Stationary entanglement entropies following an interaction quench in 1D Bose gas

Mario Collura, Márton Kormos and Pasquale Calabrese

Dipartimento di Fisica dell’Università di Pisa and INFN, I-56127 Pisa, Italy
E-mail: mario.collura@df.unipi.it, marton.kormos@df.unipi.it and calabres@df.unipi.it

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Abstract. We analyze the entanglement properties of the asymptotic steady state after a quench from free to hard-core bosons in one dimension. The Rényi and von Neumann entanglement entropies are found to be extensive, and the latter coincides with the thermodynamic entropy of the generalized Gibbs ensemble (GGE). Computing the spectrum of the two-point function, we provide exact analytical results for both the leading extensive parts and the subleading terms for the entropies as well as for the cumulants of the particle-number fluctuations. We also compare the extensive part of the entanglement entropy with the thermodynamic ones, showing that the GGE entropy equals the entanglement one and it is twice the diagonal entropy.

Keywords: correlation functions, integrable quantum field theory, entanglement in extended quantum systems (theory), quantum gases

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1. Introduction

Entanglement is a fundamental characteristic of quantum mechanics and it is the main feature distinguishing the quantum from the classical world. Over the last decade, it has become clear that quantum entanglement provides important information in many-body systems, for example in connection with criticality and topological order, see e.g. [1] for reviews. For example, it has been understood that the amount of entanglement contained in a quantum system is the main limitation to the efficiency of numerical methods based on tensor network states [2, 3], such as the celebrated density matrix renormalization group.

In pure quantum states, von Neumann and Rényi entanglement entropies of the reduced density matrix $\hat{\rho}_A$ of a subsystem $A$ turned out to be very useful measures of entanglement. Rényi entanglement entropies are defined as

$$S_A^{(\alpha)} = \frac{1}{1-\alpha} \ln \text{Tr} \hat{\rho}_A^\alpha.$$  \hspace{1cm} (1)

For $\alpha \to 1$ this definition gives the most commonly used von Neumann entropy

$$S_A = -\text{Tr} \hat{\rho}_A \ln \hat{\rho}_A,$$  \hspace{1cm} (2)

while for $\alpha \to \infty$ it is the logarithm of the largest eigenvalue of $\hat{\rho}_A$, also called single copy entanglement [4]. Furthermore, knowledge of the $S_A^{(\alpha)}$ for different $\alpha$ characterizes...
the full spectrum of non-zero eigenvalues of $\hat{\rho}_A$ [5]. Just to quote two very important results, in the ground state of 1D systems whose continuum limit is conformally invariant, the entanglement entropy grows logarithmically with subsystem size and the pre-factor is proportional to the central charge [6]–[8], while in 2D systems displaying topological order (such as quantum Hall systems), a subleading term is the topological charge of the theory [9].

Another field where entanglement entropy has been playing a crucial role is the out of equilibrium dynamics of quantum systems, such as after a quench of a Hamiltonian parameter in a closed system. Indeed, based on results from conformal field theory [10]–[12] and on analytical [10, 13, 14] or numerical calculations [15]–[23] for specific models, it is known that the entanglement entropy grows linearly with time for a global quench, while at most logarithmically for a local one. As a consequence, a local quench can be effectively simulated with tensor networks up to large times, while for a global quench one can access only relatively short time dynamics.

Furthermore, the extensive behavior of the entanglement entropy for infinite time after a global quantum quench is reminiscent of a thermodynamic entropy. This connection is in fact rather natural: a long time after a quench, the reduced density matrix of a subsystem is commonly accepted to be the one of the mixed state compatible with all the local integrals of motion, i.e. a generalized Gibbs ensemble (GGE) for an integrable system [24]–[38] and a thermal ensemble for a generic system [39]–[46], see also [47] for a review. In this construction, in order to establish the existence of a steady state, one first takes the thermodynamic limit (TDL) for the entire system and, only after this, one can consider a large subsystem [33]. Within this construction it is basically tautological that the thermodynamic entropy (i.e. the von Neumann entropy of the ‘final mixed state’) must be equal to the entanglement one. However, other entropies have been proposed in the literature [48], which can appear more natural for finite systems (and are expected to coincide for non-integrable systems [49]). For these reasons it is worth investigating whether the infinite time entanglement entropy coincides with the thermodynamic entropy in some explicitly calculable cases. Up to now, this problem has been considered in great detail only for the 1D Ising model after a quench of the transverse field [13], [49]–[53], and only marginally in a few other cases [49], [54]–[56] (often in the equivalent formulation of the inverse participation ratio).

In this paper we study another exactly solvable instance, the quench from free to hard-core bosons in the continuum, for which an analytical solution was provided only recently by the present authors [57, 58] (see also [59]). The computation of the entanglement entropy for arbitrary times is highly non-trivial because Wick’s theorem does not apply [57]; however, for infinite time its validity is restored allowing for an exact calculation of the entropies as detailed in the following.

The paper is organized as follows. In section 2 we review some recent results on a special interaction quench in the Lieb–Liniger model, and we give a brief summary of the relevant formulas to compute entanglement entropies and particle fluctuations. We obtain our analytic results in two different ways. In section 3 we present a direct method, while in section 4 we introduce a novel approach based on the spectrum of the two-point function. After comparing the entanglement entropies with the thermodynamic entropy of the GGE in section 5, we give our conclusions in section 6.
2. The model and the quantities of interest

We consider the Lieb–Liniger model, a one-dimensional Bose gas with pairwise delta interaction on a ring of circumference $L$ with periodic boundary conditions (PBCs), i.e. with Hamiltonian 
\[
H = \int_0^L dx \left[ \partial_x \hat{\phi}^\dagger(x) \partial_x \hat{\phi}(x) + c \hat{\phi}^\dagger(x) \hat{\phi}(x) \hat{\phi}^\dagger(x) \hat{\phi}(x) \right],
\]
where $\hat{\phi}(x)$ is a canonical boson field, $c$ the coupling constant and we set $\hbar = 2m = 1$. We are interested in the TDL, when $N, L \to \infty$ with the particle density $n = N/L$ fixed. The Lieb–Liniger model is integrable for arbitrary value of the interaction parameter $c$, but, despite the many approaches and results in the literature [58, 59], [61]–[71], the general non-equilibrium quench dynamics (e.g. a quench from arbitrary $c_0$ to arbitrary $c$) is still beyond reach.

