GEOMETRY OF G–STRUCTURES VIA THE INTRINSIC TORSION

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Abstract. We study the geometry of a G–structure P inside the oriented orthonormal frame bundle SO(M) over oriented Riemannian manifold M. We assume the quotient SO(n)/G, where n = dim M, is a natural homogeneous space and we equip SO(M) with the natural Riemannian structure induced from the structure on M and the Killing form of SO(n). We show, in particular, that minimality of P is equivalent to harmonicity of a induced section of the homogeneous bundle SO(M) × SO(n)/SO(n) with the modified Riemannian metric on M and the minimality of the image of this section. We apply obtained results to the case of almost product structures, i.e. structures induced by plane fields, and to almost hermitian structures.

1. Introduction

Existence of a geometric structure on an oriented Riemannian manifold is equivalent to saying that the structure group SO(n) of the oriented orthonormal frame bundle reduces to a certain subgroup G. For example, for G equal

SO(m) × SO(n − m), U(n/2), U(n/2) × 1, Sp(n/4)Sp(1), G2, Spin(7)

we have almost product, almost Hermitian, almost contact, almost quaternion–Kähler, G2 and spin structures, respectively. It is natural to ask if the holonomy group of the Levi–Civita connection ∇ is contained in given G. On the other hand the list of possible irreducible Riemannian holonomies is limited by the Berger list [17].

The defect of the Levi–Civita connection to be a G–connection measures the intrinsic torsion ξ, which is the difference of ∇ and a G–connection ∇G (with torsion),

\[ ξ_X Y = ∇_X Y - ∇^G_X Y. \]

The study of possible intrinsic torsions, i.e. the decomposition of the space of intrinsic torsions into irreducible modules, was initiated by Gray and Hervella [14] in the case of almost Hermitian manifolds. Later, many authors considered other possible cases (see, for example, [1, 3, 8, 16, 18, 21]).

The other possible direction, initiated by Wood [22] and generalized to the general case by Gonzalez–Davila and Martin Cabrera [12], is to consider differential properties of intrinsic torsion induced by condition of harmonicity of
the unique section of the associated homogeneous bundle. More precisely, a $G$–structure $P \subset SO(M)$ induces the unique section $\sigma$ of the homogeneous associated bundle $SO(M)/G = SO(M) \times_{SO(n)} SO(n)/G$,
\[
\sigma(\pi_{SO(M)}(p)) = [p, eG], \quad p \in SO(M),
\]
where $\pi_{SO(M)}$ is the projection in the orthonormal frame bundle $SO(M)$. Assume the decomposition $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement to $\mathfrak{g}$ with respect to the Killing form, on the level of Lie algebras is reductive. Equip $SO(M)/G$ with the natural Riemannian metric induced from the Killing form on $\mathfrak{m}$ and Riemannian metric $g$ on $M$. Then we say that a $G$–structure $P$ is harmonic if the section $\sigma$ is a harmonic. The correspondence of the notion of harmonicity with the intrinsic torsion follows from the fact that the intrinsic torsion $\xi$ can be considered as a section of the bundle $T^*M \otimes \mathfrak{m}_P$, where $\mathfrak{m}_P$ is the adjoint bundle $P \times_{\text{ad}} G$. This follows from the observation that the $\mathfrak{m}$–component of the connection form $\omega$ with respect to the reductive decomposition $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ can be projected to the tangent bundle $TM$ (see the next section for the details).

To seek for the 'best’ possible non–integrable $G$–structures we consider the third possible approach. Namely, we focus on the minimality of a $G$–structure in the oriented orthonormal frame bundle $SO(M)$. It is not surprising that minimality is related with the harmonicity of a $G$–structure. More generally, we use the concept of intrinsic torsion to obtain some results on the geometry of $G$–structures. The idea comes from the results obtained by the author [19] in the case of a single submanifold. We deal with the intrinsic and extrinsic geometry of a $G$–structure. To be more precise, we define the Riemannian metric on $SO(M)$ by inducing it from the Riemannian metric $g$ on $M$ and Killing form $B$ on the structure group $SO(n)$. It is interesting that the Levi–Civita connection on $P$ depends on the $G$–connection and Levi–Civita connection $\tilde{\nabla}$, which comes from the modification of the Riemannian metric $g$. This deformation $\tilde{g}$ depends on the intrinsic torsion and equals the pull–back of the Riemannian metric on the associated homogeneous bundle $SO(M)/G$ with respect to the section $\sigma$.

The main emphasis is put on the minimality of a $G$–structure $P$ in the Riemannian structure $SO(M)$. The main theorem can be stated as follows.

**Theorem.** The following conditions are equivalent

1. $G$–structure $P$ is minimal in $SO(M)$,
2. the induced section $\sigma : (M, \tilde{g}) \to (N, \langle \cdot, \cdot \rangle)$ is a harmonic map, where $\tilde{g}$ is a deformation of $g$ such that $\tilde{g} = \sigma^* \langle \cdot, \cdot \rangle$,
3. the submanifold $\sigma(M)$ is minimal in $N$.

Whereas, the vanishing of the second fundamental form is the necesary condition for the integrability of a $G$–structure.

**Theorem.** If a $G$–structure $P$ is integrable, i.e. the intrinsic torsion vanishes, then $P$ is totally geodesic in $SO(M)$.

The article is organized as follows. In the second and third sections, we recall the notion of the intrinsic torsion and state its main properties. Although these results can be found in the literature [12] [23], to make the article self contained, we give all the proofs. Some of them are different from the oryginal ones.
In the fourth section we introduce a tensor, which transfers the Riemannian metric \( g \) to the mentioned above \( \tilde{g} \). Its properties are crucial in the main considerations. With the Riemannian metric \( \tilde{g} \) and the horizontal distribution \( \mathcal{H}' \) of the \( G \)-structure \( P \) induced by the minimal \( G \)-connection \( \nabla' \), the projection \( \pi_P : P \to M \) becomes the Riemannian submersion.

The fifth section is the main section of the general outline and deals with the geometry of \( G \)-structures. Firstly, we consider intrinsic geometry focusing on the curvatures and, secondly, we consider extrinsic geometry, with the main result concerning minimality of a \( G \)-structure.

We end the article with some relevant examples. We consider almost product and almost hermitian structures.

Throughout the paper we will use the following notation and identification: For any associated bundle \( E = P \times_G S \) with the fiber \( S \) and induced by the principal bundle \( P(M, G) \) any element in \( E \) will be denoted by \( [\![ p, s \!]\] \). Moreover, we have \( \Gamma(E) \cong C^\infty(P, S)^G \), where \( C^\infty(P, S)^G \) is the space of equivariant functions \( f : P \to S \), \( f(pg) = g^{-1}f(p) \). The identification is the following \( \Gamma(E) \ni \sigma \mapsto f \in C^\infty(P, S)^G, \sigma(\pi_P(p)) = [\![ p, f(p) \!]\] \).

2. INTRINSIC TORSION

In this section we will review the basic facts concerning intrinsic torsion of a \( G \)-structure [17, 12, 23]. To make the article self contained we present all the proofs.

Let \( (M, g) \) be oriented Riemannian manifold, \( SO(M) \) its oriented orthonormal frame bundle. Denote by \( \omega \) the connection form induced by the Levi–Civita connection \( \nabla \) on \( M \). Let \( \mathcal{H} \) and \( \mathcal{V} \) be horizontal and vertical distributions on \( SO(M) \), respectively.

Assume the structure group \( SO(n) \) reduces to a subgroup \( G \) such that the quotient \( SO(n)/G \) is normal reductive, i.e. the subspace \( m = g^\perp \subset so(n) \) defines \( ad(G) \) invariant decomposition \( so(n) = g \oplus m \), where \( g \) is the Lie algebra of \( G \) and the orthogonal part is taken with respect to the Killing form \( B \). Denote by \( P \subset SO(M) \) the reduced subbundle. The \( g \)-component \( \omega_g \) of \( \omega \) defines the connection form on \( P \). Denote by \( \mathcal{H}' \) and \( \mathcal{V}' \) the horizontal and vertical distributions on \( P \) with respect to \( \omega_g \), respectively.

Moreover, consider the following associated bundles – the adjoint bundles over \( M: so(n)_P = P \times_{adG} so(n), g_P = P \times_{adG} g \) and \( m_P = P \times_{adG} m \).

Notice that \( so(n)_P \) can be realized as the bundle \( so(M) \) of skew–symmetric endomorphisms of the tangent bundle \( TM \) via the identification \( [\![ p, A \!]\] \mapsto p^{-1} \cdot A \cdot p, \)

where we treat element \( p \in P_x \) as a linear isomorphism \( p : \mathbb{R}^n \to T_x M \). Analogously, we have the bundles \( g(M) \) and \( m(M) \). Moreover, \( so(n)_P \) is isomorphic to the bundle \( \mathcal{V}_{equiv} \) over \( M \) of equivariant vertical vector fields on \( P \). Namely, the map \( [\![ p, A \!]\] \mapsto A^*_p, \)

settles this isomorphism, where \( A^* \) denotes the vertical vector induced by the matrix \( A \in so(n) \).
Let
\[ \xi(X) = \omega_m(X^{h'}), \]
where \( \omega_m \) denotes the \( m \)-component of \( \omega \) and \( X^{h'} \) is the horizontal lift with respect to \( \mathcal{H}' \). Since
\[ \omega_m(X^{h'}) = \omega_m(R_{\xi X}X^{h'}) = \text{ad}(g^{-1})\omega_m(X^{h'}), \]
it follows that \( \xi(X) \in m_P \). Section \( \xi \in \Gamma(T^*M \otimes m_P) \) is called the intrinsic torsion of a \( G \)-structure \( P \). By above identifications \( \xi(X) = \xi_X \in m(M) \).

The properties of the intrinsic torsion are gathered in the following proposition.

