THE TRIANGULATED HOPF CATEGORY $n_+SL(2)$

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Abstract. We discuss an example of a triangulated Hopf category related to SL(2). It is an equivariant derived category equipped with multiplication and comultiplication functors and structure isomorphisms. We prove some coherence equations for structure isomorphisms. In particular, the Hopf category is monoidal.

1. Introduction

Crane and Frenkel proposed a notion of a Hopf category [1]. It was motivated by Lusztig’s approach to quantum groups – his theory of canonical bases. In particular, Lusztig obtains braided deformations $U_q\mathfrak{n}_+$ of universal enveloping algebras $U\mathfrak{n}_+$ for some nilpotent Lie algebras $\mathfrak{n}_+$ together with canonical bases of these braided Hopf algebras [6, 7, 8]. Elements of the canonical basis are identified with isomorphism classes of simple perverse sheaves – certain objects of equivariant derived categories. They are contained in the subcategory of semisimple complexes. One of the proposals of Crane and Frenkel is to study this category rather than its Grothendieck ring $U_q\mathfrak{n}_+$. Conjectural properties of this category were collected into a system of axioms of a Hopf category, equipped with functors of multiplication and comultiplication, isomorphisms of associativity, coassociativity and coherence which satisfy four equations [4]. A mathematical framework and some examples were provided by Neuchl [12].

Crane and Frenkel [4] gave an example of a Hopf category resembling the semisimple category encountered in Lusztig’s theory corresponding to one-dimensional Lie algebra $\mathfrak{n}_+$ – nilpotent subalgebra of $\mathfrak{sl}(2)$. We want to discuss an example of a related notion – triangulated Hopf category – the whole equivariant derived category equipped with multiplication and comultiplication functors and structure isomorphisms. In particular, it is a monoidal category. New feature of coherence is that additive relations of [4] are replaced with distinguished triangles. This new structure does not induce a Hopf category structure of Crane and Frenkel on the subcategory of semi-simple complexes. The missing component is a consistent choice of splitting of splittable triangles. Verification of some of the consistency equations is still an open question.

To give more details let us first recall some braided Hopf algebra $H$. As an algebra $H$ is the algebra of polynomials of one variable over $R = \mathbb{Z}[q, q^{-1}]$. More precisely, $H \subset \mathbb{Q}(q)[x]$ is an $R$-submodule spanned by the elements

$$y_n = \frac{x^n}{(n)_{q-2}!}, \quad n \geq 0,$$

$$\quad (n)_{q-2}! = \prod_{m=1}^{n} (m)_{q-2}, \quad (m)_{q-2} = \frac{1 - q^{-2m}}{1 - q^{-2}}.$$

The basis $(y_n)_{n \geq 0}$ is called a canonical basis.
Multiplication table in this basis is
\[ y_k \cdot y_l = \binom{k + l}{k} y_{k+l}, \quad \binom{k + l}{k} q^{-2} = \frac{(k+l)_{q-2}!}{(k)_{q-2}!(l)_{q-2}!} \in \mathbb{R}. \]

Comultiplication by definition is
\[ y_n = \sum_{k+l=n} y_k \otimes y_l. \]

These operations make \( H \) into a \( \mathbb{Z} \)-graded \( \mathbb{R} \)-algebra and coalgebra. We equip the category \( \mathcal{C} \) of \( \mathbb{Z} \)-graded \( \mathbb{R} \)-modules with the braiding
\[ c : M \otimes \mathbb{R} N \to N \otimes \mathbb{R} M, \quad c = \sum_{k,l} c_{k,l}, \]
and we can write the coherence axiom as an equation
\[ n \otimes \mathbb{R} m \to H_{n \otimes \mathbb{R} m} \to H_{p \otimes \mathbb{R} q}, \quad (1) \]
for all \( n, m, p, q \in \mathbb{Z}_{\geq 0} \) such that \( n + m = p + q \).

This algebra was obtained by Lusztig from the following setup:
- \( \mathcal{H}_{n_1, \ldots, n_k} \) are \( \mathbb{C} \)-linear categories, depending symmetrically on parameters \( n_1, \ldots, n_k \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}_{\geq 0} \);
- \( m_{k,l} : \mathcal{H}_{k,l} \to \mathcal{H}_{k+l} \), \( \Delta_{k,l} : \mathcal{H}_{k+l} \to \mathcal{H}_{k} \otimes \mathbb{R} \mathcal{H}_{l} \) are \( \mathbb{C} \)-linear functors of multiplication and comultiplication;
- \( c_{k,l} : \mathcal{H}_{k,l} \to \mathcal{H}_{l,k} \) are braiding functors;
- there are associativity isomorphisms
\[ \mathcal{H}_{k,l,n} \xrightarrow{\langle m_{k,l,n} \rangle} \mathcal{H}_{k,l+n} \]
\[ \mathcal{H}_{k+l,n} \xrightarrow{m_{k+l,n}} \mathcal{H}_{k+l+n} \]
where the meaning of \( \otimes \) will be specified further;
- there are similar coassociativity isomorphisms.
The category $\mathcal{H}_{n_1,\ldots,n_k}$ is $D_{GL(n_1)\times\cdots\times GL(n_k)}^{p,q}(pt)$ – the bounded constructible equivariant derived category of a point. It has a distinguished object $Y_{n_1,\ldots,n_k}$ – the constant sheaf, which is the complex

$$\ldots \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{C} \rightarrow 0 \rightarrow 0 \rightarrow \ldots$$

concentrated in degree 0. It turns out that the collection $(Y_{n_1,\ldots,n_k})$ is closed under multiplication and comultiplication (up to coefficients which are graded vector spaces):

$$m_{k,l}(Y_{k,l}) \simeq H^*(Gr_k^{k+l}(\mathbb{C}), \mathbb{C}) \otimes_C Y_{k+l}.$$  

The coefficient vector space here is de Rham cohomology of the Grassmannian $Gr_k^{k+l}(\mathbb{C})$ – manifold of $k$-dimensional subspaces of a $k+l$-dimensional space. Cohomology is concentrated in even degrees and the Betti numbers

$$\beta_i = \dim_C H^{2i}(Gr_k^{k+l}(\mathbb{C}), \mathbb{C})$$

are coefficients of the expansion of a $q$-binomial coefficient in powers of $q^{-2}$:

$$\binom{k+l}{k} q^{-2i} = \sum_{i \geq 0} \beta_i q^{-2i}.$$  

Replacing the degree with the power of $q$ we get the multiplication table for the canonical basis $(y_k)$. Comultiplication law is recovered from

$$\Delta_{k,l}(Y_{k+l}) \simeq Y_{k,l}.$$  

The braiding functor

$$c_{k,l} = (D_{GL(k)\times GL(l)}(pt) \simeq D_{GL(l)\times GL(k)}(pt) \xrightarrow{[-2kl]} D_{GL(l)\times GL(k)}(pt))$$

is essentially the degree shift by $-2kl$. It translates into multiplication by $q^{-2kl}$ for the braiding in algebra setting.

In the present paper we shall discuss coherence at the category level. If one replaces linear mappings in equation (1) with functors and $\sum$ with $\oplus$ the equation fails: the left and the right hand side functors $\mathcal{H}_{n,m} \rightarrow \mathcal{H}_{p,q}$ are, in general, not isomorphic. (Restricted to $Y_{n,m}$ they give, however, isomorphic results.) One of the results of the present paper is the following. Value of the left hand side functor on an object $X$ of $\mathcal{H}_{n,m}$ is a repeated extension of values on $X$ of summands in the right hand side in the sense of distinguished triangles. Precise analogy is as follows: a sheaf $S$ on a topological space $W$ is an extension of its quotient-sheaf $S_F$ supported on closed subset $F$ by subsheaf $S_U$ supported on its open complement $U = W - F$.

Technically, this is achieved by introducing new operations-functors with two inputs and two outputs

$$\text{observed} \begin{array}{ccc} n & m & p & q \end{array} : D_{GL(n)\times GL(m)}^{p,q}(pt) \rightarrow D_{GL(p)\times GL(q)}^{b,c}(pt),$$

which depend on a parameter $\mathcal{O} - a P_{p,q} \times P_{n,m}$-invariant subset of $GL(n+m)$, where $p+q = n+m$, the parabolic subgroup $P_{n,m} \subset GL(n+m)$ consists of matrices preserving $\mathbb{C}^n \subset \mathbb{C}^{n+m}$. Minimal such subsets $\mathcal{O}$ are double cosets – points of the double coset space $P_{p,q}\backslash GL(n+m)/P_{n,m}$. This is a finite set, it is in bijection with the set of quadruples $(i,j,k,l)$, $i,j,k,l \in \mathbb{Z}_{\geq 0}$, which satisfy the equations $i+j = n$, $k+l = m$, $i+k = p$, $j+l = q$. Hence, we may index the $P_{p,q} \times P_{n,m}$-orbits with these quadruples, say, $\mathcal{O}_{ijkl}$.
First, we prove that the left and the right hand sides of (1) are isomorphic to the above mentioned operations:

\[
\begin{array}{c}
\begin{array}{ccc}
 n & m \\
 p & q & p
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
 n & m \\
 p & q & p
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
 GL(n+m) & \\
 p & q & p
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
 n & m \\
 p & q & p
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
 n & m \\
 p & q & p
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
 GL(n+m) & \\
 p & q & p
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
 \cong & \\
 p & q & p
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
 \cong & \\
 p & q & p
\end{array}
\end{array}
\end{array}
\end{array}
\]

Since \( GL(n+m) = \biguplus_{i,j,k,l} O_{ijkl} \), the former functor above is a repeated extension of the latter functors via distinguished triangles.

This shows usefulness of operations with many inputs and outputs for our purposes. They are also used to prove an equation for associativity isomorphisms which makes \( \biguplus D^{b,c}_{GL(k)}(pt) \) into a monoidal category. Similar equation for coassociativity isomorphisms is proven as well.

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2. Preliminaries

A definition of equivariant derived categories is given by Bernstein and Lunts [2]. First we explain basic terms. With a topological space \( X \) is associated the category \( Sh(X) \) of sheaves of topological spaces. Its derived category is denoted \( D(X) \). The subcategory consisting of bounded complexes of sheaves is denoted \( D^b(X) \). If \( X \) is a complex algebraic variety, we call a sheaf constructible if it is constructible with respect to some stratification by algebraic submanifolds and stalks are finite-dimensional vector spaces. A complex is cohomologically constructible if its cohomology sheaves are constructible. Subcategory of bounded constructible complexes is denoted \( D^{b,c}(X) \).

