A GENERAL QUALITATIVE THEORY OF CONSERVATION LAWS,
THEIR VIOLATION AND OTHER SPONTANEOUS PHENOMENA
Part I

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Tell me if there will be a doomsday. When is irrelevant.

**ABSTRACT**

We formulate a general theory of conservation laws and other invariants for a physical system through equivalence relations. The conservation laws are classified according to the type of equivalence relation; group equivalence, homotopical equivalence and other types of equivalence relations giving respective kinds of conservation laws. The stability properties in the topological (and differentiable) sense are discussed using continuous deformations with respect to control parameters. The conservation laws due to the abelian symmetries are shown to be stable through application of well-known theorems.

**INTRODUCTION**

The recent discoveries of what are called "topological conservation laws" and the difficulties of interpreting certain charge conservations when there are "spontaneously" broken symmetries, led us to think that the usual concepts of conservation laws corresponding to symmetries via Noether's theorem are not broad enough to include, for example, magnetic charge ('t Hooft theory) conservation and other number conservation laws. In the case of baryon and lepton conservation laws there is no physical gauge field from whose invariance these laws follow, although current does exist. Besides these difficulties, there does not exist, at present, a qualitative theory which can give an existence theorem as to the violation of any conservation laws starting from some physically acceptable axioms. The existence theorem means a kind of yes-no theorem regarding violation of conservation laws. Since the theory we are going to give is a qualitative one, it will give only a yes or no answer to the question of violation. No attempt is made to compute the quantity of violation.

It is proposed in Ref.4a that one can study this problem of violation of conservation laws as well as the symmetry breaking (spontaneous or otherwise) phenomenon using standard techniques of differential topology and bifurcation theory. It is the purpose of this paper to generalize the concept of conservation laws, to classify them and to study their stability under various circumstances. There were some efforts in this direction, for example, by Osborn, although along a classical line of thought. However, due to mathematical sophistication, such papers are rarely read by the physicists who are the prime users of selection rules and conservation laws in their study of various phenomena. In fact we shall try, in this paper, to keep the modern mathematical concept as close to physical terminology as possible and be as self-contained as possible.

In Ref.4a we have proposed the principle of stability of invariance of the physical system, a concept analogous to that of structural stability of the dynamical system defined by Poincaré, Alexandroff and Pontrjagin, Peixoto and Thom in various contexts. Although in Ref.4a the basic axioms are the existence of a group of symmetries and that Noether's theorem relates conservation laws to symmetries, we consider now all this in a more general fashion. The usual invariance principles are now replaced by the invariance principles in a very general sense, as used in mathematics. Our basic philosophy, as stated in Ref.4a, remains unchanged, i.e. we consider the invariance principles as the fundamental laws of nature and we are interested only in the invariant properties of a system. For a closed physical system, we are interested in those measurable quantities which do not change in space, time,
space-time or some other internal phase-space of description. In the usual dynamics one is interested in quantities which are changing in time or their rate of change. However, we insist that we are not just considering the kinematics, which is obviously a small domain, and this shall be made clear in the later part of this paper.

The concept of stability has various meanings and has been used in various contexts depending upon what one is dealing with. Here we use it in the differential-topological sense and it can be defined with respect to deformation in a rather general way. This approach is very much in the spirit of Thom's theory of catastrophe but it differs basically from the point of view of dynamics.

The spontaneous symmetry breaking can, nevertheless, be described within Thom's catastrophe approach, as shown by Araki et al., as one of the six elementary catastrophes of the potential function in but it has nothing to do with the usual symmetry breaking or violation of a conservation law. However, the difficulty of defining an integral charge \( Q = \int \frac{1}{2} \nabla \phi \cdot \nabla \phi \, dx \) from the continuity equation \( \nabla \cdot \mathbf{J} = 0 \) when there ia degeneracy of vacuum, is very often attributed to symmetry breaking and violation of a conservation law in the sense that the symmetry group has changed. This is not the case, as recent discovery of topological conservation laws (due to homotopy equivalence) shows, the homotopical invariance remains as it is under catastrophe of potential function conserved and as a result certain types of charge can be defined as topological invariants (the topological degree). One well known example is magnetic charge. However, the question remains whether we integrate the currents within a simply-connected domain or not. It is also possible in some cases that the homotopy type changes abruptly due to the variation of some control parameter, and then even the topological invariance breaks down. In the following we consider such a question for various physical situations as well as in our general theory.

Since we shall be using mathematical concepts in the formulation of our theory, Sec. I of this paper is devoted to mathematical preliminaries, the formulation of theory as well as proofs of the theorems. In Sec. II we give examples of interest to physicists.

SECTION I

1.1 A relation \( R \) is a set of ordered pairs, that is, a relation is a set, each member of which is an ordered pair. If \( R \) is a relation, we write \( x R y \) and \((x,y) \in R \) interchangeably, and we say that \( x \) is \( R \)-related to \( y \) if and only if \( x R y \). An "equivalence relation" is a reflexive, symmetric and transitive relation. This means each point of \( R \) (domain) is \( R \)-related to itself, \( x R x \) whenever \( x \) is in \( R \) and whenever \( x R y \) and \( y R z \) then \( x R z \). Suppose that \( R \) is an equivalence relation and that \( X \) is the domain of \( R \). A subset \( A \) of \( X \) is an equivalence class (or an \( R \)-equivalence class) if and only if there is a member \( x \) of \( A \) such that \( A \) is identical with the set of all \( y \) such that \( x R y \). The fundamental result on equivalence relations states that the family \( \mathcal{C} \) of all equivalence classes is disjoint and that a point \( x \) is \( R \)-related to a point \( y \) if and only if both \( x \) and \( y \) belong to the same equivalence class.

Theorem 1.1 [Kelly]

A relation \( R \) is an equivalence relation if and only if there is a disjoint family \( \mathcal{C} \) such that \( \mathcal{C} = \{ A \mid A \in \mathcal{C} \} \).

