Linear Continuous Functionals on $\text{FN}$-Type Spaces

Sorin G. Gal
Department of Mathematics and Computer Sciences
University of Oradea, Romania
410087 Oradea, Romania
E-mail: galso@uoradea.ro

Abstract

By using the space of fuzzy numbers, in e.g. [5] have been considered several complete metric spaces (called here $\text{FN}$-type spaces) endowed with addition and scalar multiplication, such that the metrics have nice properties but the spaces are not linear, i.e. are not groups with respect to addition and the scalar multiplication is not, in general, distributive with respect to usual scalar addition. This paper deals with the form of linear continuous functionals defined on these spaces.

Keywords: fuzzy numbers, fuzzy-number-valued functions, $\text{FN}$-type space, linear and continuous functionals.

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1 INTRODUCTION

By using the space of fuzzy numbers, in [5] have been considered several complete metric spaces endowed with addition, and scalar multiplication, such that their
metrics have nice properties but the spaces are not linear, i.e. are not groups with respect to addition and the scalar multiplication is not, in general, distributive with respect to usual scalar addition. In Section 2 we recall some properties of these spaces and introduce the concept of abstract fuzzy-number-type (shortly FN-type) space. Section 3 contains the main results of the paper and deals with the form of linear and continuous functionals defined on the FN-type spaces in Section 2.

2 PRELIMINARIES

In this section we recall the main properties of the space of fuzzy numbers and of some other spaces based on it, all called as Fuzzy-Number-type (shortly FN-type) spaces, which have similar properties.

Given a set $X \neq \emptyset$, a fuzzy subset of $X$ is a mapping $u : X \to [0, 1]$ and obviously any classical subset $A$ of $X$ can be identified as a fuzzy subset of $X$ defined by $\chi_A : X \to [0, 1], \chi_A(x) = 1$ if $x \in A, \chi_A(x) = 0$ if $x \in X \setminus A$. If $u : X \to [0, 1]$ is a fuzzy subset of $X$, then for $x \in X$, $u(x)$ is called the membership degree of $x$ to $u$ (see e.g. [14]).

**DEFINITION 2.1** (see e.g. [4], [13]) The space of fuzzy numbers denoted by $\mathbb{R}_F$ is defined as the class of fuzzy subsets of the real axis $\mathbb{R}$, i.e. of $u : \mathbb{R} \to [0, 1]$, having the following four properties:

1. $\forall u \in \mathbb{R}_F$, $u$ is normal, i.e. $\exists x_u \in \mathbb{R}$ with $u(x_u) = 1$;
2. $\forall u \in \mathbb{R}_F$, $u$ is a convex fuzzy set, i.e.

   $$ u(tx + (1 - t)y) \geq \min\{u(x), u(y)\}, \forall t \in [0, 1], x, y \in \mathbb{R}; $$

3. $\forall u \in \mathbb{R}_F$, $u$ is upper-semi-continuous on $\mathbb{R}$;
4. $\{x \in \overline{M} : u(x) > 0\}$ is compact, where $\overline{M}$ denotes the closure of $M$.

**REMARKS.** 1) Obviously, we can consider that $\mathbb{R} \subset \mathbb{R}_F$, because any real number $x_0 \in \mathbb{R}$ can be identified with $\chi_{\{x_0\}}$, which satisfies the properties $(i) - (iv)$ in Definition 2.1.

2) For $0 < r \leq 1$ and $u \in \mathbb{R}_F$, let us denote by $[u]^r = \{x \in \mathbb{R} : u(x) \geq r\}$
and \([u]^0 = \{ x \in \mathbb{R}; u(x) > 0 \}\), the so-called level sets of \(u\). Then it is an immediate consequence of \((i)-(iv)\) that \([u]^r\) represents a bounded closed (i.e. compact) subinterval of \(\mathbb{R}\), denoted by \([u]^r = [u_-(r), u_+(r)]\), where \(u_-(r) \leq u_+(r)\) for all \(r \in [0, 1]\). Also, by e.g. [10], [13], \(u_-(r)\) is bounded nondecreasing on \([0, 1]\), \(u_+(r)\) is bounded non-increasing on \([0, 1]\), both are left continuous on \((0, 1]\) and right continuous at \(r = 0\), (from monotonicity both have right limit at each point in \([0, 1]\) ), \(u_+(0) - u_+(r) \geq 0, u_-(1) - u_-(r) \geq 0, u_-(r) \leq u_+(r), \forall r \in [0, 1]\) and \(\mathbb{R}_F\) can be embeded into the Banach space \(B = \overline{C}[0, 1] \times \overline{C}[0, 1]\), by the mapping \(j(u) = (u_-, u_+), \forall u \in \mathbb{R}_F\), where \(\overline{C}[0, 1]\) denotes the Banach space of all real-valued bounded functions \(f : [0, 1] \rightarrow \mathbb{R}\), which are left continuous at each point in \((0, 1]\), have right limit at each point in \([0, 1]\), \(f\) is right continuous at 0, endowed with the uniform norm \(||f|| = \sup\{ |f(x)|; x \in [0, 1] \}\) and the product space \(B\) is considered to be endowed with the norm \(||(f, g)|| = \max\{||f||, ||g||\}\).

Also, it is important the following "converse" result.

**THEOREM 2.2** (see e.g. [9] or [13, Lemma 1.1]) If \(\{M_r; r \in [0, 1]\}\) is a family of closed subintervals of real axis with the properties :

(i) \(M_r \subset M_s, \forall r, s \in [0, 1], s \leq r,\)
(ii) for any sequence \((r_n)_{n \in \mathbb{N}}\) converging increasingly to \(r \in (0, 1]\), we have \(\bigcap_{n=1}^{\infty} M_{r_n} = M_r,\)
then there exists a unique \(u \in \mathbb{R}_F\), such that \(M_r = [u_-(r), u_+(r)], \forall r \in (0, 1]\) and \([u]^0 \subset M_0.\)

**DEFINITION 2.3** (see e.g. [4], [13]) The addition and the product with real scalars in \(\mathbb{R}_F\) are defined by \(\oplus : \mathbb{R}_F \times \mathbb{R}_F \rightarrow \mathbb{R}_F,\)

\[(u \oplus v)(x) = \sup_{y+z=x} \min\{ u(y), v(z) \}\]

and by \(\odot : \mathbb{R} \times \mathbb{R}_F \rightarrow \mathbb{R}_F,\)

\[(\lambda \odot v)(x) = \begin{cases} u \left( \frac{x}{\lambda} \right) & \text{if } \lambda \neq 0 \\ \tilde{0} & \text{if } \lambda = 0 \end{cases},\]

where \(\tilde{0} : \mathbb{R} \rightarrow [0, 1]\) is \(\tilde{0} = \chi_{\{0\}}.\)
Also, we can write \([u \oplus v]^r = [u]^r + [v]^r\), \([\lambda \odot v]^r = \lambda [v]^r\), for all \(r \in [0, 1]\), where \([u]^r + [v]^r\) means the usual sum of two intervals (as subsets of \(\mathbb{R}\)) and \(\lambda [v]^r\) means the usual product between a real scalar and a subset of \(\mathbb{R}\).

