Decomposing Jacobians Via Galois covers

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Abstract

Let \( \phi : X \rightarrow Y \) be a (possibly ramified) cover, with \( X \) and \( Y \) of strictly positive genus. We develop tools to identify the Prym variety \( C \) of the Galois closure of a composition \( X \rightarrow Y \rightarrow \mathbb{P}^1 \) of \( \phi \) with a well-chosen map \( Y \rightarrow \mathbb{P}^1 \) that identifies branch points of \( \phi \). To our knowledge, this method recovers all previously obtained descriptions of Prym varieties as Jacobians. It also finds new decompositions, and for some of these, including one where \( X \) has genus 3, \( Y \) has genus 1 and \( \phi \) is a degree 3 map totally ramified over 2 points, we find an algebraic equation of the curve \( C \).

1. Introduction

Let \( k \) be an algebraically closed field of characteristic 0, and let \( X/k \) be a (smooth, projective, and irreducible) curve of genus \( g > 0 \). Already in the 19th century, complex geometers were interested in understanding the periods of \( X \) in terms of periods of curves of smaller genera. In modern terms, one would like to decompose the Jacobian \( \text{Jac}(X) \) of \( X \) as a product (up to isogeny) of powers of nontrivial simple Abelian subvarieties \( A_i \), and then interpret the \( A_i \) as Jacobians of suitable curves \( C_i \). This is not possible for every curve \( X \): Jacobians are generically simple, and even in those cases when \( \text{Jac}(X) \) does decompose there is no reason for the simple factors \( A_i \) to be isogenous to Jacobians of curves. Indeed, while using a suitable isogeny allows us to assume that the \( A_i \) are principally polarized, such Abelian varieties are generically not Jacobians if \( \text{dim}(A_i) \geq 4 \).

When the automorphism group \( G = \text{Aut}(X) \) is non-trivial, one can often find curves \( C_i \) as quotients of \( X \) by well-chosen subgroups of \( G \), and in certain cases one even gets all of the \( A_i \) in this way, see, for example, Lange and Recillas [34]. This strategy has been employed many times, frequently in combination with the Kani–Rosen [26] formula, to get (more or less explicit) examples of Jacobians whose isogeny factors are again Jacobians. For instance, when \( g = 2 \) and \( \#G > 2 \), this strategy always gives the full decomposition of \( \text{Jac}X \) [22], and more examples have been worked out in [25, 43, 45]. In Section 2, we show the following analogous result for \( g = 3 \):

**Theorem A.** Let \( X \) be a curve of genus 3 with automorphism group \( G = \text{Aut}(X) \). Suppose that the Jacobian of the generic point of the stratum corresponding to \( G \) is not simple. Then there are non-trivial subgroups \( H_i \) of \( G \) such that the natural maps induce an isogeny

\[
\text{Jac}(X) \sim \prod_i \text{Jac}(X/H_i),
\]

except if

\[
X \text{ is non-hyperelliptic and } G \text{ is isomorphic to either of } C_2 \text{ and } C_6.
\] (1.1)

In the exceptional case of Theorem A, the quotient \( X/G \) is a curve of genus 1, but there is no Jacobian \( \text{Jac}(Q) \) of a quotient \( Q \) of \( X \) such that we have \( \text{Jac}(X) \sim \text{Jac}(X/G) \times \text{Jac}(Q) \). Instead, we consider the (generalized) Prym variety \( P := \text{Prym}(X/Y) \) of the kernel \( \phi_* : \text{Jac}(X) \rightarrow \text{Jac}(Y) \). There is a significant body of literature dedicated to the description of \( P \) in many special situations. The case when \( \phi \) is an unramified cover of degree 2 is especially beautiful and well-understood [40]. In this case, \( P \) inherits a principal polarization from \( \text{Jac}(X) \), and in some circumstances it has been described as the Jacobian of an explicit curve (see [40, p. 346], [30] or [12]).

In the situation (1.1) with \( G = C_2 \), the morphism \( X \rightarrow X/C_2 \) is not étale. Still, it was shown in [49] that one can realize this map as the degeneration of a family of étale covering maps between curves of genus 5 and 3. Using [12] then made it possible to give an explicit equation for a curve \( C \) such that \( P \sim \text{Jac}(C) \). It remains to be seen whether this idea can be extended to other types of covers.

This article grew out of the desire to find a more generalizable alternative proof of Ritzenthaler and Romagny [49]. Our main sources of inspiration were Donagi’s [19] work on Prym varieties, based on Galois-theoretic considerations, and a specific construction by Dalaljan [16] in the setting where \( \phi : X \rightarrow Y \) is a cover of degree 2 of a hyperelliptic curve \( Y \) with the property that \( \phi \) is branched at 2 points \( Q_1 \) and \( Q_2 \) (see also [30, Th.4.1]). In this latter work, \( P \) is realized as the Jacobian of a curve \( C \) obtained as
a well-chosen quotient of the Galois closure $Z \to \mathbb{P}^1$ of the composition $X \to Y \to \mathbb{P}^1$ of $\phi$ with the hyperelliptic quotient map. Section 3 shows the following analog, which indeed recovers the results in [49].

**Theorem B.** Suppose that we are in the exceptional case of Theorem A where $X$ is a non-hyperelliptic curve with automorphism group $G = C_2$ or $G = C_4$. Let $Y = X/G$ and consider the quotient map $\phi : X \to Y$. Let $\psi : Y \to \mathbb{P}^1$ be a morphism of degree 2 that maps two of the branch points of $\phi$ to the same point of $\mathbb{P}^1$, and let $Z \to \mathbb{P}^1$ be the Galois closure of the composed map $\psi \phi : X \to \mathbb{P}^1$. Then there exists a quotient $Z/H$ of $Z$ such that

$$\text{Jac}(X) \sim \text{Jac}(Y) \times \text{Jac}(Z/H),$$

and an equation for $Z/H$ can be determined explicitly.

We then set out to see to what extent this somewhat miraculous construction of identifying branch points of $\phi : X \to Y$ using a map $\psi : Y \to \mathbb{P}^1$ could yield results for other classes of (not necessarily Galois) morphisms $\phi$. There are two main difficulties in carrying out this program. One is fundamental: given a cover $\pi_{X/Y} : X \to Y$, there is no a priori reason for the Prym variety to appear as the Jacobian of a quotient of a related Galois cover. In fact the general principally polarized abelian variety of dimension 4 is known to be a Prym variety, and by Tsimerman [51], we know that there are four-dimensional Abelian varieties over $\mathbb{C}$ that are not $\mathbb{C}$-isogenous to a Jacobian. The second difficulty lies in the choice of the morphism $\psi : Y \to \mathbb{P}^1$; the Galois closure $Z$ of the composition $\psi \phi : X \to \mathbb{P}^1$ depends very strongly on this choice, and we did not find a general principle to guide us. The results in this paper are therefore mainly exploratory and experimental.

That said, such experimentation is worthwhile, since there is little previous work concerning Prym varieties when the degree of $\pi_{X/Y}$ is greater than 2, and the present project is mainly exploratory. The existing literature has focused on two main cases: (1) the consideration of Galois or étale covers $\phi : X \to Y$ [18, 19, 32, 33, 42, 46] and (2) a top-down approach, which starts with a curve $Z$ that has a large automorphism group $G$ and decomposes of Jac($Z$) in terms of Prym varieties of subcovers $Z/H \to Z/G$ [48] (see also the more complete arXiv version [47]). There is also a vast literature concerning group actions on Abelian varieties, and Jacobians in particular, that is close in spirit to the idea (2) of working with a curve with large automorphism group; see, for example, Carocca et al. [14], Lange and Recillas [34] and the references therein. The more recent article [13] considers the case where $\phi : X \to Y$ is cyclic étale and composed with a cyclic cover $\psi : Y \to \mathbb{P}^1$ of degree $p$. In this context, the resulting compositions are still Galois (see Theorem 14 in loc. cit.) and therefore falls within the scope of the intersection of (1) and (2) above. Indeed, in [13] it is shown that up to isogeny, the Prym variety of $\phi$ can be recovered as the Jacobian of a subcover of the composition $\psi \phi$, and the curve $X$ has a large automorphism group.

Finally, some cases involving maps of low degree have been handled by means of the so-called bigonal [19, 42], trigonal [46], and tetragonal [18] constructions. These rely on auxiliary curves obtained by taking suitable fiber and symmetric products. They overlap with the previous categories to some extent, but do not coincide with them, and have Galois-theoretic interpretations that are not always apparent in the geometric presentation.

Our experimental approach combines aspects of all these previous methods for a new approach to find Prym varieties of general maps $\phi : X \to Y$ in the following way:

**Main Strategy.** Let $\phi : X \to Y$ be a morphism of curves. Find a map $\psi : Y \to \mathbb{P}^1$ of small degree, and use monodromy theory to prove that $\text{Jac}(X) \sim \text{Jac}(Y) \times \text{Jac}(Z/H)$ for a suitable quotient $C = Z/H$ of the Galois closure $Z$ of the composition $\psi \phi : Y \to \mathbb{P}^1$.

Note that, as the genus of $X$ increases, it becomes progressively harder to get explicit equations for the curves and quotients of its Galois closure $Z$. For example, the genus of $Z$ grows exponentially in terms of that of $X$, and the Galois groups of $\psi \phi$ rapidly attain considerable size. We therefore sidestep the consideration of explicit equations using powerful tools from monodromy theory. This is the first essential ingredient of the Main Strategy, which we implemented in the computer algebra system Magma [3].

Specifying covers $X \to Y \to \mathbb{P}^1$ by their ramification structure, we can describe all possible monodromy types for the branched cover $\psi \phi$, which in turn yields complete Galois-theoretic information on the Galois closure $Z \to \mathbb{P}^1$ of this map, without any need to actually write down the maps. The enumeration of possible monodromy types is a classical problem, often used in the setting of Galois covers [11, 44]. In Section 4, we recall the relevant theory and show how to adapt it to our situation, when the covers $X \to Y$ and $X \to \mathbb{P}^1$ need not be Galois. This article also functions as an exposition on the explicit aspects of this classical theory.

The same holds for the second ingredient of the Main Strategy. This is an explicit version of a beautiful result by Chevalley and Weil [15, 54], by means of which one can identify, for a given Galois quotient $\pi_{Z/C} : Z \to C$, the image of $H^0(C, \Omega_C^1)$ by $\pi_{Z/C}^*$ inside $H^0(Z, \Omega_Z^1)$. Since we can similarly describe the images under pullback of $H^0(X, \Omega_X^1)$ and $H^0(Y, \Omega_Y^1)$, intersecting these subspaces allows us to decide if Jac(C) is isogenous to Prym(X/Y), and thus to check the second part of the Main Strategy for the various quotients $C = Z/H$ of Galois closure $Z \to \mathbb{P}^1$. (Occasionally, we also use multiple quotients $Z/H_i$) Implementation details are given in Section 5, and our code can be found at [31].

By using this approach, we could (up to the limitations imposed by keeping the running time of our programs acceptable) recover all situations previously known in the literature, see Section 6, in the following sense:

**Theorem C.** Let $\phi : X \to Y$ be a map studied in one of the references [1, 4, 6, 8, 12, 16, 21, 24, 27, 28, 30, 39, 49]. Then Tables 3 to 6 give maps $Y \to \mathbb{P}^1$ such that (in the notation of the Main Strategy) we have $\text{Jac}(X) \sim \text{Jac}(Y) \times \text{Jac}(Z/H)$ for some quotient $Z/H$ of the Galois closure $Z$. 


Table 1. Decomposition of Jacobian: hyperelliptic case.

| $G$ | $X : y^2 = f(x)$ | $\text{Jac}(X) \sim \prod_{i \in I} \text{Jac}(C_i)$ | Curves $C_i$ |
|-----|------------------|---------------------------------|--------|
| $C_2$ | $x^8 + ax^6 + bx^4 + cx^2 + 1$ | $l = [1, 2]$ | $C_1 : y^2 = \frac{x^2}{(2x^2 + 1)(x^2 + 1)}$, $C_2 : y^2 = x(x^2 + 1)$. |
| $C_2^2$ | $x^8 + ax^6 + bx^4 + ax^2 + 1$ | $l = [1, 2, 3]$ | $C_1 : y^2 = x(x^2 + 1)(x^2 + 1)$, $C_2 : y^2 = x(x^2 + 1)$. |
| $C_4$ | $x(x^2 - 1)(x^2 + bx^2 + b)$ | - | $\text{Jac}(X)$ is simple with endomorphism algebra $\mathbb{Q}(\sqrt{2})$. |
| $C_2 \times C_4$ | $x^8 + ax^6 - ax^2 - 1 = (x^4 - 1)(x^4 + ax^2 + 1)$ | $l = [1, 2]$ | $C_1 : y^2 = (x^4 + ax^2 + 1)$, $C_2 : y^2 = x(x^2 + 1)$, $C_3 : y^2 = x^2 - 3x + a$. |
| $D_6$ | $x(x^6 + ax^3 + 1)$ | $l = [1, 2, 2]$ | $C_1 : y^2 = x^2 + ax + 1$, $C_2 : y^2 = x^2 - 4x^2 + (a + 1)$. |
| $C_4 \times C_4$ | $x^8 + ax^4 + 1$ | $l = [1, 2, 2]$ | $C_1 : y^2 = x^4 + ax^2 + 1$, $C_2 : y^2 = x^4 - 4x^2 + (a + 1)$. |
| $C_{14}$ | $x^7 - 1$ | - | $\text{Jac}(X)$ is simple with endomorphism algebra $\mathbb{Q}(\sqrt{2})$. |
| $U_6$ | $x(x^6 + 1)$ | $l = [1, 1, 1]$ | $C_1 : y^2 = x^3 + x^2 - 4x - 4$, $C_2 : y^2 = x^3 + 2x - 1$. |
| $C_4 \times S_4$ | $x^8 + 14x^4 + 1$ | $l = [1, 1, 1]$ | $C_1 : y^2 = x^3 + x^2 - 4x - 4$, $C_2 : y^2 = x^3 + 2x - 1$. |
| $U_8$ | $x^8 + 1$ | $l = [1, 2, 2]$ | $C_1 : y^2 = x^4 + 1$, $C_2 : y^2 = x^4 - 4x^2 + 2$, $C_3 : y^2 = x^3 + x^2 - 4x - 4$. |

Table 2. Decomposition of the Jacobian: non-hyperelliptic case.

