NOTE ON ADELIC TRIANGULATIONS AND AN ADELIC BLICHFELDT-TYPE INEQUALITY

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Abstract. We introduce a notion of convex hull and polytope into adele space. This allows to consider adelic triangulations which, in particular, lead to an adelic blichfeldt-type inequality, complementing former results.

1. Introduction

Let $K_0^m$ be the set of all 0-symmetric convex bodies in the $m$-dimensional Euclidean space $\mathbb{R}^m$ with non-empty interior, i.e., $C \in K_0^m$ is an $m$-dimensional compact convex set satisfying $C = -C$. An important subclass of convex bodies are formed by polytopes $P = \text{conv}\{v_1, \ldots, v_l\}$, i.e., the convex hull of finitely many points $v_1, \ldots, v_l \in \mathbb{R}^m$. We write $P^m$ and $P_0^m$ for the set of all, respectively the set of all 0-symmetric, $m$-dimensional polytopes in $\mathbb{R}^m$.

By a lattice $\Lambda \subset \mathbb{R}^m$ we understand a free $\mathbb{Z}$-module of rank $\text{rank}\Lambda \leq m$. The set of all lattices in $\mathbb{R}^m$ is denoted by $L^m$, and $\det \Lambda$ denotes the determinant of $\Lambda \in L^m$, that is the $(\text{rank}\Lambda)$-dimensional volume of a fundamental cell of $\Lambda$. For more detailed information on lattices we refer to [15, 16].

One of the classical inequalities relating a convex body and points of a lattice is Blichfeldt’s inequality from 1921 (see, e.g., [3]). It gives an upper bound on the number of lattice points of a lattice $\Lambda \in L^m$ contained in a convex body $C \in K^m$ under the assumption that $\dim_{\mathbb{R}}(C \cap \Lambda) = m$, i.e., $C \cap \Lambda$ contains $m + 1$ affinely independent lattice points

$$|C \cap \Lambda| \leq m! \frac{\text{vol}(C)}{\det \Lambda} + m. \quad (1.1)$$

The bound is sharp, for instance for $\Lambda = \mathbb{Z}^m$ and simplices of the form $\text{conv}\{0, \ell e_1, e_2, \ldots, e_m\}$, where $\ell \in \mathbb{N}$ and $e_1, \ldots, e_m$ are the standard unit vectors. The additional requirement on the dimension is necessary, as an axis-parallel box with very small edge length in one direction can contain a large number of lattice points, while still having arbitrarily small volume.

The usually way to prove Blichfeldt’s result, as many other results in the context of the interplay of lattice points and convex bodies, is via triangulations (cf., e.g., [1, 6]). To this end firstly notices that by replacing $C$ by $\text{conv}(C \cap \Lambda)$ it suffices to prove the bound for the class of lattice polytopes $P \in P^m$, i.e., polytopes admitting a representation as $\text{conv}\{w_1, \ldots, w_l\}$ where $w_i$ are points of a lattice $\Lambda$. The next observation is that $P$ can be triangulated in at least $|P \cap \Lambda| - m$ many lattice simplices and the volume of a lattice simplex is at least $\det \Lambda/m!$.

In this note we want to introduce the notion of convex hull and triangulations in the adele space, which has been proved in recent years as an excellent and challenging space for extensions and generalizations of classical concepts from Geometry of numbers, see e.g., [11, 14, 16, 22, 23, 24, 25] as the references within.

After a short introduction to adelic geometry in Section 2, we introduce in Section 3 our notion of the adelic convex hull and adelic polytopes. In particular, we
will prove a lower bound on the adelic volume of an adelic lattice simplex in the case of totally real fields (Lemma 3.5). In Section 4 we study adelic triangulations, which we use to prove our main result

**Theorem 1.1.** Let $K$ be a totally real number field of degree $d = [K : \mathbb{Q}]$. Let $C$ be an adelic convex body with $\dim_K(C \cap K^n) = n$. Then

$$|C \cap K^n| \leq (n!)^d \mu_{\mathbb{A}}(C) + n.$$ 

The necessary notations will be introduced in Section 2. We remark that for $d = 1$ and $n = m$ we get Blichfeldt’s inequality \((\text{4.1})\). Moreover, Theorem 1.1 improves for the special case of totally real fields on a former more general result of Gaudron [13] (see (5.2)) for 0-symmetric adelic convex bodies. This result, as well as other adelic symmetric variants of Blichfeldt’s theorem will be presented in the final Section 5.

