World-Sheet Logarithmic Operators and Target Space Symmetries in String Theory

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Abstract

We discuss the target-space interpretation of the world-sheet logarithmic operators in string theory. These operators generate the normalizable zero modes (discrete states) in target space, which restore the symmetries of the theory broken by the background. The problem of the recoil in string theory is considered, as well as some general properties of string amplitudes containing logarithmic operators.

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1. Introduction.

Recently it was discovered that in some conformal field theories, namely the $c = -2$ model \cite{1}, the gravitationally dressed CFT models \cite{2}, the $c_{p,1}$ models with central charge $c = 1 - 6(p - 1)^2/p$ \cite{3} and some critical disordered models \cite{4}, there are new objects - the so called logarithmic operators, associated with non-standard properties of operator products in the model. For information, we mention that the logarithmic operators in the $c = -2$ model were discussed recently in some detail in ref. \cite{5}. The existence of logarithmic operators was first shown by Gurarie in \cite{1}. It was pointed out that the appearance of logarithms in the $c = -2$-model’s correlation functions (which also emerged earlier in the WZNW model on the supergroup $GL(1,1)$ \cite{6}) is due to the presence of special operators, whose operator product expansions (OPE’s) display logarithmic short-distance singularities. These logarithmic operators have conformal dimensions degenerate with those of the usual primary operators, and it is this degeneracy that is at the origin of the logarithms. As a result of this degeneracy one can no longer completely diagonalize the Virasoro operator $L_0$, and the new operators together with the standard ones form the basis of the Jordan-cell for $L_0$. In the paper \cite{4} it was shown that the anomalous dimensions of logarithmic operators must be integers, and it was conjectured that they are connected with some hidden continuous symmetry (which may be a replica symmetry in case of the disorder).

In this paper we would like to address the issue of logarithmic operators in string theory and to study the connection between world-sheet logarithmic operators and target-space symmetries. It turns out that these operators are ultimately connected with the existence of normalizable zero modes for some backgrounds in target-space. These modes exist when the background violates some symmetries of the theory, in which case there is a family of backgrounds connected by the symmetry (or symmetries) which acts on the moduli space of parameters characterising the background. The simplest example of the normalizable zero modes are the zero modes for solitons or instantons arising due to translations and rotations (both in the space-time and in the internal space) of the backgrounds.

These modes are crucial in calculating the recoil effects in scattering off the soliton, when the soliton state changes during the process of the scattering. Recently this problem was considered in an interesting paper \cite{7} where the effects of recoil in string theory,
or in other words the quantization of the collective coordinates in string theory, were discussed and connected with the existence of non-local world-sheet operators violating the conformal invariance and leading to some unexpected logarithms.

We shall demonstrate in this article that these operators are nothing but the logarithmic operators mentioned above. This statement is true in general, and is not restricted to the models considered in [7]. This makes the world-sheet logarithmic operators very important objects in string theory. There is an important difference between the approaches suggested in ref. [7] and ours - in the first case the logarithmic singularities come from the infinite normalization factor of some world-sheet current, which is assumed to obey normal operator product expansion relations for currents. This is a result which we do not agree with. In our case, the extra logarithmic dependences in correlation functions of the string in the (extended-object) background arise simply by identifying these currents with logarithmic operators in the theory, which have well-defined Zamolodchikov metric [8].

Another example of a model where these operators are important is the 2d black holes. We shall argue, in this case, that the target-space symmetry connected with these operators is the $W_{\infty}$ acting on the discrete states [9] in the black hole background. It should be noted, in this context, that extra world-sheet logarithms have appeared in the amplitudes of non-critical strings propagating in black-hole backgrounds, which have been interpreted [9] as expressing back-reaction of matter on the space-time geometry, and, therefore, a change of conformal field theory (CFT). Let us note that, in general, a change of CFT can be attributed to the Liouville field, whose connection with the world-sheet Renormalization-Group flow and evolution process - the latter being viewed as a change of CFT backgrounds that may cause quantum mixing - was discussed in refs. [10]. A detailed connection of these results with the logarithmic operators discussed in this article will be discussed elsewhere.

In this work, we shall argue about the emergence of logarithmic operators in these models in two ways: first, by using the degeneracy between a vertex operator corresponding to the normalizable zero mode and the identity operator; second, by demonstrating that the conformal blocks in the corresponding Conformal Field Theory (CFT) possess logarithmic terms. These general arguments will be based on some specific properties of string amplitudes containing world-sheet logarithmic operators, as well as on the construction
of the Zamolodchikov metric \cite{Zamolodchikov} pertaining to these operators. Before addressing these problems, we consider it as useful to remind the reader some facts about the logarithmic operators.

2. Logarithmic operators

2.1. Mathematical background

First of all, let us discuss how logarithms appear in the correlation functions of the conformal field theory. Let us consider for example the four-point correlation functions of primary fields $V(z)$ and $V^{-1}(z)$ with anomalous dimensions $h$, such that $<V(z)V^{-1}(0)> = z^{-2h}$. This correlation function can be represented as

$$<V(z_1)V^{-1}(z_2)V^{-1}(z_3)V(z_4)> = \frac{1}{(z_1 - z_4)^{2h}(z_2 - z_3)^{2h}} F(x),$$

(2.1)

where $x = (z_1 - z_2)(z_3 - z_4)/(z_1 - z_4)(z_3 - z_2)$ and only the dependence on holomorphic coordinates $z_i$ has been written, omitting the obvious dependence on $\bar{z}$. In many different models the unknown function $F(x)$ is defined (up to some factors like $x^\alpha(1 - x)^\beta$ where $\alpha$ and $\beta$ are some parameters depending on the model) as a solution of the hypergeometric equation

$$x(1 - x)\frac{d^2 F}{dx^2} + [c - (a + b + 1)x] \frac{dF}{dx} - abF = 0$$

(2.2)

which in general has two independent solutions \cite{11}, \cite{12}

$$\mathcal{F}_1 = F(a, b, c; x), \quad \mathcal{F}_2 = x^{1-c} F(a - c + 1, b - c + 1, 2 - c; x)$$

