Rational approximations of $f(R)$ cosmography through Padé polynomials

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Abstract. We consider high-redshift $f(R)$ cosmography adopting the technique of polynomial reconstruction. In lieu of considering Taylor treatments, which turn out to be non-predictive as soon as $z > 1$, we take into account the Padé rational approximations which consist in performing expansions converging at high redshift domains. Particularly, our strategy is to reconstruct $f(z)$ functions first, assuming the Ricci scalar to be invertible with respect to the redshift $z$. Having the so-obtained $f(z)$ functions, we invert them and we easily obtain the corresponding $f(R)$ terms. We minimize error propagation, assuming no errors upon redshift data. The treatment we follow naturally leads to evaluating curvature pressure, density and equation of state, characterizing the universe evolution at redshift much higher than standard cosmographic approaches. We therefore match these outcomes with small redshift constraints got by framing the $f(R)$ cosmology through Taylor series around $z \simeq 0$. This gives rise to a calibration procedure with small redshift that enables the definitions of polynomial approximations up to $z \simeq 10$. Last but not least, we show discrepancies with the standard cosmological model which go towards an extension of the ΛCDM paradigm, indicating an effective dark energy term evolving in time. We finally describe the evolution of our effective dark energy term by means of basic techniques of data mining.

Keywords: dark energy theory, modified gravity, cosmological parameters from LSS, cosmological simulations

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1 Introduction

According to the current observations, the universe started accelerating at a given time and it is today dominated by an exotic accelerating component, called dark energy \([1–5]\). Attempts to determine either dark energy nature or its micro-physics have been so far unsuccessful. Essentially, the standard cosmological puzzle suggests a constant dark energy modeled through a cosmological constant \(\Lambda\) \([6–12]\). The \(\Lambda\) origin is supposed to come from quantum fluctuations at the very beginning of the universe’s evolution. Although attractive and straightforward, the corresponding model, dubbed the \(\Lambda\)CDM paradigm, does not predict cosmological constant’s magnitude in agreement with quantum field theory. Furthermore, the magnitude of \(\Lambda\) is even surprisingly close to the matter value today, indicating an unexpected coincidence between matter and dark energy \([18–20]\). These problems, together with other astronomical indications, suggest that dark energy may slightly evolve in terms of the cosmic redshift \(z\), leading to a negative pressure which becomes dominant over standard gravity after a precise time.

Instead of modifying the net equation of state (EoS) with \textit{ad hoc} assumptions on the form of dark energy, plausible modifications of Einstein’s gravity have been proposed as energy scales increase. Under this hypothesis, one can postulate modified gravitational theories aiming to describe the dark energy effects by means of \textit{first principles}. Among all, the Hilbert-Einstein action can be generalized by replacing the Ricci scalar, \(R\), with generic \(f(R)\) functions \([21–25]\). The great disadvantage of this approach is that the function \(f(R)\) is not known \textit{a priori}. Approaches towards the determinations of \(f(R)\) have been severely discussed \([26–30]\), with particular regards to matching cosmic data with the reconstructed \(f(R)\). Unfortunately, expanding \(f(R)\) in Taylor series as \(R\) tends to its observable late time value does not seem enough to describe either dark energy or dark matter at different scales. This caveat is a consequence of the expansion itself which is performed in a short range of redshifts only.
In this paper, we thus propose the use of rational approximations to reconstruct the form of $f(R)$ using constraints coming from background cosmology. We motivate such a scheme because rational approximants are in general capable of expanding the range of redshifts which are not covered by Taylor series. Hence, physically speaking the use of rational approximations overcomes bad convergence issues at high redshift domains. We here employ the Padé rational approximants which are reaching great interest during the last times \[31\]. The Padé approximations represent a treatment which is here developed to get a class of $f(R)$ models compatible with cosmic data at higher redshifts.

To figure this out, we consider the function $f(z)$ which corresponds to the function $f(R)$, with the recipe $R = R(z)$. We thus frame the numerical evolution of modified Friedmann equations and we get the numerical behaviour of $f(z)$. Since $R$ is a function of the Hubble rate, we expand it either in Padé series or in Taylor expansions and we even compare the outcomes coming from the two different approaches. As Padé rational orders\(^1\) we consider the ones which turn out to be more compatible with cosmic data. In particular, to select the orders, we consider the ones which reduce error bars in numerical analyses which make use of cosmological surveys. To guarantee that the Padé rational approximations have been well-constructed we take cosmological constraints coming from cosmography. In such a way, we fix bounds over the numerical $f(z)$ which are compatible with state-of-the-art cosmology at late times. This procedure represents a calibration of our high-redshift approximations with at lower redshifts. Once reconstructed the form of $f(z)$, we turn back to $f(R)$ and we define the corresponding cosmology associated to the modified Friedmann equations. Moreover, we find through simple tools of statistical data mining how to better formalize the effective dark energy evolution at redshifts $z \geq 1$. To this end, we reproduce the effective dark energy and its properties, showing slight departures with respect to the standard $\Lambda$CDM model.

