Continuity properties of the data-to-solution map for the two-component higher order Camassa-Holm system

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Abstract. This work studies the Cauchy problem of a two-component higher order Camassa-Holm system, which is well-posed in Sobolev spaces $H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$, $s > \frac{7}{2}$ and its solution map is continuous. We show that the solution map is Hölder continuous in $H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$ equipped with the $H^r(\mathbb{R}) \times H^{r-2}(\mathbb{R})$-topology for $1 \leq r < s$, and the Hölder exponent is expressed in terms of $s$ and $r$.

Keywords: Two-component higher order Camassa-Holm system; Cauchy problem; Well-posedness; Hölder continuity.

AMS subject classifications (2000): 35G25, 35L05, 35B30.

1 Introduction

In this paper, we consider the Cauchy problem of the following two-component higher order Camassa-Holm system

\[
\begin{align*}
    m_t &= \alpha u_x - bu_x m - um_x - \kappa \rho \rho_x, \quad m = Au, \\
    \rho_t &= -u \rho_x - (b - 1)u_x \rho, \quad b \in \mathbb{R} \setminus \{1\}, \\
    \alpha_t &= 0,
\end{align*}
\]

(1.1)

where $Au = (1 - \partial_x^2)^\sigma u$ with $\sigma > 1$, and $b, \kappa$ are real parameters. Eq. (1.1) was proposed by Escher and Lyons [16], in which they showed that the system corresponds to a metric induced geodesic flow on the infinite dimensional Lie group $\text{Diff}^\infty(S^1) \otimes C^\infty(S^1) \times \mathbb{R}$ and admits a global solution in $C^\infty([0, \infty); C^\infty(S^1) \oplus C^\infty(S^1))$ with smooth initial data in $C^\infty(S^1) \oplus C^\infty(S^1))$ when $b = 2$, where $\text{Diff}^\infty(S^1)$ denotes the group of orientation preserving diffeomorphisms of the circle and $\otimes$ denotes an appropriate semi-direct product between the pair. Recently, He and Yin [22], Chen and Zhou [4] established the local well-posedness of (1.1) in Besov spaces. Zhou [38], Zhang and Li [37] investigated the local well-posedness, blow-up criteria and Gevrey regularity of the solutions to (1.1) with $\sigma = 2$. When $\rho \equiv 0$, $\alpha = 0$ and $b = 2$, (1.1) reduces to a Camassa-Holm equation...
with fractional order inertia operator, whose geometrical interpretation and local well-posedness can be seen in [12, 13, 22], and if we further assume \(2 \leq \sigma \in \mathbb{Z}_+\), (1.1) becomes a higher order Camassa-Holm equation derived as the Euler-Poincaré differential equation on the Bott-Virasoro group with respect to the \(H^\sigma\) metric [32].

For \(\sigma = 1\), (1.1) reduces to the following nonlinear system [11]

\[
\begin{align*}
    m_t &= \alpha u_x - bu_x m - um_x - \kappa \rho_x, \quad m = u - u_{xx}, \\
    \rho_t &= -u \rho_x - (b - 1)u_x \rho, \quad b \in \mathbb{R} \setminus \{1\},
\end{align*}
\]

(1.2)

which models the two-component shallow water waves with constant vorticity \(\alpha\). In [11], Escher et al. showed the local well-posedness of (1.2) under a geometrical framework, and studied the blow-up scenarios and global strong solutions of (1.2) on the circle. In [18], Guan et al. considered the Cauchy problem of (1.2) in the Besov space and showed that the solutions have exponential decay if the initial data has exponential decay. When \(\alpha = 0\), \(b = 2\) and \(\kappa = \pm 1\), (1.2) becomes the two-component Camassa-Holm system, which admits Lax pair and bi-Hamiltonian structure, and thus is completely integrable [3]. When \(\rho \equiv 0\) and \(\alpha = 0\), (1.2) reduces to a family of equations parameterised by \(b \neq 1\), the so-called \(b\)-family equation. In particular, when \(b = 2\) and \(b = 3\), the \(b\)-family equation respectively becomes the famous completely integrable Camassa-Holm equation [2] and Degasperis-Procesi equation [10], which were introduced to model the unidirectional propagation of shallow water waves over a flat bottom. The Cauchy problem for these equations have been well-studied both on the real line and on the circle, including the well-posedness, blow-up behavior, global existence, traveling wave solutions and so on, e.g. [1, 6–9, 14, 15, 17, 19–21, 24, 25, 28–30, 36] and the references therein.

The present paper is devoted to establishing the Hölder continuity of the data-to-solution map for system (1.1) with \(\sigma = 2\) in \(H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})\), \(s > \frac{7}{2}\), which provides more information about the stability of the solution map than the one given by Corollary 3.1.2 in [37]. We mention that Hölder continuity for the \(b\)-equation was proved on the line by Chen, Liu and Zhang in [5], and for other equations were showed in [23, 26, 31, 35]. To obtain the desired result, we need to extend the estimate of \(\|fg\|_{H^{s-\frac{1}{2}}(\mathbb{R})}\) for \(0 \leq r \leq 1\) in [23], commonly used in the previous works, to that of \(\|fg\|_{H^{s-k}(\mathbb{R})}\) for \(0 \leq r \leq k\) and \(k > 1\), which plays a key role in proving the main result.

The rest of the paper is organized as follows. In Section 2, the local well-posedness for (1.1) with \(\sigma = 2\) and initial data in \(H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})\), \(s > \frac{7}{2}\), is established, an explicit lower bound for the maximal existence time \(T\) and an estimate of the solution size are provided. The Hölder continuity of the data-to-solution map is showed in Section 3.

Throughout the paper, we denote by \(\|\cdot\|_X\) the norm of Banach space \(X\), \((\cdot, \cdot)\) the inner product of Hilbert space \(L^2(\mathbb{R})\), and ” \(\lesssim\) ” the inequality up to a positive constant.

