Structure constants for $K$-theory of Grassmannians, revisited

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1 Introduction

The theory of symmetric functions supports Schubert calculus of the Grassmannian $X = \text{Gr}(k, n)$ by way of the Schur function basis. In particular, the Schubert decomposition of $X$ is determined by varieties $X_\lambda$, one for each partition $\lambda$ in a $k \times (n - k)$ rectangle. The Schubert varieties induce canonical basis elements for the cohomology ring of $X$. In turn, $H^*(X)$ is isomorphic to a certain quotient of the ring of symmetric functions $\Lambda$ in $k$ variables. The Schur functions $s_\lambda(x_1, \ldots, x_k)$ are representatives for the Schubert classes with a critical feature that structure constants in

$$[X_\lambda] \cdot [X_\mu] = \sum_v C_{\lambda\mu}^v [X_v]$$

appear as Schur coefficients in a product of Schur functions:

$$s_\lambda s_\mu = \sum_v C_{\lambda\mu}^v s_v .$$

The realization of Schur functions as weight generating functions of semi-standard Young tableaux then offers combinatorial tools for the study. The development of a rich theory of tableaux ultimately enabled the computation of the structure constants $C_{\lambda\mu}^v$. The Littlewood-Richardson rule for $C_{\lambda\mu}^v$ dictates a count of tableaux with certain restrictions [LR34], and its

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proof was settled in [Sch77] using the RSK insertion algorithm on tableaux [Rob38, Sch61, Knu70].

As a basis of symmetric functions, the Schur functions are self-dual with respect to the Hall inner product on $\Lambda$. This duality is central in symmetric function theory and in particular, gives an alternative way to access the Schubert structure constants. Namely, Littlewood-Richardson numbers arise in the Schur expansion of skew Schur functions:

$$s_{\nu/\lambda} = \sum_{\mu} C_{\nu/\lambda}^{\mu} s_{\mu}. \quad (1)$$

This viewpoint gives rise to simpler proofs of the Littlewood-Richardson rule, e.g. [RS98, Ste02].

Developments in Schubert calculus have established the importance of combinatorics and symmetric function theory to the more intricate setting of the Grothendieck ring $K^0 X$ of algebraic vector bundles on $X$. Lascoux and Schützenberger [LS83] introduced Grothendieck polynomials as representatives for the structure sheaves of the Schubert varieties in a flag variety. Fomin and Kirillov [FK94] studied the symmetric power series resulting from a limit of Grothendieck polynomials.

For the Grassmannian variety $X$, it was shown in [Buc02] that the stable limits $G_{\lambda}$ are generating series of set-valued tableaux and can be applied to $K^0 X$ in a way that mirrors the Schur role in cohomology. The classes of the Schubert structure sheaves $O_{X_\lambda}$ form a basis for the Grothendieck ring of $X$, and the structure constants in

$$[O_{X_\lambda}] \cdot [O_{X_\mu}] = \sum_{\nu} c_{\nu}^{\lambda \mu} [O_{X_\nu}]$$

appear in the product

$$G_\lambda G_\mu = \sum_{\nu} c_{\nu}^{\lambda \mu} G_\nu. \quad (2)$$

By generalizing RSK insertion, Buch proved that the expansion coefficients are determined by the number of set-valued tableaux with a Yamanouchi column word.

In contrast to (1), the Schubert structure constant $c_{\nu/\lambda}^{\mu}$ for $K^0 X$ is not the coefficient of $G_{\mu}$ in $G_{\nu/\lambda}$. Instead, the bialgebra $\Gamma = \text{span}\{G_{\lambda}\}$ gives rise to a distinct family of symmetric functions $\{g_{\lambda}\}$ defined by taking the Hopf-dual basis to $\{G_{\lambda}\}$. Nevertheless, inspired by the far-reaching applications of Schur function duality, we have discovered that duality can be used to access the $K$-theoretic constants.

A combinatorial investigation of Lam and Pylyavskyy [LP07] revealed that the basis $\{g_{\lambda}\}$ can be realized as the weight generating functions of reverse plane partitions. We study the skew generating functions,

$$g_{\nu/\lambda} = \sum_{R \in \text{RPP}(\nu/\lambda)} x^{\text{wt}(R)},$$

proving that their $g$-expansion coefficients,

$$g_{\nu/\lambda} = \sum_{\mu} a_{\lambda \mu}^{\nu} g_{\mu}, \quad (3)$$

2
match the $K$-theoretical structure constants $c_{\lambda\mu}^\nu$.

We bijectively convert the reverse plane partitions into tabloids, classical combinatorial objects dating back to Young’s study of irreducible $S_n$-representations. This identification requires an unconventional notion of tabloid weight called inflated weight which we relate to the Yamanouchi property on set-valued tableaux. From the tabloid characterization for $g_{\alpha/\beta}$, a crystal structure on tabloids [KM] leads us to its Schur expansion in terms of a distinguished subclass of semi-standard tableaux. In turn, the Schur expansion can be converted into a $g$-expansion using elegant fillings with the introduction of a sign [Len00]. A bijection $\phi_T$ on pairs of tableaux and elegant fillings leads us to an expression over certain set-valued tableaux which we then reduce by way of a sign-reversing involution $\tau$. This establishes that the $g$-expansion coefficients for $g_{\alpha/\beta}$ count the same Yamanouchi set-valued tableaux prescribed by Buch in his $K$-theoretical Littlewood-Richardson rule for $c_{\lambda\mu}^\nu$. As a by-product, (3) is equivalent to (2).

2 Preliminaries

Let $X = \text{Gr}(k, n)$ denote the Grassmannian of $k$-dimensional subspaces of $\mathbb{C}^n$ and consider the subspaces $\mathbb{C}^d$ of vectors which can have non-zero entries only in the first $d$ components. $X$ can be decomposed into a family of distinguished “Schubert cells”, indexed by non-decreasing integer partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $k$ parts, none of which are larger than $n - k$. Their closures are the Schubert varieties

$$X_\lambda = \{ V \in \text{Gr}(k, n) : \dim(V \cap \mathbb{C}^{n-k+i-\lambda_i}) \geq i, \; \forall 1 \leq i \leq k \}.$$ 

The classes of Schubert cells form a basis for the cohomology of $X$. On the other hand, $H^*(X)$ is isomorphic to a certain quotient of the ring of symmetric polynomials $\Lambda_k = \mathbb{Z}[e_1, \ldots, e_k] = \mathbb{Z}[h_1, \ldots, h_k]$, where

$$e_i = \sum_{1 \leq i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r} \quad \text{and} \quad h_i = \sum_{1 \leq i_1 \leq \cdots \leq i_r} x_{i_1} \cdots x_{i_r}.$$ 

Bases for $\Lambda$ are indexed by generic partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. A quick example is the basis of homogenous symmetric functions $\{h_i\}$, defined by $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$.