2.1. Equivariant derived categories

Assume that a complex linear algebraic group \( G \) acts algebraically on a complex algebraic variety \( X \). In this setting Bernstein and Lunts [2] define bounded constructible equivariant derived category \( D^{b,c}_G(X) \), as a fiber category.

A \( G \)-variety \( P \) is called free if \( G \) acts freely on \( P \) and the quotient map \( q : P \to G\backslash P = \overline{P} \) is a locally trivial fibration with the fiber \( G \). A \( G \)-resolution of a \( G \)-variety \( X \) is a \( G \)-map \( P \to X \), where the \( G \)-variety \( P \) is free.

Let \( j : J \to \text{Res}(X,G) \), \( P \mapsto (jP : JP \to X) \) be a functor to the category of \( G \)-resolutions. Let \( \mathcal{T} \) denote the category of complex algebraic varieties. Let us denote \( \Phi : \text{Res}(X,G) \to \mathcal{T} \), \( (R \to X) \mapsto \overline{R} = G\backslash R \) quotient functor. Consider the composite functor

\[
\Psi : J \xrightarrow{j} \text{Res}(X,G) \xrightarrow{\Phi} \mathcal{T}, \quad P \mapsto (jP : JP \to X) \mapsto \overline{JP},
\]

and define the fiber-category \( D^{b,c}(\Psi) \) as follows.

2.1.1. Definition (Bernstein and Lunts [2]). An object of \( D^{b,c}(\Psi) \) is a function \( M : \text{Res}(X,G) \ni P \mapsto M(P) \in D^{b,c}(\overline{JP}) \) equipped with isomorphisms \( \alpha_\nu : (\overline{J\nu})^*(M(R)) \to M(P) \) given for any \( \nu : P \to R \in \text{Mor} J \), such that for any pair of composable morphisms \( P \xrightarrow{\nu} R \xrightarrow{\mu} S \) we have

\[
\alpha_{\mu\nu} = \left[ (\overline{J(\mu\nu)})^* M(S) \cong \overline{J\nu} \overline{J\mu}^* M(S) \xrightarrow{\overline{J\nu}_*} \overline{J\nu}^* M(R) \xrightarrow{\alpha_\nu} M(P) \right].
\]
A morphism $\phi : M \to N$ is a collection $\phi(P) : M(P) \to N(P)$, $P \in \text{Ob} \mathcal{J}$, compatible with $\alpha_\nu$ for any $\nu : P \to R \in \text{Mor} \mathcal{J}$:

$$
(\mathcal{J}\nu)^*(M(R)) \overset{\alpha_\nu^M}{\longrightarrow} M(P)
$$

$$
(\mathcal{J}\nu)^*(\phi(R)) \quad = \quad \downarrow \phi(P)
$$

$$
(\mathcal{J}\nu)^*(N(R)) \overset{\alpha_\nu^N}{\longrightarrow} N(P)
$$

Define equivariant derived category as $D_{b,c}^G(X) = D_{b,c}(\Phi)$ in the case of identity functor $j = \text{id} : \mathcal{J} \Longrightarrow \text{Res}(X, G)$.

We shall also use the notation $X^{G} = D_{b,c}^G(X)$ for equivariant derived category. Notice that, if $X$ is $G$-free, then $X^{G}$ is equivalent to $D_{b,c}(G\backslash X)$. Without freeness assumption the former and the latter categories are not equivalent, in general.

Bernstein and Lunts compute the equivariant derived category in the case when $X$ is a point.

2.1.2. Theorem (Bernstein and Lunts [2] Theorem 12.7.2). Assume that $G$ is a connected linear algebraic group. The triangulated category $D_{b,c}^G(\text{pt})$ is equivalent to the derived category of the category of finitely generated differential graded $A$-modules, where the graded algebra $A = H^\bullet(BG, \mathbb{C})$ is equipped with zero differential.

For $G = GL(n, \mathbb{C})$ the algebra $A$ is the algebra of symmetric polynomials of $n$ variables

$$A_n = \mathbb{C}[x_1, \ldots, x_n] \cong \mathbb{C}[e_1, \ldots, e_n],$$

where $\deg x_i = 2$ and $e_j$ are elementary symmetric functions. For $G = GL(n_1, \mathbb{C}) \times \cdots \times GL(n_k, \mathbb{C})$ we have $A = A_{n_1} \otimes \mathbb{C} \cdots \otimes \mathbb{C} A_{n_k}$.

2.2. Equivariant derived functors

2.2.1. The inverse image functor. Suppose that $\phi : G \to H$ is a group homomorphism, and $f : X \to Y$ is a $\phi$-equivariant map. We want to define a functor $f^* = (f, \phi)^* : D_{b,c}^H(Y) \to D_{b,c}^G(X)$. First, denote $\mathcal{J} = \text{Res}(f, \phi)$ the category, whose objects are $\phi$-maps $f : P \to R$ of resolutions, that is,

$$
P \quad \xrightarrow{f} \quad R
$$

$$
X \quad \xrightarrow{f} \quad Y
$$

commutes. Morphisms $\nu : f \to f'$ are pairs of morphisms of resolutions $\nu_1 : P \to P'$, $\nu_2 : R \to R'$ such that

$$
P \quad \xrightarrow{\nu_1} \quad P'
$$

$$
R \quad \xrightarrow{\nu_2} \quad R'
$$

commutes. Use the functor $j : \text{Res}(f; \phi) \to \text{Res}(X, G)$, $j(f : P \to R) = P$ and $\Psi = j\Phi$. Bernstein and Lunts [2] have shown that the restriction functor $D_{b,c}^g(X) \to D_{b,c}(\Psi)$ is an equivalence.

Similarly to Bernstein and Lunts [2] we define the first version of the inverse image functor:

$$f^* : D_{b,c}^H(Y) \to D_{b,c}(\Psi : \text{Res}(f; \phi) \to T),$$

where $T$ is the category of $f$-equivariant resolutions.
\[ f^*(M : R \mapsto M(R) \in D^{b,c}(\mathbb{R})) = [f^*M : (P \xrightarrow{f} R) \mapsto \mathcal{F}'(M(R)) \in D^{b,c}(\mathcal{F})]. \]

Next thing is to choose for all \((f, \phi)\) an equivalence

\[ F_{f,\phi} : D^{b,c}(\Psi : \text{Res}(f; \phi) \to \mathcal{T}) \to D_G^{b,c}(X) \]

quasi-inverse to the canonical restriction functor

\[ \text{Can}_f : D_G^{b,c}(X) \longrightarrow D^{b,c}(\Psi : \text{Res}(f; \phi) \to \mathcal{T}), \]

\[ [P \mapsto M(P)] \mapsto [(f : P \to R) \mapsto M(P)]. \]

The chosen isomorphisms of the composition with the identity functor are denoted

\[ \eta_f : [D^{b,c}(\text{Res}(f; \phi) \to \mathcal{T})] \xrightarrow{F_{f,\phi}} D_G^{b,c}(X) \xrightarrow{\text{Can}_f} D^{b,c}(\text{Res}(f; \phi) \to \mathcal{T}) \]

\[ \longrightarrow \text{Id}, \]

\[ \epsilon_f : \text{Id} \to [D_G^{b,c}(X) \xrightarrow{\text{Can}_f} D^{b,c}(\text{Res}(f; \phi) \to \mathcal{T})] \xrightarrow{F_{f,\phi}} D_G^{b,c}(X)]. \]

Define the inverse image functor of \((f, \phi)\) as

\[ f^* = (f, \phi)^* : D_H^{b,c}(Y) \xrightarrow{f^*} D^{b,c}(\Psi : \text{Res}(f; \phi) \to \mathcal{T}) \xrightarrow{F_{f,\phi}} D_G^{b,c}(X). \]

For composable maps

\[ X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \quad \text{over} \quad G \xrightarrow{\phi} H \xrightarrow{\psi} K \xrightarrow{\chi} L \]

we define categories \(\text{Res}(f, g; \phi, \psi)\) and \(\text{Res}(f, g, h; \phi, \psi, \chi)\), whose objects are pairs \((f, g)\) (resp. triples \((f, g, h)\)) of morphisms of resolutions over \((f, g)\) (resp. \((f, g, h)\)).

\[ P \xrightarrow{f} R \xrightarrow{g} S \xrightarrow{h} Q \]

\[ X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \]

Morphisms are triples \(P \to P', R \to R', S \to S'\) (resp. quadruples \ldots, \(Q \to Q')\) of morphisms of resolutions compatible with \((f, g, h)\) and \((f', g', h')\).
2.2.2. The direct image functor. Let \( f : X \to Y \) be a \( \phi \)-map, \( \phi : G \to H \). Assume that \( X \) is \( K \)-free, where \( K = \text{Ker} \phi \subset G \). For our purposes it suffices to define \( f_* : D_G^{b,c}(X) \to D_H^{b,c}(Y) \) as a right adjoint functor to \( f^* : D_H^{b,c}(Y) \to D_G^{b,c}(X) \). Furthermore, we shall use it mainly in the quotient equivalence situation: \( H = G/K \), \( \phi : G \to G/K \) is the canonical projection, \( X \) is \( K \)-free, \( Y = K \setminus X \), \( f = \pi : X \to K \setminus X \) is the canonical projection.

2.2.3. Quotient equivalence.

**Theorem ( Bernstein and Lunts [2] ).** Let \( K \) be a normal subgroup of \( G \), let \( X \) be a \( G \)-space which is free as a \( K \)-space. Then the quotient map \( \pi : X \to K \setminus X \) gives an equivalence

\[
\pi^* : D_{G/K}^{b,c}(K \setminus X) \to D_G^{b,c}(X)
\]

with a quasi-inverse \( \pi_* \).

In this situation we shall make a concrete choice of a right adjoint (and quasi-inverse) functor to \( \pi^* \)

\[
\pi_* : D_G^{b,c}(X) \to D_{G/K}^{b,c}(K \setminus X), \quad N \mapsto \pi_* N,
\]

\[
(\pi_* N)(R \to K \setminus X) = N(R \times_{K \setminus X} X \to X)
\]

\[
\in D^{b,c}(G \setminus (R \times_{K \setminus X} X)) \simeq D^{b,c}((G/K) \setminus R),
\]

the equivalence is due to the isomorphism \( G \setminus (R \times_{K \setminus X} X) \simeq (G/K) \setminus R \).

