We study a class of one-dimensional models consisting of a frustrated (N+1)-leg spin ladder, its asymmetric doped version as a special example of a Luttinger liquid in an active environment, and the N-channel Kondo-Heisenberg model away from half-filling. It is shown that these models exhibit a critical phase with generally a non-integer central charge and belong to the class of chirally stabilized spin liquids recently introduced by Andrei, Douglas, and Jerez [Phys. Rev. B 58, 7619 (1998)]. By allowing anisotropic interactions in spin space, an exact solution in the N=2 case is found at a Toulouse point which captures all universal properties of the models. At the critical point, the massless degrees of freedom are described in terms of an effective S=1/2 Heisenberg spin chain and two critical Ising models. The Toulouse limit solution enables us to discuss the spectral properties, the computation of the spin-spin correlation functions as well as the estimation of the NMR relaxation rate of the frustrated three-leg ladder. Finally, it is shown that the critical point becomes unstable upon switching on some weak backscattering perturbations in the frustrated three-leg ladder.

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I. INTRODUCTION

One-dimensional quantum spin systems exhibit fascinating physical properties. Most of the striking features observed in these systems are pure quantum effects and give rise to many exotic ground-states or unconventional excitation spectra that are not accessible by traditional approaches like spin-wave or perturbation theory. A famous example is the prediction made by Haldane \[1\] that half-integer antiferromagnetic spin chains are gapless whereas those with integer spin have a finite energy gap in its spectrum.

The different one-dimensional spin gapped phases are usually characterized by the nature of the broken discrete symmetries (lattice translational symmetry for instance) in the ground-state, the quantum numbers associated with the massive excitations, and the structure of the dynamical magnetic susceptibility. Many different spin gapped phases have been identified. The spin-1 Heisenberg chain is certainly the most famous example of a spin liquid with a gap. It is characterized by a hidden topological order \[2\] related to the breakdown of a hidden $Z_2 \times Z_2$ symmetry \[3\]. The spin excitations are optical magnons carrying spin $S=1$ and display a single peak near $q = \pi$ in the dynamical magnetic susceptibility. The low-lying excitations of a massive phase can also have fractional quantum numbers as in the $J_1$-$J_2 S=1/2$ Heisenberg chain for a sufficiently large value of $J_2$: The ground-state is spontaneously dimerized with a two-fold degeneracy and the fundamental excitations are massive $S=1/2$ solitons (spinons) separating the two ground-states \[4,5\]. Another scenario can also be realized in a two-leg spin ladder coupled by a four spin exchange interaction where the elementary excitation is neither an optical magnon nor a massive spinon but rather a pair of propagating triplet or singlet solitons connecting the two spontaneously dimerized ground-states \[6,7\]. This special structure of the excitation will reveal itself in the dynamical magnetic susceptibility which displays a two-particle threshold instead of a sharp single magnon peak near $q = \pi$ as in the $S=1$ spin chain. This incoherent background in the dynamical structure factor gives rise to a new one-dimensional spin liquid state: the so-called non-Haldane spin liquid \[6\].

Clearly, the general classification of all possible one-dimensional spin gapped phases is still lacking. However, the integrability of some special field theoretical models has given a lot of insight and has been employed to extract the degeneracies, quantum numbers of low-energy excited states of spin liquid phases and to provide sometimes the computation of the leading asymptotics of correlation functions. This program has been realised for the sine-Gordon model associated with the low-energy description of the $S=1/2$ Heisenberg chain with alternating exchange \[8\] and the Cu benzoate \[9,10\] and also for the $O(N)$ ($N=4,6,8$) Gross-Neveu model for the description of a special frustrated two-leg ladder \[11\], weakly two-leg Hubbard ladder \[12,13\], $(N,N)$ armchair Carbon nanotubes \[13\], and spin-orbital model \[15\].

The situation is totally different for critical one-dimensional quantum spin systems like the spin-1/2 Heisenberg chain where Conformal Field Theory (CFT) allows a complete classification of one-dimensional quantum systems or finite temperature two-dimensional classical systems \[16,17\]. The low-energy spectrum of the lattice model with a continuous symmetry is described in terms of representations of a certain current algebra \[18\]. This affine symmetry determines the operator content of the theory and all possible scaling dimensions of the operators are in turn fixed by the conformal invariance of the underlying Wess-Zumino-Novikov-Witten (WZNW)
model built from the currents of the affine symmetry with the Sugawara construction [18]. The knowledge of the scaling dimensions allows the computation of correlation functions and the scaling behaviour of the energies of the ground-state and low-lying states as a function of the size of the system [19]. The different critical phases are labelled, in particular, by the central charge (c) of the underlying Virasoro algebra of the WZNW CFT which fixes the low-temperature behaviour of the specific heat of the model [20].

A simple example of this description is the Luttinger liquids, a general class of one-dimensional interacting electron models whose infrared (IR) behaviour is governed by the Luttinger model [21], which have an U(1) affine Kac Moody (KM) symmetry and a central charge c=1. The Luttinger liquids are usually described in a bosonized form with the introduction of one gapless bosonic field (c=1 CFT) and are characterized by single-particle correlation functions having branch cuts with non-universal exponents [22]. Such state provides an example of non-Fermi liquid behaviour [22]. The critical theory of quantum spin systems with non-Abelian continuous symmetries can also be analysed on the basis of their SU(2)k affine KM algebra [23,24]. Examples of such representation are the critical theory of half-integer spin antiferromagnets described by the SU(2)1 WZNW model whereas the family of integrable Hamiltonian of arbitrary spin S [25] belongs to the SU(2)2S universality class [24]. This approach has also been employed to describe the low-energy physics of Heisenberg antiferromagnets with a larger symmetry group like a SU(N) symmetry [24] on the basis of the SU(N)1 WZNW model [24,27–29].

In a recent paper, Andrei, Douglas, and Jerez [30] identified a new non-Fermi-liquid class of fixed points, the so-called chiral spin fluids, describing the IR behaviour of one-dimensional interacting chiral fermions. The chiral spin liquid (CSL) behaviour appears as a stable critical state of one-dimensional interacting fermions with unequal numbers of right-moving and left-moving particles. The phenomenological Hamiltonian, in the continuum limit, corresponding to this state reads:

\[ H_{CSL} = -iv_F \left( \sum_{r=1}^{f_R} R^\dagger_{ar} \partial_x R_{ar} - \sum_{l=1}^{f_L} L^\dagger_{al} \partial_x L_{al} \right) + g J^\alpha_R \cdot J^\alpha_L \]  

(1)

where \( R^\dagger_{ar} \) (respectively \( L^\dagger_{al} \)) is the creation operator of a right-moving (respectively left-moving) fermion with spin index \( \alpha = \uparrow, \downarrow \) and flavor index \( r \) (respectively \( l \)): \( r = 1, ..., f_R \) (respectively \( l = 1, ..., f_L \)). The interaction affects only the spin degrees of freedom and consists of a marginal relevant \((g > 0)\) current-current term where the spin currents write in terms of the fermions as follows:

\[ J^\alpha_R = \frac{1}{2} \sum_{r=1}^{f_R} R^\dagger_{ar} \sigma_\alpha^{\alpha\beta} R_{br} \]

\[ J^\alpha_L = \frac{1}{2} \sum_{l=1}^{f_L} L^\dagger_{al} \sigma_\alpha^{\alpha\beta} L_{bl} \]  

(2)

\( \sigma^a \) (\( a=x,y,z \)) being the Pauli matrices and \( J^\alpha_R \) (respectively \( J^\alpha_L \)) belongs to the SU(2)_{f_R} (respectively SU(2)_{f_L}) KM algebra. The Hamiltonian [1], in the spin sector, describes thus two SU(2)_{f_R} and SU(2)_{f_L} WZNW models marginally coupled by a current-current interaction. For \( f_R \neq f_L \), i.e. breakdown of the \( R \to L \) symmetry implying breakdown of the time-reversal symmetry, the Hamiltonian [1] belongs to a new class of problems not encountered in the
above mentioned problems of one-dimensional physics. By a combination of CFT argument and Bethe Ansatz techniques, the authors of Ref. [30] showed that the Hamiltonian $H$ for $f_R \neq f_L$ flows in the IR limit to an intermediate fixed point (chiral fixed point) characterizing by universal exponents in the leading asymptotic of the electronic Green’s functions. From the electronic point of view, such a state provides a non-trivial example of a non-Fermi-liquid state which differs from the Luttinger liquids: the CSL state. A typical example of possible realizations of CSL, given in Ref. [30], is a contact between the edges of two integer Quantum Hall Effect systems with different filling factors: $\nu_L = f_L \neq \nu_R = f_R$.

In this paper, we shall consider another route where the total Hamiltonian ($\mathcal{H}$) of a possible realization of CSL is time-reversal invariant. Indeed, in the continuum limit, the leading part of the Hamiltonian that imposes the strong coupling behaviour decomposes into two commuting marginal chirally asymmetric parts $\mathcal{H}_1$ and $\mathcal{H}_2$:

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2, \quad [\mathcal{H}_1, \mathcal{H}_2] = 0,$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are not separately time-reversal invariant but they transform into each other under the symmetry $t \rightarrow -t$. The model (3) as a whole will display a CSL behaviour in the IR limit. Such scenario suggests that other realizations of this state can be found in ladder problems. However, in the continuum limit of spin ladders, one has, apart from marginal current-current interactions, backscattering terms $n_a \cdot n_b$ ($n_a$ being the staggered magnetizations of a chain of index $a$) which have scaling dimension 1 at the ultraviolet (UV) fixed point and thus represent strongly relevant perturbations. Therefore, one has clearly to kill this contribution to have a marginal perturbation and thus to find some specific realizations of the CSL state. We shall follow, in this paper, two different routes to find some CSL behaviour in the IR limit of ladder problems. On the one hand, frustration may play its role either by suppressing geometrically the backscattering contribution as in zigzag ladders [31–34] or by allowing several independent coupling constants and the possibility to eliminate the unwanted term by a fine tuning mechanism [35,11]. In that case, as we shall see later, the CSL found at this special point will be destabilized upon switching on weak backscattering contributions but there will exist an intermediate low-energy region governed by the CSL behaviour. On the other hand, doping effects can also be useful to suppress the backscattering term by some incommensurate effects: In the continuum limit, the unwanted contribution will become a strongly oscillating perturbation that can be ignored in the long distance limit.

The remainder of the paper is organized as follows: Different possible realizations of the CSL state are introduced in Section II: the (N+1)-chain cylinder model, its asymmetric doped version, and the N-channel Kondo-Heisenberg model away from half-filling. The nature of the IR fixed point (chiral fixed point) for these models is determined in Section III. In Section IV, we present a Toulouse point solution in the $N = 2$ case that captures all universal properties of the chiral fixed point. In that case, the massless degrees of freedom are described in terms of an effective S=1/2 Heisenberg spin chain and two critical Ising models. A brief summary of this approach has already been published [35]. We shall provide here and in Section V the technical details of the Toulouse point solution and in particular the determination of the physical properties of the 3-chain cylinder model. The nature of the phase diagram of the doped models in the N=2 case will be addressed in a forthcoming publication [36]. In section VI, the
stability of the chiral fixed point of the 3-chain cylinder model is analysed under the presence of weak backscattering perturbations. Finally, our concluding remarks are presented in Section VII. Three Appendixes (A,B,C) give further technical details necessary for the computation of the leading asymptotics of the correlation functions occurring in Section V whereas in the last one the stability of the Toulouse limit solution is studied.

II. POSSIBLE REALIZATIONS OF CHIRAL SPIN LIQUID BEHAVIOUR

In this section, we shall present some lattice models that are described in the continuum limit by an Hamiltonian with only marginal current-current interactions. In the next section, we shall show that the low-energy physics of this Hamiltonian is governed by the chiral fixed point and thus represents some specific realizations of a CSL state.

A. The (N+1)-chain cylinder model

The first model that we consider below can be visualized as follows. It consists of N identical S=1/2 antiferromagnetic Heisenberg chains on the surface of a cylinder parallel to each other and to the axis of the cylinder. An additional S=1/2 Heisenberg chain, that will be called the zeroth chain or central chain in the following, resides inside the cylinder along its axis. The surface chains do not interact with each other but all of them are coupled to the central one by on-rung interchain interaction (J⊥) and an interchain interaction along diagonals of the elementary plaquettes (J×). The Hamiltonian corresponding to this model is given by:

$$H = \sum_{a=0}^{N} H_0^a + H_{\text{int}},$$

where $H_0^a$ is the Hamiltonian of a S=1/2 antiferromagnetic Heisenberg spin chain of index a:

$$H_0^a = J_\parallel \sum_j S_{a,j} \cdot S_{a,j+1}, \ J_\parallel > 0$$

$S_{a,j}$ being S=1/2 spin operators on each site $j$ of the chain of index $a = 0, 1,..,N$. These Heisenberg chains are coupled with the interaction:

$$H_{\text{int}} = J_\perp \sum_j \sum_{a=1}^{N} S_{0,j} \cdot S_{a,j} + J_\times \sum_j \sum_{a=1}^{N} \{S_{a,j} \cdot S_{0,j+1} + S_{a,j+1} \cdot S_{0,j}\}.$$  

In the following, it will be assumed that all interchain exchange interactions are positive and weak: $0 < J_\perp, J_\times \ll J_\parallel$ to describe the model by a continuum limit approach. Due to the presence of the plaquette interaction $J_\times$, the cylinder model belongs to the class of frustrated spin ladders. For $N = 1$, this model has been investigated by several groups [37,11] and reveals no CSL state. As it will be discussed in Section III, the (N+1)-chain cylinder model with $N > 1$ is a better candidate for displaying non-trivial CSL physics.

Let us first consider the continuum limit of the (N+1)-chain cylinder model. The continuum description of the S=1/2 antiferromagnetic Heisenberg spin chain is based on the Sugawara representation of the SU(2)1 WZNW model with a marginal irrelevant perturbation [20,21]:
with \( \gamma < 0 \) and \( J_{aR}, J_{aL} \) are the right and left spin currents belonging to the SU(2) \(_1\) KM algebra \([18,17,38,39]\) and verify the following operator product expansion (OPE):

\[
J_\alpha a_L(z) J_\beta b_L(w) \sim \frac{i \epsilon^{\alpha\beta\gamma\delta} \delta_{ab}}{2\pi (z-w)} J_\gamma a_L(w) + \frac{\delta_{ab}}{8\pi^2 (z-w)^2} \]

\[
J_\alpha a_R(\bar{z}) J_\beta b_R(\bar{w}) \sim \frac{i \epsilon^{\alpha\beta\gamma\delta} \delta_{ab}}{2\pi (\bar{z}-\bar{w})} J_\gamma a_R(\bar{w}) + \frac{\delta_{ab}}{8\pi^2 (\bar{z}-\bar{w})^2} \]

\[
J_\alpha a_L(z) J_\beta b_R(\bar{w}) \sim 0, \quad \alpha, \beta = x, y, z, \quad a, b = 0, 1, \ldots, N \quad (8)
\]

where \( z = v\tau + ix, \) \( \tau \) being the imaginary time, and \( v \sim J_\parallel a_0 \) denotes the spin velocity on each chain (\( a_0 \) being the lattice spacing).

The next step to achieve the continuum limit of the Hamiltonian (4) is to use the continuum representation of the spin densities of each chain \([24]\):

\[
\frac{S_{a,j}}{a_0} \rightarrow S_a(x) = J_a(x) + (-1)^{j/a_0} n_a(x), \quad a = 0, 1, \ldots, N \quad (9)
\]

with \( x = ja_0 \) and \( J_a, n_a \) denote respectively the uniform and staggered part of the spin density. The smooth part \( J_a \) can be expressed as:

\[
J_a = J_{aR} + J_{aL}, \quad (10)
\]

whereas the \( n_a \) field is the vector part of the primary field of the SU(2) \(_1\) WZNW model transforming according to the fundamental representation of SU(2). Assuming weak interchain couplings \( J_\perp, J_\times \ll J_\parallel, \) one can perform the continuum limit of the cylinder model using Eq. (7) and the continuum description (9) to obtain:

\[
\mathcal{H} = \frac{2\pi v}{3} \sum_{a=0}^{N} (J_{aR} \cdot J_{aR} + J_{aL} \cdot J_{aL}) + \gamma \sum_{a=0}^{N} J_{aR} \cdot J_{aL} + \tilde{g} n_0 \cdot n_+ + g J_0 \cdot I, \quad (11)
\]

where we have discarded all irrelevant operators and oscillatory terms. In Eq. (11), \( I \) (respectively \( n_+ \)) is the sum of all uniform (respectively staggered) magnetizations of the surface chains:

\[
I = \sum_{a=1}^{N} J_a \\
n_+ = \sum_{a=1}^{N} n_a. \quad (12)
\]

Moreover, the bare coupling constants in (11) are given by:

\[
g = a_0 (J_\perp + 2J_\times) \\
\tilde{g} = a_0 (J_\perp - 2J_\times). \quad (13)
\]

The interacting terms in (11) are of different nature. A first one with the coupling constant \( \tilde{g} \) is a relevant perturbation of scaling dimension \( d = 1 \) and corresponds to the usual interchain interaction of non-frustrated spin ladders. A second one with the coupling \( g \) describes an
interaction between the total spin current of the surface chains and that of the central chain. This interaction is marginal relevant and, as long as \( \tilde{g} \) is not too small, can be discarded. As a result, for generic values of \( \tilde{g} \) and \( g \), the low-energy physics of the model will be essentially governed by the backscattering contribution. In that case, frustration will play no role except for renormalization of spin velocities and mass gaps of the modes that will become massive in the IR limit. However, it is important to stress that, in contrast with non-frustrated spin ladders, the two coupling constants \( \tilde{g}, g \) can vary independently in the model, and there exists a vicinity of the line \( J_\perp = 2J_x (\tilde{g} = 0) \) where frustration shows up in a nontrivial way. Along this line, the low-energy properties of the model are mainly determined by current-current interactions:

\[
H = \frac{2\pi v}{3} \sum_{a=0}^{N} (\mathbf{J}_{aR} \cdot \mathbf{J}_{aR} + \mathbf{J}_{aL} \cdot \mathbf{J}_{aL}) + \gamma \sum_{a=0}^{N} \mathbf{J}_{aR} \cdot \mathbf{J}_{aL} + g \mathbf{J}_0 \cdot \mathbf{I},
\]

which displays in the IR limit CSL physics as it will be shown in Section III.

