Topology of Knotted Optical Vortices

Ji-Rong Ren, Tao Zhu and Yi-Shi Duan

Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, P. R. China

(Dated: February 2, 2008)

Abstract

Optical vortices as topological objects exist ubiquitously in nature. In this paper, by making use of the $\phi$-mapping topological current theory, we investigate the topology in the closed and knotted optical vortices. The topological inner structure of the optical vortices are obtained, and the linking of the knotted optical vortices is also given.

PACS numbers: 03.65.Vf, 02.10.Kn, 42.25.-p, 41.20.Jb

*Corresponding author. Email: zhut05@lzu.cn
I. INTRODUCTION

Optical vortices\cite{1,2,3,4}, which exist ubiquitously in nature, are one of the phase singularities existed in light beams, and are the vortices of electromagnetic energy flow. It have drawn great interest and have been studied intensively in many ways because it is of importance for understanding fundamental physics and have many important applications. Optical vortices are topological objects on wave-front surfaces and possess topological charges which can be attributed to the helicoidal spatial structure of the wave-front around a phase singularity. They occur when three or more complex scalar light waves interfere. The total wave amplitude of interference field is zero and the phase is undefined. Optical vortices are generically points in two dimensional fields and form vortex lines in three dimensions.

As topological objects, the topological and geometrical properties of optical vortices have been studied by many physicists\cite{5,6,7,8,9,10}.

It has long been known that optical vortices can be closed. In 2001, Berry and Dennis showed theoretically that specific superpositions of beams could be generated, in which optical vortices could be linked together and even knotted\cite{7}. Since the original analysis of knotted and linked optical vortex loops have been done, there are a great deal of works on the knotted topology of the optical vortices have been done\cite{8,9}. In the laboratory, the stable link and knot structures have been produced\cite{10}. The observations of knotted optical vortices demonstrate the precision control of light beams that is required to create complex regions of zero light intensity. Such regions could be used, for example, to confine cold atoms and Bose-Einstein condensates in complex topologies. Knotted objects have been studied theoretically since Lord Kelvins vortex atom hypothesis\cite{11} in a range of different physical situations, such as hydrodynamics\cite{12}, field theory\cite{13,14} and nonlinear excitable media\cite{15}. The knotted optical vortex is a new type of knotted object, the two or more such knots are called a link, i.e., a family of knots. It is known that for a knot family there are important characteristic numbers to describe its topology, such as the self-linking and linking numbers. So, it is very necessary to use topological viewpoint to study these knot characteristics.

The $\phi$-mapping topological current theory\cite{16} and the decomposed theory of gauge potential\cite{17} play an important role in studying the topological invariant and structure of a physical system and have been used to study the topological current of a magnetic monopole\cite{18}, the topological string theory\cite{19}, the topological structure of the defects of...
space-time in the early universe as well as its topological bifurcation\cite{20}, the topological structure of the Gauss-Bonnet-Chern theorem\cite{21}, the topological structure of the London equation in a superconductor\cite{22} and point defect of a vector parameter\cite{23}.

In this paper, by making use of the so-called $\phi$-mapping topological current theory, we study the topological current of the optical vortices in three dimensional fields, and the topological inner structure of these optical vortices are obtained. For the case that the optical vortices are closed and knotted curves, we discussed the topological invariant of these knotted family in details. This paper is arranged as follows. In Sec.II, we study the topological current density of the optical vortices, this topological current density don’t vanish only when the optical vortices exist, the topological charge of optical vortices are expressed by the topological quantum numbers, the Hopf indices and Brouwer degrees of the $\phi$-mapping. In Sec.III, we introduce a important topological invariant to describe the optical vortices when they are linked and knotted, it is just the sum of all the self-linking and all the linking numbers of the knot family. In Sec. IV is our concluding remarks.

