ANGELIC WAY FOR MODULAR LIE ALGEBRAS
TOWARD KIM’S CONJECTURE

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Abstract. We consider modular Lie algebras over algebraically closed
field of characteristic $p \geq 7$. This paper purports to prove the conjec-
ture that classical modular Lie algebras, in particular of $C_l$ and of $A_l$
type, should be a Park’s Lie algebra, and so a Hypo- Lie algebra.

1. Introduction

If there is a Lee’s basis except for a finite number of simple
modules for a Lie algebra[4], then we would like to say that
the Lie algebra has an angelic way.

In this paper we shall see that modular $C_l$-type and $A_l$- type
Lie algebras have angelic ways.

For this we shall proceed in the following order: Section 2
deals with modular $A_l$- type Lie algebra and its representa-
tion, followed by $C_l$-type Lie algebra and its representation in
section 3.

Finally in section 4 we shall make concluding remarks re-
lating to Park’s Lie algebra and Hypo Lie algebra.

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We shall assume throughout that $F$ denotes any algebraically closed field of characteristic $p \geq 7$ unless otherwise stated.

2. Modular $A_l$-type Lie algebra and its representation

We must recall first definitions related to modular representation theory.

**Definition 2.1.** Let $(L, [p])$ be a restricted Lie algebra over $F$ and $\chi \in L^*$ be a linear form. If a representation $\rho_\chi : L \rightarrow \mathfrak{gl}(V)$ of $(L, [p])$ satisfies $\rho_\chi(x^p - x^{[p]}) = \chi(x)^p \mathrm{id}_V$ for any $x \in L$, then $\rho_\chi$ is said to be a $\chi$-representation.

In this case we say that the representation or the corresponding module has a $p$-character $\chi$. In particular if $\chi=0$, then $\rho_0$ is called a restricted representation, whereas $\rho_\chi$ for $\chi \neq 0$ is called a nonrestricted representation.

We are well aware that we have $\rho_\chi(a)^p - \rho_\chi(a^{[p]}) = \chi(a)^p \mathrm{id}_V$ for some $\chi \in L^*$, for any $a \in L$ and for any irreducible representation $\rho_\chi$.

For an algebraically closed field $F$ of prime characteristic $p$, the $A_l$-type Lie algebra $L$ over $F$ is just the analogue over $F$ of the $A_l$-type simple Lie algebra over $\mathbb{C}$.

In other words, the $A_l$-type Lie algebra over $F$ is isomorphic to the Chevalley Lie algebra of the form $\sum_{i=1}^n \mathbb{Z} c_i \otimes \mathbb{Z} F$, where $n = \dim_F L$ and $x_\alpha = $ some $c_i$ for each $\alpha \in \Phi$, $h_\alpha = $ some $c_j$ with $\alpha$ some base element of $\Phi$ for a Chevalley basis $\{c_i\}$.
of the $A_l$-type Lie algebra over $\mathbb{C}$.

The $A_l$-type Lie algebra over $\mathbb{C}$ has its root system $\Phi=\{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq l+1\}$, where $\epsilon_i$'s are orthonormal unit vectors in the Euclidean space $\mathbb{R}^{l+1}$. The base of $\Phi$ is equal to $\{\epsilon_i - \epsilon_{i+1} | 1 \leq i \leq l\}$.

We let $L$ be an $A_l$-type simple Lie algebra over an algebraically closed field of characteristic $p \geq 7$.

For a root $\alpha \in \Phi$, we put $g_\alpha := x_{\alpha}^{p-1} - x_{-\alpha}$ and $w_\alpha := (h_\alpha + 1)^2 + 4x_{-\alpha}x_{\alpha}$.

We have seen from [4],[1] that any $A_l$-type modular Lie algebra over $F$ becomes a Park’s Lie algebra. However we would like to specify the proof when $\chi(H) \neq 0$ for a CSA $H$ of $L$.

**Theorem 2.2.** Suppose that $\chi$ is a character of any simple $L$-module with $\chi(h_\alpha) \neq 0$ for some $\alpha \in$ the base of $\Phi$, where $h_\alpha$ is an element in the Chevalley basis of $L$ such that $Fx_\alpha + Fx_{-\alpha} + Fh_\alpha = \mathfrak{sl}_2(F)$ with $[x_\alpha, x_{-\alpha}] = h_\alpha \in H$ (a CSA of $L$).

We then have that the dimension of any simple $L$-module with character $\chi = p^n = p^{\frac{(n-l)}{2}}$, where $n = \dim L = 2m + l$ for $H$ with $\dim H = l$.

**Proof.** If $\chi(x_\alpha) \neq 0$ or $\chi(x_{-\alpha}) \neq 0$, then our assertion is evident from [1],[3],[4]. So we may assume that $\chi(x_\alpha) = \chi(x_{-\alpha}) = 0$ but $\chi(h_\alpha) \neq 0$.

Furthermore we may put $\alpha = \epsilon_1 - \epsilon_2$ without loss of generality since all roots are conjugate under the Weyl group of $\Phi$. 
Since the case for $l = 1$ is trivial, we may assume $l \geq 2$. For $i = 1, 2, \ldots$, we put $B_i := b_{i1}h_{e_1 - \epsilon_2} + \cdots + b_{ii}h_{\epsilon_i - \epsilon_{i+1}}$ as in [3],[4] and we put $\mathfrak{B} := \{(B_1 + A_{\epsilon_1 - \epsilon_2})^i_1 \otimes (B_2 + A_{\epsilon_2 - \epsilon_1})^i_2 \otimes (\otimes_{j=3}^{l+1}(B_j + A_{\epsilon_j - \epsilon_j})^i_j) \otimes ((\otimes_{j=3}^{l+1}(B_{3j-3+j} + A_{\epsilon_j - \epsilon_2})^i_{3j-3+j} \otimes \cdots \otimes (B_{2m-1} + A_{\epsilon_l - \epsilon_{l+1}})^i_{2m-1} \otimes (B_{2m} + A_{\epsilon_{l+1} - \epsilon_l})^i_{2m}) \} \text{ for } 0 \leq i_j \leq p - 1,$

