Exceptional orthogonal polynomials and exactly solvable potentials in position-dependent-mass Schroedinger Hamiltonians

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October 7, 2009

Abstract

Some exactly solvable potentials in the position dependent mass background are generated whose bound states are given in terms of Laguerre- or Jacobi-type $X_1$ exceptional orthogonal polynomials. These potentials are shown to be shape invariant and isospectral to the potentials whose bound state solutions involve classical Laguerre or Jacobi polynomials.

PACS: 03.65.Fd, 03.65.Ge

Keywords: Position dependent mass Schrödinger equation, Exceptional orthogonal polynomial, Point canonical transformation, Supersymmetry, Shape invariance.

1 Introduction

In recent years, quantum mechanical systems with a position dependent mass have attracted a lot of interest due to their relevance in describing the physics of many microstructures of current interest, such as compositionally graded crystals [1], quantum dots [2], $^3$He clusters [3], metal clusters [4] etc. The concept of position dependent mass comes from the effective mass approximation [5]-[8] which is an useful tool for studying the motion of carrier electrons in pure crystals and also for the virtual-crystal approximation in the treatment of homogeneous alloys (where the actual potential is approximated by a periodic potential) as well as in graded mixed semiconductors (where the potential is not periodic). The attention to the effective mass approach stems from the extraordinary development in crystallographic growth techniques which allow the production of non uniform

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semiconductor specimen with abrupt heterojunctions. In these mesoscopic materials, the effective mass of the charge carriers are position dependent. Consequently the study of the effective mass Schrödinger equation becomes relevant for deeper understanding of the non-trivial quantum effects observed on these nanostructures. The position dependent (effective) mass is also used in the construction of pseudo-potentials which have a significant computational advantage in quantum Monte Carlo method\[9, 10\]. This large variety of applications explains the growing interest in constructing solvable cases of effective mass Schrödinger equation. Recently, several of such cases were obtained by means of point canonical transformation (PCT), Lie algebraic techniques and supersymmetric quantum mechanical methods \[11]-\[32\].

In this letter, our objective is to construct new exactly solvable potentials in a position dependent effective mass background whose bound state wavefunctions can be written in terms of recently introduced \[33, 34\] Laguerre or Jacobi type $X_1$ exceptional orthogonal polynomials. (It must be mentioned here that similar study has been done by Quesne \[35\] in the constant mass scenario) Subsequently, we shall show that these exactly solvable potentials are shape invariant by using supersymmetric quantum mechanics (SUSYQM) technique in position dependent mass case. SUSYQM allows us to obtain isospectral partner potentials of these exactly solvable potentials. The motivation for doing this, apart from enlarging the class of exactly solvable potentials for position dependent mass Schrödinger equation, comes from the fact that in different areas of possible applications of low dimensional structures e.g. quantum well, quantum dot, there is need to have energy spectrum which is predetermined. Specifically, in the quantum well profile optimization isospectral potentials are generated through supersymmetric quantum mechanics. These are necessary because a particular effect such as intersubband optical transitions in a quantum well, may be grossly enhanced by achieving the resonance conditions i.e. appropriate spacings between the most relevant states and also by tailoring the wave functions so that the (combinations of) matrix elements relevant for this particular effect are maximized \[36\]. This is particularly important for higher order nonlinear processes. Optimization of simple stepwise constant profiled quantum wells has been considered quite some time ago \[37\], while the optimization of continuously graded structure requires more sophisticated techniques like supersymmetric quantum mechanics and inverse spectral theory \[38\].

