THE PRIMITIVE EQUATIONS WITH STOCHASTIC WIND DRIVEN BOUNDARY CONDITIONS: GLOBAL STRONG WELL-POSEDNESS IN CRITICAL SPACES

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ABSTRACT. This article studies the primitive equations for geophysical flows subject to stochastic wind driven boundary conditions modeled by a cylindrical Wiener process. A rigorous treatment of stochastic boundary conditions yields that these equations admit a unique global, strong, pathwise solution within the $L^1_t-L^p_x$-setting of critical spaces. Critical spaces are established for the first time within the setting of the stochastic primitive equations.

1. Introduction

Wind driven boundary conditions for the coupled atmosphere and ocean primitive equations within the deterministic setting were introduced and studied by Lions, Temam and Wang in their fundamental article [30]. For related results concerning deterministic wind driven boundary conditions for the Navier-Stokes equations we refer to the work of Desjardins and Grenier [14], Bresch and Simon [3] and Dalibard and Saint-Raymond [9].

In order to describe the situation for the primitive equations in a simple geometric setting, we consider a cylindrical domain $D = G \times (-h,0) \subset \mathbb{R}^3$ with $G = (0,1) \times (0,1)$ and $h > 0$. Let us denote by $v: D \times (0,T) \to \mathbb{R}^2$ the horizontal velocity of the fluid and by $p_s: G \times (0,T) \to \mathbb{R}$ its surface pressure on a time interval $(0,T)$, where $T > 0$. There exist several equivalent formulations of the primitive equations, depending on whether the vertical velocity $w = w(v)$ is completely substituted by the horizontal velocity $v$ and the full pressure by the surface pressure, respectively, compare e.g. [24]. Here we consider the set of equations

\[
\begin{cases}
\partial_t v + v \cdot \nabla_H v + w(v) \cdot \partial_z v - \Delta v + \nabla H p_s = f, & \text{in } D \times (0,T), \\
\text{div}_H \tau = 0, & \text{in } D \times (0,T), \\
v(0) = v_0, & \text{in } D,
\end{cases}
\]

(1.1)

where $\tau(x,y) = \frac{1}{2} \int_{-h}^0 v(x,y,\xi)d\xi$, and the vertical velocity $w = w(v)$ with $w(x,y,-h) = w(x,y,0) = 0$ is given by $w(v)(x,y,z) = -\int_{-h}^z \text{div}_H v(x,y,\xi)d\xi$. Here $x,y \in G$ denote the horizontal coordinates and $z \in (-h,0)$ the vertical one.

The equations (1.1) are supplemented by mixed boundary conditions on $\Gamma_u = G \times \{0\}$, $\Gamma_b = G \times \{-h\}$ and $\Gamma_l = \partial G \times (-h,0)$ of the form

\[
\begin{align*}
& v, p_s \text{ are periodic on } \Gamma_l \times (0,T), \\
& v = 0 \text{ or } \partial_z v = 0 \text{ on } \Gamma_b \times (0,T), \\
& \partial_z v = c q^{air}(v^{air} - v) \cdot |v^{air} - v| \text{ on } \Gamma_u \times (0,T).
\end{align*}
\]

(1.2, 1.3, 1.4)

Here $v^{air}$ denotes the velocity of the wind, $q^{air}$ the density of the atmosphere and $c$ the drag coefficient. The boundary condition (1.4) is interpreted as the physical law describing the driving mechanism on the atmosphere-ocean interface as a balance of the shear stress of the ocean and the horizontal wind force. Indeed, the shear stress of the ocean, i.e. the tangential component of the stress tensor is given by $\partial_z v + \nabla w$, which due to the flatness of the interface, i.e. $w = 0$ on $\Gamma_u$, equals $\partial_z v$, for details see [30].

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At first glance a natural boundary condition on the interface would be an adherence condition, i.e.
$v = v^{\text{air}}$, at the interface. These conditions are, however, not being used due to the occurrence of
boundary layers in the atmosphere and in the ocean at the surface. The above condition \((1.4)\) takes
into account these boundary layers. Since the velocity of air is much slower than the one of the ocean,
the term \(v\) is frequently neglected and the condition
\[
\partial_z v = cg^{\text{air}} v^{\text{air}} \cdot |v^{\text{air}}| \text{ on } \Gamma_u \times (0, T)
\]
is used instead, see e.g. [21, 40].

The mathematical analysis of the deterministic primitive equations has been pioneered by Lions,
Teman and Wang in their articles [28–30], where the existence of a global, weak solution to the primitive
equations is proven. For global weak solutions subject to \((1.4)\) we refer to [33]. The uniqueness property
of solutions remains an open problem until today. A landmark result on the global strong well-posedness
of the deterministic primitive equations subject to homogeneous Neumann conditions for initial data in
\(H^1\) was shown by Cao and Titi in [7] by the method of energy estimates. For mixed Dirichlet-Neumann
conditions, see the work of Kukavica and Ziane [26].

A different approach to the deterministic primitive equations, based on methods of evolution
equations, has been introduced in [20, 24]. This approach is based on the hydrostatic Stokes operator \(A_p\)
and the hydrostatic Stokes semigroup defined for \(p \in (1, \infty)\) on the hydrostatic solenoidal spaces \(L^p(\Omega)\)
defined by \(L^p(\Omega) := \{ v \in L^p(\Omega)^2 : \text{div} v = 0, \text{per} \text{iodic in } x, y\text{-directions} \}.\) For a survey on results
concerning the deterministic primitive equations using the approach of energy estimates, we refer to
[27]; for a survey concerning the approach based on evolution equations, see [23].

In this article we extend the above results in at least three directions: we first introduce wind driven
stochastic boundary conditions on the surface of the ocean and analyze the primitive equation subject to
these boundary conditions as a SPDE. Stochastic wind driven boundary conditions have been considered
before within the setting of the shallow water equations e.g. by Cessi and Louazei [8] from a modeling
point of view. For numerical results and statistical analysis of wind stress time series in the context of
the Ekman equation, we refer to the work of Buffoni, Cappeletti and Picco [5]. Our result seems to be
the first rigorous result concerning stochastic boundary conditions driven by wind.

Secondly, we prove that this set of equations admits a unique, global, pathwise solution in the strong
sense not only for smooth data but thirdly for data belonging to certain critical spaces. More precisely,
given a cylindrical Wiener process \(W\) on a separable Hilbert space \(\mathcal{H}\) with respect to a filtration \(\mathcal{F}\) and
adapted functions \(H_f\) and \(h_0\), we consider for the horizontal velocity of the fluid \(V : \Omega \times D \times (0, T) \to \mathbb{R}^2\)
and the surface pressure \(P_s : \Omega \times G \times (0, T) \to \mathbb{R}\), where \((\Omega, \mathcal{A}, P)\) is a probability space endowed with
the filtration \(\mathcal{F}\), the equations
\[
\begin{aligned}
\text{d}V + V \cdot \nabla_H V + w(V) \cdot \partial_z V - \Delta V + \nabla_H P_s dt &= H_f \text{d}W_f \quad \text{in } D \times (0, T), \\
\text{div}_H V &= 0, \quad \text{in } D \times (0, T), \\
V(0) &= V_0, \quad \text{in } D,
\end{aligned}
\]
subject to boundary conditions \((1.2)\) and \((1.3)\), but where the deterministic condition \((1.4)\) or \((1.5)\) is
replaced by a stochastic boundary condition modeling the wind as
\[
\partial_z V = h_b \partial_z \omega \quad \text{on } \Gamma_u \times (0, T).
\]
Here \(h_b\) is a function defined on \(\Gamma_u \times (0, T)\) and we assume that \(\omega\) can be written as
\[
\omega(t) = \sum_{n=1}^{\infty} <g, e_n> W_b(t)e_n,
\]
where \(g\) is a suitable function defined on \(\Gamma_u\), \(W_b\) is another cylindrical Wiener process on \(\mathcal{H}\) with respect
to the filtration \(\mathcal{F}\), and \((e_n)\) is an orthonormal basis of \(\mathcal{H}\).

Our strategy to prove unique, global, strong, pathwise well-posedness for equations \((1.6)\) and \((1.7)\) is
based on a combination of stochastic and deterministic methods. It can be summarized as follows: First,
in order to eliminate the pressure term we apply the hydrostatic Helmholtz projection \(\mathbb{P}\) to equation \((1.6)\)
(see [24]) and rewrite the stochastic primitive equations as a semilinear stochastic evolution equation in
the space $L^p_w(D)$ of the form
\begin{equation}
\label{eq:1.8}
dV + A_p V \, dt = F(V,V) \, dt + H_f dW, \quad V(0) = V_0.
\end{equation}
Here $A_p$ denotes the hydrostatic Stokes operator defined in $L^p_w(D)$ as $A_p = \mathbb{P} \Delta$ with domain $D(A_p)$ defined as in Section 2.2 below (see [24] for details) and $F(\cdot, \cdot)$ is the bilinear convection term.

Secondly, we rewrite the stochastic boundary condition as a forcing term. Indeed, adapting an approach due to Da Prato and Zabczyk [10] to the given situation, a solution $V, P_b$ to equation (1.6) subject to (1.2), (1.3) and the stochastic condition (1.7) is expressed by a solution to the equation
\begin{equation}
\label{eq:1.9}
dz_b(t) + A_r z_b(t) \, dt = [\Lambda h_b(t)g] \, dW_b(t), \quad z_b(0) = 0,
\end{equation}
subject to the boundary conditions
\begin{align*}
\text{Z}_b \text{ are periodic on } \Gamma_i \times (0,T), \\
\partial_z z_b = 0 \text{ on } \Gamma_u \cup \Gamma_b \times (0,T) \quad \text{or} \\
\partial_z z_b = 0 \text{ on } \Gamma_u \times (0,T), \quad Z_b = 0 \text{ on } \Gamma_b \times (0,T),
\end{align*}
where $V_b := V - Z_b$ solves the equation
\begin{equation}
\label{eq:1.10}
dV_b + A_p V_b \, dt = F(V_b + Z_b, V_b + Z_b) \, dt + H_f dW, \quad V(0) = V_0
\end{equation}
subject to the same homogeneous boundary conditions. Here $\Lambda$ denotes the so-called Neumann operator mapping deterministic in-homogeneous boundary data to the solution of the associated stationary hydrostatic Stokes problem. This hydrostatic Neumann operator is constructed in Section 4. This construction allows us to view the stochastic boundary condition as a stochastic forcing term. For a similar approach within the setting of parabolic equations in divergence form we also refer to [38]. We note that $Z_b$ is given by
\begin{equation}
\label{eq:1.11}
z_b(t) := \int_0^t e^{(t-s)A_r} [\Lambda h_b(t)g] \, dW_b(t).
\end{equation}
Thirdly, we investigate $Z_b$ as well as the solution $Z_f$ of the linearized system with linear noise
\begin{equation}
\label{eq:1.12}
dZ_f + A_p Z_f \, dt = H_f dW, \quad Z_f(0) = 0,
\end{equation}
in the solenoidal ground space $L^p_w(D)$ by the results on maximal stochastic regularity due to Van Neerven, Veraar and Weis [31]. The latter are applicable due to the fact that $-A_p$ admits a bounded $H^\infty$-calculus in $L^p_w(D)$, see [18]. In particular, singular integrals of the form (1.11) are well-defined in stochastic maximal regularity spaces provided $\Lambda(h_b(\cdot)g)$ lies in the corresponding ground space. The initial value $Z_0$ contains the probabilistic part of the initial value $V_0$ and $v_0 := V_0 - Z_0$ is deterministic.

