BOUNDS ON THE HAUSDORFF DIMENSION OF RANDOM
ATTRACTORS FOR INFINITE-DIMENSIONAL RANDOM
DYNAMICAL SYSTEMS ON FRACTALS

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Dedicated to Peter E. Kloeden on Occasion of his Seventieth Birthday

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Abstract. We consider a stochastic nonlinear evolution equation where the
domain is given by a fractal set. The linear part of the equation is given by a
Laplacian defined on the fractal. This equation generates a random dynamical
system. The long time behavior is given by an attractor which has a finite
Hausdorff dimension. We would like to reveal the connections between upper
and lower estimates of this Hausdorff dimension and the geometry of the fractal.
In particular, the parameter which determines these bounds is the spectral
exponent of the fractal. Especially for the lower estimate we construct a local
unstable random Lipschitz manifold.

1. Introduction. In this article we will study the long time behavior of stochastic
partial parabolic differential equations defined on particular sets. Usually, a sto-
chastic partial differential equation is acting on functions which are defined on a
domain $D \subset \mathbb{R}^d$. Here the set carrying these functions is given by a fractal of a
certain class. In this context the most common and well-studied objects are self-
similar fractals like the Cantor set, the Sierpinski gasket, the Mandelbrot set or
the Menger sponge. Although these examples are purely mathematical, they have
become more and more interesting for natural scientists. Examples such as the
percolation through porous structures or the diffusion across conductive layers have
supports that are modeled with fractal sets, see e.g. the references in Freiberg [15]
and the work of Lancia et al. [19]. In the 80s and 90s there have been various
approaches to define a meaningful analysis on these sets, see Barlow [3] or Kigami
[18]. In general when studying diffusions the Laplacian is of particular interest.
In contrary to the classical definition of the Laplacian one can not use partial deriva-
tives since the typical fractal set has an empty interior. Using the theory in Kigami
[18] one can still define an operator that models properly the diffusion effects and

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one still call it Laplacian. This well elaborated theory offers the possibility to study partial differential equations on fractal sets.

Our intension is now to study the long time behavior of a reaction diffusion equation on a fractal under the influence of a noise. An object which includes the long time states of the solutions of the given stochastic reaction diffusion equation is the random attractor. To obtain a quantitative description we are interested in an estimate on the Hausdorff dimension of this random object. In particular, the main purpose of this manuscript is to investigate the relation between the properties of fractal sets carrying the solutions of our stochastic equation and the Hausdorff dimension of the random attractor. In difference to the classical result the dimension of the considered domain is not the right parameter for the estimate on the Hausdorff dimension of the random attractor. In the fractal case there exists a certain parameter of the fractal that influences the Hausdorff dimension of the attractor. This parameter is the so called spectral exponent (or spectral dimension) $d \in \mathbb{R}_+$. The spectral exponent describes the asymptotic growth of the spectrum. Suppose we have the Laplacian on a bounded sufficiently smooth domain $D \subset \mathbb{R}^d$. Then the corresponding eigenvalues $(\lambda_i)_{i \in \mathbb{N}}$ satisfy the law of Weyl

$$\lim_{x \to \infty} \frac{\rho(x)}{x^d} = C,$$

where $\rho(x)$ counts the eigenvalues of the Laplacian smaller than $x$ and the constant $C$ depends on the geometry of the set $D$, see Weyl [31]. Considering the class of fractals Kigami introduces one observes in general only

$$0 < \liminf_{x \to \infty} \frac{\rho(x)}{x^{d_S/2}} \leq \limsup_{x \to \infty} \frac{\rho(x)}{x^{d_S/2}} < \infty.$$

Note that instead of the dimension $d$ the number $d_S$ is appearing in the exponent. The bounds of the limes inferior and limes superior are depending on the geometry of the fractal.

We consider a stochastic partial differential equation (SPDE) on a fractal containing a linear multiplicative noise which generates a random dynamical system (RDS). The fact that the noise is multiplicative linear allows us to transform our equation into a random one (RPDE) which is easier to handle. Attractors for RDS generated by this kind of differential equations with multiplicative noise are studied in Caraballo et al. [5], Flandoli and Schmalfuss [14] or Schmalfuss [23]. We would like to learn more about the set that we call random attractor. As one of the primary properties of a set, we consider its dimension. The dimension tells us, for instance how much the set fills the surrounding space and it describes which possible directions a containing particle can take, i.e. its number of degrees of freedom. At first we will give an upper estimate on the dimension of the random attractor. Methods to obtain upper estimates for the Hausdorff dimension are formulated in Cramel and Flandoli [8], Debussche [9] and Schmalfuss [24]. We will apply the method by Debussche. An estimate on the Hausdorff dimension for an SPDE with linear multiplicative noise on a standard domain with linear multiplicative noise is given by Caraballo et al. [6]. However in contrast to this publication we deal with qualitatively different diffusion and reaction terms where the later one is given by a nonlinear integral kernel operator defined on a fractal. In particular we have to show the uniform differentiability for the RDS generated by our RPDE.

To obtain a lower bound for the Hausdorff dimension of the random attractor we assume that the RDS has a stationary point, say zero. For deterministic systems
it is known that the dimension of unstable manifold with respect to this stationary
point gives an lower bound for the attractor. Random stable and unstable manifolds 
were introduced by Wanner [30]. A random graph transform has been considered in 
Schmalfuss [25], [26] to find a random invariant manifold. Caraballo et al.[6] apply 
a graph transform technique to find a lower bound for the Hausdorff dimension 
of a random attractor. An alternative method to show the existence of (random) 
stable/unstable manifolds is the Lyapunov-Perron transform, see Lu et al. [11] or 
Lu and Schmalfuss [21]. We apply this transform to obtain a finite dimensional 
unstable manifold for our RPDE. The dimension of this manifold is then a lower 
bound on the dimension of the random attractor. This dimension depends on the 
eigenvalues of the Laplacian defined on the fractal and on the derivative of the 
nonlinearity in zero which is given by $K \cdot \text{id}_H$.

The article is organized as follows. In the next section we introduce the random 
basics for our purpose, we introduce terms like RDS, random attractor and random 
invariant manifolds. In Section 3 we introduce the class of fractals we are considering 
and provide the tools allowing us to define the Laplacian on these sets. The defined 
Laplacian then is the generator of an analytic $C_0$ semigroup. In addition we study 
the asymptotics of the eigenvalues. We then consider a stochastic evolution equation 
with multiplicative noise and transform this equation into a random one. Both 
equations generate a random dynamical system. Finally in Section 4 we apply well 
known techniques to estimate the Hausdorff dimension of the random attractor 
from above for the random dynamical system generated by the stochastic equations. 
These estimates depend on a parameter the so called spectral exponent. To obtain a 
lower bound for the attractor we construct an unstable manifold inside the random 
attractor. An appendix contains some technical calculations.

2. Preliminaries. At first we deal with a model for a noise. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a 
probability space and suppose that $\theta$ is a measurable flow

$$\theta : \mathbb{R} \times \Omega \to \Omega, \quad \theta_t \circ \theta_{\tau} = \theta_{t+\tau}, \quad \theta_0 = \text{id}_\Omega \quad \text{for } t, \tau \in \mathbb{R},$$

$$\theta : (\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}) \to (\Omega, \mathcal{F}) .$$

We assume that the flow $\theta$ is ergodic, that is

$$\theta_t \mathbb{P} = \mathbb{P} \quad \text{for all } t \in \mathbb{R},$$

and all $\theta$-invariant sets $A \in \mathcal{F}$ ($\theta_t A = A$ for all $t \in \mathbb{R}$) have measure 0 or 1. The 
quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a metric dynamical system.

A mapping $X : \Omega \to \mathbb{R}^+$ is called tempered (from above) if we have that

$$\lim_{t \to \pm \infty} \frac{\log^+ X(\theta_t \omega)}{|t|} = 0 .$$

A positive mapping $X : \Omega \to \mathbb{R}^+$ is called tempered from below if $1/X$ is tempered 
from above. Assume now that $X$ is a random variable having the above property 
on a $\theta$-invariant set of full measure then $X$ is called a tempered from above (from 
below) random variable. We note that if a positive random variable is not tempered 
from above then we have

$$\lim sup_{t \to \pm \infty} \frac{\log^+ X(\theta_t \omega)}{|t|} = +\infty .$$

on a set of full measure. In other words, if
\[ \limsup_{t \to +\infty} \frac{\log^+ X(\theta_t \omega)}{|t|} < \infty \quad \text{or} \quad \limsup_{t \to -\infty} \frac{\log^+ X(\theta_t \omega)}{|t|} < \infty \]

on a set of positive measure then the random variable \( X \) is tempered from above.

We need the following type of mapping.

**Definition 1.** Let \( H \) be a Polish space and \( Y \) be some metric space. A mapping \( g : \Omega \times H \to Y \)

is called Carathéodory mapping if

\[ H \ni h \mapsto g(\omega, h) \quad \text{is continuous for every } \omega \in \Omega, \]
\[ \Omega \ni \omega \mapsto g(\omega, h) \quad \text{is measurable for every } h \in H. \]

It is known that such a mapping \( g \) is \( \mathcal{F} \otimes \mathcal{B}(H), \mathcal{B}(Y) \) measurable, cf. Aliprantis et al. [1, Lemma 4.51].

