RECOGNIZING RIGHT-ANGLED COXETER GROUPS USING INVOLUTIONS

CHARLES CUNNINGHAM, ANDY EISENBERG, ADAM PIGGOTT, KIM RUANE

Abstract. We consider the question of determining whether a given group (especially one generated by involutions) is a right-angled Coxeter group. We describe a group invariant, the involution graph, and we characterize the involution graphs of right-angled Coxeter groups. We use this characterization to describe a process for constructing candidate right-angled Coxeter presentations for a given group or proving that one cannot exist. As a corollary, we provide an elementary proof of rigidity of the defining graph for a right-angled Coxeter group. We also recover a result stating that if $\Gamma$ contains no SILs, then $\text{Aut}^0(W)$ is a right-angled Coxeter group.

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1. Introduction

Given a finite simple graph $\Gamma$, the right-angled Coxeter group defined by $\Gamma$ is the group $W = W_\Gamma$ generated by the vertices of $\Gamma$. The relations of $W_\Gamma$ declare that the generators all have order 2, and adjacent vertices commute with each other. Right-angled Coxeter groups (commonly abbreviated RACG) have a rich combinatorial and geometric
history [Dav08]. The particular presentation specified by $\Gamma$ is called a right-angled Coxeter system. When encountering a group generated by involutions, a natural question is to ask whether or not this group might be a right-angled Coxeter group, and if so, how to identify the preferred presentation. In this paper, we develop a procedure that successfully answers this question for certain groups.

Given a group $G$, the involution graph $\Delta_G$ of $G$ is the group invariant defined as follows: the vertices in $\Delta_G$ correspond to the conjugacy classes of involutions in $G$; vertices are adjacent when there exist commuting representatives of the corresponding conjugacy classes. In general, this invariant is unwieldy. It may be infinite, and even when it's finite, it may be impossible to construct. Nevertheless, for certain classes of groups the invariant promises insights. Like any invariant, it can allow us to distinguish between groups. It also carries information on the automorphism group of $G$. Since an automorphism must permute conjugacy classes of involutions and must preserve commuting relations, $\text{Aut}(G)$ acts naturally on $\Delta_G$. The kernel of this action is therefore a natural normal subgroup of $\text{Aut}(G)$, and has finite index in $\text{Aut}(G)$ when $\Delta_G$ is finite.

The involution graph for a right-angled Coxeter group $W$ is easily constructed directly from $\Gamma$: the vertices in $\Delta_W$ correspond to cliques in $\Gamma$; vertices are adjacent when the union of the corresponding cliques is also a clique. When constructed in this manner, we denote the graph $\Gamma_K$ and call it the clique graph for $\Gamma$. Tits [Tit88] proved that the kernel of the action $\text{Aut}(W) \cup \Delta_W$ has a natural complement, which is therefore a finite subgroup of $\text{Aut}(\Delta_W)$. Thus the involution graphs of right-angled Coxeter groups are significantly more tractable than the involution graphs of arbitrary groups, and may be more convenient for certain purposes than the defining graph $\Gamma$.

In Section 2 below we study the class of clique graphs, because they are the involution graphs of right-angled Coxeter groups. We give combinatorial criteria for determining whether or not a finite simple graph is a clique graph, and we present an algorithm, called the collapsing algorithm, for recovering $\Gamma$ from $\Gamma_K$.

The tools developed in Section 2 offer an approach to the problem of determining whether or not a given group is a right-angled Coxeter group as follows. Suppose we can describe $\Delta_G$ for a finitely-generated group $G$. If $\Delta_G$ is not a clique graph, then we may conclude $G$ is not isomorphic to a right-angled Coxeter group. If $\Delta_G$ is a clique graph, then we may reconstruct the unique graph $\Gamma$ such that $\Delta_G = \Gamma_K$. The collapsing algorithm also suggests a candidate map $W_\Gamma \to G$. Thus we have an assurance that $W_\Gamma$ is the only right-angled Coxeter group that
might be isomorphic to $G$, and we have a head-start on a potential isomorphism.

That this approach is practical in some situations is illustrated, and consequences explored, in Section 3 below. In Section 3.2, we exhibit an example, concluding that a particular extension of a right-angled Coxeter group is again a right-angled Coxeter group. In Section 3.3, we exhibit an example of a group which we might naturally expect to be a right-angled Coxeter group, but which our procedure shows cannot be. In Section 3.4 we present more general results, giving many extensions of arbitrary right-angled Coxeter groups which are again right-angled Coxeter groups.

The existence of the collapsing algorithm may be understood in a diagrammatic way. We write $\mathcal{G}$ for the class comprising all finite simple graphs, $\mathcal{K}$ for the class comprising all clique graphs, and $\mathcal{W}$ for the class comprising all right-angled Coxeter groups. It is a consequence of the results in Section 2 that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\text{clique construction}} & \mathcal{K} \\
& \xleftarrow{\text{collapse}} & \\
\text{Coxeter presentation} & \xrightarrow{\text{involution graph}} & \mathcal{W}
\end{array}
$$

The well-known rigidity of right-angled Coxeter groups is an immediate consequence of the commutativity of the diagram.
2. Characterizing Clique Graphs

2.1. The Clique Graph and the Star Poset. A finite simple graph \( \Gamma = (V, E) \) is an ordered pair of finite sets for which \( V \), the set of vertices, is nonempty and \( E \), the set of edges, comprises two-element subsets of \( V \). We say \( a, b \in V \) are adjacent if \( \{a, b\} \in E \). For convenience we shall often omit the adjective “simple”, as all graphs considered are simple.

In this section, we describe two basic graph constructions given a finite simple graph \( \Gamma \): the clique graph \( \Gamma_K \), and the star poset \( \mathcal{P}(\Gamma) \). We define some combinatorial conditions satisfied by graphs which are clique graphs, we discuss a correspondence between the maximal cliques of \( \Gamma \) and those of \( \Gamma_K \), and we discuss some properties of the star poset of graphs we know (or suspect) to be clique graphs.

**Definition 2.1.** Let \( \Gamma \) be a finite simple graph. A clique in \( \Gamma \) is a set of pairwise adjacent vertices. The clique graph of \( \Gamma \) is the finite simple graph \( \Gamma_K = (V_K, E_K) \) whose vertices correspond to nonempty cliques in \( \Gamma \), and such that vertices are adjacent if the corresponding union of cliques is also a clique in \( \Gamma \).

![Figure 1](image_url) An example of a graph \( \Gamma \) (left) and its corresponding clique graph \( \Gamma_K \) (right).

Let \( \Gamma \) be any finite graph on \( n \) vertices. Recall that the star of a given vertex \( v \), denoted \( \text{St}(v) \), is the set comprising \( v \) and all vertices adjacent to \( v \). The relation \( v \sim w \) when \( \text{St}(v) = \text{St}(w) \) is an equivalence relation on \( V(\Gamma) \). We consider a partial ordering on the star-equivalence classes of vertices of \( \Gamma \), declaring \( [v] \preceq [w] \) if \( \text{St}(v) \subseteq \text{St}(w) \). We write \( \mathcal{P}(\Gamma) \) for the poset of star-equivalence classes of vertices in \( \Gamma \).

**Definition 2.2.** Let \( \Gamma \) be a graph and \( v \in \Gamma \). We say that \( v \) is a minimal vertex of \( \Gamma \) if \( v \) is contained in a unique maximal clique of \( \Gamma \).
Proposition 2.10 will justify the term “minimal”. We observe that if $v$ is minimal, then the unique maximal clique containing $v$ is precisely $\text{St}(v)$.

We now examine properties of a clique graph $\Gamma_K$ and the corresponding poset $\mathcal{P}(\Gamma_K)$, with the goal of being able to recognize whether or not an arbitrary graph is a clique graph. The following proposition fixes notation to be used throughout the remainder of the paper and establishes a correspondence between the intersections of maximal cliques in $\Gamma$ and those of $\Gamma_K$. Namely, that $\Gamma$ and $\Gamma_K$ have the same number of maximal cliques; that the (intersections of) maximal cliques of $\Gamma_K$ are precisely the clique graphs of the (intersections of) maximal cliques in $\Gamma$; and that the (intersections of) maximal cliques in $\Gamma_K$ have predictable sizes.

**Proposition 2.3.** Suppose $\Gamma$ is a finite graph with maximal cliques $\Gamma_1, \ldots, \Gamma_r$. For any subset $I \subset \{1, 2, \ldots, r\}$, write

$$\Gamma_I = \bigcap_{i \in I} \Gamma_i.$$ 

Similarly, write $\Gamma_{K,1}, \ldots, \Gamma_{K,s}$ for the maximal cliques of $\Gamma_K$, and write $\Gamma_{K,I}$ for the intersections of maximal cliques. Then:

1. $r = s$,
2. each $\Gamma_{K,i}$ contains at least one minimal vertex (reordering the indices if necessary, this minimal vertex corresponds to the maximal clique $\Gamma_i$),
3. $\Gamma_{K,i} = (\Gamma_i)_K$, (that is, $(\Gamma_i)_K$, which is its own labeled graph, naturally injects into $\Gamma_K$, and the image is precisely $\Gamma_{K,i}$),
4. $\Gamma_{K,I} = (\Gamma_I)_K$, and
5. if $\Gamma_I$ is a clique of size $k$, then $\Gamma_{K,I}$ is a clique of size $2^k - 1$.

**Proof.** (1) For each maximal clique $\Gamma_i$ in $\Gamma$, there is a corresponding vertex $v_i$ in $\Gamma_K$. This vertex is adjacent only to vertices representing subsets of $\Gamma_i$ since $\Gamma_i$ is maximal, and so $v_i$ is contained in the unique maximal clique $\text{St}(v_i)$ in $\Gamma_K$. Thus, $v_i$ is a minimal vertex.

