THE GLOBAL WELL-POSEDNESS OF THE RELATIVISTIC BOLTZMANN EQUATION WITH DIFFUSE REFLECTION BOUNDARY CONDITION IN BOUNDED DOMAINS

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Abstract. The relativistic Boltzmann equation in bounded domains has been widely used in physics and engineering, for example, Tokamak devices in fusion reactors. In spite of its importance, there has, to the best of our knowledge, been no mathematical theory on the global existence of solutions to the relativistic Boltzmann equation in bounded domains. In the present paper, assuming that the motion of single-species relativistic particles in a bounded domain is governed by the relativistic Boltzmann equation with diffuse reflection boundary conditions of non-isothermal wall temperature of small variations around a positive constant, and regarding the speed of light \( c \) as a large parameter, we first construct a unique non-negative stationary solution \( F^* \), and further establish the dynamical stability of such stationary solution with exponential time decay rate. We point out that the \( L^\infty \)-bound of perturbations for both steady and non-steady solutions are independent of the speed of light \( c \), and such uniform in \( c \) estimates will be useful in the study of Newtonian limit in the future.

Contents

1. Introduction 1
2. Preliminaries 10
3. Uniform in \( c \) estimates for collision operators 12
4. Steady problem 29
5. Dynamical stability under small perturbations 44
6. Local-in-time existence 57
References 59

1. Introduction

1.1. The relativistic Boltzmann equation. We consider the relativistic Boltzmann equation

\[ p^\mu \partial_\mu F = \mathcal{C}(F, F), \tag{1.1} \]

which describes the dynamics of single-species relativistic particles. The unknown \( F(t, x, p) \geq 0 \) is a distribution function for relativistic particles with position \( x = (x_1, x_2, x_3) \in \Omega \) and particle momentum \( p = (p^1, p^2, p^3) \in \mathbb{R}^3 \) at time \( t > 0 \). The collision term \( \mathcal{C}(F, G) \) is defined by

\[ \mathcal{C}(F, G) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{dq}{q^0} \frac{dp}{p^0} \frac{dq'}{q'^0} W(p, q \mid p', q') [F(p') G(q') - F(p) G(q)], \]

where the transition rate \( W(p, q \mid p', q') \) has the form

\[ W(p, q \mid p', q') = s \sigma(g, \theta) \delta(p^0 + q^0 - p'^0 - q'^0) \delta(3)(p + q - p' - q'). \tag{1.2} \]
The streaming term of the relativistic Boltzmann equation (1.1) is given by
\[ p^\mu \partial_\mu = \frac{p^0}{c} \partial_t + p \cdot \nabla_x, \]
where \( c \) denotes the speed of light and \( p^0 \) denotes the energy of a relativistic particle with
\[ p^0 = \sqrt{m^2 c^2 + |p|^2}. \]
Here \( m \) denotes the rest mass of particle. Now we can rewrite (1.1) as
\[ \partial_t F + \hat{p} \cdot \nabla_x F = Q(F, F), \tag{1.3} \]
where \( \hat{p} \) denotes the normalized particle velocity
\[ \hat{p} := \frac{c p}{p^0 \sqrt{2 \Lambda}}. \]
The collision term \( Q(F, G) \) in (1.3) has the form
\[ Q(F, G) = \frac{c}{2 p^0} \int \frac{dq}{q^0} \int \frac{dp'}{p'^0} \int \frac{dq'}{q'^0} W(p, q | p', q') [F(p') G(q') - F(p) G(q)]. \]

We denote the energy-momentum 4-vector as \( p^\mu = (p^0, p^1, p^2, p^3) \). The energy-momentum 4-vector with the lower index is written as a product in the Minkowski metric
\[ p^\mu = g^\mu_\nu p^\nu, \]
where \( g_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \). The inner product of energy-momentum 4-vectors \( p^\mu \) and \( q^\mu \) is defined via the Minkowski metric
\[ p^\mu q^\mu = -p^0 q^0 + \sum_{i=1}^3 p^i q^i. \]
Then it is clear that
\[ p^\mu p_\mu = -m^2 c^2. \]
We note that the inner product of energy-momentum 4-vectors is Lorentz invariant \( p^\mu q_\mu = \Lambda p^\mu \Lambda q_\mu \), where \( \Lambda \) is any Lorentz transformation.

The function \( \sigma(g, \theta) \) in (1.2) is called the differential cross-section or scattering kernel. The quantity \( s \) is the square of the energy in the center of momentum system, \( p + q = 0 \), and is given as
\[ s = s(p, q) = (p^\mu + q^\mu) (p_\mu + q_\mu) = 2 \left(p^0 q^0 - p \cdot q + m^2 c^2\right) \geq 4m^2 c^2. \]
And the relative momentum \( g \) in (1.2) is denoted as
\[ g = g(p, q) = \sqrt{(p^\mu - q^\mu) (p_\mu - q_\mu)} = \sqrt{2 (p^0 q^0 - p \cdot q + m^2 c^2)} \geq 0. \]
It is noted that
\[ s = g^2 + 4m^2 c^2. \]
The post-collision momentum pair \( (p'^\mu, q'^\mu) \) and the pre-collision momentum pair \( (p^\mu, q^\mu) \) satisfy the relation
\[ p^\mu + q^\mu = p'^\mu + q'^\mu. \tag{1.4} \]
One may also write (1.4) as
\[ p^0 + q^0 = p'^0 + q'^0, \tag{1.5} \]
\[ p + q = p' + q', \tag{1.6} \]
where (1.5) represents the principle of conservation of energy and (1.6) represents the conservation of momentum after a binary collision.
Using Lorentz transformations as described in [31, 57], in the center of momentum system, \( Q(F, F) \) can be written as
\[
Q(F, F) = \int_{\mathbb{R}^3} \int_{S^2} v_\phi \sigma(g, \theta) \left[ F(p') F(q') - F(p) F(q) \right] \, d\omega dq
\]
where \( v_\phi = v_\phi(p, q) \) is the Møller velocity
\[
v_\phi(p, q) := \frac{\zeta}{2} \sqrt{\frac{p - q}{p_0 - q_0}^2 - \frac{p \times q}{p_0 q_0}} = \frac{\zeta \sqrt{s}}{4 p_0 q_0}.
\]
The pre-post collisional momentum in (1.7) satisfies
\[
\begin{align*}
p' &= \frac{1}{2} (p + q) + \frac{1}{2} g \left( \omega + (\gamma_0 - 1)(p + q) \frac{(p + q) \cdot \omega}{|p + q|^2} \right), \\
q' &= \frac{1}{2} (p + q) - \frac{1}{2} g \left( \omega + (\gamma_0 - 1)(p + q) \frac{(p + q) \cdot \omega}{|p + q|^2} \right),
\end{align*}
\]
where \( \gamma_0 := (p_0 + q_0)/\sqrt{s} \). The pre-post collisional energy is given by
\[
\begin{align*}
p'^0 &= \frac{1}{2} (p^0 + q^0) + \frac{1}{2} g \frac{(p + q) \cdot \omega}{\sqrt{s}}, \\
q'^0 &= \frac{1}{2} (p^0 + q^0) - \frac{1}{2} g \frac{(p + q) \cdot \omega}{\sqrt{s}}.
\end{align*}
\]
The scattering angle \( \theta \) is defined by
\[
\cos \theta := \frac{(p' \mu - q' \mu)(p' \nu - q' \nu)}{g^2}.
\]
The angle is well defined under (1.4) and we refer to [26, Lemma 3.15.3].

1.2. A brief history of relativistic Boltzmann equation. The mathematical studies on the relativistic Boltzmann equation can date back to Lichnerowicz and Marrot [52] in 1940, where they derived the full relativistic Boltzmann equation including collisional effects. The local-in-time solution was established by Bichteler [6] for the general relativistic Boltzmann equation when the initial distribution function decays exponentially with the energy and the differential cross-section is bounded. Dudyński and Ekiel-Ježewska [24] proved the existence of global-in-time DiPerna-Lions renormalized solutions [16] of large initial data, and the causality of the relativistic Boltzmann equation is studied in [23]. We refer to [21, 22] for the study of linearized relativistic Boltzmann equation, see also [47, 48].

In 1991, Glassey and Strauss [27] obtained some properties on the collision map that carries the pre-collisional momentum of a pair of colliding particles into their momentum post-collision. Later, for the hard potential case, they [28] established the existence and uniqueness of smooth solutions in a periodic box when the initial data is close to a relativistic Maxwellian. The exponential time decay rate is also obtained. And the result was extended to the whole space [29].

In 2006, Hsiao and Yu [43] relaxed the restrictions on the cross section of [28] in the case of hard potentials. Strain [59] established the global existence and asymptotic behavior in torus for soft potential around relativistic Maxwellsians. We also refer to [60] for the case in whole space. The very interesting paper [39] is on the frontier of this topic. Duan and Yu [20] showed the existence of global-in-time solutions in the weighted \( L^\infty \) framework in the periodic box. The sub-exponential time decay rate is also obtained. Wang [64] showed the global well-posedness of the relativistic Boltzmann equation for initial data with bounded \( L^\infty \)-norm and small relative entropy. Recently, by nonlinear energy method, Bae, Jang and Yun [4] established the global well-posedness of relativistic quantum Boltzmann equation in periodic box when the initial distribution is nearby a global equilibrium. For other interesting works, we refer to [36] for the
relativistic Vlasov-Maxwell-Boltzmann equations, \[13\] for the deterministic problem on relativistic Boltzmann equation, and \[46\] for the non-cutoff relativistic Boltzmann equations and the references therein.

We would like to mention some results on the Newtonian Boltzmann equation. Under a uniform bound assumption in a strong Sobolev space, Desvillettes and Villani \[14\] obtained an almost exponential decay rate of large amplitude solutions to the global Maxwellian. The result has been recently improved by Gualdani, Mischler, and Mouhot \[32\] to a sharp exponential time decay rate. On the other hand, there are many studies on the global existence of small perturbation solutions to the Boltzmann equation, for instance, \[37\] \[53\] by using the energy method, \[35\] \[38\] \[62\] by using the \(L^2 \cap L^\infty\) approach, and \[11\] \[30\] for the non-cutoff Boltzmann equation. Recently, Duan et al \[17\] developed a \(L^\infty_x L^1_v \cap L^\infty_{x,v}\) approach and proved the global existence and uniqueness of mild solutions to the Boltzmann equation in the whole space and for a class of initial data with bounded velocity-weighted \(L^\infty\)-norm under some smallness conditions on \(L^1_x L^\infty_v\)-norm as well as defect mass, energy, and entropy. For other interesting results, see \[7\] \[39\] \[40\] \[41\] \[56\] \[61\] and the references therein.

In many important physical applications, boundaries occur naturally, and the interaction between particles and the boundary plays a crucial role both from physical and mathematical view points. A first investigation of the initial boundary value problem (IBVP) of Newtonian Boltzmann equation was made by Hamdache \[44\] in the sense of DiPerna-Lions solution \[16\] for large-data, we also refer to \[2\] \[3\] \[12\] \[55\] for extensions of such result in several directions including the case of general diffuse reflection with variable wall temperature, and \[4\] \[14\] \[15\] for the large-time behavior of weak solutions. In the perturbation framework, with the help of Vidav’s iteration \[63\], Guo \[35\] developed a new unified \(L^2 - L^\infty\) approach to treat the global existence, uniqueness and continuity of bounded solutions for four basic types of boundary conditions in rather general domains when the initial perturbation is small in \(L^\infty\). Recently, an important progress was made by Guo et al \[42\] where they established the high order Sobolev regularity of Boltzmann solutions in smooth convex domains, see also \[49\] for the formation and propagation of discontinuity of Boltzmann equation in non-convex domains. For other related works on IBVP, we would like to refer to \[7\] for Maxwell boundary condition, \[43\] \[50\] \[54\] for the global existence of solutions with weakly inhomogeneous data in the case of specular reflection, \[50\] for the specular boundary condition in convex domains with \(C^3\) smoothness, \[54\] for a direct extension of \[55\] from hard potentials to soft potentials, \[18\] \[19\] for a class of datum with bounded amplitude in \(L^\infty\) and small in \(L^p(1 < p < \infty)\) and the references therein. For the steady problem in bounded domains, Guiraud \[33\] \[34\] proved the existence of stationary Boltzmann solutions in convex bounded domains, and the positivity of obtained solutions is not clear. Via \(L^2 - L^\infty\) approach in \[35\], Esposito et al \[25\] constructed a small-amplitude stationary solution of hard potential Boltzmann equation for diffuse reflection boundary conditions of non-isothermal wall temperature with small variations around a positive constant, and further established the positivity of such stationary solutions as a consequence of the dynamical stability for the time-evolutionary Boltzmann equation. Later on, Duan et al \[18\] extend the hard potential results \[25\] to the case of soft potentials. We point out that all these works in bounded domains are about the Newtonian Boltzmann equation.

For the Newtonian limits of the relativistic particles, Calogero \[8\] established the existence of local-in-time relativistic Boltzmann solutions in periodic box, and then proved that such solutions converge, in a suitable norm, to the solution of the Newtonian Boltzmann equation as \(c \to \infty\). Later, for the case near vacuum, Strain \[58\] first obtained the unique global-in-time mild solutions and justified the Newtonian limit for arbitrary time intervals \([0, T]\).

All of the aforementioned works on the relativistic Boltzmann equation were carried out in either spatially periodic box or the whole space. The relativistic Boltzmann equation in bounded domains has been widely used in physics and engineering, for example, Tokamak devices in fusion reactors. In spite of its importance, there has, to the best of our knowledge, been no mathematical theory on the global existence of solutions to the relativistic Boltzmann equation in
bounded domains. The aim of present paper is to construct a small-amplitude stationary solution of relativistic Boltzmann equation with diffuse reflection boundary condition of non-isothermal wall temperature of small variations around a positive constant, and then study the dynamical stability of such stationary solution. Also, we notice that the above mentioned works on global solution of relativistic Boltzmann equation around the global Maxwellian usually normalize the speed of light $c$ to one, then their corresponding estimates may depend on the light speed. In the present paper, we will keep the light speed $c$ as a large parameter, and we aim to establish some uniform in $c$ estimates which may be useful for the Newtonian limit in the future.

1.3. Diffuse reflection boundary condition. Throughout this paper, we assume that $\Omega = \{\xi(x) < 0\}$ is connected and bounded with $\xi(x)$ being a smooth function in $\mathbb{R}^3$. It is obvious that the Tokamak satisfies such assumption. At each boundary point with $\xi(x) = 0$, we assume that $\nabla\xi(x) \neq 0$, then the outward unit normal vector is given by $n(x) = \nabla\xi(x)/|\nabla\xi(x)|$.

We denote the phase boundary of the phase space $\Omega \times \mathbb{R}^3$ as $\gamma = \partial\Omega \times \mathbb{R}^3$, and split $\gamma$ into three disjoint parts, outgoing boundary $\gamma_+$, the incoming boundary $\gamma_-$, and the singular boundary $\gamma_0$:

$$
\gamma_+ = \{(x, p) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot p > 0\},
$$

$$
\gamma_- = \{(x, p) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot p < 0\},
$$

$$
\gamma_0 = \{(x, p) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot p = 0\}.
$$

We supplement the relativistic Boltzmann equation (1.3) with the diffuse reflection boundary condition

$$
F(t, x, p)|_{\gamma_-} = J_T(p) \int_{n(x) \cdot q > 0} F(t, x, q)\{n(x) \cdot \hat{q}\}dq, \tag{1.8}
$$

where $J_T(p)$ is a local Maxwellian with a non-isothermal wall temperature $T = T(x) > 0$, i.e.,

$$
J_T(p) = \frac{\zeta^3 e^{\zeta} }{2\pi\zeta (mc)^3(\zeta + 1)} e^{-\zeta \sqrt{1 + \frac{|p|^2}{m^2c^2}}} \quad \text{with} \quad \zeta = \frac{mc^2}{K_B T},
$$

with $K_B > 0$ being the Boltzmann’s constant. For any $x \in \partial\Omega$, it holds that

$$
\int_{n(x) \cdot p > 0} J_T(p)\{n(x) \cdot \hat{p}\}dp = 1. \tag{1.9}
$$

In the present paper, we assume that $T$ has a small variation around a fixed positive temperature $T_e = \text{constant} > 0$. For simplicity, we normalize

$$
T_e = 1, \quad m = 1, \quad K_B = 1,
$$

throughout the present paper. For brevity, we denote the global Maxwellian

$$
J(p) := J_{T_e}(p) = \frac{e^2}{2\pi(e^2 + 1)} e^{e^2 - cp^0},
$$

which yields that

$$
J(p) \equiv e^{e^2 - cp^0}.
$$

1.4. Main results. Our first result is about the existence of boundary-value problem (BVP) on the steady relativistic Boltzmann equation with diffuse reflection boundary condition, i.e.,

$$
\begin{aligned}
&\hat{p} \cdot \nabla_x F = Q(F, F), \quad (x, p) \in \Omega \times \mathbb{R}^3, \\
&F(x, p)|_{\gamma_-} = J_T(p) \int_{n(x) \cdot q > 0} F(x, q)\{n(x) \cdot \hat{q}\}dq. \tag{1.10}
\end{aligned}
$$

For later use, we define a weight function

$$
w = w(p) := (1 + |p|^2)^\frac{2}{7} e^{\omega|p|}, \tag{1.11}
$$
where $\beta > 3$ and $0 \leq \varpi \leq \frac{1}{8}$. Here for simplicity of presentation, we have omitted the explicit dependence of $w$ on $\beta$ and $\varpi$.

The relativistic differential cross section $\sigma(q, \theta)$ measures the interactions between relativistic particles. Throughout the present paper, we consider the “hard ball” particles

$$\sigma(q, \theta) = \text{constant.} \quad (1.12)$$

Without loss of generality, we take $\sigma(q, \theta) = 1$ for simplicity. The Newtonian limit in this situation, as $\epsilon \to \infty$, is the Newtonian hard-sphere Boltzmann collision operator $[58]$. We define

$$F(x, p) = J(p) + \sqrt{J(p)} f(x, p), \quad (1.13)$$

then we can rewrite (1.10) as

$$\begin{aligned}
\dot{p} \cdot \nabla_x f + L f &= \Gamma(f, f), \quad (x, p) \in \Omega \times \mathbb{R}^3, \\
 f(x, p)|_{\gamma} &= P_\gamma f + \frac{J_f(p) - J(p)}{\sqrt{J(p)}} + \frac{J_f(p) - J(p)}{\sqrt{J(p)}} \int_{n(x) \cdot \hat{q} > 0} f(x, q) \sqrt{J(q)} \{n(x) \cdot \hat{q}\} dq,
\end{aligned} \quad (1.14)$$

where

$$P_\gamma f := \sqrt{J(p)} \int_{n(x) \cdot \hat{q} > 0} f(x, q) \sqrt{J(q)} \{n(x) \cdot \hat{q}\} dq. \quad (1.15)$$

It follows from (1.9) that $P_\gamma \sqrt{J} = \sqrt{J}$. The linearized collision operator $L$ is defined as

$$L f := -\frac{1}{\sqrt{J(p)}} \{Q(\sqrt{J} f, J) + Q(J, \sqrt{J} f)\} = \nu(p)f - Kf, \quad (1.16)$$

where the collision frequency $\nu(p)$ is given by

$$\nu(p) := \int_{\mathbb{R}^3} \int_{S^2} \frac{\xi g \sqrt{s}}{4 p^0 q^0} J(q)d\omega dq, \quad (1.17)$$

and $K = K_2 - K_1$ with

$$
\begin{aligned}
K_1 f := &\int_{\mathbb{R}^3} \int_{S^2} \frac{\xi g \sqrt{s}}{4 p^0 q^0} \sqrt{J(p)J(q)} f(q)d\omega dq, \\
K_2 f := &\int_{\mathbb{R}^3} \int_{S^2} \frac{\xi g \sqrt{s}}{4 p^0 q^0} \sqrt{J(q)J(q')} f(p')d\omega dq + \int_{\mathbb{R}^3} \int_{S^2} \frac{\xi g \sqrt{s}}{4 p^0 q^0} \sqrt{J(p')J(q)} f(q')d\omega dq. \\
\end{aligned}
$$

The nonlinear collision operator takes the form

$$\Gamma(f, f) = \frac{1}{\sqrt{J}} Q^+(\sqrt{J} f, \sqrt{J} f) - \frac{1}{\sqrt{J}} Q^-(\sqrt{J} f, \sqrt{J} f)$$

$$:= \Gamma^+(f, f) - \Gamma^-(f, f), \quad (1.18)$$

where $Q^+$ and $Q^-$ are defined in (1.7).

Denote

$$\ell = \epsilon \frac{p^0 + q^0}{2}, \quad \epsilon = \epsilon \frac{|p \times q|}{g} \quad (1.19)$$

From [21, 31, 57], we know that

$$(K_i f) (p) = \int_{\mathbb{R}^3} k_i(p, q) f(q)dq, \quad i = 1, 2,$$

with the symmetric kernels

$$k_1(p, q) = \frac{\pi}{2} \frac{g \sqrt{s} \epsilon^2}{p^0 q^0} e^{-\ell} \int_0^\pi \sin \theta d\theta = \frac{\epsilon^2}{2(c^2 + 1) p^0 q^0} \frac{g \sqrt{s} \epsilon^2 - \ell}{c^2 + 1}, \quad (1.20)$$

$$k_2(p, q) = \frac{\pi}{2} \frac{g \sqrt{s} \epsilon^2}{p^0 q^0} e^{-\ell} \int_0^\pi \sin \theta d\theta = \frac{\epsilon^2}{2(c^2 + 1) p^0 q^0} \frac{g \sqrt{s} \epsilon^2 - \ell}{c^2 + 1}.$$
and
\[
\begin{align*}
  k_2(p,q) &= \frac{\pi}{4} \frac{s^{3/2}}{g p^0 q^0} \frac{c^2}{2 \pi (c^2 + 1)} \epsilon^2 \int_0^\infty \frac{y \left( 1 + \sqrt{y^2 + 1} \right)}{\sqrt{y^2 + 1}} e^{-\ell \sqrt{y^2 + 1}} I_0(jy) \, dy \\
  &= \frac{c^2}{8(c^2 + 1)} \frac{s^{3/2}}{g p^0 q^0} \epsilon^2 \left[ J_1(\ell, j) + J_2(\ell, j) \right],
\end{align*}
\]

where \( I_0(u) := \frac{1}{2\pi} \int_0^{2\pi} e^{u \cos \varphi} \, d\varphi \), and
\[
\begin{align*}
  J_1(\ell, j) &= \frac{\ell}{\ell^2 - j^2} \left[ 1 + \frac{1}{\sqrt{\ell^2 - j^2}} \right] e^{-\sqrt{\ell^2 - j^2}}, \\
  J_2(\ell, j) &= \frac{1}{\sqrt{\ell^2 - j^2}} e^{-\sqrt{\ell^2 - j^2}}.
\end{align*}
\]

**Theorem 1.1.** Let \( \beta > 3 \). There exist constants \( \delta_0 > 0 \) and \( C_0 > 0 \), which are all independent of the speed of light \( c \) (\( \gg 1 \)), such that if
\[
\delta := |T - 1|_{L^\infty(\partial \Omega)} \leq \delta_0,
\]
then there exists a unique non-negative solution \( F_*(x, p) = J(p) + \sqrt{J(p)} f_*(x, p) \geq 0 \) to the steady problem (1.10), satisfying the mass conservation
\[
\int_\Omega \int_{\mathbb{R}^3} f_*(x, p) \sqrt{J(p)} \, dp \, dx = 0,
\]
and
\[
\|w f_*\|_{L^\infty} + \|w f_*\|_{L^\infty(\gamma)} \leq C_0\delta.
\]
We point out that the uniqueness is in the class of functions satisfying (1.24)–(1.25).

The second result is concerned with the dynamical stability of \( F_*(x, p) \). We assume that (1.3) is supplemented with initial data
\[
F(t, x, p)|_{t=0} = F_0(x, p).
\]

**Theorem 1.2.** Let \( \beta > 3 \). Assume (1.23) with \( \delta_0 > 0 \) chosen to be further suitably small. There exist constants \( \varepsilon_0 > 0 \), \( \tilde{C}_0 > 0 \) and \( \lambda_0 > 0 \), which are all independent of \( c \) (\( \gg 1 \)), such that if
\[
F_0(x, p) = F_*(x, p) + \sqrt{J(p)} f_0(x, p) \geq 0
\]
satisfies
\[
\int_\Omega \int_{\mathbb{R}^3} f_0(x, p) \sqrt{J(p)} \, dp \, dx = 0,
\]
and
\[
\|w f_0\|_{L^\infty} \leq \varepsilon_0,
\]
then the initial-boundary value problem of the relativistic Boltzmann equation (1.3), (1.8) and (1.26) admits a unique global solution \( F(t, x, p) = F_*(x, p) + \sqrt{J(p)} f(t, x, p) \geq 0 \) satisfying
\[
\int_\Omega \int_{\mathbb{R}^3} f(t, x, p) \sqrt{J(p)} \, dp \, dx = 0,
\]
and
\[
\|w f(t)\|_{L^\infty} + \|w f(t)\|_{L^\infty(\gamma)} \leq \tilde{C}_0 e^{-\lambda_0 t} \|w f_0\|_{L^\infty}, \text{ for all } t \geq 0.
\]

**Remark 1.3.** In our analysis, we need the speed of light \( c \) to be suitably large. It is worth noting that previous works on the global existence of solutions for the relativistic Boltzmann equation around the global Maxwellian usually normalize the speed of light \( c \) to one, then corresponding estimates may depend on the speed of light \( c \). While, our estimates in the present paper are
uniform in the speed of light, and such uniform in \( c \) estimates will be useful in the study of Newtonian limit in the future.

**Remark 1.4.** Previous works on the relativistic Boltzmann equation were carried out in either spatially periodic box or the whole space. Our result is the first one concerning the global well-posedness of relativistic Boltzmann equation in smooth bounded domains including Tokamak. And hence we extend the corresponding results in [25] to the case of relativistic Boltzmann equation.

**Remark 1.5.** The key points of the present paper are to establish some uniform in \( c \) estimates for the relativistic collisional operators, and choose some special test functions to control the hydrodynamic part \( P_f \) in terms of \( (I-P) f \) and \( (I-P^*) f \). We point out that all these estimates are highly nontrivial due to the structure of relativistic Boltzmann equation and the calculations on Bessel functions.

### 1.5. Main difficulties and strategy of the proof.

In this subsection, we briefly explain the key points of our analysis.

- **Uniform estimates on collision operators.** For the global solutions of relativistic Boltzmann equation mentioned above, the speed of light \( c \) was usually normalized to one, hence the estimates in these papers on the collision operators and the solutions are usually dependent on the speed of light \( c \). In order to obtain the uniform in \( c \) estimates in Theorems 1.1 and 1.2, we always keep \( c \) as a large parameter and we need to establish some uniform estimates on the collision kernels \( k(p,q) \) and uniform coercivity on linearized operator \( L \). For the kernel \( \tilde{k}(p,q) \) of the classical Boltzmann equation of hard sphere, it is clear that ([26, Lemma 3.3.1])

\[
\int_{\mathbb{R}^3} |\tilde{k}(p,q)| dq \lesssim (1 + |p|)^{-1}.
\]

For the relativistic Boltzmann equation, the uniform estimate is much more complicated due to the complex expression of \( k_1(p,q) \) and \( k_2(p,q) \), see (1.20) and (1.21) for details. Dividing the proof into three cases, i.e., \( \{|p-q| \geq c^{\frac{3}{2}}\} \), \( \{|p-q| \leq c^{\frac{3}{2}}, \|p\| \leq c^{\frac{3}{2}}\} \) and \( \{|p-q| \leq c^{\frac{3}{2}}, \|p\| \geq c^{\frac{3}{2}}\} \), we can obtain

\[
\int_{\mathbb{R}^3} |k(p,q)| dq \lesssim \begin{cases} 
(1 + |p|)^{-1}, & |p| \leq c, \\
\frac{1}{c}, & |p| \geq c,
\end{cases}
\]

(1.29)

see Lemmas 3.3, 3.5 for details. Similarly, we can also obtain

\[
\nu(p) \approx \begin{cases} 
1 + |p|, & |p| \leq c, \\
\frac{1}{c}, & |p| \geq c,
\end{cases}
\]

(1.30)

see Lemma 3.6 for details. It is from (1.29) that we need \( c \) to be suitably large, see also (4.30).

Precisely speaking, \( L \), \( P \), \( K \), \( \nu(p) \) and \( \mathcal{N} \) must depend on \( c \). So, in some places, we will denote them as \( L_c \), \( P_c \), \( K_c \), \( \nu_c(p) \) and \( \mathcal{N}_c \) to emphasize the dependence of \( c \). To get the uniform in \( c \) estimates in Theorems 1.1 and 1.2, we need to establish the following uniform in \( c \) coercivity for \( L_c \)

\[
\langle L_c g, g \rangle \geq \zeta_0 \| (I - P_c) g \|^2_{L^2}, \quad \forall \ g \in L^2_{\nu}(\mathbb{R}^3_c),
\]

(1.31)

where \( \zeta_0 > 0 \) is a positive constant independent of \( c \). For this, we first prove that

\[
\int_{\mathbb{R}^3} |k_c(p,q) - \tilde{k}(p,q)| dq \lesssim c^{-\frac{3}{2}} \to 0 \quad \text{as} \quad c \to \infty,
\]

(1.32)

see Lemmas 3.8 and 3.9 for details. Moreover, we can also prove

\[
\lim_{c \to \infty} e_\alpha = \tilde{e}_\alpha,
\]

(1.33)

where \( \{e_\alpha\} \) and \( \{\tilde{e}_\alpha\} \) are the corresponding orthonormal basis of relativistic Boltzmann equation and Newtonian Boltzmann equation respectively, see Lemma 3.10 for details. The proof of Lemma...
depends heavily on the approximate expansion of the Bessel functions \(K_j(z)\). With the help of (1.32)-(1.33), we can finally prove the uniform in \(c\) coercivity estimate (1.31) by contradiction arguments, see Lemma 3.11 for details.

- **Uniform estimate on hydrodynamic part** \(Pf\). In the small perturbation framework, only the dissipation \(\| (I - P)f \|_2^2\) is obtained directly from the equation. How to estimate the missing hydrodynamic part \(Pf\) is a well-known basic question in the Boltzmann theory. Guo [39, 41] first developed a new nonlinear energy method in high Sobolev norms to estimate \(Pf\) in terms of \(\| (I - P)f \|_2^2\) in the case of periodic box for classical Vlasov-Poisson(Maxwell)-Boltzmann equations. For the initial boundary value problem, things become more complex due to the appearance of boundary term \(P_i f\). The essence is to choose several test functions which can eliminate the \(P_i f\) contribution at the boundary, and to control the \(a, b, c\) components of \(Pf\), see [25] for details.