Subsequently, we consider pathwise the remainder term $v := V - Z$ for $Z := Z_f + Z_b$, which solves (almost surely) the system
\begin{equation}
\label{eq:1.13}
\partial_t v + A_p v = F(v + Z, v + Z), \quad v(0) = v_0.
\end{equation}
Regularity properties of $Z$ allow us to regard (1.11) as a deterministic, nonautonomous, semilinear evolution equation. Let us note that local existence results for smooth initial data for (1.11) can be achieved by standard arguments, however, the latter is not the case for global existence and uniqueness results without smallness and smoothness assumptions on the data.

Aiming for optimal conditions on the initial data lying in critical spaces, we make use of the extension of the theory of semilinear evolution equations in critical spaces due to Prüss, Simonett and Wilke [35] to the nonautonomous situation discussed in [25]. The property that $-A_p$ admits a bounded $H^\infty$-calculus on $L^p_w(D)$ enables us to prove the existence of a unique, global, strong solution to (1.11) for initial data belonging to critical spaces. These critical spaces, defined in Subsection 2.2 in detail, give optimal conditions for the initial data both for the stochastic and deterministic setting. We hereby use the theory of time weighted maximal regularity. More precisely, we prove the existence and uniqueness of a global, strong solution to (1.11) for initial values lying in critical spaces, which for the deterministic part are given for $p, q \in (1, \infty)$ as subspaces of the Besov spaces
\begin{equation}
\label{eq:2.14}
B^{2/p}_{pq} \quad \text{for } p, q \in [2, \infty]
\end{equation}
and for the stochastic part for \( p, q \in (1, \infty) \) by subspaces of
\[
B_{pq}^{2-2/q} \quad \text{for } p, q \in [2, \infty] \quad \text{and} \quad B_{pq}^{1-1/q} \quad \text{with } 1/p + 1/q < 1,
\]
depending whether one regards solutions in the \( L^2_t-L^p_x \) or \( L^2_t-D(A_p^{-1/2}) \)-setting. These spaces correspond in the situation of the Navier-Stokes equations to the critical function spaces \( B_{pq}^{n/p-1} \) introduced by Cannone [6] in the case of \( \mathbb{R}^d \).

Choosing for the deterministic part in particular \( p = q = 2 \) and \( \mu = 1 \) as weight function and noting that \( B_{22}^p = H^1 \), we rediscover the same regularity class of initial data as in the celebrated result by Cao and Titi [7] for the case of deterministic and homogeneous Neumann data. Choosing \( p > 2 \) allows us to enlarge the space of admissible initial values to \( B_{pq}^{2/p} \) and in particular to \( H^{2/p-p} \times H^{2/p-p} \subset B_{pq}^{2/p} \). Choosing \( p = q = 2 \), we obtain again \( H^1 \) as space for the stochastic initial data within the \( L^2_t-L^2_x \)-setting, and for \( q = 2 \) and \( p > 2 \) we obtain \( B_{p,q}^{1/2} \) or \( B_{2,2q}^{1-1/q} \) for \( q > 2 \) and \( p = 2 \) as space for the stochastic initial data.

For readers not being too enthusiastic about critical spaces, let us emphasize that, under certain assumptions on \( H_f \) and \( h_b(\cdot)g \), we derive the existence and uniqueness of a global, strong, pathwise solution \( \nu \to (1.11) \) with \( \nu \in H^1(0,T; L^2_x(D)) \cap L^2(0,T; D(A_2)) \) for deterministic and stochastic initial data \( \nu_0 \) and \( Z_0 \) satisfying
\[
Z_0: \Omega \to \{ \nu \in H^1_{per}(D) \cap L^2_x(D): \nu|_{\Gamma_D} = 0 \} \quad \text{and} \quad \nu_0 \in \{ \nu \in H^1_{per}(D) \cap L^2_x(D): \nu|_{\Gamma_D} = 0 \},
\]
where \( \Gamma_D \) denotes the part of the boundary where Dirichlet boundary conditions are imposed. A final comment concerning how to prove the global existence of solutions is in order: the crucial a priori bounds for solutions of (1.11) in the relevant norms can be luckily deduced – even for the stochastic setting – from the deterministic ones by comparing the solution of (1.11) to solutions of the deterministic primitive equations with force term
\[
(1.12) \quad \partial_t \nu + A_{\nu} \nu = F(\nu, \nu) + F(Z, Z), \quad \nu(0) = \nu_0.
\]
For the solution of this equation, \( L^2_t-L^2_x \) a priori bounds are already well-established, see [7,19], and like this proving a priori bounds for (1.11) can be deduced from the ones for (1.12). Using the theory of time weights does not only yield well-posedness results in critical spaces within the \( L^2_t-L^2_x \)-setting, but allows us to give by the compactness of the embeddings \( X_{1/2,2} \) into \( X_{\mu,-1/q,q} \) an elegant argument for the global well-posedness in these spaces by using \( L^2_t-L^2_x \)-bounds, only.

Let us further note that our approach works simultaneously for Dirichlet, Neumann and mixed boundary conditions on the top and bottom boundaries, for details see Section 2.

Recently, Agresti and Veraar [2] developed a local theory for critical spaces for stochastic evolution equations analogously to the ideas in [35] for deterministic equations. One major difference is that due to the weaker smoothing of the stochastic convolution, the conditions on the weights used by them are more restrictive as in the deterministic case. By using our approach when solving (1.11) we are able to allow spatially rougher data \( \nu_0 \) than one could handle by considering (1.11) in the context of stochastic critical spaces. Secondly, the main advantage of our approach is that global existence results for the stochastic setting can de deduced from the deterministic one.

The three dimensional stochastic primitive equations with deterministic boundary conditions but stochastic forcing term have been studied before by several authors. Indeed, for the situation of additive noise, there exist and uniqueness results for pathwise, strong solutions within the \( L^2_x \)-setting; see [22]. They consider deterministic initial data in \( H^1(D) \) and choose Neumann boundary conditions on the bottom and top.

A global well-posed result for pathwise strong solutions of the primitive equations with deterministic and homogeneous boundary conditions was established for multiplicative white noise in time in [11,12] and later under weaker assumptions on the noise in [4]. Here Neumann conditions are used for the top and the bottom in [12] and a Dirichlet condition on the bottom combined with a mixed Dirichlet-Neumann on the top in [11]. Further results concerning the existence of ergodic invariant measures, weak-martingale solutions and Markov selection were shown in [15] and [16]. For results in two dimensions, see e.g. [17].
2. Stochastic boundary value problems for the primitive equation

In the following, we adapt an approach due to Da Prato and Zabczyk, compare [10, Section 6], to define a notion of solution to the stochastic primitive equations with stochastic boundary conditions, formulated here in Subsection 2.1 below in (2.1) and (2.2) - (2.5), respectively. To this end, we recapitulate some facts on the linearized primitive equations in Subsection 2.2, cylindrical Wiener processes and stochastic convolutions in Subsection 2.3, and then in Subsection 2.4 we introduce the hydrostatic Neumann map which maps deterministic inhomogeneous boundary data to the solution of the associated stationary hydrostatic Stokes problem. Then we use this operator to interpret the boundary condition as a stochastic forcing in Subsection 2.5, where our notion of solution is made precise.

2.1. Primitive equations with stochastic wind forcing on the boundary.

We investigate the primitive equations in the isothermal setting and in a cylindrical spatial domain \( D \) of the form

\[ D = G \times (-h, 0) \subseteq \mathbb{R}^3 \quad \text{with} \quad G = (0, 1) \times (0, 1), \quad \text{where} \ h > 0, \]
on a time interval \((0, T)\) with \(T > 0\), and a probability space \((\Omega, \mathcal{A}, P)\). The unknowns are the horizontal velocity of the fluid \( V: \Omega \times D \times (0, T) \rightarrow \mathbb{R}^2 \) and its surface pressure \( P_s: \Omega \times D \times (0, T) \rightarrow \mathbb{R} \), which are governed by the following (already reformulated) stochastic primitive equations

\[
\begin{aligned}
\frac{dV}{dt} + V \cdot \nabla_{H} V + w(V) \cdot \partial_z V - \Delta V + \nabla_{H} P_s dt &= H_f dW, \quad \text{in} \ D \times (0, T), \\
\text{div}_{H} V &= 0, \quad \text{in} \ D \times (0, T), \\
V(0) &= V_0, \quad \text{in} \ D.
\end{aligned}
\]

The vertical velocity \( w = w(V) \) is given by

\[ w(V)(x, y, z) = -\int_{-h}^{z} \text{div}_{H} V(x, y, \xi) d\xi, \quad \text{where} \ w(x, y, -h) = w(x, y, 0) = 0. \]

The equations (2.1) are supplemented by the boundary conditions on

\[ \Gamma_u = G \times \{0\}, \quad \Gamma_b = G \times \{-h\}, \quad \text{and} \quad \Gamma_l = \partial G \times (-h, 0), \]
i.e., the upper, bottom and lateral parts of the boundary \( \partial D \), respectively, where on the lateral parts

\[
\begin{aligned}
V, P_s \quad &\text{are periodic on} \ \Gamma_l \times (0, T), \\
(2.2) &\text{on the bottom part either} \\
(2.3) \quad &\partial_z V = 0 \text{ on} \ \Gamma_b \times (0, T), \quad \text{or} \\
(2.4) \quad &V = 0 \text{ on} \ \Gamma_b \times (0, T), \\
\end{aligned}
\]

and a stochastic forcing term modeling the wind is imposed on the upper part by

\[
(2.5) \quad \partial_z V = h_b \partial_z \omega \text{ on} \ \Gamma_u \times (0, T).
\]

Here, \( h_b: \Gamma_u \times (0, T) \rightarrow \mathbb{R} \) is a given real valued function, and \( \partial_z \omega \) stands for a noise term defined by a cylindrical Wiener process which is made precise in Subsection 2.3 below.
The hydrostatic Stokes operator and relevant function spaces.

Similarly to the Navier-Stokes equations, an appropriate framework for the primitive equations are hydrostatically solenoidal vector fields defined by

$$L^p_p(D) := \{V \in C^\infty_{per}(\overline{D})^2 : \text{div}_H V = 0\}$$

cf. [24]. Here, horizontal periodicity is modeled by the function space $C^\infty_{per}(\overline{D})$ defined as in [24, Section 2] as the space of smooth functions on $\overline{D}$, which are periodic only with respect to the horizontal $x,y$-coordinates and not necessarily in the vertical $z$-coordinate.

Moreover, there exists a continuous projection, the hydrostatic Helmholtz projection,

$$P : L^p_p(D)^2 \to L^p_p(D),$$

one function of which is to annihilate the pressure term $\nabla H P_s$ in (2.1), compare [18, 24].

For $p \in (1, \infty)$ and $s \in [0, \infty)$, we define the spaces

$$H^s_p(D) := C^\infty_{per}(\overline{D}) \cap \|H^s_p(D)\|_{\mathbb{L}^p(D)}.$$ 

where the two possible choices (2.3) and (2.4) for the boundary conditions on $\Gamma_b$ are comprised by the notation

$$\Gamma_N = \begin{cases} \Gamma_u \cup \Gamma_b, & \text{if (2.3) is imposed,} \\ \Gamma_u & \text{if (2.4) is imposed,} \end{cases} \text{and } \Gamma_D = (\Gamma_u \cup \Gamma_b) \setminus \Gamma_N.$$

We drop the index $p$ and write $A = A_p$ if there is no ambiguity concerning the domain of definition.

