Two topologise on the lattice of Scott closed subsets*

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Abstract

For a poset $P$, let $\sigma(P)$ and $\Gamma(P)$ respectively denote the lattice of its Scott open subsets and Scott closed subsets ordered by inclusion, and set $\Sigma P = (P, \sigma(P))$. In this paper, we discuss the lower Vietoris topology and the Scott topology on $\Gamma(P)$ and give some sufficient conditions to make the two topologies equal. We built an adjunction between $\sigma(P)$ and $\sigma(\Gamma(P))$ and proved that $\Sigma P$ is core-compact iff $\Sigma \Gamma(P)$ is core-compact iff $\Sigma \Gamma(P)$ is sober, locally compact and $\sigma(\Gamma(P)) = \upsilon(\Gamma(P))$ (the lower Vietoris topology). This answers a question in [17]. Brecht and Kawai [2] asked whether the consonance of a topological space $X$ implies the consonance of its lower powerspace, we give a partial answer to this question at the last part of this paper.

Keywords: Scott closed sets; adjunction; distributive continuous lattice; lower powerspace; consonance.

1. Introduction

The lower powerspace over a topological space $X$ is the set of closed subsets of $X$ with the lower Vietoris topology. The lower powerspace coincides with the lower powerdomain for continuous dcpos with the Scott topology, where the latter construction is used in modelling non-deterministic computation (see [1] [19]). There naturally arise a question when the lower Vietoris topology and the Scott topology coincide on the lattice of closed subsets of a topological space. For a poset $P$, we give some sufficient conditions such that the lower Vietoris topology and the Scott topology coincide on $\Gamma(P)$. We observe that there is an adjunction between $\sigma(P)$ and $\sigma(\Gamma(P))$, which serves as a useful tool in studying the relation between $P$ and $\Gamma(P)$. Particularly, we obtain the conclusion that $\langle P, \sigma(P) \rangle$ is core-compact iff $\langle \Gamma(P), \sigma(\Gamma(P)) \rangle$ is a sober and locally compact space. This positively answers a question in [17]. We show that the two topologies generally don’t coincide on the lattice of closed subsets of a topological space that is not a Scott space.

In domain theory, the spectral theory of distributive continuous lattices and the duality theorem have been extensively studied. Hoffmann [12] and Lawson [14] independently proved that a directed complete poset (or dcpo for short) is continuous iff the lattice of its Scott open subsets is completely distributive. Gierz and Lawson [8] proved that quasicontinuous dcpos equipped with the Scott topologies are precisely the spectrum of distributive hypercontinuous lattices. The correspondence between $P$ and $\Gamma(P)$ has also been studied (see [6] [13] [14] [21] [11]). A poset $P$ is continuous (resp., quasicontinuous, algebraic, quasialgebraic) if and only if $\Gamma(P)$ is continuous (resp., quasicontinuous, algebraic, quasialgebraic). Using the pair of adjoint we can give a unified proof.

We see that if $L$ is a continuous lattice such that $\sigma(L^\text{op}) = \upsilon(L^\text{op})$, then $(L^\text{op}, \sigma(L^\text{op}))$ is a sober and locally compact space. We give two equivalent conditions of $\sigma(L^\text{op}) = \upsilon(L^\text{op})$ for continuous lattices. The previous result about core-compact posets implies that if $L$ is a distributive continuous lattice such that the hull-kernel topology of $\text{Spec}L$ is just the Scott topology then $\sigma(L^\text{op}) = \upsilon(L^\text{op})$. We wonder whether the condition $\sigma(L^\text{op}) = \upsilon(L^\text{op})$ implies that $\text{Spec}L$ is a Scott space. If not, what other conditions are needed. Actually, characterize those distributive continuous lattices for which the spectrum is a dcpo equipped with the Scott topology is an open problem [20]. The case of distributive algebraic lattices is simpler. The notion of jointly Scott continuity occurs in our proofs. It is well-known that a complete lattice is sober with respect to the Scott topology if it is jointly Scott continuous [2 Corollary II-1.12]. We give an example of a complete lattice which is sober with the Scott topology but fails to be jointly Scott continuous.

