NODAL SOLUTIONS FOR SINGULAR SEMILINEAR ELLIPTIC SYSTEMS

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Abstract. In this paper, we prove existence of nodal solutions for singular semilinear elliptic systems without variational structure where its both components are of sign changing. Our approach is based on sub-supersolutions method combined with perturbation arguments involving singular terms.

1. Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) (\( N \geq 2 \)) having a smooth boundary \( \partial \Omega \) and a positive measure such that \( \text{meas}(\Omega) > 1 \). Consider the following system of semilinear elliptic equations

\[
(P) \begin{cases}
-\Delta u + h_{\lambda, \phi_1}(u) = a_1(x) \frac{f_1(v)}{|u|^{\alpha_1}} & \text{in } \Omega \\
-\Delta v + h_{\lambda, \phi_1}(v) = a_2(x) \frac{f_2(u)}{|v|^{\alpha_2}} & \text{in } \Omega \\
u, v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Delta \) stands for the Laplace differential operator and \( h_{\lambda, \phi_1} \) is a linear function defined by

\[
h_{\lambda, \phi_1}(s) := \lambda(s + \phi_1), \quad \text{for } s \in \mathbb{R}, \quad \text{for } \lambda > 0,
\]

where \( \phi_1 \) denotes the positive eigenfunction corresponding to the principal eigenvalue \( \lambda_1 \). In the reaction terms (the right hand side) of problem \( (P) \), the function \( a_i \in L^\infty(\Omega) \) satisfies

\[
\text{H}(a): \text{There exists a constant } 1 < \rho_i < \text{meas}(\Omega) \text{ such that}
\]

\[
\begin{align*}
 a_i(x) &> 0 \text{ for a.a. } x \in \Omega_{\rho_i} \\
 a_i(x) &\leq 0 \text{ for a.e. } x \in \Omega \setminus \Omega_{\rho_i},
\end{align*}
\]

where

\[
\Omega_{\rho_i} = \{ x \in \Omega : d(x, \partial \Omega) < \rho_i \}, \quad \text{for } i = 1, 2,
\]

while the nonlinear term \( f_i \) is a continuous function satisfying the growth condition

\[
\text{H}(f): \text{There is constants } m_i, M_i > 0 \text{ and } \beta_i \in (0, 1) \text{ such that}
\]

\[
m_i \leq f_i(s) \leq M_i (1 + |s|^{\beta_i}), \quad \text{for all } s \in \mathbb{R}.
\]

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We consider the system (P) in a singular case assuming that
\[ 0 < \alpha_1, \alpha_2 < 1. \]

Our main goal is to provide a nodal solution \((u, v)\) of singular nonvariational elliptic system (P). This means that both components \(u\) and \(v\) are of sign changing. According to our knowledge, this topic is a novelty. The virtually non-existent works in the literature devoted to this subject is partly due to the fact that the existence of a nodal solution for systems is more delicate than in the case of equations. Precisely, [12] is the only paper that has addressed this issue for nonvariational systems through topological degree argument where a different concept of nodal solutions is introduced. Namely, it is shown the existence of a solution where its both components are nontrivial and are not of the same constant sign. Solutions of this type have also been studied in [8, 13] for a class of quasilinear systems with variational structure by combining variational methods with suitable truncation. However, the nodal solution in [8] is defined in more subtle way considering that either its components are of the same constant sign or at least one of them is sign changing. Thence, even for variational systems, the question regarding nodal solutions where both components are of sign changing remains open. Here, it should be noted that system (P) under assumptions above is not in variational form, so the variational methods are not applicable.

Another main technical difficulty in the present paper consists in the presence of singularities in system (P) near the origin that occur under assumption (1.2). These singularities make difficult any study attendant to nodal solutions for (P) due to their sign change forcing them to cross zero. It represents a serious difficulty to overcome and, as far as we know, is never handled in the literature even for singular problems in the scalar case. For more inquiries on the study of constant sign solutions for singular systems we refer to [11, 10, 9, 4, 5, 7] and the references therein.

To handle our problem, we show that the sets where \(u\) and \(v\) vanish are of zero measure. This is an essential point enabling nodal solutions investigation. Thereby, by a solution of problem (P) we mean a couple \((u, v) \in H^1_0 (\Omega) \times H^1_0 (\Omega)\) such that the set where \(u\) (resp. \(v\)) vanishes is negligible and
\[
\begin{align*}
\int_{\Omega} (\nabla u \nabla \varphi + h_{\lambda, \phi_1} (u) \varphi) \, dx &= \int_{\Omega} a_1 (x) \frac{f_1 (u)}{|u|^{\alpha_1}} \varphi \, dx, \\
\int_{\Omega} (\nabla v \nabla \psi + h_{\lambda, \phi_1} (v) \psi) \, dx &= \int_{\Omega} a_2 (x) \frac{f_2 (v)}{|v|^{\alpha_2}} \psi \, dx,
\end{align*}
\]
for all \((\varphi, \psi) \in H^1_0 (\Omega) \times H^1_0 (\Omega)\).

Our approach is chiefly based on sub-supersolution method. It is applied to a disturbed system \((P_{\varepsilon})\) depending on parameter \(\varepsilon > 0\) whose study is relevant for problem (P). The obtained solution \((u_{\varepsilon}, v_{\varepsilon})\) of \((P_{\varepsilon})\) is located in the rectangle formed by sub-supersolutions. A significant feature of our
result lies in the construction of the sub- and supersolution pair for \((P_\varepsilon)\). At this point, the choice of suitable functions with an adjustment of adequate constants is crucial. Namely, exploiting spectral properties of the Laplacian operator, the supersolution \((\bar{u}, \bar{v})\), constructed explicitly, is sign-changing and independent of \(\varepsilon > 0\), while the subsolution \((u_\varepsilon, v_\varepsilon)\) which, besides the dependence of \(\varepsilon\), does not have an explicit form, admits a limit as \(\varepsilon \to 0\) the couple \((u, v)\) where the component \(u\) (resp. \(v\)) is negative in \(\Omega_{\rho_1}\) (resp. \(\Omega_{\rho_2}\)) and nonnegative (not necessarily positive) in \(\Omega \setminus \Omega_{\rho_1}\) (resp. \(\Omega \setminus \Omega_{\rho_2}\)). Actually, it is worth noting that \((u_\varepsilon, v_\varepsilon)\) is a solution of an auxiliary problem \((\bar{P}_\varepsilon)\) related to \((P_\varepsilon)\). Then, the general theory of sub-supersolutions for systems of quasilinear equations (see [2]) implies the existence of a solution \((u_\varepsilon, v_\varepsilon)\) of problem \((P_\varepsilon)\) with the sets where \(u_\varepsilon\) and \(v_\varepsilon\) vanish are negligible. In particular, this establish that \(u_\varepsilon\) and \(v_\varepsilon\) cannot be identically zero in \(\Omega_{\rho_1}\) and \(\Omega_{\rho_2}\), respectively. Then, the solution \((u, v)\) of \((P)\), lying in \([\bar{u}, \bar{\bar{u}}] \times [\bar{v}, \bar{\bar{v}}]\) with the property that \(u \neq 0\) in \(\Omega_{\rho_1}\) and \(v \neq 0\) in \(\Omega_{\rho_2}\), is derived by passing to the limit as \(\varepsilon \to 0\). The argument is based on a priori estimates, dominated convergence Theorem as well as \(S_+\)-property of the negative Laplacian. Hence, \((u, v)\) turns out a nodal solution of \((P)\) with both components \(u\) and \(v\) are of sign changing.