It was shown in [18, Theorem 3.1] that $-A_p$ admits a bounded $H^\infty$-calculus of angle zero in $L^p_p(D)$. The domains of the fractional powers are hence given by the complex interpolation spaces, see [18, Corollary 3.3], as

$$D(A_p^\theta) = [X_0, X_1 |_| \theta := X_\theta \text{ for } \theta \in (0, 1),$$

where $[\cdot, \cdot]|_\theta$ denotes the complex interpolation functor. These can be determined as

$$X_\theta = \begin{cases} \{ v \in H^{2\theta}_p(D)^2 \cap L^p_p(D) : \partial_z v \mid_{\Gamma_N} = 0, v \mid_{\Gamma_D} = 0 \}, & \frac{1}{2} + \frac{1}{2p} < \theta < 1, \\ \{ v \in H^{1,\theta}_p(D)^2 \cap L^p_p(D) : v \mid_{\Gamma_D} = 0 \}, & \frac{1}{2} < \theta < \frac{1}{2} + \frac{1}{2p}, \\ \{ v \in H^{2\theta}_p(D)^2 \cap L^p_p(D) \}, & 0 < \theta < \frac{1}{2p}. \end{cases}$$

(2.6)

A general setting for a Cauchy problem of the form $(\partial_t - A) v = f, v(0) = v_0$, are time-weighted vector valued $L^q$- and Sobolev spaces which for $q \in (1, \infty), \mu \in (1/q, 1], T \in (0, \infty)$ and a Banach space $X$ are defined by

$$L^q_\mu(0, T; X) := \{ v \in L^1_{loc}(0, T; X) : t^{1-\mu} v \in L^q(0, T; X) \}$$

and

$$H^{1, q}_\mu(0, T; X) := \{ v \in L^q_\mu(0, T; X) \cap H^{1, 1}_{loc}(0, T; X) : \partial_t v \in L^q_\mu(0, T; X) \},$$

cf. [34, Section 3.2.4]. Assuming that $X_1, X_0$ are Banach spaces such that $X_1 = D(A)$ is densely embedded into $X_0$, the time weighted maximal regularity space is

$$L^q_\mu(0, T; X_1) \cap H^{1, q}_\mu(0, T; X_0),$$

and its natural trace space $X_{u-1/q, q}$ is determined by means of real interpolation spaces $X_{\theta, q} := (X_0, X_1)_{\theta, q}$ for $\theta \in (0, 1)$ and $q \in (1, \infty)$. These can be computed here explicitly in terms of Besov spaces. Here, for $p, q \in (1, \infty)$ and $s \in [0, \infty)$ one defines the Besov spaces

$$B^s_{pq, per}(D) := C^\infty_{per}(\overline{D}) \cap \|B^s_{pq}(D)\|.$$
where $B_{pq}^r(D)$ denotes Besov spaces which are defined as restrictions of Besov spaces $B_{pq}^s(\mathbb{R}^3)$ on the whole space, cf. [41, Definitions 3.2.2]. Then, compare [18, Corollary 3.5], for $X_0 = L_{D}^p(D)$ and $X_1 = D(A_p)$, and $q \in (1, \infty) \bigg(\text{is defined for functions taking values in the space of } \gamma^{D} \bigg)$

$$
X_{\theta,q} = \begin{cases}
\{v \in B_{pq,\text{per}}^2(D) \cap L_{D}^p(D) : \partial_tv|_{\Gamma_N} = 0, \, v|_{\Gamma_D} = 0, \frac{1}{2} + \frac{1}{2p} < \theta < 1, \\
\{v \in B_{pq,\text{per}}^2(D) \cap L_{D}^p(D) : v|_{\Gamma_D} = 0, \frac{1}{2} < \theta < \frac{1}{2} + \frac{1}{2p}, \\
B_{pq,\text{per}}^2(D) \cap L_{D}^p(D),
\end{cases}
$$

(2.7)

2.3. Cylindrical Wiener processes and stochastic convolutions.

Let $(\Omega, \mathcal{A}, P)$ be a probability space with a filtration $\mathcal{F} = (\mathcal{F}_t)$. An $\mathcal{F}$-cylindrical Wiener process (or cylindrical Brownian motion) on a Hilbert space $\mathcal{H}$ is a bounded linear operator

$$
\mathcal{W} : L^2((0, \infty); \mathcal{H}) \to L^2(\Omega)
$$
such that for all $f, g \in \mathcal{H}$ and $0 \leq t \leq t'$:

a) The random variable $W(t)f := W(\mathbb{I}_{[0,t]} \otimes f)$ is centered Gaussian and $\mathcal{F}_t$-measurable;

b) $\mathbb{E}[W(t')f \cdot W(t)g] = t (f, g)_{\mathcal{H}}$;

c) The random variable $W(t')f - W(t)f$ is independent of $\mathcal{F}_t$.

We will also call $\mathcal{W}$ a cylindrical Wiener process. Assuming that $\mathcal{H}$ is separable with an orthonormal basis $(e_n)_n$ of $\mathcal{H}$, then $\beta_n(t) := W(t)e_n$ is a standard $\mathcal{F}$-Brownian motion, and we have the representation

$$
W(t)f = \sum_{n=1}^{\infty} \beta_n(t) (f, e_n)_{\mathcal{H}}, \quad \text{and } W(t) : \mathcal{H} \to L^2(\Omega) \quad \text{with } W(t) = \sum_{n=1}^{\infty} \beta_n(t) \langle \cdot, e_n \rangle_{\mathcal{H}}
$$
defines a family of linear operators. From now on we fix the separable Hilbert space and the filtration. For the definition of the stochastic integral with respect to $\mathcal{W}$ see [31]. Mostly, the stochastic integral is defined for functions taking values in the space of $\gamma$-radionifying operators $\gamma(\mathcal{H}; L^q(D))$, while we will use $L^q(D; \mathcal{H})$-valued functions as in [31]. The spaces $\gamma(\mathcal{H}; L^q(D))$ and $L^q(D; \mathcal{H})$ are isomorphic and both ways to define the integral are equivalent to each other. For $p = 2$ these spaces are isomorphic to the Hilbert-Schmidt operators from $\mathcal{H}$ to $L^2(D)$.

The mild solution to the stochastic hydrostatic Stokes equations

$$
dZ(t) + A_r Z(t) dt = H_f(t) dW(t), \quad Z(0) = Z_0
$$
for $\mathcal{F}$-adapted $H_f \in L^s(\Omega; L^r(0, T; L_{D}^q(D; \mathcal{H})))$ and $Z_0 : \Omega \to X_{1/2 - 1/s, s}$ strongly $\mathcal{F}_0$-measurable is given by Proposition 2.1 below – by the stochastic convolution

$$
Z_f(t) := e^{tA_r} Z_0 + \int_0^t e^{(t-s)A_r} H_f(s) dW(s).
$$

Here, $e^{tA_r}$ stands for the hydrostatic Stokes semigroup generated by $A_r$ in $L_{D}^q(D)))$ and $L_{D}^q(D; \mathcal{H})$ is the extension of $L_{D}^q(D)$ to $\mathcal{H}$-valued functions. Note that by [39, I.8.24] bounded operators on $L_{D}^q(D)$ admit an extension to $L_{D}^q(D; \mathcal{H})$ with identical norm. With a slight abuse of notation we will still denote this extension by the same symbol. Since $-A_r$ admits an $H^{\infty}$-calculus on $L_{D}^q(D)$ of angle $0$, compare [18, Theorem 3.1], we may apply here the theory of stochastic maximal $L'$-regularity developed by van Neerven, Veraar and Weis, cf. [31, 32] to define and estimate $Z_f$ as given in 2.9. We then obtain the following result.

Proposition 2.1 (Stochastic maximal regularity for the hydrostatic Stokes equations). Let $0 < T < \infty$, $s, r \geq 2$ with $r > 2$ if $s \neq 2$, $H_f \in L^s(\Omega; L^r(0, T; L_{D}^q(D; \mathcal{H})))$ $\mathcal{F}$-adapted, and $A_r$ the hydrostatic Stokes operator in $L_{D}^q(D)$.

(a) Then for any strongly $\mathcal{F}_0$-measurable $Z_0 : \Omega \to X_{1/2 - 1/s, s}$ the stochastic convolution (2.9) is well-defined, $Z_f$ given by (2.9) is $\mathcal{F}$-adapted and defines the unique solution to 2.8 satisfying

$$
Z_f \in H^{0,s}(0, T; D(A_r^{1/2-0})) \cap C([0, T]; X_{1/2 - 1/s, s})
$$
pathwise for any $\theta \in [0, 1/2)$.
(b) If in addition \( A_r^{1/2} H_f \in L^s(\Omega; L^s((0, \infty); L^p(D; H))) \) is \( \mathcal{F}_t \)-adapted and \( Z_0: \Omega \to X_{1-1/s, s} \) is strongly \( \mathcal{F}_0 \)-measurable then
\[
Z_f \in H^{\theta,s}(0, T; D(A_r^{1/2})) \cap C([0, T]; X_{1-1/s, s})
\]
pathwise for any \( \theta \in [0, 1/2) \).

**Remark 2.2** \((L^s\text{-estimates in the probability space).** Assuming in Proposition 2.1 additionally \( Z_0 \in L^s(\Omega; X_{1/2-1/s, s}) \), it follows that in Proposition 2.1(a) even
\[
Z_f \in L^s(\Omega; H^\theta,s(0, T; D(A_r^{1/2-\theta}))) \cap C([0, T]; X_{1/2-1/s, s}), \quad \theta \in [0, 1/2),
\]
and there exists a maximal regularity constant \( C > 0 \) independent of \( H_f \) such that if \( Z_0 = 0 \)
\[
\|Z_f\|_{L^s(\Omega; H^\theta,s(0, T; D(A_r^{1/2-\theta}))) \cap L^\infty([0, T]; X_{1/2-1/s, s})} \leq C \|H_f\|_{L^s(\Omega; L^s(0, T; L^p(D; H)))}.
\]
Analogously in part (b), one obtains
\[
Z_f \in L^s(\Omega; H^\theta,s(0, T; D(A_r^{1/2-\theta}))) \cap C([0, T]; X_{1-1/s, s}), \quad \theta \in [0, 1/2).
\]
However, the pathwise regularity result of Proposition 2.1 is sufficient for our purpose to construct pathwise solution.

### 2.4. The hydrostatic Neumann map.

Let us consider in \( L^p_0(D; H) \) the stationary deterministic hydrostatic Stokes equation
\[
(2.10) \quad \begin{cases}
-\Delta V + \nabla_h P_s = 0, & \text{in } D,
\mathrm{div}_h V = 0, & \text{in } D
\end{cases}
\]
subject to the boundary conditions
\[
(2.11) \quad V, P_s \text{ are periodic on } \Gamma_1, \quad \text{and} \quad \partial_n V = g \text{ on } \Gamma_u,
\]
for given boundary data \( g \) and where either (2.3) or (2.4) holds on the bottom. Here the boundary data are taken from the \( H \)-valued Sobolev-Slobodeckij spaces
\[
W^{s,r}_\text{per}(\Gamma_u; H) := C^\infty_{\text{per}}(\Gamma_u; H)^{1/s, 1/r},
\]
where \( W^{s,r}_\text{per}(\Gamma_u; H) \) is defined as restriction of \( W^{s,r}(\mathbb{R}^2; H) \). In Section 4 we prove the following result.

