Conformal Invariance, Dynamical Dark Energy and the CMB

Emil Mottola
Theoretical Div., Los Alamos National Laboratory, Los Alamos, NM 87545 USA
E-mail: emil@lanl.gov

and

Theoretical Physics Group, PH-TH, CERN CH-1211, Geneva 23, Switzerland
E-mail: emil.mottola@cern.ch

General Relativity receives quantum corrections relevant at cosmological distance scales from conformal scalar degrees of freedom required by the trace anomaly of the quantum stress tensor in curved space. In the theory including the trace anomaly terms, the cosmological “constant” becomes dynamical and hence potentially dependent upon both space and time. The fluctuations of these anomaly scalars may also influence the spectrum and statistics of the Cosmic Microwave Background. Under the hypothesis that scale invariance should be promoted to full conformal invariance, an hypothesis supported by the exact equivalence of the conformal group of three dimensions with the de Sitter group $SO(4,1)$, the form of the CMB bispectrum can be fixed, and the trispectrum constrained. The non-Gaussianities predicted by conformal invariance differ from those suggested by simple models of inflation.

Preprint Numbers: LA-UR-10-04603, CERN-PH-TH 2011-039

1 Cosmological Dark Energy and the Effective Field Theory of Gravity

Observations of type Ia supernovae at moderately large redshifts ($z \sim 0.5$ to 1) have led to the conclusion that the Hubble expansion of the universe is accelerating. According to Einstein’s equations this acceleration is possible if and only if the energy density $\rho$ and pressure $p$ of the dominant component of the universe satisfies the inequality,

$$\rho + 3p \equiv \rho (1 + 3w) < 0.$$  (1)

A vacuum energy with $\rho_v > 0$ and $w \equiv p_v/\rho_v = -1$ leads to an accelerated expansion, a kind of “repulsive” gravity in which the relativistic effects of a negative pressure can overcome a positive energy density in (1). Taken at face value, the observations imply that some 74% of the energy in the universe is of this hitherto undetected $w = -1$ dark variety. This leads to a non-zero inferred cosmological term in Einstein’s equations of

$$\Lambda_{\text{meas}} \simeq (0.74) \frac{3H_0^2}{c^2} \simeq 1.4 \times 10^{-56} \text{ cm}^{-2} \simeq 3.6 \times 10^{-122} \frac{c^3}{\hbar G}.$$  (2)

Here $H_0$ is the present value of the Hubble parameter, approximately 73 km/sec/Mpc $\simeq 2.4 \times 10^{-18}$ sec$^{-1}$. The last number in (2) expresses the value of the cosmological term inferred from the SN Ia data in terms of Planck units, $L_{\text{pl}}^{-2} = \frac{c^2}{\hbar G}$. Explaining the value of this smallest number in all of physics is the basic form of the cosmological constant problem.

If the universe were purely classical, $L_{\text{pl}}$ would vanish and $\Lambda$, like the overall size or total age of the universe, could take on any value whatsoever without any technical problem of naturalness. On the other hand, if $G = 0$ and there are also no boundary effects to be concerned with, then the cutoff dependent zero point energy of flat space could simply be subtracted, with no observable consequences. A naturalness problem arises only when the effects of quantum zero point energy on the large scale curvature of spacetime are considered. Thus this is a problem of the gravitational energy of the quantum vacuum or ground state of the system at macroscopic distance scales, very much greater than $L_{\text{pl}}$, when both $\hbar \neq 0$ and $G \neq 0$.

