On Relaxation of Dominant Sets

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Abstract

In a graph $G = (V, E)$, a k-ruling set $S$ is one in which all vertices $V \setminus S$ are at most $k$ distance from $S$. Finding a minimum k-ruling set is intrinsically linked to the minimum dominating set problem and maximal independent set problem, which have been extensively studied in graph theory. This paper presents the first known algorithm for solving all k-ruling set problems in conjunction with known minimum dominating set algorithms at only additional polynomial time cost compared to a minimum dominating set. The algorithm further succeeds for $(\alpha, \alpha - 1)$ ruling sets in which $\alpha > 1$, for which constraints exist on the proximity of vertices $v \in S$. This secondary application instead works in conjunction with maximal independent set algorithms.

1 Introduction

A set $S \subseteq V$ of a graph $G = (V, E)$ is considered a ruling set if every vertex $v \in V \setminus S$ is maximum distance $k$ from $S$. A dominating set, $D \subseteq V$ where every vertex $v \in V \setminus D$ is adjacent to a vertex of $D$, is thus a simplification of ruling sets where distance $k = 1$. Problems surrounding dominating sets were first discussed in the 1950s and have since only increased in importance. An essential question in domination theory is that of finding the minimum dominating set of a n-vertex graph, which was first calculated in $O(2^n n)$ time. Recent literature from Fomin et al. [FGKD09] has reduced time complexity to $O(1.5137^n)$. Though ample attention has been paid to the minimum dominating set problem, there is yet to be a proposed algorithm to calculate minimum ruling sets beyond the dominating set (i.e. $k = 1$ instance). We will focus on calculating minimum ruling sets regardless on distance $k$ imposed on said graphs.

The field of domination theory, and by extension ruling sets, has seen increased focus within graph theory. The topic has many applications, the most literal of which is likely the facility location problem. The problem seeks out the most optimal location
for emergency services among people/towns, while also dealing with a constraint of the number of emergency service locations. Other applications include ad-hoc computer communication networks.

All current contributions to more generalized ruling set problems neglect basic views of the problem and instead focus on independent ruling sets. An independent set is a set $I \subseteq V$ of a Graph $G = (V,E)$ such that every vertex $v \in I$ is not adjacent to any other vertices in $I$. A common problem connecting dominance and independence among graphs is that of finding a maximal independent set (MIS) such that the MIS is not a subset of any other independent sets in $G$. Informally, a MIS can not have any more vertices added to it without becoming not independent. Though not all dominating sets are independent, all maximal independent sets are dominant. Independent ruling sets can be defined as a ruling set $S \subseteq V$ upon which the distance between vertices in $V \setminus S$ is at most $k$ and all vertices in $S$ are independent. Thus, independent ruling sets are analogously a subset of normal ruling sets. Discussions of ruling sets are mostly limited to independent ruling sets; Balliu et al. [BBO20] solely focused on ruling sets in which distance between vertices in $S$ is at minimum $\alpha$ for $(\alpha, \beta)$-ruling sets. (The distance between any two vertices in $S$ is at least $\alpha$, and all vertices in $V \setminus S$ are at most $\beta$ distance from $S$.) Even more recent literature focuses more specifically on independent ruling sets in which $k = 2$ or $k = 3$ [KP12], [PP22].

We present an algorithm that transforms $k$-ruling set problems for graphs into a minimum dominating set problem through modification of our original graph. Known minimum dominating set algorithms can then be applied to the new graph to find a set of vertices which serves as a $k$-ruling set on our original graph. Our algorithm has an upper bound of $O(n^3)$ where $n$ is the number of vertices in the graph. The NP-hard status of minimum dominating set algorithms means our polynomial time complexity has little impact on speed of overall calculations of $k$-ruling set problems. Our algorithm can further be applied to specific instances of $(\alpha, \beta)$ ruling sets: particularly for when the problem is represented as an $(\alpha, \alpha - 1)$ ruling set problem, where $\alpha > 1$. In such cases, MIS algorithms are then applied to find the correct ruling set.

2 Preliminaries

Let $G = (V,E)$ be an undirected and simple graph. The distance between two vertices in a graph is the edges present within the shortest path between the two vertices. For every vertex $v \in V$ we denote the set of adjacent vertices $N(v)$, or the neighborhood of $v$ and $N[v] = N(v) \cup \{v\}$. A clique is a set $S \subseteq V$ of pairwise adjacent vertices. A complete graph is a graph with every possible edge, or a clique imposed on the entire graph.

