Exhaustive Ghost Solutions to Einstein-Weyl Equations for Two Dimensional Spacetimes

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Abstract

Exhaustive ghost solutions to Einstein-Weyl equations for two dimensional spacetimes are obtained, where the ghost neutrinos propagate in the background spacetime, but do not influence the background spacetime due to the vanishing stress-energy-momentum tensor for the ghost neutrinos. Especially, those non-trivial ghost solutions provide a counterexample to the traditional claim that the Einstein-Hilbert action has no meaningful two dimensional analogue.

1 Introduction

It is known that ghost solutions to Einstein-Weyl equations have been obtained for various cases\cite{1, 2, 3, 4}, where the stress-energy-momentum tensor for the ghost neutrino field vanishes although the field gives rise to a non-vanishing current. Obviously, the most interesting thing about the ghost solutions is the way the neutrino field propagates in the background spacetime without changing it, which implies we could not detect the ghost neutrinos by their gravitational effects. In addition, this kind of ghost solutions may be useful to quantum field theory in curved spacetimes, where the neutrino field could be quantized while leaving the background spacetime fixed\cite{5}.

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The present work is devoted to exhaustive ghost solutions to Einstein-Weyl equations for two-dimensional spacetimes, which has never been investigated so far, as we know. It is worth noting that our result presents a counterexample for the traditional belief that Einstein-Hilbert action makes no sense in two-dimensional case\cite{6}.

This paper is organized as follows. Section 2 investigates the conformal invariance of Einstein-Weyl equations for two-dimensional spacetimes. In Section 3, exhaustive ghost neutrino solutions in two-dimensional flat spacetimes are obtained. Section 4 contains some concluding remarks and discussions. The derivation of Einstein-Weyl equations by variational principle is relegated to Appendix.

2 Conformal Invariance of Einstein-Weyl Equations for Two Dimensional Spacetimes

Start with two metrics conformally related by

\[ \tilde{g}_{ab} = \Omega^2 g_{ab}, \]  

(1)

then for convenience but without loss of generalization, we let

\[ \tilde{\sigma}_{CC'}^b = \Omega^{-1} \sigma_{CC'}^b. \]  

(2)

Later, through a routine computation and by virtue of (32), the corresponding \( \Theta_{a}^{B}_{C} \) in Appendix reads

\[ \Theta_{a}^{B}_{C} = \frac{1}{2} \varepsilon_{C}^{B, C} \nabla_a \Omega - \sigma_{aCC'} \sigma^{dDC'} \nabla_d \Omega. \]  

(3)

Thus if we let

\[ \tilde{\psi}^B = \Omega^m \psi^B, \]  

(4)

then

\[ \nabla_a \tilde{\psi}^B = \Omega^m \nabla_a \psi^B + \Omega^{m-1}[(m + \frac{1}{2})\psi^B \nabla_a \Omega - \psi^C \sigma_{aCC'} \sigma^{dDC'} \nabla_d \Omega]. \]  

(5)

According to the above identity and employing (33), it is easy to know, for two dimensional case, the Weyl neutrino field equation is conformally invariant with conformal weight \( m = -\frac{1}{2} \); furthermore, the stress-energy-momentum tensors for Weyl neutrino field are conformally invariant in the sense that

\[ \tilde{T}_{ab} = T_{ab}. \]  

(6)

In addition, the Einstein tensor is also conformally invariant for two-dimensional spacetimes. Thus for two-dimensional spacetimes the Einstein equation is conformally invariant. Furthermore, according to the well-known fact, that is, any two-dimensional spacetime metric is locally conformal.
to a flat one, we know the Einstein tensor vanishes identically. Hereby there is no existence of solutions to Einstein-Weyl equations for two dimensional spacetimes except the ghost solutions under consideration, which are conformally invariant.

Therefore, our task is reduced to searching of ghost neutrino solutions in two dimensional flat spacetimes.

