THE CONLEY-ZEHNDER INDEX OF A MINIMAL ORBIT AND EXISTENCE OF A POSITIVE HYPERBOLIC ORBIT

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Abstract. As a refinement of the Weinstein conjecture, it is a natural question whether a Reeb orbit of particular types exists. D. Cristofaro-Gardiner, M. Hutchings and D. Pomerleano showed that every nondegenerate closed contact three manifold with $b_1 > 0$ has at least one positive hyperbolic orbit by directly using the isomorphism between ECH and Seiberg-Witten Floer (co)homology. In the same paper, they also asked whether the case of $b_1 = 0$ does. Suppose that $(S^3, \lambda)$ is non-degenerate contact three sphere with infinity many orbits. In the present paper, we prove the existence of a simple positive hyperbolic orbit on $(S^3, \lambda)$ under the condition that the Conley-Zehnder index of a minimal periodic orbit induced by the trivialization of a bounding disc is larger than or equal to 3. As an immediate corollary, we have the existence of a simple positive hyperbolic orbit on a non-degenerate dynamically convex contact three sphere $(S^3, \lambda)$ with infinity many simple orbits. In particular, this implies that a $C^\infty$ generic compact strictly convex energy hypersurface in $\mathbb{R}^4$ carries a positive hyperbolic simple orbit.

1. Introduction

1.1. Statement of the main theorem. A 1-form $\lambda$ on a $2n + 1$ dimensional manifold $Y$ is called a contact form if $\lambda \land (d\lambda)^n > 0$ and the pair $(Y, \lambda)$ is called a contact manifold. A contact manifold has an unique vector field $X_{\lambda}$ called the Reeb vector field of $(Y, \lambda)$ which satisfies $\lambda(X_{\lambda}) = 1$ and $d\lambda(X_{\lambda}, \cdot) = 0$. A Reeb orbit is a map $\gamma: \mathbb{R}/\mathbb{T} \rightarrow Y$ satisfying $\dot{\gamma} = X_{\lambda}(\gamma)$ for some $T > 0$. $\gamma$ is called simple if $\gamma$ is an embedding map. In this paper, two Reeb orbits are considered equivalent if they differ by reparametrization.

The Weinstein conjecture asking if every closed contact manifold has a Reeb orbit has been studied in various situations. For example, the properties of Reeb vector fields of contact manifolds realized as contact type energy hypersurfaces in $\mathbb{R}^{2n}$ have been studied from the viewpoints of Hamiltonian dynamics. The existence of a periodic orbit on energy hypersurfaces with strictly convex $[W]$, star-shaped $[R]$, contact type $[V]$ (the Weinstein conjecture of contact type hypersurfaces in $\mathbb{R}^{2n}$), generic case $[HZ]$ was shown.

On the other hand, there are unique techniques available for studying three-dimensional Reeb flows. For example, in $[H]$ H. Hofer proved the three-dimensional Weinstein conjecture in a large case by analyzing pseudoholomorphic curves and in $[HWZ]$, H. Hofer, K. Wysocki and E. Zehnder showed that a compact strictly convex energy surface in $\mathbb{R}^4$ carries two or infinity many simple orbits by constructing

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a global surface of section for the Reeb vector field $X_{\lambda}$ from a certain pseudoholomorphic curve. After that, C. Taubes completely proved the three-dimensional Weinstein conjecture by Seiberg-Witten Floer (co)homology ([T1]). In associate with them, there is a powerful tool to study three-dimensional Reeb flows, called Embedded contact homology (ECH) which was introduced mainly by M. Hutchings (see [H4] for the introduction). ECH is a kind of homology whose boundary map is defined by counting certain holomorphic curves and isomorphic with a version of Seiberg-Witten Floer (co)homology ([T2]) (for more detail explanation, see the next section). In [CH], D. Cristofaro-Gardiner and M. Hutchings showed that every contact closed three manifold $(Y, \lambda)$ has at least two simple periodic orbits by using the volume property with respect to ECH spectrums. It will be summarized as follows.

**Theorem 1.1 ([T1] + [CH]).** A contact closed three manifold $(Y, \lambda)$ has at least two simple periodic orbit.

For a Reeb orbit $\gamma$ of $(Y, \lambda)$, if the return map $d\phi_T|_{\ker\lambda=\xi} : \xi_{\gamma(0)} \to \xi_{\gamma(0)}$ has no eigenvalue 1, we call it a non-degenerate Reeb orbit and we call $(Y, \lambda)$ non-degenerate if all Reeb orbits are non-degenerate. It is well-known that the condition of non-degenerate is a $C^\infty$ generic condition.

In three dimension, non-degenerate periodic orbits are classified according to their eigenvalues of the return maps. A periodic orbit is called negative hyperbolic if $d\phi_T|_{\xi}$ has eigenvalues $h, h^{-1} < 0$, positive hyperbolic if $d\phi_T|_{\xi}$ has eigenvalues $h, h^{-1} > 0$ and elliptic if $d\phi_T|_{\xi}$ has eigenvalues $e^{\pm i2\pi \theta}$ for some $\theta \in \mathbb{R}\setminus\mathbb{Q}$.

As a refinement of the Weinstein conjecture, the following theorem was proved.

**Theorem 1.2 ([CHP]).** Let $(Y, \lambda)$ be a non-degenerate contact three manifold. Let $\ker\lambda = \xi$. Then

1. if $c_1(\xi)$ is torsion, there exists infinity many periodic orbits, otherwise there are exactly two elliptic simple periodic orbits and $Y$ is diffeomorphic to a lens space (possibly $S^3$).
2. if $c_1(\xi)$ is not torsion, there exists at least four periodic orbit.

**Theorem 1.3 ([CHP]).** If $b_1(Y) > 0$, there exists at least one positive hyperbolic orbit.

The above results are based on the isomorphism between Embedded contact homology and Seiberg-Witten Floer (co)homology. For example, Theorem 1.3 was shown by the fact that the odd part $ECH_{\text{odd}}$ of $ECH = ECH_{\text{odd}} \oplus ECH_{\text{even}}$ which detects the existence of positive hyperbolic orbit always does not vanish if $b_1(Y) > 0$. On the contrary to $b_1(Y) > 0$, if $b_1(Y) = 0$, the part of $ECH_{\text{odd}}$ may vanish (for example, see the case of $S^3$ ([28])).

In [CHP], they also asked the next question.

**Question 1.4 ([CHP]).** Let $Y$ be a closed connected three-manifold which is not $S^3$ or a lens space, and let $\lambda$ be a nondegenerate contact form on $Y$. Does $\lambda$ have a positive hyperbolic simple Reeb orbit?

As stated above, we can not directly apply the method used in $b_1(Y) > 0$ to the case of $b_1(Y) = 0$. Here we note that the reason why the cases $S^3$ and lens spaces
are excluded in Question [1.4] is that they admit contact forms with exactly two simple elliptic orbits as stated in Theorem [1.2] (for example, see [HT3]). So more appropriate assumption of Question [1.4] is that $(Y, \lambda)$ is not a lens space or $S^3$ with exactly two elliptic orbits (this is generic condition. see [10]).

In order to study Question [1.4] the author proved the next theorem.

**Theorem 1.5** ([S1]). Let $(Y, \lambda)$ be a nondegenerate contact three manifold with $b_1(Y) = 0$. Suppose that $(Y, \lambda)$ has infinity many simple periodic orbits (that is, $(Y, \lambda)$ is not a lens space with exactly two simple Reeb orbits) and has at least one elliptic orbit. Then, there exists at least one simple positive hyperbolic orbit.

The next theorem is the main theorem of this paper.

**Theorem 1.6.** $S^3$ does not admit any non-degenerate contact forms satisfying the following conditions.

1. All simple orbits are negative hyperbolic.
2. For a periodic orbit $\gamma_0$ with the smallest period among all orbits, $\mu_{\text{disc}}(\gamma_0) \geq 3$ where $\mu_{\text{disc}}$ is a Conley-Zehnder index with respect to a trivialization induced by a bounding disc (explained just below).

For a path in symplectic matrix $S : [0, 1] \to \text{Sp}(2n)$ with $S(0) = \text{Id}$ and $\det(\text{Id} - S(1)) \neq 0$, the Conely-Zehnder index $\mu(S)$ is defined. Let $\gamma$ be a non-degenerate Reeb orbit in $(S^3, \lambda)$. To explain $\mu_{\text{disc}}(\gamma)$, take a continuous disc map $\psi : D \to S^3$ satisfying $\psi(\frac{2\pi}{n}) = \gamma(t)$ where $D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ and $\int_0^\pi \lambda = T$. We have a symplectic trivialization $\beta : \psi^*\xi \to D \times \mathbb{C}$ up to homotopy. With this notation, we define $\mu_{\text{disc}}(\gamma)$ by the Conely-Zehnder index of a path $\beta[\psi^*\xi \circ d\alpha^{\beta}]|_{\psi(0)} \circ \beta^{-1}|_1$. Note that since $\pi_2(S^3) = 0$, this is independent of the choice of a disc map $\psi$.

