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Star-Free Languages are Church-Rosser Congruential

Volker Diekert*  Manfred Kufleitner†  Pascal Weil‡

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Abstract. The class of Church-Rosser congruential languages has been introduced by McNaughton, Narendran, and Otto in 1988. A language \( L \) is Church-Rosser congruential (belongs to CRCL), if there is a finite, confluent, and length-reducing semi-Thue system \( S \) such that \( L \) is a finite union of congruence classes modulo \( S \). To date, it is still open whether every regular language is in CRCL. In this paper, we show that every star-free language is in CRCL. In fact, we prove a stronger statement: For every star-free language \( L \) there exists a finite, confluent, and subword-reducing semi-Thue system \( S \) such that the total number of congruence classes modulo \( S \) is finite and such that \( L \) is a union of congruence classes modulo \( S \). The construction turns out to be effective.

Keywords. String rewriting; Church-Rosser system; star-free language; aperiodic monoid; local divisor.

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1 Introduction

Church-Rosser congruential languages (CRCL) are a nonterminal-free form of Church-Rosser languages (CRL). Both classes have been defined in [9], and it was shown there that CRCL forms a proper subclass in CRL. Languages in CRL enjoy various nice properties. For example their word problem is decidable in linear time. A detailed discussion with links to further references can be found in the PhD-thesis of Niemann [12], see also [13]. We content ourselves to define CRCL: A language \( L \in A^* \) is called a Church-Rosser congruential language, if there is a finite, length-reducing, and confluent semi-Thue system \( S \subseteq A^* \times A^* \) such that \( L \) is a finite union of congruence classes modulo \( S \). This means that \( L \) contains a finite set \( F \) of shortest words such that we have \( w \in L \) if and only if every rewriting procedure starting on \( w \) and using \( S \) terminates in one of the finitely many words in \( F \).

It was also shown in [9] that all deterministic context-free languages are Church-Rosser. However, surprisingly it is not known whether all regular languages are CRCL. The general conjecture is “yes”, but so far only partial results have been established as in [14]. The most advanced result has been announced by Reinhardt and Thérien [15]: According to their manuscript, if a regular language has a group as its syntactic monoid, then this language is in CRCL.

In this note we consider the complementary class of group-free regular languages; and we show that they belong to CRCL. A regular language is group-free if its syntactic monoid is group-free. This means it is aperiodic. There are many other characterizations for this class. A fundamental result of Schützenberger says that the class of aperiodic language \( AP(A) \) is exactly the same as the class of star-free languages \( SF(A) \) [17]. It is the class where the Krohn-Rhodes decomposition leads to a wreath product of the three-element commutative idempotent reset-monoid \( U_2 \) [8]. It is also the class \( FO(A, <) \) of languages definable in first-order logic [10]; and this is the same as the class \( LTL(A) \) of languages definable in the linear temporal logic [7].

A proof that \( FO(A, <) = SF(A) = AP(A) = LTL(A) \) can be conveniently arranged in a cycle. The inclusion \( FO(A, <) \subseteq SF(A) \) can be explained very nicely with Ehrenfeucht-Fraïssé-games [5]. The inclusion \( SF(A) \subseteq AP(A) \) follows Schützenberger’s original idea. The inclusion \( AP(A) \subseteq LTL(A) \) is done in the survey [3] with the concept of local divisors which play a prominent role here, too. The final inclusion \( LTL(A) \subseteq FO(A, <) \) is trivial.

Coming back to the class of Church-Rosser congruential languages, our main result shows \( SF(A) \subseteq CRCL \). Actually, we prove a much stronger result. First we define subword-reducing semi-Thue systems which are a proper subclass of finite length-reducing semi-Thue systems. For every language \( L \in AP(A) \) we effectively construct a finite subword-reducing confluent semi-Thue system \( S \subseteq A^* \times A^* \) such
that the total number of congruence classes modulo \( S \) is finite and \( L \) is a union of such classes, see Theorem 6. A main tool in our proof is the notion of local divisor, see Section 3 for a definition.

In the final section of this paper, Section 5, we explain our constructions in a rigorously algebraic framework. This part is mainly intended for possible future work.

In order to give a complete positive solution to the conjecture that all regular languages are CRCL, it remains to combine our approach with the one in [15]. There are however quite a number of obstacles for a fruitful combination. So, we leave the general conjecture as a challenging research problem.