Let us first introduce necessary notions. Let \( \nabla' \) be the connection on \( M \) induced by \( \omega_\Theta \) on \( P \). We call \( \nabla' \) the minimal connection of a \( G \)-structure.

**Proposition 2.1.** Intrinsic torsion \( \xi \) has the following properties:

1. \( X^{h'} = X^h + (\xi_X)^* \)
2. \( \xi_X Y = \nabla_X Y - \nabla'_X Y \)

for \( X, Y \in \Gamma(TM) \).

**Proof.** Since
\[ \pi_P(X^{h'} - (\xi_X)^*) = X \quad \text{and} \quad \omega(X^{h'} - (\xi_X)^*) = \omega_m(X^{h'}) - \xi_X = 0, \]
(1) holds. Let \( f : P \to \mathbb{R}^n \) be the equivariant function corresponding to \( Y \in \Gamma(TM) \). Then
\[ \nabla_X Y - \nabla'_X Y = X^h f - X^{h'} f = -(\xi_X)^* f \]
\[ = -\frac{d}{dt}f(p \exp(t \xi_X)) = -\frac{d}{dt}\exp(-t \xi_X) f(p) \]
\[ = \xi_X Y, \]
which proves (2). \( \square \)

**Remark 2.2.** One can show that \( \delta \xi = -\Theta \), where \( \Theta \) is the torsion of \( \omega \) and \( \delta : \Gamma(T^*M \otimes g_P) \to \Gamma(TM) \otimes \Lambda^2(T^*M) \) is the anty–symmetrization. This justifies the name for intrinsic torsion. On the level of \( M \), by above Proposition, this is equivalent to the following equality
\[ (\text{Alt} \, \xi)(X, Y) = \xi_X Y - \xi_Y X = -(\nabla'_X Y - \nabla_Y X - [X, Y]) = -T'(X, Y), \]
where \( T' \) is the torsion of \( \nabla' \).

Intrinsic torsion and the curvature tensor \( R \) obey the following formulas.

**Proposition 2.3.** The decomposition \( \mathfrak{so}(M) = g(M) \oplus m(M) \) implies the following relations
\[ (\nabla_X \xi_Y)_\Theta = [\xi_X, \xi_Y]_\Theta, \]
\[ (\nabla_X \xi_Y)_m = \nabla_X' \xi_Y + [\xi_X, \xi_Y]_m, \]
\[ R(X, Y)_\Theta = R'(X, Y) + [\xi_X, \xi_Y]_\Theta, \]
\[ R(X, Y)_m = \nabla'_X \xi_Y - \nabla'_Y \xi_X + [\xi_X, \xi_Y]_m - \xi_{[X, Y]}. \]

**Proof.** First two relations follow by Proposition 2.1 and from the fact that \( \nabla' \) respects the decomposition \( g(M) \oplus m(M) \). They imply, immediately, the remaining ones. \( \square \)
Notice also that
\begin{equation}
R(X,Y) = R'(X,Y) + (\nabla_X \xi)_Y - (\nabla_Y \xi)_X - [\xi_X, \xi_Y],
\end{equation}
where
\begin{equation}
(\nabla_X \xi)_Y = \nabla_X \xi_Y - \xi \nabla_X Y.
\end{equation}

Let now \( N = SO(M)/G = SO(M) \times_{SO(n)} (SO(n)/G) \) be the homogeneous bundle associated with \( SO(M) \). Denote by \( \zeta : SO(M) \to N \) the natural projection. Then \( \zeta \) defines the \( G \)-principal bundle. Clearly \( \zeta \) is constant on \( P \) and hence there is a bijection between \( G \)-reductions of \( SO(M) \) and sections of \( N \). Denote by \( \sigma \in \Gamma(N) \) the section induced by \( P \). Let \( m_N \) be the adjoint bundle associated with \( \zeta \), i.e., \( m_N = SO(M) \times_{adG} m \). Then \( m_N \) is isomorphic to the bundle \( \zeta^*V_{equiv} \).

Let \( \mathcal{H}^N \) and \( \mathcal{V}^N \) be horizontal and vertical distributions on \( N \), respectively, where \( \mathcal{H}^N \) is induced by \( \omega \). We have
\begin{equation}
\mathcal{H}^N = \zeta_* \mathcal{H}, \quad \mathcal{V}^N = SO(M) \times_{SO(n)} T(SO(n)/G),
\end{equation}
where \( SO(n) \) acts on \( T(SO(n)/G) \) by the differential of the natural action of \( SO(n) \) on \( SO(n)/G \). Since \( T(SO(n)/G) = SO(n) \times_{adG} m \) it follows that \( \mathcal{V}^N \) is isomorphic to \( m_N \).

The \( g \)-component \( \omega_g \) of \( \omega \) defines the connection on \( \zeta : N \to M \). Denote the horizontal lift of the vector \( U \in TN \) by \( U^{h,\zeta} \in TSO(N) \). Since the following diagram commutes
\[
\begin{array}{ccc}
P & \xleftarrow{\pi_P} & SO(M) \\
\downarrow & & \downarrow \zeta \\
M & \xleftarrow{\pi_N} & N
\end{array}
\]
and connection forms on \( \pi_P : P \to M \) and \( \zeta : SO(M) \to N \) are both equal to \( \omega_g \) it follows that
\begin{equation}
U^{h,\zeta}_p = (\pi_N)_* U^{h'}_p, \quad p \in P.
\end{equation}
In particular, \( U^{h,\zeta} \notin \mathcal{H} \), unless \( \omega_m = 0 \).

Moreover, we will frequently identify sections of \( N \) with equivariant maps from \( SO(M) \) to \( SO(n)/G \). Denote the equivariant function corresponding to \( \sigma \) by \( f \). Then
\[
\nu \sigma_s(X) = f_s(X^h_p), \quad X \in T_x M, \pi(p) = x,
\]
where \( \nu Z \) denotes the vertical component of \( Z \in TN \). Indeed, if \( \gamma \) is a curve on \( M \) such that \( \dot{\gamma}(0) = X \), denoting by \( \gamma^h \) its horizontal lift to \( \gamma^h(0) = p \), by above isomorphisms, we have
\[
\sigma_s(X) = \frac{d}{dt} (\sigma \circ \gamma)_t = \frac{d}{dt} ([\gamma^h(t), f(\gamma^h(t))]) = [[X^h, 0]] + [[p, f_s(X^h_p)]] = X^{h,N} + f_s(X^h_p).
\]
Since $f$ is constant on $P$, by Proposition 2.1
\[\nabla^\alpha(X) = f_*(X^h) = f_*(-\langle \xi_X \rangle^*)\]
\[= -\frac{d}{dt} f(p \exp(t\xi_X)) = -\frac{d}{dt} \exp(-t\xi_X) f(p)\]
\[= \xi_X,\]
where we consider $\xi_X$ as an element of $m_N$.

Moreover, denote by $\varphi : TN \to m_N$ the following map
\[\varphi([[p, [[g, A]]]]) = [[p, A]],\]
\[\varphi(X^{h,N}) = 0,\]
i.e. $\varphi$ settles the described above isomorphism of $V^N$ onto $m_N$ and is zero elsewhere. The Riemannian metric on $N$ is induced by $g$ and the Killing form on $m$, namely,
\[(2.2) \quad \langle V, W \rangle = g(\pi_N V, \pi_N W) + B(\varphi(V), \varphi(W)), \quad V, W \in TN,\]
where $B(A, B) = -\text{tr}(AB)$ for $A, B \in m$.

Denote by $\nabla^N$ the Levi–Civita connection of the metric $\langle \cdot, \cdot \rangle$ on $N$. To study vertical properties it is convenient to introduce two natural connections $\nabla^c$ and $\nabla^m$ on the adjoint bundle $m_N$.

The connection form $\omega$ in $SO(M)$ induces the connection $\nabla^\omega$ in the bundle
\[\mathfrak{so}(n)_{SO(M)} = SO(M) \times_{\text{adSO}(n)} SO(n) = \mathfrak{so}(n)_P = \mathfrak{so}(M).\]
This connection can be described as follows
\[(2.3) \quad \nabla^\omega_X \beta = \nabla_X^\omega(\beta|P) + [\xi_X, \beta], \quad X \in TM, \quad \beta \in \Gamma(\mathfrak{so}(n)_{SO(M)}),\]
where $\nabla^\omega$ is the induced connection in $\mathfrak{so}(n)_P$ from $\omega_\mathfrak{g}$ on $P$. Indeed, if $f_\beta$ is the equivariant function corresponding to $\beta$, then, by the fact that $\beta|P$ is equivariant for $\mathfrak{so}(n)_P$ we have ($p \in P$)
\[\nabla^\omega_X \beta = f_{\beta_*} X^h = f_{\beta_*} X^h - f_{\beta_*}(\xi_X)^*\]
\[= \nabla_X^\omega(\beta|P) - \frac{d}{dt} (f_{\beta}(p \exp t\xi_X))\]
\[= \nabla_X^\omega(\beta|P) - \frac{d}{dt} \text{ad}(\exp(-t\xi_X)) f_{\beta}(p)\]
\[= \nabla_X^\omega(\beta|P) + [\xi_X, \beta].\]
Since the decomposition $\mathfrak{so}(n) = \mathfrak{g} \oplus m$ is reductive, it follows that the $m$-component of $\nabla^\omega_X(\beta|P)$ equals $\nabla^\omega_X(\beta_m|P)$, where $\beta_m$ is the $m$-component of $\beta$.

