Norm attaining operators and variational principle

by

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Abstract. We establish a linear variational principle extending Deville–Godefroy–Zizler’s one. We use this variational principle to prove that if $X$ is a Banach space having property ($\alpha$) of Schachermayer and $Y$ is any Banach space, then the set of all strongly norm attaining linear operators from $X$ into $Y$ is the complement of a $\sigma$-porous set. Moreover, we apply our results to an abstract class of (linear and nonlinear) operator spaces.

1. Introduction. This paper is devoted to establishing a new linear variational principle in the spirit of Stegall’s one (see [S78] or [P93, Theorem 5.15]), which applies to certain “small class” of subsets of Banach spaces satisfying a condition that we call “uniform separation property”. However, in our statement we do not need to assume that the Banach space has the Radon–Nikodým property. The interest of this result is that, on the one hand, it extends the nonlinear variational principle of Deville–Godefroy–Zizler and Deville–Revalski (see [DGZ93] and [DR00]) and, on the other hand, it makes it possible to show that the set of norm attaining operators (under hypothesis ($\alpha$)) is not only a dense subset of the space of all bounded linear operators but it is also large in the sense of being the complement of a $\sigma$-porous subset. Moreover, a quantitative version of the Bishop–Phelps–Bollobás theorem will be given and “norm attaining operators” is extended to “strongly norm attaining operators”.

Let $X$ and $Y$ be real Banach spaces. We denote by $B(X,Y)$ (resp. $K(X,Y)$ and $F(X,Y)$) the space of all bounded linear operators (resp. of compact operators and of finite-rank operators). An operator $T \in B(X,Y)$ is said to be norm attaining (resp. strongly norm attaining) if there is an

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$x_0 \in S_X$, the unit sphere of $X$, such that $\|T\| = \|T(x_0)\|$ (resp. $\|T(x_n)\| \rightarrow \|T\| \rightarrow \|T(x_0)\|$ implies $\|x_n - x_0\| \rightarrow 0$). We write $NAB(X,Y)$ to denote the set of norm attaining operators in $B(X,Y)$. The question whether $NAB(X,Y)$ is norm dense in $B(X,Y)$ starts in 1961 with the works of Bishop and Phelps [BP61, BP63], who proved that if $Y$ is one-dimensional then $NAB(X,Y)$ is norm dense in $X^* = B(X,Y)$ for all spaces $X$. In 1963, Lindenstrauss [L63] showed that the Bishop–Phelps theorem is no longer true for linear operators and gave some partial positive results. He introduced property $(\beta)$ and proved that if $Y$ has this property, then for every Banach space $X$, $NAB(X,Y)$ is dense in $B(X,Y)$. Partington [P82] proved that every Banach space $Y$ can be renormed to have property $(\beta)$. Schachermayer [S83] introduced property $(\alpha)$ as a sufficient condition on a Banach space $X$ such that $NAB(X,Y)$ is dense in $B(X,Y)$ for every $Y$ and he showed that every weakly compactly generated Banach space can be renormed to have property $(\alpha)$. Several authors have contributed to this domain, extending these results in different ways. There also exists a “quantitative version” of the Bishop–Phelps–Bollobás theorem [B70] given by Acosta, Aron, García and Maestre [AA+08]. Several authors have proved similar results, replacing $B(X,Y)$ by other operator spaces. For a complete story of contributions in this domain, we refer to [A06] and the references therein.

The contribution of this paper consists in replacing the density of the set of norm attaining operators by being the complement of a $\sigma$-porous set, and in giving a unified and abstract class of (linear and nonlinear) operator spaces having the “norm attaining operators property” (see Theorem 4.6). In particular, we obtain the following results:

1. If $X$ has property $(\alpha)$ (see Example B in Section 3 for the definition), then for every Banach space $Y$ and every closed subspace $R(X,Y)$ of $B(X,Y)$ containing $F(X,Y)$, the subset $NAR(X,Y)$ of norm attaining operators in $R(X,Y)$ is the complement of a $\sigma$-porous subset of $R(X,Y)$. In fact, we prove the result for strongly norm attaining operators.

2. The results of the paper also apply to nonlinear operator spaces like the space of all bounded continuous (resp. uniformly continuous) functions from a complete metric space into a Banach space, extending some real-valued results of Coban, Kenderov and Revalski [CKR89] (see also [DR00]) to the vector-valued framework. For another direction of Lipschitz norm attaining functions, we refer to [KMS16] and [G16].

3. A new quantitative version of the Bishop–Phelps–Bollobás theorem is also given in our abstract framework of (linear and nonlinear) operator spaces. Notice that, in the particular case of the space $B(X,Y)$, the Bishop–Phelps–Bollobás property in the sense given in [AA+08] is not satisfied for all Banach spaces $Y$ even if $X$ has property $(\alpha)$. However, our version in Theo-
rem 4.6 allows one to obtain a version of the Bishop–Phelps–Bollobás property working for any space $X$ with property $(\alpha)$ and any Banach space $Y$. The difference with the definition given in [AA08] is that we require the Bishop–Phelps–Bollobás property only on a certain subset (given by property $(\alpha)$) $K := \{x_\lambda : \lambda \in A\}$ of the sphere $S_X$ of $X$ which is norming, and not on the whole $S_X$.

This paper is organized as follows. In Section 2, we introduce the notion of uniform separation property (for short, $\mathcal{USP}$). We then give some examples of sets having this property. In Section 3, we prove our version of linear variational principle (Theorem 3.3) and its localized version (Theorem 3.7). We also give an extension of the Deville–Godefroy–Zizler variational principle as an immediate consequence. In Section 4, we will apply this new variational principle to obtain the $\sigma$-porosity of the set of norm nonattaining operators in Theorem 4.6 and its corollaries.

2. The uniform separation property. In this section, we introduce the notion of uniform separation property and give some examples. The variational principle given in this paper applies to general pseudometric spaces and generalized lower semicontinuous functions. We first recall the following definition.

**Definition 2.1.** Let $C$ be a nonempty set and $\gamma : C \times C \to \mathbb{R}^+$. We say that $\gamma$ is a pseudometric if

- $\gamma(x, x) = 0$ for all $x \in C$.
- $\gamma(x, y) = \gamma(y, x)$ for all $x \in C$.
- $\gamma(x, y) \leq \gamma(x, z) + \gamma(z, y)$ for all $x, y, z \in C$.

Unlike a metric space, one may have $\gamma(x, y) = 0$ for some $x \neq y$. A pseudometric induces an equivalence relation that converts the pseudometric space into a metric space. This is done by defining $x \sim y$ if $\gamma(x, y) = 0$. Let $\Gamma_\gamma : C \to C/\sim$ be the canonical surjection and let

$$d_\gamma(\Gamma_\gamma(x), \Gamma_\gamma(y)) := \gamma(x, y).$$

Then $(C/\sim, d_\gamma)$ is a well-defined metric space. We say that $(C, \gamma)$ is a complete pseudometric space if $(C/\sim, d_\gamma)$ is a complete metric space.

**Definition 2.2.** Let $X$ be a Banach space, $C$ be a subset of the dual $X^*$ and $(C, \gamma)$ be a pseudometric space. We say that $(C, \gamma)$ has the weak*-uniform separation property (for short, $w^*\mathcal{USP}$) in $X^*$ if there exists $a > 0$ such that for every $\varepsilon \in [0, a]$, there exists $\varpi_C(\varepsilon) > 0$ such that for every $p \in C$, there exists $x_{p,\varepsilon} \in B_X$ (the closed unit ball of $X$) such that

$$\langle p, x_{p,\varepsilon} \rangle - \varpi_C(\varepsilon) \geq \langle q, x_{p,\varepsilon} \rangle$$

for all $q \in C$ such that $\gamma(q, p) \geq \varepsilon$. 
If $C$ is a subset of a Banach space $X$, we say that $(C, \gamma)$ has the USP in $X$ if $(C, \gamma)$ has the $w^*\text{USP}$ in $X^{**}$ when $C$ is considered as a subset of the bidual $X^{**}$.