Having Theorem [1.5] we have

**Corollary 1.7.** Let $(S^3, \lambda)$ be a non-degenerate contact three manifold with infinity many simple Reeb orbits. Let $\gamma_0$ be a Reeb orbit with the smallest period among all orbits. Suppose that $\mu_{\text{disc}}(\gamma_0) \geq 3$. Then there is at least one positive hyperbolic simple orbit.

The condition of $\mu_{\text{disc}}(\gamma_0) \geq 3$ in Theorem [1.7] naturally appears in classical contexts. Assume that $S \subset \mathbb{R}^4$ is a compact strictly convex hypersurface containing the origin in its interior and $\lambda_0 = \frac{1}{2} \sum_{i=1,2} y_i dx_i - x_i dy_i$. Then it is well-known that $(S, \lambda_0|_S)$ is a contact manifold diffeomorphic to $S^3$. Furthermore, for any smooth function $H : \mathbb{R}^{2n} \to \mathbb{R}$ such that $S$ can be realized as a regular energy surface $H^{-1}(c)$ for some $c \in \mathbb{R}$, the Reeb vector field is parallel to the Hamiltonian vector field $X_H$ which satisfies $dH = \omega_0(X_H, \cdot)$ where $\omega_0 = dy_1 \wedge dx_1 + dy_2 \wedge dx_2$.

The following notion was introduced in [HWZ] for studying the dynamics on a compact strictly convex hypersurface.

**Definition 1.8** ([HWZ]). A contact form $\lambda$ on $S^3$ is called dynamically convex if $\tilde{\mu}_{\text{disc}}(\gamma) \geq 3$ for every periodic orbit $\gamma$ (possibly degenerate) of the Reeb vector field $X_\lambda$. Here, $\tilde{\mu}_{\text{disc}}(\gamma)$ is an extension of $\mu_{\text{disc}}(\gamma)$ in a suitable manner to degenerate orbit (see [HWZ] for more details). In particular, if $\gamma$ is non-degenerate, $\tilde{\mu}_{\text{disc}}(\gamma) = \mu_{\text{disc}}(\gamma)$.
The notion of dynamically convex is a generalization of compact strictly convex hypersurface in terms of the dynamics as follows.

Theorem 1.9 ([HWZ3]). Assume $S \subset \mathbb{R}^4$ is a compact strictly convex hypersurface containing the origin in its interior. Then every periodic orbit $\gamma$ of the Reeb vector field of $(S, \lambda_0|_S)$ satisfies $\mu_{\text{disc}}(\gamma) \geq 3$.

As an immediate corollary of the main theorem, we have the next result.

Corollary 1.10. A non-degenerate dynamically convex contact three sphere $(S^3, \lambda)$ with infinity simple orbits has at least one positive hyperbolic simple orbit.

Since the conditions of non-degenerate and the existence of infinity simple orbits are both $C^\infty$ generic, we also have the next corollary.

Corollary 1.11. A $C^\infty$ generic compact strictly convex hypersurface in $\mathbb{R}^4$ carries a positive hyperbolic simple orbit.

1.2. Idea and structure of this paper. We will prove Theorem 1.6 by contradiction. First of all, in Section 2, we introduce some results of ECH which will be needed in this paper. In Section 3, under the assumptions that $(S^3, \lambda)$ satisfies (1) and (2) of Theorem 1.6 we will give a series of results used in Section 4. In particular, we will construct a global surface of section for the Reeb vector field from the $U$-map. This enables us to calculate some indexes of orbits and restricts an existence and behaviors of certain $J$-holomorphic curves. In section 4, by using results obtained in section 3, we will determine a basis of ECH in order from smallest ECH grading and show that there will always be a contradiction at some point. As a result, we will complete Theorem 1.6.

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2. Preliminaries

For a non-degenerate contact three manifold $(Y, \lambda)$ and $\Gamma \in H_1(Y; \mathbb{Z})$, Embedded contact homology $ECH(Y, \lambda, \Gamma)$ is defined. At first, we define the chain complex $(ECH(Y, \lambda, \Gamma), \partial)$. In this paper, we consider ECH over $\mathbb{Z}/2\mathbb{Z} = F$.

Definition 2.1 ([H1 Definition 1.1]). An orbit set $\alpha = \{(\alpha_i, m_i)\}$ is a finite pair of distinct simple periodic orbit $\alpha_i$ with positive integer $m_i$. If $m_i = 1$ whenever $\alpha_i$ is hyperbolic orbit, then $\alpha = \{(\alpha_i, m_i)\}$ is called an admissible orbit set.

Set $[\alpha] = \sum m_i[\alpha_i] \in H_1(Y)$. For two orbit set $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ with $[\alpha] = [\beta]$, we define $H_2(Y, \alpha, \beta)$ to be the set of relative homology classes of 2-chains $Z$ in $Y$ with $\partial Z = \sum m_i\alpha_i - \sum n_j\beta_j$. This is an affine space over $H_2(Y)$.

Definition 2.2 ([H1 Definition 2.2]). Let $Z \in H_2(Y; \alpha, \beta)$. A representative of $Z$ is an immersed oriented compact surface $S$ in $[0,1] \times Y$ such that:

1. $\partial S$ consists of positively oriented (resp. negatively oriented) covers of $\{1\} \times \alpha_i$ (resp. $\{0\} \times \beta_j$) whose total multiplicity is $m_i$ (resp. $n_j$).
2. \(\pi(S) = Z\), where \(\pi : [0, 1] \times Y \to Y\) denotes the projection.

3. \(S\) is embedded in \((0, 1) \times Y\), and \(S\) is transverse to \([0, 1] \times Y\).

From now on, we fix a trivialization of \(\xi\) defined over every simple orbit \(\gamma\) and denote it by \(\tau\).

For a non-degenerate Reeb orbit \(\gamma\), its Conley-Zehnder index with respect to a trivialization \(\tau\) is denoted by \(\mu_\tau(\gamma)\) in this paper.

**Definition 2.3 (H1 §8.2).** Let \(\alpha_1, \beta_1, \alpha_2, \beta_2\) be orbit sets with \([\alpha_1] = [\beta_1]\) and \([\alpha_2] = [\beta_2]\). For a fixed trivialization \(\tau\), we can define
\[
Q_\tau : H_2(Y; \alpha_1, \beta_1) \times H_2(Y; \alpha_2, \beta_2) \to \mathbb{Z}.
\]

by \(Q_\tau(Z_1, Z_2) = -l_\tau(S_1, S_2) + \#(S_1 \cap S_2)\) where \(S_1, S_2\) are representatives of \(Z_1, Z_2\) for \(Z_1 \in H_2(Y; \alpha_1, \beta_1)\), \(Z_2 \in H_2(Y; \alpha_2, \beta_2)\) respectively and \(l_\tau\) is a kind of crossing number (see H1 §8.2 for details).

**Definition 2.4 (H1 Definition 1.5).** For \(Z \in H_2(Y, \alpha, \beta)\), we define ECH index by
\[
I(\alpha, \beta, Z) := c_1(\xi)_Z, \tau) + Q_\tau(Z) + \sum_i \sum_{k=1}^{n_i} \mu_\tau(\alpha_i^k) - \sum_j \sum_{k=1}^{n_j} \mu_\tau(\beta_j^k).
\]

Here, \(c_1(\xi)_Z, \tau)\) is a reactive Chern number and, \(Q_\tau(Z) = Q_\tau(Z, Z)\). Moreover this is independent of \(\tau\) (see H1 for more details).

**Proposition 2.5 (H1 Proposition 1.6).** The ECH index \(I\) has the following properties.

1. For orbit sets \(\alpha, \beta, \gamma\) with \([\alpha] = [\beta] = [\gamma] = \Gamma \in H_1(Y)\) and \(Z \in H_2(Y, \alpha, \beta)\), \(Z' \in H_2(Y, \beta, \gamma)\),
\[
I(\alpha, \beta, Z) + I(\beta, \gamma, Z') = I(\alpha, \gamma, Z + Z').
\]

2. For \(Z, Z' \in H_2(Y, \alpha, \beta)\),
\[
I(\alpha, \beta, Z) - I(\alpha, \beta, Z') = c_1(\xi) + 2PD(\Gamma), Z - Z' > 0.
\]

3. If \(\alpha\) and \(\beta\) are admissible orbit sets,
\[
I(\alpha, \beta, Z) = \epsilon(\alpha) - \epsilon(\beta) \mod 2.
\]

Here, \(\epsilon(\alpha), \epsilon(\beta)\) are the numbers of positive hyperbolic orbits in \(\alpha, \beta\) respectively.

