Exceptional points make an astroid in non-Hermitian Lieb lattice: evolution and topological protection
Yi-Xin Xiao$^{1,2}$, Kun Ding$^{2,3}$, Ruo-Yang Zhang$^{2}$, Zhi Hong Hang$^{1,4}$, C. T. Chan$^{2*}$

1School of Physical Science and Technology, Soochow University, Suzhou 215006, China
2Department of Physics, Hong Kong University of Science and Technology, Hong Kong, China
3The Blackett Laboratory, Department of Physics, Imperial College London, London SW7 2AZ, United Kingdom
4Institute for Advanced Study, Soochow University, Suzhou 215006, China
*Corresponding author: phchan@ust.hk

An astroid loop of exceptional points (EPs), comprising four cusps, is found to spawn from the triple degeneracy point in the Brillouin zone (BZ) of a Lieb lattice with nearest-neighbor hoppings when non-Hermiticity is introduced. The occurrence of the EP loop is due to the realness of the discriminant which is guaranteed by chiral symmetry. The EPs at the four cusps involve the coalescence of three eigenstates, which is the combined result of chiral symmetry and mirror-T symmetry. The EP loop is exactly an astroid in the limit of an infinitesimal non-Hermiticity. The EP loop expands from the M point with increasing non-Hermiticity and splits into two EP loops at a critical non-Hermiticity. The further increase of non-Hermiticity contracts the two EP loops towards and finally to two EPs at the X and Y points in the BZ, accompanied by the emergence of Dirac-like cones. The two EPs vanish at a larger non-Hermiticity. The EP loop disappears and several discrete EPs are found to survive when next-nearest hoppings are introduced to break the chiral symmetry. A topological invariant called the discriminant number is used to characterize their robustness against perturbations. Both discrete EPs and those on the EP loop(s) are found to show anisotropic asymptotic behaviors. Finally, the experimental realization of the Lieb lattice using a coupled waveguide array is discussed.

I. introduction

The occurrence of exceptional points (EPs) is the most prominent phenomenon in non-Hermitian systems among others [1–9], which have been investigated in optics [10,11], acoustics [12,13], electric circuits [14,15], to name a few. EPs have brought about a wealth of promising applications, such as lasing [16–18], enhanced sensitivity [9], etc. It has been the theme for many recent works to extend topological band theory to non-Hermitian Hamiltonians [19–24]. Apart from the ubiquitous doubly degenerate EPs, higher-order EPs [9,25] have also been studied extensively. It was stated that EPs of higher order are usually unstable and tend to split into EPs of order 2 in the absence of symmetry protection [26].

EPs may form a continuum of various shapes, such as EP lines [10,27], rings [11,28–31], ellipses [25], and surfaces [32–34]. Exceptional rings were found to emerge from non-defective degeneracy points (NDPs) such as two-fold degenerate Weyl points when non-Hermiticity is introduced [11,28,35,36]. It is intriguing to understand how a higher order NDP is affected by non-Hermiticity, especially whether a higher-order EP ring
will form. This motivates us to explore the non-Hermitian Lieb lattice \([37–39]\) which possesses a triple degeneracy in the Hermitian scenario.

In the Lieb lattice with only nearest-neighbor (NN) hoppings included, we found that exceptional points spawn from the triple degeneracy point and form a closed loop, which takes an astroid shape when the non-Hermiticity is small. In particular, four EPs of order 3 are found at the cusps of the astroid. The EP loop and EPs of order 3 are protected by chiral symmetry. Increasing non-Hermiticity leads to the expansion and splitting of the EP loop into two loops. Further increase in non-Hermiticity induces the two EP loops to contract towards and finally vanish at the Brillouin zone (BZ) boundary. The introduction of next-nearest-neighbor (NNN) hoppings turns the EP loop to several discrete EPs, which show robustness against various perturbations. The robustness is guaranteed by a nonzero discriminant number, which is a topological invariant defined for an EP \([26]\).

The work is organized as follows. Sec. II covers results for the system with NN hoppings, and Sec. III takes NNN hoppings into account. In Sec. II, we will see that the Lieb lattice exhibits many interesting results, such as EP loop with cusps, interesting evolution of EP trajectories, Dirac-like cones, and anisotropic EPs. Symmetry analysis and explanations of various phenomena will be presented. In Sec. III, we will show the existence of discrete EPs in the presence of NNN hoppings and their interesting evolution behaviors. The discriminant number, a generalized winding number, is used to explain the robustness and describe the relevant creation/annihilation behaviors of EPs. In Sec. IV, we will discuss the effect of perturbations on both the EP loop and discrete EPs. In Sec. V, we discuss the experimental relevance of the model systems. Finally, we conclude our results in Sec. VI.

II. Non-Hermitian Lieb lattice with NN hoppings

A. EP loop with cusps

The Lieb lattice with balanced gain and loss \((\pm \text{i} \gamma)\) is shown in Fig. 1(a). Three sites in the unit cell are labeled as \(A, B, C\). The NN and NNN hoppings are denoted by \(t\) and \(w\). We assume a unity lattice constant, namely \(a = 1\), and that the lattice is periodic in the \(xy\)-plane. The Bloch Hamiltonian is

\[
H(\mathbf{k}) = \begin{pmatrix}
0 & T_{AB} & T_{AC} \\
T_{AB} & 0 & T_{BC} \\
T_{AC} & T_{BC} & -\text{i} \gamma
\end{pmatrix},
\]

where \(T_{AB} = 2t \cos \left(\frac{k_x}{2}\right)\) and \(T_{BC} = 2t \cos \left(\frac{k_x}{2}\right)\) and \(T_{AC} = 4w \cos \left(\frac{k_x}{2}\right) \cos \left(\frac{k_y}{2}\right)\). Without loss of generality, \(t = 1\) is assumed. In this section, only NN hoppings are considered, that is, \(w = 0\) is taken. The \(w \neq 0\) scenario will be discussed in Sec. III. For convenience, we will call the Lieb lattice with (without) NNN hoppings as the NNN (NN) system.

The band structure for the Hermitian system with \(\gamma = 0\) is shown in Fig. 1(b), where a flat band exists at zero energy due to chiral (sublattice) symmetry. Chiral symmetry also dictates that the other two bands are
symmetric about $E = 0$. Interestingly, a triple degeneracy occurs at $M = (\pi, \pi)$ as a consequence of chiral symmetry and $C_{4v}$ symmetry, though $C_{4v}$ symmetry alone only admits a double degeneracy. Note the plain letters $M, \Gamma, X, Y$ without arrows are used to denote the special $\mathbf{k}$ vectors in our notation. The BZ is chosen such that the degeneracy point, i.e., $M$, is at the center of the figure. Incidentally, the triple degeneracy holds even when NNN hoppings are included because $T_{AC} = 0$ at $M = (\pi, \pi)$.

An infinitesimally small $\gamma$ would turn the triple degeneracy point into a very small loop of EPs, centered at $M$. The loop has exactly an astroid shape characterized by four cusps [40,41], as will be shown soon. The EP loop expands from $M$ with the astroid shape roughly preserved when $\gamma$ is increased gradually.

As a representative for scenarios with small and moderate values of $\gamma$, the band structure for $\gamma = 2$ is shown in Figs. 1(c) and 1(d). The colors in Fig. 1(d) are taken in accordance with Fig. 1(c) and they exhibit discontinuity due to branch cuts. For clarity, only a quarter of BZ is shown in Fig. 1(d) using the fact that $E(-k_x, k_y) = E(k_x, k_y) = E(k_x, -k_y)$ due to mirror symmetries (See Eq. (A1) in the Appendix A). An EP loop with four cusps, instead of a smooth EP ring, is observed in the BZ, which is represented by cyan and magenta arcs. The EP loop resembles an astroid which is the exact shape of the EP loop for small $\gamma$. The four intersection points marked by gray dots are EPs of order 3, and all other points on the arcs between cusps are EPs of order 2. We refer to them as EP3s and EP2s. The coalesced eigenvalues have positive (negative) imaginary parts at cyan (magenta) arcs as seen in Fig. 1(d). The EP loop is centered at $M = (\pi, \pi)$ in the BZ as shown in Fig. 1(c).

Intuitively, the existence of an EP3 necessitates stricter requirements compared with an EP2, and consequently, there are only four discrete EP3s but four arcs of EP2s. EP3s should surely satisfy the conditions of EP2s and hence they join the continuum formed by EP2s. The explicit conditions for their occurrence will be provided shortly.

### B. Chiral symmetry and $C_{4v}T$ group

In this subsection, we examine the symmetries in the non-Hermitian Lieb lattice and their consequences. There are five types of symmetries: (1) chiral symmetry; (2) inversion symmetry $i$; (3) mirror symmetries $\sigma_x$ ($\sigma_y$) in the $x$ ($y$) direction; (4) mirror-$T$ symmetries $\sigma_{dT}$ ($\sigma_d'T$) in the diagonal directions $y = x$ ($y = -x$), where $T$ denotes time-reversal and “d” stands for “diagonal”; and (5) $C_{4v}T$ symmetries. The latter four kinds of symmetries together with the unit element $E$ form a group composed of $E, i, \sigma_x, \sigma_y, \sigma_{dT}, \sigma_d'T, C_4T, C_4^{-1}T$, which is the 2D magnetic group $p4'mmm'$ [42]. We will refer to it as the $C_{4v}T$ group for convenience. It reduces to the $C_{4v}$ group comprising $E, i, \sigma_x, \sigma_y, \sigma_{d}, \sigma_d', C_4, C_4^{-1}$ when $\gamma = 0$.

The system has chiral symmetry, imposing the condition $\Sigma^{-1}H^\dagger(\mathbf{k})\Sigma = -H(\mathbf{k})$, which we note takes a different form from that in the Hermitian systems due to $H^\dagger(\mathbf{k}) \neq H(\mathbf{k})$ [43]. Here $\Sigma = diag(1, -1, 1)$ is a diagonal matrix. It belongs to the AZ symmetry class in the 38-fold symmetry classification scheme for non-Hermitian systems [43]. We note a recent work that shows exceptional rings protected by chiral
symmetry in correlated systems [29]. The exceptional loops in the chiral symmetric systems are found to be characterized by the $\mathbb{Z}$ topological invariant leading to topological protection [44]. Chiral symmetry dictates that the eigenvalues come in pairs of $(E, -E^*)$. For the three eigenvalues in our system, either $E_1 = -E_3^*$, $E_2 = -E_2^*$ or $E_i = -E_i^*$, $i = 1, 2, 3$ is satisfied, as is observed in Figs. 1(c) and 1(d). In particular the eigenvalue denoted as $E_2$ must always satisfy $E_2(k) = -E_2^*(k)$ and hence be purely imaginary, which manifests as a flat band with $Re(E) = 0$ in Fig. 1(c). Explicitly, the three eigenvalues have the form as either

$$E_{1,3} = \pm a + ib, E_2 = ic,$$  \hfill (2)\ne

or

$$E_m = i\lambda_m, m = 1, 2, 3,$$  \hfill (3)\ne

where $a, b, c, \lambda_m$ are real. As a consequence, the characteristic polynomial takes the form $f(E, k) = \prod_{i=1}^{3}[E - E_i(k)] = E^3 + A_1 E + iA_2$ with $A_1, A_2$ being real, where another condition $\sum_i E_i(k) = 0$ leads to the absence of the $E^2$ term.

More detailed discussions of the $C_{4v}T$ group symmetries can be found in the Appendix. A. Due to mirror-$T$ symmetries $\sigma_d T$ and $\sigma_d T$, the eigenvalues on the BZ diagonals $k_y = \pm k_x$ satisfy either (i) $E_1 = E_3^*$ and $E_2 = E_2^*$ or (ii) $E_i = E_i^*$, $i = 1, 2, 3$ (see the Appendix A), which appears to contradict the constraints, Eqs. (2) and (3), imposed by chiral symmetry given a moment ago. It turns out however that the constraint (i) and Eq. (3) are compatible, leading to $E_2 = 0$ and $E_1 = -E_3 = ia$, where $a$ is real. Similarly the constraint (ii) and Eq. (2) are compatible, leading to $E_2 = 0$ and $E_1 = -E_3 = b$, where $b$ is real. To sum up, chiral symmetry and mirror-$T$ symmetries together dictate the three eigenvalues to be

$$E = \pm \mu, 0$$  \hfill (4)\ne

on $k_y = \pm k_x$, where $\mu$ is either real or imaginary. The transition point between the two scenarios, $\mu$ being either real or imaginary, corresponds to $E_1 = E_2 = E_3 = 0$ and is where an EP3 occurs.

