The Limiting Distribution of the Trace of a Random Plane Partition

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Abstract

We study the asymptotic behaviour of the trace (the sum of the diagonal parts) \( \tau_n = \tau_n(\omega) \) of a plane partition \( \omega \) of the positive integer \( n \), assuming that \( \omega \) is chosen uniformly at random from the set of all such partitions. We prove that \( (\tau_n - c_0 n^{2/3})/c_1 n^{1/3} \log^{1/2} n \) converges weakly, as \( n \to \infty \), to the standard normal distribution, where \( c_0 = \zeta(2)/[2 \zeta(3)]^{2/3} \), \( c_1 = \sqrt{1/3/[2 \zeta(3)]^{1/3}} \) and \( \zeta(s) = \sum_{j=1}^{\infty} j^{-s} \).

1 Introduction

Properties of various kinds of partitions are often studied using bivariate generating functions of the following type:

\[
G(u, x; \{a_j\}_{j \geq 1}) = \prod_{j=1}^{\infty} (1 - ux^j)^{-a_j} = 1 + \sum_{n,m \geq 1} Q(m, n; \{a_j\}_{j \geq 1}) u^m x^n. \tag{1.1}
\]

Here \( u \) is finite, \( |x| < 1 \) and \( \{a_j\}_{j \geq 1} \) is a given sequence of non-negative numbers. A combinatorial interpretation of (1.1) for integer-valued sequences \( \{a_j\}_{j \geq 1} \), is
obtained expanding the geometric progressions in the left-hand side. Setting
\[ \Lambda = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{a_k} \{k\} = \{\lambda_1, \lambda_2, \ldots\}, \]
one can show that the coefficient \( Q(m, n; \{a_j\}_{j \geq 1}) \) equals the number of solutions in non-negative integers \( z_j \) of the equations
\[
\begin{align*}
\sum_{\lambda_j \in \Lambda} \lambda_j z_j &= n, \\
\sum_j z_j &= m.
\end{align*}
\]
Furthermore, a specific sequence of integers \( \{a_j\}_{j \geq 1} \) may define a particular class of partitions, so that the \( n \)th coefficient \( Q(n; \{a_j\}_{j \geq 1}) \) in the power series expansion of
\[
G(1, x; \{a_j\}_{j \geq 1}) = \prod_{j=1}^{\infty} (1 - x^j)^{-a_j} = 1 + \sum_{n \geq 1} Q(n; \{a_j\}_{j \geq 1}) x^n \tag{1.2}
\]
counts the number of partitions of \( n \) belonging to that class (see e.g. [4, Chap. 6]). A fundamental problem in the theory of partitions is to determine the asymptotic behaviour of \( Q(n; \{a_j\}_{j \geq 1}) \) as \( n \to \infty \). In its most general setting this problem was studied by Meinardus [14] who introduced a set of assumptions on the sequence \( \{a_j\}_{j \geq 1} \) and obtained an expression for the leading term in the asymptotic expansion of this coefficient. Two important classes of partitions are covered by Meinardus’ formula: (i) \( a_j = 1 \) and (ii) \( a_j = j, j = 1, 2, \ldots \). In the first case \( Q(n; \{a_j = 1\}_{j \geq 1}) \) equals the number of (linear) integer partitions of \( n \) (see [4, Section 1.2]) and Meinardus’ asymptotic formula implies the famous result of Hardy and Ramanujan [8] for the number of such partitions. In the second case \( Q(n; \{a_j = j\}_{j \geq 1}) \) equals the number of plane partitions of \( n \), whose asymptotic expression was previously obtained by Wright [27].

A basic restriction in Meinardus’ scheme states that the Dirichlet’s series
\[
D(s) = \sum_{j=1}^{\infty} a_j j^{-s}, \tag{1.3}
\]
generated by the sequence \( \{a_j\}_{j \geq 1} \) has to converge in the half-plane \( \Re s > \alpha > 0 \). It turns out that the asymptotic behaviour of the coefficients \( Q(m, n; \{a_j\}_{j \geq 1}) \) in the bivariate power series expansion [10] strongly depends on the value of the parameter \( \alpha \). Haselgrove and Temperley [9] studied the case when \( \alpha < 2 \), \( n \to \infty \) and \( m \) becomes large with \( n \) at a specific rate. They applied the classical method due to Hardy and Ramanujan [8] and obtained a result of the form of a local limit theorem for the ratio \( Q(m, n; \{a_j\}_{j \geq 1})/Q(n; \{a_j\}_{j \geq 1}) \). Their result established convergence to a non-Gaussian distribution. Haselgrove and Temperley [9, Section 3] also conjectured that the Gaussian law would appear
if the Dirichlet’s series parameter $\alpha$ is not less than 2, however, a formal proof is still lacking.

In this paper we consider the sequence $a_j = j, j = 1, 2, \ldots$, and focused on plane partitions of positive integer $n$. A plane partition $\omega$ of $n$ is a representation

$$n = \sum_{i,j \geq 1} \omega_{i,j},$$

in which the array $\omega = (\omega_{i,j})_{i,j \geq 1}$ of non-negative integer entries is such that $\omega_{i,j} \geq \omega_{i+1,j}$ and $\omega_{i,j} \geq \omega_{i,j+1}$. We may also assume that $\omega$ occupies the first quadrant of the coordinate system $iOj$. As an illustration the following is the plane partition of $n = 8$:

$$(1.4)$$

For the sake of brevity the zeroes in (1.4) are deleted, so that the abbreviation

$$1 \quad 1 \quad 3 \quad 2 \quad 1$$

presents the plane partition of 8 in a shorter way. It seems that MacMahon [13] was the first who suggested the idea of a plane partition. Various properties of plane partitions and their applications to combinatorics, algebra and analysis of algorithms may be found in [4, Chap. 11], [17, Chap. 12] and [22, Chap. 7].

