An *a posteriori* error analysis for the equations of stationary incompressible magnetohydrodynamics

J. H. Chaudhry\(^a\), A. E. Rappaport\(^b,a,\)\(^*\), J. N. Shadid\(^b,a\)

\(^a\)Department of Mathematics and Statistics, University of New Mexico, Albuquerque NM, 87123
\(^b\)Center for Computing Research, Sandia National Laboratories, Albuquerque NM, 87185

Abstract

Magnetohydrodynamics (MHD) is a continuum level model for conducting fluids subject to external magnetic fields, e.g. plasmas and liquid metals. The efficient and robust solution of the MHD system poses many challenges due to its nonlinear, non self-adjoint, and highly coupled nature. In this paper, we develop a robust and accurate *a posteriori* error estimate for the numerical solution of the MHD equations based on the exact penalty method. The error estimate also isolates particular contributions of error in a quantity of interest (QoI) to inform discretization choices to arrive at accurate solutions. The tools required for these estimates involve duality arguments and computable residuals.

1. Introduction

The magnetohydrodynamics (MHD) equations provide a continuum model for conducting fluids subject to magnetic fields and are often used to model important applications e.g. highly collisional plasmas. In this context, MHD calculations aid physicists in understanding both thermonuclear fusion and astrophysical plasmas as well as understanding the behavior of liquid metals [1, 2]. The governing equations of MHD couple Navier-Stokes equations for fluid dynamics with a reduced set of Maxwell’s equations for low frequency electromagnetic phenomenon. Structurally, the equations of MHD form highly coupled, nonlinear, non self-adjoint partial differential equations (PDEs). Analytical solutions to the MHD system cannot be obtained for practical configurations; instead numerical solutions are sought. Solution methods for the incompressible MHD system include stabilized formulations, fully implicit block preconditioning, First Order System Least Squares and operator splitting methods [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. In this article we restrict

*Corresponding author

Email addresses: jehanzeb@unm.edu (J. H. Chaudhry), aerappa@sandia.gov, arappaport@unm.edu (A. E. Rappaport), jnshadi@sandia.gov (J. N. Shadid)
ourselves to the stationary MHD equations based on the exact penalty formulation [3].

The numerical solution of complex equations like the MHD equations often have a significant discretization error. This error must be quantified for the reliable use of MHD equations in numerous science and engineering fields. Accurate error estimation is a key component of predictive computational science and uncertainty quantification [18, 19, 20]. Moreover, the error depends on a complex interaction between many contributions. Thus, the availability of accurate error estimation also offers the potential of optimizing the choice of discretization parameters in order to achieve desired accuracy in an efficient fashion. In this work we leverage adjoint based a posteriori error estimates for a QoI related to the solution of the MHD equations. These estimates provide a concrete error analysis of different contributions of error, as well as inform solver and discretization strategies.

In many scientific and engineering applications, the goal of running a simulation is to compute certain a quantity of interest (QoI) of the solution, for example the drag over a plane wing in a compressible CFD context. Adjoint based analysis for quantifying the error in a numerically computed QoI has found success for a wide variety of numerical methods and discretizations ranging from finite element, finite difference, finite volume, time integration, operator splitting techniques and uncertainty quantification [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 20]. Adjoint based a posteriori error analysis uses variational analysis and duality to relate errors to computable residuals. In particular, one solves an adjoint problem whose solution provides the residual weighting to produce the error in the QoI. The technique also naturally allows to identify and isolate different components of error arising from different aspects of discretization and solution methods, by analyzing different components of the weighted residual separately.

This paper is organized as follows. In §2, we review the equations of incompressible resistive MHD, present the exact penalty weak form and the finite element method to numerically solve the problem. In §3 we develop theoretical results for adjoint based a posteriori error analysis for an abstract problem representative of the exact penalty weak form. We apply these results to the MHD equations in §4 to develop an a posteriori error estimate. In §5 we present numerical results to demonstrate the accuracy and utility of the error estimates produced by our method. In §6 we give details of the derivation of the nonlinear operators in the weak adjoint form as well as a well-posedness argument for the adjoint problem.

2. Exact penalty formulation and discretization for incompressible MHD

In this section we describe the nondimensionalized equations of incompressible stationary MHD, a stabilized weak form of the MHD system and a finite element method for its solution.
2.1. The MHD equations

We consider a Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or $3$, with boundary $\partial \Omega$. The equations for stationary incompressible MHD in $\Omega$ are given by

\begin{align}
-\frac{1}{R_f} \Delta u + u \cdot \nabla u + \nabla p - \kappa (\nabla \times b) \times b &= f, \quad (2.1a) \\
\nabla \cdot u &= 0, \quad (2.1b) \\
\frac{\kappa}{R_m} \nabla \times (\nabla \times b) - \kappa \nabla \times (u \times b) &= 0, \quad (2.1c) \\
\nabla \cdot b &= 0, \quad (2.1d)
\end{align}

where the unknowns are the velocity $u$, the magnetic field $b$, and the pressure $p$. The nondimensionalized parameters are the fluid Reynolds number $R_f$, Magnetic Reynolds number $R_m$, and interaction parameter $\kappa = H_a^2 (R_f R_m)$, where $H_a$ is the Hartmann number. The source term $f$ is viewed as data to the problem. For $x \in \Omega$ we have $u(x) \in \mathbb{R}^d$, $b(x) \in \mathbb{R}^d$, $p(x) \in \mathbb{R}$ and $f(x) \in \mathbb{R}^d$.

We supplement the system (2.1) with boundary conditions,

\begin{align}
u &= g, \quad \text{on } \partial \Omega, \quad (2.2a) \\
b \times n &= q \times n, \quad \text{on } \partial \Omega. \quad (2.2b)
\end{align}

Referring to (2.1), we observe there are $2d + 2$ and only $2d + 1$ unknowns [5]. Effectively enforcing the solenoidal constraint (2.1d) (an involution of the transient MHD system) is an open area of research. Techniques include compatible discretizations [36, 37], vector potential [38, 4] and divergence cleaning [39, 40] as well as the exact penalty method [3, 41, 5]. In this article, we consider the exact penalty method which we further describe in §2.3.

2.2. Function spaces for the MHD system

We make use of the standard spaces $L^2(\Omega)$ and $H^m(\Omega)$ as well as their vector counterparts $L^2(\Omega)$ and $H^m(\Omega)$. The $L^2(\Omega)$ (or $L^2(\Omega)$) inner product is denoted by $(\cdot, \cdot)$ and the norm is denoted by $\| \cdot \|$, while the $H^1(\Omega)$ (or $H^1(\Omega)$) norm is denoted by $\| \cdot \|_1$. These details of these function spaces are given in Appendix A. For $b \in H^1(\Omega)$, we define $\nabla b := [\nabla b_1, \ldots, \nabla b_d]$ as a matrix whose columns are the gradients of the components of $b$. The relevant subspaces of $H^1(\Omega)$ needed to satisfy the boundary conditions (in the sense of the trace operator) are,

\begin{align}
H^1_0(\Omega) := \{ w \in H^1 : w|_{\partial \Omega} \equiv 0 \}, \quad & (2.3) \\
H^1(\Omega) := \{ w \in H^1 : (w \times n)|_{\partial \Omega} \equiv 0 \}. \quad & (2.4)
\end{align}

Finally, we define the product space,

\[ \mathcal{M}(\Omega) := H^1_0(\Omega) \times H^1(\Omega) \times L^2(\Omega). \quad (2.5) \]
We also remark that for \( d = 2 \), we use the natural inclusion of \( \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \), 
\[
[v_1, v_2]^T \mapsto [v_1, v_2, 0]^T
\]
to define the operators \( \nabla \times \) and \( \times \). Thus for \( v, w \in H^1 \), we have that
\[
\nabla \times v = \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{k}, \quad v \times w = (v_x w_y - v_y w_x) \hat{k}.
\]

### 2.3. Exact penalty formulation

In this section we present the weak form of the stationary incompressible MHD system based on the exact penalty formulation. The exact penalty method requires that the domain \( \Omega \) is bounded, convex and polyhedral. This ensures that \( H(\text{curl}, \Omega) \cap H(\text{div}, \Omega) \) is continuously embedded in \( H^1(\Omega) \) \cite{37}. We also assume homogeneous Dirichlet conditions, \( g = q = 0 \). Non-homogeneous boundary conditions can be dealt with through standard lifting arguments as discussed in §4.3. The exact penalty weak problem corresponding to (2.1) and (2.2) is: find \( U = (u, b, p) \in \mathcal{M}(\Omega) \) such that
\[
N_{EP}(U, V) = (f, v), \quad \forall V \in \mathcal{M}(\Omega),
\]
where the nonlinear form \( N_{EP} \) is defined for all \( V = (v, c, q) \in \mathcal{M}(\Omega) \) by
\[
N_{EP}(U, V) := \frac{1}{R} (\nabla u, \nabla v) + (C(u), v) - (p, \nabla \cdot v) + (q, \nabla \cdot u) + \kappa (Y(b), v) - \kappa (Z(u, b), c) + \frac{\kappa}{R_m} (\nabla \cdot b, \nabla \cdot c),
\]
and the nonlinear operators are defined by
\[
C(u) := u \cdot \nabla u, \quad Y(b) := (\nabla \times b) \times b, \quad Z(u, b) := \nabla \times (u \times b).
\]
All except the last term in the weak form arise from multiplying (2.1a)-(2.1c) by test functions and performing integration by parts. The last term, \( \frac{\kappa}{R_m} (\nabla \cdot b, \nabla \cdot c) \), effectively enforces the solenoidal involution (2.1d) since, assuming the aforementioned restrictions on the domain, there exists a function (see \cite{3, 42}) \( b_0 \in H^2(\Omega) \) such that
\[
\nabla \cdot \nabla b_0 = \nabla \cdot b, \quad \text{and} \quad \nabla b_0 \in H^1(\Omega).
\]
Thus, we choose \( V = (0, \nabla b_0, 0) \) in (2.7) and use (B.1d) so that (2.6) reduces to
\[
(\nabla \cdot b, \nabla \cdot \nabla b_0) = (\nabla \cdot b, \nabla \cdot b) = 0,
\]
and hence (2.1d) is satisfied almost everywhere in \( \Omega \).
2.4. Finite element method

We introduce the standard continuous Lagrange finite element spaces. Let $\mathcal{T}_h$ be a simplicial decomposition of $\Omega$, where $h$ denotes the maximum diameter of the elements of $\mathcal{T}_h$. The standard Lagrange space finite element space of order $q$ is then

$$P^q_h := \{ v \in C(\Omega) : \forall K \in \mathcal{T}_h, v|_K \in P^q(K) \}, \quad (2.11)$$

where $P^q(K)$ is the space of polynomials of degree at most $q$ defined on the element $K$. Additionally, our finite element space satisfies the Ladyzhenskaya-Babuška-Brezzi condition stability condition [43] for the velocity pressure pair, e.g. $\mathcal{M}_h(\Omega) = P^2_h(\Omega) \times P^1_h(\Omega) \times P^1_h(\Omega)$. Then the discrete problem to find an approximate solution $U_h = (u_h, b_h, p_h) \in \mathcal{M}_h(\Omega)$ to (2.7) is,

$$\mathcal{N}_{EP}(U_h, V_h) = (f, v_h) \quad \forall V_h \in \mathcal{M}_h(\Omega). \quad (2.12)$$

Note there is no restriction on the finite element space for $b_h$, which is an advantage of this method. The well-posedness of the continuous and discrete problems are proven in [3].

