Contact Lax pairs and associated (3+1)-dimensional integrable dispersionless systems

Maciej Błaszak $^a$ and Artur Sergiyev $^b$

$^a$ Faculty of Physics, Division of Mathematical Physics, A. Mickiewicz University
Umultowska 85, 61-614 Poznań, Poland
E-mail blaszakm@amu.edu.pl

$^b$ Mathematical Institute, Silesian University in Opava,
Na Rybníčku 1, 74601 Opava, Czech Republic
E-mail artur.sergyeyev@math.slu.cz

January 17, 2019

Abstract

We review the recent approach to the construction of (3+1)-dimensional integrable dispersionless partial differential systems based on their contact Lax pairs and the related $R$-matrix theory for the Lie algebra of functions with respect to the contact bracket. We discuss various kinds of Lax representations for such systems, in particular, linear nonisospectral contact Lax pairs and nonlinear contact Lax pairs as well as the relations among the two. Finally, we present a large number of examples with finite and infinite number of dependent variables, as well as the reductions of these examples to lower-dimensional integrable dispersionless systems.

1 Introduction

Integrable systems play an important role in modern mathematics and theoretical and mathematical physics, cf. e.g. [15, 34], and, since according to general relativity our spacetime is four-dimensional, integrable systems in four independent variables ((3+1)D for short; likewise (n+1)D is shorthand for $n+1$ independent variables) are particularly interesting. For a long time it appeared that such systems were very difficult to find but in a recent paper by one of us [39] a novel systematic and effective construction for a large new class of integrable (3+1)D systems was introduced. This construction uses Lax pairs of a new kind related to contact geometry. Moreover, later in [5] it was shown that the systems from this class are amenable to an appropriate extension of the $R$-matrix approach which paved the way to constructing the associated integrable hierarchies.

The overwhelming majority of integrable partial differential systems in four or more independent variables known to date, cf. e.g. [15, 16, 28, 29] and references therein, including the celebrated (anti-)self-dual Yang–Mills equations and (anti-)self-dual vacuum Einstein equations with vanishing cosmological constant, can be written as homogeneous first-order quasilinear, i.e., dispersionless, also known as hydrodynamic-type, systems, cf. e.g. [12, 15, 16, 48] and the discussion below for details on the latter.

Integrable (3+1)D systems from the class introduced in [39] and further studied in [5, 40, 41] also are dispersionless, and it is interesting to note that this class appears to be entirely new: it does not
seem to include any of the previously known examples of integrable dispersionless (3+1)D systems with nonisospectral Lax pairs, e.g. those from [13, 14, 16, 23].

In the present paper we review the results from [39, 5] and provide some novel examples of integrable (3+1)D systems using the approach from these papers.

The rest of the text is organized as follows. After a brief review of (3+1)D dispersionless systems and their nonisospectral Lax pairs in general in Section 2 we proceed with recalling the properties of linear and nonlinear Lax pairs in (1+1)D and (2+1)D in Section 3. In Section 4 we review, following [39], the construction of linear and nonlinear contact Lax pairs and the associated integrable (3+1)D systems and illustrate it by several examples. Finally, in Section 5 we survey, following [5], a version of the $R$-matrix formalism adapted to this setting and again give a number of examples to illustrate it.

2 Isospectral versus nonisospectral Lax pairs

Dispersionless systems in four independent variables $x, y, z, t$ by definition can be written in general form

$$A_0(u)u_t + A_1(u)u_x + A_2(u)u_y + A_3(u)u_z = 0$$

where $u = (u_1, \ldots, u_N)^T$ is an $N$-component vector of unknown functions and $A_i$ are $M \times N$ matrices, $M \geq N$.

Integrable systems of the form (1) typically have scalar Lax pairs of general form

$$\chi_y = K_1(p, u)\chi_x + K_2(p, u)\chi_z + K_3(p, u)\chi_p,$$
$$\chi_t = L_1(p, u)\chi_x + L_2(p, u)\chi_z + L_3(p, u)\chi_p,$$

where $\chi = \chi(x, y, z, t, p)$ and $p$ is the (variable) spectral parameter, cf. e.g. [10, 49, 39] and references therein; we stress that $u_p = 0$.

In general, if at least one of the quantities $K_3$ or $L_3$ is nonzero, these Lax pairs are nonisospectral as they involve $\chi_p$. The same terminology is applied in the lower-dimension case, when e.g. the dependence on $z$ is dropped. The isospectral case when both $K_3$ and $L_3$ are identically zero is substantially different from the nonisospectral one. In particular, it is conjectured in [19] that integrable systems with isospectral Lax pairs (2) are linearly degenerate while those with nonisospectral Lax pairs (2) are not, which leads to significant differences in qualitative behavior of solutions: according to a conjecture of Majda [27], in linearly degenerate systems no shock formation for smooth initial data occurs, see also the discussion in [16]. Many examples of integrable dispersionless (3+1)D systems with Lax pairs (2) in the isospectral case can be found e.g. in [23, 32, 38, 42] and references therein.

On the other hand, it appears that, among dispersionless systems, only linearly degenerate systems admit recursion operators being Bäcklund auto-transformations of linearized versions of these systems, cf. e.g. [30] and references therein for general introduction to the recursion operators of this kind, and [31, 32, 38, 42] and references therein for such operators in the context of dispersionless systems. The theory of recursion operators for integrable dispersionless systems with nonisospectral Lax pairs (2), if any exists, should be significantly different both from that of the recursion operators as auto-Bäcklund transformations of linearized versions of systems under study and from that of bilocal recursion operators, see e.g. [20] and references therein for the latter.

Finally, in the case of nonisospectral Lax pairs (2) integrability of associated nonlinear systems is intimately related to the geometry of characteristic varieties of the latter [17, 11]. On the other hand, for large classes of (1+1)D and (2+1)D dispersionless integrable systems their nonlinear Lax representations are related to symplectic geometry, see e.g. [25, 4, 6, 16, 17, 18, 33, 39, 49] and references therein, although there are some exceptions, cf. e.g. [28, 44] and references therein. As a consequence of this, in the (1+1)D case the systems under study can be written in the form of the Lax equations which take
the form of Hamiltonian dynamics on some Poisson algebras. For the (2+1)D case, the systems under study can be written as zero-curvature-type equations on certain Poisson algebras, i.e., as Frobenius integrability conditions for some pseudopotentials or, equivalently, for Hamiltonian functions from the Poisson algebra under study. Moreover, thanks to some features of symplectic geometry, the original nonlinear Lax representations in (1+1)D and (2+1)D imply linear nonisospectral Lax representations written in terms of Hamiltonian vector fields of the form (2), as discussed in the next section.

In view of the wealth of integrable (2+1)D dispersionless systems it is natural to look for new multidimensional integrable systems which are dispersionless, and it is indeed possible to construct in a systematic fashion such new (3+1)D systems using contact geometry instead of symplectic one in a way proposed in [39], and we review this construction below. In particular, we will show how, using this construction, one obtains a novel class of nonisospectral Lax pairs together with the associated zero-curvature-type equations in the framework of Jacobi algebras, i.e., as Frobenius integrability conditions for contact Hamiltonian functions from such an algebra.

In what follows we will be interested in the class of dispersionless systems possessing nonisospectral Lax representations.

3 Lax representations for dispersionless systems in (1+1)D and (2+1)D

3.1 Nonlinear Lax pairs in (1+1)D and (2+1)D

Dispersionless systems in (2+1)D have the form (1) with \( A_3 = 0 \) and \( u_z = 0 \), and these in (1+1)D have the form (1) with \( A_3 = A_2 = 0 \) and \( u_z = u_y = 0 \). For the overwhelming majority of integrable systems of this kind, see e.g. [15, 29, 49], there exists a pseudopotential \( \psi \) such that the systems under study can be written as an appropriate compatibility condition for a nonlinear (with respect to \( \psi \)) Lax pair. The said nonlinear Lax pair takes the form (cf. e.g. [22])

\[
E = \mathcal{L}(\psi_x, u), \quad \psi_t = \mathcal{B}(\psi_x, u),
\]

where \( E \) is an arbitrary constant playing the role reminiscent of that of a spectral parameter for the linear Lax pairs, while in (2+1)D the nonlinear Lax pair takes the form [49] (cf. also e.g. [17, 18, 39] and references therein)

\[
\psi_y = \mathcal{L}(\psi_x, u), \quad \psi_t = \mathcal{B}(\psi_x, u).
\]

The compatibility relations for a Lax pair, which are necessary and sufficient conditions for the existence of a pseudopotential \( \psi \), are equivalent to a system of PDEs for the vector \( u \) of dependent variables.

Let us illustrate this idea by a simple example.

Example 1. Let \( u = (v_1, v_2, u_0, u_1)^T \) and take

\[
\mathcal{L}(\psi_x, u) = \psi_x + u_0 + u_1 \psi_x^{-1}, \quad \mathcal{B}(\psi_x, u) = v_1 \psi_x + v_2 \psi_x^2.
\]

Compatibility of (3) gives

\[
0 = \frac{d\mathcal{L}}{dx} = \psi_{xx} + (u_0)_x + (u_1)_x \psi_x^{-1} - u_1 \psi_{xx} \psi_x^{-2} \Rightarrow u_1 \psi_{xx} \psi_x^{-2} = \psi_{xx} + (u_0)_x + (u_1)_x \psi_x^{-1}.
\]
and
\[
0 = \frac{dL}{dt} = \psi_{xt} + (u_0)_t + (u_1)_t \psi_x^{-1} - u_1 \psi_{xt} \psi_x^{-2}
\]
\[
= \frac{dB}{dx} + (u_0)_t + (u_1)_t \psi_x^{-1} - u_1 \frac{dB}{dx} \psi_x^{-2}
\]
\[
= (v_2)_x \psi_x^2 + [(v_1)_x - 2v_2(u_0)_x] \psi_x + [(u_0)_t - 2v_2(u_1)_x - u_1(v_2)_x - v_1(u_0)_x]
\]
\[
+ [(u_1)_t - u_1(v_1)_x - v_1(u_1)_x] \psi_x^{-1}.
\]

Thus, equating to zero the coefficients at the powers of \( \psi_x \) in the above equation we obtain the following system:
\[
\begin{align*}
(v_2)_x &= 0, \\
(v_1)_x &= 2v_2(u_0)_x, \\
(u_0)_t &= 2v_2(u_1)_x + u_1(v_2)_x + v_1(u_0)_x, \\
(u_1)_t &= u_1(v_1)_x + v_1(u_1)_x.
\end{align*}
\]

(6)

In particular, if we put \( v_2 = \text{const} = \frac{1}{2} \) and \( v_1 = u_0 \), we arrive at a two-component dispersionless system in 1+1 dimensions
\[
\begin{align*}
(u_0)_t &= (u_1)_x + u_0(u_0)_x, \\
(u_1)_t &= u_1(u_0)_x + u_0(u_1)_x.
\end{align*}
\]

(7)

Now turn to the (2+1)D Lax pair (4) with (5). Then we have
\[
\begin{align*}
\psi_{yt} &= \psi_{xt} + (u_0)_t + (u_1)_t \psi_x^{-1} - u_1 \psi_{xt} \psi_x^{-2}, \\
\psi_{ty} &= (v_1)_y \psi_x + v_1 \psi_{xy} + (v_2)_y \psi_x^2 + 2v_2 \psi_x \psi_{xy}.
\end{align*}
\]

(8) (9)

The compatibility of (4) results in
\[
0 = \psi_{yt} - \psi_{ty} = [(u_0)_t - 2v_2(u_1)_x - u_1(v_2)_x - u_1(v_2)_x - v_1(u_0)_x]
\]
\[
+ [(v_2)_x - (v_2)_y] \psi_x^2 + [(v_1)_x - (v_1)_y - 2v_2(u_0)_x] \psi_x
\]
\[
+ [(u_1)_t - u_1(v_1)_x - v_1(u_1)_x] \psi_x^{-1}.
\]

Equating to zero the coefficients at the powers of \( \psi_x \) yields the system
\[
\begin{align*}
(v_2)_y &= (v_2)_x, \\
(v_1)_y &= (v_1)_x - 2v_2(u_0)_x, \\
(u_0)_t &= 2v_2(u_1)_x + u_1(v_2)_x + v_1(u_0)_x, \\
(u_1)_t &= u_1(v_1)_x + v_1(u_1)_x.
\end{align*}
\]

(10)

If we put \( v_2 = \text{const} = \frac{1}{2} \), we arrive at a three-component dispersionless system in 2+1 dimensions:
\[
\begin{align*}
(u_0)_t &= (u_1)_x + v_1(u_0)_x = (u_1)_x + v_1(u_1)_x + v_1(u_1)_y, \\
(u_1)_t &= u_1(v_1)_x + v_1(u_1)_x, \\
(v_1)_y &= (u_0)_x - (v_1)_x.
\end{align*}
\]

(11)

3.2 Basics of Poisson geometry

Now we shall restate the compatibility conditions for Lax pairs (3) and (4) using the language of symplectic geometry but first we briefly recall the setting of the latter.
Namely, consider an even-dimensional (dim $M = 2n$) symplectic manifold $(M, \omega)$, where $\omega$ is a closed ($d\omega = 0$) differential two-form which is nondegenerate, i.e., such that the $n$th exterior power of $\omega$ does not vanish anywhere on $M$.

