Application of quantum control techniques to design broadband and ultra-broadband half-wave plates

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Abstract. Broadband and ultra-broadband half-wave plates with composite design were previously experimentally demonstrated by E. Dimova et. al., in Opt. Commun. 366, 382 (2016). To achieve broadband and ultra-broadband performance, the authors utilized the relative rotation between the individual identical half-wave plates as control parameters, thus simulating the well-known control technique of composite pulses from quantum optics. In this paper, we extend the broadband and ultra-broadband designs by relaxing the condition of having identical constituent half-wave plates and allow for their thicknesses to also be variable control parameters. We further relax the condition of having symmetric rotation angles and solve numerically for both control parameters - rotation angles and thicknesses of the birefringent plates. We show that the presented design is broadband over a much wider range of the wavelengths, compared to the original design, when using the same number of constituent half-wave plates.

1. Introduction
A half-wave plate is an optical device, which rotates the polarization direction of a linearly-polarized light at an angle $2\theta$, where $\theta$ denotes the angle between the incident light polarization direction and the fast axis of the half-wave plate [1, 2, 3]. A notable drawback of the original half-wave plate is that it works only for a narrow wavelength bandwidth of the incident light due to the strong wavelength dispersion. To solve this disadvantage, Clarke and Becker [4, 5] introduced an additional medium with reciprocal dispersion to reduce the wavelength dispersion. Another approach to reduce the dispersion is to use a stack of waveplates whose optical axes are rotated at given angles creating a sequence of phase shifts and thereby near constant retardation over a large bandwidth or for a larger set of discrete wavelengths [6, 7, 8].

Most recently, an analogy between the evolution of quantum systems and the space propagation of light allowed for the successful application of quantum control techniques to optics systems such as coupled waveguides [9, 10]. This analogy can also be extended to design broadband optical elements based on the broadband composite pulses technique widely used in NMR systems [11]. Thus far, several groups have proposed and experimentally demonstrated broadband polarization waveplates [12, 13, 14, 15, 16, 17, 18].

In this paper we reveal the applicability of the method of composite pulses to design modular broadband and ultra-broadband half-wave plates, irrespective of the optical medium used. We first give an overview of the existing designs of modular half-wave plates and show that they operate over a wide range of wavelengths and how their performance depends on the number of constituent half-wave plates.
of individual half-wave plates used. For the ultra-broadband half-wave plates, commercially available achromatic half-wave plates are utilized. Finally, we present a novel design of composite half-wave plates where the constituent birefringent plates have variable length. Using this additional variable as a control parameter, we design half-wave plates with significantly wider wavelength range of operation.

2. Theory
A single birefringent retarder rotated at an angle $\theta$ with respect to the slow and the fast axes of the retarder is described by the Jones matrix,

\[
\mathbf{J}_{\theta}(\varphi) = \mathcal{R}(\theta) \begin{bmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{bmatrix} \mathcal{R}(\theta),
\]

(1)

where

\[
\mathcal{R}(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.
\]

(2)

Here, the phase retardance is $\varphi = \frac{2\pi L}{\lambda} (n_f - n_s)$ with $n_f$ and $n_s$ being the refractive indices along the fast and slow axes of the birefringent plate, $\lambda$ being the wavelength of the input light, and $L$ - its length. When the phase retardance is $\varphi = \pi$, then the retarder is a standard half-wave plate.

We proceed to change the basis from the horizontal-vertical polarization basis (HV) to left-right circular polarization basis (LR) by using the transformation matrix $W$, such that

\[
\mathbf{J}_{\theta}(\varphi) = W^{-1} \mathbf{J}_{\theta}(\varphi) W,
\]

(3)

Thus, we obtain that the Jones matrix for a retarder with a phase retardance $\varphi$ and rotated at an angle $\theta$ is given as (in the LR basis) [12, 13, 14, 15, 16, 17],

\[
\mathbf{J}_{\theta}(\varphi) = \begin{bmatrix} \cos (\varphi/2) & i \sin (\varphi/2) e^{i2\theta} \\ i \sin (\varphi/2) e^{-2i\theta} & \cos (\varphi/2) \end{bmatrix}.
\]