B. The (N+1)-chain cylinder model with doped central chain

As already stated in the Introduction, the effect of doping might be useful to suppress the backscattering perturbation in the long distance limit. In particular, one can, for example, consider the previous (N+1)-chain cylinder model where some concentration of holes are introduced in the central chain. Clearly, the very assumption that there can exist such model with only one chain doped needs justification. Probably, the simplest one would be to assume that the intrasite single-electron energies on the central \( \epsilon_0 \) and surface \( \epsilon_S \) chains are much larger than the interchain hopping amplitude: \( t_\perp \ll |\epsilon_0 - \epsilon_S| \) so that different chains are out of resonance. Anyway, we shall assume in the following that one can dope the central chain only. In that case, this special model belongs to the class of one-dimensional electron liquid in an antiferromagnetic environment [40–43] made by the surface chains.

Due to the presence of doping, the continuum description of the spin density of the central is no longer given by Eq. (9) but takes now the following form (see for instance chapter 16 of Ref. [39]):

\[
\mathbf{S}_0 (x) = \mathbf{J}_0 (x) + \cos \left( 2k_{F,0} x + \sqrt{2\pi} \Phi_c (x) \right) \mathbf{n}_0 (x),
\]

where \( \Phi_c \) is a bosonic field associated with the charge degrees of freedom, \( k_{F,0} \) being the Fermi momentum of the central chain. In the continuum limit, the backscattering contribution \( (\mathbf{n}_0 \cdot \mathbf{n}_+) \) acquires now an oscillating factor \( \sim \exp(i(2k_{F,0} - \pi) x) \) and thus becomes suppressed in the long distance limit if the concentration of holes is sufficiently high. As a result, the Hamiltonian of this asymmetric doped model in the spin sector is given by Eq. (14) with only current-current interactions.

C. The multichannel Kondo-Heisenberg model away from half-filling

The last example that we shall give in this paper is the so-called Kondo-Heisenberg model with N-channel. This model consists of N identical Hubbard chains \( (\mathcal{H}_U) \) interacting via a
Kondo coupling with a periodic array of localized spins described by the spin-1/2 operator $S_{0,i}$. The Hamiltonian of this system reads as follows:

$$
\mathcal{H} = \mathcal{H}_U + J_K \sum_i S_{0,i} \cdot S_{c,i} + J_H \sum_i S_{0,i} \cdot S_{0,i+1} \tag{16}
$$

with:

$$
\mathcal{H}_U = -t \sum_{n,\sigma,i} \left( c_{n\sigma,i}^{\dagger} c_{n\sigma,i} + h.c. \right) + U \sum_{n,i} c_{n\uparrow,i}^{\dagger} c_{n\uparrow,i} c_{n\downarrow,i}^{\dagger} c_{n\downarrow,i}, \tag{17}
$$

where $c_{n\sigma,i}$ denotes the electron annihilation operator on the site $i$, and $\sigma = \uparrow, \downarrow, n = 1, \ldots, N$ are respectively the electron spin and channel indices. The conduction electron spin operator $S_{c,i}$ at the $i$th site is defined by:

$$
S_{c,i}^a = \frac{1}{2} \sum_{n,\alpha,\beta} c_{n\alpha,i}^{\dagger} \sigma_{\alpha\beta} c_{n\beta,i}, \quad a = x, y, z. \tag{18}
$$

In Eq. (16), the interaction between the electronic degrees of freedom and the localized spin is made by a Kondo coupling with the constant $J_K > 0$. The Hamiltonian (16) can also be viewed as a generalization of the multichannel Kondo lattice with an exchange interaction $J_H > 0$ between the impurities spins (simple RKKY interaction).

In the weak coupling limit when $J_K, U \ll t, J_H$, this Hamiltonian can be represented in a bosonized form following the same route as in Refs. [31,44] for $N = 1$. The continuum description of the localized spin operator ($S_{0,i}$) is still given by Eq. (9) whereas that of the conduction spin density operator can be expressed in terms of $N$ bosonic fields $\Phi_{ac}$ and $N$ SU(2) spin currents $J_{aR,L}$:

$$
S_{c}(x) = \sum_{a=1}^{N} \left( J_{aR}(x) + J_{aL}(x) + \cos \left( 2k_F x + \sqrt{2} \pi \Phi_{ac}(x) \right) n_a(x) \right). \tag{19}
$$

With these relations at hand, one can derive the continuum limit of the Hamiltonian (16) away from half-filling (incommensurate filling). In that case, the $2k_F$ oscillation of the spin conduction operator becomes incommensurate with the alternating localized spin operator so that far away from half-filling, only marginal current-current interaction survives in the Kondo interaction in the long distance limit:

$$
\mathcal{H} = \frac{2\pi v_s}{3} \sum_{a=1}^{N} \left( J_{aR} \cdot J_{aR} + J_{aL} \cdot J_{aL} \right) + \frac{2\pi v_0}{3} \left( J_{0R} \cdot J_{0R} + J_{0L} \cdot J_{0L} \right) + g J_{0} \cdot I + \gamma \sum_{a=1}^{N} J_{aR} \cdot J_{aL} + \gamma' J_{0R} \cdot J_{0L} \tag{20}
$$

where $I = \sum_{a=1}^{N} J_a, v_s \sim 2t a_0 - U a_0/2\pi, v_0 \sim J_H a_0, g \sim J_K a_0$, and $\gamma, \gamma' < 0 \ (U, J_H > 0)$. Therefore, the multichannel version of the Kondo-Heisenberg model away from half-filling belongs to the class of models with marginally coupled current-current interactions. One should note that it is more transparent to use a different basis, a spin-charge-flavor decomposition [23,24], to single out the charge degrees of freedom. In that case, the continuum description of the Hamiltonian of the $N$-channel Kondo-Heisenberg model away from half filling reads as follows in the spin sector:
\[ H_s = \frac{2\pi v_s}{N+2} (I_R \cdot I_R + I_L \cdot I_L) + \frac{2\pi v_0}{3} (J_{0R} \cdot J_{0R} + J_{0L} \cdot J_{0L}) \\
+ gJ_0 \cdot I + \gamma I_R \cdot I_L + \gamma' J_{0R} \cdot J_{0L} \] (21)

III. IDENTIFICATION OF THE INFRARED FIXED POINT

We investigate now the IR properties of the Hamiltonian (14) with only current-current interactions which represents the continuum limit of several lattice models as seen in the previous section. In particular, the nature of the IR fixed point that governed the low-energy physics of these models for \( N > 1 \) will be identified and belongs to the universality class of chirally stabilized fluids introduced in Ref. [30].

Let us first look at the problem from a perturbative point of view near the UV fixed point where all chains are decoupled so that the model has the \([SU(2)]_N^{N+1} \otimes [SU(2)]_L^{N+1}\) symmetry. It is conformally invariant with a total central charge: \( c_{UV} = N + 1 \) \((N + 1\) gapless bosonic modes). The one-loop Renormalization Group (RG) equations for the coupling constants \( g \) and \( \gamma \) are given by:

\[ \frac{d\gamma}{d \ln L} = \frac{\gamma^2}{2\pi v} \]
\[ \frac{dg}{d \ln L} = \frac{g^2}{2\pi v} \] (22)

The bare coupling constant \( \gamma \) is negative in all the models presented in Section II, so that the perturbation \( \gamma \sum_a J_{aR} \cdot J_{aL} \) is marginally irrelevant and can therefore be neglected: it leads to logarithmic corrections. On the other hand, the effective interchain coupling \( g \) increases upon renormalization \((g > 0)\). Usually, the development of a strong coupling regime \((g(L) \to +\infty)\) is accompanied by a dynamical mass generation and the loss of conformal invariance as for the non-linear sigma model or the Gross-Neveu model in 1+1 dimensions. This is indeed the case for \( N = 1 \) which represents a continuum version of the two-leg zigzag ladder [31,32] in the absence of the so-called twist perturbation [33]. However, for \( N \geq 2 \), the interaction flows to an intermediate fixed point \((g^* < \infty)\) where the system displays critical properties with a smaller (and generally non-integer) central charge than \( c_{UV} \) according to the Zamolodchikov c-theorem [47].

To this end, let us rewrite the Hamiltonian (14) in terms of the leading part which governs the strong coupling behaviour:

\[ \mathcal{H} = \frac{2\pi v_1}{3} \sum_{a=1}^N (J_{aR} \cdot J_{aR} + J_{aL} \cdot J_{aL}) + \frac{2\pi v_0}{3} (J_{0R} \cdot J_{0R} + J_{0L} \cdot J_{0L}) + g (J_{0R} \cdot I_L + J_{0L} \cdot I_R) \] (23)

where we have omitted the marginal irrelevant contribution and the interaction between the currents of the same chirality whose effect will be effectively taken into account by allowing the surface chains velocity \( v_1 \) to be different from that of the central chain \( v_0 \). We shall explicitly check for \( N = 2 \), in Appendix D, by our Toulouse point approach that all these discarded interactions will not destabilize the intermediate fixed point.
The structure (23) for \( N > 1 \) suggests that some degrees of freedom do not participate to the interaction and thus will decouple and remain massless. Indeed, the free part of the Hamiltonian (23) is a sum of \((N + 1)\) critical \( \text{SU}(2)_1 \) WZNW models whereas the interaction describes a coupling between the \( \text{SU}(2)_1 \) spin current of the central chain \( (\mathbf{J}_{0R,L}) \) and the total surface current \( (\mathbf{I}_{R,L}) \) which is a \( \text{SU}(2)_N \) current being the sum of \( N \) \( \text{SU}(2)_1 \) currents (see Eq. (12)). To identify the nature of the non-interacting degrees of freedom, we use the following decomposition:

\[
\prod_{i=1}^{N} \text{SU}(2)_1|_i = \text{SU}(2)_N \otimes \mathcal{G}_N
\]

(24)

where \( \mathcal{G}_N \) is some piece. The central charge of the critical model with symmetry \( \mathcal{G}_N \) is:

\[
c_{\mathcal{G}_N} = N - c_{\text{SU}(2)_N} = N - \frac{3N}{N+2} = \frac{N(N-1)}{N+2}
\]

(25)

where we have used that the central charge of the \( \text{SU}(2)_k \) WZNW model is:

\[
c = 3k/(k+2).\]

Though it requires a proof, for \( N = 2m \), \( \mathcal{G}_N \) can be viewed as a direct product of \( m \) \( \mathbb{Z}_{2m'} \)-symmetric critical models \([48]\) whereas for \( N = 2m + 1 \), it is the product of \( m \) \( \mathbb{Z}_{2m'} \)-symmetric critical models and the minimal model \( \mathcal{M}_{N+1} \). We recall here that the central charge of the minimal model series \( \mathcal{M}_p \) with \( p \geq 2 \) is: \( c_p = 1 - 6/p(p+1) \) \([16,17]\). The previous identification reproduces the value of the central charge and also all primary fields have the correct scaling dimensions but it is not sufficient to prove the previous CFT embedding. Anyway, in this work, we only need the actual value of the central charge (23) of the critical model with symmetry \( \mathcal{G}_N \) which one can note that it coincides with the sum of the central charge of the \( N-1 \) first minimal models:

\[
c_{\mathcal{G}_N} = \frac{N(N-1)}{N+2} = \sum_{m=2}^{N+1} \left(1 - \frac{6}{m(m+1)}\right).
\]

(26)

The actual model under consideration simplifies thus in the following way:

\[
\mathcal{H} = \mathcal{H}_{\mathcal{G}_N} + \bar{\mathcal{H}}
\]

(27)

where \( \mathcal{H}_{\mathcal{G}_N} \) is the Hamiltonian corresponding to the critical degrees of freedom with symmetry \( \mathcal{G}_N \) that decouple from the interaction. All non-trivial physics is incorporated in the current dependent part of the Hamiltonian (\( \bar{\mathcal{H}} \)) which can be decomposed into a sum of two commuting and \emph{chirally asymmetric} parts:

\[
\bar{\mathcal{H}} = \mathcal{H}_1 + \mathcal{H}_2, \quad ([\mathcal{H}_1, \mathcal{H}_2] = 0)
\]

(28)

where

\[
\mathcal{H}_1 = \frac{2\pi v_1}{N+2} \mathbf{I}_R \cdot \mathbf{I}_R + \frac{2\pi v_0}{3} \mathbf{J}_{0L} \cdot \mathbf{J}_{0L} + g \mathbf{I}_R \cdot \mathbf{J}_{0L}
\]

\[
\mathcal{H}_2 = \frac{2\pi v_1}{N+2} \mathbf{I}_L \cdot \mathbf{I}_L + \frac{2\pi v_0}{3} \mathbf{J}_{0R} \cdot \mathbf{J}_{0R} + g \mathbf{I}_L \cdot \mathbf{J}_{0R}.
\]

(29)

The Hamiltonian \( \mathcal{H}_1 \) thus describes two marginally coupled \( \text{SU}(2)_N \) and \( \text{SU}(2)_1 \) WZNW models and \( \mathcal{H}_2 \) is obtained from \( \mathcal{H}_1 \) by interchanging the indices \( R \) and \( L \).
The properties of chirally asymmetric models with the structure of $\mathcal{H}_1$ in Eq. (29) but with a more general symmetry group $[\text{SU}(2)_{f_R}]_R \otimes [\text{SU}(2)_{f_L}]_L$ ($f_R > f_L$) have been studied by Andrei et al. [30] (see Eq. (1)). These authors argued that such models flow in the IR limit to an intermediate fixed point $g^*$ (chiral fixed point) whose symmetry is determined by a WZNW coset [49]:

$$[\text{SU}(2)_{f_R-f_L}]_R \otimes \left[ \frac{\text{SU}(2)_{f_L} \times \text{SU}(2)_{f_R-f_L}}{\text{SU}(2)_{f_R}} \right]_L.$$ (30)

This fixed point has been found by the authors of Ref. [30] using the fact that the differences between the chiral components of the central charge ($c_R - c_L$) and the KM levels ($f_R - f_L$) are preserved under the RG flow. Moreover, since the total central charge ($c_R + c_L$) always decreases upon renormalization according to the Zamolodchikov $c$-theorem [47], the identification of the IR fixed point is equivalent to find a conformally invariant model with the lowest total central charge consistent with the fixed values of $c_R - c_L$ and $f_R - f_L$. It turns out that the solution for this problem is given by the coset model (30). The structure of this fixed point has also been checked by the computation of the low temperature behaviour of the specific heat from the Thermodynamical Bethe ansatz approach [30] since this physical quantity is a direct probe of the value of the central charge of the IR fixed point [20].

Using the result (30), one can immediately deduce the structure of the IR fixed point of the Hamiltonian (28):

$$\left[ [\text{SU}(2)_{N-1}]_R \otimes \left[ \frac{\text{SU}(2)_1 \times \text{SU}(2)_{N-1}}{\text{SU}(2)_N} \right]_L \right] \otimes \left[ [\text{SU}(2)_{N-1}]_L \otimes \left[ \frac{\text{SU}(2)_1 \times \text{SU}(2)_{N-1}}{\text{SU}(2)_N} \right]_R \right].$$ (31)

where the first (respectively second) part is the symmetry of the IR fixed point associated with $\mathcal{H}_1$ (respectively $\mathcal{H}_2$). The central charge in the IR limit ($\tilde{c}_{IR}$) corresponding to the model described by the Hamiltonian ($\tilde{\mathcal{H}}$) of Eq. (28) is thus given by:

$$\tilde{c}_{IR}(N) = \frac{2(N-1)(2N+5)}{(N+1)(N+2)}.$$ (32)

From this result, we deduce the value of the total central at the IR fixed point of the original Hamiltonian (24) taking into account the contribution of the massless degrees of freedom (25) that do not participate to the interaction:

$$c_{IR}(N) = \tilde{c}_{IR}(N) + c_{\phi N} = \frac{(N-1)(N^2 + 5N + 10)}{(N+1)(N+2)}.$$ (33)

In summary, the low-energy physics of the (N+1)-chain cylinder model, its asymmetric doped version, and the multichannel Kondo-Heisenberg model away from half-filling, are governed by an intermediate IR fixed point displaying CSL behaviour with a central charge which is not integer but rational for a generic value of $N$. The first two models, in the spin sector, have a central charge given by Eq. (33) whereas the gapless modes corresponding to the spin sector of the N-channel Kondo-Heisenberg model are fixed by Eq. (32). Moreover, from Eqs. (32,33), one can notice that the central charges vanish for $N = 1$ as it should be (spin gap phase) whereas for $N = 2$ one has the simple values: $\tilde{c}_{IR}(2) = 3/2, c_{IR}(2) = 2$ which means that a simple and independent approach should be possible in that case.
In the following, we shall focus on this $N = 2$ special case by presenting an exact solution of the $U(1)$ version of the $N = 2$ problem using a Toulouse point solution. This approach has been extremely fruitful especially in Quantum Impurity problems and gives a simple description of the two-channel Kondo model and also of the Kondo lattice\cite{50,52}. In particular, for the two-channel Kondo problem, Emery and Kivelson\cite{50} identified the residual zero point entropy stemming from the decoupling of a Majorana fermion degrees of freedom using a mapping of the model onto free fermion theory for a very special value of the interaction (Toulouse point). Although the position of the solvable point is non-universal, the Toulouse limit solution captures the physical and universal properties of the low-temperature behaviour of the model. In our problem, the Toulouse point approach provides a direct and transparent reading of the spectrum of the model for $N = 2$. It also gives a non-perturbative basis from which all leading asymptotics of spin-spin correlations for the 3-chain cylinder model can be determined.