**C. Origin of EP loop and EP3s**

The eigenvalues of $H(k)$ are the roots of its characteristic polynomial $f(E, \bar{k}) = \text{det}[E - H(k)]$. Since the trace of $H(k)$ vanishes, the eigenvalues must always satisfy $\sum_i E_i(k) = 0$. Consequently, $f(E, k)$ is a depressed cubic polynomial without the $E^2$ term. Written explicitly,

$$f(E, k) = -E^3 + a_1(k)E + a_0(k),$$  \hfill (5)\ne

where

$$a_1(k) = T_{AB}^2 + T_{BC}^2 - k^2, \quad a_0(k) = i\gamma(T_{AB}^2 - T_{BC}^2),$$  \hfill (6)\ne

which is in the form $f(E, \bar{k}) = E^3 + A_1 E + iA_2$, where $A_1, A_2$ are real, as predicted by chiral symmetry earlier. In compact notation, $f(E, k) = \sum_{m=0}^{3} a_m(k)E^m$ where $a_3(k) = -1$ and $a_2(k) = 0$.

The locations of degeneracy points (DPs) in the BZ, including EPs and non-defective degeneracies points (NDPs), can be directly determined by solving $\Delta(k) = 0$, where $\Delta(k)$ is the discriminant of $f(E, k)$.
and defined as \( \Delta(\vec{k}) = \prod_{i<j} [E_i(\vec{k}) - E_j(\vec{k})]^2 \). A degeneracy point is an EP if two or more eigenvectors coalesce. The constraints imposed by chiral symmetry, namely Eqs. (2) and (3), immediately lead to the conclusion that \( \Delta(\vec{k}) \) is purely real, which is a key ingredient resulting in the EP loop and EP3s.

For \( f(E, \vec{k}) \) in Eq. (5), \( \Delta(\vec{k}) \) takes a simple form (See the Appendix B for the derivation)

\[
\Delta(\vec{k}) = -4a_1^3 - 27a_0^2,
\]

(7)
since \( a_3 = -1 \) and \( a_2 = 0 \). \( \Delta(\vec{k}) \) is indeed purely real as dictated by chiral symmetry. All \( \vec{k} \) points that satisfy \( \Delta(\vec{k}) = 0 \) correspond to double or triple degeneracy points, which are most probably EP2s or EP3s due to the non-Hermitian nature of the Hamiltonian. Non-defective degeneracies can also occur in some rare cases (See the Appendix E for examples of NDPs). The multiplicity remains to be determined by additional constraints. We observe that the only possible triple root of \( f(E, \vec{k}) \) in Eq. (5) is \( E = 0 \), which requires \( a_1(\vec{k}) = 0 \) and \( a_0(\vec{k}) = 0 \). This is consistent with the fact that triple DPs (actually EP3s involving the coalescence of 3 eigenvectors) occur at \( E = 0 \) as shown in Figs. 1(c) and 1(d). Obviously \( \Delta(\vec{k}) = 0 \) and \( a_1(\vec{k}) = 0 \) implies \( a_0(\vec{k}) = 0 \). Therefore, the constraints for the existence of a triple DP are simply

\[
\Delta(\vec{k}) = 0 \quad \text{and} \quad a_1(\vec{k}) = 0,
\]

(8)
where \( \Delta(\vec{k}) \) has the explicit form,

\[
\Delta(\vec{k}) = 4(T_{AB}^2 + T_{BC}^2 - \gamma^2)^3 + 27\gamma^2(T_{BC}^2 - T_{AB}^2)^2,
\]

(9)
with \( T_{AB} = 2t \cos \left( \frac{k_y}{2} \right) \) and \( T_{BC} = 2t \cos \left( \frac{k_x}{2} \right) \). Incidentally Eq. (7) is satisfied at \( M = (\pi, \pi) \) when \( \gamma = 0 \), which corresponds to a triple NDP. Naturally, a point \( \vec{k} \) that satisfies \( \Delta(\vec{k}) = 0 \) and \( a_1(\vec{k}) \neq 0 \) is a double DP. For the general situation when \( a_2(\vec{k}) \neq 0 \), the constraints for triple DPs can be derived by first transforming \( f \) to a form without \( E^2 \) term using a change of variables \( \epsilon = E - a_2/3 \). The constraints are found to be \( \Delta(\vec{k}) = 0 \) and \( a_1^2(\vec{k}) - 3a_1(\vec{k})a_3(\vec{k}) = 0 \). A double (triple) DP is an EP2 (EP3) if the 2 (3) eigenvectors associated with the DP coalesce to a single eigenvector. It is confirmed that all degeneracy points marked in Fig. 1(c) correspond to EP2s except that the four cusps correspond to EP3s. In rare cases, a triple DP may have two eigenvectors when the coalesced eigenvector corresponding to an EP2 and the third eigenvector have the same eigenvalue (See the Appendix E).

We emphasize that the realness of \( \Delta(\vec{k}) \) in Eq. (6) is crucial for the occurrence of EP2 continua and EP3s, though it is complex in general (e.g., when NNN hoppings are included). EP2s can form continua only in systems with appropriate parameters where the number of degrees of freedom (DOFs), i.e., \( k_x, k_y \), exceeds that of the constraint involved, i.e., \( \Delta(\vec{k}) \neq 0 \). In contrast, the existence of EP3s necessitates one more constraint, namely \( a_1(\vec{k}) = 0 \), and therefore the number of constraints equals that of DOFs. As a consequence, EP3s only occur at four isolated \( \vec{k} \) points (\( \pi \pm q, \pi \pm q \)), where \( q = 2 \sin^{-1} \left[ \frac{\sqrt{2} \gamma}{4} \right] \). The function \( \sin^{-1} \left[ \frac{\sqrt{2} \gamma}{4} \right] \) indicates that EP3s cannot exist when \( \gamma > 2\sqrt{2} \). For the specific value \( \gamma = 2 \), 4 EP3s occur at \( \left( \frac{\pi}{2}, \frac{3\pi}{2} \right), \left( \frac{\pi}{2}, \frac{\pi}{2} \right), \left( \frac{3\pi}{2}, \frac{\pi}{2} \right) \) and \( \left( \frac{3\pi}{2}, \frac{3\pi}{2} \right) \) as are shown in Figs. 1(c) and 1(d). All three eigenvalues as
well as eigenvectors coalesce at an EP3. For example, the coalesced eigenvalue and coalesced right
eigenvector at \( \vec{k} = \left( \frac{\pi}{2}, \frac{\pi}{2} \right) \) are \( E = 0 \) and \( |\psi^R| = \left( -1, i\sqrt{2}, 1 \right)^T \), respectively. All the EP3s are located at the
two BZ diagonals \( k_y = k_x \) and \( k_y = 2\pi - k_x \), which is protected by chiral symmetry and mirror-T
symmetries as discussed in the previous subsection. EP3s occur as special solutions to the EP2
constraint \( \Delta(\vec{k}) = 0 \), so that they together form a continuum, i.e., the EP loop, as reflected in Figs. 1(c) and
1(d). As a supplement, the contour lines of \( \Delta(\vec{k}) = 0 \) and \( a_y(\vec{k}) = 0 \) are illustrated in the Appendix C.

Since \( \Delta(\vec{k}) \) is a continuous real periodic function of \( \vec{k} \) for the NN system, the EPs given by the contour
line \( \Delta(\vec{k}) = 0 \) must form one or several loops in the BZ if solutions exist, as is demonstrated in Figs. 1(c) and
Fig. 2(a). We note there are also situations when EP2s or EP3s do not exist if the relevant constraints
could not be satisfied by any \( \vec{k} \) in the BZ.

We note that chiral symmetry, i.e., \( \Sigma^{-1}H^+(\vec{k})\Sigma = -H(\vec{k}) \), is preserved in the more general situation
when the three onsite energies \( \epsilon_A, \epsilon_B, \epsilon_C \) take arbitrary imaginary values. The characteristic polynomial
generally contains the \( E^2 \) term since the trace of \( H(\vec{k}) \) is nonzero. EP3s exist only when the
constraints \( \Delta(\vec{k}) = 0 \) and \( a_y^2(\vec{k}) - 3a_xa_3(\vec{k}) = 0 \) can be simultaneously satisfied in the BZ. The presence
of mirror-T symmetries is not a necessary condition for the existence of EP3s. EP2s exist only when \( \Delta(\vec{k}) = 0 \) has solutions in the BZ. If EP2s exist in such systems they would also form continuas (See the Appendix E
about some other chiral symmetric systems). The NN system is thus a subset of chiral symmetric systems
that may admit EP loops and EP3s.

**D. EP astroid and EP cusps**

We are particularly interested in the effect of a vanishingly small \( \gamma \) on the non-defective triplet
degeneracy at \( M \). It is natural to assume that EPs would occur right near \( M \) when \( \gamma \ll 1 \). For \( \vec{k} = M + \vec{q} \)
with \( |\vec{q}| \ll 1 \), \( T_{AB} = -q_y \) and \( T_{BC} = -q_x \) are small quantities. The discriminant \( \Delta(\vec{k}) \) can be well
approximated by

\[
\Delta^a(\vec{k}) = 27\gamma^2(q_x^2 - q_y^2)^2 + 4(q_x^2 + q_y^2 - \gamma^2)^3, \tag{10}
\]

where the superscript “\( a \)” stands for “approximation” and \( q_i = k_i - \pi, i = x, y \). The EP loop is then given
by \( \Delta^a(\vec{k}) = 0 \), which can be reduced to the standard equation of an astroid,

\[
Q_x^{2/3} + Q_y^{2/3} = \gamma^{2/3}, \tag{11}
\]

by employing the substitutions \( q_x = (Q_x - Q_y)/\sqrt{2} \) and \( q_y = (Q_x + Q_y)/\sqrt{2} \). Thus the EP loop is exactly
an astroid in the limit of an infinitesimal non-Hermiticity.

We observe that the vanishing of \( \Delta^a(\vec{k}) \) in Eq. (10) can be recast as

\[
27 \left( \frac{q_x^2}{\gamma^2} - \frac{q_y^2}{\gamma^2} \right)^2 + 4 \left( \frac{q_x^2}{\gamma^2} + \frac{q_y^2}{\gamma^2} - 1 \right)^3 = 0, \tag{12}
\]
which indicates that the solutions \( q_x, q_y \) are proportional to \( \gamma \) and thus the EP loops for small \( \gamma \) values should scale with \( \gamma \). A comparison between \( \Delta^a(k) = 0 \) and \( \Delta(k) = 0 \) is made in the Appendix C.

We are also interested in the asymptotic form of the EP loop near an EP3 cusp. We take the EP3 at \( \left( \frac{1}{2}, \frac{1}{2} \right) \pi \) for \( \gamma = 2 \) as an example. Near it \( \Delta(k) \) has the asymptotic form

\[
\Delta \approx 48 \left( 9k_x'^2 + 9k_y'^2 - 2k_x'^2k_y' - 2k_x'k_y'^2 - 18k_x'k_y' \right),
\]

where \( k_i' = k_i - \frac{\pi}{2}, i = x, y \) satisfy \( |k_i'| < 1 \) and only lower orders of \( k_x', k_y' \) are kept. Eq. (14) can be further reduced to

\[
\Delta \approx 48 \left( -\sqrt{2}u^3 + 18v^2 + \sqrt{2}uv^2 \right),
\]

by making the substitutions \( k_x' = \frac{1}{\sqrt{2}}(u-v) \) and \( k_y' = \frac{1}{\sqrt{2}}(u+v) \), that is, rotating the axes by \( \pi/4 \). The vanishing of Eq. (14) gives us

\[
v = \pm \frac{2^{1/4}}{\sqrt{1+\sqrt{2}u}} u^{3/2} = \pm \frac{1}{3^{1/4} \sqrt{2}} u^{3/2},
\]

for \(|u| < 1\), which is the so-called ordinary cusp [45].

