Exact deflection of a Neutral-Tachyon in the Kerr’s Gravitational field.

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Abstract

We solve in closed analytic form space-like geodesic equations in the Kerr gravitational field. Such geodesic equations describe the motion of neutral tachyons (faster than light particles) in the Kerr spacetime. More specifically we derive the closed form solution for the deflection angle of a neutral tachyon on an equatorial orbit in Kerr spacetime. The solution is expressed elegantly in terms of Lauricella’s hypergeometric function $F_D$. We applied our results to three cases: first, for the calculation of the deflection angle of a neutral tachyon on an equatorial trajectory in the gravitational field of a Kerr black hole. Subsequently, we applied our exact solutions to compute the deflection angle of equatorial spacelike geodesics in the gravitational fields of Sun and Earth assuming the Kerr spacetime geometry.

1 Introduction

Recent results from the OPERA experiment, taken at face value, seem to indicate the existence of superluminal neutrino species, whose speed $v$ as it has been determined by OPERA collaboration with respect to the speed of light, $c$, is [1]:

$$\frac{v - c}{c} = (2.48 \pm 0.28 \text{(stat)} \pm 0.30 \text{(sys)}) \times 10^{-5}$$

(1)

for a mean energy of the neutrino beam of 17GeV [1].

This result, if it will stand to further scrutiny and will be confirmed by independent measurements, will certainly constitute a fundamental discovery about Nature.

For us it means, that we are endowed with sufficient motivation to explore further properties and aspects of the general hypothesis of the existence of superluminal (tachyon) particles in fundamental physics. In particular, we intend in this Letter to explore the interaction of a neutral tachyon field with the strong gravitational field of Kerr spacetime. We shall obtain the exact analytic solution for the deflection angle of a neutral tachyon in an equatorial unbound orbit in Kerr spacetime.

The plausible existence of a superluminal particle in the framework of special relativity (SR) has been discussed by various authors [2] (see also [3]). The energy of a tachyon particle is given by [4]

$$E = \frac{|m|c^2}{\sqrt{\frac{c^2}{v^2} - 1}} \text{, for } v > c$$

(2)

where $|m|$ denotes the magnitude of the tachyon’s imaginary rest mass: $m = i|m|$. The energy momentum vector is now

$$p^\mu = \left( \frac{|m|c}{\sqrt{\frac{c^2}{v^2} - 1}} , \frac{|m|}{\sqrt{\frac{c^2}{v^2} - 1}} \frac{dx}{dt} , \frac{|m|}{\sqrt{\frac{c^2}{v^2} - 1}} \frac{dy}{dt} , \frac{|m|}{\sqrt{\frac{c^2}{v^2} - 1}} \frac{dz}{dt} \right) \text{, for } v > c$$

(3)

and the dispersion relation is valid

$$p^\mu p_\mu = (E/c)^2 - p_x^2 - p_y^2 - p_z^2 = -|m|c^2.$$ 

(4)

Assuming the gravitational mass of the tachyon is exactly equal to the magnitude of its imaginary rest mass the tachyon moves along spacelike geodesics in a gravitational field. Thus since spacelike geodesics are part of the

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1 Sometimes refer to as the metamass [2].
2 Spacelike geodesics in Kerr-de Sitter spacetime.

Taking into account the contribution from the cosmological constant $\Lambda$, the generalization of the Kerr solution \cite{8} is described by the Kerr-de Sitter metric element which in Boyer-Lindquist (BL) coordinates is given by \cite{9,10}:

$$ds^2 = \frac{\Delta_r}{\Xi r^2} (cdt - a \sin^2 \theta d\phi)^2 - \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{\Delta_\theta \sin^2 \theta}{\Xi^2 \rho^2} (acdt - (r^2 + a^2)d\phi)^2$$

(5)

$$\Delta_\theta := 1 + \frac{a^2 \Lambda}{3} \cos^2 \theta, \quad \Xi := 1 + \frac{a^2 \Lambda}{3}$$

(6)

$$\Delta_r := \left(1 - \frac{\Lambda}{3} r^2\right) (r^2 + a^2) - \frac{2GM}{c^2}, \quad \rho^2 = r^2 + a^2 \cos^2 \theta$$

(7)

We denote by $a$ the rotation (Kerr) parameter and $M$ denotes the mass of the spinning black hole. Choosing a real affine parameter $\lambda$ for the spacelike geodesic by $d\lambda^2 = -ds^2$, we have the geodesic equation in the usual notation as

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} = 0,$$

(8)

where $x^i$ denote the BL coordinates and $\Gamma^i_{jk}$ the Christoffel symbols of the second kind.