| $G$ | $X : F(x,y,z) = 0$ | $\text{Jac}(X) \sim \prod_{i \in I} \text{Jac}(C_i)$ | Curves $C_i$ |
|-----|------------------|---------------------------------|--------|
| [id] | $x : F(x,y,z) = 0$ | $\text{Jac}(X)$ is simple with endomorphism algebra $\mathbb{Q}$. |
| $C_2$ | $x^8 + y^4 + z^4 + rxy^2z^2 + sy^2z^2 + rz^2x^2$ | $l = [1, 2, 3]$ | $C_1 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$, $C_2 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$. |
| $C_3$ | $x^8 + y^4 + z^4 + rxy^2z^2 + sy^2z^2 + rz^2x^2$ | $l = [1, 2, 2]$ | $C_1 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$, $C_2 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$. |
| $D_4$ | $x^8 + y^4 + z^4 + rxy^2z^2 + sy^2z^2 + rz^2x^2$ | $l = [1, 2, 2]$ | $C_1 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$, $C_2 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$. |
| $G_{16}$ | $x^8 + y^4 + z^4 + rxy^2z^2 + sy^2z^2 + rz^2x^2$ | $l = [1, 2, 2]$ | $C_1 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$, $C_2 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$. |
| $S_4$ | $x^8 + y^4 + z^4 + rxy^2z^2 + sy^2z^2 + rz^2x^2$ | $l = [1, 2, 2]$ | $C_1 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$, $C_2 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$. |
| $C_9$ | $x^8 + y^4 + z^4 + rxy^2z^2 + sy^2z^2 + rz^2x^2$ | $l = [1, 2, 2]$ | $C_1 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$, $C_2 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$. |
| $G_{48}$ | $x^8 + y^4 + z^4 + rxy^2z^2 + sy^2z^2 + rz^2x^2$ | $l = [1, 2, 2]$ | $C_1 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$, $C_2 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$. |
| $G_{96}$ | $x^8 + y^4 + z^4 + rxy^2z^2 + sy^2z^2 + rz^2x^2$ | $l = [1, 2, 2]$ | $C_1 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$, $C_2 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$. |
| $G_{168}$ | $x^8 + y^4 + z^4 + rxy^2z^2 + sy^2z^2 + rz^2x^2$ | $l = [1, 2, 2]$ | $C_1 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$, $C_2 : y^2 = (x^8 + rxy^2z^2 + sy^2z^2 + rz^2x^2)$. |

We also found some new cases. For example, consider a cover $X \to Y$ of degree $d$, where $Y$ is an elliptic curve, totally ramified over 2 points $Q_1, Q_2$ of $Y$, and compose it with the map $Y \to \mathbb{P}^1$ which identifies $Q_1$ and $Q_2$. For $d = 3, 4$, and for some cases with $d = 5$ (see Table 8), we have been able to check within reasonable time that the abelian variety Prym($X/Y$) is indeed isogenous to the Jacobian of a quotient $C$ of the Galois closure $Z$ of $X \to Y \to \mathbb{P}^1$. When $d = 3$ and $X \to Y$ is a non-Galois cover, it turns out that $X$ and the corresponding Galois closure $Z$ of genus 5 are hyperelliptic. Using this information, we were able to write down equations for $X$ and $Y$, and to find an explicit equation for a curve $C$ for which $\text{Jac}(C) \sim \text{Prym}(X/Y)$, see Section 6.3. In another direction, we were able to generalize the example of $[12]$ (a genus 5 étale cover of degree 2 of a genus 3 curve) to genus $g_X = 2g + 1$ étale covers of degree 2 of genus $g_Y = g$ curves with $g = 4, 5$ or 6, where $Y$ is a generic trigonal curve. In particular, for $g_Y = 4$ we cover all generic non-hyperelliptic cases in this way: This turns out to be a special case of a result of Recillas [46], that our method allowed us to rediscover by experimentation.

We conclude with some remarks. First, if there exists a curve $C$ with $\text{Jac}(C) \sim \text{Prym}(X/Y)$, then there is a correspondence $C \to Z \to X$, and in particular we get a map $Z \to X \to Y$. One of our goals is to find a map $\psi : Y \to \mathbb{P}^1$ such that the composition
Table 3. Recovering Ritzenthaler–Romagny.

| Case   | \(g_X, g_Y, d_X\) | Ramification | \#G, g_Z | \(X\) nhyp/hyp | Prym dims | \(\deg Z \to C_i\) |
|--------|--------------------|--------------|----------|----------------|-----------|-------------------|
| rr-spec | 3, 1, 2            | 4, 2         | 4, 3     | [3, 0], [0, 0] | [1, 1]    | [2]               |
| rr-gen  | 3, 1, 2            | 4, 2         | 8, 7     | [8, 0], [0, 0] | [2]       | [2]               |

Table 4. Genus 2 to genus 1.

| Case   | \(g_X, g_Y, d_X\) | Ramification | \#G, g_Z | \(X\) nhyp/hyp | Prym dims | \(\deg Z \to C_i\) |
|--------|--------------------|--------------|----------|----------------|-----------|-------------------|
| g2-2   | 2, 1, 2            | 4, 2         | [0, 0], [4, 0] | [1]       | [2]               |
| g2-3   | 2, 1, 3            | 12, 4        | [0, 0], [16, 0] | [1]       | [2]               |
| g2-4   | 2, 1, 4            | 16, 13       | [0, 0], [48, 0] | [1]       | [2]               |
| g2-5   | 2, 1, 5            | 240, 61      | [0, 0], [160, 0] | [1]       | [12]              |
| g2-6   | 2, 1, 6            | 48, 13       | [0, 0], [72, 0] | [1]       | [4]               |
| g2-7   | 2, 1, 7            | 10080, 2521  | [0, 0], [672, 0] | [1]       | [240]             |

Z \to X \to Y \to \mathbb{P}^1 is a subquotient of the Galois closure of X \to Y \to \mathbb{P}^1. As one of the referees has pointed out, it is to be expected that for a sufficiently general element \(z\) of the function field of \(Z\), the trace \(\text{tr}_{k(Z)/k(Y)}(z)\) generates a rational subfield of \(Y\) that gives rise to such a map \(\psi\). However, this construction is a mere existence statement, and assumes that \(C\) and the relevant correspondence have already been found. By contrast, our algorithms find the map \(\psi\) more explicitly.

Second, our construction can also be seen through the lens of Belyi’s theorem. Indeed, Belyi’s result guarantees that all algebraic curves arise as covers of \(\mathbb{P}^1\) ramified over just 3 points, and our approach usually consists in finding a morphism \(Y \to \mathbb{P}^1\) such that the branch locus of \(X \to Y \to \mathbb{P}^1\) is smaller than it would be for a generic choice of the map \(Y \to \mathbb{P}^1\). In fact, this suggests that we are only scratching the surface, handling just the easiest of cases, and it might even be possible to always recover the curves \(C_i\), if they exist, by choosing a suitable morphism \(\psi : Y \to \mathbb{P}^1\) that identifies suitable branch points of \(\phi : X \to Y\). This point of view toward decomposing the Prym variety of \(X \to Y\), which finds the map \(\psi : Y \to \mathbb{P}^1\) by an explicit identification of branch points, seems to be genuinely new. We think that these experiments and theoretical motivations are sufficiently intriguing for the relations between Prym varieties and Galois constructions to merit further study.
2. Decomposing Jacobians of curves of genus 3 with nontrivial automorphism group

Let $k$ be an algebraically closed field of characteristic 0. We will implicitly assume that $k = \mathbb{C}$ throughout this article, especially in Section 4 when using the theory of covers. We do note that the results thus obtained will still be valid when the characteristic of $k$ does not divide the order of the intervening Galois groups — for example, the results in the current section continue to apply when $k$ has finite characteristic strictly larger than 7.

In this section, we consider the decomposition of Jacobians of curves of genus 3 induced by the action of their automorphism group. Most of these results are folklore. Note that this approach does not always yield the full decomposition of the Jacobian, nor can it guarantee that the higher dimensional factors found in this way are indecomposable.

### 2.1. Hyperelliptic case

There is a stratification of the moduli space of hyperelliptic curves of genus 3 by their automorphism group (see [9, 35] and the references in the latter). The inclusions between the different strata are summarized in the following diagram, where we write $C_n$ for the cyclic group of order $n$, $D_n$ for the dihedral group of order $2n$, $S_n$ for the symmetric on $n$ elements and $U_6$ and $U_8$ for certain

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**Table 5.** Testing Dalaljan’s results.

| Case | $g_X, g_Y, d_X$ | Ramification | #G, gZ | X nhyp/hyp | Prym dims | deg $Z \rightarrow C_i$ |
|------|----------------|--------------|--------|------------|-----------|-------------------|
| g2   | 4, 2, 2        | 6, 2         | 8      | 9          | [32, 0, 0] | [0, 0] | [2] | [2] |
| g3   | 6, 3, 2        | 8, 2         | 8      | 13         | [128, 0, 0]| [0, 0] | [3] | [2] |
| g4   | 8, 4, 2        | 10, 2        | 8      | 17         | [512, 0, 0]| [0, 0] | [4] | [2] |

**Table 6.** Generalizing Bruin’s results.

| Case | $g_X, g_Y, d_X$ | Ramification | #G, gZ | X nhyp/hyp | Prym dims | deg $Z \rightarrow C_i$ |
|------|----------------|--------------|--------|------------|-----------|-------------------|
| 3-orig | 5, 3, 2        | 10, 2        | 24     | 37         | [24, 0, 0] | [0, 0] | [2] | [6] |
| 3-g7 | 7, 4, 2        | 12, 2        | 24     | 49         | [7, 0, 0]  | [0, 0] | [2] | [6] |
| 3-g9 | 9, 5, 2        | 14, 2        | 24     | 61         | [7, 0, 0]  | [0, 0] | [2] | [6] |
| 3-g11 | 11, 6, 2       | 16, 2        | 24     | 73         | [7, 0, 0]  | [0, 0] | [2] | [6] |
groups with, respectively, 24 and 32 elements.

Proposition 2.1. Suppose $X$ is a hyperelliptic curve of genus 3 whose automorphism group contains a group $G$ appearing in the previous diagram. Then the Jacobian of $X$ decomposes up to isogeny as the product of Jacobians of quotients of $X$ described in Table 1.

2.2. Non-hyperelliptic case

A similar analysis can be carried out in the non-hyperelliptic case, with the notable exception of the group $C_2$ and its specialization $C_6$ which shall be reviewed in Section 3. There is a stratification of the moduli space of non-hyperelliptic genus 3 curves according to their automorphism group (see [23, 2.88], [53, p.62], [38], [2], and [17]; the groups $G_i$ are certain groups of order $i$). The inclusions between the different strata are summarized in the following diagram:

Proposition 2.2. Suppose $X$ is a non-hyperelliptic curve of genus 3 whose automorphism group contains a group $G$ appearing in the previous diagram. Then the Jacobian of $X$ decomposes up to isogeny as the product of Jacobians of quotients of $X$ described in Table 2.

Together, Propositions 2.1 and 2.2 imply Theorem A.

3. Plane quartics with automorphism group $C_2$ or $C_6$

Let $X/k$ be a non-hyperelliptic curve of genus 3 with automorphism group $C_2$. The action of the automorphism induces a map $\pi : X \to Y$ of degree 2, where $Y$ is an elliptic curve. Hence we know that $\text{Jac} \sim Y \times A$, but $A$ is not the Jacobian of a subcover of $X$. Indeed, the Riemann-Hurwitz formula shows that any morphism $X \to C$ with $g(C) = 2$ must be of degree 2, and therefore would be a quotient by another involution of $X$, which does not exist. The problem of describing $A$ up to isogeny as the Jacobian of an explicit curve $C$ of genus 2 was solved in [49]. A related result was previously obtained in [20] in a more analytic setting. In addition, one of the referees observed that this situation is also a special case of the tetragonal construction [18], where one of the maps is simply the
quotient by the hyperelliptic involution. The solution in [49] relies on a suitable deformation of $\pi$ to an étale covering map between curves of genus 5 and 3. This result is recalled below.

**Proposition 3.1 (Ritzenthaler-Romagny [49]).** Let $X$ be a smooth, non-hyperelliptic genus 3 curve defined by

$$X : y^4 - h(x, z) y^2 + f(x, z) g(x, z) = 0$$

in $\mathbb{P}^2_k$, where

$$f = f_2 x^2 + f_1 x z + f_0 z^2, \quad g = g_2 x^2 + g_1 x z + g_0 z^2, \quad h = h_2 x^2 + h_1 x z + h_0 z^2$$

are homogeneous polynomials of degree 2 over a field $k$ of characteristic different from 2. The involution $(x : y : z) \mapsto (x : -y : z)$ induces a cover $\pi$ of degree 2 of the genus 1 curve

$$Y : y^2 - h(x, z) y + f(x, z) g(x, z) = 0$$

in the weighted projective space $\mathbb{P}(1, 2, 1)$. Let

$$M = \begin{bmatrix} f_2 & f_1 & f_0 \\ h_2 & h_1 & h_0 \\ p_2 & p_1 & p_0 \end{bmatrix}$$

and assume that $M$ is invertible. Let

$$M^{-1} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$ 

Then $\text{Jac}(X) \sim \text{Jac}(Y) \times \text{Jac}(C)$ with $C : y^2 = b \cdot (b^2 - ac)$ in $\mathbb{P}(1, 3, 1)$ where

$$a = a_1 + 2a_2 x + a_3 x^2, \quad b = b_1 + 2b_2 x + b_3 x^2, \quad c = c_1 + 2c_2 x + c_3 x^2.$$ 

In the special case, when the automorphism group of $X$ is $C_6$, it can be realized as a plane quartic

$$X : x^2 z + y^4 + r y^2 z^2 + z^4 = 0$$

for some $r \in k$, and we similarly find $\text{Jac}(X) \sim \text{Jac}(C_1) \times \text{Jac}(C_2)$ with

$$\begin{cases} C_1 : & y^2 = -x^3 + r^2/4 - 1 \\ C_2 : & y^2 = (x^2 - 2x - 2)(x^4 - 4x^3 + (-2r^2 + 8)x - r^2 + 4) \end{cases}$$

Since over the algebraically closed field $k$ every non-hyperelliptic genus 3 curve with automorphism group $C_2$ (resp. $C_6$) can be written in the form (3.1) (resp. (3.2)), this completes the tables of Section 2. In the next subsection we explain a different approach to handle the case of non-hyperelliptic curves with automorphism group $C_2$. This will serve as motivation for the generalization discussed in Section 4.

### 3.1. A Galois approach

Let $X$ be as in the previous section, that is, a non-hyperelliptic genus 3 curve with an involution. The corresponding quotient is a curve $Y$ of genus 1, and the morphism $\pi_{X/Y} = \pi : X \rightarrow Y$ of degree 2 is branched over 4 distinct points $Q_1, Q_2, Q_3$ and $Q_4$. Let us consider a morphism $\pi_{Y/P^1} : Y \rightarrow P^1$ which maps $Q_1$ and $Q_2$ to the same point $[\beta, 1] \in P^1$ with $\beta \neq 0$. Choosing an origin on $Y$, and thereby giving it the structure of an elliptic curve, this morphism can be constructed by taking the quotient of the elliptic curve $Y$ by the involution $P \mapsto Q_1 + Q_2 - P$. Composing with an automorphism of $P^1$ if necessary, we can and will assume that $\pi_{Y/P^1}(Q_3) = [0, 1]$. Additionally, we write $\pi_{Y/P^1}(Q_4) = [y, 1]$.

**Remark 3.2.** Consider the special case $\gamma = 0$, that is, the morphism $Y \rightarrow P^1$ identifies $Q_1$ with $Q_2$, as well as $Q_3$ with $Q_4$. The methods developed later on in Section 4 will enable us to show that this happens if and only if the composite map $X \rightarrow P^1$ is Galois, with Galois group $C_2^2$ (see Table 3). In this case, $\text{Aut}(X)$ contains a copy of the Klein four-group $C_2^2$, and we see from Table 2 that $\text{Jac}(X)$ decomposes as the product of three elliptic curves, each of which is a quotient of $X$. 
From now on we restrict to the case \( \gamma \neq 0 \). By Enolskii and Richter [20], Levin [30], we have the following equations: we may write \( Y : y^2 = f(t) = (t - \alpha_1)(t - \alpha_2)(t - \alpha_3) \) and

\[
X = \begin{cases} 
  y^2 = f(t), \\
  x^2 = (t - \beta)(p_2(t) + \gamma),
\end{cases}
\]

where \( p_2 \) is a polynomial of degree 2 such that \( p_2(t)^2 - f(t) = t(t - \gamma)p_1(t)^2 \) with \( p_1(t) \) a polynomial of degree 1.