Therefore we have
\[(2.4) \quad \nabla^\omega_X \beta_m = (\nabla^\omega_X \beta)_m = [\xi_X, \beta_m]_\mathfrak{g} - [\xi_X, \beta]_m.\]
In particular, if $\alpha \in \Gamma(m_P)$, then
\[\nabla^\omega_X \alpha = (\nabla^\omega_X \alpha)_m + [\xi_X, \alpha]_\mathfrak{g}.\]
In $m_P$ we can consider the connection $\nabla^m$ induced from the connection form $\omega_\mathfrak{g}$ on $P$. Thus, by above,
\[\nabla^\omega \alpha = \nabla^m_X \alpha + [\xi_X, \alpha].\]
In order to study the geometry of $N$ it is necessary to consider sections of the bundle $\mathcal{V}^N$ of vertical vector fields on $N$. This bundle is isomorphic to $m_N$. Let $\nabla^c$ be the connection in $m_N$ associated with the connection form $\omega_\theta$ of the bundle $\zeta : SO(M) \to N$. Let us compare $\nabla^c$ with the Levi–Civita connection $\nabla^N$ on $N$. For this purpose, define the homogeneous curvature form $\Phi$, which is a 2–form on $N$ with values in $m_N$, by the formula

$$\Phi(U, V) = \Omega_m(\tilde{U}, \tilde{V}),$$

where $\zeta, \tilde{U} = U, \zeta, \tilde{V} = V$ and $\Omega_m$ is the $m$–component of the curvature form $\Omega$ of $\omega$. We have

$$(2.5) \quad \varphi(\nabla^N_U V) - \nabla^c_U \varphi(V) = \frac{1}{2} ([\varphi(U), \varphi(V)]_m - \Phi(U, V)).$$

Indeed, by the Koszul formula for the Levi–Civita connection $\nabla^N$ and the fact that $\nabla^c$ is metric for $B$ we have

$$2B(\varphi(\nabla^N_U V) - \nabla^c_U \varphi(V), \varphi(W)) = B(T^c(U, W), \varphi(V)) + B(T^c(V, W), \varphi(U)) - B(T^c(U, V), \varphi(W)),$$

where

$$T^c(U, V) = \nabla^c_U \varphi(V) - \nabla^c_V \varphi(U) - \varphi([U, V]).$$

Moreover, taking the $m$–component of the structure equation $\Omega = d\omega + [\omega, \omega]$ and by the fact that equivariant function corresponding to $\varphi(U)$ equals $\omega_m(\tilde{U})$, we get

$$T^c(U, V) = U^h \zeta \omega_m(V) - V^h \zeta \omega_m(U) - \omega_m([U, V])$$

$$= d \omega_m(\tilde{U}, \tilde{V})$$

$$= \Omega_m(\tilde{U}, \tilde{V}) - [\omega_m(\tilde{U}), \omega_m(\tilde{V})]_m$$

$$= \Phi(U, V) - [\varphi(U), \varphi(V)]_m.$$

If one of $U$ and $V$ is horizontal and the other one vertical then $T^c(U, V) = 0$. If both $U$ and $V$ are vertical, then $T^c(U, V) = -[\varphi(U), \varphi(V)]_m$, whereas if both are horizontal, then $T^c(U, V) = \Phi(U, V)$. Since we may choose $W$ to be vertical,

$$2B\varphi(\nabla^N_U V) - \nabla^c_U \varphi(V), \varphi(W)) = -B([\varphi(U), \varphi(W)]_m, \varphi(V))$$

$$- B([\varphi(V), \varphi(W)]_m, \varphi(U))$$

$$+ B([\varphi(U), \varphi(V)]_m, \varphi(W)) - B(\Phi(U, V), \varphi(W)).$$

Finally, by the natural reductivity implied by the normal reductivity of $SO(n)/G$ we get (2.5).

Gathering all above results we have the following usefull relations.

**Proposition 2.4.** The following relations between $\nabla^\omega$, $\nabla^m$ and $\nabla^N$, $\nabla^c$ hold

$$\nabla_X^c \alpha = \nabla_X^m \alpha + [\alpha, \alpha],$$

$$\varphi(\nabla^N_U V) = \nabla^c_U \varphi(V) + \frac{1}{2} ([\varphi(U), \varphi(V)]_m - \Phi(U, V)).$$

**Remark 2.5.** Notice that $\pi_N^* m_P = m_N$ and the pull–back connection $\pi_N^* \nabla^m$ equals $\nabla^c$. 
3. Harmonic $G$–reductions

Let $(M, g)$ be an oriented Riemannian manifold, $\pi_{SO(M)} : SO(M) \to M$ its orthonormal frame bundle with the connection form $\omega$ inducing Levi–Civita connection $\nabla$ on $M$. Let $G \subset O(n)$, $n = \dim M$, be a closed subgroup and $P \subset SO(M)$ the reduced subbundle. The associated homogeneous bundle $N = SO(M) \times_{SO(n)} (SO(n)/G) \to M$ admits unique section $\sigma \in \Gamma(N)$ corresponding to the reduction $P$ of $SO(M)$. The decomposition

$$TN = H^N \oplus V^N,$$

described in the previous section, defines projections $h : TN \to H^N$ and $v : TN \to H^N$. We have showed that

$$(3.1) \quad v(\sigma^* X) = \xi^\sigma_X \in m_N, \quad X \in TM.$$  

In this section we will derive the formula for the derivative $\nabla(\sigma^*)$ and the tension field of $\sigma$.

Equip $N$ with the Riemannian metric $\langle \cdot, \cdot \rangle$ given by (2.2) and $M$ with a Riemannian metric $\tilde{g}$ on $M$, which may vary from $g$. We will consider $\sigma$ as a map $\sigma : (M, \tilde{g}) \to (N, \langle \cdot, \cdot \rangle)$.

Let $E(\sigma)$ denotes the energy functional

$$(3.2) \quad E(\sigma) = \frac{1}{2} \int_M \|\sigma_*\|^2 \, d\text{vol}_M,$$

where the norm $\| \cdot \|$ is taken with respect to $\tilde{g}$ and $\langle \cdot, \cdot \rangle$, i.e.,

$$\|\sigma_*\|^2 = \sum_i \langle \sigma_*(\tilde{e}_i), \sigma_*(\tilde{e}_i) \rangle,$$

where $(\tilde{e}_i)$ is a $\tilde{g}$–orthonormal basis on $M$.

In order to study harmonic sections it is convenient to study variations of the functional (3.2) in the class of all sections of $N$. If $\tilde{g} = g$, then critical points of this functional are called harmonic $G$–structures [12]. We will consider the general case $\tilde{g} \neq g$ in the class of all maps from $M$ to $N$. In other words, we will seek for $\sigma$ to be a harmonic map. One can show that harmonicity of $\sigma$ is equivalent to vanishing of the Euler–Lagrange equation

$$\tau_{\tilde{g}}(\sigma) = \text{tr}_{\tilde{g}} \nabla \sigma_* = \sum_i \nabla^\sigma_{\tilde{e}_i} \sigma_* \tilde{e}_i - \sigma_*(\tilde{\nabla}_{\tilde{e}_i} \tilde{e}_i) = 0,$$

where $\tilde{\nabla}$ is the Levi–Civita connection of $\tilde{g}$ and $\nabla^\sigma$ is the pull–back connection in the bundle

$$\sigma^*TN = \sigma^*H^N \oplus \sigma^*V^N = TM \oplus m_P = TM \oplus m(M).$$

We call $\tau(\sigma)$ the tension field of $\sigma$. Taking the decomposition of $\tau(\sigma)$ with respect to above decomposition, harmonicity of $\sigma$ is equivalent to vanishing of $h\tau(\sigma)$ and $v\tau(\sigma)$.

Denote by $\Pi_{\tilde{g}}$ and $\Pi_g$ the differential $\nabla \sigma_*$ with respect to $\tilde{g}$ and $g$, respectively. Moreover, let $S$ be the difference between $\tilde{\nabla}$ and $\nabla$, i.e. $S(X, Y) = \tilde{\nabla}_XY - \nabla_XY$. Then

$$\tau_{\tilde{g}}(\sigma) = \text{tr}_{\tilde{g}} \Pi_{\tilde{g}} = \text{tr}_{\tilde{g}} (\Pi_g - \sigma_* S).$$
We will show (see [23, 12]) that

\begin{equation}
\varphi(\Pi_g(X, Y)) = \frac{1}{2}(\nabla_X \xi_Y + (\nabla_Y \xi)_X) \in \mathfrak{m}(M),
\end{equation}

\begin{equation}
g(\pi_N \Pi_g(X, Y), Z) = \frac{1}{2}\left(\mathcal{B}(\xi_X, R_m(Y, Z)) + \mathcal{B}(\xi_Y, R_m(X, Z))\right),
\end{equation}

where \((\nabla_X \xi)_Y = \nabla_X \xi_Y - \xi \nabla_X Y\). For the proof of the first equality, by Proposition 2.4, Remark 2.5 and (3.1) we have

\begin{align*}
\varphi(\Pi_g(X, Y)) &= \nabla^\ast \varphi(\sigma_* Y) + \frac{1}{2}[\varphi(\sigma_* X), \varphi(\sigma_* Y)] - \frac{1}{2}\Phi(\sigma_* X, \sigma_* Y) - \varphi(\sigma_*(\nabla_X Y)) \\
&= \nabla^\ast \varphi(\sigma_* Y) - [\xi_X, \varphi(\sigma_* Y)] + \frac{1}{2}[\varphi(\sigma_* X), \varphi(\sigma_* Y)] \\
&\quad - \frac{1}{2}\Phi(\sigma_* X, \sigma_* Y) - \varphi(\sigma_*(\nabla_X Y)) \\
&= \nabla_X \xi_Y - [\xi_X, \xi_Y] + \frac{1}{2}[\xi_X, \xi_Y]_m - \frac{1}{2}\Phi(\sigma_* X, \sigma_* Y) - \xi \nabla_X Y,
\end{align*}

since

\begin{equation}
\sigma_* X = X^{h,N} + \nu(\sigma_* X) = X^{h,N} + \xi_X.
\end{equation}

Moreover,

\[\Phi(\sigma_* X, \sigma_* Y) = \Omega_m(X^h, Y^h) = R_m(X, Y).\]

