Abstract. In this paper we study duoidal structures on ∞-categories of operadic modules. Let $O^\otimes$ be a small coherent ∞-operad and let $P^\otimes$ be an ∞-operad. If a $P \otimes O$-monoidal ∞-category $C^\otimes$ has a sufficient supply of colimits, then we show that the ∞-category $\text{Mod}^O_A(C)$ of $O$-$A$-modules in $C^\otimes$ has a structure of $(P, O)$-duoidal ∞-category for any $P \otimes O$-algebra object $A$.

1. Introduction

A duoidal category is a category equipped with two monoidal structures in which one is lax monoidal with respect to the other. In [8, 9] we have introduced generalizations of duoidal categories in the setting of ∞-categories. The goal of this paper is to show that ∞-categories of operadic modules have duoidal structures.

The notion of duoidal category was introduced by Aguiar-Mahajan [1] by the name of 2-monoidal category. There is a 2-category $\text{Mon}^{oplax}(\text{Cat})$ of monoidal categories, oplax monoidal functors, and natural transformations between them. Note that $\text{Mon}^{oplax}(\text{Cat})$ is a monoidal 2-category under Cartesian product. A duoidal category is identified with a pseudo-monoid in the monoidal 2-category $\text{Mon}^{oplax}(\text{Cat})$.

We can consider the ∞-category $\text{Mon}^{oplax}(\text{Cat}_\infty)$ of $O$-monoidal ∞-categories and oplax $O$-monoidal functors for an $O$-operad $O^\otimes$. Since it has finite products, $\text{Mon}^{oplax}(\text{Cat}_\infty)$ is a Cartesian symmetric monoidal ∞-category. For an $O$-operad $P^\otimes$, we say that a $P$-monoid object in the ∞-category $\text{Mon}^{oplax}(\text{Cat}_\infty)$ is a $(P, O)$-duoidal ∞-category.

Now we assume that $O^\otimes$ is coherent. For an $O$-monoidal ∞-category $C^\otimes$ which has a sufficient supply of colimits, Lurie [7] constructed an $O$-monoidal structure on the ∞-category of $O$-$A$-modules in $C^\otimes$ for each $O$-algebra object $A$. The main theorem in this paper is to extend this $O$-monoidal structure to a $(P, O)$-duoidal ∞-category.

**Theorem 1.1** (cf. Theorem 5.6). Let $\kappa$ be an uncountable regular cardinal and let $O^\otimes$ be an essentially $\kappa$-small coherent ∞-operad. Let $P^\otimes$ be an ∞-operad and let $C^\otimes$ be a $P \otimes O$-monoidal ∞-category which is compatible with $\kappa$-small colimits. Then the ∞-category $\text{Mod}^O_A(C)$ of $O$-$A$-modules in $C^\otimes$ has a structure of a $(P, O)$-duoidal ∞-category for any $P \otimes O$-algebra object $A$.

The important case is when $(P^\otimes, O^\otimes) = (E^\otimes_m, E^\otimes_n)$, where $E_k$ is the little $k$-cubes operad, and $C^\otimes$ is a presentable symmetric monoidal ∞-category. In this case we have the following corollary.

**Corollary 1.2** (cf. Theorem 6.2). Let $(P^\otimes, O^\otimes) = (E^\otimes_m, E^\otimes_n)$ and let $C^\otimes$ be a presentable symmetric monoidal ∞-category. Then the ∞-category $\text{Mod}^O_A(C)$ of $E_n$-$A$-modules in $C^\otimes$ has a structure of an $(E_m, E_n)$-duoidal ∞-category for any $E_m$-$n$-algebra object $A$.

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The organization of this paper is as follows: In §2 we study ∞-categories of operadic modules. First, we recall the construction of a map of generalized ∞-operads which encodes ∞-categories of operadic modules and restriction functors. Then we give a description of left adjoints to the restriction functors. In §3 we study adjointable diagrams of (op)lax ∞-monoidal functors. Then we study a functoriality of this construction. In §4 we construct dual structures on ∞-categories of operadic modules and prove the main theorem (Theorem 3.0). In §5 we consider the case in which \((\mathcal{P},\mathcal{O}) = (\mathcal{E},\mathcal{E}^+_{\infty})\) and \(\mathcal{O}\) is a presentable symmetric monoidal ∞-category.

**Notation 1.3.** For ∞-operads \(\mathcal{O}\) and \(\mathcal{P}\), we denote by \(\mathcal{O} \otimes \mathcal{P} = (\mathcal{O} \otimes \mathcal{P})^\otimes\) the Boardman-Vogt tensor product of ∞-operads (cf. [7, §2.2.5]). For an ∞-operad \(\mathcal{O}\), we denote by \(\text{Mon}_{\mathcal{O}}(\text{Cat}_{\infty})\) the ∞-category of small ∞-monoidal ∞-categories and lax ∞-monoidal functors. We also denote by \(\text{Mon}_{\mathcal{O}}^{\text{plax}}(\text{Cat}_{\infty})\) the ∞-category of small ∞-monoidal ∞-categories and op-lax ∞-monoidal functors.

2. Operadic Modules

In this section we study ∞-categories of operadic modules. Let \(p : \mathcal{O} \rightarrow \text{Fin}_{\ast}\) be a coherent ∞-operad and let \(q : \mathcal{C} \rightarrow \mathcal{O}^\otimes\) be a map of ∞-operads. In 2.1 we recall a generalized ∞-operad \(\text{Mod}^\ell(\mathcal{C})^\otimes\) constructed by Lurie, in which the underlying ∞-category \(\text{Mod}^\ell(\mathcal{C})\) consists of pairs \((A,M)\) of an \(\mathcal{O}\)-algebra object \(A\) and an \(\mathcal{O}\)-A-module \(M\) in \(\mathcal{C}\). We would like to construct a free functor \(\text{Alg}_{\mathcal{O}}(\mathcal{C}) \times \mathcal{C}_X^\otimes \rightarrow \text{Mod}^\ell(\mathcal{C})_X^\otimes\) for \(X \in \mathcal{O}\), which is left adjoint to the forgetful functor. For this purpose, in 2.2 and 2.3 we introduce ∞-operads \(\mathcal{X} \mathcal{M}_\mathcal{O}^\otimes\) and \(\mathcal{X} \mathcal{M}_\mathcal{O}^{\text{tr} \otimes}\) such that the ∞-categories of \(\mathcal{X} \mathcal{M}_\mathcal{O}\)-algebras and of \(\mathcal{X} \mathcal{M}_\mathcal{O}^{\text{tr} \otimes}\)-algebras in \(\mathcal{C}\) are equivalent to \(\text{Mod}^\ell(\mathcal{C})_X^\otimes\) and \(\text{Alg}_{\mathcal{O}}(\mathcal{C}) \times \mathcal{C}_X^\otimes\), respectively. In 2.4 we construct a left adjoint \(f_{1X}: f_1\) to the restriction functor \(f_{1X}^\otimes\) for a map \(f : A \rightarrow B\) in \(\text{Alg}_{\mathcal{O}}(\mathcal{C})\). We also study the monad \(T_f\) associated to the adjunction \((f_{1X}, f_1^\otimes)\) and give a description of \(T_f(M)\) as a colimit of certain diagram for \(M \in \text{Mod}^\ell(\mathcal{C})_X^\otimes\).

2.1. ∞-categories of operadic modules. Let \(p : \mathcal{O}^\otimes \rightarrow \text{Fin}_{\ast}\) be a coherent ∞-operad and let \(q : \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes\) be a map of ∞-operads. In 3.3.3.3 Lurie introduced a generalized ∞-operad \(\text{Mod}^\ell(\mathcal{C})^\otimes\) where the underlying ∞-category \(\text{Mod}^\ell(\mathcal{C})\) consists of pairs \((A,M)\), where \(A\) is an \(\mathcal{O}\)-algebra and \(M\) is an \(\mathcal{O}\)-A-module in \(\mathcal{C}\). In this subsection we recall the construction of \(\text{Mod}^\ell(\mathcal{C})^\otimes\) and a map \((\Psi,\sigma) : \text{Mod}^\ell(\mathcal{C})^\otimes \rightarrow \text{Alg}_{\mathcal{O}}(\mathcal{C}) \times \mathcal{O}^\otimes\) of generalized ∞-operads.

First, we recall the construction of the generalized ∞-operad \(\text{Mod}^\ell(\mathcal{C})^\otimes\). Let \(K_{\mathcal{O}}\) be a full subcategory of \(\text{Fun}(\mathcal{C};\mathcal{O}^\otimes)\) spanned by semi-inert morphisms in \(\mathcal{O}^\otimes\) (see [7, Definition 3.3.1.1] for the definition of semi-inert morphisms). We have the projections \(e_{\mathcal{V}_0}, e_{\mathcal{V}_1} : K_{\mathcal{O}} \rightarrow \mathcal{O}^\otimes\) given by evaluation at 0, 1 \(\in \mathcal{C}\), respectively. A morphism in \(K_{\mathcal{O}}\) is said to be inert if the images under \(e_{\mathcal{V}_0}\) and \(e_{\mathcal{V}_1}\) are inert morphisms in \(\mathcal{O}^\otimes\).

We have an ∞-category \(\text{Mod}^\ell(\mathcal{C})^\otimes\) equipped with a map \(\text{Mod}^\ell(\mathcal{C})^\otimes \rightarrow \mathcal{O}^\otimes\) so that there is an equivalence

\[\text{Map}_{\text{Cat}_{\infty}/\mathcal{O}^\otimes}(\mathcal{X}, \text{Mod}^\ell(\mathcal{C})^\otimes) \simeq \text{Map}_{\text{Cat}_{\infty}/\mathcal{O}^\otimes}(\mathcal{X} \times \mathcal{O}^\otimes, e_{\mathcal{V}_0}, K_{\mathcal{O}}, \mathcal{C}^\otimes)\]

for any map \(\mathcal{X} \rightarrow \mathcal{O}^\otimes\) of ∞-categories. An object of \(\text{Mod}^\ell(\mathcal{C})^\otimes\) over \(Y \in \mathcal{O}^\otimes\) corresponds to a functor \(\{Y\} \times \mathcal{O}^\otimes, e_{\mathcal{V}_0}, K_{\mathcal{O}} \rightarrow \mathcal{C}^\otimes\) over \(\mathcal{O}^\otimes\). We let

\[\text{Mod}^\ell(\mathcal{C})^\otimes\]

be the full subcategory of \(\text{Mod}^\ell(\mathcal{C})^\otimes\) spanned by those functors \(\{Y\} \times \mathcal{O}^\otimes, e_{\mathcal{V}_0}, K_{\mathcal{O}} \rightarrow \mathcal{C}^\otimes\) which preserve inert morphisms. The induced map \(\text{Mod}^\ell(\mathcal{C})^\otimes \rightarrow \mathcal{O}^\otimes\) is a map of generalized ∞-operads by [7, Proposition 3.3.3.10].
Next, we recall a map $\text{Mod}^\circ_\mathcal{O}(\mathcal{C}) \to \text{Alg}_/\mathcal{O}(\mathcal{C}) \times \mathcal{O}^\circ$ of generalized $\infty$-operads and its properties. Let $\mathcal{K}_{\mathcal{O}}^\circ$ be the full subcategory of $\mathcal{K}_{\mathcal{O}}$ spanned by null morphisms (see [2] Definition 3.3.1.1 for the definition of null morphisms). By [2] Remark 3.3.3.16, the inclusion $\mathcal{K}_{\mathcal{O}}^\circ \hookrightarrow \mathcal{K}_{\mathcal{O}}$ induces a map

$$
(\Phi, \sigma) : \text{Mod}^\circ_\mathcal{O}(\mathcal{C}) \to \text{Alg}_/\mathcal{O}(\mathcal{C}) \times \mathcal{O}^\circ
$$

of generalized $\infty$-operads. By taking fibers at $A \in \text{Alg}_/\mathcal{O}(\mathcal{C})$, we obtain a functor

$$
\sigma_A : \text{Mod}^\circ_\mathcal{O}(\mathcal{C}) \to \mathcal{O}^\circ,
$$

which is a map of $\infty$-operads by [2] Theorem 3.3.3.9. By [2] Corollary 3.4.3.4, the map $\Phi : \text{Mod}^\circ_\mathcal{O}(\mathcal{C}) \to \text{Alg}_/\mathcal{O}(\mathcal{C})$ is a Cartesian fibration, and the induced functor

$$
f^* : \text{Mod}^\circ_\mathcal{O}(\mathcal{C}) \to \text{Mod}^\circ_\mathcal{O}(\mathcal{C})^\circ,
$$

is a map of $\infty$-operads over $\mathcal{O}^\circ$ for any map $f : A \to B$ in $\text{Alg}_/\mathcal{O}(\mathcal{C})$.

If $\mathcal{C}^\circ$ has a sufficient supply of colimits, then $\sigma_A : \text{Mod}^\circ_\mathcal{O}(\mathcal{C}) \to \mathcal{O}^\circ$ is a coCartesian fibration of $\infty$-operads for any $A \in \text{Alg}_/\mathcal{O}(\mathcal{C})$. Let $\kappa$ be an uncountable regular cardinal. We assume that $\mathcal{O}^\circ$ is an essentially $\kappa$-small coherent $\infty$-operad and that $\mathcal{C}^\circ$ is an $\mathcal{O}$-monoidal $\infty$-category which is compatible with $\kappa$-small colimits (see [2] Definition 3.1.1.18 and Variant 3.1.1.19 for the definition of $\mathcal{O}$-monoidal $\infty$-categories compatible with $\kappa$-small colimits). Then $\sigma_A : \text{Mod}^\circ_\mathcal{O}(\mathcal{C}) \to \mathcal{O}^\circ$ is an $\mathcal{O}$-monoidal $\infty$-category which is compatible with $\kappa$-small colimits by [2] Corollary 3.4.4.6.

2.2. The $\infty$-operad $\mathcal{X}\mathcal{M}_\mathcal{O}^\circ$. We would like to construct a free functor $\text{Alg}_/\mathcal{O}(\mathcal{C}) \times \mathcal{C}_X^\circ \to \text{Mod}^\circ_\mathcal{O}(\mathcal{C})^\circ_X$ for $X \in \mathcal{O}$, which is left adjoint to the forgetful functor. For this purpose, in this subsection we introduce an $\infty$-operad $\mathcal{X}\mathcal{M}_\mathcal{O}^\circ$ such that the $\infty$-category $\text{Alg}_{\mathcal{X}\mathcal{M}_\mathcal{O}}/\mathcal{O}(\mathcal{C})$ of $\mathcal{X}\mathcal{M}_\mathcal{O}$-algebras in $\mathcal{C}^\circ$ is equivalent to $\text{Mod}^\circ_\mathcal{O}(\mathcal{C})^\circ_X$.