The function $\varpi_C$ will be called the modulus of uniform separation of $(C, \gamma)$. If $x \in X$, we denote by $\hat{x} : X^* \to \mathbb{R}$ the evaluation map at $x$ given by $x^* \mapsto \langle x^*, x \rangle$ for all $x^* \in X^*$.

**Remark 2.3.** (1) If $A \subset C$ and $(C, \gamma)$ has the $w^*\text{USP}$ (resp. the USP), then clearly $(A, \gamma)$ also has the $w^*\text{USP}$ (resp. the USP).

(2) Two interesting cases are when $\gamma$ is the norm of $X^*$ or the distance associated to the weak-star topology if $X$ is separable, but working with the general pseudometric has its applications as we will see in the context of norm attaining linear operators.

The following proposition is easy to establish; the proof is left to the reader.

**Proposition 2.4.** Let $X$ be a Banach space and $C \subset X^*$ (resp. $C \subset X$). Suppose that $(\overline{C}, \gamma)$ is a pseudometric space (where $\overline{C}$ denotes the norm closure of $C$) and the identity map $i : (C, \| \cdot \|) \to (C, \gamma)$ is continuous. Then $(C, \gamma)$ has the $w^*\text{USP}$ (resp. USP) if and only if $(\overline{C}, \gamma)$ has the $w^*\text{USP}$ (resp. USP).

We give some examples of sets having the USP or $w^*\text{USP}$.

**A. Uniformly convex spaces.** Recall that a Banach space $(L, \| \cdot \|)$ is uniformly convex if for each $\varepsilon \in [0, 2]$, 

$$\delta(\varepsilon) := \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in S_L, \|x - y\| \geq \varepsilon \right\} > 0.$$ 

**Proposition 2.5.** Let $L$ (resp. $L^*$) be a uniformly convex Banach space. Then the sphere $(S_L, \| \cdot \|)$ (resp. $(S_L^*, \| \cdot \|)$) has the USP in $L$ (resp. $w^*\text{USP}$ in $L^*$).

**Proof.** Let $\varepsilon \in [0, 2]$. For each $x, y \in S_L$ such that $\|x - y\| \geq \varepsilon$ we have 

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta(\varepsilon).$$

Thus, for all $p \in S_{L^*}$ we have 

$$\left\langle p, \frac{x + y}{2} \right\rangle \leq \left\| \frac{x + y}{2} \right\| \leq 1 - \delta(\varepsilon).$$

Now, let us fix an arbitrary $x \in S_L$ and choose $p_{x, \varepsilon} \in S_{L^*}$ such that $\langle p_{x, \varepsilon}, x \rangle > 1 - \delta(\varepsilon)/2$. Using the above inequality, we find that $\langle p_{x, \varepsilon}, y \rangle \leq 2 - 2\delta(\varepsilon) - \langle p_{x, \varepsilon}, x \rangle \leq \langle p_{x, \varepsilon}, x \rangle - \delta(\varepsilon)$ for all $y \in S_L$ such that $\|x - y\| \geq \varepsilon$. Hence, $(S_L, \| \cdot \|)$ has the USP with modulus of uniform separation $\varpi_{S_L}(\varepsilon) = \delta(\varepsilon)$ for all $\varepsilon \in [0, 2]$ (the same proof works for $(S_{L^*}, \| \cdot \|)$).
B. Property (α). Recall property (α) introduced by Schachermayer [S83]. A Banach space \(X\) has property (α) if there exist \(\{x_\lambda : \lambda \in \Lambda\} \subset X\) and \(\{x_\lambda^* : \lambda \in \Lambda\} \subset X^*\) such that:

1. \(\|x_\lambda\| = \|x_\lambda^*\| = \langle x_\lambda^*, x_\lambda \rangle = 1\) for all \(\lambda \in \Lambda\).
2. There exists a constant \(\rho\) with \(0 < \rho < 1\) such that \(|\langle x_\lambda^*, x_\mu \rangle| \leq \rho\) for any distinct \(\lambda, \mu \in \Lambda\).
3. The absolute convex hull of \(\{x_\lambda : \lambda \in \Lambda\}\) is dense in \(B_X\).

Clearly, conditions (1) and (2) imply that the closed set \(\{\{x_\lambda : \lambda \in \Lambda\}, \|\cdot\|_X\}\) has the USP in \(X\).

C. The Dirac measures. Let \((L, d)\) be a metric space and \((X, \|\cdot\|_X)\) be a Banach space included in \(C_b(L)\) (the space of all real-valued bounded continuous functions equipped with the sup-norm). Suppose that \(X\) separates the points of \(L\) and satisfies \(\|\cdot\|_X \geq \alpha\) \(\|\cdot\|_\infty\) on \(X\) for some \(\alpha > 0\). Recall that the Dirac measure associated to the point \(x \in L\) is the continuous linear functional \(\delta_x : h \mapsto h(x), h \in X\). Since \(\|\cdot\|_X \geq \alpha\) \(\|\cdot\|_\infty\), it follows that \(\|\delta_x\| \leq 1/\alpha\) for all \(x \in L\). Thus, the subset \(\delta(L) := \{\delta_x : x \in L\}\) is norm bounded in \(X^*\). We equip the set \(\delta(L)\) with the following complete metric:

\[
\tilde{d}(\delta_x, \delta_y) := d(x, y).
\]

Notice that the map \(\tilde{d}\) is well defined since \(X\) separates the points of \(L\). Let \(h\) be a real-valued function on \(L\) and \(A\) be a subset of \(L\). We denote by \(\text{supp}(h) := \{x \in L : h(x) \neq 0\}\) the support of \(h\) and by \(\text{diam}(A)\) the diameter of \(A\). We consider the following hypothesis:

\((H)\) for every \(\varepsilon > 0\) there exists \(\varpi_X(\varepsilon) > 0\) such that, for every \(x \in L\), there exists a function \(b_{x, \varepsilon} \in B_X\) such that

\[
b_{x, \varepsilon}(x) - \varpi_X(\varepsilon) \geq \sup_{y \in L : d(y, x) \geq \varepsilon} b_{x, \varepsilon}(y).
\]

It is clear that hypothesis \((H)\) is equivalent to \((\delta(L), \tilde{d})\) having the \(w^*\text{USP}\) in \(X^*\).

Now, consider the following hypothesis used by Deville–Revalski [DR00]:

\((\text{DR})\) for every natural number \(n\), there exists a positive constant \(M_n\) such that for any point \(x \in L\) there exists a function \(h_{x,n} : L \to [0,1]\) such that \(h_{x,n} \in X\), \(\|h_{x,n}\| \leq M_n\), \(h_{x,n}(x) = 1\) and \(\text{diam}(\text{supp}(h_{x,n})) < 1/n\).

The fact that \((\text{DR}) \Rightarrow (H)\) is seen by taking

\[
b_{x, \varepsilon} := \frac{h_{x,[1/\varepsilon]+1}}{M_{[1/\varepsilon]+1}} \in B_X \quad \text{and} \quad \varpi_X(\varepsilon) = \frac{1}{M_{[1/\varepsilon]+1}}
\]

for all \(\varepsilon > 0\), where \([\cdot]\) denotes integer part. However, \((H) \not\Rightarrow (\text{DR})\) in general. Indeed, for a bounded complete metric space \((L, d)\), consider \((X, \|\cdot\|_X) = \ldots\)
(\text{Lip}_0(L, \| \cdot \|), the space of all Lipschitz continuous functions that vanish at some point \( x_0 \in L \), equipped with its natural norm

\[ \|g\| := \sup_{x, y \in L: x \neq y} \frac{|g(x) - g(y)|}{d(x, y)}, \quad \forall g \in X. \]

Then hypothesis (H) is trivially satisfied with \( \omega_L(\varepsilon) = \varepsilon \) for all \( \varepsilon > 0 \) and

\[ b_{x, \varepsilon}(y) := d(x, x_0) - d(x, y) \]

for all \( x, y \in L \). However, (DR) is never satisfied for \( X = \text{Lip}_0(L) \) since \( f(x_0) = 0 \) for all \( f \in \text{Lip}_0(L) \). Thus, the condition that \( (\delta(L), d) \) has the \( w^*\text{USP} \) in \( X^* \) \((\Leftrightarrow (H))\) is more general than (DR).