Now, (3.3) follows by Proposition 2.3 (see also (2.1)). For the proof of the second equality, computing the covariant derivative of the equality \(\pi_N \sigma_* = \text{id}_{TM}\) we get

\[\pi_N(\nabla_X \sigma_*) Y = -\nabla_X \pi_N \sigma_* Y.\]

Since \(\pi_N : N \to M\) is a Riemannian submersion with totally geodesic fibers, by the following equations (see [1, 20, 23])

\[g((\nabla_U \pi_N)_V, \pi_N W) = \frac{1}{2}(V, [U, W]), \quad U, W \in \mathcal{H}^N, \quad V \in \mathcal{V}^N,\]

\[\nabla_U (\pi_N)_V = 0, \quad U, V \in \mathcal{V}^N \text{ or } U, V \in \mathcal{H}^N,\]

and by (3.5)

\[g(\pi_N(\nabla_X \sigma_*) Y, Z) = -g((\nabla_{\sigma_* X} \pi_N)_Y) \sigma_* Y, Z)\]

\[= -g(\nabla_{X^{h,N}} \pi_N)_Y \sigma_* Y, \pi_N \sigma_* Z^{h,N})\]

\[= -g(\nabla_{\nu(\sigma_* X)} \pi_N)_Y X^{h,N}, \pi_N \sigma_* Z^{h,N})\]

\[= -\frac{1}{2}(\nu(\sigma_* Y), [X^{h,N}, Z^{h,N}]) - \frac{1}{2}(\nu(\sigma_* X), [Y^{h,N}, Z^{h,N}]).\]

Now, using (3.1) and the formula

\[[X^{h,N}, Y^{h,N}] = [X, Y]^{h,N} - \xi \sigma R(X, Y)_m^*\]

we get (3.4).

In order to describe explicitly the condition for the harmonicity of \(\sigma\) let us introduce certain curvature operator [19]. For any \(\alpha \in \mathfrak{so}(M)\) let

\begin{equation}
R_\alpha(X) = \sum_i R(e_i, \alpha(e_i)) X \in TM, \quad X \in TM.
\end{equation}
Then $R_\alpha \in so(M)$ and the following formula holds
\begin{equation}
(3.7) \quad g(R_\alpha(X), Y) = B(\alpha, R(X, Y)), \quad \alpha \in so(M), \quad X, Y \in TM.
\end{equation}
Indeed,
\[ g(R_\alpha(X), Y) = \sum_i g(R(e_i, \alpha(e_i))X, Y) = \sum_i g(\alpha(e_i), R(X, Y)e_i) = B(\alpha, R(X, Y)). \]

Now, we can state the formula for the harmonicity of $\sigma$.

**Theorem 3.1.** $G$—structure $\sigma : (M, \tilde{g}) \to (N, \langle \cdot, \cdot \rangle)$ is a harmonic map if and only if the following conditions hold
\begin{equation}
(3.8) \quad \sum_i (\nabla_{\tilde{e}_i}\xi)_{\tilde{e}_i} - \xi S(\tilde{e}_i, \tilde{e}_i) = 0 \quad \text{and} \quad \sum_i R_{\xi_{\tilde{e}_i}}(\tilde{e}_i) - S(\tilde{e}_i, \tilde{e}_i) = 0,
\end{equation}
where $(\tilde{e}_i)$ is an orthonormal basis for $\tilde{g}$.

**Proof.** Follows by (3.4), (3.3) and (3.7). $\Box$

4. Properties of certain transfer tensor

In this section we will introduce invertible tensor induced by the intrinsic torsion, the Riemannian metric defined by this tensor and state the properties of the Levi–Civita connection of this new metric. Results in this section are generalizations of the results obtained by the author in [19].

Adopt the notation from the previous section. For $\alpha \in m(M)$ put
\[ \xi \cdot \alpha = -\sum_i B(\xi_{e_i}, \alpha) e_i \in TM \]
where we consider the intrinsic torsion as an element of $m(M)$. It is easy to show that
\begin{equation}
(4.1) \quad g(\xi \cdot \alpha, X) = -B(\alpha, \xi X), \quad \alpha \in m(M), \quad X \in TM
\end{equation}

Let $L$ be the endomorphism of the tangent bundle of the form
\[ L(X) = X - \xi \cdot \xi X, \quad X \in TM. \]

In order to derive some properties of $L$ recall the definition of the Riemannian metric on the bundle $SO(M)$ induced by the metric $g$ on $M$ and by the Killing form $B$ on the structure group $SO(n)$. We define the Riemannian metric $g_{SO(M)}$ on $SO(M)$ as follows
\[ g_{SO(M)}(X^h, Y^h) = g(X, Y), \]
\[ g_{SO(M)}(X^h, \alpha^*) = 0, \]
\[ g_{SO(M)}(\alpha^*, \beta^*) = B(\alpha, \beta), \]
where $X, Y \in TM, \alpha, \beta \in so(M) = so(n)_{SO(M)}$. Recall that we identify element $\alpha \in so(M)$ with the equivariant vertical vector field denoted by $\alpha^*$. Then maps $\pi_{SO(M)} : SO(M) \to M$ and $\zeta : SO(M) \to N$ are Riemannian submersions.
Proposition 4.1. We have
\[ g_{SO(M)}(X^{h'}, Y^{h'}) = g(X, LY). \]

In particular, \( L \) is a symmetric automorphism of \( TM \). Moreover, the covariant derivative of \( L \) is related with the intrinsic torsion \( \xi_X \) by the formula
\[ g((\nabla_X L)Y, Z) = \mathcal{B}((\nabla_X \xi)Y, \xi Z) + \mathcal{B}((\nabla_X \xi)Z, \xi Y). \]

Proof. By (4.1) and Proposition 2.1,
\[ g(X, LY) = g(X, Y) + \mathcal{B}(\xi_X, \xi_Y) = g_{SO(M)}(X^{h} + \xi_X, Y^{h} + \xi_Y) = g_{SO(M)}(X^{h'}, Y^{h'}). \]
Thus \( L \) is symmetric and positive definite. Moreover, by the fact that \( \nabla^\omega \) is metric for \( \mathcal{B} \) and equals usual connection \( \nabla \), we have
\[ g((\nabla_X L)Y, Z) = g(\nabla_X (LY), Z) - g(L(\nabla_X Y), Z) = Xg(LY, Z) - g(LY, \nabla_X Z) - g(\nabla_X Y, L) \]
\[ = Xg(Y, Z) + XB(\xi_Y, \xi_Z) - g(Y, \nabla_X Z) - B(\xi_Y, \xi\nabla_X Z) - g(\nabla_X Y, Z) - B(\xi_X Y, \xi_Z) \]
\[ = B(\nabla_X \xi_Y - \xi_X \nabla_Y, \xi_Z) + B(\nabla_X \xi_Z - \xi_X \nabla_Z, \xi_Y). \]
\[ (4.2) \]

By Proposition 4.1, the symmetric and bilinear form
\[ \tilde{g}(X, Y) = g(X, LY), \quad X, Y \in TM, \]
defines a Riemannian metric on \( M \). We call \( L \) the transfer tensor between \( g \) and \( \tilde{g} \). Notice, that the projection \( \pi_P : P \to M \) is a Riemannian submersion with respect to \( g_{SO(M)} \) on \( P \) and \( \tilde{g} \) on \( M \). Denote by \( \tilde{\nabla} \) the Levi–Civita connection of \( \tilde{g} \). One can show that \[ (4.3) \]
\[ 2g(\tilde{\nabla}_X Y - \nabla_X Y, LZ) = g((\tilde{\nabla}_X L)Y, Z) + g((\nabla Y L)X, Z) - g(X, (\nabla Z L)Y). \]
Thus by Proposition 4.1 we get
\[ 2g(\tilde{\nabla}_X Y - \nabla_X Y, LZ) = B((\tilde{\nabla}_X \xi)Y + (\nabla Y \xi)_X, \xi_Z) \]
\[ + B((\tilde{\nabla}_X \xi)_Z - (\nabla_Z \xi)_X, \xi_Y) \]
\[ + B((\nabla Y \xi)_Z - (\nabla_Z \xi)_Y, \xi_X), \]

5. Geometry of \( G \)-structures

In this section we study the geometry of a \( G \)-structure \( P \) in \( SO(M) \). In this case we need to consider the Levi–Civita connection \( \nabla^{SO(M)} \) of the Riemannian
metric $g_{SO(M)}$. One can show [19] that
\[
\nabla^{SO(M)}_{X^h} X^h = (\nabla_X Y)^h - \frac{1}{2} R(X, Y)^h,
\]
\[
\nabla^{SO(M)}_{X^h} \alpha^* = \frac{1}{2} R_\alpha(X)^h + (\nabla_X \alpha)^*,
\]
\[
\nabla^{SO(M)}_{\alpha^*} Y^h = \frac{1}{2} R_\alpha(Y)^h,
\]
\[
\nabla^{SO(M)}_{\alpha^*} \beta^* = -\frac{1}{2} [\alpha, \beta]^*,
\]
for $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \mathfrak{so}(M)$, or equivalently, $\alpha, \beta \in \mathfrak{so}(n)_{SO(M)}$, where $R_\alpha$ is defined by (3.4).

It is convenient to find orthogonal projections
\[
\text{TSO}(M) \hookrightarrow TP \quad \text{and} \quad \text{TSO}(M) \hookrightarrow T^\perp P,
\]
in order to derive the Levi–Civita connection and second fundamental form for $P$ in $SO(M)$.

