The dimensionless dissipation rate and the Kolmogorov (1941) hypothesis of local stationarity in freely decaying isotropic turbulence

W.D. McComb and R.B. Fairhurst

SUPA, School of Physics and Astronomy, University of Edinburgh, James Clerk Maxwell Building, The King’s Buildings, Edinburgh EH9 3JZ, United Kingdom

October 9, 2018

Abstract

An expression for the dimensionless dissipation rate was derived from the Kármán-Howarth equation by asymptotic expansion of the second- and third-order structure functions in powers of the inverse Reynolds number. The implications of the time-derivative term for the assumption of local stationarity (or local equilibrium) which underpins the derivation of the Kolmogorov ‘4/5’ law for the third-order structure function were studied. It was concluded that neglect of the time-derivative cannot be justified by reason of restriction to certain scales (the inertial range) nor to large Reynolds numbers. In principle, therefore, the hypothesis cannot be correct, although it may be a good approximation. It follows, at least in principle, that the quantitative aspects of the hypothesis of local stationarity could be tested by a comparison of the asymptotic dimensionless dissipation rate for free decay with that for the stationary case. But in practice this is complicated by the absence of an agreed evolution time $t_e$ for making the measurements during the decay. However, we can assess the quantitative error involved in using the hypothesis by comparing the exact asymptotic value of the dimensionless dissipation in free decay calculated on the assumption of local stationarity to the experimentally determined value (e.g. by means of direct numerical simulation), as this relationship holds for all measuring times. Should the assumption of local stationarity lead to significant error, then the ‘4/5’ law needs to be corrected. Despite this, scale invariance in wavenumber space appears to hold in the formal limit of infinite Reynolds numbers, which implies that the ‘-5/3’ energy spectrum does not require correction in this limit.
## Contents

1 Introduction ................................................. 3

2 The dimensionless dissipation rate ......................... 4
   2.1 The choice of an evolution time $t_e$ in free decay .... 5

3 The Kármán-Howarth Equation (KHE) ....................... 7
   3.1 The dimensionless Karman-Howarth equation for free decay 8

4 Kolmogorov’s ‘4/5’ law for freely decaying turbulence .... 8

5 The asymptotic expansions .................................. 10
   5.1 Taking the limit of infinite Reynolds numbers .......... 10

6 Implications for the ‘4/5’ law .............................. 11

7 Implications for the Kolmogorov ‘2/3’ and ‘-5/3’ laws .... 13

8 Conclusions .................................................. 14
1 Introduction

In this paper we extend the techniques used by McComb et al [1], to calculate the dimensionless dissipation rate for stationary isotropic turbulence, to the case of free decay. In the process we are able to shed some light on aspects of the Kolmogorov (1941) theory which remain controversial. We begin with a brief reconsideration of this theory.

As is well known, Kolmogorov’s theory was put forward in the context of turbulence in general. He argued that the result of the cascade would be that one could consider the turbulence to be *locally* homogeneous, *locally* isotropic; and, in time-varying situations, *locally* stationary. Regions in which this could hold, would be restricted to a range of scales, and would necessarily be remote from boundaries. Since that time, it has become usual to study turbulence which is *globally* homogeneous, isotropic and indeed stationary. This work belongs to the topic of mathematical physics, where the problem is well-posed, and its applicability to any particular situation, be it computer simulation or laboratory experiment, requires some further consideration.

In his first paper [2], Kolmogorov derived the well known expression for the second-order structure function,

\[ S_2(r) = C_2 \varepsilon^{2/3} r^{2/3}, \]  

where \( C_2 \) is a constant. This work relied on the formulation of two similarity principles, followed by dimensional analysis. We will refer to it as K41A for conciseness. In his second paper [3], his starting point was the Kármán-Howarth equation, from which he derived the inertial-range expression for the third-order structure function as:

\[ S_3 = -\frac{4}{5} \varepsilon r. \]  

We will refer to this as K41B. We shall return to a fuller consideration of the ‘4/5’ law in a later section. Here, for completeness, we note that the ‘2/3’ law is perhaps better known in its spectral form:

\[ E(k) = \alpha \varepsilon^{2/3} k^{-5/3}, \]  

where \( k \) is the wavenumber and \( \alpha \) is the famous Kolmogorov constant. This result was first given by Obukhov [4], but that was based on a closure approximation. The first generalisation of K41A to the spectral case appears to have been due to Onsager [5] in 1945.

After these preliminaries, the rest of the paper is organised into the following sections:

**Section 2** A review of recent work on the dimensionless dissipation rate.

**Section 3** We use the KHE to derive expressions for the dissipation rate in both stationary and freely-decaying turbulence. This involves the dimensionless KHE, as expressed in terms of dimensionless structure functions.

**Section 3** We revisit the derivation of the ‘4/5’ law, with the assumption of local stationarity, and consider the implications of this assumption for the calculation of the asymptotic dimensionless dissipation rate.
Section 4 We use asymptotic expansions in inverse powers of the Reynolds numbers to derive the an expression for the dimensionless dissipation rate for free decay, and compare this to our previous result for stationary turbulence [1].

Section 5 We consider the implications for the ‘4/5’ law and discuss various investigations of the effect of finite viscosity and the retention of the time-derivative term, which generally conclude that it should be recovered in the limit of large Reynolds numbers. In contrast, our analysis suggests that this cannot be true as the time-derivative does not depend on either scale or Reynolds number.

Section 6 We then consider the implications of our analysis for the ‘2/3’ and ‘-5/3’ laws.

