Contextual trace refinement for concurrent objects: Safety and progress

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Abstract. Correctness of concurrent objects is defined in terms of safety properties such as linearizability, sequential consistency, and quiescent consistency, and progress properties such as wait-, lock-, and obstruction-freedom. These properties, however, only refer to the behaviours of the object in isolation, which does not tell us what guarantees these correctness conditions on concurrent objects provide to their client programs. This paper investigates the links between safety and progress properties of concurrent objects and a form of trace refinement for client programs, called contextual trace refinement. In particular, we show that linearizability together with a minimal notion of progress are sufficient properties of concurrent objects to ensure contextual trace refinement, but sequential consistency and quiescent consistency are both too weak. Our reasoning is carried out in the action systems framework with procedure calls, which we extend to cope with non-atomic operations.

1 Introduction

Concurrent objects provide operations that can be executed simultaneously by multiple threads, and provide a layer of abstraction to programmers by managing thread synchronisation on behalf of client programs, which in turn improves safety and efficiency. Correctness of concurrent objects is usually defined in terms of the possible histories of invocation and response events generated by executing the operations of a sequential specification object. There are several notions of safety for concurrent objects [12, 7]: sequential consistency, linearizability, and quiescent consistency being the most widely used. Similarly, there are many different notions of progress [12, 13], e.g., wait-, lock- and obstruction-freedom are popular non-blocking conditions.

Both safety and progress properties are stated in terms of a concurrent object in isolation, and disregard their context, i.e., the client programs that use them. Programmers (i.e., client developers) have therefore relied on informal “folk theorems” to link correctness conditions on concurrent objects and substitutability of objects within client programs. We seek to provide a formal account of this relationship, addressing the question: “Provided concurrent object \( OC \) is correct with respect to sequential object \( OA \), how are the behaviours of \( C[OA] \) related to those of \( C[OC] \)?”, where \( C[O] \) denotes a client program \( C \) that uses object \( O \), for different notions of correctness. One of the first formal answers to this question was given by the abstraction theorems of Filipović et al. [9], who link safety properties sequential consistency and linearizability to a contextual notion of correctness called observational refinement, which defines substitutability with respect to the initial and final state of a system’s execution. For terminating clients, linearizability is shown to be equivalent to observational refinement, while sequential consistency is shown to be equivalent to observational refinement provided that clients only communicate via shared concurrent objects.
Since non-termination is common in many concurrent systems, e.g., operating systems and real-time controllers, our work aims to understand substitutability for potentially non-terminating clients. Related to this aim is the work of Gotsman and Yang [10] and Liang et al. [15], who link observational refinement to safety and progress properties of concurrent objects. However, both [10] and [15] assume that the concurrent objects in question are already linearizable. Furthermore, [10] aims to understand compositionality of progress properties, while [15] develops characterisations of progress properties based on the observational guarantees they provide.

The motivation for our work differs from [10, 15] in that we take contextual trace refinement as the underlying correctness condition when substituting OC for OA in C, then aim to understand the safety/progress properties on OC that are required to guarantee trace refinement between C[OA] and C[OC]. To this end, we develop an action systems framework that integrates and extends existing work [18, 1] from the literature, building on our preliminary results on this topic [8]. As part of our contributions we (i) extend Sere and Waldén’s treatment of action systems with procedures [18] with non-atomic procedures; (ii) develop a theory for contextual trace refinement, adapting Back and von Wright’s [1] theory for trace refinement of action systems, then reduce system-wide proof obligations (i.e., properties of the client and object together) to proof obligations on the objects only; (iii) show that linearizability [14] and minimal progress [13] together are sufficient to guarantee contextual trace refinement; and (iv) show that both sequential consistency and quiescent consistency are too weak for contextual trace refinement, even when client threads only communicate through the shared object.

2 Concurrent objects and their clients

2.1 Client-object systems

We consider concurrent systems where a client consists of multiple threads which interact with one or more concurrent objects and shared variables. For example, the following client program consists of threads 1 and 2 using a shared stack s, and variables x, y and z.

Init x, y, z = 0, 0, 0
Thread 1:
   T1: s.push(1);
   T2: s.push(2);
   T3: s.pop(x);
Thread 2:
   U1: s.pop(y);
   U2: z := x;

Thread 1 pushes 1 then 2 onto the stack s, then pops the top element of s and stores it in x. Concurrently, thread 2 pops the top element of s and stores it in y, then reads the value of x and stores it in z.

The abstract behaviour of a stack is defined in terms of a sequential object, as shown in Fig. 1. The abstract stack consists of a sequence of elements S together with two operations push and pop ('(' and ')') delimit sequences, '{ }' denotes the empty sequence, and '∐' denotes sequence concatenation. Note that when the stack is empty, pop returns a special value empty that cannot be pushed onto the stack.

If concurrent objects are implemented using fine-grained concurrency, the call statements in their clients are not necessarily atomic because they may invoke non-atomic operations. Furthermore, depending on the implementation of s, we will get different traces of the client program because the effects of the concurrent operations on s may take effect in different orders. For example, Fig. 2 presents a simplified version of a non-blocking stack example due to Treiber [19]. In this implementation, each line of the push and pop corresponds to a single atomic step, except H1,
Init: $S = \langle \rangle$

$$\text{push}(v) \Rightarrow \begin{array}{l}
\text{atomic} \{ S := \langle v \rangle \} \\
\text{atomic} \\
\text{if } S = \langle \rangle \text{ then return empty else} \\
\text{lv := head}(S) ; \\
S := \text{tail}(S) ; \\
\text{return lv} \}
\end{array}$$

$\text{pop} \Rightarrow$

$$\begin{array}{l}
\text{atomic} \\
\text{if } S = \langle \rangle \text{ then return empty else} \\
\text{lv := head}(S) ; \\
S := \text{tail}(S) ; \\
\text{return lv} \\
\end{array}$$

Fig. 1. Abstract stack

which may be regarded as being atomic because a thread can signal to other threads that a node has been taken in a single atomic step. Synchronisation of $\text{push}$ and $\text{pop}$ operations is achieved using a compare-and-swap (CAS) instruction, which takes as input a (shared) variable $g$ and an expected value $lv$ and a new value $nv$:

$$\text{CAS}(g, lv, nv) \equiv \text{atomic} \{ \text{if } (g = lv) \text{ then } g := nv ; \text{return true} \\
\text{else return false} \}$$

With this stack implementation, the executions of, say $T_1$ and $U_1$, in the above client may overlap, and different behaviours may be observed according to the order in which steps of the different threads are executed. Treiber's stack is linearizable with respect to the abstract stack in Fig. 1, so the effect of each operation call takes place between its invocation and its response. If a different stack implementation is used which satisfies a more permissive correctness condition, such as sequential consistency or quiescent consistency [12], a wider range of behaviours may be observed.

