Gribov process advected by the synthetic compressible velocity ensemble: Renormalization Group Approach

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Abstract

The direct bond percolation process (Gribov process) is studied in the presence of irrotational velocity fluctuations with long-range correlations. The perturbative renormalization group is employed in order to analyze the effects of finite correlation time on the long-time behavior of the phase transition between an active and an absorbing state. The calculation is performed to the one-loop order. Stable fixed points of the renormalization group and their regions of stability are obtained within the three-parameter \((\varepsilon, y, \eta)\)-expansion. Different regimes corresponding to the rapid-change limit and frozen velocity field are discussed.

1 Introduction

For a long time non-equilibrium continuous phase transitions \([1]\) have been an object of intense research activity. Underlying dynamic laws are responsible for diverse behavior with respect to their equilibrium counterparts. One of the most prominent example is the directed bond percolation \([2, 3]\) process, also known as Schlögl first reaction \([4, 5]\). In particle physics this process has been introduced by Gribov \([6]\) in order to explain hadron interactions at very high energies (Reggeon field theory) \([7]\). Further it can serve for a description of stochastic reaction-diffusion processes on a lattice \([8]\) and spreading of infection diseases \([9]\) among others.
It has been known that phase transitions are quite sensitive with respect to additional disturbances such as quenched disorder [10] or long-range interactions [8]. From practical point of view this might be a reason why there are not so many experimental realizations for the percolation process [11, 12]. Majority of realistic reaction-diffusion processes occur in some fluid environment, e.g., vast majority of chemical reactions is realized at finite temperature, which is inevitable accompanied with the presence of thermal fluctuations. In this paper we assume that the effect of environment can be simulated by advective velocity fluctuations [13]. Dynamics of the fluids is governed by the Navier-Stokes equation [14]. A general solution of these equations still remains an open question [15, 16]. Kraichnan model appears as a more tractable problem. In this model velocity field is assumed to obey a Gaussian distribution law with prescribed statistical properties [13, 17]. Though at a first sight too oversimplified with respect to the realistic flows, it nevertheless captures an essential physical information about advection processes [13]. Moreover some properties as intermittency are even more pronounced than for the Navier Stokes equation itself.

Recently, there has been increased interest in different advection problems in compressible turbulent flows [18, 19, 20, 21]. These studies show that compressibility plays an important role for population dynamics or chaotic mixing of colloids. In this work we consider a generalization of the original Kraichnan model proposed in [22]. There advection-diffusion problem of non-interacting admixture was studied in the presence of velocity field with finite correlation in time and compressibility taken into account. Our main motivation is to use this model and determine what influence it has on the critical properties of the directed bond percolation process. We note that in our model there is no backward influence of percolating field on the velocity fluctuations, i.e., our model corresponds to the passive advection of the scalar quantity. As initial steps in this direction have already been undertaken [23, 24, 25, 26], our main aim here is to elucidate in detail the differences between incompressible and compressible velocity field. The main theoretical tool is the field-theoretic approach [27] with subsequent Feynman diagrammatic technique and renormalization group (RG) approach, which allows us to determine large-scale behavior.

The paper is organized as follows. In Sec. 2 we introduce a field-theoretic version of the problem. In Sec. 3 we describe main steps of the perturbative RG procedure. In Sec. 4 we present an analysis of possible regimes involved in the model. We analyze numerically and to some extent analytically fixed points’ structure. In Sec. 5 we give a concluding summary.

2 Field-theoretic model

The effective field-theoretic action [28] for directed percolation can be obtained either from the Langevin formulation or via reaction-diffusion scheme employing Doi formalism [29]. However, at the very end one arrives at the same De Dominicis-Janssen action [30, 31, 32]

$$J_{\text{per}}[\tilde{\psi}, \psi] = \tilde{\psi}[\partial_t + D_0(\tau_0 - \nabla^2)]\psi + \frac{D_0 \lambda_0}{2}[\psi - \tilde{\psi}]\tilde{\psi}\psi, \quad (1)$$

