Target assignment for robotic networks: asymptotic performance under limited communication

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Abstract—We are given an equal number of mobile robotic agents, and distinct target locations. Each agent has simple integrator dynamics, a limited communication range, and knowledge of the position of every target. We address the problem of designing a distributed algorithm that allows the group of agents to divide the targets among themselves and, simultaneously, leads each agent to reach its unique target. We do not require connectivity of the communication graph at any time. We introduce a novel assignment-based algorithm with the following features: initial assignments and robot motions follow a greedy rule, and distributed refinements of the assignment exploit an implicit circular ordering of the targets. We prove correctness of the algorithm, and give worst-case asymptotic bounds on the time to complete the assignment as the environment grows with the number of agents. We show that among a certain class of distributed algorithms, our algorithm is asymptotically optimal. The analysis utilizes results on the Euclidean traveling salesperson problem.

I. INTRODUCTION

Consider a group of \(n\) mobile robotic agents and \(n\) target locations, all lying in \(\mathbb{R}^d, d \geq 1\). Each agent has a limited communication range, and knows the location of some subset (possibly all) of the \(n\) targets through GPS coordinates or a map of the environment. The target assignment problem we consider is to design a distributed algorithm that allows the group of agents to divide the \(n\) targets among themselves and, simultaneously, leads each agent to reach its unique target. Such a problem could arise in several applications. For example, one could think of the agents as UAV’s on a surveillance mission, and the targets as the centers of their desired loitering patterns. Or in the context of formation control, the target positions could describe the desired formation for a group of robots.

The first question is, how do we divide the targets among the agents in a centralized fashion? A reasonable strategy would be to minimize the sum of the distances traveled by each agent to arrive at its target. The problem of optimally dividing \(n\) persons among \(n\) objects, subject to a linear cost function, is a problem in combinatorial optimization [1]. It is referred to as the assignment problem, or the minimum weight perfect matching problem in bipartite graphs. The assignment problem can be written as an integer linear program. Unlike some integer linear programs, such as the Euclidean traveling salesperson problem, optimal solutions for the assignment problem can be computed in polynomial time.

In 1955 Kuhn [2] developed the Hungarian method—the first polynomial solution for the assignment problem. Kuhn’s method solves the problem in \(O(n^3)\) computation time (see Section III for a definition of the \(O\) notation).

Another approach to the assignment problem is the auction algorithm [3], [4], [5], first proposed by Bertsekas. This method solves the problem in \(O(n^3)\) computation time, but can be computed in a parallel fashion, with one processor for each person. Recently, Moore and Passino [6] modified the auction algorithm to assign mobile robots to spatially distributed tasks in the presence of communication delays. However, in order to exchange bids on a particular object (task), the auction algorithm, and thus the work in [6], requires that the communication graph between processors (robots) is complete.

In this paper we address the target assignment problem when each agent has knowledge of all target positions, and a limited communication range \(r > 0\). We introduce a class of distributed algorithms, called assignment-based motion, which provide a natural approach to the problem. Following the recent interest in determining the time complexity of distributed algorithms for robotic networks, as in [7] and [8], we study the worst-case asymptotic performance of the assignment-based motion class as the environment grows with \(n\). We show that for a \(d\)-dimensional cube environment, \([0, \ell(n)]^d, d \geq 1\), if the side length \(\ell(n)\) grows at a rate of at least \((1 + \epsilon)rn^{1/d}, \epsilon > 0\), then the completion time is in \(\Omega(n^{(d-1)/d}\ell(n))\), for all algorithms in this class.

In Section V we introduce a novel control and communication algorithm, called ETSP ASSIGNMENT. In this algorithm, each agent computes an ETSP tour through the \(n\) targets, turning the cloud of target points into an ordered ring. Agents then move along the ring, looking for the next available target. When agents communicate, they exchange messages of \(O(\log n)\) size, containing information on how far it is to the next available target along the ring. In Section V.A we verify the correctness of this algorithm for any communication graph which contains, as a subgraph, the \(r\)-disk graph. In Section V.B we show that when \(\ell(n) \geq (1 + \epsilon)rn^{1/d}\), among all algorithms in the assignment-based motion class, the ETSP ASSIGNMENT algorithm is asymptotically optimal (i.e., a constant factor approximation of the optimal). Finally, in Section V.F we note that ETSP ASSIGNMENT solves the target assignment problem even when there are \(n\) agents and \(m\) targets, \(n \neq m\).
II. BACKGROUND

In this section we introduce notation and review some relevant results in combinatorial optimization.

A. Notation

We let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_{>0}$ denote the set of positive real numbers, and $\mathbb{N}$ denote the set of positive integers. For a set finite $A$ we let $|A|$ denote its cardinality. For two functions $f, g : \mathbb{N} \to \mathbb{R}_{>0}$, we write $f(n) \in O(g)$ (respectively, $f(n) \in \Omega(g)$) if there exist $N \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$ such that $f(n) \leq cg(n)$ for all $n \geq N$ (respectively, $f(n) \geq cg(n)$ for all $n \geq N$). If $f(n) \in O(g)$ and $f(n) \in \Omega(g)$ we say $f(n) \in \Theta(g)$.

Finally, we use the notation (mod $n$) to denote arithmetic performed modulo $n \in \mathbb{N}$. Thus, for an integer $n \in \mathbb{N}$ we have $n + 1 \equiv 1 \pmod{n}$ and $0 = n \pmod{n}$, and $\{n - 1, n, n + 1\} = \{n - 1, n, 1\} \pmod{n}$.

B. The assignment problem

Following [4], the classical assignment problem can be described as follows. Consider $n$ persons who wish to divide themselves among $n$ objects. For each person $i$, there is a nonempty set $Q[i]$ of objects that $i$ can be assigned to, and cost $c_{ij} \geq 0$ associated to each object $j \in Q[i]$. An assignment $S$ is a set of person-object pairs $(i, j)$ such that $j \in Q[i]$ for all $(i, j) \in S$. For each person $i$ (likewise, object $j$), there is at most one pair $(i, j) \in S$. We call the assignment complete if it contains $n$ pairs. The goal is to find the complete assignment which minimizes $\sum_{(i,j) \in S} c_{ij}$.