The rest of this article is organized as follows. Section 2 contains the proof of the existence of solutions for regularized system \((P_\varepsilon)\) as well as the construction of its sign changing sub-supersolutions. Section 3 presents the proof of the existence of nodal solutions of system \((P)\).

2. THE REGULARIZED SYSTEM

For \(\varepsilon > 0\), let consider the auxiliary system

\[
(P_\varepsilon) \quad \begin{cases} 
-\Delta u + h_{\lambda, \phi_1}(u) = a_1(x) \frac{f_1(u)}{|u| + \varepsilon} \quad &\text{in } \Omega \\
-\Delta v + h_{\lambda, \phi_1}(v) = a_2(x) \frac{f_2(v)}{|v| + \varepsilon} \quad &\text{in } \Omega \\
u, v = 0 \quad &\text{on } \partial \Omega.
\end{cases}
\]

Employing sub-supersolution method we shall prove that problem \((P_\varepsilon)\) admits a nontrivial solution.

We recall that a sub-supersolution for \((P_\varepsilon)\) is any pairs \((\underline{u}, \underline{v}) \in (H^1_0(\Omega) \cap L^\infty(\Omega))^2\) and \((\bar{u}, \bar{v}) \in (H^1(\Omega) \cap L^\infty(\Omega))^2\) for which there hold \((\bar{u}, \bar{v}) \geq (\underline{u}, \underline{v})\) a.e. in \(\Omega\),

\[
\begin{align*}
\int_\Omega (\nabla \underline{u} \nabla \varphi + h_{\lambda, \phi_1}(\underline{u}) \varphi) \ dx &\ - \int_\Omega a_1(x) \frac{f_1(u)}{|u| + \varepsilon} \varphi dx \leq 0, \\
\int_\Omega (\nabla \underline{v} \nabla \psi + h_{\lambda, \phi_1}(\underline{v}) \psi) \ dx &\ - \int_\Omega a_2(x) \frac{f_2(v)}{|v| + \varepsilon} \psi dx \leq 0, \\
\int_\Omega (\nabla \bar{u} \nabla \varphi + h_{\lambda, \phi_1}(\bar{u}) \varphi) \ dx &\ - \int_\Omega a_1(x) \frac{f_1(u)}{|u| + \varepsilon} \varphi dx \geq 0, \\
\int_\Omega (\nabla \bar{v} \nabla \psi + h_{\lambda, \phi_1}(\bar{v}) \psi) \ dx &\ - \int_\Omega a_2(x) \frac{f_2(v)}{|v| + \varepsilon} \psi dx \geq 0,
\end{align*}
\]

for all \(\varphi, \psi \in H^1_0(\Omega)\) with \(\varphi, \psi \geq 0\) a.e. in \(\Omega\) and for all \(u, v \in H^1_0(\Omega)\) satisfying \(u \leq \underline{u} \leq \bar{u}\) and \(v \leq \underline{v} \leq \bar{v}\) a.e. in \(\Omega\).
2.1. A constant sign sub-supersolution pair. In what follows \( \phi_1 \) denotes the positive eigenfunction corresponding to the principal eigenvalue \( \lambda_1 \), that is,

\[ -\Delta \phi_1 = \lambda_1 \phi_1 \text{ in } \Omega, \quad \phi_1 = 0 \text{ on } \partial \Omega. \]

which is well known to verify

\[ l^{-1}d(x) \leq \phi_1(x) \leq ld(x) \text{ for all } x \in \Omega, \]  

\[ |\nabla \phi_1| \geq \eta \text{ as } d(x) \to 0, \]

with constants \( l > 1 \) and \( \eta > 0 \), where \( d(x) \) denotes the distance from a point \( x \in \Omega \) to the boundary \( \partial \Omega \) and \( \Omega = \Omega \cup \partial \Omega \) is the closure of \( \Omega \subset \mathbb{R}^N \).

Let \( \tilde{\Omega} \) be a bounded domain in \( \mathbb{R}^N \) with a smooth boundary \( \partial \tilde{\Omega} \) such that \( \bar{\Omega} \subset \tilde{\Omega} \). Denote \( \tilde{d}(x) := d(x, \partial \tilde{\Omega}) \). By the definition of \( \tilde{\Omega} \) there exists a constant \( \mu > 0 \) small enough such that

\[ \tilde{d}(x) > \mu \text{ in } \bar{\Omega}. \]

Let \( \tilde{e} \in C^1(\bar{\Omega}) \) be the unique solution of the Dirichlet problem

\[ \begin{cases} -\Delta \tilde{e} = 1 & \text{in } \tilde{\Omega} \\ \tilde{e} = 0 & \text{on } \partial \tilde{\Omega}, \end{cases} \]

which is known to satisfy the estimate

\[ c^{-1}\tilde{d}(x) \leq \tilde{e}(x) \leq c\tilde{d}(x) \text{ in } \tilde{\Omega}, \]

for certain constant \( c > 1 \) (see [3, proof of Lemma 3.1]).

**Theorem 2.1.** Under assumptions \( H(f), H(a) \) and \( (1.2) \), system \( (P_{\varepsilon}) \) possesses a solution \( (u_{\varepsilon}, v_{\varepsilon}) \in (H^1_0(\Omega) \cap L^\infty(\Omega))^2 \) within \([-C\tilde{e}, C\tilde{e}] \times [-C\tilde{e}, C\tilde{e}]\), with a constant \( C > 1 \) large, for all \( \varepsilon \in (0, 1) \) and all \( \lambda \geq 0 \).