**Proposition 5.1.** Let $X \in TM$ and $\alpha \in \mathfrak{so}(M)$.

1. The orthogonal projection $\text{TSO}(M) \hookrightarrow TP$ equals
   \[
   X^h \mapsto L^{-1}(X)^{h'} \quad \text{and} \quad \alpha^* \mapsto \alpha^*_g - L^{-1}(\xi \cdot \alpha_m)^{h'}.
   \]

2. The orthogonal projection $\text{TSO}(M) \hookrightarrow T^\perp P$ equals
   \[
   X^h \mapsto -(\xi_{L^{-1}(X)})^* - (\xi \cdot \xi_{L^{-1}X})^h \quad \text{and} \quad \alpha^* \mapsto \alpha^*_m + (\xi_{L^{-1}(\xi \cdot \alpha_m)})^* + L^{-1}(\xi \cdot \alpha_m)^h.
   \]

**Proof.** We will use frequently Proposition 2.3 and formula (4.1). Recall that $TP = \mathcal{H}' \oplus m_P$. For any $Y \in TM$
\[
g_{SO(M)}(X^h - L^{-1}(X)^{h'}, Y^{h'}) = g_{SO(M)}(X^h - L^{-1}(X)^{h} - \xi_{L^{-1}(X)}^* Y^h + \xi Y)
\]
\[
= g(X - L^{-1}(X), Y) - B(\xi_{L^{-1}(X)}, \xi Y)
\]
\[
= 0
\]
and for any $\beta \in \mathfrak{g}_P$
\[
g_{SO(M)}(X^h - L^{-1}(X)^{h'}, \beta^*) = -B(\xi_{L^{-1}(X)}, \beta) = 0.
\]
Clearly $L^{-1}(X)$ is tangent to $P$ and $X^h - L^{-1}(X)^{h'} = -(\xi_{L^{-1}(X)})^* - (\xi \cdot \xi_{L^{-1}X})^h$, which proves the desired decomposition for $X^h$. Analogously,
\[
\alpha^* - (\alpha^*_g - L^{-1}(\xi \cdot \alpha_m)^{h'}) = \alpha^*_m + L^{-1}(\xi \cdot \alpha_m)^{h'},
\]
and
\[
g_{SO(M)}(\alpha^*_m + L^{-1}(\xi \cdot \alpha_m)^{h'}, Y^{h'}) = B(\alpha_m, \xi_Y) + g(L^{-1}(\xi \cdot \alpha_m), Y)
\]
\[
+ B(\xi_{L^{-1}(\xi \cdot \alpha_m)}, \xi Y)
\]
\[
= -g(\xi \cdot \alpha_m, Y) + g(L^{-1}(\xi \cdot \alpha_m), Y)
\]
\[
- g(\xi \cdot \xi_{L^{-1}(\xi \cdot \alpha_m}), Y)
\]
\[
= 0
\]
Moreover, for $\beta \in \mathfrak{g}_P$
\[
g_{SO(M)}(\alpha^*_m + L^{-1}(\xi \cdot \alpha_m)^{h'}, \beta^*) = B(\alpha_m, \beta) + B(\xi_{L^{-1}(\xi \cdot \alpha_m)}, \beta) = 0
\]
Proposition follows from the fact that
\[ L^{-1}(\xi \cdot \alpha_m)^{h'} = L^{-1}(\xi \cdot \alpha_m)^{h} + (\xi L^{-1}(\xi \cdot \alpha_m))^{*} \]
and \( \alpha_g^{*} - L^{-1}(\xi \cdot \alpha_m)^{h'} \) is tangent to \( P \).

5.1. **Intrinsic geometry.** For the intrinsic geometry we will compute the Levi–Civita connection \( \nabla^P \) of \((P, \mathfrak{so}(M))\), the curvature tensor, Ricci tensor and sectional and scalar curvatures. We will compare obtained geometry with the geometry of the base manifold \( M \).

Let us first introduce operator \( Q_\alpha \) and establish some relations. For \( X \in \mathcal{T}M \) and \( \alpha \in \mathfrak{so}(M) \) put
\[ Q_\alpha(X) = L^{-1}(R_\alpha(X) - \xi \cdot (\nabla_X \alpha)_m). \]

Generalizing Proposition 2.3 we have the following lemma.

**Lemma 5.2.** For any \( \alpha \in \mathfrak{so}(M) \) we have the following relations with respect to the decomposition \( \mathfrak{so}(M) = \mathfrak{g}(M) \oplus \mathfrak{m}(M) \):
\[ \nabla_X \alpha_g = \nabla_X^{\prime} \alpha_g + [\xi_X, \alpha_g], \]
\[ \nabla_X \alpha_m = [\xi_X, \alpha_m]_g + (\nabla_X^{\prime} \alpha_m + [\xi_X, \alpha_m]_m). \]

**Theorem 5.3.** The Levi–Civita connection \( \nabla^P \) of \( P \) with the induced Riemannian metric \( G_{SO(M)} \) from \( SO(M) \) is of the following form
\[ \nabla^P_X Y^{h'} = (\tilde{\nabla}_X Y)^{h'} - \frac{1}{2} R^{h'}(X, Y)^{*,} \]
\[ \nabla^P_{X^{h'}} \beta^{*} = \frac{1}{2} Q_\beta(X)^{h'} + (\nabla^{\prime}_X \beta)^{*,} \]
\[ \nabla^P_\alpha Y^{h'} = \frac{1}{2} Q_\alpha(Y)^{h'}, \]
\[ \nabla^P_\alpha \beta^{*} = -\frac{1}{2} [\alpha, \beta]^{*}. \]

where \( X, Y \in \mathcal{T}M, \alpha, \beta \in \mathfrak{g}_P. \)

**Proof.** First, by (4.1) and (3.7) we have
\[ \hat{g}(Q_\alpha(X), Z) = B(R(X, Z), \alpha) + B(\nabla_X \alpha, \xi_Y). \]

Thus, using (2.1) and (4.3)
\[ \hat{g}(Q_{\xi_X}(Y) + Q_{\xi_Y}(X) + \xi \cdot (\nabla_X Y + \nabla_Y X), Z) = B((\nabla_X \xi)_Z - (\nabla_Z \xi)_X, \xi_Y) + B((\nabla_Y \xi)_Z - (\nabla_Z \xi)_Y, \xi_X) \]
\[ - B([\xi_X, \xi_Z], \xi_Y) - B([\xi_Y, \xi_Z], \xi_X) + B((\nabla_X \xi)_Y + (\nabla_Y \xi)_X, \xi_Z) \]
\[ = 2\hat{g}(\nabla_X Y - \nabla_Y X, Z). \]

We showed that
\[ (5.1) \quad P(X, Y) = \frac{1}{2}(Q_{\xi_X}(Y) + Q_{\xi_Y}(X) + \xi \cdot (\nabla_X Y + \nabla_Y X)). \]
By the formula for the connection $\nabla^{SO(M)}$

$$\nabla^P_{X^h} Y^{h'} = \nabla^{SO(M)}_{X^h} Y^h + \nabla^{SO(M)}_{(\xi_x)^*} Y^h + \nabla^{SO(M)}_{X^h} (\xi_Y)^* + \nabla^{SO(M)}_{(\xi_x)^*} (\xi_Y)^*$$

$$= \left( \nabla_X Y + \frac{1}{2} R_{\xi_Y} (X) + \frac{1}{2} R_{\xi_Y} (Y) \right)^h$$

$$+ \left( -\frac{1}{2} R(X, Y) + \nabla_X \xi_Y - \frac{1}{2} [\xi_X, \xi_Y] \right)^*$$

Since, by Proposition 2.3

$$\left( -\frac{1}{2} R(X, Y) + \nabla_X \xi_Y - \frac{1}{2} [\xi_X, \xi_Y] \right)_g = -\frac{1}{2} R'(X, Y)$$

and

$$\left( -\frac{1}{2} R(X, Y) + \nabla_X \xi_Y - \frac{1}{2} [\xi_X, \xi_Y] \right)_m = \frac{1}{2} \left( \nabla_X \xi_Y + \nabla_Y \xi_X + [\xi_X, \xi_Y] \right) ,$$

by Proposition 5.1 we obtain

$$\nabla^P_{X^h} Y^{h'} = L^{-1} \left( \frac{1}{2} R_{\xi_Y} (X) + \frac{1}{2} R_{\xi_Y} (Y) - \frac{1}{2} \xi \cdot (\nabla_X \xi_Y + \nabla_Y \xi_X) \right)^h$$

$$+ L^{-1} \left( \nabla_X Y - \frac{1}{2} \xi \cdot [\xi_X, \xi_Y] \right)^h - \frac{1}{2} R'(X, Y)^* .$$

Moreover $\nabla_X Y - \frac{1}{2} \xi \cdot [\xi_X, \xi_Y] = L(\nabla_X Y) + \xi \cdot [\nabla_X Y, \nabla_Y X]$. Thus, by the definition of $Q_\alpha$ and formula (5.1) we get the desired formula for $\nabla^P_{X^h} Y^{h'}$.