Then for an arbitrary smooth function $H$ on $M$ there exists a unique vector field $\mathfrak{x}_H$ (the Hamiltonian vector field) defined by

$$i_{\mathfrak{x}_H} \omega = dH \iff \mathfrak{x}_H = \mathcal{P}dH,$$

where $i_{\mathfrak{x}_H} \omega$ is the interior product of vector field $\mathfrak{x}_H$ with $\omega$, and $\mathcal{P}$ is the associated symplectic bivector, i.e., a nondegenerate Poisson bivector; recall that a bivector is a skew-symmetric twice contravariant tensor field. Then $\mathfrak{x}_H$ is referred to as a Hamiltonian vector field with the Hamiltonian $H$.

Note that a symplectic manifold is a particular case of the more general Poisson manifold. A Poisson manifold is a pair $(M, \mathcal{P})$ where $\mathcal{P}$ is a bivector (i.e., a contravariant rank two skew-symmetric tensor field) satisfying the following identity:

$$[\mathcal{P}, \mathcal{P}]_S = 0,$$

where $[\cdot, \cdot]_S$ is the Schouten bracket, cf. e.g. [46, 50].

The Poisson structure $\mathcal{P}$ induces a bilinear map

$$\{\cdot, \cdot\}_\mathcal{P} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M),$$

in the associative algebra $\mathcal{F}(M)$ of smooth functions on $M$ given by

$$\{F, G\}_\mathcal{P} := \mathcal{P}(dF, dG),$$

which endows $\mathcal{F}(M)$ with the Lie algebra structure and also satisfies the Leibniz rule, i.e., the bracket is also a derivation with respect to multiplication in the algebra of functions. Such a bracket is called a Poisson bracket.

It is readily checked that once (13) holds we indeed have

1. $\{F, G\}_\mathcal{P} = -\{G, F\}_\mathcal{P}$, (antisymmetry),
2. $\{F, GH\}_\mathcal{P} = \{F, G\}_\mathcal{P}H + G\{F, H\}_\mathcal{P}$, (the Leibniz rule),
3. $\{F, \{H, G\}_\mathcal{P}\} + \{H, \{G, F\}_\mathcal{P}\} + \{G, \{F, H\}_\mathcal{P}\} = 0$, (the Jacobi identity).

For a $2n$-dimensional symplectic manifold, by the Darboux theorem there exist local coordinates $(x^i, p_i), \ i = 1, \ldots, n$, known as the Darboux coordinates, such that $\omega = d\eta$, where $\eta = \sum_{i=1}^n p_i dx^i$, and hence

$$\omega = d\eta = \sum_{i=1}^n dp_i \wedge dx^i, \quad P = \sum_{i=1}^n \partial x^i \wedge \partial p_i,$$

$$\mathfrak{x}_H = \frac{\partial H}{\partial p_i} \partial x^i - \frac{\partial H}{\partial x^i} \partial p_i,$$

(15)

and

$$\{H, F\}_\mathcal{P} = \mathfrak{x}_H(F) = \frac{\partial H}{\partial p_i} \frac{\partial F}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial F}{\partial p_i}.$$  

(16)

For any $H, F \in \mathcal{F}(M)$ we also have

$$[\mathfrak{x}_H, \mathfrak{x}_F] = \mathfrak{x}_{\{H,F\}_\mathcal{P}},$$

(17)

where $[\cdot, \cdot]$ is the usual Lie bracket (commutator) of vector fields.
3.3 Compatibility conditions for Lax pairs via Poisson geometry

Now let us return to the Lax pair (3)

\[ E = \mathcal{L}(\psi_x, u) \quad \psi_t = \mathcal{B}(\psi_x, u) \tag{18} \]

for the (1+1)D case when \( u = u(x,t) \).

We have

\[ 0 = \frac{d \mathcal{L}}{dx} = \frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{L}}{\partial \psi_x} \psi_{xx} \implies \psi_{xx} = - \left( \frac{\partial \mathcal{L}}{\partial \psi_x} \right)^{-1} \frac{\partial \mathcal{L}}{\partial x} \tag{19} \]

and so

\[ 0 = \frac{d \mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \psi_x} \psi_{xt} + \frac{\partial \mathcal{L}}{\partial \psi_x} \frac{\partial B}{\partial \psi_x} + \frac{\partial \mathcal{L}}{\partial \psi_x} \frac{\partial B}{\partial \psi_x} \frac{\partial \mathcal{L}}{\partial \psi_x} d\mathcal{x} \]

\[ = \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \psi_x} \left( \frac{\partial B}{\partial x} + \frac{\partial B}{\partial \psi_x} \psi_{xx} \right) \]

\[ = \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \psi_x} \frac{\partial B}{\partial \psi_x} \frac{\partial \mathcal{L}}{\partial \psi_x} \frac{\partial B}{\partial \psi_x}. \tag{20} \]

Thus, the compatibility condition for Lax pair (3) is equivalently expressed via the so-called Lax equation

\[ L_t = \{B, L\}_P, \tag{21} \]

for a pair of functions \( L = \mathcal{L}(p, u), B = \mathcal{B}(p, u) \), where now \( P = \partial_x \wedge \partial_p \) is a Poisson bivector associated to the symplectic two-form \( dp \wedge dx \) on a two-dimensional symplectic manifold with global Darboux coordinates \((x, p)\). Here \( p \) is an additional independent variable, which in the context of linear Lax pairs will be identified as a \textit{variable spectral parameter}, see next subsection.

Now turn to the nonlinear Lax pair (4)

\[ \psi_y = \mathcal{L}(\psi_x, u) \quad \psi_t = \mathcal{B}(\psi_x, u) \tag{22} \]

for the (2+1)-dimensional case when \( u = u(x, y, t) \).

We have

\[ \psi_{yt} = \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \psi_x} \psi_{xt}, \quad \psi_{ty} = \frac{\partial \mathcal{L}}{\partial \psi_x} \psi_{xt}, \]

\[ = \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \psi_x} \psi_{xy}, \tag{23} \]

\[ \psi_{tx} = \psi_{xt} = \frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{L}}{\partial \psi_x} \psi_{xx}, \quad \psi_{ty} = \psi_{xy} = \frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{L}}{\partial \psi_x} \psi_{xx}, \tag{24} \]

and thus,

\[ 0 = \psi_{yt} - \psi_{ty} \Rightarrow \mathcal{L}_t - B_y + \frac{\partial \mathcal{L}}{\partial \psi_x} \frac{\partial B}{\partial \psi_x} - \frac{\partial \mathcal{L}}{\partial \psi_x} \frac{\partial B}{\partial \psi_x}. \tag{25} \]

The compatibility condition for the Lax pair (4) can be now written as the so-called zero-curvature-type equation of the form

\[ L_t - B_y + \{L, B\}_P = 0, \tag{26} \]

for a pair of Lax functions \( L = \mathcal{L}(p, u), B = \mathcal{B}(p, u) \).

For an illustration of this alternative form of the compatibility conditions for our nonlinear Lax pairs let us return to our example.

\textbf{Example 1a.} Let \( L(p, u) = p + u_0 + u_1 p^{-1}, B(p, u) = v_1 p + v_2 p^2 \), where \( u = (v_1, v_2, u_0, u_1)^T \) and \( u = u(x,t) \). Then (27) gives

\[ 0 = L_t - \{B, L\}_P \]

\[ = (v_2)_x p^2 + [(v_1)_x - 2v_2(u_0)_x] p + [(u_0)_t - 2v_2(u_1)_x - v_1(u_0)_x - u_1(v_2)_x] \]

\[ + [(u_1)_t - v_1(u_1)_x - u_1(v_1)_x] p^{-1}, \]

\textit{Example 1b.} Let \( L(p, u) = p + u_0 + u_1 p^{-1}, B(p, u) = v_1 p + v_2 p^2 \), where \( u = (v_1, v_2, u_0, u_1)^T \) and \( u = u(x,t) \). Then (27) gives

\[ 0 = L_t - \{B, L\}_P \]

\[ = (v_2)_x p^2 + [(v_1)_x - 2v_2(u_0)_x] p + [(u_0)_t - 2v_2(u_1)_x - v_1(u_0)_x - u_1(v_2)_x] \]

\[ + [(u_1)_t - v_1(u_1)_x - u_1(v_1)_x] p^{-1}, \]

\textit{Example 1c.} Let \( L(p, u) = p + u_0 + u_1 p^{-1}, B(p, u) = v_1 p + v_2 p^2 \), where \( u = (v_1, v_2, u_0, u_1)^T \) and \( u = u(x,t) \). Then (27) gives

\[ 0 = L_t - \{B, L\}_P \]

\[ = (v_2)_x p^2 + [(v_1)_x - 2v_2(u_0)_x] p + [(u_0)_t - 2v_2(u_1)_x - v_1(u_0)_x - u_1(v_2)_x] \]

\[ + [(u_1)_t - v_1(u_1)_x - u_1(v_1)_x] p^{-1}, \]
and equating to zero the coefficients at the powers of \( p \), we again obtain the system (6), where we can put \( v_2 = \text{const} = \frac{1}{2} \) and \( v_1 = u_0 \), and then again arrive at the two-component dispersionless system (7).

On the other hand, in the (2+1)D case, when \( u = u(x, y, t) \), the zero-curvature-type equation (26) gives

\[
0 = L_t - B_y + \{L, B\}_P
\]

\[
= [(v_2)_x - (v_2)_y] p^2 + [(v_1)_x - 2v_2(u_0)_x - (v_1)_y] p
\]

\[
+ [(u_0)_t - 2v_2(u_1)_x - v_1(u_0)_x - u_1(v_2)_x] + [(u_1)_t - v_1(u_1)_x - u_1(v_1)_x] p^{-1}.
\]

Again, equating to zero the coefficients at the powers of \( p \) reproduces the system (10), and we can put \( v_2 = \text{const} = \frac{1}{2} \) and recover the system (11).