In the present paper, we follow the ideas in [25] to control the hydrodynamic part \(Pf\)

\[
Pf = \left\{ a + b \cdot p + c \cdot \frac{p^0 - A_3}{\sqrt{A_2 - A_3}} \right\} \sqrt{J},
\]

for our relativistic Boltzmann equation, where \(A_2, A_3\) are coefficients involving Bessel functions, see section 3.1 for more details. The situation becomes slightly more complex due to the appearance of factor \(\frac{c}{p^0}\) which indeed comes from the effect of special relativity. We point out that the test functions are very similar to the ones in [25], but with complex constants \(\beta_c, \beta_b\) which indeed involve Bessel functions, see Propositions 4.7 and 5.4. However, for the unsteady case, to control the extra term \(\int_{\mathbb{R}^3} \partial_t \psi f dp\), one should be more careful to calculate the coefficients.

We briefly explain some key points for the unsteady case in the following. To control \(c\) first, we choose the test function as \(\psi = \psi_c = (|p|^2 - \beta_c) \sqrt{J}(p) \cdot \nabla_x \phi_c(x)\) with \(\phi_c\) defined in (4.48), and \(\beta_c\) such that

\[
\int_{\mathbb{R}^3} \frac{c}{p^0} (|p|^2 - \beta_c) p_i^2 \sqrt{J}(p) dp = 0.
\]

Since

\[
\int_0^t \int_{\Omega \times \mathbb{R}^3} \partial_t \psi_c Pf dp dx d\tau = \sum_{i=1}^3 \int_0^t \int_{\Omega \times \mathbb{R}^3} (|p|^2 - \beta_c) \sqrt{J}(p) p_i \left\{ a + b \cdot p + \frac{p^0 - A_3}{\sqrt{A_2 - A_3}} c \right\} \partial_t \partial_i \phi_c dp dx d\tau,
\]

we notice that the \(a\) and \(c\) contributions vanish due to the oddness in \(p\). While, the \(b\) contribution does not vanish, which is different from [25]. In fact, by using (1.34), we have

\[
\left\| \int_0^t \int_{\Omega \times \mathbb{R}^3} (|p|^2 - \beta_c) p_i^2 \sqrt{J}(p) \partial_t \partial_i \phi_c dp dx d\tau \right\| \leq \int_0^t \| \partial_t \nabla_x \phi_c \|_{L^2} \cdot \| b(\tau) \|_{L^2} d\tau \cdot \int_{\mathbb{R}^3} (|p|^2 - \beta_c) p_i^2 \sqrt{J}(p) dp \leq C \frac{1}{c^2} \int_0^t \| \partial_t \nabla_x \phi_c \|_{L^2} \cdot \| b(\tau) \|_{L^2} d\tau.
\]

Noting the coefficient \(\frac{1}{c^2}\) is small for suitably large \(c\), then the term \(\int_0^t \| b(\tau) \|_{L^2}^2 d\tau\) can be absorbed by \(\int_0^t \| b(\tau) \|_{L^2}^2 d\tau\) on LHS, which helps us to close the estimate of \(c\).
To control $b$, we choose the test function as $\psi = \psi_b^i = (p_i^2 - \beta_b)\sqrt{\mathcal{J}(p)} \partial_j \phi^i_b$ with $\phi^i_b$ defined in (4.56), and $\beta_b$ such that
\[
\int_{\mathbb{R}^3} \frac{c}{p_i^2} (p_i^2 - \beta_b) p_i^2 \mathcal{J}(p) dp = 0, \quad k \neq i.
\] (1.37)
Using Lemma 3.1, it is clear that $\beta_b = \frac{K_1(c^2)}{\sqrt{x^2}}$. Then one has
\[
\int_0^t \int_{\Omega \times \mathbb{R}^3} \partial_t \psi Pf dp dx d\tau = \int_0^t \int_{\Omega \times \mathbb{R}^3} (p_i^2 - \beta_b) \mathcal{J}(p) \partial_i \partial_j \phi^j_b \{ a + b \cdot p + c \frac{p^0 - A^3}{\sqrt{A_2 - A^3}} \} dp dx d\tau.
\] (1.38)
For the terms on the RHS of (1.38), the $b$ contribution vanishes due to the oddness in $p$. For the $a$ contribution, fortunately, with the help of Lemma 3.1, it is direct to check that
\[
\int_{\mathbb{R}^3} (p_i^2 - \beta_b) \mathcal{J}(p) dp = 0,
\]
which yields that the $a$ contribution also vanishes. It is indeed a crucial point in the estimate of $a$. Without the zero contribution of $a$, it is hard for us to close the estimate of $b$ for the unsteady case.

The estimate of $a$ is similar to the one of $c$. Therefore, combining the estimates in Proposition 4.7 and the estimates of $\int_0^t \int_{\Omega \times \mathbb{R}^3} \partial_t \psi Pf dp dx d\tau$, we can finally close the estimate of $c$, $b$ and $a$ in the unsteady case and finish the proof of Proposition 5.4.

1.6. **Organization of the paper.** In section 2, we present some results about the Bessel function and Green’s identity for later use. In section 3, we establish a series of uniform estimates with respect to the speed of light $c$ for the collision operators. In section 4, we study the steady problem and give the proof of Theorem 1.1. In section 5, we study the large-time asymptotic stability of the stationary solution under small perturbations and give the proof of Theorem 1.2. In section 6, we present the local-in-time existence of solutions for the relativistic Boltzmann equations and give the sketch of proof.

1.7. **Notations.** Throughout this paper, $C$ denotes a generic positive constant which is independent of $c$ and $C_0, C_b, \ldots$ denote the generic positive constants depending on $a, b, \ldots$, respectively, which may vary from line to line. $A \lesssim B$ means that there exists a constant $C > 0$, which is independent of the speed of light $c$, such that $A \leq CB$. $A \eqsim B$ means that both $A \lesssim B$ and $B \lesssim A$ hold. $\| \cdot \|_{L^2}$ denotes the standard $L^2(\Omega \times \mathbb{R}^3)$-norm and $\| \cdot \|_{L^\infty}$ denotes the $L^\infty(\Omega \times \mathbb{R}^3)$-norm. We denote $\langle \cdot, \cdot \rangle$ as the $L^2(\Omega \times \mathbb{R}^3)$ inner product or $L^2(\mathbb{R}^3)$ inner product. Moreover, we denote $\| \cdot \|_*: = \| \sqrt{\partial} \cdot \|_{L^2}$. For the phase boundary integration, we define $d\gamma := |n(x) \cdot \hat{p}| dS_x dp$, where $dS_x$ is the surface measure and define $|f|^2_{L^2(\gamma^\pm)} = \int_{\gamma^\pm} |f(x, p)|^2 d\gamma$ and the corresponding space is denoted as $L^2(\partial \Omega \times \mathbb{R}^3) = L^2(\partial \Omega \times \mathbb{R}^3; d\gamma^\pm)$. Furthermore, we denote $|f|_{L^\infty(\gamma^\pm)} = |f \mathbf{1}_{\gamma^\pm}|_{L^\infty(\gamma^\pm)}$ and $|f|_{L^\infty(\gamma^\pm)} = |f \mathbf{1}_{\gamma^\pm}|_{L^\infty(\gamma^\pm)}$.

2. Preliminaries

We first present some results on Bessel functions which will be used frequently. We define the modified Bessel function of the second kind (see (3.19))
\[
K_j(z) = \frac{z^j}{(2j - 1)!!} \int_1^\infty e^{-zs} (s^2 - 1)^{\frac{j}{2}} ds, \quad j \in \mathbb{N}, \ z > 0.
\]

**Lemma 2.1.** Denote
\[
B_n(z) := \int_0^\infty x^{2n} e^{-z\sqrt{1 + x^2}} dx, \quad n \in \mathbb{N}, \ z > 0,
\]
Then it holds that
\[ B_n(z) = \frac{(2n-1)!}{z^n}K_{n+1}(z), \quad n \in \mathbb{N}, \quad z > 0. \]

**Proof.** Let \( s = \sqrt{1 + x^2} \). A direct calculation shows that
\[ B_n(z) = \int_0^\infty x^{2n}e^{-x\sqrt{1+x^2}}dx = \int_1^\infty e^{-zs}(s^2 - 1)^{\frac{2n-1}{2}}ds \]
\[ = \frac{z}{2n+1} \int_1^\infty e^{-zs}(s^2 - 1)^{(n+1)-\frac{1}{2}}ds = \frac{(2n-1)!}{z^n}K_{n+1}(z). \]

Therefore the proof of Lemma 2.1 is completed. \( \square \)

**Lemma 2.2.** For the modified Bessel functions \( K_2(z) \) and \( K_3(z) \), we have
\[ \left( \frac{K_3(z)}{K_2(z)} \right)^2 - \frac{5}{2} \frac{K_3(z)}{K_2(z)} + \frac{1}{z^2} - 1 < 0, \quad z > 0. \] (2.1)

**Proof.** Noting (3.13) of [11], one can also express \( K_j(z) \) as
\[ K_j(z) = \int_0^\infty \cosh(jr)e^{-z\cosh(r)}dr, \quad j \in \mathbb{N}, \quad z > 0, \]
which yields immediately that
\[ K_j(z) < K_{j+1}(z), \quad j \in \mathbb{N}, \quad z > 0. \]

It is noted that these Bessel functions obey the following recurrence relation:
\[ K_{j+1}(z) = \frac{2j}{z}K_j(z) + K_{j-1}(z), \quad j \in \mathbb{N}, \quad z > 0. \] (2.2)

It follows from (2.2) that
\[ \frac{K_3(z)}{K_2(z)} = \frac{4}{z} + \frac{K_1(z)}{K_2(z)}. \]

Thus it suffices to show that
\[ \left( \frac{K_1(z)}{K_2(z)} \right)^2 + \frac{3}{2} \frac{K_1(z)}{K_2(z)} - \frac{3}{z^2} - 1 < 0. \] (2.3)

Noting \( K_1(z) < K_2(z) \), we get for \( 0 < z < 1 \)
\[ \left( \frac{K_1(z)}{K_2(z)} \right)^2 + \frac{3}{2} \frac{K_1(z)}{K_2(z)} - \frac{3}{z^2} - 1 = \left[ \left( \frac{K_1(z)}{K_2(z)} \right)^2 - 1 \right] + \frac{3}{z} \left[ \frac{K_1(z)}{K_2(z)} - \frac{1}{z} \right] < 0. \]

From (26) of [9], one has
\[ \frac{K_1(z)}{K_2(z)} \leq \frac{8z^2 + 3z}{8z^2 + 15z + 6}, \quad z \geq \frac{1}{2}. \]

Hence, one only needs to show that
\[ \left( \frac{8z^2 + 3z}{8z^2 + 15z + 6} \right)^2 + \frac{3}{2} \frac{8z^2 + 3z}{8z^2 + 15z + 6} < \frac{3}{z^2} + 1, \quad \text{for} \ z \geq \frac{1}{2}. \] (2.4)

A direct calculation shows that (2.4) is equivalent to
\[ 64z^6 + 240z^5 + 441z^4 + 279z^3 + 54z^2 < 64z^6 + 240z^5 + 513z^4 + 900z^3 + 999z^2 + 540z + 108, \]
which is obviously true for all \( z > 0 \). Thus (2.3) holds for all \( z > 0 \), which implies (2.1). Therefore the proof of Lemma 2.2 is completed. \( \square \)

For later use, we also need the following asymptotic expansions for \( K_j(z) \).
Lemma 2.3. ([11]) For large $z > 0$, we have
\[
K_j(z) = \sqrt{\frac{\pi}{2z}} e^z \left[ 1 + \sum_{k=1}^{n} a_{j,k} z^{-k} + O(z^{-(n+1)}) \right], \quad j \in \mathbb{N},
\]
where
\[
a_{j,k} := \frac{1}{k!} (4j^2 - 1)(4j^2 - 3^2) \cdots (4j^2 - (2k - 1)^2).
\]

Finally, we give the Green identity for both steady and unsteady cases, which is a relativistic version of [25, Lemma 2.2].

Lemma 2.4. For the steady case, assume that $f(x, p), g(x, p) \in L^2(\Omega \times \mathbb{R}^3), \hat{p} \cdot \nabla_x f, \hat{p} \cdot \nabla_x g \in L^2(\Omega \times \mathbb{R}^3)$ and $f_{\gamma}, g_{\gamma} \in L^2(\partial \Omega \times \mathbb{R}^3)$. Then
\[
\left\{ \begin{align*}
\int_{\Omega \times \mathbb{R}^3} \{ \hat{p} \cdot \nabla_x f \} g + \{ \hat{p} \cdot \nabla_x g \} f \, dpdx &= \int_{\gamma} f g \, d\gamma. \\
\end{align*} \right.
\]

For the unsteady case, assume that $f(t, x, p), g(t, x, p) \in L^2 \left( [0, T]; L^2(\Omega \times \mathbb{R}^3) \right), \partial_t f + \hat{p} \cdot \nabla_x f, \
\partial_t g + \hat{p} \cdot \nabla_x g \in L^2 \left( [0, T] \times \Omega \times \mathbb{R}^3 \right)$ and $f_{\gamma}, g_{\gamma} \in L^2 \left( [0, T] \times \partial \Omega \times \mathbb{R}^3 \right)$. Then, for almost all $t, s \in [0, T]$, it holds that
\[
\left\{ \begin{align*}
\int_{s}^{t} \int_{\Omega \times \mathbb{R}^3} \{ \partial_t f + \hat{p} \cdot \nabla_x f \} g \, dpdxdt &= \int_{s}^{t} \int_{\Omega \times \mathbb{R}^3} \{ \partial_t g + \hat{p} \cdot \nabla_x g \} f \, dpdxdt \\
&\quad + \int_{s}^{t} \int_{\Omega \times \mathbb{R}^3} f(t)g(t) \, dpdx - \int_{s}^{t} \int_{\Omega \times \mathbb{R}^3} f(s)g(s) \, dpdx + \int_{s}^{t} \int_{\gamma} f g \, d\gamma dt.
\end{align*} \right.
\]

Proof. The proof is very similar to the one in Chap. 9 of [10], Eq. (2.18), and we omit the details here for brevity. □

3. Uniform in $\epsilon$ estimates for collision operators

In this section, we shall establish some uniform in $\epsilon$ estimates for the operators $K, L$ and $\Gamma$ which will be used later.

3.1. The macroscopic operator and null space. For later use, we define the normalized global Maxwellian
\[
\hat{J}(p) := \frac{1}{4\pi K_2(\epsilon)} e^{-\epsilon p^0} \approx e^{\epsilon^2 - \epsilon p^0},
\]
where $K_2(z)$ is the Bessel function defined above. Using Lemma 2.1, we know that
\[
\int_{\mathbb{R}^3} \hat{J}(p) \, dp = 1.
\]

Lemma 3.1. For the normalized global Maxwellian $\hat{J}(p)$, it holds that
1. $\int_{\mathbb{R}^3} \rho^2 \hat{J}(p) \, dp = \frac{K_3(\epsilon^2)}{K_2(\epsilon^2)} := A_1$.
2. $\int_{\mathbb{R}^3} (\rho^0)^2 \hat{J}(p) \, dp = \epsilon^2 + 3 \frac{K_3(\epsilon^2)}{K_2(\epsilon^2)} := A_2$.
3. $\int_{\mathbb{R}^3} \rho^0 \hat{J}(p) \, dp = \frac{\epsilon K_3(\epsilon^2)}{K_2(\epsilon^2)} - \frac{1}{\epsilon} := A_3$.
4. $\int_{\mathbb{R}^3} \frac{\rho^2}{\rho^0} \hat{J}(p) \, dp = \frac{1}{\epsilon} := A_4$. 
Lemma 3.2. \[ \frac{K_4(c^2)}{K_2(c^2)} - \frac{1}{c} \frac{K_3(c^2)}{K_2(c^2)} =: A_5. \]

Lemma 3.1, one can obtain the\[ \frac{1}{c} \frac{K_3(c^2)}{K_2(c^2)} =: A_6. \]

(5) \[ \int_{\mathbb{R}^3} p_i^2 p^0 J(p) dp = \frac{c}{3} K_2(c^2) \int_{\mathbb{R}^3} \frac{|p|^4}{p^0} J(p) dp = 5 \frac{K_3(c^2)}{K_2(c^2)}.)

(6) \[ \int_{\mathbb{R}^3} \frac{|p|^4}{p^0} J(p) dp = \frac{1}{c} \frac{K_3(c^2)}{K_2(c^2)} =: A_7, \]

(i) \[ i, k = 1, 2, 3, i \neq k. \]

(ii) \[ \int_{\mathbb{R}^3} p_i^2 p^2 J(p) dp = \frac{7}{2} \frac{K_3(c^2)}{K_2(c^2)} =: A_8, \]

(iii) \[ \int_{\mathbb{R}^3} \frac{p_i^2}{p^0} J(p) dp = \frac{3}{2} \frac{K_3(c^2)}{K_2(c^2)} =: A_9. \]

(iv) \[ \int_{\mathbb{R}^3} |p|^4 J(p) dp = 15 \frac{K_2(c^2)}{K_2(c^2)} =: A_{10}. \]

Proof. We only give a proof for (i) and the others can be proved similarly. To prove (i), using Lemma 2.1, one has

\[ \int_{\mathbb{R}^3} p_i^2 J(p) dp = \frac{1}{12 \pi c^2 K_2(c^2)} \int_{\mathbb{R}^3} |p|^2 e^{-|p|^2/c^2} dp = \frac{c^4}{3} K_2(c^2) \int_0^\infty x^4 e^{-c^2 x^2} dx. \]

Therefore the proof of Lemma 3.1 is completed. \( \square \)

From Lemma 2.2 it is clear that

\[ A_2 - A_3 = -c^2 \left\{ \left( \frac{K_3(c^2)}{K_2(c^2)} \right)^2 - \frac{5}{c^2} \frac{K_3(c^2)}{K_2(c^2)} + \frac{1}{c^4} \right\} > 0. \] (3.1)

It is well-known that \( L \) is a self-adjoint non-negative definite operator in \( L^2_p \) space with the kernel

\[ \mathcal{K} = \text{span} \left\{ \sqrt{J}, p_i \sqrt{J} (i = 1, 2, 3), p^0 \sqrt{J} \right\}. \]

Using Lemma 3.1 one can obtain the orthonormal basis for the null space \( \mathcal{N} = \text{span} \{ e_0, e_1, e_2, e_3, e_4 \} \),

with \( e_0 = \sqrt{J}, e_1 = \frac{p_0}{\sqrt{A_1}} \sqrt{J}, e_2 = \frac{p_0 - A_3}{\sqrt{A_2 - A_3}} \sqrt{J}, \) (3.2)

where \( A_1, A_2 \) and \( A_3 \) are defined in Lemma 3.1. Let \( P \) be the orthogonal projection from \( L^2_p \) onto \( \mathcal{N} \). For given \( f \), we denote the macroscopic part \( Pf \) as

\[ Pf = \left\{ a + b \cdot p + c \cdot \frac{p^0 - A_3}{\sqrt{A_2 - A_3}} \right\} \sqrt{J}, \]

and further denote \( (I - P)f \) to be the microscopic part of \( f \). For the orthonormal basis in (3.2), due to (3.1), hence (3.2) is well-defined.

3.2. Estimates on \( K \). We present a similar lemma whose estimates are minor refinements of (27, Lemma 3.1) (\( c \) is normalized to one). Here we keep the light speed \( c \) as a large parameter.

Lemma 3.2. \( (27) \) Recall \( \ell \) and \( j \) defined in (1.19), then it holds

(i) \[ \sqrt{|p \times q|^2 + c^2 |p - q|^2} \leq \ell \leq |p - q|, \quad \text{and} \quad g^2 < s \leq 4p^0 q^0. \]

Proof. We only give a proof for (i) and the others can be proved similarly. To prove (i), using Lemma 2.1, one has
(ii) \( v_\phi = \frac{\xi g \sqrt{s}}{4 p^0 q_0} \leq \min \{ \varepsilon, \frac{|p - q|}{2} \} \).

(iii) \( \ell^2 - j^2 = \frac{s c^2}{4 g^2} |p - q|^2 = \frac{c^2 g^2 + 4 c^4}{4 g^2} |p - q|^2 \geq c^4 + \frac{c^2}{4} |p - q|^2 \).

(iv) \( \lim_{\varepsilon \to \infty} \frac{g}{|p - q|} = \lim_{\varepsilon \to \infty} \frac{s}{4 c^2} = \lim_{\varepsilon \to \infty} \frac{\ell}{c^2} = \lim_{\varepsilon \to \infty} \frac{\ell^2 - j^2}{c^4} = 1 \).

Note that we need \( p \neq q \), fixed in (iii) and (iv).

We establish some delicate estimates for the operators so that they are uniform in the light speed \( c \). This is crucial for us to obtain our main theorems where the amplitude perturbation is independent of light speed.

**Lemma 3.3.** For \( k_1(p, q) \) and \( k_2(p, q) \), it holds that

\[
k_1(p, q) \lesssim |p - q| e^{-\frac{|p| + |q|}{2}},
\]

and

\[
k_2(p, q) \lesssim \left[ \frac{1}{c} + \frac{1}{|p - q|} \right] e^{-\frac{|p - q|}{2}}.
\]

**Proof.** We notice that

\[
c^2 - c p_0 = c^2 \left( 1 - \sqrt{1 + \frac{|p|^2}{c^2}} \right) = -\frac{|p|^2}{1 + \sqrt{1 + \frac{|p|^2}{c^2}}},
\]

which yields

\[
-\frac{|p|^2}{2} \leq c^2 - c p_0 \leq -\frac{|p|^2}{1 + \sqrt{1 + |p|^2}} = -\sqrt{1 + |p|^2} + 1 \leq |p| + 1.
\]

It follows from (1.19), (1.20), (3.6) and Lemma 3.2 that

\[
k_1(p, q) \lesssim \frac{g \sqrt{s}}{p^0 q^0} e^{c^2 - \frac{5}{2} c(p^0 + q^0)} \lesssim \frac{c |p - q| \sqrt{p^0 q^0}}{p^0 q^0} e^{-\frac{|p|}{2} - \frac{|q|}{2}} \lesssim |p - q| e^{-\frac{|p|}{2} - \frac{|q|}{2}}.
\]

Now we consider (3.4). Noting \( J_2(\ell, j) \leq J_1(\ell, j) \), we have from (1.21) that

\[
k_2(p, q) \lesssim \frac{s^{3/2}}{g p^0 q^0} c^2 \left( 1 + \frac{1}{\sqrt{\ell^2 - j^2}} \right) e^{c^2 - \sqrt{\ell^2 - j^2}} \]

\[
\lesssim \frac{s^{3/2}}{g p^0 q^0} \frac{\ell}{\ell^2 - j^2} e^{c^2 - \sqrt{\ell^2 - j^2}}.
\]

It follows from Lemma 3.2 that

\[
c^2 - \sqrt{\ell^2 - j^2} \leq c^2 - \sqrt{c^4 + \frac{c^2}{4} |p - q|^2} \leq -\frac{c^2}{4} e^2 + \sqrt{c^4 + \frac{c^2}{4} |p - q|^2} \]

\[
= -\frac{1}{4} \frac{|p - q|^2}{1 + \sqrt{1 + \frac{1}{4} |p - q|^2}} \leq -\frac{1}{4} \frac{|p - q|^2}{1 + \sqrt{1 + \frac{1}{4} |p - q|^2}} \]

\[
= -\sqrt{1 + \frac{1}{4} |p - q|^2} + 1 \leq -\frac{|p - q|}{2} + 1,
\]

then we have

\[
k_2(p, q) \lesssim \frac{s^{3/2}}{g p^0 q^0} \frac{c(p^0 + q^0)}{2} \frac{1}{se^{\frac{|p - q|^2}{4}}} e^{-\frac{|p - q|}{2}} \lesssim \frac{s^{1/2} (p^0 + q^0)}{p^0 q^0} \frac{g}{|p - q|^2} e^{-\frac{|p - q|}{2}}
\]
following, we mainly focus on the case of which can be absorbed by the exponential part $e^{-|p-q|}$. Therefore the proof of Lemma 3.3 is completed.

Based on Lemma 3.3 we can show the following key estimate.

Lemma 3.4. Let $\alpha \geq 0$, $0 \leq \varpi \leq \frac{1}{8}$ and denote $w_{\alpha}(p) := (1 + |p|^2)^{\frac{\alpha}{8}} e^{\varpi|p|}$. Then it holds

$$\int_{\mathbb{R}^3} k_1(p, q) w_{\alpha}(p) w_{\alpha}(q) dp dq \lesssim \frac{1}{1 + |p|},$$

and

$$\int_{\mathbb{R}^3} k_2(p, q) w_{\alpha}(p) w_{\alpha}(q) dq \lesssim \begin{cases} \frac{1}{1 + |p|}, & |p| \leq c, \\ \frac{1}{c}, & |p| \geq c. \end{cases}$$

Proof. For $\alpha > 0$, notice that

$$w_{\alpha}(p) = (1 + |p|^2)^{\frac{\alpha}{8}} e^{\varpi|p|} \lesssim (1 + |p - q|^2)^{\frac{\alpha}{8}} e^{\varpi|p - q|},$$

which can be absorbed by the exponential part $e^{-|p-q|}$ of $k_1$ and $k_2$ since $\varpi \in [0, \frac{1}{8}]$. In the following, we mainly focus on the case of $\alpha = \varpi = 0$.

For (3.8), it follows from (3.3) that

$$\int_{\mathbb{R}^3} k_1(p, q) dq \lesssim \int_{\mathbb{R}^3} |p - q| e^{-\frac{|p - q|^2}{4}} dq \lesssim \int_{\mathbb{R}^3} e^{-\frac{|p - q|^2}{4}} dq \lesssim e^{-\frac{|p|^4}{4}} \lesssim \frac{1}{1 + |p|}.$$  

Next we consider (3.9). If $|p| \leq 1$, we have from (3.4) that

$$\int_{\mathbb{R}^3} k_2(p, q) dq \lesssim \int_{\mathbb{R}^3} \left( \frac{1}{c} + \frac{1}{|p - q|} \right) e^{-\frac{|p - q|^2}{4}} dq \lesssim 1 \lesssim \frac{1}{1 + |p|}.$$  

For $|p| \geq 1$, it follows from the third inequality of (5.7) that

$$k_2(p, q) \lesssim \sqrt{g^2 + 4c^2(p^0 + q^0)} \frac{1}{|p - q|} e^{-\frac{|p - q|}{4}}.$$  

Since the proof is complicated, we divide it into three cases.

Case 1: $|p - q| \geq c^\frac{1}{6}$. It follows from (11) that

$$\int_{|p - q| \geq c^\frac{1}{6}} k_2(p, q) dq \lesssim \int_{|p - q| \geq c^\frac{1}{6}} \frac{\sqrt{g^2 + 4c^2(p^0 + q^0)} \frac{1}{c}}{|p - q|} e^{-\frac{|p - q|}{4}} dq \lesssim e^{-\frac{\alpha}{4}} \int_{|p - q| \geq c^\frac{1}{6}} \frac{\sqrt{|p - q|^2 + 4}}{|p - q|} \cdot e^{-\frac{|p - q|}{4}} dq \lesssim e^{-\frac{\alpha}{4}}.$$  

Case 2: $|p - q| \leq c^\frac{1}{6}$ and $|p| \leq c$. It is direct to check that

$$g^2 - |p - q|^2 = 2p^0 q^0 - 2c^2 - 2p \cdot q - |p - q|^2$$

$$= 2c^2 (|p|^2 + |q|^2) + |p|^2 |q|^2 - (|p|^2 + |q|^2)$$

$$= \frac{2}{p^0 q^0 + c^2} \left( c^2 (|p|^2 + |q|^2) + |p|^2 |q|^2 - \frac{1}{2} (|p|^2 + |q|^2) (p^0 q^0 + c^2) \right).$$
which, together with Lemma 3.2, yields that

\[ c^2 - \sqrt{\ell^2 - \bar{j}^2} = \frac{c^4 - (\bar{c}^2 - \bar{j}^2)}{c^2 + \sqrt{\ell^2 - \bar{j}^2}} = \frac{c^4 - \frac{c^2}{4\bar{c}^2} |p - q|^2}{c^2 + \frac{c^2}{4\bar{c}^2} |p - q|} = \frac{c^2 - \frac{c^2}{4\bar{c}^2} |p - q|^2}{1 + \sqrt{\frac{4\bar{c}^2}{c^2} |p - q|^2}} \]

\[ = \frac{c^2 - \frac{c^2 + 4\bar{c}^2}{4\bar{c}^2} |p - q|^2}{1 + \sqrt{\frac{4\bar{c}^2}{c^2} |p - q|^2}} = \frac{1}{1 + \sqrt{\frac{4\bar{c}^2}{c^2} |p - q|^2}} \left\{ - \frac{1}{4} |p - q|^2 + \frac{c^2}{2\bar{c}^2} (g^2 - |p - q|^2) \right\} \]

\[ = \frac{1}{1 + \sqrt{\frac{4\bar{c}^2}{c^2} |p - q|^2}} \frac{c^2 (q^0 |p|^2 - p^0 |q|^2)^2}{(p^0 q^0 + c^2)^2}. \quad (3.13) \]

Hence, it follows from (3.11) and (3.13) that

\[
\int_{|p - q| \leq c} k_2(p,q) dq \lesssim \int_{|p - q| \leq c} \frac{\sqrt{g^2 + 4c^2 (p^0 + q^0)}}{p^0 q^0} \frac{1}{|p - q|} e^{\frac{1}{2} (c^2 - \sqrt{\ell^2 - \bar{j}^2})} dq
\]

\[
\lesssim \int_{|p - q| \leq c} \frac{1}{|p - q|} \left( \frac{1}{|p - q|} e^{\frac{1}{2} (c^2 - \sqrt{\ell^2 - \bar{j}^2})} \right) dq
\]

\[
\lesssim \int_{|p - q| \leq c} \frac{1}{|p - q|} e^{-\frac{|p - q|}{c + \frac{1}{4}\frac{g}{c}} M(p,q)} dq,
\]

where

\[ M(p, q) := -\frac{1}{1 + \sqrt{\frac{4\bar{c}^2}{c^2} |p - q|^2}} \frac{c^2 (q^0 |p|^2 - p^0 |q|^2)^2}{(p^0 q^0 + c^2)^2}. \quad (3.15) \]

Noting \(|p - q| \leq c^\frac{1}{2}\) and \(|p| \leq c\), one has

\[ p^0 = \sqrt{c^2 + |p|^2} \leq c, \quad q^0 = \sqrt{c^2 + |q|^2} \leq \sqrt{c^2 + 2 |p - q|^2 + 2|p|^2} \lesssim c, \]

which yields that

\[ (1 + \sqrt{\frac{s}{4\bar{c}^2}} \frac{|p - q|}{g}) g^2 (p^0 q^0 + c^2)^2 \leq (1 + \sqrt{1 + \frac{g^2}{4\bar{c}^2}}) |p - q|^2 (p^0 q^0 + c^2)^2 \lesssim c^4 |p - q|^2. \]