(2.3)

where $F(a, b, c; x)$ is a hypergeometric function and these two independent solutions correspond to the two primary fields in the OPE of

$$V(z)V^{-1}(0) = \frac{1}{z^{2h}} [I + z^{c-1}O + ...]$$

(2.4)

one of which is an identity operator $I$ and the second operator $O$ (adjoint operator in the case of the WZNW model, for example, see \cite{13}) has an anomalous dimension $1 - c$. In this way we recover the conformal blocks in a generic CFT \cite{14}. However, this is not
true anymore if the parameter $c$ in the hypergeometric equation (2.2) is an integer (which takes place in all examples in [1] - [4]). There is a general theorem in the theory of the second order differential equations which deals with the expansion of the solution near the regular point $x = 0$

$$x^\alpha \sum_n a_n x^n$$ \hspace{1cm} (2.5)

This theorem tells us how to calculate the coefficients $a_n$ if one knows the two roots $\alpha_1$ and $\alpha_2$, which are the solutions of the so-called indicial equation [11]. However, if the difference $\alpha_1 - \alpha_2$ is an integer, the second solution either equals the first one (when $\alpha_1 = \alpha_2$) or some of the coefficients are undefined. In both cases the second solution has logarithmic terms $x^n \ln x$ in the expansion (2.5), besides the usual terms $x^n$. For hypergeometric equation the indicial equation is

$$\alpha (\alpha - 1 + c) = 0$$ \hspace{1cm} (2.6)

and the two roots are $\alpha_1 = 0$ and $\alpha_2 = c - 1$, i.e. for integer $c = 1 + m$ the second solution has logarithmic terms.

Let us note that in this case the second operator $O$ in the OPE (2.4) is degenerate either with the identity operator (when $m = 0$ and $c = 1$) or with one of its Virasoro descendants (for negative integer $m$ when $O$ has a positive dimension $|m|$). For positive integer $m$ (in which case $O$ itself has a negative dimension $-m$) one of its descendants will be degenerate with $I$.

The condition that $c = 1 + m, \; m \in Z$ is necessary, but not sufficient if $m \neq 0$. To have logarithms in this case one has to impose additional constraints on $a$ and $b$ [11],[12].

If $c = 1 + m$, where $m$ is a natural number, the two independent solutions are

$$\mathcal{F}_1 = F(a, b, 1 + m; x),$$
$$\mathcal{F}_2 = \ln x \; F(a, b, 1 + m; x) + H(x),$$ \hspace{1cm} (2.7)

where $H(x) = x^{-m} \sum_{k=0}^{\infty} h_k x^k$ and $h_m = 0$, unless either $a$ or $b$ equal $1 + m'$ with $m'$ a natural number $m' < m$. In this case the second solution is only a polynomial in $x^{-1}$ and there are no logarithms at all.
If \( c = 1 - m \), where \( m \) is a natural number, the two independent solutions are

\[
F_1 = x^m F(a + m, b + m, 1 + m; x),
\]

\[
F_2 = \ln x \ x^m F(a + m, b + m, 1 + m; x) + H(x),
\]

where \( H(x) \) is again some regular expansion without logarithms, unless either \( a \) or \( b \) equal \(-m'\) with an integer \( m' \) such that \( 0 \leq m' < m \), in which case both solutions do not have logarithmic terms also.

### 2.2. Amplitudes in string theory with logarithmic operators

The logarithmic terms in the conformal blocks can not be explained by usual the OPE and it is necessary to introduce new operators in the theory - the logarithmic operators.

The OPE for primary fields \( V \) and \( V^{-1} \) takes the form

\[
V(z_1) V^{-1}(z_2) = ... + (z_1 - z_2)^{h_C - 2h} \left( \bar{z}_1 - \bar{z}_2 \right)^{h_C - 2\bar{h}} \left[ D + C \ln |z_1 - z_2|^2 \right] + ....,
\]

where the dimension \( h_C \) of the operators \( C \) and \( D \) is determined by the leading logarithmic terms in the conformal block (2.1). We have written here both \( z- \) and \( \bar{z} - \) dependences explicitly and it is important to note that the logarithmic term depends on \(|z|\), even for chiral fields, because in the full conformal blocks actually \( \ln |z| \) appears, as was shown in [4].

Some general properties of these operators were studied in [1] and [4]. The simplest example of a logarithmic operator one can consider is the puncture operator, which was discussed in [15] in the context of the Liouville model with the action

\[
S = \frac{1}{8\pi} \int d^2 \xi \sqrt{g(\xi)} \left[ \partial_\mu \phi(\xi) \partial^\mu \phi(\xi) + QR^{(2)}(\xi) \phi(\xi) \right]
\]

where \( R^{(2)} \) is a world-sheet curvature. The central charge equals

\[
c_L = 1 + 3Q^2
\]

and the primary fields \( \exp(\alpha \phi) \) have dimensions

\[
h_\alpha = \frac{\alpha(Q - \alpha)}{2}
\]

This means that there are two cosmological constant operators with the same dimension \( h_\alpha = 1 \), namely \( V_\pm = \exp(\alpha_\pm \phi) \), where

\[
\alpha_\pm = \frac{Q}{2} \pm \frac{1}{2} \sqrt{Q^2 - 8}
\]
If $Q^2 = 8$, i.e. $c_L = 25$, there is a degeneracy $\alpha_+ = \alpha_- = \sqrt{2}$ and instead of two exponential primary fields we have only one $C = \exp(\sqrt{2}\phi)$. The second field with the same dimension (which is called the puncture operator) turns out to be $D = (1/\sqrt{2})\phi \exp(\sqrt{2}\phi)$, here $(1/\sqrt{2})$ is the normalization factor.

