Non-integrability of dominated splitting on $\mathbb{T}^2$

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Received 9 April 2014, revised 10 September 2014
Accepted for publication 23 October 2014
Published 17 November 2014

Recommended by D V Treschev

Abstract
We construct a diffeomorphism $f$ on a 2-torus with dominated splitting $E \oplus F$ such that there exists an open neighbourhood $\mathcal{U} \ni f$ satisfying that for any $g \in \mathcal{U}$, neither $E_g$ nor $F_g$ is integrable.

Keywords: dominated splitting, integrable, DA
Mathematics Subject Classification: 37D30

1. Introduction

According to the theory of ordinary differential equations, Lipschitz vector fields are uniquely integrable. However, the bundles that appear in dynamics are mostly Hölder [PSW]. Due to hyperbolicity, the stable and unstable bundles are uniquely integrable. Particularly for two-dimensional $C^2$ Anosov diffeomorphisms, the two hyperbolic bundles are $C^1$ [AS]! But, we know little on the integrability of centre bundles, which is an exceedingly challenging problem [BBI].

In this paper, we focus on the diffeomorphisms on a 2-torus $\mathbb{T}^2$ with dominated splitting. Firstly, we recall some related definitions.

Let $E$ be a one-dimensional continuous sub-bundle of $\mathbb{T}^2$.

Definition 1.1. $E$ is said to be integrable if there exists a 1-foliation (continuous partition consisting of immersed one-dimensional sub-manifolds) of the $\mathbb{T}^2$ tangent to $E$.

Definition 1.2. $E$ is said to be uniquely integrable if there exists exactly one 1-foliation of the $\mathbb{T}^2$ tangent to $E$.
Definition 1.3. A $Df$-invariant continuous bundle splitting $E \oplus F = \mathbb{T}^2$ with $\dim E = \dim F = 1$ is said to be a dominated splitting, if for any $x \in \mathbb{T}^2$, any unitary $u \in E_x$ and any unitary $v \in F_x$, $|Df(u)| < |Df(v)|$.

Both the two bundles in the splitting are uniquely defined. Additionally, the dominated splitting is $C^1$ robust: there exists a $C^1$ neighbourhood $U \ni f$ such that for any $g \in U$, $g$ has the dominated splitting $E_g \oplus F_g$.

According to Peano’s Theorem, for a continuous vector field, through every point $x$ there exists an integral curve. But, can these curves form a foliation?

**Question 1.** Let $f$ be a diffeomorphism on $\mathbb{T}^2$ with a dominated splitting $E \oplus F$. Are these two sub-bundles integrated to foliations? Moreover, if $f$ is $C^2$, are the two bundles Lipschitz ($C^1$)?

For partially hyperbolic systems (the one of $E$ and $F$ is uniformly hyperbolic), Pujals and Sambarino have firstly given a positive answer for the former question. For the latter, it is still unclear.

**Theorem 1.4.** [PS, Po] For partially hyperbolic diffeomorphisms on 2-torus $\mathbb{T}^2$, the two bundles in the dominated splitting are uniquely integrable.

In this paper, we give a negative answer for the above question:

**Theorem 1.5.** There exists a diffeomorphism $f$ on 2-torus with dominated splitting $E \oplus F$, such that there is a $C^1$ open neighbourhood $U \ni f$ satisfying that for any $g \in U$, neither $E_g$ nor $F_g$ is integrable and hence neither of them are Lipschitz.

In our construction, the non-integrability happens in a small neighbourhood of sink (source). On the contrary, in [PS], it has an interesting corollary that ‘for any $C^2$ diffeomorphism on 2-torus with dominated splitting, if periodic points are all hyperbolic saddles, then the two bundles are uniquely integrable’. How about $C^1$ systems:

**Problem 1.** Given a $C^1$ diffeomorphism on 2-torus with dominated splitting, if periodic points are all hyperbolic saddles, are the two bundles integrable?

Between the integrability and unique integrability, there exists such a surprising phenomenon for a Hölder continuous vector field on the plane: there are uncountable distinct foliations tangent to some given vector field [BF]. So, it is natural to ask:

**Problem 2.** Is there such a diffeomorphism with dominated splitting $E \oplus F$, satisfying that $E(F)$ is integrated to different foliations?