In the following we assume that \( H \) is a separable Banach space. A setvalued mapping \( \omega \mapsto D(\omega) \) where \( D(\omega) \neq \emptyset \) and closed, is called a closed random set in \( H \)

if

\[ \omega \mapsto \sup_{x \in D(\omega)} \|x - y\| \]

is measurable for all \( y \in H \). Any closed random set can be represented by a sequence of random variables \((x_k)_{k \in \mathbb{N}}\) as follows

\[ D(\omega) = \bigcup_{k} \{x_k(\omega)\}^H. \]

A closed random set \( D \) set is called tempered if

\[ \omega \mapsto \sup_{x \in D(\omega)} \|x\| \]

is a tempered random variable. The set of all tempered closed random sets \( D \) is denoted by \( \mathcal{D} \). By the separability of \( H \) this expression defines a random variable.

A measurable mapping

\[ \varphi : \mathbb{R}^+ \times \Omega \times H \to H \]

is called a cocycle if

\[ \varphi(0, \omega, \cdot) = \text{id}_H, \]
\[ \varphi(t + \tau, \omega, \cdot) = \varphi(t, \theta_{\tau} \omega, \cdot) \circ \varphi(t, \omega, \cdot) \quad \text{for all } t, \tau \in \mathbb{R}^+. \]

If the cocycle is measurable

\[ \varphi : (\mathbb{R}^+ \times \Omega \times H, \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(H)) \to (H, \mathcal{B}(H)) \]

then \( \varphi \) is called a random dynamical system (RDS). An RDS \( \varphi \) is called continuous if the mapping

\[ v \mapsto \varphi(t, \omega, v) \]

is continuous on \( H \).

RDS are generated by random differential equations

\[ u' = f(\theta_t \omega, u), \quad u(0) = u_0 \in H. \]

assuming that \( f \) is sufficiently regular such that we have a unique global solution. In particular suppose that \( f \) is a Carathéodory mapping which is Lipschitz continuous with respect to the \( u \) argument. Later we will study an infinite dimensional version of this equation.
We are interested in studying the longtime behavior of stochastic/random evolution equations or more special stochastic/random partial differential equations (SPDE, RPDE). For this purpose we introduce the term random attractor.

**Definition 2.** A closed random set $A \in \mathcal{D}$ such that $A(\omega), \omega \in \Omega$ is compact is called random attractor for $\varphi$ if

1. $\varphi(t, \omega, A(\omega)) = A(\theta_t \omega)$ for all $t \geq 0, \omega \in \Omega$
2. $\lim_{t \to \infty} d_H(\varphi(t, \theta_{-t} \omega, D(\theta_{-t} \omega)), A(\omega)) = 0$ for all $D \in \mathcal{D}, \omega \in \Omega$

where $d_H(A, B) = \sup \inf_{a \in A, b \in B} \|a - b\|$ is the Hausdorff semi distance.

The following conditions ensure the existence of a unique random attractor for an RDS $\varphi$ on $H$.

**Theorem 3.** Let $\varphi$ be a continuous RDS on $H$ and suppose that there exists a pullback absorbing set $B \in \mathcal{D}$ such that $B(\omega)$ is compact: For all $\omega \in \Omega$ and $D \in \mathcal{D}$ we have a $T(\omega, D) > 0$ such that

$\varphi(t, \theta_{-t} \omega, D(\theta_{-t} \omega)) \subset B(\omega)$ for $t \geq T(\omega, D)$.

Then the RDS $\varphi$ has a unique random attractor.

For existence theorems of random attractors we refer to Crauel et al. [8], Flandoli [8] and Schmalfuss [23]. In addition to the pullback convergence, the random attractor has the property with respect to forward convergence:

$P - \lim_{t \to \infty} d_H(\varphi(t, \omega, D(\omega))^H, A(\theta_t \omega)) = 0$.

Let zero be a nonrandom stationary point of the cocycle $\varphi$: $\varphi(t, \omega, 0) = 0$ for all $t \geq 0, \omega \in \Omega$.

We have then $0 \in A(\omega)$ for all $\omega \in \Omega$.

We call a family of random variables $(x_t)_{t \in \mathbb{R}}$ such that $x_t(\omega) \in H$ a complete trajectory (of random variables) for the cocycle $\varphi$ if

$\varphi(s, \theta_t \omega, x_t(\omega)) = x_{t+s}(\omega)$ for all $t \in \mathbb{R}, s \in \mathbb{R}^+, \omega \in \Omega$.

**Lemma 4.** Let $\varphi$ be an RDS having a random attractor $A$ and let $(x_t)_{t \in \mathbb{R}}$ be a complete trajectory for $\varphi$. Suppose there exists a $\hat{D} \in \mathcal{D}$ such that for every $\omega \in \Omega$ there exists a $T(\omega) > 0$ we have that $x_t(\omega) \in \hat{D}(\theta_t \omega)$ for $t \leq -T(\omega)$. Then $x_0(\omega) \in A(\omega)$.

**Proof.** Let $t_\epsilon(\omega)$ be the time such that for all $\epsilon > 0$

$d_H(\varphi(t, \theta_{-t} \omega, \hat{D}(\theta_{-t} \omega)), A(\omega)) < \epsilon$ for $t \geq t_\epsilon(\omega)$

and in particular for $t \geq T(\omega) + t_\epsilon(\omega)$. Hence $d_H(x_0(\omega), A(\omega)) < \epsilon$. Since $A(\omega)$ is closed we have $x_0(\omega) \in A(\omega)$.

\end{proof}
Definition 5. A closed random set $M = (M(\omega))_{\omega \in \Omega}$, $M(\omega) \subset H$ is called the unstable set of $\varphi$ if for every random variable $x(\omega) \in M(\omega)$ there exists a complete trajectory $x_t(\omega)$ such that $x_0(\omega) = x(\omega)$ and
\[ \lim_{t \to -\infty} x_t(\omega) = 0. \] (1)

We have straightforwardly the following result.

Lemma 6. The inclusion $M(\omega) \subset A(\omega)$ holds.

The latter is an easy consequence of Lemma 4 and (1).

The observation of this lemma was given by Crauel [7] where he assumes that the RDS is two sided. However for our applications we cannot assume that the RDS is two-sided. In particular we will deal with RDS that are generated by parabolic SPDE.

We also note that the unstable random set $M$ can have particular subsets, where the complete trajectories have an exponential convergence to zero, if $t \to -\infty$. We will show under certain conditions that these random sets take the form of finite dimensional Lipschitz manifolds.

Definition 7. Given an RDS with a non random stationary point zero. The random set $M^\nu$ is called unstable set of exponential decay order $\nu > 0$ if for every random variable $x(\omega) \in M^\nu(\omega)$ there exists a complete trajectory $x_t(\omega)$ for the RDS $\varphi$ with $x_0(\omega) = x(\omega)$ and
\[ \lim_{t \to -\infty} e^{(-\nu + \epsilon)t} \|x_t(\omega)\| = 0 \quad \text{for every } \epsilon > 0. \] (2)

We are interested in an unstable set which has the structure of a manifold.

Definition 8. A random set $M$ is called random finite dimensional Lipschitz manifold if there exists a splitting of $H$:
\[ H = H^+ \oplus H^- \]
where $H^+$ is finite dimensional and $M(\omega)$ has the representation
\[ M(\omega) = \{ x^+ + m(\omega, x^+) : x^+ \in H^+ \} \]
where $m : \Omega \times H^+ \to H^-$ is a Carathéodory mapping and
\[ x^+ \mapsto m(\omega, x^+) \] is a Lipschitz mapping for every $\omega \in \Omega$.

In the following we will assume that $M^\nu$ is the random manifold of exponential decay order $\nu$.

In some situations it makes sense to consider a weaker definition of the random unstable manifold for $\varphi$ of exponential decay order $\nu$. In particular, we would like only to assume that the mapping $m$ from the last definition is defined on a closed random neighborhood $U(\omega)$ at zero. However it should be complicated to define a Carathéodory mapping with respect to a random domain $U$. To avoid these problems we give the following definition:

Definition 9. A random finite dimensional manifold $M^\nu(\omega)$ is called local random unstable manifold of exponential decay $\nu$ for the RDS $\varphi$ with non random stationary point zero if there exists a closed random set $V$ such that $V(\omega)$ is a neighborhood at zero and for any random variable $x(\omega) \in M(\omega) \cap V(\omega)$ there exists a complete trajectory of $\varphi$ denoted by $(x_t)_{t \in \mathbb{R}}$ and $x_0(\omega) = x(\omega)$ such that (2) holds. Note that $M(\omega) \cap V(\omega)$ is a random set by Hu and Papageorgiou [17] Proposition 2.1.43.
The local character of this manifold is given by the fact that we only find complete trajectories of \( \varphi \) passing states of the manifold which are in the neighborhood \( V(\omega) \). For these states we have \( x_t(\omega) \in U(\theta_t\omega) \) for \( t \leq 0 \). This has not to be true for all states (random variables) contained in the manifold \( M \).

3. Random evolution equations. In this setup we consider the following SPDE with a linear multiplicative Brownian noise,

\[
    dv(t) = (\Delta_m v(t) + \mathcal{F}(v(t))) dt + v(t) \circ d\omega(t), \quad t > 0,
\]

\[
    v(0) = v_0 \in H.
\]  

Here \( \omega \) is a two sided one dimensional standard Brownian motion and \( \mathcal{F} \) is some nonlinearity introduced below. \( \Delta_m \) is not a differential operator in the standard sense. It is an unbounded operator defined on a fractal set of a certain class of fractals. This operator has similar properties like the standard Laplacian on bounded sufficiently smooth domains. Especially this operator has a negative discrete spectrum with a particular growth order depending on the structure of the fractal and in addition this operator generates an analytic semigroup on some Hilbert space. In the following we describe this operator in more detail.