3.4 Linear nonisospectral Lax pairs in (1+1)D and (2+1)D

The relation (17) among the Poisson algebra of functions on \( M \) and the Lie algebra of Hamiltonian vector fields gives rise to alternative linear nonisospectral Lax pairs written in terms of Hamiltonian vector fields.

In the (1+1)D case such a linear Lax pair takes the form

\[
\mathcal{X}_L(\phi) = \{L, \phi\}_P = 0, \quad \phi_t = \mathcal{X}_B(\phi) = \{B, \phi\}_P,
\]

(27)

where \( \phi = \phi(x, t, p) \), and in the (2+1)-dimensional case the form

\[
\phi_y = \mathcal{X}_L(\phi) = \{L, \phi\}_P, \quad \phi_t = \mathcal{X}_B(\phi) = \{B, \phi\}_P,
\]

(28)

where now \( \phi = \phi(x, y, t, p) \).

Here \( p \) is an additional independent variable known as the variable spectral parameter, cf. e.g. [6, 8, 10, 15, 28] for details; recall that \( u_p \equiv 0 \) by assumption.

Since the Hamiltonian vector field with a constant Hamiltonian is identically zero, the Lax equation (21) implies the compatibility of (27), and the zero-curvature-type equation (26) implies the compatibility of (28), but not vice versa.

Indeed, the compatibility condition for (27) reads

\[
[\partial_t - \mathcal{X}_B, \mathcal{X}_L](\phi) = 0
\]

\[
\Updownarrow \quad (17)
\]

\[
\mathcal{X}_{L_t - \{B, L\}_P}(\phi) = \{L_t - \{B, L\}_P, \phi\} = 0,
\]

while the compatibility condition for (28) takes the form

\[
[\partial_t - \mathcal{X}_B, \partial_y - \mathcal{X}_L](\phi) = 0
\]

\[
\Updownarrow \quad (17)
\]

\[
\mathcal{X}_{L_t - B_y + \{L, B\}_P}(\phi) = \{L_t - B_y + \{L, B\}_P, \phi\} = 0.
\]

For an explicit illustration of this we return to our example.

Example 1b. Again let \( u = (v_1, v_2, u_0, u_1)^T \) and

\[
L(p, u) = p + u_0 + u_1 p^{-1}, \quad B(p, u) = v_1 p + v_2 p^2.
\]

(29)
Then in the (1+1)D case, when \( u = u(x, t) \), the Lax pair (27) reads

\[
\begin{align*}
(1 - u_1/p^2)\phi_x - ((u_0)_x + (u_1)_x/p)\phi_p &= 0, \\
\phi_t &= (v_1 + 2pv_2)\phi_x - (p(v_1)_x + p^2(v_2)_x)\phi_p, 
\end{align*}
\]

(30)

which can be equivalently written as

\[
\begin{align*}
\phi_x &= \frac{p}{p^2 - u_1} [(u_1)_x + p(u_0)_x] \phi_p, \\
\phi_t &= \frac{p}{p^2 - u_1} [(v_1 + 2pv_2) [(u_1)_x + p(u_0)_x] - (p^2 - u_1) [(v_1)_x + (v_2)_x p]] \phi_p.
\end{align*}
\]

(31)

The compatibility condition for (31) is just \( (\phi_x)_t - (\phi_t)_x = 0 \) but we cannot reproduce directly (6) by equating to zero the coefficients at the powers of \( p \). Instead, we get a set of linear combinations of differential consequences of the latter.

Now turn to the (2+1)D case with the same \( L \) and \( B \) given by (27) but with \( u = u(x, y, t) \). The associated linear nonisospectral Lax pair (28) takes the form (2), i.e.

\[
\begin{align*}
\phi_y &= (1 - u_1/p^2)\phi_x - ((u_0)_x + (u_1)_x/p)\phi_p, \\
\phi_t &= (v_1 + 2pv_2)\phi_x - (p(v_1)_x + p^2(v_2)_x)\phi_p, 
\end{align*}
\]

(32)

and, in complete analogy with the (1+1)D case, it is readily checked that its compatibility condition \( (\phi_y)_t - (\phi_t)_y = 0 \) holds by virtue of the zero-curvature-type equation \( L_t - B_y + \{L, B\}_p = 0 \) but not the other way around. Besides, just as in the (1+1)D case above, equating to zero the coefficients at the powers of \( p \) in \( (\phi_y)_t - (\phi_t)_y = 0 \), yields a system being a mix of algebraic and differential consequences of (14).

4 Lax representations for dispersionless systems in (3+1)D

4.1 Nonlinear Lax pairs in (3+1)D

In [39] the following generalization of the (2+1)D nonlinear Lax pair (11) to (3+1)D was found:

\[
\psi_y = \psi_z \mathcal{L} \left( \frac{\psi_x}{\psi_z}, u \right), \quad \psi_t = \psi_z \mathcal{B} \left( \frac{\psi_x}{\psi_z}, u \right),
\]

(33)

where now \( \psi = \psi(x, y, z, t) \). The Lax pairs of the form (33) are called nonlinear contact Lax pairs.

The above generalization leads to large new classes of integrable (3+1)D dispersionless systems for suitably chosen \( \mathcal{L} \) and \( \mathcal{B} \), e.g. rational functions or polynomials in \( \psi_x/\psi_z \) of certain special form.

The compatibility conditions for the Lax pair (33), which are necessary and sufficient conditions for the existence of a nontrivial pseudopotential \( \psi \), are equivalent to a system of PDEs for \( u \) in (3+1)D.

Let us illustrate this idea again on our simple example.

**Example 1c.** Let

\[
\begin{align*}
\psi_y &= \psi_z \mathcal{L} \left( \frac{\psi_x}{\psi_z}, u \right) = \psi_z \left( \frac{\psi_x}{\psi_z} + u_0 + u_1 \left( \frac{\psi_x}{\psi_z} \right)^{-1} \right) = \psi_x + u_0 \psi_z + u_1 \psi_z^2 \psi_x, \\
\psi_t &= \psi_z \mathcal{B} \left( \frac{\psi_x}{\psi_z}, u \right) = \psi_z \left( v_1 \psi_x^2 \psi_z + v_2 \left( \frac{\psi_x}{\psi_z} \right)^2 \right) = v_1 \psi_x + v_2 \psi_z,
\end{align*}
\]

(34)
where $u = (v_1, v_2, u_0, u_1)^T$. Then we have

\[ \begin{aligned}
    \psi_{yt} &= \psi_{xt} + (u_0)_t \psi_z + u_0 \psi_{zt} + (u_1)_t \frac{\psi_z^2}{\psi_x} + 2u_1 \frac{\psi_z \psi_{zt}}{\psi_x} - u_1 \frac{\psi_z^2 \psi_{zt}}{\psi_x^2}, \\
    \psi_{ty} &= (v_1)_y \psi_x + v_1 \psi_{xy} + (v_2)_y \frac{\psi_z^2}{\psi_x} + 2v_2 \frac{\psi_z \psi_{xy}}{\psi_x^2} - v_2 \frac{\psi_z^2 \psi_{xy}}{\psi_x^2},
\end{aligned} \]

(35)

and, the compatibility of (35) results in

\[ \begin{aligned}
    0 &= \psi_{yt} - \psi_{ty} = [(v_2)_x + u_0 (v_2)_z - (v_2)_y + v_2 (u_0)_z] \frac{\psi_z^2}{\psi_x} \\
    &\quad + [(u_1)_x - u_1 (v_1)_x - v_1 (u_1)_x] \frac{\psi_z^2}{\psi_x} \\
    &\quad + [(v_1)_x + u_0 (v_1)_z + 2u_1 (v_2)_z - (v_1)_y - 2v_2 (u_0)_x + v_2 (v_1)_z] \psi_x \\
    &\quad + [(u_0)_x - u_1 (v_2)_x + 2u_1 (v_1)_z - v_1 (u_0)_x - 2v_2 (u_1)_x] \psi_z
\end{aligned} \]

(36)

and we arrive at a four-component dispersionless (3+1)D integrable system

\[ \begin{aligned}
    (u_1)_x &= u_1 (v_1)_x + v_1 (u_1)_x, \\
    (u_0)_x &= u_1 (v_2)_x - 2u_1 (v_1)_z + v_1 (u_0)_x + 2v_2 (u_1)_x, \\
    (v_1)_y &= (v_1)_x + u_0 (v_1)_z + 2u_1 (v_2)_z - 2v_2 (u_0)_x + v_2 (v_1)_z, \\
    (v_2)_y &= (v_2)_x + u_0 (v_2)_z + v_2 (u_0)_z.
\end{aligned} \]

(37)

The fields $u_0$ and $u_1$ are dynamical variables, which evolve in time, while the remaining equations can be seen as nonlocal constraints on $u_0$ and $u_1$ which define the variables $v_1$ and $v_2$. The same situation takes place in (2+1)D case. In the (1+1)D case all fields $v_i$ are expressible via the dynamical fields $u_j$.

### 4.2 Basics of contact geometry

Now let us restate the compatibility conditions for Lax pairs (33) in the language of contact geometry, and to this end we first recall the basics of the latter.

Consider an odd-dimensional ($\dim M = 2n + 1$) contact manifold $(M, \eta)$ with a contact one-form $\eta$ such that $\eta \wedge (d\eta)^{\wedge n} \neq 0$, cf. e.g. [9] and references therein.

For a given contact form $\eta$ there exists a unique vector field $Y$, called the Reeb vector field, such that

\[ i_Y d\eta = 0, \quad i_Y \eta = 1. \]  

(38)

For any function on $M$, there exists a unique vector field $X_H$ (the contact vector field) defined by the formula

\[ i_{X_H} \eta = H, \quad i_{X_H} d\eta = dH - i_Y dH \cdot \eta \iff X_H = \mathcal{P} dH + HY, \]

(39)

where $\mathcal{P}$ is the associated bivector.

Contact manifold is a special case of the so-called Jacobi manifold. A Jacobi manifold [26] is a triple $(M, \mathcal{P}, Y)$ where $\mathcal{P}$ is a bivector and $Y$ a vector field satisfying the following conditions:

\[ [\mathcal{P}, \mathcal{P}]_S = 2Y \wedge \mathcal{P}, \quad [Y, \mathcal{P}]_S = 0. \]

(40)

The Jacobi structure induces a bilinear map $\{ \cdot, \cdot \}_J : \mathcal{F}(M) \times \mathcal{F}(M) \rightarrow \mathcal{F}(M)$ in the associative algebra $\mathcal{F}(M)$ of smooth functions on $M$ through the Jacobi bracket

\[ \{ F, G \}_J := \mathcal{P}(dF, dG) + FY(G) - GY(F), \]

(41)

which turns $\mathcal{F}(M)$ into a Lie algebra and satisfies the generalized Leibniz rule, i.e., we have
1. \( \{F, G\}_J = -\{G, F\}_J \) \text{ (antisymmetry)},

2. \( \{F, GH\}_J = \{F, G\}_J H + G\{F, H\}_J - \{F, 1\}_J GH \) \text{ (the generalized Leibniz rule)},

3. \( \{F, \{H, G\}_J\}_J + \{H, \{G, F\}_J\}_J + \{G, \{F, H\}_J\}_J = 0 \) \text{ (the Jacobi identity)}.

