Entropy numbers and spectral radius type formula for composition operators on the polydisk

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Abstract. We give estimates of the entropy numbers of composition operators on the Hardy space of the disk and of the polydisk.

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1 Introduction

This short paper was motivated by a question of J. Wengenroth ([19]) about entropy numbers of composition operators on Hardy spaces $H^2$, which stand a little apart in the jungle of “$s$-numbers”, even though they seem the most natural for the study of compactness, since their membership in $c_0$ characterizes compactness, even in the general framework of arbitrary Banach spaces. Indeed, in various papers (see [1, 10, 11, 12, 13]), we studied in detail the approximation numbers of composition operators, and here we will essentially transfer those results to entropy numbers thanks to the polar (Schmidt) decomposition and a general result on entropy numbers of diagonal operators on $\ell^2$.

So, the proofs are easy, but the statements feature a very different behavior of those entropy numbers. In particular, we will investigate a few properties related with a so-called “spectral radius type formula” which we obtained, in dimension one through a result of Widom ([12]), and, partially in dimension $N$ ([13, 14]), through a result of Nivoche ([16]) and Zakharyuta ([22]).
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2 Entropy numbers

We begin by recalling some facts on $s$-numbers.

Given an operator $T: X \to Y$ between Banach spaces, recall ([4]) that we can attach to this operator five non-increasing sequences $(a_n)$, $(b_n)$, $(c_n)$, $(d_n)$, $(e_n)$ of non-negative numbers (depending on $T$), respectively the sequences of approximation, Bernstein, Gelfand, Kolmogorov, and entropy numbers of $T$. The latter are defined ([4, Chapter 1], or [17, Chapter 5]), for $n \geq 1$ by:

\begin{equation}
(2.1) \quad e_n(T) = \inf \{ \varepsilon > 0 \mid N(T(B_X), \varepsilon B_Y) \leq 2^{n-1} \},
\end{equation}

where $B_X$ and $B_Y$ are the respective closed unit balls of $X$ and $Y$, and where, for $A, B \subseteq Y$, $N(A, B)$ denotes the smallest number of translates of $B$ needed to cover $A$.

All those sequences $(a_n)$, $(b_n)$, $(c_n)$, $(d_n)$, $(e_n)$, say $(u_n)$, share the ideal property:

$$u_n(ATB) \leq \|A\| u_n(T) \|B\|. $$

For Hilbert spaces, it turns out that $a_n = b_n = c_n = d_n = s_n$, where $(s_n)$ designates the sequence of singular numbers, but entropy numbers stay a little apart.

For general Banach spaces $X$ and $Y$ and $T: X \to Y$, we have, in general ([3, Theorem 1], see also [17, Theorem 5.2]), for $\alpha > 0$:

$$\sup_{1 \leq k \leq n} k^\alpha e_k(T) \leq C \alpha \sup_{1 \leq k \leq n} k^\alpha a_k(T),$$

and, if $X$ and $Y^*$ are of type 2:

$$a_n(T) \leq K e_n(T), \quad \text{for all } n \geq 1$$

([7, Corollary 1.6]), where $K = \kappa \left[ T_2(X)T_2(Y^*) \right]^2$; in particular, if $T$ acts between Hilbert spaces (see [17, Theorem 5.3]):

$$a_n(T) \leq 4 e_n(T), \quad \text{for all } n \geq 1.$$

Those inequalities indicate that entropy numbers are always bigger than singular numbers, up to a constant, and that, as far as the scale of powers $n^\alpha$
is implied, they are dominated by approximation numbers in a weak sense. But it turns out that, individually, they can be much bigger than the latter for composition operators, as we shall see.

We will rely on the following estimate (see [4, p. 17]), in which \( \ell^2 \) denotes the space of square-summable sequences \( x = (x_k)_{k \geq 1} \) of complex numbers. This estimate is given for the sequence \((\varepsilon_n)\) of covering numbers and with the scale of powers of 2, but \( e_n = \varepsilon_{2^{n-1}} \), by definition, and the change of 2 to \( e \) only affects constants.