2 The dimensionless dissipation rate

There continues to be much interest in the fundamentals of turbulent dissipation, as characterised by the mean dissipation rate:

\[ \varepsilon(t) = \frac{\nu}{2} \sum_{\alpha,\beta=1}^{3} \left\langle \left( \frac{\partial u_\alpha(t)}{\partial x_\beta} + \frac{\partial u_\beta(t)}{\partial x_\alpha} \right)^2 \right\rangle, \]  

(4)

where \( \nu \) is the kinematic viscosity, \( u_\alpha \) is a component of the velocity field \( \mathbf{u}(\mathbf{x}, t) \) expressed in cartesian tensor notation, with the indices taking the values \( \alpha, \beta = 1, 2, 3 \). The angle brackets \( \langle \ldots \rangle \) denote the operation of taking an ensemble average. For isotropic turbulence, this reduces to the form:

\[ \varepsilon(t) = \nu \sum_{\alpha,\beta=1}^{3} \left\langle \left( \frac{\partial u_\alpha(t)}{\partial x_\beta} \right)^2 \right\rangle. \]  

(5)

Most of this work is based on the expression

\[ \varepsilon = C_\varepsilon \frac{U^3}{L}, \]  

(6)

which was proposed on dimensional grounds by Taylor in 1935 as an approximate form for the dissipation rate [6], where \( U \) is the root-mean-square velocity and \( L \) is the integral scale. Many workers in the field refer to this as the Taylor dissipation surrogate. However, there is a growing tendency to rearrange it as

\[ C_\varepsilon = \frac{\varepsilon}{U^3/L}, \]  

(7)

and work with the dimensionless dissipation rate \( C_\varepsilon \).

As early as 1953, Batchelor [7] (in the first edition of this book) presented evidence to suggest that \( C_\varepsilon \) tends to a constant value (nowadays denoted by \( C_{\varepsilon,\infty} \)), with increasing Reynolds number. However, the growth of activity in this topic stems from the seminal papers of Sreenivasan [8, 9], who established that for grid turbulence \( C_\varepsilon \) became constant for Taylor-Reynolds numbers greater than about 50. This has inspired numerous papers reporting experimental (including numerical) studies of the dependence of the dissipation on Reynolds number. An account of this work, with many references, can be found in Chapter Seven of the book by McComb [10].
Attempts to establish a theoretical relationship between the dimensionless dissipation rate and the Reynolds number have been based on the Kármán-Howarth equation [11] (or KHE, for short). Lohse [12] used a mean-field closure of the KHE to obtain an approximate expression for the dependence of $C_\varepsilon$ on $R_\lambda$, whereas Doering and Foias [13] established upper and lower bounds which must be satisfied by any such relationship.

More recently McComb et al [1], starting from the KHE with forcing, have used an asymptotic expansion of the structure functions in inverse powers of the Reynolds number, leading to a rigorous form for $C_\varepsilon,\infty$ and, with the aid of numerical simulations, obtained a relationship of the form $C_\varepsilon = C_\varepsilon,\infty + C/R_L + O(1/R^2_L)$, where $C$ is a constant and $R_L$ is the Reynolds number based on the integral scale. This result is for stationary turbulence. Djenidi et al [14] also took the KHE as their starting point and invoked the concept of self-preservation to assess the dependence of dimensionless dissipation on Reynolds number. An interesting feature of this work is their discussion at the end of Section One of its potential importance in applications. Their analysis is for decaying turbulence.

In this article, we extend the analysis of McComb et al [1] to the case of freely decaying turbulence. We obtain rigorous results in the limit of large Reynolds numbers and explore the implication of the time-dependence for Kolmogorov’s hypothesis of local stationarity (also known as local equilibrium [7]). This is a topic of continuing concern in turbulence research: see, for instance, the recent paper by George [15]. However, we do not present a general expression for the dependence of the dimensionless dissipation on Reynolds number, as this requires numerical validation. We have no a priori way of truncating the asymptotic expansion at small values of the Reynolds number. In the stationary case [1] that was done by comparison with the results of a numerical simulation. For the freely decaying case that requires further work.

### 2.1 The choice of an evolution time $t_e$ in free decay

Both stationary and freely decaying isotropic turbulence can be thought of as initial value problems in mathematical physics. At time $t = 0$ the velocity field is chosen to be a random function of the spatial coordinates and to have a Gaussian probability distribution. It is characterised by an arbitrarily chosen energy spectrum that is confined to small wavenumbers. Then, in numerical simulation of such problems, the coupling term of the Navier-Stokes equations will lead to the energy spreading out to higher wavenumbers, and quantities such as skewness, energy flux and dissipation will rise to some characteristic value such that the turbulence can be said to be evolved. The question then arises: what is the evolution time $t_e$ for a given simulation?

For forced turbulence this is not a difficult question to answer. In practice, the energy is found to fluctuate about a mean value, with fluctuations in the dissipation rate lagging behind: e.g. see figure 3 in McComb et al [16]. The mean value of the energy is determined by the energy input rate from the stirring forces and at this stage the turbulence can be regarded as stationary. Choosing a value for $t_e$ is simply a matter of carrying on the simulation until one achieves satisfactory statistics.

In the case of free decay, we face the obvious problem that we cannot simply carry on the simulation, because the turbulence is dying away. For the most part, free
decay has been studied in the context of an assumed power-law dependence of the total energy on time. For this reason, the onset of power-law behaviour is often taken as the criterion for the turbulence to be well developed: see, for example, Chapter 7 of the book [10]. Although, the onset of power-law decay is a traditional criterion for the flow to be well-developed and this was used by, for example, Wang et al [17] and Bos et al [18], it does occur rather late in the decay and so there is a possibility of choosing other criteria. For example, Fukayama et al [19] used the peak value of the dissipation rate as a criterion for choosing $t_e$. But unfortunately the dissipation does not have a peak with time when results are taken for low Reynolds numbers. This regions is of crucial importance in establishing the curve of dissipation against Reynolds number.

A study of the effects on the asymptotic dissipation rate of adopting different criteria for the evolution time $t_e$ has been made by Yoffe [20]. (This thesis can be downloaded from arXiv:1306.3408v1 [physics.flu-dyn].) As well as considering the traditional method of taking the onset of power-law behaviour, Yoffe studied the effects of the following criteria:

$t_s$ The time taken for the skewness to reach its peak value.

