Interplay between evolutionary game and network structure: the coevolution of social net, cooperation and wealth distribution

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(Dated: March 31, 2022)

We study the interplay between evolutionary game and network structure and show how the dynamics of the game affect the growth pattern of the network and how the evolution of the network influence the cooperative behavior in the game. Simulation results show that the payoff-based preferential attachment mechanism leads to the emergence of a scale-free structural property, $P(k) \sim k^{-\gamma}$. Moreover, we investigate the average path length and the assortative mixing features. The obtained results indicate that the network has small-world and positive assortative behaviors, which are consistent with the observations of some real social networks. In parallel, we found that the evolution of the underlying network structure effectively promotes the cooperation level of the game. We also investigate the wealth distribution obtained by our model, which is consistent with the Pareto law in the real observation. In addition, the analysis of the generated scale-free network structure is provided for better understanding the evolutionary dynamics of our model.

PACS numbers: 89.75.Fb, 02.50.Le, 89.65.-s, 87.23.Ge

I. INTRODUCTION

Game theory provides a useful framework for describing the evolution of systems consisting of selfish individuals [1, 2, 3]. The prisoner’s dilemma game (PDG) as a metaphor for investigating the evolution of cooperation has drawn considerable attention [4, 5]. In the PDG, two players simultaneously choose whether to cooperate or defect. Mutual cooperation results in payoff $T$ for both players, whereas mutual defection leads to payoff $P$ gained both. If one cooperates while the other defects, the defector gains the highest payoff $R$, while the cooperator bears a cost $S$. This thus gives a simply rank of four payoff values: $T > R > P > S$. One can see that in the PDG, it is best to defect regardless of the co-player’s decision to gain the highest payoff $T$. However, besides the widely observed selfish behavior, many natural species and human being show the altruism that individuals bear cost to benefit others. These observations bring difficulties in evaluating the fitness payoffs for different behavioral patterns, even challenge the rank of payoffs in the PDG. Since it is not suitable to consider the PDG as the sole model to discuss cooperative behavior, the snowdrift game (SG) has been proposed as possible alternative to the PDG, as pointed out in Ref [6]. The main difference between the PDG and the SG is in the order of $P$ and $S$, as $T > R > S > P$ in the SG. This game, equivalent to the hawk-dove game, is also of much biological interest [1, 3, 4]. However, the original PDG and SG cannot satisfyingly reproduce the widely observed cooperative behavior in nature and society. This thus motivates numerous extensions of the original model to better mimic the evolution of cooperation in the real world [5, 10, 11, 12].

Since the spatial structure is introduced into the evolutionary games by Nowak and May [13], there has been a continuous effort on exploring effects of spatial structures on the cooperation [14, 15]. It has been found that the spatial structure promotes evolution of cooperation in the PDG [13], while in contrast often inhibits cooperative behavior in the SG [8]. In recent years, extensive studies indicate that many real networks are far different from regular lattices, instead, show small-world and scale-free topological properties. Hence, it is naturally to consider evolutionary games on networks with these kinds of properties [10, 17, 18, 19, 20, 21, 22]. An interesting result found by Santos and Pacheco is that “Scale-free networks provide a unifying framework for the emergence of cooperation” [23]. So far, most studies of evolutionary games over networks are based on static network structure. However, it has been pointed out that the network structure may coevolve with the game [24, 23, 26, 27, 28], where each individual would choose its co-players to gain more benefits, inducing the evolution of their relationship network. Some previous works about weighted networks suggest that it is indeed the traffic increment spurs the evolution of the network to maintain the system’s normal and efficient functioning [24, 30]. From this perspective, in the present paper we propose an evolutionary model with respect to the interplay between the evolutions of the game and the network for characterizing the dynamics of some social and economic systems.

In our model, the SG is adopted for its more general representation of the realism and evolutionary cooperative behavior. Since growth is a common feature among networked systems [31], we assume that the network continuously grows by adding new agents to the existent network on the basis of the payoff preferential attachment. We focus on the evolution of the network structure together with the emergence and persistence of coopera-
tion. Simulation results show that the obtained networks follow a power-law distribution, $p(k) \sim k^{-\gamma}$, with exponent $\gamma$ tuned by a model parameter. The average distance of the network scales logarithmically with the network size, which indicates the network has a small-world effect. Interestingly, the assortative mixing properties generated by our model demonstrate that the model can well mimic social networks. In parallel, with the extension of the network, the density of cooperators increases and approaches a stable value, which gives a new explanation for the emergence and persistence of cooperation. We also explore the wealth distribution, where the so-called wealth is the accumulated payoff distribution of each individual. The Pareto law is well reproduced by our model. At last, we provide analyses for the obtained scale-free network structures.

