Lateral-Torsional Buckling of Beams Elastically Restrained and Loaded with Concentrated Moments at Supports

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Abstract. The study presents the results of theoretical investigations into lateral-torsional buckling (LTB) of free supported bi-symmetric I-beams, elastically restrained against warping at supports. Beams with loaded by moments focused at supports (with a variable the ratio of moments) and with different types of ribs were taken into account. To determine the critical moment \( M_{cr} \) of LTB of beams, the \( MLTB,EL \) program [7] and proposed in this article formula were used. The obtained results were compared with the results received from approximation formulas [6] and with FEM (LTBeam, Abaqus).

1. Introduction
The study of lateral-torsional buckling (LTB) of beams are performed in a wide range, and despite this there is a number of issues whose solution would improve the design of these structural elements. The development of simplified analytical methods for accurate the calculations at the stage of initial design and verification of results from numerical analysis is advisable. Such methods due to their simplicity and in many cases satisfactory accuracy, can be successfully used in the basic designing.

The problem of lateral-torsional buckling of steel rod elements with I-sections, loaded with moments focused at supports, occurs inter alia in compression and bending columns, in frames and in continuous beams, e.g. from combinations of variable loads. The issues it could be supplemented include among others the possibility of taking into account the degree of elastic mounting of the beam against warping at support.

The influence of the elastically restrained of beams at supports was the subject of relatively few theoretical studies [1, 2, 3] and experimental works [2, 4].

In the case of bi-symmetric I-beams, which are stiffened with the end plates \( t_p \) at supports, the critical moment \( M_{cr} \) of LTB can be determined by the formulas given by [4-7]. However, from own [7] and other authors’ research [3], follows, that the classic thickness of end plate \( t_p \approx 1.5t_f \) gives a small indicator of elastic restrained against warping and relatively slightly influences to increase of the critical load.

In study [7], there are proposed the possibility of estimating \( M_{cr} \) for beams, having regard to the different variants of ribs at supports. For the beams loaded with a constant bending moment along the length, the influence of different type of the ribs on the value of the critical moment of LTB was presented in [3]. A similar analysis for closed ribs was carried out in [8]. In turn, in [9], the theoretical determination of torsional stiffness of various types of support ribs was focused. The advantages of use the intermediate ribs, that can restrained warping along the length of beams is shown, among
others, in [2, 10]. Selected analytical results have been verified by experimental studies e.g. [2, 10] and in numerical calculations using FEM (LTBeam [11], Abaqus) e.g. [3, 7].

In the paper the LTB of bi-symmetric I-beams loaded with moments focused at supports (with a variable the ratio of moments) and elastically restrained against warping at support, was taking into account. The critical moments were calculated by the \( M_{LTB,EL} \) program [7]. In addition, approximation formula to determine the elastic critical moment of lateral-torsional buckling (\( M_c \)) were proposed. The results were compared with the values obtained from the formulas available in [6] and from the FEM.

2. The critical moment of lateral-torsional buckling of beams with end plates

The critical moment of LTB of simply supported beam, with the end plates and loaded with moments focused at supports (Figure 1), can be calculated using the formula [6]:

\[
M_{kl} = \zeta \cdot \frac{\pi^2 EI_z}{L^2} \cdot \frac{1}{I_z} \left( \frac{I_{wz} + GI_z L^2}{\beta_0^2 + \frac{\pi^2 E}{I_z}} \right)
\]  

where: \( \zeta \) – coefficient dependent on the bending moment distribution [5],

\[
\zeta = 1,77 - 1,04 \psi + 0,27 \psi^2 < 2,60
\]  

\( \psi \) – ratio of concentrated support moments (-1 ≤ \( \psi \) ≤ 1),

\( \beta_0 \) – coefficient of the torsional buckling length:

\[
\beta_0 = 1 - \frac{0.5}{1 + \frac{2EI_{wz}}{c_{\omega} \cdot L}}
\]  

where: \( c_{\omega} \) – St. Venant stiffness of the end plate [6]:

\[
c_{\omega} = \frac{1}{3} Gr_t^2 b (h - t_f)
\]

where: \( t_p, b \) – thickness and width of end plate (Fig. 1),

\( h, t_f \) – height of the beam section, thickness of the beam flange (Fig. 1)

Figure 1. Static scheme of beam: a) view, b) section A-A, c) section B-B.

