A CONSTRUCTION OF SURFACES OF GENERAL TYPE
WITH $p_g = 1$ AND $q = 0$

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Abstract. In this paper we reconstruct minimal complex surfaces of general type with $p_g = 1$, $q = 0$, and $1 \leq K^2 \leq 2$ using a $\mathbb{Q}$-Gorenstein smoothing theory. Furthermore, we also construct a new family of simply connected surfaces with $p_g = 1$, $q = 0$, and $3 \leq K^2 \leq 6$ using the same method.

1. Introduction

Complex surfaces of general type with $p_g = 1$ and $q = 0$ have been drawn attention because they provide counterexamples to the Torelli problems. All such surfaces are constructed by classical methods: Quotient and covering. Kynev [6] constructed a surface with $K^2 = 1$ as a quotient of the Fermat sextic in $\mathbb{P}^3$ by a suitable action of a group of order 6. According to Catanese [2], all minimal surfaces of general type with $p_g = 1$ and $K^2 = 1$ are diffeomorphic and simply connected. Catanese and Debarre [3] constructed surfaces with $K^2 = 2$ by double coverings of the projective plane $\mathbb{P}^2$ or smooth minimal K3 surfaces. In fact, they classified such surfaces into five classes according to the degree and the image of the bicanonical map. Four of them are simply connected and the other one has a torsion $\mathbb{Z}/2\mathbb{Z}$. Todorov [13] constructed surfaces with $2 \leq K^2 \leq 8$ by considering double covers of K3 surfaces. His examples with $3 \leq K^2 \leq 8$ have big fundamental groups.

In this paper we construct minimal complex surfaces of general type with $p_g = 1$, $q = 0$, and $1 \leq K^2 \leq 6$ by a $\mathbb{Q}$-Gorenstein smoothing theory. In particular, we construct simply connected surfaces with $1 \leq K^2 \leq 6$ and a surface with $K^2 = 2$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$. We use a similar method in [8]. We blow up an elliptic K3 surface in a suitable set of points so that we obtain a surface with a very special configuration of rational curves. Inside this configuration we find some disjoint chains which can be contracted to special quotient singularities. These singularities admit a local $\mathbb{Q}$-Gorenstein smoothing, which is a smoothing whose relative canonical class is $\mathbb{Q}$-Cartier. Finally, we prove that these local smoothings can be glued to a global $\mathbb{Q}$-Gorenstein smoothing of the whole singular surface by showing that the obstruction space of a global smoothing is zero. Then it is not difficult to show that a general fiber of the smoothing is the desired surface. The key ingredient of this paper is to develop a new method for proving that the obstruction space is zero because the Lee-Park’s method in [8] for a computation of the obstruction space cannot be applied to our cases; Section 3.
In Section 2 we reconstruct complex surfaces with \( p_g = 1, q = 0, \) and \( K^2 = 2 \): A simply connected surface and a surface with \( H_1 = \mathbb{Z}/2\mathbb{Z} \) by using a (generalized) rational blow-down surgery. We prove in Section 3 that the obstruction space of a global smoothing of the singular surface constructed in Section 2 is zero. In the final section, we also construct various examples of simply connected complex surfaces satisfying \( p_g = 1, q = 0, \) and \( 1 \leq K^2 \leq 6 \).

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2. A Construction of Surfaces with \( p_g = 1, q = 0, \) and \( K^2 = 2 \)

In this section we construct a simply connected minimal surface of general type with \( p_g = 1, q = 0, \) and \( K^2 = 2 \). We start with a special rational elliptic surface \( E(1) \). Let \( L_1, L_2, L_3, \) and \( A \) be lines in \( \mathbb{P}^2 \) and let \( B \) be a smooth conic in \( \mathbb{P}^2 \) intersecting as in Figure 1(A). We consider a pencil of cubics \( \{ \lambda(L_1 + L_2 + L_3) + \mu(A + B) \mid [\lambda : \mu] \in \mathbb{P}^1 \} \) in \( \mathbb{P}^2 \) generated by two cubic curves \( L_1 + L_2 + L_3 \) and \( A + B \), which has 4 base points, say, \( p, q, r \) and \( s \). In order to obtain an elliptic fibration over \( \mathbb{P}^1 \) from the pencil, we blow up three times at \( p \) and \( r \), respectively, and twice at \( s \), including infinitely near base-points at each point. We perform one further blowing-up at the base point \( q \). By blowing-up totally nine times, we resolve all base points (including infinitely near base-points) of the pencil and we then get a rational elliptic surface \( E(1) \) with an \( I_8 \)-singular fiber, an \( I_2 \)-singular fiber, two nodal singular fibers, and four sections; Figure 1(B).

