COMPLEX LIE ALGEBRAS CORRESPONDING TO WEIGHTED PROJECTIVE LINES

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Abstract. The aim of this paper is to give an alternative proof of Kac’s theorem for weighted projective lines (§5) over the complex field. The geometric realization of complex Lie algebras arising from derived categories (§8) is essentially used.

1. Introduction

It is well known that the dimension vectors of indecomposable representations of quiver \( Q \) correspond 1 − 1 to the positive roots of the Kac-Moody algebra associated to \( Q \).

In [5], Crawley-Boevey proved an analogue of Kac’s Theorem as follows:

**Theorem 1.1.** If \( \mathbb{X}_{p,\Delta} \) is a weighted projective line over an algebraically closed field \( K \) and \( \alpha \in \hat{Q} \), there is an indecomposable sheaf in \( \text{Coh}(\mathbb{X}_{p,\Delta}) \) of type \( \alpha \) if and only if \( \alpha \) is a positive root. Moreover, there is a unique indecomposable for a real root, infinitely many for an imaginary root.

This theorem describes the possible dimension vectors of indecomposable sheaves. In order to prove it, Crawley-Boevey reduced to the case when \( K \) is the algebraic closure of a finite field. He worked over a finite field \( F_q \) and associated a Lie algebra \( L \) to the category of coherent sheaves on a weighted projective line over this finite field. We note that the Lie algebra \( L \) is defined over a field \( \mathbb{F} \), which has characteristic \( l \) such that \( q = 1 \) in \( \mathbb{F} \).

We find that the proof can be simplified when \( K \) is changed to the complex field \( \mathbb{C} \). Using [8] and the derived equivalence between the category of coherent sheaves on a weighted projective line and the module category of the corresponding canonical algebra, we construct a Lie algebra \( L \) on the category of coherent sheaves on a weighted projective line over \( \mathbb{C} \) and find elements which satisfy the relations of the loop algebra. We calculate the Euler characteristics instead of counting numbers.

Let \( v \) be a vertex of the star-shaped graph (see §3.2) and write \( \alpha_v \) for the simple root corresponding to \( v \). Let \( e \in L_{\alpha_v}, f \in L_{-\alpha_v} \), using the standard arguments in Lie algebra over the base field \( \mathbb{C} \), we have the isomorphism \( L_\phi \simeq L_{s_v(\phi)} \), i.e, the simple reflection induces isomorphism. Finally, we reduce to three simple cases by a sequence of reflections which were solved in [6].

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We note that in the process of the proof of the Kac Theorem on weighted projective lines, the operator \( \theta = \exp(ad_e) \exp(-ad_f) \exp(ad_e) \) in the \( sl_2 \)-representation can be defined directly and the definition trouble occurring in the case of the finite field is avoided. Moreover, the process of finding a suitable field as the base field of the Lie algebra can be omitted. This simplifies the proof.

2. Lie Algebras Arising from Derived Categories

2.1. Let \( \Lambda \) be a finite dimensional and finite global dimensional associative algebra over \( \mathbb{C} \). We can write (up to Morita equivalence) \( \Lambda = \mathbb{C}Q/J \), where \( Q \) is a quiver and \( J \) is the admissible ideal generated by a set \( R \) of relations.

Consider the category \( \text{mod}\Lambda \) of finite dimensional \( \Lambda \)-modules and its bounded derived category \( D^b(\Lambda) \). In [8], Xiao, Xu and Zhang obtained a geometric realization of complex Lie algebras arising from derived categories.

2.2. We fix \( \{P_1, P_2, \ldots, P_l\} \) to be a complete set of indecomposable projective \( \Lambda \)-modules. A complex \( C^* \) of \( \Lambda \)-modules is called a period-2 complex if it satisfies \( C^*[2] = C^* \). Let \( P^* = (P^0, P^1, \partial_0, \partial_1) \) be a period-2 complex of projective \( \Lambda \)-modules such that each \( P^i \) has the decomposition \( P^i = \bigoplus_{j=1}^l e_j^i P_j \). We denote by \( \mathfrak{e}(P^i) \) the vector \( (e_1^i, e_2^i, \ldots, e_l^i) \), then \( \mathfrak{e} = (\mathfrak{e}(P^0), \mathfrak{e}(P^1)) \) is called the projective dimension sequence of \( P^* \). We define \( \mathcal{P}_2(\Lambda, \mathfrak{e}) \) to be the subset of

\[
\text{Hom}_\Lambda(P^0, P^1) \times \text{Hom}_\Lambda(P^1, P^0)
\]

which consists of \((\partial_0, \partial_1)\) such that \( \partial_0 \partial_1 = 0 \) and \( \partial_1 \partial_0 = 0 \).

