Topological Wannier cycles induced by sub-unit-cell artificial gauge flux in a sonic crystal

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Gauge fields play a major role in understanding quantum effects. For example, gauge flux insertion into single unit cells is crucial towards detecting quantum phases and controlling quantum dynamics and classical waves. However, the potential of gauge fields in topological materials studies has not been fully exploited. Here, we experimentally demonstrate artificial gauge flux insertion into a single plaquette of a sonic crystal with a gauge phase ranging from 0 to 2π. We insert the gauge flux through a three-step process of dimensional extension, engineering a screw dislocation and dimensional reduction. Additionally, the single-plaquette gauge flux leads to cyclic spectral flows across multiple bandgaps that manifest as topological boundary states on the plaquette and emerge only when the flux-carrying plaquette encloses the Wannier centres. We termed this phenomenon as the topological Wannier cycle. This work paves the way towards sub-unit-cell gauge flux, enabling future studies on synthetic gauge fields and topological materials.

The Aharonov–Bohm effect1 states that a system comes back to itself if a gauge flux of 2π is inserted (here, for simplicity, gauge flux is redefined as the gauge phase accumulated when going around the flux tube once anticlockwise); however, this is true only for trivial systems. Inserting a gauge flux of 2π into a topological system leads to adiabatic pumping that yields spectral flows across the topological bandgap through the topological boundary states, as first revealed in quantum Hall effects2–4 and later in other systems5. To date, local gauge flux insertion for electrons in solid-state materials has been realized only in areas much larger than a unit cell, as limited by the size of the magnetic flux tubes in the current technology5–8. For photons, phonons and ultracold atoms, although artificial gauge fields have been realized in lattice systems using various methods9–21, the realization of gauge flux in an individual unit cell, which requires extreme control of physical parameters in real space, is still difficult in many cases.

Here, we use an approach to achieve single-plaquette artificial gauge flux in a sonic crystal. Such extremely localized gauge flux is made possible here due to the excellent controllability of sonic crystals, which also exists in other synthetic materials and systems6–12,22,23. We discover that the gauge flux insertion leads to topological Wannier cycles in crystalline insulators with a filling anomaly, which are often termed higher-order topological insulators or obstructed atomic insulators24–27. Interestingly, the topological Wannier cycles found here emerge only when the Wannier centres are enclosed by the flux-carrying plaquette. Therefore, the sub-unit-cell gauge flux provides an experimental probe of the Wannier centres.

Artificial gauge flux in a sonic crystal

We use a two-dimensional (2D) four-fold rotation (C4) symmetric sonic crystal (Fig. 1a) as the motherboard to work on. The 2D sonic crystal realizes the 2D Su–Schrieffer–Heeger (SSH) model28–30, which exhibits two bandgaps (denoted as gaps I and II; Fig. 1b). These bandgaps are acoustic analogues of higher-order topological bandgaps25, as reflected by the fact that all the bands have their Wannier centres at the unit-cell corner (the Wyckoff position 1a in Fig. 1c and the analysis in Supplementary Note 1). By contrast, the case with the Wannier centres at the unit-cell centre (the Wyckoff position 1b in Fig. 1c) is denoted as trivial. Figure 1c gives the Wannier orbitals of the four bulk bands, which are the s-like, p±, ±ip±, d-like and d-like orbitals.

To achieve the artificial gauge flux, we employ three procedures as illustrated in Fig. 1d: First, by periodic stacking, we extend the dimension of the sonic crystal from being 2D to three dimensional (3D). The interlayer couplings are realized by the air tubes with a diameter d = 3 mm (Fig. 1e). The periodicity along the z direction is H = 36 mm. Second, we create a step screw dislocation (SSD) at the centre of the 3D sonic crystal with a Burgers vector B = (0,0,H). The SSD divides the system into four flat quarter sectors that are related by the screw rotation S_Lz := (x,y,z) → (y, −x, z + 2π/4). Third, via dimensional reduction31, we map the 3D system into many kz-dependent 2D systems (kz is the wave vector along z) via a Fourier transformation in the z direction. Now, each coupling between adjacent quarter sectors picks up a gauge phase of ±kzH due to the SSD, yielding a hopping pattern in Fig. 1f and a gauge flux Φ = kzH for the central plaquette. According to Bloch’s theorem, this artificial gauge flux Φ = kzH embraces the full phase range from 0 to 2π.

