On the entropy of coverable subshifts

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Abstract. A coloration \( w \) of \( \mathbb{Z}^2 \) is said to be coverable if there exists a rectangular block \( q \) such that \( w \) is covered with occurrences of \( q \), possibly overlapping. In this case, \( q \) is a cover of \( w \). A subshift is said to have the cover \( q \) if each of its points has the cover \( q \). In a previous article, we characterized the covers that force subshifts to be finite (in particular, all configurations are periodic). We also noticed that some covers force subshifts to have zero topological entropy while not forcing them to be finite. In the current paper we work towards characterizing precisely covers which force a subshift to have zero entropy, but not necessarily periodicity. We give a necessary condition and a sufficient condition which are close, but not quite identical.

Keywords: Subshifts · SFT · entropy · quasiperiodicity · coverability

1 Introduction

A subshift is a language of infinite words defined by forbidden factors; for instance, the set of infinite words over \( \{a, b\} \) that do not contain the factors \( bb \) nor \( aaa \) is a subshift. This specific kind of language was initially introduced in the context of dynamical systems [13]; indeed, a subshift equipped with the “shift” action (translate each letter one step to the left) is a topological dynamical system. On one hand, subshifts viewed as dynamical systems can be studied with the tools of combinatorics on words; on the other hand, several systems of interest are conjugate to subshifts (so they share the same topological invariants). Thus subshifts make a useful connection between these two fields.

The definition of a subshift is very easy to generalize to higher dimensions, e.g. to \( \mathbb{Z}^2 \)-words. The two-dimensional case gives rise to a rich theory, that is connected with tilings and computability. Two-dimensional subshifts may also be used to model dynamical systems with two commuting actions.

If a subshift is defined by finitely many forbidden factors, then it has a finite description. Such languages are called subshifts of finite type, or SFT for short. The restriction to finite type is worth considering: two-dimensional SFTs are rich enough to encode objects such as, for instance, Wang tiles, Turing machines [15], or physical models such as the square ice [10]. Most questions that we could ask

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on SFTs (e.g., emptiness, equality) are decidable in dimension 1, but undecidable in dimension 2 and higher.

In this article, we focus on two-dimensional SFTs.

A classical problem is to compute the topological entropy of two-dimensional SFTs. From a dynamics point of view, entropy is the average number of “information bits” encoded in each point of the system. From the combinatorial perspective, on the other hand, entropy is connected with factor complexity, i.e., the number of factors of length \( n \) (or squares of size \( n \times n \) in 2D) occurring in the subshift in function of \( n \). Finally, topological entropy is connected with the notion of residual entropy in physics.

It is not possible to compute the entropy of an arbitrary SFT \( X \), because we need some information about \( X \). This paper considers the class of coverable subshifts. A \( \mathbb{Z}^2 \)-word \( w \) has the cover \( q \) if \( q \) is a rectangular block and \( w \) is covered with occurrences of \( q \), possibly overlapping. A subshift has the cover \( q \) if each of its elements has the cover \( q \). (Note that the term quasiperiodic is sometimes used for coverable in the context of one-dimensional words.)

Our motivations to study coverability are threefold. 1. Overlaps in general are an important part of combinatorics on one-dimensional words. This article is motivated by the larger project to build a combinatorics on two-dimensional words. Other work considering two-dimensional overlaps is also conducted [1], although not in the context of subshifts. 2. The definition of coverability may be later relaxed; for instance, we might allow several covers, where each point of a \( \mathbb{Z}^2 \)-word would have to be covered by one or the other of those covers. By relaxing the definition more and more, we might understand the entropy of larger and larger families of SFTs. However, we have to start this project with the most constrained definition: one cover of rectangular shape. 3. The famous Penrose tilings [14] can be described in terms of a single tile that overlaps itself [9]. In one dimension, the family of standard Sturmian words (which are right-infinite words) can be characterized in terms of covers [5]. These examples show that coverable phenomena are found in otherwise natural examples of words, so it makes sense to port this definition to SFTs.

Coverability was initially defined on finite words, in the context of text algorithms [2]; it was subsequently generalized to infinite words and 1D-subshifts [11,12]. In parallel, its connections with morphisms on words and Sturmian words were thoroughly studied [8] and its references]. All this work in one dimension will provide us with ideas and techniques to study the two-dimensional case, but the generalization to higher dimensions is far from trivial and new ideas are also required.

