Double bracket structures on Poisson manifolds

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Abstract
On a Poisson manifold endowed with a Riemannian metric we will construct a vector field that generalizes the double bracket vector field defined on semi-simple Lie algebras. On a regular symplectic leaf we will construct a generalization of the normal metric such that the above vector field restricted to the symplectic leaf is a gradient vector field with respect to this metric.

1 Introduction
We will recall the classical case of double bracket vector field on a semi-simple Lie algebra. Let \((\mathfrak{g}, [\cdot, \cdot])\) be a semi-simple Lie algebra with \(k : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}\) the Killing form, i.e. a non-degenerate, symmetric, \(\text{Ad}\)-invariant bilinear form. The following vector field, called the double bracket vector field, has been introduced by Brockett [7], [8], see also [4], in the context of dynamical numerical algorithms and linear programming

\[
\dot{L} = [L, [L, N]], \tag{1.1}
\]

where \(N \in \mathfrak{g}\) is a regular element in \(\mathfrak{g}\) and \(L \in \mathfrak{g}\). It turns out that the double bracket vector field is tangent to the adjoint orbits of the Lie algebra \(\mathfrak{g}\), orbits that are symplectic leaves for the canonical Lie-Poisson bracket on \(\mathfrak{g}\) induced by the K–K–S Poisson bracket on \(\mathfrak{g}^*\) and the identification between \(\mathfrak{g}\) and \(\mathfrak{g}^*\) being given by the Killing form.

More precisely, by identifying the Lie algebra \(\mathfrak{g}\) with its dual \(\mathfrak{g}^*\) using the Killing form, the K–K–S Poisson bracket on \(\mathfrak{g}^*\) transforms into the Poisson bracket on \(\mathfrak{g}\),

\[
\{F, G\}_\mathfrak{g}(L) = k(L, [\nabla F(L), \nabla G(L)]), \tag{1.2}
\]

where \(F, G : \mathfrak{g} \to \mathbb{R}\) are smooth functions. Also, the Hamiltonian vector field associated to a smooth function \(H : \mathfrak{g} \to \mathbb{R}\) is given by

\[
\dot{L} = -[L, \nabla H(L)],
\]
see [13], [14] and [12] for a pedagogical exposition of the above construction.

In the case of a compact Lie algebra $\mathfrak{g}$, it has been proved that the double bracket vector field (1.1) when restricted to a regular adjoint orbit $\Sigma \subset \mathfrak{g}$ is a gradient vector field with respect to the normal metric. We recall briefly this construction, for details see [5], [6].

For every $L \in \Sigma$ consider the orthogonal decomposition with respect to the minus Killing form $\mathfrak{k}$, $\mathfrak{g} = \mathfrak{g}_L \oplus \mathfrak{g}^L$, where $\mathfrak{g}_L := \{ X \in \mathfrak{g} \mid [L, X] = 0 \}$ and $\mathfrak{g}^L = \text{Im}(\text{ad}_L)$.

The linear space $\mathfrak{g}_L$ can be identified with the tangent space $T_L \Sigma$ and $\mathfrak{g}^L$ with the normal space. One can endow the adjoint orbit $\Sigma$ with the normal metric [2], or standard metric [1],

$$n^\Sigma([L, X], [L, Y]) = -k(X^L, Y^L),$$

where $X^L, Y^L$ are the normal components according to the above orthogonal decomposition of $X$, respectively $Y$.

**Theorem 1.1 ([5], [6]).** Let $H : \mathfrak{g} \to \mathbb{R}, H(L) = k(L, N)$. Then

$$\nabla_{n^\Sigma} H|_{\Sigma}(L) = [L, [L, N]].$$

The purpose of this paper is to obtain an analog of the above theorem in the setting of a general Poisson manifold $(M, \{\cdot, \cdot\})$. For this, first we need to construct a generalized double bracket vector field tangent to the symplectic leaves of an arbitrary Poisson manifold that corresponds to the double bracket vector field (1.1) for the case when Poisson manifold is $\mathfrak{g}$ endowed with the Poisson bracket (1.2). Following an idea presented in [3] we will construct a co-metric tensor that couples the Poisson structure and the Riemannian structure. Next step is to construct a generalized normal Riemannian metric on regular symplectic leaves such that the restriction of the generalized double bracket vector field to a regular leaf is the gradient of a smooth function with respect to this generalized normal Riemannian metric.