II. TOPOLOGICAL CHARGE CURRENT DENSITY OF OPTICAL VORICES

It is known that the three-dimensional fields of interfering light beams can be denoted by the complex scalar function $\psi(\vec{r}) = \psi(x^1, x^2, x^3)$, $\psi(\vec{r})$ is maps from space to the complex numbers, so $\psi : R^3 \rightarrow C$. The $\psi(\vec{r})$ is superpositions of free space optical modes with the same frequency, so the optical vortices placed where the phase singularities exist are temporally stable. In the theoretically analysis done by Berry and Dennis\cite{7}, $\psi(\vec{r})$ is the complex scalar solution of the Helmholtz equation

$$\nabla^2 \psi(\vec{r}) + \psi(\vec{r}) = 0,$$

and it possess optical vortices (phase singularities) in the form of knots or links. In the laboratory\cite{10}, the optical vortices can be produced by the special superpositions of laser beams (Laguerre-Gaussian beams), so the $\psi(\vec{r})$ is just this superposition field. Here in our discussions, $\psi(\vec{r})$ is a generic complex scalar optical field which possesses closed and knotted optical vortices.

The complex scalar optical field $\psi(\vec{r})$ is space dependent, and it can be written as

$$\psi(\vec{r}) = ||\psi|| e^{i\chi} = \phi^1(\vec{r}) + i\phi^2(\vec{r}),$$

(2)
where $\phi^1(\vec{r})$ and $\phi^2(\vec{r})$ can be regarded as complex representation of a two-dimensional vector field $\vec{\phi} = (\phi^1, \phi^2)$, $\|\psi\| = \sqrt{\psi^*\psi}$ is the modulus of $\psi$, and $\chi$ is the phase factor. The current density $\vec{J}$ associated with $\psi(\vec{r})$ is defined as

$$\vec{J} = Im(\psi^* \nabla \psi) = \|\psi\|^2 \vec{v},$$

the $\vec{v}$ is the velocity field. From the expressions in Eq.(2) and Eq.(3), the velocity field $\vec{v}$ can be rewritten as

$$\vec{v} = \frac{1}{2i} \|\psi\|^2 (\psi^* \nabla \psi - \nabla \psi^* \psi) = \nabla \chi,$$

it becomes the gradient of the phase factor $\chi$. The vorticity field $\vec{\Omega}$ of the velocity field $\vec{v}$ is defined as

$$\vec{\Omega} = \frac{1}{2\pi} \nabla \times \vec{v}.$$  

Form Eq.(4), we directly obtain a trivial curl-free result: $\vec{\Omega} = \frac{1}{2\pi} \nabla \times \nabla \chi = 0$. But in topology, because of the existence of optical vortices in the interference optical fields $\psi$, the vorticity $\vec{\Omega}$ does not vanish[17]. So in the following discussions, we will study that what the exact expression for $\vec{\Omega}$ is in topology.

Introducing the unit vector $n^a = \phi^a/\|\phi\| (a = 1, 2; n^a n^a = 1)$, one can reexpress the velocity field $\vec{v}$ as

$$\vec{v} = \epsilon_{ab} n^a \nabla n^b,$$

and the vorticity $\vec{\Omega}$ is

$$\Omega^i = \frac{1}{2\pi} \epsilon^{ijk} \epsilon_{ab} \partial_j n^a \partial_k n^b, \quad i, j, k = 1, 2, 3.$$  

Obviously, it is just the topological current of the optical vortices in three dimensional optical fields.

Using the $\phi$-mapping theory[16], the topological current $\vec{\Omega}$ is rewritten as

$$\Omega^i = \delta^2(\vec{\phi}) D^i(\frac{\vec{\phi}}{x}),$$

where the $D^i(\vec{\phi}/x)$ is the vector Jacobians of $\psi(\vec{r})$, and it is defined as

$$D^i(\frac{\vec{\phi}}{x}) = \frac{1}{2} \epsilon^{ijk} \epsilon_{ab} \partial_j \phi^a \partial_k \phi^b,$$

we can see from the expression (8) that the vorticity $\vec{\Omega}$ is non-vanishing only if $\vec{\phi} = 0$, i.e., the existence of the optical vortices, so it is necessary to study these zero solutions of $\vec{\phi}$. In
these solutions are some isolate zero lines, which are the so-called optical vortices in three dimensional space.

Under the regular condition

\[ D^i(\phi/x) \neq 0, \]

the general solutions of

\[ \phi_1(x^1, x^2, x^3) = 0, \quad \phi_2(x^1, x^2, x^3) = 0 \]

(10)
can be expressed as

\[ x^1 = x^1_k(s), \quad x^2 = x^2_k(s), \quad x^3 = x^3_k(s), \]

(11)
which represent \( N \) isolated singular strings \( L_k \) with string parameter \( s (k = 1, 2, \cdots, N) \). These singular strings solutions are just the optical vortices solutions in three dimensions space.