where we set

\begin{align*}
A_{\epsilon_1 - \epsilon_2} &= g_\alpha = g_{\epsilon_1 - \epsilon_2} = x_{\epsilon_1 - \epsilon_2}^p - x_{\epsilon_2 - \epsilon_1}, \\
A_{\epsilon_2 - \epsilon_1} &= c_{\epsilon_2 - \epsilon_1} + (h_\alpha + 1)^2 + 4^{-1}x_{-\alpha}x_\alpha, \\
A_{\epsilon_1 - \epsilon_3} &= g_\alpha^2(c_{\epsilon_1 - \epsilon_3} + x_{\epsilon_1 - \epsilon_2}x_{\epsilon_3 - \epsilon_2} \pm x_{\epsilon_1 - \epsilon_3}x_{\epsilon_3 - \epsilon_1}), \\
A_{\epsilon_3 - \epsilon_1} &= g_\alpha^3(c_{\epsilon_3 - \epsilon_1} + x_{\epsilon_3 - \epsilon_2}x_{\epsilon_1 - \epsilon_2} \pm x_{\epsilon_3 - \epsilon_1}x_{\epsilon_1 - \epsilon_3}) \text{ or } x_{\epsilon_3 - \epsilon_4}(c_{\epsilon_3 - \epsilon_1} + x_{\epsilon_3 - \epsilon_2}x_{\epsilon_2 - \epsilon_3} \pm x_{\epsilon_3 - \epsilon_1}x_{\epsilon_1 - \epsilon_3}), \\
A_{\epsilon_2 - \epsilon_j} &= g_\alpha^4(c_{\epsilon_2 - \epsilon_3} + x_{\epsilon_2 - \epsilon_3}x_{\epsilon_3 - \epsilon_2} \pm x_{\epsilon_1 - \epsilon_3}x_{\epsilon_3 - \epsilon_1}) \text{ (if } j = 3 \text{) or } x_{\epsilon_4 - \epsilon_j}(c_{\epsilon_2 - \epsilon_3} + x_{\epsilon_2 - \epsilon_3}x_{\epsilon_3 - \epsilon_2} \pm x_{\epsilon_1 - \epsilon_3}x_{\epsilon_3 - \epsilon_1}), \\
A_{\epsilon_j - \epsilon_2} &= g_\alpha^5(c_{\epsilon_3 - \epsilon_2} + x_{\epsilon_3 - \epsilon_2}x_{\epsilon_2 - \epsilon_3} \pm x_{\epsilon_1 - \epsilon_3}x_{\epsilon_3 - \epsilon_1}) \text{ (if } j = 3 \text{) or } x_{\epsilon_j - \epsilon_4}(c_{\epsilon_3 - \epsilon_2} + x_{\epsilon_3 - \epsilon_2}x_{\epsilon_2 - \epsilon_3} \pm x_{\epsilon_1 - \epsilon_3}x_{\epsilon_3 - \epsilon_1}), \\
A_{\epsilon_2 - \epsilon_4} &= x_{\epsilon_3 - \epsilon_4}^2(c_{\epsilon_2 - \epsilon_4} + x_{\epsilon_2 - \epsilon_4}x_{\epsilon_4 - \epsilon_2} \pm x_{\epsilon_1 - \epsilon_4}x_{\epsilon_4 - \epsilon_1}), \\
A_{\epsilon_4 - \epsilon_2} &= x_{\epsilon_4 - \epsilon_3}(c_{\epsilon_4 - \epsilon_2} + x_{\epsilon_4 - \epsilon_2}x_{\epsilon_2 - \epsilon_4} \pm x_{\epsilon_4 - \epsilon_1}x_{\epsilon_1 - \epsilon_4}), \\
A_{\epsilon_1 - \epsilon_j} &= x_{\epsilon_3 - \epsilon_j}^2(c_{\epsilon_1 - \epsilon_j} + x_{\epsilon_1 - \epsilon_j}x_{\epsilon_j - \epsilon_1} \pm x_{\epsilon_2 - \epsilon_j}x_{\epsilon_j - \epsilon_2}).
\end{align*}
\[ A_{\epsilon_j-\epsilon_1} = x_{\epsilon_j-\epsilon_3}^2 (c_{\epsilon_j-\epsilon_1} + x_{\epsilon_1-\epsilon_j} x_{\epsilon_j-\epsilon_1} \pm x_{\epsilon_2-\epsilon_j} x_{\epsilon_j-\epsilon_2}), \]

\[ A_{\epsilon_i-\epsilon_j} = x_{\epsilon_i-\epsilon_j}^2 \text{ or } x_{\epsilon_i-\epsilon_j}^3 \text{ for other roots } \epsilon_i - \epsilon_j, \]

where signs are chosen so that they may commute with \( x_\alpha \) and \( c_\beta \) are chosen so that \( A_{\epsilon_2-\epsilon_1} \) and parentheses are invertible in \( U(L)/\mathfrak{m}_\chi \) for the kernel \( \mathfrak{m}_\chi \) in \( U(L) \) of any given simple representation of \( L \) with the character \( \chi \).

We may see without difficulty that \( \mathfrak{B} \) is a linearly independent set in \( U(L) \) by virtue of P-B-W theorem.

We shall prove that a nontrivial linearly dependent equation leads to absurdity. We assume first that we have a dependence equation which is of least degree with respect to \( h_{\alpha_j} \in H \) and the number of whose highest degree terms is also least.

In case it is conjugated by \( x_\alpha \), then there arises a nontrivial dependence equation of lower degree than the given one, which contradicts to our assumption.

Otherwise we have to prove that

(i) \( x_{\epsilon_l-\epsilon_k} K + K' \in \mathfrak{m}_\chi \) with \( l, k \neq 1, 2 \)

(ii) \( g_\alpha K + K' \in \mathfrak{m}_\chi \)

lead to a contradiction, where both \( K \) and \( K' \) commute with \( x_{\pm \alpha} \) modulo \( \mathfrak{m}_\chi \). In particular \( K \) commute with \( g_\alpha \).
For the case (i), we may change it to the form \( x_\alpha K + K'' \in \mathfrak{M}_\chi \) for some \( K'' \) commuting with \( x_\alpha = x_{\epsilon_1 - \epsilon_2} \) modulo \( \mathfrak{M}_\chi \).

So we have \( x^p_\alpha K + x^{p-1}_\alpha K'' \equiv 0 \), thus \( x^{p-1}_\alpha K'' \equiv 0 \).

Subtracting from this \( x_{-\alpha} x_\alpha K + x_{-\alpha} K'' \equiv 0 \), we get

\[-x_{-\alpha} x_\alpha K + g_\alpha K'' \equiv 0.\]

Recall here that \( g_\alpha \) is invertible and \( w_\alpha \) belongs to the center of \( U(\mathfrak{sl}_2(F)) \) according to [7].