Further, for the sake of brevity, we also let

$$Q(n) = Q(n; \{a_j = j\}_{j \geq 1}), \quad G(u, x) = \prod_{j=1}^{\infty} (1 - ux^j)^{-j}. \quad (1.5)$$

In terms of notations (1.5) the generating function (1.2) becomes

$$G(1, x) = \prod_{j=1}^{\infty} (1 - x^j)^{-j} = 1 + \sum_{n \geq 1} Q(n)x^n. \quad (1.6)$$

It is also easily seen that the sequence $a_j = j, j = 1, 2, \ldots$, implies that

$$D(s) = \zeta(s - 1) = \sum_{j=1}^{\infty} j^{-(s-1)} \quad (1.7)$$

(see [13]). Therefore, in this case we have $\alpha = 2$. 
The main goal of our paper is to prove Haselgrove and Temperley’s conjecture [9, Section 3] that the Gaussian law determines the asymptotic behaviour of \( Q(m,n) = Q(m,n; \{a_j = j \}_{j \geq 1}) \) if \( m \) becomes large with \( n \) at a specific rate.

It turns out that this problem has an interesting probabilistic interpretation. If we introduce the uniform probability measure \( P = P_n \) on the set of all plane partitions of \( n \) assuming that the probability \( 1/Q(n) \) is assigned to each plane partition \( \omega \), then each conceivable numerical characteristic of \( \omega \) becomes a random variable. An exact combinatorial expression for the numbers \( Q(n) \) does not exist. As it was previously mentioned their asymptotic was determined by Wright [27] and subsequently his result was confirmed by Meinardus’ general theorem [14]. It was shown that

\[
Q(n) \sim \frac{[\zeta(3)]^{7/36}}{2^{11/36}3^{1/2} \pi^{1/2}} n^{-25/36} \exp \left\{ 3[\zeta(3)]^{1/3} (n/2)^{2/3} + 2c \right\}, \quad (1.8)
\]

where

\[
c = \int_0^\infty \frac{y \log y}{e^{2\pi y} - 1} dy.
\]

(In the statement of Wright’s main result [27, p.179] the constant \( 3^{1/2} \) in the denominator of (1.8) is missing, however, it is included in his final result at the end of the proof of his theorem on p.189. Further results on the asymptotic expansion of \( Q(n) \) can be also found in [2, 3].)

We will consider here the trace \( \tau_n \) of a partition \( \omega \) defined as the sum of its diagonal parts:

\[
\tau_n = \sum_{j=1}^n \omega_{j,j}.
\]

To study the asymptotic behaviour of \( \tau_n \) as \( n \to \infty \) we use the following generating function identity (see [21] or [4, Chap. 11, Problem 5]):

\[
G(u,x) = 1 + \sum_{n=1}^\infty Q(n) x^n \sum_{m=1}^n P(\tau_n = m) u^m = 1 + \sum_{n=1}^\infty Q(n) \varphi_n(u) x^n = \prod_{j=1}^\infty (1 - ux^j)^{-j}, \quad (1.9)
\]

where

\[
\varphi_n(u) = \sum_{m=1}^n P(\tau_n = m) u^m \quad (1.10)
\]

is the probability generating function of the trace \( \tau_n \) and \( G(u,x) \) is the generating function defined by (1.5). It is now clear that if Haselgrove and Temperley’s conjecture, that we stated above, is valid, then the trace \( \tau_n \) of a random plane partition has to be asymptotically normal as \( n \to \infty \). The main result of this paper is the following limit theorem.
Theorem 1 For any real and finite $z$, we have

$$\lim_{n \to \infty} P \left( \frac{\tau_n - c_0 n^{2/3}}{c_1 n^{1/3} \log^{1/2} n} \leq z \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy,$$

where

$$c_0 = \frac{\zeta(2)}{[2\zeta(3)]^{2/3}} = .916597104,$$

$$c_1 = \frac{\sqrt{1/3}}{[2\zeta(3)]^{1/3}} = .430977269.$$

This result confirms Haselgrove and Temperley’s conjecture when $\alpha = 2$ and $D(s) = \zeta(s-1)$ (see (1.7)). We believe that our method of proof can be utilized to study the general case $\alpha \geq 2$.

Our study is also partially motivated by similar results obtained for linear partitions of $n$. Erdős and Lehner [5] were apparently the first who have studied random linear partitions using a probabilistic approach. As a matter of fact, they showed that the number of summands, after an appropriate normalization, converges weekly, as $n \to \infty$, to a random variable having the extreme value distribution. Subsequent work by a number of authors provides considerable information about the structure of a "typical" linear partition of a large integer. (We refer the reader e.g. to [23], [24], [25], [6], [7], [18], [11], [15] and the references therein.) To this effect, Theorem 1 continues the study of partitions initiated by the probabilistic approach.

The proof of our main result is based on a method developed by Hayman [10]. To get an estimate for the Cauchy integral stemming from (1.9) we use the fact that the generating function $G(1, x)$ defined by (1.6) satisfies Hayman’s admissibility conditions in a neighborhood of its main singularity $x = 1$ and outside it (see Lemmas 1 and 2, respectively). A relevant approach to problems related to other characteristics of random plane partitions may be found in [19] and [17].

We organize our paper as follows. Section 2 contains auxiliary facts on the admissibility of the generating function $G(1, x)$ defined by (1.6) and on the asymptotic behaviour of the coefficients $Q(n)$ in its power series expansion. In Section 3 we present the proof of Theorem 1.

2 Preliminary Asymptotics

In order to get an asymptotical estimate for $G(1, x)$ around its main singularity $x = 1$, we first introduce the analytic scheme of assumptions on the sequence $\{a_j\}_{j \geq 1}$ of non-negative numbers, which is due to Meinardus [14]. The following three conditions must be satisfied:

(M1) The Dirichlet series $D(s)$ (see (1.3)) converges in the half-plane $\Re s > \alpha > 0$, and can be analytically continued into the half-plane $\Re s \geq -\alpha_0$, $\alpha_0 \in (0, 1)$. In $\Re s \geq -\alpha_0$, $D(s)$ is analytic except for a simple pole at $s = \alpha$ with residue $A$.  