Consonance is an important topological property ([2] [4] [5] [18]). It is proved in [5] that every regular Čech-complete space is consonant. Thus every completely metrizable space is consonant. Bouziad [4] proved

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that the set of rational numbers with subspace topology from $\mathbb{R}$ is not consonant. A space $X$ is consonant if and only if, for every topological space $Y$, the compact-open and the Isbell topologies agree on the space of continuous maps from $X$ to $Y$ (see [9]). Brecht and Kawai [2] proved that the consonance of a $T_0$ space $X$ is equivalent to the commutativity of the upper and lower powerspaces of $X$. There they asked whether the consonance of $X$ implies the consonance of its lower powerspace. We will give a partial answer at the last part of this paper.

2. Preliminaries

We refer to ([1 7 9]) for some concepts and notations of domain theory that will be used in the paper.

For a poset $P$ and $A \subseteq P$, let $\downarrow A = \{ x \in P : \exists a \in A, x \leq a \}$ and $\uparrow A = \{ x \in P : \exists a \in P, a \leq x \}$. For $x \in P$, we write $\downarrow x$ for $\downarrow \{ x \}$ and $\uparrow x$ for $\uparrow \{ x \}$. A set $A$ is called saturated if and only if $A = \uparrow A$. If $F$ is a finite subset of $P$, we denote by $F \subseteq^f P$. A poset $P$ is called a directed complete poset (dcpo, for short) if every directed subset $D$ of $P$ has a supremum.

For a poset $P$, the upper topology $\nu(P)$ is the topology generated by taking the collection of sets $\{ P \setminus \downarrow x : x \in P \}$ as a subbase, the lower topology $\omega(P)$ on $P$ is defined dually. A subset $A$ of $P$ is called Scott closed if $A = \uparrow A$ and for any directed set $D \subseteq A$, $\bigvee D \in A$ whenever the least upper bound $\bigvee D$ exists. The Scott topology $\sigma(P)$ consists of the complements of all Scott closed sets of $P$. The topology $\lambda(P) = (\sigma(P) \vee \omega(P))$ is called the Lawson topology on $P$.

Given a topological space $X$, let $\mathcal{O}(X)$ be the topology on $X$, then $\mathcal{O}(X)$ is a complete lattice ordered by inclusion. We denote by $(C(X), \nu(C(X)))$ the lattice of closed subsets of $X$ equipped with the lower Vietoris topology, which is generated by $\bigtriangleup U = \{ A \in C(X) : A \cap U \neq \emptyset \}$ as a subbase, where $U$ ranges over $\mathcal{O}(X)$. A topological space is core-compact if and only if $\mathcal{O}(X)$ is a continuous lattice. An arbitrary nonempty subset $A$ of a topological space $X$ is irreducible if $A \subseteq B \cup C$ for closed subsets $B$ and $C$ implies $A \subseteq B$ or $A \subseteq C$. $X$ is called sober if for every irreducible closed set $A$, there exists a unique $x \in X$ such that $\{ x \} = A$.

For a poset $P$ and $x, y \in P$, we say that $x$ is way-below $y$, in symbols $x \ll y$, if for any directed subset $D$ of $P$ that has a least upper bound in $P$, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$. For any non-empty subsets $F, G$ of $P$, we say that $F$ is way-below $G$ and write $F \ll G$ if for any directed subset $D$ of $P$ that has a least upper bound in $P$, $\bigvee D \in \uparrow G$ implies $\bigvee \uparrow D \neq \emptyset$. $P$ is continuous if the set $\{ d \in D : d \ll x \}$ is directed and $x = \bigvee \{ d \in P : d \ll x \}$ for all $x \in P$. Let $K(P) = \{ x \in P : x \ll x \}$. $P$ is algebraic if for all $x \in P$ the set $\downarrow x \cap K(P)$ is directed and $x = \bigvee (\downarrow x \cap K(P))$.

A poset $P$ is called quasicontinuous poset (resp., quasialgebraic poset) if for all $x \in P$ and $U \in \sigma(P)$, $x \in U$ implies that there is a non-empty finite set $F \subseteq P$ such that $x \in \text{int}_\sigma \uparrow F \subseteq U$ (resp., $x \in \text{int}_\sigma \uparrow F = \uparrow F \subseteq U$).

