A simplified version of the ‘Axis of Evil Theorem’ 
for distinct points.

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Abstract

Given a finite set $X$ of distinct points, Marinari-Mora’s ‘Axis of Evil 
Theorem’ states that a combinatorial algorithm and interpolation enable 
to find a ‘linear’ factorization for a lexicographical minimal Groebner 
basis $G(I(X))$ of the zero-dimensional radical ideal $I(X)$.
In this work we provide such algorithm, showing that it ends in a finite 
number of steps and that it actually provides the correct result.
The ‘Axis of Evil’ algorithm takes as input the monomial basis of the 
initial ideal $T(I(X))$ but its starting point is the (finite) Groebner escalier $N$ (obtained via Cerlienco-Mureddu correspondence) so we will also define 
the ‘potential expansion’ algorithm, a combinatorial algorithm which computes the minimal basis from a finite Groebner escalier.

Keywords: Groebner basis , Combinatorial algorithm, Interpolation.

1 Introduction.

Marinari-Mora in [10], [9], [11] gave a deep description of the structure of a 
zero-dimensional ideal $I$ described by giving its Macaulay basis $B(I)$ ([16]); in 
picular they enhanced the description of the Grobner basis of an ideal in $K[X,Y]$ given by Lazard in [8] proving that in a restricted case which includes 
the radical one, for each monomial $\tau := X_{d_1} \cdots X_{d_n}$ belonging to the mini-
mal basis $G(I)$ of the initial ideal of $I$, it is possible to produce linear factors 
$\gamma_{m\delta\tau} := X_i - f(X_1,\ldots,X_i-1), 1 \leq m \leq n, 1 \leq \delta \leq d_m$ such that the polynomi-
als $f_\tau := \prod_{m=1}^{n} \prod_{\delta=1}^{d_m} \gamma_{m\delta\tau}$ form a minimal lexicographical Groebner basis of $I$;

Such algorithm is reported and proved in [16]; later Mora in a series of lecture 
notes labelled the restriction of this decomposition and interpolation to the case 
of a set of distinct points as ‘Axis-of-Evil’ theorem and gave a precise description, 
but no simple proof, of the result stated in [16]; S. Steidel implemented 
the procedure in Singular [6], [18].

We give here such explicit algorithm that, given a finite set $X$ of distinct points, 
provides a complete decomposition $X = \bigsqcup_{m=1}^{n} \bigsqcup_{\delta=1}^{d_m} S_{m\delta}(\tau)$ on which, applying Cerlienco-Mureddu algorithm and interpolation, produces the required linear
factorization for a lexicographical minimal Groebner basis $F = \{f_1, ..., f_r\}$ of the ideal $I(\mathbf{X})$ and thus a very simple proof of the ‘Axis-of-Evil’ theorem in this particular situation. This algorithm arranges the $r$ terms $t_i$ belonging to $G(I(\mathbf{X}))$ with respect to lex ($t_1 \leq ... \leq t_r$) and constructs the factorization of each $f_i \in F$ through a suitable interpolation on a subset $S_{\Delta \delta}(t_i)$ of $\mathbf{X}$ depending on the exponents of the corresponding $t_i$. More precisely, Cerlienco-Mureddu give an algorithm that enables to find the Groebner escalier $N(I(\mathbf{X}))$ and the minimal basis $G(I(\mathbf{X}))$ of the monomial ideal $T(I(\mathbf{X}))$.

The ‘Axis-of-Evil’ algorithm’s starting point are the elements of $\mathbf{X}$ and the monomials of the finite Groebner escalier $N$ (computed using Cerlienco-Mureddu algorithm), but the algorithm requires as input the monomial basis of $T(I(\mathbf{X}))$, we also define the ‘potential expansion’ algorithm.

It takes $N$ and computes the minimal basis.

I noted here that Marinari-Mora explicitly deduced, as trivial corollaries of their ‘Axis-of-Evil’ procedure, Lazard theorem ([8]), Elimination theorem ([2]), Kalkbrener theorem ([13]), part of Gianni-Kalkbrener theorem ([7], [12]); they however remarked that, having being strongly influenced by Gianni-Kalkbrener result, they cannot dismiss the possibility that Gianni-Kalkbrener argument is an essential tool of their proof of the ‘Axis-of-Evil’ theorem.

## 2 Notation.

Let $P := k[x_1, ..., x_n] = \bigoplus_{d \in \mathbb{N}} P_d$ be the ring of polynomials in $n$ variables and coefficients in the base field $k$. For all $M \subseteq P$, $M_d = M \cup P_d$ is its degree $d$ part. Call $\mathcal{T}$ the semigroup of terms, generated by the set $\{x_1, ..., x_n\}$:

$\mathcal{T} := \{x_1^{a_1} \cdots x_n^{a_n}, (a_1, ..., a_n) \in \mathbb{N}^n\}$.

Letting $\alpha = (a_1, ..., a_n) \in \mathbb{N}^n$, we will often write $x^\alpha$ instead of $x_1^{a_1} \cdots x_n^{a_n}$.

Define also the set

$T[m] := \mathcal{T} \cap k[x_1, ..., x_m] = \{x_1^{a_1} \cdots x_m^{a_m}, (a_1, ..., a_m) \in \mathbb{N}^m\}$.

For each semigroup ordering $<$ on $\mathcal{T}$ (i.e. a total ordering such that $t_1 < t_2 \Rightarrow t_1 < tt_2$, $\forall t, t_1, t_2 \in \mathcal{T}$) we can represent a polynomial $f \in P$ as a linear combination (with coefficients in $k$) of monomials arranged w.r.t. $<$:

$$f = \sum_{t \in \mathcal{T}} c(f, t) = \sum_{i=1}^s c(f, t_i) t_i : c(f, t_i) \in k^*, t_i \in \mathcal{T}, t_1 < ... < t_s.$$ 

We will call $T(f) = Lt(f) := t_1$ the leading term of $f$ and $tail(f) = f - T(f)$ the tail of $f$.

We can also express it in a unique way as

$$f = \sum_{i=0}^\delta g_i x_i \in k[x_1, ..., x_{n-1}][x_n], g_i \in k[x_1, ..., x_{n-1}], g_\delta \neq 0$$

(where $\delta := deg_n(f)$ is the degree w.r.t. $x_n$).

We denote $L_p(f) := g_\delta$, the leading polynomial of $f$.
Definition 2.1. For each monomial \( t \in T \) and \( x_j | t \), the only \( u \in T \) such that \( t = x_j u \) is called \( j \)-th predecessor of \( t \).

A subset \( N \subseteq T \) is an order ideal if
\[
t \in N \Rightarrow s \in N \forall s | t.
\]
Let \( N \subset T \) an order ideal. A subset \( N \subseteq T \) is an order ideal if and only if \( T \setminus N = J \) is a semigroup ideal (i.e. \( \tau \in J \Rightarrow t\tau \in J, \forall t \in T \)).
We set \( N(J) := T \setminus T(J) = N \).