IV. THE TOULOUSE POINT SOLUTION IN THE $N=2$ CASE

In this section, we present the solution, by a Toulouse point approach, of the model \cite{23} with $N = 2$:

$$\mathcal{H} = \frac{2\pi v_1}{3} (J_{1R} J_{1R} + J_{1L} J_{1L} + J_{2R} J_{2R} + J_{2L} J_{2L}) + \frac{2\pi v_0}{3} (J_{0R} J_{0R} + J_{0L} J_{0L}) + g (I_{R} J_{0L} + I_{L} J_{0R}).$$  \hspace{1cm} (34)

A. Decoupling of a $Z_2$ non-magnetic excitation

As discussed in section III A, the structure of the interaction of Eq. (24) suggests an alternative representation of the two SU(2) WZNW models of the surface chains in a SU(2)$\otimes Z_2$ way reflecting the global spin rotational symmetry as well as the discrete $Z_2$ related to the interchange symmetry ($1 \rightarrow 2$). The critical model with the symmetry group $G_2$ in the description \cite{24} for $N = 2$ identifies with a single critical Ising model (or a free Majorana fermion in the continuum limit) since on has from the Goddard-Kent-Olive construction \cite{49}:

$$SU(2)_1 \otimes SU(2)_1 \sim SU(2)_2 \otimes Z_2.$$  \hspace{1cm} (35)

A simple way to take into account this non-magnetic $Z_2$ excitation (Ising degrees of freedom) is to use the representation of two SU(2)$_1$ spin currents in terms of four Majorana fermions $\xi^0$ and $\xi$ \cite{53,32,39}:

$$I_{\alpha} = J_{1\alpha} + J_{2\alpha} = \frac{-i}{2} \bar{\xi}_{\alpha} \wedge \xi_{\alpha}, \quad \alpha = R, L,$$

$$J_{1\alpha} - J_{2\alpha} = i \bar{\xi}_{\alpha} \xi_{\alpha}, \quad \alpha = R, L$$  \hspace{1cm} (36)

where the four right and left moving Majorana fermions have the following anticommutation relations ($a, b = 0, 1, 2, 3$):

$$\{\xi^a_{R,L}(x), \xi^b_{R,L}(y)\} = \delta^{ab} \delta(x - y)$$

$$\{\xi^a_{R}(x), \xi^b_{L}(y)\} = 0$$  \hspace{1cm} (37)
and are normalized according to:
\[ \xi^a_L(z) \xi^b_L(\omega) \sim \frac{\delta^{ab}}{2\pi(z-\omega)} \]
\[ \xi^a_R(\bar{z}) \xi^b_R(\bar{\omega}) \sim \frac{\delta^{ab}}{2\pi(\bar{z}-\bar{\omega})} \]
\[ \xi^a_L(\bar{z}) \xi^b_L(\omega) \sim 0. \]  
(38)

Using the correspondence \[ (36) \], the Hamiltonian \[ (34) \] simplifies as
\[ H = -\frac{\bar{v}_1}{2} (\xi^0_R \partial_x \xi^0_R - \xi^0_L \partial_x \xi^0_L) + \bar{H}[I, J_0] \]  
(39)

where the first part is the free Hamiltonian of the Majorana fermion \( \xi^0 \) with central charge \( c = 1/2 \) and

\[ \bar{H} = \frac{\pi v_1}{2} (I_R \cdot I_R + I_L \cdot I_L) + \frac{2\pi v_0}{3} (J_{0R} \cdot J_{0R} + J_{0L} \cdot J_{0L}) + g (I_R \cdot J_{0L} + I_L \cdot J_{0R}). \]  
(40)

All non-trivial physics is incorporated in the current dependent part of the Hamiltonian \( \bar{H} \) which decomposes into a sum of two commuting and chirally asymmetric parts as in Eqs. \((28), (29)) for \( N = 2 \):

\[ \bar{H} = H_1 + H_2, \quad ([H_1, H_2] = 0) \]  
(41)

\[ H_1 = \frac{\pi v_1}{2} I_R \cdot I_R + \frac{2\pi v_0}{3} J_{0L} \cdot J_{0L} + g I_R \cdot J_{0L} \]
\[ H_2 = \frac{\pi v_1}{2} I_L \cdot I_L + \frac{2\pi v_0}{3} J_{0R} \cdot J_{0R} + g I_L \cdot J_{0R}. \]  
(42)

B. Identification of the Toulouse point

We consider now the SU(2) broken version of model \((11) \) \( (g \to g_1, g_\perp) \) where we allow \( g_\parallel \) to be different from \( g_\perp \):

\[ H_1 = \frac{\pi v_1}{2} I_R \cdot I_R + \frac{2\pi v_0}{3} J_{0L} \cdot J_{0L} + g_\parallel I_R^z J_{0L}^z + \frac{g_\perp}{2} (I_R^+ J_{0L}^- + H.c.) \]  
(43)

and \( H_2 \) is obtained from \( H_1 \) by inverting chiralities of all the spin currents.

In this section, we present an exact solution of the Hamiltonian \((13) \) using Abelian bosonization. The solution is based on a mapping onto Majorana fermions and exploits the existence of a Toulouse-like, exactly solvable point, at a special value (though non-universal) of \( g_\parallel \) where the fermions are free.

The scaling equations for the model \((13) \) are of the Kosterlitz-Thouless form:

\[ \frac{dg_\parallel}{d\ln L} = \frac{g_\parallel^2}{\pi (v_1 + v_0)}, \quad \frac{dg_\perp}{d\ln L} = \frac{g_\parallel g_\perp}{\pi (v_1 + v_0)}, \]  
(44)

indicating the increase of the coupling constants upon renormalization for \( g_\parallel > -|g_\perp| \). Since both the exactly solvable point and the isotropic strong-coupling separatrix \( g_\parallel = |g_\perp| \) occur within this range, the exact solution of the anisotropic model \((13) \) is expected to exhibit generic properties of the original, SU(2) symmetric, model \((12) \).
The starting point of our approach is the Abelian bosonization of the component of the SU(2) current \( J_0 \) associated with the spin excitation of the central chain. We introduce a massless bosonic field \( \varphi \) and write (see for instance Appendix A of Ref. [34] or chapter 13 of Ref. [3]):

\[
J^z_{0R} = \frac{1}{\sqrt{2\pi}} \partial_x \varphi_R, \quad J^z_{0L} = \frac{1}{\sqrt{2\pi}} \partial_x \varphi_L \\
J^\pm_{0R} = \frac{1}{2\pi a_0} e^{\pm i \sqrt{2} \xi \varphi_R}, \quad J^\pm_{0L} = \frac{1}{2\pi a_0} e^{\pm i \sqrt{2} \xi \varphi_L}.
\]  

(45)

It is a simple matter to show that the representation (45) indeed reproduces the OPEs (8) of the SU(2)\(_1\) WZNW model.

On the other hand, the SU(2)\(_2\) current \( I \) can be expressed in terms of a triplet of massless Majorana fermions \( \xi^a, a=1,2,3 \) (see Eq. (36)). Since we consider the U(1)\(\otimes\)Z\(_2\)-symmetric version of the model, it is natural to discriminate \( (\xi^1, \xi^2) \) from the \( \xi^3 \) field. The two Majorana fields \( (\xi^1, \xi^2) \) can be combined to form a single Dirac field \( \chi \) which can in turn be bosonized with the introduction of a massless bosonic field \( \Phi \):

\[
\chi_R = \frac{\xi^2_R + i \xi^1_R}{\sqrt{2}} = \frac{\kappa}{\sqrt{2\pi a_0}} e^{i\sqrt{4\pi} \varphi_R} \\
\chi_L = \frac{\xi^2_L + i \xi^1_L}{\sqrt{2}} = \frac{\kappa}{\sqrt{2\pi a_0}} e^{-i\sqrt{4\pi} \varphi_L},
\]

(46)

where the anticommutation relation between \( \chi_R \) and \( \chi_L \) is insured by the choice \( [\Phi_R, \Phi_L] = i/4 \).

On the other hand, to take into account the correct anticommutation with the third Majorana fermion \( \xi^3 \), one needs to introduce an additional real fermionic degree of freedom \( \kappa \) (\( \kappa^2 = 1 \)). Using Eqs. (36, 46), the components of the SU(2)\(_2\) spin current \( I \) transform to:

\[
I^z_R = \frac{1}{\sqrt{\pi}} \partial_x \Phi_R, \quad I^z_L = \frac{1}{\sqrt{\pi}} \partial_x \Phi_L \\
I^\pm_R = \frac{1}{\sqrt{\pi a_0}} \xi^3_R e^{\pm i \sqrt{4\pi} \varphi_R}, \quad I^\pm_L = \frac{1}{\sqrt{\pi a_0}} \xi^3_L e^{\pm i \sqrt{4\pi} \varphi_L}.
\]

(47)

The chiral asymmetric parts of the total Hamiltonian \( \hat{H} \) (3), can then be written in the following bosonized form:

\[
\mathcal{H}_1 = v_0 \left( \partial_x \varphi_R \right)^2 + v_1 \left( \partial_x \Phi_R \right)^2 - i \frac{v_1}{2} \xi^3_R \partial_x \xi^3_R \\
+ \frac{g}{\sqrt{2\pi}} \partial_x \varphi_L \partial_x \Phi_R + \frac{ig_\perp}{2 (\pi a_0)^{3/2}} \xi^3_R \kappa \cos \left( \sqrt{4\pi} \Phi_R + \sqrt{8\pi} \varphi_L \right)
\]

(48)

and

\[
\mathcal{H}_2 = v_0 \left( \partial_x \varphi_L \right)^2 + v_1 \left( \partial_x \Phi_L \right)^2 + i \frac{v_1}{2} \xi^3_L \partial_x \xi^3_L \\
+ \frac{g}{\sqrt{2\pi}} \partial_x \varphi_R \partial_x \Phi_L + \frac{ig_\perp}{2 (\pi a_0)^{3/2}} \xi^3_L \kappa \cos \left( \sqrt{4\pi} \Phi_L + \sqrt{8\pi} \varphi_R \right).
\]

(49)

One can eliminate the cross terms \( \partial_x \varphi \partial_x \Phi \) by performing a canonical transformation:

\[
\begin{pmatrix}
\varphi_L \\
\Phi_R
\end{pmatrix} =
\begin{pmatrix}
\text{ch} \alpha & \text{sh} \alpha \\
\text{sh} \alpha & \text{ch} \alpha
\end{pmatrix}
\begin{pmatrix}
\bar{\Phi}_{2L} \\
\bar{\Phi}_{1R}
\end{pmatrix}
\]

(50)
and

\[
\begin{pmatrix}
\Phi_L \\
\varphi_R
\end{pmatrix} = \begin{pmatrix}
\text{cho} & \text{sho} \\
\text{sho} & \text{cho}
\end{pmatrix}
\begin{pmatrix}
\bar{\Phi}_1L \\
\bar{\Phi}_2R
\end{pmatrix}
\]

(51)

with

\[
\text{th}2\alpha = -\frac{g\parallel}{\pi\sqrt{2}(v_1 + v_2)}.
\]

(52)

Under the transformation (50) and (51), the arguments of the cosines in Eqs. (48, 49) become:

\[
\sqrt{4\pi}\Phi_R + \sqrt{8\pi}\varphi_L \to \sqrt{4\pi}
\left[
\left(\sqrt{2}\text{cho} + \text{sho}\right)\bar{\Phi}_2L + \left(\text{cho} + \sqrt{2}\text{sho}\right)\bar{\Phi}_1R
\right]
\]

(53)

\[
\sqrt{4\pi}\Phi_L + \sqrt{8\pi}\varphi_R \to \sqrt{4\pi}
\left[
\left(\sqrt{2}\text{cho} + \text{sho}\right)\bar{\Phi}_2R + \left(\text{cho} + \sqrt{2}\text{sho}\right)\bar{\Phi}_1L
\right].
\]

(54)

We immediately observe that for a special value of \(\alpha\) given by

\[
\text{th}2\alpha = -\frac{1}{\sqrt{2}},
\]

(55)

we are at the free fermion point where the two cosine terms acquire respectively conformal weights \((0,1/2)\) and \((1/2,0)\) and can be expressed in terms of the left and right components of a single Dirac fermion \(\psi\). When \(\alpha\) satisfies both Eqs. (52, 55), \(g\parallel\) takes a special positive value

\[
g\parallel^* = \frac{4\pi}{3} (v_1 + v_0),
\]

(56)

and the total Hamiltonian \((\mathcal{H})\) reads as follows:

\[
\mathcal{H} = -\frac{v_1}{2} \left(\xi_R^2 \partial_x \xi_R^2 - \xi_L^2 \partial_x \xi_L^2\right) + \frac{u_1}{2} \left[ \left(\partial_x \bar{\Phi}_1\right)^2 + \left(\partial_x \bar{\Theta}_1\right)^2 \right] + \frac{u_2}{2} \left[ \left(\partial_x \bar{\Phi}_2\right)^2 + \left(\partial_x \bar{\Theta}_2\right)^2 \right]
\]

\[
+ \frac{ig\perp}{2(\pi a_0)^{3/2}} \frac{\xi_R^3}{\sqrt{\pi}} \kappa \cos \left(\sqrt{4\pi} \bar{\Phi}_{2L}\right) + \frac{ig\perp}{2(\pi a_0)^{3/2}} \frac{\xi_L^3}{\sqrt{\pi}} \kappa \cos \left(\sqrt{4\pi} \bar{\Phi}_{2R}\right)
\]

(57)

where we have reestablished the complete bosonic fields \(\bar{\Phi}_a = \bar{\Phi}_{aL} + \bar{\Phi}_{aR}\) \((a = 1,2)\) and their dual counterpart \(\bar{\Theta}_a = \bar{\Phi}_{aL} - \bar{\Phi}_{aR}\). The associated velocities \(u_1\) and \(u_2\) are given at the Toulouse point by:

\[
u_1 = \frac{2v_1 - v_0}{3}, \quad u_2 = \frac{2v_0 - v_1}{3}.
\]

(58)

The Toulouse point solution is thus stable provided that all the velocities of the modes are positive, i.e. \(1/2 \leq v_0/v_1 \leq 2\). Since we are at the free fermion point, we can refermionize the two cosine terms in Eq. (57). Introducing a pair of Majorana fields \(\eta\) and \(\zeta\) and using the correspondence:

\[
\psi_{R,L} = \frac{\eta_{R,L} + i\zeta_{R,L}}{\sqrt{2}} = \frac{\kappa}{\sqrt{2\pi a_0}} e^{\pm i\sqrt{4\pi} \bar{\Phi}_{2R,L}},
\]

(59)

the two cosine terms become simply:

\[
\frac{\kappa}{\sqrt{\pi a_0}} \cos \left(\sqrt{4\pi} \bar{\Phi}_{2R,L}\right) = \eta_{R,L}.
\]

(60)
Therefore, the Hamiltonian ($\tilde{H}$) takes the following final form:

$$\tilde{H} = \frac{u_1}{2} \left[ (\partial_x \Phi_1)^2 + (\partial_x \Theta_1)^2 \right] - \frac{i v_2}{2} \left[ \xi_R \partial_x \xi_R - \xi_L \partial_x \xi_L \right] - \frac{i v_n}{2} \left[ \xi_R^3 \partial_x \xi_R^3 - \xi_L^3 \partial_x \xi_L^3 \right]$$

$$- \frac{i v_2}{2} \left[ \eta_R \partial_x \eta_R - \eta_L \partial_x \eta_L \right] + \text{im} \left[ \xi_R^3 \eta_L - \eta_R \xi_L^3 \right]$$

with $m = g_\perp / 2a_0 \pi$. Taking into account the contribution of the $Z_2$ Majorana fermion $\xi^0$ which has decoupled from the beginning, the original Hamiltonian \[ (33) \] reads at the Toulouse point:

$$H = \frac{u_1}{2} \left[ (\partial_x \Phi_1)^2 + (\partial_x \Theta_1)^2 \right] - \frac{i v_2}{2} \left[ \xi_R \partial_x \xi_R - \xi_L \partial_x \xi_L \right] - \frac{i v_n}{2} \left[ \xi_R^0 \partial_x \xi_R^0 - \xi_L^0 \partial_x \xi_L^0 \right]$$

$$- \frac{i v_2}{2} \left[ \xi_R^3 \partial_x \xi_R^3 - \xi_L^3 \partial_x \xi_L^3 \right] - \frac{i v_2}{2} \left[ \eta_R \partial_x \eta_R - \eta_L \partial_x \eta_L \right] + \text{im} \left[ \xi_R^3 \eta_L - \eta_R \xi_L^3 \right].$$

At the Toulouse point, the massless degrees of freedom are thus described in terms of an effective $S=1/2$ Heisenberg spin chain associated with the $\Phi_1$ field and two decoupled critical Ising models ($\xi^0, \zeta$) so that the total central charge in the IR limit is: $c_{IR} = 2$ in full agreement with Eq. \[ (33) \] for $N = 2$. One should also notice that the nature of the critical fields found in the Toulouse limit approach reproduces, for $N = 2$, the structure of the symmetry group \[ (31) \] of the fixed point corresponding to the Hamiltonian ($\tilde{H}$). Indeed, substituting the value $N = 2$ in Eq. \[ (31) \], we see using Eq. \[ (33) \] that this symmetry group simplifies as:

$$[[SU(2)]_R \otimes [Z_2]_L] \otimes [[SU(2)]_L \otimes [Z_2]_R]$$

which is in full agreement with the structure of the Hamiltonian \[ (31) \] at the Toulouse point.

The remaining part of the Hamiltonian \[ (32) \] has a spectral gap and describe the hybridization of the Majorana $\xi^3$ and $\eta$ fields with different chiralities. Indeed, it can be written as a sum of two commuting part $\mathcal{H} + \mathcal{H}^-$ with

$$\mathcal{H}^+ = \frac{i v_1}{2} \xi_L^3 \partial_x \xi_L^3 - \frac{i v_2}{2} \eta_R \partial_x \eta_R - \frac{\text{im}}{2} (\eta_R \xi_L^3 - \xi_L^3 \eta_R)$$

$$\mathcal{H}^- = -\frac{i v_1}{2} \xi_R^3 \partial_x \xi_R^3 + \frac{i v_2}{2} \eta_L \partial_x \eta_L + \frac{\text{im}}{2} (\xi_R^3 \eta_L - \eta_L \xi_R^3).$$

The resulting spectrum is given by:

$$E^+ (k) = ku_1 \pm \sqrt{k^2 \left( \frac{v_1 + v_0}{3} \right)^2 + m^2}$$

$$E^- (k) = -ku_1 \pm \sqrt{k^2 \left( \frac{v_1 + v_0}{3} \right)^2 + m^2},$$

which, apart from the velocity anisotropy, is reminiscent of the structure of excitations found in the Majorana approach for the Kondo lattice \[ (53) \]. One should stress that each Majorana fermion hybridizes with the other with the opposite chirality and that there is no coupling between Majorana fermion of the same chirality. This reflects the chiral nature of the fixed point. One should finally note that our solution contains the SU(2) point when $g_\perp = g_\parallel$ and the mass at the Toulouse point reads in that case: $m = g^* / 2a_0 = 2(v_1 + v_0)/3a_0$.

The exact solution based on the existence of a Toulouse point allows us to get a simple understanding of the structure of the critical elementary excitations and to estimate the leading asymptotics of the correlation functions. In the following, we shall realize this program by directly focusing on the physical properties of the 3-chain cylinder model.
V. PHYSICAL PROPERTIES OF THE 3-CHAIN CYLINDER MODEL

In this section, the spectral properties of the 3-chain cylinder model are discussed and we compute the leading asymptotics of all spin-spin correlation functions at the chiral fixed point. To this end, we first express the smooth and staggered magnetization of each chain in terms of the suitable fields describing the long distance physics at the chiral fixed point.