$t_\Pi$ The time taken for the inertial transfer rate to reach its peak value.

$t_\varepsilon$ The time taken for the dissipation rate to reach its peak value.

$t_{e|\Pi}$ A composite time equal to $t_\varepsilon$, if peak $\varepsilon$ exists; but equal to $t_\Pi$ otherwise.

We may briefly summarise these results as follows. Note that the curve of $C_\varepsilon$ versus the Taylor-Reynolds number $R_\lambda$ was used as a standard of comparison in assessing these results. Both $t_s$ and $t_\Pi$ took values of less than one eddy turnover time and, when used as criteria, led to dissipation curves which fell off more rapidly than usual, and implied an asymptotic value of $C_{\varepsilon,\infty}(t_\varepsilon) = 0$. For sake of completeness, Yoffe also took four values of evolution time in the range $3.0 \leq t_\varepsilon \leq 30$, which corresponded to the power-law regime, with time measured in units of initial eddy turnover time. The resulting values of $C_\varepsilon$ clustered together quite well and lay about 50% above the results for forced turbulence at lower Reynolds numbers. At higher Reynolds numbers (i.e. $R_\lambda \geq 50$ the results suggested asymptotic behaviour converging on the curve for forced turbulence.

The most interesting results were obtained from a consideration of the dissipation rate as providing a criterion. Yoffe found that for Taylor-Reynolds numbers below about $R_\lambda = 15$, the variation of dissipation with time did not pass through a peak, but instead seemed to have a point of inflection. He also noticed that at these low Reynolds numbers the inertial transfer rate appeared to go through a peak at the same time as the dissipation passed through an inflection. On this basis, he proposed and tested the use of a composite time $t_{e|\Pi}$, as listed above. He found that taking the dissipation rate at $t = t_\Pi$, for $R_\lambda$ less than about 15, and at $t = t_\varepsilon$ for larger values of the Reynolds number, led to a continuous variation of dissipation coefficient with increasing Reynolds number. This criterion was used in our earlier publication [21] and more recently has been found to lead to a dissipation curve which is in good agreement with the result for forced turbulence [1] up to Taylor-Reynolds numbers of $R_\lambda = 358.6$ [20].
3 The Kármán-Howarth Equation (KHE)

Our analysis is based on the KHE which, in terms of the second- and third-order structure functions, may be written as [10]:

\[- \frac{3}{2} \frac{\partial U^2}{\partial t} + \frac{3}{4} \frac{\partial S_2(r, t)}{\partial t} = -\frac{1}{4r^4} \frac{\partial}{\partial r} \left( r^4 S_3(r, t) \right) + \frac{3\nu}{2r^4} \frac{\partial}{\partial r} \left( r^4 \frac{\partial S_2(r, t)}{\partial r} \right) - I(r) ,
\]

where the structure functions are defined by

\[ S_n(r, t) = \langle [u(r, t) - u(0, t)]^n \rangle , \]

and \( I(r) \) represents an input term. In the main part of the present work, we will concentrate on free decay, so the input term must then be set equal to zero. However, at this point we will briefly consider both free decay and forced stationary turbulence, in order to facilitate later comparisons. We will also find it convenient to introduce new symbols (e.g see [10, 1]), \( \varepsilon_D(t) \) for the energy decay rate, thus:

\[ \varepsilon_D(t) = -\frac{3}{2} \frac{\partial U^2(t)}{\partial t} ; \]

and \( \varepsilon_W \) for the energy injection rate from forcing. A discussion of the relationship of the latter quantity to the stirring forces used in an accompanying numerical simulation may be found in [1], but as we do not report any numerical simulations here we will not pursue that aspect.

It should be noted that the KHE does not actually contain the dissipation rate as such. We may introduce it by means of the identity

\[ -\varepsilon_D = -\varepsilon + \varepsilon_W \]

as derived from the Lin equation, which is the equivalent of the KHE in wavenumber space [1]. Evidently, for the stationary case, this identity becomes \( \varepsilon = \varepsilon_W \); while, for free decay, it becomes \( \varepsilon_D = \varepsilon \).

For completeness, we state the stationary case of the KHE as:

\[ \varepsilon = \varepsilon_W = -\frac{1}{4r^4} \frac{\partial}{\partial r} \left( r^4 S_3(r) \right) + \frac{3\nu}{2r^4} \frac{\partial}{\partial r} \left( r^4 \frac{\partial S_2(r)}{\partial r} \right) , \]

for scales below the forcing scale, where \( I(r) \rightarrow \varepsilon_W \), and we have invoked equation (11). Note that for stationarity the dissipation is equal to the injection rate, and all time derivatives vanish. For further discussion, see reference [1].

We shall return to this result as required, but now we concentrate on the main work of this paper. For free decay, we set the input term and the injection rate equal to zero and use the equivalence of the dissipation rate and the decay rate to write

\[ \varepsilon(t) = \varepsilon_D(t) = -\frac{3}{4} \frac{\partial S_2(r, t)}{\partial t} - \frac{1}{4r^4} \frac{\partial}{\partial r} \left( r^4 S_3(r, t) \right) + \frac{3\nu}{2r^4} \frac{\partial}{\partial r} \left( r^4 \frac{\partial S_2(r, t)}{\partial r} \right) . \]
3.1 The dimensionless Karman-Howarth equation for free decay

In order to obtain the dimensionless dissipation rate we scale the dissipation on $U$ and $L$. As both of these quantities depend on the time of decay, we perform this scaling with respect to their value at some fiducial time $t = t_e$, as discussed in Section 2.1. We should also note that we are considering a well-posed mathematical initial-value problem, which can be realised in practice to a good approximation by direct numerical simulation. We are not discussing grid turbulence, which is normally a stationary flow which decays in the streamwise direction. As is well known, the description of such a flow as decaying in time relies on a Galilean transformation [22].

