THE SPECTRUM OF DIAGONAL PERTURBATION OF WEIGHTED SHIFT OPERATOR

M. L. SAHARI, A. K. TAHIA, AND L. RANDRIAMIHAMISON

Abstract. This paper provides a description of the spectrum of diagonal perturbation of weighted shift operator acting on a separable Hilbert space.

1. Introduction

Let $X$ be a separable complex Hilbert space with an orthonormal basis $\{e_i\}_{i \in \mathbb{Z}} \subset X$. We define the weighted shift operator in $X$ by

$$S e_i = w_i e_{i+1}, \quad i = 0, \pm 1, \pm 2, \ldots$$

The sequence $\{w_i\}_{i \in \mathbb{Z}} \subset \mathbb{C}$ represents the weights of the operator $S$. The matrix of such operator can be written as

$$S = \begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
\ddots & 0 & 0 & 0 & 0 & 0 & \\
\ddots & w_{-1} & 0 & 0 & 0 & 0 & \\
\vdots & 0 & w_0 & 0 & 0 & 0 & \\
\vdots & 0 & 0 & w_1 & 0 & 0 & \\
\vdots & 0 & 0 & 0 & w_2 & 0 & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
\end{pmatrix}.$$

In [1, 2, 9], it is shown that if $S$ is bounded, then there exists $0 \leq r^- \leq r^+$ such that the spectrum $\sigma(S)$ of $S$ is given by

$$\sigma(S) = \{\lambda \in \mathbb{C} : r^- \leq |\lambda| \leq r^+\}.$$ 

In this work, we propose to extend this type of result to the case of the perturbed operator $S + D$, where $D$ is a diagonal operator.

2. Preliminary notions

Let $\mathcal{L}(X)$ denote the algebra of all bounded linear operators acting on a complex Banach space $X$. The norm on $X$ and the associated operator norm on $\mathcal{L}(X)$ are both denoted by $\|\cdot\|$. For $T \in \mathcal{L}(X)$, we denote by $\sigma(T)$, $\rho(T)$ and $r(T)$ the spectrum, the resolvent and the spectral radius of $T$ respectively. Recall that $\sigma(T)$ is a non-empty compact subset of $\mathbb{C}$, $r(T) \leq \|T\|$ and $r(T) = \lim \|T^k\|^\frac{1}{k} = \inf \|T^k\|^\frac{1}{k}$. If $T$ is invertible, the inverse is denoted by $T^{-1}$ and we have $\sigma(T^{-1}) = \left\{\frac{1}{\lambda} : \lambda \in \sigma(T)\right\}$. Moreover, $\frac{1}{r(T^{-1})} = \inf \{|\lambda| : \lambda \in \sigma(T)\}$ (see [2, 3, 4, 5, 6]).

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3. Some properties of weighted shift

In the following, $X$ is a separable complex Hilbert space and $\{e_i\}_{i \in \mathbb{Z}}$ an orthonormal basis of $X$. Let $S$ be a weighted shift operator on $X$ with a weight sequence $\{w_i\}_{i \in \mathbb{Z}}$. The boundedness of the operator $S$ is a consequence of the boundedness of the weight sequence $\{w_i\}_{i \in \mathbb{Z}}$. However, we have a more general results

**Proposition 1** ([2, 4, 9]). The operator $S$ is bounded if and only if the weight sequence $\{w_i\}_{i \in \mathbb{Z}}$ is bounded. In this case,

$$
\|S^k\| = \sup_{i \in \mathbb{Z}} \left| \prod_{m=0}^{k-1} w_{i+m} \right|, \quad k = 1, 2, \ldots
$$

**Remark 1.** If $S$ is invertible, then the inverse $S^{-1}$ is given by

$$
S^{-1}e_i = \frac{1}{w_{i-1}} e_{i-1}
$$

and in this case

$$
\|S^{-k}\| = \sup_{i \in \mathbb{Z}} \left| \prod_{m=0}^{k-1} \frac{1}{w_{i+m}} \right| = \sup_{i \in \mathbb{Z}} \left| \prod_{m=0}^{k-1} \frac{1}{w_{i+m}} \right| = \left[ \inf_{i \in \mathbb{Z}} \left| \prod_{m=0}^{k-1} w_{i+m} \right| \right]^{-1}, \quad k = 1, 2, \ldots
$$

**Proposition 2.** The operator $S$ is invertible if and only if the sequence $\left\{ \frac{1}{w_i} \right\}_{i \in \mathbb{Z}}$ is bounded.