It is easy to show that the OPE of the stress-energy tensor $T$ with these fields is the follows

$$
T(z)C(0) = \frac{h}{z^2}C(0) + \frac{1}{z}\partial_z C(0) + ...
$$

$$
T(z)D(0) = \frac{h}{z^2}D(0) + \frac{1}{z^2}C(0) + \frac{1}{z}\partial_z D(0) + ...
$$

(2.13)

which is written here for general logarithmic pair $C$ and $D$ with anomalous dimensions $h$. This OPE obviously leads to a mixing between $C$ and $D$. The Virasoro operator $L_0$ which is defined through the Laurent expansion $T(z) = \sum_n Lnz^{-n-2}$ is not diagonal and mixes $C$ and $D$ inside a $2 \times 2$ Jordan cell

$$
L_0|C > = h|C > ; \quad L_0|D > = h|D > + |C >
$$

(2.14)

The OPE (2.13) leads to a modification of the Ward identity for the string amplitudes

$$
\int d^2z_1...d^2z_N < T(z)V_1(z_1)...V_N(z_N)>
$$

(2.15)

where $V_i(z_i)$ are primary fields with dimensions $h_i$. In the usual case one can show [10] that for on-shell states, i.e. when dimensions $h_i = 1$, these amplitudes are zero, which means that the descendants $L_{-n_1}...L_{-n_k}V_i$ decouple from the physical amplitudes. This is not true in case when some of the fields $V_i$ are logarithmic operators. In this case, using the OPE (2.13) one can see that (all $h_i = 1$)

$$
\int d^2z_1...d^2z_N < T(z)V_1(z_1)...D_j(z_j)...V_N(z_N)>
$$

$$
= \int d^2z_1...d^2z_N \sum_i \frac{\partial}{\partial z_i} \left( \frac{1}{(z - z_i)} < V_1(z_1)...D_j(z_j)...V_N(z_N) > \right)
$$

(2.16)

$$
+ \int d^2z_1...d^2z_N \sum_j \frac{1}{(z - z_j)^2} < V_1(z_1)...C_j(z_j)...V_N(z_N) >
$$

The first sum is a total derivative and after integration gives zero, but the second one, where the summation is performed only on $j$ corresponding to the $D$ operators, is non-zero.
and this Ward identity connects amplitudes of the descendants of $D$ with with amplitudes without logarithmic operators (but including currents $C$). One can derive a Ward identity for $\bar{T}$, which also will be nontrivial. Several examples of these identities were discussed in [4].

The important lesson we learn from this is that both $T$ and $\bar{T}$ descendants of logarithmic operators do not decouple from physical amplitudes - this will be important below. Some examples of this type of Ward identities were considered in [4].

Substituting the OPE (2.9) into the four-point correlation function one can derive (see details in [4]) the following two-point correlation functions for the fields $C$ and $D$

$$
\langle C(x)D(y) \rangle = \langle C(y)D(x) \rangle = \frac{\kappa}{2(x-y)^{2\Delta_C}}
$$

$$
\langle D(x)D(y) \rangle = \frac{1}{(x-y)^{2\Delta_C}} \left( -\kappa \ln |x-y|^2 + d \right)
$$

$$
\langle C(x)C(y) \rangle = 0
$$

(2.17)

Here the constant $d$ can be made arbitrary by shifting $D \to D + \text{const } C$. The coefficient $\kappa$ is defined by the leading logarithmic term in the conformal block ( $-2c$ in the notation of ref. [4]).

This mixing also affects the string propagator on the world-sheet cylinder between states $|m>$ and $|n>$, which is defined as

$$
\int dq d\bar{q} |n|q^{L_0-1}\bar{q}^{\bar{L}_0-1}|m>
$$

(2.18)

where $q = \exp(2\pi i \tau)$ and $\tau$ is the modular parameter. In the usual case, when $L_0$, $\bar{L}_0$ are diagonal, one gets after integrating over $\tau$

$$
< n| \frac{1}{L_0 + \bar{L}_0 - 2} \delta(L_0 - \bar{L}_0) |m> ,
$$

(2.19)

where $\delta(L_0 - \bar{L}_0)$ enforce the condition $h = \bar{h}$ for all propagating states. However in the case of logarithmic operators one must take into account the Jordan cell structure of $L_0$, $\bar{L}_0$, which in the sector of $|CD>$ and $|\bar{C}D>$ states leads to

$$
q^{L_0} = q^{h_C} \begin{pmatrix} 1 & \ln q \\ 0 & 1 \end{pmatrix} ; \quad \bar{q}^{\bar{L}_0} = \bar{q}^{\bar{h}_C} \begin{pmatrix} 1 & \ln \bar{q} \\ 0 & 1 \end{pmatrix}
$$

(2.20)
We have the new logarithmic factors here which will be very important soon. Let us also note that \( \ln q \) factors arise also in the characters \( \text{Tr}_h q^L \) as was discussed by Flohr in [3].

It is important to consider two separate cases. The first one is when states are the left-right logarithmic states \( (C, D)(\bar{C}, \bar{D}) \), then the propagator takes the form

\[
\int dq d\bar{q} q^{hc-1}\bar{q}^{hc-1} < CD | \begin{pmatrix} 1 & \ln q \\ 0 & 1 \end{pmatrix} | CD > < \bar{C}\bar{D} | \begin{pmatrix} 1 & \ln \bar{q} \\ 0 & 1 \end{pmatrix} | \bar{C}\bar{D} > \tag{2.21}
\]

One has either \( \ln q \) or \( \ln \bar{q} \) terms for transitions when either \( D \) goes to \( C \) or \( \bar{D} \) goes to \( \bar{C} \) and \( \ln q \ln \bar{q} = 4\pi^2|\tau|^2 \) for transition \( D\bar{D} \) to \( CC \). This type of states was considered in [3].

There is another type of states which we shall call chiral logarithmic states, when the logarithmic operators are present only in left or right sectors, like \( L^{-1}\bar{D} + \bar{L}^{-1}D \) when \( h = 1 \), or more general descendants for other integer \( h \). It was shown in the discussion following the Ward identity (2.16) that these states in general do not decouple from the physical spectrum. In the case of the chiral logarithmic states only one Jordan cell - either \( L_0 \) or \( \bar{L}_0 \) will work and we shall have the sum of \( \ln q \) and \( \ln \bar{q} \), i.e.

\[
\int dq d\bar{q} q^{hc-1}\bar{q}^{hc-1} \ln |q|^2 \tag{2.22}
\]

for transition between \( D \) and \( C \). As we shall see later these states will be the central issue of the recoil problem, which we shall address now.