The algebraic group \( G_\mathfrak{e} = \text{Aut}_\Lambda(P^0) \times \text{Aut}_\Lambda(P^1) \) acts on \( \mathcal{P}_2(\Lambda, \mathfrak{e}) \) by conjugation action. Thus two projective complexes in \( \mathcal{P}_2(\Lambda, \mathfrak{e}) \) are in the same orbit under the \( G_\mathfrak{e} \)-action if and only if they are quasi-isomorphic.

Let \( K_0 \) be the Grothendieck group of \( D^b(\Lambda) \), also of \( D^b(\Lambda)/(T^2) \). There is a canonical surjection from the abelian group of projective dimension sequences to \( K_0 \), which will be denoted by \( \dim \). We define \( \mathcal{P}_2(\Lambda, \mathfrak{d}) = \bigcup_{\mathfrak{e} \in \dim^{-1}(\mathfrak{d})} \mathcal{P}_2(\Lambda, \mathfrak{e}) \) for any \( \mathfrak{d} \in K_0 \). Then \( \mathcal{P}_2(\Lambda, \mathfrak{d}) \) has a natural topological structure induced by that of \( \mathcal{P}_2(\Lambda, \mathfrak{e}) \), see [8] for details. Thus \( G_\mathfrak{d} = \bigcup_{\mathfrak{e} \in \dim^{-1}(\mathfrak{d})} G_\mathfrak{e} \) partially acts on \( \mathcal{P}_2(\Lambda, \mathfrak{d}) \).

Moreover, we set

\[
T_\mathfrak{d} = \{ t^\pm_{ij} \mid x \in \mathcal{P}_2(\Lambda, \mathfrak{e}) \text{ is constructible} \}
\]

and \( T = \bigcup_{\mathfrak{e} \in \dim^{-1}(0)} T_\mathfrak{e} \) whose action on \( \mathcal{P}_2(\Lambda, \mathfrak{d}) \) is also partially defined. With the groupoid \( \langle G_\mathfrak{d}, T \rangle \) acting on \( \mathcal{P}_2(\Lambda, \mathfrak{d}) \), we have that

\[
Q\mathcal{P}_2(\Lambda, \mathfrak{d}) = \mathcal{P}_2(\Lambda, \mathfrak{d})/\sim = \mathcal{P}_2(\Lambda, \mathfrak{d})/\langle G_\mathfrak{d}, T \rangle
\]

where \( x \sim y \) in \( \mathcal{P}_2(\Lambda, \mathfrak{d}) \) if and only if their corresponding complexes are quasi-isomorphic.

2.3. We denote by \( M(X) \) the set of all constructible functions on an algebraic variety \( X \) with values in \( \mathbb{C} \). The set \( M(X) \) is naturally a \( \mathbb{C} \)-linear space. Let \( G \) be an algebraic group acting on \( X \). Then we denote by \( M_G(X) \) the subspace of \( M(X) \) consisting of all \( G \)-invariant functions.

Let \( \mathfrak{d} \) be a dimension vector in \( K_0 \) and \( \mathcal{O} \) be a \( \langle G_\mathfrak{d}, T \rangle \)-invariant and support-bounded constructible subset of \( \mathcal{P}_2(\Lambda, \mathfrak{d}) \). Here support-bounded means there exists a projective dimension sequence \( \mathfrak{e} \) such that \( \mathcal{O} = \langle G_\mathfrak{d}, T \rangle (\mathcal{O} \cap \mathcal{P}_2(\Lambda, \mathfrak{e})) \) and \( \mathfrak{e} \) is called a support projective dimension sequence of \( \mathcal{O} \).
We define the function $1_{\mathcal{O}} : P_2(\Lambda, \mathbf{d}) \to \mathbb{C}$ given by taking values 1 on each point in $\mathcal{O}$ and 0 otherwise. A function $f$ on $P_2(\Lambda, \mathbf{d})$ is called $\langle G_d, T \rangle$-invariant if $f$ can be written as a sum of finite sums $\sum m_i 1_{\mathcal{O}_i}$, where $m_i \in \mathbb{C}$ and any $\mathcal{O}_i$ is $\langle G_d, T \rangle$-invariant and support-bounded constructible subset of $P_2(\Lambda, \mathbf{d})$. Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be projective dimension sequences in $\dim^{-1}(\mathbf{d})$.