3. Program \( M_{LTB,EL} \)

The critical moment of LTB of beams elastically restrained against warping at supports, for typical static schemes, can be determined using the \( M_{LTB,EL} \) calculation program presented in [7]. At the
numerous verification analyses of the $M_{LTBE,EL}$ program a very good compliance of $M_{cr}$ in comparison with FEM ($LTBeam$, $Abaqus$), were achieved [7].

In $M_{LTBE,EL}$ program, in order to take into account cases of the asymmetric bending moment distribution (relative to the transverse axis of the beam), the function of the torsion angle ($\phi$) of beam was approximated with three terms ($i = 1, 2, 3$) of the "coupled" polynomial series [7]:

$$\phi(x) = \sum_{i=1}^{3} a_i ((1 - \kappa) W_{Pi} + \kappa W_{Ui})$$  \hspace{1cm} (5)

where: $a_i$ – free parameters of the torsion angle function,
$W_{Pi}$, $W_{Ui}$ – polynomials describing the "deflection" function of the simply supported ($P_i$) and the restrained ($U_i$) beam [7],
$\kappa$ – fixity factor:

$$\kappa = \frac{\alpha_{\omega} L}{2EI_{\omega} + \alpha_{\omega} L} \quad \text{and} \quad \alpha_{\omega} = -\frac{B}{d\phi/dx}$$  \hspace{1cm} (6ab)

where: $\alpha_{\omega}$ – stiffness of the elastic restraint,
$B$ – bimoment at the site of the beam support,
$d\phi/dx$ – warping in the support section.

The fixity factor $\kappa$ (6a) ranges from $\kappa = 0$, for the end free to warp, to $\kappa = 1$, for warping fully restrained. Formulas allowing to determine $\alpha_{\omega}$ (6b) for the most often analysed types of ribs are given by research of Kowal [8, 9], and others [2, 3 and 10]. Table 1 presents selected type of ribs and the formulas on $\alpha_{\omega}$ according to work [3].

4. Approximation formulas on $M_{cr}$

In study [7], relatively simple approximation formulas for the critical moment of LTB of beam for the most commonly used static schemes, were derived. For this purpose, a program to symbolic calculation was developed in the Mathematica® package environment. Deriving the formulas to $M_{cr}$, only the first term of the "coupled" polynomial series (5) were used, which means that $i = 1$ (one half-wave describing the axis of the deformed beam during lateral-torsional buckling). A very good agreement between the results obtained in that way, and those produced by FEM ($LTBeam$, $Abaqus$) was achieved.

However, in the case of beams loaded with moments focused at supports (Figure 1), this approach allows correct approximation of the form of LTB only in the range $0 \leq \psi \leq 1$. For the full variability the ratio of moments, i.e. $-1 \leq \psi \leq 1$, this solution is insufficient, due to the occurrence in the range $-1 \leq \psi < 0$ of the more asymmetrical form of beam deformation.

In order to determine the critical moment of LTB of beam loaded with moments focused at supports (Figure 1) in the full variability range (for $-1 \leq \psi \leq 1$), two terms ($i = 1, 2$) of the "coupled" polynomials series (5) can be used. However, such an approximation of the torsion angle function gives a very complicated form of the analytical formula, which is not useful for practical application in engineering calculations.
Table 1. Selected ways of ribbing the beams at the supports [3].

| Item | Types of ribs | Formula |
|------|---------------|---------|
| 1    |               | \( \alpha_{\omega} = GI_d h_0 \) |
|      |               | \( I_d = br^3_p / 3 \) |
| 2    |               | \( \alpha_{\omega} = 2GI_d I_{ef} + b \) |
|      |               | \( I_{ef} = l_0^3 / 3 \) |
|      |               | \( I_d = l_0^3 / 3 \) |
| 3    |               | \( \alpha_{\omega} = E I_d l_0^3 \cos^3 \alpha + 2GI_d h_0 \) |
|      |               | \( I_1 = b^3 l_1 / 12 \) |
|      |               | \( I_d = br^3_p / 3 \) |
| 4    |               | \( \alpha_{\omega} = GI_d h_0 \) |
|      |               | \( I_d = 2l_0^3 / b^3 \) |