where all irrelevant contributions from the RG point of view have been neglected. Field $\psi$ corresponds to the fluctuating density of percolating agents, $\tilde{\psi}$ stands for auxiliary Martin-Siggia-Rose field (MSR), $\partial_t = \partial/\partial t$ is the time derivative, $\nabla^2$ is the Laplace operator, $D_0$ is the diffusion constant, $g_0$ is the coupling constant and $\tau_0$ measures a deviation from the
threshold value for injected probability. It can be thought as an analog to the temperature variable in the standard $\phi^4$-theory \[28, 33\]. For the future RG use we have extracted a dimensional part from the interaction terms in the action \[1\]. In this paper we use a condensed notation in which expressions are viewed as matrices or vectors with respect to component indices, the spatial variable $x$ and the time variable $t$, respectively. For example, the first term in the action \[1\] actually reads

$$\int dt \int d^d x \dot{\psi}(t, x) \partial_t \psi(t, x),$$

where $d$ is the dimensionality of the $x$ space. Here and henceforth we distinguish between unrenormalized (with a subscript “0”) quantities and renormalized terms (without a subscript “0”).

The next step consists of an incorporation of the velocity fluctuations into the model. The standard route \[14\] is based on the replacement

$$\partial_t \rightarrow \partial_t + (v_i \partial_i),$$

where the summation over the spatial index $i$ is implied. In accordance with \[17, 22\] we assume that the velocity field is a random Gaussian variable with zero mean and a translationally invariant correlator \[22\] given in the Fourier representation

$$\langle v_i v_j \rangle_0(\omega, k) = \left[ P_{ij}^k + \alpha Q_{ij}^k \right] \frac{g_{10} u_{10} D_0^3 k^{4-d-y-\eta}}{\omega^2 + u_{10}^2 D_0^2 (k^2-\eta)^2}.$$  \(4\)

where $P_{ij}^k = \delta_{ij} - k_i k_j / k^2$ is a transverse and $Q_{ij}^k = k_i k_j / k^2$ a longitudinal projection operator, respectively. Further $k = |k|$ and a positive parameter $\alpha > 0$ can be interpreted as the simplest possible deviation \[34\] from the incompressibility condition $\partial_i v_i = 0$. The incompressible case, $\alpha = 0$, has been analyzed in previous works \[23, 25, 26\]. The coupling constant $g_{10}$ and the exponent $y$ describe the equal-time velocity correlator or, equivalently, the energy spectrum \[15, 17, 22\] of the velocity fluctuations. The constant $u_{10} > 0$ and the exponent $\eta$ are related to the characteristic frequency $\omega \approx u_{10} D_0 k^{2-\eta}$ of the mode with wavelength $k$.

The kernel function for the correlator \[4\] has been chosen in a universal form and as such it contains different limits: rapid-change model, frozen velocity ensemble and turbulence advection (see \[17, 22\]). In this paper our main goal is to analyze the case of purely potential (irrotational) velocity field. To this end one more rescaling of the variable $g_1$ according to

$$\alpha g_1 \rightarrow g_1, \quad \alpha \rightarrow \infty.$$  \(5\)

is needed. Then, in the perturbation theory we effectively work with the following velocity propagator

$$\langle v_i v_j \rangle_0(\omega, k) \rightarrow k_i k_j \frac{g_{10} u_{10} D_0^3 k^{4-d-y-\eta}}{\omega^2 + u_{10}^2 D_0^2 (k^2-\eta)^2},$$

where we have relabeled $g_1 \alpha \rightarrow g_1$. In the functional language the Gaussian nature of velocity field reveals in the following quadratic action

$$\mathcal{J}_{vel}[v] = \frac{1}{2} v D^{-1} v,$$  \(7\)

\(\)
which has to be added to the complete field-theoretical functional.

3 Renormalization group analysis

In order to apply the dimensional regularization for an evaluation of renormalization constants, an analysis of possible superficial divergences must be performed. For translationally invariant systems, it is sufficient \[33, 35\] to analyze 1-particle irreducible (1PI) graphs only. In contrast to static models, dynamical models \[27, 36\] contain two independent scales: a frequency scale \(d_\omega\) and a momentum scale \(d_k\) for each quantity \(Q\). The corresponding dimensions are found using the standard normalization conditions

\[
d_k = -d_x = 1, \quad d_\omega = d_t = 0, \quad d_\nu = d_x = 0, \quad d_\omega = -d_\nu = 1
\]

together with a condition field-theoretic action to be a dimensionless quantity. Using values \(d_\nu\) and \(d_k\), the total canonical dimension \(d_Q\),

\[
d_Q = d_k + 2d_\nu
\]

can be introduced, whose precise form is obtained from a comparison of IR most relevant terms (\(\partial_t\) must scale as \(\nabla^2\)) in the action \(1\). The total dimension \(d_Q\) for the dynamical models plays the same role as the conventional (momentum) dimension does in static problems. Dimensions of all quantities for the model are summarized in Table 1.