**Proof.** Using (2.3)-(2.5) furnishes

\[ -\Delta(C\tilde{e}) + h_{\lambda, \phi_1}(C\tilde{e}) = C + \lambda(C\tilde{e} + \phi_1) \geq C(1 + \lambda \tilde{e}) \geq C(1 + \lambda \mu) \text{ in } \Omega \]

and

\[ -\Delta(-C\tilde{e}) + h_{\lambda, \phi_1}(-C\tilde{e}) = -C(1 + \lambda \tilde{e}) + \lambda \phi_1 \leq -C(1 + \lambda \mu) + \lambda \|\phi_1\|_\infty \text{ in } \Omega. \]

By \( H(f), H(a), (1.2) \) and (2.3), it hold

\[ a_1(x) \frac{f(x)}{(|C\tilde{e}| + \varepsilon)^{\alpha_1}} \leq M_1 \|a_1\|_\infty \frac{1 + |\varepsilon|^{\beta_1}}{(|C\tilde{e}| + \varepsilon)^{\alpha_1}} \leq M_1 \|a_1\|_\infty \frac{1 + |C\tilde{e}|^{\beta_1}}{|C\tilde{e}|^{\alpha_1}} \]

\[ \leq M_1 \|a_1\|_\infty ((C\mu)^{-\alpha_1} + (\varepsilon \|\tilde{e}\|_\infty)^{\beta_1 - \alpha_1}) \leq C^{\beta_1 - \alpha_1} M_1 \|a_1\|_\infty (1 + \|\tilde{e}\|^{\beta_1 - \alpha_1}) \text{ in } \bar{\Omega}, \]
and
\[ a_1(x) \frac{f_1(v)}{|(\pm \xi \bar{C} + \varepsilon)|^{\rho_1}} \geq - \|a_1\|_\infty \frac{f_1(v)}{|(\pm \xi \bar{C} + \varepsilon)|^{\rho_1}} \geq - M_1 \|a_1\|_\infty \frac{1 + |v|^{\beta_1}}{|(\pm \xi \bar{C} + \varepsilon)|^{\rho_1}} \]
(2.9)
\[ \geq - M_1 \|a_1\|_\infty \frac{1 + |\xi \bar{C}|^{\beta_1}}{|\xi \bar{C}|^{\rho_1}} \geq - M_1 \|a_1\|_\infty \frac{1 + |\bar{C}|^{\beta_1}}{|\bar{C}|^{\rho_1}} \]
\[ \geq - C^{\beta_1 - \alpha_1} M_1 \|a_1\|_\infty \frac{1 + \|\varepsilon\|^{\beta_1 - \alpha_1}}{\|\varepsilon\|^{\rho_1}} \text{ in } \overline{\Omega}, \]
provided that \( C > 0 \) is sufficiently large, for all \((u, v) \in [-\xi \bar{C}, \bar{C}] \times [-\xi \bar{C}, \bar{C}] \) and \( \varepsilon \in (0, 1) \). Then gathering (2.8)-(2.9) together leads to
\[ -\Delta(C\bar{v}) + h_{\lambda, \phi_1}(C\bar{v}) \geq a_1(x) \frac{f_1(v)}{|(\pm \xi \bar{C} + \varepsilon)|^{\rho_1}} \text{ in } \overline{\Omega} \]
and
\[ -\Delta(-C\bar{v}) + h_{\lambda, \phi_1}(-C\bar{v}) \leq a_1(x) \frac{f_1(v)}{|(\pm \xi \bar{C} + \varepsilon)|^{\rho_1}} \text{ in } \overline{\Omega}, \]
for all \((u, v) \in [-\xi \bar{C}, \bar{C}] \times [-\xi \bar{C}, \bar{C}] \) and \( \varepsilon \in (0, 1) \), provided that \( C > 0 \) is sufficiently large. Likewise, a quite similar argument provides
\[ -\Delta(C\bar{v}) + h_{\lambda, \phi_1}(C\bar{v}) \geq a_2(x) \frac{f_2(u)}{|(\pm \xi \bar{C} + \varepsilon)|^{\rho_2}} \text{ in } \overline{\Omega} \]
and
\[ -\Delta(-C\bar{v}) + h_{\lambda, \phi_1}(-C\bar{v}) \leq a_2(x) \frac{f_2(u)}{|(\pm \xi \bar{C} + \varepsilon)|^{\rho_2}} \text{ in } \overline{\Omega}, \]
for all \((u, v) \in [-\xi \bar{C}, \bar{C}] \times [-\xi \bar{C}, \bar{C}] \) and \( \varepsilon \in (0, 1) \), with \( C > 0 \) is sufficiently large.

This proves that \((-C\bar{v}, -C\bar{v}) \) and \((C\bar{v}, C\bar{v}) \) are a sub-supersolution pair for \((P_\varepsilon)\) for all \( \varepsilon \in (0, 1) \). Consequently, we may apply the general theory of sub-supersolutions for systems (see, e.g., [2, Section 5.5]) which ensures the existence of solutions \((u_\varepsilon, v_\varepsilon) \in (H^1(\Omega) \cap L^\infty(\Omega))^2\) of \((P_\varepsilon)\) within \([-\xi \bar{C}, \bar{C}] \times [-\xi \bar{C}, \bar{C}]\), for all \( \varepsilon \in (0, 1) \). The proof is now completed. \( \square \)

2.2. A sign-changing sub-supersolution pair. Assume in \( H(a) \) that
\[ (2.10) \quad \rho_i < \frac{1}{2} \max_{\overline{\Omega}} \phi_1, \]
and fix \( \gamma_i \in (0, 1) \) such that
\[ (2.11) \quad \rho_i := \gamma_i^{\frac{1}{\gamma_i - 1}}, \]
which is possible since \( \rho_i > 1 \).