The structure of the paper is the following: in section 2, we introduce the $f(R)$ theories of gravity and the modified Friedmann equation based on the assumption of the cosmological principle. In section 3, we review the method of the Padé polynomials upon which we will build our analysis. In section 4, we show how to reconstruct the form of $f(R)$ in a model-independent way by means of the cosmographic method. In section 5, we study the cosmological properties of the obtained $f(R)$ model. In section 6, we compare the differences between the standard Taylor approach and the Padé approximation. We also describe the evolution of our effective dark energy term with its implications in the redshift domain that we considered. Finally, in section 7 we draw conclusions and perspectives of our formalism.

2 $f(R)$ gravity

The standard gravitational Lagrangian makes use of the first order invariant Ricci scalar. Replacing the Lagrangian with a generic function of $R$, i.e. $f(R)$, implies that the gravitational action leads to a new class of models, by means of: \[32–38\]:

$$S = \int d^4x \sqrt{-g} \left[ f(R) + \mathcal{L}_m \right],$$

(2.1)

where $g$ is the metric determinant, whereas $\mathcal{L}_m$ represents the matter Lagrangian.\(^2\) Varying the action with respect to the metric, we obtain the field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{(\text{curv})}_{\mu\nu} + T^{(m)}_{\mu\nu},$$

(2.2)

\(^1\)As it will be clarified later, Padé series employs two orders instead than one, as Taylor treatments do.

\(^2\)Here, we use units such that $8\pi G = 1 = c$. 


in which \( T^{(m)}_{\mu\nu} \) is the energy-momentum tensor of matter and

\[
T^{(\text{curv})}_{\mu\nu} = \frac{1}{f'} \left[ \frac{1}{2} g_{\mu\nu} (f - R f') + (g_{\alpha\mu} g_{\beta\nu} - g_{\mu\nu} g_{\alpha\beta}) \nabla^\alpha f' \right] \tag{2.3}
\]

refers to as an effective curvature energy momentum tensor. The former can be interpreted as a sort of curvature fluid characterised by a density

\[
\rho_{\text{curv}} = \frac{1}{f'} \left[ \frac{1}{2} (f - R f') - 3 H \dot{R} f'' \right], \tag{2.4}
\]

and a pressure

\[
p_{\text{curv}} = \frac{1}{f'} \left[ 2 H \dot{R} f'' + \ddot{R} f'' + R^2 f''' - \frac{1}{2} (f - R f') \right]. \tag{2.5}
\]

Throughout the text, we use the convention to denote with ‘\( \dot{\cdot} \)’ the derivative with respect to \( R \).

According to the cosmological principle, we assume the spatially flat, homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) metric \([2,12]\). We thus have \( ds^2 = dt^2 - a(t)^2 \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \), which implies the constraint over the Ricci scalar and the Hubble parameter:

\[
R = -6 (\dot{H} + 2 H^2). \tag{2.6}
\]

Further, assuming standard matter, i.e. composed by baryons and cold dark matter, with the ansatz that the corresponding EoS is pressureless, we can write

\[
H^2 = \frac{1}{3} \left[ \frac{1}{f'} \rho_m + \rho_{\text{curv}} \right]. \tag{2.7}
\]

In the Jordan frame, where matter and curvature terms are uncoupled, the continuity equation for the matter fields reads \( \dot{\rho}_m + 3H \rho_m = 0 \), which preserves the standard behaviour \( \rho_m = \rho_{m0} a^{-3} = 3H_0^2 \Omega_{m0} (1 + z)^3 \), having \( \Omega_{m0} \) the value of the matter density today. On the other hand, the continuity equation for the curvature can be written as

\[
\dot{\rho}_{\text{curv}} + 3 H (1 + w_{\text{curv}}) \rho_{\text{curv}} = 3H_0^2 \Omega_{m0} (1 + z)^3 \frac{\dot{R} f''}{(f')^2}, \tag{2.8}
\]

leading to the curvature EoS:

\[
w_{DE} = \frac{p_{\text{curv}}}{\rho_{\text{curv}}} = -1 + \frac{\dot{R} f'' + \ddot{R} f'' - H \dot{R} f''}{(f - R f')/2 - 3 H R f''}, \tag{2.9}
\]

which can be supposed to fuel the effective dark energy fluid associated to the curvature.