The combinatorial potential of $\Lambda$ is met with the unique association of each partition $\lambda$ with its Ferrers shape, a left-and bottom-justified array of $1 \times 1$ square cells in the first quadrant of the coordinate plane, with $\lambda_i$ cells in the $i^{th}$ row from the bottom. Given a partition $\lambda$, its conjugate $\lambda'$ is the partition obtained by reflecting the shape of $\lambda$ about the line $y = x$. For partitions $\mu, \lambda$, the property that every cell of $\mu$ is also a cell of $\lambda$ is denoted by $\mu \subset \lambda$. For $\mu \subset \lambda$, the skew shape $\lambda/\mu$ is defined by the cells in $\lambda$ but not in $\mu$. The skew shape obtained by placing the diagram of a partition $\lambda$ southeast and caty-corner to a partition $\mu$ is denoted by $\mu \ast \lambda$. For example,

$$\mu = (2, 2, 1), \lambda = (3, 1) \implies \mu \ast \lambda = (5, 3, 2, 2, 1)/(2, 2) = \begin{array}{ccc} & & \circ \circ \\ \circ & & \\ & \circ & \circ \circ \end{array}.$$ 

More generally, a composition is a sequence of non-negative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, and $\alpha - \beta$ is defined by usual vector subtraction for any compositions $\alpha, \beta$. For any composition $\alpha$, let $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_k$. 

3
A *(semi-standard) tableau of shape* $\lambda$ is a positive integer filling of the cells of $\lambda$ such that entries weakly increase from left to right in rows and are increasing from the bottom to top of each column. The *weight* of a tableau $T$ denoted $\text{wt}(T)$ is the composition $\alpha = (\alpha_i)_{i \geq 1}$ where $\alpha_i$ is the number of cells containing $i$ (it is customary to omit trailing 0’s). For any partition $\lambda$, there is a unique tableau of both shape and weight $\lambda$. We denote this tableau by $T_\lambda$. We use $\text{SSYT}(\lambda)$ to denote the set of all semi-standard tableaux of shape $\lambda$, and $\text{SSYT}(\lambda, \mu)$ to denote the set of all semi-standard tableaux of shape $\lambda$ and weight $\mu$.

Schur functions may be defined as the weight generating functions of semi-standard tableaux; for any partition $\lambda$,

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^{\text{wt}(T)},$$

where $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2} \cdots$. They are symmetric functions and form a basis for $\Lambda$. As such, it is convenient to collect terms into the basis of monomial symmetric functions, formed by $m_\lambda = \sum_\alpha x^\alpha$ over all distinct rearrangements $\alpha$ of partition $\lambda$. The monomial expansion of a Schur function is

$$s_\lambda = \sum_\mu K_{\lambda \mu} m_\mu,$$

where the Kostka coefficients, $K_{\lambda \mu}$, enumerate tableaux of shape $\lambda$ and weight $\mu$.

The Schur basis is orthonormal with respect to the Hall inner product $\langle , \rangle$ on $\Lambda$, defined by

$$\langle m_\lambda, h_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

An immediate consequence of (4) and duality is

$$h_\mu = \sum_\lambda K_{\mu \lambda} s_\lambda.$$

Our methods rely on fundamental operations on words and tableaux including jeu de taquin and RSK-insertion [Sch77, Rob38, Sch61, Knu70]. We briefly recall several important results here; full details can be found in a variety of texts such as [LS81, Sta99, Ful97]. A word $w$ is *Yamanouchi* when each factorization $w = uv$ satisfies $\text{wt}_i(v) \geq \text{wt}_{i+1}(v)$ for all $i$, where $\text{wt}_i(v)$ is the number of times letter $i$ appears in $v$. For a given partition $\lambda$, a word $w$ is $\lambda$-Yamanouchi if $\text{wt}_i(v) + \lambda_i \geq \text{wt}_{i+1}(v) + \lambda_{i+1}$.

The *row word* $w(T) = w_1w_2 \cdots w_n$ of tableau $T$ is defined by listing elements of $T$ starting from the top-left corner, reading across each row, and then continuing down the rows. We say that tableau $T$ is row Yamanouchi when its row word is Yamanouchi. The RSK insertion algorithm uniquely identifies a word $w$ with a pair of same shaped tableaux $(P(w), Q(w))$. For any tableau $T$,

$$P(w(T)) = T,$$

and when two words $w$ and $u$ have the same insertion tableau $P(w) = P(u)$, they are *Knuth equivalent*, denoted by $w \sim u$. 

4
3 \(K\)-theoretic Littlewood-Richardson coefficients

Because the Schubert cells form a cell decomposition of the Grassmannian \(X\), the classes of the structure sheaves \(O_{X_\lambda}\) form a basis for the Grothendieck ring of \(X\). A combinatorial description for the structure constants of \(K^*X\) with respect to its basis of Schubert structure sheaves, appearing in

\[ [O_{X_\lambda}] \cdot [O_{X_\mu}] = \sum_v c^v_{\lambda \mu} [O_{X_v}], \]

was first given in [Buc02].

3.1 Combinatorial \(K\)-theoretic background

Buch’s work initiated a framework for \(K\)-theoretic Schubert calculus using a set-valued generalization of tableaux. A set-valued tableau of shape \(\nu/\lambda\) is defined to be a filling of each cell in the diagram of \(\nu/\lambda\) with a non-empty set of positive integers such that each subfilling created from the choice of a single element in each cell is a semi-standard tableau. The weight of a set-valued tableau \(S\) is the composition \(\text{wt}(S) = (\alpha_i)_{i \geq 1}\) where \(\alpha_i\) is the total number of times \(i\) appears in \(S\) and the excess of \(S\) is defined by

\[ \varepsilon(S) = |\text{wt}(S)| - |\text{shape}(S)|. \]

A multicell refers to a cell in \(S\) that contains more than one letter. When \(S\) has no multicells, it is viewed as a semi-standard tableau. In this case, \(|\text{wt}(S)| = |\text{shape}(S)|\) and \(\varepsilon(S) = 0\).

The collection of all set-valued tableaux of shape \(\nu/\lambda\) is denoted by \(SVT(\nu/\lambda)\) and the subset of these with weight \(\alpha\) is \(SVT(\nu/\lambda, \alpha)\). For any skew shape \(\nu/\lambda\), the stable Grothendieck polynomial is defined by

\[ G_{\nu/\lambda} = \sum_{\mu} k_{\nu/\lambda, \mu} m_\mu, \]

where

\[ k_{\nu/\lambda, \mu} = (-1)^{|\lambda| - |\nu| - |\mu|} \sum_{S \in SVT(\nu/\lambda, \mu)} 1. \]

Through the study of the vector space \(\Gamma = \text{span}[G_\lambda]\) over all partitions \(\lambda\), Buch proved that the set-valued generating functions give a set of representatives for the classes of \(O_{X_\lambda}\).