The treatment of quantum effects at distances much larger than any ultraviolet cutoff is precisely the context in which effective field theory (EFT) techniques should be applicable. This
means that we assume that we do not need to know every detail of physics at extremely short distance scales of \(10^{-33}\) cm or even \(10^{-13}\) cm in order to discuss cosmology at \(10^{28}\) cm scales. In extending Einstein’s classical theory to take account of the quantum properties of matter, the classical stress-energy tensor of matter \(T^a_b\) becomes a quantum operator, with an expectation value \(<T^a_b>\). In this semi-classical theory with both \(\hbar\) and \(G\) different from zero, quantum zero-point and vacuum energy effects first appear, while the spacetime geometry can still be treated classically. Since the expectation value \(<T^a_b>\) suffers from the quartic divergence, a regularization and renormalization procedure is necessary in order to define the semi-classical EFT. The result is that General Relativity can be viewed as a low energy quantum EFT of gravity, provided that the classical Einstein-Hilbert classical action is augmented by the additional terms required by the renormalization program for quantum fields and their vacuum energy in curved space is is that General Relativity can be viewed as a low energy quantum EFT of gravity, provided that the classical Einstein-Hilbert classical action is augmented by the additional terms required by the trace anomaly when \(\hbar \neq 0\).

Massless quantum matter or radiation fields have stress-energy tensors which are traceless classically. However it is impossible to maintain both conservation and tracelessness of the quantum expectation value \(<T^a_b>\). Instead a well-defined conformal or trace anomaly for this expectation value in curved spacetime is obtained \([2]\) i.e.

\[
<T^a_b> = bF + b' \left( E - \frac{2}{3} \Box R \right) + b'' \Box R , \tag{3}
\]

where

\[
E \equiv R_{abcd} R^{abcd} = R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2 , \tag{4a}
\]

\[
F \equiv C_{abcd} C^{abcd} = R_{abcd} R^{abcd} - 2 R_{ab} R^{ab} + \frac{R^2}{3} . \tag{4b}
\]

in terms of the Riemann curvature tensor \(R_{abcd}\). The coefficients \(b\) and \(b'\) in (3) do not depend on any ultraviolet short distance cutoff, but instead are determined only by the number and spin of massless fields via

\[
b = \frac{\hbar}{120(4\pi)^2} (N_S + 6 N_F + 12 N_V) , \tag{5a}
\]

\[
b' = -\frac{\hbar}{360(4\pi)^2} (N_S + \frac{11}{2} N_F + 62 N_V) , \tag{5b}
\]

with \((N_S, N_F, N_V)\) the number of massless fields of spin \((0, \frac{1}{2}, 1)\) respectively. The number of massless fields of each spin is a property of the low energy effective description of matter, having no direct connection with physics at the ultrashort Planck scale. Indeed massless fields fluctuate at all distance scales and do not decouple in the far infrared, relevant for cosmology.

One can find a covariant action functional whose variation gives the trace anomaly \([3]\). This functional is non-local in terms of just the curvature, and hence describes long distance infrared physics. The non-local anomaly action may be put into a local form, but only by the introduction of two new scalar fields \(\varphi\) and \(\psi\). Then the local effective action of the anomaly in a general curved space may be expressed in the form \([4]\)

\[
S_{\text{anom}} = b' S_{\text{anom}}^{(E)} + b S_{\text{anom}}^{(F)} , \tag{6}
\]

where

\[
S_{\text{anom}}^{(E)} = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ - (\Box \varphi)^2 + 2 \left( R^{ab} - \frac{R}{3} g^{ab} \right) (\nabla_a \varphi)(\nabla_b \varphi) + \left( E - \frac{2}{3} \Box R \right) \varphi \right\} ;
\]

\[
S_{\text{anom}}^{(F)} = \int d^4x \sqrt{-g} \left\{ - (\Box \varphi)(\Box \psi) + 2 \left( R^{ab} - \frac{R}{3} g^{ab} \right) (\nabla_\mu \varphi)(\nabla_\mu \psi) + \frac{1}{2} C_{abcd} C^{abcd} \varphi + \frac{1}{2} \left( E - \frac{2}{3} \Box R \right) \psi \right\} . \tag{7}
\]
The free variation of the local action (6)-(7) with respect to $\psi$ and $\phi$ yields their eqs. of motion. Each of these terms when varied with respect to the background metric gives a stress-energy tensor in terms of these anomaly scalar fields $\varphi$ and $\psi$. The scalar fields of the local form (7) of the anomaly effective action describe massless scalar degrees of freedom of low energy gravity, not contained in classical General Relativity. The effective action of low energy gravity is thus

$$S_{\text{eff}}[g] = S_{\text{EH}}[g] + S_{\text{anom}}[g; \varphi, \psi]$$

with $S_{\text{EH}}$ the Einstein-Hilbert action of classical General Relativity and $S_{\text{anom}}$ the anomaly action given by (6)-(7).