**Proposition 2.3** \((\text{Hydrostatic Neumann maps}). Let \( H \) be a separable Hilbert space and \( r \geq 2 \).

(a) For \( g \in W^{1-1/r, r}_\text{per}(\Gamma_u; H)^2 \cap L^0_\text{per}(D; H)^2 \) there exist unique \( V \in H^{2 \times 2}_\text{per}(D; H)^2 \cap L^0_\text{per}(D; H)^2 \) \( P_s \in H^{1-1/r}_\text{per}(\Gamma_u; H) \cap L^0_\text{per}(D; H) \)
solving (2.10) with (2.3) and (2.11). The Neumann map \( \Lambda \) given by
\[
\Lambda: W^{1-1/r, r}_\text{per}(\Gamma_u; H)^2 \cap L^0_\text{per}(\Gamma_u; H)^2 \to L^0_\text{per}(D; H)^2 \cap L^0_\text{per}(D; H), \quad \varphi \mapsto -\text{P} \Delta V.
\]
is continuous, and
\[
\Lambda(W^{2-1/r, r}_\text{per}(\Gamma_u; H)^2 \cap L^0_\text{per}(\Gamma_u; H)^2) \subset D(A_r^{1/2}),
\]
where the restricted map is continuous as well.

(b) For \( g \in W^{1-1/r, r}_\text{per}(\Gamma_u; H)^2 \) there exist unique \( V \in H^{2 \times 2}_\text{per}(D; H)^2 \cap L^r_\text{per}(D; H) \) \( P_s \in H^{1-1/r}_\text{per}(\Gamma_u; H) \cap L^0_\text{per}(D; H) \)
solving (2.10) with (2.4) and (2.11). The Neumann map \( \Lambda \) given by
\[
\Lambda: W^{1-1/r, r}_\text{per}(\Gamma_u; H)^2 \to L^r_\text{per}(D; H)^2, \quad g \mapsto -\text{P} \Delta V.
\]
is continuous, and
\[
\Lambda(W^{2-1/r, r}_\text{per}(\Gamma_u; H)^2) \subset D(A_r^{1/2}),
\]
where the restricted map is continuous as well.

For the Neumann map in the setting of diffusion equations, we refer to [1].
2.5. **Rewriting the stochastic boundary condition as a forcing term.**

To give a precise definition of a solution to (2.1) with boundary conditions (2.2) - (2.5), we assume that there are functions

\[ g : \Gamma_u \to \mathcal{H}^2 \quad \text{and} \quad h_b : \Gamma_u \times (0, T) \to \mathbb{R} \quad \text{with} \quad \int_{\Gamma_u} h_b(\cdot)g = 0 \quad \text{if (2.3) holds}, \]

and a cylindrical Wiener process \( W_b \) defined on \( \mathcal{H} \) with respect to the filtration \( \mathcal{F} \) such that the noise term can be written as

\[ \omega(t) = \sum_{n=1}^{\infty} < g, e_n > W_b(t)e_n = W_b(t)g \]

with an orthonormal basis \( (e_n) \). Examples for such an \( \omega \) can be found in [38]. For \( s \geq 2 \) let

\[ h_b(t)g \in W_{per}^{1-1/s,s}(\Gamma_u; \mathcal{H})^2 \quad \text{for} \quad t \in (0, T). \]

Considering the equation

\[
\begin{aligned}
\frac{d}{dt}Z_b(t) + A_rZ_b(t) \, dt &= 0, \quad Z_b(0) = 0, \\
\end{aligned}
\]

subject to the inhomogeneous stochastic boundary conditions

\[
\partial_z Z_b = h_b \partial_\omega \quad \text{on} \quad \Gamma_u \times (0, T)
\]

and (2.2), (2.3) or (2.4), we call \( Z_b \) a **strong pathwise solution**, if \( Z_b \) is a strong pathwise solution to the stochastic hydrostatic Stokes equations

\[
\begin{aligned}
\frac{d}{dt}Z_b(t) + A_rZ_b(t) \, dt &= [\Lambda(h_b(t)g)] \, dW_b(t), \quad Z_b(0) = 0.
\end{aligned}
\]

subject to the homogeneous boundary conditions

\[ \partial_z Z_b = 0 \quad \text{on} \quad \Gamma_u \times (0, T) \]

and (2.2), (2.3) or (2.4), where \( \Lambda \) denotes the hydrostatic Neumann operator defined in Subsection 2.4. Note, that \( Z_b \) is given by

\[ Z_b(t) := \int_0^t e^{(t-s)A_r}[\Lambda h_b(t)g] \, dW_b(t), \]

where as in [2.3] [39, I.8.24] is applied to obtain the necessary \( \mathcal{H} \)-valued extensions of operators in \( L_p^\infty(\mathcal{D}) \). Since \( -A_p \) admits a bounded \( H^\infty \)-calculus on \( L_p^\infty(\mathcal{D}) \), the above singular integral is well-defined provided \( h_b \) is a pointwise multiplier for \( g \in W_{per}^{1-1/s,s}(\Gamma_u; \mathcal{H}) \) by Proposition 2.3. This generalizes the original construction of \( Z_b \) by Da Prato and Zabczyk in [10] using the theory of Hilbert-Schmidt operators and certain assumptions on the covariance operator \( Q \) to the above setting. This leads us to the following definition of strong pathwise solutions to the stochastic primitive equations (2.1) with boundary conditions (2.2) - (2.5).

**Definition 2.4 (Solution of the inhomogeneous boundary value problem).** We call \( V \) and \( P_s \) a **strong, pathwise solution** to (2.1) subject to the inhomogeneous stochastic boundary condition (2.5), and (2.2), either (2.3) or (2.4), with initial condition \( V_0 : \Omega \to L_p^\infty(\mathcal{D}) \) provided that

\[ V, Z_b : \Omega \to L_p^0([0, T]; D(A_p)) \cap C([0, T]; L_p^\infty(\mathcal{D})), \quad \text{and} \quad P_s : \Omega \to L_p^0([0, T]; H^{1-p}(G)), \]

where \( Z_b \) is given by (2.13), and

\[ V_b := V - Z_b \]

and \( P_s \) are adapted and solve pathwise the equation

\[
\begin{aligned}
\frac{dV_b}{dt} + (V_b + Z_b) \cdot \nabla_H V_b + \Delta V_b + \nabla_W P_s &\, dt = h_f \, dW_b, \quad \text{in} \quad \mathcal{D} \times (0, T), \\
\text{div}_H V_b &= 0, \quad \text{in} \quad \mathcal{D} \times (0, T), \\
V_b(0) &= V_0, \quad \text{in} \quad \mathcal{D},
\end{aligned}
\]

subject to the boundary conditions

\[ P_s, V_b \text{ are periodic on } \Gamma_l \times (0, T), \]

\[ Z_b \text{ is given by (2.13).} \]
and either
\[ \partial_2 V_b = 0 \text{ on } \Gamma_u \cup \Gamma_b \times (0, T), \quad \text{or} \quad \partial_2 V_b = 0 \text{ on } \Gamma_u \times (0, T) \text{ and } V_b = 0 \text{ on } \Gamma_b \times (0, T). \]

### 3. Main Results

We are now in the position to formulate the main result of this article. To this end, we set
\[ H_b(\cdot) := \Lambda(h_b(\cdot)g), \]
where \( h_b \) and \( g \) are as in \ref{2.12} and \( \Lambda \) is the Neumann map from Proposition \ref{2.3}.

**Theorem 3.1** (Global strong pathwise well-posedness in \( L^q_p \)-spaces).
Let \( 0 < T < \infty, p, q \in [2, \infty) \) and \( \mu \in [1/p + 1/q, 1]. \) Assume that
\[
H_b, A_p^{1/2} H_b, \quad H_f, A_p^{1/2} H_f \in L^q(\Omega \times (0, T); L^p_\mu(D; \mathcal{H})), \quad Z_0 : \Omega \rightarrow X_{1-1/q,q} \quad \text{and} \quad v_0 \in X_{\mu-1/q,q}
\]
with \( H_f, A_p^{1/2} H_f, \mathcal{F}_\theta \)-adapted and \( Z_0 \) strongly \( \mathcal{F}_0 \)-measurable.

Then there exists a unique, strong, pathwise solution
\[
V = v + Z_f + Z_b \quad \text{and} \quad P_s
\]
to the stochastic primitive equations \ref{2.1} subject to \ref{2.5}, \ref{2.2}, and either \ref{2.3} or \ref{2.4}, in the sense of Definition \ref{2.4}, where \( Z_f \) given by \ref{2.4} and \( Z_b \) by \ref{2.3}, and where pathwise
\[
v \in H^{\theta,s}_\mu(0, T; L^p_\theta(D)) \cap L^q_\mu(0, T; D(A_p)), \quad P_s \in L^q_\mu(0, T; H^{1-p}(G) \cap L^2_\mu(G) \text{ and } Z_b, Z_f \in H^{\theta,s}(0, T; D(A^{1-\theta}_p)) \text{ for any } \theta \in [0, 1/2).
\]
Moreover, \( v \) and \( P_s \) depend continuously on \( v_0 \).

**Remark 3.2** (Regularity assumptions in Theorem 3.1). The mapping properties of \( \Lambda \), see Proposition \ref{2.3}, imply that the condition on \( H_b \) in Theorem 3.1 is in particular fulfilled if for \( 1/p + 1/q \leq 1 \)
\[ h_b(\cdot)g \in L^q(0, T; W^{2-1/p,p}_{per}(\Gamma_u; \mathcal{H})^2), \]
cf. Proposition \ref{2.3}. The conditions on the initial conditions are by \ref{2.7}
\[ v_0 \in \{ v \in B^{2/p}_{q,p,q}(D)^2 \cap L^q_\mu(D); \quad v|_{\Gamma_D} = 0 \} \text{ and } Z_0 : \Omega \rightarrow \{ z \in B^{2(1-1/q)}_q(D)^2 \cap L^q(D); \quad z|_{\Gamma_D} = 0 \}. \]