So we get \( 4^{-1} \{(h_\alpha + 1)^2 - w_\alpha\} K + g_\alpha K'' \equiv 0 \), and hence

\((*) g_\alpha^{p-1} 4^{-1} \{(h_\alpha + 1)^2 - w_\alpha\} K + c K'' \equiv 0 \)

is obtained and from the start equation we have

\((**) c x_\alpha K + c K'' \equiv 0 \), where \( g^p_\alpha - c \equiv 0 \).

Subtracting \((**)\) from \((*)\), we have \( 4^{-1} g_\alpha^{p-1} \{(h_\alpha + 1)^2 - w_\alpha\} K - c x_\alpha K \equiv 0 \).

Multiplying this equation by \( g_\alpha^{1-p} \) to the right, we obtain

\( 4^{-1} g_\alpha^{p-1} \{(h_\alpha + 1)^2 - w_\alpha\} g_\alpha^{1-p} K - c x_\alpha g_\alpha^{1-p} K \equiv 0 \)

We thus have \( 4^{-1} \{(h_\alpha + 1 - 2)^2 - w_\alpha\} K - x_\alpha g_\alpha K \equiv 0 \).

So it follows that \( 4^{-1} \{(h_\alpha - 1)^2 - w_\alpha\} K + x_\alpha x_{-\alpha} K \equiv 0 \).

Next multiplying \( x_{-\alpha}^{p-1} \) to the right of this last equation, we obtain \( \{(h_\alpha - 1)^2 - w_\alpha\} K x_{-\alpha}^{p-1} \equiv 0 \). Now multiply \( x_\alpha \) in turn consecutively to the left of this equation until it becomes of
the form

\[(\text{a nonzero polynomial of degree } \geq 1 \text{ with respect to } h_\alpha)K \in \mathcal{M}_\chi.\]

By making use of conjugation and subtraction consecutively, we are led to a contradiction. $K \in \mathcal{M}_\chi$.

Finally for the case (ii), we consider $K + g_\alpha^{-1}K' \in \mathcal{M}_\chi$. So we have $x_\alpha K + x_\alpha g_\alpha^{-1}K' \equiv 0$ modulo $\mathcal{M}_\chi$.

By analogy with the argument as in the case (i), we obtain a contradiction $K \in \mathcal{M}_\chi$. □

3. MODULAR $C_l$-TYPE LIE ALGEBRA AND ITS REPRESENTATION

We note first that the root system of $C_l$-type Lie algebra over $\mathbb{C}$ is just $\Phi = \{\pm 2\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j)|1 \leq i \neq j \leq l \geq 3\}$ with a base $\{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{l-1} - \varepsilon_l, 2\varepsilon_l\}$, where $\varepsilon_i$ and $\varepsilon_j$ are linearly independent orthonormal unit vectors in $\mathbb{R}^l$.

For a root $\alpha \in \Phi$, we also put $g_\alpha := x_\alpha^{p-1} - x_{-\alpha}$ and $w_\alpha := (h_\alpha + 1)^2 + 4x_{-\alpha}x_\alpha$ as in section 2, where $[x_\alpha, x_{-\alpha}] = h_\alpha$.

For an algebraically closed field $F$ of prime characteristic $p$, the $C_l$-type Lie algebra $L$ over $F$ is just the analogue over $F$ of the $C_l$-type simple Lie algebra over $\mathbb{C}$. 
In other words the $C_l-$ type Lie algebra over $F$ is isomorphic to the Chevalley Lie algebra of the form $\sum_{i=1}^{n} \mathbb{Z}c_i \otimes_{\mathbb{Z}} F$, where $n = \dim_F L$ and $x_\alpha = \text{some } c_i$ for each $\alpha \in \Phi$, $h_\alpha = \text{some } c_j$ with $\alpha$ some base element of $\Phi$ for a Chevalley basis $\{c_i\}$ of the $C_l$ - type Lie algebra over $\mathbb{C}$.

We shall compute in this section the dimension of some simple modules of the $C_l$-type Lie algebra $L$ with a CSA $H$ over an algebraically closed field $F$ of characteristic $p \geq 7$.

Let $L$ be a $C_l$-type simple Lie algebra over an algebraically closed field $F$ of characteristic $p \geq 7$. Let $\chi$ be a character of any simple $L$-module with $\chi(x_\alpha) \neq 0$ for some $\alpha \in \Phi$, where $x_\alpha$ is an element in the Chevalley basis of $L$ such that $F x_\alpha + F h_\alpha + F x_{-\alpha} = \mathfrak{sl}_2(F)$ with $[x_\alpha, x_{-\alpha}] = h_\alpha$.

Then we have conjectured in [4] that any simple $L$-module with character $\chi$ is of dimension $p^m = p^{\frac{n-1}{2}}$, where $n = \dim L = 2m + l$ for a CSA $H$ with $\dim H = l$.

In this section we intend to clarify this conjecture for modular $C_l$-type Lie algebra $L$.

**Proposition 3.1.** Let $\alpha$ be any root in the root system $\Phi$ of $L$. If $\chi(x_\alpha) \neq 0$, then $\dim_F \rho_\chi(U(L)) = p^{2m}$, where $[Q(U(L)) : Q(\mathfrak{z}(L))] = p^{2m} = p^{n-1}$ with $\mathfrak{z}$ the center of $U(L)$ and $Q$ denotes the quotient algebra.

So the simple module corresponding to this representation has $p^m$ as its dimension.
Proof. Let $\mathfrak{M}_\chi$ be the kernel of this irreducible representation, i.e., a certain (2-sided) maximal ideal of $U(L)$.

(I) Assume first that $\alpha$ is a short root; then we may put $\alpha = \epsilon_1 - \epsilon_2$ without loss of generality since all roots of a given length are conjugate under the Weyl group of the root system $\Phi$.

First we let

$$B_i := b_{i1} h_{\epsilon_1 - \epsilon_2} + b_{i2} h_{\epsilon_2 - \epsilon_3} + \cdots + b_{i,l-1} h_{\epsilon_{l-1} - \epsilon_l} + b_{il} h_{\epsilon_l}$$

for $i = 1, 2, \cdots, 2m$, where $(b_{i1}, b_{i2}, \cdots, b_{il}) \in F^l$ are chosen so that any $(l + 1)-B_i$’s are linearly independent in $\mathbb{P}^l(F)$, the $B$ below becomes an $F-$ linearly independent set in $U(L)$ if necessary and $x_{\alpha} B_i \not\equiv B_i x_{\alpha}$ for $\alpha = \epsilon_1 - \epsilon_2$.