We may introduce the dimensionless structure functions $h_n(x, \tau)$ by means of the relationship:

$$S_n(r, t) = U^n(t_e) h_n(x, \tau),$$

where dimensionless time, dimensionless distance and characteristic time, respectively, are given by:

$$\tau = \frac{t}{T(t_e)}; \quad x = \frac{r}{L(t_e)}; \quad T(t_e) = \frac{L(t_e)}{U(t_e)}.$$ (15)

In terms of the dimensionless structure functions, equation (13) becomes

$$\varepsilon(\tau) = \frac{3}{4} U^2 \frac{\partial h_2(x, \tau)}{\partial \tau} + \frac{U^3}{L} \left( -\frac{1}{4x^4} \frac{\partial x^4 h_3(x, \tau)}{\partial x} + \frac{3}{2x^4} \nu \frac{\partial}{\partial x} \left( x^4 \frac{\partial h_2(x, \tau)}{\partial x} \right) \right)$$ (16)

where we leave the dependence of $U$ and $L$ on $t_e$ implicit, in the interests of simplicity. Then, with some re-arrangement, and substituting $R_L$ for $LU/\nu$, and from (6) for $\varepsilon$ in terms of $C_\varepsilon$ on the right hand side of equation (16), we obtain

$$C_\varepsilon^{\text{decay}}(\tau) = \frac{L \varepsilon(\tau)}{U^3} = \frac{3}{4} \frac{\partial h_2(x, \tau)}{\partial \tau} - \frac{1}{4x^4} \frac{\partial}{\partial x} \left( x^4 h_3(x, \tau) \right) + \frac{3}{2x^4 R_L} \frac{\partial}{\partial x} \left( x^4 \frac{\partial h_2(x, \tau)}{\partial x} \right).$$ (17)

Note the introduction of the superscript $\text{decay}$ to distinguish this result from the stationary case which will be denoted in the present work by the superscript $\text{stat}$.  

4 Kolmogorov’s ‘4/5’ law for freely decaying turbulence

Before we proceed to the asymptotic expansion, it is of interest to revisit the Kolmogorov theory for the third-order structure function $S_3$. This is the well known Kolmogorov’s ‘4/5’ law. It can be obtained by integrating equation (13) with respect to $r$, and taking the infinite Reynolds number limit. Then, with some rearrangement of terms, one obtains:

$$\lim_{\nu \to 0} S_3(r, t) = -\frac{4}{5} \varepsilon_D(t) r - \lim_{\nu \to 0} \frac{4}{r^4} \int_0^r 3r'^4 \frac{\partial S_2(r', t)}{\partial t} dr'.$$ (18)

Two points should be noted about this. First, we have written it in terms of the decay rate $\varepsilon_D$, in order to keep in mind its origins. But of course this is equal to the
dissipation rate, which is how Kolmogorov expressed it. Secondly, the term involving the time-derivative on the right hand side was dropped by Kolmogorov, who argued that in the inertial range of scales the turbulence could be \textit{locally stationary}. It is worth pointing out that this was not a specific assumption made in the second paper K41B. In fact it was introduced as part of the definition of local isotropy in K41A. For a discussion, see Section 4.6 of the book [10]. This property was later referred to by Batchelor [7] as \textit{local equilibrium}. With these two steps, equation (18) reduces to the familiar form:

$$\lim_{\nu \to 0} S_3(r) = -\frac{4}{5} \varepsilon r,$$  

where $r$ is in the inertial range of scales.

We now take the limit of infinite Reynolds numbers in equation (17), and \textit{continue, for the present, to make Kolmogorov’s assumption of local stationarity}. It should perhaps be emphasised that in this discussion the concept of local stationarity explicitly implies the neglect of the time-derivative in equation (17). In this way, we obtain the asymptotic result:

$$\lim_{\nu \to 0} C_{\text{decay}} \equiv C_{\varepsilon, \infty} = -\frac{1}{4x^4} \frac{\partial}{\partial x} \left( x^4 \lim_{\nu \to 0} h_3(x) \right),$$  

where we have introduced the usual notation for the asymptotic dimensionless dissipation rate $C_{\varepsilon, \infty}$, here decorated by the superscript \textit{decay}. We should note the important fact that, since the left hand side of this equation does not depend on scale, the right hand side must be constant with respect to $x$. This is, of course, consistent with the ‘$4/5$’ law, as may be readily seen by substituting from (14) and (15) for $h_3(x)$ in terms of $S_3(r)$ and performing the differentiation.

We may also note from [1] that the equivalent result for the stationary case is:

$$C_{\varepsilon, \infty}^{\text{stat}} = -\frac{1}{4x^4} \frac{\partial}{\partial x} \left( x^4 \lim_{\nu \to 0} h_3(x) \right),$$  

which is just equation (35) of that paper [1], with the addition of the superscript \textit{stat}, and the notational change of the name for the independent variable from $\rho$ to $x$. Thus we have the result that the expression for the asymptotic dimensionless dissipation rate for free decay is exactly the same as that for forced stationary turbulence (albeit evaluated at some specific time) \textit{provided that Kolmogorov’s assumption of local stationarity is correct}. Strictly Kolmogorov’s theory requires one to take the limit of large Reynolds numbers, and when we take this into account more formally in the next section, we find that this result leads on to an unambiguous test of the validity of the assumption of local stationarity.

