POLAR ORBITS AROUND BINARY STARS

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Abstract. Oks proposes the existence of a new class of stable planetary orbits around binary stars, in the shape of a helix on a conical surface whose axis of symmetry coincides with the interstellar axis, and rotates with the same orbital frequency as the binary pair. We show that this claim relies on the inappropriate use of an effective potential that is only applicable when the stars are held motionless. We also present numerical evidence that the only planetary orbits whose planes are initially orthogonal to the interstellar axis that remain stable on the time scale of the stellar orbit are ordinary polar orbits around one of the stars, and that the perturbations due to the binary companion do not rotate the plane of the orbit to maintain a fixed relationship with the axis.

1. Introduction

The restricted three-body problem, in which two bodies execute a Kepler orbit while a third body of negligible mass moves in the resulting potential, has been the subject of intensive investigation since the time of Newton, with important contributions by Euler, Jacobi, Lagrange and Poincaré [2].

Oks [3] proposes the existence of a new class of solutions, in which a planet orbits, approximately, in a plane that remains orthogonal to the axis between the two stars as they move in their own orbit, while undergoing perturbations on the surface of a cone that is aligned with the same axis.

The novel feature of the proposed solutions would be the rotation of the plane of the planetary orbit in step with the orbital motion of the stars. Our analysis does not support the existence of solutions of this kind. Rather, we find that for the relevant initial conditions, when there are stable, quasiperiodic orbits they are, essentially, polar orbits around one star, and while the other member of the binary pair will perturb the orbit, it will not induce a complete rotation of the orbital plane on the time scale of a single stellar orbit.

2. Analysis in the original paper

Oks [3] begins by considering planetary orbits around the interstellar axis between two motionless stars. Such systems can be characterized by a two-dimensional parameter space \((w; b)\), where \(w\) is the scaled distance of the plane of the orbit along the interstellar axis from one of the stars, and \(b\) is the mass ratio of the stars. An effective potential \(U_{\text{eff}}\) is derived, associated with a conserved quantity \(M\), the projection of the planet’s orbital angular momentum on the interstellar axis, and the
regions of the parameter space that allow points of stable equilibrium for $U_{\text{eff}}$ are identified.

Further restrictions on $(w, b)$ are derived which force the frequency of the planetary orbit to be much faster than the Kepler frequency of the binary pair. The aim of these restrictions is to allow a separation of the system into rapid and slow subsystems, but while the paper identifies a range of orbits that meet this condition on the orbital frequencies, it neither cites nor proves any result to the effect that such a separation is a valid method of approximation in this particular context.

The problem of the orbit when the binary pair rotates is studied in a non-inertial frame in which the stars remain fixed and the dynamics is modified by centrifugal and Coriolis forces. This analysis proceeds by treating the planet’s degrees of freedom in a plane through the interstellar axis as a two-dimensional harmonic oscillator, driven by the Coriolis force, around a stable equilibrium point in the $U_{\text{eff}}$ derived under the assumption that the stars are motionless. If the amplitudes of the driven oscillator are small, the full motion of the planet will be a helix lying on the surface of a cone whose axis of symmetry is the interstellar axis.

However, we believe this analysis is not correct, as $U_{\text{eff}}$ depends on $M$, which is not conserved, even approximately, as the binary pair rotates.

3. Torques and angular momentum

In Section 3.2 of [3], the centrifugal and Coriolis forces are obtained by approximating the planetary orbit as circular, with its center on the interstellar axis and its plane orthogonal to that axis. This is not intended as a detailed description of the motion, but the departures from it are assumed to be small and cyclical, with periods much shorter than the time scale associated with the stellar orbit.

Our analysis will extend that in [3] by computing the torques acting on the planet, and rather than assuming that the plane of the orbit remains orthogonal to the axis, we will allow for the possibility that its normal vector makes an angle $\alpha(t)$ with the axis, while remaining in the plane of the stellar orbit. We will also allow for the possibility that the center of the orbit moves along the axis.

We will work in Cartesian coordinates in a rotating frame fixed to the stars, with the origin at the center of mass of the binary pair, the $x$-axis as the interstellar axis, and the $z$-axis as the axis of rotation for the stellar orbit. Wlog, we will assume that the planet has unit mass.