Two constructible functions $f_i \in M_{G_\Lambda}(P_2(\Lambda, \mathbf{g}_i))$, $i = 1, 2$ are equivalent if there exists a $\langle G_d, T \rangle$-invariant constructible $F$ over $P_2(\Lambda, \mathbf{d})$ such that $f_i = F|_{P_2(\Lambda, \mathbf{g}_i)}$, $i = 1, 2$. Let $f \in M_{G_\Lambda}(\mathbf{g})$ and $\mathbf{g} \in \dim^{-1}(\mathbf{d})$. The equivalent class of $f$ is denoted by $\hat{f}$. Let $M_{GT}(P_2(\Lambda, \mathbf{d}))$ be the space of the equivalence classes $\hat{f}$ of constructible functions $f$ over $P_2(\Lambda, \mathbf{g})$ for any $\mathbf{g} \in \dim^{-1}(\mathbf{d})$.

An equivalence class $\hat{f} \in M_{GT}(P_2(\Lambda, \mathbf{d}))$ is called indecomposable if any point in $\text{supp}(f)$ is indecomposable in the (relative) homotopy category of all period-2 complexes of projective modules. Let $I_{GT}(\mathbf{d})$ be the $\mathbb{C}$-space of all indecomposable equivalence classes in $M_{GT}(P_2(\Lambda, \mathbf{d}))$.

Let $\mathcal{O}_1 \subset P_2(\Lambda, \mathbf{g}') \subset P_2(\Lambda, \mathbf{d}_1)$ and $\mathcal{O}_2 \subset P_2(\Lambda, \mathbf{g}') \subset P_2(\Lambda, \mathbf{d}_2)$ be $G_{\mathbf{g}'}$- and $G_{\mathbf{g}'}$-invariant constructible sets, respectively. For $L \in P_2(\Lambda, \mathbf{g}' + \mathbf{g}'')$, we set

$$W(\mathcal{O}_1, \mathcal{O}_2; L) = \{(f, g, h) | Y \xrightarrow{f} L \xrightarrow{g} X \xrightarrow{h} Y[1] \text{ is a distinguished triangle}$$

with $X \in \mathcal{O}_1, Y \in \mathcal{O}_2$, then the quotient space $W(\mathcal{O}_1, \mathcal{O}_2; L)/G_{\mathbf{g}'} \times G_{\mathbf{g}'}$ is independent of choices of support projective dimension sequences of both $\langle G_d, T \rangle\mathcal{O}_1$ and $\langle G_d, T \rangle\mathcal{O}_2$. So we denote it by $V(\mathcal{O}_1, \mathcal{O}_2; L)$.

Thus the convolution multiplication $\hat{1}_{\mathcal{O}_1} \ast \hat{1}_{\mathcal{O}_2} \in M_{GT}(P_2(\Lambda, \mathbf{d}_1 + \mathbf{d}_2))$ can be defined as follows:

$$\hat{1}_{\mathcal{O}_1} \ast \hat{1}_{\mathcal{O}_2}(L) = F_{\hat{1}_{\mathcal{O}_1}\mathcal{O}_2} := \chi(V(\mathcal{O}_1, \mathcal{O}_2; L))$$

where $\chi$ denotes the quasi Euler characteristic of quotient space as in $[\mathbb{S}]$.

We set $n = \bigoplus_{d \in K_0} I_{GT}(d)$ and $\mathfrak{n} = K_0 \otimes \mathbb{C}$ which is spanned by $\{h_d | d \in K_0\}$. The symmetric Euler bilinear form on $\mathfrak{n}$ is given as

$$(h_d, h_d) = \dim_{\mathbb{C}} \text{Hom}(X, Y) - \dim_{\mathbb{C}} \text{Hom}(X, Y[1])$$

$$+ \dim_{\mathbb{C}} \text{Hom}(Y, X) - \dim_{\mathbb{C}} \text{Hom}(Y, X[1])$$

for any $X \in P_2(\Lambda, \mathbf{d}_1), Y \in P_2(\Lambda, \mathbf{d}_2)$.