Let \( H := L^2(K, \mu) \) where \( (K, B(k), \mu) \) be a finite measure space of a bounded subset of \( \mathbb{R}^d \), \( d \geq 1 \). In particular we choose for the basic set \( K \) a so called post critically finite fractal, or short p.c.f. fractal. These sets show typically a certain self-similarity, have an empty interior, a non-smooth boundary and often non-integer Hausdorff dimension. For more details concerning the definition of these irregular sets we refer to the book of Kigami, [18, Chapter 1].

The standard example of this class of fractals is the Sierpinski gasket (SG). The iterated function system (IFS) giving the SG consists of three contractive mappings \( (F_i)_{i=1,2,3} \) in \( \mathbb{R}^2 \). The SG is the unique non-empty compact set that satisfies the self-similar property with respect to its contractions, i.e. \( SG = \bigcup_{i=1}^{3} F_i(SG) \), see [18, Theorem 1.1.4 & Example 1.2.8]. Other examples for p.c.f. fractals are the Koch curve, Hata’s tree-like set, the Peano curve and counterexamples are the Sierpinski carpet or Cantor-like sets.

A standard method obtaining a p.c.f. fractal \( K \) is based on an iterative sequence of graphs \( (G_m)_{m \geq 0} \) approximating the fractal with respect to the used IFS. In the case \( K = SG \) the approximation can be seen in Figure 1.

Although a p.c.f. fractal \( K \) possesses the mentioned unfavorable properties, there exists a well elaborated theory of the analysis on these sets [18, 28, 16, 4]. We give here only a short summary of the most important ideas.

The associated vertex sets of the graphs are \( (V_m)_{m \geq 0} \) with \( V_m \subset V_{m+1} \) and we define \( V_* = \bigcup_{m=0}^{\infty} V_m \). One can show that \( \nabla_* = K \) with respect to the Euclidean metric in \( \mathbb{R}^2 \), [18, Lemma 1.3.11].

For a function \( f : V_* \to \mathbb{R} \) we assign a non-decreasing sequence of energy forms \( (\mathcal{E}_m(f|_{V_m}))_{m \geq 0} \subset [0, \infty) \) which possesses a limit

\[
    \mathcal{E}(f) := \lim_{m \to \infty} \mathcal{E}_m(f|_{V_m}) \in [0, \infty]
\]

following for instance [28, Section 1.4] or [18, Section 2.2]. We define the domain of the energy by \( D(\mathcal{E}) = \{ f : V_* \to \mathbb{R} : \mathcal{E}(f) < \infty \} \) and in particular \( D^0(\mathcal{E}) = \{ f \in D(\mathcal{E}) : f|_{V_0} = 0 \} \). Any function \( f \in D(\mathcal{E}) \) can be uniquely extended to a function of the space of continuous functions on the SG: \( C(SG) \). For the extension we keep the notation \( f \in C(SG) \) and denote by \( \mathcal{D} = \{ f \in C(SG) : \mathcal{E}(f) < \infty \} \) where \( \mathcal{E}(f) := \)
The infinite iteration is the fractal $SG$

Figure 1. An approximation of the Sierpinski gasket using a sequence of graphs.

$\mathcal{E}(f|_{V_{\epsilon}})$. Similar we define, using the polarization identity, an energy bilinear form for two functions $f, g \in C(SG)$. To measure the fractal sets appropriately we choose the so called self-similar measure $\mu : B(\mathbb{R}^d) \rightarrow [0, 1]$ a certain probability measure with $\text{supp}(\mu) = K$ and weights $(\mu_i)_{i=1,\ldots,N} \in (0, 1)$ where $N$ assigns the number of possible contractions for the IFS, cf. [13, Theorem 2.8] and [18, Section 1.4] for more details.

The following result is similar to the classical potential theory, see also [15, Section 3.3].

**Theorem 10** (Corollary 3.4.7 and Theorem 3.4.6, [18]). Suppose $K$ is a p.c.f. fractal. Then using the above definitions, the quadratic form $(\mathcal{E}, D^0(\mathcal{E}))$ is a local regular Dirichlet form on $H := L^2(K, \mu)$ and there exists a positive self-adjoint operator $B$ on $L^2(K, \mu)$ with $D(B^{1/2}) = D^0(\mathcal{E})$ and $\mathcal{E}(f, g) = (B^{1/2}f, B^{1/2}g)$ for every $f, g \in D^0(\mathcal{E})$.

Moreover the operator $B$ has a compact resolvent and $-B =: \Delta_\mu$ is called the Dirichlet Laplacian on $L^2(K, \mu)$. An element $f \in D(-\Delta_\mu)$ if and only if there exists $h \in D^0(\mathcal{E})$, such that $\mathcal{E}(f, g) = \int_K h(x)g(x)\,d\mu(x)$ for all $g \in D^0(\mathcal{E})$. In this case we identify $h = -\Delta_\mu f$.

In particular we mention, $D(-\Delta_\mu) \subset D^0(\mathcal{E})$ is dense in $L^2(K, \mu)$. Note that it is possible, with the energy method described above, to obtain several ‘different’ Dirichlet Laplacians on a p.c.f. fractal as a consequence of the chosen weights of the measure $\mu$. Together with the resolvent condition we know that $\Delta_\mu$ has a negative discrete spectrum of eigenvalues having finite multiplicity

$$0 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \rightarrow -\infty.$$ 

Having specific spectral asymptotics is one of the unique properties a fractal set possesses. In the classical case of a Dirichlet Laplacian on a bounded domain in $\mathbb{R}^d$ with a smooth boundary the eigenvalue counting function,

$$\rho(x) := \# \{-\lambda_k \leq x\}, \quad \lambda_k - k\text{th eigenvalue of the Laplacian},$$

behaves like $\rho(x) \sim x^{d/2}$, for $x \to \infty$ according to well-known result of Weyl [31] or see also Lapidus [20]. The fractal counterpart for the Laplacian defined in Theorem
10 is given by the results in [18, Theorem 4.1.5] and states in short,

\[ 0 < \liminf_{x \to \infty} \frac{\rho(x)}{x^{d_S/2}} \leq \limsup_{x \to \infty} \frac{\rho(x)}{x^{d_S/2}} < \infty. \]

The positive number \( d_S \) is called the spectral exponent resp. spectral dimension and it typically does not coincide with the Hausdorff dimension \( \dim_H(\cdot) \) of the considered set. We want to mention, that in some cases (the so called non-lattice ones) it is possible that the limit: \( \lim_{x \to \infty} \frac{\rho(x)}{x^{d_S/2}} \) exist. Since we are considering self-similar measures the number \( d_S \) is obtained by the unique solution of the equation

\[ \sum_{i=1}^{N} (r_i \mu_i)^{d_S/2} = 1 \]

where \( (r_i)_{i=1,...,N} \) are positive rescaling factors of the used energy form and we assume \( r_i \mu_i < 1 \) for every \( i \in \{1,...,N\} \). For example in the case of the Sierpinski gasket \( d_S = \frac{\log 5}{\log 3} \) while the Hausdorff dimension is given by \( \dim_H(SG) = \frac{\log 3}{\log 2} \), see e.g. Barlow et al. [4] or Kigami [18, Example 4.1.6].

As a consequence of Theorem 10 we have the subsequent version of Theorem 32.1 in Sell\&You [27], which tells us that the considered operator is the generator of an analytic semigroup.

**Corollary 1.** The linear operator \( \Delta_\mu \) is the generator of an analytic semigroup \( \{S(t)\}_{t \geq 0} \subset L(H) \) with

\[ \|S(t)\|_{L(H)} \leq e^{\lambda_1 t} \]

for all \( t \geq 0 \) and \( \lambda_1 \in \sigma(\Delta_\mu) \) being the largest eigenvalue of \( \Delta_\mu \).

From the generalization of Weyl’s law one derive the following lemma, cf. Kigami [18, Lemma 5.1.3].

**Corollary 2.** The eigenvalues of \( \Delta_\mu \) have a growth order: There exist \( c_S \leq C_S < 0 \) such that for every \( i \in \mathbb{N} \)

\[ c_S i^{2/d_S} \leq \lambda_i \leq C_S i^{2/d_S} \]

where \( c_S, C_S \) depends on the geometry of the fractal.

We now define the nonlinearity \( \mathcal{F} \). Assume for the following that \( K \) is a p.c.f. fractal equipped with a self-similar measure \( \mu \). Let \( f_1 : \mathbb{R} \to \mathbb{R} \) be a function and \( f_2 : K \times K \times \mathbb{R} \to \mathbb{R} \)

be a measurable function. Now we are in a position to define for \( u \in H \)

\[ \mathcal{F}(u)[\xi] = f_1(\|u\|^2)u(\xi) + \int_K f_2(\xi, \eta, u(\eta)) d\mu(\eta) = \mathcal{F}_1(u)[\xi] + \mathcal{F}_2(u)[\xi]. \] (4)

**Lemma 11.** Let \( K \) be a p.c.f. fractal embedded in \( \mathbb{R}^d \). Consider a finite measure \( \mu \) on \( B(\mathbb{R}) = K \cap B(\mathbb{R}^d) \). Consider the mapping in (4) with \( \xi \in K \) and \( u \in H \). Suppose that \( f_1 \) is twice continuously differentiable with a compact support and \( f_2(\xi, \eta, 0) \in L_2(K \times K, \mu \times \mu) \) and

\[ \sup_{v \in \mathbb{R}} |D_3 f_2(\cdot, \cdot, v)| \in L_2(K \times K, \mu \times \mu), \quad \sup_{\eta \in K, v \in \mathbb{R}} |D_3^2 f_2(\cdot, \eta, v)| \in H. \]

Then \( \mathcal{F} \) is uniformly differentiable on \( H \)

\[ \|\mathcal{F}(u + h) - \mathcal{F}(u) - D\mathcal{F}(u)h\| \leq C\|h\|^2 \]

for all \( u, h \in H \).
for \( u, h \in H \) and the positive constant \( C \) is independent of \( u, h \). The derivative of \( F \) is given by

\[
D_F(u)h[\xi] = 2Df_1(\|u\|^2)(u, h)u(\xi) + f_1(\|u\|^2)h(\xi) + \int_K D_3f_2(\xi, \eta, u(\eta))h(\eta)d\mu(\eta).
\]

In particular, \( F \) has a continuous and bounded Fréchet derivative and \( D_F \) is Lipschitz continuous. \( F_1 \) is bounded. The (uniform) Lipschitz constant of \( F \) is given by \( \bar{L} \) and of \( D_F \) is given by \( L' \).