For $X \in \mathcal{O}$, we set

$$
\mathcal{X}\mathcal{K}_{\mathcal{O}} = \{X\} \times \mathcal{O}^\circ,\text{ev}_0 \mathcal{K}_{\mathcal{O}}.
$$

The $\infty$-category $\mathcal{X}\mathcal{K}_{\mathcal{O}}$ is equipped with a map $\mathcal{X}\mathcal{K}_{\mathcal{O}} \to \mathcal{O}^\circ$ induced by $\text{ev}_1$. We notice that the $\infty$-category $\text{Mod}^\circ_\mathcal{O}(\mathcal{C})^\circ_X$ is a full subcategory of $\text{Fun}_{\mathcal{O}^\circ}(\mathcal{X}\mathcal{K}_{\mathcal{O}}, \mathcal{C}^\circ)$ spanned by those functors which preserve inert morphisms.

Let $\text{Triv}^\circ$ be the trivial $\infty$-operad, which is the subcategory of $\text{Fin}_*$ spanned by inert morphisms ([7] Example 2.1.1.20). The inclusion map $\{\{1\}\} \hookrightarrow \text{Triv}^\circ$ induces an equivalence $\text{Alg}_{\text{Triv}}(\mathcal{O}) \simeq \mathcal{O}$ by [2] Remark 2.1.3.6. Hence we have a map of $\infty$-operads $X : \text{Triv}^\circ \to \mathcal{O}^\circ$ for $X \in \mathcal{O}$. We set

$$
\mathcal{X}\tilde{\mathcal{M}}_{\mathcal{O}}^\circ = \text{Triv}^\circ \times \mathcal{X}_{/\mathcal{O}_*},\text{ev}_0 \mathcal{K}_{\mathcal{O}}.
$$

We define

$$
\mathcal{X}\mathcal{M}_{\mathcal{O}}^\circ
$$

to be the full subcategory of $\mathcal{X}\tilde{\mathcal{M}}_{\mathcal{O}}^\circ$ spanned by those vertices which correspond to semi-inert morphisms $\oplus_m X \to Y_1 \oplus \cdots \oplus Y_n$ in $\mathcal{O}^\circ$ such that the image $\langle m \rangle \to \langle n \rangle$ in $\text{Fin}_*$ is an order-preserving injection.

We will show that the composite map $r : \mathcal{X}\mathcal{M}_{\mathcal{O}}^\circ \overset{\text{ev}_0}{\to} \mathcal{O}^\circ \overset{X}{\to} \text{Fin}_*$ exhibits $\mathcal{X}\mathcal{M}_{\mathcal{O}}^\circ$ as an $\infty$-operad, and that $\text{ev}_1 : \mathcal{X}\mathcal{M}_{\mathcal{O}}^\circ \to \mathcal{O}^\circ$ is a map of $\infty$-operads.

**Proposition 2.1.** The map $r : \mathcal{X}\mathcal{M}_{\mathcal{O}}^\circ \to \text{Fin}_*$ exhibits $\mathcal{X}\mathcal{M}_{\mathcal{O}}^\circ$ as an $\infty$-operad.

**Proof.** Let $\alpha : \oplus_m X \to Y \simeq Y_1 \oplus \cdots \oplus Y_n$ be an object of $\mathcal{X}\mathcal{M}_{\mathcal{O}}^\circ$ over $\alpha : \langle m \rangle \to \langle n \rangle$ in $\text{Fin}_*$. We show that there is an $r$-coCartesian morphism $\rho : \alpha \to \beta$ over $\rho' : \langle n \rangle \to \langle 1 \rangle$. We decompose the composite $\mathcal{X} \circ \alpha : \oplus_m X \to Y \to Y_1$ as an inert morphism $\oplus_m X \to Z$ followed by an active morphism $Z \to Y_1$, where $\mathcal{X} : Y \to Y_1$ is a $p$-coCartesian morphism in $\mathcal{O}^\circ$ over $\rho'$. This
determines a morphism $\rho^i_i : \alpha \to \beta$ in $\mathcal{X}\mathcal{M}_{\mathcal{O}}^\otimes$ over $\rho^i$. We can verify that $\rho^i_i : \alpha \to \beta$ is an $\mathcal{r}$-coCartesian morphism.

By using the fact that $p : \mathcal{O}^\otimes \to \text{Fin}_*$ is an $\infty$-operad, we can verify that the $\mathcal{r}$-coCartesian morphisms $\rho^i_i$ for $1 \leq i \leq n$ induces an equivalence $\prod_{1 \leq i \leq n} \rho^i_i : (\mathcal{X}\mathcal{M}_{\mathcal{O}}^\otimes)_n \to \prod_{1 \leq i \leq n} (\mathcal{X}\mathcal{M}_{\mathcal{O}}^\otimes)_i$ of $\infty$-categories for any $n \geq 0$, and that the family $\{\rho^i_i \circ \rho^i_i \circ \rho^i_i\}_{1 \leq i \leq n}$ of morphisms is an $\mathcal{r}$-limit diagram in $\mathcal{X}\mathcal{M}_{\mathcal{O}}^\otimes$.

**Corollary 2.2.** The map $\text{ev}_1 : \mathcal{X}\mathcal{M}_{\mathcal{O}}^\otimes \to \mathcal{O}^\otimes$ is a map of $\infty$-operads.

**Proof.** Let $\alpha : \oplus_m X \to Y \simeq Y_1 \oplus \cdots \oplus Y_n$ be an object of $\mathcal{X}\mathcal{M}_{\mathcal{O}}^\otimes$, and let $\rho^i : (n) \to (1)$ be the inert morphism in $\text{Fin}_*$. By the proof of Proposition 2.1 we have $\text{ev}_1(\rho^i : \alpha \to \rho^i(\alpha)) = (\overline{\rho}^i : Y \to Y_i)$. Hence we see that $\text{ev}_1$ preserves inert morphisms. □

Now, we will introduce an $\infty$-category $\mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1}$ which is equivalent to $\mathcal{X}\mathcal{K}_\mathcal{O}$ and show that $\mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1}$ is an approximation to the $\infty$-operad $\mathcal{X}\mathcal{M}_{\mathcal{O}}^\otimes$.

We let $\tilde{\mathcal{M}}^{\leq 1}_{\mathcal{O}}$ be the full subcategory of $\tilde{\mathcal{M}}^{\otimes}_{\mathcal{O}}$ spanned by those vertices which correspond to maps $\oplus_m X \to Y_1 \oplus \cdots \oplus Y_n$ for $0 \leq m \leq 1$. Note that $\mathcal{X}\mathcal{K}_\mathcal{O}$ is a full subcategory of $\tilde{\mathcal{M}}^{\leq 1}_{\mathcal{O}}$. We define $\mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1}$ to be the full subcategory of $\mathcal{X}\mathcal{M}_{\mathcal{O}}^{\otimes}$ spanned by those vertices which correspond to maps $\oplus_m X \to Y_1 \oplus \cdots \oplus Y_n$ for $0 \leq m \leq 1$. Note that $\mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1}$ is a full subcategory of $\tilde{\mathcal{M}}^{\leq 1}_{\mathcal{O}}$.

We consider a right Kan extension of the identity functor $\mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1} \to \mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1}$ along the inclusion functor $\mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1} \hookrightarrow \tilde{\mathcal{M}}^{\leq 1}_{\mathcal{O}}$:

$$
\begin{array}{ccc}
\mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1} & \xrightarrow{R} & \mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1} \\
\downarrow & & \\
\tilde{\mathcal{M}}^{\leq 1}_{\mathcal{O}} & & \\
\end{array}
$$

Let $\alpha : X \to Y \simeq Y_1 \oplus \cdots \oplus Y_n$ be an object of $\tilde{\mathcal{M}}^{\leq 1}_{\mathcal{O}}$, which is not contained in $\mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1}$. Note that $\alpha$ is a null morphism (see [7], Definition 3.3.1.1) for the definition of null morphisms). We observe that the $\infty$-category $\mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1} \times \tilde{\mathcal{M}}^{\leq 1}_{\mathcal{O}}$ has an initial object $(0) \to Y$. Hence there exists a right Kan extension $R : \tilde{\mathcal{M}}^{\leq 1}_{\mathcal{O}} \to \mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1}$ by [4], Lemma 4.3.2.13).

By restricting $R$ to $\mathcal{X}\mathcal{K}_\mathcal{O}$, we obtain a functor $R : \mathcal{X}\mathcal{K}_\mathcal{O} \to \mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1}$.

We will show that the functor $R$ gives an equivalence of $\infty$-categories.

**Lemma 2.3.** The functor $R : \mathcal{X}\mathcal{K}_\mathcal{O} \to \mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1}$ is an equivalence of $\infty$-categories.

**Proof.** By the same argument as above, we see that there exists a left Kan extension $L : \tilde{\mathcal{M}}^{\leq 1}_{\mathcal{O}} \to \mathcal{X}\mathcal{K}_\mathcal{O}$ of the identity functor $\mathcal{X}\mathcal{K}_\mathcal{O} \to \mathcal{X}\mathcal{K}_\mathcal{O}$ along the inclusion functor $\mathcal{X}\mathcal{M}_{\mathcal{O}} \to \mathcal{X}\tilde{\mathcal{M}}_{\mathcal{O}}^{\leq 1}$. By restricting $L$ to the full subcategory $\mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1} \subset \tilde{\mathcal{M}}^{\leq 1}_{\mathcal{O}}$, we obtain a functor $L : \mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1} \to \mathcal{X}\mathcal{K}_\mathcal{O}$. We can easily verify that the pair $(L, R)$ of functors gives an equivalence of $\infty$-categories. □

Next, we will show that $\text{Mod}^\mathcal{O}(\mathcal{C})_{\mathcal{X}}$ is equivalent to the $\infty$-category $\text{Alg}_{\mathcal{X}\mathcal{M}_{\mathcal{O}}^\otimes}(\mathcal{C})$ of $\mathcal{X}\mathcal{M}_{\mathcal{O}}$-algebra objects in $\mathcal{C}^\otimes$.

**Lemma 2.4.** The inclusion functor $i : \mathcal{X}\mathcal{M}_{\mathcal{O}}^{\leq 1} \hookrightarrow \tilde{\mathcal{M}}_{\mathcal{O}}^{\otimes}$ is an approximation to the $\infty$-operad $\mathcal{X}\mathcal{M}_{\mathcal{O}}^{\otimes}$. 

Proof. We will verify the conditions in [7 Definition 2.3.6]. Since $X \mathcal{M}_O^{\leq 1}$ is a full subcategory of $X \mathcal{M}_O^{\leq 1}$, we see that condition (1) holds. Let $\alpha : \oplus_m X \to Y$ be an object of $X \mathcal{M}_O^{\leq 1}$ and let $\beta : \oplus_n X \to Z$ be an object of $X \mathcal{M}_O^{\leq 1}$. We suppose that there exists an active morphism $\phi : \beta \to \alpha$. Since $X \mathcal{M}_O^{\leq 1}$ is a full subcategory of $X \mathcal{M}_O$, in order to show that condition (2) holds, it suffices to show that $\beta$ is an object of $X \mathcal{M}_O^{\leq 1}$. The existence of the active morphism $\phi$ implies that $m = n$. Thus, we see that $\beta \in X \mathcal{M}_O^{\leq 1}$.

By [7 Definition 2.3.3.20], we have an $\infty$-category $\text{Alg}_{X \mathcal{M}_O^{\leq 1}}(X)$ for an $\infty$-operad $X^\otimes \to \text{Fin}_*$, which is a full subcategory of $\text{Fun}_{\text{Fin}_*}(X \mathcal{M}_O^{\leq 1}, X^\otimes)$ spanned by those functors which preserve inert morphisms. The map $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ of $\infty$-operads induces a map $q_* : \text{Alg}_{X \mathcal{M}_O^{\leq 1}}(\mathcal{C}) \to \text{Alg}_{X \mathcal{M}_O^{\leq 1}}(\mathcal{O})$ of $\infty$-categories. We define

$$\text{Alg}_{X \mathcal{M}_O^{\leq 1}/\mathcal{O}}(\mathcal{C})$$



to be the fiber of the map $q_*$ at $ev_1 \circ i : X \mathcal{M}_O^{\leq 1} \to X \mathcal{M}_O^{\leq 1} \to \mathcal{O}^\otimes$.

**Lemma 2.5.** The inclusion functor $i : X \mathcal{M}_O^{\leq 1} \hookrightarrow X \mathcal{M}_O^{\leq 1}$ induces an equivalence

$$i^* : \text{Alg}_{X \mathcal{M}_O/\mathcal{O}}(\mathcal{C}) \xrightarrow{\simeq} \text{Alg}_{X \mathcal{M}_O^{\leq 1}/\mathcal{O}}(\mathcal{C}).$$

**Proof.** We have a commutative diagram

$$
\begin{array}{ccc}
\text{Alg}_{X \mathcal{M}_O/\mathcal{O}}(\mathcal{C}) & \xrightarrow{q_*} & \text{Alg}_{X \mathcal{M}_O}(\mathcal{O}) \\
\downarrow{i^*} & & \downarrow{i^*} \\
\text{Alg}_{X \mathcal{M}_O^{\leq 1}/\mathcal{O}}(\mathcal{C}) & \xrightarrow{q_*} & \text{Alg}_{X \mathcal{M}_O^{\leq 1}}(\mathcal{O})
\end{array}
$$

of $\infty$-categories. The lemma follows from the fact that the vertical arrows are equivalences by [7 Theorem 2.3.3.23].

**Proposition 2.6.** The composite $i \circ R : X K_\mathcal{O} \to X \mathcal{M}_O^{\leq 1} \to X \mathcal{M}_O^{\leq 1}$ induces an equivalence

$$\text{Alg}_{X \mathcal{M}_O/\mathcal{O}}(\mathcal{C}) \xrightarrow{\simeq} \text{Mod}^\mathcal{O}(\mathcal{C})^\otimes_X.$$

**Proof.** Since $R$ is an equivalence by Lemma 2.3 and preserves inert morphisms, it induces an equivalence $R^* : \text{Mod}^\mathcal{O}(\mathcal{C})^\otimes_X \xrightarrow{\simeq} \text{Alg}_{X \mathcal{M}_O^{\leq 1}/\mathcal{O}}(\mathcal{C})$. The proposition follows from Lemma 2.5.\[\square\]

### 2.3. The operad $X \mathcal{M}_O^{\text{tr}, \otimes}$

We would like to construct a free functor $\text{Alg}_{/\mathcal{O}}(\mathcal{C}) \times C_X^\otimes \to \text{Mod}^\mathcal{O}(\mathcal{C})^\otimes_X$ for $X \in \mathcal{O}$, which is left adjoint to the forgetful functor. For this purpose, in this subsection we introduce an $\infty$-operad $X \mathcal{M}_O^{\text{tr}, \otimes}$ such that the $\infty$-category $\text{Alg}_{X \mathcal{M}_O^{\text{tr}, \otimes}//\mathcal{O}}(\mathcal{C})$ of $X \mathcal{M}_O^{\text{tr}, \otimes}$-algebras in $C^\otimes$ is equivalent to $\text{Alg}_{/\mathcal{O}}(\mathcal{C}) \times C_X^\otimes$.