3 Exhaustive Ghost Neutrino Solutions in Two Dimensional Flat Spacetimes

Given a two dimensional flat spacetime
\[ ds^2 = dt^2 - dx^2. \] (7)

Perform a coordinate transformation
\[ v = t + r, \]
\[ u = t - r; \] (8)

then
\[ ds^2 = dudv. \] (9)

Construct the null tetrad-frame in N-P formalism as
\[ k = \sqrt{2} \frac{\partial}{\partial v}, \]
\[ l = \sqrt{2} \frac{\partial}{\partial u}; \] (10)

then the corresponding spin coefficients all vanish. Therefore the Weyl neutrino field equation reads
\[ D\psi^1 = 0, \] (11)
\[ \Delta \psi^2 = 0; \] (12)

which implies
\[ \psi^1 = \psi^1(u), \] (13)
\[ \psi^2 = \psi^2(v). \] (14)

Thus the vanishing stress-energy-momentum tensor for ghost neutrinos can be expressed as
\[ T_{00} = -\frac{\sqrt{2}\alpha_N}{32\pi} (\psi^2 \frac{\partial^2 \psi^2}{\partial v^2} - \psi^2 \frac{\partial^2 \psi^2}{\partial v}) = 0, \] (15)
\[ T_{02} = 0, \quad (16) \]

\[ T_{22} = -\frac{\sqrt{2} \alpha_N}{32 \pi} (\bar{\psi} \frac{\partial \psi}{\partial u} - \psi \frac{\partial \bar{\psi}}{\partial u}) = 0. \quad (17) \]

Let

\[ \psi^2 = f(v) + i g(v), \quad (18) \]

with \( f \) and \( g \) both real. Thus (16) can be reduced to

\[ fg' = gf', \quad (19) \]

where the prime denotes differentiating with respect to \( v \). After a straightforward calculation, there exist three classes of solutions to the above equation: firstly, \( f = 0 \) and \( g \) is an arbitrary function of \( v \); secondly, \( f \) is an arbitrary function of \( v \) with \( g = 0 \); finally \( g = cf \) where \( c \) is an arbitrary non-zero real constant. Correspondingly, we have three classes of solutions for \( \psi^2 \) as follow

\[ \psi^2 = h(v), \quad (20) \]

\[ \psi^2 = i h(v), \quad (21) \]

and

\[ \psi^2 = (1 + i a) h(v), \quad (22) \]

where \( h \) is an arbitrary real function of \( v \), with \( a \) an arbitrary non-zero real constant. Similarly there are three classes of solutions for \( \psi^1 \) as follow

\[ \psi^1 = z(u), \quad (23) \]

\[ \psi^1 = i z(u), \quad (24) \]

and

\[ \psi^1 = (1 + i b) z(u), \quad (25) \]

where \( z \) is an arbitrary real function of \( u \), with \( b \) an arbitrary non-zero real constant.

It is obvious that there exist many non-trivial ghost neutrino solutions in two dimensional flat spacetimes.
4 Discussion

We have obtained exhaustive ghost neutrino solutions to Einstein-Weyl equations for two dimensional flat spacetimes. Due to conformal invariance of the ghost solutions for two dimensional case, the exhaustive ghost solutions to Einstein-Weyl equations for two dimensional curved spacetimes can yield under conformal transformation. Especially those non-trivial ghost neutrino solutions present a counterexample to nonsense of two dimensional analogue for the Einstein-Hilbert action.

We conclude with two interesting problems worthy of further investigation. Firstly, although it is obvious that there is no non-trivial solution to gravity-scalar and gravity-electromagnetic system, there seem to exist non-trivial solutions to gravity-spin 3/2 field coupling system. In addition, up to now, all the ghost solutions obtained are within the classical framework; it is interesting to search of ghost states in semi-classical framework, where matter fields are quantized with gravity treated classically.

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Appendix: The Derivation of Einstein-Weyl Equations by Variational Principle

This appendix serves to present the derivation of Einstein-Weyl Equations by variational approach, especially the derivation of stress-energy-momentum tensor for Weyl neutrino field.