**Theorem 2.1.** (see [4, p. 17]) There exist absolute constants \( 0 < a < b \) such that, for any diagonal compact operator \( \Delta : \ell^2 \to \ell^2 \) with positive and non-increasing eigenvalues \((\sigma_k)_{k \geq 1}\), namely \( \Delta((x_k)_k) = (\sigma_k x_k)_k \), we have, for all \( n \geq 1 \):

\[
(2.2) \quad a \sup_{k \geq 1} \left[ e^{-n/k} \left( \prod_{j=1}^{k} \sigma_j \right)^{1/k} \right] \leq e_n(\Delta) \leq b \sup_{k \geq 1} \left[ e^{-n/k} \left( \prod_{j=1}^{k} \sigma_j \right)^{1/k} \right].
\]

A useful corollary of Theorem 2.1 is the following.

**Theorem 2.2.** Let \( T : H_1 \to H_2 \) be a compact operator between the Hilbert spaces \( H_1 \) and \( H_2 \), and let \((a_n)_{n \geq 1}\) be its sequence of approximation numbers. Then, for all \( n \geq 1 \):

\[
(2.3) \quad \alpha \sup_{k \geq 1} \left[ e^{-n/k} \left( \prod_{j=1}^{k} a_j \right)^{1/k} \right] \leq e_n(T) \leq \beta \sup_{k \geq 1} \left[ e^{-n/k} \left( \prod_{j=1}^{k} a_j \right)^{1/k} \right],
\]

where \( \alpha \) and \( \beta \) are positive numerical constants.

**Proof.** Let \( T x = \sum_{n=1}^{\infty} s_n(x \mid u_n) v_n \) the Schmidt decomposition of \( T \), where \((u_n)_n\) and \((v_n)_n\) are orthonormal sequences of \( H_1 \) and \( H_2 \), respectively, and \((s_n)_n\) is the sequence of singular numbers of \( T \). Let \( \Delta : \ell_2 \to \ell_2 \) the diagonal operator with diagonal values \( s_n, n \geq 1 \). Then \( T = V_1 \Delta U_1 \) and \( \Delta = V_2 T U_2 \), with \( U_1 x = ((x \mid u_n))_n, V_1 ((t_n)_n) = \sum_n t_n v_n, U_2 ((t_n)_n) = \sum_n t_n u_n \) and \( V_2 x = ((x \mid v_n)_n). We have \( \|U_1\|, \|V_1\|, \|U_2\|, \|V_2\| \leq 1 \); hence the result follows from Theorem 2.1 and the ideal property. \( \square \)

This theorem might be thought useless, because we don’t know better the \( a_n \)'s than the \( e_n \)'s! In our situation, this is not the case, since we made a more or less systematic study of the \( a_n \)'s for composition operators in [11, 10, 12] for example.

We now pass to applications to composition operators \( C_\varphi \), defined as \( C_\varphi(f) = f \circ \varphi \) when they act on the Hardy space \( H^2(\mathbb{D}^N) \) (which is always
the case if \( N = 1 \). Here, \( \varphi \) denotes an analytic and non-degenerate self-map of \( \mathbb{D}^N \). For clarity, we separate the cases of dimension \( N = 1 \) and of dimension \( N \geq 2 \).

3 Applications in dimension 1

3.1 General results

In [12], we had coined the parameter:

\[
\beta_1(T) = \lim_{n \to \infty} \left[a_n(T)\right]^{1/n}
\]

and its versions \( \beta_1^+(T), \beta_1^-(T) \) with a upper limit and a lower limit respectively. The following result ([12]) shows in particular that no lower or upper limit is needed for \( \beta = \beta_1 \), and provides a simpler proof of the second item in Theorem 3.1 than in our initial proof of [10].

For the definition of the Green capacity \( \text{Cap} (A) \) of a Borel subset \( A \) of \( \mathbb{D} \), \( 0 \leq \text{Cap} (A) \leq \infty \), we refer to [12].