2.2 Observability and contextual trace refinement

With an example client-object system in place, we return to the main question for this paper: What guarantees do correctness conditions on concurrent objects provide to clients that use the objects? Furthermore, how can one address divergence, termination and reactivity of a client? To address these, we first pin down the aspects of the system being developed that are visible to an external observer. Following Filipović et al. [9], we take the state of the client variables to be observable, and the state of the objects they use to be unobservable. Therefore, for the client program in Section 2.1, variables $x$, $y$ and $z$ are observable, but none of the variables of the stack implementation $s$ are observable. This allows us to reason about a client with respect to different implementations of $s$. Second, we define when a system may be observed. Unlike Filipović et al. [9] who only observe the state at the beginning and end of a client's execution, we assume that the states throughout a client's execution are visible. This allows us to accommodate, for example, reactive clients, which interact with an observer in some way even if they are potentially non-terminating.

Therefore, our notion of correctness for the combined system will be a form of observational refinement that holds iff every (observable) trace of a client using a concurrent object is equivalent to some (observable) trace of the same client using the corresponding abstract specification of the object. The end result is that from the perspective of a client program, it will be impossible to tell whether it is using the concurrent object, or its abstract (sequential) specification.
Example 1. Let $\mathcal{D}$ denote the client program in Section 2.1, $\mathcal{TS}$ denote the Treiber stack in Fig. 2, and $\mathcal{AS}$ denote the abstract stack in Fig. 1. Suppose $s$ in $\mathcal{D}$ is an instance of $\mathcal{TS}$. Then the following is a possible observable trace of $\mathcal{D}[\mathcal{TS}]$:

$tr \triangleq ((x, y, z) \mapsto (0, 0, 0), (x, y, z) \mapsto (0, 1, 0), (x, y, z) \mapsto (2, 1, 0), (x, y, z) \mapsto (2, 1, 1))$

where $(x, y, z) \mapsto (0, 0, 0)$ is shorthand for the state $\{x \mapsto 0, y \mapsto 0, z \mapsto 0\}$, and we ignore stuttering. Trace $tr$ is obtained by initialising as specified by $\text{Init}$, then executing $T1, T2, U1, T3$, then $U2$ to completion; i.e. they each execute their operation call without interruption. It is straightforward to see that $tr$ can also be generated by $\mathcal{D}[\mathcal{AS}]$, i.e., when using the abstract stack for $s$. Thus $tr$ can be accepted as being correct. Executions can, of course, be much more complicated than $tr$ — $\mathcal{TS}$ consists of non-atomic operations, hence, executions of $T1, T2$ or $T3$ may overlap with $U1$ or $U2$. \(\Box\)

We say that $\mathcal{TS}$ contextually trace refines $\mathcal{AS}$ with respect to the client program $\mathcal{C}$ iff every trace of $\mathcal{C}[\mathcal{TS}]$ is a possible trace of $\mathcal{C}[\mathcal{AS}]$. In this paper, we wish to know whether contextual refinement holds for every client program. To this end, we say $\mathcal{TS}$ contextually trace refines $\mathcal{AS}$ iff $\mathcal{TS}$ contextually trace refines $\mathcal{AS}$ with respect to every client program $\mathcal{C}$.

2.3 Correctness conditions on concurrent objects

There are many notions of correctness for concurrent objects, and these are defined in terms of histories of invocation and response events corresponding to operation calls on the object [12].

Concurrent histories may consist of both overlapping and non-overlapping operation calls, inducing a partial order on events. Safety properties define how, if at all, this partial order is preserved by the corresponding abstract histories generated by the corresponding sequential object [12, 7]. We will consider three different safety properties. Sequential consistency is a simple condition requiring the order of operation calls in a concrete history for a single process to be preserved. Operation calls performed by different processes may be reordered in the abstract history even if the operation calls do not overlap in the concrete history. Linearizability strengthens sequential consistency by requiring the order of non-overlapping operations to be preserved. Operation calls that overlap in the concrete history may be reordered when mapping to an abstract history. Quiescent consistency is weaker than linearizability, but is incomparable to sequential consistency. A concurrent object is said to be quiescent at some point in its history if none of its operations are executing at that point. Quiescent consistency requires the order of operation calls that are separated by a quiescent point to be preserved. Operation calls that are not separated by a quiescent point may be reordered, including operations performed by the same process.

Progress conditions on concurrent objects are necessary to ensure that clients will eventually be able to continue execution after calling operations on the objects they use. We consider a notion of progress called minimal progress [13], which guarantees that after some finite number of steps, some operation of the concurrent object terminates.

3 Modelling client-object systems

Our formal framework for reasoning about contextual trace refinement is based on existing work on action systems with procedures [18], extended to cope with potentially non-atomic operations. We let $\text{Var}$ and $\text{Val}$ denote the types of variables and values, respectively. A state is a function $\Sigma_V \triangleq V \rightarrow \text{Val}$, where $V \subseteq \text{Var}$, and a predicate of type $K$ is of type $\mathcal{P}K \triangleq K \rightarrow \mathbb{B}$, e.g., a state predicate over $V$ is of type $\mathcal{P}\Sigma_V$. 4
The abstract syntax of an action system is of the form:
\[
\mathcal{A} := [\textvar u L; \textvar o G; \textproc ph_1 = P_1 \ldots \textproc ph_n = P_n; I; \textdo A \textod]
\]
where \( L \subseteq \textvar \) is a set of unobservable variables and \( G \subseteq \textvar \) a set of observable variables such that \( L \cap G = \emptyset \); each \( ph_i = P_i \) is a (non-recursive) procedure declaration; \( I \) is an action modelling initialisation; and \( A \) is the main action. Within each \( ph_i = P_i \), \( P_i \) is an action and \( ph_i \) is a procedure heading \( p_i(\textval v, \textres x) \) with procedure name \( p_i \) and optional call-by-value and call-by-result parameters \( v \) and \( x \). Procedure declarations may additionally be parameterised by thread identifiers.

The abstract syntax of actions is of the form:
\[
A := \textvar x | \textrav x | \textskip \mid x : \subseteq E \mid x := e \mid p(e, x) \mid A_1; A_2 \mid b \rightarrow A \mid A_1 \sqcap A_2
\]
where \( x \) is a variable, \( E \) is a set-valued expression, \( e \) is an expression, \( p \) is a procedure name and \( b \) is a predicate. Actions \( \textvar x \) and \( \textrav x \) introduce and remove variable \( x \) from the state space, respectively, \textskip is an action that leaves the state unchanged, \( x : \subseteq E \) denotes non-deterministic assignment, \( x := e \) denotes assignment, \( p(e, x) \) is a procedure call with value parameter \( e \) and result parameter \( x \), \( A_1 \); \( A_2 \) is sequential composition of \( A_1 \) and \( A_2 \), \( b \rightarrow A \) is a guarded action, and \( A_1 \sqcap A_2 \) is (demonic) choice between \( A_1 \) and \( A_2 \).

