Effects of tunneling and multiphoton transitions on squeezed states generation in bistable driven systems

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Bistability of nonlinear resonantly driven oscillator in the presence of external noise is analyzed by means of classical Fokker–Planck equation in quasienergy space with account for tunneling effects and by quantum master equation in quasienergy states representation. Two time scales responsible for different stages of bistable system relaxation have been obtained. We found out that the slow relaxation rate caused by fluctuation–induced transitions between different stable states can be enhanced by several orders due to tunneling effects. It was also revealed that tunneling between nearly degenerate quasienergy states and resonant multiphoton transitions between the genuine eigenstates of the nonlinear oscillator are just the similar effects. It was demonstrated that the quasienergy states in the bistability region corresponding to higher amplitude are squeezed. The degree of squeezing is determined by the ratio between nonlinearity and detuning, so the uncertainty of one quadrature can be considerably smaller than the quantum limit. It was found out that tunneling effects can enhance the generation of output oscillator squeezed states. It is shown that 1D Fokker–Planck equation is a quasiclassical limit of a quantum master equation.

I. INTRODUCTION

Complex systems with two or more stable states appear in many fields of science from biology and chemistry to quantum optics and electronics [1] [2] [3] [4]. To understand the dynamical and statistical properties of such complex systems one should analyse the following aspects: what stable states the system has, how to control and manipulate them, what perturbations cause transitions between them, and how robust are these states to environment fluctuations.

First of all, it is necessary to understand the behaviour of bistable systems interacting with environment. Bistable systems in optics and electronics are widely used as switching elements in optics communications systems, basic elements of memory devices, logic gates, optical turnstiles, etc. So, the investigation of fluctuation–induced transitions between different stable states is crucial for stability enhancement of optical and electronic devices and for controlling their switching rates.

Due to advance in modern technology, the optics and electronics devices become so small that the quantum effects appear to be of great importance [5] [6] and it is impossible to study bistability without accounting for quantum effects. Thus, it is necessary to trace the correspondence between the classical and quantum regimes [7]. Moreover, in quantum case it is important to understand whether quantum fluctuations provide fundamental limit to stability of optical and electronic devices.

Now it is quite clear that in nonlinear optical and electronic interface systems there exist a set of quantum states — squeezed states — which have less uncertainty in one quadrature than a coherent state [8]. Generation of squeezed states is one of the crucial points in implementation of quantum information and stability enhancement of quantum optics devices [9]. Moreover, the generation of squeezed states is a key ingredient for implementation of quantum information protocols. Bistable quantum optics systems are promising candidates to realize the squeezed states. Recently the squeezed exciton–polariton field has been observed in pillar–shaped semiconductor microcavities in bistable regime near the critical point of bistable curve [10].

Nonlinear driven oscillator interacting with the bath is the minimal model describing fluctuation–induced transitions in bistable system out of equilibrium. Dynamics of various microcavities with nonlinear media coherently driven by external field including exciton–polaritons in semiconductor microresonators with external pumping can show bistable behaviour and can be well–described by the model of driven nonlinear oscillator. It was observed in recent experiments that with increasing external coherent pumping the occupied exciton–polariton mode show strong sudden jumps from one state to another. Such behaviour is caused by fluctuation–induced transitions between the stationary states. These transitions could also lead to decreasing of hysteresis area under S–shaped response curve of internal microcavity field to the external pumping [11].

Another experimental realization which can be analyzed by means of nonlinear oscillator model is a mesoscopic Josephson junction array resonator [12]. In such device, the anharmonicity can be of the same order as the linewidth, and the dynamics of bistability were measured experimentally by observing the jumps between different stable states. It was shown experimentally that the switching rate strongly depends on pumping intensity.

In addition, it is necessary to mention that the model of the driven nonlinear oscillator is applicable to investigation of highly excited molecular vibration modes with
account on anharmonicity.

The model of nonlinear driven oscillator was extensively studied theoretically since 1980th. But the fluctuation–induced transitions between two stable states were analyzed by means of classical 1D Fokker–Planck equation (FPE) in quasienergy space without accounting for quantum tunneling \[13, 14, 15\].

The ultra–quantum limit of dispersive bistability was analyzed by Drummond, Walls \[16\], where the kinetic equation for generalized Glauber function was solved analytically for the case of zero bath temperature. The same model was analysed numerically by means of quantum master equation \[17\].

Nevertheless, there is no detailed analysis of kinetics of the nonlinear driven oscillator which allows to trace the transition between the classical and quantum descriptions. So, it is necessary to analyze the interplay between of classical and quantum fluctuations. Moreover, the structure of the quasienergy states of quantum nonlinear driven oscillator and the influence of their degeneracy \[18\] on kinetics is weakly studied nowadays. Thus, for understanding the underlying rich physical properties of bistable systems it is highly important to examine the minimal model of driven nonlinear oscillator.

In present work, we derive quasiclassical kinetic equations with account for tunneling effects. These equations can be obtained as a quasiclassical limit of quantum master equation for density matrix of the quantum driven nonlinear oscillator. We show that in the quasiclassical limit tunneling transitions lead to decreasing of the stable states. We also show that tunneling between trajectories in different regions of the phase space and multi–photon resonant transitions between the states of the nonlinear oscillator are the same effects. In quantum case we explore the structure of eigenstates and show that the quasienergy states corresponding to the higher amplitude stable state are squeezed, and the uncertainty in one of the quadratures can be much lower than the usual quantum limit.