**Proposition 3** ([2, 4, 9]). If $\{\lambda_i\}_{i \in \mathbb{Z}}$ are complex numbers of modulus 1, then $S$ is unitary equivalent to the weighted shift operator with weight sequence $\{\lambda_i w_i\}$.

**Corollary 1.** The operator $S$ is unitary equivalent to the weighted shift operator of weight $\{|w_i|\}_{i \in \mathbb{Z}}$.

**Corollary 2** ([1, 4]). If $|c| = 1$, then $S$ and $cS$ are unitary equivalent.

**Remark 2.** From the last corollary, the spectrum of the operator $S$ have circular symmetry about the origin. In particular, $\{\lambda \in \mathbb{C} : |\lambda| = r(S)\} \subset \sigma(S)$ and $\left\{ \lambda \in \mathbb{C} : |\lambda| = \frac{1}{r(S^{-1})} \right\} \subset \sigma(S^{-1})$.

In the following section, we state our main result.

4. The spectrum of perturbed weighted shift

Let $T \in \mathcal{L}(X)$ be a perturbed operator given by

$$
T = S + D,
$$

where $D$ is a diagonal operator with diagonals $\{d_i\}_{i \in \mathbb{Z}}$.

**Lemma 1.** If $T$ is invertible, then we have at least one of the following two inequalities

$$
R_{S+D}^+ = \lim_{k \to \infty} \left[ \sup_{i \in \mathbb{Z}} \left| \prod_{m=0}^{k-1} \frac{1}{w_{i+m}} \prod_{m=0}^{k-1} d_{i+m} \right| \right]^{\frac{1}{k}} \leq 1
$$

or

$$
R_{S+D}^- = \lim_{k \to \infty} \left[ \sup_{i \in \mathbb{Z}} \left| \prod_{m=1}^{k} \frac{1}{w_{i-m}} \prod_{m=1}^{k} d_{i+m} \right| \right]^{\frac{1}{k}} \leq 1.
$$
Proof. Let $T$ be invertible and set $x_i = \sum_{j \in \mathbb{Z}} a_j^i e_j = T^{-1}e_i$. Thus, we have

$$\begin{cases} w_{j-1}a_{j-1}^i + d_ja_j^i = 1, & \text{if } j = i, \\ w_{j-1}a_{j-1}^i + d_ja_j^i = 0, & \text{otherwise.} \end{cases}$$

(4)

and

$$\begin{cases} w_i a_i^{i+1} + d_ia_i^i = 1, & \text{if } j = i, \\ w_i a_i^{i+1} + d_ia_i^i = 0, & \text{otherwise}. \end{cases}$$

(5)

The first equation of (4) and of (5) implies for all $i \in \mathbb{Z},$

$$a_i^i d_i = a_0^0 d_0$$

(6)

From (4), we get, for all $i \in \mathbb{Z}$ and $k > 0,$

$$a_{i-k}^i = \langle T^{-1}e_i, e_{i-k} \rangle = (-1)^{k+1} \frac{(1 - d_ia_i^i) \prod_{m=1}^{k-1} d_{i-m}}{\prod_{m=1}^k w_{i-m}},$$

assuming that $\prod_{m=1}^0 d_{i-m} = 1$. Cauchy-Schwarz inequality gives us

$$\left| \frac{(1 - d_0a_0^0) \prod_{m=1}^{k-1} d_{i-m}}{\prod_{m=1}^k w_{i-m}} \right| \leq \|T^{-1}\|, \text{ for every } i \in \mathbb{Z} \text{ and } k \geq 0. \quad (7)$$