Note that for $i \neq j$, $\Gamma_i \neq \Gamma_j$ and so $v_i \neq v_j$. If $\text{St}(v_i) = \text{St}(v_j)$, then $v_i$ and $v_j$ would be adjacent, corresponding to $\Gamma_i$ and $\Gamma_j$ being both contained in a clique. But that would contradict the maximality of these cliques, and so $\text{St}(v_i) \neq \text{St}(v_j)$. Associating $\Gamma_i$ to $\text{St}(v_i)$ now shows that $r \leq s$.

Now any maximal clique $\Gamma_{K,i}$ contains vertices $w_j$ each corresponding to a clique in $\Gamma$. Take all of the vertices $u_k$ in $\Gamma$ that appear in any of these cliques, and let $\Lambda$ be the subgraph of $\Gamma$ induced by the union of all these vertices. If any two $u_{k_1}$ and $u_{k_2}$ are
not connected in Γ, then they couldn’t be contained in a clique and no set containing them could be contained in a clique either. But they each came from a clique corresponding to some $w_{j_1}$ and $w_{j_2}$, and if these two cliques are not contained in any larger clique, then $w_{j_1}$ and $w_{j_2}$ would not be connected in $\Gamma_{K,i}$, which contradicts the fact that $\Gamma_{K,i}$ is a clique. Thus, $\Lambda$ is a clique in $\Gamma$.

If $\Lambda$ were not a maximal clique in $\Gamma$, then we could extend it to a maximal clique, which would also then extend the clique $\Gamma_{K,i}$ in $\Gamma_K$. But since this clique was maximal, no such extension is possible. So $\Lambda = \Gamma_j$ for some $j$. Since $\Lambda$ was uniquely determined, associating $\Gamma_{K,i}$ to $\Gamma_j$ in this manner shows that $s \leq r$, and so $r = s$.

(2) Following the notation above, and reordering if necessary, every $\Gamma_{K,i}$ is uniquely associated to $\Gamma_i$. But then $v_i$ is a minimal vertex in $\text{St}(v_i)$, which is just $\Gamma_{K,i}$ by the uniqueness of our associations.

(3) Given any $\Gamma_i$, the corresponding vertex $v_i$ in $\Gamma_K$ is contained in the unique maximal clique $\Gamma_{K,i} = \text{St}(v_i)$. Any clique in $\Gamma_i$ corresponds to a vertex in $\Gamma_K$ adjacent to $v_i$ and thus contained in $\text{St}(v_i) = \Gamma_{K,i}$. Vertices in $\Gamma_{K,i}$ corresponding to two cliques contained in $\Gamma_i$ are adjacent because their union is contained in $\Gamma_i$, a clique. Thus by definition, $\Gamma_{K,i} = (\Gamma_i)_K$.

(4) A vertex is in $\Gamma_{K,I}$ if and only if it is in $\Gamma_{K,i}$ for all $i \in I$ if and only if it corresponds to a clique contained in $\Gamma_i$ for all $i \in I$ if and only if it corresponds to a clique in $\Gamma_I$. So vertices in $\Gamma_{K,I}$ are exactly the cliques in $\Gamma_I$. Since the adjacency relation is still the same, $\Gamma_{K,I} = (\Gamma_I)_K$.

(5) If $\Gamma_I$ is a clique of size $k$, then every non-empty subset of vertices induces a clique, and so corresponds to a vertex in $\Gamma_{K,I}$. There are $2^k - 1$ of these subsets, which correspond to $2^k - 1$ vertices in $\Gamma_{K,I}$. Since all of these cliques are contained in the larger clique $\Gamma_I$, all of their corresponding vertices are connected in $\Gamma_K$, and so $\Gamma_{K,I}$ is a clique as well.

Let $\Gamma$ be a finite graph with maximal cliques $\Gamma_1, \ldots, \Gamma_r$. As before, write $\Gamma_I$ for the intersections of the maximal cliques, and suppose $|\Gamma_I| = k_I$. Then

$$\sum_{I \subseteq J} (-1)^{|I| + 1} k_I \leq k_J.$$ 

This is a direct application of the inclusion-exclusion principle, since the left hand side of the inequality counts the number of vertices in
We make this observation because the proposition essentially says that the maximal cliques of $\Gamma$ and of $\Gamma_K$ are in bijective correspondence and have the same intersection structure. If we are faced with some graph which we do not know to be a clique graph, we can check directly that the intersections of maximal cliques have sizes of the form $n_I = 2^{k_I} - 1$, and we can check directly that the system of integers $k_I$ satisfies the inclusion-exclusion inequalities.

**Definition 2.4.** Let $\Gamma$ be a graph with maximal cliques $\Gamma_i$, and write $\Gamma_I$ for the intersections of maximal cliques as above. A vertex $v \in \Gamma_J$ is $J$-minimal if there is no $J' \supseteq J$ such that $v \in \Gamma_{J'} \subsetneq \Gamma_J$.

We observe that there may be supersets $J' \supset J$ such that $\Gamma_J = \Gamma_{J'}$. The definition of a $J$-minimal vertex considers only further intersections of maximal cliques in which the intersection strictly shrinks in size. This generalizes the notion of minimal vertex, and the previous proposition is easily adjusted to show that each $\Gamma_{K,J}$ contains a $J$-minimal vertex.

**Definition 2.5.** Let $\Gamma$ be a graph with maximal cliques $\Gamma_i$, and write $\Gamma_I$ for the intersections of maximal cliques.

1. We say that $\Gamma$ satisfies the maximal clique condition if, for all $I$, there exists an integer $k_I$ such that $|\Gamma_I| = 2^{k_I} - 1$.
2. If $\Gamma$ satisfies the maximal clique condition, we will say that $\Gamma$ satisfies the inclusion-exclusion condition if, for each $J$,

$$\sum_{I \supseteq J} (-1)^{|I \setminus J|+1} k_I \leq k_J.$$ 

3. We say that $\Gamma$ satisfies the minimal vertex condition if every nonempty $\Gamma_J$ contains a $J$-minimal vertex $v_J$.

We have seen that the conditions in the definition are satisfied in any clique graph. The following example illustrates that the conditions are easily checked for any graph.

**Example 2.6.** For the graph with vertices $\{A, B, C, D, E, F, G\}$ and maximal cliques $\{A, B, E\}, \{B, C, F\}, \{C, D, G\}$, we collate the following data.
We see that the conditions in Definition 2.5 hold.

Example 2.7. The graph $\Gamma$ with vertex set $\{A, B, C, D, E, F\}$ and maximal cliques $\{A, B, C\}, \{A, D\}, \{B, E\}, \{C, F\}$, fails the maximal clique condition because $|\{A, D\}| = 2$.

We will later show that the conditions in Definition 2.5 characterize clique graphs. That is, given $\Gamma'$ which satisfies these three conditions, there is a (unique up to isomorphism) $\Gamma$ so that $\Gamma' = \Gamma_K$. First, we must establish some properties of the star poset structure, especially for clique graphs. We’ll make use of these facts in our collapsing algorithm in Section 2.2.

Lemma 2.8. Let $[v] \in P(\Gamma)$. Then the vertices

$$S = \bigcup_{[v] \leq [w]} [w]$$

form a clique in $\Gamma$. If this clique is maximal, then $[v]$ is minimal in $P(\Gamma)$.

Proof. If $w, w' \in S$ are any vertices, then $w \in \text{St}(v) \subseteq \text{St}(w')$, so $w$ and $w'$ are adjacent. Thus $S$ forms a clique.

We now suppose $[v]$ is not minimal. Then there is some $[w] < [v]$. In particular, $w \notin S$, but $w \in \text{St}(w) \subseteq \text{St}(s)$ for any $s \in S$, hence $w$ is a vertex outside of $S$ adjacent to all of $S$. Thus $S$ is not maximal. $\Box$

Definition 2.9. For $[v] \in P(\Gamma)$, we call the clique $S$ defined in the lemma the clique above $[v]$. We will use the notation $S_v$ if we need to keep track of the vertex $v$.

Simple computation using the graph in Example 2.7 confirms $[A] = \{a\}$ is minimal in $P(\Gamma)$, but $S_A = \{A\}$ is not a maximal clique. Hence the converse of the lemma (i.e., that minimality of $[v]$ implies maximality of $S_v$) is false in general. However, we claim that the converse does hold for those $\Gamma$ which are clique graphs. Namely:
Proposition 2.10. Suppose $\Gamma$ satisfies the minimal vertex condition. Then $[v]$ is a minimal element of $\mathcal{P}(\Gamma)$ if and only if $v$ is a minimal vertex of $\Gamma$. In this case, $S_v$ is the unique maximal clique containing $v$.

Proof. Suppose $v$ is a minimal vertex of $\Gamma$. Then $\text{St}(v)$ is the unique maximal clique containing $v$. Since $S_v$ is a clique containing $v$, it is clear that $S_v \subseteq \text{St}(v)$. Conversely, if $x \in \text{St}(v)$, then $\text{St}(v) \subseteq \text{St}(x)$, hence $[v] \leq [x]$ and $x \in S_v$. Thus $\text{St}(v) = S_v$ is maximal. By the previous lemma, since $S_v$ is maximal, $[v]$ is minimal.

Conversely, suppose $v$ is not minimal. Then $v$ is contained in the intersection of two distinct maximal cliques, $\Gamma_1$ and $\Gamma_2$. Since $\Gamma_i$ are maximal cliques, they contain minimal vertices $w_i$. By the above argument, $[w_i] \leq [v]$, and this must be a strict inequality since, e.g., $w_2 \in \text{St}(v) \setminus \text{St}(w_1)$. Thus $[v]$ is not minimal. □

Proposition 2.11. For any finite graph $\Gamma$ and $[v] \in \mathcal{P}(\Gamma)$, $S_v$ is an intersection of maximal cliques.

Proof. Let $\Gamma_1, \ldots, \Gamma_k$ be all the maximal cliques of $\Gamma$ containing $S_v$. It is clear that $S_v \subseteq \bigcap \Gamma_i$.