In two dimensions, an efficient algorithm to find the covers of any finite, square-word is known [3]. Besides, a few properties of coverable 2D SFTs were proven in previous articles, notably about entropy, minimality and uniform frequencies [4,6].

This paper is structured as follows. In Section 2, after a quick review of basic definitions and notation, we prove that the language of \( q \)-coverable configurations is an SFT for all \( q \). Then we try to compute its entropy in function of \( q \).
Since exact values are difficult to obtain, we engage in a simpler task: characterize which \( q \)'s yield zero entropy and which yield strictly positive entropy. In Section 3, we give a necessary condition for strictly positive entropy. Then, in Section 4, we define interchangeable pairs, and show how this notion is useful to compute entropies of coverable subshifts. Finally, in Section 5, we give a sufficient condition for positive entropy, which is close to (but not quite) the negation of the necessary condition; we also give a lower bound on the entropy for subshifts that satisfy the sufficient condition. Our conclusion is Section 6: we give a few open problems and state our acknowledgements.

2 Preliminaries

We start by reviewing definitions and notation. A configuration is a coloring of \( \mathbb{Z}^2 \) whose colors are taken from some finite alphabet \( \Sigma \). A domain is a finite subset of \( \mathbb{Z}^2 \) and a fragment is a coloring of a domain. When we consider a fragment up to translation, i.e., we are not interested in its position in the plane, we call it a pattern. A block is a pattern \( p \) whose domain is a rectangle, i.e., there exists natural (nonnegative) integers \( m, n \) such that \( \text{dom}(p) = \{0, \ldots, m-1\} \times \{0, \ldots, n-1\} \). The number \( m \) is the width and the number \( n \) the height of the rectangle. The position of a block is the position of its bottom, left-hand corner. We note \( \Sigma^{m \times n} \) the set of all blocks of size \( m \times n \) over alphabet \( \Sigma \). Here are the intuitive correspondences with the unidimensional case:

- configuration \( \iff \) infinite word
- pattern, block \( \iff \) finite word
- fragment \( \iff \) occurrence of a finite word in an infinite word

If \( D \) is a set (in particular a domain), then \( |D| \) denotes the cardinality of \( D \). If \( u \) is either a pattern or a fragment, then \( |u| \) denotes the cardinality of \( \text{dom}(u) \).

Let \( u \) denote a pattern, \( w \) a configuration or a pattern, and \( D \) a domain. The notation \( w(D) \) refers to the restriction of \( w \) to domain \( D \). If \( u = w(D) \), (so in particular \( \text{dom}(u) = D \) up to translation), then we say that \( u \) occurs in \( w \).

Let \( f \) denote a fragment. Elements of \( \mathbb{Z}^2 \) are often called positions; if \((i, j)\) belongs to \( \text{dom}(f) \), then we say that \( f \) covers the position \((i, j)\). Two fragments said to be neighbouring if the union of their domains is simply connected (counting only vertical and horizontal neighbours) and if they agree on the intersections of their domains (which might be empty). If moreover the intersection of their domains is not empty, then we say that they overlap.

Definition 1. Let \( q \) denote a block. A fragment, pattern, or a configuration \( w \) is said to be \( q \)-coverable if each position of its domain is covered by a copy of \( q \). Formally, there exist domains \( D_1, \ldots, D_n \) (possibly \( n = \infty \)) such that \( \text{dom}(w) = \bigcup_{i=1}^n D_i \) and \( w(D_i) \) is equal to \( q \) up to translation for all \( i \).

Definition 2. A set of configurations \( X \) is a subshift if and only if there exists a set of patterns \( \mathcal{F} \) such that \( X \) is the set of configurations in which no element of \( \mathcal{F} \) occur, i.e., \( X = \{ x \in \Sigma^\mathbb{Z}^2 \mid \forall D \subseteq \mathbb{Z}^2, x(D) \notin \mathcal{F} \} \).
Note that two different sets of forbidden patterns might yield the same subshift. If $\mathcal{F}$ can be made finite, then $X$ is said to be a subshift of finite type (or SFT for short).

Each subshift is stable by translation: if $X$ is a subshift, $x \in X$, $y \in \Sigma^{\mathbb{Z}^2}$ and there is a vector $v$ such that $y(u) = x(u + v)$ for all $u$, then $y \in X$.

