SPATIAL REGULARITY FOR A CLASS OF DEGENERATE KOLMOGOROV EQUATIONS

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Abstract. We establish spatial a priori estimates for the solution $u$ to a class of dilation invariant Kolmogorov equation, where $u$ is assumed to only have a certain amount of regularity in the diffusion’s directions, i.e. $x_1, \ldots, x_{m_0}$. The result is that $u$ is also regular with respect to the remaining directions. The approach we propose is based on the commutators identities and allows us to obtain a Sobolev exponent that does not depend on the integrability assumption of the right-hand side. Lastly, we provide an alternative proof to that of Theorem 1.5 of [9] for the optimal spatial regularity.

Key words: Kolmogorov equation, ultraparabolic, Sobolev spaces, hypoelliptic, regularity, a priori estimates

AMS subject classifications: 35K70, 35B45, 35Q84

1. Introduction

The aim of this work is to prove spatial a priori estimates for solutions to a class of strongly degenerate ultraparabolic equations

$$\mathcal{L}u(x, t) = Yu(x, t) - \sigma \Delta_0 u$$

$$:= \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x, t) - \partial_t u(x, t) - \sigma \sum_{i=1}^{m_0} \partial^2_{x_i^2} u(x, t) = g(x, t),$$

where $(x, t) = (x_1, \ldots, x_N, t) \in \mathbb{R}^{N+1}$, $1 \leq m_0 \leq N$ and $\sigma > 0$. Moreover, the matrix $B = (b_{ij})_{i,j=1,\ldots,N}$ satisfies the following structural assumption.

(H) For some basis on $\mathbb{R}^N$, the matrix $B$ takes the following form

$$B = \begin{pmatrix} \circ & \circ \\ B_1 & \circ \end{pmatrix}$$

where $B_1$ is a $m_1 \times m_0$ matrix of rank $m_1$, with

$$m_0 \geq m_1 \geq 1 \quad \text{and} \quad m_0 + m_1 = N. \quad (1.3)$$

In particular, we provide spatial a priori estimates of the solution $u$ in terms of Sobolev spaces when the right-hand side is in the $L^2$ space and $u$ is assumed to have a certain amount of regularity in the first $m_0$ directions. The result is that $u$ is also regular in the remaining $N - m_0$ space variables and this is related to a commutator identity introduced by Hörmander for hypoelliptic operators. Results of this type where firstly proved in [15] thanks to a representation formula via the fundamental solution. Later on, they were
improved in [21] through a condition that considers brackets of the vector fields, but that leads to involved estimates. Recently, we recall the works [9, 8], where the authors deal with the classical kinetic Fokker-Planck equation and the class of homogeneous Hörmander’s operators that can be written as a sum of squares, i.e. \( \sum X_i^2 \), respectively. Our aim is to extend the results of [9] to the operator \( (1.1) \), via an elegant approach based on the commutators identities that allows us to obtain a Sobolev exponent that does not depend on the integrability assumption of the right-hand side. Lastly, we provide an alternative proof to that of Theorem 1.5 of [9] for the optimal spatial regularity. The technique we consider is not based on any representation formula through the fundamental solution, or Fourier transform method, and allows us to obtain a first extension of the results in [9] in the direction of the more general class of ultraparabolic operators

\[ \mathcal{K} u(x, t) := Y u(x, t) + \text{div}(A Du) = g(x, t), \quad (x, t) \in \mathbb{R}^{N+1}, \]

known as Kolmogorov operators in divergence form, for which the matrix \( B = (b_{ij})_{i,j=1,\ldots,N} \) that defines the transport operator \( Y \) has the following form for some basis of \( \mathbb{R}^N \):

\[ B = \begin{pmatrix}
0 & \cdots & 0 \\
B_1 & 0 & \cdots & 0 \\
& \ddots & \ddots & \ddots \\
& & 0 & B_\kappa \\
& & & \cdots & 0 
\end{pmatrix} \tag{1.4} \]

where \( B_i \) is a \( m_i \times m_i-1 \) matrix of rank \( m_i \) for every \( i = 1, \ldots, \kappa \) and

\[ m_0 \geq m_1 \geq \ldots \geq m_\kappa \geq 1, \quad \sum_{i=0}^{\kappa} m_i = N. \]

