One-party Quantum Error Correcting Codes for Unbalanced Errors: Principles and Application to Quantum Dense Coding and Quantum Secure Direct Communications

Kai Wen$^1$ and Gui Lu Long$^{1,2,3}$

$^1$ Key Laboratory For Quantum Information and Measurements and Department of Physics, Tsinghua University, Beijing 100084, China
$^2$ Key Laboratory for Atomic and Molecular NanoSciences, Tsinghua University, Beijing 100084, China
$^3$ Center of Quantum Information Science and Technology, Tsinghua National Laboratory For Information Science and Technology, Beijing 100084, China

(Dated: 28 January 2007)

In this article, we present the unbalanced quantum error correcting codes (one-party-QECC), a novel idea for correcting unbalanced quantum errors. In some quantum communication tasks using entangled pairs, the error distributions between two parts of the pairs are unbalanced, and one party holds the whole entangled pairs at the final stage, and he or she is able to perform joint measurements on the pairs. In this situation the proposed one-party-QECC can improve error correction by allowing a higher tolerated error rate. We have established the general correspondence between linear classical codes and the one-party-QECC, and we have given the general definition for this type quantum error correcting codes. It has been shown that the one-party-QECC can correct errors as long as the error threshold is not larger than 0.5. The one-party-QECC works even for fidelity less than 0.5 as long as it is larger than 0.25. We give several concrete examples of the one-party-QECC. We provide the applications of one-party-QECC in quantum dense coding so that it can function in noisy channels. As a result, a large number of quantum secure direct communication protocols based on dense coding is also able to be protected by this new type of one-party-QECC.

PACS numbers: 03.67.Hk, 03.65.Ud, 03.67.Dd, 03.67.-a

I. INTRODUCTION

Quantum communication emerges as an important technology in communication. Its key advantage lies in the unconditional security benefited from the principles of quantum mechanics. As is well-known however, quantum system is more likely to be affected by environment than classical system, and leads to phenomena such as decoherence. Thus, how to efficiently build reliable quantum communication through noisy quantum channels is one of the primary tasks for scientists. Quantum error correcting codes (QECC) are such a technique aiming towards this goal[1, 2, 3, 4, 5]. QECC make use of redundant qubits to encode quantum information. To correct errors using QECC, we measure the qubit system to obtain error syndromes and then choose appropriate recovering operations. Entanglement purification protocol (EPP) is another quantum error correction method to produce high fidelity entangled pairs from low fidelity entangled pairs by sacrificing a number of them[6, 7]. Both QECC and EPP play an important role in quantum communication through noisy quantum channels. Especially, under assistance of QECC and EPP, we are able to perform unconditional secure quantum key distribution[8, 9, 10, 11, 12, 13, 14, 15].

Investigating quantum communication, we discover that the distribution of quantum errors can be classified into two kinds: balanced and unbalanced errors. Most quantum error correction methods are dealing with balanced errors. In quantum communication especially using entangled pairs, there are two kinds of qubits: flying qubits, which are transmitted from one party to the other, and home qubits, which remain in the same party. Most quantum error correction methods do not distinguish the errors between these two kinds of qubits and their capabilities of correcting errors on both of them are identical. In this case, we define these errors as balanced errors between flying and home qubits, and these quantum error correction methods as balanced quantum error correction methods.

For example, in quantum dense coding[16], Alice and Bob first share a sequence of two-qubit Einstein-Podolsky-Rosen (EPR) pairs, each in one of the four Bell-basis states, namely,

\begin{align}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).
\end{align}

(1)
The two qubits in every EPR pair are distributed between Alice and Bob. Then Alice sends her qubits to Bob through the noisy channel and these qubits are the flying qubits. The qubits initially in Bob’s hand are home qubits because they remain in the same place all through the protocol. After Bob received all flying qubits, he can perform EPP to correct the errors introduced in the process. In particular, there are three kinds of errors on qubits, namely, bit-flip errors, phase-flip errors and both bit-flip and phase-flip errors. The three kinds of errors are represented by acting three different Pauli operators

\[
X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\]

(2)

on the qubits respectively. The central purpose of EPP is to produce high fidelity pairs from such low fidelity pairs by sacrificing a number of them. Because EPP does not distinguish the errors on flying qubits and home qubits, he is able to correct errors on both of them with the same capabilities. Therefore, EPP in quantum dense coding is a kind of balanced quantum error correction method.

In this paper, we propose a novel kind of quantum error correcting codes, called one-party quantum error correcting codes, and one-party-QECC for short. The unbalanced errors are defined as those whose distribution between the flying and home qubits are unbalanced. Practically, home qubits can be stored in some quantum memory with low errors. They can be easily preserved in very high fidelity using conventional quantum error correcting codes with low errors. On the contrary, the flying qubits are probably affected by great channel noise. The high error rates on these qubits may sometimes exceed the capability of conventional quantum error correction methods. The significant difference of error distribution on the two kinds of qubits motivate us to propose a kind of QECC, named one-party-QECC. The one-party-QECC that makes use of this feature is proved in this paper to achieve much higher error correction capability.

The essential of the high capability of one-party-QECC lies in the quantum correlation in the entanglements between flying and home qubits. It employs joint operations in the final error detection and recovering process such as Bell measurements between them. This feature leads to the requirement that in the final stage of the protocol, one of the communication parties should hold both flying and home qubits. Thus he or she can then easily apply Bell measurements to measure error syndromes without classically informing the other party. Timely, many kinds of quantum communication satisfy this requirement for our one-party-QECC, especially protocols based on quantum dense coding.

Quantum dense coding is one of the most important protocols that one-party-QECC is able to apply. It is because in dense coding Alice’s flying qubits and Bob’s home qubits are initially entangled and in the final stage, Bob obtains both kind of qubits. It is reasonable that Bob’s home qubits is of much higher fidelity than flying qubits all through the protocol and thus the protocol contains unbalanced errors. We present in this paper the application of one-party-QECC in quantum dense coding and show its advantages in protecting message against very high initial and channel error rates.

Dense coding and its generalization [16, 17] have extensive application in quantum information processing. Based on quantum dense coding, one-party-QECC can be applied to many more other protocols, for instance quantum secure direct communication(QSDC) [18, 19, 20, 21, 22, 23]. A QSDC communicates secret messages directly through the quantum channel without first establishing a secret key to encrypt them. Thus QSDC has the advantages of directly read-out of the secret message by the legitimate user, and providing the eavesdropper only blind results under any circumstance [21]. A large number of QSDC protocols use entangled states and also make use of dense coding. So we also employ one-party-QECC in this kind of QSDC protocols to enhance their performance. In some secret sharing schemes, dense coding technique is also exploited, for example, in Ref. [24]. In such case, the proposed one-party-QECC may also work.

In this paper, we first give two example one-party-QECC: the [[6, 2, 1]] one-party-QECC in section II and the concatenated [[6,2,1]] one-party-QECC in section III. Then we give a general theory of one-party-QECC in section IV and elaborate on its properties. In addition, we apply one-party-QECC to the quantum dense coding protocol and QSDC and demonstrate its advantages in quantum communication in section V and VI.

II. A SIMPLE [[6, 2, 1]] QECC WITH JOINT MEASUREMENTS

In this section, we give a concrete and simple example of the one-party-QECC. The example is [[6,2,1]] one-party-QECC, which encodes 2 bit information into 3 EPR pairs and protects the state against at most one bit-flip and one phase-flip errors. Here we use double brackets to emphasize that the code is quantum. We describe the definition and operations of the codes. Then based on the example, we discuss the restrictions and application conditions of this code in quantum communication.
The motivation of designing the new one-party-QECC comes from quantum communication using entangled pairs, namely, EPR pairs. Several protocols, including quantum dense coding protocol, quantum key distribution and quantum secure direct communications \[13,19,20,21,22,23\] belong to such type of quantum communication. Especially, due to the relationship between QECC and EPP \[8\], previous work has proved that the fact that two-way classical communications in which Alice and Bob compare the error syndromes and decide the next operations provides higher capability of entanglement purification than one-way classical communications \[12\]. If we can further throw the classical communications in the post-processing step, we may obtain even higher capability. Indeed, this has been achieved in this work.

