Degree bounds for Gröbner bases of modules

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Abstract

Let $F$ be a non-negatively graded free module over a polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by $m$ basis elements. Let $M$ be a submodule of $F$ generated by elements with degrees bounded by $D$ and $\dim F/M = r$. We prove that if $M$ is graded, the degree of the reduced Gröbner basis of $M$ for any term order is bounded by $2^{\left[1/2 \left( (D^m - m + D)^{n-r} \right) \right]^{2r-1}}$. If $M$ is not graded, the bound is $2^{\left[1/2 \left( ((D^m - 1)(n-r) + 1)^{n-r} + D \right) \right]^{2r}}$. This is a generalization of Dubé (1990) and Mayr-Ritscher (2013)’s bounds for ideals in a polynomial ring.

Keywords: Gröbner bases, degree bound, cone decompositions, Hilbert functions

1. Introduction

Gröbner bases play an important role in computational commutative algebra and computational algebraic geometry. To analyze the complexity of Gröbner bases computation, it is essential to give an upper bound on degrees of elements in Gröbner bases. In 1990, Dubé used a purely combinatorial argument and gave the following doubly exponential upper bound to the Gröbner bases degrees of ideals for any term order. For earlier results, see Moller and Mora (2011).

Theorem 1 (Dubé (1990)). Let $I$ be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$ generated by polynomials with maximum degree $d$. Then for any monomial order $\prec_S$, the degree of the reduced Gröbner basis $G$ of $I$ is bounded by

$$\deg(G) \leq 2^{\left( \frac{d^2}{2} + d \right)^{2^{n-1}}}.$$

In 2013, Mayr and Ritscher incorporated the ideal dimension into Dubé’s construction and proved the following dimension-dependent bound. Based on Mayr-Meyer ideals, they also constructed a family of examples that showed any bound must be at least exponential in the codimension and doubly exponential in the dimension (see Mayr and Ritscher, 2013, §4).

Theorem 2 (Mayr and Ritscher (2013)). Let $\mathbb{K}$ be an infinite field and $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal of dimension $r$ generated by polynomials $F = \{f_1, \ldots, f_s\}$ of degrees $d_1 \geq \cdots \geq d_s$. Then for any monomial order $\prec_S$, the degree of the reduced Gröbner basis $G$ of $I$ is bounded by

$$\deg(G) \leq 2^{\left[ \frac{1}{2} \left( (d_1 \cdots d_{n-r})^{2^{n-r}} + d_1 \right) \right]^{2^r}}.$$
If \( f_1, \ldots, f_s \) are homogeneous, then
\[
\text{deg}(G) \leq 2 \left[ \frac{1}{2} (d_1 \cdots d_{n-r} + d_1) \right]^{2^{r-1}}.
\]

Despite the fact that Gröbner basis computations of modules are the most commonly used ones in computer algebra systems like Macaulay2, CoCoA, and Singular (just to name a few), surprisingly uniform bounds of the above kind are only known for ideals. Generalizing Dubé (1994) and Mayr and Ritscher (2013), we prove the following two bounds on the Gröbner basis degrees of graded and non-graded submodules of a free module, respectively.

The three theorems below are the main theorems of this paper. In the following, let \( F \) be a free module over \( K[x_1, \ldots, x_n] \) with basis elements \( e_1, \ldots, e_m \) so that \( \text{deg}(e_j) \geq 0 \) for all \( j \) and \( l = \max\{\text{deg}(e_j) : j = 1, \ldots, m\} \). If \( G \) is a Gröbner basis, let \( \text{deg}(G) \) denote the maximum degree of the elements in \( G \).

**Theorem 49** Let \( M \subset F \) be a graded submodule generated by homogeneous elements with maximum degree \( D \geq l \) and \( \dim_F/M = r \). Then the degree of the reduced Gröbner basis \( G \) of \( M \) for any monomial order on \( F \) is bounded by
\[
\text{deg}(G) \leq \begin{cases} 
Dmn - n + 1, & \text{if } r = 0 \\
2 \left[ \frac{1}{2} ((Dm)^{n-r}m + D) \right]^{2^{r-1}} & \text{if } r \geq 1
\end{cases}
\]

Mayr and Ritscher (2013) deduced their bound for inhomogeneous ideals by proving a lemma that says there exist a homogeneous regular sequence in the homogenization of \( I \) with degrees bounded by \( (d_1 \cdots d_{n-r})^2 \). This lemma is recently improved in Binaei et al. (2018) where this bound is sharpened to \( d_1 \cdots d_{n-r} \). Both of the proofs in Mayr and Ritscher (2013) and Binaei et al. (2018) used the effective Noether normalization. In this paper we sharpen the bound further to \( d_1 + \cdots + d_{n-r} - n + r + 1 \), by using generic initial ideals and Hashemi and Lazard’s Gröbner bases degree bound for zero-dimensional ideals w.r.t. the degree reverse lexicographical order (see Hashemi and Lazard (2005) or Theorem 51). In particular for inhomogeneous ideals, we use our new lemma to get the bound

**Theorem 53** Under the assumptions of Theorem 2, the degree of the reduced Gröbner basis \( G \) of \( I \) is bounded by
\[
\text{deg}(G) \leq 2 \left[ \frac{1}{2} ((d_1 + \cdots + d_{n-r} - n + r + 1)^{n-r}m + d_1) \right]^{2^r}.
\]

Notice that this is a significant improvement of Mayr and Ritscher’s bound. The bound for non-graded modules also follows from the lemma.

**Theorem 54** Let \( M \subset F \) be a submodule generated by elements of maximum degree \( D \geq l \) and \( \dim_F/M = r \). Then the degree of the reduced Gröbner basis \( G \) of \( M \) for any monomial order is bounded by
\[
\text{deg}(G) \leq 2 \left[ \frac{1}{2} ((Dm - 1)(n - r) + 1)^{n-r}m + D) \right]^{2^r}.
\]

The paper is organized as follows. We review basic notations and preliminaries in Section 2. Then in Section 3-5, we generalize Dubé’s constructions to modules. In Section 3, cone
decompositions (also called Stanley decompositions) are introduced to decompose a set of normal forms of a submodule \( M \) of \( F \). In Section 4, we use a module version of Dubé’s algorithm to construct a special cone decomposition of the set of normal forms of a module, so that the Gröbner basis degree can be bounded by the degree of the cone decomposition. In Section 5, the cone decomposition that we constructed before is refined into an exact cone decomposition whose degree can be bounded by a Macaulay constant.

We switch to Mayr and Ritscher’s approach in Section 6-8. Imitating their construction, we find a regular sequence of length equal to the dimension of \( F/M \) in the zeroth Fitting ideal of \( F/M \). Using this regular sequence, we perform two reductions in Section 6. We first reduce to a submodule \( IF \) of \( M \) where \( I \) is the ideal generated by the regular sequence, and then reduce to a monomial module generated by pure powers that share the same degrees as the regular sequence. In Section 7, we inductively construct an exact cone decomposition of the monomial module, bound its Macaulay constants, thereby deduce the bounds for graded modules. Finally in Section 8, we prove that for an ideal \( I \) there exist a homogeneous regular sequence in the homogenization of \( I \) with degrees bounded by \( d_1 + \cdots + d_{n+r} - n + r + 1 \). Using this lemma, we reduce the non-graded case to the graded case and deduce the bound for non-graded modules.

2. Preliminaries

2.1. Notation

The purpose of this subsection is to set up notations that will be used throughout the paper. For a more detailed introduction to Gröbner bases and other related topics, the reader can refer to Eisenbud (1995), Greuel and Pfister (2008), Kreuzer and Robbiano (2000), and Kreuzer and Robbiano (2005). We switch to Mayr and Ritscher’s approach in Section 6-8. Imitating their construction, we find a regular sequence of length equal to the dimension of \( F/M \) in the zeroth Fitting ideal of \( F/M \). Using this regular sequence, we perform two reductions in Section 6. We first reduce to a submodule \( IF \) of \( M \) where \( I \) is the ideal generated by the regular sequence, and then reduce to a monomial module generated by pure powers that share the same degrees as the regular sequence. In Section 7, we inductively construct an exact cone decomposition of the monomial module, bound its Macaulay constants, thereby deduce the bounds for graded modules. Finally in Section 8, we prove that for an ideal \( I \) there exist a homogeneous regular sequence in the homogenization of \( I \) with degrees bounded by \( d_1 + \cdots + d_{n+r} - n + r + 1 \). Using this lemma, we reduce the non-graded case to the graded case and deduce the bound for non-graded modules.

Let \( S = \mathbb{K}[x_1, \ldots, x_n] \) denote the ring of polynomials in the variables \( X = \{x_1, \ldots, x_n\} \). Let \( F = S e_1 \oplus \cdots \oplus S e_m \) denote a graded free \( S \)-module with \( l = \max\{\deg(e_j) : j = 1, \ldots, m\} \). Without loss of generality, we may assume that all \( e_j \) have nonnegative degrees and the one with smallest degree has degree 0.

Let \( \prec_F \) be a monomial order on \( F \), whenever there is no confusion we abbreviate \( \prec_F \) as \( \prec \). Let \( f \in F \), then the initial monomial of \( f \), denoted by \( \text{in}_\prec(f) \), is the greatest monomial among the monomials belonging to \( \text{supp}(f) \) with respect to \( \prec \). If \( M \) is a submodule of \( F \), then \( \text{in}_\prec(M) \) is the monomial submodule generated by \( \{\text{in}_\prec(f) : f \in M\} \). A subset \( G = \{g_1, \ldots, g_l\} \) in \( M \) is called a Gröbner basis of \( M \) if \( \text{in}_\prec(g_1), \ldots, \text{in}_\prec(g_l) \) generates \( \text{in}_\prec(M) \). Denote \( \text{in}_\prec(G) = \{\text{in}_\prec(g) : g \in G\} \).

Let \( M \) be a submodule of \( F \), fix a monomial order \( \prec_F \) on \( F \) and a Gröbner basis \( G \) of \( M \). For any \( f \in F \), let \( nf_G(f) \) denote the unique remainder (or normal form) of \( f \) with respect to \( G \). Collecting all such remainders of \( f \in F \), we denote

\[
N_M = \{nf_G(f) : f \in F\}.
\]

By Macaulay’s theorem (Eisenbud 1995, Theorem 15.3), we have

\[
N_M = \text{span}\{u \in \text{Mon}(F) : u \notin \text{in}_\prec(M)\} = \text{N}_{\text{in}_\prec(M)}.
\]

In particular \( N_M \) only depends on the monomial order \( \prec_F \).