The proof of the lemma can be found in the appendix, see Section 5.

Let now \( \omega \) be a two sided one dimensional continuous standard Brownian motion. Defining the filtration \( (F_t)_{t \in \mathbb{R}} \) where

\[
F_t = \sigma(\omega(r), r \leq t)
\]

is the natural filtration of \( \omega \). Investigating the dynamics of (3) we do not work directly with this equation. We will consider the following random differential equation

\[
\frac{dv(t)}{dt} = \Delta \mu v(t) + z(\theta_t \omega) v(t) + e^{-z(\theta_t \omega)} F \left( v(t) e^{z(\theta_t \omega)} \right), \quad t > 0,
\]

\[
v(0) = v_0 \in H.
\]

Here \( (t, \omega) \mapsto z(\theta_t \omega) \) is a continuous stationary Ornstein Uhlenbeck process presented by the random variable

\[
z(\omega) = \int_{-\infty}^{0} e^{\theta r} \omega(r) dr
\]

for \( \omega \) in a \( \theta \)-invariant set of full measure and outside this set we define \( z(\omega) \) by 0, see Lu et al. [10].

**Lemma 12.** We have \( \mathbb{E}z = 0 \). In addition the random variables \( e^{z(\omega)} \), \( e^{-z(\omega)} \) are tempered.

Indeed we know that \( z \) is a centered Gaussian random variable such that \( t \mapsto z(\theta_t \omega) \) solves the equation

\[
dz + z dt = d\omega.
\]

We obtain by the Burkholder inequality

\[
\mathbb{E} \sup_{t \in [0,1]} |z(\theta_t \omega)| < \infty.
\]

Now we can apply Arnold [2] Proposition 4.1.3 to see that \( |z(\omega)| \) is a random variable having sublinear growth which gives the conclusion.

We look for mild solutions of (5)

\[
v(t) = S(t)e^{\int_0^t z(\theta_r \omega) dr} v_0 + \int_0^t S(t-r)e^{\int_0^r z(\theta_s \omega) ds} e^{-z(\theta_r \omega)} F \left( v(r) e^{z(\theta_r \omega)} \right) dr
\]

where the analytic semigroup \( S \) is given in Corollary 1. In particular we know that \( v \in C([0,T], H) \) for any \( T > 0 \).

For the following we set

\[
F(\omega, v) = e^{-z(\omega)} F \left( v e^{z(\omega)} \right).
\]

In particular, \( F \) is a Carathéodory mapping from \( \Omega \times H \to H \). From the uniform boundedness of the first derivative of \( F \) we know that \( F \) is Lipschitz continuous.
where the Lipschitz constant can be chosen independently of $\omega$. Hence we conclude the following result.

**Lemma 13.** (5) has a unique mild solution in $H$ which depends continuously on the initial condition. In particular, this mapping generates a continuous RDS on $H$. For every $t \geq 0$ and $x \in H$ the mapping

$$\omega \mapsto \varphi(t, \omega, x)$$

is $\mathcal{F}_t$ measurable.

The content of this theorem can be proven under weaker assumptions than the ones we suppose. In particular sometimes we can assume that the Lipschitz constant $L$ of the nonlinearity depends on $\omega$ such that $t \mapsto L(\theta_t \omega)$ and $r \mapsto \|F(\theta_r \omega, 0)\|$ is integrable on any compact interval. The same will be true for results in the following section.

### 4. Estimates of the Hausdorff dimension of the random attractor.

In the following we mention the existence of a random attractor of the RDS generated by (6) and later for the original system (3). Assuming that the Lipschitz constant $t \mapsto L(\theta_t \omega)$ is integrable over compact intervals we have a continuous RDS. For this RDS we can formulate:

**Lemma 14.** Assume we have a nonnegative random variable $l \in L^1(\Omega)$ and a nonnegative tempered random variable $a : \Omega \to \mathbb{R}^+$ such that

$$\|F(\omega, v)\| \leq l(\omega)\|v\| + a(\omega)$$

and suppose that $\lambda_1 + \mathbb{E}l < 0$. Then the RDS $\varphi$ generated by (5) has a unique random attractor.

**Proof.** We sketch the proof. In particular applying the Gronwall lemma we obtain for the norm of the mild solution to (6)

$$\|v(t)\| \leq e^{\lambda_1 t + \int_0^t l(\theta_q \omega) dq + \int_0^t a(\theta_q \omega) dq} \|v_0\| + \int_0^t e^{(\lambda_1 t - r) + \int_0^r l(\theta_q \omega) dq + \int_0^r a(\theta_q \omega) dq} a(\theta_r \omega) dr.$$

Comparing $\|v\|$ with the unique stationary point of

$$\frac{dw}{dt} = b(\theta_t \omega) w + a(\theta_t \omega), \quad w(0) = w_0.$$  

(7)

given by

$$w_s(\omega) = \int_{-\infty}^0 e^{\int_0^t b(\theta_q \omega) dq} a(\theta_q \omega) dq,$$

where we assume that $b(\omega) = \lambda_1 + l(\omega)$, $\mathbb{E}b < 0$ we obtain that the random ball with center 0 and radius $2w_s(\omega)$ gives us a pullback absorbing which is forward invariant and in $\mathcal{D}$. In particular $|w_s|$ is a tempered random variable. In addition

$$B(\omega) = \varphi(1, \theta_{-1} \omega, B_H(0, 2w_s(\theta_{-1} \omega)))^H$$

is a measurable pullback absorbing set which is in $\mathcal{D}$ by the forward invariance of $B_H(0, 2w_s(\theta_{-1} \omega))$. $B(\omega)$ is compact because for $\lambda_1 < 0$ the operator $-\Delta_\mu$ is positive such that $D((-\Delta_\mu)^\alpha)$ is well defined and compactly embedded in $H$. Now we obtain an estimate for

$$\sup_{v_0 \in B_H(0, 2w_s(\theta_{-1} \omega))} \|(-\Delta)^\alpha v(1)\| < \infty, \quad \alpha < 1.$$
by the analyticity of $S$, see Pazy [22] Theorem 2.6.13. In addition we have
\[
\left\| (-\Delta)^\alpha \int_0^1 S(1-r) e^{\int_0^r z(\theta_t, \omega) \, ds} e^{-i z(\theta_r, \omega)} \mathcal{F} \left( v(r) e^{i z(\theta_r, \omega)} \right) \right\|
\leq \int_0^1 \frac{c_\alpha}{(1-r)^\alpha} e^{-i z(\theta_r, \omega)} \| \mathcal{F}(v(r) e^{i z(\theta_r, \omega)}) \| \, dr
\]
where the right hand side is uniformly bounded for initial conditions $v_0 \in B_H(0, 2\omega_s(\theta_1 \omega))$.

Now we will discuss the upper bound on the Hausdorff dimension of the random attractor. We start by recalling the definition of the Hausdorff dimension.

**Definition 15.** Let $F$ be a compact set in the Banach space $H$. The $\epsilon$-approximate $d$-dimensional Hausdorff measure of $F$ is given by
\[
\mu_d^\epsilon(F, d, \epsilon) = \inf \left\{ \sum_i r_i^d : r_i \leq \epsilon, F \subset \bigcup_i B_H(x_i, r_i), x_i \in F \right\}
\]
where $\epsilon > 0$, $d \geq 0$ and $B(x_i, r_i)$ denotes an open or closed ball in $H$ with center $x_i$ and radius $r_i$. Then the $d$-dimensional Hausdorff measure of $F$ is given by
\[
\mu_H(F, d) = \inf_{\epsilon > 0} \mu_d^\epsilon(F, d, \epsilon).
\]
And finally we define the Hausdorff dimension of $F$ by
\[
\dim_H(F) = \inf \{ d : \mu_H(F, d) = 0 \}.
\]

As a first main result of this section we show the upper bound on the Hausdorff dimension of the random attractor discussed in Lemma 14. need the following conditions to hold for the result concerning the estimate of the Hausdorff dimension.

(C1) For $t > 0$ there exists a random variable $C_t$, such that $\varphi$ is uniformly differentiable on $\mathcal{A}(\omega)$ with Fréchet derivative $D\varphi(t, \omega, \cdot)$ for $t \in \mathbb{R}_+$ and $\omega \in \Omega$, i.e.
\[
\sup_{u, v \in \mathcal{A}(\omega), \|u - v\| \leq \epsilon} \frac{\| \varphi(t, \omega, u) - \varphi(t, \omega, v) - D\varphi(t, \omega, v)[u - v] \|}{\| u - v \|} \leq C_t(\omega) \epsilon.
\]
and additionally it holds
\[
\mathbb{E}(\log C_t) < \infty.
\]

(C2) For $t > 0$ there exists a random variable $C_{1t}(\omega) \geq 1$ such that,
\[
\sup_{u \in \mathcal{A}(\omega)} \| D\varphi(t, \omega, u) \|_{L(H)} \leq C_{1t}(\omega) \quad \text{and} \quad \mathbb{E}(\log C_{1t}) < \infty.
\]

(C3) For a suitable $m \in \mathbb{N}$ we have: for every $t > 0$ there exists a random variable $\bar{w}_m(t, \cdot)$ with
\[
\sup_{u \in \mathcal{A}(\omega)} w_m(D\varphi(t, \omega, u)) \leq \bar{w}_m(t, \omega)
\]
and there exists a time $t_0(\omega) > 0$ such that,
\[
\mathbb{E}(\log \bar{w}_m(t, \cdot)) < 0,
\]
for $t \geq t_0(\omega)$ and every $\omega \in \Omega$. 

\[
\square
\]
Remark 16. The original conditions by Debussche [9] are described in a discrete time setting. Since we are interested in the time continuous case, we adapted the conditions to our situation.

