On strong-coupling correlation functions of circular Wilson loops and local operators

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Abstract
Motivated by the problem of understanding 3-point correlation functions of gauge-invariant operators in $\mathcal{N} = 4$ super Yang–Mills theory we consider correlators involving Wilson loops and a ‘light’ operator with fixed quantum numbers. At leading order in the strong-coupling expansion such correlators are given by the ‘light’ vertex operator evaluated on a semiclassical string world surface ending on the corresponding loops at the boundary of $\text{AdS}_5 \times S^5$. We study in detail the example of a correlator of two concentric circular Wilson loops and a dilaton vertex operator. The resulting expression is given by an integral of combinations of elliptic functions and can be computed analytically in some special limits. We also consider a generalization of the minimal surface ending on two circles to the case of non-zero angular momentum $J$ in $S^5$ and discuss a special limit when one of the Wilson loops is effectively replaced by a ‘heavy’ operator with charge $J$.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The strong-coupling limit of a 3-point correlation function of ‘long’ primary operators in dual planar $\mathcal{N} = 4$ super YM theory should correspond, in the ‘semiclassical’ approximation, to a correlator of three closed string vertex operators in $\text{AdS}_5 \times S^5$ string theory each carrying large quantum numbers of order of string tension $T = \sqrt{\lambda}/2\pi$. This brings to light a challenging problem of how to construct minimal surfaces in $\text{AdS}_5 \times S^5$ that effectively ‘end’ on three distinct points at the boundary of the Poincaré patch of $\text{AdS}_5$. While this general problem [1]

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still awaits its solution, progress in understanding 3-point correlators at strong coupling was achieved [2–6] in a special limit when only two of the three operators carry large charges (i.e. are ‘heavy’, \( V_H \)) while the third one has a fixed charge (i.e. is ‘light’, \( V_L \)), i.e. for the correlators of the type \( \langle V_H V_H V_L \rangle \). In this case the corresponding semiclassical minimal surface has cylindrical topology (i.e. is a sphere with two punctures) and is the same one that saturates the 2-point correlator \( \langle V_H V_H \rangle \) of the two ‘heavy’ operators [7, 8, 1, 5, 9]. A similar idea can be applied also to the case of higher-point functions with only two ‘heavy’ operators \( \langle V_H V_H V_L \ldots V_L \rangle \) [4, 5].

Before trying to generalize to the interesting case of \( \langle V_H V_H V_H \rangle \) one may step back and ask if a correlator with just one ‘heavy’ operator, e.g., \( \langle V_H V_L V_L \rangle \), may be also be computed by using a similar semiclassical approximation at large \( \sqrt{\lambda} \), at least for some types of ‘heavy’ operators. At first sight, the answer to this question is not obvious, as ‘heavy’ string vertex operators are naturally associated with classical string solutions which in turn describe a semiclassical approximation to a 2-point function \( \langle V_H V_H \rangle \). However, instead of considering a cylindrical world surface one may start with a disc-like (Euclidean) surface and think of specific boundary conditions that it should end on a contour \( C \) at the boundary of AdS as representing a particular closed string state. Extracting an on-shell (primary-operator) state is, of course, non-trivial: specifying a contour \( C \) at the boundary will, in general, correspond to an off-shell string state describing an infinite superposition of local operators.

To begin with, one may just consider a simpler problem by replacing a ‘heavy’ operator by a boundary state represented by some Wilson loop \( W[C] \), i.e. by replacing the question about \( \langle V_H V_L V_L \rangle \) by the one about \( \langle W[C] V_L V_L \rangle \). The semiclassical surface for the latter correlator will then be the same as the one that determines the strong-coupling expectation value of \( W[C] \) (as, e.g., in some basic examples discussed in [10, 11]).

More generally, one may be interested in minimal surfaces that end on several different closed contours at the boundary, representing correlators of Wilson loops (WLs) and local operators in the dual gauge theory. Expanding in size of a contour, a WL may be viewed, in the OPE sense [11, 12], as an infinite sum of local operators (and vice versa). Thus, this problem is, at least in principle, related to that of correlators involving several ‘heavy’ vertex operators.

With the eventual goal to understand how to compute generic 3-point correlators or to construct minimal surfaces ending on three separate contours, here we shall consider (as in the vertex operator case [2, 4]) an ‘intermediate’ case of a correlator of one or two ‘large’ WLs and one ‘light’ vertex operator. In this case the semiclassical surface will be determined just by the WLs. Our aim will be to study such correlators with the WLs being the simplest circular ones.

Let us first discuss the case of a correlator \( \langle W[C[V(x)] \rangle \) of one WL and a local operator. It can be represented by a string path integral over disc-like surfaces (with boundary conditions specified by the contour \( C \)) of a string vertex operator \( V(x) \), labeled by a point \( x^\mu \) of the boundary of the Poincaré patch of AdS\(_5\).\(^5\) If the operator is ‘light’, \( V = V_L \), i.e. it does not carry a large charge of order of string tension (so that its dimension does not depend on \( \sqrt{\lambda} \) in the BPS case or grows at most as \( (\sqrt{\lambda})^{1/2} \) in the non-BPS case) it can be ignored in finding the stationary surface in the path integral. Then the leading contribution to the normalized correlator \( \langle W[C[V(x)] \rangle \) will be given just by the value of \( V_L(x) \) on the same minimal surface that determines \( \langle W[C] \rangle \). An example was considered in [11, 13] where the WL was a

\(^5\) In our notation the AdS\(_5\) metric is \( ds^2 = \frac{1}{s}(dx^2 + du^2 + dv^2) \). Dependence of the correlator on \( C \) and \( x^\mu \) may be (partially) determined by the conformal SO(2, 4) symmetry considerations (see below); for example, in the case when \( C \) is a circle we may set \( x_\mu \) to be at 0 or at infinity.
circular one [14, 11, 15] and the vertex operator was a BPS one with an angular momentum \( J \) in \( S^5 \).

In the case when \( J \) is of order \( \sqrt{\lambda} \), i.e. the corresponding vertex operator is ‘heavy’, its contribution (proportional to \( \varepsilon^d X^d \) where \( X = X_1 + iX_2 = \cos \theta \, e^\phi \) represents an \( S^5 \) rotation plane) is to be taken into account in determining the stationary surface in the string path integral. The resulting Euclidean surface found in [13] (see also [16]) is a simple modification of the ‘semisphere’ to include a non-trivial background for the \( S^5 \) angle \( \phi = i J \tau \), \( \mathcal{J} = \frac{J}{\sqrt{\lambda}} \), which describes the back reaction to a ‘source’ provided by the ‘heavy’ vertex operator insertion. One is also to modify the \( J \) to take into account the contribution from the vertex operator (reflecting its scaling dimension)\(^8\).

One may also consider other ways of adding angular momentum \( J \) to a WL surface. In the case when \( J \) is carried by the vertex operator the corresponding R-charge is localized at the single point \( i \), while one may also spread it over the WL contour. One suggestion [17] to include rotation in \( S^5 \) is by choosing the unit 6-vector as \( \theta^m = (\cos \mathcal{J} \sigma, \sin \mathcal{J} \sigma, 0, 0, 0, 0) \) (corresponding to \( \phi = J \sigma \)) thus getting complex scalar coupling terms \( e^{-i J \sigma Z} + e^{i J \sigma \bar{Z}} \) in the exponent of the supersymmetric WL. In this case there will be a local R-charge density, but the total R-charge will be zero. Another possibility is to smear a non-zero R-charge along \( C \) by introducing \( Z^I \) factors along the contour, as, e.g., in \( W[J, C] = \text{Tr} \left[ \mathcal{P} \int d\sigma \mathcal{Z}^I (x(\sigma)) \exp \mathcal{A} \right] \) where \( \mathcal{A} = \int d\sigma [\partial_\mu (x(\sigma)) x^\mu (\sigma) + |x'(\sigma)| \Phi_m (x(\sigma)) \theta^m (\sigma)] \). Yet another (presumably equivalent to leading order) option is to introduce a linear combination of scalars including a \( Z \)-term directly into the exponent \( \mathcal{A} \).\(^9\) In this case the \( J \)-dependence of corresponding minimal surface will still be described by \( \phi = i J \tau \) but there will be no modification of the dilatation charge; the surface will still be \( z^2 + x^2 = R^2 \) but the expression for \( z(\tau) \) and the induced metric determining \( (W[C, J]) \) will be different than in the \( (W[C] V_L(0)) \) case. One may then compute \( (W[C, J] V_L(x)) \) by evaluating \( V_L \) on this surface and, again, the result will be different from that for \( (W[C] V_L(0)) V_L(x) \).