Setting
\[ (2.12) \quad \overline{\pi} = \phi_1^{\gamma_1} - \gamma_1 \phi_1, \quad \overline{\nu} = \phi_1^{\gamma_2} - \gamma_2 \phi_1, \]
observe that
\[ (2.13) \quad \overline{\pi} \geq 0 \text{ (resp. } \leq 0) \text{ if } 0 \leq \phi_1 \leq \rho_1 \text{ (resp. } \phi_1 \geq \rho_1) \]
and
\[ (2.14) \quad \overline{\nu} \geq 0 \text{ (resp. } \leq 0) \text{ if } 0 \leq \phi_1 \leq \rho_2 \text{ (resp. } \phi_1 \geq \rho_2). \]

Lemma 2.2. Under assumptions \( H(f), H(a) \) and \((1.3)\), the pair \((\overline{\pi}, \overline{\nu})\) is a supersolution for problem \((P_\varepsilon)\), for \( \lambda > 0 \) big enough and for every \( \varepsilon \in (0, 1) \).
We shall prove that 

\[ -\Delta (\phi_i^\gamma) = \gamma_i \lambda_1 \phi_i^\gamma + \gamma_i (1 - \gamma_i) \phi_i^{\gamma - 2} |\nabla \phi_i|^2 \text{ in } \Omega, \text{ for } i = 1, 2. \]

Hence

\[ -\Delta (\phi_i^\gamma - \gamma_i \phi_i) = \gamma_i \lambda_1 \phi_i^\gamma + \gamma_i (1 - \gamma_i) \phi_i^{\gamma - 2} |\nabla \phi_i|^2 - \gamma_i \lambda_1 \phi_i \]

\[ = \lambda_1 \gamma_i (\phi_i^\gamma - \phi_i) + \gamma_i (1 - \gamma_i) \phi_i^{\gamma - 2} |\nabla \phi_i|^2 \text{ in } \Omega, \text{ for } i = 1, 2. \]

We shall prove that \((\bar{\pi}, \bar{\nu})\) is a supersolution for problem \((P_\varepsilon)\). To this end, set

\[ \Omega_\delta := \{ x \in \overline{\Omega} : d(x) < \delta \}, \text{ with a constant } \delta > 0. \]

From (2.12), (2.13) and for \(\delta > 0\) small enough, we have \(\bar{\pi} \geq 0, h_{\lambda, \phi_1}(\bar{\pi}) \geq 0\) as well as \((\phi_i^\gamma - \phi_i) \geq 0\) in \(\Omega_\delta\). Thus, by (1.2), (2.15), we get

\[
\begin{align*}
(\bar{\pi} + \varepsilon)^{\alpha_1} (-\Delta \bar{\pi} + h_{\lambda, \phi_1}(\bar{\pi})) & \geq (|\phi_i^\gamma - \gamma_i \phi_1| + \varepsilon)^{\alpha_1} (-\Delta \bar{\pi}) \\
& \geq \gamma_i (1 - \gamma_1) \left( |\phi_i^\gamma - \gamma_i \phi_1| + \varepsilon \right)^{\alpha_1} (1 - \gamma_i)^{\gamma - 2} |\nabla \phi_i|^2 \\
& \geq \gamma_i (1 - \gamma_1) \left( |\phi_i^\gamma (1 - \gamma_1 \phi_1^{\gamma - 2})| + \varepsilon \right)^{\alpha_1} (1 - \gamma_i)^{\gamma - 2} |\nabla \phi_i|^2 \\
& \geq \gamma_i (1 - \gamma_1) \left( |\phi_i^\gamma (1 - \gamma_1 \phi_1^{\gamma - 2})| + \varepsilon \right)^{\alpha_1} (1 + \alpha_1) |\nabla \phi_i|^2 \text{ in } \Omega_\delta.
\end{align*}
\]

Since \(0 < \gamma_1 < 1\) we have \(\gamma_1 < \frac{2}{1 + \alpha_1}\) and so \(\gamma_1 (1 + \alpha_1) - 2 < 0\) for every \(\alpha_1 \in (0, 1)\). Fix \(C > 0\) such that the conclusion of Theorem (2.7) holds true. By (2.1), (2.2) and \(H(f)\), we infer that

\[
\begin{align*}
(\bar{\pi} + \varepsilon)^{\alpha_1} (-\Delta \bar{\pi} + h_{\lambda, \phi_1}(\bar{\pi})) & \geq (|\phi_i^\gamma - \gamma_i \phi_1| + \varepsilon)^{\alpha_1} (1 - \gamma_i) (f_i(\bar{\pi} + \varepsilon)) \\
& \geq \gamma_i (1 - \gamma_1) (f_i(\bar{\pi} + \varepsilon)) \\
& \geq \gamma_i (1 - \gamma_1) \delta^{1 - \gamma_1} |\nabla \phi_i|^2 \\
& \geq (1 + (C \|\phi_i\|_\infty))^{\alpha_1} \geq |a_1|_\infty \phi_i \geq a_1(x) f_i(v) \text{ in } \Omega_\delta,
\end{align*}
\]

for all \(v \in [-C\bar{e}, C\bar{e}]\), for all \(\varepsilon \in (0, 1)\), provided \(\delta > 0\) is sufficiently small. This shows that

\[ -\Delta \bar{\pi} + h_{\lambda, \phi_1}(\bar{\pi}) \geq a_1(x) \frac{f_i(v)}{(\bar{\pi} + \varepsilon)^{\alpha_1}} \text{ in } \Omega_\delta, \]

for all \(v \in [-C\bar{e}, C\bar{e}]\), for all \(\varepsilon \in (0, 1)\).

Next, we examine the case when \(x \in \Omega_\rho_1 \setminus \overline{\Omega}_\delta\). From (1.1), (2.15) and (2.12), we have

\[ -\Delta \bar{\pi} + h_{\lambda, \phi_1}(\bar{\pi}) \geq \gamma_1 \lambda_1 (\phi_i^\gamma - \phi_1) + \lambda (\phi_i^\gamma + (1 - \gamma_1) \phi_i) \]

\[ = (\gamma_1 \lambda_1 + \lambda) \phi_i^\gamma + \lambda (1 - \gamma_1) - \gamma_1 \lambda_1 \phi_1 \text{ in } \Omega. \]

On account of \(H(f)\), (2.15), (1.2) and recalling that

\[ |\phi_i^\gamma - \gamma_i \phi_1|^{\alpha_1} > 0 \text{ in } \Omega_\rho_1 \setminus \overline{\Omega}_\delta, \]

we get

\[
\begin{align*}
(\bar{\pi} + \varepsilon)^{\alpha_1} (-\Delta \bar{\pi} + h_{\lambda, \phi_1}(\bar{\pi})) & \geq (|\phi_i^\gamma - \gamma_i \phi_1| + \varepsilon)^{\alpha_1} [\gamma_1 (\lambda_1 + \lambda) \phi_i^\gamma + \lambda (1 - \gamma_1) \phi_1] \\
& \geq (\gamma_1 \lambda_1 + \lambda) |\phi_i^\gamma - \gamma_i \phi_1|^{\alpha_1} |\phi_i^\gamma - \gamma_i \phi_1|^{\alpha_1} \geq \lambda |\phi_i^\gamma - \gamma_i \phi_1|^{\alpha_1} \delta^{1 - \gamma_1} \\
& \geq |a_1|_\infty (1 + (C \|\phi_i\|_\infty))^{\alpha_1} \geq a_1(x) f_i(v) \text{ in } \Omega_\rho_1 \setminus \overline{\Omega}_\delta,
\end{align*}
\]
for all \( v \in [-C\bar{e}, C\bar{e}] \), provided \( \lambda > 0 \) big enough. Hence, it turns out that

\[
-\Delta \overline{\nu} + \lambda \overline{\nu} \geq a_1(x) \frac{f_1(v)}{|\overline{\nu}| + \varepsilon} \quad \text{in} \quad \Omega_{\rho_1} \setminus \overline{\Omega}_{\delta},
\]

for all \( v \in [-C\bar{e}, C\bar{e}] \), for all \( \varepsilon \in (0, 1) \).