We must remind the reader that the above underlined result is one of the axioms in our theory later on. Let \( S = \{ s_1, s_2, \ldots, s_n, \ldots \} \) and \( S' = \{ s'_1, s'_2, \ldots, s'_n, \ldots \} \) be two countable sets of systems. Let \( S \) and \( S' \) be two equivalence relations on \( S \) and \( S' \), respectively. That is, \( s_i R s_j \) and \( s'_{i'} R s'_{j'} \) is true for all \( i, j \) in an indexed set \( I \). All the systems \( s_1, s_2, \ldots, s_n, \ldots \) are equivalent to each other with respect to the equivalence relation \( R \) and similarly \( s'_1, s'_2, \ldots, s'_{n'}, \ldots \) etc. with respect to \( R' \). The sets of systems \( S \) and \( S' \) are not equivalent either with respect to \( R \) or \( R' \).

Now we define a property \( P \) associated with an equivalence relation \( R \). This is true in general, i.a. we associate with each equivalence relation defined on a set, a property which remains invariant on the elements of this set. This is going to be another axiom of this theory. Before defining what we call "quasi-catastrophe", we give some physical examples to clarify our point.

Examples:

1) Consider a trajectory (or world line) in space-time of a baryon. Denote each point of this world-line in space time as \( s_1, s_2, \ldots, s_n, \ldots \) representing this baryon. All points \( s_i, s_j, \ldots \) are equivalent in the sense that they are representing the "same" system under the equivalence relation, that is \( s_i \) and \( s_j \) are baryons. The property which remains invariant under
evolution of this trajectory is, say, baryon number $B$. We must note at this point that only with such a definition can a spontaneous change be described. One can define many equivalence relations on a given set of systems.

2) Consider now fermions having the property that they have half-integer spin. Also consider two sets $E = \{e_1, e_2, \ldots, e_n\}$ and $F = \{f_1, f_2, \ldots, f_n\}$ representing states of electron (spin 1/2) and spin 3/2 particles. Now the sets $E$ and $F$ are $R$-related if the associated property (to $R$) is half-integer spin but they are inequivalent with respect to property 1/2-spin and 3/2-spin. That is, $e_1 R' e_2$ and $f_1 R'' f_2$, $i, j \in I$, where $I$ is an indexed set, then $E$ is not equivalent to $F$ through $R'$ and $R''$. $E$ and $F$ are disjoint with respect to $R'$ and $R''$ but $E$ and $F$ are $R$-related. The $R$-relation is not of much importance. Now we define a quasi-catastrophe. Let us two sets $E$ and $F$ as above, each forming an equivalence class of set-elements and having an invariant property associated with $R'$ and $R''$. We call such a property "invariant" because it does not change over set-elements, qualitatively or quantitatively. The question is now, can we jump from the set $E$ to $F$ in the sense that the limit of a sequence of elements $E$ belongs to $F$. If such a limit exists, then it means that at some point our system changed abruptly. Physicists have been using the concept of spontaneous change very often, but without defining it. We call such a spontaneous change of equivalence classes a "quasi-catastrophe", to distinguish it from Thom's definition. In some cases quasi-catastrophe and catastrophe are the same concept. Also, we stress that in the above example 2, the equivalence relation $R$ does not represent quasi-catastrophe.

**Definition**

Let $S = \{S_1, S_2, \ldots\}$ and $S' = \{S'_1, S'_2, \ldots\}$ be two sets of systems having equivalence relations $R$ and $R'$ as above. (Physically $S_1, S_2, \ldots$ and $S'_1, S'_2, \ldots$ may represent a point of system trajectory in space-time, or even "internal space".) Then the limit of the sequence of systems $S_1, S_2, \ldots, S_n, \ldots$, if it exists, may be in $S$ or $S'$. If the limit of such a sequence $S_1, \ldots, S_n, \ldots$ belongs to $S'$ then such a limiting operation is called a "quasi-catastrophe". Later on we shall discuss the various kinds of equivalence relations, such as group equivalence (or G-equivalence), homotopy equivalence etc., and shall classify invariant types accordingly. At this stage the definition is so general that it looks almost trivial, but when we consider each type of equivalence relation separately, the structure becomes richer and gives non-trivial results.
point \( x_0 \) give the unit element of the group. Since in physical application we shall very often be using \( n \)-sphere \( S^n \), we give the following assertions without proofs.

**Theorem 2.2**

The following are equivalent for a path-wise connected space:

1) \( X \) is simply-connected;
2) \( X \) is contractible;
3) Every closed path in \( X \) is freely null homotopic;
4) The homotopy class of an arbitrary path in \( X \) is completely determined by its initial and its end points.

**Theorem 3.2**

The \( n \)-sphere \( S^n \) is simply connected for \( n > 1 \).

**Remark**

It is necessary in physical problems, for various reasons, that the space relevant to the problem be simply-connected or at least contractable. Otherwise one cannot obtain results like Green's theorem, Gauss' theorem and Stokes theorem. The reason is that non-contractibility of the space implies the existence of a singular point and the integral diverges or is undefined on a non-contractable domain. The sphere \( S^n \) is not contractable although it is simply-connected for \( n > 1 \).

**Theorem 4.2**

The fundamental group of a path-wise connected space is an invariant of homotopy type.

**Remark**

We shall show later that to prove the existence of homotopical conservation laws one has to prove that the configuration space of the physical system is "contractable" or simply-connected and there does exist a boundary of this domain. A compact domain implies stable systems. [At least that one can join any two points in a configuration space by a continuous path in it.] We list some more statements about sphere \( S^n \), which one can find in Refs. 10 and 12.

- 1) Spheres of different dimensions are not homotopic to each other, to \( S^n \) or to a point where \( S^n \) is the equatorial n-cell

\[ S^n = \{ x | x \in \mathbb{R}^{n+1}, |x| < 1, x_n = 0 \} , \]

where \( \mathbb{R}^{n+1} \) is an \((n+1)\)-dimensional space.

2) Let \( g:S^n \to S^n \) be an orthogonal transformation. The determinant, \( \det g \), is the only homotopic invariant. A vector field is said to be defined on \( S^n \) if and only if with each point \( x \in S^n \) is associated a vector \( a(x) \neq 0 \) so that the function is continuous and \( a(x) \) is orthogonal to \( x \).

