\section{Introduction}

The \( j \)-invariant \( j(\tau) \) is the \( \text{SL}_2(\mathbb{Z}) \)-invariant holomorphic function on the complex upper half-plane \( \mathfrak{H} \) with Fourier series expansion at the cusp \( j(\tau) = e^{-2\pi i \tau} + 744 + 196884 e^{2\pi i \tau} + \cdots \), which induces an isomorphism from the moduli space of elliptic curves to \( \mathbb{C} \). The \( j \)-invariant is fundamental in many branches of mathematics such as the elliptic function theory, number theory, the theory of automorphic forms, and monstrous moonshine... Besides the \( j \)-invariant itself, it has been discovered that the difference \( j(\tau) - j(\tau') \) enjoys beautiful properties as well: Gross–Zagier’s result \cite{11} on singular moduli and the denominator formula for the monster Lie algebra (see Borcherds \cite{3} and \cite{4}), for example. In this paper, we show that the difference \( j(\tau) - j(\tau') \) is closely related to the Borcherds \( \Phi \)-function \cite{5}, a remarkable automorphic form on the period domain for Enriques surfaces.

For \( \tau \in \mathfrak{H} \), we set \( E_\tau = \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z} \). Let \( \mathfrak{D} \) be the \( \Gamma(2) \times \Gamma(2) \)-orbit of the diagonal locus of \( \mathfrak{H} \times \mathfrak{H} \), where \( \Gamma(2) \subset \text{SL}_2(\mathbb{Z}) \) is the principal congruence subgroup of level 2. Then, for any \((\tau, \tau') \in \mathfrak{H} \times \mathfrak{H} \setminus \mathfrak{D} \), one can construct 15 fixed-point-free involutions on \( \text{Km}(E_\tau \times E_{\tau'}) \), the Kummer surface of product type associated to \( E_\tau \times E_{\tau'} \), which induce 15 distinct Enriques surfaces: Indeed, if \((\tau, \tau') \in \mathfrak{H} \times \mathfrak{H} \) is very general, then, up to conjugacy, these 15 involutions are the only fixed-point-free involutions on \( \text{Km}(E_\tau \times E_{\tau'}) \), the Kummer surface of product type associated to \( E_\tau \times E_{\tau'} \), which induce 15 distinct Enriques surfaces: Indeed, if \((\tau, \tau') \in \mathfrak{H} \times \mathfrak{H} \) is very general, then, up to conjugacy, these 15 involutions are the only fixed-point-free involutions on \( \text{Km}(E_\tau \times E_{\tau'}) \) (see \cite{15}, \cite{17}, \cite{18}, \cite{21}). By \cite{17}, \cite{21}, there is a one-to-one correspondence between the set of these 15 conjugacy classes of involutions and the set of non-zero elements of the discriminant group \( A_K \) of the lattice \( K = U(2) \oplus U(2) \) (see Sect. \ref{2.1} for the notation about lattices). Recall that the discriminant form of \( K \) takes its values in \( \mathbb{Z}/2\mathbb{Z} \). According to the corresponding value of the discriminant form of \( A_K \), the 15 conjugacy classes of involutions are divided into the 6 odd involutions and the 9 even involutions.

Let \( \Lambda = U(2) \oplus U \oplus E_8(2) \) be the Enriques lattice. Let \( \Omega^+_\Lambda \) denote the period domain for Enriques surfaces (cf. \( \text{2.2} \)). In \cite{5}, Borcherds discovered a remarkable automorphic form \( \Phi \), called the Borcherds \( \Phi \)-function, of weight 4 on \( \Omega^+_\Lambda \) vanishing exactly on the discriminant locus. For \( \ell \in \{1, 2\} \), once a primitive isotropic vector \( e_\ell \in U(\ell) \) and an isomorphism of \( \Omega^+_\Lambda \) with the tube domain \( M_\ell \otimes \mathbb{R} + iC^*_M \) (see \( \text{2.10} \)) are fixed, where \( M_\ell = e_\ell^T/e_\ell = U(2/\ell) \oplus E_8(2) \), then \( \Phi \) is identified with...
the holomorphic function given by an explicit infinite product, denoted by \( \Phi_\gamma \), on \( \mathbb{M}_4 \otimes \mathbb{R} + iC_{M_4}^+ \) (see Sect. \( \PageIndex{2} \) for details).

For each \( \gamma \in A_k \smallsetminus \{0\} \), let \( \varphi_\gamma \) be the attached fixed-point free involution on \( \text{Km}(E_\tau \times E_{\tau'}) \). One can naturally associate an integer \( \ell(\gamma) \in \{1, 2\} \), and a period map \( \varphi_\gamma : \mathcal{H} \to \mathbb{M}_{\ell(\gamma)} \otimes \mathbb{R} + iC_{M_4}^+ \) for the family of Kummer surfaces \( \text{Km}(E_\tau \times E_{\tau'}) \) over \( \mathcal{H} \) by fixing a normalized marking (see Definition \( \PageIndex{3.2} \)). We set \( \Phi_\gamma = \Phi_{\ell(\gamma)} \circ \varphi_\gamma \). Then \( \Phi_\gamma^2 \) is independent of the choice of a normalized marking, and is an automorphic form on \( \mathcal{H} \) of weight 8 for \( \Gamma(2) \times \Gamma(2) \). (See Sect. \( \PageIndex{3.2} \) and Sect. \( \PageIndex{3.3} \).)

The main result of this paper states that the difference \( j(\tau) - j(\tau') \) is related to the Borcherds \( \Phi \)-function in the following simple way.

**Theorem 1.1.** For any \( (\tau, \tau') \in \mathcal{H} \smallsetminus \mathcal{D} \),

\[
2^{-96} (j(\tau) - j(\tau'))^{12} = \prod_{\gamma \text{ odd}} \Phi_\gamma(\tau, \tau')^6 \prod_{\gamma \text{ even}} \Phi_\gamma(\tau, \tau')^4.
\]

We will prove Theorem 1.1 by first showing (1.1) up to a constant and then showing that the constant is equal to \( 2^{-96} \).

To prove (1.1) up to a constant, since the logarithm of the absolute value of each side of (1.1) is shown to be an \( \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \)-invariant pluriharmonic function on \( \mathcal{H} \), it suffices to compare their singularities. This will be done by analyzing the period map \( \varphi_\gamma \) and its extension to the Baily-Borel compactification, in particular its intersection with the discriminant locus and the boundary locus of the moduli space of Enriques surfaces. Using lattice theory, we will show that such a geometric property of \( \varphi_\gamma \) is determined by the parity of \( \gamma \).

Determining the constant in (1.1) is a different and somewhat a more delicate problem. We will do this by computing the leading terms of the denominator and the numerator of the right-hand side of (1.1) near the cusp \(+i\infty, +i\infty\). For the computation of \( \prod_{\gamma \text{ even}} \Phi_\gamma(\tau, \tau')^4 \), we use the formula in \( \PageIndex{13} \) Cor. 7.6], which is a consequence of an algebraic expression of \( \Phi \). For the computation of \( \prod_{\gamma \text{ odd}} \Phi_\gamma(\tau, \tau')^6 \), we carefully study the Borcherds products with respect to the 0-dimensional cusps of levels one and two. For the left-hand side of (1.1), we use the denominator formula for the monster Lie algebra \( \mathfrak{m} \) (see also \( \PageIndex{13} \)). All of these enable us to obtain the constant \( 2^{-96} \) on the left-hand side of (1.1).

For the results on the the discriminant locus and the boundary locus of the moduli space of Enriques surfaces, which we obtain in the course of the proof of Theorem 1.1 and may be of independent interest, see e.g. Lemma \( \PageIndex{2.2} \) Propositions \( \PageIndex{3.13} \) and \( \PageIndex{3.14} \).

The organization of this paper is as follows. In Sect. \( \PageIndex{2} \) we recall the moduli space of Enriques surfaces and the Borcherds \( \Phi \)-function. In Sect. \( \PageIndex{3} \) we recall the period mapping \( \varpi_\gamma \) for the Enriques surfaces \( \text{Km}(E_\tau \times E_{\tau'})/\iota_\gamma \), prove the automorphy of \( \Phi_\gamma \) and study the intersection of the period mapping \( \varpi_\gamma \) with the discriminant locus of \( \Omega_A^2 \). In Sect. \( \PageIndex{4} \) we give an explicit formula for the denominator of the right hand side of (1.1). In Sect. \( \PageIndex{5} \) we recall involutions of odd type and compute \( dd^c \log(\cdot) \) of the Petersson norm of the numerator of the right hand side of (1.1) as a current on the second symmetric product of the compactified modular curve. In Sect. \( \PageIndex{6} \) we compute the leading term of \( \Phi_\gamma \) near \((+i\infty, +i\infty)\) for all odd \( \gamma \in A_k \smallsetminus \{0\} \). In Sect. \( \PageIndex{7} \) we prove Theorem 1.1. We also list some open problems. An appendix is accompanied to give some technical results on lattices.
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2. Enriques surfaces and the Borcherds $\Phi$-function

2.1. Lattices. A lattice $L$ is a pair of a free $\mathbb{Z}$-module of rank $r$ and a non-degenerate, integral, symmetric bilinear form $\langle \cdot, \cdot \rangle_L$ on it. When there is no possibility of confusion, we often write $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle_L$. A lattice $L$ is said to be even if $x^2 := \langle x, x \rangle_L \equiv 0 \mod 2$ for any $x \in L$. For any integer $m$, we denote by $L(m)$ the lattice $L$ equipped with the rescaled bilinear form $m \langle \cdot, \cdot \rangle_L$. The group of isometries of $L$ is denoted by $O(L)$. The set of roots of $L$ is defined by $\Delta_L := \{ d \in L \mid \langle d, d \rangle_L = -2 \}$. Let $L^\vee = \text{Hom}(L, \mathbb{Z}) \subset L \otimes \mathbb{Q}$ be the dual lattice of $L$. The discriminant group of $L$ is the finite abelian group $A_L := L^\vee / L$. For a sublattice $S \subset L$, we write $S^\perp$ (and $S^\perp$ if no confusion is likely) for the orthogonal complement of $S$ in $L$.

Let $L$ be an even lattice. For $\lambda \in L^\vee$, we write $\bar{\lambda} := \lambda + L \in A_L$. The discriminant form and the discriminant bilinear from on $A_L$ are denoted by $q_L$ and $b_L$, respectively. We set $F_2 := \mathbb{Z}/2\mathbb{Z}$. If $A_L \cong F_2^{\oplus l}$ for some $l \in \mathbb{Z}_{\geq 0}$, then $L$ is said to be 2-elementary. For an even 2-elementary lattice $L$, we define its parity as $\delta(L) := 0$ if $q_L$ is $\mathbb{Z}/2\mathbb{Z}$-valued and $\delta(L) := 1$ otherwise. We refer to [20] for more about lattices and discriminant forms.

In this paper, the following lattices are important. Let $U$ be the hyperbolic plane, i.e., the even unimodular lattice of signature $(1, 1)$ and let $E_8$ be the negative-definite even unimodular lattice of rank 8. The $K3$ lattice is the even unimodular lattice $L_{K3} := U \oplus U \oplus U \oplus E_8 \oplus E_8$. Then $L_{K3}$ has rank 22 with signature $(3, 19)$. The Enriques lattice is the even 2-elementary lattice of rank 12 defined as

$$\Lambda := U(2) \oplus U \oplus E_8(2).$$

Then $\Lambda$ has signature $(2, 10)$ and discriminant group $A_\Lambda \cong F_2^{\oplus 10}$. We fix a primitive embedding $\Lambda \subset L_{K3}$. Then $\Lambda^{\perp \perp}_{K3} \cong \mathbb{Z}(2) \oplus E_8(2)$ (see [2 VIII, §19]).

2.2. The moduli space of Enriques surfaces and the Baily–Borel compactification. Recall that a compact connected complex surface is a $K3$ surface if it is simply connected and has trivial canonical bundle. For a $K3$ surface $X$, the second cohomology group $H^2(X, \mathbb{Z})$ endowed with the cup-product is isometric to $L_{K3}$ (see [2 p.241]). The Néron–Severi and transcendental lattices of $X$ are defined as $NS_X := H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R})$ and $T_X := NS_X^{\perp H^2(X, \mathbb{R})}$, respectively.

A compact connected complex surface is an Enriques surface if it is not simply connected and its universal covering is a $K3$ surface. Let $Y$ be an Enriques surface with universal covering $K3$ surface $X$. Then $\pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$ and the non-trivial covering transformation $\iota_Y : X \to X$ is then an anti-symplectic fixed-point-free involution. We set

$$H^2(X, \mathbb{Z})_{\pm} := \{ l \in H^2(X, \mathbb{Z}) \mid \iota_Y^*(l) = \pm l \}.$$
By e.g. [2] VIII, §19], there is an isometry of lattices \( \alpha : H^2(X, \mathbb{Z}) \cong \mathbb{L}_{K,3} \) such that
\[
\alpha(H^2(X, \mathbb{Z})_+) = \mathbb{L}_{K,3}^+, \quad \alpha(H^2(X, \mathbb{Z})_-) = \mathbb{L}.
\]
Such an isometry \( \alpha \) is called a marking, and the pair \((Y,\alpha)\) satisfying (2.1) is called a marked Enriques surface.

We set
\[
\Omega_\Lambda := \{ [\omega] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0 \}.
\]

Then \( \Omega_\Lambda \) consists of two disjoint connected complex manifolds of dimension 10, both of which is a bounded symmetric domain of type IV. We fix one connected component \( \Omega^+_\Lambda \) of \( \Omega_\Lambda \). On \( \Omega_\Lambda \) acts \( O(\Lambda) \) projectively. Let \( O^+(\Lambda) \subset O(\Lambda) \) be the subgroup of index 2 preserving \( \Omega^+_\Lambda \). Then \( O^+(\Lambda) \) acts on \( \Omega^+_\Lambda \) properly discontinuously. The period domain for Enriques surfaces is defined as
\[
\mathcal{M} := \Omega^+_\Lambda / O^+(\Lambda) = \Omega_\Lambda / O(\Lambda).
\]

The period of a marked Enriques surface \((Y,\alpha)\) is defined as
\[
\varpi(Y,\alpha) := [\alpha(H^0(X, \Omega^+_\Lambda))] \in \Omega^+_\Lambda
\]
and the period of an Enriques surface \(Y\) is defined as the \( O^+(\Lambda) \)-orbit of \( \varpi(Y,\alpha) \):
\[
\varpi(Y) := [\varpi(Y,\alpha)] \in \mathcal{M}.
\]

For \( d \in \Lambda \otimes \mathbb{R} \), we set
\[
H_d := \{ \omega \in \Omega^+_\Lambda \mid \langle \omega, d \rangle = 0 \}.
\]
The discriminant locus of \( \Omega^+_\Lambda \) is the \( O^+(\Lambda) \)-invariant divisor \( \mathcal{H} := \bigcup_{d \in \Delta_\Lambda} H_d \). Set
\[
\mathcal{D} := \mathcal{H} / O^+(\Lambda).
\]

Then \( \varpi(Y) \notin \mathcal{D} \) for any Enriques surface \(Y\). Via the period map, the coarse moduli space of Enriques surfaces is isomorphic to the analytic space ([2] VIII, §19])
\[
(\Omega^+_\Lambda \setminus \mathcal{H}) / O^+(\Lambda) = \mathcal{M} \setminus \mathcal{D}.
\]

Let \( \mathcal{M}^* \) be the Baily–Borel compactification of \( \mathcal{M} \). By Sterk [25] Props. 4.5, 4.6, 4.7], the boundary \( \mathcal{M}^* \setminus \mathcal{M} \) consists of two 1-dimensional components, one of which is isomorphic to the modular curve \( X(1) := (\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H})^* \) and the other to the curve \( X^1(2) := (\Gamma^1(2) \setminus \mathbb{H})^* \). Here \( \Gamma^1(2) = \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z}) \mid b \equiv 0 \mod 2 \right\} \) and the asterisk * denotes the Baily–Borel compactification. By slight abuse of notation, the components of \( \mathcal{M}^* \setminus \mathcal{M} \) corresponding to \( X(1) \) and \( X^1(2) \) are denoted by the same symbols. Further, \( \mathcal{M}^* \setminus \mathcal{M} \) has two 0-dimensional cusps: The curves \( X(1) \) and \( X^1(2) \) intersect at one point, and this point gives one 0-dimensional cusp; The other 0-dimensional cusp lies on \( X^1(2) \setminus X(1) \).

By [23] Sect. 2], from lattice-theoretical terms, the 1-dimensional components of \( \mathcal{M}^* \setminus \mathcal{M} \) correspond to the \( O(\Lambda) \)-orbits of the primitive totally isotropic sublattices of rank 2 of \( \Lambda \), and the 0-dimensional cusps of \( \mathcal{M}^* \setminus \mathcal{M} \) correspond to the \( O(\Lambda) \)-orbits of the primitive isotropic vectors of \( \Lambda \). The above results of Sterk in particular say that, up to the \( O(\Lambda) \)-action, there are exactly two distinct such sublattices of rank 2 of \( \Lambda \) and two distinct primitive isotropic vectors of \( \Lambda \). The following lemma gives their explicit representatives.

**Lemma 2.1.** Let \( \mathbb{L}_{2,9} \) be an odd unimodular lattice of signature \((2,9)\).
(1) For any root \( d \in \Delta_\Lambda \), we have \( d^{1,\Lambda} \cong \mathbb{I}_{2,9}(2) \). Let \( \{e_1, e_2, \ldots, e_{11}\} \) be a free basis of \( \mathbb{I}_{2,9}(2) \) with Gram matrix \( 2 \text{diag}(1_2, -1_9) \). Let \( F_1 \) be the sublattice of \( \Lambda \) that corresponds to \( \mathbb{Z}(e_1 + e_3) + \mathbb{Z}(e_2 + e_4) \) via \( d^{1,\Lambda} \cong \mathbb{I}_{2,9}(2) \). Then \( F_1 \) is a primitive totally isotropic sublattice of rank 2 of \( \Lambda \), and corresponds to \( X(1) \).

(2) Let \( e, e' \) be the standard free basis of the left lattice \( \mathbb{U}(2) \) of \( \Lambda \) with Gram matrix \( (2^0) \), and let \( f, f' \) be the standard free basis of the middle lattice \( \mathbb{V} \) of \( \Lambda \) with Gram matrix \( \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \). We set \( F_2 := Ze + Zf \). Then \( F_2 \) is a primitive totally isotropic sublattices of rank 2 of \( \Lambda \), and corresponds to \( X^1(2) \).

Proof. By [23, Th. 3.6.2], up to isometries of lattices, an even indefinite 2-elementary lattice is determined by its signature, the rank of its discriminant group, and its parity. It follows that \( d^{1,\Lambda} \cong \mathbb{I}_{2,9}(2) \) for any \( d \in \Delta_\Lambda \). Recall that \( X(1) \) has one 0-dimensional cusp and \( X^1(2) \) has two 0-dimensional cusps. Since \( e \) and \( f \) are primitive vectors of \( F_2 \) with \( \langle e, \Lambda \rangle = 2\mathbb{Z} \) and \( \langle f, \Lambda \rangle = \mathbb{Z} \), the above results of Sterk imply that \( F_2 \) corresponds to \( X^1(2) \), so \( F_1 \) corresponds to \( X(1) \).

For a lattice \( L \) of signature \((2, r - 2)\), we define \( \Omega_L, \Omega_L^+, O(L), O^+(L) \), and \( H_d \) in the same way as those for \( \Lambda \), and we set \( M_L := \Omega_L^+ / O^+(L) = M_d / O(L) \). The Baily–Borel compactification of \( M_L \) is denoted by \( M_L^\ast \). Recall that for any linearly independent vectors \( d_1, \ldots, d_k \in L \otimes \mathbb{R} \), we have

\[
(2.5) \quad H_{d_1} \cap \cdots \cap H_{d_k} \neq \emptyset \iff (d_i, d_j)_{1 \leq i, j \leq k} \text{ is a negative-definite matrix.}
\]

For a primitive isotropic vector \( v \in L \), the level of \( v \) is defined as the positive integer \( \ell \) with \( \langle v, L \rangle = \ell \mathbb{Z} \). In this case, we say that the 0-dimensional cusp of \( M_L^\ast \) corresponding \( v \) has level \( \ell \). For example, in the case of \( \Lambda \), by Sterk [23, Props. 4.5, 4.6, 4.7] (see also the proof of Lemma [2.1]), one of the two 0-dimensional cusps of \( \mathcal{M}^\ast \), which lies on \( X^1(2) \times X(1) \), has level 1, and the other 0-dimensional cusp, which is the intersection of \( X^1(2) \) and \( X(1) \), has level 2.

Lemma 2.2. Let \( \mathcal{D}^\ast \) be the closure of \( \mathcal{D} \) in \( \mathcal{M}^\ast \). Then \( \mathcal{D}^\ast \setminus \mathcal{D} = X(1) \).

Proof. We fix a root \( d \in \Delta_\Lambda \). To simplify the notation, we write \( d^{\perp} \) for \( d^{1,\Lambda} \subset \Lambda \). We choose the connected component \( \Omega_{d^\perp}^+ \) of \( \Omega_{d^\perp} \) such that \( \Omega_{d^\perp}^+ = H_d \subset \Omega_{d^\perp}^+ \).

Step 1. We first relate \( \mathcal{M}_d^\ast \setminus \mathcal{M}_d \), \( \mathcal{D} \setminus \mathcal{D} \), and \( \mathcal{M}^\ast \setminus \mathcal{M} \). Let \( O^+(\Lambda)_d := \{ g \in O^+(\Lambda) \mid g(d) = d \} \) be the stabilizer of \( d \) in \( O^+(\Lambda) \). Then both \( O^+(\Lambda)_d \) and \( O^+(d^\perp) \) act on \( \Omega_{d^\perp}^+ \). We will show in the appendix (see Lemma [A.1]) that the restriction map \( O^+(\Lambda)_d \ni g \mapsto g|_{d^\perp} \in O^+(d^\perp) \) is surjective. It follows that \( \mathcal{M}_d := \Omega_{d^\perp}^+ / O^+(d^\perp) \). We have the natural surjective map \( \Omega_{d^\perp}^+ / O^+(\Lambda)_d \to (O^+(\Lambda) \cdot \Omega_{d^\perp}^+) / O^+(\Lambda) \).

