Jacob’s Ladders, 
the Structure of the Hardy–Littlewood Integral 
and Some New Class of Nonlinear Integral Equations

Jan Moser

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Abstract—In this paper we obtain new formulae for short and microscopic parts of the 
Hardy–Littlewood integral, and the first asymptotic formula for the sixth-order expression 
$|\zeta(\frac{1}{2} + it)|^4 |\zeta(\frac{1}{2} + it)|^2$. These formulae cannot be obtained in the theories of Balasubramaniam, Heath-Brown and Ivić.

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1. INTRODUCTION

Let us recall that Hardy and Littlewood started to study the following integral in 1918:

$$
\int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \, dt = \int_0^T Z^2(t) \, dt,
$$

where

$$
Z(t) = e^{i\vartheta(t)} \zeta \left( \frac{1}{2} + it \right), \quad \vartheta(t) = -\frac{t}{2} \ln \pi + \text{Im} \ln \Gamma \left( \frac{1}{4} + \frac{it}{2} \right),
$$

and they have derived the following formula [2, pp. 122, 151–156]:

$$
\int_0^T Z^2(t) \, dt \sim T \ln T, \quad T \to \infty.
$$

We have shown in paper [8] that along with the asymptotic formula (1.3) possessing an unbounded error, there is an infinite family of other asymptotic representations of the Hardy–Littlewood integral (1.1). Each member of this family is an almost exact representation of the integral (1.1). Namely, the following formula takes place:

$$
\int_0^T Z^2(t) \, dt = \frac{\varphi(T)}{2} \ln \frac{\varphi(T)}{2} + (c - \ln 2\pi) \frac{\varphi(T)}{2} + c_0 + O \left( \frac{\ln T}{T} \right),
$$

$\vartheta(t)$ and $\varphi(T)$ are defined in papers [8, 17].

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where \( \varphi(T) \) is Jacob’s ladder, i.e., an arbitrary solution to the nonlinear integral equation

\[
\mu_{x(T)} \int_0^T Z^2(t)e^{-\varphi_0(T)} dt = \int_0^T Z^2(t) dt,
\]

and \( \mu(y) \geq 7y \ln y; \) each function \( \mu(y) \) generates a solution \( y = \varphi_\mu(T) = \varphi(T) \) (see [8]).

We obtain the following new properties of the signal (1.2) generated by the Riemann zeta-function:

(A) We obtain the multiplicative asymptotic formula

\[
\int_0^{T + U} Z^2(t) dt \sim U \ln T \tan[\alpha(T, U)], \quad U \in \left(0, \frac{T}{\ln T}\right], \quad T \to \infty,
\]

in Sections 2–7 of this paper. The application to microscopic \((0 < U < 1)\) and short \((1 \leq U < T^{\frac{1}{2} + 2\epsilon})\) parts of the Hardy–Littlewood integral (1.1) is the main aim of this formula.

(B) We also obtain, in Sections 8–10 of this work, the formula

\[
\int_T^{T + U_1} \left| \zeta\left(\frac{1}{2} + i\varphi_1(t)\right)\right|^4 \left| \zeta\left(\frac{1}{2} + it\right)\right|^2 dt \sim \frac{1}{2\pi} U_1 \ln^5 T, \quad T \to \infty,
\]

where \( U_1 = T^{\frac{2}{5} + 2\epsilon} \) and \( \varphi_1(t) = \frac{1}{2} \varphi(t) \). This formula is the first integral asymptotic formula in the theory of the Riemann zeta-function for the sixth-order expression \( |i\left(\frac{1}{2} + i\varphi_1(t)\right)|^4 |i\left(\frac{1}{2} + it\right)|^2 \).

(C) In Section 11 of this work the following property, for example, is noticed: Jacob’s ladder \( \varphi_1 \) is an asymptotic solution of the nonlinear integral equation

\[
\int_{x(T)}^{x(T+2)} \frac{\left[T_n(x(t) - T - 1)\right]^2}{\sqrt{1 - (x(t) - T - 1)^2}} \left| \zeta\left(\frac{1}{2} + it\right)\right|^2 dt = \frac{\pi}{2} \ln T,
\]

where \( T_n(t) \), \( t \in [-1, 1] \), is the Chebyshev polynomial of the first kind; i.e., the following asymptotic formula holds true:

\[
\int_{\varphi_1^{-1}(T)}^{\varphi_1^{-1}(T+2)} \frac{\left[T_n(\varphi_1(t) - T - 1)\right]^2}{\sqrt{1 - (\varphi_1(t) - T - 1)^2}} \left| \zeta\left(\frac{1}{2} + it\right)\right|^2 dt \sim \frac{\pi}{2} \ln T, \quad T \to \infty.
\]

2. NECESSITY OF A NEW EXPRESSION FOR SHORT AND MICROSCOPIC PARTS OF THE HARDY–LITTLEWOOD INTEGRAL

Balasubramanian’s formula

\[
\int_0^T Z^2(t) dt = T \ln T + (2c - 1 - \ln 2\pi) T + O(T^{\frac{1}{4} + \epsilon})
\]

implies (cf. [8, (2.5), (8.3)])

\[
\int_T^{T + U_0} Z^2(t) dt = U_0 \ln T + (2c - \ln 2\pi) U_0 + O(T^{\frac{1}{4} + \epsilon}), \quad U_0 = T^{\frac{1}{5} + 2\epsilon}, \quad (2.1)
\]
where \( c \) is Euler’s constant. Furthermore, let us recall Heath-Brown’s estimate (see [5, p. 178, (7.20)]):

\[
\int_{T-G}^{T+G} Z^2(t) \, dt = \mathcal{O}\left( G \ln T + G \sum_K (TK)^{-1/2} \left( S(K) + K^{-1} \int_0^K |S(x)| \, dx \right) e^{-G^2K/T} \right)
\]

(2.2)

(for the definition of the used symbols see [5, (7.21)–(7.23)], uniformly for \( T^\epsilon \leq G \leq T^{\frac{1}{2}}+\epsilon \). Finally, we add Ivić’s estimate [5, (7.62)]

\[
\int_{T-G}^{T+G} Z^2(t) \, dt = \mathcal{O}(G \ln^2 T), \quad G \geq T^{\frac{1}{4}-\epsilon_0}, \quad \epsilon_0 = \frac{1}{108} \approx 0.009.
\]

(2.3)

Remark 1. It is obvious that the short intervals \([T - G, T + G]\) with \( G = 1000 \), for example, are not included in the methods of Balasubramanian, Heath-Brown and Ivić, leading to (2.1)–(2.3).