Now we give the framework of the quantum communication that is suitable for our new one-party-QECC. In a typical scheme of quantum communication using entangled pairs, the errors distributed on both halves of the pairs are unbalanced: one half of the pairs are usually kept in one party, called home qubits, and assumed to have no error, while the other half are traveling, called flying qubits, through the noisy channels and may err. Suppose the kind of quantum communication in which Bob obtains both qubit of an EPR pair in the final stage, contrast to distributing the two qubits in a pair between Alice and Bob in EPP. This kind of communication includes the protocols such as quantum dense coding. In this kind of communication, when Bob obtains both qubit, he can make use of joint operations, especially Bell measurements. Bell measurements can successfully distinguish the four different Bell states. Bell measurements on a single EPR pair can be described as measuring its \(X_1X_2\) and \(Z_1Z_2\) and each measurement has two different outcomes. The combined 4 measurement results tell us about the state of the EPR pairs as shown in Table \[4\]. Using Bell measurements, one can compare the difference between the travelling qubit which may be in error and the local qubit which has no error, from the quantum correlation in the pair instead of classical correlation using local operations and classical communications. This will provide higher capability in error correction. Moreover, as the process of the communication is different from EPP, we define this new kind of error correction process using EPR pairs and Bell measurements as quantum error correcting codes with joint measurements, or one-party-QECC for short.

Following the above idea, we present a simple example of the one-party-QECC. We use EPR pairs, instead of qubits, as the logical qubits. In the simplest classical error correcting code – the \([3,1]\) error correcting code which encodes 1 bit information into 3 physical bits and protects 1 single bit-flip error in any of the 3 physical bits, the logical 0 and 1 is encoded into 000 and 111 respectively. Therefore, we define our logical states as the following:

\[
\begin{align*}
|00\rangle &= |\Phi^+\rangle_{12}|\Phi^+\rangle_{34}|\Phi^+\rangle_{56}, \\
|01\rangle &= |\Phi^-\rangle_{12}|\Phi^-\rangle_{34}|\Phi^-\rangle_{56}, \\
|10\rangle &= |\Psi^+\rangle_{12}|\Psi^+\rangle_{34}|\Psi^+\rangle_{56}, \\
|11\rangle &= |\Psi^-\rangle_{12}|\Psi^-\rangle_{34}|\Psi^-\rangle_{56}.
\end{align*}
\]

(3)

Generally, any logical state can be represented as

\[
\begin{align*}
& a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \\
= & a|\Phi^+\rangle_{12}|\Phi^+\rangle_{34}|\Phi^+\rangle_{56} + \\
& b|\Phi^-\rangle_{12}|\Phi^-\rangle_{34}|\Phi^-\rangle_{56} + \\
& c|\Psi^+\rangle_{12}|\Psi^+\rangle_{34}|\Psi^+\rangle_{56} + \\
& d|\Psi^-\rangle_{12}|\Psi^-\rangle_{34}|\Psi^-\rangle_{56}.
\end{align*}
\]

(4)

In the definition of the logical states, we encode the four logical Bell states, defined as the four logical states, in the left side of Eq. (3) with 3 physical Bell states given in the right side of Eq. (3) which contain 6 physical qubits with the subscripts from 1 to 6. Thus, we equivalently encode 2 logical qubit into 6 physical qubits and name such type of one-party-QECC as the \([6,2]\) one-party-QECC.

As in any scheme of quantum error corrections, we should measure some operators to get error syndromes in order to detect the error patterns. For the definition of Eq. (3) is induced by the classical \([3,1]\) error correcting code, we can find the error syndromes in a similar way. In the classical \([3,1]\) error correcting code, the error syndromes are measured by \(Z_1Z_2\) and \(Z_2Z_3\). Thus, we measure

\[
\begin{align*}
g_1 &= Z_1Z_2Z_3Z_4, \\
g_2 &= X_1X_2X_3X_4, \\
g_3 &= Z_3Z_4Z_5Z_6, \\
g_4 &= X_3X_4X_5X_6.
\end{align*}
\]

(5)
where the measurements act on the encoded state. We call the results of these 4 measurements as error syndromes which represent the error pattern of the EPR pairs. According to the stabilizer coding theory [25], we have constructed an error detection circuit of the [[6, 2]] one-party-QECC as Fig. (1). In Fig. (1), we use four ancillary qubits which are initially at \( |0\rangle \) state to measure the error syndromes \( Z_1Z_2Z_3Z_4, X_1X_2X_3X_4, Z_3Z_4Z_5Z_6 \) and \( X_3X_4X_5X_6 \) of the input 6 qubits. Based on the measurements results of the four test qubits, we can select proper operations to transform the corrupted states to the correct ones.

![FIG. 1: Error detection quantum circuit of [[6, 2, 1]] one-party-QECC. The first four ancillary qubits of initial \(|0\rangle\) states are used to measure error syndromes \(X_1X_2X_3X_4\), \(X_3X_4X_5X_6\), \(Z_1Z_2Z_3Z_4\) and \(Z_3Z_4Z_5Z_6\) respectively. The next six input qubits represent the input 3 EPR pairs.](image)

We now analyze the error model of the [[6, 2]] one-party-QECC with some restriction and present its capability in quantum error correction. As described in the framework of quantum communication with unbalanced errors that is suitable for one-party-QECC, only the flying qubits of the EPR pairs are transmitted through the noisy channel, while the home qubits are kept in Bob’s side, and stored in some quantum storages with significantly low error rates compared to the quantum channels. Thus in the [[6, 2]] one-party-QECC, we assume that only one half qubit of the logical states are subject to error. Particularly, in each logical state containing 3 EPR pairs, the 1st, 3rd and 5th physical qubits are in Bob’s hands, while the 2nd, 4th and 6th physical qubits are transmitted through the noisy channel. Therefore, in our assumption, after the transmission, only the 2nd, 4th and 6th physical qubits in each logical EPR pair may have error, i.e., single bit-flip or phase-flip or both bit-flip and phase-flip error. As a result, all error syndromes together with the error patterns and correcting operations are shown in Table II. From the table we can find that these measurement are capable of detecting all errors in our example model of one-party-QECC which is at most a single bit-flip and/or a single phase-flip on the second half of EPR pairs. In conclusion, this kind of one-party-QECC can successfully correct at most one bit-flip and one phase-flip errors occurring at the second half of the 3 physical EPR pairs of each logical state and we name it as the [[6, 2, 1]] one-party-QECC.

Furthermore, we define the logical operations of the [[6, 2, 1]] one-party-QECC which enable Bob to encode the information into the logical states. The logical operations are defined as

\[
\begin{align*}
\bar{X}_1 &= X_2X_4X_6 \\
\bar{Z}_1 &= Z_1Z_2Z_3Z_4Z_5Z_6 \\
\bar{X}_2 &= Z_1Z_3Z_5 \\
\bar{Z}_2 &= X_1X_2X_3X_4X_5X_6
\end{align*}
\]

(6)

The subscripts \( \bar{1} \) and \( \bar{2} \) in the left of the equations indicates which logical qubits in Eq. (3) the logical operators act on. These logical operations are able to transform one encoded state into another encoded state, as shown in Table III. In all, we have successfully constructed an [[6, 2, 1]] one-party-QECC, which encodes 2 bit information and are capable of correcting at most a single bit-flip and a single phase-flip errors on the second halves of the physical EPR pairs.
In this section, we extend the [[6,2,1]] one-party-QECC into longer one-party-QECC by concatenating it several rounds. The concatenated [[6,2,1]] one-party-QECC can thus be used to encode longer sequences. Then we analyze the error threshold of the concatenated [[6,2,1]] one-party-QECC.