2.5. Quantity of interest (QoI)

The goal of a numerical simulation is often to compute some functional of the solution, that is, the QoI. In particular, QoIs considered in this article have the generic form,

$$\text{QoI} = \int_\Omega \Psi \cdot U \, dx = (\Psi, U) \quad (2.13)$$

where $\Psi \in L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \equiv [L^2(\Omega)]^{2d+1}$. For example in 2D, to compute the average of the $y$ component of velocity $u_y$ over a region $\Omega_c \subset \Omega$, set $\Psi = \frac{1}{|\Omega_c|} (0, 1_{\Omega_c}, 0, 0, 0)^T$, where $1_S$ denotes the characteristic function over a set $S$. In the examples presented later, the QoIs physically represent quantities representative of the average flow rate, or the average induced magnetic field. We seek to compute error estimates in the QoI using duality arguments as presented in the following subsection.

3. Abstract a posteriori error analysis

In this section we consider an abstract variational setting for a posteriori analysis based on the ideas from [21, 31, 22, 23, 24]. Given a Banach space $\mathcal{Y}$, we denote its dual space by $\mathcal{Y}'$ and represent the duality pairing by $\langle \cdot, \cdot \rangle$. We consider generic QoI as bounded linear functionals of the form,

$$Q(w) = \langle \psi, w \rangle,$$

where $\psi \in \mathcal{Y}'$ and $w \in \mathcal{Y}$. For example, in (2.13), $\langle \psi, u \rangle = (\Psi, U)$, the duality pairing is represented by the $L^2$ inner product. We want to evaluate $Q(u)$ where $u$ is the solution to the variational problem: find $u$ in $\mathcal{V}$ such that

$$\mathcal{N}(u, v) = (f, v), \quad \forall v \in \mathcal{V}, \quad (3.1)$$

Note there is no restriction on the finite element space for $b_h$, which is an advantage of this method. The well-posedness of the continuous and discrete problems are proven in [3].
and $\mathcal{N}: \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is linear in the second argument but may be nonlinear in the first argument. Throughout this section $u$ refers to the true solution to (3.1). An example of such a variational problem is the exact penalty problem as described in §2.3. Given a numerical approximation $u_h \in \mathcal{V}_h$, in some finite dimensional subspace $\mathcal{V}_h \subset \mathcal{V}$, we define the error as $e = u - u_h$. The aim of the $a$ posteriori analysis is to compute the error in the QoI, $Q(u) - Q(u_h) = \langle \psi, u \rangle - \langle \psi, u_h \rangle = \langle \psi, e \rangle$. For nonlinear forms, the choice of an adjoint form is not straightforward. However, a common choice useful for various kinds of analysis is based on linearization [44, 45, 46, 47, 32, 30]. This choice enables the definition of a bilinear form $\mathcal{N}^\ast(v, w)$ which satisfies the useful property,

$$\mathcal{N}^\ast(v, e) = \mathcal{N}(u, v) - \mathcal{N}(u_h, v),$$  \hspace{1cm} (3.2)

for all $v \in \mathcal{V}$.

Now let $\mathcal{V} = \prod_{i=1}^n \mathcal{V}_i$ be a product space of Banach spaces $\mathcal{V}_i = \mathcal{V}_i(\Omega)$, with $\Omega \subset \mathbb{R}^d$. The problem (3.1) with $\mathcal{N}: \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is now more specifically given by

$$\mathcal{N}(u, v) = \sum_{i=1}^m \langle N_i(u), v_{\ell_i} \rangle + a(u, v),$$  \hspace{1cm} (3.3)

where $a(u, v)$ is a bilinear form and $\ell_i \in \{1, \ldots, n\}$ and $N_i: \mathcal{V} \to \mathcal{V}^\prime_{\ell_i}$. For a solution/approximation pair $(u/u_h)$, define the matrix $J$, by

$$J_{ij} = \int_0^1 \frac{\partial N_i}{\partial u_j} (su + (1 - s)u_h) \, ds,$$

(3.4)

where $\frac{\partial N_i}{\partial u_j}(\cdot)$ denotes the partial derivative of $N_i$ with respect to the argument $u_j$. Define the linearized operator $\bar{N}_i$ for $w \in \mathcal{V}$ by

$$\bar{N}_i w = \int_0^1 \frac{\partial N_i}{\partial u_j} (su + (1 - s)u_h) \, ds \cdot w$$

$$= \sum_{j=1}^n \int_0^1 \frac{\partial N_i}{\partial u_j} (su + (1 - s)u_h) \, ds w_j = \sum_{j=1}^n \bar{J}_{ij} w_j.$$ 

Now since each $\bar{N}_i$ is linear, we may define the bilinear forms,

$$\mathcal{V}_i(u, v) = \langle \bar{N}_i(u), v_{\ell_i} \rangle = \left\langle \sum_{j=1}^n \bar{J}_{ij} u_j, v_{\ell_i} \right\rangle = \sum_{j=1}^n \langle \bar{J}_{ij} u_j, v_{\ell_i} \rangle.$$ 

Define $\mathcal{V}_i^\ast(v, w) = \mathcal{V}_i(v, w)$, adjoint operators $\bar{J}_{ij}^\ast$ to $\bar{J}_{ij}$ satisfying

$$\langle \bar{J}_{ij}^\ast w, v \rangle = \langle w, \bar{J}_{ij}^\ast v \rangle$$  \hspace{1cm} (3.5)

for $w \in \mathcal{V}_j^\ast$ and $v \in \mathcal{V}_i^\ast$ and $a^*(w, v) := a(v, w)$. With these definitions, the adjoint bilinear weak form is,

$$\mathcal{N}^\ast(\phi, w) = \sum_{i=1}^m \mathcal{V}_i^\ast(\phi, v) + a^*(\phi, v) = \sum_{i=1}^m \sum_{j=1}^n \langle \psi_j, \bar{J}_{ij}^\ast \phi_{\ell_i} \rangle + a^*(\phi, v).$$  \hspace{1cm} (3.6)
Then if $\phi$ solves the dual problem,

$$\mathcal{N}^*(\phi, v) = \langle \psi, v \rangle, \forall v \in \mathcal{V},$$

we then have the following abstract error representation.

**Theorem 1.** *The error in a QoI represented by $Q(u) = \langle \psi, u \rangle$ is computable as $\langle \psi, e \rangle = \langle f, v \rangle - \mathcal{N}(u_h, \phi)$."

**Proof.** Unpacking the definitions,

$$\langle \psi, e \rangle = \mathcal{N}^*(\phi, e) = \sum_{i=1}^{m} \sum_{j=1}^{n} \langle e_j, J_{ij}^* \phi_{\ell_i} \rangle + a^*(\phi, e)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \langle J_{ij} e_j, \phi_{\ell_i} \rangle + a(e, \phi) = \sum_{i=1}^{m} \langle N_i e, \phi_{\ell_i} \rangle + a(e, \phi)$$

$$= \sum_{i=1}^{m} \langle N_i(u) - N_i(u_h), \phi_{\ell_i} \rangle + a(u, \phi) - a(u_h, \phi)$$

$$= \sum_{i=1}^{m} \langle N_i(u), \phi_{\ell_i} \rangle + a(u, \phi) - \sum_{i=1}^{m} \langle N_i(u_h), \phi_{\ell_i} \rangle - a(u_h, \phi)$$

$$= \mathcal{N}(u, \phi) - \mathcal{N}(u_h, \phi) = (f, \phi) - \mathcal{N}(u_h, \phi).$$

The main takeaway of this theorem is that computing the adjoint to a non-linear form is reduced to computing the adjoint for the averaged entries, $J_{ij}$.

**4. A posteriori error estimate for the MHD equations**

The analysis in §3 applies directly to the MHD equations. The duality pairing of the last section is represented by the $[L^2(\Omega)]^{2d+1}$ inner product $(\cdot, \cdot)$. The linear and nonlinear terms in the exact penalty weak form (2.6) are mapped to match (3.3). The mapping between the abstract formulation and MHD equation is shown in Table 1.

For the exact penalty weak form, we have that

$$\mathcal{N}_{EP}(U, V) = \sum_{i=1}^{3} (N_{EP,i}(U), V_{\ell_i}) + a_{EP}(U, V),$$

where

$$\begin{align*}
(N_{EP,1}(U), V_1) &= (Z(u, b), v), \\
(N_{EP,2}(U), V_2) &= (Y(b), c), \\
(N_{EP,3}(U), V_2) &= (C(u), v),
\end{align*}$$

(4.2)
| Abstract | MHD |
|----------|-----|
| \langle , \rangle | (f, v) |
| m \ | u_1 |
| N \ | u_2 |
| u \ | u_3 |
| v \ | v_1 |
| N_i \ | v_2 |

(b) \[
N_{EP,i} \equiv (a) \]

(c) \[
N_{EP} \equiv (b) \]

Table 1: Mapping between the abstract framework in §3 and the MHD equation in §4. \( N_{EP} \) is given in (4.1), \( N_{EP,i} \) in (4.2), \( a_{EP} \) in (4.3) and \( Z^* \), \( Y^* \), \( C^* \) are given in (4.4).

\[ Z^*_u \]
\[ Z^*_b \]
\[ Y^* \]
\[ C^* \]

\[ \mathcal{J}^*_{11} V_2 = \mathcal{Z}^*_u, \mathcal{J}^*_{12} V_2 = \mathcal{Z}^*_b, \mathcal{J}^*_{21} V_1 = \mathcal{Y}^* v \]

\[ \mathcal{J}^*_{31} V_1 = \mathcal{C}^* v \]

while the remaining \( \mathcal{J}^*_{ij} \) entries are zero. The details of the derivation are given in §6.1.