However, for later convenience, we introduce a generalisation of equation (21), as follows. We note that, from a purely mathematical point of view, the left hand side is just a name for the expression on the right hand side. So, if we choose to generalise the structure functions to the time-dependent case, then we may change the ‘name’ on the left hand side accordingly, thus:

$$C_{\varepsilon, \infty}^{\text{stat}}(t) = -\frac{1}{4x^4} \frac{\partial}{\partial x} \left( x^4 \lim_{\nu \to 0} h_3(x, \tau) \right) = -\frac{1}{4r^4} \frac{L}{U^3} \frac{\partial}{\partial r} \left( r^4 \lim_{\nu \to 0} S_3(r, t) \right),$$  

where in the second equality we have restored the structure function to its usual form.
5 The asymptotic expansions

The asymptotic expansion of the dimensionless structure functions in powers of the inverse Reynolds number is discussed in reference [1]. Here, the structure functions also depend on time, but the procedure is the same and we may write for the \( n \)-order (reduced) structure function

\[
h_n(x, \tau) = h_n^{(0)}(x, \tau) + \frac{1}{R_L} h_n^{(1)}(x, \tau) + \mathcal{O}\left(\frac{1}{R_L}\right)^2. \tag{23}
\]

Substituting from this for the second- and third-order reduced structure functions into equation (17) we obtain:

\[
C_{\varepsilon, \text{decay}}^{\text{decay}}(\tau) = -\frac{1}{4x^4} \frac{\partial}{\partial x} \left(x^4 h_3^{(0)}(x, \tau)\right) - \frac{3}{4} \frac{\partial h_2^{(0)}(x, \tau)}{\partial \tau} + \frac{1}{R_L} \left\{ -\frac{1}{4x^4} \frac{\partial}{\partial x} \left(x^4 h_3^{(1)}(x, \tau)\right) \right. \\
+ \left. \frac{3}{2x^4} \frac{\partial}{\partial x} \left(x^4 \frac{\partial h_2^{(0)}(x, \tau)}{\partial x}\right) - \frac{3}{4} \frac{\partial h_2^{(1)}(x, \tau)}{\partial \tau} \right\} + \mathcal{O}\left(\frac{1}{R_L}\right)^2. \tag{24}
\]

By analogy with the analysis in reference [1], we may write this as:

\[
C_{\varepsilon, \infty, \text{decay}}^{\text{decay}}(\tau) = C_{\varepsilon, \infty, \text{decay}}^{\text{decay}}(\tau) - \frac{3}{4} \frac{\partial h_2^{(0)}(x, \tau)}{\partial \tau} + \frac{C_{\varepsilon, \text{decay}}^{\text{decay}}(\tau)}{R_L(\tau)} + \mathcal{O}\left(\frac{1}{R_L}\right)^2, \tag{25}
\]

where the coefficients are given by

\[
C_{\varepsilon, \infty}^{\text{decay}}(\tau) = -\frac{1}{4x^4} \frac{\partial}{\partial x} \left(x^4 h_3^{(0)}(x, \tau)\right), \tag{26}
\]

and

\[
C_{\varepsilon, \infty}^{\text{decay}}(\tau) = -\frac{1}{4x^4} \frac{\partial}{\partial x} \left(x^4 h_3^{(1)}(x, \tau)\right) + \frac{3}{2x^4} \frac{\partial}{\partial x} \left(x^4 \frac{\partial h_2^{(0)}(x, \tau)}{\partial x}\right) - \frac{3}{4} \frac{\partial h_2^{(1)}(x, \tau)}{\partial \tau}. \tag{27}
\]

Comparison with the stationary case, i.e. equations (40) to (42) of reference [1], shows that the dissipation relation (25) differs from its stationary counterpart by the presence of the time-derivative of the zero-order part of the normalised structure function \( h_3(x, \tau) \), while the coefficient \( C_{\varepsilon, \text{decay}}^{\text{decay}}(\tau) \) is of the same form as in the stationary case but has the additional term in the time-derivative of the first-order part, that is, \( h_2^{(1)}(x, \tau) \).

5.1 Taking the limit of infinite Reynolds numbers

Taking the infinite Reynolds number limit of each term in equation (24) we find that

\[
\lim_{\nu \to 0} C_{\varepsilon, \text{decay}}^{\text{decay}}(\tau) = -\frac{1}{4x^4} \frac{\partial}{\partial x} \left(x^4 h_3^{(0)}(x, \tau)\right) - \frac{3}{4} \frac{\partial h_2^{(0)}(x, \tau)}{\partial \tau}. \tag{28}
\]

If local stationarity is again assumed, all time dependences can be dropped, and the constant \( C_{\varepsilon, \infty}^{\text{decay}} \) in equation (20) can be identified as:

\[
\lim_{\nu \to 0} C_{\varepsilon, \text{decay}}^{\text{decay}} = C_{\varepsilon, \infty}^{\text{decay}} = -\frac{1}{4x^4} \frac{\partial}{\partial x} \left(x^4 h_3^{(0)}(x)\right). \tag{29}
\]
That is, if Kolmogorov’s assumption of local stationarity is valid, the asymptotic dimensionless dissipation rate should take the same functional form for freely decaying turbulence as it does for the forced, stationary case.

However, if local stationarity is not assumed, the dimensionless dissipation rate instead becomes

\[
\lim_{\nu \to 0} C_{\epsilon}^{\text{decay}}(\tau) \equiv C_{\epsilon,\infty}^{\text{decay}}(\tau) = C_{\epsilon,\infty}^{\text{stat}}(\tau) - \frac{3}{4} \frac{\partial h_2^{(0)}(\tau)}{\partial \tau}.
\]

(30)

Note that we continue to use \( C_{\epsilon,\infty}^{\text{stat}}(\tau) \) to indicate that the functional form of this term is the same as in the stationary case, but that in the case of free decay its value can depend on time. Or, invoking equations (14) and (15), it may be written as:

\[
C_{\epsilon,\infty}^{\text{decay}}(t) = C_{\epsilon,\infty}^{\text{stat}}(t) - \frac{3}{4} \frac{L}{U^3} \frac{\partial S_2^{(0)}(t)}{\partial t} = C_{\epsilon,\infty}^{\text{stat}}(t) + \Delta(t),
\]

(31)

where

\[
\Delta(t) = -\frac{3}{4} \frac{L}{U^3} \frac{\partial S_2^{(0)}(t)}{\partial t},
\]

(32)
is the error made by assuming local stationarity. As before, we point out that, since the other terms in equation (29) are independent of scale, the second term on the right-hand side of this equation must also have no dependence on \( r \). We should emphasise that these conclusions apply in the limit of infinite Reynolds numbers, as \( S_2^{(0)} \) is, by definition, that part of the structure function which does not depend on the Reynolds number.