(M2) There exists an absolute constant $\alpha_1 > 0$ such that $D(s) = O(\|\Im m\ s\|^{\alpha_1})$ uniformly for $\Re e s \geq -\alpha_0$ as $\|\Im m\ s\| \to \infty$.

(M3) Define $g(v) = \sum_{j=1}^{\infty} a_j e^{-jv}$, where $v = y + 2\pi i w$ and $y$ and $w$ are real numbers. If $|\arg v| > \pi/4$ and $|w| \leq 1/2$, then $\Re e g(v) - g(y) \leq -\alpha_2 y^{-\chi}$ for sufficiently small $y$, where $\chi > 0$ is an arbitrary number, and $\alpha_2 > 0$ is suitably chosen and may depend on $\chi$.

We shall be concerned, in the first instance, with the behavior of $G(1, x; \{a_j\}_{j \geq 1})$ (see 1.2) as $x$ becomes close to 1. Meinardus [14] (see also Andrews [4, Lemma 6.1] proved that under the assumptions (M1) and (M2)

$$G(1, e^{-v}; \{a_j\}_{j \geq 1}) = \exp \left\{ A \Gamma(\alpha) \zeta(\alpha + 1) v^{-\alpha} - D(0) \log v + D'(0) + O(y^{\alpha_0}) \right\}$$

as $y \to 0$ uniformly for $|\arg v| \leq \pi/4$ and $|w| \leq 1/2$. (Here $\Gamma(\alpha)$ denotes Euler’s gamma function and $\log (\cdot)$ presents the main branch of the logarithmic function satisfying $\log v < 0$ for $0 < v < 1$.)

Let us take now a sequence $\{r_n\}$ which, as $n \to \infty$, satisfies

$$r_n = 1 - \frac{2 \zeta(3)}{n^{1/3}} + \frac{2 \zeta(3)^2}{2n^{2/3}} - \frac{\zeta(3)}{3n} + O(n^{-4/3}). \quad (2.2)$$

For the sake of brevity we also set

$$b(r) = \frac{6 \zeta(3)}{(1 - r)^2}, \quad (2.3)$$

where $0 < r < 1$. It is easy to check that (2.2) and (2.3) imply

$$b(r_n) = \frac{3n^{4/3}}{2 \zeta(3) |1/3} + O(n) \quad (2.4)$$

as $n \to \infty$.

The next lemma suggests a tool that we shall subsequently use in Section 3 to obtain the main term in our asymptotics.

**Lemma 1** If $r_n$ satisfies (2.2) for large $n$, then

$$G(1, r_n e^{i\theta}) e^{-i\theta n} = G(1, r_n) e^{-\theta^2 b(r_n)/2} \left[ 1 + O(1/ \log^3 n) \right]$$

as $n \to \infty$ uniformly for $|\theta| \leq \delta_n$, where

$$\delta_n = \frac{n^{-5/9}}{\log n} \quad (2.5)$$

and $b(r_n)$ is determined by (2.3).

**Proof.** Our starting point here will be Meinardus’ general asymptotic formula (2.1). We apply it for the sequence $a_j = j, j = 1, 2, \ldots$. It is not difficult to show that $A = 1$ and $D(0) = -1/12$. Classical result on the $\zeta$ function implies
condition (M2) (see e.g. [26, Section 13.51]). Therefore, for \( v = y + 2\pi i w \), we get
\[
G(1, e^{-v}) = \exp \left\{ \zeta(3)v^{-2} + \frac{1}{12} \log v + D'(0) + O(y^{\alpha_0}) \right\}
\] (2.6)
as \( y \to 0 \) uniformly for \( |w| \leq 1/2 \) and \( |\arg v| \leq \pi/4 \). Setting
\[
e^{-v} = r_n e^{iy},
\] (2.7)
we see that \( y = y_n = -\log r_n \) and \( w = -\theta/2\pi \). The asymptotic behaviour of \(-\log r_n\) can be determined with aid of (2.2) as follows:
\[
y_n = -\log r_n = \frac{[2 \zeta(3)]^{1/3}}{n^{1/3}} - \frac{[2 \zeta(3)]^{2/3}}{2 n^{2/3}} + \frac{\zeta(3)}{3n} + O\left(\frac{1}{n^{4/3}}\right)
\] (2.8)
Combining (2.6) - (2.8), we observe that
\[
G(1, r_n e^{i\theta}) G(1, r_n) e^{-i\theta n} = \left[ 1 + O(\delta_n n^{1/3}) \right] \exp \left\{ \zeta(3) \left[ (y_n - i\theta)^{-2} - y_n^{-2} \right] - i\theta n + O\left(\frac{1}{n}\right) \right\}
\] (2.9)
A Taylor’s formula expansion for \( |\theta| \leq \delta_n \) yields
\[
(y_n - i\theta)^{-2} - y_n^{-2} = 2i\theta y_n^{-3} - \frac{\theta^2}{2} y_n^{-4} + O(|\theta|^3 y_n^{-5})
\] (2.10)
Using (2.8), we also get the following estimate for the factor outside the exponent in (2.9):
\[
\left( \frac{y_n - i\theta}{y_n} \right)^{1/12} = \left\{ \frac{[2 \zeta(3)]^{1/3} - i\theta n^{1/3} + O(n^{-1})}{[2 \zeta(3)]^{1/3} + O(n^{-1})} \right\}^{1/12}
= 1 + O(\delta_n n^{1/3}).
\] (2.11)
Finally, we notice that (2.8) implies the bound
\[
y_n^{\alpha_0} = O(n^{-\alpha_0/3}).
\] (2.12)
Hence, inserting (2.4), (2.8), (2.10) - (2.12) into (2.9), we obtain
\[
\frac{G(1, r_n e^{i\theta})}{G(1, r_n)} e^{-i\theta n} = \left[ 1 + O(\delta_n n^{1/3}) \right] \exp \left\{ \zeta(3) \left[ \frac{2i\theta n}{2 \zeta(3)} \right] + O(n^{-1}) \right\}
\]
This completes the proof.