First, we introduce an $\infty$-operad $X \mathcal{M}_O^{\text{tr}, \otimes}$ such that the $\infty$-category $\text{Alg}_{X \mathcal{M}_O^{\text{tr}, \otimes}//\mathcal{O}}(\mathcal{C})$ of $X \mathcal{M}_O^{\text{tr}, \otimes}$-algebras in $C^\otimes$ is equivalent to $\text{Alg}_{X \mathcal{M}_O}/\mathcal{O}(\mathcal{C})$. We define

$$X \mathcal{M}_O^{\text{tr}, \otimes}$$

to be a subcategory of $X \mathcal{M}_O^{\text{tr}, \otimes}$ as follows: The objects of $X \mathcal{M}_O^{\text{tr}, \otimes}$ are the same as those of $X \mathcal{M}_O^{\text{tr}, \otimes}$. A morphism

$$\begin{array}{ccc}
\oplus_m X & \xrightarrow{\alpha} & Y_1 \oplus \cdots \oplus Y_n \\
\downarrow{\beta} & & \downarrow{\gamma} \\
\oplus_m X & \xrightarrow{\beta} & Z_1 \oplus \cdots \oplus Z_n
\end{array}$$

in $X\mathcal{M}_O^{\oplus}$ is a morphism in $X\mathcal{M}_O^{\text{tr}\oplus}$ if and only if the image

\[ \begin{array}{ccc}
    \langle m \rangle & \xrightarrow{\alpha} & \langle n \rangle \\
    \downarrow & & \downarrow \gamma \\
    \langle m' \rangle & \xrightarrow{\beta} & \langle n' \rangle
\end{array} \]

in $\text{Fin}_*$ satisfies the condition that the cardinality of the set $\gamma^{-1}(\beta(i))$ is just one for each $i \in \langle m' \rangle^\circ$. We can verify that $X\mathcal{M}_O^{\text{tr}\oplus}$ supports a structure of $\infty$-operad such that the inclusion functor $X\mathcal{M}_O^{\text{tr}\oplus} \hookrightarrow X\mathcal{M}_O^{\oplus}$ is a map of $\infty$-operads.

Next, we will show that the $\infty$-category $\text{Alg}_{X\mathcal{M}_O^{\text{tr}\oplus}/(C)}$ of $X\mathcal{M}_O^{\text{tr}\oplus}$-algebras in $C^\oplus$ is equivalent to $\text{Alg}_{/(C)\times C_X^\oplus}$. For this purpose, we construct an equivalence $O^\oplus \boxtimes \text{Triv}^\oplus \xrightarrow{\sim} X\mathcal{M}_O^{\text{tr}\oplus}$ of $\infty$-operads, where the left hand side is a coproduct of $\infty$-operads.

We define

\[ X\mathcal{M}_O^{0,\oplus} = \{0\} \times_{\text{Triv}^\oplus} X\mathcal{M}_O^{\oplus}. \]

We can verify that the map $r : X\mathcal{M}_O^{\oplus} \to \text{Fin}_*$ induces a map $r : X\mathcal{M}_O^{0,\oplus} \to \text{Fin}_*$ which exhibits $X\mathcal{M}_O^{0,\oplus}$ as an $\infty$-operad. Furthermore, the restriction of the map $\text{ev}_1 : X\mathcal{M}_O^{\oplus} \to O^\oplus$ induces an equivalence $\text{ev}_1 : X\mathcal{M}_O^{0,\oplus} \xrightarrow{\sim} O^\oplus$ of $\infty$-operads.

We notice that $X\mathcal{M}_O^{0,\oplus}$ is a full subcategory of $X\mathcal{M}_O^{\text{tr}\oplus}$ and that the inclusion functor $X\mathcal{M}_O^{0,\oplus} \hookrightarrow X\mathcal{M}_O^{\text{tr}\oplus}$ is a map of $\infty$-operads. Thus, we obtain a map $O^\oplus \to X\mathcal{M}_O^{\text{tr}\oplus}$ of $\infty$-operads since $X\mathcal{M}_O^{0,\oplus}$ is equivalent to $O^\oplus$. Furthermore, since there is an equivalence $\text{Alg}_{\text{Triv}}(X\mathcal{M}_O^{0,\oplus}) \simeq (X\mathcal{M}_O^{\text{tr}\oplus})_{(1)}$ and $(\text{id}_X : X \to X)$ is an object of $(X\mathcal{M}_O^{\text{tr}\oplus})_{(1)}$, there is a map $\text{Triv}^\oplus \to X\mathcal{M}_O^{\text{tr}\oplus}$ of $\infty$-operads corresponding to $\text{id}_X$. Combining these two maps, we obtain a map

\[ O^\oplus \boxtimes \text{Triv}^\oplus \longrightarrow X\mathcal{M}_O^{\text{tr}\oplus} \]

of $\infty$-operads.

By the definition of coproducts of $\infty$-operads ([7, §2.2.3]), we easily obtain the following lemma.

**Lemma 2.7.** The map $O^\oplus \boxtimes \text{Triv}^\oplus \xrightarrow{\sim} X\mathcal{M}_O^{\text{tr}\oplus}$ is an equivalence of $\infty$-operads.

**Corollary 2.8.** There is an equivalence

\[ \text{Alg}_{X\mathcal{M}_O^{\text{tr}\oplus}/(C)} \xrightarrow{\sim} \text{Alg}_{/O}(C) \times C_X^\oplus. \]

**Remark 2.9.** The inclusion functors $X\mathcal{M}_O^{0,\oplus} \hookrightarrow X\mathcal{M}_O^{\text{tr}\oplus} \hookrightarrow X\mathcal{M}_O^{\oplus}$ induce the following commutative diagram

\[ \begin{array}{ccc}
    \text{Alg}_{X\mathcal{M}_O^{\text{tr}\oplus}/(C)} & \xrightarrow{\simeq} & \text{Alg}_{X\mathcal{M}_O^{0,\oplus}/(C)} \\
    \downarrow \simeq & & \downarrow \simeq \\
    \text{Mod}^\oplus_{/X}(C) & \xrightarrow{\simeq} & \text{Alg}_{/O}(C) \times C_X^\oplus
\end{array} \]

**2.4. Free operadic modules.** For a map $f : A \to B$ in $\text{Alg}_{/O}(C)$ and $X \in O$, we have the restriction functor $f_X^* : \text{Mod}^\oplus_{/O}(C)_{/X} \to \text{Mod}^\oplus_{/O}(C)_{/X}$. In this subsection we first construct a left adjoint $f_{X!} : f_X^*$ under some conditions. After that, we study the monad $T_f$ associated to the adjunction $(f_{X!}, f_X^*)$ and describe $T_f(M)$ as a colimit of certain diagram for $M \in \text{Mod}^\oplus_{/O}(C)_{/X}$.

Let $\kappa$ be an uncountable regular cardinal. In this subsection we assume that $O^\oplus$ is an essentially $\kappa$-small coherent $\infty$-operad, and that $q : C^\oplus \to O^\oplus$ is an $O$-monoidal $\infty$-category which is compatible with $\kappa$-small colimits in the sense of [7, Definition 3.1.1.18 and Variant 3.1.1.19].
We set \( D^\otimes = \text{Mod}_A^O(C)^\otimes \) for simplicity. By [7 Corollary 3.4.4.6], the map \( D^\otimes \to O^\otimes \) is an \( O \)-monoidal infinity-category compatible with \( \kappa \)-small colimits. We can regard \( A \) as an object of \( D^\otimes \), and it is a unit object of the \( O \)-monoidal structure. Furthermore, there is an equivalence \( \text{Alg}_O(D) \simeq \text{Alg}_O(C)_A \) of infinity-categories by [6 Corollary 3.4.1.7]. Thus, we can regard \( f \) as an \( O \)-algebra object of \( D^\otimes \). By [6 Corollary 3.4.1.9], there is an equivalence \( \text{Mod}_f^O(D)^\otimes \to \text{Mod}_B^O(C)^\otimes \) of \( O \)-monoidal infinity-categories.

Let \( j : \mathcal{M}_{\mathcal{O}}^\otimes \to \mathcal{M}_{\mathcal{O}}^\otimes \) be the inclusion map, which is a map of \( \infty \)-operads. The map \( j \) induces a functor

\[
j^* : \text{Mod}_{\mathcal{O}}^O(D)^\otimes_X \to \text{Alg}_{\mathcal{O}}(D) \times D_X^\otimes.
\]

By [7 Corollary 3.1.5], the functor \( j^* \) admits a left adjoint

\[
j_! : \text{Alg}_{\mathcal{O}}(D) \times D_X^\otimes \to \text{Mod}_{\mathcal{O}}^O(D)^\otimes_X
\]

which is obtained by the operadic left Kan extension along \( j \). Note that \( j_! \) makes the following diagram commute

\[
\begin{array}{ccc}
\text{Alg}_{\mathcal{O}}(D) \times D_X^\otimes & \xrightarrow{j_!} & \text{Mod}_{\mathcal{O}}^O(D)^\otimes_X \\
\downarrow & & \downarrow \\
\text{Alg}_{\mathcal{O}}(D).
\end{array}
\]

By taking fibers at \( f \in \text{Alg}_{\mathcal{O}}(D) \) in diagram (2.1), we obtain a functor \( f_{\mathcal{X}} : D_X^\otimes \to \text{Mod}_{\mathcal{O}}^O(D)^\otimes_X \), which is identified with a functor

\[
f_{\mathcal{X}} : \text{Mod}_{\mathcal{O}}^O(A)^\otimes_X \to \text{Mod}_{\mathcal{O}}^O(B)^\otimes_X.
\]

**Proposition 2.10.** The functor \( f_{\mathcal{X}} \) is a left adjoint to \( f_{\mathcal{X}}^* \).

**Proof.** For any \( (f, M) \in \text{Alg}_{\mathcal{O}}(D) \times D_X^\otimes \) and \( (f, N) \in \text{Mod}_{\mathcal{O}}^O(D)^\otimes_X \), we have a map

\[
\text{Map}_{\text{Mod}_{\mathcal{O}}^O(D)^\otimes_X}(j_!(f, M), (f, N)) \to \text{Map}_{\text{Alg}_{\mathcal{O}}(D) \times D_X^\otimes}((f, M), j^*(f, N))
\]

of mapping spaces over \( \text{Map}_{\text{Alg}_{\mathcal{O}}(D)}(f, f) \), which is an equivalence since \((j_!, j^*)\) is an adjoint pair.

By taking fibers at \( \text{id}_f \in \text{Map}_{\text{Alg}_{\mathcal{O}}(D)}(f, f) \), we obtain a natural equivalence

\[
\text{Map}_{\text{Mod}_f^O(D)^\otimes_X}(f_{\mathcal{X}}(M), N) \simeq \text{Map}_{D_X^\otimes}(M, f_{\mathcal{X}}^*(N))
\]

of mapping spaces, which completes the proof. \( \square \)

We let

\[
\mathcal{T}_f = f_{\mathcal{X}}^* \circ f_{\mathcal{X}} : \text{Mod}_{\mathcal{O}}^O(A)^\otimes_X \to \text{Mod}_{\mathcal{O}}^O(A)^\otimes_X
\]

be the monad associated to the adjunction \((f_{\mathcal{X}}, f_{\mathcal{X}}^*)\). Next, we study the monad \( \mathcal{T}_f \) and describe \( \mathcal{T}_f(M) \) as a colimit of certain diagram.

We set

\[
I(O) = \left((\mathcal{M}_{\mathcal{O}}^\otimes)^{\text{act}} \times (\mathcal{M}_{\mathcal{O}}^\otimes)^{\text{act}}\right)/(\text{id}_X).
\]

For \( (f, M) \in \text{Alg}_{\mathcal{O}}(D) \times D_X^\otimes \), we will construct a functor

\[
\mathcal{T}(f, M) : I(O)^\otimes \to D_X^\otimes \simeq \text{Mod}_{\mathcal{O}}^O(A)^\otimes_X,
\]

which is a colimit diagram and carries the cone point to \( \mathcal{T}_f(M) \).

By Corollary 2.8, a pair \((f, M) \in \text{Alg}_{\mathcal{O}}(D) \times D_X^\otimes\) determines a map

\[
(f, M) : \mathcal{M}_{\mathcal{O}}^\otimes \to D^\otimes
\]
natural transformations \(\alpha\) over \(O\). By the operadic left Kan extension \(j\), we have an operadic \(q\)-colimit diagram

\[
\begin{array}{ccc}
I(O) & \rightarrow & (x \mathcal{M}_O^{tr,\otimes})_{act} \\
\downarrow & & \downarrow \\
I(O)^\circ & \rightarrow & (x \mathcal{M}_O^{\otimes})_{act/\text{id}_X} \end{array}
\]

\[
\rightarrow \mathcal{D}^\otimes
\]

by \([7, \text{Proposition 3.1.3.3}]\), where the bottom arrows carries the cone point of \(I(O)^\circ\) to \(X \in O\). By the coCartesian pushforward of the map \(I(O)^\circ \rightarrow D^\otimes\), we obtain a functor \(\overline{f}\) : \(I(O)^\circ \rightarrow D^\otimes\), which is a colimit diagram and carries the cone point to \(T_f(M)\).

By the functoriality of construction, we obtain the following proposition.

**Proposition 2.11.** We have a functor

\[
D : \text{Alg}_{/O}(C)_{A_J} \times \text{Mod}_O^\circ(C)_{X}^\otimes \rightarrow \text{Fun}(I(O), \text{Mod}_O^\circ(C)_{X}^\otimes)
\]

which assigns to a pair \((f, M) \in \text{Alg}_{/O}(C)_{A_J} \times \text{Mod}_O^\circ(C)_{X}^\otimes\) a diagram \(D(f, M) : I(O) \rightarrow \text{Mod}_O^\circ(C)_{X}^\otimes\) such that

\[
T_f(M) \simeq \text{colim}_{I(O)} D(f, M).
\]

### 3. Adjointable diagrams of monoidal \(\infty\)-categories

In this section we introduce \(\infty\)-categories \(\text{Mod}_O^{\otimes, L}(\text{Cat}_\infty)^{R\text{Ad}}\) and \(\text{Mod}_O^{\otimes, R}(\text{Cat}_\infty)^{L\text{Ad}}\), and study their properties. In \([86]\) we construct \(\text{Mod}_O^{\otimes, L}(\text{Cat}_\infty)^{R\text{Ad}}\) equipped with a Cartesian fibration \(\text{Mod}_O^{\otimes, L}(\text{Cat}_\infty)^{R\text{Ad}} \rightarrow \text{Cat}_\infty\) encoding diagrams of \(O\)-monoidal \(\infty\)-categories, left adjoint \(O\)-monoidal functors, and right adjointable commutative squares. We also construct \(\text{Mod}_O^{\otimes, R}(\text{Cat}_\infty)^{L\text{Ad}}\) equipped with a Cartesian fibration \(\text{Mod}_O^{\otimes, R}(\text{Cat}_\infty)^{L\text{Ad}} \rightarrow \text{Cat}_\infty\) encoding diagrams of \(O\)-monoidal \(\infty\)-categories, right adjoint \(O\)-monoidal functors, and left adjointable commutative squares. We show that there is an equivalence of \(\infty\)-categories between \(\text{Mod}_O^{\otimes, L}(\text{Cat}_\infty)^{R\text{Ad}}\) and \(\text{Mod}_O^{\otimes, L}(\text{Cat}_\infty)^{R\text{Ad}}\) by taking left adjoints to right adjoint lax \(O\)-monoidal functors. In \([86, \text{Section 3.2}]\) we show that \(\text{Mod}_O^{\otimes, R}(\text{Cat}_\infty)^{L\text{Ad}}\) is equivalent to a subcategory of \(\text{Op}^\text{gen}_{\infty/\text{O}}\), where \(\text{Op}^\text{gen}_{\infty/\text{O}}\) is the \(\infty\)-category of generalized \(\infty\)-operads. Dually, in \([86, \text{Section 3.3}]\) we show that \(\text{Mod}_O^{\otimes, L}(\text{Cat}_\infty)^{R\text{Ad}}\) is equivalent to a subcategory of \(\text{Op}^\text{gen}_{\infty/\text{O}}\).