For a complete lattice $L$ and $x, y \in L$, we say that $x \prec y$ iff whenever the intersection of a non-empty collection of upper sets is contained in $\uparrow y$, then the intersection of finitely many is contained in $\uparrow x$. $x \prec y$ in $\text{int}_\sigma \uparrow x$ and $x \prec y$ implies $x \ll y$. $L$ is hypercontinuous if $\{ d \in L : d \prec x \}$ is directed and $x = \bigvee \{ d \in L : d \prec x \}$ for all $x \in L$. $L$ is hyperalgebraic provided $x = \bigvee \{ y \in L : y \ll x \}$ for all $x \in L$.

Let $L$ be a complete lattice, we say that $x$ is way-below $y$, in notation $x \ll y$, if for any subset $A$ of $L$, $y \leq \bigvee A$ implies $x \leq a$ for some $a \in A$. $L$ is prime continuous if $x = \bigvee \{ d \in L : d \ll x \}$ for all $x \in L$.

The next two well-known propositions give various equivalent formulations for completely distributive lattice and hypercontinuous lattice separately.

Proposition 2.1. ([7]) Let $L$ be a complete lattice, the following conditions are equivalent:

1. $L$ is prime-continuous,
2. $L$ is a completely distributive lattice,
3. $L$ is distributive and both $L$ and $L^{op}$ are continuous lattices,

Theorem 2.2. ([8]) Let $L$ be a complete lattice, then the following conditions are equivalent:

1. $L$ is a hypercontinuous lattice,
2. $L$ is a continuous lattice and $\sigma(L) = \nu(L)$,
3. $L$ is a continuous lattice, $L^{op}$ is a quasicontinuous lattice, and the bi-Scott topology agrees with the Lawson topology($\lambda(L) = \sigma(L) \vee \sigma(L^{op})$).

The following lemma characterizes the binary relation $\prec$ on the complete lattice of Scott open subsets. It is easy to see that this conclusion is also true for any topological space.

Lemma 2.3. Let $P$ be a poset and $U, V \in \sigma(P)$. $U \prec V$ if and only if $U \subseteq \uparrow F \subseteq V$ for some $F \subseteq^f P$. 

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Proof. \((\Leftarrow)\) Let \(F = \{x_1, x_2, \ldots, x_n\}\), then \(V \not\subseteq P \downarrow x_i\), for each \(1 \leq i \leq n\). We claim that \(V \not\subseteq \bigcap_{i=1}^{n} (\sigma(P) \downarrow \sigma(P)(P \downarrow x_i)) \subseteq \uparrow_{\sigma(P)} U\). Indeed, for each \(W \in \bigcap_{i=1}^{n} (\sigma(P) \downarrow \sigma(P)(P \downarrow x_i))\), \(W \cap \downarrow x_i \not= \emptyset\) for each \(1 \leq i \leq n\), then \(U \subseteq F \subseteq W\).

\((\Rightarrow)\) If \(U \not\subseteq V\), then there exists \(\{W_i \in \sigma(P) : 1 \leq i \leq n\}\) such that \(V \not\subseteq \bigcap_{i=1}^{n} (\sigma(P) \downarrow \sigma(P)(P \downarrow x_i)) \subseteq \uparrow_{\sigma(P)} U\), which implies that \(V \not\subseteq \sigma(P)\) for each \(i\). Let \(F = \{x_1, x_2, \ldots, x_n\}\), where \(x_i \in V \setminus W_i\) for each \(1 \leq i \leq n\). Now we show that \(U \subseteq F\). If not, there exists \(x \in U \setminus \uparrow F\), i.e., \(F \subseteq P \setminus \downarrow x\). Then \(P \setminus \downarrow x \not\subseteq W_i\) for each \(1 \leq i \leq n\), but \(U \not\subseteq P \setminus \downarrow x\), which is a contradiction. \(\Box\)

M. Erné \[6\] and T. Yokoyama \[23\] respectively give the following Lemma, and prove the result that a spectral Scott space (i.e. a poset on which the Scott topology is spectral) is a quasialgebraic domain.