In this paper we present a new method to deal with short and microscopic parts

\[
\int_T^{T+U} Z^2(t) \, dt
\]

of the Hardy–Littlewood integral (1.1). In order to attain this goal, we will use only elementary geometric properties of Jacob’s ladders. The basic idea is expressed in the following theorem.

Theorem 1. For \( \mu[\varphi] = a \varphi \ln \varphi, a \in [7, 8], \) the following is true:

\[
\int_T^{T+U} Z^2(t) \, dt = \left\{ 1 + \mathcal{O}\left( \frac{\ln \ln T}{\ln T} \right) \right\} U \ln T \tan[\alpha(T, U)], \quad U \in \left( 0, \frac{T}{\ln T} \right), \quad T \geq T_0[\varphi],
\]

(3.1)

where \( \alpha = \alpha(T, U) \) is the angle of the chord of the curve \( y = \frac{1}{2} \varphi(T) \) that binds the points \( [T, \frac{1}{2} \varphi(T)] \) and \( [T + U, \frac{1}{2} \varphi(T + U)] \).

3. A GEOMETRIC CRITERION FOR THE VALIDITY

OF THE USUAL MEAN-VALUE THEOREM

3.1. First of all we will show the canonical equivalence that follows from (2.5). Let us recall (see [8, (8.3)]) that we call the chord binding the points

\[
\left[ T, \frac{1}{2} \varphi(T) \right] \quad \text{and} \quad \left[ T + U_0, \frac{1}{2} \varphi(T + U_0) \right]
\]

(3.2)

of Jacob’s ladder \( y = \frac{1}{2} \varphi(T) \) the fundamental chord. By comparing formulae (2.1) and (2.5), \( U = U_0 \), we obtain the following asymptotic formula:

\[
\tan[\alpha(T, U_0)] = \frac{\varphi(T + U_0) - \varphi(T)}{2U_0} = 1 + \mathcal{O}\left( \frac{\ln \ln T}{\ln T} \right).
\]

(3.3)

Definition. The chord binding the points

\[
\left[ N, \frac{1}{2} \varphi(N) \right] \quad \text{and} \quad \left[ M, \frac{1}{2} \varphi(M) \right], \quad [N, M] \subset [T, T + U_0],
\]

(3.4)

which satisfies the property

\[
\tan[\alpha(N, M - N)] = 1 + o(1), \quad T \to \infty
\]

(cf. (3.2)), is called an almost parallel chord to the fundamental chord. This property will be denoted by the symbol \( \parallel \).
Corollary 1. Let $[N, M] \subset [T, T + U_0]$. Then
\[ \frac{1}{M - N} \int_{N}^{M} Z^2(t) \, dt \sim \ln T \quad \Leftrightarrow \quad /fatslash. \] (3.4)

Remark 2. We see that the analytic property (the usual mean-value theorem) is equivalent to the geometric property $/fatslash$ of Jacob's ladder $y = \frac{1}{2} \varphi(T)$.

3.2. Let us consider the set of all chords of the curve $y = \frac{1}{2} \varphi(T)$ which are almost parallel to the fundamental chord. Let the generic chord of this set bind the points (3.3). Then, from Corollary 1, we obtain

Corollary 2. There is a continuum of intervals $[N, M] \subset [T, T + U_0]$ such that the following asymptotic formula holds true:
\[ \int_{N}^{M} Z^2(t) \, dt \sim (M - N) \ln T. \] (3.5)

Remark 3. There is, for example, a continuum of intervals $[N, M]$: $0 < M - N < 1$ such that the asymptotic formula (3.5) holds true (chords approaching zero).

4. ON THE MICROSCOPIC PARTS OF THE HARDY–LITTLEWOOD INTEGRAL IN THE NEIGHBOURHOODS OF ZEROES OF THE FUNCTION $\zeta(\frac{1}{2} + it)$

Let $\gamma, \gamma'$ be a pair of neighbouring zeroes of the function $\zeta(\frac{1}{2} + it)$. The function $\frac{1}{2} \varphi(T)$ is necessarily convex on some right neighbourhood of the point $T = \gamma$, and this function is necessarily concave on some left neighbourhood of the point $T = \gamma'$. Therefore, there exists a minimal value $\rho \in (\gamma, \gamma')$ such that $[\rho, \frac{1}{2} \varphi(\rho)]$ is an inflexion point of the curve $y = \frac{1}{2} \varphi(t)$. At this point, by the properties of Jacob’s ladders, we have $\varphi'(\rho) > 0$. Let furthermore $\beta = \beta(\gamma, \rho)$ be the angle of the chord binding the points
\[ \left[ \gamma, \frac{1}{2} \varphi(\gamma) \right] \quad \text{and} \quad \left[ \rho, \frac{1}{2} \varphi(\rho) \right]. \] (4.1)

Then we obtain from Theorem 1

Corollary 3. For every sufficiently large zero $T = \gamma$ of the function $\zeta(1 + \frac{1}{2} i T)$ the following formulae describing the microscopic parts of the Hardy–Littlewood integral hold true:

(a) the continuum of formulae
\[ \int_{\gamma}^{\gamma + U} Z^2(t) \, dt \sim U \ln \gamma \tan \alpha, \quad \alpha \in (0, \beta(\gamma, \rho)), \quad U = U(\gamma, \alpha) \in (0, \rho - \gamma), \] (4.2)

where $\alpha$ is the angle of the rotating chord binding the points $[\gamma, \frac{1}{2} \varphi(\gamma)]$ and $[\gamma + U, \frac{1}{2} \varphi(\gamma + U)]$;

(b) the continuum of formulae for the chords parallel to the chord given by the points (4.1),
\[ \int_{N}^{M} Z^2(t) \, dt \sim (M - N) \ln N \tan[\beta(\gamma, \rho)], \quad \gamma \leq N < M < \gamma'. \] (4.3)
Remark 4. The notion of microscopic parts of the Hardy–Littlewood integral has its natural origin in the following: by Karatsuba’s Selberg-type estimate (see [6, p. 265]), for almost all intervals $[\gamma, \gamma'] \subset [T, T + T^{4+2\varepsilon}]$ we have

$$\gamma' - \gamma < A \frac{\ln \ln T}{\ln T} \rightarrow 0, \quad T \rightarrow \infty.$$  \hfill (4.4)

Remark 5. In connection with (4.4) we can recall that if the Riemann hypothesis is true, then Littlewood’s estimate takes place (see [7]):

$$\gamma' - \gamma < A \frac{\ln \ln T}{\ln T} \rightarrow 0,$$  \hfill (4.5)

5. Second Class of Formulae for Parts of the Hardy–Littlewood Integral Beginning at Zeroes of the Function $\zeta(\frac{1}{2} + it)$

Let $\gamma, \tilde{\gamma}$ be a pair of zeroes of the function $\zeta(\frac{1}{2} + it)$ such that $\tilde{\gamma}$ obeys the following conditions:

$$\tilde{\gamma} = \gamma + \gamma \frac{1}{2} + 2\varepsilon + \Delta(\gamma), \quad 0 \leq \Delta(\gamma) = O(\gamma^{\frac{1}{4} + \varepsilon})$$

(it is sufficient to use the classical Hardy–Littlewood estimate for the distance between the neighbouring zeroes [2, pp. 125, 177–184]). Consequently,

$$U(\gamma) = \gamma \frac{1}{2} + 2\varepsilon + \Delta(\gamma) \sim \gamma \frac{1}{2} + 2\varepsilon, \quad \gamma \rightarrow \infty.$$  \hfill (5.1)

For the chord binding the points

$$\left[\gamma, \frac{1}{2} \varphi(\gamma)\right] \quad \text{and} \quad \left[\tilde{\gamma}, \frac{1}{2} \varphi(\tilde{\gamma})\right]$$

we have by (3.1) and (5.1)

$$\tan[\alpha(\gamma, U(\gamma))] = 1 + O\left(\frac{1}{\ln \gamma}\right).$$  \hfill (5.2)

The continuous curve $y = \frac{1}{2} \varphi(T)$ lies below the chord given by points (5.2) on some right neighbourhood of the point $T = \gamma$, and this curve lies above that chord on some left neighbourhood of the point $T = \tilde{\gamma}$. Therefore, there exists a common point of the curve and of the chord. Let $[\bar{\gamma}, \frac{1}{2} \varphi(\bar{\gamma})], \bar{\rho} \in (\gamma, \tilde{\gamma})$, be such a common point that is the closest one to the point $[\gamma, \frac{1}{2} \varphi(\gamma)]$. Then we obtain from Theorem 1 the following

Corollary 4. For every sufficient big zero $T = \gamma$ of the function $\zeta(\frac{1}{2} + iT)$ we have the following formulae for parts (2.4) of the Hardy–Littlewood integral (1.1):

(a) the continuum of formulae for the rotating chord,

$$\int_{\gamma}^{\gamma+U} Z^2(t) \, dt \sim U \ln \gamma \tan \alpha, \quad \alpha \in [\eta, 1 - \eta], \quad U = U(\gamma, \alpha) \in (0, \bar{\rho} - \gamma),$$  \hfill (5.4)

where $\alpha = \alpha(\gamma, U)$ is the angle of the rotating chord binding the points $[\gamma, \frac{1}{2} \varphi(\gamma)]$ and $[\gamma + U, \frac{1}{2} \varphi(\gamma + U)]$, and $0 < \eta$ is an arbitrarily small number;

(b) the continuum of formulae for the chords parallel (and almost parallel) to the chord binding the points (5.2),

$$\int_{N}^{M} Z^2(t) \, dt \sim (M - N) \ln N, \quad \gamma \leq N < M \leq \tilde{\gamma}.$$  \hfill (5.5)
Remark 6. For example, in the case $\alpha = \pi/6$ we have from (5.4)
\[
\gamma + U(\gamma, \pi/6) \int_\gamma Z^2(t) \, dt \sim \frac{1}{\sqrt{3}} U \ln \gamma
\]
for every sufficiently large zero $T = \gamma$ of the function $\zeta(1/2 + iT)$.

Remark 7. It is clear that the asymptotic formulae (2.5), (3.4), (4.2), (4.3), (5.4) and (5.5) cannot be derived within the complicated methods of Balasubramanian, Heath-Brown and Ivić.