The geodesic equations in Kerr spacetime in the presence of the cosmological constant $\Lambda$ were derived in \cite{11} by solving the Hamilton-Jacobi differential equations by separation of variables. The tachyon motion, as we mentioned earlier, is described by the spacelike geodesics (which are the first integrals of (5)) which take the form\footnote{The tachyon metamass $|m|$, which is one of the first integrals of (5), was set equal to unity without loss of generality.}.

$$\int \frac{dr}{\sqrt{\Delta_r}} = \int \frac{d\theta}{\sqrt{\Theta}}$$

$$\rho^2 \frac{d\phi}{d\lambda} = -\frac{\Xi^2}{\Delta_\theta \sin^2 \theta} (aE \sin^2 \theta - L) + \frac{a\Xi^2}{\Delta_r} (r^2 + a^2) E - aL$$

$$c^2 \rho^2 \frac{dt}{d\lambda} = \frac{\Xi^2 (r^2 + a^2)(r^2 + a^2) E - aL}{\Delta_r} - \frac{a\Xi^2 (aE \sin^2 \theta - L)}{\Delta_\theta}$$

$$\rho^2 \frac{dr}{d\lambda} = \pm \sqrt{\Delta_r}$$

$$\rho^2 \frac{d\theta}{d\lambda} = \pm \sqrt{\Theta}$$

(9)
where
\[
R := \Xi^2((r^2 + a^2)E - aL)^2 - \Delta_r(-r^2 + Q + \Xi^2(L - aE)^2)
\]
\[
\Theta := [Q + (L - aE)^2\Xi^2 + a^2\cos^2\theta]\Delta\phi - \Xi^2(aE \sin^2\theta - L)\]
\[
\sin^2\theta
\]  
\[(10)\]

The constants of motion $E, L$ are associated with the isometries of the Kerr metric while $Q$ denotes Carter’s constant, the fourth constant of integration. The spacelike Kerr geodesics are obtained by setting $\Lambda = 0$ in Eqs.(9)-(10).

3 Exact solution of equatorial spacelike geodesics in Kerr spacetime

We now proceed to determine the exact solution for the deflection of a neutral tachyon in an equatorial spacelike orbit in Kerr spacetime assuming $\Lambda = 0$. The more general case with the cosmological constant present will be a subject of a separate publication \[15\]. We have
\[
r^2(\dot{r}) = \sqrt{R}.
\]
This can be rewritten as
\[
\dot{r}^2 = E^2 + \frac{a^2E^2}{r^2} - \frac{L^2}{r^2} + \frac{2GM}{c^2}r^3(L - aE)^2 + \frac{\Delta}{r^2}.
\]
where $\Delta$ is obtained by setting $\Lambda = 0$ in equation (7) for $\Delta_r$. By defining a new variable $u := 1/r$ we obtain the following expression:
\[
\dot{u}^2 = u^4 \frac{A(u)}{D^2(u)}
\]
where
\[
A(u) := L + \alpha_S (aE - L), \quad D(u) := 1 + a^2u^2 - \alpha_S u, \quad \alpha_S := \frac{2GM}{c^2}.
\]
Thus we derive the orbital equation
\[
\frac{d\phi}{du} = \frac{A(u)}{D(u)} \frac{1}{\sqrt{B_{\text{tac}}(u)}}.
\]
(15)

We now use the technique of partial fractions from integral calculus in order to calculate the deflection of the neutral tachyon’s equatorial orbit in the Kerr gravitational field from equation (15). We write:
\[
\frac{A(u)}{D(u)} = \frac{A_+}{u_+ - u} + \frac{A_-}{u_+ - u}
\]
(16)

where $u_+ = \frac{r^+}{\alpha_+}$, $u_- = \frac{r^-}{\alpha_-}$ and
\[
r_{\pm} = \frac{GM}{c^2} \pm \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2}
\]
(17)
denote the radii of the event and Cauchy horizons respectively for the case of a Kerr black hole. Also the quantities $A_+, A_-$ are given by
\[
A_+ = \frac{L + \alpha_S (aE - L)u_+}{u_+ - u_-}, \quad A_- = \frac{\frac{L}{\alpha_+} - \frac{\alpha_S}{\alpha_+} (aE - L)u_-}{u_- - u_+}
\]
(18)

In order to calculate the angle of deflection for the tachyon it is necessary to calculate the integral: $\Delta \phi_{\text{tachyon}} = 2 \int_0^{\pi/2} d\phi$. Using the formalism developed in references \[11\]-\[13\], for computing hyperelliptic integrals in closed

\[3\] i.e. they are related to the energy and angular momentum per unit metamass of the tachyon at infinity.

\[4\] For equatorial geodesics $\theta = \pi/2$, $Q = 0$. 

