On the second mixed moment of the characteristic polynomials of 1D band matrices

Tatyana Shcherbina*

IAS, Princeton, USA
e-mail: t_shcherbina@rambler.ru

Abstract
We consider the asymptotic behavior of the second mixed moment of the characteristic polynomials of 1D Gaussian band matrices, i.e. of the Hermitian $N \times N$ matrices $H_N$ with independent Gaussian entries such that $\langle H_{ij}H_{lk} \rangle = \delta_{ik}\delta_{jl}J_{ij}$, where $J = (-W^2\Delta + 1)^{-1}$. Assuming that $W^2 = N^{1+\theta}$, $0 < \theta \leq 1$, we show that the moment’s asymptotic behavior (as $N \to \infty$) in the bulk of the spectrum coincides with that for the Gaussian Unitary Ensemble.

1 Introduction

The Hermitian Gaussian random band matrices (RBM) are Hermitian $N \times N$ matrices $H_N$ (we enumerate indices of entries by $i, j \in \mathcal{L}$, where $\mathcal{L} = [-n, n]^d \cap \mathbb{Z}^d$, $N = (2n + 1)^d$) whose entries $H_{ij}$ are random Gaussian variables with mean zero such that

$$\mathbb{E}\{H_{ij}H_{lk}\} = \delta_{ik}\delta_{jl}J_{ij},$$

where $J_{ij}$ is a symmetric function which is small for large $|i - j|$ and

$$\sum_{i=-n}^{n} J_{ij} = 1.$$

In this paper we consider the especially convenient choice of $J_{ij}$, which is given by the lattice Green’s function

$$J_{ij} = (-W^2\Delta + 1)^{-1},$$

where $\Delta$ is the discrete Laplacian on $\mathcal{L}$. For the case $d = 1$

$$(-\Delta f)_j = \begin{cases} -f_{j-1} + 2f_j - f_{j+1}, & j \neq -n, n, \\ -f_{j-1} + f_j - f_{j+1}, & j = -n, n \end{cases}$$

*Supported by NSF grant DMS 1128155
with \( f_{n-1} = f_{n+1} = 0 \) (i.e. we consider the discrete Laplacian with Neumann boundary conditions). The advantage of (1.2) is that its inverse, which appears in the integral representation (see (2.15) below), is only three-diagonal matrix.

Note that \( J_{ij} \approx C_1 W^{-1} \exp\{-C_2 |i - j|/W\} \) for \( J \) of (1.2) with \( d = 1 \), and so the variance of matrix elements is exponentially small when \( |i - j| \gg W \). Hence \( W \) can be considered as the width of the band.

It should be noted also that the odd size of the matrices is chosen only because it is more convenient to have the symmetric segment \([-n, n]\) and it does not play any role in the consideration below.

The probability law of 1D RBM \( H_N \) can be written in the form

\[
P_n(dH_N) = \prod_{-n \leq i < j \leq n} \frac{dH_{ij}d\overline{H}_{ij}}{2\pi J_{ij}} e^{-\frac{|u_{ij}|^2}{4J_{ij}}} \prod_{i=-n}^{n} \frac{dH_{ii}}{\sqrt{2\pi J_{ii}}} e^{-\frac{u_{ii}^2}{2J_{ii}}}.
\]

(1.4)

Varying \( W \), we can see that random band matrices are natural interpolations between random Schrödinger matrices \( H_{RS} = -\Delta + \lambda V \), in which the randomness only appears in the diagonal potential \( V \) (\( \lambda \) is a small parameter which measures the strength of the disorder) and mean-field random matrices such as \( N \times N \) Wigner matrices, i.e. Hermitian random matrices with i.i.d elements. Moreover, random Schrödinger matrices with parameter \( \lambda \) and RBM with the width of the band \( W \) are expected to have some similar qualitative properties when \( \lambda \approx W^{-1} \) (for more details on these conjectures see [26]).

The key physical parameter of these models is the localization length, which describes the typical length scale of the eigenvectors of random matrices. The system is called delocalized if the localization length \( \ell \) is comparable with the matrix size, and it is called localized otherwise. Delocalized systems correspond to electric conductors, and localized systems are insulators.

In the case of 1D RBM there is a physical conjecture (see [7, 13]) stating that \( \ell \) is of order \( W^2 \) (for the energy in the bulk of the spectrum), which means that varying \( W \) we can see the crossover: for \( W \gg \sqrt{N} \) the eigenvectors are expected to be delocalized and for \( W \ll \sqrt{N} \) they are localized. In terms of eigenvalues this means that the local eigenvalue statistics in the bulk of the spectrum changes from Poisson, for \( W \ll \sqrt{N} \), to GUE (Hermitian matrices with i.i.d Gaussian elements), for \( W \gg \sqrt{N} \). At the present time only some upper and lower bounds for \( \ell \) are proven rigorously. It is known from the paper [23] that \( \ell \leq W^8 \). On the other side, in the resent papers [10, 11] it was proven first that \( \ell \gg W^{7/6} \), and then that \( \ell \gg W^{5/4} \).

The questions of the order of the localization length are closely related to the universality conjecture of the bulk local regime of the random matrix theory, which we briefly outline now.

Let \( \lambda_1^{(N)}, \ldots, \lambda_N^{(N)} \) be the eigenvalues of \( H_N \). Define their Normalized Counting Measure (NCM) as

\[
\mathcal{N}_N(\sigma) = \#\{\lambda_j^{(N)} \in \sigma, j = 1, \ldots, N\}/N, \quad \mathcal{N}_N(\mathbb{R}) = 1,
\]

(1.5)

where \( \sigma \) is an arbitrary interval of the real axis. The behavior of \( \mathcal{N}_N \) as \( N \to \infty \) was studied for many ensembles. For 1D RBM it was shown in [3, 22] that \( \mathcal{N}_N \) converges weakly, as \( N, W \to \infty \), to a non-random measure \( \mathcal{N} \), which is called the limiting NCM of the ensemble. The measure \( \mathcal{N} \) is absolutely continuous and its density \( \rho \) is given by
the well-known Wigner semicircle law (the same result is valid for Wigner ensembles, in particular, for Gaussian ensembles GUE, GOE):

$$\rho(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2}, \quad \lambda \in [-2, 2].$$ (1.6)

Much more delicate result about the density of states at arbitrarily short scales is proven in [8] for $d = 3$.

These results characterize the so-called global distribution of the eigenvalues.

The local regime deals with the behavior of eigenvalues of $N \times N$ random matrices on the intervals whose length is of the order of the mean distance between nearest eigenvalues. The main objects of the local regime are $k$-point correlation functions $R_k$ ($k = 1, 2, \ldots$), which can be defined by the equalities:

$$E \left\{ \sum_{j_1 \neq \ldots \neq j_k} \varphi_k(\lambda_{j_1}^{(N)}, \ldots, \lambda_{j_k}^{(N)}) \right\} = \int_{\mathbb{R}^k} \varphi_k(\lambda_1^{(N)}, \ldots, \lambda_k^{(N)}) R_k(\lambda_1^{(N)}, \ldots, \lambda_k^{(N)}) d\lambda_1^{(N)} \ldots d\lambda_k^{(N)},$$ (1.7)

where $\varphi_k : \mathbb{R}^k \to \mathbb{C}$ is bounded, continuous and symmetric in its arguments and the summation is over all $k$-tuples of distinct integers $j_1, \ldots, j_k \in \{1, \ldots, N\}$.

According to the Wigner – Dyson universality conjecture (see e.g. [19]), the local behavior of the eigenvalues does not depend on the matrix probability law (ensemble) and is determined only by the symmetry type of matrices (real symmetric, Hermitian, or quaternion real in the case of real eigenvalues and orthogonal, unitary or symplectic in the case of eigenvalues on the unit circle). For example, the conjecture states that for Hermitian random matrices in the bulk of the spectrum and in the range of parameters for which the eigenvectors are delocalized

$$\lim_{N \to \infty} \frac{1}{(N \rho(\lambda_0))^k} R_k \left( \lambda_0 + \frac{\xi_1}{\rho(\lambda_0) N}, \ldots, \lambda_0 + \frac{\xi_k}{\rho(\lambda_0) N} \right) = \det \left\{ \frac{\sin \pi (\xi_i - \xi_j)}{\pi (\xi_i - \xi_j)} \right\}_{i,j=1}^k$$ (1.8)

for any fixed $k$, and the limit is uniform in $\xi_1, \xi_2, \ldots, \xi_k$ varying in any compact set in $\mathbb{R}$. This means that the limit coincides with that for GUE.

In this language the conjecture about the crossover for 1D RBM states that we get (1.8) for $W \gg \sqrt{N}$ (which corresponds to delocalized states), and we get another behavior, which is determined by the Poisson statistics, for $W \ll \sqrt{N}$ (and corresponds to localized states). For the general Hermitian Wigner matrices (i.e. $W = N$) bulk universality (1.8) has been proved recently in [12, 28]. However, in the general case of RBM the question of bulk universality of local spectral statistics is still open even for $d = 1$.

Other more simple objects of the local regime of the random matrix theory are the correlation functions (or the mixed moments) of characteristic polynomials.

Characteristic polynomials of random matrices have been actively studied in the last years (see e.g. [11, 5, 6, 14, 15, 16, 18, 20, 21, 23, 25, 27, 29]). The interest to this topic is
stimulated by its connections to the number theory, quantum chaos, integrable systems, combinatorics, representation theory and others.

An additional source of motivation for the current work is the development of the supersymmetric method (SUSY) in the context of random operators with non-trivial spatial structures. This method is widely used in the physics literature (see e.g. [9, 21]) and is potentially very powerful but the rigorous control of the integral representations, which can be obtained by this method, is difficult and so far for the band matrices it has been performed only for the density of states (see [8]). From the SUSY point of view characteristic polynomials correspond to the so-called fermionic sector of the supersymmetric full model, which describes the correlation functions \( R_k \). So the analysis of the local regime of correlation functions of the characteristic polynomial is an important step towards the proof of (1.8).

The correlation function of the characteristic polynomials is

\[
F_{2k}(\Lambda) = \int \prod_{s=1}^{2k} \det(\lambda_s - H_N) P_n(dH_N),
\]

where \( P_n(dH_N) \) is defined in (1.4), and \( \Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_{2k}\} \) are real or complex parameters that may depend on \( N \).

The asymptotic local behavior in the bulk of the spectrum of the \( 2k \)-point mixed moment for GUE is well-known (see e.g. [27]):

\[
F_{2k}\left(\Lambda_0 + \hat{\xi}/N\rho(\lambda_0)\right) = C_N \frac{\det\left\{\sin\frac{\pi}{2}(\xi_i - \xi_{j+k})\right\}_{i,j=1}^{k}}{\Delta(\xi_1, \ldots, \xi_k)\Delta(\xi_{k+1}, \ldots, \xi_{2k})} \times e^{\lambda_0(\xi_1 + \cdots + \xi_{2k})/2\rho(\lambda_0)}(1 + o(1)),
\]

where \( \Delta(\xi_1, \ldots, \xi_k) \) is the Vandermonde determinant of \( \xi_1, \ldots, \xi_k \), \( \hat{\xi} = \text{diag}\{\xi_1, \ldots, \xi_{2k}\} \), \( \Lambda_0 = \lambda_0 \cdot I \).

The similar result for the \( \beta \)-ensembles with \( \beta = 2 \) was obtained in [4, 27] (see the reference for the more precise statement). In the case of general Hermitian Wigner matrices it was proven that constant \( C_N \) depends only on the first four moments of the matrix elements distribution and does not depend on any higher moments (see [15] for the case \( k = 1 \) and [24] for any \( k \)). The same result was obtained for general Hermitian sample covariance matrices (see [17] for the case \( k = 1 \) and [25] for any \( k \)). This shows that the local regime of correlation functions of characteristic polynomials is universal up to the first four moments.

In this paper we are interested in the asymptotic behavior of (1.9) with \( k = 1 \) for matrices (1.1) – (1.4) as \( N, W \to \infty \), \( W^2 = N^{1+\theta} \), \( 0 < \theta \leq 1 \) (i.e. \( W \gg \sqrt{N} \)), and for

\[
\lambda_j = \lambda_0 + \frac{\xi_j}{N\rho(\lambda_0)}, \quad j = 1, 2,
\]

where \( N = 2n + 1 \), \( \lambda_0 \in (-2, 2) \), \( \rho \) is defined in (1.6), and \( \hat{\xi} = \text{diag}\{\xi_1, \xi_2\} \) are real parameters varying in any compact set \( K \subset \mathbb{R} \).
Set also
\[ D_2 = \prod_{l=1}^{2} F_2^{1/2} \left( \lambda_0 + \frac{\xi_l}{N \rho(\lambda_0)}, \lambda_0 + \frac{\xi_l}{N \rho(\lambda_0)} \right). \]  

(1.11)

The main result of the paper is the following theorem:

**Theorem 1.** Consider the random matrices (1.1) – (1.4) with 
\[ W^2 = N^{1+\theta}, \quad 0 < \theta \leq 1. \]
Define the second mixed moment \( F_2 \) of the characteristic polynomials as in (1.9). Then we have
\[ \lim_{n \to \infty} D_2^{-1} F_2(B_0 + \hat{\xi}/(N \rho(\lambda_0))) = \frac{\sin(\pi(\xi_1 - \xi_2))}{\pi(\xi_1 - \xi_2)}, \]

and the limit is uniform in \( \xi_1, \xi_2 \) varying in any compact set \( K \subset \mathbb{R} \). Here \( \rho(\lambda) \) and \( D_2 \) are defined in (1.6) and (1.11), \( B_0 = \text{diag} \{ \lambda_0, \lambda_0 \}, \lambda_0 \in (-2, 2), \hat{\xi} = \text{diag} \{ \xi_1, \xi_2 \} \).

The theorem shows that the limit above for the second mixed moment of characteristic polynomials for 1D Gaussian random band matrices (with \( W^2 = N^{1+\theta}, \quad 0 < \theta \leq 1 \)) coincides with that for the Gaussian unitary ensemble, i.e., the local behavior of the second mixed moment in the bulk of the spectrum is universal. In the case \( W \ll \sqrt{N} \) the limit is expected to be different from (1.12), but we will not discuss it in this paper.

The paper is organized as follows. In Section 2 we obtain a convenient integral representation for \( F_2 \), using the integration over the Grassmann variables. The method is a generalization of that of [4, 5] and is an analog of the method of [24, 25], where the Hermitian Wigner and general sample covariance matrices were considered. In Section 3 we give the sketch of the proof of Theorem 1. Section 4 deals with the most important preliminary results needed for the proof. In Section 5 we prove Theorem 1 applying the steepest descent method to the integral representation. Section 6 is devoted to the proofs of the auxiliary statements.

### 1.1 Notation

We denote by \( C, C_1, \text{etc.} \) various \( W \) and \( N \)-independent quantities below, which can be different in different formulas. Integrals without limits denote the integration (or the multiple integration) over the whole real axis, or over the Grassmann variables.

Moreover,

- \( N = 2n + 1; \)
- \( J = (-W^2 \Delta + 1)^{-1}; \)
- \( E\{ \ldots \} \) is an expectation with respect to the measure (1.4);
- \( U_\varepsilon(x) = (x - \varepsilon, x + \varepsilon) \subset \mathbb{R}; \)
- \( a_\pm = \pm \sqrt{4 - \lambda_0^2} = \pm \pi \rho(\lambda_0), \quad \bar{a}_\pm = (a_+, \ldots, a_-) \in \mathbb{R}^N, \)  

where \( \rho \) is defined in (1.6);
- \( \Lambda_0 = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \hat{\xi} = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}, \quad L = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}; \)
• \(d\mu\) is the Haar measure on \(U(2)\);
• \(f(x) = (x + i\lambda_0/2)^2/2 - \log(x - i\lambda_0/2),\)
  \[f_*(x) = \Re(f(x) - f(a_\pm));\]

• \(\Omega_\delta\) is a union of
  \[
  \begin{align*}
  \Omega^+_\delta &= \{(a_j), \{b_j\} : a_j, b_j \in U_\delta(a_+) \forall \ j\}, \\
  \Omega^-_\delta &= \{(a_j), \{b_j\} : a_j, b_j \in U_\delta(a-) \forall \ j\}, \\
  \Omega^\pm_\delta &= \{(a_j), \{b_j\} : (a_j \in U_\delta(a_+), b_j \in U_\delta(a_-)) \\
  &\quad \text{or } (a_j \in U_\delta(a_-), b_j \in U_\delta(a_+)) \forall \ j\},
  \end{align*}
  \]

where \(\delta = W^{-\kappa}\) and \(\kappa < \theta/8\).
• \(\Sigma\) is an integral over \(\Omega_\delta\), \(\Sigma_c\) is an integral over its complement, and \(\Sigma\pm, \Sigma_+\) and \(\Sigma_-\)
  are integrals over \(\Omega^\pm_\delta, \Omega^+_\delta\) and \(\Omega^-_\delta\).