6. AN ESTIMATE FOR $\Phi''_{\varphi t}[\varphi(T)]$

Let us recall (see [8, (3.5), (3.9)]) that
\[
\Phi'_{\varphi t}[\varphi(t)] = \frac{\mu[\varphi]}{\varphi^2} + \frac{4}{\varphi^2} \int_0^t \left( \frac{t}{\varphi} - 1 \right) e^{-\frac{2}{\varphi} t} Z^2(t) \, dt + Q[\varphi],
\]
where
\[
\Phi''_{\varphi t}[\varphi(t)] = \frac{2}{\varphi^2} \int_0^t \left( \frac{t}{\varphi} - 1 \right) e^{-\frac{2}{\varphi} t} Z^2(t) \, dt + Z^2[\mu[\varphi]] e^{-\frac{2}{\varphi} t} \frac{d\mu[\varphi]}{d\varphi} \frac{d\varphi}{d\varphi}.
\]

The following lemma is true.

Lemma 1. If $\mu[\varphi] = a \varphi \ln \varphi$, $a \in [7, 8]$, then
\[
\Phi''_{\varphi t}[\varphi(T)] = O\left( \frac{1}{\varphi} \ln \varphi \ln \ln \varphi \right), \quad T \geq T_0[\varphi],
\]
uniformly with respect to $a$.

Remark 8. The segment $[7, 8]$ is sufficient for our purpose since the continuum of Jacob’s ladders corresponds to this segment.

Proof of Lemma 1. First of all, we have (see (6.2))
\[
\Phi''_{\varphi t}[\varphi] = \frac{4}{\varphi^2} \int_0^t \left( \frac{t}{\varphi} - 1 \right) e^{-\frac{2}{\varphi} t} Z^2(t) \, dt + Q[\varphi],
\]
where
\[
Q[\varphi] = e^{-\frac{2}{\varphi} t} \left\{ \frac{4}{\varphi^2} Z^2[\mu[\varphi]] \mu[\varphi] \frac{d\mu[\varphi]}{d\varphi} - \frac{2}{\varphi} Z^2[\mu[\varphi]] \left( \frac{d\mu[\varphi]}{d\varphi} \right)^2 \right. \\
+ \left. 2 Z[\mu[\varphi]] Z'[\mu[\varphi]] \left( \frac{d\mu[\varphi]}{d\varphi} \right)^2 + Z^2[\mu[\varphi]] \frac{d^2\mu[\varphi]}{d\varphi^2} \right\}.
\]

Let
\[
g(t) = t \left( \frac{t}{\varphi} - 1 \right) e^{-\frac{2}{\varphi} t}, \quad t \in [0, \mu[\varphi]].
\]
We apply the following elementary facts:
\[
g(0) = g(\varphi) = 0, \quad g' \left[ \left( 1 - \frac{1}{\sqrt{2}} \right) \varphi \right] = g' \left[ \left( 1 + \frac{1}{\sqrt{2}} \right) \varphi \right] = 0,
\]
\[
\min \{g(t)\} = -\frac{1}{\sqrt{2}} \left( 1 - \frac{1}{\sqrt{2}} \right) e^{-2+\sqrt{2}} \varphi, \quad \max \{g(t)\} = \frac{1}{\sqrt{2}} \left( 1 + \frac{1}{\sqrt{2}} \right) e^{-2-\sqrt{2}} \varphi,
\]
\[
g(t) \leq g(\varphi \ln \ln \varphi) < \varphi \left( \frac{\ln \ln \varphi}{\ln \varphi} \right)^2, \quad t \in [\varphi \ln \ln \varphi, 8 \varphi \ln \varphi],
\]
\[
Z(t), Z'(t) = O(t^{1/4}),
\]

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and the Hardy–Littlewood formula (1.3). We have
\[
\frac{4}{\varphi^3} \int_0^T t \left( \frac{t}{\varphi} - 1 \right) e^{-\frac{2}{\varphi} t} Z^2(t) \, dt = O \left( \frac{\varphi \ln \varphi}{\ln \ln \varphi} \right),
\]
(6.7)
\[
\frac{4}{\varphi^3} \int_0^T t \left( \frac{t}{\varphi} - 1 \right) e^{-\frac{2}{\varphi} t} Z^2(t) \, dt = O \left( \frac{\varphi \ln \varphi}{\ln \ln \varphi} \right)^2 \varphi \ln^2 \varphi = O \left( \frac{1}{\varphi} \ln \ln \varphi \right)^2
\]
by (1.3) and (6.6), and (see (6.5))
\[
Q[\varphi] = O(\varphi^{-13}) \to 0, \quad T \to \infty.
\]
(6.8)
Finally, we obtain (6.3) from (6.4) by (6.7) and (6.8). □

7. PROOF OF THEOREM 1

By (6.1) we have
\[
\int_T^{T+U} Z^2(t) \, dt = \Phi'[\varphi(t_1)] \int_T^{T+U} \varphi \, d\varphi = \Phi'[\varphi(t_1)] \{ \varphi(T + U) - \varphi(T) \},
\]
i.e.,
\[
\int_T^{T+U} Z^2(t) \, dt = 2U \Phi'[\varphi(t_1)] \tan(\alpha(T,U)), \quad t_1 = t_1(U) \in (T, T + U),
\]
(7.1)
\[
\tan(\alpha(T,U)) = \frac{\varphi(T + U) - \varphi(T)}{2U}.
\]
Next, we have
\[
\int_T^{T+U_0} Z^2(t) \, dt = 2U_0 \Phi'[\varphi(t_2)] \left\{ 1 + O \left( \frac{1}{\ln T} \right) \right\}, \quad t_2 = t_2(U_0) \in (T, T + U_0),
\]
(7.2)
by (3.2) and (7.1). Hence, by the comparison of formulae (2.1) and (7.2) we obtain
\[
\Phi'[\varphi(t_2)] = \frac{1}{2} \ln T + O(1).
\]
(7.3)
Next, from the formula (see [8, (6.2)])
\[
T - \frac{\varphi(T)}{2} \sim (1 - c)\pi(T), \quad T \sim \frac{\varphi(T)}{2},
\]
(7.4)
we obtain
\[
\varphi(t_1) - \varphi(t_2) = 2(t_1 - t_2) + O \left( \frac{T}{\ln T} \right) = O \left( \frac{T}{\ln T} \right), \quad U \in \left( 0, \frac{T}{\ln T} \right),
\]
(7.5)
and subsequently (see (6.3))
\[
\Phi'[\varphi(t_1)] - \Phi'[\varphi(t_2)] = O \left\{ |\Phi''[\varphi(T)]| \cdot |\varphi(t_1) - \varphi(t_2)| \right\} = O(\ln T).
\]
(7.6)
Therefore, we obtain
\[
\Phi'[\varphi(t_1)] = \frac{1}{2} \ln T + O(\ln \ln T),
\]
(7.7)
by (7.3) and (7.6). Finally, (2.5) follows from (7.1) and (7.7).
Remark 9. Similarly to (7.6) we have
\[ \Phi'_\varphi[\varphi(t_1)] - \Phi'_\varphi[\varphi(t)] = O(\ln \ln T), \quad t \in [T, T + U], \] (7.8)
and obtain (see (6.1), (7.7), (7.8))
\[ Z^2(t) = \frac{1}{2} \left( 1 + O\left( \frac{\ln t}{\ln T} \right) \right) \ln t \frac{d\varphi(t)}{dt}, \quad t \in [T, T + U], \quad U \in \left( 0, \frac{T}{\ln T} \right). \] (7.9)