Similarly if \( I \) is an ideal of \( S \), \( \prec_S \) is any monomial order on \( S \), and \( G \) is any Gröbner basis of \( I \), then we denote

\[
N_I = \{nf_G(p) : p \in S\}.
\]
Let $M$ be a submodule of $F$, we denote $\dim(F/M) = \dim(S/M :_S F)$ to be the Krull dimension of the $S$-module $F/M$.

If $T$ is a vector space over $\mathbb{K}$, let $\dim_\mathbb{K}(T)$ denote the vector space dimension of $T$.

2.2. Hilbert function

Let $T \subseteq F$ be a graded $\mathbb{K}$-vector space with graded components

$$T_z = \{ f \in T : f \text{ is homogeneous of degree } z \} \cup \{0\},$$

then the Hilbert function of $T$ is defined as

$$HF_T(z) = \dim_\mathbb{K}(T_z).$$

The Hilbert series of $T$ is defined as

$$HS_T(t) = \sum_{z \geq 0} HF_T(z) t^z.$$

Let $M$ be a submodule of $F$, then there exists a unique polynomial which is equal to $HF_{N/M}(z)$ for sufficiently large $z$. This polynomial is called the Hilbert polynomial of $N/M$ and will be denoted as $HP_{N/M}(z)$. The Hilbert regularity of $N/M$ is defined as $\min\{z_0 \in \mathbb{Z} : HF_{N/M}(z) = HP_{N/M}(z) \forall z \geq z_0\}$.

Notice that $F/M$ and $N/M$ have the same Hilbert function, hence $\dim(F/M) = \deg(HP_{N/M}) + 1$ (with the convention that $\deg(0) = -1$).

2.3. Regular sequence

Recall that a sequence $g_1, \ldots, g_t$ of elements in $S$ is called a regular sequence if

1. $g_k$ is a non-zerodivisor on $S/(g_1, \ldots, g_{k-1})$ for all $k = 1, \ldots, t$ and
2. $(g_1, \ldots, g_t) / nequal S$.

One of the many nice properties that homogeneous regular sequences have is that the submodules they generate have the same Hilbert functions if they have the same degrees.

**Lemma 3.** Let $J = (g_1, \ldots, g_t)$ be an ideal generated by a homogeneous regular sequence in $S$ with degrees $d_1, \ldots, d_t$. Fix an arbitrary monomial ordering on $F$, consider $J F \subset F$, then $F/JF$ or equivalently $N_J F$ has the Hilbert series

$$HS_{N_J F}(t) = \sum_{z \geq 0} HF_J(z) t^z.$$

The Hilbert regularity of $N_J F$ is $d_1 + \cdots + d_t + l - n + 1$.

**Proof.** See [Kreuzer and Robbiano (2005), Corollary 5.2.17].

It is a well-known fact that given a polynomial ideal of dimension $r$ over an infinite field, then we may assume $n - r$ generators of $I$ form a regular sequence. This will be one of the key constructions to achieve a dimension-dependent bound.
Lemma 4. Let \( K \) be an infinite field and \( I \subseteq S \) an ideal generated by homogeneous polynomials \( p_1, \ldots, p_t \) with degrees \( d_1 \geq \cdots \geq d_t \) such that \( \dim(S/I) \leq r \). Then there are a strictly decreasing sequence \( t \geq j_1 > \cdots > j_{n-r} \geq 1 \) and homogeneous \( a_{ij} \in S \) such that

\[
g_k = \sum_{i=j_k}^t a_{ij}p_i \quad \text{for } k = 1, \ldots, n-r
\]

form a homogeneous regular sequence, \( \dim(S/(p_{j_1}, \ldots, p_t)) = n-k \), and \( \deg(g_k) = d_{j_k} \).

Proof. See [Binaei et al. 2018, Proposition 4.17]. \( \square \)

Remark 5. For simplicity we will use the weaker version of the above lemma, that is we may assume the regular sequence \( g_1, \ldots, g_{n-r} \) have (the largest \( n - r \)) degrees \( d_{n-r}, \ldots, d_1 \). So one could potentially get a better bound in Theorem 48 by using the degrees \( d_{j_k} \) in Lemma 4.

Note that we can reduce to the weaker version since for each \( i \), there exists a regular element \( y_i \in S_1 \) on \( S/(g_1, \ldots, \hat{g}_i, \ldots, g_{n-r}) \), so we can replace \( g_i \) by \( g_i y_i^{j_k - \deg(g_i)} \).

2.4. Fitting ideal

Definition 6. Let \( M \) be an \( S \)-module and choose a presentation

\[
S^{s} \xrightarrow{\varphi} S^{t} \twoheadrightarrow M \longrightarrow 0
\]

of \( M \). Let \( I_{s}(\varphi) \) denote the \( S \)-ideal generated by all \( k \times k \) minors of \( \varphi \) (set \( I_{s}(\varphi) = S \) for \( k \leq 0 \) and \( I_{s}(\varphi) = 0 \) for \( t > \min(s, t) \)). Then the \( i \)th Fitting ideal of \( M \) is defined as \( \text{Fitt}_i(M) = I_{s-i}(\varphi) \), which is independent of the choice of the presentation (see Eisenbud 1995, Corollary-Definition 20.4).

If \( I \) is an ideal in \( S \), let

\[
\sqrt{I} = \{ x \in S : x^k \in I \text{ for some } k \in \mathbb{N} \}
\]

be the radical of \( I \), then \( \dim S/I = \dim S/\sqrt{I} \).

Lemma 7. Let \( M \) be an \( S \)-module, then the 0th Fitting ideal \( \text{Fitt}_0(M) \) satisfies \( \text{Fitt}_0(M) \subseteq \text{ann}_S(M) \) and \( \sqrt{\text{Fitt}_0(M)} = \sqrt{\text{ann}_S(M)} \).

Proof. See Eisenbud 1995, Proposition 20.7). \( \square \)

Applying Remark 5 to the 0th Fitting ideal of \( F/M \), we have

Lemma 8. Let \( M \) be a graded submodule of \( F \) with \( \dim(F/M) = \dim(S/M) \cap F = r \). Let \( \text{Fitt}_0(F/M) \) be the 0th Fitting ideal of \( F/M \) generated by polynomials \( p_1, \ldots, p_t \) of degrees \( d_1 \geq \cdots \geq d_t \). Then \( \text{Fitt}_0(F/M) \subseteq M \cap F \) contains a regular sequence \( g_1, \ldots, g_{n-r} \) of degrees \( d_1 \geq \cdots \geq d_{n-r} \).

Proof. Since \( \sqrt{M \cap F} = \sqrt{\text{Fitt}_0(F/M)} \), we have \( r = \dim(S/M) \cap F = \dim(S/\sqrt{\text{Fitt}_0(F/M)}) \). Now apply Remark 5 to \( \text{Fitt}_0(F/M) \). \( \square \)

Notice that we need the Fitting ideal \( \text{Fitt}_0(F/M) \) in replacement of the annihilator \( M \cap F \) because the generating degree of \( \text{Fitt}_0(F/M) \) can be bounded linearly by the generating degree of \( M \), while the generating degree of \( M \cap F \) is usually much larger. We will need both the graded and non-graded versions of the following lemma.
Lemma 9. Let $M$ be a submodule of $F$ generated by (homogeneous) elements $f_1, \ldots, f_s$, with degrees $D = D_1 \geq \cdots \geq D_s$. If $\text{Fitt}_0(F/M) \neq 0$, then the maximum degree of a minimal (homogeneous) generating set of $\text{Fitt}_0(F/M)$ is bounded by $D_1 + \cdots + D_m - \sum_{j=1}^{m} \deg(e_j) \leq D_m$.

Proof. For each $i = 1, \ldots, s$, we can write $f_i = a_{ij}e_1 + \cdots + a_{in}e_n$ for some $a_{ij} \in S$, $\deg(a_{ij}) \leq D_i - \deg(e_j)$ (or $\deg(a_{ij}) = D_i - \deg(e_j)$ if we are in the homogeneous setting and $a_{ij} \neq 0$). Consider the presentation

$$S^* \xrightarrow{\varphi} S^m \cong F \xrightarrow{\iota} F/M \xrightarrow{\pi} 0$$

of $F/M$ where $\varphi = (a_{ij})_{i=1,\ldots,m,j=1,\ldots,n}$. Then $\text{Fitt}_0(F/M) = I_n(\varphi)$ is the ideal generated by all the $m \times m$ minors of $\varphi$. Every such minor corresponds to $m$ elements $f_1, \ldots, f_s$ among the generators of $M$, so if the minor is nonzero, it will be a (homogeneous) element of degree $\leq D_1 + \cdots + D_m - \sum_{j=1}^{m} \deg(e_j)$. Hence the largest degree of a minimal (homogeneous) generating set of $\text{Fitt}_0(F/M)$ is bounded by $D_1 + \cdots + D_m - \sum_{j=1}^{m} \deg(e_j) \leq D_m$. $\square$

2.5. Homogenization

Let $f \in F$ be a nonzero element and consider its associated components $f_j$. Let $t$ be a new variable, then the homogenization of $f$ is defined as $f^t = \sum_{i} f_i^{\deg(f_i) - \deg(f)}$. For a module $M \subseteq F$, the homogenization of $M$ is the submodule generated by $(f^t : f \in M)$, denoted as $M^t = \langle f^t : f \in M \rangle$.

If $f \in F^t$, let $f^{\text{deh}}$ denote the dehomogenization of $f$ which is obtained from $f$ by substituting $t$ by 1. If $G$ is a subset of $F^t$, let $G^{\text{deh}} = \{f^{\text{deh}} : f \in G\}$.

Given $< t$ a monomial order on $F$, we extend it to a monomial order $< t$ on $F^t$ as follows:

$$x^a t^e e_j < t x^b t^f e_j \iff \deg(x^a t^e e_j) < \deg(x^b t^f e_j),$$

or $\deg(x^a t^e e_j) = \deg(x^b t^f e_j)$ and $x^a e_j < x^b e_j$.

It is easy to check that $\text{in}_<(f^{\text{deh}}) = \text{in}_<(f)^{\text{deh}}$ for all homogeneous $f \in F$.