2 Dynamical Dark Energy

In order to understand the dynamical effects of the kinetic terms in the anomaly effective action (7), one can consider simplest case of the quantization of the conformal factor in the case that the fiducial metric is flat, i.e. $g_{ab} = e^{2\sigma} \eta_{ab}$. In this case the Wess-Zumino form of the effective anomaly action (6) simplifies to

$$S_{\text{anom}}[\sigma] = -\frac{Q^2}{16\pi^2} \int d^4x \left( \Box \sigma \right)^2,$$

where

$$Q^2 \equiv -32\pi^2 b'.$$

This action quadratic in $\sigma = \varphi/2$ is the action of a free scalar field in flat space, with a kinetic term that is fourth order in derivatives.

The classical Einstein-Hilbert action for a conformally flat metric $g_{ab} = e^{2\sigma} \eta_{ab}$ is

$$S_{\text{EH}}[g = e^{2\sigma} \eta] = \frac{1}{8\pi G} \int d^4x \left[ 3e^{-2\sigma} (\partial_a \sigma)^2 - \Lambda e^{4\sigma} \right],$$

which has derivative and exponential self-interactions in $\sigma$. It is remarkable that these complicated interactions can be treated systematically using the the fourth order kinetic term of (9). These interaction terms are renormalizable and their anomalous scaling dimensions due to the fluctuations of $\sigma$ can be computed in closed form. Direct calculation of the conformal weight of the Einstein curvature term shows that it acquires an anomalous dimension $\beta_2$ given by the quadratic relation,

$$\beta_2 = 2 + \frac{\beta_2^2}{2Q^2}.$$  

In the limit $Q^2 \to \infty$ the fluctuations of $\sigma$ are suppressed and we recover the classical scale dimension of the coupling $G^{-1}$ with mass dimension 2. Likewise the cosmological term in (11) corresponding to the four-volume acquires an anomalous dimension given by

$$\beta_0 = 4 + \frac{\beta_0^2}{2Q^2}.$$

Again as $Q^2 \to \infty$ the effect of the fluctuations of the conformal factor are suppressed and we recover the classical scale dimension of $\Lambda/G$, namely 4. The solution of the quadratic relations (12) and (13) determine the scaling dimensions of these couplings at the conformal fixed point at other values of $Q^2$.

The positive corrections of order $1/Q^2$ (for $Q^2 > 0$) in (12) and (13) show that both $G^{-1}$ and $\Lambda/G$ flow to zero at very large distances. Because both of these couplings are separately dimensionful, at a conformal fixed point one should properly speak only of the dimensionless
combination $\hbar G A / c^3 = \lambda$. By normalizing to a fixed four volume $V = \int d^4 x$ one can show that the finite volume renormalization of $\lambda$ is controlled by the anomalous dimension,

$$2\delta - 1 \equiv 2 \frac{\beta_2}{\beta_0} - 1 = \frac{\sqrt{1 - \frac{8}{Q^2} - \sqrt{1 - \frac{4}{Q^2}}}}{1 + \sqrt{1 - \frac{4}{Q^2}}} \leq 0.$$  \hfill (14)

This is the anomalous dimension that enters the infrared renormalization group volume scaling relation

$$V \frac{d}{dV} \lambda = 4 (2\delta - 1) \lambda.$$  \hfill (15)

The anomalous scaling dimension (14) is negative for all $Q^2 \geq 8$. This implies that the dimensionless cosmological term $\lambda$ has an infrared fixed point at zero as $V \to \infty$. Thus the cosmological term is dynamically driven to zero as $V \to \infty$ by infrared fluctuations of the conformal part of the metric described by (9).