Following the explanation of Chapter 5 in the work of Temam [29] we know that the natural number that will be our upper bound for the Hausdorff dimension.

\[
\|D \varphi(t, \omega, u)\| = \sup_{\xi_1, \ldots, \xi_m \in H} \|D \varphi(t, \omega, u)\xi_1 \wedge \cdots \wedge D \varphi(t, \omega, u)\xi_m\|_{\Lambda^m} H
\]

where the last expression is an exterior product of \( m \) linearizations \( V_i = D \varphi(t, \omega, u)\xi_i \), \( i = 1, \ldots, m \) of the solution in (6) with \( m \) initial conditions.

We use the well known theorem presented entirely by Debussche in [9, Theorem 2.4].

Theorem 17. Let \( \varphi : \mathbb{R}_+ \times \Omega \times H \to H \) be a random dynamical system on a Hilbert space \( H \). Moreover assume that the RDS possesses a unique random attractor \( \mathcal{A} \) and the conditions (C1)–(C3) are satisfied. Then we conclude for all \( \omega \in \Omega \),

\[
\dim_H(\mathcal{A}(\omega)) \leq m,
\]

where \( m \in \mathbb{N} \) is the smallest number satisfying (C3).

We proof the conditions (C1) – (C3) for the equation (5) and derive a rule for the natural number that will be our upper bound for the Hausdorff dimension.

We start with the conditions (C1) and (C2): Let \( V \) be the mild solution of the operator differential equation given by

\[
\frac{dV}{dt} = \Delta_V V + z(\theta_t \omega)V + DF(v(t)e^{z(\theta_t \omega)})V
\]

\[
V(0) = \xi \in H
\]

given by

\[
V(t) = S(t)e^{\int_0^t z(\theta_r \omega)dr} \xi + \int_0^t S(t-r)e^{\int_r^t z(\theta_r \omega)dr}DF(v(r)e^{z(\theta_r \omega)})V(r)dr.
\]

Taking the operator norm of \( V \) we have by application of the Gronwall Lemma

\[
\log \|V(t)\|_{L(H)} \leq \lambda_1 t + \int_0^t z(\theta_r \omega)dr + \bar{L} t
\]

having in mind that \( DF \) is bounded by \( \bar{L} \), see Lemma 11. Similar, we can derive that

\[
\|\varphi(t, \omega, v_0 + h) - \varphi(t, \omega, v_0)\|
\]

is bounded by

\[
\|h\| e^{\lambda_1 t + \int_0^t z(\theta_r \omega)dr + \bar{L} t}.
\]

Now we prove that \( V \) is the derivative of the RDS \( \varphi \). For this we consider

\[
\Xi(t) := \varphi(t, \omega, v_0 + h) - \varphi(t, \omega, v_0) - D \varphi(t, \omega, v_0)h = v_h(t) - v(t) - V(t)h
\]

such that

\[
\|\Xi(t)\| \leq \int_0^t e^{\lambda_1(t-r)} e^{\int_r^t z(\theta_r \omega)dr} \times e^{-z(\theta_r \omega)} \|F(v_h(r)e^{z(\theta_r \omega)}) - F(v(r)e^{z(\theta_r \omega)}) - D \varphi(e^{z(\theta_r \omega)}v(r))(e^{z(\theta_r \omega)} \times (v_h(r) - v(r)))\|dr.
\]
Following Lemma 11 the term giving the norm can be estimated by
\[ C e^{2z(\theta, \omega)} \| v_h(r) - v(r) \|^2 \]
and hence by (9)
\[ \| \Xi(t) \| \leq C \| h \|^2 e^{3 \int_0^t |z(\theta, \omega)| \, dq} \left( \int_0^t e^{z(\theta, \omega) + 2Lr} \, dr \right) \]
which proves the uniform differentiability of \( \varphi(t, \omega, \cdot) \) on \( H \). Condition (C2) follows from (8).

Following the classical approach in Temam [29] page 362ff. we can estimate \( w_m(D\varphi(t, \omega, v_0)) \) by
\[ \sup_{\xi_1, \ldots, \xi_m \in \cal{H} \atop \|\xi\| \leq 1, \forall i} \exp \left( \int_0^t \text{Tr}(\Delta_{\mu} + z(\theta, \omega)\text{id} + D\varphi(\theta, \omega, v_0)) \circ Q_m(r) \, dr \right) \]
where \( Q_m(r) := Q_m(r, \omega; \xi_1, \ldots, \xi_m) \) is the orthogonal projection mapping \( H \) to \( \text{span}[D\varphi(r, \omega, v_0)\xi_1, \ldots, D\varphi(r, \omega, v_0)\xi_m] \). Estimating this expression uniformly for \( v_0 \in \cal{A}(\omega) \) we have
\[ \bar{w}_m(t, \omega) = \exp \left( \int_0^t \sup_{v_0 \in \cal{A}(\omega)} \sup_{Q_m} \text{Tr}(\Delta_{\mu} + z(\omega)\text{id} + D\varphi(\omega, v_0)) \circ Q_m) \, dr \right) \]
In particular to obtain (C3) it is sufficient to find an \( m \in \mathbb{N} \) (as small as possible) such that
\[ \mathbb{E} \left( \sup_{v_0 \in \cal{A}(\omega)} \sup_{Q_m} \text{Tr}(\Delta_{\mu} + z(\omega)\text{id} + D\varphi(\omega, v_0)) \circ Q_m) \right) \]
where the second supremum in the above expressions is taken with respect to all orthoprojections onto \( m \) dimensional subspaces. We can estimate the latter expectation by a similar discussion as in Temam [29, p. 387 ff.],
\[ \mathbb{E} \left( \sum_{i=1}^m \lambda_i + m(\bar{L} + z(\omega)) \right) = \sum_{i=1}^m \lambda_i + m\bar{L}. \quad (10) \]
Now applying Birkhoff’s ergodic theorem
\[ \lim_{t \to \pm \infty} \frac{1}{t} \int_0^t H_m(\theta, \omega) \, dr \leq \sum_{i=1}^m \lambda_i + m\bar{L} \]
\[ H_m(\omega) = \sup_{v_0 \in \cal{A}(\omega)} \sup_{Q_m} \text{Tr}(\Delta_{\mu} + z(\omega)\text{id} + D\varphi(\omega, v_0)) \circ Q_m) \]
we can find an \( m \in \mathbb{N} \) (as small as possible) such that (C3) is fulfilled which gives the conclusion of Theorem 17. Using Corollary 2 we can see in a more direct way the dependence on the fractal properties,
\[ \sum_{i=1}^m \lambda_i \leq \frac{C_S}{1 + 2/d_S} m^{1+2/d_S}. \]
Therefore the bound on the Hausdorff dimension depends on the spectral exponent \( d_S \) which was introduced in Section 3.

Let us also note that the Hausdorff dimension of \( \cal{A}(\omega) \) for an ergodic noise is almost surely independent of \( \omega \), see for instance Schmalfuss [24].
We now construct a local Lipschitz random manifold. To ensure that the local manifold \( M^\nu \) will be unstable for exponential decay \( \nu > 0 \) we will assume that the spectrum of \( A \) contains a positive subset. In particular, we will consider an operator

\[
A = \Delta_\mu + K \mathbb{I}_H
\]

with \( K > 0 \) sufficiently large. Then we assume for the spectrum of \( A \)

\[
\lambda_1 + K \geq \lambda_2 + K \geq \cdots \geq \lambda_N + K > 0 > \lambda_{N+1} + K \geq \lambda_{N+2} + K \cdots \rightarrow -\infty
\]

where \((\lambda_i)_{i \in \mathbb{N}}\) denotes the spectrum of \( \Delta_\mu \). We consider the RDS generated by the mild solution to

\[
\frac{dv}{dt} = Av + z(\theta_t \omega)v + Q(\theta_t \omega, v), \quad v(0) = v_0 \in H.
\]

Since \( \Delta_\mu \) generates an analytic semigroup so does \( A \). The generated semigroup is denoted by \( S_A \). The main issue of this part is to proof that \( N \) is a lower bound for the Hausdorff dimension of the random attractor of the RDS generated by (5). Let us consider the splitting

\[
H = H^+ \oplus H^-
\]

where \( H^+ \) is the span of the eigenvectors \( e_i \) of \( A \) related to positive eigenvalues. \( H^- \) is the closure of the span of the other eigenvectors \( e_i \). Recall that \((e_i)_{i \in \mathbb{N}}\) is a complete orthonormal system. Then the projections \( P^+, P^- \) on these spaces are orthoprojections which have operator norm one. The semigroup \( S_A \) commutes with \( P^\pm \):

\[
P^\pm S_A(t) = S_A(t)P^\pm = : S^\pm_A(t).
\]

We consider two numbers \( \hat{\lambda} \) and \( \lambda \) such that

\[
\lambda_N + K > \hat{\lambda} > 0 > \lambda \geq \lambda_{N+1} + K.
\]

In particular, we have

\[
\|S^+_A(t)\|_{L(H)} \leq e^{\lambda t} \quad \text{for } t \leq 0, \quad \|S^-_A(t)\|_{L(H)} \leq e^{\hat{\lambda} t} \quad \text{for } t \geq 0.
\]

We consider here a nonlinear \( Q \) fulfilling particular conditions. We assume that \( Q \) is Lipschitz continuous with a Lipschitz constant \( L \) depending on \( \omega \), but is as well uniformly bounded by \( \hat{L} \). Assuming that zero is a stationary point for \( q \) which follows by \( Q(\omega, 0) = 0 \) and that \( Q \) is supposed to be bounded with respect to both arguments. We set \( Q^\pm = P^\pm Q \).