For this reason, we consider the easiest quench dynamics in the model, which is the one from initial $c = 0$ (free bosons) to final $c = \infty$ (impenetrable bosons). To be more specific, we prepare the many-body system in the $N$-particle ground state of the free boson Hamiltonian given by equation (3) with $c = 0$. At time $t = 0$, we suddenly turn on an infinitely strong interaction, and the evolution is governed by the Hamiltonian (3) with $c = \infty$. The Jordan–Wigner transformation
\[
\hat{\Psi}(x) = \exp \left\{ i \pi \int_0^x dz \hat{\Phi}^\dagger(z) \hat{\Phi}(z) \right\} \hat{\Phi}(x)
\]
maps the hard-core boson Hamiltonian onto the free fermionic one [72]. This dynamics has been studied numerically in [63] and analytically in [57]. One of the main results of the latter is that for finite times Wick’s theorem does not apply and each multi-point correlator should be calculated separately. However, for infinite time, since the stationary state is described by the density matrix $\rho_{GGE} = Z^{-1} \exp[-\int dk \lambda(k)\hat{n}(k)/2\pi]$ which is diagonal in momentum modes $\hat{n}(k)$, Wick’s theorem is restored and all multi-point correlators can be determined in terms of the fermionic two-point function [57, 58]
\[
C(x - y) \equiv \langle \Psi^\dagger(x, t = \infty) \Psi(y, t = \infty) \rangle = C_{GGE}(x - y) = ne^{-2n|x - y|}
\]
where $n$ is the particle density.

It is important to stress that since the Jordan–Wigner mapping (4) guarantees that the fermions in a given interval are functions only of the bosons in the same interval, the entanglement entropies of a single interval for the impenetrable bosons and for free fermions do coincide. This is not true anymore in the case of more disjoint intervals because of the presence of a bosonization string in equation (4), analogously to what happens in spin-chains [73].

2.1. Entanglement entropies and particle fluctuations

For a one-dimensional quantum gas that can be mapped to a non-interacting fermion system, Wick’s theorem allows for an exact representation of the bipartite entanglement
entropies of any spatial subsystem in terms of the two-point fermion correlator \[74, 75\]. Let us denote this two-point function by \(C(z, z') \equiv \langle \hat{\Psi}^\dagger(z) \hat{\Psi}(z') \rangle\). The reduced density matrix of a spatial subsystem \(A\) is

\[
\hat{\rho}_A \propto \exp \left( - \int_A dy_1 dy_2 \, \Psi^\dagger(y_1) \mathcal{H}(y_1, y_2) \Psi(y_2) \right),
\]

where \(\mathcal{H} = \ln[(1 - C)/C]\) and the normalization constant is fixed requiring \(\text{Tr} \hat{\rho}_A = 1\). This equation can be straightforwardly seen as the continuum limit of the formula for lattice free fermions \[76, 77\], but has also been obtained directly in the continuum path integral formalism \[76\]. At this point the integer powers, and hence the R´enyi entropies, of this reduced density matrix are given by Wick’s theorem as

\[
S_A^{(\alpha)} = \frac{1}{1 - \alpha} \text{Tr} \ln [C_A^\alpha + (1 - C_A)^\alpha],
\]

where \(C_A\) is the restriction of the fermionic correlation to the subsystem \(A\).

In order to be more explicit, the trace of the powers of the restricted correlation function is defined as

\[
\text{Tr} C_A^k \equiv \int_A dz_1 \ldots dz_k \, C(z_1, z_2) C(z_2, z_3) \ldots C(z_k, z_1).
\]

Introducing the matrix

\[
E_A = C_A(1 - C_A),
\]

the trace log in equation (7) can be recast in the form \[78\]

\[
\text{Tr} \ln [C_A^\alpha + (1 - C_A)^\alpha] = \sum_{k=1}^{\infty} \frac{4^k}{k^2} \text{Tr} E_A^{[\alpha/2]} \sum_{p=1}^{\infty} \cos^{2k} \left( \frac{2p - 1}{2\alpha} \pi \right).
\]

The previous formulas permit us to write all R´enyi entropies of integer order in terms of the integer powers \(\text{Tr} C_A^k\) in equation (8). When a closed analytic form for all integer \(\alpha\) has been found, one can use the replica trick and search for an analytic continuation to non-integer \(\alpha\), whose limit for \(\alpha \to 1\) would give the desired von Neumann entanglement entropy. This is the first approach we will use in the following to determine the entanglement entropy.

An alternative way to obtain the entanglement entropies directly for any real \(\alpha\) is provided by finding the spectrum of the reduced correlation matrix. Indeed, if one knows all the eigenvalues \(\lambda_m\) of \(C_A\) (which we assume to be discrete for simplicity, as it will be in the case of our interest), equation (7) can be simply rewritten as

\[
S_A^{(\alpha)} = \sum_m e_\alpha(\lambda_m), \quad e_\alpha(\lambda) \equiv \frac{1}{1 - \alpha} \ln[\lambda^\alpha + (1 - \lambda)^\alpha].
\]

For \(\alpha = 1\) this formula gives the von Neumann entropy

\[
S_A = - \sum_m [\lambda_m \ln \lambda_m + (1 - \lambda_m) \ln(1 - \lambda_m)],
\]

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and for $\alpha \to \infty$ the single copy entanglement
\begin{equation}
S_A^{(\infty)} = - \sum_m \ln \left( \frac{1}{2} + |\lambda_m - \frac{1}{2}| \right).
\end{equation}

Usually [75, 79], the spectrum of the reduced correlation matrix is calculated by introducing the Fredholm determinant
\begin{equation}
D_A(\lambda) = \det[\lambda \delta_A(z - z') - C_A(z, z')],
\end{equation}
where also the identity $\delta(z - z')$ is restricted to the subsystem $A$. If one is able to calculate (the asymptotic behavior of) $D_A(\lambda)$, the entanglement entropies are given by equation (7) as the integral [75, 79]
\begin{equation}
S_A^{(\alpha)} = \oint \frac{d\lambda}{2\pi i} c_{\alpha}(\lambda) \frac{d}{d\lambda} \ln D_A(\lambda),
\end{equation}
over a contour which encircles the segment $[0, 1]$. However, we will not exploit this method in the following because in the present case we found it easier to directly diagonalize the reduced correlation matrix.