No fine tuning is involved here and no free parameters enter except $Q^2$, which is determined by the trace anomaly coefficient $b'$ by (10). Once $Q^2$ is assumed to be positive, then $2\delta - 1$ is negative, and $\lambda$ is driven to zero at large distances by the conformal fluctuations of the metric, with no additional assumptions. Thus the fluctuations of the conformal scalar degrees of freedom of the anomaly generated effective action $S_{\text{anom}}$ may be responsible for the observed small value of the cosmological dark energy density inferred from the supernova data. Note also that the fields $\varphi$ and $\psi$ are scalar degrees of freedom in cosmology which arise naturally from the effective action of the trace anomaly in the Standard Model, without the ad hoc introduction of an inflaton field. Recent progress in evaluating their effects of these anomaly scalars in de Sitter space indicate that they have potentially large effects at the cosmological horizon. Even in the absence of a complete theory of dynamical cosmological vacuum energy, it is reasonable to assume that the conformal fluctuations of $S_{\text{anom}}$ could be observable in the signatures of conformal invariance should be imprinted on the spectrum and statistics of the CMB.

### 3 Conformal Invariance and the CMB

Our earlier studies of fluctuations in de Sitter space suggest that the fluctuations responsible for the screening of $\lambda$ take place at the horizon scale. In that case then the microwave photons in the CMB reaching us from their surface of last scattering should retain some imprint of the effects of these fluctuations. It then becomes natural to extend the classical notion of scale invariant cosmological perturbations to full conformal invariance. In that case the classical spectral index of the perturbations should receive corrections due to the anomalous scaling dimensions at the conformal phase. In addition to the spectrum, the statistics of the CMB should reflect the non-Gaussian correlations characteristic of conformal invariance.

Consider first the two-point function of any observable $O_\Delta$ with dimension $\Delta$. Conformal invariance requires

$$\langle O_\Delta(x_1) O_\Delta(x_2) \rangle \sim |x_1 - x_2|^{-2\Delta}$$  \hfill (16)

at equal times in three dimensional flat spatial coordinates. In Fourier space this gives

$$G_2(k) \equiv \langle \hat{O}_\Delta(k) \hat{O}_\Delta(-k) \rangle \sim |k|^{2\Delta - 3}.$$  \hfill (17)

Thus, we define the spectral index of this observable by

$$n \equiv 2\Delta - 3.$$  \hfill (18)
In the case that the observable is the primordial density fluctuation $\delta \rho$, and in the classical limit where its anomalous dimension vanishes, $\Delta \to p = 2$, we recover the Harrison-Zel’dovich spectral index of $n = 1$.

In order to convert the power spectrum of primordial density fluctuations to the spectrum of fluctuations in the CMB at large angular separations we follow the standard treatment relating the temperature deviation to the Newtonian gravitational potential $\varphi$ at the last scattering surface, $\frac{\delta T}{T} \sim \delta \varphi$, which is related to the density perturbation in turn by

$$\nabla^2 \delta \varphi = 4\pi G \delta \rho \ .$$

Hence, in Fourier space,

$$\frac{\delta T}{T} \sim \delta \varphi \sim \frac{1}{k^2} \frac{\delta \rho}{\rho} \ ,$$

and the two-point function of CMB temperature fluctuations is determined by the conformal dimension $\Delta$ to be

$$C_2(\theta) \equiv \left\langle \frac{\delta T}{T}(\hat{r}_1) \frac{\delta T}{T}(\hat{r}_2) \right\rangle \sim \int d^3k \left( \frac{1}{k^2} \right)^2 G_2(k) e^{ik \cdot r_{12}} \sim \Gamma(2 - \Delta)(r_{12}^2)^{2-\Delta} \ ,$$

where $r_{12} \equiv (\hat{r}_1 - \hat{r}_2)r$ is the vector difference between the two positions from which the CMB photons originate. They are at equal distance $r$ from the observer by the assumption that the photons were emitted at the last scattering surface at equal cosmic time. Since $r_{12}^2 = 2(1 - \cos \theta)r^2$, we find then

$$C_2(\theta) \sim \Gamma(2 - \Delta)(1 - \cos \theta)^{2-\Delta}$$

for arbitrary scaling dimension $\Delta$.