Thus $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ becomes a Lie algebra over $\mathbb{C}$ with the Lie bracket $[-, -]$ defined below.

$$[\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}] = [\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}]_n + \chi(\overline{\mathcal{O}_1 \cap \mathcal{O}_2[1]})h_d,$$

where $\overline{\mathcal{O}_1 \cap \mathcal{O}_2[1]} := (\mathcal{O}_1 \cap \mathcal{O}_2[1]) / G_{\mathbf{g}}$ for a support projective dimension sequence of $\mathcal{O}_1 \cap \mathcal{O}_2[1]$.

$$[\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}]_n(L) := F_{\hat{1}_{\mathcal{O}_1}\mathcal{O}_2} - F_{\hat{1}_{\mathcal{O}_2}\mathcal{O}_1}$$

$$[h_d, \hat{1}_{\mathcal{O}_2}] := (h_d, h_d) \hat{1}_{\mathcal{O}_2}, \; [\hat{1}_{\mathcal{O}_2}, h_d] := -(h_d, h_d) \hat{1}_{\mathcal{O}_2}$$

$$[h_d, h_d] := 0.$$
3. The category of coherent sheaves on weighted projective lines

3.1. Weighted projective lines. Let \( p = (p_1, p_2, \ldots, p_n) \in (\mathbb{N}^*)^n \) and \( \Delta = \{\lambda_1, \ldots, \lambda_n\} \) be a collection of distinct closed points on the projective line \( \mathbb{P}^1(\mathbb{C}) \).

Instead of giving the definition, we give a description of the structure of the category \( \text{Coh}(\mathbb{P}_p^N) \) (see [2] for details).

Let \( \mathcal{F} \) and \( \mathcal{F}' \) be two full extension-closed subcategories of \( \text{Coh}(\mathbb{P}_p^N) \). For any sheaf \( \mathcal{M} \in \text{Coh}(\mathbb{P}_p^N) \), it can be decomposed as \( \mathcal{M}_t \oplus \mathcal{M}_t' \) where \( \mathcal{M}_t \in \mathcal{F} \) and \( \mathcal{M}_t' \in \mathcal{F}' \) and \( \text{Hom}(\mathcal{M}_t, \mathcal{M}_t') = \text{Ext}^1(\mathcal{M}_t, \mathcal{M}_t') = 0 \) for any \( \mathcal{M}_t \in \mathcal{F} \) and \( \mathcal{M}_t' \in \mathcal{F}' \).

The category \( \mathcal{F} \) decomposes as a coproduct \( \mathcal{F} = \bigsqcup_{x \in \mathbb{P}^N_p}\mathcal{T}_x \), where \( \mathcal{T}_x \) is equivalent to the category \( \text{rep}_{\mathbb{p}}(C_{r_x}) \) consisting of nilpotent representations of the cyclic quiver with \( r_x \) vertices, where \( r_x = p_i \) if \( x = \lambda_i, 1 \leq i \leq n \), and \( r_x = 1 \) otherwise.

The category \( \mathcal{F} \) has a filtration by objects of the form \( \mathcal{O}(\bar{x}) \), where \( \bar{x} \in L(p) = \mathbb{Z} \bar{x}_1 \oplus \mathbb{Z} \bar{x}_2 \oplus \cdots \oplus \mathbb{Z} \bar{x}_n \) where \( J \) is the submodule generated by \( \{p_1 \bar{x}_1 - p_s \bar{x}_s | s = 2, \ldots, n\} \). Set \( \bar{c} = p_1 \bar{x}_1 = \cdots = p_n \bar{x}_n \in L(p) \). For \( \mathcal{O}(\bar{r}c) \), there is a unique simple objects \( S_{i,j} \) in each \( \mathcal{T}_{\lambda_i} \) with \( \text{dimHom}(\mathcal{O}(\bar{r}c), S) = 1 \). The simple objects are \( S_a \) (\( a \in \mathbb{P}^N \)) and \( S_{i,j} \) (\( 1 \leq i \leq n, 0 \leq j \leq p_i - 1 \)), which satisfy the relations \( \text{dimExt}(S_{i,j}, S_{i,j-1}) = 1 \).