3) \( S^{2n-1} \) has a vector field.

4) \( S^n \) has no vector field (the Cowlick theorem).

5) Let \( a(x) \) be a vector field on \( S^{2n-1} \). Then the function

\[ x \mapsto \frac{a(x)}{|a(x)|} \]

defines a continuous function \( S^{2n-1} \to S^{2n-1} \), which is homotopic to its identity.

Let us define the degree of a function \( f:S^n \to S^n \), which is a topological invariant and corresponds to magnetic charge, a conserved quantity, in the 't Hooft model. Let \( f:S^n \to S^n \) be continuous and \( R \) be a ring of integers. Then the \( n \)-th homology group of sphere \( H_0(S^n) = \mathbb{Z} \) and \( H_n(S^n) \) is a free \( Z \)-module with one generator, \( e \). For the homomorphism \( f_*:H_0(S^n) \to H_0(S^n) \) it follows that \( f_*(e) = m e \) for some \( m \in \mathbb{Z} \). Then \( f_*(we) = wf_*(e) \). The integer \( m \) is the degree of \( f \). It is roughly what physicists called a "wrapping number."

The degree of orthogonal transformation \( f:S^n \to S^n \) is \( \det f \). All homotopic functions have the same degree.

6) If \( f:S^n \to S^n \) and \( f \in C^0 \) (i.e. continuous) and \( f(x) \neq 0 \) then \( \deg f = 0 \).

7) If \( f \) and \( g \) on \( S^n \) are \( C^0 \)-functions and \( f(x) \neq g(x) \forall x \in S^n \) then \( \deg f = -\deg g \).

8) If \( f:S^n \to S^n \) has no fixed points, i.e. there is no \( x \) for which \( f(x) = x \), then \( \deg f = (-1)^{n+1} \).

All of the above statements are standard theorems in algebraic topology and one can find their proofs in any of the books on this subject. We shall only be using these theorems to prove results of physical importance.

1.3 We shall now give some results from the deformation theory of Lie algebra which we shall be using to study the stability properties of conservation laws which are due to symmetries of physical systems via the theorem of Emmy Noether.

Let \( G \) be the symmetry group of the physical system and \( \mathfrak{g} \) be the corresponding Lie algebra. We define a Lie algebra \( \mathfrak{g} \) on a field \( k \) as a pair \( (\mathfrak{g}, k) \), formed by a linear vector space \( \mathfrak{g} \), whose coefficients are in \( k \),
and a bilinear mapping \( \mu: V \otimes V \to V \) which satisfies
\[
\sum_{P(x,y,z)} \mu(x,y,z) = 0
\]
\( x, y, z \in V \). Here \( P(x,y,z) \) is the cyclic permutation over \( x, y \) and \( z \).

The mappings \( \mu \) are bilinear alternating maps. Let the dimension of \( g = (V,V) \) be \( m \). \( \mu \) is actually what physicists call Lie commutator \( [ , , ] \).

For any positive integer \( n \leq m \), let the dimension of \( V \), the projective algebraic variety of \( n \)-dimensional subspaces of Lie algebras \( g \) is a Grassmannian variety denoted by \( \Pi_n(g) \) and has a natural structure of algebraic transformation space for the algebraic group \( GL(g) \) or \( GL(V) \).

[See for details Ref. 1,2.] A rational representation of an algebraic group \( G \) is a morphism \( \rho: G \to GL(V) \), where characters are defined by \( \chi: G \to GL(V) \). The orbits of \( G \) at \( x \in V \) is denoted as \( G(x) \) and is a set \( G(x) = \{ g \cdot x | g \in G \} \). We write \( g \cdot x \) for \( \rho(g) \cdot x \), where \( \rho: G \to GL(V) \) is a rational representation of \( G \). A one-parameter subgroup of an algebraic group \( G \) is a morphism \( \lambda: \mathbb{R} \to G \) of algebraic groups.

Let \( f: G \to X \) be a morphism of algebraic varieties, then one may define \( f(t) = \lambda(t) \cdot x \in \mathbb{R} \). If \( f \) extends to a morphism \( f: G \to X \), then \( y = f(0) \) is called the specialization of \( f(t) \) at \( t = 0 \). This is written as \( f(t) \to y \) as \( t \to 0 \). A specialization in algebraic invariant theory corresponds to "quasi-catastrophe" in the present theory. Now let \( a \) and \( b \) be two points of \( P(V) \), then \( b \) is a specialization of \( a \), if \( a + b = a \cdot x \) as \( t \to 0 \), where \( a = \lambda(t) \cdot a \). Correspondingly, a Lie algebra \( b \) is a specialization of Lie algebra \( a \), if in the limit \( t \to 0 \) it is approximated by \( a \). We can describe the "quasi-catastrophe" for Lie algebra corresponding to the symmetries of physical systems in this fashion.

We also need the concept of an algebraic set which can be defined as follows: let \( \mathbb{k} \) be an algebraically closed field \( 13 \) and \( A^n \) be an affine space. Let \( M(x_1, \ldots, x_n) \) denote the ring of polynomials in \( x_1, x_2, \ldots, x_n \) with coefficients in \( \mathbb{k} \). Denote this ring as \( \mathbb{k}[A^n] \). If \( B \) is a proper subset of \( A^n \), then the set \( F \in \mathbb{k}[A^n] \) such that \( F(x) = 0 \forall x \in B \) is an ideal in \( \mathbb{k}[A^n] \), denoted as \( I(B) \). A subset \( M \) of \( A^n \) is called an algebraic set in \( A^n \) if and only if \( M = V(F) \), where \( V(F) \) is the set of zeroes of \( F \) called the locus of \( F \). This means that \( M \) is an algebraic set in \( A^n \) if and only if \( V \) is the locus of some ideal in \( \mathbb{k}[A^n] \). Also, we can write \( M = \bigcap_{F \in I(V)} V(F) \). Now, applying these ideas to the case of Lie algebras of symmetry groups of a physical system, we consider the set \( M \) of all Lie algebra multiplication, with the underlying vector space \( V \) which forms an algebraic set in the space \( A^2(V,V) \) of all bilinear alternate maps \( \mu: V \otimes V \to V \). In a given