For the proof of the second formula we will use Lemma 5.2 and Proposition 5.1. We have

$$\nabla^P_{X^h} \beta^* = \left( \nabla^{SO(M)}_{X^h} \beta^* + \nabla^{SO(M)}_{(\xi_x)^*} \beta^* \right)^T$$

$$= \left( \frac{1}{2} R_{\beta} (X)^h + (\nabla_X \beta)^* - \frac{1}{2} [\xi_X, \beta]^* \right)^T$$

$$= L^{-1} \left( \frac{1}{2} R_{\beta} (X) - \xi \cdot (\nabla_X \beta)_m + \frac{1}{2} \xi \cdot [\xi_X, \beta]_m \right)^h + (\nabla_X \beta)^*_g$$

$$= Q_{\beta} (X)^h + (\nabla_X \beta)^* .$$

We prove the remaining relations analogously. □
Proposition 5.4. The curvature tensor \( R^P \) of \( \nabla^P \) equals

\[
R^P(X^{h'}, Y^{h'})Z^{h'} = (\tilde{R}(X, Y)Z)^{h'} - \frac{1}{4}(Q_{R'(Y,Z)}(X) - Q_{R'(X,Z)}(Y) - 2Q_{R(X,Y)}(Z))^{h'}
\]

\[- \frac{1}{2}(D_X R')(Y, Z)^* + \frac{1}{2}(D_Y R')(X, Z)^* ,
\]

\[
R^P(X^{h'}, Y^{h'})\gamma^* = \frac{1}{2}((D_X Q)_{\gamma}(Y) - (D_Y Q)_{\gamma}(X))^{h'} + \frac{1}{2}[R'(X, Y), \gamma]^*
\]

\[- \frac{1}{4}(R'(X, Q_{\gamma}(Y)) - R'(Y, Q_{\gamma}(X))^* ,
\]

\[
R^P(X^{h'}, \beta^*)Z^{h'} = \frac{1}{2}(D_X Q)_{\beta}(Z)^{h'} - \frac{1}{4}(R'(X, Q_{\beta}(Z)) + [\beta, R'(X, Z)])^* ,
\]

\[
R^P(X^{h'}, \beta^*)\gamma^* = - \frac{1}{4}(Q_{[\alpha, \gamma]}(X) + Q_{\beta}Q_{\gamma}(X))^{h'} ,
\]

\[
R^P(\alpha^*, \beta^*)Z^{h'} = \frac{1}{4}[Q_{\alpha}, Q_{\beta}](Z)^{h'} + \frac{1}{2}Q_{[\alpha, \beta]}(Z)^{h'} ,
\]

\[
R^P(\alpha^*, \beta^*)\gamma^* = - \frac{1}{4}[\alpha, \beta], \gamma]^* ,
\]

where

\[
(D_X R')(Y, Z) = \nabla'_X R'(Y, Z) - R'(\nabla_X Y, Z) - R'(Y, \nabla_X Z)
\]

\[
(D_X Q)_{\alpha}(Y) = \tilde{\nabla}_X Q_{\alpha}(Y) - Q\nabla_X Q_{\alpha}(Y) - Q(\tilde{\nabla}_X Y).
\]

Proof. Follows directly by Theorem 5.3, Jacobi identity and the relations

\[
R'(X, Y)\gamma = [R'(X, Y), \gamma] , \quad \nabla'_X[\beta, \gamma] = [\nabla'_X \beta, \gamma] + [\beta, \nabla'_X \gamma].
\]

Notice that \([X^{h'}, \beta^*] = (\nabla'_X \beta)^* \) and \([\alpha^*, \beta^*] = -[\alpha, \beta]^* \). \(\square\)

Corollary 5.5. The Ricci curvature tensor \( \text{Ric}^P \) equals

\[
\text{Ric}^P(X^{h'}, Y^{h'}) = \text{Ric}(X, Y) - \frac{3}{4} \sum_i B(R'(X, \tilde{e}_i), R'(Y, \tilde{e}_i))
\]

\[+ \frac{1}{4} \sum_A \tilde{g}(Q_{\alpha A}(X), Q_{\alpha A}(Y)) ,
\]

\[
\text{Ric}^P(X^{h'}, \gamma^*) = \frac{1}{2} \left((\text{div} Q)_{\gamma}(X) - \text{tr}_{\tilde{g}}(D_X Q)_{\gamma}\right) ,
\]

\[
\text{Ric}^P(\beta^*, \gamma^*) = \frac{1}{4} \left(\sum_i \tilde{g}(Q_{\beta}(\tilde{e}_i), Q_{\gamma}(\tilde{e}_i)) + \sum_A B([\alpha A, \beta], [\alpha A, \gamma])\right) ,
\]

where

\[
(\text{div} Q)_{\gamma}(X) = \sum_i \tilde{g}((D_{\tilde{e}_i} Q)_{\gamma}(X), \tilde{e}_i)
\]

and \((\tilde{e}_i)\) is an orthonormal basis with respect to \(\tilde{g}\) on \(M\) and \((\alpha A)\) is an orthonormal basis of \(\mathfrak{g}_P\) with respect to \(B\).

Proof. First, notice that

\[
\tilde{g}(Q_{\alpha}(X), Y) = B(R'(X, Y), \alpha) ,
\]

\(\text{div} Q)_{\gamma}(X) = \sum_i \tilde{g}((D_{\tilde{e}_i} Q)_{\gamma}(X), \tilde{e}_i)\)
which follows by the definition of $Q_\alpha$, Lemma \[5.2] and Proposition \[2.3]. Hence $Q_\alpha$ is skew–symmetric with respect to $\tilde{g}$. Now, it suffices to use the formulas for the curvature tensor $R^P$.

\[\Box\]

\textbf{Corollary 5.6.} The sectional curvatures of $\nabla^P$ are given by the following formulas

\begin{align*}
\kappa^P(X^h, Y^h) &= \tilde{\kappa}(X, Y) - \frac{3}{4}\|R'(X, Y)\|^2, \\
\kappa^P(X^h, \beta^*) &= \frac{1}{4}\|Q_{\beta}(X)\|^2_{\tilde{g}}, \\
\kappa^P(\alpha^*, \beta^*) &= \frac{1}{4}\|[\alpha, \beta]\|^2,
\end{align*}

where $X, Y$ are orthonormal with respect to $\tilde{g}$ and $\alpha, \beta \in \mathfrak{g}_P$ orthonormal with respect to $B$.

\textit{Proof.} First and last relations follow immediately by the formulas for the curvature tensor $R^P$ and by (5.2). For the proof of the second one it suffices to use the skew–symmetry of $Q_{\beta}$ with respect to $\tilde{g}$ (see the proof of Corollary 5.5). \[\Box\]

\textbf{Corollary 5.7.} The scalar curvature of $\nabla^P$ equals

\begin{align*}
s^P = \tilde{s} - \frac{3}{4}\sum_{i,j} \|R'(\tilde{e}_i, \tilde{e}_j)\|^2 + \frac{1}{2}\sum_{i,a} \|Q_{\alpha_a}(\tilde{e}_i)\|^2_{\tilde{g}} + \frac{1}{4}\sum_{A,B} \|[\alpha_A, \alpha_B]\|^2.
\end{align*}

By the above formulas for the curvatures we have the following relations between the geometry of $(P, g_{SO(M)})$ and $M$.

\textbf{Corollary 5.8.} We have:

1. If $\dim M > 2$, then $P$ is never of constant sectional curvature.
2. If the intrinsic torsion $\xi$ vanishes and $(M, g)$ is of constant sectional curvature $0 \leq \kappa \leq \frac{2}{3}$, then $P$ has non–negative sectional curvatures.
3. $\text{Ric}^P(\alpha^*)$ is non–negative in any direction $\alpha \in \mathfrak{g}_P$. Moreover, if $\text{Ric}(X) \geq \frac{3}{4}\sum \|R'(X, \tilde{e}_i)\|^2$, then $\text{Ric}^P(X)$ is non–negative.
4. If $R' = 0$ and scalar curvature of $(M, \tilde{g})$ is positive, then the scalar curvature of $P$ is positive.

\section*{5.2. Extrinsic geometry.} The properties of the extrinsic geometry are encoded in the second fundamental form, which we will derive explicitly. Moreover, we will compute the mean curvature vector of $P$ in $SO(M)$ and relate the minimality of $P$ with the harmonicity of an induced section $\sigma$ with appropriate Riemannian structure.

Adopt the notation from the previous sections. For $\alpha \in \mathfrak{m}_P$ let

$$\alpha^+ = \alpha^* + (\xi \cdot \alpha)^h.$$ 

Then $\alpha^+ \in T^\perp P$. 
Theorem 5.9. The second fundamental form \( \Pi^P \) of \( P \) in \( SO(M) \) satisfies the following relations

\[
g_{SO(M)}(\Pi^P(X^{h'}, Y^{h'}), \alpha^+) = \frac{1}{2} B((\nabla_X \xi)_Y + (\nabla_Y \xi)_X - \xi R_{\xi X}(Y) + R_{\xi Y}(X), \alpha),
\]
\[
g_{SO(M)}(\Pi^P(X^{h'}, \gamma^*), \alpha^+) = \frac{1}{2} B([\xi X, \gamma]_m - \xi R_\gamma(X), \alpha),
\]
\[
g_{SO(M)}(\Pi^P(\beta^*, \gamma^*), \alpha^+) = 0.
\]

Proof. By Proposition 2.3 (see also proof of Theorem 5.3) we have

\[
\nabla_{SO(M)}^X Y^h = \left( \nabla_X Y + \frac{1}{2} R_{\xi X}(Y) + \frac{1}{2} R_{\xi Y}(X) \right)^h
\]
\[
+ \frac{1}{2} (\nabla_X \xi_Y + \nabla_Y \xi_X + \xi_{[X,Y]})^* + g_P\text{-component}.
\]

Thus

\[
g_{SO(M)}(\Pi^P(X^{h'}, Y^{h'}), \alpha^+) = g(\nabla_X Y + \frac{1}{2} (R_{\xi X}(Y) + R_{\xi Y}(X)), \xi \cdot \alpha)
\]
\[
+ \frac{1}{2} B(\nabla_X \xi_Y + \nabla_Y \xi_X + \xi_{[X,Y]}, \alpha),
\]

which implies the first equality. Moreover,

\[
\nabla_{SO(M)}^X \gamma^* = \frac{1}{2} R_\gamma(X)^h + \frac{1}{2} [\xi X, \gamma]^* + g_P\text{-component},
\]

which proves the second equality. Since \([g, g] \subset g\), it follows that \(m_P\text{-component of } \nabla_{\ast\ast}^M \beta^* \) vanishes. \qed

By above theorem we get the following implication.

Corollary 5.10. If a \( G \)-structure \( P \) is integrable, i.e. the intrinsic torsion \( \xi \) vanishes), then \( P \) is totally geodesic in \( SO(M) \).

Now, we can state the main theorem of this article.