To retain the standard notation we have introduced \(\varepsilon\) via relation \(d = 4 - \varepsilon\). It follows that the model is logarithmic (when coupling constants are dimensionless) at \(\varepsilon = y = \eta = 0\), and the UV divergences are in principle realized as poles in these parameters. For the RG analysis it is of crucial importance that the couplings become logarithmic at the same time. Otherwise, one would have to discard IR irrelevant ones and some scaling regimes will be absent. The total canonical dimension of an arbitrary 1–irreducible Green function is given by the relation

\[
d_\Gamma = d_k + 2d_\nu = d + 2 - \sum \mathcal{N}_\phi d_\phi, \phi \in \{\tilde{\psi}, \psi, v\}.
\]

The total dimension \(d_\Gamma\) in the logarithmic theory is a formal degree of the UV divergence \(\delta_\Gamma = d_\Gamma|_{\varepsilon=y=\eta=0}\). Superficial UV divergences, whose removal requires counterterms, could be present only in those functions \(\Gamma\) for which \(\delta_\Gamma\) is a non-negative integer \[27\].

Dimensional analysis should be augmented by certain additional considerations. In dynamical models with MSR response fields \[36\], all the 1-irreducible diagrams without

| \(Q\) | \(\psi, \tilde{\psi}\) | \(v\) | \(D_0\) | \(\tau_0\) | \(g_{10}\) | \(\lambda_0\) | \(u_{10}\) | \(u_{20}, a_0, \alpha\) |
|-----|-----------------|-----|-----|-----|-----|-----|-----|-----|
| \(d_\nu\) | \(d/2\) | \(-1\) | \(-2\) | 2 | \(y\) | \(\varepsilon/2\) | \(\eta\) | 0 |
| \(d_k\) | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| \(d_Q\) | \(d/2\) | 1 | 0 | 2 | \(y\) | \(\varepsilon/2\) | \(\eta\) | 0 |

Table 1: Canonical dimensions of the bare fields and bare parameters for the total field-theoretic action given by the sum of actions for percolation process \(1\), velocity field \(7\) and advection process \(16\).
the fields \( \tilde{\psi} \) vanish, and it is sufficient to consider functions with \( N_\tilde{\psi} \geq 1 \). As was shown in \[24\] the rapidity symmetry \( \psi(t) \rightarrow -\tilde{\psi}(-t), \tilde{\psi} \rightarrow -\psi(-t) \) requires also inequality \( N_\psi \geq 1 \) to hold. Using these considerations together with relation \[10\], possible UV divergent structures are expected only for the 1PI Green functions listed in Table 2.

In what follows we employ the perturbative RG approach, which allows us to calculate universal quantities in formal series in small parameter of theory. In contrast to the standard \( \varphi^4 \)-theory our model contains three small expansion parameters \( (\varepsilon, \eta, y) \). Also we would like to make the following remark. The real expansion parameters in a perturbative sense are the charges \( g_1 \) and \( g_2 = \lambda^2 \) only (the latter fact is a consequence of rapidity symmetry). The parameters \( u_1 \) and \( \alpha \) correspond to the non-perturbative quantities, and there is no physical restriction on their values. Therefore one can study also a limiting case such as \( u_{10} \rightarrow 0 \) or \( u_{10} \rightarrow \infty \).

Before we embark on results of the RG approach, let us first discuss in detail profound differences caused by compressibility and lack of Galilei invariance \[37, 38, 39\] in our model. As shown in \[24\] instead of relation (3) the following replacement is necessary

\[
\partial_t \rightarrow \partial_t + (v_i \partial_i) + a_0(\partial_i v_i),
\]

where \( a_0 \) is an additional positive parameter, whose significance can be explained as follows. For pure advection-diffusion problem \[22\] the choice \( a_0 = 1 \) corresponds to the conserved quantity \( \psi \) (density), whereas for \( a_0 = 0 \) auxiliary field \( \tilde{\psi} \) is conserved. For the whole model nor \( \psi \) nor \( \tilde{\psi} \) is conserved, hence both fields \( \tilde{\psi} \) and \( \psi \) are fluctuating quantities and RG procedure will give a birth to both counterterms \( \tilde{\psi}(v_i \partial_i)\psi \) and \( \tilde{\psi}\partial_i(v_i\tilde{\psi}) \).