For a semigroup ideal \( J \), \( G(J) \) denotes its minimal basis and
\[
G(J) := \{ \tau \in J \mid \text{each predecessor of } \tau \in N(J) \} = \\
\{ \tau \in T \mid N(J) \cup \{ \tau \} \text{ order ideal, } \tau \notin N(J) \}.
\]
For all subsets \( G \subset P \), we define \( T\{G\} := \{ t | g \in G \} \) and we call \( T(G) \) the semigroup ideal \( \{ \tau T(g), \tau \in T, g \in G \} \). For any ideal \( I \subset P \) consider the semigroup ideal \( T(I) = T\{I\} \), denoting by abuse of notation \( G(I) \) its minimal basis \( G(I) \) and the border ideal of \( I \)
\[
B(I) := \{ x_h t, 1 \leq h \leq n, t \in N(I) \} \setminus N(I) = \\
= T(I) \cap (\{1\} \cup \{ x_h t, 1 \leq h \leq n, t \in N(I) \}).
\]
We will always consider the lexicographic order induced by \( x_1 < ... < x_n \), i.e:
\[
x_1^{a_1} \cdots x_n^{a_n} < x_1^{b_1} \cdots x_n^{b_n} \Leftrightarrow \exists j | a_j < b_j, a_i = b_i, \forall i > j.
\]
This is a term order, that is a semigroup ordering such that 1 lower to every variable or, equivalently, it is a well ordering.

Lemma / Definition 2.2. We have:
1. \( P \cong I \oplus k[N(I)] \);
2. \( P/I \cong k[N(I)] \);
3. \( \forall f \in P, \exists g := \text{Can}(f, I) = \sum_{t \in N(I)} \gamma(f, t, <) t \in k[N(I)] \), called canonical form of \( f \) with respect to \( I \), such that \( f - g \in I \).

Definition 2.3. Given a term order \( < \) on \( T \):
1. a Groebner basis of \( I \) is a set \( G \subset I \) such that \( T(G) = T\{I\} \), that is \( T\{G\} \) generates the semigroup ideal \( T(I) = T\{I\} \);
2. a minimal Groebner basis is a Groebner basis such that divisibility relations among the leading monomials of its members do not exist;
3. the unique reduced Groebner basis of \( I \) is the set:
\[
\mathcal{G}(I) := \{ \tau - \text{Can}(\tau, I) : \tau \in G(I) \}.
\]
Each member of the reduced Groebner basis has a leading term which does not divide any monomial of another member.
Let \( X = \{P_1, ..., P_N\} \subset k^n \) be a finite set of distinct points
\( P_i := (a_{i1}, ..., a_{in}), i = 1, ..., N \).

We call
\[
I(X) := \{ f \in P : f(P_i) = 0, \forall i \},
\]
the \textit{ideal of points} of \( X \).

Finally, we define the projection maps:
\[
\pi_m : k^n \to k^m \quad (X_1, ..., X_n) \mapsto (X_1, ..., X_m)
\]
and, for \( P \in k^n, X \subset k^n \), let
\[
\Pi_s(P, X) := \{ P_i \in X / \pi_s(P_i) = \pi_s(P) \},
\]
\[
\Pi^s(P, X) := \{ P_i \in X / \pi^s(P_i) = \pi^s(P) \},
\]

extending in the obvious way the meanings of \( \pi_s(d), \pi^s(d), \Pi_s(d, D), \Pi^s(d, D) \)
to \( d \in \mathbb{N}^n \subset k^n \) and \( D \subset \mathbb{N}^n \subseteq \mathbb{N}^n \).

With the same notation \( \pi_m \) we indicate also
\[
\pi_m : T \cong \mathbb{N}^n \to \mathbb{N}^m \cong T[m]
\]
\[
x_1^{a_1} \cdots x_n^{a_n} \mapsto x_1^{a_1} \cdots x_m^{a_m}.
\]

3 Cerlienco-Mureddu Correspondence.

Cerlienco and Mureddu ([3], [4], [5]) provided an algorithm which solves the following
\textbf{Problem:} Given finite ordered set of distinct points
\[
\underline{X} := (P_1, ..., P_N) \subset k^n; \; P_i := (a_{i1}, ..., a_{in})
\]
compute a monomial basis (w.r.t. the lexicographic order induced by \( x_1 < ... < x_n \)) of the quotient \( k[x_1, ..., x_n]/I(\underline{X}) \), where \( \underline{X} \) denotes the support \( \{P_1, ..., P_N\} \) of \( \underline{X} \).

More precisely, they

- define the operator \( \Phi \), associating to \( \underline{X} \) an ordered set
\[
\Phi(\underline{X}) := (d_1, ..., d_N) \subset \mathbb{N}^n
\]
such that \( |\Phi(\underline{X})| = |\underline{X}| = N \) and such that, for all \( m < N \) the subset \( (d_1, ..., d_m) \) is exactly \( \Phi((P_1, ..., P_m)) \).

- define the \textit{\( \sigma \)-value} w.r.t. \( \underline{X} \) \( s = \sigma(P, \underline{X}) \) of a point \( P \in k^n \setminus \underline{X} \) as the maximal integer such that \( \Pi_{s-1}(P, \underline{X}) \neq \emptyset \) (by convention, \( \forall P, \underline{X}, \Pi_0(P, \underline{X}) \neq \emptyset \)).

For \( P \notin \underline{X} \), they define the set
\[
\Sigma(P, \underline{X}) := \{ P_i \in \underline{X} / \pi_{s-1}(P_i) = \pi_{s-1}(P), \; s = \sigma(P, \underline{X}) \} \]
containing all the points of $\mathbf{X}$ having the first $s - 1$ coordinates equal to those of $P \notin \mathbf{X}$. They extend the notation to the case $P = P_j \in \mathbf{X}$ in the following way:

$$\sigma(P, \mathbf{X}) := \sigma(P, \{P_1, \ldots, P_{j-1}\})$$

$$\Sigma(P, \mathbf{X}) := \Sigma(P, \{P_1, \ldots, P_{j-1}\}).$$

**Remark 3.1.** Given a term order $\triangleright$, a monomial basis for $A := k[x_1, \ldots, x_n]/I(\mathbf{X})$,

$$[x_1^{i_1}], \ldots, [x_n^{i_n}], \text{ with } x_1^{i_1} \preceq \ldots \preceq x_n^{i_n}$$

is called minimal with respect to the term order if, for every other monomial basis $[x_1^{j_1}], \ldots, [x_n^{j_n}]$, with $x_1^{j_1} \preceq \ldots \preceq x_n^{j_n}$ for the $A$ it holds

$$\forall j = 1, \ldots, N, \, x_1^{i_1} \preceq x_1^{j_1}.$$ 

In [3], they state that the computed monomial basis is the minimal one.

**Proposition 3.2.** (3)

Let $D := \Phi(\mathbf{X})$. Then $\{[x^d] | d \in D\}$ is a monomial basis for $k[x_1, \ldots, x_n]/I(\mathbf{X})$.

Such a monomial basis is minimal with respect to the given $<$. 

Once the Groebner escalier $N$ is known, it is very simple to compute the minimal basis $G$ of $T(I(\mathbf{X})) = T \setminus N$. Given the set $\mathbf{X}$, the first step to compute the linear factorization of a minimal Groebner basis will be to apply Cerlienco-Mureddu algorithm to $\mathbf{X}$ and compute $N_i$ in order to obtain $G$.