### II. CLASSICAL BISTABILITY

#### A. The basic model

Let us consider a model system consisting of a single oscillator mode with Kerr–like nonlinearity excited by a resonant field. Its key feature is bistability in certain range of external pumping intensity: the presence of two different classical stable states. The effective Hamiltonian for such a model is \[14, 19, 20\]:

\[
H_{\text{eff}} = -\Delta|a|^2 + \frac{\alpha}{2}|a|^4 - f(a + a^*),
\]

where \(a, a^*\) are slowly varying amplitudes of the internal oscillator field; \(\Delta \equiv \Omega - \omega_0\) is the detuning between the external field frequency \(\Omega\) and the frequency of the resonance \(\omega_0\); \(\alpha\) is the anharmonicity parameter; \(f\) is the interaction strength with external field (proportional to its amplitude). Such a model can arise for various systems in rotating–wave approximation such as microcavity with a nonlinear medium coherently driven by an external field. For example, this effective Hamiltonian can be derived for the Janes–Cummings model after excluding adiabatically the atomic variables. Also it describes the microcavity exciton–polaritons driven by an external field. This Hamiltonian also appears in description of strongly excited vibration modes of molecules in the presence of external resonant field.

For later discussion, we will use the normalized field amplitude, \(A \equiv \sqrt{\alpha/\Delta} a\), and dimensionless time, \(\tau = \Delta t\). The only dimensionless parameter which governs the system dynamics is \(\beta \equiv f^2/\Delta^3\). In new variables, the dimensionless Hamiltonian \(\mathcal{H} = \frac{\hbar}{\Delta} \mathcal{H}\):

\[
\mathcal{H} = -|A|^2 + \frac{1}{2}|A|^4 + \sqrt{\beta}(A + A^*)
\]

The equation of motion reads

\[
i\frac{\partial A}{\partial \tau} = -A + A|A|^2 + \sqrt{\beta}
\]

The classical phase trajectories of the nonlinear oscillator in the plane of \((A, A^*)\) are the contour lines of the classical Hamiltonian function \((1)\) (Fig. 1(a)). Let us focus on the structure of \((1)\) as the function of two variables, \(\text{Re} A\) and \(\text{Im} A\).

At \(\beta = 0\), the function has a shape of Mexican hat potential. It is radially symmetric, and its contour lines are concentric circles. At nonzero \(\beta\), \(0 < \beta < \beta_{\text{crit}} = 4/27\), the hat is deformed, as on Fig. 1(b). Instead of infinitely many local minima, two extrema rise: a true local minimum and a saddle point.

The stationary values of \(a\) are given by the stationary solutions of the equation \((3)\), which defines the S–shaped response curve (Fig. 2) of the internal field amplitude to external field.

In the bistability region \(0 < \beta < \beta_{\text{crit}}\), there are two stable stationary states 1,2 and one unstable, S, which lies on a self–intersecting trajectory called separatrix. It divides the phase plane into three regions: the two inner regions 1,2 with the corresponding stable states inside and the outer region 3. The stable state in the region 1 has lower field amplitude, and the stable state in the region 2 has higher field amplitude.

#### B. Fokker–Planck equation in the presence of white noise

In any realistic system, noise and damping due to interaction with the environment are always present. They result in appearance of damping term with dimensionless
The effect of damping is that the field amplitude relaxes to one of the stable stationary states. Noise has the opposite effect. First, it results in small random deviations from the stationary states. Second, it can induce transitions between the stationary states. At weak noise intensity, these transitions are exponentially rare.

In the case of white noise \[ \langle \xi(\tau)\xi^*(\tau') \rangle = Q\delta(\tau - \tau'), \] the S–shaped response curve of the normalized amplitude to the external field. At first, relaxation to the quasi–stationary distribution occurs in each region of phase space at time scales determined by inverse damping constant. Then, with much slower rate, the probability distribution evolves to true stationary distribution due to noise–induced transitions between the stable states.

As the transitions between the stationary states are very rare, the relaxation consists of two stages. At first, relaxation to the quasi–stationary distribution occurs in each region of phase space determined by inverse damping constant. Then, with much slower rate, the probability distribution evolves to true stationary distribution due to noise–induced transitions between the stable states.

At small damping \( \vartheta \ll 1 \) and weak noise \( \vartheta Q \gg 1 \), a significant simplification of the 2D FPE is possible. Weak damping and noise give only a small correction to the motion along the phase trajectories. So, it is natural to average the distribution function in each region of the phase space along the trajectory and define approximate \( P_i(t,H(A,A^*)) \), \( i = 1,2,3 \).

Different trajectories with the same quasienergies can exist in regions 1,3 (Fig. 1) By averaging the full FPE, one obtains the 1D FPE in quasienergy space in region \( i \),

\[
\frac{\partial P_i}{\partial t} = \frac{1}{T_i(E)} \frac{\partial J_i}{\partial E}
\]

The expressions for \( K_i(E) \), \( D_i(E) \), and \( T_i(E) \) are obtained in \[ 19,14,15,20 \] and are given in Appendix. \( T_i(E) \) is a period of motion along the trajectory with quasienergy \( E \) in region \( i \), and \( K_i(E) \) and \( D_i(E) \) are drift and diffusion coefficients in quasienergy space in region \( i \).

This Fokker–Planck equation should be solved in every region of the phase space. The full solution should be obtained by applying the boundary conditions near the separatrix, which include continuity of the probability distribution and the flow conservation:

\[
P_1(E_{sep}) = P_2(E_{sep}) = P_3(E_{sep}) \]

\[
J_1(E_{sep}) = J_2(E_{sep}) = J_3(E_{sep})
\]

The stationary distribution can be obtained by setting the flow \( J_i(E) \) to zero, if the tunneling effects are neglected (the discussion of the tunneling effects is below).