**Theorem 1** ([Buc02]). The structure constants of (7) occur as coefficients in the expansion of the symmetric function product

\[ G_\lambda G_\mu = \sum_{\nu | \lambda \leq \nu \leq \mu} c^\nu_{\lambda \mu} G_\nu. \]

This inspired Buch to define a column insertion algorithm on set-valued tableaux and to give combinatorial meaning to \(c^\nu_{\lambda \mu}\); they count the subset of set-valued tableaux with a certain Yamanouchi condition [Buc02]. Precisely, the column word \(w(S)\) of a set-valued tableau \(S\) is defined by reading cells from top to bottom in columns and taking letters within a cell from smallest to largest, starting with the leftmost column moving right. When \(w(S)\) is \(\lambda\)-Yamanouchi, we say \(S\) is a column \(\lambda\)-Yamanouchi set-valued tableau.
Example 2.

\[
S = \begin{array}{ccc}
5 & 4 & 57 \\
12 & 2 & 23
\end{array}
\implies w(S) = 541257223
\]

**Theorem 3 ([Buc02]).** For partitions \( \lambda, \mu \), \( c^\nu_{\lambda\mu} \) is equal to \((-1)^{|\nu|-|\lambda|-|\mu|}\) times the number of column \( \lambda \)-Yamanouchi set-valued tableaux of shape \( \mu \) and weight \( \nu - \lambda \).

The result has since been reproven in a variety of other ways. Various combinatorial approaches include [McN06, IS14, PY15].

### 3.2 Duality

As discussed, the Littlewood-Richardson coefficients in a product of Schur functions are the same as the Schur expansion coefficients of a skew Schur function (1). In contrast, the constant \( c^\nu_{\lambda\mu} \) in a product of stable Grothendieck polynomials is not the coefficient of \( G_\mu \) in \( G_{\nu/\lambda} \). However, we have discovered that there is a natural remedy.

Lam and Pylyavskyy [LP07] proved that \( \Gamma \) is a Hopf-algebra, and they studied the family \( \{g_\lambda\} \) which is Hopf-dual to \( \{G_\lambda\} \). They proved that this basis, defined by inverting the triangular system

\[
h_\mu = \sum_\lambda k_{\lambda\mu} g_\lambda.
\]  

has a direct combinatorial characterization.

Define a reverse plane partition \( R \) of shape \( \nu/\lambda \) to be a filling of cells in \( \nu/\lambda \) with positive integers which are weakly increasing in rows and columns. The weight of \( R \) is the composition \( \text{wt}(R) = (a_i)_{i \geq 1} \) where \( a_i \) is the total number of columns of \( R \) in which \( i \) appears. The collection of all reverse plane partitions of shape \( \nu/\lambda \) is denoted by \( \text{RPP}(\nu/\lambda) \) and the subset of these with weight \( \alpha \) is \( \text{RPP}(\nu/\lambda, \alpha) \). Let \( r_{\nu/\lambda, \alpha} = |\text{RPP}(\nu/\lambda, \alpha)| \).

**Theorem 4 ([LP07]).** For any partition \( \lambda \),

\[
g_\lambda = \sum_{R \in \text{RPP}(\lambda)} x^{\text{wt}(R)} = \sum_\mu r_{\lambda\mu} m_\mu.
\]

We appeal to duality to realize the \( K \)-theoretic Littlewood-Richardson numbers instead as coefficients in the \( g \)-expansion of the skew \( g \)-functions, defined by

\[
g_{\nu/\lambda} = \sum_\mu r_{\nu/\lambda, \mu} m_\mu.
\]

**Proposition 5.** For partitions \( \lambda \subseteq \nu \),

\[
g_{\nu/\lambda} = \sum_\mu c^\nu_{\lambda\mu} g_\mu.
\]
Proof. Equivalently we show that \( \langle G_\lambda G_\mu, g_\nu \rangle = \langle g_{\nu/\lambda}, G_\mu \rangle \). Consider (11) when \( \lambda = \emptyset \) and take the inverse to get
\[
m_\mu = \sum_\nu \bar{r}_{\nu \mu} g_\nu \iff G_\nu = \sum_\mu \bar{r}_{\nu \mu} h_\mu
\]
by duality. From here,
\[
\langle g_{\nu/\lambda}, G_\mu \rangle = (\sum_\beta r_{\nu/\lambda \beta} m_\beta, \sum_\alpha \bar{r}_{\mu \alpha} h_\alpha) = \sum_\alpha \bar{r}_{\mu \alpha} r_{\nu/\lambda \alpha}.
\]
On the other hand, the Pieri rule for \( \{ G_\lambda \} \) [Len00, Theorem 3.2] is
\[
G_\lambda h_\alpha = \sum_\nu r_{\nu/\lambda \alpha} G_\nu,
\]
implicating that
\[
\langle G_\lambda G_\mu, g_\nu \rangle = (\sum_\alpha \bar{r}_{\mu \alpha} G_\lambda h_\alpha, g_\nu) = (\sum_\alpha \bar{r}_{\mu \alpha} \sum_\beta r_{\beta/\lambda \alpha} G_\beta, g_\nu) = \sum_\alpha \bar{r}_{\mu \alpha} r_{\nu/\lambda \alpha}.
\]
\( \square \)

A by-product of Theorem 3 and Proposition 5 is that the coefficient of \( g_\mu \) in \( g_{\nu/\lambda} \) is determined by counting column \( \lambda \)-Yamanouchi set-valued tableaux. This begs for a direct proof, starting from the weight generating functions
\[
g_{\nu/\lambda} = \sum_\alpha r_{\nu/\lambda \alpha} x^\alpha,
\]
and combinatorially collecting terms to identify the \( g \)-expansion coefficients without relying on Theorem 3 and Proposition 5. Our method is to use a crystal structure which converts (13) into a Schur expansion and to then use the transition between Schur functions and the \( g \)-basis given by Lenart [Len00]. That is, strict elegant fillings are skew tableaux with strictly increasing entries up columns and across rows, and with the additional property that entries in row \( i \) are not larger than \( i - 1 \). The set of all strict elegant fillings of shape \( \lambda/\mu \) is denoted by \( \text{EF}(\lambda/\mu) \) and we set \( F_\lambda^4 = |\text{EF}(\lambda/\mu)| \).

**Theorem 6** ([Len00]). For any partition \( \lambda \),
\[
G_\mu = \sum_{\lambda, \mu \subset \lambda} (-1)^{|\lambda|-|\mu|} F_\mu^4 s_\lambda.
\]
An immediate corollary comes out of duality relations:
\[
s_\lambda = \sum_{\mu, \mu \subset \lambda} (-1)^{|\lambda|-|\mu|} F_\mu^4 g_\mu.
\]
4 Tabloids

The process of combinatorially tracing this through seems daunting upon first glance because two of the transitions involve signs. However, converting from the setting of reverse plane partitions to one involving a combinatorial structure called tabloid enables us to carry it through seamlessly.