### 3. Logarithmic operators, background symmetries and the recoil problem in string theory

Let us first review the situation in critical string theories in a given background \( \{g^i\} \), where \( g^i \) denotes generic background couplings/deformations that have vanishing \( \beta^i(g) \) functions. To be more specific one considers the following conformal field theory

\[
S = \int d^2z g^i(X) V_i(\partial_\alpha X, \mathcal{J}); \quad \beta^i(g) \equiv dg^i/d\ln \mu = 0 \tag{3.1}
\]

where \( V_i(\partial_\alpha X, \mathcal{J}) \) are vertex operators (deformations) turning on the background \( g^i(X) \), probably depending on some Kac-Moody currents of the exact conformal model representing string propagation in this background, \( \{X\} \) are target space-time co-ordinates of the string, and \( \partial_\alpha X \) are world-sheet derivatives of the ‘fields \( X \). As we shall argue below,
the form of the vertex operators, depending only on world-sheet derivatives of $X$, and not on $X$ itself, will be quite crucial for the demonstration of the existence of non-trivial zero modes \cite{7}.

Let us concentrate on the soliton background of ref. \cite{7} for definiteness, keeping in mind that the same can be applied to any other extended object background. Our aim is to identify the deformations that can cause a change of state of the monopole background during a scattering event (‘recoil’). As we have said earlier this problem is equivalent to identifying the relevant deformations of the $\sigma$-model that cause a change in conformal field theory \cite{9}. The important point to notice is that in the presence of the soliton background of ref. \cite{7} target space translational invariance is broken by the center of mass of the soliton. This effect can be seen in the $\sigma$-model path-integral formalism by performing a constant shift of the spatial coordinates $X$ of space-time

$$X^\mu(z, \bar{z}) \rightarrow X^\mu(z, \bar{z}) + q^\mu; \quad q^\mu = \text{const} \quad (3.2)$$

By expanding $g^i(X + q)$ in Taylor series and taking into account the specific dependence of $V_i(\partial_\alpha X, J)$ on world-sheet derivatives of $X$ only, one observes that the effect of the translation (3.2) is simply to induce a deformation in the $\sigma$-model action of the form

$$\delta S = q \int d^2z \frac{\delta g^i}{\delta X} V_i(\partial_\alpha X, J) \quad (3.3)$$

where appropriate summation over spatial indices in $X$ is assumed. Such a deformation can be expressed as a total world-sheet derivative by using the equations of motion stemming from the deformed world-sheet action

$$\frac{\delta S}{\delta X(z, \bar{z})} = \partial_\alpha \left( \frac{\delta S}{\delta (\partial_\alpha X(z, \bar{z}))} \right) \quad (3.4)$$

The result is \cite{8}

$$\mathcal{O} = \mathcal{N}(\text{ghosts}) \otimes \partial_\alpha \{ g^i(X) \frac{\delta}{\delta (\partial_\alpha X)} V_i(\partial_\beta X; J) \} \equiv \mathcal{N}(\text{ghosts}) \otimes \partial_\alpha J^\alpha \quad (3.5)$$

where $J^\alpha$ is a two-dimensional Noether current associated with the translation (3.2), and the ghost insertions must be included to absorb the ghost zero modes.

The constant $\mathcal{N}$ is a normalization factor which is found to be logarithmically divergent \cite{8}. This was found by considering the (normalized) Zamolodchikov metric corre-
sponding to $\mathcal{O}$, which is by construction [8]

$$G_{\mathcal{O}\mathcal{O}} = 4\pi \delta^{ij} = |z|^4 < \mathcal{O}^i(z, \bar{z}), \mathcal{O}^j(0) > =$$

$$\mathcal{N}^2 |z|^4 < (\partial \mathcal{J}(z, \bar{z}) + c.c.)(\partial \mathcal{J}(0, 0) + c.c.) > \tag{3.6}$$

If one assumes that in the soliton background of ref. [7], $J$ are usual $(1, 0)$ currents, one has

$$< J^i(z) J^j(0) > \sim \frac{\kappa \delta^{ij}}{z^2} \tag{3.7}$$

where $\kappa$ is a normalization constant, and the Latin indices $i, j$ run over the set of zero-modes pertaining to the specific background. Then one obtains

$$\mathcal{N}^2 |z|^4 (\partial^2 \frac{1}{z^2} + c.c.) = 4\pi \tag{3.8}$$

This equation appears obscure to us, and the way how the normalization constant was obtained in [7] is not mathematically clear. Indeed, the authors of ref. [7] integrate the equation over $z$ with a conformally invariant weight $\int d^2z/|z|^2$. The result is

$$\mathcal{N}^2 \int d^2z |z|^2 (\partial^2 \frac{1}{z^2} + c.c.) = 2\mathcal{N}^2 = 4\pi \int \frac{d^2z}{|z|^2} = 16\pi^2 \log \epsilon + \text{finite} \tag{3.9}$$

We do not understand completely this construction; for example, if one integrates with another weight, the answer will be different; Moreover, the cut-off $\epsilon$ is a world-sheet cut-off, which is not the same as the one which will arise later due to divergences in the integral over $q$.

These mathematical problems will disappear immediately, if we postulate that in the soliton background $J$ are not the usual, but the logarithmic currents. Then, instead of (3.7) one has to use (2.17) to obtain

$$< J^i(z) J^j(0) > = -\kappa \delta^{ij} \frac{\ln|z|^2}{z^2} \tag{3.10}$$

which immediately leads to finite $\mathcal{N}^2$ and well-defined Zamolodchikov metric

$$- \kappa \mathcal{N}^2 |z|^4 \left( \partial^2 \frac{\ln|z|^2}{z^2} + c.c. \right) = 2\kappa \mathcal{N}^2 = 4\pi;$$

$$\mathcal{N}^2 = \frac{2\pi}{\kappa} \tag{3.11}$$

which implies that $\mathcal{N}^2$ is not logarithmically divergent as in [7], but finite.
How, then, can one reproduce the logarithmic divergences found in [7], which are crucial for the recoil problem?