Topological Wannier cycles

With the single-plaquette gauge flux insertion, eigenstates of different symmetries evolve cyclically within a group of four, as depicted in Fig. 2a. Here, we label the eigenstates of a finite system by their C4 eigenvalues εn = e^{in(π/4)} with n = 0, ±1, 2, which corresponds to the ±d-like, p± = p± ±ip± -like and d-like states, separately. By inserting...
a flux quantum $Φ = 2\pi$ into the central plaquette, an eigenstate with $C_4$ eigenvalue $g_0$ evolves into an eigenstate with $C_4$ eigenvalue $e^{i(n+1)(2\pi)} = g_{n+1}$, because the gauge phase $\frac{Φ}{4}$ is accumulated upon each $C_4$ rotation (Supplementary Note 2). Although the system with a SSD breaks the chirality, the time-reversal symmetry remains intact. Depending on the evolution of $k_z$, the artificial gauge flux $Φ = k_z H$ can vary either from 0 to $2\pi$ (as shown in Fig. 2a) or from $2\pi$ to 0. These two processes are the time-reversal counterparts of each other.

Fig. 1 | Artificial gauge flux insertion in a single plaquette in acoustic systems. a, A 2D sonic crystal mimicking the 2D SSH model. Tan and dark-green structures depict the acoustic cavities and the tubes connecting these cavities, respectively. Lattice constant $a = 40$ mm. The diameter and height of the cylindrical cavities are $D = 15$ mm and $h = 25$ mm, respectively. The intra-cell (inter-cell) coupling, denoted as $t_1$ ($t_2$), is implemented by air tubes of diameter $d_1 = 5$ mm ($d_2 = 14$ mm). The green square depicts a unit cell where the Wyckoff positions 1a and 1b are labelled. The left inset shows the geometry details of a unit cell. The right inset shows the acoustic wavefunction in a cavity for the lowest mode at the Brillouin zone corner. Max, maximum. b, Acoustic Bloch bands (coloured regions) and topological bandgaps (I and II). Symmetry properties of the bands are labelled by the little group representations at the high-symmetry points of the Brillouin zone (Supplementary Note 1 for details). c, Illustration of the Wannier centres and Wannier orbitals of the Bloch bands. d, Schematic illustration of the procedures that realize the single-plaquette gauge flux. The blue line and arrow indicate the dislocation line and the Burgers vector, respectively. Four colours represent layers of different heights. The red circular arrow indicates the chiral structure of the system. e, The resultant structure when the procedures in d are applied to the sonic crystal in a. Inset shows that the interlayer couplings are realized by the tubes of diameter $d$. The periodicity along the $z$ direction is $H$. Gauge flux insertion at the central plaquette (orange) is indicated. f, Tight-binding model corresponding to the acoustic structure in e. Dots and links represent the sites and their mutual couplings, respectively. Each quarter sector (represented by dots of different colours) has $N \times N$ unit cells. Red arrows denote the intersector couplings where a gauge phase of $Φ/4$ is assigned for each hopping along a red arrow.
Fig. 2 | Topological Wannier cycles: underlying principles. a, Schematic illustration of the evolution of eigenstates due to the single-plaquette gauge flux insertion. The eigenstates are labelled by their \( C_{\varepsilon} \) eigenvalues \( g_{n} = \exp(i\varepsilon) \) with \( n = 0, \pm 1, 2 \) for the \( s \)-like, \( p_{x} \), \( p_{y} \), and \( d \)-like states, respectively. The \( C_{\varepsilon} \)-symmetry eigenvalues of the eigenstates evolve cyclically when \( \Phi \) goes from 0 to \( 2\pi \). b, Schematic illustration of the cyclic spectral flows among the bulk bands (coloured regions) as induced by the gauge flux insertion. c,d, Applying the tight-binding model in Fig. 1f for the topological (c) and trivial (d) cases, when the weak coupling for each case is set to zero (as specified by the parameter on top of each figure). Here, four colours represent sectors of different heights. Dots and links represent the sites and their mutual couplings, respectively. Red arrows denote the intersector couplings where a gauge flux is inserted.