### 1.1. Motivation and background.

The Kolmogorov equation \( \mathcal{K} u = g \) is a second order ultraparabolic PDE, that is strongly degenerate whenever \( m_0 < N \) and arises in various different research fields, such as the financial market modeling [20, 7, 1], or the kinetic theory of gas [16, 5, 11]. It is known that the first order part of \( \mathcal{K} \) may induce a strong regularizing property, and in fact was considered by Hörmander in [15] as a prototype for the family of hypoelliptic operators \( \sum X_i^2 + X_0 \). Indeed, under a particular structural assumption for the matrix \( B \) (see (1.4)), the operator \( \mathcal{K} \) is homogeneous and hypoelliptic, see [2, 19]. Namely, every distributional solution \( u \) to \( \mathcal{L} u = f \) defined in some open set \( \Omega \subset \mathbb{R}^{N+1} \) belongs to \( C^\infty(\Omega) \) and it is a classical solution to \( \mathcal{K} u = f \), whenever \( f \in C^\infty(\Omega) \). Moreover, we remark that assumption (H) is implied by the condition introduced by Hörmander in [15]:

\[ \text{rank Lie} (\partial_1, \ldots, \partial_{m_0}, Y) (x, t) = N + 1 \quad \text{for every} \quad (x, t) \in \mathbb{R}^{N+1}, \]

where \( \text{Lie} (\partial_1, \ldots, \partial_{m_0}, Y) \) denotes the Lie algebra generated by the first order differential operators \( (\partial_1, \ldots, \partial_{m_0}, Y) \) computed at \( (x, t) \).

Nowadays, the research community is mainly interested in the study of the regularity theory for solutions to \( \mathcal{K} u = g \), and in particular there are various open problems regarding the weak regularity theory, since the literature regarding the classical regularity theory is quite vast. For further information on this topic we refer to the survey paper...
and the references therein. On the other hand, the weak regularity theory has been widely investigated in the last decade and it is still evolving. Among the most recent results, we recall a series of paper devoted to the study of the Fokker-Planck operator \[ L \], i.e.,

\[ \Delta_v u(v, x, t) + v \cdot \nabla_v u(v, x, t) - \partial_t u(v, x, t) \quad \text{for every } (v, x, t) \in \mathbb{R}^{2d} \times \mathbb{R}, \]

obtained by choosing \( m_0 = m_1 = d, N = 2d \) and \( B_1 = I_d \), where \( I_d \) is the identity matrix of order \( d \). Very recently, the authors of [4, 22] extended the technique proposed in [13] for the Fokker-Planck equation and in [16] for the Boltzmann equation to the case of weak solutions to \( \mathcal{K} u = g \). This method is based on the combination of a weak Harnack inequality with a log-transform, and it is the only approach we can employ up to now when dealing with the case of weak solutions to \( \mathcal{K} u = g \).

Among the aforementioned open problems, we recall the extension of the De Giorgi technique proposed in [12, 14] to weak solutions to the more general equation \( \mathcal{K} u = g \). This method is based on the combination of an \( L^2 - L^\infty \) estimate (1st Lemma of De Giorgi) for weak solutions with a measure-to-pointwise result (2nd Lemma of De Giorgi), also known as intermediate value lemma for weak solutions. As far as we are concerned with the 1st Lemma, this was proved in [3, 4] in a very general framework. On the other hand, in [12, 14] we find two different proofs for the intermediate value lemma for weak solutions to the classical Fokker-Planck equation, but no generalization is available even for the case of (1.1). In particular, in the paper [12] the proof of the 2nd Lemma is based on global fractional estimates in Sobolev spaces proved by Bouchut in [9, Theorem 1.3] via techniques from the velocity averaging theory, see [10] and the references therein. These techniques mainly consist of two different approaches - the Fourier transform and the commutators identities - and to our knowledge there is no available result of this type even for the class of (1.1). Hence, the aim of this work is to provide a first extension of the results proved in [9] to the equation (1.1) following the commutators identities approach, since it allows us to treat more general operators without considering any representation formula via the fundamental solution, but with the only assumption of extra regularity in the degeneracy directions of \( u \).