Similar considerations apply (as in [4, 5]) to the correlators involving (i) one WL (with or without \( J \)) and several ‘light’ vertex operators and (ii) one WL, one ‘heavy’ operator and several ‘light’ vertex operators. Their leading-order semiclassical expression will be given by the product of the ‘light’ vertex operators evaluated on the minimal surface determined by (i) the corresponding WL or (ii) by the WL and the ‘heavy’ operator.

A correlator of \( N \) WLs and a number of ‘light’ operators, \( W[C_1] \ldots W[C_N] V_L(x_1) \ldots V_L(x_N) \), may be evaluated in a similar way, provided the stationary surface ending on the two contours is known. The latter is known in the case of \( C_1 \) and \( C_2 \) being concentric circles in AdS\(_5 \) [12, 18] and this is the case which we will consider.

\(^6\) The resulting surface is given by a ‘semisphere’ \( z^2 + x^2 = R^2 \) ending at \( z = 0 \) on a circle with the massless bulk-to-boundary propagator connecting its point to a point \( x \) at the boundary. The leading contribution at large \( x \) is then given by the corresponding OPE coefficient \( c \), i.e. \( \left( \frac{\text{dim}(W[C])}{\text{dim}(W[C])} \right)_{|\alpha| \to \infty} = \frac{c}{\alpha^2} \). \( c \) is proportional to the string tension that enters the normalization [11] of the vertex operator, i.e. \( c = \frac{1}{\sqrt{\lambda}} J \sqrt{\mathcal{J}} \). In [13] the first subleading contribution of string fluctuations near that surface was shown to give an extra factor \( \exp(-\frac{\mathcal{J}}{\sqrt{\lambda}}) \). The derivative terms in the chiral primary vertex operator and the fermionic terms in the string action were ignored but their contribution should conspire to cancel due to the marginality of the vertex operator and supersymmetry.

\(^7\) Here, \( \tau \) is the Euclidean world sheet time coordinate, with the boundary circle \( (\chi^2 = R^2) \) parametrized by the angular coordinate \( \phi = \sigma \in (0, 2\pi) \). In this paper we use (somewhat unconventionally) \( \sigma \) to parametrize the boundary contours and \( \tau \) to describe the evolution of the surface into the bulk. This is natural if one thinks of a loop \( C \) as representing a closed-string boundary state rather than a trajectory followed by the ends of an open string attached to the boundary. We shall mostly consider Euclidean surfaces in Euclidean AdS\(_5 \times S^5 \).

\(^8\) The semiclassical contribution to the correlator is \( c \sim e^{-3 S_{\text{H}}} \cdot S_{\text{H}} = -\sqrt{3}(\sqrt{J^2 + 1} + J \ln(\sqrt{J^2 + 1} - J)) \). Here \( S_{\text{H}}(J \to 0) = \frac{1}{2} \sqrt{3} J^2 + \cdots \) reproduces the ‘subleading’ contribution in the case of small \( J \), i.e. when \( V \) is a ‘light’ operator [13].

\(^9\) We are grateful to N Drukker for an important clarifying discussion of the issue of adding R-charge to a WL.
in detail below. As was pointed out in [19], the two circles can be put, by a conformal transformation, into one plane. Furthermore, such minimal surface can be generalized to the presence of an angular momentum $J$ in $S^5$. The latter case may be interpreted as corresponding to the correlator $\langle W[C_1, J] W[C_2, J] \rangle$. One may also consider the case when $J$ is introduced instead by an insertion of a local ‘heavy’ operator, $(W[C_1] W[C_2]V(x))$; the corresponding semiclassical surface should be easy to construct (following the same logic as in [13]) in the special angular symmetric case of $V_j$ inserted at the center $x = 0$ of the circles (or at $x = \infty$, which is related by inversion).

A feature of 3-point correlation functions $\langle V(x_1) V(x_2) V(x_3) \rangle$ of primary local operators in conformal field theories is that their dependence on the positions of the operators is fixed by conformal symmetry. When considering correlation functions involving WLs, such as $\langle W[C_1] W[C_2] V(x) \rangle$, the presence of the loops makes conformal invariance much less restrictive, so the dependence on the position $x^\mu$ of the vertex operator will be much richer. It is one of the aims of this paper to compute explicitly this dependence in several special cases.

This paper is organized as follows. In section 2 we consider the simplest case of a correlator of a single circular WL with a ‘light’ dilaton operator. In section 3 we study the correlation function of a dilaton with two concentric circular WLs.

In section 4 we generalize the discussion of section 3 to the case of the WLs carrying non-zero $S^5$ angular momentum $J$. This configuration admits two interesting limits. In the first limit (section 4.2) one of the two WLs shrinks to zero size, and the corresponding surface then corresponds to a correlator of a WL with a ‘heavy’ operator of charge $J$ inserted at the position of the shrunk loop. In the second limit the two circles coincide, i.e. we obtain a single circular WL carrying angular momentum $J$.

Section 5 contains some concluding remarks. Appendix A is devoted to the study of the action of conformal symmetries on the correlators considered in this paper. In appendix B we compute the correlator of section 4 in the second special limit mentioned above.

2. Semiclassical correlation function of circular WL and dilaton operator

As discussed in the introduction, WLs play a similar role to that of ‘heavy’ operators in a correlator with ‘light’ operators—at large $\sqrt{\lambda}$ they determine the semiclassical trajectory on which the ‘light’ operators should be evaluated to obtain the leading strong-coupling contribution.

In this paper we will consider configurations embedded into the AdS$_3 \times S^5$ subspace of (Euclidean) AdS$_3 \times S^5$ with metric

$$dz^2 = z^{-2}(dz_1^2 + dx_{i_1} dx^{i_1} ) + dS_5 = z^{-2}(dz_2^2 + dr^2 + r^2 d\phi^2 + \cdots) + dp^2 + \cdots, \quad (2.1)$$

where $x_1 = r \cos \phi$, $x_2 = r \sin \phi$. As ‘light’ operator we will consider the dilaton vertex operator

$$V_L(x') = \int d^2 \zeta \ U(x(\zeta) - x', z(\zeta), \varphi(\zeta)), \quad (2.2)$$

$$U(x, z, \varphi) = k_\Delta \left( \frac{z}{z^2 + x^2} \right)^\Delta \mathcal{L}, \quad \mathcal{L} = z^{-2}(\partial z \partial \bar{z} + \partial x_{i_1} \partial x^{i_1}) + \partial \varphi \partial \bar{\varphi} + \cdots.$$ 

Here $j$ is an $S^5$ momentum and $\Delta = 4 + j$ is the marginality condition. For simplicity we shall suppress the normalization factor $[11, 2, 4] k_\Delta = \frac{2^{2J}}{2J+1} \sqrt{(j+1)(j+2)(j+3)}$ in most of the expressions for the correlators in this paper. Dots stand for terms with other $S^5$ coordinates and fermions that will not be relevant at the leading semiclassical approximation considered below. The operator is labeled by a point $x'$ at the boundary ($z = 0$) of the Poincaré patch.
In what follows we shall parametrize the location of the vertex operator by \((x'_i, x'_j) = \rho(\cos \theta, \sin \theta)\) and by \(h\)—the transverse distance in the \((x'_i, x'_j)\) plane, so that
\[
\rho^2 + (x - x')^2 = \rho^2 + r^2 - 2r \rho \cos(\phi - \theta) + \rho^2 + h^2.
\] (2.3)

Note that in the case of the dilaton operator integrated over the position of the insertion point \(x'\), i.e. \(\int d^4x' V(x')\), one is to do the replacement
\[
\left(\frac{z}{\rho^2 + (x - x')^2}\right)^\Delta \rightarrow \int d^4x' \left(\frac{z}{\rho^2 + (x - x')^2}\right)^\Delta = \frac{\Delta - \Delta}{2(\Delta - 1)(\Delta - 2)}.
\] (2.4)

This factor is trivial only if \(j = 0\), i.e. when \(\Delta = 4\).

In the case of a correlator involving one circular WL the corresponding classical solution can be represented in the form \((w, \psi)\) are the radial and angular world-sheet disc coordinates in conformal gauge) [11, 15]
\[
\rho^2 + r^2 = R^2, \quad \varphi = 0,
\] (2.5)
\[
z = R \frac{1 - w^2}{1 + w^2}, \quad r = \frac{2Rw}{1 + w^2}, \quad \phi = \psi, \quad \zeta = w e^{i\phi}, \quad 0 \leq w \leq 1.
\] (2.6)

This surface ends on a circle of radius \(R\) at the boundary \(z = 0\). Let us note that under the inversion symmetry of the AdS metric (2.1) (which becomes the standard inversion at the boundary \(z = 0\))
\[
z \rightarrow \frac{z}{z^2 + \lambda^2}, \quad x_\mu \rightarrow x_\mu \frac{z}{z^2 + \lambda^2}, \quad z^\prime + \lambda^2 \rightarrow \frac{1}{z^\prime + \lambda^2},
\] (2.7)

this surface goes into a similar one with \(R \rightarrow 1/R\).

