A theory on skyrmion size and profile

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A magnetic skyrmion is a topological object consisting of an inner domain, an outer domain, and a wall that separates the two domains. The skyrmion size and wall width are two fundamental quantities of a skyrmion that depend sensitively on material parameters such as exchange energy, magnetic anisotropy, Dzyaloshinskii-Moriya interaction, and magnetic field. However, there is no quantitative understanding of the two quantities so far. Here, we present general expressions for the skyrmion size and wall width obtained from energy considerations. The two formulas agree almost perfectly with simulations and experiments for a wide range of parameters, including all existing materials that support skyrmions. Furthermore, it is found that skyrmion profiles agree very well with the Walker-like 360° domain wall formula.

Skyrmions, topological objects originally used to describe resonance states of baryons [1], were observed in magnetic systems that involve Dzyaloshinskii-Moriya interaction (DMI) [2–13]. There are two topologically equivalent magnetic skyrmions. One is the Bloch skyrmions (also known as vortex skyrmions) often found in systems with the bulk DMI [3–6, 11]. The other is the Néel skyrmions (also known as hedgehog skyrmions) in systems with interfacial DMI [7, 9, 10]. Due to their small Néel skyrmions (also known as hedgehog skyrmions) in systems with the bulk DMI [3–6, 11]. The other is the skyrmions (also known as vortex skyrmions) often found interaction (DMI) [2–13]. There are two topological

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We consider a two-dimensional (2D) ferromagnetic film in $xy$ plane with an exchange constant $A$, an interfacial DMI coefficient $D$, a perpendicular easy-axis anisotropy $K$, and a perpendicular magnetic field $B$. The total energy $E$ of the system consists of the exchange energy $E_{ex}$, the DMI energy $E_{DM}$, the anisotropy energy $E_{an}$, and the Zeeman energy $E_{Ze}$,

$$E = E_{ex} + E_{DM} + E_{an} + E_{Ze},$$  \hspace{1cm} (1)

where $E_{ex} = A \iint |\nabla m|^2 dS, E_{DM} = D \iint m_z \nabla \cdot (m \nabla m_z) dS, E_{an} = K \iint (1 - m_z^2) dS$, and $E_{Ze} = \mu_0 M_s B \iint (1 - m_z) dS$. $m$ is the unit vector of magnetization of a constant saturation magnetization $M_s$ and the integration is over the whole film. The energy reference is chosen in such a way that the energy of single domain state of $m_z = 1$ is $E = 0$. The demagnetization energy is included in $E_{an}$ by using the effective anisotropy $K = K_u - \mu_0 M_s^2/2$ corrected by the shape anisotropy, where $K_u$ is the perpendicular magnetocrystalline anisotropy. It is convenient to use a polar coordinates so that a point $r$ in the plane is denoted by $r$ and $\phi$. Magnetization at $r$ is described by polar and azimuthal angles $\Theta(r, \phi)$ and $\Phi(r, \phi)$ so that $m = (\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta)$. A skyrmion centered at $r = 0$ can be described [15] by,

$$\Theta = \Theta(r), \quad \Phi = v\phi + \gamma, \quad (2)$$

with boundary conditions of $\Theta(0) = 0(\pi)$ and $\Theta(\infty) = \pi(0)$. $v$ is the vorticity ($v = 1$ for a skyrmion and
outer domains pointing respectively to the p so that m fits to Θ ○
m exists of an inner domain, an outer domain, and a wall and γ = φ. Lower left inset: Three typical equilibrium states obtained in the numerical simulations for different parameters. 1: isolated skyrmion. 2: single-domain state of mz = 1 (or mz = −1). 3: stripe domains. The pixel color encodes the mz component with the color bar shown in the figure.

v = −1 for an antiskyrmion), and γ is a constant classifying type of skyrmions (γ = 0 or π for Néel skyrmions and γ = ±π/2 for Bloch skyrmions). A skyrmion consists of an inner domain, an outer domain, and a wall separating the two domains. We define the skyrmion size R as the radius of the mz = 0 contour. The wall width w is another fundamental skyrmion parameter often ignored in previous studies. One can also define the skyrmion polarity as p = [mz(r = ∞) − mz(r = 0)]/2 so that p = 1 (−1) corresponds to spins in the inner and outer domains pointing respectively to the −z(+z) and +z(−z)-directions.