For \(\Gamma \in H_1(Y)\), we define \(ECC(Y, \lambda, \Gamma)\) as freely generated module over \(\mathbb{Z}/2\) by admissible orbit sets \(\alpha\) such that \([\alpha] = \Gamma\). That is,
\[
ECC(Y, \lambda, \Gamma) := \bigoplus_{\alpha:\text{admissible orbit set with } [\alpha] = \Gamma} \mathbb{Z}_2 \langle \alpha \rangle.
\]

To define the differential \(\partial : ECC(Y, \lambda, \Gamma) \to ECC(Y, \lambda, \Gamma)\), we pick a generic \(\mathbb{R}\)-invariant almost complex structure \(J\) on \(\mathbb{R} \times Y\) which satisfies \(J(\frac{d}{dt}) = X_\lambda\) and \(J_\xi = \xi\).

We consider \(J\)-holomorphic curves \(u : (\Sigma, j) \to (\mathbb{R} \times Y, J)\) where the domain \((\Sigma, j)\) is a punctured compact Riemann surface. Here the domain \(\Sigma\) is not necessarily connected. Let \(\gamma\) be a (not necessarily simple) Reeb orbit. If a puncture of
$u$ is asymptotic to $\mathbb{R} \times \gamma$ as $s \to \infty$, we call it a positive end of $u$ at $\gamma$ and if a puncture of $u$ is asymptotic to $\mathbb{R} \times \gamma$ as $s \to -\infty$, we call it a negative end of $u$ at $\gamma$ (For more details [HT1]).

Let $u : (\Sigma, j) \to (\mathbb{R} \times Y, J)$ and $u' : (\Sigma', j') \to (\mathbb{R} \times Y, J)$ be two $J$-holomorphic curves. If there is a biholomorphic map $\phi : (\Sigma, j) \to (\Sigma', j')$ with $u' \circ \phi = u$, we regard $u$ and $u'$ as equivalent.

Let $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_i, n_i)\}$ be orbit sets. Let $\mathcal{M}^J(\alpha, \beta)$ denote the set of $J$-holomorphic curves with positive ends at covers of $\alpha_i$ with total covering multiplicity $m_i$, negative ends at covers of $\beta_j$ with total covering multiplicity $n_j$, and no other punctures. Moreover, in $\mathcal{M}^J(\alpha, \beta)$, we consider two $J$-holomorphic curves to be equivalent if they represent the same current in $\mathbb{R} \times Y$. For $u \in \mathcal{M}^J(\alpha, \beta)$, we naturally have $[u] \in H_2(Y; \alpha, \beta)$ and we denote $I(u) = I(\alpha, \beta, [u])$. Moreover we define

\[(7) \quad \mathcal{M}^J_k(\alpha, \beta) := \{ u \in \mathcal{M}^J(\alpha, \beta) | I(u) = k \} \]

In this notations, we can define $\partial_J : ECC(Y, \lambda, \Gamma) \to ECC(Y, \lambda, \Gamma)$ as follows.

For admissible orbit set $\alpha$ with $[\alpha] = \Gamma$, we define

\[(8) \quad \partial_J(\alpha) = \sum_{\beta: \text{admissible orbit set with } [\beta] = \Gamma} \#(\mathcal{M}^J(\alpha, \beta)/\mathbb{R}) \cdot (\beta). \]

Note that the above counting is well-defined and $\partial_J \circ \partial_J$. We can see the reason of the former in Proposition [2.8] and the later was proved in [HT1] and [HT2]. Moreover, the homology defined by $\partial_J$ does not depend on $J$ (see Theorem 2.8, or see [HT1]).

For $u \in \mathcal{M}^J(\alpha, \beta)$, the its (Fredholm) index is defined by

\[(9) \quad \text{ind}(u) := -\chi(u) + 2\chi(\xi|_u, \tau) + \sum_k \mu_\tau(\gamma_k^+) - \sum_l \mu_\tau(\gamma_l^-). \]

Here $\{\gamma_k^+\}$ is the set consisting of (not necessarily simple) all positive ends of $u$ and $\{\gamma_l^-\}$ is that one of all negative ends. Note that for generic $J$, if $u$ is connected and somewhere injective, then the moduli space of $J$-holomorphic curves near $u$ is a manifold of dimension $\text{ind}(u)$ (see [HT1] Definition 1.3).

Let $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_i, n_i)\}$. For $u \in \mathcal{M}^J(\alpha, \beta)$, it can be uniquely written as $u = u_0 \cup u_1$ where $u_0$ are unions of all components which maps to $\mathbb{R}$-invariant cylinders in $u$ and $u_1$ is the rest of $u$.

**Proposition 2.6 ([HT1] Proposition 7.15).** Suppose that $J$ is generic and $u = u_0 \cup u_1 \in \mathcal{M}^J(\alpha, \beta)$. Then

1. $I(u) \geq 0$
2. If $I(u) = 0$, then $u_1 = \emptyset$
3. If $I(u) = 1$, then $u$ is embedded, $u_0 \cap u_1 = \emptyset$ and $\text{ind}(u_1) = 1$.
4. If $I(u) = 2$ and $\alpha$ and $\beta$ are admissible, then $u$ is embedded, $u_0 \cap u_1 = \emptyset$ and $\text{ind}(u_1) = 2$. 

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For a positive integer $m$, we define an ordered partition of $m$ by
\[
P(m) := \begin{cases} (2, \ldots, 2) & \text{if } m \text{ is even}, \\ (2, \ldots, 2, 1) & \text{if } m \text{ is odd}. \end{cases}
\]

**Proposition 2.7.** Let $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$. If $u = u_0 \cup u_1 \in \mathcal{M}^d(\alpha, \beta)$ satisfies $I(u) = 1$, then for every negative hyperbolic orbit $\alpha_i$ (resp. $\beta_j$) in $\alpha$ (resp. $\beta$), the set consisting of the multiplicities of the positive (resp. negative) ends of $u_1$ at covers of $\alpha_i$ (resp. $\beta_j$) is an initial segment of $P(m_j)$ (resp. $P(n_j)$).

**Proof of Proposition 2.7.** For example, see [HT1, Definition 7.11, Definition 7.13 and Proposition 7.15] \[\square\]

If $c_1(\xi) + 2\text{PD}(\Gamma)$ is torsion, there exists the relative $\mathbb{Z}$-grading.

\[
ECH(Y, \lambda, \Gamma) := \bigoplus_{*; \mathbb{Z}\text{-grading}} ECH_*(Y, \lambda, \Gamma).
\]

Let $Y$ be connected. Then there is degree $-2$ map $U$.

\[
U : ECH_*(Y, \lambda, \Gamma) \to ECH_{*-2}(Y, \lambda, \Gamma).
\]

To define this, choose a base point $z \in Y$ which is not on the image of any Reeb orbit and let $J$ be generic. Then define a map

\[
U_{J, z} : ECC_*(Y, \lambda, \Gamma) \to ECC_{*-2}(Y, \lambda, \Gamma)
\]

by

\[
U_{J, z}(\alpha) = \sum_{\beta; \text{admissible orbit set with } |\beta| = \Gamma} \# \{ u \in \mathcal{M}_d^1(\alpha, \beta)/\mathbb{R} \mid (0, z) \in u \} \cdot \langle \beta \rangle.
\]

The above map $U_{J, z}$ is a chain map, and we define the $U$ map as the induced map on homology. Under the assumption, this map is independent of $z$. See [HT3, §2.5] for more details. Moreover, in the same reason as $\partial$, $U_{J, z}$ does not depend on $J$ (see Theorem 2.8 and see [TT]).

The next isomorphism is important.

**Theorem 2.8 ([TT]).** For each $\Gamma \in H_1(Y)$, there is an isomorphism

\[
ECH_*(Y, \lambda, \Gamma) \cong \widetilde{HM}_*( -Y, s(\xi) + 2\text{PD}(\Gamma))
\]

of relatively $\mathbb{Z}/d\mathbb{Z}$-graded abelian groups. Here $d$ is the divisibility of $s(\xi) + 2\text{PD}(\Gamma)$ in $H_1(Y)$ mod torsion and $s(\xi)$ is the spin-c structure associated to the oriented 2-plane field as in [KM]. Moreover, the above isomorphism interchanges the map $U$ in (12) with the map

\[
U_1 : \widetilde{HM}_*(-Y, s(\xi) + 2\text{PD}(\Gamma)) \to \widetilde{HM}_{*-2}(-Y, s(\xi) + 2\text{PD}(\Gamma))
\]

defined in [KM].