Theorem 3.1. Let \( \Omega = \varphi(\mathbb{D}) \), with \( \varphi: \mathbb{D} \to \mathbb{D} \) a non-constant analytic map. Then:

1) One always has \( \beta_1^-(C_{\varphi}) = \beta_1^+(C_{\varphi}) =: \beta_1(C_{\varphi}) \) and:

\[
\beta_1(C_{\varphi}) = \exp\left[-1/\text{Cap} (\Omega)\right] > 0.
\]

2) In particular, one has the equivalence:

\[
\beta_1(C_{\varphi}) = 1 \iff \|\varphi\|_{\infty} = 1.
\]

Here, another parameter emerges.

\[
\gamma_1(T) = \lim_{n \to \infty} \left[\epsilon_n(T)\right]^{1/\sqrt{n}}
\]

and its \( \gamma_1^+(T), \gamma_1^-(T) \) versions.

Theorem 3.2. Let \( \varphi: \mathbb{D} \to \mathbb{D} \) be a symbol and \( \Omega = \varphi(\mathbb{D}) \). Then:

1) \( \gamma_1^-(C_{\varphi}) = \gamma_1^+(C_{\varphi}) =: \gamma_1(C_{\varphi}) \) and:

\[
\gamma_1(C_{\varphi}) = \exp\left[-\sqrt{2/\text{Cap} (\Omega)}\right] > 0.
\]

2) In particular, one has the equivalence:

\[
\gamma_1(C_{\varphi}) = 1 \iff \|\varphi\|_{\infty} = 1.
\]
Proof. Set $\rho = 1/\text{Cap} (\Omega)$ for simplicity of notations. Let $\varepsilon > 0$, and $C_\varepsilon$ a positive constant which depends only on $\varepsilon$ and can vary from a formula to another. Theorem 3.1 implies $a_k \leq C_\varepsilon e^{\varepsilon k} e^{-k\rho}$, whence:

$$(a_1 \cdots a_k)^{1/k} \leq C_\varepsilon e^{\varepsilon k/2} e^{-\rho k/2}.$$  

Theorem 2.2 now gives:

$$e_n(C_\varphi) \leq C_\varepsilon \sup_{k \geq 1} \left[ e^{\varepsilon k/2} e^{-(n/k + \rho k/2)} \right].$$

This supremum is essentially attained for $k = \left\lfloor \sqrt{2n/\rho} \right\rfloor$ where $\lfloor . \rfloor$ stands for the integer part, and gives:

$$e_n(C_\varphi) \leq C_\varepsilon e^{\sqrt{n/(2\rho)}} e^{-\sqrt{2\rho}}.$$  

This implies $\gamma_1^+(C_\varphi) \leq e^{\sqrt{1/(2\rho)}} e^{-\sqrt{2\rho}}$, and finally:

$$\gamma_1^+(C_\varphi) \leq e^{-\sqrt{2\rho}}.$$  

The lower bound $\gamma_1^-(C_\varphi) \geq e^{-\sqrt{2\rho}}$ is proved similarly. This clearly ends the proof, since we know from [12] that $\text{Cap} (\Omega) = \infty$ if and only if $\|\varphi\|_{\infty} = 1$. \hfill \Box

3.2 Specific results

For $0 < \theta < 1$, the lens map $\lambda_\theta$ of parameter $\theta$ is defined by:

$$(3.7) \quad \lambda_\theta(z) = \frac{(1 + z)^\theta - (1 - z)^\theta}{(1 + z)^\theta + (1 - z)^\theta}$$

(see [13] or [10]).

**Theorem 3.3.** Let $\lambda_\theta$ be the lens map with parameter $\theta$. Then, with positive constants $a$, $b$, $a'$, $b'$ depending only on $\theta$:

$$(3.8) \quad a' e^{-b' n^{1/3}} \leq e_n(C_{\lambda_\theta}) \leq a e^{-b n^{1/3}}.$$  

**Proof.** We proved in [3] Theorem 2.1] (see also [10] Proposition 6.3] that $a_k = a_k(C_{\lambda_\theta}) \leq a e^{-b\sqrt{k}}$. It follows, using Theorem 2.2, that $(a_1 \cdots a_k)^{1/k} \leq a e^{-b\sqrt{k}}$ and that, for some positive constant $C$:

$$e_n(C_{\lambda_\theta}) \leq C \exp \left[ - \left( (n/k) + bk^{1/2} \right) \right].$$