E. Evolution of EP loop as non-Hermiticity changes

We know from Eq. (12) that the EP loop expands when \( \gamma \) is increased. In particular, the four EP3s are always located at \((\pi \pm q, \pi \pm q)\), where \( q = 2 \sin^{-1}\left(\frac{\sqrt{2}}{4}\right) \) grows with increasing \( \gamma \). EP3s would vanish when \( \gamma > 2\sqrt{2} \).

Numerically we found that when \( \gamma \) is increased continuously, the EP loop expands until the EPs vanish at the BZ boundary when \( \gamma = 4 \). We illustrate the evolution of the EP loops by showing the loops for 4 typical values of \( \gamma \), namely \( \gamma = 0.5, 2, 2\sqrt{2}, 3.9 \), in Fig. 2(a). The whole evolution process is shown in the supplemental video. The EPs indeed form one or several closed loops as is discussed earlier. The astroid shape remains roughly unchanged during the expansion especially for small \( \gamma \) as expected from Eq. (12). As a supplement, the EP loops on the BZ torus is also shown in Fig. 2 (c) utilizing the periodicity. Every \( \vec{k} = (k_x, k_y) \) in the BZ is mapped to a point \( \vec{r} = (x, y, z) \) on the BZ torus, where \( x = \cos k_x \left( R + r \cos k_y \right), y = \sin k_x \left( R + r \cos k_y \right), z = r \sin k_y \), and \( R \) is the distance from the torus center to the tube center, and \( r \) is the tube radius. The toroidal and poloidal directions are parameterized by \( k_x \) and \( k_y \), respectively. Without loss of generality \( r = 1 \) and \( R = 3 \) are taken in the plotting. The four special momenta \( \Gamma, X, Y, M \) are marked on the BZ torus.

\( \gamma = 2\sqrt{2} \) is a transition point from a single EP loop centered at \( M \) to two EP loops, which are centered at \( X = (\pi, 0) \) and \( Y = (0, \pi) \), respectively. At \( \gamma = 2\sqrt{2} \), the four EP3s converge to a single EP3 at \( \Gamma \). The EP3 vanishes for \( \gamma > 2\sqrt{2} \) because Eq. (8) is no longer satisfied by any \( \vec{k} \) in the BZ. As the EPs moving away from \( M \), the two split EP loops are essentially contracting towards and finally vanish at \( X \) and \( Y \),
respectively, when \( \gamma \) is tuned from \( 2\sqrt{2} \) to 4. Thus \( \gamma = 2\sqrt{2} \) and \( \gamma = 4 \) are two transition points, where the number of EP loop changes from 1 to 2 and then to 0.

As seen in Fig. 1(c), the real parts of eigenvalues all vanish around the EP loop. As a supplement to Fig. 1 (d), the imaginary part of energy bands around the EP loop for \( \gamma = 2 \) is shown in Fig. 2(b), where \( k_1,k_2,k_3,k_4 \) denote the four EP3 cusps labeled in Fig. 2(a). The three bands are plotted by green solid, blue dashed, red dot-dashed curves, respectively. Two of the eigenvalues always coincide, signifying EP2s. All three coincide at the EP3s marked by \( k_i \). Moving along an EP2 arc across the EP3 to another EP2 arc, the degeneracy changes from \( E_1 = E_2 \) to \( E_2 = E_3 \), where \( Im(E_1) \leq Im(E_2) \leq Im(E_3) \) are assumed.

F. Exact/broken phase separation by chiral symmetry

In systems with parity-time (PT) symmetry such as \( H_0 = i\sigma_z + \eta \sigma_y \), the parameter space could be separated into an exact phase (\(|\eta| > 1\)) and a broken phase (\(|\eta| < 1\)) by the two EPs at \( \eta = \pm 1 \). The eigenvalues are real (complex conjugate pairs) in the exact (broken) phase. Moreover the eigenvectors of \( H_0 \) are also eigenvectors of the PT operator in the exact phase. In contrast, they cease to be eigenvectors of the PT operator in the broken phase, although the Hamiltonian still commutes with the PT operator [46]. Instead, the eigenvectors are related to one another by the PT operator. The Lieb lattice Hamiltonian \( H(\vec{k}) \) is PT symmetric along the diagonals, \( k_y = k_x \) and \( k_y = 2\pi - k_x \), if we regard \( H(\vec{k}) \) as a system comprising three degrees of freedom.

Due to chiral symmetry, the eigenvalues take values either like \( E_{1,3} = \pm a + ib, E_2 = ic \) and \( E_m = i\lambda_m, m = 1,2,3 \) outside of and inside the EP loop, respectively, where \( a, b, c, \lambda_m \) are real. At the EP loop, the eigenvalues have the form \( E_1 = E_2 = ib, E_2 = ic \). Chiral symmetry is called pseudo-anti-Hermiticity in earlier works [47,48], because \( iH(\vec{k}) \) respects pseudo-Hermiticity when \( H(\vec{k}) \) respects chiral symmetry [44]. Thus chiral symmetry resembles PT symmetry (or pseudo-Hermiticity) to a great extent. Since EPs form one or two loops in the BZ due to chiral symmetry, it is tempting to separate the BZ into exact and broken phases dictated by chiral symmetry, mimicking that is done in systems with PT symmetry.

Specifically, we take the \( \gamma = 2 \) case as an example. We call the region inside the EP loop with \( E_m = i\lambda_m, m = 1,2,3 \) as the chiral-exact phase, because every eigenstate conforms to the chiral symmetry, that is, the right eigenvector \( |\psi^R \rangle \) and left eigenvector \( \langle \psi^L | \) are related by \( \langle \psi^L | = (\Sigma |\psi^R \rangle)^\dagger \) where \( \Sigma \) represents chiral symmetry. The chiral-exact phase can simply be determined by taking eigenvalues of anti-Hermitian matrices as the benchmark. We refer to the outer region with \( E_{1,3} = \pm a + ib, E_2 = ic \) as the chiral-broken phase, in which a right eigenvector of \( E_i \) is generally related to a left eigenvector of a different eigenvalue \( E_j \neq E_i \) by chiral symmetry operator, analogous to that happens in the PT-broken phase. Spontaneous chiral symmetry breaking of eigenvectors hence occurs at the EPs.

To be precise, the above phase separation is restricted to two of the three eigenstates since all points on the EP loop are EP2s except the four EP3s. When crossing an EP2 from the chiral-exact phase to
chiral-broken phase, two eigenvalues change from the $i\lambda_m, i\lambda_n$ form to the $\pm a + ib$ form. And the third eigenvalue remains purely imaginary and does not involve the coalescence of states at the EP2. In contrast, the phase separation on the two diagonals is rendered by EP3s and hence involves three eigenstates. The BZ is thus divided into four regions, each of which contains a chiral-exact phase and a chiral-broken phase for two eigenstates. Two adjacent chiral-exact phase regions, which are delimited by a diagonal path, are glued smoothly by the diagonal path, as is shown in Figs. 1(c) and 1(d). The same applies to two chiral-broken phase regions delimited by a diagonal path.

Thus, the regions inside and outside of the EP loop are chiral-exact phase and chiral-broken phase, respectively, when $\gamma < 2\sqrt{2}$. On the contrary, the regions inside (outside) the EP loops for $2\sqrt{2} < \gamma < 4$ are chiral-broken (chiral-exact) phases. The chiral-exact phase region grows with increasing $\gamma$ until that the whole BZ region becomes a chiral-exact phase when $\gamma > 4$.

Intuitively, the spontaneous chiral symmetry breaking of eigenvectors, namely EPs, occurs when couplings (as a function of $\vec{k}$) are on a par with the non-Hermiticity, which becomes impossible when non-Hermiticity is strong enough, i.e. $\gamma > 4$.

**G. Anisotropic EP2s and EP3s**

The fact that EPs forming continuous curves signifies anisotropic EPs [13,25,49], which mean that the eigenvalues in the tangential direction of the curve show different asymptotic behavior near the EP from those in any other directions. They are also called hybrid points [21,50,51]. We know from the literature that the tangential path traversing the EP is special in that it does not make the familiar transition between the exact phase and broken phase, but will remain in the same phase, exact or broken, at both sides of the EP. All points on the EP2 curves are anisotropic EP2s. We will show explicitly two anisotropic EPs, namely the EP2 at $(4/3, 1)\pi$ and the EP3 at $(1/2, 1/2)\pi$, for $\gamma = 2$.

For simplicity, we will just focus on the point at $\vec{k} = (4/3, 1)\pi$, which is the intersection point of the cyan line and the magenta line marked in Fig. 3(a). The band structures for the two paths are shown in Figs. 3(b) and 3(c). The familiar square-root bifurcation behavior is seen in the real and imaginary parts of a pair of eigenvalues, as shown in Fig. 3(b). In contrast, linear crossing behavior occurs in the purely imaginary of the same eigenvalue pair if the magenta tangential path is followed. Analytical calculations show that the eigenvalue pair takes the square-root form $E_{1,2} = -i \pm 3^{1/4} \sqrt{q_x} + O(3^{3/2})$ and the linear form $E_{1,2} = -i \pm i q_y/\sqrt{3} + O(q^2)$ near the EP2 for the cases in Figs. 3(b) and 3(c), respectively, where $q_x = k_x - 4\pi/3$ and $q_y = k_y - \pi$ are small quantities. We also confirmed that, for an arbitrarily chosen EP2, eigenvalues also show anisotropic behaviors, that is, they exhibit the square-root and linear forms in the tangential and non-tangential directions, respectively.
The tangential path $k_y = k_x$ at the EP3 at $\left(\frac{1}{2}, \frac{1}{2}\right)\pi$ is represented by the red line, which is tangential to both EP2 arcs meeting at the EP3 as shown in Fig. 3(a). The path $k_y = \pi - k_x$ perpendicular to it is marked by the blue line. Distinct asymptotic behaviors of eigenvalues occur for the two paths, as shown in Figs. 3(d) and 3(e). The blue path gives the expected $\sqrt{q_x}$ asymptotic behavior near the EP3 for all three eigenvalues, where $q_x = k_x - \pi/2$. The red tangential path leads to three eigenvalues $0, \pm 2i\sqrt{q_x} + O\left(q_x^{5/2}\right)$ near the EP3, bearing the familiar square-root form. The eigenvalues of square-root form can be explicitly obtained as $-0, \pm \sqrt{2TA} - \gamma^2$ because the Hamiltonian can be recast as $H(k) = i\gamma s_x + \sqrt{2TA}s_x$ at $k_y = k_x$, where $s_x$ and $s_x$ are spin-1 matrices. In stark contrast to the anisotropic EP2s discussed previously, it is the tangential path at the EP3 that makes the exact-to-broken transition. This is one important feature that distinguishes EP cusps from other smooth EP curves such as EP rings [11] and ellipses [25], for which the tangential path at any EP exhibits the same asymptotic behaviors on both sides of the EP.

The anisotropic behavior of EP3 here is reminiscent of but different from that in Ref. [25], where eigenvalues asymptotically take square-root and linear forms, respectively. The eigenvalues shown in Fig. 3(e) take the form $\pm \mu, 0$ with $\mu$ being either real or imaginary, which is imposed by chiral symmetry and mirror-T symmetry, as discussed earlier in Sec. II. C.

Phase rigidity can well characterize the behavior of eigenstates near an EP [25,52]. It is defined as $\rho_m = |\langle \psi^l_m | \psi^r_m \rangle|$, where $\langle \psi^l_m |$ and $| \psi^r_m \rangle$ are self-normalized left and right eigenvectors for the $m$-th state. There are two limiting scenarios for the phase rigidity, which are $\rho = 0$ for eigenstates at an EP and $\rho = 1$ for eigenstates of a Hermitian system. The value of phase rigidity ranges from 0 to 1 for a general eigenstate in a non-Hermitian system. Near an EP, phase rigidity adheres to a power law $\rho_m \propto |\delta|^\chi$, where $\delta$ denotes a small deviation from an EP and the exponent $\chi$ dictates the rate that the left and right eigenvectors turn orthogonal when approaching an EP.