For the case of a compact Lie algebra $\mathfrak{g}$ we will rediscover the geometry of the double bracket vector field, where $\mathfrak{g}$ is endowed with the Poisson structure (1.2). We will exemplify our construction on the case of the non-compact semi-simple Lie algebra $\text{sl}(2, \mathbb{R})$.

## 2 Generalized double bracket vector field on Poisson manifolds

Let $(M, \mathfrak{g})$ be a (pseudo-)Riemannian manifold that is also endowed with a Poisson bracket

$$\{\cdot, \cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M).$$

**Definition 2.1.** We call the co-metric double bracket tensor the following symmetric contravariant 2-tensor $D : \Omega^1(M) \times \Omega^1(M) \to \mathcal{C}^\infty(M)$,

$$D(\alpha, \beta) := \mathfrak{g}(\#\Pi\alpha, \#\Pi\beta),$$

where $\Pi$ is the skew-symmetric contravariant 2-tensor associated with the bracket $\{\cdot, \cdot\}$.
When $\alpha = dF$ and $\beta = dG$, where $F, G \in C^\infty(M)$ we have

$$D(dF, dG) = g(X_F, X_G).$$

In the finite dimensional case the symmetric matrix associated to the contravariant tensor $D$ is given by

$$[D] = [\Pi]^T \cdot [g] \cdot [\Pi].$$

The 2-tensor $D$ is degenerate and its kernel is equal with the kernel of the Poisson bivector $\Pi$ in the case of a Riemannian metric on $M$.

We introduce the generalization of the double bracket vector field. In a different context, a similar construction that used a co-metric tensor has also been given in [3].

**Definition 2.2.** For $G \in C^\infty(M)$, the vector field

$$v_G := -i_{dG}D$$

is called the **generalized double bracket vector field**.

The vector field $v_G$ is a natural generalization of the double bracket vector field defined on a semi-simple Lie algebra.

**Theorem 2.1.** Let $(\mathfrak{g}, [\cdot, \cdot])$ be a semi-simple Lie algebra endowed with the Poisson bracket defined by (1.2). Then,

$$v_G(\xi) = [\xi, [\xi, \nabla G(\xi)]]$$

**Proof.** Let $\{e_i\}_{i=1}^n$ be a base for $\mathfrak{g}$ and $\xi^i : \mathfrak{g} \rightarrow \mathbb{R}, i = 1, n$ the corresponding coordinate functions. Then,

$$v_G(\xi) = - (i_{dG}D)(\xi) = -D^{ij}(\xi) \frac{\partial G}{\partial \xi^j} e_i,$$

where

$$-D^{ij}(\xi) = -k \left( X_{\xi^i}(\xi), X_{\xi^j}(\xi) \right) = -k \left( [\xi, \nabla \xi^i(\xi)], [\xi, \nabla \xi^j(\xi)] \right)$$

$$= -k \left( (C^\gamma_{\alpha\beta} k^{\alpha\gamma} \xi^\epsilon_{\alpha_e}, C^\tau_{s\beta} k^{\beta\tau} \xi^\epsilon_{s\tau}) \right)$$

$$= -k \left( C^\gamma_{\alpha\beta} C^\tau_{s\beta} k^{\alpha\gamma} k^{\beta\tau} \xi^\epsilon_{s\tau} \xi^\epsilon \right)$$

$$= k_{\alpha\gamma} C^\gamma_{\alpha\beta} C^\tau_{s\beta} k^{\alpha\gamma} k^{\beta\tau} \xi^\epsilon_{s\tau} \xi^\epsilon$$

$$= \delta_{\gamma\delta} C^\gamma_{\alpha\beta} C^\tau_{s\beta} k^{\alpha\gamma} k^{\beta\tau} \xi^\epsilon_{s\tau} \xi^\epsilon$$