The answer was given at the end of the previous section - the same logarithmic operators lead to extra ln $q$ terms. Let us explain this in more detail. Given that the effects associated with a change in the conformal field theory, such as back-reaction of matter on the structure of space-time etc, are purely stringy effects, the most natural way of incorporating them in a $\sigma$-model language is to go beyond a fixed genus $\sigma$-model and consider the effects of resummation over world-sheet genera. This is a very difficult procedure to be carried out analytically. However, for our purposes a sufficient analysis, which describes the situation satisfactorily (at least at a qualitative level), is that of a heavy extended object in target space, which can be treated semi-classically. From a first-quantized point of view, this implies resumming one-loop (torus) world-sheets. Non-trivial effects arise from degenerate Riemann surfaces, namely from long-thin world-sheet tubes (wormholes) that are attached to a Riemann surface of lower genus (sphere in this case). From a formal point of view, representing a Riemann surface $\Sigma$ as a tube connecting two Riemann surfaces $\Sigma_1$ and $\Sigma_2$ one can take into account the degenerate tube by inserting a complete set of intermediate string states $E_\alpha$ [17]. Then the amplitude

$$ \int dm_\Sigma \langle \prod_i \int d^2 \xi_i V_i(\xi_i) \otimes (\text{ghosts}) \rangle $$

is given by the following expression

$$ \sum_\alpha \int dq d\eta \int d^2 z_1 \int d^2 z_2 \int dm_{\Sigma_1 \oplus \Sigma_2} $$

$$ \langle \prod_i \int d^2 \xi_i V_i(\xi_i) E_\alpha(z_1) \otimes q^{L_0-1} \otimes E_\alpha(z_2) \otimes (\text{ghosts}) \rangle_{\Sigma_1 \oplus \Sigma_2} $$

(3.13)

where $\int dm$ means integration over respective moduli space, $V_i(\xi_i)$ are the vertex operators for scattering states, $E_\alpha$ are the complete set of the intermediate states with dimensions $h_\alpha, \bar{h}_\alpha$ propagating along the thin tube, connecting the world-sheet pieces $\Sigma_1$ and $\Sigma_2$ (in the case of the degenerating torus handle of interest to us, $\Sigma_1 = \Sigma_2$). The terms ‘ghosts’ indicate appropriate insertions of ghost fields.

One easily observes that extra logarithmic divergencies in (3.13), may come from states with $h_\alpha, \bar{h}_\alpha = 0$ [17], in which case one has the infrared divergence at small $q \to 0$ integral $\int dq d\eta / q^4$. A trivial example of such an operator is the identity, which however
does not lead to non-trivial effects, since it carries zero measure in the space of states, and hence it does not contribute to (3.13). On the other hand, if there are states that are separated by a gap from the continuum of states, i.e. discrete in the space of states, then they bear non-trivial contributions to the sum-over-states and lead to divergencies in the amplitude (3.13).

The logarithmic states we discussed earlier are precisely these states, and their total dimension is zero because of the ghost insertion - actually they have the form $\bar{c}c \otimes \partial_{\alpha}J^{\alpha}$ and ghosts shift the total anomalous dimension from $(1, 1)$ to $(0, 0)$.

However we have to find another logarithmic divergence, instead of the normalization factor $N_{2}$ - which is precisely the logarithmic terms in (2.24) arising due to the mixing between $C$ and $D$ states. After some algebra we obtain expressions of the type

$$\frac{1}{\kappa} \int \frac{dq dq}{qq} \left[ \ln q \int d^{2}z_{1} D(z_{1}) \int d^{2}z_{2} C(z_{2}) + c.c. \right]$$

$$\sim \frac{1}{\kappa} \left( \ln^{2} \epsilon \right) \int d^{2}z_{1} D(z_{1}) \int d^{2}z_{2} C(z_{2}) + c.c.$$  (3.14)

which gives the leading singularity $\ln^{2} \epsilon$. Besides this term there will be terms with $\ln \epsilon$ corresponding to $\int d^{2}z_{1} D(z_{1}) \int d^{2}z_{2} D(z_{2})$ and $\int d^{2}z_{1} C(z_{1}) \int d^{2}z_{2} C(z_{2})$ terms. The effects of a dilute gas of wormholes on the sphere exponentiate this bilocal operator and one can obtain a change in the world-sheet action (3.1)

$$\Delta S \sim \frac{1}{\kappa} \left( \ln^{2} \epsilon \right) \int d^{2}z_{1} D(z_{1}) \int d^{2}z_{2} C(z_{2})$$  (3.15)

This bilocal term can be written as a local world-sheet effective action term, if one employs the well-known trick of wormhole calculus [18] by writing

$$e^{\Delta S} \propto \int d\alpha_{C} d\alpha_{D} \exp \left[ G^{mn} \alpha_{m} \alpha_{n} + \alpha_{C} \frac{\ln \epsilon}{\kappa} \int d^{2}z C + \alpha_{D} \frac{\ln \epsilon}{\kappa} \int d^{2}z D \right]$$  (3.16)

and, thus, it can be represented as a local deformation on the world-sheet of the string but with a coupling constant $\alpha_{m}$, where $m = C$ or $D$ and $G^{mn}$ is the metric necessary to reproduce the initial bilocal operator.

We shall not discuss this problem in more detail; further analysis can be done along the lines of ref. [7], where logarithmic divergences $\ln \epsilon$ in the case of elastic scattering on a soliton lead to the conservation of momentum. A more detailed picture will be given in
4. Logarithmic operators in exact CFT backgrounds

4.1. CFT for string solitons

To warm up and make it clearer what we mean, we start by a simple four-dimensional effective action involving antisymmetric tensor, dilaton and graviton fields. The action reads

$$S = -\frac{1}{\kappa^2} \int d^4x \sqrt{g}\{R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12} H_{\mu\nu\rho}^2\}$$

(4.1)

A solution which could correspond to an extended object in target space corresponds to the following background configuration [19],

$$g_{\mu\nu} : \quad ds^2 = e^{\phi(x)} dx^\mu dx_\mu$$
$$B_{\mu\nu} : \quad H^*_\mu = e^{\mu\nu\rho\sigma} H_{\nu\rho\sigma} = \pm \partial_\mu e^{\phi(x)}$$

(4.2)

where we wrote only the dependence on 4 Cartesian coordinates $x^\mu$ in a 4D transverse space. The Bianchi identity for $H^*$, $\partial^\mu H^*_\mu = 0$, implies

$$\nabla^2 e^\phi = 0$$

(4.3)

which leads to a solution [19]

$$e^\phi = e^{\phi_0} + \sum_{i=1}^n \frac{Q_i}{|x - a_i|^2}$$

(4.4)