• \(c_\pm = 1 - \frac{\lambda_0^2}{4} \pm \frac{i\lambda_0}{2} \cdot \sqrt{1 - \frac{\lambda_0^2}{4}}, \quad c_0 = \Re f(a_+)\); (1.16)
• \(\mu_\gamma(x) = \exp \{-\frac{1}{2} \sum_{j=-n+1}^{n} (x_j - x_{j-1})^2 - \frac{\gamma}{W^2} \sum_{j=-n}^{n} x_j^2\};\) (1.17)

\[
\langle \ldots \rangle_0 = Z_{\delta,\gamma}^{-1} \int_{-\delta W}^{\delta W} (\ldots) \cdot \mu_\gamma(x) \prod_{q=-n}^{n} dx_q, \quad Z_{\delta,\gamma} = \int_{-\delta W}^{\delta W} \mu_\gamma(x) \prod_{q=-n}^{n} dx_q,\]
\[
\langle \ldots \rangle = Z_{\gamma}^{-1} \int (\ldots) \cdot \mu_\gamma(x) \prod_{q=-n}^{n} dx_q, \quad Z_{\gamma} = \int \mu_\gamma(x) \prod_{q=-n}^{n} dx_q,
\]

where \(\delta > 0\) and \(\gamma \in \mathbb{C}, \Re \gamma > 0\);
• \(\langle \ldots \rangle_*\) (and \(\langle \ldots \rangle_{0,*}\)) is (1.18) with \(\mu_{\Re \gamma}(x)\) instead of \(\mu_\gamma(x)\).

## 2 Integral representation

In this section we obtain an integral representation for \(F_2\) of (1.9) by using integration over the Grassmann variables. This method allows us to obtain the formula for the product of characteristic polynomials, which is very useful for the averaging because it is a Gaussian-type integral (see the formula (2.7) below). After averaging over the probability measure we can integrate over the Grassmann variables to get an integral representation (in complex variables) which can be studied by the steepest descent method.

Integration over the Grassmann variables has been introduced by Berezin and is widely used in the physics literature (see e.g. [2, 9, 21]). For the reader’s convenience we give a brief outline of the techniques.
2.1 Grassmann integration

Let us consider two sets of formal variables \( \{ \psi_j \}_{j=1}^n, \{ \overline{\psi}_j \}_{j=1}^n \), which satisfy the anticommutation conditions

\[
\psi_j \psi_k + \psi_k \psi_j = \overline{\psi}_j \psi_k + \psi_k \overline{\psi}_j = 0, \quad j, k = 1, \ldots, n. \tag{2.1}
\]

Note that this definition implies \( \psi_j^2 = \overline{\psi}_j^2 = 0 \). These two sets of variables \( \{ \psi_j \}_{j=1}^n \) and \( \{ \overline{\psi}_j \}_{j=1}^n \) generate the Grassmann algebra \( \mathfrak{A} \). Taking into account that \( \psi_j^2 = 0 \), we have that all elements of \( \mathfrak{A} \) are polynomials of \( \{ \psi_j \}_{j=1}^n \) and \( \{ \overline{\psi}_j \}_{j=1}^n \) of degree at most one in each variable. We can also define functions of the Grassmann variables. Let \( \chi \) be an element of \( \mathfrak{A} \), i.e.

\[
\chi = a + \sum_{j=1}^n (a_j \psi_j + b_j \overline{\psi}_j) + \sum_{j \neq k} (a_{jk} \psi_j \psi_k + b_{jk} \psi_j \overline{\psi}_k + c_{jk} \overline{\psi}_j \overline{\psi}_k) + \ldots. \tag{2.2}
\]

For any sufficiently smooth function \( f \) we define by \( f(\chi) \) the element of \( \mathfrak{A} \) obtained by substituting \( \chi - a \) in the Taylor series of \( f \) at the point \( a \). Since \( \chi \) is a polynomial of \( \{ \psi_j \}_{j=1}^n, \{ \overline{\psi}_j \}_{j=1}^n \) of the form (2.2), according to (2.1), there exists such \( l \) that \( (\chi - a)^l = 0 \), and hence the series terminates after a finite number of terms, and so \( f(\chi) \in \mathfrak{A} \).

For example, we have

\[
\exp\{a \overline{\psi}_1 \psi_1\} = 1 + a \overline{\psi}_1 \psi_1 + (a \overline{\psi}_1 \psi_1)^2/2 + \ldots = 1 + a \overline{\psi}_1 \psi_1;
\]

\[
\exp\{a_{11} \overline{\psi}_1 \psi_1 + a_{12} \overline{\psi}_1 \psi_2 + a_{21} \overline{\psi}_2 \psi_1 + a_{22} \overline{\psi}_2 \psi_2\} = 1 + a_{11} \overline{\psi}_1 \psi_1 + a_{12} \overline{\psi}_1 \psi_2 + a_{21} \overline{\psi}_2 \psi_1 + a_{22} \overline{\psi}_2 \psi_2 \tag{2.3}
\]

Following Berezin [2], we define the operation of integration with respect to the anticommuting variables in a formal way:

\[
\int d \psi_j = \int d \overline{\psi}_j = 0, \quad \int \psi_j d \psi_j = \int \overline{\psi}_j d \overline{\psi}_j = 1, \tag{2.4}
\]

and then extend the definition to the general element of \( \mathfrak{A} \) by the linearity. A multiple integral is defined to be a repeated integral. Assume also that the “differentials” \( d \psi_j \) and \( d \overline{\psi}_k \) anticommute with each other and with the variables \( \psi_j \) and \( \overline{\psi}_k \). Thus, according to the definition, if

\[
f(\psi_1, \ldots, \psi_k) = p_0 + \sum_{j_1=1}^k p_{j_1} \psi_{j_1} + \sum_{j_1 < j_2} p_{j_1,j_2} \psi_{j_1} \psi_{j_2} + \ldots + p_{1,2,\ldots,k} \psi_1 \ldots \psi_k,
\]

then

\[
\int f(\psi_1, \ldots, \psi_k) d \psi_k \ldots d \psi_1 = p_{1,2,\ldots,k}. \tag{2.5}
\]
Let $A$ be an ordinary matrix with a positive Hermitian part. The following Gaussian integral is well-known:

$$
\int \exp \left\{ - \sum_{j,k=1}^{n} A_{j,k} z_j z_k \right\} \prod_{j=1}^{n} \frac{d \Re z_j d \Im z_j}{\pi} = \frac{1}{\det A}. 
$$

(2.6)

One of the important formulas of the Grassmann variables theory is the analog of (2.6) for the Grassmann variables (see [2]):

$$
\int \exp \left\{ - \sum_{j,k=1}^{n} A_{j,k} \bar{\psi}_j \psi_k \right\} \prod_{j=1}^{n} d \bar{\psi}_j d \psi_j = \det A,
$$

(2.7)

where $A$ now is any $n \times n$ matrix.

For $n = 1$ and $n = 2$ this formula follows immediately from (2.3) and (2.5).

Also we will need the Hubbard-Stratonovich transform (see e.g. [26]). This is a well-known simple trick, which is just the Gaussian integration. In the simplest form it looks as following:

$$
e^{a^2/2} = (2\pi)^{-1/2} \int e^{-x^2/2+ax} dx.
$$

(2.8)

Here $a$ can be complex number or the sum of the products of even numbers of Grassmann variables.

**2.2 Formula for $F_2$**

**Lemma 1.** The second mixed moment of the characteristic polynomials for 1D Hermitian Gaussian band matrices, defined in (1.9), can be represented as follows:

$$
F_2 \left( \Lambda_0 + \frac{\xi}{N \rho(\lambda_0)} \right) = - (2\pi)^{-N} \det^{-2} J \int \exp \left\{ - \frac{W^2}{2} \sum_{j=-n+1}^{n} \text{Tr} \left( X_j - X_{j-1} \right)^2 \right\} \prod_{j=-n}^{n} \det \left( X_j - i\Lambda_0/2 \right) \prod_{j=-n}^{n} dX_j,
$$

(2.9)

where $\{X_j\}$ are $2 \times 2$ Hermitian matrices and

$$
dX_j = d(X_j)_{11} d(X_j)_{22} d\Re(X_j)_{12} d\Im(X_j)_{12}.
$$

(2.10)

Moreover, this formula can be rewritten in the form

$$
F_2 \left( \Lambda_0 + \frac{\xi}{N \rho(\lambda_0)} \right) = - \frac{C(\xi) \det^{-2} J}{(4\pi)^N} \int \exp \left\{ - \frac{W^2}{2} \sum_{j=-n+1}^{n} \text{Tr} \left( V_j^* A_j V_j - A_{j-1} \right)^2 \right\} \prod_{j=-n}^{n} \det \left( X_j - i\Lambda_0/2 \right) \prod_{j=-n}^{n} dX_j,
$$

(2.11)

$$
\times \prod_{l=-n}^{n} (a_l - b_l)^2 d \mu(U_{-n}) \prod_{q=-n+1}^{n} d\mu(V_q),
$$

where $\{X_j\}$ are $2 \times 2$ Hermitian matrices and

$$
dX_j = d(X_j)_{11} d(X_j)_{22} d\Re(X_j)_{12} d\Im(X_j)_{12}.
$$

(2.10)
where \( f \) is defined in (1.14), \( A_j = \text{diag}\{a_j, b_j\} \), \( \{V_j\} \) and \( U_{-n} \) are \( 2 \times 2 \) unitary matrices, \( d\mu(U) \) is the Haar measure on \( U(2) \), and

\[
P_k = \prod_{s=k}^{-n+1} V_s, \quad C(\xi) = \exp \left\{ \frac{\lambda_0(\xi_1 + \xi_2)}{2\rho(\lambda_0)} + \frac{\xi_1^2 + \xi_2^2}{2N\rho(\lambda_0)^2} \right\}.
\] (2.12)

Remark 1. Formula (2.9) is valid for any dimension if we change the sum \( \sum \text{Tr}(X_j - X_{j-1})^2 \) to \( \sum \text{Tr}(X_j - X_{j'})^2 \), where the last sum runs over all pairs of nearest neighbor \( j, j' \) in the volume \( L \subset \mathbb{Z}^d \) (see the definition of RBM (1.1) - (1.2)).

Proof. Using (2.7) we obtain

\[
F_2(\Lambda) = \mathbb{E} \left\{ \int e^{-\sum_{\alpha=1}^{2n} \sum_{p=-n}^{n} (\lambda_\alpha - H_N)_{jk} \overline{\psi}_{j\alpha} \psi_{k\alpha}} \prod_{\alpha=1}^{2} \prod_{q=-n}^{n} d\overline{\psi}_{qa} d\psi_{qa} \right\}
\]

\[
= \mathbb{E} \left\{ \int e^{-\sum_{\alpha=1}^{2n} \sum_{p=-n}^{n} \overline{\psi}_{p\alpha} \psi_{p\alpha}} \exp \left\{ \sum_{j<k} \sum_{\alpha=1}^{2} (\Re H_{jk} \cdot (\overline{\psi}_{j\alpha} \psi_{k\alpha} + \overline{\psi}_{k\alpha} \psi_{j\alpha}) + \sum_{j=-n}^{n} H_{jj} \cdot \sum_{\alpha=1}^{2} \overline{\psi}_{j\alpha} \psi_{j\alpha}) \right\} \prod_{\alpha=1}^{2} \prod_{q=-n}^{n} d\overline{\psi}_{qa} d\psi_{qa} \right\},
\] (2.13)

where \( \{\psi_{j\alpha}\}, j = -n, \ldots, n, \alpha = 1, 2 \) are the Grassmann variables (\( 2n + 1 \) variables for each determinant in (1.9)). Here and below we use Greek letters like \( \alpha, \beta \) etc. for the field index and Latin letters \( j, k \) etc. for the position index.

Integrating over the measure (1.4) we get

\[
F_2(\Lambda) = \int \prod_{\alpha=1}^{2} \prod_{q=-n}^{n} d\overline{\psi}_{qa} d\psi_{qa} \exp \left\{ -\sum_{\alpha=1}^{2n} \lambda_\alpha \sum_{p=-n}^{n} \overline{\psi}_{p\alpha} \psi_{p\alpha} \right\}
\]

\[
\times \exp \left\{ \sum_{j<k} J_{jk}(\overline{\psi}_{j1} \psi_{k1} + \overline{\psi}_{j2} \psi_{k2})(\overline{\psi}_{k1} \psi_{j1} + \overline{\psi}_{k2} \psi_{j2}) + \sum_{j=-n}^{n} \frac{J_{jj}}{2}(\overline{\psi}_{j1} \psi_{j1} + \overline{\psi}_{j2} \psi_{j2})^2 \right\}.
\] (2.14)

Applying a couple of times the Hubbard-Stratonovich transform (2.8), we get:

\[
\int \exp \left\{ -\frac{1}{2} \sum_{jk} J_{jk}^{-1} \text{Tr} X_j X_k - i \sum_{j} (\overline{\psi}_{j1} \psi_{j2} X_j (\psi_{j1} \overline{\psi}_{j2}) \right\} \prod_{j=-n}^{n} dX_j
\]

\[
= (2\pi)^N \det^2 J \cdot \exp \left\{ \frac{1}{2} \sum_{jk} J_{jk}(\overline{\psi}_{j1} \psi_{k1} + \overline{\psi}_{j2} \psi_{k2})(\overline{\psi}_{k1} \psi_{j1} + \overline{\psi}_{k2} \psi_{j2}) \right\},
\] (2.15)

where \( X_j \) is Hermitian \( 2 \times 2 \) matrix and \( dX_j \) is defined in (2.10).

Substituting this and (1.2) for \( J_{jk}^{-1} \) into (2.14), putting \( \Lambda = \Lambda_0 + \frac{\hat{\xi}}{N\rho(\lambda_0)} \), and using
(2.7) to integrate over the Grassmann variables, we obtain

\[ F_2 \left( \Lambda_0 + \frac{\xi}{N \rho(\lambda_0)} \right) = (2\pi^2)^{-N} \det^{-2} J \int \exp \left\{ - \frac{W^2}{2} \sum_{j=-n+1}^n \text{Tr} (X_j - X_{j-1})^2 \right\} \]

\[ \times \exp \left\{ - \frac{1}{2} \sum_{j=-n}^n \text{Tr} X_j^2 \right\} \prod_{j=-n}^n \det \left( iX_j + \Lambda_0 + \frac{\xi}{N \rho(\lambda_0)} \right) \prod_{j=-n}^n dX_j \]

\[ = -(2\pi^2)^{-N} \det^{-2} J \int \exp \left\{ - \frac{W^2}{2} \sum_{j=-n+1}^n \text{Tr} (X_j - X_{j-1})^2 - \frac{1}{2} \sum_{j=-n}^n \text{Tr} X_j^2 \right\} \]

\[ \times \prod_{j=-n}^n \det \left( X_j - i\Lambda_0 - i\frac{\xi}{N \rho(\lambda_0)} \right) \prod_{j=-n}^n dX_j, \]

which gives (2.9) after shifting \( X_j \to X_j + i\Lambda_0/2 + i\xi/N \rho(\lambda_0) \). The reason of such a shift is that we need to have saddle-points lying on the contour of the integration (see (1.11) below).

Let us change the variables to \( X_j = U_j^* A_j U_j \), where \( U_j \) is a unitary matrix and \( A_j = \text{diag} \{ a_j, b_j \}, j = -n, \ldots, n \). Then \( dX_j \) of (2.10) becomes (see e.g. [19], Section 3.3)

\[ \frac{\pi}{2} (a_j - b_j)^2 da_j \, db_j \, d\mu(U_j), \]

where \( d\mu(U_j) \) is the normalized to unity Haar measure on the unitary group \( U(2) \). Thus, we have

\[ F_2 \left( \Lambda_0 + \frac{\xi}{N \rho(\lambda_0)} \right) = - \frac{C(\xi) \det^{-2} J}{(4\pi)^N} \int d\vec{a} \, d\vec{b} \int_{U(2)^N} \prod_{j=-n}^n d\mu(U_j) \]

\[ \times \exp \left\{ - \frac{W^2}{2} \sum_{j=-n+1}^n \text{Tr} (U_j^* A_j U_j - U_{j-1}^* A_{j-1} U_{j-1})^2 \right\} \]

\[ \times \exp \left\{ - \frac{1}{2} \sum_{j=-n}^n \text{Tr} \left( A_j + \frac{i\Lambda_0}{2} \right)^2 - \frac{i}{N \rho(\lambda_0)} \sum_{j=-n}^n \text{Tr} U_j^* A_j U_j \xi \right\} \]

\[ \times \prod_{k=-n}^n (a_k - i\lambda_0/2) (b_k - i\lambda_0/2) \prod_{k=-n}^n (a_k - b_k)^2, \]

where

\[ \vec{a} = \prod_{j=-n}^n da_j, \quad \vec{b} = \prod_{j=-n}^n db_j, \quad C(\xi) = \exp \{ \lambda_0 (\xi_1 + \xi_2) / 2 \rho(\lambda_0) \}. \]  \hfill (2.16)

Now changing the “angle variables” \( U_j \) to \( V_j = U_j U_{j-1}^* \), \( j = -n + 1, \ldots, n \) (i.e. the new variables are \( U_{-n}, V_{-n+1}, V_{n+2}, \ldots, V_n \)), we get (2.11).

\[ \square \]

3 Sketch of the proof of Theorem 1

The strategy of the proof is the following.
First we will study the function \( f \) and find that expected saddle-points for each \( a_j \) and \( b_j \) are \( a_{\pm} \), which are defined in (1.1). This will be done in Section 4.1.

