TVERBERG-TYPE THEOREMS FOR INTERSECTING BY RAYS

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Abstract. In this paper we consider some results on intersection between rays and a
given family of convex, compact sets. These results are similar to the center point theorem,
and Tverberg’s theorem on partitions of a point set.

1. Introduction

In this paper we consider some results on intersection between rays and a given family
of convex, compact sets, that resemble the center point theorem of [18,19], and Tverberg’s
theorem on partitions from [23].

Let us make a definition. Consider a straight line ℓ ⊂ \mathbb{R}^d and a point p ∈ ℓ. The point
p divides ℓ into two half-lines, we call these half-lines rays starting at p. We are going to
study the questions of the following type: given a family F of convex sets in \mathbb{R}^d, find a
point p ∈ \mathbb{R}^d such that every ray starting at p intersects at least α|F| members of
F, or at most β|F| members of F. Such questions were considered before in [20,10], for the case
of hyperplanes, and in [5,11] for families of convex sets.

The following theorem is similar to the “dual” Tverberg theorem for hyperplanes from [10],
the statements of this kind (with minor differences) for hyperplanes were conjectured in [20].

Theorem 1. Let F be a family of n compact convex sets in \mathbb{R}^d, such that any point x ∈ \mathbb{R}^d
belongs to at most c sets of F. Suppose that r is a prime power and the following inequality
holds

\[ n \geq (d + 1)(r - 1) + c + 1. \]

Then F has r disjoint subfamilies F_1, . . . , F_r, such that there exists a point p ∈ \mathbb{R}^d with the
following property: for any ray ρ starting at p, and any subfamily F_i, there exists K ∈ F_i
such that ρ ∩ K = ∅.

The following theorem is a generalization of the result of [5], see also [20], where a
particular case was conjectured for families of hyperplanes. This is an analogue of the
central point theorem for finite point sets, see [18,19,6].

Corollary 2. Let F be a family of n compact convex sets in \mathbb{R}^d, such that any point
x ∈ \mathbb{R}^d belongs to at most c sets of F. Suppose that r is a positive integer and the

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following inequality holds

\[ n \geq (d + 1)(r - 1) + c + 1. \]

Then there exists a point \( p \in \mathbb{R}^d \) such that any ray \( \rho \) starting at \( p \) does not intersect at least \( r \) of the sets in \( \mathcal{F} \).

Theorem 1 is formulated for compact sets, and the compactness is essential in the proof. Still, it is possible to formulate a similar result for hyperplanes. Let us make some definitions.

**Definition 1.** A convex open subset \( G \subset \mathbb{R}^d \) is called *almost bounded*, if it does not contain an open cone. Equivalently, for any point \( p \in G \) the set of rays starting at \( p \), and lying within \( G \), has an empty interior as a subset of the unit sphere \( S^{n-1} \).

**Definition 2.** For a family of hyperplanes \( \mathcal{G} \) in \( \mathbb{R}^d \) denote by \( C(\mathcal{G}) \) the union of all almost bounded components of the complement \( \mathbb{R}^d \setminus \bigcup \mathcal{G} \).

The following theorem generalizes the dual Tverberg theorem from [10] to the case, when hyperplanes are not in general position. This statement is also a partial solution of Conjecture 2 in [20].

**Theorem 3.** Let \( \mathcal{F} \) be a family of \( n \) hyperplanes in \( \mathbb{R}^d \), such that any point \( x \in \mathbb{R}^d \) belongs to at most \( c \) hyperplanes of \( \mathcal{F} \). Suppose that \( r \) is a prime power and the following inequality holds

\[ n \geq (d + 1)(r - 1) + c + 1. \]

Then \( \mathcal{F} \) has \( r \) disjoint subfamilies \( \mathcal{F}_1, \ldots, \mathcal{F}_r \), such that

\[ \bigcap_{i=1}^r C(\mathcal{F}_i) \neq \emptyset. \]

The proofs in this paper mostly follow the proofs in [10], the essential difference is that the general position requirements are substituted by an upper bound of the covering multiplicity of a family. Such strengthening is allowed by an accurate use of the concept of the Krasnosel’skii-Schwarz genus (see Section 4 for the definition) to avoid singular configurations that give a solution of the topological problem (in terms of sections of a vector bundle), but do not correspond to the solution of the original geometric problem.