Conversely, let $v' \in \bigcap \Gamma_i$ and suppose $v' \notin S_v$. Since $\text{St}(v) \notin \text{St}(v')$, there is some $x \in \text{St}(v)$ which is not in $\text{St}(v')$. In particular, since $\bigcup \Gamma_i \subseteq \text{St}(v')$, we must have $x \notin \Gamma_i$ for any $i$. By construction of $S_v$, we must have $x \in \text{St}(w)$ for each $w \in S_v$. Now $S_v \cup \{x\}$ forms a clique which contains $S_v$ and is not equal to $\Gamma_i$ for any $i$, contradicting our assumption that the list of $\Gamma_i$ contained all maximal cliques containing $S_v$. So there can exist no such $v'$, hence $S_v = \bigcap \Gamma_i$, proving the claim. □

We observe that the previous two propositions apply to all clique graphs:

Corollary 2.12. Suppose $\Gamma_K$ is a clique graph. Since $\Gamma_K$ satisfies the minimal vertex condition:

1) $[v]$ is minimal in $\mathcal{P}(\Gamma_K)$ if and only if $v$ is minimal in $\Gamma_K$, and
2) if $[v]$ is non-minimal, then $S_v$ is the intersection of maximal cliques (and therefore has size of the form $2^k - 1$). In this case,

$$S_v = \bigcap_{[w] \leq [v]} S_w.$$  

This shows that the star poset also records information about the intersections of maximal cliques: any clique above $[v]$ is such an intersection. Finally, we prove the converse.
Proposition 2.13. Suppose $\Gamma_K$ is a clique graph. Then any intersection of maximal cliques is equal to $S_v$ for some $v$.

Proof. Since $\Gamma_K$ is a clique graph, it satisfies the minimal vertex condition. Let $\Gamma_{K,J}$ be any intersection of maximal cliques, and let $v \in \Gamma_{K,J}$ be a $J$-minimal vertex. Without loss of generality, let $J$ be the maximal index set without changing the intersection. In particular, $J$ is precisely the index set of all maximal cliques containing $v$, so that $\text{St}(v) = \bigcup_{j \in J} \Gamma_j$.

We claim that $S_v = \Gamma_{K,J}$. Let $u \in S_v$. By definition of $S_v$, $[v] \leq [u]$, so

$$\text{St}(u) \supseteq \text{St}(v) = \bigcup_{j \in J} \Gamma_j.$$ 

That is, $u$ is adjacent to every vertex in $\Gamma_j$, for each $j \in J$. Since each $\Gamma_j$ is a maximal clique, this shows $u \in \Gamma_j$ for each $j \in J$. That is, $u \in \Gamma_{K,J}$.

Conversely, let $w \in \Gamma_{K,J}$. Then $w$ is adjacent to all $\Gamma_j$ for $j \in J$, thus $\bigcup_{j \in J} \Gamma_j \subseteq \text{St}(w)$. That is, $\text{St}(v) \subseteq \text{St}(w)$, so $[v] \leq [w]$. By definition of $S_v$, $w \in S_v$. □

These results establish that, for a clique graph, the cliques above vertices are precisely the intersections of maximal cliques, and every intersection of maximal cliques is the clique above some vertex. (This is not, in general, a bijective correspondence. As remarked earlier, it may be that $\Gamma_{K,J} = \Gamma_{K,J'} = S_v$, where $J \neq J'$.) In the following section, our collapsing algorithm to recover $\Gamma$ from $\Gamma_K$ begins at the top of the poset (the deepest intersections of maximal cliques) and works downwards. The previous proposition ensures that the algorithm examines every intersection of maximal cliques.

2.2. Collapsing Clique Graphs. We now describe an algorithm which decides whether a given graph $\Gamma'$ is a clique graph, and which constructs a graph $\Gamma$ such that $\Gamma' = \Gamma_K$, if such a $\Gamma$ exists. We call this procedure collapsing $\Gamma'$, and we write $\Gamma = C(\Gamma')$. We spend the bulk of this section proving the that the output graph $\Gamma$ is well-defined up to isomorphism (in particular, that the sizes of intersections of maximal cliques uniquely identify a graph up to isomorphism), and that the algorithm produces an output graph in every case in which $\Gamma'$ is a clique graph. In particular, it will follow from the proof of correctness of this algorithm that the graph properties defined in Section 2.1 characterize those finite graphs which are clique graphs.

The algorithm is as follows:

1. If $\Gamma'$ does not satisfy the minimal vertex condition, return false.
(2) If \( \Gamma' \) does not satisfy the inclusion-exclusion condition, return false.

(3) Initialize \( V = \{\} \).

(4) Let \([w]\) be a class which has not yet been considered such that all \([v]\) with \( [w] \leq [v] \) have been considered. Since \( \Gamma' \) satisfies the minimal vertex condition, \( S_w \) is the intersection of maximal cliques. Since \( \Gamma' \) satisfies the inclusion-exclusion condition (and therefore the maximal clique condition), \(|S_w| = 2^k - 1 \) for some \( k \). Let \( k' = |\{v \in V \mid [w] \leq [v]\}| \). Choose \( k - k' \) elements from \([w]\) arbitrarily, and add them to \( V \).

(5) Repeat the previous step until all equivalence classes have been considered.

(6) Return \( \Gamma \) given by the induced subgraph of \( \Gamma' \) on the vertex set \( V \).

**Proposition 2.14.** In Step (4), \( 0 \leq k - k' \leq |[w]| \). So we can choose an appropriate number of vertices from \([w]\) to add to \( V \).

**Proof.** The clique \( S_w \) is some intersection of maximal cliques \( \Gamma'_J \) by Corollary 2.12. From this clique, we have already chosen \( k' \) vertices, and every vertex among those already chosen comes from a larger poset element, which is therefore a strictly smaller intersection of maximal cliques. By the inclusion-exclusion condition, the number of elements we could have chosen is at most \( k_J = k \), hence \( k' \leq k \). \( \square \)

**Proposition 2.15.** Given \( \Gamma' \), if the procedure above does not return false, then the isomorphism type of the graph \( \Gamma \) does not depend on the choices made in Step (4).

**Proof.** Without loss of generality, we will suppose our choices differ by a single vertex: suppose we are about to consider \([v]\) and have constructed the set \( V \) thus far. Let \( v_1, \ldots, v_{k+1} \in [v] \), where \( k > 0 \) is the number of vertices from \([v]\) which we must add to \( V \). Let
\[
V_1 = V \cup \{v_1, \ldots, v_k\} \\
V_2 = V \cup \{v_1, \ldots, v_{k-1}, v_{k+1}\}.
\]
We observe that we can make all future choices the same (since we haven’t changed the number of vertices we must pick from \([w]\) for any \( [w] \leq [v]\)), so that we create two final graphs \( \Gamma_1 \) and \( \Gamma_2 \) whose vertex sets differ only by switching \( v_k \) and \( v_{k+1} \).

We now claim that the resulting graphs \( \Gamma_1 \) and \( \Gamma_2 \) are isomorphic. By the previous observation, the vertex sets of \( \Gamma_1 \) and \( \Gamma_2 \) differ only by switching \( v_k \) and \( v_{k+1} \). So we can define a map \( \varphi: \Gamma_1 \rightarrow \Gamma_2 \) which sends each vertex other than \( v_k \) to itself, and which sends \( v_k \) to \( v_{k+1} \).
We claim that \( \varphi \) defines a graph isomorphism. Clearly any adjacency relation not involving \( v_k \) is preserved under \( \varphi \). Suppose \( w \) is a vertex of \( \Gamma_1 \) adjacent to \( v_k \). Then \( w \in \text{St}(v_k) = \text{St}(v_{k+1}) \), so \( w \) is adjacent to \( v_{k+1} \). Thus \( \varphi \) is a graph homomorphism. By the same argument, the analogous map \( \psi : \Gamma_2 \rightarrow \Gamma_1 \) is also a graph homomorphism, and the two maps are clearly inverses. Hence \( \Gamma_1 \) is isomorphic to \( \Gamma_2 \). The full result follows by induction.

The following proposition says that the intersections of maximal cliques in a graph \( \Gamma \) output by the algorithm have sizes prescribed by the sizes of corresponding intersections of maximal cliques in \( \Gamma' \).

**Proposition 2.16.** Let \( \Gamma' \) be a finite graph. Suppose that the algorithm above does not return false. In particular, this implies that we have a system of integers \( k_I \) so that \( |\Gamma'_I| = 2^{k_I} - 1 \). Let \( C(\Gamma') = \Gamma \). Then the maximal cliques of \( \Gamma \) correspond to the maximal cliques of \( \Gamma' \), and \( |\Gamma_I| = k_I \) for all \( I \).

**Proof.** Since the algorithm did not return false, \( \Gamma' \) satisfies the minimal vertex and inclusion-exclusion conditions. Each \( \Gamma'_I \) contains an \( I \)-minimal vertex \( v'_I \). We have \( |\Gamma'_I| = 2^{k_I} - 1 \), and the algorithm chooses exactly \( k_I \) vertices from \( S_{v'_I} \). Corollary 2.12 implies that the maximal cliques in \( \Gamma \) have sizes of the form \( k_i \), and Proposition 2.13 ensures that we have \( |\Gamma_I| = k_I \) for all intersections of maximal cliques (since all intersections \( \Gamma'_I \) occur as the clique above some element in the poset). \( \square \)

We have shown now that, if the algorithm returns any graph, then it returns a graph with a certain number of maximal cliques, and the intersections of the maximal cliques have certain sizes. We now establish that a finite graph is determined up to isomorphism by the sizes of the intersections of maximal cliques.

**Theorem 2.17.** Let \( \Gamma, \Lambda \) be finite graphs. Suppose, for all \( I \), that \( |\Gamma_I| = |\Lambda_I| \). That is, all intersections of maximal cliques have the same sizes in each graph. Then there is an isomorphism \( \varphi : \Gamma \rightarrow \Lambda \) which maps \( \Gamma_i \) to \( \Lambda_i \) for each \( i \).