In the language of Feynman diagrams let us consider a one-loop expansion of 1PI function \( \tilde{\psi}\psi v \) that can be formally written as

\[
\Gamma_{\tilde{\psi}\psi v} = -ip_jZ_4 - iaq_jZ_5 + \ldots
\]

where \( p \) is a momentum of the field \( \psi \) and \( q \) of the field \( v \), respectively. A direct calculation shows that a divergent part of the first graph is

\[
\frac{i\lambda_0^2}{4d(2\pi)^d} [2p_j - q_j].
\]

Let us consider such graph as a subgraph in some high-order loop for an incompressible case. In this case all velocity propagators are proportional to the transverse projector and after contraction with \[13\] the compressible part evidently drops out. However, in our model the velocity propagator \[4\] contains also a longitudinal part. Moreover in contrast to

| \( \Gamma_{1-ir} \) | \( \Gamma_{\tilde{\psi}\psi} \) | \( \Gamma_{\tilde{\psi}\psi v} \) | \( \Gamma_{\psi^2\psi} \) | \( \Gamma_{\tilde{\psi}\psi^2} \) | \( \Gamma_{\tilde{\psi}\psi^2v^2} \) |
|---|---|---|---|---|---|
| \( d_\Gamma \) | 2 | 1 | \( \varepsilon/2 \) | \( \varepsilon/2 \) | 0 |
| \( \delta_\Gamma \) | 2 | 1 | 0 | 0 | 0 |

Table 2: Canonical dimensions for the (1PI) divergent Green functions of the model.
Kraichnan model, also the second graph in (12) is divergent (due to the finite correlation in time the graph does not contain a closed loop of retarded propagators [27, 36]). Symmetries play a fundamental role in physics. In turbulent problems Galilei invariance [27] is of prominent importance. It describes an invariance with respect to transformations $\varphi \rightarrow \varphi_v$ given by

$$\varphi_v(x) = \varphi(x_v) - v(t), \quad \varphi_v'(x) = \varphi'(x_v), \quad x \equiv (t, x),$$

$$x_v \equiv (t, x + u(t)), \quad u(t) = \int_{-\infty}^{t} dt' \nu(t').$$

(14)

From Table 2 it follows that possible superficial divergences may appear also in the structure $\Gamma_{\tilde{\psi}\psi\nu^2}$. Aforementioned Galilei invariance restricts presence of such term. In our model however such term must be taken into account and considered as a new interaction parameter. The one-loop expansion for this function using only cubic interactions reads

$$\Gamma_{\tilde{\psi}\psi\nu^2} = \frac{u_2}{D} \delta_{ij} Z_8 + \frac{1}{2}.$$  

(15)

A straightforward calculation shows that the first and the third graph cancel each other, whereas second graph gives a non-zero contribution. This is again a consequence of finite correlation time, which precludes appearance of closed loops of retarded propagators. To conclude the field-theoretic action given by the sum of (1) and (7) has to be augmented by the following part describing the advection interaction

$$J_{\text{adv}}[\varphi] = -\frac{u_{20}}{2D_0} \bar{\psi} \psi v^2 + \bar{\psi}(v \partial_i) \psi + a_0 \bar{\psi}(\partial_i v_i) \psi.$$  

(16)

The field theoretic action $J = J_{\text{per}} + J_{\text{vel}} + J_{\text{adv}}$ is amenable to the Feynman diagrammatic technique with the subsequent use of perturbative RG approach. Note that the added interaction term $\bar{\psi} \psi v^2$ directly leads to additional 5 Feynman diagrams for 1PI function

$$\ldots$$  

(17)

that have to be taken into account. The multiplicative renormalization can be achieved through following renormalization prescription

$$D_0 = D \bar{Z}_D, \quad \tau_0 = \tau \bar{Z}_\tau + \tau_c, \quad a_0 = a \bar{Z}_a, \quad g_{20} = g_2 \mu^\varepsilon \bar{Z}_{g_2},$$

$$g_{10} = g_1 \mu^\varepsilon \bar{Z}_{g_1}, \quad u_{10} = u_1 \mu^\varepsilon \bar{Z}_{u_1}, \quad \lambda_0 = \lambda \mu^\varepsilon / 2 \bar{Z}_\lambda, \quad u_{20} = u_2 \bar{Z}_{u_2},$$

$$\bar{\psi} = Z_{\bar{\psi}} \bar{\psi}_R, \quad \psi = Z_{\psi} \psi_R, \quad v = Z_v \nu_R,$$

(18)

where $\mu$ is the reference mass scale in the MS scheme [33]. Note that the term $\tau_c$ is a non-perturbative effect [28], which is not captured by the dimensional regularization. Physically it describes fluctuation-induced shift of critical probability $\tau_0$. 