A. Representation of the spin currents and staggered spin fields at the Toulouse point

In this subsection, the spin currents ($J_a R, L$) and staggered spin fields ($n_a$) of the chain $a = 0, 1, 2$ are related to the two critical Majorana fermions ($\xi_0, \zeta$) and the massless bosonic field ($\bar{\Phi}_1$) at the Toulouse point. To get a better insight in the structure of excitations, we introduce the smooth part ($J^a$) and staggered part ($N^a$) corresponding to the effective $S=1/2$ Heisenberg spin chain described by the bosonic $\bar{\Phi}_1$ field. Using Eq. (45), one has the following correspondence:

$$J^z_R = \frac{1}{\sqrt{2\pi}} \partial_x \bar{\Phi}_1^R, \quad J^z_L = \frac{1}{\sqrt{2\pi}} \partial_x \bar{\Phi}_1^L$$

$$J^\pm_R = \frac{1}{2\pi a_0} e^{\pm i\sqrt{2\pi} \bar{\Phi}_1^R}, \quad J^\pm_L = \frac{1}{2\pi a_0} e^{\pm i\sqrt{2\pi} \bar{\Phi}_1^L}.$$  \hspace{1cm} (67)

The staggered part of a SU(2) spin density can also be expressed in terms of the bosonic $\bar{\Phi}_1$ and its dual field $\bar{\Theta}_1$ by (see Ref. [54] or chapter 13 of Ref. [39]):

$$N^z = -\frac{\lambda}{\pi a_0} \sin \left(\sqrt{2\pi} \bar{\Phi}_1\right)$$

$$N^\pm = \frac{\lambda}{\pi a_0} e^{\pm i\sqrt{2\pi} \bar{\Phi}_1},$$ \hspace{1cm} (68)

where $\lambda$ is non-universal constant related to the band gap for the charge excitations of the underlying Hubbard model. The WZNW primary operator $g$ transforming according to the fundamental representation then writes:

$$g = \frac{\pi}{\lambda \sqrt{2}} \left(\epsilon + \bar{\mathcal{N}} \cdot \bar{\sigma}\right)$$ \hspace{1cm} (69)

where $\epsilon = \lambda \cos(\sqrt{2\pi} \bar{\Phi}_1)/\pi a_0$ is the dimerization operator. With the bosonic representation (57, 58), one can check that the primary field $g$ (38) verifies the following OPE:

$$J^\mu (z) g (\omega, \bar{\omega}) \sim -\frac{1}{2\pi (z - \omega)} \left(\frac{\sigma^n}{2}\right) g (\omega, \bar{\omega})$$ \hspace{1cm} (70)

$$J^\mu (\bar{z}) g (\omega, \bar{\omega}) \sim \frac{1}{2\pi (\bar{z} - \bar{\omega})} g (\omega, \bar{\omega}) \left(\frac{\sigma^n}{2}\right)$$ \hspace{1cm} (71)

which corresponds (up to the normalization factor $1/2\pi$) to the definition of a WZNW primary operator of Ref. [17] (chapter 15).

1. Representation of the spin currents

We begin our analysis by considering the SU(2), spin currents $J_{DR, L}$ of the central chain. Using the canonical transformation (50, 51) with the special value (55), we obtain:
\[
\begin{align*}
\varphi_R &= -\Phi_1 + \sqrt{2}\Phi_2 \\
\varphi_L &= \sqrt{2}\Phi_2 - \Phi_1 \\
\end{align*}
\] (72)

from which together with Eq. (45) we deduce the transformation of the spin currents of the central chain at the Toulouse point:

\[
\begin{align*}
J^z_{0R,L} &= -J^z_{L,R} + \frac{1}{\sqrt{2}} \partial_x \Phi_{2R,L} \\
J^\pm_{0R} &= J^\pm_{L} e^{\pm i \sqrt{16\pi} \Phi_{2R}} \\
J^\pm_{0L} &= J^\pm_{L} e^{\pm i \sqrt{16\pi} \Phi_{2L}} \\
\end{align*}
\] (73)

where we have used (67) to express the spin currents of the central chain in terms of the "physical" current \( J^a \). Using the correspondence (59), \( \partial_x \Phi_{2R,L} \) can be related to the Majorana fermions \( \eta_{R,L} \) and \( \zeta_{R,L} \):

\[
\partial_x \Phi_{2R,L} = i \sqrt{\pi} \eta_{R,L} \zeta_{R,L}.
\] (74)

Consequently, at the Toulouse point, we deduce the expression of the z-part of the spin current of the central chain:

\[
\begin{align*}
J^z_{0R} &= -J^z_{L} + i \eta_R \zeta_R \\
J^z_{0L} &= -J^z_{L} + i \eta_L \zeta_L. \\
\end{align*}
\] (75)

Since the bilinear fields \( \eta_{R,L} \zeta_{R,L} \) are short-ranged, we can simplify further (75) and obtain the following correspondence valid for the estimation of the leading contribution of correlation functions:

\[
J^z_{0R,L} \sim -J^z_{L,R}.
\] (76)

Notice that the chiralities of the central chain and the physical spin current are opposite. We shall return to this important point when discussing the nature of elementary excitations of the model. For the transverse part, the bosonic operator \( \exp(\pm i \sqrt{16\pi} \Phi_{2R,L}) \) has a non-zero expectation value, one thus obtains ignoring short-ranged pieces:

\[
\begin{align*}
J^\pm_{0R} &\sim \gamma_R J^\pm_L \\
J^\pm_{0L} &\sim \gamma_L J^\pm_R \\
\end{align*}
\] (77)

where \( \gamma_{R,L} \) are defined as:

\[
\langle e^{\pm \sqrt{16\pi} \Phi_{2R,L}} \rangle = \gamma_{R,L}.
\] (78)

The actual value of \( \gamma_{R,L} \) is computed in Appendix A where one can show that \( \gamma_R = \gamma_L = \gamma_m \) and \( \gamma_m \) is real. One thus concludes that:

\[
J^\pm_{0R,L} \sim \gamma_m J^\pm_{L,R}.
\] (79)
The representation of the SU(2)_1 spin currents of the surface chains (J_{aR,L}, a = 1, 2) proceeds in the same way. To each chain a, we associate a massless bosonic field ϕ_a and its dual fields ϑ_a to write the currents as in Eq. (45):

\[ J^z_{aR,L} = \frac{1}{\sqrt{2\pi}} \partial_x \varphi_{aR,L}, \quad J^z_{aL,R} = \frac{1}{\sqrt{2\pi}} \partial_x \varphi_{aL,R} \]

\[ J^\pm_{aR} = \frac{1}{2\pi a_0} e^{\mp i\sqrt{8\pi} \varphi_{aR}}, \quad J^\pm_{aL} = \frac{1}{2\pi a_0} e^{\pm i\sqrt{8\pi} \varphi_{aL}}. \] (80)

To relate these fields to the bosonic field Φ of Eq. (47), let us introduce the symmetric and antisymmetric combinations:

\[ \varphi_{\pm} = \frac{1}{\sqrt{2}} (\varphi_1 \pm \varphi_2), \quad \vartheta_{\pm} = \frac{1}{\sqrt{2}} (\vartheta_1 \pm \vartheta_2), \] (81)

so that

\[ J^z_{aR,L} = \frac{1}{2\sqrt{2\pi}} (\partial_x \varphi_{+R,L} + \tau_a \partial_x \varphi_{-R,L}) \]

\[ J^\pm_{aR} = \frac{1}{2\pi a_0} e^{\mp i\sqrt{4\pi} \varphi_{+R}} e^{\pm i\tau_a \sqrt{4\pi} \varphi_{-R}} \]

\[ J^\pm_{aL} = \frac{1}{2\pi a_0} e^{\pm i\sqrt{4\pi} \varphi_{+L}} e^{\pm i\tau_a \sqrt{4\pi} \varphi_{-L}} \] (82)

where \( \tau_1 = 1 \) and \( \tau_2 = -1 \). Moreover, since we have the relation:

\[ I^\pm_\alpha = J^\pm_{1\alpha} + J^\pm_{2\alpha}, \quad \alpha = R, L \] (83)

and the expressions (47) for the bosonization of the SU(2)_2 current I, we obtain the following correspondence:

\[ \varphi_{+R,L} = \Phi_{R,L} \cos \left( \sqrt{4\pi} \varphi_{-\alpha} \right) = \sqrt{\pi a_0} i \xi^3_{\alpha} \kappa, \quad \alpha = R, L. \] (84)

The difference between the SU(2)_1 currents of the surface chains can also be expressed in terms of the Majorana fermions \( \xi^a, a = 0, 1, 2, 3 \) (see Eq. (36)). Using Eqs. (46, 82), one thus finds the identification:

\[ \partial_x \varphi_{-\alpha} = i \sqrt{\pi} \xi^3_{\alpha} \xi^0_{\alpha}, \quad \alpha = R, L \]

\[ \sin \left( \sqrt{4\pi} \varphi_{-R} \right) = \sqrt{\pi a_0} i \xi^0_{R} \kappa \]

\[ \sin \left( \sqrt{4\pi} \varphi_{-L} \right) = -\sqrt{\pi a_0} i \xi^0_{L} \kappa. \] (85)

With these relations at hand, we can now relate the spin currents of the surface chains (82) as a function of the Majorana fermions \( \xi^0, \xi^3 \) and the bosonic field Φ:

\[ J^z_{aR,L} = \frac{1}{2\sqrt{\pi}} \partial_x \Phi_{R,L} + \frac{1}{2} \tau_a \xi^3_{R,L} \xi^0_{R,L} \]

\[ J^\pm_{aR} = \frac{1}{2\pi a_0} e^{\mp i\sqrt{4\pi} \varphi_R} \left( i \xi^3_{R} \kappa \pm \tau_a \xi^0_{R} \kappa \right) \]

\[ J^\pm_{aL} = \frac{1}{2\pi a_0} e^{\pm i\sqrt{4\pi} \varphi_L} \left( i \xi^3_{L} \kappa \pm \tau_a \xi^0_{L} \kappa \right). \] (86)
Under the canonical transformation \([54, 55]\), the Majorana fermions \(\xi_0, \xi_3\) remain unchanged while the chiral bosonic fields \(\Phi_{R,L}\) transform at the Toulouse point according to:

\[
\Phi_R = \sqrt{2}\Phi_{1R} - \Phi_{2L} \\
\Phi_L = \sqrt{2}\Phi_{1L} - \Phi_{2R}. 
\]

Therefore, using the relation \([73]\) between the Majorana fields \(\zeta, \eta\) and the bosonic field \(\Phi_2\), we obtain for the z-component:

\[
J^z_{aR} = J^z_R - \frac{i}{2} \left( \eta_L \zeta_L - \tau_a \xi_3^0 \xi^0_3 \right) \\
J^z_{aL} = J^z_L - \frac{i}{2} \left( \eta_R \zeta_R - \tau_a \xi_3^0 \xi^0_3 \right), 
\]

and ignoring the short-ranged contributions, we get:

\[
J^z_{aR,L} \sim J^z_{R,L}. 
\]

In the same way, the transverse parts of the spin currents of the surface chains simplify as follows:

\[
J^\pm_{aR} = -\pi a_0 J^\pm_R \left( i\eta_L \xi_3^0 \pm \zeta_L \xi_3^0 \pm \tau_a \eta_L \xi^0_3 - i\tau_a \zeta_L \xi^0_3 \right) \\
J^\pm_{aL} = -\pi a_0 J^\pm_L \left( i\eta_R \xi_3^0 \pm \zeta_R \xi_3^0 \pm \tau_a \eta_R \xi^0_3 - i\tau_a \zeta_R \xi^0_3 \right). 
\]

Majorana bilinears, in this equation, built from \(\xi^3_{R,L}\) and \(\xi^3_{R,L}\) are short-ranged and from the hybridization \([54]\), the only non-zero vacuum expectation values are:

\[
\bar{\gamma}_{R,L} = i\pi a_0 (\xi^3_{R,L} \eta_L, R). 
\]

The value of this expectation value is estimated in Appendix A and one finds: \(\bar{\gamma}_R = \bar{\gamma}_L = \bar{\gamma}_m\) (\(\bar{\gamma}_m\) being real). With the same accuracy than in Eq. \([79]\), the transverse spin currents of the surface chains are thus given by:

\[
J^\pm_{aR} \sim J^\pm_R \left( \bar{\gamma}_m + ia_0 \tau_a \pi \zeta_L \xi^0_3 \right) \\
J^\pm_{aL} \sim J^\pm_L \left( \bar{\gamma}_m + ia_0 \tau_a \pi \zeta_R \xi^0_3 \right). 
\]

**2. Representation of the staggered spin fields**

Let us first analyse the staggered spin field of the central chain \(n_0\). As in Eq. \([88]\), this field can be expressed in terms of the bosonic field \(\varphi\) and its dual field \(\vartheta\):

\[
n^z_0 = -\frac{\lambda}{\pi a_0} \sin \left( \sqrt{2}\pi \varphi \right) \\
n^\pm_0 = \frac{\lambda}{\pi a_0} e^{\pm i\sqrt{2}\pi \vartheta}. 
\]

Using the transformation of the chiral bosonic fields \([72]\), one finds at the Toulouse point:

\[
\varphi = \sqrt{2}\Phi_2 - \Phi_1 \\
\vartheta = \sqrt{2}\Theta_2 + \Theta_1. 
\]
We deduce then the following result:

\[
\begin{align*}
  n_0^z &= -\frac{\lambda}{\pi a_0} \left( \sin \left(\sqrt{4\pi\Phi_2}\right) \cos \left(\sqrt{2\pi\Phi_1}\right) - \sin \left(\sqrt{2\pi\Phi_1}\right) \cos \left(\sqrt{4\pi\Phi_2}\right) \right), \\
  n_0^\pm &= \frac{\lambda}{\pi a_0} e^{\pm i\sqrt{2\pi\Phi_2}} e^{\pm i\sqrt{4\pi\Phi_1}}.
\end{align*}
\]  

(95)

One can simplify these expressions using the correspondence (59) to obtain the relations:

\[
\begin{align*}
  \cos \left(\sqrt{4\pi\Phi_2}\right) &= i\pi a_0 (\eta_R\eta_L + \zeta_R\zeta_L) \\
  \sin \left(\sqrt{4\pi\Phi_2}\right) &= i\pi a_0 (-\eta_R\zeta_L + \zeta_R\eta_L) \\
  e^{\pm i\sqrt{4\pi\Phi_2}} &= i\pi a_0 (-\eta_R\eta_L \pm i\zeta_R\zeta_L \pm i\eta_R\zeta_L + \zeta_R\zeta_L),
\end{align*}
\]

(96)

from which, we deduce the leading part of the representation of the staggered spin fields \(n_0\) associated with the central chain at the Toulouse point:

\[
\begin{align*}
  n_0^z &\sim -i\pi a_0 \zeta_R\zeta_L \tilde{N}_z \\
  n_0^\pm &\sim i\pi a_0 \xi_R\zeta_L \tilde{N}_\pm
\end{align*}
\]

(97)

where \(\tilde{N}_z\) corresponds to the staggered magnetization of the effective S=1/2 Heisenberg spin chain (see Eq. (68)).

The same correspondence for the staggered magnetizations \((n_a)\) of the surface chains is more difficult to obtain. The reason for this is that it involves nonlocal strings of the Majorana fermions \(\xi^a\), \(\eta\) and \(\zeta\). As we shall see, the trick will be to relate all these quantities to the order and disorder parameters of the underlying Ising models present in the problem. The Appendix B gives a review of the basic definitions for the bosonization of two Ising models and introduces also the different fields which will be useful in this subsection.

The staggered part \((n_a, a = 1, 2)\) of the spin density of the surface chains can be expressed in terms of the bosonic field \(\varphi_a\) and its dual \(\vartheta_a\) as in Eq. (93):

\[
\begin{align*}
  n_a^z &= -\frac{\lambda}{\pi a_0} \sin \left(\sqrt{\pi\varphi_a}\right) \\
  n_a^\pm &= \frac{\lambda}{\pi a_0} e^{\pm i\sqrt{\pi\vartheta_a}}.
\end{align*}
\]

(98)

Introducing the symmetric and antisymmetric combinations (81) of the bosonic fields, we get:

\[
\begin{align*}
  n_a^z &= -\frac{\lambda}{\pi a_0} \left( \sin \left(\sqrt{\pi\Phi}\right) \cos \left(\sqrt{\pi\varphi_-}\right) + \tau_a \cos \left(\sqrt{\pi\Phi}\right) \sin \left(\sqrt{\pi\varphi_-}\right) \right) \\
  n_a^\pm &= \frac{\lambda}{\pi a_0} e^{\pm i\sqrt{\pi\vartheta}} e^{\pm i\tau_a \sqrt{\pi\vartheta_-}},
\end{align*}
\]

(99)

where we have used the fact that \(\varphi_{+R,L}\) identifies to the bosonic fields \(\Phi_{R,L}\). According to the canonical transformation (77) at the Toulouse point, the bosonic fields \(\Phi, \Theta\) transform as:

\[
\begin{align*}
  \Phi &= \sqrt{2}\Phi_1 - \Phi_2 \\
  \Theta &= \sqrt{2}\Theta_1 + \Theta_2
\end{align*}
\]

(100)

whereas the fields \(\varphi_-, \vartheta_-\) are not affected by this transformation. Using Eqs. (83, 85) of the Appendix B, one can then relate the staggered spin fields \(n_a\) of the surface chains to the different order-disorder operators of the underlying Ising models of the problem:
\[ n^z_a = -\frac{\lambda}{\pi a_0} \left( \sin \left( \sqrt{2\pi} \bar{\Phi}_1 \right) \left( \mu_5 \mu_4 \mu_3 \mu_0 + \tau_a \sigma_5 \sigma_4 \sigma_3 \sigma_0 \right) + \cos \left( \sqrt{2\pi} \bar{\Phi}_1 \right) \left( -\sigma_5 \sigma_4 \mu_3 \mu_0 + \tau_a \mu_5 \mu_4 \sigma_3 \sigma_0 \right) \right) \]
\[ n^\pm_a = \frac{\lambda}{\pi a_0} e^{\pm i \sqrt{2\pi} \bar{\Phi}_1} \left( \sigma_5 \mu_4 \sigma_3 \mu_0 - \tau_a \mu_5 \sigma_4 \sigma_3 \sigma_0 \pm i \mu_5 \sigma_4 \sigma_3 \mu_0 \pm i r_a \sigma_5 \mu_4 \mu_3 \sigma_0 \right). \quad (101) \]

The Ising models labelled "3,5" are non-critical due to the hybridization (64) of the Majorana fermions \( \eta, \xi \) at the Toulouse point. As shown in Appendix C, the order-disorder operators \( (\sigma_3, 5, \mu_3, 5) \) of the corresponding Ising models are short-ranged and have the following vacuum expectation values:

\[ \langle \mu_3 \mu_5 \rangle = \langle \sigma_3 \sigma_5 \rangle = \mu \neq 0 \]
\[ \langle \sigma_3 \mu_5 \rangle = \langle \mu_3 \sigma_5 \rangle = 0. \quad (102) \]

Consequently, ignoring short-ranged contribution in Eq. (101), we finally obtain the correspondence between the staggered magnetization of the surface chains and the critical fields at the chiral fixed point:

\[ n^z_a \sim \mu N^z (\mu_4 \mu_0 + \tau_a \sigma_4 \sigma_0) \]
\[ n^\pm_a \sim \mu N^\pm (\mu_4 \mu_0 - \tau_a \sigma_4 \sigma_0). \quad (103) \]

Let us summarize our results obtained in this subsection. Using our Toulouse point approach, the spin currents and staggered fields of the three chains have been expressed in terms of the critical fields at the Toulouse point. Up to short-ranged pieces, the spin currents are given by:

\[ J^z_{0R,L} \sim -J^z_{L,R} \]
\[ J^\pm_{0R,L} \sim \gamma_m J^\pm_{L,R} \]
\[ J^-_{aR,L} \sim J^-_{R,L} \]
\[ J^\pm_{aR} \sim J^\pm_{R} \left( \bar{\gamma}_m + i a_0 \tau_a \pi \zeta_L \xi_R^0 \right) \]
\[ J^\pm_{aL} \sim J^\pm_{L} \left( \bar{\gamma}_m + i a_0 \tau_a \pi \zeta_R \xi_L^0 \right) \quad (104) \]

whereas the staggered magnetizations of the different chains write as follows

\[ n^z_0 \sim -i \pi a_0 \zeta_R \zeta_L N^z \]
\[ n^\pm_0 \sim i \pi a_0 \zeta_R \zeta_L N^\pm \]
\[ n^z_a \sim \mu N^z (\mu_4 \mu_0 + \tau_a \sigma_4 \sigma_0) \]
\[ n^\pm_a \sim \mu N^\pm (\mu_4 \mu_0 - \tau_a \sigma_4 \sigma_0). \quad (105) \]

These results enable us to discuss the spectral properties of the 3-chain cylinder model as well as the computation of the leading asymptotics of the spin-spin correlation functions at the chiral fixed point as described in the following section.