Example 10. Let $< t$ be the degree reverse lexicographic order on $S = \mathbb{K}[x_1, \ldots, x_n]$ (see Kreuzer and Robbiano (2004, definition 1.4.7)), then its extension $< t$ is the degree reverse lexicographic order on $S[t] = \mathbb{K}[x_1, \ldots, x_n, t]$.

Using the above extension, we can obtain a Gröbner basis of a non-graded module from dehomogenizing a homogeneous Gröbner basis of any graded module that dehomogenizes into $M$.

Lemma 11. Let $M = S f_1 + \cdots + S f_s$ be a submodule of $F$ and $< t$ be a monomial order on $F$. If $N$ is a graded submodule of $F^t$ where $S[t] f_1^t + \cdots + S[t] f_s^t \subseteq N \subseteq M^t$ and $G$ is a homogeneous Gröbner basis of $N$ w.r.t. $< t$, then $G^{\text{deh}}$ is a Gröbner basis of $M$ w.r.t. $< t$.

Proof. Notice that as $S[t] f_1^t + \cdots + S[t] f_s^t \subseteq M^t$, we have $N^{\text{deh}} = S f_1 + \cdots + S f_s = M$. Since $\text{in}_<(f^{\text{deh}}) = \text{in}_<(f)^{\text{deh}}$ for all homogeneous $f \in N$, we have $\text{in}_<(N^{\text{deh}}) = \text{in}_<(N)^{\text{deh}} = \{\text{in}_<(G^{\text{deh}})\} = \{\text{in}_<(G^{\text{deh}})\}$. $\square$

Under monomial orders of $t$-degrev type (see Kreuzer and Robbiano (2005, Definition 4.4.1)), the initial ideal of $F^t$ can be generated by monomials that are free of $t$’s. An example of a monomial order of $t$-degrev type is the degree revlex order.
Lemma 12. Let \( I = (p_1, \ldots, p_k) \) be an ideal in \( S \) and \( \prec \) be a monomial order on \( S[t] \) of \( t \)-degrev type. If \( \{g_1, \ldots, g_s\} \) is a homogeneous Gröbner basis of \( J = (p_1^b, \ldots, p_k^b) \) w.r.t. \( \prec \), then \( \{g_1/p_1^a, \ldots, g_s/p_s^a\} \) is a homogeneous Gröbner basis of \( I^b \) w.r.t. \( \prec \) where \( a_i = \max\{a \in \mathbb{N} : t^a \text{ divides } g_i\} \).

Proof. See Kreuzer and Robbiano (2005, Corollary 4.4.9).

2.6. Generic initial ideal

Let \( S[t] = \mathbb{K}[x_1, \ldots, x_n, t] \) and \( \prec \) be a monomial order on \( S[t] \). For the rest of this subsection and Section 8, let the general linear group \( GL_n(\mathbb{K}) \) act on \( S[t] \) by acting on the variables \( x_1, \ldots, x_n \), i.e. if \( \alpha = (\alpha_{ij}) \in GL_n(\mathbb{K}) \), then \( \alpha(x_j) = \sum_{i=1}^n \alpha_{ij}x_i \) for \( j = 1, \ldots, n \).

Definition 13. Let \( I \subseteq S[t] \) be a homogeneous ideal, then there is a nonempty Zariski open set \( U \subseteq GL_n(\mathbb{K}) \) and a monomial ideal \( J \subseteq S[t] \) such that for all \( \alpha \in U \), \( J = in_{\alpha}(I) \). Then \( J \) is called the generic initial ideal of \( I \) w.r.t \( x_1, \ldots, x_n \), written \( gin_{\prec}(I) \).

Remark 14. The reader should note that our definition is non-standard but is needed for Section 8 where we deal with homogenization. The existence of such a Zariski open set \( U \) can be proved by the same proof in Eisenbud (1995) Theorem 15.18 or Greuel (1998) Theorem 1.27.

In the following theorems, we present classical results in the context of Definition 13. The goal is to show that the generic initial ideal w.r.t \( x_1, \ldots, x_n \) contains a sequence of pure powers that has length equal to its height.

Theorem 15 (Galligo, Bayer-Stillman). If \( I \subseteq S[t] \) is a homogeneous ideal, then the generic initial ideal of \( I \) w.r.t \( x_1, \ldots, x_n \) is Borel-fixed w.r.t \( x_1, \ldots, x_n \) i.e. for all upper triangular matrices \( \beta \in GL_n(\mathbb{K}) \), \( \beta(gin_{\prec}(I)) = gin_{\prec}(I) \).

Proof. Analogous to Eisenbud (1995) Theorem 15.20.

Theorem 16 (Bayer-Stillman). If a monomial ideal \( I \subseteq S[t] \) is Borel-fixed w.r.t \( x_1, \ldots, x_n \), then \( I \) is of Borel type w.r.t. \( x_1, \ldots, x_n \), i.e. for each monomial \( u \in I \) and all integers \( i, j \) with \( 1 \leq j < i \leq n \), there exists an integer \( a \geq 0 \) such that \( x_i^a u / x_j^b \in I \) where \( b_i = \max\{b \in \mathbb{N} : x_j^b \text{ divides } u\} \).

Proof. Analogous to Herzog and Hibi (2011), Proposition 4.2.9, Theorem 4.2.10.

Lemma 17. Let \( I \subseteq S[t] \) be a monomial ideal of Borel type w.r.t \( x_1, \ldots, x_n \) and \( ht(I) = n-r \), then there exists pure powers \( x_1^{c_1}, \ldots, x_r^{c_r} \) that belong to the minimal monomial generating set of \( I \).

Proof. We will construct the sequence inductively. Let \( B \) be the minimal monomial generating set of \( I \). We may assume \( n-r > 0 \) since otherwise the claim is clear. Let \( i < n-r \), assume \( x_i^{c_i}, \ldots, x_r^{c_r} \in B \) have already been constructed and we want to find \( x_{r+1}^{c_{r+1}} \in B \). Since \( i < n-r = ht(I) \), we have \( I \not\supseteq (x_1, \ldots, x_i) \). So there exists a monomial \( x_b \in I \). Starting from this monomial, for \( j = n, \ldots, i+2 \) we can consecutively exchange \( x_j^b \) for some \( x_j^{c_j} \) to get another monomial in \( I \) that is not divisible by \( x_j, \ldots, x_n \) (we can do this because \( I \) is of Borel type w.r.t \( x_1, \ldots, x_n \)). In the end we get \( x_{r+1}^{c_{r+1}} \in I \) for some \( c_{r+1} \). As \( B \) is a generating set of \( I \), there exists an element \( x_{r+1}^{c_{r+1}} \in B \) that divides \( x_{r+1}^{c_{r+1}} \).
3. Cone decomposition

In Dubé (1990), Dubé introduced cone decompositions to express a subspace $T$ in $S$ as a finite direct sum of subspaces of the form $h\mathbb{K}[u]$, where $h$ is a polynomial in $S$ and $u$ is a subset of $X$. In this section, we will give an introduction to cone decompositions in the context of a free module.

**Definition 18.** Let $f$ be a homogeneous element in $F$ and $u$ a subset of $X$, then $C = C(h, u) = h\mathbb{K}[u]$ is called a cone. The degree of the cone is defined as $\deg(C) = \deg(h)$, the dimension of the cone is defined as $\dim(C) = |u|$, and $h$ is called the pivot of the cone.

**Definition 19.** Let $T \subseteq F$, $h_1, \ldots, h_r$ be homogeneous elements in $F$, and $u_1, \ldots, u_r$ be subsets of $X$. If as $\mathbb{K}$-vector spaces,

$$T = \bigoplus_{i=1}^{r} h_i\mathbb{K}[u_i],$$

then $P = \{C(h_1, u_1), \ldots, C(h_r, u_r)\}$ is called a cone decomposition of $T$. The degree of the cone decomposition is defined as $\deg(P) = \max\{\deg(C) : C \in P\}$.

If $T$ admits a cone decomposition $P$, then the Hilbert function of $T$ is a sum of the Hilbert functions of $h\mathbb{K}[u]$. Counting the number of monomials in $h\mathbb{K}[u]$, we get that if $u = \emptyset$, then

$$HF_{h\mathbb{K}[\emptyset]}(z) = \begin{cases} 0, & \text{if } z \neq \deg(h) \\ 1, & \text{if } z = \deg(h) \end{cases}$$

and if $|u| > 0$,

$$HF_{h\mathbb{K}[u]}(z) = \begin{cases} 0, & \text{if } z < \deg(h) \\ (z - \deg(h) + |u| - 1), & \text{if } z \geq \deg(h) \end{cases}$$

Therefore cones of the form $h\mathbb{K}[\emptyset]$ only contribute to finitely many values of the Hilbert function $HF_T$. Collecting the remaining cones, we denote

$$P^* := \{C \in P : \dim(C) > 0\}$$

**Definition 20.** A cone decomposition $P$ for $T$ is said to be $q$-standard if the following two conditions hold:

1. There is no cone $C \in P^*$ with $\deg(C) < q$.
2. For every $C \in P^*$ and degree $d$ such that $q \leq d \leq \deg(C)$, $P$ contains a cone $C'$ with $\deg(C') = d$ and $\dim(C') \geq \dim(C)$.

Notice that if $P^* = \emptyset$, then $P$ is $q$-standard for all natural numbers $q$. If $P^* \neq \emptyset$, then the only possible value for $q$ is $\min\{\deg(C) : C \in P^*\}$.

We define a special cone decomposition that splits a cone and is useful for manipulating cone decompositions.

**Definition 21.** Let $u = \{x_{i_1}, \ldots, x_{i_k}\} \subseteq X$, $h$ a homogeneous element in $F$, and $C = h\mathbb{K}[u]$. Then the fan of $C$ is defined as

$$F(C) := \{|C(h, \emptyset)| \cup |C(x_{i_1}h, [x_{i_1}, \ldots, x_{i_k}]) : i = 1, \ldots, t|\}.$$
Remark 22. The following list contains some facts of \( q \)-standard cone decompositions that are easy to verify:

1. \( \{C(e_i, X)\} \) is a \( \deg(e_i) \)-standard cone decomposition of \( S e_i \).
2. If \( C \) is a cone, \( F(C) \) is a \( (\deg(C) + 1) \)-standard cone decomposition of \( C \).
3. Let \( T = T_1 \oplus T_2 \) and let \( P_1 \) and \( P_2 \) be \( q \)-standard cone decompositions of \( T_1 \) and \( T_2 \), respectively. Then \( P_1 \cup P_2 \) is a \( q \)-standard cone decomposition of \( T \) with \( \deg(P_1 \cup P_2) \geq \deg(P_1) \).
4. Let \( T \) be a subset of \( S \). If \( P = \{C(p_1, u_1), \ldots, C(p_r, u_r)\} \) is a \( q \)-standard cone decomposition of \( T \), then for any homogeneous element \( f \in F \), the set \( Q = \{C(f p_1, u_1), \ldots, C(f p_r, u_r)\} \) is a \( (\deg(f) + \deg(f)) \)-standard cone decomposition for \( fT = \{fp : p \in T\} \).