The meaning of parameterless procedures is given by syntactically replacing each procedure call \( p \) in \( A \) by the procedure body, \( P \). Procedure parameters are handled by introducing new local variables with the same name; for call-by-value, the new variable is initialised with the value of the actual parameter, while for call-by-results, the final value is copied to the variable passed as the parameter. This can be seen in Examples 2 and 3 below.

Example 2. Consider again the client program \( D \) from Section 2.1 and suppose it uses the abstract stack object \( AS \) in Fig. 1. The action system modelling the client-object system is \( D[AS] \), which is given below. The shared stack is a sequence modelled by an unobservable variable \( S \). The client uses variables \( x, y \) and \( z \), and unobservable program counters \( pc_1 \) and \( pc_2 \) are used to model control flow. Note that all client variables are assumed to be observable, but none of the object’s variables are observable. We assume \( \textnpc_i(k) \) is an action that sets \( pc_i \) to \( k \) when the procedure being called has terminated. There are many possible ways to detect termination of the procedure being called, e.g., by checking whether a program counter for the procedure has been declared in the current state (see Example 3 below).

\[
[\textvar u S, pc_1, pc_2; \textvar o x, y, z; \textproc pushi(\textval in) = S := (\textin) \sqcap S \mid \textproc popi(\textres out) = S := (\emptyset) \sqcap \neg \textdec(\textret) \rightarrow \textvar \textret; \textret := \textempty; S := \texthead.S, \texttail.S \mid \textdec(\textret) \rightarrow \textout := \textret \mid \textrav \textret; S, pc_1, pc_2 := (\emptyset), T1, U1; x, y, z := 0, 0, 0; \mid \textdo pc_1 = T1 \rightarrow \textpushi(1); pc_1(T2) \mid pc_2(U2) \mid pc_1 = T2 \rightarrow \textpushi(2); pc_1(T3) \mid pc_2(U2) \mid pc_1 = T3 \rightarrow \textpushi(x); pc_1(\bot) \mid \textod]
\]

Note that because \( (A_1 \sqcap A_2); A = (A_1; A) \sqcap (A_2; A) \) holds, and \( b_1 \rightarrow (b_2 \rightarrow S) = b_1 \land b_2 \rightarrow S \) and \( b \rightarrow (A_1 \sqcap A_2) = (b \rightarrow A_1) \cap (b \rightarrow A_2) \), expanding the action corresponding to \( U1 \) results in the action:
\[
\textpc_2 = U1 \land S = (\emptyset) \rightarrow \textout := \textempty; pc_2(U2) \mid \textpc_2 = U1 \land S \neq (\emptyset) \rightarrow \textout, S := \texthead.S, \texttail.S ; pc_2(U2)
\]
Example 3. The push\(_t\) operation of the Treiber stack (invoked by thread \(t\)) is defined as follows, where \(\text{dec}(v) \equiv \lambda \sigma \bullet v \in \text{dom} \sigma\) holds if\( v\) is declared in the domain of the given state and \(\text{newNode}.n \equiv n :\in \text{Nodes} \); \(\text{Nodes} := \text{Nodes}\setminus \{n\}\) assigns \(n\) to be a new node from the available set of nodes \(\text{Nodes}\). For simplicity, we assume \(\text{Nodes}\) is an infinite set (e.g., the natural numbers), so a new node is always available. Further note that the object’s program counter is \(\hat{pc}_t\), which is distinguished from the client’s program counter \(pc_t\). Thus we have:

\[\begin{align*}
\text{proc} \push_t(\mathit{val \ in}) &= -\text{dec}(\hat{pc}_t) \rightarrow \mathit{var} \ \hat{pc}_t, n_t, ss_t; \ v_t := \mathit{in} \\
\land \hat{pc}_t = H1 &\rightarrow \text{newNode}.n_t; \ \hat{pc}_t := H2 \\
\ldots \\
\land \hat{pc}_t = H6 &\rightarrow \mathit{rav} \ \hat{pc}_t, v_t, n_t, ss_t
\end{align*}\]

The pop operation is similar, except that it additionally sets the output variable to the returned value.

\[\begin{align*}
\text{proc} \pop_t(\mathit{res \ out}) &= -\text{dec}(\hat{pc}_t) \rightarrow \mathit{var} \ \hat{pc}_t, ss_t, ssn_t, lv_t \\
\land \hat{pc}_t = P7 &\rightarrow \mathit{out} := lv_t; \ \mathit{rav} \ \hat{pc}_t, ss_t, ssn_t, lv_t
\end{align*}\]

The action system resulting from using the Treiber stack (which we will refer to as \(TS\)) as the shared concurrent object in Section 2.1 is \(\mathcal{D}[TS]\). It is similar to the action system in Example 2, except that the unobservable variables are \(\text{Nodes}\) (the set of all available nodes), \(\text{Head}\) (a pointer to a node, or \(\text{null}\)), \(\mathit{val}\) (a partial function of type \(\text{Nodes} \rightarrow \mathit{Val}\)), \(\mathit{next}\) (a partial function of type \(\text{Nodes} \rightarrow \text{Node}\)); the procedure declarations above are used; and initialisation of the object is \(\text{Nodes}, \text{Head}, \text{val}, \mathit{next} := \mathbb{N}, \text{null}, \emptyset, \emptyset\).

We now make the concept of an object and the notation \(\mathcal{C}[O]\) for an object \(O\) and client \(\mathcal{C}\) more precise. An object is a triple \(O \equiv (L, \{\text{ph}_{1,t} = P_{1,t}, \ldots, \text{ph}_{n,t} = P_{n,t}\}, I)\), where \(L\) is a set of variables, \(\{\text{ph}_{1,t} = P_{1,t}, \ldots, \text{ph}_{n,t} = P_{n,t}\}\) is a set of (potentially parameterised) procedure declarations, and \(I\) is an initialisation action. A client is a triple \(\mathcal{C} \equiv (G, A, J)\), where \(G\) is a set of variables, and \(A\) and \(J\) are the main and initialisation actions, respectively. Then \(\mathcal{C}[O]\) is the action system

\[\mathcal{C}[O] = \{\mathit{var}_u \ L; \ \mathit{var}_o \ G; \ \text{proc} \ \text{ph}_{1,t} = P_{1,t} \ldots \text{proc} \ \text{ph}_{n,t} = P_{n,t}; \ I; \ J; \ \mathbf{do} \ A \ \mathbf{od}\}.\]

4 Contextual trace refinement

We now give the semantics for action systems and define contextual trace refinement, which extends the existing theory on trace refinement [1]. Note that we only use part of the action systems framework, foregoing generality in favour of a subset of the theory adequate for handling with contextual trace refinement. In particular, to develop a more direct link to trace refinement, we only give a relational semantics for actions, as opposed to the usual predicate transformer semantics.