**Proof.** We first claim that the poset structures \( \mathcal{P}(\Gamma) \) and \( \mathcal{P}(\Lambda) \) are the same, and the corresponding equivalence classes have the same sizes. For each \( v \in \Gamma \), let \( J_v \) be the maximal index set so that \( v \in \Gamma_{J_v} \). Then \( \text{St}(v) = \bigcup_{j \in J_v} \Gamma_j \). The equivalence class of \( v \) consists of those vertices of \( \Gamma_{J_v} \) which are not in any smaller intersection \( \Gamma_J \) (that is, with \( J_v \subsetneq J \)). By assumption, \( |\Gamma_{J_v}| = |\Lambda_{J_v}| \). Moreover, the number of
vertices which are in some further intersection is given by the inclusion-exclusion formula:

\[
\sum_{J \supseteq J_v} (-1)^{|J \setminus J_v|+1} |\Gamma_J| = \sum_{J \supseteq J_v} (-1)^{|J \setminus J_v|+1} |\Lambda_J|
\]

Thus the sizes of star-equivalence classes of vertices in \( \Gamma \) and \( \Lambda \) are equal. Moreover, we see that each star-equivalence class \([v]\) may be represented by some index subset \( J_v \) (although note that not all index subsets need represent some equivalence class).

An equivalence class represented by \( J \) is smaller in the poset structure than another represented by \( J_1 \) if and only if \( J \triangleleft J_1 \). Since this holds in both \( \Gamma \) and \( \Lambda \), it follows that the poset structures are equivalent.

We build a map \( \phi: \Gamma \rightarrow \Lambda \) by piecing together bijections between each pair of corresponding equivalence classes. We observe that, by construction, \( \phi([v]) = [\phi(v)] \).

We also observe that \( \Gamma_i \) is mapped to \( \Lambda_i \) for each \( i \): let \( v \in \Gamma_i \), so that \( i \in J_v \). By construction, \( \phi(v) \in \Lambda_{J_v} \), which is an intersection of maximal cliques including \( \Gamma_i \). That is, \( \phi(v) \in \Lambda_i \). It follows that \( \phi \) maps \( \Gamma_I \) to \( \Lambda_I \) for each \( I \).

We must show that \( \phi \) is an isomorphism. Suppose \( v, w \in \Gamma \) are adjacent. Then the edge \( \{v, w\} \) extends to some maximal clique \( \Gamma_i \). Now \( \phi \) maps \( \Gamma_i \) to \( \Lambda_i \), so \( \phi(v) \) and \( \phi(w) \) are still adjacent.

\[\square\]

**Corollary 2.18.** Let \( \Gamma' \) be a finite graph. Then:

1. if there exists \( \Gamma \) such that \( \Gamma' = \Gamma_K \), then \( C(\Gamma') = \Gamma \), and
2. if there exists no such \( \Gamma \), then \( C(\Gamma') \) returns false.

In particular, the finite clique graphs are precisely those which satisfy the three conditions of Definition 2.5.

**Proof.** Given \( \Gamma' \), suppose there exists a \( \Gamma \) such that \( \Gamma' = \Gamma_K \). By Proposition 2.3, the intersections of maximal cliques have sizes of the form \( n_I = 2^{k_I} - 1 \), where the corresponding intersection \( \Gamma_I \) of maximal cliques in \( \Gamma \) has size \( k_I \). By construction, the intersections of maximal cliques of \( C(\Gamma') \) have sizes \( k_I \). By the previous theorem, \( C(\Gamma') \cong \Gamma \).

Conversely, suppose the algorithm does not return false. Let \( C(\Gamma') = \Gamma \). Then, in particular, \( \Gamma' \) must satisfy the maximal clique condition, so that \( n_I = 2^{k_I} - 1 \) for each \( I \). By construction, the intersections of maximal cliques of \( \Gamma \) have sizes \( k_I \). Taking the clique graph \( \Gamma_K \), the intersections of maximal cliques have sizes \( n_I \). By the previous theorem, \( C(\Gamma')_K \cong \Gamma' \). That is, if the algorithm does not return false, then there exists a \( \Gamma \) (namely \( C(\Gamma') \)) such that \( \Gamma' = \Gamma_K \). \[\square\]
3. Recognizing RACGs

3.1. Involution Graphs. In this section, we introduce our main invariant, the *involution graph*. We describe the connection between the involution graph and clique graphs, and we describe a process to use the involution graph of a given group $G$ to construct a candidate isomorphism to a right-angled Coxeter group.

**Definition 3.1.** Let $G$ be a group. The *involution graph* of $G$, denoted $\Delta_G$, is a graph defined as follows. The vertices are the conjugacy classes of involutions in $G$. Two vertices $[x]$ and $[y]$ are connected by an edge if there exist representatives $gxg^{-1}$ and $hyh^{-1}$ which commute with each other.

Note that the representatives which witness commutativity are chosen for individual edges. A priori, it could be the case that there is no system of representatives for all conjugacy classes which simultaneously witnesses all commuting relations (and indeed there are such examples). A system of representatives which witness all commutativity relations simultaneously will be called *full*.

Let $\Gamma$ be a graph with vertex set $V = V(\Gamma) = \{a_1, \ldots, a_n\}$ and edge set $E = E(\Gamma)$. The *right-angled Coxeter group* (RACG) defined by $\Gamma$ is the group

$$W_{\Gamma} = \langle a_i \mid a_i^2 = 1 \text{ for all } i, [a_i, a_j] = 1 \text{ for all } \{a_i, a_j\} \in E \rangle.$$

It is a standard fact about right-angled Coxeter groups that the only finite order elements have order 2, and these are conjugate to a product of pairwise commuting generators. A product of pairwise commuting generators is easily identified in the defining graph $\Gamma$: such generators form a clique. Thus we observe that $\Delta_{W_{\Gamma}} = \Gamma_K$.

The following was originally proven in [Gre90], and many other proofs have been presented in different contexts (see, for examples, [Dro87, Lau95, Rad03]).

**Corollary 3.2** (Rigidity). The defining graph of a right-angled Coxeter group $W_{\Gamma}$ is unique up to isomorphism.

**Proof.** The involution graph of $W_{\Gamma}$ is independent of the choice of a right-angled Coxeter presentation. It follows from Proposition 2.3 and Theorem 2.17 that the graph output by the collapsing algorithm must be isomorphic to $\Gamma$. $\square$

Moreover, the collection of products of pairwise commuting generators forms a full system of representatives. If we apply the algorithm from Section 2.2 to $\Delta_{W_{\Gamma}}$, we will recover the (unlabeled) graph $\Gamma$. We
can, in fact, go through the collapsing procedure with labels, but we cannot choose vertices arbitrarily in step 4 of the algorithm. We will discuss how to modify this step below. Once we’ve made this modification, the output of the collapsing algorithm will be a labeled graph.

This suggests the following procedure for determining the right-angled Coxeter-ness of a given group $G$:

1. Construct the involution graph $\Delta_G$.
2. Find a full system of representatives for $\Delta_G$ to use as labels.
3. Apply the amended collapsing algorithm, producing a labeled graph $\Gamma$.
4. Let $\Gamma'$ be a graph isomorphic to $\Gamma$ whose vertices are labeled generically, e.g., $a_1, \ldots, a_n$. Define a map $\varphi: W_{\Gamma'} \to G$ which takes $a_i$ to the label of the corresponding vertex in $\Gamma$. Check if $\varphi$ is an isomorphism.

We call the map $\varphi$ the candidate map or candidate isomorphism. If $\varphi$ is an isomorphism, then $G$ is a right-angled Coxeter group, and moreover we have its defining graph $\Gamma$ and a right-angled Coxeter presentation, taking the labels of $\Gamma$ as our generators.

Conversely, if $\varphi$ is not an isomorphism, we cannot conclude that $G$ is not a right-angled Coxeter group. It may be that the candidate map for one full system of representatives is an isomorphism while the candidate map for another full system is not. (Indeed, we can construct such examples.) So, if $\varphi$ is not an isomorphism, we can only conclude that the particular presentation suggested by $\varphi$ is not a right-angled Coxeter presentation for $G$.

We now discuss the modification to step 4 of the collapsing algorithm which chooses which vertices to keep when collapsing. The main problem we’re trying to avoid is something like the following:

**Example 3.3.** Consider the triangle graph $\Gamma$ with vertices $a, b, c$. The involution graph $\Delta_{\Gamma}$ consists of a 7-clique whose vertices are labeled by the products of $a, b, c$. The star poset of $\Delta_{\Gamma}$ consists of a single equivalence class, from which the collapsing algorithm as described previously says to choose three vertices at random. If we choose, for example, the vertices $a, ab, abc$, then the candidate map is, in fact, an isomorphism. On the other hand, if we choose the vertices $a, b, ab$, then the candidate map is not surjective.

The problem, then, is that our choices are not independent, and in choosing candidate generators for our right-angled Coxeter presentation, we might accidentally choose elements that share a non-trivial product relation. Checking whether or not such a relation exists...
amongst a proposed set of group elements requires a solution to the word problem in $G$. Such an algorithm may either not exist or may not be apparent from the starting presentation of $G$. However, the abelianization of $G$, $G^{ab}$, is a finitely generated abelian group and thus always has such an algorithm. Furthermore, when $G$ actually is isomorphic to a right-angled Coxeter group, then $G^{ab} \cong (\mathbb{Z}/2\mathbb{Z})^n$ and such product relations among involutions in $G$ also occur in $G^{ab}$. So in order to avoid bad choices in step 4 of our procedure, we can examine the images of our labels in $G^{ab}$ and do our computations there. Computing the abelianization will also provide a “Step 0” of our procedure, for if $G^{ab}$ is not isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$, then we know that $G$ is not a right-angled Coxeter group and our procedure can immediately halt.