We also need another lemma that will establish a uniform estimate for $G(e^{iT}, x)$, $|x| = r_n$ outside the range $-\delta_n < \arg x < \delta_n$, if $T$ is real and suitably bounded.

**Lemma 2** If $r_n$ and $\delta_n$ satisfy (2.2) and (2.5), respectively, and the function $T = T(n)$ is such that

$$T = T(n) = \Theta(n^{-1/3}/\sqrt{\log n})$$

as $n \to \infty$, then there exist two positive constants $\epsilon$ and $n_0$ so that

$$|G(e^{iT}, r_n e^{i\theta})| \leq G(1, r_n) \exp \left\{ \epsilon - 2n^{2/3}/[2\zeta(3)]^{4/3} \log^2 n \right\}$$

uniformly for $\pi \geq |\theta| \geq \delta_n$ and $n \geq n_0$.

**Proof.** By taking logarithm, for $|x| < 1$ and $|u| = 1$, we get

$$\log G(u, x) = \log \left\{ \prod_{j=1}^{\infty} (1 - ux^j)^{-j} \right\} = - \sum_{j=1}^{\infty} j \log(1 - ux^j)$$

$$= \sum_{j=1}^{\infty} j \sum_{l=1}^{\infty} \frac{x^j u^l}{l} = \sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=1}^{\infty} j (x^j) u^l.$$
\[ \exp \left\{ \sum_{j=1}^{\infty} j r_n^j e^{i(j\theta + T)} + \sum_{l=2}^{\infty} \frac{1}{l} \sum_{j=1}^{\infty} j r_n^j e^{i(j\theta + T)} \right\} \]

\[ = \exp \left\{ \sum_{j=1}^{\infty} j r_n^j \Re e^{i(j\theta + T)} + \sum_{l=2}^{\infty} \frac{1}{l} \sum_{j=1}^{\infty} j r_n^j \Re e^{i(j\theta + T)} \right\} \]

\[ = \exp \left\{ \sum_{j=1}^{\infty} j r_n^j \cos(j\theta + T) + \sum_{l=2}^{\infty} \frac{1}{l} \sum_{j=1}^{\infty} j r_n^j \cos[l(j\theta + T)] \right\} \]

\[ \leq \exp \left\{ \sum_{j=1}^{\infty} j r_n^j \cos(j\theta + T) + \sum_{l=2}^{\infty} \frac{1}{l} \sum_{j=1}^{\infty} j r_n^j \cos[l(j\theta)] \right\} \]

\[ = \exp \left\{ \sum_{j=1}^{\infty} j r_n^j \cos(j\theta + T) - 1 \right\} + \sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=1}^{\infty} j r_n^j \]

\[ = G(1, r_n) \exp \{ H_n(\theta, T) \}, \tag{2.14} \]

where

\[ H_n(\theta, T) = \sum_{j=1}^{\infty} j r_n^j \cos(j\theta + T) - 1 \]

\[ = \Re \left[ \frac{r_n e^{i(\theta + T)}}{(1 - r_n e^{i\theta})^2} \right] - \frac{r_n}{(1 - r_n)^2} \tag{2.15} \]

\[ = \frac{r_n}{(1 - 2r_n \cos(\theta) + r_n^2)} \left( \cos(\theta + T) + r_n^2 \cos(\theta - T) - 2r_n \cos(T) \right) - \frac{r_n}{(1 - r_n)^2}. \]

It is not difficult to show that the function

\[ f(\theta) = \cos(\theta + T) + r_n^2 \cos(\theta - T) \]

in the numerator of \( H_n(\theta, T) \) attains its maximum value in the range \( \delta_n \leq |\theta| \leq \pi \) at \( \theta = \pm \delta_n \). To prove this one needs to determine the behaviour of \( f(\theta) \) in the ranges \( T < \theta \leq \pi \) and \( \delta_n \leq \theta \leq T \). The first case is an easy exercise whose study avoids the asymptotic form \( (2.13) \) of \( T \). In the second case one can use the following representation of the derivative

\[ f'(\theta) = -\sin(\theta + T) - r_n^2 \sin(\theta - T) = -(1 + r_n^2)\theta - (1 - r_n^2)T + O(T^3). \tag{2.2} \]

\[ (1 - r_n^2)T = \Theta(n^{-2/3} \log^{-1/2} n), \ O(T^3) = O(n^{-1} \log^{-3/2} n). \]

\[ \text{9} \]
Therefore, for sufficiently large $n$, the value of $f'(\theta)$ becomes close to $-(1+r_n^2)\theta$. This implies that $f(\theta)$ decreases for $\delta_n \leq \theta \leq T$ and increases for $-T \leq \theta \leq -\delta_n$. It follows that one may replace $\theta$ by $-\delta_n$ in (2.15). Thus we can write

$$H_n(\theta, T) \leq \frac{r_n \left[ \cos(\delta_n + T) + r_n^2 \cos(\delta_n - T) - 2r_n \cos(T) \right]}{(1 - 2r_n \cos(\delta_n) + r_n^2)^2} - \frac{r_n}{(1 - r_n)^2},$$

(2.16)