2 Preliminaries and Notation

In the following \( A \) means a finite alphabet, an element of \( A \) is called a letter, and \( A^* \) denotes the free monoid generated by \( A \). It is the set of words over \( A \). The empty word is denoted by 1. The length of a word \( u \) is denoted by \( |u| \). We have \( |u| = n \) for \( u = a_1 \cdots a_n \) where \( a_i \in A \). The empty word has length 0. We carefully distinguish between the notion of factor and subword. Let \( u, v \in A^* \). The word \( u \) is called a factor of \( v \) if there is a factorization \( v = xuy \). It is called a subword of \( v \) if there is a factorization \( v = x_0u_1x_1 \cdots u_kx_k \) such that \( u = u_1 \cdots u_k \). A subword is also sometimes called a scattered subword in the literature.

A semi-Thue system over \( A \) is a subset \( S \subseteq A^* \times A^* \). The elements are called rules. We frequently write \( \ell \rightarrow r \) for rules \((\ell,r)\). A system \( S \) is called length-reducing if we have \( |\ell| > |r| \) for all rules \((\ell,r)\in S\). It is called subword-reducing, if \( r \) is a subword of \( \ell \) and \( \ell \neq r \) for all rules \((\ell,r)\in S\). Every subword-reducing system is length-reducing, but not vice versa.

Every system \( S \) defines the rewriting relation \( \Rightarrow_S \subseteq A^* \times A^* \) by

\[
u \Rightarrow_S v \quad \text{if} \quad u = plq, v = prq \quad \text{for some rule} \quad (\ell,r) \in S.
\]

By \( \Rightarrow^*_S \) we mean the reflexive and transitive closure of \( \Rightarrow_S \). By \( \Leftarrow^*_S \) we mean the symmetric, reflexive, and transitive closure of \( \Rightarrow_S \). We also write \( u \Leftarrow^*_S v \) whenever \( v \Rightarrow^*_S u \). The system \( S \) is confluent if for all \( u \Leftarrow^*_S v \) there is some \( w \) such that \( u \Rightarrow^*_S w \Leftarrow^*_S v \).

Note that \( u \Rightarrow^*_S v \) implies that \( |u| > |v| \) for length-reducing systems. For subword-reducing systems it implies that the set of subwords in \( v \) is a proper subset of the set of subwords in \( u \).

By \( \text{IRR}(S) \) we denote the set of irreducible words, i.e., the set of words where no left-hand side occurs as any factor. The relation \( \Leftarrow^*_S \subseteq A^* \times A^* \) is a congruence,
hence the congruence classes \([u]_S = \{ v \in A^* \mid u \overset{s}{\rightarrow} v \}\) form a monoid which is denoted by \(A^*/S\). A finite semi-Thue system \(S\) can be viewed as a finite set of defining relations. Hence, \(A^*/S\) becomes a finitely presented monoid.

**Definition 1.** A semi-Thue system \(S\) is called a Church-Rosser system if it is length-reducing and confluent. A language \(L \subseteq A^*\) is called a Church-Rosser congruential language if there is a finite Church-Rosser system \(S\) such that \(L\) can be written as a finite union of congruence classes \([u]_S\).

**Remark 2.** A semi-Thue system \(S\) is a Church-Rosser system if and only if (1) it is length-reducing and (2) every congruence class has exactly one irreducible element.

Let \(\pi : A^* \rightarrow A^*/S, u \mapsto [u]_S\) be the canonical homomorphism and \(S\) be a finite Church-Rosser system. Then \(\pi^{-1}(K)\) is a Church-Rosser congruential language as soon as \(K\) is finite.

**Conjecture 1.** Every regular language is a Church-Rosser congruential language.

**Example 3.** Consider the language \(L = (bc)^+\). A Church-Rosser system for \(L\) is given by the one-rule semi-Thue system \(S = \{ cbc \rightarrow c \}\). The monoid \(\{b, c\}^*/S\) is infinite. However \(L = [bc]_S\); and hence \(u \in L\) if and only if \(u \overset{s}{\rightarrow} bc\).

A manuscript of Reinhardt and Thérien [15] says that Conjecture [1] is true in case that the syntactic monoid of the regular language is a group. Here, we are going to prove an even stronger result for aperiodic languages, i.e., for languages where the syntactic monoid is group-free. As our proof uses subword-reducing systems in the induction hypothesis, we cannot incorporate the statement of Reinhardt and Thérien (using length-reducing rather than subword-reducing systems) as base in our induction scheme. So the status of Conjecture [1] remains open, in general.