### C. Relative occupation of two stable states

The general formula for the stationary distribution function follows immediately from (8):

\[
P_i^{st}(E) = C e^{S_i - S_i(E)}
\]
\[
S_i(E) = \frac{\partial}{\partial E} \int_{E_i}^{E} K_i(E) \, dE, \quad S_i = S_i(E_{\text{sep}})
\] (9)

The distribution has maxima in the vicinity of states 1, 2, i.e., at the corresponding energies \(E_1, E_2\). Outside the neighborhood of \(E_1, E_2\), \(P^\text{st}_i(E)\) is exponentially small. Depending on whether \(S_1 > S_2\) or \(S_1 < S_2\) the probability density is mostly concentrated around the state 1 or state 2.

Numerical evaluation of \(S_1, S_2\) shows that \(S_1 = S_2\) at \(\sqrt{\beta_0/\beta_{\text{crit}}^2} = 0.29\). Therefore, \(\beta_0\) corresponds to the threshold pumping intensity: at \(\beta < \beta_0\) the oscillator mostly remains in a state 1 with a small amplitude, and at \(\beta > \beta_0\) it mostly remains in a state 2 with large amplitude. Thus, the choice of the most probable state is defined by a single parameter \(\beta\), and the switching from one most probable state to another occurs at universal threshold value \(\beta = \beta_0\). The width of the threshold region is determined by the noise characteristics. When \(|\beta - \beta_0| \sim \frac{\sigma}{\theta}\), both states are of comparable probability.

Every solution of the FPE can be expressed as a sum over eigenfunctions:
\[
P_i(t, E) = \sum_\lambda P_\lambda^i(E) e^{-\lambda t},
\] (10)
where \(\lambda, P^\lambda\) are the solutions of the eigenvalue problem
\[
-\lambda P_\lambda^i(E) = \frac{1}{T_i(E)} \frac{\partial}{\partial E} \left[ \partial K_i(E) + QD_i(E) \frac{\partial}{\partial E} \right] P^i_\lambda(E)
\] (11)
The eigenvalues of the FPE provide a very important information about kinetics of the system. As it can be seen from the Fig. 4, the lowest nonzero eigenvalue in the bistability region is several orders below all other ones. It determines the last stage of the relaxation process which was described above.

At small \(Q/\theta\), the lowest eigenvalue \(\lambda\) is exponentially small. Thus, it is possible to use the perturbation theory for \(P^i_\lambda\). In each region of the phase space the distribution function up to the first order in \(\lambda\) reads
\[
P^\lambda_i(E) = P^\text{st}_i(E) \left[ 1 + \frac{1}{Q} \int_{E_{\text{sep}}}^{E} \Phi_i(E') dE' \right]
\] (12)
\[
\Phi_i(E) = -\lambda \int_{E_{\text{sep}}}^{E} dE' T_i(E') P^\text{st}(E')
\]
where \(P^\text{st}(E)\) is the stationary distribution [8]. Using continuity of the probability distribution and the flow conservation, the following expression for lowest eigenvalue \(\lambda\) is obtained:
\[
\lambda = \frac{\theta^2}{Q} \cdot \frac{K_1(E_{\text{sep}})K_2(E_{\text{sep}})}{K_2(E_{\text{sep}}) - K_1(E_{\text{sep}})} \left[ \frac{e^{-S_2}}{D_{\lambda_2}^2(E_2)} - \frac{e^{-S_1}}{D_{\lambda_1}^2(E_1)} \right]
\] (13)
It is clear that the analytical expression [13] fits well the numerical results everywhere in the bistability region except the vicinity of its edges (Fig. 4).

The lowest eigenvalue depends nonmonotonically on the value of \(\beta\) and achieves its minimum at \(\beta = \beta_0\). At \(\beta < \beta_0\) (\(\beta > \beta_0\)), it corresponds to the escape rate \(\lambda_{\lambda_1}\) (\(\lambda_{\lambda_2}\)) from the higher (lower) amplitude state to the lower (higher) one, which drops (rises) with the growing external field intensity. At the threshold intensity \(\beta_0\), \(\lambda_{\lambda_1}\) and \(\lambda_{\lambda_2}\) have the same values.

D. Transition rates between different stable states

It is clear that the relaxation of the nonlinear driven oscillator consists of two stages. The first stage is the fast relaxation to the quasi-stationary distribution which occurs independently in the regions 1 and 2. After that, the slow relaxation to the real stationary state occurs which is governed by rare fluctuation-induced transitions between the stable states.

E. A tunneling term in the Fokker–Planck equation

The trajectories in the region 1 and in the region 3 can have the same energy. Thus, there is a possibility of quantum tunneling between them. On a quasiclassical language, it can be described as the tunneling term in the FPE:
\[
\frac{\partial P_{(1,3)}}{\partial t} = \frac{1}{T(E)} \frac{\partial J_{(1,3)}}{\partial E} + \lambda T(E)(P_{(1,3)} - P_{(1,3)})
\] (14)
From this equation, we can determine the most probable quasienergy state which is optimal for tunneling.