To estimate the cosine functions in the right-hand side of this inequality we express $\delta_n$ and $T$ by (2.5) and (2.13), respectively. We get the following expansions:

$$\cos \delta_n = 1 - \frac{\delta_n^2}{2} + O(\delta_n^4) = 1 - \frac{n^{-10/9}}{2 \log^2 n} + O(n^{-20/9} \log^{-4} n),$$

$$\cos(\delta_n \pm T) = \cos T \pm 2 \sin(T \mp \frac{\delta_n}{2}) \sin \frac{\delta_n}{2},$$

$$\cos T = 1 - \Theta(n^{-2/3} / \log n).$$

Moreover, (2.2) implies that

$$(1 - r_n)^{-2} = \left[ \frac{n}{2 \zeta(3)} \right]^{2/3} + O(n^{1/3}).$$

Substituting these estimates in the right-hand side of (2.16), after some manipulations, we obtain

$$H_n(\theta, T) \leq \frac{r_n \left[ 1 - \Theta(n^{-2/3} / \log n) \right]}{(1 - r_n)^2 \left[ 1 + \frac{r_n n^{-10/9}}{(1 - r_n)^2 \log^2 n} + O(n^{-20/9} / (1 - r_n)^4 \log^4 n) \right]^2} - \frac{r_n}{(1 - r_n)^2}$$

$$= \frac{r_n \left[ 1 - \Theta(n^{-2/3} / \log n) \right]}{(1 - r_n)^2 \left[ 1 + 2n^{-4/9} / [2 \zeta(3)]^{2/3} \log^2 n + O(n^{-7/9} / \log^2 n) \right]} - \frac{r_n}{(1 - r_n)^2}$$

$$= \frac{r_n \left[ 1 - \Theta(n^{-2/3} / \log n) \right]}{(1 - r_n)^2} \left\{ 1 - \frac{2n^{-4/9}}{[2 \zeta(3)]^{2/3} \log^2 n} + O(n^{-7/9} / \log^2 n) \right\} - \frac{r_n}{(1 - r_n)^2}$$

$$= -r_n \left\{ \left[ \frac{n}{2 \zeta(3)} \right]^{2/3} + O(n^{1/3}) \right\} \frac{2n^{-4/9}}{[2 \zeta(3)]^{2/3} \log^2 n} + O(1 / \log n)$$

$$= -\frac{2n^{3/9}}{[2 \zeta(3)]^{4/3} \log^2 n} + O(1 / \log n).$$

Inserting this into (2.14), we obtain the required bound.

Further, we shall essentially use the asymptotic form of the numbers $Q(n)$. It is given by Wright’s formula [13], however, we need this result in a slightly different form. It is not difficult to verify that Lemmas 1 and 2, (2.2) and (2.3) imply that $G(1, x)$ belongs to the class of Hayman’s admissible functions [10] and thus, Hayman’s general asymptotic formula for the coefficients in the power
series representations of admissible functions is valid for the number of plane partitions of n, \( Q(n) \), as well. The next lemma encompasses these results. We only sketch its proof and insert a remark explaining the role of the asymptotic expansion (2.2) there.

**Lemma 3** For \( G(1,x) \), the generating function of the numbers \( Q(n) \) of plane partitions of \( n \), defined by (1.6), we have

\[
Q(n) \sim G(1,r_n)\frac{r_n^{-n}}{[2\pi b(r_n)]^{1/2}}
\]

as \( n \to \infty \), where \( r_n \) satisfies the equation

\[
rG'(1,r)/G(1,r) = n
\]

for sufficiently large \( n \) and \( b(r_n) \) is defined by (2.3).

**Sketch of the proof.** It is clear that \(|x| = 1\) is a natural boundary for \( G(1,x) \). Lemma 1 shows the behaviour of \( G(1,x) \) around its main singularity \( x = 1 \) (condition I of Hayman’s Definition [10]); Lemma 2 establishes the negligibility of the growth of \( G(1,x) \) as \( x \to x_0, \; |x_0| = 1 \) and \( x_0 \neq 1 \) (condition II of [10]). It is then easily seen that

\[
\frac{r}{G'(1,r)} = 2 \sum_{j=1}^{\infty} \frac{r^{2j}}{(1-r)^{2j}} + \sum_{j=1}^{\infty} \frac{r^j}{(1-r)^j}
\]

\[
= \frac{2\zeta(3)}{(1-r)^3} + o((1-r)^{-3})
\]

as \( r \to 1^- \). This enables one to conclude that \( r_n \), determined for sufficiently large \( n \) by (2.18), can be substituted by the asymptotic expansion (2.2). Thus one can obtain (2.17) after a direct application of Hayman’s theorem [10].

Finally, we notice that (2.2) and (2.6) imply the coincidence of the right-hand sides of (1.8) and (2.17).

### 3 Proof of the Main Result

First, we let in (1.9)

\[
G(u,x) = e^{F(u,x)},
\]

that is, we set

\[
F(u,x) = -\sum_{j=1}^{\infty} j \log(1 - ux^j).
\]

We now apply Cauchy’s coefficient formula to (1.9) on the circle \( x = r_n e^{i\theta}, \; -\pi < \theta \leq \pi \), with \( r_n \) determined by (2.2). Thus, in terms of the notations (1.10), (3.1) and (3.2), for \(|u| \leq 1\), we obtain

\[
Q(n)\varphi_n(u) = \frac{r_n^{-n}}{2\pi} \int_{-\pi}^{\pi} \exp\{F(u,r_n e^{i\theta}) - i\theta n\} \, d\theta.
\]
We break up the range of integration as follows:

\[ Q(n)\varphi_n(u) = J_1(n, u) + J_2(n, u), \quad (3.3) \]

where

\[ J_1(n, u) = \frac{r_n^{-n}}{2\pi} \int_{-\delta_n}^{\delta_n} \exp \left\{ F(u, r_ne^{i\theta}) - i\theta n \right\} d\theta, \quad (3.4) \]

\[ J_2(n, u) = \frac{r_n^{-n}}{2\pi} \int_{\delta_n \leq |\theta| \leq \pi} \exp \left\{ F(u, r_ne^{i\theta}) - i\theta n \right\} d\theta. \quad (3.5) \]

### 3.1 An asymptotic estimate for \( J_1(n, u) \). Using Taylor’s formula expansion, we can write

\[ F(u, r_ne^{i\theta}) = F(u, r_n) + r_n(e^{i\theta} - 1) \frac{\partial}{\partial x} F(u, x) \bigg|_{x = r_n} + O \left( |\theta|^3 \frac{\partial^3}{\partial x^3} F(|u|, x) \bigg|_{x = r_n} \right). \quad (3.6) \]

