Coherent perfect absorption with and without lasing in complex potentials

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Abstract
We prove that coherent perfect absorption (CPA) without lasing is not possible in the PT-symmetric domain because the $S$-matrix is such that $\pm = \text{det} (S(\pm k)) = 1$. We study coherent scattering from three complex potentials, one solved analytically and the other two numerically. We conjecture that in the domain of unbroken symmetry (when the potential has a real discrete spectrum), neither spectral singularity nor CPA can occur. We show that the Scarf II potential is a special model that can analytically and explicitly exhibit these as well as other novel phenomena and their subtleties.

Keywords: scattering from non-Hermitian Hamiltonians, spectral singularity, coherent perfect absorption, PT-symmetric quantum mechanics, complex Scarf II potential, non-reciprocity, reflection and transmission

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(Some figures may appear in colour only in the online journal)
of various domains and various parametric regimes in a complex potential becomes necessary to observe these effects. So far the analytically tractable and versatile Scarf II potential has not been utilized in this regard.

Here in this paper, we prove that CPA without lasing cannot occur in PT-symmetric domains. We show that the exactly solvable complex Scarf II potential entails and exhibits these phenomena analytically and explicitly by bringing out their subtle features. Numerically solved examples show results similar to those of Scarf II. We also find that in domains where PT symmetry is unbroken, spectral singularity (and hence CPA) does not occur. When a one-dimensional complex potential (vanishing asymptotically) is spatially asymmetric, the reflectivity is sensitive to the side of incidence of the wave whether it is left or right. It has been proved that

\[ T_{\text{left}}(k) = T_{\text{right}}(k) = T(k) \text{ but } R_{\text{left}}(k) \neq R_{\text{right}}(k). \]  

(1)

Also see [3–7]. Following the same proof, it has also been proposed [14] that for PT-symmetric potentials,

\[ T(-k) = T(k), \text{ and } R_{\text{left}}(-k) = R_{\text{right}}(k). \]  

(2)

Recently these proposals have been proved [19]. Here P means parity transformation \((x \rightarrow -x)\) and \(T\) means time reversal \((i \rightarrow -i, k \rightarrow -k)\); \(k\) is wave number defined as \(k = \sqrt{E} (\hbar^2 = 1 = 2m)\). For non-PT-symmetric cases, we have [14]

\[ T(-k) \neq T(k), R(-k) \neq R(k) \text{ and } R_{\text{left}}(-k) \neq R_{\text{right}}(k). \]  

(3)

Let \(r\) and \(t\) be reflection and transmission (complex) amplitudes with phases denoted \(\phi\) and \(\theta\), respectively. Then reflectivity \(R = |r|^2\), and transmittance \(T = |t|^2\). For complex PT-symmetric structures, it has been proved that [13]

\[ \theta - \phi_{\text{left}} = \pi/2 = \theta - \phi_{\text{right}}, \text{ if } T < 1 \text{ and } \theta - \phi_{\text{left}} = \pi/2 = \phi_{\text{right}} - \theta, \text{ if } T > 1. \]  

(4)

When two waves that are identical (coherent) in all respects are incident on a complex scattering potential from left and right, the \(S\)-matrix is given as [7]

\[ S = \begin{pmatrix} t & n_{\left| e \right|} \\ n_{\text{right}} & t \end{pmatrix}, \text{ det } S = t^2 - n_{\text{left}}n_{\text{right}} \]  

(5)

Using equation (4) in (5), we find that

\[ \text{det } S = T \pm \sqrt{R_{\text{left}}R_{\text{right}}} = 1, \]  

(6)

following sub(super)-unitarity as proposed in [13, 15]. In Hermitian quantum mechanics, it is known that \((\phi_{\text{left}} - \theta) + (\phi_{\text{right}} - \theta) = \pi\) [16]; for coherent injection at a Hermitian potential, the scattering matrix admits \(|S| = T + R = 1\). Therefore, this is yet another common feature shared by complex PT-symmetric potentials and Hermitian potentials.