The organization of this article is as follows. In section 2, we have generated two new exactly solvable potentials for one dimensional position dependent mass Schrodinger equation whose wavefunctions involve Laguerre or Jacobi-type $X_1$ extended polynomials via point canonical transformation approach (PCT) \[39, 40\]. In section 3, we have shown that the new potentials are shape invariant in the context of supersymmetric quantum mechanics in the position dependent mass background. Section 4. is kept for summary and discussions.
2 Generation of new potentials via PCT

The general position dependent mass (time independent) Schrödinger equation, initially proposed by von Roos [41] in terms of three ambiguity parameters \( r, s, t \) such that \( r + s + t = -1 \) is given by

\[
H_{\text{eff}} \psi(x) = E \psi(x), \quad H_{\text{eff}} = -\frac{d}{dx} \frac{1}{M(x)} \frac{d}{dx} + V_{\text{eff}}
\]

where the effective potential

\[
V_{\text{eff}} = V(x) + \frac{s + 1}{2} \frac{M''}{M^2} - \left[ r(r + s + 1) + s + 1 \right] \frac{M'f}{M^3}
\]

depends on some mass term \( M(x) \). Here prime denotes differentiation with respect to \( x \). \( M(x) \) is the dimensionless form of the mass function \( m(x) = m_0 M(x) \) and we have taken \( \hbar = 2m_0 = 1 \).

Let us find the solution of (1) of the form

\[
\psi(x) = f(x) F(g(x))
\]

where \( f(x), g(x) \) are two function of \( x \) to be determined and \( F(g) \) satisfies second order differential equation of the type

\[
\frac{d^2 F}{dg^2} + Q(g) \frac{dF}{dg} + R(g) F = 0
\]

Now in order to be physically acceptable, \( \psi(x) \) has to satisfy the following two conditions:

(i) Like quantum systems with constant mass it will be square integrable over domain of definition \( D \) of \( M(x) \) and \( \psi(x) \) i.e.,

\[
\int_D |\psi_n(x)|^2 dx < \infty
\]

(ii) The Hermiticity of the Hamiltonian in the Hilbert space spanned by the eigenfunctions of the potential \( V(x) \) is ensured by the following extra condition [25]

\[
\frac{|\psi_n(x)|^2}{\sqrt{M(x)}} \rightarrow 0
\]

at the end points of the interval where \( V(x) \) and \( \psi_n(x) \) are defined. This condition imposes an additional restriction whenever the mass function \( M(x) \) vanishes at any one or both the end points of \( D \).

Substituting \( \psi(x) \) from equation (3) in equation (1) leads to

\[
\frac{d^2 F}{dg^2} + \left( \frac{g''}{g'^2} + \frac{2f'}{fg'} - \frac{M'}{Mg'} \right) \frac{dF}{dg} + \left( \frac{f''}{fg'^2} + (E - V_{\text{eff}}) \frac{M}{g'^2} - \frac{M'f'}{Mfg'^2} \right) F = 0
\]

Comparing equations (4) and (5) we get

\[
Q(g(x)) = \frac{g''}{g'^2} + \frac{2f'}{fg'} - \frac{M'}{Mg'}
\]

\[
R(g(x)) = \frac{f''}{fg'^2} + (E - V_{\text{eff}}) \frac{M}{g'^2} - \frac{M'f'}{Mfg'^2}
\]
The first of Eqns. (6) gives

\[ f(x) \propto \sqrt{\frac{M}{g'}} \exp\left(\frac{1}{2} \int g'(x) Q(t) dt\right) \]  

(7)

while the latter leads to the equation

\[ E - V_{eff} = \frac{g''}{2Mg'} - \frac{3}{4M} \left(\frac{g''}{g'}\right)^2 + \frac{g^2}{M} \left( R - \frac{1}{2} \frac{dQ}{dg} - \frac{Q^2}{4}\right) - \frac{M''}{2M^2} + \frac{3M'^2}{4M^3} \]  

(8)

For equation (8) to be satisfied, we need to find some functions \( M(x), g(x) \) ensuring the presence of a constant term on the right hand side of equation (8) to compensate \( E \) on its left hand side and giving rise to an effective potential \( V_{eff}(x) \) with well behaved wavefunctions.