To evaluate the partial derivatives of \( F(u, x) \), we shall use the following expressions:

\[ \frac{\partial}{\partial x} F(u, x) \bigg|_{x = r_n} = u \sum_{j=1}^{\infty} \frac{j^2 r_n^{j-1}}{1 - ur_n^j}, \quad (3.7) \]

\[ \frac{\partial^2}{\partial x^2} F(u, x) \bigg|_{x = r_n} = u \sum_{j=2}^{\infty} \frac{j^2(j-1)r_n^{j-2}}{1 - ur_n^j} + u^2 \sum_{j=1}^{\infty} \frac{j^3 r_n^{2(j-1)}}{(1 - ur_n^j)^2}, \quad (3.8) \]

\[ \frac{\partial^3}{\partial x^3} F(u, x) \bigg|_{x = r_n} = u \sum_{j=3}^{\infty} \frac{j^3(j-1)(j-2)r_n^{j-3}}{1 - ur_n^j} \]

\[ + 3u^2 \sum_{j=2}^{\infty} \frac{j^3(j-1)r_n^{2j-3}}{(1 - ur_n^j)^2} + 2u^3 \sum_{j=1}^{\infty} \frac{j^4 r_n^{3(j-1)}}{(1 - ur_n^j)^3}. \quad (3.9) \]

We proceed further to establish the convergence of \( J_1(n, u) \) in terms of Fourier transforms by setting \( u = e^{iT}, -\infty < T < \infty \), in \([3.1], \quad (3.6)-\quad (3.9)\).

In what follows later, an application of Lévy’s continuity theorem for characteristic functions [12, Section 3.6] would specify the value of \( T \) as a function of the main parameter \( n \). Furthermore, we need notations for the following four functions:

\[ \psi_{m, k}(z, T) = \int_z^{\infty} \frac{y^m}{(e^{yT} - 1)^{m-k}} dy, \quad (3.10) \]

\( (k, m) = (0, 1), (1, 2), (1, 3), (2, 3), 0 \leq z < \infty. \)

They are closely related to the Debye functions (see e.g. [1, Section 27.1]). Moreover, formula 27.1.3 of [1] shows that

\[ \psi_{m, m-1}(0, 0) = m!\zeta(m + 1). \quad (3.11) \]
We now proceed to the asymptotic estimates of the summands in (3.6). Interpreting again the sum in (3.7) by a Riemann’s one with the same step size $y_n = - \log r_n$ as in the proof of Lemma 2 of [15] and using (3.11) and (2.8), we find that

$$r_n \frac{\partial}{\partial x} F(u, x) \bigg|_{x = r_n, u = e^{iT}} = \psi_{2,1}(y_n, T) y_n^{-3} + O(y_n^{-1})$$

$$= [\psi_{2,1}(0, T) + O(n^{-2/3})] y_n^{-3} + O(n^{1/3})$$

$$= [\psi_{2,1}(0, 0) + R(n, T)] y_n^{-3} + O(n^{1/3})$$

$$= [2\zeta(3) + R(n, T)] \left[ \frac{n}{2\zeta(3)} + O(1) \right] + O(n^{1/3})$$

$$= n + \frac{n R(n, T)}{2 \zeta(3)} + O(\|R(n, T)\|) + O(n^{1/3}),$$

where

$$R(n, T) = T \psi'_{2,1}(0, T_1) = i T \int_{0}^{\infty} \frac{y^2 e^{y - iT_1}}{(e^{y - iT_1} - 1)^2} dy$$

is the remainder term in the Taylor’s formula expansion of $\psi_{2,1}(0, T)$ and $0 < T_1 < T$. Since the last integral is finite, from (2.13) we get

$$R(n, T) = O(n^{-1/3} / \sqrt{\log n}).$$

Furthermore, note that (3.6) requires a multiplication by $e^{i\theta} - 1$. Hence by (2.2), (2.5) (3.12) (3.13)

$$r_n (e^{i\theta} - 1) \frac{\partial}{\partial x} F(u, x) \bigg|_{x = r_n, u = e^{iT}} = \left[ i \theta + O(\|\theta\|^2) \right] \left[ n + \frac{n R(n, T)}{2 \zeta(3)} + O(n^{1/3}) \right]$$

$$= i\theta n + i \theta \frac{n R(n, T)}{2 \zeta(3)} + O(\|\theta\| n^{1/3}) + O(n^{1/3})$$

$$= i\theta n + i \theta \frac{n R(n, T)}{2 \zeta(3)} + O(n^{-1/9} / \log^2 n).$$

To deal with the third term of the expansion in (3.6), one has to follow the same line of reasoning. Thus, in a similar way (3.8) becomes

$$r_n^2 \frac{\partial^2}{\partial x^2} F(u, x) \bigg|_{x = r_n, u = e^{iT}} = y_n^4 [\psi_{3,2}(y_n, T) + \psi_{3,1}(y_n, T)] + O(n).$$

On the other side, using Taylor’s formula expansion and (3.11), we obtain

$$\psi_{3,2}(y_n, T) = \psi_{3,2}(0, T) + O \left( y_n \left| \frac{\partial}{\partial z} \psi_{3,2}(z, T) \bigg|_{z = y_n} \right| \right)$$

(3.16)
So (2.2), (2.5), (2.8) and (3.15) - (3.17) imply

\[ \theta = \psi + O(1/n) = \psi(0,0) + O(|T|) + O(1/n) = 6\zeta(4) + O(n^{-1/3}/\sqrt{\log n}), \]

\[ \psi_1(0,0,0) = \psi_1(0,0) + O(|T|) + O(1/n) = 6\zeta(3) - \zeta(4) + O(n^{-1/3}/\sqrt{\log n}). \]

So (2.2), (2.5), (2.8) and (3.15) - (3.17) imply

\[ \int J_1^n e^{i\theta} = \int J_1^n e^{i\theta} + O(n^{-1/3}/\sqrt{\log n}). \]

Finally, following similar but simpler analysis as above, with the aid of (3.7) we can show the negligibility of the error term in (3.6). We have

\[ \psi_1(0,0,0) = \psi_1(0,0) + O(|T|) + O(1/n) = 6\zeta(3) - \zeta(4) + O(n^{-1/3}/\sqrt{\log n}). \]