2 Local well-posedness and estimate of the solution size

In this section, we will give the local well-posedness for Eq.(1.1) with \(\sigma = 2\), and provide an explicit lower bound for the maximal existence time and an estimate of the solution size.

Setting \(\Lambda^{-1} := (1 - \partial_x^2)^{-2}\), the initial-value problem associated to Eq.(1.1) with \(\sigma = 2\) can be
rewritten in the following form:

\[
\begin{cases}
    u_t + uu_x + \partial_x \Lambda^{-4} \left( \frac{b}{2} u^2 + (3 - b) u_x^2 - \frac{b+5}{2} u_{xx} + (b - 5) u_x u_{xxx} + \frac{9}{2} \rho^2 - \alpha u \right) = 0, & t > 0, \ x \in \mathbb{R}, \\
    \rho_t + u \rho_x + (b - 1) u_x \rho = 0, & t > 0, \ x \in \mathbb{R}, \\
    u(0,x) = u_0(x), \ \rho(0,x) = \rho_0(x) & x \in \mathbb{R},
\end{cases}
\]

(2.1)

Applying the transport equation theory combined with the method of the Besov spaces, one may obtain the following local well-posedness result for system (2.1), more details can be seen in [37, 38].

**Theorem 2.1.** Given \((u_0, \rho_0) \in H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R}), s > \frac{7}{2}\), there exist a maximal \(T = T(u_0, \rho_0) > 0\) and a unique solution \((u, \rho)\) to (2.1) such that

\[(u, \rho) \in C([0, T); H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}) \times H^{s-3}(\mathbb{R}))\]

Moreover, the solution depends continuously on the initial data, and \(T\) is independent of \(s\).

Next, we recall the following estimates which will be used later.

**Lemma 2.1.** (see [27]) If \(r > 0\), then \(H^r(\mathbb{R}) \cap L^\infty(\mathbb{R})\) is an algebra. Moreover,

\[\|fg\|_{H^r(\mathbb{R})} \leq c_r (\|f\|_{L^\infty(\mathbb{R})}\|g\|_{H^r(\mathbb{R})} + \|f\|_{H^r(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})}),\]

where \(c_r\) is a positive constant depending only on \(r\).

**Lemma 2.2.** (see [27]) If \(r > 0\), then

\[\|g[A^r; f] g\|_{L^2(\mathbb{R})} \leq c_r (\|\partial_x f\|_{L^\infty(\mathbb{R})}\|A^{r-1} g\|_{L^2(\mathbb{R})} + \|A^r f\|_{L^2(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})}),\]

where \(A^r = (1 - \partial_x^2)^{r/2}\) and \(c_r\) is a positive constant depending only on \(r\).

**Lemma 2.3.** (see [33]) If \(f \in H^s(\mathbb{R})\) with \(s > \frac{3}{2}\), then there exists a constant \(c > 0\) such that for any \(g \in L^2(\mathbb{R})\) we have

\[\| \partial_x g \|_{L^2(\mathbb{R})} \leq c \|g\|_{C^1(\mathbb{R})}, \]

in which for each \(\varepsilon \in (0, 1]\), the operator \(J_\varepsilon\) is the Friedrichs mollifier defined by

\[J_\varepsilon f(x) = j_\varepsilon * f(x),\]

where \(j_\varepsilon(x) = \frac{1}{\varepsilon^3} j(\frac{x}{\varepsilon})\) and \(j(x)\) is a nonnegative, even, smooth bump function supported in the interval \((-1, 1)\) such that \(\int_{\mathbb{R}} j(x) dx = 1\). For any \(f \in H^s(\mathbb{R})\) with \(s \geq 0\), we have \(J_\varepsilon f \to f\) in \(H^s(\mathbb{R})\) as \(\varepsilon \to 0\).

**Theorem 2.2.** Let \((u, \rho)\) be the solution of system (2.1) with initial data \((u_0, \rho_0) \in H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R}), s > \frac{7}{2}\). Then, the maximal existence time \(T\) satisfies

\[T \geq T_0 := \frac{1}{2c_s} \ln(1 + \frac{1}{\|u_0\|_{H^s(\mathbb{R})} + \|\rho_0\|_{H^{s-2}(\mathbb{R})}}),\]
where $c_s$ is a constant depending on $s$. Also, we have
\[
\|u\|_{H^{s}(\mathbb{R})} + \|\rho\|_{H^{s-2}(\mathbb{R})} \leq 2e^{c_s T_0} (\|u_0\|_{H^{s}(\mathbb{R})} + \|\rho_0\|_{H^{s-2}(\mathbb{R})}), \quad t \in [0, T_0].
\]

**Proof.** Note that the products $uu_x, u\rho_x$ only have the regularity of $H^{s-1}(\mathbb{R})$ and $H^{s-3}(\mathbb{R})$ when $(u, \rho) \in H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$. To deal with this problem, we apply the operator $J_\varepsilon$ to the system
\[
\begin{aligned}
& (J_\varepsilon u)_t + J_\varepsilon (uu_x) + \partial_x \Lambda^{-4} [\frac{b+5}{2} J_\varepsilon (u^2) + (3-b)J_\varepsilon (u_x^2)] \\
& - \frac{b+5}{2} J_\varepsilon (u_x^2) + (b-5)J_\varepsilon (u_x u_{xxx}) + \frac{\varepsilon}{2} J_\varepsilon (\rho^2) - \alpha J_\varepsilon u = 0,
\end{aligned}
\]

\[
\begin{aligned}
& (J_\varepsilon \rho)_t + J_\varepsilon (u\rho_x) + (b-1)J_\varepsilon (\rho u_x) = 0.
\end{aligned}
\]