2.2.4. Induction equivalence.

**Theorem ( Bernstein and Lunts [2] ).** Let \( H \) be a subgroup of \( G \), let \( X \) be an \( H \)-space. Then the induction map \( i : X \to G \times_H X \), \( x \mapsto (1, x) \) gives an equivalence

\[
i^* : D_G(G \times_H X) \to D_H(X)
\]

with a quasi-inverse \( i_* \).
2.2.5. **The direct image with proper supports.** The following definition belongs to Bernstein and Lunts [2].

2.2.6. **Definition.** Let \( f : X \to Y \) be a map of \( G \)-varieties. For any resolution \( \pi : P \to Y \) there is a pull-back resolution \( \bar{\pi} \) of \( X \).

\[
P \times_Y X \xrightarrow{f} P \\
\downarrow \bar{\pi} \quad \downarrow \pi \\
X \xrightarrow{f} Y
\]

The functor \( f_* : D_G^{b,c}(X) \to D_G^{b,c}(Y) \) maps an object \( M : (R \to X) \mapsto M(R \to X) \in D^{b,c}(R) \) to the object \( f_! M : (P \to Y) \mapsto f_!(M(P \times_Y X \to X)) \in D^{b,c}(P) \) equipped with an isomorphism

\[
\alpha_{\nu}^fM : \bar{\nu} \bar{f}_{!} M(R \times_Y X \to X) \xrightarrow{\beta} \bar{f}_{!} M(P \times_Y X \to X) \to \bar{f}_{!} \bar{\nu} M(P \times_Y X \to X)
\]

for \( \nu : P \to R \). Here \( \beta \) is a base change isomorphism obtained from the top square of the following prism.

Required property of \( \alpha \) follows from Lemma B.2.1 (see Appendix).

2.2.7. **The equivariant base change isomorphism.** Let the pull-back square

\[
\begin{array}{ccc}
W & \xrightarrow{e} & X \\
\downarrow h & & \downarrow f \\
Z & \xrightarrow{g} & Y
\end{array}
\]
consist of equivariant maps: $h$ is a $G$-map, $f$ is an $H$-map and $e$, $g$ are $\phi$-maps, where $\phi : G \rightarrow H$ is a group homomorphism. There is a commutative cube

\[
\begin{array}{c}
P \times Z W \\
\downarrow \downarrow \downarrow \\
P \downarrow \downarrow \downarrow \\
W \downarrow \downarrow \downarrow \\
Z \downarrow \downarrow \downarrow \\
\end{array}
\]

where the left and the right walls and the bottom are pull-back squares. It follows that the top is also a pull-back square. We define a version of direct image functor with proper support:

\[h_! : Db_c(Res e \rightarrow T) \rightarrow Db_c(Res g \rightarrow T), \quad N \mapsto h_! N,\]

where $N(\epsilon) \in Db_c(P \times Z W)$. Notice that the functor $h_!$ depends on $f$ as well, which is not reflected in notation.

The base change isomorphism

\[
\begin{array}{c}
Db_c(Res e \rightarrow T) \leftrightarrow e^* Db_H(X) \\
\downarrow \downarrow \downarrow \\
Db_c(Res g \rightarrow T) \leftrightarrow g^* Db_H(Y) \\
\end{array}
\]

comes from the standard one for quotient spaces. The collection

\[(g^* f_! M)(g) = \overline{g}^*[f_! M(R \rightarrow Y)] = \overline{g}^* \overline{f_!} M(R \times_Y X \rightarrow X) \in Db_c(P)\]

is isomorphic to the collection

\[(h_! e^* M)(\epsilon) = \overline{h_!} [e^* M(\epsilon)] = \overline{h_!} \overline{e}^* M(R \times_Y X \rightarrow X) \in Db_c(P)\]

via base change isomorphism $\beta : \overline{g}^* \overline{f_!} \rightarrow \overline{h_!} \overline{e}^*$.

Finally, we define a full form of base change isomorphism as the following diagram suggests:

\[
\begin{array}{c}
Db_G(W) \leftrightarrow F_{\gamma_{\phi}} D_{h_!} \downarrow \downarrow \downarrow \\
\downarrow \downarrow \downarrow \\
Db_G(Z) \downarrow \downarrow \downarrow \\
\end{array}
\]

Namely, a full base change isomorphism is:

\[
\beta : g_{\text{full}}^* f_1 = F_{g_! \phi} g^* f_1 \xrightarrow{F_{h_!} \eta_1} F_{g_! \phi} h_! e^* \xrightarrow{F_{h_!} \eta_1^{-1}} F_{g_! \phi} h_! \text{Can}_{e} F_{f_! \phi} e^* \\
= F_{g_! \phi} \text{Can}_{g} h_! F_{f_! \phi} e^* \xrightarrow{\epsilon_{\text{full}}^{-1}} h_! F_{f_! \phi} e^* = h_! e_{\text{full}}^*.
\]
3. The Hopf category \( n_+SL(2) \)

3.1. Setup and notations

We partially follow Lusztig \cite{lusztig1} and \cite{lusztig2}, Chapter 9 in notations. Let \( V \) be a vector space and 
\[ G = G_V = \text{GL}(V). \]

Let us make the product of \( D_G(pt) \) over varying \( \text{dim} \ V \) into a sort of a graded Hopf category.

Assume we are given a decomposition 
\[ V : V^1 \oplus V^2 \oplus \cdots \oplus V^k = V \]
into vector subspaces. Associate with it a filtration of 
\[ 0 = V^{(0)} \subset V^{(1)} \subset \cdots \subset V^{(n)} = V, \quad V^{(m)} = V^1 \oplus \cdots \oplus V^m. \]

\( P_V \) is the corresponding parabolic group 
\[ P_V = \{ g \in G_V \mid \forall m \, g(V^{(m)}) \subset V^{(m)} \} = \{ g \in G_V \mid \forall m \, g(V^m) \subset V^{(m)} \} \]
and \( U_V \) is its unipotent radical. The group 
\[ L_V = \{ g \in G_V \mid \forall m \, g(V^m) \subset V^m \} = \prod_{m=1}^{k} G_{V^m} \]
is a Levi subgroup of \( P_V \).

Notice that \( P_V, U_V \) need only a filtration to be defined unlike \( L_V \), which requires a direct sum decomposition.

3.2. Suggestions for a monoidal 2-category

To provide a final framework for Hopf categories one would be interested to have a symmetric monoidal 2-category of equivariant derived categories. Tensor product of categories would be similar to that of abelian \( k \)-linear categories introduced by Deligne \cite{deligne}. In particular, \( D_G^{b,c}(X) \boxtimes D_H^{b,c}(Y) \simeq D_{G \times H}^{b,c}(X \times Y) \) is desirable. However, this wish does not look realistic. To achieve it, possibly, one has to replace equivariant derived categories with some other kind of categories, so that inverse image functors and direct image functors make sense, and the usual relations still hold.

Let us consider a question of tensor product of functors. Let \( f : X \to Y, g : Z \to W \) be maps of algebraic varieties. Denote by \( f^*, g^*, f_!, g_! \) the corresponding equivariant derived functors. It is explained in \cite{deligne} that one choice for \( f^* \boxtimes g^* \) is as good as another as long as they are isomorphic. If in the isomorphism class of \( f^* \boxtimes g^* \) there is a functor \( (f \times g)^* \), we can modify the definition of \( \boxtimes \) and set \( f^* \boxtimes g^* = (f \times g)^* \). If \( f_! \boxtimes g_! \) is isomorphic to \( (f \times g)_! \), we can set \( f_! \boxtimes g_! = (f \times g)_! \). The isomorphism 
\[ (\text{Id} \boxtimes g^*) \circ (f_! \boxtimes \text{Id}) \simeq (f_! \boxtimes \text{Id}) \circ (\text{Id} \boxtimes g^*) \]
can be chosen as the base change isomorphism \( \beta : (Y \times g)^*(f \times W)_! \to (f \times Z)_!(X \times g)^* \), constructed for the pull-back square
\[
\begin{array}{ccc}
X \times Z & \xrightarrow{X \times g} & X \times W \\
\downarrow & & \downarrow \quad (f \times W) \\
Y \times Z & \xrightarrow{Y \times g} & Y \times W
\end{array}
\]
We stress again that \( \boxtimes \) is far from being constructed. Nevertheless, we prove some statements, which can be interpreted as axioms of a Hopf category.
3.3. Braiding

Pretending that the categories $D_{G_a \times \ldots \times G_k}(pt)$ form a monoidal 2-category, where $a_i$ are some vector spaces, we define a braiding in it via functor

$$c : D_{\prod G_a \times \prod G_b}(pt) [\to -2d] \to D_{\prod G_{a_i} \times \prod G_{b_j}}(pt) \to D_{\prod G_{a_i} \times \prod G_{b_j}}(pt),$$

where $d = (\sum_i \dim_C a_i)(\sum_j \dim_C b_j)$ and $\sigma$ is a permutation isomorphism of groups. The shift functor $K \to K[-2d]$ (the $R$-matrix) is related to other functors that we are using via the following lemma.

3.3.1. Lemma. Let $h : E \to B$ be an affine linear $C^\infty$-bundle. Then there is a (canonical in $B$) isomorphism of functors

$$(D^{b,c}(B) \xrightarrow{h^*} D^{b,c}(E) \xrightarrow{h_!} D^{b,c}(B)) \simeq T^{-2 \dim_C h}.$$  

Proof. Using Propositions 10.1 and 10.8(2) from [3] we find for any $K \in D^{b,c}(B)$

$$h_! h^* K \simeq h_! h^*(C \otimes K) \simeq h_! (h^* C \otimes h^* K) \simeq (h_! h^* C) \otimes K.$$  

Thus we have to prove that $h_! h^* C_B \simeq h_! C_E$ is isomorphic to $C[-2d]$ for $d = \dim_C h$.

Choose a flat connection $\nabla$ on the bundle $h$. By definition this is a $C^\infty(B)$-linear homomorphism of Lie algebras of vector fields

$$\nabla : \text{Vect}(B) \to \text{Vect}(E), \quad \xi \mapsto \nabla \xi,$$

such that for each point $e \in E$ we have $Th(\nabla_{\xi e}) = \xi_{h(e)}$, where $Th : T_e E \to T_{h(e)} B$ is the tangent map to $h$. The fields $\nabla_{\xi e}$ (the horizontal vector fields) form a $2b$-dimensional (over $\mathbb{R}$) distribution in $E$, where $b = \dim_C B$. This distribution is involutive, therefore, by Frobenius theorem, locally there exist coordinates $(z_i, u_j), i = 1, 2b, j = 1, 2d$ in $E$ in which leaves of the obtained foliation are described by equations $u_1 = \text{const}, \ldots, u_{2d} = \text{const}$. Moreover, $z_i$ can be chosen as $z_i = z_i \circ h$, where $z_i$ are local coordinates on the base $B$.