Lemma 2.4. Let \(P\) be a poset. A Scott open subset \(U\) of \(P\) is Scott compact if and only if there exists \(F \subseteq P\) such that \(U = \text{int}_{\sigma} \uparrow F = \uparrow F\).

The correspondence between \(P\) and \(\sigma(P)\) is enumerated below, where \((2)\) and \((3)\) is a direct consequences of Lemma 2.3 and 2.4.

Theorem 2.5. \([7, 16, 22]\) Let \(P\) be a poset,

\begin{enumerate}
  \item \(P\) is continuous (algebraic) iff the lattice \(\sigma(P)\) of all Scott open sets is a completely distributive lattice (completely distributive algebraic lattice).
  \item \(P\) is quasicontinuous iff the lattice \(\sigma(P)\) of all Scott open sets is a hypercontinuous lattice.
  \item \(P\) is quasialgebraic iff the lattice \(\sigma(P)\) of all Scott open sets is an algebraic lattice iff \(\sigma(P)\) is a hyperalgebraic lattice.
\end{enumerate}

3. The correspondent properties between \(P\) and \(\Gamma(P)\)

Recently, H. Miao et al \[17\] proved that for a well-filtered dcpo \(L\), \(\Sigma L\) is locally compact if and only if \(\Sigma \Gamma(L)\) is a locally compact space. They further asked the following question \[17\] Problem 5.9:

For a poset \(P\), if \(\Sigma P\) is core-compact, must \(\Sigma \Gamma(P)\) be locally compact?

In this section, we investigate conditions when the lower Vietoris topology and the Scott topology on \(\Gamma(P)\) coincide for a poset \(P\) and give an positive answer to the above question. Particularly, we show that for a poset \(L\), \(\Sigma P\) is core-compact iff \(\Sigma \Gamma(P)\) is core-compact iff \(\Sigma \Gamma(P)\) is sober, locally compact and \(\sigma(\Gamma(P)) = \nu(\Gamma(P))\).

When we are working on continuous lattices, we notice that the condition \(\sigma(L^{op}) = \nu(L^{op})\) is very important.

Firstly, several well-known lemmas are needed.

Lemma 3.1. \([8, 21]\) A complete lattice \(L\) is a quasicontinuous (quasialgebraic) lattice iff \(\omega(L)\) is a continuous (algebraic) lattice.

Lemma 3.2. \([7\] Theorem V-5.6.\)] For a sober space \(X\), the lattice \(\mathcal{O}(X)\) of open subsets is continuous if \(X\) is locally compact.

The following lemma is easy to verify.

Lemma 3.3. For a complete lattice \(L\), both \((L, \nu(L))\) and \((L, \omega(L))\) are sober.

Proposition 3.4. Let \(L\) be a continuous lattice, if \(L\) satisfies the condition that \(\nu(L^{op}) = \sigma(L^{op})\), then \((L^{op}, \sigma(L^{op}))\) is a sober and locally compact space.

Proof. By Lemma 3.1 \(\sigma(L^{op}) = \nu(L^{op}) = \omega(L)\) are continuous lattices. Thus \((L, \sigma(L^{op}))\) is a sober and locally compact space by Lemma 3.2 and 3.3. \(\Box\)

We give a sufficient condition such that \(\sigma(\Gamma(P)) = \nu(\Gamma(P))\) for a poset \(P\), which is crucial for further discussion.

Proposition 3.5. Let \(P\) be a poset. If \(\Sigma(\prod P) = \prod(\Sigma P)\) for each \(n \in \mathbb{N}\), then \(\sigma(\Gamma(P)) = \nu(\Gamma(P))\).

Proof. We only need to prove that \(\sigma(\Gamma(P)) \subseteq \nu(\Gamma(P))\).