If we define \(D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R} \cup \{0\}\) by

\[
D(u, v) = \sup_{r \in [0,1]} \max \{ |u_-(r) - v_-(r)|, |u_+(r) - v_+(r)| \},
\]

where \([u]^r = [u_-(r), u_+(r)]\), \([v]^r = [v_-(r), v_+(r)]\), then we have the following:

**THEOREM 2.4** (see e.g. [10], [13]). \((\mathbb{R}_F, D)\) is a complete metric space and in addition, \(D\) has the following three properties:

(i) \(D(u \oplus w, v \oplus w) = D(u, v), \) for all \(u, v, w \in \mathbb{R}_F\);

(ii) \(D(k \odot u, k \odot v) = |k| D(u, v), \) for all \(u, v \in \mathbb{R}_F, k \in \mathbb{R}\);

(iii) \(D(u \odot v, w \odot e) \leq D(u, w) + D(v, e), \) for all \(u, v, w, e \in \mathbb{R}_F\).

Also, the following result is known:

**THEOREM 2.5** (see e.g. [1], [4]).

(i) \(u \odot v = v \odot u, u \odot (v \odot w) = (u \odot v) \odot w;\)

(ii) If we denote \(\tilde{0} = \chi_{\{0\}}\), then \(u \oplus \tilde{0} = \tilde{0} \oplus u = u, \) for any \(u \in \mathbb{R}_F;\)

(iii) With respect to \(\tilde{0}\), none of \(u \in \mathbb{R}_F \setminus \mathbb{R}\) has an opposite element (regarding \(\oplus\) in \(\mathbb{R}_F;\))

(iv) For any \(a, b \in \mathbb{R}\) with \(a, b \geq 0\) or \(a, b \leq 0\) and any \(u \in \mathbb{R}_F, \) we have

\[(a + b) \odot u = a \odot u \oplus b \odot u.\]

For general \(a, b \in \mathbb{R}, \) the above property does not hold.

(v) \(\lambda \odot (u \odot v) = \lambda \odot u \odot \lambda \odot v, \) for all \(\lambda \in \mathbb{R}, u, v \in \mathbb{R}_F;\)

(vi) \(\lambda \odot (\mu \odot u) = (\lambda \mu) \odot u, \) for all \(\lambda, \mu \in \mathbb{R}, u \in \mathbb{R}_F;\)

(vii) If we denote \(\|u\|_F = D(u, \tilde{0}), u \in \mathbb{R}_F, \) then \(\|u\|_F\) has the properties of an usual norm on \(\mathbb{R}_F, \) i.e. \(\|u\|_F = 0\) iff \(u = \tilde{0}, \|\lambda \odot u\|_F = |\lambda| \|u\|_F, \|u + v\|_F \leq \|u\|_F + \|v\|_F, \|u\|_F - \|v\|_F \leq D(u, v);\)

(viii) \(D(\alpha \odot u, \beta \odot u) = |\alpha - \beta| D(\tilde{0}, u), \) for all \(\alpha, \beta \geq 0, u \in \mathbb{R}_F. \) If \(\alpha, \beta \leq 0\) then the equality is also valid. If \(\alpha\) and \(\beta\) are of opposite signs, then the equality is not valid.
REMARKS. 1) Theorem 2.5 shows that \( (\mathbb{R}, \oplus, \odot) \) is not a linear space over \( \mathbb{R} \) and consequently \( (\mathbb{R}, \|\cdot\|) \) cannot be a normed space.

2) On \( \mathbb{R}_F \) we also can define a substraction \( \ominus \), called \( H \)-difference (see e.g. [3]) as follows: \( u \ominus v \) has sense if there exists \( w \in \mathbb{R}_F \) such that \( u = v + w \). Obviously, \( u \ominus v \) does not exist for all \( u, v \in \mathbb{R}_F \) (for example, \( \tilde{0} \ominus v \) does not exists if \( v \neq \tilde{0} \)).

In what follows, we define some usual spaces of fuzzy-number-valued functions, which have similar properties to \( (\mathbb{R}_F, D) \).

Denote \( C([a, b]; \mathbb{R}_F) = \{ f : [a, b] \to \mathbb{R}_F; \ f \ \text{is continuous on } [a, b] \} \), endowed with the metric \( D^* (f, g) = \sup \{ D(f(x), g(x)) ; x \in [a, b] \} \). Because \( (\mathbb{R}_F, D) \) is a complete metric space, by standard technique we obtain that \( (C([a, b]; \mathbb{R}_F), D^*) \) is a complete metric space. Also, if we define \( (f \oplus g)(x) = f(x) + g(x), (\lambda \odot f)(x) = \lambda \odot f(x) \) (for simplicity, the addition and scalar multiplication in \( \mathbb{R}_F \) are denoted as in \( \mathbb{R}_F \)), also \( \tilde{0} : [a, b] \to \mathbb{R}_F, \tilde{0}(t) = \tilde{0}_{\mathbb{R}_F} \), for all \( t \in [a, b] \),

\[
\|f\|_F = \sup \left\{ D\left(\tilde{0}, f(x)\right) ; x \in [a, b] \right\},
\]

then we easily obtain the following properties.

THEOREM 2.6 (see [5]) (i) \( f \oplus g = g \oplus f, (f \oplus g) \oplus h = f \oplus (g \oplus h) \);

(ii) \( f \oplus \tilde{0} = \tilde{0} \oplus f \), for any \( f \in C([a, b]; \mathbb{R}_F) \);

(iii) With respect to \( \tilde{0} \) in \( C([a, b]; \mathbb{R}_F) \), any \( f \in C([a, b]; \mathbb{R}_F) \) with \( f([a, b]) \cap \mathbb{R}_F \neq \emptyset \) has no an opposite member (regarding \( \oplus \)) in \( C([a, b]; \mathbb{R}_F) \);

(iv) for all \( \lambda, \mu \in \mathbb{R} \) with \( \lambda, \mu \geq 0 \) or \( \lambda, \mu \leq 0 \) and for any \( f \in C([a, b]; \mathbb{R}_F) \),

\[
(\lambda + \mu) \odot f = (\lambda \odot f) \odot (\mu \odot f);
\]

For general \( \lambda, \mu \in \mathbb{R} \), this property does not hold.

(v) \( \lambda \odot (f \oplus g) = \lambda \odot f \oplus \lambda \odot g, \lambda \odot (\mu \odot f) = (\lambda \mu) \odot f \), for any \( f, g \in C([a, b]; \mathbb{R}_F) \), \( \lambda, \mu \in \mathbb{R} \);

(vi) \( \|f\|_F = 0 \iff f = \tilde{0}, \|\lambda \odot f\|_F = |\lambda| \|f\|_F, \|f \oplus g\|_F \leq \|f\|_F + \|g\|_F, \|f\|_F - \|g\|_F \leq D^* (f, g), \) for any \( f, g \in C([a, b]; \mathbb{R}_F) \), \( \lambda \in \mathbb{R} \);

(vii) \( D^* (\lambda \odot f, \mu \odot f) = |\lambda - \mu| D^* \left(\tilde{0}, f\right), \) for any \( f \in C([a, b]; \mathbb{R}_F) \), \( \lambda \mu \geq 0 \);
(viii) 
\[ D^* (f \oplus h, g \oplus h) = D^* (f, g), \]
\[ D^* (\lambda \odot f, \lambda \odot g) = |\lambda| D^* (f, g), \]
\[ D^* (f \oplus g, h \oplus e) \leq D^* (f, h) + D^* (g, e), \]

for any \( f, g, h, e \in C ([a, b]; \mathbb{R}_F) \), \( \lambda \in \mathbb{R} \).