2. Two basic lemmas

At first, we introduce some notations used through the paper. Take

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^2.$$

Let $0 < \lambda < 1 < \mu$ be the two eigenvalues of $A$, $E^s$ the contracting eigenspace of $A$, and $E^u$ the expanding eigenspace of $A$. Let $f_A$ be the hyperbolic automorphism on 2-torus induced by $A$, which has two fixed points at least. $E^s$ and $E^u$ induce the hyperbolic splitting of $f_A$, still denoted as $E^s \oplus E^u = \mathbb{T}^2$ and the two eigenspaces induce the coordinate system $\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \}$ on the 2-torus $\mathbb{T}^2$.

Let $f$ be a diffeomorphism on $\mathbb{T}^2$, the norm of $Df$ is denoted by

$$\|Df\| = \sup\{|Df(v)|/|v| : 0 \neq v \in \mathbb{T}^2\}.$$
The norm of $Df$ restricted on a sub-bundle $E$, is denoted by $\|Df|_E\|$. Let $r > 0$. For a hyperbolic fixed non-sink $x$ with dominated splitting $T_xM = E(x) \oplus F(x)$, we define the strong unstable manifold $W^{uu}(x, f)$ as:

\[
\{ y : d(f^{-n}(y), x) < r, \text{ and } \exists N, \text{ s.t. } \frac{d(f^{-n}(y), x)}{\|Df^{-n}|E(y)\|} < \frac{1}{2}, \forall n > N \}.
\]

Similarly, we can define a strong stable manifold $W^{ss}(x, f)$ for a hyperbolic fixed non-source $x$ with dominated splitting.

Now we give two basic lemmas. Firstly, we recall the DA-operation [Wi]:

**Lemma 2.1.** Let $p = (0, 0)$ be a fixed point of $f_A$. Then, for any $\varepsilon > 0$, there exists $C^0$-perturbation $f$ of $f_A$ such that:

1. $f(x) = f_A(x)$ outside the $\varepsilon$-ball $B(p, \varepsilon)$;
2. for any $x \in M$,
\[
Df(x) = \begin{pmatrix}
a(x) & b(x) \\
0 & \mu
\end{pmatrix},
\]

where $\lambda^2 < a(x) < \sqrt{\mu}$, $|b(x)| < \varepsilon$;
3. In $W^u(p, f)$, $f$ has exactly three periodic points contained in $B(p, \varepsilon)$: one fixed source $p$ and two fixed saddles;
4. $Df$ are constant diagonal matrices in some neighbourhoods of the two saddles above;
5. $W^{uu}(p, f) = \{0\} \times (-\frac{1}{\mu}, \frac{1}{\mu})$.

Similarly, there exists a symmetrical DA-operation of $f_A$: to do the same DA-operation of $f^{-1}$.

For completion, we give a proof of this basic lemma in the following.

**Proof.** Let $I_1 \times I_2 \subset B(p, \varepsilon)$, where both $I_1$ and $I_2$ are intervals centred at 0, and
\[
\ell(I_1) = \frac{\varepsilon}{3\mu} \ell(I_2).
\]

Take a smooth bump function $\alpha$ satisfying the following conditions:

1. $\alpha(x)$ is an odd function and $\alpha(x) = 0$, for $x \notin I_1$;
2. $\lambda^2 - \lambda < \alpha'(x) < \sqrt{\mu} - \lambda$;
3. $\alpha(x) + \lambda x$ has exactly three periodic points all contained in $I_1$: one fixed source 0 and two fixed sinks;
4. $\alpha'(x)$ is constant in some neighbourhoods of the above two sinks.

Take another bump function $\beta$ satisfying that,
\[
\begin{align*}
\beta(x) &= 1, \quad x \text{ in a small neighbourhood of } 0; \\
\beta(x) &= 0, \quad x \notin I_2; \\
0 &\leq \beta(x) \leq 1; \\
|\beta'(x)| &< 3/\ell(I_2)
\end{align*}
\]

Let
\[
f(x) = f(x_1, x_2) = (\alpha(x_1)\beta(x_2) + \lambda x_1, \mu x_2).
\]

Then,
\[
Df(x) = \begin{pmatrix}
\alpha'(x_1)\beta(x_2) + \lambda & \alpha(x_1)\beta'(x_2) \\
0 & \mu
\end{pmatrix}.
\]