A direct calculation shows that

\[ p^0 |q|^2 - q^0 |p|^2 = c(|q|^2 - |p|^2) + (p^0 - c)|q|^2 - (q^0 - c)|p|^2 \]

\[ = c(|q|^2 - |p|^2) + \frac{|p|^2 |q|^2}{p^0 + c} - \frac{|p|^2 |q|^2}{q^0 + c} \]

\[ = c(|q|^2 - |p|^2) + \frac{|p|^2 |q|^2}{(p^0 + c)(q^0 + c)} (q^0 - p^0) \]

\[ = c(|q|^2 - |p|^2) + \frac{|p|^2 |q|^2 (|q|^2 - |p|^2)}{(p^0 + c)(q^0 + c)(p^0 + q^0)}. \quad (3.17) \]
Thus, from (3.16)–(3.17), there exists a positive constant $c_0$ which is independent of $c$ such that
\[
M(p, q) \leq -c_0 \frac{1}{c^2} \frac{1}{|p - q|^2} \left( c(|q|^2 - |p|^2) + \frac{|p|^2 |q|^2 ((|q|^2 - |p|^2)}{(p^0 + c)(q^0 + c)(p^0 + q^0)} \right)^2 \\
\leq -c_0 \frac{(|q|^2 - |p|^2)^2}{|p - q|^2}.
\] (3.18)

Combining (3.14) and (3.18), one has that
\[
\int_{|p - q| \leq \varepsilon} k_2(p, q) dq \lesssim \int_{|p - q| \leq \varepsilon} \frac{1}{|p - q|} e^{-\frac{|p - q|^2}{4(|q|^2 - |p|^2)^2}} dq.
\]

By taking similar arguments as in [26, Lemma 3.3.1] (see also Case 3 below), we obtain
\[
\int_{|p - q| \leq \varepsilon} k_2(p, q) dq \lesssim \frac{1}{1 + |p|}, \quad \text{for } |p| \leq c.
\] (3.19)

**Case 3:** $|p - q| \leq \varepsilon$ and $|p| \geq c$. Noting
\[
|q| \leq |q - p| + |p| \leq 2|p|, \quad |q| \geq |p| - |q - p| \geq \frac{1}{2} |p|,
\]
then we have
\[
|p| \cong |q|, \quad p^0 \cong q^0.
\]

Hence it is clear that
\[
(1 + \sqrt{\frac{s}{4c^2} \frac{|p - q|}{g}}) g^2 (p^0 q^0 + c^2)^2 \leq (1 + \sqrt{1 + \frac{g^2}{4c^2}})|p - q|^2 (p^0 q^0 + c^2)^2 \\
\lesssim |p - q|^2 (c^2 + |p|^2)^2.
\] (3.20)

For $|p| \geq c$, it holds that
\[
c + \frac{|p|^2 |q|^2}{(p^0 + c)(q^0 + c)(p^0 + q^0)} \cong c + \frac{|p|^4}{(p^0)^3} \cong c + \frac{|p|^4}{(c^2 + |p|^2)^2} \cong c + |p|,
\]
which, together with (3.17), yields that
\[
(p^0 |q|^2 - q^0 |p|^2)^2 \cong (|q|^2 - |p|^2)^2 (c^2 + |p|^2)^2.
\] (3.21)

Combining (3.15), (3.20) and (3.21), for some positive constant $c_2$, we have
\[
M(p, q) \leq -2c_2 \frac{c^2}{c^2 + |p|^2} \frac{|q|^2 - |p|^2}{|p - q|^2}.
\] (3.22)

Hence, for $|p| \geq c$, it follows from (3.14) and (3.22) that
\[
\int_{|p - q| \leq \varepsilon} k_2(p, q) dq \lesssim \int_{|p - q| \leq \varepsilon} \frac{1}{|p - q|} e^{-\frac{|p - q|^2}{4M(p, q)}} dq \\
\lesssim \int_{|p - q| \leq \varepsilon} \frac{1}{|p - q|} e^{-\frac{|p - q|^2}{c_2 |p|^2} \frac{(|q|^2 - |p|^2)^2}{|p - q|^2}} dq.
\]

Following the arguments as in [26, Lemma 3.3.1], we can make a change of variables
\[
|p - q| = r, \quad (q - p) \cdot p = |p||r| \cos \theta, \quad 0 \leq r < \infty, \quad 0 \leq \theta \leq \pi,
\]
which yields that
\[
|q|^2 = |q - p|^2 + |p|^2 + 2(q - p) \cdot p = r^2 + |p|^2 + 2r |p| \cos \theta.
\]

Denoting $a^2 := c_2 \frac{c^2}{c^2 + |p|^2}$ and $u = a(r + 2|p| \cos \theta)$, one has
\[
\int_{|p - q| \leq \varepsilon} k_2(p, q) dq \lesssim \int_0^\infty e^{-\frac{r}{a}} dr \int_0^\pi e^{-a^2(r + 2|p| \cos \theta)^2} \sin \theta d\theta
\]
\[ \int_{\mathbb{R}^3} k^2_1(p, q) dq \lesssim \frac{1}{1 + |p|}, \]

and
\[
\int_{\mathbb{R}^3} k^2_2(p, q) dq \lesssim \begin{cases} 
\frac{1}{1 + |p|}, & |p| \leq c, \\
\frac{1}{c}, & |p| \geq c.
\end{cases}
\]

Denote \( k(p, q) := k_2(p, q) - k_1(p, q) \) and
\[ k_w(p, q) := k(p, q) \frac{w(p)}{w(q)}. \]

It is clear that
\[ \frac{w(p)}{w(q)} \lesssim (1 + |p - q|)^{\frac{5}{2}} e^{(\omega + \frac{1}{\omega})|p - q|}, \]

which can be absorbed by the exponential part of \( k_1 \) and \( k_2 \). Hence, by similar arguments as in Lemma 3.4, one can also obtain
\[
\int_{\mathbb{R}^3} k_w(p, q) e^{\frac{1}{2} |p-q|} dq \lesssim \begin{cases} 
\frac{1}{1 + |p|}, & |p| \leq c, \\
\frac{1}{c}, & |p| \geq c.
\end{cases}
\]

Next we estimate the collision frequency \( \nu(p) \).

**Lemma 3.6.** It holds that
\[ \nu(p) \approx \begin{cases} 
1 + |p|, & |p| \leq c, \\
1, & |p| \geq c.
\end{cases} \]

**Proof.** Recall
\[ \nu(p) = \int_{\mathbb{R}^3} \frac{c^3}{8\pi(c^2 + 1)} \frac{g\sqrt{s}}{p^0 q^0} e^{c^2 - cu_0} d\omega dq. \]

Since the proof is complicated, we split it into four cases.

**Case 1:** \(|q| \geq c^\frac{1}{4}\). Using Lemma 3.2 and \( \nu(p) \), one has
\[
\int_{|q| \geq c^\frac{1}{4}} \int_{\mathbb{S}^2} \frac{c^3}{8\pi(c^2 + 1)} \frac{g\sqrt{s}}{p^0 q^0} e^{c^2 - cu_0} d\omega dq \lesssim \int_{|q| \geq c^\frac{1}{4}} \frac{cg\sqrt{s}}{p^0 q^0} e^{c^2 - cu_0} dq \]
\[ \lesssim \int_{|q| \geq c^\frac{1}{4}} c e^{-|q|} dq \lesssim e^{-c^\frac{1}{4}}. \]

**Case 2:** \(|q| \leq c^\frac{1}{4}\) and \(|p| \leq c^\frac{1}{4}\). Using \( \nu(p) \) again, we have
\[
\int_{|q| \leq c^\frac{1}{4}} \int_{\mathbb{S}^2} \frac{c^3}{8\pi(c^2 + 1)} \frac{g\sqrt{s}}{p^0 q^0} e^{c^2 - cu_0} d\omega dq
\]
\[ \int_{|q| \leq \epsilon^k \frac{1}{2}} \int_{\mathbb{S}^2} \frac{cg}{8\pi p^0 q^0} \left( \frac{c^2}{c^2 + 1} - 1 \right) e^{c^2 - c q^0} d\omega dq \]

\[ + \int_{|q| \leq \epsilon^k \frac{1}{2}} \int_{\mathbb{S}^2} \frac{1}{4\pi} \left( \frac{cg}{2p^0 q^0} |p - q| \right) e^{c^2 - c q^0} d\omega dq \]

\[ + \int_{|q| \leq \epsilon^k \frac{1}{2}} \int_{\mathbb{S}^2} \frac{1}{4\pi} |p - q| e^{c^2 - c q^0} d\omega dq \]

\[ =: J_1 + J_2 + J_3. \] (3.26)

It follows from Lemma 3.2 and (3.6) that

\[ |J_1| \lesssim \int_{|q| \leq \epsilon^k \frac{1}{2}} \frac{c}{c^2 + 1} e^{-|q|} dq \lesssim c^{-1}. \] (3.27)

Using (3.6), we have

\[ J_3 \gtrsim \int_{|q| \leq \epsilon^k \frac{1}{2}} |p - q| e^{-\frac{|q|^2}{2}} dq \lesssim 1 + |p|. \] (3.28)

For \( J_2 \), notice that

\[ g^2 = 2p^0 q^0 - 2p \cdot q - 2c^2 = |p - q|^2 + 2p^0 q^0 - 2c^2 - |q|^2 \]

\[ = |p - q|^2 + \frac{4(|p|^2 + c^2)(|q|^2 + c^2) - (2c^2 + |p|^2 + |q|^2)^2}{2p^0 q^0 + (2c^2 + |p|^2 + |q|^2)} \]

\[ = |p - q|^2 - \frac{(p^2 - |q|^2)^2}{2p^0 q^0 + (2c^2 + |p|^2 + |q|^2)}. \] (3.29)

then one has

\[ \frac{cg}{2p^0 q^0} - |p - q| = \frac{1}{2p^0 q^0} \left( cg - 2p^0 q^0 |p - q| \right) \]

\[ = c^2 g^2 (g^2 + 4c^2) - 4|p - q|^2 (|p|^2 + c^2)(|q|^2 + c^2) \]

\[ = 4c^4 (g^2 - |p|^2 + c^2) + 2c^2 (p^2 - |q|^2) \]

\[ \lesssim O(c^{-2}). \]

which implies that

\[ |J_2| \lesssim \int_{|q| \leq \epsilon^k \frac{1}{2}} c^{-\frac{1}{2}} e^{-|q|} dq \lesssim c^{-\frac{1}{2}}. \] (3.30)

It follows from (3.26)–(3.28) and (3.30) that

\[ \int_{|q| \leq \epsilon^k \frac{1}{2}} \int_{\mathbb{S}^2} \frac{c^3}{8\pi (c^2 + 1)} g \frac{\sqrt{s}}{p^0 q^0} e^{c^2 - c q^0} d\omega dq \approx 1 + |p|. \] (3.31)

Case 3: \( |q| \leq \epsilon^k \frac{1}{2} \) and \( \epsilon \geq |p| \geq \epsilon^k \). It follows from Lemma 3.2 that

\[ g \geq \frac{c|p - q|}{\sqrt{p^0 q^0}} \gtrsim \frac{c|p|}{\epsilon} = |p|, \]

and

\[ g \leq |p - q| \lesssim |p|, \]

which yields that \( g \approx |p|. \) Thus we have

\[ \int_{|q| \leq \epsilon^k \frac{1}{2}} \int_{\mathbb{S}^2} \frac{c^3}{8\pi (c^2 + 1)} g \frac{\sqrt{s}}{p^0 q^0} e^{c^2 - c q^0} d\omega dq \approx \int_{|q| \leq \epsilon^k \frac{1}{2}} \int_{\mathbb{S}^2} |p| e^{c^2 - c q^0} d\omega dq \approx 1 + |p|. \] (3.32)
Case 4: \( |q| \leq c^\frac{1}{2} \) and \( |p| \geq c \). It is obvious that
\[
\int_{|q| \leq c^\frac{1}{2}} \int_{\mathbb{S}^2} \frac{c^3}{8\pi(c^2 + 1)} \frac{g\sqrt{s}}{p^0q^0} e^{c^2-cq^0} d\omega dq \lesssim \int_{|q| \leq c^\frac{1}{2}} ce^{c^2-cq^0} dq \lesssim c. \tag{3.33}
\]
On the other hand, since \( |p| \geq c \), one has
\[
g \geq \frac{c|p-q|}{\sqrt{p^0q^0}} \gtrsim \frac{c|p|}{(|p| + c^2)^\frac{1}{2} \sqrt{c}} \gtrsim \sqrt{c}|p|.
\]
Thus we have
\[
\int_{|q| \leq c^\frac{1}{2}} \int_{\mathbb{S}^2} \frac{c^3}{8\pi(c^2 + 1)} \frac{g\sqrt{s}}{p^0q^0} e^{c^2-cq^0} d\omega dq \gtrsim \int_{|q| \leq c^\frac{1}{2}} \frac{\sqrt{c}|p|}{\sqrt{c + |p|}} \frac{e^{c^2-cq^0}}{p^0} dq \gtrsim c. \tag{3.34}
\]
It follows from (3.33) and (3.34) that
\[
\int_{|q| \leq c^\frac{1}{2}} \int_{\mathbb{S}^2} \frac{c^3}{8\pi(c^2 + 1)} \frac{g\sqrt{s}}{p^0q^0} e^{c^2-cq^0} d\omega dq \gtrsim c. \tag{3.35}
\]
Combining (3.25), (3.31), (3.32) and (3.35), we conclude (3.24). Therefore the proof of Lemma 3.6 is completed. \(\square\)

**Remark 3.7.** By similar arguments as in Lemma 3.6, we can obtain
\[
\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} v_\eta J^\eta(q) d\omega dq \equiv \nu(p), \quad \text{for } \eta > 0. \tag{3.36}
\]

3.3. **Uniform coercivity estimate on \( L \).** In this subsection, we shall derive a uniform in \( c \) coercivity estimate for the relativistic linear operator \( L \). For later use, we denote
\[
\begin{align*}
\hat{k}_1(p,q) &= |p-q|e^{-\frac{|p|^2}{4} - \frac{|q|^2}{4}}, \\
\hat{k}_2(p,q) &= \frac{2}{|p-q|} e^{-\frac{|p|^2}{4} - \frac{|q|^2}{4}},
\end{align*}
\]
which are indeed the corresponding kernels of Newtonian Boltzmann equation.

**Lemma 3.8.** It holds that
\[
\int_{\mathbb{R}^3} |k_1(p,q) - \hat{k}_1(p,q)| dq \lesssim c^{-\frac{3}{2}}, \quad p \in \mathbb{R}^3. \tag{3.39}
\]

**Proof.** We split the proof into three cases.

**Case 1.** \( |p| \geq c^\frac{1}{2} \). It follows from (3.37) and Lemma 3.3 that
\[
\int_{\mathbb{R}^3} |k_1(p,q) - \hat{k}_1(p,q)| dq \lesssim \int_{\mathbb{R}^3} |p-q|e^{-\frac{|p|^2}{4} - \frac{|q|^2}{4}} dq + \int_{\mathbb{R}^3} |p-q|e^{-\frac{|p|^2 - |q|^2}{4}} dq \\
\lesssim e^{-\frac{3}{4}c^\frac{1}{2}}. \tag{3.40}
\]

**Case 2.** \( |p| \leq c^\frac{1}{2} \) and \( |q| \geq c^\frac{1}{2} \). Using Lemma 3.3 again, one has
\[
\int_{|q| \geq c^\frac{1}{2}} |k_1(p,q) - \hat{k}_1(p,q)| dq \lesssim \int_{|q| \geq c^\frac{1}{2}} |p-q|e^{-\frac{|p|^2}{4} - \frac{|q|^2}{4}} dq + \int_{|q| \geq c^\frac{1}{2}} |p-q|e^{-\frac{|p|^2 - |q|^2}{4}} dq \\
\lesssim e^{-\frac{3}{4}c^\frac{1}{2}}. \tag{3.41}
\]

**Case 3.** \( |p| \leq c^\frac{1}{2} \) and \( |q| \leq c^\frac{1}{2} \). Using Lemma 3.2, one has
\[
|k_1(p,q) - \hat{k}_1(p,q)| \leq \frac{c^3}{2(c^2 + 1)} \frac{g\sqrt{s}}{p^0q^0} \left( e^{\frac{1}{2}(c^2-cq^0)} e^{\frac{1}{2}(c^2-cq^0)} - e^{-\frac{|p|^2 - |q|^2}{4}} \right) \\
+ \left| \frac{c^3}{2(c^2 + 1)} \frac{g\sqrt{s}}{p^0q^0} - |p-q|e^{-\frac{|p|^2 - |q|^2}{4}} \right|
\]
For the second term on the right hand side of (3.42), we notice that

\[ \left| p - q e^{-|p|^2 - |q|^2} e^{\frac{|p|^2}{4} + \frac{1}{2}(c^2 - \epsilon p^0) + \frac{|q|^2}{4} + \frac{1}{2}(c^2 - \epsilon q^0)} - 1 \right| + \left| \frac{e^{\frac{|p|^2}{4} + \frac{1}{2}(c^2 - \epsilon p^0) + \frac{|q|^2}{4} + \frac{1}{2}(c^2 - \epsilon q^0)} - 1}{2(c^2 + 1)p^0 q^0} \right| \]

A direct calculation shows that

\[
\begin{aligned}
\left| \frac{|p|^2}{4} + \frac{1}{2}(c^2 - \epsilon p^0) \right| &= \frac{|p|^4}{4c^2} \left( \frac{1 + |p|^2}{c^2} + 1 \right)^{-2} \lesssim c^{-\frac{3}{2}}, \\
\left| \frac{|q|^2}{4} + \frac{1}{2}(c^2 - \epsilon q^0) \right| &= \frac{|q|^4}{4c^2} \left( \frac{1 + |q|^2}{c^2} + 1 \right)^{-2} \lesssim c^{-\frac{3}{2}},
\end{aligned}
\]

which yields that

\[
\begin{aligned}
&\int_{|q| \leq \frac{1}{c}} |p - q| e^{-|p|^2 - |q|^2} e^{\frac{|p|^2}{4} + \frac{1}{2}(c^2 - \epsilon p^0) + \frac{|q|^2}{4} + \frac{1}{2}(c^2 - \epsilon q^0)} \left| dq \right| \\
&\lesssim c^{-\frac{3}{2}} \int_{|q| \leq \frac{1}{c}} |p - q| e^{-|p|^2 - |q|^2} \left| dq \right| \\
&\lesssim c^{-\frac{3}{2}}.
\end{aligned}
\]

For the second term on the right hand side of (3.42), we notice that

\[
\begin{aligned}
\frac{e^{\sqrt{\epsilon} - 2p^0 q^0}}{2p^0 q^0} - 1 &= \frac{e^{\sqrt{\epsilon} - 2p^0 q^0}}{2p^0 q^0} = \frac{2c^2 - 4c^2((|p|^2 + |q|^2) - 4|p|^2 |q|^2)}{2p^0 q^0(e^{\sqrt{\epsilon}} + 2p^0 q^0)} \lesssim O(c^{-\frac{3}{2}}).
\end{aligned}
\]

It follows from (3.29) that

\[
\begin{aligned}
g^2 - |p - q|^2 &= -\frac{(|p|^2 - |q|^2)^2}{2p^0 q^0 + (2c^2 + |p|^2 + |q|^2)} \lesssim O(c^{-\frac{3}{2}}),
\end{aligned}
\]

which yields that

\[
\begin{aligned}
|g - |p - q|| &= \frac{|g^2 - |p - q|^2|}{g + |p - q|} \lesssim \frac{O(c^{-\frac{3}{2}})}{g + |p - q|} \lesssim \frac{O(c^{-\frac{3}{2}})}{|p - q|}.
\end{aligned}
\]

Using (3.44) and (3.46), one has

\[
\begin{aligned}
\frac{e^{\sqrt{\epsilon} - 2p^0 q^0}}{2(c^2 + 1)p^0 q^0} - |p - q| &\leq \frac{e^{\sqrt{\epsilon} - 2p^0 q^0}}{2(c^2 + 1)p^0 q^0} - 1 |g^2 - (\frac{c^2}{c^2 + 1}) - 1| + |g - |p - q||
\lesssim (g + \frac{1}{|p - q|})c^{-\frac{3}{2}} \lesssim (|p - q| + \frac{1}{|p - q|})c^{-\frac{3}{2}},
\end{aligned}
\]

which implies that

\[
\begin{aligned}
&\int_{|q| \leq \frac{1}{c}} \left| \frac{e^{\sqrt{\epsilon} - 2p^0 q^0}}{2(c^2 + 1)p^0 q^0} - |p - q| e^{-|p|^2 - |q|^2} \left| dq \right| \\
&\lesssim c^{-\frac{3}{2}} \int_{|q| \leq \frac{1}{c}} \left( \frac{e^{\sqrt{\epsilon} - 2p^0 q^0}}{2(c^2 + 1)p^0 q^0} + \frac{1}{|p - q|} \right) e^{-|p|^2 - |q|^2} \left| dq \right| \\
&\lesssim c^{-\frac{3}{2}}.
\end{aligned}
\]

Combining (3.42), (3.43) and (3.47), we have that

\[
\int_{|q| \leq \frac{1}{c}} |k_1(p, q) - \tilde{k}_1(p, q)| dq \lesssim c^{-\frac{3}{2}}, \quad \text{for} \ |p| \leq \frac{1}{c^2}.
\]

Hence, we conclude (3.39) from (3.40), (3.41) and (3.48). Therefore the proof of Lemma 3.8 is completed. □
Lemma 3.9. It holds that
\[
\int_{\mathbb{R}^3} |k_2(p, q) - \tilde{k}_2(p, q)| dq \lesssim \epsilon^{-\frac{\bar{k}}{2}}, \quad p \in \mathbb{R}^3.
\]

Proof. Since the proof is complicated, we split the proof into three cases.

Case 1. \(|p - q| \geq \epsilon^{\frac{3}{2}}\). It follows from Lemma 3.3 that
\[
\int_{|p-q| \geq \epsilon^{\frac{3}{2}}} |k_2(p, q) - \tilde{k}_2(p, q)| dq \lesssim \int_{|p-q| \geq \epsilon^{\frac{3}{2}}} \frac{1}{|p-q|} e^{-\frac{|p-q|}{\epsilon^{\frac{3}{2}}}} dq + \int_{|p-q| \geq \epsilon^{\frac{3}{2}}} \frac{1}{|p-q|} e^{-\frac{|p-q|^2}{s(p-q)^2}} dq \\
\lesssim e^{-\frac{\bar{k}}{16}}.
\]
(3.49)

Case 2. \(|p - q| \leq \epsilon^{\frac{3}{2}}\) and \(|p| \geq \epsilon^{\frac{3}{2}}\). By Lemma 3.4 and the classical estimate for \(\tilde{k}\) as in [26] Lemma 3.3.1, one has
\[
\int_{|p-q| \leq \epsilon^{\frac{3}{2}}} |k_2(p, q) - \tilde{k}_2(p, q)| dq \lesssim \frac{1}{c} + \frac{1}{1 + |p|} \lesssim \epsilon^{-\frac{\bar{k}}{2}}.
\]
(3.50)

Case 3. \(|p - q| \leq \epsilon^{\frac{3}{2}}\) and \(|q| \leq \epsilon^{\frac{3}{2}}\). In this case, we have \(|p| \leq \epsilon^{\frac{3}{2}}\) and \(|q| \leq \epsilon^{\frac{3}{2}}\), then it follows from (1.21), (3.38) and Lemma 3.2 that
\[
|k_2(p, q) - \tilde{k}_2(p, q)| \lesssim \frac{c^3}{8(c^2 + 1) gp^0 q^0} \frac{s^2}{(\ell^2 - j^2)^\frac{3}{2}} \left( \epsilon^{c^2 - \sqrt{\ell^2 - j^2}} - e^{-\frac{|p-q|^2}{s(p-q)^2}} \right) \\
+ \frac{c^3}{8(c^2 + 1) gp^0 q^0} \frac{s^2}{(\ell^2 - j^2)^\frac{3}{2}} \left( \epsilon^{c^2 - \sqrt{\ell^2 - j^2}} - \frac{2}{|p-q|} e^{-\frac{|p-q|^2}{s(p-q)^2}} \right) \\
\lesssim \frac{1}{|p-q|} e^{-\frac{|p-q|^2}{s(p-q)^2}} \left( \epsilon^{c^2 - \sqrt{\ell^2 - j^2}} - \frac{2}{|p-q|} e^{-\frac{|p-q|^2}{s(p-q)^2}} \right) \\
+ \frac{1}{c^2 + 1 g^{0}\bar{q}^0 |p-q|^3} \left( \epsilon^{c^2 - \sqrt{\ell^2 - j^2}} - \frac{2}{|p-q|} e^{-\frac{|p-q|^2}{s(p-q)^2}} \right) \\
:= I_1 + I_2.
\]

For \(I_2\), it follows from Lemma 3.2 that
\[
\frac{1}{c^2 + 1 g^{0}\bar{q}^0 |p-q|^3} \left( \epsilon^{c^2 - \sqrt{\ell^2 - j^2}} - \frac{2}{|p-q|} e^{-\frac{|p-q|^2}{s(p-q)^2}} \right) \\
\lesssim 1 + \frac{c^2}{c^2 + 1 |p-q|} \left( \frac{4 p^0 q^0 |p-q|}{s p^0 q^0} \right) - \frac{2}{|p-q|} \\
\lesssim I_{21} + I_{22} + I_{23} + I_{24}.
\]

It is direct to see that
\[
|I_{23}| + |I_{24}| \lesssim \frac{1}{|p-q|} e^{-2}.
\]
(3.51)

For \(I_{22}\), we notice that
\[
\frac{s}{4 p^0 q^0} - 1 = \frac{s - 4 p^0 q^0}{4 p^0 q^0} = \frac{(g^2 + 4 c^2)^2 - 16(c^2 + |p|^2)(c^2 + |q|^2)}{4 p^0 q^0 (s + 4 p^0 q^0)} \\
= \frac{g^4 + 8 g^2 c^2 - 16 c^2 (|p|^2 + |q|^2) - 16 |p|^2 |q|^2}{4 p^0 q^0 (s + 4 p^0 q^0)}
\]
Thus we can obtain

\[ |I_{22}| \lesssim \frac{1}{|p - q|} \epsilon^{-\frac{5}{2}}. \tag{3.52} \]

For \( I_{21} \), we notice that

\[
\sqrt{s}(p^0 + q^0) - 1 = \frac{\sqrt{s}(p^0 + q^0) - 4p^0q^0|p - q|}{4p^0q^0|p - q|}
= \frac{\sqrt{s} - 2p^0|p - q|}{4q^0|p - q|} + \frac{\sqrt{s} - 2p^0|p - q|}{4p^0|p - q|}.
\]

Using (3.45), one has

\[
\frac{\sqrt{s} - 2p^0|p - q|}{4q^0|p - q|} = \frac{(g^2 + 4c^2)g^2 - 4(|q|^2 + \epsilon^2)|p - q|^2}{4q^0|p - q| (\sqrt{s} + 2q^0|p - q|)}
= \frac{4c^2(g^2 - |p - q|^2) + g^4 - 4|q|^2|p - q|^2}{4q^0|p - q| (\sqrt{s} + 2q^0|p - q|)}
= - \frac{4g^4|p - q| (\sqrt{s} + 2q^0|p - q|)(2p^0q^0 + (2c^2 + |p|^2 + |q|^2))}{g^4 - 4|q|^2|p - q|^2}
\quad + \frac{4q^0|p - q| (\sqrt{s} + 2q^0|p - q|)}{4q^0|p - q| (\sqrt{s} + 2q^0|p - q|)}
\lesssim O(\epsilon^{-\frac{5}{2}}),
\]

and

\[
\frac{\sqrt{s} - 2p^0|p - q|}{4q^0|p - q|} \lesssim O(\epsilon^{-\frac{5}{2}}).
\]

Thus we can obtain

\[ |I_{21}| \lesssim \frac{1}{|p - q|} \epsilon^{-\frac{5}{2}}. \tag{3.53} \]

Combining (3.51), (3.52) and (3.53), one obtains, for \(|p| \leq \epsilon^2\), that

\[ \int_{|p - q| \leq \epsilon^2} |I_2| dq \lesssim \epsilon^{-\frac{5}{2}} \int_{|p - q| \leq \epsilon^2} \frac{e^{-\left| \frac{|p - q|^2}{8} - \frac{(|p|^2 - |q|^2)^2}{4|p - q|^2} \right|}}{|p - q|} dq \lesssim \epsilon^{-\frac{5}{2}}. \tag{3.54} \]

Next, we consider \( I_1 \). It follows from (3.13) and (3.29) that

\[
\frac{|p - q|^2}{8} + \frac{(|p|^2 - |q|^2)^2}{8|p - q|^2} + \epsilon^2 - \sqrt{\epsilon^2 - J^2}
= \left[ \frac{(|p|^2 - |q|^2)^2}{8|p - q|^2} - \frac{1}{1 + \frac{\epsilon^2}{4 \epsilon^2} \frac{|p - q|^2}{g^2}} \frac{2p^0q^0 + (2c^2 + |p|^2 + |q|^2)}{g^4 - 4|q|^2|p - q|^2} \right]
+ \left[ \frac{|p - q|^2}{8} - \frac{1}{1 + \frac{\epsilon^2}{4 \epsilon^2} \frac{|p - q|^2}{g^2}} \right]
=: I_{11} + I_{12}. \tag{3.55}
\]

For \( I_{11} \), we have from (3.29) that

\[
I_{11} = \frac{|p - q|^2}{8} \left( 1 - \frac{2}{1 + \frac{\epsilon^2}{4 \epsilon^2} \frac{|p - q|^2}{g^2}} \right)
= \frac{|p - q|^2}{8} \frac{\sqrt{\frac{\epsilon^2}{4 \epsilon^2} \frac{|p - q|^2}{g^2}} - 1}{1 + \frac{\epsilon^2}{4 \epsilon^2} \frac{|p - q|^2}{g^2}}
= \frac{|p - q|^2}{8} \frac{\sqrt{s} |p - q| - 2 \epsilon g}{\sqrt{s} |p - q| + 2 \epsilon g}.
\]
\begin{align*}
\frac{\lvert p - q \rvert^2 g^2 |p-q|^2 + 4c^2 (|p-q|^2 - g^2)}{8} & = \frac{|p-q|^2}{8} \left\{ \frac{g^2 |p-q|^2}{\sqrt{8}|p-q| + 2cg^2} + \frac{4c^2}{\sqrt{8}|p-q| + 2cg^2} \right\} \\
& \leq \mathcal{O}(c^{-1}).
\end{align*}

For \( I_{12} \), it can be written as
\begin{align*}
\frac{(|p|^2 - |q|^2)^2}{8|p-q|^2} & \left\{ 1 - \frac{1}{\sqrt{\frac{|p|^2 - |q|^2}{g}}} \right\} \frac{g^2}{2p^0 q^0 + (2c^2 + |p|^2 + |q|^2)} \\
& = \frac{(|p|^2 - |q|^2)^2 (2cg^2 + \sqrt{8}|p-q|)(2p^0 q^0 + (2c^2 + |p|^2 + |q|^2)) - 16c^3|p-q|^2}{8|p-q|^2} \\
& = \frac{8|p-q|^2}{(2cg^2 + \sqrt{8}|p-q|)(2p^0 q^0 + (2c^2 + |p|^2 + |q|^2))} \\
& + \frac{(|p|^2 - |q|^2)^2 (2\sqrt{8}p^0 q^0)|p-q|^2 - 4c^3(p - q)^2 + (2\sqrt{8}|p-q|g)^2}{2c^2 + \sqrt{8}|p-q|}(2p^0 q^0 + (2c^2 + |p|^2 + |q|^2)) \\
& = \frac{|p|^2 - |q|^2}{8|p-q|^2} (I_{121} + I_{122} + I_{123} + I_{124} + I_{125}).
\end{align*}

