Copositivity criteria, scalar mass eigenstates, and custodial symmetry parameter $\Delta \rho$ for the $S_3 \otimes Z_3$ model

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ABSTRACT: We present the copositivity criteria for the scalar sector of our $S_3 \otimes Z_3$ model, which ensures that it is bounded from below. We also show the scalar mass eigenstates, and identify the Standard Model Higgs boson among these by imposing that its Yukawa couplings are the same in our model and the Standard Model. The $\Delta \rho$ parameter, related to the custodial symmetry, is calculated as well.
1 Introduction

The Standard Model of elementary particles (SM), governed by the local $SU(3) \times SU(2) \times U(1)$ gauge symmetry which allows us to obtain the most general Lagrangian that describes the dynamics of the fields, has only 19 free parameters. One of those parameters was finally measured by the LHC. The ATLAS [1] and CMS [2] collaborations showed the results that show the discovery of a new scalar boson, whose properties are comparable to those of the Standard Model Higgs boson [3] with mass of 125.3 GeV. However, there are still 18 parameters, namely: 6 quark masses, 3 masses of the charged leptons, 3 mixing angles in the quark sector and one CP violating phase (that give rise to the Cabibbo-Kobayashi-Maskawa, CKM, matrix), 3 coupling constants, the vacuum expectation value and the $\theta_{\text{QCD}}$.

Now, after the discovery of the massive neutrinos, we have more free parameters and some open questions that the SM is not able to solve. Among these parameters we have: three more masses for the three neutrinos, three new mixing angles for the leptonic sector, and possibly a CP violating phase in the lepton mixing matrix. Other open question are:
1. The neutrino oscillation data provides the knowledge of a squared mass difference $\Delta m^2_{ij}$, but the absolute values of $m_{\nu_1}$, $m_{\nu_2}$, $m_{\nu_3}$ are still unknown. The main constraint on the absolute value of these masses comes from cosmology data and closing of the universe: $m_{\nu_i} < 0.136$ eV.

2. What is the hierarchy of masses in the neutrino sector? From the neutrino data, it is not possible to determine the sign in $\Delta m^2_{ij}$, that is, it is possible that $m_{\nu_1}$ is larger or smaller than $m_{\nu_2}$.

3. Is there CP violation in the neutrino sector, as we have in quark sector?

4. The quark sector mixing matrix hierarchy is completely different from the hierarchy of the leptonic sector mixing matrix, does this have an explanation?

5. Is the neutrino a Dirac or a Majorana particle?

Another important evidence that the SM needs an extension is the observed dark matter (DM) from galaxy rotation curves, once there is no viable candidate in the SM as it is today. Within many possibilities, maybe the simplest DM models corresponds to coupling the DM sector to SM sector, with the SM Higgs scalar as the interaction mediator, the Higgs-portal.

In the context of dark matter, the simplest models that provide good scalar DM candidate is the model (I(1+1)HDM), one Higgs SM-like and one inert. This doublet, is called inert because it does not couple to fermions, which provides a stable DM candidate, so this is an example of the Higgs-portal [6] type of DM model, where the DM sector communicates with the SM sector through the Higgs boson exchange. As a result, the DM-Higgs coupling, $g_{DMh}$, governs the DM annihilation rate $\langle \sigma v \rangle$, the DM-nucleon scattering cross-section $\sigma_{DM-N}$ and the Higgs invisible decays.

Although the experimental constraints for these three types of processes when simultaneously considered be is a difficult task to solve, one possible solution to this problem is to introduce coannihilation processes between DM and other inert particles which are close in mass. Coannihilation processes lead to an increase or decrease of the effective annihilation cross-section, which in turn gives, respectively, smaller or larger DM relic density values. In the I(1+1)HDM, for example, the DM candidate could coannihilate with neutral and/or charged $Z_2$-odd particles. In models with a richer particle spectrum, more coannihilation processes could come to play [7].

Therefore, despite the fact that the SM is an experimentally proven theory with a high predictive power, it is still incomplete. In this context, we have proposed a model with a $S_3 \otimes Z_3$ [4, 5] discrete symmetries added to the SM symmetries. The model consists of two scalar doublets and two scalar singlets plus the SM particles, with their potential having the most general lagrangian allowed by the chosen symmetry.

The two new scalar doublets, that are inert due to the $S_3$ symmetry, act as dark matter candidates, and the $Z_3$ symmetry ensures that the right-handed neutrinos generate active neutrino masses at one-loop via the scotogenic mechanism. All the mass spectra in the scalar and lepton sectors are accommodated, and the leptonic mixing matrix as well.
Constraints on the parameters of the model coming from the lepton flavor violating decay $\mu \to e\gamma$ were obtained in previous works, using the current and the upcoming experimental limits, and those constraints were also used to predict the induced $\mu \to ee\bar{e}$ channel at 1-loop [5].

In this work we will continue the study of this model starting with the analysis of the copositivity criteria for the potential, which ensures that it is bounded from below. Also, we will present the mass eigenstates for the scalar sector and, by imposing that its Yukawa couplings are the same as the SM, identify the SM Higgs boson among the available CP-even scalars of the model. Another study presented here is the $\Delta \rho$ parameter, related to the custodial symmetry. This symmetry relates the masses of the $W$ and $Z$ bosons and is violated in multi-Higgs models.

The outline of the paper is as follows: in Secs. 2 and 3 we present the model and its particle content, in Sec. 4 we present the copositivity criteria of the scalar potential and in Secs. 5 and 6 we present the scalar mass eigenstates and identify the SM Higgs boson among those. Afterwards, we calculate $\Delta \rho$ in Sec. 7. Finally, in Sec. 8, we present our conclusions.

2 The $S_3 \otimes Z_3$ model

Here we will use the $S_3$ discrete symmetry in order to obtain a model with 3 Higgs doublets, being two of them inert. The $S_3$ symmetry consists of all permutations among three objects. However, the representation of order 3 is reducible and is decomposed in two irreducible representations: $3 = 1 \oplus 2$. Here we will write only the multiplications involving two doublets and two singlets (which will be used here for obtaining the Yukawa interactions) and the scalar potential that is invariant under the full symmetry, $SU(2)_L \otimes U(1)_Y \otimes S_3 \otimes Z_3$.

Let $[x_1, x_2]$ and $[y_1, y_2]$ be two doublets of $S_3$, the multiplication $2 \otimes 2$ is given by

$$
\begin{pmatrix}
  x_1 \\
  x_2 
\end{pmatrix}
\otimes
\begin{pmatrix}
  y_1 \\
  y_2 
\end{pmatrix}_2 = [x_1 y_1 + x_2 y_2]_1 + [x_1 y_2 - x_2 y_1]_1' + [x_1 y_1 + x_2 y_2]_2' = 1 \oplus 1' \oplus 2', \tag{2.1}
$$

being $1$ and $1'$ singlets and $2'$ a doublet. Besides we have that $1 \otimes 1 = 1$ and $1' \otimes 1' = 1$. For more details about this and other discrete symmetries see Ref. [8]. The $Z_N$ group, that is Abelian, can be represented as discrete rotations, whose generators corresponds to a $2\pi/N$ rotation.

| Symmetry | $L$ | $l_{jR}$ | $N_a$ | $N_d$ | $S$ | $D$ | $\zeta_d$ |
|----------|-----|----------|-------|-------|-----|-----|----------|
| $S_3$    | 1   | 1        | $1'$  | 2     | 1   | 2   | $\pm 1$ |
| $Z_3$    | $\omega$ | $\omega$ | 1     | 1     | 1   | $\omega^2$ | 1       |

Table 1. Transformation properties of the fermion and scalar fields under $S_3$ and $Z_3$ symmetries. Quarks and charged leptons are singlets of $S_3$ and even under $Z_3$. 

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3 Scalar sector

The scalar sector of the model is presented as follows:

\[
S = \left( \frac{1}{\sqrt{2}}(v_{SM} + \text{Re}S^0 + i\text{Im}S^0) \right), \quad D_{1,2} = \left( \frac{1}{\sqrt{2}}(\eta_{1,2} + i\chi_{1,2}) \right),
\]

plus two real the singlets \( \zeta_i = \frac{v_i + \xi_i}{\sqrt{2}}, \ i = 1, 2 \).

The scalar potential invariant under the gauge and \( S_3 \otimes \mathbb{Z}_3 \) symmetries is

\[
V_{S_3} = \mu_3^2 S^0 S + \mu_3^2[D^\dagger \otimes D]_1 + \mu_3^2[\zeta_3 \otimes \zeta_3]_1 + \mu_2^2[\zeta_1 \otimes \zeta_1]_2 + a_1([D^\dagger \otimes D]_1)^2 + a_2([D^\dagger \otimes D]_1[D^\dagger \otimes D]_1)
\]

\[
+ a_3([D^\dagger \otimes D]_2[D^\dagger \otimes D]_2) + a_4(S^0 S)^2 + a_5([D^\dagger \otimes D]_1[S^0 S] + a_6([S^0 S]_1[D^\dagger \otimes D]_1)
\]

\[
+ H.c. + a_7 S^0[D^\dagger \otimes D]_1) + b_1[S^0 S(\zeta_3 \otimes \zeta_3)]_1 + b_2[D^\dagger \otimes D]_1(\zeta_3 \otimes \zeta_3)_1 + b_3([D^\dagger \otimes D]_2[S^0 S]_1)
\]

\[+ b_4([D^\dagger \otimes D]_1[S^0 S]_1) + c_1((\zeta_3 \otimes \zeta_3)]_1^2 + c_2(\zeta_3 \otimes \zeta_3)_2]_1, \quad (3.2)
\]

with \( \mu_3^2 > 0 \) since \( \langle D^\dagger_1 \zeta_3 \rangle = 0 \) is guaranteed by the \( S_3 \) symmetry. The parameter \( a_6 \) has been chosen real without loss of generality.

We can write Eq. (3.2) explicitly as

\[
V(S, D, \zeta_3) = V(2) + V^{(4a)} + V^{(4b)} + V^{(4c)},
\]

where

\[
V^{(2)} = \mu_3^2 S^0 S + \mu_3^2(D^\dagger_1 D_1 + D^\dagger_2 D_2) + \mu_2^2(\zeta_3^2 + \bar{\zeta}_3^2) + \mu_2^2\zeta_1 \zeta_2,
\]

\[
V^{(4a)} = a_1(D^\dagger_1 D_1 + D^\dagger_2 D_2)^2 + a_2(D^\dagger_1 D_2 - D^\dagger_2 D_1)^2
\]

\[
+ a_3([D^\dagger_1 D_1 + D^\dagger_2 D_2])^2 + a_4(S^0 S)^2 + a_5(D^\dagger_1 D_1 + D^\dagger_2 D_2)S^0 S
\]

\[
+ a_6([D^\dagger_1 D_1 + S^0 S]^2 + H.c.) + a_7([S^0 S]_1[D^\dagger_1 S^0 S] + [S^0 S]_1[D^\dagger_2 S^0 S]),
\]

\[
V^{(4b)} = b_1 S^0 S(\zeta_3^2 + \bar{\zeta}_3^2) + b_2(D^\dagger_1 D_1 + D^\dagger_2 D_2)(\zeta_3^2 + \bar{\zeta}_3^2) + b_3([D^\dagger_1 D_1 + D^\dagger_2 D_1](\zeta_3 \zeta_2 + \zeta_1 \zeta_2)
\]

\[
+ [D^\dagger_1 D_1 - D^\dagger_2 D_2](\zeta_3^2 - \bar{\zeta}_3^2) + H.c.] + b_4([D^\dagger_1 D_2 - D^\dagger_2 D_1](\zeta_3 \zeta_2 - \zeta_1 \zeta_2)
\]

\[
V^{(4c)} = c_1(\zeta_3^2 + \bar{\zeta}_3^2)^2 + c_2((\zeta_3 \zeta_2 + \zeta_2 \zeta_1) + (\zeta_3^2 - \bar{\zeta}_3^2)^2), \quad (3.4)
\]

where we have used \( |\zeta_3 \zeta_2|_2^2 = (\zeta_1 \zeta_2 + \zeta_2 \zeta_1, \zeta_1 \zeta_2 - \zeta_2 \zeta_1) \). Here, we will consider all the couplings to be real parameters i.e., there is no \( CP \) violation in the scalar sector. The \( S_3 \) symmetry forbids linear terms with the doublets \( D_1, D_2 \) in the scalar potential and also some of the Yukawa interactions with charged leptons. This ensures the inert character of the these doublets after the \( S_3 \) symmetry is introduced. Notice that, although the term \( \mu_3^2 \) breaks softly the \( S_3 \) symmetry, it happens in the sector of the singlets \( \zeta_1, \zeta_2 \) and does not spoil the inert character of the doublets.

From Eq. (3.4), we obtain the following stability conditions for the potential (i.e., setting its derivatives to zero):

\[
\frac{1}{2}v_{SM} (2a_4 v_{SM}^2 + b_1 (v_1^2 + v_2^2) + 2\mu_3^2) = 0,
\]

\[
\frac{1}{2} (b_1 v_1 v_{SM}^2 + 2v_1(c_1 + c_2) (v_1^2 + v_2^2) + \mu_2^2 v_2) + \mu_2^2 v_1 = 0,
\]

\[
\frac{1}{2} b_1 v_2 v_{SM}^2 + v_2(c_1 + c_2) (v_1^2 + v_2^2) + \frac{\mu_2^2 v_1}{2} + \mu_2^2 v_2 = 0, \quad (3.5)
\]
From Eq. 3.5 we find three sets of solutions
\[ v_1 = v_2 = 0, \quad \mu_{SM} = -a_4 v_{SM}^2 \] (3.6)
\[ v_2 = -v_1, \quad \mu_2^2 = \frac{1}{2} (-b_1 v_{SM}^2 - 4v_1^2(c_1 + c_2) + \mu_{12}^2), \quad \mu_{SM} = -a_4 v_{SM}^2 - b_1 v_1^2 \] (3.7)
\[ v_2 = v_1, \quad \mu_2^2 = \frac{1}{2} (-b_1 v_{SM}^2 - 4v_1^2(c_1 + c_2) - \mu_{12}^2), \quad \mu_{SM} = -a_4 v_{SM}^2 - b_1 v_1^2 \] (3.8)
These solutions will be used in the analyses presented throughout this work, except in the the next section, where it is not necessary.

4 Copositivity criteria

From Eq. 3.3 all we found so far are the conditions for the potential stability (Eq. 3.5), which guarantees that the vacua are in a stable point, but do not guarantee that it is a minimum. To do so we will follow the method presented in Ref. [9] also using Ref. [10]. A scalar potential with the form \( \lambda_{ab} \phi_a \phi_b \) is bounded from below if its matrix of quartic couplings is copositive, which means that its eigenvalues are all non-negative. Therefore, we must build our matrix using the basis \( (S^I S, D_1^I D_1, D_2^I D_2, \zeta_1^2, \zeta_2^2) \). Although we have a 5 × 5 matrix from our basis, two of its columns/rows are equal, which reduces to the following 4 × 4 matrix
\[
A = \begin{pmatrix}
a_4 & a_5 + a_6 + 2a_7 & a_5 + a_6 + 2a_7 & b_1 \\
a_5 + a_6 + 2a_7 & a_1 + a_3 & 2(a_1 + a_3) & b_2 + b_4 \\
a_5 + a_6 + 2a_7 & 2(a_1 + a_3) & a_1 + a_3 & b_2 - b_4 \\
b_1 & b_2 + b_4 & b_2 - b_4 & c_1 + c_3
\end{pmatrix}.
\] (4.1)

Below are shown the different conditions under which the potential is bounded from below. All indices \( i, j, k, l, \ldots \) are fixed and different from each other. Also, we want to remind the reader that all diagonal elements from matrix \( A \) given in Eq. (4.1) should be positive for \( A \) to be copositive.