**Proposition 3.** Given a block $q$, the set of all $q$-coverable configurations is a subshift of finite type, that we note $X_q$.

**Remark.** Let $w$ denote a fragment or a configuration and $(x, y)$ the position of a copy of $q$ in $w$. Then that copy of $q$ covers the position $(0, 0)$ if and only if $-|w| + 1 \leq x \leq 0$ and $-|h| + 1 \leq y \leq 0$.

**Proof (of Proposition 3).** Let $(w, h)$ denote the dimensions of dom($q$). Define the domain $D = \{-|w| + 1, \ldots, |w| - 1\} \times \{-|h| + 1, \ldots, |h| - 1\}$ and $\mathcal{F}$ the set of patterns with domain $D$ that do not contain any occurrence of $q$. We show that $q$-coverable configurations are exactly the configurations that avoid the patterns in $\mathcal{F}$.

In an arbitrary configuration $w$ covered by $q$, each block of size $(2|w| - 1, 2|h| - 1)$ contains at least one occurrence of $q$. Suppose not; we can assume without loss of generality that our faulty block is at position $(-|w| + 1, -|h| + 1)$. Then, by the remark above, the position $(0, 0)$ is not covered by $q$ in $w$: a contradiction.

Conversely, if $w$ is a configuration not covered by $q$, then some pattern from $\mathcal{F}$ occurs in it. Indeed, there exists a position in $w$ which is not covered by $q$; assume without loss of generality that this position is $(0, 0)$. Then by the remark above, $w(D)$ does not contain any occurrence of $q$, so it belongs to $\mathcal{F}$.

As a conclusion, $X_q$ is the set of configurations that do not contain any pattern in $\mathcal{F}$. Since all forbidden blocks have domain $D$, which is a finite set, the set $\mathcal{F}$ itself is finite, so $X_q$ is a subshift of finite type.

**Definition 4.** If $r$ is a block and $m, n$ natural integers, we call $r^{m \times n}$ the pattern made of $m$ copies of $r$ concatenated horizontally, repeated $n$ times vertically. If $q$ can be written $r^{n \times m}$ for some strictly positive integers $m, n$, then we say that $r$ is a root of $q$. Since $q = q^{1 \times 1}$, the block $q$ is always a root of $q$. If $q$ has no root besides itself, we say that it is primitive.

When $q$ is nonprimitive, it has an unique primitive root which is root of all other roots [4, Lemma 4].

**Definition 5.** If $w$ and $u$ are two different blocks such that $u$ occurs in two opposite corners of $w$, then $u$ is called a border of $w$.

For instance, $a, b$ and $b^{2 \times 2}$ are borders of $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$. Observe that a block is never a border of itself, but the empty block is a border of all blocks.

**Theorem 6 (See [4, Theorem 5]).** Let $q$ denote a block; then $X_q$ is infinite if and only if the primitive root of $q$ has a nonempty border.
Generally speaking we are interested in getting more results which describe $X_q$, given properties on $q$. In this paper, we focus on the topological entropy which is, intuitively, the average number of bits necessary to encode one cell of one configuration in $X$.

**Definition 7.** Let $X$ denote a subshift and $L(X)$ the set of all blocks occurring in a configuration of $X$, i.e., $L(X) = \{p \mid \exists x \in X, \ p \text{ occurs in } x\}$. Define $L_{m,n}(X)$ to be the set of blocks of size $m \times n$ occurring in $X$, that is to say, $L_{m,n}(X) = L(X) \cap \Sigma^{m \times n}$. The topological entropy of $X$ is the number:

$$h(X) = \lim_{n \to \infty} \frac{\log_2(|L_{n,n}(X)|)}{n^2}$$

This limit always exists. (In the one-dimensional case, the sequence $\log(|L_n(X)|)$ is subadditive, so by the subadditivity lemma $\log(|L_n(X)|)/n$ converges. The passage to higher dimensions boils down to a computation.) Observe that if $|L_{n,n}(X)| \sim \varepsilon n^2$ for some positive $\varepsilon$, then $h(X) = \log_2 \varepsilon$. Otherwise, if $|L_{n,n}(X)|$ grows slower than any $\varepsilon n^2$, then $h(X) = 0$. Thus the maximal value for the topological entropy is $\log_2 |\Sigma|$, because there is at most $|\Sigma|^{n^2}$ elements in $L_{n,n}(X)$.