6 Implications for the ‘4/5’ law

Since unsustained turbulence decays with time at all Reynolds numbers, it follows that the time derivative term in equation (31) must satisfy the constraint

\[
-\frac{3}{4} \frac{L}{U^3} \frac{\partial S_2^{(0)}(t)}{\partial t} \geq 0.
\]

(33)

This can be seen by, for example, differentiating Kolmogorov’s ‘2/3’ law for the second order structure function with respect to time and noting that the decay rate is negative. From this, it is tempting to conclude that equation (31), taken jointly with equation (33), implies that the asymptotic dissipation rate for free decay must be greater than, or equal to, the asymptotic rate for stationary turbulence. However, we must bear in mind that, although \( C_{\epsilon,\infty}^{\text{stat}}(t) \) is of the same functional form as for the stationary case, it is being evaluated for the time-varying case of free decay. So we should go back to the earlier idea of a fiducial time \( t_e \), and evaluate the terms of (31) at this time. Accordingly, we have:

\[
C_{\epsilon,\infty}^{\text{decay}}(t_e) = C_{\epsilon,\infty}^{\text{stat}}(t_e) - \frac{3}{4} \frac{L}{U^3} \frac{\partial S_2^{(0)}(t)}{\partial t} \bigg|_{t=t_e}.
\]

(34)

The experimental position is unclear, but there is a view that experimental results suggest \( C_{\epsilon,\infty}^{\text{decay}} \geq C_{\epsilon,\infty}^{\text{stat}} \). The best evidence for this is the investigation of Bos et al.
who compared the freely decaying and stationary cases using direct numerical simulation, large-eddy simulation and a closure model (EDQNM). In turn, this has implications for the assumption of local stationarity in deriving the ‘4/5’ law. If that is the case, then the time derivative term in equation (18) should be retained.

This is not a new concern. Previously, in the context of the ‘4/5’ law, there has been some recognition of the possible effect of the time-derivative term when compared to the stationary case. Lindborg [23] used a simple model (the $k-\varepsilon$ model) to estimate the magnitude of the unsteady term. He found the term not to be negligible. Both Antonia & Burattini [24] and Tchoufag et al [25] studied the approach of $S_3/\varepsilon r$ to 4/5 for both stationary and decaying turbulence and both found that the onset of the 4/5 law was at a much lower Reynolds number in the stationary case. Antonia and Burattini [24] suggested that Taylor-Reynolds numbers of $10^3$ and $10^6$ were needed for forced and decaying turbulence, respectively.

Tchoufag et al [25] came to a similar conclusion. Their DNS results for forced turbulence were found to agree quite well with models proposed by Moisy et al [26], thus:

$$\frac{S_3(r,t)}{\varepsilon r} = 4\frac{1}{5}\left[1 - \left(\frac{R_\lambda}{R_{\lambda 0}}\right)^{-5/6}\right],$$

where $R_{\lambda 0} \approx 30$; while the results for free decay agreed well with a model of Lundgren [27]:

$$\frac{S_3(r)}{\varepsilon r} = 4\frac{4}{5} - 8.45R_{\lambda}^{-2/3}.$$   

These authors further concluded that the 4/5 law was recovered at Taylor-Reynolds numbers exceeding 5,000 in the forced case and 50,000 in the free-decay case.

We also note the recent work of Boschung et al [28] who considered only free decay and who studied the interplay between finite viscous effects and the unsteady term, using direct numerical simulation and a model closure which allowed them to extend their results up to Taylor-Reynolds numbers of $10^4$. Even at such high Reynolds numbers, these authors found that the inertial range is quite short. They also found that the viscous term acts as a sink of energy while the unsteady term acts as a source, at all scales. They concluded that the inertial range is the region surrounding the point where these two effects cancel out.

The implication of these investigations is that the Kolmogorov ‘4/5’ law should be recovered in the limit of infinite Reynolds numbers in the case of free decay. Yet, in view of our present results, this cannot be entirely true. As we have seen, the time-derivative term in equation (18) does not depend on either scale or Reynolds number. Accordingly, unless it is inherently zero for some other reason, this is the limiting case. That is to say, if the time derivative is indeed non-zero, $S_3/\varepsilon r$ should never reach 4/5; or:

$$|S_3(r,t)| < \frac{4}{5}\varepsilon(t)r \quad \forall R.$$   

In wavenumber space, this would suggest that the peak flux (through wavenumber) would never equal the dissipation rate, as previously pointed out by Sagaut and Cambon [29] and McComb et al [21]. This result could have implications for other aspects of the Kolmogorov-Richardson phenomenology and we discuss these in the next section.
7 Implications for the Kolmogorov ‘2/3’ and ‘-5/3’ laws

As noted in the Introduction, the main outcome of the Kolmogorov (1941) theory is the ‘2/3’ law for the second-order structure function, \( S_2(r,t) = C_2 \varepsilon^{2/3} r^{2/3} \), and the corresponding energy spectrum, \( E(k,t) = \alpha \varepsilon^{2/3} k^{-5/3} \), where in both cases the independent variables are restricted to their inertial range. We will concentrate on the spectral case, while bearing in mind that the real space case can be recovered by Fourier transformation. We begin by Fourier transforming the KHE in order to obtain the Lin equation (e.g. see reference [10]). This may be written as:

\[
\frac{\partial E(k,t)}{\partial t} = W(k) + T(k,t) - D(k,t),
\]

where \( k \) is the wavenumber, \( E(k,t) \) is the energy spectrum (obtained from Fourier transformation of \( S_2(r,t) \); \( W(k) \) is an energy input term, arising from stirring forces; \( T(k,t) \) is the transfer spectrum; and \( D(k,t) = 2\nu k^2 E(k,t) \) is the dissipation spectrum. An expression for \( T(k,t) \) can be found, for instance, as equation (3.14) in the book [10], but that will not be needed here.