Taking $k = \left\lfloor n^{2/3} \right\rfloor$ gives the claimed upper bound. The lower bound is proved similarly, using the left inequality in Theorem 2.2, since we know (12] that $a_k \geq a' e^{-b' \sqrt{k}}$. \hfill \Box
We refer to [11, Section 4.1] for the definition of the cusp map $\chi$. We have:

**Theorem 3.4.** Let $\chi$ be the cusp map. Then, with positive constants $a$, $b$, $a'$, $b'$:

$$a' e^{-b' \sqrt{n/\log n}} \leq e_n(C_{\chi}) \leq a e^{-b \sqrt{n/\log n}}. \quad (3.9)$$

**Proof.** We proved in [11] that:

$$a' e^{-b' k/\log k} \leq a_k(C_{\chi}) \leq a e^{-b k/\log k}. \quad (3.10)$$

The proof then follows the same lines as in Theorem 3.3 with the choice $k = [\sqrt{n \log n}]$. \qed

### 4 The multidimensional case

#### 4.1 General results

Let $\varphi: \mathbb{D}^N \to \mathbb{D}^N$ be an analytic map. We will say that $\varphi$ is non-degenerate if $\varphi(\mathbb{D}^N)$ has non-empty interior, equivalently if $\det \varphi'(z) \neq 0$ for at least one point $z \in \mathbb{D}^N$.

Let now $\varphi: \mathbb{D}^N \to \mathbb{D}^N$ be a non-degenerate analytic map inducing a bounded composition operator $C_{\varphi}: H^2(\mathbb{D}^N) \to H^2(\mathbb{D}^N)$ (this is not always the case as soon as $N > 1$, even if $\varphi$ is injective and hence non-degenerate, see for example [5, p. 246], when the polydisk is replaced by the ball; but similar examples exist for the polydisk). Assume moreover that $C_{\varphi}$ is a compact operator.

**Theorem 4.1.** Let $C_{\varphi}: H^2(\mathbb{D}^N) \to H^2(\mathbb{D}^N)$ be a compact composition operator, with $\varphi$ non-degenerate. We have:

1) $e_n(C_{\varphi}) \geq c \exp\left(-C n^{1/N}\right)$, for some constants $C > c > 0$, depending on $\varphi$;

2) if $\|\varphi\|_{\infty} < 1$, then $e_n(C_{\varphi}) \leq C \exp\left(-c n^{1/N}\right)$, with $C > c > 0$ depending on $\varphi$.

**Proof.** 1) It is proved in [11 Theorem 3.1] that, for a non-degenerate map $\varphi$, it holds:

$$a_k(C_{\varphi}) \geq a' e^{-b' k^{1/N}}. \quad (3.11)$$
As in the previous section, it follows from Theorem 2.2, that \((a_1 \cdots a_k)^{1/k} \geq e^{-b''k^{1/N}}\), and then, taking \(k = \lceil n^{N/(N+1)} \rceil\), that:

\[ e_n(C_\varphi) \geq c e^{-Cn^{1/(N+1)}}. \]

2) Similarly, for \(\|\varphi\|_\infty < 1\), it is proved in [1, Theorem 5.2] that:

\[ a_k(C_\varphi) \leq C e^{-ck^{1/N}}; \]

and we get the result from Theorem 2.2.

Those estimates motivate the introduction of the parameter:

\[ (4.1) \quad \gamma_N(C_\varphi) = \lim_{n \to \infty} \left[ e_n(C_\varphi) \right]^{1/n^{1/(N+1)}}. \]

We define similarly \(\gamma_N^+(C_\varphi)\), and will say more on it in next section.

4.2 Specific results

4.2.1 Multi-lens maps

Let \(\lambda_\theta\) be lens maps with parameter \(\theta\). We define the multi-lens map \(\Lambda_\theta\) of parameter \(\theta\) on the polydisk \(\mathbb{D}^N\) as:

\[ (4.2) \quad \Lambda_\theta(z_1, \ldots, z_N) = (\lambda_\theta(z_1), \lambda_\theta(z_2), \ldots, \lambda_\theta(z_N)); \]

for \((z_1, \ldots, z_N) \in \mathbb{D}^N\).