The turning points $q_1$, $q_2$, $q_e$ are defined by conditions $p_{1,3}^2(q_1) = 0$, $p_{3}^2(q_2) = 0$, and $p_{3}^2(q_e) = p_{3}^2(q_e)$. The turning points are defined by conditions $E_{sep} = 2\Delta$. At small $q_0 < \Delta$, the «total action» $S_{total}$ has a minimum near $E \sim E_1$. Therefore, tunneling transitions occur directly between the lower–amplitude stable state and the corresponding state from the region 3. On the contrary, at $\frac{q_0}{Q} \ll \frac{\Delta}{h\alpha}$ the «total action» $S_{total}$ has a minimum near $E \sim E_{sep}$. So, tunneling occurs between the states with energy close to $E_{sep}$, and noise–induced transitions dominate.

We concentrate on the case $\frac{q_0}{Q} \gg \frac{\Delta}{h\alpha}$. In this case, the leading term in the tunneling action at $E = E_1$ is

$$S_{tunn} = \sqrt{1 + 2E} \ln \frac{1}{\beta} + O(1)$$

Now from expression (17) we can estimate the quasienergy state which is optimal for tunneling.

The pre–exponential factor has the order of $\Delta$. It can be evaluated by matching the quasiclassical solutions near the turning points.

Tunneling between the quasiclassical trajectories effectively occurs when the energies obtained from the Bohr–Sommerfeld quantization rule become almost equal. In this case, tunneling leads to exponentially small splitting between them. As it will be shown below, this occurs when $2\Delta/h\alpha$ is exactly integer even at finite $\beta$. In this case, the tunneling rate between the classical trajectories in the regions 1 and 3 with closest energies is estimated as

$$\lambda_T \propto \frac{\Delta^2}{\gamma} \beta^{\frac{2\Delta}{h\alpha}}$$

When $2\Delta/h\alpha$ is not integer and $\gamma \ll \Delta$, $\lambda_T \propto \gamma \beta^{\frac{2\Delta}{h\alpha}}$.

In both cases, the tunneling rate is proportional to $\beta^{\frac{2\Delta}{h\alpha}}$. At integer $\frac{2\Delta}{h\alpha} = m$, the tunneling rate can be treated as the probability of $m$–photon resonant transition between real energy states of the nonlinear oscillator. So, tunneling processes in the presence of resonant external field and multi–photon transitions between the
energy states of a nonlinear oscillator are the similar effects [22].

The same expression for the tunneling amplitude in the lowest nonvanishing order can be also obtained in the framework of quantum–mechanical perturbation theory for multi–photon transitions [23]:

\[ A_{k,m-k} = \frac{\Delta}{m} \left( \frac{\beta m^3}{2} \right)^{\frac{k}{2}} \frac{\sqrt{(m-k)!}}{(m-2k-1)!} \sqrt{k!} \]  

(26)

For \( k = 0 \)

\[ |A_{0,m}|^2 \propto \Delta \beta^m \]  

(27)

The state with \( k = 0 \) corresponds to the point 1 on the phase portrait. So, for the driven bistable system the probability of \( m \)–photon transition calculated quantum–mechanically [26] is the same as tunneling probability between degenerate quasienergy states in the quasiclassical treatment.

The same nature of tunneling effects and multi–photon ionization of atoms in strong electromagnetic field was first demonstrated by L. V. Keldysh [22].

The presence of tunneling modifies both the distribution function and the relaxation rate. If \( \lambda_T \) is small, its effect can be taken into account within perturbation theory. The ratio of probability densities the states 1, 2 modifies as follows:

\[ \frac{P_2^{\text{st}}(E_2)}{P_1^{\text{st}}(E_1)} = e^{S_1} \left( e^{-S_1} + \frac{\lambda_T(E_1)}{\hat{Q}} \frac{D_1'(E_1)}{K_1(E_{\text{sep}})} \right) \]  

(28)

![Figure 5.](image)

Figure 5. The relaxation rate in units of \( \vartheta \) without and with tunneling term. Here \( Q = 0.015 \), \( \frac{\hbar \alpha}{\Delta} = 0.2 \).

According to this formula, tunneling leads to decreasing probability to be in state 1. Tunneling also changes the total transition rate between the stable states:

\[ \lambda_{\text{total}} = \lambda_{\text{noise}} + \lambda_T(E_1) \]  

(29)

Here \( \lambda_{\text{noise}} \) is defined by [13].

The behavior of the transition rate between the stable states in the presence of tunneling is depicted on the Fig. [5]. Tunneling transitions shift the threshold value of the external field intensity to the lower values and increase the threshold values of transition rate.

### III. BISTABILITY IN QUANTUM OSCILLATOR

#### A. Quantum quasienergy states and squeezing

The Hamiltonian for quantum bistable oscillator in the rotating–wave approximation is as follows:

\[ \hat{H}_0 = -\Delta \hat{a}^{\dagger} \hat{a} + \frac{\alpha}{2} \hat{a}^{\dagger} \hat{a}^{\dagger} \hat{a} + f(\hat{a} + \hat{a}^{\dagger}) \]  

(30)

The operators \( \hat{a}, \hat{a}^{\dagger} \) are the creation and annihilation operators of the internal oscillator field. In the quasiclassical limit, \( \sqrt{\hbar} \hat{a} \) and \( \sqrt{\hbar} \hat{a}^{\dagger} \) correspond to classical field amplitudes. In the following, we set \( \hbar = 1 \).

The exact eigenstates of \( \hat{H}_0 \) should be obtained numerically by diagonalization of the Hamiltonian matrix. However, qualitatively the structure of eigenstates can be understood using the classical analogy.

From the Bohr–Sommerfeld quantization rule one concludes that the eigenstates of the Hamiltonian correspond to the discrete set of trajectories on the classical phase portrait (Fig. [1]). However, the real picture is a bit more complicated because the quantum tunneling should also be taken into account. This is because the classical phase portrait has different regions with the same energy, i.e. regions I and III. So, the real eigenstates may correspond not only to single trajectories but also to superpositions of the trajectories with the same energy.