3.2. Star-shaped graph and loop algebra. Associating to the weight type \((p, \Delta)\), we have a star-shaped graph \( \Gamma \):

\[
\begin{array}{cccc}
[1,1] & [1,2] & \cdots & [1,p_n-1] \\
\bullet & \bullet & \cdots & \bullet \\
[2,1] & [2,2] & \cdots & [2,p_n-1] \\
\bullet & \bullet & \cdots & \bullet \\
\vdots & \vdots & \cdots & \vdots \\
n[1,n] & \cdots & \cdots & [n,n] \\
\bullet & \bullet & \cdots & \bullet \\
\end{array}
\]

whose vertex set \( I \) consists of the central vertex \( * \) and vertices in \( n \) branches which are denoted by \( [i,j] \), \( 1 \leq i \leq n, 1 \leq j \leq p_i - 1 \).

Consider the Kac-Moody algebra \( g = g(\Gamma) \) associated to the graph \( \Gamma \). We have the loop algebra of \( g \), denoted by \( \mathbb{L}g \), which is defined to be the complex Lie algebra generated by \( h_{i,k}, e_{i,k}, f_{i,k} : i \in I, k \in \mathbb{Z} \) and \( c \) subject to the following relations:

\[
\begin{align*}
[h_{i,k}, h_{j,l}] &= k\delta_{k,-l}a_{ij}c, \\
[e_{i,k}, f_{j,l}] &= \delta_{i,j}h_{i,k+l} + k\delta_{k,-l}c, \\
[h_{i,k}, e_{j,l}] &= a_{ij}e_{j,l+k}, \quad [h_{i,k}, f_{j,l}] = -a_{ij}f_{j,l+k}, \\
[e_{i,k}, e_{i,l}] &= 0, \quad [f_{i,k}, f_{i,l}] = 0, \quad \text{central} \\
[e_{i,k}, e_{i,k+1}, \ldots, e_{i,n}, e_{j,l}] &= 0, \quad \text{for } n = 1 - a_{ij}, \\
[f_{i,k}, f_{i,k+1}, \ldots, f_{i,n}, f_{j,l}] &= 0, \quad \text{for } n = 1 - a_{ij}.
\end{align*}
\]
The root systems of \(\mathfrak{g}\) and \(L_\mathfrak{g}\) are denoted by \(\Delta\) and \(\hat{\Delta}\) respectively and the root lattices are denoted by \(Q\) and \(\hat{Q} = Q \oplus \mathbb{Z} \delta\). In view of the graph \(\Gamma\), the simple roots in \(\Delta\) are denoted by \(\alpha_s\) and \(\alpha_{ij}\) for \(1 \leq i \leq n\) and \(1 \leq j \leq p_i - 1\). We also know that \(\Delta = \mathbb{Z}^* \delta \cup (\Delta + \mathbb{Z} \delta)\).

There is a natural identification of \(\mathbb{Z}\)-modules \(K_0(\text{Coh}(\mathcal{X})) \cong \hat{Q}\) given by \([S_{i,j}] \mapsto \alpha_{ij}\), for \(j = 1, \ldots, p_i - 1\), \([S_{i,0}] \mapsto \delta - \sum_{j=1}^{p_i-1} \alpha_{ij}\), \([\mathcal{O}(k\delta)] \mapsto \alpha_s + k \delta\).

Naturally, the non-negative combinations of the elements \(\alpha_{ij}\), \(\delta - \sum_{j=1}^{p_i-1} \alpha_{ij}\), \(\alpha_s + k \delta\) form the positive cone \(\hat{\Delta}_+\).

### 3.3. Derived equivalence and the Lie algebra.

In [3], Ringel introduced the class of canonical algebras attached to \((\Lambda, \Delta)\). It is well known that there is a triangle equivalence \(D^b(\text{Coh}(\mathbf{X}_{p, \Delta})) \simeq D^b(\text{mod}(\Lambda^\text{sp}_{p, \Delta}))\) where \(\text{Coh}(\mathbf{X}_{p, \Delta})\) is a hereditary abelian category. Therefore, their root categories are equivalent. We simply write \(\Lambda\) for \(\Lambda^\text{sp}_{p, \Delta}\). Then by 2.3 we can define a \(\hat{Q}\)-graded complex Lie algebra \(L\) on the root category of \(\Lambda\).