Effectively computing $G^{ab}$ proceeds as follows:

1. $G$ is initially given in a finite presentation

$$G = \langle s_1, s_2, \ldots, s_m \mid r_1, r_2, \ldots, r_k \rangle.$$

2. A presentation for $G^{ab}$ is thus

$$G^{ab} \cong \langle \overline{s_1}, \overline{s_2}, \ldots, \overline{s_m} \mid \overline{r_1}, \overline{r_2}, \ldots, \overline{r_k}, [\overline{s_i}, \overline{s_j}] \text{ for } 1 \leq i, j \leq m \rangle,$$

where each $\overline{r_j}$ can now be rearranged (writing additively) to be a linear expression in the variables $\overline{s_i}$ with integer coefficients:

$$\overline{r_j} = a_{1,j} \overline{s_1} + a_{2,j} \overline{s_2} + \cdots + a_{m,j} \overline{s_m}.$$

Collecting the coefficients $(a_{i,j})$ into an $k \times m$ integer matrix $R$, we get the relation matrix for $G^{ab}$.

3. Using (and keeping track of) elementary row and column operations, we diagonalize $R$ to its Smith normal form, $S$, where $R = PSQ$ for invertible matrices $P$ and $Q$. Interpreting this relation matrix as a presentation, we have that $G^{ab}$ is in a canonical form as a direct product of cyclic groups. Normal forms are immediate and computations in $G^{ab}$ are much easier.

4. We now have an effective quotient map from $G$ to $G^{ab}$ in this canonical form. For any $g \in G$, $g = \Pi s_j$, $\overline{g} = \Sigma \overline{s_j} = \Sigma_{i=1}^m b_i \overline{s_i}$. Then $(b_1 \ b_2 \ \cdots \ b_m) Q$ will give the coefficients of the image of $g$ in the Smith normal form presentation of $G^{ab}$. In particular, product relations are easy to compute.

An important fact about right-angled Coxeter groups is that the abelianization is injective on conjugacy classes of involutions.

**Proposition 3.4.** Consider a right-angled Coxeter group $W_\Gamma$ and $x, y \in W_\Gamma$ such that $x^2 = y^2 = 1$. Then $\overline{x} = \overline{y}$ in $G^{ab}$ if and only if $x$ and $y$ are conjugate in $W_\Gamma$. 
Proof. If $x = g y g^{-1}$ for some $g \in G$, then trivially $\overline{x} = \overline{y}$ in $G^{ab}$.

Conversely, suppose that $x$ and $y$ are not conjugate in $W_\Gamma$. Let $(W, S)$ be a right-angled Coxeter system for $W_\Gamma$. In a right-angled Coxeter system, every involution is conjugate to a product of distinct commuting generators. So there exists $g, h \in G$, commuting generators $a_1, a_2, \ldots, a_k \in S$, and commuting generators $b_1, b_2, \ldots, b_l \in S$ such that $x = g a_1 a_2 \cdots a_k g^{-1}$ and $y = h b_1 b_2 \cdots b_l h^{-1}$. Without loss of generality, since $x$ and $y$ are not conjugate, there is a $b_j$ that does not appear among the $a_i$. But since it is a generator, there is a $\mathbb{Z}/2\mathbb{Z}$ direct factor in $G^{ab}$ corresponding to that $\overline{b_j}$. $y$ will have a 1 in this factor and $x$ will have a 0. Thus, $\overline{x} \neq \overline{y}$ in $G^{ab}$. \qed

Corollary 3.5. For a right-angled Coxeter group $W_\Gamma$, if $H$ is a subgroup generated by distinct, commuting involutions, then $H \cong H^{ab}$ injects into $G^{ab}$.

Proof. $H$ is a finite subgroup of $W_\Gamma$ and so is conjugate to a special subgroup $H'$. Each element of $H'$ is a distinct product of commuting generators from $\Gamma$ and so each gets sent to a distinct element of $G^{ab}$. Thus, no two elements of $H'$ can be conjugate in $W_\Gamma$ and so neither can any two elements of $H$. By Proposition 3.4, $H$ injects into $G^{ab}$. \qed

Proposition 3.6. If $G$ is a right-angled Coxeter group, then in step 4 of our collapsing procedure, we can choose the $k - k'$ involutions of $G$ so that the elements chosen of $V$ never exhibit a non-trivial product relation.

Proof. In step 4 of our collapsing procedure, we consider an equivalence class $[w]$ of $\Delta_G$ and the clique above it, $S_w$, where $|S_w| = 2^k - 1$ for some $k$. If $(G, S)$ is a right-angled Coxeter system for $G$ and the labels are distinct, pairwise commuting involutions, they generate a finite subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ in $G$ where $n \geq k$. We need to show that $n = k$. $S_w$ is a generating set for this vector space, and so let $B \subset S_w$ be a basis for it. By definition, $S_w \cup \{e\} \subset \langle B \rangle$. For each $b \in B \subset S_w$, $[w] \leq [b]$ and so $\text{St}(w) \subset \text{St}(b)$. So for any $x \in \text{St}(w)$, $x$ commutes with each $b \in B$ and so commutes with every element $g \in \langle B \rangle$. Every such $g \neq e$ is an involution and $x \in \text{St}(g)$. So $\text{St}(w) \subset \text{St}(g)$, i.e., $[w] \leq [g]$. Therefore, $g \in S_w$ and so $\langle B \rangle \subset S_w \cup \{e\}$.

Thus, $\langle B \rangle = S_w \cup \{e\}$ which has $2^k$ elements and so $n = k$.

By Corollary 3.5, this subgroup projects injectively as a vector subspace into $G^{ab}$. Inductively, we assume that there exists a choice of a right-angled Coxeter system $(G, S)$ such that $\overline{V}$ is a set of standard basis elements for $(G, S)^{ab} \cong (\mathbb{Z}/2\mathbb{Z})^k$, i.e., each element has only one non-zero component in the representation for the abelianization given
by our choice of right-angled Coxeter system \((G, S)\). (The base case is \(\mathcal{V} = \emptyset\) and any choice of \((G, S)\).

It follows that \(\mathcal{V} \cap S_w\) is a linearly independent set in the \(\mathbb{Z}/2\mathbb{Z}\)-vector space \(G_{ab}\). We can then choose \(k-k'\) labels in \(S_w - V\) to extend this linearly independent set to a basis \(\mathcal{B}\) of \(\langle S_w \rangle\). (It’s possible that \(k-k'=0\).) Since the subgroup projects injectively, choosing a basis for \(\langle S_w \rangle\) is the same as choosing a basis for \(\langle S_w \rangle\). We need to show that \(VYB\) is linearly independent as well.

To clarify, we are now keeping track of two different representations of the abelianization. \(G_{ab}\) is the form calculated from the Smith Normal Form in step 0 of the procedure, and \((G, S)^{ab}\) is the form wherein each element of \(V\) is a standard basis element. We will show that such a form must exist if \(G\) is a right-angled Coxeter group, but it will not be directly computable during the procedure itself. The existence of this form will be used to show that any choice of \(\mathcal{B}\) during our procedure will result in no non-trivial product relations.

Since \(\langle S_w \rangle\) is a finite subgroup of \((G, S)\), it is conjugate to a special subgroup: \(g\langle S_w\rangle g^{-1} = \langle a_1, a_2, \ldots, a_k \rangle\) for \(\{a_1, a_2, \ldots, a_k\} \in S\). Consider \(b \in B \subset S_w\). Reordering the vertices of \(S\) if necessary, then \(gbg^{-1} = a_1a_2 \cdots a_m\) in \((G, S)\). By the deletion condition of right-angled Coxeter groups, a product of distinct, commuting generators of \((G, S)\), \(c_1c_2 \cdots c_l\) commutes with \(a_1a_2 \cdots a_m\) if and only if \(c_j\) commutes with \(a_i\) for each \(i, j\). In particular, \([b] = [a_1a_2 \cdots a_m] \leq [a_i]\) for each \(1 \leq i \leq m\).

Suppose that \([b] \leq [a_i]\) for each \(i\). Then the procedure has already considered \([a_i]\), and a subset of \(V\) is a basis for \(\langle S_{a_i}\rangle\), which contains \(\overline{a_i}\). But by our inductive hypothesis, \(V\) is a set of standard basis elements relative to \((G, S)^{ab}\) and since \(a_i \in S\), \(\overline{a_i}\) is also a standard basis element. So the only way that \(\overline{a_i} \in \langle V \rangle\) is if \(\overline{a_i} \in \langle V \rangle\) (and so by injectivity \(g^{-1}a_ig \in V\)). Thus, \(b = g^{-1}a_1a_2 \cdots a_mg \in \langle V \rangle\) and so would not be chosen by the procedure to linearly extend \(V\).

Therefore, there must be some \(i\) such that \([b] = [a_i]\). By reordering the vertices of \(S\) if necessary, \([b] = [a_1]\). But then \(gbg^{-1} = a_1a_2 \cdots a_m\) and \(a_1\) are involutions that commute with exactly the same involutions, and so the following is an involutive automorphism of \((G, S)\):

\[
\varphi : G \to G
\]

\[
\varphi(a_j) = \begin{cases} 
    a_1a_2 \cdots a_m & \text{if } j = 1 \\
    a_j & \text{otherwise}
\end{cases}
\]

Now, \((G, \varphi(S))\) is also a right-angled Coxeter system for \(G\) with the exact same generators except for swapping \(a_1\) and the product
a_1 a_2 \cdots a_m, \overrightarrow{V} is a set of standard basis elements not including \overrightarrow{a_1} and so is unchanged under the induced map \varphi : (G, S)^{ab} \to (G, \varphi(S))^{ab}. Alternatively, \varphi(b) = g^{-1} a_1 g and so \varphi(b) = \overrightarrow{a_1}. So if we let \((G, S') = (G, \varphi(S))\) be our new right-angled Coxeter system and \(V' = V \cup \{b\}\) be our new subset of labels from our chosen full set of representatives of \(\Delta_G\), then the inductive hypothesis is still satisfied. In particular, in our Smith Normal Form \(G^{ab}, \overrightarrow{V'}\) is still linearly independent.