At first, for each \(n \in \mathbb{N}\), we define a map \(s_n : \prod P \rightarrow \Gamma(P)\) as follows:

\[\forall (x_1, x_2, \ldots, x_n) \in \prod P, \ s_n(x_1, x_2, \ldots, x_n) = \bigcup_{k=1}^{n} \downarrow x_k.\]
We claim that $s_n$ preserves existing directed sups. Let $\{ (x_{i_1}, x_{i_2}, \ldots, x_{i_n}) : i \in I \}$ be a directed subset of $\prod^n P$ such that the supremum of $\{ (x_{i_1}, x_{i_2}, \ldots, x_{i_n}) : i \in I \}$ exists in $\prod^n P$, then $\bigvee_{i \in I} (x_{i_1}, x_{i_2}, \ldots, x_{i_n}) = (\bigvee_{i \in I} x_{i_1}, \bigvee_{i \in I} x_{i_2}, \ldots, \bigvee_{i \in I} x_{i_n})$. We have

$$s_n\left(\bigvee_{i \in I} (x_{i_1}, x_{i_2}, \ldots, x_{i_n})\right) = \bigvee_{k=1}^n \left(\bigvee_{i \in I} x_{ki}\right) = \bigcup_{k=1}^n \bigcup_{i \in I} x_{ki} = \bigvee_{i \in I} s_n(x_{i_1}, x_{i_2}, \ldots, x_{i_n}).$$

Thus $s_n$ is a Scott continuous map from $\prod^n P$ into $\Gamma(P)$.

Next, let $H$ be a Scott open subset of $\Gamma(P)$ and $A \in H$. With loss of generality, we assume $A \neq \emptyset$. Note that since $A = \bigcup \{ \downarrow F : F \subseteq A \}$ and $\downarrow F : F \subseteq A$ is a directed family in $\Gamma(P)$, there exists a non-empty finite subset $F$ of $A$ such that $\downarrow F \in H$. Let $F = \{ x_1, x_2, \ldots, x_n \}$, then $s_n(x_1, x_2, \ldots, x_n) \in \downarrow F \in H$. It follows that $(x_1, x_2, \ldots, x_n) \in s_n^{-1}(H)$. Since $s_n$ is Scott continuous and $\prod^n P = \prod(\prod^I P)$, there exists a family of Scott open subset $U_k$, $k = 1, 2, \ldots, n$ of $P$ such that $(x_1, x_2, \ldots, x_n) \in U_1 \times U_2 \times \cdots \times U_n \subseteq s_n^{-1}(H)$.

Since $x_k \in A$ for $1 \leq k \leq n$, we have $A \in \bigvee_{k=1}^n \downarrow U_k = \{ B \in \Gamma(P) : B \cap U_k \neq \emptyset \}$. It follows that $A \in \bigcup_{k=1}^n \downarrow U_k$ and $\bigvee_{k=1}^n \downarrow U_k \in H(P)$). For any $B \in \bigcup_{k=1}^n \downarrow U_k$, there exists $y_k \in B \cap U_k$ for $1 \leq k \leq n$. Since $(y_1, y_2, \ldots, y_n) \in s_n^{-1}(H)$, we have $\bigcup_{k=1}^n \downarrow y_k \in H$. It follows that $B \in H$, i.e., $A \in \bigcup_{k=1}^n \downarrow y_k \subseteq H$. \qed

**Lemma 3.6.** ([7] Theorem II-4.13) Let $P$ be a dcpo, $(P, \sigma(P))$ is core-compact if and only if $\Sigma(Q \times P)$ is Scott continuous for every dcpo $Q$.

It is not difficult to see that the above lemma is still true when $P$ and $Q$ are posets.

**Proposition 3.7.** Let $P$ be a poset, if $(P, \sigma(P))$ is core-compact, then $\sigma(\Gamma(P)) = v(\Gamma(P))$. Moreover, $(\Gamma(P), \sigma(\Gamma(P)))$ is sober and locally compact.

**Proof.** If $(P, \sigma(P))$ is core-compact, then $\Sigma^n P = \Sigma(\Sigma^k P)$ for each $n \in \mathbb{N}$. It follows that $\sigma(\Gamma(P)) = v(\Gamma(P))$ by Proposition 3.5. Let $L = \sigma(P)$ which is a continuous lattice by assumption and $\Gamma(P) \cong L^{op}$. By Proposition 3.4 $(\Gamma(P), \sigma(\Gamma(P)))$ is a sober and locally compact space. \qed

We have answered Problem 5.9 raised in [17]. The following result appears in J. Goubault-Larrecq’s blog and is given by Matthew de Brecht.