In this framework, the most relevant caveat remains the difficulty of finding out explicit forms of \( f(R) \). The function \( f(R) \) is indeed unknown \textit{a priori}. Since different forms of \( f(R) \) lead to different cosmological scenarios, it follows that its determination and reconstruction become essential in order to define the correct gravitational theory underlying the universe dynamics \([39]\). One of the most consolidate approach towards reformulating the form of \( f(R) \) is to take data and to frame the \( f(R) \) dynamics by means of a back scattering procedure which defines the form of \( f(R) \) directly with observations. Unfortunately, Taylor treatments have been so far unsuccessful to determine high-redshift constraints over \( f(R) \), spanning from solutions which are either inaccurate as \( z \geq 1 \) or non-univocal. In the next section, we propose to adopt the different expansion due to the Padé approximations in order to alleviate the aforementioned problems.
3 The method of Padé approximants

To overcome the problem of convergence over \( f(R) \) Taylor expansions, one can adopt the alternative strategy of expanding series through the use of rational approximants. In this section we present the method of the Padé approximants \([40]\), which is built up from the standard Taylor definition and allows one to reduce divergences at higher redshift domains.

In particular, we can express a generic function \( f(z) \) as a power series

\[
  f(z) = \sum_{i=0}^{\infty} c_i z^i, \tag{3.1}
\]

for a given set of coefficients \( c_i \). We thus define a \((N, M)\) Padé approximant as the ratio

\[
  P_{NM}(z) = \frac{\sum_{n=0}^{N} a_n z^n}{1 + \sum_{m=1}^{M} b_m z^m}, \tag{3.2}
\]

whose Taylor expansion agrees with eq. (3.1) to the highest possible order, i.e.

\[
  P_{NM}(0) = f(0), \tag{3.3}
\]

\[
  P'_{NM}(0) = f'(0), \tag{3.4}
\]

\[
  \vdots \tag{3.5}
\]

\[
  P_{NM}^{(N+M)}(0) = f^{(N+M)}(0). \tag{3.6}
\]

The \( N + 1 \) independent coefficients in the numerator and \( M \) independent coefficients in the denominator of eq. (3.2) make \( N + M + 1 \) the number of total unknown terms. Hence, we simply can write

\[
  \sum_{i=0}^{\infty} c_i z^i = \frac{\sum_{n=0}^{N} a_n z^n}{1 + \sum_{m=1}^{M} b_m z^m} + \mathcal{O}(z^{N+M+1}), \tag{3.7}
\]

and, then,

\[
  (1 + b_1 z + \ldots + b_M z^M)(c_0 + c_1 z + \ldots) = a_0 + a_1 z + \ldots + a_N z^N + \mathcal{O}(z^{N+M+1}). \tag{3.8}
\]

Equating the terms with the same power coefficients, one obtains a set of \( N + M + 1 \) equations for the \( N+M+1 \) unknown terms \( a_i \) and \( b_i \). Depending on the way in which the approximation is built up, for \( z \gg 1 \), one can use those rational functions as viable candidates to overcome the problems of Taylor series expansions, when one handles high-\( z \) data.

The advantage of Padé rational approximations are thus summarized as:

- the series can better approximate situations with bad convergence due to data intervals;
- the series can easily reduce error bias which propagate when data surveys are not inside \( z < 1 \);
- the series can be modeled by choosing appropriate orders which can be chosen depending on each case of interest.

Clearly, the Padé polynomials also suffer from precise issues, among them:
the series convergence is not known a priori, so that the orders should be found by
directly comparing with data;

- the series can have poles inside the observational domain and this can limit the con-
vergence of the code used to implement data;

- different series can degenerate among them, for the net order of Padé series is deter-
mined by the sum between the numerator and denominator orders.

In what follows, we are interested in assuming the approach of Padé approximations to
$f(R)$ gravity and in particular, we need to fix as initial settings over the free-coefficients of the
expansions the numerical bounds which can be derived by model-independent measurements
inferred from cosmological data. A relevant technique of model-independent reconstruction is
offered by cosmography. We thus apply the basic demands of cosmography and the technique
of Padé approximation to $f(R)$ gravities.