**Case 1.** All \( A_{ij} \) positive.
All couplings positive, and all \( A_{ii} > 0 \).

**Case 2.** \( A_{ij} \leq 0 \) and the other entries positive.

1. If \( a_5 + a_6 + 2a_7 \leq 0 \), then:
\[
a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2 > 0; \] (4.2)

2. If \( a_5 + a_6 + 2a_7 \leq 0 \), then:
\[
a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2 > 0; \] (4.3)

3. If \( b_1 \leq 0 \), then:
\[
a_4(c_1 + c_3) - b_1^2 > 0; \] (4.4)
4. If \(2(a_1 + a_3) \leq 0\), then:
\[
(a_1 + a_3)^2 - (2a_1 + 2a_3)^2 > 0; \tag{4.5}
\]

5. If \(b_2 + b_4 \leq 0\), then:
\[
(a_1 + a_3)(c_1 + c_3) - (b_2 + b_4)^2 > 0; \tag{4.6}
\]

6. If \(b_2 - b_4 \leq 0\), then:
\[
(a_1 + a_3)(c_1 + c_3) - (b_2 - b_4)^2 > 0. \tag{4.7}
\]

**Case 3.** \(A_{ij} \leq 0, A_{kl} \leq 0\) and all other entries positive.

1. If \((a_5 + a_6 + 2a_7) \leq 0\) and \((2a_1 + 2a_3) \leq 0\), then:
\[
a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2 > 0, \tag{4.8a}
\]
\[
(a_1 + a_3)^2 - (2a_1 + 2a_3)^2 > 0; \tag{4.8b}
\]

2. If \((a_5 + a_6 + 2a_7) \leq 0\) and \((b_2 + b_4) \leq 0\), then:
\[
a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2 > 0, \tag{4.9a}
\]
\[
(a_1 + a_3)(c_1 + c_3) - (b_2 + b_4)^2 > 0; \tag{4.9b}
\]

3. If \((a_5 + a_6 + 2a_7) \leq 0\) and \((b_2 - b_4) \leq 0\), then:
\[
a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2 > 0, \tag{4.10a}
\]
\[
(a_1 + a_3)(c_1 + c_3) - (b_2 - b_4)^2 > 0; \tag{4.10b}
\]

4. If \((a_5 + a_6 + 2a_7) \leq 0\) and \((b_2 + b_4) \leq 0\), then:
\[
a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2 > 0, \tag{4.11a}
\]
\[
(a_1 + a_3)(c_1 + c_3) - (b_2 + b_4)^2 > 0; \tag{4.11b}
\]

5. If \((a_5 + a_6 + 2a_7) \leq 0\) and \((b_2 - b_4) \leq 0\), then:
\[
a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2 > 0 \tag{4.12a}
\]
\[
(a_1 + a_3)(c_1 + c_3) - (b_2 - b_4)^2 > 0; \tag{4.12b}
\]

6. If \(2a_1 + 2a_3) \leq 0\) and \(b_1 \leq 0\), then:
\[
(a_1 + a_3)^2 - (2a_1 + 2a_3)^2 > 0, \tag{4.13a}
\]
\[
a_4(c_1 + c_3) - b_1^2 > 0; \tag{4.13b}
\]
7. If \((2a_1 + 2a_3) \leq 0\) and \((b_2 - b_4) \leq 0\), then:

\[
(a_1 + a_3)^2 - (2a_1 + 2a_3)^2 > 0, \quad (4.14a)
\]
\[
(a_1 + a_3)(c_1 + c_3) - (b_2 - b_4)^2 > 0. \quad (4.14b)
\]

**Case 4.** \(A_{ij} \leq 0, A_{ik} \leq 0\) and all other entries positive.

1. If \((a_5 + a_6 + 2a_7) \leq 0\) and \((a_5 + a_6 + 2a_7) \leq 0\), then:

\[
\sqrt{(a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2)^2 + a_4(2a_1 + 2a_3)} > 0; \quad (4.15)
\]

2. If \((a_5 + a_6 + 2a_7) \leq 0\) and \(b_1 \leq 0\), then:

\[
\sqrt{(a_4(c_1 + c_3) - b_1^2)(a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2) + a_4(b_2 + b_4)} > 0; \quad (4.16)
\]

3. If \((a_5 + a_6 + 2a_7) \leq 0\) and \(b_1 \leq 0\), then:

\[
\sqrt{(a_4(a_1 + a_3) - b_1^2)(a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2) + a_4(b_2 - b_4)} > 0; \quad (4.17)
\]

4. If \((2a_1 + 2a_3) \leq 0\) and \((b_2 + b_4) \leq 0\), then:

\[
\sqrt{((a_1 + a_3)^2 - (2a_1 + 2a_3)^2)((a_1 + a_3)(c_1 + c_3) - (b_2 + b_4)^2) + (a_1 + a_3)(b_2 - b_4)} > 0. \quad (4.18)
\]

**Case 5.** \(A_{ij} \leq 0, A_{jk} \leq 0, A_{ik} \leq 0\) and the other entries positive.

1. If \(a_5 + a_6 + 2a_7 \leq 0\), \(2a_1 + 2a_3 \leq 0\) and \(a_5 + a_6 + 2a_7 \leq 0\), then:

\[
\sqrt{a_4(a_1 + a_3) + a_5 + a_6 + 2a_7} > 0; \quad (4.19a)
\]
\[
\sqrt{a_4(a_1 + a_3) + a_5 + a_6 + 2a_7} > 0; \quad (4.19b)
\]
\[
\sqrt{(a_1 + a_3)^2 + 2a_1 + 2a_3} > 0; \quad (4.19c)
\]
\[
(a_1 + a_3)(3a_1(a_1 + a_3) - 2(a_5 + a_6 + 2a_7)^2) < 0; \quad (4.19d)
\]

2. If \(a_5 + a_6 + 2a_7 \leq 0\), \(b_2 + b_1 \leq 0\) and \(b_1 \leq 0\), then:

\[
\sqrt{a_4(a_1 + a_3) + a_5 + a_6 + 2a_7} > 0, \quad (4.20a)
\]
\[
\sqrt{a_4(c_1 + c_3) + b_1} > 0, \quad (4.20b)
\]
\[
\sqrt{(a_1 + a_3)(c_1 + c_3) + b_2 + b_4} > 0, \quad (4.20c)
\]
\[
(c_1 + c_3)(a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2) + b_1((b_2 + b_4)(a_5 + a_6 + 2a_7) - b_1(a_1 + a_3)) - (b_2 + b_4)(a_4b_2 + a_4b_4 - a_5b_1 - a_6b_1 - 2a_7b_1) > 0; \quad (4.20d)
\]

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3. If \(a_5 + a_6 + 2a_7 \leq 0\), \(b_2 - b_1 \leq 0\) and \(b_1 \leq 0\), then:

\[
\begin{align*}
\sqrt{a_4(c_1 + c_3)} + b_1 &> 0, \\
\sqrt{a_4(c_1 + c_3)} + b_1 &> 0, \\
\sqrt{(a_1 + a_3)(c_1 + c_3)} + b_2 - b_4 &> 0, \\
(c_1 + c_3) (a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2) &+ b_1((b_2 - b_4)(a_5 + a_6 + 2a_7) - b_1(a_1 + a_3)) \\
&+ (b_2 - b_4)(a_4(b_4 - b_2) + b_1(a_5 + a_6 + 2a_7)) > 0;
\end{align*}
\]

(4.21a, 4.21b, 4.21c, 4.21d)

4. If \(2a_1 + 2a_3 \leq 0\), \(b_2 - b_4 \leq 0\) and \(b_2 + b_4 \leq 0\), then:

\[
\begin{align*}
\sqrt{(a_1 + a_3)^2} + 2a_1 + 2a_3 &> 0, \\
\sqrt{(a_1 + a_3)(c_1 + c_3)} + b_2 + b_4 &> 0, \\
\sqrt{(a_1 + a_3)(c_1 + c_3)} + b_2 - b_4 &> 0, \\
(a_1 + a_3) (3 (a_1 + a_3)(c_1 + c_3) + 2b_4^2) - 2b_2^2 &< 0.
\end{align*}
\]

(4.22a, 4.22b, 4.22c, 4.22d)

**Case 6.** \(A_{ij} \leq 0\), \(A_{ik} \leq 0\), \(A_{il} \leq 0\), and the other entries positive.

1. If \(a_5 + a_6 + 2a_7 \leq 0\), \(a_5 + a_6 + 2a_7 \leq 0\) and \(b_1 \leq 0\), then:

\[
\begin{align*}
a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2 &> 0, \\
a_4(c_1 + c_3) - b_1^2 &> 0, \\
\sqrt{(a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2)^2 + a_4(2a_1 + 2a_3)} &- (a_5 + a_6 + 2a_7)^2 > 0, \\
\sqrt{(a_4(c_1 + c_3) - b_1^2)(a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2)} &+ a_4(b_2 + b_4) + b_1(-(a_5 + a_6 + 2a_7)) > 0, \\
a_4 \left( -(a_1 + a_3) \left( a_4(c_1 + c_3) - b_1^2 \right) (3a_4(a_1 + a_3) - 2(a_5 + a_6 + 2a_7)^2) \right. \\
&\left. + (a_4(b_4 - b_2) + b_1(a_5 + a_6 + 2a_7)) \left( (a_1 + a_3)(b_1(a_5 + a_6 + 2a_7) - a_1b_2) \right. \\
&\left. + b_4 \left( 2(a_5 + a_6 + 2a_7)^2 - 3a_4(a_1 + a_3) \right) \right) \right) \\
&\left. + (a_4(b_2 + b_4) - b_1(a_5 + a_6 + 2a_7)) \left( (a_1 + a_3)(a_4b_2 - b_1(a_5 + a_6 + 2a_7)) \right. \\
&\left. + b_4 \left( 2(a_5 + a_6 + 2a_7)^2 - 3a_4(a_1 + a_3) \right) \right) \right) > 0;
\end{align*}
\]

(4.23a, 4.23b, 4.23c, 4.23d, 4.23e)
2. If \( a_5 + a_6 + 2a_7 \leq 0, 2a_1 + 2a_3 \leq 0 \) and \( b_2 + b_4 \leq 0 \), then:

\[
a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2 > 0, \\
(a_1 + a_3)^2 - (2a_1 + 2a_3)^2 > 0, \\
(a_1 + a_3)(c_1 + c_3) - (b_2 + b_4)^2 > 0, \\
\sqrt{((a_1 + a_3)^2 - (2a_1 + 2a_3)^2)(a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2)} + (a_1 + a_3)(a_5 + a_6 + 2a_7) - (2a_1 + 2a_3)(a_5 + a_6 + 2a_7) > 0,
\]

\[
\sqrt{(a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2)((a_1 + a_3)(c_1 + c_3) - (b_2 + b_4)^2)} + b_1(a_1 + a_3) - (b_2 + b_1)(a_5 + a_6 + 2a_7) > 0,
\]

\[
(a_1 + a_3)^2 ((b_2 + 3b_4) ((a_1 + a_3)(b_1(a_5 + a_6 + 2a_7) - a_4b_2) + b_4 (2(a_5 + a_6 + 2a_7)^2 - 3a_4(a_1 + a_3))) - (3a_4(a_1 + a_3) - 2(a_5 + a_6 + 2a_7)^2)((a_1 + a_3)(c_1 + c_3) - (b_2 + b_4)^2) + (3b_1(a_1 + a_3) - 2b_2(a_5 + a_6 + 2a_7))(b_1(a_1 + a_3) - (b_2 + b_4)(a_5 + a_6 + 2a_7)) > 0;
\]

3. If \( a_5 + a_6 + 2a_7 \leq 0, 2a_1 + 2a_3 \leq 0 \) and \( b_2 - b_4 \leq 0 \), then:

\[
a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2 > 0, \\
(a_1 + a_3)^2 - (2a_1 + 2a_3)^2 > 0, \\
(a_1 + a_3)(c_1 + c_3) - (b_2 - b_4)^2 > 0, \\
\sqrt{((a_1 + a_3)^2 - (2a_1 + 2a_3)^2)(a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2)} + (a_1 + a_3)(a_5 + a_6 + 2a_7) - (2a_1 + 2a_3)(a_5 + a_6 + 2a_7) > 0,
\]

\[
\sqrt{(a_4(a_1 + a_3) - (a_5 + a_6 + 2a_7)^2)((a_1 + a_3)(c_1 + c_3) - (b_2 - b_4)^2)} + b_1(a_1 + a_3) - (b_2 - b_4)(a_5 + a_6 + 2a_7) > 0,
\]

\[
\sqrt{((a_1 + a_3)^2 - (2a_1 + 2a_3)^2)((a_1 + a_3)(c_1 + c_3) - (b_2 - b_4)^2)} - (2a_1 + 2a_3)(b_2 - b_4) + (a_1 + a_3)(b_2 + b_4) > 0,
\]

\[
(a_1 + a_3)^2 ((b_2 - 3b_4) ((a_1 + a_3)(b_1(a_5 + a_6 + 2a_7) - a_4b_2) - b_4 (2(a_5 + a_6 + 2a_7)^2 - 3a_4(a_1 + a_3))) - (3a_4(a_1 + a_3) - 2(a_5 + a_6 + 2a_7)^2)((a_1 + a_3)(c_1 + c_3) - (b_2 - b_4)^2) + (3b_1(a_1 + a_3) - 2b_2(a_5 + a_6 + 2a_7))(b_1(a_1 + a_3) - (b_2 - b_4)(a_5 + a_6 + 2a_7)) > 0;
\]
4. If $b_1 \leq 0$, $b_2 + b_4 \leq 0$ and $b_2 - b_4 \leq 0$, then:
\[
\begin{align*}
a_4(c_1 + c_3) - b_2^2 &> 0, \\
(a_1 + a_3)(c_1 + c_3) - (b_2 + b_4)^2 &> 0, \\
(a_1 + a_3)(c_1 + c_3) - (b_2 - b_4)^2 &> 0, \\
\sqrt{(a_4(c_1 + c_3) - b_2^2)((a_1 + a_3)(c_1 + c_3) - (b_2 + b_4)^2)} &+ (c_1 + c_3)(a_5 + a_6 + 2a_7) - b_1(b_2 + b_4) > 0, \\
\sqrt{(a_4(c_1 + c_3) - b_2^2)((a_1 + a_3)(c_1 + c_3) - (b_2 - b_4)^2)} &+ (c_1 + c_3)(a_5 + a_6 + 2a_7) - b_1(b_2 - b_4) > 0, \\
\sqrt{((a_1 + a_3)(c_1 + c_3) - (b_2 - b_4)^2)((a_1 + a_3)(c_1 + c_3) - (b_2 + b_4)^2)} &+ (2a_1 + 2a_3)(c_1 + c_3) - (b_2 - b_4)(b_2 + b_4) > 0, \\
(c_1 + c_3)^2(3a_1^2(b_1^2 - a_1(c_1 + c_3)) + 2a_1(3a_3(b_2^2 - a_4(c_1 + c_3)) + a_4b_2^2 - 3a_4b_2^2 + a_3^2c_1 + c_3(a_5 + a_6 + 2a_7)^2 + 2a_5a_6c_1 + 4a_5a_7c_1 - 2a_5b_1b_2 + a_5c_1 + 4a_6a_7c_1 - 2a_5b_1b_2 + 4a_6a_7c_1 - 2a_6b_1b_2 + 4a_7^2c_1 - 4a_7b_1b_2 + 3a_3^2(b_1^2 - a_4(c_1 + c_3)) + 2a_3(a_4b_2^2 - 3a_4b_1^2 + a_3^2c_1 + c_3(a_5 + a_6 + 2a_7)^2 + 2a_5a_6c_1 + 4a_5a_7c_1 - 2a_6b_1b_2 + 4a_6a_7c_1 - 2a_6b_1b_2 + 4a_7^2c_1 - 4a_7b_1b_2) + 4b_1^2(a_5 + a_6 + 2a_7)^2 > 0.
\end{align*}
\]

5 Mass matrices and eigenstates

The potential in Eq. 3.2 gives us four mass matrices: one for the charged scalars, one for the CP-odd neutral scalars and two for the CP-even neutral scalars. In the sections below we shall show these matrices and their corresponding eigenvalues and eigenvectors. Also, when we have $\pm$ or $\mp$, the upper sign corresponds to the vacuum stability criteria from Eq. 3.7 and the lower sign to Eq. 3.8.