### 2.2. Particle fluctuations

The Rényi entropies characterize the non-trivial connections between different parts of an extended quantum system. For systems which can be mapped to free fermions like the present one, the entanglement entropies can be related to the even cumulant $V^{(2k)}_A$ of the particle-number distribution [80]–[83]
\begin{equation}
V^{(k)}_A = (-i\partial_\lambda)^k \ln \langle e^{i\lambda \hat{N}_A} \rangle |_{\lambda = 0},
\end{equation}
where
\begin{equation}
\hat{N}_A = \int_A dz \hat{\Psi}^\dagger(z) \hat{\Psi}(z)
\end{equation}
is the operator counting the number of particles in the interval $A = [x, y]$. Indeed, it has been shown that the following formal expansion holds [82]:
\begin{equation}
S_A^{(\alpha)} = \sum_{k=1}^{\infty} s_k^{(\alpha)} V^{(2k)}_A, \quad s_k^{(\alpha)} = \frac{(-1)^k (2\pi)^{2k} 2\zeta[-2k, (1 + \alpha)/2]}{(\alpha - 1) \alpha^{2k} (2k)!},
\end{equation}
where $\zeta[n, x] \equiv \sum_{k=0}^{\infty} (k + x)^{-n}$ is the generalized Riemann zeta function. In particular, exploiting the definition of the cumulant in equation (16), one has
\begin{equation}
V^{(k)}_A = (-i\partial_\lambda)^k G(\lambda, C_A)|_{\lambda = 0},
\end{equation}
\begin{equation}
G(\lambda, C_A) = \text{Tr} \ln[I - (1 - e^{i\lambda})C_A].
\end{equation}
Also in this case we can use the spectrum $\{\lambda_m\}$ of the correlation function obtaining
\begin{equation}
V^{(k)}_A = \sum_m (-i\partial_\lambda)^k \ln[1 - (1 - e^{i\lambda})\lambda_m] \bigg|_{\lambda = 0},
\end{equation}
which can be rewritten in terms of polylogarithm functions $\text{Li}_k(z) = \sum_{n=1}^{\infty} z^n/n^k$ as [84]
\begin{equation}
V^{(k)}_A = -\sum_m \text{Li}_{1-k} \left( \frac{\lambda_m}{\lambda_m - 1} \right) = -\text{Tr} \, \text{Li}_{1-k} \left( \frac{C_A}{C_A - 1} \right).
\end{equation}
3. Direct approach

In this section we consider the entanglement entropies and particle fluctuations of a subsystem $A$ consisting of an interval of length $\ell$ in an infinite system employing a brute force direct computation of the traces of the restricted correlation matrix, as in equation (8). For low powers, making use of the two-point function in equation (5), the integrals in equation (8) can be easily calculated and for the first five we obtain

\[
\begin{align*}
\text{Tr} \mathcal{C}_A &= n\ell, \\
\text{Tr} \mathcal{C}_A^2 &= (4n\ell - 1 + e^{-4n\ell})/8, \\
\text{Tr} \mathcal{C}_A^3 &= [6n\ell - 3 + e^{-4n\ell}(6n\ell + 3)]/16, \\
\text{Tr} \mathcal{C}_A^4 &= [40n\ell - 29 + 4e^{-4n\ell}(16n^2\ell^2 + 20n\ell + 7) + e^{-8n\ell}]/128, \\
\text{Tr} \mathcal{C}_A^5 &= 5[(42n\ell - 39) + 4e^{-4n\ell}(4n\ell + 3)(4n^2\ell^2 + 6n\ell + 3) + e^{-8n\ell}(6n\ell + 3)]/24, \\
\end{align*}
\]

where we recall that $\ell$ is the length of the interval $A$.

Based on the first few powers we are led to the conjecture

\[
\text{Tr} \mathcal{C}_A^k = \frac{1 + \frac{1}{2k-1} \left(2k - 1\right)}{2^{2k-2}} \left(\frac{\Gamma(k - 1/2)}{\sqrt{\pi} \Gamma(k)}\right) n\ell + \frac{1 - 1}{4k} + O(e^{-4n\ell})
\]

which we prove in section 4 using results on the spectrum of $\mathcal{C}_A$.

The conjecture (23) contains all ingredients to calculate the leading and subleading contributions to the Rényi entanglement entropy of arbitrary order. Indeed, equation (23) implies that for the matrix $\mathcal{E}_A$ in equation (9) we have

\[
\text{Tr} \mathcal{E}_A^k = \frac{\Gamma(k - 1/2)}{2^{2k-1}} \left(\frac{\Gamma(k)}{\sqrt{\pi} \Gamma(k + 1)}\right) n\ell + \frac{1 - 1}{4k} + O(e^{-4n\ell}) \equiv e_k^1 n\ell + e_k^0 + O(e^{-4n\ell}).
\]

By straightforward summation of equation (10), $\text{Tr} \mathcal{E}_A^k$ gives all the Rényi entropies of low integer order, as for example

\[
\begin{align*}
S_A^{(2)} &= (4 - 2\sqrt{2}) n\ell + \ln(24 - 16\sqrt{2}) + O(e^{-4n\ell}) \\
&= (4 - 2\sqrt{2}) n\ell + 2 \ln \left(\frac{\pi}{8}\right) + O(e^{-4n\ell}), \\
S_A^{(3)} &= n\ell + \ln(4/3) + O(e^{-4n\ell}), \\
3S_A^{(4)} &= \left(8 - 2\sqrt{2} + 2\sqrt{2}\right) n\ell + \ln(256) - 2 \ln \left(4 + \sqrt{2} + 2\sqrt{2}\right) + O(e^{-4n\ell}) \\
&= \left(8 - \frac{2}{\sin(\pi/8)}\right) n\ell + \ln 2 + 2 \ln \left[8 \tan \left(\frac{\pi}{16}\right) \tan \left(\frac{3\pi}{16}\right)\right] + O(e^{-4n\ell}).
\end{align*}
\]