#### 3.1. An equivalence between \(\text{Mod}_O^{\otimes, R}(\text{Cat}_\infty)^{L\text{Ad}}\) and \(\text{Mod}_O^{\otimes, L}(\text{Cat}_\infty)^{R\text{Ad}}\)

In this subsection we construct Cartesian fibrations \(\text{Mod}_O^{\otimes, L}(\text{Cat}_\infty)^{R\text{Ad}} \rightarrow \text{Cat}_\infty\) and \(\text{Mod}_O^{\otimes, R}(\text{Cat}_\infty)^{L\text{Ad}} \rightarrow \text{Cat}_\infty\) by taking left adjoints to right adjoint lax \(O\)-monoidal functors, we show that there is an equivalence of \(\infty\)-categories between \(\text{Mod}_O^{\otimes, R}(\text{Cat}_\infty)^{L\text{Ad}}\) and \(\text{Mod}_O^{\otimes, L}(\text{Cat}_\infty)^{R\text{Ad}}\).

We have a wide subcategory

\[
\text{Mon}_O^{\text{Lax}, R}(\text{Cat}_\infty)
\]

of \(\text{Mon}_O^{\text{Lax}}(\text{Cat}_\infty)\) spanned by those lax \(O\)-monoidal functors \(f^*\) such that \(f^*_X\) is right adjoint for each \(X \in O\). For an \(\infty\)-category \(S\), we denote by

\[
\text{Fun}^{\text{LAd}}(S, \text{Mon}_O^{\text{Lax}, R}(\text{Cat}_\infty))
\]

the \(\infty\)-category whose objects are functors \(F : S \rightarrow \text{Mon}_O^{\text{Lax}, R}(\text{Cat}_\infty)\) and whose functors are natural transformations \(\alpha : F \rightarrow G\) such that \(\alpha(s) : F(s) \rightarrow G(s)\) is strong \(O\)-monoidal functors
for each \( s \in S \), and that the following commutative diagram

\[
\begin{array}{ccc}
F(s)_X & \rightarrow & F(s')_X \\
\downarrow & & \downarrow \\
G(s)_X & \rightarrow & G(s')_X
\end{array}
\]

is left adjointable for any morphism \( s \rightarrow s' \) in \( S \) and any \( X \in \mathcal{O} \) (see [7, Definition 4.7.4.13] for the definition of left adjointable diagrams).

We also have a wide subcategory

\[ \text{Mon}^\text{oplax,L}_{\mathcal{O}}(\text{Cat}_\infty) \]

of \( \text{Mon}^\text{oplax}_{\mathcal{O}}(\text{Cat}_\infty) \) spanned by those oplax \( \mathcal{O} \)-monoidal functors \( f \) such that \( f|_X \) is left adjoint for each \( X \in \mathcal{O} \). We denote by

\[ \text{Fun}^\text{Rad}_{\mathcal{O}}(S, \text{Mon}^\text{oplax,L}_{\mathcal{O}}(\text{Cat}_\infty)) \]

the \( \infty \)-category whose objects are functors \( F : S \rightarrow \text{Mon}^\text{oplax,L}_{\mathcal{O}}(\text{Cat}_\infty) \) and whose functors are natural transformations \( \alpha : F \rightarrow G \) such that \( \alpha(s) : F(s) \rightarrow G(s) \) is strong \( \mathcal{O} \)-monoidal functors for each \( s \in S \), and that the following commutative diagram

\[
\begin{array}{ccc}
F(s)_X & \rightarrow & F(s')_X \\
\downarrow & & \downarrow \\
G(s)_X & \rightarrow & G(s')_X
\end{array}
\]

is right adjointable for any morphism \( s \rightarrow s' \) in \( S \) and any \( X \in \mathcal{O} \) (see [7, Definition 4.7.4.13] for the definition of right adjointable diagrams).

First, we shall show that there is a natural equivalence between \( \text{Fun}^\text{LAd}_{\mathcal{O}}(S, \text{Mon}^\text{oplax,R}_{\mathcal{O}}(\text{Cat}_\infty)) \) and \( \text{Fun}^\text{Rad}_{\mathcal{O}}(S^{\text{op}}, \text{Mon}^\text{oplax,L}_{\mathcal{O}}(\text{Cat}_\infty)) \) by taking left adjoints to right adjoint \( \mathcal{O} \)-monoidal functors.

**Proposition 3.1.** For any \( \infty \)-category \( S \), there is a natural equivalence

\[
\text{Fun}^\text{LAd}_{\mathcal{O}}(S, \text{Mon}^\text{oplax,R}_{\mathcal{O}}(\text{Cat}_\infty)) \simeq \text{Fun}^\text{Rad}_{\mathcal{O}}(S^{\text{op}}, \text{Mon}^\text{oplax,L}_{\mathcal{O}}(\text{Cat}_\infty))
\]

of \( \infty \)-categories.

**Proof.** It suffices to show that there is a natural equivalence

\[
\text{Map}_{\text{Cat}_\infty}([n], \text{Fun}^\text{LAd}_{\mathcal{O}}(S, \text{Mon}^\text{oplax,R}_{\mathcal{O}}(\text{Cat}_\infty))) \simeq \text{Map}_{\text{Cat}_\infty}([n], \text{Fun}^\text{Rad}_{\mathcal{O}}(S^{\text{op}}, \text{Mon}^\text{oplax,L}_{\mathcal{O}}(\text{Cat}_\infty)))
\]

for any \([n] \in \Delta^{\text{op}}\).

We let \( (\text{Cat}_{\infty/\mathcal{B}})_{\text{lax}}^{\text{coc}} \) be a subcategory of \( \text{Cat}_{\infty/\mathcal{B}} \) whose objects are coCartesian fibrations and whose morphisms are \( \mathcal{B} \)-parametrized right adjoints, and let \( (\text{Cat}_{\infty/\mathcal{B}})_{\text{oplax}}^{\text{cart}} \) be a subcategory of \( \text{Cat}_{\infty/\mathcal{B}} \) whose objects are Cartesian fibrations and whose morphisms are \( \mathcal{B} \)-parametrized left adjoints (see [3, Definition 2.1] for the definition of \( \mathcal{B} \)-parametrized left and right adjoints). There are inclusions

\[
\begin{align*}
\text{Map}_{\text{Cat}_\infty}([n], \text{Fun}^\text{LAd}_{\mathcal{O}}(S, \text{Mon}^\text{oplax,R}_{\mathcal{O}}(\text{Cat}_\infty))) & \hookrightarrow \text{Map}_{\text{Cat}_\infty}(S, (\text{Cat}_{\infty/\mathcal{O}}^{\text{coc}})^{\text{radj}}_{\text{lax}}), \\
\text{Map}_{\text{Cat}_\infty}([n], \text{Fun}^\text{Rad}_{\mathcal{O}}(S^{\text{op}}, \text{Mon}^\text{oplax,L}_{\mathcal{O}}(\text{Cat}_\infty))) & \hookrightarrow \text{Map}_{\text{Cat}_\infty}(S^{\text{op}}, (\text{Cat}_{\infty}^{\text{cart}})^{\text{ladj}}_{\text{oplax}})
\end{align*}
\]

of mapping spaces. By [3, Theorem 2.2], we have an equivalence

\[
\text{Map}_{\text{Cat}_\infty}(S, (\text{Cat}_{\infty/\mathcal{O}}^{\text{coc}})^{\text{radj}}_{\text{lax}}) \simeq \text{Map}_{\text{Cat}_\infty}(S^{\text{op}}, (\text{Cat}_{\infty}^{\text{cart}})^{\text{ladj}}_{\text{oplax}}),
\]

which restricts to the desired equivalence. \( \square \)
We define
\[ \text{Mon}^{\text{lax}, R}(\text{Cat}_\infty)^{L\text{Ad}} \to \text{Cat}_\infty \]
to be a Cartesian fibration which is obtained by unstraightening of the functor \( \text{Cat}_\infty^{\text{op}} \to \text{\hat{Cat}}_\infty \) given by \( S \mapsto \text{Fun}^{L\text{Ad}}(S, \text{Mon}^{\text{lax}, R}(\text{Cat}_\infty)) \). We also define
\[ \text{Mon}^{\text{oplax}, L}(\text{Cat}_\infty)^{R\text{Ad}} \to \text{Cat}_\infty \]
to be a Cartesian fibration which is obtained by unstraightening of the functor \( \text{Cat}_\infty^{\text{op}} \to \text{\hat{Cat}}_\infty \) given by \( S \mapsto \text{Fun}^{R\text{Ad}}(S, \text{Mon}^{\text{oplax}, L}(\text{Cat}_\infty)) \). By Proposition 3.1, we obtain the following theorem.

**Theorem 3.2.** There is an equivalence
\[ \text{Mon}^{\text{lax}, R}(\text{Cat}_\infty)^{L\text{Ad}} \simeq \text{Mon}^{\text{oplax}, L}(\text{Cat}_\infty)^{R\text{Ad}} \]
of \( \infty \)-categories, which fits into the following commutative diagram
\[ \begin{array}{ccc} \text{Mon}^{\text{lax}, R}(\text{Cat}_\infty)^{L\text{Ad}} & \xrightarrow{\sim} & \text{Mon}^{\text{oplax}, L}(\text{Cat}_\infty)^{R\text{Ad}} \\ \downarrow & & \downarrow \\ \text{Cat}_\infty & \xrightarrow{(-)^{\text{op}}} & \text{Cat}_\infty. \end{array} \]

**Remark 3.3.** We can construct a double \( \infty \)-category
\[ \text{Mon}^{\text{lax}, R}(\text{Cat}_\infty)^{L\text{Ad}} \]
whose objects are \( \mathcal{O} \)-monoidal \( \infty \)-categories, whose horizontal 1-morphisms are strong \( \mathcal{O} \)-monoidal functors, whose vertical 1-morphisms are right adjoint lax \( \mathcal{O} \)-monoidal functors, and whose 2-morphisms are commutative squares which are left adjointable for each \( X \in \mathcal{O} \).

We can also construct a double \( \infty \)-category
\[ \text{Mon}^{\text{oplax}, L}(\text{Cat}_\infty)^{R\text{Ad}} \]
whose objects are \( \mathcal{O} \)-monoidal \( \infty \)-categories, whose horizontal 1-morphisms are strong \( \mathcal{O} \)-monoidal functors, whose vertical 1-morphisms are left adjoint oplax \( \mathcal{O} \)-monoidal functors, and whose 2-morphisms are commutative squares which are right adjointable for each \( X \in \mathcal{O} \).

There is an equivalence
\[ \text{Mon}^{\text{lax}, R}(\text{Cat}_\infty)^{L\text{Ad}} \simeq (\text{Mon}^{\text{oplax}, L}(\text{Cat}_\infty)^{R\text{Ad}})^{v\text{-op}}, \]
where the right hand side is the double \( \infty \)-category obtained from \( \text{Mon}^{\text{oplax}, L}(\text{Cat}_\infty)^{R\text{Ad}} \) by reversing the vertical direction.

3.2. \( \text{Mon}^{\text{lax}, R}(\text{Cat}_\infty)^{L\text{Ad}} \) and \( \text{Op}^{\text{gen}}_{\mathcal{O}_0} \). In this subsection we show that \( \text{Mon}^{\text{lax}, R}(\text{Cat}_\infty)^{L\text{Ad}} \) is equivalent to a subcategory of \( \text{Op}^{\text{gen}}_{\mathcal{O}_0} \), where \( \text{Op}^{\text{gen}}_{\mathcal{O}_0} \) is the \( \infty \)-category of generalized \( \infty \)-operads. For this purpose, we study a functor
\[ (-)_{(0)} : \text{Op}^{\text{gen}}_{\mathcal{O}_0} \to \text{Cat}_\infty \]
which assigns the fiber \( \mathcal{E}^{\mathcal{O}_0}_{(0)} \) at \( (0) \in \mathcal{O}_0 \) to a map \( \mathcal{E}^{\mathcal{O}_0} \to \mathcal{O}_0 \) of generalized \( \infty \)-operads.

**Lemma 3.4.** The functor \( (-)_{(0)} : \text{Op}^{\text{gen}}_{\mathcal{O}_0} \to \text{Cat}_\infty \) is a Cartesian fibration.

**Proof.** Let \( F : \text{Op}^{\text{gen}}_{\mathcal{O}_0} \to \text{Cat}_\infty \) be the functor which associates the fiber \( \mathcal{X}^{\mathcal{O}_0} \) at \( (0) \in \text{Fin}_* \) to a generalized \( \infty \)-operad \( \mathcal{X}^{\mathcal{O}_0} \to \text{Fin}_* \). By [7] Proposition 2.3.2.9, the functor \( F \) has a right adjoint \( G : \text{Cat}_\infty \to \text{Op}^{\text{gen}}_{\mathcal{O}_0} \) which is given by \( C \mapsto C \times \text{Fin}_* \).
Let $\mathcal{E}^\otimes \to \mathcal{O}^\otimes$ be a map of generalized $\infty$-operads. We set $S = \mathcal{E}^\otimes_{(0)}$. By the unit of the adjunction $(F, G)$, we have a map $\mathcal{E}^\otimes \to S \times \text{Fin}_*$ of generalized $\infty$-operads. This induces a map $\mathcal{E}^\otimes \to (S \times \text{Fin}_*) \times \text{Fin}_*, \mathcal{O}^\otimes \simeq S \times \mathcal{O}^\otimes$ of generalized $\infty$-operads.