In $U(L)/\mathfrak{M}_\chi$ we claim that we have a basis

$$\mathfrak{B} := \{(B_1 + A_{\epsilon_1 - \epsilon_2})^{i_1} \otimes (B_2 + A_{-(\epsilon_1 - \epsilon_2)})^{i_2} \otimes \cdots \otimes (B_{2l-2} + A_{-(\epsilon_{l-1} - \epsilon_l)})^{i_{2l-2}} \otimes (B_{2l-1} + A_{2\epsilon_l})^{i_{2l-1}} \otimes (B_{2l} + A_{-2\epsilon_l})^{i_{2l}} \otimes \otimes_{j=2l+1}^{2m}(B_j + A_{\alpha_j})^{i_j}) | 0 \leq i_j \leq p - 1 \},$$

where we put

$$A_{\epsilon_1 - \epsilon_2} = x_{\alpha} = x_{\epsilon_1 - \epsilon_2},$$

$$A_{\epsilon_2 - \epsilon_1} = c_{-(\epsilon_1 - \epsilon_2)} + (h_{\epsilon_1 - \epsilon_2} + 1)^2 + 4x_{\alpha} x_{-\alpha},$$

$$A_{\epsilon_2 \pm \epsilon_3} = x_{\pm 2\epsilon_3} \left( c_{\epsilon_2 \pm \epsilon_3} + x_{\epsilon_2 \pm \epsilon_3} x_{-(\epsilon_2 \pm \epsilon_3)} \pm x_{\epsilon_1 \pm \epsilon_3} x_{-(\epsilon_1 \pm \epsilon_3)} \right),$$
\[ A_{\epsilon_1+\epsilon_2} = x_{\epsilon_1-\epsilon_2}^2 (c_{\epsilon_1+\epsilon_2} + 3x_{\epsilon_1+\epsilon_2} x_{-\epsilon_1-\epsilon_2} + 2x_{2\epsilon_1} x_{-2\epsilon_1} \pm 2x_{2\epsilon_2} x_{-2\epsilon_2}), \]

\[ A_{\epsilon_2 \pm \epsilon_k} = x_{\epsilon_3 \pm \epsilon_k} (c_{\epsilon_2 \pm \epsilon_k} + x_{\epsilon_2 \pm \epsilon_k} x_{-(\epsilon_2 \pm \epsilon_k)} \pm x_{\epsilon_1 \pm \epsilon_k} x_{-(\epsilon_1 \pm \epsilon_k)}), \]

\[ A_{-2\epsilon_1} = x_{-2\epsilon_3}^2 (c_{-2\epsilon_1} + 2x_{-2\epsilon_1} x_{2\epsilon_1} \pm 3x_{-\epsilon_1-\epsilon_2} x_{\epsilon_1+\epsilon_2} \pm 2x_{-2\epsilon_2} x_{2\epsilon_2}), \]

\[ A_{-(\epsilon_1 \pm \epsilon_3)} = x_{-(\pm \epsilon_3)} (c_{-(\epsilon_2 \pm \epsilon_3)} + x_{\epsilon_2 \pm \epsilon_3} x_{-(\epsilon_2 \pm \epsilon_3)} \pm x_{\epsilon_1 \pm \epsilon_3} x_{-(\epsilon_1 \pm \epsilon_3)}), \]

\[ A_{-(\epsilon_1 \pm \epsilon_k)} = x_{-(\epsilon_3 \pm \epsilon_k)} (c_{-(\epsilon_1 \pm \epsilon_k)} + x_{\epsilon_2 \pm \epsilon_k} x_{-(\epsilon_2 \pm \epsilon_k)} \pm x_{\epsilon_1 \pm \epsilon_k} x_{-(\epsilon_1 \pm \epsilon_k)}), \]

\[ A_{2\epsilon_1} = x_{2\epsilon_1}^2, \]

\[ A_{-2\epsilon_1} = x_{-2\epsilon_1}^2, \]

with the sign chosen so that they commute with \( x_\alpha \) and with \( c_\alpha \in F \) chosen so that \( A_{\epsilon_2-\epsilon_1} \) and parentheses are invertible. For any other root \( \beta \) we put \( A_\beta = x_\beta^2 \) or \( x_\beta^3 \) if possible. Otherwise attach to these sorts the parentheses( ) used for designating \( A_{-\beta} \) so that \( A_\gamma \forall \gamma \in \Phi \) may commute with \( x_\alpha \).

We shall prove that \( \mathcal{B} \) is a basis in \( U(L)/M_\chi \).

By virtue of P-B-W theorem, it is not difficult to see that \( \mathcal{B} \) is evidently a linearly independent set over \( F \) in \( U(L) \). Furthermore \( \forall \beta \in \Phi, A_\beta \notin M_\chi \)(see detailed proof below).

We shall prove that a nontrivial linearly dependent equation leads to absurdity. We assume first that there is a dependence equation which is of least degree with respect to \( h_{\alpha_j} \in H \) and
the number of whose highest degree terms is also least.

In case it is conjugated by $x_\alpha$, then there arises a nontrivial dependence equation of lower degree than the given one, which contradicts to our assumption.

Otherwise it reduces to one of the following forms:

(i) $x_{2\epsilon_j}K + K' \in \mathcal{M}_\chi ,$

(ii) $x_{-2\epsilon_j}K + K' \in \mathcal{M}_\chi ,$

(iii) $x_{\epsilon_j+\epsilon_k}K + K' \in \mathcal{M}_\chi ,$

(iv) $x_{-\epsilon_j-\epsilon_k}K + K' \in \mathcal{M}_\chi ,$

(v) $x_{\epsilon_j-\epsilon_k}K + K' \in \mathcal{M}_\chi ,$

where $K, K'$ commute with $x_\alpha$.

For the case (i), we deduce successively

\[
x_{\epsilon_2-\epsilon_j}x_{2\epsilon_j}K + x_{\epsilon_2-\epsilon_j}K' \in \mathcal{M}_\chi
\Rightarrow x_{\epsilon_2+\epsilon_j}K + x_{2\epsilon_j}x_{\epsilon_2-\epsilon_j}K + x_{\epsilon_2-\epsilon_j}K' \in \mathcal{M}_\chi
\Rightarrow (x_{\epsilon_1+\epsilon_j} or x_{2\epsilon_1})K + x_{2\epsilon_j} (x_{\epsilon_1-\epsilon_j} or h_{\epsilon_1-\epsilon_2})K + (x_{\epsilon_1-\epsilon_j} or h_{\epsilon_1-\epsilon_2})K' \in \mathcal{M}_\chi
\]

by $adx_{\epsilon_1-\epsilon_2}$ if $j \neq 1$ or $j=1$ respectively, so that by successive $adx_\alpha$ and rearrangement we get $x_{\epsilon_1 \pm \epsilon_j}K + K'' \in \mathcal{M}_\chi$ for some $K''$ commuting with $x_\alpha$ in view of the start equation. So (i) reduces to (iii),(iv) or (v).
Similarly as in (i) and by adjoint operations, (ii) reduces to (iii),(iv) or (v). Also (iii),(iv) reduces to the form (v) putting \( \epsilon_j = -(-\epsilon_j) \), \( \epsilon_k = -(-\epsilon_k) \).