Consequently, for all $i \in \mathbb{Z}$ and $k > 0$, we have

$$a_{i+k}^i = \langle T^{-1}e_i, e_{i+k} \rangle = (-1)^k \frac{d_ia_i^i \prod_{m=0}^{k-1} w_{i+m}}{\prod_{m=0}^k d_{i+m}},$$

Cauchy-Schwarz inequality provides the inequality

$$\left| \frac{\prod_{m=0}^{k-1} w_{i+m}}{\prod_{m=0}^k d_{i+m}} \right| \leq \|T^{-1}\|, \text{ for every } i \in \mathbb{Z} \text{ and } k > 0. \quad (8)$$

From the first equation of (4), for all $i \in \mathbb{Z}$, either $w_{i-1}a_{i-1}^i$ or $d_ia_i^i$ is not zero. Thus, we can distinguish two cases:

1st case: $d_ia_i^i \neq 0,$ from (6) and by taking the supremum over $i$ in (7), we get

$$\sup_{i \in \mathbb{Z}} \left| \frac{\prod_{m=0}^{k-1} w_{i+m}}{\prod_{m=0}^k d_{i+m}} \right| < \infty, \text{ for every } k > 0.$$

(9)
2nd case: \( w_{i-1}a_{i-1}^i \neq 0 \), from (6) and by taking the supremum over \( i \) in (8), we get

\[
\sup_{i \in \mathbb{Z}} \left| \prod_{m=1}^{k-1} \frac{d_{i-m}}{w_{i-m}} \right| < \infty, \text{ for every } k > 0.
\]  

(10)

We conclude, by taking the \( k \)th root and letting \( k \to \infty \) in (9) and (10).

In the folow, we give a converse of the previous lemma.

**Lemma 2.** If \( R_{S+D}^+ < 1 \) or \( R_{S+D}^- < 1 \) then \( T \) is invertible.

**Proof.** Note that for all \( k > 0 \)

\[
\left[ \sup_{i \in \mathbb{Z}} \left| \prod_{m=0}^{k-1} \frac{d_{i-m}}{w_{i-m}} \right| \right]^{-1} = \inf_{i \in \mathbb{Z}} \left| \prod_{m=0}^{k-1} \frac{d_{i-m}}{w_{i-m}} \right| \leq \sup_{i \in \mathbb{Z}} \left| \prod_{m=0}^{k-1} \frac{d_{i+m}}{w_{i+m}} \right|,
\]

then only one of inequality \( R_{S+D}^+ < 1 \) or \( R_{S+D}^- < 1 \) can be satisfied.

Let \( R_{S+D}^- < 1 \) and let \( F \) be a linear operator on \( X \) to \( X \), defined by

\[
F := \sum_{l=0}^{\infty} F_l,
\]

(12)

such as, for all \( l \in \mathbb{N} \), \( F_l \) is an operator given by

\[
F_l e_i = a_{i+l}^i e_{i+l}, \quad i = 0, \pm 1, \pm 2, \ldots
\]

(13)

and

\[
a_{i+l}^i = (-1)^l \prod_{m=0}^{l-1} \frac{w_{i+m}}{d_{i+m}}
\]

(14)

with the assumptions that \( \prod_{m=0}^{l-1} w_{i+m} = 1 \). The condition \( R_{S+D}^- < 1 \) implies that the operator \( F \) is well defined, \( \|F\| < \infty \) and \( \lim_{k \to \infty} a_{i+k}^{i+1} = \lim_{k \to \infty} a_{i+k}^i = 0 \). From (12)-(14), and for all \( i \in \mathbb{Z} \), we have

\[
(F \circ T) e_i = w_i F e_{i+1} + d_i F e_i,
\]

\[
= d_i a_i^i e_i + \lim_{k \to \infty} \left\{ \sum_{l=0}^{k} \left( w_i a_{i+l+1}^{i+1} + d_i a_{i+l+1}^i \right) e_{i+l+1} \right\} + w_i a_{i+k+2}^{i+1} e_{i+k+2},
\]

\[
= d_i a_i^i e_i + \sum_{l=0}^{\infty} \left( w_i a_{i+l+1}^{i+1} + d_i a_{i+l+1}^i \right) e_{i+l+1}.
\]