Applying the operator $\Lambda^s = (1 - \partial_x^2)^{s/2}$ to the first equation of (2.2), then multiplying the resulting equation by $\Lambda^s J_\varepsilon u$ and integrating with respect to $x \in \mathbb{R}$, we obtain
\[
\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|J_\varepsilon u\|_{H^s(\mathbb{R})}^2 = - \langle \Lambda^s J_\varepsilon (uu_x), \Lambda^s J_\varepsilon u \rangle \\
& - \langle \Lambda^s J_\varepsilon u, \partial_x \Lambda^s \Lambda^{-4} [\frac{b+5}{2} J_\varepsilon (u^2) + (3-b)J_\varepsilon (u_x^2)] - \frac{b+5}{2} J_\varepsilon (u_x^2) + (b-5)J_\varepsilon (u_x u_{xxx}) + \frac{\varepsilon}{2} J_\varepsilon (\rho^2) - \alpha J_\varepsilon u \rangle.
\end{aligned}
\]

In what follows next we use the fact that $\Lambda^s$ and $J_\varepsilon$ commute and that $J_\varepsilon$ satisfies the properties
\[
(J_\varepsilon f, g) = (f, J_\varepsilon g) \quad \text{and} \quad \|J_\varepsilon u\|_{H^s(\mathbb{R})} \leq \|u\|_{H^s(\mathbb{R})}.
\]

Let us estimate the first term of the right hand side of (2.3).
\[
\begin{aligned}
& \langle \Lambda^s J_\varepsilon (uu_x), \Lambda^s J_\varepsilon u \rangle \\
& = \langle \Lambda^s (uu_x), J_\varepsilon \Lambda^s J_\varepsilon u \rangle \\
& = \langle \langle \Lambda^s, u \rangle u_x, J_\varepsilon \Lambda^s J_\varepsilon u \rangle + \langle u \Lambda^s u_x, J_\varepsilon \Lambda^s J_\varepsilon u \rangle \\
& = \langle \langle \Lambda^s, u \rangle u_x, J_\varepsilon \Lambda^s J_\varepsilon u \rangle + \langle J_\varepsilon u \partial_x \Lambda^s u, \Lambda^s J_\varepsilon u \rangle \\
& = \langle \langle \Lambda^s, u \rangle u_x, J_\varepsilon \Lambda^s J_\varepsilon u \rangle + \langle J_\varepsilon u \partial_x \Lambda^s u, \Lambda^s J_\varepsilon u \rangle + \langle u J_\varepsilon \partial_x \Lambda^s u, \Lambda^s J_\varepsilon u \rangle + \langle u J_\varepsilon \partial_x \Lambda^s u, \Lambda^s J_\varepsilon u \rangle \\
& \leq \|\Lambda^s u_x\|_{L^2(\mathbb{R})} \|J_\varepsilon \Lambda^s J_\varepsilon u\|_{L^2(\mathbb{R})} + \|J_\varepsilon u\partial_x \Lambda^s u\|_{L^2(\mathbb{R})} \|\Lambda^s J_\varepsilon u\|_{L^2(\mathbb{R})} \\
& \quad + \frac{1}{2} \|u \Lambda^s u_x \Lambda^s J_\varepsilon u\| \lesssim \|u\|^3_{H^s(\mathbb{R})},
\end{aligned}
\]

where we have used Lemma 2.2 with $r = s$ and Lemma 2.3. Furthermore, we estimate the second term of the right hand side of (2.3) in the following way
\[
\begin{aligned}
& \|\partial_x \Lambda^{-4} [\frac{b+5}{2} J_\varepsilon (u^2) + (3-b)J_\varepsilon (u_x^2)] - \frac{b+5}{2} J_\varepsilon (u_x^2) + (b-5)J_\varepsilon (u_x u_{xxx}) + \frac{\varepsilon}{2} J_\varepsilon (\rho^2) - \alpha J_\varepsilon u \|_{H^s(\mathbb{R})} \|u\|_{H^s(\mathbb{R})} \\
& \leq \|\partial_x \Lambda^{-4} [\frac{b+5}{2} J_\varepsilon (u^2) + (3-b)J_\varepsilon (u_x^2)] - \frac{b+5}{2} J_\varepsilon (u_x^2) + (b-5)J_\varepsilon (u_x u_{xxx}) + \frac{\varepsilon}{2} J_\varepsilon (\rho^2) - \alpha J_\varepsilon u \|_{H^s(\mathbb{R})} \|u\|_{H^s(\mathbb{R})} \\
& \lesssim \|u\|_{H^{s-3}(\mathbb{R})} + \|u_x\|_{H^{s-3}(\mathbb{R})} + \|u_{xxx}\|_{H^{s-3}(\mathbb{R})} + \|u_{xxx}\|_{H^{s-3}(\mathbb{R})} + \|u\|_{H^{s-3}(\mathbb{R})} \|u\|_{H^s(\mathbb{R})} \\
& \lesssim \|u\|^2_{H^s(\mathbb{R})} + \|\rho\|^2_{H^{s-2}(\mathbb{R})} + \|u\|_{H^s(\mathbb{R})} \|u\|_{H^s(\mathbb{R})},
\end{aligned}
\]

where we have used Lemma 2.1 with $r = s - 3$. Thus, we have
\[
\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|J_\varepsilon u\|^2_{H^s(\mathbb{R})} \lesssim (\|u\|^2_{H^s(\mathbb{R})} + \|\rho\|^2_{H^{s-2}(\mathbb{R})} + \|u\|_{H^s(\mathbb{R})}) \|u\|_{H^s(\mathbb{R})}.
\end{aligned}
\]