The concatenating steps are described as the following: in the first round, Bob, the final receiver, divides the final raw physical EPR sequence into groups, each with 3 consecutive EPR pairs. In each group, Bob applies the [[6,2,1]] one-party-QECC on the 3 EPR pairs. If the error correction of a group is successful, Bob obtains a correct logical EPR pair. This logical EPR sequence will be the building blocks for constructing logical EPR pairs for the next round. Therefore, after applying the [[6,2,1]] one-party-QECC on all groups, Bob gets a logical EPR sequence with 1/3 original length consisting the logical states of the [[6,2,1]] one-party-QECC on all groups. If the raw error rate is low enough, after the first round, the error rate of the resulting logical EPR sequence will decrease.

In the second round, Bob divides the logical EPR sequence from the first round into groups each with 3 consecutive logical EPR pairs. In particular, the logical states of each group in the second round are defined as

\[
\begin{align*}
|00\rangle^{(2)} &= |00\rangle |00\rangle |00\rangle, \\
|01\rangle^{(2)} &= |01\rangle |01\rangle |01\rangle, \\
|10\rangle^{(2)} &= |10\rangle |10\rangle |10\rangle, \\
|11\rangle^{(2)} &= |11\rangle |11\rangle |11\rangle.
\end{align*}
\]

(7)

Then he applies the [[6,2,1]] one-party-QECC on each group and obtains the correct results as another logical EPR sequence in the same way as in the first round. Bob repeats this procedure for several rounds. For the error rate is reduced in each round, Bob will finally reach a very low error rate. When he obtains the required accuracy, he finishes the error correcting procedure. A summary of the process of concatenating [[6,2,1]] one-party-QECC is illustrated in Fig. 2.

Now we analyze the asymptotic error threshold of the concatenated [[6,2,1]] one-party-QECC with nearly infinitely long initial raw physical EPR pairs. First we assume that a quantum channel with the bit-flip and phase-flip error probabilities which are both lower than $q_0$ in the beginning. After the $k$-th round of the one-party-QECC, the bit-flip and phase-flip error rates becomes lower than $q_k$. Note that, the error rate $q$ is defined as $q = \frac{k}{n}$, where $t$ is the number of the bit-flip errors or the phase-flip errors and $n$ is the number of the physical qubits that are transmitted through the quantum channels. In this type of quantum communication and the error model, only the second half of each physical EPR pair is transmitted through the quantum channel and subject to error; so the error rate $q$ also corresponds to the error rate of the physical EPR pairs. Thus the condition of the success of the concatenated [[6,2,1]] one-party-QECC is that the error rate of $q_k$ should be less than $q_{k-1}$ for each round. Considering the $k$-th round, if there is not more than one error out of the 3 EPR pairs in each group, the [[6,2,1]] one-party-QECC can be successful in that group. Therefore, the total error rate of the outcome logical EPR pair from 3 EPR pairs after the round is

\[
q_k = 1 - [1 - (1 - q_{k-1})^3 + 3(1 - q_{k-1})^2 q_{k-1}]
\]

(8)

Then he applies the [[6,2,1]] one-party-QECC on each group and obtains the correct results as another logical EPR sequence in the same way as in the first round. Bob repeats this procedure for several rounds. For the error rate is reduced in each round, Bob will finally reach a very low error rate. When he obtains the required accuracy, he finishes the error correcting procedure. A summary of the process of concatenating [[6,2,1]] one-party-QECC is illustrated in Fig. 2.

Now we analyze the asymptotic error threshold of the concatenated [[6,2,1]] one-party-QECC with nearly infinitely long initial raw physical EPR pairs. First we assume that a quantum channel with the bit-flip and phase-flip error probabilities which are both lower than $q_0$ in the beginning. After the $k$-th round of the one-party-QECC, the bit-flip and phase-flip error rates becomes lower than $q_k$. Note that, the error rate $q$ is defined as $q = \frac{k}{n}$, where $t$ is the number of the bit-flip errors or the phase-flip errors and $n$ is the number of the physical qubits that are transmitted through the quantum channels. In this type of quantum communication and the error model, only the second half of each physical EPR pair is transmitted through the quantum channel and subject to error; so the error rate $q$ also corresponds to the error rate of the physical EPR pairs. Thus the condition of the success of the concatenated [[6,2,1]] one-party-QECC is that the error rate of $q_k$ should be less than $q_{k-1}$ for each round. Considering the $k$-th round, if there is not more than one error out of the 3 EPR pairs in each group, the [[6,2,1]] one-party-QECC can be successful in that group. Therefore, the total error rate of the outcome logical EPR pair from 3 EPR pairs after the round is

\[
q_k = 1 - [1 - (1 - q_{k-1})^3 + 3(1 - q_{k-1})^2 q_{k-1}]
\]

(8)

The condition of the success of the concatenated [[6,2,1]] one-party-QECC requires that $q_k < q_{k-1}$. Substituting Eq. (8) into the inequality, we get

\[
q_{k-1} < 0.5.
\]

(9)

The result means that if $q_{k-1} < 0.5$, the $k$-th round will actually reduce the error rate of the EPR sequence. Therefore, if the initial error rate, namely, $q_0$ is lower than 50%, all rounds in the one-party-QECC will successfully reduce the error rates.
To show explicitly that the error rate in concatenating $[[6,2,1]]$ one-party-QECC indeed converges to 0, we can find from Eq. (3) that for any $0 < q_{k-1} < 1$,

$$q_k = (3 - 2q_{k-1})q_{k-1}^2$$
$$= 3q_{k-1}^2 - 2q_{k-1}^3$$
$$< 3q_{k-1}^2.$$  (10)

Thus, after $k$ rounds,

$$q_k < (\sqrt{3}q_0)^{2^k}.$$  (11)

Because we begin with $q_0 < 0.5$, we get $\sqrt{3}q_0 < 1$ and the error rate decreases exponentially with $k$. As a result, if Bob selects a proper parameter $k$ according to $q_0$, he will obtain a total error rate of nearly 0, and the error correction is successful. In conclusion, our new one-party-QECC can tolerate the bit error rate up to 50%. Compared to the upper bound of correctable bit error rate of about 11% in Calderbank-Shor-Steane codes [2, 3], one-party-QECC has significant advantages in quantum error correction and are capable of correcting more errors. The following sections will discuss the relation of the bounds between different error correcting methods and one-party-QECC.

IV. GENERAL FORMALISM OF ONE-PARTY-QECC

In this section, with the idea of the $[[6,2,1]]$ one-party-QECC and the concatenated $[[6,2,1]]$ one-party-QECC in section III and IV, we derive a general formalism of our novel one-party-QECC. To achieve this, we establish a correspondence between the classical linear codes and the one-party-QECC. We then present the logical states, the measurements of the error syndromes and the logical operations of the one-party-QECC from the correspondence to the classical linear codes. Finally, a bound of the one-party-QECC is derived from the classical Gilbert-Vashamov bound [25] and a numerical result of the error-tolerating capacity of the one-party-QECC is presented.

In section III we make use of the classical $[3,1,1]$ code to construct the $[[6,2,1]]$ one-party-QECC. This leads us to investigate more general classical linear codes and find its correspondence to the generalized one-party-QECC. Suppose there are an $[n, k, t]$ classical linear code which encodes $k$ bits into $n$ bits and protects the state against at most $t$ bit-flip errors, we can define an $[[2n, 2k, t]]$ one-party-QECC which can encode $2k$ bits information into $n$ EPR pairs ($2n$ qubits) and protect at most $t$ bit-flip errors and $t$ phase-flip errors on the second half of the EPR pairs.

Note that we use the same error model as that in section III. For our quantum communications, there are always two halves of the entangled pairs: one half stay in Bob's hands throughout the whole process of the communications, which can be stored in some quantum storage with very low errors and successfully protected by some existing codes such as the CSS codes; the other half are transferred through the noisy quantum channels but finally back to Bob's side, which will introduce much higher errors. The existing error correcting codes may fail due to such high errors; however, our new one-party-QECC can protect the half of the EPR pairs transmitted through the quantum channels in this situation. We will show in the following text that they can tolerate much higher error rates than existing codes. Consequently, we assume that in our one-party-QECC, the errors can only happen in the second half of the physical EPR pairs.

