Gravitomagnetic time delay and the Lense-Thirring effect in Brans-Dicke theory of gravity

A. Barros\textsuperscript{a} and C. Romero\textsuperscript{b}

\textsuperscript{a}Departamento de Física, Universidade Federal de Roraima, 69310-270, Boa Vista, RR, Brazil

\textsuperscript{b}Departamento de Física, Universidade Federal da Paraíba, João Pessoa, PB, Brazil

e-mail: cromero@fisica.ufpb.br
Abstract

We discuss the gravitomagnetic time delay and the Lense-Thirring effect in the context of Brans-Dicke theory of gravity. We compare the theoretical results obtained with those predicted by general relativity. We show that within the accuracy of experiments designed to measure these effects both theories predict essentially the same result.

I. INTRODUCTION

The idea that mass currents generate a field called, by analogy with electromagnetism, the gravitomagnetic field, is no doubt a very appealing one [1]. As is well known, according to general relativity, moving or rotating matter should produce a contribution to the gravitational field that is the analogue of the magnetic field of a moving charge or magnetic dipole. A clear manifestation of this field may be found, for instance, in the Lense-Thirring precession [2], an effect predicted soon after Einstein formulated general relativity. The gravitomagnetic field also contributes to the gravitational time delay [3], among other effects.

The real possibility that gravitomagnetic effects can be measured with the current technology of laser ranged satellites (LAGEOS and LAGEOS II) has aroused great interest in the subject [4]. It should be mentioned that the Relativity Gyroscope Experiment (Gravity Probe B) at Stanford University, in collaboration with NASA and Lockheed-Martin Corporation, has a program of developing a space mission to detect gravitomagnetism effects directly. Certainly, these experimental programs will open new possibilities of testing general relativity against other metric theories of gravity [5,6], in particular the scalar-tensor theory, one of the most popular alternatives to Einstein theory of gravitation. Our aim in this paper is to investigate two gravitomagnetic effects: the time delay and the Lense-Thirring precession in the context of Brans-Dicke theory, and then compare the results with those predicted by general relativity.
To define the gravitomagnetic field in general relativity one assumes the weak field and slow motion approximation. On the other hand, it has been shown that in this approximation solutions of Brans-Dicke equations are simply related to the solutions of general relativity equations for the same matter distribution [7]. By using this fact one can easily establish a straightforward correspondence between weak field effects in both theories, in particular gravitomagnetic effects.

This paper is organized as follows. In Section II, we give a brief introduction to the basic ideas of gravitomagnetism. Then, in Section III, we show how general relativity and Brans-Dicke theory of gravity are related in the weak field approximation. The gravitomagnetic field in Brans-Dicke theory is defined in Section IV. We consider the Lense-Thirring effect and the gravitomagnetic time delay in Brans-Dicke theory in Sections V and VI, respectively. Section VII is devoted to some remarks.

II. THE GRAVITOMAGNETIC FIELD IN GENERAL RELATIVITY

Let us recall that in the weak field approximation of general relativity we assume that the metric tensor $g_{\mu\nu}$ deviates only slightly from the flat spacetime metric tensor. In other words, we assume that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ denotes Minkowski metric tensor and $h_{\mu\nu}$ is a small perturbation term. Then, by keeping only first-order terms in $h_{\mu\nu}$, the Einstein equations become

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \tag{1}$$

where $h'_{\nu} = h_{\nu} - \frac{1}{2}\delta_{\nu} h$ and we are adopting the usual harmonic coordinate gauge $(h_{\nu} - \frac{1}{2}\delta_{\nu} h)_{,\mu} = 0$.

If we assume a non-relativistic matter distribution with a mass density $\rho$ and velocity field $\vec{v}$, then (1) yields

$$\Box h_{00} = -\frac{16\pi G}{c^2} \rho \tag{2}$$
\[ \Box h_{0i} = \frac{16\pi G}{c^3} \rho v_i \]  

(3)

where \( v_i \) denotes the velocity components, and terms such as \( p \) and \( v_i v_j / c^4 \) have been neglected. Let us now specialise to the case of a stationary gravitational field of a slowly rotating body. Then (2) and (3) reduce to

\[ \nabla^2 \left( \frac{c^2 h_{00}}{4} \right) \equiv \nabla^2 (\Phi_g) = -4\pi G \rho \]  

(4)

\[ \nabla^2 h_{0i} = \frac{16\pi G}{c^3} \rho v_i \]  

(5)

where \( \Phi_g \) is the gravitoelectric scalar potential (the gravitational counterpart of the electromagnetic scalar potential). Far from the source we will have

\[ \Phi_g = \frac{GM}{r} \]  