Consider when the system is finite in the \( x-y \) plane, say, with \( N^{2} \) unit cells in each quarter sector as in Fig. 1f. If the system is a trivial insulator, then each bulk band should have \( 4N^{2} \) eigenstates. In this case, the cyclic evolution of the eigenstates is within each bulk continuum, and there is no spectral flow across the bulk bandgaps. However, if the system is topological, the numbers of bulk eigenstates in each band will deviate from the number of unit cells. This phenomenon is termed as ‘filling anomaly’ and is crucial for understanding the properties of fragile and higher-order topological insulators. In our acoustic system, we find \( 4(N^{2} \pm N) + 1 \) eigenstates in the first (\( + \)) and fourth (\( - \)) bulk bands, while the second and third bulk bands together have \( 4(2N^{2} - 1) + 2 \) eigenstates (Supplementary Note 2 for details). Because these numbers are not integer multiples of four, the cyclic evolution of eigenstates cannot be completed within each bulk continuum. There must be spectral flows across the bulk bandgaps, as schematically illustrated in Fig. 2b. Such spectral flows are termed topological Wannier cycles.

We now elaborate on the theoretical foundation of topological Wannier cycles. First, a way to simplify the scenario is to adiabatically tune the topological system into a limit where the weak intra-cell couplings vanish, \( t_{1} = 0 \) (Fig. 2c); \( t_{1} \) is the intra-cell coupling constant. In this limit, the bulk eigenstates become Lannier orbitals confined in the plaquettes formed by the strong inter-cell couplings, \( t_{2} \). When a flux quantum is pierced into the central plaquette, the cyclical evolution of the four Wannier orbitals there leads to the spectral flows traversing the bandgaps, as schematically shown in Fig. 2b. By contrast, the gauge flux has no effect for the trivial system. This becomes transparent in the limit with vanishing inter-cell coupling, \( t_{2} = 0 \), as shown in Fig. 2d (details in Supplementary Note 3).

More strictly, the topological Wannier cycles can be understood through a theoretical tool termed the real-space topological invariants (RSTIs), which was developed in previous papers(34,35). For a trivial bandgap, all eigenstates below form many groups of four to complete the cyclical evolution due to the gauge flux insertion. For a topological gap, the ungrouped eigenstates above and below the gap are dictated by its RSTIs(34,35). Here, for gap I, the RSTIs are \( \delta_{i} = \delta_{i} = -1 \), which predict that there is one ungrouped \( s \)-like state below gap I and two ungrouped \( p_{x} \)-like states above gap I (Supplementary Note 4 for details). For gap II, we find that the RSTIs are \( \delta_{i} = 0 \) and \( \delta_{i} = -1 \), values that predict one ungrouped \( d \)-like state above gap II and two ungrouped \( p_{x} \)-like states below gap II. The cyclical evolution of these ungrouped states due to the local gauge flux insertion must be fulfilled by the spectral flows traversing the bandgaps, as illustrated in Fig. 2b.

The cyclical spectral flows are confirmed numerically by finite-element simulations of the acoustic model in Fig. 1e (Methods). As shown by the spectrum and eigenstates in Fig. 3 for our acoustic system, there are cyclic spectral flows traversing gaps I and II to fulfill the cyclic evolution of the eigenstates. Due to the gauge invariance, a spectral flow always starts from a bulk band at \( \Phi = 0 \) and ends at another bulk band at \( \Phi = 2\pi \), because the system has to come back to itself when \( \Phi \) is an integer multiple of \( 2\pi \). We note from Fig. 3a that although the interlayer couplings induce weak dispersions along \( k_{z} \), the acoustic bandgaps remain open for all \( k_{x} \) and thus the topological properties of the bulk bands remain unchanged.