We can apply this equation either to stationary turbulence, with the time derivative set equal to zero, and all time dependences dropped; or to freely decaying turbulence, with \( W(k) = 0 \). Alternatively, we can write equation (38), with some rearrangement, in unified form for both cases, as [10]:

\[
-T(k,t) = I(k,t) - D(k,t),
\]

where \( I(k,t) \) stands for either the time derivative term or the stirring spectrum. In the latter case the time dependences should be omitted in order to indicate stationarity. Then, by comparison with equation (24), we may deduce the form of (39) in the infinite Reynolds number limit as:

\[
-T(k,t) \big|_{\varepsilon=\text{const}} = \lim_{\nu \to 0} I(k,t) \big|_{\varepsilon=\text{const}} - \lim_{\nu \to 0} D(k,t) \big|_{\varepsilon=\text{const}}. \tag{40}
\]

We now need more explicit forms on the right hand side. The term involving the time derivative is the Fourier transform of

\[
\lim_{\nu \to 0} I(r,t) \big|_{\varepsilon=\text{const}} = I(r,t) = \lambda(t)
\]

where \( \lambda(t) \) is independent of both scale \( r \) and Reynolds number. Hence, its Fourier transform with respect to wavenumber \( k \) is just a delta function at the origin in \( k \)-space.

Evaluating the second term on the right hand side of equation (40) is a little more tricky; but the analysis, having been given by Batchelor [7] and developed by Edwards [30], is quite well known. We take the limit of the Kolmogorov wavenumber \( k_d \), thus:

\[
\lim_{\nu \to 0} k_d \big|_{\varepsilon=\text{const}} = \lim_{\nu \to 0} \frac{\varepsilon(t)}{\nu^3} \bigg|_{\varepsilon=\text{const}} \to \infty.
\]

Thus, in this limit, the dissipation rate must be concentrated at \( k = \infty \) and can be represented by \( \varepsilon(t) \delta(k - \infty) \). In all then, we can write (40) as:

\[
- \lim_{\nu \to 0} T(k,t) \big|_{\varepsilon=\text{const}} = \lambda(t) \delta(k) - \varepsilon(t) \delta(k - \infty) = \varepsilon(t) \delta(k) - \varepsilon(t) \delta(k - \infty), \tag{41}
\]
where the equality $\lambda(t) = \varepsilon(t)$ follows from integrating both sides over all values of $k$, and invoking conservation of energy.

Equation (41) was first given for stationary turbulence by Edwards [30], in the course of testing a closure approximation based on his self-consistent field theory. Here we see that it is also the limiting case for freely decaying turbulence as well. Introducing the energy flux $\Pi(\kappa, t)$ through mode $\kappa$, by the relationship

$$\Pi(\kappa, t) = -\int_0^\kappa dk T(k, t),$$

it follows from equation (41) that

$$\Pi(\kappa, t) = \varepsilon(t), \forall \kappa.$$  \hfill (43)

Hence we have scale invariance, which is the necessary condition for an inertial range. Here this applies for all values of the wavenumber, in the limit of infinite Reynolds numbers\(^1\). It was argued by Edwards [30] that the $-5/3$ spectrum would apply for all wavenumbers under these circumstances. So it appears that the unsteady term in free decay appears only as a source term in the spectral energy balance and hence, whatever its effect on the ‘4/5’ law, does not affect the Kolmogorov energy spectrum.

8 Conclusions

Previously McComb et al [1] found that, for stationary turbulence, the dimensionless dissipation rate took the form:

$$C_\varepsilon = C_{\varepsilon,\infty} + C/R_L + \mathcal{O}(1/R_L^2) : \text{stationary case;}$$

(44)

where $C$ is a constant which depends on those parts of the second- and third-order structure functions which are independent of the Reynolds number. The asymptotic value and the coefficient were evaluated respectively as $C_{\varepsilon,\infty} = 0.468 \pm 0.006$ and $C = 18.9 \pm 1.3$ from the numerical simulation [1].

In the present article, we have derived the asymptotic form for the freely decaying case and it is tempting to go on and write the analogous expression for free decay as:

$$C_\varepsilon^{\text{decay}}(t_e) = C_{\varepsilon,\infty}^{\text{decay}}(t_e) - \frac{3}{4} \frac{L}{U^3} \frac{\partial S_2(0)}{\partial t} \bigg|_{t=t_e} + C_{\varepsilon,\infty}^{\text{decay}}(t_e)/R_L(t_e) + \mathcal{O}(1/R_L^2) : \text{free decay.}$$

(45)

However, there are two points at issue here.

First, we need to justify the truncation to the first-order term in (45). At low values of the Reynolds number, it may be necessary to take higher orders into account. For instance, with magnetohydrodynamic turbulence, it is necessary to take the next order of the expansion into account, although the effect is small [31]. However, there is quite good support from an earlier investigation at low Reynolds numbers [21] suggesting that the first-order truncation in (45) is justified.

\(^1\)Strictly we should exclude the exact values $\kappa = 0$ and $\kappa = \infty$. 

14
The second issue is the absence of an agreed criterion for choosing the evolved time $t_e$. At present this is not an aspect which is discussed much in this field and there appears to be a tacit acceptance that, for times of decay greater than some evolved time, the asymptotic dissipation rate in free decay will take some universal value. The results of Yoffe [20] suggest that this may be true, provided that $t_e$ is chosen to be something like three eddy turnover times, or larger. But this means that one is losing much of the decay process by working at long evolution times. So there is some attraction in the use of the composite time $t_{e II}$, as discussed in Section 2.1. This is a matter which deserves further study.