On the other hand, the condition for CPA without lasing is [9]

\[ \text{det } S(k_c) = 0 \]  

(7)

at a real positive energy, \(E_c = k_c^2\). One can therefore see the impossibility of CPA for complex PT-symmetric cases in view of the result, equation (6).

However, the novel possibility of PT-symmetric potentials displaying CPA with lasing is distinct and different. It occurs when at \(E = E_n\), \(T\) becomes infinity and \(|\text{det } S| = 0\) [10], such that \(|\text{det } S(E_n \pm \epsilon)| = 1\). Here \(\epsilon\) is arbitrarily small. Also, the constancy of \(|\text{det } S|\) in equation (6) indicates its invariance under time reversal as
confirms proposal (2).

Nevertheless, let us point out that the possibility of CPA without lasing is due to the change of \( |\text{det} S(k)| \) under time reversal for non-PT-symmetric potentials, which in turn is due to equation (3). Thus, the following conditions may be met at \( k = k_c \):

\[
|\text{det} S(k_c)| = 0, \quad T(-k_c) = R_{\text{left}}(-k_c) = R_{\text{right}}(-k_c) = \infty
\]

and CPA alone (without lasing) is observed. This is why coherent perfect absorbers are also called time-reversed lasers [9, 15]. In this regard, one of the claims of this paper is that these potentials cannot be PT-symmetric.

Note that unlike the first proposal for general CPA [9], the authors in [12] have been cautious about choosing the optical medium as P-symmetric. They set a less general yet simpler and intuitive condition for CPA at a real energy as \( t + \eta_{\text{left}} = 0 = t + \eta_{\text{right}} \). For P-symmetric complex potentials, the reciprocity \( (\eta_{\text{left}} = \eta_{\text{right}}) \) [1, 3] works, and the result \( \theta - \phi = \pi/2 \) [22] of real Hermitian P-symmetric potentials is favorably contradicted due to the presence of non-Hermiticity (dissipation), so CPA is feasible. This phenomenon has been called controlled CPA, which is a special case of the more general condition equation (7) [9].

Thus, we conclude that complex PT-symmetric potentials do not display coherent perfect absorption alone (without lasing).

The aforementioned phenomena [7, 9, 10] occur in a complex potential as a possibility and not as a necessity; therefore, an exactly solvable potential is all the more welcome for bringing out these phenomena with their subtleties. The Scarf II potential

\[
V(x) = P \text{sech}^2 x + Q \text{sech} x \tanh x.
\]

is a versatile potential encompassing a number of interesting parametric regimes in both PT-symmetric and non-PT-symmetric domains. By virtue of the beautiful complex transmission and reflection amplitudes that are available [2, 23], Scarf II has helped to provide simple expressions for SS [8, 14]. Recently it has revealed [17] rare (accidental) phenomena like reciprocity despite complex PT-symmetry and unitarity \( (R + T = 1) \) despite non-Hermiticity. In the following, we invoke three parametric regimes of complex Scarf II by complexifying \( P \) and \( Q \) in various ways to demonstrate the novel phenomenon [7, 9, 10] of coherent injection at optical potentials as previously discussed.

We also study scattering from several complex potentials by numerically integrating the Schrödinger equation. However, here we discuss the results of two models. These models of complex potential are the rectangular profile of compact support

\[
V_k(x) = P \Theta_1(x) - iQ \Theta_2(x),
\]

where

\[
\Theta_1(x) = \begin{cases} 
1, & |x| \leq L \\
0, & |x| > L
\end{cases}, \quad \Theta_2(x) = \begin{cases} 
0, & |x| \geq L \\
-1, & -L < x < 0 \\
1, & 0 \leq x < L
\end{cases}
\]

and the asymptotically converging Gaussian

\[
V_G(x) = Pe^{-x^2} + iQ e^{-x^2},
\]

with \( P \) and \( Q \) as parameters. In the following, we present the results of coherent scattering from complex potentials equation (10)–(12) in three domains \{A,B,C\} in light of the Scarf II potential.
A: Occurrence of CPA alone (non-PT-symmetric domain)