Let \( Y \) be a double cover of the rational elliptic surface \( E(1) \) branched along two general fibers. Then the surface \( Y \) is an elliptic K3 surface with two \( I_8 \)-singular fibers, two \( I_2 \)-singular fibers, four nodal singular fibers, and four sections; Figure 2(A). In the following construction we use only one \( I_8 \)-singular fiber, one nodal singular fiber, and three sections; Figure 2(B).

Let \( \tau : V \to Y \) be the blowing-up at the node of the nodal singular fiber and let \( E \) be the exceptional divisor of \( \tau \). Since \( K_Y = 0 \), we have \( K_V = E \).
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(a) The fibration structure of $Y$

(b) A part of $Y$

Figure 2: An elliptic K3 surface $Y$

$D = D_1 + \cdots + D_6$ be the part of the $I_8$-singular fiber. Let $S_i$ ($i = 1, 2, 3$) be the sections of the fibration $V \to \mathbb{P}^1$ and set $S = S_1 + S_2 + S_3$. Let $F$ be the proper transform of the nodal fiber of the K3 surface $Y$; Figure 3.

Figure 3: A surface $V = Y \# \mathbb{P}^2$

We blow up the surface $V$ three times totally at the three marked points • and blow up twice at the marked point ○. We then get a surface $Z$; Figure 4. There exist three disjoint linear chains of $\mathbb{P}^1$'s in $Z$:

$-3 - 6 - 2 - 3 - 2, -4 - 2 - 2 - 3 - 2, -4.$

Main construction. By applying $\mathbb{Q}$-Gorenstein smoothing theory to the surface $Z$ as in [8] [11] [12], we construct a complex surface of general type with $p_g = 1$,
Figure 4: A surface $Z = Y^6\mathbb{P}^2$

$q = 0$, and $K^2 = 2$. That is, we first contract the three chains of $\mathbb{P}^1$’s from the surface $Z$ so that it produces a normal projective surface $X$ with three permissible singular points. In Section 3, we will show that the singular surface $X$ has a global $\mathbb{Q}$-Gorenstein smoothing. Let $X_t$ be a general fiber of the $\mathbb{Q}$-Gorenstein smoothing of $X$. Since $X$ is a (singular) surface with $p_g = 1$, $q = 0$, and $K^2 = 2$, by applying general results of complex surface theory and $\mathbb{Q}$-Gorenstein smoothing theory, one may conclude that a general fiber $X_t$ is a complex surface of general type with $p_g = 1$, $q = 0$, and $K^2 = 2$. Furthermore, it is not difficult to show that a general fiber $X_t$ is minimal and simply connected by using a similar technique in [8, 11, 12].

**Remark.** Catanese and Debarre [3] proved that surfaces of general type with $p_g = 1$, $q = 0$, and $K^2 = 2$ are divided into five types according to the degree and the image of the bicanonical map. Four of them are simply connected and the other one has a torsion $\mathbb{Z}/2\mathbb{Z}$. It is an interesting problem to determine in which class the simply connected example constructed in this section is contained. We leave this question for the future research.

2.1. **An example with $K^2 = 2$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$.** We construct a minimal complex surface of general type with $p_g = 1$, $q = 0$, and $K^2 = 2$, and $H_1 = \mathbb{Z}/2\mathbb{Z}$. Let $A$, $L_i$ ($i = 1, 2, 3$) be lines on the projective plane $\mathbb{P}^2$ and $B$ a nonsingular conic on $\mathbb{P}^2$ which intersect as in Figure 5(A). Consider a pencil of cubics generated by the two cubics $A + B$ and $L_1 + L_2 + L_3$. By resolving the base points of the pencil of cubics including infinitely near base-points, we obtain a rational elliptic surface $E(1)$ with an $I_7$-singular fiber, an $I_2$-singular fiber, an cusp singular fiber, a nodal singular fiber, and five sections as in Figure 5(B).

Let $Y$ be a double cover of the rational elliptic surface $E(1)$ branched along two general fibers near the cusp singular fiber. Then $Y$ is an elliptic K3 surface with two $I_7$-singular fibers, two $I_2$-singular fibers, two cusp singular fibers, two nodal singular fibers, and five sections. We use only two $I_7$-singular fibers, two $I_2$-singular fibers, and two sections; Figure 5(A). We blow up eight times totally at the eight marked points $\bullet$. We then get a surface $Z$; Figure 6(B). There exist six disjoint linear chains of $\mathbb{P}^1$’s in $Z$:

$$-5 - 3 - 2 - 2, -5 - 3 - 2 - 2, -3 - 2 - 3, -3, -3 - 2 - 3 - 4, -4$$

We now briefly explain how to prove that the complex surface obtained by the above configuration has $H_1 = \mathbb{Z}/2\mathbb{Z}$. Let $L_i$ ($i = 1, \ldots, 6$), $M_j$ ($i = 1, \ldots, 6$), $N_k$ ($k = 1, 2$) be the parts of the $I_7$-singular fibers and the $I_2$-singular fibers of the K3
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Figure 5: A rational elliptic surface $E(1)$ for $K^2 = 2$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$