Let \( Z \to \mathbb{P}^1 \) be the Galois closure of \( X \to \mathbb{P}^1 \). The Galois group of \( Z/\mathbb{P}^1 \) is isomorphic to \( D_4 \), the dihedral group on 4 elements. We write \( D_4 = \langle r, s \mid r^4 = s^2 = 1, sr = r^3s \rangle \), assuming (as we may) that \( s \) is the non-central element of order 2 such that \( Z/\langle s \rangle \cong X \). Let \( \pi \) be any root of \( X^2 = (t - \beta)(p_2(t) - \gamma) \). We then see that \( Z \) is the smooth projective curve an affine part of which is given by

\[
\begin{align*}
  y^2 &= f(t), \\
  x^2 &= (t - \beta)(p_2(t) + \gamma), \\
  \pi^2 &= (t - \beta)(p_2(t) - \gamma).
\end{align*}
\]

Since \( X \) corresponds to the quotient of \( Z \) by \( s \), we know that \( s \) sends \( (x, \pi, y) \) to \( (x, -\pi, y) \). We can choose \( r \) to be \( (x, \pi, y) \mapsto (\pi, -x, -y) \). Direct inspection of the subgroup lattice of \( D_4 \) implies that the maps \( Z \to X \to Y \to \mathbb{P}^1 \) fit into a larger diagram of maps of degree 2:

![Diagram](image)

Knowing the action of \( s \) and \( r \) explicitly allows us to work out equations for the various quotients in the previous diagram and to compute their genera using the Riemann-Hurwitz formula. We will be mainly interested in \( C = Z/\langle sr \rangle \). Consider the \( sr \)-invariant functions \( v = x + \pi \) and \( w = \frac{x\pi}{(t - \beta)p_1(t)} \). Note that the invariant function \( z := y(x - \pi) \) also lies in the function field \( k(v, w, t) \), since \( vz = y(x^2 - \pi^2) = 2f(t)(t - \beta) \). The relations between \( v \), \( w \) and \( t \) describe the quotient curve

\[
C = \begin{cases} 
  v^2 = 2(t - \beta)(p_2(t) + wp_1(t)), \\
  w^2 = t(t - \gamma)
\end{cases}
\]

which is indeed a cover of \( \mathbb{P}^1 \) of degree 4, as it is a cover of degree 2 of a conic that is in turn a cover of degree 2 of \( \mathbb{P}^1 \).

The second equation in (3.4) describes a conic with a rational point, which may be parametrized as \( (t, w) = \left( \frac{\gamma}{1 - u}, \frac{\gamma u}{1 - u^2} \right) \). Replacing this parametrization in the first equation and setting \( s := (1 - u^2)^2 \) we then get the hyperelliptic model

\[
s^2 = 2 \left( \gamma - \beta(1 - u^2) \right) \left( (1 - u^2)^2p_2\left( \frac{\gamma}{1 - u} \right) + \gamma u(1 - u^2)p_1\left( \frac{\gamma}{1 - u^2} \right) \right).
\]

**Remark 3.3.** The model (3.5) is smooth and defines a curve of genus 2. Indeed, one may check that (under our assumptions \( \beta \neq 0, \gamma \neq 0, \beta \neq \gamma \)) the irreducible factors of the discriminant of the polynomial on the right hand side are also factors of either \( \text{disc}(f) \) or \( \text{Res}_s(f(t), (t - \beta)p_2(t)) \). This shows that (3.5) is smooth, because \( \text{disc}(f) = 0 \) (resp. \( \text{Res}_s(f(t), (t - \beta)p_2(t)) = 0 \)) would imply that \( Y \) (resp. \( X \)) is not smooth.

We now aim to show that the Prym variety of the cover \( X \to Y \) is isogenous to \( \text{Jac}(C) \) (Theorem 3.6). In order to do so, we begin by investigating the action of \( D_4 \subset \text{Aut}(Z) \) on the space of regular differentials \( H^0(Z, \Omega_Z^2) \). We will freely use some results that will be discussed in general in Section 4, see in particular Theorem 4.25. Recall that the character table of \( D_4 \) is as follows:

| \{id\} | \{r^2\} | \{s, sr^2\} | \{sr, sr^3\} | \{r, r^3\} |
|-------|---------|-----------|-----------|---------|
| (1)   | 1       | 1         | 1         | 1       |
| \( V_1 \) | 1       | 1         | -1        | -1      | 1       |
| \( V_2 \) | 1       | 1         | 1         | -1      | -1      |
| \( V_3 \) | 1       | 1         | -1        | 1       | -1      |
| (2)   | 2       | -2        | 0         | 0       | 0       |
Note in particular that $r^2$ acts trivially on the one-dimensional representations $V_1$, $V_2$, $V_3$ and as $-1$ on (2), while the fixed subspace in (2) of each of the symmetries $s$, $sr$, $sr^2$, and $sr^3$ is one-dimensional.

**Lemma 3.4.** We have $H^0(Z, \Omega_2^1) \cong V_1^{\oplus 2} \oplus V_2 \oplus (2)^{\oplus 2}$ as representations of $D_4$.

**Proof.** Write $H^0(Z, \Omega_2^1) \cong (1)^{\oplus 6} \oplus V_1^{\oplus 1} \oplus V_2^{\oplus 2} \oplus V_3^{\oplus 3} \oplus (2)^{\oplus 4}$ as representations of $D_4$. Let $H$ be any subgroup of $G$. One has

$$H^0(Z, \Omega_2^1)^H \cong H^0(Z/H, \Omega_{Z/H}^1),$$

which implies that the dimension of the subspace of $H^0(Z, \Omega_2^1)$ fixed by $H$ is the genus of $Z/H$. Applying this to $H = G$, and observing that $Z/G \cong \mathbb{P}^1$ has genus 0, we obtain that $H^0(Z, \Omega_2^1)$ does not contain any copy of the trivial representation, i.e., $e_0 = 0$. Applying the same argument with $H = \langle r^2 \rangle$ one obtains $g(Z/H) = 3 = \dim H^0(Z, \Omega_2^1)^H$, and since $r^2$ acts trivially on $V_1$, $V_2$, $V_3$ and without fixed points on (2) this implies $3 = e_1 + e_2 + e_3$. We also have the condition $e_1 + e_2 + e_3 + 2e_4 = \dim H^0(Z, \Omega_2^1) = 7$, so—combining the last two equations—we obtain $e_4 = 2$. Finally, the conditions

$$3 = g(Z/\langle s \rangle) = \dim H^0(Z, \Omega_2^1)^{\langle s \rangle} = e_2 + e_4 \quad \text{(3.7)}$$

and

$$2 = g(Z/\langle sr \rangle) = \dim H^0(Z, \Omega_2^1)^{\langle sr \rangle} = e_3 + e_4 \quad \text{(3.8)}$$

imply $e_2 = 1, e_3 = 0$ and therefore $e_1 = 2$.

**Lemma 3.5.** The correspondence

$$\pi_1 \quad \pi_2 \quad g = 3 \quad g = 2$$

$$X = Z/\langle s \rangle \quad C = Z/\langle sr \rangle \quad g = 3$$

induces a homomorphism of Abelian varieties $\text{Jac}(Z/\langle sr \rangle) \to \text{Jac}(X)$ with finite kernel. In particular, $\text{Jac}(Z/\langle sr \rangle)$ is a factor of $\text{Jac}(X)$ in the category of Abelian varieties up to isogeny.

**Proof.** We consider the action of this correspondence on regular differentials and determine the image of

$$\pi_1 \pi_2^* : H^0(C, \Omega_C^1) \to H^0(X, \Omega_X^1). \quad \text{(3.10)}$$

The image of $\pi_2^*$ is the $sr$-invariant subspace of $H^0(Z, \Omega_2^1)$; given our description of $H^0(Z, \Omega_2^1)$ as a $D_4$-representation, we see that this is precisely the $sr$-invariant subspace in (2)$^{\oplus 2}$. Identifying $H^0(X, \Omega_X^1)$ with $H^0(Z, \Omega_2^1)^{\langle s \rangle}$, the map

$$\pi_1 : H^0(Z, \Omega_2^1) \to H^0(X, \Omega_X^1) \cong H^0(Z, \Omega_2^1)^{\langle s \rangle} \quad \text{(3.11)}$$

is given by $\omega \mapsto \omega + s^*\omega$. Since the structure of the two-dimensional representation (2) shows that the map $(1 + s)$ is injective on its $sr$-invariant subspace, we obtain that $\pi_1$ is injective on the image of $\pi_2^*$. This implies that the image of $\pi_1, \pi_2^*$ is two-dimensional, which in turn means that the image of $\text{Jac}(Z/\langle sr \rangle) \to \text{Jac}(X)$ is two-dimensional as claimed.

We now prove a more specific version of Theorem B.

**Theorem 3.6.** The Jacobian of $X$ decomposes up to isogeny as

$$\text{Jac}(X) \sim Y \times \text{Jac}(Z/\langle sr \rangle). \quad \text{(3.12)}$$

As a consequence, $\text{Jac}(Z/\langle sr \rangle)$ is isogenous to the Prym variety of $\pi : X \to Y$, and a nontrivial map $\text{Jac}(Z/\langle sr \rangle) \to \text{Jac}(X)$ is induced by the correspondence $Z$ in (3.9).

**Proof.** In light of Lemma 3.5, it suffices to prove that the two subspaces $\pi_1^* H^0(Y, \Omega_Y^1)$ and $\pi_2^* H^0(C, \Omega_C^1)$ of $H^0(X, \Omega_X^1)$ generate this vector space, or equivalently (by dimension considerations) that they intersect trivially. Since $\pi_1^* : H^0(X, \Omega_X^1) \to H^0(Z, \Omega_2^1)$ is injective, it suffices to prove that they intersect trivially after pullback to $H^0(Z, \Omega_2^1)$. One can describe the subspaces $\pi_2^* H^0(Y, \Omega_Y^1)$ and $\pi_1^* \pi_1 \pi_2^* H^0(C, \Omega_C^1)$ in terms of the action of $D_4$: according to Diagram (3.3) and Lemma 3.4, $\pi_1^* \pi_2^* H^0(Y, \Omega_Y^1) = H^0(Z, \Omega_2^1)^{\langle rs \rangle} = V_2$, while

$$\pi_1^* \pi_1 \pi_2^* H^0(C, \Omega_C^1) = (1 + s)H^0(Z, \Omega_2^1)^{\langle sr \rangle}. \quad \text{(3.13)}$$

It now suffices to note that $sr$ has no nonzero fixed points in $V_1^{\oplus 2} \oplus V_2$, so $H^0(Z, \Omega_2^1)^{\langle sr \rangle}$ is contained in (2)$^{\oplus 2}$. Since (2)$^{\oplus 2}$ is a subrepresentation of $H^0(Z, \Omega_2^1)$ it follows that also $(1 + s)H^0(Z, \Omega_2^1)^{\langle sr \rangle}$ is contained in (2)$^{\oplus 2}$, hence it does not intersect $H^0(Z, \Omega_2^1)^{\langle rs \rangle}$ as claimed. We conclude as desired that $\pi_1^* \pi_2^* H^0(Y, \Omega_Y^1)$ and $\pi_2^* (C, \Omega_C^1)$ together generate $H^0(Z, \Omega_2^1)^{\langle s \rangle} = H^0(X, \Omega_X^1)$. \qed
Theorem 3.6 recovers Proposition 3.1 and also clarifies the nature of a correspondence between C and X. In addition, notice that the curve C described in Proposition 3.1 depends on the choice of a factorization \( f(x, z)g(x, z) \) of a certain polynomial of degree 4 as the product of two quadratics. Note that the zero locus of \( f(x, z)g(x, z) \) on \( Y \) describes precisely the branch locus of \( X \to Y \). In our new approach, the choice of factorization can be reinterpreted as the choice of the two points \( Q_1, Q_2 \) that are contracted by the morphism \( \pi_{Y/P^1} \).

Remark 3.7. In [49], the aforementioned choice of a partition of 4 points into 2 pairs is clearly symmetric in the pairs. By contrast, in this new approach the choice is highly asymmetric since 2 points are contracted and the other 2 are not.

4. An algorithmic approach via group theory

Our purpose in this section is to generalize the previous discussion to more complicated cases, for which explicit equations are not available. The proof of Lemma 3.4 relied strongly on the fact that we could compute the genus of any quotient of \( Z \) by direct inspection of the equations of the curves and of the action of automorphisms. In general, it is more difficult to get such information explicitly, so in this section we explain how we may reverse the process: we first describe the action of \( \text{Aut}(Z \to P^1) \) on \( H^0(Z, \Omega^1_Z) \) (Section 4.4.3), and subsequently rely on this information to completely describe the morphisms of Jacobians of curves obtained as quotients of \( Z \) (Section 4.4.4). The method has its roots in the theory of monodromy actions for branched covers of curves. While developing the main notions of this theory below, we show how it can be combined with the description of the aforementioned action, and also give some explicit references for useful statements in this context, in particular Theorem 4.10.

4.1. Preliminaries on ramification and monodromy

In this section, we fix our notation and conventions for describing the ramification of a morphism of smooth projective curves over \( \mathbb{C} \). We will freely use without further mention the fact that the category of such curves is equivalent to the category of compact Riemann surfaces, and assume that all our curves are connected. We will find it useful to introduce the following definition:

Definition 4.1. Let \( \varphi : X \to Y \) be a morphism of smooth projective curves over \( \mathbb{C} \) and let \( B = (b_1, \ldots, b_n) \) be a fixed ordered subset of \( Y \) which contains the branch locus of \( \varphi \). For \( b \in Y \), let \( \varphi^{-1}(b) = \{a_1, \ldots, a_k\} \) be the fiber of \( \varphi \) above \( b \) and suppose that this set contains \( m_i \) points of ramification index \( e_i \), with the \( e_i \) distinct and with \( i \) running from 1 to \( r \), say. Then the ramification structure of \( \varphi \) at \( b \) is the set \( R_b := \{(e_1, m_1), \ldots, (e_r, m_r)\} \). The ramification structure of \( \varphi \) is the tuple \( R := (R_b : i = 1, \ldots, n) \).

Remark 4.2. The ramification structure \( R \) depends on \( B \) and on the ordering of the points in \( B \) — even though this is not emphasized by our notation, the choice of \( b_1, \ldots, b_n \) should always be clear from the context. Note furthermore that the definition above allows one to include the ramification structure at \( b \) for points in the complement of the branch locus. In this case, the ramification structure at \( b \) is \( R_b := \{(1, \deg \varphi)\} : \) all \( \deg \varphi \) points in the fiber over \( b \) have ramification index 1.

Remark 4.3. We will connect ramification structures with the cycle type of certain permutations. We therefore agree to also write cycle types in the previous way: if the permutation \( \sigma \) contains \( m_i \) cycles of length \( e_i \), with the \( e_i \) distinct and with \( i \) running from 1 to \( r \), say, then we write its cycle type as \( \{(e_1, m_1), \ldots, (e_r, m_r)\} \).