Note that $\lambda^2 - \lambda < \alpha'(x) < \sqrt{\mu} - \lambda$, $\beta(x) \in [0, 1]$ and $\ell(I_1) = \frac{\varepsilon}{3\mu} \ell(I_2)$.
Then,
\[ |\alpha(x_1)\beta'(x_2)| < \mu \ell(I_1) \times \frac{3}{\ell(I_2)} = \varepsilon. \]
And,
\[ \lambda^2 < \alpha'(x_1)\beta(x_2) + \lambda < \sqrt{\mu}. \]
This verifies the property (2) in the lemma. From properties (3) and (4) of function \( \alpha \) and \( \beta'(x) = 0 \) in a small neighbourhood of 0, we get properties (3) and (4) of the lemma. Since \( Df \) is a diagonal matrix on the line \( \{0\} \times (-\frac{1}{10}, \frac{1}{10}) \), \( f \) satisfies property (5). 

The next lemma is a classic theorem (e.g., see appendix B in [BDV]), which gives a sufficient condition for a diffeomorphism to have dominated splitting. For any two sub-bundles \( E, F \subset T\mathbb{T}^2 \),
\[ \langle E, F \rangle \triangleq \sup \{ \langle u, v \rangle \leq \frac{\pi}{2} : u \in E_x, v \in F_x, x \in \mathbb{T}^2 \}. \]

**Lemma 2.2.** Let \( K > 0 \), \( \eta > 1 \), \( \delta > 0 \), there exists \( \varepsilon > 0 \) such that for any diffeomorphism \( f \) on \( \mathbb{T}^2 \), if under the coordinate \( \{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \} \),
\[ Df(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \]
satisfies that for any \( x \in \mathbb{T}^2 \),
- \( \min\{|a(x)|, |d(x)|\} > K \),
- \( |d(x)| > \eta |a(x)| \),
- \( \max\{|b(x)|, |c(x)|\} < \varepsilon \),
then \( f \) has the dominated splitting \( E \oplus F \) with the property
\[ \langle E, E' \rangle < \delta, \quad \langle F, F' \rangle < \delta. \]

### 3. A robustly non-integrable example

Firstly, we construct a diffeomorphism on 2-torus with dominated splitting \( E \oplus F \), such that \( E \) is robustly non-integrable. It is a special DA-map: to do the DA-operation twice.

**Example 3.1.** Let \( p \) be a fixed point of \( f_A, \varepsilon > 0 \) a very small constant (to be determined in the following construction). By DA-operation in \( B(p, \varepsilon) \), we can take a map \( g \) such that
\[ g(x) = f_A(x), \quad x \notin B(p, \varepsilon), \]
\[ Dg = \begin{pmatrix} a(x) & b(x) \\ 0 & \mu \end{pmatrix}, \]
here, \( \lambda^2 < a(x) < \sqrt{\mu}, |b(x)| < \varepsilon \). Additionally, \( g \) has two fixed points: source \( p \) and saddle \( q \in B(p, \varepsilon) \).

Also, there exists an open neighbourhood \( U \ni q \) such that for any \( x \in U \):
\[ Dg(x) = \begin{pmatrix} a(q) & 0 \\ 0 & \mu \end{pmatrix}. \]
In this smaller neighbourhood \( U \), we make another DA-operation \( f \) of \( g \) such that \( f \) has two fixed points in \( B(p, \varepsilon) \): source \( p \) and sink \( q \), both the length of two components of \( W^{uu}(p, f) - p \) equals \( \frac{1}{10} \), and
\[ Df = \begin{pmatrix} a_1(x) & b_1(x) \\ c_1(x) & d_1(x) \end{pmatrix}. \]
satisfies that there exists $K > 0$ and $\eta > 1$ such that
\[
\min\{|a_1(x)|, |d_1(x)|\} > K,
\]
\[
|d_1(x)| > \eta |a_1(x)|,
\]
\[
\max\{|b_1(x)|, |c_1(x)|\} < \varepsilon.
\]
Choose $\delta < \frac{1}{1000}$ and $\varepsilon < \frac{1}{1000}$ satisfying lemma 2.2.

Then, $f$ satisfies the following properties.

1. $f$ has dominated splitting $E \oplus F$;
2. $\angle(E^s, E) < \delta$, $\angle(E^u, F) < \delta$;
3. $f(x) = f_A(x)$, $x \notin B(p, \varepsilon)$;
4. $f$ has two fixed points in $B(p, \varepsilon)$: source $p$ and sink $q$;
5. The lengths of the two components of $W_{\pm}^{uu}(p, f)$ both equal $\frac{1}{10}$, and $2\varepsilon \|Df\| < \frac{1}{10}$.