**Definition 4.** Let \(\varphi : A^* \rightarrow M\) be a homomorphism to a finite monoid \(M\). We say \(\varphi\) factorizes through a finite Church-Rosser monoid \(A^*/S\) if there is a finite Church-Rosser system \(S\) such that \(A^*/S\) is a finite monoid and \([u]_S \subseteq \varphi^{-1}(\varphi(u))\) for all \(u \in A^*\).

A classical fact states that a language \(L \subseteq A^*\) is regular if and only if it is recognizable, i.e., there is a homomorphism \(\varphi : A^* \rightarrow M\) to a finite monoid \(M\) such that \(L = \varphi^{-1}(\varphi(L))\). We also say that \(\varphi\) (or that \(M\)) recognizes \(L\). Recall that a finite monoid \(M\) is called aperiodic if there exists some \(n \in \mathbb{N}\) such that \(x^n = x^{n+1}\) for all \(x \in M\). Accordingly, a language \(L \subseteq A^*\) is called aperiodic if it is recognized by some finite aperiodic monoid \(M\).
Note that if $\varphi$ factorizes through a finite Church-Rosser monoid, then we have

$$\varphi : A^* \xrightarrow{\pi} A^*/S \xrightarrow{\psi} M,$$

where $S$ is a Church-Rosser system such that $A^*/S$ is a finite.

**Conjecture 2.** Let $\varphi : A^* \to M$ be a homomorphism to a finite monoid $M$. Then $\varphi$ factorizes through a finite Church-Rosser monoid.

Conjecture 2 is stronger than Conjecture 1. However, we believe that a positive solution to Conjecture 1 comes through a proof of Conjecture 2. Actually, the result in [15] also announces that Conjecture 2 is true for finite groups. We are going to show here that an even stronger statement than Conjecture 2 holds for finite aperiodic monoids.

**Example 5.** Consider again the language $L = (bc)^+$ from Example 3. Another Church-Rosser system for $L$ is given by

$$S = \{bbb \rightarrow bb, bbc \rightarrow bb, cbb \rightarrow bb, ccc \rightarrow bb, ccb \rightarrow bb, bcc \rightarrow bb, bcb \rightarrow b, cbc \rightarrow c\}.$$

As in Example 3 we have $L = [bc]_S$; but here, the monoid $\{b, c\}^*/S$ is finite. It has 7 elements: $[1]_S$, $[b]_S$, $[c]_S$, $[bc]_S$, $[cb]_S$, $[bb]_S$, and $[cc]_S$. Note that $S$ is not subword-reducing.

### 3 Local divisors

The notion of *local divisor* dates back to a technical report of Meyberg where he introduced this concept in commutative algebra, see [6, 11]. In finite semigroup theory and formal language theory the explicit definition of a local divisor appeared first in [2]. Since then it turned out to be a very useful tool for simplifying classical proofs like in [3, 4] or in finding new results like in this paper. The definition of a local divisor extends the definition of a Schützenberger group for the $H$-class of an arbitrary element, [1, 16]. A category generalization is being used by Steinberg and Costa in the context of symbolic dynamics (unpublished).

In this paper we use local divisors for aperiodic monoids, only. Let $M$ be a monoid and let $c \in M$. We put on the subsemigroup $cM \cap Mc$ a monoid structure by defining a new multiplication $\circ$ as follows:

$$xc \circ cy = xcy.$$
It is straightforward to see that $\circ$ is well-defined and $(cM \cap Mc, \circ)$ is a monoid with neutral element $c$.

The following observation is crucial: If the monoid $M$ is finite and aperiodic, then $|cM \cap Mc| < |M|$ whenever $c \neq 1$. This is clear, because $1 \in cM \cap Mc$ implies that $c$ is a unit of $M$, but $c \neq 1$ and there are no non-trivial units in aperiodic monoids. The set $M' = \{x \mid cx \in Mc\}$ is a submonoid of $M$, and $c \cdot : M' \to cM \cap Mc : x \mapsto cx$ is a surjective homomorphism. In particular, if $M$ is aperiodic, then $(cM \cap Mc, \circ)$ is aperiodic, too. Since $(cM \cap Mc, \circ)$ is the homomorphic image of a submonoid it is a divisor of $M$. We therefore call $(cM \cap Mc, \circ)$ the local divisor of $M$ at $c$. Note that if $c = c^2$ is an idempotent, then $(cM \cap Mc, \circ) = (Mc, \cdot)$ is the usual local monoid defined by the subsemigroup $cMc$ of $M$. Thus, the notion of local divisor generalizes the notion of local monoid from idempotents to arbitrary elements.