An extension of the Deville–Revalski result \([\text{DR}00]\) will be given by applying our main result (Theorem 3.3) to the metric space \((\delta(L), \tilde{d})\) which has the \( w^*\text{USP} \).

3. Linear variational principle. This section is devoted to establishing a linear variational principle for \( w^*\text{USP} \) subsets of Banach spaces. We recall that a function \( f \) has a strong minimum on a metric space \((C, d)\) at some point \( p \in C \) if \( f \) attains its minimum at \( p \) and for any sequence \((p_n) \subset C \) such that \( f(p_n) \to f(p) = \inf_C f \), we have \( d(p_n, p) \to 0 \). A function \( f \) has a strong maximum if \(-f\) has a strong minimum.

To obtain our result in the more general case of pseudometric spaces, we need to introduce the following definition.

**Definition 3.1.** Let \((C, \gamma)\) be a pseudometric space. Let \( f : C \to \mathbb{R} \cup \{+\infty\} \) be a proper function bounded from below. We say that \( f \) is \( \gamma\)-strongly minimized on \( C \) at \( u \in C \) if for every sequence \((q_n) \subset C \) we have

\[ \lim_{n \to \infty} f(q_n) = \inf_C f \quad \Rightarrow \quad \lim_{n \to \infty} \gamma(q_n, u) = 0. \]

A function \( g \) is \( \gamma\)-strongly maximized on \( C \) at \( u \in C \) if \(-g\) is \( \gamma\)-strongly minimized on \( C \) at \( u \in C \).

In the general case, it may be that \( \inf_C f \neq f(u) \) above. However, \( u \) is necessarily unique up to the relation \( \sim \), that is, every other \( v \in C \) having the above property satisfies \( \gamma(v, u) = 0 \) and the converse is also true.

Note that if we moreover assume that \( f \) is lower semicontinuous with respect to the pseudometric \( \gamma \) (that is, for every sequence \((q_n) \subset C \), \( \liminf_{n \to \infty} f(q_n) \geq f(u) \) whenever \( \lim_{n \to \infty} \gamma(q_n, u) = 0 \)), then the infimum of \( f \) is attained at \( u \).

In the particular case where \( \gamma \) is a metric and \( f \) is lower semicontinuous for \( \gamma \), the notion of being \( \gamma\)-strongly minimized on \( C \) coincides with the classical notion of having a strong minimum, mentioned above.

We now recall the notion of \( \sigma\)-porosity. Below, \( \tilde{B}_X(x; r) \) stands for the open ball in \( X \) centered at \( x \) and with radius \( r > 0 \).

**Definition 3.2.** Let \((X, d)\) be a metric space and \( A \) be a subset of \( X \). The set \( A \) is said to be porous in \( X \) if there exist \( \lambda_0 \in (0, 1] \) and \( r_0 > 0 \)
such that for any $x \in X$ and $r \in (0, r_0]$ there exists $y \in X$ such that $B_X(y; \lambda_0 r) \subset B_X(x; r) \cap (X \setminus A)$. The set $A$ is called $\sigma$-porous in $X$ if it can be represented as a countable union of porous sets in $X$.

Every $\sigma$-porous set is of first Baire category. Moreover, in $\mathbb{R}^n$, every $\sigma$-porous set is of Lebesgue measure zero. However, there does exist a non-$\sigma$-porous subset of $\mathbb{R}^n$ which is of first category and of Lebesgue measure zero. For more information about $\sigma$-porosity, we refer to [Z87].

We now give the main results of this section. We will see in Corollary 3.5 how to recover and easily extend the Deville–Godefroy–Zizler and Deville–Revalski variational principles, from the following theorem (a vector-valued variational principle of Deville–Godefroy–Zizler type is also given in Theorem 4.6). Note that changing “infimum” to “supremum” and $f$ to $-f$, we obtain the “supremum version” of the following theorem which will be used in the context of norm attaining operators.

Before stating Theorem 3.3, we would like to mention the existence of a variational principle due to N. Ghoussoub and B. Maurey [GM86, Theorem II.11], which applies to subsets $C$ of a dual space $X^*$ which have the property of being strongly $w^*-H_\delta$ in their $w^*$-closed convex hull $D = \overline{\text{conv}}^{w^*}(C)$ and such that $D$ is $w^*$-metrizable or $C$ is norm-separable (we refer to [GM86] for definitions and more details). The result in [GM86, Theorem II.11] and our Theorem 3.3 are rather complementary. Anyway, Theorem 3.3 is not a consequence of the results obtained in [GM86], since on the one hand our result concerns $\sigma$-porosity instead of meager sets, and on the other hand, it applies to not necessarily separable sets $C$ for which $\overline{\text{conv}}^{w^*}(C)$ is not necessarily $w^*$-metrizable. For example, the set $C = \delta(L) := \{\delta_x : x \in L\}$ in the dual space $(C_b(L))^*$ (where $L$ is a complete metric space) has the $w^*\text{USP}$ (see Example C in Section 2) but in general it is not norm-separable, and $(\delta(L))^{w^*}$, which coincides, up to homeomorphism, with the Stone–Čech compactification $\beta L$ of $L$, is not metrizable (if $L$ is not compact).

**Theorem 3.3.** Let $X$ be a Banach space and $C$ be a norm bounded subset of $X^*$. Suppose that $(C, \gamma)$ is a complete pseudometric space having the $w^*\text{USP}$ in $X^*$. Let $f : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be any proper function bounded from below. Then there exists a $\sigma$-porous subset $F$ of $X$ such that for every $x \in X \setminus F$, $f + \hat{x}$ is $\gamma$-strongly minimized on $C$ at some $u \in C$.

**Proof.** For each $n \in \mathbb{N}^*$, let

$$O_n = \{x \in X : \exists p_n \in C, (f + \hat{x})(p_n) < \inf \{(f + \hat{x})(p) : p \in C, \gamma(p, p_n) \geq 1/n\}\}.$$

Let us prove that $O_n$ is the complement of a porous set in $X$. We prove
that for each \( n \in \mathbb{N}^* \), the requirement of Definition 3.2 is satisfied with an arbitrary \( r_n > 0 \) and

\[
\lambda_n = \min \left( \frac{1}{4}, \frac{1}{8D} \omega_C \left( \frac{1}{n} \right) \right),
\]

where \( D := \sup_{p \in C} \|p\| \) and \( \omega_C(\cdot) \) is the modulus of \( w^*\text{USP} \) of \( C \). Indeed, let \( y \in X \) and \( 0 < \varepsilon < r_n \). We want to find \( y_n \in X \) such that

\[
\hat{B}_X(y + y_n, \lambda_n \varepsilon) \subset \hat{B}_X(y, \varepsilon) \cap O_n.
\]

Let \( p_n \in C \) be such that

\[
(f + \hat{y})(p_n) \leq \inf_C (f + \hat{y}) + \lambda_n \varepsilon D.
\]

Since \((C, \gamma)\) has the \( w^*\text{USP} \) in \( X^* \), there exists \( x_n \in B_X \) such that

\[
\langle p_n, x_n \rangle - \omega_C(1/n) \geq \sup_{p \in C: \gamma(p, p_n) \geq 1/n} \langle p, x_n \rangle.
\]