Now we present the detailed definition of the $[[2n, 2k, t]]$ one-party-QECC. In details, corresponding to an element

$$g_i^c = Z_{i_1}^{cl}Z_{i_2}^{cl} \cdots Z_{i_l}^{cl}$$  (12)

in the stabilizer of the classical code (here to avoid confusion, we use the superscript of “cl” to indicate the classical operators), we define 2 error syndromes of the one-party-QECC measured by

$$g_z,i = (Z_{2i_1-1}Z_{2i_1})(Z_{2i_2-1}Z_{2i_2}) \cdots (Z_{2i_l-1}Z_{2i_l}),$$
$$g_x,i = (X_{2i_1-1}X_{2i_1})(X_{2i_2-1}X_{2i_2}) \cdots (X_{2i_l-1}X_{2i_l})$$  (13)

respectively. The subscript $i_j$ for $\forall j = 1, 2, \cdots , l$ means the operation is on the $i_j$-th EPR pair, and the index of qubits of the $i_j$-th EPR pair are $2i_j - 1$ and $2i_j$. Moreover, all $i, i_1, i_2, \cdots , i_l$ are of the same values as those in Eq. (12) because of the correspondence between the classical codes and the one-party-QECC. All error syndromes measured by the Z-type operators $g_z,i$ completely give us the bit-flip error patterns, and all error syndromes measured by the X-type operators $g_x,i$ completely give us the phase-flip error patterns. Thus we can apply corresponding operations to correct the bit-flip and phase-flip errors. Furthermore, similar to the $[[6,2,1]]$ one-party-QECC, we can create quantum circuit of error detection of $[[2n, 2k]]$ one-party-QECC by following stabilizer theory [25]. For the classical
stabilizers in Eq.(12) are capable of discriminating any error which is caused by no more than \( t \) flipped bits, the stabilizers of the generalized one-party-QECC in Eq.(13) can successfully reveal any combination of at most \( t \) bit-flip and at most \( t \) phase-flip errors occurring on the second halves of the physical EPR pairs. Therefore, with the sufficient information of the error pattern, we can completely correct the corrupted states; the one-party-QECC is called \([2n, 2k, t]\) one-party-QECC.

After defining the error syndromes, the logical encoded states can be easily worked out. However, there is another simple way: firstly, any logical state can be represented as

\[
[a_1a_2; a_3a_4; \cdots; a_{2k-1}a_{2k}]
\]

(14)

in terms of \( a_i = 0, 1 \), for \( \forall i = 1, 2, \cdots, 2k \). The representation of the logical state is divided into \( k \) groups. In each group, namely, the \( i \)-th group, there are 2 logical qubits \( |0_{2i-1}0_{2i}\rangle \) in the total four different logical qubits, namely,

\[
|0_{2i-1}0_{2i}\rangle = |\Phi^+\rangle_{2i-1,2i},
|0_{2i-1}1_{2i}\rangle = |\Phi^-\rangle_{2i-1,2i},
|1_{2i-1}0_{2i}\rangle = |\Psi^+\rangle_{2i-1,2i},
|1_{2i-1}1_{2i}\rangle = |\Psi^-\rangle_{2i-1,2i}.
\]

(15)

In order to transform between the logical encoded states, we remember that in the classical codes, the logical state are transformed by the combination of several logical bit-flip operators. The logical bit-flip operators of the classical codes are defined as,

\[
\hat{X}^{cl}_i = X^{cl}_{i_1}X^{cl}_{i_2}\cdots X^{cl}_{i_l},
\]

for any \( i = 1, 2, \cdots, k \). Each operator can flip the corresponding logical bit, namely,

\[
\hat{X}^{cl}_i|0_i\rangle = |1_i\rangle,
\hat{X}^{cl}_i|1_i\rangle = |0_i\rangle.
\]

(17)

As a result, with the correspondence between the classical linear codes and the one-party-QECC, we define the logical operators for the one-party-QECC as,

\[
\hat{X}_{2i-1} = X_{2i_1}X_{2i_2}\cdots X_{2i_l},
\hat{Z}_{2i-1} = (Z_{2i_1}Z_{2i_2})(Z_{2i_2-1}Z_{2i_3})\cdots (Z_{2i_l-1}Z_{2i_l}),
\hat{X}_{2i} = Z_{2i_1}Z_{2i_2-1}\cdots Z_{2i_l-1},
\hat{Z}_{2i} = (X_{2i_1-1}X_{2i_2})(X_{2i_2-1}X_{2i_3})\cdots (X_{2i_l-1}X_{2i_l}).
\]

(18)

Again we require that the subscripts \( i, i_1, i_2, \cdots, i_l \) are of the same values to those in Eq.(16). These logical operators act as the bit-flip and phase-flip operators on the encoded states, i.e.,

\[
\hat{X}_{2i-1}|0\rangle_{2i-1} = |1\rangle_{2i-1},
\hat{X}_{2i-1}|1\rangle_{2i-1} = |0\rangle_{2i-1},
\hat{Z}_{2i-1}|0\rangle_{2i-1} = |0\rangle_{2i-1},
\hat{Z}_{2i-1}|1\rangle_{2i-1} = |1\rangle_{2i-1},
\hat{X}_{2i}|0\rangle_{2i} = |1\rangle_{2i},
\hat{X}_{2i}|1\rangle_{2i} = |0\rangle_{2i},
\hat{Z}_{2i}|0\rangle_{2i} = |0\rangle_{2i},
\hat{Z}_{2i}|1\rangle_{2i} = |1\rangle_{2i}.
\]

(19)

Furthermore, in the classical codes, traditionally, it always holds that

\[
\begin{array}{c|c|c|c|c}
0 & 0 & \cdots & 0 & 0 \\
- & - & \cdots & - & - \\
\hline
0 & 0 & \cdots & 0 & 0 \\
\end{array}
\]

(20)

and any logical states can be created by the combination of several logical bit-flip operators acting on such logical zero state. Therefore, we begin with the logical zero state of the one-party-QECC as

\[
|00; 00; \cdots ; 00\rangle_k = |\Phi^+\rangle_{12}|\Phi^+\rangle_{34}\cdots |\Phi^+\rangle_{2n-1,2n}.
\]

(21)
worked out both the logical states and the logical operations of the generalized one-party-QECC. In this way, we have worked out both the logical states and the logical operations of the generalized one-party-QECC.

In the end of this section, we estimate the asymptotic tolerable error rate of the generalized one-party-QECC. Intuitively, from the correspondence between the classical linear codes and the generalized one-party-QECC, we can say that one-party-QECC can tolerate the error rate no less than that of classical linear codes in classical situations when the physical resources are very large, namely, \( n \rightarrow \infty \). In this case, the classical Gilbert-Varshamov bound is

\[
\frac{k}{n} \geq 1 - H\left(\frac{t}{n}\right),
\]

where the function \( H(x) \) is the Shannon entropy, \( H(x) = -x \log_2 x - (1 - x) \log_2(1 - x) \). When the following condition holds,

\[
1 - H\left(\frac{t}{n}\right) > 0,
\]

there exists a good \([n, k, t]\) classical linear code to correct at most \( t \) bit-flip errors. When finding such \([n, k, t]\) classical code, we can then work out a corresponding \([2n, 2k, 2t]\) one-party-QECC, which are able to correct at most \( t \) bit-flip and \( t \) phase-flip errors on the second halves of the EPR pairs. The numerical result of condition is \( \frac{t}{n} < 0.5 \). It means that if there are no more than one half of the EPR pairs corrupted by the errors, there exists a one-party-QECC which can successfully correct all errors. This result is consistent with the result in section III.

To sum up the discussion of this section, we have constructed a generalized series of the one-party-QECC from the corresponding classical error correcting codes. The correspondence between our one-party-QECC and the classical linear codes guarantees our one-party-QECC be able to protect quantum states from at most 50\% bit error rate of the quantum channel.