**Lemma 23.** Let \( P \) be a \( q \)-standard cone decomposition of a subspace \( T \subseteq F \). Then for any \( d \geq q \), there exists a \( d \)-standard cone decomposition \( P_d \) of \( T \) with \( \deg(P_d) \geq \deg(P) \) and \( \deg(P_d^+) \geq \deg(P^+) \).

**Proof.** If \( P^+ = \emptyset \) then the result holds trivially. Assume \( P^+ \neq \emptyset \), it suffices to show that there exists a \( (q + 1) \)-standard cone decomposition \( P_{q+1} \) of \( T \) with \( \deg(P_{q+1}) \geq \deg(P) \) and \( \deg(P_{q+1}^+) \geq \deg(P^+) \). Let \( Q = \{C \in P : \deg(C) = q\} \). Notice that \( Q \) is trivially \( q \)-standard and \( P \setminus Q \) is \( (q + 1) \)-standard as \( P \) is \( q \)-standard.

By Remark 22(2), for each \( C \in Q \), there exists a \( (q + 1) \)-standard cone decomposition \( F(C) \) of \( C \). Now apply Remark 22(2) to see that \( P_{q+1} := \bigcup_{C \in P} F(C) \cup (P \setminus Q) \) is a \( (q + 1) \)-standard cone decomposition of \( T \). Also \( \deg(P_{q+1}) \geq \deg(P) \) and \( \deg(P_{q+1}^+) \geq \deg(P^+) \) are clear from the definition of \( F(C) \).

4. Decomposing a set of normal forms

For an ideal \( I \), Dubé (1990) constructed a \( 0 \)-standard cone decomposition \( P \) of \( N_I \), so that the degree of \( I \) gives an upper bound to the Gröbner basis degree of \( I \). We will follow Dubé to construct an \( I \)-standard cone decomposition \( Q \) of \( N_M \) with \( \deg(Q) \) bounding the Gröbner basis degree of \( M \). Most of the statements in Dubé holds in the module case and the proofs can be applied directly with some slight modifications. Recall that \( N_M = N_{m,(M)} \), hence it suffices to decompose \( N_M \) for monomial modules.

**Definition 24.** Let \( P \cup Q \) be a cone decomposition of \( T \subseteq F \), and let \( M = \sum M \) be a submodule of \( F \). Then \( P \) and \( Q \) are said to split \( T \) relative to \( M \) if \( C \in P \) implies \( C \subseteq M \), and \( C \in Q \) implies \( C \cap M = \emptyset \).

**Lemma 25.** Let \( P = \{C(g_1, u_1), \ldots, C(g_r, u_r)\} \) and \( Q = \{C(h_1, v_1), \ldots, C(h_s, v_s)\} \) split \( T \) relative to a monomial module \( M \), where for each \( C(h_i, v_i) \in Q \), \( h_i \) is a monomial in \( F \). Then \( P \) is a cone decomposition for \( T \cap M \) and \( Q \) is a cone decomposition for \( T \cap N_M \).

**Proof.** See Dubé (1990, Lemma 4.1).

Using Dubé’s SPLIT algorithm (Dubé, 1990, §4), we can produce cone decompositions \( P \) and \( Q \) which split a cone \( h \mathbb{K}[a] \) relative to a monomial module \( M \). In order to formulate the SPLIT algorithm, first we will need an simple algorithm called QUOTIENT BASIS that computes a generating set of \( I : x_j \) for a monomial ideal \( I \). This is needed to ensure the termination of the SPLIT algorithm.

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Let $M$ be a monomial generating set for $I$.

**Corollary 28.** See Dubé (1990, Lemma 4.3, 4.4).

**Proof.**

**Lemma 26.** The algorithm SPLIT terminates and is correct.

**Proof.** See Dubé (1990, Lemma 4.3, 4.4). \[\square\]

The cone decomposition $Q$ produced by the SPLIT algorithm has the crucial property that its cone decomposition degree bounds the Gröbner basis degree of $M$ (see Theorem 30). To see this, we start with the following lemma.

**Lemma 27.** Let $R_j$ be a minimal monomial generating set for $I_j e_j$, where $I_j \subseteq S$ is a monomial ideal. Let $B_j$ be a monomial generating set for $I_j$, and $(P_j, Q_j) = SPLIT(e_j, X, B_j, I_j e_j)$. Then for every $f \in R_j$, $Q_j$ contains a cone $C(h, u)$ with $\deg(h) = \deg(f) - 1$.

**Proof.** We sketch the proof given in Dubé (1990, Lemma 4.8). Since $P_j$ is a cone decomposition of $I_j e_j$ and $f$ is part of a minimal generating set of $I_j e_j$, $P_j$ must contain a cone of the form $C(f, v)$. Then a cone of the form $C(h, u) \in Q_j$, with $\deg(h) = \deg(f) - 1$ could be found by tracing back and forth the recursions. \[\square\]

**Corollary 28.** Let $M = \bigoplus_{j=1}^m I_j e_j$ where $I_j$ are monomial ideals. For each $j$, let $B_j$ be a monomial generating set of $I_j$ and $(P_j, Q_j) = SPLIT(e_j, X, I_j e_j, B_j)$. If $I_j \neq S$, then $I_j e_j$ can be
generated by the set \( \{ f \in B_{j e_j} : \deg(f) \leq 1 + \deg(Q_j) \} \). Hence \( M \) can be generated by the set
\[
\bigcup_{j=1}^m \{ f \in B_{j e_j} : \deg(f) \leq \max\{1 + \deg(Q_j), \deg(e_j)\} \}.
\]

**Lemma 29.** Let \((P, Q) = \text{SPLIT}(h, u, M, B)\), then \(Q\) is a \(d(h)\)-standard cone decomposition.

**Proof.** See [Dubé (1990), 4.10].

Combining Corollary 28 and Lemma 29, we obtain our main theorem of this section.

**Theorem 30.** Let \( G \) be a homogeneous Gröbner basis of a graded submodule \( M \subseteq F \) with respect to a monomial order \( \prec \). Then \( N_M \) admits a \(l\)-standard cone decomposition \( Q \) where \( l = \max\{\deg(e_1), \ldots, \deg(e_m)\} \) and \( G' = \{ g \in G : \deg(g) \leq \max\{1 + \deg(Q), l\} \} \) is also a Gröbner basis of \( M \) with respect to \( \prec \). In particular, the degree of the reduced Gröbner basis of \( M \) is bounded by \( \max\{1 + \deg(Q), l\} \).

**Proof.** Since \( \text{in}_\prec(M) \) is a monomial submodule, \( \text{in}_\prec(M) = \bigoplus_{j=1}^m I_j e_j \) for some monomial ideals \( I_j \subseteq S \). For each \( j = 1, \ldots, m \), let \( B_{j e_j} = G \cap S e_j \), then \( B_{j e_j} \) generates \( I_j e_j \). Consider \((P_j, Q_j) = \text{SPLIT}(e_j, X, I_j e_j, B_j)\). \( Q_j \) is a \(\deg(e_j)\)-standard cone decomposition of \( N_{I_j e_j} \cap S e_j \) by Lemma 28 and Lemma 29. Then by Lemma 28 there exists a \(l\)-standard cone decomposition \( Q'_j \) for \( N_{I_j e_j} \cap S e_j \) with \( \deg(Q'_j) \geq \deg(Q_j) \).

Since \( N_M = \text{in}_\prec(M) = \bigoplus_{j=1}^m N_{I_j e_j} \cap S e_j \), we have that \( Q := \bigcup_{j=1}^m Q'_j \) is a \(l\)-standard cone decomposition for \( N_M \) and clearly \( \deg(Q) \geq \deg(Q') \). By Corollary 28 \( \text{in}_\prec(M) \) can be generated by
\[
\bigcup_{j=1}^m \{ f \in B_{j e_j} : \deg(f) \leq \max\{1 + \deg(Q_j), \deg(e_j)\} \}
\subseteq \bigcup_{j=1}^m \{ f \in B_{j e_j} : \deg(f) \leq \max\{1 + \deg(Q'_j), \deg(e_j)\} \}
\subseteq \{ f \in \bigcup_{j=1}^m B_{j e_j} : \deg(f) \leq \max\{1 + \deg(Q), l\} \}
\subseteq \{ \text{in}_\prec(g) : g \in G' \}.
\]

The last inclusion holds because \( G \) is a homogeneous Gröbner basis. Hence \( G' \) is a Gröbner basis for \( M \).

The ideal version of the above theorem, which is proved in [Dubé (1990), §4], will also be needed in later sections. But since we only need the existence of the cone decomposition, we will only quote part of the theorem.

**Theorem 31** ([Dubé (1990)]). Let \( I \) be a homogeneous ideal of \( S \) and fix a monomial order \( \prec \) on \( S \), then there exists a \(0\)-standard cone decomposition of \( N_I \).

**Proof.** See [Dubé (1990), Theorem 4.11].
5. The exact cone decomposition and Macaulay constants

One of the nice properties that a \( q \)-standard cone decomposition \( P \) has is that there are cones of every degree between \( q \) and \( \deg(P) \). However, this doesn’t give us any control over the number of cones in a certain degree. Therefore the following notion is introduced to further refine a \( q \)-standard cone decomposition.

**Definition 32.** Let \( T \) be a subspace of \( F \), then \( P \) is called an \( q \)-exact cone decomposition of \( T \) if \( P \) is a \( q \)-standard cone decomposition of \( T \), and \( \deg(C) \neq \deg(C') \) for all \( C \neq C' \in P^+ \).