The possibility of quantum tunneling is closely connected to the degeneracy of eigenstates in the Hamiltonian [30] at \( f = 0 \). At \( f = 0 \), the Hamiltonian commutes with \( \hat{a}^{\dagger} \hat{a} \), and the states with \( k \) excitation quanta are the eigenstates of the Hamiltonian. Their energy is

\[ \epsilon_k = -\Delta k + \frac{\alpha k(k-1)}{2} \]  

(31)

At integer \( \frac{\Delta a}{\alpha} \), the states with \( k \) and \( m-k \) excitation quanta become degenerate. At small but nonzero \( f \), these states can mix: the true eigenstates are the superpositions of \( |k\rangle \) and \( |m-k\rangle \). On the quasiclassical language this corresponds to tunneling between degenerate classical trajectories. Numerical diagonalization shows that such mixing occurs only when \( \frac{\Delta a}{\alpha} \) is very close to integer.

To provide some illustration to this qualitative picture, we calculated the eigenstates of the Hamiltonian [30] in the coherent basis:

\[ \psi_n(z) \equiv \langle n|z \rangle, \]  

(32)

where \( |z \rangle \) is a normalized coherent state. The function \( \psi_n(z) \) corresponding to the \( n \)–th eigenstate has maximum near the contour line of the classical Hamiltonian.
\( H(a, a^*) = E_n \). This means that the quantum state \(|n\rangle\) corresponds to the classical motion along the trajectory \( H(a, a^*) = E_n \).

Figure 6. Some eigenstates of the quantum Hamiltonian are shown in the coherent basis. For each state \(|n\rangle\), the quantity \( \langle z | n \rangle \) in complex \( z \) plane \( z = \sqrt{\Delta}/\alpha \) is shown. (a) The higher–amplitude state for \( \Delta/\alpha = 20.25 \), \( \beta/\beta_{\text{crit}} = 0.6 \). It is squeezed in \( q \)-direction. (b) The lower–amplitude state for the same parameters. (c) The eigenstate which is a superposition of the lower–amplitude state and the trajectory from classical region 3. It corresponds to \( \Delta/\alpha = 20 \), \( \sqrt{\beta}/\beta_{\text{crit}} = 0.6 \).

An important property of the quasienergy states is that the states corresponding to the higher amplitude stable point are squeezed. This can be shown by mean–field expansion:

\[
\hat{a} = \langle a \rangle_2 + \hat{a}'
\]

The mean value of \( \hat{a} \) is defined from the equation \( \partial H/\partial a_2 \langle a \rangle_2, \langle a^* \rangle_2 \rangle = 0 \), which corresponds to classical stable states \( i \). For small \( \beta \), the mean field value in the higher amplitude stable state \( 2 \) is \( \langle a \rangle_2 \approx \sqrt{\Delta/\alpha}(1 + \sqrt{\beta}/2) \).

The quadratic part of the Hamiltonian takes the form

\[
\hat{H} = H((a)_2, (a^*)_2) - (\Delta - 2\alpha|\langle a \rangle_2|^2)\hat{a}^t \hat{a}'
\]
\[
+ \frac{\alpha}{2} \langle a \rangle_2 \hat{a}^t \hat{a}' + \frac{\alpha}{2} \langle a^* \rangle_2 \hat{a}'^t \hat{a}'
\]

We diagonalize this Hamiltonian by mean of Bogolyubov transformation:

\[
\hat{a}' = \hat{b} \cosh \theta - \hat{b}^\dagger \sinh \theta \\
\tanh 2\theta = \frac{\alpha|\langle a \rangle_2|^2}{2\alpha|\langle a \rangle_2|^2 - \Delta}
\]

Let us consider the uncertainties in two quadratures \( \hat{q} \) and \( \hat{p} \), \( \hat{a} = \frac{\hat{q} + i\hat{p}}{\sqrt{2}} \). Squeezing is more pronounced in higher amplitude stable states \( \langle a \rangle_2 \).

\[
\langle q^2 \rangle - \langle q \rangle_2^2 = \frac{e^{-2\theta}}{2} = \frac{1}{2} \sqrt{\frac{\alpha|\langle a \rangle_2|^2 - \Delta}{3\alpha|\langle a \rangle_2|^2 - \Delta}}
\]
\[
\langle p^2 \rangle - \langle p \rangle_2^2 = \frac{e^{-2\theta}}{2} = \frac{1}{2} \sqrt{\frac{3\alpha|\langle a \rangle_2|^2 - \Delta}{\alpha|\langle a \rangle_2|^2 - \Delta}}
\]

The quadratic approximation is correct when \( \sqrt{\beta} \) is larger than \( \frac{\Delta}{2\alpha} \). When \( \frac{\Delta}{2\alpha} \ll 1 \), this is fulfilled almost in the whole region of bistability, and the relations \( 36 \) are valid.

The minimum possible uncertainty of \( \hat{q} \) is at \( \beta \sim (\alpha/\Delta)^2 \), where it can be estimated as

\[
\langle q^2 \rangle - \langle q \rangle_2^2 \sim \sqrt{\frac{\alpha}{\Delta}}
\]

Thus, the uncertainty in \( \hat{q} \) quadrature can be far beyond the quantum limit.

As we have shown in the previous section \( 28 \), tunneling effects increase the occupation of the stable state \( 2 \) with higher amplitude and therefore enhance the generation of squeezed states.