4 Cosmographic reconstruction of $f(R)$ cosmology

Cosmography is a powerful method that allows us to study the present-time cosmology
without the need of assuming a specific model to describe the dynamical evolution of the
universe. The cosmographic method lies only on the validity of the cosmological principle.
Indeed, this model-independent technique does not depend on the solution of the cosmic
equations [13–17]. The standard procedure is to expand the scale factor $a \equiv 1/(1 + z)$ into
a Taylor series around the present cosmic time $t_0$:

$$a(t) = 1 + \sum_{k=1}^{\infty} \left[ \frac{d^k a}{dt^k} \right]_{t=t_0} (t - t_0)^k.$$  \hspace{1cm} (4.1)

Using eq. (4.1), it is possible to define the so-called cosmographic parameters, which represent
model-independent quantities that can be directly constrained by observations [41–48]:

$$H \equiv \frac{1}{a} \frac{da}{dt}, \quad q \equiv -\frac{1}{aH^2} \frac{d^2 a}{dt^2}$$ \hspace{1cm} (4.2)

$$j \equiv \frac{1}{aH^3} \frac{d^3 a}{dt^3}, \quad s \equiv \frac{1}{aH^4} \frac{d^4 a}{dt^4}.$$ \hspace{1cm} (4.3)

These quantities are known as Hubble rate, deceleration parameter, jerk and snap parameters,
respectively [42, 49–55]. From the definition of the luminosity distance

$$d_L(z) = (1 + z) \int_0^z \frac{dz'}{H(z')},$$ \hspace{1cm} (4.4)

one can use eq. (4.1) to obtain the Taylor expansion of the Hubble rate in terms of the
cosmographic parameters [56]:

$$H(z) = H_0 \left( 1 + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \left. \frac{d^\ell H}{dz^\ell} \right|_{z=0} z^\ell \right),$$ \hspace{1cm} (4.5)
the first three orders being
\[
H_z|_{z=0} = 1 + q_0, \\
H_{zz}|_{z=0} = j_0 - q_0^2, \\
H_{zzz}|_{z=0} = \frac{1}{6} (j_0(3 + 4q_0) - 3q_0(1 + q_0) + s_0).
\] (4.6)

Here, the subscripts ‘\(z\)’ denote the derivatives with respect to the redshift. The impossibility to consider the infinite number of terms of the Taylor polynomials, which would reproduce exactly the real function, leads to truncate the series at some finite order, which is a source of errors in the analysis. Moreover, the Taylor series converges only for small \(z\) and it may be inaccurate for analysing high-redshift data. A possible solution to the convergence problem is to consider the \(p\)adé approximants. Thus, motivated by the studies done in [57], we decide to consider the \((2,1)\) Padé approximant of the Hubble rate:
\[
H_{21}(z) = 2H_0(1+z)^2 \left(3 + z + j_0z - q_0(3 + z + 3q_0z)\right)^2 \\
\times \left[18(q_0 - 1)^2 + 6(q_0 - 1)(-5 - 2j_0 + q_0(8 + 3q_0))z \\
+ \left(14 + 2j_0^2 + j_0(7 - q_0(10 + 9q_0)) + q_0(-40 + q_0(17 + 9q_0(2 + q_0)))\right)z^2\right]^{-1}.
\] (4.7)

We will be comparing the results obtained using eq. (4.7) with the ones deriving from the correspondent standard third-order Taylor expansion for \(H(z)\):
\[
H(z) \simeq H_0 \left[1 + z(1 + q_0) + \frac{z^2}{2}(j_0 - q_0^2) + \frac{z^3}{6} \left(-3q_0^2 - 3q_0^3 + j_0(3 + 4q_0) + s_0\right)\right].
\] (4.8)

It is worth noting that the \(H_{21}(z)\) contains the cosmographic parameters up to the jerk, while in the case of third-order Taylor approximation also the snap is present.

A common procedure for studying \(f(R)\) gravity is to assume a specific \(f(R)\) function and solve the modified Friedmann equation to obtain \(H(z)\). However, this method relies on the \textit{a priori} choice of \(f(R)\) and, thus, on the assumption of the cosmological model. Here, we show a method that allows us to reconstruct the function \(f(R(z)) = f(z)\) in a model-independent way. Cosmography can be used in the framework of \(f(R)\) gravity and in general in the field of any modified theory for several reasons. In fact, once the definitions of the cosmographic parameters are provided, it is possible to relate the form of \(f(R)\) in terms of this set by inverting the modified Friedmann equations. This process is \textit{exact}, i.e. does not need the Taylor approximation of \(a(t)\). In such a way, once the cosmographic series is known by direct comparison with data, in principle one can go further with \(f(R)\) to get bounds over \(f(R)\) and its derivatives. Moreover, it could be also possible to frame the \(f(R)\) evolution if the cosmographic series was known at all redshifts \(z\). Unfortunately, in the framework of \(f(R)\) gravity, inverting the Friedmann equation is only possible numerically. Moreover, the present status of cosmography defines today only background results,\(^3\) i.e. bounded at \(z \simeq 0\). Applications of cosmography to get constraints over \(f(R)\) and \(f(T)\) have been properly investigated in the literature, e.g. for example [58–60]. In all these approaches, one recovers the above motivations. The authors considered cosmography as a way to rewrite

\(^3\)A high redshift cosmography would be a future goal to understand how the universe evolves in a model-independent way.
cosmic quantities of interest and to enable a much quicker inversion of each term entering the modified Friedmann equations. In our work, we consider the Padé expansions to guarantee convergence over the cosmographic set at higher redshift, i.e. to enable the cosmographic predictions in a redshift domain much larger than the standard one predicted by Taylor series.