**Definition 18.** We introduce the Banach space

\[
(\mathcal{H}, \|\cdot\|), \quad \mathcal{H} = \{ y \in C((-\infty, 0], H), \|y\| = \sup_{t \in \mathbb{R}^-} e^{-\nu t} e^{-\int_0^t z(\theta_r \omega)dr} \|y(t)\| < \infty \},
\]

where \( \nu = (\hat{\lambda} + \lambda)/2 \). For fixed \( \omega \in \Omega \), \( \nu_0^+ \in H^+ \) we define the mapping

\[
\mathcal{T}_{\nu_0^+ \omega}(y)[t] = S^+_A(t)e^{\int_0^t z(\theta_r \omega)dr}y_0^+ - \int_0^t S^+_A(t-r)e^{\int_r^t z(\theta_q \omega)dq}Q^+(\theta_r \omega, y(r))dr
\]

\[
+ \int_{-\infty}^t S^-_A(t-r)e^{\int_r^t z(\theta_q \omega)dq}Q^-(\theta_r \omega, y(r))dr, \quad t \leq 0, \quad y \in \mathcal{H}
\]

which is called Lyapunov-Perron transformation.

We now collect some properties of the Lyapunov-Perron transform:
Lemma 19. Under the above conditions we have that
\[ \mathcal{T}_{v_0^+} : \mathcal{H} \rightarrow \mathcal{H}. \]

In addition for every \( 0 < \nu < \hat{\lambda} \) such that the gap condition holds
\[ k := \frac{L}{\hat{\lambda} - \nu} + \frac{L}{\nu - \hat{\lambda}} < 1. \]  \hfill (14)

Then the Lyapunov-Perron transform is a contraction on \( \mathcal{H} \)
\[ \left\| \mathcal{T}_{v_0^+} \omega (y_1) - \mathcal{T}_{v_0^+} \omega (y_2) \right\| \leq k \left\| y_1 - y_2 \right\|. \]

The unique fixed point in \( \mathcal{H} \) is denoted by \( \Gamma(\omega, v_0^+) \). The mapping
\( (\omega, v_0^+ ) \rightarrow \Gamma(\omega, v_0^+) \in \mathcal{H} \)
is a Carathéodory mapping. In particular, this mapping is Lipschitz continuous with respect to the second argument.

Proof. We can estimate
\[ \sup_{t \leq 0} e^{-\nu t} e^{- \int_0^t z(\theta, \omega) d\theta} \left| \begin{aligned} &- \int_t^0 S_\lambda^+ (t - r) e^{\int_0^r z(\theta, \omega) d\theta} (Q^+ (\theta, \omega, y_1 (r)) - Q^+ (\theta, \omega, y_2 (r))) d\theta \\ &+ \int_{-\infty}^t S_\lambda^+ (t - r) e^{\int_0^r z(\theta, \omega) d\theta} (Q^- (\theta, \omega, y_1 (r)) - Q^- (\theta, \omega, y_2 (r))) d\theta \end{aligned} \right| \]
\[ \leq \sup_{t \leq 0} \left| \begin{aligned} &\sup_{t \leq 0} \left( \int_t^0 e^{-(\nu - \hat{\lambda}) t} e^{- \int_0^r \lambda(\theta, \omega) d\theta} \left\| y_1 (r) - y_2 (r) \right\| d\theta \\ &+ \int_{-\infty}^t e^{-(\nu - \hat{\lambda}) t} e^{- \int_0^r \lambda(\theta, \omega) d\theta} \left\| y_1 (r) - y_2 (r) \right\| d\theta \right) \right| \right. \]
\[ \leq \hat{L} \left( \sup_{t \leq 0} \int_t^0 e^{-(\lambda - \nu) t} d\theta + \sup_{t \leq 0} \int_{-\infty}^t e^{-(\lambda - \nu) t} d\theta \right). \]

Estimating the integrals yields the existence of a fixed point. Indeed the bracket can be estimated by \( 4/(\hat{\lambda} - \lambda) \) which is bounded by 1 taking the gap condition (14) into account.

We prove that the mapping
\[ H^+ \ni v_0^+ \mapsto \Gamma(\omega, v_0^+) \in \mathcal{H} \]
is Lipschitz continuous. We have
\[ \left\| \Gamma(\omega, v_0^+) - \Gamma(\omega, v_0^+) \right\| = \left\| \mathcal{T}_{v_0^+} (\Gamma(\omega, v_0^+)) - \mathcal{T}_{v_0^+} (\Gamma(\omega, v_0^+)) \right\| \]
\[ \leq \left\| \mathcal{T}_{v_0^+} (\Gamma(\omega, v_0^+)) - \mathcal{T}_{v_0^+} (\Gamma(\omega, v_0^+)) \right\| + \left\| \mathcal{T}_{v_0^+} (\Gamma(\omega, v_0^+)) - \mathcal{T}_{v_0^+} (\Gamma(\omega, v_0^+)) \right\| \]
\[ \leq \sup_{t \leq 0} e^{-\nu t} \left\| S_\lambda^+ (t) \right\|_{L(\mathcal{H})} \left\| v_0^+ - v_0^+ \right\| + k \left\| \Gamma(\omega, v_0^+) - \Gamma(\omega, v_0^+) \right\|. \]

For the linear part we can notice
\[ e^{-\nu t} \left\| S_\lambda^+ (t) \right\|_{L(\mathcal{H})} \leq e^{(-\nu + \hat{\lambda}) t} \leq 1 \quad \text{for} \ t \in (-\infty, 0]. \]
such that $S_A^+(\cdot)v_0^+ \in \mathcal{H}$. From the above chain of inequalities we obtain
\[
\|\Gamma(\omega, v_{01}^+) - \Gamma(\omega, v_{02}^+)\| \leq LM\|v_{01}^+ - v_{02}^+\|, \quad L_M = \frac{1}{1-k}.
\]

**Remark 20.** In other publications often the parameter $\nu$ is chosen by
\[
(\lambda_N + \lambda_{N+1} + 2K)/2.
\]
By this choice we obtain the optimal gap condition which means that we can choose the largest $\hat{L}$ such that the gap condition (i.e. the contraction condition) for the Lyapunov-Perron transform is fulfilled. However, our intention is different. We would like to obtain a local unstable random manifold of dimension $N$ which is in general not the case with the above choice of $\nu$. In particular, when $\lambda_N + \lambda_{N+1} + 2K$ is negative the manifold over $H^+$ is not an unstable manifold. In addition, we would like to have a random unstable manifold with a decay rate which is close to the decay rate of the linear system on $H^+$ given by $\lambda_N + K$. To obtain this we choose $0 < \nu < \hat{\nu} = \lambda_N + K$, $\nu$ close to $\hat{\nu}$ and $\hat{\nu} < 0$ but close to zero. Then for sufficiently small $\hat{L}$ the gap condition is still satisfied.

We are now in a position to formulate

**Lemma 21.** The RDS $\varphi$ generated by (13) has an unstable random manifold $M^\nu$ given by a Lipschitz continuous graph
\[
(x^+, P^-\Gamma(\omega, x^+)[0]), \quad x^+ \in H^+.
\]
of decay order $\nu$. This manifold is forward invariant and
\[
x_t(\omega) = \begin{cases} 
\Gamma(\omega, x^+(\omega))[t] & : \ t \leq 0 \\
\varphi(t, \omega, \Gamma(\omega, x^+(\omega))[0]) & : \ t \geq 0
\end{cases}
\]
is a complete trajectory for $\varphi$ with $x_0(\omega) \in M(\omega)$. This manifold has dimension $N$.

We show the invariance of $M^\nu$ by the following property
\[
\varphi(t, \omega, \Gamma(\omega, x^+(\omega))[0]) = \Gamma(\theta_t \omega, P^+ \varphi(t, \omega, \Gamma(\omega, x^+(\omega))[0]))[0], \quad t \geq 0.
\]
For the proof we refer to Lu and Schmalfuss [21]. In a similar manner we can show that $\Gamma(\omega, x^+)$ defines a complete trajectory. We check that
\[
\varphi(s, \theta_t \omega, \Gamma(\omega, x^+(\omega))[t]) = \Gamma(\omega, x^+(\omega))[t+s] \quad \text{for } s \geq 0, \; t + s \leq 0.
\]
Since we are looking for mild solutions we have
\[
S_A(s)e^{\int_0^s }z(\theta_{\omega+t} \omega) dq\left( S_A^+(t)e^{\int_0^t }z(\theta_\omega dq)x^+(\omega)
\right.
\]
\[
- \int_0^t S_A^+(t-r)e^{\int_r^t }z(\theta_\omega dq)Q^+(\theta_r \omega, \Gamma(\omega, x^+(\omega))[r])dr
\]
\[
+ \int_t^{2t} S_A(t+s-r)e^{\int_r^{2t} }z(\theta_\omega dq)Q^-(\theta_r \omega, \Gamma(\omega, x^+(\omega))[r])dr
\]
\[
+ \int_0^s S_A(s-r)e^{\int_r^s }z(\theta_{\omega+t} \omega) dq(Q(\theta_{r+t} \omega, \Gamma(\omega, x^+(\omega))[r+t])dr.
\]
Let us consider the $H^+$ component. We obtain by a linear integral substitution
\[
S^+(s+t) e^{\int_0^s z(\theta_q \omega) dq} - \int_0^t S^+(t+s-r) e^{\int_t^s z(\theta_q \omega) dq} Q^+(\theta_r \omega, \Gamma(\omega, x^+(\omega))[r]) dr \\
+ \int_t^{s+t} S_A(t+s-r) e^{\int_t^s z(\theta_q \omega) dq} Q^+(\theta_r \omega, \Gamma(\omega, x^+(\omega))[r]) dr \\
=P^+ \Gamma(\omega, x^+(\omega))[s+t]
\]
and similar for the $H^-$ component.