Consequently,

$$v_G(\xi) = C^i_{\alpha\beta} C^\tau_{s\beta} k^{\alpha\gamma} k^{\beta\tau} \xi^\epsilon_{s\tau} \xi^\epsilon \frac{\partial G}{\partial \xi^j} e_i.$$
We have the following computation,

\[
[\xi, [\xi, \nabla G(\xi)]] = \left[ \xi^p e_p, \left[ \xi^s e_s, k^{\beta j} \frac{\partial G}{\partial \xi^j} e_\beta \right] \right] = \left[ \xi^p e_p, C^\tau_{s\beta} k^{\beta j} \xi^s \frac{\partial G}{\partial \xi^j} e_\tau \right] = C^i_{pr} C^\tau_{s\beta} k^{\beta j} \xi^s \frac{\partial G}{\partial \xi^j} e_i.
\]

For the equality (***) we used the bi-invariance of the Killing metric, \(k([X, Y], Z) + k(Y, [X, Z]) = 0\). Applying this equality to the three vectors \(e_p, e_\alpha, e_\tau\) we have

\[
k ([e_p, e_\alpha], e_\tau) + k (e_\alpha, [e_p, e_\tau]) = 0 \iff k \left( C^\gamma_{pa} e_{\gamma}, e_\tau \right) + k \left( e_\alpha, C^\gamma_{pr} e_\gamma \right) = 0 \iff -k_{\gamma \tau} C^\gamma_{pa} = k_{\alpha \gamma} C^\gamma_{pr}.
\]

On a symplectic leaf \(\Sigma \subset M\) we will construct a (pseudo-)Riemannian metric that will generalize the normal metric from the case of a compact semi-simple Lie algebra.

**Definition 2.3.** Let \(x \in \Sigma\) and \(X_{\text{tan}}, Y_{\text{tan}} \in \mathfrak{X}(\Sigma)\), then the double bracket metric \(\tau^\Sigma_{db} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \rightarrow \mathcal{C}^\infty(\Sigma)\) is defined as

\[
\tau^\Sigma_{db}(x) (X_{\text{tan}}(x), Y_{\text{tan}}(x)) := (g^\Sigma_{\text{ind}})^{-1}(x) \left( i_{X_{\text{tan}}(x)} \omega(x), i_{Y_{\text{tan}}(x)} \omega(x) \right),
\]

where \(\omega \in \Omega^2(\Sigma)\) is the induced symplectic 2-form on the symplectic leaf \(\Sigma\) and \((g^\Sigma_{\text{ind}})^{-1}\) is the co-metric tensor associated to the (pseudo-)Riemannian metric \(g^\Sigma_{\text{ind}}\) induced on \(\Sigma\) by the ambient (pseudo-)Riemannian metric \(g\).

For \(x \in \Sigma\), the matrix associated with the co-metric tensor \((g^\Sigma_{\text{ind}})^{-1}\) is given by

\[
[g^\Sigma_{\text{ind}}(x)]^{-1} = [[g(x)]_{T_x \Sigma \times T_x \Sigma}]^{-1},
\]

and consequently, the matrix associated with the double bracket metric \(\tau^\Sigma_{db}\) has the expression

\[
[\tau^\Sigma_{db}(x)] = [\omega(x)]^T \cdot [[g(x)]_{T_x \Sigma \times T_x \Sigma}]^{-1} \cdot [\omega(x)].
\]

Note that in the case of a pseudo-Riemannian metric \(g\) the co-metric tensor \((g^\Sigma_{\text{ind}})^{-1}\) does not always exist.