The 4D transverse space is asymptotically flat, at each singular point $a_i$ a semiinfinite wormhole is glued in. To see this let us restrict ourselves to the simple case $n = 1$ and neglect $\exp(\phi_0)$ term. Then, one gets for the dilaton

$$e^{\phi(x)} = Q/r^2$$

(4.5)

with $r^2 = \sum_{i=1}^4 x_i^2$. This leads to a linear dilaton [20] by a change of coordinates

$$\phi = -t/\sqrt{Q}; \quad t \equiv \sqrt{Q} \ln \sqrt{r^2/Q}$$

(4.6)
and to the metric
\[ ds^2 = dt^2 + Q^2 d\Omega_3^2 \] (4.7)
with \( S^3 \) a three sphere, and \( Q \) is the axion charge defined as \( Q = \int_{S^3} H \). The geometry implied by (4.7) is that of a half throat consisting of a \( S^3 \) sphere of radius \( \sqrt{Q} \) times the open line \( R^1 \) and a linear dilaton proportional to the coordinate on \( R^1 \), necessary for conformal invariance due to the non-criticality of the target dimension. To this order in \( \alpha' \propto 1/k \) we can see, by looking at the dilaton \( \beta \) function of the theory, that the states of the theory are those of the tensor product of a compact \( SU(2) \) WZNW model of level \( k = Q/\alpha' \rightarrow \infty, \alpha' \rightarrow 0 \), representing the sphere \( S^3 \) part and the antisymmetric tensor, of central charge
\[ c_{wzw} = \frac{3k}{k - 2} \] (4.8)
and a Feygin-Fuks representation for the linear dilaton [20] and the time coordinate \( t \) (half the real line), with central charge
\[ c_{FF} = 1 + \frac{6}{k} \] (4.9)
The sum of the two central charges amounts to 4 up to terms of order \( (1/k^2) \). In the semi-classical limit of \( k \rightarrow \infty (\alpha' \rightarrow 0) \), one can see that the conformal blocks of the WZNW theory [13] corresponds to the hypergeometric functions of the form
\[ F(0, 0, 1; x), \quad F(1, 1, 2; x), \quad F(0, 0, 0; x), \] (4.10)
For the first one we definitely have logarithms \( \ln x \) in general solution (see section (2.1)), where for the last one the hypergeometric equation (2.2) becomes
\[ x(1-x)\frac{d^2 F}{dx^2} - x\frac{dF}{dx} = 0 \] (4.11)
with the general solution \( F = A + B \ln(1-x) \), which has a logarithmic singularity at \( x = 1 \). The situation is tempting to suggest that there may be logarithmic operators in this case also, but the approximation of \( k \rightarrow \infty \) is not really an exact result so as to justify the above argumentation.

The resolution comes by including supersymmetry, as described in ref. [19]. In that case the four dimensions are pertaining to the internal dimensions of some four dimensional
superstring constructions, but the details are irrelevant for our purposes. The important point to notice is that there are anomalies associated with left-right fermions, which effectively amount to modifying the level $k$ of the supersymmetric WZNW model by $k - 2$. The resulting central charge of the WZW model is now

$$c_{WZW} = \frac{3(k - 2)}{k} + \frac{3}{2}$$

(4.12)

and that of the Feygin-Fuks/dilaton part

$$c_{FF} = 1 + \frac{6}{k} + \frac{1}{2}$$

(4.13)

where the factors of $3/2$ in (4.12) and $1/2$ in (4.13) are due to the supersymmetric fermions of the CFT, which are essentially free. The total central charge now is 6, and the expansion in powers of $1/k$ truncates, leading to exact results unlike the bosonic case.

Let us now locate the zero modes and the origin of logarithmic operators in the superconformal blocks of the WZNW model. The conformal blocks were considered, for example, in ref. [21]. Consider the four point functions of a supersymmetric WZNW model with the group $SU(N)$ - actually we need here only $N = 2$. The procedure is analogous to the bosonic case [13], and one can derive easily the analogues of the Knizhnik-Zamolodchikov equations that define the superconformal blocks, $f_A^{(p)}$, $f_B^{(q)}$ in a standard notation. In this particular case, the solution acquires the form

$$f_{AB}(x, \bar{x}) = \sum_{p, q = 1}^{2} a_{pq} f_A^{(p)}(x) f_B^{(q)}(\bar{x})$$

(4.14)

with some constants $a_{pq}$ and

$$f_1^{(1)}(x) = x^{-2\Delta}(1 - x)^{\tilde{\Delta} - 2\Delta} F\left(\frac{1}{k}, \frac{1}{k}, 1 - \frac{N}{k}; x\right)$$

$$f_1^{(2)}(x) = [x(1 - x)]^{\tilde{\Delta} - 2\Delta} F\left(\frac{N + 1}{k}, \frac{N - 1}{k}, 1 + \frac{N}{k}; x\right)$$

$$f_2^{(1)}(x) = (k - N)^{-1} x^{1 - 2\Delta}(1 - x)^{\tilde{\Delta} - 2\Delta} F\left(1 + \frac{1}{k}, 1 - \frac{1}{k}, 2 - \frac{N}{k}; x\right)$$

$$f_2^{(2)}(x) = -N[x(1 - x)]^{\tilde{\Delta} - 2\Delta} F\left(\frac{N + 1}{k}, \frac{N - 1}{k}, \frac{N}{k}; x\right)$$

(4.15)

where

$$\Delta = \frac{N^2 - 1}{2Nk}, \quad \tilde{\Delta} = \frac{N}{k}$$

(4.16)
It is also important to know that unitary highest weight representations in this model exist only for non-negative integer \( \hat{k} = k - N \) (see [21] and references therein), which means that minimal possible \( k \) in this case is \( N \), i.e. minimal \( k = 2 \) in case of \( SU(2) \). Taking into account that in the model at hand the level \( k \) is associated with the size of the throat, i.e. the axion charge \( Q \), we shall consider the “minimal” soliton.