Let $x_{ij}$ be a set of variables for $i$ and $j$ in $I := \{1, \ldots, n\}$. For an assignment $S$, we write $x_{ij} = 1$ if $(i, j) \in S$, and $x_{ij} = 0$ otherwise. Thus, the problem of determining the optimal assignment can be written as a linear program:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{n} \sum_{j \in Q[i]} c_{ij} x_{ij}, \\
\text{subject to} & \quad \sum_{j \in Q[i]} x_{ij} = 1 \quad \forall i \in I, \\
& \quad \sum_{(i,j) \in Q[i]} x_{ij} = 1 \quad \forall j \in I, \\
& \quad x_{ij} \geq 0.
\end{align*}
\]

We cannot use linear inequalities to write the constraint that $x_{ij}$’s attain only the values zero and one. However, it turns out, [4], that there always exists an optimal solution in which the $x_{ij}$’s satisfy our integer assumption.

C. The Euclidean traveling salesperson problem

Here we review some relevant results on the Euclidean traveling salesperson problem (ETSP). Let $Q$ be a set of $n$ points in a compact environment $E \subset \mathbb{R}^d$, $d \geq 1$, and let $Q_n$ be the set of all point sets $Q \subset E$ with $|Q| = n$. Let $\text{ETSP}(Q)$ denote the cost of the ETSP tour over the point set $Q$, i.e., the length of the shortest closed path through all points in $Q$. An important result, from [9], is that given a compact set $E$, there exists a finite constant $\alpha(E)$ such that, for all $Q \in Q_n$,

\[
\text{ETSP}(Q) \leq \alpha(E)n^{(d-1)/d}.
\]

In fact, we have that in the worst-case setting, the $\text{ETSP}(Q)$ belongs to $O(n^{(d-1)/d})$.

In our application of these results it will be useful to consider the case where the environment grows with the number of points. That is, we are interested in environments which are cubes, $[0, \ell(n)]^d$, $d \geq 1$, where $\ell(n)$ is the side length of the cube. Applying a simple scaling argument to the result in [1], we arrive at the following corollary.

Corollary 2.1 (ETSP tour length): Consider an environment $E = [0, \ell(n)]^d$, where $d \geq 1$. For every point set $Q \in Q_n$, $\text{ETSP}(Q) \leq O(n^{(d-1)/d} \ell(n))$.

The problem of computing an optimal tour is known to be NP-complete. However, there exist heuristics which can be computed efficiently and give a constant factor approximation to the optimal tour. The best known approximation algorithm is due to Christofides [10]. The Christofides’ algorithm computes a tour that is no more than $3/2$ times longer than the optimal. It runs in time $O(n^3)$. Another method, known as the double-tree algorithm, produces tours that are no longer than twice the optimal, in run time $O(n^2)$.

III. PROBLEM FORMULATION

To describe the target assignment problem formally, consider $n$ agents in an environment $E(n) \subset \mathbb{R}^d$, $d \geq 1$. The environment $E(n)$ is compact for each $n$ but may grow with the number of points. For ease of presentation let $E := [0, \ell(n)]^d$, where $\ell(n) > 0$ (that is, $E$ is a $d$-dimensional cube with side length $\ell(n)$). Each agent has a unique identifier (UID) taken from the set $U_{\text{UID}} := \mathbb{N}$. For simplicity, we assume that $I_{\text{UID}} := I := \{1, \ldots, n\}$. However, each agent does not know the set of UIDs being used (i.e., agent $n$ does not know it has the largest UID). Agent $i \in I$ has position $p[i] \in E$. Two agents, $i$ and $k$ in $I$, are able to communicate if and only if $\|p[i] - p[k]\| \leq r$, where $r > 0$ is called the communication range. We refer to the graph representing the communication links as the $r$-disk graph. Agent $i$’s kinematic model is $\dot{p}[i] = u[i]$, where $u[i]$ is a velocity control input bounded by $v > 0$. We assume that the agents move in continuous time and communicate according to a discrete time communication schedule consisting of an increasing sequence of time instants with no accumulation points, $\{t_k\}_{k \in \mathbb{N}}$. We assume that $|t_{k+1} - t_k| \leq t_{\text{max}}$, for all $k \in \mathbb{N}$, where $t_{\text{max}} \in \mathbb{R}_{>0}$. At each communication round, agents can exchange messages of length $O(\log n)$.

We assume that communication round $k$ occurs at time $t_k$, and that all messages are sent and received instantaneously at $t_k$. Motion then occurs from $t_k$ until $t_{k+1}$. It should be noted that in this setup we are emphasizing the time complexity due to the motion of the agents.
Let \( Q := \{q_1, \ldots, q_n\} \) be a set of distinct target locations, \( q_j \in \mathcal{E} \) for each \( j \in \mathcal{I} \). Agent \( i \) is equipped with memory \( M[i] \), of size \(|M[i]|\). In this memory, agent \( i \) stores a set of target positions, \( Q[i] \subseteq Q \). These are the targets to which agent \( i \) can be assigned. We let \( Q[i](0) \) denote agent \( i \)'s initial target set. These positions may be known through GPS coordinates, or through a map of the environment.

In this paper we assume that each agent knows the position of every target. That is, \( Q[i](0) = Q \) for each \( i \in \mathcal{I} \). We refer to this as the full knowledge assumption. To store this amount of information we must assume that the size of each agents memory, \(|M[i]|\), grows linearly with \( n \). Our goal is to solve the full knowledge target assignment problem:

Determine a control and communication law for \( n \in \mathbb{N} \) agents, with attributes as described above, satisfying the following requirement. There exists a time \( T > 0 \) such that for every agent \( i \in \mathcal{I} \), there is a unique target \( q_{j_i} \in Q[i](0) \) with \( p[i](t) = q_{j_i} \) for all time \( t \geq T \), where \( j_i = j_k \) if and only if \( i = k \).