1.2. Main results. From now on, according to (1.3) we split the coordinate \( x \in \mathbb{R}^N \) as

\[
   x = (x^{(0)}, x^{(1)}), \quad x^{(0)} \in \mathbb{R}^{m_0}, \quad x^{(1)} \in \mathbb{R}^{m_1}.
\]

Moreover, we denote by \( \Delta_0 \) the Laplacean with respect to the first \( m_0 \) directions and the partial gradient in \( \mathbb{R}^{m_0} \) with respect to \( x^{(i)} \) as

\[
   D_i = (\partial_{x_1^{(i)}}, \ldots, \partial_{x_{m_0}^{(i)}}) \quad \text{for } i = 0, 1.
\]

Lastly, we define for every \( \beta \geq 1 \) and for every \( s \geq 0 \)

\[
   D_\beta := (-\Delta_{m_0} u)^{\beta/2}, \quad D_s := (-\Delta_{m_1} u)^{s/2}.
\]

Now, we are in a position to state our main results, that respectively extend Theorem 1.6 and Theorem 1.5 of [9] to solutions to (1.1). Here, we do not explicitly express which notion of solution we consider, since our results apply to every solution of the equation (1.1) that is at least in \( L^2(\mathbb{R}^N \times \mathbb{R}) \).
Theorem 1.1. Let us consider a solution $u \in L^2(\mathbb{R}^N \times \mathbb{R})$ to
\[ Y u(x, t) = g(x, t) \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}, \]
with $g \in L^2(\mathbb{R}^N \times \mathbb{R})$ and
\[ D_0^{\max(\beta, \gamma)} u \in L^2(\mathbb{R}^N \times \mathbb{R}) \quad \text{and} \quad D_0^\beta g \in L^2(\mathbb{R}^N \times \mathbb{R}) \]
with
\[ \gamma \geq 0 \quad \text{and} \quad 0 \leq 1 - \gamma \leq \beta. \]
Then $D_1^s u \in L^2(\mathbb{R}^{N+1})$, with
\[ s = \frac{\beta}{1 - \gamma + \beta} \quad \text{and} \quad s = 1 \text{ if } 1 - \gamma = \beta = 0. \] (1.6)
Moreover, the following estimate holds
\[ \|D_1^s u\|_2 \leq C(N, B, \beta) \|D_0^\beta u\|_2\frac{1-s}{2} \|D_0^\gamma g\|_2^s. \] (1.7)

The proof we propose here is reminiscent of the hypoelliptic regularity proposed in [15] and extends also [9, Proposition 1.1] ($\gamma = 0$), with the extra assumption that $\beta \geq 1$. This is due to the fact that the proof of [9, Proposition 1.1] is based on the application of the Fourier transform to the equation $Y u = g$. From the weak regularity point of view, the most interesting case is given by the choice of $\gamma = 0$ and $\beta = 1$, since $D_0^\beta u \in L^2(\mathbb{R}^N \times \mathbb{R})$ and $g \in L^2(\mathbb{R}^N \times \mathbb{R})$ are standard regularity assumptions when studying weak solutions to (1.1). In particular, this choice for the parameters leads to
\[ D_1^2 u \in L^2(\mathbb{R}^{N+1}). \]

We point out that estimates of this type are very useful for the characterization of the space $W$ of weak solutions introduced in [6, 4], and in order to complete the proof we only rely on the geometrical properties of the hypoelliptic structure of the transport operator.