For a \((2n + 1)\)-dimensional contact manifold by the Darboux theorem there exist local coordinates \((x^i, p_i, z)\), where \(i = 1, \ldots, n\), known as the Darboux coordinates, such that we have

\[
\eta = dz + \sum_{i=1}^{n} p_i dx^i \Rightarrow d\eta = \sum_{i=1}^{n} dp_i \wedge dx^i, \quad Y = \partial_z, \quad P = \sum_{i=1}^{n} (\partial x^i \wedge \partial p_i - p_i \partial z \wedge \partial p_i),
\]

\[
X_H = H \frac{\partial}{\partial z} + \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i} - p_i \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial z} - \frac{\partial H}{\partial z} \frac{\partial}{\partial p_i} \right) \right)
\]

and the contact bracket, the relevant special case of the Jacobi bracket, reads

\[
\{H, F\}_C = X_H(F) - Y(H)F = H \frac{\partial F}{\partial z} + \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial F}{\partial x^i} - p_i \frac{\partial H}{\partial p_i} \frac{\partial F}{\partial z} \right) - (H \leftrightarrow F).
\]

We also have

\[
[X_H, X_F] = X_{\{H, F\}_C}.
\]

### 4.3 Zero-curvature-type equations in \((3+1)D\) via the contact bracket

Now return to the Lax pair \((42)\) with \(u = u(x, y, z, t)\),

\[
\psi_y = \psi_x L \left( \frac{\psi_x}{\psi_z}, u \right), \quad \psi_t = \psi_x B \left( \frac{\psi_x}{\psi_z}, u \right).
\]

Let \(\theta \equiv \psi_x / \psi_z\). Then we have

\[
\begin{align*}
\psi_{yt} &= \psi_{zt} L + \psi_z L_t + \psi_x L_\theta - \psi_z \theta L_\theta, \\
\psi_{ty} &= \psi_{zy} B + \psi_z B_y + \psi_{xy} B_\theta - \psi_{zy} \theta B_\theta.
\end{align*}
\]

Again, the compatibility of \((45)\) results in

\[
0 = \psi_{yt} - \psi_{ty} = \psi_z [L_t - B_y + L_\theta B_x - L_z B_\theta - \theta (L_\theta B_z - L_z B_\theta) + \mathcal{L} B_z - B \mathcal{L}_z].
\]

Comparing \((47)\) with \((14)\) we observe that compatibility condition for Lax pair \((42)\) is equivalently given \((39)\) by the so-called zero-curvature-type equation of the form

\[
L_t - B_y + \{L, B\}_C = 0,
\]

for a pair of Lax functions \(L = \mathcal{L}(p, u)\), \(B = \mathcal{B}(p, u)\), where the contact bracket \(\{\cdot, \cdot\}_C\) now is a special case of the contact bracket \((43)\) for the three-dimensional contact manifold with the (global) Darboux coordinates \((x, p, z)\), where \(p\) is the variable spectral parameter just as in the lower-dimensional cases.

The contact bracket in this case reads

\[
\{H, F\}_C = X_H(F) - Y(H)F = H \frac{\partial F}{\partial z} + \frac{\partial H}{\partial p} \frac{\partial F}{\partial x} - p \frac{\partial H}{\partial p} \frac{\partial F}{\partial z} - (H \leftrightarrow F).
\]

For the illustration of that alternative Lax representation let us return to our previous example.
Example 1d. Let
\[ L(p, u) = p + u_0 + u_1 p^{-1}, \quad B(p, u) = v_1 p + v_2 p^2, \]
where \( u = (v_1, v_2, u_0, u_1)^T \). Then, for \((3+1)\)-dimensional case, the contact zero-curvature-type equation \((48)\) reads
\[
0 = L_t - B_y + \{L, B\}_C = [(v_2)_x - (v_2)_y + u_0(v_2)_z + v_2(u_0)_z] p^2 \\
+ [(v_1)_x - 2v_2(u_0)_x - (v_1)_y + 2u_1(v_2)_z + v_2(u_1)_z + u_0(v_1)_z] p \\
+ [(u_0)_x - 2v_2(u_1)_x - v_1(u_0)_x - u_1(v_2)_x + 2u_1(v_1)_x] \\
+ [(u_1)_x - v_1(u_1)_x - u_1(v_1)_x] p^{-1}.
\]
and we recover the four-component \((3+1)\)-dimensional integrable dispersionless system \((37)\).

### 4.4 Linear nonisospectral Lax pairs in \((3+1)D\)

Using the above results from contact geometry we readily can construct two different kinds of linear nonisospectral Lax pairs in \((3+1)D\) generalizing \((28)\), that is,
\[
\phi_y = \mathfrak{X}_L(\phi) = \{L, \phi\}_P, \quad \phi_t = \mathfrak{X}_B(\phi) = \{B, \phi\}_P,
\]
in two different ways.

The first one replaces the Poisson bracket \(\{\cdot, \cdot\}_P\) by the contact bracket \((49)\) and gives us the Lax pair of the form
\[
\phi_y = \{L, \phi\}_C, \quad \phi_t = \{B, \phi\}_C, \quad (51)
\]
where now \(\phi = \phi(x, y, z, t, p)\).

The second one replaces the Hamiltonian vector fields \(\mathfrak{X}_H\) by their contact counterparts \(X_H\), and we obtain
\[
\chi_y = X_L(\chi), \quad \chi_t = X_B(\chi), \quad (52)
\]
where now \(\chi = \chi(x, y, z, t, p)\); here we replaced \(\phi\) by \(\chi\) in order to distinguish \((51)\) from \((52)\). The Lax pairs of the form \((52)\) are called linear contact Lax pairs \((39)\).

Recall that in our particular setting we have
\[
X_H = H \frac{\partial}{\partial z} + \frac{\partial H}{\partial p} \frac{\partial}{\partial x} - \frac{\partial H}{\partial x} \frac{\partial}{\partial p} - p \left( \frac{\partial H}{\partial p} \frac{\partial}{\partial z} - \frac{\partial H}{\partial z} \frac{\partial}{\partial p} \right), \quad Y = \frac{\partial}{\partial z}. \quad (53)
\]

In stark contrast with the \((2+1)D\) case, the two Lax pairs \((51)\) and \((52)\) no longer coincide, since we have
\[
X_H(F) = \{H, F\}_C + FH_z = \{H, F\}_C + FY(H) = \{H, F\}_C + F\{1, H\}_C \quad (54)
\]
instead of
\[
\mathfrak{X}_H(F) = \{H, F\}_P,
\]
and the behaviour of these Lax pairs is quite different too.

As for \((51)\), in complete analogy with the \((2+1)D\) case we readily find that its compatibility condition, \((\phi_y)_t - (\phi_t)_y = 0\), can be written as
\[
\{L_t - B_y + \{L, B\}_C, \phi\}_C = 0, \quad (55)
\]
where \(L = \mathcal{L}(p, u)\), \(B = \mathcal{B}(p, u)\), and thus while the zero-curvature-type equation \((48)\), or equivalently, the compatibility of nonlinear Lax pair \((53)\), implies compatibility of the Lax pair \((31)\) but the converse is not true.
On the other hand, the situation for (52) is very different. We can, by analogy with the discussion before Example 1b, show that the compatibility condition for (52), that is,

\[[\partial_t - X_B, \partial_y - X_L] = 0\]

is, by virtue of (44), equivalent to the following:

\[X_{Lt - B_y + \{L, B\}} C = 0.\]  \hspace{1cm} (56)

Using the formula (53) we immediately see that, in contrast with the (2+1)-dimensional case, (56) implies that

\[L_t - B_y + \{L, B\} C = 0,\]

e.g., (56) is equivalent to (48) rather than being just a consequence of the latter, as it is the case for (55).

Let us show the explicit form of the above nonisospectral Lax pairs (51) and (52) for our example.

**Example 1e.** Again, let

\[L(p, u) = p + u + u_1 p^{-1}, \quad B(p, u) = v_1 p + v_2 p^2,\]  \hspace{1cm} (57)

where \(u = (v_1, v_2, u_0, u_1)^T\). Then the nonisospectral Lax pair (51) reads

\[\phi_y = (1 - u_1/p^2)\phi_x + (u_0 + 2u_1/p)\phi_z + (p(u_0)z + (u_1)_z - (u_0)x - (u_1)x/p)\phi_p + (-(u_0)_z - (u_1)_z/p)\phi,\]

\[\phi_t = (2pv_2 + v_1)\phi_x - p^2v_2\phi_z + ((v_2)_z p^3 + ((v_1)_z - (v_2)_x)p^2 - (v_1)_z p)\phi_p - p((v_2)_z p - (v_1)_z)\phi,\]

while the nonisospectral linear contact Lax pair (52) has the form (2), i.e.

\[\chi_y = \left(1 - \frac{u_1}{p^2}\right)\chi_x + \left(u_0 + \frac{2u_1}{p}\right)\chi_z + \left(p(u_0)_z + (u_1)_z - (u_0)_x - \frac{(u_1)_x}{p}\right)\chi_p,\]

\[\chi_t = (2pv_2 + v_1)\chi_x - p^2v_2\chi_z + ((v_2)_z p^3 + ((v_1)_z - (v_2)_x)p^2 - (v_1)_z p)\chi_p.\]  \hspace{1cm} (58)

Spelling out the compatibility condition for this Lax pair, \((\chi_y)_t - (\chi_t)_y = 0\), and equating to zero the coefficients at \(\chi_x\) and \(\chi_y\) therein, we readily see that, in perfect agreement with general discussion above, we recover (48) for \(L\) and \(B\) given by (57), and then the system (57).

As for (51), it is readily checked that (48) and (57) imply compatibility of (51), that is, \((\phi_y)_t - (\phi_t)_y = 0\), but not the other way around, i.e., \((\phi_y)_t - (\phi_t)_y = 0\) gives us not the system (57) but merely a mix of differential and algebraic consequences thereof.

## 5  \(R\)-matrix approach for dispersionless systems with nonisospectral Lax representations

### 5.1 General construction

The \(R\)-matrix approach addresses two important problems concerning the dispersionless systems under study. First, it allows for a systematic construction of consistent Lax pairs \((L, B)\) in order to generate such systems, and second, it allows for a systematic construction of an infinite hierarchy of commuting symmetries for a given dispersionless system, proving integrability of the latter. So, let us start from some basic facts on the \(R\)-matrix formalism, see for example [8, 36, 37] and references therein.
Let \( g \) be an (in general infinite-dimensional) Lie algebra. The Lie bracket \([\cdot,\cdot]\) defines the adjoint action of \( g \) on \( g \): \( \text{ad}_a b = [a, b] \).

Recall that an \( R \in \text{End}(g) \) is called a (classical) \( R \)-matrix if the \( R \)-bracket

\[
[a,b]_R := [Ra,b] + [a,Rb]
\]

is a new Lie bracket on \( g \). The skew symmetry of (59) is obvious. As for the Jacobi identity for (59), a sufficient condition for it to hold is the so-called classical modified Yang–Baxter equation for \( R \),

\[
[Ra,Rb] - R[a,b]_R - \alpha [a,b] = 0, \quad \alpha \in \mathbb{R}.
\]

Let \( L_i \in g, i \in \mathbb{N} \). Consider the associated hierarchies of flows (Lax hierarchies)

\[
(L_n)_{t_r} = [RL_{t_r},L_n], \quad r, n \in \mathbb{N}.
\]

Suppose that \( R \) commutes with all derivatives \( \partial_{t_n} \), i.e.,

\[
(RL)_{t_n} = RL_{t_n}, \quad n \in \mathbb{N},
\]

and obeys the classical modified Yang–Baxter equation (60) for \( \alpha \neq 0 \). Moreover, let \( L_i \in g, i \in \mathbb{N} \) satisfy (61). Then the following conditions are equivalent:

i) the zero-curvature equations

\[
(RL_r)_{t_s} - (RL_s)_{t_r} + [RL_r,RL_s] = 0, \quad r, s \in \mathbb{N}
\]

hold;

ii) all \( L_i \) commute in \( g \):

\[
[L_i,L_j] = 0, \quad i, j \in \mathbb{N}.
\]

Moreover, if one (and hence both) of the above equivalent conditions holds, then the flows (61) commute, i.e.,

\[
((L_n)_{t_r})_{t_s} - ((L_n)_{t_s})_{t_r} = 0, \quad n, r, s \in \mathbb{N}
\]

The reader can find the proofs of the above results for example in [8] or in [3].