For each \(b \in B\), we can perform this procedure in succession making sure that for each \(b\), we choose different \(a_i\) such that \([b] = [a_i]\). If at any point this were not possible, it would mean that there was some \(b_n = g^{-1} a_1 a_2 \cdots a_m g\) (in the updated system \((G, S')\) with \(V'\)) such that each \(a_j\) either satisfies:

1. \([b_n] \leq [a_j]\) in which case \(\overrightarrow{a_j} \in \overrightarrow{V'}\) from a previous step in the procedure, or
2. \(b_l = \overrightarrow{a_j}\) for some \(l < n\) in which case \(\overrightarrow{a_j} \in \overrightarrow{V'}\) from a previous element of the basis.

In either case, since all of the \(\overrightarrow{a_j} \in \overrightarrow{S_w}\), this would give a linear dependence in \(\overrightarrow{S_w}\) among \(\overrightarrow{B}\), which contradicts its choice as a basis.

Thus, by induction on both elements of the poset, and then within each class on the elements of each chosen basis, it will always be the case that \(\overrightarrow{V}\) will consist of elementary basis elements in \((G, S)^{ab}\) for some choice of system \((G, S)\). Since every generator \(a_i\) of \(S\) is in \(S_{a_i}\), \(\overrightarrow{a_i} \in \langle \overrightarrow{V} \cap S_{a_i} \rangle\), but since \(\overrightarrow{V}\) are all elementary basis vectors, it must be that \(\overrightarrow{a_i} \in \overrightarrow{V}\). Thus, at the end of the procedure, \(\overrightarrow{V}\) will always be the full standard basis for some system \((G, S)^{ab}\), and in particular, \(\overrightarrow{V}\) will always be a basis of \(G^{ab}\).

Any non-trivial product relation among the elements of \(V\) would induce a linear dependence among their images in \(G^{ab}\). But since \(\overrightarrow{V}\) is a basis, this can never happen. \(\square\)

There is another way in which we can use the algebra of right-angled Coxeter groups to improve our combinatorial procedure. In general, for a given group \(G\), constructing the involution graph \(\Delta_G\) may not be computable in general. It also may fail to be computable in different ways: given a presentation for a group \(G\), it may be impossible to:

(a) Identify all the involutions of \(G\),
(b) Determine the conjugacy classes amongst these involutions,
(c) Determine whether or not these conjugacy classes should be connected by an edge in \(\Delta_G\), i.e., whether or not there exist commuting representatives of these classes,
(d) Determine a full set of representatives for $\Delta_G$, i.e., a set of representatives for these conjugacy classes that simultaneously exhibits all of the above commuting relations.

It is a priori possible that for each of these steps, there exists a finitely generated presentation for a group $G$ for which that step is uncomputable but all previous steps are computable. In other words, our procedure may fail to be a complete algorithm at any of these steps. For a right-angled Coxeter system, all of these steps are computable, and this procedure can be used whenever these steps are computable for particular examples.

The algebra of right-angled Coxeter groups can significantly simplify step (c) using the following fact.

**Proposition 3.7.** If $G$ is a right-angled Coxeter group, then two conjugacy classes of involutions $[x]$ and $[y]$ should be connected by an edge in $\Delta_G$ if and only if there exists another conjugacy class of involutions $[z]$ such that $z = x^{-1}y$ in $G^{ab}$.

**Proof.** Let $(G, S)$ be a right-angled Coxeter system and $[x]$ and $[y]$ be conjugacy classes of involutions of $G$. Since $x$ and $y$ are involutions in a right-angled Coxeter group, they are each conjugate to a product of distinct, commuting generators, i.e., there exists $a_1, a_2, \ldots, a_n \in S$, $b_1, b_2, \ldots, b_m \in S$, $g, h \in G$ such that $gxg^{-1} = a_1a_2\cdots a_n$, $hyh^{-1} = b_1b_2\cdots b_m$ where all of the $a_i$ pairwise commute as do the $b_j$. The product $w := a_1a_2\cdots a_nb_1b_2\cdots b_m = c_1c_2\cdots c_k$, where the $c_i$ are the generators that appear among either the $a_i$ or the $b_j$ but not both. The ones that appear in both cancel with each other since they can be brought to the front or back of their respective words. In the abelianization $G^{ab}$, $x = a_1a_2\cdots a_n$, $y = b_1b_2\cdots b_m$, and $w = c_1c_2\cdots c_k$.

Now suppose that $[x]$ and $[y]$ are connected by an edge in $\Delta_G$. That means that some conjugates of $x$ and $y$ commute. This implies that the product of those conjugates, $z$, is an involution. But then in $G^{ab}$, $z = x^{-1}y$.

Conversely, suppose that there exists an involution, $z$ such that $z = x^{-1}y$. Since $z$ is an involution, it must be conjugate to a product of distinct, commuting generators, each of which is mapped to its corresponding generator of $G^{ab}$ and so can be recovered directly from $z$. Thus, these generators must be exactly the $c_i$, and so they each pairwise commute. In particular, $w$ is an involution, and $gxg^{-1}$ and $hyh^{-1}$ commute. Thus, $[x]$ and $[y]$ should be connected by an edge in $\Delta_G$.

\[ \square \]
Thus, if we assume that \( G \) is a right-angled Coxeter group and use our procedure to derive a contradiction in the case that it isn’t, we don’t actually have to calculate the true edge relations in \( \Delta_G \), but instead just calculate the vertices and use the edge relations given by Proposition 3.7. Since these are determined by simple calculations in the abelianization, this is always computable and much easier. Furthermore if \( G \) actually is a right-angled Coxeter group, these will give us the correct edge relations, and so the procedure will still be as effective at finding a right-angled Coxeter system for \( G \).

With this modification, our entire procedure becomes as follows:

We are given a presentation for a finitely-generated group, \( G \). We wish to determine if it is a right-angled Coxeter group, and if so, to compute an isomorphism to a right-angled Coxeter presentation.

(0) Compute an effective quotient of \( G \) to the Smith normal form of \( G^{ab} \). If \( G^{ab} \neq (\mathbb{Z}/2\mathbb{Z})^n \) for some \( n \in \mathbb{Z}^+ \), then return false.

(1) Construct the involution graph \( \Delta_G \) (or the version of \( \Delta_G \) with adjacency determined by Proposition 3.7).

(2) Find a full system of representatives for \( \Delta_G \) to use as labels.

(3) If \( \Delta_G \) does not satisfy the minimal vertex condition, return false.

(4) If \( \Delta_G \) does not satisfy the inclusion-exclusion condition, return false.

(5) Initialize \( V = \{\} \).

(6) Let \([w]\) be a class which has not yet been considered such that all \([v]\) with \([w] \leq [v]\) have been considered. Since \( \Delta_G \) satisfies the minimal vertex condition, \( S_w \) is the intersection of maximal cliques. Since \( \Delta_G \) satisfies the inclusion-exclusion condition (and therefore the maximal clique condition), \(|S_w| = 2^k - 1 \) for some \( k \). Let \( k' = |\{v \in V \mid [w] \leq [v]\}| = |V \cap S_w| \). Then \( k' \leq k \). Choose \( k - k' \) elements from \([w]\) to extend \( V \cap S_w \) to a basis of \( S_w \) as described in Proposition 3.6 and add them to \( V \).

(7) Repeat the previous step until all equivalence classes have been considered.

(8) Return \( \Gamma \) given by the induced subgraph of \( \Delta_G \) on the vertex set \( V \).

(9) Let \( \Gamma' \) be a graph isomorphic to \( \Gamma \) whose vertices are labeled generically, e.g., \( a_1, \ldots, a_n \). Define a map \( \varphi : W_{\Gamma'} \to G \) which takes \( a_i \) to the label of the corresponding vertex in \( \Gamma \). Check if \( \varphi \) is an isomorphism.

If this procedure returns false at any time before step 9, then \( G \) cannot be a right-angled Coxeter group. If the procedure returns a
graph $\Gamma$ before step 9, then if $G$ is a right-angled Coxeter group, $\Gamma$ must be its defining graph. If the procedure returns an isomorphism $\varphi$, then $G$ is a right-angled Coxeter group and $\varphi$ determines its right-angled Coxeter system. Finally, if $\Gamma$ is returned but $\varphi$ is not an isomorphism, then either $G \cong W_\Gamma$ through some other morphism, or else $G$ is not a right-angled Coxeter group, but the procedure cannot tell which. A different full system of representatives for $\Delta_G$ must be chosen.

3.2. An Example RACG. In this section, we’ll see an explicit example in detail to demonstrate the method. Consider the following defining graph:

![Figure 2. The defining graph $\Gamma$.](image)

Write $x = \chi_{1,\{2\}}$ for the partial conjugation with acting letter $a_1$ and domain $\{a_2\}$. We consider the group $G = W_\Gamma \rtimes \langle x \rangle$, which has the following presentation:

$$G = \langle a_1, a_2, a_3, a_4, x \mid a_i^2 = x^2 = 1, [a_1, a_4] = [a_2, a_4] = [a_3, a_4] = 1,$$

$$[a_1, x] = [a_3, x] = [a_4, x] = 1, xa_2x = a_1a_2a_1 \rangle$$

This is not quite a right-angled Coxeter presentation, so we apply our procedure to see if we can find one.

First, we compute $G^{ab}$ (removing any relations that become trivial and understanding that group presentations with additive notation are assumed to be abelian):

$$G^{ab} = \langle \overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{x} \mid 2\overline{a_i} = 2\overline{x} = 0 \rangle$$

$$\cong \langle \overline{a_1} \rangle \times \langle \overline{a_2} \rangle \times \langle \overline{a_3} \rangle \times \langle \overline{a_4} \rangle \times \langle \overline{x} \rangle$$

$$\cong (\mathbb{Z}/2\mathbb{Z})^5$$
The relation matrix
\[
\begin{pmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}
\]
is already in Smith normal form, and so our canonical abelianization map is just \( G \to G^{ab} : g \mapsto \overline{g} \).