When dealing with the “complete” evolution equation (1.1), it is quite clear how we can recover an estimate of $D_0 u$ starting from the equation (see the Caccioppoli inequality proved in [3, Theorem 3.1]). However, combining this information (that corresponds to the choice of $\beta = 1$) with the previous theorem does not provide us with the best Sobolev exponent for the spatial regularity, that in [21] is showed to be $2/3$ in the direction $x^{(1)}$. As we already pointed out at the beginning of this introduction, the method proposed in [21] is very involved. For this reason, in [9, Theorem 1.5] the author proposed to show that a solution to (1.1) is such that $\Delta_0 u \in L^2$ and then apply Theorem 1.1, with $\beta = 2$. Here, we state the analogous theorem for solutions to (1.1), for whose proof we refer to [9].

Theorem 1.2. Let us consider a solution $u \in L^2(\mathbb{R}^N \times \mathbb{R})$ to
\[ Y u(x, t) - \sigma \Delta_{m_0} u(x, t) = g(x, t) \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}, \]
where $\sigma > 0$ a positive constant. If $g \in L^2(\mathbb{R}^N \times \mathbb{R})$ and $D_0^\beta u \in L^2(\mathbb{R}^N \times \mathbb{R})$, with $\beta \geq 1$. Then, $Y u$ and $\Delta_{m_0} u$ are in $L^2(\mathbb{R}^N \times \mathbb{R})$ and
\[ \|Y u\|_2 + \sigma \|\Delta_{m_0} u\|_2 \leq C(N, B)\|g\|_2. \]
Moreover, $D_1^2 u \in L^2(\mathbb{R}^N \times \mathbb{R})$ and

$$\|D_1^2 u\|_2 \leq \frac{C(N, B)}{\sigma^{1/3}} \|g\|_2.$$

Lastly, we propose an alternative proof of the maximal regularity carried out by contradiction, whose main advantage is it directly extends to the case of $\mathcal{K} u = g$ once the corresponding spatial estimates are proved.

**Theorem 1.3.** Let us consider a solution $u \in L^2(\mathbb{R}^N \times \mathbb{R})$ to

$$Yu(x, t) - \sigma \Delta_{m_0} u(x, t) = g(x, t) \quad (x, t) \in \mathbb{R}^N \times \mathbb{R},$$

with $g \in L^2(\mathbb{R}^N \times \mathbb{R})$ and $\sigma > 0$ a positive constant such that $\sigma \neq 1$. Then, both $Y u$ and $\Delta_{m_0} u$ are in $L^2(\mathbb{R}^N \times \mathbb{R})$ and the estimate (1.7) holds true with the choice of the parameter $\beta = 2$.

On one hand the contradiction argument forces us to require $\sigma \neq 1$, on the other hand it allows us to get rid of the assumption of $D_0^\alpha u \in L^2(\mathbb{R}^N \times \mathbb{R})$ that we have in Theorem 1.2. Also, we believe it is possible to extend this approach to the more general case of the Kolmogorov equation in divergence (and non-divergence form) with measurable coefficients, for which it is not possible to consider an approach in the spirit of [15] since the existence of a fundamental solution associated to it has not been proved yet.

This paper is organized as follows. Section 2 and Section 3 are respectively devoted to the proof of Theorem 1.1 and Theorem 1.3. In Section 4 we state our conclusions and we introduce the model operator of the Kolmogorov equation with two groups of spatial variables.