Here $\widetilde{HM}_*(-Y, s(\xi) + 2\text{PD}(\Gamma))$ is a version of Seiberg-Witten Floer homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients defined by Kronheimer-Mrowka [KM].

The action of an orbit set $\alpha = \{(\alpha_i, m_i)\}$ is defined by

\[
A(\alpha) = \sum m_i A(\alpha_i) = \sum m_i \int_{\alpha_i} \lambda.
\]
2. For orbit sets $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_i, n_i)\}$ have $A(\alpha) \leq A(\beta)$, then the coefficient of $\beta$ in $\partial \alpha$ is 0 because of the positivity of $J$ holomorphic curves over $\partial \alpha$ and the fact that $A(\alpha) - A(\beta)$ is equivalent to the integral value of $d\lambda$ over $J$-holomorphic punctured curves which is asymptotic to $\alpha$ at $+\infty$, $\beta$ at $-\infty$.

Suppose that $b_1(Y) = 0$. In this situation, for any orbit sets $\alpha$ and $\beta$ with $[\alpha] = [\beta]$, $H_2(Y, \alpha, \beta)$ consists of only one component since $H_2(Y) = 0$. So we may omit the homology component from the notation of ECH index $I$, that is, ECH index is just denoted by $I(\alpha, \beta)$. Furthermore, for a orbit set $\alpha$ with $[\alpha] = 0$, we set $I(\alpha) := I(\alpha, \emptyset)$.

2.1. $J_0$ index and topological complexity of $J$-holomorphic curve. In this subsection, we recall the $J_0$ index.

**Definition 2.9 ([HT3 §3.3]).** Let $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ be orbit sets with $[\alpha] = [\beta]$. For $Z \in H_2(Y, \alpha, \beta)$, we define

\begin{equation}
J_0(\alpha, \beta, Z) := -c_1(\xi[Z, \tau]) + Q_r(Z) + \sum_{i} \sum_{k=1}^{m_i-1} \mu_r(\alpha_i^k) - \sum_{j} \sum_{k=1}^{n_j-1} \mu_r(\beta_j^k).
\end{equation}

**Proposition 2.10 ([HT3 §3.3] [CHR §2.6]).** The index $J_0$ has the following properties.

1. For orbit sets $\alpha, \beta, \gamma$ with $[\alpha] = [\beta] = [\gamma] = \Gamma \in H_1(Y)$ and $Z \in H_2(Y, \alpha, \beta)$, $Z' \in H_2(Y, \beta, \gamma)$,

\begin{equation}
J_0(\alpha, \beta, Z) + J_0(\beta, \gamma, Z') = J_0(\alpha, \gamma, Z + Z').
\end{equation}

2. For $Z, Z' \in H_2(Y, \alpha, \beta)$,

\begin{equation}
J_0(\alpha, \beta, Z) - J_0(\alpha, \beta, Z') = \langle -c_1(\xi) + 2\text{PD}(\Gamma), Z - Z' \rangle.
\end{equation}

Write $J_0(u) = J_0(\alpha, \beta, [u])$.

**Proposition 2.11 ([H2 Example 6.3]).** Let $\alpha$ and $\beta$ be orbit sets such that all multiplicities of orbits at $\alpha, \beta$ are 1. Suppose that $u = u_0 \cup u_1 \in M^J(\alpha, \beta)$ is embedded. Then

\begin{equation}
J_0(u) = -\chi(u_1)
\end{equation}

In the same way as ECH index, if $b_1(Y) = 0$, we may omit the homology component from the notation of $J_0$ and $J_0$ index is just denoted by $J_0(\alpha, \beta)$ for orbit sets $\alpha, \beta$.

Finally, we mention an obvious but important property. Suppose that $Y \cong S^3$. For two orbit sets $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_i, n_i)\}$, the difference between ECH index and $J_0$ index is

\begin{equation}
I(\alpha, \beta) - J_0(\alpha, \beta) = 2c_1(\xi[\star, \tau]) + \sum_{i} \mu_r(\alpha_i^{m_i}) - \sum_{j} \mu_r(\beta_j^{n_j})
\end{equation}

where $\{\star\} = H_2(S^3, \alpha, \beta)$. Set $\{C_{\alpha_i}^{m_i}\} = H_2(S^3; (\alpha_i, m_i), \emptyset)$ and $\{C_{\beta_j}^{n_j}\} = H_2(S^3; (\beta_j, n_j), \emptyset)$. Then the right hand side can be written by

\begin{equation}
\sum_{i} (2c_1(\xi[C_{\alpha_i}^{m_i}], \tau) + \mu_r(\alpha_i^{m_i})) - \sum_{j} (2c_1(\xi[C_{\beta_j}^{n_j}], \tau) + \mu_r(\beta_j^{n_j}))
\end{equation}
But since the definition of relative Chern number $c_1$ with respect to $\tau$ (see [H1]) implies that $2c_1(\xi|_{\alpha_{\mathcal{M}_i}},\tau) + \mu_\tau(\alpha_{\mathcal{M}_i}) = \mu_{\text{disc}}(\alpha_{\mathcal{M}_i})$ and $2c_1(\xi|_{\beta_{\mathcal{M}_i}},\tau) + \mu_\tau(\beta_{\mathcal{M}_i}) = \mu_{\text{disc}}(\beta_{\mathcal{M}_i})$, we can just write
\[
I(\alpha, \beta) - J_0(\alpha, \beta) = \sum_i \mu_{\text{disc}}(\alpha_{\mathcal{M}_i}) - \sum_j \mu_{\text{disc}}(\beta_{\mathcal{M}_j}).
\]

\section{Fundamental Results}

We prove Proposition 3.6 by contradiction. From now on, we assume that $(S^3, \lambda)$ satisfies the next condition.

\textbf{Condition}

1. All simple orbits of $(S^3, \lambda)$ are negative hyperbolic.
2. The periodic orbit $\gamma_0$ with minimal action satisfies $\mu_{\text{disc}}(\gamma_0) \geq 3$.

From the isomorphism with Seiberg-Witten Floer homology ([KM], [T2]), we have
\[
\text{ECH}(S^3, \lambda, 0) = F[U^{-1}, U] / UF[U]
\]

\textbf{Lemma 3.1.} Suppose that $(S^3, \lambda)$ satisfies the conditions (1) and (2). Then, there is a sequence $\{\delta_i\}_{i=0, 1, 2, \ldots}$ consisting of admissible orbit sets satisfying the following conditions.

1. For any admissible orbit set $\alpha$, $\alpha$ is in $\{\delta_i\}_{i=0, 1, 2, \ldots}$.
2. $A(\delta_i) < A(\delta_j)$ if and only if $i < j$.
3. $I(\delta_i, \delta_j) = 2(i - j)$ for any $i, j$.

\textbf{Proof of Lemma 3.1.} See [S2]. \hfill \Box

\textbf{Lemma 3.2.} Suppose that $(S^3, \lambda)$ satisfies the conditions (1) and (2). Let $\delta_k, \delta_{k+1}$ be admissible orbit sets in Lemma 3.1. Then
\[
4 - \#\delta_k - \#\delta_{k+1} + 2\#(\delta_k \cap \delta_{k+1}) \geq \mu_{\text{disc}}(\delta_{k+1}) - \mu_{\text{disc}}(\delta_k).
\]

\textbf{Proof of Lemma 3.2.} Since $U(\delta_k) = \langle \delta_{k-1} \rangle$, there is a $J$-holomorphic curve $v$ counted by the $U$-map. Let $h(v_1)$ denote the number of punctures of $v_1$ where $v_1$ is the non-trivial part of $v$. Since $v \in \mathcal{M}(\delta_k, \delta_{k-1})$, the set consisting of orbits where the punctures of $v_1$ are asymptotic contains $\delta_k \setminus (\delta_k \cap \delta_{k+1})$ or $\delta_{k+1} \setminus (\delta_k \cap \delta_{k+1})$. This implies that $h(v_1) \geq \#\delta_k + \#\delta_{k+1} - 2\#(\delta_k \cap \delta_{k+1})$. From Proposition 2.11 we have $J_0(v) \geq h(v_1) - 2 \geq \#\delta_k + \#\delta_{k+1} - 2\#(\delta_k \cap \delta_{k+1}) - 2$. By the definition,
\[
2 - J_0(u) = I(\delta_{k+1}, \delta_k) - J_0(\delta_{k+1}, \delta_k) = \mu_{\text{disc}}(\delta_{k+1}) - \mu_{\text{disc}}(\delta_k).
\]

Therefore we have Lemma 3.2. \hfill \Box

By Lemma 3.1, $\emptyset = \delta_0$ and $\gamma_0 = \delta_1$ as orbit sets in the notation of Lemma 3.1
\[
\langle \emptyset = \delta_0 \rangle \xrightarrow{U} \langle \gamma_0 = \delta_1 \rangle \xrightarrow{U} \langle \delta_2 \rangle \xrightarrow{U} \langle \delta_3 \rangle \xrightarrow{U} \langle \delta_4 \rangle \xrightarrow{U}
\]

\textbf{Lemma 3.3.} Under the conditions, $\mu_{\text{disc}}(\gamma_0) = 3$.
Proof of Lemma 3.3 From Lemma 3.2 we have $3 \geq \mu_{\text{disc}}(\gamma_0)$. Since $\mu_{\text{disc}}(\gamma_0) \geq 3$ (the assumption), we have $\mu_{\text{disc}}(\gamma_0) = 3$. □

Lemma 3.4. Under the conditions and notations, let $u \in \mathcal{M}^J(\gamma_0, \emptyset)$ be a $J$-holomorphic curve counted by the $U$-map $U(\gamma_0) = \langle \emptyset \rangle$. Then $\pi(u) \subset S^3$ satisfies the following conditions. Here, $\pi : \mathbb{R} \times S^3 \to S^3$ is the projection.