### 4 The potential expansion’s algorithm.

Consider the polynomial ring $k[x_1, \ldots, x_n]$ with usual ordering $<$. Given a finite set of distinct points $\mathbf{X} = \{P_1, \ldots, P_N\}$, consider the ideal $I(\mathbf{X}) \triangleleft k[x_1, \ldots, x_n]$ which is radical and zerodimensional, so its Groebner escalier $N$ is a finite set.

We will compute the minimal monomial basis $G$ of $T(I(\mathbf{X}))$, given the Groebner escalier. The algorithm actually provides correct results irrespective of the given term ordering, but since we use Cerlienco-Mureddu correspondence, we will consider only our lex order.

In order to make the reasoning clear, we will represent the monomials using the same diagrams introduced in [13] to study properties of Borel ideals.

Apply Cerlienco-Mureddu correspondence to $\mathbf{X}$ in order to have $N(\mathbf{X}) = \{\tau_1, \ldots, \tau_N\}$. It is well known (see, for instance [14]) that $|N(\mathbf{X})| = |\mathbf{X}|$.

We first define the potential expansion of a subset $H \subset T$, from which the algorithm bears its name.

**Definition 4.1.** Let $H \subseteq T_j$ for some $j \in \mathbb{N}^*$ we set $C^{(0)}(H) := H$ and, for all $l \in \mathbb{N}^*$, $C^{(l)}(\tau) = T_{j+l} \setminus \{x_1, \ldots, x_n\} \cdot (T_{j+l-1} \setminus C^{(l-1)}(H)).$

We then slice the Groebner escalier by degree, having $N_0, N_1, \ldots, N_h$, where $h$ is the maximal degree of terms appearing in $N$.

The minimal monomial basis $G(I(\mathbf{X}))$ will have at most degree $h + 1$. As a matter of fact, if $\tau \in G$ with $\deg(\tau) = d > h + 1$ its predecessors will belong to $N$ and have degree $d - 1 \geq h + 1$ which is impossible.
Algorithm 1 Cerlienco-Mureddu algorithm.

1: procedure CeMu($X$) → $\Phi(X)$

2: if $N = 1$ then

3: $\Phi(X) := \{(0, ..., 0)\}.$

4: end if

5: if $1 < N$ then suppose to know by induction hypothesis $\Phi((P_1, ..., P_{N-1})) = (d_1, ..., d_{N-1})$

6: and look for $d_N = \Phi(P_N)$. 

7: for $i = n$ to 1 do

8: if $i > s$ then

9: $d_{Ni} = 0.$

10: end if

11: if $i = s$ then

12: $d_{Ns} = d_{ms} + 1.$

13: end if

14: if $i < s$ then

15: $W(P_N, X) := \{P \in X | \Phi(P) = d = (\ast, \ast, d_N, 0, ..., 0), \}$ = $\{P_j, ..., P_r\}.$

16: $Q := \pi_{s-1}(W(P_N, X)).$ 

17: $|Q| = |W(P_N, X)| = r < N.$ If $h < r \leq |W(P_N, X)|$, then $\pi_{s-1}(P_h) \neq \pi_{s-1}(P_{N}).$ Moreover, since $\Phi$ is inductive, if $h < r \leq |Q|$, then $\pi_{s-1}(P_{j}) \neq \pi_{s-1}(P_{k}).$

18: $\pi_{s-1}(d_N) = d_r.$ 

19: break.

20: end if

21: end for

22: end if

23: return $\Phi(X)$.

24: end procedure
The computation of $G$ is performed as follows. Consider $\mathcal{T}_i \forall i = 0, \ldots, h + 1$: it is well known that $|\mathcal{T}_i| = \binom{n+i}{r_i-1}$.

For each $i$, define $\text{Gen}_i(I) := \{ t \in G(I) | \text{deg}(t) \leq i \}$. Since $I$ is a proper ideal, $\text{Gen}_0(I) = \emptyset$.

Let $h$ the minimal $i$ such that $\text{Gen}_h(I) \neq \emptyset$, $\forall i \geq 1$

$$\text{Gen}_{i+h} = \text{Gen}_{h+i-1} \cup (\mathcal{T}_{h+i} \setminus (N_{h+i} \cup \bigcup_{j=h+1}^{h+i-1} C^{(h+i-j)}(G_j)))$$

We then have

**Algorithm 2** The potential expansion’s algorithm.

1: procedure $\text{PotExp}(N(I)) \rightarrow I$ \hspace{1cm} $\triangleright$ $I$ is expressed using its minimal basis.
2: \hspace{1cm} Require: $N = \{N_0, \ldots, N_h, N_{h+1}\}$, such that $N_{h+1} = \emptyset$.
3: \hspace{1cm} $C = \{0\}$. \hspace{1cm} $\triangleright$ the potential expansion’s list.
4: \hspace{1cm} $\text{Gen} = \emptyset$.
5: \hspace{1cm} $I = (0)$.
6: for $i = 0$ to $h + 1$ do
7: \hspace{1cm} $d = \binom{n+\text{deg}(N_i)}{r_i-1} - |N_i \cup C[i]|$.
8: \hspace{1cm} if $d = 0$ then
9: \hspace{2cm} $\text{Gen}_i = \{0\}$ \hspace{1cm} $\triangleright$ no new generators.
10: \hspace{1cm} else
11: \hspace{2cm} $\text{Gen}_i = \mathcal{T}_i \setminus (N_i \cup C[i])$.
12: \hspace{2cm} $C[i + 1] = \text{PotentialExpansion}(C[i])$.
13: \hspace{1cm} $I = I + \text{Gen}_i$.
14: end if
15: end for
16: return $I$
17: end procedure

The algorithm uses a subroutine $\text{PotentialExpansion}$ such that

$$\text{PotentialExpansion}(F) = C^{(1)}(F).$$

We will also have a subroutine finding $\mathcal{T}_{h+1} \setminus (N_{h+1} \cup \bigcup_{j=h+1}^{h+i-1} C^{(h+i-j)}(G_j))$.

WLOG we will think that the sets $\mathcal{T}_{h+i}$ and $N_{h+i} \cup \bigcup_{j=h+1}^{h+i-1} C^{(h+i-j)}(G_j)$ are ordered w.r.t. the same ordering, since it is enough to perform a merging with the Groebner escalier and the potential expansion previously ordered.

All these steps end: the subroutine finding the complementary can be developed performing a loop on the two ordered lists $A := \mathcal{T}_i = [a_1, \ldots, a_m]$, $m \geq n$ and $B := N_i \cup C^{(i)} = [b_1, \ldots, b_n]$ (using two indices $i, j$), keeping in mind that $B \subseteq A$ or $B = A$ and that $B[j] \geq A[i]$ at every step. Start with $b_1$: if $b_1 = a_1$ we set $i = i + 1; j = j + 1$.

If we find $a_i \neq b_j$ for a certain couple $(i, j)$, we put $A[i]$ in the complementary and $i = i + 1$ without modifying $j$.