**Remark 3.3** (Integrability in the probability space). In the proof of Theorem 3.1 it is actually sufficient to have \( Z_f, Z_b \in L^{2q}(0, T; H^{1-2\theta}(D)) \). This gives us some freedom for choosing \( s, r \) in Proposition \ref{2.1}b. Indeed,
\[ H^{\theta,s}(0, T; H^{2-2\theta,r}(D)) \hookrightarrow L^{2q}(0, T; H^{1-2\theta}(D)) \quad \text{if} \quad \frac{1}{s} - \theta < \frac{1}{2q} \quad \text{and} \quad \frac{1}{r} - \frac{2 - 2\theta}{3} < \frac{1}{2p} - \frac{1}{3}.
\]
Hence, let \( p, q \) and \( \mu \) as above, \( r, s \in [2, \infty) \) with \( s > 2 \) if \( r \neq 2 \) and
\[ \frac{3}{2r} + \frac{1}{s} \leq \frac{1}{2} + \frac{3}{4p} + \frac{1}{2q}.
\]
Assume that
\[
H_f, A_p^{1/2} H_f, H_b, A_p^{1/2} H_b \in L^q(\Omega \times (0, T); L^p_\mu(D; \mathcal{H})), \quad Z_0 : \Omega \rightarrow X_{1-1/q,q} \quad \text{and} \quad v_0 \in X_{\mu-1/q,q},
\]
with \( H_f, A^{1/2} H_f, \mathcal{F}_\theta \)-adapted and \( Z_0 \) strongly \( \mathcal{F}_0 \)-measurable. Then there exists a unique, strong, pathwise solution \( V = v + Z_f + Z_b \) to the stochastic primitive equations \ref{2.1} subject to \ref{2.2}, \ref{2.3} or \ref{2.4} and \ref{2.7} satisfying pathwise
\[ v \in H^{\theta,s}_\mu(0, T; L^p_\theta(D)) \cap L^q_\mu(0, T; D(A_p)) \quad \text{and} \quad Z_f, Z_b \in H^{\theta,s}(0, T; D(A^{1-\theta}_p)) \quad \text{for} \quad \theta \in [0, 1/2).
\]
For the convenience of the reader we rephrase the assertions of Theorem 3.1 and Remark 3.2 for the case \( q = p = 2 \). Note that in this case, we have \( X^{1/2,2} = X_{1/2} = D(A^{1/2}_2). \)
Corollary 3.4 (Global strong pathwise well-posedness in $L^2_t-L^2_x$-spaces). Let $0 < T < \infty$,
$$
H_f, A^{1/2}_f H_f \in L^2(\Omega \times (0,T); L^2_F(D;\mathcal{H}))
$$
with $H_f, A^{1/2}_f H_f \mathcal{F}$-adapted, let the stochastic boundary data satisfy
$$
 h_b(\cdot)g \in L^2(0,T; H^{3/2}_\text{per}(\Gamma_\text{w};\mathcal{H})^2),
$$
and $Z_0$ be strongly $\mathcal{F}_0$-measurable and
$$
 v_0 \in \{ H^{1/2}_\text{per}(D)^2 \cap L^2_F(D) : v|_{\Gamma_D} = 0 \}, 
\quad Z_0 : \Omega \to \{ H^{1/2}_\text{per}(D)^2 \cap L^2_F(D) : z|_{\Gamma_D} = 0 \}.
$$
Then there exists a unique, strong, global, pathwise solution
$$
 V = v + Z_f + Z_b \quad \text{and} \quad P_s
$$
to the stochastic primitive equations (2.1) subject to (2.5), (2.2), and either (2.3) or (2.4), in the sense of Definition 2.4, where $Z_f$ is given by (2.10) and $Z_b$ by (2.13) and satisfy
$$
 v \in H^{1/2}(0,T; L^2_F(D)) \cap L^2(0,T; D(A_2^{-\theta})), \quad P_s \in L^2(0,T; H^{1}(\perp) \cap L^2_D(\perp)), 
\quad Z_b, Z_f \in H^{\theta,2}(0,T; D(A_2^{-\theta}))
$$
for any $\theta \in [0,1/2)$. Moreover, $v$ and $P_s$ depend continuously on $v_0$.

We now turn our attention to pathwise solutions in $L^p_\text{per}(D(A_2^{-1/2})$-spaces. Applying the hydrostatic Helmholtz projection $\mathbb{P}$ to the equation given in Definition 2.4, we get
$$
 V_b := V - Z_b \quad \text{with} \quad Z_b \quad \text{given by (2.13), this equation becomes}
$$
(3.1) \quad \begin{cases}
    dv_b - AV_b + \mathbb{P}[(V_b + Z_b) \cdot \nabla H(V_b + Z_b) + w(V_b + Z_b) \cdot \partial_w(V_b + Z_b)]dt &= H_fdW 
    \quad \text{in} \quad (0,T), \\
    V_b(0) &= V_0.
\end{cases}
$$
Due to the regularity assumption in Definition 2.4, all terms are well-defined in $L^p_\text{per}(D)$, and $V_b, Z_b : (0,T) \to D(A_p)$ satisfy the homogeneous boundary conditions required in Definition 2.4. Moreover, the pressure $P_s$ can be reconstructed in the required regularity class.

We now regard this equality in a weaker sense than Definition 2.4, namely as an equation in the ground space $D(A_p^{-1/2})$ of negative order. Hence, if $V_b, Z_b$ satisfy (3.1) pathwise in $D(A_p^{-1/2})$, then we call $V = V_b - Z_b$ a strong pathwise solution in $D(A_p^{-1/2})$ to (2.1) subject to the inhomogeneous stochastic boundary condition (2.5), and (2.2), either (2.3) or (2.4).

When considering equation (3.1) in $D(A_p^{-1/2})$, the question naturally arises whether all individual terms are well-defined in $D(A_p^{-1/2})$. One obstacle here is that for the deterministic primitive equations no strong solution theory seems to be available for the $L^p_\text{per}(D,A_p^{-1/2})$-setting. For the case of the Navier-Stokes equations such a theory is well understood, see e.g. [37] and the references therein. One of the main differences between the Navier-Stokes and the primitive equations is the structure of the nonlinearity, which can be written as
$$
\text{div}(u \cdot u) \quad \text{and} \quad \text{div}(u(v) \cdot v), \quad \text{where} \quad u(v) = (v, w(v)),
$$
respectively. When estimating these terms in the $W^{-1,p}$-norms, we observe that for the primitive equations terms of the form $w(v)v$ remain where $v(v)$ contains first order derivatives of $v$, and therefore the strategy from [37] for the Navier-Stokes equations does not seem to be applicable for the primitive equations.

However, having as in Theorem 3.3 below a decomposition of $V = v + Z_f + Z_b$, where $v$ has additional differentiability and $Z_f$ and $Z_b$ have additional integrability properties, then each term in (3.1) can be interpreted without difficulties as an element in $D(A_p^{-1/2})$, and the pressure can be reconstructed even with $P_s \in L^p_\text{per}(0,T; H^{1-p}(\perp) \cap L^2_D(\perp))$. This allows us to weaken the assumptions on the stochastic forcing terms $H_f$ and $H_b$. More precisely, we need to verify now only integrability properties of $H_f$ and $H_b$, whereas in Theorems 3.1 - considering $A^{1/2}_f H_f$ and $A^{1/2}_b$ - also differentiability and boundary conditions are involved. Due to the absence of a solution theory for $v$ in $D(A_p^{-1/2})$ in critical spaces, the regularity of $v_0$ is not improved when moving from the $L^p_F(D)$- to the $D(A_p^{-1/2})$-setting and we thus consider in the following result only the stochastic data within the critical setting.
Theorem 3.5 (Global strong pathwise well-posedness in $L_t^q \cdot D(A_p^{-1/2})$-spaces). Let $0 < T < \infty$, and $p, q \in [2, \infty)$ and $\mu \in [1/p + 1/q, 1]$. Assume that

$$H_u, H_f \in L^2(\Omega \times (0, T); L^2(D; \mathcal{H})), \quad Z_0 : \Omega \to X_{3/2 - 1/2q, 2q}, \quad \text{and} \quad v_0 \in X_{\mu - 1/q, q}$$

with $H_f \mathcal{F}$-adapted and $Z_0$ strongly $\mathcal{F}_0$-measurable. Then there exists a unique, strong, global, pathwise solution $V = v + Z_f + Z_b$ in $D(A_p^{-1/2})$ to the stochastic primitive equations \(2.1\) subject to \(2.2\), \(2.3\), and either \(2.4\) or \(2.5\), satisfying

$$v \in H^{1/2}(0, T; L^p(D)) \cap L^q(0, T; D(A_p)) \quad \text{and} \quad Z_b, Z_f \in H^{\theta, 2q}(0, T; D(A_{2p}^{1/2 - \theta})).$$

for any $\theta \in [0, 1/2]$. Moreover, $v$ depends continuously on $v_0$.

Remarks 3.6 (Regularity assumptions). Proposition 2.3 implies that the condition on $H_b$ in Theorem 3.5 is in particular fulfilled if for $1/p + 1/q < 1$

$$h_b(\cdot)g \in L^2(0, T; W^{1-1/2p, 2p}(\Gamma_u; \mathcal{H})^2).$$

The regularity assumption on the initial conditions are satisfied for

$$v_0 \in \{v \in B^{2/p}_p(D; \mathcal{H})^2 \cap L^p(D) : v|_{\Gamma_D} = 0\} \quad \text{and} \quad Z_0 : \Omega \to \{z \in B^{1-1/q}_q(D; \mathcal{H})^2 \cap L^q(D) : z|_{\Gamma_D} = 0\}.$$

4. The deterministic stationary hydrostatic Stokes problem for inhomogeneous data

In this section we deduce the properties of the Neumann map for the hydrostatic Stokes equations stated in Proposition 2.3. As a first step we consider the Laplacian with inhomogeneous Neumann boundary conditions.

Lemma 4.1 (Inhomogeneous boundary value problem for the Laplacian). Let $r \geq 2$.

(a) For $g \in W^{-1/r}_p(\Gamma_u)^2 \cap L^0(\Gamma_u)^2$ there is a unique $U \in H^{2/r}_p(D)^2 \cap L^0(\Gamma_u)^2$ solving $-\Delta U = 0$ in $D$ with \(2.11\) and \(2.3\). If in addition $g \in W^{-1/r}_p(\Gamma_u)^2$, then $\Delta U \in H^{1/r}_p(D)^2 \cap L^0(\Gamma_u)^2$.

(b) For $g \in W^{-1/r}_p(\Gamma_u)^2$ there is a unique $U \in H^{2/r}_p(D)^2$ solving $-\Delta U = 0$ in $D$ with \(2.11\) and \(2.4\). If even $g \in W^{-1/r}_p(\Gamma_u)^2$, then $\Delta U \in \{V \in H^{1/r}_p(D)^2 : V|_{\Gamma_b} = 0\}$.

Proof. Using a partial Fourier series Ansatz with

$$\hat{U}(z, k_H) = \int_{(0,1) \times (0,1)} U(z, x_H) e^{ik_H x_H} dx_H \quad \text{and} \quad \hat{g}(k_H) = \int_{(0,1) \times (0,1)} g(x_H) e^{ik_H x_H} dx_H$$

for $x_H = (x, y) \in (0, 1) \times (0, 1)$ and for $k_H \in (2\pi \mathbb{Z} \times 2\pi \mathbb{Z})$ the problem transforms into

$$(-\partial_z^2 + |k_H|^2) \hat{U}(\cdot, k_H) = 0 \quad \text{in} \quad (-h, 0);$$

$$\partial_z \hat{U}(\cdot, k_H) = \hat{g}(k_H) \quad \text{for} \quad z = 0,$$

and

(4.1) $\partial_z \hat{U}(\cdot, k_H) = 0$ for $z = -h \quad \text{and}$

(4.2) $\hat{U}(\cdot, k_H) = 0$ for $z = -h$,

respectively. In the case of (4.1), since $\int_{\Gamma_u} g = 0$, we consider $k_H \neq 0$, and an explicit solution is given by

$$\hat{U}(z, k_H) = \frac{\cosh((z + h)|k_H|)}{|k_H| \sinh(h|k_H|)} \hat{g}(k_H), \quad z \in (-h, 0), \quad k_H \in (2\pi \mathbb{Z} \times 2\pi \mathbb{Z}) \setminus \{0\},$$

and hence

$$U(z, x_H) = \sum_{k_H \in (2\pi \mathbb{Z} \times 2\pi \mathbb{Z}) \setminus \{0\}} \frac{\cosh((z + h)|k_H|)}{|k_H| \sinh(h|k_H|)} \hat{g}(k_H) e^{ik_H x_H}. $$
Note that for \( g \in H^{1/2,2}_{per}(\Gamma_u)^2 \cap L^2_{per}(\Gamma_u)^2 \), we have
\[
\int_D |\Delta_H U(z,x_H)|^2 \geq \sum_{k_H \in (2\pi \mathbb{Z} \times 2\pi \mathbb{Z})\setminus \{0\}} |k_H|^2 |\hat{g}(k_H)|^2 \left( \int_{-1}^0 \frac{\cosh((z+h)|k_H|)}{\sinh(h|k_H|)} \right) dz
\]
where we used that \( \frac{\sinh(2h|k_H|)}{4 \sinh^2(h|k_H|)} \leq C \) for all \( k_H \in (2\pi \mathbb{Z} \times 2\pi \mathbb{Z})\setminus \{0\} \). Since \( \partial^2_x U = -\Delta_H U \), it follows that \( U \in H^{2,2}_{per}(D)^2 \).