The sheaf of differential graded algebras of fibrewise forms $\Omega^*_E$ is defined as a subsheaf of $C^\infty$ differential forms on $E$

$$\Omega^*_E(U) = \{ \omega \in \Omega^*_E(U) \mid \forall \xi \in \text{Vect}(B) \quad i_{\nabla \xi} \omega = 0 \text{ and } i_{\nabla \xi} d\omega = 0 \}.$$  

The second condition may be replaced with $L_{\nabla \xi} \omega = 0$, since Lie derivative can be computed as $L_{\nabla \xi} \omega = di_{\nabla \xi} \omega + i_{\nabla \xi} d\omega$. Therefore, in local coordinates forms $\omega \in \Omega^*_E(U)$ are written as $f(u) du^\beta$. Absence of $dz_i$ is implied by the first condition, and the Lie derivative condition implies independence of the coefficients on $z_i$ coordinates. We conclude that the complex $\Omega^*_E$ is a $c$-soft on fibers of $h$ resolution of the constant sheaf $\mathbb{C}_E$. Hence, it can be used to compute the complex $h_! C_E$.

Let $V$ be an open subset of $B$. Then $h_! C_E(V)$ is a complex of vector spaces $\Omega^*_E(h^{-1}(V))$, where $c$ indicates such forms $\omega$ that $h^{-1}(K) \cap \text{supp} \omega$ is compact for each compact subset $K$ of $B$. This complex is exact everywhere except the maximal degree $2d$. It has a map into the algebra of functions on $V$

$$\alpha(V) : \Omega^{2d}_E(h^{-1}(V)) \to C^\infty(V), \quad \omega \mapsto (v \mapsto \int_{h^{-1}(v)} \omega),$$

given by fibrewise integration. Local presentation of $\omega$ implies that the function $\alpha(V)(\omega)$ is locally constant, hence, is in $\mathbb{C}_B(V)$. The obtained chain map $\alpha : h_! C_E \to \mathbb{C}_B[-2d]$ is a quasi-isomorphism.  

Another (complex analytic) construction of $\alpha$ will be published elsewhere. It follows the lines of [10], where the case of quasicompact schemes over $\overline{\mathbb{F}_p}$ and $\ell$-adic sheaves is considered. In the
analytic setting one can also prove that for any pull-back square

\[
\begin{array}{ccc}
F & \xrightarrow{j} & A \\
g & & \downarrow f \\
E & \xrightarrow{h} & B
\end{array}
\]

where \( h \) is an affine linear bundle (and so is \( j \)), we have an equation

\[
\begin{array}{c}
D_{b,c}(E) \\
D_{b,c}(F) \\
D_{b,c}(B)
\end{array}
\xrightarrow{\alpha_B}
\begin{array}{c}
D_{b,c}(A) \\
D_{b,c}(A) \\
D_{b,c}(B)
\end{array}
\]

This is the precise meaning of canonicity of isomorphism \( \alpha \).

3.3.2. Corollary. Let \( h : E \to B \) be an affine linear \( G \)-bundle (an affine linear bundle equipped with a group homomorphism \( G \to \text{Aut}_{af.lin.bun.}(h) \)). Then there is an isomorphism of functors

\[
(D_{b,c}^G(B) \xrightarrow{h^*} D_{b,c}^G(E) \xrightarrow{h_!} D_{b,c}^G(B)) \simeq T^{-2 \dim h}.
\]

Proof. The system of isomorphisms \( \alpha \) in \( D_{b,c}(P) \) for \( G \)-resolutions \( P \to X \) is coherent due to canonicity of \( \alpha \).

\[\square\]

3.4. Operations

Let two decompositions of \( V \) into a direct sum be given:

\[
\begin{align*}
\mathcal{V} : & \quad V^1 \oplus V^2 \oplus \cdots \oplus V^k = V \\
\mathcal{W} : & \quad W^1 \oplus W^2 \oplus \cdots \oplus W^l = V
\end{align*}
\]

Let \( \mathcal{O} \subset G \) be a left \( P_{\mathcal{W}} \)-invariant and right \( P_{\mathcal{V}} \)-invariant subset. We associate with it an operation

\[
\begin{array}{c}
\mathcal{O}_{\mathcal{V}} \\
\mathcal{O}_{\mathcal{W}}
\end{array}
\begin{array}{c}
V^1 V^2 V^k \\
W^1 W^2 W^k
\end{array}
\xrightarrow{=\Phi_{\mathcal{W}} \circ \Psi_{\mathcal{V}}}
\begin{array}{c}
V^1 V^2 V^k \\
W^1 W^2 W^k
\end{array}
\]

The components of it are defined below.

Notations. In graphical notations a straight line \( X \) labeled by a vector space \( X \) denotes category \( D_{G,X}(pt) \). A dashed line \( \mathcal{X} \) labeled by a filtration \( \mathcal{X} \) of a vector space \( X \) denotes category \( D_{P_{\mathcal{X}}}(pt) \).

If a horizontal line crosses a diagram intersecting transversally with several (solid or dashed) open intervals labeled by spaces \( Y_1, Y_2, \ldots \) and filtrations \( \mathcal{X}_1, \mathcal{X}_2 \) of spaces \( X_1, X_2, \ldots \), this line represents the category \( D_{G_{Y_1 \times G_{Y_2} \times \cdots \times P_{X_1} \times P_{X_2} \times \cdots}}(pt) \). The order of factors repeats the order of intersection points on the horizontal line.

A vertex (or a horizontal row of vertices) represents a functor from the category, corresponding to horizontal line just above the vertex, to the category represented by a line just below the vertex.
3.4.1. The multiplication. Multiplication operation is

\[
\Psi_W^{i,j} = (D_{L_Y}(pt) \xrightarrow{\phi^*} D_{P_W \times L_Y}(\mathcal{O}/U_Y) \\
\xrightarrow{\pi^*} D_{P_W}(\mathcal{O}/P_Y) \xrightarrow{\alpha} D_{P_W}(pt)),
\]

Here the scheme of multiplication is similar to that in [8]:

\[
pt \xleftarrow{\phi} \mathcal{O}/U_Y \xrightarrow{\pi} \mathcal{O}/P_Y \xrightarrow{\alpha} pt,
\]

where \(\pi\) is a canonical projection. The action of \(L_Y = P_Y/U_Y\) in \(\mathcal{O}/U_Y\) is induced from the action \(p.o = op^{-1}, p \in P_Y\).

In particular, for \(l = 1\) we have \(P_W = G_V, \mathcal{O} = G_V\) and the multiplication operation is

\[
\Psi_W^1 = (D_{L_Y}(pt) \xrightarrow{\phi^*} D_{G_V \times L_Y}(G_V/U_Y) \\
\xrightarrow{\pi^*} D_{G_V}(G_V/P_Y) \xrightarrow{\alpha^*} D_{G_V}(pt)),
\]

Here the scheme of multiplication is precisely that of Lusztig [8]:

\[
pt \xleftarrow{\phi} G_V/U_Y \xrightarrow{\pi} G_V/P_Y \xrightarrow{\alpha} pt.
\]

The particular case \(k = 1, \mathcal{O} = G_V\) is also important. We have then \(L_Y = P_Y = G_V, U_Y = 1\), and

\[
\Psi_W^1 = (D_G(pt) \xrightarrow{\text{pr}_G} D_{P_W \times G}(G) \xrightarrow{d_1} D_{P_W}(pt)),
\]

where \(d_1 : G \rightarrow pt\).

3.4.2. Proposition. The functor \(\Psi_W^1\) is isomorphic to the restriction functor \(\text{Res}_{P_W,G} : D_G(pt) \rightarrow D_{P_W}(pt)\).

Proof. A map \(d_1\) is the map \(G \longrightarrow pt\). Hence, \(d_1^* : D_{P_W \times G}(G) \rightarrow D_{P_W}(pt)\) is an equivalence with a quasi-inverse \(d_1^* : D_{P_W}(pt) \rightarrow D_{P_W \times G}(G)\).

The map \(s_0 : pt \rightarrow G, s_0(pt) = 1\), is a \(\Delta\)-map with respect to the homomorphism of groups \(\Delta : P_W \rightarrow P_W \times G, \Delta(p) = (p, p)\). Furthermore, \(s_0\) is an induction map \(pt \xrightarrow{i} (P \times G)/P \xrightarrow{\rho} G\), where \(i(pt) = (1, 1), \rho(p, g) = pg^{-1}\) and \(\rho\) is a \(P \times G\)-map. Therefore, \(s_0^* : D_{P_W \times G}(G) \rightarrow D_{P_W}(pt)\) is an induction equivalence. Since \((pt \xrightarrow{s_0} G \xrightarrow{d_1} pt) = \text{id}\), we have

\[
(D_{P_W}(pt) \xrightarrow{d_1^*} D_{P_W \times G}(G) \xrightarrow{s_0^*} D_{P_W}(pt)) \simeq \text{Id}.
\]

Therefore, \(s_0^* \simeq d_1^*\). Hence,

\[
\Psi_W^1 \simeq (D_G(pt) \xrightarrow{\text{pr}_G} D_{P_W \times G}(G) \xrightarrow{s_0^*} D_{P_W}(pt)) \simeq \text{Res}_{P_W,G},
\]

for \((pt \xrightarrow{s_0} G \xrightarrow{\Delta} G) = \text{id}\) and \(P_W \xrightarrow{\Delta} P_W \times G \xrightarrow{\text{pr}_G} G\) is an inclusion. \(\square\)

3.5. Associativity

Assume that besides decomposition \([3.4.1]\) of \(V\) into direct sum of \(V^i\) we have also decompositions

\[
\bigoplus_{m=1}^{m^i} V^i_m.
\]

We can produce out of these the decomposition

\[
\bigoplus_{i,m} V^i_m,
\]
where the lexicographic order of summands is used $V_1^1, \ldots, V_1^m, V_2^1, \ldots, V_2^m, \ldots, V_l^1, \ldots, V_l^m$.