Start with the action for Einstein-Weyl system[7]

\[ S = \int_M \mathcal{L}_{\text{total}} e = \int_M (\mathcal{L}_G + \alpha_N \mathcal{L}_N) e, \]  

(26)

where \( \alpha_N \) is an imaginary constant, \( \mathcal{L}_G \) is Lagrangian density for gravitation field, defined as

\[ \mathcal{L}_G = R \sqrt{-g}, \]  

(27)

and

\[ \mathcal{L}_N = \overline{\psi}^A' \nabla_{A'} \psi^A \sqrt{-g} \]  

(28)

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is Lagrangian density for Weyl neutrino field.

The Weyl neutrino equation

$$\nabla^A \psi^A = 0$$

(29)
can be easily obtained by variation of $S$ with respect to $\psi^A$ or $\bar{\psi}^{A'}$. Similarly, variation of $S$ with respect to $g^{ab}$ yields the Einstein equation[7]

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab} = 8\pi \left(- \frac{\alpha_N}{8\pi} \sqrt{-g} \frac{\delta S_N}{\delta g^{ab}} \right),$$

(30)

where $S_N$ is the action for Weyl neutrino field, given by

$$S_N = \int_M \mathcal{L}_Ne.$$ 

(31)

To derive the explicit formula of stress-energy-momentum tensor for Weyl neutrino field, we need make some preparations.

**Lemma 1** For the hybrid vector/spinorial tensor $\sigma^a_{AA'}$, there exist the following identities:

$$\sigma_{(aAA'}\sigma_b^{B')} = \frac{1}{2} g_{ab} \delta^{A'B'},$$

(32)

$$\sigma_{(aAA'}\sigma^c_{B'B)} = g_{(a} \sigma_{b)c_{B'B)} - \frac{1}{2} g_{ab} \sigma^c_{AB'},$$

(33)

where the round brackets denote the symmetrization of $a$ and $b$.

**Proof of (32).**

$$\sigma_{(aAA'}\sigma_b^{B')} = \frac{1}{2} \left( \sigma_{aAA'}\sigma_b^{A'B'} + \sigma_{bAA'}\sigma_a^{A'B'} \right)$$

$$= \frac{1}{2} \left( \sigma_{aAA'}\sigma_b^{A'B'} - \sigma_{bAA'}\sigma_a^{A'B'} \right)$$

$$= \frac{1}{2} \left( \sigma_{(aAA'}\sigma_b^{A'B'} - \sigma_{aAB'}\sigma_b^{A'} \right),$$

(34)

which implies $\sigma_{(aAA'}\sigma_b^{A'B'}$ is antisymmetric with respect to $A'$ and $B'$. Therefore we have

$$\sigma_{(aAA'}\sigma_b^{A'B')} = \frac{1}{2} \sigma_{(aAC'}\sigma_b^{AC'} \delta^{A'B'}} = \frac{1}{2} g_{ab} \delta^{A'B'}. $$

(35)
Proof of (33).

\[
\sigma_{(aA')\sigma^{cAB}\sigma(b)BB'} = \frac{1}{2}(\sigma_{aAA'}\sigma^{cAB}\sigma_{bBB'} + \sigma_{bAA'}\sigma^{cAB}\sigma_{aBB'}) = \frac{1}{2}(\sigma_{aAA'}\sigma^{cAB}\sigma_{bBB'} + \sigma_{aBB'}\sigma^{cAB}\sigma_{bAA'}) = \sigma^{cAB}\lambda_{ab(AB)(A'B')} + \frac{1}{4}\sigma_{CC'}\sigma^{cAB}\sigma_{bCC'}\epsilon_{AB'A'B'}
\]

where \(\sigma^{cAB}\lambda_{ab(AB)(A'B')}\) is given by

\[
\sigma^{cAB}\lambda_{ab(AB)(A'B')} = \frac{1}{4}(\sigma_{aAA'}\sigma^{cAB}\sigma_{bBB'} + \sigma_{aBB'}\sigma^{cAB}\sigma_{bAA'} + \sigma_{aBB'}\sigma^{cAB}\sigma_{bAA'}) = \frac{1}{4}(\sigma_{aAA'}\sigma^{cAB}\sigma_{bBB'} + g_{a}^{c}\sigma_{bAB'} + g_{b}^{c}\sigma_{aAB'} + \sigma_{bAA'}\sigma^{cAB}\sigma_{aBB'}) = \frac{1}{2}\sigma_{aAA'}\sigma^{cAB}\sigma_{bBB'} + \frac{1}{4}g_{ab}\sigma^{c}_{AB'}, \tag{36}
\]

Thus (33) can yield easily by combining (36) and (37).