Now in PCT approach there are many options for choosing \( M(x), g(x) \), for example \( M(x) = \lambda g'(x) \), \( M(x) = \lambda \), \( M(x) = \lambda g'(x) \) [42], \( \lambda \) being a constant. Here we choose \( M(x) = \lambda g'(x) \) so that equation (8) reduces to

\[ E - V_{eff} = g' \left( R - \frac{1}{2} \frac{dQ}{dg} - \frac{Q^2}{4}\right) \]  

(9)

**Example 1:**

First we consider \( F_n(g) \propto \hat{L}_n^{(\alpha)} \), where \( \hat{L}_n^{(\alpha)} \), \( n = 1, 2, 3, ..., \alpha > 0 \) is the Laguerre type \( X_1 \) polynomials [34]. For this polynomial

\[ Q(g) = -\frac{(g - \alpha)(g + \alpha + 1)}{g(g + \alpha)} \quad R(g) = \frac{n - 2}{g} + \frac{2}{g + \alpha}. \]

For these values of \( Q(g) \) and \( R(g) \) equation (9) becomes

\[ E - V_{eff} = g' \left( \frac{2\alpha n + \alpha^2 - \alpha + 2}{2\alpha} \right) - \frac{g'}{g^2} \left( \frac{\alpha^2 - 1}{4\lambda} \right) - \frac{2g'}{\lambda(g + \alpha)} - \frac{g'}{\lambda(g + \alpha)} \]  

(10)

Taking \( \frac{g'}{\lambda g} = C \), where \( C \) is a constant, a constant term can be created on the right-hand side of the above equation which will correspond to \( E \) on the left-hand side. But \( C \) must be restricted to positive values in order to get increasing energy eigenvalues for successive \( n \) values. The solution of the above-mentioned first order differential equation for \( g(x) \) leading to a positive mass function reads

\[ g(x) = e^{-bx} \quad M(x) = e^{-bx}, \quad -\infty < x < \infty \]  

(11)

where \( C\lambda = -b, b > 0 \). This exponentially behaved mass function is often used in the study of confined energy states for carriers in semiconductor quantum well [27, 28]. This mass function was also found to be useful for understanding transport properties through semiconductor heterostructures (produced by the recent crystallographic growth techniques) which is indispensable for the prediction of the performances of these samples. In this context, this mass function has been used to compute transmission probabilities for scattering in abrupt heterostructures [46] which may be useful in the
design of semiconductor devices [47].

Now equations (7), (10), and (11) give

\[
E = \frac{b^2}{4}\left(4n + \frac{4}{\alpha} + 2\alpha - 2\right) + \bar{V}_0
\]

\[
V_{eff} = \left[\frac{b^2}{4}\left((\alpha^2 - 1)e^{bx} + e^{-bx}\right)\right] + \frac{b^2}{4}\left(\frac{4}{\alpha(1 + \alpha e^{bx})} + \frac{8e^{bx}}{(1 + \alpha e^{bx})^2}\right) + \bar{V}_0
\]

where \(\bar{V}_0\) is a constant. Thus we obtain energy eigenvalues and eigenfunctions as

\[
E_m = b^2\left(m + \frac{\alpha + 1}{2}\right) + \frac{b^2}{\alpha} + \bar{V}_0
\]

\[
\psi_m(x) = N_m \exp\left[-\frac{1}{\alpha}\left((\alpha + 1)bx + e^{-bx}\right)\right] L_m^{(\alpha)}(e^{-bx}), \quad m = 0, 1, 2...
\]

where we have reset quantum number \(n = m + 1\) and \(N_m\) is the normalization constant given by [33]

\[
N_m = \left(\frac{b m!}{(m + \alpha + 1)\Gamma(m + \alpha)}\right)^{1/2}
\]

It is to be noted that the square integrability condition(i) stated earlier does not impose any additional restriction on \(\alpha\) but for condition(ii) to be satisfied, \(\alpha\) should be greater than \(-\frac{1}{2}\). In figure 1 we have plotted the mass function \(M(x)\) given in (11), potential \(V_{eff}\) given in (12) and square of first two bound state wavefunctions.