We have from (3.29) that
\begin{align*}
I_{121} &= \frac{4c^3 g^2 - 4c^3 |p-q|^2}{2c^2 + \sqrt{8}|p-q|g}(2p^0 q^0 + (2c^2 + |p|^2 + |q|^2)) \\
& = -4c^3 (|p|^2 - |q|^2)^2 \\
& \leq \mathcal{O}(c^{-\frac{3}{2}}),
\end{align*}
where we have used \( g^2 p^0 q^0 \geq c^2 |p-q|^2 \). Similarly, one has
\begin{align*}
I_{122} &= \frac{4c^2 p^0 q^0 - 4c^3 |p-q|^2}{(2cg^2 + \sqrt{8}|p-q|g)(2p^0 q^0 + (2c^2 + |p|^2 + |q|^2))} \\
& = -4c^2 p^0 q^0 |p-q|^2 + 4c|p-q|^2 p^0 q^0 - c^2 \\
& \leq \mathcal{O}(c^{-\frac{3}{2}}),
\end{align*}
\begin{align*}
I_{123} &= \frac{2\sqrt{8}|p-q|g c^2 - 4c^3 |p-q|^2}{(2cg^2 + \sqrt{8}|p-q|g)(2p^0 q^0 + (2c^2 + |p|^2 + |q|^2))} \\
& = -2c^2 |p-q| \\
& \leq \mathcal{O}(c^{-\frac{3}{2}}),
\end{align*}
\begin{align*}
I_{124} &= \frac{2\sqrt{8}p^0 q^0 |p-q| - 4c^3 |p-q|^2}{(2cg^2 + \sqrt{8}|p-q|g)(2p^0 q^0 + (2c^2 + |p|^2 + |q|^2))} \\
& = -2|p-q| \frac{(2c^3|p-q|-\sqrt{8}p^0 q^0 g)}{2|p-q| + \sqrt{8} p^0 q^0 g} \\
& \leq \mathcal{O}(c^{-\frac{3}{2}}),
\end{align*}
\begin{align*}
I_{125} &= \frac{2\sqrt{8} |p-q| g^2}{(2cg^2 + \sqrt{8}|p-q|g)(2p^0 q^0 + (2c^2 + |p|^2 + |q|^2))} \\
& = -2|p-q| \frac{4c^3|p-q|^2 s(p^0)^2(q^0)^2 g^2}{2|p-q| + \sqrt{8} p^0 q^0 g} \\
& \leq \mathcal{O}(c^{-\frac{3}{2}}).
\end{align*}
which, together with (3.55)–(3.56), yields that

\[
\text{Lemma 3.10.}
\]

\[
\text{For any fixed } p \in \mathbb{R}^3, \text{ it holds that}
\]

\[
\lim_{\epsilon \to \infty} e_\alpha = \tilde{e}_\alpha, \quad \alpha = 0, 1, \ldots, 4.
\]

**Proof.** It is clear that

\[
\lim_{\epsilon \to \infty} \hat{J}(p) = \hat{\mu}(p).
\]

Using Lemma 2.3, one has

\[
\lim_{\epsilon \to \infty} A_1 = \lim_{\epsilon \to \infty} \frac{K_2(\epsilon^2)}{K_4(\epsilon^2)} = 1,
\]

which yields that

\[
\lim_{\epsilon \to \infty} e_\alpha = \tilde{e}_\alpha, \quad \alpha = 0, 1, 2, 3.
\]

For the convergence of \(e_4\), we only need to show the following limit

\[
\lim_{\epsilon \to \infty} \frac{p^0 - A_3}{\sqrt{A_2 - A_3^2}} = \lim_{\epsilon \to \infty} \frac{(p^0)^2 - A_3^2}{\sqrt{A_2 - A_3^2(p^0 + A_3)}} = \frac{|p|^2 - 3}{\sqrt{6}}.
\]

It follows from Lemma 2.3 that, for large \(z > 0\),

\[
\left\{\begin{array}{l}
K_3(z) = \frac{\pi}{2z e^2} \left[ 1 + \frac{35}{8z} + \frac{945}{2! 8z^2} + \frac{10395}{3! 8z^3} + \frac{135135}{4! 8z^4} + O(z^{-5}) \right], \\
K_2(z) = \frac{\pi}{2z e^2} \left[ 1 + \frac{15}{8z} + \frac{105}{2! 8z^2} - \frac{945}{3! 8z^3} + \frac{31185}{4! 8z^4} + O(z^{-5}) \right],
\end{array}\right.
\]
which yields that
\[
\begin{align*}
K_3(z) - K_2(z) &= \sqrt{\pi} \frac{1}{2\pi e^2} \left[ \frac{20}{8z} + \frac{840}{2!(8z)^2} + \frac{11340}{3!(8z)^3} - \frac{166320}{4!(8z)^4} + O(z^{-5}) \right], \\
K_3(z) + K_2(z) &= \sqrt{\pi} \frac{1}{2\pi e^2} \left[ 2 + \frac{50}{8z} + \frac{1050}{2!(8z)^2} + \frac{9450}{3!(8z)^3} - \frac{103950}{4!(8z)^4} + O(z^{-5}) \right].
\end{align*}
\] (3.68)

A direct calculation shows that
\[
\frac{K_3^2(z)}{K_2(z)} - 1 = \frac{(K_3(z) - K_2(z)(K_3(z) + K_2(z)))}{K_2(z)} = \frac{\frac{40}{8z} + \frac{1840}{64z^2} + O(z^{-3})}{1 + \frac{30}{8z} + O(z^{-2})},
\] (3.69)
and
\[
\frac{K_3(z)}{K_2(z)} - 1 = \frac{(K_3(z) - K_2(z))}{K_2(z)} = \frac{\frac{20}{8z} + O(z^{-2})}{1 + O(z^{-1})}.
\] (3.70)

Using (3.69), one obtains
\[
(p^0)^2 - A_3^2 = |p|^2 + c^2 - \left( \frac{cK_3(c^2)}{K_2(c^2)} - \frac{1}{c} \right)^2 = |p|^2 + c^2 \left( 1 - \frac{K_3(c^2)}{K_2(c^2)} \right) - \frac{1}{c^2} + 2 \frac{K_3(c^2)}{K_2(c^2)} = |p|^2 - \frac{5 + \frac{1840}{64z^2} + O(\epsilon^{-4})}{1 + \frac{30}{8z} + O(\epsilon^{-4})} - \frac{1}{c^2} + 2 \frac{K_3(c^2)}{K_2(c^2)} \to |p|^2 - 3, \quad \text{as } \epsilon \to \infty.
\] (3.71)

A direct calculation shows that
\[
\sqrt{A_2 - A_3^2(p^0 + A_3)} = \sqrt{c^2(A_2 - A_3^2) \cdot \frac{1}{c}(p^0 + A_3)} = \sqrt{c^2 \left( 1 - \frac{K_3(c^2)}{K_2(c^2)} \right) + 5 \frac{K_3(c^2)}{K_2(c^2)}} - 1 \cdot \sqrt{1 + \frac{|p|^2}{c^2} + \frac{K_3(c^2)}{K_2(c^2)} - \frac{1}{c^2}}.
\] (3.72)

It follows from (3.69)–(3.70) that
\[
c^2(A_2 - A_3^2) = c^2 \left[ - \frac{5 + \frac{1840}{64z^2} + O(\epsilon^{-4})}{1 + \frac{30}{8z} + O(\epsilon^{-4})} + 5 \frac{K_3(c^2)}{K_2(c^2)} \right] - 1 = \frac{c^2}{2} \left[ 5 \frac{K_3(c^2)}{K_2(c^2)} - 1 \right] + 5 \frac{100}{8z} + O(\epsilon^{-4}) - \frac{10}{27} + O(\epsilon^{-4}) \right] - 1 = \frac{3}{2} + O(\epsilon^{-2}) \to \frac{3}{2}, \quad \text{as } \epsilon \to \infty,
\] (3.73)
and
\[
\sqrt{1 + \frac{|p|^2}{c^2} + \frac{K_3(c^2)}{K_2(c^2)} - \frac{1}{c^2}} \to 2, \quad \text{as } \epsilon \to \infty.
\] (3.74)

Combining (3.72)–(3.74), one has that
\[
\lim_{\epsilon \to \infty} \sqrt{A_2 - A_3^2(p^0 + A_3)} = \sqrt{6},
\]
which, together with (3.71), concludes (3.66). Therefore the proof of Lemma 3.10 is completed.

With above preparations, we shall prove the coercivity estimate for the linear operator $L$. 

\[\square\]
Lemma 3.11 (Uniform coercivity estimate on \( L \)). There exists a positive constant \( \zeta_0 > 0 \), which is independent of \( c \), such that

\[
\langle Lg, g \rangle \geq \zeta_0 \| (I - P)g \|_{\nu}^2,
\]

for any \( g \in L^2_\nu(\mathbb{R}^3_+) \).

Proof. In the following, we shall replace \( L, \nu \) and \( K \) with \( L_\varepsilon, \nu_\varepsilon \) and \( K_\varepsilon \), respectively, to show the explicit dependency of \( L, \nu \) and \( K \) on \( \varepsilon \). Clearly, one only needs to show that there is a positive constant \( \zeta_0 > 0 \), which is independent of \( \varepsilon \), so that

\[
\langle L_\varepsilon g, g \rangle \geq \zeta_0 \| g \|_{\nu_\varepsilon}^2 = \zeta_0,
\]

holds for any \( \varepsilon \) and any \( g \in \mathcal{N}_\varepsilon^+ \) with \( \| g \|_{\nu_\varepsilon} = 1 \).

For any given \( \varepsilon \), using Lemma 3.4 in [59], there exists a positive constant \( \alpha_\varepsilon > 0 \), such that

\[
\langle L_\varepsilon g, g \rangle \geq \alpha_\varepsilon \| g \|_{\nu_\varepsilon}^2 = \alpha_\varepsilon,
\]

for any \( g \in \mathcal{N}_\varepsilon^+ \) with \( \| g \|_{\nu_\varepsilon} = 1 \). Denote

\[
\zeta_\varepsilon := \inf_{g \in \mathcal{N}_\varepsilon^+} \frac{\langle L_\varepsilon g, g \rangle}{\| g \|_{\nu_\varepsilon}^2},
\]

It follows from (3.76) that \( \zeta_\varepsilon \geq \alpha_\varepsilon > 0 \) for any \( \varepsilon \). To prove (3.75), it suffices to show that

\[
\inf_{\varepsilon \geq 1} \zeta_\varepsilon > 0.
\]

We prove (3.78) by contradiction. Assume that (3.78) is not true, then there exists a sequence \( \{\zeta_{\varepsilon_n}\} \) such that

\[
\lim_{n \to \infty} \zeta_{\varepsilon_n} = \infty \quad \text{and} \quad \lim_{n \to \infty} \zeta_{\varepsilon_n} = 0.
\]

For each \( n \), owing to (3.77), there exists \( g_n \in \mathcal{N}_{\varepsilon_n}^+ \) with \( \| g_n \|_{\nu_{\varepsilon_n}} = 1 \), so that

\[
\zeta_{\varepsilon_n} \leq \langle L_{\varepsilon_n} g_n, g_n \rangle < \zeta_{\varepsilon_n} + \frac{1}{n},
\]

which, together with (3.79), yields that

\[
\lim_{n \to \infty} \langle L_{\varepsilon_n} g_n, g_n \rangle = 0.
\]

It is clear that \( \{g_n\}_{n=1}^\infty \) is a bounded sequence in \( L^2(\mathbb{R}^3_+) \). Since \( L^2 \) is a Hilbert space, based on the Eberlein-Smulian theorem, we have the weakly convergent sequence (up to extracting a subsequence with an abuse of notation) \( g_n \rightharpoonup g \) in \( L^2 \). Moreover, for any fixed \( N \geq 1 \), one has

\[
\chi_{\{|p| \leq N\}} \sqrt{\nu_{\varepsilon_n}} g_n \rightharpoonup \chi_{\{|p| \leq N\}} \sqrt{\nu} g \quad \text{in} \quad L^2,
\]

where \( \nu(p) = \lim_{\varepsilon \to \infty} \nu_\varepsilon(p) \). Hence, by the weak semi-continuity, for any fixed \( N \), we have

\[
\| \chi_{\{|p| \leq N\}} \sqrt{\nu} g \|_{L^2} \leq \lim_{n \to \infty} \inf \| \chi_{\{|p| \leq N\}} \sqrt{\nu_{\varepsilon_n}} g_n \|_{L^2} \leq 1,
\]

which implies that

\[
\| \sqrt{\nu} g \|_{L^2} \leq 1.
\]

For later use, we denote

\[
\tilde{L} f := \tilde{\nu} f - \tilde{K} f,
\]

where \( \tilde{K} f := \int_{\mathbb{R}^3} \tilde{b}_2(p, q) - \tilde{b}_1(p, q) f(q) dq \) with \( \tilde{b}_1 \) and \( \tilde{b}_2 \) defined in (3.37)–(3.38). We also denote \( \tilde{N} \) as the null space of \( \tilde{L} \), that is, \( \tilde{N} := \text{span}\{\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\} \). Clearly, we have

\[
0 \leq \langle L_{\varepsilon_n} g_n, g_n \rangle = \| g_n \|_{\nu_{\varepsilon_n}}^2 - \langle (K_{\varepsilon_n} - \tilde{K}) g_n, g_n \rangle - \langle \tilde{K} g_n, g_n \rangle
\]

\[
= 1 - \langle (K_{\varepsilon_n} - \tilde{K}) g_n, g_n \rangle - \langle \tilde{K} g_n, g_n \rangle.
\]

(3.82)
Since $\tilde{K}$ is a compact operator on $L^2$, it holds that
\[
\lim_{n \to \infty} \| \tilde{K}g_n - \tilde{K}g \| = 0.
\]
Hence we have
\[
(\tilde{K}g_n, g_n) - (\tilde{K}g, g) = (\tilde{K}g_n - \tilde{K}g, g_n) + (\tilde{K}g, g_n - g) \to 0, \quad n \to \infty.
\]
It follows from Lemma 3.8 and Lemma 3.9 that
\[
\left( (K_{n} - \tilde{K})g_n, g_n \right) \to 0, \quad n \to \infty. \tag{3.83}
\]
Combining (3.80), (3.82)-(3.83), we have
\[
(\tilde{K}g, g) = 1,
\]
which, together with (3.81), yields that
\[
\| g \|^2 - (\tilde{K}g, g) \leq 0.
\]
Thus we have $g \in \tilde{N}$.

Next, we shall show that $g \in \tilde{N}^\perp$. Recall $e_\alpha^n, \tilde{e}_\alpha$ defined in (3.2) (with $\epsilon$ replaced by $\epsilon_n$) and (3.64). Notice that
\[
0 = (g_n, e_\alpha^n) = (g_n - g, e_\alpha^n - \tilde{e}_\alpha) + (g - g, e_\alpha^n - \tilde{e}_\alpha) + (g, \tilde{e}_\alpha), \quad \alpha = 0, 1, \cdots, 4. \tag{3.85}
\]
Using Lemma 3.10 and $g_n \to g$ in $L^2$, we take limit $n \to \infty$ in (3.85) to obtain
\[
\langle g, e_\alpha \rangle = 0, \quad \alpha = 0, 1, \cdots, 4,
\]
which implies that $g \in \tilde{N}^\perp$. Since we also have $g \in \tilde{N}$, one concludes that $g = 0$, which contradicts with (3.84). Therefore the proof of Lemma 3.11 is completed. \qed

**Remark 3.12.** It is worth noting that we need to obtain a uniform bound for the macroscopic part $\| Pf \|_{L^2}$ in terms of the microscopic part $\| (I-P)f \|_{L^2}$, thus the uniform positive lower bound $\zeta_0$ obtained in Lemma 3.11 plays a crucial role in section 4 and section 5.

### 3.4. Estimate on nonlinear operator

In this subsection, we consider the estimate of nonlinear operator which will be used later.

**Lemma 3.13.** It holds that
\[
\| \nu^{-1} w \Gamma(f_1, f_2) \|_{L^\infty} \lesssim \| w f_1 \|_{L^\infty} \| w f_2 \|_{L^\infty}. \tag{3.86}
\]

**Proof.** For the loss part, noting (3.36), one has
\[
|w(p) \Gamma^-(f_1, f_2)(p)| = \left| \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu \sqrt{J(q)} f_1(p) f_2(q) d\omega dq \right|
\leq \| w f_1 \|_{L^\infty} \| w f_2 \|_{L^\infty} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu \sqrt{J(q)} d\omega dq
\lesssim \nu(p) \| w f_1 \|_{L^\infty} \| w f_2 \|_{L^\infty}. \tag{3.87}
\]
For the gain part, it follows from the conservation law of momentum (1.6) that
\[
|p| = |p' + q' - q| \leq |p'| + |q'| + |q|,
\]
which implies that
\[
|w(p) \lesssim w(p') w(q') w(q).
\]
Thus it follows from (3.36) and (3.88) that
\[
|w(p) \Gamma^+(f_1, f_2)(p)| = \left| \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu \sqrt{J(q)} f_1(p') f_2(q') d\omega dq \right|
\lesssim \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu \sqrt{J(q)} w(q') w(p')(p') \cdot |w(q') f_2(q')| d\omega dq.
\]
\begin{align*}
\lesssim & \|w_1\|_{L^\infty} \|w_2\|_{L^\infty} \int_{\mathbb{R}^3} \int_{S^2} v_\phi J^{\frac{1}{2}}(q)d\omega dq \\
\lesssim & \nu(p) \|w_1\|_{L^\infty} \|w_2\|_{L^\infty}. \tag{3.89}
\end{align*}

Thus (3.86) follows from (3.87) and (3.89). Therefore the proof of Lemma 3.13 is completed. \(\square\)

4. Steady problem

To construct the solution to the steady relativistic Boltzmann equation (1.10), we first consider the approximate linearized steady problem

\begin{align*}
\varepsilon f + \hat{p} \cdot \nabla_x f + L f &= S, \\
f(x,p)|_{\gamma} &= P_\gamma f + r, \tag{4.1}
\end{align*}

where \(P_\gamma f\) is defined in (1.15). As pointed out in [18, 25], the penalization term \(\varepsilon f\) is used to guarantee the conservation of mass, that is,

\[\int_{\Omega} \int_{\mathbb{R}^3} f(x,p) \sqrt{J(p)}dpdx = 0.\]

We also define

\[h(x,p) := w(p)f(x,p),\]

then (4.1) can be rewritten as

\begin{align*}
\varepsilon h + \hat{p} \cdot \nabla_x h + \nu(p) h &= K_wh + wS, \\
h(x,p)|_{\gamma} &= \frac{1}{w(p)} \int_{n(x,q) > 0} (x,q)\hat{w}(q)d\sigma + wr(x,p), \tag{4.2}
\end{align*}

where

\[d\sigma := J(q)\{n(x, q)\}dq, \quad \hat{w}(p) := \frac{1}{w(p)\sqrt{J(p)}}, \quad K_wh := wK\left(\frac{h}{w}\right).\]

4.1. A priori \(L^\infty\) estimate. For the approximate steady relativistic Boltzmann equation (4.2), the most essential part is to obtain the \(L^\infty\)-bound.

**Definition 4.1.** Given \((t, x, p)\), let \([X(s), P(s)]\) be the backward bi-characteristics for (both steady and unsteady) relativistic Boltzmann equation, which is defined by

\[
\begin{aligned}
\frac{dX(s)}{ds} &= \frac{c}{\sqrt{c^2 + |P(s)|^2}} P(s), \\
\frac{dP(s)}{ds} &= 0,
\end{aligned}
\]

Then the solution is given by

\[X(t) = X(0) + \{n(X(s), P(s))\}ds, \quad P(t) = P(0) + \{n(X(s), P(s))\}ds, \quad [X(t), P(t)] = [x, p].\]

For each \((x, p)\) with \(x \in \bar{\Omega}\) and \(p \neq 0\), we define the backward exit time \(t_b(x, p) \geq 0\) to be the last moment at which the back-time straight line \([X(-\tau; 0, x, p), P(-\tau; 0, x, p)]\) remains in \(\Omega : t_b(x, p) = \sup\{s \geq 0 : x - \hat{p}t \in \bar{\Omega} \text{ for } 0 \leq \tau \leq s\}\). We therefore have \(x - t_b\hat{p} \in \partial\Omega\) and \(\xi(x - t_b\hat{p}) = 0\). We also define

\[x_b(x, p) = x - t_b\hat{p} \in \partial\Omega.\]

Note that \(n(x_b) \cdot p = n(x_b(x, p)) \cdot p \leq 0\) always holds true.
Let $x \in \bar{\Omega}$, $(x, p) \notin \gamma_0 \cup \gamma_-$ and $(t_0, x_0, p_0) = (t, x, p)$. For $p_{k+1} \in \mathcal{P}_{k+1} := \{n(x_{k+1}) \cdot p_{k+1} > 0\}$, the back-time cycle is defined as

\begin{align}
X_{cl}(s; t, x, p) &= \sum_k 1_{(t_{k+1}, t_k)}(s) \{x_k - \tilde{p}_k (t_k - s)\}, \\
\quad P_{cl}(s; t, x, p) &= \sum_k 1_{(t_{k+1}, t_k)}(s) p_k,
\end{align}

(4.3)

with

$$(t_{k+1}, x_{k+1}, p_{k+1}) = (t_k - t_b (x_k, p_k), x_b (x_k, p_k), p_{k+1}).$$

We also define the iterated integral

$$\int_{\Pi_{j=1}^{k-1} p_j} \Pi_{j=1}^{k-1} d\sigma_j := \int_{p_1} \cdots \left( \int_{p_{k-1}} d\sigma_{k-1} \right) \cdots d\sigma_1,$$

where

$$d\sigma_j := J(p_j) \{n(x_j) \cdot \tilde{p}_j\} dp_j, \quad j = 1, \ldots, k - 1,$$

are probability measures.

**Lemma 4.1.** Let $0 \leq \eta \leq \frac{1}{2}$. For $T_0 > 0$ sufficiently large, there exist positive constants $C_1$ and $C_2$ independent of $T_0$ and $\mathcal{C}$ such that for $k = C_1 T_0^\frac{3}{2}$ and $(t, x, p) \in [0, T_0] \times \bar{\Omega} \times \mathbb{R}^3$, it holds that

$$\int_{\Pi_{j=1}^{k-1} p_j} 1_{\{t_k > 0\}} \Pi_{j=1}^{k-1} J^{-\eta}(p_j) d\sigma_j \leq \left( \frac{1}{3} \right)^{C_2 T_0^\frac{3}{2}}.$$

**Proof.** Since the proof is very similar to [35, Lemma 3.2], we omit the details of proof for simplicity of presentation. \hfill \square

For later use, we consider the following iterative linear problems involving a parameter $\lambda \in [0, 1]$:

\begin{align}
&\begin{cases}
\varepsilon h^{i+1} + \tilde{p} \cdot \nabla_x h^{i+1} + \nu(p) h^{i+1} = \lambda K_w h^i + wS, \\
h^{i+1}(x, p) \big|_{\gamma_-} = \frac{1}{w(p)} \int_{\Pi_{n(x), q \geq 0}} h^i(x, q) \tilde{w}(q) d\sigma + w(p) r(x, p),
\end{cases} \\
\text{for } i = 0, 1, 2, \ldots, \text{ where } h^0 = h^0(x, p) \text{ is given. For the mild formulation, we have the following lemma whose proof is omitted for brevity as it is similar to [35, Lemma 24].}
\end{align}

**Lemma 4.2.** Let $0 \leq \lambda \leq 1$ and $h^i$, $i = 0, 1, 2, \ldots$, be the solutions of (4.4). Denote $\nu_\varepsilon(p) := \varepsilon + \nu(p)$. For each $t \in [0, T_0]$ and for each $(x, p) \in \bar{\Omega} \times \mathbb{R}^3 \setminus (\gamma_0 \cup \gamma_-)$, we have

$$h^{i+1}(x, p) = \sum_{n=1}^{11} J_n,$$

with

$$J_1 = 1_{\{t_1 \leq 0\}} e^{-\nu_\varepsilon(p) t_1} h^{i+1}(x - \tilde{p} t, p),$$

$$J_2 + J_3 = \int_{\max\{t_1, 0\}}^t e^{-\nu_\varepsilon(p) (t - s)} \left[ \lambda K_w h^i + wS \right] (x - \tilde{p}(t - s), p) ds,$$

$$J_4 = 1_{\{t_1 > 0\}} e^{-\nu_\varepsilon(p) (t - t_1)} w(p) r(x_1, p),$$

$$J_5 = \frac{e^{-\nu_\varepsilon(p) (t - t_1)}}{w(p)} \int_{\Pi_{j=1}^{k-1} p_j} 1_{\{t_1 > 0\}} w(p) r(x_{l+1}, p_l) d\Sigma_t(t_{l+1}).$$
Lemma 4.3. Let \( \beta > 3 \) and \( h^i, i = 0, 1, 2, \ldots \), be the solutions of (4.4), satisfying

\[
\|h^i\|_{L^\infty} + |h^i|_{L^\infty(\gamma)} < \infty.
\]

Then there exists \( T_0 > 0 \) large enough such that for \( i \geq k := C_1 T_0^{\frac{5}{7}} \), it holds that

\[
\|h^{i+1}\|_{L^\infty} + |h^{i+1}|_{L^\infty(\gamma_+)} \leq \frac{1}{8} \sup_{0 \leq l \leq k} \{\|h^{i-l}\|_{L^\infty} + |h^{i-l}|_{L^\infty(\gamma_+)}\}
\]

\[
+ C \left\{\|v^{-1}wS\|_{L^\infty} + |wr|_{L^\infty(\gamma_-)}\right\} + C \sup_{0 \leq l \leq k} \left\{\|h^{i-l}\|_{L^2}\right\}.
\]

Moreover, if \( h^i \equiv h \) for \( i = 1, 2, \ldots \), that is, \( h \) is a solution, then (4.6) is reduced to the following estimate

\[
\|h\|_{L^\infty} + |h|_{L^\infty(\gamma)} \leq C \left\{\|v^{-1}wS\|_{L^\infty} + |wr|_{L^\infty(\gamma_-)}\right\} + C \left\{\|h\|_{L^2}\right\}.
\]

Here it is emphasized that the positive constant \( C > 0 \) does not depend on \( c, \lambda \in [0, 1] \) and \( \varepsilon > 0 \).

Proof. We point out that the idea is similar to [18, Lemma 3.4]. Here we present the details since we need to obtain the uniform estimates which is independent of the light speed \( c \).