**REMARK.** It is easy to show that if \( f, g \in C ([a, b]; \mathbb{R}_F) \), then \( F : [a, b] \to \mathbb{R} \), defined by \( F (x) = D (f (x), g (x)) \) is continuous on \([a, b]\).

Now, for \( 1 \leq p < \infty \), let us define

\[ L^p ([a, b]; \mathbb{R}_F) = \left\{ f \text{ is strongly measurable on } [a, b] \text{ and } \int_a^b \left( D \left( \bar{0}, f (x) \right) \right)^p dx < +\infty \right\}, \]

where according to e.g. [8], \( f \) is called strongly measurable if, for each \( x \in [a, b] \), \( f_- (x) (r) \) and \( f_+ (x) (r) \) are Lebesgue measurable as functions of \( r \in [0, 1] \) (here \( [f (x)]^r = [f_- (x) (r), f_+ (x) (r)] \) denotes the \( r \)-level set of \( f (x) \in \mathbb{R}_F \)). The following result shows that \( L^p ([a, b]; \mathbb{R}_F) \) is well defined.

**THEOREM 2.7** (see [5]) (i) If \( f : [a, b] \to \mathbb{R}_F \) is strongly measurable then \( F : [a, b] \to \mathbb{R}_+ \) defined by \( F (x) = D \left( \bar{0}, f (x) \right) \) is Lebesgue measurable on \([a, b]; \)

(ii) For any \( f, g \in L^p ([a, b]; \mathbb{R}_F) \), \( F (x) = D (f (x), g (x)) \) is Lebesgue measurable and \( L^p \)-integrable on \([a, b] \). Moreover, if we define

\[ D_p (f, g) = \left\{ (L) \int_a^b [D (f (x), g (x))]^p dx \right\}^{\frac{1}{p}}, \]

then \( (L^p ([a, b]; \mathbb{R}_F), D_p) \) is a complete metric space (where \( f = g \) means \( f (x) = g (x) \), a.e. \( x \in [a, b] \)) and, in addition, \( D_p \) satisfies the following properties:

\[ D_p (f \oplus h, g \oplus h) = D_p (f, g), \]
\[ D_p (\lambda \odot f, \lambda \odot g) = |\lambda| D_p (f, g), \]
\[ D_p (f \oplus g, h \oplus e) \leq D_p (f, h) + D_p (g, e), \]
for any \( f, g, h, e \in \mathcal{L}^p ([a, b]; F) \), \( \lambda \in \mathbb{R} \).

Other spaces with properties similar to those of \((R_F, D)\) can be constructed as follows (see [5]).

For \( p \geq 1 \), let us define

\[
\ell^p_{R_F} = \left\{ x = (x_n)_n ; x_n \in R_F, \forall n \in \mathbb{N} \quad \text{and} \quad \sum_{n=1}^{\infty} \|x_n\|_{R_F}^p < +\infty \right\},
\]

endowed with the metric

\[
\rho_p (x, y) = \left\{ \sup_{n=1}^{\infty} [D(x_n, y_n)]^p \right\}^{1/p}, \forall x = (x_n)_n, y = (y_n)_n \in \ell^p_{R_F}.
\]

By \( D(x_n, y_n) = D(x_n \oplus \tilde{0}, \tilde{0} \oplus y_n) \leq D(x_n, \tilde{0}) + D(\tilde{0}, y_n) = \|x_n\|_F + \|y_n\|_F \), we easily get (by Minkowski’s inequality if \( p > 1 \)) \( \rho_p (x, y) < +\infty \). Also, it easily follows that \( \rho_p (x, y) \) is a metric with similar properties to \( D \) (see Theorem 2.4 and Theorem 2.5,(vii),(viii)).

Because \((R_F, D)\) is a complete metric space, by the standard technique, we easily get that \((\ell^p_{R_F}, \rho_p)\) is also a complete metric space.

Let us denote by

\[
m_{R_F} = \{ x = (x_n)_n ; x_n \in R_F, \forall n \in \mathbb{N} \quad \text{and} \quad \exists M > 0 \quad \text{such that} \quad \|x_n\|_F \leq M, \forall n \in \mathbb{N} \},
\]

endowed with the metric

\[
\mu(x, y) = \sup \{ D(x_n, y_n) ; \forall n \in \mathbb{N} \}.
\]

We easily get that \((m_{R_F}, \mu)\) is a complete metric space and, in addition, \( \mu \) has similar properties to \( D \) (see Theorem 2.4 and Theorem 2.5,(vii),(viii)). Similarly, if we denote

\[
c_{R_F} = \{ x = (x_n)_n ; x_n \in R_F, \forall n \in \mathbb{N} \quad \text{and} \quad \exists a \in R_F \quad \text{such that} \quad D(x_n, a) \xrightarrow{n \to \infty} 0 \}\]

and

\[
\hat{c}_{R_F} = \{ x = (x_n)_n ; x_n \in R_F, \forall n \in \mathbb{N} \quad \text{such that} \quad D(x_n, \tilde{0}) \xrightarrow{n \to \infty} 0 \},
\]

endowed with the metric
since \((R_F, D)\) is complete, by standard technique, it follows that \((c_{R_F}, \mu)\) and 
\((c_{\tilde{R}_F}, \mu)\) are complete metric spaces.

**REMARK.** Let \((X, \oplus, \odot, d)\) be represent any space from 
\((R_F, D), (l^p_{R_F}, \rho_p), (m_{R_F}, \mu); (c_{R_F}, \mu); (c_{\tilde{R}_F}, \mu); (L^p ([a, b]; R_F), D_p), 1 \leq p < \infty, (C ([a, b]; R_F), D^*)\), or any finite cartesian product of them. The properties in
Theorems 2.4, 2.5, 2.6, 2.7, suggest us in a natural way the following concept of
abstract space.

**DEFINITION 2.8** We say that \((X, \oplus, \odot, d)\) is a fuzzy-number type space
(shortly \(FN\)-type space), if the following properties are satisfied :

(i) \((X, d)\) is a metric space (complete or not) and \(d\) has the properties in Theorem
2.4, (i)-(iii) (where \(R_F\) is replaced by \(X\) and \(D\) by \(d\));

(ii) The operations \(\oplus, \odot\) on \(X\) have the properties in Theorem 2.5, (i),(iv),(v),(vi)
(where \(R_F\) is replaced by \(X\) ) ;

(iii) There exists a neutral element \(\tilde{0} \in X, \text{i.e. } u \oplus \tilde{0} = \tilde{0} \oplus u = u, \text{for any } u \in X \text{ and a linear subspace } Y \subset X \text{ (with respect to } \oplus \text{ and } \odot), \text{non-dense in } X, \text{such that}
\text{with respect to } \tilde{0}, \text{none of } u \in X \setminus Y \text{ has an opposite member (regarding } \oplus \text{) in } X.\)

**REMARK.** A \(FN\)-type space obviously is a more general structure that that of
Banach or Fréchet space, because it is not a linear space. However, due to the nice
properties of the metric \(d\), very many results (especially those of quantitative kind)
valid for Banach (or Fréchet) spaces, can be extended to this case too. For example,
the theories of almost periodic and almost automorphic functions with values \(FN\)-
type spaces were developed in [2] and [7], respectively. Also, the \(FN\)-type spaces
have recent applications to the study of fuzzy differential equations, which model
the real world’s problems governed by imprecision due to uncertainty or vagueness
rather than randomness. In this sense, we mention for example [5] and [6], where
basic elements of the theory of semigroups of operators on \(FN\)-type spaces with
applications in solving fuzzy partial differential equations are considered. Of course
that the theory of semigroups of operators requires basic elements of operator theory
on these spaces, as for example, the followings.
**DEFINITION 2.9** ([5]) $A : X \to \mathbb{R}$ is a linear functional if

\[
\begin{align*}
A(x \oplus y) & = A(x) + A(y), \\
A(\lambda \odot x) & = \lambda A(x),
\end{align*}
\]

for all $x, y \in X, \lambda \in \mathbb{R}$.