4.1 Inflated weight characterization for Yamanouchi tabloid

Tabloids are fillings with positive integers that are weakly increasing in rows. Although these are classical combinatorial objects going back to Young’s study of the irreducible representations of $S_n$, our purposes require an association of tabloids (and set-valued tableaux) with a less familiar weight called the inflated weight. We start with the definition and several results establishing that the notion of this particular weight is closely tied to the Yamanouchi property.

Let $T(\nu/\lambda)$ denote the set of tabloids with shape $\nu/\lambda$. The weight of a tabloid $T$ is the vector $\text{wt}(T) = \alpha$ where $\alpha_i$ records the number of times $i$ appears in $T$. The subset $T(\nu/\lambda, \alpha) \subset T(\nu/\lambda)$ contains only tabloids of weight $\alpha$. We require another statistic which can be associated to a tabloid to further our investigation.

**Definition 7.** The inflated weight of a tabloid $T$, $\text{ιwt}(T)$, is defined iteratively from its rows $T_1, T_2, \ldots, T_{\ell}$, read from bottom to top. With $r = 2$ and $\hat{T} = T_1$, modify $T_r^{*}$ by moving the last letter $e$ in $T_r$ to the rightmost empty cell of row $r$ that has no entry $e' \geq e$ below it in any row $r' \leq r - 1$. If no such cell exists, $e$ remains in place. Now ignoring $e$, repeat with the last letter in row $r$. Once all letters in row $r$ have been addressed, iterate by setting the resulting filling to $\hat{T}$ and $r = r + 1$. When $r = \ell$, the process terminates with a column-strict filling $\hat{T}$ called the inflated weight tableau of $T$. We refer to the entries at the top of each column in $\hat{T}$ as uncovered. The inflated weight is the conjugate of the partition rearrangement of the uncovered entries in $\hat{T}$.

Example 8. $T = \begin{array}{cccc}
1 & 2 & 3 & 6 \\
1 & 2 & 4 & 5 \\
6 & 5 & 4 & 3
\end{array}$ $\rightarrow$ $\begin{array}{cccc}
1 & 2 & 3 & 6 \\
1 & 2 & 4 & 5 \\
3 & 5 & 4 & 6
\end{array}$ $\rightarrow$ $\begin{array}{cccc}
1 & 2 & 3 & 6 \\
1 & 2 & 4 & 5 \\
1 & 2 & 4 & 5
\end{array}$ implying that $\text{ιwt}(T) = (7, 6, 4, 3, 2, 1)' = (6, 5, 4, 3, 2, 2, 1)$

This example demonstrates that the weight of a tabloid does not necessarily equal its inflated weight. It is possible for weight and inflated weight to match; in fact, this characterizes the Yamanouchi property on tabloid. The row word of a tabloid $T$ is defined by reading letters of $T$ from left to right in rows, starting with the top row and proceeding south. A tabloid $T$ is row Yamanouchi when its row word is Yamanouchi.

**Example 9.** For tabloid $T = \begin{array}{cc}
1 & 2 \\
1 & 2
\end{array}$, consider $T \ast T_{(3,1)}$. Its row word $1122111$ is Yamanouchi and $\text{wt}(T \ast T_{(3,1)}) = (5, 2)$. The inflated weight computation also gives $(5, 2)$.

$T \ast T_{(31)} = \begin{array}{cccc}
1 & 2 & 2 & 3 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{array}$ $\rightarrow$ $\begin{array}{cccc}
1 & 2 & 2 & 3 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{array}$ $\rightarrow$ $\begin{array}{cccc}
1 & 2 & 2 & 3 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{array}$ $\rightarrow$ $\begin{array}{cccc}
1 & 2 & 2 & 3 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{array}$ $\Rightarrow$ $\text{ιwt}(T \ast T_{(31)}) = (2, 2, 1, 1, 1)'$. 

8
**Proposition 10.** For any tabloid $T$ and partition $\lambda$, 

$$T \ast T_\lambda \text{ is row Yamanouchi } \iff \text{cwt}(T \ast T_\lambda) = \text{wt}(T \ast T_\lambda).$$

**Proof.** Consider a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. Take $T \ast T_\lambda$ to be row Yamanouchi and note that the bottom row in its inflated weight tableau $D$ is a row with $\lambda_1$ ones. Thus, to prove that $\text{cwt}(T \ast T_\lambda) = \text{wt}(T \ast T_\lambda)$, it suffices to show that $D$ has *interval valued columns*. That is, we claim that the column of an uncovered entry $x$ in $D$ contains precisely $1, 2, \ldots, x$.

This condition holds on $T_\lambda$ since the rows in its inflated weight tableau are *right* justified and row $i$ is made up of $\lambda_i$’s. Next is the construction of row $\lambda_{\ell} + 1$ in the inflated weight tableau of $T \ast T_\lambda$, beginning by sliding entry $b$ into the rightmost cell $(\lambda_{\ell} + 1, j)$ such that all entries in column $j$ are smaller than $b$. By the Yamanouchi property on $T \ast T_\lambda$, there are strictly more $b$’s than $b$’s in lower rows and at least one is uncovered since columns are interval valued. Therefore, the rightmost column with an uncovered $b - 1$ is the column $j$ into which $b$ is moved and it remains interval valued. The same argument applies for the next letter $a$ in row $\lambda_{\ell} + 1$ since $a \leq b$. By iteration, columns in the inflated weight tableau are interval valued.

On the other hand, consider $T \ast T_\lambda$ with the property that $\text{cwt}(T) = \text{wt}(T)$. Since columns of the inflated weight tableau $D$ are strictly increasing, every column with an entry $x$ must contribute to the number of uncovered entries in $D$ which are at least $x$. Therefore, columns of $D$ must be interval valued because $\text{cwt}(T) = \text{wt}(T)$ implies that the number of uncovered entries in $D$ which are at least $x$ equals the number of times $x$ occurs in $D$. Every $x > 1$ in $D$ must then lie above an $x - 1$ in the same column. This implies that every $x$ in $w(D)$ can be paired with an $x - 1$ to its right. Therefore, $w(T \ast T_\lambda) = w(D)$ is Yamanouchi as claimed. $\square$

We define the notion of inflated weight on a set-valued tableau $S$ so that it reduces to Definition 7 when $S$ has no multicells.

**Definition 11.** The **inflated weight** of a set-valued tableau $S$ is defined iteratively from its rows $S_1, S_2, \ldots, S_\ell$, read from bottom to top. With $r = 2$ and $\hat{S} = S_1$, starting from the last cell $c$ in $S_r$, modify $S_r \ast \hat{S}$ as follows:

1. move the largest entry $e$ in cell $c$ to the rightmost empty cell $(r, j)$ such that all entries in cells $(r', j)$ for all $r' < r$ are smaller than $e$. If no such cell exists, $e$ remains in place.