Figure 6: An example with $H_1 = \mathbb{Z}/2\mathbb{Z}$

surface $Y$ and let $S_l$ ($l = 1, \ldots, 4$) the four sections of $Y$; Figure 6(A). The matrix whose entries are the intersection numbers of $L_i$, $M_j$, $N_k$, and $S_l$ is invertible. Therefore the Picard number of the K3 surface $Y$ is at least 18. Furthermore, during the fiber sum surgery, four new homology elements, say $T_1, \ldots, T_4$, are constructed. Two elements among the four elements are represented by tori and the other two elements are represented by spheres. Therefore the homology $H_2(Y, \mathbb{Z})$ are generated by the eighteen $(-2)$-curves $L_i$, $M_j$, $N_k$, $S_l$, and the four new elements $T_m$. Hence the homology $H_2(Z, \mathbb{Z})$ are generated by the proper transforms of $L_i$,
$M_j, N_k, S_l, T_m,$ and the eight exceptional curves. Then, by using a similar method in \([7, 10]\), it is not difficult to show that the complex surface obtained by the above configuration has $H_1 = \mathbb{Z}/2\mathbb{Z}$.

3. The Existence of $\mathbb{Q}$-Gorenstein smoothings

This section is devoted to the proof of the following theorem.

**Theorem 3.1.** The singular surface $X$ constructed in the main construction in Section 2 has a global $\mathbb{Q}$-Gorenstein smoothing.

For this, we first briefly review some basic facts on $\mathbb{Q}$-Gorenstein smoothing theory for normal projective surfaces with special quotient singularities (refer to \([8]\) for details).

**Definition.** Let $X$ be a normal projective surface with quotient singularities. Let $X \to \Delta$ (or $X/\Delta$) be a flat family of projective surfaces over a small disk $\Delta$. The one-parameter family of surfaces $X \to \Delta$ is called a $\mathbb{Q}$-Gorenstein smoothing of $X$ if it satisfies the following three conditions:

(i) the general fiber $X_t$ is a smooth projective surface,
(ii) the central fiber $X_0$ is $X$,
(iii) the canonical divisor $K_{X/\Delta}$ is $\mathbb{Q}$-Cartier.

A $\mathbb{Q}$-Gorenstein smoothing for a germ of a quotient singularity $(X_0, 0)$ is defined similarly. A quotient singularity which admits a $\mathbb{Q}$-Gorenstein smoothing is called a singularity of class $T$.

**Proposition 3.2** \([5, 9]\). Let $(X_0, 0)$ be a germ of two dimensional quotient singularity. If $(X_0, 0)$ admits a $\mathbb{Q}$-Gorenstein smoothing over the disk, then $(X_0, 0)$ is either a rational double point or a cyclic quotient singularity of type $\frac{1}{dn}(1, dna - 1)$ for some integers $a, n, d$ with $a$ and $n$ relatively prime.

**Proposition 3.3** \([5, 9]\). (1) The singularities $-4$, $-3$, $-2$, $-2$, $-2$, and $-3$ are of class $T$.

(2) If the singularity $-b_1 \cdots -b_r$ is of class $T$, then so are $-b_1 - b_2 - \cdots - b_r$ and $-b_1 - b_2 - \cdots - b_r - 2$.

(3) Every singularity of class $T$ that is not a rational double point can be obtained by starting with one of the singularities described in (1) and iterating the steps described in (2).

Let $X$ be a normal projective surface with singularities of class $T$. Due to the result of Kollár and Shepherd-Barron \([5]\), there is a $\mathbb{Q}$-Gorenstein smoothing locally for each singularity of class $T$ on $X$. The natural question arises whether this local $\mathbb{Q}$-Gorenstein smoothing can be extended over the global surface $X$ or not. Roughly geometric interpretation is the following: Let $\bigcup V_\alpha$ be an open covering of $X$ such that each $V_\alpha$ has at most one singularity of class $T$. By the existence of a local $\mathbb{Q}$-Gorenstein smoothing, there is a $\mathbb{Q}$-Gorenstein smoothing $V_\alpha/\Delta$. The question is if these families glue to a global one. The answer can be obtained by figuring out the obstruction map of the sheaves of deformation $T^i_X = \text{Ext}^i_X(\Omega_X, O_X)$ for $i = 0, 1, 2$. For example, if $X$ is a smooth surface, then $T^0_X$ is the usual holomorphic tangent
sheaf $T_X$ and $T_X^1 = T_X^2 = 0$. By applying the standard result of deformations to a
normal projective surface with quotient singularities, we get the following