Interestingly, although both gaps I and II carry non-trivial higher-order topology, they do not support any corner or edge state due to the chiral symmetry breaking, which is consistent with recent theories(35,36) and experiments(37). These higher-order topological bandgaps are in fact characterized by filling anomaly and fractional corner charges(32) (Supplementary Notes 1 and 2), instead of the commonly believed corner states. Here, remarkably, the topological Wannier cycles can serve as an experimental probe of the higher-order topology even when the corner states fail to emerge.

We emphasize that the above analysis unveils a mechanism beyond the existing studies on the dislocation modes in topological insulators(37–40) (Supplementary Notes 1–6 for details). In particular, the widely adopted theory in the literature(35) cannot explain the findings in this work, because our system is not an acoustic analogue of the weak topological insulator since there is only one band below gap I (details in Supplementary Note 6).

Experimental verification of topological Wannier cycles

Before going into the experiments, we first show from calculations that the topological Wannier cycles emerge only in topological sonic crystals but vanish in trivial sonic crystals. This property is confirmed in Fig. 4a,b via the simulation results based on the same material parameters of the experimental sample, where the trivial sonic crystal is constructed from the topological sonic crystal by interchanging the inter-cell and intra-cell couplings.

In experiments, a sonic crystal with a SSD (Fig. 4c) is fabricated using 3D-printing technology based on the acoustic model in Fig. 1e. The sonic crystal is made of photosensitive resins, which serve as the
hard-wall boundaries for acoustic waves. Using a tiny microphone attached to a thin steel rod mounted on a translational stage, we can scan the 3D acoustic wavefunctions (specifically, the acoustic pressure fields) in the sample at various excitation frequencies (from 0.5 kHz to 8 kHz with a step of 9.4 Hz). In all the measurements, an acoustic source is enclosed in either the top or the bottom cavity in hole 1 (denoted, separately, as the top or bottom excitation set-ups; Methods). Through Fourier transformations of the detected acoustic pressure fields at each excitation frequency, we can extract the dispersions of the topological boundary states (TBSs). For bottom (top) excitation, only the TBSs with positive (negative) group velocities are excited and measured. This is because only acoustic waves with positive (negative) group velocity can propagate from the source to the detector when the source is at the bottom (top) of the sample and the detector is inserted deeply into the dislocation core.

As shown in Fig. 4d,e, the measured dispersions of the TBSs in both gaps I and II agree excellently with the calculation, except when the TBSs nearly merge into the bulk bands. In the latter case, the bulk states are also excited and their contributions to the detected acoustic signals cannot be filtered out. This mixing of signals from both the bulk states and the TBSs causes the deviation between the measured acoustic dispersions and the calculated spectrum of the TBSs. The unavoidable dissipation of acoustic waves and the finite-size effect of the sample broaden the measured dispersions in both frequency and wave vector dimensions, as analysed in detail in Supplementary Note 7. Due to the dissipation and finite-size effects, the measured results show a discretized structure in Fig. 4d,e. However, overall, the measured spectrum agrees well with the calculated eigen-spectrum within gaps I and II. Those effects thus do not spoil the experimental observation of the topological Wannier cycles.

To further characterize the topological Wannier cycles, we present the measured acoustic pressure profiles in the x–y plane in Fig. 4f for five ascending excitation frequencies across gap I in the bottom excitation set-up. The detection x–y plane is sufficiently away from the acoustic source (108 mm above, as shown by the blue plane in Fig. 4c) to ensure that the measured acoustic pressure fields are mainly determined by the eigenstates wavefunctions, instead of the evanescent waves excited by the source. Figure 4f shows that as the excitation frequency gradually increases from below gap I to above gap I, the detected acoustic wavefronts evolve from extended (bulk) states to the localized TBSs, and then evolve again to extended (bulk) states. Such evolution of eigenstates is consistent with the scenario in Fig. 3. Similar phenomena are observed across gap II (Supplementary Note 7).