The following result is proved in [1, Theorem 6.1].

**Theorem 4.2.** Let \(\Lambda_\theta\) be the multi-lens map with parameter \(\theta\). Then, for positive constants \(a, b, a', b'\) depending only on \(\theta\) and \(N\), one has:

\[ (4.3) \quad a' e^{-b'n^{1/(2N)}} \leq a_n(C_{\Lambda_\theta}) \leq a e^{-b'n^{1/(2N)}}. \]

The version of Theorem 4.2 for entropy numbers, stated without proof, is:

**Theorem 4.3.** Let \(\Lambda_\theta\) be the multi-lens map with parameter \(\theta\). Then:

\[ (4.4) \quad a' \exp(-b'n^{1/(2N+1)}) \leq e_n(C_{\Lambda_\theta}) \leq a \exp(-b'n^{1/(2N+1)}). \]
4.2.2 Multi-cusp maps

Let \( \chi : \mathbb{D} \to \mathbb{D} \) be the cusp map and \( \varphi : \mathbb{D}^N \to \mathbb{D}^N \) be the multi-cusp map defined by:

\[
\Xi(z_1, \ldots, z_N) = (\chi(z_1), \chi(z_2), \ldots, \chi(z_N)).
\]

It is proved in [1, Theorem 6.2]:

**Theorem 4.4.** Let \( \chi : \mathbb{D} \to \mathbb{D} \) be the cusp map and \( \Xi : \mathbb{D}^N \to \mathbb{D}^N \) be the multi-cusp map. Then:

\[
a' e^{-b' n^{1/N} / \log n} \leq a_n(C_\Xi) \leq a e^{-b n^{1/N} / \log n},
\]

where \( a, b, a', b' \) are positive constants depending only on \( N \).

The version of Theorem 4.4 for entropy numbers, stated without proof, is:

**Theorem 4.5.** Let \( \chi : \mathbb{D} \to \mathbb{D} \) be the cusp map and \( \Xi : \mathbb{D}^N \to \mathbb{D}^N \) be the multi-cusp map. Then:

\[
a' \exp \left[ -b' n^{1/(N+1)} (\log n)^{-N/(N+1)} \right] \leq e_n(C_\Xi) \leq a \exp \left[ -b n^{1/(N+1)} (\log n)^{-N/(N+1)} \right].
\]

5 Connections with pluricapacity and Zakharyuta’s results

Here, in dimension \( N \geq 2 \), the situation is satisfactory for upper bounds (see [13]); for lower bounds, see [14]. The notion involved is now that of pluricapacity, or Monge-Ampère capacity, coined by Bedford and Taylor in [2]. More precisely, if \( A \) is a Borel subset of \( \mathbb{D}^N \), we refer to [13] or [14] for the definition of its pluricapacity \( \text{Cap}_N(A) \), belonging to \([0, +\infty]\), and set:

\[
\tau_N(A) = \frac{1}{(2\pi)^N} \text{Cap}_N(A)
\]

\[
\Gamma_N(A) = \exp \left[ -\left( \frac{N!}{\tau_N(A)} \right)^{1/N} \right]
\]

\[
\beta_N^+(T) = \limsup_{n \to \infty} \left[ a_n(T) \right]^{1/n^{1/N}}.
\]

We temporarily assume that \( \|\varphi\|_\infty < 1 \) so that \( K = \overline{\varphi(\mathbb{D}^N)} \) is a compact subset of \( \mathbb{D}^N \). We proved in [13, Theorem 6.4], relying on positive results of Nivoche ([16]) and Zaharyuta ([22, Proposition 6.1]) on the so-called Kolmogorov conjecture, that:
Theorem 5.1. It holds:

\[ \beta_N^+(C_\varphi) \leq \Gamma_N(K). \]  

We have the following result, which extends the previous result in dimension 1.