2. move the next largest entry $e_2$ in cell $c$ to the rightmost cell $(r, \hat{j})$, where $\hat{j} \leq j$, such that all entries in cells $(r', \hat{j})$ for all $r' < r$ are less than $e_2$. Let $j = \hat{j}$ and repeat this step on all remaining entries in cell $c$.

3. repeat from step 1 on the cell $\hat{c}$ just west of cell $c$.

When all cells in row $r$ have been addressed, iterate by setting the resulting filling to $\hat{S}$ and $r := r + 1$. The process terminates when $r = \ell$, with a column-strict set-valued filling $\hat{S}$ called the **inflated weight tableau** of $S$. The inflated weight is the conjugate of the partition rearrangement of the list of maximal entries in uncovered cells of $\hat{S}$.
Example 12.

\[
S = \begin{array}{ccc}
4 & 9 & 5 \\
1 & 5 & 6 \\
\end{array} \quad \text{implies} \quad \iota \text{wt}(S) = (9, 8)' = (2, 2, 2, 2, 2, 2, 2, 2, 1)
\]

\[
S = \begin{array}{ccc}
47 & 9 & 6 \\
1 & 5 & 6 \\
\end{array} \quad \text{implies} \quad \iota \text{wt}(S) = (9, 8, 4)' = (3, 3, 3, 2, 2, 2, 2, 2, 1)
\]

Remark 13. The inflated weight of a set-valued tableau could also have been defined as an iterative process on its columns. Reading the columns \(S_1, S_2, \ldots, S_{\lambda_i}\) from left to right, first consider just the sub-diagram with columns \(S_{\lambda_i-1}\) and \(S_{\lambda_i}\). Perform the moves described above starting with the lowest cell of \(S_{\lambda_i-1}\) and working upwards. Repeat this process with the cells of \(S_{\lambda_i-2}\) and proceed in this fashion until all columns are exhausted. Note that any cell \((r, j)\) in a set-valued tableau has entries which are strictly larger than the entries in cell \((r', j')\) for \(r' \leq r, j' \leq j\). If we instead construct the inflated tableau by rows, all entries of \(c\) necessarily move further to the right than the entries in \((r', j')\). With this in mind one can see the resulting inflated weights to be equal.

Proposition 14. For partition \(\lambda\) and set-valued tableau \(S\), if \(S * T_\lambda\) is column Yamanouchi then \(\text{wt}(S * T_\lambda) = \iota \text{wt}(S * T_\lambda)\) and the columns of the inflated weight tableau of \(S * T_\lambda\) are interval valued.

Proof. Once again the results hold for \(T_\lambda\) since the rows in its inflated weight tableau are right justified and row \(i\) is made up of \(\lambda_i\)'s. We thus consider the column construction of the inflated weight tableau \(D\) of a column Yamanouchi set-valued tableau \(S * T_\lambda\). Let \(b\) be the largest entry in the most southeastern cell of \(S\). Starting with the inflated weight tableau \(\hat{D}\) of \(T_\lambda\), note that the Yamanouchi condition on \(S * T_\lambda\) implies there are strictly more \(b - 1\)'s than \(b\)'s. At least one \(b - 1\) is uncovered since columns of \(\hat{D}\) are interval valued. Therefore, the rightmost column \(j\) with an uncovered \(b - 1\) is the column into which \(b\) is moved and it remains interval valued. If \(b\) came from a multicell in \(S\), an entry \(a < b\) moves next. The order in which entries are read from a multicell when checking the Yamanouchi property ensures that there is an uncovered \(a - 1\) in \(\hat{D}\) (in some column \(\hat{j} < j\)). Entry \(a\) is thus placed in \((\lambda_j + 1, \hat{j})\) and columns retain interval values. The argument applies to entries taken from biggest to smallest in multicells and then moving up the column. Note that the columns of \(D\) are not only interval valued, but there are no multicells.

Since \(D\) has no multicells, it also corresponds to the inflated weight tableau of \(T * T_\lambda\) where \(T\) is the tabloid with row \(i\) made up of the entries of row \(i\) in \(S\). The proof of Proposition 10 shows that \(D\) has interval valued columns if and only if \(\iota \text{wt}(T * T_\lambda) = \text{wt}(T * T_\lambda)\). The claim follows by noting that \(\iota \text{wt}(T * T_\lambda) = \iota \text{wt}(D)\) and \(\text{wt}(T * T_\lambda) = \text{wt}(S * T_\lambda)\). \(\Box\)

Inflated weight also exposes a correspondence between reverse plane partitions and a distinguished subset of skew tabloid. Precisely, for composition \(\alpha\) and partitions \(\lambda\) and \(\nu\), let

\[
T_\lambda(\alpha, \iota \text{wt} = \nu) = \{T * T_\lambda : T \in T(\alpha) \text{ and } \iota \text{wt}(T * T_\lambda) = \nu\}.
\]

We refer to \(T \in T_\lambda\) as a \(\lambda\)-augmented tabloid. The subset \(SSYT_\lambda\) of elements in \(T_\lambda\) which are semi-standard tableaux will be of particular interest.
Definition 15. For any $R \in \text{RPP}(\cdot / \lambda, \alpha)$, let $T$ be the tabloid of shape $\alpha$ where entries of row $x$ record the row positions of the subset of $x \in R$ which do not lie below an $x$ in the same column, for $x = 1, \ldots, \ell(\alpha)$. Set $\partial(R) = T \ast T_\delta$.

Example 16.

\[
\partial \begin{bmatrix}
1 & 1 & 2 \\
1 & 1 & 2 \\
2 & 2
\end{bmatrix} = 
\begin{bmatrix}
1 & 1 \\
1 & 2 \\
2 & 4
\end{bmatrix}
\]

Proposition 17. For any skew partition $\nu/\lambda$ and composition $\alpha$, the map $\partial$ gives a bijection

\[\text{RPP}(\nu/\lambda, \alpha) \leftrightarrow T_\alpha(\alpha, \text{twt} = \nu) .\]

Proof. Let $R \in \text{RPP}(\nu/\lambda, \alpha)$ and $\partial(R) = T \ast T_\delta$. We will first show that $\partial$ produces a tabloid with the correct shape and inflated weight. The $i^{th}$ component of the weight $\alpha$ of $R$ is determined by counting the number of columns containing an $i$. On the other hand, one $i$ from each such column contributes to an element in row $i$ of $T$ proving the shape of $T$ is indeed $\alpha$.

We next prove that $\text{twt}(T \ast T_\delta) = \nu$ by iteration on $\partial(R')$, where $R'$ is the sub-reverse plane partition obtained by deleting all letters larger than $i$ from $R$. We shall prove that $\text{twt}(\partial(R')) = \nu'$ where $\nu'/\lambda = \text{shape}(R')$ by showing that the uncovered entries in the inflated weight tableau $T$ of $\partial(R')$ match the column heights of $\nu'$.