**Lemma 2** Let

\[
\tilde{\nabla}_{a}\psi^{B} = \nabla_{a}\psi^{B} + \Theta_{aB'C}\psi^{C}, \tag{38}
\]

where \(\tilde{\nabla}_{a}\) and \(\nabla_{a}\) are covariant derivative operators associated with the metrics \(\bar{g}_{ab}\) and \(g_{ab}\) respectively; then \(\Theta_{aB'C}\) is given by

\[
\Theta_{aB'C} = \frac{1}{2}\tilde{\sigma}_{b}^{BC'}(\nabla_{a}\tilde{\sigma}_{CC'}^{b} + C_{aCC'}^{b}), \tag{39}
\]

where \(\tilde{\sigma}_{AA'}^{a}\) is compatible with \(\tilde{\nabla}_{a}\).

**Proof.** From (38), it is obvious to yield

\[
\tilde{\nabla}_{a}\psi^{B} = \nabla_{a}\psi^{B} + \Theta_{aBC}\psi^{C}, \tag{40}
\]

\[
\tilde{\nabla}_{a}\psi^{B'} = \nabla_{a}\psi^{B'} + \Theta_{a'B'C'}\psi^{C'}, \tag{41}
\]

where we have employed

\[
\tilde{\nabla}_{a}\epsilon_{AB} = \nabla_{a}\epsilon_{AB} = 0. \tag{42}
\]
On the other hand
\[ \tilde{\nabla}_a \epsilon_{AB} = \nabla_a \epsilon_{AB} + \Theta_{aAC} \epsilon^C_B + \Theta_{aBC} \epsilon^C_A, \] (43)

hereby
\[ \Theta_{aAB} = \Theta_{aBA}. \] (44)

In addition, we have
\[ \tilde{\nabla}_a v^{BB'} = \tilde{\nabla}_a (\tilde{\sigma}_B^{BB'} v^b) = \tilde{\sigma}_B^{BB'} \tilde{\nabla}_a v^b = \tilde{\sigma}_B^{BB'} (\nabla_a v^b + C^b_{ac} v^c) \]
\[ = \tilde{\sigma}_B^{BB'} [\nabla_a (\tilde{\sigma}_C^{CC'} v^{CC'}) + C^b_{ac} v^c] \]
\[ = \nabla_a v^{BB'} + \tilde{\sigma}_B^{BB'} (\nabla_a \tilde{\sigma}_C^{CC'} + C^b_{aCC'} v^{CC'}), \] (45)
on the other hand
\[ \tilde{\nabla}_a v^{BB'} = \nabla_a v^{BB'} + \Theta_B^{BC} v^{CB'} + \Theta_C^{CC'} v^{BC'} \]
\[ = \nabla_a v^{BB'} + (\Theta_B^{BC} \tilde{\epsilon}_C^{CB'} + \Theta_C^{CC'} \epsilon_C^B) v^{CC'} \] (46)

Combining (45) and (46), we obtain
\[ \tilde{\sigma}_B^{BB'} (\nabla_a \tilde{\sigma}_C^{CC'} + C^b_{aCC'}) = \Theta_B^{BC} \tilde{\epsilon}_C^{CB'} + \Theta_C^{CC'} \epsilon_C^B, \] (47)
contracting \( B' \) with \( C' \) and using (44), (39) can yield.