**B. Spectral properties**

There are two different kinds of massless elementary excitations at the Toulouse point: magnetic excitation described by the field \( \bar{\Phi}_1 \), and non-magnetic, singlet, excitations associated
with the two Majorana fermions $\xi^0$ and $\zeta$. These two groups of excitations are decoupled at the chiral fixed point. Let us first discuss the nature of the magnetic excitations.

1. Magnetic excitations

The magnetic elementary excitations of the system correspond to the spinons of the effective S=1/2 Heisenberg chain described by the bosonic field $\Phi_1$. These spinons appear only in pairs in all physical states and carry a spin S=1/2. It is very important to notice that, due to the mixing of different degrees of freedom reflected in the canonical transformation (50,51), the “physical” spinons, i.e. those defined as $\sqrt{\pi/2}$-kinks of the field $\Phi_1$ should not be misleadingly identified as the spinons of the central chain. To get a better understanding of the structure of the spin excitations at the chiral fixed point, let us recall the expressions (72, 88) of the currents $J^z_1$ and $J^z_2$ in terms of the “physical” current $\mathcal{J}^z = (1/\sqrt{2\pi}) \partial_x \Phi_1$ at the Toulouse point obtained in the previous subsection:

$$J^z_{1R(L)} = \mathcal{J}^z_{R(L)} - \frac{i}{2} \left( \eta_{L(R)} \zeta_{L(R)} - \xi^3_{R(L)} \xi^0_{R(L)} \right),$$

$$J^z_{2R(L)} = \mathcal{J}^z_{R(L)} - \frac{i}{2} \left( \eta_{L(R)} \zeta_{L(R)} + \xi^3_{R(L)} \xi^0_{R(L)} \right),$$

$$J^z_{0R(L)} = -\mathcal{J}^z_{L(R)} + i \eta_{R(L)} \zeta_{R(L)}.$$  \hspace{1cm} (106)

At energies $|\omega| \gg m$, where all Majorana fields can be considered as massless, Eqs. (106) transform back to the standard definitions of the currents of the three decoupled chains. In this UV limit, one has a picture of three groups of independently propagating spinons. However, in the IR limit ($|\omega| \ll m$), all Majorana bilinears in Eq. (106) are characterized by short-ranged correlations (since the Majorana fermions $\eta$ and $\xi^3$ are massive), implying that strongly fluctuating parts of the currents of individual chains are no longer independent. The spinons of the individual chains turn out to be strongly correlated in the IR limit to participate at the formation of a single, physical, current $\mathcal{J}^z$. In fact, the physical spinon represents a chirally asymmetric, strongly correlated state of three spinons. Indeed, consider, for instance, a right-moving $\sqrt{\pi/2}$-kink of the field $\Phi_1$, representing a physical spinon with the spin projection $S^z = 1/2$. According to the exact relation

$$\mathcal{J}^z_{R(L)} = J^z_{1R(L)} + J^z_{2R(L)} + J^z_{0L(R)}$$ \hspace{1cm} (107)

following from Eq. (106), such an excitation is a combination of two right-moving spinons of the surface chains, each carrying the spin $S^z = 1/2$, and a left-moving antispinon of the central chain, with $S^z = -1/2$. The rigidity of such a state is ensured by a finite mass gap in the $(\eta - \xi^3)$ sector of the model. Therefore, in physical excitations, two spinons of the surface chains form a bound state with an antispinon of the central chain. Otherwise stated, the polarization of the long-wavelength magnetic excitations of the surface chains is always the same but opposite to the polarization of the central chain. This peculiar structure of the elementary spin excitations at the chiral fixed point is clearly reflected by the expression for the spin velocity $u_1$ of the bosonic field $\Phi_1$: $u_1 = (2v_1 - v_0)/3$.

These excitations correspond to the magnetic excitations of the model since a uniform magnetic field $H$ along the z direction only couples to the $\Phi_1$ field. Indeed, in the continuum limit, this magnetic field couples to the smooth part of the total spin density of the three chains:
\[ \mathcal{H}_H = -\frac{H}{\sqrt{2\pi}} (\partial_x \varphi_1 + \partial_x \varphi_1 + \partial_x \varphi_1), \quad (108) \]

and performing the Toulouse transformation \[ \{D4, 100\}, \] one obtains:

\[ \mathcal{H}_H = -\frac{H}{\sqrt{2\pi}} \partial_x \Phi_1. \quad (109) \]

The uniform susceptibility \( \chi \) of the model corresponds thus to that of a single Heisenberg S=1/2 spin chain and it is given by (in units of \( g\mu_B \)):

\[ \chi = \frac{1}{2\pi u_1} = \frac{3}{2\pi (2v_1 - v_0)}. \quad (110) \]

Notice that due to the non-trivial structure of the spinon at the Toulouse point, the uniform susceptibility of each chain does not add in the case of a truly uniform field. In contrast, the susceptibilities would add coherently if one considers a staggered magnetic field \( H_a \) in the transverse direction: \( H_a = H(\delta_{a1} + \delta_{a2} - \delta_{a0}) \), \( a \) being the chain index. In that case, one finds the contribution:

\[ \chi_{\text{stag}}^{\perp} = \frac{3}{2\pi u_1}. \quad (111) \]

2. **Pseudo-charge degrees of freedom**

Apart from the non-trivial nature of the spinon, the chiral fixed point manifests itself in the existence of massless \( Z_2 \) singlet excitations. A first non-magnetic excitation stems from the Majorana fermion \( \xi^0 \) that decouples from the rest of the spectrum. This Majorana fermion describes collective excitations of singlet pairs formed on the surface chains. The nature of the other massless Majorana fermion \( \zeta \) at the chiral fixed point is less transparent: It is a highly nonlocal object when expressed in terms of the original spin operators. A simple way to understand the role of these singlet excitations is to combine the Majorana \( \xi^0 \) and \( \zeta \) fields into a single Dirac \( \Psi_c \) fermion and then bosonize it with the introduction of a bosonic field \( \tilde{\Phi}_c \):

\[ \Psi_{cR} = \frac{\xi^0_R + i\zeta_R}{\sqrt{2}} = \frac{1}{\sqrt{2\pi a_0}} e^{i\sqrt{4\pi} \tilde{\Phi}_{cR}} \]
\[ \Psi_{cL} = \frac{\xi^0_L + i\zeta_L}{\sqrt{2}} = \frac{1}{\sqrt{2\pi a_0}} e^{-i\sqrt{4\pi} \tilde{\Phi}_{cL}}. \quad (112) \]

The corresponding massless bosonic field \( \tilde{\Phi}_c \) resembles the scalar field describing the charge degrees of freedom in the one-dimensional SU(2) Hubbard model away from half-filling in the limit \( U = \infty \). This analogy can be seen by looking at the representation \[ \{105\} \] of the staggered fields at the Toulouse point. Using the bosonization result of two Ising models \[ \{30\} \], the total staggered magnetization \( n_+ \) of the surface chains reads as follows in terms of the bosonic field \( \tilde{\Phi}_c \):

\[ n_+ = n_1 + n_2 \sim 2\mu_4\mu_0 N^\chi \]
\[ \sim \cos \left( \sqrt{\tau} \tilde{\Phi}_c \right) \bar{N}. \quad (113) \]
This bosonized version of $n^+$ is reminiscent of the staggered magnetization of the Hubbard model away from half-filling at $U = \infty$ ($K_c = 1/2$ in that case, see for instance Refs. [38,39]). Apart from a velocity anisotropy ($v_0 \neq u_2$), the fermionic fields $\xi^0$ and $\zeta$ plays the role of the charge degrees of freedom in the corresponding Hubbard model. We shall hence refer to the gapless Majorana fermions $\xi^0$ and $\zeta$ as “pseudo-charge” excitations which account for the central charge $c = 1$ at the IR fixed point. The spin and pseudo-charge separation is already manifest at the Toulouse point in the Hamiltonian (62) since these fermions fields are decoupled from the magnetic excitations described by the bosonic field $\Phi^1$. This separation of the different modes will also be clearly seen in the expression of the leading asymptotics of correlations functions that we now estimate.

C. Spin-spin correlation functions at the chiral fixed point

We are in position to compute the leading asymptotics of the spin-spin correlation functions of the 3-chain cylinder model at the chiral fixed point. Let us begin our analysis by considering the spin-spin correlation functions between the central spins.

1. Correlation functions between the central spins

Since the SU(2) symmetry is broken to an U(1) symmetry in our Toulouse point approach ($g_\parallel \neq g_L$), we shall discriminate between the z and perpendicular components of the correlation function:

\[
\langle S^z_0 (x, \tau) S^z_0 (0, 0) \rangle = \langle J^z_{0R} (x, \tau) J^{-}_0 (0, 0) \rangle + (R \rightarrow L) + (-1)^{x/a_0} \langle n^z_0 (x, \tau) n^z_0 (0, 0) \rangle,
\]

(114)

\[
\langle S^+_0 (x, \tau) S^-_0 (0, 0) \rangle = \langle J^+_{0R} (x, \tau) J^-_0 (0, 0) \rangle + (R \rightarrow L) + (-1)^{x/a_0} \langle n^+_0 (x, \tau) n^-_0 (0, 0) \rangle.
\]

(115)

Using the representation (104, 105) of the uniform and staggered parts of the spin density of the central chain at the Toulouse point, we get the following leading behaviours:

\[
\langle S^z_0 (x, \tau) S^z_0 (0, 0) \rangle \sim \langle J^z_{0R} (x, \tau) J^{-}_0 (0, 0) \rangle + (R \rightarrow L) + (-1)^{x/a_0} \pi^2 a_0^2 \langle N^z (x, \tau) N^z (0, 0) \rangle \langle \zeta_R (x, \tau) \zeta_R (0, 0) \rangle \langle \zeta_L (x, \tau) \zeta_L (0, 0) \rangle
\]

(116)

and

\[
\langle S^+_0 (x, \tau) S^-_0 (0, 0) \rangle \sim \gamma_m^2 \langle J^+_R (x, \tau) J^-_R (0, 0) \rangle + (R \rightarrow L) + (-1)^{x/a_0} \pi^2 a_0^2 \langle N^+ (x, \tau) N^- (0, 0) \rangle \langle \zeta_R (x, \tau) \zeta_R (0, 0) \rangle \langle \zeta_L (x, \tau) \zeta_L (0, 0) \rangle.
\]

(117)

Accordingly, the estimation of these correlations simply reduces to the computation of two-point functions of a massless bosonic field $\Phi^1$ and a massless Majorana fermion $\zeta$. The final expressions for the leading asymptotics of the correlation functions between spins in the central chain at the chiral fixed point are thus
where we recall that $\tau$ using the expression of the uniform and staggered spin fields (104, 105) at the Toulouse point:

$$\langle S_{0}^{\pm(\lambda)}(x,\tau)S_{0}^{\pm(\lambda)}(0,0) \rangle \sim \frac{1}{8\pi^2} \left( \frac{1}{(x+i\mu_1\tau)^2} + \frac{1}{(x-i\mu_1\tau)^2} \right)$$

$$+ (-1)^{x/\alpha_0} \frac{\lambda^2 a_0}{8\pi^2} \left( \frac{1}{(x^2+u_1^2\tau^2)^{1/2}} - \frac{1}{(x^2+u_1^2\tau^2)^{1/2}} \right) \, \gamma_{\lambda}$$

$$+ (-1)^{x/\alpha_0} \frac{\lambda^2 a_0}{4\pi^2} \left( \frac{1}{(x^2+u_1^2\tau^2)^{1/2}} - \frac{1}{(x^2+u_1^2\tau^2)^{1/2}} \right) \, \gamma_{\lambda}$$

$$+ (-1)^{x/\alpha_0} \frac{\lambda^2 a_0}{4\pi^2} \left( \frac{1}{(x^2+u_1^2\tau^2)^{1/2}} - \frac{1}{(x^2+u_1^2\tau^2)^{1/2}} \right) \, \gamma_{\lambda}$$

$$+ (-1)^{x/\alpha_0} \frac{\lambda^2 a_0}{4\pi^2} \left( \frac{1}{(x^2+u_1^2\tau^2)^{1/2}} - \frac{1}{(x^2+u_1^2\tau^2)^{1/2}} \right) \, \gamma_{\lambda}$$

(118)

where $a, b = (1, 2)$ is the chain index. We shall now compute the spin-spin correlation functions between the surface chains:

$$\langle S_{0}^{\pm(\lambda)}(x,\tau)S_{0}^{\pm(\lambda)}(0,0) \rangle \sim \langle J_{\alpha(\lambda)}^{\pm}(x,\tau)J_{\beta(\lambda)}^{\mp}(0,0) \rangle + (R \rightarrow L) + (-1)^{x/\alpha_0} \langle n_{0}^{\pm(\lambda)}(x,\tau)n_{0}^{\mp(\lambda)}(0,0) \rangle$$

(120)

where $\tau_1 = 1, \tau_2 = -1$ and the perpendicular part reads

$$\langle S_{0}^{\pm(\lambda)}(x,\tau)S_{0}^{\pm(\lambda)}(0,0) \rangle \sim \langle J_{\alpha(\lambda)}^{\pm}(x,\tau)J_{\beta(\lambda)}^{\mp}(0,0) \rangle \left( \gamma_{\lambda}^2 + \tau_3 \gamma_{\lambda}^2 \right)$$

$$+ (R, L) \rightarrow (L, R) + (-1)^{x/\alpha_0} \mu^2 \langle N^+(x,\tau)N^-(0,0) \rangle$$

$$+ (\mu_4(x,\tau)\mu_4(0,0))\langle \mu_0(x,\tau)\mu_0(0,0) \rangle + \tau_3 \gamma_{\lambda} \langle \sigma_4(x,\tau)\sigma_4(0,0) \rangle \langle \sigma_0(x,\tau)\sigma_0(0,0) \rangle$$

(122)

The Ising models labelled by “0” and “4” associated with respectively the Majorana fermion $\xi^0$ and $\zeta$ are decoupled and critical at the chiral fixed point. The correlation functions of the order and disorder operators $\sigma_0,4, \mu_0,4$ are known exactly. In the long time and long distance limit they are given by (see Appendix B):

$$\langle \mu_0(x,\tau)\mu_0(0,0) \rangle = \langle \sigma_0(x,\tau)\sigma_0(0,0) \rangle$$

$$\sim \frac{a_{0}^{1/4}}{(x^2+v_1^2\tau^2)^{1/8}} \, \gamma_{\lambda}$$

(124)

$$\langle \mu_4(x,\tau)\mu_4(0,0) \rangle = \langle \sigma_4(x,\tau)\sigma_4(0,0) \rangle$$

$$\sim \frac{a_{0}^{1/4}}{(x^2+v_1^2\tau^2)^{1/8}} \, \gamma_{\lambda}$$

(125)
This enables us to obtain the result for the asymptotics of the correlation functions between spins of the surface chains:

\[
\langle S^z_a (x, \tau) S^z_b (0, 0) \rangle \sim -\frac{1}{8\pi^2} \left( \frac{1}{(x + i\mu_1 \tau)^2} + \frac{1}{(x - i\mu_1 \tau)^2} \right) + (-1)^{x/a_0} \delta_{ab} \frac{\mu^2 \lambda^2}{\pi^2 \sqrt{\mu_0} (x^2 + u_1^2 \tau^2)^{1/2}} \frac{1}{(x^2 + v_1^2 \tau^2)^{1/8}} \frac{1}{(x^2 + u_2^2 \tau^2)^{1/8}},
\]

(126)

\[
\langle S^+_a (x, \tau) S^-_b (0, 0) \rangle \sim -\frac{\gamma_m}{4\pi^2} \left( \frac{1}{(x + i\mu_1 \tau)^2} + \frac{1}{(x - i\mu_1 \tau)^2} \right) + (-1)^{x/a_0} \delta_{ab} \frac{2\mu^2 \lambda^2}{\pi^2 \sqrt{\mu_0} (x^2 + u_1^2 \tau^2)^{1/2}} \frac{1}{(x^2 + v_1^2 \tau^2)^{1/8}} \frac{1}{(x^2 + u_2^2 \tau^2)^{1/8}}.
\]

(127)

Here again, the leading contribution of the uniform part of the correlation functions is that of a single S=1/2 Heisenberg spin chain. The other critical modes manifest themselves in subleading contributions in the uniform part and directly in the staggered part. In particular, the contributions of the Majorana fermions \( \xi^0 \) and \( \zeta^0 \) in the staggered part show up through their corresponding order and disorder Ising operators.