**Example 4.2.** There are situations in which $N$ contains monomials of degree at most $h$, but also the minimal basis shares the same property. Take $I = (x^3, y^2, z^2, xy) \triangleleft k[x, y, z]$, whose Groebner escalier is:
\(N_0 = \{1\}\)
\(N_1 = \{x, y, z\}\)
\(N_2 = \{yz, xz, x^2\}\)
\(N_3 = \{x^2z\}\):

The monomial basis does not contain elements of degree 4.

We call \(G_i\) the set of \(i\)-degree elements of the minimal basis and \(I\) the monomial ideal we want to find.

**Lemma 4.3.** For all \(i = 0, \ldots, h + 1\)

\[T_i \setminus (N_i \cup \bigcup_{j=1}^{i-1} C^{i-j}(G_j)) = G_i.\]

**Proof:** The inclusion \(T_i \setminus (N_i \cup \bigcup_{j=1}^{i-1} C^{i-j}(G_j)) \supseteq G_i\) is trivial, so we only prove \(T_i \setminus (N_i \cup \bigcup_{j=1}^{i-1} C^{i-j}(G_j)) \subseteq G_i\).

Consider \(\tau \in T_i \setminus (N_i \cup \bigcup_{j=1}^{i-1} C^{i-j}(G_j))\). Clearly \(\tau \in I\).

Let \(\sigma\) the \(i\)th predecessor of \(\tau\); if \(\sigma \in I\), \(\exists t \in G\) with \(\sigma = t \cdot m\) for a suitable \(m \in T\).

Then \(\tau = t \cdot m \cdot x_i\) i.e. \(\tau \in \bigcup_{j=1}^{i-1} C^{i-j}(G_j)\).

This lemma assures that the result obtained via the potential expansion’s algorithm is correct.

## 5 The Axis of Evil Algorithm.

A 0–dimensional radical ideal \(I \triangleleft P\) is completely determined if we know the set \(V(I)\) of its zeros.

Consider a finite set of distinct points \(X = \{P_1, \ldots, P_r\}\); we will denote indifferently the Groebner escalier of the ideal \(I(X)\) with \(N(I(X))\) or \(N\). A variation of Cerlienco-Mureddu algorithm (3) allows us to find a ‘linear factorization’ for every element of a lexicographic minimal Groebner basis in the sense of the

**Theorem 5.1.** Let \(t_i := x_1^{d_1} \cdots x_n^{d_n}, i = 1, \ldots, r\) be the generators of the minimal basis of \(T(I)\), where \(I\) is a 0–dimensional radical ideal.

A combinatorical algorithm and interpolation allow us to deduce polynomials

\[\gamma_{m\delta i} = x_m - y_{m\delta i}(x_1, \ldots, x_{m-1}),\]

\(\forall i, m, \delta, \) with \(1 \leq i \leq r, 1 \leq m \leq n, 1 \leq \delta \leq d_m\) such that

\[f_i = \prod_{m} \prod_{\delta} \gamma_{m\delta i} \forall i\]
where \( f_i, i = 1, ..., r \) are the polynomials forming a minimal Groebner basis of \( I \) with respect to the lexicographic order induced by \( x_1 < ... < x_n \).

In that algorithm we will use the projections, as we defined in section 3. The Axis of Evil algorithm works then in the following way:

- consider \( \tau_j := x_1^{d_1} \cdots x_n^{d_n} \in G \). The required polynomial \( f = \tau_j + \text{tail}(f) \) is factorized in \( \sum_{i=1}^{d_i} \) factors: \( d_1 \) polynomials whose leading term is \( x_1 \), \( d_2 \) polynomials such that their leading term is \( x_2 \) and so on;
- consider the monomials \( x_1^{a_1} x_2^{d_2} \cdots x_n^{d_n} \) such that \( a_1 < d_1 \);
- every such monomial is associated, via Cerlienco-Mureddu Correspondence, to a point of our set \( X \). Project these points with respect to the first coordinate, obtaining \( \{x_1,y_1,...,y_{d_1}\} \);
- \( x_1 = y_i, i = 1,...,d_1 \) are the first \( d_1 \) factors;
- construct the subset \( D_{20} \) of \( X \) containing all the points in which the product \( (x_1 - y_1) \cdots (x_1 - y_{d_1}) \) does not vanish. If it is empty then stop and consider the next monomial in \( G \); otherwise continue as follows;
- find the set \( N_2(\tau_j) \) of all monomials in \( T[2] \) such that \( x_1^{a_1} x_2^{d_2} < x_1^{d_1} x_2^{d_2} \);
- split the elements of \( N_2(\tau_j) \) with respect to the exponents of \( x_2 \) and construct, via Cerlienco-Mureddu Correspondence, the set \( \{\Phi^{-1}(v x_2^{d_2-\delta} x_3^{d_3} \cdots x_n^{d_n}) / v \in T[1], v x_2^{d_2-\delta} \in N_2(\tau_j)\} \);
- intersect the previous set with \( D_{20} \), project the resulting set of points \( (A_{2\delta}(\tau_j)) \) with respect to the first two coordinates and apply Cerlienco-Mureddu Correspondence, obtaining a set \( E_{2\delta \tau_j} \);
- interpolate over \( A_{2\delta}(\tau_j) \), finding \( d_2 \) factors whose leading terms are all equal to \( x_2 \). The monomials of \( E_{2\delta \tau_j} \) are the ones appearing in such factorization;
- update the set of points in which the current polynomial does not vanish and stop if it is empty;
- repeat these steps letting all the variables vary one by one;
- repeat all the steps for all \( \tau_i \in G \).

Remark 5.2. Given \( \tau_j = x_1^{d_1} \cdots x_n^{d_n} \in G \), every variable \( x_i \) will appear only \( d_i \) times in the execution of the algorithm.

Remark 5.3. The sets \( N_m(\tau_j) := \{\omega \in T[m], \tau_j > \omega x_{m+1}^{d_{m+1}} \cdots x_n^{d_n} \in N\} \) (in particular for \( m = 1 \) we have \( N_1(\tau_j) := \{x_i / i < d_i\} \) are constructed in order to determine in which points it is necessary to interpolate. Since for \( \mu > \tau_j \) the Cerlienco-Mureddu correspondence provides a point \( P_{\mu'} \) such that \( 3k \in \{1,...,n\} : \pi_k(P_{\mu'}) = \pi_k(P_{\mu'}) \), in order to obtain polynomials vanishing on all the point of \( X \) it is not necessary to interpolate in the whole \( \Phi^{-1}(N) \) as it suffices to consider only those corresponding to \( \mu \in N \) with \( \mu < \tau_j \).
Algorithm 3 The Axis of Evil algorithm.

1: procedure AoE(\(X, G(I(X)) := \{\tau_1, ..., \tau_r\}\)) \(\rightarrow \{\tau_1, ..., \tau_r\}\) \(\rightarrow R \supset R\) contains a factorized minimal Groebner basis of \(I\).

Require: the elements \(G(I(X))\) are in increasing order w.r.t the lexicographical order w.r.t. \(x_1 < ... < x_r\).