The leading semiclassical expression for the correlator of one circular WL with one ‘light’ dilaton operator is then be given by
\[
\mathcal{C} = \frac{\langle \mathcal{W}[C] V_L(x') \rangle}{\langle \mathcal{W}[C] \rangle} = \left[ V_L(x') \right]_{\text{semicl.}} = \frac{8\pi k_\Delta}{\Delta - 1 \left[ (h^2 + \rho^2 - R^2)^2 + 4R^2 h^2 \right]^{\Delta/2}},
\] (2.8)

where we plugged in the solution (2.6) into (2.2),(2.3) and performed the 2d integral (the dependence on \(\theta\) dropped out due to angular symmetry of the solution). This agrees with the expression found in [11]. A salient feature of this expression is that it diverges as \(d^{-\Delta}\) when the distance \(d = \sqrt{(\rho - R)^2 + h^2}\) between the vertex operator insertion point \(x'\) and the WL becomes small. In appendix A we show how one could find this result by using the conformal group symmetry considerations.

In the following sections we will consider several other examples of solutions involving circular WLs. The resulting expression for the correlators \(\mathcal{C}\) will be much more involved being given in terms of an integral of elliptic functions, but we will be able to analyze several limits of interest.

3. Correlation function of two concentric circular WLs and a dilaton operator

In this section we will consider the 3-point correlation function of two concentric circular WLs and a light dilaton vertex operator, i.e. \(\langle \mathcal{W}[C_i] \mathcal{W}[C_f] V_L(x) \rangle\), where \(C_i\) and \(C_f\) stand for an ‘initial’ and ‘final’ circles on which the semiclassical surface will be ending.
3.1. Classical solution for surface ending on two concentric circles

Let us start with the semiclassical solution corresponding to two concentric circular WLs [12, 18]. Using conformal transformations one may arrange the two concentric circles to lie in the same plane at the boundary [19] (see also appendix A)\(^{10}\). In this section we restrict ourselves to the case without rotation in \(S^5 (\varphi = 0)\), i.e. the surface will lie only inside of AdS\(_3\)

\[
\begin{align*}
    z &= z(\tau), & r &= r(\tau), & \phi &= \sigma,
\end{align*}
\]

(3.1)

where \(\tau\) (ranging from \(\tau_i = 0\) to \(\tau_f\)) and \(\sigma\) (having period 2\(\pi\)) are coordinates of a 2-d cylinder. The boundary conditions are

\[
\begin{align*}
    z(0) &= z(\tau_f) = 0, & r(0) &= R_i, & r(\tau_f) &= R_f, \\
\end{align*}
\]

(3.2)

where \(R_i, R_f\) are the radii of the two circular WLs. Using the conformal gauge we have the usual constraint or vanishing of the '2D energy' (dot will denote derivatives with respect to \(\tau\))

\[
\begin{align*}
    z^{-2}(\dot{z}^2 + \dot{r}^2 - r^2) = 0,
\end{align*}
\]

(3.3)

and also the following integral of motion (dilatation charge):

\[
\begin{align*}
    z^{-2}(\ddot{z} + \dot{r}) = p,
\end{align*}
\]

(3.4)

where \(p\) is a constant that will parametrize the solution. From the spacetime point of view, the solution is parametrized by the ratio \(R_i/R_f\), so there is a relation (given below) between this ratio and \(p\). Note also that under the inversion symmetry (2.7) the equation (3.4) goes into itself with \(p \rightarrow -p\), i.e. the solutions with \(p > 0\) and \(p < 0\) are related by the inversion. The special case of \(p = 0\) corresponds to the case when the two radii are the same, \(R_i = R_f\), i.e. the two circles coincide and we get back to the 'semisphere' surface.

It is convenient to perform the following change of variables:

\[
\begin{align*}
    z &= \frac{u e^v}{\sqrt{1 + u^2}}, & r &= \frac{e^v}{\sqrt{1 + u^2}}.
\end{align*}
\]

(3.5)

In terms of \(u(\tau), v(\tau)\) the integrals of motion (3.3) and (3.4) become

\[
\begin{align*}
    \dot{u}^2 &= 1 + u^2 - p^2 u^4, & \dot{v} &= \frac{pu^2}{u^2 + 1}.
\end{align*}
\]

(3.6)

The boundary conditions imply that \(u \to 0\) and \(e^v \to R_i, R_f\) at the two boundary circles. It is useful to introduce the following combinations (the roots of the polynomial \(1 + u^2 - p^2 u^4\)):

\[
\begin{align*}
    u_\pm &= \frac{1 \pm \sqrt{1 + 4p^2}}{2p^2}.
\end{align*}
\]

(3.7)

For real solutions (having \(p^2 \geq 0\)) only \(u_+^2\) is positive. The first equation in (3.6) can be solved [19] in term of the elliptic integral \(F(y|m)\) of the first kind expressing \(\tau\) as a function of \(u\)

\[
\tau = u_+ F\left(\arcsin \frac{u}{u_+}, \frac{u_+^2}{u^2}\right).
\]

(3.8)

\(u\) has a maximum (a turning point) at \(u_+\). Beyond it, we have to continue on the other branch of the arcsine function, until reaching the boundary again. The full range of \(\tau\) (or \(\tau_f\), since we have set \(\tau_i = 0\)) is given by

\[
\tau_f = 2u_+ K\left(\frac{u_+^2}{u^2}\right),
\]

(3.9)

where \(K(x)\) is the complete elliptic integral of the first kind.

\(^{10}\) The solution we discuss below is equivalent to the one in [18, 19]; note that our notations are different from those in [19].
Figure 1. The relation between \( p \) and the ratio of radii \( R_f/R_i = e^v \). Note that this ratio is bigger than 1 for \( p > 0 \) and smaller than 1 for \( p < 0 \). Furthermore, in general, there are two different values of \( p \) corresponding to the same value of \( v \) or \( R_f/R_i \).

One can then find \( v(\tau) \) in terms of elliptic integrals. Its expression, as that of \( \tau(u) \), will depend on whether we are on the first or the second branch. For the first branch (\( u \) from 0 to \( u_+ \))

\[
\tau = u_+ F \left( \arcsin \frac{u}{u_+} \left| \frac{u_+^2}{u^2} \right| \right), \quad v = v_i + \hat{v}(u),
\]

(3.10)

while for the second branch (\( u \) from \( u_+ \) to 0) we obtain

\[
\tau = \tau_f - u_+ F \left( \arcsin \frac{u}{u_+} \left| \frac{u_+^2}{u^2} \right| \right), \quad v = v_f - \hat{v}(u),
\]

(3.11)

where

\[
\hat{v}(u) = pu_+ \left[ F \left( \arcsin \frac{u}{u_+} \left| \frac{u_+^2}{u^2} \right| \right) - \Pi \left( -u_+, \arcsin \frac{u}{u_+} \left| \frac{u_+^2}{u^2} \right| \right) \right],
\]

(3.12)

\[
v_f - v_i = \log \frac{R_f}{R_i} = 2pu_+ \left[ K \left( \frac{u_+^2}{u^2} \right) - \Pi \left( -u_+, \frac{u_+^2}{u^2} \right) \right].
\]

(3.13)

The latter equation thus gives the expression for the modulus of the solution \( R_f/R_i \) in terms of the integral of motion \( p \).

Let us now analyze the spacetime structure of this solution. For definiteness, we can take \( v_i = 0 \), i.e. \( R_i = e^{v_i} = 1 \). We can then plot \( v_f \) in (3.13) as a function of \( p \), see figure 1. From this plot we can infer that \( p > 0 \) corresponds to \( R_f = e^{v_f} > 1 \), while \( p < 0 \) corresponds to \( R_f < 1 \). As was mentioned below (3.4), the two configurations are related by an inversion. We may focus, e.g., on the case of \( p > 0 \). The radius \( R_f > 1 \) cannot be chosen arbitrarily large since there is a value \( p = p_0 \approx 0.5811 \) for which \( R_f = e^{v_f} \) reaches a maximum, \( R_0 \). This indicates a phase transition [14, 15, 12, 18], in which two separate minimal surfaces, one per each WL, dominate over this solution. Note also that given some value of \( R_f \), \( 1 < R_f < R_0 \) there are two values of \( p \) corresponding to it (see figure 1).

In order to understand how these solutions look in spacetime we can go back to the original coordinates \((z, r)\) in (3.5) and plot \( z(r) \) parametrically, varying \( u \) (see figure 2). For instance, both \( p = 0.0244 \) and \( p = 35.52 \) correspond to \( v_f = 0.2 \). The dashed curve corresponds to \( p = 0.0244 < p_0 \). It starts at \( z = 0, r = R_i = 1 \), then goes up to some \( z = z_{\text{max}} \) almost reaching...
Figure 2. Two different solutions with the same value of $R_f/R_i$. The solution corresponding to $p < p_0$ represented by the dashed line tends to the single WL solution as $R_f/R_i \to 1$.