**Proposition 3.4** ([14, §4]). Let $X$ be a normal projective surface with quotient
singularities. Then

1. The first order deformation space of $X$ is represented by the global Ext 1-group
   $T_X^1 = \text{Ext}^1_X(\Omega_X, \mathcal{O}_X)$.
2. The obstruction lies in the global Ext 2-group $T_X^2 = \text{Ext}^2_X(\Omega_X, \mathcal{O}_X)$.

Furthermore, by applying the general result of local-global spectral sequence of
ext sheaves to deformation theory of surfaces with quotient singularities so that
$E_2^{p,q} = H^p(T_X^q) \Rightarrow \mathcal{T}_X^{p+q}$, and by $H^1(T_X^q) = 0$ for $i, j \geq 1$, we also get

**Proposition 3.5** ([9] [14]). Let $X$ be a normal projective surface with quotient
singularities. Then

1. We have the exact sequence
   $$0 \rightarrow H^1(T_X^0) \rightarrow T_X^1 \rightarrow \ker[H^0(T_X^1) \rightarrow H^2(T_X^0)] \rightarrow 0$$
   where $H^1(T_X^0)$ represents the first order deformations of $X$ for which the
   singularities remain locally a product.
2. If $H^2(T_X^0) = 0$, every local deformation of the singularities may be globalized.

As mentioned above, there is a local $\mathbb{Q}$-Gorenstein smoothing for each singularity
of $X$ due to the result of Kollár and Shepherd-Barron [5]. Hence it remains to show
that every local deformation of the singularities can be globalized. Note that the
following proposition tells us a sufficient condition for the existence of a global
$\mathbb{Q}$-Gorenstein smoothing of $X$.

**Proposition 3.6** ([8]). Let $X$ be a normal projective surface with singularities
of class $T$. Let $\pi : V \rightarrow X$ be the minimal resolution and let $A$ be the reduced
exceptional divisor. Suppose that $H^2(T_V(-\log A)) = 0$. Then $H^2(T_X^0) = 0$ and
there is a $\mathbb{Q}$-Gorenstein smoothing of $X$.

Furthermore, the proposition above can be easily generalized to any log resolu-
tion of $X$ by keeping the vanishing of cohomologies under blowing up at the points.
It is obtained by the following well-known result.

**Proposition 3.7** ([11, §1]). Let $V$ be a nonsingular surface and let $A$ be a simple
normal crossing divisor in $V$. Let $f : V' \rightarrow V$ be a blowing up of $V$ at a point $p$ of
$A$. Set $A' = f^{-1}(A)_{\text{red}}$. Then $h^2(T_{V'}(-\log A')) = h^2(T_V(-\log A))$.

Therefore Theorem 3.1 follows from Proposition 3.6 and the following proposi-
tion:

**Proposition 3.8.** $H^2(T_V(-\log(D + S + F))) = H^0(\Omega_V(\log(D + S + F)))(E)) = 0$.

The idea of the proof is as follows: There is an exact sequence of locally free sheaves

$$0 \rightarrow \Omega_V \rightarrow \Omega_V(\log(D + S + F)) \rightarrow \bigoplus_{i=1}^6 \mathcal{O}_{D_i} \oplus \bigoplus_{i=1}^3 \mathcal{O}_{S_i} \oplus \mathcal{O}_F \rightarrow 0.$$

By tensoring $\mathcal{O}_V(E)$, we have an exact sequence

$$0 \rightarrow \Omega_V(E) \rightarrow \Omega_V(\log(D + S + F))(E) \rightarrow \bigoplus_{i=1}^6 \mathcal{O}_{D_i} \oplus \bigoplus_{i=1}^3 \mathcal{O}_{S_i} \oplus \mathcal{O}_F(E) \rightarrow 0.$$
Since $H^0(\Omega_V(E)) = 0$, the proof of Proposition 3.8 is done if we show that the connecting homomorphism

$$
\bigoplus_{i=1}^6 H^0(\mathcal{O}_{D_i}) \oplus \bigoplus_{i=1}^3 H^0(\mathcal{O}_{S_i}) \oplus H^0(\mathcal{O}_F(E)) \to H^1(\Omega_V(E))
$$

is injective.