The notation of ruling sets in past literature is not clearly established and often changes from paper to paper. The frequent discussion of independent ruling sets result in false equivalencies created between independent ruling sets and all ruling sets, where the two terms are used interchangeably. Other definitions, as in Balliu et al. [BBO20], define an $(\alpha, \beta)$ ruling set such that the distance between vertices in $V \setminus S$ is at most $\beta$ and every vertex in $S$ is at minimum $\alpha$ distance from every other vertex in $S$. We will mostly avoid this definition as our paper does not focus on independence, but return to it in later portions when discussing applications of our algorithm in related fields.
Figure 1: Displays the construction of additional edges for a graph G to convert its
2-ruling set into a minimum dominant set problem.

We will seek to establish a clear definition for ruling sets, one that is quite similar to
Gfeller and Vicari [GV07]'s definition.

Definition 1 Given a graph $G = (V, E)$, a $k$-ruling set is a subset $S \subseteq V$ such that
every vertex in $V \setminus S$ is at most distance $k$ from $S$.

3 Simplifying K-ruling sets to Dominating sets

Rather than view the minimum dominant set problem as a specific variant of k-
ruuling sets, we seek to instead view every instance of k-ruling sets as a relaxation of a
minimum dominant set problem. Consider the relation between a 2-ruling set and an
minimum dominant set.

Consider a graph with a 2-ruling set. A vertex $v \in S$ in a 2-ruling set effectively
rules every $u \in N(v)$, which we will denote as $U$, as well as every vertex $w \in N(N(u))$
\setminus v, which we will denote as $W$. Ignoring overlap between $U$ and $W$ (if a vertex is in
$U$, it is removed from $W$), then: $U$ is the set of vertices of distance 1 from $v$ and $W$ is
a set of all vertices of distance 2 from $v$. Set $S$ dominates $U \cup W$.

Similarly consider a dominant set (1-ruling) within a graph. A vertex $v \in S$ in a
minimum dominant set similarly rules every vertex in $U$ (all vertices distance 1 away).
Thus, to re-configure the 2-ruling set $S$ into a minimum dominant set, we can add an
edge between $v$ and every vertex $w \in W$. Set $S$ would then dominate all vertices of
distance 1 away, which is identically $W \cup U$ (even though we have decreased $k$ by 1).

Now, let us consider the instance in which the vertices $v \in S$ are currently unknown
for a 2-ruling set. We can simply construct edges from every vertex $v \in V$ to vertices
$u \in N(N(v))$. There now exist edges between vertices two degrees from each other, so
finding a solution to the minimum dominant set problem would find a solution to the
2-ruling set problem on the original graph we began with. An example can be seen at
Figure 1.

The simplification of the 2-ruling set to a minimum dominant set problem is not
the only possible simplification to a minimum dominant set. For each k-ruling set, an
additional round of adding edges between neighbors of neighbors of neighbors etc. is
required. Thus, each k-ruling set requires k-1 rounds of iteration to reach a minimum
dominant set. Our analysis leads to the following algorithm:

3.1 The Algorithm

In this section, we define the algorithm for transforming a k-ruling problem into
a minimum dominating set problem. We will give a proof of correctness and further
analysis shortly after.
Algorithm 1 reducekGraph

Input: G = (V,E); k = c
Output: G = (V,E); k = 1

1: i ← k
2: E′ ← E
3: while i > 1 do
4:     E″ ← E′
5:     for all (u,v) ∈ E′ do
6:         for all w s.t (v,w) ∈ E′ do
7:             E″ ← E′ ∪ (u,w)
8:         end for
9:     end for
10: if E″ = E′ then 
11:     i ← 1
12: else 
13:     i ← i − 1
14:     E′ ← E″
15: end if
16: end while
17: G′ ← (V, E′)
18: return G′

3.2 Correctness and Analysis

Suppose our original graph has become complete after n iterations. Then, no more edges will be added to set E′; thus, iterating through all edges would be a waste of time for the remaining (k − 1) − n iterations. Thus, once E″ = E′, the algorithm has terminated and we can return our new graph.

Theorem 1 Any given vertex v would, if chosen to be added to the dominating set, dominate the same set of vertices both before and after completion of the algorithm.