It has been proved by Weinstein [16] that locally any Poisson manifold \((M, \{\cdot, \cdot\})\) is diffeomorphic to a product \(\Sigma \times N\), where \(\Sigma\) is a symplectic leaf of \(M\) and \(N \subset M\) is a transverse submanifold to \(\Sigma\). Moreover, around any point \(x \in M\) there exist a local system of coordinates \((q, p, z)\), called Darboux-Weinstein coordinates, such that the symplectic leaf \(\Sigma\) that contains the point \(x\) is locally described by \(z = 0\) and \(z\) are
local coordinates on the transverse submanifold \( N \). In this set of coordinates the matrix associated with the Poisson tensor \( \Pi \) has the local expression

\[
[\Pi] = \begin{pmatrix}
0 & I_n & 0 \\
-I_n & 0 & 0 \\
0 & \Pi'(z) & 0
\end{pmatrix},
\]

where the matrix \([\Pi'(z)]\) corresponds to a Poisson bivector \( \Pi' \) that endows the submanifold \( N \) with a Poisson structure called the transverse Poisson structure. Also, \( \Pi'(0) = 0 \) and in the case when \( \Sigma \) is a regular symplectic leaf we have \( \Pi'(z) = 0 \), for any \( z \) in the domain of Darboux-Weinstein coordinates. The transverse Poisson structure has been extensively studied in [11], [10], [9].

In analogy with the compact semi-simple Lie algebra case we have the following result.

**Theorem 2.2.** The generalized double bracket vector field is tangent to regular symplectic leaves and the restriction to a regular leaf \( \Sigma \) is the gradient vector field of the restricted function \( G|_{\Sigma} \) with respect to the double bracket metric. More precisely,

\[
(v_G)|_{\Sigma} = -\nabla_{\tau^\Sigma} G|_{\Sigma}.
\]

**Proof.** Locally the symplectic leaf \( \Sigma \) is given by \( z = 0 \), where \((q, p, z)\) is a set of Darboux–Weinstein coordinates. For an arbitrary point \((q, p, 0)\) \( \in \Sigma \) in the domain of the Darboux–Weinstein coordinates, we have the following matrices for the corresponding tensors \( g, \Pi, g_{ind}^\Sigma \), and \( \omega \):

\[
[g] = \begin{pmatrix}
g_{11}(q, p, z) & g_{12}(q, p, z) & g_{13}(q, p, z) \\
g_{21}(q, p, z) & g_{22}(q, p, z) & g_{23}(q, p, z) \\
g_{31}(q, p, z) & g_{32}(q, p, z) & g_{33}(q, p, z)
\end{pmatrix};
[\Pi] = \begin{pmatrix}
0 & I_n & 0 \\
-I_n & 0 & 0 \\
0 & \Pi'(z) & 0
\end{pmatrix};
\]

\[
[g_{ind}^\Sigma(q, p)] = \begin{pmatrix}
A_{qq}(q, p) & A_{qp}(q, p) \\
A_{pq}(q, p) & A_{pp}(q, p)
\end{pmatrix};
[\omega] = \begin{pmatrix}
0 & -I_n \\
I_n & 0
\end{pmatrix},
\]

with \( A_{qq}(q, p) = A_{qq}^T(q, p) = g_{11}(q, p, 0) \); \( A_{pp}(q, p) = A_{pp}^T(q, p) = g_{22}(q, p, 0) \); \( A_{qp}(q, p) = g_{12}(q, p, 0) \). The vector field \( v_G \) has the local expression,

\[
v_G(q, p, 0) = \left( A_{qp}^T(q, p) \frac{\partial G}{\partial p} - A_{pp}(q, p) \frac{\partial G}{\partial q} \right) \frac{\partial}{\partial q}
+ \left( -A_{qq}(q, p) \frac{\partial G}{\partial p} + A_{qp}(q, p) \frac{\partial G}{\partial q} \right) \frac{\partial}{\partial p},
\]

which is a tangent vector to the regular symplectic leaf \( \Sigma \).
The vector field $\nabla_{\tau_{d^b}^\Sigma} G|_{\Sigma} \in \mathfrak{X}(\Sigma)$ has the following local expression,

$$-\nabla_{\tau_{d^b}^\Sigma} G|_{\Sigma}(q, p) = - ([\omega]^T \cdot [g_{imd}^\Sigma(q, p)]^{-1} \cdot [\omega])^{-1} \left( \begin{array}{c} \frac{\partial G}{\partial q} \\ \frac{\partial G}{\partial p} \end{array} \right)$$