In the remainder of the paper, we will refer to this as the target assignment problem.

Remark 3.1 (Consistent knowledge): A more general assumption on the initial target sets, \( Q[i](0) \), which still ensures the existence of a complete assignment, is the consistent knowledge assumption: For each \( K \subseteq \mathcal{I} \), \(|\cup_{k \in K} Q[k](0)| \geq |K| \). In fact, it was proved by Frobenius, 1917, and Hall, 1935 that this is the necessary and sufficient condition for the existence of a complete assignment \( [1] \).

In the full knowledge assumption, each agent knows the position of all targets in \( Q \). These positions will be stored in an array within each agents memory, rather than as an unordered set. To represent this, we replace the target set \( Q \) with the target \( n \)-tuple \( q := (q_1, \ldots, q_n) \), and the local target set \( Q[i] \) with the \( n \)-tuple \( q[i] \). Thus, in the full knowledge assumption, \( q[i](0) := q \) for each \( i \in \mathcal{I} \). (It is possible that the order of the targets in the local sets \( q[i] \) may initially be different. However, given a set of distinct points in \( \mathbb{R}^d \), it is always possible to create a unique ordering.)

IV. Assignment-based algorithms with lower bound analysis

In this section we introduce and analyze a class of deterministic algorithms for the target assignment problem.

A. The assignment-based motion class

The initialization, motion, and communication for each algorithm in the assignment-based motion class have the following attributes:

Initialization: In this class of algorithms agent \( i \) initially selects the closest target in \( q[i] \), and sets the variable \( curr[i] \) (agent \( i \)'s current target), to the index of that target.

Motion: Agent \( i \) moves toward the target \( curr[i] \) at speed \( v \):

\[
\dot{p}[i] = \begin{cases} 
\frac{q[i] - curr[i]}{|q[i] - curr[i]|^2}, & \text{if } q[i] = p[i], \\
0, & \text{otherwise}, 
\end{cases}
\]

where \( v > 0 \) is a constant.

Communication: As agent \( i \) communicates with other agents, it updates the tuple \( q[i] \) “removing” targets which are assigned to other agents. If agent \( i \) must change \( curr[i] \), it selects a new target in \( q[i] \), that has not been removed. This is described more formally in the following.

Communication round for agent \( i \):

1. Broadcast a message, msg[k], based on \( q[i] \) and containing \( curr[i] \) and the UID \( i \).
2. Receive msg[k] from each agent \( k \) within communication range.
3. For all msg[k] received do
4. Based on msg[k] (possibly) remove assigned targets from \( q[i] \).
5. If \( curr[i] = curr[k] \) then
6. If agent \( i \) is farther from \( curr[i] \) than agent \( k \), or if they are the same distance but \( i < k \), remove the target given by \( curr[i] \) from \( q[i] \).
7. Set(curr[i]) to a target in \( q[i] \) (i.e., a target that has not been removed).

B. Lower bound on task complexity

In order to classify the time complexity of the assignment-based motion class of algorithms, we introduce a few useful definitions. We say that agent \( i \in \mathcal{I} \) is assigned to target \( q[i] \), \( j \in \mathcal{I} \), when \( curr[i] = j \). In this case, we also say target \( j \) is assigned to agent \( i \). We say that agent \( i \in \mathcal{I} \) enters a conflict over target \( curr[i] \), when agent \( i \) receives a message, msg[k], with \( curr[i] = curr[k] \). Agent \( i \) loses the conflict if agent \( i \) is farther from \( curr[i] \) than agent \( k \), and wins the conflict if agent \( i \) is closer to \( curr[i] \) than agent \( k \), where ties are broken by comparing UIDs.

Now we show that if agent \( i \) is assigned to the same target as another agent, it will enter a conflict in finite time.

Lemma 4.1 (Conflict in finite time): Consider any communication range \( r > 0 \), and any fixed number of agents \( n \in \mathbb{N} \). If, for two agents \( i \) and \( k \), \( curr[i] = curr[k] \) at some time \( t_1 \geq 0 \), then agent \( i \) (and likewise, agent \( k \)) will enter a conflict over \( curr[i] \) in finite time.

Proof: For each \( n \) the region \( \mathcal{E} \) is compact, and the motion for each agent is given by \( \mathfrak{B} \). Hence, agent \( i \) will reach \( curr[i] \) in no more than \( \text{diam}(\mathcal{E})/v \) time units, as will agent \( k \). The condition \( |p[i] - p[k]| \leq r \) will be satisfied within \( \text{diam}(\mathcal{E}(n))/v \) time units. At the next communication round, agent \( i \) will enter a conflict over \( curr[i] \).

With these definitions we give a lower bound on the time complexity of the task assignment problem when the environment grows with the number of agents.

Theorem 4.2 (Time complexity of target assignment): Consider \( n \) agents, with communication range \( r > 0 \), in an environment \( \mathcal{E} = [0,\ell(n)]^d \), \( d \geq 1 \). If \( \ell(n) \geq (1 + \epsilon) n^{1/d} \), where \( \epsilon \in \mathbb{R}_{>0} \), then for all algorithms in the assignment-based motion class, the time complexity of the target assignment problem is in \( \Omega(n^{(d-1)/d} \ell(n)) \).

Proof: We will construct a set of target positions and a set of initial agent positions for which the bound holds. The environment \( \mathcal{E} \) is the \( d \)-cube, \( [0,\ell(n)]^d \). Divide the cube \( \mathcal{E} \) into \((\lceil n^{1/d} \rceil)^d \) cubes, each with side length \( \ell(n)/n^{1/d} \), and place a target at the center of each of the cubes until you run out. This is shown in Fig. \( \square \) Notice that the distance between any two targets is lower.
bounded by $\ell(n)/[n^{1/d}]$, and that, for sufficiently large $n$, 
$\ell(n)/[n^{1/d}] \geq (1 + \epsilon)n^{1/d}/[n^{1/d}] > r$.