5.1 Charged scalars

From Eq. 3.3, in the basis $(S^+, D_1^+, D_2^+) MC(S^-, D_1^-, D_2^-)^T$, we find the mass matrix $M_C$ for the charged scalars to be

\[
M_C = \begin{pmatrix}
0 & 0 & 0 \\
0 & b_2v_1^2 + \frac{\alpha v_3^2}{2} + \mu_d^2 & \mp 2b_3v_1^2 \\
0 & \mp 2b_3v_1^2 & b_2v_1^2 + \frac{\alpha v_3^2}{2} + \mu_d^2
\end{pmatrix}
\]

(5.1)

The above matrix can be diagonalized as $R_C^T MC R_C$, where $R_C$ is the orthogonal rotation matrix. For the matrix in Eq. 5.1 we find that the symmetry and mass eigenstates are related as

\[
\begin{pmatrix}
S^+ \\
D_1^+ \\
D_2^+
\end{pmatrix}
= R_C
\begin{pmatrix}
G^+ \\
H_1^+ \\
H_2^+
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
\pm \frac{1}{\sqrt{2}} & \mp \frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
G^+ \\
H_1^+ \\
H_2^+
\end{pmatrix}
\]

(5.2)
And their masses are:

\[
    m_{G^+}^2 = 0, \quad m_{H_1^+}^2 = \frac{a_5 v_{SM}^2}{2} + v_1^2 (b_2 - 2b_3) + \mu_d^2, \quad m_{H_2^+}^2 = \frac{a_5 v_{SM}^2}{2} + v_1^2 (b_2 + 2b_3) + \mu_d^2. \tag{5.3}
\]

5.2 **CP-odd scalars**

From Eq. 3.3, in the basis \((Im[S^0], \chi_1, \chi_2)M_O(Im[S^0], \chi_1, \chi_2)^T\), we find the mass matrix \(M_O\) for the CP-odd scalars to be

\[
    M_O = \begin{pmatrix}
        0 & 0 & 0 \\
        0 & \frac{1}{4} \left(2b_2 v_1^2 + (a_5 - 2a_6 + a_7)v_{SM}^2 + 2\mu_d^2\right) + 2b_3 v_1^2 & \frac{1}{4} \left(2b_2 v_1^2 + (a_5 - 2a_6 + a_7)v_{SM}^2 + 2\mu_d^2\right) \\
        0 & \frac{1}{4} \left(2b_2 v_1^2 + (a_5 - 2a_6 + a_7)v_{SM}^2 + 2\mu_d^2\right) + 2b_3 v_1^2 & 0
    \end{pmatrix}.
\]

The diagonalization for this mass matrix is given by

\[
    \begin{pmatrix}
        Im[S^0] \\
        \chi_1 \\
        \chi_2
    \end{pmatrix} = R_O \begin{pmatrix}
        G_A^0 \\
        A_1^0 \\
        A_2^0
    \end{pmatrix} = \begin{pmatrix}
        \frac{1}{\sqrt{2}} \epsilon + \frac{1}{\sqrt{2}} \\
        0 \\
        0 \frac{1}{\sqrt{2}} \epsilon - \frac{1}{\sqrt{2}}
    \end{pmatrix} \begin{pmatrix}
        G_A^0 \\
        A_1^0 \\
        A_2^0
    \end{pmatrix}. \tag{5.5}
\]

With masses:

\[
    m_{G_A^0}^2 = 0, \quad m_{A_1^0}^2 = \frac{1}{4} \left(v_{SM}^2 (a_5 - 2a_6 + a_7) + 2v_1^2 (b_2 - 4b_3) + 2\mu_d^2\right), \quad m_{A_2^0}^2 = \frac{1}{4} \left(v_{SM}^2 (a_5 - 2a_6 + a_7) + 2v_1^2 (b_2 + 4b_3) + 2\mu_d^2\right). \tag{5.6}
\]

5.3 **CP-even scalars, 2 \times 2 matrix**

Eq. 3.3 gives us for the CP-even sector two matrices, one 2 \times 2 and one 3 \times 3. The 2 \times 2 matrix, in the basis \((\eta_1, \eta_2)M_{E1}(\eta_1, \eta_2)^T\), is

\[
    M_{E1} = \begin{pmatrix}
        \frac{1}{4} \left(2b_2 v_1^2 + (a_5 + 2a_6 + a_7)v_{SM}^2 + 2\mu_d^2\right) + 2b_3 v_1^2 & \frac{1}{4} \left(2b_2 v_1^2 + (a_5 + 2a_6 + a_7)v_{SM}^2 + 2\mu_d^2\right) \\
        \frac{1}{4} \left(2b_2 v_1^2 + (a_5 + 2a_6 + a_7)v_{SM}^2 + 2\mu_d^2\right) + 2b_3 v_1^2 & 0
    \end{pmatrix}.
\]

The symmetry and mass eigenstates are related as

\[
    \begin{pmatrix}
        \eta_1 \\
        \eta_2
    \end{pmatrix} = R_{E1} \begin{pmatrix}
        h_1^0 \\
        h_2^0
    \end{pmatrix} = \begin{pmatrix}
        \frac{1}{\sqrt{2}} \epsilon + \frac{1}{\sqrt{2}} \\
        0 \\
        0 \frac{1}{\sqrt{2}} \epsilon - \frac{1}{\sqrt{2}}
    \end{pmatrix} \begin{pmatrix}
        h_1^0 \\
        h_2^0
    \end{pmatrix}. \tag{5.7}
\]

The masses are:

\[
    m_{h_1^0}^2 = \frac{1}{4} \left(v_{SM}^2 (a_5 + 2a_6 + a_7) + 2v_1^2 (b_2 - 4b_3) + 2\mu_d^2\right), \quad m_{h_2^0}^2 = \frac{1}{4} \left(v_{SM}^2 (a_5 + 2a_6 + a_7) + 2v_1^2 (b_2 + 4b_3) + 2\mu_d^2\right). \tag{5.8}
\]

6 **The CP-even 3 \times 3 matrix and the Higgs boson**

Again from Eq. 3.3, we obtain the 3 \times 3 matrix for the CP-even scalars. Considering the basis \((Re[S^0], \xi_1, \xi_2)M_{E2}(Re[S^0], \xi_1, \xi_2)^T\) we find

\[
    M_{E2} = \begin{pmatrix}
        a_4 v_{SM}^2 & b_{1v1vSM} & b_{1v1vSM} + \mu_{12} \\
        b_{1v1vSM} & (c_1 + c_2) v_1^2 \pm \frac{\mu_{12}}{2} & \frac{1}{2} \left(\mu_{12} + 4(c_1 + c_2) v_1^2\right) \\
        b_{1v1vSM} + \mu_{12} & \frac{1}{2} \left(\mu_{12} + 4(c_1 + c_2) v_1^2\right) & (c_1 + c_2) v_1^2 \pm \frac{\mu_{12}}{2}
    \end{pmatrix}, \tag{6.1}
\]
where, when we have \( \pm \) or \( \mp \), the upper sign corresponds to the vacuum stability criteria from Eq. 3.7 and the lower one to Eq. 3.8. To find the mass eigenstates of this sector, we will follow Ref. [11], where we impose that the Yukawa couplings of the Higgs boson are the same in the SM and in the \( S_3 \otimes Z_3 \) model.

In the \( S_3 \otimes Z_3 \) model, the Yukawa sector for the leptons is given by

\[
-L_{\text{Yukawa}}^{\text{leptons}} = G_{ij}^L T_i R S + G_{ij}^R \overline{\ell_i} \gamma_s \ell_{ab} [N_d D_b]_1 + \frac{1}{\Lambda} G_{ij}^c \overline{L}_i \gamma_s c_{ab} [N_{s'} D_{b'}]_1 \tag{6.2}
\]

where \( a, b \) are SU(2) indices, \( i, j = e, \mu, \tau \) (we omit summation symbols), \( L_i (R) \) denote the usual left-handed lepton doublets (right-handed charged lepton singlets), \( G^s \) are the Yukawa couplings, and \( N_{s,d} \) are the right-handed neutrinos; \([D_{s'}]_1 = D_1 \zeta_2 - D_2 \zeta_1, [N_dD]_1 = N_2 R D_1 + N_3 R D_2 \), according to the \( S_3 \) multiplication rules, and \( D_{1,2} = i \tau_2 D_{1,2} \), where \( \tau_2 \) is the second Pauli matrix.

From Eq. 6.3, \( Re[S^0] \) is the only one that gives mass to the known fermions, therefore we will identify it as the SM Higgs. The matrix in Eq. 6.1 can be diagonalized by an orthogonal \( 3 \times 3 \) matrix, such that \( R_{E2}^T M_{E2} R_{E2} = \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) \), where \( H \) is the SM Higgs boson and \( H_1^0 \) are the other CP-even scalars from the \( 3 \times 3 \) CP-even mass matrix. This implies

\[
\begin{pmatrix}
H \\
H_1^0 \\
H_2^0
\end{pmatrix} = R_{E2}^T \begin{pmatrix}
Re[S^0] \\
\xi_1 \\
\xi_2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\cos \theta_2 & -\cos \theta_3 \sin \theta_2 & \sin \theta_2 \sin \theta_3 \\
\cos \theta_1 \sin \theta_2 & \cos \theta_2 \cos \theta_3 - \sin \theta_1 \sin \theta_3 & -\cos \theta_3 \sin \theta_1 - \cos \theta_1 \cos \theta_2 \sin \theta_3 \\
\sin \theta_1 \sin \theta_2 & \cos \theta_2 \cos \theta_3 \sin \theta_1 + \cos \theta_1 \sin \theta_3 & \cos \theta_1 \cos \theta_3 - \cos \theta_2 \sin \theta_1 \sin \theta_3
\end{pmatrix}
\begin{pmatrix}
Re[S^0] \\
\xi_1 \\
\xi_2
\end{pmatrix}.
\]

Since \( Re[S^0] \) is the scalar we identify as the SM Higgs, we need that \((R_{E2})_{11} = 1\) and all the other elements from the first row to be zero. To do so, we need \( \theta_2 = 0 \), which gives us \( \cos \theta_2 = 1 \) and \( \sin \theta_2 = 0 \), leaving \( R_{E2} \) as

\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_1 \cos \theta_3 - \sin \theta_1 \sin \theta_3 & \cos \theta_3 \sin \theta_1 + \cos \theta_1 \sin \theta_3 \\
0 & -\cos \theta_1 \sin \theta_3 - \cos \theta_1 \sin \theta_3 & \cos \theta_1 \cos \theta_3 - \sin \theta_1 \sin \theta_3
\end{pmatrix}.
\]

The matrix \( R_{E2} \) from Eq. 6.4 does not automatically diagonalize \( M_{E2} \), i.e., it does not lead to \( R_{E2}^T M_{E2} R_{E2} = \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) \). To have that we need to impose conditions on the parameters that make up \( M_{E2} \) and \( R_{E2} \), so that we fulfill the equation \( R_{E2}^T M_{E2} R_{E2} = \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) \). The possible solutions depend on which solution for the vacuum stability conditions (Eqs. 3.6-3.8) we choose.

1. \( v_1 = v_2 = 0 \):

This solution trivializes our mass matrix \( M_{E2} \), making it diagonal from the very beginning, leaving one scalar with mass \( m^2_{\text{Re}[S^0]} = a_4 v_{SM}^2 \) and the other two scalars
with mass $m_{\xi_1,\xi_2}^2 = -\mu_{12}^2/4$. Therefore, there is no need for the diagonalization method presented in this section.

2. $v_2 = -v_1$ and $\mu_2^2 = \frac{1}{2}(-b_1v_{SM}^2 - 4v_1^2(c_1 + c_2) + \mu_{12}^2)$:

In this case we find 52 possible solutions for the parameters $a_4$, $b_1$, $\mu_{12}^2$ and the sines and cosines of $\theta_{1,3}$. Amongst all solutions, we either have all masses equal, $m^2 = 2v_1^2(c_1 + c_2)$; or two equal masses, $m_1^2 = \frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2)$, and a third different mass $m_2^2 = 3v_1^2(c_1 + c_2) - \frac{\mu_{12}^2}{4}$. In both cases, $b_1 = 0$. When all masses are equal, $a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2}$ and $\mu_{12}^2 = 4v_1^2(c_1 + c_2)$; when we have different masses, $a_4 = \frac{3\mu_{12}^2 - 4v_1^2(c_1 + c_2)}{4v_{SM}^2}$. As for the sines and cosines, they can either be functions of each other or have values $\pm 1/\sqrt{2}$ or $\pm 1$.

3. $v_2 = v_1$, $\mu_2^2 = \frac{1}{2}(-b_1v_{SM}^2 - 4v_1^2(c_1 + c_2) - \mu_{12}^2)$:

Once again we have 52 possible solutions for the parameters $a_4$, $b_1$, $\mu_{12}^2$ and the sines and cosines of $\theta_{1,3}$. Amongst all solutions, we either have all masses equal, $m^2 = 2v_1^2(c_1 + c_2)$; or two equal masses, $m_1^2 = v_1^2(-(c_1 + c_2)) - \frac{3\mu_{12}^2}{4}$, and a third different mass $m_2^2 = 3v_1^2(c_1 + c_2) + \frac{\mu_{12}^2}{4}$. In both cases, $b_1 = 0$. When all masses are equal, $a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2}$ and $\mu_{12}^2 = -4v_1^2(c_1 + c_2)$; when we have different masses, $a_4 = \frac{-4v_1^2(c_1 + c_2) + 3\mu_{12}^2}{4v_{SM}^2}$. As for the sines and cosines, they can either be functions of each other or have values $\pm 1/\sqrt{2}$ or $\pm 1$.

The complete set of solutions are shown in Appendix A. The mass degeneracy in this sector may look like a problem at first. However, having particles with the same mass helps to reduce the parameter $\Delta \rho$, which we address in the following section.

### 7 Custodial symmetry

An accidental $\mathbb{Z}_2$ symmetry remains after the break of all the gauge and discrete symmetries in some extensions of the SM, which implies that in all interactions an inert neutral or charged scalar is always accompanied by a sterile neutrino and one known particle. For this reason, all the phenomenological consequences of the model appears through one or more loops. Multi-Higgs models are among the models that contribute to the oblique parameters, mainly to $\Delta \rho$ (or its equivalent the $T$ parameter)

$$\Delta \rho = \frac{\overline{m_W^2}}{m_Zc_W^2(M_Z)\rho} - 1 \simeq \hat{a}(M_Z)T, \quad (7.1)$$

where $m_W$ and $m_Z$ are the masses of the SM weak vector bosons. This means that the parameter $T$ govern the size of weak isospin violating corrections to the relation between $m_W$ and $m_Z$. There are still at least two other oblique parameters $S$ and $U$ which characterizes the $q^2/M^2$ corrections, while $U$ requires both effects [23–26]. These quantum effects arise because the vacuum polarization is sensitive to any field that couples with $W^\pm$.
and/or \( Z^0 \). Notwithstanding, the \( T \) parameter is usually the dominating one when the scalar masses are much larger than \( m_Z \) [26], which we expect to be true for these exotic particles. Therefore, we will not worry about the other oblique parameters in this work and shall only calculate the lowest order corrections to \( \Delta \rho \).

Following the calculations presented in [26] for multi-Higgs-doublets models, \( \Delta \rho \) is calculated as:

\[
\Delta \rho = \frac{g^2}{64\pi^2 m_W^2} \left\{ \sum_{a=2}^{n} \sum_{b=2}^{m} \left( U^\dagger V \right)_{ab} \left[ \frac{1}{2} F \left( m_a^2, \mu_b^2 \right) - \sum_{b'=b+1}^{m-1} \left[ \Im \left( V^\dagger V \right)_{bb'} \right] \right] \right. \\
-2 \sum_{a=2}^{n} \sum_{a'=a+1}^{n} \left( U^\dagger U \right)_{aa'} \left[ F \left( m_a^2, m_a^2 \right) + 3 \sum_{b=2}^{m} \left[ \Im \left( V^\dagger V \right)_{1b} \right] \right] \left[ F \left( m_Z^2, \mu_b^2 \right) - F \left( m_W^2, \mu_b^2 \right) \right] \\
-3 \left[ F \left( m_Z^2, m_H^2 \right) - F \left( m_W^2, m_H^2 \right) \right] \left\} \right. \\
\]

\( (7.2) \)

\[
F(x, y) \equiv \begin{cases} 
\frac{x+y}{2} - \frac{x-y}{2} \ln \frac{x}{y}, & \text{if } x \neq y, \\
0, & \text{if } x = y.
\end{cases} \\
\]

\( (7.3) \)

In the above equations, \( m \) denotes the number of neutral mass eigenstates (8 in our case), \( n \) denotes the number of charged mass eigenstates (3 in our case), \( m_a \) are the masses of the charged scalars, and \( \mu_b \) the masses of the neutral scalars. Also, \( m_H, m_Z \), and \( m_W \) are the masses of the SM Higgs, \( Z \), and \( W^\pm \) bosons, respectively.