The second step is to prove that the main contribution to the integral (2.11) is given by \( \Sigma \), i.e. by the integral over \( \Omega_\delta \) (see (1.13)). More precisely, we are going to prove that

\[
F_2 \left( A_0 + \frac{\xi}{N\rho(\lambda_0)} \right) = -\frac{C(\xi) \det^{-2} J}{(4\pi)^N} \cdot \Sigma \cdot (1 + o(1)), \quad W \to \infty. \tag{3.1}
\]

The bound for the complement \( |\Sigma_c| \) can be obtained by inserting the absolute value inside the integral and by performing exactly the integral over the unitary groups. After this, since we are far from the saddle-points of \( f \), one can control the integral. This will be done in Lemma 9, Section 5.1.

The next step is the calculation of \( \Sigma \) (see Section 5.2, Lemma 10). First note that shifting

\[
U_j \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_j
\]

for some \( j \), we can rotate each domain of type

\[
\{\{a_j\}, \{b_j\} : (a_j \in U_\delta(a_+), \ b_j \in U_\delta(a_-)) \text{ or } (a_j \in U_\delta(a_-), \ b_j \in U_\delta(a_+)) \forall j \}
\]

to the \( \delta \)-neighborhood of the point \((\bar{\sigma}_+, \bar{\sigma}_-)\) with \( \bar{\sigma}_\pm \) of (1.1). Thus, we can consider the contribution over \( \Omega_\delta^\pm \) as \( 2^N \) contributions of the \( \delta \)-neighborhood of the point \((\bar{\sigma}_+, \bar{\sigma}_-)\). Consider this neighborhood (or the neighborhoods of the points \( a_j = b_j = a_+ \) or \( a_\pm = b_j = a_- \) for \( \Omega_\delta^+ \) or \( \Omega_\delta^- \) correspondingly), and change the variables as

\[
a_j \to a_+ + \tilde{a}_j/W, \quad |\tilde{a}_j| \leq \delta W, \tag{3.2}\\
b_j \to a_- + \tilde{b}_j/W, \quad |\tilde{b}_j| \leq \delta W,
\]

and set \( \tilde{A}_j = \text{diag}\{\tilde{a}_j, \tilde{b}_j\} \). To compute \( \Sigma \), one has to perform first the integral over the unitary groups. This integral is some analytic in \( \{\tilde{a}_j/W\}, \{\tilde{b}_j/W\} \) function. The main idea is to prove that the leading part of this function can be obtained by replacing all \( V_s \) in the “bad” term

\[
\exp \left\{ -\frac{i}{N\rho(\lambda_0)} \sum_{j=-n}^n \text{Tr} \left( \prod_{s=j}^{n+1} V_s \cdot U_{-n} \right)^*(L + \tilde{A}_s/W) \left( \prod_{s=j}^{n+1} V_s \cdot U_{-n} \right) \xi \right\}
\]

with \( I \). To this end, we expand the “bad” term into the series and for each summand, which is analytic in \( \{\tilde{a}_j/W\}, \{\tilde{b}_j/W\} \), find the bound for its Taylor coefficients (see Lemma 12).

To integrate with respect to \( \{\tilde{a}_j\}, \{\tilde{b}_j\} \), we expand

\[
f(x) - f(a_\pm) = c_\pm(x - a_\pm)^2 + \varepsilon_3(x - a_\pm)^3 + \ldots = c_\pm(x - a_\pm)^2 + \varphi_\pm(x - a_\pm),
\]

\[
c_\pm = 1 - \frac{\lambda_0^2}{4} \pm \frac{i\lambda_0}{2} \sqrt{1 - \frac{\lambda_0^2}{4}}, \tag{3.3}
\]

and then leave only the quadratic form in the exponent in (2.11). At this step we will face with a problem to study a complex valued Gaussian \((2n + 1)\)-dimensional distribution (1.17) with \( \gamma \in \mathbb{C}, \Re \gamma > 0 \) (in our case \( \gamma = c_+ \) or \( c_- \)). The properties of the measure \( \mu \) will be studied in Section 4.2. The most important steps here are:
• to prove that
\[
\left\langle \exp \left\{ \sum_{j=-n}^{n} \varphi(x_j/W) \right\} - 1 \right\rangle_0 = o(1),
\]
where \( \varphi \) is any analytic in the neighborhood of 0 function whose Taylor expansion starts from the third order, and \( \langle \ldots \rangle_0 \) is defined in (1.18) (this will be done in Lemma 5);

• to prove that if for some function the absolute value of each coefficient of its Taylor expansion does not exceed the corresponding coefficient of the other function (majorant), then we can estimate the averaging of the first function over the complex measure by the averaging of the majorant over the positive one (see Lemma 8). This helps to integrate the function obtained after the integration over the unitary groups, since the proper majorant can be found.

These arguments show that the leading term of \( \Sigma_\pm \) is the integral over the Gaussian measures \( \mu_{c_\pm} \) in \( \{a_j\} \) and \( \{b_j\} \) variables, and the integral over the unitary group \( d\mu(U_{-n}) \) which gives the sine-kernel. This gives an asymptotic expression for \( \Sigma_\pm \) (see Lemma 11, (5.22) – (5.23)).

It will be also shown in Section 5.2.2 that the integrals \( \Sigma_+ \) and \( \Sigma_- \) over \( \Omega^{+}_\delta \) and \( \Omega^{-}_\delta \) have smaller orders than \( \Sigma_\pm \).

### 4 Preliminary results

#### 4.1 Saddle-point analysis for \( f \) of (1.14)

Considering zeros of the first derivative of the function \( f \) of (1.14), we find that the expected saddle-points are \( a_\pm \), which are defined in (1.1).

We can write in the small neighborhood of \( a_\pm \)
\[
f(x) - f(a_\pm) = c_\pm(x - a_\pm)^2 + s_3(x - a_\pm)^3 + \ldots = c_\pm(x - a_\pm)^2 + \varphi_\pm(x - a_\pm),
\]
where \( |\varphi_\pm(x - a_\pm)| = O(|x - a_\pm|^3) \).

Let us also study
\[
f^*(x) = \Re(f(x) - f(a_\pm)) = \frac{1}{2}(x^2 - \lambda_0^2/4 - \log(x^2 + \lambda_0^2/4)) - c_0,
\]
\[
c_0 = \Re f(a_\pm) = 1/2 - \lambda_0^2/4.
\]

We need

**Lemma 2.** The function \( f^*(x) \) for \( x \in \mathbb{R} \) attains its minimum at \( x = a_\pm \), where \( a_\pm \) is defined in (1.1). Moreover, \( f^*(a_\pm) = 0 \) and if \( x \not\in U_\delta(a_\pm) := (a_\pm - \delta, a_\pm + \delta) \) for sufficiently small \( \delta > 0 \), then
\[
f^*(x) \geq C\delta^2.
\]
In addition, we have for $x \in (-\infty, \delta)$
\[ f_*(x) \geq \alpha (x - a_-)^2, \quad (4.3) \]
where $\alpha$ is some positive constant. A similar inequality holds for $x \in (-\delta, +\infty)$ (with $a_+$ instead of $a_-$).

The proof of this simple lemma can be found in Section 6.

### 4.2 Analysis of the measure $\mu_\gamma$

In this section we study the properties of the complex Gaussian distribution $\mu_\gamma$ defined in (1.17). Set

\[ \mu_\gamma^{(m)}(x) = \exp \left\{ -\frac{1}{2} \sum_{j=2}^{m} (x_j - x_{j-1})^2 - \frac{\gamma}{W^2} \sum_{j=1}^{m} x_j^2 \right\}. \quad (4.4) \]

**Lemma 3.** We have for any $\gamma \in \mathbb{C}, \Re \gamma > 0$

1. \[ Z_\gamma^{(m)} := \int \mu_\gamma^{(m)}(x) \prod_{q=1}^{m} dx_q = (2\pi)^{m/2} \det^{-1/2} (-\Delta + 2\gamma/W^2) \quad (4.5) \]
\[ = (2\pi)^{m/2} \left( \frac{\sqrt{2\gamma}}{W} \sinh \frac{m\sqrt{2\gamma}}{W} \right)^{-1/2} (1 + o(1)) \]

Moreover, if we set
\[ G^{(m)}(\gamma) = \left( -\Delta + \frac{2\gamma}{W^2} \right)^{-1}, \quad (4.6) \]
then
\[ |G^{(m)}_{ii}(\gamma)| \leq \frac{C_1 W}{\sqrt{2\gamma}} \coth \frac{m\sqrt{2\gamma}}{W} (1 + o(1)). \quad (4.7) \]

2. \[ \frac{|Z_\gamma^{(m)} - Z_{\delta,\gamma}^{(m)}|}{|Z_\gamma^{(m)}|} := \left| Z_\gamma^{(m)} \right|^{-1} \int_{\max|x_i|>\delta W} \mu_\gamma^{(m)}(x) \prod_{q=1}^{m} dx_q \leq C_1 e^{-C_2 \delta^2 W}, \quad W \to \infty, \]
where $m > CW$, $\delta = W^{-\kappa}$ for sufficiently small $\kappa < \theta/8$, and
\[ Z_{\delta,\gamma}^{(m)} = \int_{-\delta W}^{\delta W} \mu_\gamma^{(m)}(x) \prod_{q=1}^{m} dx_q. \]

In addition, for any $m$
\[ \left| Z_\gamma^{(m)} \right|^{-1} \int_{|x_k-x_1|>\delta W} \mu_\gamma^{(m)}(x) \prod_{q=1}^{m} dx_q \leq C_1 e^{-C_2 \delta^2 W}, \quad W \to \infty, \]
and for $m > CW$ and any $\gamma_1, \gamma_2 \in \mathbb{C}, \Re \gamma_1, \Re \gamma_2 > 0$
\[ \frac{|Z_{\gamma_1}^{(m)}|}{|Z_{\gamma_2}^{(m)}|} \leq e^{C_1 m/W}, \quad W \to \infty. \quad (4.8) \]
Let \( m > C_1 W \), \( k \leq C m / W \), \( S = \{i_1, \ldots, i_s\} \subset \{1, \ldots, m\} \), and \( \sum_{l=1}^{s} k_{i_l} = 3k \), where \( k_l \in \{3, \ldots, k\} \). Then

\[
|Z^{(m)}|^{-1} \left| \int_{\max|x_i|>\delta W} \prod_{j \in S} (x_j/W)^{k_j} \cdot \mu_{\gamma}^{(m)}(x) \prod_{q=1}^{m} dx_q \right| \leq e^{-C_1 \delta^2 W}, \quad W \to \infty,
\]

where \( \delta = W^{-\kappa} \) for sufficiently small \( \kappa < \theta/8 \).

The proof of the lemma is rather standard and can be found in Section 6.

Let us study the properties of the averages of (1.18), where \( \delta = W^{-\kappa}, \kappa < \theta/8 \).

We will use below the following form of the Wick theorem:

**Lemma 4.** (i) For any smooth function \( f \)

\[
\langle x_{i_1} f(x_{i_1}, \ldots, x_{i_p}) \rangle = \sum_{j=1}^{p} \langle x_{i_1} \rangle \langle \partial f(x_{i_1}, \ldots, x_{i_p})/\partial x_{i_j} \rangle. \tag{4.9}
\]

The same is valid for \( \langle \ldots \rangle^* \), where \( \langle \ldots \rangle, \langle \ldots \rangle^* \) are defined in (1.18).

(ii) \n
\[
|\langle x_{i_1}^{k_1} \ldots x_{i_l}^{k_l} \rangle| \leq \langle x_{i_1}^{k_1} \ldots x_{i_l}^{k_l} \rangle^*. \tag{4.10}
\]

**Proof.** The first part of the lemma is well-known Wick’s theorem, which can be easily proven using the integration by parts.

To prove the second part set

\[
M = -\Delta + \gamma/W^2 = (2 + \gamma/W^2)I - \tilde{M}, \quad M_* = -\Delta + \Re \gamma/W^2 = (2 + \Re \gamma/W^2)I - \tilde{M}, \tag{4.11}
\]

where \( \tilde{M} = \Delta + 2I \). Then

\[
\langle x_i x_j \rangle = (M^{-1})_{ij}, \quad \langle x_i x_j \rangle_* = (M_*^{-1})_{ij}.
\]

Besides, since all entries of \( \tilde{M} \) are positive and \( \Re \gamma > 0 \),

\[
|\langle (M^{-1})_{ij} \rangle| = \left| \sum_{k=0}^{\infty} \frac{(\tilde{M}^k)_{ij}}{(2 + \gamma/W^2)^{k+1}} \right| \leq \sum_{k=0}^{\infty} \frac{(\tilde{M}^k)_{ij}}{|2 + \gamma/W^2|^{k+1}} \leq \sum_{k=0}^{\infty} \frac{(\tilde{M}^k)_{ij}}{(2 + \Re \gamma/W^2)^{k+1}} = (M_*^{-1})_{ij}.
\]

This and (4.9) yield (ii). \( \square \)

To leave only the quadratic form of (3.3) in the exponent in (2.11), we have to prove (3.3). This can be done using three ideas: (1) we can replace \( \langle \ldots \rangle_0 \) by \( \langle \ldots \rangle \) with an error which we can control; (2) using (4.10) we can estimate the averaging of some function over the complex measure by the averaging of the “changed” function (which means that we replace all coefficients in the Taylor expansion of the function by its absolute values)
over the positive measure (we take $\Re c_\pm$ instead of $c_\pm$); (3) using Wick’s theorem (4.9) we can prove (3.4) for the positive measure (see Lemma 3).

Define

$$E_n[g] := \exp\left\{-\sum_{j=-n}^{n} g(x_j/W)\right\} \quad (4.12)$$

for any function $g : \mathbb{R} \to \mathbb{C}$.

Then (3.4) can be rewritten in the form

Lemma 5. For $E_n$ of (4.12) we have

$$|\langle E_n[\varphi_{\pm}] \rangle_0 - 1| = o(1), \quad (4.13)$$

where $\varphi_{\pm}$ are defined in (3.3).

The key point in the proof of Lemma 5 is

Lemma 6. Let $g$ be a polynomial of degree $q$ with real coefficients starting from the third power, i.e. $g(x) = \sum_{j=3}^{q} c_j x^j$, $c_j \in \mathbb{R}$. Then we have

$$|\langle E_n[g] \rangle_0 - 1| = o(1), \quad n \to \infty. \quad (4.14)$$

Proof. The lower bound.

Since $e^x - 1 \geq x$, we have

$$\langle E_n[g] \rangle_0 - 1 \geq \left\langle \sum_{j=-n}^{n} g(x_j/W)\right\rangle_0 = \left\langle \sum_{j=-n}^{n} g(x_j/W)\right\rangle + o(1),$$

where we use the third assertion of Lemma 3 in the last equality. Using Wick’s theorem (4.9) and $(M_s^{-1})_{ii} = CW$ (see the assertion (1) of Lemma 3), we can write

$$\langle (x_j/W)^{2l} \rangle_s = O(W^{-l}),$$

and hence

$$\langle \sum_{j=-n}^{n} g(x_j/W)\rangle_s = O((2n + 1)/W^2) = o(1).$$

The upper bound.

Let us prove that

$$\langle E_n[g] \rangle_0 - 1 \leq \varepsilon_{1,n} \langle E_n[g] \rangle_0,$$ 

which implies

$$\langle E_n[g] \rangle_0 - 1 \leq 2\varepsilon_{1,n},$$

where $\varepsilon_{1,n} = o(1)$, as $n \to \infty$.