Our strategy consists of combining eq. (2.7) and eq. (2.4) and using eq. (2.6), once the values of the cosmographic parameters are known. To do this, we need to express the derivatives with respect to time and with respect to $R$ as derivatives with respect to $z$. Being $F(z)$ an arbitrary function depending on the redshift, one has

$$
\frac{dF}{dt} = -(1+z)HF_z,
$$

$$
\frac{\partial F}{\partial R} = \frac{1}{6}(1+z)H_z^2 + H(-3H_z + (1+z)H_{zz})^{-1}F_z.
$$

This leads to a second-order differential equation for $f(z)$:

$$
H^2f_z = \left[-(1+z)H_z^2 + H(3H_z - (1+z)H_{zz})\right] \left[-6H_0^2(1+z)^3\Omega_m - f - \frac{Hf_z(2H-(1+z)H_z)}{(1+z)H_z^2 + H(-3H_z + (1+z)H_{zz})}\right]
$$

$$
- \frac{(1+z)H^2}{[(1+z)H_z^2 + H(-3H_z + (1+z)H_{zz})]^2} \left[f_{zz}((1+z)H_z^2 + H(-3H_z + (1+z)H_{zz})) + f_z(2H_z^2 - 3(1+z)H_zH_{zz} + H(2H_z - (1+z)H_{zz}))\right].
$$

Eq. (4.11) requires two initial conditions over $f$ and $f_z$ to be solved. These can be obtained by means of eq. (2.4), (2.5) and (2.7) evaluated at the present time, together with the condition $f'(R_0) = 1$ which guaranties that the effective gravitational constant of the theory, $G_{\text{eff}} = G_N/f'(R)$, matches the Newtonian constant $G_N$ today. One, thus, finds:

$$
f_0 = R_0 + 6H_0^2(\Omega_m - 1),
$$

$$
f_z\big|_{z=0} = R_z\big|_{z=0}.
$$

Throughout our analysis, we fix $\Omega_m = 0.3$. As far as the cosmographic parameters are concerned, we use the results found in [57]. For the (2,1) Padé approximant, these read

$$
\begin{align*}
    h &= 0.7064^{\pm 0.0277}_{-0.0263}, \\
    q_0 &= -0.4712^{\pm 0.1224}_{-0.1106}, \\
    j_0 &= 0.593^{\pm 0.216}_{-0.210},
\end{align*}
$$

while, in the case of the third-order Taylor expansion, we have

$$
\begin{align*}
    h &= 0.7253^{\pm 0.0353}_{-0.0351}, \\
    q_0 &= -0.6642^{\pm 0.2050}_{-0.1963}, \\
    j_0 &= 1.223^{\pm 0.644}_{-0.664}, \\
    s_0 &= 0.394^{\pm 1.335}_{-0.731},
\end{align*}
$$

where $h \equiv H_0/(100 \text{ km/s/Mpc})$. Plugging eq. (4.7) into eq. (4.11), we are able to reconstruct $f(z)$ numerically. Due to the negative sign of $R$ as a consequence of the metric signature we have adopted, our reconstructed $f(R)$ function will be negative. This, in turns, implies that $f(z)$ must to be negative, which is consistent with choosing the upper bound values of eq. (4.14). Figure 1 shows the numerical results for the (2,1) Padé approximant.
Figure 1. Numerical reconstruction of $|f(z)|$ using the (2,1) Padé approximant.

The following test-functions have been considered in order to find an analytical form of $f(z)$ that matches the numerical results:

**Exponential**

\[
\begin{align*}
    f_1(z) &= A z + B z^3 e^{C z} \\[1pt]
    f_2(z) &= A + B z^2 \sinh(1 + C z) \\[1pt]
    f_3(z) &= A z + B z^3 \cosh(C z) \\[1pt]
    f_4(z) &= A z^2 + B z^4 \tanh(C z)
\end{align*}
\] (4.16)

**Trigonometric**

\[
\begin{align*}
    f_5(z) &= A z^3 + B z^5 \sin(1 + C z) \\[1pt]
    f_6(z) &= A z^3 + B z^4 \cos(1 + C z) \\[1pt]
    f_7(z) &= A z + B z^2 \tan(C z)
\end{align*}
\] (4.16)

**Logarithmic**

\[
\begin{align*}
    f_8(z) &= A z + B z^3 \ln(1 + C z)
\end{align*}
\] (4.16)

where the set of three constants, $A$, $B$ and $C$, includes free parameters determined through a fitting procedure.