Since \( O^+(\Lambda) \) acts on \( \Delta_\Lambda \) transitively (see [23, Remark 3.6]), we have \( O^+(\Lambda) \cdot \Omega_{d^\perp}^+ = O^+(\Lambda) \cdot H_d = \mathcal{H} \). Thus \( (O^+(\Lambda) \cdot \Omega_{d^\perp}^+) / O^+(\Lambda) = \mathcal{D} \). We have obtained the surjective morphism \( \mathcal{M}_d \to \mathcal{D} \). By the description of the Baily–Borel compactification, we have the extended surjective morphisms \( \mathcal{M}_d^\ast \to \mathcal{D} \) and \( \mathcal{M}_d^\ast \setminus \mathcal{M}_d \to \mathcal{D} \setminus \mathcal{D} \). The inclusion \( \Omega_{d^\perp}^+ = H_d \subset \Omega_{d^\perp}^+ \) induces the injective morphisms \( \mathcal{D} \hookrightarrow \mathcal{M} \), \( \mathcal{D} \hookrightarrow \mathcal{M}^\ast \), and \( \mathcal{D} \to \mathcal{D} \to \mathcal{M}^\ast \setminus \mathcal{M} \). To conclude, we have the morphisms

\[
(2.6) \quad \mathcal{M}_d^\ast \setminus \mathcal{M}_d \to \mathcal{D} \setminus \mathcal{D} \hookrightarrow \mathcal{M}^\ast \setminus \mathcal{M},
\]

where the left morphism is surjective, and the right morphism is injective.
Step 2. We show that $D^* \setminus D$ is irreducible. Indeed, by Lemma 2.1 (1), we have an isometry $d^\perp \cong I_{2,9}(2)$, with which we identify $\mathcal{M}^*_{I_{2,9}(2)} \setminus \mathcal{M}_{I_{2,9}(2)} = \mathcal{M}^*_{I_{2,9}(2)} \setminus \mathcal{M}_{I_{2,9}(2)}$.

By [23] Prop. 1.17, $I_{2,9}(2)$ has a unique primitive totally isotropic sublattice of rank 2 up to $O(I_{2,9}(2))$. It follows from [23] Sect. 2.1 that $\mathcal{M}^*_{I_{2,9}(2)} \setminus \mathcal{M}_{I_{2,9}(2)}$ is irreducible. Since the morphism $\mathcal{M}^*_{I_{2,9}(2)} \setminus \mathcal{M}_{I_{2,9}(2)} \to D^* \setminus D$ in (2.6) is surjective, $D^* \setminus D$ is irreducible.

Step 3. We show that $D^* \setminus D = X(1)$ in $\mathcal{M}^*$. Let $F \subset d^\perp$ be a primitive totally isotropic sublattice of rank 2. We set $N(F) = \{ g \in O^+(\Lambda) \mid g(F) = F \}$. Let $\overline{\Omega}^+_{d^\perp}$ be the closure of $\Omega^+_{d^\perp}$ in $\{ \omega \in P(d^\perp \otimes \mathbb{C}) \mid (\omega, \omega) = 0 \}$. Then the description of the boundary component of the Baily–Borel compactification (see [23] Sect. 4.1) implies that the closure of the boundary component corresponding to $F$ is $P(F \otimes \mathbb{C}) \cap \overline{\Omega}^+_{d^\perp}$, on which $N(F)$ acts naturally.

We will show in the appendix (see Lemma A.2) that any element of $\text{SL}(F)$ lifts to an element of $O^+(d^\perp)$. Together with the surjectivity of the restriction map $O^+(\Lambda)d \to O^+(d^\perp)$ (see Step 1), we obtain that the restriction map $N(F) \ni g \mapsto g|_F \in \text{SL}(F)$ is surjective. Then it follows from [23] Sect. 4.1 that $\mathcal{M}^*_{I_{2,9}(2)} \setminus \mathcal{M}_{I_{2,9}(2)} \to X(1)$.

> From the construction of the Baily–Borel compactification (see [23] Sect. 4.1) and Lemma 2.1 (1), the map (2.6) from $\mathcal{M}^*_{I_{2,9}(2)} \setminus \mathcal{M}_{I_{2,9}(2)}$ to $\mathcal{M}^* \setminus \mathcal{M}$ is given by the identity map on $X(1)$. Since the map (2.6) factors through $D^* \setminus D$, we get $D^* \setminus D = X(1)$.

2.3. The Borcherds $\Phi$-function. In [5], Borcherds constructed an automorphic form on $\Omega^+_A$ for $O^+(\Lambda)$ of weight 4 vanishing exactly on $\mathcal{H}$. We call this automorphic form the $\text{Borcherds}$ $\Phi$-function. Let us recall its definition. For a subset $S \subset P(\Lambda \otimes \mathbb{C})$, let $C(S) := \{ \eta \in (\Lambda \otimes \mathbb{C}) \setminus \{ 0 \}; [\eta] \in S \}$ be the cone over $S$. Up to a constant, the Borcherds $\Phi$-function is defined as the holomorphic function $\Phi$ on $C(\Omega^+_A)$ with the following properties:

- $\Phi(\lambda Z) = \lambda^{-4} \Phi(Z)$ for all $\lambda \in \mathbb{C}^*$.
- $\Phi(g(Z)) = \chi(g) \Phi(Z)$ for all $g \in O^+(\Lambda)$, where $\chi \in \text{Hom}(O^+(\Lambda), \{ \pm 1 \})$.
- The zero divisor of $\Phi$ is the cone $C(\mathcal{H})$.

By choosing a section from $\Omega^+_A$ to $C(\Omega^+_A)$ and pulling back $\Phi$ by the section, $\Phi$ is identified with a holomorphic function on $\Omega^+_A$. In the following, we give two such identifications of the Borcherds $\Phi$-function, corresponding to the choices of 0-dimensional cusps of level one and two. Then $\Phi$ will be defined without an ambiguity of constant by giving explicit infinite product expressions at those cusps.

2.3.1. A tube domain realization of $\Omega^+_A$ with respect to the 0-dimensional cusp of level $\ell$. By [23] Prop. 4.5 and [23] Sect. 2] (see also Sect. 2.3), the $O(\Lambda)$-orbits of primitive isotropic vectors of $\Lambda$, or equivalently the 0-dimensional cusps of $\mathcal{M}^*$ consist of two points: the level 1 and level 2 cusps.

Let $e_1, f_1$ (resp. $e_2, f_2$) be the standard basis of $U$ (resp. $U(2)$) of the middle (resp. left) sublattice of $\Lambda = U(2) \oplus U \oplus E_8(2)$. Note that $e_1$ (resp. $e_2$) is a primitive isotropic vector of $\Lambda$ of level 1 (resp. level 2). Let $\ell \in \{ 1, 2 \}$. We set

$$M_\ell := U(2/\ell) \oplus E_8(2).$$

Then $M_\ell$ is a Lorentzian lattice of rank 10 and $\Lambda = (Ze_\ell + Zf_\ell) \oplus M_\ell$ in the obvious way. Let $C_{M_\ell} := \{ x \in M_\ell \otimes \mathbb{R} \mid (x, x) > 0 \}$ be the positive cone of the Lorentzian lattice $M_\ell$. Let $D_\ell := \text{Im}S_\ell = (ZM_\ell) \cap C_{M_\ell}$. Let $D_\ell$ be the domain in which the Borcherds $\Phi$-function is defined.
lattice \( \mathbb{M}_\ell \). Let \( \mathcal{C}_{\mathbb{M}_\ell}^+ \) be one of the two connected components of \( \mathcal{C}_{\mathbb{M}_\ell} \) such that \( \mathbb{M}_\ell \otimes \mathbb{R} + i \mathcal{C}_{\mathbb{M}_\ell}^+ \) corresponds to \( \Omega_A^+ \) via \( (2.9) \). Let \( \mathcal{T}_{\mathbb{M}_2/\ell}^+ \) be the closure of \( \mathcal{C}_{\mathbb{M}_2/\ell}^+ \) in \( \mathbb{M}_2/\ell \otimes \mathbb{R} \). In what follows, we assume that the basis \( \{ \mathbf{e}_\ell, \mathbf{f}_\ell \} \) is chosen in such a way that

\[
(2.8) \quad \mathbf{e}_\ell, \mathbf{f}_\ell \in \mathcal{T}_{\mathbb{M}_2/\ell}^+.
\]

Then the tube domain \( \mathbb{M}_\ell \otimes \mathbb{R} + i \mathcal{C}_{\mathbb{M}_\ell}^+ \) is identified with \( \Omega_A \) via the map

\[
(2.9) \quad j_\ell: \mathbb{M}_\ell \otimes \mathbb{R} + i \mathcal{C}_{\mathbb{M}_\ell}^+ \ni u \mapsto \left[-(u^2/2) \mathbf{e}_\ell + (f_\ell/u) + (-1)^{2/\ell}u\right] \in \Omega_A.
\]

Here the sign \((-1)^{2/\ell}\) is due to the condition \( (2.8) \). By an abuse of notation, we also write \( j_\ell(u) = -(u^2/2) \mathbf{e}_\ell + f_\ell/u + (-1)^{2/\ell}u \in C(\Omega_A) \). Thus we obtain an isomorphism \( j_\ell: \mathbb{M}_\ell \otimes \mathbb{R} + i \mathcal{C}_{\mathbb{M}_\ell}^+ \to \Omega_A^+ \). On \( \mathbb{M}_\ell \otimes \mathbb{R} + i \mathcal{C}_{\mathbb{M}_\ell}^+ \) acts \( O^+(\mathbb{A}) \) via the identification \( (2.9) \). Namely, for \( g \in O^+(\mathbb{A}) \) and \( u \in \mathbb{M}_\ell \otimes \mathbb{R} + i \mathcal{C}_{\mathbb{M}_\ell}^+ \), we define

\[
g \cdot u := j_\ell^{-1}g(j_\ell(u)).
\]

### 2.3.2. Series \( \{ c(n) \}_{n \geq -1} \).

Recall that the Dedekind \( \eta \)-function is defined as

\[
(2.10) \quad \eta(\tau) := e^{2\pi i \tau/24} \prod_{n > 0} (1 - e^{2\pi in\tau}).
\]

To describe the infinite product expansions of \( \Phi \), let \( \{ c(n) \}_{n \geq -1} \subset \mathbb{Z} \) be the series defined as the generating function (see [5, p. 705])

\[
\eta(\tau)^{-8} \eta(2\tau)^8 \eta(4\tau)^{-8} = \sum_{n \geq -1} c(n) e^{2\pi in\tau} = e^{-2\pi i\tau} + 8 + 36e^{2\pi i\tau} + \cdots.
\]

#### 2.3.3. The Borcherds \( \Phi \)-function with respect to the 0-dimensional cusp of level 1.

In [6, Ex. 3.1] (see also [13 Sect. 2.2.4]), Borcherds introduced the following infinite product for \( z \in \mathbb{M}_1 \otimes \mathbb{R} + i \mathcal{C}_{\mathbb{M}_1}^+ \) with \( (\text{Im} z)^2 \gg 0 \):

\[
(2.11) \quad \Phi_1(z) := \prod_{\lambda \in \mathbb{M}_1 \cap \mathcal{C}_{\mathbb{M}_1}^+ \setminus \{0\}} \left( \frac{1 - e^{\pi i(\lambda, z)_{\mathbb{M}_1}}}{1 + e^{\pi i(\lambda, z)_{\mathbb{M}_1}}} \right)^{c(\lambda^2/2)}.
\]

Then \( \Phi_1(z) \) converges absolutely when \( (\text{Im} z)^2 \gg 0 \) and extends holomorphically to the whole \( \mathbb{M}_1 \otimes \mathbb{R} + i \mathcal{C}_{\mathbb{M}_1}^+ \). By [6, Ex. 3.1], \( \Phi_1 \circ j_1^{-1} \) is an automorphic form on \( \Omega_A^+ \) of weight 4 with zero divisor \( \mathcal{H} \). We call \( \Phi_1(z) \) the **Borcherds \( \Phi \)-function with respect to the 0-dimensional cusp of level 1**. Then \( \Phi \) is normalized in such a way that \( \Phi_1 = j_1^* \Phi \).

#### 2.3.4. The Borcherds \( \Phi \)-function with respect to the 0-dimensional cusp of level 2.

Recall that \( \{ \mathbf{e}_1, \mathbf{f}_1 \} \) is the standard free-basis of \( \mathbb{U} \). We set

\[
(2.12) \quad \Pi^+ := \{ \lambda \in \mathbb{M}_2 \mid \langle \lambda, \mathbf{e}_1 \rangle_{\mathbb{M}_2} > 0, \lambda^2 \geq -2 \}.
\]

(Our \( \Pi^+ \) is slightly different form the one in [5, p. 701] for the convenience in later use.) In [5 Sect. 3], Borcherds introduced the following infinite product for \( w \in \mathbb{M}_2 \otimes \mathbb{R} + i \mathcal{C}_{\mathbb{M}_2}^+ \) with \( (\text{Im} w)^2 \gg 0 \):

\[
(2.13) \quad \Phi_2(w) := 2^8 e^{2\pi i(\mathbf{a}, w)_{\mathbb{M}_2}} \prod_{\lambda \in \mathbb{Z} \otimes \mathbb{R} \cap \Pi^+} \left( 1 - e^{2\pi i(\lambda, w)_{\mathbb{M}_2}} \right)^{(-1)^{\langle \lambda, -\mathbf{a}_1 - \mathbf{f}_1 \rangle} c(\lambda^2/2)}.
\]
That \( \Phi_2 \) converges absolutely when \((\text{Im } w)^2 \gg 0\) and extends holomorphically to the whole \( \mathbb{M}_2 \otimes \mathbb{R} + i \mathbb{C}_{M_2}^+ \). By [6, Sect. 3], \( \Phi_2 \circ j_2^{-1} \) is an automorphic form on \( \Omega_\Lambda^+ \) for \( O^+(\mathbf{A}) \) of weight 4 with zero divisor \( \mathcal{H} \). We call \( \Phi_2(w) \) the Borcherds \( \Phi \)-function with respect to the 0-dimensional cusp of level 2. Then we have an equality \( \Phi_2 = j_2^* \Phi \) of functions on \( \Omega_\Lambda^+ \). Indeed, there is a constant \( C^* \) with \(|C^*| = 1\) such that \( \Phi_2 = C^* j_2^* \Phi \) by [6, Th. 13.3 (5)]. Then \( C^* = 1 \) by comparing [13, (2.12)] with \( C = 2^8 \) and the formula (2.13).

2.3.5. Remarks. Since \( \Phi \) is a function on \( \mathbb{M}_2 \otimes \mathbb{R} + i \mathbb{C}_{M_2}^+ \), \( \Phi_1 = j_1^* \Phi \) and \( \Phi_2 = j_2^* \Phi \) are two distinct realizations of \( \Phi \in \mathcal{O}(C_{M_2}^+) \) as a function on the tube domain of a 10-dimensional affine space. For the precise relation between them, see [13, Sect. 2.2.3]. In [6, Ex. 13.7], Borcherds used the lattice \( \Lambda' \) (2) instead of \( \Lambda \). For this reason, the infinite product expansion with respect to the level \( \ell \) cusp in [6, Ex. 13.7] corresponds to that for \( \Phi_{2/\ell} \) with respect to the level \( 2/\ell \) cusp.

2.3.6. Automorphy of the Borcherds \( \Phi \)-function. Let \( \ell \in \{1,2\} \). By the automorphy of the Borcherds \( \Phi \)-function, for any \( g \in O^+(\mathbf{A}) \) and \( u \in \mathbb{M}_\ell \otimes \mathbb{R} + i \mathbb{C}_{M_\ell}^+ \), we have

\[
\Phi_\ell(g \cdot u) = \chi(g) J_\ell(g,u)^4 \Phi_\ell(u),
\]

where \( J_\ell(g,u) := \langle g(-(u^2/2)e_\ell + (f_\ell/\ell) + (-1)^{2/\ell}u), e_\ell \rangle_\Lambda \).

Since \( \chi^2 = 1 \) by [10] proof of Prop. 5.6 (see also [13, Lem. 2.1]), we have

\[
(2.14) \quad \Phi_\ell(g \cdot u)^2 = J_\ell(g,u)^8 \Phi_\ell(u)^2.
\]

We will use the following invariance of the square of the Borcherds \( \Phi \)-function.

Lemma 2.3. Let \( \ell \in \{1,2\} \). Let \( g \in O^+(\mathbf{A}) \) be such that \( g(e_\ell) = e_\ell \). Then \( \Phi_\ell(g \cdot u)^2 = \Phi_\ell(u)^2 \) for all \( u \in \mathbb{M}_\ell \otimes \mathbb{R} + i \mathbb{C}_{M_\ell}^+ \).

Proof. Since \( g^{-1}(e_\ell) = e_\ell \), we have

\[
J_\ell(g,u) = \langle g(-(u^2/2)e_\ell + (f_\ell/\ell) + (-1)^{2/\ell}u), e_\ell \rangle_\Lambda = \langle -(u^2/2)e_\ell + (f_\ell/\ell) + (-1)^{2/\ell}u, g^{-1}(e_\ell) \rangle_\Lambda = \langle -(u^2/2)e_\ell + (f_\ell/\ell) + (-1)^{2/\ell}u, e_\ell \rangle_\Lambda = (f_\ell/\ell, e_\ell)_\Lambda = 1.
\]

Now the result follows from (2.14).

2.3.7. The Petersson norm. Let \( \ell \in \{1,2\} \). The Petersson norm of \( \Phi_\ell \) is the \( C^\infty \) function on \( \mathbb{M}_\ell \otimes \mathbb{R} + i \mathbb{C}_{M_\ell}^+ \) defined as

\[
(2.15) \quad \| \Phi_\ell(u) \|^2 := (\text{Im } u, \text{Im } u)_{M_\ell}^4 |\Phi_\ell(u)|^2.
\]

Since \( \langle \text{Im } (g \cdot u), \text{Im } (g \cdot u) \rangle_{M_\ell} = |J_\ell(g,u)|^{-2} \langle \text{Im } u, \text{Im } u \rangle_{M_\ell} \), it follows from (2.14) that \( \| \Phi_\ell \|^2 \) is \( O^+(\mathbf{A}) \)-invariant, and \( \| \Phi_\ell \|^2 \) is regarded as a \( C^\infty \) function on \( \mathcal{M} \). Moreover, if \( j_1(u_1) = j_2(u_2) \), then \( |\Phi_1(u_1)| = |\Phi_2(u_2)| \). Hence it makes sense to define \( \| \Phi(Z) \| \) as \( |\Phi_1(u_1)| = |\Phi_2(u_2)| \) when \( Z = j_1(u_1) = j_2(u_2) \). Recall that \( \Phi \) is defined as a holomorphic function on \( \mathcal{C}(S) \). As a function on \( \mathcal{C}(S) \), we have \( \| \Phi(Z) \| = 2^{-4} \langle Z, Z \rangle_\Lambda |\Phi(Z)|^2 \). For an Enriques surface \( Y \), we define

\[
(2.16) \quad \| \Phi(Y) \| := \| \Phi(\overline{\mathbb{C}}(Y)) \|.
\]
3. Fixed-point-free involutions on Kummer surfaces of product type

In this section, we recall Kummer surfaces of product type and fixed-point-free involutions on them and their parities. Then we study some properties of the period maps according to their parities.

3.1. Kummer surfaces of product type and their periods. For \( \tau \in \mathfrak{H} \), we set

\[
E_\tau := \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}.
\]

For \((\tau, \tau') \in \mathfrak{H} \times \mathfrak{H}\), let \( f: \widetilde{E_\tau \times E_{\tau'}} \to E_\tau \times E_{\tau'} \) be the blowing-up of the points of order 2 of \( E_\tau \times E_{\tau'} \). Then the involution \([-1] \cdot (x) := -x\) on \( E_\tau \times E_{\tau'} \) induces an involution on \( E_\tau \times E_{\tau'} \), which is denoted by \([-1]\). The quotient

\[
K_{\tau, \tau'} = \text{Km}(E_\tau \times E_{\tau'}) := E_\tau \times E_{\tau'}/[-1]
\]

is a K3 surface called a Kummer surface of product type.

We consider the map

\[
\phi: H^1(E_\tau, \mathbb{Z}) \otimes H^1(E_{\tau'}, \mathbb{Z}) \to H^2(E_\tau \times E_{\tau'}, \mathbb{Z}) \xrightarrow{\rho \otimes f^*} H^2(K_{\tau, \tau'}, \mathbb{Z}),
\]

where the first map is given by the cup-product and \( \rho: E_\tau \times E_{\tau'} \to K_{\tau, \tau'} \) in the second map is the projection. We set

\[
K := \phi(H^1(E_\tau, \mathbb{Z}) \otimes H^1(E_{\tau'}, \mathbb{Z})) \subset H^2(K_{\tau, \tau'}, \mathbb{Z}).
\]

By [2, VIII, Proposition (5.1)], we have an isometry of lattices \( K \cong \mathbb{U}(2) \oplus \mathbb{U}(2) \).

Since \( H^0(K_{\tau, \tau'}, \Omega^2_{K_{\tau, \tau'}}) \subset K \otimes \mathbb{C} \), we have \( T_{K_{\tau, \tau'}} \subset K \). If \((\tau, \tau') \in \mathfrak{H} \times \mathfrak{H}\) is very general, then \( T_{K_{\tau, \tau'}} = K \).

Let \( a \) (resp. \( b \)) be the element of \( H^1(E_\tau, \mathbb{Z}) \) corresponding to the line segment \([0, \tau]\) (resp. \([0, \tau']\)) of \( \mathbb{C} \). Let \( a^\vee \) (resp. \( b^\vee \)) be the Poincaré dual of \( a \) (resp. \( b \)) such that \( \int_E \eta = \int_{E_\tau} a^\vee \wedge \eta \) and \( \int_E \eta = \int_{E_{\tau'}} b^\vee \wedge \eta \) for any closed 1-form \( \eta \) on \( E_\tau \).

Then \( \{a^\vee, b^\vee\} \) is a symplectic basis of \( H^1(E_\tau, \mathbb{Z}) \) such that \([dz] = a^\vee - \tau b^\vee\) in \( H^1(E_\tau, \mathbb{Z}) \). Similarly, we define a symplectic basis \( \{a^\vee \otimes b^\vee\} \) of \( H^1(E_\tau \times E_{\tau'}, \mathbb{Z}) \).

Following [13, Sects.6.1, 6.2, 7.3], we set

\[
\Gamma^\vee_{12} := \phi(a^\vee \otimes a^\vee), \quad \Gamma^\vee_{14} := \phi(b^\vee \otimes a^\vee), \\
\Gamma^\vee_{14} := \phi(a^\vee \otimes b^\vee), \quad \Gamma^\vee_{23} := -\phi(b^\vee \otimes a^\vee).
\]

Then \( \{\Gamma^\vee_{12}, \Gamma^\vee_{14}, \Gamma^\vee_{14}, \Gamma^\vee_{23}\} \) is a basis of \( K \) with Gram matrix \(-\binom{0}{2} \oplus -\binom{0}{2}\).

Suppose that we are given a fixed-point-free involution \( \iota: K_{\tau, \tau'} \to K_{\tau, \tau'} \).