It follows from Lemma 3.6 that there exists a positive constant \( \nu_0 \), such that

\[
\nu_c(p) = \varepsilon + \nu(p) \geq \nu(p) \geq \nu_0 > 0,
\]

where \( \nu_0 \) is independent of \( c \). For \( J_1 \), it follows from (4.8) that

\[
|J_1| \leq e^{-\nu_0 t} \|h^{i+1}\|_{L^\infty}.
\]

Using (3.6), one has

\[
\frac{1}{\tilde{w}(p)} = w(p) \sqrt{J(p)} \leq Ce^{-\frac{|p|}{c}},
\]

and

\[
\int_{\mathbb{P}_j} \tilde{w}(p_j) d\sigma_j = \int_{\mathbb{P}_j} \frac{1}{w(p_j) \sqrt{J(p_j)}} J(p_j) \{n(x_j) \cdot \hat{p}_j\} dp_j \leq \int_{\mathbb{R}^3} e^{-\frac{|p_j|}{c}} |p_j| dp_j \leq C,
\]

for

\[
J_6 = \frac{e^{-\nu_c(p)(t-t_1)}}{\tilde{w}(p)} \int_{\Pi_{j=1}^{k-1} p_j} \sum_{l=1}^{k-1} 1_{\{t_{l+1} \leq 0 < t_l\}} h^{i+1-l} (x_l - \hat{p}_l t_l, p_l) d\Sigma_l(0),
\]

\[
J_7 + J_8 = \frac{e^{-\nu_c(p)(t-t_1)}}{\tilde{w}(p)} \int_{\Pi_{j=1}^{k-1} p_j} \sum_{l=1}^{k-1} \int_{0}^{t_l} 1_{\{t_{l+1} \leq 0 < t_l\}}
\]

\[
\times [\lambda K_w h^{i-l} + wS] (x_l - \hat{p}_l (t_l - s), p_l) ds d\Sigma_l(s),
\]

\[
J_0 + J_{10} = \frac{e^{-\nu_c(p)(t-t_1)}}{\tilde{w}(p)} \int_{\Pi_{j=1}^{k-1} p_j} \sum_{l=1}^{k-1} \int_{t_{l+1}}^{t_l} 1_{\{t_{l+1} > 0\}}
\]

\[
\times [\lambda K_w h^{i-l} + wS] (x_l - \hat{p}_l (t_l - s), p_l) ds d\Sigma_l(s),
\]

\[
J_{11} = \frac{e^{-\nu_c(p)(t-t_1)}}{\tilde{w}(p)} \int_{\Pi_{j=1}^{k-1} p_j} 1_{\{t_k > 0\}} h^{i+2-k} (x_k, p_{k-1}) d\Sigma_{k-1}(t_k),
\]

where we have denoted

\[
d\Sigma_l(s) = \left\{\Pi_{j=l+1}^{k-1} d\sigma_j\right\}. \left\{\tilde{w}(p_l) e^{-\nu_c(p_l)(t_l - s)} d\sigma_l\right\}. \left\{\Pi_{j=l+1}^{k-1} e^{-\nu_c(p_l)(t_{l+1} - s)} d\sigma_l\right\}.
\]
which immediately yield that

\[
\begin{aligned}
\int_{\prod_{j=1}^{k-1} P_j} \sum_{l=1}^{k-1} 1_{\{t_{i+1} \leq 0 < t_l\}} \nu(p_l) \hat{w}(p_l) \prod_{j=1}^{k-1} d\sigma_j &\leq C k, \\
\end{aligned}
\]

(4.12)

Then it follows from (4.12) that

\[
\begin{aligned}
|J_3| + |J_5| + |J_{10}| &\leq C k \| \nu^{-1} w S \|_{L^\infty}, \\
|J_4| + |J_5| &\leq C k |w r|_{L^\infty(\gamma^-)},
\end{aligned}
\]

(4.13)

and

\[
\begin{aligned}
|J_6| &\leq C e^{-\frac{|p|}{2} t} e^{-\nu_0 (t-t_1)} \int_{\prod_{j=1}^{k-1} P_j} \sum_{l=1}^{k-1} 1_{\{t_{i+1} \leq 0 < t_l\}} d\Sigma_j(0) \cdot \sup_{1 \leq i \leq k-1} \| h^{i+1-l} \|_{L^\infty} \\
&\leq C k e^{-\frac{|p|}{2} t} \sup_{1 \leq i \leq k-1} \| h^{i+1-l} \|_{L^\infty}.
\end{aligned}
\]

(4.14)

For the term $J_{11}$, it follows from (4.10) and Lemma 4.1 that

\[
|J_{11}| \leq C e^{-\frac{|p|}{2} t} \int_{\prod_{j=1}^{k-1} P_j} 1_{\{t_{k-1} > 0\}} \hat{J}^{-\frac{1}{2}} (p_{k-1}) \Pi_{j=1}^{k-1} d\sigma_j |h^{i+2-k}|_{L^\infty(\gamma^-)}
\]

\[
\leq C e^{-\frac{|p|}{2} t} \left( \frac{1}{2} \right) C_2 T_0^\frac{5}{2} \|h^{i+2-k}\|_{L^\infty(\gamma^-)},
\]

(4.15)

where we have taken $k = C_1 T_0^\frac{5}{2}$ and $T_0$ is a large constant to be chosen later. From the boundary condition (4.14), it holds that

\[
|h^{i+2-k}|_{L^\infty(\gamma^-)} \leq C |h^{i+1-k}|_{L^\infty(\gamma^+)} + |w r|_{L^\infty(\gamma^-)}.
\]

(4.16)

For $J_7$, it holds that

\[
\begin{aligned}
|J_7| &\leq C e^{-\frac{|p|}{2} t} \sum_{i=1}^{k-1} \int_{\prod_{j=1}^{k-1} P_j} d\sigma_{i-1} \cdots d\sigma_1 \int_0^{t_i} e^{-\nu_0 (t-s)} ds \\
&\times \int_{P_1} \int_{\mathbb{R}^3} 1_{\{t_{i+1} < 0 < t_i\}} \hat{w}(p_i) |k_w (p_i, p') h^{i-1} (x_l - p_l (t_l - s), p')| dp' d\sigma_l \\
&= C e^{-\frac{|p|}{2} t} \sum_{i=1}^{k-1} \int_{\prod_{j=1}^{k-1} P_j} d\sigma_{i-1} \cdots d\sigma_1 \int_0^{t_i} e^{-\nu_0 (t-s)} ds \int_{P_1 \cap \{|p_i| \geq N\}} \int_{\mathbb{R}^3} (\cdots) dp' d\sigma_l \\
&+ C e^{-\frac{|p|}{2} t} \sum_{i=1}^{k-1} \int_{\prod_{j=1}^{k-1} P_j} d\sigma_{i-1} \cdots d\sigma_1 \int_0^{t_i} e^{-\nu_0 (t-s)} ds \int_{P_1 \cap \{|p_i| \leq N\}} \int_{\mathbb{R}^3} (\cdots) dp' d\sigma_l \\
&+ C e^{-\frac{|p|}{2} t} \sum_{i=1}^{k-1} \int_{\prod_{j=1}^{k-1} P_j} d\sigma_{i-1} \cdots d\sigma_1 \int_0^{t_i} e^{-\nu_0 (t-s)} ds \int_{P_1 \cap \{|p_i| \geq N\}} \int_{\mathbb{R}^3} (\cdots) dp' d\sigma_l \\
&+ C e^{-\frac{|p|}{2} t} \sum_{i=1}^{k-1} \int_{\prod_{j=1}^{k-1} P_j} d\sigma_{i-1} \cdots d\sigma_1 \int_0^{t_i} e^{-\nu_0 (t-s)} ds \int_{P_1 \cap \{|p_i| \leq N\}} \int_{|p'| \geq 2N} (\cdots) dp' d\sigma_l \\
&+ C e^{-\frac{|p|}{2} t} \sum_{i=1}^{k-1} \int_{\prod_{j=1}^{k-1} P_j} d\sigma_{i-1} \cdots d\sigma_1 \int_0^{t_i} e^{-\nu_0 (t-s)} ds \int_{P_1 \cap \{|p_i| \leq N\}} \int_{|p'| \leq 2N} (\cdots) dp' d\sigma_l \\
&:= \sum_{i=1}^{k-1} (J_{71} + J_{72} + J_{73} + J_{74}).
\end{aligned}
\]

(4.17)
It follows from Lemma 3.4 that
\[
\sum_{l=1}^{k-1} J_{71l} \leq C e^{-\frac{|p|}{N}} \sum_{l=1}^{k-1} \int_{\mathbb{R}^{d-1}} d\sigma_{l-1} \cdots d\sigma_{1} \int_{0}^{t_{l}} e^{-v_{0}(t-s)} ds \\
\times \int_{P_{1}\cap \{|p| \geq N\}} e^{-\frac{|p|}{N}} dp_{l} \sup_{1 \leq l \leq k-1} \|h^{i-l}\|_{L^{\infty}}.
\]
and
\[
\sum_{l=1}^{k-1} J_{72l} \leq C e^{-\frac{|p|}{N}} \sup_{1 \leq l \leq k-1} \|h^{i-l}\|_{L^{\infty}}.
\]

For \( J_{73l} \), we have
\[
J_{73l} \leq C e^{-\frac{|p|}{N}} \sum_{l=1}^{k-1} \int_{\mathbb{R}^{d-1}} d\sigma_{l-1} \cdots d\sigma_{1} \int_{0}^{t_{l} - \frac{1}{N}} e^{-v_{0}(t-s)} ds \\
\times \int_{|p'| \geq 2N} |k_{w}(p_{l}, p')| e^{\frac{1}{2} |p' - p|} dp' e^{-\frac{|p|}{N}} \sup_{1 \leq l \leq k-1} \|h^{i-l}\|_{L^{\infty}}
\]
(4.18)

To estimate \( J_{74l} \), we have from Lemma 3.3 that
\[
\int_{P_{1}\cap \{|p| \leq N\}} \int_{|p'| \leq 2N} 1_{\{t_{i+1} \leq 0 < t_{i}\}} \hat{w}(p_{i}) |k_{w}(p_{i}, p') h^{i-l}(x_{i} - \hat{p}_{l} (t_{i} - s), p')| dp' dp_{i}
\]
\[
\leq C_{N} \left\{ \int_{P_{1}\cap \{|p| \leq N\}} \int_{|p'| \leq 2N} e^{-\frac{|p|}{N}} |k_{w}(p_{i}, p')|^{2} dp' dp_{i} \right\}^{\frac{1}{2}}
\times \left\{ \int_{P_{1}\cap \{|p| \leq N\}} \int_{|p'| \leq 2N} 1_{\{t_{i+1} \leq 0 < t_{i}\}} \frac{h^{i-l}(x_{i} - \hat{p}_{l} (t_{i} - s), p')}{w(p')} dp' dp_{i} \right\}^{\frac{1}{2}}
\leq C_{N} \left\{ \int_{P_{1}\cap \{|p| \leq N\}} \int_{|p'| \leq 2N} 1_{\{t_{i+1} \leq 0 < t_{i}\}} \frac{h^{i-l}(x_{i} - \hat{p}_{l} (t_{i} - s), p')}{w(p')}^{2} dp' dp_{i} \right\}^{\frac{1}{2}}.
\]
(4.21)

Let \( y_{l} = x_{l} - \hat{p}_{l} (t_{l} - s) \in \Omega \) for \( s \in [0, t_{l} - \frac{1}{N}] \). Then direct calculation shows that
\[
\left| \det \left( \frac{\partial y_{l}}{\partial p_{l}} \right) \right| = \frac{c^{5}(t_{l} - s)^{3}}{(c^{2} + |p_{l}|^{2})^{\frac{5}{2}}} \geq \frac{c^{5} N^{-3}}{(c^{2} + N^{2})^{\frac{5}{2}}} \geq \frac{N^{-3}}{(1 + N^{2})^{\frac{5}{2}}}.\]
(4.22)

Making change of variable \( p_{l} \to y_{l} \) and using (4.22), one obtains that
\[
C_{N} \left\{ \int_{P_{1}\cap \{|p| \leq N\}} \int_{|p'| \leq 2N} 1_{\{t_{i+1} \leq 0 < t_{i}\}} \frac{h^{i-l}(x_{i} - \hat{p}_{l} (t_{i} - s), p')}{w(p')}^{2} dp' dp_{i} \right\}^{\frac{1}{2}}
\leq C_{N} \left\{ \int_{\Omega} \int_{|p'| \leq 2N} \frac{h^{i-l}(y_{l}, p')}{w(p')} dp' dy_{l} \right\}^{\frac{1}{2}} \leq C_{N} \frac{h^{i-l}}{w} \| \|_{L^{2}},
\]
(4.23)
which, together with [4.18]-[4.21], yields that
\[ |J_7| \leq C \frac{e^{-\frac{t}{N}}}{N} \sup_{0 \leq t \leq k-1} \| h^{i-1} \|_{L^\infty} + C_N k e^{-\frac{t}{N}} \sup_{0 \leq t \leq k-1} \| h^{i-1} \|_{L^2}. \]  
(4.24)

By similar arguments as in (4.17)-(4.24), one has
\[ |J_9| \leq C \frac{e^{-\frac{t}{N}}}{N} \sup_{0 \leq t \leq k-1} \| h^{i-1} \|_{L^\infty} + C_N k e^{-\frac{t}{N}} \sup_{0 \leq t \leq k-1} \| h^{i-1} \|_{L^2}. \]  
(4.25)

Now substituting (4.9), (4.13)-(4.16), (4.24) and (4.25) into (4.5), we get, for \( t \in [0, T_0] \), that
\[ |h^{i+1}(x, p)| \leq \int_{\max\{t_1, 0\}}^{t} e^{-\nu_0(t-s)} \int_{\mathbb{R}^3} |k_w(p, p')| \cdot |h^i(x - \hat{p}(t-s), p')| dp'ds + A_i(t, p), \]  
(4.26)

where we have denoted
\[ A_i(t, p) := C k e^{-\frac{t}{N}} \left\{ e^{-\nu_0 t} + \left( \frac{1}{2} \right)^{C_2 T_0^\frac{2}{3}} + \frac{1}{N} \right\} \sup_{0 \leq t \leq k-1} \left\{ \| h^{i-1} \|_{L^\infty} + \| h^{i-1} \|_{L^\infty(\gamma_\pm)} \right\} \]
\[ + C_N k e^{-\frac{t}{N}} \sup_{0 \leq t \leq k-1} \| h^{i-1} \|_{L^2}. \]

We denote \( x' \equiv x - \hat{p}(t-s) \) ∈ Ω and \( t'_1 = t_1(s, x', p') \) for \( s \in (\max\{t_1, 0\}, t) \). Using the Vidav’s iteration in (4.26), we obtain that
\[ |h^{i+1}(x, p)| \leq A_i(t, p) + \int_0^t e^{-\nu_0(t-s)} \int_{\mathbb{R}^3} |k_w(p, p')| A_{i-1}(s, p') dp'ds \]
\[ + \int_0^t ds \int_0^s e^{-\nu_0(t-\tau)} \int_{\mathbb{R}^3} |k_w(p, p') k_w(p', p'')| \times 1_{\{\max\{t_1, 0\} < s < \tau \}} \cdot 1_{\{\max\{t_1, 0\}, \tau < \tau < s\}} \| h^{i-1} \|_{L^\infty} ds \]
\[ := A_i(t, p) + B_1 + B_2. \]  
(4.27)

For the term \( B_1 \), using Lemma 3.4 one has
\[ B_1 \leq C k \left\{ e^{-\frac{\tau}{N}} + \left( \frac{1}{2} \right)^{C_2 T_0^\frac{2}{3}} + \frac{1}{N} \right\} \sup_{0 \leq t \leq k-1} \left\{ \| h^{i-1} \|_{L^\infty} + \| h^{i-1} \|_{L^\infty(\gamma_\pm)} \right\} \]
\[ + C k \{ \| \nu^{-1} w S \|_{L^\infty} + \| \nu r \|_{L^\infty(\gamma_\pm)} \} + C_N k \sup_{0 \leq t \leq k} \| h^{i-1} \|_{L^2}. \]  
(4.28)

For the term \( B_2 \), we split the estimate into several cases.
\textbf{Case 1.} For \( |p| \geq N \), we have from Lemma 3.4 that
\[ B_2 \leq C \max\left\{ 1, \frac{1}{N} \right\} \| h^{i-1} \|_{L^\infty}. \]  
(4.29)

\textbf{Case 2.} For \( |p| \leq N, |p'| \geq 2N \) or \( |p'| \leq 2N, |p''| \geq 3N \). In this case, we note from (3.23) that
\[ \begin{aligned}
\int_{|p| \leq 2N, |p'| = 2N} |k_w(p, p')| e^{\frac{1}{2} |p-p'|} dp' & \leq C, \\
\int_{|p'| \leq 2N, |p''| = 3N} |k_w(p, p'')| e^{\frac{1}{2} |p'-p''|} dp'' & \leq C.
\end{aligned} \]  
(4.30)
which yields that
\[
\int_0^t ds \int_0^s e^{-\nu_0(t-\tau)} d\tau \left\{ \int_{|p| \leq N, |p'| \geq 2N} + \int_{|p'| \leq 2N, |p''| \geq 3N} \right\} (\cdots) dp'' dp' \\
\leq e^{-\frac{N}{2}} \| h^i \|_{L^\infty} \int_{|p| \leq N, |p'| \geq 2N} |k_w(p, p')| e^{\frac{N}{2}|p-p'|} \cdot |k_w(p', p'')| |dp'' dp'|
\]
\[
+ e^{-\frac{N}{2}} \| h^i \|_{L^\infty} \int_{|p'| \leq 2N, |p''| \geq 3N} |k_w(p, p')| \cdot |k_w(p', p'')| e^{\frac{N}{2}|p-p''|} |dp'' dp'|
\]
\[
\leq C e^{-\frac{N}{2}} \| h^i \|_{L^\infty}. \tag{4.31}
\]

**Case 3.** For $|p| \leq N$, $|p'| \leq 2N$, and $|p''| \leq 3N$, we note that
\[
\int_0^t ds \int_0^s e^{-\nu_0(t-\tau)} d\tau \int_{|p'| \leq 2N, |p''| \leq 3N} (\cdots) dp'' dp'
\]
\[
\leq \int_0^t ds \int_0^{\frac{2}{\nu_0}} e^{-\nu_0(t-\tau)} d\tau \int_{R^2} (\cdots) dp'' dp' + \frac{C}{N} \| h^i \|_{L^\infty}
\]
\[
\leq C_N \int_0^t ds \int_0^{\frac{2}{\nu_0}} e^{-\nu_0(t-\tau)} d\tau \left\{ \int_{|p'| \leq 2N, |p''| \leq 3N} |k(p, p') k(p', p'')|^2 dp'' dp' \right\}^{\frac{1}{2}}
\]
\[
\times \left\{ \int_{|p'| \leq 2N, |p''| \leq 3N} 1_{\{\max\{t_1, 0\} < s \leq t\}} 1_{\{\max\{t_2, 0\} < \tau < s\}} \left| \frac{h^i - 1}{w(p''')} \right|^2 \right\}^{\frac{1}{2}}
\]
\[
+ \frac{C_N}{N} \| h^i \|_{L^\infty}
\]
\[
\leq C_N \| h^i \|_{L^\infty} + C_N \int_0^t ds \int_0^{\frac{2}{\nu_0}} e^{-\nu_0(t-\tau)} d\tau \left\{ \int_{|p'| \leq 2N, |p''| \leq 3N} (\cdots) dp'' dp' \right\}^{\frac{1}{2}} \tag{4.32}
\]

where we have denoted $p' := x' - \hat{\nu}'(s-\tau) \in \Omega$ for $s \in (\max\{t_1, 0\}, t)$ and $\tau \in (\max\{t_2, 0\}, s)$. Making the change of variable $p' \mapsto y'$, then the second term on the RHS of (4.32) is bounded as
\[
C_N \int_0^t ds \int_0^{\frac{2}{\nu_0}} e^{-\nu_0(t-\tau)} d\tau \left\{ \int_{|p'| \leq 2N, |p''| \leq 3N} (\cdots) dp'' dp' \right\}^{\frac{1}{2}} \leq C_N \left\| \frac{h^i - 1}{w} \right\|_{L^2}, \tag{4.33}
\]

which, together with (4.32), yields that
\[
\int_0^t ds \int_{|p'| \leq 2N, |p''| \leq 3N} (\cdots) dp'' dp' \leq \frac{C}{N} \| h^i \|_{L^\infty} + C_N \left\| \frac{h^i - 1}{w} \right\|_{L^2}. \tag{4.34}
\]

Combining (4.29), (4.31) and (4.34), we have
\[
B_2 \leq C \max\left\{ \frac{1}{N}, \frac{1}{\epsilon} \right\} \| h^i \|_{L^\infty} + C_N \left\| \frac{h^i - 1}{w} \right\|_{L^2}. \tag{4.35}
\]

Hence it follows from (4.27), (4.28) and (4.35) that
\[
\| h^{i+1}(x, p) \| \leq C \left\{ e^{-\frac{2}{\nu_0}t} + \frac{1}{2} \right\} C^2 \| \frac{S}{\epsilon^2} \|_{L^\infty} + \frac{1}{N} \sup_{0 \leq \ell \leq k} \left\{ \| h^{i-\ell} \|_{L^\infty} + \| h^{i-\ell} \|_{L^\infty(\gamma_+)} \right\}
\]
\[
+ e^{-\nu_0 t} \left\| \frac{h^{i+1}}{w} \right\|_{L^\infty} + C_k \left\{ \| \nu^{-1} u S \|_{L^\infty} + \| u \|_{L^\infty(\gamma_+)} \right\}
\]
Lemma 4.5. Define the near-grazing set of

\[ \gamma_{\pm} = \left\{ (x, p) \in \gamma_{\pm} : |n(x) \cdot p| \leq \varepsilon \text{ or } |p| \geq \frac{1}{\varepsilon} \text{ or } |p| \leq \varepsilon \right\}. \] (4.38)

Then it holds that

\[ \left| f^1_{\gamma_{\pm} \backslash \gamma_{\pm}^\varepsilon} \right|_{L^1(\gamma)} \leq C_{\varepsilon, \Omega} \left( \| f \|_{L^1} + \| \hat{p} \cdot \nabla_x f \|_{L^1} \right), \] (4.39)

and

\[ \int_s^t \left| f(\tau) \right|_{L^1(\gamma)} \, d\tau \leq C_{\varepsilon, \Omega} \left( \| f(s) \|_{L^1} + \int_s^t \| f(\tau) \|_{L^1} + \| \{ \partial_t + \hat{p} \cdot \nabla_x \} f(\tau) \|_{L^1} \right) \, d\tau, \] (4.40)

for 0 \leq s \leq t.

As pointed out in [25], both \( \gamma_{\pm} \backslash \gamma_{\pm}^\varepsilon \) and \( \gamma_{\pm} \backslash \gamma_{\pm}^\varepsilon \) are controlled for the steady case, while only the outgoing part \( \gamma_{\pm} \backslash \gamma_{\pm}^\varepsilon \) can be estimated in the unsteady case. The proof is very similar to [25 Lemma 2.1] and so we omit the details here for brevity.

Lemma 4.6. Let \( \varepsilon > 0 \) and \( \beta > 3 \), and assume \( \| \nu^{-1} wS \|_{L^\infty} + |w_r|_{L^\infty(\gamma_-)} < \infty \). Then there exists a unique solution \( f^\varepsilon \) to solve the approximate linearized steady relativistic Boltzmann equation (4.1) with

\[ \| w f^\varepsilon \|_{L^\infty} + |w f^\varepsilon|_{L^\infty(\gamma)} \leq C_{\varepsilon} \left\{ \| w_r \|_{L^\infty(\gamma_-)} + \| \nu^{-1} wS \|_{L^\infty} \right\}, \]

where the positive constant \( C_{\varepsilon} > 0 \) depends only on \( \varepsilon \).

4.2. Approximate Sequence. To construct solutions of (4.1) or equivalently (4.2), we first consider the following approximate problem

\[
\begin{align*}
\varepsilon f^n + \hat{p} \cdot \nabla_x f^n + \nu(p) f^n - K f^n &= S, \\
f^n(x, p) |_{\gamma_-} &= (1 - \frac{1}{\beta}) P_{\gamma} f^n + r,
\end{align*}
\] (4.37)

where \( \varepsilon \in (0, 1] \) is arbitrary and \( n > 1 \) is an integer. Recall \( k = C_1 T_0^\frac{5}{2} \) with \( T_0 \) large enough. To the end, we choose \( n_0 > 1 \) large enough such that

\[ \frac{1}{8} \left( 1 - \frac{2}{n} + \frac{3}{2n^2} \right)^{-\frac{k+1}{2}} \leq \frac{1}{2}, \]

for \( n \geq n_0 \).

Lemma 4.4. Let \( \varepsilon > 0, n \geq n_0, \) and \( \beta > 3 \). Assume \( \| \nu^{-1} wS \|_{L^\infty} + |w_r|_{L^\infty(\gamma_-)} < \infty \). Then there exists a unique solution \( f^n \) to (4.37) satisfying

\[ \| w f^n \|_{L^\infty} + |w f^n|_{L^\infty(\gamma)} \leq C_{\varepsilon, n} \left( \| w_r \|_{L^\infty(\gamma_-)} + \| \nu^{-1} wS \|_{L^\infty} \right), \]

where the positive constant \( C_{\varepsilon, n} > 0 \) depends only on \( \varepsilon \) and \( n \).

Proof. Since the proof is very similar to [18 Lemma 3.5], so we omit the details for simplicity of presentation. \( \square \)
Recall that $P \square 3.6$. We omit the details of proof for simplicity of presentation.

Proof. With the help of (4.39), one can prove this lemma by similar procedure as in [18, Lemma 3.6]. We omit the details of proof for simplicity of presentation.

The next lemma states the crucial $L^2$ bound for $P f$. It provides uniform in $\varepsilon$ estimates which enable us to take the limit $\varepsilon \to 0$ in (4.1).

**Proposition 4.7.** Let $f$ be a solution, in the sense of (4.44) below, to

$$\hat{p} \cdot \nabla_x f + L f = S, \quad f|_{\gamma_-} = P_\gamma f + r,$$

with

$$\int_{\Omega \times \mathbb{R}^3} f \sqrt{\hat{J}} dpdx = \int_{\Omega \times \mathbb{R}^3} S \sqrt{\hat{J}} dpdx = \int_{\gamma_-} r \sqrt{\hat{J}} d\gamma = 0,$$

then we have

$$\|P f\|_{L^2_{\nu}} \lesssim \|(I - P)f\|_{L^2_{\nu}}^2 + \|S\|_{L^2}^2 + \|(I - P_\gamma)f\|_{L^2(\gamma_-)}^2 + |r|^2_{L^2(\gamma_-)}.$$  \hspace{1cm} (4.43)

Proof. By Green’s identity in Lemma 2.4, we have the following weak version of (4.41):

$$\int_{\gamma} \psi dfd\gamma - \int_{\Omega \times \mathbb{R}^3} \hat{p} \cdot \nabla_x \psi dpdx = -\int_{\Omega \times \mathbb{R}^3} \psi L(I - P)f dpdx + \int_{\Omega \times \mathbb{R}^3} \psi S dpdx. \hspace{1cm} (4.44)$$

Recall that $P f = \left\{a + b \cdot p + c \frac{p^0 - A_3}{\sqrt{A_2 - A_3^2}}\right\} \sqrt{\hat{J}}(p)$ on $\Omega \times \mathbb{R}^3$. It is noted that (4.42) still holds if we replace $J$ with $\hat{J}$. Motivated by [25], the key of the proof is to choose proper test functions $\psi$ to estimate $a$, $b$ and $c$. We point out that the test functions are similar to the ones in [25]. For the present proposition, we need to prove that the relativistic effect can be controlled by using the fact that $c \gg 1$. We also need to get uniform estimate independent of $c$. Throughout the proof, we use the following decomposition

$$f = P_\gamma f + 1_{\gamma_+}(I - P_\gamma)f + 1_{\gamma_-}r, \quad \text{on } \gamma,$$

$$f = \left\{a + b \cdot p + c \frac{p^0 - A_3}{\sqrt{A_2 - A_3^2}}\right\} \sqrt{\hat{J}}(p) + (I - P)f, \quad \text{on } \Omega \times \mathbb{R}^3. \hspace{1cm} (4.46)$$

**Step 1. Estimate on $c$.** We choose the test function as

$$\psi = \psi_c \equiv (|p|^2 - \beta_c) \sqrt{\hat{J}}(p) p \cdot \nabla_x \phi_c(x), \hspace{1cm} (4.47)$$

where

$$-\Delta_x \phi_c(x) = c(x), \quad \phi_c|_{\partial \Omega} = 0, \hspace{1cm} (4.48)$$

and $\beta_c$ is a constant to be determined. Clearly, we have $\|\phi_c\|_{H^2} \lesssim \|c\|_{L^2}$. Since $\hat{p} \cdot \nabla_x \psi_c = \sum_{i,j=1}^3 \frac{c}{p^0} (|p|^2 - \beta_c) \sqrt{\hat{J}}(p)p_i p_j \partial_{ij} \phi_c(x)$, the LHS of (4.44) takes the form

$$\int_{\partial \Omega \times \mathbb{R}^3} \frac{c}{p^0} \left\{n(x) \cdot p\right\} (|p|^2 - \beta_c) \sqrt{\hat{J}} \sum_{i=1}^3 p_i \partial_i \phi_c f dpdS_x$$

$$- \int_{\Omega \times \mathbb{R}^3} \frac{c}{p^0} (|p|^2 - \beta_c) \sqrt{\hat{J}} \left\{\sum_{i,j=1}^3 p_i p_j \partial_{ij} \phi_c \right\} f dpdx. \hspace{1cm} (4.49)$$

We choose $\beta_c$ such that

$$\int_{\mathbb{R}^3} \frac{c}{p^0} (|p|^2 - \beta_c) p_i^2 \hat{J}(p) dp = 0, \hspace{1cm} (4.50)$$
which, together with Lemma 3.1, yields that $\beta_c = 5\frac{K_0(c^2)}{K_0(c^2)} = 5 + O(c^{-2})$. Because of the choice of $\beta_c$, there is no $a$ contribution in the bulk and no $P_jf$ contribution at the boundary in (4.49). Due to the oddness in $p$ in the bulk, the LHS of (4.44) becomes

$$
\sum_{i,j=1}^{3} \int_{\partial \Omega \times \mathbb{R}^3} \frac{\epsilon}{p^0} (\beta_c)p^i p_j n_i \partial_j \phi_c [f_{\gamma+} + r_{\gamma+}] d\sigma_x
$$

$$
- \sum_{i=1}^{3} \int_{\mathbb{R}^3} \frac{\epsilon}{p^0} (\beta_c)p^0 - A_3 \frac{p^i p^j}{\sqrt{A_2 - A_3^2}} J(p) dp \int_{\Omega} \partial_i \phi_c(x) c(x) dx
$$

$$
- \sum_{i,j=1}^{3} \int_{\Omega \times \mathbb{R}^3} \frac{\epsilon}{p^0} (\beta_c) \sqrt{J(p)} p_i p_j \partial_j \phi_c \cdot (I - P) f dp dx.
$$

(4.51)

Using Lemma 3.1, one has

$$
\int_{\mathbb{R}^3} \frac{\epsilon}{p^0} (\beta_c)p^0 - A_3 \frac{p^i p^j}{\sqrt{A_2 - A_3^2}} J(p) dp = \frac{5\sqrt{6}}{3} + O(c^{-2}).
$$

It is clear that the RHS of (4.44) is bounded uniformly by

$$
\|c\|_{L^2} \{ \| (I - P) f \|_{L^2} + \| S \|_{L^2} \}.
$$

(4.52)

Therefore, using (4.51) and (4.52), one has the uniform bound

$$
\|c\|_{L^2}^2 \lesssim \|c\|_{L^2} \left( \| (I - P) f \|_{L^2(\gamma_+)} + \| (I - P) f \|_{L^2} + \| S \|_{L^2} + |r|_{L^2(\gamma_-)} \right),
$$

(4.53)

which yields immediately that

$$
\|c\|_{L^2}^2 \leq C \left( \| (I - P) f \|_{L^2(\gamma_+)} + \| (I - P) f \|_{L^2} + \| S \|_{L^2}^2 + |r|_{L^2(\gamma_-)}^2 \right),
$$

(4.54)

where $C > 0$ is independent of the light speed $c$.

**Step 2. Estimate on $b$.** We fix $i, j$ and choose the test function $\psi$ as

$$
\psi = \psi^{i,j}_b \equiv (p^i - \beta_b) \sqrt{J} \partial_j \phi_b^i, \quad i, j = 1, 2, 3,
$$

(4.55)

where $\beta_b$ is a constant to be determined, and

$$
-\Delta_x \phi_b^i(x) = b_j(x), \quad \phi_b^i|_{\partial \Omega} = 0.
$$

(4.56)

The standard elliptic estimate yields $\|\phi_b^i\|_{H^2} \lesssim \|b\|_{L^2}$. Substituting (4.45) and (4.46) into the LHS of (4.44), by the oddness in $p$, we note that

$$
\int \left\{ \int_{\partial \Omega \times \mathbb{R}^3} \{n(x) \cdot \hat{p}\} (p^i - \beta_b) \sqrt{J(p)} \partial_j \phi_b^i P_\gamma f d\sigma_x \right\} = 0.
$$

Thus the LHS of (4.44) takes the form

$$
\int \left\{ \int_{\partial \Omega \times \mathbb{R}^3} \{n(x) \cdot \hat{p}\} (p^i - \beta_b) \sqrt{J(p)} \partial_j \phi_b^i f d\sigma_x \right\}
$$

$$
- \int \left\{ \int_{\partial \Omega \times \mathbb{R}^3} \frac{\epsilon}{p^0} (\beta_c)p^i \sqrt{J(p)} \left\{ \sum_{l=1}^{3} p_l \partial_l \phi_b^i \right\} \right\} f d\sigma_x
$$

$$
= \int \left\{ \int_{\partial \Omega \times \mathbb{R}^3} \frac{\epsilon}{p^0} \left\{ (n(x) \cdot \hat{p}) (p^i - \beta_b) \sqrt{J(p)} \partial_j \phi_b^i \right\} [f_{\gamma+} + r_{\gamma+}] d\sigma_x \right\}
$$

$$
- \sum_{l=1}^{3} \int \left\{ \int_{\partial \Omega \times \mathbb{R}^3} \frac{\epsilon}{p^0} (p^l - \beta_b)p^i \sqrt{J(p)} \partial_j \phi_b^i (x) \cdot b_l(x) d\sigma_x \right\}.
$$
\[-\sum_{i=1}^{3} \iint_{\Omega \times \mathbb{R}^3} \frac{c}{p} \left( p_i^2 - \beta_b \right) \rho t \sqrt{J(p)} \partial_{ij} \phi_b^i(x) \cdot (I - P)f dp dx,\]

where the \( a, c \) contributions vanish due to the oddness. We choose \( \beta_b \) such that

\[\int_{\mathbb{R}^3} \frac{c}{p} \left( p_i^2 - \beta_b \right) p_k^2 J(p) dp = 0, \quad k \neq i. \quad (4.57)\]

Using Lemma 3.1 we have \( \beta_b = \frac{K_2(c^2)}{K_2(c^2)} = 1 + O(\varepsilon^{-2}). \) And a direct calculation shows that

\[\int_{\mathbb{R}^3} \frac{c}{p} \left( p_i^2 - \beta_b \right) p_j^2 J(p) dp = 2 \frac{K_2(c^2)}{K_2(c^2)} = 2 + O(\varepsilon^{-2}). \quad (4.58)\]

Thus the RHS of (4.44) can be bounded uniformly by

\[\|b\|_{L^2} \left\{ \|f(I - P)\|_{L^2} + \|S\|_{L^2} \right\}. \quad (4.59)\]

Hence, using (4.58), we have the following uniform estimate for all \( i, j, \)

\[\left| \int_{\Omega} \left( \partial_{ij} \Delta^{-1} b_j \right) b_i \right| \lesssim C \left( \|f(I - P)\|_{L^2} + \|S\|_{L^2} + \|r\|_{L^2} \right) + \varepsilon \|b\|_{L^2}, \quad (4.60)\]

for a small \( \varepsilon > 0 \) chosen later.