1. $\pi(u)$ is a global surface of section for the Reeb vector field $X_\lambda$. That is, $\pi(u) \subset S^3$ is an embedded compact surface such that $\pi(u)$ is transversal to $X_\lambda$ and has the property that every orbit of $X_\lambda$ other than the boundary component $\gamma_0$ intersects $\pi(u)$, in forward and in backward time.

2. $d\lambda$ is non-degenerate on $\pi(u)$ and the Poincare section map $\psi : \pi(u) \to \pi(u)$ is symplectic, $\psi^*d\lambda = d\lambda$.

Proof of Lemma 3.4 Since $\gamma_0$ is the minimal periodic orbit, the moduli space $\mathcal{M}^J(\gamma_0, \emptyset)$ is compact. Therefore we can apply [CHP, Proposition 3.2] to this case and so Lemma 3.4 is proved (for more previous proofs, see [HWZ3]). □

For the global surface of section $\pi(u)$ and a Reeb simple orbit $\gamma (\neq \gamma_0)$, define $w(\gamma) = \#(\gamma \cap \pi(u))$ where $\#$ is the algebraic intersection number and $\pi(u)$ is oriented by the complex structure. Here we note that $\#(\gamma \cap \pi(u)) = \#(\mathbb{R} \times \gamma \cap u) > 0$ and the correspondence $[\gamma] \to w(\gamma)$ gives an identification from $H_1(S^3 \setminus \gamma_0)$ to $\mathbb{Z}$. For an orbit set $\alpha = \{(\alpha_i, m_i)\}$, we also define $w(\alpha) = \sum_{\alpha \neq \gamma_0} m_i w(\alpha_i)$.

Lemma 3.5. Let $\alpha$ be an orbit set which does not contain $\gamma_0$. Then

$$(29) \quad I(\alpha \cup \gamma_0, \alpha) = 2 + 2w(\alpha).$$

Proof of Lemma 3.5 Let $u \in \mathcal{M}^J(\gamma_0, \emptyset)$ be a $J$-holomorphic curve counted by the $U$-map $U(\gamma_0) = \langle \emptyset \rangle$ as in Lemma 3.4. Then by the definition, we have

$$(30) \quad I(\alpha \cup \gamma_0, \alpha) = I(\gamma_0) + 2\#(\mathbb{R} \times \alpha \cap u) = 2 + 2w(\alpha).$$

This completes the proof. □

Lemma 3.6. Let $\alpha$ and $\beta$ be orbit sets. Suppose that $\alpha$ and $\beta$ do not contain $\gamma_0$. Then the following statements hold.

1. If $\emptyset \neq \mathcal{M}^J(\alpha, \beta)$, then $w(\alpha) = w(\beta) + \#(\mathbb{R} \times \gamma_0 \cap u)$ for every $v \in \mathcal{M}^J(\alpha, \beta)$. In particular, $w(\alpha) \geq w(\beta)$ and equality holds if and only if $v \cap \mathbb{R} \times \gamma_0 = \emptyset$ for every $v \in \mathcal{M}^J(\alpha, \beta)$.

2. If $\emptyset \neq \mathcal{M}^J(\alpha \cup \gamma_0, \beta)$, $w(\alpha) \geq w(\beta)$.

3. If $\emptyset \neq \mathcal{M}^J(\alpha, \beta \cup \gamma_0)$, $w(\alpha) > w(\beta)$.

Before the proof of Lemma 3.6 we recall a relation between the Conley-Zehnder index and the winding numbers of eigenfunctions of the associated operator.

Take a symplectic $C^1$ path $S : [0, 1] \to \text{Sp}(2)$ with $S(0) = \text{Id}$ and $\det(\text{Id} - S(1)) \neq 0$. For $A_S(t) := -iS(t)S^{-1}(t)$, define a self-adjoint operator

$$(31) \quad L_{A_S} := -i\partial_t - A_S : C^\infty(S^1; \mathbb{C}) \to C^\infty(S^1; \mathbb{C}).$$

Let $\sigma(L_{A_S}) \subset \mathbb{R}$ denote its spectrum. Note that $\sigma(L_{A_S})$ is unbounded below and above, and has no accumulation point. For $\kappa \in \sigma(L_{A_S})$, let $\nu_\kappa(\neq 0)$ denote its
eigenfunction. Because of the uniqueness of solutions of the ordinary differential equation, \( \eta_\kappa \) takes values in \( \mathbb{C}\setminus\{0\} \) and hence, the winding number \( wind(\eta_\kappa) \) is well-defined and independent of the choice of \( \eta_\kappa \).

There is a relation between the Conley-Zehnder index \( \mu(S) \) and the winding numbers of eigenfunctions of \( \sigma(L_{A_S}) \) as follows.

**Proposition 3.7 (HWZ2).** Under the notations above,
1. If \( \kappa \leq \kappa' \), then \( wind(\eta_\kappa) \leq wind(\eta_{\kappa'}) \).
2. Let \( \kappa_{-,\text{max}} := \sup \{ \kappa | \kappa \in \sigma(L_{A_S}) \cap (-\infty, 0) \} \). Then \( \lceil \frac{\mu(S)}{2} \rceil = wind(\eta_{\kappa_{-,\text{max}}}) \).
3. Let \( \kappa_{+,\text{min}} := \inf \{ \kappa | \kappa \in \sigma(L_{A_S}) \cap (0, +\infty) \} \). Then \( \lceil \frac{\mu(S)}{2} \rceil = wind(\eta_{\kappa_{+,\text{min}}}) \).

With this understood, we now prove Lemma 3.6.

**Proof of Lemma 3.6.** At first, for \( v \in M^J(\alpha, \beta) \) and sufficiently large \( s_\ast > 0 \), we consider \( \pi(v \cap [-s_\ast, s_\ast] \times S^3) \subset S^3 \). Since \( \pi(v \cap [-s_\ast, s_\ast] \times S^3) \) is a surface with \( \partial \pi(v \cap [-s_\ast, s_\ast] \times S^3) = \pi(v \cap \{s_\ast\} \times S^3) - \pi(v \cap \{-s_\ast\} \times S^3) \), from some topological arguments, we have

\[
(32) \quad w(\alpha) - w(\beta) = \#(\gamma_0 \cap \pi(v \cap [-s_\ast, s_\ast] \times S^3)).
\]

Here we use the fact that \( \pi(v \cap \{s_\ast\} \times S^3) \) and \( \pi(v \cap \{-s_\ast\} \times S^3) \) are sufficiently close to \( \alpha \) and \( \beta \) respectively. Therefore, we have

\[
(33) \quad w(\alpha) = w(\beta) + \#(\mathbb{R} \times \gamma_0 \cap v)
\]

Taking notice of the positivity of intersection numbers of holomorphic curves in 4-dimension, we obtain the first statement.

Set \( \int_{\gamma_0} \lambda = T_0 \). To prove the second and third statements, choose an orientation preserving identification of a tubular neighborhood of the image of \( \gamma_0 \) in \( S^3 \) with \( S^1 \times D^2 \) such that \( \gamma_0(T_0 t) = \{t\} \times \{0\} \in S^1 \times \{0\} \) and the framing of \( \xi \) induced by a fixed trivialization \( \psi : \xi \to D \times \mathbb{C} \) along \( \gamma_0 \) agree with \( D^2 \). Here, \( \psi \) is a disc map to define \( \mu_{\text{disc}} \) explained just below Theorem 1.4. Note that if a small disc is transversal to \( \gamma_0 \), its tangent plane can be naturally identified with the contact plane \( \xi \) on the intersection point.