Therefore, the approximation formula for \( M_{cr} \) for the entire interval \((-1 \leq \psi \leq 1)\) of the variability of the ratio of moments focused at supports was developed by using only the first term \((i = 1)\) of function of the torsion angle (5) and applying an correction factor \( D_1 \). In this case, the approximation formula was presented as:

\[
M_{cr} = D_1 \sqrt{EI_1 \left( C_1 GI_d L^3 + C_2 EI_{\omega} \right)} / C_3 L^2 \tag{7}
\]

where: \( C_1, C_2, C_3 \) – coefficients (Table 2)

\( D_1 \) – correction factor (Table 2), which has been calibrated using the IPE300 beam with the span \( L = 5m \) and the full range of variability coefficients \( \psi \) (from -1 to 1) and \( \kappa \) (from 0 to 1).

Practical application of the formula (7) it comes down to the elaboration of a simple Excel spreadsheet, in which the stiffness of \( \alpha_{\omega} \) (6b) can be estimated using [2, 3, 8-10], and the fixity factor \( \kappa \) can be calculated according to (6a) [7].
Table 2. Coefficients $C_1$, $C_2$, $C_3$, $D_1$.

\[
\begin{align*}
C_1 &= 35 \cdot \left(1.457 - 2.4\kappa + \kappa^2 \right) \\
C_2 &= 420 \cdot \left(1.2 - \kappa \right) \\
C_3 &= \sqrt{\left(1.462 - 2.417\kappa + \kappa^2 \right) \left(1 + \psi^2 \right) + 1.5 \left(1.495 - 2.444\kappa + \kappa^2 \right) \psi} \\
D_1 &= \beta_1 + \beta_2\psi^2 + \beta_3\psi^3
\end{align*}
\]

for $-1 \leq \psi \leq -0.5$

\[
\begin{align*}
D_1 &= \beta_1 + \beta_2\psi^2 + \beta_3\psi^3
\end{align*}
\]

where:

\[
\begin{align*}
\beta_1 &= 0.027 \cdot \left(34.065 - 0.585\kappa + \kappa^5 \right) \\
\beta_2 &= 0.324 \cdot \left(0.193 - 0.05\kappa - \kappa^5 \right) \\
\beta_3 &= 0.233 \cdot \left(0.008 - 0.135\kappa - \kappa^5 \right)
\end{align*}
\]

for $-0.5 < \psi \leq 1$

\[
\begin{align*}
D_1 &= \beta_1 + \beta_2\psi^2 + \beta_3\psi^3
\end{align*}
\]

where:

\[
\begin{align*}
\beta_1 &= 0.00247 \cdot \left(392.653 - 2.259\kappa - \kappa^5 \right) \\
\beta_2 &= 0.024 \cdot \left(2.496 + 0.659\kappa + \kappa^5 \right) \\
\beta_3 &= -0.025 \cdot \left(1.377 + 0.406\kappa + \kappa^5 \right)
\end{align*}
\]

5. FEM verification - LTBeam, Abaqus

To verify the calculation results obtained from the ML2BE,EL program [7] and approximation formulas (1) [6] and (7) numerical analyses FEM (LTBeam and Abaqus) were performed.

The LTBeam [11] program, based on finite bar elements, allows taking into account the classic (fork support or full restraint) or elastic boundary conditions. In case of warping restrained the $\alpha_{w}$ rigidity for a given type of ribs at supports should be determined, e.g. acc. to Table 1.