Then it is anti-symplectic. Let \( H^2(K_{\tau, \tau'}, \mathbb{Z})_{\pm} \) be the \pm 1-eigenspace of \( H^2(K_{\tau, \tau'}, \mathbb{Z}) \) with respect to the \( \iota \)-action. We suppose moreover that \( \iota \) satisfies the condition

\[
K \subset H^2(K_{\tau, \tau'}, \mathbb{Z})_{\pm}.
\]

Since \( H^0(K_{\tau, \tau'}, \Omega^2_{K_{\tau, \tau'}}) \subset K \otimes \mathbb{C} \), there exist no roots \( d \in \Delta_{H^2(K_{\tau, \tau'}, \mathbb{Z})_{\pm}} \) with \( K \subset d^\perp \) by (3.3). Since \( \iota \) is anti-symplectic and thus \( T_{K_{\tau, \tau'}} \subset H^2(K_{\tau, \tau'}, \mathbb{Z})_{\pm} \), (3.3) holds if \( T_{K_{\tau, \tau'}} = K \). The geometric meaning of (3.3) is given as follows.
Lemma 3.1. Let $U$ be a neighborhood of $(\tau, \tau')$ in $\mathcal{F} \times \mathcal{F}'$. Let $\pi: \mathcal{K} \to U$ be a family of Kummer surfaces such that $\pi^{-1}(\sigma, \sigma') \cong K_{\sigma, \sigma'}$ for all $(\sigma, \sigma') \in U$. Let $\theta$ be a fixed-point-free involution on $K_{\tau, \tau'}$. If $U$ is sufficiently small and contractible, then the following conditions are equivalent.

1. $\theta$ satisfies (3.4).
2. There exists an involution $\theta_{\mathcal{K}}: \mathcal{K} \to \mathcal{K}$ preserving the fibers of $\pi$ such that $\theta_{\mathcal{K}}|_{K_{\tau, \tau'}} = \theta$.

Proof. The result follows from the global Torelli theorem. The details are left to the reader. (The fixed-point-free involutions that we will need satisfy the condition (3.4) (see Theorem 3.8, Sect. 4 and Sect. 5.1).)

Recall that we have fixed a primitive embedding $\Lambda \subset \mathbb{L}_{K3}$ throughout this paper and that $\{e_1, f_1\}$ (resp. $\{e_2, f_2\}$) is the standard basis of $U$ (resp. $U(2)$) of the middle (resp. the left) sublattice of $\Lambda := U(2) \oplus U(2) \oplus E_6(2)$.

Definition 3.2. For a fixed-point-free involution $\iota$ on $K_{\tau, \tau'}$ satisfying (3.4), let $\ell \in \{1, 2\}$ denote the level of $\Gamma'_{34}$ in $H^2(K_{\tau, \tau'}, \mathbb{Z})_-$. An isometry $\alpha : H^2(K_{\tau, \tau'}, \mathbb{Z}) \to \mathbb{L}_{K3}$ is called a normalized marking for $(K_{\tau, \tau'}, \iota)$ if it satisfies (2.1), (2.3) and

\begin{equation}
\alpha(\Gamma'_{34}) = e_\ell.
\end{equation}

Lemma 3.3. If (3.4) holds, then there exists a normalized marking $\alpha$ for $(K_{\tau, \tau'}, \iota)$.

Proof. Let $\alpha' : H^2(K_{\tau, \tau'}, \mathbb{Z}) \to \mathbb{L}_{K3}$ be a marking satisfying (2.1). By (3.4), $\Gamma'_{34} \in K \subset H^2(K_{\tau, \tau'}, \mathbb{Z})_-$. Then $\ell$ is the level of $\alpha'(\Gamma'_{34}) \in \Lambda$. Set $I := \alpha' \iota^* (\alpha')^{-1}$. Then $\Lambda$ is exactly the anti-invariant subspace of $\mathbb{L}_{K3}$ with respect to the $I$-action. Since the $O(\Lambda)$-orbit of a primitive isotropic vector of $\Lambda$ is determined by its level, there exists $g \in O(\Lambda)$ such that $g(\alpha'(\Gamma'_{34})) = -e_\ell$.

Replacing $g$ by $s \circ g$ where $s \in O(M_3)$ exchanges the components $C^3_{M_3}$ if necessary, we may assume $g \in O(\Lambda(1))$. By [19, Remark 1.15], there exists $\tilde{g} \in O(M_{K3})$ with $\tilde{g} I = I \tilde{g}$ such that $\tilde{g}|_\Lambda = g$. Then $\iota := \tilde{g} \circ \alpha'$ is a marking on $K_{\tau, \tau'}$ satisfying (2.1), (2.3).

Let $\alpha : H^2(K_{\tau, \tau'}, \mathbb{Z}) \to \mathbb{L}_{K3}$ be a normalized marking for $(K_{\tau, \tau'}, \iota)$. Then the period of a (normalized) marked Enriques surface $(K_{\tau, \tau'}/\iota, \alpha)$ is given by that of $(K_{\tau, \tau'}, \alpha)$, i.e.,

\begin{equation}
\varpi(K_{\tau, \tau'}/\iota, \alpha) = \left[ \alpha \left( \phi \left( H^0(E_{\tau'}, \Omega^1_{E_{\tau'}}) \otimes H^0(E_{\tau'\iota}, \Omega^1_{E_{\tau'\iota}}) \right) \right) \right] \in \Omega^1_{\Lambda}.
\end{equation}

By (3.6), (3.8) and the relation $[dz] = a^\vee - \tau b^\vee$, the period of $(K_{\tau, \tau'}, \alpha)$ is concretely expressed as follows (see [13, Sects. 7.3, 7.4]):

\begin{equation}
\varpi(K_{\tau, \tau'}/\iota, \alpha) = \left[ \tau \alpha'(\Gamma'_{34}) + \alpha(\Gamma'_{12}) + \tau \alpha(\Gamma'_{23}) - \tau' \alpha(\Gamma'_{14}) \right].
\end{equation}

We make a crucial observation on the value of Borcherds $\Phi$-function.

Lemma 3.4. Let $\iota$ be a fixed-point-free involution on $K_{\tau, \tau'}$ satisfying (3.4), and let $\ell$ be the level of $\Gamma'_{34}$ in $H^2(K_{\tau, \tau'}, \mathbb{Z})_-$. Let $\alpha, \alpha'$ be normalized markings for $(K_{\tau, \tau'}, \iota)$. We put $u := j^{-1}_e (\varpi(K_{\tau, \tau'}/\iota, \alpha))$ and $u' := j^{-1}_e (\varpi(K_{\tau, \tau'}/\iota, \alpha'))$. Then we have $\Phi_\ell(u)^2 = \Phi_\ell(u')^2$. In other words, the value $\Phi_\ell(u)^2$ is independent of the choice of a normalized marking.
Proof. We put \( g := \alpha' \circ \alpha^{-1} \). Then \( g \in O(\mathbb{L}_{K_3}) \) satisfies \( g(A) = A \), \( g(\Omega^1_A) = \Omega^1_A \) and \( g(e_\ell) = e_\ell \). It follows that \( g|_\tau \in O^+(A) \) and \( g|_\tau (e_\ell) = e_\ell \). Since \( j_\ell(u') = \varpi(K_{\tau,\tau'})/\iota, \alpha' = g(\varpi(K_{\tau,\tau'}/\iota, \alpha)) = g(j_\ell(u)) \), the definition of the action of \( O^+(A) \) on \( M_\ell \otimes \mathbb{R} + iC^+_{M_\ell} \) gives \( u' = g|_\tau \cdot u \). Hence the result follows from Lemma 2.3. \( \square \)

By [13, Sects. 6.4 and 7.4], we can express

\[ \varpi(K_{\tau,\tau'}, \alpha) = j_\ell(u) = \left[-\frac{u^2}{2}\right]e_\ell + (\ell/\ell) + (-1)2/\ell u_0, \]

where \( u = u(\tau, \tau'; \alpha) \in M_\ell \otimes \mathbb{R} + iC^+_{M_\ell} \) is given explicitly by

\[ (-1)^{2/\ell} u = \frac{1}{2}(A + B\tau + D\tau'), \quad \begin{cases} A := \alpha(\Gamma'_1) - (\ell/\ell, \alpha(\Gamma')) A e_\ell - (2/\ell) f_\ell, \\ B := -(f_\ell/\ell, \alpha(\Gamma'_1)) A e_\ell + (\ell/\ell, \alpha(\Gamma'_1)), \\ D := (f_\ell/\ell, \alpha(\Gamma'_1)) A e_\ell - (\ell/\ell, \alpha(\Gamma'_1)). \end{cases} \]

Remark 3.5. There are several isomorphisms in [13]. In [13], (6.7), Lemma 6.2 and its proof, and in Theorem 6.3 and its proof, \( z_{(\ell)}(T) \) should be replaced by \( (-1)^{2/\ell} z_{(\ell)}(T) \), so that \( 3z_{(\ell)}(T) \in C^+_{M_\ell} \). In the last line of [13, p.1509], the formula for \( A \) should be replaced by \( A = f^* - (2/\ell) f_\ell - (f_\ell/\ell, f^*) e_\ell \). For the same reason as above, in [13, p.1514], \( z_{(\ell)}(\tau_1, \tau_2) \) should be replaced by \( (-1)^{2/\ell} z_{(\ell)}(\tau_1, \tau_2) \). In [13], the proof of Lemma 7.1, \( B \) and \( D \) should be replaced by \( (-1)^{2/\ell} B \) and \( (-1)^{2/\ell} D \), respectively.

### 3.2. The parity of an involution.

For \( \tau, \tau' \in \delta \), let \( K_{\tau,\tau'} \) be a Kummer surface of product type, and let \( \iota \) be a fixed-point-free involution on \( K_{\tau,\tau'} \), satisfying [4.4].

We define the patching element \( d_1 \in A_K \setminus \{0\} \) as follows. For simplicity, write \( H^2_\tau \) for \( H^2(K_{\tau,\tau'}, \mathbb{Z}) \). Let \( K^\perp = K^\perp H^2_\tau \) be the orthogonal complement of \( K \) in \( H^2_\tau \).

By [21, Proof of Prop. 4.3], \( K^\perp \cong \mathbb{E}_6(2) \) and we have the following inclusions

\[ K \oplus K^\perp \subset H^2_\tau \subset (H^2_\tau)^\vee \subset K^\vee \oplus (K^\perp)^\vee. \]

Since \( \dim_{\mathbb{F}_2} A_K \oplus A_K \cong 12 \) and \( \dim_{\mathbb{F}_2} A_{H^2} = 10 \), we get \( \dim_{\mathbb{F}_2} H^2_\tau / (K \oplus K^\perp) = 1 \).

Hence there exists \( d \in H^2_\tau \setminus \{0\} \) such that

\[ H^2_\tau = Zd + K \oplus K^\perp. \]

We write \( d = d_1 + d_2 \) with \( d_1 \in K^\vee \) and \( d_2 \in (K^\perp)^\vee \). Since \( d \notin K \oplus K^\perp \), the primitivity of the embeddings \( K \subset H^2_\tau \) and \( K^\perp \subset H^2_\tau \) implies that \( d_1 \neq 0 \) and \( d_2 \neq 0 \). Since \( \delta(K) = \delta(K^\perp) = 0 \), we have \( d^2_1 \in Z \) and \( d^2_2 \in Z \). Since the lattice \( H^2_\tau \) is even and hence \( d^2 = 2Z \), the equality \( d^2 = d^2_1 + d^2_2 \) implies that \( d^2_1 \equiv d^2_2 \mod 2 \). Namely, \( q_{K}(d_1) = q_{K^\perp}(d_2) \in Z/2Z \). Since \( d \mod K \oplus K^\perp \) is determined by \( \iota \), the value \( q_{K}(d_1) = q_{K^\perp}(d_2) \in Z/2Z \) depends only on \( \iota \).

**Definition 3.6** (Patching element and parity). For a fixed-point-free involution \( \iota \) on \( K_{\tau,\tau'} \), satisfying [4.4], define the patching element of \( \iota \) as \( d_1 := d_1 \in A_K \setminus \{0\} \). Then \( \iota \) is said to be of odd (resp. even) type if \( q_{K}(d_1) = 1 \) (resp. \( q_{K}(d_1) = 0 \)).

Remark 3.7. The notion of patching element given in Definition 3.6 coincides with that of Ohashi [21, Def. 4.5]. To see it, write \( H^\pm_\tau \) and \( H^2_\tau \) for \( H^2(K_{\tau,\tau'}, \mathbb{Z}) \) and \( H^2(K_{\tau,\tau'}, \mathbb{Z}) \), respectively. Let \( S := K^\perp H^2_\tau \) be the orthogonal complement of \( K \) in \( H^2_\tau \cong \mathbb{L}_{K_3} \). Then \( S \cong U \oplus \mathbb{E}_8 \oplus \mathbb{D}^{\mathbb{Z}^2}_8 \) and we have \( K^\perp = (H^2_\tau)^{2-s} \), i.e., \( K^\perp \) is also given as the orthogonal complement of \( H^2_\tau \) in \( S \). There is a subgroup \( \Gamma \subset A_{H^2_\tau} \) with \( \dim_{\mathbb{F}_2} \Gamma = 7 \) such that \( q_S = q_{H^2_\tau} \circ q_{K^\perp} |_{\Gamma / \Gamma} \), where \( \Gamma / \Gamma \) is the orthogonal
complement of $\Gamma$ with respect to the discriminant bilinear from on $A_{H^2} \oplus A_{K^L}$ (cf. \cite{21} Th. 4.2 (1)). Let $\Gamma_{K^L}$ be the image of $\Gamma$ by the obvious projection. Since $\Gamma \cong \Gamma_{K^L}$ and $\dim_{\mathbb{F}_2} K^L = 8$, there is a unique vector $z \in A_{K^L}$ with $\Gamma_{K^L} = \mathbb{F}_2 z$. Then $v := [(0, z)] \in \Gamma^L/\Gamma = A_S$ is the patching element in \cite{21}. Since $(0, z)$ represents $v$ and since $q_{H^2} = -q_S \oplus q_{K^L} |_{\Gamma^L/\Gamma}$ for some subgroup $\Gamma^L \subset A_S \oplus A_{K^L}$ (cf. \cite{21} Th. 4.2 (2)), we see that the class of $(v, z) \in \Gamma^L \subset A_S \oplus A_{K^L}$ coincides with 0 in $A_{H^2}$. Since $(A_S, -q_S) = (A_K, q_K)$ and $(A_{H^2}, q_{H^2}) = (A_{H^2}, q_{H^2})$, this implies that $(v, z) \in H^2/K^{\perp} \oplus K^L$. Since $(v, z) \neq (0, 0)$ in $A_K \oplus A_{K^L}$, we get $H^2 = \mathbb{Z}(v, z) + K \oplus K^L$. Hence $v \in A_S = A_K$ is the patching element of $\iota$.

Mukai \cite{17} and Ohashi \cite{21} Th. 0.2] classified the conjugacy classes of fixed-point-free involutions on Kummer surfaces of product type satisfying \cite[34]{21}. For $n \in \mathbb{Z}_{>0}$, let $\Gamma(n) := \{(a, b, c) \in \text{SL}_2(\mathbb{Z}) ; a \equiv d \equiv 1, b \equiv c \equiv 0 \mod n\} \subset \text{SL}_2(\mathbb{Z})$ denote the principal congruence subgroup of level $n$. Let $\Delta_\tau$ be the diagonal locus of $\mathfrak{K} \times \mathfrak{K}$. We set

$$\mathcal{D} := \bigcup_{\gamma \in \Gamma(2)} (\gamma \times 1) \Delta_\tau = \bigcup_{\gamma \in \Gamma(2)} (1 \times \gamma) \Delta_\tau.$$ 

**Theorem 3.8** (\cite{17, 21}). For $\tau, \tau' \in \mathfrak{K}$, let $K_{\tau, \tau'}$ be a Kummer surface of product type and let $\omega_{K_{\tau, \tau'}}$ be its non-zero canonical form.

1. Let $(\tau, \tau') \in \mathfrak{K} \times \mathfrak{K} \setminus \mathcal{D}$. Then there exist 15 conjugacy classes of fixed-point-free involutions on $K_{\tau, \tau'}$ satisfying \cite{34} and there exists a bijection between the set of these 15 conjugacy classes of involutions and $A_K \setminus \{0\}$ given by the assignment $\iota \mapsto \mathfrak{d}_\iota$, where $\mathfrak{d}_\iota \in A_K \setminus \{0\}$ is the patching element of $\iota$.

2. If $(\tau, \tau')$ is very general, i.e., $T_{K_{\tau, \tau'}} = K$ and $\text{Aut}(T_{K_{\tau, \tau'}}, \omega_{K_{\tau, \tau'}}) = \{\pm 1\}$, then the 15 fixed-point-free involutions in (1) are, up to conjugacy, the only fixed-point-free involutions on $K_{\tau, \tau'}$.

We will recall geometric descriptions of these 15 involutions in Sect. \cite{11} and Sect. \cite{5}. Note that $A_K \setminus \{0\}$ consists of 6 elements with $x^2 = 1 \in \mathbb{Z}/2\mathbb{Z}$ and 9 elements $x$ with norm $x^2 = 0 \in \mathbb{Z}/2\mathbb{Z}$. Thus these 15 involutions are divided into 6 odd involutions and 9 even involutions.

By Lemma \cite{5} the following definition makes sense.

**Definition 3.9.** Let $\iota : K_{\tau, \tau'} \rightarrow K_{\tau, \tau'}$ be a fixed-point-free involution with \cite{34} and as in Theorem \cite{34} Let $\gamma = \mathfrak{d}_\iota \in A_K \setminus \{0\}$ be the patching element of $\iota$. Define

$$\Phi_\gamma(\tau, \tau')^2 := \Phi_\ell \left( j^{-1} (\omega(K_{\tau, \tau'})/\iota, \alpha) \right)^2,$$

where $\alpha$ is a normalized marking for $(K_{\tau, \tau'}, \iota)$ and $\ell$ is the level of $\Gamma_{34}^\vee$ in $H^2(K_{\tau, \tau'}, \mathbb{Z})$.

Let $\gamma \in A_K \setminus \{0\}$. As we see in Sections \cite{11} and \cite{5} below, there is a family of Kummer surfaces of product type $\pi : K \rightarrow \mathfrak{K} \times \mathfrak{K}$ and fixed-point-free involutions $\iota_\gamma : K|_{\mathfrak{K} \times \mathfrak{K} \setminus \mathcal{D}} \rightarrow K|_{\mathfrak{K} \times \mathfrak{K} \setminus \mathcal{D}}$ preserving the fibers of $\pi$ and satisfying \cite{34} such that the set of representatives of the 15 conjugacy classes of Theorem \cite{34} (1) is given by $\{\iota_\gamma|_{K_{\tau, \tau'}}\}_{\gamma \in A_K \setminus \{0\}}$. Since $\mathfrak{K} \times \mathfrak{K}$ is contractible, by choosing a reference point $(\tau_0, \tau'_0) \in \mathfrak{K} \times \mathfrak{K} \setminus \mathcal{D}$ and choosing a normalized marking $\alpha_{\tau_0}$ for $(K_{\tau_0, \tau'_0}, \iota_{\tau_0})$, we have a marking $\alpha_\gamma : R^2 \pi_* \mathbb{Z} \cong L_{K_{\tau, \tau'}}$ for $\pi : K \rightarrow \mathfrak{K} \times \mathfrak{K}$ extending $\alpha_{\tau_0}$ such that $\alpha_\gamma|_{K_{\tau, \tau'}}$ is a normalized marking for $(K_{\tau, \tau'}, \iota_\gamma)$ for all $(\tau, \tau') \in \mathfrak{K} \times \mathfrak{K} \setminus \mathcal{D}$. Write
\(\varpi_\gamma\) for the period mapping for \((\pi : \mathcal{K} \to \mathfrak{H} \times \mathfrak{H}, \alpha_\gamma)\): \(\varpi_\gamma(\tau, \tau') := \varpi(K_{\tau, \tau'}/\ell_\gamma, \alpha_\gamma)\). Then \(\varpi_\gamma\) is a holomorphic map from \(\mathfrak{H} \times \mathfrak{H}\) to \(\Omega_{\alpha_\gamma}(\mathbb{K})\) such that
\[
\Phi_\gamma(\tau, \tau')^2 = \Phi_\ell(j_{\ell}^{-1} \varpi_\gamma(\tau, \tau'))^2
\]
for all \((\tau, \tau') \in \mathfrak{H} \times \mathfrak{H} \setminus \mathfrak{D}\). Since \(\Phi_\gamma^2 \circ j_{\ell}^{-1} \circ \varpi_\gamma\) is a holomorphic function on \(\mathfrak{H} \times \mathfrak{H}\), so is \(\Phi_\gamma^2\).

3.3. Automorphy of \(\Phi_\gamma^2\). In Sect. 3.3, we prove that \(\Phi_\gamma^2\) is an automorphic form of weight 8 for the principal congruence subgroup of level 2 of \(\text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z})\). We keep the notation in Sect. 3.2. For \(\gamma \in A_{\mathbb{K}} \setminus \{0\}\), we set
\[
\mathbb{K}_\gamma := \alpha_\gamma(\mathbb{K}), \quad \mathbb{E}_\gamma := \alpha_\gamma(\mathbb{K}^{1/2}).
\]
Since \(H^1(E_\tau, \mathbb{Z})\) is endowed with the basis \(\{a^\vee, b^\vee\}\), \(\text{SL}_2(\mathbb{Z})\) acts on \(H^1(E_\tau, \mathbb{Z})\) by \(g \cdot (ma^\vee + nb^\vee) := (a^\vee, b^\vee)g(m)\) for \(g \in \text{SL}_2(\mathbb{Z})\). Since this action preserves the cup-product, so does the induced \(\text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z})\)-action on \(\mathbb{K} = \phi(H^1(E_\tau, \mathbb{Z}) \otimes H^1(E_{\tau'}, \mathbb{Z}))\).

We define a map \(\rho : \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \to O^+(\mathbb{K})\) as
\[
\rho(g, h) : \phi(u \otimes v) := \phi(tg^{-1} \cdot (u) \otimes t^{-1}h^{-1} \cdot (v))
\]
for any \(g, h \in \text{SL}_2(\mathbb{Z})\) and \(u \in H^1(E_\tau, \mathbb{Z}), v \in H^1(E_{\tau'}, \mathbb{Z})\). Then \(\rho\) is a group homomorphism such that the period map
\[
\varpi : \mathfrak{H} \times \mathfrak{H} \ni (\tau, \tau') \to [\tau \tau' \Gamma_{14}^\vee + \Gamma_{12}^\vee + \tau \Gamma_{23}^\vee + \tau' \Gamma_{14}^\vee] \in \Omega_{\mathbb{K}}^+
\]
is \(\text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z})\)-equivariant, i.e.,
\[
(\text{3.12}) \quad \rho(g, g') \cdot \varpi(\tau, \tau') = \varpi(g \cdot \tau, g' \cdot \tau'),
\]
for any \(g, g' \in \text{SL}_2(\mathbb{Z})\) and \(\tau, \tau' \in \mathfrak{H}\). To summarize, for any involution \(\iota_\gamma\) on \(\mathbb{K}_{\mathfrak{H}} \times \mathfrak{H} \setminus \mathfrak{D}\), satisfying (3.3) as in Theorem 3.3 under the identification \(\alpha_\gamma : \mathbb{K} \cong \mathbb{K}_\gamma\) via a normalized marking for \((\mathbb{K}, \iota_\gamma)\), \(\tilde{\rho} := \alpha_\gamma \rho \alpha_\gamma^{-1} : \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \to O^+(\mathbb{K}_\gamma)\) is a group homomorphism such that the period map \(\varpi_\gamma = \alpha_\gamma \circ \varpi : \mathfrak{H} \times \mathfrak{H} \to \Omega_{\mathbb{K}_\gamma}^+\) in (3.7) is \(\text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z})\)-equivariant.