Letting $\varepsilon \to 0$, we get
\[
\frac{d}{dt} \|u\|_{H^s(\mathbb{R})} \lesssim \|u\|^2_{H^s(\mathbb{R})} + \|\rho\|^2_{H^{s-2}(\mathbb{R})} + \|u\|_{H^s(\mathbb{R})}. \tag{2.4}
\]
Applying the operator $\Lambda^{s-2} = (1 - \partial^2_r)^{(s-2)/2}$ to the second equation of (2.3), then multiplying the resulting equation by $\Lambda^{s-2} J_r \rho$ and integrating with respect to $x \in \mathbb{R}$, we obtain
\[ \frac{1}{2} \frac{d}{dt} \| J_r \rho \|_{H^{s-2}(\mathbb{R})}^2 = - (\Lambda^{s-2} J_r (u\rho_x), \Lambda^{s-2} J_r \rho) - (b - 1) (\Lambda^{s-2} J_r (\rho u_x), \Lambda^{s-2} J_r \rho) \]
\[ = - (\Lambda^{s-2} (u\rho_x), J_r \Lambda^{s-2} J_r \rho) - (b - 1) (\Lambda^{s-2} (\rho u_x), J_r \Lambda^{s-2} J_r \rho) \]
\[ = - (\Lambda^{s-2}, u|\rho_x| J_r \Lambda^{s-2} J_r \rho) - (|J_r, u|\Lambda^{s-2} \rho_x, \Lambda^{s-2} \rho \xi J_r \Lambda^{s-2} J_r \rho) - (u J_r \Lambda^{s-2} \rho_x, \Lambda^{s-2} J_r \rho) \]
\[ - (b - 1) (\Lambda^{s-2}, \rho u_x), J_r \Lambda^{s-2} J_r \rho) - (b - 1) (|J_r, \rho|\Lambda^{s-2} \rho_x, \Lambda^{s-2} J_r \rho) \]
\[ - (b - 1) (\rho J_r \Lambda^{s-2} u_x, \Lambda^{s-2} J_r \rho) \]
\[ \lesssim \| \Lambda^{s-2}, u \rho_x \|_{L^2(\mathbb{R})} \| J_r \Lambda^{s-2} J_r \rho \|_{L^2(\mathbb{R})} + \| |J_r, u|\Lambda^{s-2} \rho_x \|_{L^2(\mathbb{R})} \| \Lambda^{s-2} \rho \|_{L^2(\mathbb{R})} \]
\[ + \| (u_x \Lambda^{s-2} J_r \rho, \Lambda^{s-2} J_r \rho) \|_{L^2(\mathbb{R})} + \| |J_r, \rho|\Lambda^{s-2} u_x \|_{L^2(\mathbb{R})} \| \Lambda^{s-2} \rho \|_{L^2(\mathbb{R})} \]
\[ + \| J_r \rho \Lambda^{s-2} u_x \|_{L^2(\mathbb{R})} \| \Lambda^{s-2} \rho \|_{L^2(\mathbb{R})} \| \Lambda^{s-2} \rho \|_{L^2(\mathbb{R})} \]
\[ \lesssim \| u \|_{H^s(\mathbb{R})} \| \rho \|_{H^{s-2}(\mathbb{R})}^2, \]
where we have used Lemmas 2.2-2.3 and integrating by parts. Letting $\varepsilon \to 0$, we get
\[ \frac{d}{dt} \| \rho \|_{H^{s-2}(\mathbb{R})} \lesssim \| \rho \|_{H^{s-2}(\mathbb{R})} \| u \|_{H^s(\mathbb{R})}. \quad (2.5) \]

Combining (2.4) and (2.5), we have
\[ \frac{d}{dt} (\| u \|_{H^s(\mathbb{R})} + \| \rho \|_{H^{s-2}(\mathbb{R})}) \]
\[ \lesssim \| u \|_{H^s(\mathbb{R})}^2 + \| \rho \|_{H^{s-2}(\mathbb{R})}^2 \| u \|_{H^s(\mathbb{R})} + \| \rho \|_{H^{s-2}(\mathbb{R})} \]
\[ \leq (\| u \|_{H^s(\mathbb{R})} + \| \rho \|_{H^{s-2}(\mathbb{R})}^2)^2 + \| u \|_{H^s(\mathbb{R})} + \| \rho \|_{H^{s-2}(\mathbb{R})}. \]

Letting $y(t) = \| u \|_{H^s(\mathbb{R})} + \| \rho \|_{H^{s-2}(\mathbb{R})}$, then we get
\[ - \frac{d(y^{-1} + 1)}{dt} \leq c_s (y^{-1} + 1), \quad y_0 := y(0) = \| u_0 \|_{H^s(\mathbb{R})} + \| \rho_0 \|_{H^{s-2}(\mathbb{R})}, \]
which implies that
\[ y \leq \frac{1}{e^{-c_s t} (y_0^{-1} + 1) - 1}. \]
Setting
\[ T_0 := \frac{1}{2c_s} \ln(1 + \frac{1}{\| u_0 \|_{H^s(\mathbb{R})} + \| \rho_0 \|_{H^{s-2}(\mathbb{R})}}), \]
we see from the above inequality that the solution $(u, \rho)$ exists for $0 \leq t \leq T_0$ and satisfies a solution size bound
\[ \| u \|_{H^s(\mathbb{R})} + \| \rho \|_{H^{s-2}(\mathbb{R})} \leq 2e^{c_s T_0} (\| u_0 \|_{H^s(\mathbb{R})} + \| \rho_0 \|_{H^{s-2}(\mathbb{R})}), \quad \forall 0 \leq t \leq T_0, \]
which completes the proof of the theorem. \qed

3 Hölder continuity

In this section, we will show that the solution map for system (2.1) is Hölder continuous in $H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$, $s > \frac{7}{2}$, equipped with the $H^r(\mathbb{R}) \times H^{r-2}(\mathbb{R})$-topology for $1 \leq r < s$. Firstly,
we recall the following lemmas.