The proof of Proposition 3.8 begins with the following Lemma. We have a commutative diagram

$$
\begin{array}{cccc}
\bigoplus_{i=1}^6 H^0(\mathcal{O}_{D_i}) & \bigoplus_{i=1}^3 H^0(\mathcal{O}_{S_i}) & \oplus & H^0(\mathcal{O}_F(E)) \\
\downarrow & & & \downarrow \\
\bigoplus_{i=1}^6 H^0(\mathcal{O}_{D_i}) & \bigoplus_{i=1}^3 H^0(\mathcal{O}_{S_i}) & \oplus & H^0(\mathcal{O}_F(E)) \\
\end{array}
$$

Lemma 3.9. The composition map

$$
\beta_1 \circ c_1 : \bigoplus_{i=1}^6 H^0(\mathcal{O}_{D_i}) \oplus \bigoplus_{i=1}^3 H^0(\mathcal{O}_{S_i}) \oplus H^0(\mathcal{O}_F) \to H^0(\Omega_V(E))
$$

is injective.

Proof. Note that the map $c_1$ is the first Chern class map. Since the intersection matrix, whose entries are the intersection numbers of $D_i (i = 1, \ldots, 6)$, $S_j (j = 1, 2, 3)$, and $F$, is invertible, their images by the map $c_1$ are linearly independent. Therefore the map $c_1$ is injective.

Consider the commutative diagram

$$
\begin{array}{cccc}
\bigoplus_{i=1}^6 H^0(\mathcal{O}_{D_i}) & \bigoplus_{i=1}^3 H^0(\mathcal{O}_{S_i}) & \oplus & H^0(\mathcal{O}_F(E)) \\
\downarrow & & & \downarrow \\
\bigoplus_{i=1}^6 H^0(\mathcal{O}_{D_i}) & \bigoplus_{i=1}^3 H^0(\mathcal{O}_{S_i}) & \oplus & H^0(\mathcal{O}_F(E)) \\
\end{array}
$$

where the vertical sequence is induced from the exact sequence

$$
0 \to H^0(\Omega_V \otimes \mathcal{O}_E(E)) \xrightarrow{\delta} H^1(\Omega_V) \xrightarrow{\beta_1} H^1(\Omega_V(E)) \xrightarrow{\gamma} H^1(\Omega_V \otimes \mathcal{O}_E(E)) \to 0.
$$
Claim: The connecting homomorphism \( \delta : H^0(\Omega_V \otimes \mathcal{O}_E(E)) \to H^1(\Omega_V) \) is the first Chern class map of \( \mathcal{O}_V(E) \): Since \( H^0(\Omega_E \otimes \mathcal{O}_E(E)) = 0 \), we have an isomorphism

\[
H^0(\mathcal{O}_E(-E) \otimes \mathcal{O}_E(E)) \cong H^0(\Omega_V \otimes \mathcal{O}_E(E)).
\]

Here the above map is given by \( z \otimes \frac{1}{z_\alpha} \mapsto \{ \frac{dz_\alpha}{z_\alpha} \} \), where \( z_\alpha \) is a local equation of \( E \). Therefore the connecting homomorphism \( \delta : H^0(\Omega_V \otimes \mathcal{O}_E(E)) \to H^1(\Omega_V) \) is given by

\[
\left\{ \frac{dz_\alpha}{z_\alpha} \right\} \mapsto \left\{ \frac{dz_\alpha}{z_\alpha} - \frac{dz_\beta}{z_\beta} \right\} = \left\{ d \left( \frac{z_\alpha}{z_\beta} \right) / \left( \frac{z_\alpha}{z_\beta} \right) \right\},
\]

which is the first Chern class map of \( \mathcal{O}_V(E) \). This proves the claim.

Since \( E \) is the exceptional divisor, it is independent of the other divisors in \( H^1(\Omega_V) \). Therefore \( \text{im} c_1 \cap \text{im} \delta = 0 \); hence the composition map \( \beta_1 \circ c_1 \) is injective.

We now concentrate on the following restriction map \( d_1' \) of the map \( d_1 \) in (3.1):

\[
d_1' : H^0(\mathcal{O}_F(E)) \to H^1(\Omega_V(E)),
\]

which is also regarded as a connecting homomorphism induced from the exact sequence

\[
0 \to \Omega_V \to \Omega_V(\log F) \to \mathcal{O}_V \to 0
\]
tensored by \( \mathcal{O}_V(E) \).

Lemma 3.10. We have \( H^0(\Omega_V(\log F)(E)) = 0 \).