**Lemma 3.8.** ([3]) Let $P$ and $Q$ be two posets. If both $(P, \sigma(P))$ and $(Q, \sigma(Q))$ are first-countable, then $\Sigma P \times \Sigma Q = \Sigma P \times Q$.

Since a countable product of first-countable spaces is first-countable, we have the following result:

**Proposition 3.9.** Let $P$ be a poset, if $(P, \sigma(P))$ is first-countable, then $\sigma(\Gamma(P)) = v(\Gamma(P))$.

We observe that there is an adjunction between $\sigma(P)$ and $\sigma(\Gamma(P))$, which serves as a useful tool in studying the relation between $P$ and $\Gamma(P)$. For the standard theory of adjunctions or Galois connection we refer to ([7], Section O-3).

**Proposition 3.10.** Let $P$ be a poset.

1. $\eta : P \to \Gamma(P)$, $\forall x \in P, \eta(x) = \downarrow x$. Then $\eta$ is Scott continuous.
2. A map $f : \sigma(P) \to \sigma(\Gamma(P))$ is defined by $f(U) = \bigvee U = \{ A \in \Gamma(P) : A \cap U \neq \emptyset \}$. Then $f$ preserves arbitrary sups.
3. The map $\eta^{-1} : \sigma(\Gamma(P)) \to \sigma(P)$ be defined by $\eta^{-1}(U) = \{ x \in P : \downarrow x \in U \}$. Then $\eta^{-1} \circ f = 1_{\sigma(P)}, f \circ \eta^{-1} \leq 1_{\sigma(\Gamma(P))}$, which implies that $(\eta^{-1}, f)$ is an adjunction.

**Proof.** (1) Straightforward.

(2) Obviously, $f$ is monotone. Let $\{ U_i : i \in I \}$ be arbitrary subset of $\sigma(P)$. $f(\bigcup_{i \in I} U_i) = \{ A \in \Gamma(P) : A \cap \bigcup_{i \in I} U_i \neq \emptyset \} = \bigvee_{i \in I} f(U_i)$.

(3) Because $\eta$ is Scott continuous, $\eta^{-1}$ preserves arbitrary sups and finite infs. For any $U \in \sigma(P), x \in \eta^{-1}(f(U)) \Leftrightarrow \eta(x) \in f(U) \Leftrightarrow \downarrow x \cap U \neq \emptyset \Leftrightarrow x \in U$, hence $\eta^{-1} \circ f = 1_{\sigma(P)}$. For any $U \in \sigma(\Gamma(P)), A \in f \circ \eta^{-1}(U) \Leftrightarrow A \cap \eta^{-1}(U) \neq \emptyset \Rightarrow A \in U$, i.e., $f \circ \eta^{-1} \leq 1_{\sigma(\Gamma(P))}$.

For a complete lattice $L$, the condition that $\Sigma^n L = \prod^n (\Sigma L)$ for each $n \in \mathbb{N}$ relates to the concept of jointly continuous semilattice.
Definition 3.11. Let $L$ be a semilattice, the sup operation is jointly continuous with respect to the Scott topology provided that the mapping 

$$(x, y) \mapsto x \lor y : (L, \sigma(L)) \times (L, \sigma(L)) \to (L, \sigma(L))$$

is continuous in the product topology.

Proposition 3.12. ([7 Corollary II.1.12]) If $L$ is a dcpo and a semilattice such that the sup operation is jointly Scott continuous, then $(L, \sigma(L))$ is a sober space.

Now we complete the conclusion in Proposition 3.7.

Theorem 3.13. Let $P$ be a poset, then the following statements are equivalent:

1. $(P, \sigma(P))$ is core-compact.
2. $(\Gamma(P), \sigma(\Gamma(P)))$ is core-compact.
3. $(\Gamma(P), \sigma(\Gamma(P)))$ is sober and locally compact.
4. $(\Gamma(P), \sigma(\Gamma(P)))$ is sober and locally compact with $\sigma(\Gamma(P)) = \nu(\Gamma(P))$.

Proof. $(1) \Rightarrow (4) \Rightarrow (3)$. Proposition 3.7.