We would like to apply now the results of Section 2 to prove the existence of a local unstable manifold for the RDS generated by (5). For this we assume that $F(0) = 0$ and the generating function of $F$ satisfy the assumption of Lemma 11. In particular we assume that
\[
D F(0) = K \text{id}_H. \tag{15}
\]
Having our example for $F$ in mind formulated in Lemma 11 we assume that $f_1(0) = K > -\lambda_1$, $D f_1(0) = 0$, $D^3 f_2(\xi, \eta, 0) = 0$.

By Lemma 11 $D F$ is Lipschitz continuous where the Lipschitz constant is denoted by $L'$. According to the linear growth condition of Lemma 14 we note that if
\[
\int_K \int_{v \in \mathbb{R}} |D^3 f_2(\xi, \eta, v)|^2 d\mu(\eta) d\mu(\xi)
\]
is sufficiently small we obtain the existence of a random attractor. Indeed we have the linear growth by (16). Note that $f_1$ has a compact support such that $F_1$ is a bounded operator.

We then have
\[
e^{-z(\omega) F(e^{z(\omega) u}) = F(\omega, u) - DF(\omega, 0) u + DF(\omega, 0) u} \\
= F(\omega, u) - K u + K u =: G(\omega, u) + K u
\]
such that $DG(\omega, 0) = 0$. Let us consider the following function
\[
\sigma : \mathbb{R}^+ \to [0, 1], \quad \sigma(s) = \begin{cases} 
1 & : s \leq 1, \\
2 - s & : 1 < s < 2, \\
0 & : s \geq 2.
\end{cases}
\]
This function is Lipschitz continuous with Lipschitz constant 2. For a positive random variable $\rho : \Omega \to \mathbb{R}^+$ which is tempered from below we set
\[
G_\rho(\omega, u) = G\left(\omega, u \cdot \sigma\left(\frac{\|u\|}{\rho(\omega)}\right)\right).
\]

For the properties of this nonlinear mapping we notice

**Lemma 22.** $G_\rho$ is a Carathéodory function which is Lipschitz continuous
\[
\|G_\rho(\omega, u_1) - G_\rho(\omega, u_2)\| \leq L_G(\omega)\|u_1 - u_2\| \quad \text{for all } u_1, u_2 \in H, \omega \in \Omega.
\]
where
\[
L_G(\omega) = 9 L' e^{z(\omega) \rho(\omega)}.
\]
In particular we have
\[
\|G_\rho(\omega, u)\| \leq 18 L' e^{z(\omega) \rho(\omega)} \quad \text{for } u \in H, \omega \in \Omega.
\]
The proof of this lemma is given in the Appendix. Let us assume that we have a $K$ such that there exists an $N$, $\lambda$, $\lambda$ as given in Remark 20. Finally we set
\[
\rho(\omega) = \frac{e^{-z(\omega)}}{9L^L}.\]
With this setting $G_\rho(\omega, u)$ is uniformly bounded in $(\omega, u)$.

For the following we need the Lemma.

**Lemma 23.** Let $\rho$ be a random variable tempered from below and $\nu$ be a positive number. Then there exists a positive random variable $R$ such that
\[
\inf_{t \leq 0} \rho(\theta_t \omega)e^{-\nu t} = \inf_{t \leq 0, t \in \mathbb{Q}} \rho(\theta_t \omega)e^{-\nu t} =: R(\omega).
\]
Indeed, since $t \mapsto z(\theta_t \omega)$ is continuous so is $t \mapsto \rho(\theta_t \omega)$. The random variable $e^{-z}$ is tempered from below, see Lemma 12.

We then have the following theorem.

**Theorem 24.** Consider the RDS $\varphi$ generated by (6). Assume that the conditions for $\hat{\lambda}$, $\check{\lambda}$ from Remark 20 be satisfied. In addition $G_\rho(\omega, v) =: Q(\omega, v)$ has the properties stated in Lemma 22. Then there exists a local Lipschitz unstable random manifold of dimension $N$. In particular, we have
\[
\dim_H(A(\omega)) \geq \dim H^+ = N \quad \text{for all } \omega \in \Omega
\]
where $\dim_H$ describes the dimension of the finite dimensional vector space $H^+$.

**Proof.** We can conclude that the RDS generated by (13) has a global Lipschitz unstable random manifold of dimension $N$. Let $L_M$ be the Lipschitz constant of the manifold $M^\nu(\omega)$ and define
\[
R_1(\omega) = \frac{R(\omega)}{\sqrt{1 + \frac{L_M}{L}}},
\]
By the Lipschitz continuity of $\Gamma(\omega, x^+)[0]$ we have that if $\|x^+(\omega)\| \leq R_1(\omega)$ then
\[
\|\Gamma(\omega, x^+)[0]\| = \sqrt{\|x^+(\omega)\|^2 + \|P^\nu \Gamma(\omega, x^+(\omega)))[0]\|^2} \leq R(\omega)
\]
where $R$ is given in Lemma 23 and we can set $V(\omega) = B_H(0, R(\omega))$, see Section 2. Hence we have for the complete trajectory $t \mapsto \Gamma(\omega, x^+(\omega))[t]$ of the cut-off system (13) that
\[
\|\Gamma(\omega, x^+(\omega))[t]\| \leq \rho(\theta_t \omega) \quad \text{for } t \leq 0
\]
which means that this trajectory is complete for the RDS given by the uncutted equation (6). Indeed we have $Q(\omega, u) = G(\omega, u)$ for $u \in U(\omega) = B_H(0, \rho(\omega))$. \qed

We note the following consequences.

**Remark 25.** (i) Transforming the random attractor $A$ of $\varphi$ to the random attractor $A_\psi$ of the random dynamical system $\psi$ generated by (3) then $A_\psi(\omega)$ has the same Hausdorff dimension (and hence the same bounds) as $A(\omega)$. Indeed we have
\[
A_\psi(\omega) = e^{z(\omega)}A(\omega)
\]
and using the result of Falconer [12, Corollary 2.4] we obtain that the Hausdorff dimension is invariant under the transformation.

(ii) Suppose that $K > 0$ is large and choose $N$ as large as possible such that
\[
K > -c_3 N \frac{2}{d_0} \quad \text{or} \quad N < \left( \frac{K}{-c_3} \right)^{\frac{d_0}{2}}.
\]
Then \(-c_sN \frac{d}{dt} \geq -\lambda_N\), \((c_s < 0)\). Hence the random attractor has a dimension which is larger than or equal to \(N\).

(iii) To check our calculations we note that the value \(N\) of the lower estimate of the Hausdorff dimension of \(A(\omega)\) is smaller than the value \(m\) of the upper estimate. For this purpose suppose that \(N \geq m\). Recall that there exists \(N \in \mathbb{N}\) such that

\[
\lambda_1 + K \geq \lambda_2 + K \geq \cdots \geq \lambda_N + K > 0 > \lambda_{N+1} + K \geq \ldots
\]

where \(K = \|DF(0)\|_{L(H)} > -\lambda_1\). We know by the method for the upper bound and in particular condition (C3) that \(E(\log \bar{w}_m(t, \cdot)) < 0\) for \(t\) large enough, hence

\[
\sum_{i=1}^{m} \lambda_i + m\bar{L} < 0.
\]

The latter left-hand size can be rewritten to

\[
\sum_{i=1}^{m} (\lambda_i + K) + m(\bar{L} - K).
\]

Every summand of the first sum is positive due to the splitting of the spectrum and the assumption. Moreover, we know by Lemma 11 that \(K \leq \sup_{v \in H} \|DF(v)\|_{L(H)} \leq \bar{L}\) and therefore the second summand in the last line is also positive, which leads to a contradiction.

5. Appendix. Proof of Lemma 11

Proof. Let \(u, h, h_1 \in H\). First we consider \(\mathcal{F}_1\). We have

\[
DF_1(u)h = 2Df_1(\|u\|^2)(u, h)u + f_1(||u||^2)h,
\]

\[
D^2F_1(u)(h, h_1) = 4D^2f_1(\|u\|^2)(u, h)(u, h_1)u + 2Df_1(\|u\|^2)(h, h_1)u + 2Df_1(\|u\|^2)(u, h)h + 2Df_1(\|u\|^2)(u, h_1)h.
\]

Since \(f_1\) has a compact support \(\mathcal{F}_1\) is uniformly differentiable and \(DF_1\) is Lipschitz continuous.