The base case is $\partial(R^0) = T_\delta$, whose inflated weight tableau $T^0$ contains $\lambda_i$'s right justified. The construction of the inflated weight tableau $T^1$ of $\partial(R^1)$ involves rows $r_1 \geq \ldots \geq r_t$, where $r_1$ is the height of the leftmost column $c_1$ containing a 1 in $R^1$. Row positions $r_2, \ldots, r_t$ are similarly obtained by reading columns west to east. $T^1$ is obtained by successively sliding $r_1, \ldots, r_t$ into row $\ell(\lambda) + 1$ atop $T^0$ as far right as possible without violating the column increasing condition. Note that $r_1$ takes position above the entry in $T^0$ which records the height of column $c_1$ in $T^0$. The same argument applied to the remaining row positions implies that uncovered entries of $T^1$ match the column heights of $\nu^1$. The results follows by iteration.

The bijectivity of $\partial$ is seen through the construction of its inverse. A unique reverse plane partition $\bar{R}$ of shape $\nu/\lambda$ corresponds to $T \ast T_\delta$ using the following construction. Starting with row $i = 1$ of $T$, place an $i$ in the leftmost empty cell $(e, c)$ of $\nu/\lambda$, where $e$ is the rightmost entry in the first row of $T$. Fill all empty cells below $(e, c)$ in column $c$ with $i$ as well. Proceed by placing an $i$ in the leftmost empty cell $(\hat{e}, \hat{c})$ where $\hat{e}$ is the next to last entry $\hat{e}$ in row $i$ of $T$. Repeat on all entries in row $i$ of $T$ and iterate letting $i = i + 1$. \hfill \Box

4.2 Tabloids to set-valued tableaux

The jeu de taquin operation induces a crystal graph on the set of tabloids with fixed inflated weight. The highest weights are simply tabloids with strictly increasing columns.

Proposition 18 ([KM]). For each skew partition $\nu/\lambda$, a crystal graph whose vertices are augmented tabloids $T_\delta(\cdot , \text{twt} = \nu)$ supports $g_{\nu/\lambda}$. The highest weights are semi-standard tableaux in $T_\delta(\cdot , \text{twt} = \nu)$. 

11
A different perspective is studied by Galashin in [Gal14] using a crystal graph on reverse plane partitions. Proposition 18 can be proven by applying the graph isomorphism \( \partial \) to the reverse plane partition crystal. The tabloid point of view is amenable to a bijection \( \phi \) of [BM12] associating set-valued tableaux to strict elegant fillings and this route leads to the \( K \)-theoretic Littlewood-Richardson structure constants.

The map \( \phi \) on set-valued tableaux is defined by iteratively eliminating multicells through a process called \textit{dilation}. Given a set-valued tableau \( S \), let \( \text{row}(S) \) be the highest row containing a multicell. Let \( S_{\leq i} \) denote the subtableau formed by taking only rows of \( S \) above row \( i \). For the rightmost multicell \( c \) in \( \text{row}(S) \), define \( x = x(S) \) to be the largest entry in \( c \). The dilation of \( S \), \( \text{di}(S) \), is constructed from \( S \) by removing \( x \) from \( c \) and RSK-inserting \( x \) into \( S_{> \text{row}(S)} \).

**Example 19.** Since \( \text{row}(S) = 2 \) and \( x(S) = 6 \),

\[
\begin{pmatrix}
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 & 1 \\
4 & 3 & 2 & 1 & 1 & 1 & 1 & 2 \\
3 & 2 & 1 & 1 & 1 & 1 & 1 & 3 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 5 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 7
\end{pmatrix}
\]

It was proven [BM12, Property 4.4] that dilation preserves Knuth equivalence. More precisely, for any set-valued tableau \( S \),

\[
\text{sw}(S) \sim \text{sw}(\text{di}(S)),
\]

where the \textit{set-valued row word} \( \text{sw}(S) \) is the word obtained by listing the entries from rows of \( S \) as follows (starting in the highest row): first ignore the smallest entry in each cell and record the remaining entries in the row from right to left and from largest to smallest within each cell, then record the smallest entry of each cell from left to right.

**Example 20.**

\[
S = \begin{pmatrix}
3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4
\end{pmatrix}
\implies \text{sw}(S) = 653432123.
\]

Dilation expands a set-valued tableau by reducing the number of entries in a given multicell by one. The iteration of this process produces a semi-standard tableau from a set-valued tableau.

**Definition 21.** The map \( \phi \) acts on a set valued tableau \( S \) by constructing the sequence of set-valued tableaux

\[
S = S_0 \to \text{di}(S) = S_1 \to \text{di}(S_1) = S_2 \to \cdots \to S_r
\]

and defining \( \phi(S) \) to be the filling of \( \text{shape}(S_r)/\text{shape}(S) \) where cell \( \text{shape}(S_r)/\text{shape}(S_{r-1}) \) contains the difference between the row index of this cell and \( \text{row}(S_{r-1}) \). The sequence (17) is defined to terminate at the first set-valued tableau \( S_r \) with no multicell.

**Example 22.**

\[
\phi \left( \begin{pmatrix} 4 & 23 & 1234 \end{pmatrix} \right) = \begin{pmatrix} 2 & 1 \end{pmatrix}
\]

12
is constructed by recording the sequence of dilations

\[
\begin{array}{c|c|c|c|c}
4 & 2 & \text{2} & 3 & \\
1 & 1 & \text{1} & \text{123} & \\
\end{array}
\rightarrow
\begin{array}{c|c|c|c|c}
4 & 3 & \text{2} & 2 & 1 \\
1 & 1 & \text{12} & 3 & \\
\end{array}
\rightarrow
\begin{array}{c|c|c|c|c}
4 & 3 & 4 & 2 & 2 \\
1 & 1 & \text{12} & 3 & \\
\end{array}
\rightarrow
\begin{array}{c|c|c|c|c}
4 & 3 & 4 & 2 & 2 \\
1 & 1 & \text{12} & 3 & \\
\end{array}
\]

The restriction of $\phi$ to an action on a subset of set-valued tableaux determined by their set-valued row words and a tableau $T$ is a bijection $\phi_T$ between these set-valued tableaux and elegant fillings. In particular this gives a bijective proof of Lenart’s formula [Len00, Theorem 3.2].

**Proposition 23** ([BM12], Proposition 5.6). For any fixed tableau $T$ and partition $\eta \subset \text{shape}(T)$,

$$\phi_T : \{ S \in \mathcal{SVT}(\eta) : \text{sw}(S) \sim \text{w}(T) \} \leftrightarrow \text{EF}((\text{shape}(T))/\eta)$$

is a bijection.