Let us consider the non-PT-symmetric domain of Scarf II as

\[ V_j(x) = \left(d^2 - id\right) \text{sech}^2 x, \quad d \in R. \]  

(13)

This is an absorptive P-symmetric potential \((d > 0)\), and it is ideal for demonstrating controlled CPA [12]. By using the transmission and reflection amplitudes [2, 23] and by eliminating gamma functions (with complex argument) in them, in this case equation (13), we find

\[ T(k) = \frac{k - d}{k + d} \frac{\sinh^2 \pi k}{\cosh^2 \pi k - \cosh^2 \pi d} \]  

(14)

\[ f_{\text{left}}(k) = -\frac{\sinh \pi d}{\sinh \pi k} = f_{\text{right}}(k) \]  

(15)

The reflection amplitudes are calculated as \(r(k) = t(k)f(k)\); the equivalence of left/right in equation (15) is by virtue of the P symmetry of the potential equation (13) (see [1, 3]). Next we derive

\[ \left|\det \ S(k)\right| = \left|\begin{array}{c} k - d \\ k + d \end{array}\right| \frac{\cosh^2 \pi k - \cosh^2 \pi d}{\cosh^2 \pi k - \cosh^2 \pi d} \]  

(16)

Notice that in (14), \(k = -d\) is a pole (SS) and \(k = +d\) is not a pole. So here \(k_c = d\) and the energy at which CPA occurs is \(E_c = k_c^2 = d^2\). We use the L’Hospital rule to see that limit \(k \to k_c \), \(T(k) = \frac{\tanh \pi d}{\sinh \pi d}\), which is finite. So there is only one SS. The conjectured [14] properties (3) can be verified here readily. More interestingly, at \(k = k_c\), \(\left|\det \ S(k)\right|\) becomes indeterminate \(\left(\frac{0}{0}\right)\) but limit \(k \to k_c \), \(\left|\det \ S(k)\right| = 0\), in contrast with the case of CPA with lasing [10], where it ought to be 1. This, however, presents the CPA scenario [9]. The second aspect of CPA is fulfilled by noticing that \(k = -k_c\) is clearly an SS (equivalently, SS at \(k = k_c\) in time-reversed transmittance \(T(-k)\)) (14)). Figure 1 presents the results (equations (14) and (16)) for Scarf II (equation 13), which have also been recovered by numerical integration of the Schrödinger equation, to confirm our numerical method.

We confirm the existence of CPA without lasing in non-PT-symmetric domains of two numerically solved complex potentials (rectangular and Gaussian; see figures 2 and 3). However, here in these models one has to carry out a judicious search of potential parameters to observe CPA. For example, we find that for the rectangular model for \(P = 2.21 - 1.09i\), \(Q = 0\), \(L = 2\) CPA exists at \(E = E_c = 4.015\) where \(\left|\det \ S(E_c)\right| = 0\); at this energy, spectral singularity occurs in the time-reversed transmission co-efficient \(T(-k_c)\) (14)). For the Gaussian model (12), we find a similar scenario for \(P = 3.89 - 2.04i\), \(Q = 0\), this time \(E_c = 3.992\). CPA is of crucial physical significance to spectral singularity in that SS in \(T(k)\) at \(k = \pm k_c\) implies CPA at \(k = \pm k_a\) at a real discrete energy \(E = E_c\). Hence the coherent perfect absorbers are called time-reversed lasers [9, 15].