It remains to prove that the estimate holds true in \( \Omega \setminus \overline{\Omega}_{\rho_1} \). Recall from H(a) that \( a_1(x) \) is negative outside \( \overline{\Omega}_{\rho_1} \). Then, by (2.17), H(f) and for \( \lambda > 0 \) large, it follows that

\[
\begin{align*}
(\overline{\nu} + \varepsilon)^{\alpha_1} (-\Delta \overline{\nu} + h_{\lambda, \phi_1}(\overline{\nu})) \\
geq |\overline{\nu}|^{\alpha_1} [(\gamma_1 \lambda_1 + \lambda) \phi_1^{\gamma_1} + (\lambda(1 - \gamma_1) - \gamma_1 \lambda_1) \phi_1] \\
geq 0 \\
\end{align*}
\]

for all \( v \in [-C\bar{e}, C\bar{e}] \) and all \( \varepsilon \in (0, 1) \). Thus, it turns out that

\[
-\Delta \overline{\nu} + \lambda \overline{\nu} \geq a_1(x) \frac{f_1(v)}{|\overline{\nu}| + \varepsilon} \quad \text{in} \quad \Omega \setminus \overline{\Omega}_{\rho_1},
\]

Gathering together (2.16), (2.18) and (2.19) we deduce that

\[
-\Delta \overline{\nu} + h_{\lambda, \phi_1}(\overline{\nu}) \geq a_1(x) \frac{f_1(v)}{|\overline{\nu}| + \varepsilon} \quad \text{in} \quad \Omega,
\]

for all \( v \) within \([−C\bar{e}, \overline{\nu}]\), for all \( \varepsilon \in (0, 1) \). Similarly, following the same argument as above we obtain

\[
-\Delta \overline{\nu} + h_{\lambda, \phi_1}(\overline{\nu}) \geq a_2(x) \frac{f_2(v)}{|\overline{\nu}| + \varepsilon} \quad \text{in} \quad \Omega,
\]

for all \( u \) within \([−C\bar{e}, \overline{\nu}]\), for all \( \varepsilon \in (0, 1) \). Consequently, on the basis of (2.20) and (2.21) we conclude that \((\overline{\nu}, \overline{\nu})\) is a supersolution of \((P_\varepsilon)\). □

Remark 2.3. A careful inspection of the proof of Lemma 2.2 shows that for a fixed \( C > 0 \) in Theorem 2.1 constants \( \delta := \delta(C) \) and \( \lambda := \lambda(C, \delta) \) can be precisely estimated.

Lemma 2.4. Under assumptions H(f), H(a) and (1.2), problem \((P_\varepsilon)\) possesses a subsolution \((\underline{u}_\varepsilon, \underline{v}_\varepsilon) \in (H^1_0(\Omega) \cap L^\infty(\Omega))^2\), with \( \underline{u}_\varepsilon \leq \overline{u}, \underline{v}_\varepsilon \leq \overline{v}, \) for every \( \varepsilon \in (0, 1) \) and all \( \lambda \geq 0 \). Moreover, there exist functions \( \underline{u}, \underline{v} \in H^1_0(\Omega) \) verifying

\[
\underline{u} < 0 \quad \text{in} \quad \Omega \setminus \overline{\Omega}_{\rho_1}, \quad \underline{u} \geq 0 \quad \text{in} \quad \Omega_{\rho_1}
\]

and

\[
\underline{v} < 0 \quad \text{in} \quad \Omega \setminus \overline{\Omega}_{\rho_2}, \quad \underline{v} \geq 0 \quad \text{in} \quad \Omega_{\rho_2},
\]

such that

\[
\underline{u}_\varepsilon \to \underline{u} \quad \text{and} \quad \underline{v}_\varepsilon \to \underline{v} \quad \text{in} \quad H^1_0(\Omega) \quad \text{as} \quad \varepsilon \to 0.
\]

Proof. For any \( s \in \mathbb{R} \), denote by \( s_+ := \max\{s, 0\} \) and \( s_- := \max\{-s, 0\} \).

Define the truncation

\[
\chi_{\phi_1}(s) = \frac{1}{\|\phi_1\|_\infty} \begin{cases} 
\phi_1 & \text{if } s \geq 2\phi_1 \\
 s - \phi_1 & \text{if } \phi_1 \leq s \leq 2\phi_1 \\
 0 & \text{if } s \leq \phi_1 
\end{cases}
\]
and consider the problem
\[ (P_\varepsilon) \begin{cases} -\Delta u + h_{\lambda, \phi_1}(u) = f_{1, \varepsilon}(x, u, v) & \text{in } \Omega \\ -\Delta v + h_{\lambda, \phi_1}(v) = f_{2, \varepsilon}(x, u, v) & \text{in } \Omega \\ u, v = 0 & \text{on } \partial \Omega, \end{cases} \]
for \( \varepsilon \in (0, 1) \), where
\[ f_{1, \varepsilon}(x, u, v) = \begin{cases} a_{1, +}(x) \chi_{\phi_1}(u_+) \frac{f_1(v)}{|u_+|^{\mu_1}} & \text{in } \Omega_{\rho_1} \\ -a_{1, -}(x) \frac{1+|\nabla_2|}{(|u_+|+\varepsilon)^{\mu_1}} & \text{in } \Omega \setminus \Omega_{\rho_1} \end{cases} \]
and
\[ f_{2, \varepsilon}(x, u, v) = \begin{cases} a_{2, +}(x) \chi_{\phi_1}(v_+) \frac{f_2(u)}{|v_+|^{\beta_1}} & \text{in } \Omega_{\rho_2} \\ -a_{2, -}(x) \frac{1+|\nabla_2|}{(|v_+|+\varepsilon)^{\beta_1}} & \text{in } \Omega \setminus \Omega_{\rho_2}. \end{cases} \]
It is a simple matter to see that
\[ f_{1, \varepsilon}(x, u, v) \leq a_1(x) \frac{f_1(v)}{|u_+|^{\mu_1}} \quad \text{and} \quad f_{2, \varepsilon}(x, u, v) \leq a_2(x) \frac{f_2(u)}{|v_+|^{\beta_1}}, \]
for all \((u, v) \in [-C\tilde{e}, C\tilde{e}] \times [-C\tilde{e}, C\tilde{e}]\) and all \(\varepsilon \in (0, 1)\). Then, any solution of \((\tilde{P}_\varepsilon)\) within \([-C\tilde{e}, C\tilde{e}] \times [-C\tilde{e}, C\tilde{e}]\) is a subsolution of \((P_\varepsilon)\).