Equation (14), leads to \( a_i^i = \frac{1}{d_i} \) and

\[
w_i a_{i+l+1}^{i+1} = (-1)^l \frac{\prod_{m=0}^{l} w_{i+m}}{\prod_{m=0}^{l} d_{i+m+1}}.
\]

(15)
Also
\[ d_i a_{i+l+1}^i = -(-1)^l \prod_{m=0}^{l} \frac{w_{i+m}}{d_{i+m+1}}, \]  
(16)
hence
\[ w_i a_{i+l+1}^{i+1} + d_i a_{i+l+1}^i = 0, \]
which implies \((F \circ T)e_i = e_i\). Moreover, note that
\[ (T \circ F)e_i = T \left( \sum_{l=0}^{\infty} F_l e_i \right), \]
\[ = a_i d_i e_i + \lim_{k \to \infty} \left\{ \sum_{l=0}^{k} \left( a_{i+l}^i w_{i+l} + a_{i+l+1}^i d_{i+l+1} \right) e_{i+l+1} \right\} + a_{i+k+1}^i w_{i+k+1} e_{i+k+1}, \]
\[ = a_i d_i e_i + \sum_{l=0}^{\infty} \left( a_{i+l}^i w_{i+l} + a_{i+l+1}^i d_{i+l+1} \right) e_{i+l+1}. \]

From (14), we have \(a_i^i = \frac{1}{d_i}\), then
\[ w_{i+l} a_{i+l+1}^{i+1} = (-1)^l \prod_{m=0}^{l} \frac{w_{i+m}}{d_{i+m}}, \]  
(17)
and
\[ d_{i+l+1} a_{i+l+1}^i = -(-1)^l \prod_{m=0}^{l} \frac{w_{i+m}}{d_{i+m}}. \]  
(18)
Then
\[ w_{i+l} a_{i+l}^{i+1} + d_{i+l+1} a_{i+l+1}^i = 0. \]
Hence, \((T \circ F)e_i = e_i\), which lead to
\[ T \circ F = F \circ T = I, \]
where \(I\) denotes the identity operator.

If \(R_{S+D}^< \) \(<\ 1\), let \(F\) be an operator on \(X\) to \(X\), defined by
\[ F := \sum_{l=1}^{\infty} F_{-l}, \]  
(19)
and for all \(l \in \mathbb{N} - \{0\}\), \(F_{-l}\) is an operator given by
\[ F_{-l} e_j = a_{j-l}^j e_{j-l}, \quad j = 0, \pm 1, \pm 2, \ldots \]  
(20)
and

\[ a_{i-k}^j = (-1)^{k+1} \prod_{m=1}^{k-1} \frac{d_{i-m}}{m}, \quad (21) \]

with assumptions that \( \prod_{m=1}^0 d_{i-m} = 1 \). Note that, the condition \( R_{S^+D}^- < 1 \) implies that the operator \( F \) is well defined, \( \|F\| < \infty \) and \( \lim_{k \to \infty} a_{i-k}^j = 0 \). From (19)-(21), then for all \( i \in \mathbb{Z} \),

\[
(F \circ T) e_i = w_i Fe_{i+1} + d_i Fe_i, \\
= w_i a_{i+1}^l e_i + \lim_{k \to \infty} \left\{ \left[ \sum_{l=1}^{n} (w_i a_{i+1}^l + d_i a_{i-l}^i) e_{i-l} \right] + d_i a_{i-k}^l e_{i-k-1} \right\}, \\
= w_i a_{i+1}^l e_i + \sum_{l=1}^{\infty} (w_i a_{i+1}^l + d_i a_{i-l}^i) e_{i-l}.
\]

Formula (21), leads to \( a_{i+1}^l = \frac{1}{w_i} \) and

\[
w_i a_{i-l}^l = (-1)^{l+1} \frac{\prod_{m=1}^{l} d_{i-m+1}}{\prod_{m=1}^{l+1} w_{i+m+1}} = (-1)^{l+1} \frac{\prod_{m=1}^{l} d_{i-m+1}}{\prod_{m=1}^{l} w_{i-m}}. \quad (22)
\]