2: \(R = \emptyset\)
3: for \(i = 1\) to \(r\) do
4: \(N_1(\tau_j) := \{x_i^j / i < d_i\} = \{\omega \in T[1], \tau_j > \omega x_i^d \cdot ... \cdot x_n^d / i < d_i\} \subset X\).
5: \(A_1(\tau_j) := \Phi^{-1}(x_i^d \cdot ... \cdot x_n^d / i < d_i) \subset k\).
6: \(B_1(\tau_j) := \pi_1(A_1(\tau_j)) \subset k\).
7: \(\gamma_1 \tau_j := \Pi_{a \in B_1(\tau_j)}(x_1 - a)\).
8: for \(m = 2\) to \(n\) do
9: \(\zeta_{m\tau_j} := \Pi_{\nu=1}^{m-1} \gamma_{m\tau_j}\).
10: \(D_{m0} := \{P_i \in X / \zeta_{m\tau_j}(P_i) \neq 0\}\).
11: if \(|D_{m0}| = 0\) then
12: \(R = [R, \zeta_{m\tau_j}]\).
13: break.
14: end if
15: \(N_m(\tau_j) := \{\omega \in T[m], \tau_j > \omega x_i^d \cdot ... \cdot x_n^d / i < d_i\} \subset X\).
16: for \(\delta = 1\) to \(d_m\) do
17: \(A_{m\delta}(\tau_j) := \Phi^{-1}(x_i^d \cdot ... \cdot x_n^d / i < d_i) \subset k\).
18: \(E_{m\delta}(\tau_j) := \Phi(A_{m\delta}(\tau_j))\).
19: \(\gamma_{m\delta \tau_j} := x_m + \sum_{\omega \in E_{m\delta}(\tau_j)} e(\gamma_{m\tau_j}, \omega)\),

such that \(\gamma_{m\delta \tau_j}(P) = 0, \forall P \in A_{m\delta}(\tau_j)\).
20: \(\xi_{m\delta} := \Pi_{\nu=1}^{m-1} \gamma_{m\tau_j} \Pi_{\delta=1}^{d_m} \gamma_{m\delta \tau_j}\).
21: \(D_{m\delta}(\tau_j) := \{P_i \in X / \xi_{m\delta}(P_i) \neq 0\} \subset X\).
22: if \(|D_{m\delta}(\tau_j)| = 0\) then
23: \(R = [R, \xi_{m\delta}]\).
24: break.
25: end if
26: end for
27: \(\gamma_{m\tau_j} := \Pi_{\delta} \gamma_{m\delta \tau_j}\).
28: end for
29: end for
30: return \(R\).
31: end procedure
Remark 5.4. The terms smaller than $\tau_j$ mentioned before are found releasing all the variables one by one.

Imagine the monomials in $k[x_1, \ldots, x_n]$ as points in $k^n$, identifying every term to the $n$-uple of its exponents. So we can ‘draw’ them in a $n$-dimensional space and we can think our realisations as an increment by one of the ‘directions’ where we can move there.

We point out that $N_m(\tau_j) \subseteq N_j(\tau_j)$ for $m \leq h$.

If $\omega \in N_m(\tau_j)$, $\tau_j > \omega x_{m+1}^{d_{m+1}} \cdots x_n^{d_n} \in N$; as $\omega x_{m+1}^{d_{m+1}} \cdots x_n^{d_n}$ we have $\omega x_{h+1}^{d_{h+1}} \cdots x_n^{d_n} \in N$ and $\omega x_{h+1}^{d_{h+1}} \cdots x_n^{d_n} \leq x_{m+1}^{d_{m+1}} \cdots x_n^{d_n} < \tau_j$.

At each step we will count out all the points in which the polynomial already vanishes and we will stop the computation when the current factorized polynomial vanishes on the whole $X$.

We will see an example of it later.

Remark 5.5. If the number of released variables is $> 1$, we also must split the obtained monomials regarding the exponent of the maximal variable.

Consider then the loop on $\delta$ and, in particular, the set:

$$C_{m\delta}(\tau_j) := \{\Phi^{-1}(\{x_{m+1}^{d_{m+1}} \cdots x_n^{d_n}\})/ v \in T[m-1], \omega x_{m}^{d_{m}} \in N_m(\tau_j)\}.$$  

We intersect $C_{m\delta}(\tau_j)$ with the subset of $X$ containing the points not vanishing the current factorized polynomial.

We can easily notice that, performing the algorithm, we only compute the sets $C_{m1}(\tau_j), \ldots, C_{md_{m}}(\tau_j)$, but in $N_m(\tau_j)$ there are also monomials $\omega = x_{1}^{a_{1}} \cdots x_{m-1}^{a_{m-1}} x_{m}^{a_{m}}$ such that $\tau_j > \omega x_{m+1}^{d_{m+1}} \cdots x_n^{d_n} \in N$, which would be generated considering $\delta = 0$.

They are not considered in the algorithm because they are related to monomials examined in the previous step: $x_{1}^{a_{1}} \cdots x_{m-1}^{a_{m-1}} \in N_{m-1}$, so the corresponding points have already been treated. Taking $\delta = 0, \ldots, d_{m}$, the sets $C_{m\delta}(\tau_j)$ form a partition of $N_m(\tau_j)$ basing on the degree of $x_m$. As a matter of fact, in order to have $\omega \in N_m(\tau_j)$ we must have $\tau_j > \omega x_{m+1}^{d_{m+1}} \cdots x_n^{d_n}$, where $\omega x_{m+1}^{d_{m+1}} \cdots x_n^{d_n} \in N$, then the exponent of $x_m$ will be the first checked in the lexicographic test and so it will be limited by $d_m$.

According to the values of this exponent, the ones associated to smaller variables will vary.

Remark 5.6. At the beginning of the algorithm, we imposed the monomials $\tau_j$, $j = 1, \ldots, r$ to be in increasing order with respect $<$. The steps made by the algorithm on each $\tau_j$ are totally independent both on those made and on those to be made on a monomial $\tau_k$ (it is indifferent whether $j \geq k$) belonging to $G$, so we will obtain the same factorizations even if we launch the computation on a list of unordered monomials.

Clearly, the result of our computation is not the reduced Groebner basis of the given ideal, it is only one of the minimal Groebner bases but we can obtain the reduced Groebner basis via simple reduction.

We decided to put the monomials in such an order because we want every polynomial to be reduced with respect to the ‘previous’ ones.

If $f_j$ is one of our resulting polynomials and $Lt(f_j) = \tau_j$, the polynomials utilizable to reduce $f_j$ (the previous ones) must be necessarily all and only the ones having as leading terms elements in $G$ lower than the given $\tau_j$. 

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The algorithm terminates because it works on:

1. points in the finite set $X$;
2. monomials $\tau \in G$ (they are in a finite number, $[16]$);
3. a finite set of variables.

Let us study the correctness of the algorithm.

**Lemma 5.7.** The factorized polynomials obtained from our algorithm vanish on all the points of the set $X$.

**Proof:** Suppose we want to construct $\gamma_\tau$ with $\tau = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.
Let $\mu = x_1^{\beta_1} \cdots x_n^{\beta_n}$, corresponding to a point $P_\mu \in X$ through Cerlienco-Mureddu Correspondence.