### 4.1 Statistics and strategy of data mining

In order to find the best analytical approximation for our models, we perform the $F$-statistics [61], defined by:

\[
F = \frac{(TSS - RSS)/p}{RSS/(n - p - 1)},
\] (4.17)

where

\[
\begin{align*}
    TSS &= \sum_{i=1}^{n} (y_i - \bar{y})^2, \\[1pt]
    RSS &= \sum_{i=1}^{n} (y_i - \hat{y}_i)^2
\end{align*}
\] (4.18)
are the total sum of squares and the residual sum of squares, respectively. Here,

\[ \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i, \quad (4.20) \]

\( y_i \) represents the \( i \)-th observed response value and \( \bar{y}_i \) the \( i \)-th response value predicted by the model, while \( n \) is the number of observations and \( p \) the number of predictors. The \( \mathcal{F} \)-statistics provides a measure of the goodness of a model by testing the joint explanatory power of its predictors. The null hypothesis, i.e. that all the \( p \) regression coefficients are zero and the model has no explanatory power, is tested against the alternative hypothesis that at least one of the regression coefficients is different from zero. The \( \mathcal{F} \)-statistics presents some advantages compared with tests that look for any association between the individual variables and the response, such as \( t \)-statistics and \( p \)-value, or compared with \( R^2 \)-test since it adjusts the number of predictors. In fact, when \( p \) is large, there is a very high chance to observe small \( p \)-values even in absence of any real association between the predictors and the response. Further, the \( R^2 \)-test may be often misleading as \( R^2 \) always increases when more predictors are added to the model, even if those variables are only weakly associated with the response. The \( \mathcal{F} \)-statistics can also be expressed in terms of \( R^2 \) as

\[ \mathcal{F} = \frac{R^2/p}{(1 - R^2)/(n - p - 1)}. \quad (4.21) \]

If the null hypothesis is true, we expect \( R^2 \) and \( \mathcal{F} \) to be close to zero. Thus a high value of the \( \mathcal{F} \)-statistics indicates evidence for the model against the null hypothesis. In our case, \( p = 3 \) and we generate \( n = 1000 \) points from the numerical solution of \( f(z) \). As shown in table 1, the best analytical match to the numerical \( f(z) \) for the Padé approximant results to be

\[ f(z) = \mathcal{A}z + Bz^3e^{Cz}, \quad (4.22) \]

with

\[ (\mathcal{A}, B, C) = (-8.078, -0.530, 0.005). \quad (4.23) \]

\[ \begin{array}{|c|c|c|} \hline \text{Test-function} & (\mathcal{A}, B, C) & \mathcal{F} \times 10^6 \\hline f_1(z) & (-8.078, -0.530, 0.005) & 31.7 \\hline f_2(z) & (-6.147, -2.148, 0.080) & 13.5 \\hline f_3(z) & (-8.046, -0.541, 0.025) & 3.637 \\hline f_4(z) & (-3.699, 0.027, -562.2) & 4.535 \\hline f_5(z) & (-0.708, -0.001, 1.095) & 0.118 \\hline f_6(z) & (-0.717, -0.008, 0) & 0.142 \\hline f_7(z) & (-41.30, 0.002, 1.000) & 0.026 \\hline f_8(z) & (-11.69, -0.208, 1.182) & 1.484 \\hline \end{array} \]

**Table 1.** \( \mathcal{F} \)-statistics on the test-functions eqs. (4.16a)–(4.16h) for the (2,1) Padé approximant.

Figure 2 shows the comparison between the numerical solution of \( f(z) \) and the analytical function eq. (4.22) for the Padé approximant in the redshift domain \( z \in [0, 10] \).
5 Cosmological consequences

To obtain $f(R)$, we need to find $R$ as a function of the redshift $z$ and then to invert it to have $z(R)$, which will be plugged back into eq. (4.22). Unfortunately, eq. (4.7) cannot be inverted analytically, which drives us to use the numerical result. Thus, by means of eq. (2.6), we are able to find $z(R)$ (see figure 3). This can be inserted into eq. (4.22) to finally get $f(R)$ (see figure 4).