4.1. Weak form of adjoint for Incompressible MHD

We are now prepared pose a weak adjoint problem corresponding to exact penalty primal problem (2.6). Based on (4.1), (4.4) and (3.7), the weak dual problem is therefore be stated as: find \( \Phi = (\phi, \beta, \pi) \in \mathcal{M} (\Omega) \) such that

\[ \mathcal{N}_{EP}^*(\Phi, V) = (\Psi, V), \quad \forall \, V \in \mathcal{M} (\Omega) \] (4.5)

with

\[ \mathcal{N}_{EP}^*(\Phi, V) = \frac{1}{R_f} (\nabla \phi, \nabla v) + (\mathcal{C}^* \phi, v) - (\nabla \cdot v, \pi) - (\nabla \cdot \phi, q) + \frac{\kappa}{R_m} (\nabla \times \beta, \nabla \times c) + \frac{\kappa}{R_m} (\nabla \cdot \beta, \nabla \cdot c) - \kappa (\mathcal{Y}^* \phi, \mathcal{C}^* v) - \kappa (\mathcal{Z}^*_u \beta, \mathcal{C}^* v) - \kappa (\mathcal{Z}^*_b \beta, \mathcal{C}^* v). \] (4.6)

The forms of the linear operators \( \mathcal{C}^*, \mathcal{Y}^*, \mathcal{Z}^*_u, \mathcal{Z}^*_b \) are given in (4.4). We discuss the well-posedness of the adjoint weak form in §6.2.
4.2. Error representation

In order to discuss error representation we need to the following definitions

Definition 1. Define the monolithic error by

\[ E = [e_u, e_b, e_p]^T \]

with component errors

\[ e_u = u - u_h, \quad e_b = b - b_h, \quad e_p = p - p_h. \]  (4.7)

We have the following error representation.

Theorem 2 (Error representation for exact penalty). For a given QoI represented by \( \Psi = [\psi_u, \psi_b, \psi_p]^T \), the error in the numerical approximation of the QoI satisfies

\[
(\Psi, E) = (f, \phi) - \left[ \frac{1}{R_f}(\nabla u_h, \nabla \phi) + (u_h \cdot \nabla u_h, \phi) \\
- (p_h, \nabla \cdot \phi) + \kappa((\nabla \times b_h) \times b_h, \phi) + (\nabla \cdot u_h, \pi) \\
+ \frac{\kappa}{R_m}(\nabla \times b_h, \nabla \times \beta) + \kappa(\nabla \times (u_h \times b_h), \beta) \\
+ \frac{\kappa}{R_m}(\nabla \cdot b_h, \nabla \cdot \beta) \right].
\]

Proof. By Theorem 1,

\[
(\Psi, E) = \mathcal{N}_{EP}(\Phi, E) = N_{EP}(U, \Phi) - N_{EP}(U_h, \Phi) = (f, \phi) - N_{EP}(U_h, \Phi).
\]

4.3. Non-homogeneous boundary conditions for the MHD system

The analysis above easily extends to the case of non-homogeneous boundary conditions, i.e. when \( g \) or \( q \times n \) are not identically zero. First assume that the numerical solution \( U_h \) on \( \partial \Omega \) satisfies the non-homogeneous conditions exactly. That is, \( u = u_h = g \) and \( b \times n = b_h \times n = q \times n \) on \( \partial \Omega \). Then, although neither the true solution \( U \) nor the numerical solution \( U_h \) belong to \( \mathcal{M}(\Omega) \), yet the error, \( E = U - U_h \), satisfies homogeneous boundary conditions and hence belongs to \( \mathcal{M}(\Omega) \). Thus, the error analysis in the previous section applies directly in this case.

Now, if \( U_h \) belongs to \( \mathcal{M}_h(\Omega) \), then in general \( U_h \) does not satisfy the non-homogeneous boundary conditions exactly. Hence we consider the splitting of the numerical solutions as,

\[
U_h = U_h^0 + U^d, \quad \text{ (4.8)}
\]

where \( U_h^0 \in \mathcal{M}_h(\Omega) \) solves

\[
\mathcal{N}_{EP}(U_h, V_h) = \mathcal{N}_{EP}(U_h^0 + U^d, V_h) = (F, V_h), \quad \forall V_h \in \mathcal{M}_h(\Omega), \quad \text{ (4.9)}
\]

and \( U^d \) is a known function that satisfies the non-homogeneous boundary conditions accurately. That is, the unknown is now \( U_h^0 \) and the numerical solution \( U_h \) is formed through the sum in (4.8). In this article the function \( U^d \) is approximated through a finite element space of much higher dimension than \( \mathcal{M}_h(\Omega) \) to capture the boundary conditions accurately.
4.4. Error Estimate and Contributions

The error representation in Theorem 2 requires the exact solution \( \Phi = (\phi, \beta, \pi) \in \mathcal{M}(\Omega) \). Moreover, the adjoint weak form (4.6) is linearized around the true solution \( U \) and the approximate solution \( U_h \). In practice, the adjoint solution itself must be approximated in a finite element space \( \mathcal{W}_h \subset \mathcal{M}(\Omega) \) and is linearized only around the numerical solution. These approximations lead to an error estimate from the error representation 2. Let this estimate be denoted by \( \eta \). That is, \( \eta \approx (\Psi, E) \) where,

\[
\eta = E_{\text{mom}} + E_{\text{con}} + E_{\text{M}},
\]

where,

\[
E_{\text{mom}} = (f, \phi_h) - \left( \frac{1}{R_f} (\nabla u_h, \nabla \phi_h) + (u_h \cdot \nabla u_h, \phi_h) - (p_h, \nabla \cdot \phi_h) \right)
+ \kappa ((\nabla \times b_h) \times b_h, \phi_h)
\]

\[
E_{\text{con}} = -(\nabla \cdot u_h, \pi_h),
\]

\[
E_{\text{M}} = -\frac{\kappa}{R_m} (\nabla \times b_h, \nabla \times \beta_h) + \kappa (\nabla \times (u_h \times b_h), \beta_h) - \frac{\kappa}{R_m} (\nabla \cdot b_h, \nabla \cdot \beta_h).\]

(4.11)

Here \( E_{\text{mom}}, E_{\text{con}} \) and \( E_{\text{M}} \) represent the momentum error contribution, the continuity error contribution and the magnetic error contribution respectively.

To obtain an accurate error estimate we choose \( \mathcal{W}_h \) to be of much higher dimension than \( \mathcal{M}_h(\Omega) \) as is standard in adjoint based \textit{a posteriori} error estimation [25, 21, 31, 48, 35, 25, 49, 50, 34]. Moreover, the inaccuracy caused by substituting the numerical solution in place of true solution in the adjoint weak form is shown to decrease in the limit of refined discretization [25].

5. Numerical results

In this section we present numerical results to verify the accuracy of the error estimate (4.10) and the utility of the error contributions in (4.11). The effectivity ratio, denoted \( \text{Eff.} \), characterizes how well the error estimate approximates the true error,

\[
\text{Eff.} = \frac{\text{Error estimate}}{\text{True error}} = \frac{\eta}{(\Psi, E)}.
\]

(5.1)

The closer the effectivity is to 1, the better the error estimate provided by our method.

We present two numerical examples here, the Hartmann problem §5.1 and the magnetic lid driven cavity §5.2. Since there is no closed form solution for the magnetic lid driven cavity, we use an reference high order/fine mesh to get a good estimate for the true error. All the following computations were carried out using the finite element package \texttt{Dolfin} in the \texttt{FEniCS} suite [51, 52, 53].
For all experiments, we chose different polynomial orders of Lagrange spaces for the product space $\mathcal{H}_h(\Omega)$ and ensure that the adjoint space $W_h$ has a higher polynomial degree. The computational domain for all problems is chosen to be a unit length square, $\Omega := [-\frac{1}{2}, \frac{1}{2}]^2 \subset \mathbb{R}^2$. The mesh is a simplicial uniform mesh with the total number of elements $2D\ \text{Elem.} = n_e \times n_e$.

5.1. Hartmann flow in 2D

Our first results concern the so-called Hartmann problem [2]. This problem models the flow of a conducting fluid in a channel. In this case we take consider a square channel, as described in the beginning of the section. This problem admits an analytic solution [5], $u = (u_x, 0)$, $b = (B_x, 1), p$ where

$$u_x(y) = \frac{G R_f (\cosh(H_a/2) - \cosh(H_ay))}{2H_a \sinh(H_a/2)}, \quad (5.2a)$$
$$B_x(y) = \frac{G (\sinh(H_ay) - 2 \sinh(H_a/2)y)}{2\kappa \sinh(H_a/2)}, \quad (5.2b)$$
$$p(x) = -Gx - \kappa B^2_x/2, \quad (5.2c)$$

and $G = -\frac{dp}{dx}$ is an arbitrary pressure drop that we choose to normalize maximum velocity $u_x(y)$ to 1.

5.1.1. Problem parameters and QoI

The values of the nondimensionalized constants are chosen as follows: $R_f = 16, R_m = 16, \kappa = 1$ which produce a Hartmann number of $H_a = 16$. The QoI is chosen as the average velocity across the flow over a slice. To this end, define $\Omega_c := [-\frac{1}{4}, \frac{1}{2}] \times [-\frac{1}{4}, \frac{1}{4}]$ (5.3) and consequently $1_{\Omega_c}$ the characteristic function on $\Omega_c$. The monolithic quantity of interest $\Psi$ as in Theorem 2 is chosen to be $\Psi = (1_{\Omega_c}, 0, 0, 0, 0)^T$. The QoI thus reduces to

$$(\Psi, U) = (1_{\Omega_c}, u_x). \quad (5.4)$$

This has a physical interpretation of the capturing the flow rate across this slice of the channel, $\Omega_c$.