Let $\mu < \tau$, then at least one of the exponents of the variables appearing in $\mu$ is lower than the corresponding in $\tau$, say $\beta_i < \alpha_i$, so $\mu$ is linked to an element of $N_i(\tau)$ and so it can, alternatively:

- belong to $A_{i\delta}(\tau)$ for some $\delta$;
- be such that the corresponding point already annihilates the polynomial found.

If $\mu > \tau$ (since $\tau \notin N$, it is surely impossible that $\tau = \mu$) then there will be a point $P_{\mu'}$ such that

$$\pi_j(P_{\mu'}) = \pi_j(P_{\mu}),$$

corresponding to a $\mu' < \tau$.

We then use $\mu'$ and we come back to the previous situation. $\diamond$

**Corollary 5.8.** The ideal generated by these polynomials is exactly $I(X)$.

**Proof:** By the previous lemma, the polynomials vanish on all the points of the set $X$ and the equality comes out by reasons of multiplicity $\diamond$

The resulting polynomials form a minimal Groebner basis because:

- they vanish on all the points of $X$;
- their heads form exactly $G(I(X))$.

Notice that we can obtain the current interpolating polynomial applying Moeller algorithm to the projection through $\pi_m$ of the points of the current $A_{m,\delta}(\tau)$ ($[14]$).

**Example 5.9.** Let $X := \{(4,0,0), (2,1,4), (2,4,0), (3,0,1), (2,1,3), (1,3,4), (2,4,3), (2,4,2), (1,0,2)\}$.
$P_1 := (4,0,0) : it\ is\ a\ single\ point,\ so\ \Phi(\{(4,0,0)\}) = (0,0,0)$
$P_2 := (2,1,4) : s = 1, m = 1, (1,0,0)$
$P_3 := (2,4,0) : s = 2, m = 2, (0,1,0)$
$P_4 := (3,0,1) : s = 1, m = 1, (2,0,0)$
$P_5 := (2,1,3) : s = 3, m = 2, (0,0,1)$
$$P_0 := (1, 3, 4) : s = 1, m = 4, (3, 0, 0)$$
$$P_1 := (2, 4, 3) : s = 3, m = 3, W = \{(2, 1, 3), (2, 4, 3)\}, t_7 = (0, 1, 1)$$
$$P_2 := (2, 4, 2) : s = 3, m = 7, (0, 0, 2)$$
$$P_3 := (1, 0, 2) : s = 2, m = 6, W = \{(2, 4, 0), (1, 0, 2)\}, t_9 = (1, 1, 0).$$

Then \(N := \{1, x_1, x_2, x_1^2, x_2^3, x_1 x_3, x_2 x_3, x_1^2 x_2\}\) and so we can easily obtain \(G = \{x_1^4, x_1^2 x_2, x_2^3, x_1 x_3, x_2 x_3^2, x_3^3\}\).

The monomials belonging to \(G\) are exactly the input for the Axis of Evil algorithm and they are already ordered with respect to our ordering:

starting with \(\tau = x_1^4\) we obtain

\[
\begin{align*}
N_1(\tau) &= \{1, x_1, x_1^2, x_1^3\}; \\
A_1(\tau) &= \{(4, 0, 0), (2, 1, 4), (3, 0, 1), (1, 3, 4)\} : \text{these are the corresponding points via Cerlienco-Mureddu Correspondence}; \\
B_1(\tau) &= \{4, 2, 3, 1\} \\
\gamma_{1\tau} &= (x_1 - 4)(x_1 - 2)(x_1 - 3)(x_1 - 1); \\
\text{all the linear factors are only depending from } x_1 \text{ are computed in the same time.}
\end{align*}
\]

\[ m = 2: \]

\[
\zeta_{m\tau} = \gamma_{1\tau}
\]

\(D_{20}(\tau) = 0\): stop here obtaining, as first result, a polynomial having as leading term an element of \(G\) (while the other monomials belong to \(N\)) and belonging to \(I(X)\) since it vanishes in every point of \(X\) (so an element of our minimal Groebner basis).

\[
\begin{align*}
\tau &= x_1^4 x_2 \quad N_1(\tau) = \{1, x_1\}, \\
A_1(\tau) &= \{(2, 4, 0), (1, 0, 2)\}; \\
B_1(\tau) &= \{2, 1\} \\
\gamma_{1\tau} &= (x_1 - 2)(x_1 - 1)
\end{align*}
\]

\[ \delta = 1: \]

\[
\begin{align*}
A_{21}(\tau) &= \{(4, 0, 0), (3, 0, 1)\} = D_{20} \\
\end{align*}
\]

The monomials \(x_2^\delta m - 3\) are \(1, x_1, x_1^2, x_1^3\), corresponding to the points \(P_3, P_4, P_5\).

The polynomial already vanishes on \(P_2, P_6\), so we consider only the remaining two points.

\[
\begin{align*}
E_{21}(\tau) &= \{1, x_1\}, \\
\gamma_{21\tau} &= x_2; \\
\xi_{21} &= \gamma_{1\tau} \gamma_{21\tau} = (x_1 - 2)(x_1 - 1)x_2; \\
D_{21}(\tau) &= \emptyset.
\end{align*}
\]
Remark that γ_{2\tau} is actually γ_{21\tau}.

\( \tau = x_2^2 \)

\( N_1(\tau) = \emptyset; \)

\( A_1(\tau) = \emptyset; \)

\( B_1(\tau) = \emptyset; \)

\( m = 2; \)

\( D_{20}(\tau) = X \)

\( N_2(\tau) = \{1, x_1, x_1^2, x_2, x_1x_2\}; \) \( \delta = 1\):

\( A_{21}(\tau) = \{(2, 4, 0), (1, 0, 2)\}; \)

\( E_{21}(\tau) = \{1, x_1\}; \)

\( \gamma_{21\tau} = x_2 - 4x_1 + 4 \)

\( \xi_{21} = \gamma_{1\tau}\gamma_{21\tau} = x_2 - 4x_1 + 4; \)

\( D_{21}(\tau) = \{(4, 0, 0), (2, 1, 4), (3, 0, 1), (2, 1, 3), (1, 3, 4)\}; \)

\( \delta = 2; \)

\( A_{22}(\tau) = \{(4, 0, 0), (2, 1, 4), (3, 0, 1), (1, 3, 4)\} \)

The terms \( vx_{m}^{d-\delta} \) are \( 1, x_1, x_1^2, x_1^3 \) and they correspond exactly to \( P_1, P_2, P_4, P_6 \).

\( \tau = x_1x_3 \)

\( m = 2; \)

\( N_2(\tau) = \{1\}. \)

\( A_{1}(\tau) = \{(2, 1, 3)\}; \)

\( B_1(\tau) = \{2\} \)

\( \gamma_{1\tau} = (x_1 - 2) \)

\( D_{20}(\tau) = \{(4, 0, 0), (3, 0, 1), (1, 3, 4), (1, 0, 2)\} \)

\( \delta = 1; \)

\( A_{31}(\tau) = \{(4, 0, 0), (3, 0, 1), (1, 3, 4), (1, 0, 2)\} \)

The terms are \( 1, x_1, x_1^2, x_1^3, x_2, x_1x_2 \), corresponding to \( P_1, P_2, P_3, P_4, P_6, P_9 \), but we can neglect \( P_2, P_3 \).