### 3 A necessary condition for strictly positive entropy

**Definition 8.** If a block has two borders in consecutive corners that are neighbouring (see Figure 1), then we say that it satisfies the condition $(\ast)$.

![Fig. 1. Illustration of the condition $(\ast)$.](image)

Let $q$ denote an arbitrary block. If $q$ has a full-width or full-height border, i.e. a border having the same width (resp. the same height) as $q$ itself, then $q$ satisfies the condition $(\ast)$. Indeed, this full-width or full-height border occurs in two consecutive corners, and it obviously overlaps with itself.

**Theorem 9.** Let $q$ denote a primitive block. If $X_q$ has positive entropy, then $q$ satisfies the condition $(\ast)$.

The contraposition is also interesting: if $q$ does not satisfy the condition $(\ast)$, then $X_q$ has zero entropy.

If $q$ is nonprimitive, then it has a full-width or a full-height border and thus it satisfies condition $(\ast)$. Besides, if $q$ covers some pattern or configuration $w$, then
so does each root of \( q \). Therefore it makes sense to restrict ourselves to primitive covers. We could lift this restriction by replacing, in condition (\( \ast \)) and elsewhere, the term \textit{border} with \textit{border that is not a power of the primitive root}. This would require extra precautions in the proofs while not adding any significant value to our results, so we will keep the supposition that \( q \) is primitive.

The remainder of this section is devoted to the proof of Theorem \( \ref{thm:main} \).

**Definition 10.** Let \( w \) denote a configuration with a cover \( q \). We note by \( \rho(i, j) \) the topmost among the rightmost occurrences of \( q \) covering position \((i, j)\).

This means that we first select the rightmost copies of \( q \) covering position \((i, j)\), and then among them (if there are several) we select the topmost one. Observe that if \( q \) does not satisfy the condition (\( \ast \)), then the “rightmost occurrence of \( q \) covering position \((i, j)\)” is unique: we don’t need to select the topmost one. Indeed, if we had two rightmost occurrences covering \((i, j)\), they would share the same \( x \)-coordinate, and either they would be equal, or \( q \) would have a full-width border: as explained above, this implies the condition (\( \ast \)).

The vector \( \rho \) implicitly depends on \( q \) and \( w \), but there will be no ambiguity in what follows.

**Lemma 11.** Let \( q \) denote a primitive block of size \( m \times n \) that does not satisfy condition (\( \ast \)), and \( w \) a configuration covered by \( q \). Then \( \rho(0, 0) \) is either equal to \((0, 0)\), or to \( \rho(-1, 0) \), or to \( \rho(0, -1) \). In particular, \( \rho(0, 0) \) is uniquely determined by \( \rho(-1, 0) \) and \( \rho(0, -1) \).

**Proof.** Call \((x, y)\) and \((x', y')\) the positions of \( \rho(-1, 0) \) and \( \rho(0, -1) \), respectively, and \((i, j)\) the position of \( \rho(0, 0) \). We have \( i \leq 0 \) and \( j \leq 0 \), otherwise \( \rho(0, 0) \) would not cover the position \((0, 0)\). There are several cases to consider.

**Case 1.** If \((x, y) = (x', y')\), then necessarily we have \((x, y) = (x', y') = (i, j)\) by definition of \( \rho \) and because \( \text{dom}(q) \) is convex (it is a rectangle).

**Case 2.** If \( \rho(-1, 0) \) (respectively \( \rho(0, -1) \)) contains \((0, 0)\), then \( \rho(0, 0) = \rho(-1, 0) \) (respectively \( \rho(0, 0) = \rho(0, -1) \)) so \( \rho(0, 0) \) is uniquely determined.

Now we have to consider the cases where the occurrences \( \rho(0, -1), \rho(-1, 0) \) and \( \rho(0, 0) \) are all different. We will prove that, in this situation, the position of \( \rho(0, 0) \) is always \((0, 0)\). There are two disjoint cases to consider: either \( \rho(0, -1) \) and \( \rho(-1, 0) \) overlap, or they don’t.