An associativity isomorphism is obtained from the isomorphism

$$\pi' : Y \rightarrow L_V \backslash Y \simeq \mathcal{O}/U_V,$$

$$\pi'' : Y \rightarrow L_U \backslash Y \simeq X.$$ 

The maps $\pi', \pi''$ are the quotient maps

$$\pi' : Y \rightarrow L_V \backslash Y \simeq \mathcal{O}/U_V, \quad \pi'' : Y \rightarrow L_U \backslash Y \simeq X.$$ 

The canonical projection $\beta$ comes from the inclusion $P_U \subset P_V$,

$$\beta : \mathcal{O}/P_U \rightarrow \mathcal{O}/P_V.$$
3.6. The associativity equation

Assume that we have the following decompositions of finite-dimensional $\mathbb{C}$-vector spaces:

$$ W : \ V = \oplus_j W^j, \quad V : \ V = \oplus_i V^i, $$
$$ \mathcal{V}^i : \ V^i = \oplus_m V^i_m, \quad \mathcal{V}^i_m : \ V^i_m = \oplus_p V^i_{m,p}. $$

These decompositions imply also the following decompositions:

$$ U : \ V = \oplus_{i,m} V^i_m, \quad \mathcal{Y}^i : \ V^i = \oplus_{m,p} V^i_{m,p}, \quad \mathcal{X} : \ V = \oplus_{i,m,p} V^i_{m,p}. $$

Let $\mathcal{O} \subset G_V$ be a left $P_W$-invariant and right $P_V$-invariant subset. Associativity isomorphisms give two isomorphisms between the composite operation and the single operation as in diagram (3.6.1)

3.6.1. Proposition (Associativity equation). Diagram (3.6.1) is commutative.

Proof. The left-lower associativity composition is given on diagram in Figure 2. The right-upper associativity composition is given on diagram in Figure 3. We have to check that isomorphism in this figure equals to the one in Figure 2.

The subdiagram of diagram in Figure 3 between the third and the forth columns is transformed via Proposition B.2.2 to a 3-column subdiagram. The right part of it coincides with the right subdiagram in Figure 2 between the third and the fourth columns and can be canceled. The left subdiagram of Figure 2 between the first and the third columns is also transformed via Proposition B.2.2. We come to equation between the isomorphisms in Figures 4 and 5.

Figure 4 can be transformed further using Proposition B.2.2 to the form of Figure 6. We have to prove that it equals to isomorphism in Figure 5.

Two lower squares and two rightmost squares in diagrams in Figure 6 and Figure 5 cancel and we have to prove the following equation. We have to show that the isomorphism in Figure 7 is equal to the isomorphism in Figure 8.

Using Proposition B.1.1 we reduce both isomorphisms to the expressions in Figure 9 and Figure 10.

The two upper triangles in Figure 9 and Figure 10 coincide. Due to Proposition B.1.1 the lower parts are equal to isomorphisms in Figure 11.

Finally, the last two isomorphisms are equal to each other due again to Proposition B.1.1.

3.7. Comultiplication

The comultiplication functor is

$$ \eta^W_W = \left( \begin{array}{ccc} & \text{Res}_{L_W} & \text{Res}_{F_W} \\ \text{Res}_{L_W} & \eta_{L_W,F_W} & \eta_{L_W,F_W} \\ \eta_{L_W,F_W} & \eta_{L_W,F_W} & \eta_{L_W,F_W} \end{array} \right) = \eta^W_{L_W,F_W}. $$
Figure 2. First composition of associativity isomorphisms

Figure 3. Second composition of associativity isomorphisms
Together with multiplication definition of Section 3.4.1 it gives general operation (in the set-up of (3.4.1)–(3.4.2))

\[
\mathcal{O}_W = (D_{L_V}(pt) \xrightarrow{\phi^*} D_{P_W \times L_V}(\mathcal{O}/U_V) \xrightarrow{\pi^*} D_{P_W}(\mathcal{O}/P_V) \xrightarrow{\alpha_i} D_{P_W}(pt) \xrightarrow{\text{Res}_{L_W,P_W}} D_{L_W}(pt) ) \]

\[
\simeq (D_{L_V}(pt) \xrightarrow{\phi^*} D_{L_W \times L_V}(\mathcal{O}/U_V) \xrightarrow{\pi^*} D_{L_W}(\mathcal{O}/P_V) \xrightarrow{\alpha_i} D_{L_W}(pt) ).
\]

In the particular case \( k = 1 \) we have \( \mathcal{O} = L_V = P_V = G_V, U_V = 1 \), and the comultiplication operation is isomorphic to

\[
\simeq (D_{G_V}(pt) \xrightarrow{\text{Res}_{P_W,G_V}} D_{P_W}(pt) \xrightarrow{\text{Res}_{L_W,P_W}} D_{L_W}(pt) )
\]
by Proposition 3.4.2

3.8. The coassociativity isomorphism

Assume that besides decomposition (3.4.2) of $V$ into direct sum of $W^j$ we have also decompositions

$$W^j : W^j = \bigoplus_{m=1}^{m_j} W^j_m.$$ 

We can produce out of these the decomposition

$$U : V = \bigoplus_{j,m} W^j_m;$$

where the lexicographic order of summands is used $W^1_1, \ldots, W^1_{m_1}, W^2_1, \ldots, W^2_{m_2}, \ldots, W^l_1, \ldots, W^l_{m_l}$.

Coassociativity isomorphism $\text{coass} : \psi^V_{\mathcal{W}} \circ \mathcal{X}^V_{\mathcal{W}} \rightarrow \mathcal{X}^V_{\mathcal{W}}$ between functors

$$\psi^V_{\mathcal{W}} = \left( pt \xrightarrow{L_V} \prod_{G^V} \xrightarrow{\psi^V_{\mathcal{W}}} \prod_{G^V} \xrightarrow{\text{coass}} \prod_{G^V} \xrightarrow{L_V} pt \right)$$

Figure 5. Second isomorphism

$$\simeq \left( \text{coass} \left( D_{G^V}(pt) \xrightarrow{\text{Res}_{L^W,G^V}} D_{L^W}(pt) \right) \right)$$
and

\[
(\tilde{D}_{c}(pt), \tilde{D}_{c}(pt), \tilde{D}_{c}(pt), \tilde{D}_{c}(pt))
\]

is given by the diagram
3.9. The coassociativity equation

Assume that we have the following decompositions of finite-dimensional C-vector spaces:

\[ \mathcal{V} : \quad V = \oplus_i V^i, \quad \mathcal{W} : \quad V = \oplus_j W^j, \]
\[ \mathcal{W}^j : \quad W^j = \oplus_m W^j_m, \quad \mathcal{W}^j_m : \quad W^j_m = \oplus_p W^j_{m,p}. \]
These decompositions imply also the following decompositions:

\[
\mathcal{U} : V = \bigoplus_{j,m} W^j_m, \quad \mathcal{Y}^j : W^j = \bigoplus_{m,p} W^j_{m,p}, \quad \mathcal{X} : V = \bigoplus_{j,m,p} W^j_{m,p}.
\]

Let \( \mathcal{O} \subset G_V \) be a left \( P_W \)-invariant and right \( P_Y \)-invariant subset. Coassociativity isomorphisms give two isomorphisms between the composite operation and the single operation as in the diagram.
Figure 11. Final isomorphisms
3.9.1. Proposition (The coassociativity equation). Diagram (3.9.1) is commutative.

Proof. The graphically expressed equation means equality of the following two isomorphisms:

\[
\begin{array}{ccccccccc}
\phi^* & \rightarrow & \phi^* & \rightarrow & \phi^* & \rightarrow & \phi^* & \rightarrow & \phi^* \\
L_V & \rightarrow & L_V & \rightarrow & L_V & \rightarrow & L_V & \rightarrow & L_V \\
\end{array}
\]

Four triangular prisms, which follow from Proposition [B.1.1] and Proposition [B.2.2], imply that the above isomorphisms are equal. \qed

3.10. The coherence isomorphism

Assume given a vector space decomposition

\[ V = \bigoplus_{m=1}^{k} \bigoplus_{l=1}^{l} V_{r}^{m}. \]

We shall denote it \( \mathcal{Y} \) if the summands are ordered as follows

\[ V_1^1, V_2^1, \ldots, V_1^l, V_2^1, V_2^2, \ldots, V_1^1, V_2^1, \ldots, V_1^l. \]

The same decomposition with order of summands

\[ V_1^1, V_2^2, \ldots, V_1^l, V_2^1, V_2^1, \ldots, V_1^1, V_2^2, \ldots, V_1^l. \]

will be denoted \( \mathcal{X} \). We also have decompositions

\[ \mathcal{V}^{m} : V^{m} = \bigoplus_{r=1}^{l} V_{r}^{m} \]

\[ \mathcal{W}_{r} : W_{r} = \bigoplus_{m=1}^{k} V_{r}^{m} \]

which produce filtrations \( V_{r}^{m} \) of \( V^{m} \), \( V_{r}^{(m)} \) of \( V \), \( W_{r}^{(m)} \) of \( W \), and \( W_{(r)} \) of \( V \). We associate with these filtrations parabolic groups \( P_{V_{r}^{m}} \subset G_{V_{r}^{m}} \), \( P_{V} \subset G_{V} \), \( P_{W_{r}} \subset G_{W_{r}} \), and \( P_{W} \subset G_{W} \).

Notations. Let \( A, B \) be vector spaces. We denote the braiding functor as

\[ \sigma = \left( D_{G_A \times G_B}(pt) \xrightarrow{\tau} D_{G_A \times G_B}(pt) \xrightarrow{\sigma^*} D_{G_B \times G_A}(pt) \right), \]

where \( \sigma \) is the permutation isomorphism of groups and modules and the functor \( \tau \) is the shift \( \tau(L) = L[-2 \dim A \dim B] \).
We claim that for any collection of indices and for any collection of bi-invariant locally closed subsets \((\mathcal{O}'_1, \ldots, \mathcal{O}'_k, \mathcal{O}''_1, \ldots, \mathcal{O}''_l)\), which may occur in the following diagram, there exists a bi-invariant locally closed subset \(\mathcal{O}\) and a coherence isomorphism explicitly described below. Here \(\sigma_{k,l} = (s_{k,l})_{+}^{-}\) is the braid, corresponding to the permutation \(s_{k,l}\) of the set \(\{1, 2, \ldots, kl\}\),

\[
s_{k,l}(1 + t + nl) = 1 + n + tk \quad \text{for} \quad 0 \leq t < l, 0 \leq n < k, \quad (3.10.1)
\]

under the standard splitting \(S_{kl} \to B_{kl}\), which maps the elementary transpositions to the generators of the braid group. The subset \(\mathcal{O}\) is computed as follows.