In \( \delta \)-function theory \[24\], one can obtain in three dimensions space

\[ \delta^2(\vec{\phi}) = \sum_{k=1}^{N} \beta_k \int_{L_k} \frac{\delta^3(\vec{x} - \vec{x}_k(s))}{|D(\vec{x}/u)|\Sigma_k} ds, \]

(12)
where

\[ D(\vec{\phi}/u)|\Sigma_k = \frac{1}{2} \epsilon^{jkn} \frac{\partial \phi^m}{\partial u^j} \frac{\partial \phi^n}{\partial u^k}, \]

and \( \Sigma_k \) is the \( k \)th planar element transverse to \( L_k \) with local coordinates \( (u^1, u^2) \). The \( \beta_k \) is the Hopf index of \( \phi \) mapping, which means that when \( \vec{x} \) covers the neighborhood of the zero point \( \vec{x}_k(s) \) once, the vector field \( \phi \) covers the corresponding region in \( \phi \) space \( \beta_k \) times. Meanwhile the direction vector of \( L_k \) is given by

\[ \frac{dx^i}{ds}|_{x_k} = \frac{D^i(\phi/x)}{D(\phi/u)|_{x_k}}|_{x_k}. \]

(13)
Then from Eq.\[12\] and Eq.\[13\] one can obtain the inner structure of \( \Omega^i \):

\[ \Omega^i = \sum_{k=1}^{N} W_k \int_{L_k} \frac{dx^i}{ds} \delta^3(\vec{x} - \vec{x}_k(s)) ds, \]

(14)
where \( W_k = \beta_k \eta_k \) is the winding number of \( \vec{\phi} \) around \( L_k \), with \( \eta_k = sgn D(\phi/u)|_{\vec{x}_k} = \pm 1 \) being the Brouwer degree of \( \phi \) mapping. The sign of Brouwer degrees are very important, the \( \eta_k = +1 \) corresponds to the vortex, and \( \eta_k = -1 \) corresponds to the antivortex. The integer number \( W_k \) measures windings of the phase around the phase singularities, and is called the
topological charge of the optical vortex. Hence the topological charge of the optical vortices $L_k$ is

$$Q_k = \int_{\Sigma_k} \Omega_i d\sigma_i = W_k. \quad (15)$$

In the theory of the optical vortices, the topological charge plays the role of an angular momentum. The Eq.(15) shows us that the topological current $\vec{\Omega}$ describes the density of optical vortices in space. So, we call the topological current $\vec{\Omega}$ the topological charge current density of the optical vortices.

The results in this section show us the topological inner structure of the topological charge current density $\vec{\Omega}$. The topological charge of the $k$th optical vortex in three dimensions can be expressed by the topological numbers: $Q_k = W_k = \beta_k \eta_k$, and the $\eta_k = +1$ corresponds to the vortex, and $\eta_k = -1$ corresponds to the antivortex.

### III. Linking Numbers of Knot Family

Topology has played a very important role in understanding the knot configurations, so it is necessary to study the topology in the knotted optical vortices. In order to do that, we define a the helicity integral

$$H = \frac{1}{4\pi^2} \int \vec{\nu} \cdot \nabla \times \vec{v} d^3 x, \quad (16)$$

this is an important topological knot invariant and it measures the linking of the optical vortices. From the Eq.(15), the helicity integral can be changed as

$$H = \frac{1}{2\pi} \int \vec{\nu} \cdot \vec{\omega} d^3 x. \quad (17)$$

For the magnetic helicity $26$, $\vec{\nu}$ is the magnetic field $\vec{B}$ and $\vec{\omega}$ the $\vec{A}$, the helicity integral measures the linking number of the field lines, averaged over all pairs of field lines, and weighted by magnetic flux. Another application of the helicity is in fluid mechanics $12$, helicity $H$ measures the linking of the fluid vortex lines. In this paper, we apply this helicity to study the linking of the optical vortices. In Section.II, we have known that the vorticity $\vec{\Omega}$ does not vanish only when the optical vortices exist, so from the Eq.(17), when the optical vortices exist, the helicity also does not vanish. So helicity is just the optical vortices helicity.