Equivalently, multiplying by \(-\varepsilon/2\), we have

\[
\langle p_n, -\varepsilon/2 x_n \rangle \leq \inf_{p \in C: \gamma(p, p_n) \geq 1/n} \langle p, -\varepsilon/2 x_n \rangle - \varepsilon/2 \omega_C \left( \frac{1}{n} \right).
\]

Set \( y_n = -\varepsilon/2 x_n \). We prove that \( \hat{B}_X(y + y_n, \lambda_n \varepsilon) \subset \hat{B}_X(y, \varepsilon) \cap O_n \). Indeed, clearly \( \hat{B}_X(y + y_n, \lambda_n \varepsilon) \subset \hat{B}_X(y, \varepsilon) \) since \( \|y_n\| \leq \varepsilon/2 \) and \( \lambda_n \leq 1/4 \). To prove that \( \hat{B}_X(y + y_n, \lambda_n \varepsilon) \subset O_n \), let \( z \in X \) with \( \|z\| < \lambda_n \varepsilon \). From (3.3) and the definition of \( \lambda_n \), we get

\[
\langle p_n, z + y_n \rangle = \langle p_n, -\varepsilon/2 x_n \rangle + \langle p_n, z \rangle
\]

\[
< \inf_{p \in C: \gamma(p, p_n) \geq 1/n} \langle p, y_n \rangle - \varepsilon/2 \omega_C \left( \frac{1}{n} \right) + \lambda_n \varepsilon D
\]

\[
< \inf_{p \in C: \gamma(p, p_n) \geq 1/n} \langle p, y_n \rangle - \frac{\varepsilon}{2} \omega_C \left( \frac{1}{n} \right) + \frac{\varepsilon}{4} \omega_C \left( \frac{1}{n} \right)
\]

\[
= \inf_{p \in C: \gamma(p, p_n) \geq 1/n} \langle p, y_n \rangle - \frac{\varepsilon}{4} \omega_C \left( \frac{1}{n} \right)
\]

\[
\leq \inf_{p \in C: \gamma(p, p_n) \geq 1/n} \langle p, y_n \rangle - 2\lambda_n \varepsilon D
\]

\[
\leq \inf_{p \in C: \gamma(p, p_n) \geq 1/n} \langle p, z + y_n \rangle - \lambda_n \varepsilon D.
\]
Using (3.2) and (3.4), we get
\[
(f + \hat{y} + \hat{y}_n + \hat{z})(p_n) = (f + \hat{y})(p_n) + \langle p_n, z + y_n \rangle
\leq \inf_{C}(f + \hat{y}) + \lambda_n \varepsilon D + \langle p_n, z + y_n \rangle
\leq \inf_{C}(f + \hat{y}) + \inf_{p \in C: \gamma(p,p_n) \geq 1/n} \langle p, y_n + z \rangle
\leq \inf_{p \in C: \gamma(p,p_n) \geq 1/n} (f + \hat{y})(p) + \inf_{p \in C: \gamma(p,p_n) \geq 1/n} \langle p, y_n + z \rangle
\leq \inf_{p \in C: \gamma(p,p_n) \geq 1/n} (f + \hat{y} + \hat{y}_n + \hat{z})(p).
\]

This shows that \( y + y_n + z \in O_n \) for all \( \|z\| < \lambda_n \varepsilon \). Hence, \( \hat{B}_X(y + y_n, \lambda_n \varepsilon) \subset O_n \). Thus, we have proved that \( \hat{B}_X(y + y_n, \lambda_n \varepsilon) \subset \hat{B}_X(y, \varepsilon) \cap O_n \). Hence, \( O_n \) is the complement of a porous set in \( X \). Consequently, \( \bigcap_{n \in \mathbb{N}} O_n \) is the complement of a \( \sigma \)-porous set in \( X \).

To conclude the proof, we need to show that for every \( x \in \bigcap_{n \in \mathbb{N}} O_n \) (the \( \sigma \)-porous set is \( F = X \setminus \bigcap_{n \in \mathbb{N}} O_n \)), \( f + \hat{x} \) is \( \gamma \)-strongly minimized on \( C \) at some \( u \in C \). Indeed, let \( x \in \bigcap_{n \in \mathbb{N}} O_n \). Then for each \( n \geq 1 \), there exists \( p_n \in C \) such that
\[
(f + \hat{x})(p_n) < \inf_{q \in C: \gamma(q,p_n) \geq 1/n} (f + \hat{x})(q).
\]

First, we show that \( (p_n) \) is a Cauchy sequence in \((C, \gamma)\). Indeed, \( \gamma(p_k, p_n) < 1/n \) for each \( k > n \) (otherwise, by the definition of \( p_n \), we have \( (f + \hat{x})(p_n) < (f + \hat{x})(p_k) \) and since \( \gamma(p_k, p_n) \geq 1/n > 1/k \), by the definition of \( p_k \) we have \( (f + \hat{x})(p_k) < (f + \hat{x})(p_n) \), a contradiction). Thus, \( (p_n) \) converges to some \( u \in C \) (unique up to ~).

Now, we prove that \( f + \hat{x} \) is \( \gamma \)-strongly minimized on \( C \) at \( u \in C \). Indeed, let \( (q_k) \subset C \) with \( (f + \hat{x})(q_k) \to \inf_C(f + \hat{x}) \). Suppose for contradiction that \( (q_k) \) does not converge to \( u \) in the pseudometric \( \gamma \). Extracting a subsequence if necessary, we can assume that there exists \( \varepsilon > 0 \) such that \( \gamma(q_k, u) \geq \varepsilon \) for all \( k \in \mathbb{N}^* \). Thus, there exists an integer \( m \) such that \( \gamma(q_k, p_m) \geq 1/m \) for all \( k \in \mathbb{N}^* \). It follows that, for all \( k \in \mathbb{N}^* \),
\[
\inf_C(f + \hat{x}) \leq (f + \hat{x})(p_m) < \inf_{q \in C: \gamma(q,p_m) \geq 1/m} (f + \hat{x})(q) \leq (f + \hat{x})(q_k),
\]
which contradicts the fact that \( (f + \hat{x})(q_k) \) converges to \( \inf_C(f + \hat{x}) \).

Now, we investigate the case where the pseudometric \( \gamma \) is a metric. Typically, in the following corollary, the metric \( d \) can be the norm of the dual space \( X^* \) or a distance compatible with the weak-star topology if \( C \) is a weak-star metrizable subset of \( X^* \).

**Corollary 3.4.** Let \( X \) be a Banach space and \( C \) be a norm bounded subset of \( X^* \). Suppose that \((C, d)\) is a complete metric space such that the identity map \( I_C : (C, d) \to (C, \text{weak}^*) \) is continuous. Suppose that \((C, d)\) has the w*USP in \( X^* \). Let \( f : (C, d) \to \mathbb{R} \cup \{+\infty\} \) be proper, bounded from below.
and lower semicontinuous. Then there exists a \( \sigma \)-porous subset \( F \) of \( X \) such that for every \( x \in X \setminus F \), \( f + \hat{x} \) has a strong minimum on \( (C, d) \).

**Proof.** Since \( I_C \) is \( d \)-to-weak-star continuous, for every \( x \in X \), the map \( \hat{x} : x^* \mapsto (x^*, x) \) is continuous on \( C \) for the metric \( d \). Thus, \( f + \hat{x} \) is lower semicontinuous on \( (C, d) \) and so we can apply Theorem 3.3 with the complete metric space \( (C, \gamma) = (C, d) \), observing that in this case the notion of being \( \gamma \)-strongly minimized on \( C \) for \( f + \hat{x} \) coincides with the notion of strong minimum for the distance \( d \).

Recall hypothesis (H): for every \( \varepsilon > 0 \) there exists \( \omega_X(\varepsilon) > 0 \) such that, for every \( x \in L \), there exists a function \( b_{x, \varepsilon} \in B_X \) such that

\[
b_{x, \varepsilon}(x) - \omega_X(\varepsilon) \geq \sup_{y \in L : d(y, x) \geq \varepsilon} b_{x, \varepsilon}(y).
\]

As an immediate application, we obtain the following extension of the Deville–Revalski theorem \[\text{DR}00\]. Recall from Example C in Section 2 that \( (\text{DR}) \Rightarrow (H) \) but \( (H) \not\Rightarrow (\text{DR}) \) in general.