\( m = 3; \)

\( N_3(\tau) = \{1, x_1, x_1^2, x_2, x_1x_2\}; \)

\( \xi_{m7} = (x_1 - 2); \)

\( D_{30}(\tau) = \{(4, 0, 0), (3, 0, 1), (1, 3, 4), (1, 0, 2)\}; \)

\( \delta = 1; \)

\( A_{31}(\tau) = \{(4, 0, 0), (3, 0, 1), (1, 3, 4), (1, 0, 2)\} \)
\[ \gamma_{31}(\tau) = 6x_3 - 4x_2 + x_1^2 - x_1 - 12; \]
\[ \xi_{31} = (x_1 - 2)(6x_3 - 4x_2 + x_1^2 - x_1 - 12); \]
\[ D_{31}(\tau) = \emptyset. \]

The desired polynomial is \( \gamma_{3\tau} = \gamma_{31}(\tau) \).

\( \tau = x_2x_3^2 \)
\( N_1(\tau) = \emptyset; \)
\( A_1(\tau) = \emptyset; \)
\( B_1(\tau) = \emptyset; \)
\( m = 2: \)

\[ N_2(\tau) = \{1\}; \]
\[ D_{20}(\tau) = X; \]
\( \delta = 1: \)
\( A_{21}(\tau) = \{(2, 4, 2)\}; \)
\( E_{21}(\tau) = \{1\}; \)
\( \gamma_{21\tau} = x_2 - 4 \)
\( \xi_{21} = x_2 - 4; \)
\( D_{21}(\tau) = \{(4, 0, 0), (2, 1, 4), (3, 0, 1), (2, 1, 3), (1, 3, 4), (1, 0, 2)\}; \)

\( \tau = x_3 \)
\( N_1(\tau) = \emptyset; \)
\( A_1(\tau) = \emptyset; \)
\( B_1(\tau) = \emptyset; \)
\( m = 2: \)

\[ N_3(\tau) = N(X); \]
\( \delta = 1: \)
\( A_{31}(\tau) = \{(2, 1, 3)\}; \)
\( E_{31}(\tau) = \{1\}; \)
\( \gamma_{21\tau} = x_3 - 3 \)
\( \xi_{31} = (x_2 - 4)(x_3 - 3); \)
\( D_{31}(\tau) = \{(4, 0, 0), (2, 1, 4), (3, 0, 1), (1, 3, 4), (1, 0, 2)\}; \)

\( \tau = x_3 \)
\( N_1(\tau) = \emptyset; \)
\( A_1(\tau) = \emptyset; \)
\( B_1(\tau) = \emptyset; \)
\( m = 2: \)

\[ D_{20}(\tau) = X; \]
\( N_2(\tau) = \emptyset; \)
\( \delta = 1: \)
\( A_{21}(\tau) = \emptyset; \)
\( D_{21}(\tau) = X; \)
Then our minimal Groebner basis of the ideal associated to X is obtained by our polynomials by the reductions stated in the Axis of Evil Theorem. If we obtain a factorized polynomial, then it is impossible that a partial product vanishes on the whole X because of Step 5.7.

Then our minimal Groebner basis of the ideal associated to X with respect to the given order is:

\[
\mathcal{G}(I(X)) = \left\{ x_1^4 - 10x_1^3 + 35x_1^2 - 50x_1 + 24, x_2x_1^2 - 3x_2x_1 + 2x_2, \\
x_2^3 - 2x_2x_1 - x_2 + 2x^3 - 16x_1^2 + 38x_1 - 24, x_3x_2 - 2x_3 - \frac{x_2}{2}x_2x_1 + \frac{1}{2}x_2 + \\
+ \frac{1}{6}x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_1 + 4, x_3^2x_2 - 4x_3^2 - 7x_3x_2 + 28x_3 + \frac{5}{3}x_2x_1 + \\
+ \frac{20}{9}x_2 - \frac{16}{3}x_1^2 + 48x_2^2 - \frac{144}{3}x_1 + 32, x_3^3 - 5x_3^2 + \frac{8}{3}x_3x_2 - \frac{14}{3}x_3 - \frac{16}{3}x_2x_1 \\
- \frac{40}{9}x_2 + \frac{73}{9}x_3^2 + \frac{107}{9}x_2^2 + \frac{155}{9}x_1 - 72 \right\},
\]

obtained by our polynomials by the reductions stated in the Axis of Evil Theorem.

Finally, we remark that:

1. let \( \tau_j = x_1^{d_1} \cdot x_n^{d_n} \in G \). The polynomial we are looking for must contain exactly \( \sum_{i=1}^n d_i \) factors. It is impossible that the algorithm stops before, so it is impossible that a partial product vanishes on the whole X. In fact, if so, there would be a polynomial \( f \in I \) such that \( T(f) \notin (G) \) (we know the minimal basis \( G \) before starting the Axis of Evil process);

2. if we obtain a factorized polynomial \( f \) such that its leading term \( T(f) \) belongs to the minimal basis \( G \), then \( f \) vanishes over all X, because of Step 5.7.
Example 5.10. Consider the following ideal, given with its primary decomposition:

\[ J := (x_1^2, x_2 + x_1, x_3) \cap (x_1^2, x_2 - x_1, x_3 - 1) = (x_1^2, x_1 x_2, x_1 x_3 - \frac{1}{2} x_1 - \frac{1}{2} x_2, x_2 x_3 - \frac{1}{2} x_1 - \frac{1}{2} x_2, x_3^2 - x_3) \subseteq \mathbb{C}[x_1, x_2, x_3]. \]

Call its generators \( f_1, ..., f_6 \), considering them in the correct order.

It is 0-dimensional because \( x_1^2, x_2^2, x_3^2 \in \text{In}(J) \) (see [16]), but it is not radical: its radical is \( \sqrt{J} = (x_2, x_3^2 - x_3, x_1) \).

For such an ideal the Axis of Evil does not hold.

Consider the polynomial \( f_4 = x_1 x_3 - \frac{1}{2} x_1^2 - \frac{1}{2} x_2 \). By the Axis of Evil theorem (5.1), its factorization should be of the form:

\[ (x_1 + ...) (x_3 + ...) \]

and we should have

\[ x_1 x_3 - \frac{1}{2} x_1 - \frac{1}{2} x_2 + P x_1^2 + Q x_1 x_2 + R x_2^2, P, Q, R \in \mathbb{C}[x_1, x_2, x_3], \]

since we can only reduce deleting the multiples of \( x_1^2, x_1 x_2, x_2^2 \), in order to obtain \( f_4 \). In order to have the correct product we must have \( -\frac{1}{2} x_2 \) in it. We can not obtain it through reductions, so the only chance is that we have a product of the form

\[ k * h x_2, \]

with \( h, k \) constants such that \( hk = -\frac{1}{2} \), in particular both different from 0.

A priori, we can have two possibilities:

- \((x_1 + k + ...) (x_3 + h x_2 + ...)) ;
- \((x_1 + h x_2 + ...) (x_3 + k + ...) .