In the case of \( (4.3) \) an explicit solution is given by
\[
U(z,x_H) = (z+h)\hat{g}(0) + \sum_{k_H \in (2\pi \mathbb{Z} \times 2\pi \mathbb{Z})\setminus \{0\}} \frac{\sinh((z+h)|k_H|)}{|k_H| \cosh(h|k_H|)} \hat{g}(k_H) e^{ik_H x_H}.
\]

Similar to the above, one shows that for \( g \in H^{1/2,2}_{per}(\Gamma_u)^2 \), we obtain
\[
\int_D |\Delta_H U(z,x_H)|^2 \geq \sum_{k_H \in (2\pi \mathbb{Z} \times 2\pi \mathbb{Z})\setminus \{0\}} |k_H|^2 |\hat{g}(k_H)|^2 \left( \frac{\sinh(2h|k_H|)}{4 \sinh^2(h|k_H|)} + \frac{2h|k_H|}{|k_H| \cosh^2(h|k_H|)} \right)
\]
where
\[
\frac{\sinh(2h|k_H|)}{4 \sinh^2(h|k_H|)} \leq C \frac{2h|k_H|}{|k_H| \cosh^2(h|k_H|)} < \infty.
\]
Since \( \partial^2_x U = -\Delta_H U \), it follows that \( U \in H^{2,2}_{per}(D)^2 \).

To obtain the additional \( L^r \)- and \( W^{1,r} \)-regularity, note first that
\[
\text{H}^{1/2,2}_{\per}(\Gamma_u) \hookrightarrow \text{W}^{1-1/r,r}_{\per}(\Gamma_u) \quad \text{for} \quad r \geq 2.
\]

So, for \( g \in W^{1-1/r,r}_{\per}(\Gamma_u)^2 \) we obtain first an \( L^2 \)-solution \( U \in H^{2,2}_{per}(D)^2 \). Now, denote the horizontal periodic extension of \( U \) by \( EU \), and let \( \chi : \mathbb{R}^2 \rightarrow [0,1] \) be a smooth cut-off function with compact support and \( \chi \equiv 1 \) on \((0,1) \times (0,1)\), and \( \chi_u : [-h,0] \rightarrow [0,1] \) a smooth cut-off function which is equal to one close to zero, and vanishes at \(-h\), and \( \chi_b := 1 - \chi_u \). Set \( \xi_i := \chi_E \cdot \chi_b \) and \( \xi_u := \chi_E \cdot \chi_u \), then the functions \( \xi_i EU \) for \( i \in \{u,b\} \) solve on \( D_u = (-\infty,0) \times \mathbb{R}^2 \) and \( D_b = (-h,\infty) \times \mathbb{R}^2 \)
\[
\Delta \xi_i EU = -\Delta \xi_i EU - 2\nabla \xi_i \cdot \nabla EU \quad \text{on} \quad D_i, \quad i \in \{u,b\},
\]
with the boundary conditions
\[
\partial_z \xi_u EU = \xi_u EU g \quad \text{on} \quad \partial D_u \quad \text{and} \quad \xi_b EU = 0 \quad \text{on} \quad \partial D_b,
\]
for \( (2.3) \) and \( (2.4) \) respectively. Here, the right hand side of \( (4.3) \) is in \( H^{1,2}(D) \rightarrow L^r(D) \) for \( r \leq 6 \). So, using \( (4.3) \) we may apply classical results on inhomogeneous boundary value problems on half-spaces, cf. e.g. [1, Theorem 9.2] and [13]. Therefore, \( \xi_i EU \in H^{2,r}(D_i) \) for \( i \in \{u,b\} \), and hence \( U \in H^{2,r}(D) \) if \( r \geq 2 \). If \( r > 6 \), we use the embeddings \( W^{2-1/6,6}(\Gamma_u) \hookrightarrow W^{1-1/6,6}(\Gamma_u) \) and \( H^{1,6}(\Gamma_u) \hookrightarrow L^r(\Gamma_u) \), and using the above we show first that \( U \in H^{7/3,2}(D) \) and then analogously that \( U \in H^{7/3,2}(D) \). Moreover, if \( g \in W^{2-1/r,r}_{\per}(\Gamma_u)^2 \), then by [13, Theorem 3.3] \( \xi_i EU \in H^{3,r}(D_i) \) for \( i \in \{u,b\} \), and hence \( U \in H^{3,r}_{\per}(D)^2 \).

In the case of \( (2.4) \), the boundary condition follows from
\[
\Delta U(z,x_H) = \sum_{k_H \in (2\pi \mathbb{Z} \times 2\pi \mathbb{Z})\setminus \{0\}} 2|k_H|^2 \frac{\sinh((z+h)|k_H|)}{|k_H| \cosh(h|k_H|)} \hat{g}(k_H) e^{ik_H x_H}.
\]

Uniqueness follows since the difference of two solutions would solve the homogeneous boundary value problem, i.e., with \( g = 0 \) which has a unique solution in \( L^r(D) \). \( \square \)
Proof of Proposition 2.3. Assume first that $H = \mathbb{R}$. To prove the existence for the case a), assume that $V$ and $P_s$ are the desired solutions to (2.10) with (2.11), (2.3), and $U$ be the solution from Lemma 4.1(a). Then define

$$V_\delta := V - \tilde{U}, \quad \tilde{U} = (U - U),$$

where in particular $\text{div}_H \tilde{U} = 0$. Using that $\partial_z \tilde{U} = 0$, $V_\delta$ solves together with $P_s$

$$
\begin{cases}
-\Delta V_\delta + \nabla_H P_s = \Delta_H \tilde{U}, & \text{in } D, \\
\text{div}_H V_\delta = 0, & \text{in } D, \\
V_\delta, P_s & \text{are periodic on } \Gamma_t, \\
\partial_z V_\delta = 0 & \text{on } \Gamma_u \cup \Gamma_b.
\end{cases}
$$

This is a hydrostatic Stokes equation with inhomogeneous right hand side, and homogeneity Neumann boundary conditions. Since for $g \in W^{1-1/r,r} \perp (\Gamma_u)^2$, we have $U \in H^{2,r} \perp (\Gamma_u)^2$, it follows that $\tilde{U} \in H^{2,r} \perp (G)^2$ and hence $\Delta_H \tilde{U} \in L^r(D)^2$. So, there is a solution $V_\delta \in D(A_r)$ and $P_s \in H^{1,r} \perp (\Gamma_u)$, which is unique up to constants. Now, reversing the construction, we deduce that

$$V = V_\delta + (U - \tilde{U}) \in H^{2,r} \perp (D)^2 \quad \text{and} \quad P_s \in H^{1,r} \perp (\Gamma_u)$$

solve the original problem, where $\nabla = V_\delta$ and $\tilde{V} = \tilde{U}$. If even $g \in W^{2-1/r,r} \perp (\Gamma_u)^2$, then $U, \tilde{U} \in H^{3,r} \perp (D)^2$ and hence $\mathbb{P} \Delta_H \tilde{U} \in D(A_r^{1/2})$ using that $\mathbb{P} : H^{1,r} \perp (D)^2 \to H^{1,r} \perp (D)^2 \cap L^r(D)$ and (2.6). Therefore, $V_\delta \in D(A_r^{3/2})$ and hence

$$\mathbb{P} \Delta V = \mathbb{P} \Delta V_\delta + \mathbb{P} \Delta \tilde{U} \in H^{1,r} \perp (D)^2 \cap L^r(D) = D(A_r^{1/2}).$$

To prove the existence in (b), assume that $V, P_s$ solve (2.10) with (2.11), (2.3), and let $U$ be the solution from Lemma 4.1(b). Moreover let

$$\varphi \in C_0^\infty((-h,0); \mathbb{R}) \text{ with } \frac{1}{h} \int_h^0 \varphi = 1, \quad \chi \in C_0^\infty([-h,0]; \mathbb{R}) \text{ with } \int_h^0 \chi = 0, \chi(-h) = 1, \partial_z \chi(0) = 0.$$

Then set

$$U_1 := \varphi \cdot (1 - \mathbb{P}) \tilde{U} \quad \text{and} \quad U_2 := A_r^{-1}(\chi \cdot (\mathbb{P} \Delta U_1|_{\Gamma_b})).$$

Notice that since $\mathbb{P} V = \tilde{V} + \mathbb{P} V$

$$\mathbb{P} \Delta U_1 = \mathbb{P} \Delta (\varphi \cdot (1 - \mathbb{P}) \tilde{U}) = \varphi \Delta_H (1 - \mathbb{P}) \tilde{U} + \tilde{V} \mathbb{P} \Delta_H (1 - \mathbb{P}) \tilde{U} + \partial_z \varphi \mathbb{P} (1 - \mathbb{P}) \tilde{U}.$$}

Here due to the periodic boundary conditions $\mathbb{P} \Delta_H = \Delta_H \mathbb{P}$, and $\partial_z \varphi = \frac{1}{h}(\varphi (0) - \varphi (-h)) = 0$ since $\varphi \in C_0^\infty((-h,0); \mathbb{R})$, and therefore all but the first terms vanish, and hence by Lemma 4.1(b)

$$\mathbb{P} \Delta U_1 = \varphi \Delta_H (1 - \mathbb{P}) \tilde{U} + (\partial_z^2 \varphi) (1 - \mathbb{P}) \tilde{U} \in L^r(D)$$

and using that $\varphi \in C_0^\infty((-h,0); \mathbb{R})$

$$\chi \mathbb{P} \Delta U_1|_{\Gamma_b} = \chi (\varphi (-h) \Delta_H (1 - \mathbb{P}) \tilde{U}) \in L^r(D),$$

and in particular $U_2$ is well-defined. Now, set

$$V_\delta := V - U' \quad \text{with} \quad U' := U - U_1 + U_2$$

This implies, since $U_2 \in L^r(D)$ by construction, that $U'$ satisfies since $\tilde{U}_1 = (1 - \mathbb{P}) \tilde{U}$

$$\text{div}_H \tilde{U} = \text{div}_H U = \text{div}_H \mathbb{P} \tilde{U} = 0.$$}

Moreover, $U' = U = 0$ on $\Gamma_b$, and $\partial_z U' = \partial_z U = g$ on $\Gamma_u$. Then $V_\delta$ and $P_s$ solve

$$
\begin{cases}
-\Delta V_\delta + \nabla_H P_s = \Delta (U_2 - U_1), & \text{in } D, \\
\text{div}_H V_\delta = 0, & \text{in } D, \\
V_\delta, P_s & \text{are periodic on } \Gamma_t, \\
\partial_z V_\delta = 0 & \text{on } \Gamma_u, \\
V_\delta = 0 & \text{on } \Gamma_b.
\end{cases}
$$
Due to Lemma 4.1 one has $U \in H^2_{\text{per}}(D)$, and hence $\Delta U_1 = \Delta (\varphi \cdot (1 - \mathbb{P}) U) \in L^r(D)^2$. From (4.5) it follows that also $(\chi \cdot (\mathbb{P} \Delta U_1|_{\Gamma_b})) \in L^r(D)^2$ and since $\Delta A_{\nu}^{-1}$ is bounded in $L^r(D)^2$ also $\Delta U_2 \in L^r(D)^2$. So, also $\mathbb{P} \Delta (U_2 - U_1) \in L^r(D)$, and therefore there exists a unique solution $V_\delta$ in $D(A_{\nu})$ to the equation

$$-A_{\nu} V_\delta = \mathbb{P} \Delta (U_2 - U_1).$$

For the higher regularity notice first that by Lemma 4.1 (b) $U \in H^3_{\text{per}}(D)^2$, and hence \(\mathbb{P} \Delta U_1 = \mathbb{P} \Delta (\varphi \cdot (1 - \mathbb{P}) U) + (\partial_z^2 \varphi) (1 - \mathbb{P}) U \in H^1_{\text{per}}(D)^2 \cap L^r(D)^2\), and using (4.3) also

$$\mathbb{P} \Delta U_2 = (\chi \cdot (\mathbb{P} \Delta U_1|_{\Gamma_b})) = \chi \cdot \varphi (-\hat{h}) (1 - \mathbb{P}) \Delta H U \in H^1_{\text{per}}(D)^2 \cap L^r(D)^2.$$  