**Corollary 3.5.** Let \( (L, d) \) be a complete metric space and \( (X, \| \cdot \|_X) \) be a Banach space included in \( C_b(L) \) such that

(a) \( \| \cdot \|_X \geq \alpha \| \cdot \|_\infty \) on \( X \), for some \( \alpha > 0 \).

(b) \( X \) satisfies hypothesis (H).

Let \( f : L \to \mathbb{R} \cup \{ +\infty \} \) be proper, bounded from below and lower semicontinuous. Then there exists a \( \sigma \)-porous subset \( F \) of \( X \) such that for every \( h \in X \setminus F \), \( f + h \) has a strong minimum on \( L \).

**Proof.** We set \( C := \delta(L) := \{ \delta_x : x \in L \} \subset X^* \). Hypothesis (H) is equivalent to \( (C, d) \) having the \( \text{w}^* \text{USP} \) in \( X^* \), where \( d(\delta_x, \delta_y) := d(x, y) \) is a complete metric space. On the other hand, it is trivial that the identity map \( I_C : (C, d) \to (C, \text{weak}^*) \) is continuous (by the continuity of the elements of \( X \) on \( (L, d) \)). We apply Corollary 3.4 to \((C, d)\) and the proper, bounded from below, and lower semicontinuous function \( \tilde{f}_- : (C, d) \to \mathbb{R} \cup \{ +\infty \} \) defined by \( \tilde{f}(\delta_x) := f(x) \) for all \( x \in L \) (note that \( \tilde{f} \) is well defined since \( X \) separates the points of \( L \), which is a consequence of (H)).

**Example 3.6.** Let \( Z \) be a uniformly convex Banach space and \( S_Z \) its unit sphere. Denote by \( \mathcal{P}(Z) \) the space of all continuous positively homogeneous functions from \( Z \) into \( \mathbb{R} \), that is, \( f(\lambda x) = \lambda f(x) \) for all \( (\lambda, x) \in \mathbb{R}^+ \times Z \) and all \( f \in \mathcal{P}(Z) \). The space \( X := \mathcal{P}(Z) \subset C_b(S_Z) \) equipped with the norm \( \|f\| := \sup_{z \in S_Z} |f(z)|, f \in X \), is a Banach space satisfying (H) thanks to Proposition 2.5 and therefore also Corollary 3.5. Notice that hypothesis (DR) of Deville–Revalski is not satisfied for the space \( \mathcal{P}(Z) \).

In the following theorem, we give a localization of Theorem 3.3.
Theorem 3.7. Let $X$ be a Banach space and $C$ be a norm bounded subset of the dual $X^*$. Suppose that $(C, \gamma)$ is a complete pseudometric space having the $w^{*}\text{USP}$ in $X^*$. Let $f : C \to \mathbb{R} \cup \{+\infty\}$ be proper and bounded from below. Then there exists $a > 0$ such that for every $\varepsilon \in [0, a]$ and every $p^* \in C$ such that $f(p^*) < \inf_C f + \varepsilon \varpi_C(\varepsilon)$ (where $\varpi_C(\varepsilon)$ denotes the modulus of the $w^{*}\text{USP}$ of $C$, see Definition 2.2), there exist $x \in X$ and $u \in C$ such that

(i) $\gamma(p^*, u) \leq \varepsilon$,
(ii) $\|x\| < 2\varepsilon$,
(iii) $f + \hat{x}$ is $\gamma$-strongly minimized on $C$ at $u \in C$.

Proof. From the definition of the $w^{*}\text{USP}$ (see Definition 2.2), there exists $a > 0$ such that for every $\varepsilon > 0$, there exists $x_{\varepsilon} \in B_X$ such that

\[
\langle p^*, x_{\varepsilon} \rangle - \varpi_C(\varepsilon) \geq \sup \{ \langle q, x_{\varepsilon} \rangle : q \in C, \gamma(q, p^*) \geq \varepsilon \}.
\]

For every $\theta > 0$, set

\[
\lambda_{\varepsilon, \theta} := (1 + \theta)\frac{f(p^*) - \inf_C f + \theta \varpi_C(\varepsilon)}{\varpi_C(\varepsilon)}.
\]

Then clearly we have

\[
0 < \lambda_{\varepsilon, \theta} < \frac{(1 + \theta)(\varepsilon \varpi_C(\varepsilon) + \theta \varpi_C(\varepsilon))}{\varpi_C(\varepsilon)} = (1 + \theta)(\varepsilon + \theta).
\]

Now, we apply Theorem 3.3 to the function $h = f - \lambda_{\varepsilon, \theta} \hat{x}_{\varepsilon}$. Thus, there exist $y \in X$ and $u \in C$ such that $\|y\| < \frac{\theta \varpi_C(\varepsilon)}{2D}$ (where $D := \sup_{q \in C} \|q\|$) and $f - \lambda_{\varepsilon, \theta} \hat{x}_{\varepsilon} + \hat{y}$ is $\gamma$-strongly minimized on $C$ at $u \in C$. Choose a sequence $(p_n) \subset C$ such that

\[
\lim_{n \to \infty} (f - \lambda_{\varepsilon, \theta} \hat{x}_{\varepsilon} + \hat{y})(p_n) = \inf_C (f - \lambda_{\varepsilon, \theta} \hat{x}_{\varepsilon} + \hat{y}).
\]

Then

\[
\lim_{n \to \infty} \gamma(p_n, u) = 0.
\]

On the other hand,

\[
\inf_C f + \liminf_{n \to \infty} (-\lambda_{\varepsilon, \theta} \hat{x}_{\varepsilon} + \hat{y})(p_n) \leq \liminf_{n \to \infty} (f - \lambda_{\varepsilon, \theta} \hat{x}_{\varepsilon} + \hat{y})(p_n) = \inf_C (f - \lambda_{\varepsilon, \theta} \hat{x}_{\varepsilon} + \hat{y}) \leq (f - \lambda_{\varepsilon, \theta} \hat{x}_{\varepsilon} + \hat{y})(p^*).
\]

Using the above inequality and the fact that $\|y\| < \frac{\theta \varpi_C(\varepsilon)}{2D}$ (where $D := \sup_{q \in C} \|q\|$), we get

\[
\liminf_{n \to \infty} -\lambda_{\varepsilon, \theta} \hat{x}_{\varepsilon}(p_n) \leq f(p^*) - \inf_C f - \lambda_{\varepsilon, \theta} \hat{x}_{\varepsilon}(p^*) + \theta \varpi_C(\varepsilon).
\]
Equivalently,
\[
\liminf_{n \to \infty} (p^* - p_n, x_\varepsilon) \leq \frac{f(p^*) - \inf C \ f + \theta \, \varpi_C(\varepsilon)}{\lambda_{\varepsilon, \theta}} = \frac{1}{1 + \theta} \, \varpi_C(\varepsilon).
\]

CLAIM. \( \gamma(p^*, u) \leq \varepsilon. \)

Proof of Claim. Suppose that \( \gamma(p^*, u) > \varepsilon. \) Then, from \((\bullet)\), there exists an integer \( N \) such that for every \( n \geq N \), we have \( \gamma(p^*, p_n) > \varepsilon. \) Using \((\bullet)\) we see that \( \liminf_{n \to \infty} (p^* - p_n, x_\varepsilon) \geq \varpi_C(\varepsilon) \), a contradiction since \( \theta > 0. \)

Now, set \( x := y - \lambda_{\varepsilon, \theta} x_\varepsilon. \) Using the formula of \( \lambda_{\varepsilon, \theta} \) with \((3.5)\) we get (since \( x_\varepsilon \in B_X \))
\[
\|x\| \leq \|y\| + \lambda_{\varepsilon, \theta} \leq \frac{\theta \varpi_C(\varepsilon)}{2D} + \lambda_{\varepsilon, \theta} < \frac{\theta \varpi_C(\varepsilon)}{2D} + (1 + \theta)(\varepsilon + \theta).
\]
Choosing \( \theta > 0 \) so small that \( \|x\| < 2\varepsilon \) ends the proof of the theorem. \( \blacksquare \)

Similarly to Corollary \((3.4)\) using Theorem \((3.7)\) we obtain the following localization.