Hence we have only to consider the case (v). We consider

\[
x_{\epsilon_k} - x_{\epsilon_k} K + x_{\epsilon_k} - 2 K' \in \mathcal{M}_\chi,
\]

so that \( (x_{\epsilon_j} - x_{\epsilon_j} x_{\epsilon_k} + x_{\epsilon_k} - 2 K')_K + x_{\epsilon_k} - 2 K' \in \mathcal{M}_\chi \) for \( j, k \neq 1,2 \).

We thus have \( x_{\epsilon_j} - 2 K + (x_{\epsilon_j} - 2 x_{\epsilon_k} x_{\epsilon_j} - 2 K + x_{\epsilon_k} - 2 K') \in \mathcal{M}_\chi \), so that we may put this last ( ) = another \( K' \) alike as in the equation (v).

Hence we need to show that \( x_{\epsilon_j} - 2 K + K' \in \mathcal{M}_\chi \) leads to absurdity. We consider

\[
x_{\epsilon_2} x_{\epsilon_j} - 2 K + x_{\epsilon_2} - 2 K' \in \mathcal{M}_\chi \Rightarrow (h_{\epsilon_2} - 2 + x_{\epsilon_j} x_{\epsilon_2} x_{\epsilon_2} - 2 K) + x_{\epsilon_2} - 2 K' \in \mathcal{M}_\chi \text{ by adj} x_{\epsilon_1} - 2 \Rightarrow \text{ either } x_{\epsilon_1} - 2 K \in \mathcal{M}_\chi \text{ or } (x_{\epsilon_1} - 2 + x_{\epsilon_j} x_{\epsilon_2} x_{\epsilon_1} - 2) K + x_{\epsilon_1} - 2 K' \in \mathcal{M}_\chi
\]

depending on \( [x_{\epsilon_j} - 2, x_{\epsilon_1} - 2] = +x_{\epsilon_1} - 2 \) or \(-x_{\epsilon_1} - 2\). The former case leads to \( K \in \mathcal{M}_\chi \), a contradiction.

For the latter case we consider

\[
x_{\epsilon_1} - 2 K + (x_{\epsilon_j} - 2 x_{\epsilon_1} - 2 K + x_{\epsilon_1} - 2 K') \in \mathcal{M}_\chi.
\]

So we may put
\((\star)x_{\epsilon_1-\epsilon_2}K + K'' \in \mathcal{M}_\chi\),

where \(K'' = x_{\epsilon_j-\epsilon_2}x_{\epsilon_1-\epsilon_j}K + x_{\epsilon_1-\epsilon_j}K'\). Thus \(x_{\epsilon_2-\epsilon_1}x_{\epsilon_1-\epsilon_2}K + x_{\epsilon_2-\epsilon_1}K'' \in \mathcal{M}_\chi\). From \(w_{\epsilon_1-\epsilon_2} := (h_{\epsilon_1-\epsilon_2} + 1)^2 + 4x_{\epsilon_2-\epsilon_1}x_{\epsilon_1-\epsilon_2} \in \text{the center of } U(\mathfrak{sl}_2(F))\), we get \(4^{-1}\{w_{\epsilon_1-\epsilon_2} - (h + 1)^2\}K + x_{\epsilon_2-\epsilon_1}K'\) \(\equiv 0\) modulo \(\mathcal{M}_\chi\).

If \(x_{\epsilon_2-\epsilon_1}^p \equiv c\) which is a constant, then

\((\star\star)4^{-1}x_{\epsilon_2-\epsilon_1}^{p-1}\{w_{\epsilon_1-\epsilon_2} - (h_{\epsilon_1-\epsilon_2} + 1)^2\}K + cK'' \equiv 0\)

is obtained.

From \((\star), (\star\star)\), we have

\[4^{-1}x_{\epsilon_2-\epsilon_1}^{p-1}\{w_{\epsilon_1-\epsilon_2} - (h_{\epsilon_1-\epsilon_2} + 1)^2\}K - cx_{\epsilon_1-\epsilon_2}K \equiv 0 \text{ modulo } \mathcal{M}_\chi.\]

Multiplying \(x_{\epsilon_1-\epsilon_2}^{p-1}\) to this equation, we obtain

\((\star\star\star)4^{-1}x_{\epsilon_1-\epsilon_2}^{p-1}x_{\epsilon_2-\epsilon_1}^{p-1}\{w_{\epsilon_1-\epsilon_2} - (h_{\epsilon_1-\epsilon_2} + 1)^2\}K - cx_{\epsilon_1-\epsilon_2}^{p-1}K \equiv 0.\)

By making use of \(w_{\epsilon_1-\epsilon_2}\), we may deduce from \((\star\star\star)\) an equation of the form

\(\text{a polynomial of degree } \geq 1 \text{ with respect to } h_{\epsilon_1-\epsilon_2}K - cx_{\epsilon_1-\epsilon_2}K \equiv 0.\)

Finally if we use conjugation and subtraction consecutively, then we are led to a contradiction \(K \in \mathcal{M}_\chi\).

(II) Assume next that \(\alpha\) is a long root; then we may put \(\alpha = 2\epsilon_1\) because all roots of the same length are conjugate
under the Weyl group of $\Phi$.

Similarly as in (I), we let $B_i :=$ the same as in (I) except that this time $\alpha = 2\epsilon_1$ instead of $\epsilon_1 - \epsilon_2$.