2. ADELCIC GEOMETRY

In this section we will briefly introduce the notations and concepts from adelic geometry used in the following sections. For a detailed discussion we refer to [4] [15], [24].

Let $K$ be an algebraic number field of degree $d = [K : \mathbb{Q}]$. Let $r$ be the number of real and $s$ the number of pairs of complex embeddings of $K$ into $\mathbb{C}$, so $d = r + 2s$. Denote by $\mathcal{O}$ the ring of algebraic integers of $K$ and by $\Delta_K$ and $h_K$ its field discriminant and class number, respectively.

Let $M(K)$ be the set of all places of $K$. For $v \in M(K)$ we write $v \not| \infty$ for non-archimedean places and $v \mid \infty$ for the archimedean ones. We write $| \cdot |_v$ for the corresponding absolute value on $K$. We normalise it to extend either the usual absolute value on $\mathbb{Q}$ for archimedean places or the $p$-adic absolute value for a prime $p$. Then the local field $K_v$ is the completion of $K$ with respect to $v$. For $v \not| \infty$ let $\mathcal{O}_v$ be the local ring of integers of $K_v$.

Let $K_n$ be the ring of adeles of $K$ and $K_n^n$ the standard module of rank $n \geq 2$, i.e., the $n$-fold product of adeles. Recall that $K_n$ is the restricted direct product of the $K_v$ with respect to the $\mathcal{O}_v$. For any $v \in M(K)$ let $d_v = [K_v : \mathbb{Q}_v]$ be the local degree $|Q_\infty \cong \mathbb{R}|$. Then

$$d = \sum_{v \not| \infty} d_v, \quad \text{and for all non-zero } a \in K \prod_{v \in M(K)} |a|_v^{d_v} = 1. \quad (2.1)$$

For $v \not| \infty$ let $\mu_v$ be the Haar measure on $K_v$ normalized such that $\mu_v(\mathcal{O}_v) = 1$. Thus for any ideal $\alpha \mathcal{O}_v \subseteq \mathcal{O}_v$ we get $\mu_v(\alpha \mathcal{O}_v) = |\alpha|_v^{d_v}$. For $v \mid \infty$ let $\mu_v$ be the Lebesque measure on $K_v = \mathbb{R}$ resp. twice the Lebesque measure on $K_v = \mathbb{C}$. Define the Haar measure $\mu_{\mathbb{A}}$ on $K_n$ by

$$\mu_{\mathbb{A}} = \frac{1}{\sqrt{|\Delta_K|}} \prod_{v \in M(K)} \mu_v$$

and use the product measure on $K_n^n$.

**Definition 2.1** (Adelic convex body). For each $v \not| \infty$ let $C_v$ be a free $\mathcal{O}_v$-module of full rank, where $C_v = \mathcal{O}_v^n$ for all but finitely many $v$. In other words, for any $v \not| \infty$ there is an $A_v \in \text{GL}_n(\mathbb{R})$ such that $C_v = A_v \mathcal{O}_v^n$, where $A_v \in \text{GL}_n(\mathcal{O}_v)$ for all but finitely many $v$. For $v \mid \infty$ we have $K_v \cong \mathbb{R}$ or $K_v \cong \mathbb{C}$. In this case let $C_v \subseteq K_v^{\infty}$ be a compact convex body with non-empty interior in $\mathbb{R}^n$ or $\mathbb{C}^n \cong \mathbb{R}^{2n}$ respectively, i.e., $C_v \in K^n$ or $C_v \in K^{2n}$. Then the set

$$C = \prod_{v \not| \infty} C_v \times \prod_{v \mid \infty} C_v$$
is called an \textit{adelic convex body}. If $C_v$ is symmetric for $v \mid \infty$, i.e., $C_v \in \mathcal{K}_0^n$ or $C_v \in \mathcal{K}_0^{2n}$, we call $C$ a 0-symmetric adelic convex body.