Example 4.4. Let \( X \) be a smooth projective curve of genus 3, let \( Y \) be an elliptic curve, and \( \varphi : X \to Y \) be a morphism of degree 2. The Riemann-Hurwitz formula immediately implies that \( \varphi \) is ramified at exactly 4 points, each with ramification index 2. If we take \( B \) to be the branch locus of \( \varphi \) (consisting of 4 points, ordered arbitrarily), then the ramification structure of \( \varphi \) is \( \{(2, 1), (2, 1), (2, 1), (2, 1)\} \).

We now recall some basic facts about monodromy. Consider a morphism \( \varphi : X \to Y \) of smooth projective curves over \( \mathbb{C} \). Let \( B = (b_1, \ldots, b_n) \) be a finite ordered subset of \( Y \) which contains the branch locus of \( \varphi \), and fix a base point \( q \in Y - B \). Also fix loops \( \gamma_1, \ldots, \gamma_n \), based at \( q \), with the property that \( \gamma_i \) is nontrivial in \( \pi_1(Y - B, q) \) but trivial in \( \pi_1(Y - (B - \{b_i\}), q) \), and that winds precisely once in the counter-clockwise direction around \( b_i \). We will call such a loop a small loop based at \( q \) around \( b_i \). The classes \( [\gamma_1], \ldots, [\gamma_n] \) then generate the fundamental group of \( Y - B \). One can classify all maps \( \varphi \) with branch locus contained in \( B \) and of fixed degree in terms of representations of the fundamental group \( \pi_1(Y - B, q) \). More precisely, we have

Theorem 4.5 ([37, Proposition 4.9]). Let \( Y \) be a compact Riemann surface, \( B \) be a finite subset of \( Y \), and let \( q \) be a base point of \( Y - B \). There is a bijection

\[
\begin{cases}
\text{isomorphism classes of} \\
\text{holomorphic maps } \varphi : X \to Y \\
of degree d \\
\text{whose branch points} \\
\text{lie in } B
\end{cases}
\leftrightarrow
\begin{cases}
\text{group homomorphisms } \\
\rho : \pi_1(Y - B, q) \to S_d \\
\text{with transitive image} \\
\text{up to conjugacy in } S_d
\end{cases}
\]
denoted by \( \varphi, \rho \leftrightarrow \rho \). If \( \gamma_i \) is a small loop based at \( q \) around \( b_i \in B \), the ramification structure of \( \varphi, \rho \) at \( b_i \) is the cycle type of \( \sigma_i := \rho(\langle \gamma_i \rangle) \).

As an immediate consequence of the previous theorem, we have

**Corollary 4.6.** With the same notation as in the theorem, the ramification structure of \( \varphi, \rho \) : \( X \to Y \) is determined by the conjugacy classes in \( S_d \) of \( \rho(\langle \gamma_i \rangle) \) for \( i = 1, \ldots, n \).

**Definition 4.7.** In the situation of the previous theorem, we will call the tuple \( \Sigma = (\sigma_1, \ldots, \sigma_n) = (\rho(\langle \gamma_1 \rangle), \ldots, \rho(\langle \gamma_n \rangle)) \) the monodromy datum associated with \( \varphi \).

**Remark 4.8.** The monodromy datum \( \Sigma \) alone does not uniquely identify a map \( \varphi : X \to Y \) (even up to isomorphism), because one also needs to specify the ordered set of points \( (b_1, \ldots, b_n) \), any choice of such an ordered set will lead to a map \( \varphi \) with the same ramification structure. Recall from \([38, \S 3]\) that the dimension of the moduli space of covers of \( \mathbb{P}^1 \) branched over \( n \) points has dimension \( n - 3 \), so by letting the branch locus vary we get \((n - 3)\)-dimensional families of curves with fixed monodromy.

We now specialize this discussion to the case \( Y = \mathbb{P}^1 \). The fundamental group of \( \mathbb{P}^1 - B \) is generated by \([\gamma_1], \ldots, [\gamma_n]\), and by choosing the loops \( \gamma_i \) appropriately, the relations between these classes are generated by the single relation \( \prod_{i=1}^n [\gamma_i] = 1 \). Thus, given \( \sigma_1, \ldots, \sigma_n \in S_d \) that satisfy \( \prod_{i=1}^n \sigma_i = 1 \), we can define a homomorphism

\[ \rho : \pi_1(\mathbb{P}^1 - B, q) \to S_d \]

by sending \([\gamma_i]\) to \( \sigma_i \) and every homomorphism arises in this way for some \( (\sigma_1, \ldots, \sigma_n) \). Thus, we obtain the following special important case of Theorem 4.5:

**Theorem 4.9 ([37, Corollary 4.10]).** There is a bijective correspondence

\[
\begin{align*}
\text{isomorphism classes of} & \quad \text{conjugacy classes of n-tuples} \\
\text{holomorphic maps } \varphi : C \to \mathbb{P}^1 & \quad (\sigma_1, \ldots, \sigma_n) \text{ of permutations in } S_d \\
\text{of degree } d & \quad \text{such that } \sigma_1 \cdots \sigma_n = 1 \\
\text{whose branch points} & \quad \text{and the subgroup generated by the } \sigma_i \\
\text{lie in } B & \quad \text{is transitive}
\end{align*}
\]

which enjoys the following additional property: the ramification structure at \( b_i \) of the map \( \varphi \) corresponding to \( (\sigma_1, \ldots, \sigma_n) \) is the cycle type of \( \sigma_i \).

### 4.2. Galois closure of a morphism of curves

Given a non-constant morphism \( \varphi : X \to Y \) of smooth projective curves over \( \mathbb{C} \), it makes sense to consider the corresponding (finite, separable) field extension \( \varphi^* \mathbb{C}(Y) \subseteq \mathbb{C}(X) \). As with any such extension, we can then consider the Galois closure of \( \mathbb{C}(X) \) over \( \varphi^* \mathbb{C}(Y) \), which by the equivalence between smooth projective curves over \( \mathbb{C} \) and extensions of \( \mathbb{C} \) of transcendence degree 1 corresponds to a curve \( \tilde{C} \) equipped with a canonical morphism \( \tilde{C} \to X \). We call \( \tilde{C} \) (equipped with its maps \( \tilde{C} \to X \to Y \)) the Galois closure of \( X \to Y \). There is a natural action of \( G \) on \( \tilde{C} \), and for a subgroup \( H \) of \( G \) we write \( \tilde{C}/H \) for the curve corresponding to the subfield of \( \mathbb{C}(\tilde{C}) \) fixed by \( H \).

We now recall a description of the Galois closure in terms of the monodromy datum. Suppose the map \( \varphi : X \to Y \) corresponds, as in Theorem 4.5, to \( B = (b_1, \ldots, b_n) \) and to the representation \( \rho \). As in the statement of the theorem, let \( \gamma_i \) be a small loop based at \( q \) around \( b_i \). Finally, let \( \sigma_i = \rho(\langle \gamma_i \rangle) \). Then we have the following description of the Galois closure of \( \varphi \):

**Theorem 4.10.** Let \( \tilde{\varphi} : \tilde{C} \to X \to Y \) be the Galois closure of \( \varphi : X \to Y \). Then:

(i) the Galois group of \( \tilde{C}/Y \) is the subgroup \( G \) of \( S_d \) generated by the \( \sigma_i \), and the degree of \( \tilde{\varphi} \) is \( |G| \);

(ii) the branch locus of \( \tilde{\varphi} \) is contained in \( B \);

(iii) the corresponding representation \( \rho_{\tilde{\varphi}} \) is obtained as follows: identifying \( S_{|G|} \) with the group of permutations of the elements of \( G \), the class \( [\gamma_i] \) is sent to the permutation of \( G \) induced by left-multiplication by \( \sigma_i \).

This is all explained in [5], which, however, does not contain a separate statement that comprises all three items above. We therefore include a short proof with more detailed references:

**Proof.** Part (i) is in [5, §4.3.1]. Part (ii) follows from the equivalence between curves and function fields: a point \( b \in Y \) is a branch point for \( \varphi : X \to Y \) precisely when the corresponding place of \( \mathbb{C}(Y) \) ramifies in \( \mathbb{C}(X) \). Moreover, because a compositum of unramified extensions of local fields is unramified [41, II.7.3], the branched places of the extension \( \mathbb{C}(X)/\varphi^*\mathbb{C}(Y) \) coincide with those of its Galois closure. Finally, (iii) is part of the theory of \( G \)-sets [36, Chapter V], [29, Chapter 1]. More generally, if \( H \) is any
subgroup of G, then the fiber of $\tilde{G}/H \to \tilde{G}/G = Y$ is identified with $G/H$, and the monodromy action is the natural multiplication action of G on $G/H$. Applying this to $H = \{1\}$ yields the result.

\[ \square \]

\textbf{Remark 4.11.} If $\varphi : X \to Y$ corresponds to the monodromy datum $\Sigma = (\sigma_1, \ldots, \sigma_n)$, we will denote by $\rho_i$ the permutation $\rho_i([Y_i])$ and by $\tilde{\Sigma}$ the tuple $(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$.

\subsection{Statement of the problem}

We begin by describing the objects of interest:

\textbf{Definition 4.12.} Consider a 5-tuple $(g_X, g_Y, d_X, d_Y, R)$, where $g_X, g_Y$ are non-negative integers, $d_X, d_Y$ are positive integers, and $R = (R_1, \ldots, R_n)$ is a ramification structure, that is, a collection of pairs $R_i = (e_i, m_i)$ of positive integers. A \textit{diagram of type $(g_X, g_Y, d_X, d_Y, R)$} is a diagram of maps of smooth projective curves

\[ Z \longrightarrow X \longrightarrow Y \longrightarrow \mathbb{P}^1 \]

that satisfies the following properties:

\begin{itemize}
  \item[(i)] the genera of $X$ and $Y$ are $g_X, g_Y$ respectively;
  \item[(ii)] $\pi_{X/Y}$ is of degree $d_X$ and $\pi_{Y/\mathbb{P}^1}$ is of degree $d_Y$;
  \item[(iii)] the branch locus of $X \to \mathbb{P}^1$ is contained in an ordered set $B = (b_1, \ldots, b_n)$ with $n$ elements;
  \item[(iv)] the ramification structure of $X \to \mathbb{P}^1$, computed with respect to $B$, is equal to $R$; and
  \item[(v)] $Z \to \mathbb{P}^1$ is the Galois closure of $X \to \mathbb{P}^1$.
\end{itemize}

\textbf{Remark 4.13.} Note that the number of branch points of $X \to \mathbb{P}^1$ is precisely $n$ if and only if none of the $R_i$ is equal to $(1, 1)$. Indeed, this ramification structure denotes a point whose fiber contains $d_X d_Y$ points, none of which is ramified.

The map $X \to \mathbb{P}^1$ yields a monodromy datum $\Sigma$ as in Theorem 4.9. Let $G$ be the Galois group of $Z/\mathbb{P}^1$, and let $\tilde{\Sigma}$ be the corresponding monodromy datum. On the function fields side we have corresponding inclusions $\mathbb{C}(\mathbb{P}^1) \subseteq \mathbb{C}(Y) \subseteq \mathbb{C}(X) \subseteq \mathbb{C}(Z)$, and by Galois correspondence we obtain subgroups $H_X, H_Y$ of $G$ with the property that $Z/H_X = X$ and $Z/H_Y = Y$. In what follows we will be interested in 4-tuples $(G, H_X, H_Y, \Sigma)$ that arise from this construction.

\textbf{Remark 4.14.} Let $d = d_X d_Y$. The construction of $Z \to \mathbb{P}^1$ as the Galois closure of $X \to \mathbb{P}^1$ amounts to fixing a distinguished embedding of the Galois group $G$ into $S_d$, for which $H_X$ is conjugate to the stabilizer of 1. This leads to a corresponding notion of isomorphism, which is that of simultaneous conjugation of the 4-tuple $(G, H_X, H_Y, \Sigma)$ in $S_d$. That is, if $g$ is any element of $S_d$, and $\Sigma = (\sigma_1, \ldots, \sigma_n)$, then we write $g\Sigma g^{-1}$ for the tuple $(g\sigma_i g^{-1})_{i=1,\ldots,n}$ and say that the 4-tuples $(G, H_X, H_Y, \Sigma)$ and $(g\Sigma g^{-1}, gH_X g^{-1}, gH_Y g^{-1}, g\Sigma g^{-1})$ are isomorphic.

The problem we will solve is the following. Fix a 5-tuple $(g_X, g_Y, d_X, d_Y, R)$ as in Definition 4.12 and let $X \to Y \to \mathbb{P}^1$ be a diagram of type $(g_X, g_Y, d_X, d_Y, R)$. Let $(G, H_X, H_Y, \Sigma)$ be the corresponding 4-tuple constructed above. Up to isomorphism, there are only finitely many possibilities for $(G, H_X, H_Y, \Sigma)$, and our first algorithmic task is the following:

\textbf{Problem 4.15.} Given $(g_X, g_Y, d_X, d_Y, R)$ as in Definition 4.12, output a list $\mathcal{L}(g_X, g_Y, d_X, d_Y, R)$ of all isomorphism classes of 4-tuples $(G, H_X, H_Y, \Sigma)$ that can be obtained from a diagram $X \to Y \to \mathbb{P}^1$ of type $(g_X, g_Y, d_X, d_Y, R)$.

Note that a list $\mathcal{L}(g_X, g_Y, d_X, d_Y, R)$ as in the statement of Problem 4.15 gives a complete set of representatives of isomorphism classes of diagrams of type $(g_X, g_Y, d_X, d_Y, R)$, in the following precise sense. Suppose we have a diagram of type $(g_X, g_Y, d_X, d_Y, R)$; then one of the 4-tuples $(G, H_X, H_Y, \Sigma)$ in $\mathcal{L}(g_X, g_Y, d_X, d_Y, R)$ enjoys the following properties. Consider the unique (up to isomorphism) cover $X'$ of $\mathbb{P}^1$ of degree $d_X d_Y$, branched at most over $B$, and corresponding to the monodromy datum $\Sigma$. Also let $Z'$ be the Galois closure of $X'$ $\to \mathbb{P}^1$. The following holds:

\begin{itemize}
  \item[(i)] we have $\text{Aut}(\mathbb{P}^1) \cong \text{Aut}(\mathbb{P}^1) \cong G$;
  \item[(ii)] there is a canonical identification $X' = Z'/H_X$;
  \item[(iii)] the map $Z' \to \mathbb{P}^1$ is isomorphic to $Z \to \mathbb{P}^1$ as a $G$-cover;
  \item[(iv)] the $G$-isomorphism $Z' \cong Z$ can be chosen in such a way that $Z'/H_X$ is carried to $X$ and $Z'/H_Y$ is carried to $Y$;
  \item[(v)] in particular, the monodromy datum attached to $X \to \mathbb{P}^1$ is equivalent to $\Sigma$ (up to conjugacy in the symmetric group).
\end{itemize}

\textbf{Remark 4.16.} Informally, this means that a diagram of type $(g_X, g_Y, d_X, d_Y, R)$ arises from one of the monodromy data $\Sigma$ found in $\mathcal{L}(g_X, g_Y, d_X, d_Y, R)$, the only information missing being the ordered set of branch points.