5.1.2. Numerical results and discussion

Error contributions and effectivites using different order polynomial spaces are presented in Table 2, Table 3, Table 4, and Table 5. Effectivity ratio in tables Table 2 and Table 3 is quite close to 1 indicating the accuracy of the error estimate. The error estimate in Table 4 is not as accurate due to linearization error incurred by replacing the true solution by the approximate solution in the definition of the adjoint as discussed in the introduction of this section. This may be verified by linearizing the adjoint weak form around both the true (which we know for this example) and the approximate solutions. These results are shown in Table 5 and now the error estimate is again accurate.
We remark that for the lowest order tuple \((P_2, P_1, P_1)\) for the variables \((u, b, p)\) in Table 2, the error is largely dominated by the contributions \(E_{\text{con}}\) and \(E_M\). We greatly reduce the error in \(E_M\) by using a higher order space for \(b\) as demonstrated in Table 3. However, this does not reduce the magnitude of the total error much (about 5%) which is still dominated by the contribution \(E_{\text{con}}\). The contribution \(E_{\text{con}}\) is not significantly affected by the finite dimensional space for \(b\). Now finally, if we refine all three variables, Table 4 shows that the total error drops by two orders of magnitude.

| 2D Elem. | True Error | Eff. | \(E_{\text{mom}}\) | \(E_{\text{con}}\) | \(E_M\) |
|----------|------------|------|-------------------|-------------------|--------|
| 1600     | 2.80e-04   | 1     | 6.87e-06          | -3.38e-04         | 6.12e-04 |
| 6400     | 7.06e-05   | 1     | 1.95e-06          | -9.26e-05         | 1.62e-04 |
| 14400    | 3.15e-05   | 1     | 9.12e-07          | -4.24e-05         | 7.32e-05 |
| 25600    | 1.77e-05   | 1     | 5.26e-07          | -2.42e-05         | 4.15e-05 |

Table 2: Error in \((u_x, 1_{\Omega_c})\) for the Hartmann problem §5.1, with \(1_{\Omega_c} = [-\frac{1}{4}, \frac{1}{2}] \times [-\frac{1}{4}, \frac{1}{2}]\). The finite dimensional space here is \((P_2, P_1, P_1)\) for \((u, b, p)\).

| 2D Elem. | True Error | Eff. | \(E_{\text{mom}}\) | \(E_{\text{con}}\) | \(E_M\) |
|----------|------------|------|-------------------|-------------------|--------|
| 1600     | -3.35e-04  | 1.04 | 1.38e-06          | -3.37e-04         | -1.40e-05 |
| 6400     | -9.13e-05  | 1.06 | 1.39e-06          | -9.26e-05         | -5.34e-06 |
| 14400    | -4.17e-05  | 1.06 | 8.06e-07          | -4.25e-05         | -2.67e-06 |
| 25600    | -2.38e-05  | 1.06 | 5.09e-07          | -2.43e-05         | -1.59e-06 |

Table 3: Error in \((u_x, 1_{\Omega_c})\) for the Hartmann problem §5.1. The finite dimensional space here is \((P_2, P_2, P_1)\) for \((u, b, p)\).

| 2D Elem. | True Error | Eff. | \(E_{\text{mom}}\) | \(E_{\text{con}}\) | \(E_M\) |
|----------|------------|------|-------------------|-------------------|--------|
| 1600     | 1.74e-06   | 1.35 | 1.03e-06          | -1.14e-05         | 1.27e-05 |
| 6400     | 2.82e-07   | 1.59 | 2.53e-07          | -2.01e-06         | 2.20e-06 |
| 14400    | 1.10e-07   | 1.70 | 1.08e-07          | -7.56e-07         | 8.35e-07 |
| 25600    | 5.83e-08   | 1.75 | 5.95e-08          | -3.86e-07         | 4.29e-07 |

Table 4: Error in \((u_x, 1_{\Omega_c})\) for the Hartmann problem §5.1. The finite dimensional space here is \((P_3, P_2, P_2)\) for \((u, b, p)\). Here, we approximate the true solution with the computed solution which results in linearization error. For this accurate a solution, this deteriorates the quality of the estimate which in turn results in a efficiency further from 1. This is confirmed in Table 5 where we use the true solution and the effectivity is again close to 1.

5.2. Magnetic Lid Driven Cavity

5.2.1. Regularization and solution method

The magnetic lid driven cavity is another common benchmark problem for verifying MHD codes [5, 54]. However, the standard lid velocity is discontinuous and therefore has at most \(H^{1/2-\varepsilon}\) regularity in 2D with \(\varepsilon > 0\). By the converse
of the trace theorem and the Sobolev inequality [55, 56], the solution $u_x$ cannot obtain $H^1$ regularity on the interior. Indeed, in this situation, we do not even have well posedness of the primal problem, so there is not real hope for error analysis. This issue has been address in a purely fluid context [57, 58]. In both cases, a regularization of the lid velocity is proposed to mitigate theoretical issues (in the former) and the ability to achieve higher Reynold’s numbers (in the latter). In this work, we use a similar regularization to the one proposed in [58], a polynomial regularization of the lid velocity,

$$u_{\text{top}}(x) = C \left( x - \frac{1}{2} \right)^2 \left( x + \frac{1}{2} \right)^2,$$

with $C$ chosen such that

$$\int_{-1/2}^{1/2} u_{\text{top}}(x) \, dx = 1.$$

The boundary conditions are imposed as $g(x, 0.5) = [u_{\text{top}}, 0]^T$ on the top face and zero on the rest of the boundary. The boundary conditions for the magnetic field are $q = [0, 1]^T$ on the left and right ($x = -0.5$ or $x = 0.5$) and zero on the top and bottom ($y = -0.5$ or $y = 0.5$). To get a qualitative measure of the validity of the regularized problem, we show plot of the velocity profile for a fixed Reynold’s number $R_f = 5000$ and varying magnetic Reynold’s numbers $R_m$ in Figure 1. These plots are qualitatively similar to Figure 1 in [5] (for which an un-regularized lid velocity is used), which gives us a good indication that the regularized version produces qualitatively similar features.

Furthermore, since high $R_f$ flows provide difficulties for the continuous Galerkin method [59], we use a homotopic sequence of initial guesses to achieve high $R_f$. Indeed, we run the problem for a moderate value of $R_f = 200$ for example, and then use the solution produced by the solver as the initial guess for a larger value e.g. $R_f = 1000$ until we have achieved the desired value. Figure 2 shows the intermediate values in this sequence to solve a problem with $R_f = 1000$.

### 5.2.2. Problem parameters and results

We take the same QoI as for the Hartmann problem although now with

$$\Omega_c := \left[ -\frac{1}{4}, \frac{1}{4} \right] \times [0, \frac{1}{2}],$$

Table 5: Error in $(u_x, \Omega_c)$ for the Hartmann problem, §5.1. The finite dimensional space here is $(P^3, P^2, P^2)$ for $(u, b, p)$. No linearization error is present here because we use the true solution in the definition of the adjoint.

| 2d Elem. | True Error | Eff. | $E_{\text{mom}}$ | $E_{\text{con}}$ | $E_M$ |
|----------|------------|------|-----------------|-----------------|------|
| 1600     | 1.74e-06   | 0.96 | 6.41e-07        | -9.93e-06       | 1.10e-05 |
| 6400     | 2.82e-07   | 0.96 | 1.50e-07        | -1.56e-06       | 1.68e-06 |
| 14400    | 1.10e-07   | 0.95 | 6.19e-08        | -5.47e-07       | 5.90e-07 |
| 25600    | 5.83e-08   | 0.95 | 3.30e-08        | -2.65e-07       | 2.87e-07 |

13
so that the QoI \((\Psi, U) = (\mathbb{1}_{\Omega_c}, u_x)\) has the physical interpretation of capturing the recirculation in the lower half of the box.

The effectivity ratio and error contributions various values of \(R_f\) are shown in Tables 7, 6, 8 and 9. We note that the error estimate is accurate in all cases with effectivity ratios close to 1.

In terms of error contributions, for the lowest order cases, both \(R_f = 2000\) and \(R_f = 1000\) have a fairly balanced distribution of error between the components as seen in Tables 6 and 8. Similarly to the Hartmann problem, when we refine the space for \(b\), the contribution \(E_M\) decreases by several orders of magnitude, but the overall error remains dominated by \(E_{\text{con}}\) and \(E_{\text{mom}}\). In contrast to Hartmann, \(E_{\text{mom}}\) is substantial even when we refine \(b\) which we attribute to the high \(R_f\) nature of the problem.
Table 6: error in \((u_x, 1_{\Omega_c})\) for the lid driven cavity §5.2. The finite dimensional space here is \((P^2, P^1, P^1)\) for \((u, b, p)\). We use an overkill solution on a 400x400=160000 element mesh and \((P^3, P^2, P^2)\) elements. The parameters are \(R_f = 1000, R_m = 0.4, \kappa = 1\).

| 2d Elem. | True Error | Eff. | \(E_{\text{mom}}\) | \(E_{\text{con}}\) | \(E_M\) |
|----------|------------|------|-----------------|-----------------|-------|
| 1600     | 5.02e-04   | 0.97 | 8.95e-05       | 2.21e-04        | 1.75e-04 |
| 3600     | 1.61e-04   | 0.96 | 2.19e-05       | 5.27e-05        | 8.06e-05 |
| 6400     | 2.20e-05   | 0.98 | 8.65e-06       | 1.58e-05        | 4.60e-05 |
| 10000    | 4.02e-05   | 0.99 | 4.02e-06       | 6.06e-06        | 2.96e-05 |

Table 7: error in \((u_x, 1_{\Omega_c})\) for the lid driven cavity §5.2. The finite dimensional space here is \((P^2, P^2, P^1)\) for \((u, b, p)\). We use an overkill solution on a 400x400=160000 element mesh and \((P^3, P^2, P^2)\) elements. The parameters are \(R_f = 1000, R_m = 0.4, \kappa = 1\).

| 2d Elem. | True Error | Eff. | \(E_{\text{mom}}\) | \(E_{\text{con}}\) | \(E_M\) |
|----------|------------|------|-----------------|-----------------|-------|
| 1600     | 3.17e-04   | 0.99 | 9.36e-05       | 2.20e-04        | -6.79e-07 |
| 3600     | 7.92e-05   | 0.95 | 2.27e-05       | 5.28e-05        | 1.24e-07 |
| 6400     | 2.58e-05   | 0.96 | 8.95e-06       | 1.58e-05        | 3.79e-08 |
| 10000    | 1.05e-05   | 0.97 | 4.14e-06       | 6.08e-06        | 1.52e-08 |

Table 8: error in \((u_x, 1_{\Omega_c})\) for the lid driven cavity §5.2. The finite dimensional space here is \((P^2, P^2, P^1)\) for \((u, b, p)\). We use an overkill solution on a 400x400=160000 element mesh and \((P^3, P^2, P^2)\) elements. The parameters are \(R_f = 1000, R_m = 0.4, \kappa = 1\).