Bi-invariance of initial parametrising subsets means that

\[
G_{V^m} \supset \mathcal{O}'_m \in P_{V^m} - \text{sets}- P_{Y^m},
\]

\[
G_{W_r} \supset \mathcal{O}''_r \in P_{Z_r} - \text{sets}- P_{V^-_r},
\]

where \(Y^m\) is the incoming filtration of \(V^m\), and \(Z_r\) is the outgoing filtration of \(W_r\). The subsets

\[
P_{V'} \supset \mathcal{O} \overset{\text{def}}{=} U_Y \cdot \prod_m \mathcal{O}'_m = \prod_m \mathcal{O}'_m \cdot U_Y \in P_{V^-} - \text{sets}- P_{Y'},
\]

\[
P_{V'} \supset \mathcal{O} \overset{\text{def}}{=} U_W \cdot \prod_r \mathcal{O}''_r = \prod_r \mathcal{O}''_r \cdot U_W \in P_{Z^-} - \text{sets}- P_{V'^-}
\]

are locally closed. The subset

\[
G_{V} \supset \mathcal{O} \overset{\text{def}}{=} \mathcal{O} \cdot \overline{\mathcal{O}} \in P_{Z} - \text{sets}- P_{Y'}, \quad \mathcal{O} = \mathcal{O} P_{W} \times P_{V'} \overline{\mathcal{O}} = \mathcal{O} \times P_{W \cap P_{V'}} \overline{\mathcal{O}},
\]

is also locally closed. Indeed, \(\mathcal{O} \times \overline{\mathcal{O}}\) is locally closed in \(P_{W} \times P_{V'}\), hence, \(\mathcal{O} \times P_{W \cap P_{V'}} \overline{\mathcal{O}}\) is locally closed in \(P_{W} \times P_{W \cap P_{V'}} P_{V'} \simeq P_{W} \cdot P_{V'}\). The subspace \(C = P_{W} \cdot P_{V'}\) of \(G_{V}\) is locally closed, since it is an orbit of \(P_{W} \times P_{V'}\) in \(G_{V}\), and the number of such orbits is finite. Indeed, \(C\) is embedded in its closure as

\[
C = \overline{\mathcal{O}} - \bigcup \{ \overline{\mathcal{O}}_a \mid \mathcal{O}_a \subset \overline{\mathcal{O}} \text{ is an orbit of } P_{W} \times P_{V'}, \mathcal{O}_a \neq C\}
\]

due to dimension considerations.

Both associativity isomorphism and coassociativity isomorphism are particular cases of the general coherence isomorphism.

In the particular case, when the upper row of operations consists of comultiplications (operations with single input), and the lower row of operations consists of multiplications (operations with single output), we have necessarily \(\mathcal{O}'_m = G_{V^m}\) and \(\mathcal{O}''_r = G_{W_r}\). In such cases we omit the parametrising
set, since it is unique. Graphical notation here is the following:

\[
\begin{array}{c}
V^1 \to V^2 \to V^k \\
\downarrow \quad \quad \quad \downarrow \\
W_1 \to W_2 \to W_l
\end{array}
\]

By general formulas we find here \( O = P_Y, \quad \overline{O} = P_W, \) hence, \( O = P_Y P_W. \)

The general coherence isomorphism is built as the composition

\[
\text{Coher} : \quad \overline{O} = O = \text{assoc} \to \text{coass}.
\]

The three components of the coherence isomorphism are defined next.

\[
\begin{array}{c}
\prod_{G_{Y^m}} P_{Y^m} \to \prod_{P_{Y^m}} O'_{m} \to \prod_{P_{Y^m}} O'_{m}/P_{Y^m} \to \prod_{P_{Y^m}} P_{Y^m} \\
\downarrow \phi^* \downarrow \pi^* \downarrow \alpha_1 \\
\end{array}
\]

\[
\begin{array}{c}
P_Z \times P_Y \\
\downarrow \pi_* \\
\end{array}
\]

\[
\begin{array}{c}
\overline{O} = O = \text{assoc} \to \text{coass} \\
\downarrow \beta \\
\end{array}
\]

\[
\begin{array}{c}
\prod_{P_Z} P_U \\
\downarrow \alpha_1 \\
\end{array}
\]

\[
\begin{array}{c}
\text{coass} : \quad \left( P^t \prod_{r} P_{Z^r} \right) \to \left( P^t \prod_{n,r} G_{Z^r}^n \right) \sim \left( P^t \prod_{n,r} G_{Z^r}^n \right).
\end{array}
\]
Figure 12. Coherence isomorphism coher

Coherence isomorphism in Figure 12 is composed of several isomorphisms. Isomorphism (1) exists by Proposition 1.2. Isomorphism (2) follows from the fact that \( q_* \) is an equivalence.

Isomorphism \( T^{-2A} \rightarrow \rho \rho^* \) marked by (3) is obtained from a sequence of affine linear bundles

\[
U_W/(U_W \cap P_Y) \rightarrow \ldots \rightarrow U_W/[U_W^{(k+1)}(U_W \cap P_Y)] \rightarrow U_W/[U_W^{(k)}(U_W \cap P_Y)] \rightarrow \ldots \rightarrow pt,
\]

where \( U = U^{(1)} \supset U^{(2)} \supset \ldots \supset U^{(m)} = 1 \) is the lower central series of \( U \). The group \( \prod P_{W_r} \) acts by conjugation. Pull-back of this sequence along \( \prod \mathcal{O}_r'' \times F_{W_r} | \prod P_{Z_r} \times P_{W_r} \rightarrow pt | \prod P_{W_r} \) gives a sequence of affine linear \( \prod P_{Z_r} \times P_{W_r} \)-bundles

\[
\prod \mathcal{O}_r'' \times U_W/(U_W \cap P_Y) \rightarrow \ldots \rightarrow \prod \mathcal{O}_r'' \times U_W/[U_W^{(k+1)}(U_W \cap P_Y)] \rightarrow \prod \mathcal{O}_r'' \times U_W/[U_W^{(k)}(U_W \cap P_Y)] \rightarrow \ldots \rightarrow \prod \mathcal{O}_r''.
\]
Multiplication map $\prod \mathcal{O}_r'' \times U_W \to \mathcal{O}$ is an isomorphism of $\prod P_{Z_r} \times P_{W_t}$-spaces, hence, we may replace the first space with the other one. The composition of the above maps equals $\mathcal{O} / (U_W \cap P_V) \to \prod_r \mathcal{O}_r''$. The type (3) isomorphism for $\rho$ is a composition of isomorphisms, constructed for each affine bundle of the sequence in Corollary 3.3.2.

Isomorphism (4) is obtained from a commutative square of equivariant maps. Isomorphism (5) is obtained from a base change isomorphism by inverting $q^*$ and $\pi^*$ and replacing them with their quasi-inverses $q_*$ and $\pi_*$.

We use the facts

$$U_W = 1 + \oplus_{m,n;r \geq s} \text{Hom}(V^m_r, V^n_s),$$

$$U_W \cap P_V = 1 + \oplus_{m \geq n; r > s} \text{Hom}(V^m_r, V^n_s),$$

$$U_W / (U_W \cap P_V) \cong \oplus_{m < n; r > s} \text{Hom}(V^m_r, V^n_s),$$

The dimension of fiber bundle $\rho$

$$A \overset{\text{def}}{=} \dim C U_W / (U_W \cap P_V) = \sum_{m < n; r > s} \dim V^m_r \cdot \dim V^n_s$$

multiplied by $-2$ equals to the total shift, obtained from all braidings in the source functor of coherence isomorphism Coher.

### 3.11. Distinguished triangles

Consider left $P_W$-invariant and right $P_V$-invariant subsets $F \subset Y \subset G_V$, such that $F$ is closed in $Y$. Denote $S = Y - F$ and consider inclusions

$$F / U_V \overset{i_U}{\longrightarrow} Y / U_V \overset{j_U}{\longleftarrow} S / U_V,$$

$$F / P_V \overset{i_P}{\longrightarrow} Y / P_V \overset{j_P}{\longleftarrow} S / P_V.$$

Denote $\phi^X, \pi^X, \alpha^X$ maps [3.4.3] for $\mathcal{O} = X \in \{Y, F, S\}$. Let $K \in D_{L_V}(pt)$, $K_U = \phi^Y \cdot K \in D_{P_W \times L_V}(Y / U_V)$, $K_P = \pi^Y \cdot K_U \in D_{P_W}(Y / P_V)$ (see the middle row in the following diagram).

$$D_{L_V}(pt) \overset{\phi^{S_\alpha}}{\longrightarrow} D_{P_W \times L_V}(S / U_V) \overset{\pi^S}{\longrightarrow} D_{P_W}(S / P_V) \overset{\alpha^S}{\longrightarrow} D_{P_W}(pt)$$

$$D_{L_V}(pt) \overset{\phi^{Y_\alpha}}{\longrightarrow} D_{P_W \times L_V}(Y / U_V) \overset{\pi^Y}{\longrightarrow} D_{P_W}(Y / P_V) \overset{\alpha^Y}{\longrightarrow} D_{P_W}(pt)$$

$$D_{L_V}(pt) \overset{\phi^{F_\alpha}}{\longrightarrow} D_{P_W \times L_V}(F / U_V) \overset{\pi^F}{\longrightarrow} D_{P_W}(F / P_V) \overset{\alpha^F}{\longrightarrow} D_{P_W}(pt)$$

Applying $\alpha^Y$ to a standard triangle for $K_P$ we get a distinguished triangle

$$\alpha^Y \cdot j_P, j^*_P K_P \to \alpha^Y \cdot K_P \to \alpha^Y \cdot i_P, i^*_P K_P \to$$

The above diagram shows that it is isomorphic to certain following sequences:

$$\alpha^S \cdot \phi^S \cdot j_U, j^*_U K_U \to \alpha^Y \cdot \pi^Y \cdot K_U \to \alpha^F \cdot \pi^F \cdot i_U, i^*_U K_U \to$$

$$\alpha^S \cdot \pi^S \cdot \phi^S \cdot K \to \alpha^Y \cdot \pi^Y \cdot \phi^Y \cdot K \to \alpha^F \cdot \pi^F \cdot \phi^F \cdot K \to$$

This is the triangle

$$\Psi^{S, Y} \cdot K \to \Psi^{Y, Y} \cdot K \to \Psi^{F, Y} \cdot K \to$$
from which the structure triangle

\[
\begin{array}{c}
X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} \mathbb{R}
\end{array}
\]

is obtained by application of the exact functor \( \mathfrak{F}_W \).