B. Quantum kinetic equation

Let us assume that the system is weakly interacting with the environment.

\[
H_{\text{full}} = H_0 + \xi^\dagger \hat{a} + \xi \hat{a}^\dagger + H_{\text{bath}}
\]

We assume that the correlation functions of damping operators are delta–correlated:

\[
\langle \dot{\xi}(t) \xi(t') \rangle = \gamma (N + 1) \delta(t - t')
\]
\[
\langle \dot{\xi}^\dagger(t) \xi(t') \rangle = \gamma N \delta(t - t'),
\]

where \( N \) is the number of noise quanta.

With such assumptions, the density matrix evolution can be described by the master equation \( 24, 25, 26 \), \( 19, 17 \):

\[
\partial_t \rho = i[\rho, H] + \frac{\gamma}{2} (2\hat{a} \rho \hat{a}^\dagger - \rho \hat{a}^\dagger \hat{a} - \hat{a} \hat{a}^\dagger \rho + 2N[\hat{a}, \hat{a}^\dagger], \rho]
\]

If \( \gamma \) is small compared to \( \Delta \), the density matrix is almost diagonal in the basis of eigenstates \( |n\rangle \), and the master equation reduces to the rate equation for probabilities \( P_n \) to be in \( n \)-th eigenstate:

\[
\frac{dP_n}{dt} = \sum_{n'} w_{n'n} P_{n'} - w_{nn'} P_n,
\]

\[
w_{n'n} = \gamma \left( (N + 1)|\langle n | \hat{a} | n' \rangle|^2 + N |\langle n' | \hat{a}^\dagger | n \rangle|^2 \right)
\]

This equation is a quantum analog of \( 6 \). The evolution of the density matrix has the same features as the evolution of the distribution function for the classical oscillator with bistability. At infinite time, the density matrix evolves to the stationary distribution. The relaxation to \( P_n^\text{st} \) consists of two stages. The first stage corresponds to relaxation to the quasi–stationary distribution. Its typical time is \( \gamma^{-1} \). The second stage is the relaxation to the true stationary state. This stage is very slow and happens due to transitions between the classical stationary states. These transitions can be induced by quantum fluctuations and by thermal noise.

Formally, the general solution of \( 41 \) reads

\[
P_n(t) = P_n^\text{st} + \sum_{\lambda > 0} C_k P_n^\text{st} e^{-\lambda t},
\]
The lowest nonzero eigenvalue is much smaller than all other eigenvalues. Therefore, at large \( t \) only the term with the lowest nonzero \( \lambda \) should be retained on equation (42).

The density matrix relaxes to the true stationary distribution with the rate \( \lambda_{\text{min}} \), which can be interpreted as the rate of fluctuation–induced transitions between the stable states.

C. The quasiclassical limit

It is possible to show that the continuous limit of (41) is (9). As it was mentioned, every eigenstate corresponds to a trajectory on the classical phase portrait, and the hybridization of the trajectories from the regions 1 and 3 can be neglected unless \( 2\Delta/\alpha \) is very close to integer. Thus, in the quasiclassical limit the distribution function \( P_n \) slowly depends on \( n \) in each region of the phase space.

Moreover, the transition rates \( w_{n,n'} = \tilde{w}_{n,k} \tilde{n} = \frac{n+n'}{2} \), \( k = n' - n \) fastly decrease with increasing \( |k| \) and slowly depend on \( \tilde{n} \), which is close to \( n \). Then, it is possible to perform a gradient expansion of \( P_n, w_{nm} \) in (41):

\[
P(n+k) = P(n) + \frac{\partial P}{\partial n}k + \frac{1}{2} \frac{\partial^2 P}{\partial n^2}k^2 + \ldots
\]

(43)

\[
w_{n,n+k} = \tilde{w}_{n,k} + \frac{\partial \tilde{w}_{n,k}}{\partial n} \left( k \right) + \frac{1}{2} \frac{\partial^2 \tilde{w}_{n,k}}{\partial n^2} \left( k \right)^2 + \ldots
\]

(44)

Keeping the terms up to the second order in \( k \), one obtains the differential equation for \( P(n) \):

\[
\frac{\partial P_n}{\partial t} = \frac{\partial}{\partial n} \left[ A(n)P(n) + B(n)\frac{\partial P}{\partial n} \right]
\]

(45)

where the coefficients \( A(n), B(n) \) can be obtained using the expressions (41) for probabilities \( w_{nm} \):

\[
A(n) = - \sum_{n'} \tilde{w}_{n,k} = \frac{i\gamma \tilde{T}(\epsilon_n)}{2\pi} \langle n|\hat{a}\hat{b}^\dagger|n \rangle
\]

(46)

\[
B(n) = \frac{1}{2} \sum_{n'} \tilde{w}_{n,k}k^2 = \gamma \left( N + \frac{1}{2} \right) \frac{\tilde{T}(\epsilon_n)^2}{4\pi^2} \langle n|\hat{b}\hat{a}\hat{b}^\dagger|n \rangle
\]

(47)

Here \( \tilde{T}(\epsilon_n) \) is the period of the classical motion at energy \( \epsilon_n \).