Our purposes require a refinement of this result which identifies strict elegant fillings with subsets of set-valued tableaux characterized by inflated weight. To state the result we define $\lambda$-augmented set-valued tableaux in a way analogous to that for tabloids. Precisely, let

$$\mathcal{SVT}_\lambda(\eta,\text{wt} = \alpha) = \{ S \ast T_\lambda : s \in \mathcal{SVT}(\eta) \text{ and } \text{wt}(S \ast T_\lambda) = \alpha \}.$$

**Proposition 24.** For each $T \in \mathcal{SSYT}_\lambda(\mu,\text{wt} = \alpha)$ and partition $\eta \subset \mu$,

$$\phi_T : \{ S \in \mathcal{SVT}_\lambda(\eta,\text{wt} = \alpha) : \text{sw}(S) \sim \text{w}(T) \} \leftrightarrow \text{EF}((\text{shape}(T))/\eta)$$

is a bijection.

**Proof.** By definition of $\phi_T$ and Proposition 23, it suffices to show that $\text{wt}(S) = \text{wt}(d(S))$ for set-valued tableau $S$. The dilation of $S$ is an iterative process whereby a generic step involves the deletion of the largest entry $b$ from some multicell in row $r$, followed by the RSK-insertion of this entry starting in row $r+1$. Row $r$ is taken to be the highest row containing a multicell.

We proceed by showing how entries in the inflated weight tableau $D$ of a set-valued tableau $S$ differ from those in the inflated weight tableau $\tilde{D}$ of the set-valued tableau $\tilde{S}$ resulting from such a step. From this, we argue that the uncovered entries are the same in $D$ and $\tilde{D}$.

Rows lower than row $r$ in $D$ and $\tilde{D}$ are identical since these are unchanged by the step in dilation and the construction of inflated weight tableaux proceeds from bottom to top. We claim next that rows $r$ of $D$ and $\tilde{D}$ are identical except for the omission of entry $b$ in $\tilde{D}$. Let $a$ be the maximal entry smaller than $b$ in its multicell of $S$. In row $r$ of $D$, if $b$ lies in column $j_r$, then $a$ is in column $j_r - t$ for some $t \geq 0$. Note then that any cell $c$ in row $r$ between column $j_r - t$ and $j_r$ is empty and there is an entry $e > a$ below it. Therefore, deleting $b$ from row $r$ of $D$ and reconstructing the dilation leaves all other entries in this row unchanged.
Next consider the entries \( \{x_r, \ldots, x_n\} \) lying in the cells of the bumping route created by the deletion of \( x_r = b \) from row \( r \) of \( S \) followed by its insertion into row \( r + 1 \). In particular, \( x_i \) lies in row \( i \) of \( S \) whereas it lies in row \( i + 1 \) of \( \bar{S} \). Now consider the inflated tableau \( D \) and let \( c_i = (i, j_i) \) denote the leftmost cell of row \( i \) in \( D \) containing \( x_i \), for \( i = r, \ldots, n \). We prove that the only difference between row \( i + 1 \) in \( \bar{D} \) and \( D \) is

(a) entry \( x_{i+1} \) is replaced by \( x_i \) in \( c_{i+1} \) when \( j_{i+1} \leq j_i \)

(b) entry \( x_{i+1} \) is deleted and \( x_i \) is placed in cell \((i + 1, j_i)\) above \( c_i \) in \( \bar{D} \) otherwise.

The inflated tableau \( D \), with \( x_{r+1} \) in column \( j_{r+1} \), has an entry \( e_1 < x_{r+1} \) in column \( j_{r+1} \) and entry \( e_2 \geq x_{r+1} \) in \((\hat{r}, j_{r+1} + 1)\) for some \( \hat{r} \leq r + 1 \). By the previous discussion, since \( x_r < x_{r+1} \), \( e_1 \) and \( e_2 \) occur in \( \bar{D} \) in their same positions unless \( e_1 = x_r \) is in \( c_r \). Note that \( j_{r+1} \leq j_r \) occurs precisely when \( e_1 < x_r \) or \( e_1 = x_r \) is in \( c_r \). In this case, since \( x_{r+1} \) is the leftmost entry larger than \( x_r \) in row \( r + 1 \) of \( D \), the entry \( x_r \) replaces \( x_{r+1} \) in position \( c_{r+1} \) of \( \bar{D} \). When \( j_{r+1} > j_r \), \( e_1 \geq x_r \) and cells of row \( r + 1 \) and columns \( j_r, j_r + 1, \ldots, j_{r+1} - 1 \) must be empty since \( c_{r+1} \) contains the leftmost entry larger than \( x_r \), in row \( r + 1 \) of \( D \). We see then that going from \( D \) to \( \bar{D} \) in row \( r + 1 \), \( x_{r+1} \) is deleted and \( x_r \) takes the position in column \( j_r \). The claim for generic \( i \) follows by iteration.

It thus remains to show that the specified change from \( D \) to \( \bar{D} \) does not change the set of uncovered entries. Since every column \( j \neq j_i \) is the same in \( D \) and \( \bar{D} \), we need only consider columns \( j_i \). For each \( t \) such that \( j_t < j_{t+1} \), columns \( j_t, j_t + 1, \ldots, j_{t+1} - 1 \) are empty in row \( t + 1 \) of \( D \) implying that \( x_t \) lies below the empty cell \((t + 1, j_t)\) in \( D \). In \( \bar{D}, x_t \) simply slides up into this cell and the status of covered entries is unaltered. Note that an empty cell lies above \( c_{t+1} \) in \( D \), \( b) \) applies for \( i = t + 1 \) and results in sliding \( x_{t+1} \) up into this empty cell in \( \bar{D} \).

For \( r \) where \( j_r \geq j_{r+1} \), \( x_r \) lies just below an entry in row \( t + 1 \) of \( D \) whereas it lies in position \( c_{t+1} = (t + 1, j_{t+1}) \) of \( \bar{D} \). Again, our only worry then is if an empty cell lies above \( c_{t+1} \) in \( D \). However, if it does, \( b) \) applies with \( i = t + 1 \) and results in moving \( x_{t+1} \) above \( x_t \) in \( \bar{D} \). \( \square \)

5 Dual approach to the K-theoretic LR rule

The preceding maps come together to identify the \( g \)-expansion of \( g_{\nu/\lambda} \) without soliciting the help of Theorem 3 and Proposition 5. The last piece of the puzzle is an appropriate sign reversing involution on \( \lambda \)-augmented set-valued tableaux.