![Figure 1: Plot of the potential \(V_{eff}\) (solid line) given in equation (12), square of its first two bound state wave functions \(|\psi_0(x)|^2\) (dashed line) and \(|\psi_1(x)|^2\) (dotted line), for the mass function \(M(x)\) (long dashed line) given in equation (11). We have considered here \(b = 1, \alpha = 2\).](image)
Example 2:

Next we choose $F_n(g) \propto \hat{F}_n^{(\alpha, \beta)}(x)$, where $\hat{F}_n^{(\alpha, \beta)}$, $n = 1, 2, 3, \ldots$, $\alpha, \beta > -1$, $\alpha \neq \beta$ is the Jacobi type $X_1$ polynomial. For this polynomial, we have

\[
\begin{align*}
Q(g) &= -\frac{\alpha + \beta + 2g - (\beta - \alpha)}{1 - g^2} - \frac{2(\beta - \alpha)}{(\beta - \alpha)g - (\beta + \alpha)^2}, \\
R(g) &= -\frac{(\beta - \alpha)g - (\beta + \alpha)^2}{1 - g^2}
\end{align*}
\]  

(15)

For these $Q(g)$ and $R(g)$ equation (8) becomes

\[
E - V_{\text{eff}} = \frac{g'}{\lambda} \left[ \frac{A_1 g + A_2}{1 - g^2} + \frac{A_3 g + A_4}{(1 - g^2)g - (\beta - \alpha)} + \frac{A_5}{(\beta - \alpha)^2} + \frac{A_6}{(\beta + \alpha)^2} \right]
\]

(16)

where

\[
\begin{align*}
A_1 &= \frac{\beta^2 - \alpha^2}{2\alpha \beta}, & A_2 &= n^2 + (\beta + \alpha - 1)n + \frac{1}{4}(\beta + \alpha)^2 - 2(\beta + \alpha) - 4 + \frac{\beta^2 + \alpha^2}{2\alpha \beta}, \\
A_3 &= \frac{\beta^2 - \alpha^2}{2}, & A_4 &= -\frac{\beta^2 + \alpha^2 - 2}{2}, & A_5 &= \frac{(\beta + \alpha)(\beta - \alpha)^2}{2\alpha \beta}, & A_6 &= -2(\beta - \alpha)^2
\end{align*}
\]

To generate a constant term on the right-hand side, we suppose $\frac{g'}{\lambda(1 - g^2)} = C_1 > 0$, where $C_1$ is a constant. Consequently we obtain

\[
g(x) = \tanh(ax), \quad M(x) = \text{sech}^2(ax), \quad -\infty < x < \infty.
\]

(17)

where $C_1 \lambda = a$, $a > 0$. This asymptotically vanishing mass function depicts a solitonic profile \[48\]. Recently, this mass profile has been used in position dependent mass Hamiltonians of Zhu-Kroener \[44\] and BenDaniel-Duke \[43\] type and interesting connection was shown \[48\] between the discrete eigenvalues of such Hamiltonians and the stationary 1-soliton and 2-soliton solutions of the Korteweg-de-Vries (kdv) equation that match with the mass function up to a constant of proportionality. Also, in dealing with position dependent mass models controlled by a $\text{sech}^2$-mass profile, it was demonstrated \[45\] that in the framework of a first order intertwining relationship, such a mass environment generates an infinite sequence of bound states for the conventional free-particle problem. Now using (7), (16), (17) and setting quantum number $n = m + 1$, we obtain the new potential, energy eigenvalues and corresponding bound state wavefunctions as

\[
V_{\text{eff}} = \left[ \frac{a^2}{4} ((\alpha^2 - 1)e^{2ax} + (\beta^2 - 1)e^{-2ax}) \right] + \frac{a^2}{4} \left( \frac{4(\alpha - \beta)(\alpha - 3\beta)}{\alpha(\beta + \alpha \text{e}^{2ax})} - \frac{8\beta(\alpha - \beta)^2}{\alpha(\beta + \alpha \text{e}^{2ax})^2} \right) + \hat{V}_0
\]