We claim that \((-C\tilde{e}, -C\tilde{e})\) is a subsolution of \((P_\varepsilon)\). Indeed, from (2.27), (2.23), (2.5), \(H(a)\) and \(H(f)\), we have
\[ -\Delta(-C\tilde{e}) + h_{\lambda, \phi_1}(-C\tilde{e}) \leq 0 \leq \tilde{f}_\varepsilon(x, -C\tilde{e}, v) \quad \text{in } \Omega_{\rho_1} \]
and
\[ -\Delta(-C\tilde{e}) + h_{\lambda, \phi_1}(-C\tilde{e}) \leq -C(1 + \lambda \frac{\mu_1}{\alpha_1}) + \lambda \|\phi_1\|_{\infty} \leq -C^{-\alpha_1} \|a_{1, -}\|_{\infty} \|\nabla_2\|_{\infty} \leq -a_{1, -}(x) \frac{1+|\nabla_2|}{(|-C\tilde{e}|+\varepsilon)^{\alpha_1}} = \tilde{f}_{1, \varepsilon}(x, -C\tilde{e}, v) \quad \text{in } \Omega \setminus \Omega_{\rho_1}, \]
provided that \(C > 0\) is sufficiently large, for all \(v \in [-C\tilde{e}, \tilde{\tau}]\), and all \(\varepsilon \in (0, 1)\). Similarly, one derives that
\[ -\Delta(-C\tilde{e}) + h_{\lambda, \phi_1}(-C\tilde{e}) \leq \tilde{f}_{2, \varepsilon}(x, u, -C\tilde{e}) \quad \text{in } \tilde{\Omega}, \]
provided that \(C > 0\) is sufficiently large, for all \(u \in [-C\tilde{e}, \tilde{\tau}]\) and all \(\varepsilon \in (0, 1)\). This proves the claim.

On the basis of (2.28) and Lemma (2.2) \((\overline{\tau}, \overline{\tau})\) in (2.12) is a supersolution of problem \((\tilde{P}_\varepsilon)\). Consequently, owing to [2 section 5.5], problem \((\tilde{P}_\varepsilon)\) admits a solution \((u_{\varepsilon}, v_{\varepsilon})\) within \([-C\tilde{e}, \overline{\tau}] \times [-C\tilde{e}, \overline{\tau}]\) and \(u_{\varepsilon}, v_{\varepsilon}\) are both nontrivial because \(\overline{\tau}\) and \(\overline{\tau}\) are both of sign-changing in \(\Omega\). Moreover, according to (2.28), \((u_{\varepsilon}, v_{\varepsilon})\) is a subsolution of \((P_\varepsilon)\) for all \(\varepsilon \in (0, 1)\).

Now we prove (2.24). Set \(\varepsilon = \frac{1}{n}\) with any positive integer \(n > 1\). From above there exist \(u_n := u_{\varepsilon_n}\) and \(v_n := v_{\varepsilon_n}\) such that
\[ \begin{cases} \langle -\Delta u_n + h_{\lambda, \phi_1}(u_n), \varphi \rangle = \int_{\Omega} F_{1, n}(x, u_n, v_n) \varphi \, dx \\ \langle -\Delta v_n + h_{\lambda, \phi_1}(v_n), \psi \rangle = \int_{\Omega} F_{2, n}(x, u_n, v_n) \psi \, dx \end{cases} \]
for all \( \varphi, \psi \in H^1_0(\Omega) \) with

\[
- C \bar{e} \leq \frac{\varphi}{\|\varphi\|} \leq \bar{\varphi} \leq C \bar{e}, \quad - C \bar{e} \leq \frac{\psi}{\|\psi\|} \leq \bar{\psi} \leq C \bar{e} \quad \text{in} \ \Omega.
\]

Acting with \( \varphi = u_n \) in \((\ref{eq:2.29})\), by \((\ref{eq:2.26})\), \(H(a)\) and \((\ref{eq:1.2})\), bearing in mind \((\ref{eq:2.30})\), it follows that

\[
\int_{\Omega} (|\nabla u_n|^2 + h \chi_{\phi_1}(u_n)) \, dx = \int_{\Omega} (|\nabla u_n|^2 + \lambda(|u_n|^2 + \phi_1 u_n)) \, dx
\]

\[
= \int_{\Omega_{\nu_1}} a_{1,+}(x) \chi_{\phi_1}(u_n,+) \frac{f_1(u_n)}{(|u_n|^{\alpha_1} + 1)^{\frac{1}{\alpha_1}}} u_n \, dx - \int_{\Omega_{\nu_1}} a_{1,-}(x) \frac{1}{(|u_n|^{\alpha_1} + 1)^{\frac{1}{\alpha_1}}} u_n \, dx
\]

\[
\leq \int_{\Omega_{\nu_1}} \|u\|_{\infty} (1 + (C \bar{e})^{\beta_1}) u_n \, dx - \int_{\Omega_{\nu_1}} a_{1,-}(x) \frac{1}{(|u_n|^{\alpha_1} + 1)^{\frac{1}{\alpha_1}}} u_n \, dx
\]

\[
\leq \Omega \|u\|_{\infty} (1 + C \|\tilde{\varphi}\|_{\beta_1}) C \|\tilde{\varphi}\|_{\infty} < \infty.
\]

This proves that \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \). Similarly, we derive that \( \{v_n\} \) is bounded in \( H^1_0(\Omega) \). We are thus allowed to extract subsequences (still denoted by \( \{u_n\} \) and \( \{v_n\} \)) such that

\[ u_n \rightharpoonup u \quad \text{and} \quad v_n \rightharpoonup v \quad \text{in} \quad H^1_0(\Omega) \]

and

\[ -C \bar{e} \leq \varphi \leq \bar{\varphi} \leq C \bar{e}, \quad -C \bar{e} \leq \psi \leq \bar{\psi} \leq C \bar{e} \quad \text{in} \ \Omega. \]