For each path in Fig. 3(a), the phase rigidity is calculated for an arbitrarily chosen eigenstate related to the EP2/EP3 and shown in Fig. 3(f), where “(b)”,”(c)”, “(d)”,”(e)” represent the four paths. And $q_i = k_i - k_i^{EP}$, where $i = x, y$ and $(k_x^{EP}, k_y^{EP})$ is the coordinate of the EP2/EP3, stands for the deviation from the EP2/EP3 along each path. The phase rigidity vanishes at the EP2/EP3 for all four scenarios. The phase rigidities obey $\rho \propto q_x^{1/2}$ and $\rho \propto q_y$ for the EP2-related eigenstates along paths corresponding to Figs. 3(b) and 3(c), respectively. The results agree with the power laws $\rho \propto \delta^{(N-1)/N}$ and $\rho \propto \delta^{N-1}$ reported earlier for an EP of order $N$ (EPN) which exhibits $E \propto \delta^{1/N}$ and $E \propto \delta$ asymptotic behaviors, respectively, in the non-tangential and tangential directions [25]. The power laws $\rho \propto q_x^{2/3}$ and $\rho \propto q_x$ hold for the eigenstates near the EP3 along paths corresponding to Figs. 3(d) and 3(e), respectively, which correspond to $\rho \propto \delta^{(N-1)/N}$ and $\rho \propto \delta^{(N-1)/2}$ reported for an EPN exhibiting $E \propto \delta^{1/N}$ and $E \propto \sqrt{\delta}$ behaviors in the non-tangential and tangential directions, respectively [25].
H. Dirac-like cones residing at discrete EPs

Interestingly, the imaginary part of the complex energy sheets exhibit Dirac-like cones when the EP loops contract to \(X\) and \(Y\) when \(\gamma = 4\), which gives point degeneracies between energy bands. Two Dirac-like cones manifest in the purely imaginary bands and reside at \(X\) and \(Y\), which are shown in Fig. 4(a) with BZ region chosen as \([-\pi/2, 3\pi/2] \times [-\pi/2, 3\pi/2]\) for a better view. Different from a diabolical point in the real energy sheets of Hermitian Dirac-cone physics, EPs appear at the cone vertices and the Hamiltonian matrix is defective at \(X\) and \(Y\). Such a phenomenon is new although a linear dispersion is known to occur in a single direction for anisotropic EP2s, such as shown in Fig. 3(c). The purely imaginary eigenvalues are due to chiral symmetry as discussed previously.

Each Dirac-like cone involves two bands. Focusing on the cone at \(X\), the effective two-band model near \(X\) takes the form

\[
H_{\text{eff}}(\vec{q}) = \begin{pmatrix} 4i & 2 - \frac{q_x^2}{4} \\ 2 - \frac{q_y^2}{4} & -i \frac{q_y^2}{6} \end{pmatrix},
\]

(16)

where \(\vec{q} \equiv (q_x, q_y) = \vec{k} - X\), with \(|q_x|, |q_y| \ll 1\). The eigenvalues are

\[
E_{\pm}(\vec{q}) = i \left(2 \pm \frac{1}{\sqrt{3}} q_x^2 + q_y^2 \right),
\]

(17)

where terms are kept to a linear order of \(q_x, q_y\). Obviously \(E_{\pm}(\vec{q})\) describes a Dirac-like cone, of which the horizontal cross-section is an ellipse. In contrast to the anisotropic EPs discussed very recently, the eigenvalues near the two EPs at \(X\) and \(Y\) exhibit a purely linear form in all directions. And the phase rigidity is also found to follow \(\rho(\vec{q}) \propto |\vec{q}|\) isotropically.

Another difference between the Dirac-like cone and Dirac cone in graphene is that the Berry phase for the EP at the vertex is 0 rather than \(\pi\), as calculated by \(\Phi = i \oint_L \langle \psi^k | \vec{\nabla}^k | \psi^k \rangle \cdot d\vec{k}\), where \(\langle \psi^k | \psi^k \rangle\) and \(\langle \psi^k | \psi^k \rangle\) denote left and right eigenvectors of \(H(\vec{k})\) and \(L\) denotes a small closed loop on the cone around the vertex.

As a comparison, the purely imaginary band structure for \(\gamma = 4.1\) is shown in Fig. 4(b), where gap opening occurs as anticipated. Also, the band structure for \(\gamma = 3.9\) is shown in Figs. 4(c) and 4(d) for the real part and imaginary part, respectively. The two elliptical EP2 loops, shown earlier in Fig. 2(a), manifest clearly in the band structure.

Figs. 4 as a whole show the evolution of band structure from involving a broken phase to a total gapped exact phase. Its one-dimensional version occurs in systems such as the PT-symmetric model \(H(k) = i\sigma_x + k\sigma_x + \alpha\sigma_y\), where \(\sigma_i, i = x, y, z\) are Pauli matrices. The bands \(E(k) = \pm \sqrt{k^2 + \alpha^2 - 1}\) show similar evolution from the existence of a PT-broken phase to a gapped PT-exact phase when tuning from \(\alpha < 1\) to \(\alpha > 1\).
III. Systems with NNN hoppings

A. Discrete EPs

After considering the existence of EP loops in the NN system, we consider the EP behaviors when NNN hoppings are included.

The NNN hopping \( w \) couples the “A” and “C” sites as marked in Fig. 1(a), and enters the Hamiltonian via \( T_{AC} = 4w \cos \left( \frac{k_x}{2} \right) \cos \left( \frac{k_y}{2} \right) \) in Eq. (1). All the \( C_4vT \) group symmetries hold for \( w \neq 0 \). But chiral symmetry is no longer preserved for both Hermitian (\( \gamma = 0 \)) and non-Hermitian (\( \gamma \neq 0 \)) Lieb lattices, as can be seen from the fact that eigenvalues violate the constraints imposed by chiral symmetry. However, the non-defective triple degeneracy remains intact because \( T_{AC} = 0 \) at \( M \), making the degeneracy unsusceptible to the NNN hopping. The band structure for the NNN system is characterized by a dispersive middle band compared to the flat band seen in Fig. 1(b).

Instead of an EP loop, only discrete EPs appear in the BZ when non-Hermiticity is introduced, in stark contrast with the NN systems. The typical band structure is shown for \( \gamma = 1 \) and \( w = 0.2 \) in Figs. 5(a) and 5(b), where only a quarter of BZ is shown utilizing Eq. (A1). There are only discrete EP2s, which are marked by 3 dots, black, cyan, and magenta, in Figs. 5(a) and 5(b). For clarity, we represent the locations with \( Re[E_i(\vec{k})] = Re[E_j(\vec{k})] \) by yellow points and the ones with \( Im[E_i(\vec{k})] = Im[E_j(\vec{k})] \) by green points in Fig. 5(c), where the points together form several lines. Moreover, the thick (thin) lines represent where three (two) eigenvalues have identical real/imaginary parts. The 3 meeting points between yellow and green lines correspond to the 3 EP2s in Figs. 5(a) and 5(b). Thus there are 8 EP2s in the entire BZ as denoted by dots in Fig. 5(d).

We now use the discriminant \( \Delta(\vec{k}) \) function to find the degeneracies (EPs). With \( w \neq 0 \), the characteristic polynomial still takes a depressed form of \( f(E, \vec{k}) = -E^3 + a_1E + a_0 \), where

\[
\begin{align*}
a_1 &= T_{AB}^2 + T_{BC}^2 + T_{AC}^2 - \gamma^2, \\
a_0 &= 2T_{AB}T_{BC}T_{AC} + i\gamma(T_{AB}^2 - T_{BC}^2),
\end{align*}
\]

which reduces to Eq. (6) when \( T_{AC} = 0 \). And the discriminant becomes the following form,

\[
\Delta(\vec{k}) = g_6\gamma^6 + g_4\gamma^4 + g_2\gamma^2 + ig_1\gamma + g_0,
\]

where the coefficients are

\[
\begin{align*}
g_6 &= -4, \\
g_4 &= 12(T_{AB}^2 + T_{AC}^2 + T_{BC}^2), \\
g_2 &= 3(T_{AB}^2 - 2T_{AC}^2 - 5T_{BC}^2)(5T_{AB}^2 + 2T_{AC}^2 - T_{BC}^2), \\
g_1 &= 108T_{AB}T_{AC}T_{BC}(T_{BC}^2 - T_{AB}^2), \\
g_0 &= 4(T_{AB}^2 + T_{AC}^2 + T_{BC}^2)^3 - 108T_{AB}^2T_{AC}^2T_{BC}^2.
\end{align*}
\]
The discriminant $\Delta(k)$ in Eq. (18) is a complex function of $\vec{k}$ due to the breaking of chiral symmetry, which reduces to Eq. (9) when $T_{AC} = 0$. We can write it as $\Delta = \Delta_R + i\Delta_I$, where both the real part
\[
\Delta_R = g_0 \gamma^6 + g_4 \gamma^4 + g_2 \gamma^2 + g_0
\]
(21)
and the imaginary part
\[
\Delta_I = g_1 \gamma
\]
(22)
are real functions of $\vec{k}$ in the BZ.

The analysis in Sec. II tells us that the EP2 occurs only when $\Delta_R = 0$ and $\Delta_I = 0$ are satisfied simultaneously. Now with these two constraints, only discrete EP2s are possible, just like the case for EP3s in Sec. II. For illustration, the solutions to $\Delta_R = 0$ and $\Delta_I = 0$ are plotted in Fig. 5(d) as blue curves and green lines, respectively, where their intersection points give the EP2s, marked with colored dots. The solutions to $\Delta_I = 0$ occupy the high-symmetry lines, namely $\Gamma M, XM$, and $YM$ segments in the BZ. Therefore the discrete EP2s for the NNN system are restricted to $\Gamma M, XM$, and $YM$. When $\gamma$ is increased from 0, the EP2s start out from $M$ and move along the high-symmetry lines until they get annihilated in fours at $\Gamma$ or in pairs at $X$ and $Y$. These properties are due to the $C_{4v}T$ symmetry group. In particular, the four EP2s located on $XM$ and $YM$ segments are inherited from the NN system, because $T_{AC} = 4w \cos \left(\frac{k_x}{2}\right) \cos \left(\frac{k_y}{2}\right)$ vanishes on $XM$ and $YM$. EP2s may not exist when $\Delta_R = 0$ and $\Delta_I = 0$ are incompatible for certain system parameters.

For EP3s to exist in the NNN systems, three constraints need to be satisfied, namely $\Delta_R = 0$, $\Delta_I = 0$ and $a_3 = 0$. That is generally impossible because the three constraints are independent and there are only two degrees of freedom, i.e., $k_x, k_y$ (See the illustration of the three contour lines in the Appendix D).

B. Topological protection and evolution of EPs

The complex nature of $\Delta(k) = \Delta_R + i\Delta_I$ prompts us to define a vector field
\[
\vec{D}(k) = (\Delta_R, \Delta_I).
\]
(23)
The EP2s appear at the zeros of the vector field $\vec{D}(k)$. A generalized winding number, referred to as a discriminant number, can be defined for each discrete EP2 [26],
\[
\nu(\vec{k}^{EP}) = \frac{i}{2\pi} \oint_{C(\vec{k}^{EP})} d\vec{k} \cdot \nabla_k \ln \Delta(k),
\]
(24)
where $C(\vec{k}^{EP})$ denotes a circular loop encircling an EP at $\vec{k}^{EP}$. The discriminant number is equal to the vorticity defined in Ref. [21] summed over all pairs of distinct bands [26]. For illustration, the vector field $\vec{D}(k)$ is plotted as arrows in Fig. 6 for two typical values of scenarios: (a) $\gamma = 1, w = 0.2$, and (b) $\gamma = 1, w = 0.295$, which respectively contain 8 and 12 EPs as marked by blue dots. It can be inferred from Eq. (18) that $\Delta(2\pi - k_x, k_y) = \Delta(k_x, 2\pi - k_y) = \Delta(k_x, k_y)$ which results from mirror symmetry ultimately. Immediately, the vector field satisfies $\vec{D}(2\pi - k_x, k_y) = \vec{D}(k_x, 2\pi - k_y) = \vec{D}(k_x, k_y)$. Therefore
there is a sign difference between discriminant numbers of two EP2s which are mirror-symmetric about $k_x = \pi$ or $k_y = \pi$.