Substituting Eq. (14) into Eq. (17), one can obtain

$$H = \frac{1}{2\pi} \sum_{k=1}^{N} W_k \int_{L_k} \vec{v} \cdot d\vec{x}, \quad (18)$$
for closed and knotted lines, i.e., a family of knots $\xi_k (k = 1, 2, \ldots, N)$, Eq. (18) becomes

$$H = \frac{1}{2\pi} \sum_{k=1}^{N} W_k \oint_{\xi_k} \vec{v} \cdot d\vec{x}. \quad (19)$$

It is well known that many important topological numbers are related to a knot family such as the self-linking number and Gauss linking number. In order to discuss these topological numbers of knotted optical vortices, we define Gauss mapping:

$$\vec{m} : S^1 \times S^1 \rightarrow S^2, \quad (20)$$

where $\vec{m}$ is a unit vector

$$\vec{m}(\vec{x}, \vec{y}) = \frac{\vec{y} - \vec{x}}{|\vec{y} - \vec{x}|}, \quad (21)$$

where $\vec{x}$ and $\vec{y}$ are two points, respectively, on the knots $\xi_k$ and $\xi_l$ (in particular, when $\vec{x}$ and $\vec{y}$ are the same point on the same knot $\xi$, $\vec{n}$ is just the unit tangent vector $\vec{T}$ of $\xi$ at $\vec{x}$). Therefore, when $\vec{x}$ and $\vec{y}$, respectively, cover the closed curves $\xi_k$ and $\xi_l$ once, $\vec{n}$ becomes the section of sphere bundle $S^2$. So, on this $S^2$ we can define the two-dimensional unit vector $\vec{e} = \vec{e}(\vec{x}, \vec{y})$. $\vec{e}$, $\vec{m}$ are normal to each other, i.e.,

$$\vec{e}_1 \cdot \vec{e}_2 = \vec{e}_1 \cdot \vec{m} = \vec{e}_2 \cdot \vec{m} = 0,$$

$$\vec{e}_1 \cdot \vec{e}_1 = \vec{e}_2 \cdot \vec{e}_2 = \vec{m} \cdot \vec{m} = 1. \quad (22)$$

In fact, the velocity field $\vec{v}$ can be decomposed in terms of this two-dimensional unit vector $\vec{e}$: $v_i = \epsilon_{ab}e^a \partial_i e^b - \partial_i \theta$, where $\theta$ is a phase factor [17]. Since one can see from the expression $\vec{\Omega} = \frac{1}{2\pi} \nabla \times \vec{v}$ that the $(\partial_i \theta)$ term does not contribute to the integral $H$, $v_i$ can in fact be expressed as

$$v_i = \epsilon_{ab}e^a \partial_i e^b. \quad (23)$$

Substituting it into Eq.(14), one can obtain

$$H = \frac{1}{2\pi} \sum_{k=1}^{N} W_k \oint_{\xi_k} \epsilon_{ab}e^a(\vec{x}, \vec{y})\partial_i e^b(\vec{x}, \vec{y})dx^i. \quad (24)$$

Noticing the symmetry between the points $\vec{x}$ and $\vec{y}$ in Eq.[21], Eq.(21) should be reexpressed as

$$H = \frac{1}{2\pi} \sum_{k,l=1}^{N} W_k W_l \oint_{\xi_k \cup \xi_l} \epsilon_{ab}\partial_a e^a \partial_j e^b dx^i \wedge dy^j. \quad (25)$$
In this expression there are three cases: (1) $\xi_k$ and $\xi_l$ are two different optical vortices ($\xi_k \neq \xi_l$), and $\vec{x}$ and $\vec{y}$ are therefore two different points ($\vec{x} \neq \vec{y}$); (2) $\xi_k$ and $\xi_l$ are the same optical vortices ($\xi_k = \xi_l$), but $\vec{x}$ and $\vec{y}$ are two different points ($\vec{x} \neq \vec{y}$); (3) $\xi_k$ and $\xi_l$ are the same optical vortices ($\xi_k = \xi_l$), and $\vec{x}$ and $\vec{y}$ are the same points ($\vec{x} = \vec{y}$). Thus, Eq. (25) can be written as three terms:

$$H = \sum_{k=1 \atop k \neq l}^N \frac{1}{2\pi} W_k^2 \oint_{\xi_k} \varepsilon_{ab} \partial_a e^b dx^i \wedge dy^j$$

$$+ \frac{1}{2\pi} \sum_{k=1}^N W_k^2 \oint_{\xi_k} \varepsilon_{ab} e^a \partial_i e^b dx^i$$

$$+ \sum_{k,l=1 \atop k \neq l}^N \frac{1}{2\pi} W_k W_l \oint_{\xi_k} \oint_{\xi_l} \varepsilon_{ab} \partial_a e^b dx^i \wedge dy^j. \quad (26)$$

By making use of the relation $\varepsilon_{ab} \partial_a e^b = \frac{1}{2} \vec{m} \cdot (\partial_i \vec{m} \times \partial_j \vec{m})$, the Eq. (26) is just

$$H = \sum_{k=1 \atop \vec{x} \neq \vec{y}}^N \frac{1}{4\pi} W_k^2 \oint_{\xi_k} \oint_{\xi_k} \vec{m}^* (dS)$$

$$+ \frac{1}{2\pi} \sum_{k=1}^N W_k^2 \oint_{\xi_k} \varepsilon_{ab} e^a \partial_i e^b dx^i$$

$$+ \sum_{k,l=1 \atop k \neq l}^N \frac{1}{4\pi} W_k W_l \oint_{\xi_k} \oint_{\xi_l} \vec{m}^* (dS), \quad (27)$$

where $\vec{m}^* (dS) = \vec{m} \cdot (\partial_i \vec{m} \times \partial_j \vec{m}) dx^i \wedge dy^j (\vec{x} \neq \vec{y})$ denotes the pullback of the $S^2$ surface element.

In the following we will investigate the three terms in the Eq. (27) in detail. Firstly, the first term of Eq. (27) is just related to the writhing number $W_r(\xi_k)$ of $\xi_k$

$$W_r(\xi_k) = \frac{1}{4\pi} \oint_{\xi_k} \oint_{\xi_k} \vec{m}^* (dS). \quad (28)$$

For the second term, one can prove that it is related to the twisting number $T_w(\xi_k)$ of $\xi_k$

$$\frac{1}{2\pi} \oint_{\xi_k} \varepsilon_{ab} e^a \partial_i e^b dx^i = \frac{1}{2\pi} \oint_{\xi_k} (\vec{T} \times \vec{V}) \cdot d\vec{V} = T_w(\xi_k), \quad (29)$$

where $\vec{T}$ is the unit tangent vector of knot $\xi_k$ at $\vec{x}$ ($\vec{m} = \vec{T}$ when $\vec{x} = \vec{y}$) and $\vec{V}$ is defined as $e^a = \varepsilon^{ab} V^b (\vec{V} \perp \vec{T}, \vec{e} = \vec{T} \times \vec{V})$. In terms of the White formula $SL(\xi_k) = W_r(\xi_k) + T_w(\xi_k)$,
we see that the first and the second terms of Eq. (27) just compose the self-linking numbers of knots.

Secondly, for the third term, one can prove that

\[
\frac{1}{4\pi} \oint_{\xi_k} \oint_{\xi_l} \vec{m}^*(dS) = \frac{1}{4\pi} \epsilon_{ijk} \int_{\xi_k} dx^i \int_{\xi_l} dy^j \frac{(x^k - y^k)}{\|\vec{x} - \vec{y}\|^3} = Lk(\xi_k, \xi_l) \quad (k \neq l),
\]

where \( Lk(\xi_k, \xi_l) \) is the Gauss linking number between \( \xi_k \) and \( \xi_l \) [28]. Therefore, from Eqs. (28), (29), (30) and (31), we obtain the important result:

\[
H = \sum_{k=1}^{N} W_k^2 SL(\xi_k) + \sum_{k,l=1(k\neq l)}^{N} W_k W_l Lk(\xi_k, \xi_l).
\]

This precise expression just reveals the relationship between \( H \) and the self-linking and the linking numbers of the optical vortices knots family [28]. Since the self-linking and the linking numbers are both the invariant characteristic numbers of the optical vortices knots family in topology, \( H \) is an important topological invariant required to describe the linked optical vortices in optical systems.