$(3) \Rightarrow (2)$. The open subsets of a locally compact space form a continuous lattice.

$(2) \Rightarrow (1)$. If $(\Gamma(P), \sigma(\Gamma(P)))$ is locally compact, then $\sigma(\Gamma(P))$ is a continuous lattice. By Proposition 3.10 $\sigma(P)$ is also a continuous lattice.

$(2) \Rightarrow (3)$. If $(\Gamma(P), \sigma(\Gamma(P)))$ is core-compact, then $\Sigma(\Gamma(P) \times \Gamma(P)) = \Sigma(\Gamma(P)) \times \Sigma(\Gamma(P))$ by Lemma 3.6. Obviously, the sup operation is jointly Scott continuous. Thus $(\Gamma(P), \sigma(\Gamma(P)))$ is sober and locally compact by Proposition 3.12 and Lemma 3.2. \square

It is proved by different method in [17] that for a poset $P$, if $\Sigma(\Gamma(P))$ is locally compact then $\Sigma L$ is core-compact. It seems that the equivalence in Theorem 3.13 only works for posets with the Scott topology. In the following example, we give a topological space $X$ and show that $X$ is a compact Hausdorff space while $\Sigma C(X)$ is sober but not locally compact. Moreover, $\sigma(C(X)) \neq \nu(C(X))$.

Example 3.14. Consider the subset $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}^+\}$ of the real line $\mathbb{R}$, with the subspace topology. Let $C_1(X) = \{A : A \subseteq \{\frac{1}{n} : n \in \mathbb{N}^+\}\}$ and $C_2(X) = \{B \cup \{0\} : B \subseteq \{\frac{1}{n} : n \in \mathbb{N}^+\}\}$, then $C(X) = C_1(X) \cup C_2(X)$.

Clearly, $X$ is a compact Hausdorff space. We will prove that $C(X)$ is sober but not locally compact with respect to the Scott topology. Hence, $\sigma(C(X)) \neq \nu(C(X))$ also holds.

1. We show that $\uparrow_{C(X)} A$ is Scott open for each $A \in C_1(X)$. Indeed, for any directed family $\{C_i\}_{i \in I}$ of closed subsets, if $\{C_i\}_{i \in I}$ is a finite family of $C_1(X)$, then there exists a largest element; if $\{C_i\}_{i \in I}$ is an infinite family of $C_1(X)$, then there exists some $I$ such that $\bigvee_{i \in I} C_i = \bigcup_{i \in I} C_i$; otherwise there exists some $i \in I$ such that $\bigvee_{i \in I} C_i = \bigcup_{i \in I} C_i$. In all cases, $\bigvee_{i \in I} C_i \in \uparrow_{C(X)} A$.

Let $\mathcal{A}$ be a Scott closed subset of $C(X)$ and $\mathcal{A}$ be the set of maximal elements of $\mathcal{A}$. Then $\mathcal{A} = \downarrow_{C(X)} \operatorname{Max}\mathcal{A}$. We claim that $\mathcal{A}$ cannot be irreducible whenever $\mathcal{A}$ is an infinite subset of $C(X)$, which implies that $C(X)$ is sober with respect to the Scott topology.

Case 1. There exist $A_1, A_2$ in $C_1(X) \cap \operatorname{Max}\mathcal{A}$ such that $A_1 \setminus A_2 \neq \emptyset$ and $A_2 \setminus A_1 \neq \emptyset$. Then $\uparrow_{C(X)} A_1$ and $\uparrow_{C(X)} A_2$ are two Scott open subsets of $C(X)$.

Case 2. There is only one $A \in C_1(X) \cap \operatorname{Max}\mathcal{A}$. Then must be some $B \in C_2(X)$ such that $A \setminus B \neq \emptyset$ and $B \setminus (A \cup \{0\}) \neq \emptyset$. Let $x \in B \setminus (A \cup \{0\}) \neq \emptyset$, then $\uparrow_{C(X)} A$ and $\uparrow_{C(X)} \{x\}$ are two Scott open subsets of $C(X)$ which intersect with $A$. But $\uparrow_{C(X)} A \cap \uparrow_{C(X)} \{x\} \cap A = \emptyset$ by the maximality of $A_1$ and $A_2$.