Now we consider \(\mathcal{F}_2\). By the Taylor expansion we have

\[
f_2(\xi, \eta, u(\eta)) = f_2(\xi, \eta, 0) + D_3f_2(\xi, \eta, \vartheta_{u(\eta)}u(\eta))u(\eta)
\]

where \(\vartheta_{u(\eta)} \in (0, 1)\). In particular we have

\[
\int_K \left( \int_K f_2(\xi, \eta, 0)d\mu(\eta) \right)^2 d\mu(\xi) \leq \mu(K) \int_K \int_K f_2(\xi, \eta, 0)^2 d\mu(\eta)d\mu(\xi)
\]

\[
\int_K \left( \int_K D_3f_2(\xi, \eta, \vartheta_{u(\eta)}u(\eta))u(\eta)d\mu(\eta) \right)^2 d\mu(\xi)
\]

\[
\leq \int_K \int_{K_v \in \mathcal{K}} \sup_{v \in \mathcal{V}} |D_3f_2(\xi, \eta, v)|^2 d\mu(\eta)d\mu(\xi) ||u||^2
\]

which shows that \(\mathcal{F}_2(u)\) is well defined as an element in \(H\). We prove that

\[
\int_K D_3f_2(\xi, \eta, u(\eta))h(\eta)d\mu = DF_2(u)[\xi].
\]
Considering
\[
\int_{\mathcal{K}} \left( \int_{\mathcal{K}} \left( f_2(\xi, \eta, u(\eta) + h(\eta)) - f_2(\xi, \eta, u(\eta)) - D_3 f_2(\xi, \eta, u(\eta)) h(\eta) d\mu(\eta) \right) \right)^2 d\mu(\xi)
\]
\[
= \int_{\mathcal{K}} \left( \int_{\mathcal{K}} \frac{1}{2} D_3^2 f_2(\xi, \eta, u(\eta) + \partial u(\eta), h(\eta)) h(\eta)^2 d\mu(\eta) \right)^2 d\mu(\xi)
\]
\[
\leq \frac{1}{2} \int_{\mathcal{K}} \max_{\eta \in \mathcal{K}, v \in \mathbb{R}} |D_3^2 f_2(\xi, \eta, v)|^2 d\mu(\xi) |h|^4
\]
which also proves the uniform Fréchet differentiability of \( \mathcal{F}_2 \). To see that \( D\mathcal{F}_2 \) is Lipschitz continuous, we note that
\[
\| D\mathcal{F}_2(u_1) h - D\mathcal{F}_2(u_2) h \|^2
\]
\[
\leq \int_{\mathcal{K}} \left( \int_{\mathcal{K}} (D_3 f_2(\xi, \eta, u_1(\eta)) - D_3 f_2(\xi, \eta, u_2(\eta))) h(\eta) d\mu(\eta) \right)^2 d\mu(\xi)
\]
\[
\leq \int_{\mathcal{K}} \left( \int_{\mathcal{K}} D_3^2 f_2(\xi, \eta, u_1(\eta) + \partial u(\eta), u_2(\eta) - u_1(\eta)) h(\eta)^2 d\mu(\eta) \right)^2 d\mu(\xi)
\]
\[
\leq \int_{\mathcal{K}} \max_{\eta \in \mathcal{K}, v \in \mathbb{R}} |D_3^2 f_2(\xi, \eta, v)|^2 d\mu(\xi) \| u_2 - u_1 \|^2 \| h \|^2
\]
with the Lipschitz constant
\[
L' = \left( \int_{\mathcal{K}} \max_{\eta \in \mathcal{K}, v \in \mathbb{R}} |D_3^2 f_2(\xi, \eta, v)|^2 d\mu(\xi) \right)^{\frac{1}{2}}.
\]
\[
\square
\]
**Proof of Lemma 22**

Proof. We write \( \rho \) instead of \( \rho(\omega) \) since we only make \( \omega \)-wise estimates. We remark a priori
\[
\sigma \left( \frac{|u|}{\rho} \right) = \begin{cases} 1, & |u| \leq \rho \\ 2 - \frac{|u|}{\rho}, & \rho < |u| < 2\rho \\ 0, & |u| \geq 2\rho \end{cases}
\]
and let us introduce the following abbreviation, \( \tilde{u} := u \cdot \sigma \left( \frac{|u|}{\rho} \right) \). Therefore for every \( u_1, u_2 \in H, \omega \in \Omega \),
\[
\| \tilde{u}_1 - \tilde{u}_2 \| = \left\| u_1 \sigma \left( \frac{|u_1|}{\rho} \right) - u_2 \sigma \left( \frac{|u_2|}{\rho} \right) \right\| \leq L_\sigma \| u_1 - u_2 \| \tag{17}
\]
with \( L_\sigma = 3. \) To see this, we verify the cases
\[
(1^*) \; \| u_1 \| \in (\rho, 2\rho), \; \| u_2 \| \in (\rho, 2\rho) \quad \text{and}
\]
\[
(2^*) \; \| u_1 \| \in (0, \rho), \; \| u_2 \| \in (\rho, 2\rho) \quad .
\]
The other cases follow by similar arguments. Suppose we are in the case \((1^*)\) then we obtain,
\[
\| \tilde{u}_1 - \tilde{u}_2 \| = \left\| u_1 \left( 2 - \frac{|u_1|}{\rho} \right) - u_2 \left( 2 - \frac{|u_2|}{\rho} \right) \right\|
\]
\[\leq \left\| u_1 - u_2 \right\| \cdot \frac{\left\| u_1 \right\|}{\rho} + \left( 2 - \frac{\left\| u_2 \right\|}{\rho} \right) \left\| u_1 - u_2 \right\| \]
\[\leq \left\| u_1 - u_2 \right\| \cdot \frac{\left\| u_1 \right\|}{\rho} + \left\| u_1 - u_2 \right\| \leq 3\left\| u_1 - u_2 \right\|.\] (18)

where inserted the terms \( \pm u_1 \left( 2 - \frac{\left\| u_2 \right\|}{\rho} \right) \) in a first step. For \( (2^*) \) we consider the following convex combination \( u_3 := (1 - \tau)u_1 + \tau u_2, \tau \in [0, 1] \) and choose \( \tau \) such that \( \left\| u_3 \right\| = \rho \) then
\[\| \tilde{u}_1 - \tilde{u}_2 \| = \left\| u_1 - u_3 + u_3 - u_2 \left( 2 - \frac{\left\| u_2 \right\|}{\rho} \right) \right\| \leq \left\| u_1 - u_3 \right\| + \left\| u_3 \left( 2 - \frac{\left\| u_3 \right\|}{\rho} \right) - u_2 \left( 2 - \frac{\left\| u_2 \right\|}{\rho} \right) \right\|.\] (19)

The second term in the last inequality of (19) can be exactly estimated as (18). Respecting \( \left\| u_3 \right\| = \rho \) we obtain that this second term is in fact smaller or equal to \( 2\left\| u_3 - u_2 \right\| \). Hence we have with \( \tau \in [0, 1] \)
\[\| \tilde{u}_1 - \tilde{u}_2 \| \leq \| u_1 - u_3 \| + 2\| u_3 - u_2 \| \leq \left( |\tau| + 2|1 - \tau| \right) \| u_1 - u_2 \| \leq 2\| u_1 - u_2 \|.\]

To obtain the aimed estimate we use the mean value theorem, see e.g. Wouk [32] Section 12.1, Corollary 3, p.266,
\[\| G_\rho(\omega, u_1) - G_\rho(\omega, u_2) \| = \| G(\omega, \tilde{u}_1) - G(\omega, \tilde{u}_2) \| \leq \sup_{\tau \in (0, 1)} \| DG(\omega, \tilde{u}_1 + \tau [\tilde{u}_1 - \tilde{u}_2]) \|_{L(H)} \cdot \| \tilde{u}_1 - \tilde{u}_2 \|.\]

Now we observe by the assumption (15) and the definition of \( G \),
\[DG(\omega, x) = DF(\omega, x) - K \text{id}_H = DF(\omega, x) - DF(0) = DF(xe^{z(\omega)}) - DF(0)\]
for every \( x \in H \) and \( \omega \in \Omega \). Using this and the Lipschitz continuity of \( DF \) we see that
\[\| G_\rho(\omega, u_1) - G_\rho(\omega, u_2) \| \leq \sup_{\tau \in (0, 1)} \| DG(\omega, \tilde{u}_1 + \tau [\tilde{u}_1 - \tilde{u}_2]) \|_{L(H)} \cdot \| \tilde{u}_1 - \tilde{u}_2 \| \leq L'e^{z(\omega)} \sup_{\tau \in (0, 1)} \| \tilde{u}_1 + \tau [\tilde{u}_1 - \tilde{u}_2] \| \cdot \| \tilde{u}_1 - \tilde{u}_2 \| .\]

We use the Lipschitz continuity in (17) again to observe
\[\| G_\rho(\omega, u_1) - G_\rho(\omega, u_2) \| \leq L'L_e^{z(\omega)} \sup_{\tau \in (0, 1)} \| \tilde{u}_1 + \tau [\tilde{u}_1 - \tilde{u}_2] \| \cdot \| u_1 - u_2 \| \leq 3L'L_e^{z(\omega)} \rho \| u_1 - u_2 \| = 9L'e^{z(\omega)} \rho \| u_1 - u_2 \| ,\]
where the estimate for the supremum is obtained from \( \| \tilde{u} \| \leq \rho \), for every \( u \in H \).

As a consequence of \( G_\rho(\omega, 0) = 0 \) we have for \( \| u \| \leq 2\rho \)
\[\| G_\rho(\omega, u) \| \leq 9L'e^{z(\omega)} \rho \| u \| \leq 18L'e^{z(\omega)} \rho^2 .\]

In the case \( \| u \| > 2\rho \) it is clear that \( \| G_\rho(\omega, u) \| = \| G(\omega, 0) \| = 0. \)  \( \square \)
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