On the Sutherland Spin Model of $B_N$ Type and Its Associated Spin Chain

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Abstract: The $B_N$ hyperbolic Sutherland spin model is expressed in terms of a suitable set of commuting Dunkl operators. This fact is exploited to derive a complete family of commuting integrals of motion of the model, thus establishing its integrability. The Dunkl operators are shown to possess a common flag of invariant finite-dimensional linear spaces of smooth scalar functions. This implies that the Hamiltonian of the model preserves a corresponding flag of smooth spin functions. The discrete spectrum of the restriction of the Hamiltonian to this spin flag is explicitly computed by triangularization. The integrability of the hyperbolic Sutherland spin chain of $B_N$ type associated with the dynamical model is proved using Polychronakos’s “freezing trick”.

1. Introduction

Since the publication of the pioneering papers of Calogero [5] and Sutherland [32, 33], the study of solvable and integrable quantum many-body problems has become a fruitful field of research with multiple connections in many branches of contemporary mathematics and physics. From a mathematical standpoint, one of the key developments in the field was the discovery by Olshanetsky and Perelomov of an underlying $A_N$ root system structure for both the Calogero and Sutherland models [25]. The integrability of these models follows by expressing the Hamiltonian as one of the radial parts of the Laplace–Beltrami operator in a symmetric space associated with the given root system. It was also shown in this paper that the original inverse square (Calogero) and trigonometric/hyperbolic (Sutherland) potentials arise as appropriate limits of the most general potential in this class, given by the Weierstrass $\wp$-function, and that integrable models associated to other root systems also exist. The rational and trigonometric Calogero–Sutherland (CS) models are also exactly solvable, in the sense that their

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eigenfunctions and eigenvalues can be computed algebraically. In fact, the study of the eigenfunctions of these models has led to significant advances in the theory of multivariate orthogonal polynomials \cite{1, 11, 23}. Apart from their mathematical interest, CS models have found numerous applications in diverse areas of physics such as soliton theory \cite{22, 29}, fractional statistics and anyons \cite{4, 6}, random matrix theory \cite{35}, and Yang–Mills theories \cite{9, 16}, to name only a few.

During the last decade, CS models with internal degrees of freedom have been actively explored by a variety of methods, including the exchange operator formalism \cite{24}, the Dunkl operators approach \cite{2, 8, 11}, reduction by discrete symmetries \cite{30}, and construction of Lax pairs \cite{19–21}. Historically, the first CS models with spin discussed in the literature were related to the original models of $A_N$ type introduced by Calogero and Sutherland \cite{2, 17, 19, 20, 24, 34}. The integrability of these CS spin models was established in some cases by relating the Hamiltonian to a quadratic combination of Dunkl operators of $A_N$ type \cite{7, 10, 26}. The $B_N$ counterpart of the $A_N$ CS spin models mentioned above were first considered by Yamomoto \cite{38}. In this paper, the spectrum of the rational $B_N$ spin model was explicitly determined, and its integrability was shown by means of the Lax pair approach. In Ref. \cite{39}, Yamamoto and Tsuchiya presented an alternative proof of the integrability of this model using Dunkl operators of $B_N$ type. The same operators were later employed by Dunkl to construct a complete basis of eigenfunctions \cite{11}. In contrast, the trigonometric/hyperbolic $B_N$ spin model has received remarkably little attention. In our recent paper \cite{13} we proved that this model is exactly solvable in the sense of Turbiner \cite{36, 37}, meaning that its Hamiltonian leaves invariant a known infinite increasing sequence (or flag) of finite-dimensional linear spaces of smooth spin functions. In fact, in \cite{13} we developed a systematic method for constructing exactly (or in some cases partially) solvable $B_N$-type CS models with spin by combining several families of Dunkl operators. The key elements of this method – first introduced in the $A_N$ case in \cite{12} – are: i) the definition of a new family of Dunkl operators, and ii) the construction of a very wide class of quadratic combinations of these operators and those in the other two families considered by Dunkl in \cite{11}.

The interest on CS spin models has been further enhanced by their close connections with integrable spin chains of Haldane–Shastry type \cite{18, 31}. Spin chains describe a fixed arrangement of particles that interact through their spins. A well-known example is the Heisenberg spin chain, whose spins are equally spaced and interact only with their nearest neighbors. The Haldane–Shastry model was actually the first one-dimensional spin chain with long range interactions whose spectrum could be computed exactly. In this model, the spin sites are equally spaced in a circle and interact with each other with strength decreasing as the inverse square of the chord distance between the sites. The integrability of the Haldane–Shastry spin chain was proved by Fowler and Minahan in \cite{14}. Polychronakos later realized that the commuting conserved quantities of the Haldane–Shastry spin chain can be elegantly deduced from those of the (dynamical) Sutherland spin model of $A_N$ type by applying what he called the “freezing trick” \cite{27} (see also \cite{34}). This corresponds to taking the strong coupling limit in the Sutherland spin model and restricting to states with no momentum excitations, so that the internal degrees of freedom remain the only relevant variables in the problem and the particles are “frozen” at their classical equilibrium positions. This observation is, in principle, valid for any integrable spin Calogero–Sutherland model. For instance, in Ref. \cite{27} the freezing trick is applied to the spin Calogero model with rational interaction to construct a new integrable spin chain of rational type in which the sites are no longer equally spaced. The spectrum of this chain was later calculated by Frahm \cite{15} and Polychro-