3. Correlation functions between spins of the surface chains and of the central chain

The last spin-spin correlation functions to estimate at the chiral fixed point consists in correlation between a spin of the surface chain \( S_a \) (a = 1, 2) and a spin of the central chain \( S_b \). Using the representation \([104, 105]\) of the different spin fields at the Toulouse point, we obtain following the same route as for the previous correlation functions:

\[
\langle S^z_a (x, \tau) S^z_b (0, 0) \rangle \sim \frac{1}{8\pi^2} \left( \frac{1}{(x + i\mu_1 \tau)^2} + \frac{1}{(x - i\mu_1 \tau)^2} \right),
\]

(128)

\[
\langle S^+_a (x, \tau) S^-_b (0, 0) \rangle \sim -\frac{\gamma_m \tau_m}{4\pi^2} \left( \frac{1}{(x + i\mu_1 \tau)^2} + \frac{1}{(x - i\mu_1 \tau)^2} \right).
\]

(129)

It is interesting to notice that the staggered parts of these correlation functions are short-ranged at the chiral fixed point. This result stems from the very special structure of the spectrum of the Hamiltonian at the Toulouse point discussed above.

We end this section by estimating the NMR relaxation rate \( 1/T_1 \) at low temperature. The slowest correlation functions of the model at the chiral fixed point correspond to the staggered correlations between the spins of the surface chains (see Eqs. \([126, 127]\)). We emphasize that the exponents occurring in these correlation functions are universal and characterize a new universality class in spin ladders. We deduce from these correlation function \([126, 127]\) the low-temperature dependence of the NMR relaxation rate:

\[
\frac{1}{T_1} \sim \sqrt{T}
\]

(130)
in contrast with the S=1/2 Heisenberg spin chain where $1/T_1 \sim \text{const}$ \cite{38,39}. Finally, we remark that the result (130) corresponds to the leading behaviour of the NMR relaxation rate of the one-dimensional Hubbard model away from half-filling at $U = \infty$ \cite{38,39}. The main reason for this stems from the presence of the pseudo-charge degrees of freedom $\xi^0$, $\zeta$ in the correlation function (126, 127) that enter through their corresponding Ising order-disorder operators $(\sigma^0, \rho^0, \mu^0)$ with scaling dimension $1/8$.

VI. STABILITY OF THE CHIRAL FIXED POINT OF THE 3-CHAIN CYLINDER MODEL

In Sections IV and V, we have carefully analysed the Hamiltonian consisting in only current-current interaction describing two marginally coupled SU(2)$_2$ and SU(2)$_1$ WZNW models. The model with only current-current interaction has been solved exactly using a Toulouse point approach. At the strong coupling fixed point, this Hamiltonian belongs to the chiral stabilized liquids universality class with central charge $c = 2$. The critical fields at the chiral fixed point consist of two decoupled gapless modes: a magnetic sector described by an effective S=1/2 Heisenberg chain associated with the bosonic field $\bar{\Phi}_1$ and the pseudo-charge degrees of freedom stemming from the two Majorana fermions $\xi^0$ and $\zeta$. Moreover, the Toulouse point solution captures all universal properties of the chiral fixed point as shown in Appendix D.

The next step of our approach is to analyse the stability of this chiral fixed point. The non-perturbative basis provided by the Toulouse point approach enables us to investigate the stability of the chiral fixed point under several operators such as backscattering contributions. In particular, we shall first investigate the effect of the interchain backscattering operator to deduce the nature of the low energy physics of the 3-chain cylinder model in the vicinity of the line $\bar{g} = 0$.

A. Effect of the interchain backscattering perturbation at the chiral fixed point

Let us begin by studying the effect of the interchain coupling in the 3-chain cylinder model in the vicinity of the chiral fixed point:

$$\mathcal{H}_b = \tilde{g} \mathbf{n}_0 \cdot \mathbf{n}_+. \quad (131)$$

From Eq. (13), we see that the coupling constant $\tilde{g}$ is independent from the coupling $g$ of the current-current interaction. One can thus always suppose that $\tilde{g}$ is small to investigate the backscattering term (131) as a weak perturbation at the chiral fixed point: $|\tilde{g}| \ll 1$. Using the bosonized form of the staggered spin fields given by Eqs. (93, 98), we have:

$$\mathcal{H}_b = \frac{2\tilde{g} \lambda^2}{\pi^2 a_0} \left( \cos \left( \sqrt{\pi} \vartheta_- \right) \cos \left( \sqrt{2\pi} \vartheta - \sqrt{\pi} \Theta \right) + \cos \left( \sqrt{\pi} \varphi_- \right) \sin \left( \sqrt{2\pi} \varphi \right) \sin \left( \sqrt{\pi} \Phi \right) \right) \quad (132)$$

so that in terms of the different fields at the Toulouse point, the backscattering term (131) reads as follows:
\[ H_b = \frac{2\tilde{g}\lambda^2}{\pi^2a_0^2} \left( \cos(\sqrt{\pi}\theta) \cos(\sqrt{\pi}\Theta) - \frac{1}{2} \cos(\sqrt{\pi}\varphi) \cos(\sqrt{\pi}\Phi) \right) \]

\[ + \frac{1}{2} \cos(\sqrt{\pi}\varphi) \cos(3\sqrt{\pi}\Phi - \sqrt{8\pi}\Phi_1) \right) . \]  

(133)

Using the Ising dictionary (see Appendix B), it can be shown that the contribution in the magnetic sector of Eq. (133) is irrelevant. The backscattering term affects mostly the pseudo-charge sector. Indeed, using the results (C9) from the structure of the massive modes at the Toulouse point, one finds the following estimate at the chiral fixed point:

\[ H_b \sim \frac{\lambda^2\mu\tilde{g}}{\pi^2a_0^2} \mu_0\mu_4. \]  

(134)

We thus conclude that the backscattering operator opens a gap, \( \Delta_c \), in the pseudo-charge sector but has no effect on the magnetic (spinons) excitations. Standard scaling arguments give an estimate of the mass gap since the operator (134) has scaling dimension 1/4 and thus \( \Delta_c \sim \tilde{g}^{1/7} \). The chiral fixed point is thus unstable in the far IR limit, and one expects that the system will flow to the \( c = 1 \) fixed point of the standard three-leg spin ladder [54]. Of course, the very applicability of the perturbative approach to the chiral fixed point requires that \( \Delta_c \ll m \), the condition which can always be satisfied for sufficiently small \( \tilde{g} \). Under this condition, there exists an intermediate but still low-energy region \( \Delta_c \ll E \ll m \) where the \( c = 2 \) behaviour caused by frustration is dominant. The physics in this region is universal and cannot be understood without having recourse to the chiral fixed point. At lower energies, \( E \ll \Delta_c \), the system will eventually cross over to the conventional critical \( c = 1 \) behaviour.

**B. Effect of a weak interaction between surface chains at the chiral fixed point**

In this subsection, we investigate the stability of the chiral fixed point against a weak interaction between the spins of the surface chains:

\[ O_{12} = g_{12}\mathbf{S}_1 \cdot \mathbf{S}_2 \approx g_{12} (\mathbf{n}_1 \cdot \mathbf{n}_2 + \mathbf{J}_1 \cdot \mathbf{J}_2) . \]  

(135)

Such a term accounts for periodic transverse boundary condition in the 3-chain cylinder model and may change drastically the physics of the chiral fixed point. In Eq. (135), the coupling constant \( g_{12} \) is independent of the coupling \( g \) of the current-current interaction of the model so that one can always consider \( O_{12} \) as a weak perturbation at the chiral fixed point: \( |g_{12}| \ll 1 \).

Let us first analyse the \( \mathbf{n}_1 \cdot \mathbf{n}_2 \) contribution at the chiral fixed point. Using the bosonized form (98) of the staggered magnetizations of the surface chains and the Toulouse basis (100), one finds:

\[ \mathbf{n}_1 \cdot \mathbf{n}_2 = \frac{\lambda^2}{2\pi^2a_0^2} \left[ \cos(4\sqrt{\pi}\varphi) + 2 \cos(4\sqrt{\pi}\theta) - \cos(\sqrt{8\pi}\Phi_1 - \sqrt{8\pi}\Phi_2) \right] . \]  

(136)

Using the refermionization formula (84, 85, 96), this operator can be expressed in terms of the different fields occuring in the Toulouse point solution:

\[ \mathbf{n}_1 \cdot \mathbf{n}_2 = \frac{i\lambda^2}{2\pi a_0} \left[ -\xi_R^3\xi_L^3 + 3\xi_R^0\xi_L^0 - (\eta_R\eta_L + \zeta_R\zeta_L) \cos(\sqrt{8\pi}\Phi_1) \right. \]

\[ \left. - (\zeta_R\eta_L - \eta_R\zeta_L) \sin(\sqrt{8\pi}\Phi_1) \right] . \]  

(137)
Since the Majorana fermions $\xi$ and $\eta$ are short-ranged fields, we can drop their contribution so that:

$$\mathbf{n}_1 \cdot \mathbf{n}_2 \simeq \frac{i\lambda^2}{2\pi a_0} \left[ 3\xi_R^0 \xi_L^0 - \zeta_R \zeta_L \cos \left( \sqrt{8\pi} \Phi_1 \right) \right].$$  \hspace{1cm} (138)

The two terms in this equation are of different nature. One the one hand, the first contribution is a relevant perturbation (with scaling dimension 1) which identifies with the energy density operator of the Ising model corresponding to the Majorana fermion $\xi^0$ and drives the latter model out-of criticality. On the other hand, the second operator in Eq. (138) is a naively irrelevant contribution with scaling dimension 3 but, as we shall see below, it is crucial to keep it since it couples degrees of freedom of different nature: Magnetic ones described by the bosonic field $\Phi_1$ and a half of the pseudo-charge degrees of freedom stemming from the Majorana fermion $\zeta_{R,L}$.

The same analysis can be made for the current-current contribution in Eq. (135). In particular, using the results (88, 90), the leading part of this perturbation at the chiral fixed point affects mostly the magnetic degrees of freedom:

$$\mathbf{J}_1 \cdot \mathbf{J}_2 \simeq 2 \mathcal{J}_R^2 \mathcal{J}_L^2 - \frac{\bar{\gamma}^2}{2\pi a_0} \cos \left( \sqrt{8\pi} \Phi_1 \right)$$  \hspace{1cm} (139)

which can be rewritten at the SU(2) point ($\bar{\gamma}^2 = 1$, see Appendix A) in a rotationally invariant form:

$$\mathbf{J}_1 \cdot \mathbf{J}_2 \simeq 2 \mathcal{J}_R \cdot \mathcal{J}_L.$$  \hspace{1cm} (140)

With the two results (138, 140) at hand, we are now in position to investigate the stability of the chiral fixed point upon switching on a weak interaction between the surface chains. The effective interaction (133) at the chiral fixed point writes:

$$\mathcal{O}_{12} \simeq 2g_{12} \mathcal{J}_R \cdot \mathcal{J}_L + \frac{g_{12}i\lambda^2}{2\pi a_0} \left[ 3\xi_R^0 \xi_L^0 - \zeta_R \zeta_L \cos \left( \sqrt{8\pi} \Phi_1 \right) \right].$$  \hspace{1cm} (141)

Due to the first term, the effect of the operator ($\mathcal{O}_{12}$) on the chiral fixed point strongly depends on the sign of the coupling $g_{12}$.

For an antiferromagnetic interaction (i.e. $g_{12} > 0$), we expect the succession of two transitions in that case. A first one (Ising transition), as soon as the coupling $g_{12}$ is switched on, the model is still critical but with a smaller central charge $c = 3/2$ since, as already emphasized, the Majorana fermion $\xi^0$ acquires a mass. As a consequence, the $\mathbb{Z}_2$ symmetry with respect to the interchange of the two surface chains is now broken. When $g_{12}$ increases, there is a critical value when $2g_{12}$ exceeds the coupling constant $\gamma < 0$ of the marginal irrelevant current-current interaction of the effective $S=1/2$ Heisenberg chain associated with the bosonic field $\Phi_1$. In that case, the magnetic sector becomes massive and enters in a dimerized phase as in the $J_1$-$J_2$ problem [4]. Moreover, in this massive phase, the bosonic field $\Phi_1$ becomes locked and the operator $\cos \left( \sqrt{8\pi} \Phi_1 \right)$ acquires a non-zero expectation value: $\langle \cos \left( \sqrt{8\pi} \Phi_1 \right) \rangle \neq 0$. As a consequence, the magnetic excitation affects indirectly the pseudo-charge degrees of freedom by giving a mass term for the Majorana fermion $\zeta$ due to the third term in Eq. (141): $-g_{12}i\lambda^2 \cos \left( \sqrt{8\pi} \Phi_1 \right) \zeta_R \zeta_L / (2\pi a_0)$. Therefore, in the dimerized phase, all degrees of freedom are massive. In summary, the chiral fixed point is unstable against the presence
of an antiferromagnetic interaction between the surface chains. There are two distinct Ising transitions in the pseudo-charge sector whereas the spin part spontaneously dimerized for a sufficiently strong value of the interaction $g_{12}$.

For a ferromagnetic interaction ($g_{12} < 0$), the Majorana fermion $\xi^0$ still acquires a mass but now the bosonic field $\Phi_1$ and the Majorana fermion $\zeta$ remain massless excitations so that the resulting model has central charge $c = 3/2$. One should note that a pseudo-charge fractionalization occurs in this $c = 3/2$ phase: A half of the degrees of freedom, described by the Majorana fermion $\xi^0$, in the pseudo-charge sector becomes massive by an Ising transition whereas the other (i.e. the Majorana fermion $\zeta$) is still critical. A similar phenomenon for the charge degrees of freedom of a one-dimensional model has been reported in Ref. [62].

VII. CONCLUDING REMARKS

In this work, we have studied some specific models which display in the long distance limit some CSL physics and belongs to a new non-Fermi-Liquid class of fixed points describing the IR behaviour of one-dimensional interacting chiral fermions. The possible realizations of this CSL state, proposed in this paper, belong to several topics of the one-dimensional Quantum Magnetism: the (N+1)-chain cylinder as a special frustrated spin ladder, its asymmetric doped version with a doped central chain as an example of a Luttinger liquid in an active environment.
and finally the Kondo-Heisenberg model with $N$ channels away from half-filling as a generalized multichannel Kondo lattice.

The common feature of all these models stems from the fact that their Hamiltonians in the continuum limit have the same structure consisting of two marginally coupled SU(2)$_N$ and SU(2)$_1$ WZNW models with only current-current interactions. In particular, this Hamiltonian decomposes into two commuting chirally asymmetric parts which transform into each other under the time-reversal symmetry. Each part flows to an intermediate fixed point where the system as a whole displays critical properties characterized by a non-trivial symmetry group belonging to the universality class of the CSL state [30]. All the possible lattice realizations are expected to exhibit critical properties in the IR limit with, in general, non-integer central charge.

For the $N = 2$ case, by allowing anisotropic interactions in spin space, the model of two marginally coupled WZNW models can be solved using a Toulouse point approach. It provides a direct reading of the spectrum by identifying the nature of the massive and massless degrees of freedom in the IR limit. Moreover, the Toulouse limit solution captures all universal properties of the model including the SU(2) symmetric case. At the strong coupling fixed point, the massless degrees of freedom with central charge $c = 2$ consist of an effective $S=1/2$ Heisenberg spin chain and two critical Ising models which act as pseudo-charge degrees of freedom. The special nature of the chiral fixed point identified by the Toulouse point approach reveals itself in two important facts for the 3-chain cylinder model. On the one hand, the magnetic excitations consist of a strongly correlated bound-state made by two spinons of the surface chains and an antispinon of the central chain. On the other hand, the slowest spin-spin correlation functions corresponds to the staggered magnetizations contribution of the surface chains with an unusual exponent which characterizes a new universality class in spin ladders and manifest itself in physical quantity such as the low-temperature behaviour of the NMR relaxation rate: $1/T_1 \sim \sqrt{T}$.

The Toulouse limit solution provides also a non-perturbative basis to investigate the stability of the chiral fixed point under the presence of several operators such as backscattering contributions. In particular, the interchain backscattering term of the 3-chain cylinder model destabilizes the chiral fixed point and the system will cross over to a fixed point with central charge $c = 1$ which is expected to be the one for the non-frustrated 3-leg spin ladder with open transverse boundary conditions. However, there still exists an intermediate low-energy region governed by the chiral fixed point. This fixed point becomes also unstable upon switching on a weak interaction between the surface chains. For a ferromagnetic interaction, the system exhibits an interesting critical phase with a smaller central charge $c = 3/2$ where a half of the pseudo charge degrees of freedom becomes massive by an Ising transition and the low temperature behaviour of the NMR relaxation rate is now: $1/T_1 \sim T^{1/4}$. In the case of antiferromagnetic interaction (frustrated periodic transverse boundary conditions), we expect the succession of two transitions with an intermediate $c = 3/2$ phase followed by a massive phase where the magnetic sector is spontaneously dimerized.

Regarding perspectives, it is clearly of interest to further explore the phase diagram of the $(N+1)$-chain cylinder model in the vicinity of the chiral fixed point. The experience gained by the Toulouse approach for $N = 2$ leads us to expect, though it requires a proof, that the
intermediate fixed point will become unstable and the system may cross over to a fixed point presumably characterized by a central charge \( c = 1 \). A Toulouse limit solution for the general \( N \) is thus clearly desirable to shed light on this problem. Using a parafermionic description of the \( SU(2)_N \) spin current [48], one can readily find a Toulouse limit for every \( N \) with the decoupling of a bosonic field corresponding to the bosonic field \( \Phi_1 \) in the \( N = 2 \) case. Unfortunately, the other critical and massive degrees of freedom strongly interact with each other: The resulting effective theory cannot mapped onto Majorana fermions as in \( N = 2 \) case. Otherwise stated, when \( N \) exceeds two, the Toulouse limit does not correspond to a theory of free particles. The simplest case is the \( N = 4 \) case which is currently under study. With this respect, the situation is in close parallel to the multichannel Kondo problem since when the number of channels exceeds two, the Toulouse limit is described by a non-trivial field theory [36, 64].