The paper is arranged as follows. In the following section, we describe the model in detail. In Sec. III, simulation results and correspondent analytical ones are provided, and in Sec. IV, the work is concluded.

II. THE MODEL

Let us introduce briefly the SG first. Consider two drivers are trapped in two side of a snowdrift. Each driver has two possible selections, either shoving the snowdrift (cooperator-C) or remaining in the car and do nothing (defect-D). If both cooperate, they could be back home on time, so that each will gain a reward of $b$, whereas mutual defection results in still blocked by the snowdrift and each gets a payoff $P = 0$. If only one driver shovels (takes C), then both drivers can be back home. The driver taking D gets home with do nothing and hence gets a payoff $T = b$, while the driver taking C gains a “sucker” payoff of $S = b - c$. Thus, the rank of four payoff values is $T > R > S > P$. Following common practice, the SG is rescaled with $R = 1$, $T = 1 + r$ and $S = 1 - r$, where $r$ is a tunable parameter ranging from 0 to 1. Hence, the payoffs can be characterized by a single parameter for convenient study.

Our model starts from $n_0$ nodes randomly connected with probability $p$, each of which represents a player. In the following, we fix $n_0 = 10, p = 0.6$ and examine that it has no influence on our results in present work.) Initially, the nodes are randomly assigned to be either strategy C or D with 50 – 50 percentages. Players interact with all their neighbors simultaneously and get payoffs according to the preset payoff parameter. The total payoff of a certain player is the sum over all its encounters. Then, every node $i$ randomly selects a neighbor $j$ at the same time for possible updating its strategy. The probability that $i$ follows the strategy of the selected node $j$ is determined by the total payoff difference between them, i.e.,

$$p_{ij} = \frac{1}{1 + \exp[(M_i - M_j)/T]},$$

where $M_i$ and $M_j$ are the total payoffs of $i$ and $j$ at the moment of the encounter. Here, $T$ characterizes “noise”, including bounded rationality, individual trials, errors in decision, etc. It should be noted that $T$ here plays a different role comparing with the cases of adopting the normalized payoff difference. In parallel, $T$ does not play the same role for different network sizes. Since in our model, the network size gradually grows, it is not easy to keep the effect of $T$ unchanged at every time step. For simplicity, we fix $T$ to 0.1 during the evolution of the network.

Here, we adopt the synchronous updating rule. After each step that players update their strategies, a new individual is added into the network with $m \leq m_0$ (we fix $m = 3$ for convenience) links preferentially attached to existent nodes of higher payoffs, i.e.,

$$\Pi_{new \rightarrow i} = \frac{M_i + W}{\sum_j (M_j + W)},$$

where $M_i$ and $M_j$ are the total payoffs of $i$ and $j$ obtained in the interaction process. $W$ is a tunable parameter, which reflects the original payoff values of players when they join into the game system. For simplicity, we set $W$ be a constant. The payoff-based preferential selection takes into account the “rich gets richer” characteristic and couples the dynamics of the evolutionary game and the evolution of the underlying network. After a new player joins into the network, the new one randomly choose strategy C or D and all old players preserve their strategies for the game in the next round. Then, repeat the above procedures, and the network size gradually grows.
We have checked that the parameter $W$ has slight effect over 10 different network realizations. Each data point is obtained by averaging over 10 network realizations. Figure 2 shows that for all the values of $W$, $\langle L \rangle$ increases logarithmically with the growth of the network, but the slopes for different $W$ show slight differences. These results demonstrate that the small-world effect can be reproduced by the proposed model.

Another important structural feature useful for measuring the correlation among nodes of a network is the assortative mixing coefficient $A$, or called degree-degree correlation $[37, 38]$, which is defined as follows:

$$A = \frac{M^{-1} \sum_{i,j} j_i k_i - [M^{-1} \sum_{i} \frac{1}{2} (j_i + k_i)]^2}{M^{-1} \sum_{i} \frac{1}{2} (j_i^2 + k_i^2) - [M^{-1} \sum_{i} \frac{1}{2} (j_i + k_i)]^2},$$  \hspace{1cm} (4)$$

where $d_{ij}$ is the shortest path length from node $i$ to node $j$, and $N$ is the network size. We perform simulations on $\langle L \rangle$ as a function of network size for different values of parameter $W$. Each data point is obtained by averaging over 10 different network realizations. Figure 2 shows that for all the values of $W$, $\langle L \rangle$ increases logarithmically with the growth of the network, but the slopes for different $W$ show slight differences. These results demonstrate that the small-world effect can be reproduced by the proposed model.