**Corollary 3.8.** Let \( X \) be a Banach space and \( C \) be a norm bounded subset of \( X^*. \) Let \( (C, d) \) be a complete metric space such that the identity \( I_C : (C, d) \to (C, \text{weak}^*) \) is continuous. Suppose that \( (C, d) \) has the \( w^*\text{USP}. \) Let \( f : (C, d) \to \mathbb{R} \cup \{+\infty\} \) be proper, bounded from below, and lower semi-continuous. Then there exists \( a > 0 \) such that for every \( \varepsilon \in [0, a] \) and every \( p^* \in C \) such that \( f(p^*) < \inf C \ f + \varepsilon \varpi_C(\varepsilon) \) (where \( \varpi_C(\varepsilon) \) denotes the modulus of the \( w^*\text{USP} \) of \( C \)), there exist \( x \in X \) and \( u \in C \) such that

(i) \( d(p^*, u) \leq \varepsilon, \)
(ii) \( \|x\| < 2\varepsilon, \)
(iii) \( f + \hat{x} \) attains its strong minimum on \( C \) at \( u. \)

\[ \text{4. Application to norm attaining operators.} \] Let \((K, d)\) be a complete metric space, \( Y \) be a Banach space and \( S_{Y^*} \) be the unit sphere of its dual. We denote by \( C_b(K, Y) \) the Banach space of all \( Y \)-valued bounded continuous functions equipped with the sup-norm. For every \((x, y^*) \in K \times S_{Y^*}\), we define the evaluation maps \( \delta_x : T \mapsto T(x) \) and \( y^* \circ \delta_x : T \mapsto \langle y^*, T(x) \rangle \), for all \( T \in C_b(K, Y) \). For any Banach space \((Z, \| \cdot \|_Z)\) included in \((C_b(K, Y))\) and such that \( \| \cdot \|_Z \geq \| \cdot \|_\infty \), we have \( y^* \circ \delta_x \in Z^* \) for each \((x, y^*) \in K \times S_{Y^*}\).

We suppose that the space \( Z \) satisfies the following condition:

\[(4.1) \quad y^* \circ \delta_{x_1} = y^*_2 \circ \delta_{x_2} \text{ on } Z \implies x_1 = x_2 \text{ and } y^*_1 = y^*_2. \]

Let \( C_K := \{y^* \circ \delta_x : x \in K, y^* \in S_{Y^*}\} \subset Z^*. \) We define a complete pseudometric on \( C_K \) as follows:

\[ \gamma_p(y^* \circ \delta_x, z^* \circ \delta_{x'}) := d(x, x'), \quad \forall y^* \circ \delta_x, z^* \circ \delta_{x'} \in C_K. \]

The map \( \gamma_p \) is well defined thanks to \((4.1)\).
Lemma 4.1. Let \((K, d)\) be a complete metric space, \(Y\) be a Banach space and \((Z, \| \cdot \|_Z)\) be a Banach space included in \(C_b(K, Y)\) and satisfying

(a) \(\| \cdot \|_Z \geq \| \cdot \|_\infty\),
(b) for every \(\varepsilon > 0\) there exists \(\varpi_K(\varepsilon) > 0\) and a collection \(\{b_{x,\varepsilon} : x \in K\} \subset C_b(K, \mathbb{R})\) such that, for every \(e \in S_Y\) and every \(x \in K\), we have \(b_{x,\varepsilon} e \in Z\), \(\|b_{x,\varepsilon} e\|_Z \leq 1\) and \(\varepsilon > 0\)

\[
\|b_{x,\varepsilon}(x) - \varpi_K(\varepsilon)\| \geq \sup_{x' \in K : d(x', x) \geq \varepsilon} \|b_{x,\varepsilon}(x')\|.
\]  

Then \((C_K, \gamma_P)\) is a complete pseudometric space having the \(w^*USP\) in \(Z^*\).

Proof. First, (a) implies \(y^* \circ \delta_x \in Z^*\) for each \((x, y^*) \in K \times S_Y^*\).

The map \(\gamma_P\) is well defined. Indeed, we will prove that \(y^* \circ \delta_x = z^* \circ \delta_x^*\) implies that \(x = x^*\) and \(y^* = z^*\). Let \(e \in S_Y\) be such that \(\langle y^*, e \rangle = \langle z^*, e \rangle\) \((e \in \text{Ker}(y^* - z^*))\). Since \(y^* \circ \delta_x (b_{x,\varepsilon} e) = z^* \circ \delta_x^* (b_{x,\varepsilon} e)\) for every \(\varepsilon > 0\), it follows that \(b_{x,\varepsilon}(x) = b_{x,\varepsilon}(x')\) for every \(\varepsilon > 0\), which implies that \(x = x'\) by using \((4.2)\). Now, we have \(y^* \circ \delta_x (b_{x,\varepsilon} e) = z^* \circ \delta_x^* (b_{x,\varepsilon} e)\) for every \(e \in S_Y\). This implies (since \(x = x'\)) that \(\langle y^*, e \rangle = \langle z^*, e \rangle\) for all \(e \in S_Y\) and so \(y^* = z^*\). Now, it is clear that \((C_K, \gamma_P)\) is a complete pseudometric space, since \((K, d)\) is a complete metric space.

It remains to prove that \((C_K, \gamma_P)\) has the \(w^*USP\) in \(Z^*\). Indeed, for every \(y^* \in S_Y^*\) and \(\varepsilon > 0\), choose \(e_{y^*,\varepsilon} \in S_Y\) such that \(\langle y^*, e_{y^*,\varepsilon} \rangle > 1 - \frac{\varpi_K(\varepsilon)}{2(1 + \varpi_K(\varepsilon))}\)

> 0, and for each \((x, y^*) \in K \times S_Y^*\), define the operator \(T_{(x, y^*, \varepsilon)} : K \to Y\) by \(T_{(x, y^*, \varepsilon)}(x') = b_{x,\varepsilon}(x') e_{y^*,\varepsilon}\) for all \(x' \in K\). By assumption, \(T_{(x, y^*, \varepsilon)} \in Z\) and \(\|T_{(x, y^*, \varepsilon)}\|_Z \leq 1\). On the other hand, for all \((x', z^*) \in K \times S_Y^*\) such that \(d(x, x') := \gamma_P(y^* \circ \delta_x, z^* \circ \delta_x^*) \geq \varepsilon\), we have

\[
\langle y^* \circ \delta_x, T_{(x, y^*, \varepsilon)}(x) \rangle - \frac{\varpi_K(\varepsilon)}{2} = \langle y^*, T_{(x, y^*, \varepsilon)}(x) \rangle - \frac{\varpi_K(\varepsilon)}{2}
\]

\[
\geq \langle y^*, T_{(x, y^*, \varepsilon)}(x) \rangle - \varpi_K(\varepsilon) \langle y^*, e_{y^*,\varepsilon} \rangle + \frac{\varpi_K(\varepsilon)}{2(1 + \varpi_K(\varepsilon))}
\]

\[
= [b_{x,\varepsilon}(x) \langle y^*, e_{y^*,\varepsilon} \rangle - \varpi_K(\varepsilon) \langle y^*, e_{y^*,\varepsilon} \rangle] + \frac{\varpi_K(\varepsilon)}{2(1 + \varpi_K(\varepsilon))}
\]

\[
\geq [b_{x,\varepsilon}(x') \langle y^*, e_{y^*,\varepsilon} \rangle] + \frac{\varpi_K(\varepsilon)}{2(1 + \varpi_K(\varepsilon))}
\]

\[
\geq [b_{x,\varepsilon}(x')] \left( 1 - \frac{\varpi_K(\varepsilon)}{2(1 + \varpi_K(\varepsilon))} \right) + \frac{\varpi_K(\varepsilon)}{2(1 + \varpi_K(\varepsilon))}
\]

\[
\geq [b_{x,\varepsilon}(x')] \quad \text{(since} \|b_{x,\varepsilon}\|_\infty \leq 1)\]

\[
\geq [b_{x,\varepsilon}(x')] \frac{\varpi_K(\varepsilon)}{2(1 + \varpi_K(\varepsilon))}
\]

\[
= \langle z^* \circ \delta_x^*, T_{(x, y^*, \varepsilon)}(x') \rangle \geq \langle z^* \circ \delta_x^*, T_{(x, y^*, \varepsilon)}(x') \rangle.
\]