In addition, for each $(G, H_X, H_Y, \Sigma)$ we would like to extract some additional information:
**Problem 4.17.** Given \((G, H_X, H_Y, \Sigma)\), determine:

(i) for every pair of subgroups \(H_1 < H_2 < G\), the degree and ramification structure of the corresponding map \(Z/H_1 \to Z/H_2\);

(ii) for every subgroup \(H\) of \(G\), the genus of the curve \(Z/H\);

(iii) the action of \(G\) on the vector space \(H^0(Z, \Omega_Z^2)\) induced by the natural action of \(G\) on \(Z\);

(iv) for every pair of subgroups \(H_1, H_2\) of \(G\), the dimension of the image of the map on Jacobians \(\text{Jac}(Z/H_1) \to \text{Jac}(Z/H_2)\) induced by the correspondence

\[
\begin{array}{ccc}
Z & \overset{\pi_1}{\leftarrow} & Z/H_1 \\
\downarrow & & \downarrow \\
Z & \overset{\pi_2}{\rightarrow} & Z/H_2
\end{array}
\]

\hspace{1cm} (4.2)

**4.4. Theory**

We now review the theoretical tools necessary to solve **Problem 4.17**. Our input data is a 4-tuple \((G, H_X, H_Y, \Sigma)\), corresponding to a diagram of type \((g_X, g_Y, d_X, d_Y, R)\).

**4.4.1. Degree and ramification structure of \(Z/H_1 \to Z/H_2\).**

Galois theory immediately shows that the degree of the natural projection \(Z/H_1 \to Z/H_2\) is equal to \([H_2 : H_1]\).

As for the ramification structure, we begin with the special case \(H_2 = G\) and \(H_1\) arbitrary. The quotient \(Z/H_2\) is therefore equal to \(\mathbb{P}^1\), the curve \(Z/H_1\) is a branched cover of it, and we may rely on **Theorem 4.9** to describe its ramification. In fact, the theorem shows that it suffices to understand the monodromy representation corresponding to \(\pi : Z/H_1 \to \mathbb{P}^1\). Let \(B\) be the set (containing the branch locus) that defines the cover \(Z \to \mathbb{P}^1\) and let \(B_\Sigma\) (resp. \(B_{Z/H_1}\)) be the inverse images of \(B\) in \(Z\) (resp. \(Z/H_1\)). Let \(Z^0 := Z - B\) and observe that \(Z^0/H_1\) coincides with \(Z/H_1 - B_{Z/H_1}\). We have a diagram of étale maps

\[
Z^0 \to Z^0/H_1 \to \mathbb{P}^1 - B
\]

which we may study via the usual topological interpretation of coverings as \(\pi_1\)-sets. In particular, fixing a base point \(q \in \mathbb{P}^1 - B\), one may identify the fiber of \(Z^0\) over \(q\) with \(G\) and the fiber of \((Z/H_1)^0\) with \(G/H\). In this language, the monodromy datum \(\Sigma\) gives rise to a representation

\[
\rho : \pi_1(\mathbb{P}^1 - B, q) \to G
\]

the \(\pi_1\)-structure of \(G\) is then \(\gamma \cdot g := \rho(\gamma)g\) for \(\gamma \in \pi_1(\mathbb{P}^1 - B, q)\). The \(\pi_1\)-set corresponding to \(Z^0/H_1\) is then the set \(G/H_1\), equipped with the action \(\gamma \cdot gH_1 := \rho(\gamma)gH_1\). We can now translate back to the language of monodromy data: for each \(i = 1, \ldots, n\) we have a permutation of the set \(G/H_1\), defined by left-multiplication by the element \(\sigma_i\). We may then use **Theorem 4.9** to describe the ramification structure of \(Z/H_1 \to \mathbb{P}^1\), and we obtain:

**Proposition 4.18.** Let \(H_1\) be a subgroup of \(G\). The branched cover \(Z/H_1 \to \mathbb{P}^1\) is ramified at most over the points in \(B = (b_1, \ldots, b_n)\). The ramification over \(b_i\) can be determined as follows: consider the left multiplication of \(\sigma_i\) on the quotient set \(G/H_1\). This induces a permutation of \(G/H_1\), with cycle type \((e_1, m_1), \ldots, (e_k, m_k))\). Then for all \(1 \leq j \leq k\), the fiber over \(b_i\) contains exactly \(m_j\) points with multiplicity \(e_j\), and no other points beyond these.

**Remark 4.19.** More precisely, the fiber of \(Z/H_1 \to \mathbb{P}^1\) over a point \(b_i \in B\) is in bijection with the double coset space \(\langle \sigma_i \rangle G/H_1\): Fixing a point \(z_i\) of \(Z\) over \(b_i\), the class \(\langle \sigma_i \rangle gH_1\) is sent to \(g \cdot z_i\) in \(Z/H_1\).

We will also need the following straightforward generalization of **Proposition 4.18**, which follows upon replacing **Theorem 4.9** with **Theorem 4.5**:

**Proposition 4.20.** Let \(Z\) be a smooth projective curve over \(\mathbb{C}\) with an action of a group \(G\), and let \(H\) be a subgroup of \(G\). Let \(B = (b_1, \ldots, b_n) \subseteq Z/H\) be a finite ordered subset containing the branch locus of \(H\) and let \(\rho\) be the corresponding representation \(\rho : (Z/G - B, q) \to S_n\) as in **Theorem 4.5**. Finally, let \(\gamma_j\) be small loops based at \(q\) around each \(b_j\) and let \(\sigma_i = \rho(\langle \gamma_i \rangle)\). Recall from **Theorem 4.10** that \(G\) is identified with the subgroup of \(S_n\) generated by the \(\sigma_i\). The ramification of \(Z/H\) over \(b_i\) can be determined as follows. Consider the left multiplication by \(\sigma_i\) on the quotient set \(G/H\): it induces a permutation of the set \(G/H\), with cycle type \((e_1, \ldots, e_k)\). The fiber over \(b_i\) consists of \(k\) points, of multiplicities \(e_1, \ldots, e_k\).

Second, we treat the case of a Galois cover \(\pi_H : Z \to Z/H\). This is discussed for example in [10, Proposition 2.2.2] and in [38, §4], so we only recall the result. Let as before \(Z^0\) be the complement in \(Z\) of the inverse image of \(B\), and observe that we have a tower of topological covers

\[
Z^0 \overset{\pi_H}{\to} Z^0/H \overset{\varphi}{\to} \mathbb{P}^1 - B
\]
in particular, \( \pi_H \) is unramified outside of the inverse image of \( B \) in \( Z/H \). Thus the branch locus of \( \pi_H \) is contained in \( \varphi^{-1}(B) \), and we have a description of this set by the special case we treated above: by Remark 4.19, the set \( \varphi^{-1}(B) \) is in bijective correspondence with the set of pairs \((b_i, (\sigma_i)gH)\), where the first coordinate is an element of \( B \) and where the second coordinate is an element in the double coset space \((\sigma_i)G/h\). The monodromy operator given by a small loop around the point corresponding to \((\sigma_i)gH\) is obtained as follows: letting \( m_{g,i} \) be the smallest positive integer for which \( g^{-1}\sigma_i^{m_{g,i}}g \in H \), the monodromy operator is precisely \( g^{-1}\sigma_i^{m_{g,i}}g \).

The case of a general intermediate cover \( \pi : Z/H_1 \rightarrow Z/H_2 \) follows upon combining the previous two special cases: we first obtain the monodromy datum of \( Z \rightarrow Z/H_2 \) in the way just described, and then deduce that of \( Z/H_1 \rightarrow Z/H_2 \) by applying Proposition 4.20. This leads to the following algorithmic procedure to express the monodromy of \( Z/H_1 \rightarrow Z/H_2 \) in terms of \((G, H_1, H_2, \Sigma)\):

**Algorithm 4.21.** Input: \((G, H_1, H_2, \Sigma)\) with \( H_1 < H_2 \) and \( \Sigma = (\sigma_1, \ldots, \sigma_n) \).

Output: the ramification structure of \( Z/H_1 \rightarrow Z/H_2 \).

Procedure:

(i) For every \( i = 1, \ldots, n \):
   
   (a) Compute \( \ell_i = |\langle \sigma_i \rangle\backslash G/H_2| \) as well as elements \( g_{i,1}, \ldots, g_{i,\ell_i} \in G \) such that \( \langle \sigma_i \rangle\backslash G/H_2 = \{ \langle \sigma_i \rangle g_{i,j}H_2 : j \in \{1, \ldots, \ell_i\} \} \).
   
   (b) For each \( g_{i,j} \):
      
      i. Let \( m_{g,i,j} \) be the least positive integer for which \( g_{i,j}^{-1}\sigma_i^{m_{g,i,j}}g_{i,j} \) lies in \( H_2 \). Set \( \sigma_{ij} = g_{i,j}^{-1}\sigma_i^{m_{g,i,j}}g_{i,j} \).
      
      ii. Compute the permutation of \( G/H_1 \) induced by left multiplication by \( \sigma_{ij} \). Let \( R_{ij} \) be the cycle type of this permutation.

(ii) Return the tuple \( (R_{ij} \mid i = 1, \ldots, n, \langle \sigma_i \rangle g_{i,j}H_2 \in \langle \sigma_i \rangle\backslash G/H_2) \).

4.4.2. The genera of the curves \( Z/H \)

By the previous paragraph we know how to read off our data, the ramification structure of the map \( \varphi : Z/H \rightarrow Z/G = \mathbb{P}^1 \). In particular, we know the multiplicity of each ramification point \( y_i \in Z/H \), and since we also know \( \deg \varphi = [G : H] \) we can simply apply the Riemann-Hurwitz formula to obtain

\[
g(Z/H) = \frac{1}{2} \left( 2 - 2[G : H] + \sum_{j \in \mathbb{Z}/H} (e(y) - 1) \right).
\] (4.3)

4.4.3. \( G \)-module structure of \( H^0(Z, \Omega^2_Z) \)

To extract this information from \((G, H_X, H_Y, \Sigma)\), we use a beautiful theorem due to Chevalley and Weil [15, 54] that we now recall.

We need some preliminary notation. Let \( B = (b_1, \ldots, b_n) \) be the ordered branch locus of \( Z \rightarrow \mathbb{P}^1 \) and consider one of the branch points \( b_i \in B \). As part of our data, we have access to a permutation \( \sigma_i \in G \) corresponding to the branch point \( b_i \). Let \( e_i \) be the order of the permutation \( \sigma_i \) or equivalently (by Theorem 4.10) the ramification index of any point of \( Z \) lying over \( b_i \). Fix once and for all a primitive \( |G| \)-th root of unity \( \xi \in \mathbb{C} \), and, for any divisor \( e \) of \( |G| \), denote by \( \zeta_e \) the complex number \( \xi^{|G|/e} \).

Observe that \( V := H^0(Z, \Omega^2_Z) \) is a \( \mathbb{C}[G] \)-module in a natural way, and it is automatically semisimple since \( \mathbb{C} \) is of characteristic 0. In order to describe the \( \mathbb{C}[G] \)-module structure of \( V \), therefore, it suffices to give the multiplicity of each irreducible representation of \( G \) in \( V \). For a fixed linear representation \( \tau \) of \( G \), denote by \( N_{i,\alpha} = N_{i,\alpha}(\tau) \) the multiplicity of \( \zeta_e^n \) as eigenvalue of \( \tau(\sigma_i) \), where \( \sigma_i \) is the monodromy operator corresponding to the cover \( Z \rightarrow \mathbb{P}^1 \) and the point \( b_i \). With this notation, and in the special case of covers of \( \mathbb{P}^1 \), the Chevalley-Weil formula reads as follows:

**Theorem 4.22 (Chevalley-Weil).** Let \( \varphi : Z \rightarrow \mathbb{P}^1 \) be a branched Galois cover of smooth projective complex algebraic curves, let \( B \) be its branch locus, and let \( G \) be the corresponding Galois group. Let \( \tau_\chi \) be an irreducible linear complex representation of \( G \) with character \( \chi : G \rightarrow \mathbb{C} \) and define \( e_i \) and \( N_{i,\alpha} := N_{i,\alpha}(\tau_\chi) \) as above. The multiplicity \( v_\chi \) of \( \tau_\chi \) in the \( G \)-representation \( H^0(Z, \Omega^2_Z) \) is given by

\[
v_\chi = -d_\chi + \sum_{i=1}^p \sum_{\alpha=0}^{e_i-1} N_{i,\alpha} \left( -\left\lfloor \frac{\alpha}{e_i} \right\rfloor + \frac{\alpha}{e_i} \right) + \sigma,
\] (4.4)

where \( d_\chi = \chi(1) \) is the dimension of \( \tau_\chi \) and \( \sigma \) is defined as

\[
\sigma = \begin{cases} 1 & \text{if } \chi \text{ is the trivial character,} \\ 0 & \text{otherwise.} \end{cases}
\]

Here \( \langle x \rangle = x - \lfloor x \rfloor \in [0, 1) \) denotes the fractional part of the real number \( x \).

Note that the multiplicity \( v_\chi \) is determined by \((G, H_X, H_Y, \Sigma)\) and \( \chi \); we have already observed that \( e_i \) is the order of \( \sigma_i \), and explained how to obtain the monodromy datum \( \Sigma \) (see Theorem 4.10). Finally, the number \( N_{i,\alpha} \) is the multiplicity of \( \zeta_e \) (which is a known complex number) as an eigenvalue of \( \tau_\chi(\sigma_i) \), and a basic result in representation theory shows that \( \tau_\chi \) is in turn determined by \( \chi \), so that \( v_\chi \) is indeed determined by \((G, H_X, H_Y, \Sigma)\). The upshot of this discussion is that we have an isomorphism of \( \mathbb{C}[G] \)-modules \( V \cong \bigoplus_\chi v_\chi \Omega^2_Z \), where all the objects on the right hand side are determined by \((G, H_X, H_Y, \Sigma)\) as desired.
4.4.4. The maps Jac(Z/H_1) \to Jac(Z/H_2)

Our last objective is to understand the image of the maps Jac(Z/H_1) \to Jac(Z/H_2) induced by the correspondence (4.2). Note that the complex vector space \( V = H^0(Z, \Omega^1_Z) \) provides the natural analytic uniformization of Jac(Z), and that the maps Jac(Z/H_i) \to Jac(Z) are induced by the pullback \( \pi^* : H^0(Z/H_i, \Omega^1_{Z/H_i}) \to H^0(Z, \Omega^1_Z) \). Thus it suffices to study the map \( \pi_{2*} \circ \pi^*_1 : H^0(Z/H_1, \Omega^1_{Z/H_1}) \to H^0(Z/H_2, \Omega^1_{Z/H_2}) \). Note that the pushforward \( \pi_{2*} \) makes sense since \( Z \to Z/H_2 \) is a finite (albeit ramified) cover. We will need a result from representation theory:

**Theorem 4.23** ([50, Théorème 2.6.8]). Let \( \tau : G \to GL(V) \) be a finite-dimensional linear complex representation of the finite group G and let H be a subgroup of G. Define \( p_H := \frac{1}{|H|} \sum_{h \in H} \tau(h) \in \text{End}(V) \). Then \( p_H \) is a projector, that is, \( p_H^2 = p_H \), and its image is precisely the H-invariant subspace of V.

**Remark 4.24.** We will only work with the representation of G afforded by \( V = H^0(Z, \Omega^1_Z) \), so, for the sake of simplicity, given a subgroup \( H \) of G we will simply write \( p_H = \frac{1}{|H|} \sum_{h \in H} h \), omitting the representation \( \tau \).

In order to connect the maps \( \pi_{2*} \) and \( \pi^*_1 \) with representation theory, we will make use of the following result:

**Theorem 4.25.** Let \( H \) be a subgroup of G and let \( \pi : Z \to Z/H \) be the corresponding quotient map. Then:

(i) \( \pi^* : H^0(Z/H, \Omega^1_{Z/H}) \to H^0(Z, \Omega^1_Z) \) is injective, and its image is the H-invariant subspace of \( H^0(Z, \Omega^1_Z) \);

(ii) \( \pi^*_1 \circ \pi_{2*} : H^0(Z, \Omega^1_Z) \to H^0(Z, \Omega^1_Z) \) coincides with the operator \#H \cdot p_H.