In the quasiclassical limit, the averages over quantum quasienergy states transform to time–averages over classical trajectories. Thus, in the quasiclassical limit \( A(n), B(n) \) are expressed as line integrals over classical trajectories:

\[
A(n) = \frac{i\gamma}{4\pi} \int_{C(\epsilon_n)} a \, da^* - a^* \, da
\]

\[
B(n) = \frac{i\gamma \tilde{T}(\epsilon_n)}{8\pi^2} \left( N + \frac{1}{2} \right) \int_{C(\epsilon_n)} \frac{\partial H}{\partial a} \, da - \frac{\partial H}{\partial a^*} \, da^*
\]

(48)

After variable change \( \frac{\Delta}{\alpha} T(E) \, dE = 2\pi d\tilde{n}, t \Delta = \tau, \gamma/\Delta = \theta \) the equation transforms to the classical FPE (9). The coefficient \( A(n) \) transforms to \( \partial K(E) \), and \( B(n) \) transforms to \( QT(E)D(E) \), where

\[
Q = \frac{\partial \alpha}{\Delta} \left( N + \frac{1}{2} \right).
\]

(49)

\[
E = \frac{\alpha}{\Delta^2} \epsilon \text{ is the dimensionless energy, and } T(E) = \Delta T(\alpha E/\Delta^2) \text{ is the dimensionless period as in (6).}
\]

D. Results and discussion

Qualitatively, the behavior of \( P^\text{cl}_n \) in the diagonal approximation resembles the behavior of \( P^\text{cl}_n(E) \) of the classical oscillator, as \( P^\text{cl}_n(E) \) is the classical limit of \( P^\text{cl}_n \) (here \( i \) indicates the classical region of the phase space). As \( P^\text{cl}_n(E) \), it consists of two sharp peaks which can be attributed to the classical stable states 1 and 2. Below (above) threshold value of external field, the state 1 (2) dominates.

We directly compared the distributions obtained from classical FPE and from quantum master equation. In the classical limit \( P_{\text{cl}}^n \) equals \( \frac{\alpha}{\pi^2} P^\text{cl}_n(E_n), E_n = \alpha n/\Delta^2 \), where \( P^\text{cl}_n(E) \) is the classical distribution function for a dimensionless Hamiltonian (2) with number of noise quanta defined by (49). The index \( i \) corresponding to the classical region of the phase space is uniquely defined for each eigenstate unless \( \frac{\Delta}{\alpha} \) is integer. In the latter case, the classical FPE should be derived from the quantum master equation more carefully. It can be obtained only after choosing the proper basic quasienergy states. One should deal with quantum states corresponding to the trajectories in the regions of phase space 1 and 3, but not with their superposition.

For classical case, it was shown that the changing of the most probable stable state takes place at \( \sqrt{3} \equiv \sqrt{\frac{\alpha^2}{\Delta^2}} \approx 0.29 \). It is clear from Fig. [7] that for rather high number of noise quanta \( N \gg \frac{\alpha}{\Delta^2} \) and large noninteger \( \frac{\Delta}{\alpha} \gg 1 \) the probability distribution for quantum oscillator obtained from master equation (41) coincides with the classical distribution over quasienergies. For small number of noise quanta, the situation is more complicated even for large \( \frac{\Delta}{\alpha} \). Even though at large \( \frac{\Delta}{\alpha} \) the quasiclassical approximation for matrix elements of \( \hat{a} \) is valid, the quantum distribution function doesn’t coincide with the classical one due to quantum fluctuations and tunneling effects. For the quantum oscillator described by the rate equation (41), \( \beta_0 \) is no more the universal threshold parameter. Not only the parameter \( \beta \) matters, but also \( \frac{\Delta}{\alpha} \) and \( N \). The threshold value \( \beta_0 \) in quantum limit can exceed the classical value for noninteger \( \frac{\Delta}{\alpha} \), and it is below the classical value at integer \( \frac{\Delta}{\alpha} \). At small number of noise quanta, when \( N \lesssim \frac{\alpha}{2\Delta} \), the quantum effects including tunneling processes govern the transitions between different stable states. In this case the distribution function and
Figure 7. (a) The quantum analog of classical distribution function is shown for \( \Delta \alpha = 9.8 \), \( \sqrt{\beta/\beta_{\text{crit}}} = 0.3 \) and \( N = 0 \), 1. Solid (dashed) line represents the classical distribution for \( N = 1(0) \), triangles (circles) represent the quantum distribution for \( N = 1(0) \). For \( N = 1 \), the quantum distribution fits well the corresponding classical distribution. For \( N = 0 \), the quantum and classical distribution differ significantly. (b) The distribution functions at \( N = 1 \) and different close values of \( \Delta \alpha \) are shown. At \( \Delta \alpha = 9.99 \) (red triangles) the distribution fits well the classical distribution (black dashed line). At \( \Delta \alpha = 9.9999 \) (blue stars) the population in the lower–amplitude state becomes significantly smaller than the classical FPE predicts.

Figure 8. (a) The lowest eigenvalue of rate equation for \( N = 0, 0.3, 1, \Delta/\alpha = 9.9 \). (b) The dependence of the lowest eigenvalue of rate equation for \( N = 0 \) on \( \Delta/\alpha, \beta \).

To clarify the influence of fluctuation–induced transitions on statistical properties of internal oscillator field, one should calculate the second–order correlation function \( g^2(0) \):

\[
g^2(0) \equiv \frac{\langle \hat{a} \dagger \hat{a} \dagger \hat{a} \hat{a} \rangle}{\langle \hat{a} \dagger \hat{a}\rangle^2} \tag{50}
\]

When one of the stable states dominates, \( g^{(2)}(0) \sim 1 \). In a narrow region of fluctuation–induced transitions, \( g^{(2)}(0) \) strongly rises. For several values of \( N \), we have calculated \( g^2(0) \) as a function of two parameters: \( \sqrt{\beta} \) and \( \Delta/\alpha \).