4 Conjecture 2 holds for aperiodic monoids

We have the following result.

**Theorem 6.** Let $\varphi : A^* \to M$ be a homomorphism to a finite aperiodic monoid $M$. Then $\varphi$ factorizes through a finite aperiodic Church-Rosser monoid $A^*/S$ where $S$ is subword-reducing.

The rest of this section is devoted to the proof of Theorem 6. The proof is by induction on the parameter $(|M|, |A|)$ with lexicographic order. The result is true if $\varphi(A^*)$ is trivial. Note that this covers $M = \{1\}$ as well as $A = \emptyset$. In the remaining case there is a letter $c \in A$ such that $\varphi(c) \neq 1$. We let $B = A \setminus \{c\}$, and for better reading we identify $c$ and $\varphi(c) \in M$. Since $c \neq 1 \in M$ and $M$ is aperiodic, $c$ is not a unit. Hence $M_c = cM \cap Mc$ has less elements than $M$.

Since $|B| < |A|$ we find, by induction, a finite subword-reducing Church-Rosser system $R \subseteq B^* \times B^*$ such that the restriction $\varphi|_{B^*} : B^* \to M$ factorizes through a finite Church-Rosser monoid $B^*/R$. In particular, $(\ell, r) \in R$ implies $\varphi(\ell) = \varphi(r)$.

For $u \in B^*$ let $\widehat{u}$ denote the unique word such that $\widehat{u} \in \text{IRR}(R)$ and $u \overset{R}{\Rightarrow} \widehat{u}$. The subset $K = \text{IRR}(R)c \subseteq A^*$ is a finite code. This means that $K^*$ is freely generated, as a submonoid of $A^*$, by the finite set $K$. Note that $K^+ \subseteq A^*c$. Consider the homomorphism $\psi : K^* \to (M_c, \circ)$ which is given by $\psi(\widehat{u}c) = c\varphi(u)c$. We have $c\varphi(u)c = \varphi(c\widehat{u}c)$. In particular, $\psi$ is well-defined. By induction $\psi : K^* \to (M_c, \circ)$ factorizes through a finite aperiodic Church-Rosser monoid $K^*/T$, where $T \subseteq K^* \times K^*$ is a finite subword-reducing Church-Rosser system.

Consider a rule $(\ell, r) \in T$. It has the form

$$\widehat{u}_1 c \cdot \widehat{u}_m c \longrightarrow \widehat{v}_1 c \cdot \widehat{v}_n c$$
where the \( \hat{u}_i c \) and \( \hat{v}_j c \) are letters in \( K \), every right-hand side \( \hat{u}_1 c \cdots \hat{u}_n c \in K^* \) is a proper subword of \( \hat{u}_1 c \cdots \hat{u}_m c \in K^+ \). Since \( K^* \subseteq A^* \) we can read \( T \) as a semi-Thue system over \( A \) as well. Next, we define a new system \( \hat{T} \subseteq A^* \times A^* \) as follows:

\[
\hat{T} = \{ c\ell \rightarrow cr \mid (\ell, r) \in T \},
\]

We collect some important properties of \( \hat{T} \) in a remark:

**Remark 7.** The semi-Thue system \( \hat{T} \subseteq A^* \times A^* \) satisfies the following assertions.

1. \( \hat{T} \) is subword-reducing, because \( T \) has this property. This is crucial. Knowing only that \( T \) is length-reducing as a system over \( K^* \) would not be enough to conclude that \( \hat{T} \) is length-reducing as a system over \( A^* \).

2. \( \hat{T} \) is confluent. For this it is crucial that we added a letter \( c \) on the left. This allows to read the words \( \hat{u} c \) as letters in \( K \) and the confluence of \( T \) transfers to the confluence of \( \hat{T} \). If there was no \( c \) on the left, then \( T \) could contain rules \( abc \rightarrow 1 \) and \( bc \rightarrow 1 \), but \( a \) is no left-hand side in \( T \). Over \( K \) the words \( abc \) and \( bc \) are letters, hence there is no overlap in \( K^* \).