If \( P^+ \neq \emptyset \), then there is a unique value \( q > 0 \) such that \( P \) is \( q \)-standard. If \( P^+ = \emptyset \), \( P \) is trivially \( q \)-exact for all natural numbers \( q \), then we set \( P \) to be 0-exact for the following definition.

**Definition 33.** Let \( P \) be a \( q \)-exact cone decomposition of \( T \subseteq F \). Then the Macaulay constants of \( P \) are defined as

\[ b_k := \max \{ (q) \cup \{ 1 + \deg(C) : C \in P, \dim(C) \geq k \} \} \quad \text{for} \quad k = 0, \ldots, n+1. \]

It is a simple consequence of this definition that the \( b_k \)’s satisfy \( b_0 \geq b_1 \geq \cdots \geq b_{n+1} = q \), and if \( P^+ \neq \emptyset \) then \( b_0 = 1 + \deg(P) \) and \( b_1 = 1 + \deg(P^+) \).

Once we have an exact cone decomposition \( P \), then the Macaulay constants give a nearly complete picture of \( P \), i.e. they control degrees of all the cones in \( P \) but not the specific pivots.

**Lemma 34.** Let \( P \) be a \( q \)-exact cone decomposition, and let \( b_0, \ldots, b_{n+1} \) be defined as above. Then for each \( 1, \ldots, n \) and degree \( d \) such that \( b_{i+1} \leq d < b_i \), there is exactly one cone \( C \in P^+ \) such that \( \deg(C) = d \), and for that cone \( \dim(C) = i \). In particular \( b_i = b_{i+1} + |\{ C \in P^+ : \dim(C) = i \}| \) for \( i = 1, \ldots, n \).

**Proof.** See [Dubé (1990), Lemma 6.1].

One way to make a \( q \)-standard cone decomposition \( q \)-exact is as follows: whenever there are two cones of the same degree, replace the cone of lower-dimension by its fan and the resulting cone decomposition still remains \( q \)-standard. The \texttt{EXACT} algorithm is from Mayr and Ritscher (2013) and is a reformulation of \texttt{SHIFT} and \texttt{EXACT} in Dubé (1990).

**Algorithm 3: EXACT(\( Q \))**

\begin{algorithmic}
\Input \( Q \) a \( q \)-standard cone decomposition of \( T \subseteq F \)
\Output \( P \) a \( q \)-exact cone decomposition of \( T \subseteq F \)
\State \( P := Q \)
\For \( d := q, \ldots, \deg(P^+) \)
\State \( S := \{ C \in P^+ : \deg(C) = d \} \)
\While \( |S| > 1 \)
\State Choose \( C \in S \) with minimal dimension \( \dim(C) \)
\State \( S := S \setminus \{ C \} \)
\State \( P := P \setminus \{ C \} \cup F(C) \)
\EndWhile
\EndFor
\EndInput
\EndOutput
\EndAlgorithm

The \texttt{EXACT} algorithm gives the following lemma.
Lemma 35. Every q-standard cone decomposition $Q$ of a vector space $T \subseteq F$ may be refined into q-exact cone decomposition $P$ of $T$ with $\deg(Q) \leq \deg(P)$ and $\deg(Q') \leq \deg(P')$.

Proof. See Mayr and Ritscher (2013, Lemma 16).

As we have seen before, if $T$ admits a cone decomposition $P$, then its Hilbert polynomial is determined by $P$. If we further assume $P$ is exact, then by Lemma 34 $P$ is determined by the Macaulay constants of $P$. Hence it follows that the Hilbert polynomial $HP_T$ is determined by the Macaulay constants.

Lemma 36. Let $P$ be a q-exact cone decomposition of a subspace $T \subseteq F$, let $b_0, \ldots, b_{n+1}$ be the Macaulay constants of $P$. Then for $z \geq b_0$, the Hilbert function $H_T(z)$ attains the polynomial form

$$HP_T(z) = \left( \frac{z - b_{n+1} + n}{n} - 1 \right) - \sum_{i=1}^{n} \binom{z - b_i + i - 1}{i}.$$ 

In addition for $z \geq b_1$, $HF_T(z) = HP_T(z) + \|C(h, \emptyset) \in P : \deg(h) = z\|$

Proof. See Dubé (1990, §7).

The converse of the previous lemma also holds, that is the Macaulay constants are determined by the Hilbert polynomial. In particular the Macaulay constants does not depend on the chosen q-exact cone decomposition.

Lemma 37. Let $P$ be any q-exact cone decomposition for a subspace $T \subseteq F$. Then the Macaulay constants $b_1, \ldots, b_{n+1}$ are uniquely determined by $HP_T$ and $q$, and $b_0 = \min\{d \geq b_1 : HP_T(z) = HF_T(z) \forall z \geq d\}$.

Proof. See Dubé (1990, Lemma 7.1).

The above lemma also shows that if $T$ is a subspace with a known Hilbert regularity (e.g. set of normal forms of a submodule generated by a homogeneous regular sequence), then to bound $b_0$ it suffices to bound $b_1$.

6. Reduction to the complete intersection case

So far we have reduced our problem of bounding the Gröbner basis degree into bounding the Macaulay constant $b_0$. However as in Section 8 of Dubé (1990), attacking this problem directly requires a large amount of computations because our module is arbitrary. Mayr and Ritscher (2013) used Lemma 3 to reduce the problem into bounding the Macaulay constant of a complete intersection, thereby simplify the computations and improve the bound for ideals of small dimension. Adopting their approach, in this section we will show that if $IF \subseteq M$ with $I$ a complete intersection (which exists by Lemma 8 if $K$ is infinite), then the Macaulay constant of $N_{IF}$ bounds the Macaulay constant of $N_M$. Then by Lemma 5 and Lemma 5', if $I$ is generated by polynomials of degrees $d_1, \ldots, d_{n-r}$, then it suffices to bound the Macaulay constant $b_1$ of $N_{IF}$ where $J = (x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}})$.

Notice that Lemma 6.1-6.3 require a monomial order defined on $S$, but its sole purpose is to define $N_I$ for an ideal $I$ in $S$ and is irrelevant to the main theorems.
Lemma 38. Let $M$ be a submodule of $F$, $f \in F$, and $I = M :_S f$. Then for a fixed monomial order $<_S$ on $S$, 
\[ M + S f = M \oplus f \cdot N_I \]

Proof. See Dubé (1990, §2.2 Example 2).

Corollary 39. Let $M$ be a submodule of $F$ generated by elements $g_1, \ldots, g_r, f_1, \ldots, f_s \in F$ and let $L = S g_1 + \cdots + S g_r \subseteq M$. Then for a fixed monomial order $<_S$ on $S$, 
\[ M = L \oplus \bigoplus_{i=1}^s f_i \cdot N_{L_{i+1}:f_i} \]

where $L_k = (g_1, \ldots, g_r, f_1, \ldots, f_k)$ for $k = 0, \ldots, s$.

Proof. Apply Lemma 38 to get $L + S f_1 = L \oplus f_1 N_{L:f_1}$ and proceed inductively.

In order to reduce to $IF \subseteq M$ with $I$ a complete intersection, we show that for any submodule $L \subseteq M$, the Macaulay constant $b_0$ (equivalently the degree of cone decomposition) does not decrease if we replace $N_M$ by a vector space $T$ whose Hilbert function equal to the Hilbert function of $N_L$.

Lemma 40. Let $M$ be a graded submodule of $F$ generated by homogeneous elements $g_1, \ldots, g_r, f_1, \ldots, f_s \in F$, and fix a monomial order $<_S$ on $S$ and a monomial order $<_F$ on $F$. Let $L = S g_1 + \cdots + S g_r \subseteq M$ and $D = \max\{\deg(f_i) : i = 1, \ldots, s\} \geq 1$. Then if $Q$ is an $l$-standard cone decomposition of $N_M$, then there exists a vector space $T \subseteq F$ and a $D$-exact cone decomposition $P$ of $T$ such that $HF_T = HF_{N_L}$ and $\deg(Q) \leq \deg(P)$.

Proof. We use the notation defined in the previous lemma. By Theorem 31 there exists 0-standard cone decompositions $Q_k$ of $N_{L_{i+1}:f_i}$ for each $k = 0, \ldots, s$. Hence by Remark 22(4) $f_i Q_k$ is a $\deg(f_i)$-standard cone decompositions of $f_i N_{L_{i+1}:f_i}$. Then by Lemma 33 $Q, f_1 Q_1, \ldots, f_s Q_s$ can be converted into $D$-standard cone decompositions $\tilde{Q}, \tilde{Q}_1, \ldots, \tilde{Q}_s$. Define $T := \oplus_{i=1}^s f_i N_{L_{i+1}:f_i} \oplus N_M$, then we have 
\[ F = M \oplus N_M = L \oplus \bigoplus_{i=1}^s f_i N_{L_{i+1}:f_i} \oplus N_M = L \oplus T, \]

so the union $Q' = \tilde{Q} \cup \tilde{Q}_1 \cup \cdots \cup \tilde{Q}_s$ is a $D$-standard cone decomposition of $T$ and $HF_T = HF_{N_L}$ is clear. Finally by Lemma 35 $Q'$ can be refined to a $D$-exact cone decomposition $P$ of $T$. Notice that applying the two lemmas and taking union does not decrease the degree of the cone decomposition, hence $\deg(Q) \leq \deg(\tilde{Q}) \leq \deg(Q') \leq \deg(P)$.

Now it remains to reduce from a vector space $T$ with $HF_T = HF_{N_L}$ and $I$ generated by a regular sequence of degrees $d_1, \ldots, d_r$, to the monomial submodule $(x_1^{d_1}, \ldots, x_n^{d_r}) F$, using the fact that they have the same Hilbert function. We connect the two reductions to get our main theorem of this section. Recall that if $R$ is infinite, Lemma 8 guarantees that there exists an ideal $I \subseteq S$ generated by a regular sequence of degrees $d_1, \ldots, d_r$, with $IF \subseteq M$, hence the assumption of the following theorem can be satisfied for an arbitrary module with $\dim(F/M) = r$. 

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Lemma 42. Proof. It follows from the fact cone decomposition $P$ of $N$ where monomial order will be omitted in the following lemmas. D. Mayr and Ritscher (2013), in which Mayr and Ritscher bound the Macaulay constant obvious from the definition of $N$.