**REMARK.** If $A : X \to \mathbb{R}$ is linear and continuous at $\tilde{0} \in X$, then this does not imply the continuity of $A$ at each $x \in X$, because we cannot write $x_0 = (x_0 \ominus x) \oplus x$, in general, (the difference $x_0 \ominus x$ does not always exist).

However, we can prove the following theorem.

**THEOREM 2.10** ([5]) *If $A : X \to \mathbb{R}$ is linear, then it is continuous at $\tilde{0} \in X$, if and only if there exists $M > 0$ such that*

\[
|A(x)| \leq M \|x\|_F, \forall x \in X,
\]

*where $\|x\|_F = d\left(\tilde{0}, x\right)$.*

Now, for $A : X \to \mathbb{R}$ linear and continuous at $\tilde{0}$, let us denote by

\[
\mathcal{M}_A = \{M > 0; |A(x)| \leq M \|x\|_F, \forall x \in X\},
\]

Also, denote $\|A\|_F = \inf \mathcal{M}_A$.

We have

**THEOREM 2.11** (see [5]) *If $A : X \to \mathbb{R}$ is linear and continuous at $\tilde{0}$, then*

\[
|A(x)| \leq \|A\|_F \|x\|_F
\]

*for all $x \in X$ and*

\[
\|A\|_F = \sup \{ |A(x)|; x \in X, \|x\|_F \leq 1 \}.
\]

**COROLLARY 2.12** (see [5]) *If $A : X \to \mathbb{R}$ is additive (i.e. $A(x \oplus y) = A(x) + A(y)$), positive homogeneous (i.e. $A(\lambda \odot y) = \lambda A(x), \forall \lambda \geq 0$) and continuous at $\tilde{0}$, then*

\[
|A(x)| \leq \|A\|_F \|x\|_F, \forall x \in X.
\]
Also, the following uniform boundedness principle holds.

**THEOREM 2.13** (see [5]) Let \((X, \oplus, \cdot, d)\) be a **FN**-type space and \(L(X)\) be any from the spaces

\[
\mathcal{L}^+(X) = \{A \in \mathcal{L}^+_0(X) : A \text{ is continuous at each } x \in X\}, \\
\mathcal{L}(X) = \{A \in \mathcal{L}_0(X) : A \text{ is continuous at each } x \in X\},
\]

where

\[
\mathcal{L}^+_0(X) = \{A : X \to X; A \text{ is additive, positive homogeneous and continuous at } \tilde{0}\}, \\
\mathcal{L}_0(X) = \{A : X \to X; A \text{ is linear and continuous at } \tilde{0}\}.
\]

If \(A_j \in \mathcal{L}(X), j \in J\), is pointwise bounded, i.e. for any \(x \in X\), \(\|A_j(x)\|_F = d(A_j(x), \tilde{0}) \leq M_x\), for all \(j \in J\), then there exists \(M > 0\) such that

\[
\|A_j\|_F \leq M, \forall j \in J,
\]

(i.e. \((A_j)\) is uniformly bounded).

**REMARK.** It is worth to note that not all the results in operator theory on Banach spaces can be extended to **FN**-type spaces (see [5]).

### 3 FORMS OF THE LINEAR CONTINUOUS FUNCTIONALS

In this section we deal with the form of linear and continuous functionals defined on the **FN**-type spaces introduced by Section 2. The first main result is the following.

**THEOREM 3.1** \(x^* : \mathbb{R}_F \to \mathbb{R}\) is a linear continuous functional on \(\mathbb{R}_F\), if and only if there exists a linear functional \(L : \overline{C}[0,1] \to \mathbb{R}\), such that

\[
x^*(x) = L(x_- + x_+), \forall x \in \mathbb{R}_F,
\]

where \(x_-\) and \(x_+\) are the functions given by the formula \([x]^r = [x_- (r), x_+ (r)]\), (see Remark 2 after Definition 2.1) and \(L|_{\overline{C}[0,1]}\) is continuous with respect to the uniform
convergence on $IC[0, 1] = \{ f \in \overline{C}[0, 1] : f \text{ is increasing on } [0, 1] \}$. (Here $L|_A$ denotes the restriction of $L$ to $A$.)

**Proof.** Let $L$ be as in the statement and define $x^*(x) = L(x_+ + x_-), \forall x \in \mathbb{R}_F$.

First we prove that $x^*$ is linear.

Let $x, y \in \mathbb{R}_F$. From the obvious relations $(x \oplus y)_- = x_- + y_-, (x \oplus y)_+ = x_+ + y_+$ and the linearity of $L$, we immediately get $x^*(x \oplus y) = x^*(x) + x^*(y)$.

Let $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}_F$. If $\alpha \geq 0$ then by the obvious relations $(\alpha \odot x)_- = \alpha(x)_-, (\alpha \odot x)_+ = \alpha(x)_+$ and the linearity of $L$, we easily obtain $x^*(\alpha \odot x) = \alpha x^*(x)$.

If $\alpha < 0$ then by the relations $(\alpha \odot x)_- = \alpha(x)_+, (\alpha \odot x)_+ = \alpha(x)_-$ and the linearity of $L$, we again arrive at the same conclusion $x^*(\alpha \odot x) = \alpha x^*(x)$.

Next we prove that $x^*$ is continuous. For that let $x_n, x \in \mathbb{R}_F$, $n \in \mathbb{N}$ be such that $\lim_{n \to \infty} D(x_n, x) = 0$. From the definition of the metric $D$, this is obviously equivalent to

$$\lim_{n \to \infty} ||x_n^- - x_-|| = \lim_{n \to \infty} ||x_n^+ - x_+|| = 0,$$

where $|| \cdot ||$ denotes the uniform norm on $\overline{C}[0, 1]$ and $x_n^- = (x_n)_-, x_n^+ = (x_n)_+$. By the definition of $x^*$, since $L$ is linear we can write

$$x^*(x_n) = L(x_n^-) - L(-x_n^+),$$

where obviously $x_n^-, x_n^+ \in IC[0, 1], \forall n \in \mathbb{N}$. Passing to limit with $n \to \infty$ and taking into account that by hypothesis $L|_{IC[0, 1]}$ is continuous with respect to the uniform convergence on $IC[0, 1]$, we immediately get $x^*(x_n) \to L(x_-) - L(-x_+) = L(x_- + x_+) = x^*(x)$.

Conversely, let $x^* : \mathbb{R}_F \to \mathbb{R}$ be linear and continuous on $\mathbb{R}_F$.