(6)

\[ \vec{\mathbf{h}} = -\frac{2G(\vec{\mathbf{J}} \times \vec{\mathbf{r}})}{c^3 r^3} \equiv -\frac{2\vec{\mathbf{A}}_g}{c^2} \]  

(7)

where \( \vec{\mathbf{A}}_g \) is the gravitomagnetic vector potential vector (the gravitational counterpart of the electromagnetic vector potential), \( h_{0i} \) are the components of the vector \( \vec{\mathbf{h}} \), \( M \) and \( \vec{\mathbf{J}} \) are the total mass and angular momentum of the source. In close analogy with electrodynamics we define the gravitoelectric field to be \( \vec{\mathbf{E}}_g = -\nabla \Phi_g \) and the gravitomagnetic field to be

\[ \vec{\mathbf{B}}_g = \nabla \times \vec{\mathbf{A}}_g = \frac{G}{c} \left[ \frac{3\hat{\mathbf{r}} (\hat{\mathbf{r}} \cdot \vec{\mathbf{J}}) - \vec{\mathbf{J}}}{r^3} \right] \]  

(8)

It is interesting to see that the condition \( h_{\mu\nu} h^{\mu\nu} = 0 \) leads to \( \nabla \cdot \vec{\mathbf{A}}_g = 0 \) (analogous to the Coulomb gauge of electromagnetism).

Let us note that for the case of a slowly rotating sphere with angular momentum \( \vec{\mathbf{J}} = (0, 0, J) \), we obtain from (7) in spherical coordinates

\[ h_{0\varphi} = \frac{-2GJ}{c^3 r} \sin^2 \theta \]  

(9)

which is the \( g_{0\varphi} \) component of the Kerr metric in Boyer-Lindquist coordinates in the weak field and slow motion limit [8].
III. THE WEAK FIELD APPROXIMATION OF BRANS-DICKE THEORY

In Brans-Dicke theory of gravity the field equations are given by [9]
\[
G_{\mu\nu} = \frac{8\pi}{c^4} T_{\mu\nu} + \frac{\omega}{g^2} \left( \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\alpha} \phi^{,\alpha} \right) + \frac{1}{\phi} \left( \phi_{,\mu;\nu} - g_{\mu\nu} \Box \phi \right)
\] (10)

Paralleling general relativity one can linearize Brans-Dicke field equations by assuming that the metric \( g_{\mu\nu} \) and the scalar field \( \phi \) can be written as \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) and \( \phi = \phi_0 + \epsilon \), where \( \phi_0 \) is a constant and \( \epsilon = \epsilon(x) \) is a first-order term (it is assumed that both \( |h_{\mu\nu}| \) and \( |\epsilon \phi_0^{-1}| \) are \( \ll 1 \)). From these assumptions it follows that
\[
\Box h_{\mu\nu} = -\frac{16\pi}{c^4 \phi_0} \left[ T_{\mu\nu} - \frac{\omega + 1}{2\omega + 3} \eta_{\mu\nu} T \right]
\] (11)
where we have used the Brans-Dicke gauge [9] \( (h_{\nu}^{\mu} - \frac{1}{2} \delta_{\nu}^{\mu} h) \, ,_{\mu} = \epsilon,_{\nu} \phi_0^{-1} \).

It has been shown that the problem of finding solutions of Brans-Dicke equations of gravity in the weak field approximation may be reduced to solving the linearized Einstein field equations for the same energy-momentum tensor [7]. Indeed, if \( g^*_{\mu\nu}(G, x) \) is a known solution of the Einstein equations in the weak field approximation for a given \( T_{\mu\nu} \), then the Brans-Dicke solution corresponding to the same \( T_{\mu\nu} \) will be given in the weak field approximation by
\[
g_{\mu\nu}(x) = \left[ 1 - \epsilon G_0 \right] g^*_{\mu\nu}(G_0, x)
\] (12)
where \( G \) is the gravitational constant and \( G_0 = \phi_0^{-1} = \left( \frac{3\omega+3}{2\omega+1} \right) G \) and the function \( \epsilon(x) \) is a solution of the scalar field equation
\[
\Box \epsilon = \frac{8\pi T}{c^4(2\omega + 3)}
\] (13a)
\( T \) denoting the trace of \( T_{\mu\nu} \).