Now let us present a procedure for extending the systems under study by adding an extra independent variable. This procedure bears some resemblance to that of the central extension approach, see e.g. [8, 37] and references therein. Namely, we assume that all elements of \( g \) depend on an additional independent variable \( y \) not involved in the Lie bracket, so all of the above results remain valid. Consider an \( L \in g \) and the associated Lax hierarchy defined by

\[
L_{t_r} = [RL_{t_r},L] + (RL_{t_r})_y, \quad r \in \mathbb{N}.
\]

Suppose that \( L_i \in g, i \in \mathbb{N} \) are such that the zero-curvature equations (63) hold for all \( r, s \in \mathbb{N} \) and the \( R \)-matrix \( R \) on \( g \) satisfies (62). Then the flows (66) commute, i.e.,

\[
(L_{t_r})_{t_s} - (L_{t_s})_{t_r} = 0, \quad r, s \in \mathbb{N}
\]

Indeed, using the equations (66) and the Jacobi identity for the Lie bracket we obtain

\[
(L_{t_r})_{t_s} - (L_{t_s})_{t_r} = \left\{ ([RL_{t_r})_{ts} - (RL_{ts})_{tr} + [RL_{tr},RL_{ts}],L] + ([RL_{tr})_{ts} - (RL_{ts})_{tr} + [RL_{tr},RL_{ts}])_y \right\} = 0.
\]
The right-hand side of the above equation vanishes by virtue of the zero curvature equations (63).

An important question is whether there exists a systematic procedure for constructing \( R \in \text{End}(\mathfrak{g}) \) with the desired properties. Fortunately, the answer is positive. It is well known (see e.g. [36, 37, 8]) that whenever \( \mathfrak{g} \) admits a decomposition into two Lie subalgebras \( \mathfrak{g}_+ \) and \( \mathfrak{g}_- \) such that

\[
\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \quad [\mathfrak{g}_\pm, \mathfrak{g}_\pm] \subset \mathfrak{g}_\pm, \quad \mathfrak{g}_+ \cap \mathfrak{g}_- = \emptyset,
\]

the operator

\[
R = \frac{1}{2} (\Pi_+ - \Pi_-) = \Pi_+ - \frac{1}{2} \tag{68}
\]

where \( \Pi_{\pm} \) are projectors onto \( \mathfrak{g}_{\pm} \), satisfies the classical modified Yang–Baxter equation (60) with \( \alpha = \frac{1}{4} \), i.e., \( R \) is a classical \( R \)-matrix.

Next, we specify the dependence of \( L_j \) on \( y \) via the so-called Lax–Novikov equations (cf. e.g. [7] and references therein)

\[
[L_j, L] + (L_j)_y = 0, \quad j \in \mathbb{N}. \tag{69}
\]

Then, upon applying (64), (68) and (69), after elementary computations, equations (61), (63) and (66) take the following form:

\[
(L_s)_{ts} = [B_r, L_s], \quad r, s \in \mathbb{N}, \tag{70}
\]

\[
(B_r)_{ts} - (B_s)_{tr} + [B_r, B_s] = 0, \tag{71}
\]

\[
L_{tr} = [B_r, L] + (B_r)_y, \quad n, r \in \mathbb{N}, \tag{72}
\]

where \( B_i = \Pi_+ L_i \).

Obviously, if under the reduction to the case when all quantities are independent of \( y \) we put \( L = L_n \) for some \( n \in \mathbb{N} \), then the hierarchies (66) boil down to hierarchies (61) and the Lax–Novikov equations (69) reduce to the commutativity conditions (64). In particular, if the bracket \([\cdot, \cdot]\) is such that equations (66) give rise to integrable systems in \( d \) independent variables, then equations (61) yield integrable systems in \( d - 1 \) independent variables.

A standard construction of a commutative subalgebra spanned by \( L_i \) whose existence ensures commutativity of the flows (61) and (66) is, in the case of Lie algebras which admit an additional associative multiplication \( \circ \) which obeys the Leibniz rule

\[
[a, b \circ c] = [a, b] \circ c + b \circ [a, c], \tag{73}
\]

as follows: the commutative subalgebra is generated by rational powers of a given element \( L \in \mathfrak{g} \), cf. e.g. [37, 8] and references therein. This is also our case for (1+1)D and (2+1)D dispersionless systems, when the Lie algebra in question is a Poisson algebra.

However, in our (3+1)D setting, when the Leibniz rule is no longer required to hold, this construction does not work anymore. In particular, it is the case of (3+1)D dispersionless systems when the Lie algebra under study is a Jacobi algebra. In order to circumvent this difficulty, instead of an explicit construction of commuting \( L_i \), we will impose the zero-curvature constraints (63) on chosen elements \( L_i \in \mathfrak{g}, \ i \in \mathbb{N} \); it is readily seen that in the case of the Jacobi algebra that we are interested in this can be done in a consistent fashion.

Let us come back to the systems considered in the previous sections. For the (3+1)D case consider a commutative and associative algebra \( A \) of formal series in \( p \)

\[
A \ni f = \sum_i u_i p^i \tag{74}
\]

with ordinary dot multiplication

\[
f_1 \cdot f_2 \equiv f_1 f_2, \quad f_1, f_2 \in A. \tag{75}
\]
The coefficients $u_i$ of these series are assumed to be smooth functions of $x, y, z$ and infinitely many times $t_1, t_2, \ldots$

The Jacobi structure on $A$ will be induced by the contact bracket \( [f_1, f_2] \equiv \{f_1, f_2\}_C = \frac{\partial f_1}{\partial p} \frac{\partial f_2}{\partial x} - p \frac{\partial f_1}{\partial p} \frac{\partial f_2}{\partial z} + f_1 \frac{\partial f_2}{\partial z} - (f_1 \leftrightarrow f_2) \). \hspace{1cm} \( (76) \)

Notice that this bracket is independent of $y$. As the unit element $e = 1$ does not belong to the center of the Jacobi algebra, the Leibniz rule \( (73) \) does not hold anymore, and instead we have

\[ \{f_1 f_2, f_3\}_C = \{f_1, f_3\}_C f_2 + f_1 \{f_2, f_3\}_C - f_1 f_2 \{1, f_3\}_C. \] \hspace{1cm} \( (77) \)

For \((2+1)\)D and \((1+1)\)D cases, if we drop the dependence on $z$ or on $z$ and $y$, this bracket reduces to the canonical Poisson bracket \( (16) \) in one degree of freedom

\[ \{f_1, f_2\}_P = \frac{\partial f_1}{\partial p} \frac{\partial f_2}{\partial x} - \frac{\partial f_2}{\partial p} \frac{\partial f_1}{\partial x} \] \hspace{1cm} \( (78) \)

and the Jacobi algebra $g = (A, \cdot, \{,\}_C)$ reduces to the Poisson algebra $g = (A, \cdot, \{,\}_P)$ respectively.

As for the choice of the splitting of the Jacobi algebra $g = (A, \cdot, \{,\}_C)$ into Lie subalgebras $g_\pm$ with $\Pi_\pm$ being projections onto the respective subalgebras, so that $g_\pm = \Pi_\pm(g)$, it is readily checked that we have two natural choices when the $R$’s defined by \( (68) \) satisfy the classical modified Yang–Baxter equation \( (60) \) and thus are $R$-matrices. These two choices are of the form

\[ \Pi_+ = \Pi_{\geq k}, \] \hspace{1cm} \( (79) \)

where $k = 0$ or $k = 1$, and by definition

\[ \Pi_{\geq k} \left( \sum_{j=-\infty}^{\infty} a_j p^j \right) = \sum_{j=k}^{\infty} a_j p^j. \]

Note that for \((1+1)\)D and \((2+1)\)D systems, associated with the Poisson algebra $g = (A, \cdot, \{,\}_P)$, the additional choice of $k = 2$ in \( (79) \) is also admissible \[6, 7\].

### 5.2 Integrable \((3+1)\)D infinite-component hierarchies and their lower-dimensional reductions

We begin with the case of $k = 0$ and the $n$th order Lax function from $A$ of the form

\[ L \equiv L_n = u_n p^n + u_{n-1} p^{n-1} + \cdots + u_0 + u_{-1} p^{-1} + \cdots, \quad n > 0 \] \hspace{1cm} \( (80) \)

and let

\[ B_m \equiv \Pi_+ L_n = v_{m,m} p^m + v_{m,m-1} p^{m-1} + \cdots + v_{m,0}, \quad m > 0 \] \hspace{1cm} \( (81) \)

where $u_i = u_i(x, y, z, \vec{t})$, $v_{m,j} = v_{m,j}(x, y, z, \vec{t})$, and $\vec{t} = (t_1, t_2, \ldots)$.

Substituting $L$ and $B_m$ into the zero-curvature-type equations

\[ L_{tm} = \{B_m, L\}_C + (B_m)_y \] \hspace{1cm} \( (82) \)

we see that one can impose a natural constraint: $u_n = c_n$, $v_{m,m} = c_{m,m}$, where $c_n, c_{m,m} \in \mathbb{R}$.

Then, if we put $c_n = c_{m,m} = 1$, we get

\[ L = p^n + u_{n-1} p^{n-1} + \cdots + u_0 + u_{-1} p^{-1} + \cdots, \quad n > 0, \] \hspace{1cm} \( (83) \)
Thus, the final Lax pair takes the form

\[ 0 = X^m_r[u, v_m], \quad n < r < n + m, \]
\[ (u_r)_m = X^m_r[u, v_m], \quad r \leq n, \quad r \neq 0, \ldots, m - 1, \]
\[ (u_r)_m = X^m_r[u, v_m] + (v_{m,r})_g, \quad r = 0, \ldots, m - 1, \]

where \( v_m = (v_{m,0}, \ldots, v_{m,m} = 1) \) and

\[
X^m_r[u, v_m] = \sum_{s=0}^{m} [sv_{m,s}(u_{r-s+1})_x - (r-s+1)u_{r-s+1}(v_{m,s})_x - (s-1)v_{m,s}(u_{r-s})_z + (r-s-1)u_{r-s}(v_{m,s})_z],
\]

where \( u_n = 1 \) and \( u_r = 0 \) for \( r > n \). The fields \( u_r \) for \( r \leq n \) are dynamical variables while equations for \( n + m > r > n \) can be seen as nonlocal constraints on \( u_r \) which define the variables \( v_{m,s} \). Observe that the additional dependent variables \( v_{m,s} \) for different \( m \) are by construction related to each other through the zero-curvature equations (71).