In order to be systematic and give close formulas for arbitrary integer and real $\alpha$, let us expand the Rényi entropies in powers of $\ell$ as

\[
S_A^{(\alpha)} = s_\alpha^1 n\ell + s_\alpha^0 + O(e^{-4n\ell}).
\]
The series coefficients \( s^a_\alpha \) are clearly related to the factors \( e^a_k \) in equation (24). According to equation (10) this relation reads

\[
s^a_\alpha = \frac{1}{\alpha - 1} \sum_{k=1}^{\infty} \frac{4^k k^{\alpha/2}}{e^a_k} \sum_{p=1}^{\alpha/2} \cos 2k \left( \frac{2p - 1}{2\alpha} \pi \right).
\]  

A simpler analytic expression can be obtained by exchanging the order of the two summations. Let us first consider the term linear in \( n\ell \), in which we can use the formula

\[
\sum_{k=1}^{\infty} e^1_k 4^k k \cos 2k x = \sum_{k=1}^{\infty} \frac{\Gamma(k - 1/2)}{2^{2k-1} \sqrt{\pi} \Gamma(k)} 4^k k \cos 2k x = 4(1 - \sin x),
\]

which inserted in equation (27) provides

\[
s^1_\alpha = \frac{4}{\alpha - 1} \sum_{p=1}^{\alpha/2} \left( 1 + \sin \frac{\pi(1 - 2p)}{2\alpha} \right) = 2\csc \left( \frac{\pi}{2\alpha} \right) - \frac{\alpha}{1 - \alpha}, \tag{29}
\]

giving, in particular, \( s^1_1 = 2 \).

The calculation of the subleading term (order one in \( n\ell \)) is more complicated. For \( x \in (0, \pi/2] \), we have

\[
\sum_{k=1}^{\infty} e^0_k 4^k k \cos 2k x = \sum_{k=1}^{\infty} \frac{\Gamma(k - 1/2)}{\sqrt{\pi} \Gamma(k + 1)} 4^k k \cos 2k x = -4 \ln \sin \left( \frac{x}{2} + \frac{\pi}{4} \right), \tag{30}
\]

and so

\[
s^0_\alpha = \frac{4}{1 - \alpha} \sum_{p=1}^{\alpha/2} \ln \sin \left( \frac{\alpha-1+2p}{4\alpha} \right) = \frac{4}{1 - \alpha} \sum_{p=1}^{\alpha/2} \ln \cos \left( \frac{1 + \alpha - 2p}{4\alpha} \right). \tag{31}
\]

In order to perform the above sum explicitly, we use the integral [86]

\[
\ln \cos \frac{\pi a}{b} = -2 \int_0^\infty \frac{dx \sinh^2(ax)}{x \sinh(bx)}, \quad b > 2|a|, \tag{32}
\]

and obtain

\[
s^0_\alpha = -\frac{8}{1 - \alpha} \sum_{p=1}^{\alpha/2} \int_0^\infty \frac{dx \sinh^2[(1 + \alpha - 2p)x]}{x \sinh(4\alpha x)}. \tag{33}
\]

Exchanging the order of sum and integral, the sum can be performed as

\[
-4 \sum_{p=1}^{\alpha/2} \sinh^2((1 + \alpha - 2p)x) = \alpha - \csch(2x) \sinh(2\alpha x), \tag{34}
\]
so that
\[ s_0^\alpha = \frac{2}{1 - \alpha} \int_0^\infty \frac{dx}{x} \alpha - \text{csch}(x) \sinh(\alpha x) \frac{\sinh(2\alpha x)}{\sinh(2\alpha x)}. \] (35)

Now the limit \( \alpha \to 1 \) can be taken straightforwardly obtaining
\[ s_1^1 = 2 \int_0^\infty \frac{dx}{x} \frac{x \coth x - 1}{\sinh(2x)} = 2 \ln 2 - 1. \] (36)

Collecting the linear and the subleading terms, we arrive at the main result of the section, i.e. the analytic expression for the Rényi entropies
\[ S_A^{(\alpha)} = 2 \csc \left( \frac{\pi}{2} / \alpha \right) - \frac{\alpha}{1 - \alpha} n\ell + \frac{2}{1 - \alpha} \int_0^\infty \frac{dx}{x} \frac{\alpha - \text{csch}(x) \sinh(\alpha x)}{\sinh(2\alpha x)} + O(e^{-4n\ell}), \] (37)
which in the limit \( \alpha \to 1 \) gives the von Neumann entanglement entropy
\[ S_A = 2n\ell + 2\ln 2 - 1 + O(e^{-4n\ell}). \] (38)

4. Spectrum of the correlation function

An alternative way to compute the entanglement entropies and the particle fluctuations is based on the knowledge of the full spectrum of the restricted two-point fermionic correlation function \( C_A \). This allows for the numerically exact computation of the entanglement entropies for arbitrary subsystem size, and leads to exact integral formulas for the leading and subleading terms in \( n\ell \) which agree with the results of the previous section.

The spectrum of the correlation in equation (5) restricted to an interval of length \( \ell \) is given by the continuum eigenvalue problem
\[ \int_0^\ell dy n e^{-2n|x-y|} v_m(y) = \lambda_m v_m(x), \] (39)
where, thanks to translational invariance, the integral depends only on the length \( \ell > 0 \) of the integration interval. Notice that, since the exponential kernel is real and symmetric with norm smaller than one, the eigenvalues \( \lambda_m \) are real and fall in the interval [0, 1].

