Self-similar evaporation of a rigidly rotating cosmic string loop

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Abstract
The gravitational back-reaction on a certain type of rigidly rotating cosmic string loop, first discovered by Allen, Casper and Ottewill, is studied at the level of the weak-field approximation. The near-field metric perturbations are calculated and used to construct the self-acceleration vector of the loop. Although the acceleration vector is divergent at the two kink points on the loop, its net effect on the trajectory over a single oscillation period turns out to be finite. The net back-reaction on the loop over a single period is calculated using a method due to Quashnock and Spergel, and is shown to induce a uniform shrinkage of the loop while preserving its original shape. The loop therefore evolves by self-similar evaporation.

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1. Introduction
Cosmic strings are thin filaments of Higgs field energy that may have played a role in the formation of large-scale structure in the early universe. The dynamics of cosmic strings in Minkowski and Robertson–Walker backgrounds has been studied extensively [1–6], and their possible cosmogonic effects are well known (see [7] for a review), but their gravitational properties remain relatively unexplored. In particular, the motion of a string loop under the action of its own gravitational field is described by a nonlinear system of equations that can normally be solved only by numerical methods, even in the case of a zero-thickness string in the weak-field limit [8].

Quite apart from the potential cosmological ramifications, a better understanding of the gravitational field and back-reaction effects of general cosmic string configurations is important for the theoretical development of general relativity. Zero-thickness cosmic strings form a class of line singularities that are natural higher dimensional analogues of black holes, and it remains an open question whether they can be satisfactorily incorporated into the framework of general relativity. Areas that are usually regarded as problematic include the...
definition of an invariant characterization of zero-thickness cosmic strings, the development of a suitable distributional formalism for line singularities, and the demonstration that zero-thickness cosmic strings are the unique limits of well-behaved finite-thickness solutions [9]. These issues can all be resolved in the few known cases of exact self-gravitating solutions [10–13], but a general resolution awaits further work.

The uncertain status of line singularities in general relativity is underscored by the back-reaction problem. Almost nothing is known about the motion of a zero-thickness cosmic string under the action of its self-gravity. Although it can be shown that the self-gravity of a GUT string would almost everywhere be small enough to justify a weak-field treatment [8], even the weak-field dynamics is complicated and therefore little studied. Furthermore, flat-space string solutions commonly support pathological features—kinks and cusps—where the weak-field approximation is known to break down [14, 15]. Much work needs to be done to justify the claim that zero-thickness cosmic strings can move freely and self-consistently under their own gravity, and are therefore compatible with general relativity.

In this paper, I calculate the gravitational field and back-reaction force on a certain type of rigidly rotating cosmic string loop—the Allen–Casper–Ottewill solution [16]—in the weak-field limit, and show that the loop evolves by self-similar shrinkage. This is the first explicit back-reaction calculation to be published for a cosmic string loop, as all previous calculations have relied on numerical approximation [8]. The importance of the results described below lies not so much in the simple behaviour of the loop, but in the detailed features of the gravitational effects involved. In particular, the loop supports a pair of kinks near which the back-reaction force diverges but nonetheless remains integrable.

2. Preliminaries

A zero-thickness cosmic string is a line singularity which as it moves traces out a two-dimensional surface \( T \) whose dynamics and stress–energy content are governed by the Nambu–Goto action. In what follows, \( x^a \equiv [x^0, x^1, x^2, x^3] \) are local coordinates on the four-dimensional background spacetime \((M, g_{ab})\), and the metric tensor \( g_{ab} \) has signature \((+,-,-,-)\), so that timelike vectors have positive norm. The world sheet \( T \) of a general zero-thickness cosmic string is then described parametrically by a set of equations of the form \( x^a = X^a(\zeta^A) \), where the parameters \( \zeta^A \equiv (\zeta^0, \zeta^1) \) are commonly referred to as world-sheet or gauge coordinates.

The intrinsic 2-metric induced on \( T \) by a given choice of gauge coordinates is

\[
\gamma_{AB} = g_{ab} X_a^A X_b^B
\]

(2.1)

where \( X_a^A \) is shorthand for \( \partial X^a / \partial \zeta^A \). For a string composed of non-tachyonic material \( \gamma_{AB} \) is non-degenerate with signature \((+, -, -)\) almost everywhere on \( T \), although \( \gamma_{AB} \) may be degenerate at isolated points called cusps. If \( \gamma \) denotes \( |\text{det}(\gamma_{AB})| \) the Nambu–Goto action [17, 18] has the form

\[
I = -\mu \int \gamma^{1/2} d^2 \zeta
\]

(2.2)

where \( \mu \) is the mass per unit length of the string.

The Lagrangian density \( \mathcal{L} = -\mu \gamma^{1/2} \) in (2.2) is a functional of the position functions \( X^a \) and their gauge derivatives, and of the background metric \( g_{ab} \). The equation of motion of the corresponding string is recovered by setting the functional derivative \( \delta \mathcal{L} / \delta X^a \) equal to zero, while the stress–energy tensor \( T^{ab} \) at a general point on the world sheet \( T \) is

\[
-2g^{-1/2} \delta \mathcal{L} / \delta g_{ab}
\]

where \( g \equiv |\text{det}(g_{ab})| \). For the purposes of calculation, it is convenient to first fix the gauge so that \( \gamma_{AB} \) is everywhere proportional to the Minkowski tensor \( \eta_{AB} = \text{diag}(1, -1) \), a choice often
referred to as the standard gauge. The corresponding gauge coordinates are conventionally written as $\tilde{t}^0 \equiv t$ and $\tilde{t}^1 \equiv \sigma$.

In the standard gauge, with $X^a_\sigma = \partial X^a / \partial \sigma$ and $X^a_t = \partial X^a / \partial t$, the string equation of motion reads

$$X^a_{tt, \sigma} + \Gamma^a_{bc} X^b_t X^c_t = X^a_{\sigma \sigma} + \Gamma^a_{bc} X^b_\sigma X^c_\sigma$$

(2.3)

where $\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\delta_{db,c} + \delta_{dc,b} - \delta_{bc,d})$ is the Christoffel symbol on the background spacetime, evaluated at $x^a = X^a(t, \sigma)$. Also, the stress–energy tensor at a general spacetime point $x^a$ in the absence of other material sources is

$$T^{ab}(x^c) = \mu g^{-1/2} \int (X^a x^b - X^a x^b ) \delta^{(d)}(x^c - x^c) \, dt \, d\sigma.$$  

(2.4)

The strong-field version of the back-reaction problem is completely specified by combining (2.3) and (2.4) with the Einstein equation $G^{ab} = -8\pi T^{ab}$, where $G^{ab} = \frac{1}{2} g^{ab} R$ is the Einstein tensor$^1$.

To date, the only known solutions to the strong-field back-reaction problem describe either

infinite straight strings (with or without an envelope of cylindrical gravitational waves [19–22]) or infinite strings interacting with plane-fronted gravitational waves (the so-called travelling-wave solutions [23, 24]). One of the reasons why an explicit solution is possible in these cases is that the world sheets are intrinsically flat and non-dissipative. A self-gravitating string loop, by contrast, would be expected to continually radiate energy and angular momentum, and there seems little prospect of generating strong-field solutions of this type. The weak-field approximation is therefore the only tool currently available for analysing loop evolution.

In the weak-field approximation $g_{ab} = \eta_{ab} + h_{ab}$, where $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ and the components of $h_{ab}$ are all small in absolute values compared to 1. To linear order in $h_{ab}$ the string equation of motion (2.3) then reads

$$X^a_{tt, \sigma} - X^a_{\sigma \sigma} = \alpha^a$$

(5.5)

where

$$\alpha^a = (h^a_{bc} - \frac{1}{2} \eta^{ad} h_{bc, d}) (X^b_\sigma X^c_\sigma - X^b_\tau X^c_\tau)$$

(2.6)

is the local 4-acceleration of the string. Here, indices are everywhere raised and lowered using the Minkowski tensor $\eta_{ab}$ and its inverse $\eta^{ab}$. For future reference, $h \equiv h^a_a$ and inner products such as $X^2_\sigma$ and $X_\sigma \cdot X_a$ are formed using $\eta_{ab}$.

If the harmonic gauge conditions $h^b_{\tau b} = \frac{1}{2} h_{\tau a}$ are imposed, the Einstein equation $G^{ab} = -8\pi T^{ab}$ reduces to $\Box h_{ab} = -16\pi S_{ab}$, where $S_{ab} = T_{ab} - \frac{1}{2} \eta_{ab} T^c$. The flat-space d’Almbertian. With $x^a \equiv [t, x]^a$ the standard solution for $h_{ab}$ is

$$h_{ab}(t, x) = -4 \int S_{ab}(t', x') |x - x'| d^3x'$$

(2.7)

where $t' = t - |x - x|$ is the retarded time at the source point $x'$.

When the source function $S_{ab}$ corresponding to the string stress–energy tensor (2.4) is inserted into (2.7) an integral over $t, \sigma$ and the three components of $x'$ results. The distributional factor $\delta^{(d)}(x^c - x')$ can be integrated out by first transforming from $y^a \equiv [t, x]^a$ to $z^a \equiv [t', x]' - X^a(t, \sigma)$, with $\sigma, t$ and $x$ fixed. The Jacobian of this transformation is

$$|\partial y^a / \partial z^b| = |\partial x^b / \partial y^a|^{-1} = |X^b_\sigma - n^b \cdot X_\tau|^{-1},$$

(2.8)

$^1$ The convention adopted here for the Riemann tensor is that

$$R_{abcd} = \frac{1}{2} (\Gamma^e_{bacd} + \Gamma^e_{bdac} - \Gamma^e_{bdca} - \Gamma^e_{bcad}) + \text{c.f.} \left(\Gamma^f_{ae} \Gamma^f_{bd} - \Gamma^f_{bf} \Gamma^f_{ae}\right).$$

Also, geometrized units have been chosen, so that the gravitational constant $G$ and the speed of light $c$ are set to 1.
where \( n' = (\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'| \) is the unit vector from the source point \( \mathbf{x}' \) to the field point \( \mathbf{x} \), and \( X^a \equiv [X^0, X^i] \). On integrating over \( x^a \), \( x' \) is everywhere replaced by \( X^0 \) and \( \mathbf{x}' \) by \( \mathbf{X} \). In particular, the gauge coordinate \( \tau \) becomes an implicit function of \( \sigma \) through the light-cone condition

\[
X^0(\tau, \sigma) = t - |\mathbf{x} - \mathbf{X}(\tau, \sigma)|. \tag{2.9}
\]

The corresponding curve in the \( \tau-\sigma \) plane, which represents the intersection of the past light cone of the field point with the string world sheet, will be denoted by \( \Gamma \).

Since \( g^{1/2} = 1 \) to leading order in \( h_{ab} \), the solution for \( h_{ab} \) reads

\[
h_{ab}(t, \mathbf{x}) = -4\mu \int_{\Gamma} \frac{\Psi_{ab}}{|\mathbf{x} - \mathbf{X}|} |X^0_t - n \cdot X_t|^{-1} d\sigma \tag{2.10}
\]

where now \( n = (\mathbf{x} - \mathbf{X})/|\mathbf{x} - \mathbf{X}| \) and

\[
\Psi_{ab}(\tau, \sigma) = X_t^a X_t^b - X_\sigma^a X_\sigma^b - \frac{1}{2} \eta_{ab}(X^2_t - X^2_\sigma). \tag{2.11}
\]

The equation of motion (2.5) and expression (2.10) for the metric perturbations \( h_{ab} \) together constitute the weak-field back-reaction problem.

It should be stressed that the weak-field equation of motion (2.5) holds only in a flat background, and in a particular gauge. Since questions have been raised about the validity of this equation [25] it is worth taking the time to briefly review its derivation. The basic assumption underlying the weak-field back-reaction formalism is that any solution pair \( (g_{ab}, X^a) \) to the strong-field problem represented by equations (2.3) and (2.4) can—for small values of the mass per unit length \( \mu \) at least—be expanded as perturbative series of the form

\[
g_{ab} = \sum_{k=0}^{\infty} \mu^k g_{ab}^{(k)} \quad \text{and} \quad X^a = \sum_{k=0}^{\infty} \mu^k X^a_{(k)} \tag{2.12}
\]

where the functions \( g_{ab}^{(k)} \) and \( X^a_{(k)} \) are independent of \( \mu \).

In the problem at hand, \( g_{ab}^{(0)} = h_{ab} \) and \( \mu g_{ab}^{(1)} = h_{ab} \), while the position vector \( X^a \) appearing in (2.5) should be read as \( X^a_{(0)} + \mu X^a_{(1)} \). However, for consistency the occurrences of \( X^a \) in (2.6), (2.10) and (2.11) are all truncated at \( X^a_{(0)} \) only. The weak-field equation of motion (2.5) is therefore effectively two equations, namely

\[
X^a_{(0)};r^a - X^a_{(0),r} = 0 \tag{2.13}
\]

and

\[
\delta X^a;\tau - \delta X^a_{,\tau} = \alpha^a \tag{2.14}
\]

with \( \delta X^a = \mu X^a_{(1)} \).

Equation (2.5) was generated by first fixing a gauge and then inserting the series expansions (2.12) into the strong-field equation of motion (2.3). An alternative method of derivation is to first write down the perturbative equation of motion in a gauge-independent form, and then fix the gauge. A suitable gauge-independent formalism has been developed by Battye and Carter [26], and in appendix B it is shown that this second approach does indeed lead to the weak-field equations of motion (2.13) and (2.14) in the standard gauge.

The structure of the paper from this point onwards is (I hope) very straightforward. The unperturbed solution \( X^a_{(0)} \) is described in section 3 (where, however, I will write \( X^a \) in place of \( X^a_{(0)} \)), the metric perturbation \( h_{ab} \) is calculated in section 4.1, the acceleration vector \( \alpha^a \) is calculated in section 4.2, and the perturbed trajectory \( X^a = X^a_{(0)} + \delta X^a \) is discussed in section 4.3. However, many of the details of the calculation of \( \alpha^a \) are consigned to appendix A.
3. The Allen–Casper–Ottewill loop

The standard method for solving the weak-field back-reaction problem, which was pioneered numerically by Quashnock and Spergel [8] over a decade ago, is to start with a loop solution \( X^a(τ, σ) \) to the flat-space equation of motion \( X^a,ττ - X^a,σσ = 0 \), calculate the corresponding self-acceleration vector \( α^a \) at each point on the world sheet over a single period of oscillation of the loop, integrate the resulting weak-field equation of motion (2.5) over a complete period to generate a new, perturbed solution of the flat-space equation of motion, and then iterate the procedure. What results is not a continuous solution to (2.5) but rather a sequence of flat-space solutions, each member of which is constructed by debiting from the preceding member of the sequence the gravitational energy and momentum it radiates over a single period. A quasi-stationary approximation scheme of this type is valuable whenever the mass per unit length \( μ \) of the string is small—such as the case for GUT strings, which have \( μ \sim 10^{-6} \)—as the sequence of flat-space solutions is then almost continuous.

The Quashnock–Spergel method will be described in more detail in the next section. In this section, I will briefly explain the choice of an initial loop solution \( X^a(τ, σ) \) to be used in the algorithm. The general form of loop solutions to the flat-space equation of motion \( X^a,ττ - X^a,σσ = 0 \) is well known. In a flat background it is always possible to choose the gauge coordinates so that \( X^0(τ, σ) = τ \). The spatial components of the position function \( X^a \) have the form

\[
X^a(τ, σ) = \frac{1}{2}[a^a(τ + σ) + b^a(τ − σ)]
\]  

(3.1)

where for a loop in its centre-of-momentum frame the functions \( a^a \) and \( b^a \) are each periodic functions of their arguments with some parametric period \( L \). The gauge constraints \( γ^a_τ + γ^a_σ \equiv X^a_τ + X^a_σ = 0 \) and \( γ^a_τ \equiv X^a_τ \cdot X^a_τ = 0 \) together imply that \( |a|^2 = |b|^2 = 1 \), where a prime denotes differentiation with respect to the relevant argument.