In terms of Θ, four energy terms for v = 1 are

\[ E_{\text{ex}} = 2\pi A \int_0^\infty \left( \frac{d\Theta}{dr} + \frac{\sin^2 \Theta}{r^2} \right) rdr, \]
\[ E_{\text{DM}} = 2\pi D \cos \gamma \int_0^\infty \left( \frac{d\Theta}{dr} + \frac{\sin 2\Theta}{2r} \right) rdr, \]
\[ E_{\text{an}} = 2\pi K \int_0^\infty \sin^2 \Theta rdr, \]
\[ E_{\text{Ze}} = 2\pi \mu_0 M_s B \int_0^\infty (1 - \cos \Theta) rdr. \]

For a skyrmion of p = 1 as shown in Fig. 1, \( \int_0^\infty \frac{d\Theta}{dr} rdr < 0 \) because Θ decreases monotonically from Θ(0) = π to Θ(∞) = 0. \( \int_0^\infty \sin \Theta \cos \Theta rdr < 0 \) is small because sin Θ cos Θ = 0 far from the skyrmion wall and sin Θ cos Θ changes its sign from negative near the inner domain to positive near the outer domain. Thus, to lower the total energy, one needs γ = 0(π) for D > 0 (< 0). This corresponds to a Néel skyrmion. Along a radial direction, the magnetization profile looks like a 360° Néel domain wall as illustrated in Fig. 1. This leads us to model a skyrmion profile by the Walker-like 360° domain wall solution

\[ \Theta_{\text{dw}}(r) = 2 \arctan \left[ \frac{\sin(R/w)}{\sinh(r/w)} \right]. \]

To test how good ansatz [4] is for a skyrmion, we use MuMax3 [30] to simulate various magnetic stable states in a magnetic disk of diameter 512 nm and thickness 0.4 nm. The mesh size of 1 nm × 1 nm × 0.4 nm is used in our simulations. A = 15 pJ/m, Ms = 580 kA/m, and perpendicular easy-axis anisotropy \( K_u = 0.8 \text{ MJ/m}^3 \) are used to mimic Co layer in Pt/Co/MgO system. The initial state is \( m_z = 1 \) for \( r > 10 \text{ nm} \) and \( m_z = -1 \) for \( r \leq 10 \text{ nm} \). The final stable state depends on the values of D and B. The lower left inset of Fig. 1 is one typical stable states. 1 is a skyrmion for \( D = 3.7 \text{ mJ/m}^2 \) and \( B = 0 \). 2 is a single-domain state of \( m_z = 1 \) (or \( m_z = -1 \)) for \( D = 0 \) and \( B = 0 \). 3 is a stripe domain state for \( D = 5 \text{ mJ/m}^2 \) and \( B = 0 \). The upper left plot of Fig. 1 shows the spatial distribution of mz of the skyrmion in (1) along three radial directions, φ = 0 (crosses), 45° (triangles), and 90° (circles). All three sets of data are on the same smooth curve, showing mz is a function of r, but not φ. The curve can fit perfectly to Eq. (4) with \( R = 25.77 \text{ nm} \), \( w = 4.94 \text{ nm} \). We plotted also \( \Phi(\phi) \) at randomly picked spins from the simulated skyrmion. All numerical data (red circles) are perfectly on \( \Phi = \phi \). These results not only confirm the validity of skyrmion expression of Eq. (2), but also suggest that \( m_z(r) \equiv \cos[\Theta_{\text{dw}}(r)] \) follows the Walker-like 360° DW profile [4].

The energy of a skyrmion can then be obtained from Eq. (3) by using the Walker-like 360° domain wall profile \( \Theta_{\text{dw}}(r) \). The total energy is, in general, a function of R and w [instead of a function of Θ(r) and \( \Phi(\phi) \)] as

\[ E(R, w) = 4\pi \left\{ A \left[ f_1 \left( \frac{R}{w} \right) + f_2 \left( R \right) \right] + D w \left[ f_3 \left( \frac{R}{w} \right) + f_4 \left( R \right) \right] + K w^2 f_5 \left( \frac{R}{w} \right) + B w^2 f_6 \left( \frac{R}{w} \right) \right\}, \]

where \( f_i(x) \) (i = 1 ~ 6) are non-elementary functions defined in the Supplemental Material [31]. The skyrmion size and wall width R and w are the values that minimize \( E(R, w) \). Figure 2 are D− (a), A− (b), K− (c)
The solid lines are exact analytical results obtained from
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A

FIG. 2. The

D (a), A (b), K (c) and B dependences of

skyrmion size \( R \) (left axis) and wall width \( w \) (right axis).