Part (i) is well known; we include a short proof of (ii):

**Proof of (ii).** Since a curve and its Jacobian share the same space of regular differentials, it suffices to prove the same statement with \( Z, Z/H \) replaced by their Jacobians. We prove the stronger statement that the required relation is true for the divisor groups. Let \( D = \sum_i n_i P_i \) be a divisor on Z. By definition, \( \pi_* D = \sum_i n_i \pi(P_i) \). Since the fiber over \( \pi(P_i) \) is the divisor given by the sum of all the points that map to \( \pi(P_i) \), namely, \( \sum_{h \in H} h \cdot P_i \), we obtain \( \pi^*_1 \circ \pi_{2*} D = \sum_i n_i \sum_{h \in H} h \cdot P_i = \#H \cdot p_H(D) \) as desired. \( \square \)

We wish to determine the dimension of \( \text{Im} \left( \text{Jac}(Z/H_1) \to \text{Jac}(Z/H_2) \right) \), or equivalently the dimension of \( \pi_{2*} \circ \pi^*_1 (H^0(Z/H_1, \Omega^1_{Z/H_1})) \). Since \( \pi^*_1 \) is injective, we may as well study the dimension of \( \pi^*_1 \circ \pi_{2*} \circ \pi^*_1 (H^0(Z/H_1, \Omega^1_{Z/H_1})) \). By Theorem 4.25, \( \pi^*_1 (H^0(Z/H_1, \Omega^1_{Z/H_1})) \) is precisely the \( H_1 \)-invariant subspace of V, hence (by Theorem 4.23) it is the image of \( p_{H_1} \). We may easily identify this subspace, because we have already shown how to write down a representation isomorphic to V. It follows that

\[
\pi^*_1 \circ \pi_{2*} \circ \pi^*_1 (H^0(Z/H_1, \Omega^1_{Z/H_1})) = \#H_2 \cdot p_{H_2} \pi^*_1 (H^0(Z/H_1, \Omega^1_{Z/H_1})) \] = \#H_2 \cdot p_{H_2} \cdot p_{H_1} (V)

has dimension equal to the rank of the operator \( p_{H_2} \cdot p_{H_1} \). We have obtained:

**Proposition 4.26.** The dimension of the image of the map Jac(Z/H_1) \to Jac(Z/H_2) induced by the correspondence Z is equal to the rank of

\[
\left( \sum_{h_2 \in H_2} h_2 \right) \left( \sum_{h_1 \in H_1} h_1 \right) : V \to V. \tag{4.5}
\]

Since we have already shown that the action of G on V is completely determined by the monodromy datum \( \Sigma \), this allows us to express the dimension of \( \text{Im} \left( \text{Jac}(Z/H_1) \to \text{Jac}(Z/H_2) \right) \) in terms of \( (G, H_X, H_Y, \Sigma) \).

Note furthermore that the same machinery allows us to also answer a slightly different question: for example, in our application we consider diagrams of curves of the form

```
   Z ----> C
   |  \pi_1
  π_2 |   \ /
    |    \ π_3
  X ---- Y
```

and we need to understand whether or not the image of the map Jac(C) \to Jac(X) induced by the correspondence Z intersects the image of the map Jac(Y) \to Jac(X) induced by pulling back divisors from Y to X. Passing to analytic uniformizations, the question is whether the subspaces \( \pi_{2*} \circ \pi^*_1 (H^0(C, \Omega^1_C)) \) and \( \pi^*_2 \circ \pi^*_3 (H^0(Y, \Omega^1_Y)) \) of \( H^0(X, \Omega^1_X) \) intersect nontrivially. However, since \( \pi^*_1, \pi^*_2, \pi^*_3 \) are all injective, it suffices to know whether

\[
\pi^*_2 \circ \pi_{2*} \circ \pi^*_1 (H^0(C, \Omega^1_C)) \text{ and } \pi^*_2 \circ \pi^*_3 (H^0(Y, \Omega^1_Y)) = (\pi_3 \circ \pi_2)^* (H^0(Y, \Omega^1_Y))
\]
interact nontrivially inside $V$. Proceeding as above, and letting $H_X, H_Y$, and $H_C$ be the subgroups of $G$ corresponding via Galois theory to $X, Y,$ and $C$, respectively, we conclude that the image of the map $\text{Jac}(C) \to \text{Jac}(X)$ induced by $Z$ intersects the image of $\text{Jac}(Y) \to \text{Jac}(X)$ if and only if the operator $p_{HC} p_{HC}$ is nonzero.

4.4.5. Conclusion
Putting together the results of the previous paragraphs we obtain:

Proposition 4.27. Let $Z \to \mathbb{P}^1$ be a Galois branched cover with group $G$. The monodromy datum of $Z \to \mathbb{P}^1$ determines the following (in an effectively computable way): For every subgroup $H < G$, the genus of $Z/H$, and for every pair of subgroups $H_1, H_2$ of $G$, the dimension of the image of the induced map $\text{Jac}(Z/H_1) \to \text{Jac}(Z/H_2)$.

5. Implementation

We now turn to details and optimizations concerning the algorithmic implementation of our solutions to Problems 4.15 and 4.17. Since our solution to Problem 4.15 actually relies on being able to handle Part (i) of Problem 4.17, we begin with the latter.

5.1. Solving Problem 4.17

We apply the theory from Section 4 to obtain the following procedure:

Algorithm 5.1. Input: a 4-tuple $(G, H_X, H_Y, \Sigma)$ as in Problem 4.17.

Output: the structure of $H^0(Z, \Omega^2_Z)$ as a $G$-representation; for each pair of subgroups $H_1 < H_2 < G$, the ramification structure of $Z/H_1 \to Z/H_2$, the genus of $Z/H_1$, and the dimension of the image of the map $\text{Jac}(Z/H_1) \to \text{Jac}(Z/H_2)$ induced by $Z$ as in (4.2).

Procedure:

(i) Compute the genera of $Z \to Z/H_1$ and the intermediate ramification of $Z/H_1 \to Z/H_2$ using (4.3) and Algorithm 4.21.
(ii) Compute the $G$-module structure of $H^0(Z, \Omega^2_Z)$ as explained in Section 4.4.3.
(iii) Compute the dimension of $\text{Im}(\text{Jac}(Z/H_1) \to \text{Jac}(Z/H_2))$ using Theorem 4.22 and Proposition 4.27.
(iv) Return the results of Steps (i)-(iii).

The theory behind each of these steps has been laid out in Section 4.4, and it remains to explain what is required for their implementation. For Step (i), a standard implementation of symmetric groups suffices, whereas for Steps (ii)-(iii), one only requires an implementation of standard algorithms in the representation theory of subgroups of the symmetric group and the corresponding vector spaces. One computer algebra system that contains this joint functionality is MAGMA, for which we wrote our implementation [31].

Remark 5.2. (i) Our computations do not require the computation of curve equations and depend only on the specified ramification structure $\Sigma$ over the branch locus $B$ of $X \to \mathbb{P}^1$, and not on this branch locus itself. This independence of $B$ implies that all our calculations may be performed abstractly, and will be valid for any choice of $B$. This means that we actually consider families of examples of dimension $r-3$, where $r$ is the number of branch points of $X \to \mathbb{P}^1$, see Remark 4.8.
(ii) As long as the degree of the composed map $X \to \mathbb{P}^1$ is small, the computations involved in Algorithm 5.1(i), which are described in Algorithm 4.21 and Section 4.4.2, take place in a symmetric group on a small set, and therefore terminate quickly. We discuss speedups for Parts (ii) and (iii) in Section 5.2.2.

5.2. Solving Problem 4.15

Let $(g_X, g_Y, d_X, d_Y, R)$ be given. We want to find the corresponding tuples $(G, H_X, H_Y, \Sigma)$. Recall that for $d = d_X d_Y$, the group $G$ is the subgroup of $S_d$ generated by the monodromy $\Sigma$. Moreover, we want that the corresponding Galois cover is the Galois closure of the cover of degree $d$ corresponding to $H_X$. We can ensure this by fixing an embedding of $G$ into $S_d$ and letting $H_X$ be the stabilizer of $1$. In other words, given $G$, determining the possible pairs $(G, H_X)$ comes down to realizing $G$ as a conjugacy class of subgroups of $S_d$. Moreover, when the group $G$ is not specified, we can find all pairs $(G, H_X)$ up to equivalence by running through the conjugacy classes of subgroups of $S_d$. For this latter problem, efficient algorithms exist when $d$ is small.

Remark 5.3. Note that at the very least $G$ has to act transitively to correspond to a connected cover of $\mathbb{P}^1$. Moreover, we may restrict to subgroups $H_X$ with the property that $H_X$ has a normal subgroup of index at most $(d_Y - 1)!$, since only such subgroups can give rise to a diagram (4.1) with the requested properties. Indeed, the core of $H_X$ in $H_Y$ is a normal subgroup of $H_Y$ that is contained in $H_X$ and that is of index at most $d_Y!$ in $H_Y$. 


It now remains to find all the possible extensions of a given pair \((G, H_X)\) to quadruples \((G, H_X, H_Y, \Sigma)\). Once \((G, H_X)\), or alternatively (by the above) an embedding of \(G\) into \(S_d\), is given, the remaining isomorphisms on the level of covers translate into conjugation by the normalizer \(N_G\) of \(G\) in \(S_d\). We accordingly determine the subgroups \(K_Y\) of \(G\) of index \(d_Y\) up to conjugacy by \(N_G\). Having found this, we find representatives for triples \((G, H_X, H_Y)\) as follows:

**Proposition 5.4.** The simultaneous \(N_G\)-conjugacy classes of triples \((G, H_X, H_Y)\) such that \(H_X\) (resp. \(H_Y\)) is \(N_G\)-conjugate to a given subgroup \(K_X\) (resp. \(K_Y\)) of \(G\) are in bijection with the double coset space \(N_Y \setminus N_G / N_X\). Here \(N_Y\) (resp. \(N_X\)) is the normalizer of \(K_Y\) (resp. \(K_X\)) in \(N_G\), and to a double coset \(N_Y g N_X\) there corresponds the triple \((G, K_X, g^{-1} K_Y g)\).

**Proof.** The indicated map is well-defined, and it is surjective since after conjugating by a suitable element of \(N_G\) if necessary we may assume that \(K_X = H_X\). Conversely, if two pairs \((G, K_X, n_1^{-1} K_Y n_1)\) and \((G, K_X, n_2^{-1} K_Y n_2)\) are simultaneously \(N_G\)-conjugate, then \(n_1^{-1} K_Y n_1 = g n_2^{-1} K_Y n_2 g\) for some element \(g\) of \(N_X\), which implies that \(n g = h n_1\) for \(h \in N_Y\).

Applying Proposition 5.4, we find the possible triples \((G, H_X, H_Y)\) such that moreover \(H_X < H_Y < G\), all up to simultaneous conjugacy by \(N_G\). If so desired, then we can impose that \(H_X\) be maximal in \(H_Y\), and similarly for \(H_Y\) and \(G\). Moreover, our algorithms allow one to impose that the map \(X \to Y\) be indecomposable, in the sense that it does not factor as a composition of two maps of degree strictly larger than 1. (Our examples nowhere assume this indecomposability of the map \(X \to Y\).)

It then remains to find the possible monodromy data \(\Sigma\) starting from \((G, H_X, H_Y)\). For this, we have used fast and efficient code by Paulhus [44] based on work of Breuer [11]. This finds the possible \(\Sigma\) up to conjugation by elements of \(G\) once conjugacy classes in \(G\) are given. While we do not have these conjugacy classes at our disposal, we do have imposed ramification data \(R\), which above any point determines the cycle structure of the corresponding conjugacy classes (recall that our data furnish a conjugacy class of embeddings of \(G\) into \(S_d\), so that this is well-defined). This gives a finite number of explicit possibilities for the conjugacy classes above a given point. Combining the outcomes of Breuer's algorithms for all possible choices, we obtain the possible covers \(\Sigma\). If so desired, we can still reduce the set of possible \(\Sigma\) further under common \(N_G\)-conjugacy to prevent duplicates. While we usually do this, it can occasionally cost some time if there are lots of covers involved, in which case our algorithms allow this step to be skipped.

Finally, given an element \(\Sigma\), we append it to one of the triples obtained before to obtain a quadruple \((G, H_X, H_Y, \Sigma)\). If this quadruple has the correct ramification, as can be checked using Algorithm 5.1(i), then we retain this quadruple.

The above discussion motivates the following algorithm, and shows how it can be implemented in a computer algebra system containing basic functionality for symmetric groups, such as in our MAGMA implementation [31].

**Algorithm 5.5.** Input: \((g_X, g_Y, d_X, d_Y, R)\) as in Definition 4.12.

Output: a list of 4-tuples \((G, H_X, H_Y, \Sigma)\).

Procedure:

(i) Initialize \(d := d_X d_Y\) and let \(L_1\) and \(L_2\) be the empty lists.

(ii) Loop over representatives \(G\) of conjugacy classes of subgroups of \(S_d\). For each representative do:

(a) If \(G\) is not transitive, discard \(G\) and continue with the next subgroup;
(b) Set \(H_X\) to be the stabilizer of 1 in \(G\);
(c) Append to \(L_1\) all triples \((G, H_X, H_Y)\) obtained using Proposition 5.4.

(iii) Using Breuer’s algorithm as implemented by Paulhus, find all possible isomorphism classes of monodromy data \(\Sigma\), up to \(N_G\)-conjugacy if desired. Loop over these \(\Sigma\), and for a fixed such element do:

(a) Loop over the triples \((G, H_X, H_Y)\) in \(L_1\);
(b) Using Algorithm 5.1, compute the genera of \(Z/H_X\) and of \(Z/H_Y\). If \(g(Z/H_X) \neq g_X\) or \(g(Z/H_Y) \neq g_Y\), proceed to the next triple in the loop;
(c) Using Algorithm 5.1, compute the ramification structure of \(X \to \mathbb{P}^1\). If it is different from \(R\), proceed to the next triple in the loop;
(d) Add \((G, H_X, H_Y, \Sigma)\) to \(L_2\).

(iv) Return \(L_2\).