Figure 3. The two different solutions approach each other as the corresponding two values of $p$ get closer to $p_0$.

$r = 0$ and then back to the outer WL, at $z = 0, r = R_f \approx 1.2$. Note that in the limit when the inner WL approaches the outer (which corresponds to very small $p$), this world-sheet surface approaches the surface for a single WL (2.6). The solid curve corresponds to $p = 35.52 > p_0$. Note that it disappears in the limit $R_f \to 1$ as then the range of variation of $z$ shrinks to 0.

We can also study what happens when $p$ is closer to the special value $p_0$, see figure 3. Here again the dashed line corresponds to $p < p_0$ while the solid line corresponds to $p > p_0$. We see that both curves reach a minimal value of $r$, where $\dot{r}(u) = 0$. This value happens at $u = 1/p$ and is given in terms of elliptic functions. An important feature is that this minimal value increases monotonically with $p$.

To conclude, in general, we obtain two distinct solutions for a given value of $R_f/R_i$. One can evaluate the Euclidean string action on both of them; the action turns out to be negative after the subtraction of a divergence [18]. The solution with $p > p_0$ representing non-intersecting
surface (solid curve) has bigger action and thus should dominate over the solution with \( p < p_0 \) (dashed curve). A related remark appeared in [18].

In doing the semiclassical computation for a correlator with a ‘light’ vertex operator we are, in general, supposed to sum over all competing stationary points, i.e.

\[
C = \frac{\langle W[C_f] \rangle}{\langle W[C_f] \rangle} = \frac{b_1 e^{\sqrt{a_1}} + b_2 e^{\sqrt{a_2}} + \ldots}{c_1 e^{\sqrt{a_1}} + c_2 e^{\sqrt{a_2}} + \ldots} = \frac{b_1}{c_1} + \left( \frac{b_2}{c_1} - \frac{b_1 c_2}{c_1^2} \right) e^{-\sqrt{a_1-a_2}} + \ldots. \tag{3.14}
\]

Here we assumed that \( a_1 > a_2 \) and \( b_2 \sim \sqrt{\lambda} \) (taking into account the normalization of the vertex operator in (2.2)). While only one (dominant) solution will contribute to the leading term in (3.14), below we shall formally consider the case of a general \( p > 0 \), i.e. we will treat the cases of the two solutions on an equal footing.

3.2. Semiclassical correlation function

To compute the leading semiclassical correlation function (3.14) we need to evaluate the dilaton vertex operator (2.2) on the classical solution described in the previous subsection. It is convenient to express the integral over the cylindrical world-sheet coordinates \((\tau, \sigma)\) using \( u(\tau) \) instead of \( \tau \). This will introduce the following factor in the measure (see (3.6)):

\[
\dot{\tau}(u) = \frac{d\tau}{du} = \left[ \left( 1 - \frac{u^2}{u_+^2} \right) \left( 1 - \frac{u^2}{u_-^2} \right) \right]^{1/2}. \tag{3.15}
\]

Here we have chosen the sign plus for the first branch; the world-sheet integral is an integral over the two branches but the different sign in the measure will be compensated by the interchange of the limits of integration. We end up with the following expression for the correlator in (3.14)\(^{11}\):

\[
\mathcal{C} = \frac{V(x)}{\text{semicl.}} = I(R_i) + I(R_f),
\]

\[
I(R_{i,f}) = \int_0^{2\pi} d\sigma \int_0^{u_+} du \frac{2\dot{\tau}(u)}{u^2} \left( \frac{ue^{\nu_i,\xi/\lambda(u)}}{\sqrt{1 + u^2[e^{2\nu_i,\xi/\lambda(u)} + h^2 + \rho^2] - 2e^{\nu_i,\xi/\lambda(u)}\rho \cos \sigma}} \right)^\Delta, \tag{3.16}
\]

where we used (2.3) and eliminated the dependence on \( \theta \) by a shift of \( \sigma \) (the dependence on \( \theta \) drops out due to angular symmetry).

The integrand depends on four external parameters: \( v_{i,f} = \log R_{i,f} \) (i.e. the radii of the WLs) and \( \rho, h \) (the location of the vertex operator). It is useful to study how AdS\(_5\) symmetries act on this expression. For instance, if one acts with dilatations,

\[
h \rightarrow \ell h, \quad \rho \rightarrow \ell \rho, \quad v_{i,f} \rightarrow \log \ell + v_{i,f}, \tag{3.17}
\]

then the integrand picks up a factor \( \ell^{-\Delta} \). This allows us to set, e.g., \( v_i = 0 \) or \( R_i = 1 \). Then \( \mathcal{C} \) will depend on three dimensionless parameters, i.e. on \( p, \rho, h \) (and, of course, on \( \Delta \)). If one acts with the inversion (cf (2.7))

\[
h \rightarrow \frac{h}{h^2 + \rho^2}, \quad \rho \rightarrow \frac{\rho}{h^2 + \rho^2}, \quad v_{i,f} \rightarrow -v_{i,f}, \quad p \rightarrow -p, \tag{3.18}
\]

then one can explicitly check (using that \( \hat{v}(u) \) changes sign when \( p \) changes sign) that the integrand picks up a factor \((\rho^2 + h^2)^{-\Delta}\).

\(^{11}\) Here and below we suppress the normalization factor \( k_\Delta \) of the dilaton operator in the expressions for the correlators.
The integral over $\sigma$ can be done using that
\[
\frac{1}{\pi} \int_0^{2\pi} d\sigma (\cos \sigma + s)^{-\Delta} = (s - 1)^{-\Delta} \text{}_2F_1 \left( \frac{1}{2}, \Delta, 1, \frac{2}{1 - s} \right) + (1 + s)^{-\Delta} \text{}_2F_1 \left( \frac{1}{2}, \Delta, 1, \frac{2}{1 + s} \right)
\]
where we have assumed $s > 1$. The integral diverges for $s \to 1$, which will happen when the insertion point of the operator approaches one of the WLs.

The remaining integral over $u$ is hard to compute explicitly due to the dependence of the integrand on $v(u)$ given in (3.12). Still, it can be easily evaluated numerically for any choice of the external parameters. Furthermore, it can be, in principle, computed analytically in special limits that we discuss below.

### 3.2.1. Operator close to a WL

An interesting limit corresponds to the case when position of the vertex operator is close to one of the boundary circles. In this case we should expect the correlator to look like the one for the single WL discussed in section 2. Since the distance to one of the concentric loops (say $C_f$) is $d^2 = (\rho - R_f)^2 + h^2$, we expect to find that for $d \to 0$ the correlator diverges as
\[
\mathcal{C} \approx \frac{2^{-\Delta}}{\Delta - 1} \frac{1}{d^\Delta}.
\]
A numerical evaluation of (3.16) confirms this expectation. This result can also be understood directly from (3.16) by noticing that the divergence comes from the region close to the boundary $u \approx 0$. There, and for $d \approx 0$, $s$ in (3.19) approaches one.

### 3.2.2. Small $p$ limit

Small $p$ limit is possible only for the first ‘self-intersecting’ solution (which, as was mentioned above, is, in general, subdominant). In this limit the classical solution approaches the one for a single circular WL, i.e. $R_f \to R_i$. For small $p$ we have
\[
\hat{v}(u) = v^{(1)}(u) p + \mathcal{O}(p^3), \quad v^{(1)}(u) = \text{arcsinh } u - \frac{u}{\sqrt{1 + u^2}}.
\]
Let us adopt here the symmetric choice $v_f = -v_i = \frac{1}{2} \delta v$ (i.e. $R_f = 1/R_i$). For small $p$ (i.e. $R_f = 1/R_i \to 1$)
\[
\delta v = 2[\log(p/8) + 1]p + \mathcal{O}(p^3).
\]
The integral over $u$ is from $0$ to $u_+$, with $u_+ \to \infty$ for $p \to 0$. Performing the integrals in (3.16) up to $u_+$ and then expanding the result in powers of $p$ we obtain for the correlator
\[
\mathcal{C} = C^{(0)} + p^2 C^{(2)} + \mathcal{O}(p^4), \quad C^{(0)} = \frac{8\pi}{\Delta - 1} \frac{1}{[h^2 + \rho^2 - 1]^2 + 4h^2]}^{3/2}.
\]
As expected, $C^{(0)}$ matches the expression (2.8) for the correlator of a dilaton operator and a single circular WL of unit radius. For $C^{(2)}$ we find
\[
C^{(2)} = -4\pi b^{-2\Delta} + 2b^{-2\Delta} f_1(y) + 2b^{-2\Delta} - 4\Delta (b^2 - 2)^2 f_2(y) + 4b^{-2\Delta} f_3(y),
\]
\[
b^2 \equiv h^2 + \rho^2 + 1, \quad y \equiv \frac{\rho}{h^2 + \rho^2 + 1},
\]
\[
f_1(y) = \int du \frac{u^{2+\Delta}}{(1 + u^2)^{3/2}(\sqrt{1 + y^2} - 2y \cos \sigma)^{\Delta}},
\]
\[
f_2(y) = \int du \frac{u^{2+\Delta}}{(1 + u^2)^{3/2}(\sqrt{1 + y^2} - 2y \cos \sigma)^{\Delta}}.
\]