The vector field $\vec{D}(\vec{k})$ at four points surrounding each EP2 is highlighted by red arrows. The EP2s on $\Gamma M$ segments are either a vortex or a saddle point [53], corresponding to a discriminant number of $-1$ or $1$, which is confirmed by numerical calculations. Green and cyan circles are used to mark the EP2s with $\nu = 1$ and $\nu = -1$, respectively.

We found $\nu(\vec{k}_{EP}) = 0$ if $\vec{k}_{EP}$ is located on $XM$ and $YM$ segments, because $\Delta(\vec{k})$ on the loop has mirror symmetry and canceling will happen in Eq. (27), as confirmed numerically. We note that symmetries are assumed to be absent in Ref. [26], which is distinct from our situation.

The discriminant number defined in Eq. (23) can be recast as $\nu(\vec{k}_{EP}) = \frac{i}{2\pi} \oint_C(\vec{k}_{EP}) \frac{d\ln \Delta}{d\theta} d\theta$, which counts the winding number of $Arg \Delta$ for a trip around $\vec{k}_{EP}$. In the following discussions, we focus on two representative EP2s, one at $\vec{k}_{EP} = (0.7, 0.7)\pi$ with $\nu(\vec{k}_{EP}^1) = 1$ and the other at $\vec{k}_{EP} = (2 \cos^{-1}(1/\nu), \pi) \approx (1.2, 1)\pi$ with $\nu(\vec{k}_{EP}^2) = 0$. The variation of $Arg \Delta$ with increasing $\theta$ along a small circular loop encircling $\vec{k}_{EP}^{(1)}$ is shown in Fig. 6(c), which explicitly demonstrates $\nu(\vec{k}_{EP}^1) = 1$.

Similarly $\nu(\vec{k}_{EP}^2) = 0$ is shown explicitly in Fig. 6(d). Specifically, the argument of $\Delta$ is exactly $0$ or $\pi$ when $\theta = 0, \pi$ due to $Im(\Delta) = 0$. The steep jumps near $\theta = \pi/2, 3\pi/2$ come as the combined consequence of the facts that $Re(\Delta)$ changes sign there and $Im(\Delta) \geq 0$ is very small relative to $|Re(\Delta)|$ except for the very neighborhood of $Re(\Delta) = 0$. And $\theta = \pi$ is a reflection point in Fig. 6(d), which results from the mirror symmetry $\sigma_y$.

Winding numbers have been used to characterize the topological robustness of some physical phenomena such as bound states in the continuum [54], which involve a vector field description. Nonzero discriminant numbers of EP2s, which can be viewed as quantized topological charges, guarantee their stability against perturbations. The fermion doubling theorem dictates the neutrality of the charges, meaning that the sum of the discriminant numbers for all the EPs in the BZ must vanish [26]. Moreover, the neutrality should be preserved when EPs are created or annihilated upon tuning system parameters, which is demonstrated by Figs. 6(a) and 6(b). Specifically, four additional EP2s with discriminant numbers $1, 1, -1, -1$ are created from $\Gamma$ in the process of continuously tuning $w = 0.2$ to $w = 0.295$, which preserves the neutrality of total charge.

To be more specific, we show the creation and annihilation behaviors of the discrete EP2s when tuning $w$ from 0.2 to 0.302 in Fig. 7. The locations of EP2s for 4 typical values of $w$ are shown in Fig. 7(a), where the EP2s with nonzero $\nu \neq 0$ evolve on $\Gamma M$ segments accompanied by both creation and annihilation, changing their number from 4 to 8 and then to 0. And those 4 EP2s with $\nu = 0$ remain fixed due to their independence from $w$. Four additional EP2s are created from $\Gamma$ when $w$ is increased to
around 0.29, as depicted in Fig. 7(a). A further increase causes two EP2s on each of the four $\Gamma M$ segments to approach and finally annihilate in pairs when $w$ is slightly over 0.3.

The band structures along $\Gamma M$ are shown in Figs. 7(b) and 7(c) for the real parts and imaginary parts, respectively. Only the two bands relevant to the EP2s are plotted. In addition to the four cases shown in Fig. 7(a), the band structure for $w = 0.302$ is plotted at the last row in Figs. 7(b) and 7(c), which shows a band opening in $\text{Im}(E)$ after the annihilation of two EP2s. The evolution of the EP2s with $v \neq 0$ is explicitly exhibited in the variations of the band structures.

The EP2s with a nonzero discriminant number are robust against perturbation, which is demonstrated in Fig. 7, where an EP2 remains intact with changing $w$ until they get annihilated by another EP2 with an opposite charge. Such topological protection originates from the fact that the discriminant number is an integer and has to remain unchanged because they can only vary continuously when the system is changed continuously. The topologically protected robustness of EP2s with $v \neq 0$ against perturbations will be shown next.

C. Anisotropic behavior at discrete EP2s

We found that the discrete EP2s in Fig. 5(d) exhibit similar anisotropic behaviors as those for EPs residing on the EP loop, as is shown in Fig. 3. This is very surprising because there is no longer a special tangential direction of an EP loop which is different from other directions, as in the case of EPs in Fig. 3(a). This anisotropy is due to the particular band structure shown in Figs. 5(a) and 5(b), which inherited many features of the band structure of the NN system shown in Figs. 1(c) and 1(d). Recalling Dirac-like cones shown in Fig. 4(a), we have observed both anisotropic and isotropic behaviors for discrete EP2s. We know from previous work [25] that anisotropic EPs essentially mean that there is one special direction for an EP so that it feels a different type of perturbation compared to other directions, as is demonstrated in Fig. 3. Different types of perturbations may correspond to perturbations introduced to different entries of the Hamiltonian matrix, or perturbations of different form, etc. More precisely the perturbation felt in that special direction is weaker in the sense that the change of eigenvalue will not be as drastic as in other directions for perturbations of the same magnitude, for example, $\delta E \propto q_x^{1/2}$ in Fig. 3(e) compared to $\delta E \propto q_x^{1/3}$ in Fig. 3(d).

IV. Effects of perturbation

After showing the interesting EP behaviors in Sec. II and III, we want to investigate the effect of perturbations on the EPs.

For the NN system with $\gamma = 2$, the presence of EP2 curves and EP3s necessitates chiral symmetry. Now we introduce a chiral-symmetry-breaking perturbation into the NN system so that the Hamiltonian becomes $H' = H + \Delta H$. Now we take $\Delta H = \delta s_x = \delta \text{Diag}(1, 0, -1)$, which is a diagonal matrix and $\delta$ is
the perturbation strength. The EP loop is plotted in black in Fig. 8(a). An infinitesimal perturbation wipes out all the EP2 arcs immediately and splits every EP3s into two EP2s, due to the breaking of chiral symmetry. A continuous increase in \( \delta \) traces out the evolutionary trajectories of the 8 EP2s split from the 4 EP3s, which are represented by the curved arrows. The EP2s for \( \delta = 0.15, 0.3, 0.5, 0.7 \) are shown by circles with different colors. An EP2 on a red (blue) trajectory has \( \nu = 1 \) (\( \nu = -1 \)), which again shows the preservation of \( \nu \) and therefore the robustness of discrete EP2s against varying \( \delta \). The total charge keeps neutral during the evolution, as dictated by the fermion doubling theorem [26]. Pairs of EP2s with \( \nu = \pm 1 \) will annihilate at the BZ boundaries when \( \delta \) is increased to some critical value, again preserving the neutrality. Thus the discrete EP2s generated are robust though the EP loop is fragile against chiral-symmetry-breaking.

For the NNN system with \( \gamma = 1 \) and \( w = 0.2 \), the 8 discrete EP2s are shown as black dots in Fig. 8(b). Still, we introduce perturbation in the form of \( \Delta H = \delta s_z \). The 8 EP2s with different discriminant numbers behave differently for a nonzero \( \delta \). The EP2s with \( \nu = \pm 1 \), located on the BZ diagonals, only shift slightly for a small \( \delta \) as expected from the topological robustness. The EP2s with \( \nu = 0 \) originate from the preserved chiral symmetry along \( XM \) and \( YM \) and are robust against perturbations that preserve chiral symmetry, for example, \( \Delta H = i\delta \text{diag}(1,2,3) \) with \( \delta \ll 1 \). However, they are fragile under perturbations that break chiral symmetry, such as \( \Delta H = \delta s_z \), as is shown in Fig. 8(b). Without topological protection, the EP2s with \( \nu = 0 \) are either wiped out or split into two EP2s with \( \nu = \pm 1 \). The two scenarios correspond to the EP2s located at \( XM \) and \( YM \), respectively. Thus, the EP2s with \( \nu = 0 \) make a stark contrast to those EP2s with \( \nu = \pm 1 \), which do not involve chiral symmetry, when responding to perturbations that breaks chiral symmetry.

The total number of EP2s in the BZ remains to be 8 after destruction and splitting. The curved arrows represent the evolutionary trajectories of the EP2s with increasing \( \delta \), where red and blue colors mean \( \nu = 1 \) and \( \nu = -1 \), respectively. All the EP2s get annihilated in pairs at the BZ boundary for some large \( \delta \). The neutrality of topological charges is always preserved.

We mention that the introduction of hoppings from further neighbors does not bring qualitative difference compared to the NNN system. Discrete EP2s with \( \nu \neq 0 \) can exist for appropriate parameters and preserve the robustness protected by non-trivial discriminant numbers.

To conclude, discrete EP2s are ubiquitous in the non-Hermitian Lieb lattice system because the degrees of freedom and constraints are equal in number, whereas EP3s and EP loop are fragile against chiral-symmetry-breaking. The robustness of discrete EP2s against perturbations is well characterized by a nonzero discriminant number.

V. Possible realization in real systems

An evanescently coupled optical waveguide array [37,38] is a possible platform for the realization of non-Hermitian Lieb lattice. A schematic representation of a Lieb photonic lattice, comprising optical
waveguides periodically arranged in the $x$ and $y$ directions, is shown in Fig. 9(a). The axes of the waveguides are along the $z$ direction. The evolution of light in the $z$ direction of the Lieb photonic lattice can be well described by a discrete Schrödinger equation [37,38],

$$ -i \frac{d}{dz} \varphi_{\vec{m}} = \beta_{\vec{m}} \varphi_{\vec{m}} + \sum_{\vec{n}} V_{\vec{m},\vec{n}} \varphi_{\vec{n}}, \quad \text{(25)} $$

where the $\varphi_{\vec{m}}$ denotes the light amplitude at the waveguide indexed by a two-component vector $\vec{m}$. All waveguides are assumed to be identical and therefore have the same propagation constant $\beta_0$ in the absence of non-Hermiticity. The inclusion of gain and loss to “A” and “C” waveguides makes $\beta_{\vec{m}}$ to become $\beta_0 + i\gamma$ and $\beta_0 - i\gamma$, respectively. Assuming $\varphi_{\vec{m}}(z) = \varphi_{\vec{m}} e^{i\beta z}$, Eq. (25) becomes

$$ \beta \varphi_{\vec{m}} = \beta_{\vec{m}} \varphi_{\vec{m}} + \sum_{\vec{n}} V_{\vec{m},\vec{n}} \varphi_{\vec{n}}, \quad \text{(26)} $$

which mimics a tight-binding model with the propagation constant $\beta$ substituted for the eigen-energy $E$. For the Lieb photonic lattice, Eq. (26) can be recast into $M(\vec{k}) \Phi = \beta \Phi$, where $\Phi = (u_A, u_B, u_C)^T$ denotes the cell-periodic Bloch function and $M(\vec{k})$ takes the same form as $H(\vec{k})$ in Eq. (1) except for an identity matrix $\beta_0 I$, if the NN and NNN hoppings are still denoted as $t$ and $w$. The matrix $M(\vec{k})$ is thus an analog of the Hamiltonian $H(\vec{k})$.