Model parameters are \( A = 15 \) pJ/m, \( K = K_u - \mu_0 M_s^2 = 0.588 \)

MJ/m\(^3\), \( D = 3.7 \) mJ/m\(^2\), and \( B = 0 \).

In each subfigure, one of these four parameters is treated as a tuning parameter,
and the other three parameters are fixed to above values.
The symbols are the micromagnetic simulation data.
The solid lines are exact analytical results obtained from
Eq. (5). The dashed lines are approximate results of
\[ R = \frac{A}{\sqrt{2\pi w D + R}}, \quad w = \frac{\pi D}{2R} \] (subfigures (a)-(c)) and solution
of Eq. (5) (subfigure (d)). Vertical dashed lines are the upper (lower) bound of parameters above (below) which a stable
skyrmion cannot exist.

and \( B \) dependences of skyrmion size \( R \) (left y-axis) and wall width \( w \) (right y-axis), with other parameters
fixed to the values for Co mentioned earlier. The sym-

bols are the micromagnetic simulation data \([R] \) is the size of

\( m_s = 0 \) contour line and \( w \) is the fit of skyrmion pro-

goal to \( \Theta_{dw}(r) \). Solid lines are numerical results from

Eq. (5). The simulation results agree almost perfectly
(except slight deviation in the \( D \)-dependence of \( w \) for
smaller \( D \)) with our analytical results of Eq. (16). Both
micromagnetic simulations and analytical results clearly
show that skyrmion can exist for \( D < 3.8 \) mJ/m\(^2\) in the
current case. Above the upper limit, the stable state is
not a skyrmion, but stripe domains as shown in subfigures (a)-(c) and solution
of Eq. (5) (subfigure (d)). Vertical dashed lines are the upper (lower) bound of parameters above (below) which a stable
skyrmion cannot exist.

around 14 pJ/m as shown in Fig. 2(b) and a minimal
\( K \) of around 0.56 MJ/m\(^3\) as shown in Fig. 2(c), below
which skyrmion does not exist, and the stable state is
stripe domains as shown in of Fig. 1. The skyrmion
size decreases with \( B \), which is consistent with the experimental observations [11, 23].

It is still unclear how \( R \) and \( w \) depend on \( A, D \) and \( K \)

although Eq. (5) agrees almost perfectly with simulation results. Thus it is highly desirable to have a simple ap-

proximate expression for \( R \) and \( w \) in terms of material

parameters. The exchange and DMI energies come from
the spatial magnetization variation rate. For a skyrmion,

the magnetization variation rates in the radial and tan-

gent directions scale respectively as \( 1/w \) and \( 1/R \). The

exchange energy is then proportional to skyrmion wall
area of \( \pi R w \) multiplying the square of the magnetization
variation rates \( 1/R^2 + 1/w^2 \), i.e. \( E_{ex} \approx (R/w + w/R) \).

For a Néel skyrmion, the magnetization variation rate

along the tangent direction is perpendicular to \( \mathbf{m} \) and
does not contribute to the DMI energy. The DMI en-

ergy is then proportional to wall area \( (Rw) \) multiplying

the magnetization variation rate along radial direc-

tion \( (1/w) \), i.e. \( E_{DM} \propto R \). The anisotropy energy is

mainly from the skyrmion wall area. Thus, \( E_{an} \propto Rw \).
The Zeeman energy of the skyrmion comes from the

inner domain proportional to its area of \( \pi (R - cw)^2 \),

where \( c \) is a coefficient depending on the magnetiza-

tion profile, and from the wall area proportional to its

area of \( \pi Rw \). To obtain the proportional coefficients,

one needs to find approximate expressions for \( f_i (R/w) \)

\((i = 1, \ldots, 6) \) in Eq. (5). In the case of \( R \gg w \) (or \( x \gg R/w \gg 1 \)), \( \sinh(x) \approx \cosh(x) \approx e^x \).
Thus, function

\[ g(t, x) = \frac{2\sinh^2(x) \cosh^2(t)}{\sinh^2(x) + \sinh^2(t)} \]

\[ \approx \frac{2e^{2(x-t)} [e^{2(t-x)} + 1]}{2e^{2(t-x)}} \] is positive and significantly non-

zero only near \( t = x \), reflecting the fact that \( E_{ex}, E_{DM}, \) and \( E_{an} \) are mainly from skyrmion wall region that is

assumed to be very thin. Furthermore, the area bounded

by \( g(t, x) \)-curve and \( t \)-axis is 1 so that \( g(t, x) \approx \delta(t - x) \)

resembles the properties of a delta function.