Inserting \((\varphi, \psi) = (u_n, v_n - v)\) in \((\ref{eq:2.29})\) yields

\[
\left\{ \begin{array}{l}
\langle -\Delta u_n + h \chi_{\phi_1}(u_n), u_n - u \rangle = \int_{\Omega} F_{1,n}(x, u_n, v_n) (u_n - u) \, dx \\
\langle -\Delta v_n + h \chi_{\phi_1}(v_n), v_n - v \rangle = \int_{\Omega} F_{2,n}(x, u_n, v_n) (v_n - v) \, dx.
\end{array} \right.
\]

By \((\ref{eq:3.7})\), \((\ref{eq:2.30})\), \(H(a)\) and \((\ref{eq:1.2})\), we have

\[
|a_{1,+}(x) \chi_{\phi_1}(u_n,+) \frac{f_1(u_n)}{(|u_n|^{\alpha_1} + 1)^{\frac{1}{\alpha_1}}} (u_n - u)| \leq 2 \|a_1\|_{\infty} (1 + C \|\tilde{\varphi}\|_{\beta_1}) C \|\tilde{\varphi}\|_{\infty}
\]

as well as

\[
|a_{1,-}(x) \frac{1}{(|u_n|^{\alpha_1} + 1)^{\frac{1}{\alpha_1}}} (u_n - u)| \leq \|a_1\|_{\infty} \|u_n\|_{\infty}^{1-\alpha_1} (1 + |\tilde{\varphi}|^{\beta_1})
\]

\[
\leq \|a_1\|_{\infty} (C \|\tilde{\varphi}\|_{\infty})^{1+\alpha_1} (1 + (C \|\tilde{\varphi}\|_{\infty})^{\beta_1}).
\]

Then, applying Fatou’s Lemma, it follows that

\[
\limsup_{n \to \infty} \int_{\Omega} F_{1,n}(x, u_n, v_n) (u_n - u) \, dx
\]

\[
\leq \int_{\Omega_{\nu_1}} \limsup_{n \to \infty} (F_{1,n}(x, u_n, v_n) (u_n - u)) \, dx \to 0 \quad \text{as} \quad n \to +\infty,
\]

showing that \( \lim_{n \to \infty} -\Delta u_n, u_n - u \leq 0 \). Then the \( S^+ \)-property of \(-\Delta\) on \( H^1_0(\Omega) \) guarantees that \( u_n \rightharpoonup u \) in \( H^1_0(\Omega) \). Similarly, we prove that \( v_n \rightharpoonup v \) in \( H^1_0(\Omega) \). Hence, we may pass to the limit in \((\ref{eq:2.29})\) to conclude that \( (u, v) \) is a solution of problem \((P_0)\) (That is \((\tilde{P}_\varepsilon)\) with \(\varepsilon = 0\)).

It remains to prove \((\ref{eq:2.22})\) and \((\ref{eq:2.23})\). Since \( u \leq \bar{\varphi} \) and \( \bar{\varphi} \) is negative in \( \Omega \setminus \Omega_{\nu_1} \), it turns out that \( u \) is negative in \( \Omega \setminus \Omega_{\nu_1} \). Similarly, one derives that \( v \) is negative in \( \Omega \setminus \Omega_{\nu_2} \). Acting in \((\tilde{P}_0)\) with \((1_{\Omega_{\nu_1}} u_-, 1_{\Omega_{\nu_1}} v_-)\), from the definition of the truncation \( \chi_{\phi_1} \) in \((\ref{eq:2.25})\), it follows that

\[
\int_{\Omega_{\nu_1}} (|\nabla u_-|^2 + \lambda(|u_-|^2 + \phi_1 u_-)) \, dx
\]

\[
= \int_{\Omega_{\nu_1}} a_{1,+}(x) \chi_{\phi_1}(u_+ \frac{f_1(u)}{(|u|^{\alpha_1} + 1)^{\frac{1}{\alpha_1}}} u_- \, dx = 0
\]
and
\[ \int_{\Omega_{\rho_1}} \left( |\nabla v_+|^2 + \lambda (|v_+|^2 + \phi_1 v_+) \right) dx = \int_{\Omega_{\rho_2}} \Lambda_2(x) \chi_{\phi_1} (v) |u|^\alpha (v_+ + 1)^\beta v_- dx = 0. \]

Consequently, one can conclude that \( u \geq 0 \) in \( \Omega_{\rho_1} \) and \( v \geq 0 \) in \( \Omega_{\rho_2} \). This completes the proof. \( \square \)

3. Existence of solutions

Our main result is formulated as follows.

**Theorem 3.1.** Under assumptions \( H(f), H(a) \) and \( (1.2) \), problem \( (P) \) possesses a nodal solution \( (\tilde{u}, \tilde{v}) \in (H^1_0(\Omega) \cap L^\infty(\Omega))^2 \) for \( \lambda > 0 \) large.

**Proof.** Set \( \varepsilon = \frac{1}{n} \) with any positive integer \( n > 1 \). According to Lemmas 2.2 and 2.4 and thanks to [2, Section 5.5], there exists \( (u_n, v_n) := (u_{1n}, v_{1n}) \in (H^1_0(\Omega) \cap L^\infty(\Omega))^2 \) such that

\[
\begin{cases}
(\Delta u_n + h_{\lambda, \phi_1} (u_n), \varphi) = \int_{\Omega} a_1(x) \frac{f_1(v_n)}{|u_n| + \frac{1}{n}} \varphi \ dx \\
(\Delta v_n + h_{\lambda, \phi_1} (v_n), \psi) = \int_{\Omega} a_2(x) \frac{f_2(u_n)}{|v_n| + \frac{1}{n}} \psi \ dx
\end{cases}
\]  

for all \( \varphi, \psi \in H^1_0(\Omega) \) and

\[
(3.2) \quad \underbar{u}_n \leq u_n \leq \overline{u}, \quad \underbar{v}_n \leq v_n \leq \overline{v}, \quad \forall n.
\]

In addition, due to (2.22) and (2.23) one has

\[
(3.3) \quad u_n < 0 \text{ in } \Omega \setminus \overline{\Omega}_{\rho_1}, \quad u_n \geq 0 \text{ in } \Omega_{\rho_1}
\]

and

\[
(3.4) \quad v_n < 0 \text{ in } \Omega \setminus \overline{\Omega}_{\rho_2}, \quad v_n \geq 0 \text{ in } \Omega_{\rho_2}, \quad \forall n.
\]

We claim that the sets where the sequences \( \{u_n\} \) and \( \{v_n\} \) vanish are negligible. To this end, define the measurable set

\[
\Gamma_n := \{ x \in \Omega_{\rho_1} : u_n(x) = 0 \}, \quad \forall n.
\]

By a classical result of measure theory (see [14, Theorem 3.28]), \( \Gamma_n \) is a \( G_\gamma \)-set for less than a zero measure set. Thus, one can write

\[
(3.5) \quad \Gamma_n = A_n - B_n,
\]

where \( A_n - B_n \) is the relative complement of \( A_n \) in \( B_n \), \( |B_n| = 0 \) and \( A_n \) is a \( G_\gamma \)-set, that is

\[
A_n = \bigcap_k G_{n,k}, \quad G_{n,k} \text{ is open.}
\]

Without loss of generality, we may assume that \( B_n \) is not dense in \( \Omega_{\rho_1} \) because otherwise, by density and according to [1, Proposition 2.4], the term \( a_1(x) \frac{f_1(v_n)}{|u_n| + \frac{1}{n}} \) should vanish in \( \Omega_{\rho_1} \) which is absurd in view of \( H(f) \) and \( H(a) \).