However, this does not imply that the period map (3.7) extends to an \(\text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z})\)-equivariant holomorphic map from \(\mathfrak{H} \times \mathfrak{H}\) to \(\Omega^+_\Lambda\), because \(O^+(\mathbb{K}_\gamma)\) is not a subgroup of \(O^+(\Lambda)\).

Let \(\tilde{O}^+(\mathbb{K}_\gamma)\) be the kernel of the canonical homomorphism \(O^+(\mathbb{K}_\gamma) \to O(q_{\mathbb{K}_\gamma})\). Since \(g \otimes 1_{\mathbb{K}_\gamma} \in O^+(\mathbb{K}_\gamma \oplus \mathbb{E}_\gamma)\) preserves \(\Lambda\) for any \(g \in \tilde{O}^+(\mathbb{K}_\gamma)\) by the expression (3.10), we get the inclusion \(\tilde{O}^+(\mathbb{K}_\gamma) \ni g \mapsto g \otimes 1_{\mathbb{E}_\gamma} \in O^+(\Lambda)\).

Recall that \(\Gamma(2) \subset \text{SL}_2(\mathbb{Z})\) is the principal congruence subgroup of level 2. The image of \(\Gamma(2) \times \Gamma(2)\) under \(\tilde{\rho} : \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \to O^+(\mathbb{K}_\gamma)\) is contained in \(\tilde{O}^+(\mathbb{K}_\gamma)\). It follows that the period map \(\varpi : \mathfrak{H} \times \mathfrak{H} \to \Omega^+_\Lambda\), where the target space is now \(\Omega^+_\Lambda\), is \(\Gamma(2) \times \Gamma(2)\)-equivariant with respect to the homomorphism \(\Gamma(2) \times \Gamma(2) \to O^+(\Lambda)\).

**Lemma 3.10.** For any \(\gamma \in A_{\mathbb{K}} \setminus \{0\}\), \(\Phi_\gamma^2\) is an automorphic form on \(\mathfrak{H} \times \mathfrak{H}\) of weight 8 for \(\Gamma(2) \times \Gamma(2)\). Namely, the following functional equation holds for all \((g, g') = ((c, d), (c', d')) \in \Gamma(2) \times \Gamma(2)\) and \((\tau, \tau') \in \mathfrak{H} \times \mathfrak{H}:
\[
\Phi_\gamma(g \cdot \tau, g' \cdot \tau')^2 = (c \tau + d)^8(c' \tau' + d')^8 \Phi_\gamma(\tau, \tau')^2.
\]
Proof. Write \( \omega_\gamma(\tau, \tau') = \omega(K_{\tau, \gamma}, \tau, \tau') \). Let \( j_\ell(u) = \left[ -\left( u^2/2 \right) e_\ell + (f_\ell/e_\ell) + (-1)^{2/\ell} u \right] \), where \( \ell \) is the level of \( \Gamma_\gamma \) in \( H^2(K_{\tau, \gamma}, \mathbb{Z}) \). Then \( \Phi_\gamma(\tau, \tau')^2 = \Phi_\ell(u)^2 \) by the definition of \( \Phi_\gamma \). Let \( (g, g') \in \Gamma(2) \times \Gamma(2) \). By the \( \Gamma(2) \times \Gamma(2) \)-equivariance of \( \rho \), we get

\[
\Phi_\gamma(g \cdot \tau, g' \cdot \tau')^2 = \Phi_\ell(j_\ell^{-1} \omega_\gamma(g \cdot \tau, g' \cdot \tau'))^2 = \Phi_\ell(j_\ell^{-1} \rho(g, g') \omega_\gamma(\tau, \tau'))^2 = \Phi_\ell(\rho(g, g') \cdot u).
\]

By the automorphy of \( \Phi_\ell \) (see (3.14)), we get

\[
(3.13) \quad \Phi_\gamma(g \cdot \tau, g' \cdot \tau')^2 = J_\ell(\rho(g, g') \cdot u) = J_\ell(\rho(g, g') \cdot u) = J_\ell(\rho(g, g') \cdot u) = J_\ell(\rho(g, g') \cdot u) = J_\ell(\rho(g, g') \cdot u).
\]

Since \( \langle \tau \tau' \gamma(\Gamma_\gamma) + \alpha(\Gamma_\gamma) + \tau \alpha(\Gamma_\gamma) - \tau' \gamma(\Gamma_\gamma) + \alpha(\Gamma_\gamma) \rangle = -2 \) by (3.3), we get

\[
-(u^2/2)e_\ell + (f_\ell/e_\ell) + (-1)^{2/\ell} u = \frac{\tau \tau' \gamma(\Gamma_\gamma) + \alpha(\Gamma_\gamma) + \tau \alpha(\Gamma_\gamma) - \tau' \gamma(\Gamma_\gamma) + \alpha(\Gamma_\gamma)}{\langle \tau \tau' \gamma(\Gamma_\gamma) + \alpha(\Gamma_\gamma) + \tau \alpha(\Gamma_\gamma) - \tau' \gamma(\Gamma_\gamma) + \alpha(\Gamma_\gamma) \rangle} \Lambda
\]

where we used (3.7) to get the first equality. By (3.3), (3.5),

\[
J_\ell(\rho(g, g') \cdot u) = \langle \rho(g, g') \cdot u \rangle = \langle \rho(g, g') \cdot u \rangle = \langle \rho(g, g') \cdot u \rangle = \langle \rho(g, g') \cdot u \rangle.
\]

Write \( g = (\alpha, b, c) \), \( g' = (\alpha', b', c) \). By the definition of the \( \Gamma(2) \times \Gamma(2) \)-action, we get (3.14)

\[
J_\ell(\rho(g, g') \cdot u) = \left[ \phi \left( (\alpha, b) \cdot u \right) + (f_\ell/e_\ell) + (-1)^{2/\ell} u \right] \Lambda
\]

Substituting (3.14) into (3.13), we get the desired functional equation. \( \square \)

For \( u \in \mathbb{M}_e \otimes \mathbb{R} + i \mathbb{C}_e^+ \) and \( (\tau, \tau') \in \mathcal{S} \times \mathcal{S} \), we set

\[
B_{\mathcal{S}}(\tau, \tau') := \langle \text{Im} \tau, \text{Im} \tau' \rangle, \quad B_{\mathfrak{K}}(\tau, \tau') := \text{Im} \tau \cdot \text{Im} \tau'.
\]

We define the Petersson norm of \( \Phi_\gamma \) as

\[
\| \Phi_\gamma(\tau, \tau') \|^2 := \langle \text{Im} \tau \cdot \text{Im} \tau' \rangle^4 \| \Phi_\gamma(\tau, \tau') \|^2.
\]

When \( u = j_\ell^{-1} \omega_\gamma(\tau, \tau') \in \mathbb{M}_e \otimes \mathbb{R} + i \mathbb{C}_e^+ \) is given by (3.3), since \( \langle B, B \rangle = \langle D, D \rangle = 0 \) and \( \langle B, D \rangle = 2 \) by (3.3), we have \( \{ \text{Im} u, \text{Im} u \} = \text{Im} \tau \cdot \text{Im} \tau' \). By (2.13), we have

\[
(3.15) \quad \| \Phi_\ell(\omega_\gamma(\tau, \tau')) \|^2 = \langle \text{Im} \tau \cdot \text{Im} \tau' \rangle^4 \| \Phi_\ell(j_\ell^{-1} \omega_\gamma(\tau, \tau')) \|^2 = \| \Phi_\gamma(\tau, \tau') \|^2.
\]

By (3.15) and the \( O^\oplus(\mathfrak{A}) \)-invariance of \( \| \Phi_\gamma \|^2 \) or by the automorphy of \( \Phi_\gamma \), \( \| \Phi_\gamma \|^2 \) is a \( \Gamma(2) \times \Gamma(2) \)-invariant \( C^\infty \) function on \( \mathcal{S} \times \mathcal{S} \).

3.4. The period map and the discriminant locus. In this subsection, we study those involutions \( \iota \) in Theorem 3.3 with patching element \( \gamma \in \mathfrak{A}_\mathfrak{K} \setminus \{ 0 \} \) and satisfying \( \mathcal{H} \cap \Omega_{\mathfrak{K}}^+ \neq \emptyset \). Further, when \( \mathcal{H} \cap \Omega_{\mathfrak{K}}^+ \neq \emptyset \), we determine which \( d \in \Delta_\mathfrak{K} \) satisfies \( \mathcal{H} \cap \Omega_{\mathfrak{K}}^+ = \mathcal{H} \cap \Omega_{\mathfrak{K}}^- \). In (3.10), we write

\[
\mathfrak{A} = \mathbb{Z}(d_1 + d_2) + \mathbb{K}_\gamma \otimes \mathbb{E}_\gamma \subset \mathbb{K}_\gamma \otimes \mathbb{E}_\gamma, \quad d_1 + d_2 \in \mathbb{E}_\gamma \subseteq \mathbb{K}_\gamma, \quad d_2 \in \mathbb{E}_\gamma \setminus \mathbb{K}_\gamma.
\]

Since \( \mathbb{K}_\gamma \) and \( \mathbb{E}_\gamma \) are 2-elementary, we have \( 2d_1 \in \mathbb{K}_\gamma \) and \( 2d_2 \in \mathbb{E}_\gamma \).
Lemma 3.11. Let $\delta \in \Delta_\Lambda$ be any root of $\Lambda$. If $\delta = \delta_{K_\gamma} + \delta_{E_\gamma} \in K_\gamma^\vee \oplus E_\gamma^\vee$ is the orthogonal decomposition, then $\delta_{K_\gamma} - d_1 \in K_\gamma$ and $\delta_{E_\gamma} - d_2 \in E_\gamma$. In particular, $\Lambda = Z\delta + K_\gamma \oplus E_\gamma$.

Proof. Since $\delta \in \Lambda$, we can write $\delta = m(d_1 + d_2) + k + e$, where $m \in Z$, $k \in K_\gamma$ and $e \in E_\gamma$. Then $\delta_{K_\gamma} = md_1 + k$ and $\delta_{E_\gamma} = md_2 + e$. Suppose that $\delta_{K_\gamma} \in K_\gamma$. Then $m$ is even whence $\delta_{K_\gamma} \in E_\gamma$. It follows from the isometries $K_\gamma \cong U(2) \oplus U(2)$ and $E_\gamma \cong E_8(2)$ that $\delta_{K_\gamma}^2 \in 4Z$ and $\delta_{E_\gamma}^2 \in 4Z$. This contradicts the equality $-2 = \delta^2 = \delta_{K_\gamma}^2 + \delta_{E_\gamma}^2$. Thus $\delta_{K_\gamma} \not\in K_\gamma$, and $m$ is odd. This proves the result. $\square$

For $\delta \in \Delta_\Lambda$, we write $\Lambda = Z(\delta_{K_\gamma} + \delta_{E_\gamma}) + K_\gamma \oplus E_\gamma \subset K_\gamma^\vee \oplus E_\gamma^\vee$, $\delta_{K_\gamma} \in K_\gamma^\vee \setminus K_\gamma$, $\delta_{E_\gamma} \in E_\gamma^\vee \setminus E_\gamma$.

Lemma 3.12. If $i$ is of odd type with patching element $\gamma$, then there exists $\delta \in \Delta_\Lambda$ such that $\delta_{K_\gamma}^2 = -1$ and $\delta_{E_\gamma}^2 = -1$.

Proof. Any element of $A_{U(2) \oplus U(2)}$ of odd norm is represented by one of the following vectors of $(U(2) \oplus U(2))^\vee$ of norm $-1$

\[
(\begin{array}{c}
1/2, -1/2, 0, 0, 1/2, -1/2, 1/2, 0, 1/2, -1/2, 0, 1/2, -1/2, 0, 1/2, -1/2, 1/2, -1/2
\end{array})
\]

Thus there exists $d_1' \in K_\gamma^\vee$ such that $d_1'^2 = -1$ and $d_1' - d_1 \in K_\gamma$. Similarly, since any element of $A_{E_8(2)}$ of odd norm is represented by a vector of $E_8(2)$ of norm $-1$ by [11, Lemma 1.4, Cor. 1.5]), there exists $d_2' \in E_\gamma^\vee$ such that $d_2'^2 = -1$ and $d_2' - d_2 \in E_\gamma$. We set $\delta := d_1' + d_2'$. It follows from $\delta - (d_1 + d_2) \in K_\gamma \oplus E_\gamma$ that $\delta \in \Lambda$. Further, $\delta^2 = d_1'^2 + d_2'^2 = -2$, so $\delta \in \Delta_\Lambda$. $\square$

Proposition 3.13. The following hold:

1. If $i$ is an involution of odd type with patching element $\gamma$, then $H \cap \Omega_{K_\gamma}^+ \neq \emptyset$.
2. If $i$ is an involution of even type with patching element $\gamma$, then $H \cap \Omega_{K_\gamma}^+ = \emptyset$.

Proof. (1) It suffices to prove the existence of $\delta \in \Delta_\Lambda$ with $H_\delta \cap \Omega_{K_\gamma}^+ \neq \emptyset$. Since $i$ is an involution of odd type, we take $\delta \in \Delta_\Lambda$ as in Lemma 3.12. Since $\delta \in \Delta_\Lambda$, we get $H_\delta \cap \Omega_{K_\gamma}^+ = H_{\delta_{K_\gamma}} \cap \Omega_{K_\gamma}^+ \neq \emptyset$, where the non-emptiness follows from $\delta_{K_\gamma}^2 = -1 < 0$ (see (2)). This proves (1).

(2) To derive a contradiction, we assume that there exists a root $\delta \in \Delta_\Lambda$ with $H_\delta \cap \Omega_{K_\gamma}^+ \neq \emptyset$. Since $i$ is of even type, we get $\delta_{K_\gamma}^2 \in 2Z$. Since $H_\delta \cap \Omega_{K_\gamma}^+ \neq \emptyset$, we have $\delta_{K_\gamma}^2 < 0$ (see (2)). On the other hand, since $E_\gamma \cong E_8(2)$ is negative-definite, we have $\delta_{E_\gamma}^2 \leq 0$. It then follows from the equality $-2 = \delta^2 = \delta_{K_\gamma}^2 + \delta_{E_\gamma}^2$ that $-2 \leq \delta_{K_\gamma}^2 < 0$. Thus $\delta_{K_\gamma}^2 = -2$ and $\delta_{E_\gamma}^2 = 0$. Since $E_\gamma \cong E_8(2)$ is negative-definite, we get $\delta_{E_\gamma} = 0$, so $\delta = \delta_{K_\gamma}$. The primitivity of the embedding $K_\gamma \subset \Lambda$ then gives $\delta \in \Delta_\Lambda$. Since $K_\gamma \cong U(2) \oplus U(2)$, this contradicts the fact $\Delta_{U(2) \oplus U(2)} = \emptyset$. $\square$

Let $\mu \colon Km(E \times E') \to Km(E \times E')$ be the anti-symplectic holomorphic involution induced from the one on $E \times E'$ defined as $(x, y) \mapsto (-x, y)$. (We remark that $\mu$ is not fixed-point-free.) Let $I_\mu \in O(LK_3)$ be the involution defined as $I_\mu = \alpha \mu^* \alpha^{-1}$. By [17] Prop. 6.1, we get the following:

(3.16) $K_\gamma = \{ \lambda \in LK_3 \mid I_\mu(\lambda) = -\lambda \}$, $K_\gamma^{\pm-K3} = \{ \lambda \in LK_3 \mid I_\mu(\lambda) = \lambda \}$. 
Proposition 3.14. Suppose that \( \nu \) is an involution of odd type with patching element \( \gamma \).

Let \( d, \delta \in \Delta_\Lambda \) with \( \delta_\Lambda^2 < 0 \). Then

\[
H_d \cap \Omega_{\Lambda, \gamma}^+ = H_\delta \cap \Omega_{\Lambda, \gamma}^+ \quad \text{if and only if} \quad d \in \{ \pm \delta, \pm I_\delta(\delta) \}.
\]

Proof. In the proof, for simplicity, we write \( \mathbb{K} \) (resp. \( \mathbb{E} \)) for \( K_\gamma \) (resp. \( E_\gamma \)). We write \( d = d_G + d_E \) and \( \delta = \delta_G + \delta_E \), where \( d_G, \delta_G \in \mathbb{K}^\vee \) and \( d_E, \delta_E \in \mathbb{E}^\vee \). Note that, since \( \delta_\Lambda^2 < 0 \), we have \( \delta^\perp \cap \Omega_{\Lambda}^+ = \emptyset \) (see (2.5)). First we show the "if" part. We assume \( d \in \{ \pm \delta, \pm I_\delta(\delta) \} \). Since \( I_\mu(\delta) = -\delta_G + \delta_E \), we get \( H_\delta \cap \Omega_{\Lambda}^+ = H_\delta \cap \Omega_{\Lambda, \gamma}^+ = H_{I_\mu(\delta)} \cap \Omega_{\Lambda}^+ \).

Since \( H_{-\delta} = H_\delta \) and \( H_{I_\mu(\delta)} = H_{-I_\mu(\delta)} \), we obtain the assertion.

To prove the "only if" part, assume \( d \neq \pm \delta \) and \( d^\perp \cap \Omega_{\Lambda}^+ = \delta^\perp \cap \Omega_{\Lambda}^+ \). Then \( \mathbb{Z}d + \mathbb{Z}\delta \subset \Lambda \) is a sublattice of rank 2.

Step 1. We show \( \langle d, \delta \rangle = 0 \). Since \( H_d \cap H_\delta \cap \Omega_{\Lambda}^+ = H_\delta \cap \Omega_{\Lambda}^+ \neq \emptyset \), the sublattice \( \mathbb{Z}d + \mathbb{Z}\delta \subset \Lambda \) is negative-definite by (2.5). Thus \( \langle d, \delta \rangle^2 < 0 \), so \( -2 < \langle d, \delta \rangle < 2 \). Since \( \langle d, \delta \rangle \in \mathbb{Z}d \) for all \( d, \delta \in \Delta_\Lambda \) (cf. [21] proof of Th. 4.7), we get \( \langle d, \delta \rangle = 0 \).

Step 2. We show \( d_\delta = \pm \delta \). Since \( H_d \cap \Omega_{\Lambda}^+ = \Omega_{\Lambda}^+ \delta_G \cap \Omega_{\Lambda} = \Omega_{\Lambda}^+ \delta_G \cap \Omega_{\Lambda} \neq \emptyset \) by the assumption. Hence \( d_G = t \delta_G \) for some \( t \in \mathbb{Q} \).

Since \( \Omega_{\Lambda}^+ \delta_G \cap \Omega_{\Lambda} 
eq \emptyset \), we get \( d_G^2 < 0 \) and \( \delta_G^2 < 0 \) by (2.5). Since \( q_\Lambda \) is \( \mathbb{Z}/2\mathbb{Z} \)-valued, we have \( d_G^2 \in \mathbb{Z}_{< 0} \) and \( \delta_G^2 \in \mathbb{Z}_{< 0} \).

On the other hand, it follows from \( \Delta_\Lambda = \emptyset \) that \( d_G \neq 0 \) and \( \delta_G \neq 0 \). Since \( \mathbb{E} \) is negative-definite, we get \( d_G^2 < 0 \) and \( \delta_G^2 < 0 \). By the conditions \(-2 = d^2 = (d_G)^2 + \langle d_G, \delta_G \rangle + \delta_G^2 \in \mathbb{Z}_{< 0} \), we get \( d_G^2 = d_E^2 = -1 \). Similarly, \( \delta_G^2 = \delta_E^2 = -1 \).

Since \( d_G = t \delta_G \), we get \( d_G = \pm \delta_G \) by the equality \( d_G^2 = \delta_G^2 = -1 \).

Step 3. We show \( d_\delta = \pm \delta_G \). Indeed, since \( 0 = \langle d, \delta \rangle = \langle d_G, \delta_G \rangle + \langle d_E, \delta_E \rangle \) and \( d_G = \pm \delta_G \), we get \( \langle d_G, \delta_G \rangle = \mp d_G^2 = \mp 1 \) and thus

\[
\begin{pmatrix}
\langle d_G, \delta_G \rangle \\
\delta_G^2
\end{pmatrix} = \begin{pmatrix}
-1 & \pm 1 \\
\pm 1 & -1
\end{pmatrix}.
\]

If \( d_G \neq \pm \delta_G \), then \( \mathbb{Q}d_G + \mathbb{Q}\delta_G \subset \mathbb{E} \otimes \mathbb{Q} \) is a 2-dimensional subspace, which is not negative-definite by (3.17). This contradicts the fact that \( \mathbb{E} \) is negative-definite. Hence \( d_G = \pm \delta_G \). Since we assume \( d \neq \pm \delta \), we get \( d = \pm I_\delta(\delta) \) (see (3.10)). This proves \( d = \pm I_\delta(\delta) \).

\[
\square
\]

4. INVOLUTIONS OF EVEN TYPE

In this section, we consider involutions of even type on \( K_{\tau, \tau'} \). The main result of this section is Theorem 4.1 below, which is essentially shown in [13 Cor. 7.6] as a consequence of an algebraic expression of the Borcherds \( \Phi \)-function. For details of this section, we refer the reader to [13 Sect. 7.2].

For \( \tau \in \mathfrak{H} \), let \( \theta_3(\tau) := \sum_{n \in \mathbb{Z}} e^{\pi i (n + \frac{1}{2})^2} \), \( \theta_3(\tau) := \sum_{n \in \mathbb{Z}} e^{\pi in^2} \) be the theta constants. For \( \tau, \tau' \in \mathfrak{H} \), we set

\[
\lambda := \theta_3(\tau)^4 / \theta_3(\tau')^4, \quad \lambda' := \theta_3(\tau')^4 / \theta_3(\tau')^4.
\]

Let \( \mathcal{X} \to \mathfrak{H} \times \mathfrak{H} \) be the family of surfaces over \( \mathfrak{H} \times \mathfrak{H} \) defined as

\[
\mathcal{X} := \left\{ (x, (\tau, \tau')) \in \mathbb{P}^5 \times \mathfrak{H} \times \mathfrak{H} \mid \begin{array}{c}
(1 - \lambda) x_1^2 + \lambda x_2^2 - x_3^2 = 0, \\
\lambda' x_1^2 - \lambda x_2^2 - x_3^2 + x_5^2 = 0, \\
x_1^2 - x_2^2 - x_3^2 + x_5^2 = 0
\end{array} \right\}.
\]
Let $X_{\lambda,\lambda'}$ be its fiber over $(\tau,\tau') \in \mathfrak{S} \times \mathfrak{S}$. Then $\Sigma = \{ (0 : 0 : 0 : 1 : \pm 1), (1 : \pm 1 : \pm 1 : 0 : 0) \}$ is the singular locus of $X_{\lambda,\lambda'}$, which consists of ordinary double points, and the minimal resolution $\tilde{X}_{\lambda,\lambda'}$ of $X_{\lambda,\lambda'}$ is isomorphic to the Kummer surface $K_{\tau,\tau'} = \text{Km}(E_\tau \times E_{\tau'})$. Let $K \to X$ be the blowing-up of $X \subset \mathbb{P}^5 \times \mathfrak{S} \times \mathfrak{S}$ along the loci $\Pi_{p \in \Sigma} \{ p \} \times \mathfrak{S} \times \mathfrak{S}$ and let $\tau : K \to \mathfrak{S} \times \mathfrak{S}$ be the obvious projection. By construction, $\tau : K \to \mathfrak{S} \times \mathfrak{S}$ is a family of Kummer surfaces such that $\tau^{-1}(\tau,\tau') \cong K_{\tau,\tau'}$ for all $(\tau,\tau') \in \mathfrak{S} \times \mathfrak{S}$.