This analogy leads us to expect that the simple model of a Luttinger liquid in an antiferromagnetic environment considered in this work and the multichannel Kondo-Heisenberg model away from half-filling should exhibit a non-Fermi-liquid low-temperature behaviour with enhanced composite pairing correlations (odd-frequency pairing) as in the two channel Kondo problem [50] or the one-dimensional Kondo lattice [51, 52]. This question in the \( N = 2 \) case will be addressed by the Toulouse point approach in a forthcoming publication [36]. We hope that the chirally stabilized liquid state with all its physical properties will be observed in numerical simulations and in further experiments on spin ladder systems.

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Note added

When this work was completed, we became aware of a work by N. Andrei and E. Orignac [65] who have also determined the low energy theory of the one dimensional multichannel Kondo-Heisenberg lattice away from half filling and found that composite order parameters induce the dominant instabilities for a number of channels less than 4.

APPENDIX: A ESTIMATION OF \( \gamma_M \) AND \( \bar{\gamma}_M \)

In this Appendix, we shall compute the constants \( \gamma_m, \bar{\gamma}_m \) that appear in the representation (104) of the different spin current fields in terms of the current \( \vec{J} \) of the effective S=1/2 Heisenberg spin chain at the Toulouse point.

1. Estimation of \( \gamma_m \)

Let us first consider the following vacuum expectation values:

\[
\gamma_R = \langle \exp \left( \pm i \sqrt{16 \pi} \Phi_{2R} \right) \rangle \\
\gamma_L = \langle \exp \left( \pm i \sqrt{16 \pi} \Phi_{2L} \right) \rangle.
\]
In the following, we shall show that \( \gamma_R = \gamma_L = \gamma_m \), \( \gamma_m \) being real and expresses in terms of the mass \( m \) of the massive modes at the Toulouse point.

Using the definition (59) of the chiral bosonic field \( \Phi_{2R,L} \), we have:

\[
\eta_{R,L} = \frac{\kappa}{\sqrt{\pi}} : \cos \left( \sqrt{16\pi} \Phi_{2R,L} \right) :
\]

\[
\zeta_{R,L} = \pm \frac{\kappa}{\sqrt{\pi}} : \sin \left( \sqrt{16\pi} \Phi_{2R,L} \right) :
\]  

(A2)

where we have used for commodity the normal ordered form of an exponential of chiral bosonic fields:

\[
: \exp \left( i \alpha \bar{\Phi}_{2R,L} \right) : = a_{0}^{-\alpha^2/8\pi} \exp \left( i \alpha \bar{\Phi}_{2R,L} \right). 
\]  

(A3)

The next step is to express the previous exponential bosonic fields with scaling dimensions 2 in terms of the different Majorana fields. This can be done by considering the following OPE from (A2):

\[
\eta_R (\bar{z}) \eta_R (\bar{\omega}) \sim \frac{1}{2\pi (\bar{z} - \bar{\omega})} - 4 (\bar{z} - \bar{\omega}) : (\bar{\partial} \bar{\Phi}_{2R})^2 :
\]

\[
+ \frac{\bar{z} - \bar{\omega}}{2\pi} : \cos \left( \sqrt{16\pi} \Phi_{2R} \right) :
\]

\[
\zeta_R (\bar{z}) \zeta_R (\bar{\omega}) \sim \frac{1}{2\pi (\bar{z} - \bar{\omega})} - 4 (\bar{z} - \bar{\omega}) : (\bar{\partial} \bar{\Phi}_{2R})^2 :
\]

\[
- \frac{\bar{z} - \bar{\omega}}{2\pi} : \cos \left( \sqrt{16\pi} \Phi_{2R} \right) :
\]  

(A4)

where the fields in the second part of the previous equations are only function of \( \bar{\omega} \). On the other hand, since \( \eta, \zeta \) are Majorana fermions, they satisfy the following OPEs:

\[
\eta_R (\bar{z}) \zeta_R (\bar{\omega}) \sim \frac{1}{2\pi (\bar{z} - \bar{\omega})} - (\bar{z} - \bar{\omega}) : \eta_R \bar{\partial} \eta_R :
\]

\[
\zeta_R (\bar{z}) \eta_R (\bar{\omega}) \sim \frac{1}{2\pi (\bar{z} - \bar{\omega})} - (\bar{z} - \bar{\omega}) : \zeta_R \bar{\partial} \zeta_R :
\]  

(A5)

Comparing the two short distance expansions (A4, A5), we end with the equivalence:

\[
\cos \left( \sqrt{16\pi} \Phi_{2R} \right) = a_0^2 i \pi (-\eta_R \partial_x \eta_R + \zeta_R \partial_x \zeta_R),
\]  

(A6)

since for instance, in our convention, \( \bar{\partial}_{\zeta_R} = i \partial_x \zeta_R \) \( (z = u_2 \tau + ix) \). One can do the same computation for the left sector to find:

\[
\cos \left( \sqrt{16\pi} \Phi_{2L} \right) = a_0^2 i \pi (\eta_L \partial_x \eta_L - \zeta_L \partial_x \zeta_L).
\]  

(A7)

In the same way, considering the OPE \( \eta_R (\bar{z}) \zeta_R (\bar{\omega}) \), we get:

\[
\sin \left( \sqrt{16\pi} \Phi_{2R,L} \right) = \pm 2i a_0^2 \partial_x \eta_{R,L} \zeta_{R,L}.
\]  

(A8)

This proves that \( \gamma_{R,L} \) is real since at the chiral fixed point \( \zeta \) is a free and massless Majorana fermion so that \( \langle \sin \left( \sqrt{16\pi} \Phi_{2R,L} \right) \rangle = 0 \). Therefore, using Eqs. (A6, A7) and the definition (A1), we obtain the following correspondence:

\[
\gamma_{R,L} = \mp a_0^2 i \pi \left( \langle \eta_{R,L} \partial_x \eta_{R,L} \rangle - \langle \zeta_{R,L} \partial_x \zeta_{R,L} \rangle \right).
\]  

(A9)
The contribution of the massless Majorana fermion \( \zeta \) is easy to find:

\[
\langle \zeta_{R,L} \partial_x \zeta_{R,L} \rangle = \int \frac{dk}{2 \pi} \int \frac{d\omega}{2 \pi} \frac{ik e^{i \omega_0}}{i \omega + u_2 k}
\]

\[
= \pm \frac{i}{2 \pi} \int_0^\Lambda dk k = \mp \frac{i \Lambda^2}{4 \pi}
\]

(A10)

where we have used a simple cut-off regularization to cure the UV singularity. To estimate \( \langle \eta_{R,L} \partial_x \eta_{R,L} \rangle \), we need to know the Green’s function of the Majorana fermion \( \eta \) that comes from the massive part of the Hamiltonian at the Toulouse point describing the hybridization between the Majorana fermions \( \xi^3 \) and \( \eta \):

\[
\mathcal{H}_+ = \frac{i n_1}{2} \xi_L^3 \partial_x \xi_L^3 - \frac{iu_2}{2} \eta_R \partial_x \eta_R - \frac{im}{2} (\eta_R \xi_L^3 - \xi_L^3 \eta_R) \\
\mathcal{H}_- = -\frac{i n_1}{2} \xi_R^3 \partial_x \xi_R^3 + \frac{iu_2}{2} \eta_L \partial_x \eta_L + \frac{im}{2} (\xi_R^3 \eta_L - \eta_L \xi_R^3).
\]

(A11)

The Green’s functions corresponding to \( \mathcal{H}_+ \) are given by:

\[
G_+ (k, \omega) = \begin{pmatrix}
\langle \eta_R (-k, -\omega) \eta_R (k, \omega) \rangle & \langle \eta_R (-k, -\omega) \xi_L^3 (k, \omega) \rangle \\
\langle \xi_L^3 (-k, -\omega) \eta_R (k, \omega) \rangle & \langle \xi_L^3 (-k, -\omega) \xi_L^3 (k, \omega) \rangle
\end{pmatrix}
\]

(A12)

with

\[
G_+^{-1} (k, \omega) = \begin{pmatrix}
i \omega - u_2 k & im \\
-im & i \omega + v_1 k
\end{pmatrix}
\]

(A13)

whereas the ones associated with \( \mathcal{H}_- \) read as follows:

\[
G_- (k, \omega) = \begin{pmatrix}
\langle \xi_R^3 (-k, -\omega) \xi_R^3 (k, \omega) \rangle & \langle \xi_R^3 (-k, -\omega) \eta_L (k, \omega) \rangle \\
\langle \eta_L (-k, -\omega) \xi_R^3 (k, \omega) \rangle & \langle \eta_L (-k, -\omega) \eta_L (k, \omega) \rangle
\end{pmatrix}
\]

(A14)

with

\[
G_-^{-1} (k, \omega) = \begin{pmatrix}
i \omega - v_1 k & -im \\
+im & i \omega + u_2 k
\end{pmatrix}
\]

(A15)

The Majorana Green’s function for the \( \eta \) field is thus given by:

\[
\langle \eta_{R,L} (-k, -\omega) \eta_{R,L} (k, \omega) \rangle = -\frac{i \omega \pm v_1 k}{(\omega \mp ik u_1)^2 + E^2 (k)}
\]

(A16)

where \( E^2 (k) = u_*^2 k^2 + m^2 \) and \( u_* = (v_0 + v_1)/3 \). Integrating over the frequency, we obtain the following estimate:

\[
\langle \eta_{R,L} \partial_x \eta_{R,L} \rangle = \pm \frac{i u_*}{2 \pi} \int_0^\Lambda dk \frac{k^2}{E (k)}
\]

\[
\simeq \pm \frac{i \Lambda^2}{4 \pi} \pm \frac{im^2}{4 \pi u_*^2} \ln \left( \frac{\Lambda u_*}{m} \right).
\]

(A17)

Substituting (A17) and (A10) into (A9), we find the value of \( \gamma_m \) at the Toulouse point:

\[
\gamma_m = \gamma_{R,L} \simeq \frac{m^2 a_1^2}{4 u_*^2} \ln \left( \frac{\Lambda u_*}{m} \right).
\]

(A18)
2. Estimation of $\bar{\gamma}_m$

The next vacuum expectation values to compute is

$$\bar{\gamma}_{R,L} = i\pi a_0 \langle \xi_{R,L}^3 \eta_{L,R} \rangle.$$  \hspace{1cm} (A19)

Using the previous expression of the fermionic Green’s function $G_{\pm}$, we have:

$$\langle \xi_{R,L}^3 (-k, -\omega) \eta_{L,R} (k, \omega) \rangle = -\frac{i m}{(\omega \pm ik u_1)^2 + E^2 (k)}.$$ \hspace{1cm} (A20)

Integrating over $\omega$, the value of $\bar{\gamma}_{R,L}$ can easily be determined:

$$\bar{\gamma}_{R,L} = \frac{ma_0}{2} \int_0^\Lambda \frac{dk}{E(k)},$$ \hspace{1cm} (A21)

and thus

$$\bar{\gamma}_m = \bar{\gamma}_{R,L} \simeq \frac{ma_0}{2u_*} \ln \left( \frac{\Lambda a_0}{m} \right).$$ \hspace{1cm} (A22)

Let us conclude this appendix by discussing the SU(2) invariance of the spin-spin correlation functions between the central spins and also between the surface chains at the chiral fixed point. Using the Toulouse point value \[g_\parallel = 4\pi u_*\] and expressing the values of $\gamma_m, \bar{\gamma}_m$ in terms of the ratio $\bar{\kappa} = g_\perp / g_\parallel$, we have:

$$\gamma_m \simeq \bar{\kappa}^2 \ln \left( \frac{\Lambda a_0}{2\bar{\kappa}} \right),$$

$$\bar{\gamma}_m \simeq \bar{\kappa} \ln \left( \frac{\Lambda a_0}{2\bar{\kappa}} \right).$$ \hspace{1cm} (A23)

In the continuum limit, the value of the physical mass is fixed. This gives the dependence of $g_\perp$ in terms of the short distance cut-off $a_0$. The ratio of the couplings $g_\perp / g_\parallel$ is also fixed to some physical value $\bar{\kappa}$ which is a RG invariant flow of the equation (44). The two physical parameters emerging from the renormalized theory are precisely $m$ and $\bar{\kappa}$ apart from the different velocities of the problem. At the SU(2) point when $\bar{\kappa} = 1$, we must have $\gamma^2_m = \bar{\gamma}^2_m = 1$ so that the correlation functions of the central chain \[118, 119\] are rotational invariant. Using Eq. (A23), this fixes the value of the product $\Lambda a_0$ to 2.

APPENDIX: B BASIC FACTS ABOUT THE BOSONIZATION OF TWO ISING MODELS

In this Appendix, we briefly review some basic materials on the relation between two Ising models and a free bosonic field in order to fix notations for the computation of the spin-spin correlation function between spins of the surface chains (Section V).

It is well known that a theory of free massless Majorana fermion with central charge $c = 1/2$ describes the long distance properties of the critical two-dimensional Ising model. Two Ising models can be mapped onto the Gaussian model with central charge $c = 1$ (see for a review chapter 12 of Ref. \[39\]). This procedure of doubling the Ising models has been very fruitful for the computation of the correlation function of the Ising model \[51, 53, 54\], for the problem of
random impurities in the Ising model [50], and also for the calculation of the structure factor of the two-leg spin ladder [54]. This mapping stems from the fact that the two Majorana fields say \( \psi_{1,2} \) can be combined into a single Dirac fermion \( \chi \) which can be in turn be expressed in terms of a bosonic field \( \Phi \) by Abelian bosonization:

\[
\chi_R = \frac{\psi_{1R} + i\psi_{2R}}{\sqrt{2}} = e^{i\sqrt{4\pi\Phi_R}} \frac{1}{\sqrt{2\pi a_0}} e^{i\sqrt{4\pi\Phi_R}} \\
\chi_L = \frac{\psi_{1L} + i\psi_{2L}}{\sqrt{2}} = e^{-i\sqrt{4\pi\Phi_L}} \frac{1}{\sqrt{2\pi a_0}} e^{-i\sqrt{4\pi\Phi_L}}
\]

(B1)

with \([\Phi_R, \Phi_L] = i/4\) to insure the anticommutation relation between \( \chi_R \) and \( \chi_L \). The Majorana fermions \( \psi_{1,2} \) can therefore be expressed directly in terms the \( \Phi \) field as follows:

\[
\psi_{1R} = \frac{1}{\sqrt{\pi a_0}} \cos \left( \sqrt{4\pi\Phi_R} \right) \\
\psi_{1L} = \frac{1}{\sqrt{\pi a_0}} \cos \left( \sqrt{4\pi\Phi_L} \right) \\
\psi_{2R} = \frac{1}{\sqrt{\pi a_0}} \sin \left( \sqrt{4\pi\Phi_R} \right) \\
\psi_{2L} = -\frac{1}{\sqrt{\pi a_0}} \sin \left( \sqrt{4\pi\Phi_L} \right).
\]

(B2)

The energy density operator \( \epsilon_a = i\psi_{aR}\psi_{aL} \), \( a = 1, 2 \), which is a mass term for the Majorana fermion and drives the model out of criticality, can also be written down as a function of the bosonic field \( \Phi \) and its dual \( \Theta \):

\[
\cos \left( \sqrt{4\pi\Phi} \right) = i\pi a_0 (\psi_{1R}\psi_{1L} + \psi_{2R}\psi_{2L}) \\
\cos \left( \sqrt{4\pi\Theta} \right) = i\pi a_0 (-\psi_{1R}\psi_{1L} + \psi_{2R}\psi_{2L}),
\]

(B3)

and one can also relate all bilinears involving \( \psi_{1,R,L} \) and \( \psi_{2,R,L} \) to the bosonic field \( \Phi \) and its dual \( \Theta \):

\[
\sin \left( \sqrt{4\pi\Phi} \right) = i\pi a_0 (\psi_{2R}\psi_{1L} - \psi_{1R}\psi_{2L}) \\
\sin \left( \sqrt{4\pi\Theta} \right) = i\pi a_0 (\psi_{2R}\psi_{1L} + \psi_{1R}\psi_{2L}),
\]

(B4)

and

\[
\partial_x \Phi = i\sqrt{\pi} (\psi_{1R}\psi_{2R} + \psi_{1L}\psi_{2L}) \\
\partial_x \Theta = i\sqrt{\pi} (-\psi_{1R}\psi_{2R} + \psi_{1L}\psi_{2L}).
\]

(B5)

The Ising model labelled 1, for instance, contains apart from the Majorana fields \( \psi_{1,R,L} \) that have conformal dimensions \((1/2, 0)\) and \((0, 1/2)\) and the energy density operator, two other fields, with conformal dimensions \((1/8, 1/8)\): the order and disorder operators \( \sigma_1 \) and \( \mu_1 \). These fields are non local when expressed in terms of the Majorana fermion \( \psi_{1,R,L} \) [57,58,61]. The Kramers-Wannier duality transformation maps the order operator \( \sigma_1 \) onto the disorder field \( \mu_1 \). In the ordered phase \((T < T_c)\), \( \langle \sigma_1 \rangle \neq 0, \langle \mu_1 \rangle = 0 \) whereas in the disorder phase \((T > T_c)\), we have \( \langle \sigma_1 \rangle = 0, \langle \mu_1 \rangle \neq 0 \). At \( T = T_c \), both fields have a zero vacuum expectation value. At \( T = T_c \), the products \( \sigma_1\sigma_2, \sigma_1\mu_2, \mu_1\mu_2 \) and \( \mu_1\sigma_2 \) have scaling dimension \( 1/4 \) and can be related to the bosonic exponentials \( \exp(\pm i\sqrt{\pi\Phi}), \exp(\pm i\sqrt{\pi\Theta}) \) [61,54].
\[ \begin{align*}
\mu_1\mu_2 & \sim \cos(\sqrt{\pi}\Phi) \\
\sigma_1\sigma_2 & \sim \sin(\sqrt{\pi}\Phi) \\
\sigma_1\mu_2 & \sim \cos(\sqrt{\pi}\Theta) \\
\mu_1\sigma_2 & \sim \sin(\sqrt{\pi}\Theta),
\end{align*} \]  
(B6)

This correspondence should also hold at small deviations from criticality. These operators are very useful for practical computation since the correlation functions of the order and disorder fields are known exactly even out of criticality. For \( T = T_c \), we have the following power law in the long distance-long time limit:

\[ \langle \mu(x, \tau) \mu(0,0) \rangle = \langle \sigma(x, \tau) \sigma(0,0) \rangle \sim \frac{1}{(x^2 + v^2\tau^2)^{1/8}}, \]  
(B7)

\( v \) being the spin velocity of the Majorana fermion. The description of spin chains in terms of order-disorder parameters of the underlying Ising models has been used by Shelton et al. [54, 55] for the two-leg spin ladder and by Allen and Sénéchal [32] for the two-leg zigzag ladder in the absence of the twist term [33]. This approach allows a simple description of the excitations of the system and is very useful for the computation of the correlation functions and the dynamic structure factors in a massive phase.