Expanding the function $C_2(\theta)$ in multipole moments,

$$C_2(\theta) = \frac{1}{4\pi} \sum_{\ell} (2\ell + 1)c^{(2)}_\ell(\Delta)P_\ell(\cos \theta) \ ,$$

$$c^{(2)}_\ell(\Delta) \sim \Gamma(2 - \Delta) \sin \left[ \pi(2 - \Delta) \right] \frac{\Gamma(\ell - 2 + \Delta)}{\Gamma(\ell + 4 - \Delta)} \ ,$$

shows that the pole singularity at $\Delta = 2$ appears only in the $\ell = 0$ monopole moment. This singularity is just the reflection of the fact that the Laplacian in (19) cannot be inverted on constant functions, which should be excluded. Since the CMB anisotropy is defined by removing the isotropic monopole moment (as well as the dipole moment), the $\ell = 0$ term does not appear in the sum, and the higher moments of the anisotropic two-point correlation function are well-defined for $\Delta$ near 2. Normalizing to the quadrupole moment $c^{(2)}_2(\Delta)$, we find

$$c^{(2)}_\ell(\Delta) = c^{(2)}_2(\Delta) \frac{\Gamma(6 - \Delta)}{\Gamma(\Delta)} \frac{\Gamma(\ell - 2 + \Delta)}{\Gamma(\ell + 4 - \Delta)} \ ,$$

which is a standard result. Indeed, if $\Delta$ is replaced by $p = 2$ we obtain $\ell(\ell + 1)c^{(2)}_\ell(p) = 6c^{(2)}_2(p)$, which is the well-known predicted behavior of the lower moments ($\ell \leq 30$) of the CMB anisotropy where the Sachs-Wolfe effect should dominate.

Turning now from the two-point function of CMB fluctuations to higher point correlators, we find a second characteristic prediction of conformal invariance, namely non-Gaussian statistics.
for the CMB. The first correlator sensitive to this departure from gaussian statistics is the three-point function of the observable $\mathcal{O}_\Delta$, which takes the form

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3) \rangle \sim |x_1 - x_2|^{-\Delta}|x_2 - x_3|^{-\Delta}|x_3 - x_1|^{-\Delta},$$

or in Fourier space

$$G_3(k_1, k_2) \sim \int d^3 p \ |p|^{-3} |p+k_1|^{-3} |p-k_2|^{-3} \sim \frac{\Gamma\left(3 - \frac{3\Delta}{2}\right)}{\left[\Gamma\left(\frac{3-\Delta}{2}\right)\right]^3} \times \int_0^1 du \int_0^1 dv \ \frac{[u(1-u)v]^{\frac{1-\Delta}{2}}(1-v)^{-1+\frac{\Delta}{2}}}{[u(1-u)(1-v)k_1^2 + v(1-u)k_2^2 + uv(k_1 + k_2)^2]^{3 - \frac{3\Delta}{2}}}. \quad (27)$$

This three-point function of primordial density fluctuations gives rise to three-point correlations in the CMB by reasoning precisely analogous as that leading from Eqns. (17) to (21). That is,

$$C_3(\theta_{12}, \theta_{23}, \theta_{31}) \equiv \left\langle \frac{\delta T}{T}(\hat{r}_1)\frac{\delta T}{T}(\hat{r}_2)\frac{\delta T}{T}(\hat{r}_3) \right\rangle \sim \int \frac{d^3 k_1 d^3 k_2}{k_1^2 k_2^2 (k_1 + k_2)^2} G_3(k_1, k_2) e^{ik_1 \cdot r_{13}} e^{ik_2 \cdot r_{23}} \quad (28)$$

where $r_{ij} \equiv (\hat{r}_i - \hat{r}_j) r$ and $r_{ij}^2 = 2(1 - \cos \theta_{ij}) r^2$.