Theorem 5.2. The following upper bound holds:

\[ \gamma_N^+(C_\varphi) \leq \exp \left( -\beta_N \rho^{N/(N+1)} \right), \]

where:

\[ \rho = \left( \frac{N!}{\tau_N(K)} \right)^{1/N} = 2\pi \left( \frac{N!}{\operatorname{Cap}_N(K)} \right)^{1/N}, \]

and

\[ \beta_N = \left( \frac{N}{N+1} \right)^{N/(N+1)} \left( N^{-N/(N+1)} + N^{1/(N+1)} \right) \geq e^{-1/(N+1)} N^{1/(N+1)}. \]

Proof. Abbreviate \( a_n(C_\varphi) \) and \( e_n(C_\varphi) \) to \( a_n \) and \( e_n \), and set \( \alpha = N/(N+1) \).

Let \( \varepsilon > 0 \). Theorem 5.1 implies:

\[ a_k \leq C \varepsilon e^{\varepsilon k^{1/N}} e^{-\rho k^{1/N}}, \]

so:

\[ (a_1 \cdots a_k)^{1/k} \leq C \varepsilon e^{\varepsilon k^{1/N}} e^{-\rho \alpha^{k^{1/N}}}. \]

Apply once more Theorem 5.1 to obtain:

\[ e_n \leq C \varepsilon \sup_{k \geq 1} e^{\varepsilon k^{1/N}} \exp \left[ -\left( n/k + \rho \alpha^{k^{1/N}} \right) \right]. \]

The supremum is essentially attained for \( k \) the integral part of \( (N/\rho \alpha)^\alpha n^\alpha \) and then, in view of (5.7) and \( \alpha/N = 1 - \alpha \), up to a negligible term:

\[ n/k + \rho \alpha^{k^{1/N}} = n^{1-\alpha} \left( \frac{\rho \alpha}{N} \right)^{\alpha} + \rho \alpha n^{1-\alpha} \left( \frac{N}{\rho \alpha} \right)^{1-\alpha} = n^{1-\alpha} (\rho \alpha)^\alpha (N^{-\alpha} + N^{1-\alpha}). \]

Finally,

\[ e_n \leq C \varepsilon e^{\varepsilon n^{1-\alpha}} \exp \left( -\beta_N \rho^{n^{1-\alpha}} \right) = C \varepsilon e^{\varepsilon n^{1/(N+1)}} \exp \left( -\beta_N \rho^{n^{1/(N+1)}} \right). \]

This clearly ends the proof of Theorem 5.2. \( \square \)
Remark. We have so far no sharp lower bound for entropy numbers, at least when \( \| \varphi \|_\infty = 1 \), since we already fail to have one in general for approximation numbers (see however [14]).

Besides, let \( J : H^\infty(\mathbb{D}^N) \to C(K) \) be the canonical embedding, when \( K \subseteq \mathbb{D}^N \) is a “condenser”, namely a compact subset of \( \mathbb{D}^N \) such that any bounded analytic function on \( \mathbb{D}^N \) which vanishes on \( K \) vanishes identically, which is moreover “regular”. The positive solution to the Kolmogorov conjecture can be expressed in terms of the Kolmogorov numbers \( d_n(J) \) of \( J \) or equivalently, in terms of the entropy numbers \( e_n(J) \) of \( J \) ([21, Theorem 5], generalizing Erokhin’s result in dimension 1 appearing in his posthumous paper [6] and methods due to Mityagin [15] and Levin and Tikhomirov [9]; see also [22, Lemma 2.2]). The result is that, taking \( K = \varphi(\mathbb{D}^N) \), one has, with sharp constants \( c_K, c'_K \) depending on the pluricapacity of \( K \) in \( \mathbb{D}^N \):

\[
(5.8) \quad d_n(J) \approx e^{-c_K n^{1/N}} \quad \text{and} \quad e_n(J) \approx e^{-c'_K n^{1/(N+1)}}.
\]

This jump from the exponent \( 1/N \) to the exponent \( 1/(N + 1) \) is reflected in our Theorem 5.2 through the new parameter \( \gamma^+_N \).

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