| 2d Elem. | True Error | Eff. | \(E_{\text{mom}}\) | \(E_{\text{con}}\) | \(E_M\) |
|----------|------------|------|-----------------|-----------------|-------|
| 1600     | 6.64e-04   | 1.08 | 2.83e-04       | 4.27e-04        | 6.81e-06 |
| 3600     | 2.00e-04   | 0.95 | 4.34e-05       | 1.37e-04        | 9.07e-06 |
| 6400     | 7.00e-05   | 0.95 | 1.49e-05       | 4.37e-05        | 7.71e-06 |
| 10000    | 3.06e-05   | 0.97 | 6.94e-06       | 1.68e-05        | 5.68e-06 |

Table 9: error in \((u_x, 1_{\Omega_c})\) for the lid driven cavity §5.2. The finite dimensional space here is \((P^2, P^2, P^1)\) for \((u, b, p)\). We use an overkill solution on a 400x400=160000 element mesh and \((P^3, P^2, P^2)\) elements. The parameters are \(R_f = 1000, R_m = 0.4, \kappa = 1\).

| 2d Elem. | True Error | Eff. | \(E_{\text{mom}}\) | \(E_{\text{con}}\) | \(E_M\) |
|----------|------------|------|-----------------|-----------------|-------|
| 1600     | 6.18e-04   | 1.12 | 2.78e-04       | 4.17e-04        | -8.54e-07 |
| 3600     | 1.85e-04   | 0.98 | 4.24e-05       | 1.39e-04        | -2.72e-07 |
| 6400     | 6.09e-05   | 0.97 | 1.47e-05       | 4.41e-05        | -9.47e-08 |
| 10000    | 2.44e-05   | 0.98 | 6.90e-06       | 1.70e-05        | -3.90e-08 |

6. Derivation of the weak adjoint and well-posedness

In this section we provide the details of computing the adjoint to exact penalty weak form following the theory in §3. Then we use a standard saddle point argument to demonstrate the well-posedness of this new adjoint problem (4.5). We take inspiration for these proofs from [3]. To simplify notation in this section, we define

\[ s := u + u_h, \quad t := b + b_h. \] (6.1)
Finally, we use the notation \((\cdot)\) and \((\cdot)\) ≤ to denote that the equality or inequality is justified by equation (\cdot).

6.1. Derivation of the weak form of the adjoint

In this section we provide derivation for the primal linearized operators \(J_{21}^* = Y^*, J_{11}^* = Z_u^*, J_{12}^* = Z_b^*, J_{31}^* = \mathcal{C}^*\) in (4.4). We first compute the primal linearized operators, \(Y = J_{21}, Z_u = J_{11}, Z_b = J_{12}\) and \(\mathcal{C} = J_{31}\), using (3.4) and then apply (3.5) to compute the \(J_{ij}^*\)s. We have from (3.4),

\[ Y_d := \int_0^1 \frac{\partial Y}{\partial b} (sb + (1 - s)b_h) d_s, \]

\[ Z_b d := \int_0^1 \frac{\partial Z}{\partial b} (su + (1 - s)u_h) d_s, \]

\[ Z_u w := \int_0^1 \frac{\partial Z}{\partial u} (sb + (1 - s)b_h) w d_s. \]

To this end, we compute

\[ Y_d = \int_0^1 \frac{\partial Y}{\partial b} (sb + (1 - s)b_h) d_s \]

\[ = \int_0^1 \left[ \nabla \times (sb + (1 - s)b_h) \right] \times d + (\nabla \times d) \times (sb + (1 - s)b_h) d_s \quad (6.2) \]

\[ = \frac{1}{2} \left[ (\nabla \times (b_h + b)) \times d + (\nabla \times d) \times (b_h + b) \right]. \]

Similarly, for the two \(Z\) terms,

\[ Z_b d = \int_0^1 \frac{\partial Z}{\partial b} (su + (1 - s)u_h) d_s \]

\[ = \int_0^1 \nabla \times ((su + (1 - s)u_h) \times d) d_s = \frac{1}{2} [\nabla \times ((u_h + u) \times d)]. \quad (6.3) \]

An identical procedure produces,

\[ Z_u w = \frac{1}{2} [\nabla \times (w \times (b + b_h))]. \quad (6.4) \]

Now, to find the adjoints of these operators, we use (3.5), which in our case involves multiplying by a test function and then isolating the trial function using integration by parts. We also make use of the vector identities in Appendix B.

We are now prepared to compute the adjoint for \(Y\). Integrating (6.2) against
\[ v \in H^1_0(\Omega), \]
\[ (\mathcal{Y} d, v) = \frac{1}{2} \int_{\Omega} \left[ (\nabla \times t) \times d + (\nabla \times d) \times t \right] \cdot v \, dx \]
\[ = \frac{1}{2} \int_{\Omega} d \cdot [v \times (\nabla \times t)] + (\nabla \times d) \cdot [t \times v] \, dx \]
\[ = \frac{1}{2} \int_{\Omega} -d \cdot [(\nabla \times t) \times v] + d \cdot [\nabla \times (t \times v)] \, dx - \frac{1}{2} \int_{\partial \Omega} d \cdot [(t \times v) \times n] \, ds \]
\[ = \frac{1}{2} \int_{\Omega} -d \cdot [(\nabla \times t) \times v] + d \cdot [\nabla \times (t \times v)] \, dx + \frac{1}{2} \int_{\partial \Omega} (t \times v) \cdot [d \times n] \, ds \]
\[ = \frac{1}{2} \int_{\Omega} -d \cdot [(\nabla \times t) \times v] + d \cdot [\nabla \times (t \times v)] \, dx \overset{(2.4)}{=} (d, \mathcal{Y}^* v). \]

We proceed with computing the adjoint for \( Z_u \),
\[ (\mathcal{Z}_u w, c) = \frac{1}{2} (\nabla \times (w \times t), c) \]
\[ = \frac{1}{2} \int_{\Omega} (w \times t) \cdot (\nabla \times c) \, dx - \frac{1}{2} \int_{\partial \Omega} (w \times t) \cdot (c \times n) \, ds \]
\[ = \frac{1}{2} \int_{\Omega} w \cdot [t \times (\nabla \times c)] \, dx - \frac{1}{2} \int_{\partial \Omega} (w \times t) \cdot (c \times n) \, ds \]
\[ = \frac{1}{2} \int_{\Omega} w \cdot [t \times (\nabla \times c)] \, dx \overset{(4.4)}{=} (w, \mathcal{Z}_u^* c). \]

Finally we compute the adjoint to the linearized operator \( Z_b \),
\[ (\mathcal{Z}_b d, c) = \frac{1}{2} (\nabla \times (s \times d), c) \]
\[ = \frac{1}{2} \int_{\Omega} (s \times d) \cdot (\nabla \times c) \, dx - \frac{1}{2} \int_{\partial \Omega} (s \times d) \cdot (c \times n) \, ds \]
\[ = \frac{1}{2} \int_{\Omega} d \cdot [(\nabla \times c) \times s] \, dx - \frac{1}{2} \int_{\partial \Omega} d \cdot [s \times (c \times n)] - (s \times d) \cdot (c \times n) \, ds \]
\[ = \frac{1}{2} \int_{\Omega} d \cdot [(\nabla \times c) \times s] \, dx \overset{(4.4)}{=} (d, \mathcal{Z}_b^* c). \]

The operator \( C^* \) is identical to the one presented in [30].

6.2. Well posedness of the adjoint problem

In this section we prove the well-posedness of the adjoint problem \( \S 4.1 \) equation (4.5) using a saddle point type argument. To keep consistent with the standard setting of saddle point problems [55, 56], we use the notation \( X := H_0^1(\Omega) \times H_1^1(\Omega) \) and \( M := L^2(\Omega) \) so that \( \mathcal{M}(\Omega) = X \times M \). We equip the space \( X \) with the graph norm
\[ \| (v, c) \|_X := (\| v \|^2 + \| c \|^2)^{1/2}. \]
We next define the bilinear form \( a : X \to \mathbb{R} \) by
\[
a((\phi, \beta), (v, c)) = \frac{1}{R_f} (\nabla \phi, \nabla v) + (\mathcal{C}^* \phi, v) + \frac{\kappa}{R_m} (\nabla \times \beta, \nabla \times c) + \frac{\kappa}{R_m} (\nabla \cdot \beta, \nabla \cdot c)
\]
\[
-\kappa \left( \mathbf{Y}^* \phi, c \right) - \kappa \left( \mathbf{Z}_{u}^* \beta, v \right) - \kappa \left( \mathbf{Z}_{b}^* \beta, c \right),
\]
and the mixed form \( b : X \times M \to \mathbb{R} \) by
\[
b((v, c), \pi) = -\left( \pi, \nabla \cdot v \right).
\]
The weak dual problem (4.5) is then equivalent to the following mixed problem: find \((\phi, \beta, \pi) \in X \times M \) such that
\[
\begin{cases}
a((\phi, \beta), (v, c)) + b((v, c), p) = (\psi, v), & \forall (v, c) \in X, \\
b((\phi, \beta), q) = (\psi_p, q), & \forall q \in M.
\end{cases}
\]
According the theory of saddle point systems, in order to show the existence and uniqueness of solutions to (6.8), we must show three things:

(i) The bilinear forms \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) are bounded on their respective domains.

(ii) The form \( a(\cdot, \cdot) \) is coercive on \( X_0 := \{ v \in X : b(v, q) = 0, \forall q \in M \} \).

(iii) The form \( b(\cdot, \cdot) \) satisfies the inf-sup condition: \( \exists \beta > 0 \) such that
\[
\inf_{q \in M} \sup_{(v, c) \in X} \frac{b((v, c), q)}{\|v\|_X \|q\|_M} \geq \beta.
\]
We organize these parts in the following lemmas. We make frequent use of the inequalities in Appendix C in the proofs.

