genus \( \geq 2 \), and from the Faltings theorem we know that the set \( C_k(\mathbb{Q}) \) is finite. From numerical calculations we expect that besides the trivial point \( (0, \pm 1) \) there is no additional integral points on \( C_k \). We thus formulate the following.

**Conjecture 4.4.** The only values of \( n \in \mathbb{N}_{\geq 3} \) such that \( M(n) = 2 \) are \( n = 3, 4, 5, 11, 41156 \). In particular, \( \liminf_{n \to +\infty} M(n) = 3 \).

Since \( (1, n - 3, 2, n, \frac{1}{2}n(3n - 5)) \in S(n) \) for each \( n \in \mathbb{N}_{\geq 3} \), we know that
\[
\limsup_{n \to +\infty} M(n) \leq 4.
\]

We suppose that the condition \( M(n) = 3 \) is rarely satisfied.

**Conjecture 4.5.** We have \( \limsup_{n \to +\infty} M(n) = 4 \).

In this context we also prove the following.

**Theorem 4.6.** For each \( m \in \mathbb{N}_{\geq 2} \) and each sequence \( \overline{Y}_m = (y_1, y_2, \ldots, y_m) \in \mathbb{N}_{\geq 2}^m \backslash \{(2, 2), (2, 3)\} \) there are \( k, Y \in \mathbb{N}_+ \) such that for \( n = k + m + 1 \) we have \( (\overline{X}_k, \overline{Y}_m, Y) \in S(n) \). In particular,
\[
\limsup_{n \to +\infty} \max_{X_n \in S(n)} |X_n| = +\infty.
\]

**Proof.** To find suitable values of \( k, Y \) we consider the equation
\[
\sigma_2(\overline{T}_k, \overline{Y}_m, Y) = \sigma_2(\overline{Y}_m, Y) = \sigma_m(\overline{Y}_m)Y,
\]
which is equivalent with the equation
\[
\sigma_2(\overline{T}_k) + \sigma_2(\overline{Y}_m) + \sigma_1(\overline{T}_k)\sigma_1(\overline{Y}_m) + (\sigma_1(\overline{T}_k) + \sigma_1(\overline{Y}_m))Y = \sigma_m(\overline{Y}_m)Y.
\]
i.e.
\[
\binom{k}{2} + \sigma_2(Y_m) + k\sigma_1(Y_m) = (\sigma_m(Y_m) - k - \sigma_1(Y_m))Y.
\]
Thus, by taking \( k = \sigma_m(Y_m) - \sigma_1(Y_m) - 1 \) we get the corresponding value of \( Y = \binom{k}{2} + \sigma_2(Y_m) + k\sigma_1(Y_m) \). Under our assumption on \( Y_m \) it is clear that \( k > 0 \) (for an easy proof of this fact see [2]) and we get required solution.

To get the second statement it is enough to take \( Y_m = (2, \ldots, m+1) \). Then \( k = (m+1)! - \binom{m+2}{2} \) and \( n = (m+1)! - \binom{m+2}{2} + m+1 \). The corresponding element \( X_n \in S(n) \) satisfies \( |X_n| = m+2 \) and hence the result.

**Problem 4.7.** For which \( a \in \mathbb{N}_{\geq 4} \) the equation \( \max_{X_n \in S(n)} |X_n| = a \) has infinitely many solutions?

Our expectation is that for each \( a \in \mathbb{N}_{\geq 4} \) the equation \( \max_{X_n \in S(n)} |X_n| = a \) has infinitely many solutions.

We firmly believe that many results proved in this paper can be appropriately generalized in the context of the Diophantine equation

\[\sigma_i(X_n) = \sigma_n(X_n),\]

where \( i \in \{3, \ldots, n\} \) is fixed. However, even the case \( i = 3 \) will require new ideas. For example, based on numerical calculations we formulate the following.

**Conjecture 4.8.** Let \( n \in \mathbb{N}_{\geq 4} \) and suppose that \( m \in \mathbb{N} \) satisfies

\[m = \sigma_3(X_n) = \sigma_n(X_n).\]

Then

\[m \leq \frac{1}{12} \left(3021n^6 - 77575n^5 + 920361n^4 - 6235705n^3 + 24764202n^2 - 53664304n + 48986640\right)\]

and

\[x_n \leq \frac{1}{12} \left(27n^4 - 190n^3 + 471n^2 - 500n + 216\right).

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