Also

\[
d_i a_{i-l}^l = (-1)^{l+1} \frac{\prod_{m=1}^{l-1} d_{i-m}}{\prod_{m=1}^{l} w_{i-m}} = (-1)^{l+1} \frac{\prod_{m=1}^{l-1} d_{i-m+1}}{\prod_{m=1}^{l} w_{i-m}}. \quad (23)
\]

Combining (22) and (23), we obtain \( w_i a_{i+1}^l + d_i a_{i-l}^l = 0 \). Therefore \( (F \circ T) e_i = e_i \). Moreover, note that

\[
(T \circ F) e_i = T \left( \sum_{l=1}^{\infty} F_{i-l} e_i \right), \\
= w_{i-1} a_{i-1}^l e_i + \lim_{k \to \infty} \left\{ \left[ \sum_{l=1}^{k} (a_{i-l-1}^l w_{i-l-1} + a_{i-l}^l d_i e_{i-l}) \right] + a_{i-k-1} a_{i-k-1} d_i e_{i-k-1} \right\}, \\
= w_{i-1} a_{i-1}^l e_i + \sum_{l=1}^{\infty} (a_{i-l-1}^l w_{i-l-1} + a_{i-l}^l d_i e_{i-l}) e_{i-l},
\]

using (21), we obtain \( a_{i-1}^l \) and

\[
w_{i-l-1} a_{i-l-1}^l = (-1)^{l+1} \frac{\prod_{m=1}^{l} d_{i-m}}{\prod_{m=1}^{l+1} w_{i+m+1}} = (-1)^{l+1} \frac{\prod_{m=1}^{l} d_{i-m}}{\prod_{m=1}^{l} w_{i-m}}. \quad (24)
\]
Also

\[ d_{i-1}a_{i-1} = (-1)^{i+1} \frac{\prod_{m=1}^{l} d_{i-m}}{\prod_{m=1}^{l} w_{i-m}}, \]  

(25)

which leads to

\[ w_{i-1}a_{i-1} + d_{i-1}a_{i-1} = 0, \]

therefore \((T \circ F) \epsilon = \epsilon_i\) and we have

\[ T \circ F = F \circ T = I, \]

then the claim is proved. □

**Theorem 1.** Let \( T \in \mathcal{L}(X) \) be the operator given by (I) and for any \( \lambda \in \mathbb{C} \), \( R_{S+D}^+(\lambda) \), \( R_{S+D}^-(\lambda) \) are given by

\[
R_{S+D}^+(\lambda) = \lim_{k \to \infty} \left[ \sup_{n \in \mathbb{Z}} \left| \frac{\prod_{m=0}^{k} d_{i+m}}{\prod_{m=0}^{k} (d_{i+m} - \lambda)} \right| \right]^{\frac{1}{k}},
\]

and

\[
R_{S+D}^-(\lambda) = \lim_{k \to \infty} \left[ \sup_{n \in \mathbb{Z}} \left| \frac{\prod_{m=0}^{k} (d_{i+m} - \lambda)}{\prod_{m=0}^{k} w_{i+m}} \right| \right]^{\frac{1}{k}}.
\]

(i) If \( S \) is an invertible operator, then

\[
\sigma(T) = \{ \lambda \in \mathbb{C} : R_{S+D}^+(\lambda) \geq 1 \text{ and } R_{S+D}^-(\lambda) \geq 1 \};
\]

(28)

(ii) if \( S \) is a non-invertible operator, then

\[
\sigma(T) = \{ \lambda \in \mathbb{C} : R_{S+D}^+(\lambda) \geq 1 \},
\]

(29)

Proof. Let \( \lambda \in \rho(T) = \{ \lambda \in \mathbb{C} : (T - \lambda I)^{-1} \in \mathcal{L}(X) \} \). If we replace \( d_j \) by \( d_j - \lambda \) in the Lemma 1, then we get either \( R_{S+D}^+(\lambda) \leq 1 \) or \( R_{S+D}^-(\lambda) \leq 1 \). But the equality is excluded by spectrum compactness. So we have at least

\[
R_{S+D}^+(\lambda) < 1
\]