**Lemma 1.** The form \( a(\cdot, \cdot) \) is bounded on \( X \).

**Proof.** Consider the splitting
\[
a((\phi, \beta), (v, c)) = a_0((\phi, \beta), (v, c)) + a_1((\phi, \beta), (v, c))
\]
where
\[
a_0((\phi, \beta), (v, c)) = \frac{1}{R_f} (\nabla \phi, \nabla v) + \frac{\kappa}{R_m} (\nabla \times \beta, \nabla \times c) + \frac{\kappa}{R_m} (\nabla \cdot \beta, \nabla \cdot c),
\]
\[
a_1((\phi, \beta), (v, c)) = (\mathbf{C}^* \phi, v) - \kappa \left( \mathbf{Y}^* \phi, c \right) - \kappa \left( \mathbf{Z}_{u}^* \beta, v \right) - \kappa \left( \mathbf{Z}_{b}^* \beta, c \right).
\]
Then it suffices to show that both \( a_0 \) and \( a_1 \) are bounded separately. The proof for the boundedness of \( a_0 \) is given in [3]. For \( a_1 \) observe that
\[
|a_1((\phi, \beta), (v, c))| \leq \int_{\Omega} |\mathcal{C}^* \phi \cdot v| \, dx + \kappa \int_{\Omega} |\mathbf{Y}^* \phi \cdot c| \, dx
\]
\[
+ \kappa \int_{\Omega} |\mathbf{Z}_{u}^* \beta \cdot v| \, dx + \kappa \int_{\Omega} |\mathbf{Z}_{b}^* \beta \cdot c| \, dx.
\]
For the second term on the right hand side of (6.11),
\[
\int_{\Omega} |\mathbf{C}^* \phi \cdot v| \, dx = \frac{1}{2} \int_{\Omega} \left| \phi \cdot \nabla s^T - (s \cdot \nabla \phi) - (\nabla \cdot s) \phi \right| \cdot v \, dx \\
\leq \frac{1}{2} \left[ (\phi, (\nabla s)^T v) + (s, (\nabla \phi)v) + (\nabla \cdot s, \phi \cdot v) \right] \\
\leq \frac{1}{2} \left[ \|\phi\|_{L^2} \|(\nabla s)^T v\|_{L^2} + \|s\|_{L^2} \|\nabla (\nabla \phi)v\|_{L^2} + \|s\|_{L^2} \|\nabla \cdot s\|_{L^2} \|\phi \cdot v\|_{L^2} \right] \\
\overset{(B.2c)}{\leq} \frac{1}{2} \left[ \|\phi\|_{L^2} \|(\nabla s)^T v\|_{L^2} + \|s\|_{L^2} \|\nabla (\nabla \phi)v\|_{L^2} \right] \\
\overset{(C.1)}{\leq} \frac{C_\Omega}{2} \left( \|\phi\|_{L^1} \|s\|_{L^1} \|v\|_{L^1} + \|s\|_{L^1} \|\phi\|_{L^1} \|v\|_{L^1} \right) \\
\overset{(C.3)}{\leq} \frac{3\sqrt{3}C_\Omega}{2} \|s\|_{L^1} \|\phi\|_{L^1} \|v\|_{L^1},
\]
where \( \gamma \) is the square of the embedding constant of \( H^1(\Omega) \) into \( L^4(\Omega) \), see (C.1).

For the second term on the right hand side of (6.11),
\[
\kappa \left( \nabla \phi \cdot c \right) \leq \frac{\kappa}{2} \int_{\Omega} |c \cdot [(\nabla \times t) \times \phi] + |(\nabla \times c) \cdot [t \times \phi]| \, dx \\
\overset{(B.1d)}{=} \frac{\kappa}{2} \int_{\Omega} |c \cdot [(\nabla \times t) \times \phi]| + |(\nabla \times c) \cdot [t \times \phi]| \, dx \\
\leq \frac{\kappa}{2} \left[ \|c\|_{L^2} \|(\nabla \times t) \times \phi\|_{L^2} + \|\nabla \times c\|_{L^2} \|t \times \phi\|_{L^2} \right] \\
\overset{(C.3)}{\leq} \frac{\kappa C_\Omega}{2} \left( \|c\|_{L^1} \|t\|_{L^1} \|\phi\|_{L^1} + \|\nabla \times c\|_{L^2} \|t\|_{L^1} \|\phi\|_{L^1} \right) \\
\overset{(B.2d)}{\leq} \frac{\kappa \sqrt{2} C_\Omega}{2} \left( \|c\|_{L^1} \|t\|_{L^1} \|\phi\|_{L^1} \right) \\
\overset{(C.1)}{\leq} \kappa \gamma \sqrt{2} C_\Omega \|c\|_{L^1} \|t\|_{L^1} \|\phi\|_{L^1}.
\]

For the third term on the right hand side of (6.11),
\[
\kappa \left( \mathbf{Z}^a \beta, v \right) \leq \frac{\kappa}{2} \int_{\Omega} |v \cdot [t \times (\nabla \times \beta)]| \, dx \\
\leq \frac{\kappa}{2} \|v\|_{L^2} \|t \times (\nabla \times \beta)\|_{L^2} \leq \frac{\kappa}{2} \|v\|_{L^2} \|t\|_{L^1} \|\nabla \times \beta\|_{L^1} \\
\overset{(C.3),(B.2d)}{\leq} \frac{\kappa C_\Omega \sqrt{2}}{2} \|v\|_{L^1} \|t\|_{L^1} \|\beta\|_{L^1} \leq \frac{\kappa C_\Omega \gamma \sqrt{2}}{2} \|v\|_{L^1} \|t\|_{L^1} \|\beta\|_{L^1} \leq \frac{\kappa C_\Omega \gamma \sqrt{2}}{2} \|v\|_{L^1} \|t\|_{L^1} \|\beta\|_{L^1} \leq \frac{\kappa C_\Omega \gamma \sqrt{2}}{2} \|v\|_{L^1} \|t\|_{L^1} \|\beta\|_{L^1}.
\]

The fourth term follows the same argument as the third term to yield the bound,
\[
\kappa \left( \mathbf{Z}^b \beta, c \right) \leq \frac{\kappa C_\Omega \gamma \sqrt{2}}{2} \|c\|_{L^1} \|s\|_{L^1} \|\beta\|_{L^1}.
\]

(6.12)
Putting these bounds together, we conclude

\[
\begin{align*}
   a_1((\phi, \beta), (v, c)) &\leq \gamma C_\Omega \left( \frac{3\sqrt{3}}{2} \|s\|_1 \|\phi\|_1 \|v\|_1 + \frac{\kappa \sqrt{2}}{2} \|c\|_1 \|t\|_1 \|\phi\|_1 \\
   &\quad + \frac{\kappa \sqrt{2}}{2} \|v\|_1 \|t\|_1 \|\beta\|_1 + \frac{\kappa \sqrt{2}}{2} \|c\|_1 \|s\|_1 \|\beta\|_1 \right) \\
   &\leq \gamma C_\Omega \left( \frac{3\sqrt{3}}{2} \|s\|_1 \|\phi\|_1 \|v\|_1 + \frac{\kappa \sqrt{2}}{2} \|c\|_1 \|s\|_1 \|\beta\|_1 \\
   &\quad + \|t\|_1 \kappa \sqrt{2} \|\phi\|_X \|\beta\|_X \right) \\
   &\leq \gamma C_\Omega \left( \|s\|_1 \max \left\{ \frac{3\sqrt{3}}{2}, \frac{\kappa \sqrt{2}}{2} \right\} \|\phi\|_X \|\beta\|_X \right) \\
   &\quad + \|t\|_1 \|v\|_1 \|c\|_1 \|\phi\|_X \|\beta\|_X \\
\end{align*}
\]

(C.5)

\[
\leq \gamma C_\Omega \left( \|s\|_1 \max \left\{ \frac{3\sqrt{3}}{2}, \frac{\kappa \sqrt{2}}{2} \right\} \|\phi\|_X \|\beta\|_X \right) \\
\quad + \|t\|_1 \|v\|_1 \|c\|_1 \|\phi\|_X \|\beta\|_X \\
\quad \leq \alpha_b \|v\|_X \|c\|_X \|\phi\|_X \|\beta\|_X,
\]

where in turn

\[
\alpha_b = \max \left\{ \|s\|_1 \max \left\{ \frac{3\sqrt{3}}{2}, \frac{\kappa \sqrt{2}}{2} \right\}, \|t\|_1 \right\}.
\]

Now we consider the coercivity of the bilinear form \(a(\cdot, \cdot)\) on \(X\).

**Lemma 2.** There exists a constant \(\alpha_c > 0\) such that whenever

\[
\frac{k_1}{R_f} - \frac{\kappa}{R_m} \left[ \frac{3\sqrt{3}}{2} \|s\|_1 + \frac{3\kappa \sqrt{2}}{4} \|t\|_1 \right] > 0,
\]

(6.14)

and

\[
\frac{k_2\kappa}{R_m^2} - \frac{\kappa}{R_m} \left[ \frac{\kappa \sqrt{2}}{2} \|s\|_1 + \frac{3\kappa \sqrt{2}}{4} \|t\|_1 \right] > 0
\]

(6.15)

then

\[
a((\phi, \beta), (\phi, \beta)) \geq \alpha_c \|\phi\|_X^2, \quad \forall (\phi, \beta) \in X.
\]

(6.16)

**Proof.** Using the splitting established in the previous lemma,

\[
a((\phi, \beta), (\phi, \beta)) = a_0((\phi, \beta), (\phi, \beta)) - |a_1((\phi, \beta), (\phi, \beta))| \\
= \frac{1}{R_f} \langle \nabla \phi, \nabla \phi \rangle + \frac{\kappa}{R_m} \langle \nabla \times \beta, \nabla \times \beta \rangle + \frac{\kappa}{R_m} \langle \nabla \cdot \beta, \nabla \cdot \beta \rangle \\
- |a_1((\phi, \beta), (\phi, \beta))| \\
\geq \frac{k_1}{R_f} \|\phi\|_1^2 + \frac{k_2\kappa}{R_m^2} \|\beta\|_1^2 - |a_1((\phi, \beta), (\phi, \beta))| \\
\]

(6.17)
where $k_1$ comes from the Poincaré inequality (C.6), and $k_2$ is defined though

$$\|\nabla \times v\|_0^2 + \|\nabla \cdot v\|_0^2 \geq k_2\|v\|_1^2, \quad \forall v \in H^1_+(\Omega),$$  \hspace{1cm} (6.18)

which is valid under the restrictions we have imposed on the domain $\Omega$ and the continuous embedding of $H^1_+(\Omega) \hookrightarrow H^1(\Omega)$ [42]. Picking up from (6.17) and using (C.7) we conclude that,

$$a((\phi, \beta), (\phi, \beta)) \geq k_1 \|\phi\|_1^2 + k_2 \|\beta\|_1^2$$

$$- \frac{\gamma C_\Omega 3\sqrt{3}}{2} \|\phi\|_1 \|t\|_1 \|\beta\|_1$$

$$\geq \left( \frac{k_1}{R_f} - \frac{\gamma C_\Omega 3\sqrt{3}}{2} \|s\|_1 \right) \|\phi\|_1^2 + \left( \frac{k_2 \kappa}{R_m^2} - \frac{\gamma C_\Omega \kappa \sqrt{2}}{2} \|s\|_1 \right) \|\beta\|_1^2$$

$$- \frac{\gamma C_\Omega 3\kappa \sqrt{2}}{4} \|t\|_1 \left( \|\beta\|_1^2 + \|\phi\|_1^2 \right)$$

$$= \left( \frac{k_1}{R_f} - \gamma C_\Omega \left[ \frac{3\sqrt{3}}{2} \|s\|_1 + \frac{3 \kappa \sqrt{2}}{4} \|t\|_1 \right] \right) \|\phi\|_1^2$$

$$+ \left( \frac{k_2 \kappa}{R_m^2} - \gamma C_\Omega \left[ \frac{\kappa \sqrt{2}}{2} \|s\|_1 + \frac{3 \kappa \sqrt{2}}{4} \|t\|_1 \right] \right) \|\beta\|_1^2.$$ 

Thus, taking

$$\alpha_c = \min \left\{ \frac{k_1}{R_f} - \gamma C_\Omega \left[ \frac{3\sqrt{3}}{2} \|s\|_1 + \frac{3 \kappa \sqrt{2}}{4} \|t\|_1 \right], \right.$$

$$\left. \frac{k_2 \kappa}{R_m^2} - \gamma C_\Omega \left[ \frac{\kappa \sqrt{2}}{2} \|s\|_1 + \frac{3 \kappa \sqrt{2}}{4} \|t\|_1 \right] \right\},$$  \hspace{1cm} (6.19)

concludes the lemma. 

