Proof of a Conjecture of Helleseth: Maximal Linear Recursive Sequences of Period $2^{2n} - 1$ Never Have Three-Valued Cross-Correlation

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Abstract

We prove a conjecture of Helleseth that claims that for any $n \geq 0$, a pair of binary maximal linear sequences of period $2^{2n} - 1$ can not have a three-valued cross-correlation function.

1 Introduction

The binary maximal linear sequences of period $2^m - 1$ are the sequences of elements in GF(2) of the form $\{\text{Tr}(\alpha^{ni})\}_{i \in \mathbb{Z}}$ where $\alpha$ is a generator of GF($2^m$), $\text{Tr}$: GF($2^m$) → GF(2) is the absolute trace, and $d$ and $t$ are integers (or integers modulo $2^m - 1$) with $\gcd(d, 2^m - 1) = 1$. (See the Introduction of [2].) The cross-correlation of any two binary sequences $a = \{a_i\}$ and $b = \{b_i\}$ whose periods are divisors of $2^m - 1$ is the function $C_{a,b}(t) = \sum_{i=0}^{2^m-2} (-1)^{a_i-t+b_i}$. In this note, we shall take $a = \{a_i\} = \{\text{Tr}(\alpha^i)\}$ and $b = \{b_i\} = \{\text{Tr}(\alpha^{di})\}$, where the decimation $d$ has $\gcd(d, 2^m - 1) = 1$. We call decimations with $d \equiv 1, 2, \ldots, 2^{m-1}$ (mod $2^m - 1$) trivial decimations because $\{\text{Tr}(\alpha^{2^k i})\}$ is the same sequence as $\{\text{Tr}(\alpha^i)\}$. One readily shows that $C_{a,b}(t)$ is the same as

$$C_d(t) = \sum_{x \in \text{GF}(2^m)\ast} (-1)^{\text{Tr}(\alpha^{-t}x+x^d)}.$$

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For a fixed $d$, we are interested in how many different values $C_d(t)$ takes as $t$ varies over $\mathbb{Z}/(2^m - 1)\mathbb{Z}$. We say that $C_d(t)$ is $v$-valued to mean that $|\{C_d(t): t \in \mathbb{Z}/(2^m - 1)\mathbb{Z}\}| = v$. Helleseth gave the following criterion for determining whether $C_d(t)$ is two-valued.

**Theorem 1.1** (Helleseth [2], Theorem 3.1(d),(g), Theorem 4.1). If $d \equiv 1, 2, \ldots, 2^{m-1} \pmod{2^m}$, then $C_d(t) \in \{-1, 2^m - 1\}$ for all $t$. Otherwise, $C_d(t)$ takes at least three different values.

In the same paper, Helleseth conjectured the following.

**Conjecture 1.2** (Cf. Helleseth [2], Conjecture 5.2). If $m$ is a power of 2, $C_d(t)$ is not three-valued.

In view of Theorem 1.1, this conjecture says that if $m$ is a power of 2, then $C_d(t)$ is either two-valued (if $d$ is a trivial decimation) or takes four or more values (if $d$ is nontrivial). We prove this conjecture in this note.

Feng [1] recently proved the following weaker form of Conjecture 1.2.

**Theorem 1.3** (Feng [1], Theorem 2). If $m$ is a power of 2 and $C_d(t) = -1$ for some value of $t$, then $C_d(t)$ cannot be three-valued.

We prove Conjecture 1.2 by proving the following.

**Theorem 1.4.** If $C_d(t)$ is three-valued, then $C_d(t) = -1$ for at least one value of $t$.

This, combined with Theorem 1.3, immediately implies Conjecture 1.2.

**Remark 1.5.** One should note that our theorem does not assume $m$ is a power of 2, so it is much more general in scope that what is needed. In fact, one can prove the same theorem for maximal linear sequences derived from fields $\text{GF}(p^m)$ with $p$ odd: this (and more) is done in [3].

## 2 Proof of Theorem 1.4

We shall prefer to work in terms of the *Walsh transform*, defined as

$$W_d(a) = \sum_{x \in \text{GF}(2^m)} (-1)^{\text{Tr}(x^d + ax)},$$

and it is straightforward to show that

$$W_d(\alpha^{-t}) = 1 + C_d(t).$$
Thus the values of $W_d$ on $\text{GF}(2^m)^*$ are just the values of $C_d$ shifted by 1. So $C_d$ is three-valued if and only if $W_d$ is three-valued on $\text{GF}(2^m)^*$.

We need to establish a few well-known facts before proceeding to the proof of Theorem 1.4. First, we need a simple result which, in rough terms, states that a sequence cannot be perfectly correlated or anti-correlated to a nontrivial decimation of itself.

**Lemma 2.1.** If $d \not\equiv 1, 2, \ldots, 2^m-1 \pmod{2^m-1}$, then $|W_d(a)| < 2^m$.

**Proof.** From the definition of $W_d(a)$ as the sum $\sum_{x \in \text{GF}(2^m)} (-1)^{\text{Tr}(x^d + ax)}$ of $2^m$ terms in $\{1, -1\}$, it suffices to prove that the said terms are not all of the same sign. The $x = 0$ term is 1, and so the only way that all the terms can have the same sign is if

$$\text{Tr}(x^d + ax) = (x^d + x^{2d} + \cdots + x^{2^{m-1}d}) + a(x + x^2 + \cdots + x^{2^{m-1}})$$

equals 0 for all $x \in \text{GF}(2^m)$, i.e., if and only if this polynomial is zero modulo $x^{2^m} - x$. Given our assumption on $d$, all the exponents of $x$ that appear in the polynomial as expressed above are distinct modulo $2^m - 1$, so this cannot happen. \hfill \square

We consider the first few power moments of $W_d$, with the $r$th power moment defined to be

$$P_r = \sum_{a \in \text{GF}(2^m)^*} W_d(a)^r,$$

where we use the convention $0^0 = 1$ in evaluating $P_0$. The power moments of $C_d$ have been calculated by Helleseth, whence it is easy to obtain those of $W_d$.

