Wavelike urban formations

I.N. Inovenkov¹, E.Yu. Echkina, V.V. Nefedov and L.S. Ponomarenko
Lomonosov Moscow State University, Department of Computational Math & Cybernatics, Leninskie gory, bld. 1/58, Moscow, 119234, Russian Federation
E-mail: inov@cs.msu.ru

Abstract. Within the framework of the two-dimensional spatial economy, various forms of urban structures are considered. Since the number of structurally stable urban configurations is rather limited, structurally unstable formations are of great interest. For example, there are city models that behave near unstable singular points like a «running» wave.
In this paper we consider a system of quasi-linear parabolic equations for two unknown functions: population density and, so-called, quality of housing stock. After some transformations of the required functions and variables we seek a self-similar solution in the form of a running wave with an unknown amplitude and velocity.
Further the stability of the obtained solution is analyzed. To determine the role of the obtained particular (self-similar) solution, the Cauchy problem for different initial conditions is also solved numerically.
The results of computational experiments show us that the analytical wave-like solution can be interpreted is an intermediate asymptotic for the partial differential problem which is being considered this paper.

Keywords: regional and spatial economy, heterogeneous urban system, population density, housing quality, spatial dynamic approximation, partial derivative equation of parabolic type

1. Introduction
All sorts of economical activity is brought into correlation with the certain place and time, thus while considering evolutional systems we must take into account the spatial dependence. Due to the progress in transportation and communication facilities the interaction between different economical agents is getting more and more dependent on their location. It is very important to realize and to describe characteristics of these spatial interactions. As a consequence of technological progress and changes in people’s behavior urban problems have become more complicated.
The problem of the formation of urban structures due to the impact of various economic, geographical, demographic, social and political factors is of considerable theoretical and practical interest. The number of possible urban structures in structurally stable systems is rather limited. Structural instability increases the diversity of urban structures. In detail with examples from various fields of natural science, the concept of structural stability and instability is considered in the well-known monography [1]. Of great interest is the extension of these concepts to the problems of mathematical urbanistics.

¹ To whom any correspondence should be addressed.
2. Mathematical model

In the present paper a mathematical model of a city is discussed, in which solutions can arise which close to unstable singular points behave like a “running” wave. A similar axial symmetric model was proposed in [2].

It is assumed that the model is described by two main parameters (functions):
- population density in the point \( M(x, y) \) at time moment \( t \),
- housing quality (cost of realty) in the point \( M(x, y) \) at time moment \( t \)

and also \( r = \sqrt{x^2 + y^2} \) is a the destination between the “down-town” and the place of residence.

In this model it is assumed that during the time period under study, the city population does not change (or changes but not significantly). In addition, we will neglect demographic processes and migration processes between the city and the “outside world”. Then the urban system can be described by the following dynamic differential equations:

\[
\frac{\partial u}{\partial t} = \alpha (f(q) - u) + \frac{1}{r} \frac{\partial}{\partial r} \left[ rD \frac{\partial u}{\partial r} \right],
\]

\[
\frac{\partial q}{\partial t} = -\delta q + H(I(u, q)),
\]

where \( u = u(r, t) \) and \( q = q(r, t) \) with \( 0 < r < R \).

For the \( u = u(r, t) \) function we have a boundary condition of the third kind at the boundary of \( \Omega \) area \( (r = R) \) which is of the form:

\[
h_1 \frac{\partial u}{\partial r} + h_2 u |_{r=R} = \nu(t).
\]

If \( h_2 = 0 \) and \( \nu(t) = 0 \), then boundary condition \( \frac{\partial u}{\partial r} |_{r=R} = 0 \) means that during concerned period of time there is no changes in population.

If \( t = 0 \) we have additional initial conditions for functions \( u \) and \( q \):

\[
\begin{align*}
u_{t=0} &= u_0(r), \\
q_{t=0} &= q_0(r).
\end{align*}
\]

In this model the region \( \Omega = \{ r \leq R \} \) denotes the area of urban space (in fact, it is a circle), \( \alpha \) — adaptation parameter, \( D \) — population’s diffusion coefficient, \( \delta \) — housing fracture velocity. Note that coefficient \( D \) can depend on \( u \) and \( q \), and also on \( x \), \( y \) and time \( t \).