We claim that we have a basis in $\mathcal{U}(L)/\mathcal{M}_\chi$ such as

$$\mathfrak{B} := \{(B_1 + A_{2\epsilon_1})^{i_1} \otimes (B_2 + A_{-2\epsilon_1})^{i_2} \otimes (B_3 + A_{\epsilon_1 - \epsilon_2})^{i_3} \otimes (B_4 + A_{-(\epsilon_1 - \epsilon_2)})^{i_4} \otimes \cdots \otimes (B_{2l} + A_{-(\epsilon_{l-1} - \epsilon_1)})^{i_{2l}} \otimes (B_{2l+1} + A_{2\epsilon_1})^{i_{2l+1}} \otimes (B_{2l+2} + A_{-2\epsilon_1})^{i_{2l+2}} \otimes (\otimes_{j=2l+3}^{2m} (B_j + A_{\alpha_j})^{i_j}; 0 \leq i_j \leq p - 1)\},$$

where we put

$$A_{2\epsilon_1} = x_{2\epsilon_1},$$

$$A_{-2\epsilon_1} = c_{-2\epsilon_1} + (h_{2\epsilon_1} + 1)^2 + 4x_{-2\epsilon_1}x_{2\epsilon_1},$$

$$A_{-\epsilon_1 \pm \epsilon_2} = x_{-\epsilon_3 \pm 2} (c_{-\epsilon_1 \pm \epsilon_2} \pm x_{-\epsilon_1 \pm \epsilon_2} x_{\epsilon_1 \mp \epsilon_2} \pm x_{\epsilon_1 \pm \epsilon_2} x_{-\epsilon_1 \mp \epsilon_2}),$$

$$A_{-\epsilon_1 \pm \epsilon_j} = x_{-\epsilon_2 \pm \epsilon_j} (c_{-\epsilon_1 \pm \epsilon_j} + x_{\epsilon_j - \epsilon_1} x_{\epsilon_1 \mp \epsilon_j} \pm x_{\epsilon_1 \pm \epsilon_j} x_{-\epsilon_1 \mp \epsilon_j}),$$

and for any other root $\beta$ we put $A_{\beta} = x_{\beta}^2$ or $x_{\beta}^3$ if possible.

Otherwise attach to these sorts the parentheses ( ) used for designating $A_{-\beta}$. Likewise as in case (I), we shall prove that $\mathfrak{B}$ is a basis in $\mathcal{U}(L)/\mathcal{M}_\chi$.

By virtue of P-B-W theorem, it is not difficult to see that $\mathfrak{B}$ is evidently a linearly independent set over $F$ in $\mathcal{U}(L)$. Moreover $\forall \beta \in \Phi$, $A_{\beta} \notin \mathcal{M}_\chi$ (see detailed proof below).
We shall prove that a nontrivial linearly dependent equation leads to absurdity. We assume first that there is a dependence equation which is of least degree with respect to $h_{\alpha_j} \in H$ and the number of whose highest degree terms is also least.

If it is conjugated by $x_{\alpha_j}$, then there arises a nontrivial dependence equation of least degree than the given one, which contravenes our assumption.

Otherwise it reduces to one of the following forms:

(i) $x_{2\epsilon_j}K + K' \in \mathcal{M}_\chi$,

(ii) $x_{-2\epsilon_j}K + K' \in \mathcal{M}_\chi$,

(iii) $x_{\epsilon_j + \epsilon_k}K + K' \in \mathcal{M}_\chi$,

(iv) $x_{-\epsilon_j - \epsilon_k}K + K' \in \mathcal{M}_\chi$,

(v) $x_{\epsilon_j - \epsilon_k}K + K' \in \mathcal{M}_\chi$,

where $K$ and $K'$ commute with $x_{\alpha} = x_{2\epsilon_1}$.

For the case (i), we consider a particular case $j=1$ first; if we assume $x_{2\epsilon_1}K + K' \in \mathcal{M}_\chi$, then we are led to a contradiction according to the similar argument (\*) as in (I).

So we assume $x_{2\epsilon_j}K + K' \in \mathcal{M}_\chi$ with $j \geq 2$. Now we have $x_{2\epsilon_j}K + K' \in \mathcal{M}_\chi \Rightarrow x_{-\epsilon_1 - \epsilon_j}x_{2\epsilon_j}K + x_{-\epsilon_1 - \epsilon_j}K' \in \mathcal{M}_\chi \Rightarrow x_{-\epsilon_1 + \epsilon_j}K + x_{2\epsilon_j}x_{-\epsilon_1 - \epsilon_j}K + x_{-\epsilon_1 - \epsilon_j}K' \in \mathcal{M}_\chi \Rightarrow$ by $ad x_{\epsilon_1}$, $x_{\epsilon_1 + \epsilon_j}K + x_{2\epsilon_j}x_{\epsilon_1 - \epsilon_j}K + x_{\epsilon_1 - \epsilon_j}K' \in \mathcal{M}_\chi$ is obtained. Hence (i) reduces to
Similarly (ii) reduces to (iii) or (iv) or (v). So we have only to consider (iii), (iv), (v). However (iii), (iv), (v) reduce to 
\[ x_{2\epsilon_1}K + K'' \in \mathcal{M}_\chi \] after all considering the situation as in (I). Similarly following the argument as in (I), we are led to a contradiction \( K \in \mathcal{M}_\chi \).

\[ \square \]

Now we are ready to consider another nonzero character \( \chi \) different from that of proposition 3.1.

**Proposition 3.2.** Let \( \chi \) be a character of any simple \( L \)-module with \( \chi(h_\alpha) \neq 0 \) for some \( \alpha \in \Phi \), where \( h_\alpha \) is an element in the Chevalley basis of \( L \) such that \( Fx_\alpha + Fh_\alpha + Fx_{-\alpha} = \mathfrak{sl}_2(F) \) with \( [x_\alpha, x_{-\alpha}] = h_\alpha \in H \).

We then have that any simple \( L \)-module with character \( \chi \) is of dimension \( p^m = p^{n-l} \), where \( n = \text{dim}L = 2m + l \) for a CSA \( H \) with \( \text{dim}H = l \).

**Proof.** Let \( \mathcal{M}_\chi \) be the kernel of this irreducible representation, i.e., a certain (2-sided) maximal ideal of \( U(L) \).
If \( x_{\epsilon_1-\epsilon_2} \neq 0 \) or \( x_{\epsilon_2-\epsilon_1} \neq 0 \), then our assertion is evident from proposition 4.1 in [3].

So we may let \( x_{\epsilon_1-\epsilon_2} \equiv x_{\epsilon_2-\epsilon_1} \equiv 0 \) modulo \( \mathcal{M}_\chi \).