3. \( \ell \rightarrow r \in \hat{T} \) implies \( \varphi(\ell) = \varphi(r) \). This is a straightforward calculation in local divisors: Let \( \ell = cu_1 c \cdots u_m c \) and \( r = cv_1 c \cdots v_n c \) with \( u_i, v_i \in \text{IRR}(R) \). By induction, we have \( \psi(\ell) = \psi(r) \) and thus

\[
\varphi(\ell) = \varphi(cu_1 c) \circ \cdots \circ \varphi(cu_m c)
= \psi(u_1 c) \circ \cdots \circ \psi(u_m c)
= \psi(u_1 c \cdots u_m c) = \psi(v_1 c \cdots v_n c)
= \psi(v_1 c) \circ \cdots \circ \psi(v_n c)
= \varphi(cv_1 c) \circ \cdots \circ \varphi(cv_n c) = \varphi(r).
\]

The proof of Theorem 6 is now a direct consequence of the following lemma which shows that the system \( S = R \cup \hat{T} \) has the desired properties.

**Lemma 8.** The semi-Thue system \( S = R \cup \hat{T} \) over \( A \) satisfies the following assertions.

1. \( S \) is subword-reducing.
2. \( S \) is confluent.
3. \( \ell \rightarrow r \in S \) implies \( \varphi(\ell) = \varphi(r) \).
4. \( A^*/S \) is a finite aperiodic monoid.

**Proof.** Assertion 1 is clear, because \( R \) and \( \hat{T} \) are subword-reducing. Assertion 2 is clear, because there is no overlap of left-hand sides between rules of \( R \) and \( \hat{T} \).
Assertion 3 is clear, because \( R \) and \( \hat{T} \) have this property. It remains to show 4. By induction \( K^*/T \) is finite. Hence there is a maximal value \( \mu \) such that every word in \( K^* \) of length at least \( \mu \) is reducible. We conclude that:

\[
\text{IRR}(S) \subseteq \{ \hat{u}_0 \hat{c}u_1 \cdots \hat{c}u_m \mid \hat{u}_i \in \text{IRR}(R) \land 0 \leq m \leq \mu \}.
\]

Since \( \text{IRR}(R) \) is finite, we see that \( \text{IRR}(S) \) is a subset of a finite set, and thus the finiteness of \( \text{IRR}(S) \) and of \( A^*/S \) follow. This leaves us to show that \( A^*/S \) is aperiodic. We have to show that there exists some \( n \in \mathbb{N} \) such that for all \( u = \hat{u}_0 \hat{c}u_1 \cdots \hat{c}u_m \in \text{IRR}(S) \) we have \( u^{n+1} \overset{*}{\rightarrow}_S u^n \). Let \( v = \hat{u}_1 c \cdots u_m u_0 c \). Then \( u^{n+1} \overset{*}{\rightarrow}_R pcv^n q \) and \( u^n \overset{*}{\rightarrow}_R pcv^{n-1} q \) for some \( p, q \in A^* \). Therefore, it is enough to show that \( cv^n \overset{*}{\rightarrow}_S cv^{n-1} \) whenever \( n \) is large enough. The \( \hat{u}_i \)’s are code words of \( K \), hence letters in the alphabet \( K \) and we can read \( v \in K^* \). Here we can use induction, and we know \( v^n \overset{*}{\rightarrow}_T v^{n-1} \) if \( n \) is large enough, because \( K^*/T \) is aperiodic. This implies \( cv^n \overset{*}{\rightarrow}_T cv^{n-1} \) and hence the result.

This completes the proof of Theorem 6.

**Example 9.** Consider again the language \( L = (bc)^+ \) from Example 3 and Example 5. Its syntactic monoid is \( M = \{ 1, b, c, bc, cb, 0 \} \) with \( bb = cc = 0 \), \( bcb = b \), \( cbc = c \), 1 is neutral, and 0 is a zero element. In particular, \( bc \) and \( cb \) are idempotent. Here, the syntactic homomorphism \( \varphi_L : \{ b, c \}^* \rightarrow M \) is induced by \( b \mapsto b \) and \( c \mapsto c \). We apply the above algorithm for obtaining a Church-Rosser monoid factorizing \( \varphi_L \).