Lastly, as seen in Section 5.1, $\Delta(t)$ as defined by equation (32), is the error made by assuming local stationarity in the case of freely decaying turbulence. If we can show a physical basis for adopting the composite form of evolution time, then the results of Yoffe [20] indicate that $\Delta(t)$ can be evaluated from a comparison between the asymptotic results for forced and freely decaying turbulence. This will be the subject of further work. However, it may also be pointed out that this error can also be found from suitable limiting procedures applied to equation (45), as this equation holds for all times. This would be a more cumbersome procedure but could be carried out using conventional numerical simulations.

Acknowledgements

One of us (WDM) would like to thank Katepalli Sreenivasan for a helpful exchange of emails and in particular for pointing out the significance of the ‘fragment’ by Onsager.

References

[1] W. D. McComb, A. Berera, S. R. Yoffe, and M. F. Linkmann. Energy transfer and dissipation in forced isotropic turbulence. Phys. Rev. E, 91:043013, 2015.

[2] A. N. Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. C. R. Acad. Sci. URSS, 30:301, 1941.

[3] A. N. Kolmogorov. Dissipation of energy in locally isotropic turbulence. C. R. Acad. Sci. URSS, 32:16, 1941.

[4] A. M. Obukhov. On the distribution of energy in the spectrum of turbulent flow. C.R. Acad. Sci. U.R.S.S, 32:19, 1941.

[5] L. Onsager. The Distribution of Energy in Turbulence. Phys. Rev., 68:281, 1945.

[6] G. I. Taylor. Statistical theory of turbulence. Proc. R. Soc., London, Ser. A, 151:421, 1935.

[7] G. K. Batchelor. The theory of homogeneous turbulence. Cambridge University Press, Cambridge, 1st edition, 1953.

[8] K. R. Sreenivasan. On the scaling of the turbulence dissipation rate. Phys. Fluids, 27:1048, 1984.
[9] K. R. Sreenivasan. An update on the energy dissipation rate in isotropic turbulence. *Phys. Fluids*, 10:528, 1998.

[10] W. David McComb. *Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures*. Oxford University Press, 2014.

[11] T. von Karman and L. Howarth. On the statistical theory of isotropic turbulence. *Proc. Roy. Soc. Lond. A*, 164:192, 1938.

[12] Detlef Lohse. Crossover from High to Low Reynolds Number Turbulence. *Phys. Rev. Lett.*, 73(22):3223, 1994.

[13] Charles R. Doering and Ciprian Foias. Energy dissipation in body-forced turbulence. *J. Fluid Mech.*, 467:289–306, 2002.

[14] L. Djenidi, N. Lefeuvre, M. Kamruzzaman, and R. A. Antonia. On the normalized dissipation parameter $C_\epsilon$ in decaying turbulence. *J. Fluid Mech.*, 817:63, 2017.

[15] W. K. George. Reconsidering the ‘local equilibrium’ hypothesis for small scale turbulence. In M. Farge, H. K. Moffat, and K Schneider, editors, *Turbulence colloquium Marseille 2011: Fundamental problems of Turbulence, 50 years after the Marseille 1961 Conference*, page 1. EDP Science, 2014.

[16] W. D. McComb, A. Hunter, and C. Johnston. Conditional mode-elimination and the subgrid-modelling problem for isotropic turbulence. *Phys. Fluids*, 13:2030, 2001.

[17] L.-P. Wang, S. Chen, J. G. Brasseur, and J. C. Wyngaard. Examination of hypotheses in the Kolmogorov refined turbulence theory through high-resolution simulations. Part 1. Velocity field. *J. Fluid Mech.*, 309:113, 1996.

[18] W. J. T. Bos, L. Shao, and J.-P. Bertoglio. Spectral imbalance and the normalized dissipation rate of turbulence. *Phys. Fluids*, 19:45101, 2007.

[19] D. Fukayama, T. Oyamada, T. Nakano, T. Gotoh, and K. Yamamoto. Longitudinal structure functions in decaying and forced turbulence. *J. Phys. Soc. Japan*, 69:701, 2000.

[20] S. R. Yoffe. *Investigation of the transfer and dissipation of energy in isotropic turbulence*. PhD thesis, University of Edinburgh, 2012.

[21] W. David McComb, Arjun Berera, Matthew Salewski, and Sam R. Yoffe. Taylor’s (1935) dissipation surrogate reinterpreted. *Phys. Fluids*, 22:61704, 2010.

[22] W. D. McComb. *The Physics of Fluid Turbulence*. Oxford University Press, 1990.

[23] Erik Lindborg. Correction to the four-fifths law due to variations of the dissipation. *Phys. Fluids*, 11:510, 1999.

[24] R. A. Antonia and P. Burattini. Approach to the 4/5 law in homogeneous isotropic turbulence. *J. Fluid Mech.*, 550:175, 2006.
[25] J. Tchoufag, P. Sagaut, and C. Cambon. Spectral approach to finite Reynolds number effects on Kolmogorov’s 4/5 law in isotropic turbulence. *Phys. Fluids*, 24:015107, 2012.

[26] F. Moisy, P. Tabeling, and H. Willaime. Kolmogorov Equation in a Fully Developed Turbulence Experiment. *Phys. Rev. Lett.*, 82:3994–3997, 1999.

[27] Thomas S. Lundgren. Kolmogorov turbulence by matched asymptotic expansion. *Phys. Fluids*, 15:1074, 2003.

[28] J. Boschung, M. Gauding, F. Hennig, D. Denker, and H. Pitsch. Finite Reynolds number corrections of the 4/5 law for decaying turbulence. *Phys. Rev. Fluids*, 1:064403, 2016.

[29] P. Sagaut and C. Cambon. *Homogeneous Turbulence Dynamics*. Cambridge University Press, Cambridge, 2008.

[30] S. F. Edwards. Turbulence in hydrodynamics and plasma physics. In *Proc. Int. Conf. on Plasma Physics, Trieste*, page 595. IAEA, 1965.

[31] M. F. Linkmann, A. Berera, W. D. McComb, and M. E. McKay. Nonuniversality and Finite Dissipation in Decaying Magnetohydrodynamic Turbulence. *Phys. Rev. Lett.*, 114:235001, 2015.