Due to [HWZ1], for \( v \in M^J(\alpha \cup \gamma_0, \beta) \), the asymptotic behavior near the puncture asymptotic to \( \gamma_0 \) as \( \to +\infty \) can be described by

\[
(34) \quad v_+: \{0, +\infty\} \times S^1 \to \mathbb{R} \times S^1 \times D^2
\]

where \( v_+(s, t) = (s, t, \eta_+(s, t)) \).

In the same way, for \( v \in M^J(\alpha, \beta \cup \gamma_0) \), the asymptotic behavior near the puncture asymptotic to \( \gamma_0 \) as \( \to -\infty \) can be described by

\[
(35) \quad v_-: (-\infty, 0] \times S^1 \to \mathbb{R} \times S^1 \times D^2
\]

where \( v_-(s, t) = (s, t, \eta_-(s, t)) \).

Furthermore, \( \eta_{\pm} \) can be described by

\[
(36) \quad \eta_{\pm}(s, t) = e^{\kappa_\pm s} \eta_{\kappa_{\pm}}(t) + O(e^{(\kappa_\pm + \kappa_\mp) s})
\]
where \( \kappa_+ > 0 > \kappa_- \) and \( \epsilon_+ > 0 > \epsilon_- \). Moreover, \( \eta_{\kappa_+} : S^1 \to D^2 \) are eigenfunctions of the asymptotic operator associated to the trivialization \( \psi \) along \( \gamma_0 \) with eigenvalues \( \kappa_+ \) (see just below Theorem 3.4). Furthermore since \( \mu_{\text{disc}}(\gamma_0) = 3 \), from Proposition 3.7 we have the winding number of \( \eta_{\kappa_-} \) is less than or equal to 1 and the one of \( \eta_{\kappa_+} \) is more than or equal to 2.

On the other hand, \( \mu_{\text{disc}}(\gamma_0) = 3 \) implies that in the neighborhood identification, the projection to \( D^2 \) of the Reeb flow near \( \gamma_0 \) rotate approximately by \( \tfrac{\pi}{2} \) while it goes once around \( \gamma_0 \). Moreover the winding number of the dominating eigenfunction of \( u \in \mathcal{M}^I(\gamma_0, \emptyset) \) asymptotic to \( \gamma_0 \) is exactly 1 (see [CHP] Proof of Proposition 3.2, Lemma 3.4 and Lemma 3.5) or [HWZ3]). This means that for a sufficiently small eigenfunction \( \eta : S^1 \to D^2 \) of the asymptotic operator, \((t, \eta(t))\) intersect with \( \pi(u) \) to the opposite direction with \( X_\lambda \) or does not intersects with \( \pi(u) \) if the winding number of \( \eta \) is smaller than or equal to 1, and to the same direction with \( X_\lambda \) if the winding number of \( \eta \) is larger than 1.

Now, for \( v \in \mathcal{M}^I(\alpha \cup \gamma_0, \beta) \) and sufficiently large \( s_* > 0 \), we consider \( \pi(v \cap [-s_*, s_*] \times S^3) \subset S^3 \). In the same way as the proof of the first statement, from some topological arguments, we have

\[
(37) \quad w(\alpha) + (\text{wind}(\eta_{\kappa_-}) - 1) - w(\beta) = \#(\gamma_0 \cap \pi(v \cap [-s_*, s_*] \times S^3)) = \#(\mathbb{R} \times \gamma_0 \cap v).
\]

Since \( \text{wind}(\eta_{\kappa_-}) \leq 1 \), we have the second statement. Also for \( v \in \mathcal{M}^I(\alpha, \beta \cup \gamma_0) \) and sufficiently large \( s_* > 0 \), we have

\[
(38) \quad w(\alpha) - (w(\beta) + \text{wind}(\eta_{\kappa_+}) - 1) = \#(\gamma_0 \cap \pi(v \cap [-s_*, s_*] \times S^3)) = \#(\mathbb{R} \times \gamma_0 \cap v).
\]

Since \( \text{wind}(\eta_{\kappa_+}) \geq 2 \), we have the third statement and we complete the proof of Lemma 3.6. \( \square \)

**Lemma 3.8.** Let \( \delta_i, \delta_j \) be admissible orbit sets in the notation of Lemma 3.6 with \( i < j \). Suppose that \( \delta_i \) and \( \delta_j \) do not contain \( \gamma_0 \).

1. If there is \( \delta_{i'} \) with \( i < i' < j \) such that \( \delta_{i'} \) contains \( \gamma_0 \), then \( w(\delta_i) < w(\delta_j) \).

2. If all \( \delta_{i'} \) with \( i < i' < j \) do not contain \( \gamma_0 \), then \( w(\delta_i) = w(\delta_j) \).

**Proof of Lemma 3.8** For every \( k > 0 \), since \( U(\delta_k) = (\delta_{k-1}) \), we have \( \emptyset \neq \mathcal{M}^I(\delta_k, \delta_{k-1}) \). Since Lemma 3.6 this implies that \( w(\delta_k) \) is monotonically increasing with respect to \( k \).

1. If there is \( \delta_{i'} \) with \( i < i' < j \) such that \( \delta_{i'} \) contains \( \gamma_0 \).

   In this case, there is \( k \) with \( j' \leq k < j \) such that \( \delta_k \) contains \( \gamma_0 \) and \( \delta_{k+1} \) does not. Hence from Lemma 3.6 we have \( w(\delta_k) < w(\delta_{k+1}) \). By the monotonicity, we have \( w(\delta_i) < w(\delta_j) \).

2. If all \( \delta_{i'} \) with \( i < i' < j \) do not contain \( \gamma_0 \).

   In this case, we have \( A(\delta_i) < A(\delta_j) < A(\delta_i \cup \gamma_0) \) and so by the monotonicity, we have \( w(\delta_i) \leq w(\delta_j) \leq w(\delta_i) \). This implies that \( w(\delta_i) = w(\delta_j) \).

   Hence we complete the proof of Lemma 3.8. \( \square \)

**Lemma 3.9.** Let \( \delta_k, \delta_{k+1} \) with \( k > 0 \) be admissible orbit sets in the notation of Lemma 3.7. Let \( \gamma \) be a simple orbit with \( A(\gamma) > A(\delta_{k+1}) \). Then

\[
(39) \quad I(\delta_{k+1} \cup \gamma, \delta_k \cup \gamma) > 2.
\]
Proof of Lemma 3.9. Note that $\gamma \notin \delta_{k+1},\delta_k$. From the assumption, for generic $z \in Y$, there is a $J$-holomorphic curve $v^z \in \mathcal{M}t(\delta_{k+1},\delta_k)$ through $(0, z)$. By the definition of ECH index, we have

$$I(\delta_{k+1} \cup \gamma, \delta_k \cup \gamma) = I(\delta_{k+1},\delta_k) + 2#(\mathbb{R} \times \gamma \cap v^z) = 2 + 2#(\mathbb{R} \times \gamma \cap v^z).$$

Note that $#(\mathbb{R} \times \gamma \cap v^z) \geq 0$ because of positivity of intersection number of holomorphic curves in four dimension.

Suppose that $#(\mathbb{R} \times \gamma \cap v^z) = 0$. Then $\mathbb{R} \times \gamma \cap v^z = \emptyset$. Consider a sequence of $J$-holomorphic curves of $v^z$ as $z \to \gamma$. By the compactness argument (for the case of ECH, see [H1]), there is a subsequence which converges to some $v^\infty \in \mathcal{M}t(\delta_{k+1},\delta_k)$ where $v^\infty$ may split into some floors. Suppose that $v^\infty$ does not split, that is, $v^\infty \in \mathcal{M}t(\delta_{k+1},\delta_k)$. Since $v^\infty$ is a limit curve obtained from $v^z$ as $z \to \gamma$, we have $v^\infty \cap \mathbb{R} \times \gamma \neq \emptyset$. By the definition, $I(v^\infty \cup \mathbb{R} \times \gamma) = 2$, but this contradicts Proposition 2.6. So $v^\infty$ has to split. Let $v^\infty_i$ denote the holomorphic curve in the top floor of $v^\infty$. By the construction, $v^\infty_i$ must have a negative puncture asymptotic to $\gamma$. Moreover, the positive punctures of $v^\infty_i$ are asymptotic to $A(\delta_{k+1})$. Therefore, the integral value of $v^\infty_i$ over $d\lambda$ is smaller than $A(\delta_{k+1}) - A(\gamma)$. This contradicts the positivity $v^\infty_i|_{d\lambda} > 0$. So $#(\mathbb{R} \times \gamma \cap v^z) > 0$ and this complete the proof. □

4. Proof of the main result

Let $\{\gamma_i\}_{i \in \mathbb{Z}_{\geq 0}}$ be a indexed set consisting of all simple orbits in $(S^3, \lambda)$ such that if $i < j$, $A(\gamma_i) < A(\gamma_j)$. Note that since Lemma 3.1 if $i < j$, then $I(\gamma_i) < I(\gamma_j)$.