We do not consider a diffusion modification of the housing quality, so the equation for \( q \) is an ordinary differential one. Note that one-dimensional variant of this model was suggested by Zhang [2].

The form of the \( f(q) \) function could be found from the assumption of rationality of household behavior. This function determines the equilibrium value of population density for a given quality of housing stock.

Eventually the second equation (2) describes modification in time of the housing quality. Term \((−\delta q)\), which is a part of this equation, describes housing destruction effects and the function \( H(.) \) characterizes the costs of maintaining the quality of housing.

We consider that the owners who define the quantity of expenses maintain housing conditions. So the housing price depends on the rent revenue of the owners which determine the amount of financial costs for the maintenance of their homes, the cost of which in turn depends on the owner’s income from the unit of housing.
Let the total income be defined as $I$. The income in the fixed point depends on the density of population and housing quality, that is $I = I(u, q)$. The partial derivative $\frac{\partial I}{\partial u}$ of this functional has no definite sign but the derivative $\frac{\partial I}{\partial q}$ is above zero. Under the constant level of $q$ the sign of $\frac{\partial I}{\partial u}$ generally is not definite because increase or decrease of the income depends on the current situation. Partial derivative $\frac{\partial I}{\partial q}$ is above zero because housing quality improvement under fixed level of population density $u$ has to lead to the increasing of owner’s income. Further we assume that the costs of maintaining the housing stock are positively related to income which means $\frac{dH}{dI} > 0$.

Also we define function for the maintenance of the housing stock expenses as follows:

$$H(I) = \frac{\mu uq^2}{1 + \sigma u},$$

(5)

where $\mu$ and $\sigma$ are positive coefficients. If we interpret value $\frac{q^2}{1 + \sigma u}$ as a housing rent, received from a unit of housing stock, then $\frac{uq^2}{1 + \sigma u}$ will be total income of the housing owners into the current point of city area. Parameter $\mu$ can be considered as a ratio of maintenance expenses of the housing stock and total income.

3. **Dimensionless differential nonlinear model**

Let us make all the coefficients of (1)-(2) dimensionless corresponding the formulas below:

$$\alpha t = \tau, q = \frac{\mu Q}{\alpha \sigma}, u = \frac{N}{\sigma}, \gamma = \frac{D}{\alpha}, v = \frac{\delta}{\alpha}, g(Q) = \sigma f\left(\frac{\mu Q}{\alpha \sigma}\right).$$

(6)

As a result of such changes, equations (1)-(2) could be written in terms of $N$ and $Q$ functions in the following form:

$$\begin{cases}
\frac{\partial N}{\partial \tau} = g(Q) - N + \frac{1}{r} \frac{\partial}{\partial r} \left( \gamma r \frac{\partial N}{\partial r} \right) \\
\frac{\partial Q}{\partial \tau} = -vQ + \frac{NQ^2}{1 + N}
\end{cases}$$

(7)

with taking into account the corresponding boundary and initial value conditions. Suppose that there is asymptotic equilibrium of system (7). We will be interested in the existence of periodical solutions such as “running” waves near asymptotic static equilibrium. Non-stationary solutions such as “running” waves, usual for partial differential equations, are often found in some physical, chemical and biological problems, in particular, in ecology. The solution of system (7) of “running” wave type could be presented in the following form:

$$N(r, \tau) = N(r - \varepsilon \tau), Q(r, \tau) = Q(r - \varepsilon \tau),$$

(8)

where $\varepsilon > 0$ is “velocity” of perturbation diffusion. A periodic urban structure such as a “running” wave is defined as a solution that is periodic with respect to $(r - \varepsilon \tau)$.

Further introducing the function $W(r - \varepsilon \tau) = N'(r - \varepsilon \tau)$ and substituting the expressions (8) into equations (7), we have:
Further, our purpose is to prove the possibility of the existence of a limit cycle in system (9), which can be obtained by using the Poincaré-Andronov-Hopf bifurcation theory [3].