For any functor $f : T \to S$ in $\text{Cat}_\infty$, we can verify that the projection $T \times_S \mathcal{E}^\otimes \to \mathcal{E}^\otimes$ is a Cartesian morphism in $\text{Op}^\text{gen}_{\mathcal{O}^\otimes}$ covering $f$. Hence the map $(-)_{(0)} : \text{Op}^\text{gen}_{\mathcal{O}^\otimes} \to \text{Cat}_\infty$ is a Cartesian fibration.

We recall that $\text{Mon}^\text{lax,R}_{\mathcal{O}}(\text{Cat}_\infty)^{\text{LAd}} \to \text{Cat}_\infty$ is a Cartesian fibration which is associated to the functor $\text{Cat}^\text{op}_\infty \to \text{Cat}_\infty$ given by $S \mapsto \text{Fun}^{\text{LAd}}(S, \text{Mon}^\text{lax,R}_{\mathcal{O}}(\text{Cat}_\infty))$. By the definition of the $\infty$-category $\text{Fun}^{\text{LAd}}(S, \text{Mon}^\text{lax,R}_{\mathcal{O}}(\text{Cat}_\infty))$, there is a natural map $\text{Fun}^{\text{LAd}}(S, \text{Mon}^\text{lax,R}_{\mathcal{O}}(\text{Cat}_\infty)) \to \text{Fun}(S, \text{Op}^\text{gen}_{\mathcal{O}^\otimes})$.

By [8, §3], we can identify the $\infty$-category $\text{Fun}(S, \text{Op}^\text{gen}_{\mathcal{O}^\otimes})$ with a subcategory of the slice category $\text{Cat}^\text{op}_\infty/S \times \mathcal{O}^\otimes$ for any $\infty$-category $S$. For an object of $\text{Fun}(S, \text{Op}^\text{gen}_{\mathcal{O}^\otimes})$, the corresponding object is a map

$$f : \mathcal{E}^\otimes \to S \times \mathcal{O}^\otimes$$

such that

- the map $f_S : \mathcal{E}^\otimes \to S$ is a coCartesian fibration, and $f$ preserves coCartesian morphisms, where $f_S$ is the composite of $f$ with the projection $S \times \mathcal{O}^\otimes \to S$,
- the restriction $f_s : \mathcal{E}^\otimes_s \to \mathcal{O}^\otimes$ is a map of $\infty$-operads for each $s \in S$, and
- the induced map $\mathcal{E}^\otimes_s \to \mathcal{E}^\otimes_{s'}$ over $\mathcal{O}^\otimes$ preserves inert morphisms for each morphism $s \to s'$ in $S$.

We can easily verify the following lemma.

**Lemma 3.5.** The map $f : \mathcal{E}^\otimes \to S \times \mathcal{O}^\otimes$ is a morphism of $\text{Op}^\text{gen}_{\mathcal{O}^\otimes}$ such that $\mathcal{E}^\otimes_{(0)} \to S$ is an equivalence of $\infty$-categories.

By Lemma 3.5, we can identify $\text{Fun}(S, \text{Op}^\text{gen}_{\mathcal{O}^\otimes})$ with a subcategory of $\text{Op}^\text{gen}_{\mathcal{O}^\otimes} \times \text{Cat}_\infty \{S\}$. Hence we obtain a natural transformation

$$\text{Fun}^{\text{LAd}}(-, \text{Mon}^\text{lax,R}_{\mathcal{O}}(\text{Cat}_\infty)) \to \text{Op}^\text{gen}_{\mathcal{O}^\otimes} \times \text{Cat}_\infty \{S\}$$

in which $\text{Fun}^{\text{LAd}}(S, \text{Mon}^\text{lax,R}_{\mathcal{O}}(\text{Cat}_\infty))$ is a subcategory of $\text{Op}^\text{gen}_{\mathcal{O}^\otimes} \times \text{Cat}_\infty \{S\}$ for any $S \in \text{Cat}_\infty$.

By unstraightening this natural transformation, we obtain the following proposition.

**Proposition 3.6.** The $\infty$-category $\text{Mon}^\text{lax,R}_{\mathcal{O}}(\text{Cat}_\infty)^{\text{LAd}}$ is equivalent to a subcategory of $\text{Op}^\text{gen}_{\mathcal{O}^\otimes}$.

The inclusion functor $\text{Mon}^\text{lax,R}_{\mathcal{O}}(\text{Cat}_\infty)^{\text{LAd}} \hookrightarrow \text{Op}^\text{gen}_{\mathcal{O}^\otimes}$ is a map of Cartesian fibrations over $\text{Cat}_\infty$.

### 3.3. $\text{Mon}^\text{op,plax,L}_{\mathcal{O}}(\text{Cat}_\infty)^{\text{RAd}}$ and $\text{Op}^\text{gen,v}_{\mathcal{O}^\otimes,\text{op}}$.

In this subsection we introduce an $\infty$-category $\text{Op}^\text{gen,v}_{\mathcal{O}^\otimes,\text{op}}$ which is equivalent to $\text{Op}^\text{gen}_{\mathcal{O}^\otimes}$ and show that $\text{Mon}^\text{op,plax,L}_{\mathcal{O}}(\text{Cat}_\infty)^{\text{RAd}}$ is equivalent to a subcategory of $\text{Op}^\text{gen,v}_{\mathcal{O}^\otimes,\text{op}}$.

We define an $\infty$-category

$\text{Op}^\text{gen,v}_{\mathcal{O}^\otimes,\text{op}}$

to be a subcategory of $\text{Cat}_\infty/\text{Fin}^\text{op}_\infty$ whose objects are maps $\mathcal{E}^\otimes \to \text{Fin}^\text{op}_\infty$ such that the opposite $\mathcal{E}^\otimes_{\text{op}} \to \text{Fin}_\infty$ is a generalized $\infty$-operad and whose morphisms are maps $\mathcal{E}^\otimes \to \mathcal{F}^\otimes$ over $\text{Fin}^\text{op}_\infty$ such that the opposite $\mathcal{E}^\otimes_{\text{op}} \to \mathcal{F}^\otimes_{\text{op}}$ is a map of generalized $\infty$-operads. We have a functor

$$(-)_{(0)} : \text{Op}^\text{gen,v}_{\mathcal{O}^\otimes,\text{op}} \to \text{Cat}_\infty$$
which assigns the fiber $E^\otimes_{(0)}$ at $0 \in O^\otimes_{op}$ to a map $E^\otimes \to O^\otimes_{op}$ in $Op^\gen,v$. Since $Op^\gen,v$ is equivalent to $Op^\gen_\infty$, we see that the map $(-)_{(0)} : Op^\gen,v_\infty/O^\otimes_{op}$ is a Cartesian fibration by Lemma 4.3.

By definition, there is a natural functor $\Fun(S, Mon^\oplax,L(Cat_\infty)) \to \Fun(S, Op^\gen,v_\infty/O^\otimes_{op})$. In the same way as in the case of $\Mon^\oplax,L(Cat_\infty)^{Rad}$, we can identify $\Fun(S, Op^\gen,v_\infty/O^\otimes_{op})$ with a subcategory of $Op^\gen,v_\infty/O^\otimes_{op} \times_{\Cat_\infty} \{S\}$. Hence we obtain a natural transformation

$$\Fun^{Rad}(-, Mon^\oplax,L(Cat_\infty)) \to Op^\gen,v_\infty/O^\otimes_{op} \times_{\Cat_\infty} \{-\}$$

in which $\Fun^{Rad}(S, Mon^\oplax,L(Cat_\infty))$ is a subcategory of $Op^\gen,v_\infty/O^\otimes_{op} \times_{\Cat_\infty} \{S\}$ for any $S \in \Cat_\infty$.

By unstraightening this natural transformation, we obtain the following proposition.

**Proposition 3.7.** The $\infty$-category $\Mon^\oplax,L(Cat_\infty)^{Rad}$ is equivalent to a subcategory of the $\infty$-category $Op^\gen,v_\infty/O^\otimes_{op}$. The inclusion functor $\Mon^\oplax,L(Cat_\infty)^{Rad} \hookrightarrow Op^\gen,v_\infty/O^\otimes_{op}$ is a map of Cartesian fibrations over $\Cat_\infty$.

### 4. Mixed Fibrations of Operadic Modules

Let $O^\otimes$ be a coherent $\infty$-operad and let $q : C^\otimes \to O$ be a map of $\infty$-operads. In §23 we recalled the construction of the map $(\Phi, \sigma) : Mod^O(C)^\otimes \to Alg_{/O}(C) \times O^\otimes$ of generalized $\infty$-operads. If $C^\otimes$ has a sufficient supply of colimits, then this encodes a structure consisting of $O$-monoidal $\infty$-categories $\Mod_A^O(C)^\otimes$ and lax $O$-monoidal functors $f^* : \Mod_B^O(C)^\otimes \to \Mod_A^O(C)^\otimes$. In §4.1 we construct a map $(\Psi, \tau) : Mod^O(C)^{\otimes,v} \to Alg_{/O}(C) \times O^\otimes_{op}$ which encodes a structure consisting of $O$-monoidal $\infty$-categories $\Mod_A^O(C)^\otimes$ and oplax $O$-monoidal functors $f_1 : \Mod_B^O(C)^\otimes \to \Mod_A^O(C)^\otimes$.

In §4.2 we study a functoriality of the construction of $(\Psi, \tau)$ for $C^\otimes$. We construct a coCartesian fibration $\Theta : Mod^{O,\Triple}(Cat_\infty)^{\otimes,v} \to Alg_{/O}^{Pair}(Cat_\infty)$ which is associated to a functor given by $(C^\otimes, A) \mapsto Mod_A^O(C)^{\otimes,v}$.

#### 4.1. Construction of the map $(\Psi, \tau)$

Let $\kappa$ be an uncountable regular cardinal. We assume that $O^\otimes$ is an essentially $\kappa$-small coherent $\infty$-operad and that $C^\otimes$ is an $O$-monoidal $\infty$-category which is compatible with $\kappa$-small colimits. In this subsection we construct a map $(\Psi, \tau) : Mod^O(C)^{\otimes,v} \to Alg_{/O}(C) \times O^\otimes_{op}$ which encodes a structure consisting of $O$-monoidal $\infty$-categories $\Mod_A^O(C)^\otimes$ and oplax $O$-monoidal functors $f_1 : \Mod_B^O(C)^\otimes \to \Mod_A^O(C)^\otimes$.

Recall that $\Phi : Mod^O(C)^\otimes \to Alg_{/O}(C)$ is a Cartesian fibration. By the straightening functor, there is a functor

$$\St(\Phi) : Alg_{/O}(C)^{op} \to \Cat_\infty,$$

which associates to $A \in Alg_{/O}(C)$ the $\infty$-category $\Mod_A^O(C)^\otimes$. We can lift this functor to a functor

$$\St(\Phi) : Alg_{/O}(C)^{op} \to \Mon_{/O}^{\text{lax},R}(\Cat_\infty).$$

In [§4, §5, §10], we have proved that there is an equivalence

$$\Mon_{/O}^{\text{oplax},L}(\Cat_\infty) \simeq (\Mon_{/O}^{\text{lax},R}(\Cat_\infty))^{op}\text{ of }\infty\text{-categories. By using this equivalence, we obtain a functor}$$

$$\Alg_{/O}(C) \to (\Mon_{/O}^{\text{lax},R}(\Cat_\infty))^{op} \simeq \Mon_{/O}^{\text{oplax},L}(\Cat_\infty) \hookrightarrow \Mon_{/O}^{\text{oplax}}(\Cat_\infty).$$

Now, we recall a description of $\Mon_{/O}^{\text{oplax}}(\Cat_\infty)$. Let $Op^\infty$ be a full subcategory of $Op^\gen,v_\infty$ spanned by those maps $E^\otimes \to \Fin_{op}^\infty$ such that the opposite $E^\otimes_{op} \to \Fin$, is an $\infty$-operad. We regard $\Mon_{/O}^{\text{oplax}}(\Cat_\infty)$ as a full subcategory of $Op^\infty/O^\otimes_{op}$ spanned by those maps $E^\otimes \to O^\otimes_{op}$.
which is a Cartesian fibration. We also say that an object of $\text{Mon}_O^{\oplus,\text{plax}}(\text{Cat}_\infty)$ is an $O$-monoidal $\infty$-category and a morphism is an oplax $O$-monoidal functor.

For a coCartesian fibration $\mathcal{X} \to S$ which is classified by a functor $S \to \text{Cat}_\infty$, we denote by $\mathcal{X}^{\vee} \to S^{\op}$ a Cartesian fibration which is classified by the same functor. For an $O$-monoidal $\infty$-category $\mathcal{E}^\odot \to O^\odot$, the Cartesian fibration $\mathcal{E}^{\odot,\vee} \to O^{\odot,\op}$ is an object in $\text{Mon}_O^{\oplus,\text{plax}}(\text{Cat}_\infty)$.

Note that the underlying $\infty$-category $\mathcal{E}_X^{\odot,\vee}$ of $\mathcal{E}^{\odot,\vee} \to O^{\odot,\op}$ is equivalent to the underlying $\infty$-category $\mathcal{E}_X^\odot$ of $\mathcal{E}^\odot \to O^\odot$ for each $X \in O$.

The functor $\text{Alg}_{/O}(C) \to \text{Mon}_O^{\oplus,\text{plax}}(\text{Cat}_\infty)$ induces a commutative diagram

\[
\begin{array}{ccc}
\text{Mod}_O^\odot(C)^{\odot,\vee} & \xrightarrow{(\Psi,\tau)} & \text{Alg}_{/O}(C) \times O^{\odot,\op} \\
\Psi & \downarrow & \pi \\
\text{Alg}_{/O}(C),
\end{array}
\]

where $\Psi$ is a coCartesian fibration, the map $\tau$ carries $\Psi$-coCartesian morphisms to equivalences, and $\pi$ is the projection. For each $A \in \text{Alg}_{/O}(C)$, the fiber of $(\Psi,\tau)$ at $A$ determines a Cartesian fibration

$$\tau_A : \text{Mod}_A^O(C)^{\odot,\vee} \to O^{\odot,\op},$$

which is an $O$-monoidal $\infty$-category equivalent to $\sigma_A : \text{Mod}_A^O(C)^\odot \to O^\odot$. A morphism $f : A \to B$ in $\text{Alg}_{/O}(C)$ induces an oplax $O$-monoidal functor

$$f_! : \text{Mod}_A^O(C)^{\odot,\vee} \to \text{Mod}_B^O(C)^{\odot,\vee},$$

where the restriction $f_{\ast X}$ is a left adjoint to $f_!^*$. For each $X \in O$.

In [5] Definition 3.15 we introduced the notion of mixed fibrations. A map $g : X \to S \times T$ of $\infty$-categories is a mixed fibration over $(S,T)$ if it satisfies the following conditions:

- The map $g_S : X \to S$ is a coCartesian fibration and $g$ preserves coCartesian morphisms, where $g_S$ is the composite of $g$ with the projection $S \times T \to S$.
- The map $g_T : X \to T$ is a Cartesian fibration and $g$ preserves Cartesian morphisms, where $g_T$ is the composite of $g$ with the projection $S \times T \to T$.