Proof. Let \( C \) be a general fiber of the elliptic fibration \( \phi : Y \to \mathbb{P}^1 \). By the projection formula we have

\[
H^0(\Omega_V(\log F)(E)) \subseteq H^0(\Omega_V(F + E)) = H^0(\Omega_V(\tau^*C - E)) \subseteq H^0(\Omega_V(\tau^*C)) = H^0(Y, \Omega_Y(C)).
\]

On the other hand it is not difficult to show that \( H^0(Y, \Omega_Y(C)) = H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(1)) = 0 \) by a similar method in [8, Lemma 2]. Therefore the assertion follows.

By the above lemma, we have the following commutative diagram of exact sequences:

\[
\begin{array}{cccccccc}
0 & \to & H^0(\mathcal{O}_F) & \to & H^0(\mathcal{O}_F(E)) & \to & H^1(\Omega_V(\log F)) & \to & 0 \\
\downarrow & & \downarrow \delta_1 & & \downarrow \delta_1 & & \downarrow \delta_2 & & \\
0 & \to & H^0(\mathcal{O}_F(E)) & \to & d_1' & \to & H^1(\Omega_V(E)) & \to & 0 \\
\downarrow & & \downarrow & & \uparrow \gamma=d_1' & & \downarrow & & \downarrow \gamma \\
0 & \to & H^1(\Omega_V \otimes \mathcal{O}_E(E)) & \to & H^1(\Omega_V(\log F)(E)) & \to & 0 & & \\
\end{array}
\]

(3.2)
Lemma 3.11. $h^0(\Omega_V(\log F) \otimes \mathcal{O}_E(E)) = 1$.

Proof. Let
$$\varpi : \Omega_V(\log F) \otimes \mathcal{O}_E \to \Omega_E \otimes \mathcal{O}_E(F)$$
be the restriction to the subsheaf $\Omega_V(\log F)$ of the restriction map $r \otimes \text{id} : \Omega_V|_E \otimes \mathcal{O}_E(F) \to \Omega_E \otimes \mathcal{O}_E(F)$.

Claim. $\ker \varpi = \mathcal{O}_E(-E)$: Let $z_1, z_2$ be the local equations of $F, E$, respectively. Note that the restriction map $r : \Omega_V|_E \to \Omega_E$ is defined by
$$r : \varpi z_1 + \varpi z_2 = \varpi z_1.$$ 
Hence the map $\varpi$ is defined by
$$\varpi \left( \frac{\varpi z_1}{z_1} + \frac{\varpi z_2}{z_1} \right) = \frac{\varpi z_1}{z_1}.$$ 
It follows that $\ker \varpi = \ker r = \mathcal{O}_E(-E)$.

By the claim we have
$$\ker(\varpi \otimes \text{id}) : \Omega_V(\log F) \otimes \mathcal{O}_E(E) \to \Omega_E \otimes \mathcal{O}_E(F + E) = \mathcal{O}_E(-E) \otimes \mathcal{O}_E(E) = \mathcal{O}_E.$$ 
Hence we have the exact sequence
$$0 \to H^0(\mathcal{O}_E) \to H^0(\Omega_V(\log F) \otimes \mathcal{O}_E(E)) \to H^0(\ker(\varpi \otimes \text{id})).$$ 
Since $H^0(\ker(\varpi \otimes \text{id})) \subset H^0(\mathcal{O}_E \otimes \mathcal{O}_E(F + E)) = 0$, the result follows. $\square$

Lemma 3.12. The composition map
$$\gamma \circ d'_1 : H^0(\mathcal{O}_F(E)) \to H^1(\Omega_V \otimes \mathcal{O}_E(E))$$
is surjective.

Proof. On the second column in [52], we have $h^0(\Omega_V \otimes \mathcal{O}_E(E)) = 1$, $h^1(\Omega_V) = 21$, $h^1(\Omega_V \otimes \mathcal{O}_E(E)) = 2$. Hence $h^1(\Omega_V(E)) = 22$. Since $h^0(\mathcal{O}_F(E)) = 1$ and $h^0(\mathcal{O}_F(E)) = 3$, it follows from the first and the second columns that $h^1(\Omega_V(\log F)) = 20$ and $h^1(\Omega_V(\log F))(E)) = 19$. Therefore, on the third column, we have $h^1(\Omega_V(\log F) \otimes \mathcal{O}_E(E)) = 0$ because $h^0(\Omega_V(\log F) \otimes \mathcal{O}_E(E)) = 1$ by Lemma 3.11. Hence $\beta_2$ is surjective.