Since \( X(τ + L/2, σ + L/2) = X(τ, σ) \) for any values of \( τ \) and \( σ \) when \( a^a \) and \( b^a \) are periodic functions, the fundamental oscillation period \( t_p \) of the loop is \( L/2 \) rather than the parametric period \( L \). The total 4-momentum of the string loop on any surface of constant \( t = τ \) is

\[
p^a = \mu \int_0^L X^a_τ dσ = \mu \int_0^L \left[ \frac{1}{2} a^a(τ + σ) + \frac{1}{2} b^a(τ − σ) \right] dσ = \mu L [1, 0]
\]  

(3.2)

and in particular the energy of the loop (in this, the centre-of-momentum frame) is \( E = \mu L \). The corresponding angular momentum of the loop is

\[
J = \frac{1}{4} \mu \int_0^L (a \times a' + b \times b') dσ.
\]  

(3.3)

In the weak-field approximation the gravitational power per unit solid angle radiated by a periodic source out to a point \( x \) in the wave zone, where \( r \equiv |x| \) is much greater than the characteristic dimension \( L \) of the source, is given by

\[
\frac{dP}{dΩ} = -\frac{r^2}{8π} \sum_{j=1}^{3} n^j (R^{0j}),
\]  

(3.4)

where \( n = x/r \) and \( R^{ab} \) here denotes the Ricci tensor evaluated to the second order in the metric perturbations \( h_{ab} \) and to the second order in \( r^{-1} \). The angle brackets indicate coarse-grained averaging, which is performed by expanding \( h_{ab} \) as a Fourier series in the components of the field point \( x^a \) and then eliminating all but the zeroth-order Fourier mode from \( R^{ab} \).

The resulting expression for the power radiated per unit solid angle is

\[
\frac{dP}{dΩ} = \frac{α^2}{π} \sum_{m=1}^\infty m^2 \left[ \overline{T_{ab} T_{ab}} - \frac{1}{2} |T_a|^2 \right]
\]  

(3.5)
where \( \omega = 2\pi / t_p \) is the circular frequency of the source, and

\[
T^{ab}(m, n) = t_p^{-1} \int_{B^3} \int_0^{t_p} T^{ab}(t, x) e^{i\omega(t-x)} \, dt \, dx. \tag{3.6}
\]

In the case of the string stress–energy tensor (2.4), with \( X^a = \left[ t, \frac{1}{2}(a + b) \right]^\mu \) a given flat-space solution, it is more convenient to work with the light-cone coordinates \( \sigma \equiv \tau \pm \sigma \), rather than with \( \tau \) and \( \sigma \) themselves. The fundamental domain \( 0 \leq \sigma < L \) is then equivalent to \( 0 \leq \sigma - < L \) and \( 0 \leq \sigma < L \). If \( a^\nu \equiv [1, a(\sigma_+)]^\nu \) and \( b^\nu \equiv [1, b(\sigma_-)]^\nu \) then \( T^{ab} = \mu LA^{(a} B^{b)} \), where the round brackets denote symmetrization, and

\[
A^\nu(m, n) = \frac{1}{L} \int_0^L e^{i2\pi \sin[\sigma_+_n a(\sigma_+)]/L} a^\nu(\sigma_+) \, d\sigma_+ \tag{3.7}
\]

and

\[
B^\nu(m, n) = \frac{1}{L} \int_0^L e^{i2\pi \sin[\sigma_- n b(\sigma_-)]/L} b^\nu(\sigma_-) \, d\sigma_. \tag{3.8}
\]

In terms of the 4-vectors \( A^\nu \) and \( B^\nu \), therefore,

\[
\frac{dP}{d\Omega} = 8\pi \mu^2 \sum_{m=1}^{\infty} m^2 [(A \cdot A^*) (B \cdot B^*) + |A \cdot B^*|^2 - |A \cdot B|^2]. \tag{3.9}
\]

A convenient measure of the total power radiated by a string loop is the radiative efficiency \( \gamma^0 \equiv \mu^{-2} P \), where \( P = \int \frac{dP}{d\Omega} \, d\Omega \) is the total power of the loop. In terms of \( \gamma^0 \), the fractional energy radiated by the loop over the double oscillation period \( L = 2t_p \) is \( |\Delta E/E| = (P L)/(\mu L) = \mu \gamma^0 \). Numerical simulations of networks of string loops indicate that \( \gamma^0 \) is typically of order 65–70, although it may fall as low as 40 [27, 28].

The range of \( \gamma^0 \) has been examined analytically by Allen, Casper and Ottewill [16] for a large class of string loops in which the mode function \( a \) consists of two anti-parallel segments of length \( \frac{1}{2} L \) aligned along the \( z \)-axis, so that

\[
a(\sigma_+) = \begin{cases} 
\sigma_+ \hat{z} & \text{if } 0 \leq \sigma_+ < \frac{1}{2} L \\
(L - \sigma_+) \hat{z} & \text{if } \frac{1}{2} L \leq \sigma_+ < L, 
\end{cases} \tag{3.10}
\]

and the mode function \( b \) is restricted to the \( x-y \) plane but is otherwise unconstrained. The minimum possible value of \( \gamma^0 \) for loops of this class is approximately 39.0025, and occurs when

\[
b(\sigma_-) = \frac{L}{2\pi} \left[ \cos(2\pi \sigma_-/L) \hat{x} + \sin(2\pi \sigma_-/L) \hat{y} \right]. \tag{3.11}
\]

The corresponding loop consists of two helical segments which are rigidly rotating about the \( z \)-axis with an angular speed \( \omega = 4\pi / L \). The evolution of this loop is illustrated in figure 1, which shows the \( y-z \) projection of the loop at times \( \tau - \epsilon = 0 \), \( L/16 \), \( L/8 \) and \( 3L/16 \) (top row) and \( \tau - \epsilon = L/4 \), \( 5L/16 \), \( 7L/16 \) and \( 9L/16 \) (bottom row), where the time offset \( \epsilon \) is 0.02\( L \). The string has been artificially thickened for the sake of visibility, and the \( z \)-axis is also shown. The projections of the loop onto the \( x-y \) plane are circles of radius \( L/(4\pi) \). The points at the extreme top and bottom of the loop, where the helical segments meet and the modal tangent vector \( a' \) is discontinuous, are technically known as kinks. They trace out circles in the planes \( z = L/4 \) and \( z = 0 \). All other points on the loop trace out identical circles, although with varying phase lags. The net angular momentum of the loop is \( J = \frac{1}{2\pi} \mu L^2 \hat{z} \).

The importance of the solution defined by the mode functions (3.10) and (3.11), which I will henceforth call the Allen–Casper–Ottewill (ACO) loop, lies in the fact that its radiative efficiency \( \gamma^0 \approx 39.0025 \) is the lowest known of any loop solution. Casper and Allen [28] examined an ensemble of 11 625 string loops, generated by numerically evolving a large set of parent loops forward in time (in the absence of radiative effects) until only non-intersecting
child loops remained, and found that only six of the loops studied had radiative efficiencies less than 42, while none at all had $\gamma^0 < 40$. When examined more closely, all six of the low-$\gamma^0$ loops were seen to have the same general shape as the ACO loop. A second study by Casper and Allen involving another 12,830 loops yielded similar results.

This observation has interesting implications for the evolution of a network of string loops. If it can be shown that an ACO loop does not evolve secularly towards another shape with higher radiative efficiency, then it will be the longest lived loop in any ensemble of loops with a given energy $E$ (and invariant length $L$). Furthermore, if the general trend in loop evolution is from higher to lower radiative efficiencies—a trend which has not so far been demonstrated, but which would be expected if radiative dampening acts to preferentially eliminate higher multipole structure—then the ACO loop would in some sense play the role of an attractor in the ensemble of loops forming a network. However, it is conceivable that, because the energy $E$ of a string loop falls off as $E_0 - \mu \gamma^0 t$ (since $|\Delta E| = \mu \gamma^0 E$ over a time period $\Delta t = 2t_p$), but $\Delta t = E/\mu$, any fall in the radiative efficiencies of the loops in a network might proceed too slowly to have a significant effect on any but the very largest loops.

What I will demonstrate below is that, in the weak-field approximation at least, the ACO loop does indeed evaporate in a self-similar manner, and so $\gamma^0$ remains constant throughout its evolution. When the weak-field equation of motion (2.5) has been integrated over a single period $t_p$, it turns out that an initial ACO loop with energy $E$ and invariant length $L$ decays to form a loop with exactly the same shape, but with smaller values $E' = E \left(1 - \frac{1}{2} \mu \gamma^0\right)$ and $L' = L \left(1 - \frac{1}{2} \mu \gamma^0\right)$ of energy and length, plus a small rotational phase shift.

4. Back-reaction of the ACO loop

4.1. The metric perturbations

The first step in the back-reaction calculation is to evaluate the line integrals (2.7) for the metric perturbations $h_{ab}$. Let $X^a \equiv [\tau, X]^a$ be a general source point on the string loop, and
\( \vec{x}^a \equiv [\vec{r}, \vec{x}]^a \) a general field point. The mode functions of an ACO loop with invariant length \( L \) are as given in (3.10) and (3.11), but can be simplified slightly (by rescaling the gauge coordinates \( \sigma_+ \) and \( \sigma_- \)) to read
\[ a(\sigma_+) = L|v_a|\vec{z} \quad \text{if} \quad -\frac{1}{2} \leq v_a \leq \frac{1}{2} \] (4.1)
and
\[ b(\sigma_-) = \frac{L}{2\pi} [\cos(2\pi v_+)k + \sin(2\pi v_-)\hat{y}] \] (4.2)
where \( (v_+, v_-) \equiv (\sigma_+/L, \sigma_-/L) \). Note that at any point on the world sheet it is always possible to choose \( v_+ \) and \( v_- \) so that \( \tau \equiv \frac{1}{2}L(v_+ + v_-) \) remains equal to the coordinate time \( t \), but \( v_+ \in [-\frac{1}{2}, \frac{1}{2}] \).

The spacetime coordinates (3.1) of the source point are then
\[ X^a = \frac{1}{2} \left[ \frac{v_+ + v_-}{L} \frac{\cos(2\pi v_-)}{2\pi} \hat{x} + \frac{\sin(2\pi v_-)}{2\pi} \hat{y} + |v_a|\hat{z} \right]^a \] (4.3)
For a general field point \( \vec{x}^a \), with \( \vec{x} = \vec{r} + \vec{y} + \vec{z} \), the light-cone condition \( \vec{r} - \tau = |\vec{x} - X| \) reads
\[ 2\ell/L - v_+ - v_- = \left\{ \left[ \frac{2\vec{r}}{\ell} - \frac{1}{2\pi} \cos(2\pi v_-) \right]^2 + \left[ \frac{2\vec{y}}{\ell} - \frac{1}{2\pi} \sin(2\pi v_-) \right]^2 + (2\vec{z}/L - |v_a|)^2 \right\}^{1/2}, \] (4.4)
or equivalently
\[ v_+ = -\frac{1}{2}(v_- - 2\ell/L - 2s\vec{z}/L) + \frac{1}{2} \left\{ \left[ \frac{2\vec{r}}{\ell} - \frac{1}{2\pi} \cos(2\pi v_-) \right]^2 + \left[ \frac{2\vec{y}}{\ell} - \frac{1}{2\pi} \sin(2\pi v_-) \right]^2 \right\}^{1/2} \] (4.5)
where \( s = \text{sgn}(v_+) \).

In particular, if the field point itself lies on the loop, and has scale-free gauge coordinates \( (\sigma_+/L, \sigma_-/L) = (u_+, u_-) \) so that
\[ [\vec{r}, \vec{x}] = \frac{1}{2} \left[ u_+ + u_- \frac{\cos(2\pi u_-)}{2\pi} \hat{x} + \frac{\sin(2\pi u_-)}{2\pi} \hat{y} + |u_a|\hat{z} \right] \] (4.6)
then the light-cone condition (4.4) reads
\[ u_- - u_+ + u_- - v_- = \left[ \frac{1}{2\pi^2} \{1 - \cos 2\pi(u_- - v_-)\} + (\delta u_+ - s v_a)^2 \right]^{1/2} \] (4.7)
where \( \delta = \text{sgn}(u_+) \). (For future convenience, the 4-vector on the right-hand side of (4.6) will be abbreviated as \( \vec{X}^a \equiv [\vec{r}, \vec{X}]^a \).)

Recall from the previous section that the ACO loop consists of two helical segments bounded by kinks at \( \sigma_+ = -\frac{1}{2}, 0 \) and \( \frac{1}{2} \). If the source point and field point lie on the same segment \( (s\delta = 1) \) then \( s\delta u_+ - sv_a = \pm(u_+ - v_+) \) and (4.7) can be rearranged to read
\[ 2(u_+ - v_+)(u_- - v_+) + (u_- - v_-)^2 = \frac{1}{2\pi^2} \{1 - \cos 2\pi(u_- - v_-)\} \] (4.8)
For a fixed field point \( (u_+, u_-) \) this equation clearly has two possible solutions:
\[ v_- = u_- \] (4.9)
and

\[ v_+ - u_+ = -\frac{1}{2}(v_- - u_-) + \frac{1}{4\pi^2} \{ 1 - \cos 2\pi(v_- - u_-) \} / (v_- - u_-). \]  

(4.10)

In view of (4.7) the first solution is feasible if and only if \( v_+ \leq u_+ \). It is also easily verified that if the second solution holds then \( u_+ - v_+ + u_- - v_- \geq 0 \) if and only if \( v_- - u_- < 0 \), in which case (4.10) indicates that \( v_+ > u_+ \). So the two solutions are mutually exclusive and fully cover the case where the source point and field point lie on the same segment.

If on the other hand the two points lie on different segments, then \( s\vec{\tau} = -1 \) and \( s\vec{u}_+ - s\vec{v}_+ = \pm(u_+ + v_+) \). So (4.7) expands to give

\[ (2u_+ - v_- + u_-)(-2v_+ - v_- + u_-) = \frac{1}{2\pi^2} \{ 1 - \cos 2\pi(u_- - v_-) \}, \]  

(4.11)
or equivalently

\[ v_+ = -\frac{1}{2}(v_- - u_-) + \frac{1}{4\pi^2} \{ 1 - \cos 2\pi(v_- - u_-) \} / (v_- - u_- - 2u_+). \]  

(4.12)

For fixed values of \( u_+ \) and \( u_- \) this equation typically defines two branches in the \( v_+ - v_- \) plane, one with \( v_- - u_- - 2u_+ < 0 \) and the other with \( v_- - u_- - 2u_+ > 0 \). From (4.11), the combinations \( 2u_+ - v_- + u_- \) and \(-2v_+ - v_- + u_- \) are either both positive or both negative, and since their sum \( 2(u_+ - v_+ + u_- - v_-) \) must be positive or zero from (4.7), the branch with \( v_- - u_- - 2u_+ < 0 \) is the only feasible one.

Figure 2 shows the backwards light cones of two field points on the same spacelike section \( \vec{\tau} = 0 \) of the loop. The field point marked \( \Gamma \) coincides with the kink at \( u_+ = u_- = 0 \), and its past light cone consists of the straight line segment \( v_- = 0 \) (solution (4.9)) on the left, and the curve \( v_+ = -\frac{1}{2}v_- + \frac{1}{4\pi}(1 - \cos 2\pi v_-)/v_- \) (which can interchangeably be regarded as solution (4.10) or solution (4.12)) on the right. The point marked \( B \) is a general field point, and its past light cone contains all three components identified above: solution (4.12) on the left, followed by the straight-line segment (4.9), and then solution (4.10) on the right. For future reference these three segments will be designated \( \Gamma^- \), \( \Gamma^0 \) and \( \Gamma^+ \) respectively.