Theorem 41. 1 sequence of degrees $d_1, \ldots, d_{n-r}$ and $D = \max(\deg(f_j) : i = 1, \ldots, s) \geq l$. Fix a monomial order $<_F$ on $F$, if $Q$ is a $D$-standard cone decomposition of $N_M$, then

$$1 + \deg(Q) \leq \max(1 + \deg(P^+), d_1 + \cdots + d_{n-r} + l - n + 1)$$

where $P$ is a $D$-exact cone decomposition of $N_{IF}$ and $J = (x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}})$.

Proof. Let $I = (g_1, \ldots, g_{n-r}) \subseteq S$ and $L = IF \subseteq M$. By Lemma [30] we can complete any $L$-standard cone decomposition $Q$ of $N_M$ to a $D$-exact cone decomposition $\tilde{Q}$ of a vector space $T \subseteq F$ with $HF_T = HF_{\tilde{Q}}$, with $\deg(\tilde{Q}) \geq \deg(Q)$. Since $g_1, \ldots, g_{n-r}$ and $x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}}$ are both regular sequences of the same degrees, $N_I$ and $N_{IF}$ have the same Hilbert function by Lemma [5]. As $HF_T = HF_{\tilde{Q}} = HF_{N_{IF}}$, by Lemma [37] the Macaulay constant $b_0$ of $Q$ and $P$ are the same, so $\deg(Q) = \deg(P)$. Finally by Lemma [30] [5] and [37] $1 + \deg(Q) \leq 1 + \deg(\tilde{Q}) = 1 + \deg(P) = \max(1 + \deg(P^+), d_1 + \cdots + d_{n-r} + l - n + 1)$. ~\Box

7. Bounding the Macaulay constants

By Theorem [30] and Theorem [41] it remains to bound the Macaulay constant $b_1$ of $N_{IF}$ where $J = (x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}})$. As a result of the simple structure of $JF$, using induction we can explicitly construct a $D$-exact cone decomposition of $N_{IF}$ (without using the EXACT algorithm), which will give us the bound on $b_1$ easily. This section is a generalization of Section 3.3 in Mayr and Ritscher (2013), in which Mayr and Ritscher bound the Macaulay constant $b_1$ of $N_J \subseteq S$.

Since the subspace $N_{IF}$ is independent of any monomial order on $F$, the assumption on the monomial order will be omitted in the following lemmas.

Notice that $r = \dim(S/J) = \dim(F/IF)$ tells us that $b_i = D$ for all $i > r$. In particular if $r = 0$ then $b_1 = D$, so it suffices to bound $b_1$ for $r \geq 1$.

Lemma 42. Let $J = (x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}}) \subseteq S$ and $b_0, \ldots, b_{r+1}$ be the Macaulay constants of a $D$-exact cone decomposition $P$ of $N_{IF}$. Then

$$b_{r+1} = b_r = \cdots = b_{r+1} = D.$$  

Proof. It follows from the fact $\deg(HP_{N_{IF}}) = r - 1$ and Lemma [54] ~\Box

The following lemma presents the base case of the induction needed for Lemma [55] and is obvious from the definition of $N_{IF}$.

Lemma 43. Let $J = (x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}})$ be an ideal of $S$. Then $N_{IF}$ may be decomposed as

$$N_{IF} = T_r \times \mathbb{K}[x_{n-r+1}, \ldots, x_n] = \bigoplus_{h \in \text{Mon}(T_r)} hI_{\mathbb{K}[x_{n-r+1}, \ldots, x_n]},$$

where the vector space $T_r$ is given by

$$T_r = \text{span}_{\mathbb{K}}\{x^\alpha e_j \in F : j = 1, \ldots, m, x^\alpha \in \mathbb{K}[x_1, \ldots, x_{n-r}], 0 \leq \alpha_i < d_i \text{ for } i = 1, \ldots, n-r\}.$$
Lemma 46. Let $T_k \subseteq F$ be a finite dimensional vector space generated by monomials and $P_k$ a cone decomposition of $T_k \times \mathbb{K}[x_{n-k+1}, \ldots, x_n]$. Then $P_k$ has exactly $\dim(T_k)$ cones of dimension $k$.

Proof. See Mavry and Ritscher 2013, Lemma 29). \hfill \square

Corollary 45. Let $J = (x_1^{d_1}, \ldots, x_n^{d_n})$ be an ideal of $S$ and $b_0, \ldots, b_{n+1}$ the Macaulay constants of a $D$-exact cone decomposition $P$ of $N_{JF}$.

If $r \geq 1$, then

$$b_r = d_1 \cdots d_{n-r} m + D.$$ \hfill \square

Proof. Apply Lemma 44 and the previous three lemmas to get $b_r = b_{r+1} + \dim(T_r) = D + d_1 \cdots d_{n-r} m$.

Now we construct a $D$-exact cone decomposition $P$ of $N_{JF}$ layer by layer. That is to say in the $k$th inductive step, we ”peel off” from $T_k \times \mathbb{K}[x_{n-k+1}, \ldots, x_n]$ dimension $T_k$ many cones of dimension $k$, which will become all the dimension $k$ cones in $P$, and take $T_{k-1} \times \mathbb{K}[x_{n-k+2}, \ldots, x_n]$ to be the complement.

Lemma 46. Let $J = (x_1^{d_1}, \ldots, x_n^{d_n})$ be an ideal of $S$. Then for any $D \geq \max\{2, l\}$ and $k = 0, \ldots, r$, there exists a $D$-exact cone decomposition $P_k$ and a finite-dimensional subspace $T_k \subseteq N_{JF} \cap \bigoplus_{j=1}^m \mathbb{K}[x_1, \ldots, x_{n-k}] e_j$ which have a monomial basis such that

$$N_{JF} = (T_k \times \mathbb{K}[x_{n-k+1}, \ldots, x_n]) \oplus \bigoplus_{C \in P_k} C.$$ Let $b_0, \ldots, b_{n+1}$ be the Macaulay constants of $P_0$, then $b_k - 1 \leq \frac{k}{k} b_{n+1}^2$ for $k = 2, \ldots, r$.

Proof. We will first construct $P_0, \ldots, P_r$ first and then bound $b_1, \ldots, b_{r-1}$. Inductively, we construct $P_{k-1} \supseteq P_k$ and $T_{k-1} \times \mathbb{K}[x_{n-k+2}, \ldots, x_n] \subseteq T_k \times \mathbb{K}[x_{n-k+1}, \ldots, x_n]$, so that the following three conditions hold:

1. $P_{k-1} \setminus P_k$ consists of $\dim(T_{k-1})$ cones of dimension $k$.
2. If $\{h_1, \ldots, h_s\}$ is a monomial basis of $T_{k-1}$ with $\deg(h_1) \leq \cdots \leq \deg(h_s)$, then $\deg(h_i) \leq \deg(h_{i-1}) + 1$ whenever $\deg(h_i) \geq l + 1$.
3. $N_{JF} = (T_{k-1} \times \mathbb{K}[x_{n-k+2}, \ldots, x_n]) \oplus \bigoplus_{C \in P_{k-1}} C$.

Notice that condition (1) implies that $b_{k-1} = b_k + \dim(T_{k-1})$ for $k = 2, \ldots, r$.

The inductions starts with $k = r$. Let $P_r = 0$ and $T_r$ be as in Lemma 44 then it is easy to see that they satisfy the three conditions. Let $1 \leq k \leq r$ and assume $P_k$ and $T_k$ have been constructed, we want to construct $P_{k-1}$ and $T_{k-1}$. Notice that $P_k \subseteq P_0$ contains all cones of dimension larger than $k$, therefore $b_0, \ldots, b_{k+1}$ are fixed. Let $\{h_1, \ldots, h_s\}$ is a monomial basis of $T_k$ with $\deg(h_1) \leq \cdots \leq \deg(h_s)$ and choose

$$C_i = h_i^{b_{k+1} - \deg(h_i) - 1} \mathbb{K}[x_{n-k+1}, \ldots, x_n]$$ for $i = 1, \ldots, s$. \hfill \square
In order for these $C_i$’s to be well-defined, we need to show that $b_{k+1} + i - \deg(h_i) - 1 \geq 0$. Since $b_{k+1} \geq b_{n+1} = d \geq 1$, if $\deg(h_i) \leq l$, then $b_{k+1} + i - \deg(h_i) - 1 \geq l + i - l - 1 \geq 0$. If $\deg(h_i) \geq l + 1$, then as $T_k$ satisfies condition (2),

$$0 \leq b_{k+1} + (i - 1) - \deg(h_{k+1}) - 1$$

$$\leq b_{k+1} + (i - 1) - (\deg(h_i) - 1) - 1$$

$$= b_{k+1} + i - \deg(h_i) - 1.$$

Hence the $C_i$’s are well-defined. It is easy to see that $C_i \subseteq T_k \times K[x_{n-k+1}, \ldots, x_n]$, $\deg(C_i) = b_{k+1} + i - 1$, and $\dim(C_i) = k$. Thus $P_{k-1} = P_k \cup \{C_1, \ldots, C_s\}$ is a $D$-exact cone decomposition. Define

$$T_{k-1} = \text{span}_{K}[h, x_{n-k+1} : i = 1, \ldots, s, c = 0, \ldots, b_{k+1} + i - \deg(h_i) - 2]$$

$$\subseteq \bigoplus_{j=1}^{m} K[x_1, \ldots, x_{n-k+1}]e_j.$$

Notice that $T_{k-1}$ satisfies condition (2) as $T_k$ does, and

$$T_k \times K[x_{n-k+1}, \ldots, x_n] = C_1 \oplus \cdots \oplus C_s \oplus (T_{k-1} \times K[x_{n-k+2}, \ldots, x_n]),$$

hence it follows by induction that

$$N_{JF} = (T_{k-1} \times K[x_{n-k+2}, \ldots, x_n]) \oplus \bigoplus_{C \in P_{k-1}} C.$$

Inductively we have constructed $P_0, \ldots, P_r$, now we turn to the computation of the Macaulay constants $b_1, \ldots, b_r$ of $P_0$. Let $2 \leq k \leq r$ and we want to prove $b_{k-1} \leq \frac{1}{2}b_k^2$. As $b_{k-1} = b_k + \dim_T T_{k-1}$ (from condition (1)), it suffices to bound $\dim_T T_{k-1}$. By definition of $T_{k-1}$,

$$\dim_T T_{k-1} = \sum_{i=1}^{\dim} (b_{k+1} + i - \deg(h_i) - 1) \leq \sum_{i=1}^{\dim} (b_{k+1} + i - 1) = sb_{k+1} + \frac{1}{2}s(s - 1).$$