By [5] Proposition 3.25, we obtain the following proposition.

**Proposition 4.1.** There is a mixed fibration

$$(\Psi,\tau) : \text{Mod}_O^\odot(C)^{\odot,\vee} \longrightarrow \text{Alg}_{/O}(C) \times O^{\odot,\op}$$

over $(\text{Alg}_{/O}(C),O^{\odot,\op})$. For $A \in \text{Alg}_{/O}(C)$, the fiber of $(\Psi,\tau)$ at $A$ is a Cartesian fibration $\tau_A : \text{Mod}_A^O(C)^{\odot,\vee} \to O^{\odot,\op}$, which is an $O$-monoidal $\infty$-category equivalent to $\sigma_A$. For $f : A \to B$ in $\text{Alg}_{/O}(C)$, the induced functor $f_! : \text{Mod}_A^O(C)^{\odot,\vee} \to \text{Mod}_B^O(C)^{\odot,\vee}$ is an oplax $O$-monoidal functor, in which the restriction $f_{\ast X}$ is a left adjoint to $f_!^*$ for each $X \in O$.

**4.2. A functoriality of the construction of $(\Psi,\tau)$.** In this subsection we consider a functoriality of the construction of the map $(\Psi,\tau)$ for $C^\odot$. We construct a coCartesian fibration $\Theta : \text{Mod}^{O,\text{Triple}}(\text{Cat}_\infty^\omega)^{\odot,\vee} \to \text{Alg}_{/O}^\text{Pair}(\text{Cat}_\infty^\omega)$ which is associated to a functor given by $(C^\odot, A) \mapsto \text{Mod}_A^O(C)^{\odot,\vee}$.

Let $\text{Cat}_\infty^\omega$ be the subcategory of $\text{Cat}_\infty$ spanned by those small $\infty$-categories which have $\kappa$-small colimits, and those functors which preserve $\kappa$-colimits ([7] Definition 4.8.1.1). By [7] Corollary 4.8.1.4,
the $\infty$-category $\text{Cat}^{(\kappa)}_\infty$ inherits a symmetric monoidal structure from $\text{Cat}_\infty$, that is, there is a coCartesian fibration

$$\text{Cat}^{(\kappa)}_\infty \to \text{Fin}_*$$

of $\infty$-operads such that the inclusion functor $\text{Cat}^{(\kappa)}_\infty \simeq \text{Cat}_\infty$ is a map of $\infty$-operads. We have an $\infty$-category $\text{Alg}_O(\text{Cat}^{(\kappa)}_\infty)$ of $O$-algebra objects in $\text{Cat}^{(\kappa)}_\infty$. The objects are identified with $O$-$\kappa$-monoidal $\infty$-categories which are compatible with $\kappa$-small colimits by [7, Remark 4.8.1.9].

We have a functor

$$\text{Alg}/_O(-) : \text{Alg}_O(\text{Cat}^{(\kappa)}_\infty) \to \text{Cat}_\infty$$

which associates to $C\otimes$ the $\infty$-category $\text{Alg}/_O(C)$. By unstraightening, we obtain a coCartesian fibration

$$v : \text{Alg}_O^{\text{Pair}}(\text{Cat}^{(\kappa)}_\infty) \to \text{Alg}_O(\text{Cat}^{(\kappa)}_\infty),$$

where the objects of $\text{Alg}_O^{\text{Pair}}(\text{Cat}^{(\kappa)}_\infty)$ are pairs $(C\otimes, A)$ of an $O$-$\kappa$-monoidal $\infty$-category $C\otimes$ that is compatible with $\kappa$-small colimits and $A$ is an $O$-$A$-module object in $C\otimes$.

We have a functor

$$\text{Mod}^O(-)_{\otimes,\lor} : \text{Alg}_O(\text{Cat}^{(\kappa)}_\infty) \to \text{Cat}_\infty$$

which associates to $C\otimes$ the $\infty$-category $\text{Mod}^O(C\otimes)_{\otimes,\lor}$. We denote by

$$\omega : \text{Mod}^O_{\text{Triple}}(\text{Cat}^{(\kappa)}_\infty)_{\otimes,\lor} \to \text{Alg}_O(\text{Cat}^{(\kappa)}_\infty)$$

the associated coCartesian fibration by unstraightening. The objects of $\text{Mod}^O_{\text{Triple}}(\text{Cat}^{(\kappa)}_\infty)_{\otimes,\lor}$ are triples $(C\otimes, A, M)$ of an $O$-$\kappa$-monoidal $\infty$-category $C\otimes$ that is compatible with $\kappa$-small colimits, an $O$-$A$-algebra object $A$, and an $O$-$A$-module $M$ in $C\otimes$.

The functor $\text{Mod}^O(-)_{\otimes,\lor}$ lifts to a functor

$$\text{Mod}^O(-)_{\otimes,\lor} : \text{Alg}_O(\text{Cat}^{(\kappa)}_\infty) \to \text{Op}_{0,\infty}^{\text{gen},\lor},$$

which fits into the following commutative diagram

$$\begin{array}{ccc}
\text{Alg}_O(\text{Cat}^{(\kappa)}_\infty) & \xrightarrow{\omega} & \text{Alg}_O(\text{Cat}^{(\kappa)}_\infty) \\
\downarrow & & \downarrow \\
\text{Alg}_O(\text{Cat}^{(\kappa)}_\infty) & \xrightarrow{\text{Mod}^O(-)_{\otimes,\lor}} & \text{Cat}_\infty.
\end{array}$$

By unstraightening, this induces the following commutative diagram

$$\begin{array}{ccc}
\text{Alg}_O^{\text{Pair}}(\text{Cat}^{(\kappa)}_\infty) & \xrightarrow{\Theta, \rho} & \text{Alg}_O^{\text{Pair}}(\text{Cat}^{(\kappa)}_\infty) \times \text{Op}_{0,\infty}^{\text{gen},\lor} \\
\omega \downarrow & & \downarrow \\
\text{Alg}_O^{\text{Pair}}(\text{Cat}^{(\kappa)}_\infty) & \xrightarrow{v} & \text{Alg}_O(\text{Cat}^{(\kappa)}_\infty), \\
\varpi \downarrow & & \\
\text{Alg}_O(\text{Cat}^{(\kappa)}_\infty) & \xrightarrow{\pi} & \text{Alg}_O(\text{Cat}^{(\kappa)}_\infty),
\end{array}$$

where the map $\Theta$ carries $\omega$-coCartesian morphisms to $v$-coCartesian morphisms, the map $\rho$ carries $\omega$-coCartesian morphisms to equivalences, and $\pi$ is the projection.
We would like to show that $\Theta$ is a coCartesian fibration. For this purpose, we consider the following situation. Suppose that we have a commutative diagram
\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
p & & q \\
\downarrow & & \downarrow \\
\mathcal{Z} & & \\
\end{array}
\]
of $\infty$-categories. We assume that $p$ and $q$ are coCartesian fibrations, that $f_z : \mathcal{X}_z \to \mathcal{Y}_z$ is a coCartesian fibration for each $z \in \mathcal{Z}$, and that $f$ carries $p$-coCartesian morphisms to $q$-coCartesian morphisms. In this situation we obtain a commutative diagram
\[
\begin{array}{ccc}
\mathcal{X}_z & \xrightarrow{\phi^*} & \mathcal{X}_{z'} \\
f_z & & f_{z'} \\
\downarrow & & \downarrow \\
\mathcal{Y}_z & \xrightarrow{\phi^*} & \mathcal{Y}_{z'} \\
\end{array}
\]
for any morphism $\phi : z \to z'$ in $\mathcal{Z}$.

The dual form of the following lemma was proved in [3, Proposition 8.3].

**Lemma 4.2** ([3, Proposition 8.3]). If $\phi^* : \mathcal{X}_z \to \mathcal{X}_{z'}$ carries $f_z$-coCartesian morphisms to $f_{z'}$-coCartesian morphisms for any morphism $\phi : z \to z'$ in $\mathcal{Z}$, then $f$ is a coCartesian fibration.

**Proposition 4.3.** The functor $\Theta$ is a coCartesian fibration.

**Proof.** For any $C^\otimes \in \text{Alg}_O(\text{Cat}_\infty^{(c)})$, the map $\Theta_C : \text{Mod}^{\otimes, \text{Triple}}(\text{Cat}_\infty^{(c)})^\otimes_{\otimes, \nabla} \to \text{Alg}_O(\text{Cat}_\infty^{(c)})$ is identified with the map $\Psi : \text{Mod}^O(\mathcal{C})^{\otimes, \nabla} \to \text{Alg}_{/O}(\mathcal{C})$ which is a coCartesian fibration. For a strong $O$-monoidal functor $F : C^\otimes \to D^\otimes$, we have a commutative diagram
\[
\begin{array}{ccc}
\text{Mod}^O(\mathcal{C})^{\otimes, \nabla} & \xrightarrow{F_*} & \text{Mod}^O(\mathcal{D})^{\otimes, \nabla} \\
\Theta_C & & \Theta_D \\
\text{Alg}_{/O}(\mathcal{C}) & \rightarrow & \text{Alg}_{/O}(\mathcal{D}). \\
\end{array}
\]
By Lemma [4.2] in order to prove the proposition, it suffices to show that $F_*$ carries $\Theta_C$-coCartesian morphisms to $\Theta_D$-coCartesian morphisms. This follows from the following commutative diagram
\[
\begin{array}{ccc}
\text{Mod}^O_A(\mathcal{C})^{\otimes, \nabla} & \xrightarrow{F_*} & \text{Mod}^O(\mathcal{D})^{\otimes, \nabla} \\
\downarrow f_1 & & \downarrow F(f)_1 \\
\text{Mod}^O_B(\mathcal{C})^{\otimes, \nabla} & \xrightarrow{F_*} & \text{Mod}^O_{F(B)}(\mathcal{C})^{\otimes, \nabla} \\
\end{array}
\]
for any $f : A \to B$ in $\text{Alg}_{/O}(\mathcal{C})$. \qed

5. **Duoidal $\infty$-Categories of Operadic Modules**

In this section we will construct duoidal structures on $\infty$-categories of operadic modules. We fix an uncountable regular cardinal $\kappa$ and an essentially $\kappa$-small coherent $\infty$-operad $O^\otimes$ throughout this section unless otherwise stated. Let $P^\otimes$ be an $\infty$-operad. We take a $P \otimes O$-monoidal $\infty$-category $C^\otimes$ that is compatible with $\kappa$-small colimits, and a $P \otimes O$-algebra object $A$ in $C^\otimes$. We shall show that the $\infty$-category $\text{Mod}_A^O(\mathcal{C})$ of $O$-$A$-modules in $C^\otimes$ has a structure of a $(P, O)$-duoidal $\infty$-category (Theorem 5.6).
5.1. **Mixed \((P, Q)\)-monoidal \(\infty\)-categories.** Let \(P^\otimes\) and \(Q^\otimes\) be \(\infty\)-operads. In this subsection we recall a formulation of \((P, Q)\)-duoidal \(\infty\)-categories as mixed \((P, Q)\)-monoidal \(\infty\)-categories.

We defined a \((P, Q)\)-duoidal \(\infty\)-category as a \(P\)-monoid object in the Cartesian symmetric monoidal \(\infty\)-category \(\text{Mon}_{Q,\text{monoidal}}^\text{plax}(\text{Cat}_\infty)\). There are three formulations of the \(\infty\)-category of \((P, Q)\)-duoidal \(\infty\)-categories depending on which kinds of functors we choose. In this paper it is convenient to formulate \((P, Q)\)-duoidal \(\infty\)-categories as mixed \((P, Q)\)-monoidal \(\infty\)-categories.

**Definition 5.1** (cf. [9] Definition 3.11). A mixed \((P, Q)\)-monoidal \(\infty\)-category is a mixed fibration 
\[
D^\otimes \longrightarrow P^\otimes \times Q_{Q^{\text{op}}}^\otimes
\]
over \((P^\otimes, Q_{Q^{\text{op}}}^\otimes)\) which satisfies the following conditions:

- For each \(p \simeq p_1 \oplus \cdots \oplus p_m \in P^\otimes\), the Segal morphism
  \[
  D_p^\otimes \longrightarrow D_{p_1}^\otimes \times Q_{Q^{\text{op}}}^\otimes \cdots \times Q_{Q^{\text{op}}}^\otimes D_{p_m}^\otimes
  \]
is an equivalence.
- For each \(q \simeq q_1 \oplus \cdots \oplus q_n \in Q^\otimes\), the Segal morphism
  \[
  D_q^\otimes \longrightarrow D_{q_1}^\otimes \times P_{P^{\text{op}}}^\otimes \cdots \times P_{P^{\text{op}}}^\otimes D_{q_n}^\otimes
  \]
is an equivalence.

We can define a bilax \((P, Q)\)-monoidal functor between mixed \((P, Q)\)-monoidal \(\infty\)-categories. By [9] Theorem 3.12, the \(\infty\)-category \(\text{Mon}_{P}^{\text{plax}}(\text{Mon}_{Q}^{\text{monoidal}}(\text{Cat}_\infty))\) is equivalent to the \(\infty\)-category of mixed \((P, Q)\)-monoidal \(\infty\)-categories and bilax \((P, Q)\)-monoidal functors. In this paper we identify a \((P, Q)\)-duoidal \(\infty\)-category with the corresponding mixed \((P, Q)\)-monoidal \(\infty\)-category.

5.2. **Construction of a functor \(\Theta^\otimes\).** In this subsection we construct a map \(\Theta^\otimes\) of generalized \(\infty\)-operads which encodes a multiplicative structure on the triples \((C^\otimes, A, M)\), where \(C^\otimes\) is an \(\mathcal{O}\)-monoidal \(\infty\)-category that is compatible with \(\kappa\)-small colimits, \(A\) is an \(\mathcal{O}\)-algebra object, and \(M\) is an \(\mathcal{O}\)-\(A\)-module object in \(C^\otimes\). In the next subsection we will show that \(\Theta^\otimes\) is a coCartesian fibration of generalized \(\infty\)-operads.

First, we shall construct a coCartesian fibration \(v^\otimes: \text{Alg}_{\mathcal{O}}^\text{Pair}(\text{Cat}_\infty)^\otimes \rightarrow \text{Alg}_{\mathcal{O}}(\text{Cat}_\infty)^\otimes\) of \(\infty\)-operads, which is an extension of the map \(v: \text{Alg}_{\mathcal{O}}^\text{Pair}(\text{Cat}_\infty) \rightarrow \text{Alg}_{\mathcal{O}}(\text{Cat}_\infty)\).