We assume expressions are functions from states to values. A relation is of type \(\mathcal{R}(K, K') \equiv K \rightarrow \mathcal{P}K'\), thus a state relation is of type \(\mathcal{R}(\Sigma_V, \Sigma_{V'})\), where \(V, V' \subseteq \text{Var}\). Assume \(r, r_1\) and \(r_2\) are state relations, \(b\) is a predicate and \(S\) is a set. We let:

\[\begin{align*}
- \ (r_1 \circ r_2).\gamma.\gamma' &\equiv \exists \gamma'', \ r_1.\gamma.\gamma'' \land r_2.\gamma''.\gamma' \text{ denote relational composition,} \\
- \ (b \land r).\gamma.\gamma' &\equiv b.\gamma \land r.\gamma.\gamma' \text{ denote domain restriction, and} \\
- \ S \land r &\equiv \{(\gamma, \gamma') \in r \ | \ \gamma \notin S\} \text{ denote domain anti-restriction.}
\end{align*}\]
For a function \( f \), we let \( f \{ x \mapsto v \} \triangleq \lambda z \in \text{dom} \ f \bullet \text{if} \ z = x \ \text{then} \ v \ \text{else} \ f.z \) denote functional overriding.

**Definition 1.** The (relational) semantics of an action system \( \mathcal{A} \) is given by \( \text{rel.} \mathcal{A} \):

\[
\begin{align*}
\text{rel.}(\text{var} \ x) & \triangleq \lambda \sigma \bullet \lambda \sigma' \bullet (\{ x \} \triangleq \sigma') = \sigma \wedge \text{dec}(x).\sigma' & \text{rel.}\text{skip} & \triangleq \text{id} \\
\text{rel.}(\text{rav} \ x) & \triangleq \lambda \sigma \bullet \lambda \sigma' \bullet (\{ x \} \triangleq \sigma) = \sigma' & \text{rel.}(b \rightarrow A_1) & \triangleq b \triangleleft \text{rel.} A_1 \\
\text{rel.}(x := e) & \triangleq \lambda \sigma \bullet \lambda \sigma' \bullet \sigma' = \sigma \uplus \{ x \mapsto e.\sigma \} & \text{rel.} (A_1; A_2) & \triangleq \text{rel.} A_1 \circ \text{rel.} A_2 \\
\text{rel.}(x \in E) & \triangleq \lambda \sigma \bullet \lambda \sigma' \bullet \exists k; E.\sigma \bullet \sigma' = \sigma \uplus \{ x \mapsto k \} & \text{rel.} (A_1 \cap A_2) & \triangleq \text{rel.} A_1 \lor \text{rel.} A_2
\end{align*}
\]

Recall that the semantics of a procedure call is given by substitution as described in Section 3.

We let \( \text{grd}.A.\gamma \triangleq \gamma \in \text{dom}(\text{rel.} A) \) denote the guard of \( A \). Because an action system is a loop with a non-deterministic choice over actions, we frequently use iteration in our reasoning. Formally, finite iteration of relation \( r \) (denoted \( r^* \)) is defined as follows:

\[
r^0 \triangleq \text{id} \quad r^{k+1} \triangleq r \circ r^k \quad r^* \triangleq \exists k \in \mathbb{N} \bullet r^k
\]

The semantics of an iterated action is defined by lifting from iteration defined on relations, namely, \( \text{rel.} A^* \triangleq (\text{rel.} A)^* \). We say an iterated execution of \( A \) terminates from state \( \gamma \) iff \( \text{term.} A.\gamma \triangleq \exists k \bullet \forall \gamma' \bullet (\text{rel.} A)^k.\gamma' \rightarrow \neg \text{grd.} A.\gamma' \). Note that \( \neg \text{grd.} A.\gamma \Rightarrow \text{term.} A.\gamma \) holds for all actions \( A \) and states \( \gamma \).

We use \( \text{seq} X \) to denote (possibly infinite) sequences of elements of type \( X \), and assume indices start from 0.

**Definition 2.** A possibly infinite sequence of states \( s \) is a trace of an action system \( \mathcal{A} \) iff \( \exists \sigma \bullet \text{rel.} I.\sigma.(s.0) \land \forall i \in \text{dom} \ s \{ 0 \} \bullet \text{rel.} A.(s.(i - 1)).(s.i) \) holds. □

A trace is complete iff either the trace is of infinite length or the guard of \( A \) does not hold in the last state of the trace. The set of all complete traces of an action system \( \mathcal{A} \) is denoted \( [\mathcal{A}] \).

Traces (Definition 2) provide a conceptually simple model for a system’s execution, and trace refinement provides a conceptually simple notion of substitutability [1]. Typically, because a concrete system is more fine-grained than the abstract, one must remove stuttering from a trace, i.e., consecutive states that leave the observable state unchanged. An action system may also exhibit infinite stuttering by generating a trace that ends with an infinite sequence of consecutive stuttering steps. After infinite stuttering, one will never be able to observe any state changes, and hence, we treat infinite stuttering as divergence, which is denoted by a special symbol ‘\( \uparrow \notin \Sigma' \). For any trace \( s \in [\mathcal{A}] \), we define \( \text{Tr}.s \) to be the non-stuttering observable sequence of states, possibly followed by \( \uparrow \), which is obtained from \( s \) as follows. First, we obtain a sequence \( s' \) by removing all finite stuttering in \( s \) and replacing any infinite stuttering in \( s \) by \( \uparrow \). Second, for each \( i \in \text{dom} \ s' \), we let \( (\text{Tr}.s).i = \text{if} \ s'.i \neq \uparrow \ \text{then} \ L \leftarrow s'.i \ \text{else} \ \uparrow \). It is straightforward to define functions that formalise the steps and above (see for example [6]).

**Definition 3.** We say abstract action system \( \mathcal{A} \) is trace refined by concrete action system \( \mathcal{C} \) (denoted \( \mathcal{A} \sqsubseteq \mathcal{C} \)) iff \( \forall s' \in [\mathcal{C}] \bullet \exists s \in [\mathcal{A}] \bullet \text{Tr}.s = \text{Tr}.s' \) holds. □

Back and von Wright have developed simulation rules (details elided due to lack of space) for verifying trace refinement of action systems [1], which we adapt to reason about client-object systems in Lemmas 1 and 2. First, we formalise the meaning of contextual trace refinement. The notion is similar to the notion of data refinement given by He et al. [11,3], but extended to traces, which enables one to cope with non-terminating reactive systems.
Definition 4. An abstract object OA is contextually trace refined by a concrete object OC, denoted $\text{OA} \sqsubseteq \text{OC}$, if for any client $C$ we have $C[\text{OA}] \sqsubseteq C[\text{OC}]$.