In order to verify the boundary condition at $\Gamma_b$ in (2.6), observe that since $\mathbb{P} \Delta (U_2) \big|_{\Gamma_b} = \mathbb{P} \Delta U_1 \big|_{\Gamma_b}$

$$\mathbb{P} \Delta (U_2 - U_1) \big|_{\Gamma_b} = \mathbb{P} \Delta (U_2) \big|_{\Gamma_b} - \mathbb{P} \Delta (U_1) \big|_{\Gamma_b} = 0.$$

So, by (2.6) one concludes that $\mathbb{P} \Delta (U_2 - U_1) \in D(A_{\nu}^{1/2})$, and hence $V_\delta \in D(A_{\nu}^{3/2})$ and $\mathbb{P} \Delta V_\delta \in D(A_{\nu}^{1/2})$. As above one concludes that also $\mathbb{P} \Delta U' = \mathbb{P} \Delta (U_2 - U_1) \in D(A_{\nu}^{1/2})$, and therefore $\Delta V = \mathbb{P} \Delta V \in D(A_{\nu}^{1/2})$.

Solutions to the problem with (2.8) are unique up to constants, because having two solutions $V_1, P_1$ and $V_2, P_2$, the difference $V_1 - V_2, P_1 - P_2$ solves the homogeneous primitive equations with homogeneous Neumann boundary conditions and hence these are constant, see e.g. [18]. Hence $V - \int_D V$ and $P_s - \int_{\Gamma_s} P_s$, are the unique average free solutions. For (2.4), the difference solves the homogeneous primitive equations with homogeneous Neumann boundary conditions on the top and Dirichlet boundary conditions on the bottom of the solution of which is zero, see e.g. [18].

Moreover, the Neumann maps $\Lambda$ are continuous as composition of continuous operators. By [39, I.8.24], the operator $\Lambda$ admits an $\mathcal{H}$-valued extension with norm identical to the one of $\Lambda$. We denote this extension, with a slight abuse of notation, again by $\Lambda$. \hfill $\Box$  

5. Local strong well-posedness

Similarly to [24, Section 5], we define for $1 < p < \infty$ the bilinear map $F$ by

$$F(v, v') := \mathbb{P} (v \cdot \nabla_H v' + w(v) \partial_z v'),$$

and set $F(v) := F(v, v)$. Then, applying the hydrostatic Helmholtz projection $\mathbb{P}$, the primitive equations with force term and stochastic boundary conditions from Definition 2.4 may be reformulated to become

$$dV_\delta + AV_\delta dt = F(V_\delta + Z_\delta) dt + f + H_f dW, \quad V_\delta(0) = V_0,$$

where one recalls that $V_\delta = V - Z_\delta$. Given $Z_f$ as in (2.9), one considers

$$v := V_\delta - Z_f = V + Z \quad \text{with} \quad Z := Z_f + Z_\delta$$

which solves the deterministic equation

$$\partial_t v + A v = f + F(v + Z, v + Z), \quad v(0) = v_0, \quad \text{where} \quad v_0 = V_0 - Z_0.$$  

The regularity of $Z$ needed to solve this equation in the strong sense, and hence also the regularity we have to impose on $H_f$ and $H_b$ will become apparent in the following lemma.

**Lemma 5.1** (Estimate on the non-linearity). Let $p, q \in (1, \infty)$ such that $1/p + 1/q \leq 1, \mu \in [1/p + 1/q, 1], 0 < T < \infty$. Then there exists a constant $C > 0$ such that for all

$$v \in L^p_T(0, T; D(A)) \cap H^1_T(0, T; L^r(D)) \quad \text{and} \quad Z \in L^q_T(0, T; H^1; L^r(D))$$

the following estimates hold:

$$\|F(v, v)\|_{L^p_T(0, T; L^r(D))} \leq C (\|v\|_{L^p_T(0, T; H^2(D))} + \|v\|_{H^1_T(0, T; L^r(D))})^2,$$

$$\|F(Z, v) + F(v, Z)\|_{L^p_T(0, T; L^r(D))} \leq C (\|v\|_{L^p_T(0, T; H^2(D))} + \|v\|_{H^1_T(0, T; L^r(D))}) \|Z\|_{L^q_T(0, T; H^1; L^r(D))},$$

$$\|F(Z, Z)\|_{L^p_T(0, T; L^r(D))} \leq C \|Z\|_{L^r_T(0, T; H^1; L^r(D))}^2.$$
and for all \(v, Z \in H^{1/2-1/2p,q}_\mu(0,T;H^{1+1/p,p}(\mathcal{D}))\) one has
\[
\|F(Z,v) + F(v,Z)\|_{L^q_\mu(0,T;L^p(\mathcal{D}))} \leq C \|Z\|_{H^{1/2-1/2p,q}_\mu(0,T;H^{1+1/p,p}(\mathcal{D}))} \|v\|_{H^{1/2-1/2p,q}_\mu(0,T;H^{1+1/p,p}(\mathcal{D}))}.
\]

**Proof.** As in [24, Proof of Lemma 5.1] we show that
\[
\|F(v,v')\|_{L^p(\mathcal{D})} \leq C \|v\|_{H^{1+1/p,p}} \|v'\|_{H^{1+1/p,p}} \quad \text{and} \quad \|F(v,v')\|_{L^p(\mathcal{D})} \leq C \|v'\|_{H^{1+1/p,p}} \|v\|_{H^{1+1/p,p}},
\]
where one uses the anisotropic estimates
\[
\|F(v,v')\|_{L^p(\mathcal{D})} \leq C \|v\|_{L^{p,p}_H \rightarrow L^{p,p}_H} \|v'\|_{L^{p,p}_H},
\]
and then applies the embeddings
\[
(5.2) \quad H^{1,2p}(\mathcal{D}) \hookrightarrow H^{1,p}_z L^{2p}_y, \quad H^{1,2p}(\mathcal{D}) \hookrightarrow L^p_z H^{1,2p}_y, \quad \text{and} \quad (5.3) \quad H^{1+1/p,p}(\mathcal{D}) \hookrightarrow H^{1,p}_z L^{2p}_y, \quad H^{1+1/p,p}(\mathcal{D}) \hookrightarrow L^p_z H^{1,2p}_y,
\]
where \(H^{s_1,p_1}_z H^{s_2,p_2}_y := H^{s_1,p_1}(-h,0;H^{s_2,p_2}(G))\) for \(s_1, s_2 \geq 0\) and \(p_1, p_2 \in [1,\infty]\). For the time-norms we estimate as in [36]
\[
\|F(v,v')\|_{L^q_\mu(0,T;L^p(\mathcal{D}))} \leq C \|v\|_{L^q_\mu(0,T;H^{1+1/p,p}(\mathcal{D}))} \|v'\|_{L^q_\mu(0,T;H^{1+1/p,p}(\mathcal{D}))},
\]
\[
\|F(v,v')\|_{L^q_\mu(0,T;L^p(\mathcal{D}))} \leq C \|v\|_{L^q_\mu(0,T;H^{1+1/p,p}(\mathcal{D}))} \|v'\|_{L^q_\mu(0,T;H^{1+1/p,p}(\mathcal{D}))},
\]
\[
\|F(v,v')\|_{L^q_\mu(0,T;L^p(\mathcal{D}))} \leq C \|v\|_{L^q_\mu(0,T;H^{1+1/p,p}(\mathcal{D}))} \|v'\|_{L^q_\mu(0,T;H^{1+1/p,p}(\mathcal{D}))},
\]
and again as in [36] we may apply Sobolev’s and Hardy’s inequalities to obtain the embedding
\[
L^q_\mu(0,T;H^{2,p}(\mathcal{D})) \cap H^{1,q}(0,T;L^p(\mathcal{D})) \hookrightarrow L^{2q}_\sigma(0,T;H^{1+1/p,q}(\mathcal{D})).
\]
The last inequality uses the embeddings [5.3] and concludes, once more as in [36], that
\[
H^{1/2-1/2p,q}_\mu(0,T;H^{1+1/p,p}(\mathcal{D})) \hookrightarrow L^{2q}_\sigma(0,T;H^{1+1/p,q}(\mathcal{D})). \quad \Box
\]

Since \(-A\) admits a bounded \(H^\infty\)-calculus on \(L^p_\sigma(\mathcal{D})\), we may use Lemma 5.1 and the theory of non-autonomous semilinear equations in critical spaces, (see [36] and [25, Proposition 2.3] for the adaptation to the non-autonomous case) to obtain the following local existence result. In particular, we use here the embedding
\[
L^q_\mu(0,T;H^2(\mathcal{D})) \cap H^{1,q}_\mu(0,T;L^p(\mathcal{D})) \hookrightarrow H^{1/2-1/2p,q}_\mu(0,T;H^{1+1/p,q}(\mathcal{D})).
\]

**Proposition 5.2 (Local well-posedness in \(L^q_\mu-L^q_\sigma\)-spaces).** Let \(p,q \in (1,\infty)\) with \(1/p + 1/q \leq 1\), \(\mu \in [1/p + 1/q,1]\) and \(T > 0\). Assume that
\[
v_0 \in X^{1-1/q,q}_{\mu} \quad \text{and} \quad Z \in H^{1/2-1/2p,q}_\mu(0,T;D(A^{1/2}_{1/2})), \quad \text{or} \quad Z \in L^{2q}_\sigma(0,T;D(A^{1/2}_{1/2})) \quad \text{for} \quad \sigma = 1/2 + \mu/2.
\]
Then there exists \(T' = T'(v_0,f,Z) < 0 < T' \leq T\) and a unique, strong solution \(v\) to the deterministic equation (6.1) on \((0,T')\)
\[
v \in H^{1,q}_\mu(0,T';L^p(\mathcal{D})) \cap L^q_\mu(0,T';D(A_p))
\]
Moreover, the solution depends continuously on the data.

6. **GLOBAL WELL-POSEDNESS**

In this section we deduce the crucial a priori bounds by comparing the solution of (5.1) to solutions of the *deterministic* primitive equations with force term
\[
(6.1) \quad \partial_t v + A_p v = F(v,v) + F(Z,Z), \quad v(0) = v_0.
\]
For the solution of the latter equation, \(L^2_\mu-L^2_\sigma\) a priori bounds are already well-established (see [7] for Neumann boundary conditions and [26] and [19, 24] for mixed boundary conditions), and hence a priori bounds for solutions of (5.1) will be deduced from the ones for (6.1). Then, using the theory of time weights does not only yield local well-posedness results in critical spaces within the \(L^2_\mu-L^2_\sigma\)-setting, but
allows us to give by the compactness of the embeddings \( X_{1/2,2} \hookrightarrow X_{\mu, -1/q,q} \) an elegant argument for the global well-posedness in these spaces by using \( L^2_t L^2_x \)-bounds, only.