Figure 4. Analytic expression for the correlator $C(\rho)$ in the large $p$ limit (solid line) versus numerical result (dots). We have chosen a large value of $p$ and $h = 0.2$ in order to smooth out the singularity at $\rho = 1$.

\[ f_2(y) = \int \frac{du \, d\sigma}{\sqrt{1 + u^2 - 2y \cos \sigma}} \frac{\Delta u^{\Delta - 2}}{(\sqrt{1 + u^2 - 2y \cos \sigma})^{\Delta + 2}} [v^{(1)}(u) + 1 + \log(p/4)]^2, \]

\[ f_3(y) = \int \frac{du \, d\sigma}{\sqrt{1 + u^2 - 2y \cos \sigma}} \frac{\Delta u^{\Delta - 2}}{(\sqrt{1 + u^2 - 2y \cos \sigma})^{\Delta + 2}} [v^{(1)}(u) + 1 + \log(p/4)]^2. \]

For $\Delta = 4$ the functions $f_k(y)$ entering $C^{(2)}$ simplify and can be computed, for instance, as an expansion in powers of $y$ (i.e. in the limit of small $\rho$ or large $h$).

### 3.2.3. Large $p$ limit

In a similar way, one may also compute $C$ for large values of $p$. This limit is possible only for the dominant non-self-intersecting solution. It corresponds to a ’small’ world-sheet surface between two close circles with radii approaching 1. It is useful to change the variable $u \to \tilde{u} = u/\Delta$, so that $\tilde{u}$ runs from zero to one. As in the previous case, we shall set $v_f = -v_i = \frac{1}{2} \delta v$. For large $p$ we find (here $E$ and $F$ are the elliptic integrals)

\[ \delta v = 2\pi^{1/2} \frac{\Gamma\left[\frac{3}{4}\right]}{\Gamma\left[\frac{1}{4}\right]} p^{-1/2} + \cdots, \]

\[ \hat{v}(u) = [E(\arcsin \tilde{u}, -1) - F(\arcsin \tilde{u}, -1)] p^{-1/2} + \cdots, \]

and at the leading order we obtain

\[ C^{(0)} = p^{1/2 - \Delta/2} \frac{\sqrt{\pi} \Gamma\left[\frac{1}{4}\right]}{\Gamma\left[\frac{1}{4}\right] (\Delta + 1)} \int_0^{2\pi} \frac{1}{(1 + h^2 + \rho^2 - 2\rho \cos \sigma)^{\Delta + 1}}. \]

which can easily be solved in terms of hypergeometric functions using (3.19). This result is in good agreement with the numerics, see figure 4.

We see that the result is smooth except for a singularity at $h^2 + (\rho - 1)^2 = 0$. This singularity is, of course, expected, since there we approach the location of the WLs. By looking at the singularity of (3.19) as $s \to 1$ we obtain that in this limit

\[ C \approx [h^2 + (\rho - 1)^2]^{-\Delta + 1/2}. \]

This is a milder singularity than in the case of a single WL (3.20). This is to be expected, since in this limit the world-sheet also disappears.
Finally, let us mention that, with some effort, the subleading terms can also be computed. Explicitly, for $\Delta = 4$ we obtain for the first correction to (3.30) ($C = C^{(0)} + C^{(1)} + \cdots$)

$$C^{(1)} = p^{-5/2} \frac{128\sqrt{2\pi}b^2}{3(b^2 - 4\rho^2)^{11/2}(\Gamma[\frac{1}{2}])} \left( \pi^2 [b^8(5 - 5b^2 + b^4) + b^4(100 - 100b^2 + 23b^4)\rho^2 + 6(25 - 25b^2 + 8b^4)\rho^4 + 6\rho^6] + 640\left( \Gamma\left[\frac{5}{4}\right]\right)^8 (b^{12} + 14b^8\rho^2 - 25b^4\rho^4 - 96\rho^6) \right),$$

where $b^2 = h^2 + \rho^2 + 1$ as in (3.24).

### 4. Case of circular WLs with angular momentum in $S^5$

For the classical solution ending on two concentric circles discussed in the previous section the ratio of radii $\frac{R_i}{R_f} \equiv \frac{b_{\text{out}}}{b_{\text{in}}} < 1$ cannot be made arbitrarily small—there is a critical value after which the dominant surface is a combination of disjoint ‘cups’ ending separately on each of the two circles. Here we shall consider, following [19], a generalization of the above solution to the case of non-constant angle of $S^5$ in $S^5$, which, after continuation to Minkowski signature, may be interpreted as an $S^5$ angular momentum carried by the WL. In this case there will be a nontrivial $\frac{R_i}{R_f} \to 0$ limit of the corresponding surface which will be equivalent to the surface [13] which ends on the ‘outer’ circle and is also ‘sourced’ by a ‘heavy’ vertex operator with a semiclassical angular momentum $J$.

#### 4.1. Classical solution and general form of the correlator with dilaton operator

We start with the following generalization [19] of the ansatz in (3.1):

$$z = z(\tau), \quad r = r(\tau), \quad \phi = \sigma, \quad \varphi = i\mathcal{J}\tau. \quad (4.1)$$

The conformal constraint (3.3) is then replaced by

$$z^{-2}(\dot{z}^2 + \dot{r}^2 - \dot{r}^2) = \mathcal{J}^2, \quad (4.2)$$

while the second integral of motion is the same as in (3.4). Then the first equation in (3.6) also keeps a similar form:

$$\dot{u}^2 = 1 + (1 + \mathcal{J}^2)u^2 - (p^2 - \mathcal{J}^2)u^4. \quad (4.3)$$

In this section we will consider the case in which $p^2 - \mathcal{J}^2 > 0$. In this case the resulting solution is analogous to the one with $\mathcal{J} = 0$, e.g., the structure of the world-sheet surface, with two branches, is exactly the same. The only difference is that now $\mathcal{u}_{\pm}$ in (3.7) should be replaced by

$$\mathcal{u}_{\pm} = \frac{1 + \mathcal{J}^2 \pm \sqrt{((\mathcal{J}^2 - 1)^2 + 4p^2)}}{2(p^2 - \mathcal{J}^2)}. \quad (4.4)$$

The resulting expression for $R_i/R_f$ in terms of $p$ and $\mathcal{J}$ is then

$$\log \frac{R_i}{R_f} = -2p\mathcal{u}_{\pm} \left[ K \left( \frac{u_{\pm}^2}{u_{\pm}^2} \right) - \Pi \left( -u_{\pm}^2, \frac{u_{\pm}^2}{u_{\pm}^2} \right) \right]. \quad (4.5)$$

We now have an interesting possibility of taking $p^2 - \mathcal{J}^2 \to 0$ (so that $\mathcal{u}_{\pm}$ become very large), while keeping $p$ finite which corresponds to $\frac{R_i}{R_f} \to 0$. In this limit $p = \pm \mathcal{J}$ the solution

$^{12}$ A special case where $p^2 - \mathcal{J}^2 < 0$ will be considered in appendix B.
becomes equivalent to the one found in [13]; written in terms of $(u, v)$ in (3.5) it takes the following simple form:

$$u = \frac{1}{\sqrt{1 + p^2}} \sinh(\sqrt{1 + p^2} \tau), \quad v = pt - \arctanh \left[ \frac{1}{\sqrt{1 + p^2}} \tanh(\sqrt{1 + p^2} \tau) \right].$$  \hspace{1cm} (4.6)

In this limit the configuration of two circular WLs degenerates to that of a single WL plus an effective local operator, which is located at the position of the ‘shrunk’ loop, i.e. at $0$ for $p < 0$ or at infinity for $p > 0$ (the two cases are, of course, related by an inversion). This is a ‘heavy’ operator $V_{\ell}^{(j)}$ parametrized by $p = J$ which has an interpretation of a semiclassical angular momentum $J = \sqrt{\ell} J$. The spacetime geometry can be visualized by plotting $(z, r)$ for different values of $p$, for $\tau \in (0, \infty)$. If $p$ is positive, then the solution starts at $(r, z) = (1, 0)$, where the WL is located, and ends at $(0, \infty)$, where the operator is located. If $p$ is negative, the solution ends at $(0, 0)$ instead, i.e. the operator is located at the origin.