$M(t)$, the projection of the planet’s angular momentum on the interstellar axis, is given by:

\begin{equation}
M(t) = y(t)z'(t) - z(t)y'(t)
\end{equation}

If the angular velocity of the stellar orbit is $\omega$, the Coriolis force is:

\begin{equation}
F_1(t) = 2\omega(y'(t), -x'(t), 0)
\end{equation}

The component of the torque along the $x$-axis due to the Coriolis force is:

\begin{equation}
T_1(t) = y(t)F_{1,z}(t) - z(t)F_{1,y}(t) = 2\omega z(t)x'(t)
\end{equation}

The centrifugal force is:
The component of the torque along the $x$-axis due to the centrifugal force is:

\[ T_2(t) = y(t)F_{2,z}(t) - z(t)F_{2,y}(t) \]

The gravitational force exerted by either star, the displacement of the planet from the center of mass, and the interstellar axis all lie in the same plane, so the torque due to the gravitational force is always orthogonal to the interstellar axis, and has no effect on $M(t)$.

We approximate the motion of the planet with a circular orbit of angular velocity $f$ and radius $\rho$, centered on the point $(x_1(t), 0, 0)$, and tilted at an angle $\alpha(t)$.

\[ x(t) = x_1(t) + \rho \sin(\alpha(t)) \cos(f t) \]
\[ y(t) = \rho \cos(\alpha(t)) \cos(f t) \]
\[ z(t) = \rho \sin(f t) \]

Substituting (6) into (1), (3) and (5) we obtain:

\[ M(t) = \frac{1}{2} \rho^2 \sin(2 f t) \sin(\alpha(t)) \alpha'(t) + f \rho^2 \cos(\alpha(t)) \]

\[ T_1(t) = \rho^2 \omega \sin(2 f t) \cos(\alpha(t)) \alpha'(t) + \]
\[ f \rho^2 \omega \cos(2 f t) \sin(\alpha(t)) - \rho \omega (f \rho \sin(\alpha(t)) - 2 \sin(f t)x_1'(t)) \]

\[ T_2(t) = -\frac{1}{2} \rho^2 \omega^2 \sin(2 f t) \cos(\alpha(t)) \]

Differentiating (7), we obtain:

\[ M'(t) = \frac{1}{2} \rho^2 \sin(2 f t) \left( \sin(\alpha(t)) \alpha''(t) + \cos(\alpha(t)) \alpha'(t) \right) + \]
\[ \frac{f \rho^2 \sin(\alpha(t)) \alpha'(t) (\cos(2 f t) - 1)}{f \rho^2 \sin(\alpha(t)) \alpha'(t) (\cos(2 f t) - 1)} \]

The rate of change of the planet’s angular momentum must be equal to the total torque it experiences, so for the components projected on the interstellar axis we have:

\[ M'(t) - T_1(t) - T_2(t) = 0 \]

If we substitute (8), (9) and (10) into (11), and require the terms with factors of $\sin(2 f t)$, $\cos(2 f t)$ and $\sin(f t)$ to be equal to zero, we obtain the differential equations:

\[ \alpha''(t) \sin(\alpha(t)) + \cos(\alpha(t)) (\omega - \alpha'(t))^2 = 0 \]
\[ \sin(\alpha(t)) (\omega - \alpha'(t)) = 0 \]
\[ x_1'(t) = 0 \]
The second of these equations is satisfied by either \( \alpha(t) = n\pi \) for some integer \( n \), or \( \alpha(t) = \omega t + \alpha_0 \), but the first equation is satisfied only by the latter choice. So the only solution to the full system of equations is:

\[
\begin{align*}
\alpha(t) &= \omega t + \alpha_0 \\
x_1(t) &= x_0
\end{align*}
\]  

(13)

If we substitute this solution into (8), (9) and (10), then equation (11) is satisfied, because the only remaining terms that we did not use to construct (12) are a multiple of the left-hand side of the second equation in (12).

Substituting this solution into (7), we obtain:

\[
M(t) = f \rho^2 \cos(\omega t + \alpha_0) + \frac{1}{2} \rho^2 \omega \sin(2ft) \sin(\omega t + \alpha_0)
\]  

(14)

The regime of interest in [3] is \( f \gg \omega \), so the first term here will dominate, and \( M(t) \) will be approximately sinusoidal, with a period equal to that of the stellar orbit. So \( M(t) \) is not conserved, even approximately, on the time scale of the stellar orbit.