Step 1. Replacing $\langle \ldots \rangle_0$ with $\langle \ldots \rangle_s$

Note that if we choose $s_\kappa > 3$ such that (recall that $\delta = W^{-\kappa}$, $\kappa < \theta/8$)

$$W^{-\kappa s_\kappa} \leq W^{-2}, \quad (4.16)$$

then...
then for any \( p > s_\kappa/2 \) and for \( x_j \in (-\delta W, \delta W) \)
\[
\sum_{j=-n}^n (x_j/W)^{2p} < N/W^2 = o(1),
\]
and thus if we replace \( g(x) \) by \( g(x) + C x^{2p} \) with any \( C \), then \( E_n[g] \) will be changed by \( E_n[g](1+o(1)) \). Since it is easy to see that we can choose \( C \) such that \( c_0 x^2/2 + g(x) + C x^{2p} \) has only one minimum \( x = 0 \) in \( \mathbb{R} \), without loss of the generality we can assume that \( c_0 x^2/2 + g(x) \geq c_0 x^2/4 \). Moreover, \( c_0 x^2/2 + g(x) \leq c_0 x^2 \) for \( x \in (-\delta, \delta) \). This and assertions (1), (2) of Lemma 3 give
\[
\int_{\max |x| \leq \delta W} E_n[g] \mu_{c_0}(x) dx \leq \int_{\max |x| \leq \delta W} \mu_{c_0/2}(x) dx \leq C_n/\theta = o(1),
\]
because \( \delta = W^{-\kappa} \) with \( \kappa < \theta/8 \). Thus,
\[
\left< E_n[g] \right>_{0, \ast} = \left< E_n[g] \right>_{\ast} + o(1).
\]

(4.17)

**Step 2. Application of Wick’s theorem (Lemma 4 (i))**

Since for \( x \in \mathbb{R} \)
\[
e^x \leq 1 + xe^x,
\]
we can write using Wick’s theorem (4.9)
\[
\left< E_n[g] \right>_{\ast} - 1 \leq \sum_{i_1} \left< g(x_{i_1}/W) \cdot E_n[g] \right>_{\ast} = \sum_{i_1} \sum_{l=3}^q \left< c_l x_{i_1}^l \cdot E_n[g] \right>_{\ast}
\]
\[
\leq \sum_{i_1} \sum_{l=3}^q \frac{(l-1)c_l}{W^2} \left< x_{i_1}^{l-2} \cdot E_n[g] \right>_{\ast}
\]
\[
+ \sum_{i_1, i_2} \sum_{l=3}^q \frac{c_l}{W^2} \left< x_{i_1}^{l-1} \cdot g \left( \frac{x_{i_2}}{W} \right) \cdot E_n[g] \right>_{\ast}
\]
\[
\leq \sum_{i_1} \sum_{l=4}^q \frac{(l-1)(l-3)c_l}{W^4} \left< x_{i_1}^{l-4} \cdot E_n[g] \right>_{\ast}
\]
\[
+ \sum_{i_1, i_2} \sum_{l=3}^q \frac{(2l-3)c_l}{W^4} \left< x_{i_1}^{l-3} \cdot g \left( \frac{x_{i_2}}{W} \right) \cdot E_n[g] \right>_{\ast}
\]
\[
+ \sum_{i_1, i_2} \sum_{l=3}^q \frac{c_l}{W^4} \left< x_{i_1}^{l-2} \cdot g'' \left( \frac{x_{i_2}}{W} \right) \cdot E_n[g] \right>_{\ast}
\]
\[
+ \sum_{i_1, i_2, i_3} \sum_{l=3}^q \frac{c_l}{W^4} \left< x_{i_1}^{l-2} \cdot g' \left( \frac{x_{i_2}}{W} \right) g' \left( \frac{x_{i_3}}{W} \right) \cdot E_n[g] \right>_{\ast} = \ldots
\]
For every term \( \langle x_{i_1}^{m_1} \ldots x_{i_k}^{m_k} E_n[g] \rangle_s \), we take \( x_{i_l} \) with the smallest index \( l \) and find its pair according to (4.9). We repeat this procedure until we get \( \langle E_n[g] \rangle_s \) or until the number of steps becomes bigger than \( s_k \), where \( s_k \) is defined in (4.16). All terms have the form

\[
\sum_{i_1, \ldots, i_{p+l}} G(x_{i_1}, \ldots, x_{i_{p+l}}) \left\langle \frac{x_{i_{p+1}}^{\alpha_{p+1}} x_{i_{p+2}}^{\alpha_{p+2}} \ldots x_{i_{p+l}}^{\alpha_{p+l}}}{W^\alpha} E_n[g] \right\rangle_s,
\]

where \( \alpha_{p+1}, \ldots, \alpha_{p+l} \in \mathbb{N} \) are bounded by some absolute constant (since in any case we make a finite number of steps), \( \alpha = \alpha_{p+1} + \ldots + \alpha_{p+l} \). Here \( G(x_{i_1}, \ldots, x_{i_{p+l}}) \) is the product of the expectations of some pairing \( x_{i_1}^{k_1} x_{i_2}^{k_2} \ldots x_{i_{p+l}}^{k_{p+l}} \) with \( k_j \geq 1, j = 1, \ldots, p \) and \( k_j \geq 1, j = p + 1, \ldots, p + l \) (all \( \{k_j\} \) are bounded by some absolute constant) divided by \( W^{k_1+\ldots+k_{p+l}} \), with some bounded positive coefficient.

We can visualize these pairings as connected multigraphs (i.e. graphs which may contain multiple edges and loops) with vertices \( i_1, \ldots, i_{p+l} \), where \( p + l \leq s_k \). The degree of \( i_j \) is at least 3 for \( j \leq p \) and is at least 1 for \( j = p + 1, \ldots, p + l \). The multigraphs are connected, since one can proceed to a different connected component only if we obtained \( \langle E_n[g] \rangle_s \) before.

Let \( H \) be one of such multigraphs. Any \( \langle x_{i} x_{j} \rangle_s \) gives \( (M_{ij}^{-1})_{ij} \). Thus, any loop gives a factor \( (M_{ij}^{-1})_{ij} = CW(1 + o(1)) \) (see the assertion (1) of Lemma 3). Moreover, according to the Cauchy-Schwarz inequality, we have

\[
(M_{ij}^{-1})_{ij} \leq (M_{ii}^{-1})_{ij}^{1/2} (M_{jj}^{-1})_{ij}^{1/2}.
\]

Hence, we can remove the edge \((j_1, j_2)\) from any cycle \((j_1, j_2, \ldots, j_r, j_1)\) \( (r \neq 1) \) and replace it with two semiloops \((j_1, j_1), (j_2, j_2)\) (a “semiloop” is a loop counted with the coefficient \(1/2\), i.e. the contribution of a semiloop is \((M_{ij}^{-1})_{ii}^{1/2}\) instead of \( (M_{ij}^{-1})_{ii} \); one semiloop adds 1 to the degree of the vertex, and two semiloops add up to one loop.) In this way we transform the multigraph \( H \) to a tree \( H_0 \) with some loops and semiloops (the degree of each vertex is still the same as in \( H \)).

Since we make a finite number of steps, there is only a finite number of graphs \( H \) such that corresponding graphs \( H_0 \) are equal to each other. Hence, we can consider the sum over \( H_0 \) instead of \( H \). Let \( G_0(x_{i_1}, \ldots, x_{i_{p+l}}) \) be the function, which corresponds to the new graph \( H_0 \).

Note that, according to (4.17),

\[
\left| \left\langle \frac{x_{i_{p+1}}^{\alpha_{p+1}} x_{i_{p+2}}^{\alpha_{p+2}} \ldots x_{i_{p+l}}^{\alpha_{p+l}}}{W^\alpha} E_n[g] \right\rangle_s \right| \leq \left| \left\langle \frac{x_{i_{p+1}}^{\alpha_{p+1}} x_{i_{p+2}}^{\alpha_{p+2}} \ldots x_{i_{p+l}}^{\alpha_{p+l}}}{W^\alpha} E_n[g] \right\rangle_{0,s} + o(1) \right|
\]

\[
\leq \frac{1}{W^{\kappa s}} \left\langle E_n[g] \right\rangle_{0,s} + o(1) = \frac{1}{W^{\kappa s}} \langle E_n[g] \rangle_s + o(1).
\]

Therefore, we are left to prove

**Lemma 7.** Let \( H_0 \) be a tree with loops and semiloops, whose vertices \( i_1, \ldots, i_{p+l} \) admit the following condition: the degree of each vertex \( i_j \) is at least 3 for \( j \leq p \) and is at least 1 for \( j = p + 1, \ldots, p + l \). Denote by \( m \) the sum of degrees of all vertices and let also \( G_0(x_{i_1}, \ldots, x_{i_k}) \) be the function of the pairing, which corresponds to \( H_0 \). Then we have

\[
\sum_{i_1, \ldots, i_{p+l}} G_0(x_{i_1}, \ldots, x_{i_{p+l}}) \leq N/W^{m/2-(p+l)+1}.
\]
Moreover,
\[
\sum_{i_1, \ldots, i_{p+l}} W^{-\alpha} G_0(x_{i_1}, \ldots, x_{i_{p+l}}) = o(1). \tag{4.19}
\]

**Proof.** Since \((1, \ldots, 1)\) is an eigenvector for \(M_*\) of (4.11) with eigenvalue \(W^2/\Re\gamma\), we have
\[
\sum_j (M_*^{-1})_{ij} = \frac{W^2}{\Re\gamma}, \quad i = -n, \ldots, n. \tag{4.20}
\]

Let us consider the sum over \(i_1, \ldots, i_{p+l}\). Any loop or semiloop gives \(W\) or \(W_1/2\) respectively. Thus, since the tree has \(p + l - 1\) edges, all loops and semiloops give the contribution \(W_{m/2} - (p+l)+1\). Using (4.20), we obtain that the contribution of the tree edges is \(N \cdot W^{m/2} - (p+l)+1\). Evidently \(m\) is even, and hence \(m/2 - p + 1\) is integer.

Now let us prove (4.19). Consider two cases

1. **The case** \(l = 0\).

   First consider the case when we get \(\langle E_n[g]\rangle_*\) at some step. Then \(l = 0\) and \(H_0\) is a tree with \(p\) vertices and some loops and semiloops, where the degree of each vertex is at least 3. Hence, \(m \geq 3p\), where \(m\) is the sum of all degrees, and thus \(m/2 - p + 1 \geq m/6 + 1 > 1\), which means \(m/2 - p + 1 \geq 2\). Therefore, \(N/W^{m/2-k+1} = N/W^2 = o(1)\), and hence (4.18) implies (4.19).

2. **The case** \(l > 0\).

   Let now \(l > 0\). Set \(k = p + l\). Using (4.18), we get that the sum over all vertices is not greater than \(N/W^{m/2-k+1}\). Since \(H_0\) is a connected graph, we have \(m \geq 2(k-1)\), i.e. \(m/2 - k + 1 \geq 0\). The sum in (4.19) has a factor \(W^{-\alpha}\). In addition, \(m + \alpha \geq 3k\) and \(m = 2s_\kappa\), since we did \(s_\kappa\) steps and thus obtained \(s_\kappa\) edges. If \(m/2 - k + 1 \geq 2\), then the sum can be bounded by \(W^{-\alpha} N/W^2 = o(1)\). Hence, we are left to consider the case \(0 \leq m/2 - k + 1 \leq 1\), i.e. \(2(k-1) \leq m \leq 2k\). Therefore, we get \(k \geq s_\kappa\) (because \(m = 2s_\kappa\)), thus

   \[
   2s_\kappa + \alpha = m + \alpha \geq 3k \geq 3s_\kappa,
   \]

   which implies \(\alpha > s_\kappa\). Hence, the sum in (4.19) is bounded by

   \[
   W^{-\alpha} N/W^{m/2-k+1} \leq N/W^{s_\kappa} = o(1),
   \]

   which gives (4.19).

   Now (4.19) implies (4.15) with \(\langle \ldots \rangle_*\) and hence with \(\langle \ldots \rangle_{0,*}\) (see (4.17)).

**Proof of Lemma 5.**

We can write for \(x \in (-\delta, \delta)\)

\[
\check{\varphi}_\pm(x) := \exp\{-\varphi_\pm(x)\} - 1 = \sum_{l=3}^\infty \phi_l x^l,
\]
where \(|\phi_t| \leq (C_0)^t\). Thus,

\[
|\langle E_n[\varphi_{\pm}] \rangle_0 - 1| = \left| \left\langle \prod_{j=-n}^{n} (1 + \sum_{l=3}^{\infty} \phi_l x^l) \right\rangle_0 - 1 \right| = \left| \sum_{k=3}^{\infty} \Sigma_k^0 \right|,
\]

(4.21)

where \(\Sigma_k^0, \Sigma_k\) are the sums of all terms \(\langle \prod_{i=1}^{s} (\phi_{k_1} x_{i_1}^{k_1}/W^{k_1}) \rangle_0\) and \(\langle \prod_{i=1}^{s} (\phi_{k_1} x_{i_1}^{k_1}/W^{k_1}) \rangle\) respectively with \(k_1 + \ldots + k_s = k\), \(k_i \in \{3, \ldots, k\}\) (\(\Sigma_{k,s}^0, \Sigma_{k,s}\) are defined by the same way with \(\langle \ldots \rangle_s\) instead of \(\langle \ldots \rangle\) and with \(|\phi_t|\) instead of \(\phi_t\)).

Also denote

\[
S_s^0 = \sum_{i_1 < \ldots < i_s} \langle x_{i_1}/W \rangle_3^{3*},
\]

According to (4.8), we have

\[
|\langle (\phi_{k_1} x_{i_1}^{k_1}/W^{k_1}) \ldots (\phi_{k_s} x_{i_s}^{k_s}/W^{k_s}) \rangle_0| \leq (C_0)^k \delta^{k-3s} e^{Cn/W} \langle \prod_{l=1}^{s} |x_{i_l}/W|^3 \rangle_0,\]

Hence, since the number of partitions of \(k\) to \(s\) non-zero summands is not greater than \(\binom{k}{s}\), we obtain

\[
|\Sigma_k^0| \leq e^{Cn/W} (C_0)^k \sum_{s=1}^{k/3} \binom{k}{s} \delta^{k-3s} S_s^0 \leq e^{Cn/W} (2C_0)^k \sum_{s=1}^{k/3} \delta^{k-3s} S_s^0.
\]

(4.22)

Note now that

\[
|x|^3 \leq \frac{p^{-1}x^2 + px^4}{2},
\]

(4.23)

and hence, again according to (4.8), we get for any \(p > 0\)

\[
S_s^0 \leq \sum_{i_1 < \ldots < i_s} \left\langle \prod_{l=1}^{s} \frac{p^{-1}x_{i_l}^2/W^2 + px_{i_l}^4/W^4}{2} \right\rangle_0 := \tilde{S}_s^0.
\]

(4.24)

Besides,

\[
1 + q \cdot \frac{p^{-1}x^2 + px^4}{2} \leq (1 + \frac{qx^2}{2p})(1 + \frac{px^4}{2}) \leq e^{qx^2/2p}(1 + \frac{px^4}{2}),
\]

and thus, taking into account (4.8), we have for any \(p, q > 0\) such that \(q/p < c_0\) with \(c_0\) of (4.1)

\[
1 + \sum_{k=1}^{2n+1} q^k \tilde{S}_k^0 = \left\langle \prod_{j=-n}^{n} \left(1 + q \cdot \frac{p^{-1}x_{j}^2/W^2 + px_{j}^4/W^4}{2} \right) \right\rangle_0,\]

\[
\leq \left\langle e^{q/2p \sum_{j} x_{j}^2/W^2} \prod_{j=-n}^{n} \left(1 + \frac{px_{j}^4/W^4}{2} \right) \right\rangle_0,\]

\[
\leq e^{C_{p,q}n/W} \langle \prod_{j=-n}^{n} \left(1 + \frac{px_{j}^4/W^4}{2} \right) \rangle_0,\]

\[
\leq Ce^{C_{p,q}n/W},
\]

19
where the last inequality holds in view of Lemma 6 \((\ldots)_{0,c_0-q/p} \text{ means } (1.18) \text{ with } \gamma = c_0 - q/p\). This gives
\[
\tilde{S}_k^0 \leq e^{C_p,qn/W} q^{-k},
\]
and we have from (4.24) for \(k > C_n/W\) with sufficiently big \(C\)
\[
S_k^0 \leq e^{(C_{p,q}+C_1)n/W} q^{-k}.
\] (4.25)

Take \(q > (2|C_0|e)^3\). Then (4.26) and (4.25) yield for \(k > C_1 n/W\)
\[
|\Sigma_k^0| \leq e^{C_n/W} \sum_{s=1}^{k/3} (2C_0)^k \delta^{k-3s} q^{-s} \leq e^{C_n/W} (2C_0)^k \sum_{s=1}^{k/3} (\delta^3 q)^{k/3-s} \leq 2 e^{C_n/W-k}. \] (4.26)

This and (4.21) imply
\[
|\langle E_n[\varphi_{\pm}]_0 - 1 \rangle| \leq \left| \sum_{k=3}^{C_n/W} \Sigma_k^0 \right| + e^{-C_1 n/W}. \] (4.27)

Taking into account that the number of distributions of \(k\) items into \(n\) boxes is \(\binom{n+k-1}{k}\) and using the assertion (3) of Lemma 3 we get
\[
|Z_\gamma|^{-1} \left| \int_{\max_{|x_i| > \delta W}} \sum_{k=1}^{k/3} \sum_{s=1}^{k/3} \sum_{i_1, \ldots, i_s} \left| \frac{x_1^{k_1} \cdots x_s^{k_s}}{W^k} \right| \mu_\gamma(x) dx \right| \leq e^{-C_3^2 W/4} \left( n + \frac{k-1}{k} \right) \leq e^{2k \log(n/k) - C_3^2 W} \leq e^{-C_3^2 W/4},
\]
where the second sum in the first line is over all collections \(\{k_i\}_{i=1}^s, \sum k_i = k, k_i \in \{3, \ldots, k\}\). This yields
\[
\Sigma_k = \Sigma_k^0 + e^{-C_3^2 W/4}, \quad \Sigma_{k,*} = \Sigma_{k,*}^0 + e^{-C_3^2 W/4}, \quad k \leq C_n/W,
\]
and thus by (4.10) we have
\[
\left| \sum_{k=1}^{C_n/W} \Sigma_k^0 \right| = \left| \sum_{k=1}^{C_n/W} \Sigma_k \right| + e^{-C_3^2 W/4} \leq \sum_{k=1}^{C_n/W} \Sigma_{k,*} + e^{-C_3^2 W/4} \] (4.28)
\[
\leq \sum_{k=1}^{C_n/W} \Sigma_{k,*} + 2e^{-C_3^2 W/4} \leq (1 + \sum_{l=3}^{\infty} |\phi_l| x^{l}/W^l) \left( \prod_{i=-n}^{n} (1 + \sum_{l=3}^{\infty} |\phi_l| x^{l}/W^l) \right)_{0,*} + 2e^{-C_3^2 W/4}.
\]

Since \(|\phi_l| \leq (C_0)^l\), there exists \(C\) such that
\[
1 + \sum_{l=3}^{\infty} |\phi_l| x^{l}/W^l \leq e^{C(x^3/W^3 + x^4/W^4)}, \quad x \in (-\delta W, \delta W). \] (4.29)

This, Lemma 6 and (4.27) yield (4.13). □
Define the following partial ordering. Let $\Phi_1(x_1, \ldots, x_n)$, $\Phi_2(x_1, \ldots, x_n)$ be two analytic functions in some ball centered at 0, and let the coefficients of the Taylor expansion of $\Phi_2$ be non-negative. Then we write

$$
\Phi_1 \prec \Phi_2
$$

(4.30)

if the absolute value of each coefficient of the Taylor expansion of $\Phi_1$ does not exceed the corresponding coefficient of $\Phi_2$.