In the general case of three different angles, this expression for the non-Gaussian three-point correlation function (28) is quite complicated, although it can be rewritten in parametric form analogous to (27) to facilitate numerical evaluation. In the special case of equal angles, it follows from its global scaling properties that the three-point correlator is

$$C_3(\theta) \sim (1 - \cos \theta)^{\frac{3}{2}(2-\Delta)}. \quad (29)$$

Expanding the function $C_3(\theta)$ in multiple moments as in (23) with coefficients $c^{(3)}_\ell$, and normalizing to the quadrupole moment, we find

$$c^{(3)}_\ell(\Delta) = c^{(3)}_2(\Delta) \frac{\Gamma(4 + \frac{3}{2}(2 - \Delta))}{\Gamma(2 - \frac{3}{2}(2 - \Delta))} \frac{\Gamma(\ell - \frac{3}{2}(2 - \Delta))}{\Gamma(\ell + 2 + \frac{3}{2}(2 - \Delta))}. \quad (30)$$

In the limit $\Delta \to 2$, we obtain $\ell(\ell + 1)c^{(3)}_\ell = 6c^{(3)}_2$, which is the same result as for the moments $c^{(2)}_\ell$ of the two-point correlator but with a different quadrupole amplitude. The value of this quadrupole normalization $c^{(3)}_2(\Delta)$ cannot be determined by conformal symmetry considerations alone, and requires more detailed dynamical information about the origin of conformal invariance in the spectrum.

For higher point correlations, conformal invariance does not determine the total angular dependence. Already the four-point function takes the form,

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\mathcal{O}_\Delta(x_4) \rangle \sim \frac{A_4}{\prod_{i<j} r_{ij}^{2\Delta/3}}, \quad (31)$$

where the amplitude $A_4$ is an arbitrary function of the two cross-ratios, $r_{13}^2 r_{24}^2 / r_{12}^2 r_{34}^2$ and $r_{14}^2 r_{23}^2 / r_{12}^2 r_{34}^2$. Analogous expressions hold for higher $p$-point functions.

An important point to emphasize is that all of these results depend upon the hypothesis of conformal invariance on the spatially homogeneous and isotropic flat spatial sections of geometries. This is only one way in which conformal invariance may be realized, for example, if the universe went through a de Sitter like inflationary expansion. That this is actually related to the geometric symmetries of de Sitter space is shown next.

*Note that (27) corrects two minor typographical errors in eq. (16) of Ref.
4 Conformal Invariance as a Consequence of de Sitter Invariance

In cosmology the line element of de Sitter space is usually expressed in the form

$$ds^2 = -d\tau^2 + a^2(\tau) \, d\vec{x} \cdot d\vec{x} = -d\tau^2 + e^{2H\tau} \,(dx^2 + dy^2 + dz^2)$$  \hspace{1cm} (32)

with flat spatial sections, and the Hubble parameter $H = \sqrt{\Lambda/3}$. This de Sitter geometry has an $SO(4,1)$ symmetry group with 10 Killing generators satisfying

$$\nabla_a \xi^{(\alpha)}_b + \nabla_b \xi^{(\alpha)}_a = 0\,, \quad \alpha = 1, \ldots, 10\,,$$  \hspace{1cm} (33)

which leave the de Sitter metric invariant. In coordinates (32), (33) becomes

$$\partial_\tau \xi^{(\alpha)}_i = 0\,, \quad (34a)$$
$$\partial_\tau \xi^{(\alpha)}_i + \partial_i \xi^{(\alpha)}_\tau - 2H \xi^{(\alpha)}_i = 0\,, \quad (34b)$$
$$\partial_i \xi^{(\alpha)}_j + \partial_j \xi^{(\alpha)}_i - 2Ha^2 \delta_{ij} \xi^{(\alpha)}_\tau = 0\,. \quad (34c)$$

For $\xi_\tau = 0$ we have the three translations, $\alpha = T_j$,

$$\xi^{(T_j)} = 0\,, \quad \xi^{(T_j)}_i = a^2 \delta_i^j\,, \quad j = 1, 2, 3\,.$$  \hspace{1cm} (35)

and the three rotations, $\alpha = R_\ell$,

$$\xi^{(R_\ell)} = 0\,, \quad \xi^{(R_\ell)}_i = a^2 \epsilon_{\ell m} x^m\,, \quad \ell = 1, 2, 3\,.$$  \hspace{1cm} (36)