The band structure of $\beta$ as a function of $k_x$ and $k_y$, obtained from a full-wave simulation, is shown in Figs. 9(b) and 9(c) with data points represented by dots. The lattice constant and radii of the cylinders are $44\mu m$ and $10\mu m$, respectively. The working wavelength is $\lambda = 1\mu m$. The refractive indices for the background medium and the waveguides are 1.473 and 1.4737. The refractive indices for the “A” and ”C” waveguides contain imaginary parts of $-10^{-5}$ and $10^{-5}$, respectively, corresponding to gain and loss. The red curves, from fitting to the tight-binding model $M(\vec{k})$, agree well with the simulation results. They correspond to the situation in Fig. 6(a) where there are 8 EP2s in the BZ. The case with 12 EP2s shown in Fig. 6(b) and the evolution behaviors shown in Fig. 7 can also be realized in the waveguide array platform. Couplings between further neighbors could be safely ignored, and they actually do not affect much due to the topologically protected robustness. The discrete EP2s, discussed in Sec. III, appear at $\Gamma M, MX$, and $MY$ as expected, which shows that the Lieb photonic lattice is a good candidate for the realization of the physics discussed.

The material loss of the fabricated waveguide array is often negligible. Practically gain/loss could be replaced by staggered losses added to different waveguides. The loss could be engineered, for example, by creating breaks during the fabrication of waveguides [36].

With great flexibility in tailoring the geometric and material parameters and choosing working frequency, the Lieb photonic lattice should be realizable experimentally. EP2s are guaranteed to occur for appropriate system parameters due to their topologically protected robustness against perturbations. Even features of EP3s and EP2 loop can be traced out by specially tailoring system parameters so that NNN hoppings are negligible compared to NN hoppings. The band structures and the EPs could be extracted from
the output light field by a Fourier transform. Other variants discussed in the Appendix E can also be realized in the waveguide array platform.

In addition to Lieb photonic lattice, coupled acoustic cavities [12,13] are also good candidates as prototypical systems with and without NNN hopping, which can be realized by engineering the connections between the relevant acoustic cavities.

VI. Conclusions

In this work, we showed the interesting phenomenon that EPs form an astroid loop with EP3 cusps in NN coupled non-Hermitian Lieb lattices. Mathematically, the occurrence of the EP loop and EP3s for the NN system results from the realness of the discriminant guaranteed by non-Hermitian chiral symmetry. The EP loop trajectory exhibits interesting evolution behavior when non-Hermiticity is changed. Utilizing chiral symmetry, the BZ is separated into chiral-exact and chiral-broken phases delimited by the EP loop. Moreover, the remarkable phenomena of EP2s and EP3s exhibiting anisotropic behavior are shown. Dirac-like cones were found with EPs residing at the cone vertices. The EP loop is found to be fragile against perturbations that break chiral symmetry. For example, the EP loop is destroyed by the inclusion of NNN hoppings, resulting in discrete EP2s in the BZ. The discrete EP2s are robust against perturbations, as they are topologically protected by a nonzero discriminant number. The effect of perturbations on both the NN and NNN systems are studied in detail. The interesting behaviors of annihilation, splitting as well as shifting in the BZ manifest the topological protection of discrete EP2s by nontrivial discriminant numbers and manifest the so-called fermion doubling theorem in the two dimensional non-Hermitian systems. The experimental realization in the coupled optical waveguide array is discussed and demonstrated by simulation results.

ACKNOWLEDGMENTS

This work is supported by Hong Kong RGC (16303119, AoE/P-02/12, N_HKUST608/17). It is also supported by the National Natural Science Foundation of China (Grant No. 11847205) and China Postdoctoral Science Foundation (Grant No. 2018M630597). K.D. acknowledges funding from the Gordon and Betty Moore Foundation.

APPENDIX A: $C_{4v}T$ GROUP SYMMETRIES

Here we discuss the $C_{4v}T$ group symmetries. For illustration, an arbitrary point $\vec{k}_0 = (k_{0x}, k_{0y})$ with $k_{0x}, k_{0y} \in [0, \pi]$ is chosen. The $C_{4v}T$ group reduces to the $C_{4v}$ group when $\gamma = 0$. We show the star for $\vec{k}_0$ in the BZ ($[-\pi, \pi] \times [-\pi, \pi]$) by 8 dots/squares marked with numbers 0,1,⋯,7, which stand for $\vec{k}_0, \vec{k}_1, \cdots, \vec{k}_7$, in Fig. 10. The points $\vec{k}_1, \cdots, \vec{k}_7$ are related to $\vec{k}_0$ by $i, \sigma_x, \sigma_y, \sigma_z, \sigma_y', C_4, C_4^{-1}$, respectively. The 4 squares marked with 4,5,6,7 represent the symmetry operators $\sigma_d, \sigma_d', C_4, C_4^{-1}$ which are replaced
by \( \sigma_d T, \sigma_d' T, C_4 T, C_4^{-1} T \) when \( \gamma \neq 0 \). In addition, the points labeled as \( 1', 2', 3', 5', 6', 7' \) stand for \( \tilde{k}'_i \) with \( i = 1, 2, 3, 5, 6, 7 \) that reside in \([0,2\pi] \times [0,2\pi]\). A point \( \tilde{k}'_i \) differs from the corresponding \( \tilde{k}_i \) by a reciprocal lattice vector that is denoted by a line between them.

With inversion symmetry \( i \) and mirror symmetries \( \sigma_x, \sigma_y \), we have the relation
\[
E(\tilde{k}_m) = E(\tilde{k}_0),
\]
where \( m = 1, 2, 3 \), for the band structure. The band structure in the whole BZ just repeats \( E(\tilde{k}), \tilde{k} \in [0,\pi] \times [0,\pi] \) four times following the relation given by Eq. (A1).

The symmetries \( \sigma_d T, \sigma_d' T, C_4 T, C_4^{-1} T \) dictate that
\[
E(\tilde{k}_n) = E^*(\tilde{k}_0),
\]
where \( n = 4, 5, 6, 7 \) and the complex conjugation is due to the time-reversal operator. Explicitly we have \( H(\tilde{k}_n) = U^{-1} H^*(\tilde{k}_0) U \), which implies Eq. (A2), where \( U \) = \[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\].

All the constraints manifest themselves in the band structure and particularly the locations of EPs, as shown in the Figs. 1(c) and 1(d), recalling that \( E(\tilde{k}'_i) \equiv E(\tilde{k}_i + \hat{\gamma}_i) = E(\tilde{k}_i) \).

**APPENDIX B: DERIVATION OF EQ. (7)**

A degenerate eigenvalue of \( H(\tilde{k}) \) is the common root of \( f(E, \tilde{k}) = \sum_{m=0}^{3} a_m(\tilde{k}) E^m \) and \( f'(E, \tilde{k}) = \partial f/\partial E = \sum_{n=0}^{n} b_n(\tilde{k}) E^n \), where \( b_n = (n + 1) a_{n+1} \). This fact leads to a practical formula \( \Delta(\tilde{k}) \), which involves only the coefficients of \( f(E, \tilde{k}) \) and \( f'(E, \tilde{k}) \) [26],
\[
\Delta(\tilde{k}) = -\det[Syl(f, f')],
\]
where
\[
Syl(f, f') = \begin{pmatrix}
a_3 & a_2 & a_1 & a_0 & 0 \\
0 & a_3 & a_2 & a_1 & a_0 \\
b_2 & b_1 & b_0 & 0 & 0 \\
0 & b_2 & b_1 & b_0 & 0 \\
0 & 0 & b_2 & b_1 & b_0
\end{pmatrix}
\]
is the Sylvester matrix of \( f \) and \( f' \). After some algebra, the discriminant for \( f(E, \tilde{k}) \) in Eq. (5) is derived to be
\[
\Delta(\tilde{k}) = -4a_3^2(\tilde{k}) - 27a_0^2(\tilde{k}),
\]
which Eq. (7) in the main text. The discriminant of any general polynomial can also be calculated similarly.

**APPENDIX C: GRAPHICS of EP LOOP AND EP3s**

For illustration, the contour lines for \( \Delta(\tilde{k}) = 0 \) and \( a_1(\tilde{k}) = 0 \) with \( \gamma = 1 \) are shown in Fig. 11(a), where the four intersection points are EP3s according to Eq. (8) and the fact that all three eigenvectors coalesce. All other points on the blue contour line are EP2s. \( a_1 = 0 \), namely \( \gamma^2 - T_{AB}^2 - T_{BC}^2 = 0 \), can be approximated by \( (k_x - \pi)^2 + (k_y - \pi)^2 = \gamma^2 \) for small \( \gamma \) and therefore gives a circle-like contour line.
For comparison, the exact and approximate EP loops, computed from $\Delta(k) = 0$ and $\Delta^a(k) = 0$, are shown in Fig. 11(b) for $\gamma = 1$ and $\gamma = 2$. The discrepancy between the exact and approximate EP loops is negligible for $\gamma = 1$. The EP loop for $\gamma < 1$ is approximately a linear scale-down of the one shown for $\gamma = 1$, noting that $\Delta^a(k) = 0$ can be recast as

$$27 \left( \frac{q_x^2}{y^2} - \frac{q_y^2}{y^2} \right)^2 + 4 \left( \frac{q_x^2}{y^2} + \frac{q_y^2}{y^2} - 1 \right)^3 = 0,$$

which indicates that the solutions $q_x, q_y$ are proportional to $\gamma$.

**APPENDIX D: GRAPHICS of DISCRETE EP2s**

To ensure the existence of EP3s in the NNN systems, three constraints need to be satisfied, namely $\Delta_R = 0, \Delta_I = 0$ and $a_1 = 0$. That is generally impossible because the three constraints are independent and there are only two degrees of freedom, i.e., $k_x, k_y$.

For illustration of the absence of EP3s, contour lines for $\Delta_R = 0, \Delta_I = 0$ and $a_1 = 0$ are shown in Fig. 12 for $\gamma = 1, w = 0.2$. There are obviously no common intersection points among the three and therefore no EP3s occur. The contour line $\Delta_R = 0$ has quite a peculiar shape due to its complicated dependence on $k_x$ and $k_y$. The 8 intersections between the contour line of $\Delta_R = 0$ and that of $\Delta_I = 0$ are the EP2s.

For comparison the contour $\Delta = 0$ for the NN system is also shown by dashed curves, which almost coincide with some parts of $\Delta_R = 0$. Incidentally, the intersection points of $\Delta_I = 0$ and $a_1 = 0$ are close to but not exactly located at the EP3 cusps for the NN system. We can imagine that they will tend to coincide totally when $w$ approaches zero. Moreover the contours $\Delta_R = 0, \Delta_I = 0$ and $a_1 = 0$ for small $w$ are qualitatively the same as that is shown in Fig. 12 for $w = 0.2$.

**APPENDIX E: OTHER CHIRAL-SYMMETRIC VARIANTS of THE NN SYSTEM**

Here we consider the Hamiltonian

$$H(k) = \begin{pmatrix} i\gamma_A & T_{AB} & 0 \\ T_{AB} & 0 & T_{BC} \\ 0 & T_{BC} & i\gamma_C \end{pmatrix},$$

where the onsite energies at sites “A”, “B”, “C” are assumed to be $\epsilon_A = i\gamma_A, \epsilon_B = 0, \epsilon_C = i\gamma_C$ without loss of generality. Mirror-T symmetries are absent now. The chiral symmetry is still present for real $\gamma_A$ and $\gamma_C$. The three eigenvalues thus satisfy the constraints given by Eqs. (2) and (3). And they add up to $i(\gamma_A + \gamma_C)$ instead of 0 at an arbitrary $k$. The characteristic polynomial has the following form

$$f(E, k) = -E^3 + iE^2(\gamma_A + \gamma_C) + E(T_{AB}^2 + T_{BC}^2 + \gamma_A\gamma_C) - i(\gamma_AT_{BC}^2 + T_{AB}^2\gamma_C).$$

in which all the coefficients are generally nonzero.