For $J = \{ j_1, j_2, j_3 \} \subset \{ 1, \ldots, 6 \}$ with $j_1 < j_2 < j_3$, let $\langle J \rangle$ denote the partition $\{ 1, \ldots, 6 \} = J \sqcup J'$. We write $J' = \{ j_4, j_5, j_6 \}$ with $j_4 < j_5 < j_6$. Note that $\langle J \rangle = \langle J' \rangle$. We also denote $\langle J \rangle$ by $(j_1j_2j_3)$. For a partition $\langle J \rangle = (j_1j_2j_3)$, we define the involution $\iota_{\langle J \rangle}$ on $\mathbb{P}^5$ as

\[ \iota_{\langle J \rangle}(x_{j_1}, x_{j_2}, x_{j_3}, x_{j_4}, x_{j_5}, x_{j_6}) := (x_{j_1}, x_{j_2}, x_{j_3}, -x_{j_4}, -x_{j_5}, -x_{j_6}). \]

Then $\iota_{\langle J \rangle}$ acts on $X_{\lambda,\lambda'}$. For $\langle J \rangle \neq (\frac{123}{2456})$, $\iota_{\langle J \rangle}$ is fixed-point-free on $X_{\lambda,\lambda'}$. Since $K_{\tau,\tau'} = \tilde{X}_{\lambda,\lambda'}$, the involution $\iota_{\langle J \rangle}$ acts on $K$ and preserves the fibers of $\pi$. By Lemma 3.1, $\iota_{\langle J \rangle}$ satisfies (3.3). By [13] Sect. 7.2, the involution on $K_{\tau,\tau'}$ induced by $\iota_{\langle J \rangle}$ is an involution of even type, which is again denoted by $\iota_{\langle J \rangle}$, and these 9 involutions $\{ \iota_{\langle J \rangle} \}_{\langle J \rangle \neq (\frac{123}{2456})}$ give the 9 conjugacy classes of involutions of even type on $K_{\tau,\tau'}$ (cf. Theorem 3.8). Since $\iota_{\langle J \rangle}$ satisfies (3.3), we define $\ell(J) \in \{ 1, 2 \}$ as the level of the primitive isotropic vector $\Gamma_{34}^\vee$ in $H^2(K_{\tau,\tau'}, \mathbb{Z})$.

By Lemma 3.10 $K_{\tau,\tau'}$ admits a normalized marking. By the triviality of the local system $R^2\pi_*\mathbb{Z}$, this marking extends to the one for the family $\pi : K \to \mathfrak{S} \times \mathfrak{S}$, which is fiberwise normalized with respect to $\iota_{\langle J \rangle}$. Let $\alpha_{\langle J \rangle} : R^2\pi_*\mathbb{Z} \cong \mathbb{L}_{K,3}$ be a normalized marking obtained in this way. We set $\mathbb{K}_{\langle J \rangle} := \mathbb{L}_{\iota_{\langle J \rangle}} = \alpha_{\langle J \rangle}(\mathbb{K})$. Then $\Omega_{\mathbb{K}_{\langle J \rangle}} = \{ \omega \in \Omega_{\mathbb{X}}^\vee \mid \omega \in \mathbb{K}_{\langle J \rangle} \otimes \mathbb{C} \}$ is the period domain for the marked family of Enriques surfaces $(\pi : \tilde{X} / \iota_{\langle J \rangle} \to \mathfrak{S} \times \mathfrak{S}, \alpha_{\langle J \rangle})$. Let

\[ \varpi_{\langle J \rangle} : \mathfrak{S} \times \mathfrak{S} \to \Omega_{\mathbb{K}_{\langle J \rangle}} \]

be its period map. Let $\gamma \in A_{\mathbf{K}} \setminus \{ 0 \}$ be the patching element of $\iota_{\langle J \rangle}$. Then

\[ \Phi_\gamma(\tau,\tau') = \Phi_{\iota_{\langle J \rangle}} \left( j_{\iota_{\langle J \rangle}}^{-1}(\varpi_{\langle J \rangle}(\tau,\tau')) \right)^2 = \Phi_{\iota_{\langle J \rangle}} \left( j_{\iota_{\langle J \rangle}}^{-1}(\varpi_{\langle J \rangle}(K_{\tau,\tau'}/\iota_{\langle J \rangle}, \alpha_{\langle J \rangle})) \right)^2 \]

by Definition 3.3. By Lemma 3.3 (2), $\Phi_\gamma^2$ is independent of the choice of a normalized marking $\alpha_{\langle J \rangle}$ and is a holomorphic function on $\mathfrak{S} \times \mathfrak{S}$, which is an automorphic form for $\Gamma(2) \times \Gamma(2)$ of weight 8 by Lemma 3.10. By [13], we have the following

**Theorem 4.1.** For any $(\tau,\tau') \in \mathfrak{S} \times \mathfrak{S}$, we have

\[ \prod_{\gamma \text{ even}} \Phi_\gamma(\tau,\tau')^2 = \prod_{\langle J \rangle \neq (\frac{123}{2456})} \Phi_{\iota_{\langle J \rangle}} \left( j_{\iota_{\langle J \rangle}}^{-1}(\varpi_{\langle J \rangle}(\tau,\tau')) \right)^2 = 2^{96} \eta^{144}(\tau) \eta^{144}(\tau'). \]

**Proof.** The result follows from [13 Cor. 7.6].

\[ \square \]

5. **Involutions of Odd Type**

In this section, for involutions of odd type, we give their geometric realizations and study the extended period map. Then we study behavior of the Borcherds $\Phi$-function along the boundary of the extended period map.
5.1. Realization of the involutions of odd type. For \( \lambda \in \mathbb{C} \setminus \{0, 1\} \), let \( E(\lambda) \) be the elliptic curve defined as the double covering of \( \mathbb{P}^1 \) with ordered four branch points \((0, 1, \lambda, \infty)\). Let \( \lambda, \lambda' \in \mathbb{C} \setminus \{0, 1\} \) and set \( p_1 := (0, 0), p_2 := (1, 1), p_3 := (\lambda, \lambda'), p_4 := (\infty, \infty) \in \mathbb{P}^1 \times \mathbb{P}^1 \). Then \( \mathbb{P}^1 \times \mathbb{P}^1 \) with ordered four points \( p_1, p_2, p_3, p_4 \) is isomorphic to the quadric \( Q_{\lambda, \lambda'} \) of \( \mathbb{P}^4 \) with ordered four points \((1 : 0 : 0 : 0), (0 : 1 : 0 : 0), (0 : 0 : 1 : 0), (0 : 0 : 0 : 1)\), where \( Q_{\lambda, \lambda'} \) is defined by the equation
\[
(5.1) \quad \lambda' x_2 x_3 + x_1 x_3 + (1 - \lambda)(1 - \lambda') x_1 x_2 + x_4 (x_1 + x_2 + x_3) = 0.
\]
When \( \lambda \neq \lambda' \), \( Q_{\lambda, \lambda'} \) is a non-singular quadric. When \( \lambda = \lambda' \), \( Q_{\lambda, \lambda'} \) is a singular quadric with a unique ordinary double point.

Let \( W \) be the subvariety of \( \mathbb{P}^1(1 : 1 : 1 : 2) \times (\mathbb{C} \setminus \{0, 1\})^2 \) defined by the equations \((5.1) \) and \( w^2 = x_1 x_2 x_3 x_4 \), which is endowed with the obvious projection \( W \to (\mathbb{C} \setminus \{0, 1\})^2 \). Let \( W_{\lambda, \lambda'} \) be the fiber of \( W \) over \((\lambda, \lambda') \in (\mathbb{C} \setminus \{0, 1\})^2 \). Then \( W_{\lambda, \lambda'} \) is the double covering of \( Q_{\lambda, \lambda'} \) with branch divisor \( x_1 x_2 x_3 x_4 = 0 \). When \( \lambda \neq \lambda' \), \( \text{Sing } W_{\lambda, \lambda'} \) consists of four \( D_4 \)-singularities. When \( \lambda = \lambda' \), \( \text{Sing } W_{\lambda, \lambda'} \) consists of four \( D_4 \)-singularities and two \( A_1 \)-singularities. As is easily verified, we can take a simultaneous resolution of \( W \to (\mathbb{C} \setminus \{0, 1\})^2 \), which we write \( \pi : \mathcal{Y} \to (\mathbb{C} \setminus \{0, 1\})^2 \).

We set \( Y_{\lambda, \lambda'} := \pi^{-1}(\lambda, \lambda') \). When \( \lambda \neq \lambda' \), we have \( Y_{\lambda, \lambda'} \cong \text{Km}(E(\lambda) \times E(\lambda')) \) by \(17\) Lemma 11. Thus for any \((\lambda, \lambda') \in (\mathbb{C} \setminus \{0, 1\})^2 \), the weak Torelli theorem implies that \( Y_{\lambda, \lambda'} \cong \text{Km}(E(\lambda) \times E(\lambda')) \).

Let \( Z \subset (\mathbb{C} \setminus \{0, 1\})^2 \) be the diagonal locus. Let \( \iota \) be the involution on \( \mathcal{Y} |_{\mathbb{C} \times \{0, 1\}^2 \setminus \mathbb{Z}} \) induced by the rational involution on \( W |_{\mathbb{C} \times \{0, 1\}^2 \setminus \mathbb{Z}} \)
\[
(5.2) \quad (x, w) \mapsto \left( \frac{\lambda'}{x_1}, \frac{1}{x_2}, \frac{(1 - \lambda)(1 - \lambda')}{x_3}, \frac{\lambda\lambda'(1 - \lambda)(1 - \lambda')}{x_4}, \frac{\lambda\lambda'(1 - \lambda)(1 - \lambda')}{w} \right).
\]

Then \( \iota \) preserves the fibers of \( \pi \). By \(17\) Lemma 16 (2), \( \iota \) is an involution of odd type in the sense of Definition \(5.6\). By Lemma 3.11, \( \iota \) satisfies \(3.1\).

Remark 5.1. When \( \lambda = \lambda' \), it follows from \(6.2\) that \( \iota |_{W_{\lambda, \lambda}} \) has two fixed points, i.e., the two \( A_1 \)-singularities of \( W_{\lambda, \lambda} \) lying over \( \text{Sing } Q_{\lambda, \lambda} \). Hence the involution on \( Y_{\lambda, \lambda} \) induced by the rational involution \((5.2) \) has non-empty fixed locus consisting of two disjoint \(-2\)-curves. This implies that \( \iota \) does not lift to an involution on \( \mathcal{Y} \).

Recall that \( E(\lambda) \) is endowed with the ordered four points \((0, 1, \lambda, \infty)\). We set \( z_1 = 0, z_2 = 1, z_3 = \lambda \). It is classical that if we change the order of the points \( z_1, z_2, z_3, z_4 \) by a permutation \( \varphi \in S_3 \), then the pair \((E(\lambda), (z_{\varphi(1)}, z_{\varphi(2)}, z_{\varphi(3)}, \infty)) \) is isomorphic to \((E(\varphi(\lambda)), (0, 1, \varphi(\lambda), \infty)) \), where \( \varphi(\lambda) := (z_{\varphi(3)} - z_{\varphi(1)})/(z_{\varphi(2)} - z_{\varphi(1)}) /\) is a linear fractional transformation preserving \( \mathbb{C} \setminus \{0, 1\} \). Since \( E(\varphi(\lambda)) \cong E(\lambda) \), there is an isomorphism \( \psi_{\varphi, \varphi'} : Y_{\varphi(\lambda), \varphi'(\lambda')} \to \text{Km}(E(\lambda) \times E(\lambda')) \), so that \( \psi_{\varphi, \varphi'} \circ \psi_{\varphi', \varphi}^{-1} \) is a fixed-point-free involution on \( \text{Km}(E(\lambda) \times E(\lambda')) \) when \( \varphi(\lambda) \neq \varphi'(\lambda) \). By \(5.4, 5.2\), we easily see that \((W_{\varphi(\lambda), \varphi'(\lambda')}, \iota) \cong (W_{\lambda, \lambda'}, \iota) \) for all \( \sigma \in S_3 \). Hence we have
\[
(5.3) \quad (Y_{\varphi(\lambda), \varphi'(\lambda')}, \iota) \cong (Y_{\lambda, \lambda'}, \iota) \quad (\forall \lambda, \lambda' \in \mathbb{C} \setminus \{0, 1\}, \forall \sigma \in S_3).
\]
So we have the following isomorphism for all \( \varphi, \varphi', \sigma \in S_3 \):
\[
(\text{Km}(E(\lambda) \times E(\lambda')), \psi_{\varphi, \varphi'} \circ \psi_{\varphi', \varphi}^{-1}) \cong (\text{Km}(E(\lambda) \times E(\lambda'))), \psi_{\varphi, \varphi'} \circ \psi_{\varphi', \varphi}^{-1} \).
\]
We set \( \psi_{\varphi} := \psi_{1, \varphi} \). By \(17\) Lemma 16 (2), \(21\) Th. 4.1, the involutions \( \{\psi_{\varphi} \circ \psi_{\varphi}^{-1} \}_{\varphi \in S_3} \) give complete representatives of involutions of odd type on \( \text{Km}(E(\lambda) \times E(\lambda')) \).
For $g \in \mathcal{S}_3$, let $\pi_g : \mathcal{Y}_g \to (\mathcal{C} \setminus \{0,1\})^2$ be the family induced from $\mathcal{Y}$ by the map $id \times g : (\mathcal{C} \setminus \{0,1\})^2 \to (\mathcal{C} \setminus \{0,1\})^2$. Hence $\pi_g^{-1}(\lambda, \lambda') = Y_{\lambda, \phi(\lambda')}$. We set

$$Z_g := \{(\lambda, \lambda') \in (\mathcal{C} \setminus \{0,1\})^2 \mid \lambda = \phi(\lambda')\}.$$ 

Then the family $\pi_g : \mathcal{Y}_g|_{(\mathcal{C} \setminus \{0,1\})^2 \smallsetminus \mathcal{Z}_g} \to (\mathcal{C} \setminus \{0,1\})^2 \smallsetminus \mathcal{Z}_g$ is endowed with the fixed-point-free involution $\iota_g$ such that

$$\tag{5.4} (Y_{\lambda, \lambda'}, \iota_g) \cong (Y_{\lambda, \phi(\lambda')}, \iota).$$

### 5.2. The period map.

For $n \in \mathbb{Z}_{\geq 0}$, recall that $\Gamma(n) \subset SL_2(\mathbb{Z})$ is the principal congruence subgroup of level $n$, which acts projectively on $\mathfrak{H}$. Note that $\Gamma(1) = SL_2(\mathbb{Z})$. Recall from (3.11) that $\mathfrak{D} = \bigcup_{\gamma \in \Gamma(2)} (\gamma \times 1) \Delta_\mathfrak{H} = \bigcup_{\gamma \in \Gamma(2)} (1 \times \gamma) \Delta_\mathfrak{H}$. For $g \in \mathcal{S}_3$, we define the reduced divisor on $\mathfrak{H} \times \mathfrak{H}$ as

$$\tag{5.5} \mathfrak{D}_g := (1 \times g) \mathfrak{D} = (g^{-1} \times 1) \mathfrak{D},$$

where $\mathcal{S}_3$ acts on $\mathfrak{H} / \Gamma(2)$ via the standard isomorphism $\mathcal{S}_3 \cong \Gamma(1) / \Gamma(2)$.

Recall that $\lambda(\tau) = \theta_2(\tau)^4 / \theta_3(\tau)^4$. Let $\Pi : \mathfrak{H} \times \mathfrak{H} \to (\mathcal{C} \setminus \{0,1\})^2$ be the holomorphic map defined as $\Pi(\tau, \tau') := (\lambda(\tau), \lambda(\tau'))$. Then $\Pi^{-1}(\mathcal{Z}_g) = \mathfrak{D}_g$. For $g \in \mathcal{S}_3$, we define a family of Kummer surfaces

$$\pi_g : \mathcal{K}_g \to \mathfrak{H} \times \mathfrak{H}$$

as the pullback of $\pi_g : \mathcal{Y}_g \to (\mathcal{C} \setminus \{0,1\})^2$ by $\Pi$. Then it is a “universal family” in the sense that $\pi_g^{-1}(\tau, \tau') = Y_{\lambda(\tau), \phi(\lambda(\tau'))} \cong Km(E(\lambda(\tau)) \times E(\phi(\lambda(\tau')))) \cong K_{\tau, \tau'}$, for all $(\tau, \tau') \in \mathfrak{H} \times \mathfrak{H}$. By Sect. 5.1, $\mathcal{K}_g|_{(\mathfrak{H} \times \mathfrak{H}) \smallsetminus \mathfrak{D}_g}$ is endowed with the involution $\iota_g$, which does not extend to an involution on $\mathcal{K}_g$ by Remark 5.1.

As in Sect. 4, we take a normalized marking $\alpha_g : R^2 \pi_g, \mathbb{Z} \cong \mathbb{L}_{K3}$ for $(\mathcal{K}_g, \iota_g)$. Using the map $\phi$ in (3.13), we set

$$\mathcal{K}_g := \alpha_g(\mathcal{K}) \subset \Lambda, \quad \mathcal{E}_g := \mathcal{K}_g^{\perp}.\Lambda.$$

For $g \in \mathcal{S}_3$, let

$$\varpi_g : \mathfrak{H} \times \mathfrak{H} \to \Omega_{\mathcal{K}_g}^+ \subset \Omega_\Lambda^+$$

be the period map (3.6) for the marked family $(\pi_g : \mathcal{K}_g \to \mathfrak{H} \times \mathfrak{H}, \alpha_g)$. Since $\varpi_g$ is given explicitly by (3.3), under the identification $3.3$, $\varpi_g$ induces an isomorphism between $\mathfrak{H} \times \mathfrak{H}$ and $\Omega_{\mathcal{K}_g}^+$. Since $\varpi_g(\tau, \tau') = \varpi(K_{\tau, \tau'}/\iota_g, \alpha_g) \in \Omega_\Lambda^+ \cap \mathfrak{H}$, we get

$$\varpi_g^{-1}(\mathfrak{H} \cap \Omega_{\mathcal{K}_g}^+) \subset \mathfrak{D}_g.$$

Let $\gamma \in \mathcal{A}_{K \setminus \{0\}}$ be the patching element of $\iota_g$. Let $\ell(g)$ be the level of $\Gamma_{34}$ in $H^2(K_{\gamma, \gamma'}, \mathbb{Z})$. By Lemma 3.14,

$$\Phi_\gamma(\tau, \tau')^2 = \Phi_{\ell(g)} \left( j_{\ell(g)}(\varpi_g(\tau, \tau')) \right)^2 = \Phi_{\ell(g)} \left( j_{\ell(g)}(\varpi(K_{\tau, \tau'}/\iota_g, \alpha_g)) \right)^2$$

is an automorphic form for $\Gamma(2) \times \Gamma(2)$ of weight 8. In particular, $\|\Phi_\gamma\|^2$ is a $\Gamma(2) \times \Gamma(2)$-invariant $C^\infty$ function on $\mathfrak{H} \times \mathfrak{H}$. Indeed, we have the following equality of functions on $\mathfrak{H} \times \mathfrak{H}$ for all $g \in \mathcal{S}_3$:

$$\|\Phi_\gamma(\tau, \tau')\|^2 = \|\Phi(Km(E(\lambda(\tau)) \times E(\phi(\lambda(\tau'))))/i)\|^2.$$

In this section, we study the Petersson norm $\|\Phi_\gamma\|^2$. For this sake, we study the behavior of the period mapping at the boundary.
5.3. Behavior of the period map at the boundary. For $n \in \mathbb{Z}_{>0}$, we set $Y(n) := \Gamma(n) \backslash \mathcal{H}$, $X(n) := (\Gamma(n) \backslash \mathcal{H})^*$ and

$$B_{X(n)} := (X(n) \times X(n)) \setminus \Sigma(n)$$

where the asterisk * denotes the Daini–Borel compactification. It is classical that the modular $\lambda$-function induces an $\mathcal{S}_2$-equivariant isomorphism from $Y(n)$ to $\mathbb{C} \setminus \{0, 1\}$, where the $\mathcal{S}_2$-action on $Y(n)$ is given by the isomorphism $\mathcal{S}_2 \cong \Gamma(1)/\Gamma(2)$. Since $X(n)$ has three 0-dimensional cusps, $B_{X(n)}$ is the union of $9 \mathbb{P}^1$'s.

Let $p: Y(n) \rightarrow Y(1)$ be the projection. Then $p: Y(n) \rightarrow Y(1)$ is a Galois covering with Galois group $\mathcal{S}_3 \cong \Gamma(1)/\Gamma(2)$, which induces a $\mathcal{S}_3 \times \mathcal{S}_3$-action on $Y(n)$.

Let $\Delta_Y$ be the diagonal locus of $Y(n)$ and $\Delta_Y(n)$ be the diagonal locus of $Y(n)$ and $\Delta_Y(n)$ and define the divisor $Z := \Delta_Y(n)$ as

$$Z := (p \times p)^{-1}(\Delta_Y(n)) = \bigcup_{g \in \mathcal{S}_3} (1 \times g)\Delta_Y(n).$$

Under the identification $Y(n) \cong \mathbb{C} \setminus \{0, 1\}$ via the $\lambda$-invariant, we have

$$Z_n = (1 \times g)\Delta_Y(n)$$

for $g \in \mathcal{S}_3$. Since $\Delta_Y(n) = \mathcal{D}/\Gamma(2) \times \Gamma(2) \setminus \mathcal{D} \times \mathcal{D}$ (cf. (5.11)), we have $Z_n = \mathcal{D}_n/\Gamma(2) \times \Gamma(2)$ for $n \in \mathcal{S}_3$ (cf. (5.3)), and thus $Z = \bigcup_{g \in \mathcal{S}_3} \mathcal{D}_n$.