We now want to list all conjugacy classes of involutions in \( G \). The classes of involutions in \( W_\Gamma \) are straightforward to read off of \( \Gamma \): \( a_i \) for each \( i \), and \( a_j a_4 \) for each \( 1 \leq j \leq 3 \). The new generator \( x \) is also an involution, and the products of \( x \) with the other generators that commute with it give new involutions: \( xa_1, xa_3, xa_4 \). There are two remaining conjugacy classes of involutions, namely \( xa_1 a_2 \) and \( xa_1 a_2 a_4 \).

These are all of the conjugacy classes of involutions in \( G \). We could go through similar details as in Section 3.3 to prove this, but it will also end up following from the fact that our procedure in this case does in fact construct an explicit isomorphism with a right-angled Coxeter group. Thus, we can omit the details.

We claim that the following is the involution graph \( \Delta_G \). The given system of representatives is a full system, and the commuting relations are straightforward to check. (If they weren’t, we could easily construct the edge relations given by Proposition 3.7, which in this case are the same.)

The brackets in the involution graph represent conjugacy classes. Since we now have a full system of representatives, we may stop writing these brackets. For the remainder of the calculation, brackets around a vertex label will denote its star equivalence class. Before calculating the star poset structure, we observe that this graph clearly satisfies the maximal clique condition and the minimal vertex condition, and the inclusion-exclusion condition is straightforward to verify.

The equivalence classes in the star poset are the following (identified by the dashed ellipses in the figure):

\[
\begin{align*}
[a_1] &= \{a_1, a_1 a_4\} \\
[a_2] &= \{a_2, a_2 a_4, xa_1 a_2, xa_1 a_2 a_4\} \\
[a_3] &= \{a_3, a_3 a_4, xa_3, xa_3 a_4\} \\
[a_4] &= \{a_4\} \\
x &= \{x, xa_4\} \\
[xa_1] &= \{xa_1, xa_1 a_4\}
\end{align*}
\]

The Hasse diagram for this poset is as follows:

The element \([a_4]\) is maximal in the poset structure and contains a single element. We add \( a_4 \) to \( V \). Next, we consider \([x]\) (or \([xa_1]\), the order in which we consider these classes is irrelevant). The clique
above $[x]$ has size 3, so 2 of its vertices must be added to $V$. We have already added 1, so we must pick one more from $[x]$. Examining the abelianization, $\langle \overline{a_4}, x \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$ and either of $x$ or $xa_1$ will extend $\overline{a_4}$ into a basis. So we choose to add $x$ to $V$. Similarly, we consider $[xa_1]$ and add $xa_1$ to $V$.

The remaining three classes are all minimal. Suppose we take $[a_2]$ next. The clique above $[a_2]$ has size 7, so we must choose 3 elements from it. We have already chosen 2, so we need to choose 1 more. Checking the abelianization again, we see that any choice of the 4 elements in $[a_2]$ will extend to a basis, and so we add $a_2$ to $V$. Similarly, from $[a_3]$, we add $a_3$ to $V$.

Finally, we consider $[a_1]$. The clique above $[a_1]$ has size 7, and we have already chosen 3 of these vertices, so we choose no more. This
leaves us with \( V = \{ a_2, a_3, a_4, x, xa_1 \} \). We take the induced subgraph \( \Lambda \) of \( \Delta_G \) on these vertices (Figure 5).

![Figure 5](image)

**Figure 5.** On the left is the collapsed graph \( \Lambda \). On the right is an isomorphic graph with generic labels.

We now have a candidate map \( \phi: W_\Lambda \to G \). It is straightforward to check that the map \( \psi \) below is the inverse, and that \( \phi \) and \( \psi \) are isomorphisms:

\[
\begin{align*}
\phi: & b_1 \mapsto a_3 & & \psi: & a_1 \mapsto b_2b_4 \\
& b_2 \mapsto x & & & a_2 \mapsto b_5 \\
& b_3 \mapsto a_4 & & & a_3 \mapsto b_1 \\
& b_4 \mapsto xa_1 & & & a_4 \mapsto b_3 \\
& b_5 \mapsto a_2 & & & x \mapsto b_2
\end{align*}
\]

Thus, \( G \) is a right-angled Coxeter group.

### 3.3. A Group which is not a RACG

Let \( G \) denote the group presented as follows:

\[
\langle a, b, c, x, y \mid a^2, b^2, c^2, x^2, y^2, \\
xax = a, xbx = b, xcex = aca, \\
yay = a, yby = b, ycy = bcb \rangle.
\]

Let \( W = \langle a, b, c \rangle \) and \( H = \langle x, y \rangle \). Then \( W = \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}, \)

\[ H \cong \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z} \] and \( G \cong W \rtimes H \).

**Proposition 3.8.** The involution graph \( \Delta_G \) is not a clique graph, hence \( G \) is not a right-angled Coxeter group.

**Proof.** To construct \( \Delta_G \), we must understand the involutions in \( G \). Since \( G = W \rtimes H \), each \( g \in G \) may be written uniquely in the form \( g = wh \), where \( w \in W \) and \( h \in H \). Further, \( g^2 = whw(h^{-1}h)h = wwh^{-1}h^2 \). Since every element in \( G \) can be uniquely written as a product of an element of \( G \) and an element of \( H \), if \( g \) is an involution, then \( h \) is an involution and \( w^{h^{-1}} = w^h = w^{-1} \). \( H \) is a right-angled Coxeter group (in fact, \( D_\infty \)), and so every non-trivial
involution in $H$ is conjugate to either $x$ or $y$; it follows that, up to conjugation, we may suppose $g$ has one of the following forms:

1. $w$ such that $w^2 = 1$,
2. $wx$ such that $w^x = w^{-1}$, or
3. $wy$ such that $w^y = w^{-1}$.

Every element of the first type is conjugate to either $a$, $b$, or $c$. Now we'll try to list elements of the second type (elements of the third type will be analogous).

Suppose $g = wx$ with $w^x = w^{-1}$. We further suppose that, within the collection of words of this form in the conjugacy class of $g$, we choose the shortest possible $w$. The element $w$ can be written uniquely in the form $u_0bu_1b\cdots u_{m-1}bu_m$ where $m \geq 0$, each $u_i$ is a geodesic word in \{a, c\}$^*$, and only $u_0$ and $u_m$ may be trivial. Then $w^x = u_0^xbu_1^xb\cdots u_{m-1}^xbu_m^x = w^{-1}$ implies that $u_0^x = u_1^{-1}$, $u_1^x = u_1^{-1}$, and so on. We now consider a few subcases.

If $m > 0$ and $u_0$ is not trivial, then

$$u_0^{-1}(wx)u_0 = u_0^{-1}(u_0bu_1b\cdots u_{m-1}bu_mx)u_0 = bu_1b\cdots u_{m-1}bu_mu_0x = bu_1b\cdots u_{m-1}bx.$$  

This contradicts the minimality of the length of $w$, so either $m = 0$ or $u_0$ is trivial. If $u_0$ is trivial and $m > 1$, then $w$ begins and ends with $b$, in which case $|b(wx)b| < |wx|$. Again, this contradicts minimality, hence either $m = 0$ or $w = b$.

If $m = 0$, then $w = u_0 \in \langle a, c \rangle$ is geodesic and so is an alternating string of $a$ and $c$. If $|w| > 1$ and $|w|$ is odd, then $w$ begins and ends with the same letter. If $w$ begins and ends with $a$, then $|awxa| = |awx| < |wx|$; if $w$ begins and ends with $c$ then $w^x$ begins and ends with $a$, hence $w^x \neq w^{-1}$. In either case, we have a contradiction, so $|w| = 1$, in which case $w = a$ or $w = c$, or else $|w|$ is even. If $w = (ac)^n$ and $n > 1$, then $|aca(wx)aca| < |wx|$; if $w = (ca)^n$ and $n > 1$, then $|cwxc| < |wx|$. In both cases, we have a contradiction. Our only case left is $m = 0, n = 1$, which corresponds to $w = ac$ or $w = ca$. Therefore, our only non-trivial possibilities for $w$ are $w = b, a, c, ac, ca$.

Note that $a(cax)a = acx$, so these cases fall into the same conjugacy classes. In summary, we have that each involution of the form $wx$ is conjugate to exactly one of the elements $x, ax, bx, acx$. (We observe that the final option $cx$ is not, in fact, an involution. In this case, $w = c$, and $w^x \neq w^{-1}$.) We also observe that none of these involutions are conjugate to each other since they all map to distinct elements in $G^{ab}$. 
Similarly, each involution of the form \( wy \) is conjugate to exactly one of the elements \( y, ay, by, bcy \). Therefore the following is the complete list of conjugacy classes in \( G \), and hence serves as the list of vertex labels in \( \Delta_G \):

\[
[a], [b], [c], [x], [ax], [bx], [acx], [y], [ay], [by], [bcy].
\]

We now consider pairs of distinct conjugacy classes, to see whether or not they should be adjacent in \( \Delta_G \). By Proposition 3.7, we can just check the product relations among the images of the involutions in \( G^{ab} \). We omit the actual calculation and show the resulting involution graph in Figure 6.

![Figure 6](image_url)

**Figure 6.** An involution graph which cannot be a clique graph. The labeled triangles \( \Gamma_i \) are the maximal cliques.

Now \( \Delta_G \) is not a clique graph, since, for example, the inclusion-exclusion condition fails. (The reader can check this directly for the maximal cliques labeled \( \Gamma_3 \) and \( \Gamma_4 \) in the figure.)

3.4. Extensions of Right-angled Coxeter Groups. The main result of this section, Lemma 3.9, was obtained by using the method described in Section 3.1. Below, we have omitted the calculations of involutions graphs and given a more concise proof which works in the full generality of the claim.