Let us consider the set

$$A = \{ u \in \overline{C}[0, 1] ; \text{ there exist } f, g \in \overline{C}[0, 1], f \leq g, f \text{ is increasing on } [0, 1],$$

$$g \text{ is decreasing on } [0, 1], \text{ such that } u = f + g \}. $$

The set $A$ is a linear subspace of $\overline{C}[0, 1]$. Indeed, for $u, v \in A$, $u = f + g, v = h + l$ we have $u + v = (f + h) + (g + l)$, where by hypothesis we easily obtain $f + h \leq g + l$, so $u + v \in A$. Therefore $A$ is a linear subspace of $\overline{C}[0, 1]$. 

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\( f + h \in \overline{C}[0, 1], f + h \) is increasing on \([0, 1], g + l \in \overline{C}[0, 1], g + l \) is decreasing on \([0, 1], \) which implies that \( u + v \in A. \) For \( \alpha \in \mathbb{R} \) and \( u = f + g \in A, \) we have \( \alpha u = \alpha f + \alpha g, \) where \( \alpha f, \alpha g \in \overline{C}[0, 1]. \) If \( \alpha \geq 0 \) then \( \alpha f \) is increasing and \( \alpha g \) is decreasing, while if \( \alpha < 0 \) then \( \alpha g \) is increasing and \( \alpha f \) is decreasing but as a consequence, in both cases it follows \( \alpha u \in A. \)

First we observe that \( IC[0, 1] \subset A. \) Indeed, for \( u \in IC[0, 1] \), we have two possibilities : a) \( u(1) \leq 0 \) or b) \( u(1) > 0. \) In the case a) we can write \( u = u + 0 \in A, \) while in the case b) we can write \( u = f + g, \) where \( f(t) = u(t) - u(1), g(t) = u(1), \forall t \in [0, 1], \) which proves that \( u \in A. \) Also, if we denote by \( DC[0, 1] \) the set of all \( u \in \overline{C}[0, 1] \) which are decreasing on \([0, 1], \) then \( DC[0, 1] \subset A. \) Indeed, for \( u \in DC[0, 1] \) we have two possibilities : a) \( u(1) \geq 0 \) when we write \( u = f + g \) with \( f = 0, g = u \) and b) \( u(1) < 0 \) when we write \( u = f + g \) with \( f(t) = u(1), \forall t \in [0, 1] \) and \( g(t) = u(t) - u(1), \forall t \in [0, 1]. \)

Now, let us define \( L_0 : A \to \mathbb{R} \) by \( L_0(u) = x^*(x), \forall u = f + g \in A, \) where \( x \in \mathbb{R}_F \) is the unique fuzzy number existing by Theorem 2.2, such that \( x_- = f \) and \( x_+ = g. \) Notice that if \( u = f + g \in A, \) then for all \( \varepsilon \geq 0 \) we also have the representation \( u = (f - \varepsilon) + (g + \varepsilon), \) where obviously \( f - \varepsilon \leq g + \varepsilon. \) It follows that for given \( u \in A, \) we can choose an infinity of such \( x \in \mathbb{R}_F, \) which means that in fact we can define an infinity of mappings \( L_0 \) as above. For our purposes, we choose only one, intimately connected to the chosen representations of the elements \( u \in A, \) such that for \( u \in IC[0, 1] \) and \( u \in DC[0, 1] \) we choose the above representations.

First we show that \( L_0 \) is linear on \( A. \) Indeed, for \( u = f + g \in A, v = h + l \in A, \) we have \( L_0(u) = x^*(x), L_0(v) = x^*(y), \) where \( x_- = f, x_+ = g \) and \( y_- = h, y_+ = l. \) But \( (x + y)_- = x_- + y_- = f + h, (x + y)_+ = x_+ + y_+ = g + l, \) which by the linearity of \( x^* \) implies

\[
L_0(u + v) = x^*(x + y) = x^*(x) + x^*(y) = L_0(u) + L_0(v).
\]

Now, let \( \alpha \in \mathbb{R}, u = f + g \in A \) and \( x \in \mathbb{R}_F \) with \( x_- = f, x_+ = g, \) i.e. \( L_0(u) = x^*(x). \) If \( \alpha \geq 0 \) then \( (\alpha x)_- = \alpha(x)_- = \alpha f, (\alpha x)_+ = \alpha(x)_+ = \alpha g, \) so \( L_0(\alpha u) = x^*(\alpha x) = \alpha x^*(x) = \alpha L_0(u). \) If \( \alpha < 0 \) then \( \alpha u = \alpha f + \alpha g = \alpha g + \alpha f, \)
where \((\alpha x)_- = \alpha g, (\alpha x)_+ = \alpha f\), which again implies \(L_0(\alpha u) = \alpha L_0(u)\).

As a conclusion, \(L_0\) is linear on \(A\) and by a well-known result in Functional Analysis (see e.g. [11, pp. 56-57, Proposition 1.1]), \(L_0\) can be prolonged to a linear functional \(L : C[0, 1] \to \mathbb{R}\).

It remains to prove that the restriction of \(L_0\) to \(IC[0, 1]\) is continuous on \(IC[0, 1]\). Thus, let \(u_n, u \in IC[0, 1], n \in \mathbb{N}\) be such that \(u_n \to u\), uniformly on \([0, 1]\). We have three possibilities: a) \(u(1) < 0\); b) \(u(1) > 0\); c) \(u(1) = 0\).

Case a). We get \(u(t) \leq u(1) < 0, \forall t \in [0, 1]\) and by Theorem 2.2, there is a unique \(x \in \mathbb{R}_x\) with \(x_-(t) = u(t), x_+(t) = 0, \forall t \in [0, 1]\). From \(u_n \to u\) uniformly on \([0, 1]\), for \(\varepsilon = \frac{-u(1)}{2} > 0\), there is \(n_0 \in \mathbb{N}\), such that \(u_n(t) - u(t) < \frac{-u(1)}{2}, \forall t \in [0, 1], n \geq n_0\), which implies that for all \(t \in [0, 1]\) and \(n \geq n_0\) we have

\[
\begin{align*}
  u_n(t) < u(t) - \frac{u(1)}{2} \leq u(1) - \frac{u(1)}{2} = \frac{u(1)}{2} \leq 0.
\end{align*}
\]

Therefore, for any \(n \geq n_0\) there is \(x_n \in \mathbb{R}_x\) such that \(x_n^-(t) = u_n(t), x_n^+(t) = 0, \forall t \in [0, 1]\). As a conclusion, \(u_n \to u\) uniformly on \([0, 1]\), implies \(x_n^- \to x_-\) and \(x_n^+ \to x_+\), uniformly on \([0, 1]\), which by e.g. [13, p. 524] implies \(D(x_n, x) \to 0\) (when \(n \to \infty\)) and together with the continuity of \(x^*\) we get

\[
L_0(u_n) = x^*(x_n) \to x^*(x) = L_0(u).
\]

Case b). There is a unique \(x \in \mathbb{R}_x\) with \(x_-(t) = u(t) - u(1), x_+(t) = u(1), \forall t \in [0, 1]\). Since \(\lim_{n \to \infty} u_n(1) = u(1) > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(u_n(1) > 0, \forall n \geq n_0\). Therefore, for any \(n \geq n_0\), there is a unique \(x_n \in \mathbb{R}_x\) with \(x_n^-(t) = u_n(t) - u_n(1), x_n^+(t) = u_n(1), \forall t \in [0, 1]\). Then, obviously again we obtain \(x_n^- \to u(t) - u(1) = x_-\) and \(x_n^+ \to u(1) = x_+\), uniformly on \([0, 1]\), which implies \(D(x_n, x) \to 0\) (when \(n \to \infty\)) and by the continuity of \(x^*\) it follows

\[
L_0(u_n) = x^*(x_n) \to x^*(x) = L_0(u).
\]

Case c). From \(u(t) \leq u(1) = 0, \forall t \in [0, 1]\), there exists a unique \(x \in \mathbb{R}_x\) with \(x_-(t) = u(t), x_+(t) = 0, \forall t \in [0, 1]\). Concerning each term of the sequence \((u_n)\), we have two possibilities:
(i) $u_n(1) \leq 0$ or (ii) $u_n(1) > 0$.