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where $j_i$ and $k_i$ are the degrees of the two nodes at the end of the $i$th edge, with $i = 1, \cdots, M$ ($M$ is the total number of edges of the observed graph). Two main classes of possible correlations have been observed in the real world: assortative behavior if $A > 0$, which indicates that large-degree nodes are preferentially connected with other large-degree nodes, and disassortative if $A < 0$, which denotes that links are more easily built between large-degree nodes and small-degree ones. As demonstrated in Ref $[37]$, almost all social networks show positive values of $A$, while others, including technological and biological networks, show negative $A$. However, the mechanism that leads to the basic difference between these two classes networks remains unclear $[39]$. We calculate the assortative mixing coefficient $A$ to check whether the generated networks by our model are suitable representations of social systems. Figure 3 shows $A$.
as a function of parameter $r$ for different values of $W$. One can see that assortative behavior occurs as $W$ increases. For each value of $W$, $A$ shows slight changes in the cases of low values of $r$, while for large $r$, assortative mixing is enhanced with the same value of $W$. The reported results demonstrate that the networks generated by our model can well capture the key distinguished structural property of social networks.

### B. Cooperation and wealth distribution

So far we have studied the evolution of the underlying network structure influenced by the game. Next, we turn to the effect of changing network structure on the cooperative behavior in the game. The key quantity for measuring the cooperation level of the game is the density of cooperators, $\rho_c$. Figure 4 shows the time series of $\rho_c$ for different values of $W$ and $r$. One can find that after a short period of temporary behavior, $\rho_c$ reaches a stable value with small fluctuations around the average value. Hence, $\rho_c$ for each pair of $W$ and $r$ can be calculated by averaging a period time after the system enter a steady state. Here, the network size $N$ is equal to the evolutionary time step, since each time an individual joins into the system. In Fig. 5, we report the $\rho_c$ depending on $r$ for different values of $W$. Each data point is obtained by averaging over 10 different simulations with an average from $N = 5000$ to 10000 for each simulation. One can see in Fig. 5, in the case of $r \leq 0.4$, $\rho_c$ shows no difference for distinct $r$. While for $r > 0.4$, the lower the value of $r$, the higher the cooperation level. It has been known that scale-free networks favor the emergence and persistence of cooperation \[23\]. Thus, the fact that $\rho_c$ in our model is larger than that of well-mixed cases is attributed to the emergence of scale-free structural properties. At the very beginning, the network evolves from a core of random-like structure, in which cooperation cannot dominate in the game. While as the network gradually grows, power-law degree distribution emerges, which leads to a sharp increase of $\rho_c$, as shown in Fig. 4 from $N = 0$ to 500. The inhibited cooperation by the increment of $W$ is also ascribed to the weakened heterogeneity of degree distribution. As displayed in Fig. 1, lower value of $W$ corresponds to stronger heterogeneity of degree distribution reflected by the longer fat tail. The above discussion gives a thorough picture that it is the growth and the payoff-based preferential attachment that produce scale-free network structures and meanwhile, the generated heterogeneity of degree distribution effectively promotes the emergence and persistence of cooperation.

Generally speaking, cooperation and defection are prototypical actions in economical systems. Hence, evolutionary games may be suitable paradigms for studying and characterizing the phenomena observed in economical systems with players represented by agents. In such systems, a well-known and extensively studied phenomenon is the wealth distribution of agents which follows the Pareto law in the high-income group \[10, 11, 12\]. In order to check the validity of our model for understanding economical behavior, we investigate the wealth distribution by adopting the present evolutionary model, where the wealth of an agent is naturally represented by the accumulated payoff over time steps. Figure 6 reports the accumulated distribution of accumulated payoff $P_c$ in the whole population for different model parameters $r$ and $W$. One can see that power-law distribution can be observed in a wide range of $P_c$, while the wealth distributions behave as exponential corrections in the zone of...
linear function of $k$ with degree simulations. Using the mean-field approximation, a node probability $\rho$ dent degrees $k$ accumulated payoff attachment via considering the correlation between the free network structure induced by the payoff-preferential attachment. Hence, the real observations can be reproduced by our model by tuning the value of $W$.

low values of $P_c$, which is in accordance with the empirical evidence. Moreover, in the left panel of Fig. 6, $W$ has strong influence on the exponent of power-law distribution and higher value of $W$ corresponds to larger exponent. In contrast, in the right panel, $r$ nearly has no effect on the wealth distribution. The correlation between $W$ and the exponent of wealth distribution makes our model general for reproducing the empirical observation.