The ansatz applied in Section 3.2 of [3] is a restricted version of the one we have used, imposing the assumption that \( \alpha(t) = 0 \), but we have seen that this assumption is not compatible with the relationship between the torques that the planet experiences and the evolution of its orbital angular momentum. This means that the subsequent treatment of the orbits, which relies on \( M(t) \) being at least approximately conserved in order that the same effective potential can be used throughout the stellar orbit, is not correct.

4. Numerical Analysis

To examine the behavior of orbits that are initially orthogonal to the interstellar axis, we carried out numerical integration for a sample of 540 points in the \((w,b)\) parameter space, in each case over a single stellar orbit. These calculations were performed in Mathematica version 11.1, using 200 decimal digits of working precision.

The results allow us to discriminate between four possibilities, over the time scale of a single stellar orbit: the plane of the planetary orbit rotates in such a way that it remains (approximately) orthogonal to the interstellar axis, as proposed in [3]; the plane of the orbit remains orthogonal to an (approximately) fixed direction in an inertial frame; the orbit becomes unbound; or the orbit exhibits some more complex behavior. The long-term stability of these orbits is beyond the scope of our analysis; see [1] for a detailed investigation of the stability of high-inclination orbits around binary stars.

To determine which of our samples serve as suitable test cases for the claims in [3], we note that the analysis there relies on four prerequisites: (1) that a circular orbit around the interstellar axis exists for the choice of \((w,b)\) in the case where the two stars are held motionless; (2) that this orbit is stable against small perturbations; (3) that the frequency of this orbit is at least 10 times greater than the frequency of the stellar orbit; and (4) that the radial perturbations in the orbit are reasonably small compared to the radius itself.
Figure 1. Sample points in the $(w, b)$ parameter space for orbits close to the lighter of the two stars. Here $w$ is the scaled distance from the lighter star, and $b$ is the mass ratio of the stars. To the left of the unbroken curve is the region where stable orbits around the interstellar axis are possible when the stars are held motionless, to the left of the dashed curve is the region where the planetary orbit has an angular frequency at least 10 times as fast as the stellar orbit, and to the right of the dot-dashed curve is the region where the radial perturbations described by equation (15) are smaller than half the radius of the planetary orbit.

All the sample points we chose met the first two prerequisites, and for each choice of $(w, b)$ we simply transformed the initial conditions for the stable circular orbit that exists when the stars are held fixed into the rotating frame attached to the stars when they are allowed to orbit. We chose some points that met the third prerequisite, along with some that did not, to allow for a comparison.

In determining which of our sample points met the fourth prerequisite, one complication is that both the original paper and a subsequent erratum, [4], contain errors in the algebra leading up to the final statements of the claimed perturbations from the equilibrium orbits, as given by equation (51) in both versions. The original paper omitted a factor of twice the radius of the planetary orbit, and while the erratum correctly inserts this factor, it also corrects a sign error in the change of coordinates from equation (48) to equation (50), but does not correct a sign error in the second line of equation (48), which was previously canceling out the first error. The result is that the erratum gives an incorrect version of equation (51) that claims the perturbations from the equilibrium would be entirely axial. This makes no sense physically, since it amounts to saying that the response of a
Figure 2. Sample points in the \((w, b)\) parameter space for orbits close to the heavier of the two stars. Here \(1 - w\) is the scaled distance from the heavier star, and \(b\) is the mass ratio of the stars. To the left of the unbroken curve is the region where stable orbits around the interstellar axis are possible when the stars are held motionless, and to the left of the dashed curve is the region where the planetary orbit has an angular frequency at least 10 times as fast as the stellar orbit. Note that there are no points here where the radial perturbations described by equation (15) are smaller than half the radius of the planetary orbit.

two-dimensional driven harmonic oscillator is colinear with the driving force (the Coriolis force, which is directed along the interstellar axis) even though the interstellar axis is not a principal axis of the oscillator.