**Proposition 2.2** (See Helleseth [2]). We have

(a) $P_0 = 2^m - 1$,

(b) $P_1 = 2^m$,

(c) $P_2 = 2^{2m}$, and

(d) $P_3 = 2^{2m}|V|$, where $V$ is the set of roots of $1 + x^d + (1 + x)^d$ in $\text{GF}(2^m)$.

From these one can readily deduce the following, which also appears as calculations in [1].
Proposition 2.3. Suppose that $W_d(a)$ is three-valued on $GF(2^m)^*$ with values $A$, $B$, and $C$, and that $W_d(a) = C$ for $N_C$ values of $a \in GF(2^m)^*$. Then

$$N_C = \frac{2^{2m} - 2^m(A + B) + (2^m - 1)AB}{(C - A)(C - B)}$$

and

$$2^{2m}|V| = 2^{2m}(A + B + C) - 2^m(AB + BC + CA) + (2^m - 1)ABC,$$

where $V$ is the set of roots of $1 + x^d + (1 + x)^d$ in $GF(2^m)$.

Proof. To get $N_C$, compute $\sum_{a \in GF(2^m)^*}(W_d(a) - A)(W_d - B)$. On the one hand, $W_d(a) \in \{A, B, C\}$ implies that the sum is $N_C(C - A)(C - B)$. On the other hand, one can also calculate the sum in terms of power moments as $P_2 - (A + B)P_1 + ABP_0$, and then use the values given in Proposition 2.2. To get $|V|$, one can employ the same approach, this time with the sum $\sum_{a \in GF(2^m)^*}(W_d(a) - A)(W_d(a) - B)(W_d(a) - C)$: on the one hand, it is zero, and on the other, it can be expressed in terms of $P_0$, $P_1$, $P_2$, and $P_3$. \qed

This can be used to prove an interesting result about the 2-divisibility of the values assumed by $W_d(a)$.

Lemma 2.4. Suppose that $W_d(a)$ takes precisely three values $A$, $B$, and $C$ for $a \in GF(2^m)^*$. If all three values are non-zero, then $2^{m+1} | AB$.

Proof. From Proposition 2.3 we have

$$2^{2m}|V| = 2^{2m}(A + B + C) - 2^m(AB + BC + CA) + (2^m - 1)ABC,$$  \hspace{1cm} (1)

where $V$ is the set of roots of $1 + x^d + (1 + x)^d$ in $GF(2^m)$. Suppose that $A, B, C \neq 0$; then Lemma 2.1 shows that $A, B, C \not\equiv 0 \pmod{2^m}$. (We clearly have a nontrivial decimation by Theorem 1.1 since $W_d$ is three-valued on $GF(2^m)^*$, and hence $C_d$ is three-valued.) Then the term $(2^m - 1)ABC$ is divisible by fewer powers of 2 than the other terms on the right hand side of (1), so $2^{2m}|V|$ and $ABC$ have exactly the same power of 2 in their respective prime factorizations, and so $2^{2m}|ABC$. Since $C \not\equiv 0 \pmod{2^m}$, this means that $2^{m+1} | AB$. \qed

Now we are ready to prove Theorem 1.4. We assume that $C_d$ is three-valued and that none of these values is $-1$ in order to show a contradiction. Then $W_d(a)$ is three-valued for $a \in GF(2^m)^*$ with the three nonzero values $A$, $B$, $C$. Note that Proposition 2.2(b) shows that

$$\sum_{a \in GF(2^m)^*} W_d(a) = 2^m,$$
so we cannot have $A, B, C < 0$. Furthermore, by parts (b) and (c) of the same proposition,

$$\left( \sum_{a \in \text{GF}(2^m)^*} W_d \right)^2 = 2^{2m} = \sum_{a \in \text{GF}(2^m)^*} W_d(a)^2,$$

so we cannot have $A, B, C > 0$. Then without loss of generality, we may take $A < 0 < B$ and $C$ not between $A$ and $B$. Then by Proposition 2.3, the number $N_C$ of $a \in \text{GF}(2^m)^*$ such that $W_d(a) = C$ is

$$N_C = \frac{2^{2m} - 2^m(A + B) + (2^m - 1)AB}{(C - A)(C - B)}.$$

Since $C$ is not between $A$ and $B$, the denominator is positive, so

$$2^{2m} - 2^m(A + B) + (2^m - 1)AB > 0.$$

We use Lemma 2.4 and the fact that $A < 0$ and $B > 0$ to see that

$$2^{2m} - 2^m(-2^m - 1 + 1) + (2^m - 1)AB > 0,$$

so that $AB > -2^{m+1}$. But by Lemma 2.4 and the fact that $A < 0 < B$, we have that $AB \leq -2^{m+1}$, which gives the contradiction that completes the proof of Theorem 1.4.

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References

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