By Sect. 3.3, the period map $\varphi_\sigma$ is a $\Gamma(2) \times \Gamma(2)$-equivariant holomorphic map from $\mathcal{H} \times \mathcal{H}$ to $\mathcal{M}$. Thus $\varphi_\sigma$ descends to a morphism of modular varieties $\varphi_\sigma: Y(2) \times Y(2) \rightarrow \mathcal{M}$ such that $\varphi_\sigma(\lambda(\tau), \lambda(\tau')) = \varphi(K_{\tau, \tau'/\sigma})$. Since $\varphi_\sigma(Y(2) \times Y(2) \setminus \mathcal{D}_n \times \mathcal{D}_n) \subset \mathcal{M}$, it follows from Borel's extension theorem [17] that $\varphi_\sigma$ extends to a holomorphic map from $\mathbb{C} \times \mathbb{C}$ to $\mathcal{M}$, which is again denoted by $\varphi_\sigma$. Recall from Sect. 2.2 that $\mathcal{M} \times \mathcal{M}$ consists of two modular curves $X(1)$ and $X(2)$.

**Lemma 5.2.** The following inclusion holds:

$$\varphi_\sigma(B_{X(n)} \times X(n)) \subset X(2).$$

**Proof.** Via [3.7], $\varphi_\sigma$ sends the boundary $B_{X(n)} \times X(n)$ to the boundary $\mathcal{M} \times \mathcal{M}$. Assume that there is an irreducible component $C$ of $B_{X(n)} \times X(n)$ with $\varphi_\sigma(C) \subset X(1)$. Since $X(1) = \mathcal{D} \setminus \mathcal{D}$ by Lemma 2.2, we get $\varphi_\sigma(C) \subset \mathcal{D} \setminus \mathcal{D}$. In lattice-theoretical terms, by Sect. 2.2, an irreducible component of $B_{X(n)} \times X(n)$ corresponds to a primitive totally isotropic sublattice of $\mathbb{K}_p$ of rank 2. Thus there is a primitive totally isotropic sublattice $L \subset \mathbb{K}_p$ of rank 2 and a root $\delta \in \Delta_\Lambda$ such that $L \subset \delta^{\perp_\Lambda}$.

By Lemma 3.11 we have $\Lambda = \mathbb{Z}\delta + \mathbb{K}_p \oplus \mathbb{E}_p$. We write $\delta = \delta_1 + \delta_2$ with $\delta_1 \in \mathbb{K}_p^\vee$ and $\delta_2 \in \mathbb{E}_p^\vee$. Since $\nu_\sigma$ is of odd type, we get $\delta_1 \equiv 1 \mod 2$, so $\delta_2 \neq 0$. Thus $\delta^{\perp_\Lambda} \cap \mathbb{K}_p = \delta^{\perp_\Lambda} \cap \mathbb{K}_p$ has signature $(1, 2)$ or $(2, 1)$, which cannot contain a totally isotropic sublattice of rank 2. This contradicts the assumption $L \subset \delta^{\perp_\Lambda} \cap \mathbb{K}_p$. Thus $\varphi_\sigma(C) \subset X(2)$ for any component $C$ of $B_{X(n)} \times X(n)$. \hfill $\square$

5.4. Involution of odd type and the Borchers $\phi$-function. Let $\Sigma^2 Y(1) := Y(1) \times Y(1)/\mathcal{S}_2$ (resp. $\Sigma^2 X(1) := X(1) \times X(1)/\mathcal{S}_2$ denote the second symmetric product of $Y(1)$ (resp. $X(1)$), where $\mathcal{S}_2$ acts on $Y(1) \times Y(1) \times X(1)$ as the permutation of coordinates. Since $Y(1) \cong \mathbb{C}$ and $X(1) \cong \mathbb{P}^1$, we have isomorphisms $\Sigma^2 Y(1) \cong \mathbb{C}^2$ and $\Sigma^2 X(1) \cong \mathbb{P}^2$. Let $\Delta \subset \Sigma^2 X(1)$ be the image of the diagonal locus $\Delta_X(1)$ by the projection $X(1) \times X(1) \rightarrow \Sigma^2 X(1)$, i.e.,

$$\Delta := \Delta_X(1)/\mathcal{S}_2 \cong \mathbb{P}^1.$$
We set
\[ B := \Sigma^2 X(1) \setminus \Sigma^2 Y(1). \]
Under the identifications \( X(1) \cong \mathbb{P}^1 \) and \( Y(1) \cong \mathbb{P}^1 \setminus \{ \infty \}, \) \( B \) is the divisor at infinity of \( \mathbb{P}^2: \)
\[ B = ((\infty) \times X(1) \amalg X(1) \times (\infty))/\mathfrak{S}_2 \cong X(1) \cong \mathbb{P}^1. \]

In the rest of this section, we are going to compute \(-dd^c \left[ \log \prod_{e \in \mathfrak{S}_3} \varpi^e_0 \| \Phi \|^2 \right]\) as a current on \( \Sigma^2 X(1). \) Since \( \varpi_0(Y(2) \times Y(2) \setminus Z_0) \subset \mathcal{M} \setminus \mathcal{D} \) and \( \text{div}(\Phi) = \mathcal{H}, \)
\( \prod_{e \in \mathfrak{S}_3} \varpi^e_0 \| \Phi \|^2 \) is a nowhere vanishing \( C^\infty \) function on \( (Y(2) \times Y(2)) \setminus Z. \) By \((5.3), (5.7), (5.8), \) \( \prod_{e \in \mathfrak{S}_3} \varpi^e_0 \| \Phi \|^2 \) is invariant under the actions of \( \mathfrak{S}_3 \times \mathfrak{S}_3 \) and \( \mathfrak{S}_2. \) Thus we regard \( \prod_{e \in \mathfrak{S}_3} \varpi^e_0 \| \Phi \|^2 \) as a function on \( \Sigma^2 Y(1) \setminus \Delta. \)

Let \( \omega_{\hat{K} \times \hat{K}} \) be the Kähler form of the Poincaré metric on \( \hat{K} \times \hat{K}, \) i.e.,
\[ \omega_{\hat{K} \times \hat{K}} := -dd^c \log |\tau| - dd^c \log |\tau'|. \]
Let \( \omega_{\Sigma^2 Y(1)} \) (resp. \( \omega_{\Sigma^2 Y(2)} \)) be the Kähler form on \( \Sigma^2 Y(1) \) (resp. \( \Sigma^2 Y(2) \)) in the sense of orbifolds induced from \( \omega_{\hat{K} \times \hat{K}}. \) Since the area of \( \Sigma^2 Y(1) \) (resp. \( \Sigma^2 Y(2) \)) with respect to \( \omega_{\Sigma^2 Y(1)} \) (resp. \( \omega_{\Sigma^2 Y(2)} \)) is finite, the \((1, 1)\)-current \( \omega_{\Sigma^2 Y(1)} \) (resp. \( \omega_{\Sigma^2 Y(2)} \)) extends trivially to a closed positive \((1, 1)\)-current \( \omega_{\Sigma^2 Y(1)} \) on \( \Sigma^2 X(1). \)

**Proposition 5.3.** The following equation of currents on \( \Sigma^2 X(1) \) holds:
\[-dd^c [\log \prod_{e \in \mathfrak{S}_3} \varpi^e_0 \| \Phi \|^2] = 24 \omega_{\Sigma^2 Y(1)} - \delta.\]

**Proof.** Step 1. Set \( F := \log \prod_{e \in \mathfrak{S}_3} \varpi^e_0 \| \Phi \|^2 = \sum_{e \in \mathfrak{S}_3} \varpi^e_0 \log \| \Phi \|^2. \) By \((6.13), \)
we have the equality of \((1, 1)\)-forms on \( \Sigma^2 Y(1) \)
\[ -dd^c F = 24 \omega_{\Sigma^2 Y(1)}. \]
Since \( \log \| \Phi \|^2 \) has logarithmic singularities along \( \mathcal{D} \cup X(1), \) \( F \) has at most logarithmic singularities along \( \Delta \cup B, \) which, together with \((5.9), \) and the irreducibility of \( \Delta \) and \( B, \) yields the following equation of currents on \( \Sigma^2 X(1) \)
\[ -dd^c F = 24 \omega_{\Sigma^2 Y(1)} + \alpha \delta_\Delta + \beta \delta_B, \]
where \( \alpha, \beta \in \mathbb{R} \) are some constants. We are going to verify that \( \alpha = -1 \) and \( \beta = 0. \)

Step 2. Let \( \varpi^*_e(\mathcal{H}) \) denote the pull-back of the divisor \( \mathcal{H}. \) Since \( \hat{H} \cap \varpi^*_e(\hat{K} \times \hat{K}) \setminus \mathcal{D}_e = \emptyset, \) we have \( \text{Supp}(\varpi^*_e(\hat{H})) \subset \mathcal{D}_e. \) Since the map \( \varpi_0 \) is \( \Gamma(2) \times \Gamma(2) \)-equivariant and since the divisor \( Z_0 = \mathcal{D}_e / (\Gamma(2) \times \Gamma(2)) \) of \( Y(2) \times Y(2) \) is irreducible, the inclusion \( \text{Supp}(\varpi^*_e(\hat{H})) \subset \mathcal{D}_e \) implies the existence of an integer \( \nu \in \mathbb{Z}_{\geq 0} \) with
\[ \varpi^*_e(\hat{H}) = \nu \mathcal{D}_e. \]
Let \( i: \Omega^+_{K^e} \to \Omega^+_{K^e} \) be the inclusion. By Proposition \((3.14), \) \( E := \sum_{d \in \Delta \cup \{ \pm 1, \pm I_e \}} H_d \cap \Omega^+_{K^e} \) is a reduced divisor on \( \Omega^+_{K^e} \) with
\[ i^* \mathcal{H} = \sum_{d \in \Delta \cup \{ \pm 1, \pm I_e \}} (H_d \cap \Omega^+_{K^e} + H_{I_e(d)} \cap \Omega^+_{K^e}) = 2E. \]
Since \( \varpi_0 \) is an isomorphism from \( \hat{K} \times \hat{K} \) to \( \Omega^+_{K^e}, \) we obtain the equality of divisors
\[ \varpi^*_e(\mathcal{H}) = 2 \varpi^*_e(E), \]
where $\mathbb{w}_\psi^* (E)$ is a reduced divisor on $\mathfrak{H} \times \mathfrak{H}$. Comparing (5.11) and (5.12), we get the equality of divisors on $\mathfrak{H} \times \mathfrak{H}$

\begin{equation}
\mathbb{w}_\psi^* (\mathcal{H}) = 2 \mathcal{D}_e.
\end{equation}

Since $\text{div}(\Phi) = \mathcal{H}$, we deduce from (5.13) that $\mathbb{w}_\psi^* \| \Phi \|$ has zeros of order 2 along $\mathcal{D}_e$. Hence $F$ has zeros of order 2 along $\mathcal{D} = \sum_{v \in \mathcal{E}} \mathcal{D}_v$. This, together with (5.9), implies the following equation of currents on $\mathfrak{H} \times \mathfrak{H}$:

\begin{equation}
- dd^c F = 24 \omega_{\mathfrak{H} \times \mathfrak{H}} - 2 \delta_\mathcal{D}.
\end{equation}

Since the projection $p: Y(1) \times Y(1) \to \Sigma^2 Y(1)$ is doubly ramified along $\Delta_Y(1)$, we get $\alpha = -2/ \deg p = -1$ by (5.14).

**Step 3.** By Lemma 5.2, $\mathbb{w}_\psi^* (B_{X(2) \times X(2)})$ is not contained in the boundary component $X(1) = D^* \setminus D$. Thus the pullback $F$ does not vanish identically on $B$. Since $B$ is an irreducible divisor on $\Sigma^2 X(1)$, we get $\beta = 0$. \hfill $\square$

### 6. Involutions of Odd Type: The Leading Term of $\Phi$

Let $(\psi: K \to \mathfrak{H} \times \mathfrak{H}, \alpha)$ be the marked family of Kummer surfaces defined in Sect. 5 such that $\alpha$ is normalized for $(K, \iota)$. So far, we have fixed the Enriques lattice $\Lambda$ in $\text{L}_{K,3}$, and we have considered sublattices $\alpha(K)$ of $\Lambda$ associated to the involutions $\iota$ on $K_{\tau,\tau'}$. Here $\alpha(K)$ is isometric to $\mathbb{U}(2) \oplus \mathbb{U}(2)$. In this section, we compute the leading term of $\Phi$, for odd $\gamma$. To do this, however, descriptions become much simpler if we fix the lattice $\mathbb{U}(2) \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2)$ and vary the lattice $H^2(K_{\tau,\tau'}, \mathbb{Z})$ inside its dual lattice.

In Sect. 6.1, we set the stage. Then we compute the the leading term of $\Phi_\gamma (\tau, \tau')$ near $(\tau, \tau') = (+i\infty, +i\infty)$ for odd $\gamma$. To ease the notation, we write

\[ K := \mathbb{U}(2) \oplus \mathbb{U}(2), \]

and we set

\[ v := (1, 0, 0, 0) \in K. \]

Recall that $K$ is the sublattice of $H^2(K_{\tau,\tau'}, \mathbb{Z})$ defined in (3.2). We define the isometry $\psi: K \to \mathbb{K}$ as

\[ \psi(-\Gamma_{34}') = (1, 0, 0, 0), \quad \psi(\Gamma_{23}') = (0, 0, 1, 0), \]
\[ \psi(\Gamma_{12}') = (0, 1, 0, 0), \quad \psi(-\Gamma_{14}') = (0, 0, 0, 1). \]

In what follows, we identify $K$ with $\mathbb{K}$ via $\psi$.

#### 6.1. Set up

Let $\gamma \in A_{\mathbb{K}} \setminus \{0\}$ and $\delta_\gamma \in A_{\mathbb{E}_8(2)} \setminus \{0\}$ be such that $\gamma^2 = \delta_\gamma^2$ in $\mathbb{Z}/2\mathbb{Z}$. Let $d_1 \in \mathbb{K}^\vee$ and $d_2 \in \mathbb{E}_8(2)^\vee$ be vectors such that $d_1 = \gamma$ and $d_2 = \delta_\gamma$. As in Sect. 3.3, the anti-invariant sublattice with respect to the fixed-point-free involution corresponding to $\gamma$ is realized as

\begin{equation}
\Lambda_{\gamma} = \mathbb{Z}(d_1, d_2) + \mathbb{K} \oplus \mathbb{E}_8(2).
\end{equation}

Let $(\tau, \tau') \in \mathfrak{H} \times \mathfrak{H} \setminus \mathcal{D}$. Let $\iota: K_{\tau,\tau'} \to K_{\tau,\tau'}$ be an involution satisfying (3.3) with patching element $\psi^{-1}(\gamma) \in A_{\mathbb{K}} \setminus \{0\}$.

**Lemma 6.1.** There exists an isometry $\bar{\psi}: H^2(K_{\tau,\tau'}, \mathbb{Z}) \to \Lambda_{\gamma}$ extending $\psi$. 

Proof. Let \( \{e_1, \ldots, e_8\} \) (resp. \( \{e'_1, \ldots, e'_8\} \)) be a basis of \( K^1 = K^{1,2}_+ \) (resp. \( E_8(2) \)) whose Gram matrix is the (negative-definite) Cartan matrix of type \( E_8 \). We define an isometry \( \psi': K^1 \to E_8(2) \) as \( \psi'(e_i) := e'_i \) \( (i = 1, \ldots, 8) \) and we set \( \Lambda' := (\psi \circ \psi')(H^2(K_{\tau', \tau}, Z)_-). \) Since \( \Lambda' \cong \Lambda \), we can express \( \Lambda' = Z(d'_1, d'_2) + K \oplus E_8(2) \) with \( d'_1 \in K^1, \, d'_2 \in E_8(2). \) Since \( \psi^{-1}(\tau) \in A_K \setminus \{0\} \) is the patching element of \( \iota \), we have \( d'_1 = \tau \in A_K \setminus \{0\}. \) If \( a \in O(E_8(2)) \) and if we define \( \psi'_a : K^1 \cong E_8(2) \) as \( \psi'_a(e_i) := a(e'_i) \) \( (i = 1, \ldots, 8) \) and set \( \Lambda'_a := (\psi \circ \psi'_a)(H^2(K_{\tau', \tau}, Z)_-), \) then we have \( \Lambda'_a = Z(\tau, \pi(d'_2)) + K \oplus E_8(2). \) Since the natural homomorphism \( O(E_8(2)) \to O(q_{E_8(2)}) \) is surjective by e.g. \([11, 1.7]\), we have \( \pi(d'_2) = \delta \) by choosing \( a \in O(E_8(2)) \) suitably. Then \( \psi := \psi \circ \psi'_a \) is the desired isometry. \( \square \)

Let \( \ell \in \{1, 2\} \) be the level of \( \Gamma^\vee_{34} \) in \( H^2(K_{\tau', \tau}, Z)_- \). Via \( \psi, \ell \) is then the level of \( v \) in \( \Lambda_{\gamma} \). Since the \( O(\Lambda) \)-orbit of a primitive isotropic vector of \( \Lambda_{\gamma} \) is determined by its level, we can take a primitive isotropic vector \( v' \in \Lambda_{\gamma} \) of level \( \ell \) with

\[
\langle v, v' \rangle = \ell.
\]

We set \( U(\ell)_{\gamma} := Z(v) + Z(v') \) and

\[
M_{\gamma} := v^\perp \cap v'^\perp \cap \Lambda_{\gamma} = U(\ell)_{\gamma} \cap \Lambda_{\gamma}.
\]

Then \( \Lambda_{\gamma} = U(\ell)_{\gamma} \cap M_{\gamma}, \) where \( U(\ell)_{\gamma} \cong U(\ell) \) and \( M_{\gamma} \cong M_{\ell} \) (cf. \([27]\)). Recall that \( U(1), U(2) \subset \Lambda \) are endowed with the basis \( \{e_1, f_1\}, \{e_2, f_2\}, \) respectively.

**Lemma 6.2.** Let \( \rho, \rho' \) be primitive isotropic vectors of \( M_{\gamma} \) such that

\[
\langle \rho, \rho' \rangle = 2/\ell, \quad (1/\ell)^2 \langle \rho, ((0,0,1,0), 0) \rangle > 0, \quad (1/\ell)^2 \langle \rho, ((0,0,0,1), 0) \rangle > 0.
\]

Then there exists a normalized marking \( \alpha : H^2(K_{\tau', \tau}, Z) \to \mathbb{L}_{K_3} \) for \( (K_{\tau', \tau}, \iota) \) such that, if we define the isometry \( \beta_{\gamma} : \Lambda_{\gamma} \to \Lambda \) as \( \beta_{\gamma} := \alpha \circ \psi^{-1} \), then

\[
\beta_{\gamma}(v) = e_{\ell}, \quad \beta_{\gamma}(v') = f_{\ell}, \quad \beta_{\gamma}(\rho) = e_{2/\ell}, \quad \beta_{\gamma}(\rho') = f_{2/\ell}.
\]

**Proof.** Let \( \alpha' \) be a normalized marking for \( (K_{\tau', \tau}, \iota) \), and we set \( \beta'_{\gamma} := \alpha' \circ \psi^{-1}. \) Let \( g \in O(\Lambda) \) be such that \( g(e_{\ell}) = e_{\ell} \) and we set \( \beta_{\gamma} := g\beta'_{\gamma}. \) Let us see that by choosing \( g \) appropriately, \( \beta_{\gamma} \) satisfies \([6.2]\). We set

\[
e_{\ell} := \beta'_{\gamma}(v) = e_{\ell}, \quad f_{\ell} := \beta'_{\gamma}(v'), \quad e_{2/\ell} := \beta'_{\gamma}(\rho), \quad f_{2/\ell} := \beta'_{\gamma}(\rho').
\]

Then \( e_{\ell}, f_{\ell} \) are primitive isotropic vectors of \( \Lambda \) of level \( \ell \) and \( e_{2/\ell}, f_{2/\ell} \) are primitive isotropic vectors of \( \Lambda \) of level \( 2/\ell \) such that the Gram matrix of \( \{e_{\ell}, f_{\ell}, e_{2/\ell}, f_{2/\ell}\} \) is given by \( \begin{pmatrix} e_{\ell} & f_{\ell} \\ f_{\ell} & e_{\ell} \end{pmatrix} \). Set \( L_1 := Z e_{\ell} + Z f_{\ell} + Z e_{2/\ell} + Z f_{2/\ell} \) and \( L_2 := L_1^+ \). It is easy to see that \( \Lambda = L_1 \oplus L_2 \). Hence \( L_2 \) is an even 2-elementary lattice of rank 8 with \( \dim_{\mathbb{F}_2} A_{L_2} = 8 \) and \( \delta(L_2) = 0 \), which implies \( L_2 \cong E_8(2) \). Let \( \theta : L_2 \to E_8(2) \) be an isometry of lattices. We define \( g \in O(\Lambda) \) as

\[
g(\alpha e_{\ell} + b f_{\ell} + c e_{2/\ell} + d f_{2/\ell} + x) := a e_{\ell} + b f_{\ell} + c e_{2/\ell} + d f_{2/\ell} + \theta(x).
\]

Then \( \beta_{\gamma} = g\beta'_{\gamma} \) satisfies \([6.2]\).