Subcase (i). We have $u_n(t) \leq 0, \forall t \in [0, 1]$ and there is a unique $x_n \in \mathbb{R}_L$ with $x_n^-(t) = u_n(t), x_n^+(t) = 0, \forall t \in [0, 1]$.

Subcase (ii). There is a unique $x_n \in \mathbb{R}_L$ with $x_n^-(t) = u_n(t) - u_n(1), x_n^+(t) = u_n(1), \forall t \in [0, 1]$.

From both subcases we obtain that $u_n(t) \rightarrow u(t)$ uniformly on $[0, 1]$ implies $x_n^-(t) \rightarrow u(t) = x_-(t), x_n^+(t) \rightarrow u(1) = 0 = x_+(t)$, uniformly on $[0, 1]$, i.e. \( \lim_{n \rightarrow \infty} D(x_n, x) = 0 \) and reasoning as for the above cases we obtain the continuity of the restriction of $L_0$ to $IC[0, 1]$ in this last subcase too. The theorem is proved.

**REMARK.** A natural family of functionals $L$ in the statement of Theorem 3.1 can be defined as follows. For any fixed continuous function $h : [0, 1] \rightarrow \mathbb{R}$, first define $L_0 : A \rightarrow \mathbb{R}$, as the Riemann-Stieltjes integral $L_0(u) = \int_0^1 h(t)d[u(t)]$. Then $L$ will be a linear extension of $L_0$ to $\overline{C}[0, 1]$. Obviously that for any $u = f + g \in A$, $L_0(u)$ has sense and we have $L_0(u) = \int_0^1 h(t)d[f(t)] + \int_0^1 h(t)d[g(t)]$.

It is easy to prove that $L_0$ is linear on $A$. Also, the restriction of $L_0$ to $IC[0, 1]$ is continuous on $IC[0, 1]$. Indeed, let $u_n, u \in IC[0, 1], n \in \mathbb{N}, u_n \rightarrow u$ uniformly on $[0, 1]$ (when $n \rightarrow \infty$). We have : $\bigvee_n u_n = u_n(1) - u_n(0) \rightarrow u(1) - u(0)$, when $n \rightarrow \infty$.

This means that there exists $M > 0$ such that $\bigvee_0^1 u_n \leq M, \forall n \in \mathbb{N}$, which by the classical Helly-Bray theorem (see e.g. [12, p. 38]) implies

$$\lim_{n \rightarrow \infty} L_0(u_n) = \lim_{n \rightarrow \infty} \int_0^1 h(t)d[u_n(t)] = \int_0^1 h(t)d[u(t)] = L_0(u),$$

proving the desired continuity. Therefore, a class of linear continuous functionals $x^* : \mathbb{R}_L \rightarrow \mathbb{R}$ are of the form

$$x^*(x) = \int_0^1 h(t)d[x_-(t)] + \int_0^1 h(t)d[x_+(t)], \forall x \in \mathbb{R}_L,$$

where $h : [0, 1] \rightarrow \mathbb{R}$ is continuous on $[0, 1]$.

Notice that from a well-known formula for the Riemann-Stieltjes integral (see e.g. [12, p. 30]), we also can write

$$x^*(x) = h(1)[x_-(1) + x_+(1)] - h(0)[x_-(0) + x_+(0)] - \int_0^1 [x_-(t) + x_+(t)]d[h(t)].$$
It is natural the following.

**OPEN QUESTION 1.** Are all the linear continuous functionals $x^*: \mathbb{R}_F \to \mathbb{R}$ of the form in the previous remark?

**REMARK.** A method to answer the above open question would be to use the ideas in the proof of classical Riesz’s result concerning the form of linear continuous functionals on $C[0, 1]$. Unfortunately, it seems that these ideas do not work in our case, since if we define $h(t) = L_0(z_t), t \in [0, 1]$, (where $z_t(x) = 1, x \in [0, t], z_t(x) = 0, x \in [t, 1]$ and the restriction of $L_0$ to $IC[0, 1]$ is supposed to be continuous on $IC[0, 1]$ with respect to the uniform convergence), then $h$ is not, in general, continuous on $[0, 1]$.

In order to prove our second main result, we need the following notations:

$SIC[0, 1] = \{(s_n)_n; s_n \in IC[0, 1], \forall n \in \mathbb{N}, (s_n)_n \text{ is uniformly convergent}\}$,

$SC[0, 1] = \{(s_n)_n; s_n \in C[0, 1], \forall n \in \mathbb{N}, (s_n)_n \text{ is uniformly convergent}\}$.

**THEOREM 3.2** $x^*: c_{\mathbb{R}_F} \to \mathbb{R}$ is a linear continuous functional on $c_{\mathbb{R}_F}$, if and only if there exists a linear functional $L: SC[0, 1] \to \mathbb{R}$ such that

$$x^*(x) = L(x_- + x_+), \forall x = (x_n)_n \in c_{\mathbb{R}_F},$$

where the restriction of $L$ to $SIC[0, 1]$ is continuous on $SIC[0, 1]$ with respect to the convergence induced by the metric on $SC[0, 1]$ defined by

$$\Phi[(s_n)_n, (t_n)_n] = \sup\{|s_n - t_n|; n \in \mathbb{N}\}.$$

Here $|\cdot|$ represents the uniform norm, for $x = (x_n)_n \in c_{\mathbb{R}_F}$, we have denoted $x_- = (x_n^-)_n, x_+ = (x_n^+_n)_n$ and obviously that the convergence of the sequence $x = (x_n)_n$ in the metric $\mu$ in $c_{\mathbb{R}_F}$, implies that $x_-$ and $x_+$ are uniformly convergent sequences of functions.

**Proof.** First, let us suppose that $x^*: c_{\mathbb{R}_F} \to \mathbb{R}$ is of the form in statement. The linearity of $x^*$ follows exactly as in the proof of Theorem 3.1. To prove the continuity of $x^*$, let $x = (x_n)_n \in c_{\mathbb{R}_F}, x^m = (x_n^{(m)})_n \in c_{\mathbb{R}_F}, m = 1, 2, \ldots$, be such that
\[ \lim_{m \to \infty} \mu(x^m, x) = 0. \]

From the definition of \( \mu \) (see Section 2) and \( \Phi \), it is immediate that
\[ \Phi[(x^m)_-, x_-] \to 0, \Phi[(x^m)_+, x_+] \to 0, \]
when \( m \to \infty \). Since we can write \( x^*(x) = L(x_-) - L(-x_+) \) and by \( (x^m)_-, x_-, -(x^m)_+, -x_+ \in SIC[0, 1] \), obviously that the continuity of \( L \) implies the continuity of \( x^* \). (Above we have denoted \( (x^m)_- = ((x^m)_n)_n, (x^m)_+ = ((x^m)_n)_n, \forall m = 1, 2, ... \).