V. APPLICATION: QUANTUM DENSE CODING WITH ONE-PARTY-QECC

In this section, we present an application of our novel one-party-QECC in quantum communication: quantum dense coding with one-party-QECC. Quantum dense coding can transmit more information with a small number of information carrier, and has important applications in quantum communication. Suppose that the server, namely Bob, first sends the second halves of entangled pairs to a user, namely Alice, and they use other quantum error correction methods such as EPP to distribute high-fidelity EPR pairs between them. Then, Alice stores her qubits into the quantum memory for future use. However, as the conditions in the user’s side are always worse than those in the server’s side. Alice, the user, may soon find that her qubits may have more errors, while the qubits in Bob, the server’s side, are still nearly correct. If they want to make use of these impure EPR pairs in quantum communication, conventionally they should first distill the pairs into nearly pure using EPP. However, when the error rate in Alice’s side is higher than the threshold of EPP, namely, the fidelity of the impure state is less than the minimum of 0.5 for EPP, no successful purification can be done and the state is totally a mixed state without any entanglement. Are these pairs actually useless? Indeed, we will demonstrate that Alice and Bob still can make use of the quantum correlation in the very low fidelity entanglement resources in quantum dense coding using one-party-QECC.

First we give a brief introduction of the process of the quantum dense coding with initial pure EPR pairs and through the noiseless channels. Quantum dense coding makes use of previous established entangled pairs to send two bits classical information over only one qubit. Suppose that before the process, Alice and Bob have already shared a sequence of pure EPR pairs, each in the state \( |\Phi^+\rangle_{AB} \); in each pair, one qubit is held by Alice while the other is at the Bob’s. Alice wants to transfer some information to Bob. She makes use of the entangled qubits to send as much as 2 bits of information with 1 qubit. To be more specific, for each qubit on her side, she can perform one of the four possible operations: \( I \) (Identity), \( X, Y, Z \). With simple calculations, these operations effectively transform the initial state \( |\Phi^+\rangle_{AB} \) into one of the four possible Bell states: \( |\Phi^+\rangle_{AB}, |\Psi^+\rangle_{AB}, |\Psi^-\rangle_{AB}, |\Phi^-\rangle_{AB} \). After encoding all information into the sequence of the qubits, she sends them to Bob. If there is no error in the channel, after receiving the qubits from Alice, Bob obtains both halves of the EPR pairs in different Bell states. Then he performs the Bell measurements which distinguish the four orthogonal Bell states and retrieves the encoded information from Alice. In summary, the quantum dense coding employs 1 EPR pair to encode 2 bit information while only 1 qubit travels through the channel.

Now, let us examine the quantum dense coding in the noisy situation. The protocol contains two sources of errors. One comes from the initial impure EPR pairs. The impurity may be brought by the errors in Alice’s qubits which can be characterized by bit-flip error, phase-flip error or both of them. The other comes from the channel noise. When the
quantum channels are noisy, any transmitted qubits may be also altered by bit-flip error, phase-flip error or both of them. One may consider using EPP to correct both errors. But if the errors are high enough, EPP will fail. Even if the channel is noiseless but the fidelity of the initial impure EPR pairs is lower than 0.5, no EPR pairs can be purified and this method fails. However, if we make use of one-party-QECC, quantum dense coding is still feasible.

To be more specific, to protect the qubit against error from both sources, we should introduce error correction procedure. Remembering the criteria of the suitable quantum communications of the one-party-QECC, we learn that if one of the communication parties finally obtains both halves of the EPR pairs, the home qubits, the joint measurements and operations in one-party-QECC can be applied. In addition, the different storage conditions of Alice and Bob and the noisy channels brings much more errors to the flying qubits sent from Alice to Bob. Thus the errors in the flying and home qubits are unbalanced. The process of the quantum dense coding that Alice encodes the message into her qubits and then sends them to Bob who finally holds the whole pairs is suitable for the application of the one-party-QECC.

With this idea, we transform the quantum dense coding into error-tolerating one. Normally, Alice and Bob should first share the initial encoding states(not required to be pure). Then Alice encodes the information by the logical operations and sends them to Bob. Bob receives the qubits from Alice and measures the stabilizer to obtain the error syndromes. With the error syndromes, Bob identifies the error pattern and chooses appropriate recovering operation. Finally he performs the logical measurements on the logical states and retrieves the message. In a word, all the operations and measurements are changed into the logical ones. Reviewing section IX, suppose that there are $2k$ bit information to be transmitted and $[[2n, 2k, t]]$ one-party-QECC is applied. The initial encoding state is given by Eq. (21), which is equal to a physical EPR pairs each in the state of $|\Phi^\pm\rangle$. Note that the qubits with odd index are held by Alice and those with even index are held by Bob. Then it comes to the encoding operations, Eq. (15). At first sight, there is only one kind of logical operations, $X_{2i}$, containing only the operations on the physical qubits in Alice’s (the qubits with odd index). This fact means that Alice would have only two choices, $I$ and $X_{2i}$, to transforms one logical EPR pair. However, there are two other possible choices corresponding to each $X_{2i} = Z_{2i-1}Z_{2i-2} \cdots Z_{2i-1}$:

$$\bar{U}_{2i} = X_{2i-1}X_{2i-2}\cdots X_{2i-1},$$
$$\bar{V}_{2i} = Y_{2i-1}Y_{2i-2}\cdots Y_{2i-1},$$

(24)

where the subscripts $i_1, i_2, \cdots, i_t$ are the same to those in the definition of $X_{2i-1}$. These $\bar{U}_{2i}$ and $\bar{V}_{2i}$, though do not commute with the logical operations in Eq. (15), commute with the stabilizers in Eq. (13) and successfully transform the initial encoded state. Particularly, $X_{2i}$, $U_{2i}$ and $V_{2i}$ can transform the encoded state $|00\rangle_{2i-1,2i} = |\Phi^+\rangle_{2i-1,2i}$ to the states $|00\rangle_{2i-1,2i} = |\Phi^-\rangle_{2i-1,2i}$, $|01\rangle_{2i-1,2i} = |\Psi^+\rangle_{2i-1,2i}$ and $|11\rangle_{2i-1,2i} = |\Psi^-\rangle_{2i-1,2i}$ respectively, up to a global phase. Consequently, Alice still has four choices, $I, X_{2i}, U_{2i}, V_{2i}$ to encode $2$ bits information into $1$ logical EPR pair. After encoding the message, Alice sends all her qubits to Bob. Bob receives them and measures the stabilizers in Eq. (13). If the bit-flip and phase-flip errors are no more than $t$ respectively on the transmitted qubits, the measurement results reveal the exact positions of each error and Bob chooses proper bit-flip and phase-flip operations on those positions to recover the correct state. With the correct state, Bob performs the logical Bell measurements, i.e., measures $X_{2i-1}X_{2i}$ and $Z_{2i-1}Z_{2i}$ on the $i$-th logical EPR pair, and fully retrieves the message of Alice.

Furthermore, as the depolarizing channel in six-state quantum key distribution provides higher tolerable channel bit error rate[11], in order to make use of the same advantage in quantum dense coding with one-party-QECC, we should make sure that the final bit, phase and both bit and phase errors are symmetric, as an effective depolarizing channel. Firstly, to deal with the initial impure EPR pairs, we employ the similar method in Ref. [9] to obtain the pairs with symmetric errors. In particular, Alice first applies $Y$ to her qubits in order to transform $|\Phi^\pm\rangle$ into $|\Psi^\pm\rangle$. Then Alice and Bob employ bilateral $\pi/2$ rotations $B_x, B_y, B_z[9]$ to their pairs. These operations are capable of obtaining a rotationally symmetric mixture, i.e., a Werner state [27],

$$W_F = F|\Psi^\pm\rangle\langle\Psi^\pm| + \frac{1-F}{3}|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}|\Phi^-\rangle\langle\Phi^-| + \frac{1-F}{3}|\Psi^+\rangle\langle\Psi^+|,$$

(25)

where $F$ is the initial fidelity of the impure pairs. Finally, Alice applies $Y$ to her qubits to revert $|\Psi^\pm\rangle$ into $|\Phi^\pm\rangle$ and thus obtains a sequence of impure pairs with symmetric errors.