**5.2.1. Action of \(G\) on \(H^2(Z, \Omega^1_Z)\) and calculation of image dimensions**

Given a finite group \(G\), one can compute its character table, for example by using the Dixon–Schneider algorithm, or the LLL-based induce/reduce algorithm of Unger [52]. Once the character table of \(G\) is known, in order to fully describe the \(G\)-representation \(V\) we merely need to determine the multiplicity with which each character \(\chi\) of \(G\) appears in \(V\). Such multiplicities can be obtained by applying Theorem 4.22 to the map \(Z \to \mathbb{P}^1\). Indeed, given a character \(\chi\) corresponding to a representation \(\tau_G\), the only information we need to determine \(v_\chi\) are the numbers \(e_\sigma\) and \(N_{i,a}\). We have already observed that \(e_\sigma\) is the order of \(\sigma\). Furthermore, by definition, \(N_{i,a}\) is the multiplicity of \(\xi e_\sigma\) as an eigenvalue of \(\tau_G(\sigma)\). This multiplicity can be read off the characteristic polynomial of \(\tau_G(\sigma)\), whose coefficients are the elementary symmetric functions in the eigenvalues of \(\tau_G(\sigma)\). As we are in characteristic zero, the symmetric
functions of $\lambda_1, \ldots, \lambda_k$ are determined by the Newton sums
\[
\sum_{i=1}^{k} \lambda_i = \text{tr} \tau_x(\sigma_i) = \chi(\sigma_i), \quad \sum_{i=1}^{k} \lambda_i^2 = \text{tr} \tau_x(\sigma_i^2) = \chi(\sigma_i^2), \ldots, \quad \sum_{i=1}^{k} \lambda_i^k = \text{tr} \tau_x(\sigma_i^k) = \chi(\sigma_i^k).
\]
This shows that the knowledge of the character $\chi$ is enough to determine the characteristic polynomial of $\tau_x(\sigma_i)$, hence we may compute the numbers $N_{i,a}$ from the knowledge of $\chi$ without even having to describe the $G$-module $\tau_x$. This solves Part (ii) of Algorithm 5.1. Part (iii) can then be obtained by calculating the relevant irreducible representations $\tau_x$ explicitly and summing the dimensions of the images of the maps obtained in Proposition 4.26, multiplied by the relevant multiplicities. In our application, we often use the following more specific procedure:

**Algorithm 5.6.** *Input:* a 4-tuple $(G, H_X, H_Y, \Sigma)$ as in Problem 4.17.
*Output:* A subgroup $H_C$ of $G$ (if it exists) for which the corresponding curve $C = Z/H_C$ has the following properties:

- $0 < g_C \leq g_X - g_Y$;
- The map $\text{Jac}(C) \to \text{Jac}(X)$ induced by $X \leftrightarrow Z \to C$ has finite kernel;
- The image of $\text{Jac}(C) \to \text{Jac}(X)$ does not intersect the image of $\text{Jac}(Y) \to \text{Jac}(X)$.

*Procedure:*

(i) Run through the subgroups of $H$ of $G$;
(ii) Compute the genus $g_C$ of the curve $C := Z/H_C$. If we do not have that $0 < g_C \leq g_X - g_Y$, then move on to the next $H$, otherwise proceed to (iii);
(iii) Compute the dimension of the image of the induced map $\text{Jac}(C) \to \text{Jac}(X)$ using the $G$-module $H^0(Z, \mathcal{O}_Z^1)$ and Proposition 4.26. If it is non-zero, then move on to the next $H$, otherwise proceed to (iv);
(iv) Using similar methods, compute the dimension of the image of the induced map $\text{Jac}(C) \to \text{Jac}(X)$. If its dimension does not equal $g_C$, then move on to the next $H$, otherwise return $H$.

Our discussion shows that as soon as one has the representation-theoretic prerequisites Algorithm 5.1 are at one’s disposal, Algorithm 5.6 can also be implemented, with our implementation [31] once more using the MAGMA version of this functionality.

**Remark 5.7.** Step (iv) of Algorithm 5.6 insists on the map $\text{Jac}(C) \to \text{Jac}(X)$ having finite kernel because otherwise we would have to deal with another decomposition problem in order to describe the part of the Prym variety thus obtained as a Jacobian (up to isogeny).

If Algorithm 5.6 returns a group $H_C$ for which moreover $g_C = g_X - g_Y$, then $\text{Jac}(X) \sim \text{Jac}(C) \times \text{Jac}(Y)$, so that (up to isogeny) we have realized the Prym variety of the cover $X \to Y$ as the Jacobian of the curve $C$. If for all $(G, H_X, H_Y, \Sigma)$ that we consider we can find a group $H_C$ and a corresponding curve $C$ as above, then we know that for every diagram $X \to Y \to \mathbb{P}^1$ of type $(g_X, g_Y, d_X, d_Y, R)$, the abelian variety $\text{Prym}(X \to Y)$ is isogenous to the Jacobian of a quotient $C$ of the Galois closure of $X \to \mathbb{P}^1$. Even if this does not happen, it is still possible that we are successful for, say, all quadruples for which $G$ is in a certain specified isomorphism class. To our surprise, we have discovered several types $(g_X, g_Y, d_X, d_Y, R)$ for which this construction gives non-trivial information on the Prym variety, and we report on these findings in Section 6.3 below.

If there is no single quotient $C$ in Algorithm 5.6 such that $\text{Jac}(C)$ is isogenous to $\text{Prym}(X \to Y)$, then it may still happen that the latter Prym variety is isogenous to a product of Jacobians obtained in this fashion, as can be ascertained by determining the sum of the corresponding subspaces in $\text{Jac}(X)$. An example of this is given in the entry $\text{rr-spec}$ of Table 3, as explained in Section 6.

### 5.2.2. Fine print and speedups

This final section contains a smattering of more detailed remarks on our implementation, calculations, and results. To start, we note that the calculation in Algorithm 5.6(ii) is possible from the knowledge of the modules $\tau_x$ and their multiplicities $n_x$, which we need only calculate once given $G$ and $\Sigma$. As we run through the possible $\Sigma$, we store the different intervening representations $\tau_x$ so that we do not have to recalculate them later for different $\Sigma$. (We do have to calculate new multiplicities $n_x$, but this is fortunately far less laborious.) This is worthwhile because our implementation works with the Chevalley-Weil decomposition throughout: All dimension calculations involving Proposition 4.26 are done for these irreducible representation $\tau_x$, after which the corresponding results are summed with the relevant multiplicities $n_x$.

When looking for a single curve $C$ to furnish the complement of $\text{Jac}(Y)$ in $\text{Jac}(X)$, we can in fact do better than running over all possible $H$. Indeed, reasoning as in Proposition 5.4 shows that it suffices to consider candidate subgroups $K$ up to $G$-conjugacy at first, as whether the condition in Algorithm 5.6(ii) holds depends only on the $G$-conjugacy class of $H$. Given a representative $K$ of such a conjugacy class, the argument from Proposition 5.4 then shows that we can work with the tuples $(G, H_X, H_Y, \Sigma, n^{-1} Kn)$, where $n$ runs through the double coset $N_C \backslash G / (N_X \cap N_Y)$ for $N_C$ (resp. $N_X, N_Y$) the normalizer of $K$ (resp. $H_X, H_Y$) in $G$. Since the pairs $(H_X, H_Y)$ in our quadruples $(G, H_X, H_Y, \Sigma)$ stem from a fixed list, we can store these double coset representatives beforehand so as not to have to keep recalculating them.
The same uniformity in the pairs \((H_X, H_Y)\) ensures that only a relatively small number of triples \((H_X, H_Y, H)\) is encountered for a fixed group \(G\), albeit for many different \(\Sigma\) and with many different multiplicities. This makes it worthwhile to store all the ranks and projectors in Proposition 4.26 that are calculated when working with a fixed representation \(\tau_X\) in a hash table, as considerable time is gained when using a lookup instead of a recalculation. In fact, in practice our calculations show that most time is spent constructing the explicit projectors in Proposition 4.26 on the larger irreducible subrepresentations of \(H^0(Z, \Omega^1_Z)\). Similarly, we can ensure that the rank of composition of these projectors does not need be calculated for \(\tau_X\) when we encounter a previously stored triple \((H_X, H_Y, H)\), which is often the case in practice.

6. Results

6.1. Presentation of the tables

The tables in the appendix describe results obtained by running our algorithms. We describe is how to read an entry in these tables by means of the concrete case \(\text{total4}\):

- \(g_X, g_Y, d_X\) For the case \(\text{total4}\), the genus \(g_X\) of \(X\) equals 4, the genus \(g_Y\) of \(Y\) equals 1 and the degree \(d_X\) of the cover \(X \to Y\) equals 4.

- \(\text{Ramification}\) This describes the ramification structure of the composition \(X \to Y \to \mathbb{P}^1\). The degree of \(Y \to \mathbb{P}^1\) usually equals 2. If not, the name of the case starts with the degree \(\text{deg}(Y \to \mathbb{P}^1)\) (for instance \(3\)-\text{orig} in Table 6). A thin line represents an unramified point. A thick line without a number on its side a totally ramified point; for a thick line with a number on its side, this number specifies the ramification index. For all ramification types thus displayed, the number over them represents the number of copies in the total ramification structure. In the case \(\text{total4}\), we see that the map \(Y \to \mathbb{P}^1\) has 4 ramification points, all of which split totally in the cover \(X \to Y\). Moreover, the 2 total ramification points of \(X \to Y\) are merged under the map \(X \to \mathbb{P}^1\).

- \(\#G, g_Z\) This lists the different possible pairs \#\(G, g_Z\), where \#\(G\) = \(\text{deg}(Z \to \mathbb{P}^1)\) is the cardinality of the monodromy group \(G\) and where \(g_Z\) is the genus of \(Z\). In the case \(\text{total4}\) there turns out to be only one such pair.

- \(X\) nhyp/hyp’’ Running through the possible cases from the previous item, we consider the isomorphism classes of curves \(X\) for which an automorphism of the Galois closure induces a hyperelliptic involution. Given such a class, we use our algorithms to check whether a piece of the Prym variety of \(X \to Y\) is given by the Jacobian of a quotient of \(Z\). The number of curves for which this happens (resp. does not happen) is the second bracket entry of the case listed in this column. The first entry does the same, but instead for those isomorphism class of curves \(X\) for which no hyperelliptic involution is induced by the Galois closure. In the case \(\text{total4}\), we obtain 48 possibly non-hyperelliptic and 16 hyperelliptic curves in this way for the single possible pair \#\(G, g_Z\), for all of which we can indeed generate a piece of the Prym variety as the Jacobian of a quotient of \(Z\).

- \(\text{Prym dims}\) For the entries above, we give the dimensions of the disjoint pieces of the Prym variety that we found as Jacobians of quotients of \(Z\), separated into the non-hyperelliptic and hyperelliptic case (if one, or both, of these cases never yields a piece of the Prym, it is omitted). In the case \(\text{total4}\), we always find a curve \(C\) of genus 3 such that \(\text{Jac} C \sim P(X/Y)\) in the non-hyperelliptic case. By contrast, in the hyperelliptic case, we find two curves \(C_1\) and \(C_2\) of genus 1 and 2 such that \(\text{Jac} C_1 \times \text{Jac} C_2 \sim P(X/Y)\). It is possible that there are multiple cases with different resulting dimensions. This is illustrated in the case \(\text{total15}\).

- \(\text{deg} Z \to C_i\) The last column gives the degrees of the maps \(Z \to C_i\) obtained in the previous entry, separated into the non-hyperelliptic and hyperelliptic case as before.

Remark 6.1. (i) Our implementation allows the determination of more information, like the ramification of intermediate covers. (ii) Given a certain ramification structure, our programs can equally well calculate results for “specializations” of it, for instance those obtained by collapsing two ramification points of the cover \(Y \to \mathbb{P}^1\). We did not try to do this systematically, but we did often observe that if one recovers the Prym as the Jacobian of a quotient of \(Z\) in the generic initial case, this continues to hold for the specializations. This observation may be explained by the compactification arguments in [19]. A notable example of this is furnished by Table 6.

6.2. Comments on the tables

Let us start with examples that already appear in the existing literature and that we could recover and extend. The following discussion proves Theorem C.

- Table 3 recovers the [49] case we looked at in Section 3 and which was the starting point of this article.
- Table 4 gathers covers of genus 2 of curves genus 1 by a map of degree \(d_X\) with \(2 \leq d_X \leq 7\). In all these cases, the Prym (which is a curve of genus 1) appears as a quotient of the Galois closure \(Z\). Note that when \(2 \leq d_X \leq 11\), there are direct construction of the Prym as an explicit curve of genus 1. The case \(d_X = 2\) goes back to the work of Jacobi on Abelian integrals, (see the references in
Table 5 gathers degree 2-covers of hyperelliptic curves ramified over exactly 2 points. We recover the results of Dalaljan [16] and Levin [30, Th.4.1].

Table 6 gathers étale covers of degree 2 of curves of genus \(g_Y \geq 3\). Recillas’ [46] trigonal construction shows that in this case the Prym variety is always isogenous to a Jacobian that can be obtained in a Galois-theoretic way. When \(g_X = 5\), the same result is also obtained by Bruin [12] in very explicit terms. We can still use this case as a benchmark for our programs, which we have tested by taking a sample of several hundred random covers. In each case, our algorithms confirm that the Prym variety is isogenous to a quotient of the Galois closure \(Z\). Note that here, the map to \(\mathbb{P}^1\) is of degree 3, which for \(g_Y = 3\) and \(g_Y = 4\) is actually the smallest degree that is generically possible.

Here are some new ramification patterns:

- Table 7 gathers étale covers of curves of genus 2 by maps of degrees 3, 4, and 5. The situation is more checkered since certain cases give positive results and others do not, even for the same ramification structure. A particular subcase of \(\text{etale-4}\) where the morphism \(X \to Y\) is Galois with group the Klein four-group and which corresponds to the first case in this entry is studied in [7] (see Section 4 of op. cit., and Proposition 4.9 in particular). There it is shown that one can indeed recover the Prym variety in terms of a suitable explicit Jacobian. However, our results are more general, as can be seen by the presence of additional entries. The reason for this is that we do not assume that \(X \to Y\) is Galois. We still recover the Prym variety whenever possible as the Jacobian \(\text{Jac}(C)\) of a quotient of the Galois closure \(Z\).

- Table 8 is a new situation which does not appear in the literature. One sees that when the degree of the cover from \(X\) to the curve \(Y\) of genus 1 is 3 or 4 we only get positive cases (that is, cases in which our strategy can describe the Prym as a Jacobian up to isogeny), but as soon as the degree is 5, we only get very few favorable situations.

- Table 9 gathers some miscellaneous cases.

### 6.3. Explicit equations

As an application of our programs, we consider the first case of Table 8: \(g_X = 3\), \(g_Y = 1\), \(d_X = 3\) and \(d_Y = 2\) with the ramification data

\[
R = \{(3, 2), (2, 3), (2, 3), (2, 3), (2, 3), (2, 3)\}.
\]

The cover \(X \to \mathbb{P}^1\) may be Galois with \(G \simeq S_3\); in this case, \(X\) is non-hyperelliptic, and the result already appears in Table 2. We therefore concentrate on the second case where \(X\) is hyperelliptic. The programs show in the same way that \(Z\) is also hyperelliptic, is equipped with an action of \(C_2 \times S_3\), and admits both \(X\) (of genus 3) and \(C\) (of genus 2) as quotients by involutions. Since the action of \(S_3\) commutes with that of the hyperelliptic involution, we may assume that the automorphism group is generated by \(\sigma : (x, y) \mapsto (\xi_3 x, y), \tau : (x, y) \mapsto (1/x, y/x^6)\) and \(t : (x, y) \mapsto (x, -y)\). This means that the hyperelliptic curve \(Z\), of genus 5, is given by

\[
y^2 = x^{12} + ax^9 + bx^6 + ax^3 + 1,
\]

with discriminant \(3^{12} (a^2 - 4b + 8)^6 ((b + 2)^2 - 4a)^3 \neq 0\). The quotient \(Z/(\langle \tau \rangle)\) is the genus 3 curve \(X\), and by taking the fixed functions \(u = x + 1/x\) and \(v = y(x - 1/x)\), we obtain

\[
X : v^2 = (u^6 - 6u^4 + au^3 + 9u^2 - 3au + (b - 2))(u^2 - 4),
\]

with discriminant \(-2^4 \cdot 3^6 \cdot (a^2 - 4b + 8)^3 (4a^2 - (b + 2)^2)^3 \neq 0\). The map \(X \to Y\) can then be recovered by considering the quotient \(Z/(\langle \tau, \sigma \rangle)\): we then obtain \(Y : t^2 = (s^2 - 4)(s^2 + as + (b - 2))\) and a 3-to-1 map

\[
X
\]

\[
\rightarrow
\]

\[
Y
\]

\[
(u, v) \mapsto (u^3 - 3u, v(u^2 - 1)).
\]

The Prym variety of \(X \to Y\) is isogenous to the Jacobian of \(C = Y/\langle \tau \rangle\). By taking the fixed functions \(u = x + 1/x\) and \(v' = y/x^3\), we obtain:

\[
C : v'^2 = (u^6 - 6u^4 + au^3 + 9u^2 - 3au + (b - 2)).
\]

The discriminant is \(-729 (a^2 - 4b + 8)^3 (4a^2 - (b + 2)^2) \neq 0\), so this is indeed a smooth curve of genus 2.