(4)

Designing half-wave plates that are robust to variations in the phase retardance $\varphi$ around values $\varphi = m\pi$, ($m \in \mathbb{N}$), is done in an analogy to composite pulses [13, 14, 15, 16], which is a widely-used technique for robust control in quantum optics [19, 20]. We replace the birefringent retarder (half-wave plate) with a stack of an odd number $N = (2n + 1)$ of birefringent retarders, thus creating a half-wave plate with composite or modular design. Each individual birefringent retarder has a phase retardance $\varphi = \pi$. However, each is rotated at a different angle $\theta_k$. To obtain the total Jones matrix in the LR basis of the above described stack of birefringent retarders (half-wave plates), we multiply the individual Jones matrices (4) according to (read from right to left),

\[
\mathbf{J}^{(N)} = \mathbf{J}_{\theta_N}(\varphi) \mathbf{J}_{\theta_{N-1}}(\varphi) \cdots \mathbf{J}_{\theta_1}(\varphi).
\]

(5)

We require that the total effect of the modular half-wave plate (5) is identical to one from an ideal half-wave plate propagator, which is given by the Jones matrix $\mathbf{J}_0$ in the LR basis (up to a global phase factor),

\[
\mathbf{J}_0 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.
\]

(6)

Mathematically, this corresponds to setting $\mathbf{J}^{(N)} \equiv \mathbf{J}_0$ at $\varphi = \pi$, which allows us to use the $n$ independent angles $\theta_k$ as control parameters. After calculations [15, 18], we obtain that the following symmetry condition for the rotation angles $\theta_k = \theta_{N+1-k}$, ($k = 1, 2, ..., n$) ensures
Table 1. For different number \( N \) of the individual half-wave plates, rotation angles \( \theta_k \) (in degrees) assuring a broadband behaviour of the modular half-wave plate [15, 18].

| \( N \) | Rotation angles (\( \theta_1; \theta_2; \cdots; \theta_{N-1}; \theta_N \)) |
|---------|--------------------------------------------------|
| 3       | (60; 120; 60)                                    |
| 5       | (51.0; 79.7; 147.3; 79.7; 51.0)                  |
| 7       | (68.0; 16.6; 98.4; 119.8; 98.4; 16.6; 68.0)      |
| 9       | (99.4; 25.1; 64.7; 141.0; 93.8; 141.0; 64.7; 25.1; 99.4) |

Figure 1. Calculated fidelity of the Jones matrix of the modular half-wave plate, \( F = tr(J_{0}^{-1}J^{(N)}) \), as a function of the wavelength \( \lambda \). The number of individual half-wave plates (WPMQ10M-780, Thorlabs) is \( N=\{3, 5, 7, 9\} \) and their respective angles of rotation are given in Table 1 [18]. The fidelity of a single half-wave plate is labeled by 1 and is added for easy comparison.

that the total effect of the modular half-wave plate is that of an ideal half-wave plate. Several examples of modular broadband half-wave plates arrangements are presented in Table 1.

As shown in Fig.1, the modular designs of half-wave plates have significantly extended spectral range of operation. To achieve an even wider range of operation, one should increase the number of individual half-wave plates used to assemble the modular half-wave plate. These results are proven in the experimental paper of Ref. [18] and all experimental details can be found there.

3. Modified broadband half-wave plate

We propose a new design of broadband and ultra-broadband half-wave plates, which is different in two significant ways. First, additionally to the rotation angles, we also assume to have the freedom to vary the length of the individual half-wave plates as control parameters, \( L_n \). Thus, the individual half-wave plates have a variable phase retardance, \( \varphi \) by using \( \varphi_n = 2\pi L_n(n_f - n_s)/\lambda \). Secondly, we relax the condition of having symmetry in the rotation angles, as shown in Table 1, and allow for arbitrary rotation angles. We assume that we use the same single half-wave plates (WPMQ10M-780, Thorlabs), as in Section 2. This scheme is given in Figure 2.