Since $s = \dim_T T_k = b_k - b_{k+1}$, the induction hypothesis and $b_{k+1} \geq D \geq 2$ imply

$$b_{k-1} = \dim_T T_{k-1} + b_k \leq (b_k - b_{k+1})b_{k+1} + \frac{1}{2}(b_k - b_{k+1})(b_k - b_{k+1} - 1) + b_k$$

$$\leq \frac{1}{2}(b_k^2 - b_{k+1}^2 + b_k + b_{k+1})$$

$$\leq \frac{1}{2}(b_k^2 - b_{k+1}^2 + \frac{1}{2}b_{k+1}^2 + b_{k+1}) \leq \frac{1}{2}b_k^2.$$

Notice that Lemma 46 shows that if $[h_1, \ldots, h_s]$ is a monomial basis of $T_0$, then $P = P_0 \cup \{C(h_i, \emptyset) : i = 1, \ldots, s\}$ is a $D$-exact cone decomposition of $N_{JF}$ with Macaulay constants $b_1, \ldots, b_{n+1}$ equal to those of $P_0$. Combining Corollary 45 and Lemma 46, we have

**Corollary 47.** Let $J = (x_1^d_1, \ldots, x_n^d_n) \subseteq S$ and $b_0, \ldots, b_{n+1}$ the Macaulay constants of a $D$-exact cone decomposition $P$ of $N_{JF}$ where $D \geq \max(2, l)$. Then if $r \geq 1$,

$$b_k \leq 2 \left[ \frac{1}{2}(d_1 \cdots d_m + D) \right]^{2^{n-k}}$$

for $k = 1, \ldots, r$. 

□
Finally, we combine all the previous results to obtain a bound for the Gröbner basis degree of an arbitrary graded module. Recall that $F = S e_1 \oplus \cdots \oplus S e_m$ with $\deg(e_j) \geq 0$ for all $j$ and $l = \max\{\deg(e_j) : j = 1, \ldots, m\}$. Without loss of generality we may assume the maximum degree $D$ of a generating set of $M$ is greater or equal to $l$, since otherwise $M$ and the summand with largest degree (say $S e_1$) are irrelevant so we may replace $F$ by $S e_2 \oplus \cdots \oplus S e_m$.

We first prove Theorem 48 which uses assumption on the generating degrees of the 0th Fitting ideal, and then prove Theorem 49 which uses only the generating degree of $M$. Notice that if $M = I$ is an ideal in $S$, then as $I = \text{Fitt}_0(S/I)$, Theorem 48 gives the same bound as the bound of Mayr and Ritscher (see Theorem 2).

**Theorem 48.** Let $M \subseteq F$ be a graded submodule generated by homogeneous elements of maximum degree $D$ where $D \geq 1$ and $\dim F/M = r$. Let $\text{Fitt}_0(F/M)$ be generated by homogeneous polynomials $p_1, \ldots, p_k$ of degrees $d_1 \geq \cdots \geq d_k$.

If $r = 0$, then the degree of the reduced Gröbner basis $G$ of $M$ for any monomial order on $F$ is bounded by

$$\deg(G) \leq d_1 + \cdots + d_n + l - n + 1.$$  

If $r \geq 1$, then the degree of the reduced Gröbner basis $G$ of $M$ for any monomial order on $F$ is bounded by

$$\deg(G) \leq 2 \left[ \frac{1}{2}(d_1 \cdots d_n, m + D) \right]^{2^{-r}}.$$

**Proof.** Without loss of generality we may assume $K$ is infinite. If $r = 0$, notice that $d_1 + \cdots + d_n + l - n + 1 \geq D \geq 1$, so we are done by Theorem 30, Theorem 41, and Lemma 42.

Assume $r \geq 1$. Let $f_1, \ldots, f_s$ be a generating set of $M$ with $D = \max\{\deg(f_i) : i = 1, \ldots, s\}$. We may assume $D \geq 2$, since otherwise it is easy to see that the Gröbner basis degree of $M$ is bounded by 1. By Lemma 8, $\text{Fitt}_0(F/M) \subseteq M$; so $F$ contains a regular sequence $g_1, \ldots, g_{n-r}$ of degrees $d_1 \geq \cdots \geq d_{n-r}$. Then $g_i e_j \in M$ for $i = 1, \ldots, n - r$ and $j = 1, \ldots, m$, and so $(g_i e_j : i = 1, \ldots, n - r, j = 1, \ldots, m) \cup \{f_1, \ldots, f_s\}$ is a generating set of $M$. Let $J = (a_1^0, \ldots, a_{n-r}^0)$ and let $P$ be a $D$-exact cone decomposition of $N_J$ with Macaulay constants $b_0, \ldots, b_{n+1}$. By Corollary 44, $1 + \deg(P^*) = b_1$ is bounded by $2 \left[ \frac{1}{2}(d_1 \cdots d_{n-r}, m + D) \right]^{2^{-r}}$, which is greater than $d_1 + \cdots + d_{n-r} + l - n + 1$ and $l$. Hence by Theorem 30 and Theorem 41 the reduced Gröbner basis degree is bounded by

$$\deg(G) \leq \max\{1 + \deg(P^*), d_1 + \cdots + d_{n-r} + l - n + 1, l\}$$

$$\leq 2 \left[ \frac{1}{2}(d_1 \cdots d_{n-r}, m + D) \right]^{2^{-r}}.$$

\(\square\)

**Theorem 49.** Let $M \subseteq F$ be a graded submodule generated by homogeneous elements $f_1, \ldots, f_s$ with degrees $D_1 \geq \cdots \geq D_m = D_1 \geq 1$, and $\dim F/M = r$.

If $r = 0$, then the degree of the reduced Gröbner basis $G$ of $M$ for any monomial order on $F$ is bounded by

$$\deg(G) \leq \left( D_1 + \cdots + D_m - \sum_{j=1}^{m} \deg(e_j) \right) n + l - n + 1 \leq Dmn - n + 1.$$
If \( r \geq 1 \), then the degree of the reduced Gröbner basis \( G \) of \( M \) for any monomial order on \( F \) is bounded by

\[
\deg(G) \leq 2 \left\lceil \frac{1}{2} \left( (D_1 + \cdots + D_m - \sum_{j=1}^m \deg(e_j))^{n-r} m + D \right) \right\rceil^{2^{n-1}} \\
\leq 2 \left\lceil \frac{1}{2} \left( (Dm)^{n-r} m + D \right) \right\rceil^{2^{n-1}}.
\]

**Proof.** Choose a minimal homogeneous generating set of \( \text{Fit}_{t_0}(M) \) with degrees \( d_1 \geq \cdots \geq d_k \) and use Lemma 59 to bound \( d_1, \ldots, d_{n-r} \), then apply Theorem 48.

A bound only depending on \( n, m, \) and \( D \) can be easily deduced from Theorem 49. If \( r = n \), then \( \deg(G) \leq 2 \left\lceil \frac{1}{2} (m + D) \right\rceil^{2^{n-1}} \). If \( r \leq n-1 \), the bound decreases when \( r \) decreases, so \( \deg(G) \leq 2 \left\lceil \frac{1}{2} (Dm^2 + D) \right\rceil^{2^{n-1}} \). Picking a bound that is greater than both, we have:

**Corollary 50.** Let \( M \subseteq F \) be a graded submodule generated by homogeneous elements with maximum degree \( D \geq 1 \). Then the degree of the reduced Gröbner basis \( G \) for any monomial order on \( F \) is bounded by

\[
\deg(G) \leq 2 (Dm)^{n-1}.
\]

8. Non-graded case

To solve the non-graded case, it seems natural to homogenize \( M \) using an additional variable \( t \) and deduce the non-graded bound from the graded bound simply by replacing \( n \) with \( n + 1 \). However this approach is false for a dimension-dependent bound since homogenizing an ideal may increase the dimension (see Mayr and Ritscher [2013, Example 3.4]). We fixed this problem by using generic initial ideal and Hashemi and Lazard’s Gröbner basis degree bound for zero-dimensional ideals w.r.t the degree revlex order. For reference we present Hashemi and Lazard’s bound in the lemma below.

Notice that our linear bound in Lemma 52 is a significant improvement of Binaei et al. [2018, Theorem 4.19] in which the regular sequence has degrees bounded by \( d_1 \cdots d_{n-r} \).

**Lemma 51 (Hashemi and Lazard [2003]).** Let \( I \) be a zero-dimensional ideal in \( \mathbb{K}[x_1, \ldots, x_n] \) generated by polynomials of degrees \( d_1 \geq \cdots \geq d_k \). Then the elements of the reduced Gröbner basis of \( I \) w.r.t. the degree revlex order have a degree at most \( d_1 + \cdots + d_n - n + 1 \).

**Lemma 52.** Let \( \mathbb{K} \) be an infinite field and \( I \subseteq S = \mathbb{K}[x_1, \ldots, x_n] \) be an ideal with \( \text{dim} S/ I = r \) generated by polynomials \( p_1, \ldots, p_k \) of degrees \( d_1 \geq \cdots \geq d_k \). Then there are polynomials \( g_1, \ldots, g_{n-r} \in I \) such that \( g_1^{b_1}, \ldots, g_{n-r}^{b_{n-r}} \in \mathbb{K}^r \) form a regular sequence and \( \deg(g_i) \leq d_1 + \cdots + d_{n-r} - n + r + 1 \) for \( i = 1, \ldots, n-r \).

**Proof.** Fix \( < \) to be the degree revlex order on \( S \) and \( <' \) the degree revlex order on \( S[I] \). Let \( J = (p_1^0, \ldots, p_k^0) \subseteq S[I] \). Let \( \alpha \in \text{GL}_n(\mathbb{K}) \) such that the generic initial ideal of \( J \) w.r.t. \( x_1, \ldots, x_n \) is equal to \( \text{in}_{<'}(\alpha J) \) (see the definition given in Definition 13). Notice that \( \alpha \) fixes \( t \) so it commutes with homogenization and dehomogenization, therefore \( \alpha J = (\alpha(p_1^0), \ldots, \alpha(p_k^0)) = ((\alpha p_1)^0, \ldots, (\alpha p_k)^0) \) and \( \alpha(t^0) = (\alpha t)^0 \) which we may denote as \( \alpha t^h \). Let \( G = \{f_1, \ldots, f_s\} \) be a
homogeneous Gröbner basis of \(aJ\) w.r.t. \(\prec\), then by Lemma 12 \((f_1/ t^{k_1}, \ldots, f_s/ t^{k_s})\) is a homogeneous Gröbner basis of \((aI)^h\) w.r.t. \(\prec\) where \(a_i = \max_{a \in \mathbb{N}} : t^e\) divides \(f_i = \max_{a \in \mathbb{N}} : t^e\) divides \(\in_{\prec}(f_i)\). Notice that \(\in_{\prec}(f_i/ t^{k_i}) = \in_{\prec}(f_i)/ t^{k_i}\).