For $(x_v)_v \in K^n_v$, we define the scalar multiple $(y_v)_v = \lambda(x_v)_v$ for $\lambda \in \mathbb{R}^+$ by

$$y_v := \begin{cases} x_v & \text{if } v \nmid \infty, \\ \lambda x_v & \text{if } v \mid \infty. \end{cases}$$

Denote by $\sigma_i$, $1 \leq i \leq r$, the embeddings of $K$ into $\mathbb{R}$ and by $\sigma_{r+i} = \tau_{r+i}$, $1 \leq i \leq s$, the pairs of embeddings of $K$ into $\mathbb{C}$. If $s = 0$, then $K$ is called a \textit{totally real} field. For instance, $\mathbb{Q}[\sqrt{2}]$ is totally real, but not $\mathbb{Q}[\sqrt{2}]$.

Let $\tau$ denote complex conjugation in $\mathbb{C}$, cf. \cite{2}. Then

$$\iota: x \mapsto (\sigma_1(x), \ldots, \sigma_r(x), \sigma_{r+1}(x), \ldots, \sigma_{r+s}(x))$$

are embeddings of $K$ into $K_\infty = \prod_{v \mid \infty} K_v$. There is a canonical isomorphism $\rho: K_\infty \to \mathbb{R}^d$ with

$$\rho(x_1, \ldots, x_r, x_{r+1}, \ldots, x_{r+s}) = (x_1, \ldots, x_r, \Re(x_{r+1}), \Im(x_{r+1}), \ldots, \Re(x_{r+s}), \Im(x_{r+s})). \quad (2.2)$$

Here $\Re$ and $\Im$ denote real and imaginary parts, respectively.

Together we get a map $(\rho \circ \iota): K \to \mathbb{R}^d$, that sends a field element to the vector whose entries are the images under the real and complex embeddings, splitting the latter points into real and imaginary part,

$$x \mapsto (\sigma_1(x), \ldots, \sigma_r(x), \Re(\sigma_{r+1}(x)), \Im(\sigma_{r+1}(x)), \ldots, \Re(\sigma_{r+s}(x)), \Im(\sigma_{r+s}(x))).$$

In the rank-$n$-case let $K^n_\infty = \prod_{v \mid \infty} K^n_v$,

$$x^n := (\sigma^n_1, \ldots, \sigma^n_r, \sigma^n_{r+1}, \ldots, \sigma^n_{r+s}): K^n \to K^n_\infty,$$

and $\tau^n$ analogously, where the $\sigma_i$ act component-wise. Similarly $\rho^n: K^n_\infty \to \mathbb{R}^{nd}$. To simplify notation, we usually write $\rho$ and $\iota$ instead of $\rho^n$ and $\iota^n$.

Throughout the paper, we use the following notation

$$C_\infty = \prod_{v \mid \infty} C_v \quad \text{and} \quad \mathfrak{M} = \bigcap_{v \mid \infty} (C_v \cap K^n). \quad (2.3)$$

Observe that our standard embedding $\rho \circ \iota: K^n \to \mathbb{R}^{nd}$, cf. (2.2) and thereafter, is injective, and therefore

$$|C \cap K^n| = \left|\rho\left(\prod_{v \mid \infty} C_v \right) \cap \rho(\mathfrak{M})\right|. \quad (2.4)$$

We will use this important connection for some of the proofs below.

### 3. Adelic Polytopes

We start by giving local definitions of \textit{convex hull} of points $\overline{\pi}_0, \ldots, \overline{\pi}_m \in K^n_\infty$, where $\overline{\pi}_{k,v}$ is the $v$-entry of $\overline{\pi}_k$ for $v \in M(K)$. To exclude degenerate cases, we always require that for all $v \in M(K)$ we have

$$\text{lin}_{K_v} \{ \overline{\pi}_{k,v} \mid 0 \leq k \leq m \} = K^n_v,$$

and for all but finitely many $v \nmid \infty$, the entries of $\overline{\pi}_{k,v}$ are in $\mathcal{O}_v^n$, where, as usual, $\mathcal{O}_v^n$ denotes the group of units in $\mathcal{O}_v$.