First we choose to localize at \( c \). Then \( N = \{ 1, b, 0 \} \) is the submonoid generated by \( b \). The restriction of \( \varphi_L \) to \( b^* \) factorizes through the Church-Rosser monoid defined by the system

\[
R = \{ bbb \rightarrow bb \}.
\]

This leads to the irreducible elements \( \text{IRR}(R) = \{ 1, b, bb \} \). Now, the homomorphism \( \psi : \{ c, bc, bbc \}^* \rightarrow M_c \) is defined by \( x \mapsto cx \) for \( x \in \{ c, bc, bbc \} \). Note that we consider \( \{ c, bc, bbc \} \) as a three-letter alphabet. In particular, \( M_c = \{ c, 0 \} \) and \( c \mapsto 0 \), \( bbc \mapsto 0 \), and \( bc \mapsto c \).

For \( \psi \) we obtain the rules

\[
T = \{(c)(c) \rightarrow (c),
(b)(bc) \rightarrow 1,
(bbc)(bbc) \rightarrow (bbc),
(c)(bbc)(c) \rightarrow (c)\}
\]

This completes the proof of Theorem 6.
The parenthesis are for identifying letters of the alphabet of $\psi$. This leads to the system

\[
\hat{T} = \{ \begin{array}{c}
ccc \rightarrow cc, \\
cbc \rightarrow c, \\
ccbbcc \rightarrow cc,
\end{array}
\]

and $S = R \cup \hat{T}$ is the system for $\varphi$. In $\text{IRR}(S)$ there are 65 irreducible elements and $bbcbbccbbcbb$ is the longest one.

\[\Diamond\]

5 Algebraic constructions

The aim of this section is to place the explicit constructions from the previous Section 4 into a broader algebraic context. It shows that the quotient monoid $A^*/S$ in Lemma 8 has an algebraic interpretation.

5.1 Rees-extension monoids and Church-Rosser systems

Let $\rho : P \rightarrow Q$ be a mapping between two monoids $P$ and $Q$. We are going to define the Rees-extension monoid of $\rho$ which we shall denote by $E(\rho)$. If $\rho$ is chosen properly, then $E(\rho)$ coincides with the monoid $A^*/S$ where $S \subseteq A^* \times A^*$ is the subword-reducing confluent semi-Thue system of Lemma 8, see Proposition 10. As a carrier set for the monoid $E(\rho)$ we choose the disjoint union $P \dot{\cup} (P \times Q \times P)$. The multiplication is as follows:

- $u \cdot v = uv$ for $u, v \in P$.
- $x \cdot (u, q, v) \cdot y = (xu, q, vy)$ for $x, u, v, y \in P$ and $q \in Q$.
- $(u, q, v) \cdot (x, r, y) = (u, q \rho(vx) r, y)$ for $u, v, x, y \in P$ and $q, r \in P$.

Now, $P$ is a submonoid of $E(\rho)$ and $P \times Q \times P$ is an ideal. As a semigroup, $P \times Q \times P$ is a special case of the Rees-matrix construction, see e.g. [[1, 16]]. The mapping $\rho$ defines a $P \times P$ matrix $\mathcal{R}$ with coefficients in $Q$ by $\mathcal{R}(v, x) = \rho(vx)$; and the multiplication in $P \times Q \times P$ can be written as $(u, p, v) \cdot (x, q, y) = (u, p \mathcal{R}(v, x) q, y)$.

In the following we let $c = \rho(1) \in Q$. Multiplying triples $(1, q, 1)$ and $(1, r, 1)$ yields $(1, q, 1) \cdot (1, r, 1) = (1, q \rho(1) r, 1) = (1, qcr, 1)$. In particular, the sandwich construction $(Q, \#_c)$ appears as a subsemigroup, where $\#_c$ denotes the standard sandwich-multiplication defined by $q \#_c r = qcr$. We have $(1, 1, 1)^n = (1, c^{n-1}, 1)$ and, more general, $(u, q, v)^n = (u, (q \rho(vu))^{n-1} q, v)$ for all $n \geq 1$. It follows that $E(\rho) = P \cup (P \times Q \times P)$ is aperiodic if both $P$ and $Q$ are aperiodic.
For Proposition 10 below, we apply the Rees-extension monoid to the setting in Section 4. We start with a homomorphism \( \varphi : A^* \to M \) to a finite aperiodic monoid \( M \), the alphabet \( A \) is the disjoint union of \( B \) and \( \{ c \} \). \( P \) is the quotient \( B^*/R \), \( Q \) is the quotient \( K^*/T \) for \( K = \text{IRR}(R)c \). Since we can identify \( P = B^*/R \) and \( \text{IRR}(R) \), we define \( \rho : P \to Q \) by \( \rho(u) = [uc]_T \) for \( u \in \text{IRR}(R) \). Now, \( A^*/S \) from Section 4 and \( E(\rho) \) coincide:

**Proposition 10.** In the situation above, \( A^*/S \) and \( E(\rho) \) are isomorphic.

**Proof.** Let \( \sigma : \text{IRR}(S) \to E(\rho) \) be defined by

\[
\sigma(u_0) = u_0 \quad \text{and} \quad \sigma(u_0cu_1 \cdots cu_{k+1}) = (u_0, \rho(u_1) \cdots \rho(u_k), u_{k+1})
\]

for \( k \geq 0 \) and \( u_i \in B^* \cap \text{IRR}(R) \). Here, we indentify \( P \) with \( \text{IRR}(R) \), and \( Q \) with \( \text{IRR}(T) \). In particular, by definition of \( P \) and \( Q \), the mapping \( \sigma \) is surjective. Suppose \( \sigma(u_0cu_1 \cdots cu_{k+1}) = \sigma(v_0cv_1 \cdots cv_{\ell+1}) \) for \( k, \ell \geq 0 \) and \( u_i, v_i \in B^* \cap \text{IRR}(R) \). Then \( u_0 = v_0 \) and \( u_{k+1} = v_{\ell+1} \). Moreover, \( cu_1 \cdots u_{k}c \in \text{IRR}(S) \) and thus \( (u_1c) \cdots (u_kc) \in K^* \cap \text{IRR}(T) \). Similarly, \( (v_1c) \cdots (v_\ell c) \in K^* \cap \text{IRR}(T) \). Now, \( \rho(u_1) \cdots \rho(u_k) = \rho(v_1) \cdots \rho(v_\ell) \) implies \( (u_1c) \cdots (u_kc) = (v_1c) \cdots (v_\ell c) \) in \( K^* \) and thus \( cu_1 \cdots u_kc = cv_1 \cdots v_\ell c \) in \( A^* \). This shows \( u_0cu_1 \cdots cu_{k+1} = v_0cv_1 \cdots cv_{\ell+1} \).

We conclude that \( \sigma \) is injective.

It remains to show that \( \sigma \) is a homomorphism. Let \( u, v \in \text{IRR}(S) \) and \( uv \xrightarrow S w \in \text{IRR}(S) \), i.e., \( [u]_S[v]_S = [w]_S \). If \( u, v \in B^* \), then \( \sigma(u)\sigma(v) = w = \sigma(w) \). Let now \( w, w \in A^*cA^* \) and \( w = w_0cw_1 \cdots cw_{m+1} \). If \( u \in B^* \), \( v = v_0cv_1 \cdots cv_{\ell+1} \) and \( uv \xrightarrow S x \in \text{IRR}(S) \), then \( w_0 = x \) and \( cw_1 \cdots cw_{m+1} = cv_1 \cdots cv_{\ell+1} \). It follows \( \sigma(u)\sigma(v) = (x, \rho(w_1) \cdots \rho(w_m), w_{m+1}) = \sigma(w) \). The case \( v \in B^* \) is symmetric.

Let now \( u = u_0cu_1 \cdots cu_{k+1} \in A^*cA^* \) and \( v = v_0cv_1 \cdots cv_{\ell+1} \in A^*cA^* \) with \( u_i, v_i \in B^* \). Let \( u_{k+1}v_0 \xrightarrow S x \in \text{IRR}(S) \). Then \( u_{k+1}v_0 \xrightarrow R x \). We have \( cu_1 \cdots cu_{k+1}v_0c \cdots v_kc \xrightarrow S cw_1c \cdots w_mc \). By construction of \( S \) we see that

\[
ucu_1 \cdots cu_{k}cxv_1c \cdots v_{\ell}c \xrightarrow T cw_1c \cdots w_mc,
\]

and hence

\[
(u_1c) \cdots (u_kc)(xc)(v_1c) \cdots (v_{\ell}c) \xrightarrow T (w_1c) \cdots (w_mc),
\]

i.e., \( \rho(u_1) \cdots \rho(u_k)\rho(x)\rho(v_1) \cdots \rho(v_{\ell}) = \rho(w_1) \cdots \rho(w_m) \) in \( Q \). We conclude that \( \sigma(u)\sigma(v) = \sigma(w) \). \( \square \)
5.2 Rees-extension monoids and local divisors