In this paper, for simplicity, we assume that (atomic) actions do not abort [3], therefore the proof obligations for aborting actions do not appear in Lemmas 1 and 2 below – it is straightforward to extend our results to take aborting behaviour into account. However, like Back and von Wright [1], our notion of refinement ensures total correctness of the systems we develop, i.e., the concrete system may only deadlock (or diverge) if the abstract system deadlocks (or diverges). Thus, in addition to the standard step correspondence proof obligations for ensuring safety of the concrete system, we include Back and von Wright’s proof obligations that ensure progress.

Because the entire state of the client is observable, the proof obligations pertaining to the client can be trivially discharged, leaving one with proof obligations that only refer to the object. For procedure declarations $P \equiv \{p_{1,t} = P_{1,t}, \ldots, p_{n,t} = P_{n,t}\}$, we let

$$\text{act.} P \equiv \cap_{v,x,t} p_{1,t}(v, x) \cap \cdots \cap p_{n,t}(v, x)$$

denote the action corresponding to the potential procedure calls in $P$, then define the following action, where $tt_t$ is assumed to be a fresh variable for all threads $t$.

$$\text{rem.} P \equiv \cap_{v,x,t} \neg \text{dec} (\widehat{p_{c_t}}) \rightarrow (p_{1,t}(v, x) \cap \cdots \cap p_{n,t}(v, x))$$

The guard $\neg \text{dec} (\widehat{p_{c_t}})$ is used to detect whether the procedure being executed by thread $t$ has terminated. Upon termination of procedure $p_{i,t}$ for some $1 \leq i \leq n$, $\neg \text{dec} (\widehat{p_{c_t}})$ will hold. The intention is to use $\text{rem.} P$ in (4) below, which attempts to execute the remaining steps of the operation invoked by thread $t$ to completion.

Lemma 1 (Forward simulation). Suppose $\text{OA} = (L_A, P_A, I_A)$ and $\text{OC} = (L_C, P_C, I_C)$ are objects. Then $\text{OA} \sqsubseteq \text{OC}$ if there exists a relation $R$ and the following hold for any states $\sigma$, $\tau$ and $\tau'$:

$$\text{rel.} I_C.\tau' \Rightarrow \exists \sigma \cdot R.\sigma.\tau' \land \text{rel.} I_A.\sigma \quad (1)$$

$$R.\sigma.\tau \land \text{rel.}(\text{act.} P_C).\tau' \Rightarrow \exists \sigma' \cdot R.\sigma'.\tau' \land \text{rel.}(\text{act.} P_A)\tau' \land \text{rel.}(\text{act.} P_A).\sigma \quad (2)$$

$$R.\sigma.\tau \land \neg \text{grd.}(\text{act.} P_C).\tau \Rightarrow \neg \text{grd.}(\text{act.} P_A).\sigma \quad (3)$$

$$\tau' = (\tau \uplus \bigcup_{t:T} \{tt_t \mapsto \text{false}\}) \Rightarrow \text{term.}(\text{rem.} P_C).\tau' \quad (4)$$

The first three proof obligations are straightforward. Proof obligation (4) requires that the main action of the concrete object $OC$ terminates if threads do not invoke new operations after the operation currently being executed has terminated. Note that (4) does not rule out infinite stuttering within the program $C[OC]$, but it does ensure that any infinite stuttering is caused by the client as opposed to the object $OC$, and hence, this infinite stuttering must also be present within $C[OA]$. Therefore, if (4) holds, so does Back and von Wright’s non-termination condition.

Dually to forward simulation, there exists a method of backward simulation, which requires that the abstract action system under consideration is continuous. An action system $A$ with main action $A$ is continuous iff for all $\sigma$, the set $\{\sigma' \mid \text{rel.} A.\sigma.\sigma'\}$ is finite, i.e., $A$ does not exhibit infinite non-determinism.

Lemma 2 (Backward simulation). Suppose $\text{OA} = (L_A, P_A, I_A)$ and $\text{OC} = (L_C, P_C, I_C)$ are objects and $C$ is a client such that $C[OA]$ is continuous. Then $C[OA] \sqsubseteq C[OC]$ holds if there exists a total relation $R$ and for any states $\sigma'$ and $\tau, \tau'$ (4) holds and each of the following hold:
\[
rel.I_C.\tau' \land R.\sigma'.\tau' \Rightarrow rel.I_A.\sigma' \\
rel.(act.P_C).\tau' \land R.\sigma'.\tau' \Rightarrow \exists \sigma \bullet R.\sigma.\tau \land rel.(act.P_A)^* .\sigma.\sigma' \\
\neg \text{grd.} (act.P_C).\tau \Rightarrow \exists \sigma \bullet R.\sigma.\tau \land \neg \text{grd.} (act.P_A).\sigma
\]

Lemmas 1 and 2 reduce the proof obligations for trace refinement of client-object systems to the level of objects only. This provides us with the opportunity to explore properties of objects in isolation to guarantee contextual trace refinement.

5 Events and histories

This section provides background for defining safety (e.g., linearizability) and progress (e.g., lock-freedom) properties of concurrent objects [12]. We define both types of properties in terms of histories of invocation and response events [14, 12] that record the externally visible interaction between a client and the object it uses. The type of an event is Event, which is defined as follows [4]:

\[
\text{Event} ::= \text{inv}\langle \text{N} \times \text{Op} \times (\text{Val} \cup \{\bot\}) \rangle | \text{ret}\langle \text{N} \times \text{Op} \times (\text{Val} \cup \{\bot\}) \rangle
\]

The components of each event are the thread identifier, the operation name and input/output values. We use \( \bot \not\in \text{Val} \) to denote an invocation (return) event that has no input (output). Thus, for example, \( \text{inv}(1, \text{push}, 2) \) denotes a push invocation by thread 1 with value 2, and \( \text{ret}(1, \text{push}, \bot) \) denotes a return from this invocation.

The history of an object is a (potentially infinite) sequence of events, i.e., \( \text{History} \equiv \text{seq Event} \). A history of an object is generated by an execution of a most-general client for the object [5]. We formalise the concept of a most general client in our framework in Definition 5 below, but first we describe how invocations and responses are recorded in a history. For an object \( O \triangleq (L, \{p_{h,t} = P_{t,t}, \ldots, p_{h,t} = P_{t,t}\}, I) \) assuming \( H \not\in L \) is a history variable, we let \( P_{t,t}^H \) be the history-extended action derived from \( P_{t,t} \) by additionally recording invocation and response events in \( H \) (also see [4]).