By \([19\text{ Cor. 2.6]} \), there exists \( \overline{g} \in O(\mathbb{L}_{K_3}) \) with \( \overline{g}(\Lambda) = \Lambda \) such that \( \overline{g}|_A = g \). Set \( \alpha := \overline{g}\alpha' \). Then \( \alpha(H^2(K_{\tau', \tau}, Z)_-) = \Lambda \). Since \( \alpha \circ \psi^{-1} = \overline{g}\alpha' \circ \psi^{-1} = g\beta'_{\gamma} = \beta_{\gamma} \), and since \( \beta_{\gamma} \) satisfies \([6.2]\), we get \( \alpha(-\Gamma_{34}^\vee) = \beta_{\gamma} \circ \psi(-\Gamma_{34}^\vee) = \beta_{\gamma}(v) = e_{\ell}. \) Let us verify
Recall that \(u, B, D\) were defined in (3.9). Then
\[
\langle (-1)^{2/\ell} B, e_{2/\ell} \rangle = \langle (-1)^{2/\ell} \alpha, e_{2/\ell} \rangle = \langle (-1)^{2/\ell} \beta, \varphi(\Gamma_{23}^\vee), e_{2/\ell} \rangle = \langle (-1)^{2/\ell} \beta, \varphi(\Gamma_{23}^\vee), \beta(\rho) \rangle = \langle (-1)^{2/\ell} \varphi(\Gamma_{23}^\vee), \rho \rangle = \langle -1 \rangle^{2/\ell} \varphi((0, 0, 1, 0), 0), \rho \rangle > 0.
\]
Similarly, \(\langle (-1)^{2/\ell} D, e_{2/\ell} \rangle = \langle -1 \rangle^{2/\ell} \varphi((0, 0, 1, 0), 0) > 0\). Since \(e_{2/\ell} \in \mathbb{C}_M^+\) by (2.3), we get \((-1)^{2/\ell} B \in \mathbb{C}_M^+\) and \((-1)^{2/\ell} D \in \mathbb{C}_M^+\). By (3.8), this implies \(\exists u \in \mathbb{C}_M^+\). Since \(\varphi(K_{\tau, \tau'}, \alpha) = \{-u^2/2, 2/\ell + (-1)^{2/\ell} u\} \) by (5.3) and since \(\exists u \in \mathbb{C}_M^+\), we see that \(\alpha\) satisfies (2.3). This completes the proof. \(\Box\)

In the situation of Lemma (5.2) via \(\beta, \) let \(\Omega_A^+\) and \(\mathbb{C}_M^+\) correspond to \(\Omega_\mathbb{A}_\gamma^+\) and \(\mathbb{C}_{M_\gamma}^+\), respectively. Then, similar to (2.9), the tube domain \(M_\gamma \otimes \mathbb{R} + i \mathbb{C}_{M_\gamma}^+\) is identified with \(\Omega_\mathbb{A}_\gamma^+\) through the map
\[
j_\gamma : M_\gamma \otimes \mathbb{R} + i \mathbb{C}_{M_\gamma}^+ \ni z \rightarrow \left[ \left(-\frac{z^2}{2}\right) + \frac{\nu}{\ell} + (-1)^{2/\ell} z \right] \in \Omega_\mathbb{A}_\gamma^+.
\]
Let
\[
h_\gamma : \Omega_K \ni [\eta] \rightarrow [(\eta, 0)] \in \Omega_\mathbb{A}_\gamma^+\]
be the embedding of domains induced by the inclusion of lattices \(\mathbb{K} \subseteq \mathbb{A}_\gamma\), and let \(\Omega_\mathbb{A}_\gamma^+\) be one of the two (isomorphic) connected components of \(\Omega_K^+\) such that \(h_\gamma(\Omega_K^+) \subseteq \Omega_\mathbb{A}_\gamma^+\). Then we have the following commutative diagram:
\[
\begin{array}{ccc}
\mathcal{D} \times \mathcal{D} & \xrightarrow{\varpi} & \Omega_\mathbb{A}_\gamma^+ \\
j \downarrow & & \beta_\gamma \uparrow \\
\Omega_K^+ & \xrightarrow{h_\gamma} & \Omega_\mathbb{A}_\gamma^+ \\
\end{array}
\]
where \(\varpi\) is the period mapping for \((\pi : \mathcal{K} \rightarrow \mathcal{D} \times \mathcal{D}, \alpha)\) given by (5.7). \(j\) is the isomorphism given by \(j(\tau, \tau') := \left[ \left(-\frac{\tau}{\ell}, \frac{\nu}{\ell}, \frac{\nu}{\ell}, \frac{\nu}{\ell} \right) \right], \) and \(\beta_\gamma : M_\gamma \otimes \mathbb{R} + i \mathbb{C}_{M_\gamma}^+ \rightarrow M_\gamma \otimes \mathbb{R} + i \mathbb{C}_{M_\gamma}^+\) is the restriction of the linear map \(\beta_\gamma : M_\gamma \otimes \mathbb{C} \rightarrow M_\gamma \otimes \mathbb{C}\). Now, we define a map \(\varphi_\gamma : \mathcal{D} \times \mathcal{D} \rightarrow M_\gamma \otimes \mathbb{R} + i \mathbb{C}_{M_\gamma}^+\) as
\[
\varphi_\gamma := \beta_\gamma^{-1} \circ j_\ell^{-1} \circ \varpi = j_\ell^{-1} \circ h_\gamma \circ j.
\]
For \((\tau, \tau') \in \mathcal{D} \times \mathcal{D},\) in Definition (5.3) we defined \(\Phi_\gamma(\tau, \tau')^2 = \Phi_\ell \left( j_\ell^{-1}(\varpi(K_{\tau, \tau'/\ell}, \alpha)) \right)^2 \).

From the above commutative diagram, Lemma (5.3) and Lemma (5.2) we have
\[
\Phi_\gamma(\tau, \tau')^2 = \Phi_\ell \left( \beta_\gamma \circ \varphi_\gamma(\tau, \tau') \right)^2.
\]
Recall that the elements of \(A_\mathbb{K} \times \{0\}\) of odd norm are represented by the following vectors of \(\mathbb{K}^\vee\) of norm \(-1\) (see the proof of Lemma 5.12):
\[
(1/2, -1/2, 0, 0), (1/2, -1/2, 1/2, 0), (1/2, -1/2, 0, 1/2), (0, 0, 1/2, -1/2), (1/2, 0, 1/2, -1/2), (0, 1/2, 1/2, -1/2).
\]

Lemma 6.3. Regard \(v = (1, 0, 0, 0)\) as a primitive isotropic vector of \(\mathbb{A}_\gamma\).

(1) If \(\gamma \equiv (1/2, 1/2, 0, 0), (1/2, 1/2, 0, 0), (1/2, 1/2, 0, 1/2), (1/2, 1/2, 0, 1/2) \mod \mathbb{K}, \) then \(v\) has level 1 in \(\mathbb{A}_\gamma\).
(2) If \( \gamma \equiv (0,0,\frac{1}{2}, \frac{1}{2}), (\frac{1}{2},0, \frac{1}{2}, \frac{1}{2}) \) mod \( \mathbb{K} \), then \( \mathbf{v} \) has level 2 in \( \Lambda_{\gamma} \).

Proof. In (6.3), we may assume that \( d_1 \in \mathbb{K}^\vee \) is one of the 6 vectors in (1), (2) as above. Since \( \langle \mathbf{v}, \mathbb{K} \rangle_k = 2\mathbb{Z} \), the generator of \( \langle \mathbf{v}, \Lambda_{\gamma} \rangle_{\Lambda_{\mathbb{K}}} \) has the same parity as that of \( \langle \mathbf{v}, d_1 \rangle_{\Lambda_{\mathbb{K}}} \), >From this, we can verify the assertion. \( \square \)

For \( \tau, \tau' \in \mathfrak{S}_1 \), we set \( p^{1/2} := e^{\pi i \tau}, q^{1/2} := e^{\pi i \tau'} \). Let \( \mathbb{Z}\{p^{1/2}, q^{1/2}\} \) denote the ring of convergent series of \( p^{1/2}, q^{1/2} \) with coefficients in \( \mathbb{Z} \). Let \( \mathfrak{m} \) be the ideal generated by \( p^{1/2}, q^{1/2} \):

\[ \mathfrak{m} := \mathbb{Z}\{p^{1/2}, q^{1/2}\} + q^{1/2}\mathbb{Z}\{p^{1/2}, q^{1/2}\} \]

6.2. The leading term of \( \Phi_{\gamma} \) for odd \( \gamma \): level 1 case. The rest of this section is devoted to determining the leading term of \( \Phi_{\gamma}(\tau, \tau') \) near \( (+i\infty, +i\infty) \) for each odd \( \gamma \) as above in Lemma [6.3]. In this subsection, we assume that the level of \( \mathbf{v} \) in \( \Lambda_{\gamma} \) is 1. Thus, modulo \( \mathbb{K} \), \( \gamma \) is one of the four vectors in Lemma [6.3] (1).

Lemma 6.4. If \( \gamma \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 0, 0), (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) mod \( \mathbb{K} \), then

\[ \Phi_{\gamma}(\tau, \tau') \equiv 1 \mod \mathfrak{m} \]

Proof. Since \( \mathbf{v} \in \Lambda_{\gamma} \) is an isotropic vector of level 1 by Lemma [6.3], the proof is the same as that of [13] Lemma 7.1. \( \square \)

6.3. The leading term of \( \Phi_{\gamma} \) for odd \( \gamma \): level 2 case. In this subsection, we assume that the level of \( \mathbf{v} = (1,0,0,0) \) in \( \Lambda_{\gamma} \) is 2, i.e., \( \ell = 2 \). Thus \( \gamma \equiv (0,0, \frac{1}{2}, \frac{1}{2}) \) or \( (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}) \) mod \( \mathbb{K} \). We take a vector \( \mathbf{r} \in \mathbb{E}_8(2)^\vee \) with \( \mathbf{r}^2 = -1 \) and define

\[ \Lambda_{\gamma} := \begin{cases} \mathbb{Z}\{(0,0,\frac{1}{2},\frac{1}{2}), r\} \oplus \mathbb{E}_8(2) & \text{if } \gamma \equiv (0,0,\frac{1}{2},\frac{1}{2}) \text{ mod } \mathbb{K}, \\ \mathbb{Z}\{(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}), r\} \oplus \mathbb{E}_8(2) & \text{if } \gamma \equiv (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}) \text{ mod } \mathbb{K}. \end{cases} \]

We set

\[ \mathbf{v}^\prime := \begin{cases} (0,1,0,0,0) & \text{if } \gamma \equiv (0,0,\frac{1}{2},\frac{1}{2}) \text{ mod } \mathbb{K}, \\ (0,1,1,0,0) & \text{if } \gamma \equiv (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}) \text{ mod } \mathbb{K}. \end{cases} \]

Then \( \mathbf{v}^\prime \in \Lambda_{\gamma} \) is a primitive isotropic vector of \( \Lambda_{\gamma} \) with \( \langle \mathbf{v}, \mathbf{v}^\prime \rangle = 2 \) and \( \langle \mathbf{v}^\prime, \mathbf{r} \rangle = 0 \).

We define \( \mathbf{w} \in \mathbf{v}^\perp \cap \mathbf{v}^\perp \cap \Lambda_{\gamma} \) and \( \mathbf{w}^\prime \in \mathbf{v}^\perp \cap \mathbf{v}^\perp \cap \Lambda_{\gamma}^\vee \) as

\[ \mathbf{w} := (0,0,-1,0,0), \quad \mathbf{w}^\prime := \begin{cases} (0,0,\frac{1}{2},-\frac{1}{2},0) & \text{if } \gamma \equiv (0,0,\frac{1}{2},\frac{1}{2}) \text{ mod } \mathbb{K}, \\ (\frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}, 0) & \text{if } \gamma \equiv (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}) \text{ mod } \mathbb{K}. \end{cases} \]

Then \( \mathbf{w}^2 = 0, \mathbf{w}^2 = -1, \langle \mathbf{w}, \mathbf{w}^\prime \rangle = 1 \) and \( \langle \mathbf{w}, \mathbf{r} \rangle = \langle \mathbf{w}^\prime, \mathbf{r} \rangle = 0 \).

We are going to express elements of \( \mathfrak{M}_{\gamma} = \mathbf{v}^\perp \cap \mathbf{v}^\perp \cap \Lambda_{\gamma} \), concretely. We note that an element of \( \mathbf{v}^\perp \cap \mathbf{v}^\perp \) in \( \mathbb{K}^\vee \oplus \mathbb{E}_8(2)^\vee \) is written as

\[ (a+b)\mathbf{w} + 2b\mathbf{w}^\prime + (0, x) = \begin{cases} (0,0,-a, b, x) & \text{if } \gamma \equiv (0,0,\frac{1}{2},\frac{1}{2}) \text{ mod } \mathbb{K}, \\ (\frac{1}{2}, 0, -a, b, x) & \text{if } \gamma \equiv (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}) \text{ mod } \mathbb{K}. \end{cases} \]

where \( a, b \in (1/2)\mathbb{Z} \) and \( x \in \mathbb{E}_8(2)^\vee \).

We define \( \rho, \rho^\prime \in \mathbf{v}^\perp \cap \mathbf{v}^\perp \cap \Lambda_{\gamma} \) as

\[ \rho := \mathbf{w} + \mathbf{w}^\prime + (0, \mathbf{r}) = \begin{cases} (0,0, -\frac{1}{2}, -\frac{1}{2}, \mathbf{r}) & \text{if } \gamma \equiv (0,0,\frac{1}{2},\frac{1}{2}) \text{ mod } \mathbb{K}, \\ (\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, \mathbf{r}) & \text{if } \gamma \equiv (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}) \text{ mod } \mathbb{K}. \end{cases} \]

\[ \rho^\prime := \mathbf{w}. \]
Lemma 6.5. The following equality holds:

\[
\varphi_\gamma(\tau, \tau') = \begin{cases}
((0, 0, 0, 0, 0), 0) & \text{if } \gamma \equiv (0, 0, 0, 1, 1) \mod \mathbb{K}, \\
((\frac{1}{2}, 0, 0, 1, 1), 0) & \text{if } \gamma \equiv (\frac{1}{2}, 0, 0, 1, 1) \mod \mathbb{K}, \\
(\gamma/2, (0, 0, 0, 0, 0)) & \text{if } \gamma \equiv (0, 0, 0, 1, 1) \mod \mathbb{K}, \\
((0, 0, 0, 0, 0)) & \text{if } \gamma \equiv (0, 0, 0, 1, 1) \mod \mathbb{K}.
\end{cases}
\]

Proof. To compute \(\varphi_\gamma(\tau, \tau')\), write \(\varphi(K_{\tau, \tau'}, \alpha) = j_\ell(u)\) with \(u \in \mathbb{M}_\ell \otimes \mathbb{R} + i \mathbb{C}_M^+\). By \(\text{(3.2)}\), we can express \(\beta_\gamma^{-1}(u) = -(\tilde{A} + \tilde{B} \tau + \tilde{D} \tau')/2\), where

\[
\tilde{A} = \beta_\gamma^{-1}(A) = \beta_\gamma^{-1}(\Gamma_{12}) - \langle \beta_\gamma^{-1}(f_2/2), \beta_\gamma^{-1} \alpha(\Gamma_{12}) \rangle \beta_\gamma^{-1} e_2 - \beta_\gamma^{-1} f_2
\]

\[
= \psi(\Gamma_{12}) - \langle \beta_\gamma^{-1}(f_2/2), \psi(\Gamma_{12}) \rangle v - v'
\]

\[
= \left\{ \begin{array}{ll}
((0, 0, 0, 0, 0)) & \text{if } \gamma \equiv (0, 0, 0, 1, 1) \mod \mathbb{K}, \\
((0, 0, 0, 0, 0)) & \text{if } \gamma \equiv (0, 0, 0, 1, 1) \mod \mathbb{K}.
\end{array} \right.
\]

Here we used \(\text{(3.2)}\) to get the fourth equality. Similarly, we get

\[
\tilde{B} = \beta_\gamma^{-1}(B) = -\langle \beta_\gamma^{-1}(f_2/2), \beta_\gamma^{-1} \alpha(\Gamma_{12}) \rangle \beta_\gamma^{-1} e_2 + \beta_\gamma^{-1} f_2
\]

\[
= -\langle v'/2, (0, 0, 1, 0, 0) \rangle v + ((0, 0, 1, 0, 0)) = ((0, 0, 1, 0, 0)),
\]

\[
\tilde{D} = \beta_\gamma^{-1}(D) = \langle \beta_\gamma^{-1}(f_2/2), \beta_\gamma^{-1} \alpha(\Gamma_{12}) \rangle \beta_\gamma^{-1} e_2 - \beta_\gamma^{-1} \alpha(\Gamma_{12})
\]

\[
= \langle v'/2, (0, 0, 0, -1, 0) \rangle v + ((0, 0, 0, 1, 0))
\]

\[
= \left\{ \begin{array}{ll}
((0, 0, 0, 1, 0)) & \text{if } \gamma \equiv (0, 0, 0, 1, 1) \mod \mathbb{K}, \\
((0, 0, 0, 1, 0)) & \text{if } \gamma \equiv (0, 0, 0, 1, 1) \mod \mathbb{K}.
\end{array} \right.
\]

Substituting these formulae into the following equation (cf. \(\text{(3.1)}\))

\[
\varphi_\gamma(\tau, \tau') = \beta_\gamma^{-1} \circ j_\ell^{-1} \circ \varphi(\tau, \tau') = \beta_\gamma^{-1}(u) = -(\tilde{A} + \tilde{B} \tau + \tilde{D} \tau')/2,
\]

we get the result. \(\square\)

We recall from Sect. \(2.3.4\) and the relation \(\beta_\gamma^{-1}(e_1) = \rho\) that

\[
\Pi^+_\gamma := \beta_\gamma^{-1}(\Pi^+) = \{ \lambda \in \mathbb{M}_\gamma \mid \langle \lambda, \rho \rangle > 0, \lambda^2 \geq -2 \}.
\]
By (5.5), \( \Phi_\gamma(\tau, \tau') = \pm \Phi_\epsilon(\beta, \varphi_\gamma(\tau, \tau')) \) is given by
\begin{align}
\pm \Phi_\gamma(\tau, \tau') &= 2^8e^{2\pi i(\epsilon_1, \beta, \epsilon_1, \varphi_\gamma(\tau, \tau'))} \prod_{\mu \in \mathbb{Z}_{>0} \cup \Pi^+} \left( 1 - e^{2\pi i(\mu, \beta, \epsilon_1, \varphi_\gamma(\tau, \tau'))} \right)^{-(1)(\mu, \epsilon_1, \epsilon_1)} c(\mu^2/2) \\
&= 2^8e^{2\pi i(\rho, \varphi_\gamma(\tau, \tau'))} \prod_{\lambda \in \mathbb{Z}_{>0, \rho} \cup \Pi^+} \left( 1 - e^{2\pi i(\lambda, \varphi_\gamma(\tau, \tau'))} \right)^{-(1)(\lambda, \rho, \rho')} c(\lambda^2/2) 
\end{align}
for \( \tau, \tau' \in \mathfrak{H} \), where we used \( \beta_\gamma^{-1}(\epsilon_1) = \rho, \beta_\gamma^{-1}(\epsilon_1) = \rho' \) to get the second equality.

We will compute the leading term of \( \Phi_\gamma(\tau, \tau') \) near \( (\tau, \tau') = (\pm i\infty, \pm i\infty) \) by dividing \( \prod_{\lambda \in \mathbb{Z}_{>0, \rho} \cup \Pi^+} \) into three parts: \( \prod_{\lambda \in \mathbb{Z}_{>0, \rho} \cup \Pi^+} \neq 0 \), \( \prod_{\lambda \in \mathbb{Z}_{>0, \rho} \cup \Pi^+} = 0 \), and \( \prod_{\lambda \in \mathbb{Z}_{>0, \rho} \cup \Pi^+} = 0 \), which will be respectively treated in the following.

First we consider the case \( \prod_{\lambda \in \mathbb{Z}_{>0, \rho} \cup \Pi^+} = 0 \) of (6.8).

**Lemma 6.6.** If \( \lambda \in \Pi^+_\gamma \) satisfies \( H_\lambda \cap \Omega^+_K \neq \emptyset \), then \( \lambda = \pm w' - r \).

**Proof.** Let \( \lambda \in \Pi^+_\gamma \) be such that \( H_\lambda \cap \Omega^+_K \neq \emptyset \). Then \( \lambda^2 < 0 \) (see (5.5)). Since \( \lambda^2 \in 2\mathbb{Z} \) and \( \lambda \in \Pi^+_\gamma \), we obtain \( \lambda^2 = -2 \), i.e., \( \lambda \in \Delta_{\mathfrak{K}, \gamma} \subset \Delta_{\mathfrak{A}, \gamma} \). We write \( \lambda = \lambda_1 + \lambda_2 \), where \( \lambda_1 \in \mathbb{K}^\top, \lambda_2 \in \mathbb{E}_8(2)^\top \). As in the proof of Step 2 of Proposition 5.1, we have \( \lambda_1^2 = \lambda_2^2 = -1 \). By Lemma 5.11 \( \Lambda_\gamma = \mathbb{Z}_\lambda + \mathbb{K} \oplus \mathbb{E}_8(2) \), so \( \lambda + \rho \in \mathbb{K} \oplus \mathbb{E}_8(2) \). Since \( \langle \lambda, \nu \rangle = 0 \), we can write \( \lambda_1 = (c, 0, \frac{a}{2}, \frac{a}{2}) \) with \( a \equiv b \equiv 1 \) mod 2 and \( c \in (1/2)\mathbb{Z} \). Since \( -1 = \lambda_1^2 = ab \), we get \( \lambda_1 = c' \nu \pm w' \) with \( c' \in (1/2)\mathbb{Z} \). Since \( \langle \lambda, \nu' \rangle = 0 \), we get \( c' = 0 \), so \( \lambda_1 = \pm w' \). It follows from \( \lambda + \rho \in \mathbb{K} \oplus \mathbb{E}_8(2) \) that \( \lambda_2 + \rho \in \mathbb{E}_8(2) \). Since \( 2\mathbb{R} \in \mathbb{E}_8(2) \), we can write \( \lambda_2 = -r + x \) with \( x \in \mathbb{E}_8(2) \). Since \( \lambda_2^2 = -1 \) and \( r^2 = 1 \), we get \( \langle r, x \rangle = x^2/2 \). Noting that \( \lambda \in \Pi^+_\gamma \), we have
\[
0 < \langle \rho, \lambda \rangle = \langle w + w' + r, \pm w' - r + x \rangle = \pm \langle w + w', w' \rangle + 1 + \langle r, x \rangle = 1 + x^2/2.
\]
Since \( x^2/2 \in 2\mathbb{Z}_{\leq 0} \), we have \( x = 0 \). Thus \( \lambda = \pm w' - r \).

**Lemma 6.7.** The following equality holds:
\begin{align}
\prod_{\lambda \in \Pi^+_\gamma, H_\lambda \cap \Omega^+_K \neq \emptyset} \left( 1 - e^{2\pi i(\lambda, \varphi_\gamma(\tau, \tau'))} \right)^{-(1)(\lambda, \rho, \rho')} c(\lambda^2/2) 
\end{align}

**Proof.** We have \( \langle \lambda, \rho - \rho' \rangle = \langle \pm w' - r, w' + r \rangle = \pm w'^2 - r^2 = 1 + 1 \) and \( c(\lambda^2/2) = c(-1) = 1 \), so \( -(1)(\lambda, \rho - \rho') c(\lambda^2/2) = 1 \). Since
\[
\langle \lambda, \varphi_\gamma(\tau, \tau') \rangle = \begin{cases} 
\pm (\frac{x - w'}{2}) & \text{if } \lambda = \pm w' - r, \gamma \equiv (0, 0, \frac{1}{2}, \frac{1}{2}) \mod \mathbb{K}, \\
\pm (\frac{x - w'}{2} - 1) & \text{if } \lambda = \pm w' - r, \gamma \equiv (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}) \mod \mathbb{K}
\end{cases}
\]
by (5.7), we get the result.

Next we consider the part \( \prod_{\lambda \in \Pi^+_\gamma, H_\lambda \cap \Omega^+_K \neq \emptyset} \) of (6.8).
Lemma 6.8. Let $\lambda \in \Pi_+^+$ be such that $H_\lambda \cap \Omega_+^+ = \emptyset$. We write $\lambda = \lambda_1 + \lambda_2 \in \Pi_+^+$, where $\lambda_1 = (c,0,a,b) \in \mathbb{K}_+^\vee$, $a,b,c \in (1/2)\mathbb{Z}$, $\lambda_2 \in \mathbb{E}_8(2)^\vee$. Then $a \leq 0$, $b \leq 0$, and $(a,b) \neq (0,0)$.