base \( \{ e_1, e_2, \ldots, e_n \} \) for \( V \), the bilinear maps \( B: V \otimes V \to V \) can be written as
\[
B(e_i, e_j) \sum_{k=1}^{n} c_{ij}^k e_k , \quad \text{where } i, j = 1, 2, \ldots, n.
\]
For \( B: V \times V \to V \), \( c_{ij}^k \in \mathbb{k}^3 \) and for \( \mu: V \otimes V \to V ; \mu(x,y) = x \cdot y \), where \( \mathbb{k}^3 = \mathbb{k} \times \mathbb{k} \times \mathbb{k} \) and \( \mathbb{k}^6(n-1)/2 = \mathbb{k} \times \mathbb{k} \times \mathbb{k} \times \mathbb{k} \) are affine spaces. The \( c_{ij}^k \), so called structure constants, satisfy
\[
c_{ij}^k = -c_{kj}^i , \quad \mu(x,y) = 0 \quad \text{and} \quad \sum_{P(x,y,z)} \mu(x,y,z) = 0 .
\]

The equation
\[
\sum_{P(x,y,z)} \mu(x,y,z) = 0 .
\]
is a polynomial equation and defines an algebraic variety in \( \mathbb{k}^3 \), which is a cone. Denote this variety as \( M \). The group \( G = GL(V) \) acts on \( M \) as follows.

Let \( \varphi \in GL(V) \) and \( \mu \in M \). Then \( \varphi \cdot \mu \in M \) with \( \varphi \cdot \mu(x,y) = \varphi \cdot (x \cdot y) = \varphi \cdot x \varphi^{-1} \cdot \varphi \cdot y \) for \( x, y \in V \). All \( \mu' = \varphi \cdot \mu \) are the lie algebra multiplication which gives Lie algebra \( (V,\mu') \) isomorphic to \( (V,\mu) \), i.e. the diagram
\[
V \times V \xrightarrow{\mu} V \xrightarrow{\varphi} V \xrightarrow{\varphi} V
\]
is commutative.

In the algebraic variety, all points \( \mu' \) are deformed Lie algebra of \( \mu \). This corresponds to what physicists usually do when they make structure constants \( c_{ij}^k \) parameter-dependent as \( c_{ij}^k(\lambda) \), which in fact is an action of \( GL(V) \). However, we shall formally define what deformation means in the case of Lie algebras and also "contraction", as it is called by the physicists in this context. The topology we shall use in this case is called Zariski topology. By taking closed sets \( A^n \) together with \( 1 \), the affine one-space and empty set, one can topologize algebraic sets. Then \( F \in \mathbb{k}[A^n] \) is \( F(A^n) = A^n \).
On $\mathbb{A}^n$, the coarsest topology, such that all these mappings are continuous, is called the Zariski topology, and if the algebraic set $M \subset \mathbb{A}^n$ is a closed set in $\mathbb{A}^n$, then $M$ inherits a topology from $\mathbb{A}^n$. Therefore the Zariski topology is closed if and only if there exists a closed subset $L \subset \mathbb{A}^n$ such that $M = M \cap L$. This is also a Zariski topology.

A simplex is a convex object which includes a point, a triangle, a pyramid etc., and a complex is a linear combination of simplexes of various order, i.e., of higher dimensional simplexes also. A cohomology group on a complex $c^n$ (or what is called co-chain) can be defined as follows. Consider the sequence of mappings,

$$
\delta^n c^n \rightarrow \delta^{n-1} c^{n-1} \rightarrow \cdots \rightarrow \delta^2 c^2 \rightarrow \delta^1 c^1 \rightarrow 0 .
$$

Let $\ker \delta_n$ (i.e., the kernel of mapping $\delta_n$ are those elements of $c^n$ which are mapped to identity (or zero) element of $c^{n+1}$) be the co-cycle operator and $\text{Im} \delta_n$ (i.e., the image of mapping $\delta_n$) be the co-boundary operator. Suppose $\text{Im} \delta_{n-1} \subset \ker \delta_n$; then define a coset $H^n = c^n/\ker \delta_n$ is the n-th cohomology group for complex $c^n$. The exactness of the sequence, which means $\ker \delta_n = \text{Im} \delta_{n-1}$, implies that $\gamma^n = \{0\}$ or nth cohomology group is trivial for all $n$. For Lie algebra $\mathfrak{g}$ it can be defined as follows.

Let $\delta^n c^n \rightarrow \delta^{n-1} c^{n-1} \rightarrow \cdots$ be a multilinear antisymmetric form and $\omega: \mathfrak{g} \times \mathfrak{g} \times \cdots \mathfrak{g} \rightarrow V$ a function called n-co-chain. The set of all these co-chains form a vector space denoted as $c^n$. A co-chain $c^n$ is a co-cycle if the exterior derivative $d_{\omega} = 0$ where $dc^n = c^{n+1}$. If there exists an (n-l) form $\omega$ such that $\omega = d\omega$ , for $\omega \in c^{n-2}$, it is a co-cycle called co-boundary if such a $\omega$ exists. The set of co-cycles form a vector space $Z^n$ and the co-boundary a subspace denoted as $d c^n$. Now the cohomology group is $H^n = Z^n/d c^n$ .

Why do we want to use the cohomology groups? Since we are going to define stability with respect to deformation or a limiting operation (quasi-catastrophe), the vanishing of the cohomology group gives the stability condition and extremely good and well-defined criteria for stability.