6
4 IR stable regimes

The large scale behavior with respect to spatial and time variables is governed by the attractive IR stable fixed points \( g^* \). Here and henceforth the asterisk refers to a coordinate of the fixed point (FP). Their coordinates are determined from the zeros of RG flow equations \[27\, 33\, 35\]

\[
\beta_{g_1}(g^*) = \beta_{g_2}(g^*) = \beta_{u_1}(g^*) = \beta_{u_2}(g^*) = \beta_a(g^*) = 0. \tag{19}
\]

The eigenvalues of the matrix of first derivatives \( \Omega = \{\Omega_{ij}\} \) determine whether given FP is IR stable. Such points are proper candidates for macroscopic regimes and thus can be observed experimentally. The matrix \( \Omega \) is defined as

\[
\Omega_{ij} = \frac{\partial \beta_i}{\partial g_j}, \quad i, j \in \{g_1, g_2, u_1, u_2, a\}. \tag{20}
\]

The explicit form of beta functions follows from \[18\] using definition \( \beta_g = \mu \partial_\mu g|_0 \) and a straightforward calculation yields

\[
\beta_{g_1} = g_1(-y + 2\gamma_D - 2\gamma_\mu), \quad \beta_{g_2} = g_2(-\varepsilon - \gamma_{g_2}), \quad \beta_a = -a\gamma_a,
\]

\[
\beta_{u_1} = u_1(-\eta + \gamma_D), \quad \beta_{u_2} = -u_2\gamma_{u_2}, \tag{21}
\]

where \( \gamma_x \equiv \mu \partial_\mu \ln Z_x|_0 \) are the anomalous dimensions \[27\]. In the 1-loop approximation they are given by the following expressions

\[
\gamma_{g_1} = -\frac{g_1 u_1 (1 - 2u_2)}{2(1 + u_1)^2} - \frac{g_2}{4}, \quad \gamma_D = \frac{g_1}{4(1 + u_1)} \left[ \frac{u_1 - 1}{u_1 + 1} + \frac{4a(1-a)}{(1+u_1)^2} \right] + \frac{g_2}{8},
\]

\[
\gamma_a = (1 - 2a) \left[ \frac{g_1 (1-a)}{2(1 + u_1)^3} + \frac{g_1 u_2 (u_1 - 1)}{4a(1 + u_1)^2} + \frac{g_2}{8a} \right],
\]

\[
\gamma_{u_2} = \frac{g_1 (1 - 2u_2)}{4(1 + u_1)} \left[ \frac{u_1 - 1}{u_1 + 1} + \frac{2a(1-a)}{u_2(1 + u_1)^2} \right] - \frac{g_2}{8},
\]

\[
\gamma_{g_2} = \frac{g_1}{1 + u_1} \left[ \frac{(1 - 2a)^2}{2} + \frac{1 - 3a(1-a)}{(1 + u_1)} + \frac{2a(1-a)u_1}{(1 + u_1)^2} \right] - \frac{3g_2}{2},
\]

\[
\gamma_\tau = -\frac{g_1}{4(1 + u_1)^2} \left[ u_1 - 1 + \frac{4a(1-a)}{1 + u_1} \right] - \frac{3g_2}{8}. \tag{22}
\]

The fields \( \psi, \tilde{\psi} \) and \( v \) also have to be renormalized and therefore corresponding anomalous dimensions are nontrivial

\[
\gamma_\psi = \frac{g_1}{2(1 + u_1)^2} \left[ -a(1-a) + (1 + u_1)(2a - 1) \right] - \frac{g_2}{8},
\]

\[
\gamma_{\tilde{\psi}} = \frac{g_1}{2(1 + u_1)^2} \left[ -a(1-a) + (1 + u_1)(1 - 2a) \right] - \frac{g_2}{8},
\]

\[
\gamma_v = \frac{g_1}{4(1 + u_1)^2} \left[ \frac{4a(1-a)}{1 + u_1} - 1 \right] + \frac{g_1 u_1 u_2}{2(1 + u_1)^2}. \tag{23}
\]
In order to simplify the analysis it is convenient to introduce new charges $g_1', u_1'$ and $a'$

\[
\frac{g_1}{1 + u_1} = g_1' \rightarrow g_1, \quad \frac{1}{1 + u_1} = u_1' \rightarrow u_1, \quad (1 - 2a)^2 = a' \rightarrow a. \quad (24)
\]