$$= - ([\omega]^T \cdot [g_{imd}^\Sigma(q, p)] \cdot [\omega]) \left( \begin{array}{c} \frac{\partial G}{\partial q} \\ \frac{\partial G}{\partial p} \end{array} \right)$$

$$= \left( A_{qp}^T(q, p) \frac{\partial G}{\partial p} - A_{pp}(q, p) \right) \frac{\partial}{\partial q} + \left( -A_{qq}(q, p) \frac{\partial G}{\partial p} + A_{qp}(q, p) \right) \frac{\partial}{\partial p}.$$

\[ \square \]

Remark 2.1. In analogy with the result in [3], we have that for any $x \in \Sigma$,

$$([D(x)]_{T_x^* \Sigma \times T_x^* \Sigma})^{-1} = \tau_{d^b}^\Sigma(x).$$

More precisely, for an arbitrary $x = (p, q, 0) \in \Sigma$ in the domain of Darboux–Weinstein local coordinates,

$$[D(x)]_{T_x^* \Sigma \times T_x^* \Sigma} = \left( \begin{array}{cc} A_{pp}(q, p) & -A_{qp}(q, p) \\ -A_{qp}(q, p) & A_{qq}(q, p) \end{array} \right) = [\tau_{d^b}^\Sigma(x)]^{-1}.$$

Next, we will show that for the case of a compact semi-simple Lie algebra the double bracket metric and the normal metric coincide up to a sign.

**Theorem 2.3.** Let $\mathfrak{g}$ be a semi-simple compact Lie algebra and $\Sigma \subset \mathfrak{g}$ a regular adjoint orbit. Then

$$\tau_{d^b}^\Sigma = -n^\Sigma.$$

**Proof.** Let $(p, q, z)$ be a system of Darboux–Weinstein local coordinates adapted to the regular adjoint orbit $\Sigma$ and we make the notations $u = (p, q)$. If $x_0 = (u_0, 0)$ is an arbitrary point of $\Sigma$ that belongs to the domain of the adapted local coordinates system, then by Theorem 2.2 we have

$$[\tau_{d^b}^\Sigma(u_0)]^{-1} \cdot du^i(u_0) = \nabla_{\tau_{d^b}^\Sigma} u^i(u_0) = -v_{u^i}(u_0, 0), \quad \text{all} \quad i = 1, \dim \Sigma.$$

By Theorem 1.1 and Theorem 2.1 we also have

$$[n^\Sigma(u_0)]^{-1} \cdot du^i(u_0) = \nabla_{n^\Sigma} u^i(u_0) = [x_0, [x_0, \nabla u^i(x_0)]] = v_{u^i}(u_0, 0), \quad \text{all} \quad i = 1, \dim \Sigma.$$

As $\{du^i(u_0)\} \mid i = 1, \dim \Sigma$ is a base for $T_{u_0}^* \Sigma$ we obtain the equality

$$[\tau_{d^b}^\Sigma(u_0)] = -[n^\Sigma(u_0)].$$

\[ \square \]
On a regular symplectic leaf $\Sigma$, a sufficient condition for the matrix $[[g(x)]_{|T_x \Sigma \times T_x \Sigma}]^{-1}$ to be equal with the matrix $[g^{-1}(x)]_{|T_x \Sigma \times T_x \Sigma}$ is the compatibility condition that the Poisson bivector $\Pi$ is $g$-parallel, i.e.

\[
\nabla \Pi = 0,
\]

where $\nabla$ is the covariant derivative on the Riemannian manifold $(M, g)$, see [15].

3 Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ and hyperbolic geometry

In this section we show the connection between the structure of the semi-simple non-compact Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ and the Poincaré open disc model for the hyperbolic geometry. More precisely, we prove that double bracket metric $\tau_{db}^H$ on the connected component of the adjoint orbit (symplectic leaf) given by the upper-sheet $H^2$ of the two-sheeted hyperboloid is the hyperbolic metric in the Poincaré open disc model.