As done in [85] for a different kernel, taking the first two derivatives with respect to \( x \) of the integral equation (39), one can recast it as the following second order differential equation:
\[ \partial_x^2 v_m(x) = -\omega_m^2 v_m(x), \] (40)
with \( \omega_m^2 \equiv 4n^2(1/\lambda_m - 1) \in \mathbb{R} \). The solutions of equation (40) are
\[ v_m(x) = A_m \cos \omega_m x + B_m \sin \omega_m x. \] (41)
The coefficients $A_m$ and $B_m$ as well as the ‘frequencies’ $\omega_m$ are determined by the boundary conditions at $x = 0$ and $x = \ell$ that are imposed by the integral equation. In particular, one has

$$\partial_x v_m(0) = 2n v_m(0), \quad \partial_x v_m(\ell) = -2n v_m(\ell),$$

which leads to the linear system

$$\mathbf{M} \begin{pmatrix} A_m \\ B_m \end{pmatrix} = 0,$$

with

$$\mathbf{M} \equiv \begin{pmatrix} 2n & -\omega \\ 2n \cos \omega \ell - \omega \sin \omega \ell & 2n \sin \omega \ell + \omega \cos \omega \ell \end{pmatrix}.$$ (44)

In order to have a set of eigenfunctions building up the full eigensubspace, we have to impose the condition $\det \mathbf{M} = 0$ which leads to the following equation for the rescaled frequencies $\Omega_m \equiv \omega_m/2n$:

$$\tan(2n\ell \Omega) = \frac{2 \Omega}{\Omega^2 - 1}.$$ (45)

Using the trigonometric identity for $\tan(2x)$, all the solutions of equation (45) can be rewritten as the union of the solutions of the following two independent equations:

$$\tan(n\ell \Omega) = -\Omega, \quad \tan(n\ell \Omega) = \frac{1}{\Omega}.$$ (46)

Finally, the eigenvalues of the integral equations are related to $\Omega_m$ by

$$\lambda_m = \frac{1}{1 + \Omega_m^2}.$$ (47)

The asymptotic behavior for large $m$ is easily obtained from equation (46):

$$\lambda_m \sim \Omega_m^{-2} \sim \left( \frac{\pi m}{2n\ell} \right)^{-2}, \quad m \gg 1.$$ (48)

In figure 1(a) we give a pictorial representation of the first solutions of equation (46), and in (b) we report the numerically obtained eigenvalues $\lambda_m$ as a function of $n\ell$.

### 4.1. Numerical results

The eigenvalues $\lambda_m$ are easily found for arbitrary $n\ell$ by solving the two equations in (46), as pictorially depicted in figure 1. In order to check the correctness of the solution for the spectrum of the correlation function, we first compute the traces $\text{Tr} C_k^\Lambda$ for the lowest values of $k$ obtained summing over the eigenvalues $\lambda_m$. In figure 2(a), these are compared with the analytical expressions in equations (22), showing a perfect agreement for all values of $n\ell$. 

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Figure 1. (a) Pictorial representation of the first few solutions of equation (45). The trivial solution \( \Omega = 0 \) is a spurious root which does not correspond to an eigenvalue of the original problem. (b) The eigenvalues \( \lambda_m \) versus \( m \) for different values of \( n\ell \). The full lines are the asymptotic behavior in equation (48). In the inset we report the behavior of the first four non-trivial eigenvalues as a function of the rescaled interval length \( n\ell \). All the eigenvalues flow to 1 for \( n\ell \to \infty \).

In the same way, in figure 2(b), we compare the cumulants numerically evaluated via equation (21) with the following analytic form for the first five cumulants:

\[
V_A^{(1)} = \text{Tr} C_A, \\
V_A^{(2)} = \text{Tr} C_A - \text{Tr} C_A^2, \\
V_A^{(3)} = \text{Tr} C_A - 3\text{Tr} C_A^2 + 2\text{Tr} C_A^3, \\
V_A^{(4)} = \text{Tr} C_A - 7\text{Tr} C_A^2 + 12\text{Tr} C_A^3 - 6\text{Tr} C_A^4, \\
V_A^{(5)} = \text{Tr} C_A - 15\text{Tr} C_A^2 + 50\text{Tr} C_A^3 - 60\text{Tr} C_A^4 + 24\text{Tr} C_A^5.
\]

In figure 3 we report the numerically evaluated entanglement entropies \( S_A^{(1)} \), \( S_A^{(2)} \), \( S_A^{(5)} \), and \( S_A^{(\infty)} \) for a subsystem of length \( \ell \). It is evident that for large enough \( n\ell \), i.e. for \( n\ell \geq 3 \), the numerical results are very well described by the asymptotic formula with the linear (in \( \ell \)) and constant term reported in equation (26). The asymptotic formula agrees quite well with the data also for relatively low values of \( n\ell \) because, as in equation (26), the corrections to this are exponentially small in \( n\ell \). This is very different from what is usually found in the ground state where power law corrections give sizable effects also for much larger values of \( \ell \). It is also worth noticing that for \( \alpha = \infty \), there is an infinite sequence of non-analytic points in \( n\ell \) which reflects the presence of an absolute value in the sum of the eigenvalues \( \lambda_m \) in equation (13). For finite \( \alpha \), these singularities are smoothed out but some signs of their appearance are clear (see e.g. the curve for \( \alpha = 5 \) in figure 3).
Figure 2. (a) The trace of the first five powers of the reduced correlation matrix as a function of $n\ell$. The solid lines are the analytical formulas in (22), while the symbols represent the numerical evaluation using the spectrum $\{\lambda_m\}$ as in equation (47). (b) The first five cumulants $V_A^{(k)}$; the lines and symbols are the same as in panel (a) comparing numerical and analytic results.

Figure 3. Exact numerical evaluation of the Rényi entanglement entropies for $\alpha = 1, 2, 5, \infty$ as a function of $n\ell$. For large $n\ell$ these are perfectly described by the extensive asymptotic results reported as full (red) lines. Notice the presence of some non-analytic points for $\alpha = \infty$. 

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Figure 4. Exact entanglement entropy profiles (solid black lines) in the stationary state for a subsystem of length $\ell$ compared with the cumulant expansion given by equation (18) truncated at a given order (dashed lines). While for all integer $\alpha \geq 2$ the convergence of the cumulant expansion is very fast, for $\alpha = 1$ (and all non-integer $\alpha$) the series is only asymptotic and the sum does not converge, as is clear from panel (a).