## 2. Proof of Theorem 1.1

By suitable smoothing and cut-off, we can reduce ourselves to the case where $u, g$ are smooth functions, i.e. $u, g \in C^\infty_c(\mathbb{R}^{N+1})$. Following the notation of [15], from now on we denote the $L^2$ pairing by

$$\langle f, g \rangle = \int_{\mathbb{R}^{N+1}} fg \, dx \, dt. \quad (2.1)$$

By assumption (H), the block $B_1$ of the matrix $B$ is of maximum rank $m_1$, since by (1.3) we have that $m_1 \leq m_0$. Without loss of generality, we may assume the block $B_1$ is in the canonical form with $m_1$ pivots, i.e.

$$B_1 = \begin{pmatrix}
  b_{11} & b_{12} & \cdots & b_{1m_1} & \cdots & b_{1m_0} \\
  0 & b_{22} & \cdots & b_{2m_1} & \cdots & b_{2m_0} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  0 & \cdots & b_{m_1m_1} & \cdots & b_{m_1m_0} 
\end{pmatrix} \quad (2.2)$$

The proof of Theorem 1.1 is based on the representation of the derivative along a certain direction $x_j^{(1)}$, through a commutators identity between $\partial x_j^{(0)}$ and $Y$. In particular, thanks
to the explicit expression of \( Y \), we have that
\[
[\partial_{x_i}^{(0)}, Y]u = \partial_{x_i}^{(0)} Y u - Y \partial_{x_i}^{(0)} u = b_{1i}^{1j} \partial_{x_j}^{(1)} u
\]  
(2.3)
\[
[\partial_{x_j}^{(0)}, Y]u = \partial_{x_j}^{(0)} Y u - Y \partial_{x_j}^{(0)} u = b_{1j}^{1i} \partial_{x_i}^{(1)} u + b_{2j}^{1i} \partial_{x_i}^{(1)} u
\]
\[
\vdots
\]
\[
[\partial_{x_m}^{(0)}, Y]u = \partial_{x_m}^{(0)} Y u - Y \partial_{x_m}^{(0)} u = \sum_{j=1}^{m_1} b_{jm_1}^{1i} \partial_{x_i}^{(1)} u.
\]

First of all, this procedure is well posed since \( m_0 \geq m_1 \) and the matrix \( B_1 \) is of maximum rank \( m_1 \). Hence, thanks to the first commutators identity of (2.3), we have
\[
\partial_{x_i}^{(1)} u = \frac{1}{b_{1i}^{1j}} [\partial_{x_j}^{(0)}, Y]u.
\]

Then, there exists two constants such that
\[
\partial_{x_j}^{(1)} u = c_1 \partial_{x_i}^{(1)} u + c_2 [\partial_{x_i}^{(0)}, Y]u.
\]  
(2.4)

Generally speaking, for a fixed \( i = 1, \ldots, m_1 \), there exist \( c_1, \ldots, c_i \) constants such that
\[
\partial_{x_i}^{(1)} u = \sum_{j=1}^{i-1} c_j \partial_{x_j}^{(1)} u + c_i [\partial_{x_i}^{(0)}, Y]u.
\]  
(2.5)

Now, the idea behind the proof of this theorem is to consider the commutators identities above and to estimate the \( L^2 \) bracket of each of them against a suitable derivative. Let us begin with the estimate of the derivative \( \partial_{x_i}^{(1)} u \):

\[
\| D_{x_i}^{(1-s)} \partial_{x_i}^{(1)} u \|_2^2 = \langle D_{x_i}^{(1-s)} \partial_{x_i}^{(1)} u, \partial_{x_i}^{(1)} u \rangle
\]  
(2.6)
\[
= \langle D_{x_i}^{(1-s)} \partial_{x_i}^{(1)} u, \partial_{x_i}^{(0)} Y u - Y \partial_{x_i}^{(0)} u \rangle
\]
\[
= \langle D_{x_i}^{(1-s)} \partial_{x_i}^{(1)} u, \partial_{x_i}^{(0)} g - Y \partial_{x_i}^{(0)} u \rangle
\]
\[
= -\langle \partial_{x_i} Y D_{x_i}^{(1-s)} \partial_{x_i}^{(1)} u, g \rangle + \langle Y D_{x_i}^{(1-s)} \partial_{x_i}^{(1)} u, \partial_{x_i}^{(0)} u \rangle
\]
\[
= -2Re(D_{x_i}^{(1-s)} \partial_{x_i}^{(1)} u, \partial_{x_i}^{(0)} u)
\]
\[
\leq C(N, B) \| D_{x_i}^{(1-s)} D_0 u \|_2 \| g \|_2.
\]