8. THE INTEGRAL ASYMPTOTIC FORMULA THAT CONTAINS THE SIXTH-ORDER EXPRESSION \(|\zeta(\frac{1}{2} + i\varphi_1(t))| \frac{1}{4} |\zeta(\frac{1}{2} + it)|^2\)

8.1. Let us recall that Hardy and Littlewood started to study the following integral in 1922:
\[ \int_1^T \left| \zeta\left( \frac{1}{2} + it \right) \right|^4 dt = \int_1^T Z^4(t) dt, \]
and they derived the following estimate (see [3, pp. 41, 59; 10, p. 124]):
\[ \int_1^T \left| \zeta\left( \frac{1}{2} + it \right) \right|^4 dt = O(T \ln^4 T). \]

In 1927 Ingham derived the asymptotic formula
\[ \int_1^T \left| \zeta\left( \frac{1}{2} + it \right) \right|^4 dt = \frac{1}{2\pi^2} T \ln^4 T + O(T \ln^3 T) \] (8.1)
(see [4, p. 277; 10, p. 129]). Let us recall, finally, the Ingham–Heath-Brown formula (see [5, p. 129])
\[ \int_0^T Z^4(t) dt = T \sum_{k=0}^4 C_k \ln^{4-k} T + O\left( T^{\frac{7}{8} + \epsilon} \right), \quad C_0 = \frac{1}{2\pi^2}, \] (8.2)
which improves the Ingham formula (8.1) (the small improvements of the exponents 1/3 and 7/8 (see (2.1), (8.2)) are irrelevant for our purpose).

8.2. In this direction, the following theorem holds true.

Theorem 2.
\[ \int_T^{T+U_1} \left| \zeta\left( \frac{1}{2} + i\varphi_1(t) \right) \right|^4 \left| \zeta\left( \frac{1}{2} + it \right) \right|^2 dt \sim \frac{1}{2\pi^2} U_1 \ln^5 T, \] (8.3)
\[ U_1 = T^{\frac{7}{8} + 2\epsilon}, \quad \varphi_1(t) = \frac{1}{2} \varphi(t), \quad T \to \infty, \]
and the distance of the interaction of the functions
\[ \left| \zeta\left( \frac{1}{2} + i\varphi_1(t) \right) \right|^4, \quad \left| \zeta\left( \frac{1}{2} + it \right) \right|^2 \]
is
\[ t - \varphi_1(t) \sim (1 - c) \pi(t), \] (8.4)
where \( c \) is Euler’s constant and \( \pi(t) \) is the prime-counting function.

Remark 10. Formula (8.3) is the first integral asymptotic formula in the theory of the Riemann zeta-function for the sixth-order expression \(|\zeta(\frac{1}{2} + i\varphi_1(t))| \frac{1}{4} |\zeta(\frac{1}{2} + it)|^2\). This formula cannot be obtained by the methods of Balasubramanian, Heath-Brown and Ivić.
8.3. Since (see (8.4))

\[ T + U_1 - \varphi_1(T + U_1) \sim (1 - c)\pi(T + U_1), \quad U_1 = T^{\frac{2}{3} + 2\varepsilon}, \]

we obtain

\[ T - \varphi_1(T + U_1) \sim (1 - c)\pi(T + U_1) - U_1 \sim (1 - c)\pi(T), \]

and consequently

\[ \rho\{[\varphi_1(T), \varphi_1(T + U)], [T, T + U]\} \sim (1 - c)\pi(T), \quad \varphi_1(T + U_1) < T, \quad \text{(8.5)} \]

where \( \rho \) denotes the distance between the corresponding segments. Next, by using the mean-value theorem in (8.3), we obtain

**Corollary 5.**

\[ \left| \zeta\left( \frac{1}{2} + i\varphi_1(\omega) \right) \right|^4 \left| \zeta\left( \frac{1}{2} + i\omega \right) \right|^2 \sim \frac{1}{2\pi^2 \ln^5 T}, \quad \text{(8.6)} \]

where

\[ \omega \in (T, T + U_1), \quad \varphi_1(\omega) \in (\varphi_1(T), \varphi_1(T + U_1)), \quad \omega = \omega(T, U_1, \varphi_1). \]

**Remark 11.** Some nonlocal interaction of the functions

\[ \left| \zeta\left( \frac{1}{2} + i\varphi_1(\omega) \right) \right|^4 \left| \zeta\left( \frac{1}{2} + i\omega \right) \right|^2 \]

is expressed by formula (8.6). This interaction is connected with two segments unboundedly receding from each other (see (8.5); \( \rho \to \infty \) as \( T \to \infty \))—like mutually receding galaxies in the Friedman expanding universe.