Finally, in figure 4 we compare the entanglement entropies reported in figure 3, i.e. $\alpha = 1, 2, 5, \infty$, with the corresponding approximations given by the cumulant expansion (18) calculated as a sum up to a given finite order. As an important difference from the ground-state results [81, 82, 78, 87] and some other non-equilibrium situations [80, 88], all cumulants contribute to the leading behavior of the entanglement entropies and the expansion (18) does not get effectively truncated at the second order. As already stressed
elsewhere [74, 78, 84], when all cumulants contribute to the expansion (18), such a series is well defined and convergent only for integer \( \alpha > 1 \). For all other values, and in particular for \( \alpha = 1 \), the coefficients \( s_\alpha^{(k)} \) grow too quickly with \( k \), the resulting series is only asymptotic, and adequate resummation schemes should be used to extract quantitative information from it. In fact, panel (a) in figure 4 shows that by adding more terms to the expansion (18) for \( \alpha = 1 \), we have worse and worse results.

### 4.2. Analytic evaluation of entanglement entropies from the correlation spectrum

Interestingly, both the leading and subleading behavior of all the traces, cumulants and entropies can be analytically extracted by analyzing the roots of equation (46) in the limit \( n\ell \gg 1 \). For \( m \ll n\ell \) one gets \( \Omega_m \sim \pi m/(2n\ell) \). However, for any fixed \( n\ell \) for large \( m \) we have \( \Omega_m \sim \pi(m-1)/(2n\ell) \), i.e. there is a \( \pi/(2n\ell) \) shift of the roots as \( m \) grows. To capture this behavior, we start by solving equation (45) perturbatively for large \( n\ell \):

\[
\Omega_m = \frac{m\pi}{2n\ell} - \frac{1}{n\ell} \left( \frac{m\pi}{2n\ell} - \frac{m\pi}{2(n\ell)^2} + \frac{m\pi/2 - (1/24)(m\pi)^3}{(n\ell)^3} - \frac{m\pi/2 - (1/6)(m\pi)^3}{(n\ell)^4} \right.
+ \left. \frac{m\pi/2 - (5/12)(m\pi)^3 + (1/160)(m\pi)^5}{(n\ell)^5} + \cdots \right) .
\]

The subseries corresponding to the highest powers of \( m \) in each term can be summed up:

\[
\Omega_m = \frac{m\pi}{2n\ell} - \frac{1}{n\ell} \arctan \left( \frac{m\pi}{2n\ell} \right) + \frac{1}{(n\ell)^2} \arctan \left( \frac{m\pi/2n\ell}{1 + (m\pi/2n\ell)^2} \right) + \cdots
\approx \frac{m\pi}{2n\ell} - \frac{1}{n\ell} \arctan \left( \frac{m\pi}{2n\ell} \right) .
\]

Note that the \( \arctan \) function interpolates between 0 and \( \pi/2 \), reproducing the expected shift of the solutions as \( m \) grows. In the large \( n\ell \) limit \( \Omega \) becomes a continuous variable and all the sums over \( m \) can be replaced by integrals. We can compute the density of roots of the equation:

\[
\sigma(\Omega_m) = \frac{1}{\Omega_{m+1} - \Omega_m} = \frac{n\ell}{\pi/2 + \arctan (m\pi/2n\ell) - \arctan ((m+1)\pi/2n\ell)}
\approx \frac{2n\ell}{\pi} \left( 1 + \frac{1}{n\ell} \frac{1}{1 + (m\pi/2n\ell)^2} \right) \approx \frac{2n\ell}{\pi} \left( 1 + \frac{1}{n\ell} \frac{1}{1 + \Omega_m^2} \right) .
\]

Therefore for \( n\ell \gg 1 \), using \( \lambda_m = (1 + \Omega_m^2)^{-1} \) (cf. equation (47)), one has

\[
\text{Tr} C_A^k = \int_0^{\infty} \frac{d\Omega}{(1 + \Omega^2)^k} \sigma(\Omega)
\approx \frac{2n\ell}{\pi} \int_0^{\infty} \frac{d\Omega}{(1 + \Omega^2)^k} \left( 1 + \frac{1}{n\ell} \frac{1}{1 + \Omega^2} \right) - \frac{1}{2}
= \frac{\Gamma(k - 1/2)}{\sqrt{\pi} \Gamma(k)} \left( n\ell + 1 - \frac{1}{2k} \right) - \frac{1}{2} .
\]

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where the extra $-1/2$ comes from the $m = 0$ boundary term which needs to be taken into account when converting the sum to an integral according to the Euler–Maclaurin formula. The above result coincides with the conjectured equation (23), providing an explicit proof of it.

For the Rényi entropies, using equation (11), we find

$$S_A^{(\alpha)} = \int_0^\infty d\Omega \sigma(\Omega)e_\alpha \left( \frac{1}{1 + \Omega^2} \right)$$

$$\approx \frac{2n\ell}{\pi(1 - \alpha)} \int_0^\infty d\Omega \left( 1 + \frac{1}{n\ell 1 + \Omega^2} \right) \ln \left( \frac{1}{(1 + \Omega^2)^\alpha} + \left( \frac{\Omega^2}{1 + \Omega^2} \right)^\alpha \right)$$

$$= \frac{2n\ell}{\pi(1 - \alpha)} \int_0^\infty d\Omega \ln \frac{1 + \Omega^{2\alpha}}{(1 + \Omega^2)^\alpha} + \frac{2}{\pi(1 - \alpha)} \int_0^\infty d\Omega \frac{1}{1 + \Omega^2} \ln \frac{1 + \Omega^{2\alpha}}{(1 + \Omega^2)^\alpha},$$

(54)

and for the von Neumann entropy

$$S_A \approx \frac{2n\ell}{\pi} \int_0^\infty d\Omega \left( 1 + \frac{1}{n\ell 1 + \Omega^2} \right) \left[ \frac{1}{(1 + \Omega^2)^\alpha} \ln \left( \frac{1}{1 + \Omega^2} \right) + \frac{\Omega^2}{1 + \Omega^2} \ln \left( \frac{\Omega^2}{1 + \Omega^2} \right) \right]$$

$$= 2n\ell + (2 \ln 2 - 1).$$

(55)

The single copy entanglement is

$$S_A^{\infty} \approx \frac{2n\ell}{\pi} \int_0^\infty d\Omega \left( 1 + \frac{1}{n\ell 1 + \Omega^2} \right) \left( \frac{1}{2} + \left| \frac{1}{1 + \Omega^2} - \frac{1}{2} \right| \right)$$

$$= \left( 2 - \frac{4}{\pi} \right) n\ell + 2 \ln 2 - \frac{4}{\pi} C,$$

(56)

where $C \approx 0.915966$ is Catalan’s constant.