**2. Facts from topology**

In this section some topological facts, that arise in the proof of Theorem 1 are given. In fact, the first part of the proof follows the proof of Theorem 1.1 in [11], this and the following sections restate the needed lemmas.

We consider topological spaces with continuous (left) action of a finite group \( G \) and continuous maps between such spaces that commute with the action of \( G \). We call them \( G \)-spaces and \( G \)-maps. In this paper we actually consider groups \( G = (\mathbb{Z}_p)^k \) for prime \( p \), called usually \( p \)-tori, but most of the definitions are valid for arbitrary finite group \( G \).

For basic facts about (equivariant) topology and vector bundles the reader is referred to the books [9, 13, 17]. The cohomology is taken with coefficients \( \mathbb{Z}_p \) (\( p \) is the same as in
the definition of $G$), in notations we omit the coefficients. Let us start from some standard definitions.

**Definition 3.** Denote by $EG$ the classifying $G$-space, which can be thought of as an infinite join $EG = G ∗ ⋯ ∗ G ∗ ⋯$ with diagonal left $G$-action. Denote $BG = EG/G$. For any $G$-space $X$ denote $X_G = (X × EG)/G$, and put (equivariant cohomology in the sense of Borel) $H^*_G(X) = H^*(X_G)$. It is easy to verify that for a free $G$-space $X$, the space $X_G$ is homotopy equivalent to $X/G$.

Consider the algebra of $G$-equivariant cohomology of the point $A_G = H^*_G(pt) = H^*(BG)$. For a group $G = (\mathbb{Z}_p)^k$, the algebra $A_G = H^*_G(\mathbb{Z}_p)$ has the following structure (see [9]). In the case $p$ odd it has $2k$ multiplicative generators $v_i, u_i$ with dimensions $\dim v_i = 1$ and $\dim u_i = 2$ and relations $v_i^2 = 0, \beta v_i = u_i$.

We denote by $\beta(x)$ the Bockstein homomorphism.

In the case $p = 2$ the algebra $A_G$ is the algebra of polynomials of $k$ one-dimensional generators $v_i$.

Any representation of $G$ can be considered as a vector bundle over the point pt, and it has corresponding characteristic classes in $H^*_G(pt)$. We need the following lemma, that follows from the results of [9], Chapter III §1.

**Lemma 1.** Let $G = (\mathbb{Z}_p)^k$, and let $I[G]$ be the subspace of the group algebra $\mathbb{R}[G]$, consisting of elements

$$\sum_{g \in G} a_g g, \sum_{g \in G} a_g = 0.$$  

Then the Euler class $e(I[G]) \neq 0 \in A_G$ and is not a divisor of zero in $A_G$.

Note that in this lemma the fact that $G = (\mathbb{Z}_p)^k$ is essential.

### 3. Topology of Tverberg’s theorem

This paper reproduces some lemmas from [11]. In Tverberg’s theorem and its topological generalizations (see [2] [24] for example) it is important to consider the configuration space of $r$-tuples of points $x_1, \ldots, x_r \in \Delta^N$ with pairwise disjoint supports. Here $\Delta^N$ is a simplex of dimension $N$. Let us make some definitions, following the book [15].

**Definition 4.** Let $K$ be a simplicial complex. Denote by $K^r_\Delta$ the subset of the $r$-fold product $K^r$, consisting of the $r$-tuples $(x_1, \ldots, x_r)$ such that every pair $x_i, x_j$ ($i \neq j$) has disjoint supports in $K$. We call $K^r_\Delta$ the $r$-fold deleted product of $K$.

**Definition 5.** Let $K$ be a simplicial complex. Denote by $K^{r*}_\Delta$ the subset of the $r$-fold join $K^{r*}$, consisting of convex combinations $w_1 x_1 + \cdots + w_r x_r$ such that every pair $x_i, x_j$ ($i \neq j$) with weights $w_i, w_j > 0$ has disjoint supports in $K$. We call $K^{r*}_\Delta$ the $r$-fold deleted join of $K$. 
Note that the deleted join is a simplicial complex again, while the deleted product has no natural simplicial complex structure, although it has some cellular complex structure.

The $r$-fold deleted product of the simplex $\Delta^{(r-1)(d+1)}$ is the natural configuration space in Tverberg’s theorem, but sometimes it is simpler to use the deleted join. Denote by $[r]$ the set $\{1, \ldots , r\}$, with the discrete topology.