**Remark 12.** Since \( T \sim \omega, \omega \in (T, T + U_1) \), from (8.6) we obtain

\[ \left| \zeta\left( \frac{1}{2} + i\omega \right) \right| \sim \frac{1}{\sqrt{2\pi}} \frac{\ln^{5/2} \omega}{\left| \zeta\left( \frac{1}{2} + i\varphi_1(\omega) \right) \right|}; \quad \text{(8.7)} \]

i.e., we have the prediction of the values \( \left| \zeta\left( \frac{1}{2} + i\omega \right) \right|, \omega \in (T, T + U) \), by means of the values \( \left| \zeta\left( \frac{1}{2} + i\varphi_1(\omega) \right) \right| \) corresponding to the argument \( \varphi_1(\omega) \in (\varphi_1(T), \varphi_1(T + U)) \), which descend from the very distant past (see (8.5), (8.7)), and vice versa.

9. CONTACT POINT OF \(|\zeta(\frac{1}{2} + it)|^2\) WITH THE CLASS OF L-INTEGRABLE FUNCTIONS OF CONSTANT SIGN

Let

\[ \tilde{Z}^2(t) = \frac{d\varphi_1(t)}{dt}, \quad \varphi_1(t) = \frac{1}{2} \varphi(t), \quad t \geq T_0[\varphi], \quad \text{(9.1)} \]

where

\[ \tilde{Z}^2(t) = \frac{Z^2(t)}{2\Phi_{\varphi}(\varphi(t))} = \frac{\left| \zeta\left( \frac{1}{2} + it \right) \right|^2}{1 + O\left( \frac{\ln^2 t}{\ln^4 T} \right)} \ln t, \quad t \in [T, T + U], \quad U \in \left( 0, \frac{T}{\ln T} \right) \quad \text{(9.2)} \]

(see (6.1), (7.7), (7.8)). The following lemma holds true (see (9.1)).

**Lemma 2.** For every integrable function (in the Lebesgue sense) \( f(x), x \in [\varphi_1(T), \varphi_1(T + U)] \), the following is true:

\[ \int_T^{T + U} f[\varphi_1(t)]\tilde{Z}^2(t) \, dt = \int_{\varphi_1(T)}^{\varphi_1(T + U)} f(x) \, dx, \quad T \geq T_0[\varphi], \quad \text{(9.3)} \]
where
\[ t - \varphi_1(t) \sim (1 - c)\pi(t). \tag{9.4} \]

**Remark 13.** Formula (9.3) is also true in the case of relatively convergent improper Riemann integral on its right-hand side.

If \( \varphi_1\left(\left[\frac{T}{G}, T + U\right]\right) = [T, T + U] \), then we have the following formula (see (9.3)):

**Lemma 3.**
\[
\int_T^{T+U} f[\varphi_1(t)] \frac{Z^2(t)}{T} \, dt = \int_T^{T+U} f(x) \, dx, \quad T \geq T_0[\varphi].
\tag{9.5}
\]

Next, the following lemma holds true.

**Lemma 4.** If \( f(x) \geq 0 (\leq 0), \ x \in [\varphi_1(T), \varphi_1(T + U)] \), then
\[
\int_T^{T+U} f[\varphi_1(t)] \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \, dt = \left\{ 1 + O\left(\frac{\ln T}{T}\right) \right\} \ln T \int_T^{\varphi_1(T+U)} f(x) \, dx, \quad U \in \left(0, \frac{T}{\ln T}\right],
\tag{9.6}
\]
and
\[
\int_T^{T+U} f[\varphi_1(t)] \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \, dt = \left\{ 1 + O\left(\frac{\ln T}{T}\right) \right\} \ln T \int_T^{T+U} f(x) \, dx, \quad U \in \left(0, \frac{T}{\ln T}\right].
\tag{9.7}
\]

**Proof.** By using the mean-value theorem on the left-hand side of (9.3) we directly obtain (9.6) (see (9.2)). If we make use of the mean-value theorem on the left-hand side of (9.5), we obtain, by (9.2),
\[
\int_T^{T+U} f[\varphi_1(t)] \frac{Z^2(t)}{T} \, dt = \frac{1}{\left\{ 1 + O\left(\frac{\ln T}{T}\right) \right\} \ln t_1} \int_T^{T+U} f[\varphi_1(t)] \left| \frac{1}{2} + it \right|^2 \, dt,
\tag{9.8}
\]
where \( t_1 \in (T, T + U) = (\varphi_1^{-1}(T), \varphi_1^{-1}(T + U)) \) and
\[
t_1 = \varphi_1^{-1}(T_1), \quad T_1 \in (T, T + U).
\tag{9.9}
\]

Next, we obtain from (9.4) by (9.9) \( (t_1 \to \infty \Leftrightarrow T \to \infty) \)
\[
t_1 - T_1 = O\left(\frac{1}{\ln t_1}\right) \quad \Rightarrow \quad 1 - \frac{T_1}{t_1} = O\left(\frac{1}{\ln t_1}\right) \to 0, \quad T \to \infty,
\tag{9.10}
\]
i.e.,
\[
t_1 - T_1 \sim T_1 \sim T, \quad T \to \infty,
\tag{9.11}
\]
and (see (9.10), (9.11))
\[
t_1 - T = t_1 - T_1 + T_1 - T = O\left(\frac{1}{\ln t_1}\right) + O(U) = O\left(\frac{T}{\ln T}\right),
\tag{9.12}
\]
where \( U \leq \frac{T}{\ln T} \) by the condition of Lemma 4. Now (see (9.12))
\[
\ln t_1 = \ln T + O\left(\frac{t_1 - T}{T}\right) = \ln T + O\left(\frac{U}{T}\right) = \ln T + O\left(\frac{1}{\ln T}\right).
\tag{9.13}
\]