\( V \) and \( U \) are matrices that relate the symmetry and mass eigenstates, in our case they are defined as

\[
\begin{pmatrix}
Re[S^0] + iIm[S^0] \\
\xi_1 \\
\xi_2 \\
\eta_1 + i\chi_1 \\
\eta_2 + i\chi_2
\end{pmatrix} = V \\
\begin{pmatrix}
G^0_A \\
H \\
H^0_I \\
H^0_J \\
A^0_I \\
A^0_J
\end{pmatrix},\]

\( (7.4) \)

\[
V = \begin{pmatrix}
i & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cos \theta_1 \cos \theta_3 - \sin \theta_1 \sin \theta_3 & \cos \theta_3 \sin \theta_1 + \cos \theta_1 \sin \theta_3 & 0 & 0 & 0 & 0 \\
0 & 0 & -\cos \theta_3 \sin \theta_1 - \cos \theta_1 \sin \theta_3 & \cos \theta_1 \cos \theta_3 - \sin \theta_1 \sin \theta_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \pm \frac{1}{\sqrt{2}} & \pm \frac{1}{\sqrt{2}} & \pm \frac{i}{\sqrt{2}} & \pm \frac{i}{\sqrt{2}} \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}}
\end{pmatrix}.
\]

\( (7.5) \)

Here we used the rotation matrix \( R_{E2} \) from Eq. 6.4 for the mixing of the CP-even eigenstates (i.e., we are not using any of the solutions presented in Appendix A), while the upper signs of \( \pm \) and \( \mp \) corresponds to the vacuum stability criteria from Eq. 3.7 and the lower signs corresponds to the solution presented in Eq. 3.8.
As for the $U$ matrix we have:

\[
\begin{pmatrix}
S^+ \\
D_1^+ \\
D_2^+ \\
\xi_1^+ = 0 \\
\xi_2^+ = 0
\end{pmatrix}
= U
\begin{pmatrix}
G^+ \\
H_1^+ \\
H_2^+
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & \pm \frac{1}{\sqrt{2}} & \mp \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
G^+ \\
H_1^+ \\
H_2^+
\end{pmatrix}.
\] (7.6)

Despite not having the fields $\xi_{1,2}$ in our model (that is why we have indicated them as equal to zero), we have to add them in Eq. 7.6 so that the matrices $U$ and $V$ have the proper dimensions to multiply each other in Eq. 7.2. Nonetheless, the projection of these non-existent fields over the mass eigenstates is zero, not influencing the calculation of $\Delta \rho$. Also, it is worth mentioning that we set the Goldstone bosons as the first elements of our column matrices of mass eigenstates. That is because in Eq. 7.2 the sums in $m^2_{\phi}$ that are non-existent fields over the mass eigenstates is zero, not influencing the calculation of $\Delta \rho$. As for the

\[
\Delta \rho = \frac{g^2}{256 \pi^2 m_W^2} \left\{ 4m^2_{A_1} m^2_{H_1} \ln \left( \frac{m^2_{H_1}}{m^2_{A_1}} \right) + 4m^2_{A_2} m^2_{H_2} \ln \left( \frac{m^2_{H_2}}{m^2_{A_2}} \right) - 2 \left( m^2_{A_1} + m^2_{A_2} + m^2_{H_1} + m^2_{H_2} \right) \right\}
+ \frac{1}{m^2_{H_1} - m^2_{H_1}^+} \left( m^4_{H_1} + 2m^2_{H_1} m^2_{H_1} \ln \left( \frac{m^2_{H_1}}{m^2_{H_1}} \right) - m^4_{H_1} \right) (\sin \left( 2(\theta_1 + \theta_3) \right)) \pm 1
+ \frac{1}{m^2_{H_2} - m^2_{H_2}^+} \left( m^4_{H_2} + 2m^2_{H_2} m^2_{H_2} \ln \left( \frac{m^2_{H_2}}{m^2_{H_2}} \right) - m^4_{H_2} \right) (\sin \left( 2(\theta_1 + \theta_3) \right)) \pm 1
+ \frac{1}{m^2_{H_1} - m^2_{H_2}^+} \left( m^4_{H_1} - 2m^2_{H_1} m^2_{H_2} \ln \left( \frac{m^2_{H_1}}{m^2_{H_2}} \right) - m^4_{H_2} \right) (\sin \left( 2(\theta_1 + \theta_3) \right)) \pm 1
+ \frac{1}{m^2_{H_2} - m^2_{H_1}} \left( m^4_{H_2} + 2m^2_{H_2} m^2_{H_1} \ln \left( \frac{m^2_{H_2}}{m^2_{H_1}} \right) - m^4_{H_1} \right) (\sin \left( 2(\theta_1 + \theta_3) \right)) \pm 1 \right\}.
\] (7.7)

Now, if we assume that the masses of $H, H_1^0$ and $H_2^0$ are the same (as it happens in some
of the solutions from Appendix A), we find

$$\Delta \rho = \frac{g^2}{128\pi^2 m_W^2} \left\{ \frac{2m^2_{A_1^+}m^2_{h_0^+}}{m^2_{h_0^+} - m^2_{A_1^0}} \ln \left( \frac{m^2_{h_0^+}}{m^2_{A_1^0}} \right) - m^2_{A_1^0} + \frac{2m^2_{A_2^0}m^2_{h_2^0}}{m^2_{h_2^0} - m^2_{A_2^0}} - m^2_{A_2^0} \right\} - \frac{2m^2_{m_0^+}m^2_{m_2^+}}{m^2_{m_2^+} - m^2_{m_0^+}} \ln \left( \frac{m^2_{m_2^+}}{m^2_{m_0^+}} \right) + \frac{2m^2_{m_1^+}m^2_{m_3^+}}{m^2_{m_3^+} - m^2_{m_1^+}} \ln \left( \frac{m^2_{m_3^+}}{m^2_{m_1^+}} \right) + 2m^2_{m_1^+} - m^2_{h_0^+} + m^2_{h_2^+} - m^2_{m_0^+} + m^2_{m_2^+} \right\},$$

with the same result considering either Eq. 3.7 or Eq. 3.8.

8 Conclusions

We have presented the copositivity criteria for the scalar sector of our $S_3 \otimes Z_3$ model, which ensures that it is bounded from below. We have also shown the scalar mass eigenstates, and identified the SM Higgs boson among these. With the results we found one can further explore the model, where the mass eigenstates allows the calculation of the interaction terms necessary for phenomenological studies, while the copositivity criteria can have their consistency verified once phenomenological results impose future limits on the model parameters. The $\Delta \rho$ parameter adds another constraint to the model, imposing limits on the scalar masses. However, the complicated result (even when considering $m_H = m_{h_0} = m_{h_2}$), and the several possibilities to be considered from the copositivity criteria and the mass eigenstates solutions that we presented here, leaves us with too many possible routes to pursue in imposing limits over the model parameters at the moment. Therefore it seems wise to leave these as they are until other phenomenological studies bring different constraints on the model parameters, which can give us a direction to follow among the many possibilities here presented.

Some possible phenomenological studies are: the electron and muon anomalous magnetic dipole moments [14], the neutron electric dipole moment [15, 16], and Flavor-Changing Neutral Currents [17]. All these observables are well suited for studies in multi-Higgs models. Also, the works presented in [4] and [5] can be revisited, to check whether the values for the model parameters agree with the copositivity criteria here presented.

Models with a $S_3$ symmetry have been studied in a variety of previous works. Among these, some are: the model’s scalar potential, including its mass eigenstates and self-couplings [18–20], the quark sector of these models [21], and the neutrino masses and their mixing [22]. Despite some of these topics overlapping our work, we feel that what we present here is relevant. In our model two scalar singlets are added, which further increases the scalar sector and its complexity, leading to different copositivity conditions, mass eigenstates and Yukawa sectors. Therefore, our results are not the same as the ones shown in the works just cited.
A Diagonalization solutions for the $3 \times 3$ CP-even mass matrix

In this appendix we show all the solutions for the $3 \times 3$ CP-even mass matrix discussed in Sec. 6, except the ones where $v_1 = v_2 = 0$. In the first subsection we consider the vacuum stability condition from Eq. 3.7, and in the second the condition from Eq. 3.8. All solutions are presented in the same format: first we show the relations that the parameters must obey, then the diagonalized matrix with the masses squared, and finally the orthogonal diagonalization matrix.

A.1 Using $v_2 = -v_1$, $\mu^2_\chi = \frac{1}{2}(-b_1 v^2_{SM} - 4v_1^2(c_1 + c_2) + \mu^2_{12})$, and $\mu_{SM} = -a_4 v^2_{SM} - b_1 v_1^2$.

- Solution 1: $a_4 = \frac{3\mu_{12}^2 - 4v_1^2(c_1 + c_2)}{4v^2_{SM}}$, $b_1 = 0$, $\cos \theta_3 = -\sqrt{\frac{1}{2} - \cos \theta_1 \sqrt{1 - \cos^2 \theta_1}}$, $\sin \theta_1 = \sqrt{1 - \cos^2 \theta_1}$, $\sin \theta_3 = \frac{2 \cos \theta_1^2 - 1}{\sqrt{2 - 4 \cos \theta_1 \sqrt{1 - \cos^2 \theta_1}}}$;

$$\text{diag}(m_H^2, m_{H^0}^2, m_{H^0}^2) = \begin{pmatrix} 3\mu_{12}^2 - v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 3\mu_{12}^2 - v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 3v_1^2(c_1 + c_2) - \frac{\mu_{12}^2}{4} \end{pmatrix}$$

$$R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\sqrt{1 - \cos^2 \theta_1 - \cos \theta_3}}{\sqrt{2 - 4 \cos \theta_1 \sqrt{1 - \cos^2 \theta_1}}} & \frac{\cos \theta_1 - \sqrt{1 - \cos^2 \theta_3}}{\sqrt{2 - 4 \cos \theta_1 \sqrt{1 - \cos^2 \theta_1}}} & 0 \\ 0 & \frac{\sqrt{1 - \cos^2 \theta_1 - \cos \theta_3}}{\sqrt{2 - 4 \cos \theta_1 \sqrt{1 - \cos^2 \theta_1}}} & \frac{\cos \theta_1 - \sqrt{1 - \cos^2 \theta_3}}{\sqrt{2 - 4 \cos \theta_1 \sqrt{1 - \cos^2 \theta_1}}} \end{pmatrix}.$$
• Solution 3: \[ a_4 = \frac{3\nu_1^2 - 4\nu_2^2 (c_1 + c_2)}{4v_{SM}} \], \( b_1 = 0 \), \( \cos\theta_3 = -\sqrt{\frac{1}{2} - \cos\theta_1 \sqrt{1 - \cos^2 \theta_1}} \), \( \sin\theta_1 = -\sqrt{1 - \cos^2 \theta_1} \), \( \sin\theta_3 = \frac{1 - 2\cos^2 \theta_1}{\sqrt{2 - 4\cos\theta_1 \sqrt{1 - \cos^2 \theta_1}}} \);

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} \frac{3\nu_1^2}{4} - v_1^2 (c_1 + c_2) & 0 & 0 \\ 0 & 3v_1^2 (c_1 + c_2) - \frac{\nu_2^2}{4} & 0 \\ 0 & 0 & \frac{3\nu_2^2}{4} - v_1^2 (c_1 + c_2) \end{pmatrix} \];

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1 - \cos^2 \theta_1 - \cos \theta_1} & \sqrt{1 - \cos^2 \theta_1 - \cos \theta_1} \\ 0 & \sqrt{2 - 4\cos\theta_1 \sqrt{1 - \cos^2 \theta_1}} & \sqrt{2 - 4\cos\theta_1 \sqrt{1 - \cos^2 \theta_1}} \end{pmatrix} \].

• Solution 4: \[ a_4 = \frac{3\nu_1^2 - 4\nu_2^2 (c_1 + c_2)}{4v_{SM}} \], \( b_1 = 0 \), \( \cos\theta_3 = \sqrt{\frac{1}{2} - \cos\theta_1 \sqrt{1 - \cos^2 \theta_1}} \), \( \sin\theta_1 = \sqrt{1 - \cos^2 \theta_1} \), \( \sin\theta_3 = \frac{1 - 2\cos^2 \theta_1}{\sqrt{2 - 4\cos\theta_1 \sqrt{1 - \cos^2 \theta_1}}} \);

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} \frac{3\nu_1^2}{4} - v_1^2 (c_1 + c_2) & 0 & 0 \\ 0 & \frac{3\nu_2^2}{4} - v_1^2 (c_1 + c_2) & 0 \\ 0 & 0 & 3v_1^2 (c_1 + c_2) - \frac{\nu_2^2}{4} \end{pmatrix} \];

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 - \sqrt{1 - \cos^2 \theta_1} & \sqrt{1 - \cos^2 \theta_1 - \cos \theta_1} \\ 0 & \sqrt{2 - 4\cos\theta_1 \sqrt{1 - \cos^2 \theta_1}} & \sqrt{2 - 4\cos\theta_1 \sqrt{1 - \cos^2 \theta_1}} \end{pmatrix} \].

• Solution 5: \[ a_4 = \frac{3\nu_1^2 - 4\nu_2^2 (c_1 + c_2)}{4v_{SM}} \], \( b_1 = 0 \), \( \cos\theta_3 = -\sqrt{1 - \cos^2 \theta_1 \cos \theta_1 + \frac{1}{2}} \), \( \sin\theta_1 = \sqrt{1 - \cos^2 \theta_1} \), \( \sin\theta_3 = \frac{1 - 2\cos^2 \theta_1}{\sqrt{4 \sqrt{1 - \cos^2 \theta_1} \cos \theta_1 + 2}} \);

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} \frac{3\nu_1^2}{4} - v_1^2 (c_1 + c_2) & 0 & 0 \\ 0 & 3v_1^2 (c_1 + c_2) - \frac{\nu_2^2}{4} & 0 \\ 0 & 0 & \frac{3\nu_2^2}{4} - v_1^2 (c_1 + c_2) \end{pmatrix} \];

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{1 - \cos^2 \theta_1 - \cos \theta_1} & -\sqrt{1 - \cos^2 \theta_1 - \cos \theta_1} \\ 0 & \sqrt{4 \sqrt{1 - \cos^2 \theta_1} \cos \theta_1 + 2} & \sqrt{4 \sqrt{1 - \cos^2 \theta_1} \cos \theta_1 + 2} \end{pmatrix} \].

• Solution 6: \[ a_4 = \frac{3\nu_1^2 - 4\nu_2^2 (c_1 + c_2)}{4v_{SM}} \], \( b_1 = 0 \), \( \cos\theta_3 = \sqrt{1 - \cos^2 \theta_1 \cos \theta_1 + \frac{1}{2}} \), \( \sin\theta_1 = -\sqrt{1 - \cos^2 \theta_1} \), \( \sin\theta_3 = \frac{1 - 2\cos^2 \theta_1}{\sqrt{4 \sqrt{1 - \cos^2 \theta_1} \cos \theta_1 + 2}} \).
\[ \text{Solution 9:} \quad a = \frac{3\mu^2_2}{4} - v^2_1(c_1 + c_2), \quad b = 0, \quad \cos \theta_3 = 0, \quad \sin \theta_1 = \frac{3\mu^2_2 - v^2_1(c_1 + c_2)}{2v^2_1}, \quad \sin \theta_2 = v^2_1(c_1 + c_2) - \frac{3\mu^2_2}{4} \]
\[ R_{E2} = \begin{pmatrix} 1 & 0 \\ 0 & \cos \theta_1 \cos \theta_3 - \sqrt{1 - \cos \theta_1^2} \sqrt{1 - \cos \theta_3^2} \\ 0 & \sqrt{1 - \cos \theta_1^2} \cos \theta_3 + \cos \theta_1 \sqrt{1 - \cos \theta_3^2} \cos \theta_3 - \sqrt{1 - \cos \theta_1^2} \cos \theta_3 \end{pmatrix} \]

- **Solution 10:** \( a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2} \), \( b_1 = 0 \), \( \sin \theta_1 = -\sqrt{1 - \cos \theta_1^2} \), \( \sin \theta_3 = \sqrt{1 - \cos \theta_3^2} \), \( \mu_{t2}^2 = 4v_1^2(c_1 + c_2) \);

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix}.
\]

\[ R_{E2} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \cos \theta_1^2} \sqrt{1 - \cos \theta_3^2} + \cos \theta_1 \cos \theta_3 \sqrt{1 - \cos \theta_1^2} \cos \theta_3 - \sqrt{1 - \cos \theta_1^2} \cos \theta_3 \end{pmatrix} \]

- **Solution 11:** \( a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2} \), \( b_1 = 0 \), \( \sin \theta_1 = \sqrt{1 - \cos \theta_1^2} \), \( \sin \theta_3 = -\sqrt{1 - \cos \theta_3^2} \), \( \mu_{t2}^2 = 4v_1^2(c_1 + c_2) \);

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix}.
\]

\[ R_{E2} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \cos \theta_1^2} \sqrt{1 - \cos \theta_3^2} + \cos \theta_1 \cos \theta_3 \sqrt{1 - \cos \theta_1^2} \cos \theta_3 - \sqrt{1 - \cos \theta_1^2} \cos \theta_3 \end{pmatrix} \]

- **Solution 12:** \( a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2} \), \( b_1 = 0 \), \( \sin \theta_1 = \sqrt{1 - \cos \theta_1^2} \), \( \sin \theta_3 = \sqrt{1 - \cos \theta_3^2} \), \( \mu_{t2}^2 = 4v_1^2(c_1 + c_2) \);

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix}.
\]

\[ R_{E2} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \cos \theta_1^2} \sqrt{1 - \cos \theta_3^2} + \cos \theta_1 \cos \theta_3 \sqrt{1 - \cos \theta_1^2} \cos \theta_3 - \sqrt{1 - \cos \theta_1^2} \cos \theta_3 \end{pmatrix} \]

- **Solution 13:** \( a_4 = \frac{3\mu_{t2}^2 - 4v_1^2(c_1 + c_2)}{4v_{SM}^2} \), \( b_1 = 0 \), \( \cos \theta_1 = -\frac{1}{\sqrt{2}} \), \( \cos \theta_3 = 0 \), \( \sin \theta_1 = -\frac{1}{\sqrt{2}} \), \( \sin \theta_3 = -1 \);

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 3\mu_{t2}^2 - v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 3\mu_{t2}^2 - v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 3v_1^2(c_1 + c_2) - \mu_{t2}^2 \end{pmatrix}.
\]