If $r$ is a prime power $r = p^k$, then the group $G = (\mathbb{Z}_p)^k$ can be somehow identified with $[r]$, so a $G$-action on $K^*_\Delta$ and $K^*_\Delta$ by permuting $[r]$ arises. The following lemma is well-known, see [24] for example.

**Lemma 2.** The deleted join of the simplex $(\Delta^N)^*_{\Delta} = [r]^{*N+1}$ is $N - 1$-connected, and the natural map $A^l_G \to H^l_G((\Delta^N)^*_{\Delta})$ is injective for $l \leq N$.

Let us say a few words about the proof. There is the Leray-Serre spectral sequence that relates the ordinary cohomology of a $G$-space $X$ to its equivariant cohomology, the bottom row of $E_2$ in this spectral sequence being $A^*_G$. The connectedness hypothesis implies that the corresponding part of the bottom row survives in $E_\infty$, that is the statement of the lemma.

The next lemma is used in [24] too, a proof of this lemma can be found in [11], for example.

**Lemma 3.** Let $r = p^k$, $G = (\mathbb{Z}_p)^k$, and let $K$ be a simplicial complex. If the natural map $A^l_G \to H^l_G(K^*_{\Delta})$ is injective for $l \leq N$, then the similar map $A^l_G \to H^l_G(K^*_{\Delta})$ is injective for $l \leq N - r + 1$.

4. **The genus of $G$-spaces**

In this section we describe some measure of complexity for a $G$-space. Let $X$ be a paracompact free $G$-space, $G$ being a finite group. Informally, the main idea is that this measure can be estimated from the equivariant cohomology of $X$, by the statements like those in Lemmas 2 and 3. Let us make a definition.

**Definition 6.** The free genus of a free $G$-space $X$ is the least number $n$ such that $X$ can be covered by $n$ open subsets $X_1, \ldots , X_n$ so that every $X_i$ can be $G$-mapped to $G$. Denote the free genus by $g_{\text{free}}(X)$.

There are several kinds of genus for a $G$-space, here we only use the free genus, and call it simply “genus”. The free genus was introduced in [13, 21, 22], different versions of this definition for non-free action are discussed in [3].

Let us explain the definition of the genus. The set $X_i$ in the definition can be $G$-mapped to $G$ iff the group $G$ acts on connected components of $X_i$ freely, we call such spaces inessential in the sequel. In fact, for paracompact $X$ the sets $X_i$ in the definition of genus may be taken closed instead of open.

Let us state the properties of the genus, valid for paracompact spaces, following [25].

(1) (Monotonicity) If there is a $G$-map $f : X \to Y$, then $g_{\text{free}}(X) \leq g_{\text{free}}(Y)$;

(2) (Subadditivity) Let $X = A \cup B$, where $A$, $B$ are closed or open $G$-invariant subspaces. Then $g_{\text{free}}(X) \leq g_{\text{free}}(A) + g_{\text{free}}(B)$;
(3) (Dimension upper bound) $g_{\text{free}}(X) \leq \dim X + 1$;

(4) (Cohomology lower bound) If the natural map $A^n_G \to H^n_{\partial}(X, M)$ is nonzero for some $G$-module $M$, then $g_{\text{free}}(X) \geq n + 1$.

Take the deleted join $(\Delta^N)^*_{\Delta}$ and the deleted product $(\Delta^N)_{\Delta}$, considered in the previous section for $r$ being a prime power, with an action of the corresponding $p$-torus. Then the cohomology lower bound and the dimension upper bound, with Lemmas 2 and 3 give

$$g_{\text{free}}((\Delta^N)^*_{\Delta}) = N + 1, \quad g_{\text{free}}((\Delta^N)_{\Delta}) = N - r + 2.$$ 

We need the following lemma, that can be considered a strengthening of the definition of genus. A particular case of this lemma for $G = Z_2$ was proved in [12, Theorem 9].

**Lemma 4.** Let $X$ be a paracompact $G$-space, let $U = \{U_i\}_{i=1}^N$ be some open (or closed) covering of $X$ by inessential invariant subsets. Then there exist a point $x \in X$, that is covered by at least $g_{\text{free}}(X)$ sets of $U$.