In this case one can see immediately that the dimension of the adjoint field is \( \tilde{\Delta} = 1 \) and we have a degeneracy with all the consequences. Using (2.7) and (2.8) one can see that at \( k = N \) we have the set of solutions similar to what was considered in [4]

\[
\begin{align*}
  f^{(1)}_1(x) &= [x(1-x)]^{1/N^2} \ln x F \left( \frac{N+1}{N}, \frac{N-1}{N}, 2; x \right) + H_2(x) \\
  f^{(2)}_1(x) &= [x(1-x)]^{1/N^2} F \left( \frac{N+1}{N}, \frac{N-1}{N}, 2; x \right) \\
  f^{(1)}_2(x) &= [x(1-x)]^{1/N^2} \ln x F \left( 1 + \frac{N+1}{N}, \frac{N-1}{N}, 1; x \right) + H_1(x) \\
  f^{(2)}_2(x) &= [x(1-x)]^{1/N^2} F \left( \frac{N+1}{N}, \frac{N-1}{N}, 1; x \right)
\end{align*}
\] (4.17)

where \( H_1 \) and \( H_2 \) are two functions discussed in section (2.1) and we took into account that \( 1 - 2\Delta = 1/N^2 \) for \( k = N \). Using the same methods as in [4] we can get the OPE for two primary fields in this model which has the form

\[
V(z) V^{-1}(0) = |z|^{2/N^2 - 2} \left\{ I + z\overline{z} \left[ \mathcal{D}(0) + \mathcal{C}(0) \ln |z|^2 \right] + \ldots \right\} \] (4.18)

From this OPE one can see immediately that the logarithmic pair \( \mathcal{D}, \mathcal{C} \) has anomalous dimension \( h = \bar{h} = 1 \). These fields can be constructed either from the adjoint field with \( h = \tilde{\Delta} = 1 \) and the corresponding Kac-Moody currents, or from objects like \( \partial(\mathcal{C}, D) \) and their complex conjugates. This situation is slightly different from the chiral situation of ref. [4]. At present we are not in a position to determine the precise form of the logarithmic pair. Anyhow, the above analysis supports our conjecture that in the CFT corresponding to the solitonic background there are logarithmic operators with dimension 1 which generate the symmetry transformations.

It appears from the above considerations that logarithmic operators appear only for certain values of the level \( k \), namely \( k = 2 \) for the model of 5-brane. This is contrary to the generic arguments given above on the existence of zero modes due to the broken translational invariance by the center of mass of the solitonic solution (instanton, monopole...
etc), which appears to be a general argument. The resolution is provided by the fact that so far we have looked at part of the problem, only the supersymmetric WZNW model that refers to the $S^3$ and the antisymmetric part of the dilaton. The independence of $k$ may come from combining (tensoring) this model with the Feygin-Fuks conformal field theory, and then repeating the Kniznik-Zamolodchikov equation for the tensored CFT. This hope is based on the fact that only for $k = 2$ we got the second operator with dimension 1.

Let us note that in the case of the CFT model of disorder there were also two types of fields: a $SU(r)$ WZNW field and a $U(1)$ scalar field, and only by taking into account both of these fields one was able to recover the logarithmic operators in a general case. We hope the same is true in this model also. One, then, has to find the analogue of the hypergeometric functions appearing in the conformal blocks of the tensor model in this case. Once this is done, one should hopefully be able to show the existence of logarithmic operators as a result of the existence of a degenerate (zero) mode in the spectrum, other than the identity, corresponding to ‘recoil’ of the soliton center of mass.

### 4.2. 2D Black Holes

As we discussed earlier there are logarithmic operators in the Liouville $c = 1$ theory which are

$$C = \exp(\sqrt{2}\phi); \quad D = \frac{1}{\sqrt{2}}\phi \exp(\sqrt{2}\phi) \quad (4.19)$$

It is interesting to know if this pair exists in a two-dimensional black hole model which is a generalization of the $c = 1$ string theory. 2D black holes can be described as *exact* conformal field theories of gauged WZNW type over the coset $SL(2, R)/U(1)$ with the central charge $c = 3k/(k - 2) - 1$, where $k$ is the level of the $SL(2, R)$ Kac-Moody algebra. At $k = 9/4$ one has the critical string with $c = 26$. One can see that the factor $1/(k - 2) = 4$ in this case, and for pure $SL(2, R)$ conformal blocks one again has integer $c$ in the corresponding hypergeometric functions. However the conformal blocks for 2D black holes correspond to the coset $SL(2, R)/U(1)$ and their actual form is yet unknown. So, instead of attacking this problem in the same way as before, we shall try to find some marginal deformations of the background which far from the horizon, where the space-time corresponds to the $c = 1$ Liouville theory, will have logarithmic terms. The $\sigma$-model
action describing a Euclidean black hole can be written in the form

\[ S = \frac{k}{4\pi} \int d^2 z \frac{1}{1 + |w|^2} \partial_\mu \bar{w} \partial^\mu w + \ldots \quad (4.20) \]

where the conventional radial and angular coordinates \((r, \theta)\) are given by \(w = \sinh r e^{-i\theta}\) and the target space \((r, \theta)\) line element is

\[ ds^2 = \frac{dw \bar{dw}}{1 + w \bar{w}} = dr^2 + \tanh^2 r d\theta^2 \quad (4.21) \]

The corresponding exactly-marginal deformation, which turns on matter backgrounds in this geometry is constructed by \(W_\infty\) symmetry considerations [23, 9], and is given by [24]

\[ L^1_0L^1_0 \propto F^{c-c}_{\frac{1}{2},0,0} + i(\psi^{++} - \psi^{--}) + \ldots \quad (4.22) \]

where the \(\psi\) denote higher-string-level operators [24], and the ‘tachyon’ operator is given by

\[ F^{c-c}_{\frac{1}{2},0,0}(r) = \frac{1}{\cosh r} F^{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \tanh^2 r} \quad (4.23) \]

There is an integral representation for the hypergeometric function [11]

\[ F(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a} dt \quad (4.24) \]

which for the hypergeometric function \(F(1/2, 1/2, 1; x)\) gives us

\[ F(1/2, 1/2, 1; x) = \int_0^{\pi/2} (1 - x \sin^2 \theta)^{-1/2} d\theta \quad (4.25) \]

and one sees immediately that at \(x \to 1\) there is a logarithmic singularity. Actually one can show that

\[ F(1/2, 1/2, 1; x) = F(1/2, 1/2, 1; 1 - x) \ln(1 - x) + H(x) \quad (4.26) \]

where \(H(x)\) is regular at \(x \to 1\). Let us note that precisely this function appears as a conformal block in \(c = -2\) model [11].