It is evident from the foregoing discussion that the integration contour \( \Gamma \) is a piecewise-smooth spacelike or null curve that wraps once around the world sheet \( T \) of the loop. When calculating the metric perturbation \( h_{ab} \) or its spacetime derivatives \( h_{ab,\kappa} \) at a given field point \( [\vec{\tau}, \vec{X}] \) it is possible in principle to use \( \sigma \) or \( v_+ \) or \( v_- \) as the integration variable on \( \Gamma \), as circumstances warrant. From (2.9) and the fact that \( \sigma = \frac{1}{2}L(v_+ - v_-) \) and \( \vec{\tau} - \vec{\tau} = |\vec{X} - \vec{X}_0| \), the relevant transformation rules are

\[ |\vec{X} - \vec{X}_0|^{-1} |1 - n \cdot X_0|^{-1} d\sigma = L \Delta_0^{-1} dv_+ = -L \Delta_0^{-1} dv_- \]  

(4.13)
where
\[ \Delta_{\pm} \equiv \bar{t} - \tau - 2(\hat{\mathbf{s}} - \mathbf{X}) \cdot \mathbf{X}_{\pm} \] (4.14)
and it is understood that \( v_+ \) is varied with \( v_- \) held constant, and vice versa. Because an explicit formula (4.5) exists for \( v_+ \) as a function of \( v_- \) on \( \Gamma \), the most useful choice of the integration variable will normally be \( v_- \), in which case (2.10) becomes
\[ h_{ab}(\bar{t}, \mathbf{X}) = 4\mu L \int_{\Gamma} \Delta_{\pm}^{-1} \Psi_{ab} d\nu_. \] (4.15)

Now, in terms of the derivatives of the light-cone coordinates \( \sigma_+ \) and \( \sigma_- \) the formula (2.11) for the source function \( \Psi^{ab} \) reads
\[ \Psi^{ab} = 2(X^a_\pm X^b_\pm + X^a_\mp X^b_\mp) - 2\eta^{ab} X_+ \cdot X_- \] (4.16)
where \( X^a_\pm = \left[ \frac{1}{2}, \mathbf{X}_\pm \right] \), and \( X_+ = \frac{1}{2} s\hat{\mathbf{x}} \) and \( X_- = \frac{1}{2}(-\sin 2\pi v_- \hat{\mathbf{x}} + \cos 2\pi v_- \hat{\mathbf{y}}) \) are orthogonal vectors. Hence, \( X_+ \cdot X_- = \frac{1}{4} \) and
\[ \Psi_{ab}(s, v_-) = \frac{1}{2} \begin{bmatrix} 1 & \sin 2\pi v_- & -\cos 2\pi v_- & -s \\ \sin 2\pi v_- & 1 & 0 & -s \sin 2\pi v_- \\ -\cos 2\pi v_- & 0 & 1 & s \cos 2\pi v_- \\ -s & -s \sin 2\pi v_- & s \cos 2\pi v_- & 1 \end{bmatrix}_{ab} \] (4.17)

Also, since
\[ -L \Delta_{\pm}^{-1} = 2(v_- - 2\bar{t}/L + 2s\hat{\mathbf{z}}/L)^{-1} \] (4.18)
the integral (4.15) for \( h_{ab} \) reads
\[ h_{ab}(\bar{t}, \mathbf{X}) = -8\mu \int_{\Gamma} (v_- - 2\bar{t}/L + 2s\hat{\mathbf{z}}/L)^{-1} \Psi_{ab} d\nu_. \] (4.19)
where the limits on \( v_- \) are fixed by the requirement that \( v_+ \) range from \(-\frac{1}{2}\) to 0 (with \( s = -1 \)) and then from 0 to \( \frac{1}{2} \) (with \( s = 1 \)). Note here that in view of (4.13) and the fact that \( \Delta_+ \) and \( \Delta_- \) are non-negative functions, \( v_- \) is a non-increasing function of \( v_+ \) on \( \Gamma \).

If \( V_{-1}, V_0 \) and \( V_1 \) denote the values of \( v_- \) when \( v_+ = -\frac{1}{2}, 0 \) and \( \frac{1}{2} \) respectively, then equation (4.4) for \( \Gamma \) indicates that
\[ 2\bar{t}/L - V_k - \frac{k}{2} = \left\{ \left[ \frac{2\hat{\mathbf{z}}}{L} - \frac{1}{2\pi} \cos(2\pi V_k) \right]^2 + \left[ \frac{2\hat{\mathbf{y}}}{L} - \frac{1}{2\pi} \sin(2\pi V_k) \right]^2 \right\}^{1/2} \] (4.20)
in each of the three cases \( k = -1, 0 \) and 1. In particular, it is evident that if \( V_1 \) satisfies (4.20) with \( k = 1 \) then \( V_{-1} = V_1 + 1 \) satisfies (4.20) when \( k = -1 \). So \( V_1 \leq V_0 \leq V_{-1} \equiv V_1 + 1 \), and (4.15) becomes
\[ h_{ab}(\bar{t}, \mathbf{X}) = 8\mu \int_{V_1}^{V_0} (v_- - \psi_-)^{-1} \Psi_{ab} |_{s=1} d\nu_- + 8\mu \int_{V_0}^{V_1+1} (v_- - \psi_-)^{-1} \Psi_{ab} |_{s=1} d\nu_- \] (4.21)
where \( \psi_\pm = 2\bar{t}/L \pm 2\hat{\mathbf{z}}/L \).
In the case of greatest interest, where the field point \([\vec{t}, \vec{x}]\) is a point (4.6) on the loop, if \(u_+ > 0\) then \(V_0 = u_+\) and (from (4.7))

\[
V_1 = u_+ + W_1(u_+) \quad \text{where} \quad u_+ = \frac{1}{2} + \frac{1}{2} W_1 - \frac{1}{4 \pi^2} (1 - \cos 2\pi W_1) / W_1
\]  

(4.22)

while if \(u_+ < 0\) then \(V_1 = u_- - 1\) and

\[
V_0 = u_- + W_0(u_+) \quad \text{where} \quad u_+ = \frac{1}{2} W_0 - \frac{1}{4 \pi^2} (1 - \cos 2\pi W_0) / W_0.
\]  

(4.23)

Note that \(W_1 \in (-1, 0)\) for \(u_+ \in (0, \frac{1}{2})\) and \(W_0 \in (-1, 0)\) for \(u_+ \in (-\frac{1}{2}, 0)\). In fact, \(W_1(u_+) = W_0(u_+) - \frac{1}{2}\) for all \(u_+ \in (0, \frac{1}{2})\). At each of the kink points \(u_+ = -\frac{1}{2}, 0\) and \(\frac{1}{2}\) the values of \((V_1, V_0)\) tend to \((u_-, u_-)\) as \(u_+\) approaches the kink value from below, and tend to \((u_- - 1, u_-)\) as \(u_+\) approaches the kink value from above.

Collecting together the results of the preceding paragraphs gives the formula

\[
h_{ab}(\vec{t}, \vec{x}) = 8\mu \int_{V_1 - \psi_-}^{V_0 - \psi_-} w^{-1} \Psi_{ab}(1, w + \psi_-) \, dw + 8\mu \int_{V_1 - \psi_-}^{V_0 - \psi_-} w^{-1} \Psi_{ab}(-1, w + \psi_-) \, dw
\]  

(4.24)

where the components (4.17) of \(\Psi_{ab}\) are all smooth, and all except \(\Psi_{xy}\) and \(\Psi_{yx}\) are non-zero. The two integrals in (4.24) are guaranteed to converge if the field point \([\vec{t}, \vec{x}]\) lies off the string, but not if it lies on the string. In the second case, \(\psi_+ = u_- + u_\pm |u_+|\) and so \(V_0 - \psi_- = 0\) if \(u_+ > 0\) while \(V_1 + 1 - \psi_- = 0\) if \(u_+ < 0\). Thus at least one of the integrals in (4.24) diverges.

To investigate more fully the behaviour of the metric perturbations near the world sheet \(T\), suppose that

\[
[\vec{t}, \vec{x}] = [\vec{t}, \vec{X}] + [\delta \vec{t}, \delta \vec{x}]
\]  

(4.25)

where \(\delta \vec{t}\) and \(\delta \vec{x}\) are assumed to be small. Then if \(u_+ > 0\) equation (4.20) with \(k = 0\) can be solved for \(V_0\) to the second order in \([\delta \vec{t}, \delta \vec{x}]\) to give

\[
V_0 - \psi_- \approx \frac{2}{u_+ L^2} [\delta \vec{x}^2 + \delta \vec{y}^2 + (\delta \vec{t} - \delta \vec{x})^2 + 2(\delta \vec{t} - \delta \vec{y})[\delta \vec{x} \sin(2\pi u_-) - \delta \vec{y} \cos(2\pi u_-)]]
\]  

(4.26)

whereas if \(u_+ < 0\) equation (4.20) with \(k = 1\) can be solved for \(V_1\) to give

\[
V_1 + 1 - \psi_+ \approx -\frac{2}{(u_+ + \frac{1}{2}) L^2} [\delta \vec{x}^2 + \delta \vec{y}^2 + (\delta \vec{t} + \delta \vec{x})^2]
\]

\[
+ 2(\delta \vec{t} + \delta \vec{y})[\delta \vec{x} \sin(2\pi u_-) - \delta \vec{y} \cos(2\pi u_-)].
\]  

(4.27)

Here, the quadratic forms

\[
Q_\pm = \delta \vec{t}^2 + \delta \vec{y}^2 + (\delta \vec{t} \pm \delta \vec{x})^2 + 2(\delta \vec{t} \pm \delta \vec{y})[\delta \vec{x} \sin(2\pi u_-) - \delta \vec{y} \cos(2\pi u_-)]
\]

\[
\equiv 2\Psi_{ab}(\mp 1, u_-) [\delta \vec{t}, \delta \vec{x}]^a [\delta \vec{t}, \delta \vec{x}]^b
\]  

(4.28)

are positive semi-definite and vanish on the 2-surface in \([\delta \vec{t}, \delta \vec{x}]\) space spanned by the vectors \([1, \mp \vec{z}]\) and \([1, -\sin(2\pi u_-) \vec{x} + \cos(2\pi u_-) \vec{y}]\), which is just the tangent plane to the string at the point \([\vec{t}, \vec{X}]\). Thus, if the field point \([\vec{t}, \vec{x}]\) approaches a point on the string from any direction outside the tangent plane, the divergent contribution to \(h_{ab}\) has the form

\[
h_{ab} \approx 8\mu \Psi_{ab}(\pm 1, \psi_{\pm}) \ln Q_\mp
\]  

(4.29)

where the upper sign applies when \(u_+ > 0\) and the lower sign when \(u_+ < 0\). Note also from (4.26) and (4.27) that the divergence is more complicated at the kink points \(u_+ = 0\) and \(u_+ = -\frac{1}{2}\).
The divergence of the metric perturbation $h_{ab}$ on the world sheet is to be expected (see, for example, [7], p 216), and is simply a reflection of the conical singularity that is present at every ordinary point on a zero-thickness cosmic string. For example, the exact metric about a static straight string lying along the $z$-axis has the ‘isotropic’ form [29]

$$dx^2 = dr^2 - d\phi^2 - \rho^{-8\mu}(dx^2 + dy^2)$$

(4.30)

where $\rho = (x^2 + y^2)^{1/2}$. In the limit of small $\mu$ the only non-zero metric perturbations are $h_{1x} = h_{yy} = 8\mu \ln \rho$, which satisfy the Lorentz gauge conditions $h_{a,b}^{\mu} = \frac{1}{2} h_{ab}$ but also diverge logarithmically as $\rho \to 0$.

Two quantities that will play an important role in section 4.3 in fixing the initial data for the perturbation $\delta X^a$ are the values of the functions $h_{ab}\bar{X}_a^\mu\bar{X}_b^\mu$ and $h_{ab}\bar{X}_a^\mu\bar{X}^b\mu$ on the world sheet $T$. Since $\Psi_{ab}\bar{X}_b^\mu = \Psi_{ab}\bar{X}_b^\mu = 0$ at all ordinary points on $T$, the leading-order divergence (4.29) in $h_{ab}$ makes no contribution to these functions.

At the next highest order in $[\delta\bar{\delta} X]$, the metric perturbation has the form

$$h_{ab} = 4\mu\left[
\begin{array}{cccc}
I_+ + I_- & S_+ + S_- & -C_+ - C_- & -I_+ + I_- \\
S_+ + S_- & I_+ + I_- & 0 & -S_+ + S_- \\
-C_+ - C_- & 0 & I_+ + I_- & C_+ - C_- \\
-I_+ + I_- & -S_+ + S_- & C_+ - C_- & I_+ + I_- \\
\end{array}
\right]$$

(4.31)

where, for $0 < \mu_+ < \frac{1}{2}$,

$$I_+ = -\ln |W_1|, \quad I_- = \ln D_+ - \ln 2\mu_+$$

(4.32)

and

$$(S_+, C_+) = (\cos 2\pi u_- - \sin 2\pi u_-)\text{Si}(2\pi |W_1|)$$

$$- (\sin 2\pi u_- \cos 2\pi u_-)\text{Ci}(2\pi |W_1|) - \gamma_E - \ln 2\pi$$

(4.33)

and

$$(S_-, C_-) = (-\cos 2\pi (2\mu_+ + u_-), \sin 2\pi (2\mu_+ + u_-))\text{Si}(2\pi D_+) - \text{Si}(4\pi u_-)$$

$$+ (\sin 2\pi (2\mu_+ + u_-), \cos 2\pi (2\mu_+ + u_-))\text{Ci}(2\pi D_+) - \text{Ci}(4\pi u_-)$$

(4.34)

with $\text{Si}(x) = \int_0^x w^{-1} \sin w dw$ and $\text{Ci}(x) = -\int_x^\infty w^{-1} \cos w dw$ the usual sine and cosine integrals, and $\gamma_E \approx 0.5772$ Euler’s constant. (In fact, $\text{Ci}(2\pi x) - \gamma_E - \ln 2\pi$ = $\ln x - \int_0^x x^{-1}(1 - \cos 2\pi x) dx$. Also, the function $D_+(u_+)$ is defined by

$$2\pi D_+ = 2\pi (2\mu_+ - W_1 - 1) \equiv \frac{1}{\pi} (1-\cos 2\pi W_1)/W_1$$

(4.35)

while $W_1$ is defined implicitly as a function of $u_+$ through (4.22). Substituting $\bar{X}_a^\mu = \frac{1}{2}[1, \hat{x}]^\mu$ and $\bar{X}_a^\mu = \frac{1}{2}[1, -\sin 2\pi u_- \hat{x} + \cos 2\pi u_- \hat{y}]^\mu$ gives

$$h_{ab}\bar{X}_a^\mu\bar{X}_b^\mu = 4\mu I_- = 4\mu (\ln D_+ - \ln 2\mu_+)$$

(4.36)

and

$$h_{ab}\bar{X}_a^\mu\bar{X}_b^\mu = 2\mu (I_+ + I_-) - 2\mu [(S_+ + S_-) \sin 2\pi u_- + (C_+ + C_-) \cos 2\pi u_- - 2\mu \ln W_1 + 2\mu \text{Ci}(2\pi |W_1|) - \gamma_E - \ln 2\pi]$$

$$- 2\mu [\text{Si}(2\pi D_+) - \text{Si}(4\pi u_-)] \sin(4\pi u_-)$$

$$- 2\mu [\text{Ci}(2\pi D_+) - \text{Ci}(4\pi u_-)] \cos(4\pi u_-)$$

(4.37)

when $0 < \mu_+ < \frac{1}{2}$. The function $h_{ab}\bar{X}_a^\mu\bar{X}_b^\mu$ diverges as $4\mu \ln \mu_+ \to 0$, and as $\frac{1}{2} \mu \ln (1-2\mu_+)$ as $\mu_+ \to \frac{1}{2}$. However, the function $h_{ab}\bar{X}_a^\mu\bar{X}_b^\mu$ is bounded, and converges to $2\mu [\text{Ci}(2\pi) - \gamma_E - \ln 2\pi] \approx -4.8753\mu$ at both $u_+ = 0$ and $u_+ = \frac{1}{2}$. 
In the case where \(-\frac{1}{2} < u_+ < 0\), it turns out that \(I_+(u_+) = I_-(u_+ + \frac{1}{2})\) and \(I_-(u_+) = I_+(u_+ + \frac{1}{2})\), and analogous identities hold for \(S_{\pm}\) and \(C_{\pm}\). The functions \(h_{ab} \vec{X}_a^\tau \vec{X}_b^\tau\) and \(h_{ab} \vec{X}_a^\epsilon \vec{X}_b^\epsilon\) are therefore periodic functions of \(u_+\) with period \(\frac{1}{2}\).