Let \( \Gamma \) be a finite graph. If \( D \) is a union of connected components of \( \Gamma \setminus \text{St}(a_i) \) for some \( i \), then the automorphism of \( W_\Gamma \) given by

\[
\chi_{i,D}(a_j) = \begin{cases} 
a_i a_j a_i & a_j \in D \\a_j & \text{otherwise} \end{cases}
\]
is called the partial conjugation with acting letter $a_i$ and domain $D$. Note that this terminology is not entirely consistent in the literature. Other papers have reserved partial conjugation for the case in which $D$ is a single connected component [GPR12, CRSV10], while Laurence has use the term locally inner automorphism [Lau95] before the term partial conjugation became common. We have preferred here to allow for multiple connected components in the domain of a partial conjugation, and we would propose the term elementary partial conjugation for the case in which $D$ consists of a single connected component.

Lemma 3.9. Suppose $W_\Gamma$ is a right-angled Coxeter group. If $\alpha_1, \ldots, \alpha_k$ are partial conjugations of $W$ with the same acting letter and pairwise disjoint domains, then $G = W \rtimes \langle \alpha_1, \ldots, \alpha_k \rangle$ is a right-angled Coxeter group.

Proof. Without loss of generality, we may assume each $\alpha_j$ has acting letter $a_1$. Let $D_i$ denote the domain of $\alpha_i$ for each $1 \leq i \leq k$. Now $G$ is generated by the elements $\{a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_k\}$ and presented by the following relations:

(R1) $a_i^2 = 1$ for $1 \leq i \leq n$,
(R2) $[a_i, a_j] = 1$ for $\{a_i, a_j\} \in E(\Gamma)$,
(R3) $\alpha_i^2 = 1$ for $1 \leq i \leq k$,
(R4) $[\alpha_i, \alpha_j] = 1$ for $1 \leq i < j \leq k$
(R5) $[\alpha_i, a_j]$ for $a_j \notin D_i$, and
(R6) $a_i a_j a_i a_j a_1$ for $a_j \in D_i$.

Let $H$ be the group generated by $\{b_1, \ldots, b_n, \beta_1, \ldots, \beta_k\}$ and presented by the relations:

(S1) $b_i^2 = 1$ for $1 \leq i \leq n$,
(S2) $[b_i, b_j] = 1$ for $\{a_i, a_j\} \in E(\Gamma)$,
(S3) $\beta_i^2 = 1$ for $1 \leq i \leq k$,
(S4) $[\beta_i, \beta_j] = 1$ for $1 \leq i < j \leq k$
(S5) $[\beta_i, b_j]$ for $a_j \notin D_i$, and
(S6) $[b_1, b_i]$ for $2 \leq i \leq n$ and $a_i \in D_1 \cup \cdots \cup D_k$. 

We note that the given presentation for $H$ is a right-angled Coxeter presentation. We define maps

$$\hat{\varphi}: \{a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_k\} \to \{b_1, \ldots, b_n, \beta_1, \ldots, \beta_k\}$$

- $a_1 \mapsto b_1 \beta_1 \ldots \beta_k$
- $\alpha_i \mapsto \beta_i \ (1 \leq i \leq k)$
- $a_i \mapsto b_i \ (2 \leq i \leq n)$

$$\hat{\psi}: \{b_1, \ldots, b_n, \beta_1, \ldots, \beta_k\} \to \{a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_k\}$$

- $b_1 \mapsto a_1 \alpha_1 \ldots \alpha_k$
- $\beta_i \mapsto \alpha_i \ (1 \leq i \leq k)$
- $b_i \mapsto a_i \ (2 \leq i \leq n)$

It is straightforward to check that $\hat{\varphi}$ and $\hat{\psi}$ preserve the relations (R1)-(R6) and (S1)-(S6), respectively, so they induce homomorphisms $\varphi: G \to H$ and $\psi: H \to G$. (Note that the preservation of the relation (S6) uses the assumption that the domains $D_i$ are pairwise disjoint.) Finally, it is straightforward to see that $\varphi$ and $\psi$ are inverses to each other, hence $G$ and $H$ are isomorphic. That is, $G$ is a right-angled Coxeter group.

**Corollary 3.10.** We note as a special case of the previous lemma that any extension $W_{\Gamma} \rtimes \langle \chi_i, D \rangle$ by a partial conjugation yields a new right-angled Coxeter group.

**Lemma 3.11.** Suppose $W, \Gamma, a_1, \alpha_1, \ldots, \alpha_k, H$ and $G$ are as in the lemma and proof above. Let $\gamma$ be a partial conjugation of $W$ with acting letter $a_2 \neq a_1$ and such that $\gamma$ commutes with each of the automorphisms $\alpha_1, \ldots, \alpha_k$. Then $\gamma$ acts on $G$ as a partial conjugation.

**Proof.** Without loss of generality we may assume $\gamma$ has acting letter $a_2$ and domain $D$. Recall that $a_2 = b_2$. To show that $\gamma$ acts on $G$ as a partial conjugation we shall consider the result of conjugation by $\gamma$ on each of the generators $b_1, \ldots, b_n, \beta_1, \ldots, \beta_k$. Firstly we note: $\gamma b_i \gamma = \beta_i$ for $1 \leq i \leq k$; $\gamma b_i \gamma = b_i$ for $1 \leq i \leq n$ and $a_i \notin D$; $\gamma b_i \gamma = b_2 b_i b_2$ for $2 \leq i \leq n$ and $a_i \in D$. If $a_1 \notin D$, then $\gamma b_1 \gamma = \gamma a_1 \gamma = b_1$. Suppose $a_1 \in D$. Since $\gamma$ commutes pairwise with $\alpha_1, \ldots, \alpha_k$, we have that
We compute
\[ \gamma b_1 \gamma = a_1 \gamma a_1 \cdots \alpha_k \gamma = a_2 a_1 a_2 \gamma a_1 \cdots \alpha_k = a_2 a_1 \alpha_1 \alpha_2 \cdots \alpha_k a_2 = b_2 b_1 b_2. \]

Since \( \gamma \) is an automorphism of \( G \), and \( \gamma \) takes each generator to either itself, or the conjugate of itself by \( b_2 \), \( \gamma \) is a partial conjugation of \( G \).

Write \( \varphi : \langle a_1, \ldots, a_n \rangle \to \langle b_1, \ldots, b_n \rangle \) for the map \( \varphi(a_i) = b_i \). From the calculations above, the domain of \( \gamma \) acting on \( G \) is \( \varphi(D) \). \( \square \)

**Corollary 3.12.** Suppose \( W_\Gamma \) is a right-angled Coxeter group and \( \chi_1, \ldots, \chi_k \) are pairwise commuting partial conjugations such that whenever \( \chi_i \) and \( \chi_j \) have the same acting letter, their domains don’t intersect. Then \( G = W \rtimes \langle \chi_1, \ldots, \chi_k \rangle \) is a right-angled Coxeter group. Further, writing \( S_i \subset \{ \chi_1, \ldots, \chi_k \} \) for the set comprising those partial conjugations with acting letter \( a_i \), we have that
\[
\left\{ a_1 \prod_{\chi_i \in S_i} \chi_i, \ldots, a_n \prod_{\chi_i \in S_k} \chi_i \right\} \cup \{ \chi_1, \ldots, \chi_k \}
\]
is a Coxeter generating set for \( G \).

**Proof.** The proof is by induction, applying the lemmas above at each step. Let \( \alpha_1, \ldots, \alpha_k \) be those \( \chi_i \) with acting letter 1. By assumption, they have pairwise disjoint domains. By Lemma 3.9, \( W_\Gamma \rtimes \langle \alpha_1, \ldots, \alpha_k \rangle \) is a RACG.

Moreover, by Lemma 3.11, the remaining \( \chi_i \)'s still act like partial conjugations, and their domains do not intersect, since they didn’t before the extension. Now take \( \beta_1, \ldots, \beta_k \) among the remaining \( \chi_i \) to be those which have acting letter 2, and extend by \( \langle \beta_1, \ldots, \beta_k \rangle \).

Continuing inductively, we extend at the \( i \)th step by all remaining partial conjugations with acting letter \( i \). The result follows. \( \square \)

In [GPR12], the authors investigate the automorphism groups of graph products of cyclic groups. In the case that \( W \) is a right-angled Coxeter group, the authors recover a result from [Tit88] which shows \( \text{Aut}(W) = \text{Aut}^0(W) \rtimes \text{Aut}^1(W) \) with \( \text{Aut}^1(W) \) finite. Thus \( \text{Aut}^0(W) \) (sometimes denoted \( \text{Aut}^{PC}(W) \)) which is the subgroup of \( \text{Aut}(W) \) generated by all partial conjugations of \( W \), is a finite index subgroup of \( \text{Aut}(W) \). Furthermore, they show that \( \text{Aut}^0(W) \) splits as \( \text{Inn}(W) \rtimes \text{Out}^0(W) \). Finally, they give a condition on \( \Gamma \) called \( \text{no SILs} \) which
characterizes exactly when $\text{Out}^0(W)$ is finite and is thus isomorphic to $\mathbb{Z}_2^n$. In this case, $\text{Aut}^0(W)$ is a finite extension of $\text{Inn}(W)$ which is known to be a right-angled Coxeter group. In [CRSV10], it is shown that $\text{Aut}^0(W)$ is again a right-angled Coxeter group in that case. We arrive at this same result as a direct application of the previous corollary.

**Corollary 3.13.** If $\Gamma$ contains no SILs, then $\text{Aut}^0(W)$ is a right-angled Coxeter group and thus $\text{Aut}(W)$ contains a right-angled Coxeter group as a subgroup of finite index.

**Proof.** Without loss of generality we may assume $W$ has trivial center. Suppose $\Gamma$ contains no SILs. Then

$$\text{Aut}^0(W) = \text{Inn}(W) \rtimes \text{Out}^0(W) \cong W \rtimes \text{Out}^0(W),$$

and $\text{Out}^0(W)$ is generated by pairwise commuting partial conjugations which satisfy the condition in the corollary above. $\square$

**Remark 3.14.** In general, one should not expect $\text{Aut}(W)$ to be right-angled Coxeter. The elements of $\text{Aut}^1(W)$ include graph symmetries, which could then introduce torsion elements of order other than 2.
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