Lemma 4.1. In the notation of Lemma 3.9 and above,

1. $\delta_2 = \gamma_1$ and $w(\gamma_1) = 1$
2. $\delta_3 = \gamma_2$ and $w(\gamma_2) = 1$
3. $\delta_4 = \gamma_0 \cup \gamma_1$
4. $\delta_5 = \gamma_0 \cup \gamma_2$
5. $\delta_6 = \gamma_3$ and $w(\gamma_3) = 2$
6. $\delta_9 = \gamma_0 \cup \gamma_3$

Proof of Lemma 4.1. The admissibility of orbit sets implies that $\gamma_1 = \delta_2$. Because of the existence of a periodic orbit $\gamma$ with $w(\gamma) = 1$ (this can be proved by the Brouwer’s translation theorem. See [HWZ3]) and the monotonicity of $w$ (Lemma 3.9), we have $w(\gamma_1) = 1$. From Lemma 3.9 we have $I(\gamma_0 \cup \gamma_1,\gamma_1) = 4$ and hence $\gamma_0 \cup \gamma_1 = \delta_4$. By the admissibility and the monotonicity, $\gamma_2 = \delta_3$ and $w(\gamma_2) = 1$, also we have $\gamma_0 \cup \gamma_2 = \delta_5$. Taking notice of $A(\gamma_2) > A(\gamma_1)$, from Lemma 3.9 we have $\gamma_0 \cup \gamma_2 \neq \delta_6$ and hence $\gamma_3 = \delta_6$. Moreover, the monotonicity $1 = w(\gamma_3) < w(\gamma_5) \leq w(\gamma_1 \cup \gamma_2) = 2$ implies that $w(\gamma_3) = 2$ and so $\delta_9 = \gamma_0 \cup \gamma_3$. This completes the proof. □
Lemma 4.2. In the notation above,
\[ \mu_{\text{disc}}(\gamma_1) = 5, \quad \mu_{\text{disc}}(\gamma_2) = 7 \]
and moreover
\[ A(\gamma_1) < 2A(\gamma_0) < A(\gamma_2). \]

Proof of Lemma 4.2. We set \( \mu_{\text{disc}}(\gamma_1) = n \). Since \( \mu_{\text{disc}}(\gamma_0) = 3 \), we have \( \mu_{\text{disc}}(\gamma_0 \cup \gamma_1) = n + 3 \). By (20), we have
\[ 2 \geq \mu_{\text{disc}}(\gamma_2) - \mu_{\text{disc}}(\gamma_1) = \mu_{\text{disc}}(\gamma_2) - n. \]
By (20), we also have
\[ 1 \geq \mu_{\text{disc}}(\gamma_0 \cup \gamma_1) - \mu_{\text{disc}}(\gamma_2) = n + 3 - \mu_{\text{disc}}(\gamma_2). \]
Therefore we have \( \mu_{\text{disc}}(\gamma_2) = n + 2 \).

Since \( U(\gamma_2) = \langle \gamma_1 \rangle \) and the \( U \)-map is defined for generic \( z \in Y \), there is a \( v^z \in \mathcal{M}^I(\gamma_2, \gamma_1) \) through generic \( (0, z) \in \mathbb{R} \times Y \). Moreover,
\[ 0 = I(\gamma_2, \gamma_1) - (\mu_{\text{disc}}(\gamma_2) - \mu_{\text{disc}}(\gamma_1)) = J_0(\gamma_2, \gamma_1) = J_0(v^z). \]
This implies that \( v^z \) is of genus 0 (see Proposition 2.11). On the other hand, since \( w(\gamma_1) = w(\gamma_2) = 1 \) and Theorem 5.0 we have \( \mathbb{R} \times \gamma_0 \cap v^z = \emptyset \) and so \( I(\mathbb{R} \times \gamma_0 \cup v^z) = 2 \).

Consider a sequence of holomorphic curves \( v^z \) as \( z \to \gamma_0 \). By the compactness argument, the sequence of \( v^z \) converges to some \( v^\infty \in \mathcal{M}^I(\gamma_2, \gamma_1) \) which may split into some floors up to subsequence. Suppose that \( v^\infty \) does not split, that is, \( v^\infty \in \mathcal{M}^I(\gamma_2, \gamma_1) \). Since \( v^\infty \) is a limit curve obtained from \( v^z \) as \( z \to \gamma_0 \), we have \( v^\infty \cap \mathbb{R} \times \gamma_0 \neq \emptyset \). By the definition, \( I(\mathbb{R} \times \gamma_0 \cup v^\infty) = 2 \), but this contradicts Proposition 2.6. So \( v^\infty \) splits and thus is in \( \mathcal{M}^I(\gamma_2, \gamma_1) \setminus \mathcal{M}^I(\gamma_2, \gamma_1) \).

Let \( v^\infty_+ \) and \( v^\infty_- \) be the holomorphic curves in top and bottom floors of \( v^\infty \) respectively. By additivity of ECH index, we have \( I(\mathbb{R} \times \gamma_0 \cup v^\infty) = I(\mathbb{R} \times \gamma_0 \cup v^\infty) = 1 \) and so \( \text{ind}(v^\infty) = \text{ind}(v^\infty) = 1 \), \( \mathbb{R} \times \gamma_0 \cap v^\infty = \emptyset \), \( \mathbb{R} \times \gamma_0 \cap v^\infty = \emptyset \). Furthermore, since each of \( v^z \) is of genus 0 with two punctures, both of \( v^\infty \) are also the same. By the construction, the negative end of \( v^\infty_+ \) is asymptotic to \( \gamma_0 \) and also the positive end of \( v^\infty_- \) is the same (see the below picture). Furthermore, \( v^\infty \) splits into only two components \( v^\infty_\pm \). By the partition conditions (Proposition 2.7), both multiplicities are two. This implies that \( A(\gamma_1) < 2A(\gamma_0) < A(\gamma_2) \) and so we have (43).

Next, see (9). Since \( \mu_{\text{disc}}(\gamma_0) = 2\mu_{\text{disc}}(\gamma_0) = 6 \) (see [H1, Proposition 2.1]), we have
\[ 1 = \text{ind}(v^\infty) = \mu_{\text{disc}}(\gamma_2) - 6 = n - 4 \]
This implies that \( n = 5 \). Therefore, we have \( \mu_{\text{disc}}(\gamma_2) = 7 \) and \( \mu_{\text{disc}}(\gamma_1) = 5 \). This completes the proof of lemma.
Lemma 4.3. In the notation, there is no simple orbit $\gamma_i$ satisfying the following conditions at the same time.

1. $i > 2$
2. $I(\gamma_{i+1}, \gamma_i) = 2$
3. $\mu_{\text{disc}}(\gamma_{i+1}) - \mu_{\text{disc}}(\gamma_i) = 2$.

Proof of Lemma 4.3 we prove this by contradiction. Since $I(\gamma_{i+1}, \gamma_i) = J_0(\gamma_{i+1}, \gamma_i) = \mu_{\text{disc}}(\gamma_{i+1}) - \mu_{\text{disc}}(\gamma_i) = 2$, we have $J_0(\gamma_{i+1}, \gamma_i) = 0$. This implies that all $J$-holomorphic curves counted by $U(\gamma_{i+1}) = \langle \gamma_i \rangle$ are of genus 0. Furthermore, from Lemma 3.8, we have $w(\gamma_i) = w(\gamma_{i+1})$ and hence all holomorphic curves in $\mathcal{M}^4(\gamma_{i+1}, \gamma_i)$ do not intersect with $\mathbb{R} \times \gamma_0$. These mean that there is a sequence of holomorphic curves through generic $(0, z) \in \mathbb{R} \times S^3$ as $z \to \gamma_0$ which do not intersect with $\mathbb{R} \times \gamma_0$ and are of genus 0. In the same way as Lemma 4.2, we have $A(\gamma_i) < 2A(\gamma_0) < A(\gamma_{i+1})$, but this contradicts $A(\gamma_1) < 2A(\gamma_0) < A(\gamma_2)$ and $A(\gamma_2) < A(\gamma_i)$. This completes the proof.

Considering Lemma 3.8 with Lemma 4.1, we have $w(\delta_7) = w(\delta_8) = 2$ and moreover since $w(\gamma_1 \cup \gamma_2) = 2$, we can find that the pair $(\delta_7, \delta_8)$ is either $(\gamma_1 \cup \gamma_2, \gamma_4)$ or $(\gamma_4, \gamma_1 \cup \gamma_2)$. Here we note that $w(\gamma_4) = 2$.