IV. CONCLUSION

In this paper, we consider the topology of the closed and knotted optical vortices in optical systems. By making use of the \( \phi \)-mapping topological current theory, we obtain the topological inner structure of the optical vortices, the topological charge of the vortices can be expressed by the topological numbers: Hopf indices and Brouwer degrees. Furthermore for the closed and knotted optical vortices, we introduce the helicity to study the knotted topology, the helicity is a topological invariant of the knots family, and we find that it is just the total sum of all the self-linking and all the linking number of the knotted optical vortices family.
Acknowledgments

This work was supported by the National Natural Science Foundation of China.

[1] A. S. Desyatnikov, Y. S. Kivshar, and L. Torner, Optical Vortices and Vortex Solitons, Progress in Optics Vol. 47 (North-Holland, Amsterdam, 2005).

[2] J. F. Nye, Natural focusing and fine structure of light (Institute of Physics Publishing, 1999).

[3] M. V. Berry and M. R. Dennis, Proc. R. Soc. A 456 (2000) 2059.

[4] M. V. Berry and J. F. Nye, Proc. R. Soc. A 336 (1974) 165.

[5] Y. S. Duan, G. Jia and H. Zhang, J. Phys. A: Math. Gen. 32 (1999) 4943.

[6] Y. S. Duan, H. Zhang and G. Jia, Phys. Letts. A 253 (1999) 57.

[7] M. V. Berry and M. R. Dennis, Proc. R. Soc. A 457 (2001) 2251.

[8] M. R. Dennis, J. Opt. A: Pure. Appl. Opt. 6 (2004) S202.

[9] K. O. Holleran, M. J. Padgett, and M. R. Dennis, Opt. Exp 14 (2006) 3039.

[10] J. Leach, M. R. Dennis, J. Courtial and M. J. Padgett, Nature 432, (2004)165, New. J. Phys. 7 (2005) 55.

[11] W. Thompson, Phil. Mag. 34 (1867) 15.

[12] H. K. Moffatt, J. Fluid. Mech. 35 (1969) 117.

[13] Y. S. Duan, X. Liu, and L. B. Fu, Phys. Rev. D 67 (2003) 085022.

[14] F. Faddeev and A. J. Niemi, Nature 387 (1997) 58.

[15] A. T. Winfree, Nature 371 (1994) 233.

[16] Y. S. Duan, SLAC-PUB-3301, 1984; Y. S. Duan, H. Zhang, G. Jia, Phys. Letts. A 253 (1999) 57.

[17] Y. S. Duan, X. Liu and P. M. Zhang, J. Phys: Condens. Mather 14 (2002) 7941.

[18] Y. S. Duan and M. L. Ge, Sci. Sin. 11 (1979) 1072; G. H. Yang and Y. S. Duan, Int. J. Theor. Phys. 37 (1998) 2435.

[19] Y. S. Duan and J. C. Liu, in Proceedings of Johns Hopkins Workshop II, edited by Y. S. Duan et al. (World Scientific, Singapore, 1988).

[20] Y. S. Duan and Y. Jiang, Int. J. Theor. Phys. 38 (1999)563; Y. S. Duan, G. H. Yang, and Y. Jiang, Gen. Rel. Grav. 29 (1997) 715.
[21] Y. S. Duan, S. Li and G. H. Yang, Nucl. Phys. B 514 (1998) 705.

[22] Y. S. Duan, H. Zhang, and S. Li, Phys. Rev. B 58 (1998) 125.

[23] Y. S. Duan, H. Zhang, and L. B. Fu, Phys. Rev. E 59 (1999) 528.

[24] J. A. Schouten, Tensor Analysis for Physicists (Clarendon Press, Oxford, 1951).

[25] H. Aref and I. Zawadzki, Nature 354 (1991) 50.

[26] A. F. Rañada, Eur. J. Phys. 13 (1992) 70.

[27] N. D. Mermin, T. L. Ho, Phys. Rev. Lett. 36 (1976) 594.

[28] E. Witten, Commun. Math. Phys. 121 (1989) 351; A. M. Polyakov, Mod. Phys. Lett. A 3 (1988) 325.

[29] D. Rolfsen, Knots and Links (Publish or Perish, Berkeley, CA, 1976).