(30)

or

\[
R_{S+D}^-(\lambda) < 1.
\]

(31)

If \( S \) is invertible, then from (11), only one of inequality (30) and (31) can be satisfied. Thus,

\[
\{ \lambda \in \mathbb{C} : R_{S+D}^+(\lambda) \geq 1 \text{ and } R_{S+D}^-(\lambda) \geq 1 \} \subset \sigma(T).
\]

Therefore, if \( S \) is non invertible, \( \sup_{n \in \mathbb{Z}} \left| \frac{\prod_{m=0}^{k} (d_{i+m} - \lambda)}{\prod_{m=0}^{k} w_{i+m}} \right| \) is not bounded and we have only \( R_{S+D}^+(\lambda) < 1 \). So,

\[
\{ \lambda \in \mathbb{C} : R_{S+D}^+(\lambda) \geq 1 \} \subset \sigma(T).
\]
Conversely, in order to show that
\[ \sigma(T) \subset \{ \lambda \in \mathbb{C} : R_{S+D}^+(\lambda) \geq 1 \text{ and } R_{S}^- (\lambda) \geq 1 \}, \]
we take \( \lambda \in \mathbb{C} \), such that
\[ R_{S+D}^+(\lambda) < 1 \]  \hspace{1cm} (32)

or
\[ R_{S+D}^-(\lambda) < 1 \]  \hspace{1cm} (33)

and we show that \( \lambda \notin \sigma(T) \). From Lemma 2, \( T - \lambda I \) is invertible and there will exist an operator \( (T - \lambda I)^{-1} \in \mathcal{L}(X) \) such that
\[ I = (T - \lambda I)^{-1}(T - \lambda I) = (T - \lambda I)(T - \lambda I)^{-1}. \]  \hspace{1cm} (34)

Therefore, \( \lambda \notin \sigma(T) \). Similarly one can show that if \( S \) is not invertible, then
\[ \sigma(T) \subset \{ \lambda \in \mathbb{C} : R_{S+D}^+(\lambda) \geq 1 \}. \]

\[ \square \]

Remark 3. In the previous theorem, if we take \( d_i = 0 \) for all \( i \in \mathbb{Z} \), then we obtain a result already shown in [1, 2, 8, 9] about the spectrum of the operator \( S \). That is
\[ \sigma(S) = \left\{ \lambda \in \mathbb{C} : \frac{1}{r(S^{-1})} \leq |\lambda| \leq r(S) \right\}. \]

5. Remark about the spectrum of perturbed weighted \( n \)-shift

For a strictly positive integer \( n \), we define the weighted \( n \)-shift operator in \( X \) by
\[ S_n e_i = w_i e_{i+n}, \quad i = 0, \pm 1, \pm 2, ... \]
The sequence \( \{w_i\}_{i \in \mathbb{Z}} \subset \mathbb{C} \) represents the weights of the operator \( S_n \). It is clear that the weighted 1-shift coincide with weighted shift (in the usual sense, see [10]).

Remark 4. Let \( T_n \in \mathcal{L}(X) \) be the operator given by
\[ T_n = S_n + D, \]  \hspace{1cm} (35)

where \( D \) is a diagonal operator defined in (1). For every \( j \in \{0, ..., n-1\} \) and \( i \in \mathbb{Z} \), let that \( e^j_i := e_{i+in}, w^j_i := w_{j+in} \) and \( S^j_n e^j_i = w^j_i e^j_{i+1} \). Where \( S^j_n \) is the restriction of \( S_n \) on \( X_j \), the \( S_n \)-invariant closed linear subspace spanned by \( \{e^j_i : i \in \mathbb{Z}\} \).