For $v \mid \infty$ define the module

$$C_v = \text{conv}_v \{ \overline{\pi}_{0,v}, \ldots, \overline{\pi}_{m,v} \} = \mathcal{O}_v \overline{\pi}_{0,v} + \ldots + \mathcal{O}_v \overline{\pi}_{m,v}. \quad (3.1)$$
Note that $C_v$ is an $\mathcal{O}_{v'}$-module in $K_v^n$ of full rank. In fact, we have $C_v = \text{lin}_{\mathcal{O}_{v'}}(\overline{a}_{0,v}, \ldots, \overline{a}_{m,v})$, the minimal $\mathcal{O}_{v'}$-module in $K_v^n$ containing all the points. Since the $K_v$ are local fields, there exist $A_v \in \text{GL}_n(K_v)$, such that $C_v = A_v \mathcal{O}_{v'}^n$. Note that in general for $t \in K^n$

$$\text{conv}_v \{ \overline{a}_{0,v} + t, \ldots, \overline{a}_{m,v} + t \} \neq \text{conv}_v \{ \overline{a}_{0,v}, \ldots, \overline{a}_{m,v} \} + t,$$

but if $t \in \{-\overline{a}_{0,v}, \ldots, -\overline{a}_{m,v}\}$, we certainly have

$$\text{conv}_v \{ \overline{a}_{0,v} + t, \ldots, \overline{a}_{m,v} + t \} \subseteq \text{conv}_v \{ \overline{a}_{0,v}, \ldots, \overline{a}_{m,v} \},$$

as the points on the left are contained in the $\mathbb{Z}$-span of the points on the right.

For $v | \infty$ real, let

$$C_v = \text{conv}_v \{ \overline{a}_{0,v}, \ldots, \overline{a}_{m,v} \}$$

(3.3)

and

$$C_0^v = \text{conv}_v \{ \pm \overline{a}_{0,v}, \ldots, \pm \overline{a}_{m,v} \}$$

(3.4)

These are the standard convex hull of points in real space and its symmetric variant. They are equivalent to defining the bodies as the intersection of all (symmetric) convex bodies containing the points $\overline{a}_{0,v}, \ldots, \overline{a}_{m,v}$.

For $v | \infty$ complex, we only define the symmetric body

$$C_0^v = \text{conv}_v \{ \overline{a}_{0,v}, \ldots, \overline{a}_{m,v} \}$$

(3.5)

$$= \left\{ \sum_{i=0}^m \lambda_i \overline{a}_{i,v} \left| \lambda_i \in \mathbb{C}, 0 \leq |\lambda_i| \leq 1, \sum_{i=0}^m |\lambda_i| = 1 \right\} \subseteq \mathbb{C}^n.$$

By construction, this is the intersection of all symmetric convex bodies in complex space containing the points $\overline{a}_{0,v}, \ldots, \overline{a}_{m,v}$. We are not aware of any more general notion of convex hull in complex spaces. When identifying $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we can use the definitions used in the real case, but the bodies obtained in this way lie in a real (affine) subspace of $\mathbb{R}$-dimension $n$ in $\mathbb{R}^{2n}$ and do thus not define an adelic convex body. This is the reason why we will consider arbitrary adelic polytopes only in the case of totally real fields.

Using our constructions of convex hull, we can now define the following special classes of adelic convex bodies.

**Definition 3.1** (Adelic convex hull, polytopes, simplices and cross-polytopes). Given $\overline{a}_{0,v}, \ldots, \overline{a}_{m,v} \in K_v^n$ as before, we define

$$C_v^0 = \text{conv}_v \{ \overline{a}_{0,v}, \ldots, \overline{a}_{m,v} \} = \prod_{v | \infty} A_v \mathcal{O}_{v'}^n \times \prod_{v | \infty} C_v^0$$

with $A_v$ implicitly defined by (3.1) above and $C_v^0$ as in (3.3) and (3.5), as the symmetric adelic convex hull of $\overline{a}_{0,v}, \ldots, \overline{a}_{m,v}$. If $K$ is totally real, we define the adelic convex hull of $\overline{a}_{0,v}, \ldots, \overline{a}_{m,v}$ as

$$C = \text{conv}_K \{ \overline{a}_{0,v}, \ldots, \overline{a}_{m,v} \} = \prod_{v | \infty} A_v \mathcal{O}_{v'}^n \times \prod_{v | \infty} C_v,$$

where $C_v$ is defined as in (3.3). In case $m = n$, we speak of the adelic cross-polytope and adelic simplex respectively. All of these bodies will also be called adelic.
polytopes and if $C$ is an adelic polytope, it can be written as $C = \prod_{v \in M(K)} C_v$ with local bodies $C_v$ defined as above.