It follows that \((C_K, \gamma_P)\) has the \(w^*USP\) in \(Z^*\).
Example 4.2. Let $X$ be a Banach space with property $(\alpha)$ and $Y$ be any Banach space. Let $\{x_\lambda : \lambda \in \Lambda\} \subset X$ and $\{x_\lambda^* : \lambda \in \Lambda\} \subset X^*$ be as in the definition of property $(\alpha)$ (see Example B in Section 2). Set $K := \{x_\lambda : \lambda \in \Lambda\}$ (a norm-closed set). It is easy to see, thanks to parts (1) and (2) of property $(\alpha)$, that for every $\varepsilon > 0$ there exists $\varpi_K(\varepsilon) > 0$ such that, for every $x \in K$, there exists $b_{x,\varepsilon} \in \{x_\lambda^* : \lambda \in \Lambda\} \subset S_{X^*}$ such that

$$\langle b_{x,\varepsilon}, x \rangle - \varpi_K(\varepsilon) \geq \sup_{x' \in K : \|x' - x\| \geq \varepsilon} |\langle b_{x,\varepsilon}, x' \rangle|.$$ 

Thus, every closed subspace $R(X, Y)$ of $B(X, Y) \subset (C_b(K, Y), \| \cdot \|_\infty)$, containing $F(X, Y)$, satisfies (a) and (b) of Lemma 4.1.

Example 4.3. We denote by $C_b^u(K, Y)$ the Banach space of all bounded uniformly continuous operators from a complete metric space $(K, d)$ to a Banach space $Y$ equipped with the sup-norm. It is easy to see that $(C_b^u(K, Y), \| \cdot \|_\infty)$ and $(C_b(K, Y), \| \cdot \|_\infty)$ satisfy (a) and (b) of Lemma 4.1 with $b_{x,\varepsilon} : z \mapsto \max(0, 1 - d(z, x)/\varepsilon)$.

Example 4.4. Let $X$ be a Banach space such that there exists a Lipschitz $C^1$-bump function from $X$ into $\mathbb{R}$, and $Y$ be a Banach space. We denote by $C_b^1(X, Y)$ the Banach space of all bounded continuously Fréchet differentiable functions from $X$ to $Y$ equipped with the norm

$$\|f\| := \max(\|f\|_\infty, \|f'\|_\infty).$$

The above lemma applies to $Z = C_b^1(X, Y)$.

Remark 4.5. In the nonlinear operators case, the hypothesis in (4.2) of Lemma 4.1 can be replaced by the following strong but fairly general and useful condition (the existence of “bump function” in $Z$): For every $\varepsilon > 0$ there exists a collection $\{b_{x,\varepsilon} : x \in K\} \subset C_b(K, Y)$ such that $b_{x,\varepsilon} \in Z$ and $\|b_{x,\varepsilon}\| \leq 1$ for every $e \in S_Y$, $x \in K$, $\varepsilon > 0$, and satisfying

$$b_{x,\varepsilon} \geq 0, \quad b_{x,\varepsilon}(x) = 1, \quad b_{x,\varepsilon}(y) = 0 \quad \text{whenever } d(y, x) \geq \varepsilon.$$ 

A general and abstract statement on operators (linear or not) attaining their sup-norm is given in the following result. Lemma 4.1 gives a general criterion for when the following theorem applies. Examples 4.2, 4.4 are particular cases.

Theorem 4.6. Let $(K, d)$ be a complete metric space and $Y$ be a Banach space. Let $(Z, \| \cdot \|_Z)$ be a Banach space included in $C_b(K, Y)$ such that $\| \cdot \|_Z \geq \| \cdot \|_\infty$. Suppose that $(C_K, \gamma_P)$ is a complete pseudometric space having the $w^*\text{USP}$ in $Z^*$. Then, for every $h \in C_b(K, Y)$, the set

$$\mathcal{N}(h) := \{g \in Z : h + g \text{ does not strongly attain its sup-norm}\}$$

is a $\sigma$-porous subset of $(Z, \| \cdot \|_Z)$. Moreover, the following “quantitative version” of the Bishop–Phelps–Bollobás theorem holds: for every $\varepsilon > 0$, there
exists $\lambda(\varepsilon) > 0$ such that for every $f \in Z$, $\|f\|_\infty = 1$ and every $x \in K$ satisfying $\|f(x)\| > 1 - \lambda(\varepsilon)$, there exists $k \in Z$, $\|k\|_\infty = 1$ and $\bar{x} \in K$ such that

(i) $x \mapsto \|k(x)\|$ strongly attains its maximum on $K$ at $\bar{x}$,

(ii) $d(\bar{x}, x) < \varepsilon$ and $\|f - k\|_\infty < \varepsilon$.

**Proof.** Since $(C_K, \gamma_P)$ is a complete pseudometric space having the $w^*\text{USP}$ in $Z^*$, by applying Theorem 3.3 (with “minimized” replaced by “maximized”) to $(C, \gamma_P)$ with the function

$$\hat{h} : C_K \to \mathbb{R}, \quad y^* \circ \delta_x \mapsto \langle y^*, h(x) \rangle,$$

we get a $\sigma$-porous subset $\mathcal{N}(h)$ of $Z$ such that for every $f \in Z \setminus \mathcal{N}(h)$, $\hat{h} + \hat{f}$ is $\gamma_P$-strongly maximized on $C_K$ at some $y^*_n \circ \delta_{x_f} \in C_K$. This implies that, for every $f \in Z \setminus \mathcal{N}(h)$, the function $\|(h + f)(\cdot)\|$ strongly attains its maximum on $K$ at $x_f \in K$. Indeed, let $(u_n) \subset K$ be such that $\|(h + f)(u_n)\| \to \|h + f\|_\infty$. By the Hahn–Banach theorem, there exists $(y^*_n) \subset S_{Y^*}$ such that $\|(h + f)(u_n)\| = \langle y^*_n, h + f \rangle$. Thus $\langle y^*_n \circ \delta_{u_n}, h + f \rangle \to \|h + f\|_\infty = \sup_{x \in K} \sup_{y^* \in S_{Y^*}} \langle y^* \circ \delta_x, h + f \rangle$, which implies that $d(u_n, x_f) := \gamma_P(y^*_n \circ \delta_{u_n}, y^*_f \circ \delta_{x_f}) \to 0$, since $\hat{h} + \hat{f}$ is $\gamma_P$-strongly maximized on $C_K$ at $y^*_f \circ \delta_{x_f}$. By the continuity of $\|(h + f)(\cdot)\|$, we have $\|h + f\|_\infty = \|(h + f)(x_f)\|$. Hence, $\|(h + f)(\cdot)\|$ strongly attains its maximum on $K$ at $x_f$.