The spectral features can also be revealed by transmission measurements as shown in Fig. 4g–i. Results in Fig. 4g (for hole 1) show that the acoustic waves propagate persistently along the z direction in gaps I and II. By comparison, outside gaps I and II, acoustic waves decay rapidly in the z direction. In these measurements, the source is placed at the bottom of the dislocation core, while the detector is also in the dislocation core (in hole 1). The large overlap between the TBSs and the source as well as the robust wave guiding due to the TBSs along the dislocation core make the detected acoustic signals strong within gaps I and II. By contrast, outside gaps I and II, the bulk eigenstates have small overlap with the source, and there is no wave guiding along the dislocation core. Therefore, the excitation efficiency of the bulk states is lower, while the acoustic waves propagating between the source and the detector are also reduced. This is why the detected acoustic signals are weaker outside gaps I and II. Therefore, the measurement set-up in Fig. 4 is convenient for us to identify the experimental signatures of the TBSs. Results in Fig. 4h (for hole 3 near the dislocation core) show similar features as in Fig. 4g. These observations indicate that the acoustic wave propagation near the core of the SSD is dominated by the modes in gaps I and II. By contrast, the results in Fig. 4i (for hole 6 away from the dislocation core) exhibit opposite trends, indicating that the acoustic wave propagation away from the dislocation core is mainly due to the bulk modes outside gaps I and II. These experimental findings, together with the phase profiles and other results in Supplementary Note 7, support the predicted and calculated features of the TBSs.

Fig. 3 | Topological Wannier cycles: acoustic model. a, Spectral flows (coloured lines with arrows) traversing gaps I and II as obtained from the finite-element simulation based on the acoustic model in Fig. 1e with 6 × 6 unit cells in the x–y plane and periodic boundaries in the z direction. Here, the arrows indicate the spectral flow directions when the gauge flux \( \Phi = k_H \) goes from 0 to 2\( \pi \). The starting and ending points of the spectral flows are labelled by their \( C_4 \)-symmetry eigenvalues and examined in b. The bulk bands are represented by the coloured regions and labelled by the texts. Insets show the 4-symmetry eigenvalues and examined in b. The bulk bands are represented by the coloured regions and labelled by the texts. Insets show the 4-symmetry eigenvalues and examined in b. The bulk bands are represented by the coloured regions and labelled by the texts.
Note 7, are consistent with the physical picture that, in gaps I and II, acoustic wave propagation is dominated by the TBSs bound to the dislocation core, whereas outside these gaps, acoustic wave propagation is mainly through the extended bulk states.

Visualizing the TBSs and the artificial gauge flux

Here, we visualize experimentally the TBSs and characterize the artificial gauge flux in the central plaquette. For this purpose, we use the set-up with the bottom excitation as depicted in Fig. 5a. In gaps I and II, the TBSs are localized around and propagate along the central plaquette (that is, the core of the SSD; Fig. 4a,b). By scanning the acoustic pressure field in the whole sample in the bottom excitation set-up, we give the detected 3D acoustic wavefunction at the excitation frequency 3.0 kHz in Fig. 5b to visualize the TBSs directly. The acoustic structure in Fig. 5b is cut in a way to facilitate the 3D visualization of localized TBSs. The TBSs are indeed
localized around and propagate along the core of the SSD. Meanwhile, the localized acoustic wavefunction around the core of the SSD in Fig. 5a, which is obtained from the eigenstates calculation corresponding to the TBS at 3.0 kHz, agree with the measured acoustic wavefunction in Fig. 5b. The swirling of the acoustic energy flows in the inset of Fig. 5a indicates the non-vanishing angular momentum of the TBSs. Numerical calculations confirm that the acoustic TBSs indeed carry notable angular momentum (details in Supplementary Note 9).

The artificial gauge flux inserted into the central plaquette can be calibrated by directly measuring the acoustic phase winding around the dislocation core. We focus particularly on the acoustic phases in the five cavities, A, B, C, D and A\textsuperscript{′}, at the centre of the sample (Fig. 5c). From these phases, we can determine quantitatively the gauge phase accumulation when circulating the central plaquette once anticlockwise. In particular, the phase difference, \(\varphi_{A\textsuperscript{′}} - \varphi_A\), which is the gauge phase accumulated by circulating the central plaquette once anticlockwise, gives exactly the artificial gauge flux inserted into the central plaquette, \(\Phi = kzH\). Therefore, the dependence of the artificial gauge flux \(\Phi\) on the excitation frequency should be consistent with the dispersion of the TBSs. As shown in Fig. 5d, the experimental results indeed show such consistency. In the bottom excitation set-up, only the branch of the TBSs with positive group velocities are excited in each bandgap. The agreement between the calculated dispersion of the TBSs and the measured dependence of the artificial gauge flux \(\Phi\) on the excitation frequency is reasonably good, except when the group velocity of the TBSs becomes negative (and thus the TBSs become difficult to excite) or when the dispersion of the TBSs is close to the bulk continua. In the latter case, the simultaneously excited TBSs and bulk states mess up the measured phase difference and thus lead to the deviation between the measured gauge flux \(\Phi = \varphi_{A\textsuperscript{′}} - \varphi_A\) and the theoretical gauge flux.