Now we are prepared to prove the main result.

**Theorem 3.** Under the conditions of Lemma 2 there exists a unique solution to the dual problem (4.5).

**Proof.** The boundedness and inf-sup condition for $b(\cdot, \cdot)$ are standard see e.g. [56]. The boundedness of $a(\cdot, \cdot)$ follows from Lemma 1, and Lemma 2 proves $a(\cdot, \cdot)$ is coercive on $X$ so in particular on $X_0$. 

\hfill \Box
Appendix A. Standard function spaces

We denote by \( L^2(\Omega) \) the set of all square Lebesgue integrable functions on \( \Omega \subset \mathbb{R}^d \) with associated inner product \((\cdot, \cdot)\) and norm \(\|\cdot\|\). This extends naturally to vector valued functions, denoted by \( L^2(\Omega) \), where the inner product is given by,

\[
(u, v) = \sum_{i=1}^{d} (u_i, v_i).
\]

The Sobolev norm for \( p = 2 \) is,

\[
\|v\|_m := \left( \sum_{|\alpha|=0}^{m} \|D^{\alpha}v\|^2 \right)^{1/2},
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_m) \) is a multi-index of length \( m \) and

\[
D^{\alpha}v := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m} v,
\]

where the partial derivatives are taken in the weak sense. Thus, the Hilbert spaces \( H^m \) for \( m = 0, 1, 2, \ldots \) is simply be defined as functions with bounded \( m \)-norm,

\[
H^m(\Omega) := \{ v : \|v\|_m < \infty \}.
\]

The space \( H^0(\Omega) \) is identified with \( L^2(\Omega) \). For vector valued functions, the Hilbert space \( H^m \) is defined as,

\[
H^m(\Omega) := \{ v : v_i \in H^m(\Omega), i = 1, \ldots, d \}.
\]

Appendix B. Vector identities and inequalities

In this section we have collected all relevant identities to perform the necessary integration by parts arguments for the readers convenience,

\[
A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B), \tag{B.1a}
\]

\[
\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B), \tag{B.1b}
\]

\[
\nabla \cdot ([\nabla \cdot A]B) = (\nabla \cdot A)(\nabla \cdot B) + B \cdot [\nabla (\nabla \cdot A)], \tag{B.1c}
\]

\[
\int_{\Omega} A \cdot (\nabla \times B) \, dx = \int_{\Omega} B \cdot (\nabla \times A) \, dx - \int_{\partial \Omega} B \cdot (A \times n) \, ds, \tag{B.1d}
\]

\[
\int_{\Omega} B \cdot [\nabla (\nabla \cdot A)] \, dx = - \int_{\Omega} (\nabla \cdot A)(\nabla \cdot B) \, dx + \int_{\partial \Omega} (\nabla \cdot A)B \, ds. \tag{B.1e}
\]

One should note that the integral identities (B.1d) and (B.1e) follow from the component-wise identities (B.1a)-(B.1c) and the divergence theorem. We also
make use of the following inequalities for \( u, v \in H^1 \),

\[
|u \cdot v| \leq \|u\|_{R^3} \|v\|_{R^3}, \quad (B.2a)
\]
\[
\|u \times v\| \leq \|u\|_{R^3} \|v\|_{R^3}, \quad (B.2b)
\]
\[
\|\nabla \times u\| \leq \sqrt{2}\|\nabla u\|_{R^3}, \quad (B.2c)
\]
\[
\|\nabla \cdot u\| \leq \sqrt{3}\|\nabla u\|_{R^3} \|v\|_{R^3}, \quad (B.2d)
\]
\[
\|Av\|_{R^3} \leq \|A\|_{R^3} \|v\|_{R^3}, \quad (B.2e)
\]

and finally the equality

\[
\|\nabla v^T\|_{R^3} = \|\nabla v\|_{R^3}. \quad (B.3)
\]

**Appendix C. Useful inequalities from analysis**

1. The space \( H^1(\Omega) \) embeds continuously in \( L^4(\Omega) \) with constant \( \sqrt{\gamma} \). That is, \( H^1(\Omega) \hookrightarrow L^4(\Omega) \) such that,

\[
\|v\|_{L^4} \leq \sqrt{\gamma}\|v\|_{H^1}. \quad (C.1)
\]

2. We have

\[
\|u \cdot v\|_{L^2} \leq \|u\|_{L^4} \|v\|_{L^4}. \quad (C.2)
\]

This is seen as follow,

\[
\|u \cdot v\|_{L^2} = \left( \int_\Omega (u \cdot v)^2 \, dx \right)^{1/2} \leq \left( \int_\Omega \|u\|_{R^3}^2 \|v\|_{R^3}^2 \, dx \right)^{1/2} \\
\leq \left( \left( \int_\Omega \|u\|_{R^3}^4 \right)^{1/2} \left( \int_\Omega \|v\|_{R^3}^4 \right)^{1/2} \right)^{1/2} = \|u\|_{L^4} \|v\|_{L^4}.
\]

3. For \( v \in L^4 \), we have

\[
\|v\|_{L^2} \leq C_\Omega \|v\|_{L^4}. \quad (C.3)
\]

Here \( C_\Omega \) is a constant that depends on \( \Omega \). The last inequality follows from the Hölder inequality. Suppose \( r \) satisfies \( \frac{1}{q} + \frac{1}{r} = \frac{1}{p} \). From the Hölder inequality,

\[
\|f\|_p \leq \|1\|_p \|f_p\|_1 \|1^p\|_{r/p} \|1\|_{r/p} = \|f_p\|_q \|1\|_p \\
\Rightarrow \|f\|_p \leq \text{mes}(\Omega)^{\frac{1}{q}} \|f\|_q.
\]

In this specific case, \( q = r = 4 \) and \( p = 2 \).

4. We have

\[
\|u \times v\|_{L^2} \leq \sqrt{3}\|u\|_{L^4} \|v\|_{L^4}. \quad (C.4)
\]
as shown below,

\[
\|u \times v\|_{L^2} = \left( \int_{\Omega} \|u \times v\|_{\mathbb{R}^3}^2 \, dx \right)^{1/2} \overset{(B.2b)}{\leq} \left( \int_{\Omega} \sqrt{3} \|u\|_{\mathbb{R}^3}^2 \|v\|_{\mathbb{R}^3}^2 \, dx \right)^{1/2}
\]

\[
\leq \left( \sqrt{3} \left( \int_{\Omega} \|u\|_{\mathbb{R}^3 \times \mathbb{R}^3}^4 \right)^{1/2} \left( \int_{\Omega} \|v\|_{\mathbb{R}^3}^4 \right)^{1/2} \right)^{1/2} = \sqrt{3} \|u\|_{L^4} \|v\|_{L^4}.
\]

5. The Cauchy-Schwarz inequality for \([a, b], [c, d] \in \mathbb{R}^2\),

\[
ac + bd = [a, b] [c, d]^T \leq \sqrt{a^2 + c^2} \sqrt{b^2 + d^2}, \quad (C.5)
\]

6. The Poincaré inequality is,

\[
\|\nabla v\|_{L^2}^2 \geq k_1 \|v\|_{H^1_{0}}^2, \quad \forall v \in H^1_{0}(\Omega), \quad (C.6)
\]

7. For \(x, y \in \mathbb{R}\),

\[
-xy \geq -\frac{1}{2} (x^2 + y^2), \quad (C.7)
\]

Acknowledgements

J. Chaudhry’s work is supported by the NSF-DMS 1720402. The work of Ari. E. Rappaport and John N. Shadid was partially supported by the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research, Applied Mathematics Program and by the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research and Office of Fusion Energy Sciences, Scientific Discovery through Advanced Computing (SciDAC) program. Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energys National Nuclear Security Administration under contract DE-NA0003525. This paper describes objective technical results and analysis. Any subjective views or opinions that might be expressed in the paper do not necessarily represent the views of the U.S. Department of Energy or the United States Government.

References

[1] J. P. H. Goedbloed, S. Poedts, A. Mills, S. Romaine, Principles of Magnetohydrodynamics: With Applications to Laboratory and Astrophysical Plasmas, Cambridge University Press, 2003.

[2] M. Ulrich, B. Leo, Magnetofluiddynamics in Channels and containers, Springer, 2010.
[3] M. D. Gunzburger, A. J. Meir, J. S. Peterson, On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics, Mathematics of Computation 56 (194) (1991) 523–523.

[4] J. Shadid, R. Pawlowski, J. Banks, L. Chacón, P. Lin, R. Tuminaro, Towards a scalable fully-implicit fully-coupled resistive MHD formulation with stabilized FE methods, Journal of Computational Physics 229 (20) (2010) 7649–7671.

[5] E. G. Phillips, H. C. Elman, E. C. Cyr, J. N. Shadid, R. P. Pawlowski, A block preconditioner for an exact penalty formulation for stationary MHD, SIAM Journal on Scientific Computing 36 (6) (2014).