Next, place agent 2 at $q_2$, agent 3 at $q_3$ and so on so
that $p[i] = q_i$, for all $i \in \{2, \ldots, n\}$. From the initialization,
we have that $\text{curr}^{[i]} = i$ for each $i \in \{2, \ldots, n\}$. Now, if
we place agent 1 in $E \setminus \{q_1, \ldots, q_n\}$, it will lose a conflict
over any of the $n - 1$ occupied targets $q_2, \ldots, q_n$. Thus, the
assignment will not be complete until agent 1 reaches target $q_1$. Since the
distance between targets is greater than $r$, communication between agents $i$ and $k$ is not possible
for any $i, k \in \{2, \ldots, n\}$. So, agent $i \in \{2, \ldots, n\}$ will
communicate only with agent 1. Thus, prior to communication with agent 1, each agent $i \in \{2, \ldots, n\}$ will have $q^{[i]} = q$. The first time agent 1 comes within distance $r$ of a target $j \in \{2, \ldots, n\}$, in the best-case, agent 1 will remove target $j$ from $q^{[i]}$. Now, for any deterministic method of selecting $\text{curr}^{[i]}$, we can place agent 1 in $E \setminus \{q_1, \ldots, q_n\}$ such that target $q_1$ is the last target for which agent 1 will come within
distance $r$. Therefore, agent 1 must come within distance $r$ of each of the $n - 1$ assigned targets, before finally arriving at $q_1$.

Now we will lower bound the distance traveled by agent 1.
To do this, split the large $d$-cube into $[n/3^d]$ smaller $d$-cubes,
or blocks, where each block contains $3^d$ targets. An example
is shown in Fig. 1. There is one target at the center of each
of these blocks, and agent 1 must come within distance $r$ of
it. The distance between the center target of each block is
lower bounded by the distance between targets, $\ell(n)/[n^{1/d}]$. Agent 1 must travel this distance at least $[n/3^d] - 1$ times. So we have

$$\text{Path length} \geq \left(\left\lceil \frac{n}{3^d} \right\rceil - 1\right) \frac{\ell(n)}{[n^{1/d}]} \in \Omega(n^{(d-1)/d}\ell(n)).$$

Hence, the path length is in $\Omega(n^{(d-1)/d}\ell(n))$. Since $v \in \mathbb{R}_{>0}$, the time complexity is also in $\Omega(n^{(d-1)/d}\ell(n))$. □

Remark 4.3 ($\ell(n) \leq \ell_{\text{crit}}$): We have lower bounded the
time complexity when $\ell(n)$ grows faster than some critical value, $\ell_{\text{crit}} = r n^{1/d}$. This same type of bound appears in
percolation theory and the study of random geometric graphs,
where it is referred to as the thermodynamic limit [11].
When $\ell(n) \leq \ell_{\text{crit}}$, congestion issues in both motion and
communication become more prevalent, and a more complex
communication and motion model would ideally be used. •

In the next section we introduce an asymptotically optimal
algorithm in the assignment-based motion class.

V. THE ETSP ASSIGNMENT ALGORITHM

In this section we introduce the ETSP ASSIGNMENT
algorithm—an algorithm within the assignment-based motion
class. We will show that when $\ell(n)$ grows more quickly
than a critical value, this algorithm is asymptotically optimal.
The algorithm can be described as follows.

For each $i \in I$, agent $i$ computes a constant factor
approximation of the optimal ETSP tour of the $n$ targets in
$q^{[i]}$, denoted $\text{tour}(q^{[i]})$. We can think of tour as a map which
reorders the indices of $q^{[i]}$: $\text{tour}(q^{[i]}) = (q^{[i]}_{\sigma(1)}, \ldots, q^{[i]}_{\sigma(n)})$, where $\sigma : I \to I$ is a bijection. Notice that this map is
independent of $i$ since all agents use the same method. An
example is shown in Fig. 2. Agent $i$ then replaces its $n$-tuple $q^{[i]}$ with $\text{tour}(q^{[i]})$. Next, agent $i$ computes the index
of the closest target in $q^{[i]}$, and calls it $\text{curr}^{[i]}$. Agent $i$ also
maintains the index of the next target in the tour which may
be available, $\text{next}^{[i]}$, and first target in the tour before $\text{curr}^{[i]}
which may be available, $\text{prev}^{[i]}$. Thus, $\text{next}^{[i]}$ is initialized
to $\text{curr}^{[i]} + 1$ (mod $n$) and $\text{prev}^{[i]}$ to $\text{curr}^{[i]} - 1$ (mod $n$).
This is depicted in Fig. 3. In order to “remove” assigned
targets from the tuple $q^{[i]}$, agent $i$ also maintains the $n$-tuple,
status$^{[i]}$. Letting status$^{[i]}(j)$ denote the $j$th entry in the
\[ n \text{-tuple, the entries are given by} \]
\[
\text{status}^{[i]}(j) = \begin{cases} 
0, & \text{if agent } i \text{ knows } q^{[i]}_j \text{ is assigned to another agent,} \\
1, & \text{otherwise.} 
\end{cases}
\]
\[
\text{Thus, status}^{[i]} \text{ is initialized as the } n \text{-tuple } (1, \ldots, 1). \text{ The initialization is summarized in Table I.}
\]

| Initialization for agent } i.
| \hline
| Assumes: } q^{[i]} := q \text{ for each } i \in I. \\
| 1: Compute a TSP tour of } q^{[i]}, \text{ tour}(q^{[i]}), \text{ and set } q^{[i]} := \text{tour}(q^{[i]}). \\
| 2: Compute the closest target in } q^{[i]}, \text{ and set } curr^{[i]} \text{ equal to its index: } \\
| curr^{[i]} := \arg \min_{j \in I} \{||q^i_j - p||\}. \\
| 3: Set next^{[i]} := curr^{[i]} + 1 \text{ (mod } n). \\
| 4: Set prev^{[i]} := curr^{[i]} - 1 \text{ (mod } n). \\
| 5: Set status^{[i]} := 1_n \text{ (i.e., an } n \text{-tuple containing } n \text{ ones).} \\
| \hline