**Example 4.** The history-extended action for \( \text{push}_t \) from Example 2 is:

\[
H := H \cap \langle \text{inv}(t, \text{push}, \text{in}) \rangle; \quad S := \langle \text{in} \rangle \cap S; \quad H := H \cap \langle \text{ret}(t, \text{push}, \bot) \rangle
\]

while the history-extended version of \( \text{push}_t \) procedure from Example 3 is:

\[
\neg \text{dec}(\hat{p}_c) \rightarrow \text{var} \; \hat{p}_c, v_t, n_t, ss_t; \quad v_t := \text{in}; \quad H := H \cap \langle \text{inv}(t, \text{push}, \text{in}) \rangle
\]

\[
\neg \hat{p}_c = H6 \rightarrow H := H \cap \langle \text{ret}(t, \text{push}, \bot) \rangle; \quad \text{rav} \; \hat{p}_c, v_t, n_t, ss_t
\]

**Definition 5.** The most general client of \( O \) is the action system below, where \( H \not\in L \) is its history and \( tt \not\in L \) is a fresh variable that models termination.

\[
\mathcal{M}[O] \equiv \langle [\text{var}_u \; L \cup \{H, tt\}; \; \text{var}_o \; \emptyset; \; \text{proc} \; p_{h,t} = P_{h,t}^H \ldots \text{proc} \; p_{h,t} = P_{h,t}^H; \; I; \; H, tt : \langle \rangle, \text{false}; \; \text{do} \neg tt \rightarrow \bigcap p_{t,x}(v, x) \cap \cdots \cap p_{t,x}(v, x) \cap tt : \text{true} \; \text{od} \rangle
\]

Thus, \( \mathcal{M}[O] \) includes unobservable variables \( H \) (initially \( \langle \rangle \)) and \( tt \) (initially \( \text{false} \)), which model the history and termination of \( \mathcal{M}[O] \), respectively. Provided \( tt \) is false, the history-extended procedures of \( O \) are executed, or the system decides to terminate by setting \( tt \) to \( \text{true} \). The intention of \( \mathcal{M}[O] \) is to model all possible client behaviours, including for instance faults (where a thread stops running) or a divergence (where a thread repeatedly executes the same operation).

**Definition 6.** The set of histories of an object \( O \) is given by

\[
\{ h \in \text{seq Event} | \exists s: [\mathcal{M}[O]] \bullet \exists i: \text{dom} s \bullet h = (s.i).H \}.
\]
6 Contextual trace refinement: Progress

The progress condition we will consider is minimal progress, which guarantees system-wide progress, even though there may be individual threads that may not make progress [13]. To formalise minimal progress, we say event \( e_1 \) matches \( e_2 \) iff \( \text{matches}(e_1, e_2) \equiv \exists t, o, u, v \cdot e_1 = \text{inv}(t, o, u) \land e_2 = \text{ret}(t, o, v) \) holds, i.e., \( e_1 \) is an invocation of an operation by a thread and \( e_2 \) is the corresponding return. We say \( m \in \text{dom} \ h \) is a pending invocation iff \( \text{pi}(m, h) \equiv \forall n \in \text{dom} \ h \cdot m < n \Rightarrow \neg \text{matches}(h.m, h.n) \) holds.

An object \( O \) satisfies minimal progress iff for every trace \( tr \) of the \( M[O] \), it is always the case that in the future, either \( M[O] \) terminates, or there is some pending operation invocation that completes and returns.

**Definition 7.** An object \( O \) satisfies minimal progress iff for every \( s \in [M[O]] \) and \( i \in \text{dom} \ s \), there exists a \( j \in \text{dom} \ s \) such that \( i \leq j \) and

\[
(s.j).tt \lor \exists m \cdot \text{pi}(m, (s.j)).H) \land \neg \text{pi}(m, (s.(j + 1))).H)
\]

That is, for any trace \( s \) of \( M[O] \) and index \( i \in \text{dom} \ s \) there is a state \( s.j \) (where \( j \geq i \)) from which some pending operation in \( s.j \) completes. There are a variety of objects that satisfy minimal progress, e.g., wait-, lock-free objects under any scheduler, and obstruction-free objects under isolating schedulers (see [13] for details). Objects that do not satisfy minimal progress include obstruction free implementations that are executed using a weakly fair scheduler.

The lemma below states that any object that satisfies minimal progress does not suffer from deadlock, and is guaranteed to terminate if no additional operations are invoked.

**Lemma 3.** If \( O = (L, P, I) \) satisfies minimal progress, then for any \( \gamma \in [M[O]] \) and \( i \in \text{dom} \ \gamma \), both \( \text{grd}.(\text{act}.P).(\gamma, i) \) and condition (4) hold.

Using Lemma 3, we simplify and combine Lemmas 1 and 2. In particular, we are left with the proof obligations for safety only as in the theorem below.

**Theorem 1.** Suppose \( OA = (L_A, P_A, I_A) \) and \( OC = (L_C, P_C, I_C) \) are objects, \( OC \) satisfies minimal progress, and \( R \in \mathcal{R}(\Sigma_{L_A}, \Sigma_{L_C}) \). Then

1. \( OA \sqsubseteq OC \) if both (1) and (2) hold, and
2. for any client \( C \) such that \( C[OA] \) is continuous, \( C[OA] \subseteq C[OC] \) holds if \( R \) is total and both (5) and (6) hold.

7 Safety and contextual trace refinement

We give the formal definition of safety properties using the nomenclature in [7] and [4]. We say \( m, n \in \text{dom} \ h \) form a matching pair in \( h \) iff \( mp(m, n, h) \) holds, where \( mp(m, n, h) \equiv m < n \land \text{matches}(h.m, h.n) \land \forall i \cdot m < i < n \Rightarrow \pi_1.(h.i) \neq \pi_1.(h.m) \) and \( \pi_1 \) is the projection function returning the \( i \)th element of the given tuple.

Following [7], safety properties are defined in terms of a history \( h \) and a mapping function \( f \) between indices. The sequential history corresponding to \( h \) and \( f \) is obtained using \( \text{map}(h, f) \equiv \{ f(k) \mapsto h(k) \mid k \in \text{dom} \ f \} \). Different safety properties are defined by placing different types of restrictions on \( f \). The most basic restriction is validity of a mapping. We say a function \( f \) is a valid mapping function if, for any history \( h \), (a) the domain of \( f \) is contained in the domain of \( h \), (b)
the range of \( f \) is a consecutive sequence starting from 0, (c) \( f \) only maps matching pairs in \( h \), and (d) matching pairs in \( h \) are mapped to consecutive events in the target abstract history. Assuming \([m, n]\) is the set of integers from \( m \) to \( n \) inclusive, we formalise validity for mapping functions using \( \text{VMF}(h,f) \), where
\[
\text{VMF}(h,f) \equiv \text{dom } f \subseteq \text{dom } h \land (\exists n: \mathbb{N} \bullet \text{ran } f = [0, n - 1]) \land \text{injective}(f) \land \\
(\forall m, n: \text{dom } h \bullet \text{mp}(m, n, h) \Rightarrow (m \in \text{dom } f \Leftrightarrow n \in \text{dom } f)) \land \\
(\forall m, n: \text{dom } f \bullet \text{mp}(m, n, h) \Rightarrow f.n = f.m + 1)
\]

When formalising correctness conditions, one must also consider incomplete histories, which have pending operation invocations that may or may not have taken effect. To cope with these, like Herlihy and Wing [14], we use history extensions, which are constructed from a history \( h \) by concatenating a sequence of returns corresponding to some of the pending invocations of \( h \).