The \(\infty\)-category \(\text{Alg}_{\mathcal{O}}(\text{Cat}_\infty)^\otimes\) has a symmetric monoidal structure by pointwise multiplication ([7] Example 3.2.4.4)). We denote by
\[
\text{Alg}_{\mathcal{O}}(\text{Cat}_\infty)^\otimes \longrightarrow \text{Fin}_\kappa
\]
the associated coCartesian fibration of \(\infty\)-operads. The functor \(\text{Alg}_{/\mathcal{O}}(-): \text{Mon}_{\mathcal{O}}(\text{Cat}_\infty) \rightarrow \text{Cat}_\infty\) preserves finite products and thus induces a functor \(\text{Alg}_{/\mathcal{O}}(-)^\otimes: \text{Mon}_{\mathcal{O}}(\text{Cat}_\infty)^\otimes \rightarrow \text{Cat}_\infty\), which is a \(\text{Mon}_{\mathcal{O}}(\text{Cat}_\infty)\)-monoid object in \(\text{Cat}_\infty\). The map \(\text{Cat}_\infty^\otimes \rightarrow \text{Cat}_\infty\) of \(\infty\)-operads induces a lax symmetric monoidal functor \(\text{Alg}_{\mathcal{O}}(\text{Cat}_\infty)^\otimes \rightarrow \text{Alg}_{\mathcal{O}}(\text{Cat}_\infty)^\times \simeq \text{Mon}_{\mathcal{O}}(\text{Cat}_\infty)^\times\). By composing these two functors, we obtain a functor \(\text{Alg}_{/\mathcal{O}}(-)^\otimes: \text{Alg}_{\mathcal{O}}(\text{Cat}_\infty)^\otimes \rightarrow \text{Cat}_\infty\), which is an \(\text{Alg}_{\mathcal{O}}(\text{Cat}_\infty)^\otimes\)-monoid object in \(\text{Cat}_\infty\). We define a map
\[
v^\otimes: \text{Alg}_{\mathcal{O}}(\text{Cat}_\infty)^\otimes \longrightarrow \text{Alg}_{/\mathcal{O}}(\text{Cat}_\infty)^\otimes
\]
of \(\infty\)-operads to be the associated coCartesian fibration by unstraightening.

Next, we shall construct a functor \(\text{Mod}^\mathcal{O}_\kappa(-)^\otimes : \text{Alg}_{\mathcal{O}}(\text{Cat}_\infty)^\otimes \rightarrow \text{Op}_{\text{gen,}\kappa}^\text{plax,\otimes} \otimes_{\mathcal{O}_{\kappa}^\otimes},\) which is an extension of the functor \(\text{Mod}^\mathcal{O}_\kappa(-)^\otimes : \text{Alg}_{\mathcal{O}}(\text{Cat}_\infty)^\otimes \rightarrow \text{Op}_{\text{gen,}\kappa}^\text{plax,\otimes} \otimes_{\mathcal{O}_{\kappa}^\otimes},\) and show that it is an \(\text{Alg}_{\mathcal{O}}(\text{Cat}_\infty)^\otimes\)-monoid object in \(\text{Op}_{\text{gen,}\kappa}^\text{plax,\otimes} \otimes_{\mathcal{O}_{\kappa}^\otimes}.\)
We have a functor \( \text{Mod}^\mathcal{O}(-)^\otimes : \text{Mon}^\mathcal{O}_\infty(\mathcal{C}_\infty) \to \text{Op}^\text{gen}_{\mathcal{O}^\otimes} \) which associates to \( \mathcal{C} \) the map \( \sigma : \text{Mod}^\mathcal{O}(\mathcal{C})^\otimes \to \mathcal{O}^\otimes \) of generalized \( \infty \)-operads. The functor \( \text{Mod}^\mathcal{O}(-)^\otimes \) preserves finite products and hence it extends to a functor \( \text{Mod}^\mathcal{O}(-)^\otimes : \text{Mon}^\mathcal{O}_\infty(\mathcal{C}_\infty)^\times \to \text{Op}^\text{gen}_{\mathcal{O}^\otimes} \), which is a \( \text{Mon}^\mathcal{O}(\mathcal{C}_\infty) \)-monoid object in \( \text{Op}^\text{gen}_{\mathcal{O}^\otimes} \). By composing \( \text{Mod}^\mathcal{O}(-)^\otimes \) with the lax symmetric monoidal functor \( \text{Alg}_\mathcal{O}(\mathcal{C}_\infty)^\Sigma \to \text{Alg}_\mathcal{O}(\mathcal{C}_\infty)^\times \simeq \text{Mon}^\mathcal{O}_\infty(\mathcal{C}_\infty)^\times \), we obtain a functor

\[
\text{Mod}^\mathcal{O}(-)^\otimes : \text{Alg}_\mathcal{O}(\mathcal{C}_\infty)^\Sigma \to \text{Op}^\text{gen}_{\mathcal{O}^\otimes},
\]

which is an \( \text{Alg}_\mathcal{O}(\mathcal{C}_\infty) \)-monoid object in \( \text{Op}^\text{gen}_{\mathcal{O}^\otimes} \).

The \( \infty \)-category \( \text{Mon}^\text{lax.R}_{\mathcal{O}^\otimes}(\mathcal{C}_\infty)^\text{LAd} \) is equivalent to a subcategory of \( \text{Op}^\text{gen}_{\mathcal{O}^\otimes} \) by Proposition 3.4. The functor \( \text{Mod}^\mathcal{O}(-)^\otimes \) factors through the subcategory \( \text{Mon}^\text{lax.R}_{\mathcal{O}^\otimes}(\mathcal{C}_\infty)^\text{LAd} \). By Propositions 3.1 and 3.7, we obtain a functor

\[
\text{Mod}^\mathcal{O}(-)^\otimes : \text{Alg}_\mathcal{O}(\mathcal{C}_\infty)^\Sigma \to \text{Op}^\text{gen}_{\mathcal{O}^\otimes}(\mathcal{C}_\infty)^\otimes \text{op}.
\]

**Lemma 5.2.** The functor \( \text{Mod}^\mathcal{O}(-)^\otimes \) is an \( \text{Alg}_\mathcal{O}(\mathcal{C}_\infty) \)-monoid object of \( \text{Op}^\text{gen}_{\mathcal{O}^\otimes}(\mathcal{C}_\infty)^\otimes \text{op} \).

**Proof.** Let \((\mathcal{C}_1^\otimes, \ldots, \mathcal{C}_n^\otimes) \in \text{Alg}_\mathcal{O}(\mathcal{C}_\infty)^\Sigma \). By definition, the image of \((\mathcal{C}_1^\otimes, \ldots, \mathcal{C}_n^\otimes) \) under the functor \( \text{Mod}^\mathcal{O}(-)^\otimes : \text{Alg}_\mathcal{O}(\mathcal{C}_\infty)^\Sigma \to \text{Op}^\text{gen}_{\mathcal{O}^\otimes}(\mathcal{C}_\infty)^\otimes \text{op} \) is given by a Cartesian fibration

\[
\text{Mod}^\mathcal{O}(-)^\otimes : \text{Alg}_\mathcal{O}(\mathcal{C}_\infty)^\Sigma \to \text{Op}^\text{gen}_{\mathcal{O}^\otimes}(\mathcal{C}_\infty)^\otimes \text{op}
\]

of \( \infty \)-operads. We can see that it is equivalent to a product of the Cartesian fibrations \( \text{Mod}^\mathcal{O}(-)^\otimes \) in \( \text{Op}^\text{gen}_{\mathcal{O}^\otimes}(\mathcal{C}_\infty)^\otimes \text{op} \) for \( 1 \leq i \leq n \).

By unstraightening the functor \( \text{Mod}^\mathcal{O}(-)^\otimes \), we obtain the following commutative diagram

\[
\begin{array}{ccc}
\text{Mod}^\mathcal{O}(-)^\otimes & \xrightarrow{(\Theta^\otimes, \tau^\otimes)} & \text{Alg}_\mathcal{O}^\text{pair}(\mathcal{C}_\infty)^\Sigma \\
\Theta^\otimes & \downarrow & \pi \\
\text{Alg}_\mathcal{O}^\text{pair}(\mathcal{C}_\infty)^\Sigma & \xrightarrow{\omega^\otimes} & \text{Alg}_\mathcal{O}(\mathcal{C}_\infty)^\Sigma \end{array}
\]

where the maps \( \omega^\otimes \) and \( v^\otimes \) are coCartesian fibrations of \( \infty \)-operads, the map \( \Theta^\otimes \) is a map of \( \infty \)-operads which carries \( \omega^\otimes \)-coCartesian morphisms to \( v^\otimes \)-coCartesian morphisms, and the map \( \tau^\otimes \) carries \( \omega^\otimes \)-coCartesian morphisms to equivalences.

### 5.3. The coCartesian fibration \( \Theta^\otimes \)

In Lemma 5.2, we have constructed a map

\[
\Theta^\otimes : \text{Mod}^\mathcal{O}(-)^\otimes \to \text{Alg}_\mathcal{O}^\text{pair}(\mathcal{C}_\infty)^\Sigma
\]

of \( \infty \)-operads. In this subsection, we shall show that \( \Theta^\otimes \) is a coCartesian fibration.

For simplicity, we set \( \mathcal{M}^T = \text{Mod}^\mathcal{O}(-)^\otimes \), \( \mathcal{A}^P = \text{Alg}_\mathcal{O}^\text{pair}(\mathcal{C}_\infty)^\Sigma \), and \( \mathcal{A} = \text{Alg}_\mathcal{O}(\mathcal{C}_\infty)^\Sigma \). Recall that we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}^T & \xrightarrow{\Theta^\otimes} & \mathcal{A}^P \\
\omega^\otimes & \downarrow & \nu^\otimes \\
\mathcal{A} & \xrightarrow{\pi} & \mathcal{A}
\end{array}
\]
where $\omega^\otimes$ and $\nu^\otimes$ are coCartesian fibrations, and $\Theta^\otimes$ carries $\omega^\otimes$-coCartesian morphisms to $\nu^\otimes$-coCartesian morphisms. Furthermore, for any object $C \in A$, the induced functor $\Theta^\otimes_C : M^\otimes_C \to A^\otimes_C$ is a coCartesian fibration.

Let $\phi : (C^\otimes_1, \ldots, C^\otimes_n) \to D^\otimes$ be an active morphism in $A$, where $C^\otimes_1, \ldots, C^\otimes_n, D^\otimes$ are objects in the underlying $\infty$-category $\text{Alg}_O(\text{Cat}_\infty^\otimes)$. Then we have a commutative diagram

$$
\begin{array}{ccc}
\text{Mod}^\otimes(C_1 \times_O \cdots \times_O C_n)^\otimes \nu & \xrightarrow{\phi_*} & \text{Mod}^\otimes(D)^\otimes \nu \\
\text{Alg}^O_O(C_1 \times_O \cdots \times_O C_n) & \xrightarrow{\phi} & \text{Alg}^O_O(D).
\end{array}
$$

In order to prove that $\Theta^\otimes$ is a coCartesian fibration, it suffices to show that $\phi_*$ carries $p$-coCartesian morphisms to $q$-coCartesian morphisms for any $p$ by Lemma 4.2.

First, we consider the case in which $D = C_1 \times \cdots \times C_n$ and $\phi : (C_1, \ldots, C_n) \to C_1 \times \cdots \times C_n$ is a multiplication map in the symmetric monoidal $\infty$-category $\text{Alg}_O(\text{Cat}_\infty^\otimes)$.

**Lemma 5.3.** When $\phi : (C_1, \ldots, C_n) \to C_1 \times \cdots \times C_n$ is a multiplication map, the functor $\phi_*$ carries $p$-coCartesian morphisms to $q$-coCartesian morphisms.

**Proof.** Let $f_i : A_i \to B_i$ be a map in $\text{Alg}^O_O(C_i)$, and let $M_i$ be an object of $\text{Mod}^O_O(C_i)^\otimes \nu$ for $1 \leq i \leq n$ and $X \in O^\otimes$. We have to show that the canonical map

$$(f_1 \boxtimes \cdots \boxtimes f_n)(M_1 \boxtimes \cdots \boxtimes M_n) \to (f_1M_1) \boxtimes \cdots \boxtimes (f_nM_n)$$

is an equivalence in $\text{Mod}^O_O(B_1 \boxtimes \cdots \boxtimes B_n)(C_1 \boxtimes \cdots \boxtimes C_n)^\otimes \nu$.

In order to prove this, it suffices to show the case in which $f_2, \ldots, f_n$ are identities. Thus, we will prove that the canonical map

$$(f_1 \boxtimes \text{id} \boxtimes \cdots \boxtimes \text{id})(M_1 \boxtimes M_2 \boxtimes \cdots \boxtimes M_n) \to (f_1M_1) \boxtimes M_2 \boxtimes \cdots \boxtimes M_n$$

is an equivalence.

For simplicity, we assume that $n = 2$. Since the functor $(f_1 \boxtimes \text{id})^* : \text{Mod}^O_O(B_1 \boxtimes B_2)(C_1 \boxtimes C_2)^\otimes \nu \to \text{Mod}^O_O(A_1 \boxtimes A_2)(C_1 \boxtimes C_2)^\otimes \nu$ is conservative, it suffices to show that the map

$$(f_1 \boxtimes \text{id})^*(f_1 \boxtimes \text{id})(M_1 \boxtimes M_2) \to (f_1 \boxtimes \text{id})^*((f_1M_1) \boxtimes M_2)$$

is an equivalence. Note that there is an equivalence $(f_1 \boxtimes \text{id})^*((f_1M_1) \boxtimes M_2) \simeq \text{colim}_{\pi(u)}D(B_1, M_1)$. Since $\pi$ preserves small colimits separately in each variable, we obtain an equivalence $(f_1^*f_1M_1) \boxtimes M_2 \simeq \text{colim}_{\pi(u)}D(B_1, M_1) \boxtimes M_2$.

By Proposition 4.4, we have an equivalence $f_1^*f_1M_1 \simeq \text{colim}_{\pi(u)}D(B_1, M_1)$. Since $\pi$ preserves small colimits separately in each variable, we obtain an equivalence $(f_1^*f_1M_1) \boxtimes M_2 \simeq \text{colim}_{\pi(u)}D(B_1, M_1) \boxtimes M_2$.

On the other hand, we have an equivalence $(f_1 \boxtimes \text{id})^*(f_1 \boxtimes \text{id})(M_1 \boxtimes M_2) \simeq \text{colim}_{\pi(u)}D(B_1 \boxtimes A_2, M_1 \boxtimes M_2)$ by Proposition 4.4. Since we have an equivalence $D(B_1 \boxtimes A_2, M_1 \boxtimes M_2) \simeq D(B_1, M_1) \boxtimes M_2$ of functors, we obtain the desired equivalence. \qed

For any active morphism $\phi : (C^\otimes_1, \ldots, C^\otimes_n) \to D^\otimes$ in $\text{Alg}_O(\text{Cat}_\infty^\otimes)$, where $C^\otimes_1, \ldots, C^\otimes_n, D^\otimes$ are objects in the underlying $\infty$-category $\text{Alg}_O(\text{Cat}_\infty^\otimes)$, we can decompose $\phi$ as

$$(C^\otimes_1, \ldots, C^\otimes_n) \to C^\otimes_1 \boxtimes \cdots \boxtimes C^\otimes_n \to D^\otimes,$$

where the first map is a multiplication map and the second map is a strong $O$-monoidal functor that preserves $\kappa$-small colimits. By Lemma 5.3 and the proof of Proposition 4.3, we obtain the following proposition.