5.1 Viability conditions for $f(R)$ models

For a viable explanation to dark energy, $f(R)$ models have to satisfy certain conditions. In the context of the metric formalism, the first condition is

$$f'(R) > 0, \ R \geq R_0$$

if $R_0 > 0$. This condition is required in order to avoid negative values of the effective gravitational constant. Then, the second condition reads

$$f''(R) > 0, \ R \geq R_0$$

Figure 2. Comparison between the numerical reconstruction of $|f(z)|$ and the functional form eq. (4.22) for the (2,1) Padé approximant.

Figure 3. Numerical solution of $z(R)$ for the (2,1) Padé approximant.
if $R_0 > 0$. This arises from the constraints of tests of gravity in the solar system [62, 63], and the consistency with the presence of a standard matter-dominated epoch [64]. Moreover, this condition guarantees the stability of cosmological perturbations [65–67].

To verify whether our model fulfills the above conditions, we display $f'(R)$ and $f''(R)$ in figure 5 and figure 6, respectively. We note that both conditions are satisfied, which ensures the cosmological viability of our model.

Constraints obtained from the Cosmic Microwave Background require that a viable $f(R)$ model approaches to ΛCDM for large curvatures. To ensure this behavior, one expects to fulfill the requirement that $f'(R) \to 1$ for $R \gg 1$ [68–71]. However, figure 5 indicates that $f'(R)$ slightly exceeds unity in the limit of large curvatures. This is due to the fact that the asymptotic value of $f'(R)$ depends on the initial settings adopted for $f'(R_0)$. To demonstrate this fact, one can weakly relax the assumption $f'(R_0) = 1$, allowing small departures from the Newtonian gravitational constant, i.e. requiring that $G_{\text{eff}}$ is not exactly equivalent to the Newton $G$. This choice does not violate the limits on $\dot{G}/G$ as imposed in

Figure 4. Reconstructed $|f(R)|$ for the $(2,1)$ Padé approximant in the redshift domain $z \in [0, 10]$.

Figure 5. Functional behaviour of $df/dR$ as result of the $(2,1)$ Padé approximant in the redshift domain $z \in [0, 10]$. 
Figure 6. Functional behaviour of \( d^2 f/dR^2 \) as result of the (2,1) Padé approximant in the redshift domain \( z \in [0, 10] \). We show \( -f''(R) \) for display purposes.

Figure 7. Functional behaviour of \( f'(R) \) for different values of the effective gravitational constant.

current literature \cite{72}. We thus infer that eq. (4.12) and (4.13) take the following expressions:

\[
\begin{align*}
  f_0 &= f'(R_0)(6H_0^2 + R_0) - 6H_0^2\Omega_m, \\
  f_{z|z=0} &= f'(R_0) R_e|_{z=0}.
\end{align*}
\]  

Indeed, using the above relations as initial values to get the auxiliary function \( f(z) \), we soon obtain the results displayed in figure 7. Last but not least, a further factor which affects the asymptotic value of \( f'(R) \) is related to the determination of the cosmographic series. Bearing in mind that the predictive power of the cosmographic method degrades as the redshift increases, one should consider higher-order Padé polynomials to improve the convergence radii of the cosmographic series. This issue is also known as cosmographic convergence problem \cite{73} and clearly influences any treatments at high-redshift domains. By adopting the aforementioned settings, the difference \( f'(R)_{\text{numerical}} - f'(R)_{\text{exact}} \) is small at larger curvatures and it is clearly due to the approximations made on the orders, initial values, and convergence of the Padé polynomials.
5.2 Dark energy equation of state

The reconstructed $f(R)$ we have obtained can be used to find $\rho_{\text{curv}}$ and $p_{\text{curv}}$ as functions of the Ricci scalar and, therefore, to study the dark energy EoS, i.e. $w_{DE}$ (cf. eq. (2.9)). To improve the error propagation in our numerical analyses, we re-scale eq. (4.22) as follows

$$f(z) \rightarrow \lambda + f(z).$$

(5.5)

Here, the parameter $\lambda$ means that we are using test-functions to numerically reconstruct the form of $f(z)$. Indeed, its role is to tune the numerical result and to enable a Taylor expansion over the expressions for $f(z)$, here involved for understanding the evolution of $f(R)$. To figure this out, it is simple to check that it does not come as vacuum energy contribution since it acts as a scaling constant to guarantee that at $z = 0$ the value of $f(z)$ is always compatible with the fact that $f'(R_0) = 1$. Moreover, its magnitude is ten times higher than the critical density, being outer the limit which enables $H(z = 0) = H_0$ and then cannot be considered as a vacuum energy entering the weak energy condition.