By Theorem 43 and Theorem 45 \(\in_{\prec}(aI) = (\in_{\prec}(f_1), \ldots, \in_{\prec}(f_s))\) is of Borel type w.r.t. \(x_1, \ldots, x_n\), therefore \(\in_{\prec}(aI^h) = (\in_{\prec}(f_1)/ t^{k_1}, \ldots, \in_{\prec}(f_s)/ t^{k_s})\) is also of Borel type w.r.t. \(x_1, \ldots, x_n\). Since \(n^r = n^r = n^r = n^r = n^r\), by Lemma 17 the set of generators \(B = (\in_{\prec}(f_1)/ t^{k_1}, \ldots, \in_{\prec}(f_s)/ t^{k_s})\) of \(\in_{\prec}(aI^h)\) contains pure powers \(x_1^d, \ldots, x_n^d\) which are part of the minimal monomial generating set of \(\in_{\prec}(aI^h)\). After renumbering we may assume \(x_1^d = \in_{\prec}(f_1)/ t^{k_1}, \ldots, x_n^d = \in_{\prec}(f_s)/ t^{k_s}\). Let \(g_i^h = f_i/ t^{k_i} \in aI^h\) for \(i = 1, \ldots, s\). Notice that \(x_1^d = \in_{\prec}(g_i^h), \ldots, x_n^d = \in_{\prec}(g_s^h)\) from a regular sequence implies that \(g_i^h, g_s^h\) form a regular sequence. Now we claim that \(\deg(g_i^h) = c_i \leq d_1 + \cdots + d_{n-r} - n + r + 1\) for \(i = 1, \ldots, n-r\). Notice that once we have shown this claim, we are done since we can pick \(g_i := (x_1^d(g_i^h))^d_i \in I\), then \(g_i^h = x_1^d(g_i^h) \in I^h\) and \(g_1^h, \ldots, g_s^h\) form a regular sequence.

To prove the claim, let \(f\) denote \(f_{n-r+1}, \ldots, x_n\) if \(f \in S\), and by an abuse of notation let \(\overline{f}\) denote \(g_{n-r+1}, \ldots, x_n\) if \(g \in S[I]\). Also let \(\overline{\prec}\) denote the degree reverse order on \(K[x_1, \ldots, x_{n-r}]\). Notice that \(\in_{\prec}(g) = \in_{\overline{\prec}}(\bar{g})\) for all homogeneous \(g \in S[I]\), so as \((g_1^h, \ldots, g_s^h)\) is a Gröbner basis of \(aI^h\) w.r.t. \(\prec\), we have that \((g_1^h, \ldots, g_s^h)\) is a Gröbner basis of \(aI^h\) w.r.t. \(\overline{\prec}\). Then because \(\in_{\overline{\prec}}(f) = \in_{\overline{\prec}}(f') = \in_{\overline{\prec}}(aI) = (\in_{\overline{\prec}}(g_1^h), \ldots, \in_{\overline{\prec}}(g_s^h))\). For \(i = 1, \ldots, n-r\), \(\in_{\overline{\prec}}(g_i^h) = x_i^d\), so in particular this tells us that \(aI^h \in K[x_1, \ldots, x_n]\) is a zero-dimensional ideal. Notice that \((\overline{\in}_{\overline{\prec}}(g_1^h), \ldots, \overline{\in}_{\overline{\prec}}(g_s^h))\) must be part of the minimal monomial generating set of \(\in_{\overline{\prec}}(aI)\) by minimality of the \(c_i\)'s. Finally by Lemma 51 we have \(\deg(\overline{g_i^h}) = c_i \leq d_1 + \cdots + d_{n-r} - n + r + 1\) for \(i = 1, \ldots, n-r\).

We will apply Lemma 52 to the ideal \(F_{n-r}^h M\) to get a regular sequence \(g_1^h, \ldots, g_{n-r}^h \in F_{n-r}^h M\). If \(M\) is generated by elements \(f_1, \ldots, f_s\), we consider the graded module \(M \subseteq \mathbb{N}^h\) generated by \(g_1^h e_j : i = 1, \ldots, n-r, j = 1, \ldots, m\) \(\cup \{f_1^h, \ldots, f_s^h\}\). Then since the homogenization of a Gröbner basis of \(M\) is a Gröbner basis of \(M\), it suffices to bound the Gröbner basis degree of \(M\). Now all of our previous tools can be applied as \(M\) is graded.

Notice that if \(M = I\) is an ideal in \(S\), then as \(I = F_{n-r}^h S[I]\), Theorem 53 gives a bound that is sharper than Mayr and Ritscher’s bound (see Theorem 2).

Recall that \(F\) is a free module over \(K[x_1, \ldots, x_n]\) with basis elements \(e_1, \ldots, e_m\) with \(\deg(e_j) \geq 0\) for all \(j\) and \(l = \max\{\deg(e_j) : j = 1, \ldots, m\}\).

**Theorem 53.** Let \(M \subseteq F\) be a submodule generated by elements of maximum degree \(D\) with \(D \geq l + \dim F/M = r\). Let \(F_{n-r}^h M\) be generated by polynomials \(p_1, \ldots, p_s\) of degrees \(d_1 \geq \cdots \geq d_s\). Then the degree of the reduced Gröbner basis \(G\) for any monomial order on \(F\) is bounded by

\[
\deg(G) \leq 2 \left( \frac{1}{2} ((d_1 + \cdots + d_{n-r} - n + r + 1)^{n-r} m + D)^2 \right).
\]

**Proof.** Without loss of generality we may assume \(K\) is infinite. Fix a monomial order \(\prec\) on \(F\) and let \(\prec^{'\prime}\) be its extension on \(F^h\) defined in Section 2.5. Let \(M\) be generated by \(f_1, \ldots, f_s \in F\) with \(D = \max_{i=1}^s \deg(f_i) : i = 1, \ldots, s\), and let \(F_{n-r}^h M\) be generated by \(p_1, \ldots, p_s \in S\) of degrees \(d_1 \geq \cdots \geq d_s\). By Lemma 52 there exists polynomials \(g_1, \ldots, g_{n-r} \in F_{n-r}^h M\) with \(\deg(g_i) := d_i \leq d_1 + \cdots + d_{n-r} - n + r + 1\) for \(i = 1, \ldots, n-r\), and \(g_1^h, \ldots, g_{n-r}^h\) form a regular sequence. Consider the graded module \(M \subseteq \mathbb{N}^h\) generated by \(\{g_j^h e_j : i = 1, \ldots, n-r, j = 1, \ldots, m\} \cup \{f_1^h, \ldots, f_s^h\}\), and let \(\bar{G}\) be a reduced Gröbner basis of \(\bar{M}\) w.r.t. \(\prec^{'\prime}\). Notice that we have...
the inclusions ⊗_{j=1}^m ⊗_{i=1}^{n-r} S[t(\mathcal{G}_j)]h \subseteq \hat{M} \subseteq M^h. By Lemma \ref{lem:subset}, \hat{G}^{deh} is a Gröbner basis of \(M\) whose degree is clearly bounded by the degree of \(\hat{G}\), so it suffices to bound \(\deg(\hat{G})\).

By Theorem \ref{thm:standard}, there exists a \(l\)-standard cone decomposition \(Q\) of \(N_{\hat{M}}\) with \(\deg(\hat{G}) \leq \max(1 + \deg(Q), l)\). Let \(J = (x_1^{d_1}, \ldots, x_n^{d_{(n-r)+1}})\), then by Theorem \ref{thm:standard} \(1 + \deg(Q) \leq \max(b_1, d_1 + \cdots + d_{n-r} + l - (n + 1) + 1)\) where \(b_1\) is the Macaulay constant of a \(D\)-exact cone decomposition \(P\) of \(N_{J \circ F}\). Notice that \(r + 1 \geq 1\), so by Corollary \ref{cor:constant} \(b_1\) is bounded by \(2^{\lfloor \frac{1}{2}(d_1 + \cdots + d_{n-r} + 1) / (n + 1) + 1 \rfloor}\), which is greater than \(d_1 + \cdots + d_{n-r} + l - (n + 1) + 1 + l\).

Finally we combine the above inequalities to get

\[
\deg(G) \leq \deg(\hat{G}) \leq \max\{b_1, \tilde{d}_1 + \cdots + \tilde{d}_{n-r} + l - (n + 1) + 1, l\}
\leq 2^{\left\lfloor \frac{1}{2}(d_1 + \cdots + d_{n-r} + D) \right\rfloor},
\leq 2^{\left\lfloor \frac{1}{2}(d_1 + \cdots + d_{n-r} - n + r + 1 + D) \right\rfloor},
\]

\(\square\)

**Theorem 54.** Let \(M \subseteq F\) be a submodule generated by elements of maximum degree \(D\) with \(D \geq l\) and \(\dim F/M = r\). Then the degree of the reduced Gröbner basis \(G\) for any monomial order on \(F\) is bounded by

\[
\deg(G) \leq 2^{\left\lfloor \frac{1}{2}((Dm - 1)(n - r) + D) \right\rfloor}.
\]

**Proof.** Choose a minimal generating set of \(F\)(it\(M\)) with degrees \(d_1 \geq \cdots \geq d_k\) and use Lemma \ref{lem:degrees} to bound \(d_1, \ldots, d_{n-r}\), then apply Theorem \ref{thm:standard} 

To get a bound that does not depend on the dimension, we replace \(n\) by \(n + 1\) in the bound given by Corollary \ref{cor:constant}.

**Corollary 55.** Let \(M \subseteq F\) be a submodule generated by elements of maximum degree \(D\) with \(D \geq l\). Then the degree of the reduced Gröbner basis \(G\) of \(M\) for any monomial order on \(F\) is bounded by

\[
\deg(G) \leq 2(Dm)^{2^r}.
\]

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