A similar calculation of the leading-order contributions to \(h_{ab}\) near the kink point \(u_+ = 0\) gives

\[
(I_+, S_+, C_+) \approx (1, \sin 2\pi u_-, \cos 2\pi u_-) \ln(L^{-1} Q_\tau/Q_0) \\
+ (0, \cos 2\pi u_-, -\sin 2\pi u_-) \sin(2\pi) \\
- (0, \sin 2\pi u_-, \cos 2\pi u_-) [\text{C}(2\pi) - \gamma_E] - \ln 2\pi]
\]

and

\[
(I_-, S_-, C_-) \approx (1, \sin 2\pi u_-, \cos 2\pi u_-) \ln(4L^{-1} Q_0)
\]

where

\[
Q_0 = \delta t + \delta \chi \sin 2\pi u_+ - \delta \chi \cos 2\pi u_+ = 2h_{ab} \vec{X}_a^\tau \vec{X}_b^\tau [\delta t, \delta \chi]^{a,b}.
\]

Hence, \(h_{ab} \vec{X}_a^\tau \vec{X}_b^\tau\) is continuous at the kink point and the function \(h_{ab} \vec{X}_a^\tau \vec{X}_b^\epsilon\) has a jump discontinuity at the kink point, and \(h_{ab} \vec{X}_a^\epsilon \vec{X}_b^\epsilon\)

as well as containing formal logarithmic divergences is undefined even for non-zero \([\delta t, \delta \chi]\).

### 4.2. The self-acceleration vector

Although \(h_{ab}\) diverges at all points on the loop, the self-acceleration vector \(\alpha^a\) defined in (2.6) typically does not. This result was first established by Quashnock and Spergel [8] for a generic point on a smooth loop, but does not extend to non-generic points such as cusps and kinks. In terms of the light-cone derivatives \(\vec{X}_a^\tau\) and \(\vec{X}_a^\epsilon\) of \(\vec{X}^a = [\vec{\tau}, \vec{\chi}]^a\), expression (2.6) for \(\alpha^a\) at a point \((u_+, u_-)\) on the string loop can be rewritten as

\[
\alpha^a(u_+, u_-) = -\eta^{ad} (2h_{db,c} - h_{bc,d}) \{ \vec{X}_a^\tau \vec{X}_b^c + \vec{X}_a^\epsilon \vec{X}_b^c \}
\]

where it is understood that the right-hand side is evaluated by taking the limit \([\vec{\tau}, \vec{\chi}] \to [\vec{\tau}, \vec{\chi}]\) from any direction outside the tangent plane, in the metric derivatives \(h_{ab,c}\).

Now the ACO loop, as described by (4.1) and (4.2), rotates uniformly about the z-axis, and so the self-acceleration vector \(\alpha^a\) is uniformly rotating as well. In order to simplify the calculation of \(\alpha^a\), it is useful to pre-multiply \(\alpha^a\) by the matrix

\[
M^a_b = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos 2\pi u_+ & \sin 2\pi u_- & 0 \\
0 & -\sin 2\pi u_+ & \cos 2\pi u_- & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}_{ab}
\]

the effect of which is to map each horizontal radius vector joining the rotation axis to a point on the loop into a vector parallel to the \(x\)-axis. In other words, \(\alpha^L = M^a_b \alpha^b\) is the lateral component of the loop’s horizontal acceleration and \(\alpha^N = M^a_b \alpha^b\) is its normal component.

To demonstrate that \(\alpha^a\) is well defined away from the kink points at \(u_+ = 0\) and \(\pm \frac{1}{2}\), and to explicitly calculate the components of the co-rotating acceleration vector \(M^a_b \alpha^b\), is

\[2\] It should be emphasized that the Quashnock–Spergel result holds only to linear order in the expansion parameter \(\mu\). Copeland, Haws and Hindmarsh [30] have made the claim that the integral for the self-acceleration vector \(\alpha^a\) contains a local divergence of the form \(\rho^2 \ln \rho\), where \(\rho\) is the normal distance from the world sheet. Carter and Battey [25] have more recently disputed this claim, maintaining that the divergence is cancelled by a counterterm that was previously neglected. Since neither analysis is based on a full second-order expansion of the equations of motion ((2.3) and the equation \(\kappa^a = 0\) of appendix B in the respective cases), this dispute remains unresolved. However, the question has no relevance for the purely first-order treatment of the back-reaction problem offered here.
straightforward but tedious. The details can be found in appendix A. It turns out that $\alpha', \alpha^L$ and $\alpha^R$ are periodic functions of $u_+$, and $\alpha^z$ is an anti-periodic function, with period $\frac{1}{2}$.

For $0 < u_+ < \frac{1}{2}$, the components of $M_{ab}^\mu \alpha^b$ are

$$\alpha' = \alpha^z = 32\pi^2 \mu L^{-1} F(W_1) \cos 2\pi W_1 (1 - \cos 2\pi W_1)$$

(4.43)

and

$$\begin{pmatrix} \alpha^L \\ \alpha^N \end{pmatrix} = -32\pi \mu L^{-1}
\times \begin{pmatrix}
[\text{Si}(4\pi u_+ - \text{Si}(2\pi D_+)) \sin 4\pi u_+ + \text{Ci}(4\pi u_+) - \text{Ci}(2\pi D_+) \cos 4\pi u_+ ] \\
-\{\text{Si}(4\pi u_+) - \text{Si}(2\pi D_+) \cos 4\pi u_+ + \text{Ci}(4\pi u_+) - \text{Ci}(2\pi D_+) \sin 4\pi u_+ 
\end{pmatrix}
+ 32\pi^2 \mu L^{-1} F(W_1) W_1 \cos 2\pi W_1 \sin 2\pi W_1 
\cos 2\pi W_1 - 1 + 2\pi W_1 \sin 2\pi W_1 
\end{pmatrix}$$

(4.44)

where

$$F(W_1) = \left(2\pi^2 W_1^2 + 1 - \cos 2\pi W_1 - 2\pi W_1 \sin 2\pi W_1 \right)^{-1},$$

(4.45)

$$2\pi D_+ = -\frac{1}{\pi} (1 - \cos 2\pi W_1) / W_1,$$

(4.46)

as before, and $W_1$ is defined implicitly as a function of $u_+$ through

$$4\pi u_+ = 2\pi (1 + W_1) - \frac{1}{\pi} (1 - \cos 2\pi W_1) / W_1$$

(4.47)

(see (4.22)).

The dependence of the acceleration component $\alpha'$ on $u_+$ is shown in figure 3. The interval $u_+ \in (0, \frac{1}{2})$ corresponds to $W_1 \in (-1, 0)$. Since $\alpha'$ diverges as $32\mu L^{-1} W_1^{-1}$ near $W_1 = 0$ while $u_+ = \frac{1}{2} \approx \frac{1}{6} \pi^2 W_1^2$ for small values of $W_1 < 0$, the singularity in $\alpha'$ at $u_+ = 0$ has $\alpha' \approx -32(\frac{1}{6} \pi^2)^{1/3} \mu L^{-1}/|u_+|^{1/3}$ for $u_+ \lesssim 0$. Note, however, that the singularity in $\alpha'$ is one-sided, as $\alpha' \approx -32\pi^2 \mu L^{-1} (1 + W_1)^2$ for $W_1$ near $-1$, while $u_+ \approx \frac{1}{2} (1 + W_1)$, and so $\alpha' \approx -128\pi^2 \mu L^{-1} u_+^2$ for $u_+ \gtrsim 0$. Similar remarks apply to $\alpha^z$, except that $\alpha^z > 0$ for $u_+ \in (-\frac{1}{2}, 0)$ and $\alpha^z < 0$ for $u_+ \in (0, \frac{1}{2})$.

The point $u_+ = \bar{\tau} + \bar{\sigma} = 0$ of course marks the location of the lower of the two kinks on the string, which propagates around the string in the direction of increasing $\bar{\sigma}$. For any fixed value of $\bar{\tau}$, points with $u_+ \gtrsim 0$ lie ahead of the kink and have yet to be disturbed by its passing.
whereas points with \( u_+ \lesssim 0 \) lie just behind the kink and so have recently been subject to severe gravitational stresses. This is the reason for the one-sided nature of the singularity in \( \alpha^t \) and \( \alpha^z \) at \( u_+ = 0 \). Also, \( \alpha^t \leq 0 \) because the string is radiating energy, whereas \( \alpha^z \) has different signs on the two branches of the loop because it is directed counter to the \( z \)-component of the string’s 4-velocity \( X_a^\tau \), which is negative for \( u_+ \in (-\frac{1}{2}, 0) \) and positive for \( u_+ \in (0, \frac{1}{2}) \).

The two remaining components of the acceleration vector, \( \alpha^L \) and \( \alpha^N \), are plotted in figures 4 and 5 respectively. Both are divergent at \( u_+ = 0 \), with \( \alpha^L \approx \frac{32\pi}{3} \mu L^{-1} \ln |u_+| \) for \( u_+ \lesssim 0 \) and \( \alpha^L \approx \frac{32\pi}{3} \mu L^{-1} \ln |u_+| \) for \( u_+ \gtrsim 0 \), while \( \alpha^N \approx -32\pi^2 \mu L^{-1}/|u_+|^{1/3} \) for \( u_+ \lesssim 0 \) and \( \alpha^N \approx 128\pi^2 \mu L^{-1} u_+ \ln u_+ \) for \( u_+ \gtrsim 0 \). The normal component \( \alpha^N \) is negative almost everywhere because it acts to torque down the rotational motion of the loop, and like \( \alpha^t \) and \( \alpha^z \), it is strongest in the immediate wake of the two kinks. The lateral component \( \alpha^L \) is also negative, but its singularity at \( u_+ = 0 \) is more symmetric. The lateral acceleration is presumably a response to the imbalance between the string tension and the centripetal force caused by the rotational torquing. Its net effect is to induce lateral shrinkage of the loop as it radiates.

Although all four components of the acceleration vector \( \alpha^a \) are divergent at the kink points \( u_+ = -\frac{1}{2}, 0 \) and \( \frac{1}{2} \), the singularities are all integrable, in the sense that \( \int \alpha^a \, du_+ \) exists on any closed sub-interval of \( \left[ -\frac{1}{2}, \frac{1}{2} \right] \). This in turn means that the weak-field equations of motion for the ACO loop are integrable as well, as will be seen shortly.
It could be argued that the method adopted here, of evaluating $\alpha^a$ away from the kink points, then integrating the equations of motion through the kinks, ignores the possible presence of other localized ($\delta$-function or non-integrable) singularities in $\alpha^a$ at the kink points themselves. There seems to be no way of ruling out this possibility short of solving the full set of field equations for a finite-thickness string and taking the limit as the thickness goes to zero. However, it should be noted that a formal calculation of $\alpha^a$ at the kink point $u_+=0$ using only the logarithmically divergent contributions (4.38) and (4.39) to $h_{ab}$ gives $\alpha^a=0$ irrespective of the value chosen for $\delta$ in the tangent vector $X^a_+ = \frac{1}{2} [t, \delta \hat{k}]^a$. (This remark should not be taken to mean that $\alpha^a$ actually vanishes at the kink points, as terms of order $[\delta \delta, \delta \hat{k}]$ in $h_{ab}$ contribute additional terms to $\alpha^a$ which have no well-defined limits when $[\delta \delta, \delta \hat{k}] \rightarrow 0$.)

Furthermore, as will be seen, the energy loss of the loop over a parametric period exactly matches the radiative flux calculated for the ACO loop in [16], so there are good reasons for believing that the acceleration vector $\alpha^a$ has been represented accurately.

### 4.3. Secular evolution of the ACO loop

Once the acceleration vector

$$\alpha^a = [\alpha', (\alpha^t \cos 2\pi u_- - \alpha^N \sin 2\pi u_+)\hat{k} + (\alpha^t \sin 2\pi u_- + \alpha^N \cos 2\pi u_-)\hat{y} + \alpha^z\hat{z}]^a$$

(4.48) is known as a function of $u_+$ and $u_-$, it is a straightforward matter to use (2.5) to calculate the secular perturbations in the trajectory of the ACO loop due to gravitational back-reaction. In terms of the derivatives with respect to the scaled gauge coordinates $\sigma_+ = Lu_+$ and $\sigma_- = Lu_-$ the first-order equation of motion (2.14) reads

$$\delta X^a_+ = \frac{1}{4} \alpha^a (\sigma_+, \sigma_-).$$

(4.49)

The initial data for this equation on the curve $\tau \equiv \frac{1}{2} (\sigma_+ + \sigma_-) = 0$ are fixed by requiring $\delta X^a_+ |_{\tau=0}$ and $\delta X^a_- |_{\tau=0}$ to satisfy the first-order gauge constraint (B.13), while preserving the initial configuration $X^a_+ |_{\tau=0}$ of the loop and the tangent space to the loop at each point on the initial curve. This can be done by writing

$$\delta X^a_+ |_{\tau=0} = k^+ X^a_+ |_{\tau=0} + l^+ X^a_- |_{\tau=0}$$

(4.50)

and

$$\delta X^a_- |_{\tau=0} = k^- X^a_+ |_{\tau=0} + l^- X^a_- |_{\tau=0}$$

(4.51)

where the coefficients $k^\pm$ and $l^\pm$ are functions of $\sigma \equiv \frac{1}{2} (\sigma_+ - \sigma_-)$ to be determined, while

$$X^a_+ |_{\tau=0}(\sigma) = \frac{1}{2} [1, \text{sgn}(\sigma) \hat{k}]^a$$

(4.52)

and

$$X^a_- |_{\tau=0}(\sigma) = \frac{1}{2} [1, \sin(2\pi \sigma/L)\hat{k} + \cos(2\pi \sigma/L)\hat{y}]^a.$$  

(4.53)

More explicitly, an expansion of $h_{ab}$ about the kink point $u_+ = 0$ reads, to linear order in $\delta \tilde{x}^a = [\delta \tilde{t}, \delta \hat{k}]^a$,

$$h_{ab} = 8\mu (\Psi_{ab}(1, u_-) \ln(L^{-1} Q_0/Q_0) + \Psi_{ab}(-1, u_-) \ln(4L^{-1} Q_0)) \delta \tilde{x}^a + \frac{F_{ab}(-1, u_-)}{2L} \ln(L^{-1} Q_0/Q_0) - L^{-1} Q_0/Q_0 \delta \tilde{x}^a + \frac{F_{ab}(1, u_-)}{2L} \ln(L^{-1} Q_0/Q_0) + L^{-1} Q_0/Q_0 \delta \tilde{x}^a + \cdots$$

where the ellipses ($\cdots$) includes terms of the first order in $\delta \tilde{x}^a$ even more complicated than those shown. The corresponding acceleration vector $\alpha^a$, calculated from (4.41), contains no contributions from the first two, logarithmically divergent, terms. However, the terms proportional to $F_{ab}(-1, u_-)$ and the other linear terms, do formally contribute to $\alpha^a$, with coefficients that typically depend on $\sigma = \text{sgn}(u_+)$. The derivatives with respect to $\delta \tilde{x}^a$ of all these terms are undefined in the limit as $\delta \tilde{x}^a \rightarrow 0$.

The bars on $\hat{X}^a$, $t$ and $\sigma$ can henceforth be dropped without risk of confusion.
In the light-cone gauge, the constraint (B.13) reads simply
\[ h_{ab} X^a X^b + 2 \eta_{ab} \delta X^a X^b = 0 \quad \text{and} \quad h_{ab} X^a X^b + 2 \eta_{ab} \delta X^a X^b = 0 \quad (4.54) \]
and (since \( \eta_{ab} X^a X^b = \frac{1}{4} \)) is satisfied on the initial curve \( \tau = 0 \) if
\[ l^+ = -2 (h_{ab} X^a X^b) \big|_{\tau=0} \quad \text{and} \quad l^- = -2 (h_{ab} X^a X^b) \big|_{\tau=0} \quad (4.55) \]
where \( h_{ab} X^a X^b \) and \( h_{ab} X^a X^b \) are known functions of \( u_+ = (\tau + \sigma)/L \) given by (4.36) and (4.37) respectively.