**Definition 8.** A concurrent object \( \text{OC} \) implementing an abstract object \( \text{OA} \) is correct with respect to a correctness condition \( Z \), denoted \( \text{OC} \models_{\text{OA}} Z \), iff for any history \( h \) of \( \text{OC} \), there exists an extension \( \text{he} \) of \( h \), a valid mapping function \( f \) such that \( Z(\text{he},f) \) holds and \( \text{map}(\text{he},f) \) is a history of \( \text{OA} \).

### 7.1 Linearizability

We now show that linearizability is a sufficient safety condition for discharging the remaining proof obligations in Theorem 1. Linearizability is a total condition, which means that all completed (i.e., returned) operation calls in a given history \( h \) must be mapped by \( f \).\(^3\) In addition, it must satisfy an order condition \( \text{lin} \), which states that the return of an operation may not be reordered with an invocation that occurs after it. For an event \( e \), we use \( \text{inv}(e) \equiv \exists t, o, v \bullet e = \text{inv}(t, o, v) \) to determine whether \( e \) is an invocation event and \( \text{ret}(e) \equiv \exists t, o, v \bullet e = \text{ret}(t, o, v) \) to determine whether \( e \) is a response event.

\[
total(h,f) \equiv \forall m: \text{dom } h \bullet \neg \text{pi}(m, h) \Rightarrow m \in \text{dom } f \\
\text{lin}(h,f) \equiv \forall m, n: \text{dom } f \bullet m < n \land \text{ret}(h.m) \land \text{inv}(h.n) \Rightarrow f.m < f.n
\]

**Definition 9.** An object \( \text{OC} \) is linearizable with respect to \( \text{OA} \) iff \( \text{OC} \models_{\text{OA}} \text{lin} \land \text{total} \).

First, we show contextual trace refinement for canonical implementation [16, 2, 17], which splits each sequential abstract operation call into three actions: an invocation, a effect action and a response.

**Definition 10.** For an abstract procedure \( p_t(\text{val } \text{in }, \text{res } \text{out}) = P_t \), the canonical implementation of the procedure is:
\[
P^\xi_t \equiv \neg \text{dec}(p_t) \rightarrow \text{var } p_t; p_t := 1; H \leftarrow \langle \text{inv}(t, p, \text{in}) \rangle \\
\square p_t = 1 \rightarrow p_t(\text{in }, \text{out}); p_t := 2 \\
\square p_t = 2 \rightarrow \text{ra}v p_t; H \leftarrow \langle \text{ret}(t, p, \text{out}) \rangle
\]

Invocation and response actions modify the auxiliary history variable by recording the corresponding event, while the effect action has the same effect as the abstract operation call. Unlike the abstract object, the histories of a canonical implementation are potentially concurrent.

**Theorem 2 (Canonical contextual trace refinement).** Suppose \( \text{OA} \) and \( \text{OB} \) are objects, where \( \text{OB} \) is a canonical implementation of \( \text{OA} \). Then \( \text{OA} \sqsubseteq \text{OB} \).

\(^3\) This is in contrast to partial conditions defined for relaxed memory (see [7] for details).
Proof. We use Lemma 1 because OB may not satisfy minimal progress. Here, rel.act.OB trivially satisfies (4) because by nature each procedure of a canonical object terminates. The proof of (3) requires further consideration because rel.act.OB may deadlock. For example, OB may be a stack with a pop operation that blocks when the stack is empty. In such cases, because no data refinement is performed, the guard of the canonical object is false when the guard of the abstract object is false, allowing one to discharge (3). The remaining proof obligations are straightforward.

Next, we restate a result by Schellhorn et al. [17], who have shown completeness of backward simulation for verifying linearizability. In particular, provided OC is a linearizable implementation of OA, they show that it is always possible to construct a backward simulation relation between the OC and the canonical implementation of OA.

Lemma 4 (Completeness of backward simulation [17]). Suppose OA, OB and OC are objects and M[OA] is continuous. If OC |=OA lin ∧ total and OB is a canonical implementation of OA, then there exists a total relation R such that both (5) and (6) hold between M[OB] and M[OC].

Finally, we prove our main result for linearizability, i.e., that linearizability and minimal progress together preserve contextual trace refinement.

Theorem 3. Suppose object OC is linearizable with respect to OA, OC satisfies minimal progress, and M[OA] is continuous. If C is a client such that C[OA] is continuous then C[OA] ⊑ C[OC].

Proof. Construct a canonical implementation OB of OA. By transitivity of ⊑, the proof holds if both (a) C[OA] ⊑ C[OB] and (b) C[OB] ⊑ C[OC]. Condition (a) holds by Theorem 2, and (b) holds by Theorem 1 (part 2), followed by Lemma 4. Application of Theorem 1 (part 2) is allowed because if C[OA] is continuous then C[OB] is continuous, whereas application of Lemma 4 is allowed because if R satisfies (5) and (6) for M[OB] and M[OC], then R also satisfies (5) and (6) for C[OB] and C[OC].

7.2 Sequential and quiescent consistency

We now consider contextual trace refinement for concurrent objects that satisfy sequential consistency and quiescent consistency, both of which are weaker than linearizability. Both conditions are total [7]. Additionally, sequential consistency disallows reordering of operation calls within a thread (see sc below), while quiescent consistency (see qc below) disallows reordering across a quiescent point (defined by qp below).

\[
sc(h, f) \triangleq \forall m, n: \text{dom } f \bullet m < n \land \pi_1.(h.m) = \pi_1.(h.n) \land
\]
\[
\text{ret?}(h.m) \land \text{inv?}(h.n) \Rightarrow f.m < f.n
\]
\[
qp(m, h) \triangleq \forall n: \text{dom } h \bullet (n \leq m \Rightarrow \neg \pi_1.(h.m[0..m]))
\]
\[
qc(h, f) \triangleq \forall m, k, n: \text{dom } f \bullet m < k < n \land qp(k, h) \Rightarrow f.m < f.n
\]

Definition 11. An object OC is sequentially consistent with respect to OA iff OC |=OA sc ∧ total, and OC quiescent consistent with respect to OA iff OC |=OA qc ∧ total.

Our results for sequential consistency and quiescent consistency are negative — neither condition guarantees trace refinement of the underlying clients, regardless of whether the client program in question is data independent, i.e., the state spaces of the client threads outside the shared object are pairwise disjoint.
Theorem 4. Suppose object \( OC \) is sequentially consistent with respect to object \( OA \). Then it is not necessarily the case that \( OA \preceq OC \) holds.