**Proof.** Since every $U_i$ can be mapped to $G$, then from the partition of unity, corresponding to $U$, arises a map $f : X \to G^*$. Consider the contrary: the covering $U$ has multiplicity at most $g_{\text{free}}(X) - 1$. Then the image of $f$ is within the $(g_{\text{free}}(X) - 2)$-dimensional skeleton of $G^*$. Now from the dimension upper bound and the monotonicity of the genus it follows that $g_{\text{free}}(X) \leq g_{\text{free}}(X) - 1$, which is a contradiction. □

Note that this lemma is true if we consider the fixed-point-free genus $g_G(X)$ (see [3, 25]) of a fixed point free $G$-space, and call a subset inessential if none of its connected components is stabilized by the whole group $G$. This follows from the dimension upper bound for fixed-point-free genus.

5. **Proof of Theorem 1**

Consider the simplex $\Delta = \Delta^{n-1}$, along with some identification of its vertices with $\mathcal{F}$. Take some large enough ball $B \subset \mathbb{R}^d$, containing all the sets of $\mathcal{F}$ in its interior. The configuration space that we study is $\Delta^r \times B$, denote its elements by $(\alpha_1, \alpha_2, \ldots, \alpha_r, p)$. The points $\alpha_i$ in the simplex $\Delta$ will be considered as functions $\alpha_i : \mathcal{F} \to \mathbb{R}^+$ with unit sum.

Denote for brevity $\mathbb{R}^d = V$. Now let us map our configuration space to $V^r$ by the following rule. Let $\pi_K(p)$ be the orthogonal projection of $p$ to $K \in \mathcal{F}$. Put

$$f(\alpha_1, \alpha_2, \ldots, \alpha_r, p) = \bigoplus_{i=1}^r \sum_{K \in \mathcal{F}} \alpha_i(K)(\pi_K(p) - p).$$

This map is evidently continuous and $G$-equivariant, if we identify $V^r$ with $V[G]$ ($V$-valued functions on $G$ with $G$-action by right multiplication by $g^{-1}$).

Denote the zero set of $f$ by $Z$. Similar to [11], the map $f$ can be considered as a section of $G$-equivariant vector bundle, its Euler class being

$$e(f) = w^d \times u \in H^d_G(\Delta^r \times B, \Delta^r \times \partial B),$$
where \( w \) is the image of the \( e(I[G]) \), \( u \) is the generator of \( H^d(B, \partial B) \). By Lemmas 1 and 3 \( w^d \neq 0 \in H^d_G(\Delta^r) \), and \( e(f) \neq 0 \).

Similar to the proof of Lemma 3 in [11], we conclude that the natural map \( A^l_G \to H^l_G(Z) \) is injective in dimensions \( l \leq n - r - (r - 1)d = n - 1 - (r - 1)(d + 1) \). Let us sketch the proof of this claim. Suppose that some \( \xi \in A^l_G \) maps to zero in \( H^l_G(Z) \) by the natural map \( \pi^*_G : A^*_G \to H^*_G(Z) \), then by the properties of the cohomology multiplication

\[
\xi w^d \times u = 0 \in H^{r+1}_G(\Delta^r \times B, \Delta^r \times \partial B),
\]

which contradicts with Lemma 3.

It follows from the cohomology lower bound on the genus that \( g_{\text{free}}(Z) \geq n - (r - 1)(d + 1) \geq c + 1 \). Now we are going to use this fact and show that the point \( p \) is not contained in any \( K \in \mathcal{F} \) with \( \alpha_i(K) > 0 \).

We can find small enough \( \varepsilon > 0 \) so that the family of \( \varepsilon \)-neighborhoods \( \mathcal{F}(\varepsilon) = \{ K(\varepsilon) \}_{K \in \mathcal{F}} \) has covering multiplicity at most \( c \). Now consider the following open subsets of \( Z \): for any \( K \in \mathcal{F} \) denote

\[
U_K = \{ (\alpha_1, \alpha_2, \ldots, \alpha_r, p) \in Z : \exists i \in [r] \text{ such that } \alpha_i(K) > 0 \text{ and } p \in K(\varepsilon) \}.
\]

Note that for any \((\alpha_1, \alpha_2, \ldots, \alpha_r, p) \in U_K\) there is only one \( i \in [r] \) such that \( \alpha_i(K) > 0 \), since we consider the deleted product \( \Delta^r \). Hence the set \( U_K \) is partitioned into connected components, that are permuted by \( G \) freely, i.e. it is inessential. The family \( \{ U_K \} \) covers \( Z \) with multiplicity at most \( c \). If it does cover \( Z \), than \( g_{\text{free}}(Z) \leq c \), that was shown above to be false.