Conversely, let \( x^*: \mathcal{C}_R \to R \) be linear continuous functional on \( \mathcal{C}_R \). We use a similar idea to that in the proof of Theorem 3.1. Firstly, it is easy to show by standard procedure that with respect to usual addition and scalar multiplication of the sequences of functions and the norm \( ||(s_n)_n|| = \sup \{ ||s_n(t)||; n \in \mathbb{N} \} \), \( \overline{SIC}[0, 1] \) becomes a real Banach space. Also, we have \( \Phi[(s_n)_n, (t_n)_n] = ||(s_n)_n - (t_n)_n||. \)

Now, let us define the set
\[ SA = \{ u = (u_n)_n; u_n = f_n + g_n \in A, \ \text{such that} \ f = (f_n)_n, g = (g_n)_n \}

are uniformly convergent \},

where the set \( A \) is defined in the proof of Theorem 3.1. As in the previous proof, we easily obtain that \( SA \) is a linear subspace of \( \overline{SIC}[0, 1] \). We define \( L_0 : SA \to R \) by \( L_0(u) = x^*(x), \forall u = f + g \in SA \), where \( x = (x_n)_n \in \mathcal{C}_R \) is the unique sequence of fuzzy numbers satisfying the relations \( (x_n)_- = f_n, (x_n)_+ = g_n, \forall n \in \mathbb{N} \).

The linearity of \( L_0 \) follows exactly as in the proof of Theorem 3.1 and therefore there exists a linear extension of \( L_0 \), denoted by \( L : \overline{SIC}[0, 1] \to R \).

On the other hand, from the inclusion \( IC[0, 1] \subset A \), we immediately get the inclusion \( SIC[0, 1] \subset SA \). It remains to prove the continuity on \( SIC[0, 1] \) of the restriction of \( L_0 \) to \( SIC[0, 1] \). Thus, let \( u^m, u \in SIC[0, 1] \) be such that \( \lim_{m \to \infty} \Phi(u^m, u) = 0 \). Denoting \( u^m = (u^m_n)_n, u = (u_n)_n \), this means \( u^m_n(t) \to u_n(t) \) (with \( m \to \infty \)), uniformly with respect to \( t \in [0, 1] \) and \( n \in \mathbb{N} \). According to the proof of Theorem 3.1, there exists (in a unique way) \( x_n \in \mathcal{C}_R \) (depending on \( u_n \)) and \( x^m_n \in \mathcal{C}_R \) (depending on \( u^m_n \), such that \( D(x^m_n, x_n) \to 0 \), (with \( m \to \infty \)), uniformly respect to \( n \in \mathbb{N} \), where
\(x = (x_n)_n, x^m = (x^m_n)_n \in c_{RF},\) for all \(m = 1, 2, \ldots,\) and
\[
L_0(u^m) = x^*(x^m), x^*(x) = L_0(u).
\]
From the continuity of \(x^*\) it follows the continuity of \(L_0.\) Note that as in the proof of Theorem 3.1, we define here \(L_0\) under the hypothesis that for \(u \in IC[0,1]\) and \(u \in DC[0,1]\) we choose the representations in the proof of Theorem 3.1, which implies the corresponding representation for the elements in \(SIC[0,1].\)

**REMARK.** A natural class of linear continuous functionals \(x^* : c_{RF} \to R\) is given by the form
\[
x^*(x) = \int_0^1 h_1(t)dz_-(t) + z_+(t)] + \sum_{j=1}^{\infty} \alpha_j \int_0^1 h_2(t)d[(x_j)_-(t) + (x_j)_+(t)],
\]
for all \(x = (x_j)_j \in c_{RF}\) with \(\lim_{j \to \infty} D(x_j, z) = 0,\) where \(h_1, h_2 : [0,1] \to R\) are arbitrary continuous functions on \([0,1]\) and \((\alpha_j)_j\) is an arbitrary sequence of real numbers satisfying \(\sum_{j=1}^{\infty} |\alpha_j| < +\infty.\)

Taking into account the well-known ”inversion” formula for the Riemann-Stieltjes integral, \(x^*(x)\) in the above formula can be written by
\[
x^*(x) = h_1[z_-(1) + z_+(1)] - h_1(0)[z_-(0) + z_+(0)] - \int_0^1 [z_-(t) + z_+(t)]d[h_1(t)] + \sum_{j=1}^{\infty} \alpha_j \{h_2(1)[(x_j)_-(1) + (x_j)_+(1)] - h_2(0)[(x_j)_-(0) + (x_j)_+(0)] - \int_0^1 [(x_j)_-(t) + (x_j)_+(t)]d[h_2(t)]\}.
\]

**OPEN QUESTION 2** It is an open question if all the linear continuous functionals on \(c_{RF}\) are of the above form.

In a similar manner can be proved the following four theorems.

**THEOREM 3.3** \(x^* : m_{RF} \to R\) is a linear continuous functional, if and only if there exists a linear functional \(L : \overline{MC}[0,1] \to R\) such that
\[
x^*(x) = L(x_- + x_+), \forall x = (x_n)_n \in m_{RF},
\]
where the restriction to \(MIC[0,1]\) of \(L\) is continuous with respect to the metric \(\Phi\) in Theorem 3.2.
Here

\[ M\overline{C}[0,1] = \{ (s_n)_n; s_n \in \overline{C}[0,1], \forall n \in \mathbb{N}, \text{is uniformly bounded} \}, \]

\[ MIC[0,1] = \{ s_n \in IC[0,1], \forall n \in \mathbb{N}, \text{is uniformly bounded} \}, \]

and for \( x = (x_n)_n \in m_{R,F} \), we have denoted \( x_- = ((x_n)_n)_n, x_+ = ((x_n)_n)_n \), which also are uniformly bounded.

**THEOREM 3.4** Let \( 1 \leq p < +\infty \). Then \( x^* : l^p_{R,F} \rightarrow \mathbb{R} \) is linear and continuous functional, if and only if there exists a linear functional \( L : S^p\overline{C}[0,1] \rightarrow \mathbb{R} \) such that

\[ x^*(x) = L(x_- + x_+), \forall x = (x_n)_n \in l^p_{R,F}, \]

where the restriction to \( S^p IC[0,1] \) of \( L \) is continuous with respect to the metric on \( S^p\overline{C}[0,1] \)

\[ \Psi[(s_n)_n, (t_n)_n] = \left\{ \sum_{n=1}^{\infty} ||s_n - t_n||^p \right\}^{1/p}. \]

Here \( ||\cdot|| \) denotes the uniform norm, for \( x = (x_n)_n \in l^p_{R,F} \) the sequences of functions \( x_- = ((x_n)_n)_n, x_+ = ((x_n)_n)_n \) satisfy

\[ \sum_{n=1}^{\infty} ||(x_n)_-||^p < +\infty, \sum_{n=1}^{\infty} ||(x_n)_+||^p < +\infty, \]

and we have the notations

\[ S^p\overline{C}[0,1] = \{ (s_n)_n; s_n \in \overline{C}[0,1], \sum_{n=1}^{\infty} ||s_n||^p < +\infty \}, \]

\[ S^p IC[0,1] = \{ (s_n)_n; s_n \in IC[0,1], \sum_{n=1}^{\infty} ||s_n||^p < +\infty \}. \]