The set of indecomposable objects of \(\text{R}_{p, \Delta} = D^b(\text{Coh}(\mathbf{X}_{p, \Delta}))/\langle T^2 \rangle\) is ind\(\text{R}_{p, \Delta} = (\text{indCoh}(\mathbf{X}_{p, \Delta})) \cup \{TY | Y \in \text{indCoh}(\mathbf{X}_{p, \Delta})\}\). For any simple object \(S\), \(S[r]\) denotes the unique object \(S[r]\) with length \(r\) and \(\text{Hom}(S, S[j]) = 0\) for all \(1 \leq i \leq n, 1 \leq j \leq p_i - 1\).

**Lemma 3.1.** (i) For any \(X \in \text{indR}_{p, \Delta}\), the image of \(X\) in the root category of the canonical algebra \(\Lambda\) is denoted by \(\hat{F}(X)\). Assume \(F(X) \in \mathcal{P}_2(\Lambda, \underline{e}(\alpha))\), \(\hat{\mathcal{F}}_{G(F(X))}\) is the equivalence class of the characteristic function of the orbit \(G(F(X))\). Then \(\hat{\mathcal{F}}_{G(F(X))} \in I_{G(F(X))}(\dim \underline{e}(\alpha))\).

(ii) The set \(F(H_r) \subset \mathcal{P}_2(\Lambda, \underline{e}(\alpha))\), \(\hat{\mathcal{F}}_{G(F(H_r))} \in I_{G(F(H_r))}(\dim \underline{e}(\alpha))\). Moreover, \(\chi(\underline{e}(\alpha)F(H_r)/G(e(\alpha)) = 2\).

**Proof.** (i) is trivial because \(F(X)\) is also indecomposable in the root category of \(\Lambda\).

(ii) The Serre subcategory generated by \(\mathcal{O}(k\delta)\) for \(k \in \mathbb{Z}\), \(S_{i,0}[a] (a \in \mathbb{P}^1 \setminus \Delta, l \geq 1)\) and \(S_{i,0}[p_i] (1 \leq i \leq n, l \geq 1)\) is equivalent to the category \(\text{Coh}(\mathbb{P}^1)\). Therefore, it is enough to prove the non-weighted case. We have \(D^b(\text{Coh}(\mathbb{P}^1)) = D^b(\text{rep} \hat{Q})\), where \(\hat{Q}\) is the Kronecker quiver. There exists \(e(\alpha)\) such that \(F(H_r) \subset \mathcal{P}_2(\Lambda, \underline{e}(\alpha))\).

The results in the Kronecker quiver case imply \(\hat{\mathcal{F}}_{G(F(H_r))} \in I_{G(F(H_r))}(\dim \underline{e}(\alpha))\) and \(\chi(\underline{e}(\alpha)F(H_r)/G(e(\alpha)) = 2\).

4. NEW PROOF

4.1. Main result.

**Theorem 4.1.** If \(\mathbf{X}_{p, \Delta}\) is a weighted projective line over the complex field \(\mathbb{C}\) and \(\alpha \in \hat{Q}\), there is an indecomposable sheaf in \(\text{Coh}(\mathbf{X}_{p, \Delta})\) of type \(\alpha\) if and only if \(\alpha\) is a positive root. Moreover, there is a unique indecomposable for a real root, infinitely many for an imaginary root.

This theorem is proved in [5] over any algebraically closed field. In the case of the complex field \(\mathbb{C}\), we find a new proof as follows, which also uses the Hall
algebras. We define a $\dot{Q}$-graded complex Lie algebra $L$ on the root category $\mathcal{R}_{\mathcal{P},\lambda}$ (section 3.3) and there is a subalgebra satisfying the relations of the loop algebra.

Set $l(r) = 1$, for $r \geq 0$ and $l(r) = -1$, for $r < 0$. For any $X \in \text{indR}_{\mathcal{P},\lambda}$, we write $\hat{1}_X = \hat{1}_G(X)$ and $\hat{1}_{(H_r)} = \hat{1}_{G_{(r)}F(H_r)}$ for short.