In new variables the rapid change model corresponds to the choice $u_1 = 0$, whereas frozen velocity field to $u_1 = 1$. The final expressions for $\beta$-functions read

\[
\begin{align*}
\beta_{g_1} &= \frac{g_1}{8} \left\{ -8y + 8\eta(1 - u_1) + 2g_1(1 - u_1)[1 + u_1(2 + u_1(a - 1)) - 4u_2] + g_2(1 + u_1) \right\}, \\
\beta_{g_2} &= \frac{g_2}{4} \left\{ -4\varepsilon + g_1u_1(2u_1 - 3) - ag_1(2 + u_1 + 2u_1^2) + 6g_2 \right\}, \\
\beta_{u_1} &= \frac{u_1(1 - u_1)}{8} \left\{ 8\eta + 2g_1[2 + u_1(-1 + a)] - 1 \right\} - g_2, \\
\beta_{u_2} &= \frac{1}{8} \left\{ g_1(1 - 2u_2)[u_1^2(a - 1) - 2u_2 + 4u_1u_2] + g_2u_2 \right\}, \\
\beta_{a} &= \frac{a}{2} \left\{ g_1[u_1(1 - au_1 - 4u_2) + 2u_2] + g_2 \right\}. \quad (25)
\end{align*}
\]

4.1 Rapid change model

The rapid change model \cite{22} is characteristic by $u_1^* = 0$ which, having in mind the replacement (24), corresponds to velocity propagator (6) with short range correlations in time. For this case seven FPs were found. Out of them only four (FP$_I^1$, FP$_I^2$, FP$_I^5$ and FP$_I^6$) are IR stable. Coordinates of all fixed points are listed in Table 3. As we can see the coordinates depend only on the difference $y - \eta$, which confirms previous expectations \cite{17, 22}. Here NF stands for Not Fixed, i.e., the corresponding value of a charge coordinate could not be unambiguously determined. In that case the given FP, rather than to a point, corresponds to the whole line of FPs. From the explicit expression for $\beta_{a}$ in (25) we can draw a conclusion about points for which $a^* = 0$. For them this relation is exact and is fulfilled to all orders in a perturbation theory. The schematic depiction of the phase space structure can be found in Fig. 1. The phase boundaries for the FP$_I^6$ can be obtained only

| FP$_I^1$ | $g_1$ | $g_2$ | $u_2$ | $a$ |
|---------|-------|-------|-------|-----|
| FP$_I^1$ | 0     | 0     | NF    | NF  |
| FP$_I^2$ | 0     | $\frac{2}{3} \varepsilon$ | 0     | 0   |
| FP$_I^3$ | $4(y - \eta)$ | 0     | 0     | NF  |
| FP$_I^4$ | $4(\eta - y)$ | 0     | $\frac{1}{2}$ | 0   |
| FP$_I^5$ | $\frac{1}{3}[12(y - \eta) - \varepsilon]$ | $\frac{2}{3} \varepsilon$ | 0     | 0   |
| FP$_I^6$ | $\varepsilon + 4(\eta - y)$ | $\frac{2}{3} \varepsilon$ | $\frac{\varepsilon + 6(\eta - y)}{3[\varepsilon + 4(\eta - y)]}$ | 0   |
| FP$_I^7$ | $\eta - y$ | $2(y - \eta)$ | 1     | $-6 + \frac{2\varepsilon}{\eta - \eta}$ |

Table 3: Coordinates of the fixed points for the rapid change model.
4.2 Thermal fluctuations

Now we analyze a special case of the rapid-change model, which describes thermal fluctuations \([40]\). They are characterized by quadratic dispersion law and in our choice of velocity correlator (4), an additional condition

\[
\eta = 6 + y - \varepsilon \quad (26)
\]

has to be met. A phase structure in the plane \((\varepsilon, y)\) is depicted in Fig. 2. For physical space dimensions \(d = 3\) (\(\varepsilon = 1\)) and \(d = 2\) (\(\varepsilon = 2\)) the only stable regime is that of pure DP. The nontrivial regimes \(FP_5^I\) and \(FP_6^I\) are realized only in the nonphysical region for large values of \(\varepsilon\). This numerical result confirms our previous expectations \([23, 24]\). It was pointed out \([41]\) that genuine thermal fluctuations can change IR stability of the given universality class. However, this is not realized for the percolation process itself.