Moreover, thanks to (2.4) we also have
\[
\| D_{x_i}^{(1-s)} \partial_{x_j}^{(1)} u \|_2^2 = \langle D_{x_j}^{(1-s)} \partial_{x_j}^{(1)} u, \partial_{x_j}^{(1)} u \rangle
\]
\[
= c_1 \langle D_{x_j}^{(1-s)} \partial_{x_j}^{(1)} u, \partial_{x_j}^{(0)} Y u - Y \partial_{x_j}^{(0)} u \rangle
\]
\[
+ c_2 \langle D_{x_j}^{(1-s)} \partial_{x_j}^{(1)} u, \partial_{x_j}^{(0)} Y u - Y \partial_{x_j}^{(0)} u \rangle
\]
\[
\leq C(N, B) \| D_{x_j}^{(1-s)} D_0 u \|_2 \| g \|_2.
\]
Taking into consideration (2.5), the above argument holds true also for every other direction $x_i^{(1)}$, with $i = 2, \ldots, m_1$. For this reason, from now on we restrict ourselves to the case where $B_1$ is of the form (2.2), where every component is zero except $b_{1i}^1 = 1$, with $i = 1, \ldots, m_1$, i.e.

$$B_1 = \begin{pmatrix}
1 & 0 & \ldots & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & 1 & \ldots & 0 \\
\end{pmatrix},$$

and we keep in mind that, from now, every constant we consider depends on the matrix $B$. By noticing that $\frac{1}{2} \leq s < 1$ and that (2.6) can be computed for every $\partial_{x_i^{(1)}} u$, with $i = 1, \ldots, m_1$, we have that:

$$\|D_s^1 u\|_2^2 \leq C(N, B) \|D_1^{1-2(1-s)} D_0^{1-\gamma} u\|_2 \|D_0^\beta g\|_2.$$  

Then we define $\theta = 2 - 1/s \in [0, 1]$ and notice that $1 - 2(1-s) = \theta s$ and $1 - \gamma = (1-\theta)\beta$. Then if we apply the H"older inequality in the Fourier variables we consider to define the fractional derivative in $x^{(1)}$ (see equation (3.6) of [9]), we get

$$\|D_s^1 u\|_2^2 \leq C(N, B) \|D_1^\theta u\|_2 \|D_0^\beta u\|_2^{1-\theta} \|D_0^\beta g\|_2. \tag{2.7}$$

Lastly, by collecting together all of the corresponding terms, we obtain

$$\|D_s^1 u\|_2^{2-\theta} \leq C(N, B) \|D_0^\beta u\|_2^{1-\theta} \|D_0^\beta g\|_2.$$  

Hence, the proof is complete.  

\section{3. Proof of Theorem 1.3}

We proceed by contradiction. Thus, we assume $Y u \notin L^2(\mathbb{R}^{N+1})$ and $\Delta_{m_0} u \notin L^2(\mathbb{R}^{N+1})$. By computing the $L^2$ norm on both sides of (1.1) and by applying the inverse triangle inequality we have that

$$\|Y u\|_2 - \|\Delta_{m_0} u\|_2 \leq \|Y u - \sigma \Delta_{m_0} u\|_2 = \|g\|_2.$$  

Thus, by considering that $\|Y u\|_2$ is unbounded, the above inequality leads us to a contradiction:

$$+\infty \leq \|g\|_2.$$  

The same reasoning applies if we assume $\Delta_{m_0} u \notin L^2(\mathbb{R}^{N+1})$ and $Y u \notin L^2(\mathbb{R}^{N+1})$. Lastly, if we assume both $\Delta_{m_0} u, Y u \notin L^2(\mathbb{R}^{N+1})$, the only interesting case is when their asymptotic behavior is comparable. Still, since by assumption $\sigma > 0$ and $\sigma \neq 1$, what we obtain is a contradiction again. Indeed, if we denote by $\omega = \|\Delta_{m_0} u\|_2 = \|Y u\|_2$ and we consider the reverse triangle inequality once more, we obtain

$$+\infty = \lim_{\omega \to +\infty} |1 - \sigma| \omega \leq \|g\|_2.$$
4. Conclusions