To estimate \( \left( \partial_{ij} \Delta^{-1} b_j \right) b_i \) for \( i \neq j \), we choose the test function as

\[\psi = |p|^2 p_i p_j \sqrt{J} \partial_{ij} \phi_b^i(x), \quad i \neq j, \quad (4.61)\]

where \( \phi_b^i \) is the one defined in (4.56). It is clear that the RHS of (4.44) is again bounded by (4.59). Substituting (4.45) and (4.46) into the LHS of (4.44), we notice that the \( P_j f \) contribution and \( a, c \) contributions vanish again due to the oddness. Then the LHS of (4.44) becomes

\[\int_{\partial\Omega \times \mathbb{R}^3} \left\{ n(x) \cdot \rho \right\} |p|^2 p_i p_j \sqrt{J} \partial_{ij} \phi_b^i f dp dS_x \]

\[= \int_{\Omega \times \mathbb{R}^3} \frac{c}{p} \left| n(x) \cdot \rho \right| |p|^2 p_i p_j \sqrt{J} \partial_{ij} \phi_b^i \left[ (I - P_j) f 1_{\gamma_+} + r 1_{\gamma_-} \right] dp dS_x \quad (4.62)\]

\[= \int_{\Omega \times \mathbb{R}^3} \frac{c}{p} \left| n(x) \cdot \rho \right| |p|^2 p_i p_j \sqrt{J} \partial_{ij} \phi_b^i \left[ \partial_{ij} \phi_b^i b_j + \partial_{jj} \phi_b^i(x) b_i \right] dp dx \quad (4.63)\]

\[- \sum_{k=1}^{3} \int_{\Omega \times \mathbb{R}^3} \frac{c}{p} \left| p_i p_j p_k \sqrt{J} \partial_{kj} \phi_b^i(x) \cdot (I - P)f \right| dp dx. \quad (4.64)\]

It is clear that (4.63) is evaluated as

\[cA_8 \int_{\Omega} \left\{ \left( \partial_{ij} \Delta^{-1} b_j \right) b_i + \left( \partial_{jj} \Delta^{-1} b_j \right) b_i \right\} dx.\]

Using Lemma 3.1 we have

\[cA_8 := \int_{\mathbb{R}^3} \frac{c}{p} |p|^2 p_i p_j \sqrt{J} J(p) dp = 7 \frac{K_4(c^2)}{K_2(c^2)} = 7 + O(\varepsilon^{-2}), \quad i \neq j. \]

Noting \( \left| \partial_{ij} \phi_b^i \right|_{L^2} \lesssim \|\phi_b^i\|_{L^2} \lesssim \|b\|_{L^2} \), then we have

\[\|4.62\| + \|4.64\| \lesssim \|b\|_{L^2} \left\{ \|f(I - P_j) f\|_{L^2(\gamma_+)} + \|r\|_{L^2(\gamma_-)} + \|S\|_{L^2} \right\}. \]
Combining with (4.60), for $i \neq j$, one has
\[
\left| \int_{\Omega} (\partial_{ij} \Delta^{-1} b_i) b_j \right| \lesssim \left| \int_{\Omega} (\partial_{ij} \Delta^{-1} b_i) b_j \right| + \varepsilon \|b\|_{L^2}^2 \\
+ C\varepsilon \left( |(I - P_{\gamma}) f|_{L^2(\gamma_+)}^2 + \|(I - P) f\|_{L^2}^2 + \|S\|_{L^2}^2 + |r|_{L^2(\gamma_-)}^2 \right) \\
\lesssim C\varepsilon \left( |(I - P_{\gamma}) f|_{L^2(\gamma_+)}^2 + \|(I - P) f\|_{L^2}^2 + \|S\|_{L^2}^2 + |r|_{L^2(\gamma_-)}^2 \right) + \varepsilon \|b\|_{L^2}^2.
\]

(4.65)

On the other hand, it follows from (4.60) that
\[
\left| \int_{\Omega} (\partial_{ii} \Delta^{-1} b_i) b_i \right| \lesssim C\varepsilon \left( |(I - P_{\gamma}) f|_{L^2(\gamma_+)}^2 + \|(I - P) f\|_{L^2}^2 + \|S\|_{L^2}^2 + |r|_{L^2(\gamma_-)}^2 \right) + \varepsilon \|b\|_{L^2}^2.
\]

(4.66)

Combining (4.65) and (4.66) and taking $\varepsilon$ suitably small, one has
\[
\|b\|_{L^2}^2 \leq C\left( |(I - P_{\gamma}) f|_{L^2(\gamma_+)}^2 + \|(I - P) f\|_{L^2}^2 + \|S\|_{L^2}^2 + |r|_{L^2(\gamma_-)}^2 \right),
\]

(4.67)

where the positive constant $C > 0$ is independent of $c$.

**Step 3. Estimate on $a$.** It follows from (4.42) that $\int_{\Omega} x(p) \sqrt{\bar{J}(p)} dp = 0$, which yields that $\int_{\Omega} a(x) dx = 0$. Now we choose the test function as
\[
\psi = \psi_a \equiv (|p|^2 - \beta_a) p \cdot \nabla_x \phi_a \sqrt{\bar{J}} = \sum_{i=1}^3 \left( |p|^2 - \beta_a \right) p_i \partial_i \phi_a \sqrt{\bar{J}},
\]

(4.68)

where
\[
-\Delta_x \phi_a(x) = a(x), \quad \frac{\partial \phi_a}{\partial n} \bigg|_{\partial \Omega} = 0.
\]

(4.69)

Using the standard elliptic estimate and the fact $\int_{\Omega} a(x) dx = 0$, one has
\[
\|\phi_a\|_{H^2} \lesssim \|a\|_{L^2}.
\]

We choose $\beta_a > 0$ so that, for all $i$,
\[
\int_{\mathbb{R}^3} \frac{c}{p_i} \left( |p|^2 - \beta_a \right) \frac{p_i^2}{\sqrt{A_2 - A_3^2}} \bar{J}(p) dp = 0.
\]

Using Lemma 3.1 one gets that
\[
\beta_a = 35 \frac{K_3(c^2)}{K_2(c^2)} - 5c^2 \left\{ \left( \frac{K_3(c^2)}{K_2(c^2)} \right)^2 - 1 \right\} = 10 + O(c^{-2}).
\]

Plugging $\psi_a$ into (4.44), the RHS of (4.44) is bounded by
\[
\|a\|_{L^2} \left\{ \|I - P\|_{L^2} + \|S\|_{L^2} \right\}.
\]

We note that $c$ contribution vanishes in the LHS of (4.44) due to the choice of $\beta_a$ and $b$ contribution vanishes in the LHS of (4.44) due to the oddness. Hence the LHS of (4.44) takes the form
\[
\sum_{i=1}^3 \int_{\Omega \times \mathbb{R}^3} \left\{ \n(x) \cdot \hat{p} \right\} \left( |p|^2 - \beta_a \right) p_i \sqrt{\bar{J}(p)} \partial_i \phi_a(x) \left[ P_{\gamma} f + (I - P_{\gamma}) f 1_{\gamma_+} + r 1_{\gamma_-} \right] dp dS_x \quad (4.70)
\]

\[
- \sum_{i,k=1}^3 \int_{\Omega \times \mathbb{R}^3} \frac{c}{p_i} \left( |p|^2 - \beta_a \right) p_i p_k \partial_{ik} \phi_a(x) a(x) \sqrt{\bar{J}(p)} dp dx
\]

(4.71)

\[
- \sum_{i,k=1}^3 \int_{\Omega \times \mathbb{R}^3} \frac{c}{p_i} \left( |p|^2 - \beta_a \right) p_i p_k \partial_{ik} \phi_a(x) \sqrt{\bar{J}(p)} \cdot (I - P) f dp dx.
\]

(4.72)
As in [25], we make an orthogonal decomposition at the boundary $p_i = (p \cdot n)n_i + (n_i)_i$ and denote
\[
z_\gamma(x) := \int_{n(x) \cdot q = 0} f(q, x) \sqrt{J(q)} \{n(x) \cdot q\} dq,
\]
then the contribution of $P_\gamma f = z_\gamma(x) \sqrt{J(p)}$ in (4.70) is
\[
\int_{\partial \Omega \times \mathbb{R}^3} \frac{c}{p} \left|\frac{|p|^2 - \beta_a}{p} \right| p \cdot \nabla_x \phi_a(x) (p \cdot n) \sqrt{J(p)} \sqrt{J(p)} z_\gamma(x) dp \frac{dS_x}{dx}
\]
\[
= \int_{\partial \Omega \times \mathbb{R}^3} \frac{c}{p} \left(\frac{|p|^2 - \beta_a}{p} (p \cdot n)^2 \frac{\partial \phi_a}{\partial n} \sqrt{J(p)} \sqrt{J(p)} z_\gamma(x) dp \frac{dS_x}{dx}
\]
\[
+ \int_{\partial \Omega \times \mathbb{R}^3} \frac{c}{p} \left(\frac{|p|^2 - \beta_a}{p} (p \cdot n) n_i \cdot \nabla_x \phi_a \sqrt{J(p)} \sqrt{J(p)} z_\gamma(x) dp \frac{dS_x}{dx}
\]
\[
= 0,
\]
where we have used the fact $\frac{\partial \phi_a}{\partial n}|_{\partial \Omega} = 0$ and the oddness of $p \cdot n(n_i)_i$ for all $i$. Hence, we can control (4.70) and (4.72) by
\[
\|a\|_{L^2} \left\{ \| (I - P) f \|_{L^2} + \| (I - P_\gamma) f \|_{L^2(\gamma_+)} + \| r \|_{L^2(\gamma_-)} \right\}.
\]
For (4.71), due to the oddness, we only have the $k = i$ contribution:
\[
- \sum_{i=1}^{3} \int_{\Omega \times \mathbb{R}} \frac{c}{p} \left(\frac{|p|^2 - \beta_a}{p} p_i^2 \sqrt{J(p)} \partial_i \phi_a(x) dp \right)
\]
\[
= \|a\|_{L^2} \int_{\mathbb{R}} \frac{c}{p} \left(\frac{|p|^2 - \beta_a}{p} p_i^2 \sqrt{J(p)} dp \right) = \left( -30 \frac{K_3(e^2)}{K_2(e^2)} + 5e^2 \left\{ \frac{K_3(e^2)}{K_2(e^2)} - 1 \right\} \right) \|a\|_{L^2}^2
\]
\[
= (-5 + O(e^{-2})) \|a\|_{L^2}^2.
\]
Thus we obtain the uniform bound
\[
\|a\|_{L^2} \leq C \left( \| (I - P) f \|_{L^2} + \| (I - P_\gamma) f \|_{L^2(\gamma_+)} + \| r \|_{L^2(\gamma_-)} + \| S \|_{L^2} \right),
\]
where $C > 0$ is independent of the light speed $c$. Combining (4.54), (4.67) and (4.73), we obtain (4.43). Therefore the proof of Proposition 4.7 is completed.

**Lemma 4.8.** Assume that
\[
\int_{\Omega \times \mathbb{R}^3} S \sqrt{J} dp dx = \int_{\gamma_-} r \sqrt{J} d\gamma = 0.
\]
Let $\beta > 3$, and assume $\|\nu^{-1} w S\|_{L^\infty} + \| w r \|_{L^\infty(\gamma_-)} < \infty$. Then there exists a unique solution $f = f(x, p)$ to the linearized steady Boltzmann equation
\[
\hat{p} \cdot \nabla_x f + L f = S, \quad f(x, p)|_{\gamma_-} = P_\gamma f + r,
\]
such that
\[
\int_{\Omega} \int_{\mathbb{R}^3} f \sqrt{J} dp dx = 0,
\]
and
\[
\|wf\|_{L^\infty} + \|wf\|_{L^\infty(\gamma_-)} \leq C \left\{ \| wr \|_{L^\infty(\gamma_-)} + \| \nu^{-1} w S \|_{L^\infty} \right\}.
\]

**Proof.** Let $f^\varepsilon$ be the solution of (4.1) constructed in Lemma 4.6 for $\varepsilon > 0$. Multiplying (4.1) by $\sqrt{J}$, integrating the resultant equation over $\Omega \times \mathbb{R}^3$, and noting (4.74), we obtain
\[
\int_{\Omega} \int_{\mathbb{R}^3} f^\varepsilon(x, p) \sqrt{J(p)} dp dx = 0, \quad \text{for any } \varepsilon > 0.
\]
Multiplying (4.1) by $f_\varepsilon$ and integrating the resultant equation over $\Omega \times \mathbb{R}^3$, one has
\[
\varepsilon \|f_\varepsilon\|_{L^2}^2 + \frac{1}{2} \| (I - P_\gamma) f_\varepsilon \|_{L^2(\gamma_+)}^2 + \zeta_0 \| (I - P) f_\varepsilon \|_{L^2}^2 \\
\leq \eta \| P_\gamma f_\varepsilon \|_{L^2(\gamma_+)}^2 + C\eta \| r \|_{L^2(\gamma_-)}^2 + \| f_\varepsilon \|_{L^2} \cdot \| S \|_{L^2}. \tag{4.79}
\]
Applying Proposition 4.3 to $f_\varepsilon$, we obtain
\[
\| P f_\varepsilon \|_{L^2} \leq C \left\{ \| (I - P) f_\varepsilon \|_{L^2} + \| (I - P_\gamma) f_\varepsilon \|_{L^2(\gamma_+)} + \| S \|_{L^2} + \| r \|_{L^2(\gamma_-)} \right\},
\]
which, together with (4.79), implies that
\[
\| f_\varepsilon \|_{L^2}^2 + \| (I - P_\gamma) f_\varepsilon \|_{L^2(\gamma_+)}^2 + \| (I - P) f_\varepsilon \|_{L^2}^2 \\
\leq C \eta \| P_\gamma f_\varepsilon \|_{L^2(\gamma_+)}^2 + C\eta \| r \|_{L^2(\gamma_-)}^2 + C \| S \|_{L^2}^2, \tag{4.80}
\]
where $\eta > 0$ is a small positive constant to be determined later.

To control the term $| P_\gamma f_\varepsilon \|_{L^2(\gamma_+)}^2$ on the RHS of (4.80), we should be careful since we do not have the uniform bound on $\| f_\varepsilon \|_{L^2}$. Denote
\[
z^\varepsilon_\gamma (x) := \int_{n(x) \cdot q > 0} f_\varepsilon (x, q) \sqrt{J(q)} \{ n(x) \cdot \dot{q} \} \, dq,
\]
then one has $P_\gamma f_\varepsilon = z^\varepsilon_\gamma (x) \sqrt{J(p)}$. A direct calculation shows that
\[
\int_{n(x) \cdot p \geq \varepsilon', \varepsilon' \leq |p| \leq \frac{1}{\varepsilon}} J(p) | n(x) \cdot \dot{p} | \, dp \geq \zeta_1 > 0, \tag{4.81}
\]
provided that $0 < \varepsilon' \ll 1$, where $\zeta_1 > 0$ is a positive constant independent of $\varepsilon'$ and the light speed $c$. Using (4.81), we have that
\[
| P_\gamma f_\varepsilon \|_{L^2(\gamma_+)}^2 = \int_{\partial \Omega} \left| z^\varepsilon_\gamma (x) \right|^2 \, dS_x \cdot \int_{n(x) \cdot p \geq \varepsilon', \varepsilon' \leq |p| \leq \frac{1}{\varepsilon}} J(p) | n(x) \cdot \dot{p} | \, dp \\
\geq \zeta_1 \| P_\gamma f_\varepsilon \|_{L^2(\gamma_+)}^2,
\]
which, together with (4.80), yields that
\[
| P_\gamma f_\varepsilon \|_{L^2(\gamma_+)}^2 \leq C | P_\gamma f_\varepsilon \|_{L^2(\gamma_+)}^2 \\
\leq C \left\{ \| f_\varepsilon \|_{L^2(\gamma_+)}^2 + \| (I - P_\gamma) f_\varepsilon \|_{L^2(\gamma_+)}^2 \right\} \\
\leq C \| f_\varepsilon \|_{L^2(\gamma_+)}^2 + C\eta \| r \|_{L^2(\gamma_-)}^2 + C \| S \|_{L^2}^2. \tag{4.82}
\]
It follows from (4.1) that
\[
\frac{1}{2} \dot{\theta} \cdot \nabla_x \left| f_\varepsilon \right|^2 = -\varepsilon \left| f_\varepsilon \right|^2 - f_\varepsilon \mathbf{L} f_\varepsilon + f_\varepsilon g,
\]
which implies that
\[
\| \dot{\theta} \cdot \nabla_x \left| f_\varepsilon \right|^2 \|_{L^1} \leq C \left\{ \| f_\varepsilon \|_{L^2}^2 + \| (I - P) f_\varepsilon \|_{L^2}^2 + \| S \|_{L^2}^2 \right\}. \tag{4.83}
\]
Using (4.83) and (4.39), one has
\[
\frac{1}{2} \| f_\varepsilon \|_{L^2(\gamma_+)}^2 = \frac{1}{2} \| (f_\varepsilon)\|_{L^2(\gamma_+)}^2 + \| \dot{\theta} \cdot \nabla_x \left| f_\varepsilon \right|^2 \|_{L^1} \\
\leq C \left\{ \| f_\varepsilon \|_{L^2}^2 + \| \dot{\theta} \cdot \nabla_x \left| f_\varepsilon \right|^2 \|_{L^1} \right\} \\
\leq C \left\{ \| f_\varepsilon \|_{L^2}^2 + \| (I - P) f_\varepsilon \|_{L^2}^2 + \| S \|_{L^2}^2 \right\},
\]
which, together with (4.82), and by taking $\eta > 0$ suitably small, yields that
\[
| P_\gamma f_\varepsilon \|_{L^2(\gamma_+)} \leq C \| f_\varepsilon \|_{L^2}^2 + C \| (I - P) f_\varepsilon \|_{L^2}^2 + C \| S \|_{L^2}^2 + C \| r \|_{L^2(\gamma_-)}^2. \tag{4.84}
\]
Combining (4.84) and (4.80), then taking \( \eta > 0 \) small, one gets that
\[
\|f^\varepsilon\|_{L^2(\gamma_+)}^2 + \|(I - P_\gamma)f^\varepsilon\|_{L^2(\gamma_+)}^2 + \|(I - \mathbf{P})f^\varepsilon\|_{L^2(\gamma_+)}^2 \leq C \|\mathbf{S}\|_{L^2(\gamma_+)}^2 + C\|\mathbf{r}\|_{L^2(\gamma_+)}^2.
\] (4.85)

Applying (4.7) to \( f^\varepsilon \) and using (4.85), then we obtain
\[
\|w f^\varepsilon\|_{L^\infty(\gamma)} + \|w f^\varepsilon\|_{L^\infty(\gamma)} \leq C \left[ \|\nu^{-1} w\mathbf{S}\|_{L^\infty} + |w r|_{L^\infty(\gamma_+)} \right].
\] (4.86)

Next we consider the convergence of \( f^\varepsilon \) as \( \varepsilon \to 0^+ \). For any \( \varepsilon_1, \varepsilon_2 > 0 \), we consider the difference \( f^{\varepsilon_2} - f^{\varepsilon_1} \) satisfying
\[
\begin{align*}
\varepsilon_2 (f^{\varepsilon_2} - f^{\varepsilon_1}) + \hat{p} \cdot \nabla_x (f^{\varepsilon_2} - f^{\varepsilon_1}) + \mathbf{L} (f^{\varepsilon_2} - f^{\varepsilon_1}) &= (\varepsilon_1 - \varepsilon_2) f^{\varepsilon_1}, \\
(f^{\varepsilon_2} - f^{\varepsilon_1})|_{\gamma_+} &= P_\gamma (f^{\varepsilon_2} - f^{\varepsilon_1}).
\end{align*}
\] (4.87)

Multiplying (4.87) by \( f^{\varepsilon_2} - f^{\varepsilon_1} \), by similar arguments as in (4.79)-(4.85), one gets
\[
\begin{align*}
\|f^{\varepsilon_2} - f^{\varepsilon_1}\|_{L^2(\gamma_+)}^2 + \|(I - P_\gamma)(f^{\varepsilon_2} - f^{\varepsilon_1})\|_{L^2(\gamma_+)}^2 + \|(I - \mathbf{P})(f^{\varepsilon_2} - f^{\varepsilon_1})\|_{L^2(\gamma_+)}^2 &\leq C\|\varepsilon_1 - \varepsilon_2\|_{L^2(\gamma_+)}^2 + \|w f^{\varepsilon_1}\|_{L^\infty}^2 \\
&\leq C (\varepsilon_1^2 + \varepsilon_2^2) \cdot \left[ \|\nu^{-1} w\mathbf{S}\|_{L^\infty} + |w r|_{L^\infty(\gamma_+)} \right]^2 \to 0,
\end{align*}
\] (4.88)

as \( \varepsilon_1, \varepsilon_2 \to 0^+ \), where we have used (4.86) in the last inequality. Finally, applying (4.7) to \( f^{\varepsilon_2} - f^{\varepsilon_1} \), then we obtain
\[
\begin{align*}
\|w (f^{\varepsilon_2} - f^{\varepsilon_1})\|_{L^\infty(\gamma)} + \|w (f^{\varepsilon_2} - f^{\varepsilon_1})\|_{L^\infty(\gamma)} &\leq C \|\nu^{-1} w\|_{L^\infty} + C \|f^{\varepsilon_2} - f^{\varepsilon_1}\|_{L^2(\gamma_+)} \\
&\leq C (\varepsilon_1 + \varepsilon_2) \cdot \left[ \|\nu^{-1} w\|_{L^\infty} + |w r|_{L^\infty(\gamma_+)} \right] \to 0,
\end{align*}
\] (4.89)

as \( \varepsilon_1, \varepsilon_2 \to 0^+ \), where we have used (4.86) and (4.88) above. With (4.89), we know that there exists a function \( f \) so that \( \|w (f^{\varepsilon} - f)\|_{L^\infty(\gamma)} \to 0 \) as \( \varepsilon \to 0^+ \). And it is direct to see that \( f \) solves (4.75). Also, (4.76) and (4.77) follows immediately from (4.78) and (4.86), respectively. Therefore the proof of Lemma 4.3 is completed.

### 4.3. Proof of Theorem 1.1

We consider the following iterative sequence
\[
\begin{align*}
\begin{cases}
\hat{p} \cdot \nabla_x f^{j+1} + \mathbf{L} f^{j+1} &= \Gamma (f^j, f^j), \\
(f^{j+1})|_{\gamma_+} &= P_\gamma f^{j+1} + \frac{J_T(p) - J(p)}{\sqrt{J(p)}} \int_{n(x,q) > 0} f^j(x,q) \sqrt{J(q)} \{n(x) \cdot \hat{q}\} dq,
\end{cases}
\end{align*}
\] (4.90)

for \( j = 0, 1, 2, \cdots \) with \( f^0 = 0 \). A direct computation shows that
\[
\int_{\Omega} \int_{\mathbb{R}^3} \Gamma (f^j, f^j) \sqrt{J(p)} dp dx = 0, \quad \int_{n(x,q) < 0} |J_T(p) - J(p)| \{n(x) \cdot \hat{p}\} dp = 0,
\] (4.91)

which yields that
\[
\int_{\gamma_+} \left\{ \frac{J_T(p) - J(p)}{\sqrt{J(p)}} + \frac{J_T(p) - J(p)}{\sqrt{J(p)}} \int_{n(x,q) > 0} f^j(x,q) \sqrt{J(q)} \{n(x) \cdot \hat{q}\} dq \right\} \sqrt{J(p)} d\gamma = 0.
\]

It follows from Lemma 3.13 that
\[
\|\nu^{-1} \Gamma (f^j, f^j)\|_{L^\infty(\gamma)} \leq C \|w f^j\|_{L^\infty}^2.
\] (4.92)

It is clear that
\[
\begin{align*}
\|w \left( \frac{J_T(p) - J(p)}{\sqrt{J(p)}} + \frac{J_T(p) - J(p)}{\sqrt{J(p)}} \int_{n(x,q) > 0} f^j(x,q) \sqrt{J(q)} \{n(x) \cdot \hat{q}\} dq \right)\|_{L^\infty(\gamma_+)} &\leq C \delta + C \delta \|f^j\|_{L^\infty(\gamma_+)}. \quad (4.93)
\end{align*}
\]
Noting (4.91)-(4.93), and using Lemma 4.8, we can solve (4.90) inductively for \( j = 0, 1, 2, \ldots \). Moreover, it follows from (4.77), (4.92) and (4.93) that
\[
\|w f^{j+1}\|_{L^\infty} + |w f^{j+1}|_{L^\infty(\gamma)} \leq \tilde{C}_1 \delta + \tilde{C}_1 \delta \|f^j\|_{L^\infty(\gamma_+)} + \tilde{C}_1 \|w f^j\|_{L^\infty}^2. \tag{4.94}
\]
We aim to prove
\[
\|w f^j\|_{L^\infty} + |w f^j|_{L^\infty(\gamma)} \leq 2 \tilde{C}_1 \delta, \quad \text{for } j = 1, 2, \ldots. \tag{4.95}
\]
Indeed, for \( j = 0 \), it follows from \( f^0 \equiv 0 \) and (4.94) that
\[
\|w f^1\|_{L^\infty} + |w f^1|_{L^\infty(\gamma)} \leq \tilde{C}_1 \delta. \tag{4.96}
\]
Now we assume that (4.95) holds for \( j = 1, 2, \ldots, l \), then we consider the case for \( j = l + 1 \). It follows from (4.94) that
\[
\|w f^{l+1}\|_{L^\infty} + |w f^{l+1}|_{L^\infty(\gamma)} \leq \tilde{C}_1 \delta + \tilde{C}_1 \delta |f|_{L^\infty(\gamma)} + \tilde{C}_1 \|w f^j\|_{L^\infty}^2 \\
\leq \tilde{C}_1 \delta (1 + 2 \tilde{C}_1 \delta + 4 \tilde{C}_1^2 \delta) \leq \frac{3}{2} \tilde{C}_1 \delta,
\]
where we have used (4.95) for \( j = l \), and chosen \( \delta > 0 \) small enough such that \( 2 \tilde{C}_1 \delta + 4 \tilde{C}_1^2 \delta \leq 1/2 \). Therefore we have proved (4.95) by induction.

Finally we consider the convergence of sequence \( f^j \). It follows from (4.91) that
\[
\begin{align*}
\left\{ \hat{p} \cdot \nabla_x (f^{j+1} - f^j) + L (f^{j+1} - f^j) = & \Gamma (f^j - f^{j-1}, f^j) + \Gamma (f^{j-1}, f^j - f^{j-1}), \\
(f^{j+1} - f^j) |_{\gamma_-} = & P_\gamma (f^{j+1} - f^j), \\
+ & J_T(p) - J(p) \right\} \\
= & \int_{n(x) < 0} \left( f^j - f^{j-1} \right)(x, q) \sqrt{J(q)} \{n(x) \cdot \hat{q}\} dq. \tag{4.97}
\end{align*}
\]
Using Lemma 3.13 and applying (4.77) to (4.97), we have that
\[
\begin{align*}
\|w \{f^{j+1} - f^j\}\|_{L^\infty} + |w \{f^{j+1} - f^j\}|_{L^\infty(\gamma)} \\
\leq C \left\{ \|w^{-1} \Gamma (f^j - f^{j-1}, f^j)\|_{L^\infty} + \|w^{-1} \Gamma (f^{j-1}, f^j - f^{j-1})\|_{L^\infty} \right\} \\
+ C \left\{ \|w \left\{ J_T - J \sqrt{J} \right\} \int_{n(x) < 0} (f^j - f^{j-1})(x, q) \sqrt{J(q)} \{n(x) \cdot \hat{q}\} dq \right\}_{L^\infty(\gamma_+)} \\
\leq C \left[ \delta + \|w f^j\|_{L^\infty} + \|w f^{j-1}\|_{L^\infty} \cdot \left\{ \|w (f^j - f^{j-1})\|_{L^\infty} + \|w (f^j - f^{j-1})\|_{L^\infty(\gamma_+)} \right\} \right] \\
\leq \frac{1}{2} \left\{ \|w (f^j - f^{j-1})\|_{L^\infty} + \|w (f^j - f^{j-1})\|_{L^\infty(\gamma_+)} \right\}, \tag{4.98}
\end{align*}
\]
where we have used (4.96) and taken \( \delta > 0 \) small such that \( C \delta \leq 1/2 \). Hence \( f^j \) is a Cauchy sequence in \( L^\infty \), then we obtain the solution by taking the limit \( f_* = \lim_{j \to \infty} f^j \). The uniqueness can also be obtained by using the inequality as (4.98).