4.3 Frozen velocity field

Frozen velocity limit is obtained for the choice of \(u_1 = 1\). In the formulation of an advection of density field \([22]\) it corresponds to the model of random walks in a random environment with long-range correlations \([12]\). In this case five fixed points can found and their coordinates are listed in Table 4. Only the points \(FP_1^{II}\), \(FP_2^{II}\) and \(FP_4^{II}\) are IR stable. The fixed point \(FP_1^{II}\) describes the free (Gaussian) theory. It is stable in the region

\[
y < 0, \quad \varepsilon < 0, \quad \eta < 0. \quad (27)
\]
Figure 2: Phase portrait for the percolation process in the presence of thermal fluctuations in the \((\varepsilon, y)\)-plane. The notations for the fixed points agrees with that of rapid-change model in Table 3.

| FP\(II\) | \(g_1\) | \(g_2\) | \(u_2\) | \(a\) |
|----------|--------|--------|--------|--------|
| FP\(II\)_1 | 0 | 0 | Not Fixed | Not Fixed |
| FP\(II\)_2 | 0 | \(\frac{2}{3}\varepsilon\) | 0 | 0 |
| FP\(II\)_3 | \(\varepsilon - y\) | 4y | 1 | \(\frac{5y - \varepsilon}{\varepsilon - y}\) |
| FP\(II\)_4 | 4(6y - \(\varepsilon\)) | 4y | \(\frac{4\varepsilon - 25y + \sqrt{-8\varepsilon y + 49y^2}}{8(\varepsilon - 6y)}\) | 0 |
| FP\(II\)_5 | 4(6y - \(\varepsilon\)) | 4y | \(\frac{4\varepsilon - 25y - \sqrt{-8\varepsilon y + 49y^2}}{8(\varepsilon - 6y)}\) | 0 |

Table 4: Coordinates of the fixed points for the frozen velocity ensemble.

For the FP\(II\)_2 the velocity field is irrelevant and the only relevant interaction is the nonlinearity of the percolation process. This FP is stable in the region

\[
\varepsilon > 6y, \quad \varepsilon > 0, \quad \varepsilon > 12\eta.
\]  

(28)

FP\(II\)_4 embodies a nontrivial regime for which both velocity and percolation interactions are relevant. The regions of stability for the FP\(II\)_1 and FP\(II\)_2 are depicted in Fig. 3. Because for these two points the velocity field could be effectively neglected, it directly follows that given boundaries could not depend on the value of parameter \(\alpha\). The stability region of FP\(II\)_4 can be computed only numerically and it turns out that it depends on \(\alpha\). From Fig. 3 we observe that the correlation parameter \(\eta\) crucially affects boundaries between FP\(II\)_2 and FP\(II\)_4.

5 Conclusions

In this paper we have studied percolation spreading in the presence of irrotational velocity field with long-range correlations. The coarse grained model was formulated and multiplicative renormalizability of the field theoretic model was discussed in detail.
Figure 3: On the left picture region of stability for $\eta = 0$ in the plane ($\varepsilon, y$) is depicted and on the right picture phase structure for the choice $\eta = 4/3$.

We have found that depending on the values of a spatial dimension $d = 4 - \varepsilon$, scaling exponents $y$ and $\eta$, describing statistics of velocity fluctuations, the model exhibits various universality classes. Some of them are already well-known: the Gaussian (free) fixed point, a directed percolation without advection and a passive scalar advection. The remaining points correspond to new universality classes, for which an interplay between advection and percolation is relevant.

It was shown [43] that anomalous scaling behavior is destroyed when $\alpha$ and $y$ are large enough. Therefore only relatively small values of $\alpha$ are allowed ($\alpha \ll 1$) in our model. They correspond to small fluctuations of the density $\rho$, what is tacitly supposed in our investigation. In other words, it is assumed that the velocity of the fluid is much smaller than the velocity of the sound in the system (the Mach number $Ma \ll 1$). Nevertheless we believe that a qualitative picture for large values of compressibility should remain the same. A possible further investigation should take into account additional effects such as feedback on the dynamics of the advecting field, anisotropies or broken mirror symmetry.

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