We have fully addressed the first issue we encounter when extending the results of [9] to the Kolmogorov equation $\mathcal{K} u = g$, that is represented by the choice of the matrix $B$ as in (H), i.e. with $m_0 \geq m_1$, $m_0$ possibly different from $m_1$, and $B_1$ a generic matrix of order $m_1 \times m_0$, not necessarily equal to the identity.

The next issue we have to face is to consider a prototype equation for the family of $\mathcal{K} u = g$ that introduces the mixing structure of the transport operator $Y$. In particular, we are interested in the preliminary study of the following toy model equation, also known as Kolmogorov equation with two groups of spatial variables:

$$\tilde{Y} u(x, t) - \sigma \Delta_0 u(x, t) := \sum_{j=1}^d x_j^{(1)} \partial_{x_j^{(0)}} u(x, t) + \sum_{j=1}^d x_j^{(2)} \partial_{x_j^{(1)}} u(x, t) \quad (4.1)$$

$$- \partial_t u(x, t) - \sigma \sum_{i=1}^{m_0} \partial_i^2 u(x, t) = g(x, t),$$

where $(x, t) = (x^{(0)}, x^{(1)}, x^{(2)}, t) \in \mathbb{R}^{d+1}$ and $x^{(i)} \in \mathbb{R}^d$ for every $i = 1, 2, 3$. The toy model (4.1) is obtained by considering the following choice for the matrix $B$

$$B = \begin{pmatrix} O_d & O_d & O_d \\ B_1 & O_d & O_d \\ O_d & B_2 & O_d \end{pmatrix},$$

where for $i = 1, 2$ every $B_i = I_d$. This toy model can be seen as a first extension of the Fokker-Planck equation where we have one velocity direction $x^{(0)}$ and two spatial directions $x^{(1)}$ and $x^{(2)}$ and, from a geometrical point of view, presents all of the difficulties we encounter when studying the more general Kolmogorov equation $\mathcal{K} u = g$. Indeed, even if the matrix $B$ is simply made of $B_1$, $B_2$ and two non-zero blocks, it is immediately clear how the structure of the transport operator

$$\tilde{Y} u = \sum_{j=1}^d x_j^{(1)} \partial_{x_j^{(0)}} u(x^{(1)}, x^{(2)}, t) + \sum_{j=1}^d x_j^{(2)} \partial_{x_j^{(1)}} u(x^{(1)}, x^{(2)}, t) - \partial_t u(x^{(1)}, x^{(2)}, t)$$

mixes the spatial direction $x^{(1)}$ with the velocity direction $x^{(0)}$, but also the spatial direction $x^{(2)}$ with the velocity direction $x^{(1)}$. On one hand, this means it is not straightforward to adapt the kinetic theory’s techniques to (4.1). On the other hand, it is clear that it presents all the difficulties one has to deal with in the more general case of $\mathcal{K} u = g$. For a recent bibliography on the study of this particular model we refer to [17, 18] and the reference therein.

In the future, our aim is to prove a spatial regularity result in the spirit of Theorem 1.1 for the toy model (4.1), and then to consider it to prove analogous results to that of Theorem 1.2 and Theorem 1.3. A first attempt to the proof of this result would be to consider the technique presented in Section 2, with $s = \frac{\beta}{\beta + 1}$ that plays the same role as $\beta$ in the estimate (2.7) when computing the iteration for the second space variable $x^{(2)}$. Still, this is not well posed since the derivative we would obtain in the last line of (2.6)
would have a negative exponent. On the other hand, we conjecture it is possible to extend the Fourier technique considered in [9, Theorem 1.3] to the toy model (4.1).

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