Let \( \rho : P \to Q \) be arbitrary again. Observe that \( c \neq 1 \in Q \), in general. In the remainder of this section, we draw a connection between local divisors and the Rees-extension monoid. We define an alphabet \( C \) by the disjoint union \( C = (P \setminus \{1\}) \cup \{c\} \). The mapping \( \rho \) induces a homomorphism \( \tau : C^* \to E(\rho) \) by defining \( \tau(x) = x \) for \( x \in P \setminus \{1\} \) and \( \tau(c) = (1,1,1) \). By considering \( (P \setminus \{1\})^*c \) as an infinite alphabet, \( \rho \) also induces a homomorphism \( \sigma : ((P \setminus \{1\})^*c)^* \to Q \) by \( \sigma(uc) = \rho(\varepsilon(u)) \) for \( u \in (P \setminus \{1\})^* \). Here, \( \varepsilon : (P \setminus \{1\})^* \to P \) is the evaluation homomorphism.

Consider a homomorphism \( \gamma : C^* \to M \) with \( \gamma(c) = c \in M \). The aim is to find a condition such that \( \gamma \) factorizes through \( \tau : C^* \to E(\rho) \). This means we wish to write \( \gamma = \tau \psi \) for some suitable homomorphism \( \psi : E(\rho) \to M \). The condition we are looking for is statement 1 of Proposition 11.

**Proposition 11.** Let \( \gamma : C^* \to M \) be a homomorphism with \( \gamma(c) = c \in M \). If \( Q \) is generated by \( \rho(P) \), then the following assertions are equivalent.

1. For \( w, w' \in ((P \setminus \{1\})c)^* \) the equality \( \sigma(w) = \sigma(w') \in Q \) implies \( c\gamma(w) = c\gamma(w') \in cM \cap MC \).

2. There exists a homomorphism \( \psi : Q \to M_c \) with \( M_c = (cM \cap MC,c) \) such that the following diagram commutes.

\[
\begin{array}{ccc}
((P \setminus \{1\})^*c)^* & \xrightarrow{\sigma} & Q \\
\downarrow{\gamma} & & \downarrow{\psi} \\
M_c \cup \{1\} & \xrightarrow{c} & M_c
\end{array}
\]

3. There exists a homomorphism \( \psi : E(\rho) \to M \) such that the following diagram commutes.

\[
\begin{array}{ccc}
C^* & \xrightarrow{\tau} & E(\rho) \\
\downarrow{\gamma} & & \downarrow{\psi} \\
M
\end{array}
\]

**Proof.** 1 \( \Rightarrow \) 2. We define \( \psi_c(\sigma(w)) = c\gamma(w) \). Condition 1 says that \( \psi_c : Q \to M_c \) is well-defined. It is a homomorphism because \( \gamma \) and the left-shift \( c : M_c \cup \{1\} \to M_c, x \mapsto cx \) are homomorphisms and \( Q \setminus \{1\} \subseteq \sigma(C^*c) \).

2 \( \Rightarrow \) 3. For \( u \in P \subseteq E(\rho) \) we define \( \psi(u) = \gamma(u) = u \in P \subseteq M \). All other elements in \( E(\rho) \) have the form \( (u, \sigma(\alpha), v) \) with \( u, v \in P \) and \( \alpha \in ((P \setminus \{1\})^*c)^* \).
Define $\psi(u, \sigma(\alpha), v) = u\psi_c(\sigma(\alpha))v$. This in an element in $M$ because $M_c \subseteq M$. Now, $\psi_c(\sigma(\alpha)) = c\gamma(\alpha)$. Hence, $\psi(u, \sigma(\alpha), v) = \gamma(u\alpha v)$. Since $\gamma$, $\tau$ are homomorphisms and $\tau$ is surjective, $\psi$ is a homomorphism, too.

3 $\Rightarrow$ 1: Consider $w \in (P^c)^*$. We have $\tau(cw) = (1, \sigma(w), 1)$. By 3 we have $\gamma(cw) = \psi(1, \sigma(w), 1)$. In particular, $\sigma(w) = \sigma(w') \in Q$ implies $c\gamma(w) = c\gamma(w') \in M_c$.

\[ \square \]

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