The second part of the theorem follows from Theorem 3.7. Indeed, let $\varepsilon > 0$, $\lambda(\varepsilon) = \frac{\varepsilon}{4} \overline{w}_C K(\varepsilon/4) > 0$ (where $\overline{w}_C K$ is the modulus of uniform $w^*\text{USP}$ of $(C_K, \gamma_P)$ in $Z^*$). Let, $f \in Z$, $\|f\|_\infty = 1$ and $x \in K$ be such that

$$\|f(x)\| > 1 - \frac{\varepsilon}{4} \overline{w}_C K(\varepsilon/4) = \|f\|_\infty - \frac{\varepsilon}{4} \overline{w}_C K(\varepsilon/4).$$

We have $1 = \|f\|_\infty = \sup_{y^* \circ \delta_z \in C_K} \langle y^* \circ \delta_z, f \rangle$. Moreover, by the Hahn–Banach theorem there exists $y^*_x \in S_{Y^*}$ such that

$$\langle y^*_x \circ \delta_x, f \rangle := \langle y^*_x, f(x) \rangle = \|f(x)\|.$$

Thus, the above inequality can be written as follows:

$$\langle y^*_x \circ \delta_x, f \rangle > \sup_{y^* \circ \delta_z \in C_K} \langle y^* \circ \delta_z, f \rangle - \frac{\varepsilon}{4} \overline{w}_C K(\varepsilon/4).$$

We apply Theorem 3.7 (replacing “minimized” by “maximized”) with the function $\hat{f}$ and the set $C_K$ to obtain some $g \in Z$ and a point $\bar{y}^* \circ \delta_{\bar{x}} \in C_K$ such that

(a) $\gamma_P(\bar{y}^* \circ \delta_{\bar{x}}, y^*_x \circ \delta_x) := d(\bar{x}, x) < \varepsilon/4$,

(b) $\|g\|_\infty \leq \|g\|_Z < \varepsilon/2$,

(c) $\hat{f} - \hat{g}$ is $\gamma_P$-strongly maximized on $C_K$ at $\bar{y}^* \circ \delta_{\bar{x}}$.

This implies, as we have shown above, that $\|(f - g)(\cdot)\|$ strongly attains its maximum at $\bar{x}$. Equivalently, the function $k := \frac{f - g}{\|f - g\|_\infty}$ is such that $\|k(\cdot)\|
strongly attains its maximum on $K$ at $\bar{x}$, $\|k\|_\infty = 1$ and we have (using the triangular inequality)

$$\|f - k\|_\infty = \left\| f - \frac{f - g}{\|f - g\|_\infty} \right\|_\infty = \left\| g + \left( f - g - \frac{f - g}{\|f - g\|_\infty} \right) \right\|_\infty \leq 2\|g\|_\infty < \varepsilon.$$ 

This concludes the proof. ■

As a direct consequence of Theorem 4.6 and Lemma 4.1 we obtain the following result on norm attaining linear operators, which generalizes some old results, passing from density to being the complement of a $\sigma$-porous set and from norm attained to strongly norm attained.

**Corollary 4.7.** Let $X$ be a Banach space having property $(\alpha)$. Then, for every Banach space $Y$, every $S \in B(X, Y)$ and every closed subspace $R(X,Y)$ of $B(X,Y)$ containing $F(X,Y)$, the set

$$\mathcal{N}(S) := \{ T \in R(X,Y) : S + T \text{ does not strongly attain its norm} \}$$

is a $\sigma$-porous subset of $R(X,Y)$. In particular (with $S = 0$), NAR$(X,Y)$ is the complement of a $\sigma$-porous subset of $R(X,Y)$.

**Proof.** Let $\{x_\lambda : \lambda \in \Lambda\} \subset X$ and $\{x_\lambda^* : \lambda \in \Lambda\} \subset X^*$ have property $(\alpha)$ (see the definition in Example B in Section 2). Let $K := \{x_\lambda : \lambda \in \Lambda\}$ (a norm-closed set). It is easy to see, thanks to property $(\alpha)$, that for every $\varepsilon > 0$ there exists $\varpi_K(\varepsilon) > 0$ such that, for every $x \in K$, there exists $b_{x,\varepsilon} \in \{x_\lambda^* : \lambda \in \Lambda\} \subset S_{X^*}$ such that

$$\langle b_{x,\varepsilon}, x \rangle - \varpi_K(\varepsilon) \geq \sup_{x' \in K : \|x' - x\| \geq \varepsilon} |\langle b_{x,\varepsilon}, x' \rangle|.$$ 

On the other hand, since the absolute convex hull of $\{x_\lambda : \lambda \in \Lambda\}$ is dense in the unit ball of $X$, for every $T \in B(X, Y)$ we have $\|T\| = \sup_{x \in K} \|T(x)\|$. Considering $Z := R(X,Y)$ as a closed subspace of $(C_b(K,Y), \| \cdot \|_\infty)$, it is clear that $R(X,Y)$ satisfies parts (a) and (b) of Lemma 4.1. Thus, the conclusion follows from Theorem 4.6. ■

**Remark 4.8.** Under the hypothesis of Corollary 4.7 (and thanks to Theorem 4.6), we also have the following version of the Bishop–Phelps–Bollobás theorem for the space $R(X,Y)$: for every $\varepsilon > 0$, there exists $\lambda(\varepsilon) > 0$ such that for every $T \in R(X,Y)$ with $\|T\| = 1$ and every $x \in K := \{x_\lambda : \lambda \in \Lambda\}$ (as in property $(\alpha)$) satisfying $\|T(x)\| > 1 - \lambda(\varepsilon)$, there exist $S \in R(X,Y)$ with $\|S\| = 1$ and $\varpi \in K$ such that

(i) $S$ strongly attains its norm at $\varpi$,
(ii) $\|\varpi - x\| < \varepsilon$ and $\|T - S\| < \varepsilon$.

**Corollary 4.9.** Let $X$ be a Banach space having property $(\alpha)$. Let $(T_n) \subset B(X,Y)$ be a sequence of bounded linear operators. Then, for every $\varepsilon > 0$, there exists a compact operator $T$ which is the norm-limit of a
sequence of finite-rank operators such that \( \|T\| < \varepsilon \) and \( T_n + T \) strongly attains its norm for every \( n \in \mathbb{N} \).

Proof. We apply Corollary 4.7 with \( R(X, Y) = F(X, Y) \) and \( S = T_n \) for each \( n \in \mathbb{N} \) to get \( \sigma \)-porous sets \( \mathcal{N}(T_n) \) such that for \( T \in F(X, Y) \setminus \mathcal{N}(T_n) \), \( T_n + T \) strongly attains its norm. The set \( \bigcup_n \mathcal{N}(T_n) \) is also a \( \sigma \)-porous set. Thus, in particular, \( F(X, Y) \setminus \bigcup_n \mathcal{N}(T_n) \) is dense in \( F(X, Y) \). Hence, for every \( \varepsilon > 0 \), there exists \( T \in F(X, Y) \setminus \bigcup_n \mathcal{N}(T_n) \) such that \( \|T\| < \varepsilon \) and \( T_n + T \) strongly attains its norm for all \( n \in \mathbb{N} \).

Since \( (C^u_b(K, Y), \| \cdot \|_\infty) \) satisfies (a) and (b) of Lemma 4.1 with \( b_{x,\varepsilon} : z \mapsto \max(0, 1 - d(z, x)/\varepsilon) \), using Theorem 4.6 we immediately obtain the following result.

**Corollary 4.10.** Let \((K, d)\) be a complete metric space and \( Y \) be a Banach space. Then the subset of \( C_b(K, Y) \) (resp. of \( C^u_b(K, Y) \)) of all bounded continuous (resp. uniformly continuous) operators strongly attaining their sup-norm, is a complement of a \( \sigma \)-porous subset of the Banach space \( (C_b(K, Y), \| \cdot \|_\infty) \) (resp. of \( (C^u_b(K, Y), \| \cdot \|_\infty) \)).

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