We can evaluate \( f_i \)’s under this approximation (See

Supplemental Materials [5]). For example, \( f_1(x) \) is

\[ f_1(x) = \int_0^\infty g(t, x)tdt \approx \int_0^\infty \delta(t - x)tdt = x. \] (6)

The total energy is then

\[ E(R, w) = 4\pi \left[ A \left( \frac{R}{w} + \frac{w}{R} \right) - \frac{\pi}{2} DR \right. \]

\[ + K w R + \mu_0 M_s B \left( \frac{R^2}{2} + \frac{\pi^2}{24} w^2 \right) \]. \] (7)

Due to the specific form of the magnetization pro-

file of \( \Theta_{dw}(r) \), \( R w \)-term in \( E_{ex} \) vanishes and \( E_{ex} \approx 4\pi \mu_0 M_s B \left( \frac{R^2}{2} + \frac{\pi^2}{24} w^2 \right) \). The skyrmion size and wall

width are then the values that make \( E(R, w) \) minimal,
or

$$A \left( \frac{1}{w} - \frac{w}{R^2} \right) - \frac{\pi}{2} D + Kw + \mu_0 M_s BR = 0, \quad (8)$$

$$A \left( - \frac{R}{w^2} + \frac{1}{R} \right) + KR + \frac{\pi^2}{12} \mu_0 M_s B w = 0. \quad (9)$$

For $B = 0$, Eqs. (8) and (9) can be analytically solved. The results are

$$R = \pi D \sqrt{\frac{A}{16AK^2 - \pi^2 D^2 K^2}}, \quad w = \frac{\pi D}{4K}. \quad (10)$$

The dashed lines in Fig. 2(a)-(c) are the approximate formulas that compare quite well with simulation results too. For $B \neq 0$, it is difficult to analytically solve Eqs. (8) and (9), but their numerical solutions are easily obtained that are plotted as dashed lines in Fig. 2(d). In summary, our approximate formula agrees very well with the simulations for $R \gg w$ as expected from our approximation. For smaller skyrmions, the approximation is still not bad, and qualitatively gives correct parameter dependence. We can also determine the upper limit of $D$ and lower limits of $A$, $K$, and $B$ from the approximate formula. Since $R$ must be real and finite, we have

$$D < \frac{4}{\pi \sqrt{AK}}, \quad A > \frac{\pi^2 D^2}{16K}, \quad K > \frac{\pi^2 D^2}{16A}. \quad (11)$$

for $B = 0$. Note that these limits are consistent with the criteria of the existence of chiral domain walls.

These critical values are plotted in Fig. 2(a)-(c) as vertical dashed lines that agree also well with simulations.

We compare our theoretical results of skyrmion size with the experimental results for PdFe/Ir [7, 22, 34] and W/Co$_{90}$Fe$_{50}$B$_{20}$/MgO [10, 32], in which isolated skyrmions are observed. For PdFe/Ir, the parameters are $M_s = 961 \sim 1160$ kA/m, $A = 2 \sim 4.87$ pJ/m, $K = 2.5$ MJ/m$^3$, $D = 3.4 \sim 3.9$ mJ/m$^2$, and $B = 1.15 \sim 2.97$ T. [7, 23]. Our theory gives small skyrmion size of 0.53 \sim 1.59 nm that compares well with the experimental results of 0.9 \sim 1.9 nm in Ref. [7]. For W/Co$_{90}$Fe$_{50}$B$_{20}$/MgO, the parameters are $M_s = 650$ kA/m, $A = 10$ pJ/m, $K = 0.02275$ MJ/m$^3$, $D = 0.68 \sim 0.73$ mJ/m$^2$, and $B = 0.00025 \sim 0.0005$ T. [10, 32]. Our theory gives large skyrmion size of 356 \sim 1484 nm, consistent with the experimental results 700 \sim 2000 nm in Ref. [11]. Our theoretical results show good agreement with the experiments although some of the material parameters can only be roughly estimated from different literatures.