IV. THE GRAVITOMAGNETIC FIELD IN BRANS-DICKE THEORY

Let us now proceed to the definition of the gravitomagnetic field in Brans-Dicke theory. We start by defining \( \bar{T}_{\mu\nu} \) as
\[ \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h - \epsilon G_0 \eta_{\mu\nu} \]  

(14)

It is easily seen from (11) that

\[ \Box \bar{h}_{\mu\nu} = -\frac{16\pi G_0}{c^4} T_{\mu\nu} \]  

(15)

Thus, in close analogy to the general relativity case, if we again restrict ourselves to slow motion and stationary sources, we immediately arrive at the following equations, which are supposed to hold far from the source

\[ \bar{h}_{00} \equiv \frac{4\Phi_{BD}}{c^2} = \frac{4G_0M}{c^2r} \]

\[ \bar{h} = -\frac{2G_0}{c^3r^3} \left( \frac{\vec{r} \times \vec{J}}{r^3} \right) \equiv -\frac{2\vec{A}^{BD}_g}{c^2} \]

Therefore, in the context of Brans-Dicke theory we define the gravitoelectric field to be

\[ \vec{E}^{BD}_g = -\nabla \Phi_{BD} \]  

and the gravitomagnetic field to be

\[ \vec{B}^{BD}_g = \nabla \times \vec{A}^{BD}_g = \frac{G_0}{c} \left[ 3\vec{r} \cdot \vec{J} - \vec{J} \right] \]  

(16)

Another way of deriving the results above is to start with the Kerr metric in the weak field and slow motion approximation given in isotropic coordinates

\[ ds^2 = -\left( 1 - \frac{2GM}{c^2r} \right) c^2 dt^2 + \left( 1 + \frac{2GM}{c^2r} \right) \left( dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right) - \frac{4GJ}{c^3r} \sin^2 \theta d\varphi cdt \]

Following the prescription given by the theorem discussed in Section III the corresponding solution in Brans-Dicke theory in the same approximation will be given by

\[ ds^2 = [1 - \epsilon G_0] \left[ -\left( 1 - \frac{2G_0M}{c^2r} \right) c^2 dt^2 + \left( 1 + \frac{2G_0M}{c^2r} \right) \left( dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right) \right] - \frac{4G_0J}{c^3r} \sin^2 \theta d\varphi cdt \]

with the function \( \epsilon \) satisfying the equation (13a) (recall that in this case there is no contribution from the angular momentum of the rotating body to the trace of the energy-momentum tensor). Therefore, for a slowly rotating sphere with angular momentum \( \vec{J} = (0, 0, J) \) we will have

\[ \bar{h}_{0\varphi} = h_{0\varphi} = -\frac{2G_0J}{c^3r} \sin^2 \theta \]  

(17)
V. THE LENSE-THIRRING EFFECT IN BRANS-DICKE THEORY

As is well known, the Lense-Thirring effect consists in a precession of gyroscopes relative to distant stars, or, equivalently, a dragging of inertial frames, an effect caused by the gravitomagnetic field. Denoting the angular momentum and the angular velocity of the precession by $\vec{S}$ and $\vec{\Omega}$, then the torque acting on the gyroscope predicted by general relativity is given by

$$\vec{\tau} = \frac{1}{2} \vec{S} \times \left( -\frac{2}{c^2} \vec{B}_g \right) = \frac{d\vec{S}}{dt} = \vec{\Omega} \times \vec{S}$$

with

$$\vec{\Omega} = \frac{1}{c^2} \vec{B}_g = G \left( \frac{3\hat{r} \cdot \hat{J}}{c^3 r^3} - \hat{J} \right)$$

Thus in the case of Brans-Dicke theory the equation (19) becomes

$$\vec{\Omega}^{BD} = \frac{1}{c^2} \vec{B}_g^{BD} = G_0 \left( \frac{3\hat{r} \cdot \hat{J}}{c^3 r^3} - \hat{J} \right)$$

To compare the value of $\Omega$ predicted by general relativity with $\Omega^{BD}$, predicted by Brans-Dicke theory, we must ascribe values for $\omega$, the scalar field coupling constant. According to the latest experimental results (VLBI measurements [6]) the current value for $\omega$ is 3500. On the other hand, for a polar orbit at about 650 km altitude the axis of a gyroscope is predicted to undergo a precession rate of 42 milliarcsec per year. The expected accuracy of the experiment under these conditions (Gravity Probe B) is about 0.5 milliarcsec per year. Since $G_0 = \left( \frac{2\omega+3}{2\omega+4} \right) G$ the predicted value of Brans-Dicke theory is

$$\Omega^{BD} = \frac{7003}{7004} \Omega \simeq 41.99 \text{ milliarcsec per year}.$$ 

Therefore we see that within the precision of the experiment one cannot distinguish one theory from another.
VI. THE GRAVITOMAGNETIC TIME DELAY IN BRANS-DICKE THEORY