Then formula (9.7) follows from (9.8) by (9.1) and (9.13). □
10. PROOF OF THEOREM 2

10.1. Putting

\[ f(t) = \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 \]

in (9.6), we obtain

\[
\int_T^{T+U_1} \left| \zeta \left( \frac{1}{2} + i \varphi_1(t) \right) \right|^4 \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim \ln T \int_{\varphi_1(T)}^{\varphi_1(T+U_1)} Z^4(t) dt; \quad (10.1)
\]

i.e., we have to consider the integral (see (8.2))

\[
\int_{\varphi_1(T)}^{\varphi_1(T+U_1)} Z^4(t) dt = \left\{ \varphi_1(t) \sum_{k=0}^{4} C_k \ln^{4-k} \varphi_1(t) \right\}_{t=T}^{t=T+U_1} + O(T^{\frac{7}{2} + \epsilon}) = \sum_{k=0}^{4} C_k V_k + O(T^{\frac{7}{2} + \epsilon}), \quad (10.2)
\]

where

\[ V_k = \varphi_1(T + U_1) \ln^{4-k} \varphi_1(T + U_1) - \varphi_1(T) \ln^{4-k} \varphi_1(T) \]

and, for example,

\[
V_0 = \varphi_1(T + U_1) \ln^4 \varphi_1(T + U_1) - \varphi_1(T) \ln^4 \varphi_1(T) \\
= \left[ \varphi_1(T + U_1) - \varphi_1(T) \right] \frac{d}{d\varphi_1} \left[ \varphi_1(t) \ln^4 \varphi_1(t) \right] \bigg|_{t=d_0} \\
= \left[ \varphi_1(T + U_1) - \varphi_1(T) \right] \ln^4 \varphi_1(d_0) + 4 \ln^3 \varphi_1(d_0) \\
= U_1 \frac{\varphi_1(T + U_1) - \varphi_1(T)}{U_1} \ln^4 \varphi_1(d_0) \left\{ 1 + O \left( \frac{1}{\ln \varphi_1(d_0)} \right) \right\}, \quad (10.4)
\]

\[ \varphi_1(d_0) \in (\varphi_1(T), \varphi_1(T + U_1)). \]

10.2. By the Ingham formula (see [4, p. 294; 10, p. 120])

\[
\int_0^T Z^2(t) dt = T \ln T + (2c - 1 - \ln 2\pi) T + O(T^{\frac{3}{2} + \epsilon})
\]

we have in the case \( U_1 = T^{\frac{3}{2} + 2\epsilon} \)

\[
\int_T^{T+U_1} Z^2(t) dt = U_1 \ln T + (2c - \ln 2\pi) U_1 + O(T^{\frac{7}{2} + \epsilon}). \quad (10.5)
\]

Comparing formulae (2.5) and (10.5), we obtain

\[
\frac{\varphi_1(T + U_1) - \varphi_1(T)}{U_1} = \tan[\alpha(T, U_1)] = 1 + O \left( \frac{\ln \ln T}{\ln T} \right). \quad (10.6)
\]

Since (see (10.6))

\[ \varphi_1(d_0) - \varphi_1(T) \leq \varphi_1(T + U_1) - \varphi_1(T) = O(U_1), \]

we have (see (9.4); \( \varphi_1(T) \sim T \))

\[
\ln \varphi_1(d_0) = \ln \varphi_1(T) + \ln \left( 1 + \frac{\varphi_1(d_0) - \varphi_1(T)}{\varphi_1(T)} \right) = \ln \varphi_1(T) + O \left( \frac{U_1}{T} \right) \sim \ln T. \quad (10.7)
\]
Hence, we obtain from (10.4) by (10.6) and (10.7)

\[ V_0 \sim U_1 \ln^4 T, \]  

(10.8)

and similarly

\[ V_l = O(U_1 \ln^{4-l} T), \quad l = 1, 2, 3, 4. \]  

(10.9)

Finally, formula (8.3) follows from (10.1) by (10.2), (10.8) and (10.9).

11. JACOB’S LADDERS AND A NEW CLASS OF NONLINEAR INTEGRAL EQUATIONS; CONCLUDING REMARKS

11.1. The proof of Theorem 2 is simultaneously a proof of the following theorem.

\textbf{Theorem 3.} Every Jacob’s ladder \( \varphi_1(t) = \frac{1}{2} \varphi(t) \), where \( \varphi(t) \) is an exact solution of the nonlinear integral equation

\[
\mu[x(T)] \int_0^T Z^2(t) e^{-\frac{x(T)}{2}} dt = \int_0^T Z^2(t) dt,
\]

is an asymptotic solution of the following nonlinear integral equation:

\[
\int_T^{T+U_1} \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 \left| \frac{1}{2} + it \right|^2 dt = \frac{1}{\pi^2} U_1 \ln^5 T, \quad U_1 = T^{\frac{7}{2}+2\epsilon}, \quad (11.1)
\]

where \( x(t) = x(t; T, \epsilon) \), for every fixed \( T \geq T_0[\varphi] \); i.e., the following asymptotic formula (see (8.5)) holds true:

\[
\frac{2\pi^2}{U_1 \ln^5 T} \int_T^{T+U_1} \left| \zeta \left( \frac{1}{2} + i\varphi_1(t) \right) \right|^4 \left| \frac{1}{2} + it \right|^2 dt \sim \left( \frac{2k}{k!} \right)^2 U_2 \ln T \ln \ln T, \quad T \to \infty.
\]

(11.2)