Let \( \tilde{\varphi} \in C_0^\infty(\Omega_n) \) such that

\[ \tilde{\varphi} > 0 \text{ in } A_n \text{ and } \tilde{\varphi} = 0 \text{ in } \Omega_{\rho_1} \setminus A_n. \]
Testing with $\tilde{\varphi}$ in (3.1) we have

$$\int_{A_n} (\nabla u_n \nabla \tilde{\varphi} + h_{\lambda, \phi_1}(u_n)\tilde{\varphi}) \, dx = \int_{A_n} a_1(x) \frac{f_1(v_n)}{|u_n| + \frac{1}{n})^{\alpha_1}} \tilde{\varphi} \, dx$$

In view of (3.5) we get

$$(3.6) \quad \int_{\Gamma_n} (\nabla (u_n \nabla \tilde{\varphi} + h_{\lambda, \phi_1}(u_n)\tilde{\varphi}) \, dx = \int_{\Gamma_n} a_1(x) \frac{f_1(v_n)}{|u_n| + \frac{1}{n})^{\alpha_1}} \tilde{\varphi} \, dx.$$  

Since $u_n \in W^{1,1}_{\text{loc}}(\Omega)$ it follows, by [6, Lemma 7.7], that $\nabla u_n = 0$ on $\Gamma_n$. Replacing in (3.6) leads

$$\int_{\Gamma_n} \lambda \phi_1 \tilde{\varphi} \, dx = n^{\alpha_1} \int_{\Gamma_n} a_1(x) f_1(v_n) \tilde{\varphi} \, dx, \forall n \geq 1,$$

which, by $H(f)$, $H(a)$, forces that $|\Gamma_n| = 0$. The same conclusion can be drawn for the set $\tilde{\Gamma_n} := \{ x \in \Omega_{\rho_2} : v_n(x) = 0 \}$, for all $n \geq 1$. This shows that $u_n$ and $v_n$ cannot be identically zero in $\Omega_{\rho_1}$ and $\Omega_{\rho_2}$, respectively.

Now, taking $\varphi = u_n$ in (3.1), by $H(f)$, $H(a)$ and since

$$(3.7) \quad |u_n|, |v_n| \leq C\varepsilon \quad \text{in} \quad \Omega,$$

we get

$$\int_{\Omega} (|\nabla u_n|^2 + h_{\lambda, \phi_1}(u_n)u_n) \, dx \leq \int_{\Omega} \|a_1\|_{\infty} \frac{1}{(|u_n| + \frac{1}{n})^{\alpha_1}} u_n \, dx$$

$$\leq \|a_1\|_{\infty} \int_{\Omega} |u_n|^{1-\alpha_1} (1 + |v_n|^{\beta_1}) \, dx$$

$$\leq \|a_1\|_{\infty} \int_{\Omega} (C\varepsilon)^{1-\alpha_1} (1 + (C\varepsilon)^{\beta_1}) \, dx < \infty,$$

showing that $\{u_n\}$ is bounded in $H^1_0(\Omega)$. Similarly, we prove that $\{v_n\}$ is bounded in $H^1_0(\Omega)$. We are thus allowed to extract subsequences (still denoted by $\{u_n\}$ and $\{v_n\}$) such that

$$(3.9) \quad u_n \rightharpoonup \bar{u} \quad \text{and} \quad v_n \rightharpoonup \bar{v} \quad \text{in} \quad H^1_0(\Omega)$$

Moreover, by (2.24) and (3.2) one has

$$(3.10) \quad \underline{\varphi} \leq \bar{u} \leq \bar{\varphi}, \quad \underline{\varphi} \leq \bar{v} \leq \bar{\varphi} \quad \text{in} \quad \Omega.$$  

Then, analysis similar to that of the proof of (2.24) in Lemma 2.4 leads to

$$u_n \rightharpoonup \bar{u} \quad \text{and} \quad v_n \rightharpoonup \bar{v} \quad \text{in} \quad H^1_0(\Omega).$$  

Setting

$$\Gamma := \{ x \in \Omega_{\rho_1} : \bar{u}(x) = 0 \} \quad \text{and} \quad \tilde{\Gamma} := \{ x \in \Omega_{\rho_2} : \bar{v}(x) = 0 \},$$

by (3.7) and the dominated convergence theorem, it follows that

$$(3.11) \quad |\Gamma| = \lim_{n \to +\infty} |\Gamma_n| = 0 \quad \text{and} \quad |\tilde{\Gamma}| = \lim_{n \to +\infty} |\tilde{\Gamma}_n| = 0,$$

showing that $\bar{u}$ and $\bar{v}$ cannot be identically zero in $\Omega_{\rho_1}$ and $\Omega_{\rho_2}$, respectively.

Finally, we may pass to the limit in (3.1) to conclude that $(\bar{u}, \bar{v})$ is a solution of problem (P) within $[\underline{u}, \bar{\varphi}] \times [\underline{v}, \bar{\varphi}]$. On account of (2.10)-(2.14)
and (3.11) together with Lemma 2.4, we infer that $\bar{u}, \bar{v}$ are both of sign-changing and satisfying

$$\bar{u} \leq \nabla < 0 \text{ in } \Omega \setminus \Omega_{\rho_1} \text{ and } \bar{u} \geq 0 \geq 0, \bar{u} \neq 0 \text{ in } \Omega_{\rho_1},$$

$$\bar{v} \leq \nabla < 0 \text{ in } \Omega \setminus \Omega_{\rho_2} \text{ and } \bar{v} \geq 0 \geq 0, \bar{v} \neq 0 \text{ in } \Omega_{\rho_2}.$$ 

This completes the proof. $\square$

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