Another special case corresponds to $p = 0$ for $J \neq 0$. In this case (3.4) implies that $z^2 + r^2 = R^2 = \text{const}$ or $v$ in (3.6) is constant. That means $R_i = R_f$, i.e. the two circles coincide and the form of the surface is again the ‘semisphere’. At the same time, $z(\tau)$ or $u(\tau)$ in (3.5) is non-trivial as (4.3) becomes $\dot{u}^2 = 1 + (1 + J^2) u^2 + J^2 u^4$. This case is equivalent to that of a single circular WL with an addition of angular momentum $J$ and will be considered in detail in appendix B.\textsuperscript{13}

Let us now consider the correlation function of two generalized ($J \neq 0$) circular WLs and a dilaton vertex operator (2.2) (located at $(\rho, h)$) with fixed $S^5$ angular momentum $J$, i.e. $\Delta = 4 + j$. The integrand of the correlator simplifies if we use the first-order constraints (3.4), (4.2) satisfied by $z(\tau)$ and $r(\tau)$. After introducing the variables $u(\tau)$, $v(\tau)$ as in (3.5) we end up with the following expression for the corresponding analog of (3.16):\textsuperscript{14}

$$\mathcal{C} = \int_0^{2\pi} d\sigma \int_0^{\ell_f} d\tau \frac{2}{[u(\tau)]^2} \left[ \frac{e^{i(1)\rho(\tau)}}{(e^{2i\rho(\tau)} + h^2 + \rho^2)\sqrt{1 + u^2(\tau)} - 2e^{i\rho(\tau)}\rho \cos \sigma} \right]^{\Delta}. \hspace{1cm} (4.7)$$

In the following section we shall study this correlator in the special case (4.6).

### 4.2. Special case of one degenerate circle ($p = \pm J$)

Let us consider the special case of $p = \pm J$ (we shall assume that $J > 0$). For $p > 0$ the radius of the ‘outer’ circle goes to infinity. For $p < 0$ the ‘inner’ circle shrinks to zero, i.e. is effectively replaced by a local operator $V^{(j)}_{\ell}$. After some simplifications, we obtain the following expression for the corresponding correlators (below we set $R_f = 1$):

$$\mathcal{C}^{(\pm)} = \frac{[W[C_j][V^{(j)}_{\ell}][V^{(j)}_{\ell}]]}{[W[C_j][V^{(j)}_{\ell}]]} = 2 \int_0^{2\pi} d\sigma \int_0^{\infty} d\tau \frac{1 + p^2}{\sinh^2(\sqrt{1 + p^2} \tau)} \times \left[ \sqrt{1 + p^2} \tanh g(\tau) - p \sqrt{h^2 + \rho^2 + e^{2\rho} (1 + 2p^2)} - 2e^{\rho} \sqrt{1 + p^2 (\cosh g(\tau) \rho \cos \sigma + p e^{\rho} \sinh g(\tau) \rho \cos \sigma)} \right]^{\Delta},$$

$$g(\tau) = \sqrt{1 + p^2} \tau + \arcsinh p. \hspace{1cm} (4.8)$$

\textsuperscript{13} Note that while in the first limiting case the angular momentum density is singular as $\frac{1}{\sqrt{1 - r^2}}$, i.e. $J$ is concentrated near $r = 0$ where the operator is located, in the second limit the density is $\frac{1}{\sqrt{1 - r^2}}$ which is integrable, i.e. $J$ is spread along the surface (cf discussion in the introduction).

\textsuperscript{14} In (4.1) we have chosen the initial condition for $\psi$ so that it vanishes at $\tau = 0$. Keeping $q_0 = \psi(0)$ non-zero will introduce an extra phase factor $e^{\rho \psi}$ in $\mathcal{C}$. 

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The ± choice comes from the sign in \( p = \pm J \). Since under the inversion \( p \to -p \), \( C^{(+)} \) and \( C^{(-)} \) should also be related by an inversion transformation. Indeed, as one can show explicitly,

\[
C^{(\pm)}(p, \Delta, \rho, h) = (h^2 + \rho^2)^{-\Delta} C^{(\mp)} \left( -p, \Delta, \frac{\rho}{\rho^2 + h^2}, \frac{h}{\rho^2 + h^2} \right). \tag{4.9}
\]

The integrand in (4.8) depends on the momentum \( j \) of the 'light' dilaton operator (\( \Delta = 4 + j \)), on \( p \) or the momentum \( J \) of the effective ‘heavy’ vertex operator, and on the location of the dilaton operator, parametrized by \( \rho \) and \( h \).

Below we will compute the correlators (4.8) in several special limits.

4.2.1. Small \( p \) limit. The small \( p \) limit is very simple but interesting nonetheless. In this limit the ‘heavy’ operator parametrized by \( p \) becomes effectively ‘light’ and the solution (4.6) tends to the circular WL solution plus a wiggle going from the center of the world-sheet to the origin, see figure 5.

In this limit the dependence on \( j \) is not seen, and we simply obtain, as expected, the same result as in the single circular WL case.

4.2.2. Large \( p \) limit. Taking \( |p| \) large at leading order we obtain

\[
C^{(\pm)} = |p|^{\pm j - 1} 2^{\pm j + 1} \int_0^\infty du \int_0^{2\pi} d\sigma \frac{u^{\Delta \pm j - 3}}{(1 + h^2 + \rho^2 + u^2 - 2\rho \cos \sigma)^\Delta}. \tag{4.10}
\]

Performing the integrals gives

\[
C^{(\pm)} = \pi |p|^{\pm j - 1} 2^{\pm j + 1} (1 + h^2 + \rho^2)^{1/2(\pm j - 2 - \Delta)} \frac{\Gamma[\frac{1}{2}(2 + \Delta \mp j)] \Gamma[\frac{1}{2}(-2 + \Delta \pm j)] \Gamma[\Delta]}{\Gamma[j + 1]} \times \frac{4\rho^2}{(1 + h^2 + \rho^2)^2}. \tag{4.11}
\]

These results are in perfect agreement with the numerics. Using the marginality condition \( \Delta = 4 + j \), we obtain

\[
C^{(+)} = 2^{j + 1} |p|^{-j} \frac{\Gamma[j + 1]}{\Gamma[j + 4]} \frac{(1 + h^2)^2 + 2(h^2 + 2)\rho^2 + \rho^4}{(1 + h^2)^2 + 2(h^2 - 1)\rho^2 + \rho^4} \frac{3 + j}{3} \frac{4 + j}{2} \frac{1}{1}, \tag{4.12}
\]

\[
C^{(-)} = 2^{j - 1} |p|^{-j - 1} \frac{(1 + h^2 + \rho^2)^{j - 3}}{3 + j} \frac{4 + j}{2} \frac{1}{1}, \tag{4.13}
\]

Note that \( C^{(\pm)} \) diverges in the limit \( h^2 + (\rho - 1)^2 \to 0 \) since the argument of the hypergeometric function becomes one. This is, of course, the expected divergence as then the location of the vertex operator approaches the WL (of radius \( R = 1 \)).
4.2.4. \textit{OPE limits.} When the location of the dilaton operator approaches the remaining finite WL, we expect a singularity that is the same as in the case of a single WL \((3.20)\), i.e. \(d^{-\Delta}\), where \(d\) is the distance to the WL. This is readily confirmed by a numerical evaluation of the correlator \((4.8)\).

Another interesting limit is when the location of the dilaton operator becomes very close to the location of the effective ‘heavy’ operator, which is at the origin for \(p < 0\) (the case we consider below). This happens in the limit \(d = \sqrt{h^2 + p^2} \ll 1\). In this case we find the leading singularity to be
\[
C(-) \sim \pm \frac{f(p, j)}{d^\gamma}, \quad \gamma = 4 + 2j + 2\sqrt{1 + p^{-2}}.
\]

\(f(p, j) = -4[1 + 2(p^2 - |p|\sqrt{1 + p^2})]^{4(j + \sqrt{1 + p^2})} \frac{1}{\Gamma(2 - \sqrt{1 + p^{-2}}) \Gamma[2 + j + \sqrt{1 + p^{-2}}]}\)
\[
\frac{\Gamma[2 - \sqrt{1 + p^{-2}}] \Gamma[2 + j + \sqrt{1 + p^{-2}}]}{\Gamma[4 + j]}.
\]

It would be interesting to understand the interpretation of the power \(\gamma\). On general grounds, one may expect this limit to be related to an OPE involving the dilaton operator \(V^{(j)}(x)\) and an effective ‘heavy’ operator \(V^{(j)}(0)\) and indeed, the leading contribution to \((4.14)\) comes from the region close to the heavy operator, or large \(\tau\) in \((4.8)\), see [2] for a related discussion.