Denote by $\sigma_v : K \to K_v$ the inclusion of $K$ into $K_v$ and by abuse of notation also $\sigma_v : K^n \to K^n_v$ for all places $v$ of $K$.

**Definition 3.2** (Adelic lattice polytopes, simplices, cross-polytopes). Given points $a_0, \ldots, a_m \in K^n$ that span $K^n$, identify

$$\sigma_v(a_k) = (\sigma_v(a_k)|v \in M(K)).$$

Then $C = \text{conv}_K\{a_0, \ldots, a_m\}$ and $C^\circ = \text{conv}^\circ_K\{a_0, \ldots, a_m\}$ are the **adelic lattice polytope** and symmetric **adelic lattice polytope** generated by $a_0, \ldots, a_m$, respectively. The body $C$ is, of course, again only defined for $K$ totally real, and for $m = n$ we call $C$ an **adelic lattice simplex**. If additionally $a_0 = 0$, $C^\circ$ will be called an **adelic lattice cross-polytope**.

**Remark 3.3.** The intersection of two adelic polytopes is again an adelic polytope, since this property holds for all $v$ and two adelic polytopes differ only for finitely many $v$ and for arbitrary sets $X$, $Y$ and $Z$ we have

$$(X \times Y) \cap (X \times Z) = \{ (x, y) | x \in X, y \in Y, y \in Z \} = X \times (Y \cap Z).$$

The intersection of two adelic lattice polytopes, however, is not an adelic lattice polytope in general.

Adelic polytopes are not as nice as their classical real counterparts. In Euclidean space, a polytope $P \in \mathcal{P}^m$ can be written as both the convex hull of a finite number of points or as the intersection of a finite number of closed half-spaces. Given a linear functional $\ell : \mathbb{R}^m \to \mathbb{R}$, the kernel of $\ell$ is a hyperplane $H$ and using the ordering of $\mathbb{R}$, we decide whether two points $x_1, x_2 \in \mathbb{R}^m$ lie on the same side of $H$ by comparing the signs of $\ell(x_1)$ and $\ell(x_2)$ and thereby also defining two half-spaces.

Such a construction is not possible in the adelic setting, as we do not have an ordering on $K$.

Unfortunately, we also cannot expect to have an adelic counterpart to the classical Ehrhart-theory for lattice polytopes. To this end we recall that for a lattice polytope $P = \text{conv}\{v_1, \ldots, v_s\} \in \mathcal{P}^m$, where $v_i$ are lattice points of a rank $m$ lattice $\Lambda$, we know by a theorem of Ehrhart [1] that the number of lattice points in $kP$ for a positive integer $k$ is given by a polynomial of degree $m$, the Ehrhart polynomial

$$|kP \cap \Lambda| = \sum_{i=1}^m G_i(P, \Lambda)k^i, \quad k \in \mathbb{N}. $$

The polynomial is unique and the coefficients depend only on $P$ and $\Lambda$. The behaviour and properties of this polynomial have been studied intensively, see Beck and Robins [1] for an overview as well as e.g. McMullen [20] and Linke [19] for more specific results.

Now consider an adelic lattice polytope $C = \prod_v C_v$, and let $C_\infty$ and $\mathcal{M}$ be as before, cf. (2.3). Then $\rho(C_\infty)$ is a polytope in $\mathbb{R}^{nd}$, as the factors of $C_\infty$ are polytopes. On account of (2.4), the number $|kC \cap K^n|$ has to grow like $k^{nd}$. On the other hand, the body $\rho(C_\infty)$ is in general not a lattice polytope with respect to the lattice $\rho(\nu(\mathcal{M}))$.