The time delay of light is considered a classical test of general relativity and its measurement was first proposed by Shapiro [10]. It can be shown that this effect can be separated into two parts: the Shapiro time delay and the gravitomagnetic time delay, the latter due to the gravitomagnetic field. The gravitomagnetic time delay has been investigated recently by Ciufolini et al [3]. Assuming again the weak field and slow motion approximation of general relativity one can show that the gravitational time delay $\Delta$ of a light signal travelling between two points $P_1$ and $P_2$ is given by

$$\Delta = \frac{1}{2c} \int_{P_1}^{P_2} h_{\mu\nu}(x) k^\mu k^\nu dl$$

(21)

where $k^\mu = (1, \hat{k})$, $\hat{k}$ denotes the light propagation unit vector and $dl = |d\vec{r}|$ is the Euclidean length element along the straight line that joins $P_1$ to $P_2$. Now from (4), (7) and (21) it follows that $\Delta = \Delta_{ge} + \Delta_{gm}$, where

$$\Delta_{ge} = \frac{2}{c^3} \int_{P_1}^{P_2} \Phi_g dl$$

(22)

is the Shapiro delay and

$$\Delta_{gm} = -\frac{2}{c^3} \int_{P_1}^{P_2} \vec{A}_g \cdot d\vec{r}$$

(23)

is the gravitomagnetic time delay.

Clearly, the above equations keep exactly the same form when we go from general relativity to Brans-Dicke theory, the only change needed is the substitution $\Phi_g \rightarrow \Phi_g^{BD}$ and $\vec{A}_g \rightarrow \vec{A}_g^{BD}$. Thus we have

$$\Delta_{ge}^{BD} = \left(\frac{2\omega + 3}{2\omega + 4}\right) \Delta_{ge}$$

(24)

$$\Delta_{gm}^{BD} = \left(\frac{2\omega + 3}{2\omega + 4}\right) \Delta_{gm}$$

(25)

At this point two comments are in order. Firstly, it should be noted that analogously to the general relativity approach the gravitomagnetic echo delay vanishes [3]. Secondly, if
the light rays travel along a closed loop around a rotating body (this can be arranged with the help of “mirrors”), then the time delay due to the gravitomagnetic field depends on the direction the rays go around the loop. Similarly to the general relativity case, the total time difference between two opposite-oriented paths is given by

$$\delta t^{BD} = -\frac{4}{c^3} \oint \mathbf{A}_g \cdot d\mathbf{r} = -\frac{4}{c^3} \left( \frac{2\omega + 3}{2\omega + 4} \right) \oint \mathbf{A}_g \cdot d\mathbf{r}$$

VII. FINAL REMARKS

In this article, we have examined two effects associated with the so-called gravitomagnetism, namely, the Lense-Thirring effect and the gravitomagnetic time delay in Brans-Dicke theory of gravity. Following the same line of reasoning employed in this article it can easily be shown that the equations for the gravitational time delay in different images due to gravitational lensing [3] in Brans-Dicke theory may be obtained again from the corresponding equations in general relativity by using the correction factor $\frac{2\omega + 3}{2\omega + 4}$. We believe that subsequent analyses of other weak field effects where the gravitomagnetic field can play a role may benefit of the simple theorem of Section III, which establishes a direct correspondence between general relativity and Brans-Dicke theory of gravity in the weak field approximation.

It is interesting to note that a major motivation that has led to the formulation of Brans-Dicke theory was the quest for a Machian theory of gravity [9]. In this respect, one could say that the prediction of the gravitomagnetism, or the dragging of inertial frames, by both general relativity and Brans-Dicke theory alike make these theories, at least in a “weak sense” (as pointed out by Ciufolini et al [4]), satisfy the Principle of Mach.
REFERENCES

[1] I. Ciufolini, J. A. Wheeler, Gravitation and Inertia, Princeton University Press, Princeton, 1995.

[2] J. Lense, H. Thirring, Phys. Z. 19 (1918) 156 (English translation by B. Mashhoon, F.W. Hehl, D. S. Theiss, Gen. Rel. Grav. 16 (1984) 711)

[3] I. Ciufolini, S. Kopeikin, B. Mashhoon and F. Ricci, Phys. Lett. A 308 (2003) 101

[4] I. Ciufolini, E. Pavlis, F. Chieppa, E. Fernandes-Vieira, J. Pérez-Mercader, Science, 279 (1998) 2100

[5] A. Camacho, Gen. Rel. Grav. 34 (2002) 1403

[6] C. M. Will, Theory and experiment in gravitational physics, Cambridge University Press, Cambridge (1993). C. M. Will, Living Rev. Relativity 4 (2001) 4, www.livingreviews.org/Articles/Volume4/2001-4will

[7] A. Barros, C. Romero, Phys. Lett. A 245 (1998) 31

[8] R. H. Boyer, R. W. Lindquist, J. Math. Phys. 8 (1967) 265

[9] C. Brans, R. H. Dicke, Phys. Rev. 124 (1961) 925

[10] I. I. Shapiro, Phys. Rev. Lett. 13 (1964) 789