MHD turbulence has been studied in great detail in the past [1], in particular due to its relevance for space applications such as solar wind turbulence [2]. In the absence of external, or self generated mean magnetic fields, MHD turbulence tends to be isotropic [3]. While in nature mean magnetic fields abound, the statistically isotropic case, is interesting in its own right, which may be relevant when the background fields are sufficiently weak.

Nested polyhedra models were introduced recently as self-similar, spherically symmetric decimations of Fourier space using complete triangles in Navier-Stokes formulations such as solar wind turbulence [2]. In these, the wave-vector domain is discretized using nested, alternating icosahedron dodecahedron pairs that are organized in such a way that wavevectors that are represented by the vertices of these objects always form complete triads between neighboring scales. They naturally respect the conservation laws of the original system and since the discretization is separated from the formulation of the equations, they are straightforward to develop for different systems. Here we show a similar model developed for MHD system of equations. The result is a model that describes the three dimensional spectral evolution of MHD turbulence, which in principle has the ability to represent anisotropy. Since there is no source of anisotropy however, the resulting turbulence is isotropic.

The Model - The nested polyhedra model of incompressible MHD equations can be written as:

\[ \partial_t u^i_n + iM^{ijκ}_n \sum_{(n',n'')} (u^{κ*}_{n'} u^{jκ}_{n''} - b^{κ*}_{n'} b^{jκ}_{n''}) = -\nu k^2_n u^i_n \]
\[ \partial_t b^i_n + i\delta M^{ijκ}_n \sum_{(n',n'')} (u^{κ*}_{n'} b^{jκ}_{n''} - b^{κ*}_{n'} u^{jκ}_{n''}) = -\eta k^2_n b^i_n \]

where \( M^{ijκ}_n = (M^{ijκ}_n + M^{ijκ}_{n−1}) \), \( \delta M^{ijκ}_n = (M^{ijκ}_n - M^{ijκ}_{n−1}) \) and

\[ M^{ijκ}_n (k) = k^κ_n \left[ \delta_{ij} - \frac{k^i_n k^j_n}{k^2_n} \right]. \]

Here the Einstein summation convention is used over repeated indices and the sums are computed over the set of pairs \( p_n \) that form a triad with the node \( n \), which is determined by the geometry of nested polyhedra representation -independent from the equations- as described in detail in Ref. [4]. Note that if the node belongs to the \( m \)th polyhedron in the nested hierarchy, it can form triads with pairs of nodes from neighboring polyhedra \( m−2, m−1, m+1 \) and \( m+2 \). Thus, the requirement of exact triads and the choice of the nodes on the vertices of nested polyhedra makes the interactions “local”, with a constant about 62% (i.e. \( 1/\varphi \) where \( \varphi = (1 + \sqrt{5})/2 \) is the golden ratio) for the ratio between the smallest to largest wavenumber of the interacting triad. Note that in this model this ratio is not a separate choice but imposed by the choice of the nested polyhedra geometry. The notation in [4] is such that \( n \) corresponds to the node number. The node numbers 0 to 5 belong to the first icosahedron (i.e. \( m = 0 \)), while 6 to 15 correspond...
and using a fixed time step solver. Random forcing is implemented averaged over the nodes and from $t = 460$ to $t = 500$. The results are averaged over the nodes and from $t = 460$ to $t = 500$. The model is implemented using an adaptive time stepping.