B: The occurrence of CPA with lasing (broken PT symmetry)

However, the situation changes dramatically when spectral singularity is present in the potential. Now let us consider the following parameterization of the Scarf II potential for \(c \in R\):

\[ V_j(x) = \left[2c^2 - 1/4\right] \text{sech}^2 x - i \left[2c^2 + 1/2\right] \text{sech} x \tanh x. \]  

(17)
For this case, we obtain

\[
T(k) = \frac{\sinh^2 \pi k \cosh^2 \pi k}{(\cosh^2 \pi k - \cosh^2 \pi \epsilon)^2}
\]

(18)

and

\[
f_{\text{left}}(k) = i [e^{-\pi k} - e^{\pi k} \cosh 2\pi \epsilon] \cosech 2\pi k,
\]

\[
f_{\text{right}}(k) = i [e^{\pi k} - e^{-\pi k} \cosh 2\pi \epsilon] \cosech 2\pi k.
\]

(19)

One can readily notice self-dual SS [15] in transmission co-efficient (18) (poles at \( k = \pm \epsilon \)); i.e., at \( E = \epsilon^2 \) both \( T(\epsilon) \) and \( T(-\epsilon) \) are infinity. \( r(k) \) is calculated as \( r(k) = f(k) r(k) \). Next, using (18, 19) in equation (5), we obtain
Thus $|\text{det } S| = \left( \frac{\cosh^2 \pi k - \cosh^2 \pi \xi}{\cosh^2 \pi k - \cosh^2 \pi \xi} \right)^2$. (20)

In Figure 4, the scenario of CPA with lasing is shown for $c = 2$. Here we present the results of numerical integration of the Scarf II potential. Notice a kinky behavior in $|\text{det } S(E)|$.
at \( E = E_a = c^2 = 4 \) representing indeterminacy. However, in the neighborhood of this energy, \(|S| = 1\) is retained.

In figure 5, we take the rectangular potential (11) with \( P = 2.7, Q = -0.9, L = 2 \). Here the kinky behavior in \(|S(k)|\) in (a) and common spectral singularity in \( T(k) \) and \( T(-k) \) are displayed at \( E_a = 3.448 \). For the Gaussian potential (12), for \( P = 4.0, Q = -6.25 \), we get \( E_a = 3.808 \) and the same scenario in figures 6 except that the kinky behavior in figure 3(a) is depicted as merely a dot at \( E = 3.380 \). The indeterminacy of \(|\det S(k)| = E_a\) depicted as a kinky behavior in figures 4(a), 5(a), and 6(a) is the most subtle feature and is best displayed by the Scarf II potential analytically equations (18)–(20) and not displayed so well by numerical computation presented graphically (see figure 4(a)).

**{C}**: Non-occurrence of spectral singularity and CPA (unbroken PT symmetry)

When in (12) \( P = -V_i, V_i > 0, \) and \( Q = iV_2 \) (both \( V_i, V_2 \in \mathbb{R} \)), it has been shown \[20\] that if \(|V_2| \leq \frac{1}{2} \), the potential entails a real discrete spectrum wherein the energy eigenstates are also eigenstates PT (PT-symmetry exact (unbroken) \[20\]); otherwise, the real discrete eigenvalues disappear and transition to non-real complex conjugate pairs, and the PT symmetry is said to be spontaneously broken. Therefore, for all real values of \( a, b \) the potential

\[
V_{a,b}(x) = -\left(a^2 + b^2 + a\right) \text{sech}^2 x - i b \left(2a + 1\right) \text{sech} x \tanh x
\]

(21)

can be verified to have a finite number of real discrete eigenvalues and the PT symmetry remains unbroken. Using the available scattering amplitudes \[2, 23\], the following results follow from there \[17\].

\[
T(k) = \frac{\sinh^2 \pi k \cos \frac{\pi k}{2}}{\left(\sinh^2 \pi k + \sin^2 \pi a\right)\left(\sinh^2 \pi k + \cos^2 \pi b\right)},
\]

(22)

and

\[
f_{a,b}(k) = i \left[\frac{-\cos \pi a \sin \frac{\pi b}{2}}{\cosh \frac{\pi k}{2}} + \frac{\sin \pi a \cos \frac{\pi b}{2}}{\sinh \frac{\pi k}{2}}\right]
\]