The marginal deformation [11,22] contains the generalization of the cosmological constant operator. Asymptotically in space, \(r \to \infty\), this operator reduces to the flat-space \(c = 1\) exactly marginal deformation of the \(c = 1\) matrix model discussed above,
which has logarithmic operators. To see this let us consider $r \rightarrow \infty$, which means that $x = \tanh^2 r \rightarrow 1$. From (4.26) one observes that

$$
\mathcal{F}_{c,c,1,0,0}(r) = \frac{1}{\cosh r} F\left(\frac{1}{2}, \frac{1}{2}; 1, \tanh^2 r\right) = \frac{1}{\cosh r} \ln \frac{1}{\cosh^2 r} F\left(\frac{1}{2}, \frac{1}{2}; 1, 1/\cosh^2 r\right) + \frac{1}{\cosh r} H(\tanh^2 r)
$$

$$
\rightarrow r \exp(-r) + A \exp(-r) + O(\exp(-2r)),
$$

where $A$ is some constant. After appropriate change of variables $r = \sqrt{2\phi}$ we see that at $r \rightarrow \infty$ the background $\mathcal{F}_{c,c,1,0,0}(r)$ becomes the $c = 1$ cosmological constant background including the logarithmic operator $T(\phi) \sim (\phi - \frac{1}{2} \ln \mu) \sqrt{\mu} e^{-\sqrt{2}\phi}$.

Thus the exactly marginal deformation of the black hole background also have logarithmic operators and one can hope that they can be observed in the corresponding conformal blocks. As we saw, in the case of black hole background the logarithmic operators are closely related to the hidden $W_\infty$ symmetry, however the full picture needs further clarifications. It is an interesting problem to see how these new operators will affect the scattering amplitude in the black hole background.

5. Conclusions

In this paper we have demonstrated that in the general class of solitonic string backgrounds a new phenomenon takes place - the emergence of logarithmic operators associated with a target space symmetry.

The physical meaning of the target space symmetry is the existence of the normalizable zero modes corresponding to the deformations of the string background under the symmetry of the full theory. The simplest examples we considered support this picture and this is a pleasant fact.

We think that the phenomenon we discussed is very general and it is important for studying the general problem of back-reaction in string theory - the recoil problem we considered is only one of the problems of this type. It will be interesting to expand our
methods to the case of general $P$- and $D$-branes. We hope to return to this, as well as to other related problems, such as black holes in string theory etc, in the near future.

It will be also interesting to find a general classification of all possible logarithmic operators and to use this as a world-sheet approach to classify the possible target space symmetries in string theory. At present this remains the biggest challenge.

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References

[1] V. Gurarie, Nucl. Phys. B410, 535 (1993); hep-th/9303160.

[2] A. Bilal and I.I. Kogan, PUPT-1482, hep-th@/9407151 (unpublished); Nucl. Phys. B449, 569 (1995); hep-th@/9503209.

[3] M.A. Flohr, CSIC-IMAFF-42; hep-th/9509166.

[4] J.S. Caux, I.I. Kogan and A. Tsvelik, *Logarithmic operators and Hidden Continuum Replica Symmetry in the Exactly Solvable Model with Disorder*, Oxford preprint (1995), OUTP-95-62, hep-th/9511130.

[5] H.G. Kausch, DAMTP 95-52; hep-th/9510149.

[6] L. Rozansky and H. Saleur, Nucl. Phys. B376, 461 (1992).

[7] W. Fischler, S. Paban and M. Rozali, Phys. Lett. B352 (1995), 298; hep-th/9503072.

[8] A.B. Zamolodchikov, JETP Lett. 43 (1986), 730; Sov. J. Nucl. Phys. 46 (1987), 1090.

[9] J. Ellis, N.E. Mavromatos and D.V. Nanopoulos, CERN-TH.7195/94, ENS-LAPP-A-463/94, ACT-5/94, CTP-TAMU-13/94, lectures presented at the Erice Summer School, 31st Course: From Supersymmetry to the Origin of Space-Time, Ettore Majorana Centre, Erice, July 4-12 1993; hep-th/9403133, to be published in the proceedings (World Sci.)

In a different context, see: N.E. Mavromatos and D.V. Nanopoulos, ACT-19/95, CTP-TAMU-55/95, OUTP-52P, quant-ph/9512021.

[10] I.I. Kogan, Phys. Lett. B265 (1991), 269; preprint UBCTP-91-13 (June 1991); Proc. *Particles and Fields 91*, Vancouver 18-22 April 1991 (eds. Axen, Bryman and Comyn, World. Sci. 1992);

J. Ellis, N.E. Mavromatos and D.V. Nanopoulos, Phys. Lett. B293 (1992), 37.

[11] E.T. Whittaker and G.N. Watson, *A course of Modern Analysis*, Cambridge University Press (1927).
[12] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, San Diego, p. 1046, (1992).

[13] V.G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B247, 83 (1984).

[14] A.A. Belavin, A.M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B241, 333 (1984).

[15] J. Polchinski, Nucl. Phys. B346, 253 (1990);

[16] A.M. Polyakov, *Gauge Fields and Strings*, Harwood Academic Publishers (1987).

[17] J. Polchinski, Nucl. Phys. B307 (1988), 61; *ibid* B357 (1995), 241.

[18] S. Coleman, Nucl. Phys. B310, 643 (1988).

[19] C. Callan, J. Harvey and A. Strominger, Nucl. Phys. B359, 611 (1991); Nucl. Phys. B367, 60 (1991).

[20] I. Antoniades, C. Bachas, J. Ellis, and D.V. Nanopoulos, Phys. Lett. B211 (1988), 393; Nucl. Phys. B328 (1989), 117.

[21] J. Fuchs, Nucl. Phys. B286 (1987), 455.

[22] E. Witten, Phys. Rev. D44 (1991), 344.

[23] I. Bakas and E. Kiritsis, Int. J. Mod. Phys. A7 (Suppl. 1A) (1992), 55.

[24] S. Chaudhuri and J. Lykken, Nucl. Phys B396 (1993), 270.