**REMARK.** A class of linear continuous functionals \( x^* : l^p_{R,F} \rightarrow \mathbb{R} \) is given by the formula

\[ x^*(x) = \sum_{j=1}^{\infty} \alpha_j \int_{0}^{1} h(t)d[(x_j)_-(t) + (x_j)_+(t)], \forall x = (x_j)_j \in l^p_{R,F}, \]

where \( h : [0,1] \rightarrow \mathbb{R} \) is continuous on \([0,1]\) and \((\alpha_j)_j\) is a sequence of real numbers satisfying : (i) \( |\alpha_j| \leq M, \forall j \in \mathbb{N} \) if \( p = 1 \) and (ii) \( \sum_{j=1}^{\infty} |\alpha_j|^q < +\infty, 1/p + 1/q = 1, \) if \( 1 < p < +\infty. \)
OPEN QUESTION 3  It is an open question if the linear continuous functionals on $b^p_{RF}$ are all of the above form.

**THEOREM 3.5** $x^* : C([a, b]; R_F) \to R$ is linear continuous functional if and only if there exists a linear functional $L : C([a, b]; \overline{C}[0, 1]) \to R$, such that

$$x^*(x) = L(x_+ + x_-), \forall x \in C([a, b]; R_F),$$

where the restriction to $CIC[0, 1]$ of $L$ is continuous on $CIC[0, 1]$ with respect to the metric on $C([a, b]; \overline{C}[0, 1])$ given by

$$\Delta(F, G) = \sup\{||F(t) - G(t)||; t \in [a, b]\},$$

for all $F, G : [a, b] \to \overline{C}[0, 1]$ continuous on $[a, b]$, where $|| \cdot ||$ is the uniform norm on $\overline{C}[0, 1]$ and $\overline{C}[0, 1]$ is considered endowed with the uniform metric $\Gamma(f, g) = ||f - g||$.

Here

$$C([a, b]; \overline{C}[0, 1]) = \{F : [a, b] \to \overline{C}[0, 1]; F \text{ is continuous on } [a, b]\},$$

$$CIC[0, 1] = \{F \in C([a, b]; \overline{C}[0, 1]); F(t) \in IC[0, 1], \forall t \in [a, b]\},$$

and for $x \in C([a, b]; R_F)$, we define $x_-, x_+ : [a, b] \to \overline{C}[0, 1]$ by

$$[x_-(t)](r) = [x(t)]_-(r), [x_+(t)](r) = [x(t)]_+(r), \forall t \in [a, b], r \in [0, 1],$$

which obviously satisfy $x_-, x_+ \in C([a, b]; \overline{C}[0, 1])$.

**REMARK.** A class of linear continuous functionals $x^* : C([a, b]; R_F) \to R$ is given by the formula

$$x^*(x) = \int_a^b \left\{ \int_0^1 h_1(s) d[x(t)_- - (s) + x(t)_(s)] \right\} d[h_2(t)],$$

where $h_1; [0, 1] \to R$ is continuous on $[0, 1]$ and $h_2; [a, b] \to R$ is of bounded variation, arbitrary.

OPEN QUESTION 4  Remains an open question if all the linear continuous functionals on $C([a, b]; R_F)$ are of this form.
**THEOREM 3.6** Let $1 \leq p < +\infty$. Then $x^* : L^p([a, b]; \mathbb{R}_F) \to \mathbb{R}$ is linear continuous functional, if and only if there exists a linear functional $L : L^p([a, b]; \overline{C}[0, 1]) \to \mathbb{R}$, such that

$$x^*(x) = L(x_- + x_+),$$

where the restriction to $L^p IC[0, 1]$ of $L$ is continuous on $L^p IC[0, 1]$ with respect to the metric on $L^p([a, b]; \overline{C}[0, 1])$ given by

$$\Delta_p(F, G) = \left\{ \int_a^b ||F(t) - G(t)||^p dt \right\}^{1/p},$$

for all $F, G \in L^p([a, b]; \overline{C}[0, 1])$, with $|| \cdot ||$ the uniform norm on $\overline{C}[0, 1]$ and $\int_a^b$ the Lebesgue-kind integral.

Here

$$L^p([a, b]; \overline{C}[0, 1]) = \{ F : [a, b] \to \overline{C}[0, 1]; \int_a^b ||F(t)||^p dt < +\infty \},$$

$$L^p IC[0, 1] = \{ F \in L^p([a, b]; \overline{C}[0, 1]); F(t) \in IC[0, 1], \forall t \in [0, 1] \},$$

and $x_-, x_+$ are defined as in the statement of Theorem 3.5.

**REMARKS.** 1) A class of linear continuous functionals $x^* : L^p([a, b]; \mathbb{R}_F) \to \mathbb{R}$ is given by the formula

$$x^*(x) = \int_a^b \left\{ \int_0^1 h_1(s) d[x(t)_-(s) + x(t)_+(s)] \right\} h_2(t) dt,$$

where $h_1 : [0, 1] \to \mathbb{R}$ is continuous on $[0, 1]$, $h_2 \in L^q([a, b]; \mathbb{R})$ with $1/p + 1/q = 1$ if $1 < p < +\infty$ and $h_2$ is a.e. bounded on $[a, b]$, in the case when $p = 1$.

**OPEN QUESTION 5** Remains an open question if all the linear continuous functionals on $L^p([a, b]; \mathbb{R}_F)$ are of the form in Remark 1.

2) A crucial step in the proofs of Theorems 3.1 and 3.2 is the construction of the set $A$ and the linear functional $L_0 : A \to \mathbb{R}$ and of $SA$ and $L_0 : SA \to \mathbb{R}$, respectively.

In the case of Theorem 3.5, the corresponding constructions are the set

$$FA = \{ U \in C([a, b]; \mathbb{R}_F); \text{ there exist } F, G \in CIC[0, 1] \text{ such that}$$

...
\[ F(t)(r) \leq G(t)(r), \forall t \in [a, b], r \in [0, 1] \text{ and } U(t) = F(t) + G(t), \forall t \in [a, b], \]

and \( L_0 : FA \to R \) is defined by \( L_0(U) = x^*(V) \), where \( x^* : C([a, b]; R_\mathcal{F}) \to R \) is given and \( V \) is the unique function \( V : [a, b] \to R_\mathcal{F} \) obtained (by Theorem 2.2) from the relations

\[
[V(t)]_- = F(t), [V(t)]_+ = G(t), \forall t \in [a, b].
\]

In the case of Theorem 3.6, the construction of \( FA \) and \( L_0 \) is similar, with the difference that \( U \in L^p([a, b]; R_\mathcal{F}) \) and \( x^* : L^p([a, b]; R_\mathcal{F}) \to R \).

Of course that in both cases (of Theorems 3.5 and 3.6), the mapping \( L_0 \) is defined under the hypothesis that for \( u \in IC[0, 1] \) (and \( u \in DC[0, 1] \)) we adopt the representations in the proof of Theorem 3.1.

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