The following proposition is our main result.

**Proposition 3.2.** For any diffeomorphism $f$ on $\mathbb{T}^2$ satisfying the above five properties, there exists a $C^1$ open neighbourhood $U \ni f$ such that for any $g \in U$, $g$ has dominated splitting $E_g \oplus F_g$, but $E_g$ is non-integrable.

A curve $\gamma^E$ is said to be an $E$-curve, if $\gamma^E$ is tangent to $E$ everywhere. Similarly, we define the $F$-curve. The non-integrability of $E$ results from the following fact:

**Lemma 3.3.** Let $B(q)$ be the intersection of $B(p, \varepsilon)$ and the basin of the sink $q$. For any $x \in B(q)$ and any $E$-curve $\gamma^E$ of length $3\varepsilon$ centred at $x$, we have that $p \in \gamma^E$.

**Proof.** We give the natural order on the curve $W_{\pm}^{uu}(p, f)$. By dominated splitting it is not difficult to show that the tangent space
\[
TW_{\pm}^{uu}(p, f) = F|W_{\pm}^{uu}(p, f).
\]
Let $I$ be the set of the intersections of $W_{\pm}^{uu}(p, f)$ and $E$-curves $\gamma^E$ of length $3\varepsilon$ centred at some $x \in B(q)$. Note that
\[
q \in B(p, \varepsilon), \angle(E^s, E) < \delta, \angle(E^u, F) < \delta, \text{ and } \frac{1}{10} \gg \varepsilon.
\]
Then, it is not difficult to deduce that every intersection above is exactly one point. Also, the lower bound $a$ and upper bound $b$ of $I$ satisfy that
\[
\max\{|d(a, p)|, |d(b, p)|\} < 2\varepsilon.
\]
Suppose, on the contrary, that $I \neq \{p\}$, say $b \neq p$. Then, we can take a point $y \in I$ close enough to $b$. By the definition of $I$, we take an $E$-curve $\gamma^E$ starting from $y$ to $B(q)$ of length smaller than $3\varepsilon$ (see the following picture).
Note that $f = f_A$ outside $B(p, \varepsilon)$, and $\zeta(E^c, E) < \delta$. Then,
$$\ell(f(\gamma^E)) < 3\varepsilon.$$ By $2\varepsilon \|Df\| < \frac{1}{10}$, we see that
$$f([a, b]) \subset W^{ss}_{\pm}(p, f).$$ Then, $f(y) \in I$. By the uniform expansion of $f$ on the curve $W^{uu}(p, f)$, we have that the intersection $f(y) \notin [a, b]$. This contradiction finishes the proof of the lemma. \qed

**Proof of the robust non-integrability of $E$ (proposition 3.2).** Note that the above five properties of $f$ are all $C^1$ robust. Then, by the above lemma, there exists a $C^1$ open neighbourhood $U \ni f$ such that for any $g \in U$, $g$ has dominated splitting $E_g \oplus F_g$, but $E_g$ is non-integrable.

**Remark 3.4.** Consider another saddle $q'$ in the $W^s(p, f_A)$. Then, the phenomenon in the above lemma also happens between the saddle $q'$ and sink $q$.

**Proof of theorem 1.5.** Let $p_1$ and $p_2$ be the two fixed points of $f_A$. Take a small enough $\varepsilon > 0$. We construct the map $f$ as follows:

1. make the same perturbation in $B(p_1, \varepsilon)$ as the example above;
2. make the symmetrical perturbation in $B(p_2, \varepsilon)$: for $f_A^{-1}$, we make the same perturbation as the example above;
3. $B(p_1, \varepsilon)$ and $B(p_2, \varepsilon)$ are disjointed. Also, $f$ has that
   - the lengths of the two components of $W^{uu}_{\pm}(p_1, f) - p_1$ both equal $\frac{1}{10}$,
   - the lengths of the two components of $W^{uu}_{\pm}(p_2, f) - p_2$ both equal $\frac{1}{10}$.

Now, $f$ satisfies all the properties in the theorem.

**Acknowledgments**

BH is supported by CPSF 2013M540805. SG is supported by 973 project 2011CB808002, NSFC 11025101 and 11231001.

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