Figure 2. The high resolution case with $Pr_m = 1.0$, $\nu = 10^{-10}$, $N = 60$ and $h_f = 10^{-3}$ kinetic (solid blue line) and magnetic (dotted line) energy spectra, together with the low resolution case with $N = 30$ and $\nu = 10^{-6}$ kinetic (solid red line) and magnetic (dotted red line) energy spectra. The kinetic energy spectrum becomes dissipative. However when the Prandtl number is decreased further this is shown to be non-universal feature. The result is averaged over the nodes and from $t = 80$ to $t = 100$.

\[
F_n^r = \left( \delta_{ij} - \frac{k_n^i k_n^j}{k_n^2} \right) \xi_j \tag{2}
\]

where $\xi_j$ is a vector random variable. The expression guarantees that $k_n^i F_n^r = 0$ and that no helicity is injected. Indeed a preliminary attempt with $\xi_j = (1 + i) \times 10^{-2}$, a constant, which is a standard choice in shell models, lead to the development of large imbalances between $z^+ = u + b$ and $z^- = u - b$ asymptotically. Partly expected from the fact that such a forcing leads to strong correlation between $u$ and $b$, which modifies the spectrum strongly. Question of the relation between alignment and forcing and the relevance to real world MHD turbulence is an important one. However, the simplest possible mathematical approach is to choose a forcing that eliminate velocity-magnetic field correlation. This urged us to implement the random forcing discussed above, which removed the accumulation of imbalance. In a sense the imbalance should have been expected, since a constant forcing would lead to an accumulation of the alignment (or anti-alignment) between $u$ and $b$.

Results- Three dimensional incompressible MHD spectra can be computed with little difficulty up to $N = 60$, where $N$ is the total number of polyhedra in the nested polyhedra model. Starting from $k_0 = 1.0$, one gets $k_{max} = k_0 \varphi^{N/2}$. This means that a three dimensional wavenumber spectrum covering a range of more than 6 decades can easily be simulated with such a model. This is particularly useful if a clear identification of two or more different power laws are desired, such as the case with large or small magnetic Prandtl numbers.
The reference case corresponding to parameters \( Pr_m = 1, \nu = 10^{-9}, N = 60 \) and \( \eta_f = 10^{-3} \) is shown in figure 1. Indeed this case is rather similar to a regularly discretized numerical simulation with the same parameters, except such a run with regular discretization would be hideously costly. One interesting aspect of the nested polyhedra models is that the dissipative range can be eliminated as shown in figure 2 in this case by taking \( \nu = 10^{-10} \). Note that a smaller \( \nu \) with the same \( N \) would lead to an increasing spectrum around the maximum \( k \). This particular feature of the nested polyhedra model has the advantage that it does not need a subgrid model (such as large eddy simulation or LES) to push the dissipation range outside the simulation domain. Choosing the right value of dissipation is sufficient. A lower resolution case with \( N = 30 \) is also shown in figure 2. In fact even the case \( N = 30 \) is sufficiently resolved when the dissipative range is eliminated by the choice of \( \nu \). This is helpful because when one needs very good statistics such is the case for instance, when computing structure functions for intermittency corrections (typically runs up to \( t = 25000 \) may be needed) one can use lower resolution without loosing any important features of the solution.

We have also considered different values of the magnetic Prandtl number \( Pr_m \). The case \( Pr_m = 10^{-2} \) is shown in blue in figure 3 representing the small magnetic Prandtl number behavior. We can see that while there appears to be a secondary range where the magnetic energy is dissipated and the kinetic energy seemingly displays a \( k^{-8/3} \) power law scaling. However when the magnetic Prandtl number is decreased further to \( Pr_m = 10^{-4} \), this behavior is lost and one recovers a \( k^{-5/3} \) scaling also in this range as shown in figure 3. Note that, the model slows down when treating large or small Prandtl number cases, due to explicit treatment of linear terms. It is possible to alleviate this by using an implicit scheme or other more advance techniques such as exponential time integration schemes. Therefore the case with \( Pr_m = 10^{-4} \) was integrated only up to \( t = 100 \).

\[
Pr_m \equiv \nu/\eta.
\]

### Table I.

\( n = 8m + \ell^m \) is interacting with \( p_n = \{n', n''\} = \{8m - 16 + \ell^{m-2}, 8m - 10 + \ell^{m-1}\} \), \( \{8m - 10 + \ell^{m-1}, 8m + 6 + \ell^{m+1}\} \) and \( \{8m + 6 + \ell^{m+1}, 8m + 16 + \ell^{m+3}\} \) for an even \( m \) (i.e. an icosahedron node) \( \ell^m, \ell^{m+1} \) and \( \ell^{m+2} \) are to be taken from the values given above, where if the integer value \( n' \) has a bar over it we replace \( \{u_{n'}^*, b_{n'}^*\} \rightarrow \{u_{n'}^*, b_{n'}^*\} \) in the interaction term in [1].