This triangle replaces non-existing isomorphism \( X \cong Y \cong Z \) (this would be too much to ask for).

3.11.1. Proposition. For any pair of closed embeddings \( F \subset Z \subset Y \) set \( S = Y - F \), \( Q = Z - F \), \( R = Y - Z \). The following diagram made with distinguished triangles (3.11.1) is an octahedron.

That is, the four triangles marked “=” commute, as well as two squares

\[
\begin{array}{ccc}
X^R \xrightarrow{a} X^S \xrightarrow{b} X^F & = & X^R \xrightarrow{a} X^S \xrightarrow{b} X^F \\
\downarrow c \xi \quad \quad \quad \downarrow c \xi & = & \downarrow c \xi \quad \quad \quad \downarrow c \xi
\end{array}
\]

The proof follows from Corollary C.2.2.

Appendix A. Technical results

A.1. An equivalence of equivariant derived categories

Let \( P \) be a linear complex algebraic group, let \( U, L \) be its subgroups such that \( U \) is normal in \( P \) and the manifold \( U \) is an affine space. Assume that \( P \) is a semidirect product of \( U \) and \( L \), that is, the multiplication map \( \cdot : U \times L \to P \) is an isomorphism of manifolds. Denote by \( \iota : L \longrightarrow P \) and \( \kappa : P \twoheadrightarrow L \) the natural inclusion and projection. Let \( Y \) be a \( P \)-variety, let \( E \) be an \( L \)-variety, let \( i : E \to Y \) be a \( \iota \)-map and let \( p : Y \to E \) be a \( \kappa \)-map identifying \( E \) with the quotient \( U \setminus Y \) such that \( p \circ i = \text{id}_E \). Consider

\[
W = P \times_L E = (P \times E)/L, \quad (p, e) \cdot (l, e) = (pl, l^{-1}e).
\]

The variety \( W \) is a \( P \)-variety via left translations on \( P \). As a \( P \)-variety \( W \) is isomorphic to \( U \times E \), where \( P \) acts by

\[
(u, l) \cdot (v, e) = (u \cdot (lv^{-1}), l \cdot e).
\]

There is a surjective map of \( P \)-varieties

\[
\pi : W \twoheadrightarrow U \times E \to Y, \quad (u, e) \mapsto u \cdot i(e).
\]

Assume furthermore that \( \pi \) is a locally trivial fibration with affine fibers. Then \( \pi : W \to Y \) is \( \infty \)-acyclic in the sense of [2].

A.1.1. Lemma. In the above assumptions the functor \( \pi^* : D^b_P(Y) \to D^b_P(W) \) is fully faithful.
Proof. Choose an interval \( I = [a, b] \subset \mathbb{Z} \) and a number \( m \geq |I| \). Let \( M = M_m \) be an \( m \)-acyclic free \( P \)-manifold. Then \( R = M \times W \xrightarrow{pr} W \) is an \( m \)-acyclic \( P \)-resolution of \( W \) and \( r : R \xrightarrow{pr} W \xrightarrow{\pi} Y \) is such of \( Y \). We know that

\[
\pi^* : D^I(Y) \to D^I(W)
\]

is full and faithful, see Proposition 1.9.2 of \cite{2}. Therefore, functor from Section 2.1.6 of \cite{2}

\[
\pi^* : D^L_p(Y, R) \to D^L_p(W, R),
\]

\( (\alpha_Y, \overline{\alpha}) \) is also full and faithful, where \( q : R \to \overline{\overline{I}}, \ r : R \xrightarrow{pr} W \xrightarrow{\pi} Y \). Indeed, a morphism of \( D^L_p(Y, R) \) is a pair \((\alpha_Y : F_Y \to H_Y, \overline{\alpha} : \overline{F} \to \overline{H})\), which makes commutative the right square in the following diagram

\[
\begin{array}{ccc}
pr^* \pi^* F_Y & \sim & r^* F_Y \\
\downarrow & & \downarrow \\
pr^* \pi^* H_Y & \sim & r^* H_Y
\end{array}
\]

\( \beta \). The morphism \((\alpha_Y, \overline{\alpha})\) is sent by \( \pi^* \) to a morphism \((\pi^* \alpha_Y, \overline{\alpha})\). This shows faithfulness of functor \[A.1.1].\) This functor is also full because any morphism \((\pi^* F_Y, \overline{F}, \phi) \to (\pi^* H_Y, \overline{H}, \eta)\) has to be of the form \((\pi^* \alpha_Y, \overline{\alpha})\) for some \( \alpha_Y \). The commutativity of the exterior of diagram \[A.1.2\] implies that the pair \((\alpha_Y, \overline{\alpha})\) is a morphism.

Assign \( D^L_p(Y) = D^L_p(Y, R), D^L_p(W) = D^L_p(W, R) \) and vary \( R = M_m \times W \), increasing \( m \). Since \( D^L_p(Y) \) (resp. \( D^L_p(W) \)) is a colimit of a system of fully faithful functors \( D^L_p(Y) \to D^L_p(Y), J \subset I \), (by Definition 2.2.4 of \cite{2}), we see that the colimits are also fully and faithfully embedded.

\[A.1.2, \text{Proposition.} \] The functors

\[
\begin{align*}
\pi^* : & D^{b,c}_p(Y) \to D^{b,c}_p(W), \\
p^* : & D^{b,c}_L(E) \to D^{b,c}_L(Y), \\
i^* : & D^{b,c}_L(Y) \to D^{b,c}_L(E)
\end{align*}
\]

are equivalences, and the last two are quasi-inverse to each other.

Proof. Notice that \( W \simeq U \times E \) is a free \( U \)-variety. Denote by \( \phi : P \longrightarrow L = P/U \) the canonical projection. The \( \phi \)-map \( s : W \xrightarrow{\pi} Y \xrightarrow{p} E \) is the projection \( W \to U \setminus W = E \).

Therefore, the functor \( s^* : D^{b,c}_L(E) \to D^{b,c}_P(W) \) is an equivalence by Theorem 2.6.2 of \cite{2}. It decomposes as

\[
s^* = (D^{b,c}_L(E) \xrightarrow{p^*} D^{b,c}_P(Y) \xrightarrow{\pi^*} D^{b,c}_P(W)).
\]

We conclude that \( \pi^* \) is essentially surjective on objects, hence, an equivalence by Lemma \[A.1.1\]. Therefore, \( p^* \) is also an equivalence. Since

\[
\begin{align*}
(D^{b,c}_L(E) \xrightarrow{p^*} D^{b,c}_P(Y) \xrightarrow{i^*} D^{b,c}_L(E)) & \simeq \text{Id}
\end{align*}
\]

\( i^* \) is a quasi-inverse of \( p^* \) (and also an equivalence).
Appendix B. Properties of standard isomorphisms

B.1. Inverse image isomorphisms

B.1.1. Proposition. Isomorphisms $\iota$ from (2.2.1) satisfy the cocycle condition:

\[
\begin{align*}
D^b_{K}(Z) & \xrightarrow{g^*} D^b_{H}(Y) \\
D^b_{L}(W) & \xrightarrow{(hg)^*} D^b_{G}(X)
\end{align*}
\]

\[
\begin{align*}
D^b_{K}(Z) & \xrightarrow{g^*} D^b_{H}(Y) \\
D^b_{L}(W) & \xrightarrow{(hg)^*} D^b_{G}(X)
\end{align*}
\]

(B.1.1)

Proof. Compose this equation with the functor

\[
D^b_{G}(X) \to D^b_{c}(\text{Res}(f, g, h) \to \mathcal{T})
\]

and substitute the definition of $\iota$. Its interpretation might be the following: $\iota$ and $i$ are gauge equivalent via gauge transformation $\eta$. The required equation reduces to equality of the following two isomorphisms:

\[
\begin{align*}
D^b_{c}(\text{Res}(h) \to \mathcal{T}) & \xrightarrow{g^*} D^b_{c}(\text{Res}(g, h) \to \mathcal{T}) \\
D^b_{c}(\text{Res}(hg) \to \mathcal{T}) & \xrightarrow{(gf)^*} D^b_{c}(\text{Res}(f, hg) \to \mathcal{T})
\end{align*}
\]

This equation follows from (B.1.1) for $D^b_{c}(T)$, $T \in \mathcal{T}$, which in turn follows from the same equation for $D^b(T)$ and, in turn, for the category of sheaves. □

B.2. Properties of base change isomorphisms

B.2.1. Lemma. Whenever squares 1 and 2 below are pull-back squares of algebraic varieties

\[
\begin{align*}
X & \xrightarrow{\kappa} Y \xrightarrow{\lambda} Z \\
\downarrow f & \quad \downarrow g & \quad \downarrow h \\
U & \xrightarrow{\nu} V \xrightarrow{\mu} W
\end{align*}
\]

(B.2.1)
the following equation holds

\[
\nu^* \mu^* (R\mu) \xrightarrow{\nu^* \beta_2} \nu^* (Rg_f) \lambda^* \xrightarrow{\beta_1} (Rf_1) \kappa^* \lambda^* \\
\downarrow_{i_{\mu, \nu}} \quad = \quad \downarrow_{Rf_1 \kappa_{\lambda, \kappa}} \quad .
\]

(B.2.2)

The statement can be reformulated as an equation

\[
\begin{array}{ccc}
D^{b,c} (X) & \xleftarrow{(\lambda\kappa)^*} & D^{b,c} (Z) \\
\downarrow Rf_1 & & \downarrow Rh_1 \\
D^{b,c} (U) & \xleftarrow{(\mu\nu)^*} & D^{b,c} (W) \\
& \downarrow \epsilon_{\ast} & \downarrow \nu^* \\
D^{b,c} (V) & & \\
\end{array}
\]

\[
\begin{array}{ccc}
D^{b,c} (X) & \xleftarrow{(\lambda\kappa)^*} & D^{b,c} (Z) \\
\downarrow Rf_1 & & \downarrow Rh_1 \\
D^{b,c} (Y) & \xleftarrow{\mu^*} & D^{b,c} (W) \\
& \downarrow \lambda^* & \downarrow \nu^* \\
D^{b,c} (V) & & \\
\end{array}
\]

Proof. First we prove analogous to (B.2.2) equation for sheaves. We use the definition of the base change isomorphism of sheaves, corresponding to the pull-back square

\[
\begin{array}{ccc}
& e & \\
h & \downarrow \eta & \downarrow i \\
& k & \\
\end{array}
\]