4.2.5. \textit{Correlator with integrated dilaton operator.} If we consider the case of the correlator with the dilaton operator integrated over the insertion point (see \((2.4)\)) then it turns out that \(C_{\text{int}}(-)\) can be computed explicitly as a function of \(p\) and \(j\)
\[
C_{\text{int}}(-) = \int_0^\infty du \frac{2u^{i-2}}{\sqrt{1 + (1 + p^2)u^2}} (1 + u[p + 2p^2u + 2p\sqrt{1 + (1 + p^2)u^2}])^{-j/2}.
\]

This integral is convergent for \(j > 1\). The easiest way to compute it is by first expanding in powers of \(p\) and then integrating term by term. For the first few terms we obtain
\[
C_{\text{int}}(-) = \frac{2}{j - 1} - 2p + (j - 1)p^2 - \frac{1}{3}j(j - 2)p^3 + \cdots.
\]

\(^{15}\) The correlator \(C(+)\) is regular in this limit (it is singular for \(d \gg 1\).
One can then guess the general form of these terms to be \((-1)^n 2^n \frac{\Gamma\left(\frac{1}{2}(j+n-1)\right)}{\Gamma\left(\frac{1}{2}(j-n+1)\right)} p^n\), and finally resume the series. We arrive at the following simple result valid for all values of \(p\) and \(j > 1\):

\[
C^{(-)}_{\text{int}} = \frac{2}{j-1} (p + \sqrt{1 + p^2})^{1-j}, \quad j > 1.
\]  \tag{4.20}

For \(j = 0\) the insertion into a correlator of the integrated dilaton operator (2.2), (2.4) is equivalent to an insertion of the string action and thus to a differentiation of the correlator over the string tension:

\[
C_{\text{int}}(\epsilon) = \int \frac{2u^2}{\sqrt{1 + (1 + p^2)u^2}} = 2 \int_{\tau_i}^{\tau_f} \frac{1 + p^2}{\sinh^2(\sqrt{1 + p^2} \tau)} \frac{d\tau}{\tau} \rightarrow -\sqrt{1 + p^2},
\]  \tag{4.21}

where we first changed back to the world-sheet coordinate \(\tau\), introduced a cutoff \(\tau_i\) (that should correspond to a cutoff in \(z = \epsilon \rightarrow 0\)) and finally dropped the singular term. This expression is the same (up to an overall normalization factor) as the corresponding ‘bulk’ string action in equation (4.15) of [13] or equation (3.27) of [19].

5. Concluding remarks

In this paper we studied the leading semiclassical approximation to correlators of Wilson loops and local operators. One motivation is to learn more about how to represent local operators by (limits of) particular Wilson loops and also to shed light on the structure of 3-point correlation functions of local operators. We have seen that correlators involving two Wilson loops and a local operator have rather non-trivial dependence on the quantum numbers and position of the operator.

It would be useful to generalize our discussion to other more complicated Wilson loops like in [20] and also study other examples of simple Wilson loops for which minimal surfaces are known explicitly, as, e.g., in [19, 21, 22].

Another particularly interesting case is that of Wilson loops built out light-like segments [23–25]. With a special choice of a contour (as in weak coupling picture [26]) correlators of such loops with local operators may be used to represent correlators involving large-spin twist 2 operators [27]. Similar correlators may be related also to the form factor problem [28].

More generally, it would be interesting to understand whether leading semiclassical results for such correlators may be found indirectly by some combination of integrability-based methods as in [6, 29].

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Appendix A. Consequences of conformal symmetry

In this appendix we shall discuss restrictions imposed on the correlators we considered in this paper by the AdS$_5$ isometry group or conformal symmetry of the boundary theory. The symmetries of the boundary configurations extend to symmetries of the corresponding minimal surface in AdS$_5$ that ends on them\textsuperscript{16}.

A.1. Single circle

Let us start with a circular loop
\[ x_1^2 + x_2^2 = R^2, \quad x^3 = x^0 = 0 \]  
(A.1)
and identify which symmetries leave it invariant up to reparametrizations. The infinitesimal generators of the conformal group $(P_\mu, J_{\mu\nu}, D, K_\mu)$ have the following action (with parameters $(a_\mu, \omega_{\mu\nu}, \ell, b_\mu)$) on the boundary coordinates:
\[ \delta x^\mu = a_\mu + \omega_{\mu\nu} x^\nu + \ell x^\mu + (x^2 \eta^{\mu\nu} - 2 x^\mu x^\nu) b_\nu. \]  
(A.2)

Following [30], let us introduce the notation $x_l = (x^1, x^2)$ and $x_t = (x^3, x^0)$ and note that along the loop the transformations become
\[ \delta x^l = d^l + \omega^{lm} x_m + \ell x^l + R^2 b^l - 2 b^m x^m x^l, \]  
(A.3)
\[ \delta x^t = d^t + \omega^{tj} x_j + R^2 b^t. \]  
(A.4)

The conditions $\delta x^l = 0$ and $x_l \delta x^l = 0$ imply
\[ d^t = -R^2 b^l, \quad \omega^{tj} = 0, \quad d^l = R^2 b^t, \quad \ell = 0. \]  
(A.5)

That gives a total of $2 + 2 + 2 + 1 = 9$ constraints, leaving six transformations that leave the loop invariant. These transformations are generated by rotations in the plane of the loop, $J_{12}$, rotations in the transverse plane, $J_{30}$ and four additional symmetries
\[ \Pi^+_l = R P_l + R^{-1} K_l, \quad \Pi^-_l = R P_l - R^{-1} K_l. \]  
(A.6)

These generators form an $so(2, 2)$ subalgebra\textsuperscript{17}. One can also work out the finite action of these generators on the spacetime coordinates (see [30]).

A.2. Two concentric circles

To leave two concentric circular Wilson loops invariant, we need stronger constraints. In particular, in (A.5) we should have $d^t = b^l = d^l = b^t = 0$. This leaves us with two generators: rotations in the plane of the loop and rotations in the orthogonal plane. That means, in particular, that a correlator of two concentric circles with a local operator at an arbitrary position will depend only on the radial directions $\rho$ and $h$ in the two orthogonal planes.

In [12, 18] the correlator of two concentric circular Wilson loops lying on different parallel planes was considered. This configuration is, in fact, related [19] by a conformal transformation to the configuration where the circles lie in the same plane. To see this, let us start with two concentric circles (with radii 1 and $R_0$) in the same plane and consider the transformation $K_l - P_l$ along one of the transverse coordinates $x_l$, which leaves the unit circle invariant. From the above expressions for the infinitesimal transformations its easy to deduce that the loop of

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\textsuperscript{16} Below we do not distinguish between the cases of the Euclidean and Minkowski signature of the boundary.

\textsuperscript{17} If we were in two dimensions, then we would have three generators, giving basically by the Mobius transformations that leave a unit (after a dilatation) circle invariant.
radius \( R_0 \) at \( x_\mu = 0 \) is mapped, by a finite transformation with parameter \( b \), to a concentric loop of radius \( R(b) \) at \( x_\mu(b) \) where (here we assume that, e.g., \( x_1 = x_3 \))

\[
\dot{x}_\mu(b) = 2x_\mu(x)\dot{R}(b), \quad \dot{R}(b) = 2x_\mu(b)R(b), \quad x_\mu(b) = 1 + x_\mu^2 - R^2(b).
\]

(A.7)

These equations can be easily integrated; assuming \( R(0) = R_0, x_\mu(0) = 0 \) we obtain

\[
R(b) = \frac{2R_0}{1 + R_0^2 - (R_0^2 - 1)\cos 2b}, \quad x_\mu(b) = \frac{(R_0^2 - 1)\sin 2b}{1 + R_0^2 - (R_0^2 - 1)\cos 2b}.
\]

(A.8)

This gives us two concentric circles of radii 1 and \( R(b) \) separated by \( x_\mu(b) \). This implies, in particular, that the configuration of two concentric circles of equal radius \( R = 1 \) separated by a distance \( d \) is equivalent to two concentric circles in one plane with radii equal to 1 and

\[
R_0^\pm = 1 + \frac{1}{2}d(d \pm \sqrt{d^2 + 4}), \quad R_0^+R_0^- = 1,
\]

(A.9)

where the two sign options are related by an inversion.