Note that $H^0(\mathcal{O}_F) \subseteq \beta_1(H^1(\Omega_V)) \cap \ker \alpha_2$ in $H^1(\Omega_V(E))$. On the other hand, since $\alpha_1$ and $\beta_2$ are surjective, we have
$$(\alpha_2 \circ \beta_1)(H^1(\Omega_V)) = (\beta_2 \circ \alpha_1)(H^1(\Omega_V)) = H^1(\Omega_V(\log F))(E)).$$
Therefore $\dim(\beta_1(H^1(\Omega_V)) \cap \ker \alpha_2) = 1$; hence $H^0(\mathcal{O}_F) = \beta_1(H^1(\Omega_V)) \cap \ker \alpha_2$ in $H^1(\Omega_V(E))$. Thus
$$\dim(\beta_1(H^1(\Omega_V)) \cap H^0(\mathcal{O}_F(E))) = 1.$$ 
Since $\ker \gamma = \im \beta_1$, we have $\dim(\ker \gamma \cap H^0(\mathcal{O}_F(E))) = 1$; hence the composition $\gamma \circ d'_1$ is surjective because $h^0(\mathcal{O}_F(E)) = 3$ but $h^1(\Omega_V \otimes \mathcal{O}_E(E)) = 2$. $\square$
Proof of Proposition 3.8. Set $K = H^0(\Omega_V(\log(D + S + F))(E))$. We want to show that $K = 0$. Consider the commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & H^0(\Omega_V \otimes O_E(E)) & \overset{\delta}{\longrightarrow} & H^0(\Omega_V) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 6 \bigoplus_{i=1}^6 H^0(O_{D_i}) \oplus 3 \bigoplus_{i=1}^3 H^0(O_{S_i}) \oplus H^0(O_F(E)) & \overset{e_1}{\longrightarrow} & H^1(\Omega_V) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K & \overset{\delta_1}{\longrightarrow} & 6 \bigoplus_{i=1}^6 H^0(O_{D_i}) \oplus 3 \bigoplus_{i=1}^3 H^0(O_{S_i}) \oplus H^0(O_F(E)) & \overset{d_1}{\longrightarrow} & H^1(\Omega_V(E)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1(\Omega_V \otimes O_E(E)) & \overset{\gamma}{\longrightarrow} & H^1(\Omega_V) \\
\end{array}
$$

Let $\{1, v_1, v_2\}$ be a basis for $H^0(O_F(E))$, where $1$ is a basis for $H^0(O_F) \cong \mathbb{C}$. By Lemma 3.12 the images $\gamma \circ d_1(v_1)$ and $\gamma \circ d_1(v_2)$ span $H^1(\Omega_V \otimes O_E(E))$. Therefore they are linearly independent in $H^1(\Omega_V(E))$ and they are not contained in the kernel of the map $\gamma$. On the other hand

$$
\bigoplus_{i=1}^6 H^0(O_{D_i}) \oplus \bigoplus_{i=1}^3 H^0(O_{S_i}) \oplus H^0(O_F) \subset \ker \gamma.
$$

Therefore the map $d_1$ is injective; hence $K = \ker d_1 = 0$. □

4. Simply connected surfaces with $p_g = 1$, $q = 0$, and $1 \leq K^2 \leq 6$

In this section we construct various examples of simply connected minimal surfaces of general type with $p_g = 1$, $q = 0$ and $1 \leq K^2 \leq 6$. Since all the proofs are basically the same as the case of the main construction, we describe only complex surfaces $Z$ which make it possible to get singular surfaces $X$ with permissible singularities.

4.1. An example with $K^2 = 1$. Let $Y$ be the K3 surface described in Section 2. We blow up the surface $Y$ totally three times at the three marked points $\bullet$: Figure 7(A). We then get a surface $Z$; Figure 7(B). There exist two disjoint linear chains of $\mathbb{P}^1$’s in $Z$:

$$
\begin{array}{cccc}
& -3 & -2 & -2 \\
\circ & \circ & \circ & \circ \\
\end{array},
\begin{array}{cccc}
& -2 & -5 & -3 \\
\circ & \circ & \circ & \circ \\
\end{array}.
$$

Remark. According to Catanese [2], all minimal surfaces of general type with $p_g = 1$ and $K^2 = 1$ are diffeomorphic and simply connected. Hence the example above is automatically simply connected. However we can prove the simply connectedness directly by using similar techniques – rational blow-down surgery, Milnor fiber theory, and Van-Kampen Theorem – in [8, 11, 12].
4.2. **An example with** $K^2 = 3$. Let $A, L_i$ ($i = 1, 2, 3$) be lines on the projective plane $\mathbb{P}^2$ and $B$ a nonsingular conic on $\mathbb{P}^2$ which intersect as in Figure 8(A). Consider a pencil of cubics generated by the two cubics $A + B$ and $L_1 + L_2 + L_3$. Blow up the five base points $\bullet$ of the pencil of cubics including infinitely near base-points at each point. Then we obtain a rational elliptic surface $E(1)$ with an $I_6$-singular fiber, $I_3$-singular fiber, three nodal singular fibers, and five sections which intersect as in Figure 8(B), where we omit one nodal singular fiber and one section which are not used in the following construction.