Note that
\[ X = X_0 \oplus X_1 \oplus \cdots \oplus X_{n-1} \]
and
\[ S_n = S^0_n \oplus S^1_n \oplus \cdots \oplus S^{n-1}_n \]

Also, since each \( S^j_n \) is a weighted 1-shift, then the spectra of \( S_n \) is the union of the spectra of all \( S^j_n \), \( j = 0, ..., n-1 \) (see [4]). In particular,
\[ \sigma(S_n) = \sigma(S^0_n) \cup \sigma(S^1_n) \cup \cdots \cup \sigma(S^{n-1}_n). \]

Moreover, if we denote by \( D^j \) the restriction of \( D \) to \( X_j \), then
\[ S_n + D = (S^0_n + D^0) \oplus (S^1_n + D^1) \oplus \cdots \oplus (S^{n-1}_n + D^{n-1}) \]
and thus
\[ \sigma(S_n + D) = \sigma(S^0_n + D^0) \cup \sigma(S^1_n + D^1) \cup \cdots \cup \sigma(S^{n-1}_n + D^{n-1}) \]
Furthermore, for \( j \in \{0, ..., n - 1\} \), \( R^+_j(\lambda) \) and \( R^-_j(\lambda) \) are given by

\[
R^+_j(\lambda) = \lim_{k \to \infty} \left[ \sup_{i \in \mathbb{Z}} \left| \frac{\prod_{m=0}^{k-1} w_{j+(i+m)n}}{\prod_{m=0}^{k} (d_{j+(i+m)n} - \lambda)} \right|^{\frac{1}{k}} \right]
\] 

(36)

and

\[
R^-_j(\lambda) = \lim_{k \to \infty} \left[ \sup_{i \in \mathbb{Z}} \left| \frac{\prod_{m=0}^{k} (d_{j+(i+m)n} - \lambda)}{\prod_{m=0}^{k} w_{j+(i-m)n}} \right|^{\frac{1}{k}} \right].
\] 

(37)

By Theorem 1, if \( S^j_n \) is invertible operator, then we have

\[
\sigma(S^j_n + D^j) = \{ \lambda \in \mathbb{C} : R^+_j(\lambda) \geq 1 \text{ and } R^-_j(\lambda) \geq 1 \},
\]

and if \( S^j_n \) is non-invertible operator then

\[
\sigma(S^j_n + D^j) = \{ \lambda \in \mathbb{C} : R^+_j(\lambda) \geq 1 \}.
\]

REFERENCES

[1] A. Bourhim, Spectrum of Bilateral Shifts with Operator-Valued Weights, Proc. Amer Math Soc, Vol. 134, No. 7 (2006).
[2] J. B. Conway, A Course in Functional Analysis. Springer, 1997.
[3] H.R. Dowson, Spectral Theory of Linear Operators, Academic Press, London New York San Francisco 1978.
[4] P. Halmos, A Hilbert Space Problem Book, Second Edition, Springer-Verlag, New York, 1982.
[5] P. D. Hislop and I. M. Sigal, Introduction to spectral theory: With applications to Schrödinger operators (Vol. 113). Springer Science & Business Media, 2012.
[6] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, 1980.
[7] G. Krishna Kumar and S. H. Lui, On some properties of the pseudospectral radius, Electronic Journal of Linear Algebra, Volume 27. (2014)
[8] W. C. Ridge, Approximate point spectrum of a weighted shift, Trans. Amer. Math. Soc. 147 (1970).
[9] A.L. Shields, Weighted Shift Operators and Analytic Function Theory. Math. Surveys. Vol 13, Amer. Math. Soc., Providence, R.I., 1974.
[10] J. Stochel and F. H. Szafraniec, Unbounded weighted shifts and subnormality, J. Funct. Anal. 159 (1998), 432-491.

M.L. Sahari: LANOS Laboratory, Department of Mathematics, Badji Mokhtar-Annaba University, P. O. Box 12, 23000 Annaba, Algeria
E-mail address: mohamed-lamine.sahari@univ-annaba.dz; mlsahari@yahoo.fr

A.K. Taha: INSA, University of Toulouse, 135 Avenue de Rangueil, 31077 Toulouse Cedex 4, France
E-mail address: taha@insa-toulouse.fr

L. Randriamihamon: IPST-Cnam, Institut National Polytechnique de Toulouse, University of Toulouse, 118, route de Narbonne, 31062 Toulouse Cedex 9, France
E-mail address: louis.randriamihamon@ipst.fr