It is easy to see that

$$
\Phi_3 \prec \Phi_1, \quad \Phi_4 \prec \Phi_2 \Rightarrow \Phi_3 \Phi_4 \prec \Phi_1 \Phi_2.
$$

(4.31)

We will need

**Lemma 8.** (i) Let $|\phi_1| \leq CW^{-1}$, $|\phi_2| = o(1)$ and $|\phi_k| \leq C_k$ for some absolute constant $C > 0$. Then

$$
\langle \prod_{i=-n}^{n} (1 + \sum_{l=1}^{\infty} |\phi_l|x_i^l/W^l) \rangle_{0,*} \leq \exp\{C|\phi_2|n/W\}.
$$

(4.32)

(ii) If

$$
\Phi_1(s_1, \ldots, s_n) - \Phi_1(0, \ldots, 0) \prec \prod_{j=1}^{n} (1 + q(s_i)) - 1,
$$

where $s_i = s(\tilde{a}_i/W, \tilde{a}_{i+1}/W, \ldots, \tilde{a}_{i+k}/W, \tilde{b}_i/W, \tilde{b}_{i+1}/W, \ldots, \tilde{b}_{i+k}/W)$ is a polynomial with $s(0, \ldots, 0) = 0$, $k$ is an $n$-independent constant, and $q(s) = \sum_{j=1}^{\infty} |c_j|s^j$ with $|c_1| \leq CW^{-1}$, $|c_2| = o(1)$, $|c_l| \leq (C_0)^l$, $l \geq 3$, then

$$
|\langle \Phi_1(s_1, \ldots, s_n) - \Phi_1(0, \ldots, 0) \rangle_0 | \leq \langle \prod_{j=1}^{n} (1 + q(s_i^*)) - 1 \rangle_{0,*} + e^{-Cn/W},
$$

where $s_i^*$ is obtained from $s_i$ by replacing the coefficients of $s$ with their absolute values.

**Proof.** The proof of the lemma essentially repeats the proof of Lemma 5. Indeed, using

$$
|\phi_2| x^2 \leq \frac{p^{-1}x^2 + px^4}{2}, \quad |\phi_2| = o(1),
$$

$$
|\phi_1| |x| \leq |x|/W \leq \frac{n^{-1} + (n/W^2)|x|^2}{2} \leq \frac{p^{-1}x^2 + px^4}{2} + Cn^{-1}
$$

instead of (4.23), we can prove (4.28) for the series started from $l = 1$ with $|\phi_1| \leq CW^{-1}$, $|\phi_2| = o(1)$. Besides, in view of (4.29)

$$
1 + \sum_{l=1}^{\infty} |\phi_l|x^l/W^l \leq e^{p|x|/W + (|\phi_2| - |\phi_1|^2/2)x^2/W^2+C(x^3/W^3+x^4/W^4)}, \quad x \in (-\delta W, \delta W),
$$

21
and hence the Cauchy-Schwarz inequality yields
\[
\left\langle \prod_{i=-n}^{n} \left(1 + \sum_{l=1}^{\infty} |\phi_l|^2 x_i^l/W^l\right)_{0,*} \leq \left\langle \exp \left\{ \sum_{i=-n}^{n} 2C(x_i^3/W^3 + x_i^4/W^4) \right\} \right\rangle_{0,*}^{1/2}
\]
\[
\times \left\langle \exp \left\{ \sum_{i=-n}^{n} (2|\phi_1| x/W + (2|\phi_2| - |\phi_1|^2) x_i^2/W^2) \right\} \right\rangle_{0,*}^{1/2}
\]
\[
\leq \exp\{Cn|\phi_1|^2 + C_2|\phi_2|n/W\}(1 + o(1)) \leq \exp\{C|\phi_2|n/W\},
\]
where, to obtain the third line, we use Lemma 6 for the first factor and take the Gaussian integral for the second factor. This proves (4.32).

The second assertion of the lemma follows from the fact that if \( \Phi_1(s_1, \ldots, s_k) \preceq \Phi_2(s_1, \ldots, s_k) \), then, putting \( s_i = P_i(x_1, \ldots, x_k) \) for some polynomials \( P_i \), we get
\[
\Phi_1(P_1(x_1, \ldots, x_k), \ldots, P_k(x_1, \ldots, x_k)) \preceq \Phi_2(P_1^*(x_1, \ldots, x_k), \ldots, P_k^*(x_1, \ldots, x_k)),
\]
where \( P_i^* \) is obtained from \( P_i \) by replacing the coefficients of \( P_i \) with their absolute values.

In addition, there exist polynomials \( S_a, S_b \) such that \( S_a(0) = S_b(0) = 0 \) and
\[
s(a_i/W, a_{i+1}/W, \ldots, a_{i+k}/W, b_i/W, b_{i+1}/W, \ldots, b_{i+k}/W) \preceq i+k \prod_{j=i}^{i+k} S_a(a_j/W) \prod_{j=i}^{i+k} S_b(b_j/W).
\]

Using this two facts, one can repeat the argument of the proof of Lemma 5 \( \square \)

### 4.3 Integration over the unitary group \( U(2) \)

The integral over the unitary group \( U(2) \) can be computed using the well-known Harish Chandra/Itsikson-Zuber formula (see e.g. [19], Appendix 5)

**Proposition 1.** Let \( C \) be a normal \( p \times p \) matrix with distinct eigenvalues \( \{c_i\}_{i=1}^{p} \) and \( D = \text{diag}\{d_1, \ldots, d_p\} \), \( d_i \in \mathbb{R} \). Then
\[
\int \exp\{t \text{Tr } C^*DU\} d\mu(U) = \left( \prod_{j=1}^{p-1} j! \right)^{-2} \frac{\det[\exp\{tc_jd_j\}]_{j=1}^{p}}{\text{tr}^{p-1/2}(C)\Delta(C)\Delta(D)},
\]
where \( t \) is some constant and \( \Delta(C) \), \( \Delta(D) \) are the Vandermonde determinants for the eigenvalues \( \{c_i\}_{i=1}^{p} \), \( \{d_i\}_{i=1}^{p} \) of \( C \) and \( D \).

Moreover,
\[
\int_{U(p)} \int_{\Omega} \exp\left\{ -\frac{t}{2} \text{Tr } (C - U^*DU)^2 \right\} \Delta^2(D)f(D) d\mu(U)dD
\]
\[
= \left( \prod_{j=1}^{p} j! \right) t^{-p(p-1)/2} \int_{\Omega} \exp\left\{ -\frac{t}{2} \sum_{j=1}^{p} (c_j - d_j)^2 \right\} \Delta(D)/\Delta(C)f(d_1, \ldots, d_p)dD, \tag{4.33}
\]
where \( f(D) \) is any symmetric function of \( \{d_j\}_{j=1}^{p} \) in the symmetric domain \( \Omega \), \( dD = \prod_{j=1}^{p} dd_j \).
The proof of the proposition can be found in [19]. Moreover, it follows from the properties of the Haar measure on the unitary group $U(2)$ that
\[
\int_{U(2)} V^{p_{ls}} V^{q_{ls}} \exp \left\{ \text{Tr} \, CV^*DV \right\} d\mu(V) \neq 0
\]
only only if \( p_{11} - q_{11} = p_{22} - q_{22} = -(p_{12} - q_{12}) = -(p_{21} - q_{21}) \). Since
\[
|(V_j)_{12}|^2 = |(V_j)_{21}|^2, \\
|(V_j)_{11}|^2 = |(V_j)_{22}|^2 = 1 - |(V_j)_{12}|^2 \\
V_{11} V_{12} = -V_{21} V_{22},
\]
this means that all non-zero moments of the measure \( \exp \left\{ \text{Tr} \, CV^*DV \right\} d\mu(V) \) can be expressed via expectations of \( |(V_j)_{12}|^{2s} \). In addition,
\[
\int_{U(2)} |V_{12}|^{2s} e^{i(\text{Tr} \, CV^*DV - \text{Tr} \, CD)} d\mu(V) = (-1)^s \frac{d^s}{dx^s} \frac{1 - e^{-x}}{x} \bigg|_{x=t(c_1-c_2)(d_1-d_2)}. 
\] (4.34)

5 Proof of the main theorem

In this section we will prove Theorem 1 applying the steepest descent method to the integral representation (2.11).

5.1 The bound for \( \Sigma_c \)

Lemma 9. Let \( \Sigma_c \) be the part of the integral in (2.11) over the complement of the domain \( \Omega_\delta \), which is defined in (1.15). Then
\[
|\Sigma_c| \leq C_1 W^{-8n-2} (4\pi)^N e^{-2Nc_0} e^{-C_2 W^{1-2\kappa}}, \quad (5.1)
\]
where \( \kappa < \frac{\theta}{8} \) and \( c_0 = \Re f(a_\pm) \).

Proof. According to (2.11), we have
\[
|\Sigma_c| \leq e^{-2Nc_0} \cdot \int_{\Omega_\delta} \exp \left\{ - \sum_{j=-n}^{n} (f_*(a_j) + f_*(b_j)) \right\} \\
\times \exp \left\{ - \frac{W^2}{2} \sum_{j=-n+1}^{n} \text{Tr} \left( V_j A_j V_j^* - A_{j-1} \right)^2 \right\} \\
\times \prod_{l=-n}^{n} (a_l - b_l)^2 d \mu(U_{-n}) \, d\overline{a} \, d\overline{b} \prod_{p=-n+1}^{n} d\mu(V_p),
\]
where \( f_\ast \) and \( c_0 \) are defined in (4.1). Here we insert the absolute value inside the integral and use that
\[
\left| \exp \left\{ - \frac{i}{N \rho(\lambda_0)} \sum_{j=-n}^{n} \text{Tr} \left( P_j U_{-n} \right)^* A_j (P_j U_{-n}) \xi \right\} \right| = 1.
\]
To simplify formulas below, set
\[ I_0 = W^{-8n-2}(4\pi)^N e^{-2Nc_0} \cdot \det^{-1}(-\Delta + 2c_+/W^2). \] (5.2)
As we will see below, \( I_0 \) is an order of \( \Sigma \) (see Lemma [10]). Also recall that, according to Lemma 3, eq. (4.5),
\[ e^{-C_1N/W} \leq \det^{-1}(-\Delta + 2c_+/W^2) \leq e^{-C_2N/W}, \] (5.3)
and that \( W^2 = N^{1+\theta}, \kappa < \theta/8 \), and hence \( CN/W \ll W^{1-2\kappa} \).
We are going to prove that
\[ |\Sigma_c/I_0| \leq e^{-CW^{1-2\kappa}}. \] (5.4)
Using Harish Chandra/Itzykson – Zuber formula (4.33), we get (recall that \( A_j = \text{diag} \{a_j, b_j\}, j = -n, \ldots, n \) and \( \Omega^C_\delta \) is still a symmetric domain)
\[ I_0^{-1} \cdot |\Sigma_c| \leq \frac{2^{2n}e^{-2Nc_0}}{W^{4n}I_0} \int_{\Omega^C_\delta} \exp \left\{ -\frac{W^2}{2} \sum_{j=-n+1}^{n} \left( (a_j - a_{j-1})^2 + (b_j - b_{j-1})^2 \right) \right\} \times \exp \left\{ -\sum_{j=-n}^{n} \left( f_s(a_j) + f_s(b_j) \right) \right\} |(a_{-n} - b_{-n})(a_n - b_n)| d\alpha d\beta \] (5.5)
\[ \leq CW^{-2}(2\pi)^{-N}e^{C_1N/W} \int_{\Omega^C_\delta} \exp \left\{ -\frac{1}{2} \sum_{j=-n+1}^{n} \left( (a_j - a_{j-1})^2 + (b_j - b_{j-1})^2 \right) \right\} \times \exp \left\{ -\sum_{j=-n}^{n} \left( f_s(a_j/W) + f_s(b_j/W) \right) \right\} |(a_{-n} - b_{-n})(a_n - b_n)| d\alpha d\beta, \]
where \( f_s \) and \( c_0 \) are defined in (4.1). The first line here is obtained performing recursively the integral over \( V_j \) and \( A_j \) starting from \( j = n \) and going backwards. At each step the integral can be written in the form (4.33), with a suitable choice of the function \( f \). In the third line we did the change \( a_j \to a_j/W, b_j \to b_j/W \) and used (5.2) – (5.3).

Consider \( a_{-n}, \ldots, a_n \) (for \( b_{-n}, \ldots, b_n \) we have the same). Let us divide all configurations of \( \{a_j\} \) into two parts.

(i) **First part:** configurations where there is at least one local large scale fluctuation.

This means that there exists an index \( j_0 \) for which \( |a_{j_0} - a_{j_0-1}| \geq Wn^{-\theta/4}, \) where \( \theta \) is defined in the condition of Theorem [1]. Let us prove that the integral (5.5) over such configuration obey (5.1).

Indeed, in this case
\[ \frac{1}{2} \sum_{j=-n+1}^{n} (a_j - a_{j-1})^2 \geq CW^2n^{-\theta/2} = Cn^{1+\theta/2}. \]
Besides, Lemma [2] yields
\[ f_s(x) \geq \alpha (x - a_-)^2, \quad x \leq a_-, \]
\[ f_s(x) \geq \alpha (x - a_+)^2, \quad x \geq a_+, \] (5.6)
and hence the integral in (5.3) over $\prod_{q=-n}^{n} da_q$ can be bounded by

$$
\exp\{-Cn^{1+\theta/2}\} \int_{a_{j_0}-a_{j_0-1} \geq Wn^{-\varepsilon}} \exp\left\{-\sum_{j=-n}^{n} f_s(a_j/W)\right\} \prod_{q=-n}^{n} da_q
\leq \exp\{-Cn^{1+\theta/2}\} \int \exp\left\{-\sum_{j=-n}^{n} f_s(a_j/W)\right\} \prod_{q=-n}^{n} da_q
\leq (C_1 W)^N \exp\{-Cn^{1+\theta/2}\} \leq \exp\{-Cn^{1+\theta/2}/2\}.
$$

Here we use (5.6) to estimate the integral in the second line (after change $a_j \to Wa_j$ the integral over $da_j$ converges and thus can be bounded by the constant). By the same way in view of (5.6) the integral over $\prod_{q=-n}^{n} db_q$ can be bounded by $(CW)^N$ (because $\sum(b_j - b_{j-1})^2 \geq 0$). Hence, the integral (5.5) over the configuration with at least one local large scale fluctuation (in $\{a_j\}$ or $\{b_j\}$) is bounded by (recall $W^2 = N^{1+\theta}$, $0 < \theta \leq 1$)

$$(C_1 W)^N W^{-2} e^{C_2 N/W} \cdot \exp\{-Cn^{1+\theta/2}/2\} \leq \exp\{-Cn^{1+\theta/2}/4\},$$

which obeys (5.1). Note that the expression $|(a_n - b_n)(a_{n-1} - b_{n-1})|$ does not play any important role in the bounds.