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.
\]
• Solution 14: \( a_4 = \frac{3\mu_3^2-4v_1^2(c_1+c_2)}{4v_3^2 S_M} \), \( b_1 = 0 \), \( \cos\theta_1 = \frac{1}{\sqrt{2}} \), \( \cos\theta_3 = 0 \), \( \sin\theta_1 = -\frac{1}{\sqrt{2}} \), \( \sin\theta_3 = -1 \);
\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
\frac{3\mu_3^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 3v_1^2(c_1 + c_2) - \frac{\mu_1^2}{4} & 0 \\
0 & 0 & \frac{3\mu_2^2}{4} - v_1^2(c_1 + c_2)
\end{pmatrix};
\]
\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

• Solution 15: \( a_4 = \frac{3\mu_3^2-4v_1^2(c_1+c_2)}{4v_3^2 S_M} \), \( b_1 = 0 \), \( \cos\theta_1 = -\frac{1}{\sqrt{2}} \), \( \cos\theta_3 = 0 \), \( \sin\theta_1 = \frac{1}{\sqrt{2}} \), \( \sin\theta_3 = -1 \);
\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
\frac{3\mu_3^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 3v_1^2(c_1 + c_2) - \frac{\mu_1^2}{4} & 0 \\
0 & 0 & \frac{3\mu_2^2}{4} - v_1^2(c_1 + c_2)
\end{pmatrix};
\]
\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

• Solution 16: \( a_4 = \frac{3\mu_3^2-4v_1^2(c_1+c_2)}{4v_3^2 S_M} \), \( b_1 = 0 \), \( \cos\theta_1 = \frac{1}{\sqrt{2}} \), \( \cos\theta_3 = 0 \), \( \sin\theta_1 = \frac{1}{\sqrt{2}} \), \( \sin\theta_3 = -1 \);
\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
\frac{3\mu_3^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & \frac{3\mu_2^2}{4} - v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 3v_1^2(c_1 + c_2) - \frac{\mu_1^2}{4}
\end{pmatrix};
\]
\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

• Solution 17: \( a_4 = \frac{3\mu_3^2-4v_1^2(c_1+c_2)}{4v_3^2 S_M} \), \( b_1 = 0 \), \( \cos\theta_1 = -\frac{1}{\sqrt{2}} \), \( \cos\theta_3 = -1 \), \( \sin\theta_1 = -\frac{1}{\sqrt{2}} \), \( \sin\theta_3 = 0 \);
\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
\frac{3\mu_3^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 3v_1^2(c_1 + c_2) - \frac{\mu_1^2}{4} & 0 \\
0 & 0 & \frac{3\mu_2^2}{4} - v_1^2(c_1 + c_2)
\end{pmatrix};
\]
\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

• Solution 18: \( a_4 = \frac{3\mu_3^2-4v_1^2(c_1+c_2)}{4v_3^2 S_M} \), \( b_1 = 0 \), \( \cos\theta_1 = \frac{1}{\sqrt{2}} \), \( \cos\theta_3 = -1 \), \( \sin\theta_1 = -\frac{1}{\sqrt{2}} \), \( \sin\theta_3 = 0 \);
Solution 21:
$$\begin{align*}
& \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \\
& \begin{pmatrix}
\frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & \frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 3v_1^4(c_1 + c_2) - \frac{\mu_{12}^2}{4}
\end{pmatrix}; \\
& R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\end{align*}$$

• Solution 19: $a_4 = \frac{3\mu_{12}^2 - 4v_1^2(c_1 + c_2)}{4v_{SM}^2}$, $b_1 = 0$, $\cos\theta_1 = -\frac{1}{\sqrt{2}}$, $\cos\theta_3 = -1$, $\sin\theta_1 = \frac{1}{\sqrt{2}}$, $\sin\theta_3 = 0$;

$$\begin{align*}
& \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \\
& \begin{pmatrix}
\frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & \frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 3v_1^4(c_1 + c_2) - \frac{\mu_{12}^2}{4}
\end{pmatrix}; \\
& R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}.
\end{align*}$$

• Solution 20: $a_4 = \frac{3\mu_{12}^2 - 4v_1^2(c_1 + c_2)}{4v_{SM}^2}$, $b_1 = 0$, $\cos\theta_1 = \frac{1}{\sqrt{2}}$, $\cos\theta_3 = -1$, $\sin\theta_1 = \frac{1}{\sqrt{2}}$, $\sin\theta_3 = 0$;

$$\begin{align*}
& \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \\
& \begin{pmatrix}
\frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 3v_1^2(c_1 + c_2) - \frac{\mu_{12}^2}{4} & 0 \\
0 & 0 & \frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2)
\end{pmatrix}; \\
& R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\end{align*}$$

• Solution 21: $a_4 = \frac{3\mu_{12}^2 - 4v_1^2(c_1 + c_2)}{4v_{SM}^2}$, $b_1 = 0$, $\cos\theta_1 = -\frac{1}{\sqrt{2}}$, $\cos\theta_3 = 1$, $\sin\theta_1 = -\frac{1}{\sqrt{2}}$, $\sin\theta_3 = 0$;

$$\begin{align*}
& \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \\
& \begin{pmatrix}
\frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 3v_1^2(c_1 + c_2) - \frac{\mu_{12}^2}{4} & 0 \\
0 & 0 & \frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2)
\end{pmatrix}; \\
& R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\end{align*}$$

• Solution 22: $a_4 = \frac{3\mu_{12}^2 - 4v_1^2(c_1 + c_2)}{4v_{SM}^2}$, $b_1 = 0$, $\cos\theta_1 = \frac{1}{\sqrt{2}}$, $\cos\theta_3 = 1$, $\sin\theta_1 = -\frac{1}{\sqrt{2}}$, $\sin\theta_3 = 0$;

$$\begin{align*}
& \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \\
& \begin{pmatrix}
\frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & \frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 3v_1^2(c_1 + c_2) - \frac{\mu_{12}^2}{4}
\end{pmatrix};
\end{align*}$$
\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

- **Solution 23:** \(a_4 = \frac{3\mu_{12}^2 - 4v_1^2(c_1 + c_2)}{4v_{SM}^2}, b_1 = 0, \cos\theta_1 = -\frac{1}{\sqrt{2}}, \cos\theta_3 = 1, \sin\theta_1 = \frac{1}{\sqrt{2}}, \sin\theta_3 = 0;\)

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^0) = \begin{pmatrix}
\frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & \frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 3v_1^2(c_1 + c_2) - \frac{\mu_{12}^2}{4}
\end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

- **Solution 24:** \(a_4 = \frac{3\mu_{12}^2 - 4v_2^2(c_1 + c_2)}{4v_{SM}^2}, b_1 = 0, \cos\theta_1 = \frac{1}{\sqrt{2}}, \cos\theta_3 = 1, \sin\theta_1 = \frac{1}{\sqrt{2}}, \sin\theta_3 = 0;\)

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^0) = \begin{pmatrix}
\frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 3v_1^2(c_1 + c_2) - \frac{\mu_{12}^2}{4} & 0 \\
0 & 0 & \frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2)
\end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

- **Solution 25:** \(a_4 = \frac{3\mu_{12}^2 - 4v_3^2(c_1 + c_2)}{4v_{SM}^2}, b_1 = 0, \cos\theta_1 = -\frac{1}{\sqrt{2}}, \cos\theta_3 = 0, \sin\theta_1 = -\frac{1}{\sqrt{2}}, \sin\theta_3 = 1;\)

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^0) = \begin{pmatrix}
\frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & \frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 3v_1^2(c_1 + c_2) - \frac{\mu_{12}^2}{4}
\end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

- **Solution 26:** \(a_4 = \frac{3\mu_{12}^2 - 4v_4^2(c_1 + c_2)}{4v_{SM}^2}, b_1 = 0, \cos\theta_1 = \frac{1}{\sqrt{2}}, \cos\theta_3 = 0, \sin\theta_1 = -\frac{1}{\sqrt{2}}, \sin\theta_3 = 1;\)

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^0) = \begin{pmatrix}
\frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 3v_1^2(c_1 + c_2) - \frac{\mu_{12}^2}{4} & 0 \\
0 & 0 & \frac{3\mu_{12}^2}{4} - v_1^2(c_1 + c_2)
\end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}.
\]
• Solution 27: $a_4 = \frac{3v_1^2 - 4v_1^2(c_1 + c_2)}{4v_2^2}$, $b_1 = 0$, $\cos \theta_1 = -\frac{1}{\sqrt{2}}$, $\cos \theta_3 = 0$, $\sin \theta_1 = \frac{1}{\sqrt{2}}$, $\sin \theta_3 = 1$;

$$\text{diag}(m_H^2, m_{H_0}^2, m_{H_2}^2) = \begin{pmatrix}
\frac{3v_1^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 3v_1^2(c_1 + c_2) - \frac{v_1^2}{4} & 0 \\
0 & 0 & 3\frac{v_1^2}{4} - v_1^2(c_1 + c_2)
\end{pmatrix};$$

$$R_{E2} = \begin{pmatrix}1 & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}.$$

• Solution 28: $a_4 = \frac{3v_2^2 - 4v_2^2(c_1 + c_2)}{4v_3^2}$, $b_1 = 0$, $\cos \theta_1 = \frac{1}{\sqrt{2}}$, $\cos \theta_3 = 0$, $\sin \theta_1 = \frac{1}{\sqrt{2}}$, $\sin \theta_3 = 1$;

$$\text{diag}(m_H^2, m_{H_0}^2, m_{H_2}^2) = \begin{pmatrix}
\frac{3v_1^2}{4} - v_1^2(c_1 + c_2) & 0 & 0 \\
0 & \frac{3v_1^2}{4} - v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 3v_1^2(c_1 + c_2) - \frac{v_1^2}{4}
\end{pmatrix};$$

$$R_{E2} = \begin{pmatrix}1 & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.$$

• Solution 29: $a_4 = \sqrt{v_1^2(c_1 + c_2)}$, $b_1 = 0$, $\cos \theta_3 = -\sqrt{\frac{1}{2} - \cos \theta_1 \sqrt{1 - \cos \theta_1^2}}$, $\sin \theta_1 = \sqrt{1 - \cos \theta_1^2}$, $\sin \theta_3 = \frac{2\cos \theta_1^2 - 1}{\sqrt{2 - 4\cos \theta_1 \sqrt{1 - \cos \theta_1^2}}}$, $\mu_{12} = 4v_1^2(c_1 + c_2)$;

$$\text{diag}(m_H^2, m_{H_0}^2, m_{H_2}^2) = \begin{pmatrix}2v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 2v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 2v_1^2(c_1 + c_2)
\end{pmatrix};$$

$$R_{E2} = \begin{pmatrix}1 & 0 & 0 \\
0 & \sqrt{\frac{1 - \cos \theta_1^2}{1 - \cos \theta_1^2} - \cos \theta_1} & \cos \theta_1 - \sqrt{\frac{1 - \cos \theta_1^2}{1 - \cos \theta_1^2}} \\
0 & \sqrt{\frac{2 - 4\cos \theta_1 \sqrt{1 - \cos \theta_1^2}}{1 - \cos \theta_1^2} - \cos \theta_1} & \sqrt{\frac{2 - 4\cos \theta_1 \sqrt{1 - \cos \theta_1^2}}{1 - \cos \theta_1^2} - \cos \theta_1} \\
0 & \sqrt{\frac{1 - \cos \theta_1^2}{1 - \cos \theta_1^2} - \cos \theta_1} & \sqrt{\frac{1 - \cos \theta_1^2}{1 - \cos \theta_1^2} - \cos \theta_1}
\end{pmatrix}.$$
\[ R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{\cos\theta_1 - \sqrt{1 - \cos^2\theta_1}}{\sqrt{1 - \cos^2\theta_1}} & \frac{\cos\theta_1 - \sqrt{1 - \cos^2\theta_1}}{\sqrt{1 - \cos^2\theta_1}} \\
0 & \sqrt{2 - 4\cos\theta_1\sqrt{1 - \cos^2\theta_1}} & \sqrt{2 - 4\cos\theta_1\sqrt{1 - \cos^2\theta_1}} \\
0 & \frac{\sqrt{1 - \cos^2\theta_1}}{\sqrt{1 - \cos^2\theta_1}} & \frac{\sqrt{1 - \cos^2\theta_1}}{\sqrt{1 - \cos^2\theta_1}}
\end{pmatrix} \]

- **Solution 31:** \( a_4 = \frac{2v_f^2(c_1 + c_2)}{v_{SM}^2}, \) \( b_1 = 0, \) \( \cos\theta_3 = -\sqrt{\frac{1}{2}} - \cos\theta_1 \sqrt{1 - \cos^2\theta_1}, \) \( \sin\theta_1 = -\sqrt{1 - \cos^2\theta_1}, \) \( \sin\theta_3 = \frac{1 - \cos^2\theta_1}{\sqrt{2 - 4\cos\theta_1\sqrt{1 - \cos^2\theta_1}}}, \) \( \mu_{12}^2 = 4v_f^2(c_1 + c_2); \)

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
2v_f^2(c_1 + c_2) & 0 & 0 \\
0 & 2v_f^2(c_1 + c_2) & 0 \\
0 & 0 & 2v_f^2(c_1 + c_2)
\end{pmatrix} \]

- **Solution 32:** \( a_4 = \frac{2v_f^2(c_1 + c_2)}{v_{SM}^2}, \) \( b_1 = 0, \) \( \cos\theta_3 = \sqrt{\frac{1}{2}} - \cos\theta_1 \sqrt{1 - \cos^2\theta_1}, \) \( \sin\theta_1 = \sqrt{1 - \cos^2\theta_1}, \) \( \sin\theta_3 = \frac{1 - \cos^2\theta_1}{\sqrt{2 - 4\cos\theta_1\sqrt{1 - \cos^2\theta_1}}}, \) \( \mu_{12}^2 = 4v_f^2(c_1 + c_2); \)

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
2v_f^2(c_1 + c_2) & 0 & 0 \\
0 & 2v_f^2(c_1 + c_2) & 0 \\
0 & 0 & 2v_f^2(c_1 + c_2)
\end{pmatrix} \]

- **Solution 33:** \( a_4 = \frac{2v_f^2(c_1 + c_2)}{v_{SM}^2}, \) \( b_1 = 0, \) \( \cos\theta_3 = -\sqrt{\sqrt{1 - \cos^2\theta_1} \cos\theta_1 + \frac{1}{2}}, \) \( \sin\theta_1 = \sqrt{1 - \cos^2\theta_1}, \) \( \sin\theta_3 = \frac{1 - \cos^2\theta_1}{\sqrt{4\sqrt{1 - \cos^2\theta_1} \cos\theta_1 + 2}}, \) \( \mu_{12}^2 = 4v_f^2(c_1 + c_2); \)

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
2v_f^2(c_1 + c_2) & 0 & 0 \\
0 & 2v_f^2(c_1 + c_2) & 0 \\
0 & 0 & 2v_f^2(c_1 + c_2)
\end{pmatrix} \]
• Solution 34: \( a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}^2}, \quad b_1 = 0, \quad \cos \theta_3 = \sqrt{1 - \cos \theta_1^2 \cos \theta_1 + \frac{1}{2}}, \quad \sin \theta_1 = -\sqrt{1 - \cos \theta_1^2}, \quad \sin \theta_3 = \frac{1 - 2 \cos \theta_1^2}{\sqrt{4(1 - \cos \theta_1^2 \cos \theta_1 + 2)}}, \quad \mu_{12}^2 = 4v_1^2(c_1 + c_2); \)

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
2v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 2v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 2v_1^2(c_1 + c_2)
\end{pmatrix},
\]

\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \sqrt{1 - \cos \theta_1^2 + \cos \theta_1} & -\sqrt{1 - \cos \theta_1^2 - \cos \theta_1} \\
0 & 4\sqrt{1 - \cos \theta_1^2 \cos \theta_1 + 2} & \sqrt{4(1 - \cos \theta_1^2 \cos \theta_1 + 2)}
\end{pmatrix}.
\]

• Solution 35: \( a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}^2}, \quad b_1 = 0, \quad \cos \theta_3 = -\sqrt{1 - \cos \theta_1^2 \cos \theta_1 + \frac{1}{2}}, \quad \sin \theta_1 = -\sqrt{1 - \cos \theta_1^2}, \quad \sin \theta_3 = \frac{2 \cos \theta_1^2 - 1}{\sqrt{4(1 - \cos \theta_1^2 \cos \theta_1 + 2)}}, \quad \mu_{12}^2 = 4v_1^2(c_1 + c_2); \)

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
2v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 2v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 2v_1^2(c_1 + c_2)
\end{pmatrix},
\]

\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \sqrt{1 - \cos \theta_1^2 - \cos \theta_1} & \sqrt{1 - \cos \theta_1^2 + \cos \theta_1} \\
0 & 4\sqrt{1 - \cos \theta_1^2 \cos \theta_1 + 2} & \sqrt{4(1 - \cos \theta_1^2 \cos \theta_1 + 2)}
\end{pmatrix}.
\]