Therefore, there exists a combination \((\alpha_1, \alpha_2, \ldots, \alpha_r, p)\) with the following property: if \( \alpha_i(K) > 0 \), then \( p \notin K(\varepsilon) \). Put

\[
\mathcal{F}_i = \{ K \in \mathcal{F} : \alpha_i(K) > 0 \},
\]

the families \( \mathcal{F}_i \) are disjoint. For any \( i \in [r] \) the point \( p \) is in the convex hull of the points \( X_i = \{ \pi_K(p) \}_{K \in \mathcal{F}_i} \), reducing the family \( \mathcal{F}_i \) if needed, we may assume that \( p \) is in the relative interior of \( X_i \). It is clear, that for any ray \( \rho \) starting at \( p \), some of the angles \( \angle(\rho, \pi_K(p) - p) (K \in \mathcal{F}_i) \) is at least 90°, and \( \rho \) cannot intersect the corresponding set \( K \).

6. Proof of Corollary 2

If \( r \) is a prime power, then the statement follows from Theorem 1. Otherwise choose a positive integer \( k \) so that \( R = k(r - 1) + 1 \) is prime, such \( k \) exists by the Dirichlet theorem on arithmetic progressions. Now consider the family \( \mathcal{G} \) of \( kn \) sets, that is obtained from \( \mathcal{F} \) by taking each member of \( \mathcal{F} \) exactly \( k \) times. Any point in \( \mathbb{R}^d \) belongs to at most \( kc \) sets of \( \mathcal{G} \). The inequality

\[
kn \geq (d + 1)(R - 1) + kc + 1 = k(d + 1)(r - 1) + kc + 1
\]

holds since \( kn \geq k(d + 1)(r - 1) + kc + k \). Hence there exists a point \( p \in \mathbb{R}^d \) such that any ray \( \rho \) starting at \( p \) does not intersect at least \( R \) members of \( \mathcal{G} \). In this case it is clear that \( \rho \) does not intersect at least \( r \) members of \( \mathcal{F} \).
7. Proof of Theorem 3

The proof mainly follows the proof of Theorem 1, though some changes are required. Denote again

\[ f(\alpha_1, \alpha_2, \ldots, \alpha_r, p) = \bigoplus_{i=1}^{r} \sum_{K \in \mathcal{F}} \alpha_i(K)(\pi_K(p) - p), \]

to use the above reasonings, the map \( f \) should not have zeros on \( \Delta^r_\Delta \times \partial B \) for large enough ball \( B \). But in the case of hyperplanes this is not true. We need the following lemma from [1].

Lemma 5. Suppose \( \mathcal{F} = \{h_1, \ldots, h_n\} \) is a set of hyperplanes in \( \mathbb{R}^d \), consider the orthogonal projections \( \pi_1, \ldots, \pi_n \) onto the respective hyperplanes. Then there exists a convex body \( P \), such that

\[ \forall i = 1, \ldots, n, \; \pi_i(P) \subseteq P. \]

Take the convex body \( P \) from Lemma 5. Denote the zero set of \( f \) on \( \Delta^r_\Delta \times P \) by \( Z \), this set still can have nonempty intersection with \( \Delta^r_\Delta \times \partial P \).

Suppose that \( P \) contains the origin, and approximate the map \( f \) on \( P \) by

\[ f_\varepsilon(\alpha_1, \alpha_2, \ldots, \alpha_r, p) = \bigoplus_{i=1}^{r} \sum_{K \in \mathcal{F}} \alpha_i(K)((1 - \varepsilon)\pi_K(p) - p), \]

denote its zero set by \( Z_\varepsilon \). It is clear that \( Z_\varepsilon \cap \Delta^r_\Delta \times \partial P = \emptyset \), and, similar to the proof of Theorem 1, \( g_{\text{free}}(Z_\varepsilon) \geq c + 1 \).

Suppose that \( g_{\text{free}}(Z) \leq c \), then its open cover by \( c \) inessential sets should be an open cover for \( Z_\varepsilon \), for small enough \( \varepsilon \). Hence, \( g_{\text{free}}(Z_\varepsilon) \leq c \), that is not true. Therefore, \( g_{\text{free}}(Z) \geq c + 1 \), and the end of the reasoning is the same as in the proof of Theorem 1.

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