A base for Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is given by

\[
e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

For the Lie algebra structure we have

\[
[e_1, e_2] = e_3; \quad [e_1, e_3] = -2e_1; \quad [e_2, e_3] = 2e_2,
\]

and the associated adjoint operators are

\[
ad e_1 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad ad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}; \quad ad e_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The matrix corresponding to the Killing metric in this base is given by

\[
[k] = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}.
\]

We make the following change of base

\[
e_x = \frac{1}{2\sqrt{2}}(e_1 + e_2); \quad e_y = \frac{1}{2\sqrt{2}}e_3; \quad e_z = \frac{1}{2\sqrt{2}}(e_1 - e_2).
\]

For the Lie bracket we have

\[
[e_x, e_y] = -\frac{1}{\sqrt{2}}e_z; \quad [e_x, e_z] = -\frac{1}{\sqrt{2}}e_y; \quad [e_y, e_z] = \frac{1}{\sqrt{2}}e_x,
\]

and the adjoint operators are

\[
ad e_x = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad ad e_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad ad e_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
In this new base the Killing metric becomes
\[
\kappa(x, y, z) = dx \otimes dx + dy \otimes dy - dz \otimes dz,
\]
where \((x, y, z)\) are the coordinates on \(\mathbb{R}^3\) corresponding to the base \(\{e_x, e_y, e_z\}\). The gradient vector fields with respect to the Killing metric corresponding to coordinate functions are
\[
\nabla x = e_x; \quad \nabla y = e_y; \quad \nabla z = -e_z.
\]
Using the formula (1.2) for the Lie-Poisson bracket on \(\mathfrak{sl}(2, \mathbb{R})\) we have
\[
\{x, y\} = \kappa((x, y, z), [\nabla x, \nabla y]) = \kappa(x e_x + y e_y + z e_z, [e_x, e_y]) = \frac{1}{\sqrt{2}} z.
\]
Analogously,
\[
\{x, z\} = \frac{1}{\sqrt{2}} y; \quad \{y, z\} = -\frac{1}{\sqrt{2}} x.
\]
Consequently, the matrix associated to the Poisson tensor is
\[
\Pi = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & z & y \\
-z & 0 & -x \\
-y & x & 0
\end{pmatrix},
\]
and a Casimir function is \(C(x, y, z) = x^2 + y^2 - z^2\).
The matrix corresponding to the double bracket contravariant tensor \(D\) is given by
\[
D = [\Pi^T][\kappa][\Pi] = \frac{1}{2} \begin{pmatrix}
-y^2 + z^2 & xy & xz \\
x y & -x^2 + z^2 & yz \\
x z & y z & x^2 + y^2
\end{pmatrix}.
\]
Let \(\mathbb{H}^2\) be the upper-sheet of the two-sheeted hyperboloid \(x^2 + y^2 - z^2 = -1\) and \(\mathbb{D}^2 = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\}\) be the open disc in \(\mathbb{R}^2\). Using the stereographic projection with center \((0, 0, -1)\) we obtain the set of local coordinates for \(\mathbb{H}^2\), \(\Phi: \mathbb{D}^2 \rightarrow \mathbb{H}^2 \subset \mathbb{R}^3\):
\[
\begin{cases}
u = \frac{y}{1 + z} \\
u = \frac{x}{1 + z}.
\end{cases}
\]
The two 1-forms \(du = \frac{1}{1 + z} dx - \frac{x}{(1 + z)^2} dz\) and \(dv = \frac{1}{1 + z} dy - \frac{y}{(1 + z)^2} dz\) generate a base for the cotangent space \(T^*_{(x,y,z)} \mathbb{H}^2\), where \((x, y, z) \in \mathbb{H}^2\). We have the following computations:
\[
D(x, y, z)(du, du) = \frac{1}{2} \begin{pmatrix}
1 + x^2 & xy & xz \\
x y & 1 + y^2 & yz \\
x z & y z & z^2 - 1
\end{pmatrix} \begin{pmatrix}
\frac{1}{1 + z} \\
0 \\
-x
\end{pmatrix} = \frac{1}{2(1 + z)^2}.
\]
Analogously,
\[ D(x, y, z)(du, dv) = D(x, y, z)(dv, du) = 0, \quad D(x, y, z)(dv, dv) = \frac{1}{2(1 + z)^2}. \]