There is one more constraint in the Lax pair (80) and (81). The first equation from the system (85), i.e., the one for \( r = n + m - 1 \), takes the form

\[(n-1)(v_{m,m-1})_z - (m-1)(u_{n-1})_z = 0,\]

so the system under study for \( n > 1 \) admits a further constraint

\[ v_{m,m-1} = \frac{(m-1)}{(n-1)}u_{n-1}. \]

Thus, the final Lax pair takes the form

\[ L = p^n + u_{n-1}p^{n-1} + \cdots + u_0 + u_{-1}p^{-1} + \cdots, \quad n > 0, \]
\[ B_m = p^m + \frac{(m-1)}{(n-1)}u_{n-1}p^{m-1} + \cdots + v_{m,0}, \quad m > 0 \]

It is readily seen that for \( n = 1 \) the constraint (87) should be replaced by \( u_0 = \text{const} \). Let us consider this case in more detail. Upon taking \( u_0 = 0 \), consider the Lax equation (82) for

\[ L = p + u_{-1}p^{-1} + u_{-2}p^{-2} + \cdots, \]
\[ B_m = p^m + v_{m,m-1}p^{m-1} + \cdots + v_{m,1}p + v_{m,0}, \quad m > 0; \]

then the related system reads

\[ 0 = (v_{m,r})_y + X^m_r[u, v_m], \quad r = 0, \ldots, m - 1, \]
\[ (u_r)_m = X^m_r[u, v_m], \quad r < 0. \]

Thus, the simplest nontrivial case is \( m = 2 \), so

\[ B_2 = p^2 + v_1p + v_0 \]

and generates the following infinite-component system \footnote{5} \footnote{5}

\[
(v_1)_y = (v_1)_x + (u_{-1})_z, \\
(v_0)_y = (v_0)_x + (u_{-2})_z - 2(u_{-1})_x + 2u_{-1}(v_1)_z, \\
(u_r)_{t_2} = 2(u_{r-1})_x - (u_{r-2})_z - (r+1)u_{r+1}(v_0)_x + v_0(u_r)_z + (r-1)u_r(v_0)_z + v_1(u_r)_x - ru_r(v_1)_x + (r-2)u_{r-1}(v_1)_z,
\]

\[ B_2 = p^2 + v_1p + v_0 \]
where \( r < 0 \) and \( v_{2,s} \equiv v_s, s = 0, 1. \)

We have a natural \((2+1)\)D reduction of \((93)\) when \( u_r, v_0 \) and \( v_1 \) are independent of \( y, \)

\[
0 = (v_1)_x + (u_{-1})_z,
0 = (v_0)_x + (u_{-2})_z - 2(u_{-1})_x + 2(u_{-1}(v_1)_z,
(u_r)_t = 2(u_{r-1})_x - (u_{r-2})_z - (r + 1)u_{r+1}(v_0)_x + v_0(u_r)_x
+ (r - 1)u_r(v_0)_z + v_1(u_r)_x - ru_r(v_1)_x + (r - 2)u_{r-1}(v_1)_z,
\]

another \((2+1)\)D reduction

\[
(v_1)_y = (u_{-1})_z,
(v_0)_y = (u_{-2})_z + 2u_{-1}(v_1)_z,
(u_r)_t = -(u_{r-2})_z + v_0(u_r)_z + (r - 1)u_r(v_0)_z + (r - 2)u_{r-1}(v_1)_z,
\]

when \( u_r, v_0 \) and \( v_1 \) are independent of \( x, \) and yet another \((2+1)\)D reduction

\[
(v_1)_y = (v_1)_x,
(v_0)_y = (v_0)_x - 2(u_{-1})_x,
(u_r)_t = 2(u_{r-1})_x - (r + 1)u_{r+1}(v_0)_x + v_1(u_r)_x
\]

when \( u_r, v_0 \) and \( v_1 \) are independent of \( z. \)

Moreover, system \((96)\) admits a further reduction \( v_1 = 0 \) to the form

\[
(v_0)_y = (v_0)_x - 2(u_{-1})_x,
(u_r)_t = 2(u_{r-1})_x - (r + 1)u_{r+1}(v_0)_x + v_1(u_r)_x.
\]

The system \((97)\) reduces to \((1 + 1)\)-dimensional system

\[
(u_r)_t = 2(u_{r-1})_x - 2(r + 1)u_{r+1}(u_{-1})_x, \quad r < 0,
\]

when \( u_i \) are independent of \( y \) and we put \( v_0 = 2u_{-1}. \)

Notice that \((98)\) is the well-known \((1+1)\)D Benney system a.k.a. the Benney momentum chain \([3,24]\). From that point of view, the systems \((97)\) and \((94)\) can be seen as natural \((2+1)\)D extensions of the Benney chain while the system \((93)\) represents a \((3+1)\)D extension of the Benney system.

On the other hand, system \((95)\) admits no reductions to \((1 + 1)\)-dimensional systems. Note that for systems \((93)\)–\((98)\) there are no obvious finite-component reductions.

For systems \((80), (81)\) and \((83), (84)\) we have \((2+1)\)-dimensional and \((1+1)\)-dimensional reductions of the same types as above.

Now pass to the case of \( k = 1, \) when \( \Pi_+ = \Pi_{\geq 1}, \) and consider the general case when

\[
L = u_ne^n + u_n^{-e^n} - u_0 + u_0 + u_0^{-1} + \ldots, \quad n > 0,
B_m = v_m,p^m + v_m^{-p^m} - v_m + v_m + v_m^{-p} + \ldots, \quad m > 0,
\]

from which we again obtain the hierarchies of infinite-component systems

\[
0 = X_r[u,v], \quad n < r \leq n + m,
(u_r)_t = X_r[u,v], \quad r \leq n, \quad r \neq 1, \ldots, m,
(u_r)_t = X_r[u,v] + (v_r)_y, \quad r = 1, \ldots, m,
\]
where \( v_m = (v_{m,1}, \ldots, v_{m,m}) \) and

\[
X^m_r [u, v_m] = \sum_{s=1}^{m} s v_{m,s} (u_{r-s+1})_x - (r-s+1) u_{r-s+1} (v_{m,s})_x - (s-1) v_{m,s} (u_{r-s})_z + (r-s-1) u_{r-s} (v_{m,s})_z.
\]  

(101)

For \( n > 1, m > 1 \) there is an additional constraint imposed on the Lax pair \([99]\). The first equation from the system \(101\), i.e., the one for \( r = n + m \), takes the form

\[
(n-1) u_n (v_{m,m})_z - (m-1) v_{m,m} (u_n)_z = 0,
\]

and hence, for \( n > 1, m > 1 \), admits the constraint

\[
v_{m,m} = (u_n)^{\frac{m-1}{m-1}}.
\]

(102)

So, the final Lax pair takes the form

\[
L = u_n p^n + u_{n-1} p^{n-1} + \cdots + u_0 + u_{-1} p^{-1} + \cdots, \quad n > 1,
\]

\[
B_m = (u_n)^{\frac{m-1}{m-1}} p^m + v_{m,m-1} p^{m-1} + \cdots + v_{m,1} p, \quad m > 0.
\]

(103)

For \( n = 1 \) the constraint in question is replaced by \( u_1 = \text{const} \). Thus, consider again in detail the simplest case when \( n = 1 \) and \( u_1 = 1 \)

\[
L = p + u_0 + u_{-1} p^{-1} + \cdots,
\]

\[
B_m = v_{m,m-1} p^m + v_{m,m-2} p^{m-1} + \cdots + v_{m,1} p, \quad m > 0,
\]

(104)

when the associated system reads

\[
0 = (v_{m,r})_y + X^m_r [u, v_m], \quad r = 1, \ldots, m,
\]

\[
(u_r)_t = X^m_r [u, v_m], \quad r < 0.
\]

(105)

Thus, the simplest nontrivial case is \( m = 2 \), so \( B_2 = v_2 p^2 + v_1 p \) generates the following infinite-component system \([5]\)

\[
(v_2)_y = (v_2)_x + u_0 (v_2)_z + v_2 (u_0)_z,
\]

\[
(v_1)_y = (v_1)_x + u_0 (v_1)_z + v_2 (u_{-1})_z + 2 u_{-1} (v_2)_z - 2 v_2 (u_0)_z,
\]

\[
(u_r)_t = v_1 (u_r)_x - r u_r (v_1)_x + (r - 2) u_{r-1} (v_1)_z + 2 v_2 (u_{r-1})_x - (r - 1) u_{r-1} (v_2)_x - v_2 (u_{r-2})_z + (r - 3) u_{r-2} (v_2)_z,
\]

(106)

where \( r < 1 \) and \( v_{2,s} \equiv v_s, s = 1, 2 \).

We have a natural \((2+1)\)-dimensional reduction of \((106)\) when \( u_r, v_1 \) and \( v_2 \) are independent of \( y \),

\[
0 = (v_2)_x + u_0 (v_2)_z + v_2 (u_0)_z,
\]

\[
0 = (v_1)_x + u_0 (v_1)_z + v_2 (u_{-1})_z + 2 u_{-1} (v_2)_z - 2 v_2 (u_0)_z,
\]

\[
(u_r)_t = v_1 (u_r)_x - r u_r (v_1)_x + (r - 2) u_{r-1} (v_1)_z + 2 v_2 (u_{r-1})_x - (r - 1) u_{r-1} (v_2)_x - v_2 (u_{r-2})_z + (r - 3) u_{r-2} (v_2)_z,
\]

(107)

On the other hand, if \( u_r, v_1 \) and \( v_2 \) are independent of \( x \), we obtain from \((106)\) another \((2+1)\)-dimensional system

\[
(v_2)_y = u_0 (v_2)_z + v_2 (u_0)_z,
\]

\[
(v_1)_y = u_0 (v_1)_z + v_2 (u_{-1})_z + 2 u_{-1} (v_2)_z,
\]

\[
(u_r)_t = (r - 2) u_{r-1} (v_1)_z - v_2 (u_{r-2})_z + (r - 3) u_{r-2} (v_2)_z.
\]

(108)
Next, if \( u_r, v_1 \) and \( v_2 \) in (106) are independent of \( z \), we arrive at the third \((2 + 1)\)-dimensional system

\[
\begin{align*}
(v_2)_y &= (v_2)_x, \\
(v_1)_y &= (v_1)_x - 2v_2(u_0)_x, \\
(u_r)_{t_2} &= v_1(u_r)_x - ru_r(v_1)_x + 2v_2(u_{r-1})_x - (r - 1)u_{r-1}(v_2)_x,
\end{align*}
\]  

(109)

whence, after the substitution \( v_2 = \text{const} = 1 \), we obtain

\[
\begin{align*}
(v_1)_y &= (v_1)_x - 2(u_0)_x, \\
(u_r)_{t_2} &= v_1(u_r)_x - ru_r(v_1)_x + 2(u_{r-1})_x.
\end{align*}
\]  

(110)

If \( u_r, v_1 \) and \( v_2 \) are independent of both \( y \) and \( z \), we can put \( v_1 = 2u_0 \) and obtain

\[
(u_r)_{t_2} = 2(u_{r-1})_x + 2u_0(u_r)_x - 2ru_r(u_0)_x.
\]  

(111)

Finally, when \( u_r, v_1 \) and \( v_2 \) are independent of both \( y \) and \( x \), we have

\[
(u_r)_{t_2} = (r - 2)u_{r-1}(v_1)_x - v_2(u_{r-2})_x + (r - 3)u_{r-2}(v_2)_x,
\]  

(112)

where a reduction

\[
v_2 = au_0^{-1}, \quad v_1 = -au_{-1}u_0^{-2},
\]

was performed, and \( a \in \mathbb{R} \) is an arbitrary constant. Thus, in this case the system under study is rational (rather than polynomial) in \( u_0 \).

### 5.3 Finite-component reductions

For \( k = 0 \), in contrast with the simplest case (90), we do have natural reductions to finite-component systems. Namely, they are of the form

\[
\begin{align*}
L &= u_np^n + u_{n-1}p^{n-1} + \cdots + u_rp^r, \quad r = 0, 1, \\
B_m &= (u_n)^{\frac{n-1}{n-1}}p^m + v_{m,m-1}p^{m-1} + \cdots + v_{m,0},
\end{align*}
\]  

(113)

and

\[
\begin{align*}
L &= p^n + u_{n-1}p^{n-1} + \cdots + u_rp^r, \quad r = 0, 1, \\
B_m &= p^m + (m-1)u_{n-1}p^{m-1} + \cdots + v_{m,0}.
\end{align*}
\]  

(114)

The case (114) for \( r = 0 \) was considered for the first time in [39], while the remaining cases was analyzed in [3]. Notice that in (113) and (114) for \( r = 0 \) we have \( L = B_n \), and hence the variable \( y \) can be identified with \( t_n \). Then equations (82) coincide with the zero-curvature equations (71), and the Lax–Novikov equation (69) reduces to equation (70).