The cohomology of $\mathfrak{g}$ is a Lie algebra $\mathfrak{g}_t$ over the underlying vector space $V_K = \{V_k = V_k \otimes k((t))\}$, where $K$ is an (algebraic) extension of the field $k$, given by a mapping $f_t : V_K \rightarrow V_k$ expressed as

$$
f_t(a,b) = (a,b) + tF_1(a,b) + t^2F_2(a,b) + \cdots ,
$$

where $F_1$ are bilinear functions of $V \times V$ into $V$ defined over $k$. $F_2 \in Z^2(\mathfrak{g},\mathfrak{g})$ is the co-cycle and gives rise to deformation if and only if it is integrable, i.e., if

$$
\sum_{\mu + \nu = n} F_{\mu}(F_{\nu}(a,b),c) + F_{\nu}(F_{\mu}(a,b),c) = 0 .
$$

This is the formal deformation by Gerstenhaber which can be put into a more appropriate way so as to define the set of catastrophe or quasi-catastrophe points. Let $\mathfrak{u}$ and $\mathfrak{v}$ be as above. Set $\varphi = \mu - \mu$; then $\varphi$ is also an alternating bilinear map. Now $\mu'$ is a Lie algebra multiplication if it satisfies the Jacobi identity,

$$
\delta^n c^n = \text{Im} \delta_{n-1} \subset \ker \delta_n .
$$

The first term (r.h.s.) vanishes, while the second and third take the familiar form in cohomology. They give the co-boundary $\delta \varphi$ of the 2-co-chain $\varphi$ ;

$$
\delta \varphi(x,y,z) = \sum_{cyc} \varphi(x,\varphi(y,z)) + \sum_{cyc} \varphi(x,\varphi(y,z)) .
$$

Denote $\frac{1}{2} [\varphi, \varphi](x,y,z) = \sum_{cyc} \varphi(\varphi(x,y),z)$ and the deformation equation becomes

$$
\delta \varphi = \frac{1}{2} [\varphi, \varphi] .
$$

where $\varphi$ is a deformation of $\mu$ (i.e., $\mu' = \mu + \varphi$ is a Lie product if and only if $\varphi$ is a solution of the above equation(2)). The solution to the linearized version of (2), i.e.,

$$
\delta \varphi = 0 ,
$$


[Gerstenhaber]
is called the "infinitesimal deformation" of \(\mu\). Thus the infinitesimal deformations are just 2-co-cycles. Equivalent deformations define isomorphic Lie algebras.

The problem of stability we are going to study below is therefore related to whether or not one can obtain non-equivalent deformations and contractions or the boundary of deformation. It is well known from the work of Gerstenhaber, Kodaira and Spencer, Richardson and Sijthoff that when

\[ H^2(\mathfrak{g}, \mathfrak{g}) = 0 \text{ where } \mathfrak{g} = (\mathcal{V}, \mu) , \]

then all deformations are equivalent. In other words, \(\mathfrak{g} = (\mathcal{V}, \mu)\) is rigid or stable. However, whether or not the deformation set \(\mathfrak{g}_\mu\) is a dense set in \(\mathcal{M}\) is related to the occurrence of quasi-catastrophe. Quasi-catastrophe occurs if the bifurcation point exists and there exists not-trivial contraction of \(\mathfrak{g}\).

Let \(\mathfrak{g} = (\mathcal{V}, \mu)\) be a deformation of \(\mathfrak{g}\) and \(\mathfrak{h} = (\mathcal{V}, \mu')\), then \(\mathfrak{h}\) is a contraction of \(\mathfrak{g}\) in \(\mathfrak{g}\) if \(\mu'\) lies on the Zariski closure of the orbit of \(\mathfrak{GL}(\mathcal{V})\) at \(\mu\).

Denote \(\mathcal{G}(\mu)\) as the orbit of \(\mathfrak{GL}(\mathcal{V})\) and \(\overline{\mathcal{G}(\mu)}\) as the Zariski closure, then \(\mathcal{G}(\mu) = \overline{\mathcal{G}(\mu)}\) is the bifurcation set of points describing the quasi-catastrophe. If \(\mu' \in \overline{\mathcal{G}(\mu)}\), then \(\mu\) and \(\mu'\) are related through quasi-catastrophe and the Lie algebra \((\mu', \mathcal{V})\) are non-isomorphic, representing two non-equivalent systems from the point of view of symmetry.

**Formulation of theory**

Let us remind the reader that a physical theory consists of the following three ingredients:

1. A set of classes, such as the class of particles, the class of events, the class of electrons etc.
2. A set of relations that holds between the elements of these classes, such as the time relation between events, etc.
3. A set of propositions that specify the properties of these relations, such as the proposition that the coincidence relation between events is an equivalence relation.

The elements of the first two items are called "physical concepts" and those of the third item "physical laws". In effect, the aim of a physical theory is to explain the meaning of the physical concepts and to justify the physical laws. A physical concept is "explained" if it is defined in terms of understood concepts, and a physical law is justified if it is deduced from the accepted ones. But this can be a never-ending process and the only way out is to start with a set of concepts called the "primitive concepts" and a set of laws called "the postulates" (or axioms); then define all concepts in terms of the primitive concepts and deduce all laws from the postulates. The primitive concepts and postulates must satisfy certain experimental conditions and that at least one consequence of the postulates can be tested experimentally. In the case of our theory, the ingredients 1), 2) and 3) are given exactly in the way described above and the consequences of our postulates which can be tested experimentally is the violation of conservation laws in nature and abrupt changes like phase-transition, spontaneous symmetry breaking and symmetry restoration etc.

As described in the Introduction, we consider sets of systems such that with respect to equivalence relations (and hence associated invariant properties) they are classes. Our basic philosophy is to classify systems according to type of equivalence relations and study the breakdown of these equivalence relations. This in turn corresponds to studying invariant properties and their breakdown. At this point we would like to clarify the relation between the usual conservation laws and our invariant properties associated with an equivalence relation[11,12] Usually a conservation law means that the time-derivative of some quantity, say \(Q\), is zero, i.e. \(\partial Q / \partial t = 0\) or \([Q, \mathcal{H}] = 0\), where \(\mathcal{H}\) is a Hamiltonian operator. This concept is too restrictive and obviously non-covariant because of preferential treatment given to time, it is not adequate for treating complicated processes of modern relativistic physics. When we use the words "conservation law", it means that quantity, say again \(Q\), remains invariant under the equivalence relation which one specifies. Consequently, the derivative of this \(Q\) in time, space, space-time or some other internal parameter space, is zero,

\[ \partial Q = 0 , \quad \partial^2 Q = 0 , \quad \partial^3 Q = 0 , \quad \partial^4 Q = 0 , \quad ... \]

The usual conservation laws are a subset in this set of conservation laws. If we specify equivalence relations, then we get a definite type of conservation laws. There can be many types of conservation laws, out of which three kinds are known to us in nature.