**Lemma 3.1.** (see [34]) If \( s > \frac{1}{2} \) and \( 0 \leq \sigma + 1 \leq s \), then there exists a constant \( c > 0 \) such that
\[
\| \Lambda^s \partial_x f \|_{L^2(\mathbb{R})} \leq c \| f \|_{H^r(\mathbb{R})} \| v \|_{H^s(\mathbb{R})}.
\]

**Lemma 3.2.** (see [25]) If \( r > \frac{1}{2} \), then there exists a constant \( c_r > 0 \) depending only on \( r \) such that
\[
\| f g \|_{H^{r-1}(\mathbb{R})} \leq c_r \| f \|_{H^r(\mathbb{R})} \| g \|_{H^{r-1}(\mathbb{R})}.
\]

Lemma 3.2 gives the estimate of \( \| f g \|_{H^r(\mathbb{R})} \) for \( s > -\frac{1}{2} \), the other cases are provided in the following lemma.

**Lemma 3.3.** If \( 0 \leq r \leq k \), \( j > \frac{1}{2} \) and \( j \geq k-r \) with \( k \in \mathbb{Z}_+ \), then there exists a constant \( c_{r,j,k} > 0 \) depending on \( r, j \) and \( k \) such that
\[
\| f g \|_{H^{r-k}(\mathbb{R})} \leq c_{r,j,k} \| f \|_{H^j(\mathbb{R})} \| g \|_{H^{r-k}(\mathbb{R})}.
\]

**Proof.** The proof can be done by adapting analogous methods as in [23], in which they only considered the case \( k = 1 \). For the reader’s convenience, we provide the arguments with obvious modifications. Similar as the proof of Lemma 3 on \( \mathbb{R} \) in [23], we can obtain
\[
\| f g \|_{H^{r-k}(\mathbb{R})}^2 = \int_{\mathbb{R}} (1 + \xi^2)^{r-k} | \int_{\mathbb{R}} \hat{f}(\eta) \hat{g}(\xi - \eta) d\eta |^2 d\xi
\]
\[
= \int_{\mathbb{R}} (1 + \xi^2)^{r-k} | \int_{\mathbb{R}} (1 + \eta^2)^{\frac{j}{2}} \hat{f}(\eta) \cdot (1 + \eta^2)^{-\frac{j}{2}} \hat{g}(\xi - \eta) d\eta |^2 d\xi
\]
\[
\leq \| f \|_{H^j(\mathbb{R})}^2 \| \hat{g}(\eta) \|_{L^2}^2 \int_{\mathbb{R}} (1 + \xi^2)^{r-k} (1 + (\xi - \eta)^2)^{-j} d\xi d\eta,
\]
in which we have applied the Cauchy-Schwartz inequality in \( \eta \), a change of variables, and changed the order of summation. To get the desired result, it is sufficient to show that there exists a constant \( c_{r,j,k} > 0 \) such that
\[
\int_{\mathbb{R}} (1 + \xi^2)^{r-k} (1 + (\xi - \eta)^2)^{-j} d\xi \leq c_{r,j,k} (1 + \eta^2)^{r-k}.
\]
In fact, we can check the inequality under the conditions \( j > \frac{1}{2} \) and \( j \geq k-r \), the main difference with proof of Lemma 5 in [23] is replacing the discussions on \( \frac{1}{2} < r \leq 1 \) (\( 0 \leq r < \frac{1}{2}, r = \frac{1}{2} \), respectively) by \( k - \frac{1}{2} < r \leq k \) (\( 0 \leq r < k - \frac{1}{2}, r = k - \frac{1}{2} \), respectively) for the cases \( \xi \in [\frac{1}{2}, \eta] \) and \( \xi \in [\eta, \frac{3}{2}] \).

**Remark 3.1.** Lemma 3.3 is more general than (iii) of Proposition 2.4 in [21] when considering the Sobolev norm of \( f g \) with negative index, since it covers the case \( j = k-r \) here.

**Theorem 3.1.** Assume \( s > \frac{1}{2} \) and \( 1 \leq r < s \). Then the solution map for system (2.1) is Hölder continuous with exponent
\[
\beta = \begin{cases} 
1, & \text{if } 1 \leq r \leq s - 1 \text{ and } s + r \geq 5, \\
\frac{2s-3}{s-r}, & \text{if } \frac{5}{2} < s < 4 \text{ and } 1 \leq r \leq 5 - s, \\
s - r, & \text{if } s - 1 < r < s \end{cases}
\]
as a map from $B(0, h) := \{(u, \rho) \in H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R}) : \|u\|_{H^s(\mathbb{R})} + \|\rho\|_{H^{s-2}(\mathbb{R})} \leq h\}$ with $H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$-norm to $C([0, T_0]; H^r(\mathbb{R}) \times H^{r-2}(\mathbb{R}))$, where $T_0 > 0$ is defined as in Theorem 2.2. More precisely, we have

$$\|\|(u(t), \rho(t)) - (v(t), \theta(t))\|_{C([0, T_0]; H^r(\mathbb{R}) \times H^{r-2}(\mathbb{R}))} \leq c\|(u(0), \rho(0)) - (v(0), \theta(0))\|_{H^r(\mathbb{R}) \times H^{r-2}(\mathbb{R})}^\beta,$$

for all $(u(0), \rho(0)), (v(0), \theta(0)) \in B(0, h)$ and $(u(t), \rho(t)), (v(t), \theta(t))$ the solutions corresponding to the initial data $(u(0), \rho(0)), (v(0), \theta(0))$, respectively. The constant $c$ depends on $s, r, T_0$ and $h$.