At each communication round agent } i \text{ executes the algorithm COMM-RD displayed in Table I at the end of the paper. The following is an informal description.}

| Informal description of COMM-RD for agent } i.
| \hline
| Assumes: } status^{[i]}(s) = 0 \text{ for each } s \in \{prev^{[i]} + 1, prev^{[i]} + 2, \ldots, next^{[i]} - 1\} \mod n. \\
| 1: Broadcast msg^{[i]}, \text{ consisting of the target indices, prev}^{[i]}, \text{ curr}^{[i]}, \text{ and next}^{[i]}, \text{ the UID } i, \text{ and the distance to the current target, dist}^{[i]}. \\
| 2: for all messages, msg^{[k]}, received do \\
| 3: Set status^{[i]}(j) = 0 \text{ for each target } j \text{ from prev}^{[k]} + 1 \text{ to next}^{[k]} - 1 \text{ (mod } n) \text{ not equal to curr}^{[i]} \text{.} \\
| 4: if prev^{[k]} = next^{[k]} = curr^{[i]} \text{ then} \\
| 5: Set the status of curr^{[k]} to 0 \text{ (because it was missed in the previous step).} \\
| 6: if curr^{[i]} = curr^{[k]} \text{ and agent } i \text{ is farther from curr}^{[i]} \text{ than agent } k \text{ (ties broken by UIDs) then} \\
| 7: Set the status of curr}^{[i]} \text{ to assigned } (0'). \\
| 8: if curr}^{[i]} = curr^{[k]} \text{ and agent } i \text{ is closer than agent } k \text{ then} \\
| 9: Leave curr}^{[i]} \text{ unchanged. However, agent } k \text{ will set curr}^{[k]} \text{ to a new target. This target will be at least as far along the tour as the nearer of } curr^{[i]} \text{ and } curr^{[k]} \text{. So, set the status of next}^{[k]} \text{ and next}^{[i]} \text{ to assigned } (0'). \\
| 10: if the status of every target is assigned } (0') \text{ then} \\
| 11: Exit ETSP ASSIGNMENT and stop motion. (This can occur only if there are more agents than targets and every target is assigned.) \\
| 12: else \text{ Update next}^{[i]} \text{ to the next target in the tour with status available } (1'), \text{ next}^{[i]} \text{ to the next available target in the tour after curr}^{[i]} \text{, and prev}^{[i]} \text{ to the first available target in the tour before curr}^{[i]} \text{.} \\
| \hline

Fig. 4 gives an example of COMM-RD resolving a conflict between agents } i \text{ and } k, \text{ over curr}^{[i]} = curr^{[k]}. \text{ In this figure, all other agents are omitted.}

We are now ready to define the algorithm ETSP ASSIGNMENT for solving the target assignment problem.

**Definition 5.1 (ETSP ASSIGNMENT):** The ETSP ASSIGNMENT algorithm is the triplet consisting of the initialization of each agent (see Table I), the motion law in 2, and COMM-RD (see Table I), which is executed at each communication round.

**A. Correctness of ETSP ASSIGNMENT**

We will now prove the correctness of ETSP ASSIGNMENT. It should be noted that this result is valid for any communication graph which contains the } r \text{-disk graph as a subgraph. In order to prove correctness, let us first present some properties of the algorithm.}

**Lemma 5.2 (ETSP ASSIGNMENT properties):** During an execution of ETSP ASSIGNMENT the following statements hold:

(i) Once target } j \in I, \text{ is assigned to some agent, the assignment may change, but target } j \text{ remains assigned for all time.}

(ii) Agent } i \text{ is assigned to the target curr}^{[i]} \text{ which satisfies } status^{[i]}(curr^{[i]}) = 1.

(iii) For agent } i, \text{ status}^{[i]}(j) = 0, \text{ for each } j \in \{prev}^{[i]} + 1, prev}^{[i]} + 2, \ldots, next}^{[i]} - 1\} \mod n.

(iv) For agent } i, \text{ status}^{[i]}(j) = 0 \text{ only if target } j \text{ is assigned to some agent } k \neq i.

(v) If, for agent } i, \text{ status}^{[i]}(j) = 0 \text{ at some time } t_1, \text{ then status}^{[i]}(j) = 0 \text{ for all } t \geq t_1.

(vi) If agent } i \text{ receives msg}^{[k]} \text{ during a communication round, agent } i \text{ will set status}^{[i]}(j) = 0 \text{ for each } j \in \{prev}^{[k]} + 1, \ldots, next}^{[k]} - 1\} \mod n.

Proof: Statements (III) and (V) and (VI) follow directly from the initialization and the algorithm COMM-RD.

To see (I) consider an agent } i \text{, who is assigned to target } j. \text{ Agent } i \text{’s assignment can change only if it loses a conflict over target } j. \text{ In every conflict there is a winner and the winner remains assigned to target } j.

Statement (III) is initially satisfied since } prev}^{[i]} + 1 = curr}^{[i]} = next}^{[i]} - 1 \text{ implies that } \{prev}^{[i]} + 1, \ldots, next}^{[i]} - 1\} \mod n = \emptyset. \text{ Assume that statement (III) is satisfied before the execution of COMM-RD. At the end of COMM-RD, prev}^{[i]} \text{ is updated to the first target before curr}^{[i]} \text{ in the tour}
with status available (‘1’). If status\(^{[i]}\)(curr\(^{[i]}\)) = 1 then curr\(^{[i]}\) remains unchanged. If status\(^{[i]}\)(curr\(^{[i]}\)) = 0 then curr\(^{[i]}\) is increased to the first target with status available (‘1’). Finally, next\(^{[i]}\) is set to the first target after curr\(^{[i]}\) which is available. Thus, at the end of COMM-RD the status of prev\(^{[i]}\), curr\(^{[i]}\) and next\(^{[i]}\) are available, and status\(^{[i]}\)(j) = 0 for each target j \in \{prev\(^{[i]}\) + 1, ..., next\(^{[i]}\) − 1\} \mod n.