The Jones matrix \( J^{(N)} \) of the above described arrangement of waveplates in the LR basis is given by (read from right to left):

\[
J^{(N)} = J_{\theta_N} (\varphi_N) J_{\theta_{N-1}} (\varphi_{N-1}) \cdots J_{\theta_1} (\varphi_1).
\] (7)

We proceed as in Section 2 and require that \( \dot{J}^{(N)} \equiv J_0 \). We seek to nullify as many lowest order derivatives of \( J^{(N)} \) vs the optical retardance \( \varphi \) at \( \varphi = \pi \) as possible. Thus, we can obtain the
The schematics of the modified broadband half-wave plate with variable lengths $L_n$ of the individual half-wave plates and different angles of rotation $\theta_n$.

Following nonlinear algebraic equations for the modified Jones matrix,

\[
\left[ \mathbf{\partial}^k_{\varphi} \mathbf{J}^{(N)}_{ij} \right]_{\varphi=\pi} = 0, \quad \left[ \mathbf{\partial}^k_{\varphi} \mathbf{J}^{(N)}_{ij} \right]_{\varphi=\pi} = 0,
\]

where $(k = 1, 2, ..., n)$.

We can solve these equations to obtain the rotation angles and the lengths of the individual half-wave plates. This method gives more freedom than the one described in Section 2. However, the trade off is that it is very difficult to find analytical solutions of Eqs. (8) and we proceed to solve them numerically. For simplicity, we express the lengths of the half-wave plates in terms of the phase retardance $\varphi_n$. Table 2 lists our results for the rotation angles $\theta_n$ and the phase retardance $\varphi_n$ for $N = \{3, 5\}$.

**Table 2.** Rotation angles $\theta_k$ (in degrees) and phase retardance $\varphi_k$ (in radians) of $N$ individual half-wave plates to design broadband modular half-wave plates.

| $N$ | Rotation angles ($\theta_1; \cdots; \theta_N$) | Lengths of birefringence ($\varphi_1; \cdots; \varphi_N$) |
|-----|-----------------------------------------------|-------------------------------------------------|
| 3   | (23.17; 148.70; 209.54)                       | (0.735\pi; 0.682\pi; 0.693\pi)                  |
| 5   | (233.56; 9.06; 316.09; 46.66; 204.29)          | (0.73\pi; 0.74\pi; 1.78\pi; 1.04\pi; 1.47\pi)   |

We compare the performances of the broadband half-wave plates presented in Sections 2 and 3. We assume that the constituent half-wave plates have the same single half-wave plate fidelities, which have been extracted from experimental measurements (WPMQ10M-780, Thorlabs). In Fig. 3 we plot the results for $N = \{3, 5\}$ individual half-wave plates. It is clearly shown that the modified design of modular half-wave plate with variable length of the individual half-wave plates is broadband over a much wider range of the wavelength of the incident light compared to the previously described designs.

**4. Conclusions**

We demonstrated modular broadband half-wave plates and we proposed a novel design of even more broadband modular half-wave plates, which is achieved due to a large freedom in setting the variable control parameters. We calculated numerically the values of these parameters – rotation
Figure 3. Fidelity $F = tr(J_0^{-1}J^{(N)})$ of the Jones matrix of the modified broadband half-wave plate as a function of the wavelength $\lambda$. The number of individual half-wave plates is $N = \{3, 5\}$ and we compare with the results of the existing broadband half-wave plates from Fig. 1 also with $N = \{3,5\}$.

angles and lengths of the individual half-wave plates – for $N = \{3, 5\}$ number of constituent half-wave plates. We showed that by using the same number of individual half-wave plates we obtain a half-wave plate with a larger bandwidth compared to previous designs. Finally, our results confirm the applicability of the composite pulses control technique from quantum optics to design robust optical elements, irrespective of the birefringent materials used.

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