(18)

\[
E_m = a^2 \left( m + \frac{\alpha + \beta}{2} \right) \left( m + \frac{\alpha + \beta + 2}{2} \right) + a^2 \left( \frac{\beta}{\alpha} - \frac{\alpha^2 + \beta^2 - 2}{4} \right) + \tilde{V}_0
\]

(19)

\[
\psi_m(x) = N_m \frac{(1 - \tanh(ax))^{\frac{a+1}{2}} (1 + \tanh(ax))^{\frac{a+1}{2}}}{\alpha + \beta + (\alpha - \beta) \tanh(ax)} \hat{F}_{m+1}^{(\alpha, \beta)}(\tanh(ax)) \quad m = 0, 1, 2, \ldots
\]

(20)

where $\tilde{V}_0$ is a constant and the normalization constant $N_m$ is given by \[33\],

\[
N_m = \left( \frac{a(\alpha - \beta)^2}{2^{\alpha+\beta-1}m!(2m + \alpha + \beta + 1)\Gamma(m + \alpha + \beta + 1)} \Gamma(m + \alpha + 1) \Gamma(m + \beta + 1) \right)^{1/2}
\]
An additional restriction $\alpha, \beta > -1/2$ is to be imposed to satisfy condition(ii) stated before whereas the square integrability condition does not require any extra restriction on the parameters. In figure 2 we have plotted the mass function given in (17) and potential $V_{eff}$ given in (18) and square of its first two bound state wavefunctions.

Figure 2: Plot of the potential $V_{eff}$ (solid line) given in equation (18), square of its first two bound state wave functions $|\psi_0(x)|^2$ (dashed line) and $|\psi_1(x)|^2$ (dotted line), for the mass function $M(x)$ (long dashed line) given in equation (17). We have considered here $a = .2, \alpha = 2, \beta = 2.5$.

It should be mentioned here that for the suitable choices of the constants $\bar{V}_0$ and $\hat{V}_0$, the expressions within the square bracket of the obtained potentials given in (12) and (18), coincides with previously obtained potentials in [42] using classical Laguerre or Jacobi polynomial. For the same choices of $\bar{V}_0$ and $\hat{V}_0$, the potentials obtained here also become isospectral with the previously obtained potentials in [42].

3 Supersymmetric quantum mechanics approach

Defining $A\psi = \frac{1}{\sqrt{M}} \frac{d\psi}{dx} + B\psi$ and $A^\dagger\psi = -\frac{d}{dx} \left( \frac{\psi}{\sqrt{M}} \right) + B\psi$

where $B(x) = -\frac{1}{\sqrt{M}} \frac{\psi_0'}{\psi_0}$ is the superpotential, the Hamiltonian of equation (1), can be factorized in the following way

$$H_{eff} = A^\dagger A = -\frac{d}{dx} \left( \frac{1}{M(x)} \right) \frac{d}{dx} + V_{eff}$$

(21)

It’s supersymmetric partner Hamiltonian is given by

$$H_{1,eff} = AA^\dagger = -\frac{d}{dx} \left( \frac{1}{M(x)} \right) \frac{d}{dx} + V_{1,eff}$$

(22)
where $V_{\text{eff}}$ and $V_{1,\text{eff}}$ are the supersymmetric partner potentials given by

\[
V_{\text{eff}} = -\left(\frac{B}{\sqrt{M}}\right)' + B^2
\]

\[
V_{1,\text{eff}} = V_{\text{eff}} + \frac{2B'}{\sqrt{M}} - \left(\frac{1}{\sqrt{M}}\right)\left(\frac{1}{\sqrt{M}}\right)''
\] (23)