**Example 3.4.** Consider for example $K = \mathbb{Q}[\sqrt{2}]$ for $n = 1$ and the body $C = \prod_{v|\infty} \mathcal{O}_v \times [-1,1]^2$, which is an adelic lattice polytope in the sense above, as $C = \text{conv}_K\{ \pm 1 \}$. Observe, that $\mathcal{M} = \mathcal{O}$. Figure 4 shows the embedding of $C$ into real space $\mathbb{R}^{nd}$. It is evident from the figure, that of the four vertices of $\rho([-1,1]^2)$ only
(1, 1) and (−1, −1) are lattice points but not (1, −1) and (−1, 1) and thus $\rho(C_\infty)$ is not a lattice polytope with respect to $\rho(i(O))$. However, the infinite part of any adelic convex body has to be of the form $C_\infty = [-a, a] \times [-b, b]$ for $a, b \in \mathbb{R}$. But since $\rho(i(O))$ is generated by $(1, 1), (−\sqrt{2}, \sqrt{2}) \in \mathbb{R}^2$, no box can have only lattice points as vertices. Thus the image of $C_\infty$ under the embedding into $\mathbb{R}^2$ can not be a lattice polytope, and we can therefore not find an adelic analogue to the Ehrhart polynomial.

Next we deal with the adelic volume of lattice simplices. Let $e_1, \ldots, e_n$ be any basis of $K^n$, then it is known that the volume of the adelic lattice cross-polytope $C^\Diamond = \text{conv}_A \{ e_1, \ldots, e_n \}$ is (cf, e.g., [5])

$$\text{vol}_A(C^\Diamond) = \frac{2^{dn} \pi^{sn}}{(n!^d (!2n)!^s)}.$$  (3.6)

If $K$ is totally real and $S = \text{conv}_A \{ 0, e_1, \ldots, e_n \} = \prod_{v \in M(K)} S_v$ is an adelic lattice simplex, then for the local simplices $S_v$ at the infinite places we have

$$\text{vol}_v(S_v) = \frac{1}{n!} \quad \text{for } v \mid \infty \text{ real.}$$  (3.7)

**Lemma 3.5.** Let $K$ be a totally real number field of degree $d = [K : \mathbb{Q}]$ and let $S = \text{conv}_A \{ a_0, \ldots, a_n \}$ be an adelic lattice simplex. Then

$$\text{vol}_A(S) \geq \frac{1}{(n!)^d}.$$  

**Proof.** On account of (3.6), we may assume w.l.o.g. $a_0 = 0$, possibly switching to a subset at some finite places. Let $A = (a_1 \ldots a_n)$ be the matrix whose columns are $a_1, \ldots, a_n$, and let $S = \prod_{v \in M(K)} C_v$.

For $v \mid \infty$ we get from (3.7) that

$$\text{vol}_v(C_v) = \text{vol}_v(\text{conv}_A \{ \sigma_v(a_1), \ldots, \sigma_v(a_n) \}) = \frac{|\det(\sigma_v(a_1) \ldots \sigma_v(a_n))|_\infty}{n!} = \frac{|\sigma_v(\det(A))|_\infty}{n!}.$$  

On the other hand, for $v \nmid \infty$, we have

$$C_v = \mathcal{O}_v \sigma_v(a_1) + \ldots + \mathcal{O}_v \sigma_v(a_n) = (\sigma_v(a_1) \ldots \sigma_v(a_n)) \mathcal{O}_v^n$$

and thus

$$\text{vol}_v(C_v) = \left| \det(\sigma_v(a_1) \ldots \sigma_v(a_n)) \right|_v^{d_v}.$$  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Embedding of an adelic convex body into real space.}
\end{figure}
Therefore, in view of (2.1) we get
\[ \text{vol}_A(S) = \prod_{v \nmid \infty} \text{vol}_v(C_v) \cdot \prod_{v \mid \infty} \text{vol}_v(C_v) \]
\[ = \prod_{v \mid \infty} |\det(A)|^d_v \cdot \frac{1}{(n!)^d_v} \prod_{v \nmid \infty} |\det(A)|^{1/d_v} \]
\[ = \frac{1}{(n!)^d} \cdot 1. \]
\[ = \frac{1}{(n!)^d} \cdot 1. \]

4. ADELC TRANGULATIONS

Throughout this section we assume that \( K \) is totally real, i.e., \( K_v = \mathbb{R} \) for all \( v \nmid \infty \), and we start with the proof of Theorem 1.1.