The initial configuration of the loop will be preserved if
\[ 0 = \delta X^a \big|_{\tau=0} = \delta X^a \big|_{\tau=0} - \delta X^a \big|_{\tau=0} = (k^+ l^- - l^- k^-) X^a \big|_{\tau=0} \quad (4.56) \]
at all points on the initial curve, and so \( k^+ = l^- \) and \( l^- = k^- \). Together with \( \delta X^a \big|_{\tau=0} = 0 \), the initial data for the perturbation \( \delta X^a \) therefore have the form
\[ \delta X^a \big|_{\tau=0}(\sigma) = \delta X^a \big|_{\tau=0}(\sigma) = -2 (h_{cd} X^c X^d) \big|_{\tau=0} X^a \big|_{\tau=0} - 2 (h_{cd} X^c X^d) \big|_{\tau=0} X^a \big|_{\tau=0}. \quad (4.57) \]

It may seem at first glance that replacing \( X^a \big|_{\tau=0} \) with \( X^a \big|_{\tau=0} + \delta X^a \big|_{\tau=0} \) will induce a change in the initial velocity of the loop. However, the effect of the changed initial conditions is to simply realign the gauge coordinates in response to the rotation of the light cones caused by the metric perturbation \( h_{ab} \). In the standard gauge, the tangent vector \( X_t^a \) at any point \( P \) on a given spacelike section \( \tau = \) constant of the world sheet is uniquely determined by the tangent space at \( P \) and the choice of gauge coordinate \( \sigma \) on the section, as \( X_t^a \) is just the future-pointing projection of the tangent space orthogonal to \( X_t^a \), normalized so that \( X_t^2 = - X_t^2 \).

The equation of motion (4.49) can now be integrated forward to give
\[ \delta X^a(\sigma, \sigma_-) = \frac{1}{4} \int_{-\sigma_-}^{\sigma} \alpha^a(\sigma, \theta) \, d\theta + \delta X^a \big|_{\tau=0}(\sigma), \quad (4.58) \]
and
\[ \delta X^a(\sigma, \sigma_-) = \frac{1}{4} \int_{-\sigma_-}^{\sigma} \alpha^a(\theta, \sigma_-) \, d\theta + \delta X^a \big|_{\tau=0}(-\sigma_-). \quad (4.59) \]

at all points with \( \tau = \frac{1}{2} (\sigma_+ + \sigma_-) > 0 \).

Since
\[ \int \alpha^L \, du_+ = 8 \mu L^{-1} \{ \ln |W_1| - \text{Ci}(2\pi |W_1|) \}, \quad (4.60) \]
\[ \int \alpha^L \, du_+ = 8 \mu L^{-1} [\text{Si}(4\pi u_+) - \text{Si}(2\pi D_+)] \cos 4\pi u_+ \]
\[ \quad \quad \quad - [\text{Ci}(4\pi u_+) - \text{Ci}(2\pi D_+)] \sin 4\pi u_+ - 2\pi W_1 \] \quad (4.61)
and
\[ \int \alpha^N \, du_+ = 8 \mu L^{-1} [\text{Si}(4\pi u_+) - \text{Si}(2\pi D_+)] \sin 4\pi u_+ \]
\[ \quad \quad \quad + [\text{Ci}(4\pi u_+) - \text{Ci}(2\pi D_+)] \cos 4\pi u_+ - \ln(4\pi u_+) + \ln D_\tau, \quad (4.62) \]
it is possible to write down explicit expressions for \( \delta X^a_+ \) and \( \delta X^a_- \) at all points on the perturbed trajectory. These in turn can be used to verify directly that the gauge constraints (4.54) are satisfied at all ordinary points on the trajectory. For example,
\[ \frac{1}{4} \eta_{ab} X^a(\sigma_-) \int_{-\sigma_-}^{\sigma_-} \alpha^p(\theta, \sigma_-) \, d\theta = \frac{1}{8} \int_{-\sigma_-}^{\sigma_-} \alpha^p \, d\sigma_- = \frac{1}{8} \int_{-\sigma_-}^{\sigma_-} \alpha^N \, d\sigma_- \]
\[ \quad = - \frac{1}{2} (h_{cd} X^c X^d) \big|_{\tau=0}(\sigma_-) + \frac{1}{2} (h_{cd} X^c X^d) \big|_{\tau=0}(-\sigma_-) \quad (4.63) \]
and
\[ \eta_{ab} \delta X^a_\tau(\sigma) \big|_{\tau=0}(\sigma) = -\frac{1}{2} (h_{cd} X^c_{V_\tau} X^d_{V_\tau}) \big|_{\tau=0}(\sigma) \] (4.64)
and so \( \eta_{ab} \delta X^a_\tau(\sigma_\tau, \sigma_-) = -\frac{1}{2} (h_{cd} X^c_{V_\tau} X^d_{V_\tau}) \) as required. Similarly,
\[ \frac{1}{4} \eta_{ab} \delta X^a_\tau(\sigma) \int_{\sigma}^{\sigma_\tau} \alpha_t(\sigma, \theta) d\theta = \frac{1}{4} \tau [\alpha_t(\sigma) - \text{sgn}(\sigma) \alpha_t(\sigma)] = 0 \] (4.65)
and
\[ \eta_{ab} \delta X^a_\tau(\sigma_\tau) \big|_{\tau=0}(\sigma_\tau) = -\frac{1}{2} (h_{cd} X^c_{V_\tau} X^d_{V_\tau}) \big|_{\tau=0}(\sigma_\tau) \] (4.66)
and so \( \eta_{ab} \delta X^a_\tau(\sigma_\tau, \sigma_-) = -\frac{1}{2} (h_{cd} X^c_{V_\tau} X^d_{V_\tau}) \). 

To track the cumulative change in the trajectory over a single oscillation period \( \Delta \tau = t_p = L/2 \), all that is necessary is to calculate the position \( X^a + \delta X^a \) of the loop and the tangent vectors \( X^a_\tau + \delta X^a_\tau \) on the spacelike section \( \tau = L/2 \), turn off the metric perturbation \( h_{ab} \), then evolve the trajectory forwards in flat space subject to the new initial conditions. The section \( \tau = L/2 \) corresponds to \( \sigma_\pm = L/2 \pm \sigma \), or equivalently (since \( \alpha^a \) and \( \delta X^a_\tau \) are periodic functions of their arguments with period \( L \)) to \( \sigma_\pm = \pm (\sigma - L/2) \). So the perturbed null tangent vectors after an oscillation period are
\[ \delta X^a_\tau \big|_{\tau=L/2}(\sigma) = \frac{1}{4} \int_{-L/2}^{L/2} \alpha^a(\sigma - L/2, \theta) d\theta + \delta X^a_\tau \big|_{\tau=0}(\sigma - L/2) \] (4.67)
and
\[ \delta X^a_\tau \big|_{\tau=L/2}(\sigma) = \frac{1}{4} \int_{-L/2}^{L/2} \alpha^a(\theta, L/2-\sigma) d\theta + \delta X^a_\tau \big|_{\tau=0}(\sigma - L/2). \] (4.68)

The only dependence of \( \alpha^a \) on \( \sigma_- \) is through the explicit trigonometric functions in (4.48), and
\[ \frac{1}{4} \int_{-L/2}^{L/2} \alpha^a(\sigma - L/2, \theta) d\theta = \frac{1}{4} L [\alpha^a, \alpha^a] \] (4.69)
where \( \alpha^a \) and \( \alpha^a \) are now understood to be functions of \( u_+ = (\sigma - L/2)/L \). Also, in view of (4.60), (4.61) and (4.62),
\[ \frac{1}{4} \int_{-L/2}^{L/2} \alpha^a(\sigma - L/2, \theta) d\theta = -4 \mu [\gamma_H + \ln 2 \pi - \text{Ci}(2 \pi)] \approx -9.756 \mu, \] (4.70)
\[ \frac{1}{4} \int_{-L/2}^{L/2} \alpha^a(\theta, L/2-\sigma) d\theta = -4 \mu [2 \pi - \text{Si}(2 \pi)] \approx -19.460 \mu \] (4.71)
and
\[ \frac{1}{4} \int_{-L/2}^{L/2} \alpha^a(\sigma - L/2, \theta) d\theta = -4 \mu [\gamma_H + \ln 2 \pi - \text{Ci}(2 \pi)] \approx -9.756 \mu, \] (4.72)
where \( \gamma_H \approx 0.5772 \) is Euler’s constant as before. By inspection, \( \frac{1}{4} \int_{-L/2}^{L/2} \alpha^a \alpha^a d\sigma_+ = 0 \).

Hence, if \( \Delta X^a_\tau(\sigma) = \delta X^a_\tau \big|_{\tau=L/2}(\sigma) - \delta X^a_\tau \big|_{\tau=0}(\sigma - L/2) \) are the net changes in the perturbations \( \delta X^a_\tau \) over an oscillation period then
\[ \Delta X^a_\tau(\sigma) = \frac{1}{4} L [\alpha^a(\sigma), \alpha^a(\sigma_\tau) \hat{a}^a] \] (4.73)
and
\[ \Delta X^a(\sigma) = [-\kappa, (-\lambda \cos 2 \pi u_- + \kappa \sin 2 \pi u_-) \hat{a}^a - (\lambda \sin 2 \pi u_- + \kappa \cos 2 \pi u_-) \hat{a}^a] \] (4.74)
where \( u_+ = (\sigma - L/2)/L \) in the first equation and \( u_- = -(\sigma - L/2)/L \) in the second, while \( \kappa = 4\mu[\gamma + \ln 2\pi - \text{Ci}(2\pi\tau)] \) and \( \lambda = 4\mu[2\pi - \text{Si}(2\pi\tau)] \).

The perturbed solution can now be generated by integrating the Nambu–Goto equation forward from the spacelike slice \( \tau = L/2 \) with the initial data \((X^a_0 + \delta X^a_0)|_{\tau=L/2}(\sigma)\) and \((\dot{X}^a_0 + \delta \dot{X}^a_0)|_{\tau=L/2}(\sigma)\), and \( h_{ab} \) assumed to be zero. Since \( \delta X^a|_{\tau=0} = \delta X^a|_{\tau=0} \), the tangent vector to the initial slice is

\[
T^a(\sigma) = (X^a_0 + \delta X^a_0)|_{\tau=L/2} - (X^a_0 + \delta X^a_0)|_{\tau=L/2} = \frac{1}{2}[0, \sin 2\pi u_- \hat{x} + \cos 2\pi u_- \hat{y} + \text{sgn}[u_+] \hat{z}]^a + \Delta X^a - \Delta \dot{X}^a. \quad (4.75)
\]

The vector in the span of \( (X^a_0 + \delta X^a_0)|_{\tau=L/2} \) and \((X^a_0 + \delta X^a_0)|_{\tau=L/2} \) that is orthogonal to \((X^a_0 + \delta X^a_0)|_{\tau=L/2}\) (to linear order in \( \delta X^a_0 \)) is

\[
N^a(\sigma) = (X^a_0 + \delta X^a_0)|_{\tau=L/2} + (X^a_0 + \delta X^a_0)|_{\tau=L/2} = \frac{1}{2}[2, -\sin 2\pi u_- \hat{x} + \cos 2\pi u_- \hat{y} + \text{sgn}[u_+] \hat{z}]^a + \Delta X^a - \Delta \dot{X}^a \quad (4.76)
\]

Note here from (4.57) that \( X_\pm|_{\tau=L/2} \cdot \delta X_\pm|_{\tau=0} (Lu_+) = -\frac{1}{2} [h_{cd} X^c_\pm X^d_\pm] |_{\tau=0} (Lu_+) \) and so the terms involving \( \delta X^a|_{\tau=0} \) in \( N^a \) all cancel, leaving

\[
N^a(\sigma) = \frac{1}{2}[2, -\sin 2\pi u_- \hat{x} + \cos 2\pi u_- \hat{y} + \text{sgn}[u_+] \hat{z}]^a + \Delta X^a + \Delta \dot{X}^a \quad (4.77)
\]

as \( X_\pm \cdot \Delta X_\pm = X_- \cdot \Delta X_- = 0 \) on the slice \( \tau = L/2 \).

Furthermore, the net displacement of the loop over an oscillation period can be measured by tracking the motion of the centre-of-mass of the loop, which for any constant-\( \tau \) slice is defined to be

\[
x_0(\tau) = L^{-1} \int_{-L/2}^{L/2} [X(\tau, \sigma) + \delta X(\tau, \sigma)] \, d\sigma. \quad (4.78)
\]

In view of (4.3), and the fact that \( (h_{ab} X^a X^b)|_{\tau=0} \) and \( (h_{ab} X^a X^b)|_{\tau=0} \) are periodic functions of \( \sigma \) with period \( L/2 \) while \( X_\pm|_{\tau=0} \) and \( X_-|_{\tau=0} \) are anti-periodic functions with the same period (and so \( X_\pm|_{\tau=0} \) and \( \delta X_\pm|_{\tau=0} \) are also anti-periodic),

\[
x_0(0) = \frac{1}{L} \hat{L} \quad \text{and} \quad x_0(0) = 0. \quad (4.79)
\]

The subsequent motion of the centre-of-mass is governed by the equation

\[
x_0''(\tau) = L^{-1} \int_{-L/2}^{L/2} (X + \delta X),_{\tau\tau} \, d\sigma = L^{-1} \int_{-L/2}^{L/2} [(\alpha^L \cos 2\pi u_- - \alpha^N \sin 2\pi u_-) \hat{x}
\]

\[
+ (\alpha^L \sin 2\pi u_- + \alpha^N \cos 2\pi u_-) \hat{y} + \alpha^Z \hat{z}] \, d\sigma \quad (4.80)
\]

where the second line follows because \( (X^a + \delta X^a),_{\tau\tau} = \alpha^a + (X^a + \delta X^a),_{\sigma\sigma} \) and \( X_\sigma + \delta X_\sigma \) is a periodic function of \( \sigma \) on \([-L/2, L/2]\). For fixed values of \( \tau \) the acceleration components \( \alpha^L \) and \( \alpha^N \) are periodic functions of \( \sigma \) with period \( L/2 \), whereas \( \alpha^c \), \( \cos 2\pi u_- \) and \( \sin 2\pi u_- \) are anti-periodic with period \( L/2 \). So \( x_0''(\tau) = 0 \), and the centre-of-mass remains unperturbed.
by the back-reaction, a feature which is in any case obvious from the symmetry of the ACO loop.

Let $X^a(\tau, \sigma)$ now denote the flat-space solution generated by the initial conditions $X^a_\sigma = T^a(\sigma)$ and $X^a_\tau = N^a(\sigma)$ on the spacelike slice $\tau = L/2$. The equation of motion $X^a_{\tau\tau} = X^a_{\sigma\sigma}$ is then satisfied by taking

$$X^a_+ = \frac{1}{2} [N^a(\sigma_+) + T^a(\sigma_+)]$$

(4.81)

and

$$X^a_- = \frac{1}{2} [N^a(-\sigma_-) - T^a(-\sigma_-)]$$

(4.82)

with $\sigma_\pm = \tau \pm \sigma$ as usual. The position vector $X$ itself can be found on each constant-$\tau$ section by integrating the equation $X_\sigma = X_+ - X_-$ subject to the constraint $x_0(\tau) = \frac{1}{8} L \hat{z}$.

However, there is no simple relationship between the gauge coordinate $\tau$ in the new solution and the Minkowski time coordinate $t$, as

$$X^0_\tau |_{\tau = L/2} = N^0(\sigma) = 1 + \frac{1}{2} L a'(u_+) - \kappa \neq 1.$$  

(4.83)

In order to facilitate comparison between successive stages in the evaporation of the loop, it is useful to realign the coordinates $\sigma_+$ and $\sigma_-$ by replacing them with two new coordinates

$$\hat{\sigma}_+ = \int_0^{\sigma_+} [N^0(\sigma) + T^0(\sigma)] d\sigma = \sigma_+ + \frac{1}{2} L \int_0^{\sigma_+} \alpha'(\theta/L) d\theta$$

(4.84)

and

$$\hat{\sigma}_- = -\int_0^{\sigma_-} [N^0(\sigma) - T^0(\sigma)] d\sigma = (1 - 2\kappa)\sigma_-.$$  

(4.85)

defined so that $\partial X^\prime / \partial \hat{\sigma}_+ = \partial X^\prime / \partial \hat{\sigma}_- = \frac{1}{2}$ at all points on the new trajectory. Note that because $\hat{\sigma}_\pm$ is a function of $\sigma_\pm$ only, the form of the flat-space equation of motion $X^a_{\hat{\sigma}_+\hat{\sigma}_+} = 0$ is preserved.