C. Analysis

In the following, we provide some analysis for the scale-free network structure induced by the payoff-preferential attachment via considering the correlation between the accumulated payoff $P_c$ of individuals and their correspondent degrees $k$. As shown in Fig. 7, $P_c(k)$ is a good linear function of $k$ with slope depending on $r$ in the simulations. Using the mean-field approximation, a node with degree $k$ may have $k\rho_c$ cooperative neighbors and $k(1 - \rho_c)$ defectors and itself may be cooperator with probability $\rho_c$ and defector with $1 - \rho_c$. Thus, at time step $t$, its payoff can be calculated as

$$M_k(t) = \rho_c(t) \cdot k\rho_c(t) \cdot R + \rho_c(t) \cdot k(1 - \rho_c(t)) \cdot S + (1 - \rho_c(t)) \cdot k\rho_c(t) \cdot T + (1 - \rho_c(t)) \cdot k(1 - \rho_c(t)) \cdot P$$

(5)

By substituting the elements of payoff matrix of the SG in Eq. (5), where $R = 1$, $S = 1 - r$, $T = 1 + r$ and $P = 0$, Eq. (5) is simplified to

$$M_k(t) = k\rho_c(t)(2 - \rho_c(t)).$$

(6)

Then, we get the accumulated payoff

$$P_c(k) = \sum_t M_k(t) = k \sum_t \rho_c(t)(2 - \rho_c(t)) \propto k\rho_c(2 - \rho_c),$$

(7)

where the approximation is ascribed to the fact that $\rho_c(t)$ quickly reaches a stable value which is almost independent of $t$ (as shown in Fig. 4), so that $\rho_c(t)$ are replaced by $\rho_c$ for simplicity. The inset of Fig. 7 illustrates the comparison between the simulation results and the analytical ones on the the normalized slopes vs $r$. The theoretical predictions are calculated by substituting the simulation results of $\rho_c$ (in Fig. 5) into Eq. (7). The theoretical results are consistent with simulations.

Accordingly, considering the relation between the payoff $M_i$ of an individual $i$ and its degree $k_i$ in Eq. (6), we
give the evolution equation of the degree of a given node
\[
\frac{dk_i}{dt} = \frac{m(M_i + W)}{\sum_{j=1}^{m}(M_j + W) - m}\left(\frac{k_i}{m} - \rho_c(2 - \rho_c) + W\right) = \frac{2m\rho_c(2 - \rho_c) + W}{m}\left(\frac{W}{m\rho_c(2 - \rho_c)}\right)^t.
\]

Thus, we get \(k_i(t) \sim t^\alpha\) with \(\alpha = 1/(2 + \frac{W}{m\rho_c(2 - \rho_c)})\).

Then, in the infinite size, the degree probability distribution can be acquired by \(P(k) \sim \int_0^\infty \delta(k - k_i(t))dt \sim k^{-\gamma}\) with
\[
\gamma = 1 + 1/\alpha = 3 + \frac{W}{m\rho_c(2 - \rho_c)}.
\]

Though this expression is somewhat rough because of several approximations, such as \(\rho_c \approx \rho_c(t), t \to \infty\), Eq. (9) can qualitatively describe the power-law degree distribution of the generated network.

In addition, we should mention a network model proposed by Dorogovtsev and Mendes [43], which is related to the present work. In such model, by introducing the “initial attractiveness” (IA) to each node, power-law degree distributions can be generated together with the exponent of the distribution controlled by strength of the IA. Very interestingly, the IA plays a significant role in the emergence of the assortative mixing property [44]. In our model, the parameter \(W\) may play the same role as that of the IA in the perspective of assortative feature. The introduction of \(W\) enlarges the probability of poor players being connected by the new one in the growth process. Moreover, there is an approximately positive correlation between the payoff and the degree of a given individual. Hence, \(W\) enhances the connecting probability between small-degree individuals that results in the assortative mixing behavior. However, in our model, the degree distribution is not only controlled by \(W\) and \(m\), but also by \(\rho_c\), as obtained in Eq. (9). Our model couples the dynamical process of the SG and the evolution of the network, which leads to the difference between our model and the network model of Dorogovtsev and Mendes.

IV. CONCLUSION

In summary, we have studied the interplay of the evolutionary game and the relevant network structure. Simulation results indicate that both scale-free structural property and high cooperation level result from the interplay between the game and the network. Moreover, the resultant networks reproduce some typical features of social networks, including small-world and positive assortative mixing properties. The investigation of the wealth distribution of players indicates the validity of our model in mimicking the dynamical behavior of economical systems.

However, some issues still remain unclear and deserve further study, such as the evolution of connections among existing nodes, as discussed in previous works [45, 46]. On the other hand, in the present work, we only consider the case of “birth” of new players, which leads to the growth of the network. While, in social systems, “death” and “aging” are also important events and a previous work has already pointed out that the aging effect plays a significant role in the evolution of network structures [47]. Therefore, there is a need to consider the death and aging processes for better characterizing the evolutionary dynamics of social systems in the future study.

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