• Solution 36: \( a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}^2}, \quad b_1 = 0, \quad \cos \theta_3 = \sqrt{1 - \cos \theta_1^2 \cos \theta_1 + \frac{1}{2}}, \quad \sin \theta_1 = \sqrt{1 - \cos \theta_1^2}, \quad \sin \theta_3 = \frac{2 \cos \theta_1^2 - 1}{\sqrt{4(1 - \cos \theta_1^2 \cos \theta_1 + 2)}}, \quad \mu_{12}^2 = 4v_1^2(c_1 + c_2); \)

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
2v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 2v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 2v_1^2(c_1 + c_2)
\end{pmatrix},
\]

\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \sqrt{1 - \cos \theta_1^2 + \cos \theta_1} & \sqrt{1 - \cos \theta_1^2 - \cos \theta_1} \\
0 & 4\sqrt{1 - \cos \theta_1^2 \cos \theta_1 + 2} & \sqrt{4(1 - \cos \theta_1^2 \cos \theta_1 + 2)}
\end{pmatrix}.
\]

• Solution 37: \( a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}^2}, \quad b_1 = 0, \quad \cos \theta_1 = -\frac{1}{\sqrt{2}}, \quad \cos \theta_3 = 0, \quad \sin \theta_1 = -\frac{1}{\sqrt{2}}, \quad \sin \theta_3 = -1, \quad \mu_{12}^2 = 4v_1^2(c_1 + c_2); \)

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
2v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 2v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 2v_1^2(c_1 + c_2)
\end{pmatrix}.
\]
\[ R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 - \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 - \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}. \]

- Solution 38: \( a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}^2}, b_1 = 0, \cos \theta_1 = \frac{1}{\sqrt{2}}, \cos \theta_3 = 0, \sin \theta_1 = -\frac{1}{\sqrt{2}}, \sin \theta_3 = -1, \mu_{12}^2 = 4v_1^2(c_1 + c_2); \)

\[ \text{diag}(m_{H_1}^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
2v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 2v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 2v_1^2(c_1 + c_2)
\end{pmatrix}; \]

\[ R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 - \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 - \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}. \]

- Solution 39: \( a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}^2}, b_1 = 0, \cos \theta_1 = -\frac{1}{\sqrt{2}}, \cos \theta_3 = 0, \sin \theta_1 = \frac{1}{\sqrt{2}}, \sin \theta_3 = -1, \mu_{12}^2 = 4v_1^2(c_1 + c_2); \)

\[ \text{diag}(m_{H_1}^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
2v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 2v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 2v_1^2(c_1 + c_2)
\end{pmatrix}; \]

\[ R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 - \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}. \]

- Solution 40: \( a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}^2}, b_1 = 0, \cos \theta_1 = \frac{1}{\sqrt{2}}, \cos \theta_3 = 0, \sin \theta_1 = \frac{1}{\sqrt{2}}, \sin \theta_3 = -1, \mu_{12}^2 = 4v_1^2(c_1 + c_2); \)

\[ \text{diag}(m_{H_1}^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
2v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 2v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 2v_1^2(c_1 + c_2)
\end{pmatrix}; \]

\[ R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 - \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}. \]

- Solution 41: \( a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}^2}, b_1 = 0, \cos \theta_1 = -\frac{1}{\sqrt{2}}, \cos \theta_3 = -1, \sin \theta_1 = -\frac{1}{\sqrt{2}}, \sin \theta_3 = 0, \mu_{12}^2 = 4v_1^2(c_1 + c_2); \)

\[ \text{diag}(m_{H_1}^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix}
2v_1^2(c_1 + c_2) & 0 & 0 \\
0 & 2v_1^2(c_1 + c_2) & 0 \\
0 & 0 & 2v_1^2(c_1 + c_2)
\end{pmatrix}; \]

\[ R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 - \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}. \]
• Solution 42: $a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{5M}^2}$, $b_1 = 0$, $\cos\theta_1 = \frac{1}{\sqrt{2}}$, $\cos\theta_3 = -1$, $\sin\theta_1 = -\frac{1}{\sqrt{2}}$, $\sin\theta_3 = 0$, $\mu_{12}^2 = 4v_1^2(c_1 + c_2)$:
\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix}; \]
\[
R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \]

• Solution 43: $a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{5M}^2}$, $b_1 = 0$, $\cos\theta_1 = -\frac{1}{\sqrt{2}}$, $\cos\theta_3 = -1$, $\sin\theta_1 = \frac{1}{\sqrt{2}}$, $\sin\theta_3 = 0$, $\mu_{12}^2 = 4v_1^2(c_1 + c_2)$:
\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix}; \]
\[
R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \]

• Solution 44: $a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{5M}^2}$, $b_1 = 0$, $\cos\theta_1 = \frac{1}{\sqrt{2}}$, $\cos\theta_3 = -1$, $\sin\theta_1 = \frac{1}{\sqrt{2}}$, $\sin\theta_3 = 0$, $\mu_{12}^2 = 4v_1^2(c_1 + c_2)$:
\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix}; \]
\[
R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \]

• Solution 45: $a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{5M}^2}$, $b_1 = 0$, $\cos\theta_1 = -\frac{1}{\sqrt{2}}$, $\cos\theta_3 = 1$, $\sin\theta_1 = -\frac{1}{\sqrt{2}}$, $\sin\theta_3 = 0$, $\mu_{12}^2 = 4v_1^2(c_1 + c_2)$:
\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix}; \]
\[
R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \]

• Solution 46: $a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{5M}^2}$, $b_1 = 0$, $\cos\theta_1 = \frac{1}{\sqrt{2}}$, $\cos\theta_3 = 1$, $\sin\theta_1 = -\frac{1}{\sqrt{2}}$, $\sin\theta_3 = 0$, $\mu_{12}^2 = 4v_1^2(c_1 + c_2)$;
\[
\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.
\]

- Solution 47: \(a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2}, b_1 = 0, \cos\theta_1 = -\frac{1}{\sqrt{2}}, \cos\theta_3 = 1, \sin\theta_1 = \frac{1}{\sqrt{2}}, \sin\theta_3 = 0, \mu_{12}^2 = 4v_1^2(c_1 + c_2);\)

\[
\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.
\]

- Solution 48: \(a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2}, b_1 = 0, \cos\theta_1 = \frac{1}{\sqrt{2}}, \cos\theta_3 = 1, \sin\theta_1 = \frac{1}{\sqrt{2}}, \sin\theta_3 = 0, \mu_{12}^2 = 4v_1^2(c_1 + c_2);\)

\[
\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.
\]

- Solution 49: \(a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2}, b_1 = 0, \cos\theta_1 = -\frac{1}{\sqrt{2}}, \cos\theta_3 = 0, \sin\theta_1 = -\frac{1}{\sqrt{2}}, \sin\theta_3 = 1, \mu_{12}^2 = 4v_1^2(c_1 + c_2);\)

\[
\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.
\]

- Solution 50: \(a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2}, b_1 = 0, \cos\theta_1 = \frac{1}{\sqrt{2}}, \cos\theta_3 = 0, \sin\theta_1 = -\frac{1}{\sqrt{2}}, \sin\theta_3 = 1, \mu_{12}^2 = 4v_1^2(c_1 + c_2);\)

\[
\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix};
\]
\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \].

- Solution 51: \( a_4 = \frac{2v_2^2(c_1 + c_2)}{\mu_{SM}} \), \( b_1 = 0 \), \( \cos \theta_1 = -\frac{1}{\sqrt{2}} \), \( \cos \theta_3 = 0 \), \( \sin \theta_1 = \frac{1}{\sqrt{2}} \), \( \sin \theta_3 = 1 \), \( \mu_{l_2}^2 = 4v_1^2(c_1 + c_2) \);

\[ \text{diag}(m_{H_2}^2, m_{H_1}^2, m_{H_0}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix} \];

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \).

- Solution 52: \( a_4 = \frac{2v_2^2(c_1 + c_2)}{\mu_{SM}} \), \( b_1 = 0 \), \( \cos \theta_1 = \frac{1}{\sqrt{2}} \), \( \cos \theta_3 = 0 \), \( \sin \theta_1 = \frac{1}{\sqrt{2}} \), \( \sin \theta_3 = 1 \), \( \mu_{l_2}^2 = 4v_1^2(c_1 + c_2) \);

\[ \text{diag}(m_{H_2}^2, m_{H_1}^2, m_{H_0}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix} \];

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \).

**A.2 Using** \( v_2 = v_1 \), \( \mu_\xi^2 = \frac{1}{2} (-b_1v_{SM}^2 - 4v_1^2(c_1 + c_2) - \mu_{l_2}^2) \) and \( \mu_{SM} = -a_4v_{SM}^2 - b_1v_1^2 \).  

- Solution 1: \( a_4 = \frac{4v_1^2(c_1 + c_2) + 3v_2^2}{4v_{SM}} \), \( b_1 = 0 \), \( \cos \theta_3 = \sqrt{1 - \cos \theta_1^2} \), \( \sin \theta_3 = \frac{2\cos \theta_1^2 - 1}{\sqrt{2 - 4\cos \theta_1\sqrt{1 - \cos \theta_1^2}}} \);

\[ \text{diag}(m_{H_1}^2, m_{H_0}^2, m_{H_0}^2) = \begin{pmatrix} v_1^2((-c_1 + c_2)) - \frac{3v_2^2}{4} & 0 & 0 \\ 0 & 3v_1^2(c_1 + c_2) + \frac{3v_2^2}{4} & 0 \\ 0 & 0 & v_1^2((-c_1 + c_2)) - \frac{3v_2^2}{4} \end{pmatrix} \];

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{1 - \cos \theta_1^2} - \cos \theta_1}{\sqrt{2 - 4\cos \theta_1\sqrt{1 - \cos \theta_1^2}}} & \frac{\cos \theta_1 - \sqrt{1 - \cos \theta_1^2}}{\sqrt{2 - 4\cos \theta_1\sqrt{1 - \cos \theta_1^2}}} \\ 0 & \frac{\sqrt{1 - \cos \theta_1^2} - \cos \theta_1}{\sqrt{2 - 4\cos \theta_1\sqrt{1 - \cos \theta_1^2}}} & \frac{\cos \theta_1 - \sqrt{1 - \cos \theta_1^2}}{\sqrt{2 - 4\cos \theta_1\sqrt{1 - \cos \theta_1^2}}} \end{pmatrix} \).

- Solution 2: \( a_4 = \frac{4v_1^2(c_1 + c_2) + 3v_2^2}{4v_{SM}} \), \( b_1 = 0 \), \( \cos \theta_3 = \sqrt{\frac{1}{2} - \cos \theta_1\sqrt{1 - \cos \theta_1^2}} \), \( \sin \theta_1 = -\sqrt{1 - \cos \theta_1^2} \), \( \sin \theta_3 = \frac{2\cos \theta_1^2 - 1}{\sqrt{2 - 4\cos \theta_1\sqrt{1 - \cos \theta_1^2}}} \);
\[
diag(m_H^2, m_{H_1^0}^2, m_{H_2^0}^2) = \begin{pmatrix}
v_1^2(-(c_1 + c_2)) - \frac{3\mu_2^2}{4} & 0 & 0 \\
0 & v_1^2(-(c_1 + c_2)) - \frac{3\mu_2^2}{4} & 0 \\
0 & 0 & 3v_1^2(c_1 + c_2) + \frac{\mu_2^2}{4}
\end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_1 - \frac{\sqrt{1 - \cos^2 \theta_1}}{2 - 4\cos \theta_1} & \cos \theta_1 - \frac{\sqrt{1 - \cos^2 \theta_1}}{2 - 4\cos \theta_1} \\
0 & \frac{\sqrt{1 - \cos^2 \theta_1}}{2 - 4\cos \theta_1} & \frac{\sqrt{1 - \cos^2 \theta_1}}{2 - 4\cos \theta_1}
\end{pmatrix}.
\]

• Solution 3: \(a_4 = -\frac{4\mu_2^2(c_1 + c_2) + 3\mu_2^2}{4\mu_3^2}, b_1 = 0, \cos \theta_3 = -\sqrt{\frac{1}{2} - \cos \theta_1 \sqrt{1 - \cos^2 \theta_1}}, \sin \theta_1 = -\sqrt{1 - \cos^2 \theta_1}, \sin \theta_3 = \frac{1 - \cos \theta_1}{\sqrt{2 - 4\cos \theta_1} \sqrt{1 - \cos^2 \theta_1}};\)

\[
diag(m_H^2, m_{H_1^0}^2, m_{H_2^0}^2) = \begin{pmatrix}
v_1^2(-(c_1 + c_2)) - \frac{3\mu_2^2}{4} & 0 & 0 \\
0 & v_1^2(-(c_1 + c_2)) - \frac{3\mu_2^2}{4} & 0 \\
0 & 0 & 3v_1^2(c_1 + c_2) + \frac{\mu_2^2}{4}
\end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \sqrt{1 - \cos^2 \theta_1} - \cos \theta_1 & \sqrt{1 - \cos^2 \theta_1} - \cos \theta_1 \\
0 & \frac{\sqrt{1 - \cos^2 \theta_1}}{2 - 4\cos \theta_1} & \frac{\sqrt{1 - \cos^2 \theta_1}}{2 - 4\cos \theta_1}
\end{pmatrix}.
\]

• Solution 4: \(a_4 = -\frac{4\mu_2^2(c_1 + c_2) + 3\mu_2^2}{4\mu_3^2}, b_1 = 0, \cos \theta_3 = \sqrt{\frac{1}{2} - \cos \theta_1 \sqrt{1 - \cos^2 \theta_1}}, \sin \theta_1 = \sqrt{1 - \cos^2 \theta_1}, \sin \theta_3 = \frac{1 - \cos \theta_1}{\sqrt{2 - 4\cos \theta_1} \sqrt{1 - \cos^2 \theta_1}};\)

\[
diag(m_H^2, m_{H_1^0}^2, m_{H_2^0}^2) = \begin{pmatrix}
v_1^2(-(c_1 + c_2)) - \frac{3\mu_2^2}{4} & 0 & 0 \\
0 & 3v_1^2(c_1 + c_2) + \frac{\mu_2^2}{4} & 0 \\
0 & 0 & v_1^2(-(c_1 + c_2)) - \frac{3\mu_2^2}{4}
\end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_1 - \frac{\sqrt{1 - \cos^2 \theta_1}}{2 - 4\cos \theta_1} & \cos \theta_1 - \frac{\sqrt{1 - \cos^2 \theta_1}}{2 - 4\cos \theta_1} \\
0 & \frac{\sqrt{1 - \cos^2 \theta_1}}{2 - 4\cos \theta_1} & \frac{\sqrt{1 - \cos^2 \theta_1}}{2 - 4\cos \theta_1}
\end{pmatrix}.
\]

• Solution 5: \(a_4 = -\frac{4\mu_2^2(c_1 + c_2) + 3\mu_2^2}{4\mu_3^2}, b_1 = 0, \cos \theta_3 = -\sqrt{\frac{1}{2} - \cos \theta_1 \cos \theta_1 + \frac{1}{2}}, \sin \theta_1 = \sqrt{1 - \cos^2 \theta_1}, \sin \theta_3 = \frac{1 - \cos \theta_1}{\sqrt{4(1 - \cos^2 \theta_1) \cos \theta_1 + 2}};\)

\[
diag(m_H^2, m_{H_1^0}^2, m_{H_2^0}^2) = \begin{pmatrix}
v_1^2(-(c_1 + c_2)) - \frac{3\mu_2^2}{4} & 0 & 0 \\
0 & v_1^2(-(c_1 + c_2)) - \frac{3\mu_2^2}{4} & 0 \\
0 & 0 & 3v_1^2(c_1 + c_2) + \frac{\mu_2^2}{4}
\end{pmatrix};
\]
$R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{1 - \cos^2 \theta_2 - \cos^2 \theta_1} & -\sqrt{1 - \cos^2 \theta_2 - \cos^2 \theta_1} \\ 0 & \sqrt{1 - \cos^2 \theta_2 + \cos^2 \theta_1} & \sqrt{1 - \cos^2 \theta_2 + \cos^2 \theta_1} \\ 0 & \sqrt{1 - \cos^2 \theta_2 + \cos^2 \theta_1} & -\sqrt{1 - \cos^2 \theta_2 - \cos^2 \theta_1} \end{pmatrix}$.