**Case 3.** Suppose that \((x, y), (x', y')\), and \((i, j)\) are all different and that the occurrences of \( q \) at positions \((x, y)\) and \((x', y')\) overlap. We are in the situation of Figure \( \ref{fig:overlap1} \) where coordinate \((0, 0)\) is marked by the symbol \( \circ \). If \( i = 0 \) and \( j < 0 \), then Figure \( \ref{fig:overlap2} \) shows how \( q \) would satisfy the condition (\( \ast \)): the hatched areas are the two consecutive borders which are neighbouring. If \( i < 0 \) and \( j = 0 \), then Figure \( \ref{fig:overlap3} \) shows how \( q \) would satisfy the condition (\( \ast \)). Finally, if both \( j < 0 \) and \( i < 0 \), then we are in the situation of Figure \( \ref{fig:overlap4} \). Consider \((x'', y'') = \rho(x' - 1, y - 1)\): the coordinates \((x' - 1, y - 1)\) are shown by the symbol \( \bullet \) on Figure \( \ref{fig:overlap4} \). Then the figure shows how the condition (\( \ast \)) would be satisfied again. In this figure,
Case 3. The origin of the occurrence covering $\circ$ is forced to be $\circ$, otherwise the condition (*) is satisfied (see hatched lines).

Case 4. The origin of the occurrence covering $\circ$ is forced.

Case 5. The origin of the occurrence covering $\circ$ is forced.

Case 6. The origin of the occurrence covering $\circ$ is forced.
We prove the contraposition of the theorem: if \( \rho \) and \( \rho \) satisfy the condition \((\ast)\) as shown by Figure 16. The only remaining possibility is on Figure 15. If either \( k < s \) or \( y = n \) (as in Figure 10), or both (as in Figure 14). As before, we prove that \( \rho(0,0) = (0,0) \).

**Case 4.** Suppose that \( \rho(-1,0), \rho(0,-1), \) and \( \rho(0,0) \) are all different, that \( \rho(-1,0) \) and \( \rho(0,-1) \) do not overlap and that \( x' = x = m \). We are in the situation of Figure 8. If \( i < 0 \) and \( y \leq j < 0 \) then the condition \((\ast)\) is satisfied, as shown on Figure 7. If \( i < 0 \) and \( j < y \), then let \( (x'',y'') \) denote the position of \( \rho(x'-1,y-1) \); the coordinates \((x'-1,y-1)\) are marked by the symbol \( \bullet \) on Figure 8. This figure shows how the condition \((\ast)\) is satisfied (on the figure, the minimal \( x'' \) and \( y'' \) are represented, but the argument still works if \( x'' \) or \( y'' \) or both are larger). If \( i = 0 \) and \( j < 0 \) then \( q \) has a full-width border, as shown on Figure 9, so the condition \((\ast)\) is also satisfied. The only remaining case is \((i,j) = (0,0)\).

**Case 5.** Suppose that \( \rho(-1,0), \rho(0,-1), \) and \( \rho(0,0) \) are all different, that \( \rho(-1,0) \) and \( \rho(0,-1) \) do not overlap and that \( y' - y = n \). We are in the situation of Figure 10. The reasoning is similar to the previous case, but the roles of \( x \) and \( y \) are swapped; see Figures 11 and 12.

**Case 6.** \((x,y) = (m,0)\) and \((x',y') = (0,-n)\); we are in the situation of Figure 14. We show that \((i,j) = (0,0)\), so suppose towards a contradiction that \((i,j) \neq (0,0)\), as on Figure 15. Let \( (k,\ell) = \rho(i+m,0) \) and \( (k',\ell') = \rho(0,j+n) \); the coordinates \((i+m,0)\) and \((0,j+n)\) are respectively shown as a \( \bullet \) and a \( \ast \) on Figure 15. If either \( k < 0 \) or \( \ell' < 0 \), then the condition \((\ast)\) would be satisfied, as shown by Figure 16. The only remaining possibility is \( k = \ell' = 0 \). But then the condition \((\ast)\) is satisfied again, as shown on Figure 17.

**Conclusion.** We showed that, in each case, either \( \rho(0,0) = (0,0) \), or \( \rho(0,0) = \rho(-1,0), \) or \( \rho(0,0) = \rho(0,-1) \). Thus, \( \rho(0,0) \) is uniquely determined by \( \rho(-1,0) \) and \( \rho(0,-1) \), and the lemma is proved.