Proof. Since $H_\lambda \cap \Omega_+^+ = \emptyset$, we have $\lambda_1^2 \geq 0$ (see (2.5)). It follows that $ab \geq 0$. Hence “$a \leq 0$ and $b \leq 0$” or “$a \geq 0$ and $b \geq 0$.” To derive a contradiction, we assume that $a \geq 0$ and $b \geq 0$. We set $a' := -a$, $b' := -b$. Then $a' \leq 0$ and $b' \leq 0$. We set

$$\nu := \langle \rho, \lambda \rangle = \begin{cases} \langle (0,0,-\frac{1}{2},-\frac{1}{2}) + r, (c,0,a,b) + \lambda_2 \rangle & \text{if } \gamma \equiv (0,0,\frac{1}{2},\frac{1}{2}) \mod \mathbb{K}, \\ \langle (\frac{1}{2},0,-\frac{1}{2},-\frac{1}{2}) + r, (c,0,a,b) + \lambda_2 \rangle & \text{if } \gamma \equiv (\frac{1}{2},0,\frac{1}{2},\frac{1}{2}) \mod \mathbb{K}, \\ = -a - b + \langle r, \lambda_2 \rangle. \end{cases}$$

We also set $2k := \lambda^2$. Since $\lambda \in \Pi_+^+$, we have $\nu > 0$ and $k \geq -1$. Note that $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Since the sublattice $\mathbb{Z}r + \mathbb{Z}\lambda_2 \subset \mathbb{E}_8(2)^\vee$ is negative-definite, the matrix

$$\begin{pmatrix} r^2 & \langle r, \lambda_2 \rangle \\ \langle r, \lambda_2 \rangle & \lambda_2^2 \end{pmatrix} = \begin{pmatrix} -1 & \nu - a' - b' \\ \nu - a' - b' & 2k - 4a'b' \end{pmatrix}$$

is semi-negative, so $(\nu - a' - b')^2 \leq 4a'b' - 2k$. Thus we get the inequality

$$(a' - b')^2 + \nu^2 - 2\nu(a' + b') \leq -2k.$$  

Since $a' \leq 0$, $b' \leq 0$ and $\nu > 0$, we get $k < 0$. It follows from $k \in \mathbb{Z}$ and $k \geq -1$ that $k = -1$. Then, by (6.10), we conclude that $\nu = 1$ and $a' = b' = 0$. Since $a' = b' = 0$ and $\lambda \in \mathbb{M}_\gamma \subset \mathbb{V}^\perp$, we get $0 = \langle \lambda, \nu \rangle = c$. Hence $\lambda_1 = 0$, so $\lambda = \lambda_2 \in \mathbb{E}_8(2)$. Since $\lambda_2^2 = 2k = -2$, this contradicts $\Delta_{\mathbb{E}_8(2)} = \emptyset$. Thus we conclude $a \leq 0$, $b \leq 0$, and $(a,b) \neq (0,0)$.

Lemma 6.9. The following holds:

$$\prod_{\lambda \in \Pi_+^+, H_\lambda \cap \Omega_+^+ = \emptyset} \left( 1 - e^{2\pi i \langle \lambda, \varphi_\gamma(\tau, \tau') \rangle} \right)^{(-1)^{\langle \lambda, \rho \rangle - \rho}} c(\lambda^2/2) \in 1 + m.$$  

Proof. It suffices to show that for any $\lambda \in \Pi_+^+$ with $H_\lambda \cap \Omega_+^+ = \emptyset$, we have

$$1 - e^{2\pi i \langle \lambda, \varphi_\gamma(\tau, \tau') \rangle} \in 1 + m \subset \mathbb{Z}\{p^{1/2}, q^{1/2}\}.$$  

We write $\lambda = (c,0,a,b) + \lambda_2 = c\nu - (a+b)w - 2bw' + \lambda_2 \in \mathbb{M}_\gamma$ as in Lemma 6.9. By (6.6), (6.7), we get

$$\langle \lambda, \varphi_\gamma(\tau, \tau') \rangle = \begin{cases} -bt - a\tau' & \text{if } \gamma \equiv (0,0,\frac{1}{2},\frac{1}{2}) \mod \mathbb{K}, \\ -bt - a\tau' + b & \text{if } \gamma \equiv (\frac{1}{2},0,\frac{1}{2},\frac{1}{2}) \mod \mathbb{K}. \end{cases}$$

Since $a \leq 0$ and $b \leq 0$ with $(a,b) \neq (0,0)$ by Lemma 6.8, we have $e^{2\pi i \langle \lambda, \varphi_\gamma(\tau, \tau') \rangle} \in m$ and we get the result.

Finally we consider the part $\prod_{\lambda \in \mathbb{Z}_{2\rho \neq 0}}$.  

Lemma 6.10. The following equality holds:

$$e^{2\pi i \langle \rho, \varphi_\gamma(\tau, \tau') \rangle} = \begin{cases} p^{1/2}q^{1/2} & \text{if } \gamma \equiv (0,0,\frac{1}{2},\frac{1}{2}) \mod \mathbb{K}, \\ -p^{1/2}q^{1/2} & \text{if } \gamma \equiv (\frac{1}{2},0,\frac{1}{2},\frac{1}{2}) \mod \mathbb{K}. \end{cases}$$

Proof. Since $\langle \rho, \varphi_\gamma(\tau, \tau') \rangle_{\mathbb{M}_\gamma} = \frac{\tau + \tau'}{2}$ if $\gamma \equiv (0,0,\frac{1}{2},\frac{1}{2})$ and since $\langle \rho, \varphi_\gamma(\tau, \tau') \rangle_{\mathbb{M}_\gamma} = \frac{\tau + \tau'}{2}$ if $\gamma \equiv (0,0,\frac{1}{2},\frac{1}{2})$ by the definition of $\rho$ and (6.7), we get the result.
Lemma 6.11. The following holds:
\[ e^{2\pi i(n\rho,\varphi_\gamma(\tau,\tau'))} \prod_{n>0} \left(1 - e^{2\pi i(n\rho,\varphi_\gamma(\tau,\tau'))}\right)^{(-1)^{(n\rho,\varphi_\gamma(\tau,\tau'))} c((n\rho)^2/2)} \]
\[ \in \begin{cases} p^{1/2}q^{1/2}(1+m) & \text{if } \gamma \equiv (0,0,\frac{1}{2},\frac{1}{2}) \pmod{\mathbb{K}}, \\ -p^{1/2}q^{1/2}(1+m) & \text{if } \gamma \equiv (\frac{1}{2},0,\frac{1}{2},\frac{1}{2}) \pmod{\mathbb{K}}. \end{cases} \]

Proof. We have \( \langle n\rho,\rho - \rho' \rangle = \langle n\rho,\rho' \rangle = -n \) and \( c((n\rho)^2/2) = c(0) = 8 \) (see Sect. 5.4), so \( (-1)^{(n\rho,\varphi_\gamma(\tau,\tau'))} c((n\rho)^2/2) = 8 \cdot (-1)^n. \) By Lemma 6.10 we get \( \left(1 - e^{2\pi i(n\rho,\varphi_\gamma(\tau,\tau'))}\right)^{8(-1)^n} \in 1+m \) and the assertion (6.13).

All together, we obtain the leading term of (6.8).

Lemma 6.12. The following holds:
\[ \Phi_\gamma(\tau,\tau') \in \begin{cases} -2^8(p^{1/2} - q^{1/2})^2(1+m) & \text{if } \gamma \equiv (0,0,\frac{1}{2},\frac{1}{2}) \pmod{\mathbb{K}}, \\ -2^8(p^{1/2} + q^{1/2})^2(1+m) & \text{if } \gamma \equiv (\frac{1}{2},0,\frac{1}{2},\frac{1}{2}) \pmod{\mathbb{K}}. \end{cases} \]

Proof. Suppose that \( \gamma \equiv (0,0,\frac{1}{2},\frac{1}{2}). \) Then substituting (6.9), (6.11), (6.13) into (6.8), we obtain \( \Phi_\gamma(\tau,\tau') \in 2^8(1-p^{-1/2}q^{1/2})(1-p^{1/2}q^{-1/2})p^{1/2}q^{1/2}(1+m) = -2^8(p^{1/2} - q^{1/2})^2(1+m). \)
This proves the first case. The second case is shown similarly.

6.4. The leading term of \( \prod_{\gamma \text{ odd}} \Phi_\gamma. \)

Proposition 6.13. The following holds:
\[ \prod_{\gamma \text{ odd}} \Phi_\gamma(\tau,\tau') \in 2^{16}(p-q)^2(1+m). \]

Proof. The result follows from Lemmas 6.11 and 6.12.

7. Proof of Theorem 1.1

We first show (1.1) up to a constant, and then determine the constant.

7.1. The formula (1.1) up to a constant. Recall that the \( j \)-invariant \( j(\tau) \) is the \( \text{SL}_2(\mathbb{Z}) \)-invariant holomorphic function on \( \mathfrak{H} \) defined as
\[ j(\tau) = \frac{(1 + 240 \sum_{n>0} \sigma_3(n)p^n)^3}{p \prod_{n>0} (1-p^n)^{24}} = \frac{1}{p} + 744 + 196884p + \cdots, \]
where \( p = \exp(2\pi i \tau) \) and \( \sigma_3(n) = \sum_{d|n} d^3. \) Then \( Y(1) = \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{H} \) is isomorphic to \( \mathbb{C} \) and \( X(1) = Y(1)^* \) is isomorphic to \( \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \) via \( j \).

Let \( \text{pr}_i: X(1) \times X(1) \to X(1) \) be the projection to the \( i \)-th factor. Under the identification \( X(1) \cong \mathbb{P}^1 \) via \( j \), we get the equality of divisors on \( X(1) \times X(1) \)
\[ \text{div}(\text{pr}_1^* j - \text{pr}_2^* j) = \Delta_{X(1) \times X(1)} - (\{\infty\} \times \mathbb{P}^1) - (\mathbb{P}^1 \times \{\infty\}). \]

Recall from Sect. 5.4 that \( \Sigma^2 X(1) := (X(1) \times X(1))/\mathfrak{S}_2 \cong \mathbb{P}^2 \), that \( \Delta \) is the image of the diagonal of \( X(1) \times X(1) \) in \( \Sigma^2 X(1) \), and that \( B \) is the line at infinity of \( \Sigma^2 X(1) \cong \mathbb{P}^2 \). Since the projection \( X(1) \times X(1) \to \Sigma^2 X(1) \) is a double covering with ramification divisor \( \Delta \), we get
\[ \text{div}(\text{pr}_1^* j - \text{pr}_2^* j)^2 = \Delta - 2B \]
on $\Sigma^2 X(1)$. Thus we get the following equation of currents on $\Sigma^2 X(1)$:

\[ -dd^c \log \left| (\text{pr}_1^* j - \text{pr}_2^* j)^2 \right| = -\delta_\Delta + 2\delta_B. \]  

(7.1)

On the other hand, since the Dedekind \( \eta \)-function $\eta(\tau)$ is a modular function of half weight on $\frak{H}$, we have

\[ -dd^c \log \| \eta(\tau)^{24} \|^2 = 12\tilde{\omega}_{Y(1)} - \delta_\infty \]

as a current on $X(1)$, where $\tilde{\omega}_{Y(1)}$ is the extension of the Kähler form of $Y(1)$ induced from the Poincaré metric on $\frak{H}$. It follows from Theorem 1.1 that as a current on $\Sigma^2 X(1)$ we have

\[ -dd^c \log \left( \prod_{\gamma \text{ even}} \| \Phi_\gamma \|^2 \right) = -dd^c \log \left( \prod_{\langle j \rangle \neq \langle 12 \rangle \langle 156 \rangle} \| \Phi_{\langle j \rangle} \|^2 \right) = 36\tilde{\omega}_{\Sigma^2 Y(1)} - 3\delta_B. \]

Together with Proposition 6.3, we get the following equation of currents on $\Sigma^2 X(1)$:

\[ -dd^c \log \left( \prod_{\gamma \text{ even}} \| \Phi_\gamma \|^2 \right) = -dd^c \log \left( \prod_{\langle j \rangle \neq \langle 12 \rangle \langle 156 \rangle} \| \Phi_{\langle j \rangle} \|^2 \right) = -3\delta_\Delta + 6\delta_B. \]

(7.2)

Combining (7.1) and (7.2), we get the following equation of currents on $\Sigma^2 X(1)$

\[ -dd^c \log \left( \prod_{\gamma \text{ odd}} \| \Phi_\gamma \|^2 \right) \cdot \left| (\text{pr}_1^* j - \text{pr}_2^* j)^2 \right|^{-6} = 0. \]

(7.3)

Since $\Sigma^2 X(1)$ is compact, there exists by (7.3) a nonzero constant $C$ such that

\[ \prod_{\gamma \text{ odd}} \Phi_\gamma^6 / \prod_{\gamma \text{ even}} \Phi_\gamma^4 = C (\text{pr}_1^* j - \text{pr}_2^* j)^{12}. \]

(7.4)

7.2. **Determination of the constant.** Let $a(n)$ be the $n$-th Fourier coefficient of $j(\tau) - 744$. As before, we put $p = \exp(2\pi i \tau)$ and $q = \exp(2\pi \tau')$. The denominator formula for the Monster Lie algebra [3] Lemma 7.1] states that

\[ j(\tau) - j(\tau') = (p^{-1} - q^{-1}) \prod_{m,n > 0} (1 - p^m q^n)^{a(mn)} \]

for all $\tau, \tau' \in \frak{H}$. Since $\eta(\tau) = p^{1/24} \prod_{n > 0} (1 - p^n)$, we get by Theorem 4.1

\[ \prod_{\gamma \text{ even}} \Phi_\gamma(\tau, \tau')^2 = \prod_{\langle j \rangle \neq \langle 12 \rangle \langle 156 \rangle} \Phi_{\ell(\langle j \rangle)} \left( j^{-1}_{\ell(\langle j \rangle)} \varpi(\langle j \rangle, \tau') \right)^2 \in 2^{96}(pq)^6(1 + m). \]

Together with Proposition 6.3, we get

\[ (p^{-1} - q^{-1})^{-12} \prod_{\gamma \text{ odd}} \Phi_\gamma(\tau, \tau')^6 \prod_{\gamma \text{ even}} \Phi_\gamma(\tau, \tau')^4 \equiv 2^{-96} (p^{-1} - q^{-1})^{-12} (p - q)^{12} (pq)^{12} (1 + m) = 2^{-96} \mod m. \]

(7.6)

Comparing (7.3) and (7.6), we get

\[ (j(\tau) - j(\tau'))^{-12} \prod_{\gamma \text{ odd}} \Phi_\gamma(\tau, \tau')^6 \prod_{\gamma \text{ even}} \Phi_\gamma(\tau, \tau')^4 \equiv 2^{-96} \mod m. \]

(7.7)

By (7.4), (7.7), we get $C = 2^{-96}$. This completes the proof of Theorem 1.1. \qed
7.3. Open problems. We pose some problems, which may merit further study.

Problem 1. We study the lattice embeddings $\mathcal{K} \hookrightarrow \Lambda$ to relate the difference of the $j$-invariants and the Borcherds $\Phi$-function. With other lattice embeddings, does our method produce other relations between seemingly unrelated modular functions? Two particularly interesting cases seem

1) the Enriques structures on very general Jacobian Kummer surfaces, whose classification was given by Ohashi [22], and

2) the Enriques surfaces with cohomologically trivial involutions, studied by Horikawa [12], Barth-Peters [1], Mukai-Namikawa [18] and Mukai [17, Appendix A].

The former is parametrized by Siegel 3-folds and the latter by $X^1(2) \times X^1(2)$ minus the diagonal.

Problem 2. Our computation of the restrictions of $\Phi$ for various embeddings $\mathcal{K} \hookrightarrow \Lambda$ is achieved by geometric considerations of the period mapping for Enriques surfaces. Is there an alternate (non-geometric) proof using techniques of Borcherds products such as Schofer’s formula [24] or Ma’s formula [16]?

Problem 3. By Freitag–Salvati-Manni [8], $\Phi$ can be viewed as a theta series, and our calculation in Sect. 11 may be viewed as the determination of the leading terms of the theta series of the orthogonal complement of $\mathcal{K}$ in $\Lambda$. It is also natural to expect that $\Phi_\gamma$ may be expressed as some type of theta series. For even $\gamma$, this is indeed the case by [13]. For odd $\gamma$, we do not know any explicit formula for $\Phi_\gamma$.

Problem 4. By [26], $\Phi$ is the equivariant analytic torsion of $K3$ surfaces with fixed-point-free involution, and analytic torsions are firmly tied with Gillet–Soulé’s arithmetic Riemann–Roch formula [9]. Is it possible to deduce the formula (1.1) from the arithmetic Riemann–Roch formula or its equivariant extension [17]?

Appendix: Some properties of the Enriques lattice

We prove some technical results used in the proof of Lemma 2.2. To simplify the notation, we write $d^\perp$ for $d^\perp_{\Lambda}$. By Lemma 2.1, we have $d^\perp \cong \mathbb{Z}/2(2)$.

Lemma A.1. Let $O^+(\Lambda)_d$ be the stabilizer of $d$ in $O^+(\Lambda)$. Then the restriction map $O^+(\Lambda)_d \ni g \mapsto g|_{d^\perp} \in O^+(d^\perp)$ is surjective.

Proof. Recall that for an even 2-elementary lattice $L$, a vector $\lambda \in L^\vee$ is said to be characteristic if $(\lambda, x)_L \equiv x^2 \mod \mathbb{Z}$ for all $x \in L^\vee$. By [20, p. 150], a characteristic vector always exists. Let $\lambda_1 \in (\mathbb{Z}d)^\vee$ and $\lambda_2 \in (d^\perp)^\vee$ be characteristic vectors of the 2-elementary lattices $\mathbb{Z}d$ and $d^\perp$, respectively. Note that $(\mathbb{Z}d)^\vee = \mathbb{Z}(d/2)$, so $\lambda_1 \in \mathbb{Z}(d/2)$. The choice of $\lambda_1$ (resp. $\lambda_2$) is unique up to a vector of $\mathbb{Z}d$ (resp. $d^\perp$).

We define $L := \mathbb{Z}(\lambda_1 + \lambda_2) + \mathbb{Z}d \oplus d^\perp$.

Claim. The lattice $L$ is isometric to $\Lambda$.

Since $d/2$ is a characteristic vector of $\mathbb{Z}d$, we take $\lambda_1 = d/2$. We see that $(3, -1, \ldots, -1)/2 \in \mathbb{Z}/2(2)^\vee$ is a characteristic vector of $\mathbb{Z}/2(2)$, which we take as $\lambda_2$ via the identification of $d^\perp \cong \mathbb{Z}/2(2)$. Then $\lambda_1^2 = -\frac{1}{2}$ and $\lambda_2^2 = \frac{1}{2}$. Any element of $L$ can be expressed as $x_1 + \lambda_1 + x_2 + \lambda_2$, where $x_1 \in \mathbb{Z}d$ and $x_2 \in d^\perp$. Since

\[(x_1 + \lambda_1 + x_2 + \lambda_2)^2 = (x_1 + \lambda_1)^2 + (x_2 + \lambda_2)^2 \equiv \lambda_1^2 + \lambda_2^2 = 0 \mod 2\mathbb{Z},\]
L is an even lattice. Considering the inclusions of lattices $\mathbb{Z}d \oplus (d^\perp) \subset L \subset L^\vee \subset \mathbb{Z}(d/2) \oplus (d^\perp)^\vee$, $A_2$ is a quotient of a subspace of $A_2 \mathbb{Z}d \oplus A_2 \mathbb{Z}d \cong F_{2,2}$. Hence $L$ is 2-
 by Lemma 2.1, it suffices to prove that, if $\delta \in O(d^\perp)$, $g$ preserves $L$. By the above claim, we have $g \in O(A_d)$ and $g|_{d^\perp} = \gamma$. Thus we get $\delta(L) = 0$. All together, we obtain the claim $L \cong A$ by [20, Th. 3.6.2].

Let $\gamma \in O(d^\perp)$ be an arbitrary element. We set $g := 1_{d^\perp} \oplus \gamma \in O(\mathbb{Z}d \oplus d^\perp)$. Since, by [20, Lemma 3.9.1], $1_{d^\perp}$ and $\gamma$ respectively preserve the set of characteristic vectors of $Z(d/2)$ and $(d^\perp)^\vee$, $g$ preserves $L$. By the above claim, we have $g \in O(A_d)$ and $g|_{d^\perp} = \gamma$. If $\gamma \in O^+(d^\perp)$, then $\gamma$ preserves the connected components of $\Omega_{d^\perp}$. Since $g$ is an extension of $\gamma$, $g$ preserves the connected components of $\Omega_A$. That is, if $\gamma \in O^+(d^\perp)$, then $g \in O^+(A_d)$. $\square$

**Lemma A.2.** Let $d \in \Delta_A$. If $F \subset d^\perp$ is a totally isotropic primitive sublattice of rank 2, then any element of $SL(F)$ lifts to an element of $O^+(d^\perp)$.

**Proof.** Since $d^\perp \cong \mathbb{Z}_2(2)$ by Lemma 2.1, it suffices to prove that, if $F$ is a totally isotropic primitive sublattice of $\mathbb{Z}_2(2)$ of rank 2, then any element of $SL(F)$ lifts to an element of $O^+(\mathbb{Z}_2(9), 2)$. We take an identification $\mathbb{Z}_2(9) \cong U \oplus U \oplus \mathbb{Z}_2(-1)$, where $U$ is the positive-definite unimodular lattice of rank 7 given by the Gram matrix $1_7$. Let $e, e'$ (resp. $f, f'$) be the standard free basis of the left (resp. middle) lattice $U$ of $U \oplus U \oplus \mathbb{Z}_2(-1)$. By [20, Prop. 1.17.1], $I_2, 9$ has a unique primitive totally isotropic sublattice of rank 2 up to $O(\mathbb{Z}_2, 9)$. Thus we may assume that $F = Z_0 + Zf$. Let $\gamma = (x, y, z) \in SL(F)$ be any element. Since $O^+(U \oplus U) \subset O^+(U \oplus U \oplus \mathbb{Z}_2(-1))$ in the obvious way, it suffices to show that $\gamma$ lifts to an element of $O^+(U \oplus U)$. We define the lattice homomorphism $g: U \oplus U \to U \oplus U$ as

$$g(e) = ae + cf, \quad g(f) = be + df, \quad g(e') = de' - bf', \quad g(f') = -ce' + af'. \quad$$

Then $g \in O(U \oplus U)$ and $g|_U = \gamma$. Further, with the identification of $x e + y e' + z f + w f' \in \mathbb{P}(U \oplus U \oplus \mathbb{C})$ with $(x : y : z : w) \in \mathbb{P}(\mathbb{C}^2)$, we regard

$$\Omega_{U \oplus U} = \{(x : y : z : w) \in \mathbb{P}(\mathbb{C}^4) | xy + zw = 0, x\overline{y} + \overline{x}y + z\overline{w} + \overline{z}w > 0\}$$

and $\Omega^+_{U \oplus U} = \Omega_{U \oplus U} \cap \{\text{Im}(z/x) > 0, \text{Im}(w/x) > 0\}$. Then we see that $g$ maps $(1 : 1 : i : i) \in \Omega^+_{U \oplus U}$ to an element of $\Omega^+_{U \oplus U}$. We conclude that $\gamma$ lifts to $g \in O^+(U \oplus U)$. $\square$

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