The positivity of \( F_* := J + \sqrt{J} f_* \) will be proved in section 5 by the large-time behavior. Therefore we complete the proof of Theorem 1.1 \( \square \)

5. Dynamical stability under small perturbations

In this section, we are concerned with the large-time asymptotic stability of the steady solution \( F_* \) obtained in Theorem 1.1. We introduce the perturbation
\[
f(t, x, p) := \frac{F(t, x, p) - F_*(x, p)}{\sqrt{J(p)}},
\]
then the initial-boundary value problem \([1.3], [1.8], [1.26]\) can be written as
\[
\begin{aligned}
&\frac{\partial f}{\partial t} + \hat{p} \cdot \nabla_x f + Lf = -L_s f + \Gamma(f, f), \\
f(t, x, p)|_{t=0} = f_0(x, p) := \frac{F(0, x, p) - F_s(x, p)}{\sqrt{J(p)}}, \\
f(t, x, p)|_{\gamma_-} = P_\gamma f + \frac{J_T - J}{\sqrt{J}} \int_{n(x) \cdot q > 0} f(t, x, q) \sqrt{J(q)} n(x) \cdot dq.
\end{aligned}
\tag{5.1}
\]
Here \(P_\gamma f\) is defined in \([1.15]\), the linearized collision operator \(L\) is defined in \([1.16]\), the nonlinear term \(\Gamma(f, f)\) is defined in \([1.18]\) and
\[
L_s f := -\frac{1}{\sqrt{J}} \left[ Q \left( \sqrt{J} f_s, \sqrt{J} f \right) + Q \left( \sqrt{J} f, \sqrt{J} f_s \right) \right].
\]
Recall \([1.11]\), and denote
\[
h(t, x, p) := w(p)f(t, x, p).
\]
Then one can reformulate \((5.1)\) as
\[
\begin{aligned}
&\frac{\partial h}{\partial t} + \hat{p} \cdot \nabla_x h + \nu(p) h - K_w h = -wL_s f + w\Gamma(f, f), \\
h|_{t=0} = w f_0 := h_0, \\
h|_{\gamma_-} = \frac{1}{w(p)} \int_{n(x) \cdot q > 0} h(q) \tilde{w}(q) dq + w(p) \frac{J_T - J}{\sqrt{J}} \int_{n(x) \cdot q > 0} h(q) \tilde{w}(q) dq,
\end{aligned}
\tag{5.2}
\]
where \(d\sigma\), \(\tilde{w}\) and \(K_w h\) are the same ones defined in section 4. For simplicity, we denote
\[
r(t, x, p) := \frac{J_T - J}{\sqrt{J}} \int_{n(x) \cdot q > 0} h(t, x, q) \tilde{w}(q) d\sigma.
\tag{5.3}
\]

5.1. Linear problem. We first study the following linear inhomogeneous problem:
\[
\begin{aligned}
&\frac{\partial h}{\partial t} + \hat{p} \cdot \nabla_x h + \nu(p) h - K_w h = wS, \\
h|_{t=0} = w f_0 := h_0, \\
h|_{\gamma_-} = \frac{1}{w(p)} \int_{n(x) \cdot q > 0} h(q) \tilde{w}(q) dq + w(p) \frac{J_T - J}{\sqrt{J}} \int_{n(x) \cdot q > 0} h(q) \tilde{w}(q) dq,
\end{aligned}
\tag{5.4}
\]
where \(S\) is a given function. Then the equation of \(h = wf\) becomes
\[
\begin{aligned}
&\frac{\partial h}{\partial t} + \hat{p} \cdot \nabla_x h + \nu(p) h - K_w h = wS, \\
h|_{t=0} = \frac{1}{w(p)} \int_{n(x) \cdot q > 0} h(q) \tilde{w}(q) dq + w(p) \frac{J_T - J}{\sqrt{J}} \int_{n(x) \cdot q > 0} h(q) \tilde{w}(q) dq.
\end{aligned}
\tag{5.5}
\]

Recall the stochastic cycle defined in \([1.3]\). The following lemma gives the mild formulation for \((5.4)\). Since its proof is almost the same as for \([33\text{ Lemma 24}]\), we omit details for brevity.

**Lemma 5.1.** Let \(k \geq 1\) be an integer and \(h(t, x, p) \in L^\infty\) satisfy \((5.4)\). For any \(t > 0\), for almost every \((x, p) \in \Omega \times \mathbb{R}^3 \setminus \gamma_0 \cup \gamma_-\) and for any \(0 \leq s \leq t\), it holds that
\[
h(t, x, p) = \sum_{n=1}^{11} K_n,
\]
with
\[
K_1 = 1_{\{t_1 \leq s\}} e^{-\nu(p)(t-s)} h(s, x - \hat{p}(t-s), p), \\
K_2 + K_3 = \int_{\max\{t_1, s\}}^{t} e^{-\nu(p)(t-\tau)} \left[ K_w h + wS \right](\tau, x - \hat{p}(t-\tau), p) d\tau, \\
K_4 = 1_{\{t_1 > s\}} e^{-\nu(p)(t-t_1)} w(p)r(t_1, x_1, p),
\]
\[ K_5 = \frac{e^{-\nu(p)(t-t_1)}}{\hat{w}(p)} \int_{\Pi_{j=1}^{k-1} \Pi_j} \sum_{l=1}^{k-2} \mathbf{1}_{\{t_{l+1} > s\}} w(p_l) r(t_{l+1}, x_{l+1}, p_l) d\Sigma_l(t_{l+1}), \]
\[ K_6 = \frac{e^{-\nu(p)(t-t_1)}}{\hat{w}(p)} \int_{\Pi_{j=1}^{k-1} \Pi_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1} \leq s < t_l\}} h(s, x_l - \hat{p}_l t_l, p_l) d\Sigma_l(s), \]
\[ K_7 + K_8 = \frac{e^{-\nu(p)(t-t_1)}}{\hat{w}(p)} \int_{\Pi_{j=1}^{k-1} \Pi_j} \sum_{l=1}^{k-1} \int_{t_l}^{t_{l+1}} \mathbf{1}_{\{t_{l+1} \leq s < t_l\}} \times [K_u h + w S](\tau, x_1 - \hat{p}_l (t_1 - \tau), p_l) d\tau d\Sigma_l(\tau), \]
\[ K_9 + K_{10} = \frac{e^{-\nu(p)(t-t_1)}}{\hat{w}(p)} \int_{\Pi_{j=1}^{k-1} \Pi_j} \sum_{l=1}^{k-1} \int_{t_l}^{t_{l+1}} \mathbf{1}_{\{t_{l+1} > s\}} \times [K_u h + w S](\tau, x_1 - \hat{p}_l (t_1 - \tau), p_l) d\tau d\Sigma_l(\tau), \]
\[ K_{11} = \frac{e^{-\nu(p)(t-t_1)}}{\hat{w}(p)} \int_{\Pi_{j=1}^{k-1} \Pi_j} \mathbf{1}_{\{t_{k+1} > s\}} h(t_k, x_k, p_k) d\Sigma_{k-1}(t_k), \]
where we have denoted
\[ d\Sigma_l(\tau) = \left\{ \Pi_{j=1}^{l-1} d\sigma_j \right\} \cdot \left\{ \hat{w}(p_l) e^{-\nu(p_l)(t_1 - \tau)} d\sigma_l \right\} \cdot \left\{ \Pi_{j=1}^{l-1} e^{-\nu(p_j)(t_j - t_{j+1})} d\sigma_j \right\}. \]

By similar arguments as in Lemma 4.1, we have the following lemma.

**Lemma 5.2.** Let \( 0 \leq \eta \leq \frac{1}{4} \). For \( T_0 > 0 \) sufficiently large, there exist constants \( C_3 \) and \( C_4 \) independent of \( T_0 \) and \( c \) such that for \( k = C_3 T_0^5 \) and \( (t, x, p) \in [s, s + T_0] \times \Omega \times \mathbb{R}^3 \), it holds that
\[ \int_{\Pi_{j=1}^{k-1} \Pi_j} \mathbf{1}_{\{t_{k+1} > s\}} \Pi_{j=1}^{k-1} J^{-\eta}(p_j) d\sigma_j \leq \left( \frac{1}{2} \right)^{C_4 T_0^5}. \]

**Lemma 5.3.** Let \( h(t, x, p) \) be the \( L^\infty \) mild solution of the linear problem (5.4). Then for any \( s \geq 0 \), for any \( s \leq t \leq s + T_0 \) with \( T_0 > 0 \) sufficiently large, and for almost everywhere \( (x, p) \in \Omega \times \mathbb{R}^3 \setminus \gamma_0 \), it holds that
\[ |h(t, x, p)| \leq C T_0^5 e^{-\lambda_1 t} \left\{ \delta + \left( \frac{1}{2} + \frac{1}{N} \right) \sup_{\tau \leq \tau \leq s} \{ \| e^{\lambda_1 \tau} h \|_{L^\infty} + \| e^{\lambda_1 \tau} h \|_{L^\infty(\gamma)} \} \right. \]
\[ + C T_0^5 e^{-\nu_0(t-s)} \| h(s) \|_{L^\infty} + C T_0^5 e^{-\lambda_1 t} \sup_{\tau \leq \tau \leq s} \| e^{\lambda_1 \tau} \nu^{-1} w S(\tau) \|_{L^\infty} \]
\[ + C N T_0 e^{-\lambda_1 t} \sup_{\tau \leq \tau \leq s} \| e^{\lambda_1 \tau} f(\tau) \|_{L^2} + C \max \left\{ \frac{1}{c}, \frac{1}{N} \right\} e^{-\lambda_1 t} \sup_{\tau \leq \tau \leq s} \| e^{\lambda_1 \tau} h \|_{L^\infty}, \]
where \( \nu_0 > 0 \) is the one given in (4.8), and \( 0 < \lambda_1 < \frac{\nu_0}{2} \) is a constant to be chosen later. \( N > 0 \) can be chosen arbitrarily large.

**Proof.** We bound the right hand side of (5.5) term by term. For \( K_1 \), it holds that
\[ |K_1| \leq e^{-\nu_0(t-s)} \| h(s) \|_{L^\infty}. \]

For any \( 0 < \lambda_1 < \frac{\nu_0}{2} \), we have
\[ |K_3| \leq \int_{s \leq \tau \leq t} e^{-\nu(p)(t-t)} |w S(\tau, x - \hat{p}(t - \tau), p)| d\tau \]
\[ \leq \int_{s \leq \tau \leq t} e^{\frac{\nu(p)}{2}(t-t)} \nu(p) e^{-\lambda_1 t} \nu^{-1}(p) w S(\tau, x - \hat{p}(t - \tau), p)| d\tau \]
\[ \leq C e^{-\lambda_1 t} \sup_{s \leq \tau \leq t} \| e^{\lambda_1 \tau} \nu^{-1} w S(\tau) \|_{L^\infty}. \]
For $K_4$ and $K_5$, we note from (5.2) and (4.11) that
\[
|w(p)r(t, x, p)| \leq w(p) \frac{|J_T(p) - J(p)|}{\sqrt{J(p)}} \int_{n(x) - q > 0} \|h(t, x, q)|\tilde{w}(q)|d\sigma
\leq C\delta e^{-\frac{|p|}{T}} e^{-\lambda_1 t} \sup_{s \leq \tau \leq t} |e^{\lambda_1 \tau} h(\tau)|_{L^\infty(\gamma_+)}
\]
which yields that
\[
|K_4| \leq C\delta e^{-\frac{|p|}{T}} e^{-\lambda_1 t} \sup_{s \leq \tau \leq t} |e^{\lambda_1 \tau} h(\tau)|_{L^\infty(\gamma_+)}
\leq C\delta e^{-\frac{|p|}{T}} e^{-\lambda_1 t} \sup_{s \leq \tau \leq t} |e^{\lambda_1 \tau} h(\tau)|_{L^\infty(\gamma_+)}
\]
and
\[
|K_5| \leq C\delta e^{-\frac{|p|}{T}} \sum_{k=1}^{k+2} \int_{l=1}^{k+2} e^{-\lambda_1 t} \tilde{w}(p_k) \int_{s \leq \tau \leq t} \|e^{\lambda_1 \tau} h(\tau)|_{L^\infty(\gamma_+)}
\leq Ck\delta e^{-\frac{|p|}{T}} e^{-\lambda_1 t} \sup_{s \leq \tau \leq t} |e^{\lambda_1 \tau} h(\tau)|_{L^\infty(\gamma_+)},
\]
(5.9)

For $K_6$, one has
\[
|K_6| \leq Ce^{-\frac{|p|}{T}} \sum_{l=1}^{k+2} \int_{l=1}^{k+2} e^{-\lambda_1 t} \tilde{w}(p_l) \int_{s \leq \tau \leq t} \|e^{\lambda_1 \tau} h(\tau)|_{L^\infty(\gamma_+)}
\leq Cke^{-\frac{|p|}{T}} e^{-\lambda_1 t} \sup_{s \leq \tau \leq t} |e^{\lambda_1 \tau} h(\tau)|_{L^\infty(\gamma_+)}.
\]
(5.10)

By similar arguments as in (5.8), one obtains that
\[
|K_8| \leq Ce^{-\frac{|p|}{T}} \sum_{l=1}^{k+2} \int_{l=1}^{k+2} e^{-\lambda_1 t} \tilde{w}(p_l) \int_{s \leq \tau \leq t} \|e^{\lambda_1 \tau} h(\tau)|_{L^\infty(\gamma_+)}
\leq Cke^{-\frac{|p|}{T}} e^{-\lambda_1 t} \sup_{s \leq \tau \leq t} |e^{\lambda_1 \tau} h(\tau)|_{L^\infty(\gamma_+)}.
\]
(5.12)

and
\[
|K_{10}| \leq Cke^{-\frac{|p|}{T}} e^{-\lambda_1 t} \sup_{s \leq \tau \leq t} |e^{\lambda_1 \tau} h(\tau)|_{L^\infty(\gamma_+)}.
\]
(5.13)

For $K_{11}$, it follows from Lemma 5.2 that
\[
|K_{11}| \leq Ce^{-\frac{|p|}{T}} \int_{l=1}^{l+2} \int_{l=1}^{l+2} 1_{\{t_k > s\}} e^{-\lambda_1 t_k} \tilde{w}(p_k) \int_{s \leq \tau \leq t} \sup_{s \leq \tau \leq t} |e^{\lambda_1 \tau} h(\tau)|_{L^\infty(\gamma_+)}
\leq Ce^{-\frac{|p|}{T}} e^{-\lambda_1 t} \int_{l=1}^{l+2} \int_{l=1}^{l+2} \|e^{\lambda_1 \tau} h(\tau)|_{L^\infty(\gamma_+)}.
\]
(5.14)

By similar arguments as in (4.17) - (4.24), we can bound $K_7$ and $K_9$ as
\[
|K_7| + |K_9| \leq \frac{Ck}{N} e^{-\frac{|p|}{T}} e^{-\lambda_1 t} \sup_{s \leq \tau \leq t} |e^{\lambda_1 \tau} h(\tau)|_{L^\infty} + CNke^{-\frac{|p|}{T}} e^{-\lambda_1 t} \sup_{s \leq \tau \leq t} |e^{\lambda_1 \tau} f(\tau)|_{L^2}.
\]
(5.15)

Combining (5.5), (5.7) - (5.15), one has
\[
|h(t, x, p)| \leq \int_{\max\{t_1, \}} \int_{\mathbb{R}^3} e^{-\nu(p)(t-\tau)} k_w(p, p') h(\tau, x - p(t - \tau), p') |dp'd\tau + \tilde{A}(t, p),
\]
(5.16)
where

\[ \hat{A}(t,p) := C_ke^{-\frac{|p|}{2}}e^{-\lambda_1t}\left\{ \delta + \left( \frac{1}{2} \right) \sup_{s \leq \tau \leq t} \| e^{\lambda_1\tau}h \|_{L^\infty} + \| e^{\lambda_1\tau}h \|_{L^\infty(\gamma_+)} \right\} + C_k e^{-\nu_0(t-s)}\| h(s) \|_{L^\infty} + C_k e^{-\lambda_1t} \sup_{s \leq \tau \leq t} \| e^{\lambda_1\tau}f(\tau) \|_{L^2}. \]

We denote \( x' = x - \hat{p}(t - \tau) \in \Omega \) and \( t'_1 = t_1(\tau, x', p') \) for \( \tau \in (\max\{t_1, s\}, t) \). Using the Vidav's iteration in (5.16), then we obtain that

\[ |h(t,x,p)| \leq \hat{A}(t,p) + \int_{\max\{t_1,s\}}^t \int_{\mathbb{R}^3} e^{-\nu(p)(t-\tau)}|k_w(p,p')\hat{A}(\tau, p')|dp'd\tau \]
\[ + \int_s^t \int_s^\tau \int_{\mathbb{R}^3} 1_{\max\{t_1,s\} \leq \tau \leq t} 1_{\max\{t'_1,s\} \leq \tau'} e^{-\nu(p)(t-\tau)} e^{-\nu(p')(\tau-\tau')} \times |k_w(p,p') k_w(p'', p''') h(\tau', x' - (\tau - \tau')p', p''')| dp'' dp' dp' d\tau \]

By similar arguments as in (4.28)-(4.35), for \( (x,p) \in \Omega \times \mathbb{R}^3 \setminus \gamma_0 \cup \gamma_- \), we have

\[ |h(t,x,p)| \leq C T_0^3 e^{-\lambda_1t}\left\{ \delta + \left( \frac{1}{2} \right) \frac{1}{N} \sup_{s \leq \tau \leq t} \| e^{\lambda_1\tau}h \|_{L^\infty} + \| e^{\lambda_1\tau}h \|_{L^\infty(\gamma_+)} \right\} + C T_0^3 e^{-\nu_0(t-s)}\| h(s) \|_{L^\infty} + C T_0^3 e^{-\lambda_1t} \sup_{s \leq \tau \leq t} \| e^{\lambda_1\tau}f(\tau) \|_{L^2} + C \max\{1, \frac{1}{c} \} e^{-\lambda_1t} \sup_{s \leq \tau \leq t} \| e^{\lambda_1\tau}h \|_{L^\infty}. \]  

(5.17)

Since \( |h(t)|_{L^\infty(\gamma_-)} \lesssim |h(t)|_{L^\infty(\gamma_+)} \), we conclude (5.6) from (5.17). Therefore the proof of Lemma 5.3 is completed.

Proposition 5.4. Let \( f \) be a solution of the following IBVP

\[ \begin{cases} \partial_t f + \hat{p} \cdot \nabla_x f + Lf = S, & f(0) = f_0, \\ f|_{\gamma_-} = P_\gamma f + r, \end{cases} \]  

(5.18)

such that for all \( t > 0 \),

\[ \iint_{\Omega \times \mathbb{R}^3} f(t,x,p)\sqrt{J}dpdx = \iiint_{\Omega \times \mathbb{R}^3} S(t,x,p)\sqrt{J}dpdx = \int_{\gamma_-} r \sqrt{J}d\gamma = 0. \]  

(5.19)

Then there exists a function \( G(t) \) such that, for all \( 0 \leq \tau \leq t \), \( G(\tau) \lesssim \| f(\tau) \|_{L^2}^2 \) and

\[ \int_s^t \| Pf(\tau) \|_{L^2}^2 d\tau \lesssim \int_s^t \| (I - P) f(\tau) \|_{L^2}^2 d\tau + \int_s^t \| (I - P_\gamma) f(\tau) \|_{L^2(\gamma_+)}^2 d\tau + \int_s^t \| S(\tau) \|_{L^2}^2 + \| r(\tau) \|_{L^2(\gamma_-)}^2 d\tau + |G(t) - G(s)|. \]  

(5.20)

Proof. By Green's identity in Lemma 2.4, the weak version of (5.18) takes the form

\[ \int_s^t \int_{\gamma_+} \psi f d\gamma - \int_s^t \int_{\gamma_-} \psi f d\gamma - \int_s^t \int_{\Omega \times \mathbb{R}^3} \hat{p} \cdot \nabla_x \psi f - \int_s^t \int_{\Omega \times \mathbb{R}^3} \partial_t \psi f \]
\[ = - \int_{\Omega \times \mathbb{R}^3} \psi f(t) + \int_{\Omega \times \mathbb{R}^3} \psi f(s) + \int_s^t \int_{\Omega \times \mathbb{R}^3} \{ -\psi L(I - P) f + \psi S \}. \]  

(5.21)

Without loss of generality we only prove the case of \( s = 0 \). We note that (5.18) and (5.19) are all invariant under a standard \( t \)-mollification for all \( t > 0 \). The estimates in Step 1 to Step 3 below are obtained via a \( t \)-mollification so that all the functions are smooth in \( t \). For notational
we have from Lemma 3.1 that \( \phi \) with \( \nabla \phi \) depending only on \( x \). Then it follows from (5.19) and (5.21) that

\[
\int_\Omega [a(t + \varepsilon) - a(t)] \phi(x) = \int_t^{t + \varepsilon} \int_\Omega b \cdot \nabla_x \phi + \int_t^{t + \varepsilon} \int_\gamma \nabla_\gamma \phi \sqrt{J} + \sum_{j=1}^3 \int_t^{t + \varepsilon} \int_{\Omega \times \mathbb{R}^3} \frac{c}{p^p} p_j \sqrt{J} \partial_j \phi(I - P)f.
\]

Taking the difference quotient, we can obtain

\[
\int_\Omega \partial_t a = \int_\Omega b \cdot \nabla_x \phi + \int_\gamma \nabla_\gamma \phi \sqrt{J} + \sum_{j=1}^3 \int_\Omega \int_{\Omega \times \mathbb{R}^3} \frac{c}{p^p} p_j \sqrt{J} \partial_j \phi(I - P)f(t).
\]  

(5.22)

For \( \phi = 1 \), it follows from (5.19) and (5.22) that

\[
\int_\Omega \partial_t a(t)dx = 0, \quad t > 0.
\]

On the other hand, noting \( |\phi|_{L^2(\partial \Omega)} \lesssim \|\phi\|_{H^1(\Omega)} \), we get from (5.22) that

\[
\left| \int_\Omega \phi(x) \partial_t a dx \right| \lesssim |r|_{L^2(\partial \Omega)} \|\phi\|_{L^2(\partial \Omega)} + \|b\|_{L^2} \|\phi\|_{H^1} + \|\phi\|_{H^1} \lesssim \{ \|b(t)\|_{L^2} + |r|_{L^2(\partial \Omega)} + \|\phi\|_{H^1} \} \|\phi\|_{H^1}.
\]  

(5.23)

Hence, for all \( t > 0 \), one has

\[
\|\partial_t a(t)\|_{(H^1)^*} \lesssim \|b(t)\|_{L^2} + |r|_{L^2(\partial \Omega)} + \|\phi\|_{H^1} \lesssim \|\phi\|_{H^1}.
\]

where \((H^1)^*(\Omega)\) is the dual space of \( H^1(\Omega) \) with respect to the dual pair \((A, B) = \int_\Omega A(x)B(x)dx, \) for \( A \in H^1 \) and \( B \in (H^1)^* \).

By the standard elliptic theory, we can solve \(-\Delta \Phi_a = \partial_t a(t), \partial_\partial \phi_a = 0 \) on \( \partial \Omega \), with the crucial condition \( \int_\Omega \partial_t a(t, x)dx = 0 \) for all \( t > 0 \). Noting that \( \Phi_a = -\Delta^{-1} \partial_t a = \partial_\partial \phi_a \) where \( \phi_a \) is defined in (4.69), thus we have

\[
\|\nabla_\partial \partial \phi_a\|_{L^2} \leq \|\Delta^{-1} \partial_t a(t)\|_{H^1} = \|\Phi_a\|_{H^1} \lesssim \|\partial_t a(t)\|_{(H^1)^*} \leq C \left( \|b(t)\|_{L^2} + |r|_{L^2(\partial \Omega)} + \|\phi\|_{H^1} \right),
\]  

(5.24)

where the constant \( C > 0 \) is independent of the light speed \( c \).

**Step 2.** Estimate on \( \nabla_x \Delta^{-1} \partial_t b_i = -\nabla_x \partial_\partial \phi_b \) in (5.21). Choosing the test function \( \psi = \phi(x)p_i \sqrt{J} \) with \( \phi(x) \) depending only on \( x \) and substituting it into (5.21) (with time integration over \([t, t + \varepsilon]\)), we have from Lemma 3.1 that

\[
A_1 \int_\Omega [b_i(t + \varepsilon) - b_i(t)] \phi = -\int_t^{t + \varepsilon} \int_\Omega \phi \nabla_\partial \phi \sqrt{J} + \int_t^{t + \varepsilon} \int_\Omega \partial_\partial \phi \left[ a + \frac{1}{\epsilon \sqrt{A_2 - A_3}} \right] \phi - \frac{3}{p^p} \int_\Omega \int_{\Omega \times \mathbb{R}^3} \frac{c}{p^p} p_j \sqrt{J} \partial_j \phi(I - P)f + \int_t^{t + \varepsilon} \int_\Omega \int_{\Omega \times \mathbb{R}^3} \phi \nabla_\partial S \sqrt{J}.
\]

Taking the difference quotient, we can obtain

\[
A_1 \int_\Omega \partial_t b_i(t) \phi = -\int_\Omega \phi \nabla_\partial \phi \sqrt{J} + \int_\Omega \partial_\partial \phi \left[ a + \frac{1}{\epsilon \sqrt{A_2 - A_3}} \right] \phi.
\]
\begin{equation}
+ \sum_{j=1}^{3} \int_{\Omega \times \mathbb{R}^3} \frac{c}{p_j} b_{pj} \sqrt{\partial_j f(I - P)} \phi(t) + \int_{\Omega \times \mathbb{R}^3} \phi_b S(t) \sqrt{J}.
\end{equation}

For fixed $t > 0$, we choose $\phi = \Phi_{b}^i$ with $-\Delta \Phi_{b}^i = \partial_t b_i(t), \Phi_{b}^i|_{\partial \Omega} = 0$. It is clear that $\Phi_{b}^i = -\Delta^{-1} \partial_t b_i = \partial_t \phi_{b}^i$, where $\phi_{b}^i$ is defined in (4.56). Thus the boundary term of (5.25) vanishes due to $\Phi_{b}^i|_{\partial \Omega} = 0$. Since $A_1 = 1 + O(\varepsilon^{-2})$ and $c \sqrt{A_2 - A_3^2} = \frac{\varepsilon}{2} + O(\varepsilon^{-2})$, then it holds that

$$
\int_{\Omega} |\nabla \Delta^{-1} \partial_t b_i(t)|^2 dx = \int_{\Omega} |\nabla \Phi_{b}^i|^2 dx = - \int_{\Omega} \Delta \Phi_{b}^i \Phi_{b}^i dx
$$

where we have used the Poincaré inequality in the last step. Taking $\varepsilon$ suitably small, for all $t > 0$, we have

$$
\|\nabla \partial_t \phi_{b}^i(t)\|_{L^2} \leq C \left( \|(a, c)(t)\|_{L^2} + \|(I - P)f(t)\|_{L^2} + \|S(t)\|_{L^2} \right),
\end{equation}

where the constant $C > 0$ is independent of the light speed $c$.

**Step 3. Estimate on $\nabla \Delta^{-1} \partial_t c = -\nabla \partial_t \phi_{c}$ in (5.21).** We choose the test function $\psi = \phi(x) \frac{p_0 - A_3}{\sqrt{A_2 - A_3^2}} \sqrt{J}$ in (5.21), then it follows from Lemma 3.1 that

\begin{align*}
\int_{\Omega} \phi(x)[c(t + \varepsilon) - c(t)] &= - \int_{\Omega} \frac{p_0 - A_3}{\sqrt{A_2 - A_3^2}} \sqrt{J} \phi f + \int_{t}^{t + \varepsilon} \int_{\Omega \times \mathbb{R}^3} \frac{c}{p_j} \frac{p_0 - A_3}{\sqrt{A_2 - A_3^2}} \sqrt{J} b \cdot \nabla \phi
\end{align*}

Taking the difference quotient, we obtain

\begin{align*}
\frac{\phi(x) \partial_t c(t, x)}{dx} &= - \int_{\Omega} \frac{p_0 - A_3}{\sqrt{A_2 - A_3^2}} \sqrt{J} \phi f + \int_{t}^{t + \varepsilon} \int_{\Omega \times \mathbb{R}^3} \frac{c}{p_j} \frac{p_0 - A_3}{\sqrt{A_2 - A_3^2}} \sqrt{J} b(t) \cdot \nabla \phi
\end{align*}

For fixed $t > 0$, we define a test function $\phi = \Phi_{c} = \partial_t \phi_{c}$ where $\phi_{c}$ is defined in (4.48). Then we have, for $t > 0, -\Delta \Phi_{c} = \partial_t c(t), \Phi_{c}|_{\partial \Omega} = 0$. The boundary terms vanish due to $\Phi_{c}|_{\partial \Omega} = 0$. Notice that $\Phi_{c} = -\Delta^{-1} \partial_t c(t) = \partial_t \phi_{c}(t)$. By similar arguments as in Step 2, it holds that

\begin{equation}
\|\nabla \Delta^{-1} \partial_t c(t)\|_{L^2}^2 = \int_{\Omega} |\nabla \Phi_{c}(x)|^2 dx = \int_{\Omega} \Phi_{c}(x) \partial_t c(t, x) dx
\end{equation}

\begin{equation}
\lesssim \varepsilon \|\nabla \Phi_{c}\|_{L^2}^2 + C_{\varepsilon} \left( \|b(t)\|_{L^2}^2 + \|(I - P)f(t)\|_{L^2}^2 + \|S(t)\|_{L^2}^2 \right).
\end{equation}

By taking $\varepsilon$ suitably small, we have, for all $t > 0$,

\begin{equation}
\|\nabla \partial_t \phi_{c}\|_{L^2} \leq C \left( \|b(t)\|_{L^2} + \|(I - P)f(t)\|_{L^2} + \|S(t)\|_{L^2} \right),
\end{equation}

where the constant $C > 0$ is independent of the light speed $c$. 
Step 4. Estimate on $a$, $b$, $c$ contributions in (5.21). The key of the estimate is to use the same choices of test functions as in Proposition 4.7 (with extra dependence on time), i.e., (4.47), (4.55), (4.61) and (4.68), to bound the new term $\int_{\Omega} \int f(t) \partial_t \psi f$. We choose $G(\tau) := - \int_{\Omega \times \mathbb{R}^3} \psi f(t)$, which yields that $|G(\tau)| \lesssim \|f(t)\|_{L^2}^2$.