**Type I**

If we take group equivalence as the equivalence relation of our theory, then we obtain the usual conservation laws due to symmetries of physical system via the theorem of Emmy Noether. In the terminology given above, all
the invariants of a given group are conserved quantities with respect to some parameter, only a few of them with respect to time-parameter. As an example, given $G = O(n)$, then the invariants of $O(n)$ are conserved quantities and only angular momentum is the usual Noether-type invariant. The reason we take all invariants is that these invariants determine completely the trajectory of the system without any reference to the equation of motion. We must remember again that we measure only invariant properties of a physical system and neither the changing quantities nor their rate of change in time.

Diagrammatically the evolution of a system looks like this:

The equivalence relation actually maps $S_1$ to $S_2$, $S_3$, etc. We are not assuming any ordering for $S_1, S_2, ...$, since what we measure is the invariant property with respect to the equivalence relation and hence it is irrelevant how one goes from, say, $S_1$ to $S_2$ as the property $P$ remains as it is. For a non-idealized natural system, one obviously has simultaneously many equivalence relations and invariants. In fact, the usual dynamics fits into this general approach, as time or space-time evolution can be described using equivalence relations which are displacements, translations or rotations in space, time or space-time as the case may be. In fact, when Newton invented his equation of motion, he took displacement of system in space and time separately and calculated the ratio of the two at each instant in space-time "continuously" to see the evolution of the system. We are doing the same thing, except we do not just stop at displacement in space and time and calculate ratio at each "instant" (a point in parameter-space) but we take in general any type of equivalence relation and calculate invariants instead of ratio. Certainly, in a parameter-space we can also define all kinds of derivatives (or "continuous ratios") but we do not need them to study the stability properties of invariants or conservation laws.

**Type II**

If the equivalence relation is a homotopy equivalence, then we obtain other kinds of invariants or conservation laws. In fact, the magnetic charge conservation law recently discovered (theoretically) by 't Hooft, Freund et al. belongs to this type. They label it "topological conservation law" resulting from the topological properties of gauge fields. [Details are given below.]

**Type III**

If we take cohomological equivalence as our equivalence relation, then we get some kind of super-selection rules telling us whether or not a particular kind of Type I conservation law is violable. For example, in Ref. we have shown that for $\mathbb{Z}^3(g,g) = 0$, where $g$ is the Lie algebra of the symmetry group, implies stability under catastrophe (catastrophe under deformation) but instability under quasi-catastrophe. One gets another classification of what is violable and what is not.

**The question of stability**

What kind of stability are we talking about in this paper? Given the definition of "quasi-catastrophe" and catastrophe, stability means that for an equivalence class a quasi-catastrophe does not occur at all. Roughly, this means that the equivalence class is complete in itself and no limit of its sequence goes out of the class. The system is then stable. We should like to distinguish between stability under catastrophe and quasi-catastrophe, however, in both cases it is related to stability under deformation or contraction of Lie algebra (or Lie group) for Type I conservation laws and stability under deformation for Type II conservation laws also. It will become clear from the examples given later exactly how stability under deformation is related to violation or non-violation of conservation laws. In fact, deformation of, say, a symmetry group with respect to a parameter corresponds to evolution or change of that system as some parameter, such as temperature, magnetic field etc., is varying. Correspondingly, conservation laws evolve and if one has non-trivial deformation then conservation laws change drastically, which means their spontaneous violation.

An important theorem we shall use for the case of Type I and Type II conservation laws is given below.

**Theorem** [Golubitsky and Guillemin, p.141]

Let $f:X \rightarrow Y$ be a smooth map and $X$ be compact. The following are equivalent:

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**Theorem** [Golubitsky and Guillemin, p.141]

Let $f:X \rightarrow Y$ be a smooth map and $X$ be compact. The following are equivalent:

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a) $f$ is stable,
b) $f$ is transverse stable,
c) $f$ is infinitesimally stable,
d) $f$ is homotopically stable,
e) $f$ is stable under $k$-parameter families of deformations.

A useful definition in this connection is:

Let $f$ and $g$ be relevant structures (Lie algebra, vector fields or so) and they are said to be "structurally stable" if, given $\varepsilon > 0$, one may find $\delta > 0$ such that whenever $\rho(g,f) < \delta$, $g$ is equivalent to $f$ and the corresponding homeomorphism is within $\varepsilon$ from the identity. Here $\rho$ is a metric in the space of the relevant structure.

This definition of stability will be used only for catastrophe, while for quasi-catastrophe the stability means that the bifurcation set $E = G(u) - G(u)$ is empty for Type I cases and non-existence of limit in general.

Stability under deformation can be defined more generally by this $c,\delta$-definition. Let $f: X \rightarrow Y$ be smooth and let $I_{\varepsilon} = (-\varepsilon, \varepsilon)$. Then

a) Let $F: X \times I_{\varepsilon} \rightarrow Y \times I_{\varepsilon}$ be smooth. $F$ is a deformation of $f$ if for each $s \in (-\varepsilon, \varepsilon)$, $F_s: X \rightarrow Y$ is defined by $F_s(x) = (F(x,s),s)$. Then $F$ is a deformation if there exists a deformation $G: X \times I_{\varepsilon} \rightarrow Y \times I_{\varepsilon}$ such that $G$ and $H$ are deformations of $f$ and $f$ respectively, and such that the diagram



commutes.

b) Let $F: X \times I_{\varepsilon} \rightarrow Y \times I_{\varepsilon}$ be the deformation of $f$. Then $F$ is trivial if there exists a diffeomorphism $G: X \times I_{\varepsilon} \rightarrow X \times I_{\varepsilon}$ such that $G$ and $H$ are deformation of $f$ and $f$ respectively, and such that the diagram



commutes.

c) $f$ is stable under deformations (or homotopically stable) if every deformation of $f$ is trivial.