Statement (iv) is also initially satisfied since status\(^{[i]}\) = 1\(_n\) for each i \in \mathcal{J}. Assume Statement (iv) is satisfied before the execution of COMM-RD and that during this communication round agent i changes the status of a target j to assigned (‘0’). We will show that Statement (iv) is still satisfied upon completion of the execution of COMM-RD. In order for status\(^{[i]}\)(j) to be changed, agent i must have received a message, msg\(^{[k]}\), for which one of the following cases is satisfied: (1) Target j \neq curr\(^{[i]}\) lies between prev\(^{[k]}\) and next\(^{[k]}\) on the tour; (2) There is a conflict between agents i and k over target j which agent i loses; or, (3) There is a conflict between agents i and k which agent i wins and next\(^{[i]}\) = j or next\(^{[k]}\) = j.

In Case (1) either status\(^{[k]}\)(j) = 0 or curr\(^{[k]}\) = j, and thus target j is assigned. In Case (2) agent k won the conflict implying curr\(^{[k]}\) = j entering the communication round. Thus after the communication round, curr\(^{[i]}\) \neq j and target j is assigned to another agent. In Case (3), curr\(^{[i]}\) = curr\(^{[k]}\) \neq j, and agent k loses the conflict. In this case, agent k will change curr\(^{[k]}\) to the next available target on its tour. All targets from prev\(^{[k]}\) + 1 to next\(^{[k]}\) − 1 have been assigned. Also, during the communication round, agent k will receive msg\(^{[i]}\) and determine that all targets from prev\(^{[k]}\) + 1 to next\(^{[k]}\) − 1 are assigned. Thus, the next available target is at least as far along the tour as the farthest of next\(^{[k]}\) and next\(^{[i]}\). Thus, after the communication round, both next\(^{[i]}\) and next\(^{[k]}\) are assigned.

With these properties we are now ready to present the main result of this section.

**Theorem 5.3 (Correctness of ETSP ASSIGNMENT):** For any fixed n \in \mathbb{N}, ETSP ASSIGNMENT solves the target assignment problem.

**Proof:** Assume by way of contradiction that at some time \(t_1 \geq 0\) there are \(J \in \{1, ..., n-1\}\) targets unassigned, and for all time \(t \geq t_1\), J targets remain unassigned. By Lemma 5.2(ii) the n − J assigned targets remain assigned for all time, and thus it must be the same J targets which remain unassigned for all \(t \geq t_1\). Let \(\mathcal{J}\) denote the index set of the J unassigned targets. From our assumption, and by Lemma 5.2(iv) for every \(t \geq t_1\) and for every \(i \in \mathcal{I}\), status\(^{[i]}\)(j) = 1 for each \(j \in \mathcal{J}\). Now, among the n − J assigned targets, there is at least one target to which two or more agents are assigned. Consider one such target, call it \(j_1\), and consider an agent \(i_1\) with curr\(^{[i_1]}\) = \(j_1\). By Lemma 4.1 agent \(i_1\) will enter a conflict over \(j_1\) in finite time. Let us follow the loser of this conflict. The losing agent, call it \(i_2\), will set status\(^{[i_2]}\)(j_1) = 0, and will move to the next target in the tour it believes may be available, call it \(j_2\). Now, we know \(j_2\) is not in \(\mathcal{J}\), for if it were \(J−1\) targets would be unassigned contradicting our assumption. Moreover, by Lemma 5.2(ii), \(j_2 \neq j_1\). Thus, agent \(i_2\) will enter a conflict over \(j_2\) in finite time. After this conflict, the losing agent, call it \(i_3\), will set status\(^{[i_3]}\)(j_2) = 0 (because it lost the conflict), and from Lemma 5.2(vi) status\(^{[i_3]}\)(j_1) = 0. Again, agent \(i_3\)’s next target, \(j_3\) must not be in \(\mathcal{J}\), for if it were we would have a contradiction. Thus, repeating this argument \(n - J\) times we have that agent \(i_{n - J}\) loses a conflict over \(j_{n - J}\). After this conflict, we have status\(^{[n - J]}\)(j_k) = 0 for each \(k \in \{1, ..., n - J\}\), where \(j_{k_1} = j_{k_2}\) if and only if \(k_1 = k_2\). In other words, agent \(i_{n - J}\) knows that all \(n - J\) assigned targets have indeed been assigned. Also, by our initial assumption, status\(^{[n - J]}\)(j) = 1 for each \(j \in \mathcal{J}\). By Lemma 5.2(ii) agent \(i_{n - J}\)’s new current target must have status available (‘1’). Therefore, it must be that agent \(i_{n - J}\) will set curr\(^{[n - J]}\) to a target in \(\mathcal{J}\). Thus, after a finite amount of time, \(J−1\) targets are unassigned, a contradiction.

The following remark displays that the ETSP ASSIGNMENT algorithm does not solve the target assignment under the consistent knowledge assumption.

**Remark 5.4 (Consistent knowledge: cont’d):** Consider as in Remark 5.1 the consistent knowledge assumption for each agent’s target set. Specifically, consider two agents, 1 and 2, with initial target sets Q\(^{[1]}\)(0) = \{q_1, 2\}, Q\(^{[2]}\)(0) = \{q_1, q_2\}, and any initial positions such that p\(^{[1]}\)(0) = q_2. We will have curr\(^{[i]}\) = curr\(^{[j]}\) = 2. However, agent 2 will win the conflict over target 2. Thus, agent 1 will set status\(^{[1]}\)(2) = 0, and a complete assignment will not be possible.

**B. Time complexity for ETSP ASSIGNMENT**

In this section we will give an upper bound on the time complexity for ETSP ASSIGNMENT. We will show that \(\ell(n) \geq (1 + \epsilon)n^{1/d}\), for some \(\epsilon \in \mathbb{R}_{>0}\), ETSP ASSIGNMENT is asymptotically optimal among algorithms in the assignment-based motion class. Before doing this, let us first comment on the lower bound when the environment grows at a slower rate.