As mentioned in Sec. II. C, EP2s exist as long as $\Delta(k) = 0$ has solutions in the BZ. And triple degeneracies exist as long as $\Delta(k) = 0$ and $p(k) = 0$ can be simultaneously satisfied in the BZ, where

$$p = a_2^2(k) - 3a_1(k)a_3(k).$$
We note $\gamma_A$ and $\gamma_C$ appear in a nearly symmetric way in Eq. (A8). We thus fix $\gamma_A = 2.8$ and consider five typical cases that correspond to varied values of $\gamma_C$, namely $-1.5, 0, 0.2, 1.4, 2.4$. The contour lines of $\Delta(\vec{k}) = 0$ and $p(\vec{k}) = 0$ for the five cases are shown in Figs. 13 (a)-(e), where each square frame denotes the BZ with $\vec{k} \in [0,2\pi] \times [0,2\pi]$. There are EP2 continua denoted by navy curve in all five cases. Triple (defective or non-defective) degeneracies occur at the intersections between the contour lines of $\Delta(\vec{k}) = 0$ and $p(\vec{k}) = 0$. The numbers of triple degeneracies are $4, 4, 4, 2, 0$, respectively, in Figs. 13 (a)-(e). The EP loop in Fig. 13(a) is similar to the NN system with $\gamma_A = -\gamma_C$ except for being squeezed in the $k_x$ direction. As $\gamma_C$ is increased, the left and right arcs shift in opposite directions, leading to the behaviors of (b) touching, (c) crossing, (d) forming a loop with the other two arcs eliminated, and (e) disappearance of triple degeneracies.

When $\gamma_C = 0$, the touching point $M = (\pi, \pi)$ in Fig. 13(b) corresponds to a two-fold non-defective degeneracy instead of an EP2, resulting from the vanishing of $T_{AB}$ and $T_{BC}$. When $\gamma_C = 0.2$ and $k_x = \pi$, the Hamiltonian becomes

$$H(\vec{k}) = \begin{pmatrix} i\gamma_A & T_{AB} & 0 \\ T_{AB} & 0 & 0 \\ 0 & 0 & i\gamma_C \end{pmatrix}.$$  \hspace{1cm} (A10)

which can sustain both two EP2s, corresponding to an eigenvalue $E = i\gamma_A/2$, at $k_y \approx 1.59$ and $k_y \approx 4.69$ and two non-defective degeneracies, corresponding to an eigenvalue $E = i\gamma_C$, at the two self-intersection points of the contour line $\Delta(\vec{k}) = 0$. When $\gamma_C = \gamma_A/2$ and $k_x = \pi$, the Hamiltonian becomes

$$H(\vec{k}) = \begin{pmatrix} i\gamma_A & T_{AB} & 0 \\ T_{AB} & 0 & 0 \\ 0 & 0 & i\gamma_A/2 \end{pmatrix}.$$  \hspace{1cm} (A11)

There are two triple degeneracies at $\vec{k} = \left(\pi, \pi \pm 2 \sin^{-1}\left(\frac{\gamma_A}{4}\right)\right)$ with eigenvalue $E = i\gamma_A/2$, which correspond to the two intersections in Fig. 13 (d). However, only two eigenvectors coalesce as can be inferred from the decoupled form of Eq. (A11). Thus EP3s only occur in Figs. 13 (a)-(c) at the intersections of $\Delta(\vec{k}) = 0$ and $p(\vec{k}) = 0$, and some points at the contour line $\Delta(\vec{k}) = 0$ correspond to non-defective degeneracies instead of being EP2s.

[1] L. Feng, R. El-Ganainy, and L. Ge, *Non-Hermitian Photonics Based on Parity–Time Symmetry*, Nature Photonics 11, 752 (2017).

[2] R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z. H. Musslimani, S. Rotter, and D. N. Christodoulides, *Non-Hermitian Physics and PT Symmetry*, Nature Physics 14, 11 (2018).

[3] Ş. K. Özdemir, S. Rotter, F. Nori, and L. Yang, *Parity–Time Symmetry and Exceptional Points in Photonics*, Nature Materials 18, 783 (2019).

[4] M.-A. Miri and A. Alù, *Exceptional Points in Optics and Photonics*, Science 363, eaar7709 (2019).
[5] C. Dembowski, B. Dietz, H.-D. Gräf, H. L. Harney, A. Heine, W. D. Heiss, and A. Richter, *Observation of a Chiral State in a Microwave Cavity*, Phys. Rev. Lett. **90**, 034101 (2003).

[6] M. V. Berry, *Physics of Nonhermitian Degeneracies*, Czech. J. Phys. **54**, 1039 (2004).

[7] J. Doppler, A. A. Mailybaev, J. Böhm, U. Kuhl, A. Girschik, F. Libisch, T. J. Milburn, P. Rabl, N. Moiseyev, and S. Rotter, *Dynamically Encircling an Exceptional Point for Asymmetric Mode Switching*, Nature (London) **537**, 76 (2016).

[8] X.-L. Zhang, S. Wang, B. Hou, and C. T. Chan, *Dynamically Encircling Exceptional Points: In Situ Control of Encircling Loops and the Role of the Starting Point*, Phys. Rev. X **8**, 021066 (2018).

[9] H. Hodaei, A. U. Hassan, S. Wittek, H. Garcia-Gracia, R. El-Ganainy, D. N. Christodoulides, and M. Khajavikhan, *Enhanced Sensitivity at Higher-Order Exceptional Points*, Nature (London) **548**, 187 (2017).

[10] K. Ding, Z. Q. Zhang, and C. T. Chan, *Coalescence of Exceptional Points and Phase Diagrams for One-Dimensional PT-Symmetric Photonic Crystals*, Phys. Rev. B **92**, 235310 (2015).

[11] B. Zhen, C. W. Hsu, Y. Igarashi, L. Lu, I. Kaminer, A. Pick, S.-L. Chua, J. D. Joannopoulos, and M. Soljačić, *Spawning Rings of Exceptional Points out of Dirac Cones*, Nature (London) **525**, 354 (2015).

[12] K. Ding, G. Ma, M. Xiao, Z. Q. Zhang, and C. T. Chan, *Emergence, Coalescence, and Topological Properties of Multiple Exceptional Points and Their Experimental Realization*, Phys. Rev. X **6**, 021007 (2016).

[13] K. Ding, G. Ma, Z. Q. Zhang, and C. T. Chan, *Experimental Demonstration of an Anisotropic Exceptional Point*, Phys. Rev. Lett. **121**, 085702 (2018).

[14] X.-X. Zhang and M. Franz, *Non-Hermitian Exceptional Landau Quantization in Electric Circuits*, Phys. Rev. Lett. **124**, 046401 (2020).

[15] T. Stehmann, W. D. Heiss, and F. G. Scholtz, *Observation of Exceptional Points in Electronic Circuits*, J. Phys. A: Math. Gen. **37**, 7813 (2004).

[16] B. Peng, Ş. K. Özdemir, M. Liertzer, W. Chen, J. Kramer, H. Yilmaz, J. Wiersig, S. Rotter, and L. Yang, *Chiral Modes and Directional Lasing at Exceptional Points*, PNAS **113**, 6845 (2016).

[17] L. Jin and Z. Song, *Incident Direction Independent Wave Propagation and Unidirectional Lasing*, Phys. Rev. Lett. **121**, 073901 (2018).

[18] J. Zhang, B. Peng, Ş. K. Özdemir, K. Pichler, D. O. Krimer, G. Zhao, F. Nori, Y. Liu, S. Rotter, and L. Yang, *A Phonon Laser Operating at an Exceptional Point*, Nat. Photonics **12**, 479 (2018).

[19] T. E. Lee, *Anomalous Edge State in a Non-Hermitian Lattice*, Phys. Rev. Lett. **116**, 133903 (2016).

[20] D. Leykam, K. Y. Bliokh, C. Huang, Y. D. Chong, and F. Nori, *Edge Modes, Degeneracies, and Topological Numbers in Non-Hermitian Systems*, Phys. Rev. Lett. **118**, 040401 (2017).

[21] H. Shen, B. Zhen, and L. Fu, *Topological Band Theory for Non-Hermitian Hamiltonians*, Phys. Rev. Lett. **120**, 146402 (2018).
[22] S. Yao and Z. Wang, *Edge States and Topological Invariants of Non-Hermitian Systems*, Phys. Rev. Lett. **121**, 086803 (2018).

[23] S. Yao, F. Song, and Z. Wang, *Non-Hermitian Chern Bands*, Phys. Rev. Lett. **121**, 136802 (2018).

[24] F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, *Bipartite Bulk-Boundary Correspondence in Non-Hermitian Systems*, Phys. Rev. Lett. **121**, 026808 (2018).

[25] Y.-X. Xiao, Z.-Q. Zhang, Z. H. Hang, and C. T. Chan, *Anisotropic Exceptional Points of Arbitrary Order*, Phys. Rev. B **99**, 241403 (2019).

[26] Z. Yang, A. P. Schnyder, J. Hu, and C.-K. Chiu, *Fermion Doubling Theorems in 2D Non-Hermitian Systems for Fermi Points and Exceptional Points*, ArXiv:1912.02788 (2020).

[27] K. Moors, A. A. Zyuzin, A. Yu. Zyuzin, R. P. Tiwari, and T. L. Schmidt, *Disorder-Driven Exceptional Lines and Fermi Ribbons in Tilted Nodal-Line Semimetals*, Phys. Rev. B **99**, 041116 (2019).

[28] Y. Xu, S.-T. Wang, and L.-M. Duan, *Weyl Exceptional Rings in a Three-Dimensional Dissipative Cold Atomic Gas*, Phys. Rev. Lett. **118**, 045701 (2017).

[29] T. Yoshida, R. Peters, N. Kawakami, and Y. Hatsugai, *Symmetry-Protected Exceptional Rings in Two-Dimensional Correlated Systems with Chiral Symmetry*, Phys. Rev. B **99**, 121101 (2019).

[30] R. Okugawa and T. Yokoyama, *Topological Exceptional Surfaces in Non-Hermitian Systems with Parity-Time and Parity-Particle-Hole Symmetries*, Phys. Rev. B **99**, 041202 (2019).

[31] J. C. Budich, J. Carlström, F. K. Kunst, and E. J. Bergholtz, *Symmetry-Protected Nodal Phases in Non-Hermitian Systems*, Phys. Rev. B **99**, 041406 (2019).

[32] X. Zhang, K. Ding, X. Zhou, J. Xu, and D. Jin, *Experimental Observation of an Exceptional Surface in Synthetic Dimensions with Magnon Polaritons*, Phys. Rev. Lett. **123**, 237202 (2019).

[33] K. Kimura, T. Yoshida, and N. Kawakami, *Chiral-Symmetry Protected Exceptional Torus in Correlated Nodal-Line Semimetals*, Phys. Rev. B **100**, 115124 (2019).

[34] H. Zhou, J. Y. Lee, S. Liu, and B. Zhen, *Exceptional Surfaces in PT-Symmetric Non-Hermitian Photonic Systems*, Optica, OPTICA **6**, 190 (2019).

[35] A. Cerjan, M. Xiao, L. Yuan, and S. Fan, *Effects of Non-Hermitian Perturbations on Weyl Hamiltonians with Arbitrary Topological Charges*, Phys. Rev. B **97**, 075128 (2018).

[36] A. Cerjan, S. Huang, M. Wang, K. P. Chen, Y. Chong, and M. C. Rechtsman, *Experimental Realization of a Weyl Exceptional Ring*, Nat. Photonics **13**, 623 (2019).

[37] R. A. Vicencio, C. Cantillano, L. Morales-Inostroza, B. Real, C. Mejía-Cortés, S. Weimann, A. Szameit, and M. I. Molina, *Observation of Localized States in Lieb Photonic Lattices*, Phys. Rev. Lett. **114**, 245503 (2015).

[38] S. Mukherjee, A. Spracklen, D. Choudhury, N. Goldman, P. Öhberg, E. Andersson, and R. R. Thomson, *Observation of a Localized Flat-Band State in a Photonic Lieb Lattice*, Phys. Rev. Lett. **114**, 245504 (2015).
[39] S. M. Zhang and L. Jin, *Flat Band in Two-Dimensional Non-Hermitian Optical Lattices*, Phys. Rev. A **100**, 043808 (2019).