Secondly, the errors introduced by the noise channels can also be symmetrized. It can be done by the same method of six-state quantum key distribution[11]. To be more specific, Alice randomly applies $I$ (Identity), $T, T^2$ to the qubits after encoding the message, where

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

(26)

These operations effectively transform the qubit into $Z$, $X$ and $Y$ basis. When Bob receives the qubits, Alice tells her operations and then Bob uses $I, T^2, T$ to recover all the qubits to the $Z$ basis, corresponding to Alice’s operations.
So with this manipulation, the three kinds of error probabilities of the quantum channel is averaged. Thus the final errors that comes from both sources become symmetric. In summary, the process of the quantum dense coding with one-party-QECC is presented as following:

*Protocol 1: Quantum dense coding with one-party-QECC*

1. Alice and Bob initially share a group of \(n\) impure EPR pairs, with fidelity \(F_0\) to the pure state of \(\left| \Phi^+ \right\rangle^\otimes n\).

2. Alice and Bob first make the impure state into a rotational symmetric mixture. Alice applies \(Y\) to her qubits. Then Alice and Bob employ bilateral \(\pi/2\) rotations \(B_x, B_y, B_z\) to their pairs. After that, Alice applies \(Y\) to her qubits again and the errors of the impure state are symmetrized.

3. Alice wants to transmit \(2k\) bit message. She and Bob agree on a \([2n, 2k, t]\) one-party-QECC which fully protect the qubits from the errors of the channel.

4. Alice and Bob make use of the shared \(n\) EPR pairs from the initial prepared collection. The initial state is described as the logical state of \(\Phi^+\) of the qubits, where the odd qubits are held by Alice and the even ones are held by Bob.

5. Then Alice encodes the message into the EPR pairs. For the \(i\)-th logical pair \(\left| 0 \right\rangle_{2i−1,2i}\), according to the corresponding 2 bit values of the message, Alice chooses one of the four operations, \(I, \tilde{X}_{2i}, \tilde{U}_{2i}, \tilde{V}_{2i}\) to transform the pair into one of the four states, \(\left| 0 \right\rangle_{2i−1,2i}, \left| 1 \right\rangle_{2i−1,2i}, \left| \Phi^+ \right\rangle_{2i−1,2i}, \left| \bar{\Phi}^+ \right\rangle_{2i−1,2i}\).

6. Alice randomly apply \(I, T, T^2\) operations to all \(n\) qubits on her side, then sends them to Bob through the noisy quantum channels.

7. Bob receives the qubits and then there are total \(2n\) qubits on his side.

8. Alice tells Bob her choices of \(I, T, T^2\) operations. Accordingly Bob applies \(I, T^2, T\) to the received qubits, in order to reverse Alice’s operations. If they are conducting key distribution in this protocol, they can also random permute the sequence of the EPR pairs and publicly notify each other.

9. Bob measures the stabilizers of Eq. (13) of the \([2n, 2k, t]\) one-party-QECC on the corrupt state. With the measurement results as the error syndrome, he knows the positions of the errors and applies proper recovering operations to obtain the correct state.

10. Bob performs the logical Bell measurements on each logical EPR pair and retrieves the \(2k\) bit message.

Finally, we demonstrate the advantages of one-party-QECC in quantum dense coding. Generally, a noisy quantum channel is defined with three parameters \((p_X, p_Y, p_Z)\) which fully characterize the error probabilities of each qubit. \(p_X\) means the probability of only a single bit-flip error occurring at a qubit which is transmitted through the noisy quantum channel. \(p_Z\) means the probability of only a single phase-flip error and \(p_Y\) the probability of both a bit-flip error and a phase-flip error. So the fidelity of the channel is \(F = 1 − p_X − p_Y − p_Z\). In the quantum dense coding above, the effective channel is a depolarizing channel, the three probabilities are symmetric, namely, \(p_X = p_Y = p_Z = p\), and the fidelity is \(F = 1 − 3p\). In section IV, we showed that a \([2n, 2k, t]\) one-party-QECC can tolerate at most \(t\) bit-flip errors and \(t\) phase-flip errors. Thus the success of the \([2n, 2k, t]\) one-party-QECC requires that

\[
\frac{t}{n} \geq p_Z + p_Y = 2p, \quad \frac{t}{n} \geq p_X + p_Y = 2p.
\]

It follows that

\[
F \geq 1 − \frac{3}{2} \frac{t}{n}.
\]

In asymptotic situation, because \(t/n < 0.5\), so \(F > 0.25\) and the correctable fidelity of one-party-QECC reaches 0.25, which is much lower than the lower fidelity bound of 0.5 in EPP. As indicated in previous works, the situation that the fidelity reaches 0.5 means that the entangled pairs are a total mixture state without any entanglement, and that no advantage of entanglement is helpful in quantum communication. However, using one-party-QECC, the quantum dense coding that makes use of established quantum correlation is still able transmit classical information, although the fidelity is much lower. Moreover, the correspondence between one-party-QECC and classical linear codes
In section IV shows that under the same channel bit error rate, quantum dense coding with one-party-QECC can transmit classical bit information twice as much as that by classical communication, although only the same amount of information carriers (qubits and bits) are transmitted. This is also the result of quantum correlation established prior to the communication. The application of one-party-QECC also extend the requirement of quantum memory: although the errors introduced by Alice’s quantum memory make the fidelity lower than 0.5 and forbid the conventional EPP, quantum dense coding with one-party-QECC is still feasible if the combined errors of both the memory and the channels guarantee the final fidelity upper to 0.25.

VI. APPLICATION: QUANTUM SECURE DIRECT COMMUNICATIONS

Based on quantum dense coding, one-party-QECC can be applied to many more protocols due to the extensive use of dense coding in quantum information processing. In this section, we present the application of one-party-QECC in a kind of quantum secure direct communications (QSDC), that are based on quantum dense coding. We also show that the application can increase the performance of QSDC through noisy quantum channels.

A typical QSDC protocol through noiseless channels consists of two transmission phases of qubits. In the first phase, Bob creates a sequence of EPR pairs all in $|\Phi^+\rangle$ state and distributes the second halves of the pairs to Alice’s side. In the second phase, Alice encodes her message using certain operations on her received qubits, then sends them back to Bob. Finally, Bob employs Bell measurements to retrieve Alice’s encoded message. As the requirement of security, Alice and Bob performs error check after both phases. The concrete protocol is described as:

Protocol 2: QSDC protocol through noiseless channel

1. Bob first prepares a sequence of $3n$ EPR pairs in the state of $|\Phi^+\rangle$.

2. Bob chooses a random $3n$ bit binary string $b$, applied Hadamard transformation $H$ to the second halves of the pairs in which the corresponding bits of $b$ are 1. Then he sends the second halves to Alice.

3. Alice receives the qubits and publicly acknowledges her receipt.

4. Alice and Bob randomly chooses an $n$ subset of the EPR pairs as first-round check pairs. Bob tells Alice the bit values of $b$ in the place of the check pairs. Then Alice applies $H$ to the qubits of the first-round check pairs in her part where the corresponding bits of $b$ are 1. They both measure the check qubits in their halves of the check pairs respectively in the $Z$-basis. Note that because $H \otimes H = I (Identity)$, the results of Alice and Bob in each pairs will be the same if there is no error. Therefore, if they find that there are too many inconsistencies, they know that the transmitting qubits are eavesdropped and abort the protocol.

5. Alice randomly selects $n$ subset of the rest $2n$ EPR pairs as second-round check pairs; the rest are served as code pairs.

6. Alice wants to send a $n$ bit binary sequence of message $M$. She encodes $M$ to the $n$ qubits in her halves of the code pairs by applying $S = -iY = XZ$ to them where the corresponding bit of $M$ is 1. Then she returns the $2n$ qubits to Bob.