Before going over to examples which have motivated this general approach, we should like to give below a summary of ideas and general results.

To include all kinds of conservation laws in nature and not only to describe the violation of these conservation laws but also to study phenomena of abrupt changes or what is called "spontaneous breakdown" (or spontaneous restoration of symmetries) we give a deductive physical theory where we start with a very general definition of conservation laws as the "invariants" of a given equivalence class of systems. A jump or spontaneous change from one equivalence class to another is described through quasi-catastrophes, catastrophes which are in any case limiting operators. When we take equivalence classes as group, homotopical and (co-)homological equivalences, then we obtain various types of conservation laws. The deformation (or limiting operation) of a set of equivalence relations gives the stability properties of the respective "invariants". The results obtained which are of physical importance are given in Sec. II, where only examples are given and comparison is made with the general theory.

SECTION II

Examples

Coming back to physical examples where situations like spontaneous change occur, we need a more precise definition of a physical system with symmetry. There are many equivalent descriptions, but we take a system specified by a function called Lagrangian for the sake of illustration to physicists, otherwise it is not necessary for the theory to be valid.

Let $M$ be a manifold which can be phase-space of description; $F(M)$ its ring of real valued functions, $\mathcal{V}(M)$ the set of its vector field, and $\mathcal{T}(M) = \bigcup_{\Phi} \mathcal{T}(\Phi)$ its tangent bundle. A Lagrangian $L$ on $M$ is just a real valued function on $\mathcal{T}(M) \times \mathbb{R}$ parametrized by $t$. The function $L$ enables one to define a real-valued function $\sigma = L(\dot{\sigma}(t), t)$ on curves $\sigma$ in $M$. If $\Sigma(a,b) \subset M$, then $L = \int_a^b L(\dot{\sigma}(t), t) dt$, where $t = \sigma'(t) \in M(t)$ is the tangent vector field to $\sigma$. $\sigma(a)$ and $\sigma(b)$ are the extremal points of $\sigma$. Now suppose that $L$ and $L'$ are Lagrangians on manifolds $M$ and $M'$; that $\phi: M' \rightarrow M$ is a map and $\phi_*: \mathcal{T}(M') \rightarrow \mathcal{T}(M)$ is the differential of $\phi$. Suppose that $(\phi_*)^*L = L'$, e.g. if $\phi$ is a diffeomorphism between $M$ and $M'$, it is clear that this implies that $\phi$ maps the extremal of $L'$ into an extremal of $L$. If $M = M'$ and $L = L'$, then

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such a map $\phi$ can be regarded as a symmetry of the variational problem. A Lie group $G$ acting on $M$ as diffeomorphisms of $M$ can be regarded as a group of symmetries if each individual transformation is a symmetry. Assuming that the function $L$ specifies the physical system, the action of $G$ will give a series of $L', L'', \ldots$, etc., which are equivalent to $L$, and as a result there will be invariant properties, which we call "conservation laws" in our theory of Sec.I. Given this, we shall now give examples.

Example 1 [Ref.1]

In this example we should like to describe the stability of invariance under quasi-catastrophe. Let the equivalence relations be the group action of $G$ and let $\mathfrak{g} = (V, \mu)$ be the corresponding Lie algebra of dimension $n$ and underlying space $V$. Let $\mathcal{M}_G$ be the set of all $\mu$ and the orbit of $GL(V) = G$ at $\mu$ as $G(\mu)$. $G(\mu)$ gives the isomorphism class of Lie algebras. Suppose now that there exists a point $\mu \in \mathcal{O}(\mu) - G(\mu)$ where $G(\mu)$ is the closure of $G(\mu)$ in the Zariski topology defined on $\mathcal{M}_G$, then the corresponding Lie algebra $(\mu, V) = \mathfrak{g}_1$ is not isomorphic to $\mathfrak{g}$. We define a system to be invariance stable if and only if the orbits of $G$ at $\mu$ are closed. The abelian algebras usually belong to the boundary of the orbit and hence abelian Lie algebras remain unaltered under this kind of operation. The consequence of this is that all conservation laws which are due to abelian symmetries are stable and one cannot deform or contract them to a simpler one. Two trivial consequences of proposition 1 of Ref. 4 are that:

1) Electric charge conservation cannot be violated by any means,
2) Energy-momentum conservation cannot be violated by any means.

However, angular-momentum conservation can be violated for a closed system by the variation of some parameter of the system. The angular-momentum conservation law is violated at a fixed value of the parameter. The conservation law holds at other values. We shall discuss as an example a recent work of Salam and Strathdee [10] where it is shown that the variation of high intensity magnetic field "restores" the symmetry which was spontaneously broken. Although the terminology is slightly different, we are saying the same thing. In our theory, if we keep varying some parameter, for example, continuously increasing entropy of the system, we shall end up with a system which is completely homogeneous (corresponding to a very simple kind of disordered system at some future time) and has only translation symmetry. In such a system, perhaps our universe, the conservation of energy-momentum, charge, etc. still remains valid.

Another example of this kind of abrupt change or quasi-catastrophe is the change of a relativistic system to action-at-a-distance system as the velocity of light goes to infinity [Wigner-Inon'U]. The Lorentz group (or Poincaré) goes to Galilean group and hence several conservation laws are broken into simpler conservation laws when one passes from relativistic to non-relativistic physics.