- Solution 6: $a_4 = -\frac{4\nu_1^2(c_1 + c_2) + 3\mu_1^2}{4\nu_2^2 M}$, $b_1 = 0$, $\cos \theta_3 = \sqrt{1 - \cos^2 \theta_1 \cos \theta_1 + \frac{1}{2}}$, $\sin \theta_1 = -\sqrt{1 - \cos^2 \theta_2}$, $\sin \theta_3 = \frac{1 - 2\cos^2 \theta_2}{4\sqrt{1 - \cos^2 \theta_2 \cos \theta_1 + 2}}$;

$\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-(c_1 + c_2)) - \frac{3\mu_1^2}{4} & 0 & 0 \\ 0 & 3v_1^2(c_1 + c_2) + \frac{\mu_1^2}{2} & 0 \\ 0 & 0 & v_1^2(-(c_1 + c_2)) - \frac{3\mu_1^2}{4} \end{pmatrix}$;

$R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{1 - \cos^2 \theta_2 - \cos^2 \theta_1} & -\sqrt{1 - \cos^2 \theta_2 - \cos^2 \theta_1} \\ 0 & \sqrt{1 - \cos^2 \theta_2 + \cos^2 \theta_1} & \sqrt{1 - \cos^2 \theta_2 + \cos^2 \theta_1} \\ 0 & \sqrt{1 - \cos^2 \theta_2 + \cos^2 \theta_1} & -\sqrt{1 - \cos^2 \theta_2 - \cos^2 \theta_1} \end{pmatrix}$.

- Solution 7: $a_4 = -\frac{4\nu_1^2(c_1 + c_2) + 3\mu_1^2}{4\nu_2^2 M}$, $b_1 = 0$, $\cos \theta_3 = -\sqrt{1 - \cos^2 \theta_1 \cos \theta_1 + \frac{1}{2}}$, $\sin \theta_1 = -\sqrt{1 - \cos^2 \theta_2}$, $\sin \theta_3 = \frac{2\cos^2 \theta_2 - 1}{4\sqrt{1 - \cos^2 \theta_2 \cos \theta_1 + 2}}$;

$\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-(c_1 + c_2)) - \frac{3\mu_1^2}{4} & 0 & 0 \\ 0 & 3v_1^2(c_1 + c_2) + \frac{\mu_1^2}{2} & 0 \\ 0 & 0 & v_1^2(-(c_1 + c_2)) - \frac{3\mu_1^2}{4} \end{pmatrix}$;

$R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{1 - \cos^2 \theta_2 - \cos^2 \theta_1} & -\sqrt{1 - \cos^2 \theta_2 - \cos^2 \theta_1} \\ 0 & \sqrt{1 - \cos^2 \theta_2 + \cos^2 \theta_1} & \sqrt{1 - \cos^2 \theta_2 + \cos^2 \theta_1} \\ 0 & \sqrt{1 - \cos^2 \theta_2 + \cos^2 \theta_1} & -\sqrt{1 - \cos^2 \theta_2 - \cos^2 \theta_1} \end{pmatrix}$.

- Solution 8: $a_4 = -\frac{4\nu_1^2(c_1 + c_2) + 3\mu_1^2}{4\nu_2^2 M}$, $b_1 = 0$, $\cos \theta_3 = \sqrt{1 - \cos^2 \theta_1 \cos \theta_1 + \frac{1}{2}}$, $\sin \theta_1 = \sqrt{1 - \cos^2 \theta_2}$, $\sin \theta_3 = \frac{1 - 2\cos^2 \theta_2}{4\sqrt{1 - \cos^2 \theta_2 \cos \theta_1 + 2}}$;

$\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-(c_1 + c_2)) - \frac{3\mu_1^2}{4} & 0 & 0 \\ 0 & 3v_1^2(c_1 + c_2) + \frac{\mu_1^2}{2} & 0 \\ 0 & 0 & v_1^2(-(c_1 + c_2)) - \frac{3\mu_1^2}{4} \end{pmatrix}$;

$R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{1 - \cos^2 \theta_2 - \cos^2 \theta_1} & -\sqrt{1 - \cos^2 \theta_2 - \cos^2 \theta_1} \\ 0 & \sqrt{1 - \cos^2 \theta_2 + \cos^2 \theta_1} & \sqrt{1 - \cos^2 \theta_2 + \cos^2 \theta_1} \\ 0 & \sqrt{1 - \cos^2 \theta_2 + \cos^2 \theta_1} & -\sqrt{1 - \cos^2 \theta_2 - \cos^2 \theta_1} \end{pmatrix}$.
• Solution 9: $a_4 = \frac{2v_{2}^{2}(c_1+c_2)}{v_{3M}^{2}}$, $b_1 = 0$, $\sin\theta_1 = -\sqrt{1 - \cos\theta_1^2}$, $\sin\theta_3 = -\sqrt{1 - \cos\theta_3^2}$,
$\mu_{12}^2 = -4v_{2}^{2}(c_1+c_2);
\begin{align*}
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) &= \begin{pmatrix}
2v_{1}^{2}(c_1+c_2) & 0 & 0 \\
0 & 2v_{1}^{2}(c_1+c_2) & 0 \\
0 & 0 & 2v_{1}^{2}(c_1+c_2)
\end{pmatrix}; \\
R_{E2} &= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos\theta_1\cos\theta_3 - \sqrt{1 - \cos\theta_1^2}\sqrt{1 - \cos\theta_3^2} & \cos\theta_1 \left( -\sqrt{1 - \cos\theta_1^2} - \sqrt{1 - \cos\theta_1^2}\cos\theta_3 \right) \\
0 & \sqrt{1 - \cos\theta_1^2}\cos\theta_3 + \cos\theta_1\sqrt{1 - \cos\theta_3^2} & \cos\theta_3 - \sqrt{1 - \cos\theta_1^2}\sqrt{1 - \cos\theta_3^2}
\end{pmatrix}.
\end{align*}$

• Solution 10: $a_4 = \frac{2v_{2}^{2}(c_1+c_2)}{v_{3M}^{2}}$, $b_1 = 0$, $\sin\theta_1 = -\sqrt{1 - \cos\theta_1^2}$, $\sin\theta_3 = \sqrt{1 - \cos\theta_3^2}$,
$\mu_{12}^2 = -4v_{2}^{2}(c_1+c_2);
\begin{align*}
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) &= \begin{pmatrix}
2v_{1}^{2}(c_1+c_2) & 0 & 0 \\
0 & 2v_{1}^{2}(c_1+c_2) & 0 \\
0 & 0 & 2v_{1}^{2}(c_1+c_2)
\end{pmatrix}; \\
R_{E2} &= \begin{pmatrix}
1 & 0 & 0 \\
0 & \sqrt{1 - \cos\theta_1^2}\sqrt{1 - \cos\theta_3^2} + \cos\theta_1\cos\theta_3 & \cos\theta_3 - \sqrt{1 - \cos\theta_1^2}\sqrt{1 - \cos\theta_3^2}
\end{pmatrix}.
\end{align*}$

• Solution 11: $a_4 = \frac{2v_{2}^{2}(c_1+c_2)}{v_{3M}^{2}}$, $b_1 = 0$, $\sin\theta_1 = \sqrt{1 - \cos\theta_1^2}$, $\sin\theta_3 = -\sqrt{1 - \cos\theta_3^2}$,
$\mu_{12}^2 = -4v_{2}^{2}(c_1+c_2);
\begin{align*}
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) &= \begin{pmatrix}
2v_{1}^{2}(c_1+c_2) & 0 & 0 \\
0 & 2v_{1}^{2}(c_1+c_2) & 0 \\
0 & 0 & 2v_{1}^{2}(c_1+c_2)
\end{pmatrix}; \\
R_{E2} &= \begin{pmatrix}
1 & 0 & 0 \\
0 & \sqrt{1 - \cos\theta_1^2}\sqrt{1 - \cos\theta_3^2} + \cos\theta_1\cos\theta_3 & \cos\theta_3 - \sqrt{1 - \cos\theta_1^2}\sqrt{1 - \cos\theta_3^2}
\end{pmatrix}.
\end{align*}$

• Solution 12: $a_4 = \frac{2v_{2}^{2}(c_1+c_2)}{v_{3M}^{2}}$, $b_1 = 0$, $\sin\theta_1 = \sqrt{1 - \cos\theta_1^2}$, $\sin\theta_3 = \sqrt{1 - \cos\theta_3^2}$,
$\mu_{12}^2 = -4v_{2}^{2}(c_1+c_2);
\begin{align*}
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) &= \begin{pmatrix}
2v_{1}^{2}(c_1+c_2) & 0 & 0 \\
0 & 2v_{1}^{2}(c_1+c_2) & 0 \\
0 & 0 & 2v_{1}^{2}(c_1+c_2)
\end{pmatrix}; \\
R_{E2} &= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos\theta_1\cos\theta_3 - \sqrt{1 - \cos\theta_1^2}\sqrt{1 - \cos\theta_3^2} & \sqrt{1 - \cos\theta_1^2}\cos\theta_3 + \cos\theta_1\sqrt{1 - \cos\theta_3^2}
\end{pmatrix}.
\end{align*}$

• Solution 13: $a_4 = \frac{-4v_{2}^{2}(c_1+c_2) + 3\mu_{12}^{2}}{4v_{3M}^{2}}$, $b_1 = 0$, $\cos\theta_1 = -\frac{1}{\sqrt{2}}$, $\cos\theta_3 = 0$, $\sin\theta_1 = -\frac{1}{\sqrt{2}}$, $\sin\theta_3 = -1$;
Solution 17: \[ \sin \theta \]

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-(c_1 + c_2)) - \frac{3\mu_{12}^2}{4} & 0 & 0 \\ 0 & 3v_1^2(c_1 + c_2) + \frac{\mu_{23}^2}{4} & 0 \\ 0 & 0 & v_1^2(-(c_1 + c_2)) - \frac{3\mu_{12}^2}{4} \end{pmatrix}; \]

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \]

- Solution 14: \[ a_4 = -\frac{4\nu^2(c_1+c_2)+3\mu_{12}^2}{4\nu_{SM}^2}, \ b_1 = 0, \ \cos \theta_1 = \frac{1}{\sqrt{2}}, \ \cos \theta_3 = 0, \ \sin \theta_1 = -\frac{1}{\sqrt{2}}; \]

\[ \sin \theta_3 = -1; \]

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-(c_1 + c_2)) - \frac{3\mu_{12}^2}{4} & 0 & 0 \\ 0 & v_1^2(-(c_1 + c_2)) - \frac{3\mu_{12}^2}{4} & 0 \\ 0 & 0 & 3v_1^2(c_1 + c_2) + \frac{\mu_{23}^2}{4} \end{pmatrix}; \]

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \]

- Solution 15: \[ a_4 = -\frac{4\nu^2(c_1+c_2)+3\mu_{12}^2}{4\nu_{SM}^2}, \ b_1 = 0, \ \cos \theta_1 = -\frac{1}{\sqrt{2}}, \ \cos \theta_3 = 0, \ \sin \theta_1 = \frac{1}{\sqrt{2}}; \]

\[ \sin \theta_3 = -1; \]

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-(c_1 + c_2)) - \frac{3\mu_{12}^2}{4} & 0 & 0 \\ 0 & v_1^2(-(c_1 + c_2)) - \frac{3\mu_{12}^2}{4} & 0 \\ 0 & 0 & 3v_1^2(c_1 + c_2) + \frac{\mu_{23}^2}{4} \end{pmatrix}; \]

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \]

- Solution 16: \[ a_4 = -\frac{4\nu^2(c_1+c_2)+3\mu_{12}^2}{4\nu_{SM}^2}, \ b_1 = 0, \ \cos \theta_1 = \frac{1}{\sqrt{2}}, \ \cos \theta_3 = 0, \ \sin \theta_1 = \frac{1}{\sqrt{2}}; \]

\[ \sin \theta_3 = -1; \]

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-(c_1 + c_2)) - \frac{3\mu_{12}^2}{4} & 0 & 0 \\ 0 & 3v_1^2(c_1 + c_2) + \frac{\mu_{23}^2}{4} & 0 \\ 0 & 0 & v_1^2(-(c_1 + c_2)) - \frac{3\mu_{12}^2}{4} \end{pmatrix}; \]

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \]

- Solution 17: \[ a_4 = -\frac{4\nu^2(c_1+c_2)+3\mu_{12}^2}{4\nu_{SM}^2}, \ b_1 = 0, \ \cos \theta_1 = -\frac{1}{\sqrt{2}}, \ \cos \theta_3 = -1, \ \sin \theta_1 = -\frac{1}{\sqrt{2}}; \]

\[ \sin \theta_3 = 0; \]

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-(c_1 + c_2)) - \frac{3\mu_{12}^2}{4} & 0 & 0 \\ 0 & v_1^2(-(c_1 + c_2)) - \frac{3\mu_{12}^2}{4} & 0 \\ 0 & 0 & 3v_1^2(c_1 + c_2) + \frac{\mu_{23}^2}{4} \end{pmatrix}; \]
\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \]

• Solution 18: \( a_4 = -\frac{4\nu_1^2(c_1 + c_2) + 3\mu_2^2}{4\nu_{SM}^2} \), \( b_1 = 0 \), \( \cos \theta_1 = \frac{1}{\sqrt{2}} \), \( \cos \theta_3 = -1 \), \( \sin \theta_1 = -\frac{1}{\sqrt{2}} \), \( \sin \theta_3 = 0 \);

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-c_1 + c_2) - \frac{3\mu_2^2}{4} & 0 & 0 \\ 0 & 3v_1^2(c_1 + c_2) + \frac{\nu_2^2}{4} & 0 \\ 0 & 0 & v_1^2(-c_1 + c_2) - \frac{3\mu_2^2}{4} \end{pmatrix} ; \]

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} . \]

• Solution 19: \( a_4 = -\frac{4\nu_1^2(c_1 + c_2) + 3\mu_2^2}{4\nu_{SM}^2} \), \( b_1 = 0 \), \( \cos \theta_1 = -\frac{1}{\sqrt{2}} \), \( \cos \theta_3 = -1 \), \( \sin \theta_1 = \frac{1}{\sqrt{2}} \), \( \sin \theta_3 = 0 \);

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-c_1 + c_2) - \frac{3\mu_2^2}{4} & 0 & 0 \\ 0 & 3v_1^2(c_1 + c_2) + \frac{\nu_2^2}{4} & 0 \\ 0 & 0 & v_1^2(-c_1 + c_2) - \frac{3\mu_2^2}{4} \end{pmatrix} ; \]

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \]

• Solution 20: \( a_4 = -\frac{4\nu_1^2(c_1 + c_2) + 3\mu_2^2}{4\nu_{SM}^2} \), \( b_1 = 0 \), \( \cos \theta_1 = \frac{1}{\sqrt{2}} \), \( \cos \theta_3 = -1 \), \( \sin \theta_1 = \frac{1}{\sqrt{2}} \), \( \sin \theta_3 = 0 \);

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-c_1 + c_2) - \frac{3\mu_2^2}{4} & 0 & 0 \\ 0 & v_1^2(-c_1 + c_2) - \frac{3\mu_2^2}{4} & 0 \\ 0 & 0 & 3v_1^2(c_1 + c_2) + \frac{\nu_2^2}{4} \end{pmatrix} ; \]

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} . \]

• Solution 21: \( a_4 = -\frac{4\nu_1^2(c_1 + c_2) + 3\mu_2^2}{4\nu_{SM}^2} \), \( b_1 = 0 \), \( \cos \theta_1 = -\frac{1}{\sqrt{2}} \), \( \cos \theta_3 = 1 \), \( \sin \theta_1 = -\frac{1}{\sqrt{2}} \), \( \sin \theta_3 = 0 \);

\[ \text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-c_1 + c_2) - \frac{3\mu_2^2}{4} & 0 & 0 \\ 0 & v_1^2(-c_1 + c_2) - \frac{3\mu_2^2}{4} & 0 \\ 0 & 0 & 3v_1^2(c_1 + c_2) + \frac{\nu_2^2}{4} \end{pmatrix} ; \]