It is defined as the composition

\[
\beta : k^* l_1 \xrightarrow{\eta} k^* h_1 e_* e^* \xleftarrow{\epsilon} k^* k_* h_1 e^* \xrightarrow{\epsilon} h_1 e^*
\]

or as

\[
\begin{array}{cccc}
& e^* & \\
h_1 & \downarrow \eta & \downarrow e_* \\
& k_* & \downarrow \kappa_* \\
\end{array}
\]

\[
\beta = \quad ,
\]
where $\epsilon$ and $\eta$ are adjunction maps. Equation (B.2.2) for sheaves which we want to prove becomes an equation between the following two isomorphisms

\[
\begin{array}{ccc}
Sh(X) & \xleftarrow{(\lambda\kappa)^*} & Sh(Z) \\
\downarrow_{f_1} & & \downarrow_{\eta} \\
Sh(U) & \xleftarrow{(\mu\nu)^*} & Sh(Z) \\
\downarrow_{\zeta} & & \downarrow_{h_1} \\
Sh(U) & \xleftarrow{\lambda\kappa} & Sh(W) \\
\downarrow_{\zeta} & & \\
Sh(V) & \xleftarrow{\mu\kappa} & Sh(W) \\
\end{array}
\]

\[\text{(B.2.3)}\]

The parallelogram in diagram (B.2.3) equals to the parallelogram in the right diagram above since the embedding $h_!(\lambda\kappa)_* \supseteq (\mu\nu)_* f_!$ can be presented as a composition of two embeddings

\[h_!(\lambda\kappa)_* \supseteq \mu_* g_! k_* \supseteq \mu_* \nu_* f_* .\]

Indeed, all these functors are subfunctors of $h_! \lambda_! \kappa_* = \mu_* \nu_* f_*$. From equation (B.2.2) for sheaves we deduce it for derived functors.

**B.2.2. Proposition.** Let $f : X \to U$, $g : Y \to V$ and $h : Z \to W$ be $G$-equivariant, $H$-equivariant and $K$-equivariant maps of algebraic varieties. Let $X \xrightarrow{\kappa} Y \xrightarrow{\lambda} Z$ and $U \xrightarrow{\nu} V \xrightarrow{\mu} W$ be $G \xrightarrow{\phi} H \xrightarrow{\psi} K$-equivariant. Assume that squares 1 and 2 of commutative diagram (B.2.1) are pull-back squares. Then the following equation holds

\[
\begin{array}{cc}
\nu^* \mu^* h_! \xrightarrow{\nu^* \beta_2} & \nu^* g_! \lambda^* \\
\downarrow_{i_{\mu,K}} & \downarrow_{f_! \lambda_! \kappa_*} \\
(\mu\nu)_! h_! & \xrightarrow{\beta} f_!(\lambda\kappa)_! \\
\end{array}
\]
The statement can be reformulated as an equation

Proof. The statement reduces to equality between the following two isomorphisms:

Substituting definitions of isomorphisms $\beta$ from Section 2.2.7 and similar, we see that the equation between the above isomorphisms follows from Lemma B.2.1, that is, from the non-equivariant version.

Appendix C. Distinguished triangles and octahedrons

C.1. Standard distinguished triangles.

Let us consider a closed $G$-invariant subspace $i : X \subset \rightarrow Y$ with the complement $S = Y - X$, $j : S \subset \rightarrow Y$. Let $P \rightarrow Y$ be a resolution of $Y$. As explained in [2] resolutions of $X$ of the form $X \times_Y P \rightarrow X$ taken from the pull-back square
suffice to determine an object of $D^b_G(X)$ up to an isomorphism. Therefore, the functor $\tilde{i}i^*$ is canonically isomorphic to the functor

$$\tilde{i}i^*: D^b_G(Y) \to D^b_G(Y)$$

$$[P \mapsto K(P)] \mapsto [P \mapsto \tilde{i}i^*(K(P))]$$

Similarly for $j_!j^*$.

There is a distinguished triangle in $D^b_G(Y)$

$$\tilde{j}_!j^*K \xrightarrow{a} K \xrightarrow{b} \tilde{i}\tilde{i}^*K \xrightarrow{c}$$

given by a collection of standard distinguished triangles

$$\tilde{j}_!j^*(KP) \xrightarrow{akp} KP \xrightarrow{bkp} \tilde{i}\tilde{i}^*(KP) \xrightarrow{ckp}$$

for $P \in \mathrm{Res}(Y,G)$. The canonically isomorphic to it distinguished triangle in $D^b_G(Y)$ is denoted

$$j_!j^*K \xrightarrow{a} K \xrightarrow{b} \tilde{i}i^*K \xrightarrow{c}$$

C.2. Standard octahedron associated with a triple of spaces.

Suppose that $X \subset Z \subset Y$ are closed embeddings of stratified subspaces of some stratified space and $S = Y - X$, $Q = Z - X$, $R = Y - Z$. With a closed embedding $i$ and the complementary embedding $j$ is associated a standard triangle in $D(X)$

$$\tilde{j}_!j^*K \xrightarrow{a} K \xrightarrow{b} \tilde{i}i^*K \xrightarrow{c}$$

Denote by $i_{YZ}$, $i_{YX}$, $i_{ZX}$, $i_{SQ}$ the closed embeddings and $j_{RY}$, $j_{QZ}$, $j_{SY}$, $j_{RY}$ the open ones. Add corresponding indices to the triangle maps $a$, $b$, $c$.

C.2.1. Proposition. The obvious identification of vertices makes the following diagram in $D(Y)$ into an octahedron
That is, the following “triangles” and “squares” commute.

\[
\begin{align*}
T_{jSY} & jRS jRS jSY K \xrightarrow{\sim} T_{jRY} jRY K \\
& \Downarrow_{jSY =^{QR}} \downarrow_{j = SY} \downarrow_{jRS = SY} \Downarrow_{\sim} \\
& jSY jSQ jSQ jSY K \xrightarrow{\sim} i_{YZ} jQZ jQZ i_{YZ} K \\
& \downarrow_{jSY =^{SQ}} \downarrow_{j = SY} \\
& j_{SY} jRS jRS jSY K \xrightarrow{\sim} j_{SY} jSY K \\
\end{align*}
\]
Proof. This octahedron in a derived category of sheaves follows from the commutative diagram with exact lines in the abelian category of sheaves.

Here \( k_{QY} : Q \rightarrow Y \) denotes the embedding. The ambiguous north-north-east line is isomorphic to the exact sequence

\[
0 \rightarrow j_{SY}!j_{RS}^*j_{SY}^*F \overset{j_{SY}!a_{RS}^{RS}}{\longrightarrow} j_{SY}!j_{SY}^*F \overset{j_{SY}!b_{SQ}^{SQ}}{\longrightarrow} j_{SY}!i_{SQ}^*i_{SY}^*F \rightarrow 0.
\]

The ambiguous south-south-east line is isomorphic to the exact sequence

\[
0 \rightarrow i_{YZ}!i_{QZ}^*i_{YZ}^*F \overset{i_{YZ}!a_{QZ}^{QZ}}{\longrightarrow} i_{YZ}!i_{YZ}^*F \overset{i_{YZ}!b_{ZX}^{ZX}}{\longrightarrow} i_{YZ}!i_{ZX}^*i_{YZ}^*F \rightarrow 0.
\]

This shows exactness of the lines of the diagram. It remains to show commutativity of the two triangles and the quadrangle.

Together with a hidden isomorphism the left triangle is square 3:

\[
\begin{array}{ccc}
j_{SY}!j_{RS}^*j_{SY}^*F & \overset{j_{SY}!a_{RS}^{RS}}{\longrightarrow} & j_{SY}!j_{SY}^*F \\
\downarrow & & \downarrow a_{SY}^{SY} \\
j_{RY}!j_{RY}^*F & \overset{j_{RY}!a_{RY}^{RY}}{\longrightarrow} & F
\end{array}
\]

Recall that for any sheaf \( F \) on \( Y = S \cup X \) we have \( j_{SY}!j_{SY}^*F = F_S = (F|_S)^Y \) and \( i_{XY}!i_{XY}^*F = F_X = (F|_X)^Y \), the sheaves extended by 0 from \( S \) (resp. from \( X \)). The embedding \( a_{SY}^{SY} : j_{SY}!j_{SY}^*F \rightarrow F \) is the embedding of éspaces étalés \( F_S \hookrightarrow F \). Therefore, (C.2) states simply that the triangle of embeddings

\[
\begin{array}{ccc}
F_S & \hookrightarrow & F \\
\downarrow & & \downarrow \\
F_R & \hookrightarrow & F
\end{array}
\]
is commutative, which is obvious.

Similarly, the right triangle of diagram (C.2.1) is square 4.

Since the surjection \( b^X_F : \mathcal{F} \to i_{YZ}^*i_Y^*X \mathcal{F} \) is interpreted as the projection \( \mathcal{F} \to \mathcal{F}_X \) of \( \text{'espaces \ ' \ 'étalés} \), the commutativity of this square reduces to commutativity of

\[
\begin{array}{c}
\mathcal{F} \\
\downarrow \\
\mathcal{F}_Z \\
\downarrow \\
\mathcal{F}_X
\end{array}
\]

which is obvious.

The quadrangle in diagram (C.2.1) is hexagon 6.

Due to identifications

\[
\begin{align*}
\left( (\mathcal{F}|_Q)_Y \right)^S = (\mathcal{F}|_Q)^Y &= \mathcal{F}_Q = q_{Y!}k^*_Q \mathcal{F}, \\
\left( (\mathcal{F}|_Q)_Z \right)^X = (\mathcal{F}|_Q)^Y &= \mathcal{F}_Q = q_{Y!}k^*_Q \mathcal{F}
\end{align*}
\]

this hexagon is in fact the square

\[
\begin{array}{c}
\mathcal{F}_S \\
\downarrow \\
\mathcal{F}_Q \\
\downarrow \\
\mathcal{F}_Z
\end{array}
\]

Considering mappings of stalks of these sheaves at a point of \( Y \) we deduce commutativity of this square. \( \square \)

C.2.2. Corollary. The octahedron of Proposition C.2.1 holds true in \( G \)-equivariant case.

Proof. The octahedron for closed \( G \)-subspaces \( X \subset Z \subset Y \) in \( D_G(Y) \) follows from that for closed subspaces \( X \times_Y P \subset Z \times_Y P \subset P \), where \( P \) is a \( G \)-resolution of \( Y \), see Section C.1. \( \square \)

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