We also compare our analytical results with micromagnetic simulations for PdFe/Ir [7, 22, 34], MnSi [22, 36], and W/Co$_{90}$Fe$_{50}$B$_{20}$/MgO [10]. The skyrmion sizes range from several nanometers to about 2 micrometers. All the comparisons give quite good agreement (See Supplemental Materials [31]). Our results show that skyrmion size increases with $D$, and decreases with $A$ and $K$. Our results also show that not only DMI, but also magnetic anisotropy or perpendicular magnetic field is necessary for the formation of isolated skyrmions, which is consistent with all previous experiments and simulations [2, 11, 14, 16, 22, 23, 28].

It is natural to extend our approach to Bloch skyrmions in the systems with bulk inversion symmetry broken. The bulk DMI energy $E_{DM} = D \int \mathbf{m} \cdot (\nabla \times \mathbf{m}) \, dS$ can be rewritten as

$$E_{DM} = 2\pi D \sin \gamma \int_0^{\infty} \left( \frac{d\Theta}{dr} + \sin \frac{2\Theta}{2r} \right) r \, dr, \quad (12)$$

where $\gamma = \pi/2$ gives minimal energy. Since all other discussions are the same as those for Néel skyrmions, the results about $R$ and $w$ are applicable for the Bloch skyrmions.

In conclusion, we found a single skyrmion can be well described by a 360° domain wall profile parametrized by two fundamental quantities, skyrmions size and wall width. Through the minimization of total energy with respect to skyrmion size and wall width, analytical formulas for skyrmion size and wall width as a function of exchange stiffness, anisotropy coefficient, DMI strength and external field are obtained. The formulas agree very well with simulations and experiments.

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SUPPLEMENTAL MATERIAL

Derivation of energy expressions

To derive the functions $f_i(x)$ in the energy expression, we substitute Eq. (1) into Eq. (3). For the exchange energy, by defining $x = R/w$, $t = r/w$, we have

$$E_{ex} = 2\pi A \int_0^\infty \left[ \frac{d\Theta}{dr} + \frac{\sin^2 \Theta}{r^2} \right] r dr$$

$$= 4\pi A \int_0^\infty \left\{ \frac{2 \sinh^2(x) \cosh^2(t)}{\sinh^2(x) + \sinh^2(t)} t + \frac{2 \sinh^2(x) \sinh^2(t)}{t \sinh^2(x) + \sinh^2(t)} \right\} dt$$

Thus, we define

$$f_1(x) = \int_0^\infty \frac{2 \sinh^2(x) \cosh^2(t)}{\sinh^2(x) + \sinh^2(t)} t dt,$$

$$f_2(x) = \int_0^\infty \frac{2 \sinh^2(x) \sinh^2(t)}{t \sinh^2(x) + \sinh^2(t)} dt.$$  

While $R \gg w$ ($x \gg 1$), we have $\sinh(x) \approx \cosh(x) \approx e^x$, so that

$$f_1(x) \approx \int_0^\infty \frac{2e^{2(x-t)}}{[e^{2(x-t)} + 1]^2} t dt$$

The function $\frac{2e^{2(x-t)}}{[e^{2(x-t)} + 1]^2}$ is non-zero only for $x \approx t$. Approximately, we have

$$\frac{2e^{2(x-t)}}{[e^{2(x-t)} + 1]^2} \approx I_1 \delta(x - t),$$

where the coefficient $I_1$ is determined by $I_1 = \int_{-\infty}^{\infty} \frac{2e^x}{(e^x + 1)^2} dx = 1$. Thus, $f_1(x) \approx \int_0^\infty \delta(x - t) dt = x$. Similarly, $f_2(x) \approx \int_0^\infty \delta(t - x) dt = 1/x$.

For the DM energy, we have

$$E_{DM} = 2\pi D \cos \gamma \int_0^\infty \left( \frac{d\Theta}{dr} + \frac{\sin 2\Theta}{2r} \right) r dr$$

$$= 4\pi D w \int_0^\infty \left[ -\frac{\sinh x \cosh t}{\sinh^2 x + \sinh^2 t} t - \frac{\sinh x \sinh t (\sinh^2 x - \sinh^2 t)}{\sinh^2 x + \sinh^2 t} \right] dt.$$  

We define

$$f_3(x) = -\int_0^\infty \frac{t \sinh(x) \cosh(t)}{\sinh^2(x) + \sinh^2(t)} dt,$$

$$f_4(x) = -\int_0^\infty \frac{t \sinh(x) \sinh(t) [\sinh^2(x) - \sinh^2(t)]}{[\sinh^2(x) + \sinh^2(t)]^2} dt.$$  