(ii) **Second part: no large local scale fluctuations.**

Let now $|a_j - a_{j-1}| \leq Wn^{-\theta/4}$, $j = -n+1,\ldots,n$ (and the same is valid for $\{b_j\}$). Without loss of generality let $a_{n} < 0$. Let $l_1$ be the first number such that $a_{n} > W^\delta$, where $\delta > 0$ is sufficiently small. Consider the nearest to $l_1$ indices $p_1 < l_1$ and $q_1 > l_1$ such that $a_{p_1} \leq 0$, $a_{q_1} \leq 0$. We will call the sequence $a_{p_1+1},\ldots,a_{q_1-1}$ a “peak”. Remove from the sum $\sum_{j=-n+1}^{n} (a_j - a_{j-1})^2$ the terms $(a_{p_1+1} - a_{p_1})^2$ and $(a_{q_1} - a_{q_1-1})^2$ (the integral becomes larger). Then take the first number $l_2 > q_1$ such that $a_{l_2} > W^\delta$ and the nearest to $l_2$ indices $p_2 < l_2$ and $q_2 > l_2$ such that $a_{p_2} \leq 0$, $a_{q_2} \leq 0$ and again remove the terms $(a_{p_2+1} - a_{p_2})^2$ and $(a_{q_2} - a_{q_2-1})^2$, and so on (the last peak can be from $a_{p_j+1}$ to $a_{n}$). Assume that we obtain $k$ of such peaks.

Consider one of them. Let it consist of $m+1$ positive numbers $a_{p_1+1},\ldots,a_{p_r+m+1} = a_{q_r-1}$. Since $|a_j - a_{j-1}| \leq Wn^{-\theta/4}$, we have $m \geq n^{\theta/4}\delta$ and taking into account that $a_{p_r} < 0$, we have $|a_{p_r+1}/W - a_+| \geq \delta$, $|a_{l_r} - a_{p_r+1}| \geq \delta W/2$. Let $Q_{p_r,m}$ be the domain of configurations such that $a_{p_r+1},\ldots,a_{p_r+m+1}$ form the peak.

Since $a_{p_r+1},\ldots,a_{p_r+m+1} > 0$, according to Lemma 2 we can write

$$f_s(a_{p_r+s}/W) \geq \alpha (a_{p_r+s}/W - a_+)^2, \quad s = 1,\ldots,m + 1.$$

Using the inequality in the r.h.s. of (5.3) and applying Lemma 3 to the integral over
\[ \int_{Q_{pr,s}} \exp \left\{ -\frac{1}{2} \sum_{j=p_{r+2}}^{p_{r+m+1}} (a_j - a_{j-1})^2 - \sum_{j=p_{r+1}}^{p_{r+m+1}} f_s(a_j/W) \right\} da_{p_{r+1}} \ldots da_{p_{r+m+1}} \]

\[ \leq \int_{|a_{p_{r+1}} - a_{s_r}| > \delta W/2} \exp \left\{ -\frac{1}{2} \sum_{j=p_{r+2}}^{p_{r+m+1}} (a_j - a_{j-1})^2 - \sum_{j=p_{r+1}}^{p_{r+m+1}} \alpha a_j^2/W^2 \right\} da_{p_{r+1}} \ldots da_{p_{r+m+1}} \]

\[ \leq (2\pi)^{m/2} \cdot (2\alpha)^{-1/4} W^{1/2} \cdot \left( \sinh \frac{m\sqrt{2\alpha}}{W} \right)^{-1/2} \cdot e^{-C_2\delta^2 W} \]

\[ \leq (2\pi)^{m/2} \cdot C_1 W \cdot e^{-m\sqrt{2\alpha}/(2W) - C_2\delta^2 W}. \]

The last inequality holds since for \( m \geq 1 \) and large \( W \)

\[ (1 - e^{-2a(W)})^{-1/2} \leq C_1 W^{1/2}. \]

Hence, for the integrals over \( k \) peaks of the length \( m_1, \ldots, m_k \) we obtain the bound

\[ (2\pi)^{\sum m_i/2} (C_1 W)^k \exp \{-\sqrt{2\alpha} \sum m_i/(2W)\} \exp \{-C_2\delta^2 W k\}. \]

By the same way we can estimate the integral over \( da_{q_1}, \ldots, da_{q_{k+1}} \) (i.e. over \( a_j \)’s that lie between two peaks) by \((2\pi)^{s/2}(C_1 W) \exp \{-s\sqrt{2\alpha}/(2W)\}\), where \( s = p_{k+1} - q_i + 1 \). Finally, the whole integral over \( \{a_j\} \) configurations with \( k \) peaks which begin at \( p_1, \ldots, p_k \) and end at \( q_1, \ldots, q_k \) can be bounded by

\[ (2\pi)^{N/2}(C_1 W)^{2k+1} \exp \{-\sqrt{2\alpha} N/(2W)\} \exp \{-\delta^2 W k\}. \]

The number of \( \{a_j\} \) configurations with \( k \) peaks is smaller than \((2^{n+1})^{2k}\) (since the number of choices of the “beginnings” and “ends” of \( k \) peaks is \((2^{n+1})^{2k}\) and not all choices are suitable). Hence, we get the bound for the integral \((5.5)\) over all \( \{a_j\} \) configurations which have at least one peak:

\[ (2\pi)^{N/2} \exp \{-\sqrt{2\alpha} N/(2W)\} \sum_{k=1}^{n} \binom{2n+1}{2k} (C_1 W)^{2k+1} \exp \{-2^2 W k\} \]

\[ \leq (2\pi)^{N/2} \exp \{-\sqrt{2\alpha} N/(2W)\} \cdot W \cdot e^{-C_2\delta^2 W ((1 + C_1 W e^{-\delta^2 W})^n - 1)} \]

\[ \leq (2\pi)^{N/2} \exp \{-\sqrt{2\alpha} N/(2W)\} e^{-C_2\delta^2 W}. \]

Moreover, for the configurations of \( \{a_j\} \) without peaks \( f_s(a_j) \geq \alpha (a_j - a_+)^2 \) for each \( j \), and so, according to Lemma 3 the integral over \( \Omega_{\delta}^C \) over such configurations can be bounded by

\[ (2\pi)^{N/2} C_1 W^{1/2} \exp \{-\sqrt{2\alpha} N/(2W)\} e^{-C_2\delta^2 W}. \]

By the same way estimating the integral over \( \{b_j\} \) and substituting the bounds to (5.5), we get Lemma 9.
5.2 Calculation of $\Sigma$

**Lemma 10.** For the integral $\Sigma$ over the domain $\Omega_\delta$ (see (1.18)) we have

$$
\Sigma = \frac{e^{-2N\rho(\lambda_0)^2(4\pi)^N\pi^2}}{W^{8n+2}} \cdot \frac{\sin \pi (\xi_1 - \xi_2)}{\pi (\xi_1 - \xi_2)} \cdot \left| \det^{-1} \left( -\Delta + \frac{2c_+}{W^2} \right) \right| (1 + o(1)) \quad (5.8)
$$

where $I_0$ is defined in (5.2).

Note that (5.8) together with (5.4) yield

$$
|\Sigma c| \leq e^{-CW^{1-2n}} |\Sigma|,
$$

which gives (5.1).

Now using (5.1) and (5.8) we get Theorem 1.

Thus, we are left to compute $\Sigma$. We are going to show that the leading term in $\Sigma$ is given by $\Sigma_\pm$, i.e. that the contributions of $\Sigma_+$ and $\Sigma_-$ are smaller.

5.2.1 Calculation of $\Sigma_\pm$

Consider the $\delta$-neighborhood of the point $(\vec{a}_+, \vec{a}_-)$ with $\vec{a}_\pm$ of (1.11) and $\delta = W^{-\kappa}$.

Let us show that

**Lemma 11.** For the integral $\Sigma_\pm$ over the domain $\Omega_\delta^\pm$ of (1.19) we have

$$
\Sigma_\pm = W^{-2N^2} e^{-2N\rho(\lambda_0)^2} e^{i\pi (\xi_1 - \xi_2)} \int_{|a_j|,|b_j| \leq W^{1-\kappa}} \int_{U(2)^N} \mu_+ (a) \mu_- (b) \times e^{W^2 \sum_{j=-n+1}^n \text{Tr}_j \left( V_j^* (L+\tilde{A}_j) W V_j (L+\tilde{A}_{j-1}) (L+\tilde{A}_j) - (L+\tilde{A}_{j-1}) \right) - \frac{1}{N\rho(\lambda_0)} \sum_{k=-n}^n \left( \text{Tr}_k P_k U_n^* (L+\tilde{A}_k) P_k U_n - \text{Tr}_k L_k \right) \times \prod_{l=-n}^n (a_+ - a_- + (\tilde{a}_l - \tilde{b}_l)/W)^2 d\mu(U_n) \prod_{q=-n+1}^n d\mu(V_q) d\tilde{a} d\tilde{b},
$$

where $L = \text{diag} \{a_+,a_-\}$, $\tilde{A}_j = \text{diag} \{\tilde{a}_j,\tilde{b}_j\}$, and $\mu_\gamma (a)$ is defined in (1.17).

Now we are going to integrate over $\{V_j\}$.  

\[ \]
Denote
\[ F(\overline{a}, \overline{b}, V) = -\frac{i}{\rho(\lambda_0)} \sum_{k=-n}^{n} \left( \text{Tr} \left( P_k U_{-n} \right)^* (L + \tilde{A}_k/W) (P_k U_{-n}) \right) \hat{\xi} - \text{Tr} L \hat{\xi}, \] (5.10)
\[ d\tilde{\mu}(V, \tilde{A}) = e^{W^2} \sum_{j=-n+1}^{n} \text{Tr} \left( V_j (L+\tilde{A}_j/W) V_j (L+\tilde{A}_{j-1}/W) (L+\tilde{A}_{j-1}/W) \right) \prod_{q=-n+1}^{n} d\mu(V_q), \]
\[ I_{\tilde{\mu}}(\tilde{A}) = \int d\tilde{\mu}(V, \tilde{A}). \]

According to the Itsykson-Zuber formula (see Proposition 1)
\[ I_{\tilde{\mu}}(\tilde{A}) = W^{-4n} \prod_{q=-n+1}^{n} \frac{1 - e^{-W^2(a_+ - a_+ + (\tilde{a}_q - \tilde{a}_q)/W)(a_+ - a_+ + (\tilde{b}_q - \tilde{b}_q)/W)}}{a_+ - a_+ + (\tilde{a}_q - \tilde{b}_q)/W(a_+ - a_+ + (\tilde{a}_q - \tilde{b}_q)/W)}. \] (5.11)

We want to integrate the r.h.s. of (5.10) over \( d\tilde{\mu}(V, \tilde{A}) \). To this end, we expand \( \exp \left\{ F(\overline{a}, \overline{b}, V) \right\} \) into a series in \( |(V_j)_{12}|^2 \) (note that \(|(V_j)_{12}|^2 = |(V_j)_{21}|^2; |(V_j)_{11}|^2 = |(V_j)_{22}|^2 = 1 - |(V_j)_{12}|^2\)). Formula (4.34) implies
\[ \int |(V_j)_{12}|^2 d\tilde{\mu}(V, \tilde{A}) = W^{-4n} \prod_{q \neq j} \frac{1 - e^{-W^2(a_+ - a_+ + (\tilde{a}_q - \tilde{a}_q)/W)(a_+ - a_+ + (\tilde{a}_q - \tilde{a}_q)/W)}}{a_+ - a_+ + (\tilde{a}_q - \tilde{b}_q)/W(a_+ - a_+ + (\tilde{a}_q - \tilde{b}_q)/W)} \]
\[ \times (-1)^s \frac{dx}{dx} \left| \frac{1 - e^{-x}}{x} \right|_{x=W^2(a_+ - a_+ + (\tilde{a}_j - \tilde{b}_j)/W)(a_+ - a_+ + (\tilde{a}_j - \tilde{b}_j)/W)} . \] (5.12)

We are going to show that the leading term of the integral is given by the summands without \(|(V_j)_{12}|^2\).

**Lemma 12.** In the notations of (5.10)
\[ \left| \left\langle \left( \exp \left\{ \left( F(\overline{a}, \overline{b}, V) - F(0, 0, I) \right)/N \right) - 1 \right) \cdot \Pi_1 \cdot \Pi_2 \right|_{0/\tilde{\mu}} \right \rangle \right| = o(1), \quad N \to \infty, \] (5.13)
where \( \Pi_1, \Pi_2 \) are the products of the Taylor’s series for \( \exp \{ \varphi_+(\tilde{a}_j/W) \} \) and for \( \exp \{ \varphi_-(\tilde{b}_j/W) \} \) and
\[ \langle \ldots \rangle_{\tilde{\mu}} = I_{\tilde{\mu}}(\tilde{A})^{-1} \int \langle \ldots \rangle d\tilde{\mu}(V, \tilde{A}). \] (5.14)

**Proof.** Since \( \hat{\xi} = \frac{\xi_1 + \xi_2}{2} I + \frac{\xi_1 - \xi_2}{2a_+} L \), we have
\[ \text{Tr} \left( (P_k U_{-n})^* (L + \tilde{A}_k/W) (P_k U_{-n}) \hat{\xi} - \text{Tr} (L + \tilde{A}_k/W) \hat{\xi} \right) \]
\[ = \frac{\xi_1 - \xi_2}{2a_+} \text{Tr} \left( (P_k U_{-n})^* (L + \tilde{A}_k/W) (P_k U_{-n}) L - (L + \tilde{A}_k/W) L \right) \]
\[ = 2a_+ (\xi_2 - \xi_1) \cdot |(P_k U_{-n})_{12}|^2 (1 + (\tilde{a}_k - \tilde{b}_k)/(a_+ - a_-) W), \]
thus
\[
F(\vec{a}, \vec{b}, V) - F(0, 0, I) = \frac{2ia_+((\xi_1 - \xi_2))}{p(\lambda_0)} \sum_{k=-n+1}^n (|(P_k U_{-n})|_2^2 - |(U_{-n})|_2^2) \cdot \left(1 + \frac{\tilde{a}_k - \tilde{b}_k}{(a_+ - a_-)W}\right).
\] (5.15)

We can write
\[
\exp\left\{\frac{1}{N}(F(\vec{a}, \vec{b}, V) - F(0, 0, I))\right\} - 1 = \sum_{p=1}^{\infty} \frac{C^p}{p! N^p} \sum_{k_1, \ldots, k_p} \left\langle \prod_{j=1}^p \left([(|(P_{k_j} U_{-n})|_2^2 - |(U_{-n})|_2^2) \cdot \left(1 + \frac{\tilde{a}_{k_j} - \tilde{b}_{k_j}}{(a_+ - a_-)W}\right)\right)\right\rangle_{\tilde{\mu}},
\] where \(\langle \ldots \rangle_{\tilde{\mu}}\) is defined in (5.14). Hence, we have to study
\[
\Phi_{k_1, \ldots, k_p}(\vec{a}, \vec{b}) = \left\langle \prod_{j=1}^p \left([(|(P_{k_j} U_{-n})|_2^2 - |(U_{-n})|_2^2) \cdot \left(1 + \frac{\tilde{a}_{k_j} - \tilde{b}_{k_j}}{(a_+ - a_-)W}\right)\right)\right\rangle_{\tilde{\mu}}.
\] (5.16)

Let \(p < Cn/W\) for some constant \(C\). Introduce i.i.d \(\{t_j\}\) such that the density of the distribution has the form
\[
\rho(t_j) = \frac{(a_+ - a_-)^2}{2} t_j \exp\{-t_j^2(a_+ - a_-)^2\} \cdot 1_{0 < t_j < W/2}.
\] (5.17)

Consider the unitary matrices
\[
\tilde{V}_j = \begin{pmatrix}
\tilde{r}_j e^{i\theta_j} & \tilde{v}_j e^{i\tilde{\theta}_j} \\
-\tilde{v}_j e^{-i\tilde{\theta}_j} & \tilde{r}_j e^{-i\theta_j}
\end{pmatrix},
\] (5.18)

where
\[
\tilde{v}_j = \frac{t_j}{W} \left(1 + \frac{\tilde{a}_j - \tilde{b}_j}{W(a_+ - a_-)}\right)^{-1/2} \left(1 + \frac{\tilde{a}_{j-1} - \tilde{b}_{j-1}}{W(a_+ - a_-)}\right)^{-1/2},
\]
\[
\tilde{r}_j = (1 - \tilde{v}_j^2)^{1/2},
\]
and \(\theta_j, \tilde{\theta}_j \in [-\pi, \pi]\).