It can be shown that the system of equations (9) admits the existence of solutions that can be interpreted as periodic urban structures of the type of a “running” wave of the following form.

\[
\begin{align*}
N' &= W, \\
\gamma W' &= N - \varepsilon W - \frac{\gamma}{r} W - g(Q), \\
\varepsilon Q' &= vQ \frac{-NQ^2}{1 + N}.
\end{align*}
\]

where the value \( \omega \) is determined by the parameters of the problem and the value \( \varepsilon \) is sufficiently small.

4. Computational simulation

The question concerning unique existence of the system (7) solution is not very simple but for the wide class of nonlinear coefficients unique existence theorems of the classical problem’s solution are still valid. It seems very interesting to study a solution’s behavior of the system (7) against initial conditions, especially the existence of the intermediate asymptotic forms. They exist when the solution quickly “forgets” the parts of the initial conditions and develops according to the internal structure of the model.

For the computational solution of the system (7) we used the second order approximation difference scheme upon the \( r \) coordinate and with the first order approximation upon the time \( t \) [4].

Let us describe computational experiment with \( g(Q) = Q \) and \( \gamma(U) = U \). Homogeneous initial distributions of \( U \) and \( Q \) functions were taken across the \( r \) coordinate that is \( U(r,t)|_{t=0} = 1 \) and \( Q(r,t)|_{t=0} = 0 \). Bifurcation parameter \( \nu = 0,5 \).

Computational modelling shows that starting with \( t = 1,5 \) solution comes to the intermediate asymptotic form. That means normalized profile of population density \( U \) is not changing as time goes by. At the same time function is monotonously tending to zero. If bifurcation parameter \( \nu \) is equal to 0,1 with the same initial and boundary conditions, then solution becomes unstable. It was received the so called “sawtooth” regime, which proves that solution becomes unstable. Thus, it could be concluded that there is no solution of this problem in the late times.

Let take the following initial value conditions:

\[
\begin{align*}
U(r,t)|_{t=0} = 1 + 0,9 \sin(10\pi r) \\
Q(r,t)|_{t=0} &= 1
\end{align*}
\]

with bifurcation parameter \( \nu \) is equal again to 0,5.

As time goes on, the solution of system (7) “quickly” forgets details of the initial value conditions (11). Here there is a clear transformation of sinusoid initial profile into the curve which form similar to another one from the case, where \( U(r,t)|_{t=0} = 1 \). During the later point of time solution again comes to the intermediate asymptotic form, which can be proved by the diagram of the normalized density.
5. Conclusion
We have made a computational simulation and theoretical research of the urban population density dynamics with the consideration of the housing quality modification. For the wide class of coefficients we have shown that problem’s solution quickly “forgets” details of initial conditions and comes to the intermediate asymptotic form, which characterizes by operator of the system only. In fact this means that urban system does not depend on the external circumstances and defines by the internal structure of the model. The investigation of all possible variations of mathematical model seems to be separate and rather difficult problem. The solution of this problem will be the next step to the understanding such problems as distribution of the population inside urban area, urban structure evaluation and other problems of modern urbanistics and also demographical processes.

6. References
[1] Tim Poston and Ian Stewart, Catastrophe Theory and Its Applications. Published by Courier Corporation. 1996. 491 P.
[2] Zhang W.-B. Synergetic economics: Time and change in nonlinear economics (Springer series in Synergetics) // Berlin, Germany: Springer-Verlag. 1991. 246 P.
[3] V.I. Arnold, V.S. Afraimovich, Yu.S. Il’yashenko, L.P. Shil’nikov. Theory of bifurcation // Dynamical systems — 5. Results of Science and Technology. Modern problems of mathematics. Fundamental research. — Moscow. Published by VINITI. 1986. — Vol.5. — P.P. 5—218. (in Russian).
[4] E.Yu. Echkina, O.I. Inovenkov and D.P. Kostomarov, A selfsimilar behavior of the urban structure in the spatially inhomogeneous model // The European Physical Journal (B), Vol.50, 2006, P.P.215-220.