Definition 25. (see also [IS14, Lemma 3]). For partitions \( \lambda \) and \( \eta \), the map

\[ \tau : SVT_\lambda(\eta) \rightarrow SVT_\lambda(\eta) \]

acts on \( S \ast T_\lambda \) as follows; when \( S \) is column \( \lambda \)-Yamanouchi, let \( \hat{S} = S \). Otherwise, define \( c \) to be the rightmost column such that \( S_{\geq c} \) is not column \( \lambda \)-Yamanouchi, where \( S_{\geq c} \) is the set-valued tableau made up of only those cells which are weakly right of column \( c \). Let \( y \) be the rightmost letter in the column word \( w(S_{\geq c})w(T_\lambda) \) with the property that there are more \( y \)'s than \( y - 1 \)'s and take \( r \) to be the row of the cell \((r, c)\) in \( S \) containing this \( y \). Denote by \( \text{cell}_{\min} \) the leftmost cell in row \( r \) containing \( y \). The image \( \hat{S} = \tau(S) \) is defined by deleting \( y - 1 \) if it is present in \( \text{cell}_{\min} \) and otherwise \( \hat{S} \) is obtained by adding \( y - 1 \) to \( \text{cell}_{\min} \).
Example 26.

\[
\begin{bmatrix}
& 2 & 1 \\
1 & 1 & \\
\end{bmatrix}
\quad \leftrightarrow \quad
\begin{bmatrix}
& 2 & 1 \\
1 & 1 & \\
\end{bmatrix}
\]

Lemma 27. For partitions \(\eta, \nu\), and \(\lambda \subseteq \nu\), \(\tau\) is a sign-reversing involution on \(SVT_\lambda(\eta, \iwt = \nu)\) where set-valued tableaux with a Yamanouchi column word are fixed points.

Proof. For \(S \in SVT_\lambda(\eta, \iwt = \nu)\), we first show that \(\hat{S} = \tau(S)\) is a set-valued tableau. Certainly deleting a letter maintains the column-strict condition. On the other hand, if \(\hat{S}\) is obtained by adding \(y - 1\), since it is added to the leftmost cell containing a \(y\) in some row, the rows remain non-decreasing. Moreover, since \(c\) is the rightmost column where \(S_{>j}\) fails to be column \(\lambda\)-Yamanouchi (and it fails at \(y\)), either \(y - 1\) is not in column \(c\) or it lies in the cell with \(y\) in row \(r\). Therefore, the entries in cell \((r - 1, c)\) is strictly smaller than \(y - 1\). This implies that entries below the \(cell_{\text{min}}\) are also smaller than \(y - 1\) and we see the columns remain strictly increasing under the action of \(\tau\).

Observe further that \(cell_{\text{min}}(\hat{S}) = cell_{\text{min}}(S)\) and \(y(S) = y(\hat{S})\) since any change in column word due to \(\tau\) occurs to the left of \(y\). Thus by definition of \(\tau\), if \(S\) is not column \(\lambda\)-Yamanouchi, neither is \(\hat{S}\).

It thus remains to show that \(\tau\) preserves inflated weight. By assumption, \(S * T_\lambda\) first fails to be column Yamanouchi with an entry \(y\) in cell \((r, c)\). Therefore, \(S_{>c} * T_\lambda\) is column Yamanouchi and contains the same number of \(y\)'s as \(y - 1\)'s. By Proposition 14, the inflated weight tableau of \(S_{>c} * T_\lambda\) has interval valued columns implying that each \(y - 1\) is covered. This is also true in the continued column construction of the inflated weight tableau with entries in column \(c\) until reaching cell \((r, c)\). However, at cell \((r, c)\), \(y\) slides over and takes a position above an entry strictly smaller than \(y - 1\) since there are no uncovered \(y - 1\)'s. Since there are no \(y - 1\)'s in rows below \(r\) and in any column west of \(c + 1\), this is true for all the \(y\)'s in row \(r\). Note that the inflated weight tableaux before and after applying \(\tau\) are the same up until this point. Then, if \(cell_{\text{min}}\) contains a \(y\) and \(y - 1\), it shares a cell in the inflated weight tableau \(S * T_\lambda\) and thus the deletion of that \(y - 1\) would not affect the inflated weight. On the other hand, when \(cell_{\text{min}}\) does not contain a \(y - 1\) but the action of \(\tau\) introduces a one, we see that in the corresponding inflated weight tableau it would slide into the cell with \(y\) again preserving inflated weight. \(\square\)

Theorem 28. For partitions \(\mu\) and \(\lambda \subseteq \nu\), the coefficient of \(g_\mu\) in \(g_{\nu/\lambda}\) is \((-1)^{\nu - |\lambda| - |\mu|}\) times the number of column \(\lambda\)-Yamanouchi set-valued tableaux of shape \(\mu\) and weight \(\nu/\lambda\).

Proof. Proposition 17 allows us to convert reverse plane partitions of fixed shape into a class of augmented tabloids of fixed inflated weight, implying that

\[
g_{\nu/\lambda} = \sum_{A \in \Delta_{\lambda} : \iwt = \nu} x^{\text{shape}(A)}.
\]

The crystal graph of Proposition 18 implies that

\[
g_{\nu/\lambda} = \sum_{T \in SSYT(\lambda, \iwt = \nu)} S^{\text{shape}(T)}.
\]

(18)
Lenart’s elegant formula (15) then converts this identity to an expansion in terms of the basis of $g$-functions:

$$g_{\nu/\lambda} = \sum_{T \in \text{SSYT}_{\lambda}(\cdot, \iota \text{wt} = \nu)} \sum_{\eta \subseteq \text{shape}(T)} \sum_{E \in \text{EF}(\text{shape}(T)/\eta)} (-1)^{|\text{shape}(T)|-|\eta|} g_{\eta}.$$ 

For each tableau $T$, we use the bijection $\phi_T$ and Proposition 24 to replace elegant fillings by set-valued tableaux.

$$g_{\nu/\lambda} = \sum_{T \in \text{SSYT}_{\lambda}(\cdot, \iota \text{wt} = \nu)} \sum_{\eta \subseteq \text{shape}(T)} \sum_{S \in \text{SVT}_{\lambda}(\eta, \iota \text{wt} = \nu)} (-1)^{|\text{shape}(T)|-|\eta|} g_{\eta}.$$ 

We can replace $|\text{shape}(T)| = |\text{wt}(T)|$ by $|\text{wt}(S)|$ since any tableau $T$ and set-valued tableau $S$ where $w(T) \sim sw(S)$ must have matching weights. In fact, we can drop the dependence on $T$ altogether. Namely, the expansion of a given set-valued tableau $S$ by iterated dilation gives rise to a unique tableau $T$ and $sw(S) \sim w(T)$ by (16). Moreover, our proof of Proposition 24 shows that the inflated weight of $T$ matches $\iota \text{wt}(S)$ under dilation. Therefore,

$$g_{\nu/\lambda} = \sum_{\eta \subseteq \text{shape}(T)} \sum_{S \in \text{SVT}_{\lambda}(\eta, \iota \text{wt} = \nu)} (-1)^{|\text{wt}(S)|-|\eta|} g_{\eta}.$$ 

Finally, we apply the the sign-reversing involution $\tau$ on $\text{SVT}_{\lambda}(\eta, \iota \text{wt} = \nu)$ and use Lemma 27. Proposition 14 then assures us that the fixed points $S$ satisfy $\text{wt}(S \ast T_{\lambda}) = \nu$. □

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