Theorem 5.11. A \( G \)-structure \( P \) is minimal in \( SO(M) \) if and only the induced section \( \sigma : M \rightarrow N \) is a harmonic map with respect to Riemannian metrics \( \tilde{g} \) and \( \langle \cdot, \cdot \rangle \) on \( M \) and \( N \), respectively.

Proof. Recall, that by Theorem 5.1 section \( \sigma : (M, \tilde{g}) \rightarrow (N, \langle \cdot, \cdot \rangle) \) is a harmonic map if and only if

\[
(H1) \quad \sum_i (\nabla_{\xi_i} \xi)_{\tilde{e}_i} - \xi S(\tilde{e}_i, \tilde{e}_i) = 0,
\]
\[
(H2) \quad \sum_i R_{\xi_i} (\tilde{e}_i) - S(\tilde{e}_i, \tilde{e}_i) = 0,
\]

whereas, by Theorem 5.9 minimality of \( P \) is equivalent to the following condition

\[
(M) \quad \sum_i (\nabla_{\xi_i} \xi)_{\tilde{e}_i} - \xi R_{\xi_i} (\tilde{e}_i) = 0.
\]
Claerly, (H1) and (H2) imply (M). Conversely, assume (M) holds. It suffices to show that (H2) holds. By (3.3) and (M) we have

\[ g(\sum_i R_{\xi_i}(\tilde{\xi}_i) - S(\tilde{\xi}_i, \tilde{\xi}_i), L Z) = - \sum_i B((\nabla_{\xi_i} \xi_i), \xi Z) + B((\nabla_{\xi_i} \xi_i)Z - (\nabla_{\xi_i} \xi_i)\xi_i, \xi_i) + \sum_i g(R_{\xi_i}(\xi_i), Z - \xi \cdot \xi Z) = - \sum_i B((\nabla_{\xi_i} \xi_i)Z - (\nabla_{\xi_i} \xi_i)\xi_i, \xi_i) - B(R(\xi_i, Z), \xi_i). \]

By Proposition 2.3 and natural reductivity of SO(n)/G we get the desired equality. □

6. Minimality of related structures

Let, as before, \( \pi : SO(M) \to M \) be an oriented orthonormal frame bundle over a Riemannian manifold \( (M, g) \). Let \( G \subset SO(n) \), \( n = \dim M \), be the closed subgroup such that the quotient \( SO(n)/G \) is normal reductive. Let \( P \) be the reduced \( G \)-structure and denote by \( \sigma \) the induced section of the associated bundle \( N = SO(M) \times SO(n)SO(n)/G \) over \( M \). We equip \( SO(M) \) and \( N \) in natural Riemannian metrics as in previous sections. Namely, the Riemannian metric \( \langle \cdot, \cdot \rangle \) on \( N \), by (2.2), equals

\[ \langle X^{h,\xi}, Y^{h,\xi} \rangle = g(X, Y), \]
\[ \langle X^{h,\xi}, \zeta^{*} \rangle = 0, \]
\[ \langle \zeta^{*}, \zeta^{*} \rangle = B(\alpha, \beta), \]

where \( X, Y \in TM, \alpha, \beta \in m(M) \). For simplicity denote the element \( \zeta^{*} \) by \( \alpha^{\dagger} \).

Recall that the Riemannian metric \( \tilde{g} \) on \( M \) equals

\[ \tilde{g}(X, Y) = g(X, LY) = g(X, Y) + B(\xi X, \xi Y), \quad X, Y \in TM. \]

By (3.4) and (3.5) for any \( X, Y \in TM \) we have \( \langle \sigma_{*}(X), \alpha^{*} \rangle = \tilde{g}(X, Y) \). Thus \( \sigma^{*}(\cdot, \cdot) = \tilde{g} \) and the map \( \sigma : (M, \tilde{g}) \to (N, \langle \cdot, \cdot \rangle) \) is an isometric imbedding.

Therefore the harmonicity of \( \sigma \) is equivalent to its minimality [6].

For the sake of completeness, let us compute the minimality condition of an image \( \sigma(M) \) in \( N \). Notice that the distribution \( \sigma_{*}(TM)^{\perp} \) in \( TN \) is spanned by the elements

\[ \alpha^{\dagger} + (\xi \cdot \alpha)^{h,\xi}, \quad \alpha \in m(M). \]

Fix a vector \( X \in TM \). Then the second fundamental form \( \Pi \) of \( \sigma(M) \subset N \) on the same vector \( \sigma_{*}(X) \), by (3.4) and (3.5) and using (3.4), (4.1), equals

\[ \langle \Pi(\sigma_{*}(X), \sigma_{*}(X)), \alpha^{\dagger} + (\xi \cdot \alpha)^{h,\xi} \rangle = \langle (\nabla_{X} \sigma_{*})X, \alpha^{\dagger} + (\xi \cdot \alpha)^{h,\xi} \rangle = B(\xi X, R_{\alpha_{*}}(X, \xi \cdot \alpha)) + B((\nabla_{X} \xi)X, \alpha) \]
\[ = g(R_{\xi X}(X), \xi \cdot \alpha) + B((\nabla_{X} \xi)X, \alpha) \]
\[ = -B(\xi_{R_{\xi X}(X), \alpha}) + B((\nabla_{X} \xi)X, \alpha). \]

Therefore, \( \sigma(M) \) is minimal if and only if

\[ \sum_{i}(\nabla_{\xi_{i} \xi} \xi_{i} - \xi R_{\xi_{i} \xi_{i}}) = 0, \]
where $\tilde{e}_i$ is a $\tilde{g}$--orthonormal basis on $M$, which equals $(M)$. Thus, together with Theorem 5.11 we have the following result.

**Theorem 6.1.** The following conditions are equivalent

1. a $G$--structure $P$ is minimal in $SO(M)$,
2. the induced section $\sigma : (M, \tilde{g}) \rightarrow (N, \langle \cdot, \cdot \rangle)$ is a harmonic map,
3. the submanifold $\sigma(M)$ is minimal in $(N, \langle \cdot, \cdot \rangle)$.

7. Examples

In this section we illustrate obtained results for the $G$--structures, where $G = SO(m) \times SO(n - m)$ or $G = U(n)$. The case of other possible $G$--structures, for example coming from the Berger list of possible Riemannian holonomy groups, will be studied by the author independently.

7.1. Almost product structures. Let $(M, g)$ be an oriented $n$--dimensional Riemannian manifold and let $G \subset SO(n)$ be the closed subgroup of the form $G = SO(m) \times SO(n - m)$ for some $m = 1, \ldots, n - 1$. The quotient $SO(n)/G$ is a symmetric space, which is the oriented Grassmannian $G_m^o(\mathbb{R}^n)$ of oriented $m$--dimensional subspaces in the Euclidean space $\mathbb{R}^n$. The reduction of the oriented orthonormal frame bundle $SO(M)$ to the subbundle $P$ with the structure group $G$ is equivalent to the existence of $m$--dimensional distribution $E$ on $M$ and, hence, its orthogonal complement $F = E^\perp$. We call $M$ with the distinguished distribution $E$ an almost product structure.

The connection $\nabla'$ induced by the connection form $\omega_\theta$, where $\omega$ is the connection form of the Levi–Civita connection $\nabla$, takes the form

$$\nabla'_{X}Y = (\nabla_{X}Y^\perp)^\perp + (\nabla_{X}Y^\perp)^\perp,$$

where the decomposition $X = X^\perp + X^\perp$ is taken with respect to $TM = E \oplus F$. In other words it is the sum of two connections induced by $\nabla$–connections in the vector bundles $E$ and $F$ over $M$. The intrinsic torsion of almost product structure equals

$$\xi_{X}Y = (\nabla_{X}Y^\perp)^\perp + (\nabla_{X}Y^\perp)^\perp.$$ 

The associated bundle $N = SO(M)/G = SO(M) \times_{SO(n)} (SO(n)/G)$ is the Grassman bundle $Gr_m^o(TM)$ of $m$--dimensional oriented subspaces of tangent spaces to $M$ and the induced section $\sigma : M \rightarrow N$ is just the Gauss map of the distribution $E$. Therefore the main result (Theorem 6.1) can be rewritten in this case in the following form

**Corollary 7.1.** Let $P \subset SO(M)$ be an almost product structure on an oriented Riemannian manifold $(M, g)$ induced by the distribution $E$. The following conditions are equivalent

1. $P$ is minimal in $SO(M)$,
2. the Gauss map $\sigma : (M, \tilde{g}) \rightarrow (Gr^o_m(TM), \langle \cdot, \cdot \rangle)$ of $E$ is a harmonic map,
3. the image $E = \sigma(M) \subset Gr^o_m(TM)$ is minimal in the oriented Grassmann bundle $Gr^o_m(TM)$. 

Let us provide some examples by considering examples known in the literature satisfying the third condition of above corollary (see [10] and bibliography therein). To fit these examples to our context we will analyze each in more detail concentrating on the intrinsic torsion.

(1) Let \((M, g, X_0)\) be a K–contact manifold with the Reeb vector field \(X_0\), i.e. \(X_0\) is a unit Killing vector field and there exist one–form \(\eta\) and endomorphism \(\varphi\) such that

\[
\eta(X) = g(X, X_0), \quad \varphi^2 X = -X + \eta(X)X_0,
\]

\[
d\eta(X, Y) = g(X, \varphi Y), \quad \iota_{X_0} d\eta = 0
\]

for all \(X, Y \in \Gamma(TM)\). One can show that \([2, 13]\)

\[
\nabla_X X_0 = -\varphi X,
\]

\[
R(X_0, X)Y = g(X, Y)X_0 - g(Y, X_0)X,
\]

\[
(\nabla_X \varphi)Y = R(X_0, X)Y,
\]

for \(X, Y \in \Gamma(TM)\).