7. Bob receives the qubits from Alice, applies $H$ to the received qubits where the corresponding bits of $b$ are 1. Then he publicly announces his receipt.

8. Alice publicly announces the places of second-round check pairs. The same fact holds that if Bob measures both the qubits in each check pairs in $Z$-basis respectively, he will get the same results if there is no error. Thus if Bob gets too many errors, the protocol is aborted.

9. Bob measures both the qubits of the rest $n$ code pairs in $Z$-basis. Because $S|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and $(H \otimes S \otimes H)|\Phi^+\rangle = -\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, the different results of the measurements on the two qubits in one pair tell Bob that the corresponding bit of $M$ is 1 while the same results tell that the corresponding bit of $M$ is 0. Therefore, Bob can retrieve the full information of $M$.

Analyzing protocol 2, we find that the two phases are related to entanglement distribution and dense coding respectively. In particular, in the first phase, the sequence Bob creates is a pure state and he sends the second halves of the EPR pairs to Alice. Thus they are able to perform EPP on the distributed states in order to correct potential errors. We note that in order to perform EPP, Bob needs to reveal his random Hadamard transformations on the rest $2n$ pairs to Alice and thus Alice should perform again the random Hadamard transformations before the second phase. As $H_1H_2|\Phi^+\rangle_{12} = |\Phi^+\rangle_{12}$, Bob can perform the same Hadamard transformations on the qubits in his side.
before the EPP and then perform again after the EPP. We also note that the random selection of first-round check pairs gives the correlation between the check pairs and the rest pairs, which guarantees that the estimation of error rates in the rest pairs are of great probability bounded by the error rates in the check pairs. As a result, they should make sure the estimation of error rates in the rest pairs does not exceed the capability of EPP. A general EPP with two-way classical communications is able to correct corrupted EPR pairs with fidelity $F < 0.5 \[3, 7\]$. As Bob’s random Hadamard transformation makes the bit error rates and phase error rates symmetric, namely, $p_x + p_y = p_z + p_y$. Because the check procedure does not distinguish $p_y$ from $p_x$ or $p_z$, the worst case for the errors is $p_y = 0$ and $p_x = p_z \[12\]$. In this situation, the channel bit error rate $p_{bit} = p_z + p_y = F/2 < 25\%$. Therefore, Alice and Bob can use EPP to correct errors as long as the tolerable channel bit error rates $p_{bit}$ is less than 25\%.

In the second phase, Alice first encodes her message $M$ using $S$ and then sends her halves of the EPR pairs distributed in the first phase back to Bob. This process is effectively a dense coding protocol. As shown in section V, there are two choices for Bob to correct the errors on the qubits sent by Alice. The first choice is to use EPP, which is also able to achieve this as long as the channel error rates is less than 25\%. On the other hand, if Bob makes use of the alternative solution of one-party-QECC, he can enhance the performance. The qubits sent from Alice to Bob is flying qubits, while the qubits on Bob’s hand is home qubits. It is also obvious that Bob in the final stage receives both halves of the EPR pairs. As a result, the condition for applying one-party-QECC is satisfied. Reviewing section V, the operation $S$ Alice uses in encoding is related to the operation $-iY$, thus the logical operation of Alice is $\tilde{S}_i = -i\tilde{V}_2i$ in Eq. 24 in correspondence of the $i$-th bit of $M$. In this way, one-party-QECC can be applied and the detailed protocol through noisy channels is:

**Protocol 3: QSDC protocol with one-party-QECC**

1. Bob first prepares a sequence of $3n$ EPR pairs in the state of $|\Phi^+\rangle$.
2. Bob chooses a random $3n$ bit binary string $b$, applies Hadamard transformation $H$ to the second halves of the pairs in which the corresponding bits of $b$ are 1. Then he sends the second halves to Alice.
3. Alice receives the qubits and publicly acknowledges her receipt. Bob tells Alice the bit values of $b$. Then Alice applies $H$ to the qubits in her part where the corresponding bits of $b$ are 1.
4. Alice and Bob randomly chooses an $n$ subset of the EPR pairs as first-round check pairs. They both measure the check qubits in their halves of the check pairs respectively in the Z-basis. Note that because $H \otimes H = I (\text{Identity})$, the results of Alice and Bob in each pair will be the same if there is no error. Therefore, if they find that there are too many inconsistencies, they know that the transmitting qubits are eavesdropped and abort the protocol.
5. Alice and Bob uses a suitable EPP to purify their rest EPR pairs.
6. Alice randomly selects $m$ subset of the rest $2m$ first-level logical EPR pairs as second-round check pairs; the rest are served as code pairs. She also randomly chooses a $2m$ bit binary string $b'$, applies first-level logical Hadamard transformation $H$ to the second halves of the pairs in which the corresponding bits of $b$ are 1. Then he sends the second halves to Alice.
7. Alice wants to send a $k$ bit binary sequence of message $M$. She picks a $[2m; 2k, t]$ one-party-QECC that can correct the errors in the second transmission. In the view of one-party-QECC, there are $k$ second-level logical EPR pairs in the code pairs. She encodes $M$ to her halves of the second-level logical qubits in the code pairs by applying $\tilde{S}_{2i} = -i\tilde{Y}_{2i}$ to them where the corresponding bit of $M$ is 1. Then she returns all her qubits to Bob.
8. Bob receives the qubits from Alice and publicly announces his receipt. Then Alice announces $b'$, and Bob applies the first-level logical $H$ to the received first-level qubits where the corresponding bits of $b'$ are 1.
9. Alice publicly announces the places of second-round check pairs and the one-party-QECC she chooses. The same fact holds that if Bob measures the both qubits in each check pairs in Z-basis respectively, he will get the same results if there is no error. Thus if Bob gets too many errors, the protocol is aborted.
10. Bob uses the $[2m, 2k, t]$ one-party-QECC to correct the errors on the rest $m$ first-level logical EPR pairs and obtains $k$ second-level logical code pairs.
11. Bob measures both the qubits of the rest $k$ second-level logical code pairs in Z-basis. Therefore, from the comparison of the measurements on corresponding pairs, Bob can retrieve the full information of $M$.

The maximal tolerable channel bit error rate in the second phase using one-party-QECC is obviously 50\%, according to section III and IV. Compared to the result of 25\% for EPP, this result clearly show the enhancement of one-party-QECC in the application of QSDC.
VII. CONCLUSION

In this paper, we first present and discuss a novel type of one-party quantum error correcting codes. We consider a kind of quantum communication with unbalanced errors on flying and home qubits. In this kind of communication, instead of distributing the entangled pairs, Bob finally receive both qubits of the entangled pairs. Thus the feasibility of joint measurements and operations between the flying and home qubits in error correction enlightens us to introduce them to quantum error correcting code, called one-party-QECC. Therefore, in section II similarly to the classical [3,1] linear code, we pair up 3 EPR pairs to form a [[6,2,1]] one-party-QECC. Then we concatenate the [[6,2,1]] one-party-QECC in section III in order to encode longer codes. In section IV we derive an generalized [[2\, n, 2k, t]] one-party-QECC which covers a large series of the one-party-QECC. Consequently we establish the formalism of the novel one-party-QECC. At the same time, we analyze the correcting process of one-party-QECC and derive its tolerable channel bit error rate 50% which is higher than previous codes. In section V and VI we give an application of one-party-QECC in the quantum dense coding and quantum secure direct communications and show that they can be conducted with much higher capacity due to the advantages of one-party-QECC.

Recently, some other proposals on entanglement-assisted quantum error correcting codes have been discussed[28, 29]. They follow the same idea to make use of entanglement to construct quantum codes. But such codes are partial entanglement-assisted while our one-party-QECC provides quantum coding from only entanglement. In addition, we present a physical picture of the one-party-QECC and thorough analysis of its capacity and applications.