Example 2

Another example of great interest to particle physics is "spontaneous" breaking of symmetry. The terminology is a little awkward because there is no breaking of symmetry in our sense, except that the ground state of the potential function degenerates. Such a phenomenon can be described by Thom's restricted approach to the gradient dynamical system or conservative system. Consider the Lagrangian density

$$L = \frac{1}{2}(\mathbf{\phi}^2 - \mathbf{\phi}_\mathbf{n}^2) - \frac{1}{2} \mu^2 \mathbf{\phi}^2 - \frac{1}{2} \lambda \mathbf{\phi}^4,$$

where the potential $V(\mathbf{\phi}) = \frac{1}{4} \lambda \mathbf{\phi}^4 + \frac{1}{2} \mu^2 \mathbf{\phi}^2 (\lambda > 0)$. In the usual terminology, responsible for degeneracy of the vacuum state. Thom has given an expansion, called universal unfolding around the point where one started, of a potential $V(x, \mathbf{c}_1)$ function of position and control parameters (which we call deformation parameters) $\mathbf{c}_1$. Araki et al. [8] have explained that the potential responsible for catastrophe (which leads to degeneracy of the vacuum) is

$$V(\mathbf{\phi}) = \frac{1}{4} \mathbf{\phi}^2 + \frac{\mu^2}{2} \mathbf{\phi}^2 + \frac{\lambda}{X} \mathbf{\phi}^4,$$

and it corresponds to cusp-catastrophe in Thom's classification. For detailed proof we refer the reader to their paper.

Example 3 [Salam and Strathdee] [10,17]

In the Landau-Ginzburg theory of superconductivity it is known that at a critical value of the magnetic field the symmetry (complete disorder) is restored. Restoration of symmetry means as follows. Consider a ferromagnet with all spin-up or all spin-down electrons. This state is not considered as a symmetrical state since there is a definite preferred direction. We should rather say that the "ordered" state is of higher symmetrical type than the disordered state. The disordered state has the simplest kind of symmetry. In fact "restoration" of symmetry in Salam-Strathdee terminology corresponds to our "quasi-catastrophe" towards the "simpler" kind of symmetry. For example, in our model if a system has $O(n)$ symmetry and supposing we start heating it so that $T = T_1, T_2, T_3, \ldots$, etc.

$$0(n) + I_1 0(n-1) + I_2 0(n-2) + \cdots + I_n,$$

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where $I_n$ is the translational part, eventually at some temperature the system has only $I_n$, the $n$-dimensional translational symmetry. This is also the consequence of the theorem [propositions 1 and 2 in Ref. 1] that by varying some parameter (or parameters) we shall have as end product an extremely homogeneous system. In their paper, Salam and Strathdee consider the problem of CP violation and symmetry restoration of $O(3)$ symmetric Lagrangian,

$$\mathcal{L} = \frac{1}{4} F_{\mu \nu}^2 + \frac{i}{2} \left( \bar{\psi}_i \gamma^\mu \gamma^5 \psi_i \right) + \frac{g^2}{2} \psi^2 - \frac{1}{4} \left( \psi^2 \right)^2,$$

where scalar and vector fields $\psi_i$ and $A_\mu$ are triplets with respect to $O(3)$. The critical value of the field for which the system is a simpler kind of symmetry is

$$H_0 = \frac{1}{a} \left( \frac{\partial \phi}{\partial \mu} - a \right) \frac{\partial^2 \phi \psi}{\psi} ; \quad b < 0 \quad \text{and} \quad \lambda^2 n > 2n/a .$$

**Example**

We shall now briefly give an example of topological conservation laws, the details of which can be found in Refs. 1, 2, 3 and 7.

Consider a gauge theory based on $SO(3)$, with a triplet of Yang-Mills fields $A^a_\mu$ ($\mu = 0,1,2,3$ and $a = 1,2,3$) and a triplet of Higgs fields $\phi^a$. The 't Hooft electromagnetic tensor $F_{\mu \nu} = \frac{\partial A^a_\nu}{\partial \mu} - \frac{\partial A^a_\mu}{\partial \nu} + \frac{1}{2} \epsilon_{abc} A^b_\mu A^c_\nu$,

where $\phi^a = (\phi^a_0, \phi^a_1, \phi^a_2, \phi^a_3)$, $D_\mu \phi^a = \partial_\mu \phi^a + \epsilon_{abc} \epsilon^{ab} \omega^b_\mu \phi^c$,

$\epsilon_{abc} \epsilon^{ab} \omega^b_\mu \phi^c$ is given. The tensor $F_{\mu \nu} = M_{\mu \nu} + H_{\mu \nu}$,

where $M_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ with $B_\mu = \frac{\partial A^a_\mu}{\partial \phi^a}$ and $H_{\mu \nu} = (1/\epsilon) \epsilon_{abc} A^a_\mu \partial_\nu \phi^b$.

The magnetic current can be defined as $k_\mu = \frac{1}{2} \epsilon_{a \nu \rho \sigma} \partial^\nu \phi^a \rho^\sigma$, which is conserved, i.e. $\mu k_\mu = 0$.

One may note that the magnetic current is specified completely in terms of Higgs fields. The magnetic charge $M = \frac{1}{16 \pi^2} \int k_\mu \frac{\partial^2}{\partial x^\mu \partial x^\mu}$ obeys $\lambda^2 M = 0$.

It is shown by Freund et al. [1] that the equation $\lambda^2 M = 0$ is true no matter what determines the dynamics of the fields $\phi^a$. This equation does not generate any symmetry. It is shown that the magnetic charge is related to the degree of a mapping of sphere on field space to sphere in phase space. The degree of topological function is an invariant and its invariance does not follow from the usual symmetry argument. This type of conservation law comes under Type II of our classification. In fact the magnetic charge is equal to the indexed index of the normalized vector fields $\phi(x)$ defined over a large sphere $S^2 \rightarrow T^2$ in configuration space. The fields $\phi$ map $S^2_R \rightarrow S^2_{\phi}$ and the topological invariants are conserved quantities.

The stability properties of such conservation laws we shall discuss in a forthcoming paper. These "topological" conservation laws are stable if the domain of the functions is compact. In general, in the forthcoming paper, we shall study the properties and existence of the bifurcation set, which is a boundary of the usually simply-connected domain of our field function.

The present theory is applied to prove the existence of tachyons in Refs. 20 and 21. Some conjectures regarding black hole physics are also given.

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