Furthermore, since $X^a_\pm$ and $\Delta X^a_\pm$ are periodic functions of $\sigma_\pm$ with period $L$, the net change in $\hat{\sigma}_+$ or $\hat{\sigma}_-$ over a parametric period is

$$\Delta \hat{\sigma}_\pm = L + \frac{1}{2} \int_0^L \int_0^L \alpha'(\sigma_+) d\sigma_+ d\sigma_- = L - 2\kappa L.$$  

(4.86)

Thus, in a gauge coordinate system aligned with the Minkowski time coordinate $t$, the parametric period changes over a single oscillation period $t_p$ from $L$ to $L' = L + \Delta L$, where

$$\Delta L = -2\kappa L \approx -19.501 \mu L.$$  

(4.87)

Since the total energy of the string loop is $E = \mu L$, the energy radiated by the loop over a complete parametric period $2t_p$ is $\Delta E = 2\mu \Delta L \approx -39.002\mu^2 L$, in agreement with the original calculation of Allen, Casper and Ottewill [16].

In terms of the new gauge coordinates $\hat{\sigma}_+$ and $\hat{\sigma}_-$, the initial data on the constant-time slice $\tau = L/2$ become

$$\frac{\partial X^a}{\partial \hat{\sigma}_+} = \left( X^a_\tau \frac{d\sigma_+}{d\hat{\sigma}_+} \right)_{\tau = L/2} = \frac{1}{2} \left[ 1, \text{sgn}(\sigma - L/2) \hat{z} \right]^\mu + \frac{1}{4} L [\alpha'(u_+), \alpha^2(u_+)\hat{z}]^\mu \left\{ 1 + \frac{1}{2} L a'(u_+) \right\}^{-1}$$

$$= \frac{1}{2} \left[ 1, \text{sgn}(\sigma - L/2) \hat{z} \right]^\mu$$

(4.88)
as \( \alpha'(u_\ast) = \text{sgn}(u_\ast)\alpha''(u_\ast) \); and

\[
\frac{\partial X^a}{\partial \sigma} = \frac{1}{2}[1, \{-2\lambda(1 - 2\kappa)^{-1}\cos 2\pi u_- - \sin 2\pi u_\ast\} \hat{x} + \{-2\lambda(1 - 2\kappa)^{-1}\sin 2\pi u_- + \cos 2\pi u_\ast\} \hat{y}]^a.
\]

(4.89)

The final step in demonstrating that the ACO loop evaporates by self-similar shrinkage is to show that the tangent vectors (4.88) and (4.89) are the same functions (to order \( \mu \)) of \( \hat{\sigma} \) and \( \hat{\sigma}_- \), respectively, as \( X^a_\ast \) and \( X^a_\ast \) are of \( \sigma \) in the initial configuration (4.52) and (4.53), save for a change in parametric period from \( L \) to \( L' \), and a rotational phase shift in (4.89).

In the case of (4.88) note that \( \text{sgn}(\sigma - L/2) \) is the restriction to the interval \((0, L)\) of its \( L \)-periodic extension. As \( \sigma \) varies from 0 to \( L \) on the constant-time slice \( \tau = L/2, \sigma_\ast \) varies from \( L/2 \) to \( 3L/2 \) and \( \hat{\sigma}_\ast \) varies from \( L/2 - \kappa L \) to \( 3L/2 - 3\kappa L \). Thus, the periodic extension of

\[
\text{sgn}(\sigma - L/2) \equiv \text{sgn}(\sigma_\ast - L)
\]

(4.90)

with period \( L \) is the same as the periodic extension of

\[
\text{sgn}(\sigma_\ast - L + 2\kappa L) \equiv \text{sgn}(\sigma_\ast - L')
\]

(4.91)

on \( (L/2 - \kappa L, 3L/2 - 3\kappa L) \) with period \( L' = (1 - 2\kappa)L \). But this in turn is the same as the \( L' \)-periodic extension of \( \text{sgn}(\sigma_\ast) \) on \((-L'/2, L'/2)\), and

\[
\frac{\partial X^a}{\partial \hat{\sigma}_\ast} = \frac{1}{2}[1, \text{sgn}(\hat{\sigma}_\ast)\hat{z}]^a,
\]

(4.92)

as required.

Referring now to (4.89), the constants \( \kappa \) and \( \lambda \) are both of order \( \mu \), while

\[
u_\ast = \sigma_\ast/L = \hat{\sigma}_\ast L^{-1}(1 - 2\kappa)^{-1} = \hat{\sigma}_\ast /L'
\]

(4.93)

on the slice \( \tau = L/2 \). So

\[
\frac{\partial X^a}{\partial \sigma_-} \approx \frac{1}{2}[1, \{-2\kappa \cos 2\pi \hat{u}_- + \sin 2\pi \hat{u}_\ast\} \hat{x} + \{-2\kappa \sin 2\pi \hat{u}_- + \cos 2\pi \hat{u}_\ast\} \hat{y}]^a
\]

(4.94)

where \( \hat{u}_- = \hat{\sigma}_- /L' \). Note here that

\[
\eta_{ab} \frac{\partial X^a}{\partial \sigma_-} \frac{\partial X^b}{\partial \sigma_-} = -\lambda^2
\]

(4.95)

and therefore \( \frac{\partial X^a}{\partial \hat{\sigma}_-} \) remains a null vector to first order in \( \mu \). In fact,

\[
-(2\kappa \cos 2\pi \hat{u}_- + \sin 2\pi \hat{u}_\ast) = -A \sin(2\pi \hat{u}_\ast + \psi)
\]

(4.96)

and

\[
-2\lambda \sin 2\pi \hat{u}_- + \cos 2\pi \hat{u}_\ast = A \cos(2\pi \hat{u}_\ast + \psi)
\]

(4.97)

where

\[
\psi = \tan^{-1}(2\kappa) \approx 2\kappa \quad \text{and} \quad A = (1 + 4\lambda^2)^{1/2} \approx 1
\]

(4.98)

to first order in \( \mu \). It follows that

\[
\frac{\partial X^a}{\partial \hat{\sigma}_-} \approx \frac{1}{2}[1, -\sin(2\pi \hat{u}_- + 2\lambda) \hat{x} + \cos(2\pi \hat{u}_- + 2\lambda) \hat{y}]^a
\]

(4.99)

and \( \frac{\partial X^a}{\partial \hat{\sigma}_-} \) is the same function of \( \hat{u}_- \) as \( X^a_\ast \) of \( \sigma/L \) in (4.53), apart from the phase shift of \( 2\lambda \approx 38.92\mu \).

Note finally that if (4.88) and (4.89) are used as initial data for a new oscillation of the loop, \( \hat{\tau} \) re-zeroed so that the initial surface \( t = L/2 \) corresponds to \( \hat{\tau} = 0 \), then

\[
\frac{\partial X^a}{\partial \hat{\tau}} \bigg|_{\hat{\tau} = 0} = \left( \frac{\partial X^a}{\partial \hat{\sigma}_\ast} - \frac{\partial X^a}{\partial \hat{\sigma}_-} \right) \bigg|_{\hat{\tau} = 0}
\]

\[
= \frac{1}{2}[1, \text{sgn}(\hat{\sigma})\hat{z}]^a - \frac{1}{2}[1, -\sin(-2\pi \hat{\sigma}/L' + 2\lambda) \hat{x} + \cos(-2\pi \hat{\sigma}/L' + 2\lambda) \hat{y}]^a
\]

(4.100)
and so
\[ X^a(\hat{\sigma})|_{\hat{t}=0} = \left[ \frac{1}{2} |\hat{\sigma}| + \frac{1}{8} |\Delta L| \right] \hat{z}^a + \frac{L'}{4\pi} \left[ 0, \cos(-2\pi \hat{\sigma}/L' + 2\lambda) \hat{x} + \sin(-2\pi \hat{\sigma}/L' + 2\lambda) \hat{y} \right]^a. \]

Hence, the kink point at \( \hat{\sigma} = 0 \) corresponds to the spacetime point
\[ X^a = \left[ L/2, \frac{1}{8} |\Delta L| \hat{z} \right]^a + \frac{L'}{4\pi} \left[ 0, \cos(2\lambda) \hat{x} + \sin(2\lambda) \hat{y} \right]^a \]

which is in the same spatial position as the kink point was in the unperturbed trajectory at the slightly later time \( \tau = -\sigma = \lambda L/2\pi \) (except for a uniform contraction towards the centre-of-mass point).

This observation throws some light on a question that was raised in section 6.11 of [31], namely whether gravitational radiation in the weak-field approximation would be expected to force the overall pattern of a loop (and in particular salient features such as kinks and cusps) to advance or precess. In the case of the ACO loop it can be seen that the pattern advances. The phase shift here is due entirely to the form of the lateral acceleration \( \alpha_L \). If the lateral acceleration is everywhere inwards, and the \( x-y \) position vector is a single-mode harmonic function of \( \sigma \) (or \( \sigma + \)), then it is easily seen that advance of the pattern will always occur. But advance of the pattern does not seem to be a general feature of all loops.

5. Conclusion

In this paper, I have generated explicit formulae for the weak-field metric perturbations induced by an Allen–Casper–Ottewill (ACO) loop of a cosmic string and calculated the corresponding weak-field back-reaction acceleration vector. Although the acceleration vector diverges at the two kink points on the loop, it turns out that the net acceleration and radiative energy loss of the loop are finite. Using a method first described by Quashnock and Spergel, I have determined the net effect of the back-reaction on the motion of the loop over a single oscillation period, and shown that the new, perturbed trajectory of the loop is identical to the original, except for a uniform contraction of scale and a small rotational phase shift. The ACO loop, which is rigidly rotating and has the lowest radiative efficiency of any known loop solution, therefore evolves by self-similar evaporation at the weak-field level. This in turn means that, in any ensemble of loops with a given energy \( E \), the ACO loops (and presumably any loops similar in shape) will be the most stable and longest lived.

The conclusions drawn here depend of course on the assumption that the Quashnock–Spergel method and the weak-field approximation accurately model the secular evolution of the ACO loop. Although there is no reason for suspecting that either approximation introduces spurious features into the calculation, it would be a useful check to be able to generate a continuously self-similar solution to the back-reaction equations at the weak-field level or (even better) within the full framework of general relativity.

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Appendix A. Calculation of the self-acceleration vector

As was mentioned in section 4.1, if the field point \([\bar{t}, \bar{X}]\) is a point \([\bar{\tau}, \bar{\hat{X}}]\) on the string the contour \( \Gamma \) in the integral (4.15) for \( h_{ab} \) divides naturally into three segments, which have been
labelled as \( \Gamma^-, \Gamma^0 \) and \( \Gamma^+ \). When \( u_+ > 0 \) the contour \( \Gamma \) consists first of \( \Gamma^- \) then \( \Gamma^0 \) and then \( \Gamma^+ \), with \((v_-, v_-)\) ranging from \((-\frac{1}{2}, u_- + W_1 + 1)\) to \((0, u_-)\) to \((u_+, u_-)\) to \((\frac{1}{2}, u_- - 1)\) along the segments. On the other hand, if \( u_- < 0 \) then the order of the segments is first \( \Gamma^0 \) then \( \Gamma^- \) and then \( \Gamma^+ \), with \((v_+, v_-)\) ranging from \((-\frac{1}{2}, u_-)\) to \((u_+, u_-)\) to \((0, u_- + W_0)\) to \((\frac{1}{2}, u_- - 1)\). The divergence in \( h_{ab} \) at an arbitrary point on the string is due to the contribution near \((v_+, v_-) = (u_+, u_-)\), which lies on the boundary between \( \Gamma^0 \) and \( \Gamma^+ \) if \( u_+ > 0 \), and on the boundary between \( \Gamma^0 \) and \( \Gamma^- \) if \( u_- < 0 \).

Now, if the field point \([\vec{t}, \vec{x}]\) lies off the world sheet \( T \), the line integral for \( h_{ab} \) takes the form previously given in (4.21), namely

\[
h_{ab}(\vec{t}, \vec{x}) = -4\mu L \int_{V_{\downarrow}}^{V_{\uparrow}} \left( \Delta_{\tau}^{-1} \Psi_{ab} \right)_{\mid_{\tau=\bar{r}}} \, dv_- - 4\mu L \int_{V_{\downarrow}}^{V_{\uparrow}} \left( \Delta_{\tau}^{-1} \Psi_{ab} \right)_{\mid_{\tau=-\bar{r}}} \, dv_-
\]

(A.1)

where the limit functions \( V_{\downarrow} \) and \( V_{\uparrow} \) depend on \( \bar{x}^\tau = [\bar{t}, \bar{x}]^\tau \) through equation (4.20), while the integrand \( \Delta_{\tau}^{-1} \Psi_{ab} \) depends on \( \bar{x}^\tau \), \( v_- \), and \( v_+ \), which is itself a function of \( \bar{x}^\tau \) and \( v_- \) on \( \tau \) through (4.4). However, the dependence of \( \Psi_{ab} \) and \( \Delta_{\tau} \) on \( v_+ \) involves a dependence on \( s = \text{sgn}(v_+) \), and \( s \) is constant on the two sub-integrals in (A.1).

The spacetime derivatives of the metric perturbations therefore have the form

\[
h_{ab,c}(\vec{t}, \vec{x}) = -4\mu L \left[ H_{abc}(0^+, V_0) - H_{abc} \left( \frac{1}{2}, V_1 \right) + H_{abc} \left( \frac{1}{2}, V_1 + 1 \right) - H_{abc}(0^-, V_0) \right] + 4\mu L \int_{V_{\downarrow}}^{V_{\uparrow}} \left( \Delta_{\tau}^{-2} \Psi_{ab} \Delta_{\tau,c} \right)_{\mid_{\tau=\bar{r}}} \, dv_- + 4\mu L \int_{V_{\downarrow}}^{V_{\uparrow}} \left( \Delta_{\tau}^{-2} \Psi_{ab} \Delta_{\tau,c} \right)_{\mid_{\tau=-\bar{r}}} \, dv_-
\]

(A.2)

where

\[
\Delta_{\tau,c} = [1, -s\hat{z}]_c
\]

(A.3)

and

\[
H_{abc}(v_+, V) = \left( \Delta_{\tau}^{-1} \Psi_{ab} V_{\tau} \right)_{\mid_{\tau=\bar{r}}.}
\]

(A.4)

Furthermore, differentiation of the light-cone condition \( \frac{1}{2} L(v_+ + V) = \bar{t} - \bar{x}_\cdot (Lv_+ + LV) \) with \( v_\cdot \) constant gives

\[
V_{\tau} = 2L^{-1} \Delta_{\tau}^{-1} [\bar{t} - \tau, -(\bar{x} - X)]_\tau
\]

(A.5)

with \( \tau = \frac{1}{2} L(v_+ + V) \) and \( \Delta_{\tau} = [\bar{t} - \tau - 2(\bar{x} - X) \cdot X_\tau] \) as before.