**Proof (of Theorem 9).** We prove the contraposition of the theorem: if \( q \) does not satisfy the condition \((\ast)\), then \( X_q \) has zero topological entropy. Recall that \( q \) is of size \( m \times n \). Consider an arbitrary square occurring in a configuration of \( X_q \), i.e., an element of \( L_{k,k}(X_q) \) for some \( k \in \mathbb{N} \). This square appears in a configuration \( w \) in \( X_q \), and we can assume without loss of generality that it has position \((0,0)\). (Indeed, if a configuration belongs to \( X_q \), so do all the translations of that configuration.) We will bound the number of possibilities for such a square.

Let \( I \) denote \( \{(-k,k),(-k+1,k-1),\ldots,(k,-k)\} \) (the darkest cells in Figure 18). Suppose that we know \( \rho(i) \) for each \( i \) in \( I \). By applying Lemma 11 several times, we can uniquely determine \( \rho(i') \) for each \( i' \) in \( I' = \{(-k+1,k),\ldots,(k,-k+1)\} \) (this is \( I \) shifted one cell to the right, minus the bottommost cell). This information, in turn, determines \( \rho(i'') \) for each \( i'' \) in \( I'' = \{(-k+2,k),\ldots,(k,-k+2)\} \), and so on. By iterating this process, we deduce the contents of the whole square \( \{0,\ldots,k-1\} \times \{0,\ldots,k-1\} \), and even a bit more (see Figure 18). The area that we can determine is not shaped like a square.
in general, and there might have several \( k \times k \) squares in it. We can locate the desired square with two coordinates \((x, y)\) satisfying \(0 \leq x, y \leq 2k + 1\).

![Fig. 18](image-url)

\textbf{Fig. 18.} From \(\rho(I)\) we deduce \(\rho(I'), \rho(I''), \ldots\) and in the end we know the contents of the square.

For each \(i\) in \(I\), there are \(nm\) possibilities for \(\rho(i)\), so we can compute a bound on the number of \(k \times k\) squares in \(X_q\); we have not more than \(u_k = (2k + 1)^2 \times (nm)^{2k+1}\) such squares. The sequence \(\log(u_k)/k^2\) converges to 0 as \(k\) grows to infinity, so the entropy of \(X_q\) must be zero. The theorem is proved.

\section{Interchangeable pairs}

Now our goal is to give a sufficient condition on a block \(q\) to force \(X_q\) to have strictly positive entropy. We use a tool called \textit{interchangeable pairs}, which we define now. In what follows, a \textit{q-patch} is a \(q\)-coverable pattern.

\textbf{Definition 12.} An \textit{interchangeable pair} for \(q\) is a pair of different \(q\)-patches with the same domain.

Let \(p_1, p_2\) be an interchangeable pair for \(q\) and \(w\) a configuration in \(X_q\). Any occurrence of \(p_1\) in \(w\) can be replaced with an occurrence of \(p_2\); the result would still be a configuration of \(X_q\) (and different from \(w\)). Hence the name interchangeable pair.

\textbf{Proposition 13.} Let \(q\) denote a primitive block. If the subshift \(X_q\) has strictly positive entropy, then there exists an interchangeable pair for \(q\).

\textit{Proof.} By contraposition: suppose that there is no interchangeable pair for \(q\). Let \(n\) denote an integer and \(Y\) the set of \(q\)-patches whose domains contain a square of size \(n \times n\) and minimal for this property. In other terms, each \(y\) in \(Y\) contains a square of size \(n \times n\), but if we remove one occurrence of \(q\) from \(y\), then it is not the case anymore. Since there is no interchangeable pair for \(q\), each element of \(Y\) is uniquely determined by the shape of its domain, thus \(Y\) contains not more than \(|q|^{4n}\) elements. Ineed, the shape can be uniquely determined by attaching,
for each cell in the frontier of the \( n \times n \)-square, the position of an occurrence of \( q \) relative to the cell. Any block \( c \) in \( L_{n,n}(X_q) \) is uniquely determined by an element of \( Y \) and a pair of coordinates \((x, y)\) satisfying \( 0 \leq x, y \leq n + 2|q| - 1 \), so there are not more than \( v_n = (n + 2|q|)^2 \times |q|^{4n} \) possibilities for \( w \). The sequence \( \log(v_n)/n^2 \) converges to 0 as \( n \) grows to infinity, thus the entropy of \( X_q \) is zero.