A correct calculation of the amplitudes of the perturbations yields a result that agrees with equation (51) in the original paper, except for the insertion of the missing factor of twice the radius of the planetary orbit:

\[
\begin{align*}
\delta w(\tau) &= \frac{4\omega_s f_p v_0(w, b)}{\omega_+^2 - \omega_-^2} \cos(2\alpha) \cos(f_p \tau) \\
\delta v(\tau) &= \frac{4\omega_s f_p v_0(w, b)}{\omega_+^2 - \omega_-^2} \sin(2\alpha) \cos(f_p \tau)
\end{align*}
\]

Here we have used the notation of [3], where \(w\) and \(v\) are scaled axial and radial coordinates in a cylindrical coordinate system whose axis is the interstellar axis, \(v_0(w, b)\) is the scaled radius of the equilibrium orbit, \(\tau\) is a scaled time coordinate, \(f_p\) is the scaled primary frequency of the planetary motion, \(\omega_s\) is the scaled Kepler frequency of the stars’ orbit, \(\omega_+\) and \(\omega_-\) are scaled eigenfrequencies of the
two-dimensional harmonic oscillator that approximates the well in the effective potential, and \( \alpha \) is the angle between the coordinate directions and the principal axes of the harmonic oscillator. The scaling corresponds to a choice of units where the distance between the stars, the gravitational constant \( G \), and the mass of the lighter of the two stars, are all set equal to 1.

Figures 1 and 2 show the sample points we used, close to the lighter and heavier of the stars, respectively. In each case, the unbroken curve marks the farthest distance from the star for which stable orbits around the interstellar axis exist when the stars are held motionless, while the dashed curve marks the farthest distance from the star for which the frequencies of these orbits are at least 10 times faster than the frequency of the stellar orbit. In Figure 1, the dot-dashed curve marks the nearest distance to the star for which the radial perturbations described by (15) are no more than half the radius of the planetary orbit. Note that all the sample points close to the heavier star fail to meet that condition. So, the subset of the sample points which meet all four prerequisites consists of the 59 points in the shaded region of Figure 1.

The results of each integration were assessed as follows. (1) If, after one stellar orbit, the kinetic energy of the planet exceeded its potential energy in relation to the star to which it was originally bound, the orbit was classified as unbound. (2) Otherwise, \( M(t) \), the projection of the orbital angular momentum on the interstellar axis, was examined throughout the stellar orbit, to see if it remained approximately constant. However, for every orbit considered, either \( M(t) \) changed sign, or it fell to less than one per cent of its original value. (3) The planet’s displacement vector from the closest star was checked throughout the stellar orbit, to see if it remained within 15° on either side of its original plane; if it did, the orbit was classified as planar.

It is apparent from the results shown in Figures 1 and 2 that the majority of orbits examined were planar by this definition, including 49 of the 59 orbits in the shaded region of Figure 1 that meet the four prerequisites for the analysis in [3]. So, at least on the time scale of a single stellar orbit, these 49 orbits are, essentially, ordinary polar orbits around one of the stars, perturbed to some degree by the companion star.

When the orbits are located farther from one of the stars, they begin to exhibit more complex, nonplanar motion, and close to the boundary where the orbits cease to be stable when the stars are held motionless, they become unbound within one stellar orbit. We should emphasize that in none of these cases are we claiming to have demonstrated the long-term stability of the orbit; our primary aim is simply to show that none of the orbits we considered exhibited the novel features claimed in [3], which would be apparent within a single stellar orbit.

Figures 3, 4 and 5 show the trajectory of the planet, and plots of \( M(t) \), for an example of each kind of orbit. The trajectory is shown in a non-rotating frame centered on the closest star. For the plots of \( M(t) \) both axes are scaled, by dividing \( M(t) \) by \( M(0) \), and dividing the time \( t \) by the period of the stellar orbit, \( \tau_S \). The example in Figure 3 is for initial conditions that meet all four prerequisites for the analysis in [3]. Similar plots for all 540 cases can be viewed at http://www.gregegan.net/SCIENCE/PolarOrbits/PolarOrbits.html.
Figure 3. Trajectory of planet and plot of $M(t)$ for an example of a planar orbit.

Figure 4. Trajectory of planet and plot of $M(t)$ for an example of a nonplanar orbit.

5. Conclusions

The claims in [3] in regard to a proposed new class of stable orbits in the restricted three-body problem are based on an incorrect assumption: that the effective potential that governs a planet orbiting the axis between two motionless stars can still be employed when the stars are rotating. An analysis of the torques acting on the planet shows that its orbital angular momentum will change in a manner that is inconsistent with this assumption.

Furthermore, our numerical results show that for 540 choices of initial conditions, including 59 that meet all the prerequisites for the claims in [3] to apply, the plane of the planet’s orbit is not rotated to maintain even approximate orthogonality with the interstellar axis.
Figure 5. Trajectory of planet and plot of $M(t)$ for an example of an unbound orbit. The heavier line shows the orbit of the second star.

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References

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