According to Lemma 2, with respect to the variation of the metric, the corresponding variation of \( \Theta_B^{BC} \) is given by
\[ \delta \Theta_B^{BC} = \frac{1}{2} \sigma_B^{BC'} (\nabla_a \delta \sigma_c^{CC'} + \delta C^b_{aCC'}), \] (48)
where, for convenience but without loss of generalization, the variation of \( \sigma^b_{CC'} \) can be gauge fixed as
\[ \delta \sigma^b_{CC'} = \frac{1}{2} \sigma_{eCC'} \delta g^{be}. \] (49)
and
\[ \delta C^b_{aCC'} = \frac{1}{2} \sigma_{eCC'} g^{bd} (\nabla_a \delta g_{cd} + \nabla_c \delta g_{ad} - \nabla_d \delta g_{ac}). \] (50)

Using
\[ g_{ac} \delta g^{bc} = -g^{bc} \delta g_{ac}, \] (51)
we have
\[ \delta C^b_{aCC'} = \frac{1}{2} \sigma_{eCC'} (g_{ad} \nabla_b \delta g^{ed} - \nabla_a \delta g^{be} - g_{ad} \nabla_e \delta g^{bd}). \] (52)
Corollary 3  The final version for the variation of $\Theta_a^B C$ is given by
\[
\delta \Theta_a^B C = \frac{1}{4} \sigma_b^{BC'} \sigma_{CDE} g_{ad} (\nabla^b \delta g^{ed} - \nabla^e \delta g^{bd}).
\] (53)

With the above preparations, we can derive the stress-energy-momentum tensor for Weyl neutrino field.

Theorem 4  The stress-energy-momentum tensor for Weyl neutrino field is given by
\[
T_{ab} = -\frac{\alpha_N}{32\pi} (\bar{\psi}^A \delta \sigma^{AB} \nabla_b \psi^A - \nabla_b (\bar{\psi}^A \sigma_{bA} \psi^A),
\] (54)

where the round brackets denote the symmetrization of $a$ and $b$.

Proof. Start with
\[
\delta S_N = \int_M \delta L_N e = \int_M (\bar{\psi}^A \delta \sigma^A \nabla_b \psi^A \sqrt{-g} + \bar{\psi}^A \sigma^A \delta \Theta_a^B \sqrt{-g} + \bar{\psi}^A \nabla_b \psi^A \delta \sqrt{-g}) e,
\] (55)

using (49), the first integrand gives
\[
\frac{1}{2} \bar{\psi}^A \sigma_{A'B} \nabla_b \psi^A \delta g^{ab} \sqrt{-g},
\] (56)

similarly, employing Corollary 3 and making a partial integration, the second term in the integrand yields
\[
\frac{1}{4} [\sigma_{aB'} \sigma_b^{BCB'} \nabla_{C'B'} (\bar{\psi}^{B'} \psi^C) - \sigma_{aCC'} \sigma_{cC'B} \sigma_{bC'} \nabla_c (\bar{\psi}^{B'} \psi^C)] \delta g^{ab} \sqrt{-g};
\] (57)

using Lemma 1 and the Weyl neutrino field equation, which can be written as
\[
-\frac{1}{4} (\bar{\psi}^A \sigma_{(aA'} \psi^{A')} \nabla_b \psi^A - \nabla_b (\bar{\psi}^A \sigma_{bA} \psi^A)) \delta g^{ab} \sqrt{-g},
\] (58)

the third integrand vanishes by virtue of the Weyl neutrino field equation.

Thus, combining (56) and (58), we have
\[
\frac{\delta S_N}{\delta g^{ab}} = \frac{1}{4} (\bar{\psi}^A \sigma_{(aA'} \nabla_b \psi^{A')} - \nabla_b (\bar{\psi}^A \sigma_{bA} \psi^A)) \sqrt{-g},
\] (59)

therefore
\[
T_{ab} = -\frac{\alpha_N}{32\pi} (\bar{\psi}^A \sigma_{(aA'} \nabla_b \psi^{A')} - \nabla_b (\bar{\psi}^A \sigma_{bA} \psi^A)),
\] (60)
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