Proof. Consider the program in Fig. 3, where the client threads are data independent — \( x \) is local to thread 1, while \( y \) and \( z \) are local to thread 2 — and \( s \) is assumed to be sequentially consistent. Suppose thread 1 is executed to completion, and then thread 2 is executed to completion. Because \( s \) is sequentially consistent, the first \( \text{pop} \) (at \( T3 \)) may set \( x \) to 1, the second (at \( U2 \)) may set \( y \) to 2. This gives the execution:

\[
\langle (x, y, z) \mapsto (0, 0, 0), \quad (x, y, z) \mapsto (1, 0, 0), \quad (x, y, z) \mapsto (1, 0, 1), \quad (x, y, z) \mapsto (1, 2, 1) \rangle
\]

which cannot be generated when using the abstract stack \( \text{AS} \) from Fig. 1 for \( s \).

Figure 3. Counter example for contextual trace refinement and sequential consistency

Theorem 5. Suppose object \( OC \) is quiescent consistent with respect to object \( OA \). Then it is not necessarily the case that \( OA \preceq OC \) holds.

Proof. Consider the program Fig. 4, where the client threads are data independent — \( x \) and \( y \) are local to thread 1, while \( z \) is local to thread 2 — and \( s \) is a quiescent consistent stack. The concrete program may generate the following observable trace:

\[
\langle (x, y, z) \mapsto (0, 0, 0), \quad (x, y, z) \mapsto (1, 0, 0), \quad (x, y, z) \mapsto (1, 2, 0), \quad (x, y, z) \mapsto (1, 2, 3) \rangle
\]

Note that the \( \text{pop} \) operations at \( T3 \) and \( T4 \) have been reordered, which could happen if the execution of \( \text{pop} \) at \( U1 \) overlaps with \( T1, T2, T3 \) and \( T4 \). The trace above is not possible when the client uses the abstract stack \( \text{AS} \) from Fig. 1.

Figure 4. Counter example for contextual trace refinement and quiescent consistency

8 Conclusions

In this paper, we have developed a framework, based on action systems with procedures, for studying the link between the correctness conditions for concurrent objects and contextual trace refinement, which guarantees substitutability of objects within potentially non-terminating reactive clients. Thus, we bring together the previously disconnected worlds of correctness for concurrent objects and trace refinement within action systems. We have shown that linearizability and minimal progress
together ensure contextual trace refinement, but sequential consistency and quiescent consistency are inadequate for guaranteeing contextual trace refinement regardless of whether clients communicate outside the concurrent object. The sequential consistency result contrasts earlier results for observational refinement, where sequential consistency is adequate when clients only communicate through shared objects [9].

We have derived the sufficient conditions for contextual trace refinement using the proof obligations for forwards and backward simulation. However, neither of these conditions have been shown to be necessary, leaving open the possibility of using weaker correctness conditions on the underlying concurrent objects. Studying this relationship remains part of future work — areas of interest include the study of how the correctness conditions for safety of concurrent objects under relaxed memory models [7] can be combined with different scheduler implementations for progress (e.g., extending [15, 13]) to ensure contextual trace refinement.

**Acknowledgements** We thank John Derrick and Graeme Smith for helpful discussions. Brijesh Dongol is supported by EPSRC grant EP/N016661/1 “Verifiably correct high-performance concurrency libraries for multi-core computing systems”.

**References**

1. R. J. R. Back and J. von Wright. Trace refinement of action systems. In *CONCUR ’94: Proceedings of the Concurrency Theory*, pages 367–384. Springer-Verlag, 1994.
2. R. Colvin, S. Doherty, and L. Groves. Verifying concurrent data structures by simulation. *Electr. Notes Theor. Comput. Sci.*, 137(2):93–110, 2005.
3. W. P. de Roever and K. Engelhardt. *Data Refinement: Model-oriented proof methods and their comparison*. Cambridge Tracts in Theor. Comp. Sci. Cambridge Univ. Press, 1996.
4. J. Derrick, G. Schellhorn, and H. Wehrheim. Mechanically verified proof obligations for linearizability. *ACM Trans. Program. Lang. Syst.*, 33(1):4, 2011.
5. S. Doherty. Modelling and verifying non-blocking algorithms that use dynamically allocated memory. Master’s thesis, Victoria University of Wellington, 2003.
6. B. Dongol. *Progress-based verification and derivation of concurrent programs*. PhD thesis, The University of Queensland, 2009.
7. B. Dongol, J. Derrick, G. Smith, and L. Groves. Defining correctness conditions for concurrent objects in multicore architectures. In J. T. Boyland, editor, *ECOOP*, volume 37 of *LIPIcs*, pages 470–494. Dagstuhl, 2015.
8. B. Dongol and L. Groves. Towards linking correctness conditions for concurrent objects and contextual trace refinement. REFINe workshop, 2015. To appear.
9. I. Filipović, P. W. O’Hearn, N. Rinetzky, and H. Yang. Abstraction for concurrent objects. *Theor. Comput. Sci.*, 411(51-52):4379–4398, 2010.
10. A. Gotsman and H. Yang. Liveness-preserving atomicity abstraction. In L. Aceto, M. Henzinger, and J. Sgall, editors, *ICALP(2)*, volume 6756 of *LNCS*, pages 453–465, 2011.
11. J. He, C. A. R. Hoare, and J. W. Sanders. Data refinement refined. In B. Robinet and R. Wilhelm, editors, *ESOP*, volume 213 of *LNCS*, pages 187–196. Springer, 1986.
12. M. Herlihy and N. Shavit. *The Art of Multiprocessor Programming*. Morgan Kauf., 2008.
13. M. Herlihy and N. Shavit. On the nature of progress. In A. Fernández Anta, G. Lipari, and M. Roy, editors, *OPODIS*, volume 7109 of *LNCS*, pages 313–328. Springer, 2011.
14. M. P. Herlihy and J. M. Wing. Linearizability: a correctness condition for concurrent objects. *ACM Trans. Program. Lang. Syst.*, 12(3):463–492, 1990.
15. H. Liang, J. Hoffmann, X. Feng, and Z. Shao. Characterizing progress properties of concurrent objects via contextual refinements. In P. R. D’Argenio and H. C. Melgratti, editors, *CONCUR*, volume 8052 of *LNCS*, pages 227–241. Springer, 2013.
16. N. A. Lynch. *Distributed Algorithms*. Morgan Kaufmann, 1996.
17. G. Schellhorn, J. Derrick, and H. Wehrheim. A sound and complete proof technique for linearizability of concurrent data structures. *ACM TOCL*, 15(4):31:1–31:37, 2014.

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18. K. Sere and M. A. Waldén. Data refinement of remote procedures. *Formal Asp. Comput.*, 12(4):278–297, 2000.
19. R. K. Treiber. Systems programming: Coping with parallelism. Technical Report RJ 5118, IBM Almaden Res. Ctr., 1986.