In what follows we show that if an agent arrives and remains at its assigned target for sufficiently long time, then it stays there for all subsequent times.

**Lemma 5.5:** Consider \(n\) agents executing ETSP ASSIGNMENT with communication range \(r > 0\) and assume the time delay between communication rounds, \(t_{max}\), satisfies \(t_{max} < r/v\). If there exists a time \(t_1\) and an agent \(i\) such that p\(^{[i]}\)(t) = curr\(^{[i]}\) for all \(t \in [t_1, t_1 + t_{max}]\), then p\(^{[i]}\)(t) = curr\(^{[i]}\) for all \(t > t_1 + t_{max}\).

**Proof:** Consider agent \(i\), who has been at target curr\(^{[i]}\) during the entire time interval \([t_1, t_1 + t_{max}]\). By the definition of \(t_{max}\) there was a communication round at some time \(t_2 \in [t_1, t_1 + t_{max}]\). Agent \(i\) must have won any conflicts it entered during this communication round since we have assumed that p\(^{[i]}\)(t_1 + t_{max}) = curr\(^{[i]}\). Thus every agent \(k\) within distance \(r\) of curr\(^{[i]}\) will have set status\(^{[k]}\)(curr\(^{[i]}\)) = 0. After the communication round at \(t_2\), every agent \(k\) with curr\(^{[k]}\) = curr\(^{[i]}\) must be a distance greater than \(r\) from curr\(^{[i]}\). Since \(t_{max} < r/v\), any agent \(k\) that enters a conflict with agent \(i\) at time \(t > t_2\) will be at a distance dist\(^{[k]}\) \in [0, r] from curr\(^{[i]}\). Agent \(k\) will lose the
conflict since dist\([k]\) > 0 = dist\([i]\). Thus, agent \(i\) will remain at curr\([t]\) for all \(t > t_1 + t_{\text{max}}\). □

With this lemma we are now able to provide an upper bound on the time complexity of our scheme.

**Theorem 5.6 (Time complexity for ETSP ASSIGNMENT):** Consider an environment \(E = [0, \ell(n)]^d\), \(d \geq 1\). If \(t_{\text{max}} < r/v\), then ETSP ASSIGNMENT solves the target assignment problem with time complexity in \(O(n^{(d-1)/d}\ell(n) + n)\). If, in addition, \(\ell(n) \geq (1 + \epsilon)rn^{1/d}\), where \(\epsilon \in \mathbb{R}_{>0}\), the time complexity is in \(\Theta(n^{(d-1)/d}\ell(n))\), and ETSP ASSIGNMENT is asymptotically optimal among algorithms in the assignment-based motion class.

**Proof:** Consider any initial agent positions, \(p^{(0)}\), and any \(n\)-tuple of target positions, \(q\). In the worst-case, some agent must travel around its entire ETSP tour, losing a conflict at each of the first \(n - 1\) targets in the tour. By Lemma 5.5 this agent can spend no more than \(t_{\text{max}}\) time units at each of the \(n - 1\) targets, before losing a conflict. Since each agent’s tour is a constant factor approximation of the optimal, the tour length is \(O(n^{(d-1)/d}\ell(n))\) (see Theorem 4.2). The agent will not follow the ETSP tour exactly because it will enter a conflict over each of the \(n - 1\) targets before actually reaching the target. However, the resulting path is no longer than the ETSP tour (since the agent could just follow the ETSP tour exactly if that happened to be the shortest path). Hence, the time complexity is \(O(n^{(d-1)/d}\ell(n) + t_{\text{max}}(n - 1)) = O(n^{(d-1)/d}\ell(n) + n)\). If \(\ell(n) = (2 + \epsilon)rn^{1/d}\), with \(\epsilon \in \mathbb{R}_{>0}\), we can combine this with Theorem 4.2 to get a time complexity in \(\Theta(n^{(d-1)/d}\ell(n))\). □

Notice that when \(\ell(n)\) satisfies the bound in Theorem 5.6 and \(\ell(n) \in O(n^{1/d})\), the time complexity is in \(O(n)\).

We have given complexity bounds for the case when \(r\) and \(v\) are fixed constants, and \(\ell(n)\) grows with \(n\). We allow the environment \(E(n)\) to grow with \(n\) so that, as more agents are involved in the task, their workspace is larger. An equivalent setup would be to consider \(\ell\) to be fixed, and allow \(r\) and \(v\) to vary inversely with the \(n\). That is, we can introduce a set of parameters, \(\ell, 1, v\) and \(\gamma(n)\) such that the time complexity will be the same as for the parameters \(r, v, \ell(n)\).

**Corollary 5.7 (Scaling radius and speed):** Consider \(n\) agents in the environment \(E = [0, 1]^d\), with speed \(\gamma(n) := v/\ell(n)\), and communication radius \(\gamma(n) := r/\ell(n)\), where \(\ell(n) \geq (1 + \epsilon)rn^{1/d}\), and \(\epsilon \in \mathbb{R}_{>0}\). Then ETSP ASSIGNMENT solves the target assignment problem with time complexity in \(\Theta(n^{(d-1)/d}\ell(n))\).

Scaling the communication radius \(r\) inversely with the number of agents arises in the study of wireless networks [12]. In wireless applications there are interference and media access problems between agents in the network. Since the agents are in a compact environment, the only way to limit this interference is to scale the communication radius inversely with the number of agents. Scaling the agent speed inversely with \(n\) appears in the study of the vehicle routing problem in [7]. The inverse scaling is required to avoid collisions in the presence of traffic congestion.