Notice that the analytical forms as a function of $\alpha$ of equations (54) and (37) are apparently very different both for the leading and the subleading term. We checked by explicit numerical computation of the two integrals for many different (integer and arbitrary real) values of $\alpha$ that they are the same as they should be.

5. Comparison with thermodynamic entropies

In this section we explore the connection of the entanglement entropy with the thermodynamic one. At a first look this relation can sound awkward since we are dealing with the time evolution from a pure state, which is always a pure state and its global entropy should be just zero. However, the expectation values of local operators and their correlation functions can be obtained as averages over a proper statistical ensemble which is expected to be thermal for generic systems and the GGE for an integrable one. For the quench studied in this paper, the convergence to the GGE has been established in [57]. Before presenting the calculation of the thermodynamic entropies let us recall how and in which sense a thermodynamic ensemble describes the steady state.
For a quench in a general integrable model, the GGE for the whole system is defined as [24]

$$\hat{\rho}_{\text{GGE}} = \frac{e^{-\sum \lambda \hat{I}_i}}{Z},$$

(57)

where $\{\hat{I}_i\}$ is a complete set of \textit{local} [33, 36] integrals of motion, $Z = \text{Tr} e^{-\sum \lambda \hat{I}_i}$ is a normalization constant, and the Lagrange multipliers $\lambda_i$ are fixed by the initial condition $|\psi_0\rangle$ as $\langle \psi_0 | \hat{I}_i | \psi_0 \rangle = \text{Tr}[\hat{\rho}_{\text{GGE}} \hat{I}_i]$. For the quench we are studying in this paper, the final Hamiltonian has a simpler infinite set of conserved charges, formed by the fermionic mode occupations, $\hat{n}(k)$, which are not local. Fortunately, the local conserved charges can be expressed as \textit{linear} combinations of the $\hat{n}(k)$ [36, 84], so the GGEs built from $\hat{n}(k)$ and $\{\hat{I}_i\}$ are equivalent.

As discussed in [28, 29, 33, 36, 51], the steady state after a quantum quench in an integrable system is described by $\hat{\rho}_{\text{GGE}}$ in equation (57) in the sense that in the TDL, a long time limit of the reduced density matrix of any finite subsystem $A$ exists and it is equal to the reduced density matrix of $\hat{\rho}_{\text{GGE}}$. In formulas, one defines (when it exists)

$$\hat{\rho}_A^{\infty} = \lim_{t \to \infty} \text{Tr}_B \left[ \lim_{L \to \infty} |\psi(t)\rangle \langle \psi(t)| \right],$$

(58)

and

$$\hat{\rho}_{A,\text{GGE}} = \text{Tr}_B \hat{\rho}_{\text{GGE}},$$

(59)

where, in both cases, $B$ is the complement of $A$. At this point, it is usually said that a system is described by the GGE if $\hat{\rho}_{A,\text{GGE}} = \hat{\rho}_A^{\infty}$ [33]. In equation (58), the order of limits and partial trace is fundamental for the existence of a stationary value.

Alternatively, the steady state can be described by the so-called diagonal ensemble [48]

$$\hat{\rho}_d = \sum_j |c_j|^2 |j\rangle \langle j|,$$

(60)

where $c_j = \langle j| \Psi_0 \rangle$ are the overlaps of the eigenstates $j$ of the post-quench Hamiltonian with the initial state. The diagonal ensemble clearly describes the time-averaged values of all observables, including non-local and non-stationary ones. In some sense, $\hat{\rho}_d$ contains much more information about the quench than the GGE which knows only about local observables. Indeed it has been argued that in general $\hat{\rho}_d \neq \hat{\rho}_{\text{GGE}}$ [51] while $\text{Tr}_B \hat{\rho}_d = \hat{\rho}_{A,\text{GGE}}$, for any finite $A$.

The inequivalence of the diagonal and GGE ensembles is indeed captured in an easy way by their entropies, reflecting the fact that the crucial difference is the information loss in passing from diagonal to GGE ensembles. The diagonal and GGE entropies are simply the von Neumann entropies of the corresponding density matrices, i.e.

$$S_d = -\text{Tr} \hat{\rho}_d \ln \hat{\rho}_d = -\sum_j |c_j|^2 \ln |c_j|^2,$$

(61)

$$S_{\text{GGE}} = -\text{Tr} \hat{\rho}_{\text{GGE}} \ln \hat{\rho}_{\text{GGE}}.$$
In order to calculate these two thermodynamic entropies, we exploit the fundamental property that in integrable models, the summation over states in the expectation values of an observable can be recast as a functional integral over the Bethe ansatz root densities $\rho(\lambda)$, as in the Yang–Yang approach to equilibrium thermodynamics [89]. For the stationary values of (some) observables after a quantum quench, it has been shown by Caux and Essler [67] that, in the thermodynamic limit, only the saddle-point over these roots contributes, i.e.

\[
\lim_{t \to \infty} \langle O(t) \rangle = \lim_{L \to \infty} \langle \Phi_s | O | \Phi_s \rangle,
\]

where $| \Phi_s \rangle$ is the saddle-point state, represented in the Bethe ansatz by a proper saddle-point Bethe root density $\rho_s(\lambda)$.