Throughout these sections, two main ideas play an important role. The first one is the joint measurements and operations between two kinds of qubits, based on the quantum entanglements and correlations. The second one is the correspondence between the one-party-QECC and the classical linear codes. In the one-party-QECC, we effectively treat the EPR pairs as the basic elements with four different values, |00⟩, |01⟩, |10⟩ and |11⟩, which play the same role of the bits in the classical codes. The errors on only one part of each pair causes its value changes rather than the phase changes, similar to the classical situation. With such correspondence to the classical linear codes, we easily find the elements of the one-party-QECC, i.e., the stabilizers, logical states and operations. Moreover, the correspondence guarantees the same high error-tolerating capability and efficiency of the one-party-QECC to that of the classical codes, which is the most intriguing result of the discussion.

Moreover, the idea of the one-party-QECC is also valuable to the implementation of quantum computation. In the one-party-QECC and its applications in quantum communication, one part of the state, i.e., the home qubits in Bob’s hands all through the protocol, are stored in some storage which provide low error environment, while the other part of the state, i.e., the flying qubits transmitted from Alice to Bob, are affected by much more errors in an open environment. It is stated in section V the introduction of one-party-QECC loose the requirement of quantum memory in Alice, the user’s side. Thus, in more general quantum computing schemes, sometimes there are also unbalanced errors on two different parts of qubits. We divide these two parts into two layers. In the first layer, the qubits stored in the storages with very low errors are protected by some error correction scheme, simpler but tolerating fewer errors. In the second layer, it is feasible to protect the qubits in much more noisy environments by the one-party-QECC, for in the view of this layer, the errors only happen in this part of the qubits with much higher error rate than the part handled in the first hierarchy. Such idea of one-party quantum error correction may have many other promising applications in the quantum computers with devices of different error levels.

In conclusion, we establish a series of one-party-QECC and demonstrate its applications in quantum dense coding and quantum secure direct communications. The one-party-QECC reveals the amazing strength of the quantum entanglements and joint operations in error correction. It also brings forth some interesting ideas in designing error correction schemes. More deep research in the applications of the one-party-QECC in the quantum communication and computation should be carried out in the future.

Acknowledgment

This work is supported by the 973 Program Grant No. 2006CB921106, China National Natural Science Foundation Grant No. 10325521, 60433050, 60635040, the SRDFP program of Education Ministry of China, No. 20060003048 and the Key grant Project of Chinese Ministry of Education No.306020.

[1] P. W. Shor, Phys. Rev. A 52, 2493 (1995).
[2] A. R. Calderbank and P. W. Shor, Phys. Rev. A, 54,1098 (1996).
[3] A. M. Steane, Proc. R. Soc. London A 452, 2551(1996).
[4] K. Q. Feng, S. Ling and C. Xing, IEEE Trans. Inf. Theory 52, 986 (2006)
[5] H. Chen, S. Ling and C. Xing, IEEE Trans. Inf. Theory, 51, 2915 (2005)
[6] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin and W. K. Wootters, Phys. Rev. Lett., 76, 772 (1996).
[7] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu and A. Sanpera, Phys. Rev. Lett., 77, 2818 (1996).
[8] Charles H. Bennett, David P. DiVincenzo, John A. Smolin and William K. Wootters, Phys. Rev. A, 54, 3824 (1996).
[9] H.-K. Lo and H. F. Chau, Science 283, 2050 (1999).
[10] P. W. Shor and J. Preskill, Phys. Rev. Lett. 85, 441 (2000).
[11] H.-K. Lo, Quantum Information and Computation, 1, No. 2, 81 (2001).
[12] D. Gottesman and H.-K. Lo, IEEE Trans. Inf. Theory 49, 457 (2003).
[13] H. F. Chau, Phys. Rev. A 66, 060302(R) (2002).
[14] W.-Y. Hwang, X.-B. Wang, K. Matsumoto, J. Kim and H.-W. Lee, Phys. Rev. A 67, 012302 (2003).
[15] K. Wen and G. L. Long, Phys. Rev. A 72, 022336 (2005).
[16] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 68, 3121 (1992).
[17] X S Liu, G L Long, D M Tong and Feng Li, Phys. Rev. A65, 022304 (2002)
[18] G. L. Long and X. S. Liu, Phys. Rev. A 65, 032302 (2002)
[19] K. Boström and T. Felbinger, Phys. Rev. Lett. 89, 187902 (2002)
[20] F. G. Deng and G. L. Long, Phys. Rev. A68, 042315 (2003)
[21] F. G. Deng, G. L. Long, and X. S. Liu, Phys. Rev. A 68, 042317 (2003).
[22] Q. Y. Cai and B. W. Li, Chin. Phys. Lett. 21, 601 (2004).
[23] C. Wang, F. G. Deng, Y. S. Li, X. S. Liu and G. L. Long Phys. Rev. A71 044305 (2005)
[24] F. G. Deng, G. L. Long, H. Y. Zhou Phys. Lett. A 340 43 (2005)
[25] M. A. Nielsen and I. L. Chuang, “Quantum Computation and Quantum Information”, Cambridge University Press (2000), pp. 472-474.
[26] F. J. MacWilliams and N. J. A. Sloane, The Theory of Error-Correcting Codes (North-Holland, Amsterdam, 1977), pp. 557-558 (proof of Thm.31 of Chap.17).
[27] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[28] G. Bowen, Phys. Rev. A 66, 052313 (2002).
[29] T. Brun, I. Devetak and M.-H. Hsieh, quant-ph/0608027
### TABLE I: Results of Bell measurements

| $X_1X_2$ | $Z_1Z_2$ | Bell state |
|----------|----------|------------|
| 1 1      |          | $|\Phi^+\rangle$ |
| 1 −1     |          | $|\Psi^+\rangle$ |
| −1 1     |          | $|\Phi^-\rangle$ |
| −1 −1    |          | $|\Psi^-\rangle$ |

### TABLE II: Error syndromes of [[6,2,1]] one-party-QECC

| $X_1X_2X_3X_4$ | $X_5X_6Z_1Z_2Z_3Z_4Z_5Z_6$ | Corresponding error | Correcting operations |
|----------------|-----------------------------|---------------------|----------------------|
| 1 1 1 1        |                             | no error            | $I$ (Identity)       |
| 1 1 1 −1       |                             | bit 5 flip          | $X_5$                |
| 1 1 −1 1       |                             | bit 1 flip          | $X_1$                |
| 1 −1 −1 −1     |                             | bit 3 flip          | $X_3$                |
| −1 1 1 1       |                             | phase 5 flip        | $Z_5$                |
| 1 −1 1 −1      |                             | both bit and phase 5 flip | $Z_5X_5$            |
| 1 −1 −1 −1     |                             | bit 3 and phase 5 flip | $Z_5X_3$            |
| −1 −1 1 1      |                             | phase 1 flip        | $Z_1$                |
| −1 1 1 −1      |                             | both bit and phase 1 flip | $Z_1X_1$            |
| −1 1 −1 −1     |                             | bit 3 and phase 1 flip | $Z_1X_3$            |
| −1 −1 −1 1     |                             | phase 3 flip        | $Z_3$                |
| 1 −1 1 −1      |                             | bit 5 and phase 3 flip | $Z_3X_5$            |
| −1 −1 −1 1     |                             | bit 1 and phase 3 flip | $Z_3X_1$            |
| −1 −1 −1 −1    |                             | both bit and phase 3 flip | $Z_3X_3$            |

### TABLE III: Transform between encoded states

| $\bar{X}_1$ | $\bar{Z}_1$ | $X_2$ | $\bar{Z}_2$ |
|-------------|-------------|-------|-------------|
| $|00\rangle$ | $|00\rangle$ | $|01\rangle$ | $|01\rangle$ |
| $|01\rangle$ | $|01\rangle$ | $|10\rangle$ | $|10\rangle$ |
| $|10\rangle$ | $|00\rangle$ | $|00\rangle$ | $|01\rangle$ |
| $|11\rangle$ | $|00\rangle$ | $|00\rangle$ | $|01\rangle$ |