**C. Communication and computation complexity**

In our notion of time complexity we have emphasized the complexity due to the motion of the agents. Here we will briefly classify the complexity of computation and communication for ETSP ASSIGNMENT. (i) Initialization: As reviewed in Section II.C, we can compute a constant factor approximation ETSP tour in time \(O(n^2)\). This is the most expensive computation and thus the complexity of initialization is in \(O(n^2)\). (ii) Communication complexity per round: At each round agent \(i\) broadcasts a message of length \(O(\log n)\), msg\([i]\), and we consider this to be one unit of communication. In the worst-case, each agent receives \(n\) messages, and so, the worst-case communication complexity is in \(O(n)\) [8]. (iii) Computation complexity per round: For each message received, agent \(i\) sets status\([i]\)\([s]\) = 0 for \(s\) from prev\([k]\) + 1 to next\([k]\) - 1. In the worst-case, this operation is \(O(n)\) and must be performed for \(n\) messages. This is the dominant computation in COMM-RD and thus the worst-case computation complexity in each round is \(O(n^2)\).

It should be noted that in the case when the communication graph is not even connected (let alone complete as is required to achieve these worst-case bounds), the complexity will be considerably lower.

**D. Simulations**

We have simulated ETSP ASSIGNMENT in \(\mathbb{R}^2\) and \(\mathbb{R}^3\). To compute the ETSP tour we have used the concorde TSP solver.\(^2\) A representative simulation for 15 agents in \([0,100]^3 \subset \mathbb{R}^3\) with \(r = 15\) and \(v = 1\) is shown in Fig. 5. The initial configuration shown in Fig. 5(a) consists of uniformly randomly generated target and agent positions.

**E. The case of \(n\) agents and \(m\) targets**

It should be noted that the ETSP ASSIGNMENT algorithm works without any modification when there are \(n\) agents and \(m\) targets. If \(m \geq n\), at completion, \(n\) targets are assigned and \(m - n\) targets are not. When, \(m < n\), at completion, all \(m\) targets are assigned, and the \(n - m\) unassigned agents come to a stop after losing a conflict at each of the \(m\) targets. The complexity bounds are changed as follows.

The lower bound on the assignment-based motion class in Theorem 4.2 holds when \(m \geq n\), and \(\ell(n) \geq (1 + \epsilon)rn^{1/d}\) (notice the \(m\) instead of \(n\)). The bound becomes \(\Omega(\ell(n)m^{1/d})\). If \(m = Cn\) where \(C \in \mathbb{R}_{\geq 1}\), (i.e., \(m \geq n\) but they grow at the same rate), then the bound becomes \(\Omega(\ell(n)n^{(d-1)/d})\).

The upper bound on ETSP ASSIGNMENT holds for any \(n\) and \(m\), and becomes \(O(\ell(n)n^{(d-1)/d})\), where \(N := \min\{n, m\}\). So our final result would be that if \(m = Cn\) where \(C \in \mathbb{R}_{\geq 1}\) and when \(\ell(n) \geq (1 + \epsilon)rn^{1/d}\), then ETSP ASSIGNMENT solves the target assignment problem in \(\Omega(\ell(n)n^{(d-1)/d})\). That is, among all algorithms in the assignment-based motion class, ETSP ASSIGNMENT is asymptotically optimal.

\(^2\)The concorde TSP solver is available for research use at http://www.tsp.gatech.edu/concorde/index.html
VI. CONCLUSIONS

We have developed the ETSP ASSIGNMENT algorithm for solving the full knowledge target assignment problem. We derived worst-case asymptotic bounds on the time complexity, and we showed that among a certain class of algorithms, ETSP ASSIGNMENT is asymptotically optimal. There are many possible extensions of this work. We have not given a lower bound on the time complexity of ETSP ASSIGNMENT when \( \ell(n) \leq \ell_{\text{exit}} \). Also, the problem is unsolved under the more general consistent knowledge assumption. We would like to extend the ETSP ASSIGNMENT algorithm to agents with nonholonomic motion constraints. Also, it would be interesting to consider a sensor based version of this problem, where agents acquire target positions through local sensing. Finally, to derive asymptotic time bounds, we made some assumptions on the communication structure at each communication round. An interesting avenue for future study would be to more accurately address the communication issues in robotic networks.

REFERENCES

[1] B. Korte and J. Vygen, Combinatorial Optimization: Theory and Algorithms, No. 21 in Algorithmics and Combinatorics, New York: Springer Verlag, 3 ed., 2005.
[2] H. W. Kuhn, “The Hungarian method for the assignment problem,” Naval Research Logistics, vol. 2, pp. 83–97, 1955.
[3] D. P. Bertsekas and D. A. Castaño, “Parallel synchronous and asynchronous implementations of the auction algorithm,” Parallel Computing, vol. 17, pp. 707–732, 1991.
[4] D. P. Bertsekas and J. N. Tsitsiklis, Parallel and Distributed Computation: Numerical Methods, Belmont, MA: Athena Scientific, 1997.
[5] D. A. Castaño and C. Wu, “Distributed algorithms for dynamic reassignment,” in IEEE Conf. on Decision and Control, (Maui, HI), pp. 13–18, Dec. 2003.
[6] B. J. Moore and K. M. Passino, “Distributed task assignment for mobile agents,” IEEE Transactions on Automatic Control, 2006. to appear.
[7] V. Sharma, M. Savchenko, E. Frazzoli, and P. Voulgaris, “Time complexity of sensor-based vehicle routing,” in Robotics: Science and Systems (S. Thrun, G. Sukhatme, S. Schaal, and O. Brock, eds.), pp. 297–304, Cambridge, MA: MIT Press, 2005.
[8] S. Martínez, F. Bullo, J. Cortés, and E. Frazzoli, “On asynchronous robotic networks – Part I: Models, tasks and complexity notions, & Part II: Time complexity of rendezvous and deployment algorithms,” IEEE Transactions on Automatic Control, Apr. 2005. Submitted.
[9] N. Christofides, “Worst-case analysis of a new heuristic for the traveling salesman problem,” Tech. Rep. 388, Carnegie-Mellon University, Pittsburgh, PA, Apr. 1976.
[10] M. Penrose, Random Geometric Graphs. Oxford Studies in Probability, Oxford, UK: Oxford University Press, 2003.
[11] P. Gupta and P. R. Kumar, “The capacity of wireless networks,” IEEE Transactions on Information Theory, vol. 46, no. 2, pp. 388–404, 2000.