This accounts for 6 of the 10 de Sitter isometries which are self-evident in the spatially flat homogeneous and isotropic Robertson-Walker coordinates (32) with $\xi_\tau = 0$. The 4 additional solutions of (34) have $\xi_\tau \neq 0$. They are the three special conformal transformations of $\mathbb{R}^3$, $\alpha = C_n$,

$$\xi^{(C_n)} = -2H x^n\,, \quad \xi^{(C_n)}_i = H^2 a^2 (\delta_i^k \delta_{jk} x^j x^k - 2\delta_{ij} x^j x^n) - \delta_i^n\,, \quad n = 1, 2, 3\,,$$  \hspace{1cm} (37)

and the dilation, $\alpha = D$,

$$\xi^{(D)} = 1\,, \quad \xi^{(D)}_i = Ha^2 \delta_{ij} x^j\.$$  \hspace{1cm} (38)

This last dilational Killing vector is the infinitesimal form of the finite dilational symmetry,

$$\vec{x} \rightarrow \lambda \vec{x}\,,$$  \hspace{1cm} (39a)
$$a(\tau) \rightarrow \lambda^{-1} a(\tau)$$  \hspace{1cm} (39b)

of de Sitter space. The existence of this symmetry explains why Fourier modes of different $|\vec{k}|$ leave the de Sitter horizon at a shifted RW time $\tau$, so in an eternal de Sitter space, in which there is no preferred $\tau$, one expects a scale invariant spectrum.

The existence of the three conformal modes of $\mathbb{R}^3$ (37) implies in addition that any $SO(4,1)$ de Sitter invariant correlation function must decompose into representations of the conformal group of three dimensional flat space. Fundamentally this is because the de Sitter group $SO(4,1)$ is the conformal group of flat Euclidean $\mathbb{R}^3$, as eqs. (33)-(38) shows explicitly. Moreover, because of the exponential expansion in de Sitter space, the decomposition into representations of the conformal group become simple at distances large compared to the horizon scale $1/H$. Thus if the universe went through an exponentially expanding de Sitter phase for many e-foldings when the fluctuations responsible for the CMB were generated, then the CMB should exhibit full conformal invariance in addition to simple scale invariance. Neither the form nor magnitude
of the CMB power or bispectrum depend upon an inflaton or “slow-roll” parameters as in conventional scalar models of inflation.

Another quite distinct possibility for realizing conformal invariance from de Sitter space is if the fluctuations due to the anomaly scalars are generated in the vicinity of the cosmological horizon at \( r = 1/H \) in the static coordinates of de Sitter space, \( i.e. \)

\[
ds^2 = -(1 - H^2 r^2) dt^2 + \frac{dr^2}{(1 - H^2 r^2)} + r^2 d\Omega^2.
\]

Conformal invariance on the sphere \( r = 1/H \) leads to a different characteristic form of the non-Gaussian bispectrum and higher angular correlations. This form and additional relevant details will be presented in a forthcoming article.\( ^9 \)

References

1. A. G. Riess et. al., *Astron. J.* 116, 1009 (1998); *Astron. J.* 607 665 (2004);
   S. Perlmutter et. al., *Astrophys. J.* 517 565 (1999);
   J. L. Tonry et. al., *Astrophys. J.* 594, 1 (2003).
2. N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982), and references therein.
3. E. Mottola and R. Vaulin, Phys. Rev. D 74, 064004 (2006).
4. I. Antoniadis, P. O. Mazur and E. Mottola, New J. Phys. 9, 11 (2007).
5. I. Antoniadis and E. Mottola, Phys. Rev. D 45, 2013 (1992).
6. P. R. Anderson, C. Molina-Páris, and E. Mottola, Phys. Rev. D 80, 084005 (2009).
7. I. Antoniadis, P. O. Mazur and E. Mottola, Phys. Rev. Lett. 79, 14 (1997).
8. I. Antoniadis, P. O. Mazur and E. Mottola, e-print arxiv:astro-ph/9705200.
9. I. Antoniadis, P. O. Mazur and E. Mottola, “Conformal Invariance, Dark Energy, and CMB Non-Gaussianity,” LA-UR-11-10115, to appear Mar. 2011.