Figure 9. The correlation function \( g^2(0) \) at different values of \( N \) depending on \( \Delta/\alpha, \beta \). (a) \( N = 0 \) (b) \( N = 0.01 \) (c) \( N = 0.05 \) (d) \( N = 0.1 \)

For each \( \Delta/\alpha \) the correlation function has similar dependence on \( \sqrt{\beta} \): there is a sharp peak at some value of \( \sqrt{\beta/\beta_{\text{crit}}} \) between 0 and 1 which indicates changing of the most probable state. However, the peak is not placed at \( \sqrt{\beta/\beta_{\text{crit}}} \approx 0.29 \) as the classical FPE predicts. Its position is an oscillating function of \( \Delta/\alpha \) with sharp minima.
at $\frac{\Delta}{\alpha}$ is integer, which corresponds to enhanced tunneling between degenerate quasienergy states. The amplitude of oscillations decreases at increasing $\frac{\Delta}{\alpha}$, when one approaches the quasiclassical limit. This fact points to quantum nature of these oscillations.

IV. CONCLUSION

Quasiclassical kinetic equations for the probability distribution over quasienergy states of a nonlinear driven oscillator with account for tunneling effects have been obtained. The stationary distribution for wide range of the system parameters and the typical relaxation rate have been determined. It was shown that the relaxation consists of two stages. At first, relaxation to the quasi–stationary distribution occurs in each region of phase space at time scales determined by inverse damping constant. Then, at exponentially large times, the probability distribution evolves to true stationary one. The relaxation to the real stationary state happens due to multi–photon resonant transitions between the quasiclassical stable states. In classical limit, if tunneling is neglected, there exists a universal threshold value of field intensity responsible for switching between the most probable stable states of the system. With account for tunneling effects, this value becomes no more universal. Tunneling transitions lead to decreasing of the threshold value of external field intensity and to increasing up to one order of the fluctuation–induced transition rate between the stable states in threshold area.

For the quantum nonlinear driven oscillator, it is shown that the quasienergy state corresponding to the classical higher–amplitude stable state is squeezed. The degree of squeezing is determined by the ratio of nonlinearity and detuning, and the uncertainty of one of the oscillator quadratures can be much lower than the usual quantum limit.

As tunneling transitions increase occupation of higher–amplitude stable state, the generation of squeezed states can be enhanced in the presence of tunneling effects.

Also we show that the quasienergy states become superpositions of trajectories from different regions of the phase space when the detuning is an integer or half–integer multiple of nonlinear shift per quantum. This happens due to multi–photon resonance between the real eigenstates of the nonlinear oscillator. It was shown that such resonance can be described as tunneling between quasienergy states in different regions of the classical phase space.

The kinetics of the quantum oscillator is investigated using the quantum master equation. It is shown that at large detuning–nonlinearity ratio, large number of thermal photons and absence of multi–quantum resonance the classical FPE in quasienergy space is a continuous limit of the quantum master equation. It is important to notice that the large detuning–nonlinearity ratio is not enough for validity of the classical FPE, because at weak noise the quantum effects become especially important.

The relaxation rate and the threshold intensity of external field are very sensitive to the detuning–nonlinearity ratio. At integer and half–integer detuning–nonlinearity ratio, the relaxation rate can increase up to several orders of magnitude and the threshold value of external field intensity shifts towards lower values. In this case, tunneling between degenerate quasienergy states and multi–photon resonant transitions between the original states of the nonlinear oscillator can be treated as the same effect.

It was demonstrated that second–order correlation function of the system internal field strongly rises near the threshold pumping intensity, which indicates to super–Poissonian statistics of the internal oscillator field.

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Appendix: The coefficients of the classical Fokker–Planck equation

The coefficients of the classical FPE are defined as follows:

\[ K(E) = \frac{i}{2} \int a \, da^* - a^* \, da \]

\[ D(E) = \frac{i}{2} \int \frac{\partial H}{\partial a} \, da - \frac{\partial H}{\partial a^*} \, da^* \]

\[ T(E) = \int da^* \, da \, \delta(E - H(a^*, a)) \]  \hspace{1cm} (A.1)

For them, we obtained the following integral representations:

\[ T(E) = \int \frac{dt}{\sqrt{2f^2 t - (E + \frac{t^2}{2} - \frac{t^2}{8})^2}} \]  \hspace{1cm} (A.2)

\[ K_i(E) = \int \frac{3t^2/16 - t/4 + E/2}{\sqrt{2f^2 t - (E + \frac{t^2}{2} - \frac{t^2}{8})^2}} \, dt \]  \hspace{1cm} (A.3)

\[ D_i(E) = \int \frac{t^3/16 - t^2/8 + Et/2 + f^2 - E}{\sqrt{2f^2 t - (E + \frac{t^2}{2} - \frac{t^2}{8})^2}} \, dt \]  \hspace{1cm} (A.4)

The limits of integration are the roots or the equation

\[ 2f^2 t - \left( E + \frac{t^2}{2} - \frac{t^2}{8} \right)^2 = 0 \]

For energies \( E_2 < E < E_{\text{sep}} \) corresponding to the classical region 2, this equation has only two real roots. For energies \( E_{\text{sep}} < E < E_1 \) corresponding to regions 1, 3, there are four real roots. \( t_1 < t_2 < t_3 < t_4 \). To obtain \( T_1, K_1, D_1 \), the limits of integration should be \( t_1, t_2 \), and for \( T_3, K_3, D_3 \), they should be \( t_3, t_4 \). Finally, for \( E > E_1 \), which corresponds only to region 3, there are two real roots again.