The constant $\lambda$ is found requiring the accelerated universe today, i.e.

$$-1 \leq w_{DE}\big|_{z=0} < -\frac{1}{3},$$

(5.6)

getting the constraint

$$\lambda \gtrsim 19.3.$$  

(5.7)

Figure 8 and figure 9 show the behaviours of curvature density and pressure for an indicative value of $\lambda = 100$. In figure 10, we show the effective dark energy EoS parameter for different outcomes of $\lambda$, satisfying the condition (5.7).

6 Taylor expansion vs Padé approximation

To better check the benefits of our analysis, based on Padé approximations with respect to the standard series approach, we present the results one would obtain using the Taylor method. Using eq. (4.8), we can solve eq. (4.11) by adopting the best-fit results of (4.15). Figure 11 shows the comparison between the best-fit results based on the Padé and the Taylor approximations in the redshift interval $z \in [0, 1]$. We can see that the Taylor approximation stops being predictive already at $z \simeq 0.3$. At higher redshifts, the two approaches have very
Figure 9. Effective curvature pressure for the Padé approximant with $\lambda = 100$.

Figure 10. Effective dark energy EoS parameter for the Padé approximant for different values of the constant $\lambda$.

Figure 11. Comparison between the numerical reconstruction of $|f(z)|$ using the (2,1) Padé (solid red) and the third-order Taylor (dashed blue) approximations.
different behaviours. The numerical inversion of eq. (4.8) by means of eq. (2.6) yields $z(R)$ (see figure 12), which inserted back in $f(z)$, allows us to find the form of $f(R)$ for the Taylor approximation (see figure 13).

We shall study the dark energy EoS parameter (cf. eq. (2.9)) using the Taylor approach and compare the results with those we have obtained for the Padé approximation. In the case of the Taylor approximation, the rescaling (4.1) together with the condition (5.5) lead to

$$\lambda \gtrsim 1196.$$ (6.1)

As shown in figure 14, the dark energy EoS parameter crosses the phantom line ($w_{DE} = -1$) at $z \sim 0.3$, confirming the inability of the Taylor method to account for observations at higher redshifts.

7 Conclusions and perspectives

The $f(R)$ gravity models have been revised here, considering a strategy to reconstruct at high redshift the functional forms of $f(R)$. In particular, we presented a technique for reconstructing the form of $f(R)$ in a model-independent way, without resorting to any $a$
Figure 14. Effective dark energy EoS parameter for the third-order Taylor approximation for different values of the constant \( \lambda \).

priori assumptions. To do so, we applied the cosmographic method with the use of Padé rational polynomials, showing the advantages of this treatment with respect to the standard method based on Taylor expansions, especially when the redshift domain exceeds \( z \simeq 1 \). To figure this out, expanding the Hubble rate \( H(z) \) in terms of the cosmographic series and using the relation \( R = -6(\dot{H} + 2H^2) \) in a FRW universe, we found \( f(z) \equiv f(R(H)) \) by numerically solving the modified Friedmann equations. We compared the results obtained for a (2,1) Padé approximant, involving up to the jerk parameter, with the outcomes of the third-order Taylor approach, when also the snap comes into the analysis. The \( F \)-statistics applied to several test-functions showed that the most suitable choice for \( f(z) \) is the form \( f(z) = A z + B z^3 e^{Cz} \), where the free parameters \( A, B, \) and \( C \) were found in order to match the cosmographic parameters to the values suggested by the most recent observations. Then, through a back-scattering procedure, we inverted the constraint \( R = -6(\dot{H} + 2H^2) \) to find \( z(R) \) and we finally reconstruct the term \( f(R) \). We showed that the so-obtained \( f(R) \) model satisfies all the conditions required from solar system tests of gravity and cosmological perturbations theory and, therefore, it represents a viable model to explain dark energy. The cosmological implications of our model have been investigated by studying the EoS of the effective dark energy fluid in the redshift interval \( z \in [0,10] \). We performed statistical analyses based on basics demands of data mining, employing in particular the \( F \) test and other strategies which overcome problems related to the \( p \)-value analysis and \( \chi^2 \) square procedure. We thus selected the effective dark energy behaviour and we reproduced its functional evolution at both the small and high redshift domains. By increasing the approximation order of the Padé series and slightly changing the initial settings on the differential equation, it is possible to reduce the discrepancy between \( f'(R)_{\text{numerical}} \) and \( f'(R)_{\text{exact}} \) in order to show that the condition \( f'(R) \to 1 \) for \( R \to \infty \) is naturally guaranteed. Future analyses will also involve tests on the Cosmic Microwave Background in order to check the accuracy of our numerics.

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