Consequently, the matrix associated to the contravariant tensor \( D \) restricted to the
adjoint orbit \( \mathbb{H}^2 \) is given by
\[
\begin{bmatrix}
D(x, y, z)_{T^*_x \times T^*_y \mathbb{H}^2}(x, y, z) \\
\end{bmatrix}^{-1} = \frac{1}{2} \begin{pmatrix}
\frac{1}{(1+z)^2} & 0 \\
0 & \frac{1}{(1+z)^2}
\end{pmatrix}
\]
and
\[
\begin{bmatrix}
D(x, y, z)_{T^*_x \times T^*_y \mathbb{H}^2}(x, y, z) \\
\end{bmatrix}^{-1} = \frac{1}{2} \begin{pmatrix}
(1+z)^2 & 0 \\
0 & (1+z)^2
\end{pmatrix}.
\]

Having \( z = \frac{1+u^2+v^2}{1-u^2-v^2} \) we obtain
\[
\tau_{\mathbb{H}^2}^{db}(u, v) = \frac{2}{1-(u^2+v^2)^2}(du^2 + dv^2),
\]
which is twice the hyperbolic metric for the Poincaré open disc model. If we drop the
coefficient \( \frac{1}{\sqrt{2}} \) in the expression of the Poisson tensor \( \Pi \), then we obtain the hyperbolic
metric.

Note that the induced metric on \( \mathbb{H}^2 \subset (\mathbb{R}^3, k) \) is
\[
k_{ind}^{\mathbb{H}^2}(u, v) = \frac{4}{1-(u^2+v^2)^2}(du^2 + dv^2),
\]
which is the hyperbolic metric for the Poincaré open disc model.

Next we show that the induced metric and double bracket metric differ on the upper-
sheet of the two-sheeted hyperboloids \( \mathbb{H}^2_c := \{(x, y, z) \in \mathbb{R}^3 | x^2+y^2-z^2 = -c^2 \} \subset (\mathbb{R}^3, k) \). Using the following set of local coordinates for \( \mathbb{H}^2_c \):
\[
\begin{align*}
x &= c \sinh(\nu) \cos(u) \\
y &= c \sinh(\nu) \sin(u) \\
z &= c \cosh(\nu)
\end{align*}
\]
we obtain
\[
\tau_{\mathbb{H}^2_c}^{db}(u, v) = 2[\cosh(\nu)^2 - 1]du^2 + 2d\nu^2,
\]
and
\[
k_{ind}^{\mathbb{H}^2_c}(u, v) = c^2[\cosh(\nu)^2 - 1]du^2 + c^2d\nu^2.
\]

We will repeat the above computations for the regular adjoint orbits given by the
one-sheeted hyperboloid \( \mathcal{H}_l := \{(x, y, z) \in \mathbb{R} | x^2+y^2-z^2 = l^2 \} \subset (\mathbb{R}^3, k) \). Using the following set of local coordinates for \( \mathcal{H}_l \):
\[
\begin{align*}
x &= l \cosh(\nu) \cos(u) \\
y &= l \cosh(\nu) \sin(u) \\
z &= l \sinh(\nu)
\end{align*}
\]
we obtain
\[ \tau_{db}^{H}(u, \nu) = -2 \cosh^2(\nu) du^2 + 2 d\nu^2, \]
and
\[ k_{\text{ind}}^{H}(u, \nu) = l^2 \cosh^2(\nu) du^2 - l^2 d\nu^2. \]

The semi-simple Lie algebra \( sl(2, \mathbb{R}) \) has another two regular orbits given by the connected components of the cone \( \mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 = 0\} \) without its vertex \((0, 0, 0)\). On this two symplectic leaves the induced metric by the pseudo-Riemannian metric \( k \) is degenerate and the construction given in Definition 2.3 does not apply.

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