For $f_3(x)$, the function inside the integral $\frac{\sinh(x) \cosh(t)}{\sinh^2(x) + \sinh^2(t)} \approx \frac{e^{(x-t)}}{e^{2(x-t)} + 1}$ is localize at $x = t$ so that we have the approximation

$$f_3(x) \approx -\int_0^\infty \frac{e^{(x-t)}}{e^{2(x-t)} + 1} t dt$$

$$\approx -\int_0^\infty I_3 \delta(x - t) dt$$

$$\approx I_3 x,$$
where $I_3$ is determined by $I_3 = \int_{-\infty}^{\infty} \frac{x^2}{e^{x^2}+1} \, dx = \pi/2$. The integrand in $f_4$ is 0 at $r = 0$, $r = \infty$ and $r = R$. Furthermore, it has opposite signs for $r < R$ and $r > R$. So $f_4 \sim 0$ after the integration.

For the anisotropy energy, we have

$$f_5(x) = \int_0^\infty \frac{2t\sinh^2(x)\sinh^2(t)}{[\sinh^2(x) + \sinh^2(t)]^2} \, dt.$$ 

Similar to the exchange energy and DM energy, the anisotropy energy is only non-zero near $r = R$, too. The approximate form of function $f_5$ is the same as $f_1$,

$$f_5(x) \approx \int_0^\infty \frac{2te^{2(x-t)}}{[e^{2(x-t)} + 1]^2} \, dt \approx \int_0^\infty t\delta(x-t) \, dt = x.$$ 

The Zeeman energy is non-zero for both the wall region and the inner domain. The function $f_6$ is

$$f_6(x) = \frac{\sinh^2(x)}{\sinh^2 t + \sinh^2 x} - t \, dt$$ 

Again, we replace sinh functions by exponential functions to obtain

$$f_6(x) \approx \int_0^\infty \frac{1}{e^{2(t-x)} + 1} \, dt = -\frac{1}{4} \text{Li}_2(-e^{2x}),$$

where $\text{Li}_s(x)$ is the polylogarithm function defined by $\text{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}$. The asymptotic form of $-\frac{1}{4} \text{Li}_2(-e^{2x})$ is

$$\lim_{x \to \infty} -\frac{1}{4} \text{Li}_2(-e^{2x}) = \frac{\pi^2}{2} + \frac{\pi^2}{24}.$$ 

So we have $f_6(x) \approx \frac{\pi^2}{2} + \frac{\pi^2}{24}$.

**Numerical verification of theoretical results for different materials**

We compare the theoretical results with micromagnetic simulations. Similar to Fig. 2 in the main text, we compare the $D$-, $A$-, $K$-, and $B$-dependencies of skyrmion size $r$ and skyrmion wall width $w$. The sample size ranges from 256 nm to 2048 nm, and the sample thickness is fixed to 0.1 nm. In each subfigure, one of $D$, $A$, $K$, and $B$ is treated as a tuning parameter, and the other three parameters are fixed to the values listed in Table S1.

| Material                  | $A$ (pJ/m) | $K$ (MJ/m$^3$) | $D$ (mJ/m$^2$) | $M_s$ (kA/m) | $B$ (T) |
|---------------------------|------------|----------------|----------------|--------------|---------|
| PdFe/Ir (IDMI)            | 4.87       | 2.5            | 3.43           | 961          | 1.15    |
| MnSi (BDMI)               | 0.845      | -0.0334        | 0.338          | 163          | 1       |
| W/Co$_{20}$Fe$_{60}$B$_{20}$/MgO (IDMI) | 10         | 0.0228         | 0.7            | 650          | 0.0005  |

**TABLE S1.** Parameters used for verification of our results. IDMI (BDMI) means the DMI is interfacial (bulk) type.
FIG. S1. Comparison between numerical and theoretical results for PdFe/Ir parameters. The meanings of lines and symbols are the same as those in Fig. 2 of the main text.
FIG. S2. Comparison between numerical and theoretical results for MnSi parameters. The skyrmions are Bloch-type. $R \gg w$ cannot be satisfied for parameters near the MnSi parameter, so the approximate formulas (dashed lines) do not agree well with the exact formulas (solid lines). Nevertheless, the numerical simulation results agree well with the exact formulas.
FIG. S3. Comparison between numerical and theoretical results for W/Co$_{20}$Fe$_{60}$B$_{20}$/MgO parameters.