Let $Y$ be a double cover of the rational elliptic surface $E(1)$ branched along two general fibers. Then $Y$ is an elliptic K3 surface with two $I_6$-singular fibers, two $I_3$-singular fibers, six nodal singular fibers, and five sections. We use only two $I_6$-singular fibers, one $I_3$-singular fiber, one nodal singular fiber, and four sections; Figure 9(A).

We blow up the surface $Y$ six times at the six marked points $\bullet$ and twice at the marked point $\bigcirc$. We then get a surface $Z$; Figure 9(B). There exist four disjoint linear chains of $\mathbb{P}^1$’s in $Z$:

\[
\begin{align*}
&-5 -2 -6 -2 -2 -2, & -2 -3 -4, & -2 -3 -4, & -3 -3 \\
&0 -0 -0 -0 -0, & 0 -0 -0, & 0 -0 -0, & 0 -0.
\end{align*}
\]
A CONSTRUCTION OF SURFACES OF GENERAL TYPE WITH $p_g = 1$ AND $q = 0$

4.3. An example with $K^2 = 4$. Let $A$ and $L$ be lines on the projective plane $\mathbb{P}^2$ and $B$ a nonsingular conic on $\mathbb{P}^2$ which intersect as in Figure 10(A). Consider a pencil of cubics generated by the two cubics $A + B$ and $3L$. Blow up the three base points of the pencil of cubics including infinitely near base-points at each point. Then we obtain a rational elliptic surface $E(1)$ with an $\tilde{E}_6$-singular fiber, an $I_2$-singular fiber, two nodal singular fibers, and three sections; Figure 10(B).

Let $Y$ be a double cover of the rational elliptic surface $E(1)$ branched along two general fibers. Then $Y$ is an elliptic K3 surface with two $\tilde{E}_6$-singular fibers, two $I_2$-singular fibers, four nodal singular fibers, and three sections. We use only...
two $E_6$-singular fibers, two $I_2$-singular fibers, one nodal singular fibers, and three sections; Figure 11(A).

We blow up the surface $Y$ totally 16 times at the marked points. We then get a surface $Z$; Figure 11(B). There exist six disjoint linear chains of $\mathbb{P}^1$’s in $Z$:

$$
\begin{align*}
-2 & -10 -2 -2 -2 -2 -3 -2 -8 -2 -2 -2 -2 -2 -3 \\
0 & 0 0 0 0 0 0 0 0 0 0 0 0
\end{align*}
$$

$$(a) \ Y
$$

$$
\begin{align*}
-3 & -2 -2 -3 -2 -5 -3 -4 -4 \\
0 & 0 0 0 0 0 0 0 0
\end{align*}
$$

$$(b) \ Z = Y^{\sharp}16\mathbb{F}^2
$$

Figure 11: An example with $K^2 = 4$

4.4. **An example with $K^2 = 5$.** Let $Y$ be the K3 surface used in Section 2.1. We use only two $I_7$-singular fibers, two $I_2$-singular fibers, one nodal singular fibers, and three sections; Figure 12(A).

We blow up the surface $Y$ totally 15 times at the marked points. We then get a surface $Z$; Figure 12(B). There exist seven disjoint linear chains of $\mathbb{P}^1$’s in $Z$:

$$
\begin{align*}
-2 & -3 -8 -2 -2 -3 -3 -7 -2 -2 -2 -2 \\
0 & 0 0 0 0 0 0 0 0 0 0 0
\end{align*}
$$

$$
\begin{align*}
-3 & -5 -3 -2 -3 -3 -4 -4 -4 -4 \\
0 & 0 0 0 0 0 0 0 0 0
\end{align*}
$$

4.5. **An example with $K^2 = 6$.** Let $Y$ be the elliptic K3 surface described in Section 2. We use only two $I_8$-singular fibers, two $I_2$-singular fibers, one nodal singular fibers, and three sections; Figure 13(A).
A CONSTRUCTION OF SURFACES OF GENERAL TYPE WITH \( p_g = 1 \) AND \( q = 0 \)

We blow up the surface \( Y \) totally 18 times at the marked points. We then get a surface \( Z \); Figure 13(B). There exist five disjoint linear chains of \( \mathbb{P}^1 \)'s in \( Z \):

\[
\begin{align*}
&-2 -2 -3 -9 -2 -2 -2 -3 -4 , \\
&-2 -3 -7 -2 -2 -3 -3 -7 -2 -2 -2 , \\
&-4 -3 -2 -4 .
\end{align*}
\]

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Figure 13: An example with $K^2 = 6$

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