The one dimensional distribution \(E\) tangent to \(X_0\) defines the almost product structure on \(M\). Since \(X_0\) is geodesic vector field, the distribution \(E\) is totally geodesic. It can be shown, that \([2, 13]\)

\[
\nabla_X X_0 = -\varphi X,
\]

\[
R(X_0, X)Y = g(X, Y)X_0 - g(Y, X_0)X,
\]

\[
(\nabla_X \varphi)Y = R(X_0, X)Y,
\]

for \(X, Y \in \Gamma(TM)\).

Denote by \(X^\top\) and \(X^\perp\) the tangent and orthogonal parts of the vector \(X\) with respect to the decomposition \(TM = E \oplus E^\perp\). By the above properties, the intrinsic torsion \(\xi\) takes the form

\[
\xi_X Y = -g(Y, X_0)\varphi X + g(Y, \varphi X)X_0 = R(X_0, \varphi X)Y.
\]

In particular, \(\xi_{X_0} = 0\). Thus \(LX = X + 2X^\perp\) and

\[
\tilde{g}(X, Y^\top) = g(X, Y^\top), \quad \tilde{g}(X, Y^\perp) = 3g(X, Y^\perp).
\]

If \((e_i)\) is an orthonormal basis with respect to \(g\), where \(e_1 = X_0\), then the \(\tilde{g}\)–orthonormal basis \(\tilde{e}_i\) equals

\[
\tilde{e}_1 = e_1 = X_0, \quad \tilde{e}_A = \frac{1}{\sqrt{3}} e_A, \quad A = 2, 3, \ldots, n.
\]

Notice, moreover, that

\[
R_{\xi_X}(X) = 2R(\varphi X, X_0)X = -2\xi_X X, \quad X \in \Gamma(TM).
\]

Therefoee,

\[
\sum_i (\nabla_{\tilde{e}_i} \xi)_{\tilde{e}_i} - \xi_{R_{\xi_X}(\tilde{e}_i)} = \frac{1}{3} \sum_A (\nabla_{\tilde{e}_A} \xi)_{\tilde{e}_A} - \xi_{R_{\xi_X}(\tilde{e}_A)} = \frac{1}{3} \sum_A (\nabla_{\tilde{e}_A} \xi)_{\tilde{e}_A} + 2\xi_{\xi_X(\tilde{e}_A)}.
\]
Evaluating on the vector $X$, and using the identities
\[
\sum_A (\nabla_{e_A} \varphi)e_A = (n - 1)X_0, \quad \sum_A \xi_{e_A}(e_A) = 0,
\]
we get
\[
\sum_A (\nabla_{e_A} \xi)e_A(X) + 2\xi_{e_A}(e_A)(X) = -\sum_A g(X, \nabla_{e_A} X_0)\varphi e_A + g(X, \varphi e_A)\nabla_{e_A} X_0 + \sum_A -g(X, X_0)(\nabla_{e_A} \varphi)e_A + g(X, (\nabla_{e_A} \varphi)e_A)X_0
\]
\[
= -\varphi^2 X - \nabla \varphi X_0
\]
\[
= 0,
\]
which implies that the $G$–structure $P$ is minimal.

(2) Consider the sphere $S^{4n-1}$ in the Euclidean space $\mathbb{R}^{4n}$ and let $I, J, K$ be the usual quaternionic structure on $\mathbb{R}^{4n}$. The 3–dimensional subspaces spanned by $IN, JN, KN$, where $N$ is the unit outward vector field to $S^{4n-1}$, determine the Hopf distribution $E$ on $S^{4n-1}$. Since the Hopf distribution defines the minimal immersion of $S^{4n-1}$ into the Grassmann bundle $Gr_3^3(TS^{4n-1})$, it follows that the $SO(3) \times SO(4n-4)$–structure $P \subset SO(S^{4n-1})$ is minimal.

Let us derive the formula for the tension field and check the minimality of $P$. The Hopf distribution defines totally geodesic foliation. For simplicity let
\[
e_1 = IN, \quad e_2 = JN, \quad e_3 = KN.
\]
Then we have the following well known conditions,
\[
[e_1, e_2] = 2e_3, \quad [e_2, e_3] = 2e_1, \quad [e_3, e_1] = 2e_2,
\]
which imply
\[
\nabla_{e_1} e_1 = 0, \quad \nabla_{e_2} e_2 = -\nabla_{e_1} e_2 = e_3, \quad \nabla_{e_1} e_2 = -\nabla_{e_1} e_2 = e_3, \quad \nabla_{e_1} e_2 = -\nabla_{e_3} e_1 = e_2.
\]
Since the Hopf distribution is determined by the fibration $S^3 \to S^{4n-1} \to \mathbb{H}P^{n-1}$ over the quaternionic projective space, we may lift the orthonormal basis $(f_i)$, such that $\nabla f_if_j = 0$ and $I f_i = f_{i+4}, Jf_i = f_{i+2}, Kf_i = f_{i+3}$ to the orthonormal basis $(\vec{e}_i)$, where $\vec{e}_i$ is the lift of $f_i$. Moreover, the vectors $[e_i, \vec{e}_j]$ and $\nabla \vec{e}_j \vec{e}_k$ are, by O’Neill equations [20], vertical. Then (see [15])
\[
\nabla_{\vec{e}_{4i+1}} e_1 = -\vec{e}_{4i+2}, \quad \nabla_{\vec{e}_{4i+2}} e_1 = -\vec{e}_{4i+1}, \quad \nabla_{\vec{e}_{4i+3}} e_1 = -\vec{e}_{4i+4}, \quad \nabla_{\vec{e}_{4i+4}} e_1 = -\vec{e}_{4i+3}
\]
\[
\nabla_{\vec{e}_{4i+1}} e_2 = -\vec{e}_{4i+3}, \quad \nabla_{\vec{e}_{4i+3}} e_2 = -\vec{e}_{4i+4}, \quad \nabla_{\vec{e}_{4i+4}} e_2 = -\vec{e}_{4i+1}, \quad \nabla_{\vec{e}_{4i+1}} e_2 = -\vec{e}_{4i+2}
\]
\[
\nabla_{\vec{e}_{4i+1}} e_3 = -\vec{e}_{4i+4}, \quad \nabla_{\vec{e}_{4i+4}} e_3 = -\vec{e}_{4i+1}, \quad \nabla_{\vec{e}_{4i+1}} e_3 = -\vec{e}_{4i+2}, \quad \nabla_{\vec{e}_{4i+4}} e_3 = -\vec{e}_{4i+3}
\]
and $\nabla_{e_i} \vec{e}_{4j+k} = \nabla_{\vec{e}_{4j+k}} e_i$. Using these relations we get
\[
\xi_{e_i} = 0, \quad \xi_{\vec{e}_j} = \nabla_{\vec{e}_j}.
Thus \( \tilde{g} = g \) on the hopf distribution (fibers) and \( \tilde{g} = 4g \) on the distribution orthogonal to the fibers. Moreover, since \( S^{4n-1} \) has constant sectional curvature equal to one,

\[
R_{\xi_X}(X) = -2\xi_X X, \quad X \in \Gamma(TS^{4n-1}).
\]

In particular, \( R_{\xi_e}(e_i) = 0 \) and \( R_{\xi_{\tilde{e}_j}}(\tilde{e}_j) = -2\nabla_{\tilde{e}_j} \tilde{e}_j \). Therefore, the condition for the minimality of \( P \) in \( SO(M) \) takes the form

\[
\frac{1}{2} \sum_j (\nabla_{\tilde{e}_j} \xi)_{\tilde{e}_j} = \nabla_{\tilde{e}_j} \xi_{\tilde{e}_j} - \xi_{\nabla_{\tilde{e}_j} \tilde{e}_j} = \nabla_{\tilde{e}_j} \xi_{\tilde{e}_j},
\]

Evaluating on a vector \( X \) we get

\[
\frac{1}{2} \sum_j (\nabla_{\tilde{e}_j} \xi_{\tilde{e}_j})(X) = \sum_j (\nabla_{\tilde{e}_j} \nabla_{\tilde{e}_j} X - \xi_{\tilde{e}_j} (\nabla_{\tilde{e}_j} X)) = 0,
\]

which implies that \( P \) is minimal.

7.2. **Almost Hermitian structures.** Let \((M, g)\) be a \(2n\)-dimensional manifold with the almost complex structure \(J\), i.e. \(J\) is an endomorphism of the tangent bundle \(TM\) such that \(J^2 = -I\). Assume \(J\) is hermitian with respect to \(g\),

\[
g(JX, JY) = g(X, Y), \quad X, Y \in TM.
\]

Let \(P\) be a \(U(n)\)-structure with respect to \(J\). The holonomy of \(M\) reduces to \(U(n)\) if \((M, g, J)\) is integrable. The intrinsic torsion of \(U(n)\)-structure \(P\) is given by the formula \[7\]

\[
\xi_X Y = \frac{1}{2} J(\nabla_X J)Y.
\]

Denote by \(Z(M)\) the twistor bundle over \(M\), i.e. the associated bundle to \(SO(M)\) with the fiber of the totality \(SO(2n)/U(n)\) of almost complex structures in \(\mathbb{R}^{2n}\).

Theorem \[5.11\] can be stated as follows.

**Corollary 7.2.** The following conditions are equivalent

1. the almost hermitian structure \(P\) is minimal in \(SO(M)\)
2. the induced section \(\sigma : (M, \tilde{g}) \to Z(M)\) of the twistor bundle \(Z(M)\) is a harmonic map,
3. the image \(\sigma(M)\) in the twistor bundle \(Z(M)\) is minimal.

Let us provide one example. Consider the 6–sphere \(M = S^6\) with the standard almost complex structure. Then the submanifold \(\sigma(M) \subset Z(M)\) is minimal \[3\]. Therefore, by Corollary \[7.2\] almost hermitian structure \(P\) is minimal in \(SO(M)\). Since this structure is compatible with the orthogonal structure, it follows that it cannot be integrable. Thus the intrinsic torsion is non–trivial.

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