In turn, in the Abaqus program (Figure 2), to model the analysed beams together with the physical interpretation of the rib type at the supports, volumetric elements (C3D8) were used, eight-node with six degrees of freedom in the node. The beam models were discretized by the finite element mesh, with mesh size is not exceed 10mm. Boundary conditions at the supports prevented displacements with respect to the principal axes of the section inertia, and along the longitudinal axis for one of the supports. The load with moments focused at supports (Figure 1) was defined in the form of pairs of concentrated forces, directed in opposite directions and applied at the extreme nodes of the cross-section of beams. The calculations were carried out in the elastic range, using the "buckling" procedure. Figure 2 shows the results of analysis in the Abaqus program for the IPE300 beam, the span $L = 5$m and the end plate $t_p = 20$mm.
6. Calculation examples

The analysis includes beams from the IPE300 section ($I_z = 604 \text{cm}^4$, $I_\omega = 125900 \text{cm}^6$, $I_t = 20.7 \text{cm}^4$, $E = 210 \text{GPa}$, $G = 81 \text{GPa}$) with span of $L = 5 \text{m}$, which were loaded by concentrated moments at supports (acc. to Figure 1). The critical moment values were calculated for the full range of variability of the moment ratio $\psi$ from -1 to 1.

6.1. Influence of the stiffening of the end plate ($t_p$) on the value of $M_{cr}$

In Table 3, the influence of the warping stiffness of the end plates ($t_p$) on the critical moment of LTB was analysed. Calculations were made using (1) [6] and (7) formula, $M_{LT,EL}$ [7] program and FEM ($LTBeam, Abaqus$). Critical moments received from the $LTBeam$ program were taken as a reference.

| Item | $\psi$ | $t_p$ [mm] | LTBeam | Abaqus | $\%$ 5-4 | $\%$ 7-4 | $\%$ 9-4 | $\%$ 11-4 |
|------|-------|------------|--------|--------|---------|---------|---------|---------|
| 1    | 0     | 5          | 116.7  | 117.7  | 0.8     | 116.7   | 0       | -0.1    |
| 2    | 0.75  | 10         | 118.7  | 121.3  | 2.2     | 118.7   | 0       | 0.3     |
| 3    | 20    | 30         | 148.0  | 154.4  | 4.3     | 148.0   | 0       | -0.4    |
| 4    | 40    | 50         | 164.1  | 168.4  | 2.6     | 164.0   | -0.1    | -3.8    |
| 5    | 50    | 70         | 174.7  | 177.7  | 1.7     | 174.8   | 0.1     | -6.5    |
| 6    | 100   | 90         | 133.1  | 134.2  | 0.8     | 133.1   | 0       | 132.6   |
| 7    | 0     | 20         | 148.1  | 155.2  | 4.8     | 148.2   | 0.1     | 150.2   |
| 8    | 0.75  | 30         | 168.9  | 176.1  | 4.3     | 168.8   | -0.1    | -0.3    |
| 9    | 20    | 50         | 187.2  | 192.1  | 2.6     | 187.1   | -0.1    | 180.2   |
| 10   | 40    | 70         | 199.4  | 202.7  | 1.7     | 199.5   | 0.1     | 186.6   |

Table 3 a) Comparison of critical moments for lateral-torsional buckling.
The results in Table 3 show that the critical moments of LTB, determined with $M_{LTE,EL}$ program [7] (column 7) confirm the results obtained from $LTBeam$. Maximum differences did not exceed +0.2%. Such a good agreement of the results was obtained by approximation the torsion angle ($\phi$) of the beam by use the first three terms of the series (5) [7]. In the case of the $Abaqus$ program, the differences in