11.2. Let us recall Selberg’s formula [9, p. 128]

\[
\int_T^{T+U_2} \left\{ S(t) \right\}^{2k} dt \sim \left( \frac{2k!}{k!} \right)^2 U_2 \ln T \ln T, \quad (11.2)
\]

where \( U_2 = T^{\frac{7}{2}+\epsilon} \), \( k \) is a fixed positive number, and

\[
S(t) = \frac{1}{\pi} \arg \zeta \left( \frac{1}{2} + it \right)
\]

(where \( \arg \) is defined in the usual way). From (11.2) by (9.7) one obtains

\[
\int_{\varphi_1^{-1}(T+U_2)}^{\varphi_1^{-1}(T)} \left\{ \arg \zeta \left( \frac{1}{2} + i\varphi_1(t) \right) \right\}^{2k} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim \left( \frac{2k!}{k!} \right)^2 U_2 \ln T \ln T, \quad T \to \infty. \quad (11.3)
\]

\textbf{Remark 14.} This formula cannot be obtained in the classical theory of A. Selberg and, least of all, in the theories of Balasubramanian, Heath-Brown and \\Ivić.

Some nonlocal interaction of the functions

\[
\left\{ \arg \zeta \left( \frac{1}{2} + i\varphi_1(t) \right) \right\}^{2k}, \quad \left| \zeta \left( \frac{1}{2} + it \right) \right|^2
\]

is expressed by formula (11.3).
Remark 15. Every Jacob’s ladder $\varphi_1(t)$ is an asymptotic solution (see (11.3)) of the nonlinear integral equation

$$\int_{x^{-1}(T)}^{x^{-1}(T+U_2)} \left\{ \arg \zeta \left( \frac{1}{2} + ix(t) \right) \right\}^{2k} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = \frac{(2k)!}{k! 2^k} U_2 \ln T \ln \ln T^k. \quad (11.4)$$

11.3. Since

$$\int_T^{T+U} \pi(x) dx \sim \frac{UT}{\ln T}, \quad U \leq \frac{T}{\ln T},$$

holds true, we obtain (see (9.7), $f(t) = \pi(t)$)

$$\int_{\varphi_1^{-1}(T+U)}^{\varphi_1^{-1}(T)} \pi(\varphi(t)) \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim \frac{UT}{\ln T}, \quad (11.5)$$

where $t - \varphi_1(t) \sim (1 - c) \pi(t)$.

Remark 16. Every Jacob’s ladder $\varphi_1(t)$ is an asymptotic solution (see (11.5)) of the following nonlinear integral equation:

$$\int_{x^{-1}(T)}^{x^{-1}(T+U)} \pi(x(t)) \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = \frac{UT}{\ln T}, \quad 0 < U \leq \frac{T}{\ln T}. \quad (11.6)$$

11.4. Another source of the integrals containing the function $\left| \zeta \left( \frac{1}{2} + it \right) \right|^2$ is, for example, the system of Chebyshev polynomials $T_n(x)$, $x \in [-1, 1]$, $n = 0, 1, 2, \ldots$, of the first kind. We obtain, from the well-known formula

$$\frac{1}{\sqrt{1 - x^2}} \int_{-1}^{1} [T_n(x)]^2 dx = \begin{cases} \frac{\pi}{2}, & n \geq 1, \\ \pi, & n = 0, \end{cases}$$

the following one:

$$\int_T^{T+2} \frac{[T_n(t - T - 1)]^2}{\sqrt{1 - (t - T - 1)^2}} dt = \frac{\pi}{2}, \quad n \geq 1.$$

Next, putting

$$f(t) = \frac{[T_n(t - T - 1)]^2}{\sqrt{1 - (t - T - 1)^2}}$$

in (9.7), $T = \varphi_1^{-1}(T)$ and $\varphi_1^{-1}(T+2)$, we obtain

$$\int_{\varphi_1^{-1}(T+2)}^{\varphi_1^{-1}(T+2)} \frac{[T_n(\varphi_1(t) - T - 1)]^2}{\sqrt{1 - (\varphi_1(t) - T - 1)^2}} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim \frac{\pi}{2} \ln T, \quad n \geq 1. \quad (11.7)$$
Remark 17. Jacob’s ladder $\varphi_1(t)$ is an asymptotic solution of the nonlinear integral equation (see (11.2))

$$
\int_{x^{-1}(T)}^{x^{-1}(T+2)} \frac{[T_n(x(t) - T - 1)]^2}{1 - (\varphi_1(t) - T - 1)^2} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \, dt = \frac{\pi}{2} \ln T, \quad n \geq 1. \quad (11.8)
$$

11.5. Let us recall the Lyapunov equation

$$
\gamma \int (a) \rho'(z')^3 + \frac{1}{2} \omega_i k \psi_j (x_i - a_i)(x_j - a_j) = V_a
$$

(cf. [1, Ch. VI, §73]) for determining the form of the integration domain $(a)$ (the density $\rho$ is prescribed), i.e., the equilibrium figures of the rotating body.

Analogously to the case (11.9), we will call the segment $[x^{-1}(T), x^{-1}(T + 2)]$ entering equation (11.8), for example, the equilibrium segment, and the segment $[\varphi_1^{-1}(T), \varphi_1^{-1}(T + 2)]$ will be called the asymptotical equilibrium segment.

Remark 18. By (11.7), for every fixed $T \geq T_0[\varphi]$, there is a continuum of asymptotic equilibrium segments $[\varphi_1^{-1}(T), \varphi_1^{-1}(T+2)]$. However, is there any equilibrium segment $[x^{-1}(T), x^{-1}(T+2)]$ for some $T \geq T_0[\varphi]$?

Remark 19. There are fixed-point methods and other methods of functional analysis used to study nonlinear equations. What can be obtained by using these methods in the case of nonlinear integral equations of type (11.1), (11.4), (11.6) or (11.8)?

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