**Theorem 4.2.** The following elements satisfy the relations in $\mathcal{L}_0$.

$$
eq_{v,r} = \begin{cases} l(r)\hat{1}_{(S_{i,j}[r_{p_1}+1])} & v = [i,j] \\ l(r)\hat{1}_{(\theta(r\bar{c})}) & v = * \end{cases} \quad f_{v,r} = \begin{cases} l(r-1)\hat{1}_{(S_{i,j-1}[r_{p_1}+1])} & v = [i,j] \\ l(r)\hat{1}_{(\theta(-r\bar{c})}) & v = * \end{cases}$$

$$c = -\delta \quad h_{v,r} = \begin{cases} -\alpha_v & r = 0 \\ l(r)\hat{1}_{(S_{i,j}[r_{p_1}])} - l(r)\hat{1}_{(S_{i,j-1}[r_{p_1}])} & r \neq 0, v = [i,j] \\ l(r)\hat{1}_{(H_r)} & r \neq 0, v = * \end{cases}$$

4.2. **Proof of Theorem 4.2.** We note that $[\hat{1}_{\Omega_1}, \hat{1}_{\Omega_2}](M) = 0$ for $M$ decomposable and the triangles $X \rightarrow Y \rightarrow Z \rightarrow 0$ with $X, Y, Z \in \text{indR}_{\mathcal{P},\lambda}$ are in 1-1 correspondence with short exact sequences in $\text{Coh}(X_{\mathcal{P},\lambda})$. The section 3 of [5] is still true for the complex field. However, we calculate the Euler characteristics instead of counting numbers.

(i) 

$$[l(r)\hat{1}_{(S_{i,j}[r])}, l(s)\hat{1}_{(S_{i,s}[s])}] = \begin{cases} \delta_{j-r,k}(l(s)s)\hat{1}_{(S_{i,j}[r+s])} & r + s \neq 0 \\ -\delta_{j-r,k}(S_{i,j}[r]) & r + s = 0 \end{cases}$$

Proof of (i): In one tube, if $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is a short exact sequence of indecomposable objects, then there is a unique short exact sequence with the same terms up to automorphisms of any two of $X, Y, Z$. Using the fact $\chi$ (one point) = 1, we complete the proof.

Note that we can prove all relations in one tube by (i) now.

(ii) $[h_{s,r}, h_{s,-r}] = [l(r)\hat{1}_{(H_r)}, l(-r)\hat{1}_{(H_{-r})}] = -r\delta(\mathcal{G}(r\mathcal{c})/\mathcal{G}(r\bar{c})) = -2r\delta = 2rc$

(iii) For $[e_{s,r}, f_{s,s}]$,

if $r + s = 0$, $[e_{s,r}, f_{s,s}] = -[\hat{1}_{(\mathcal{c}(r\bar{c})\mathcal{c})}, \hat{1}_{(\theta(-r\bar{c})})] = -\chi((\mathcal{c}(r\bar{c})\mathcal{c})) = -\mathcal{c}(r\bar{c}) = h_{s,0} + rc$

if $r + s \neq 0$, assume $r + s > 0$, we get the short exact sequence $0 \rightarrow \mathcal{c}((r + s)^\mathcal{c}) \rightarrow \mathcal{c} \rightarrow Y \rightarrow 0$ with $Y \in H_{r+s}$, $\text{dimHom}(\mathcal{c}, Y) = r + s$. The non-epimorphisms form a subspace of dimension $r + s - 1$ and each short exact sequence is determined by an epimorphism up to an automorphism of $\mathcal{c}((r + s)^\mathcal{c})$.

$[e_{s,r}, f_{s,s}](Y) = \chi(\mathcal{c}((r + s)^\mathcal{c})) = 1$. That implies $[e_{s,r}, f_{s,s}] = h_{s,s}$. $r + s$.

(iv) We assume $r > 0$. The support of the function $[h_{[i,1],r}, e_{s,s}]$ is the orbit of $\mathcal{c}((r + s)^\mathcal{c})$. For $X \in (\mathcal{c}((r + s)^\mathcal{c}))$, $[h_{[i,1],r}, e_{s,s}][X] = -\chi$(one point) = -1, then $[h_{[i,1],r}, e_{s,s}] = -e_{s,s}$.

(v) We assume $r > 0$. The support of the function $[h_{s,r}, e_{s,s}]$ is the orbit of $\mathcal{c}((r + s)^\mathcal{c})$. For $X \in (\mathcal{c}((r + s)^\mathcal{c}))$, $[h_{s,r}, e_{s,s}][X] = \chi(\mathcal{P}^1) = 2$, then $[h_{s,r}, e_{s,s}] = 2e_{s,s}$. ■
4.3. **Proof of Theorem 4.4**  

$L$ is a $Q$-graded complex Lie algebra with $L_0 = Q \otimes_{\mathbb{Z}} \mathbb{C}$. For $\phi \in Q_+$, if there is an indecomposable sheaf $X$ in $\text{Coh}(X_{P,\lambda})$ of type $\phi$, then $\hat{1}_X \in L_\phi$ and $L_\phi \neq 0$. If there is no indecomposable sheaf of type $\phi$, $L_\phi = 0$. The case of $-\phi \in Q_+$ is similar.