[40] M. Am-Shalam, R. Kosloff, and N. Moiseyev, *Exceptional Points for Parameter Estimation in Open Quantum Systems: Analysis of the Bloch Equations*, New J. Phys. **17**, 113036 (2015).

[41] D. Dizdarevic, D. Dast, D. Haag, J. Main, H. Cartarius, and G. Wunner, *Cusp Bifurcation in the Eigenvalue Spectrum of PT-Symmetric Bose-Einstein Condensates*, Phys. Rev. A **91**, 033636 (2015).

[42] Daniel. B. Litvin, *Magnetic Group Tables: 1-, 2- and 3-Dimensional Magnetic Subperiodic Groups and Magnetic Space Groups* (International Union of Crystallography, Chester, England, Chester, England, 2013).

[43] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, *Symmetry and Topology in Non-Hermitian Physics*, Phys. Rev. X **9**, 041015 (2019).

[44] K. Kawabata, T. Bessho, and M. Sato, *Classification of Exceptional Points and Non-Hermitian Topological Semimetals*, Phys. Rev. Lett. **123**, 066405 (2019).

[45] J. W. Bruce and P. J. Giblin, *Curves and Singularities: A Geometrical Introduction to Singularity Theory* (Cambridge University Press, Cambridge, 1992).

[46] S. Bittner, B. Dietz, U. Günther, H. L. Harney, M. Miski-Oglu, A. Richter, and F. Schäfer, *PT Symmetry and Spontaneous Symmetry Breaking in a Microwave Billiard*, Phys. Rev. Lett. **108**, 024101 (2012).

[47] K. Esaki, M. Sato, K. Hasebe, and M. Kohmoto, *Edge States and Topological Phases in Non-Hermitian Systems*, Physical Review B **84**, 205128 (2011).

[48] S. Lieu, *Topological Phases in the Non-Hermitian Su-Schrieffer-Heeger Model*, Phys. Rev. B **97**, 045106 (2018).

[49] X. Cui, K. Ding, J.-W. Dong, and C. T. Chan, *Exceptional Points and Their Coalescence of PT-Symmetric Interface States in Photonic Crystals*, Phys. Rev. B **100**, 115412 (2019).

[50] X.-L. Zhang and C. T. Chan, *Hybrid Exceptional Point and Its Dynamical Encircling in a Two-State System*, Phys. Rev. A **98**, 033810 (2018).

[51] L. Jin, H. C. Wu, B.-B. Wei, and Z. Song, *Hybrid Exceptional Point Created from Type-III Dirac Point*, Phys. Rev. B **101**, 045130 (2020).

[52] I. Rotter, *A Non-Hermitian Hamilton Operator and the Physics of Open Quantum Systems*, J. Phys. A: Math. Theor. **42**, 153001 (2009).

[53] T. Needham, *Visual Complex Analysis* (Clarendon, Oxford, 2000).

[54] B. Zhen, C. W. Hsu, L. Lu, A. D. Stone, and M. Soljačić, *Topological Nature of Optical Bound States in the Continuum*, Phys. Rev. Lett. **113**, 257401 (2014).
Fig. 1 Band structures and exceptional loop. (a) The Lieb lattice with non-Hermiticity brought by onsite gain/loss ($\pm i\gamma$). (b) The band structure of the Hermitian Lieb lattice ($\gamma = 0$) features a triple degeneracy at $M = (\pi, \pi)$. (c) Real part and (d) imaginary part of the band structure for the non-Hermitian system with NN hoppings. For good visualization only a quarter of the Brillouin zone is shown for the imaginary part utilizing the symmetry $E(k_x, k_y) = E(2\pi - k_x, k_y) = E(k_x, 2\pi - k_y)$. The magenta and cyan intersection curves correspond to exceptional points of order 2, and the four cusps where they meet correspond to exceptional points of order 3 in the BZ. The gain/loss parameter is chosen as $\gamma = 2$. 
Fig. 2 The evolution of the exceptional loops with increasing non-Hermiticity. (a) An exceptional loop emerges at $M$ for small non-Hermiticity, and then expands with increasing non-Hermiticity. The values of $\gamma$ are labeled at each EP loop. (b) The bands along the EP loop for $\gamma = 2$ is shown, where the ticks $k_1, k_2, k_3, k_4$ denote the four EP3 cusps as labeled in (a). The twofold bands are marked with “2”. (c) The EP loops for $\gamma = 2, 2\sqrt{2}, 3.9$ are plotted on the BZ torus, where $\Gamma, Y, M, X$ are denoted by dots.

Fig. 3 The anisotropic behaviors of an EP2 at $(4/3,1)\pi$ and an EP3 at $(1/2,1/2)\pi$. (a) The perpendicular/tangential paths traversing the EP2 (EP3) are denoted by cyan/magenta (blue/red) lines. They correspond to subfigures (b)-(e) and are marked with “(b)”, “(c)”, “(d)”, “(e)” accordingly. (b)-(e) Energy
bands along the corresponding paths in (a). The asymptotic forms of eigenvalues are respectively $q_x^{1/2}, q_y, q_x^{1/3}$ and $q_x^{1/2}$ for (b), (c), (d), (e), respectively, where $q_x (q_y)$ denotes small deviations from the EP2/EP3. (f) Phase rigidity $\rho$ near the EP2/EP3 with critical exponents $\frac{1}{2}, 1, \frac{2}{3}, 1$ is shown for paths corresponding to (b), (c), (d), (e), respectively. The values of $\rho$ for (c) and (e) are multiplied by 10.

Fig. 4 (a) Two Dirac-like cones reside at $X$ and $Y$ in the purely imaginary bands when $\gamma = 4$. EP2s occur at the cone vertices. (b) The EP2s at $X$ and $Y$ are gapped when $\gamma = 4.01$. (c), (d) The bands exhibit two elliptical EP2 loops centered $X$ and $Y$ when $\gamma = 3.9$.  

Fig. 5 The band structure and exceptional points in the Lieb lattice with NNN hoppings. (a) The real part and (b) imaginary part of the band structure in a quarter of BZ are shown. The three colored dots correspond to discrete EP2s. (c) The thick (thin) yellow lines represent where 3 (2) eigenvalues have the same real part. The green lines similarly represent where eigenvalues have identical imaginary parts. Discrete EP2s occur at the three joints. (d) The trajectories of \( \text{Re}(\Delta) = 0 \) and \( \text{Im}(\Delta) = 0 \) in the BZ are represented by a navy-blue curved loop and 4 green lines, respectively. Their 8 intersections are EP2s as are marked by dots. The black, cyan, magenta dots correspond to the three dots in (a) and (b). Here \( \gamma = 1, w = 0.2 \) is assumed.

Fig. 6 The vector field \( \vec{D} = (\Delta_R, \Delta_I) \) for two typical cases: (a) \( w = 0.2 \) and (b) \( w = 0.295 \). The field distributions near the EP2s are highlighted by red arrows. The discriminant numbers of EP2s enclosed by
green and cyan circles are $\nu = 1$ and $\nu = -1$, respectively. The discriminant numbers of the four EP2s at $k_x = \pi$ and $k_y = \pi$ are all 0. The variations of $\text{Arg}(\Delta)$ around (c) an EP2 with $\nu = 1$ and (d) an EP2 with $\nu = 0$.

Fig. 7 The evolution of discrete EP2s in the BZ with increasing $w$. (a) The discrete EP2s are shown for 4 typical values of NNN hoppings, namely $w = 0.25, 0.29, 0.925, 0.3$. The 4 EP2s at $k_x = \pi$ and $k_y = \pi$ remain invariant when $w$ is varied. There are 4, 8, 8, 8 EP2s at the BZ diagonals for the 4 scenarios, respectively. EP2s are created in fours at $\Gamma$ (when $w \approx 0.29$) or annihilated in pairs (when $w$ is just above 0.3) at the BZ diagonals, when $w$ is continuously increased. (b), (c) The two bands pertaining to EP2s along $\Gamma M$ are shown in the first 4 rows, each row corresponding to a value of $w$. The last row displays the bands for $w = 0.302$, where the imaginary parts are gapped after the annihilation of two EP2s.

Fig. 8 The evolution of the EP loop (NN system) and discrete EP2s (NNN system) under perturbation. (a) The EP loop for the unperturbed NN system with $\gamma = 2$ is colored in black. The chiral-symmetry-breaking perturbation causes all EP2s on the loop to collapse and every EP3 to split into two discrete EP2s. The 8 EP2s split from EP3s trace out 8 trajectories when the perturbation strength $\delta$ is increased. The discriminant number is 1 ($-1$) for an EP2 on a red (blue) trajectory. The colored circles mark $\delta = 0.15, 0.3, 0.5, 0.7$. (b) The EP2s for the unperturbed NNN system with $\gamma = 1, w = 0.2$ are denoted by black dots. The 4 EP2s with $\nu = \pm 1$ move towards $k_y = 0$ and $k_y = 2\pi$ and finally annihilate there in pairs when $\delta$ is increased from 0 up to a certain value. Two EP2s with $\nu = 1$ and $\nu = -1$ are split from each of the two EP2s.
on $k_x = \pi$ with $\nu = 0$, and they finally annihilate with each other at $k_x = 0$. The two EP2s at $k_y = \pi$ collapse for an infinitesimal $\delta$. The colored circles mark $\delta = 10^{-3}$, $3 \times 10^{-3}$, 0.01, 0.1, 0.3, 0.5, 0.7, 0.97, 1.1, 1.2, 1.34.

Fig. 9 (a) A coupled optical waveguide array forms a non-Hermitian Lieb lattice, where gain/loss is illustrated by red/blue colors. (b), (c) The band structure of the mode propagation constant $\beta$ as a function of Bloch momenta $\vec{k}$. The dots denote full-wave simulation results, and the red curves are the fitted band structure using only the data points at $\Gamma$, $X$, $M$, $Y$. The lattice constant and radii of the cylinders are $44 \mu m$ and $10 \mu m$, respectively. The refractive indices for the background medium and the cylinders are 1.473 and 1.4737. The refractive indices for the “A” cylinders and “C” cylinders contain imaginary parts of $-10^{-5}$ and $10^{-5}$, respectively.

Fig. 10 The star for an arbitrarily chosen $\vec{k}_0$ in the Brillouin zone of the Hermitian Lieb lattice is represented by 8 dots/squares marked with numbers 0, 1, $\ldots$, 7 that correspond to $\vec{k}_0, \vec{k}_1, \ldots, \vec{k}_7$. Dots marked with 1, 2, 3 correspond to the symmetry operators $i, \sigma_x, \sigma_y$. Squares marked with 4, 5, 6, 7 represent the symmetry operators $\sigma_d, \sigma_d', C_4, C_4^{-1}$ which are replaced by $\sigma_d T, \sigma_d' T, C_4 T, C_4^{-1} T$ when $\gamma \neq 0$. The points $\vec{k}'_i$ with $i = 1, 2, 3, 5, 6, 7$ are labeled by $1', 2', 3', 5', 6', 7'$. Each $\vec{k}'_i$ differs from the corresponding $\vec{k}_i$ by a reciprocal lattice vector which is denoted by a line.
Fig. 11 (a) Contour lines for $\Delta(\vec{k}) = 0$ and $a_1(\vec{k}) = 0$ cross at EP3s. $\gamma = 1$ is assumed. (b) The exact and approximate EP contours for $\gamma = 1$ and $\gamma = 2$. The difference is imperceptible even for $\gamma = 1$.

Fig. 12 Contour lines $\Delta_R = 0$, $\Delta_I = 0$ and $a_1 = 0$ are shown for the NNN system with $\gamma = 1$ and $w = 0.2$. The contour line $\Delta = 0$ for the NN system with $\gamma = 1$ is shown for comparison.

Fig. 13 The contour lines of $\Delta(\vec{k}) = 0$ and $p(\vec{k}) = 0$ for five typical cases of non-Hermiticity that $\gamma_A = 2.8$ and $\gamma_C$ takes varied values. (a) $\gamma_C = -1.5$; (b) $\gamma_C = 0$; (c) $\gamma_C = 0.2$; (d) $\gamma_C = 1.4$; (e) $\gamma_C = 2.4$. 