We need

**Lemma 13.**
\[
\tilde{\Phi}_{k_1, \ldots, k_p}(\vec{a}, \vec{b}) := \left\langle \prod_{j=1}^p \left([\prod_{l=k_j}^{-n+1} \tilde{V}_l \cdot U_{-n})|_2^2 - |(U_{-n})|_2^2)\right)\right\rangle_{t_j, \theta_j, \tilde{\theta}_j} = \Phi_{k_1, \ldots, k_p}(\vec{a}, \vec{b}) + O(e^{-cW^2}),
\] (5.19)

where \(\langle \ldots \rangle_{t_j, \theta_j, \tilde{\theta}_j}\) means the expectation over \(\{t_j\}\) with respect to the measure with the distribution (5.17) and over \(\{\theta_j\}, \{\tilde{\theta}_j\}\) from \(-\pi\) to \(\pi\).
The proof of the lemma can be found in Section 6. Denote

\[ s_j = 1 - \left( 1 + \frac{a_j - \bar{b}_j}{W(a_+ - a_-)} \right) \left( 1 + \frac{a_{j-1} - \bar{b}_{j-1}}{W(a_+ - a_-)} \right). \] (5.20)

Expanding \( \tilde{V}_j \) with respect to \( s_j \) we get

\[ \tilde{V}_j = \tilde{V}_j(0) + \frac{t_j}{W} ((1 - s_j)^{-1/2} - 1) V^1_j + \frac{t^2_j}{W^2} \sum_{r=1}^{\infty} V_j^{(r)} s_j^r, \]

where \( \tilde{V}_j(0) \) is a unitary matrix (and hence \( \| \tilde{V}_j(0) \| \leq 1 \)),

\[ \tilde{V}_j^1 = \begin{pmatrix} 0 & e^{i\theta_j} \\ -e^{-i\theta_j} & 0 \end{pmatrix}, \quad \| \tilde{V}_j^{(r)} \| \leq C^r \quad (r = 1, 2, \ldots), \]

and \{\( \tilde{V}_j^{(r)} \)\} are diagonal matrices.

Since the integrals of \( e^{im\theta_j} \) equal 0 for \( m \neq 0 \) and \( 2\pi \) for \( m = 0 \), we conclude that if we replace the coefficients in front of \( e^{i\theta_j} \) and \( e^{-i\theta_j} \) with the bounds for their absolute values, then, after the averaging with respect to \( \theta_j \), the resulting coefficients in front of \( s_j^k \) will grow. Hence,

\[ \tilde{\Phi}_{k_1, \ldots, k_p}(a, b) - \tilde{\Phi}_{k_1, \ldots, k_p}(0, 0) \prec 4 \left( \prod \left| 1 + \frac{t_j}{W} e^{i\theta_j} s_j^* g(s_j^*) + \frac{t^2_j}{W^2} s_j^* g(s_j^*) \right|^{2p} \right)_{t_j, \theta_j} - 1, \]

where \( g(t) = C_0/(1 - Ct) \) with some \( n \)-independent \( C, C_0 \) and

\[ s_j^* = \frac{\tilde{a}_j + \tilde{b}_j + \tilde{a}_{j-1} + \tilde{b}_{j-1}}{W(a_+ - a_-)} + \frac{(\tilde{a}_{j-1} + \tilde{b}_{j-1})(\tilde{a}_j + \tilde{b}_j)}{W^2(a_+ - a_-)^2}. \]

Moreover,

\[ \left( \frac{t^2_j}{W^{2k}} \right)_{t_j} \leq \frac{k!}{(a_+ - a_-)^{2k} W^{2k}}, \]

and thus we conclude

\[ \left( \prod \left| 1 + \frac{t_j}{W} e^{i\theta_j} s_j^* g(s_j^*) + \frac{t^2_j}{W^2} s_j^* g(s_j^*) \right|^{2p} \right)_{t_j, \theta_j} \prec \prod \left( 1 + \frac{2p}{W^2} s_j^* g_1(s_j^*) + \frac{p^2}{W^2} (s_j^*)^2 g(s_j^*)^2 \right). \]

Set

\[ \Pi_3 = \prod_{j=1}^{p} \left( 1 + \frac{\tilde{a}_{k_j} - \tilde{b}_{k_j}}{(a_+ - a_-) W} \right), \quad \Pi_{3, \ast} = \prod_{j=1}^{p} \left( 1 + \frac{\tilde{a}_{k_j} + \tilde{b}_{k_j}}{(a_+ - a_-) W} \right). \]

Since \( p \leq Cn/W \), we have \( 2p/W^2 \leq W^{-1}, p^2/W^2 = o(1) \). In addition, \( \Pi_3 \) has degree \( p < Cn/W, |\Pi_3| \leq (1 + \delta)^p \) and thus does not spoil the bounds (4.21) - (4.28). Thus,
Lemma 8 yields
\[
\left| \left( \Phi_{k_{1},...,k_{p}}(\tilde{a}, \tilde{b}) - \Phi_{k_{1},...,k_{p}}(0, 0) \right) \cdot \Pi_{1} \cdot \Pi_{2} \cdot \Pi_{3} \right|_{0}
\]
\[
\leq 4 \left( \prod_{j=-n}^{\infty} \left( 1 + \frac{2p}{W} s_{j} g(s_{j}) + \frac{p^{2}}{W^{2}} s_{j}^{2} g(s_{j})^{2} \right) - 1 \right) \cdot \Pi_{1,\ast} \cdot \Pi_{2,\ast} \cdot \Pi_{3,\ast} \right|_{0,\ast} + e^{-C_{n}/W}
\]
\[
\leq 4(1 + \delta)^{p} \left( \exp \left\{ \sum_{i=-n}^{n} \left( \frac{C_{p}}{W^{2}} \cdot \tilde{a}_{i} + \tilde{b}_{i} \right) + \frac{p^{2}c}{W^{2}} \cdot \tilde{a}_{i}^{2} + \tilde{b}_{i}^{2} \right) \right) - 1 \right) \cdot \Pi_{1,\ast} \cdot \Pi_{2,\ast} \right|_{0,\ast} + e^{-C_{n}/W}
\]
\[
\leq 4e^{\delta p} \left( \exp \left\{ \frac{c p^{2} n}{W^{2}} \right\} - 1 \right) \leq 4e^{\delta p} \left( \exp \left\{ \frac{c p n^{2}}{W^{4}} \right\} - 1 \right),
\]
and thus, since \( p < C_{n}/W \),
\[
\sum_{p=1}^{C_{n}/W} \frac{(C_{1})^{p}}{p! N_{p}} \sum_{k_{1},...,k_{p}} \left| \left( \Phi_{k_{1},...,k_{p}}(\tilde{a}, \tilde{b}) - \Phi_{k_{1},...,k_{p}}(0, 0) \right) \cdot \Pi_{1} \cdot \Pi_{2} \cdot \Pi_{3} \right|_{0}
\]
\[
\leq \exp \left\{ e^{C_{2}n^{2}/W^{4} + C_{1}\delta} \right\} - e^{C_{1}\delta} = o(1). \quad (5.21)
\]
If \( p \gg n/W \), then \( 1/\sqrt{p!} \ll e^{-C_{n}/W} \), and hence we can replace \( \langle \ldots \rangle_{0} \) with \( \langle \ldots \rangle_{0,\ast} \) (see Lemma 3) and then take the absolute value under the integral and get the bound
\[
e^{C_{1}n/W} \left( \sqrt{C_{n}/W} \right)_{-1} \sum_{p=C_{n}/W}^{\infty} \frac{(C_{2})^{p}}{\sqrt{p!}} = o(1).
\]
Let us prove now that
\[
\tilde{\Phi}_{k_{1},...,k_{p}}(0, 0) = \left\langle \prod_{j=1}^{p} \left( |(\tilde{P}_{k_{j}}(0)U_{-n})_{12}|^{2} - |(U_{-n})_{12}|^{2} \right) \right\rangle_{t_{j}, \theta_{j}, \tilde{\theta}_{j}} = o(1),
\]
where
\[
\tilde{P}_{k_{j}}(0) = \prod_{l=k_{j}}^{-n+1} \tilde{V}_{l}(0).
\]
To this end, we write

\[
\langle \prod_{j=1}^{p} |(\tilde{P}_{k_j}(0)U_{-n})_{12}|^2 - |(U_{-n})_{12}|^2 \rangle_{t_j,\theta_j,\tilde{\theta}_j} \leq \langle |(\tilde{P}_{k_1}(0)U_{-n})_{12}|^2 - |(U_{-n})_{12}|^2 \rangle_{t_1,\theta_1,\tilde{\theta}_1} + \langle |(\tilde{V}_{k_1}(0))_{12}(\tilde{P}_{k_1-1}(0)U_{n})_{22} + (\tilde{V}_{k_1}(0))_{11}(\tilde{P}_{k_1-1}(0)U_{n})_{12}|^2 - |(U_{-n})_{12}|^2 \rangle_{t_2,\theta_2,\tilde{\theta}_2} + \ldots \leq \frac{CN}{W^2} = o(1).
\]

This yields

\[
\sum_{p=1}^{Cn/W} \frac{(C_1)^p}{p!N_p} \sum_{k_1,\ldots,k_p} \left| \langle \tilde{\Phi}_{k_1,\ldots,k_p}(0,0) \cdot \Pi_1 \cdot \Pi_2 \cdot \Pi_3 \rangle \right|_{0}
\leq \frac{CN}{W^2} \sum_{p=1}^{Cn/W} \frac{(C_1)^p(1+\delta)^p}{p!} \leq C_1N/W^2 = o(1),
\]

which together with (5.21) completes the proof of Lemma 12. \[\square\]

Thus, we can change \(F(\tilde{a},\tilde{b},V)\) to \(F(0,0,I)\) in (5.9), and then integrate over \(\tilde{\mu}\), according to (5.11). We obtain

\[
\Sigma_+ = W^{-8n-22n}e^{-2Nc_0} \int_{U(2)} \int_{|\tilde{a}_j|,|\tilde{b}_j|\leq W^{1-\kappa}} \mu_{e_+}(a) \mu_{e_-}(b)
\times \exp \left\{ - \sum_{j=-n}^{n} \varphi_+ (\tilde{a}_j/W) - \sum_{j=-n}^{n} \varphi_- (\tilde{b}_j/W) \right\}
\times e^{-\frac{1}{\rho(\theta_0)}} \text{Tr} U_{-n}^{\dagger} \mu(U_{-n}) \prod_{q=-n}^{n} d\tilde{a}_q d\tilde{b}_q (1 + o(1))
\]

Integrating over \(U_{-n}\) by the Itsykson-Zuber formula (see Proposition II) and using Lemma 3 we get finally

\[
\Sigma_+ = \frac{W^{-8n-22n}e^{-2Nc_0}(e^{i\pi(\xi_1 - \xi_2)} - e^{i\pi(\xi_2 - \xi_1)})}{2i\pi(\xi_1 - \xi_2)} \int_{|\tilde{a}_j|,|\tilde{b}_j|\leq W^{1-\kappa}} \prod_{q=-n}^{n} d\tilde{a}_q d\tilde{b}_q \cdot \mu_{e_+}(a) \mu_{e_-}(b)
\times (a_+ - a_- + (\tilde{a}_n - \tilde{b}_n)/W)(a_+ - a_- + (\tilde{a}_n - \tilde{b}_n)/W)(1 + o(1))
\]

\[
= \frac{2\pi^2 e^{-2Nc_0} \rho(\lambda_0)^2 (4\pi)^N \sin(\pi(\xi_1 - \xi_2))}{W^{8n+2} \cdot \pi(\xi_1 - \xi_2)} \left| \det^{-1} \left( - \Delta + \frac{2c_+}{W^2} \right) \right| (1 + o(1)).
\]

\[\square\]
5.2.2 $\Sigma_+$ and $\Sigma_-$.  

In this section we prove that the integrals $\Sigma_+$ and $\Sigma_-$ over $\Omega^+ \delta$ and $\Omega^- \delta$ have smaller orders than $\Sigma_{\pm}$.

Similarly to (5.9) we get
\[
\Sigma_+ = W^{-8n-4} e^{-2N f(a_+)} e^{-i\pi(\xi_1+\xi_2)} \int_{\lambda \in \mathbb{R}} \int \mu_{\xi_1}(a) \mu_{\xi_2}(b) \\
\times \sum_{k=-n}^{n} \text{Tr} (V_j^* A_j V_j A_j - A_j A_j^{-1}) \\
\times e^{j=-n+1} (\varphi_+ (\tilde{a}_k/W) + \varphi_+ (\tilde{b}_k/W)) - \frac{1}{N^p(\lambda_0)} \sum_{k=-n}^{n} \text{Tr} (P_k U_n)^* (\tilde{A}_k/W) (P_k U_n) \xi \\
\times \prod_{l=-n}^{n} (\tilde{a}_l - \tilde{b}_l)^2 d\mu(U_{-l}) \prod_{q=-n+1}^{n} d\mu(V_q) d\tilde{a} d\tilde{b} (1 + o(1)).
\]

By the same argument as for $\Sigma_{\pm}$ we get
\[
\Sigma_+ = 2^n W^{-8n-4} e^{-2N f(a_+)} e^{-i\pi(\xi_1+\xi_2)} \int \prod_{q=-n}^{n} d\tilde{a}_q d\tilde{b}_q \\
\times \mu_{\xi_1}(a) \mu_{\xi_2}(b) (\tilde{a}_n - \tilde{b}_n) (\tilde{a}_n - \tilde{b}_n) (1 + o(1)) \\
= 2^n W^{-4(2n+1)} e^{-2N f(a_+)} e^{-i\pi(\xi_1+\xi_2)} \int \prod_{q=-n}^{n} d\tilde{a}_q d\tilde{b}_q \\
\times \mu_{\xi_1}(a) \mu_{\xi_2}(b) (\tilde{a}_n - \tilde{b}_n) (\tilde{a}_n - \tilde{b}_n) (1 + o(1)) \\
= (4\pi)^N W^{-8n-4} e^{-2N f(a_+)} e^{-i\pi(\xi_1+\xi_2)} D^{-1} \det^{-1} D,
\]

where
\[
D = -\Delta + \frac{2c_+}{W^2}.
\]

It is easy to see (see the proof of Lemma 3) that for $W^2 = N^{1+\theta}$, $0 < \theta \leq 1$
\[
|D^{-1}_{n,n}| = 1/|\det D| \leq CW.
\]

Hence, since $\Re f(a_+) = c_0$, we get
\[
|\Sigma_+| \leq C (4\pi)^N W^{-8n-3} e^{-2N c_0} |\det^{-1} D| \leq CW^{-1} |\Sigma_{\pm}|,
\]

and thus the order of $\Sigma_+$ is smaller than the order of $\Sigma_{\pm}$. This completes the proof of Lemma 10.

6 Auxiliary result

Proof of Lemma 2. Note that
\[
f_\pm(a) = 0, \quad \frac{d}{dx} f_\pm(x) \bigg|_{x=a_\pm} = 0, \quad \frac{d^2}{dx^2} f_\pm(x) \bigg|_{x=a_\pm} = 2(1 - \lambda_0^2/4) > 0.
\]
Thus, function $f_\ast(x)$ attains its minimum at $a_\pm$ and expanding $f_\ast(x)$ in $x \in (a_\pm - \delta, a_\pm + \delta)$ we get

$$f_\ast(x) = (1 - \lambda_0^2/4)(x - a_\pm)^2 + O(\delta^3).$$

(6.1)

This yields (4.2). Besides, it is easy to see, that if we take $\alpha = \frac{1}{2}(1 - \lambda_0^2/4)$, then we obtain (4.3) for some sufficiently small $\delta > 0$. \Box

**Proof of Lemma 3**

1) Set $-\Delta_1 = -\Delta + E_0$, where $E_0$ is an $N \times N$ matrix whose elements are zeros except $(E_0)_{-n,-n} = 1$.