6.1. **Global strong well-posedness in \( L^2_t L^2_x \)-spaces.**

The global, strong well-posedness result for \( \delta \xi \) in the \( L^2_t L^2_x \)-setting reads as follows.

**Proposition 6.1 (\( L^2_t L^2_x \) a priori bounds).** Let \( 0 < T < \infty \) and

\[
 v_0 \in \{ H^{1/2}_{\perp} (\mathcal{D}) \cap L^2_{\sigma}(\mathcal{D}): v\big|_{\Gamma_D} = 0 \}, \quad \mathbb{P} f \in L^2(0, T; L^2_{\sigma}(\mathcal{D})), \quad \text{and} \quad Z \in H^{1/4, 2}(0, T; D(A^{3/4}_2)) \quad \text{or} \quad Z \in L^4(0, T; D(A^{1/2}_2)).
\]

Then there exists a continuous function \( B: [0, \infty)^4 \to [0, \infty) \) such that

\[
 \| v \|_{L^2(0, T; D(A^2_2))} + \| v \|_{H^{1/2}(0, T; L^{2}_{\sigma}(\mathcal{D}))} \leq B(\| v_0 \|_{H^{1/2}(\mathcal{D})}, \| f \|_{L^2(0, T; L^{2}_{\sigma}(\mathcal{D}))}, \| Z \|, T),
\]

where \( \| Z \| = \| Z \|_{L^4(0, T; H^{1/4, 2}(\mathcal{D}))} \) or \( \| Z \| = \| Z \|_{H^{1/4, 2}(0, T; H^{3/4, 2}(\mathcal{D}))} \), respectively. In particular, the local strong solution to the perturbed primitive equations \( \delta \xi \) extends to a unique global strong solution

\[
 v \in H^1(0, T; L^2_{\sigma}(\mathcal{D})) \cap L^2(0, T; D(A^2_2)),
\]

where the solution depends continuously on the data.

**Proof.** First, we solve the primitive equations with forcing term

\[
 \partial_t u - A_2 u = F(u, u) + f(Z), \quad u(0) = v_0, \quad \text{where} \ f(Z) := F(Z, Z) + f,
\]

This equation has a unique, global, strong solution (compare for instance [19, 24])

\[
 u \in H^1(0, T; L^2_{\sigma}(\mathcal{D})) \cap L^2(0, T; D(A^2_2)),
\]

and there exists an a priori bound by means of a continuous function \( B_u: [0, \infty)^3 \to [0, \infty) \) such that

\[
 \| u \|_{L^2(0, T; D(A^2_2))} + \| u \|_{H^{1/2}(0, T; L^{2}_{\sigma}(\mathcal{D}))} \leq B_u(\| v_0 \|_{H^{1/2}(\mathcal{D})}, \| f \|_{L^2(0, T; L^{2}_{\sigma}(\mathcal{D}))}, T).
\]

We then consider the perturbed primitive equations

\[
 \partial_t \xi - A_2 \xi = F(\xi + Z, \xi + Z) + g(u, Z), \quad \xi(0) = v_0, \quad \text{where} \ g(u, Z) = f - F(u, Z) - F(Z, u).
\]

Using Lemma 5.1 we see that this equation has a unique, local solution \( \xi \) within the maximal regularity space. Now, the difference \( \delta := u - \xi \) solves

\[
 \partial_t \delta - A_2 \delta = F(u, u) + F(Z, Z) - F(\xi + Z, \xi + Z) + F(u, Z) + F(Z, u), \quad \delta(0) = 0,
\]

and using the bi-linearity of \( F(\cdot, \cdot) \) and inserting \( \xi = \delta + u \) this simplifies to

\[
 \partial_t \delta - A_2 \delta = F(\delta, u + Z) + F(u + Z, \delta) - F(\delta, \delta), \quad \delta(0) = 0,
\]

where the terms \( F(u, Z) + F(Z, u) \) and \( f \) cancel out. Lemma 5.1 implies that this equation has a unique, local solution, and in fact this solution is \( \delta \equiv 0 \) which exists even globally on \( (0, T) \), i.e., \( u \equiv \xi \). Hence, by (6.2)

\[
 \| \xi \|_{L^2(0, T; H^2(\mathcal{D}))} + \| \xi \|_{H^1(0, T; L^2(\mathcal{D}))} \leq B_u(\| v_0 \|_{H^{1/2}(\mathcal{D})}, \| f \|_{L^2(0, T; L^2(\mathcal{D}))}, T).
\]

Solving now (6.3) for

\[
 f = F(u, Z) + F(Z, u) \quad \text{we have} \quad g(u, Z) = 0,
\]

it follows that its solution \( v \) solves (5.1) and moreover by Lemma 5.1

\[
 \| v \|_{L^2(0, T; H^2(\mathcal{D}))} + \| v \|_{H^1(0, T; L^2(\mathcal{D}))} \leq B_u(\| v_0 \|_{H^{1/2}(\mathcal{D})}, C(B'_u \cdot \| Z \| + \| f \|_{L^2(0, T; L^2(\mathcal{D}))}, T)
\]

\[
 =: B(\| v_0 \|_{H^{1/2}(\mathcal{D})}, \| f \|_{L^2(0, T; L^2(\mathcal{D}))}, \| Z \|, T),
\]

where \( B'_u := B_u(\| v_0 \|_{H^{1/2}(\mathcal{D})}, \| f \|_{L^2(0, T; L^2(\mathcal{D}))} + C \| Z \|^2, T) \). Here we used the fact that the function \( B_u \) can be chosen to be monotone in each variable which follows from the explicit derivation of the a priori bound, compare e.g. [19, 24]. Hence, the maximal regularity norm is uniformly bounded and it follows that the local solution \( v \) extends to a global solution (see e.g. [34] and [25] for the local-to-global argument). 

\[ \square \]
6.2. Global strong well-posedness in $L^q_tL^p_x$-spaces for the stochastic primitive equations.

Proof of Theorem 3.1. By Proposition 2.1, the function $Z = Z_f + Z_b$ has almost surely the regularity needed in Propositions 5.2 and 6.1. Now, consider the situation of Proposition 6.2 and note that for any $\delta \in (0, T)$

$$H^{1,q}_\delta(0, T; L^p_x(D)) \cap L^q_\delta(0, T; D(A_p)) \hookrightarrow H^{1,q}(\delta, T; L^p_x(D)) \cap L^q(\delta, T; D(A_p)) \hookrightarrow BUC(0, T; X_{1-1/q,q}).$$

So, for $\delta' \in (\delta, T')$ it follows from (2.7) and the embedding $B^{2-2/q}_p(D) \hookrightarrow H^1(D)$ for $p, q \geq 2$ that

$$v(\delta') \in X_{1-1/q,q} \hookrightarrow \{H^{1,q}_\delta(D) \cap L^q_{\mu}(D); \ v|_{\Gamma_D} = 0\} = D(A^{1/2}_2),$$

and hence the local solution $v$ to the perturbed primitive equations (5.1) from Propositions 5.1 extends to a unique global solution in the maximal $L^q_tL^p_x$-regularity space. Moreover, since the case $p = 2$ is already covered by Proposition 6.1, consider the case $p > 2$. In this case the embedding

$$X_{1/2,2} \hookrightarrow \{H^{1,q}_\delta(D) \cap L^q_{\mu}(D); \ v|_{\Gamma_D} = 0\} \hookrightarrow X_{1/q+1/q,q} \subset B^{2/p}_q(D)^2,$$

is compact, see e.g. [41, Remark 4.3.2.1]. Hence, we may use the $L^q_tL^p_x$-a priori bound to apply [34, Theorem 5.7.1], and the solution extends even to a global solution

$$v \in H^{1,q}_\mu(\delta, T; L^p_x(D)) \cap L^q_{\mu}(\delta, T; D(A_p)).$$

This solution lies in the desired regularity class since on $(0, T')$ it is in the $\mu$-time weighted space and on $(\delta, T)$ it is in the (possibly larger) $\mu$-time weighted space.

Eventually, going back to the stochastic setting, one verifies that $V = v + Z$ solves pathwise the original equation in $D(A^{1/2}_p)$. Moreover, fixing $Z$, $v$ is governed by the deterministic equation (5.1), and hence depends continuously on $v_0$. \hfill $\square$

Proof of Theorem 3.1. By Proposition 2.1 b) the function $Z = Z_f + Z_b$ has for $\delta \in [0, 1/2]$ almost surely the regularity $Z \in H^{\theta,q}(0, T; D(A^{1-\theta}_p))$. Applying Proposition 5.2 with $\theta = 1/2 - 1/2p$ yields the local well-posedness and global well-posedness follows as above in the proof of Theorem 3.1. One can reconstruct the pressure by

$$\nabla_H P_s = -(1 - p)\Delta v + (1 - p)(v + Z) \cdot \nabla_H (v + Z) + w((v + Z)\partial_z(v + Z)),$$

and since the right hand side lies in $L^q_\mu(0, T; L^2(\mathbb{G}))$, there is a unique $P_s \in L^q_\mu(0, T; H^{1,p}_\mu(G) \cap L^p_\mu(G))$ solving this equation. Eventually, one verifies that $V = v + Z$ and $P_s$ are a strong pathwise solution in the sense of Definition 2.1.

Moreover, for fixed $Z$, $v$ and $P_s$ are governed by the deterministic equation (5.1) and hence depend continuously on $v_0$. \hfill $\square$

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References

[1] H. Amann. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In Function, operators, and nonlinear analysis (Friedrichroda, 1992), Teubner, Stuttgart, 1993. doi:10.1007/bf00353866

[2] A. Agresti, M. Veraar. Nonlinear parabolic stochastic evolution equations in critical spaces Part I. Stochastic maximal regularity and local existence. Preprint arXiv:2001.00512, 2020.

[3] D. Bresch, J. Simon. On the effect of friction on wind driven shallow lakes. J. Math. Fluid Mech. 3:231-258, 2001. doi:10.1007/s000210050018

[4] Z. Brzeźniak, J. Slavík. Well-Posedness of the 3D stochastic primitive equations with transport noise. Preprint arXiv:2008.00274, 2020.

[5] G. Buffoni, A. Cappelletti, P. Picco. On the Ekman equations for ocean currents driven by stochastic wind. Stoch. Anal. Appl. 33:356-382, 2015. doi:10.1080/07362994.2014.998770

[6] M. Cannone. A generalization of a theorem of Kato on the Navier-Stokes equations. Rev. Mat. Iberoamericana 13:515-541, 1997.

[7] Ch. Cao, E. Titi. Global well–posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics. Annals of Mathematics, 166:245–267, 2007. doi:10.4007/annals.2007.166.245
[38] R. Schnaubelt and M.C. Veraar Stochastic equations with boundary noise. In: Nonlinear Parabolic Problems: Herbert Amann Festschrift, Progress in Nonlinear Differential Equations and Their Applications, Vol. 80, Birkhäuser Verlag, 2011. doi:10.1007/978-3-0348-0075-4

[39] E.M. Stein Harmonic Analysis Princeton University Press, Princeton, 1993.

[40] R. Temam, M. Ziane. Navier-Stokes equations in thin domains with various boundary conditions. Adv. Diff. Equ., 1:499-546, 1996

[41] H. Triebel. Theory of Function Spaces. (Reprint of 1983 edition) Springer AG, Basel, 2010. doi:10.1007/978-3-0346-0416-1

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