| Item | $\psi$ | $\psi_{5-4}$ | $\psi_{7-4}$ | $\psi_{9-4}$ | $\psi_{11-4}$ |
|------|--------|-------------|-------------|-------------|-------------|
| 13   | 0      | 0.8         | 0.1         | -0.2        | 153.2       |
| 14   | 0      | 2.1         | 0.2         | 0.7         | 158.8       |
| 15   | 0      | 4.7         | 0.1         | 1.1         | 170.5       |
| 16   | 0      | 4.3         | 0.6         | -0.6        | 194.5       |
| 17   | 0      | 2.6         | 0.0         | -4.0        | 215.9       |
| 18   | 0      | 1.6         | 0.1         | -6.7        | 230.4       |
| 19   | 0      | 0.7         | 0.1         | -1.3        | 179.4       |
| 20   | 0      | 4.7         | 0.0         | -0.1        | 199.6       |
| 21   | 0      | 4.2         | 0.1         | -1.8        | 227.8       |
| 22   | 0      | 2.5         | 0.2         | -5.3        | 253.0       |
| 23   | 0      | 1.5         | 0.1         | -8.1        | 270.2       |
| 24   | 0      | 0.5         | 0.1         | -1.3        | 212.9       |
| 25   | 0      | 1.9         | 0.1         | -3.2        | 216.4       |
| 26   | 0      | 4.5         | 0.1         | -2.3        | 236.9       |
| 27   | 0      | 4.1         | 0.1         | -1.8        | 237.8       |
| 28   | 0      | 2.5         | 0.1         | -6.7        | 270.2       |
| 29   | 0      | 1.4         | 0.1         | -11.0       | 321.8       |
| 30   | 0      | 0.2         | 0.1         | -6.1        | 253.8       |
| 31   | 0      | 0.2         | 0.1         | -5.8        | 258.0       |
| 32   | 0      | 4.2         | 0.1         | -5.0        | 269.2       |
| 33   | 0      | 4.5         | 0.1         | -2.3        | 236.9       |
| 34   | 0      | 4.1         | 0.1         | -4.2        | 270.7       |
| 35   | 0      | 2.5         | 0.1         | -5.3        | 253.0       |
| 36   | 0      | 1.5         | 0.1         | -8.1        | 270.2       |
| 37   | 0      | 0.3         | 0.1         | -7.4        | 295.3       |
| 38   | 0      | 1.1         | 0.1         | -7.1        | 300.4       |
| 39   | 0      | 3.9         | 0.1         | -6.7        | 330.5       |
| 40   | 0      | 3.5         | 0.1         | -9.3        | 381.4       |
| 41   | 0      | 1.7         | 0.1         | -14.2       | 429.0       |
| 42   | 0      | 0.5         | 0.1         | -17.4       | 463.1       |
| 43   | 0      | 0.8         | 0.1         | -7.6        | 326.2       |
| 44   | 0      | 0.7         | 0.1         | -7.4        | 324.2       |
| 45   | 0      | 3.6         | 0.1         | -7.6        | 368.9       |
| 46   | 0      | 3.3         | 0.1         | -11.0       | 429.8       |
| 47   | 0      | 1.2         | 0.1         | -16.4       | 487.3       |
| 48   | 0      | 0.1         | 0.2         | -21.0       | 530.1       |
| 49   | 0      | -1.0        | 0.1         | -4.3        | 315.2       |
| 50   | 0      | 0.5         | 0.1         | -4.2        | 321.0       |
| 51   | 0      | 3.7         | 0.1         | -5.0        | 359.7       |
| 52   | 0      | 3.3         | 0.1         | -9.8        | 422.7       |
| 53   | 0      | 1.1         | 0.1         | -16.0       | 483.0       |
| 54   | 0      | -0.4        | 0.2         | -20.0       | 528.9       |
results compared to LTBeam did not exceed +4.8% (column 6, \( \psi = \{0.75, 1\}, t_p = 20\text{mm} \)). In turn, the critical moments estimated acc. Eq. (7), gave a sufficient engineering approximation when compared with LTBeam (column 12, differences from -1.1 to -0.1%). While, the critical moments obtained from Eq. (1) [6] were differed to FEM (LTBeam) from -21% to +1.4% (column 10).

Figure 3 presents the graphs of the values of the critical moment determined on the basis of Table 3 (columns 4, 5, 9 and 11) as a function of the stiffness of the end plate, for \( \psi = \{-0.75, -0.25, 0.25, 0.75\} \).

![Figure 3. Critical moments as a function of the stiffness of the end plate.](image)

The comparison of the graphs in Figure 3 shows, that the critical moments of LTB estimated with the (7) (thick solid line) practically coincide with the values determined from the LTBeam program (thin continuous line). The highest critical moments were obtained from Abaqus program (thick dotted line), with there are some differences to values obtained from LTBeam, especially for end plates \( t_p = 20\text{mm} \) (up to +4.8%) and \( t_p = 30\text{mm} \) (up to +4.3%). In turn, (1) [6] (thin dotted line) it gave good approximation of the critical moment for \( \psi = \{0.25, 0.75\} \) and \( t_p \leq 40\text{mm} \). While, for \( \psi = \{-0.75, -0.25\} \) and \( t_p \geq 30\text{mm} \) a significant lowering of the critical moment were obtained, according to FEM.