In view of (4.41), (A.2) and (A.5), the acceleration vector \( \alpha^a \) can be written in the form

\[
\alpha^a(u_+, u_-) = -8\mu L \left[ A^a(0^+, V_0) - A^a \left( \frac{1}{2}, V_1 \right) + A^a \left( \frac{1}{2}, V_1 + 1 \right) - A^a(0^-, V_0) \right] + 8\mu L \int_{V_{\downarrow}}^{V_{\uparrow}} B^a_{\mid_{\tau=\bar{r}}} \, dv_- + 8\mu L \int_{V_{\downarrow}}^{V_{\uparrow}} B^a_{\mid_{\tau=-\bar{r}}} \, dv_-
\]

(A.6)

where

\[
A^a(v_+, V) = -\Delta_{\tau}^{-1} \left\{ (\Psi_{b_0}^c \hat{X}_b^c \hat{X}_c + \Psi_{b_0}^c \hat{X}_b^c \hat{X}_c) V_{\tau} - \Psi_{b_0}^b \hat{X}_b^c \hat{X}_c \eta^{ad} V_{\tau} \right\} \mid_{\tau=\bar{r}}
\]

(A.7)

and

\[
B^a = -\Delta_{\tau}^{-2} \left\{ (\Psi_{b_0}^c \hat{X}_b^c \hat{X}_c + \Psi_{b_0}^c \hat{X}_b^c \hat{X}_c) \Delta_{\tau,c} - \Psi_{b_0}^b \hat{X}_b^c \hat{X}_c \eta^{ad} \Delta_{\tau,d} \right\}
\]

(A.8)

and again it is understood that the limit \([\bar{t}, \bar{x}] \to [\bar{t}, \bar{x}]\) is taken on the right-hand side of (A.6).
Now,\[ X_b^b = \frac{1}{2}[1, s\hat{z}]^b \quad \text{and} \quad \tilde{X}^c = \frac{1}{2}[1, -\sin(2\pi u)\hat{x} + \cos(2\pi u)\hat{y}]^c, \]while the components of $\Psi_{ab}$ are given in (4.17). Hence,\[ \Psi_{\hat{y}} X^b = \frac{1}{2}(1 - s\hat{s})[1, -\sin(2\pi v)\hat{x} + \cos(2\pi v)\hat{y} + s\hat{z}]^b \]\[ \Psi_{\hat{y}} \tilde{X}_1 = \frac{1}{2}(1 - s\hat{s})[1, -\sin(2\pi v)\hat{x} + \cos(2\pi v)\hat{y}] + \frac{1}{2}[0, -\sin(2\pi v) - \sin(2\pi u)\hat{x} + \cos(2\pi v) - \cos(2\pi u)\hat{y}]^b. \]

and\[ \Psi_{bc} X_b^c = \frac{1}{8}(1 - s\hat{s})[1 + \cos 2\pi(v - u)]. \]

Furthermore,\[ \tilde{X}_c^c \Delta_{+,c} = \frac{1}{2} \quad \text{and} \quad \tilde{X}_c^c \Delta_{-,c} = \frac{1}{2}(1 - s\hat{s}) \]
while the general limiting values of $\Delta_+$ and $\Delta_-$ at $[\hat{r}, \hat{x}] = [\bar{r}, \hat{X}]$ are\[ \Delta_+(v_+, v_-) = \frac{1}{2}L[u_+(1 - s\hat{s}) + u_- - v_+] \]
and\[ \Delta_-(v_+, v_-) = \frac{1}{2}L[u_+ - v_+ + u_- - v_- + \frac{1}{2\pi} \sin 2\pi(v_+ - u_-)]. \]

For the moment, attention will be restricted to the case where $u_+ > 0$ and so $\hat{s} = 1$. It was seen in section 4.1 that the integral for $h_{ab}(\hat{r}, \hat{x})$ diverges in the limit as $[\hat{r}, \hat{x}] \rightarrow [\bar{r}, \hat{X}]$, because $V_0 \rightarrow u_-$ and so $\Delta_+(0^+, V_0) \rightarrow 0$, as is evident from (14.4). It is clear from (7.9) and (A.8) that there may be similar divergences in $A^a(0^+, V_0)$ and $\int V_0 B^a_{1=1} d\nu$. However, it turns out that both functions are in fact convergent, and therefore that $\alpha^a$ is well defined.

To see this, note first that if $u_+ > 0$ and $v_+ = 0^+$ then $s = \hat{s} = 1$ and so the terms (A.10) and (A.12) vanish identically. Therefore\[ A^a(0^+, V_0) = -\Delta_{-1}(\Psi_{\hat{y}} X^b, \tilde{X}_c^c V_{+c})|_{0^+, V_0} \]
where, at any field point $[\bar{r}, \hat{X}]$,$\tilde{X}_c^c V_{+c} |_{0^+, V_0} = L^{-1}\Delta_{-1}\left[[\hat{r} - \frac{1}{2}L(v_+ + v_-) - \hat{s}(\hat{r} - \frac{1}{2}Ls v_+)] |_{0^+, V_0} \right.$\[ = L^{-1}(\Delta_{-1}(\Delta_+))|_{0^+, V_0} \]
an so\[ A^a(0^+, V_0) = -L^{-1}\left(\Delta_{-1}(\Psi_{\hat{z}} X^b_{-})\right)|_{0^+, V_0} \]
In the limit as $[\hat{r}, \hat{x}] \rightarrow [\bar{r}, \hat{X}]$, it can be seen from (15.15) that $(\Delta_{-1})|_{0^+, V_0} \rightarrow 2(Lu_+)^{-1}$ and from (11.11) that $(\Psi_{\hat{y}} \tilde{X}_c^c)|_{0^+, V_0}$ vanishes. Hence, $A^a(0^+, V_0)$ also vanishes in the limit.

Similarly, because $\Psi_{\hat{z}} X^b_{+}$ and $\tilde{X}_c^c \Delta_{+,c}$ and $\Psi_{\hat{y}} \tilde{X}_c^c \tilde{X}_c^c$ are all identically zero when $s = \hat{s}$, it follows that $B^a_{1=1}$ is also identically zero, and therefore that $\int V_0 B^a_{1=1} d\nu \rightarrow 0$ at any field point $[\hat{r}, \hat{x}]$. Furthermore, all other limiting values of $\Delta_+$ and $\Delta_-$ are non-zero, as $V_1 \rightarrow W_1(u_+, u_-)$ in the limit as $[\hat{r}, \hat{x}] \rightarrow [\bar{r}, \hat{X}]$ and so$\Delta_+ \rightarrow \frac{1}{2}L(2u_+ - 1 - W_1)$ \[ \Delta_- \rightarrow \frac{1}{2}L\left(u_+ - \frac{1}{2} - W_1 + \frac{1}{2\pi} \sin 2\pi W_1\right) \]
at $(-\frac{1}{2}, V_1 + 1)$,\[ \Delta_+ \rightarrow Lu_+ \quad \text{and} \quad \Delta_- \rightarrow \frac{1}{2}Lu_+ \].
at \((0^-, V_0)\), and
\[
\Delta_+ \rightarrow - \frac{1}{2} L W_1 \quad \text{and} \quad \Delta_- \rightarrow \frac{1}{2} L \left( u_+ - \frac{1}{2} - W_1 + \frac{1}{2}\pi \sin 2\pi W_1 \right)
\]
at \((\frac{1}{2}, V_1)\). It is easily verified that these limiting values are all positive if \(u_+ \in (0, \frac{1}{2})\).

The limiting values of the three non-zero \(M_{ab}^a a^b\) terms contributing to the co-rotating acceleration vector \(M_{ab}^a a^b\) through (A.6) can now be found by direct substitution. If \(u_+ > 0\) then
\[
M_{a}^a A^b \left( \frac{1}{2}, V_1 \right) \rightarrow - \frac{1}{2} L^{-2} D_{-1}^{-1} \left( \frac{1}{2} - \cos 2\pi W_1 \right) [1, \hat{Z}]^a + [0, -\sin 2\pi W_1 \hat{\sigma} - (1 - \cos 2\pi W_1) \hat{z}]^a \tag{A.19}
\]
\[
M_{a}^a A^b (0^-, V_0) \rightarrow - \frac{1}{2} L^{-2} u^a_+ [1, \hat{Z}]^a \tag{A.20}
\]
and
\[
M_{a}^a A^b \left( -\frac{1}{2}, V_1 + 1 \right) \rightarrow \frac{1}{2} L^{-2} D_{-1}^{-1} \left( 1 - \cos 2\pi W_1 \right) [1, \hat{Z}]^a - L^{-2} D_{-1}^{-1} [1, -\sin 2\pi W_1 \hat{\sigma} + \cos 2\pi W_1 \hat{z}]^a
= \frac{1}{2\pi} L^{-2} (1 - \cos 2\pi W_1) D_{-1}^{-1} [0, -(1 - \cos 2\pi W_1) \hat{\sigma} + \sin 2\pi W_1 \hat{z}]^a + \frac{1}{2} L^{-2} D_{-1}^{-1} W_1 [0, -\sin 2\pi W_1 \hat{\sigma} - (1 - \cos 2\pi W_1) \hat{z}]^a \tag{A.21}
\]
where
\[
D_+ = 2u_+ - 1 - W_1 \quad \text{and} \quad D_- = u_+ - \frac{1}{2} - W_1 + \frac{1}{2}\pi \sin 2\pi W_1. \tag{A.22}
\]

The one remaining contribution to \(M_{ab}^a a^b\) is \(\int_{V_0}^{V_{i+1}} M_{a}^a B^b |_{v=-} \, dv_-\). If \(s = -1\)
\[
\Psi_{a}^a A_{c}^b X_{a}^c \Delta_{+, c} = \frac{1}{4} [1 - \sin(2\pi v_-) \hat{\sigma} + \cos(2\pi v_-) \hat{z}]^a \tag{A.23}
\]
\[
\Psi_{a}^a A_{c}^b X_{a}^c \Delta_{+, c} = \frac{1}{4} [1 - \cos 2\pi (v_- - u_-)] [1, -\hat{Z}]^a + \frac{1}{4} [0, -\sin(2\pi v_-) - \sin(2\pi u_-) \hat{\sigma} + \cos(2\pi v_-) - \cos(2\pi u_-) \hat{z}]^a \tag{A.24}
\]
and
\[
\Psi_{bc} X_{a}^b X_{a}^c \eta^{ad} \Delta_{+, d} = \frac{1}{4} [1 - \cos 2\pi (v_- - u_-)] [1, -\hat{Z}]^a. \tag{A.25}
\]
so
\[
M_{ab}^b |_{v=-} \rightarrow - L^{-2} (v_- - 2\hat{t}/L - 2\hat{z}/L)^{-2} [1, -\hat{Z}]^a + 2 L^{-2} (v_- - 2\hat{t}/L - 2\hat{z}/L)^{-2} [0, \sin 2\pi (v_- - u_-) \hat{\sigma} - \cos 2\pi (v_- - u_-) \hat{z}]^a \tag{A.26}
\]
Hence,
\[
\int_{V_0}^{V_{i+1}} M_{a}^a B^b \big|_{v=-} \, dv_- = L^{-2} \{ (v_- - \psi_+)^{-1} [1, -\hat{Z}]^a \} \big|_{V_0}^{V_{i+1}}
- 2 L^{-2} (v_- - \psi_+)^{-1} [0, \sin 2\pi (v_- - u_-) \hat{\sigma} - \cos 2\pi (v_- - u_-) \hat{z}]^a \big|_{V_0}^{V_{i+1}}
+ 4\pi L^{-2} [0, \{ \text{Ci}(2\pi |v_- - \psi_+|) \} \cos 2\pi (u_- - \psi_+)
- \text{Si}(2\pi |v_- - \psi_+|) \sin 2\pi (u_- - \psi_+) \hat{\sigma} + \{ \text{Ci}(2\pi |v_- - \psi_+|) \} \sin 2\pi (u_- - \psi_+)
- \text{Si}(2\pi |v_- - \psi_+|) \cos 2\pi (u_- - \psi_+) \hat{\sigma}] \big|_{V_0}^{V_{i+1}} \tag{A.27}
\]
where \( \text{Si}(x) = \int_0^x \frac{\sin w}{w} \, dw \) and \( \text{Ci}(x) = -\int_x^\infty \frac{\cos w}{w} \, dw \) are again sine and cosine integrals, and \( \psi_+ = 2l/L + 2s/L \) as before.

In the limit as \([\tilde{t}, \tilde{x}] \rightarrow [t, \tilde{X}],\) therefore,

\[
\int_{V_0}^{V_{1+1}} M_{\mu}^{\nu} B^{\mu} |_{r=-1} \, d\nu \rightarrow L^{-2} \left[ (2u_+)^{-1} - D_+^{-1} \right] [1, \bar{\nu} - \bar{\theta}]^{\nu} + L^{-2} u_+^{-1} [0, \bar{\theta}]^{\nu}
\]

+ \( 2L^{-2} D_+^{-1} [0, \sin 2\pi W_1 \tilde{y} - \cos 2\pi W_1 \tilde{y}]^{\nu} \)

+ \( 4\pi L^{-2} [0, -\{\text{Si}(4\pi u_-) - \text{Si}(2\pi D_+)\} \cos 4\pi u_+ \)

- \( \{\text{Si}(4\pi u_-) - \text{Si}(2\pi D_+)\} \sin 4\pi u_+ \)

- \( \{\text{Ci}(4\pi u_-) - \text{Ci}(2\pi D_+)\} \sin 4\pi u_+ \)

+ \( \{\text{Si}(4\pi u_-) - \text{Si}(2\pi D_+)\} \cos 4\pi u_+ \)\( \tilde{y}^{\nu} \). \quad (A.28)

Collecting together (A.19), (A.20), (A.21) and (A.28), plus the definition (4.22) of \( W_1 \) in terms of \( u_+ \), gives finally

\[
\alpha' = \alpha^c = 32\pi^2 \mu L^{-1} F(W_1) W_1 (1 - \cos 2\pi W_1)
\]

(A.29)

and

\[
\begin{bmatrix}
\alpha^L \\
\alpha^N
\end{bmatrix}
= -32\pi \mu L^{-1} \times \left[ \begin{array}{c}
\{\text{Si}(4\pi u_-) - \text{Si}(2\pi D_+)\} \sin 4\pi u_+ + \{\text{Ci}(4\pi u_-) - \text{Ci}(2\pi D_+)\} \cos 4\pi u_+ \\
-\{\text{Si}(4\pi u_-) - \text{Si}(2\pi D_+)\} \cos 4\pi u_+ + \{\text{Ci}(4\pi u_-) - \text{Ci}(2\pi D_+)\} \sin 4\pi u_+
\end{array} \right]

+ 32\pi^2 \mu L^{-1} F(W_1) W_1 \left[ 2\pi W_1 \cos 2\pi W_1 - \sin 2\pi W_1 \cos 2\pi W_1 - 1 + 2\pi W_1 \sin 2\pi W_1 \right] \quad (A.30)
\]

where

\[
F(W) = (2\pi^2 W^2 + 1) - \cos 2\pi W - 2\pi W \sin 2\pi W)^{-1}.
\]

(A.31)

\[
4\pi u_+ = 2\pi (1 + W_1) - \frac{1}{\pi} (1 - \cos 2\pi W_1) / W_1
\]

(A.32)

and

\[
2\pi D_+ = -\frac{1}{\pi} (1 - \cos 2\pi W_1) / W_1.
\]

(A.33)

A similar calculation in the case where \( u_+ < 0 \) (and so \( \tilde{s} = -1 \)) gives

\[
\alpha' = -\alpha^c = 32\pi^2 \mu L^{-1} F(W_0) W_0 (1 - \cos 2\pi W_0)
\]

(A.34)

and

\[
\begin{bmatrix}
\alpha^L \\
\alpha^N
\end{bmatrix}
= -32\pi \mu L^{-1} \times \left[ \begin{array}{c}
\{\text{Si}(2\pi (1 + 2u_+)) - \text{Si}(2\pi D_+)\} \sin 4\pi u_+ + \{\text{Ci}(2\pi (1 + 2u_+))
\end{array} \right]

- \{\text{Si}(2\pi (1 + 2u_+)) - \text{Si}(2\pi D_+)\} \cos 4\pi u_+

- \{\text{Ci}(2\pi (1 + 2u_+)) - \text{Ci}(2\pi D_+)\} \sin 4\pi u_+

+ 32\pi^2 \mu L^{-1} F(W_0) W_0 \left[ 2\pi W_0 \cos 2\pi W_0 - \sin 2\pi W_0 \cos 2\pi W_0 - 1 + 2\pi W_0 \sin 2\pi W_0 \right] \quad (A.35)
\]

where now

\[
4\pi u_+ = 2\pi W_0 - \frac{1}{\pi} (1 - \cos 2\pi W_0) / W_0
\]

(A.36)
and
\[ 2\pi D_s = -\frac{1}{\pi} (1 - \cos 2\pi W_0) / W_0. \]  
(A.37)

As can be seen from (A.29) and (A.34), \( \alpha' \) is the same function \( \alpha'(W) \) of \( W_0 \) for \( u_+ \in (-\frac{1}{2}, 0) \) as it is of \( W_1 \) for \( u_+ \in (0, \frac{1}{2}) \). Since \( W_1(u_+) = W_0(u_+ - \frac{1}{2}) \) for all \( u_+ \in (0, \frac{1}{2}) \) the acceleration component \( \alpha' \) is therefore a periodic function of \( u_+ \) with period \( \frac{1}{2} \). The same is true of the lateral and normal components \( \alpha^L \) and \( \alpha^N \), while the vertical component \( \alpha^z \) is anti-periodic.