(48) \[
\langle \gamma_0 \cup \gamma_2 \rangle \overrightarrow{\mu} \langle \gamma_3 \rangle \overrightarrow{\mu} \langle \delta_7 \rangle \overrightarrow{\mu} \langle \delta_8 \rangle \overrightarrow{\mu} \langle \gamma_0 \cup \gamma_3 \rangle
\]

Lemma 4.4. Under the assumption,
1. If $(\delta_7, \delta_8) = (\gamma_1 \cup \gamma_2, \gamma_4)$, then $\mu_{\text{disc}}(\gamma_3) = 11$ and $\mu_{\text{disc}}(\gamma_4) = 13$.
2. If $(\delta_7, \delta_8) = (\gamma_4, \gamma_1 \cup \gamma_2)$, then $\mu_{\text{disc}}(\gamma_3) = \mu_{\text{disc}}(\gamma_4) = 11$.

Proof of Lemma 4.4.
1. If $(\delta_7, \delta_8) = (\gamma_1 \cup \gamma_2, \gamma_4)$.

From (29) and Lemma 4.2 we have

\[ 1 \geq \mu_{\text{disc}}(\gamma_1 \cup \gamma_2) - \mu_{\text{disc}}(\gamma_3) = 12 - \mu_{\text{disc}}(\gamma_3) \]

and also

\[ 1 \geq \mu_{\text{disc}}(\gamma_3) - \mu_{\text{disc}}(\gamma_0 \cup \gamma_2) = \mu_{\text{disc}}(\gamma_3) - 10. \]

These inequalities imply that $\mu_{\text{disc}}(\gamma_3) = 11$. In the same way, we have $\mu_{\text{disc}}(\gamma_4) = 13$.

2. If $(\delta_7, \delta_8) = (\gamma_4, \gamma_1 \cup \gamma_2)$.

From (26) and Lemma 4.2 we have

\[ 1 \geq \mu_{\text{disc}}(\gamma_1 \cup \gamma_2) - \mu_{\text{disc}}(\gamma_4) = 12 - \mu_{\text{disc}}(\gamma_4) \]

and also

\[ 1 \geq \mu_{\text{disc}}(\gamma_3) - \mu_{\text{disc}}(\gamma_0 \cup \gamma_2) = \mu_{\text{disc}}(\gamma_3) - 10. \]

Furthermore from (26), we have

\[ 2 \geq \mu_{\text{disc}}(\gamma_4) - \mu_{\text{disc}}(\gamma_3). \]

From these inequalities, we can see that the pair $(\mu_{\text{disc}}(\gamma_3), \mu_{\text{disc}}(\gamma_4))$ has three possibilities, $(\mu_{\text{disc}}(\gamma_3), \mu_{\text{disc}}(\gamma_4)) = (9, 11), (11, 11), (11, 13)$. But we can exclude the cases $(\mu_{\text{disc}}(\gamma_3), \mu_{\text{disc}}(\gamma_4)) = (9, 11), (11, 13)$ by Lemma 4.3. This completes the proof.

\[ \square \]

If $(\delta_7, \delta_8) = (\gamma_1 \cup \gamma_2, \gamma_4)$, the diagram is the below.

\[ \begin{array}{c}
\gamma_3 \quad \gamma_1 \cup \gamma_2 \quad \gamma_4 \quad \gamma_0 \cup \gamma_3 \\
\gamma_0 \cup \gamma_1 \cup \gamma_2 \quad \gamma_0 \cup \gamma_4 \quad \delta_{12} \quad \delta_{13} \quad \delta_{14} \\
\delta_{15} \quad \delta_{16} \quad \delta_{17} \\
\end{array} \]

If $(\delta_7, \delta_8) = (\gamma_4, \gamma_1 \cup \gamma_2)$, the diagram is the below.

\[ \begin{array}{c}
\gamma_3 \quad \gamma_4 \quad \gamma_0 \cup \gamma_3 \\
\gamma_0 \cup \gamma_4 \quad \gamma_0 \cup \gamma_1 \cup \gamma_2 \quad \delta_{12} \quad \delta_{13} \quad \delta_{14} \\
\delta_{15} \quad \delta_{16} \quad \delta_{17} \\
\end{array} \]

**Lemma 4.5.** In any cases,

\[ (\delta_{12}, \delta_{13}, \delta_{14}, \delta_{15}) = (\gamma_1 \cup \gamma_3, \gamma_1 \cup \gamma_4, \gamma_2 \cup \gamma_3, \gamma_2 \cup \gamma_4) \]
Proof of Lemma 4.5. Note that
\[(57) \quad w(\gamma_1 \cup \gamma_3) = w(\gamma_2 \cup \gamma_3) = w(\gamma_1 \cup \gamma_4) = w(\gamma_2 \cup \gamma_4) = 3.\]
From the discussion so far, we can see that \(\delta_{12}\) does not contain \(\gamma_0\) and \(w(\delta_{12}) = 3.\)
From Lemma 3.5, we have \(I(\gamma_0 \cup \delta_{12}, \delta_{12}) = 2w(\delta_{12}) + 2 = 8\) and so \(\delta_{16} = \gamma_0 \cup \delta_{12}.\)
These imply that \((\delta_{12}, \delta_{13}, \delta_{14}, \delta_{15})\) is correspond to \((\gamma_1 \cup \gamma_3, \gamma_1 \cup \gamma_4, \gamma_2 \cup \gamma_3, \gamma_2 \cup \gamma_4)\) as a set. Since \(A(\delta_{12}) < A(\delta_{13}) < A(\delta_{14}) < A(\delta_{15})\) (see Lemma 3.4), we have \(\delta_{12} = \gamma_1 \cup \gamma_3\) and \(\delta_{15} = \gamma_2 \cup \gamma_4.\)
Now, two possibilities \((\delta_{13}, \delta_{14}) = (\gamma_2 \cup \gamma_3, \gamma_1 \cup \gamma_4),\)
\((\gamma_1 \cup \gamma_4, \gamma_2 \cup \gamma_3)\) still remain. Suppose that \((\delta_{13}, \delta_{14}) = (\gamma_2 \cup \gamma_3, \gamma_1 \cup \gamma_4).\)
Then, \(I(\delta_{13}, \delta_{12}) = I(\gamma_2 \cup \gamma_3, \gamma_1 \cup \gamma_3) = 2.\)
But \(I(\gamma_2, \gamma_1) = 2\) and \(A(\gamma_3) > A(\gamma_2)\) contradict Lemma 3.9. So \((\delta_{13}, \delta_{14}) = (\gamma_1 \cup \gamma_4, \gamma_2 \cup \gamma_3)\) and we complete the proof. \(\Box\)

Proof of Theorem 1.6.
1. If \((\delta_7, \delta_8) = (\gamma_1 \cup \gamma_2, \gamma_4).\)
   Since \(\mu_{\text{disc}}(\gamma_3) = 11\) and \(\mu_{\text{disc}}(\gamma_4) = 13\) (Lemma 4.4), we have
   \[(58) \quad I(\delta_{13}, \delta_{12}) - J(\delta_{13}, \delta_{12}) = \mu_{\text{disc}}(\gamma_1 \cup \gamma_4) - \mu_{\text{disc}}(\gamma_1 \cup \gamma_3) = 2\]
   and so \(J(\gamma_1 \cup \gamma_4, \gamma_1 \cup \gamma_3) = 0.\) This means that non trivial parts of \(J\)-holomorphic curves counted by the \(U\)-map \(U(\gamma_1 \cup \gamma_4) = (\gamma_1 \cup \gamma_3)\) are of genus 0 with two punctures and hence in \(\mathcal{M}^J(\gamma_1, \gamma_3)\). Moreover they does not intersect with \(\mathbb{R} \times \gamma_1.\)
   This implies that \(I(\gamma_4, \gamma_3) = 2.\) But since \(I(\gamma_4, \gamma_3) = I(\delta_8, \delta_6) = 4,\) this is a contradiction.
2. If \((\delta_7, \delta_8) = (\gamma_4, \gamma_1 \cup \gamma_2).\)
   Since \(\mu_{\text{disc}}(\gamma_3) = 11\) and \(\mu_{\text{disc}}(\gamma_4) = 11\) (Lemma 4.4), we have
   \[(59) \quad \mu_{\text{disc}}(\delta_{14}) - \mu_{\text{disc}}(\delta_{13}) = \mu_{\text{disc}}(\gamma_2 \cup \gamma_3) - \mu_{\text{disc}}(\gamma_1 \cup \gamma_4) = 2\]
   This contradicts Lemma 3.2.
   In any cases, contradiction arises. Combining the arguments so far, we complete the proof of the main theorem. \(\Box\)

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