In our problem, we shall use this approach for the computation of the spin-spin correlation functions of spins of the surface chains and for the study of the stability of the chiral fixed point perturbed by some operators. One has to express, at the Toulouse point, all the order-disorder operators of the underlying Ising models. First of all, the bosonic field \( \varphi_- = (\varphi_1 - \varphi_2)/\sqrt{2} \) is not affected by the canonical transformation (50, 51) and is built from the two Majorana fermions \( \xi^0, \xi^3 \). The two corresponding Ising models are labelled “0” and “3” respectively and using Eq. (B6) with the identification (84, 85), we have:

\[ \begin{align*}
\mu_3\mu_0 & \sim \cos(\sqrt{\pi}\varphi_-) \\
\sigma_3\sigma_0 & \sim \sin(\sqrt{\pi}\varphi_-) \\
\sigma_3\mu_0 & \sim \cos(\sqrt{\pi}\vartheta_-) \\
\mu_3\sigma_0 & \sim \sin(\sqrt{\pi}\vartheta_-).
\end{align*} \]  
(B8)

A second couple of Ising models (noted “4” and “5”) stems from the two Majorana fermions \( \zeta, \eta \) respectively and are associated with the \( \Phi_2 \) bosonic field (see Eq. (59)). In that case, the correspondence (B6) gives

\[ \begin{align*}
\mu_5\mu_4 & \sim \cos(\sqrt{\pi}\Phi_2) \\
\sigma_5\sigma_4 & \sim \sin(\sqrt{\pi}\Phi_2) \\
\sigma_5\mu_4 & \sim \cos(\sqrt{\pi}\Theta_2) \\
\mu_5\sigma_4 & \sim \sin(\sqrt{\pi}\Theta_2).\n\end{align*} \]  
(B9)

Since the \( \xi^0 \) field is decoupled from the interaction from the beginning and the Majorana fermion \( \zeta \) is massless at the Toulouse point, the Ising models noted “0” and “4” decouple from the other and remains critical. The correlation function of their order-disorder operators are
therefore given by Eq. (B7). However, the two Ising models “3” and “5” strongly interact with each other due to the hybridization of the Majorana $\xi^3$ and $\eta$ fields at the Toulouse point (see Eq. (F4)). As shown by the spectra (F3, F4), these two Ising models are non-critical and the order-disorder operators have short-ranged correlation functions characterized by the mass $m$ of the massive modes at the Toulouse point.

APPENDIX: C HYBRIDIZED ISING MODELS

The aim of this appendix is to describe more precisely the consequence of the hybridization between the Majorana fermions $\xi^3$ and $\eta$ on their corresponding order-disorder ($\sigma^3, \mu^3, \mu^5$) Ising operators. In particular, we shall prove, in the following, that at the Toulouse point, one has

\begin{equation}
\langle \mu^3 \mu^5 \rangle = \langle \sigma^3 \sigma^5 \rangle = \mu \neq 0
\end{equation}

\begin{equation}
\langle \mu^3 \sigma^5 \rangle = \langle \sigma^3 \mu^5 \rangle = 0.
\end{equation}  \tag{C1}

Let us begin by recalling the effective Hamiltonian associated with the massive modes at the Toulouse point:

\begin{equation}
H_{hyb} = -\frac{i v_1}{2} (\xi^3_R \partial_x \xi^3_R - \xi^3_L \partial_x \xi^3_L) - \frac{i u_2}{2} (\eta_R \partial_x \eta_R - \eta_L \partial_x \eta_L)
+ \text{im} (\xi^3_R \eta_L - \eta_R \xi^3_L).
\end{equation} \tag{C2}

For the computation of ground-state expectation values, we notice that the velocity anisotropy present in the Hamiltonian (C2) can be ignored for several reasons. On the one hand, we have two phenomenological parameters in our approach: $v_1$ and $v_0$. One can always choose $v_0 = v_1/2$ so that all spin velocities $u_1, u_2$ (58) of the modes remain positive and $v_1 = u_2$. We expect that the results for the vacuum expectation values (C1) obtained with this particular fine-tuning represent the generic situation. On the other hand, one can justify further this proposition by studying the structure of the equal-time fermionic Green’s functions of the Majorana fermions. The nice thing is that the equal-time Green’s functions corresponding to the Hamiltonian (C2) are in fact equal to that of a model with a single spin velocity (the average spin velocity):

\begin{equation}
\tilde{H}_{hyb} = -\frac{iu_*}{2} (\xi^3_R \partial_x \xi^3_R - \xi^3_L \partial_x \xi^3_L) - \frac{iu_*}{2} (\eta_R \partial_x \eta_R - \eta_L \partial_x \eta_L)
+ \text{im} (\xi^3_R \eta_L - \eta_R \xi^3_L).
\end{equation} \tag{C3}

where $u_*$ is the average of the velocities $v_1$ and $u_2$: $u_* = (v_1 + u_2)/2 = (v_1 + v_0)/3$. The very reason for this equivalence on the equal-time Green’s functions stems from the fact that both Hamiltonian (C2, C3) have the same eigenvectors although different spectra. Therefore, there is a one-to-one correspondence between the static ground-state correlation functions of the model (C2) and that of the Hamiltonian (C3). The velocity anisotropy will be important when considering dynamical properties. Consequently, our problem is reduced to the computation of the ground-state expectation values (C1) with the Hamiltonian (C3). To this end, let us introduce a bosonic field $\phi$ and its dual $\Phi$:

\begin{equation}
\frac{\xi^3_R + i \eta_R}{\sqrt{2}} = \frac{1}{\sqrt{2\pi a_0}} e^{i \sqrt{2\pi} \phi_R},
\frac{\xi^3_L + i \eta_L}{\sqrt{2}} = \frac{1}{\sqrt{2\pi a_0}} e^{-i \sqrt{2\pi} \phi_L}.
\end{equation} \tag{C4}
Using the results (B6) of the Appendix B, the different products of order and disorder Ising operators can be expressed in terms of the bosonic fields:

\[\mu_3\mu_5 \sim \cos \left(\sqrt{\pi}\tilde{\Phi}\right)\]
\[\sigma_3\sigma_5 \sim \sin \left(\sqrt{\pi}\tilde{\Phi}\right)\]
\[\sigma_3\mu_5 \sim \cos \left(\sqrt{\pi}\tilde{\Theta}\right)\]
\[\mu_3\sigma_5 \sim \sin \left(\sqrt{\pi}\tilde{\Theta}\right)\]. \quad (C5)

The Hamiltonian (C3) can also be rewritten in the following bosonized form:

\[\tilde{H}_{hyb} = \frac{u_s}{2} \left( \left(\partial_x \tilde{\Phi}\right)^2 + \left(\partial_x \tilde{\Theta}\right)^2 \right) - \frac{m}{\pi a_0} \sin \left(\sqrt{4\pi}\tilde{\Phi}\right). \quad (C6)\]

Using the substitution:

\[\tilde{\Phi} \rightarrow \tilde{\Phi} + \frac{\sqrt{\pi}}{4}\]
\[\tilde{\Theta} \rightarrow \tilde{\Theta}, \quad (C7)\]

\(\tilde{H}_{hyb}\) identifies with the conventional form of a sine-Gordon model at \(\beta^2 = 4\pi\):

\[\tilde{H}_{hyb} = \frac{u_s}{2} \left( \left(\partial_x \tilde{\Phi}\right)^2 + \left(\partial_x \tilde{\Theta}\right)^2 \right) - \frac{m}{\pi a_0} \cos \left(\sqrt{4\pi}\tilde{\Phi}\right). \quad (C8)\]

In this model, the bosonic field \(\tilde{\Phi}\) is locked such as \(\langle \tilde{\Phi} \rangle = 0 \) \((m > 0)\). Consequently, we have \(\langle \cos \left(\sqrt{\pi}\tilde{\Phi}\right) \rangle \neq 0\) and \(\langle \sin \left(\sqrt{\pi}\tilde{\Phi}\right) \rangle = 0\) whereas the dual field exponents \(\exp \left(\pm i\sqrt{\pi}\tilde{\Theta}\right)\) have zero vacuum expectation values. Using Eq. (C5) and the substitution (C7), we thus obtain the above mentioned result (C1):

\[\langle \mu_3\mu_5 \rangle = \langle \sigma_3\sigma_5 \rangle = \mu \neq 0\]
\[\langle \mu_3\sigma_5 \rangle = \langle \sigma_3\mu_5 \rangle = 0. \quad (C9)\]

**APPENDIX: D STABILITY OF THE TOLUOUSE POINT SOLUTION**

The consistency of the Toulouse approach to describe the universal properties of the chiral fixed point is investigated in this Appendix.

We first begin by analysing the effect of the operators that we have neglected from the beginning in the Toulouse point approach. These operators are the current-current interaction of same chirality and the in-chain marginal irrelevant contribution that appears in the continuum limit of each S=1/2 Heisenberg spin chain (see Eq. (7)):

\[O_{cc} = J_{0R} \cdot (J_{1R} + J_{2R}) + R \rightarrow L\]
\[O_{sc} = J_{0R} \cdot J_{0L} + J_{1R} \cdot J_{1L} + J_{2R} \cdot J_{2L}. \quad (D1)\]

Let us first consider the current-current interaction of same chirality \(O_{cc}\). Using the bosonization of a SU(2)\(_1\) spin current \(\text{[13]}\), we can express the operator \(O_{cc}\) in terms of the bosonic fields associated with each chain of the model:
\[ O_{cc} = \frac{1}{\sqrt{2\pi}} (\partial_x \varphi_R \partial_x \Phi_R + \partial_x \varphi_L \partial_x \Phi_L) + \frac{1}{2\pi^2 a_0^2} \left( \cos\left(\sqrt{8\pi} \varphi_R - \sqrt{4\pi} \Phi_R\right) \cos\left(\sqrt{4\pi} \varphi_- R\right) + R \to L \right). \] (D2)

Using the canonical transformation \[^5\, [1]\), the current-current operator can then be written as a function of the different fields of the Toulouse point solution:

\[ O_{cc} = \frac{1}{\sqrt{2\pi}} \left( 3\partial_x \Phi_2 R \partial_x \Phi_1 R + 3\partial_x \Phi_2 L \partial_x \Phi_1 L - 2 \sqrt{2} \partial_x \Phi_1 L \partial_x \Phi_1 R - 2 \sqrt{2} \partial_x \Phi_2 L \partial_x \Phi_2 R \right) + \frac{1}{2\pi^2 a_0^2} \left( \cos\left(\sqrt{4\pi} \varphi_- R\right) \cos\left(\sqrt{8\pi} \Phi_1 \right) \cos\left(\sqrt{16\pi} \Phi_2 R + \sqrt{4\pi} \Phi_2 L\right) + \cos\left(\sqrt{4\pi} \varphi_- R\right) \sin\left(\sqrt{8\pi} \Phi_1 \right) \sin\left(\sqrt{16\pi} \Phi_2 R + \sqrt{4\pi} \Phi_2 L\right) + R \to L \right). \] (D3)

The next step of the calculation is to extract the leading contribution of this operator in terms of the different critical fields at the chiral fixed point. Using Eqs. \[^5\, [1]\), the non-zero expectation values \[^7\, [1]\), and the hybridization \[^4\) of the massive fields, the leading part of the current-current operator of same chirality \[^3\) at the chiral fixed point is given by:

\[ O_{cc} \sim -\frac{1}{\pi} \partial_x \Phi_1 L \partial_x \Phi_1 R + \frac{\gamma_m \bar{\gamma}_m}{\pi^2 a_0^2} \cos\left(\sqrt{8\pi} \Phi_1 \right). \] (D4)

up to contributions that will give renormalization of mass and spin velocities. One can express the operator \[^3\) in terms of the physical spin current \( \tilde{J} \) associated with the bosonic field \( \tilde{\Phi}_1 \) (see Eq. \[^5\)):

\[ O_{cc} \sim -\frac{\gamma_m \bar{\gamma}_m}{2} \left( J_R^+ J_L^- + H.c. \right). \] (D5)

At the SU(2) point, one has \( \gamma_m \bar{\gamma}_m = 1 \) (see Appendix A) and the expression \[^3\) can be recasted in a full rotational form:

\[ O_{cc} \sim -4 \tilde{J}_R \cdot \tilde{J}_L. \] (D6)

The operator \[^3\) coincides thus with the usual marginal irrelevant term of the continuum limit of the effective S=1/2 Heisenberg spin chain corresponding to the bosonic field \( \tilde{\Phi}_1 \). The contribution \[^3\) gives logarithmic corrections in the spin-spin correlation functions at the chiral fixed point.

The second operator that we have neglected in our Toulouse solution is the marginally irrelevant in-chain current-current interaction:

\[ O_{ic} = J_{0 R} \cdot J_{0 L} + J_{1 R} \cdot J_{1 L} + J_{2 R} \cdot J_{2 L}. \] (D7)

We proceed in the same way as for the previous operator by first rewriting \( O_{ic} \) in terms of the different bosonic fields of each chain:

\[ O_{ic} = \frac{1}{8\pi} \left( (\partial_x \varphi)^2 - (\partial_x \vartheta)^2 \right) + \frac{1}{8\pi} \left( (\partial_x \Phi)^2 - (\partial_x \Theta)^2 \right) + \frac{1}{8\pi} \left( (\partial_x \varphi_-)^2 - (\partial_x \vartheta_-)^2 \right) - \frac{1}{4\pi^2 a_0^2} \cos\left(\sqrt{8\pi} \varphi \right) - \frac{1}{2\pi^2 a_0^2} \cos\left(\sqrt{4\pi} \Phi \right) \cos\left(\sqrt{4\pi} \varphi_- \right). \] (D8)

At the Toulouse point, this operator takes the following form:
that case, the Hamiltonian (61) at the Toulouse point picks up an extra term:

\[ O_{ic} = \frac{1}{8\pi} \left( (\partial_x \varphi_-)^2 - (\partial_x \varphi_+)^2 \right) + \frac{3}{8\pi} \left( (\partial_x \Phi_1)^2 - (\partial_x \Theta_1)^2 \right) + \frac{3}{8\pi} \left( (\partial_x \Phi_2)^2 - (\partial_x \Theta_2)^2 \right) \]

\[ - \frac{1}{\sqrt{2\pi}} \partial_x \Phi_1 \partial_x \Phi_2 - \frac{1}{\sqrt{2\pi}} \partial_x \Theta_1 \partial_x \Theta_2 - \frac{1}{4\pi^2 a_0^2} \cos \left( \sqrt{8\pi} \Phi_1 \right) \cos \left( \sqrt{16\pi} \Phi_2 \right) \]

\[ - \frac{1}{4\pi^2 a_0^2} \sin \left( \sqrt{8\pi} \Phi_1 \right) \sin \left( \sqrt{16\pi} \Phi_2 \right) - \frac{1}{2\pi^2 a_0^2} \cos \left( \sqrt{8\pi} \Phi_1 \right) \cos \left( \sqrt{4\pi} \Phi_2 \right) \cos \left( \sqrt{4\pi} \varphi_- \right) \]

\[ - \frac{1}{2\pi^2 a_0^2} \sin \left( \sqrt{8\pi} \Phi_1 \right) \sin \left( \sqrt{4\pi} \Phi_2 \right) \cos \left( \sqrt{4\pi} \varphi_- \right). \] (D9)

Using Eqs. (59, 84), and the non-zero expectation values (78, 91), the leading contribution of the operator \( O_{ic} \) is at the chiral fixed point up to short-ranged parts and terms giving a renormalization of the spin velocities:

\[ O_{ic} \sim \frac{3}{2\pi} \partial_x \Phi_1 \partial_x \Phi_{1R} - \frac{(\gamma_m^2 + 2\pi m)}{4\pi^2 a_0^2} \cos \left( \sqrt{8\pi} \Phi_1 \right) \] (D10)

and at the SU(2) symmetric point when \( \gamma_m^2 = \tilde{\gamma}_m^2 = 1 \) (see Appendix A), it can be reduced to the simple following form:

\[ O_{ic} \sim 3 \vec{J}_R \cdot \vec{J}_L. \] (D11)

Since the original coupling constant of the in-chain current-current interaction \( O_{ic} \) is negative, the effective operator (D11) gives additional logarithm corrections of the spin-spin correlation at the chiral fixed point.

The next check of the correctness of the Toulouse solution is to investigate the effect of small deviation of the coupling constant \( g_\parallel \) from its Toulouse-point value: \( g^*_\parallel = 4\pi(v_0 + v_1)/3 \). In that case, the Hamiltonian (61) at the Toulouse point picks up an extra term:

\[ \mathcal{H}_{cor} = \frac{\delta g_\parallel}{\sqrt{2\pi}} \left( \partial_x \varphi_L \partial_x \Phi_R + \partial_x \varphi_R \partial_x \Phi_L \right) \] (D12)

with \( \delta g_\parallel = g_\parallel - g^*_\parallel \). Using the canonical transformation (50, 51), this operator transforms into:

\[ \mathcal{H}_{cor} = -\frac{\delta g_\parallel}{\pi} \left( (\partial_x \Phi_{1L})^2 + (\partial_x \Phi_{1R})^2 + (\partial_x \Phi_{2L})^2 + (\partial_x \Phi_{2R})^2 \right) \]

\[ + \frac{3\delta g_\parallel}{\sqrt{2\pi}} \left( \partial_x \Phi_{1L} \partial_x \Phi_{2R} + \partial_x \Phi_{1R} \partial_x \Phi_{2L} \right). \] (D13)

Neglecting the renormalization of the velocities \( u_{1,2} \), we end with using Eq. (74):

\[ \mathcal{H}_{cor} \sim \frac{3\delta g_\parallel \sqrt{\pi}}{\sqrt{2}} i \left( \partial_x \Phi_{1L} \zeta_R \eta_R + \partial_x \Phi_{1R} \zeta_L \eta_L \right) . \] (D14)

Since the field \( \eta \) is massive, the expansion in \( \delta g_\parallel \) does not introduce new IR singularities, implying that the long-distance behaviour of the correlation functions will not be modified except for velocities and mass renormalization. Therefore, we can conclude that the solution at the Toulouse point captures all universal properties of the chiral fixed point.
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