These two potentials will be called shape invariant if they satisfy the condition $[18, 49]$,

\[
V_{1,\text{eff}}(x, a_1) = V_{\text{eff}}(x, a_2) + R(a_1)
\] (24)

where $a_1$ is a set of parameters, $a_2$ is some function of $a_1$ and $R(a_1)$ is independent of $x$. In case of unbroken supersymmetry, the energy spectrum and wavefunctions of two such shape invariant effective mass potentials are related by $[18]$,

\[
E_n^{(\text{eff})} = 0, \quad E_n^{(1,\text{eff})} = E_n^{(\text{eff})}, n = 0, 1, 2, ...
\] (25)

and

\[
\psi_n^{(1,\text{eff})} = \frac{A\psi_n^{(\text{eff})}}{\sqrt{E_n^{(\text{eff})}}}, \quad \psi_n^{(\text{eff})} = \frac{A\psi_n^{(1,\text{eff})}}{\sqrt{E_n^{(1,\text{eff})}}}, n = 0, 1, 2...
\] (26)

Now the superpotential $B(x)$ for the potentials obtained in the Example 1 and Example 2, can be written as

\[
B(x) = \frac{b}{2} \left[(\alpha + 1)e^{bx} - e^{-bx}\right] - \frac{b}{\alpha(\alpha + 1)e^{2bx} + (2\alpha + 1)e^{bx} + 1}
\] (27)

and

\[
B(x) = \frac{a}{2} \left[\{\alpha - \beta)cosh(\alpha x) + (\alpha + \beta + 2)sinh(\alpha x)\}
+ \frac{2a(\alpha - \beta)}{[(\alpha + \beta)cosh(\alpha x) + (\alpha - \beta)sinh(\alpha x)] [(\alpha + \beta + 2)cosh(\alpha x) + (\alpha - \beta)sinh(\alpha x)]}
\] (28)

respectively.

Now using (23) and (27) we obtain

\[
V_{\text{eff}} = \frac{b^2}{4} \left[(\alpha^2 - 1)e^{bx} + e^{-bx} + \frac{4}{(1 + \alpha e^{bx})} + \frac{8e^{bx}}{\alpha(1 + \alpha e^{bx})^2}\right] - b^2 \left(\frac{\alpha + 1}{2} + \frac{1}{\alpha}\right)
\] (29)

\[
V_{1,\text{eff}} = \frac{b^2}{4} \left[(\alpha(\alpha + 2)e^{bx} + e^{-bx} + \frac{4}{(\alpha + 1)(1 + (\alpha + 1)e^{bx})} + \frac{8e^{bx}}{(1 + (\alpha + 1)e^{bx})^2}\right] - b^2 \left(\frac{\alpha^2 + \alpha + 2}{\alpha + 1}\right)
\] (30)

Also using (23) and (28) we obtain

\[
V_{\text{eff}} = \frac{a^2}{4} \left[(\alpha^2 - 1)e^{2ax} + (\beta^2 - 1)e^{-2ax} + \frac{4(\alpha - \beta)(\alpha - 3\beta)}{\alpha(\beta + \alpha e^{2ax})} - \frac{8\beta(\alpha - \beta)^2}{\alpha(\beta + \alpha e^{2ax})^2}\right]
- \frac{a^2}{4} \left(2\alpha\beta + 2\alpha + 2\beta + 2 + \frac{4\beta}{\alpha}\right)
\] (31)

\[
V_{1,\text{eff}} = \frac{a^2}{4} \left[(\alpha(\alpha + 2)e^{2ax} + (\beta(\beta + 2))e^{-2ax} + \frac{4(\alpha - \beta)(\alpha - 3\beta - 2)}{\alpha(\alpha + 1)((\alpha + 1)e^{2ax} + \beta + 1)}\right]
- \frac{a^2}{4} \left(\frac{8(\beta + 1)(\alpha - \beta)^2}{(\alpha + 1)((\alpha + 1)e^{2ax} + \beta + 1)^2} + 2\alpha\beta + \frac{4(\beta + 1)}{\alpha + 1}\right)
\] (32)
The potentials \( V_{\text{eff}} \) obtained in (29) and (31) are same with the potentials (12) and (18) for 
\[
\tilde{V}_0 = -b^2 \left( \frac{\alpha+1}{2} + \frac{1}{\alpha} \right) \quad \text{and} \quad \hat{V}_0 = -a^2 \left( 2\alpha\beta + 2\alpha + 2\beta + 2 + \frac{4\beta}{\alpha} \right) \]
respectively.