**Fig. 5 | Visualizing the TBSs and the artificial gauge flux.**

- **a.** Illustration of the TBSs localized around the central plaquette (that is, the core of the SSD) as excited by an acoustic source in the bottom cavity in hole 1. Inset gives a zoomed-in picture around the system centre. Hot colours indicate the acoustic pressure amplitude |\(p|\), while the green arrows indicate the distribution of the acoustic energy flows.
- **b.** The measured acoustic pressure profile |\(p|\) (represented by the colour bar, in dB) in the whole sample at the excitation frequency of 3.0 kHz (within gap I).
- **c.** Calculated 3D acoustic phase profile around the core of the SSD. Inset gives the zoomed-in structure.
- **d.** The acoustic phase difference between the A\textsuperscript{′} and A cavities, \(\varphi_{A\textsuperscript{′}} - \varphi_A\), as a function of the excitation frequency for the set-up illustrated in a. The calculated spectrum of the acoustic system versus \(kzH\) is also presented in the figure for comparison; the blue curves give one branch of the TBSs in each bandgap, and the coloured regions represent the bulk bands. The brown square labels the excited TBS that is studied in a-c.
\[ \Phi = k \cdot H \] (more data and analysis in Supplementary Note 10). The measured eigenstates also have very high fidelity with the calculated eigenstates within gaps I and II (Supplementary Note 7).

The direct measurements and visualization of the TBSs in 3D space and the transmission characterization of the 3D system in Figs. 4 and 5, which have not been achieved in previous experimental studies of dislocations, demonstrate the advantages of the versatile measurements in acoustic systems.

**Discussions and outlook**

Our study unveils a concept of topological Wannier cycles, which are manifested as cyclical spectral flows traversing the bandgaps. This phenomenon provides a powerful tool for the study of topological crystalline materials. Particularly, it gives access to the measurement of Wannier centres with sub-unit-cell spatial resolution, which is crucial for the experimental investigation of various topological phases in crystalline solids for electrons and in metamaterials for phonons and photons. For instance, the mechanism discovered here can be generalized to a photonic system, such as photonic crystalline fibres, which can yield robust one-dimensional (1D) wave guiding in optical systems without breaking time-reversal symmetry. This mechanism can also be applied to electronic systems, which will provide an experimental probe of a large category of topological crystalline insulators with filling anomaly that widely exist in crystalline compounds, as shown by recent works based on ab initio calculations. These topological crystalline insulators are found to be valuable for catalytic and energy applications due to their topologically induced large surface charge density or surface density of states. For acoustic systems, the discovered topological helical modes propagating along the screw dislocation can serve as robust 1D wave guiding in 3D acoustic systems, as demonstrated in Fig. 4 and elaborated in Supplementary Note 11.

In addition, the single-plaquette gauge flux realized here is valuable in the manipulation of phonons as well as in the engineering of topological states. Here, the local gauge flux \( \Phi = k \cdot H \) can be expressed as the inner product between the Burgers vector of the SSD and the wave vector of the system. When multiple SSDs exist in a system, they lead to multiple local gauge fluxes. By varying the Burgers vectors of the SSDs locally, pseudo-magnetic fields and other effects can be created. Finally, we expect that the underlying physics and topological phenomena would be much richer if our study were extended to 3D topological crystalline materials, since the topological crystalline insulator phases in three dimensions are much richer than those in two dimensions. However, this would require a synthetic fourth dimension to create the artificial gauge flux, which can be realized in some photonic and acoustic systems (for example, as in one previous study).