Appendix B. Deriving the equation of motion from the Battye–Carter equation

In this appendix, I briefly describe the formalism behind the gauge-independent Battye–Carter equation of motion (B.5), and demonstrate that the equation reduces to the weak-field equations of motion (2.13) and (2.14) in the standard gauge.

If \((g_{ab}, X^a)\) is a general solution pair to the strong-field back-reaction problem then the projection tensor \(p_{ab}\) corresponding to the induced 2-metric \(\gamma^{AB} = g_{ab}X^a,A X^b,B\) is
\[ p_{ab} = \gamma_{AB} X^a,A X^b,B \]  
(B.1)
with \(\gamma^{AB}\) the inverse of \(\gamma_{AB}\). The extrinsic curvature tensor of the world sheet \(T\) is therefore
\[ K_{abc} = p_{ab} \nabla_c p_{ac} \]  
(B.2)
where \(\nabla_n\) is the derivative operator associated with \(g_{ab}\).

In terms of the derivatives of the position function \(X^a\) the curvature tensor \(K_{abc}\) and its trace \(K^c\)\[\equiv g_{ab} K^{abc}\] can be written as
\[ K^{abc} = q_{cd} \left( \gamma^{AC} \gamma^{BD} X^a,A X^b,B X^d,CD + p_{mn} \Gamma^d_{mn} \right) \]  
(B.3)
and
\[ K^c = q_{cd} \left( \gamma^{CD} X^d,CD + p^{mn} \Gamma^d_{mn} \right) \]  
(B.4)
with \(q^{ab} = g_{ab} - p_{ab}\) the orthogonal complement of \(p_{ab}\), and \(\Gamma^a_{bc}\) the Christoffel symbol associated with \(g_{ab}\). The equation of motion of the string, obtained by varying the Lagrangian density \(L \equiv -\mu \gamma^{1/2}\), is simply \(K^c = 0\).

If the equation of motion \(K^c = 0\) is perturbed by replacing \(g_{ab}\) with \(g_{ab}^{(0)} + h_{ab}\) then, to linear order in \(h_{ab}\),
\[ K^c - (K^c + K^a h_{ac}) h_{ab} + q^{cd} p_{ab} \left( \nabla_a h_{bd} - \frac{1}{2} \nabla_d h_{ab} \right) = 0 \]  
(B.5)
with all the geometrical quantities in this equation, except \(h_{ab}\), now evaluated using \(g_{ab}^{(0)}\) in place of \(g_{ab}\) [26, 25]. In particular, \(K^c\) is no longer identically zero but instead of order \(h_{ab}\). However, the term \(K^c h_{ac} h_{ab}\) is of the second order in \(h_{ab}\) and can be ignored.

The Battye–Carter equation (B.5) can be specialized to the problem at hand by first setting \(g_{ab}^{(0)} = \eta_{ab}\), so that \(\Gamma^{ac}_{ab} = 0\). Then to linear order in \(h_{ab}\) (B.5) reads
\[ q^c_d (\gamma^{CD} X^d,CD - \gamma^{AC} \gamma^{BD} h_{ab} X^a,A X^b,B X^d,CD) = -q^d_c \gamma^{AC} X^a,A X^b,B \left( h_{bd,a} - \frac{1}{2} h_{ab,d} \right) \]  
(B.6)
where \(q^c_d\) and \(\gamma^{AC}\) are of course evaluated using \(\eta_{ab}\) rather than \(g_{ab}\).

The next step in the perturbation is to replace \(X^a\) with \(X^a_{(0)} + \delta X^a\), where \(\delta X^a\) is of linear order in \(h_{ab}\). The only substantial change is to the term \(q^c_d \gamma^{CD} X^d,CD\), which becomes
\[ q^c_d \gamma^{CD} \delta X^d,CD + q^c_d \gamma^{CD} X^d,CD + q^c_d (\gamma^{CD} \delta X^d,CD - 2 \gamma^{AC} \gamma^{BD} \eta_{ab} X^a_{(0),A} \delta X^b,B X^d_{(0),CD} \) \]  
(B.7)
where $q_\mu^c$ and $\gamma^{AC}$ are now evaluated using $X_0^a$ rather than $X^a$, and
\[ \delta q_\mu^c = -\delta p_\mu^c = -2\eta_{db}\gamma^{AB} q_d^b X_0^{(0)A} \delta X^a, B \]
with round brackets on spacetime indices denoting symmetrization.

Equation (B.6) therefore splits into two parts,
\[ q_\mu^c \gamma^{CD} X_0^{(0)C} = 0 \]  
and
\[ q_\mu^c \gamma^{CD} \delta X^d, CD - 2\eta_{db} \gamma^{AB} q_d^b X_0^{(0)A} \delta X^a, B \gamma^{CD} X_0^{(0)C} = 0 \]
(\[\text{B.8}\])

at zeroth order and first order in $h_{ab}$ respectively.

The final step is to fix the gauge. To keep the analysis as general as possible, suppose that the full solution pair $(g_{ab}, X^a)$ satisfies a gauge condition of the form
\[ \gamma^{AB} = \gamma^{1/2} \kappa^{AB} \]  
\[\text{B.11}\]
where $\kappa_{AB}$ is a constant, symmetric $2 \times 2$ matrix independent of $\mu$, with determinant $-1$.

The standard gauge, for example, has $\kappa_{AB} = \eta_{AB}$. Replacing $g_{ab}$ with $\eta_{ab} + h_{ab}$ and $X^a$ with $X_0^a + \delta X^a$ then gives a zeroth-order gauge constraint
\[ \gamma^{-1/2} \gamma^{AB} = \kappa_{AB} \]  
\[\text{B.12}\]
and a first-order gauge constraint
\[ 0 = \Delta \kappa_{AB} \equiv \gamma^{-1/2} \left[ 2\eta_{ab} X_0^a (\gamma \delta X^b, B) + h_{ab} X_0^a, A X_0^b, B \right. \]
\[ \left. - \gamma^{CD} \left( \eta_{ab} X_0^a, C X_0^b, D \gamma^{CD} X_0^{(0)C} \right) \right] \]
(\[\text{B.13}\])
where, as before, all geometric quantities are evaluated using $\eta_{ab}$ and $X_0^a$ in place of $g_{ab}$ and $X^a$.

If the zeroth-order constraint (B.12) is substituted into the zeroth-order equation of motion (B.9) then, since $q_\mu^c = \delta_\mu^c - p_\mu^c$ and
\[ p_\mu^c \gamma^{CD} X_0^{(0)C} = -\gamma^{-1/2} \gamma^{CD} X_0^{(0)C} \]
(\[\text{B.14}\])
it follows that
\[ 0 = q_\mu^c \gamma^{CD} X_0^{(0)C} = \gamma^{-1/2} \left( \gamma^{1/2} \gamma^{CD} X_0^{(0)C} \right) = \gamma^{CD} X_0^{(0)C} \]
\[\text{B.15}\]
as $\gamma^{1/2} \gamma^{CD} = \kappa^D$, the inverse of $\kappa_{CD}$. The zeroth-order equation of motion is therefore simply $\gamma^{CD} X_0^{(0)C} = 0$, which in the standard gauge reduces to (2.13) as required. It might seem odd that the equation $q_\mu^c \gamma^{CD} \delta X^d, CD = 0$, which contains only two algebraically independent components, should be equivalent to an equation $\gamma^{CD} X_0^{(0)C} = 0$ with four independent components. However, as is evident from (B.14), the equation $\gamma^{CD} X_0^{(0)C} = 0$ contains information about the gauge choice as well as the equation of motion.

The first-order equation of motion (B.10) now reduces to
\[ q_\mu^c \gamma^{CD} \delta X^d, CD - q_\mu^c \gamma^{AC} \gamma^{BD} (2\eta_{ab} X_0^a, A \delta X^b, B + h_{ab} X_0^a, A X_0^b, B) X_0^{(0)C} = 0 \]
\[\text{B.16}\]
where
\[ \gamma^{AC} \gamma^{BD} (2\eta_{ab} X_0^a, A \delta X^b, B + h_{ab} X_0^a, A X_0^b, B) X_0^{(0)C} \]
\[ = \gamma^{1/2} \gamma^{AC} \gamma^{BD} \Delta \kappa_{AB} X_0^{(0)C} + \gamma^{EF} (\eta_{ab} X_0^a, E \delta X^b, F \]
\[ + h_{ab} X_0^a, E X_0^b, F) \gamma^{CD} X_0^{(0)C} \]
\[ = 0 \]
(\[\text{B.17}\])
as $\Delta \kappa_{AB} = 0$ and $\gamma^{CD}X^d_{(0),C} = 0$. So the first-order equation of motion becomes

$$q^d\gamma^{CD}\delta X^d_{(0),C} = -q^{cd}\gamma^{AB}X^a_{(0),A}X^b_{(0),B} \left( h_{bd,a} - \frac{1}{2}h_{ab,d} \right).$$

(B.18)

Finally, after considerable algebraic rearrangement it can be seen that

$$p^d_\gamma\gamma^{CD}\delta X^d_{(0),C} + p^{cd}\gamma^{AB}X^a_{(0),A}X^b_{(0),B} \left( h_{bd,a} - \frac{1}{2}h_{ab,d} \right) = \gamma^{1/2}\gamma^{AB}X^a_{(0),B}\gamma^{CD}(\Delta \kappa_{AC})_{,D}$$

$$+ \gamma^{AB}X^a_{(0),B} \left( h_{ab}X^b_{(0),A} + \eta_{ab}\delta X^a_{,A} \right) \gamma^{CD}X^d_{(0),C}D$$

$$- \gamma^{1/2}\gamma^{AB}X^a_{(0),B}\gamma^{EF}\eta_{ab}X^b_{(0),E} \left( X^b_{(0),AD}\Delta \kappa_{CF} - X^b_{(0),DF}\Delta \kappa_{CA} \right)$$

$$+ \gamma^{1/2}\gamma^{AB}X^a_{(0),B}\gamma^{EF}X^b_{(0),E} \left( \frac{1}{2}h_{ab}X^b_{(0),F} + \eta_{ab}\delta X^a_{,F} \right) \gamma^{CD}(\gamma^{-1/2}\gamma_{AC})_{,D} = 0$$

(B.19)

(again because $\gamma^{-1/2}\gamma_{AB}$ is constant, $\Delta \kappa_{AB} = 0$ and $\gamma^{CD}X^d_{(0),C} = 0$), and so, given that

$q^{cd} = \eta^{cd} - p^{cd}$, the first-order equation of motion (B.18) takes the form

$$\gamma^{CD}\delta X^c_{(0),C} = -\eta^{cd}\gamma^{AB}X^a_{(0),A}X^b_{(0),B} \left( h_{bd,a} - \frac{1}{2}h_{ab,d} \right)$$

(B.20)

which in the standard gauge is just (2.14). As with the zeroth-order equation of motion, the fact that (B.20) has four independent components does not mean that the system of equations is overdetermined, as the tangential components (B.19) only contain information about the choice of gauge.

Conversely, if the zeroth-order gauge condition (B.12) is imposed then

$$\kappa^{BC}(\gamma^{1/2}\Delta \kappa_{AB})_{,C} = \gamma^{1/2}\eta_{ce}X^c_{(0),A} \left[ \gamma^{CD}\delta X^c_{(0),C} + \eta^{cd}\gamma^{AB}X^a_{(0),A}X^b_{(0),B} \left( h_{bd,a} - \frac{1}{2}h_{ab,d} \right) \right]$$

$$+ \gamma^{1/2}\left( \eta_{ad}\delta X^a_{,A} + \hbar_{cd}X^c_{(0),A} \right) \gamma^{CD}X^d_{(0),C}$$

(B.21)

and so if $X^a_{(0)}$ and $\delta X^a$ satisfy the zeroth- and first-order equations of motion $\gamma^{CD}X^d_{(0),C} = 0$ and (B.20), the evolution of the first-order constraint matrix $\Delta \kappa_{AB}$ is governed by the simple system of equations $\kappa^{BC}(\gamma^{1/2}\Delta \kappa_{ABC})_{,C} = 0$.

To analyse this system, first rotate the gauge coordinates $\xi^A$ so that the symmetric matrix $\kappa^{BC}$ is diagonal, by setting $\xi^A = \Lambda^A_A \xi^A$ where the orthogonal matrix $\Lambda^A_A$ is defined by

$$\Lambda^B_A \Lambda^C_A \kappa^{BC} = \text{diag}(\lambda_1, \lambda_2).$$

(B.22)

Then the function $U_{A'B'} = (\Lambda^B_A \Lambda^A_A)^{-1} \gamma^{1/2}\Delta \kappa_{AB}$ satisfies the equation $\kappa^{BC}U_{A'B',C} = 0$. Furthermore, the matrix $\Delta \kappa_{AB}$ is by definition (B.13) trace-free, and so

$$\kappa^{AB}U_{A'B'} = \gamma^{1/2}\kappa^{AB}\Delta \kappa_{AB} = 0.$$  

(B.23)

On setting $u = \frac{1}{2}(\lambda_1 U_{1'1} - \lambda_2 U_{2'2})$ and $v = U_{1'2}$, the equation $\kappa^{BC}U_{A'B',C} = 0$ reduces to the linear system

$$Au + Bu, z = 0$$

(B.24)

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \lambda_2 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad u = (u, v)^T.$$  

(B.25)

The classification and characteristics of this system of equations are determined by the roots $\xi = \pm \sqrt{-\lambda_1/\lambda_2}$ of the equation $\det(A - \xi B) = 0$. Since the induced metric $\gamma^{AB} = \gamma^{-1/2}\kappa^{AB}$ has signature $(+, -)$, the roots $\xi$ are real and distinct, and the system of equations is hyperbolic. Moreover, the characteristics satisfy the equation $d\xi^1/d\xi^2 = \xi$, and so have tangent directions $t^A = [\xi, 1]$. Hence,

$$\kappa_{AB}t^A t^B = \kappa_{AB}t^A t^B = \xi^2 \lambda_1^{-1} + \lambda_2^{-1} = 0$$

(B.26)
and the characteristics are null curves on the world sheet. The equation $\kappa^{BC}(\gamma_1/\Delta_1\kappa_{AB}),C = 0$ therefore has the unique solution $\gamma_1/\Delta_1\kappa_{AB} = 0$ (and satisfies the first-order gauge constraint (B.13)) at all points on the world sheet, provided that $\Delta_1\kappa_{AB} = 0$ on the non-null initial curve $\tau = 0$.

It is always possible to ensure that $\Delta_1\kappa_{AB} = 0$ on the initial curve by choosing the initial data $\delta X^{a,A}$ so that (B.13) is satisfied at all points on the curve. The details of this procedure as it applies to the ACO loop are explained at the beginning of section 4.3. In the present case, the above discussion is modified slightly by the presence of the kinks, because $\gamma_{AB}$ is undefined (that is, discontinuous) at the kink points, and, strictly speaking, so are $\kappa_{AB}$ and $\Delta_1\kappa_{AB}$ (although continuous extensions of these functions exist trivially). Nonetheless, as is shown explicitly in section 4.3, the first-order constraint $\Delta_1\kappa_{AB} = 0$ is still satisfied at all ordinary points on the world sheet.

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