From the equation (29) and (30) we observe that the potential \( V_{\text{eff}} \) and its supersymmetric partner potentials \( V_{1,\text{eff}} \) satisfy the following relation

\[
V_{1,\text{eff}}(x, \alpha) = V_{\text{eff}}(x, \alpha + 1) + b^2 \quad (33)
\]

Also from equation (31) and (32) it is clear that the potentials \( V_{\text{eff}} \) and \( V_{1,\text{eff}} \) are related by

\[
V_{1,\text{eff}}(x, \alpha, \beta) = V_{\text{eff}}(x, \alpha + 1, \beta + 1) + a^2 (\alpha + \beta + 2) \quad (34)
\]

From the above two relations we observe that the potentials obtained in Example 1 and Example 2 satisfy the condition (24). So we conclude that the new potentials are shape invariant.

Now using (68), (73) of [34] and (26) we have derived the eigenstates of the partner potential of the potential (12), as

\[
\psi_{m}^{(1,\text{eff})}(x) \propto \frac{\text{Exp}\left[\frac{-1}{2}\left((\alpha+1)bx + e^{-bx}\right)\right]}{(\alpha + 1)e^{\frac{ax}{2}} + e^{-\frac{ax}{2}}} \hat{L}_{m+1}^{(\alpha+1)}(e^{-bx}) , m = 0, 1, 2...
\]

and using (42),(47) of [34] and (26) we obtain the eigenstates of the partner potential of the potential (18) as

\[
\psi_{m}^{(1,\text{eff})}(x) \propto \frac{(1 + \text{tanh}(ax))^{\frac{\alpha+\beta}{2}}(1 \text{ - tanh}(ax))^{\frac{\alpha}{2}}}{(\alpha - \beta)\text{sinh}(ax) - (\alpha + \beta + 2)\text{cosh}(ax)} \hat{P}_{m+1}^{(\alpha+1,\beta+1)}(\text{tanh}(ax)) , \quad m = 0, 1, 2...
\]

4 Summary and Outlook

We have obtained exactly solvable potentials for position dependent (effective) mass Schrödinger equation whose bound state solutions are given in terms of Laguerre or Jacobi type \( X_1 \) exceptional orthogonal polynomials. As mentioned earlier, the obtained potentials are the generalizations (by some rational functions) of the previously obtained potentials [42] whose bound state solutions involve classical Laguerre or Jacobi orthogonal polynomials. The method discussed here can be used for other choices of the function \( g(x) \) in order to generate other type of exactly and quasi exactly solvable potentials for the one dimensional Schrödinger equation with position dependent mass. We have shown that these potentials are shape invariant and are isospectral to the previously obtained potentials in position dependent mass background whose solutions are given in terms of classical Laguerre or Jacobi type orthogonal polynomials. Though the origin of such isospectrality in the constant mass scenario has recently been shown in [50], it will be worthwhile to study the origin of such isospectrality in the position dependent mass background. As to the possible physical applications of our obtained potentials in the position dependent mass background, let us mention that it will be interesting to use supersymmetric quantum mechanics to generate isospectral potentials that depend on a specified number of scalar parameters by multiple deletion and restoring of some levels.
of the original potential in the position dependent mass scenario so as to make them suitable for multiparameter optimization of optical nonlinearities in semiconductor quantum wells. This particular problem was dealt in ref [51], taking the constant effective mass $m^*$.

Acknowledgement: It is a pleasure to thank Rajkumar Roychoudhury for many valuable comments and suggestions. We also thank the referees for their valuable suggestions towards improving the manuscript.

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