To estimate $c$ in (5.21), we plug (4.47) into (5.21). It holds that

$$\int_0^t \int_{\Omega \times \mathbb{R}^3} \partial_t \psi_c f = \sum_{i=1}^3 \int_0^t \int_{\Omega \times \mathbb{R}^3} (|p|^2 - \beta_c) p_i \sqrt{J(p)} \partial_i \partial_t \phi_c f$$

$$= \sum_{i=1}^3 \int_0^t \int_{\Omega \times \mathbb{R}^3} (|p|^2 - \beta_c) p_i \sqrt{J(p)} \partial_i \partial_t \phi_c \left\{ a + b \cdot p + c \frac{p^0 - A_3}{\sqrt{A_2 - A_3^2}} \right\}$$

$$+ \sum_{i=1}^3 \int_0^t \int_{\Omega \times \mathbb{R}^3} (|p|^2 - \beta_c) p_i \sqrt{J(p)} \partial_i \partial_t \phi_c \cdot (I - P) f. \tag{5.28}$$

For the first term on the RHS of (5.28), the $a$, $c$ contributions vanish due to the oddness in $p$. The $b$ contribution takes the form

$$\sum_{i=1}^3 \int_0^t \int_{\Omega \times \mathbb{R}^3} (|p|^2 - \beta_c) p_i^2 \sqrt{J(p)} \partial_i \partial_t \phi_c b_i. \tag{5.29}$$

Using (4.50), one has

$$\left| \int_{\mathbb{R}^3} (|p|^2 - \beta_c) p_i^2 \sqrt{J(p)} dp \right| = \left| \int_{\mathbb{R}^3} \left( 1 - \frac{c}{p^0} \right) (|p|^2 - \beta_c) p_i^2 \sqrt{J(p)} dp \right| \lesssim \frac{1}{c^2}. \tag{5.30}$$

Thus the $b$ contribution can be bounded by

$$\frac{1}{c^2} \int_0^t \| \partial_t \nabla_x \phi_c \|_{L^2} \cdot \| b \|_{L^2} d \tau. \tag{5.31}$$

It follows from (5.27), (5.28) and (5.31) that

$$\left| \int_0^t \int_{\Omega \times \mathbb{R}^3} \partial_t \psi_c f \right| \lesssim \int_0^t \left( \| b(\tau) \|_{L^2} + \| (I - P) f(\tau) \|_{L^2} + \| S(\tau) \|_{L^2} \right) \cdot \| (I - P) f(\tau) \|_{L^2} + \frac{1}{c^2} \| b(\tau) \|_{L^2} d \tau$$

$$\leq Ce \int_0^t \left( \| (I - P) f(\tau) \|_{L^2}^2 + \| S(\tau) \|_{L^2}^2 \right) d \tau + C \| b(\tau) \|_{L^2}^2 + C \frac{1}{c^2} \int_0^t \| b(\tau) \|_{L^2}^2 d \tau, \tag{5.32}$$

which, together with (4.54), yields that

$$\int_0^t \| c(\tau) \|_{L^2}^2 d \tau \leq |G_c(t) - G_c(0)| + C \| b(\tau) \|_{L^2}^2 + C \frac{1}{c^2} \int_0^t \| b(\tau) \|_{L^2}^2 d \tau$$

$$+ C \| c(\tau) \|_{L^2}^2 d \tau + \| S(\tau) \|_{L^2}^2 + \| (I - P) f(\tau) \|_{L^2(\gamma_+)}^2 + \| r(\tau) \|_{L^2(\gamma_-)}^2 \} \right) d \tau. \tag{5.33}$$

To estimate $b$ in (5.21), we use (4.55) to get

$$\int_0^t \int_{\Omega \times \mathbb{R}^3} \partial_t f = \int_0^t \int_{\Omega \times \mathbb{R}^3} (p_i^2 - \beta_b) \sqrt{J(p)} \partial_i \partial_t \phi_i f$$

$$= \int_0^t \int_{\Omega \times \mathbb{R}^3} (p_i^2 - \beta_b) \sqrt{J(p)} \partial_i \partial_t \phi_i \left\{ a + b \cdot p + c \frac{p^0 - A_3}{\sqrt{A_2 - A_3^2}} \right\}$$

$$+ \int_0^t \int_{\Omega \times \mathbb{R}^3} (p_i^2 - \beta_b) \sqrt{J(p)} \partial_i \partial_t \phi_i \cdot (I - P) f. \tag{5.34}$$
For the first term on the RHS of (5.34), the $b$ contribution vanishes due to the oddness in $p$. Since $\beta_b = \frac{K_3(c)}{K_2(c)}$, it is direct to check that
\[
\int_{\mathbb{R}^3} (p_i^2 - \beta_b) \tilde{J}(p) dp = 0,
\]
which yields that the $a$ contribution also vanishes. For the $c$ contribution, a direct calculation shows that
\[
\int_{\mathbb{R}^3} (p_i^2 - \beta_b) \tilde{J}(p) \frac{p^0 - A_3}{\sqrt{A_2 - A_3^2}} dp = \frac{A_5 - A_1 A_3}{\sqrt{A_2 - A_3^2}} = 1 + O(c^{-2}).
\]
Thus the $c$ contribution can be bounded by
\[
\int_0^t \| \partial_t \nabla_x \phi_b \|_{L^2} \cdot \| c \|_{L^2} d\tau. 
\tag{5.35}
\]
Using (5.26), (5.34) and (5.35), we have
\[
\left| \int_0^t \int_{\Omega \times \mathbb{R}^3} (p_i^2 - \beta_b) \sqrt{\tilde{J}(p) \partial_i \partial_j \phi_b^j} f \right| 
\lesssim \int_0^t \left( \| (a, c)(t) \|_{L^2} + \| (I - P) f(t) \|_{L^2} + \| S(t) \|_{L^2} \right) \cdot \left( \| (I - P) f(\tau) \|_{L^2} + \| c(\tau) \|_{L^2} \right) d\tau 
\leq C_\varepsilon \int_0^t \left( \| (I - P) f(\tau) \|_{L^2}^2 + \| S(\tau) \|_{L^2}^2 + \| c(\tau) \|_{L^2}^2 \right) d\tau + C_\varepsilon \int_0^t \| a(\tau) \|_{L^2}^2 d\tau. 
\tag{5.36}
\]
Combining (4.60) and (5.36), for all $i, j$, we have,
\[
\left| \int_0^t \int_{\Omega} (\partial_{ij} A^{-1} b_j) b_i \right| \leq |G_0(t) - G_0(0)| + C_\varepsilon \int_0^t \| a(\tau) \|_{L^2}^2 d\tau 
+ C_\varepsilon \int_0^t \left\{ \| (I - P) f \|_{L^2}^2 + \| (I - P) f \|_{L^2}^2 + \| r \|_{L^2}^2 \right\} d\tau. 
\tag{5.37}
\]
For $i \neq j$, we use (4.61) and (5.26) to obtain
\[
\left| \int_0^t \int_{\Omega \times \mathbb{R}^3} |p|^2 p_i p_j \sqrt{\tilde{J}} \partial_i \partial_j \phi_b^j f \right| 
\lesssim \int_0^t \left( \| (a, c) \|_{L^2} + \| (I - P) f \|_{L^2} + \| S \|_{L^2} \right) \| (I - P) f \|_{L^2} 
\lesssim \int_0^t C_\varepsilon \left\{ \| (I - P) f \|_{L^2}^2 + \| S \|_{L^2}^2 \right\} + \varepsilon \| (a, c) \|_{L^2}^2, 
\tag{5.38}
\]
where the $P f$ contribution vanishes due to the oddness in $p$. Combining (4.65) and (5.38), for $i \neq j$, we have
\[
\left| \int_0^t \int_{\Omega} (\partial_{ij} A^{-1} b_j) b_i \right| \leq \int_0^t C_\varepsilon \left\{ \| (I - P) f \|_{L^2}^2 + \| (I - P) f \|_{L^2}^2 + \| r \|_{L^2}^2 \right\} 
+ |G_0(t) - G_0(0)| + C_\varepsilon \int_0^t \| (a, c) \|_{L^2}^2. 
\tag{5.39}
\]
Combining (5.37) and (5.39), we obtain the uniform estimate
\[
\int_0^t \| b(\tau) \|_{L^2}^2 d\tau \leq |G_0(t) - G_0(0)| + C_\varepsilon \int_0^t \| a(\tau) \|_{L^2}^2 d\tau 
+ C_\varepsilon \int_0^t \left\{ \| (I - P) f \|_{L^2}^2 + \| (I - P) f \|_{L^2}^2 + \| r \|_{L^2}^2 \right\} d\tau. 
\tag{5.40}
\]
To estimate $a$ in (5.21), we plug (4.67) into (5.21). It holds that
\[
\int_0^t \iint_{\Omega \times \mathbb{R}^3} \partial_t \psi_a f = \sum_{i=1}^3 \int_0^t \iint_{\Omega \times \mathbb{R}^3} (|p|^2 - \beta_a) p_i \sqrt{J(p)} \partial_t \partial_i \phi_a f = \sum_{i=1}^3 \int_0^t \iint_{\Omega \times \mathbb{R}^3} (|p|^2 - \beta_a) p_i \sqrt{J(p)} \partial_t \partial_i \phi_a \left\{ a + b \cdot p + c \frac{p^0 - A_3}{\sqrt{A_2 - A_3^2}} \right\}
+ \sum_{i=1}^3 \int_0^t \iint_{\Omega \times \mathbb{R}^3} (|p|^2 - \beta_a) p_i \sqrt{J(p)} \partial_t \partial_i \phi_a \cdot (1 - \mathbf{P}) f.
\] (5.41)
The $a$ and $c$ contribution vanish due to the oddness of $p$. For the contribution of $b$, it holds that
\[
\int_{\mathbb{R}^3} (|p|^2 - \beta_a) p_i^2 \sqrt{J(p)} dp = -5 + O(\varepsilon^{-2}).
\] (5.42)
Using (5.24), (5.41) and (5.42), one has
\[
\begin{align*}
&\left| \int_0^t \iint_{\Omega \times \mathbb{R}^3} (|p|^2 - \beta_a) p_i \sqrt{J(p)} \partial_t \partial_i \phi_a f \right|
\leq \int_0^t \left\{ \|b\|_{L^2} + |r|_{L^2(\gamma_-)} + \|(1 - \mathbf{P}) f\|_{L^2} \right\} \left\{ \|b\|_{L^2} + \|(1 - \mathbf{P}) f\|_{L^2} \right\}
\leq \int_0^t \left\{ \|(1 - \mathbf{P}) f\|_{L^2}^2 + \|b\|_{L^2}^2 + |r|_{L^2(\gamma_-)}^2 \right\},
\end{align*}
\] (5.43)
which, together with (4.73), yields that
\[
\int_0^t \|a(t)\|_{L^2}^2 d\tau \leq |G_a(t) - G_a(0)|
+ C \int_0^t \left\{ \|(1 - \mathbf{P}) f(\tau)\|_{L^2}^2 + \|\Sigma(\tau)\|_{L^2}^2 + \|(1 - P_{\gamma}) f(\tau)\|_{L^2(\gamma_+)}^2 + |r(\tau)|_{L^2(\gamma_-)}^2 + \|b(\tau)\|_{L^2}^2 \right\} d\tau.
\] (5.44)
By choosing $\varepsilon$ suitably small and $\varepsilon$ suitably large, we obtain (5.20) from (5.33), (5.40), and (5.44). Therefore the proof of Proposition 5.3 is completed. 

**Lemma 5.5.** Let $f$ be a solution of the linear problem (5.3). If
\[
\int_{\Omega \times \mathbb{R}^3} f_0(x, p) \sqrt{J(p)} dp dx = \int_{\Omega \times \mathbb{R}^3} \mathcal{S}(t, x, p) \sqrt{J(p)} dp dx = 0,
\] (5.45)
and $|T - 1|_{L^{\infty}(\partial \Omega)} = \delta$ is sufficiently small, then there exists a constant $\lambda_2 > 0$ such that for any $t \geq 0$,
\[
\|f(t)\|_{L^2}^2 \leq C e^{-\lambda_2 t} \left\{ \|f_0\|_{L^2}^2 + \int_0^t e^{\lambda_2 s} \|\mathcal{S}(s)\|_{L^2}^2 ds \right\}.
\] (5.46)

**Proof.** Multiplying (5.3) by $\sqrt{J}$ and using (5.45), we have
\[
\int_{\Omega \times \mathbb{R}^3} f(t, x, p) \sqrt{J(p)} dp dx = \int_{\Omega \times \mathbb{R}^3} f_0(x, p) \sqrt{J(p)} dp dx = 0.
\] (5.47)
Denote $y(t) := e^{\lambda t} f(t)$ with $\lambda > 0$ being a suitably small constant to be determined. We multiply (5.3) by $e^{\lambda t}$ to obtain
\[
\partial_t y + \hat{p} \cdot \nabla_x y + L_y = \lambda y + e^{\lambda t} \mathcal{S}, \quad y|_{\gamma_-} = P_{\gamma} y + e^{\lambda t} r.
\] (5.48)
Multiplying (5.48) by $y$, one gets
\[
\frac{1}{2} \|y(t)\|_{L^2}^2 + \frac{1}{2} \int_0^t |y(s)|_{L^2(\gamma_+)}^2 ds + \int_0^t (L_y(s), y(s)) ds
\]
\[ y(t) = \frac{1}{2} \| y_0 \|_{H^2}^2 + \frac{1}{2} \int_0^t |P_\gamma y(s) + e^{\lambda s} r(s)|_{L^2(\gamma)}^2 \, ds + \lambda \int_0^t \| y(s) \|_{H^2}^2 \, ds + \int_0^t \iint_{\Omega \times \mathbb{R}^3} e^{\lambda s} S(s) y(s) \, dp \, dx \, ds. \]  

(5.49)

From Lemma 3.11, it holds that

\[ \int_0^t \langle Ly(s), y(s) \rangle \, ds \geq \zeta_0 \int_0^t \| (I - P) y(s) \|_{H^2}^2 \, ds. \]  

(5.50)

Since

\[ \frac{1}{2} \int_0^t \| y(s) \|_{H^2}^2 \, ds - \frac{1}{2} \int_0^t |P_\gamma y + e^{\lambda s} r|_{L^2(\gamma)}^2 \, ds = \frac{1}{2} \int_0^t |(I - P_\gamma) y(s)|_{L^2(\gamma)}^2 \, ds - \frac{1}{2} \int_0^t e^{2\lambda s} |r(s)|_{L^2(\gamma)}^2 \, ds, \]

which, together with (5.49) and (5.50), yields that

\[ \| y(t) \|_{H^2}^2 + \int_0^t |(I - P_\gamma) y(s)|_{L^2(\gamma)}^2 \, ds + 2\zeta_0 \int_0^t \| (I - P) y(s) \|_{H^2}^2 \, ds \leq \| y_0 \|_{H^2}^2 + C\delta \int_0^t \| y(s) \|_{H^2}^2 \, ds + C\lambda \int_0^t e^{2\lambda s} \| S(s) \|_{H^2}^2 \, ds. \]  

(5.51)

Recall the definition of \( \gamma_+^+ \) in (4.38), then we have

\[ |P_\gamma y(s)|_{L^2(\gamma_+)}^2 = \int_{\gamma_+} J(p) \left( \int_{n(x),q>0} y(s,x,q) \left\{ 1_{\gamma_+^+} + 1_{\gamma_+ \setminus \gamma_+^+} \right\} \sqrt{J(q)}|n(x) \cdot \hat{q}| \, dq \right)^2 \, d\gamma \]

\[ \leq C \int_{\partial \Omega} dS_x \left( \int_{n(x),q>0} y(s,x,q) |n(x) \cdot \hat{q}| \, dq \cdot \int_{n(x),q>0} 1_{\gamma_+^+} J(q)|n(x) \cdot \hat{q}| \, dq \right) \]

\[ + C \int_{\partial \Omega} dS_x \left( \int_{n(x),q>0} y(s,x,q) 1_{\gamma_+ \setminus \gamma_+^+} |n(x) \cdot \hat{q}| \, dq \cdot \int_{n(x),q>0} J(q)|n(x) \cdot \hat{q}| \, dq \right) \]

\[ \leq C\varepsilon \| y(s) \|_{L^2(\gamma_+)}^2 + C \| y(s) \|_{L^2(\gamma_+)}^2 \]  

(5.52)

Note that

\[ \frac{1}{2} (\partial_t + \hat{p} \cdot \nabla_x) y^2 = -yL \gamma y + \lambda y^2 + e^{\lambda t} S y, \]

which implies that

\[ \int_0^t \| (\partial_t + \hat{p} \cdot \nabla_x) y^2(s) \|_{L^1} \, ds \leq C \int_0^t \| (I - P) y(s) \|_{H^2}^2 \, ds + C\lambda \int_0^t \| y(s) \|_{H^2}^2 \, ds + C \lambda \int_0^t e^{2\lambda s} \| S(s) \|_{H^2}^2 \, ds. \]  

(5.53)

It follows from (5.52), (5.53) and (4.40) that

\[ \int_0^t \| y(s) \|_{L^2(\gamma_+)}^2 \, ds \leq \int_0^t (I - P_\gamma) y(s)|_{L^2(\gamma_+)}^2 \, ds + \int_0^t |P_\gamma y(s)|_{L^2(\gamma_+)}^2 \, ds \]

\[ \leq \int_0^t (I - P_\gamma) y(s)|_{L^2(\gamma_+)}^2 + C\varepsilon \int_0^t \| y(s) \|_{L^2(\gamma_+)}^2 \, ds + C \int_0^t \| y(s) \|_{L^2(\gamma_+)}^2 \, ds + C \| y_0 \|_{L^1} \]

\[ + C \int_0^t \{ \| y^2(s) \|_{L^1} + \| (\partial_t + \hat{p} \cdot \nabla_x) y^2(s) \|_{L^1} \} \, ds \]
\[
\int_0^t |(I - P_\gamma)y(s)|_{L^2(\gamma_+)}^2 \, ds + C \varepsilon \int_0^t |y(s)|_{L^2(\gamma_+)}^2 \, ds + C \|y_0\|_{L^2}^2 + C \int_0^t \|(I - P)y(s)\|_{L^2}^2 \, ds
\]
\[
\quad + C \int_0^t \|y(s)\|_{L^2}^2 \, ds + C \lambda \int_0^t e^{2\lambda s}\|S(s)\|_{L^2}^2 \, ds
\]
\[
\leq C \int_0^t |(I - P_\gamma)y(s)|_{L^2(\gamma_+)}^2 \, ds + C \|y_0\|_{L^2}^2 + C \int_0^t \|(I - P)y(s)\|_{L^2}^2 \, ds + C \int_0^t \|y(s)\|_{L^2}^2 \, ds
\]
\[
\quad + C \lambda \int_0^t e^{2\lambda s}\|S(s)\|_{L^2}^2 \, ds,
\]
where we have used the smallness of \(\varepsilon\).

From (5.45) and (5.47) we know that
\[
\int_{\Omega \times \mathbb{R}^3} y \sqrt{J} = \int_{\Omega \times \mathbb{R}^3} (\lambda y + e^{\lambda t}S) \sqrt{J} = 0, \quad \int_{\gamma} e^{\lambda t}r \sqrt{J} d\gamma = 0.
\]

Applying Proposition 5.4 to (5.48), we deduce
\[
\int_0^t \|P_y(s)\|_{L^2}^2 \, ds \leq C|G(t) - G(0)| + C \int_0^t \|(I - P)y(s)\|_{L^2}^2 \, ds + C \int_0^t e^{2\lambda s}\|S(s)\|_{L^2}^2 \, ds
\]
\[
\quad + C \lambda \int_0^t \|y(s)\|_{L^2}^2 \, ds + C \int_0^t (1 - P_\gamma) y(s) |_{L^2(\gamma_+)}^2 \, ds + C \int_0^t e^{2\lambda s}|r(s)|_{L^2(\gamma_+)}^2 \, ds
\]
\[
\leq C|G(t) - G(0)| + C \int_0^t \|(I - P)y(s)\|_{L^2}^2 \, ds + C \int_0^t e^{2\lambda s}\|S(s)\|_{L^2}^2 \, ds
\]
\[
\quad + C \lambda^2 \int_0^t \|y(s)\|_{L^2}^2 \, ds + C \int_0^t (1 - P_\gamma) y(s) |_{L^2(\gamma_+)}^2 \, ds + C \delta^2 \int_0^t \|y(s)\|_{L^2(\gamma_+)}^2 \, ds,
\]
where \(G(t) \lesssim \|y(t)\|_{L^2}^2\).

Combining (5.51), (5.54), (5.55) and taking \(\lambda, \delta\) suitably small, we can obtain
\[
\|y(t)\|_{L^2}^2 + \int_0^t |(I - P_\gamma)y(s)|_{L^2(\gamma_+)}^2 \, ds + \int_0^t \|y(s)\|_{L^2}^2 \, ds \leq C \|y_0\|_{L^2}^2 + C \int_0^t e^{2\lambda s}\|S(s)\|_{L^2}^2 \, ds,
\]
which implies that
\[
\|f(t)\|_{L^2}^2 \leq Ce^{-2\lambda t} \left\{ \|f_0\|_{L^2}^2 + \int_0^t e^{2\lambda s}\|S(s)\|_{L^2}^2 \, ds \right\}.
\]
Take \(\lambda_2 := 2\lambda\) and we complete the proof of Lemma 5.5. \(\square\)

**Proposition 5.6.** Let \(\beta > 3\). Assume that
\[
\int_{\Omega \times \mathbb{R}^3} \sqrt{J(p)} \, dpdx = \int_{\Omega \times \mathbb{R}^3} S(t, x, p) \sqrt{J(p)} \, dpdx = 0,
\]
and
\[
\|w f_0\|_{L^\infty} + \sup_{s \geq 0} e^{\lambda_0 s} \|\nu^{-1} wS(s)\|_{L^\infty} < \infty,
\]
where \(\lambda_0 > 0\) is a small constant to be chosen in the proof. Then the linear IBVP problem (5.3) admits a unique solution \(f(t, x, p)\) satisfying
\[
\sup_{0 \leq s \leq t} e^{\lambda_0 s} \left\{ \|w f(s)\|_{L^\infty} + |w f(s)|_{L^\infty(\gamma)} \right\} \leq C \|w f_0\|_{L^\infty} + C \sup_{0 \leq s \leq t} e^{\lambda_0 s} \|\nu^{-1} wS(s)\|_{L^\infty}
\]
for any \(t \geq 0\).
Proof. The local existence and uniqueness of solutions to the linear inhomogeneous problem (5.3) can be proved by similar arguments as in [18, Proposition 6.2] and [19, Theorem 4.1]. We omit the details for brevity.

Recall the finite-time estimate (5.6). We define
\[ \lambda_0 = \min\left\{ \frac{\nu_0}{4}, \frac{\lambda_2}{4} \right\}. \]

Let \( \eta > 0 \) be suitably small to be determined later. Then we choose \( T_0 > 0 \) suitably large and \( \delta > 0 \) suitably small, and also take \( N, \epsilon \) suitably large, such that
\[ CT_0^2 \left\{ \delta + 2^{-\epsilon_0} + \frac{1}{N} \right\} \leq \frac{\eta}{2}, \quad C \max\left\{ \frac{1}{N}, \frac{1}{\epsilon} \right\} \leq \frac{\eta}{2}, \quad CT_0^2 e^{-\frac{\eta}{4} T_0} \leq 1. \quad (5.56) \]

For any \( s \geq 0 \) and \( t \in [s, s + T_0] \), it follows from (5.6) with \( \lambda_1 = \lambda_0 \) that
\[ \|h(t)\|_{L^\infty} + \|h(t)\|_{L^\infty(\gamma)} \leq \begin{cases} e^{-\frac{\eta}{4}(t-s)}\|h(s)\|_{L^\infty} + e^{-\lambda_0 t}D(t, s), & t - s = T_0, \\ e^{-\nu_0 T_0} e^{-\nu_0 (t-s)}\|h(s)\|_{L^\infty} + e^{-\lambda_0 t}D(t, s), & s \leq t < s + T_0, \end{cases} \quad (5.57) \]
where we have defined
\[ D(t, s) = \eta \sup_{s \leq \tau \leq t} e^{\lambda_0 \tau} \left\{ \|h(\tau)\|_{L^\infty} + \|h(\tau)\|_{L^\infty(\gamma)} \right\} + C_{T_0} \sup_{s \leq \tau \leq t} \|e^{\lambda_0 \tau} f(\tau)\|_{L^2} + C \sup_{\epsilon \leq \tau \leq t} \|e^{\lambda_0 \tau} \nu^{-1} wS(\tau)\|_{L^\infty}. \quad (5.58) \]

It follows from (5.56) that
\[ \|f(t)\|_{L^2}^2 \leq C e^{-4\lambda_0 t} \|w_{f_0}\|_{L^\infty}^2 + C \int_0^t e^{-4\lambda_0 (t-s)} \|\nu^{-1} wS(s)\|_{L^\infty}^2 ds \\ \leq C e^{-4\lambda_0 t} \|w_{f_0}\|_{L^\infty}^2 + C e^{-2\lambda_0 t} \sup_{0 \leq s \leq t} \|e^{\lambda_0 s} \nu^{-1} wS(s)\|_{L^\infty}^2, \]
which implies that
\[ e^{\lambda_0 t} \|f(t)\|_{L^2} \leq C \|w_{f_0}\|_{L^\infty} + C \sup_{0 \leq s \leq t} \|e^{\lambda_0 s} \nu^{-1} wS(s)\|_{L^\infty}. \quad (5.59) \]

Combining (5.58) and (5.59), we have
\[ D(t, 0) \leq \eta \sup_{0 \leq \tau \leq t} e^{\lambda_0 \tau} \left\{ \|h(\tau)\|_{L^\infty} + \|h(\tau)\|_{L^\infty(\gamma)} \right\} + C_{T_0} \|h_0\|_{L^\infty} + C_{T_0} \sup_{0 \leq \tau \leq t} \|e^{\lambda_0 \tau} \nu^{-1} wS(\tau)\|_{L^\infty}. \quad (5.60) \]

For any \( t > 0 \), there exists an integer \( n \geq 0 \) such that \( nT_0 < t < (n + 1)T_0 \). Then applying (5.57) to \([0, T_0], [T_0, 2T_0], \ldots, [(n-1)T_0, nT_0] \) inductively, we have
\[ \|h(T_0)\|_{L^\infty} + \|h(T_0)\|_{L^\infty(\gamma)} \leq e^{-\frac{\eta}{4} T_0} \|h_0\|_{L^\infty} + e^{-\lambda_0 T_0} D(T_0, 0), \]
\[ \|h(2T_0)\|_{L^\infty} + \|h(2T_0)\|_{L^\infty(\gamma)} \leq e^{-\frac{\eta}{4} T_0} \|h(T_0)\|_{L^\infty} + e^{-2\lambda_0 T_0} D(2T_0, 0) \]
\[ \leq e^{-\nu_0 T_0} \|h_0\|_{L^\infty} + e^{-2\lambda_0 T_0} \left( 1 + e^{-\lambda_0 T_0} \right) D(2T_0, 0), \]
\[ \ldots \]
\[ \|h(nT_0)\|_{L^\infty} + \|h(nT_0)\|_{L^\infty(\gamma)} \leq e^{-\frac{\eta}{4} nT_0} \|h((n-1)T_0)\|_{L^\infty} + e^{-n\lambda_0 T_0} D(nT_0, (n-1)T_0) \]
\[ \leq e^{-\frac{\eta}{4} nT_0} \|h_0\|_{L^\infty} + e^{-n\lambda_0 T_0} \sum_{k=0}^{n-1} e^{-k\lambda_0 T_0} D(nT_0, 0) \]
\[ \leq e^{-\frac{\eta}{4} nT_0} \|h_0\|_{L^\infty} + C e^{-n\lambda_0 T_0} D(nT_0, 0). \quad (5.61) \]

Finally applying (5.57) in \([nT_0, (n+1)T_0]\), then using (5.60) and (5.61), we have
\[ \|h(t)\|_{L^\infty} + \|h(t)\|_{L^\infty(\gamma)} \]
Applying Proposition 5.6, we have (1.8) and (1.26) has a unique non-negative solution (1.28). Therefore the proof of Theorem 1.2 is completed. The global existence follows from the standard continuity argument and the uniform estimate a priori, where we have used Lemma 3.13 and Theorem 1.1. Now we make the assumption that problem (5.1) is provided in section 6. In the following, we aim to show (1.28). It is clear that

Proof of Theorem 1.2. Since the obtained time-dependent solution $w_f(x,t)$ is non-negative for all $t \geq 0$ and converges to the stationary solution $F(x,p)$ in the large time, then one can obtain the non-negativity of $F(x,p)$.

6. Local-in-time existence

Proposition 6.1. Recall $w(p)$ in 1.11. Assume $|T - 1|_{L^\infty(\partial D)} = \delta \ll 1$, $F_0(x,p) = F_s(x,p) + \sqrt{J(p)} f_0(x,p) \geq 0$, and $\|w_0\|_{L^\infty} := M_0 < \infty$. Then there exists a positive time $t > 0$, such that the IBVP (1.3), (1.8) and (1.26) has a unique non-negative solution

$$F(t,x,p) = F_s(x,p) + \sqrt{J(p)} f(t,x,p) \geq 0,$$
in $[0, t]$ satisfying
\[
\sup_{0 \leq t \leq \hat{t}} \{ \|wf(t)\|_{L^\infty} + |wf(t)|_{L^\infty(\gamma)} \} \leq 2\hat{C} (M_0 + 1).
\]
Here $\hat{C} > 1$ is a generic constant independent of $M_0$. We point out that $\hat{t}$ may depend on the light speed $c$.

**Proof.** We consider the following iteration scheme:
\[
\begin{cases}
\partial_t F^{n+1} + \hat{p} \cdot \nabla_x F^{n+1} + R(F^n) F^{n+1} = Q^+ (F^n, F^n), \\
F^{n+1}(t, x, p)|_{t=0} = F_0(x, p) \geq 0, \\
F^{n+1}(t, x, p) |_{\gamma^-} = J_T(x, p) \int_{n(x) > 0} F^{n+1}(t, x, q) \{ n(x) \cdot \hat{q} \} dq,
\end{cases}
\]
where
\[
F^0(t, x, p) = J(p),
\]
and
\[
R(F^n)(t, x, p) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu_{\phi} F^n(t, x, q) d\omega dq.
\]
Let
\[
f^n(t, x, p) = \frac{F^n(t, x, p) - F_\phi(t, x, p)}{\sqrt{J(p)}}, \quad h^n(t, x, p) = w(p) f^n(t, x, p), \quad n = 0, 1, 2, \ldots
\]
Then the equation of $h^{n+1}$ reads as
\[
\begin{cases}
\partial_t h^{n+1} + \hat{p} \cdot \nabla_x h^{n+1} + R(F^n) h^{n+1} = w(p) K_* f^n + w(p) \Gamma^+ (f^n, f^n), \\
h^{n+1}(t, x, p)|_{t=0} = h_0(x, p) = w(p) J_0(x, p), \\
h^{n+1} |_{\gamma^-} = \frac{1}{w(p)} \int_{n(x) > 0} h^{n+1}(q) d\sigma + w(p) \frac{J_T - J}{\sqrt{J}} \int_{n(x) > 0} h^{n+1}(q) d\sigma, \\
h^0(t, x, p) = -w(p) J_\phi(t, x, p),
\end{cases}
\]
where we have denoted
\[
K_* f^n := -J^{-\frac{1}{2}} R \left( \sqrt{J} f^n \right) F_* + \Gamma^+ \left( \frac{F_*}{\sqrt{J}}, f^n \right) + \Gamma^+ \left( f^n, \frac{F_*}{\sqrt{J}} \right).
\]
By similar arguments as in [18, Proposition 6.2] and [19, Theorem 4.1], we can use the induction on $n = 0, 1, \ldots$ to obtain that there exists a positive time $\hat{t}_1 > 0$ which is independent of $n$, such that [6.1] or equivalently [6.2] admits a unique mild solution on the time interval $[0, \hat{t}_1]$, and the following uniform bound and positivity hold true:
\[
\|h^n(t)\|_{L^\infty} + |h^n(t)|_{L^\infty(\gamma)} \leq 2\hat{C} \{ \|h_0\|_{L^\infty} + 1 \},
\]
and
\[
F^n(t, x, p) \geq 0,
\]
for $0 \leq t \leq \hat{t}_1$ and suitably chosen constants $\hat{C} > 0$ independent of $t$.

With the uniform estimates [6.3] in hand, we can use similar arguments as one in [19, Theorem 4.1] to show that there is a positive time $\hat{t}$ with $0 < \hat{t} < \hat{t}_1$, so that $\frac{1}{\sqrt{1+|p|^2}} h^n, n = 0, 1, 2, \ldots$, is a Cauchy sequence in $L^\infty$ for $0 \leq t \leq \hat{t}$. We omit the details here for brevity. Thus the solution is obtained by taking the limit $n \to \infty$. The uniqueness is standard. Therefore the proof of Proposition 6.1 is completed. \(\square\)

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