Define

$$T_n(x) = \det (-\Delta_1 + x \cdot I), \quad S_n(x) = \det (-\Delta + x \cdot I).$$

(6.2)

It is easy to check that

$$T_n(x) = (2 + x)T_{n-1}(x) - T_{n-2}(x), \quad T_1(x) = 1 + x, \quad T_2(x) = x^2 + 3x + 1,$$

(6.3)

$$S_n(x) = (1 + x)T_{n-1}(x) - T_{n-2}(x).$$

(6.4)

Solving the recurrent relation (6.3), we get

$$T_m(x) = \frac{\zeta^{m+1} - \zeta^{-m}}{\zeta + 1}, \quad S_m(x) = \frac{(\zeta^m - \zeta^{-m})(\zeta - 1)}{\zeta + 1}$$

(6.5)

where

$$\zeta = \frac{2 + x + \sqrt{x^2 + 4x}}{2}.$$

For $x = 2\gamma/W^2$

$$\zeta = 1 + \sqrt{2\gamma/W + \gamma/W^2 + O(W^{-3})}, \quad W \to \infty.$$

This and (6.4) – (6.5) yield

$$T_m(2\gamma/W^2) = \cosh \frac{m\sqrt{2\gamma}}{W}(1 + o(1)), \quad S_m(2\gamma/W^2) = \frac{\sqrt{2\gamma}}{W} \sinh \frac{m\sqrt{2\gamma}}{W}(1 + o(1)),$$

and thus (4.5). Also it is easy to see that

$$G^{(m)}_{ii}(\gamma) = \frac{T_{i-1}(2\gamma/W^2)T_{m-i}(2\gamma/W^2)}{S_m(2\gamma/W^2)} \leq \frac{C_\gamma W}{\sqrt{2\gamma}} \coth \frac{m\sqrt{2\gamma}}{W}(1 + o(1)).$$

Moreover,

$$C^{(m)}_{11}(\gamma) - C^{(m)}_{1m}(\gamma) = \frac{T_{m-1}(2\gamma/W^2) - 1}{S_m(2\gamma/W^2)}$$

$$= C_\gamma W \coth \frac{m\sqrt{2\gamma}}{2W}(1 + o(1)) \leq C_\gamma \min\{m, W\}.$$
2) Take $m \geq CW$ and $\alpha \in \mathbb{R}, \alpha > 0$. Note that for any sufficiently small $\delta > 0$ and $\varepsilon > 0$

$$Z^{(m)}_{\alpha} - Z^{(m)}_{\delta, \alpha} = \int e^{-\frac{1}{2} \sum_{j=2}^{m} (x_j - x_{j-1})^2 - \frac{\alpha}{2} \sum_{j=1}^{m} x_j^2 \prod_{q=1}^{m} dx_q} \max_{|x_i| > \delta W}$$

$$\leq \sum_{i=1}^{m} \int e^{-\frac{2}{\delta^2} (x_i^2 - W^2 \varepsilon^2) - \frac{\alpha}{2} \sum_{j=2}^{m} (x_j - x_{j-1})^2 - \frac{\alpha}{2} \sum_{j=1}^{m} x_j^2 \prod_{q=1}^{m} dx_q}$$

$$= \sum_{i=1}^{m} \int \sqrt{2\pi} dt \ e^{-t^2/2} \int \prod_{q=1}^{m} dx_q \ e^{\varepsilon t x_1 - \frac{1}{2} \sum_{j=2}^{m} (x_j - x_{j-1})^2 - \frac{\alpha}{2} \sum_{j=1}^{m} x_j^2}$$

$$= \sum_{i=1}^{m} \frac{m e^{-\varepsilon^2 \delta^2 W^2/2}}{\sqrt{2\pi}} \cdot Z^{(m)}_{\alpha} \sum_{i=1}^{\delta, \alpha} \int e^{-t^2/2 + \varepsilon^2 G^{(m)}(\alpha) t^2/2} dt,$$

where $G^{(m)}$ is defined in (4.6).

Let us take $\varepsilon^2 = (C_{\delta, \alpha}^{(m)}(\alpha))^{-1/2}$ in (6.6). Then taking into account (4.7) and $CW \leq m \leq 2n + 1$, we obtain for $\alpha \in \mathbb{R}, \alpha > 0$

$$\frac{Z^{(m)}_{\alpha} - Z^{(m)}_{\delta, \alpha}}{Z^{(m)}_{\alpha}} \leq C_1 e^{-C_2 \delta^2 W}.$$  (6.7)

Since $m \leq 2n + 1$, according to the first assertion of the lemma, we get

$$\left| \frac{Z^{(m)}_{\gamma}}{Z^{(m)}_{\gamma_2}} \right| = (1 + C/W)^m \leq e^{C_1 m/W}, \quad m, W \rightarrow \infty,$$

which gives (4.8). This and (6.7) yield for $m \geq CW, \gamma \in \mathbb{C}, \Re \gamma > 0$

$$\left| \frac{Z^{(m)}_{\gamma} - Z^{(m)}_{\delta, \gamma}}{Z^{(m)}_{\gamma}} \right| \leq \frac{Z^{(m)}_{\gamma}}{Z^{(m)}_{\gamma_2}} \cdot \frac{Z^{(m)}_{\delta, \gamma}}{Z^{(m)}_{\gamma}} \leq C_1 e^{-C_2 \delta^2 W + C_1 m/W} \leq C_1 e^{-C_3 \delta^2 W}.$$  (6.7)

Since $W^2 = N^{1+\theta}$, we can take $\delta = W^{-\kappa}$ with $\kappa < \theta/(1 + \theta)$.

Take now any $m$. Using the assertion (1) of the lemma, we can write for any $\varepsilon > 0$

$$(Z^{(m)}_{\alpha})^{-1} \int_{x_k - x_1 > \delta W} e^{-\frac{1}{2} \sum_{j=2}^{m} (x_j - x_{j-1})^2 - \frac{\alpha}{2} \sum_{j=1}^{m} x_j^2 \prod_{q=1}^{m} dx_q}$$

$$\leq (Z^{(m)}_{\alpha})^{-1} \int e^{\varepsilon (x_k - x_1 - \delta W) - \frac{1}{2} \sum_{j=2}^{k} (x_j - x_{j-1})^2 - \frac{\alpha}{2} \sum_{j=1}^{k} x_j^2 \prod_{q=1}^{k} dx_q}$$

$$\times \int e^{\frac{1}{2} \sum_{j=k+1}^{m} (x_j - x_{j-1})^2 - \frac{\alpha}{2} \sum_{j=k+1}^{m} x_j^2 \prod_{q=k+1}^{m} dx_q}$$

$$\leq \frac{Z^{(m-k)}_{\alpha}}{Z^{(m)}_{\alpha}} \cdot e^{-\varepsilon \delta W + C \varepsilon^2 (G^{(k)}_{11} - G^{(k)}_{1k})} \leq W e^{-\varepsilon \delta W + C \varepsilon^2 \min \{m, W\}} \leq e^{-C_1 \delta^2 W}.$$
3) It is easy to see that
\[-\frac{\alpha x^2}{2} + k_i \log |x| \leq -\frac{\alpha x^2}{4} + \frac{k_i}{2} \log \frac{2k_i}{\alpha}.
\]
Thus, using the assertions (1) – (2) of the lemma, we obtain
\[
|Z^{(m)}_\gamma|^{-1} \int_{\max |x_i|>\delta W} \prod_{j \in S} x_j^{k_j} \cdot \mu^{(m)}_\gamma(x) \prod_{q=1}^m dx_q 
\leq |Z^{(m)}_\gamma|^{-1} e_{\alpha_1} \sum_{k_i} k_i \log \frac{2k_i}{\alpha} \cdot \mu^{(m)}_{\gamma/2}(x) \prod_{q=1}^m dx_q 
\leq e^{C_1 k \log k + C_2 m/W} \frac{|Z^{(m)}_{\delta, \gamma/2} - Z^{(m)}_{\delta, \gamma/2}|}{|Z^{(m)}_{\delta, \gamma/2}|} \leq e^{-CW^2},
\]
where the last inequality holds since \(k \leq C m/W \ll W\). □

**Proof of Lemma 13.** Recall that all non-zero moments of measure \(\tilde{\mu}\) can be expressed via expectations of \(|(V_j)_{12}|^{2s}\) (see Section 4.3). In addition, according to (5.11),
\[
\langle |(V_j)_{12}|^{2s} \rangle_{V_j} = \frac{s!}{W^{2s}(a_+ - a_-)^{2s}} \left(1 + \frac{\bar{a}_{j-1} - \bar{b}_{j-1}}{W(a_+ - a_-)} \right)^{-s} \left(1 + \frac{\bar{a}_j - \bar{b}_j}{W(a_+ - a_-)} \right)^{-s} + O(e^{-C_1 W^2}).
\]
Besides,
\[
\int \prod_{l,s=1}^2 \tilde{V}_{ts}^{p_{ls}} \tilde{V}_{ts}^{q_{ls}} \rho(t) dt d\theta d\tilde{\theta} \neq 0
\]
only if \(p_{11} - q_{11} = p_{22} - q_{22}, p_{12} - q_{12} = p_{21} - q_{21}\), and all non-zero moments of the measure with respect to \(t_j, \theta_j, \tilde{\theta}_j\) can be expressed via the expectations of \(|(V_j)_{12}|^{2s}\). Moreover,
\[
\langle |(\tilde{V}_j)_{12}|^{2s} \rangle_{t_j, \theta_j, \tilde{\theta}_j} = \frac{s!}{W^{2s}(a_+ - a_-)^{2s}} \left(1 + \frac{\bar{a}_{j-1} - \bar{b}_{j-1}}{W(a_+ - a_-)} \right)^{-s} \left(1 + \frac{\bar{a}_j - \bar{b}_j}{W(a_+ - a_-)} \right)^{-s} + O(e^{-C_2 W^2}).
\]
Hence, if \(\sum p_{ls} = \sum q_{ls}, 0 \leq p_{ls}, q_{ls} \leq 2p\), then
\[
\langle \prod_{l,s=1}^2 V_{ts}^{p_{ls}} V_{ts}^{q_{ls}} \rangle_{V} = \langle \prod_{l,s=1}^2 \tilde{V}_{ts}^{p_{ls}} \tilde{V}_{ts}^{q_{ls}} \rangle_{t_j, \theta_j, \tilde{\theta}_j} + O(e^{-C W^2}).
\]
Now let \(E_k\) be the averaging with respect to the product of the measures \(t_j, \theta_j, \tilde{\theta}_j\) for \(j\) from \((-n+1)\) to \((-n+k)\) and the measures \(d\mu(V_j)\) for \(j\) from \((-n+k+1)\) to \(n\). Thus, if
\[
\Psi_{k_1,\ldots,k_s} = \prod_{j=1}^{s} |(P_{k_j} U_{-n})_{12}|^2,
\]
then it suffices to estimate
\[
|\langle E_0 - E_{2n} \rangle \{\Psi_{k_1,\ldots,k_s}\}| \leq e^{-c W^2}
\]
for $s \leq p$. Note that
\[
|\langle E_0 - E_{2n} \{ \Psi_{k_1, \ldots, k_s} \rangle \rangle | \leq \sum_i |\langle E_{i-1} - E_i \{ \Psi_{k_1, \ldots, k_s} \rangle \rangle |.
\]
In each summand we write for $\gamma = i - 1, i$ (we assume that all $k_j \geq (-n + i)$)
\[
E_\gamma \{ \Psi_{k_1, \ldots, k_s} \} = E_\gamma \left\{ \prod_{j=1}^s |(P_{-n+i-1}V_{-n+i}(P_{-n+i}^*P_{k_j}U_{-n}))_{12}|^2 \right\}
\]
\[
= E_\gamma \left\{ \sum_{\alpha, \alpha' = 1, 2} (P_{-n+i-1})_{1\alpha}(V_{-n+i})_{\alpha\alpha'}(P_{-n+i}^*P_{k_j}U_{-n}))_{\alpha'2}|^2 \right\}
\]
\[
= \sum_{l=1}^s C_l E_\gamma \{|(V_i)_{12}|^{2l}\},
\]
where the coefficients $C_l$ are the same for $\gamma = i$ and $\gamma = i - 1$ and can be bounded by $C^p$, since $|(P_{-n+i-1})_{1\alpha}| \leq 1$ and $|(P_{-n+i}^*P_{k_j}U_{-n}))_{\alpha'2}| \leq 1$. Moreover, since
\[
|E_i \{|(V_i)_{12}|^{2l}\} - E_{i-1} \{|(V_i)_{12}|^{2l}\}| \leq C^p p! e^{-CW^2},
\]
we obtain
\[
|\langle E_0 - E_{2n} \{ \Psi_{k_1, \ldots, k_s} \rangle \rangle | \leq nC_1^p p! e^{-CW^2}
\]
Then the summation with respect to $s$ gives the bound $nC_1^{C_{2n}/W} e^{-CW^2} = O(e^{-CW^2})$. This yields Lemma [13] since the expression under the expectation in [5,16] has the same number of elements of $V_j$ and $V_j^*$. \(\square\)

**Acknowledgements.** I am grateful to Thomas Spencer who drew my attention to this problem. Also I would like to thank both referees for their substantial efforts and useful comments which help to make the presentation of the paper much more clear.

**References**

[1] Baik, J.; Deift, P.; Strahov, E.: Products and ratios of characteristic polynomials of random Hermitian matrices. J. Math. Phys., 44, 3657 – 3670 (2003)

[2] Berezin, F.A.: Introduction to the algebra and analysis of anticommuting variables. Moscow State University Publ., Moscow (1983) (Russian)

[3] Bogachev, L. V., Molchanov, S. A., and Pastur, L. A.: On the level density of random band matrices. Mat. Zametki, 50:6, 31 – 42(1991)

[4] Brezin, E., Hikami, S.: Characteristic polynomials of random matrices. Commun. Math. Phys. 214, 111 – 135 (2000)

[5] Brezin, E., Hikami, S.: Characteristic polynomials of real symmetric random matrices. Commun. Math. Phys. 223, 363 – 382 (2001)
[6] Borodin, A., Strahov, E.: Averages of characteristic polynomials in random matrix theory. Comm. Pure Appl. Math., 59, 161 – 253 (2006)

[7] Casati, G., Molinari, L., Israilev, F.: Scaling properties of band random matrices, Phys. Rev. Lett. 64 (1990), 1851–1854.

[8] Disertori, M., Pinson, H., and Spencer, T.: Density of states for random band matrices. Comm. Math. Phys. 232, 83 – 124 (2002)

[9] Efetov, K.: Supersymmetry in disorder and chaos. Cambridge university press, New York (1997)

[10] Erdős, L., Knowles, A.: Quantum diffusion and eigenfunction delocalization in a random band matrix model. Comm. Math. Phys. 303, 509 – 554 (2011).

[11] Erdős, L., Knowles, A., Yau, H.-T., Yin, J.: Delocalization and diffusion profile for random band matrices. arXiv:1205.5669v1

[12] Erdős, L., Yau, H.-T., Yin, J.: Bulk universality for generalized Wigner matrices, Preprint arXiv:1001.3453.

[13] Fyodorov, Y.V., Mirlin, A.D.: Scaling properties of localization in random band matrices: a σ-model approach, Phys. Rev. Lett. 67, 2405 – 2409 (1991).

[14] Fyodorov, Y. V., Strahov, E.: An exact formula for general spectral correlation functions of random matrices, J. Phys. A 36, 3203 – 3213 (2003)

[15] Götze, F., Kösters, H.: On the second-ordered correlation function of the characteristic polynomial of a Hermitian Wigner matrix. Commun. Math. Phys. 285, 1183 – 1205 (2008)

[16] Hughes, C., Keating, J., O’Connell, N. On the characteristic polynomials of a random unitary matrix. Comm. Math. Phys., 220, 429 – 451 (2001)

[17] Kösters, H.: Characteristic polynomials of sample covariance matrices: the nonsquare case, Cent. Eur. J. Math. 8, 763 – 779 (2010)

[18] Keating, J.P., Snaith, N.C.: Random matrix theory and ζ(1/2 + it). Commun. Math. Phys. 214, 57 – 89 (2000)

[19] Mehta, M.L.: Random Matrices. Academic Press, New York (1991)

[20] Mehta, M.L., Normand, J.-M.: Moments of the characteristic polynomial in the three ensembles of random matrices. J.Phys A: Math.Gen. 34, 4627 – 4639 (2001)

[21] Mirlin, A. D.: Statistics of energy levels. New Directions in Quantum Chaos, (Proceedings of the International School of Physics Enrico Fermi, Course CXLIII), ed. by G.Casati, I.Guarneri, U.Smilansky, IOS Press, Amsterdam, 223-298 (2000)

[22] Molchanov, S. A., Pastur, L. A., Khorunzhii, A. M.: Distribution of the eigenvalues of random band matrices in the limit of their infinite order, Theor. Math. Phys. 90, 108 – 118 (1992)
[23] Schenker, J.: Eigenvector localization for random band matrices with power law band width, Comm. Math. Phys. 290, 1065 – 1097 (2009)

[24] Shcherbina, T.: On the correlation function of the characteristic polynomials of the Hermitian Wigner ensemble. Commun. Math. Phys. 308, p. 1 – 21 (2011)

[25] Shcherbina, T.: On the correlation functions of the characteristic polynomials of the Hermitian sample covariance ensemble, preprint: arXiv: 1105.3051v2.pdf (2011)

[26] Spencer, T.: SUSY statistical mechanics and random band matrices. Quantum many body system, Cetraro, Italy 2010, Lecture notes in mathematics 2051 (CIME Foundation subseries) (2012)

[27] Strahov, E., Fyodorov, Y.V.: Universal results for correlations of characteristic polynomials: Riemann-Hilbert Approach. Commun. Math. Phys. 241, 343 – 382 (2003)

[28] Tao, T., Vu, V.: Random matrices: Universality of the local eigenvalue statistics. Acta Math. 206, 127 – 204 (2011).

[29] Vanlessen, M.: Universal Behavior for Averages of Characteristic Polynomials at the Origin of the Spectrum. Comm. Math. Phys., 253, 535 – 560 (2003)