The analysis of the influence of the stiffness of the end plates \( t_p \) on the elastic critical moments \( (\text{Figure 3}) \) allowed stating, that:

- a) a clear trend of increasing the critical load capacity of beams is associated with the increase in the end plate thickness \( t_p \), especially for \( t_p > 20\text{mm} \),
- b) optimal advantages related to restrain warping of beams at support can be obtained for end plates with a thickness \( t_p \geq 1.5 \times t_f \),
- c) for typical end plate thicknesses (ie. \( t_p = 10 \div 20\text{mm} \)), the increase of the critical moment is small.

### 6.2. Influence of the stiffening of the selected types of ribs on the value of \( M_{cr} \)

In Table 4 compares the influence of selected types of ribs (Table 1) on the value of \( M_{cr} \), for the end moments ratio \( \psi = \{1, 0, -1\} \). The elastic critical moments of LTB were estimated with Eq. (7).
components of all variants of the ribs were made of plates of thickness \( t_p = 10 \text{mm} \). In brackets of columns 5, 6 and 7 given the percentage increases in the critical moments of beams in relation to the fork support (row 1).

In column 4 of Table 4 gives the equivalent thickness of end plate \( t_p \) that should be adopted to obtain the stiffness of the elastic restrained \( \alpha_\omega \) corresponding to the selected type of rib (column 2). Thicknesses of the end plates as a function of \( \alpha_\omega \) stiffness or \( \kappa \) fixity factor were determined using the formulas:

\[
t_p = \sqrt{\frac{3\alpha_\omega}{Gbh_0}} \quad t_p = \sqrt{\frac{6\kappa El_\omega}{(1-\kappa)LGb/h_0}}
\]

(8ab)

Table 4. Comparison of the influence of selected types of ribs on the value of \( M_{cr} \) for IPE300 and \( L = 5\text{m} \).

| Item | Types of ribs | \( t_p \) [mm] | \( M_{cr} \) [kNm] acc. (7) |
|------|---------------|----------------|----------------------------|
|      |               | 4              | \( \psi = 1 \) (%), \( \psi = 0 \) (%), \( \psi = -1 \) (%) |
| 1    | fork support  | -              | 116.2, 212.9, 315.2         |
| 2    | \( t_p = 10\text{mm} \) \( \alpha_\omega = 1.17\text{kJm}^3/\text{rad} \) \( \kappa = 0.1 \) | 10             | 118.2 (+2), 216.4 (+2), 321.6 (+2) |
| 3    | \( l = 150\text{mm} \) \( \alpha_\omega = 6.81\text{kJm}^3/\text{rad} \) \( \kappa = 0.392 \) | 18             | 126.2 (+9), 231.2 (+9), 349.1 (+11) |
| 4    | \( l_i = t_p = 10\text{mm} \) \( l = 130\text{mm} \) \( \alpha_\omega = 31.05\text{kJm}^3/\text{rad} \) \( \kappa = 0.746 \) | 30             | 146.9 (+26), 270.1 (+27), 421.6 (+34) |
| 5    | \( t_p = 10\text{mm} \) \( l = 150\text{mm} \) \( \alpha_\omega = 790.9\text{kJm}^3/\text{rad} \) \( \kappa = 0.987 \) | 88             | 187.4 (+61), 349.0 (+64), 597.7 (+90) |

From the comparison (Table 4) of the elastic critical moments of LTB follows that as the stiffness \( \alpha_\omega \) of the ribs increases, the critical load capacity of the beam rises (columns 5, 6 and 7). Higher values of the critical moments in individual variants of the ribs were obtained along with decreased of the coefficient \( \psi \), i.e. from 1 to -1 (Table 4). The most effective (+61%, +64%, +90%) from the analysed stiffeners are the closed section ribs (row 5). The lowest increase in the critical moments (+2%) were obtained using end plates \( t_p = 10\text{mm} \) (row 2). In order to obtain an equivalent increase in the critical moment of beam, in comparison with ribs of closed section (row 5), it is necessary to use end plates with a thickness \( t_p \geq 88\text{mm} \), which is not rational from a technical point of view.
7. Conclusions
The investigations confirmed that elastic restraint could significantly increase the elastic critical moment of LTB of beam against warping at supports of beams.