**Proposition 14.** Suppose that there exists an interchangeable pair \((p_1, p_2)\) for \( q \). Suppose further that there are a configuration \( w \) in \( X_q \), strictly positive integers \( k, \ell \), and domains \( (D_i)_{i \in \mathbb{N}} \) such that \( \cup_i D_i = \mathbb{Z}^2 \), that each \( D_i \) is a rectangle of size \((k, \ell)\), and that for each \( i \) the fragment \( w(D_i) \) contains either an occurrence of \( p_1 \) or of \( p_2 \). Then the entropy of \( X_q \) is at least \((k\ell)^{-1}\).

Note that the condition of this proposition is satisfied, in particular, when the shape of the interchangeable pair tiles the plane by translation.

**Proof (of Proposition 14).** Let \( t \) denote a natural integer and \( c \) an arbitrary block of size \( t \times t \) in \( w \). By the assumptions on \( w \), there is at least \( u_t = (t^2 - 2) \times (t^2 - 2) \) occurrences of \( \{p_1, p_2\} \) in \( c \). Since each occurrence of \( p_1 \) can be swapped to an occurrence of \( p_2 \), and vice-versa, without leaving \( L(X_q) \), we have at least \( 2^{u_t} \) blocks of size \( t \times t \) in \( L(X_q) \). Compute:

\[
\lim_{t \to \infty} \frac{\log(2^{u_t})}{t^2} = \lim_{t \to \infty} \frac{1}{t^2} \left( \frac{t^2}{k\ell} - \frac{2t}{k} - \frac{2t}{\ell} + 4 \right) = (k\ell)^{-1}.
\]

The proposition is proved.

## 5 A sufficient condition for strictly positive entropy

Ideally we would like to prove the converse of Theorem 9, in order to have a necessary and sufficient condition on the cover to get a strictly positive entropy coverable subshift. However, it is not clear whether the condition of this theorem is actually sufficient; it might be too weak. We define another condition on \( q \) which is stronger than the condition \((\ast)\) and which is sufficient for positive entropy. We prove the following theorem.

**Theorem 15.** Let \( q \) denote a primitive block with size \((w, h)\). Suppose that \( q \) has two borders \( b_1 \) and \( b_2 \) in opposite corners, such that:

1. either \( \text{width}(b_1) = \text{width}(b_2) \) and \( \text{height}(b_1) + \text{height}(b_2) \geq \text{height}(q) \),
2. or \( \text{height}(b_1) = \text{height}(b_2) \) and \( \text{width}(b_1) + \text{width}(b_2) \geq \text{width}(q) \);

then \( X_q \) has entropy at least \((9wh)^{-1}\).

Figure 19 illustrates the condition of Theorem 15.

**Proof (of Theorem 15).** Without loss of generality, suppose \( q \) satisfies Condition 1 of the theorem. Figure 20 shows an interchangeable pair for \( q \). Indeed, one easily checks that both patterns have the same domain and are \( q \)-coverable.
Moreover they are different, otherwise we would have \((b_1 \oplus b_2) \oplus q = q \oplus (b_2 \oplus b_1)\), with \(\oplus\) denoting horizontal concatenation and \(\oplus\) denoting vertical concatenation. This situation implies that \(q\) is not primitive by [7, Theorem 3]. This pair tiles the plane (see Figure 21), so Proposition 14 implies that \(X_q\) has strictly positive entropy. Moreover, the tiling on Figure 21 shows that there is at least one occurrence of the pair in each rectangle of size \((3w, 3h)\), hence the bound on the entropy.

6 Conclusion

Results. We showed that the set of \(q\)-coverable \(\mathbb{Z}^2\)-words is a subshift of finite type, for all \(q\). Then we gave a necessary condition and a sufficient condition on \(q\) for this subshift to have strictly positive entropy. These conditions were not quite identical, but close; we also gave a lower bound on the entropy when the sufficient condition is satisfied. This lower bound used the concept of interchangeable pair; we showed that any coverable subshift with strictly positive entropy has interchangeable pairs.
Open problems. Our work may be extended in various directions. The first direction is to generalize the notion of coverability: allow non-rectangular shapes, allow two covers instead of one (as a disjunction of covers), or even allow some minimal distance between the covers ("negative overlaps", in a sense). Connections with the recurrence function could be established.

Another direction of further work is to close the gap between our necessary and our sufficient condition, and to give more precise bounds on the value of the entropy in function of $q$.

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