Proof We prove this theorem by induction on n, where n is the number of iterations of the while loop. Exclusively consider n > 0; n = 0 is trivial. We prove the theorem by showing it is not only true at its end but that it holds throughout all iterations. When n = 1 any vertices v, w s.t the distance between v, is less than or equal to 2, will have an edge between them; k similarly decreases to k − 1. As discussed in the simple case of 2-ruling set to 1-ruling set above, these graphs would have the same ruling set.

Let us prove P(n) implies P(n + 1). Immediately apply the inductive hypothesis; thus, every every vertex within distance n of each other have an edge between them and k := k − n. An additional iteration would create an additional edge between vertex and any other vertex of distance 2 away from it. Thus, every vertex within n + 1 distance of the original graph is now adjacent and k := k − (n + 1). When n = k − 1 the algorithm will terminate and every vertex within k distance of each other will now have an edge between them.

Theorem 2 The algorithm reducekGraph solves our problem in \( O\left(\frac{k-n(n-1)^2}{2}\right) \) worst-case time.
The maximum number of edges is \( \frac{n(n-1)}{2} \) upon which a complete clique graph occurs. This results in a time complexity of \( O\left(\frac{n(n-1)^2}{2}\right) \) per round. This round will only occur once because the algorithm will then terminate. Thus, the entire algorithm has an upper bound of \( O\left(\frac{kn(n-1)^2}{2}\right) \) for the entire algorithm with constant \( k \).

### 3.3 Independent Sets

Let us consider the applications of the algorithm to independent set problems utilizing \((\alpha, \beta)\) notation [BBO20]. Consider a \((2, 2)\) ruling set; each vertex in set \( S \) is minimum distance 2 from each other. Thus, the ruling set is independent. Edges would be produced between every vertex within distance 2 of each other by algorithm \( reducekGraph \). Following the algorithm’s application, running an MIS algorithm, such as Luby’s Algorithm [Lub86], would prevent vertices which were not initially adjacent to each other from joining the independent set due to the construction of new edges. Similarly, applying a minimum dominating set algorithm would not work as vertices adjacent in the original graph may be added together, and our solution set would no longer be independent. Thus, for many \((\alpha, \beta)\) ruling sets where \( \alpha \neq 1 \), our algorithm will not work.

**Theorem 3** The algorithm \( reducekGraph \) succeeds in all \((\alpha, \alpha - 1)\) ruling set problems in conjunction with MIS algorithms for \( \alpha > 1 \).

**Proof** Consider an \((\alpha, \alpha - 1)\) ruling set. Following termination of \( reducekGraph \), all vertices of distance \( \alpha - 1 \) will be adjacent to each other. Then, applying a MIS algorithm will ensure no adjacent vertices join the independent set together, so only vertices of distance \( \alpha \) or greater from each other on the original graph will possibly join the ruling set, as required. So, if \( \beta = \alpha - 1 \), the ruling set can be found using our algorithm.

Consider if \( \beta < \alpha - 1 \). Like our example above of a \((2,2)\) ruling set, the termination of the \( reducekGraph \) algorithm will result in adjacency of all vertices within distance \( \alpha - 1 \), but our ruling set allows for the inclusion of more vertices; that is, those of \( \beta \) distance from each other. This does not occur, excluding said vertices from joining the ruling set together.

Also consider if \( \beta > \alpha - 1 \). Once again, termination the \( reducekGraph \) algorithm will result in adjacency of all vertices of within distance \( \alpha - 1 \), but we require a minimum distance of \( \beta \) between two vertices in the ruling set; thus, running an MIS would possibly include vertices that are not at minimum \( \beta \) apart and thus do not belong in the solution set.

### 4 Future Work

A large number of open questions remain within k-ruling set problems. A natural one is that of whether the creation of an efficient algorithm exists for all \((\alpha, \beta)\) ruling sets, not only those in which \( \beta = \alpha - 1 \). One may also wonder whether one could work backwards from dominating set problems to ruling set ones instead of traversing in the opposite direction as we have done. One may consider the synthesis of our universal algorithm with the examination of specific ruling sets by other authors like
[BBO20], [PP22], [KP12]. More generalized possibilities include expanding this paper’s crux—that of simplifying an unknown problem to a known one—into other subtopics of graph theory.
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