\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} . \]
• Solution 22: $a_4 = -\frac{4v^2(c_1 + c_2) + 3v_{SM}^2}{4v_{SM}^2}$, $b_1 = 0$, $\cos\theta_1 = \frac{1}{\sqrt{2}}$, $\cos\theta_3 = 1$, $\sin\theta_1 = -\frac{1}{\sqrt{2}}$, $\sin\theta_3 = 0$;

$$\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-(c_1 + c_2)) - \frac{3v_{SM}^2}{4} & 0 & 0 \\ 0 & 3v_1^2(c_1 + c_2) + \frac{v_{SM}^2}{2} & 0 \\ 0 & 0 & v_1^2(-(c_1 + c_2)) - \frac{3v_{SM}^2}{4} \end{pmatrix};$$

$$R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 1 \end{pmatrix}.$$

• Solution 23: $a_4 = -\frac{4v^2(c_1 + c_2) + 3v_{SM}^2}{4v_{SM}^2}$, $b_1 = 0$, $\cos\theta_1 = -\frac{1}{\sqrt{2}}$, $\cos\theta_3 = 1$, $\sin\theta_1 = \frac{1}{\sqrt{2}}$, $\sin\theta_3 = 0$;

$$\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-(c_1 + c_2)) - \frac{3v_{SM}^2}{4} & 0 & 0 \\ 0 & 3v_1^2(c_1 + c_2) + \frac{v_{SM}^2}{2} & 0 \\ 0 & 0 & v_1^2(-(c_1 + c_2)) - \frac{3v_{SM}^2}{4} \end{pmatrix};$$

$$R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & 1 \end{pmatrix}.$$

• Solution 24: $a_4 = -\frac{4v^2(c_1 + c_2) + 3v_{SM}^2}{4v_{SM}^2}$, $b_1 = 0$, $\cos\theta_1 = \frac{1}{\sqrt{2}}$, $\cos\theta_3 = 1$, $\sin\theta_1 = \frac{1}{\sqrt{2}}$, $\sin\theta_3 = 0$;

$$\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-(c_1 + c_2)) - \frac{3v_{SM}^2}{4} & 0 & 0 \\ 0 & v_1^2(-(c_1 + c_2)) - \frac{3v_{SM}^2}{4} & 0 \\ 0 & 0 & 3v_1^2(c_1 + c_2) + \frac{v_{SM}^2}{2} \end{pmatrix};$$

$$R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}.$$

• Solution 25: $a_4 = -\frac{4v^2(c_1 + c_2) + 3v_{SM}^2}{4v_{SM}^2}$, $b_1 = 0$, $\cos\theta_1 = -\frac{1}{\sqrt{2}}$, $\cos\theta_3 = 0$, $\sin\theta_1 = -\frac{1}{\sqrt{2}}$, $\sin\theta_3 = 1$;

$$\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} v_1^2(-(c_1 + c_2)) - \frac{3v_{SM}^2}{4} & 0 & 0 \\ 0 & 3v_1^2(c_1 + c_2) + \frac{v_{SM}^2}{2} & 0 \\ 0 & 0 & v_1^2(-(c_1 + c_2)) - \frac{3v_{SM}^2}{4} \end{pmatrix};$$

$$R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 1 \end{pmatrix}.$$

• Solution 26: $a_4 = -\frac{4v^2(c_1 + c_2) + 3v_{SM}^2}{4v_{SM}^2}$, $b_1 = 0$, $\cos\theta_1 = \frac{1}{\sqrt{2}}$, $\cos\theta_3 = 0$, $\sin\theta_1 = -\frac{1}{\sqrt{2}}$, $\sin\theta_3 = 1$;
\[
\text{Solution 27: } a_4 = -\frac{4\nu_1^2(c_1 + c_2) + 3\mu_1^2}{4v_{SM}^2}, \quad b_1 = 0, \quad \cos\theta_1 = -\frac{1}{\sqrt{2}}, \quad \cos\theta_3 = 0, \quad \sin\theta_1 = \frac{1}{\sqrt{2}}, \quad \sin\theta_3 = 1; \\
\begin{align*}
\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) &= \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
\end{pmatrix}.
\end{align*}
\]

\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\end{pmatrix}.
\]

\[
\text{Solution 28: } a_4 = -\frac{4\nu_1^2(c_1 + c_2) + 3\mu_1^2}{4v_{SM}^2}, \quad b_1 = 0, \quad \cos\theta_1 = \frac{1}{\sqrt{2}}, \quad \cos\theta_3 = 0, \quad \sin\theta_1 = \frac{1}{\sqrt{2}}, \quad \sin\theta_3 = 1; \\
\begin{align*}
\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) &= \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
\end{pmatrix}.
\end{align*}
\]

\[
R_{E2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\end{pmatrix}.
\]

\[
\text{Solution 29: } a_4 = \frac{2\nu_1^2(c_1 + c_2)}{v_{SM}^2}, \quad b_1 = 0, \quad \cos\theta_3 = -\sqrt{\frac{1}{2} - \cos\theta_1 \sqrt{1 - \cos\theta_1}}, \quad \sin\theta_1 = \sqrt{1 - \cos\theta_1}, \quad \sin\theta_3 = \frac{2\cos\theta_1 - 1}{\sqrt{2-4\cos\theta_1 \sqrt{1 - \cos\theta_1}}}, \quad \mu_1^2 = -4\nu_1^2(c_1 + c_2); \\
\begin{align*}
\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) &= \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
\end{pmatrix}.
\end{align*}
\]

\[
R_{E2} = \begin{pmatrix}
\frac{\sqrt{1 - \cos\theta_1} - \cos\theta_1}{\sqrt{2 - 4\cos\theta_1 \sqrt{1 - \cos\theta_1}}}, & \frac{\cos\theta_1 - \sqrt{1 - \cos\theta_1}}{\sqrt{2 - 4\cos\theta_1 \sqrt{1 - \cos\theta_1}}} \\
\frac{\sqrt{1 - \cos\theta_1} - \cos\theta_1}{\sqrt{2 - 4\cos\theta_1 \sqrt{1 - \cos\theta_1}}}, & \frac{\cos\theta_1 - \sqrt{1 - \cos\theta_1}}{\sqrt{2 - 4\cos\theta_1 \sqrt{1 - \cos\theta_1}}} \\
\frac{\sqrt{1 - \cos\theta_1} - \cos\theta_1}{\sqrt{2 - 4\cos\theta_1 \sqrt{1 - \cos\theta_1}}}, & \frac{\cos\theta_1 - \sqrt{1 - \cos\theta_1}}{\sqrt{2 - 4\cos\theta_1 \sqrt{1 - \cos\theta_1}}} \\
\end{pmatrix}.
\]

\[
\text{Solution 30: } a_4 = \frac{2\nu_1^2(c_1 + c_2)}{v_{SM}^2}, \quad b_1 = 0, \quad \cos\theta_3 = \sqrt{\frac{1}{2} - \cos\theta_1 \sqrt{1 - \cos\theta_1}}, \quad \sin\theta_1 = -\sqrt{1 - \cos\theta_1}, \quad \sin\theta_3 = \frac{2\cos\theta_1 - 1}{\sqrt{2-4\cos\theta_1 \sqrt{1 - \cos\theta_1}}}, \quad \mu_1^2 = -4\nu_1^2(c_1 + c_2); \\
\begin{align*}
\text{diag}(m_H^2, m_{H_1}^2, m_{H_2}^2) &= \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
\end{pmatrix}.
\end{align*}
\]
\[
\text{Solution 31: } a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}}, b_1 = 0, \cos \theta_3 = -\sqrt{\frac{1}{2} - \cos \theta_1 \sqrt{1 - \cos \theta_1^2}}, \sin \theta_1 = -\sqrt{1 - \cos \theta_1^2}, \sin \theta_3 = \frac{1 - 2\cos \theta_1^2}{\sqrt{2 - 4\cos \theta_1 \sqrt{1 - \cos \theta_1^2}}}, \mu_{12}^2 = -4v_1^2(c_1+c_2);
\]
\[
\text{Solution 32: } a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}}, b_1 = 0, \cos \theta_3 = \sqrt{\frac{1}{2} - \cos \theta_1 \sqrt{1 - \cos \theta_1^2}}, \sin \theta_1 = \sqrt{1 - \cos \theta_1^2}, \sin \theta_3 = \frac{1 - 2\cos \theta_1^2}{\sqrt{2 - 4\cos \theta_1 \sqrt{1 - \cos \theta_1^2}}}, \mu_{12}^2 = -4v_1^2(c_1+c_2);
\]
\[
\text{Solution 33: } a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}}, b_1 = 0, \cos \theta_3 = -\sqrt{\frac{1}{2} - \cos \theta_1 \sqrt{1 - \cos \theta_1^2} + \frac{1}{2}}, \sin \theta_1 = \sqrt{1 - \cos \theta_1^2}, \sin \theta_3 = \frac{1 - 2\cos \theta_1^2}{\sqrt{4(1 - \cos \theta_1^2) \cos \theta_1^2 + 2}}, \mu_{12}^2 = -4v_1^2(c_1+c_2);
\]
\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{1 - \cos^2 \theta_i - \cos \theta_1} & -\sqrt{1 - \cos^2 \theta_i - \cos \theta_1} \\ 0 & \sqrt{4 \sqrt{1 - \cos^2 \theta_i} \cos \theta_1 + 2} & \sqrt{4 \sqrt{1 - \cos^2 \theta_i} \cos \theta_1 + 2} \\ 0 & \sqrt{4 \sqrt{1 - \cos^2 \theta_i} \cos \theta_1 + 2} & -\sqrt{1 - \cos^2 \theta_i - \cos \theta_1} \end{pmatrix}. \]

- Solution 34: \( a_4 = \frac{2v^2_i(c_1 + c_2)}{v^2_M}, b_1 = 0, \cos \theta_3 = \sqrt{1 - \cos^2 \theta_i \cos \theta_1 + \frac{1}{2}}, \sin \theta_1 = -\sqrt{1 - \cos^2 \theta_i}, \sin \theta_3 = \frac{1 - \cos \theta_i^2}{\sqrt{4 \sqrt{1 - \cos^2 \theta_i} \cos \theta_1 + 2}}, \mu^2_{12} = -4v^2_i(c_1 + c_2); \)

\[
\text{diag}(m_{H_i}^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v^2_i(c_1 + c_2) & 0 & 0 \\ 0 & 2v^2_i(c_1 + c_2) & 0 \\ 0 & 0 & 2v^2_i(c_1 + c_2) \end{pmatrix};
\]

- Solution 35: \( a_4 = \frac{2v^2_i(c_1 + c_2)}{v^2_M}, b_1 = 0, \cos \theta_3 = -\sqrt{1 - \cos^2 \theta_i \cos \theta_1 + \frac{1}{2}}, \sin \theta_1 = -\sqrt{1 - \cos^2 \theta_i}, \sin \theta_3 = \frac{2 \cos \theta_i^2 - 1}{\sqrt{4 \sqrt{1 - \cos^2 \theta_i} \cos \theta_1 + 2}}, \mu^2_{12} = -4v^2_i(c_1 + c_2); \)

\[
\text{diag}(m_{H_i}^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v^2_i(c_1 + c_2) & 0 & 0 \\ 0 & 2v^2_i(c_1 + c_2) & 0 \\ 0 & 0 & 2v^2_i(c_1 + c_2) \end{pmatrix};
\]

- Solution 36: \( a_4 = \frac{2v^2_i(c_1 + c_2)}{v^2_M}, b_1 = 0, \cos \theta_3 = \sqrt{1 - \cos^2 \theta_i \cos \theta_1 + \frac{1}{2}}, \sin \theta_1 = \sqrt{1 - \cos^2 \theta_i}, \sin \theta_3 = \frac{2 \cos \theta_i^2 - 1}{\sqrt{4 \sqrt{1 - \cos^2 \theta_i} \cos \theta_1 + 2}}, \mu^2_{12} = -4v^2_i(c_1 + c_2); \)

\[
\text{diag}(m_{H_i}^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v^2_i(c_1 + c_2) & 0 & 0 \\ 0 & 2v^2_i(c_1 + c_2) & 0 \\ 0 & 0 & 2v^2_i(c_1 + c_2) \end{pmatrix};
\]
• Solution 37: \( a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2} \), \( b_1 = 0 \), \( \cos \theta_1 = -\frac{1}{\sqrt{2}} \), \( \cos \theta_3 = 0 \), \( \sin \theta_1 = -\frac{1}{\sqrt{2}} \), \( \sin \theta_3 = -1 \), \( \mu_{12}^2 = -4v_1^2(c_1 + c_2) \);

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.
\]

• Solution 38: \( a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2} \), \( b_1 = 0 \), \( \cos \theta_1 = \frac{1}{\sqrt{2}} \), \( \cos \theta_3 = 0 \), \( \sin \theta_1 = -\frac{1}{\sqrt{2}} \), \( \sin \theta_3 = -1 \), \( \mu_{12}^2 = -4v_1^2(c_1 + c_2) \);

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.
\]

• Solution 39: \( a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2} \), \( b_1 = 0 \), \( \cos \theta_1 = -\frac{1}{\sqrt{2}} \), \( \cos \theta_3 = 0 \), \( \sin \theta_1 = \frac{1}{\sqrt{2}} \), \( \sin \theta_3 = -1 \), \( \mu_{12}^2 = -4v_1^2(c_1 + c_2) \);

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.
\]

• Solution 40: \( a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2} \), \( b_1 = 0 \), \( \cos \theta_1 = \frac{1}{\sqrt{2}} \), \( \cos \theta_3 = 0 \), \( \sin \theta_1 = \frac{1}{\sqrt{2}} \), \( \sin \theta_3 = -1 \), \( \mu_{12}^2 = -4v_1^2(c_1 + c_2) \);

\[
diag(m_H^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix};
\]

\[
R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.
\]

• Solution 41: \( a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2} \), \( b_1 = 0 \), \( \cos \theta_1 = -\frac{1}{\sqrt{2}} \), \( \cos \theta_3 = -1 \), \( \sin \theta_1 = -\frac{1}{\sqrt{2}} \), \( \sin \theta_3 = 0 \), \( \mu_{12}^2 = -4v_1^2(c_1 + c_2) \);
Solution 42: $a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{3M}^2}$, $b_1 = 0$, $\cos\theta_1 = \frac{1}{\sqrt{2}}$, $\cos\theta_3 = -1$, $\sin\theta_1 = -\frac{1}{\sqrt{2}}$, $\sin\theta_3 = 0$, $\mu_{12}^2 = -4v_1^2(c_1 + c_2)$;

$$\text{diag}(m_{H}^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix};$$

$$R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$
\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \]

- Solution 46: \[ a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}^2}, \quad b_1 = 0, \quad \cos\theta_1 = \frac{1}{\sqrt{2}}, \quad \cos\theta_3 = 1, \quad \sin\theta_1 = -\frac{1}{\sqrt{2}}, \quad \sin\theta_3 = 0, \]
\[ \mu_{12}^2 = -4v_1^2(c_1+c_2); \]
\[ \text{diag}(m_{H}^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1+c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1+c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1+c_2) \end{pmatrix}; \]
\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \]

- Solution 47: \[ a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}^2}, \quad b_1 = 0, \quad \cos\theta_1 = -\frac{1}{\sqrt{2}}, \quad \cos\theta_3 = 1, \quad \sin\theta_1 = \frac{1}{\sqrt{2}}, \quad \sin\theta_3 = 0, \]
\[ \mu_{12}^2 = -4v_1^2(c_1+c_2); \]
\[ \text{diag}(m_{H}^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1+c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1+c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1+c_2) \end{pmatrix}; \]
\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \]

- Solution 48: \[ a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}^2}, \quad b_1 = 0, \quad \cos\theta_1 = \frac{1}{\sqrt{2}}, \quad \cos\theta_3 = 1, \quad \sin\theta_1 = \frac{1}{\sqrt{2}}, \quad \sin\theta_3 = 0, \]
\[ \mu_{12}^2 = -4v_1^2(c_1+c_2); \]
\[ \text{diag}(m_{H}^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1+c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1+c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1+c_2) \end{pmatrix}; \]
\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \]

- Solution 49: \[ a_4 = \frac{2v_1^2(c_1+c_2)}{v_{SM}^2}, \quad b_1 = 0, \quad \cos\theta_1 = -\frac{1}{\sqrt{2}}, \quad \cos\theta_3 = 0, \quad \sin\theta_1 = -\frac{1}{\sqrt{2}}, \quad \sin\theta_3 = 1, \quad \mu_{12}^2 = -4v_1^2(c_1+c_2); \]
\[ \text{diag}(m_{H}^2, m_{H_1}^2, m_{H_2}^2) = \begin{pmatrix} 2v_1^2(c_1+c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1+c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1+c_2) \end{pmatrix}; \]
\[ R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \]
• Solution 50: \(a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2}\), \(b_1 = 0\), \(\cos\theta_1 = \frac{1}{\sqrt{2}}\), \(\cos\theta_3 = 0\), \(\sin\theta_1 = -\frac{1}{\sqrt{2}}\), \(\sin\theta_3 = 1\), \(\mu_{12}^2 = -4v_1^2(c_1 + c_2)\);

\[\text{diag}(m_H^2, m_{H_0}^2, m_{H_0}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix};\]

\[R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.\]

• Solution 51: \(a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2}\), \(b_1 = 0\), \(\cos\theta_1 = -\frac{1}{\sqrt{2}}\), \(\cos\theta_3 = 0\), \(\sin\theta_1 = \frac{1}{\sqrt{2}}\), \(\sin\theta_3 = 1\), \(\mu_{12}^2 = -4v_1^2(c_1 + c_2)\);

\[\text{diag}(m_H^2, m_{H_0}^2, m_{H_0}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix};\]

\[R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.\]

• Solution 52: \(a_4 = \frac{2v_1^2(c_1 + c_2)}{v_{SM}^2}\), \(b_1 = 0\), \(\cos\theta_1 = \frac{1}{\sqrt{2}}\), \(\cos\theta_3 = 0\), \(\sin\theta_1 = \frac{1}{\sqrt{2}}\), \(\sin\theta_3 = 1\), \(\mu_{12}^2 = -4v_1^2(c_1 + c_2)\);

\[\text{diag}(m_H^2, m_{H_1}^2, m_{H_0}^2) = \begin{pmatrix} 2v_1^2(c_1 + c_2) & 0 & 0 \\ 0 & 2v_1^2(c_1 + c_2) & 0 \\ 0 & 0 & 2v_1^2(c_1 + c_2) \end{pmatrix};\]

\[R_{E2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.\]

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