Of the types of ribs examined (Table 4), the most effective were proved ribs with a closed cross-section ($M_{cr}$ increase to +90%). In turn, the efficiency of commonly used end plates is small and increases with a significant increase of their thickness (Table 3). In order to obtain a comparable stiffening in relation to the closed rib, it would be necessary to use plates of considerable thickness (Table 4, column 4), which is not rational due to their heavy weight and unfavourable welding conditions.

The best accuracy of $M_{cr}$ compared to $LTBeam$, was obtained from the $MLTB_{EL}$ program [7]. The maximum differences did not exceed +0.2% (Table 3, column 8). A very good compliance of the results is related to the approximation of the torsion angle function ($\varphi$) by the three ($i = 1, 2, 3$) terms of the "coupled" polynomial series (5) [7].

A very good estimation of $M_{cr}$ was achieve by the formula Eq. (7) which was proposed in the article, and which was derived using one terms (containing “coupled” polynomials $WP_1$ and $WU_1$) of the function of the torsion angle (5) [7]. Differences in the results oscillated between -1.1% and -0.1% (Table 3, Figure 3) in relation to $M_{cr}$ obtained from the $LTBeam$ program. While $M_{cr}$ obtained from Eq. (1) [6] gave bigger differences (from -21% to +1.4%) in relation to FEM ($LTBeam$).

The approximate formulas for estimating the critical moment of LTB of beam, developed in the article, may be an important aid in the design of metal structures, due to their simple application and "technically" sufficient accuracy. It seems that a good standard of engineering calculations should be obtaining more important computational parameters, e.g. critical moments, by at least two independent methods.

References
[1] M. Giżejowski, “Lateral buckling of steel beams with limited rotation ability at supports”, Inżynieria i Budownictwo, vol. 10, pp. 589–594, 2001 (in Polish).
[2] B. Gosowski, “Spatial stability of longitudinally and laterally braced full-walled members of metal structures”, Wrocław, 1992 (in Polish).
[3] Y.-L. Pi, and N. S. Trahair, “Distortion and warping at beam supports”, Journal of Structural Engineering, vol. 11, pp. 1279–1287, 2000.
[4] J. Lindner, “Influence of constructional details on the load carrying capacity of beams”, Engineering Structures, vol. 18, pp. 752–758, 1996.
[5] U. Kuhlmann, “Stahlbau kalender”, 2009.
[6] J. Lindner, and R. Gietzelt, “Stabilisierung von Biegeträgern mit I-Profil durch angeschweißte Kopfplatten”, Stahlbau, vol. 3, pp. 69–74, 1984.
[7] R. Piotrowski, and A. Szychowski, “Lateral-torsional buckling of beams elastically restrained against warping at supports”, Archives of Civil Engineering, vol. LXI (4), pp. 155–174, 2015.
[8] Z. Kowal, and M. Malec, “Critical resistance of beams with support ribs with closed section”, Inżynieria i Budownictwo, vol. 2, pp. 71–74, 1989 (in Polish).
[9] R. Bijak, Z. Kowal, and M. Malec, “Critical load capacity of thin-walled beams stiffened with ribs”, XXXVIII KN KILiW PAN i KN PZITB, Łódź - Krynica, pp. 13–18, 1992 (in Polish).
[10] B. Gosowski, “Non-uniform torsion of stiffened open thin-walled members of steel structures”, Journal of Constructional Steel Research, vol. 63, pp. 849–865, 2007.
[11] Y. Galéa, “Moment critique de déversement élastique de pouters fléchies. Présentation du logiciel LTBEAM”, Revue Construction Métallique, CTICM, 2003.