![Figure 4](image-url)

Figure 4. Different modes of the system as a function of time showing that the dynamo effect kicks in some time after the large scales are saturated. Here we can see the effects of random forcing on \( |u_{12}^2| \), which then couples to other nodes.

![Figure 5](image-url)

Figure 5. Equipartition time \( \delta \tau_k = \tau_k - \tau_0 \) with respect to the equipartition time of the small scales (i.e. \( \tau_0 \)) seems to roughly follow a \( \delta \tau_k \propto k^{-1/2} \) scaling.
initial conditions are shown in figure 5. In this model, the interactions are "local"

\[ \ell \rightarrow \text{model the MHD system of equations with no external} \]

and dodecahedra, such that the wave-vectors that cor-

\[ \text{establishment of the equipartition} \]

get established via \( k \) to \( k \)-space of the kinetic energy appears, and fills the whole spectral\[ \text{domain. As this front reaches high-} k \text{-end of the inertial range (roughly about} \]

\( t \approx 30 \) for the reference case above), equipartition between kinetic and magnetic energies gets established at high- \( k \). Then another front (this time of the magnetic energy density) fills up the \( k \)-range moving towards smaller \( k \). We can define the time it takes for the establishment of the equipartition \( \tau_k \) which is a function of the wave-number \( k \), which in general is a function of the initial conditions. The case of the very small seed initial conditions are shown in figure [5].

\[ \text{Dynamo- The simulations that are presented above, are all driven with a} \]

large scale random forcing of the velocity field. The resulting spectra however, present and almost perfect equipartition of kinetic and magnetic energies. When one studies how these final steady state spectra are established, one observes that it happens in stages. First, as the large scale kinetic energy reaches roughly its final maximum values a front in \( k \)-space of the kinetic energy appears, and fills the whole spectral domain.

\[ \text{Then another front (this time of the magnetic energy density) fills up the} \]

\( k \)-range moving towards smaller \( k \). We can define the time it takes for the establishment of the equipartition \( \tau_k \) which is a function of the wave-number \( k \), which in general is a function of the initial conditions. The case of the very small seed initial conditions are shown in figure [5].

\[ \text{Conclusion- We show that a nested polyhedra model, obtained from "decimating" the wave-number space using self-similarly scaled nested, alternating icosahedra and dodecahedra, such that the wave-vectors that correspond to two nodes of the system can combine to give a third one that also falls on a resolved node, can be used to model the MHD system of equations with no external magnetic field. In this model, the interactions are "local" in } k \text{-space (i.e. the ratio } k_{n-2}/k_n \text{ of the smallest to the largest wavenumbers of the interacting triad is about } 62\%). \]

\[ \text{Considering isotropic MHD turbulence with no background magnetic field or rotation, and random large scale forcing on the velocity component, we find that the model} \]

can display a clear Kolmogorov power law scaling of the form \( k^{-5/3} \) over 6 decades in wave-number space with very good statistics, which allows considering large or small magnetic Prandtl number cases. Moreover, with a careful choice of the high- \( k \) dissipation, the apparent inertial range can extend all the way up to the end of the resolved range in \( k \)-space due to perfect self-similarity. Finally, since the random forcing was applied only on velocity, the magnetic energy spectrum gets established through the dynamo effect that starts from the small scales. It was observed that the time scale \( \delta \tau_k = \tau_k - \tau_0 \) for the equipartition, offset with the time of equipartition of the smallest scales shows a \( k^{-1/2} \) scaling.

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