For $\phi \in Q_+$, we want to determine whether or not $L_\phi = 0$. We need the following two lemmas:

**Lemma 4.3.** Let $v$ be a vertex of the star-shaped graph. The operators $\text{ad } e_{v,0}$ and $\text{ad } f_{v,0}$ are locally nilpotent.

**Proof.** For any $\psi \in \hat{Q}$ and $f \in L_\psi$, we need to show $(\text{ad } e_{v,0})^n(f) = (\text{ad } f_{v,0})^n(f) = 0$, for some $n$. It is enough to prove $(\text{ad } \hat{1}_X)^n(1_Y) = 0$ for any two indecomposable sheaves $X, Y$ with $\text{Ext}^1(X,Y) = 0$.

If $Z$ is in the support of $(\text{ad } \hat{1}_X)(1_Y)$, then $Z$ is the middle term of a nonsplit exact sequence whose end terms are $X$ and $Y$, so

$$\dim \text{Ext}^1(X,Z) + \dim \text{Ext}^1(Z,X) \leq \dim \text{Ext}^1(X,Y) + \dim \text{Ext}^1(Y,X),$$

thus $(\text{ad } \hat{1}_X)^n(1_Y) = 0$ for $n > \dim \text{Ext}^1(X,Y) + \dim \text{Ext}^1(Y,X)$. \qed

**Lemma 4.4.** Let $v$ be a vertex of the star-shaped graph and write $\alpha_v$ for the simple root corresponding to $v$. For any $\phi \in Q_+$, we have $L_\phi \simeq L_{s_v(\phi)}$.

**Proof.** As proved in [4,2] $e_{v,0} \in L_{\alpha_v}$ and $f_{v,0} \in L_{-\alpha_v}$ satisfy $[e_{v,0}, f_{v,0}] = h_{v,0}$ and for $f \in L_\psi$, $(\text{ad } h_{v,0})(f) = (\alpha_v, \psi)f$. From Lemma 4.3 ad $e_{v,0}$ and ad $f_{v,0}$ are locally nilpotent. So the operator $\theta = \exp(\text{ad } e_{v,0}) \exp(\text{ad } f_{v,0}) \exp(\text{ad } e_{v,0})$ acts on $h_{v,0}$ as multiplication by $-1$. For $f \in L_\phi$, we have $\theta(f) = \sum_{r \in \mathbb{Z}} f_r$ with $f_r \in L_{\phi + r\alpha_v}$.

$$\sum_{r \in \mathbb{Z}} (\alpha_v, \phi) f_r = \theta([h_{v,0}, f]) = [\theta(h_{v,0}), \theta(f)] = [-h_{v,0}, \theta(f)]$$

Comparing the coefficients of the above equation, we get $\theta(f) = f_r$ with $r = -(\alpha_v, \phi)$, which means $\theta(L_\phi) \subseteq L_{\phi - (\alpha_v, \phi) \alpha_v}$. Similarly $\theta^{-1}(L_{\phi - (\alpha_v, \phi) \alpha_v}) \subseteq L_\phi$. Thus the operator $\theta = \exp(\text{ad } e_{v,0}) \exp(\text{ad } f_{v,0}) \exp(\text{ad } e_{v,0})$ induces an isomorphism $L_\phi \simeq L_{s_v(\phi)}$. \qed

For $\phi \in \hat{Q}$, we can reduce to the following three cases by a sequence of reflections:

$\pm \alpha_v + r\delta$;  
$\alpha + r\delta$, with $\alpha$ in the fundamental region;  
$\alpha + r\delta$, where $\alpha$ is not positive or negative, or has disconnected support.

For the first case: $\dim L_\phi = \dim L_{\pm \alpha_v + r\delta} = 1$, there is a unique indecomposable sheaf;  
the second case: $\dim L_\phi = \dim L_{\alpha + r\delta} = \infty$, there are infinitely many indecomposable sheaves (see [5]);  
the last case: $\dim L_\phi = \dim L_{\alpha + r\delta} = 0$, there is no indecomposable object.

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