We are now ready to substitute (3.6), (3.14), (3.18) and (3.19) into the integral of \( J_1(n,e^{iT}) \) (see the (3.14)). We can write

\[ J_1(n,e^{iT}) = \frac{r_n n^{-1/3}}{2\pi} \int_{-\delta_n}^{\delta_n} \exp \left\{ i\theta n + i\theta n R(n,T) \frac{2\zeta(3)}{2\zeta(3)} \right\} d\theta \]

\[ -\delta_n \left[ \frac{n}{2\zeta(3)} \right]^{4/3} \frac{6\zeta(3)}{6\zeta(3)} + O(n^{-1/3}/\log n) + O(1/\log^2 n) - i\delta_n \]
\[
\frac{r^{-n} e^{F(e^{iT}, r_n)}}{2 \pi b^{1/2}(r_n)} \left[ 1 + O \left( \frac{1}{\log^3 n} \right) \right] \int_{-\delta_n b^{1/2}(r_n)}^{\delta_n b^{1/2}(r_n)} \exp \left\{ \frac{itn^{1/3} R(n, T)}{2 \zeta(3)^{5/6} \sqrt{3}} - \frac{t^2}{2} \right\} dt.
\]

Note that we substituted \( \theta = t/b^{1/2}(r_n) \), \(-\infty < t < \infty\), with \( b^{1/2}(r_n) \) defined by (2.3) and (2.4). In addition, (2.4) and (2.5) justify the computation
\[
\delta_n b^{1/2}(r_n) \sim dn^{1/9} \log n, \quad d = \sqrt{3}/[2\zeta(3)]^{1/6}.
\]

So, the bounds of the last integral in (3.20) tend to \( \pm \infty \). The estimate (3.13) for \( R(n, T) \) and the Lebesgue's dominated convergence theorem allow the passage to the limit under the integral. Moreover, (3.13) shows that the limit of the integrand equals \( e^{-t^2/2} \) with an error term of order \( O(1/\sqrt{\log n}) \). The additional error term that we get replacing \( \pm \delta_n b^{1/2}(r_n) \) by \( \pm \infty \) can be estimated using the asymptotic expansion of the function \( (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-t^2} dt \) as \( z \to \infty \) (see [11 Chap.7]). Therefore, (3.21) implies that this error term is at most \( O(n^{-1/9}(\log n) \exp \{-d^2n^{2/9}\log^2 n\}) = O(1/\sqrt{\log n}) \). This shows in turn that
\[
J_1(n, e^{iT}) = \frac{r^{-n} e^{F(e^{iT}, r_n)}}{2 \pi b^{1/2}(r_n)} \left[ 1 + O(1/\sqrt{\log n}) \right] \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt
\]
\[
= \frac{r^{-n} e^{F(e^{iT}, r_n)}}{\sqrt{2\pi b(r_n)}} \left[ 1 + O(1/\sqrt{\log n}) \right].
\]

This completes the required estimate for \( J_1(n, e^{iT}) \).

### 3.2 An asymptotic estimate for \( J_2(n, u) \)

To estimate this integral asymptotically we only need to apply directly Lemmas 2 and 3. So for \( n \geq n_0 \) and bounded \( T \), we get from (3.4) and (2.4):
\[
|J_2(n, u)| \leq \frac{r^{-n}}{2\pi} \int_{\pi \geq |\theta| \geq \delta_n} \left| G(e^{iT}, r_n e^{i\theta}) \right| d\theta
\]
\[
\leq \frac{r^{-n} G(1, r_n)(1 + \varepsilon) \exp \{-2n^{2/9}/[2\zeta(3)]^{4/3} \log^2 n\}}{\sqrt{2\pi b(r_n)}} (1 + \varepsilon) \exp \{-2n^{2/9}/[2\zeta(3)]^{4/3} \log^2 n + \log \sqrt{2\pi b(r_n)} \}
\]
\[
\sim Q(n)(1 + \varepsilon) \exp \{-2n^{2/9}/[2\zeta(3)]^{4/3} \log^2 n + \frac{2}{3} \log n + O(1) \} = o(Q(n)).
\]

### 3.3 Final estimate for \( J_1(n, e^{iT}) \): the choice of \( T = T(n) \)

We start our estimation with a Taylor’s formula expansion:
\[
F(e^{iT}, r_n) = F(1, r_n) + (e^{iT} - 1) \frac{\partial}{\partial u} F(u, r_n) \bigg|_{u=1}
\]
+ (e^T - 1)^2 \frac{\partial^2}{\partial u^2} F(u, r_n) \bigg|_{u=1} + O \left( |T|^3 \left| \frac{\partial^3}{\partial u^3} F(u, r_n) \right|_{u=1} \right)

= F(1, r_n) + i T \frac{\partial}{\partial u} F(u, r_n) \bigg|_{u=1} - T^2 \frac{\partial^2}{\partial u^2} F(u, r_n) \bigg|_{u=1}

+ O \left( |T|^3 \left[ \left| \frac{\partial^3}{\partial u^3} F(u, r_n) \right|_{u=1} \right] + |T|^2 \left| \frac{\partial}{\partial u} F(u, r_n) \right|_{u=1} \right). \quad (3.24)

The partial derivatives of $F(u, r_n)$ can be again evaluated by Riemann's integral in a way similar to that presented in the proof of Lemma 2 of [15]. We have by (2.38), (3.10) and (3.11) that

$$\frac{\partial}{\partial u} F(u, r_n) \bigg|_{u=1} = \sum_{j=1}^{\infty} \frac{j^n r_j}{1 - r_j^n} = y_n^{-2} \int_{y_n}^{\infty} \frac{ve^{-v}}{1 - e^{-v}} dv + O(y_n^{-1})$$

$$= \left[ \frac{n}{2\zeta(3)} \right]^{2/3} \left[ 1 + O(n^{-1}) \right] \psi_{1,0}(0,0) \left[ 1 + O(n^{-1/3}) \right] + O(n^{1/3})$$

$$= \left[ \frac{n}{2\zeta(3)} \right]^{2/3} \zeta(2) + O(n^{1/3}). \quad (3.25)$$