Proof of Theorem 1.1. Let \( C \cap K^n = \{ a_1, \ldots, a_{n+m} \} \). According to our assumption \( \dim_K(C \cap K^n) = n \) we have \( m \geq 1 \), and let \( P = \text{conv}_A\{ a_1, \ldots, a_{n+m} \} \). It suffices to prove the theorem for \( P \). To this end fix an embedding \( v : K \to \mathbb{R} \).

Then \( \tilde{P} = \text{conv}_A\{ a_1, \ldots, a_{n+m} \} \subset \mathbb{R}^n \) is a polytope and there exists a triangulation \( T_1, \ldots, T_k \) of \( \tilde{P} \) with \( k \geq m \) full-dimensional simplices, whose vertices are among the \( \tilde{v}(a_i) \), see e.g. Section 2.2 of the book [6], to which we refer also for more details on triangulations. An element \( T_j \) of this triangulation, i.e. an \( (n+1) \)-element set from the \( n+m \) points, gives rise to an adelic simplex
\[ S_j = \text{conv}_A \{ a_i \mid i \in T_j \} = \text{conv}_A \{ a_i \mid \tilde{v}(a_i) \text{ is a vertex of } T_j \} \]
with \( S_{j,v} = T_j \). Since the triangulation fulfills \( \dim(T_j \cap T_i) < n \) for \( j \neq i \), we get \( \text{vol}_v(S_{j,v} \cap S_{i,v}) = 0 \) and thus \( \text{vol}_v(S_j \cap S_i) = 0 \) for \( i \neq j \). On the other hand,
\[ P = \text{conv}_A\{ a_1, \ldots, a_{n+m} \} \supset S_1 \cup \ldots \cup S_k. \]
Hence by Lemma 3.3 we conclude
\[ \text{vol}_A(P) \geq k \cdot \frac{1}{(n!)^d} \geq \frac{m}{(n!)^d}. \] (4.1)
\[ \square \]

Remark 4.1. The construction in the proof above, however, does not give a full triangulation of \( P \), since in general
\[ S_1 \cup \ldots \cup S_k \nsubseteq P, \]
see Examples 4.2 and 4.3 below. Example 4.3 does also show that the dependence on \( m \) can not be improved in general, whereas a minimal triangulation of \( \tilde{P} \) does not necessarily give rise to a minimal set of adelic simplices, as Example 4.2 shows.

Example 4.2. Let \( K = \mathbb{Q}[\sqrt{2}] \) and \( n = 2 \). Let \( P \) be the adelic convex hull of
\[ a = (\sqrt{2}, 1), \quad b = (1, 3), \quad c = (2, 3) \quad \text{and} \quad d = (1, \sqrt{2}) \in K^2. \]
Then for \( v \nmid \infty \) we get \( P_v = O_2^2 \) and the two convex bodies at the infinite places \( v_1 \) and \( v_2 \) with corresponding real embeddings \( \sigma_1 \) and \( \sigma_2 \) are \( P_{v_1} \) and \( P_{v_2} \) as depicted in the figure.
The adelic simplices

\[ S_1 = \text{conv}_A \{ a, b, c \} \]
\[ S_3 = \text{conv}_A \{ a, c, d \} \]
\[ S_2 = \text{conv}_A \{ a, b, d \} \]
\[ S_4 = \text{conv}_A \{ b, c, d \} \]

all satisfy \( S_j, v = O^2_v \) for \( v \mid \infty \) and \( \text{vol}(S_j \cap S_i) = 0 \) for all \( j \neq i \) since the intersection at one infinite place is always lower-dimensional. Therefore (4.1) is not optimal in this case as \( P \) contains four disjoint simplices, even though the indicated triangulations of \( P_{v_1} \) and \( P_{v_2} \) are minimal.

The point \( z = (z_1, z_2) \) indicated in the picture is not contained in any \( S_j \).