(23)

\[
R_{\text{left}}(k) = T(k) |f_{a,b}(k)|^2, R_{\text{right}}(k) = T(k) |f_{a,-b}(k)|^2
\]

(24)

Verify that the reflection and transmission (24) coefficients have common relevant poles at real discrete energies:

\[
E_n = -(n - a)^2, E_m = -(m - b - 1)^2,
\]

(25)

where \(0 \leq n < a\) and \(0 \leq m < b + 1/2\), which are two branches of the well-known discrete eigenvalues \[21, 24\] of (21). The invariances given in equation (2) can be readily checked using (22)–(24). Furthermore, we can write

\[
|\det S(k)| = T(k) \left[1 - f_{a,b}(k)f_{a,-b}(k)\right].
\]

(26)

Using (22, 24), we eventually find that

\[
|\det S(k)| = \frac{\sinh^4 \pi k + \sinh^2 \pi k \left(\sin^2 \pi a + \cos^2 \pi b\right) + \sin^2 \pi a \cos^2 \pi b}{\left(\sinh^2 \pi k + \sin^2 \pi a\right)\left(\sinh^2 \pi k + \cos^2 \pi b\right)} = 1.
\]

(27)

One can immediately verify that \( T(k) \) (24) does not have any pole at a real \( k \) and that it cannot become infinity (absence of SS) at any positive or negative real value of \( k \). In several other
numerically complex PT symmetric potentials that possess a real discrete spectrum, we have found the absence of SS and hence CPA.

We summarize our findings as follows.

- It is interesting to see that Hermitian and complex PT-symmetric potentials share yet another common feature, which is $|\text{det } S(E)| = 1$ (6) for coherent scattering at these potentials. However, a novel dissimilarity arises at $E = E_s$ (spectral singularity) in the latter, where $|\text{det } S| = \frac{0}{0}$ (indeterminate); occur and at this energy, CPA and lasing occur simultaneously [10], giving rise to new types of lasers.

- In figures 4, 5, and 6 parts (b) and (c) for various complex scattering potentials confirm our previous conjecture [14] that for complex PT-symmetric potentials or domains, $T(-k) = T(k)$ (this has been proved recently [19]).

- Importantly, it turns out that apart from its amenability to analysis, as displayed amply in the new expressions (14)–(20), (27), the complex Scarf II potential is in no way special. These (11), (12) numerically solved potentials behave in a qualitatively similar manner in bringing out CPA (see figures 2 and 3) and CPA with lasing (see figures 5 and 6).

- As previously discussed, a complex non-Hermitian potential can have three parametric domains: {A}: non-PT-symmetric, {B}: PT-symmetric with broken PT symmetry, and {C}: PT-symmetric with unbroken PT symmetry. Without undermining the novel proposals and revelations of spectral singularity [7], CPA [9], and CPA with lasing [10], we add that the necessary domain(s) of the potential could not be pinpointed due to the intuitive nature of these proposals. We elaborate this in the following.

For example, spectral singularity was proposed [7] for complex non-Hermitian potentials. Here we find that spectral singularity occurs only in the {A} and {B} domains and does not occur in the {C} domain. CPA has been claimed [9] to occur in non-Hermitian potentials; here we have argued and found that CPA cannot occur in a complex PT-symmetric domain owing to the result brought out here: $|\text{det } S| = 1$ (6). CPA with lasing was proposed [10] as a property of complex PT-symmetric potentials; later it was found [11] that it is actually the property of the broken PT-symmetric ({B}) domain.

Finally, we hope that the amenability to analysis of Scarf II for studying coherent scattering at a complex potential has been well noted. We conclude that coherent perfect absorption cannot occur in complex PT-symmetric potentials. We have also conjectured that when PT symmetry is unbroken (the potential has a real discrete spectrum), spectral singularity (and hence CPA with or without lasing) does not arise. This, however, requires proof. We hope that our work presented here strengthens the recent novel, intuitive concepts of wave propagation through non-Hermitian complex mediums/potentials.

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