**Proof.** Define $w = u - v$ and $\eta = \rho - \theta$, then $(w, \eta)$ satisfies that

$$\begin{cases}
    w_t + \partial_x \left( \frac{1}{2} w(u + v) + \partial_x \Lambda^{-4} \frac{3}{2} w(u + v) + (3 - b)w_x(u_x + v_x) - \frac{4 + 5}{2} w_{xx}(u_{xx} + v_{xx}) \right) \\
    + (b - 5)w_x u_{xxx} + (b - 5)v_x w_{xxx} + \frac{5}{2} \eta(\rho + \theta) - \alpha w = 0, \quad t > 0, \ x \in \mathbb{R}, \\
    \eta_t + w \partial_x \eta = -(b - 1)(w_x \rho + v_x \eta), \quad t > 0, \ x \in \mathbb{R}, \\
    w(0, x) = u_0 - v_0, \ \eta(0, x) = \rho_0 - \theta_0, \ \ x \in \mathbb{R}.
\end{cases} \tag{3.1}$$

(i) We first consider the case $1 \leq r \leq s - 1$ and $r + s \geq 5$, where $s > \frac{5}{2}$. Applying $\Lambda^r$ to the first equation of (3.1), then multiplying both sides by $\Lambda^r w$ and integrating over $\mathbb{R}$ with respect to $x$, we get

$$\frac{1}{2} \frac{d}{dt} \|w\|_{H^r(\mathbb{R})}^2 = - (\Lambda^r \partial_x (\frac{1}{2} w(u + v)), \Lambda^r w)
\quad - (\Lambda^r \partial_x \Lambda^{-4} \frac{3}{2} w(u + v) + (3 - b)w_x(u_x + v_x) - \frac{4 + 5}{2} w_{xx}(u_{xx} + v_{xx})
\quad + (b - 5)w_x u_{xxx} + (b - 5)v_x w_{xxx} + \frac{5}{2} \eta(\rho + \theta) - \alpha w, \Lambda^r w)
\quad := E_1 + E_2.$$

To get the desired result, we need to estimate $E_1$ and $E_2$.

**Estimate $E_1$.** By using Lemma 3.1, integrating by parts and the Sobolev embedding theorem $H^r(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ for $r > \frac{5}{2}$, we have

$$\|w\|_{H^r(\mathbb{R})} \leq \|u + v\|_{H^s(\mathbb{R})} \|w\|_{H^r(\mathbb{R})}.$$

**Estimate $E_2$.** It is easy to show that

$$|E_2| = | - (\Lambda^r \partial_x \Lambda^{-4} \frac{3}{2} w(u + v) + (3 - b)w_x(u_x + v_x) - \frac{4 + 5}{2} w_{xx}(u_{xx} + v_{xx})
\quad + (b - 5)w_x u_{xxx} + (b - 5)v_x w_{xxx} + \frac{5}{2} \eta(\rho + \theta) - \alpha w, \Lambda^r w)|
\leq \|\partial_x \Lambda^{-4} \frac{3}{2} w(u + v) + (3 - b)w_x(u_x + v_x) - \frac{4 + 5}{2} w_{xx}(u_{xx} + v_{xx})
\quad + (b - 5)w_x u_{xxx} + (b - 5)v_x w_{xxx} + \frac{5}{2} \eta(\rho + \theta) - \alpha w\|_{H^r(\mathbb{R})} \|w\|_{H^r(\mathbb{R})}.$$
Using integrating by parts, we have
\[
\|\partial_r A^{-1}\left(\frac{b}{2}w(u + v) + (3 - b)w_x(u_x + v_x) - \frac{b+5}{2}w_{xx}(u_{xx} + v_{xx})\right)
+ (b - 5)v_xw_{xxx} + (b - 5)v_xw_{xxx} + \frac{b}{2}\eta(\rho + \theta) - \alpha w\|_{H^{1}(\mathcal{R})}
= \|\partial_r A^{-1}\left(\frac{b}{2}w(u + v) + (3 - b)w_x(u_x + v_x) + w_{xx}(\frac{5-b}{2}u_{xx} - \frac{b+5}{2}v_{xx})\right)
+ (b - 5)v_xw_{xxx} + \frac{b}{2}\eta(\rho + \theta) - \alpha w\|_{H^{1}(\mathcal{R})}
\lesssim \|w(u + v)\|_{H^{r-\gamma}(\mathcal{R})} + \|w_x(u_x + v_x)\|_{H^{r-\gamma}(\mathcal{R})} + \|w_{xx}(u_{xx} + v_{xx})\|_{H^{r-\gamma}(\mathcal{R})}
+ \|v_xw_{xxx}\|_{H^{r-\gamma}(\mathcal{R})} + \|\eta(\rho + \theta)\|_{H^{r-\gamma}(\mathcal{R})} + \|w\|_{H^{r-\gamma}(\mathcal{R})} + \|w_{xx}u_{xx}\|_{H^{r-\gamma}(\mathcal{R})}
:= F_1 + F_2.
\]

For $F_1$, if $r > \frac{\gamma}{2}$, we have
\[
F_1 \lesssim \|u\|_{H^{r-\gamma}(\mathcal{R})} \|u + v\|_{H^{r-\gamma}(\mathcal{R})} + \|w_x\|_{H^{r-\gamma}(\mathcal{R})} \|u_x + v_x\|_{H^{r-\gamma}(\mathcal{R})} + \|w_{xx}\|_{H^{r-\gamma}(\mathcal{R})} \|u_{xx} + v_{xx}\|_{H^{r-\gamma}(\mathcal{R})}
+ \|\eta\|_{H^{r-\gamma}(\mathcal{R})} \|\rho + \theta\|_{H^{r-\gamma}(\mathcal{R})} + \|w\|_{H^{r-\gamma}(\mathcal{R})}
\lesssim \|w\|_{H^{r}(\mathcal{R})} (\|u\|_{H^{r}(\mathcal{R})} + \|v\|_{H^{r}(\mathcal{R})} + 1) + \|\eta\|_{H^{r}(\mathcal{R})} (\|\rho\|_{H^{r}(\mathcal{R})} + \|\theta\|_{H^{r}(\mathcal{R})})
\]
by using Lemma 3.2 and the fact $r \leq s - 1$.