Example 4.3. Consider again \( K = \mathbb{Q}[\sqrt{2}] \) and let \( P \) be the adelic convex hull of

\[ a = (1, 1), \quad b = (2, 1), \quad c = (2, 2) \quad \text{and} \quad d = (1, 2) \in K^2. \]

Then for \( v \nmid \infty \) we get \( P_v = O^2_v \) and the two convex bodies at the infinite places \( v_1 \) and \( v_2 \) with corresponding real embeddings \( \sigma_1 \) and \( \sigma_2 \) are

As before, for an adelic simplex \( S \) with vertices from \( a, b, c, d \) it still holds \( S_v = O^2_v \) for \( v \nmid \infty \). Any selection of more than two simplices will contain a pair whose infinite parts have non-trivial intersection, thus (4.1) is best possible. Again, the point \( z = (z_1, z_2) \) indicated in the picture is not contained in any of the four adelic simplices.

5. The symmetric case

Blichfeldt’s inequality has recently been improved for symmetric \( C \in K^m_0 \) by Henze [17 (2.4)]. He proved that for \( C \in K^m_0 \) and \( \Lambda \in \mathcal{L}^m \) with \( \dim_{\mathbb{R}}(C \cap \Lambda) \geq m \) it holds

\[ |C \cap \Lambda| \leq \frac{m!}{2^m} L_m(2) \frac{\text{vol}_m(C)}{\det \Lambda}, \quad (5.1) \]

where \( L_m \) is the \( m \)-th Laguerre polynomial, \( L_m(x) = \sum_{k=0}^{m} \frac{m!}{k!} \frac{x^k}{k!} \). The bound is asymptotically sharp for certain cross-polytopes. It was also pointed out by Henze that for any \( 0 < \epsilon < 1 \) and \( m = m(\epsilon) \) large enough we have \( L_m(2) / 2^m \leq 1 / (2 - \epsilon)^m \). Hence (5.1) is an exponential improvement on Blichfeldt’s inequality for symmetric bodies.

In this section, we show an adelic version of Henze’s inequality, again for an arbitrary number field \( K \).
Proposition 5.1. Let $K$ be an algebraic number field of degree $d$ and let $C$ be a symmetric adelic convex body with $\dim_{\mathbb{Q}}(C \cap K^n) = nd$. Then
\[
|C \cap K^n| \leq \frac{(nd)!}{2^{nd}} L_{nd}(2) \frac{\text{vol}_h(C)}{(\sqrt{|\Delta_K|})^n}.
\]

Proof. We use the embedding $\rho \circ \iota : K^n \to \mathbb{R}^{nd}$ and $[2.4]$. By Henze's Blichfeldt-type inequality (5.1), with
\[
\dim_{\mathbb{Q}}(C \cap K^n) = \dim_{\mathbb{R}}(\rho(C_{\infty}) \cap \rho(\iota(\mathcal{M}))) \geq nd,
\]
we get
\[
|\rho(\iota(\mathcal{M})) \cap \rho(C_{\infty})| \leq \frac{(nd)!}{2^{nd} L_{nd}(2) \det(\rho(\iota(\mathcal{M})))^n} \frac{\text{vol}_h(C)}{(\sqrt{|\Delta_K|})^n}.
\]

Due to the embedding argument into $\mathbb{R}^{nd}$ we have to assume that $\dim_{\mathbb{Q}}(C \cap K^n) = nd$. A more adelic version, i.e., only with the assumption $\dim_{K}(C \cap K^n) = n$, and with a better bound for large degrees was proved by Gaudron, [13, p. 173], using the language of heights and vector bundles. He showed that for a symmetric adelic convex body $C$ with $\dim_{K}(C \cap K^n) = n$ it holds
\[
|C \cap K^n| < (5n)^n \text{vol}_h(C).
\]

For arbitrary, i.e., not necessarily 0-symmetric, adelic convex bodies $C$ we are not aware of any results except our Theorem 1.1 in the case of totally real fields and an “embedded version” of Blichfeldt’s inequality (1.1)
\[
|C \cap K^n| \leq (nd)! \frac{\text{vol}_h(C)}{(\sqrt{|\Delta_K|})^n} + nd,
\]
which can be proved analogously to Proposition 5.1. Here we have to assume, again, $\dim_{\mathbb{Q}}(C \cap K^n) = nd$.

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