3
analytic form, in terms of Lauricella’s hypergeometric function $F_D$, we compute:

$$\Delta \phi_{Tachyon}^{GTR} = \frac{2}{\sqrt{u_1' - u_3'}} \frac{1}{\sqrt{\frac{\pi}{\alpha (L-aE)^2}} \left( \frac{A_+}{G\alpha r_+ - u_3'} \left( F_1 \left( \frac{1}{2}, \beta_A, 1, z_A^r \right) \right) \right) - 2 \sqrt{-u_3' / u_2' - u_3'} F_D \left( \frac{1}{2}, \beta_3, 3, 2, z_D^r \right) + \frac{A_-}{G\alpha r_- - u_3'} \left( F_1 \left( \frac{1}{2}, \beta_4, 1, z_A^r \right) \right) - 2 \sqrt{-u_3' / u_2' - u_3'} F_D \left( \frac{1}{2}, \beta_3, 3, 2, z_D^r \right) \right)} \right) \right),$$

(19)

where

$$\beta_A = \left( \frac{1}{2}, 1 \right), \quad \beta_3 = \left( \frac{1}{2}, \frac{1}{2}, 1 \right)$$

and

$$z_A^r = \left( \frac{u_3 - u_3'}{c^2 + a^2 u_3'}, u_2' - u_3', u_1' - u_3' \right), \quad z_D^r = \left( -u_3' / c^2 + a^2 u_3', u_1' - u_3', u_2' - u_3' \right).$$

(21)

We have also defined: $u' = u \frac{GM}{c^2}$. The roots of the cubic in this case are real and organized in the ascending order

$$u_1' > u_2' > 0 > u_3'.$$

(22)

The angle of deflection $\delta$ of a neutral tachyon equatorial trajectory from the gravitational field of a rotating black hole or a rotating central mass is defined to be the deviation of $\Delta \phi_{Tachyon}^{GTR}$ from the transcendental number $\pi$

$$\delta = \Delta \phi_{Tachyon}^{GTR} - \pi.$$

(23)

We shall apply our closed form solution, Eq. (19) for the deflection angle of an equatorial neutral tachyon’s orbit in the gravitational field of Kerr spacetime in two cases. First, we shall compute the deflection angle $\delta$, for a tachyon in the gravitational field of the Sun (assuming the Kerr spacetime geometry). Second, the deflection angle of a neutral tachyon in an equatorial orbit around a Kerr black hole for various values of the involved physical parameters. The physical parameters are the velocity of the tachyon particle $v$ (at large distances from the central mass), the energy per unit mass (in units of $c^2$) $E = \frac{1}{\sqrt{\alpha^2 / \alpha - 1}}$, the parameter $L$ and the spin $a$ (Kerr parameter) of the black hole.

### 3.1 Deflection of neutral tachyon in an equatorial orbit in the Kerr (Sun’s) gravitational field.

Assuming that the gravitational field of the spinning Sun is described by the Kerr spacetime geometry we compute the deflection angle of a tachyon in an unbound equatorial orbit using our closed form solution Eq. (22). We repeat the calculation for three values of the velocity of the tachyon particle. Namely: 1) $v = (1 + 2.48 \times 10^{-5})c$ (same as OPERA’s experimental value) 2) $v = \sqrt{2}c$ 3) $v = 10^6c$. For the parameter $E$ we use the formula $E = \frac{1}{\sqrt{\alpha^2 / \alpha - 1}}$.

Taking the point of closest approach $r_0 = R_\odot = 6.9551 \times 10^8$ m the parameter $L$ is chosen to have the value: $L = 4.71013.2781620c GM_{\odot} / c$. Our results are summarized in Table 1. We note that the deflection angle ranges from half the value of a photon’s deflection from the Sun’s gravitational field (case 3 high velocities, low energies for the tachyon) to a value equal to the deflection of light for the case 1) i.e. tachyon velocity as the one determined by OPERA experiment (1).

Our findings are consistent with the results of [3], in which, a perturbative calculation of the deflection angle for a tachyon in a Schwarzschild field of the Sun was performed.