For the quench considered in this paper the saddle-point density of roots function $\rho_s(\lambda)$ has already been computed as [57, 59]

\[
\rho_s(\lambda) = \frac{1}{2\pi} \frac{1}{1 + \lambda^2/(2n)^2} = \frac{1}{2\pi} \tilde{\rho}_s(\lambda).
\]

(64)

In the thermodynamic Bethe ansatz, the entropy of a Bethe state (defined by the density of particles $\tilde{\rho}(\lambda)$ and holes $\tilde{\rho}_h(\lambda)$ with $\tilde{\rho}_t(\lambda) = \tilde{\rho}(\lambda) + \tilde{\rho}_h(\lambda)$) is given by [89]

\[
S[\tilde{\rho}] = L \int \frac{d\lambda}{2\pi} [\tilde{\rho}_t(\lambda) \ln \tilde{\rho}_t(\lambda) - \tilde{\rho}(\lambda) \ln \tilde{\rho}(\lambda) - \tilde{\rho}_h(\lambda) \ln \tilde{\rho}_h(\lambda)].
\]

(65)

In the case at hand we have $\tilde{\rho}(\lambda) = \tilde{\rho}_s(\lambda)$ and $\tilde{\rho}_h(\lambda) = 1 - \tilde{\rho}_s(\lambda)$, so that the thermodynamic entropy of the GGE is given by

\[
S_{\text{GGE}} = S[\tilde{\rho}_s] = -L \int \frac{d\lambda}{2\pi} [\tilde{\rho}_s(\lambda) \ln \tilde{\rho}_s(\lambda) + (1 - \tilde{\rho}_s(\lambda)) \ln (1 - \tilde{\rho}_s(\lambda))] = 2nL.
\]

(66)

The diagonal entropy can also be calculated in the thermodynamic limit using the explicit expression for the overlaps found in [63, 59]. Denoting by $c_\rho$ the overlap with a state with root density $\rho(\lambda)$, we have

\[
S_d = -\int D\rho e^{S_Q[\rho]} c^2_\rho \ln c^2_\rho = -\int D\rho e^{S_Q[\rho] + 2 \ln c_\rho} 2 \ln c_\rho,
\]

(67)

where $S_Q[\rho]$ is the quench action of [67, 90] whose exponential gives the density of states of a root configuration. For our case the logarithm of the overlaps is [59]

\[
-2 \ln c_\rho = L \left[ \int_0^\infty \frac{d\lambda}{2\pi} \tilde{\rho}(\lambda) \ln \frac{\lambda^2}{4n^2} + n \right].
\]

(68)

In the TDL, also the diagonal entropy is dominated by the saddle-point

\[
S_Q[\rho_s] = -2 \ln c_{\rho_s},
\]

(69)

whose contribution is

\[
S_d = -\int D\rho e^{S_Q[\rho] + 2 \ln c_\rho} 2 \ln c_\rho \approx -2 \ln c_{\rho_s} = S_Q[\rho_s],
\]

(70)
showing in particular that the diagonal entropy equals the saddle-point quench action (as expected). Thus we finally have

$$S_d = S_Q[\rho_s] = nL + L \int_0^\infty \frac{d\lambda}{2\pi} \tilde{\rho}(\lambda) \ln \left( \frac{\lambda^2}{4n^2} \right) = nL.$$  \hspace{1cm} (71)

At this point some comments are in order. The GGE and diagonal entropies are both extensive as they should be, being thermodynamic entropies. They are however different from each other. The GGE entropy has exactly the same density as the entanglement entropy in equation (38), confirming the expectation that for large time after the quench the entanglement entropy does become the thermodynamic entropy. The diagonal entropy is instead exactly half of the entanglement and GGE entropies. Once again this reflects the fact that $\hat{\rho}_d$ contains much more information than that needed to describe the expectation values of local observables. Furthermore, the same ratio of 2 between the GGE and the diagonal entropies has been also observed in previous studies on the transverse field Ising chain [54, 49, 51] and it is natural to wonder what the precise physical origin and the value of this ratio are for other quenches in integrable models. At a qualitative level, the ratio between the diagonal and GGE entropies has been explained by the following argument in the Ising chain [49, 91]. For free fermions, the quench creates excitations in pairs of opposite momenta $k$ and $-k$, but in the GGE such correlations are neglected (which in equation (66) is encoded in the integral with $\lambda$ going from $-\infty$ to $\infty$). Indeed they have no influence on the reduced density matrix of a finite subsystem $A$: if a particle with momentum $k$ is in $A$, for long enough time, the $-k$ partner is surely outside of $A$ [49]. Quasi-particle excitations are created in pairs even for the quench considered here [63, 92, 59] (and arguably in more general circumstances [10, 26]), so the previous qualitative argument still applies.

6. Conclusions

We calculated Rényi entanglement entropies in the stationary state after a quench from free to hard-core bosons in one dimension, exploiting the knowledge of the two-point fermionic correlation function obtained in [57] and the restoration of Wick’s theorem for infinite time. The Rényi entanglement entropies are calculated using two different methods. First, following the approach introduced in [74, 75, 78], we directly sum over the powers of the reduced correlation matrix obtaining the integer order Rényi entropies. In this way, at the end of the calculation we find the analytical continuation to real $\alpha$ which provides, among other things, the von Neumann entropy. The second approach is based on the computation of the spectrum of the reduced correlation matrix, which, in the present case, is enormously simplified by mapping it into an eigenvalue problem of a second order differential equation. Both methods allow us to obtain an explicit and analytic form for the leading and subleading terms in $\ell$ of the Rényi entanglement entropies for arbitrary $\alpha$. In particular, for the von Neumann entropy we find the very simple result

$$S_A = 2n\ell + 2 \ln 2 - 1 + O(e^{-4n\ell}).$$  \hspace{1cm} (72)

The approach to the asymptotic behavior is exponentially fast and so it is already reached for relatively small values of $\ell$. From the technical point of view it would be very interesting
to understand whether the problem of the spectrum of the reduced correlation matrix can be mapped into a simple differential equation even in other instances, as e.g. recently done in [85].

We also compared the von Neumann entanglement entropy with the thermodynamic entropies in both the GGE and the diagonal ensembles. We found that while the entanglement and GGE entropies coincide (as expected), the diagonal entropy is half of the other two. This can be easily interpreted as the loss of non-local information passing from the diagonal to the GGE ensemble. The same factor of two was previously found also for the transverse field Ising chain [49, 51] and it is surely an interesting open problem (numerically initiated in [54]) to understand the relation between diagonal and GGE entropies for more complicated quantum quenches in integrable models.

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