(I) Assume first that \( \alpha \) is a short root; then we may put \( \alpha = \epsilon_1 - \epsilon_2 \) without loss of generality since all roots of a given length are conjugate under the Weyl group of the root system \( \Phi \).
First we let $B_i := b_{i1} h_{\epsilon_1 - \epsilon_2} + b_{i2} h_{\epsilon_2 - \epsilon_3} + \cdots + b_{i,l-1} h_{\epsilon_{l-1} - \epsilon_l} + b_{il} h_{\epsilon_l}$ for $i = 1, 2, \cdots, 2m$, where $(b_{i1}, b_{i2}, \cdots, b_{il}) \in F^l$ are chosen so that any $(l + 1)-B_i$’s are linearly independent in $\mathbb{P}^l(F)$, the $\mathfrak{B}$ below becomes an $F-$ linearly independent set in $U(L)$ if necessary and $x_\alpha B_i \neq B_i x_\alpha$ for $\alpha = \epsilon_1 - \epsilon_2$.

In $U(L)/\mathfrak{M}_\chi$ we claim that we have a basis

$$\mathfrak{B} := \{(B_1 + A_{\epsilon_1 - \epsilon_2})^{i_1} \otimes (B_2 + A_{-(\epsilon_1 - \epsilon_2)})^{i_2} \otimes \cdots \otimes (B_{2l-2} + A_{-(\epsilon_{l-1} - \epsilon_l)})^{i_{2l-2}} \otimes \cdots \otimes (B_{2l-1} + A_{2\epsilon_l})^{i_{2l-1}} \otimes (B_{2l} + A_{-2\epsilon_l})^{i_{2l}} \otimes (\otimes_{j=2l+1}^{2m} (B_j + A_{\alpha_j})^{i_j}) | 0 \leq i_j \leq p - 1 \},$$

where we put

$A_{\epsilon_1 - \epsilon_2} = g_\alpha = g_{\epsilon_1 - \epsilon_2}$,

$A_{\epsilon_2 - \epsilon_1} = 3^{-1}(h_{\epsilon_1 - \epsilon_2} + 1)^2 + 4x_\alpha x_\alpha$,

$A_{\epsilon_2 - \epsilon_2} = x_{\epsilon_2 - \epsilon_2} \left( c_{\epsilon_2 - \epsilon_2} \pm x_{\epsilon_2 - \epsilon_3} x_{-(\epsilon_2 - \epsilon_3)} \pm x_{\epsilon_1 - \epsilon_3} x_{-(\epsilon_1 - \epsilon_3)} \right)$,

$A_{\epsilon_1 + \epsilon_2} = g_{\epsilon_1 - \epsilon_2}^2 \left( c_{\epsilon_1 + \epsilon_2} + 2^{-1} x_{\epsilon_1 + \epsilon_2} x_{-(\epsilon_1 - \epsilon_2)} \pm 3^{-1} x_{2\epsilon_1} x_{-2\epsilon_1} \pm 3^{-1} x_{2\epsilon_2} x_{-2\epsilon_2} \right)$,

$A_{-\epsilon_1 - \epsilon_2} = g_{\epsilon_1 - \epsilon_2}^3 \left( (c_{-\epsilon_1 - \epsilon_2} + 2^{-1} x_{\epsilon_1 + \epsilon_2} x_{-(\epsilon_1 - \epsilon_2)} \pm 3^{-1} x_{2\epsilon_1} x_{-2\epsilon_1} \pm 3^{-1} x_{2\epsilon_2} x_{-2\epsilon_2} \right)$,

$A_{\epsilon_2 - \epsilon_k} = x_{\epsilon_2 - \epsilon_k} \left( c_{\epsilon_2 - \epsilon_k} \pm x_{\epsilon_2 - \epsilon_k} x_{-(\epsilon_2 - \epsilon_k)} \pm x_{\epsilon_1 - \epsilon_k} x_{-(\epsilon_1 - \epsilon_k)} \right)$,

$A_{2\epsilon_2} = g_{\epsilon_1 - \epsilon_2}^6 \left( c_{2\epsilon_2} + 3^{-1} x_{2\epsilon_2} x_{-2\epsilon_2} \pm 2^{-1} x_{\epsilon_1 + \epsilon_2} x_{-(\epsilon_1 - \epsilon_2)} \pm 3^{-1} x_{2\epsilon_1} x_{-2\epsilon_1} \right),$
\begin{align*}
A_{-2\epsilon_2} &= g_\alpha A_{\epsilon_2-\epsilon_1} (c_{-2\epsilon_2} + 3^{-1}x_{2\epsilon_2}x_{-2\epsilon_2} \pm 2^{-1}x_{\epsilon_1+\epsilon_2}x_{-\epsilon_1-\epsilon_2} + 3^{-1}x_{2\epsilon_1}x_{-2\epsilon_1}) \\
A_{2\epsilon_1} &= g_{\epsilon_1-\epsilon_2} (c_{2\epsilon_1} + 3^{-1}x_{-2\epsilon_1}x_{2\epsilon_1} \pm 2^{-1}x_{-\epsilon_1-\epsilon_2}x_{\epsilon_1+\epsilon_2} \pm 3^{-1}x_{-2\epsilon_2}x_{2\epsilon_2}) \\
A_{-2\epsilon_1} &= g_{\epsilon_1-\epsilon_2} (c_{-2\epsilon_1} + 3^{-1}x_{-2\epsilon_1}x_{2\epsilon_1} \pm 2^{-1}x_{-\epsilon_1-\epsilon_2}x_{\epsilon_1+\epsilon_2} \pm 3^{-1}x_{-2\epsilon_2}x_{2\epsilon_2}), \\
A_{-(\epsilon_1\pm\epsilon_3)} &= x_{-(\pm\epsilon_3)} (c_{-(\epsilon_2\pm\epsilon_3)} + x_{\epsilon_2\pm\epsilon_3}x_{-(\epsilon_2\pm\epsilon_3)} \pm x_{\epsilon_1\pm\epsilon_3}x_{-(\epsilon_1\pm\epsilon_3)}) \\
A_{-(\epsilon_1\pm\epsilon_k)} &= x_{-(\epsilon_1\pm\epsilon_k)} (c_{-(\epsilon_2\pm\epsilon_k)} + x_{\epsilon_2\pm\epsilon_k}x_{-(\epsilon_2\pm\epsilon_k)} \pm x_{\epsilon_1\pm\epsilon_k}x_{-(\epsilon_1\pm\epsilon_k)}), \\
A_{2\epsilon_1} &= x_{2\epsilon_1}^2 (\text{if } l \neq 1, 2), \\
A_{-2\epsilon_1} &= x_{-2\epsilon_1},
\end{align*}

with the sign chosen so that they commute with \( x_\alpha \) and with \( c_\alpha \in F \) chosen so that \( A_{\epsilon_2-\epsilon_1} \) and parentheses are invertible. For any other root \( \beta \) we put \( A_\beta = x_\beta^2 \) or \( x_\beta^3 \) if possible.