**Proposition 5.4.** The map $\Theta^\otimes : \text{Mod}^O_O(\text{Cat}_\infty^\otimes)^\otimes \nu \to \text{Alg}^O_O(\text{Cat}_\infty^\otimes)^\otimes$ is a coCartesian fibration of generalized $\infty$-operads.
5.4. Construction of duoidal ∞-categories of operadic modules. In this subsection we prove the main theorem (Theorem 5.6). For any ∞-operad \( P \), any \( P \otimes O \)-monoidal ∞-category \( C^{\otimes} \) that is compatible with \( \kappa \)-small colimits, and any \( P \otimes O \)-algebra object \( A \) in \( C^{\otimes} \), we show that the ∞-category \( \text{Mod}_A^O(C) \) of \( O-A \)-modules in \( C^{\otimes} \) has a structure of \((P,O)\)-duoidal ∞-category.

First, we consider the universal case in which \( P^{\otimes} = \text{Alg}_O^{\text{Pair}}(\text{Cat}_{\infty}^{(\kappa)})^{\otimes} \). We have a symmetric monoidal ∞-category \( \text{Alg}_O^{\text{Pair}}(\text{Cat}_{\infty}^{(\kappa)})^{\otimes} \) and regard it as an ∞-operad. We have a functor
\[
\text{Mod}^{\circ,O,\text{Triple}}(\text{Cat}_{\infty}^{(\kappa)})^{\otimes,\vee} : \text{Alg}_O^{\text{Pair}}(\text{Cat}_{\infty}^{(\kappa)}) \rightarrow \text{Mon}^{\text{oplax}}(\text{Cat}_{\infty}^{(\kappa)})
\]
which assigns to a pair \((C^{\otimes}, A)\) the \( O \)-monoidal ∞-category \( \text{Mod}_A^O(C^{\otimes})^{\otimes,\vee} \rightarrow O^{\otimes,\text{op}} \). By Proposition [5.4] and the straightening functor, we obtain the following theorem.

**Theorem 5.5.** Let \( \kappa \) be an uncountable regular cardinal and let \( O^{\otimes} \) be an essentially \( \kappa \)-small coherent ∞-operad. Then we have a mixed \( (\text{Alg}_O^{\text{Pair}}(\text{Cat}_{\infty}^{(\kappa)}), O) \)-monoidal ∞-category
\[
\text{Mod}^{\circ,O,\text{Triple}}(\text{Cat}_{\infty}^{(\kappa)})^{\otimes,\vee} \rightarrow \text{Alg}_O^{\text{Pair}}(\text{Cat}_{\infty}^{(\kappa)}) \times O^{\otimes,\text{op}},
\]
in which the underlying ∞-category \( \text{Mod}^{\circ,O,\text{Triple}}(\text{Cat}_{\infty}^{(\kappa)})^{\otimes,\vee} \) is equivalent to \( \text{Mod}_A^O(C^{\otimes})^{\otimes,\vee} \) for \((C^{\otimes}, A) \in \text{Alg}_O^{\text{Pair}}(\text{Cat}_{\infty}^{(\kappa)})\) and \( X \in O \).

**Proof.** By Proposition [5.4] and the straightening functor, we can extend \( \text{Mod}^{\circ,O,\text{Triple}}(\text{Cat}_{\infty}^{(\kappa)})^{\otimes,\vee} \) to a functor
\[
\text{Mod}^{\circ,O,\text{Triple}}(\text{Cat}_{\infty}^{(\kappa)})^{\otimes,\vee} : \text{Alg}_O^{\text{Pair}}(\text{Cat}_{\infty}^{(\kappa)})^{\otimes} \rightarrow \text{Mon}^{\text{oplax}}(\text{Cat}_{\infty}^{(\kappa)})
\]
which is an \( \text{Alg}_O^{\text{Pair}}(\text{Cat}_{\infty}^{(\kappa)}) \)-monoid in \( \text{Mon}^{\text{oplax}}(\text{Cat}_{\infty}^{(\kappa)}) \). \( \square \)

Next, we consider the case in which \( P^{\otimes} \) is any ∞-operad. We set
\[
\text{Alg}_P^{\text{Pair}}(\text{Cat}_{\infty}^{(\kappa)}) = \text{Alg}_P(\text{Alg}_O^{\text{Pair}}(\text{Cat}_{\infty}^{(\kappa)})),
\]
where the objects of \( \text{Alg}_P^{\text{Pair}}(\text{Cat}_{\infty}^{(\kappa)}) \) are pairs \((C^{\otimes}, A)\) of a \( P \otimes O \)-monoidal ∞-category \( C^{\otimes} \) that is compatible with \( \kappa \)-small colimits and a \( P \otimes O \)-algebra object \( A \) in \( C^{\otimes} \).

The functor \( \text{Mod}^{\circ,O,\text{Triple}}(\text{Cat}_{\infty}^{(\kappa)})^{\otimes,\vee} \) induces a lax symmetric monoidal functor
\[
\text{Alg}_P^{\text{Pair}}(\text{Cat}_{\infty}^{(\kappa)})^{\otimes} \rightarrow \text{Mon}^{\text{oplax}}(\text{Cat}_{\infty}^{(\kappa)})^{\otimes}.
\]
By applying the functor \( \text{Alg}_P(-) \) to this functor, we obtain the following theorem.

**Theorem 5.6.** Let \( \kappa \) be an uncountable regular cardinal and let \( O^{\otimes} \) be an essentially \( \kappa \)-small coherent ∞-operad. For any pair \((C^{\otimes}, A)\) of a \( P \otimes O \)-monoidal ∞-category \( C^{\otimes} \) that is compatible with \( \kappa \)-small colimits and a \( P \otimes O \)-algebra object \( A \) in \( C^{\otimes} \), we have a mixed \( (P,O) \)-monoidal ∞-category
\[
\text{Mod}_A^P(C^{\otimes})^{\otimes,\vee} \rightarrow P^{\otimes} \times O^{\otimes,\text{op}}.
\]
Furthermore, we have a functor
\[
\text{Alg}_P^{\text{Pair}}(\text{Cat}_{\infty}^{(\kappa)}) \rightarrow \text{Mon}_P(\text{Mon}^{\text{oplax}}(\text{Cat}_{\infty}^{(\kappa)})),
\]
which associates to a pair \((C^{\otimes}, A)\) the mixed \( (P,O) \)-monoidal ∞-category \( \text{Mod}_A^P(C^{\otimes})^{\otimes,\vee} \rightarrow P^{\otimes} \times O^{\otimes,\text{op}} \).

We denote by \( \text{Pr}^L \) the large ∞-category of presentable ∞-categories and colimit-preserving functors. By [7, Proposition 4.8.1.15], the ∞-category \( \text{Pr}^L \) has a symmetric monoidal structure in which the inclusion \( \text{Pr}^L \rightarrow \text{Cat}_{\infty}(K) \) is strong symmetric monoidal, where \( K \) is the set of all small simplicial sets. By Theorem [5.6] we obtain the following corollary.
Corollary 5.7. Let $\mathcal{O}^\otimes$ be a small coherent $\infty$-operad and let $\mathcal{P}^\otimes$ be an $\infty$-operad. For a presentable $\mathcal{P} \otimes \mathcal{O}$-monoidal $\infty$-category and a $\mathcal{P} \otimes \mathcal{O}$-algebra object $A$ in $\mathcal{C}^\otimes$, we have a mixed $(\mathcal{P}, \mathcal{O})$-monoidal $\infty$-category

$$\text{Mod}^\Delta_A(\mathcal{C})^{\otimes,\otimes,\vee} \longrightarrow \mathcal{P}^\otimes \times \mathcal{O}^{\otimes,\text{op}}.$$

Furthermore, we have a functor

$$\text{Alg}^\text{Pair}_{\mathcal{P} \otimes \mathcal{O}}(\mathcal{P}_L) \longrightarrow \text{Mon}_\mathcal{P}(\text{Mon}^\text{plax}_{\mathcal{O}}(\text{Cat}_{\infty})),$$

which associates to a pair $(\mathcal{C}^\otimes, A)$ the mixed $(\mathcal{P}, \mathcal{O})$-monoidal $\infty$-category $\text{Mod}^\Delta_A(\mathcal{C})^{\otimes,\otimes,\vee} \rightarrow \mathcal{P}^\otimes \times \mathcal{O}^{\otimes,\text{op}}$.

6. $(\mathbb{E}_m, \mathbb{E}_n)$-duoidal $\infty$-categories of $\mathbb{E}_m$-modules

Let $\mathbb{E}_k^\otimes$ be the little $k$-cubes operad for $k \geq 0$. In this section we consider the important case in which $(\mathcal{P}^\otimes, \mathcal{O}^\otimes) = (\mathbb{E}_m^\otimes, \mathbb{E}_n^\otimes)$ and $\mathcal{C}^\otimes$ is a presentable symmetric monoidal $\infty$-category. The $\infty$-category $\text{Mod}^\Delta_{\mathbb{E}_k}(\mathcal{C})$ of $\mathbb{E}_n$-$A$-modules has a structure of an $(\mathbb{E}_m, \mathbb{E}_n)$-duoidal $\infty$-category for any $A \in \text{Alg}_{\mathbb{E}_m, \mathbb{E}_n}(\mathcal{C})$ by Corollary 5.7. The key to prove this was lemma 5.3. We will give another proof of lemma 5.3 in this case by using enveloping algebras.

We recall that $\mathbb{E}_k^\otimes$ is coherent by [7] Theorem 5.1.1.1, and that there is an equivalence $\mathbb{E}_{m+n}^\otimes \simeq \mathbb{E}_m^\otimes \otimes \mathbb{E}_n^\otimes$ for $m, n \geq 0$ by Dunn-Lurie Additivity Theorem (7 Theorem 5.1.2.2). In this section we assume that $\mathcal{C}^\otimes$ is a presentable symmetric monoidal $\infty$-category. Under this assumption, there is an equivalence $\text{Mod}^\Delta_{\mathbb{E}_k}(\mathcal{C}) \simeq \text{LMod}_{U(A)}(\mathcal{C})$ of $\infty$-categories for $A \in \text{Alg}_{\mathbb{E}_m, \mathbb{E}_n}(\mathcal{C})$, where $\text{LMod}_{U(A)}(\mathcal{C})$ is the $\infty$-category of left modules over the enveloping algebra $U(A)$ of $A$. Note that $U(A)$ is equivalent to the factorization homology $\int_{S_n-1} A$ by [2] Proposition 3.16 (see also [7] Example 5.5.1.16).

A map $f : A \rightarrow B$ in $\text{Alg}_{\mathbb{E}_m, \mathbb{E}_n}(\mathcal{C})$ induces a map $U(A) \rightarrow U(B)$ of enveloping algebras in $\text{Alg}(\mathcal{C})$. If we regard $M \in \text{Mod}^\Delta_{\mathbb{E}_n}(\mathcal{C})$ as a left $U(A)$-module, then there is an equivalence

$$f_!(M) \simeq U(B) \otimes_{U(A)} M$$

in $\text{LMod}_{U(B)}(\mathcal{C})$.

Now, we recall that symmetric monoidal structures on $\text{Alg}_{\mathbb{E}_m, \mathbb{E}_n}(\mathcal{C})$ and $\text{Mod}^\Delta_{\mathbb{E}_k}(\mathcal{C})$. The $\infty$-category $\text{Alg}_{\mathbb{E}_m, \mathbb{E}_n}(\mathcal{C})$ is a symmetric monoidal $\infty$-category by pointwise tensor product (7 Example 3.2.4.4). Furthermore, $\text{Mod}^\Delta_{\mathbb{E}_k}(\mathcal{C})$ is also a symmetric monoidal $\infty$-category by pointwise tensor product such that the projection $\text{Mod}^\Delta_{\mathbb{E}_k}(\mathcal{C}) \rightarrow \text{Alg}_{\mathbb{E}_k}(\mathcal{C})$ is a strong symmetric monoidal functor.

The following proposition is a counterpart of Lemma 5.3 in the setting of this section.

Proposition 6.1. Let $\mathcal{C}^\otimes$ be a presentable symmetric monoidal $\infty$-category. Suppose that we have maps $f_i : A_i \rightarrow B_i$ in $\text{Alg}_{\mathbb{E}_m, \mathbb{E}_n}(\mathcal{C})$ and $M_i \in \text{Mod}^\Delta_{\mathbb{E}_n}(\mathcal{C})$ for $i = 1, \ldots, n$. Then the canonical map

$$(f_1 \otimes \cdots \otimes f_n)(M_1 \otimes \cdots \otimes M_n) \longrightarrow (f_1)_!(M_1) \otimes \cdots \otimes (f_n)_!(M_n)$$

is an equivalence for each $n \geq 0$.

Proof. It suffices to prove the case $n = 2$. We have an equivalence

$$(f_1)_!(M_1) \otimes (f_2)_!(M_2) \simeq (U(B_1) \otimes U(B_2)) \otimes_{U(A_1) \otimes U(A_2)(M_1 \otimes M_2)}$$

By [7] Theorem 5.3.2, $U(A \otimes B) \simeq U(A) \otimes U(B)$ for any $A, B \in \text{Alg}_{\mathbb{E}_m, \mathbb{E}_n}(\mathcal{C})$. This implies an equivalence

$$(U(B_1) \otimes U(B_2)) \otimes_{U(A_1) \otimes U(A_2)} (M_1 \otimes M_2) \simeq (f_1 \otimes f_2)_!(M_1 \otimes M_2),$$

which completes the proof. \(\square\)

By using Proposition 6.1 we obtain the following theorem in the same way as in 5.4.
Theorem 6.2. Let $C^\otimes$ be a presentable symmetric monoidal $\infty$-category. For any $E_{m+n}$-algebra object $A$ in $C^\otimes$, we have a mixed $(E_m, E_n)$-monoidal $\infty$-category
\[
\text{Mod}^{E_n}_A (C)^{\otimes, \otimes, \vee} \longrightarrow E^\otimes_m \times E^\otimes_n^\text{op},
\]
in which the underlying $\infty$-category is equivalent to the $\infty$-category $\text{Mod}^{E_n}_A (C)$ of $E_n$-$A$-modules in $C^\otimes$. For any map $f : A \to B$ in $\text{Alg}_{E_n} (C)$, the induced functor $f_! : \text{Mod}^{E_n}_A (C) \to \text{Mod}^{E_n}_B (C)$ is a bilax $(E_m, E_n)$-monoidal functor that is strong monoidal with respect to the $E_m$-monoidal product.

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Department of Mathematics, Okayama University, Okayama 700–8530, Japan
Email address: torii@math.okayama-u.ac.jp