For $1 \leq r \leq \frac{\gamma}{2}$, applying Lemma 3.3 with $k = 3$ to the term $F_1$, we have
\[
F_1 \lesssim \|w\|_{H^{r-\gamma}(\mathcal{R})} \|u + v\|_{H^{r}(\mathcal{R})} + \|w_x\|_{H^{r-\gamma}(\mathcal{R})} \|u_x + v_x\|_{H^{r}(\mathcal{R})} + \|w_{xx}\|_{H^{r-\gamma}(\mathcal{R})} \|u_{xx} + v_{xx}\|_{H^{r}(\mathcal{R})}
+ \|w_{xxx}\|_{H^{r-\gamma}(\mathcal{R})} \|v_x\|_{H^{r}(\mathcal{R})} + \|\theta\|_{H^{r}(\mathcal{R})} + \|w\|_{H^{r-\gamma}(\mathcal{R})}
\lesssim \|w\|_{H^{r}(\mathcal{R})} (\|u\|_{H^{r}(\mathcal{R})} + \|v\|_{H^{r}(\mathcal{R})} + 1) + \|\eta\|_{H^{r}(\mathcal{R})} (\|\rho\|_{H^{r}(\mathcal{R})} + \|\theta\|_{H^{r}(\mathcal{R})})
\]
where $j$ satisfies $j > \frac{1}{2}$ and $j \geq 3 - r$. Since $s \geq 5 - r$, we can take $j = s - 2$, and then
\[
F_1 \lesssim \|w\|_{H^{r}(\mathcal{R})} (\|u\|_{H^{r}(\mathcal{R})} + \|v\|_{H^{r}(\mathcal{R})} + 1) + \|\eta\|_{H^{r}(\mathcal{R})} (\|\rho\|_{H^{r}(\mathcal{R})} + \|\theta\|_{H^{r}(\mathcal{R})}).
\]

For $F_2$, if $r > \frac{\gamma}{2}$, we have
\[
F_2 \lesssim \|w_x\|_{H^{r-\gamma}(\mathcal{R})} \|u_{xx}\|_{H^{r-\gamma}(\mathcal{R})} \leq \|w\|_{H^{r}(\mathcal{R})} \|u\|_{H^{r}(\mathcal{R})},
\]
by using Lemma 3.2 and the fact $r \leq s - 1$.

For $1 \leq r \leq \frac{3}{2}$, applying Lemma 3.3 with $k = 2$ to the term $F_2$, we have
\[
F_2 \lesssim \|w_x\|_{H^{r-\gamma}(\mathcal{R})} \|u_{xx}\|_{H^{r-\gamma}(\mathcal{R})} \leq \|w\|_{H^{r-\gamma}(\mathcal{R})} \|u\|_{H^{r+2}(\mathcal{R})},
\]
where $l$ satisfies $l > \frac{1}{2}$ and $l \geq 2 - r$. Since $s \geq 5 - r$, we can take $l = s - 3$, and then
\[
F_2 \lesssim \|w\|_{H^{r}(\mathcal{R})} \|u\|_{H^{r}(\mathcal{R})}.
\]
Thus,
\[
\frac{5}{2} \|w\|_{H^{r}(\mathcal{R})} \|w\|_{H^{r}(\mathcal{R})} \lesssim \|w\|_{H^{r}(\mathcal{R})} (\|u\|_{H^{r}(\mathcal{R})} + \|v\|_{H^{r}(\mathcal{R})} + 1)
+ \|w\|_{H^{r}(\mathcal{R})} \|\eta\|_{H^{r}(\mathcal{R})} (\|\rho\|_{H^{r}(\mathcal{R})} + \|\theta\|_{H^{r}(\mathcal{R})}).
\]
On the other hand, applying $\Lambda^{-2}$ to the second equation of (3.1), then multiplying both sides by $\Lambda^{-2}\eta$ and integrating over $\mathbb{R}$ with respect to $x$, we get

$$\frac{1}{2} \frac{d}{dt} \| \eta \|^2_{H^{-2}(\mathbb{R})} = - (\Lambda^{-2} (u\eta_x), \Lambda^{-2} \eta) - (\Lambda^{-2} (u\theta_x), \Lambda^{-2} \eta) - (b-1) (\Lambda^{-2} (w_x \rho + v_x \eta), \Lambda^{-2} \eta)
$$

By Lemma 3.1, we know

$$\| [\Lambda^{-2} \partial_x, u] \eta \|_{L^2(\mathbb{R})} \leq \| u \|_{H^r(\mathbb{R})} \| \eta \|_{H^{-2}(\mathbb{R})}.$$ 

If $r > \frac{3}{2}$, by Lemma 3.2, we have

$$\| u \|_{H^r(\mathbb{R})} \| \theta \|_{H^{-3}(\mathbb{R})} \leq \| u\theta_x - u_x \eta \|_{H^{-2}(\mathbb{R})} + \| w_x \rho + v_x \eta \|_{H^{-2}(\mathbb{R})} + \| u_x \|_{H^r(\mathbb{R})} \| \eta \|_{H^{-2}(\mathbb{R})} + \| u \|_{H^r(\mathbb{R})} \| \theta \|_{H^{-3}(\mathbb{R})} + \| \rho \|_{H^{r-2}(\mathbb{R})} + \| \eta \|_{H^{r-2}(\mathbb{R})} \| u \|_{H^r(\mathbb{R})} + \| v \|_{H^r(\mathbb{R})}.$$ 