The second one is impossible: the polynomial having \( x_1 \) as head can not contain variables greater than \( x_1 \), so we consider only:

\[ (x_1 + k + ...) (x_3 + h x_2 + ...) . \]

We will then obtain

\[ x_1 x_3 + h x_1 x_2 + k x_3 - \frac{1}{2} x_2 + ... \]

We can delete the term \( x_1 x_2 \) but it remains \( k x_3 \) which can not be reduced.

6 Corollaries.

We enumerate here some famous theorems which can be easily proved as corollaries of the Axis of Evil Theorem. For more details see, for example, [16]. Here we provide the general statements of these results, but clearly they can only be deduced under the hypothesis of the Axis of Evil theorem.

The first one is Lazard Structural Theorem, which describes the structure of a minimal lexicographical Groebner basis of an \( I \triangleleft k[x_1, x_2] \).

The original proof considers \( P = k[x_1, x_2] = k[x_1][x_2] \) and it is based on the fact that \( k[x_1] \) is a Principal Ideal Domain (PID).

Norton-Sălăgean [17] reformulated it using, more generally, \( R[x] \) with \( R \) PIR.

We briefly recall the following
Definition 6.1. The content \( r_f \in R \), with \( R \) PIR, of a polynomial \( f(x) \in R[x] \) is the GCD of its coefficients. A polynomial \( f(x) \in R[x] \) is called primitive if \( r_f = 1 \). The primitive part of \( f(x) \in R[x] \) is the polynomial \( p_0(x) \in R[x] \) such that
\[
f(x) = r_f p_0(x).
\]

Let \( R \) be a PIR, \( P := R[x] \). Let \( I \subset P =: \{ f_0, \ldots, f_s \} \) a minimal Groebner basis of \( I \) ordered in such a way that, called \( d(i) := \deg(f_i) \), \( 0 \leq i \leq s \)
\[
d(0) \leq \ldots \leq d(s).
\]
Define then \( c_i = \text{lc}(f_i) \), \( r_i \in R \setminus \{0\} \) \( e \) \( p_i \in P \) the leading coefficient, the content and the primitive part of \( f_i \), for all \( 1 \leq i \leq n \).

Theorem 6.2 (Lazard). If, moreover, \( R \) is a PID, then:

- \( f_0 = PG_1 \cdots G_{s+1} \);
- \( f_j = PH_j G_{j+1} \cdots G_{s+1} \), \( 1 \leq j \leq s \).

where

1. \( d(1) < \ldots < d(s) \);
2. \( G_i \in R \), \( 1 \leq i \leq s + 1 \) is such that \( c_{i-1} = G_i c_i \)
3. \( P = p_0 \) (the primitive part of \( f_0 \in R[x] \));
4. \( H_i \in R[x] \) is a monic polynomial of degree \( d(i) \) in \( x \), for all \( i \);
5. for all \( i \) we have \( H_{i+1} = G_1 \cdots G_i, H_1 G_2 \cdots G_{i+1}, \ldots, H_{i-1} G_i, H_i \);
6. \( r_i = G_{i+1} \cdots G_s \)

Theorem 6.3 (Norton-Sălăgean). With the previous notation, each
\[
p_i \in (f_j, j < i) : r_i.
\]

In fact, we have \( r_i = \prod_{t=1}^{n-i} \prod_{s=1}^{d_t} \gamma_{mst} \), and \( p_i = \prod_{s=1}^{d_n} \gamma_{nst} \).

The second well-known result which can be straightforwardly derived from the Axis of Evil Theorem is the well known Elimination Theorem (see \([2]\) for details)

Theorem 6.4 ([13]). Let \( I \subset k[x_1, \ldots, x_n] \) an ideal, take the lexicographical ordering induced by \( x_1 < \ldots < x_n \) and call \( I_j \) the \( j \)-th elimination ideal \( I_j = I \cap k[x_1, \ldots, x_j] \). Let \( \mathcal{G} \) be a Groebner basis of \( I \), then \( \mathcal{G}_j = \mathcal{G} \cap k[x_1, \ldots, x_j] \) is a Groebner basis of \( I_j \).

The following result, Kalkbrener theorem ([13], [14]), is another consequence of the Axis of Evil Theorem and it is a stronger characterization of the lexicographical ordering.

For each subset \( L \subset k[x_1, \ldots, x_n] \), \( i = 1, \ldots, n \), \( \forall \delta \in \mathbb{N} \) set
\[
L_{i, \delta} = \{ p \in L, |p| \in k[x_1, \ldots, x_i], \deg_i(p) \leq \delta \}
\]
and
\[
L_{p_i, \delta} = \{ Lp(p), p \in L_{i, \delta} \}.
\]
Theorem 6.5 (Kalkbrenner). With the previous notations, considered an ideal $I \triangleleft k[x_1, \ldots, x_n]$ and a Groebner basis $G$ of it, these forms are equivalent:

- $G$ is a Groebner basis of $I$ w.r.t. the lexicographical order $< \text{ induced by } x_1 < \ldots < x_n$;
- $Lp_{i, \delta}(G)$ is a Groebner basis of $Lp_{i, \delta}(I)$, $i = 1, \ldots, n$, $\forall \delta \in \mathbb{N}$.

Let us now mention Gianni-Kalkbrenner theorem, whose situation is a bit more complicated (see [12], [7], [16]).

Theorem 6.6 (Gianni-Kalkbrenner). Let $I \triangleleft k[x_1, \ldots, x_n]$ an ideal and $G$ w.r.t. the lexicographical order $< \text{ induced by } x_1 < \ldots < x_n$. As before we define also

$G_d = G \cap k[x_1, \ldots, x_d]$.

Consider $\alpha = (b_1, \ldots, b_d) \in V(I_d)$ and define the projection map

$\Phi_\alpha : k[x_1, \ldots, x_n] \rightarrow k[x_{d+1}, \ldots, x_n]$

$f(x_1, \ldots, x_n) \mapsto f(b_1, \ldots, b_d, x_{d+1}, \ldots, x_n)$.

Let $\sigma$ be the minimal value such that $\Phi_\alpha(Lp(g_\sigma)) \neq 0$ and $j, \delta$ the values such that $g_\sigma = Lp(g_\sigma)x_\delta^{j+1} + \ldots \in k[x_1, \ldots, x_j] \setminus k[x_1, \ldots, x_{j-1}]$.

Then

1. $j = \delta + 1$
2. $\forall g \in G_d$, $\Phi_\alpha(g) = 0$;
3. $\forall g \in G_{d+\delta}$, $\Phi_\alpha(g) = 0$;
4. $\Phi_\alpha(g_\sigma) = \gcd(\Phi_\alpha(g), g \in G_{d+1}) \in k[x_{d+1}]$;
5. $\forall b \in k$, $(b_1, \ldots, b_d) \in V(I_{d+1}) \Leftrightarrow \Phi_\alpha(g_\sigma)(b) = 0$.

Clearly (1 – 3) are essentially a corollary of theorem 6.3 on the other side, (4 – 5) apparently cannot be deduced from the Axis of Evil Theorem.

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