Otherwise attach to these sorts the parentheses( ) used for designating \( A_{-\beta} \) so that \( A_\gamma, \forall \gamma \in \Phi \) may commute with \( x_\alpha \).

We shall prove that \( \mathfrak{B} \) is a basis in \( U(L)/\mathfrak{M}_\chi \). By virtue of P-B-W theorem, it is not difficult to see that \( \mathfrak{B} \) is evidently a linearly independent set over \( F \) in \( U(L) \). Furthermore \( \forall \beta \in \Phi, A_\beta \notin \mathfrak{M}_\chi \) (see detailed proof below).

We shall prove that a nontrivial linearly dependent equation leads to absurdity.
We assume first that there is a dependence equation which is of least degree with respect to $h_{\alpha_j} \in H$ and the number of whose highest degree terms is also least.

In case it is conjugated by $x_\alpha$, then there arises a nontrivial dependence equation of lower degree than the given one, which contradicts our assumption.

Otherwise it reduces to one of the following forms:

(i) $x_{\pm 2\epsilon_j} K + K' \in \mathcal{M}_\chi$,

(ii) $x_{\pm \epsilon_j \pm \epsilon_k} K + K' \in \mathcal{M}_\chi$,

(iii) $g_{\epsilon_1 - \epsilon_2} K + K' \in \mathcal{M}_\chi$,

where $K, K'$ commute with $x_\alpha$ and $x_{-\alpha}$ modulo $\mathcal{M}_\chi$.

By making use of proofs of proposition 4.1 in [3] and theorem 2.1 in [5], we may reduce (i) and (ii) to the equation of the form

$x_{\epsilon_1 - \epsilon_2} K + K' \in \mathcal{M}_\chi$,

where $K$ commute with $x_{\pm(\epsilon_1 - \epsilon_2)}$ and $K'$ commute with $x_{\epsilon_1 - \epsilon_2}$ modulo $\mathcal{M}_\chi$.

We have $x_{\epsilon_1 - \epsilon_2}^p K + x_{\epsilon_1 - \epsilon_2}^{p-1} K' \equiv 0$, so we get $x_{\epsilon_1 - \epsilon_2}^{p-1} K' \equiv 0$.

Subtracting $x_{\epsilon_2 - \epsilon_1} x_{\epsilon_1 - \epsilon_2} K + x_{\epsilon_2 - \epsilon_1} K' \equiv 0$ from this equation, we obtain $-x_{\epsilon_2 - \epsilon_1} x_{\epsilon_1 - \epsilon_2} K + g_\alpha K' \equiv 0$. We should remember
that $g_\alpha$ is invertible in $U(L)/\mathfrak{M}_\chi$ by virtue of [8].

By the way we use $w_\alpha := (h_\alpha + 1)^2 + 4x_\alpha x_\alpha \in$ the center of $U(\mathfrak{sl}_2(F))$. Hence we have $-4^{-1}\{w_\alpha - (h_\alpha + 1)^2\}K + g_\alpha K' \equiv 0$. So we obtain

$$4^{-1}g_\alpha^{-1}\{(h_\alpha + 1)^2 - w_\alpha\} + cK' \equiv 0 \cdots (\ast)$$

and from the start equation we get

$$cx_\alpha K + cK' \equiv 0 \cdots (**).$$

Subtracting (** from (\ast), we get $4^{-1}g_\alpha^{-1}\{(h_\alpha + 1)^2 - w_\alpha\}K - cx_\alpha K \equiv 0$. Multiplying this equation by $g_\alpha^{1-p}$ to the right, we have

$$4^{-1}g_\alpha^{p-1}\{(h_\alpha + 1)^2 - w_\alpha\}g_\alpha^{1-p}K - cx_\alpha g_\alpha^{1-p}K \equiv 4^{-1}g_\alpha^{p-1}\{(h_\alpha + 1)^2 - w_\alpha\}g_\alpha^{1-p}K + x_\alpha x_\alpha K \equiv 0.$$

Conjugation of the brace of this equation $(p - 1)$-times by $g_\alpha$ gives rise to $4^{-1}\{(h_\alpha - 1)^2 - w_\alpha\}K + x_\alpha x_\alpha K \equiv 0$. Next multiplying $x_{-\alpha}^{p-1}$ to the right of the last equation, we obtain

$$\{(h_\alpha - 1)^2 - w_\alpha\}Kx_{-\alpha}^{p-1} \equiv 0 \text{ modulo } \mathfrak{M}_\chi.$$

Now we multiply $x_\alpha$ to the left of this equation consecutively until it becomes of the form

$$(a \text{ nonzero polynomial of degree } \geq 1 \text{ with respect to } h_\alpha)K \equiv 0 \text{ modulo } \mathfrak{M}_\chi.$$
If we make use of conjugation and subtraction consecutively, then we arrive at a contradiction \( K \equiv 0 \).

Next for the case (iii), we change it to the form (iii)' \( K + g^{-1}_\alpha K' \in \mathcal{M}_\chi \).

We thus have an equation

\[
x_{\epsilon_1-\epsilon_2}K + x_{\epsilon_1-\epsilon_2}g_{\epsilon_1-\epsilon_2}K' \equiv 0 \text{ modulo } \mathcal{M}_\chi.
\]

According to the above argument, we are also led to a contradiction \( K \in \mathcal{M}_\chi \).

4. CONCLUDING REMARK

We have considered up to now the relationship of \( C_l \) and \( A_l \)-type modular Lie algebras with Hypo- Lie algebra.

So we may recapitulate the arguments in this paper as follows.

**Theorem 4.1.** Let \( F \) be any algebraically closed field of characteristic \( p \geq 7 \). Let \( L \) be any \( C_l \) or \( A_l \)-type modular Lie algebra over \( F \). We then assert that \( L \) is a Park's Lie algebra, and so a Hypo- Lie algebra.

**Proof.** Combining theorem 2.2, proposition 3.1 and proposition 3.2 gives rise to our assertion.

□

We are looking forward to claiming that any \( B_l \) and \( D_l \)-type modular Lie algebras also become a Hypo Lie algebra over any
algebraically closed field of characteristic $p \geq 7$.

The prime number 7 is important since all modular Lie algebras of classical type are simple for $p \geq 7$ if we disregard their centers.

Furthermore all modular simple Lie algebras are known to be either of classical type or of Cartan type over any algebraically closed field of characteristic $p \geq 7$.

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