### 3.2 Deflection of a neutral tachyon in an equatorial orbit in the Kerr black hole spacetime

Again for different choices of values for the velocity of a tachyon we compute the deflection angle for different values for the spin of the black hole and the remaining parameters. For the three different values of the tachyon particle we choose first $L = 100vE^{\frac{GM}{c^2}}$. We present our results in Table 2.

We repeat the analysis for $L = 40vE^{\frac{GM}{c^2}}$. Our results are displayed in Table 3.
This is fine as long as their modules are less than 1.

Thus we transform our integral onto the integral representation of Lauricella’s function $F_D$ of four-variables (see also (25))

$$u' = u_2'(1 - t),$$

we have:

$$\frac{GM_\odot r_+}{c^2 a^2} - u' = \left(\frac{GM_\odot r_+}{c^2 a^2} - u_2\right) \left[1 + \frac{tu'_2}{\frac{GM_\odot r_+}{c^2 a^2} - u_2}\right], \quad u'_2 - u' = u_2t,$$

$$u'_1 - u' = (u'_1 - u'_2) \left[1 + \frac{w'_2t}{u'_1 - u'_2}\right], \quad u' - u'_3 = (u'_2 - u'_3) \left[1 - \frac{w'_2t}{u'_2 - u'_3}\right]$$

thus we transform our integral onto the integral representation of Lauricella’s function $F_D$ of four-variables (see also (26))

$$\begin{array}{c|c|c|c}
\text{a} & \text{Case 1: } v = (1 + 2.48 \times 10^{-5})c & \text{Case 2: } v = \sqrt{2}c & \text{Case 3: } v = 10^6c \\
0.52 & \delta = 0.04099735 = 8456.31 \text{arcsec} & \delta = 0.0305724 = 6306.01 \text{arcsec} & \delta = 0.0202396 = 4174.71 \text{arcsec} \\
0.99616 & \delta = 0.04079425 = 8414.42 \text{arcsec} & \delta = 0.0304312 = 6276.89 \text{arcsec} & \delta = 0.02024108 = 4175.02 \text{arcsec} \\
\end{array}$$
Appendix):

\[
\int d\phi = \Delta \phi^{\text{GTR}}_{\text{NT}}\]

\[
= \frac{2}{a^2 \left( \frac{GM_M}{c^2 a^2} - u_2' \right) \left( \frac{GM_M}{c^2 a^2} - u_2'' \right) \sqrt{u_2'(u_2'' - u_3')(u_1' - u_2') \sqrt{\frac{\alpha (L-aE)^2}{(GM_M/c^2)^2}}}} \times
\]

\[
\alpha_S(aE-L)u_2^2 \frac{\Gamma(1/2)\Gamma(2)}{\Gamma(5/2)} F_D \left( \frac{1}{2}, \beta_1, \frac{5}{2}, z_2^\oplus \right) +
\]

\[
Lu_2^2 \frac{\Gamma(1/2)\Gamma(1)}{\Gamma(3/2)} F_D \left( \frac{1}{2}, \beta_1, \frac{3}{2}, z_2^\oplus \right)
\]

where

\[
\beta_1 = \left( 1, 1, \frac{1}{2}, \frac{1}{2} \right), \quad z_2^\oplus = \left( \frac{-u_2'}{GM_M/c^2}, \frac{-u_2'}{GM_M/c^2} - u_2'', \frac{u_2'}{u_2'' - u_3'}, \frac{u_2'}{u_1' - u_2'} \right)
\]

We apply our analytic solution eq. (27) for computing the deflection angle of a neutral tachyon in the gravitational field of Earth. We choose as value for the velocity of the tachyon the central value of OPERA experiment \( v = (1+2.48\times10^{-5})c \). Taking the point of closest approach as the radius (equatorial) of Earth \( r_0 = R_\oplus = 6.378137\times10^6 \) m we have that the parameter \( L = vE \times 1.438127796068894 \times 10^9 \). For the Kerr parameter of Earth we choose the above mentioned value. Then we determine \( \delta = \Delta \phi^{\text{GTR}}_{\text{NT}} - \pi = 2.78 \times 10^{-9} \) rad \( \sim 0.000573 \) arcsec.