If $1 \leq r \leq \frac{3}{2}$, similar to (3.2), we have

$$\| u \|_{H^r(\mathbb{R})} \| \theta \|_{H^{-3}(\mathbb{R})} \leq \| u \|_{H^r(\mathbb{R})} \| \theta \|_{H^{-2}(\mathbb{R})} + \| w_x \rho + v_x \eta \|_{H^{r-2}(\mathbb{R})} + \| u \|_{H^r(\mathbb{R})} \| \theta \|_{H^{-3}(\mathbb{R})} + \| \rho \|_{H^{r-2}(\mathbb{R})} + \| \eta \|_{H^{r-2}(\mathbb{R})} \| u \|_{H^r(\mathbb{R})} + \| v \|_{H^r(\mathbb{R})}.$$ 

Moreover, it is easy to get

$$\| u \|_{L^\infty(\mathbb{R})} \| \Lambda^{-2} \eta \|_{L^2(\mathbb{R})} \leq \| u \|_{H^r(\mathbb{R})} \| \eta \|_{H^{-2}(\mathbb{R})}.$$ 

Thus, we have

$$\frac{1}{2} \frac{d}{dt} \| \eta \|^2_{H^{-2}(\mathbb{R})} \leq \| w \|_{H^r(\mathbb{R})} \| \eta \|_{H^{-2}(\mathbb{R})} \| \theta \|_{H^{-2}(\mathbb{R})} + \| \rho \|_{H^{r-2}(\mathbb{R})} + \| \eta \|_{H^{r-2}(\mathbb{R})} \| u \|_{H^r(\mathbb{R})} + \| v \|_{H^r(\mathbb{R})}.$$ 

Combing (3.3) and (3.4), and using the solution size estimate in Theorem 2.2, we get

$$\| w \|_{H^r(\mathbb{R})} + \| \eta \|_{H^{-2}(\mathbb{R})} \leq C T_0 (\| u \|_{H^r(\mathbb{R})} + \| \rho \|_{H^{r-2}(\mathbb{R})}),$$

(3.5)
where \( C \) is a constant depending on \( s, r, T_0 \) and \( h \).

(iii) Next, we consider the case \( \frac{7}{2} < s < 4 \) and \( 1 \leq r \leq 5 - s \). By the condition \( r \leq 5 - s \) and (3.5), we have

\[
\|w\|_{H^r(\Omega)} + \|\eta\|_{H^{s-r}(\Omega)} \leq \|w\|_{H^{s-r}(\Omega)} + \|\eta\|_{H^{s-r}(\Omega)} \leq C_\beta \left( \|w_0\|_{H^{s-r}(\Omega)} + \|\eta_0\|_{H^{s-r}(\Omega)} \right).
\]

Using interpolation inequalities, we obtain

\[
\|w_0\|_{H^{s-r}(\Omega)} + \|\eta_0\|_{H^{s-r}(\Omega)} \leq \left( \|w_0\|_{H^{s-r}(\Omega)} + \|\eta_0\|_{H^{s-r}(\Omega)} \right) \leq \left( \|w_0\|_{H^{s-r}(\Omega)} + \|\eta_0\|_{H^{s-r}(\Omega)} \right)^{2-r}.\]

Thus, we get

\[
\|w\|_{H^r(\Omega)} + \|\eta\|_{H^{s-r}(\Omega)} \leq \left( \|w_0\|_{H^r(\Omega)} + \|\eta_0\|_{H^{s-r}(\Omega)} \right)^{2-r}.\]

(iii) Now we consider the case \( s - 1 < r < s \), where \( s > \frac{7}{2} \). Using interpolation inequalities, we obtain

\[
\|w\|_{H^r(\Omega)} + \|\eta\|_{H^{s-r}(\Omega)} \leq \left( \|w\|_{H^{s-r}(\Omega)} + \|\eta\|_{H^{s-r}(\Omega)} \right)^{s-r} \leq \left( \|w\|_{H^{s-r}(\Omega)} + \|\eta\|_{H^{s-r}(\Omega)} \right)^{s-r}.\]

By applying inequality (3.5), we have

\[
\|w\|_{H^{s-r}(\Omega)} + \|\eta\|_{H^{s-r}(\Omega)} \leq C_\beta \left( \|w_0\|_{H^{s-r}(\Omega)} + \|\eta_0\|_{H^{s-r}(\Omega)} \right)\]  

(3.6)

Also, using the solution size estimate in Theorem 2.2, we get

\[
\|w\|_{H^r(\Omega)} + \|\eta\|_{H^{s-r}(\Omega)} \leq \left( \|w\|_{H^{s-r}(\Omega)} + \|\rho\|_{H^{s-r}(\Omega)} \right)^{s-r} \leq 2C_\beta \left( \|w_0\|_{H^r(\Omega)} + \|\rho_0\|_{H^{s-r}(\Omega)} \right)\]  

(3.7)

Combining (3.6), (3.7) and (3.8) gives

\[
\|w\|_{H^r(\Omega)} + \|\eta\|_{H^{s-r}(\Omega)} \leq \left( \|w_0\|_{H^r(\Omega)} + \|\eta_0\|_{H^{s-r}(\Omega)} \right)^{s-r}.\]

This completes the proof of Theorem 3.1.

\[\square\]

**Remark 3.2.** If \( \rho \equiv 0 \), then the condition \( 1 \leq r < s \) in Theorem 3.1 can be extended to \( 0 \leq r < s \), and the exponent \( \beta \) is defined as follows

\[
\beta = \begin{cases} 
1, & \text{if } 0 \leq r \leq s - 1 \text{ and } s + r \geq 5, \\
\frac{2s-5}{s-r}, & \text{if } \frac{7}{2} < s < 5 \text{ and } 0 \leq r \leq 5 - s, \\
|s - r|, & \text{if } s - 1 < r < s.
\end{cases}
\]

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