Similarly

$$\frac{\partial^2}{\partial u^2} F(u, r_n) \bigg|_{u=1} = \sum_{j=1}^{\infty} \frac{j^{n^2} r_j}{(1 - r_j^n)^2} = y_n^{-2} I(y_n) + o(y_n^2 I(y_n)), \quad (3.26)$$

where

$$I(y_n) = \int_{y_n}^{\infty} \frac{ve^{-v}}{(1 - e^{-v})^2} dv = \frac{y_n}{e^{y_n} - 1} - \psi_{1,0}(y_n, 0) + \int_{y_n}^{\infty} \frac{dv}{e^v - 1}.$$}

Since the first two summands above have finite limits and

$$\int_{y_n}^{\infty} \frac{dv}{e^v - 1} - \log y_n = 1 + o(1)$$

as $y_n \to 0$, we observe that

$$I(y_n) = O(1) - \log y_n [1 + o(1)] = [1 + o(1)](- \log y_n).$$

Substituting this into (3.26) and invoking the asymptotic (2.8) for $y_n$, we find that

$$\frac{\partial^2}{\partial u^2} F(u, r_n) \bigg|_{u=1} = \left[ \frac{n}{2\zeta(3)} \right]^{2/3} \frac{\log n}{3} + o \left( n^{2/3} \log n \right). \quad (3.27)$$

In the same way one can deduce the following estimate for the third partial derivative:

$$\frac{\partial^3}{\partial u^3} F(u, r_n) \bigg|_{u=1} = O(n). \quad (3.28)$$
Putting (3.25), (3.27) and (3.28) into (3.24), we obtain

\[
F\left(e^{iT}, r_n\right) = F(1, r_n) + iT \left[ \frac{n}{2\zeta(3)} \right]^{2/3} \zeta(2) + O(|T|) - \frac{T^2}{2} \left[ \frac{n}{2\zeta(3)} \right]^{2/3} \frac{\log n}{3} \\
+ o \left( T^2 n^{2/3} \log n \right) + O(n|T|^3) + O(n^{2/3}T^2).
\]

(3.29)

It is now clear how to choose the variable \( T = T(n) \) which satisfies (2.13). Note that this choice determines the scaling factor in our limit theorem. Setting

\[
T = w \left/ \left[ \frac{n}{2\zeta(3)} \right]^{1/3} \sqrt{\frac{\log n}{3}} \right., \quad -\infty < w < \infty,
\]

(3.30)

(in agreement with (2.13)) we can rewrite (3.29) in the following form

\[
F\left(\exp \left\{ iw \left/ \left[ \frac{n}{2\zeta(3)} \right]^{1/3} \sqrt{\frac{\log n}{3}} \right.\right), r_n\right) = F(1, r_n) + iw \left[ \frac{n}{2\zeta(3)} \right]^{1/3} \zeta(2) \left/ \sqrt{\frac{\log n}{3}} - \frac{w^2}{2} + o(1)\right.
\]

(3.31)

The convergence here is uniform with respect to \( w \) belonging to any finite interval. Going back to the estimate (3.22) for \( J_1(n, e^{iT}) \) and replacing there (3.30) and (3.31), we find that

\[
J_1\left( n, \exp \left\{ iw \left/ \left[ \frac{n}{2\zeta(3)} \right]^{1/3} \sqrt{\frac{\log n}{3}} \right.\right) = \frac{r_n e^{F(1, r_n)}}{\sqrt{2\pi b^3 r_n}} \right.
\]

\[
\times \left[ 1 + O\left( \frac{1}{\sqrt{\log n}} \right) \right] \exp \left\{ iw \left/ \left[ \frac{n}{2\zeta(3)} \right]^{1/3} \zeta(2) \left/ \sqrt{\frac{\log n}{3}} - \frac{w^2}{2} + o(1)\right.\right\}
\]

\[
\sim Q(n) \exp \left\{ iw \left/ \left[ \frac{n}{2\zeta(3)} \right]^{1/3} \zeta(2) \left/ \sqrt{\frac{\log n}{3}} - \frac{w^2}{2} \right.\right\}
\]

(3.32)

where the reduction in the last asymptotic equivalence is due to a direct application of Lemma 3. To handle with \( J_1 \) in a more appropriate way we may rewrite (3.32) as follows:

\[
J_1\left( n, \exp \left\{ iw \left/ \left[ \frac{n}{2\zeta(3)} \right]^{1/3} \sqrt{\frac{\log n}{3}} \right.\right) \right.
\]

\[
\times \exp \left\{ -iw \left/ \left[ \frac{n}{2\zeta(3)} \right]^{1/3} \zeta(2) \left/ \sqrt{\frac{\log n}{3}} \right.\right\} \sim Q(n)e^{-w^2/2}.
\]

(3.33)
It remains now to deal with the characteristic function of \( (\tau_n - c_0 n^{2/3})/c_1 n^{1/3} \log^{1/2} n \).
The expressions for \( c_0 \) and \( c_1 \) in the Theorem and (1.9) imply that it is equal to
\[
\exp \left\{ -iw \left[ \frac{n}{2\zeta(3)} \right]^{1/3} \frac{\zeta(2)}{3} \sqrt{\log \frac{n}{3}} \right\} \\
\times \varphi_n \left( \exp \left\{ iw \left[ \frac{n}{2\zeta(3)} \right]^{1/3} \sqrt{\log \frac{n}{3}} \right\} \right) = \Phi_n(w)
\]
Setting \( u = e^{iT} \) in (3.3) with \( T \) given by (3.30) and substituting estimates (3.32) and (3.23) in it, we obtain
\[
Q(n) \Phi_n(w) = Q(n)e^{-w^2/2} + o(Q(n)).
\]
The required weak convergence follows from Lévy’s continuity theorem for characteristic functions [12, Section 3.6].

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