We remark that once this example, and similar ones in higher genus, were brought to our attention by the output of our programs, we were able to spot a generalization, which allowed us to recover some of the hyperelliptic cases in Table 8. Fix an integer \(k \geq 2\) and consider the hyperelliptic curve \(Z : y^2 = f(x)\), where \(f(x) = x^{4k} + ax^{3k} + bx^{2k} + ax^k + 1\) for generic \(a\) and \(b\). Factor...
Table 7. Postcomposing étale covers of curves of genus 2.

| Case | $g_X, g_Y, d_X$ | Ramification | $\#G, g_Z$ | $X_{nhyp/hyp}$ | Prym dims | $\deg Z \to C_i$ |
|------|-----------------|--------------|------------|----------------|-----------|----------------|
| étale-3 | 4, 2, 3 | 6, 4 | [40, 0], [0, 0] | [1, 1] | [2, 2] |
| étale-4 | 5, 2, 4 | 24, 13 | [0, 0], [0, 120] | [2, 1], [3] | [2, 4], [2] |
| étale-5 | 6, 2, 5 | 120, 61 | [4032, 0], [0, 0] | [2] or [1] | [24] or [8] |

Table 8. Merging total ramification above two points.

| Case | $g_X, g_Y, d_X$ | Ramification | $\#G, g_Z$ | $X_{nhyp/hyp}$ | Prym dims | $\deg Z \to C_i$ |
|------|-----------------|--------------|------------|----------------|-----------|----------------|
| total-3 | 3, 1, 3 | 6, 3 | [9, 0], [0, 0] | [2] | [2] |
| total-4 | 4, 1, 4 | 16, 7 | [24, 0], [24, 0] | [3] | [2] |
| total-5 | 5, 1, 5 | 120, 49 | [380, 0], [0, 0] | [2] or [1] | [24] or [8] |

Table 9. Genus 3 to genus 1, degree 4.

| Case | $g_X, g_Y, d_X$ | Ramification | $\#G, g_Z$ | $X_{nhyp/hyp}$ | Prym dims | $\deg Z \to C_i$ |
|------|-----------------|--------------|------------|----------------|-----------|----------------|
| 3131 | 3, 1, 4 | 24, 9 | [0, 0], [0, 27] | [2], [1, 1] | [8], [4, 6] |
| 4211 | 3, 1, 4 | 192, 73 | [64, 0], [0, 0] | [2] | [16] |

The polynomial $(u^2 - 4)(q^2(u) + ag(u) + b - 2)$ has $2k + 2$ different roots, and $X$ is a smooth hyperelliptic curve of genus $k$. Similarly, we consider the quotient $\pi_{Z/C} : Z \to C := Z/(\tau)$, and to get an equation for $C$ we consider the invariant functions $u = x + \frac{1}{x}$.
and $w = \frac{x^4}{0.7}$. We write first $f(x) = x^{2k}(x^4 + \frac{1}{x^4}) + a(x^k + \frac{1}{x^k}) + b - 2)$, and note that there exists a polynomial $g$ of degree $k$ such that $g(x + \frac{1}{x}) = x^k + \frac{1}{x^k}$. We get then $C : w^2 = g^2(u) + ag(u) + b - 2$. The roots of $g^2(u) + ag(u) + b - 2$ are $\zeta^i u_j + \frac{1}{\zeta^i u_j}$ with $i = 0, 1, \ldots, k - 1$ and $j = 1, 2$. This yields a smooth hyperelliptic curve $C$ of genus $g(C) = k - 1$. Furthermore, we consider the quotient $\pi_{X/Y} : X \to Y = \mathbb{Z}/(\ell, \sigma)$: we take the invariant functions $U = x^k + \frac{1}{x^k} = g(u)$ and $V = \frac{1}{\ell} (x^k - \frac{1}{x^k}) = vh(u)$, where $h$ is a degree $k - 1$ polynomial such that $h(x + \frac{1}{x}) = x^k - \frac{1}{x^k}$. We get then the equation $Y : V^2 = (U^2 - 4)(U^2 + aU + b - 2)$. The roots of $(U^2 - 4)(U^2 + aU + b - 2)$ are all different, and $Y$ is a smooth curve genus 1. Computing the pullbacks to $Z$ of the regular differentials of $X$, $Y$ and $C$ leads to the following proposition:

**Proposition 6.2.** With the notation above, the Prym variety $\text{Prym}(X/Y)$ is isogenous to the Jacobian of the curve $C$.

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**Declaration of Interest**

No potential conflict of interest was reported by the author(s).

**References**

[1] Baker, H. F. (1995). *Abelian Functions*. Cambridge: Cambridge Mathematical Library, Cambridge University Press. Abel’s theorem and the allied theory of theta functions, Reprint of the 1897 original, With a foreword by Igor Krichever.

[2] Bars, F. (2006). *Automorphism groups of genus 3 curves*. Notes del seminari Corbes de Gèneres 3.

[3] Bosma, W., Cannon, J., Playoust, C. (1997). The Magma algebra system. I. The user language. *J. Symbolic Comput.* 24(3/4): 235–265. Computational algebra and number theory (London, 1993).

[4] Bruin, N., Doerksen, K. (2011). The arithmetic of genus two curves with $(4,4)$-split Jacobians. *Canad. J. Math.* 63(5): 992–1024.

[5] Bertin, J. (2013). Algebraic stacks with a view toward moduli stacks of covers. In Pierre Débes, Michel Emsealam, Matthieu Romagny and A. Mmhammed Uludag (Editors), *Arithmetic and Geometry Around Galois Theory*, Vol. 304 of *Progr. Math*. Basel: Birkhäuser/Springer, pp. 1–148.

[6] Bröker, R., Lauter, R., Stevenhagen, P. (2015). Genus-2 curves and Jacobians with a given number of points. *LMS J. Comput. Math.* 18(1): 170–197.

[7] Borówka, P., Ortega, A. (2020). Klein coverings of genus 2 curves. *Trans. Amer. Math. Soc.* 373(3): 1885–1907.

[8] Bolza, O. (1887). Über die Reduktion hyperelliptischer Integrale erster Ordnung und erster Gattung auf elliptische durch eine Transformation vierten Grades. *Math. Ann.* 28: 447–456.

[9] Bouw, I. I. (1998). Tame covers of curves: $p$-ranks and fundamental groups. PhD thesis, Utrecht University.

[10] Bertin, J., Romagny, M. (2011). Champs de Hurwitz. *Mém.Soc.Math.Fr.(N.S.)* 125–126: 219.

[11] Breuer, T. (2000). *Characters and Automorphism Groups of Compact Riemann Surfaces*, Vol. 280 of London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press.

[12] Bruin, N. (2008). The arithmetic of Prym varieties in genus 3. *Compos. Math.* 144(2): 317–338.

[13] Carocca, A., Hidalgo, R., Rodriguez, R. E. (2020). $q$-étale covers of cyclic $p$-gonal covers. Preprint available at arXiv:2002.12082.

[14] Carocca, A., Recillas, S., Rodriguez, R. (2002). Dihedral groups acting on Jacobians. In Clifford J. Earle, William J. Harvey and Sevin Recillas-Pishmish (Editors), *Complex Manifolds and Hyperbolic Geometry (Guanajuato, 2001)*, Vol. 311 of *Contemp.Math.. Providence, RI: Amer. Math. Soc.*, pp. 41–77.

[15] Chevalley, C., Weil, A., Hecke, E. (1934). Über das Verhalten der Integrale I. Gattung bei Automorphismen des Funktionenkörpers. *Abh. Math. Sem. Univ. Hamburg* 10(1): 358–361.

[16] Dalaljan, S. G. (1975) The Prym variety of a two-sheeted covering of a hyperelliptic curve with two branch points. *Mat. Sh. (N.S.)* 98(140): (2 (10)).

[17] Dolgachev, I. V. (2012). *Classical Algebraic Geometry*. Cambridge: Cambridge University Press. A modern view.

[18] Donagi, R. (1981). The tetragonal construction. *Bull. Amer. Math. Soc. (N.S.)* 4(2): 181–185.

[19] Donagi, R. (1992). The fibers of the Prym map. In Ron Donagi (Editor), *Curves, Jacobians, and Abelian Varieties* (Amherst, MA, 1990), Vol. 136 of *Contemp. Math.. Providence, RI: Amer. Math. Soc.*, pp. 55–125.

[20] Enolskii, V., Richter, P. (2008). Periods of hyperelliptic integrals expressed in terms of $\theta$-constants by means of Thomae formulae. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 366(1867): 1005–1024.

[21] Goursat, E. (1885). Sur la réduction des intégrales hyperelliptiques. *Bull. Soc. Math. France* 13: 143–162.

[22] Gaudry, P., Schost, E. (2001). On the invariants of the quotients of the Jacobian of a curve of genus 2. In Serdar Boztas and Igor E. Shparlinski (Editors), *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes (Melbourne, 2001)*, Vol. 2227 of *Lecture Notes in Comput. Sci*. Berlin: Springer, pp. 373–386.
[23] Henn, P.-G. (1976). Die Automorphismengruppen der algebraischen Funktionenkörper vom Geschlecht 3. PhD thesis, Heidelberg.

[24] E. W. Howe, F. Leprévost, and B. Poonen. (2000). Large torsion subgroups of split Jacobians of curves of genus two or three. Forum Math. 12(3): 315–364.

[25] Izquierdo, M., Jiménez, L., Rojas, A. (2019). Decomposition of Jacobian varieties of curves with dihedral actions via equisymmetric stratification. Rev. Mat. Iberoam. 35(4): 1259–1279.

[26] Kani, E., Rosen, M. (1989). Idempotent relations and factors of Jacobians. Math. Ann. 284(2): 307–327.

[27] Kuhn, R. M. (1988). Curves of genus 2 with split Jacobian. Trans. Amer. Math. Soc. 307(1): 41–49.

[28] Kumar, A. (2015). Hilbert modular surfaces for square discriminants and elliptic subfields of genus 2 function fields. Res. Math. Sci. 2: Art. 24, 46.

[29] Lenstra, Jr., H. (2008). Galois theory for schemes. Available at: http://websites.math.leidenuniv.nl/algebra/.

[30] Levin, A. (2012). Siegel’s theorem and the Shafarevich conjecture. J. Théor. Nombres Bordeaux 24(3): 705–727.

[31] Lombardo, D., Lorenzo García, E., Ritzenthaler, C., Sijsling, J. (2020). Prym_decomposition, a MAGMA package for realizing Prym varieties as Jacobians. Available at: https://github.com/JRSijsling/prym_decomposition.

[32] Lange, H., Ortega, A. (2011). Prym varieties of triple coverings. Int. Math. Res. Not. IMRN (22): 5045–5075.

[33] Lange, H., Ortega, A. (2018). Prym varieties of étale covers of hyperelliptic curves. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 18(2): 467–482.

[34] Lange, H., Recillas, S. (2004). Abelian varieties with group action. J. Reine Angew. Math. 575: 135–155.

[35] Lercier, R., Ritzenthaler, C. (2012). Hyperelliptic curves and their invariants: geometric, arithmetic and algorithmic aspects. J. Algebra 372: 595–636.

[36] Massey, W. S. (1991). A basic Course in Algebraic Topology, Vol. 127 of Graduate Texts in Mathematics. New York: Springer-Verlag.

[37] Miranda, R. (1995). Algebraic curves and Riemann surfaces, Vol. 5 of Graduate Studies in Mathematics. Providence, RI: American Mathematical Society.

[38] Magaard, K., Shaska, T., Shpекторов, S., Völklein, H. (2002). The locus of curves with prescribed automorphism group. Surikaisekikenkyusho Kokyuroku (1267): 112–141. Communications in arithmetic fundamental groups (Kyoto, 1999/2001).

[39] Magaard, K., Shaska, T., Völklein, H. (2009). Genus 2 curves that admit a degree 5 map to an elliptic curve. Forum Math. 21(3): 547–566.

[40] Mumford, D. (1974). Prym varieties. I. In Lars V. Ahlfors, Irwin Kra, Bernard Maskit and Louis Nirenberg (Editors), Contributions to analysis (a collection of papers dedicated to Lipman Bers), pp. 325–350.

[41] Neukirch, J. (1999). Algebraic Number Theory, Vol. 322 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Berlin: Springer-Verlag.

[42] Pantazis, S. (1986). Prym varieties and the geodesic flow on SO(n). Math. Ann. 273(2): 297–315.

[43] Paulhus, J. (2008). Decomposing Jacobians of curves with extra automorphisms. Acta Arith. 132(3): 231–244.

[44] Paulhus, J. (2015). Branching data for curves up to genus 48. Preprint and code. Available at https://paulhus.math.grinnell.edu/monodromy.html

[45] Paulhus, J., Rojas, A. M. (2017). Completely decomposable Jacobian varieties in new genera. Exp. Math. 26(4): 430–445.

[46] Recillas, S. (1974). Jacobians of curves with $g^1_4$‘s are the Prym’s of trigonal curves. Bol. Soc. Mat. Mexicana (2) 19(1): 9–13.

[47] Recillas, S., Rodrigo, R. E. (2006). Prym varieties and fourfold covers. II. The dihedral case. In The Geometry of Riemann Surfaces and Abelian Varieties, Vol. 397 of Contemp. Math., Providence, RI: Amer. Math. Soc., pp. 177–191.

[48] Recillas, S., Rodríguez, R. E. (2006). Prym varieties and fourfold covers. II. The dihedral case. In The Geometry of Riemann Surfaces and Abelian Varieties, Vol. 397 of Contemp. Math., Providence, RI: Amer. Math. Soc., pp. 177–191.

[49] Ritzenthaler, C., Romagny, M. (2018). On the Prym variety of genus 3 covers of genus 1 curves. Épijournal Geom. Algébrique 2: Art. 2, 8.

[50] Serre, J.-P. (1978). Représentations linéaires des groupes finis, revised ed. Paris: Hermann.

[51] Tsimerman, J. (2012). The existence of an Abelian variety over $\mathbb{Q}$ isogenous to no Jacobian. Ann. Math. (2) 176(1): 637–650.

[52] Unger, W. R. (2006). Computing the character table of a finite group. J. Symbolic Comput. 41(8): 847–862.

[53] Vermeulen, A. (1983). Weylstrass points of weight two on curves of genus three. PhD thesis, University of Amsterdam, Amsterdam.

[54] Weil, A. (1935). Über Matrizenringe auf Riemannschen Flächen und den Riemann - Rochsehen Satz. Abh. Math. Sem. Univ. Hamburg 11(1): 110–115.