**Online content**

Any methods, additional references, Nature Research reporting summaries, source data, extended data, supplementary information, acknowledgements, peer review information; details of author contributions and competing interests; and statements of data and code availability are available at [https://doi.org/10.1038/s41563-022-01200-w](https://doi.org/10.1038/s41563-022-01200-w).

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Methods

Simulations. We performed systematic finite-element simulations for the acoustic waves in the 3D-printed sonic crystal structures using COMSOL Multiphysics with the pressure acoustic module. Due to the huge acoustic impedance mismatch between air and the photosensitive resin used in the 3D printing, the latter can be treated as sound hard boundaries in the simulation. Sound waves propagate in air with a mass density of 1.25 kg m\(^{-3}\) at a speed of 343 m s\(^{-1}\) at room temperature (23°C). The bulk band dispersions of the acoustic waves in the 2D sonic crystal (Fig. 1b in the main text) are calculated using a single unit cell (shown in the inset of Fig. 1e in the main text) with Floquet–Bloch boundary conditions in both the x and y directions. However, to calculate the eigenstates of the structure with a SSD, we treat the system as periodic in the z direction but finite in the x and y directions. Specifically, using the supercell shown in Fig. 1e in the main text, we calculate the acoustic dispersions with the Floquet–Bloch boundary condition in the z direction and closed boundary conditions in the x and y directions. Simultaneously, the acoustic wavefunctions (that is, the amplitude and phase profiles of the acoustic pressure field) of the eigenstates are obtained.

Experiments. The sample was manufactured by 3D-printing technology using photosensitive resin and was assembled through layer-by-layer stacking up to 13 layers. All geometry parameters in the main text refer to the air regions where the acoustic waves propagate and reside. The printed sample had an average thickness of several millimetres to encapsulate the air regions for the confinement of the acoustic waves in the designed structure. To measure the TBSs, a headphone of diameter 6 mm was used for acoustic excitations with the frequency sweeping from 0.5 kHz to 8 kHz at a step of 9.4 Hz. The headphone was placed and enclosed in either the top or the bottom cavity of hole 1 in the sample (denoted as the top or bottom excitation, respectively). A tiny microphone (2.5 mm × 1.1 mm × 3.3 mm) mounted on a steel rod was connected with the network analyser (Keysight E5061B) and inserted into the sample to detect the acoustic pressure profile (Supplementary Note 7). Automatic scanning with 9 mm steps in the z direction in each hole as driven by a translational stage could be used to measure the acoustic pressure profiles in the whole sample if needed. Such measurements contain both the amplitude and phase profiles of the acoustic pressure field, thanks to the data processing by the network analyser. By fast Fourier transformations of the detected acoustic pressure profiles at each excitation frequency, we can extract the dispersions of the acoustic waves in the sample. Specifically, this method is used to determine experimentally the dispersions of the TBSs. Such acoustic pump–probe measurements are also used to analyse the phase accumulations around the central plaquette (Supplementary Note 10). To directly visualize the acoustic wavefunctions of the TBSs inside the sample (as shown in Fig. 5b), we measure the acoustic pressure profiles in the whole sample in the bottom excitation set-up.

Data availability

All relevant data are presented in detail in the manuscript and the Supplementary Information. Additional information is available from the corresponding authors through reasonable request.

Code availability

We use the commercial software COMSOL Multiphysics to perform the acoustic wave simulations and eigenstate calculations based on finite-element methods. Reasonable requests to have the computation details can be addressed to the corresponding authors.

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Author contributions

J.-H.J. initiated the project and guided the research. J.-H.J. and Z.-K.L. established the theory. Y.W., Z.-K.L., B.J. and Y.L. performed the numerical calculations and simulations. Y.W., Z.-K.L., S.-Q.W., J.-H.J. and F.L. designed and performed the experiments. All the authors contributed to the discussions of the results and the manuscript preparation. J.-H.J., Z.-K.L. and Y.W. wrote the manuscript and the Supplementary Information.

Competing interests

The authors declare no competing interests.

Additional information

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