Another class of natural reductions to finite-component systems arises for \( k = 1 \) [5]. Indeed, for \( n > 1 \) we have

\[
\begin{align*}
L &= u_np^n + u_{n-1}p^{n-1} + \cdots + u_rp^r, \quad r = 1, 0, -1, \ldots \\
B_m &= (u_n)^{\frac{m-1}{n-1}}p^m + v_{m,m-1}p^{m-1} + \cdots + v_{m,1}p
\end{align*}
\]  

(115)

while for \( n = 1 \)

\[
\begin{align*}
L &= p + u_0 + u_{-1}p^{-1} + \cdots + u_rp^r, \quad r = 0, 1, -1, \ldots \\
B_m &= v_{m,m}p^m + v_{m,m-1}p^{m-1} + \cdots + v_{m,1}p, \quad m > 1.
\end{align*}
\]  

(116)
In closing we point out a large class of finite-component reductions of the hierarchy associated with \((83)\) and \((84)\) for \(k = 0\). The reductions in question for \(L\) \((83)\) are given by rational Lax functions, cf. \[22, 43\] and references therein for the \((1+1)\)D case, namely,

\[ L = p^n + \sum_{j=0}^{n-1} u_j p^j + \sum_{i=1}^{k} \frac{a_i}{(p - r_i)}, \quad n > 1, \quad k > 0, \tag{117} \]

where \(u_j, a_i\) and \(r_i\) are unknown functions; in this case \(B_m\) are still given by \((84)\).

**Example 2.** First let us begin with the case of \(m = 3\) and \(k = 0\) and the simplest Lax pair from \((114)\) when \(n = 2\) and \(m = 3\)

\[ L = p^2 + u_1 p + u_0, \quad B = p^3 + 2u_1 p^2 + v_1 p + v_0, \tag{118} \]

The zero-curvature-type Lax equation

\[ L_t = \{B, L\}_C + (B)_y \]

generates a four-component system \[39\]

\[
\begin{align*}
(u_0)_t &= (v_0)_y + v_0(u_0) + v_1(u_1)x - u_0(v_0)z - u_1(v_0), \\
(u_1)_t &= (v_1)_y - 2(v_0)x + 4u_1(u_0) - u_1(v_1)x + v_1(u_1)x + v_0(u_1)z - u_0(v_1), \\
0 &= (v_1)_z - (u_1)_x - 2(u_0)z - 2u_1(u_1)_z, \\
0 &= (v_0)_z + 3(u_0)x + 2(u_1)_y - 2(v_1)_x + 2u_1(u_1)x - 2u_1(u_0)z - 2u_0(u_1)_z.
\end{align*} \tag{119} \]

which is a natural \((3+1)D\) extension of the \((2+1)D\) dispersionless Kadomtsev-Petviashvili \((dKP)\) equation. Indeed, upon assuming that all fields are independent of \(z\), and that \(u_1 = 0\) and \(v_1 = \frac{3}{2}u_0\), and denoting \(u_0 \equiv u\) and \(v_0 \equiv v\), the system \((119)\) reduces to the form

\[ u_t = v_y + \frac{3}{2}u u_x, \quad 3u_y = 4v_x \quad \Rightarrow \quad (u_t - \frac{3}{2}u u_x)_x = \frac{3}{4}u_{yy} \tag{120} \]

where the last equation in \((120)\) is, up to a suitable rescaling of independent variables, nothing but the celebrated \(dKP\) equation, also known as the three-dimensional Khokhlov–Zabolotskaya \[47\] equation.

**Example 3.** Consider again our system \((50)\), which is a particular case of \((110)\) for \(k = 1\) and \(r = -1\), with notation \(u_{-1} \equiv u_1\), being the first member of the hierarchy \((110)\) generated by the Lax functions

\[ L = p + u_0 + u_1 p^{-1}, \quad B_2 = v_2 p^2 + v_1 p, \]

and takes the known form \[37\]

\[
\begin{align*}
(u_1)_t &= u_1(v_1)_x + v_1(u_1)_x, \\
(u_0)_t &= -2u_1(v_1)_x + v_1(u_0)_x + u_1(v_2)_x + 2v_2(u_1)_x, \\
(v_1)_y &= (v_1)_x + 2u_1(v_2)_x + v_2(u_1)_x + u_0(v_1)_x - 2v_2(u_0)_x, \\
(v_2)_y &= (v_2)_x + u_0(v_2)_x + v_2(u_0)_x.
\end{align*} \tag{121} \]

The second member of the hierarchy is generated by

\[ L = p + u_0 + u_1 p^{-1}, \quad B_3 = w_3 p^3 + w_2 p^2 + w_1 p \]

and has the form

\[
\begin{align*}
(u_1)_t &= u_1(w_1)_x + w_1(u_1)_x, \\
(u_0)_t &= w_1(u_0)_x - 2u_1(w_1)_x + u_1(w_2)_x + 2w_2(u_1)_x, \\
(w_1)_y &= (w_1)_x + w_2(u_1)_x - u_1(w_3)_x - 2w_2(u_0)_x + 2u_1(w_2)_x + u_0(w_1)_x - 3w_3(u_1)_x, \\
(w_2)_y &= (w_2)_x - 3w_3(u_0)_x + 2w_3(u_1)_x + w_2(u_0)_x + u_0(w_2)_x + 2u_1(w_3)_x, \\
(w_3)_y &= (w_3)_x + u_0(w_3)_x + 2w_3(u_0)_x.
\end{align*} \tag{122} \]
Commutativity of the flows associated with \(t_2\) and \(t_3\), i.e.

\[
((u_i)_{t_2})_{t_3} = ((u_i)_{t_3})_{t_2}, \quad i = 0, 1,
\]

can be checked using the set of relations

\[
\begin{align*}
(v_1)_z &= -\frac{v_2}{w_3}(w_3)_x - \frac{v_2w_2}{4w_3^2}(w_3)_z + \frac{v_2}{2w_3}(w_2)_z + \frac{3}{2}(v_2)_x, \quad (v_2)_z = \frac{v_2}{2w_3}(w_3)_z, \\
(w_1)_{t_2} &= v_1(w_1)_x - w_1(v_1)_x + (v_1)_{t_3}, \\
(w_2)_{t_2} &= v_1(w_2)_x - w_1(v_2)_x + 2v_2(w_1)_x - 2w_2(v_1)_x + (v_2)_{t_3}, \\
(w_3)_{t_2} &= \frac{v_2w_2}{2w_3}(w_2)_z - \frac{w_2}{2}(v_2)_x - \frac{v_2w_2}{4w_3^2}(w_3)_z + \frac{v_1w_3 - v_2w_2}{w_3}(w_3)_x - v_2(w_1)_z + 2v_2(w_2)_x - 3w_3(v_1)_x,
\end{align*}
\]

which is equivalent to the zero-curvature equation

\[
(B_2)_{t_3} - (B_3)_{t_2} + \{B_2, B_3\}_C = 0. \tag{123}
\]

Moreover, the compatibility conditions

\[
((v_i)_y)_{z} = ((v_i)_z)_y, \quad i = 1, 2,
\]

are also satisfied by virtue of (121) and (123).

When \(u_r\) and \(v_j\) are independent of \(z\), we obtain \((2 + 1)D\) systems with additional constraints \(v_2 = \text{const} = \frac{1}{2}, w_3 = \text{const} = \frac{1}{3}\)

\[
\begin{align*}
(u_1)_{t_2} &= u_1(v_1)_x + v_1(u_1)_x, \quad (u_1)_{t_3} = u_1(w_1)_x + w_1(u_1)_x, \\
(u_0)_{t_2} &= v_0(u_0)_x + (u_1)_x, \quad (u_0)_{t_3} = w_1(u_0)_x + u_1(w_2)_x + 2w_2(u_1)_x, \\
(v_1)_y &= (v_1)_x - (u_0)_x, \quad (w_1)_y = (w_1)_x - (u_1)_x - 2w_2(u_0)_x, \\
(v_2)_y &= (v_2)_x - (u_0)_x, \quad (w_2)_y = (w_2)_x - (u_0)_x.
\end{align*}
\tag{124}

see also (11).

When \(u_r\) and \(v_j\) are independent of \(x\), we obtain other \((2 + 1)D\) systems making use of a naturally arising extra constraint \(u_1 = \frac{1}{2}\), namely,

\[
\begin{align*}
(u_0)_{t_2} &= -(v_1)_z, \quad (u_0)_{t_3} = -(w_1)_z, \\
(v_1)_y &= (v_2)_x + u_0(v_1)_x, \quad (w_1)_y = (w_2)_z + u_0(w_1)_z, \\
(v_2)_y &= (u_0v_2)_z, \quad (w_2)_y = (w_3)_x + (u_0w_2)_z, \\
(w_3)_y &= u_0(w_3)_z + 2w_3(u_0)_z. \tag{125}
\end{align*}
\]

Further reduction of (124) and (125) by assuming that \(u_r, v_j\) and \(w_k\) are independent of \(y\) leads to \((1 + 1)D\) systems of the form

\[
\begin{align*}
(u_1)_{t_2} &= (u_1u_0)_x, \quad (u_1)_{t_3} = (u_1u_0^2 + u_1^2)_x, \\
(u_0)_{t_2} &= (u_1 + \frac{1}{2}u_0^2)_x, \quad (u_0)_{t_3} = (\frac{3}{2}u_0^3 + 2u_0u_1)_x,
\end{align*} \tag{126}
\]

where we put \(v_1 = u_0\), \(w_2 = u_0^2\) and \(w_1 = u_0^2 + u_1\).

Likewise, the reduction of (125) and (126) by assuming that \(u_r, v_j\) and \(w_k\) are independent of \(y\) leads to \((1 + 1)D\) systems of the form

\[
\begin{align*}
(u_0)_{t_2} &= \frac{1}{2}(u_0^{-2})_z, \quad (u_0)_{t_3} = -\frac{3}{4}(u_0^{-4})_z,
\end{align*} \tag{127}
\]

thanks to the relations

\[
v_2 = u_0^{-1}, \quad v_1 = -u_0^2, \quad w_3 = u_0^{-2}, \quad w_2 = -u_0^{-3}, \quad w_1 = \frac{3}{4}u_0^{-4}.
\]
Example 4. Consider the first member of the hierarchy \((116)\) for \(r = 1\), generated by the Lax pair

\[
L = wp^3 + wp^2 + vp, \quad B_2 = u^2 p^2 + sp,
\]

and takes the form

\[
\begin{align*}
t & = su_x - 3us_x + ws_x + 2u^2 w_x - u^2 v_x - wu^{-\frac{1}{2}} u_x, \\
t & = sw_x - 2ws_x + 2u^2 v_x - \frac{1}{2} vu^{-\frac{1}{2}} u_x + \frac{1}{2} u - \frac{1}{2} u_y, \\
t & = u_x + s_y - vs_x, \\
0 & = 2us_z - u^\frac{1}{2} w_x + \frac{1}{2} u^\frac{1}{2} u_x + \frac{1}{2} wu^\frac{1}{2} u_z,
\end{align*}
\]

(128)

where we put \(u_3 = u, \ u_2 = w, \ u_1 = v, \ v_{2,1} = s\) and \(t_2 = t\). The \((2 + 1)D\) reduction with all fields independent of \(z\) and \(u = 1, \ s = \frac{2}{3} w\) reads

\[
\begin{align*}
t & = 2w_x - \frac{2}{3} w w_x, \\
t & = v_x - \frac{2}{3} v w x + \frac{2}{3} w y,
\end{align*}
\]

(129)

while the case when all fields are independent of \(x\) takes the form

\[
\begin{align*}
t & = ws_z - u^\frac{1}{2} v_x, \\
t & = (u^\frac{1}{2})_y, \\
t & = s_y, \\
0 & = 2us_z - u^\frac{1}{2} w_x + w(u^\frac{1}{2})_{z},
\end{align*}
\]

(130)

or equivalently

\[
\begin{align*}
2a_t a_{tt} + a_t b_{y z} - a_y b_{t z} = 0, \\
2a_t^2 b_{t z} + a_y a_{t z} - a_t a_{y z} = 0,
\end{align*}
\]

(131)

where \(a_t = u^\frac{1}{2}, \ a_y = w, \ b_t = s, \ b_y = v\). The \((1 + 1)D\) reductions of \((129)\) and \((137)\), when all fields are additionally independent of \(y\), take the form

\[
\begin{align*}
t & = 2v_x - \frac{2}{3} w w_x, \\
t & = v_x - \frac{2}{3} v w x,
\end{align*}
\]

and

\[
t + u^{-\frac{2}{3}} u_x = 0,
\]

where \(v = 0\) end \(w = 2\).