[6] P. T. Lin, J. N. Shadid, P. H. Tsuji, On the performance of krylov smoothing for fully coupled AMG preconditioners for VMS resistive MHD, International Journal for Numerical Methods in Engineering 120 (12) (2019) 1297–1309.

[7] P. Lin, J. Shadid, J. Hu, R. Pawlowski, E. Cyr, Performance of fully-coupled algebraic multigrid preconditioners for large-scale VMS resistive MHD, Journal of Computational and Applied Mathematics 344 (2018) 782–793.

[8] J. Shadid, R. Pawlowski, J. Banks, L. Chacón, P. Lin, R. Tuminaro, Towards a scalable fully-implicit fully-coupled resistive MHD formulation with stabilized FE methods, Journal of Computational Physics 229 (20) (2010) 7649–7671.

[9] L. Chacón, A. Stanier, A scalable, fully implicit algorithm for the reduced two-field low-$\beta$ extended mhd model, Journal of Computational Physics 326 (2016) 763–772.

[10] E. C. Cyr, J. N. Shadid, R. S. Tuminaro, R. P. Pawlowski, L. Chacón, A new approximate block factorization preconditioner for two-dimensional incompressible (reduced) resistive MHD, SIAM Journal on Scientific Computing 35 (3) (2013) B701–B730.

[11] J. H. Adler, T. A. Manteuffel, S. F. McCormick, J. W. Ruge, First-order system least squares for incompressible resistive magnetohydrodynamics, SIAM Journal on Scientific Computing 32 (1) (2010) 229–248.

[12] J. H. Adler, T. A. Manteuffel, S. F. McCormick, J. W. Ruge, G. D. Sanders, Nested iteration and first-order system least squares for incompressible, resistive magnetohydrodynamics, SIAM Journal on Scientific Computing 32 (3) (2010) 1506–1526.

[13] J. H. Adler, M. Brezina, T. A. Manteuffel, S. F. McCormick, J. W. Ruge, L. Tang, Island coalescence using parallel first-order system least squares
on incompressible resistive magnetohydrodynamics, SIAM Journal on Scientific Computing 35 (5) (2013) S171–S191.

[14] P.-W. Hsieh, S.-Y. Yang, A bubble-stabilized least-squares finite element method for steady MHD duct flow problems at high hartmann numbers, Journal of Computational Physics 228 (22) (2009) 8301–8320.

[15] C. Trenchea, Unconditional stability of a partitioned IMEX method for magnetohydrodynamic flows, Applied Mathematics Letters 27 (2014) 97–100.

[16] R. Codina, N. H. Silva, Stabilized finite element approximation of the stationary magneto-hydrodynamics equations, Computational Mechanics 38 (4-5) (2006) 344–355.

[17] J. Li, H. Yang, E. Machorro (Eds.), Recent Advances in Scientific Computing and Applications, American Mathematical Society, 2013.

[18] D. Estep, V. Ginting, D. Ropp, J. N. Shadid, S. Tavener, An a posteriori–a priori analysis of multiscale operator splitting, SIAM Journal on Numerical Analysis 46 (3) (2008) 1116–1146.

[19] D. Estep, A. Mååqvist, S. Tavener, Nonparametric density estimation for randomly perturbed elliptic problems I: Computational methods, a posteriori analysis, and adaptive error control, SIAM J. Sci. Comput. 31 (2009) 2935–2959.

[20] J. H. Chaudhry, N. Burch, D. Estep, Efficient distribution estimation and uncertainty quantification for elliptic problems on domains with stochastic boundaries, SIAM/ASA Journal on Uncertainty Quantification 6 (3) (2018) 1127–1150.

[21] D. Estep, A posteriori error bounds and global error control for approximation of ordinary differential equations, SIAM Journal on Numerical Analysis 32 (1) (1995) 1–48.

[22] M. B. Giles, E. Sli, Adjoint methods for PDEs: a posteriori error analysis and postprocessing by duality, Acta Numerica 2002 (2002) 145236.

[23] M. Ainsworth, T. Oden, A posteriori error estimation in finite element analysis, John Wiley-Teubner, 2000.

[24] W. Bangerth, R. Rannacher, Adaptive Finite Element Methods for Differential Equations, Birkhauser Verlag, 2003.

[25] D. J. Estep, M. G. Larson, R. D. Williams, A. M. Society, Estimating the error of numerical solutions of systems of reaction-diffusion equations, American Mathematical Society, 2000.

[26] R. Becker, R. Rannacher, An optimal control approach to a posteriori error estimation in finite element methods: Acta numerica (Jan 2003).
[27] Y. Cao, L. Petzold, A posteriori error estimation and global error control for ordinary differential equations by the adjoint method, SIAM Journal on Scientific Computing 26 (2) (2004) 359–374.

[28] J. H. Chaudhry, J. N. Shadid, T. Wildey, A posteriori analysis of an IMEX entropy-viscosity formulation for hyperbolic conservation laws with dissipation, Applied Numerical Mathematics 135 (2019) 129–142.

[29] J. H. Chaudhry, J. Collins, J. N. Shadid, A posteriori error estimation for multi-stage runge–kutta IMEX schemes, Applied Numerical Mathematics 117 (2017) 36–49.

[30] D. Estep, S. Tavener, T. Wildey, A posteriori error estimation and adaptive mesh refinement for a multiscale operator decomposition approach to fluid-solid heat transfer, Journal of Computational Physics 229 (11) (2010) 4143–4158.

[31] K. Eriksson, D. Estep, P. Hansbo, C. Johnson, Introduction to adaptive methods for differential equations, Acta Numerica 4 (1995) 105–158.

[32] J. H. Chaudhry, A posteriori analysis and efficient refinement strategies for the poisson–boltzmann equation, SIAM Journal on Scientific Computing 40 (4) (2018) A2519–A2542.

[33] K. Eriksson, D. Estep, P. Hansbo, C. Johnson, Computational Differential Equations, Cambridge University Press, Cambridge, 1996.

[34] T. J. Barth, A posteriori Error Estimation and Mesh Adaptivity for Finite Volume and Finite Element Methods, Vol. 41 of Lecture Notes in Computational Science and Engineering, Springer, New York, 2004.

[35] J. H. Chaudhry, D. Estep, V. Ginting, J. N. Shadid, S. Tavener, A posteriori error analysis of imex multi-step time integration methods for advection–diffusion–reaction equations, Computer Methods in Applied Mechanics and Engineering 285 (2015) 730–751.

[36] D. Schotzau, Mixed finite element methods for stationary incompressible magneto-hydrodynamics, Numerische Mathematik 96 (4) (2004) 771800.

[37] J. C. Nedelec, Mixed finite elements in $\mathbb{R}^3$, Numerische Mathematik 35 (3) (1980) 315341.

[38] J. H. Adler, Y. He, X. Hu, S. P. MacLachlan, Vector-potential finite-element formulations for two-dimensional resistive magnetohydrodynamics, Computers & Mathematics with Applications 77 (2) (2019) 476–493.

[39] A. Dedner, F. Kemm, D. Kröner, C.-D. Munz, T. Schnitzer, M. Wesenberg, Hyperbolic divergence cleaning for the mhd equations, Journal of Computational Physics 175 (2) (2002) 645–673.
[40] D. Kuzmin, N. Klyushnev, Limiting and divergence cleaning for continuous finite element discretizations of the MHD equations, Journal of Computational Physics 407 (2020) 109230.

[41] J.-F. Gerbeau, A stabilized finite element method for the incompressible magnetohydrodynamic equations, Numerische Mathematik 87 (1) (2000) 83–111.

[42] V. Girault, P.-A. Raviart, Finite Element Methods for Navier-Stokes Equations, Springer Berlin Heidelberg, 1986.

[43] D. Boffi, F. Brezzi, M. Fortin, et al., Mixed finite element methods and applications, Vol. 44, Springer, 2013.

[44] G. I. Marchuk, V. I. Agoshkov, V. P. Shutyaev, Adjoint equations and perturbation algorithms in nonlinear problems, CRC Press, 1996.

[45] G. I. Marchuk, Adjoint Equations and Analysis of Complex Systems, Springer Nature, 1995.

[46] J. H. Chaudhry, J. N. Shadid, T. Wildey, A posteriori analysis of an imex entropy-viscosity formulation for hyperbolic conservation laws with dissipation (Aug 2018).

[47] J. H. Chaudhry, J. Collins, J. N. Shadid, A posteriori error estimation for multi-stage rungekutta imex schemes (Feb 2017).

[48] J. H. Chaudhry, D. Estep, S. Tavener, V. Carey, J. Sandelin, A posteriori error analysis of two-stage computation methods with application to efficient discretization and the Parareal algorithm, SIAM Journal on Numerical Analysis 54 (5) (2016) 2974–3002.

[49] J. M. Connors, J. W. Banks, J. A. Hittinger, C. S. Woodward, Quantification of errors for operator-split advection–diffusion calculations, Computer Methods in Applied Mechanics and Engineering 272 (2014) 181–197.

[50] V. Carey, D. Estep, S. Tavener, A posteriori analysis and adaptive error control for multiscale operator decomposition solution of elliptic systems I: One way coupled systems, SIAM Journal on Numerical Analysis 47 (1) (2009) 740–761.

[51] M. S. Alnæs, J. Blechta, J. Hake, A. Johansson, B. Kehlet, A. Logg, C. Richardson, J. Ring, M. E. Rognes, G. N. Wells, The fenics project version 1.5, Archive of Numerical Software 3 (100) (2015).

[52] A. Logg, K.-A. Mardal, G. N. Wells, et al., Automated Solution of Differential Equations by the Finite Element Method, Springer, 2012.

[53] A. Logg, G. N. Wells, J. Hake, DOLFIN: a C++/Python Finite Element Library, Springer, 2012.
S. Sivasankaran, A. Malleswaran, J. Lee, P. Sundar, Hydro-magnetic combined convection in a lid-driven cavity with sinusoidal boundary conditions on both sidewalls, International Journal of Heat and Mass Transfer 54 (1-3) (2011) 512–525.

A. Ern, J.-L. Guermond, Theory and practice of finite elements, Springer, 2011.

S. C. Brenner, L. R. Scott, The mathematical theory of finite element methods, Springer, 2011.

M. Hamouda, R. Temam, L. Zhang, Modeling the lid driven flow: Theory and computation, International Journal of Numerical Analysis and Modeling 14 (2017) 313–341.

M. W. Lee, E. H. Dowell, M. J. Balajewicz, A study of the regularized lid-driven cavity’s progression to chaos (Nov 2018).

J. Donea, A. Huerta, Finite element methods for flow problems, Wiley, 2005.