4 Conclusions

In this paper we investigated tachyon orbits (spacelike geodesics) in Kerr spacetime. More specifically, we derived the closed form solution for the deflection angle of a neutral tachyon in an equatorial orbit in the gravitational field of Kerr spacetime. The solution was expressed elegantly in terms of Lauricella’s multivariable hypergeometric function \( F_D \). We applied our exact solutions in three cases: 1) we calculated the deflection of a neutral tachyon by the gravitational field of a rotating Kerr black hole, for different values of the velocity of the tachyon and the spin of the black hole. We note the strong dependence of the deflection angle on the spin of the spinning black hole for low tachyon velocities, especially for lower values for the parameter \( L \). Large magnitudes for the deflection angle were produced see table 3. 2) we calculated the deflection of equatorial neutral tachyon orbits by the gravitational field of our Sun assuming a curved Kerr spacetime geometry. For low tachyon velocities (such as the ones reported by the OPERA collaboration) the deflection angle was calculated to have a value \( \sim 1.75 \) arcsec a value equals to the amount of deflection that light experiences by the gravitational field of our Solar system star. For high tachyon velocities \( v \gg c \) the calculated deflection angle decreases to half the value of \( \delta \) at low superluminal velocities. 3) we calculated the deflection angle of an equatorial tachyon trajectory in the gravitational field of Earth assuming a Kerr geometry. There is a further novelty in the calculation. The solution for \( \delta \) is expressed in terms of Lauricella’s hypergeometric function \( F_D \) of four variables two of which are complex-conjugates. The neutral tachyon undergoes a small deflection of \( 2.78 \times 10^{-9} \) radians \( \sim 0.000573 \) arcsec. Thus if tachyons do exist and move on spacelike geodesics they undergo a deflection by the gravitational field of the rotating central mass. The deflection exhibits a strong dependence on the superluminal velocity and the spin of the rotating mass. This gravitational effect is in principle measurable. It will be interesting to generalize our results to the case of finding the exact solutions for generic non-equatorial spacelike orbits in the presence of the cosmological constant \( \Lambda \) (spacelike non-equatorial Kerr-de Sitter orbits). However, such an investigation is beyond the scope of the current paper and it will be the subject of a separate publication [15].

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5 Appendix

We introduce the Lauricella function \( F_D \) and its integral representation

\[
F_D(\alpha, \beta, \gamma, z) = \sum_{n_1, n_2, \ldots, n_m = 0}^{\infty} \frac{(\alpha)_{n_1+\ldots+n_m}(\beta_1)_{n_1} \cdots (\beta_m)_{n_m} z_1^{n_1} \cdots z_m^{n_m}}{(\gamma)_{n_1+\ldots+n_m}(1)_{n_1} \cdots (1)_{n_m}}
\]

(29)
where
\[
\mathbf{z} = (z_1, \ldots, z_m), \\
\beta = (\beta_1, \ldots, \beta_m).
\] (30)

The Pochhammer symbol \((\alpha)_m = (\alpha, m)\) is defined by
\[
(\alpha)_m = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)} = \begin{cases} 
1, & \text{if } m = 0 \\
\alpha(\alpha + 1) \cdots (\alpha + m - 1), & \text{if } m = 1, 2, 3
\end{cases}
\] (31)

With the notations \(\mathbf{z}^n := z_1^n \cdots z_m^n\), \((\beta)_n := (\beta_1)_{n_1} \cdots (\beta_m)_{n_m}\), \(n! = n_1! \cdots n_m!\), \(\mathbf{n} := n_1 + \cdots n_m\) for \(m\)-tuples of numbers in \([0, \infty)\) and of non-negative integers \(\mathbf{n} = (n_1, \ldots, n_m)\) the Lauricella series \(F_D\) in compact form is
\[
F_D(\alpha, \beta, \gamma, \mathbf{z}) := \sum_{\mathbf{n}} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \mathbf{n}!} \mathbf{z}^n.
\] (32)

The series admits the following integral representation:
\[
F_D(\alpha, \beta, \gamma, \mathbf{z}) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha-1}(1 - t)^{\gamma-\alpha-1}(1 - z_1 t)^{-\beta_1} \cdots (1 - z_m t)^{-\beta_m} dt
\] (33)

which is valid for \(\text{Re}(\alpha) > 0, \text{Re}(\gamma - \alpha) > 0\). It converges absolutely inside the \(m\)-dimensional cuboid:
\[
|z_j| < 1, (j = 1, \ldots, m).
\] (34)

For \(m = 2\), \(F_D\) becomes Appell’s \(F_1\) two variable hypergeometric function \(F_1(\alpha, \beta, \beta', \gamma, x, y)\) with integral representation
\[
\int_0^1 u^{\alpha-1}(1 - u)^{\gamma-\alpha-1}(1 - ux)^{-\beta}(1 - yu)^{-\beta'} du = \frac{\Gamma(\alpha)\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} F_1(\alpha, \beta, \beta', \gamma, x, y).
\] (35)

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