5.4 Acknowledgments

The research of AS was supported in part by the Ministry of Education, Youth and Sport of the Czech Republic (MŠMT ČR) under RVO funding for IČ47813059 and the Grant Agency of the Czech Republic (GA ČR) under grant P201/12/G028.

AS gratefully acknowledges warm hospitality extended to him in the course of his visit to the Adam Mickiewicz University where a substantial part of the present paper was written.

A number of computations in the present paper were performed using the software Jets for Maple \cite{2} whose use is acknowledged with gratitude.

The authors are pleased to thank B.M. Szablikowski for helpful comments. AS also thanks A. Borowiec and R. Vitolo for stimulating discussions.

22
References

[1] Adler V E, Shabat A B, Yamilov R I, Symmetry approach to the integrability problem, *Theor. Math. Phys.* **125**, no. 3, 1603–1661, 2000.

[2] Baran H, Marvan M, *Jets. A software for differential calculus on jet spaces and diffieties*, http://jets.math.slu.cz

[3] Benney D J, Some properties of long nonlinear waves, *Stud. Appl. Math.* **52**, 45–50, 1973.

[4] Błaszak M, Classical $R$-matrices on Poisson algebras and related dispersionless systems, *Phys. Lett. A* **297**, no. 3-4, 191–195, 2002.

[5] Błaszak M, Sergiyev A, Dispersionless (3+1)-dimensional integrable hierarchies, *Proc. R. Soc. A* **473**, no. 2201, 20160857, 2017, [arXiv:1605.07592](https://arxiv.org/abs/1605.07592).

[6] Błaszak M, Szablikowski B, Classical $R$-matrix theory of dispersionless systems. I. (1+1) dimension theory, *J. Phys. A: Math. Gen.* **35**, no. 48, 10325–10345, 2002, [arXiv:nlin/0211008](https://arxiv.org/abs/nlin/0211008).

[7] Błaszak M, Szablikowski B, Classical $R$-matrix theory of dispersionless systems. II. (2+1) dimension theory, *J. Phys. A: Math. Gen.* **35**, no. 48, 10345–10364, 2002, [arXiv:nlin/0211018](https://arxiv.org/abs/nlin/0211018).

[8] Błaszak M, Szablikowski B, Classical $R$-matrix theory for bi-Hamiltonian field systems, *J. Phys. A: Math. Theor.* **42** (2009) 404002

[9] Bravetti A, Contact Hamiltonian dynamics: the concept and its use, *Entropy* **19**, no. 10, art. 535, 2017.

[10] Burtsev S P, Zakharov V E, Mikhailov A V, Inverse scattering method with variable spectral parameter, *Theor. Math. Phys.* **70**, no. 3, 227–240, 1987.

[11] Calderbank D M J, Kruglikov B, Integrability via geometry: dispersionless equations in three and four dimensions, preprint [arXiv:1612.02753](https://arxiv.org/abs/1612.02753)

[12] Dubrovin B A, Novikov S P, The Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov–Whitham averaging method, *Sov. Math. Dokl.* **270**, no. 4, 665–669, 1983.

[13] Dunajski M, Anti-self-dual four-manifolds with a parallel real spinor, *Proc. R. Soc. A* **458**, 1205–1222, 2002, [arXiv:math/0102225](https://arxiv.org/abs/math/0102225).

[14] Dunajski M, Ferapontov E V and Kruglikov B, On the Einstein–Weyl and conformal self-duality equations, *J. Math. Phys.* **56**, no. 8, art. 083501, 2015, [arXiv:1406.0018](https://arxiv.org/abs/1406.0018).

[15] Dunajski M, *Solitons, Instantons and Twistors*, Oxford Univ. Press, Oxford, 2010.

[16] Ferapontov E V, Khusnutdinova K R, Klein C, On linear degeneracy of integrable quasilinear systems in higher dimensions, *Lett. Math. Phys.* **96**, no. 1-3, 5–35, 2011.

[17] Ferapontov E, Kruglikov B, Dispersionless integrable systems in 3D and Einstein–Weyl geometry, *J. Diff. Geom.* **97**, 215–254, 2014, [arXiv:1208.2728](https://arxiv.org/abs/1208.2728).

[18] Ferapontov E V, Moro A, Novikov V S, Integrable equations in 2 + 1 dimensions: deformations of dispersionless limits, *J. Phys. A: Math. Gen.* **42**, no. 34, art. 345205, 2009.

[19] Ferapontov E V, private communication.

[20] Fokas A S, Symmetries and integrability, *Stud. Appl. Math.* **77**, 253–299, 1987.

[21] Konopelchenko B G, Martínez Alonso L, Dispersionless scalar integrable hierarchies, Whitham hierarchy, and the quasiclassical $\partial$-dressing method, *J. Math. Phys.* **43**, no. 7, 3807–3823, 2002, [arXiv:nlin/0105071](https://arxiv.org/abs/nlin/0105071).
[22] Krichever I M, The $\tau$-function of the universal Whitham hierarchy, matrix models and topological field theories, *Comm. Pure Appl. Math.* **47**, 437–475, 1994.

[23] Kruglikov B, Morozov O, Integrable dispersionless PDEs in 4D, their symmetry pseudogroups and deformations, *Lett. Math. Phys.* **105**, no. 12, 1703–1723, 2015, arXiv:1410.7104.

[24] Kupershmidt B A, Manin Yu I, Long wave equations with a free surface. I. Conservation laws and solutions, *Funkt. Anal. i Pril.* **11**, no. 3, 31–42, 1977; Kupershmidt B A, Manin Yu I, Long wave equations with a free surface. II. The Hamiltonian structure and the higher equations, *Funkt. Anal. i Pril.* **12**, no. 1, 25–37, 1978.

[25] Li L-C, Classical $r$-matrices and compatible Poisson structures for Lax equations in Poisson algebras, *Commun. Math. Phys.* **203** 573–592, 1999.

[26] Lichnerowicz A, Les variétés de Jacobi et leurs algèbres de Lie associées, *J. Math. Pures Appl.* **57**, no. 4, 453–488, 1978.

[27] Majda A J, *Compressible fluid flows and systems of conservation laws in several space variables*, Springer, N.Y., 1984.

[28] Manakov S V, Santini P M, Integrable dispersionless PDEs arising as commutation condition of pairs of vector fields, *J. Phys.: Conf. Ser.* **482**, paper 012029, 2014, arXiv:1312.2740.

[29] Manakov S V, Santini P M, Solvable vector nonlinear Riemann problems, exact implicit solutions of dispersionless PDEs and wave breaking, *J. Phys. A: Math. Theor.* **44**, no. 34, paper 345203, 2011, arXiv:1011.2619.

[30] Marvan M, Another look on recursion operators, in *Differential geometry and applications (Brno, 1995)*, Masaryk Univ., Brno, 1996, 393–402.

[31] Marvan M, Sergiyev A, Recursion operators for dispersionless integrable systems in any dimension, *Inverse Problems* **28**, art. 025011, 2012.

[32] Morozov O I, Sergiyev A, The four-dimensional Martínez Alonso–Shabat equation: reductions and nonlocal symmetries, *J. Geom. Phys.* **85**, 40–45, 2014, arXiv:1401.7942.

[33] Odesskii A V, Sokolov V V, Integrable pseudopotentials related to generalized hypergeometric functions, *Sel. Math. (N.S.)* **16**, 145–172, 2010, arXiv:0803.0086.

[34] Olver P J, *Applications of Lie groups to differential equations*, 2nd ed., Springer, New York, 2000.

[35] Plebański J F, Przanowski M, The Lagrangian of a self-dual gravitational field as a limit of the SDYM Lagrangian, *Phys. Lett. A* **212**, no. 1-2, 22–28, 1996.

[36] Semenov-Tian-Shansky M A, What is a classical $r$-matrix?, *Func. Anal. Appl.* **17** (1983), 259–272.

[37] Semenov-Tian-Shansky M, Integrable systems: the $r$-matrix approach, Preprint RIMS-1650, Kyoto, 2008.

[38] Sergiyev A, A simple construction of recursion operators for multidimensional dispersionless integrable systems. *J. Math. Anal. Appl.* **454**, 468–480, 2017, arXiv:1501.01955.

[39] Sergiyev A, New integrable (3+1)-dimensional systems and contact geometry, *Lett. Math. Phys.* **108**, no. 2, 359–376, 2018, arXiv:1401.2122.

[40] Sergiyev A, Integrable (3+1)-dimensional systems with rational Lax pairs, Nonlinear Dynamics **91** (2018), no. 3, 1677–1680, arXiv:1711.07395.

[41] Sergiyev A, Integrable (3+1)-dimensional system with an algebraic Lax pair, arXiv:1812.02263.

[42] Sergiyev A, Recursion Operators for Multidimensional Integrable PDEs, arXiv:1710.05907.
[43] Szablikowski B M, Błaszak M, Meromorphic Lax representations of (1+1)-dimensional multi-Hamiltonian dispersionless systems, *J. Math. Phys.* 47, 092701, 2006, arXiv:nlin/0510068

[44] Szablikowski B M, Hierarchies of Manakov-Santini type by means of Rota-Baxter and other identities, *SIGMA* 12, art. 022, 2016, arXiv:1512.05817.

[45] Takasaki K, Takebe T, Integrable hierarchies and dispersionless limit, *Rev. Math. Phys.* 7 (1995), 743–808, arXiv:hep-th/9405096

[46] Vaisman I, *Lectures on the geometry of Poisson manifolds*, Birkhäuser Verlag, Basel, 1994.

[47] Zabolotskaya E A, Khokhlov R V, Quasi-plane waves in the nonlinear acoustics of confined beams, *Sov. Phys. Acoust.* 15, 35–40, 1969.

[48] Zakharov V E, Multidimensional integrable systems, in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, PWN, Warsaw, 1984, 1225–1243.

[49] Zakharov V E, Dispersionless limit of integrable systems in (2+1) dimensions, in *Singular Limits of Dispersive Waves*, ed. by N.M. Ercolani et al., Plenum Press, N.Y., 1994, 165–174.

[50] Zarraga J A, Perelomov A M and Perez Bueno J C, The Schouten–Nijenhuis bracket, cohomology and generalized Poisson structures, *J. Phys. A: Math. Gen.* 29, 7993–8009, 1996.