### A.3. Correlator of a circular Wilson loop and a local operator

Another configuration we considered in this paper is that of a circular Wilson loop with its center at the origin and a local operator inserted at some arbitrary point (see also [32]). It is clear that by rotations we can always bring the operator to the location \( \rho, h, 0, 0 \). It turns out that one can act with the generator \( \Pi_1^+ \) in (A.6) so that after the transformation \( \rho \rightarrow 0 \), i.e. the operator is at a new location \( 0, 0, h', 0 \). Then, we can act with the generator \( \Pi_1^- \) to achieve that \( h' \rightarrow 0 \), i.e. that the operator is located at the origin. This shows that we can use four of the symmetries of the circular Wilson loop in order to bring an operator from an arbitrary location to the origin.

Once the operator is at the origin, we can ask which generators leave the resulting configuration invariant. An infinitesimal translation of the origin simply acts as \( \delta x^\mu = a^\mu \) so that we need to require \( \rho' = \rho = 0 \). Hence, again, this configuration will be left invariant by two transformations: rotation in the plane of the loop and rotation in the orthogonal plane.

The fact that a local operator can always be brought to the origin by conformal transformations that leave a given circular Wilson loop invariant suggests that we should be able to fix the form of their correlator (2.8) using symmetry considerations. This is indeed the case. Let \( \rho, h, 0, 0 \) be the location of the operator in the original correlator. According to the above discussion, a 2-parameter family of transformations \( \Pi_{a,b} \) generated by \( \Pi_1^+ \) with parameter \( a \) and by \( \Pi_1^- \) with parameter \( b \) that leaves the Wilson loop invariant acts on \( \rho, h \) as

\[
\delta \rho = R^2a + (\rho^2 + h^2)a - 2a\rho^2 + 2bh, \quad \delta h = R^2b - (\rho^2 + h^2)b + 2bh^2 - 2a\rho h.
\]

(A.10)

Naively, we would expect \( \Pi_{a,b} \) to leave the correlator

\[
\mathcal{C} = \frac{\langle W[C] V_2(\rho, h) \rangle}{\langle W[C] \rangle}
\]

(A.11)

invariant. This is almost the case. Translations are, of course, a manifest symmetry of the problem, but conformal transformations are not: one finds that \( \mathcal{C} \) transforms as

\[
\Pi_{a,b}: \quad \log \mathcal{C} \rightarrow \log \mathcal{C}' = \log \mathcal{C} + 2\Delta(a\rho - bh).
\]

(A.12)

The origin of this transformation is due to an anomaly under inversions, since special conformal transformations are obtained by a combination of translations \( P \) and two inversions \( I \), i.e. \( K = IPI \). Indeed, one can check that under the inversions

\[
I: \quad \log \mathcal{C} \rightarrow \log \tilde{\mathcal{C}} = \Delta \log \frac{h^2 + \rho^2}{R^2} + \log \mathcal{C}.
\]

(A.13)
Appendix B. Case of coinciding circular Wilson loops with $J \neq 0$

In this appendix we consider a special limit of the solution (4.1) in which $p \to 0$, while $\mathcal{J}$ is kept finite. In this limit (3.4) implies that $x^2 + r^2 = R^2$, so that the surface corresponds to a semisphere, as for a single circular Wilson loop. Here $v = 0$ (i.e. $R_i = R_f = 1$) while the equation for $u(\tau)$ in (4.3) becomes

$$\ddot{u}^2 = 1 + (1 + \mathcal{J}^2)u^2 + \mathcal{J}^2 u^4.$$  \hspace{1cm} (B.1)

As discussed in [19], the structure of the solution of this equation is different from the case of $p^2 - \mathcal{J}^2 > 0$ we discussed in section 4.1. Now the solution does not have turning points but again has two branches. On the first branch the solution starts at the boundary $r = 1$ and then reaches the point $r = 0$; after that we should continue to the second branch, in which the solution goes from $r = 0$ back to the boundary at $r = 1$. Again, $\tau$ can be found as a function of $u$. For the first and the second branch we obtain, respectively,

$$\tau = \tilde{u}+F \left( \arctan \frac{u}{\tilde{u}+} 1 - \frac{\tilde{u}^2}{\tilde{u}^2} \right), \quad \tau = \tau_f - \tilde{u}-F \left( \arctan \frac{u}{\tilde{u}-} 1 - \frac{\tilde{u}^2}{\tilde{u}^2} \right),$$ \hspace{1cm} (B.2)

$$\tau_f = 2\tilde{u}+K \left( 1 - \frac{\tilde{u}^2}{\tilde{u}^2} \right), \quad \tilde{u}+ = \sqrt{\mathcal{J}}, \quad \tilde{u}- = 1.$$ \hspace{1cm} (B.3)

$u$ runs from zero to infinity for the first branch and then back to zero for the second. The string action computed on this solution is (after removing a linear divergence) [19]

$$S = -2\sqrt{\lambda} \frac{1}{\tilde{u}+} E \left( 1 - \frac{\tilde{u}^2}{\tilde{u}^2} \right) = -2\sqrt{\lambda} \mathcal{J} E(1 - \mathcal{J}^{-2}) = -2\sqrt{\lambda} \mathcal{J} E(1 - \mathcal{J}^2).$$ \hspace{1cm} (B.4)

$$S_{\mathcal{J} \to 0} = -2\sqrt{\lambda} \left[ 1 + \mathcal{J}^2 \left( -\frac{1}{2} \log \mathcal{J} + \log 2 - \frac{1}{4} \right) + O(\mathcal{J}^3) \right].$$ \hspace{1cm} (B.5)

$$S_{\mathcal{J} \to \infty} = -2\sqrt{\lambda} \left[ \mathcal{J} + \frac{1}{\mathcal{J}} \left( \frac{1}{2} \log \mathcal{J} + \log 2 - \frac{1}{4} \right) + O \left( \frac{1}{\mathcal{J}^2} \right) \right].$$ \hspace{1cm} (B.6)

This action has a rather remarkable ‘duality symmetry’ under $\tilde{u}+ \leftrightarrow \tilde{u}-$ due to the elliptic function identity $E(1 - \frac{1}{\mathcal{J}}) = \frac{1}{\mathcal{J}} E(1 - \mathcal{J})$.\footnote{See [33] for a very neat symmetry argument.} For small $\mathcal{J}$ the action starts with twice the value of the action of the circular Wilson loop (here we have two coinciding loops as a limit of self-intersecting surface and that doubles the semisphere surfaces and thus the action) while for large $\mathcal{J}$ the action has familiar ‘near-BPS’ behavior.

\footnote{Note that for $\mathcal{J} > 1$ we have $\tilde{u}+ < \tilde{u}-$, so that $\tilde{u}+$ and $\tilde{u}-$ should be interchanged but that leaves the form of the action invariant. The same will happen with all the expressions below, i.e. by our choice of $\tilde{u}+$ we cover all the values of $\mathcal{J}$.}
To compute the correlator in this limit we need to evaluate the dilaton vertex operator on this classical solution. Using $\Delta = 4 + j$, we obtain

$$C = e^{-\frac{1}{2}J_4 r^2} \int_0^{2\pi} d\sigma \int_0^\infty du \frac{4u^{2j+1}}{\sqrt{1 + u^2 \sqrt{1 + J_4^2 u^2}}} \times \frac{\cosh(jF(\arctan J_4 u, 1 - J_4^{-2}) - K(1 - J_4^{-2}))}{(\sqrt{1 + u^2}(1 + h^2 + \rho^2) - 2\rho \cos \sigma)^{2j+1}} ,$$  \hspace{1cm} (B.7)

where we assume that $j, J \geq 0$. For the particular case of $j = 0$ the above integral can be computed explicitly, e.g., by expanding in powers of $\rho$, integrating term by term and then resuming. We obtain

$$C = -8\pi \frac{J_r}{3(1 - J_4^2)^2} [((J_4^4 - 1)b^4 + 2J_4^2(7 + J_4^2)\rho^2)E(1 - J_4^{-2})]
- 2[(J_4^2 - 1)b^4 + (3 + 5J_4^2)\rho^2]K(1 - J_4^{-2}) ,$$  \hspace{1cm} (B.8)

where we have introduced

$$b^2 = 1 + h^2 + \rho^2 , \quad J_r = J_r \frac{b^2}{\sqrt{b^4 - 4\rho^2}} .$$  \hspace{1cm} (B.9)

$E$ and $K$ denote the complete elliptic integrals. For general $j$, the integral (B.7) can be computed in some special limits. For instance, for $J_r \to 0$ we reproduce the well-known result for a single circular Wilson loop without angular momentum.

Another special case is the large $j$ limit in which we can compute the integral by a saddle point approximation. For instance, if $h = \rho = 0$, then the saddle point is at $u = 1/\sqrt{J}$, giving

$$C = \frac{4J_4^{1/2}\pi^{3/2}}{(1 + J_4^{2j+2})^{1/2}} e^{-1/2}K(1, J_4^{-2}) ,$$  \hspace{1cm} (B.10)

where $K$ is the complete elliptic integral. This result is in perfect agreement with the numerics.

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