Case 3. $\mathcal{A} \subseteq C_2(X)$. Let $B_1 \cup \{0\}$ and $B_2 \cup \{0\}$ are two different elements in $\operatorname{Max}\mathcal{A}$. Then $B_1 \setminus B_2 \neq \emptyset$ and $B_2 \setminus B_1 \neq \emptyset$. For any $F_1 \subseteq f B_1$ and $F_2 \subseteq B_2$, then $\uparrow_{C(X)} F_1$ and $\uparrow_{C(X)} F_2$ are two Scott open subsets of $C(X)$ which intersect with $A$. Suppose that $\mathcal{A}$ is an irreducible closed subset. We have that $\uparrow_{C(X)} F_1 \cap \uparrow_{C(X)} F_2 \cap \mathcal{A} \neq \emptyset$, which implies that $F_1 \cup F_2 \cup \{0\} \in \mathcal{A}$. Since $F_1, F_2$ are arbitrary and $\mathcal{A}$ is a Scott closed subset, it follows that $B_1 \cup B_2 \cup \{0\}$ must be in $\mathcal{A}$. This contradicts with the maximality of $B_1 \cup \{0\}$.

(2) Suppose that $(C(X), \sigma(C(X)))$ is locally compact. Let $U$ be any Scott open set containing $\{0\}$ but not $\emptyset$. From the above, it exists a compact saturated subset $K$ such that $\{0\} \in \sigma(C(X)) \subseteq K \subseteq U$. It is easy to see that $K = \downarrow_{C(X)} \operatorname{Min}K$ and for each $A \in \operatorname{Min}K$, $A = \{0\}$ or $A \subseteq \{\frac{1}{n} : n \in \mathbb{N}^+\}$. We claim that $\mathcal{K}$ is Scott open. Let $\{C_i\}_{i \in I}$ be any directed family of closed subsets with $\bigvee_{i \in I} C_i \subseteq K$, then $\bigvee_{i \in I} C_i \subseteq C(X)$ for some $A \in \operatorname{Min}K$. If $A = \{0\}$, then there exists some $i \in I$ such that $C_i \in \sigma(C(X)) \subseteq K$. Otherwise, $\{\frac{1}{n} : n \in \mathbb{N}^+\}$, there exists some $C_i \supseteq A$ by previous proof. Then by Lemma 3.4, $\operatorname{Min}K$ is finite. But this is impossible, which may contain some $\{0\} \subseteq X \setminus \operatorname{Min}K$.

Moreover, according to the previous proof $(C(X), \sigma(C(X)))$ is core-compact while $X$ is core-compact. By Lemma 3.2 and 3.3, $(C(X), \nu(C(X)))$ is sober and locally compact. From this we get that $\sigma(C(X)) \neq \nu(C(X))$. Indeed, we have the following result:
Proposition 3.15. Let $Y$ be a $T_1$ space, denote by $C(Y)$ the lattice of all closed subsets of $Y$ with inclusion order, if $\sigma(C(Y)) = v(C(Y))$, then $Y$ has the discrete topology.

Proof. Suppose not, then there exists an infinite subset $A$ of $Y$ such that $\overline{A} \setminus A \neq \emptyset$. Let $x \in \overline{A} \setminus A$ and $U = C(Y) \setminus (\{x\} \cup \{y : y \in A\})$, then $\{x\} \in U$ and $U \in \sigma(C(Y))$. According to the assumptions that $\sigma(C(Y)) = v(C(Y))$, there exists finitely many open neighborhoods $\{U_i : 1 \leq i \leq n\}$ of $x$ such that $\{x\} \in \bigtriangleup U_1 \cap \bigtriangleup U_2 \cap \ldots \cap \bigtriangleup U_n \subseteq U$. Let $V = U_1 \cap U_2 \cap \ldots \cap U_n$. Then $\{x\} \in \bigtriangleup V \subseteq \bigtriangleup U_1 \cap \bigtriangleup U_2 \cap \ldots \cap \bigtriangleup U_n \subseteq U$ and $C(Y) \setminus U \subseteq \{H \in C(Y) : H \cap V = \emptyset\}$. It follows that $A \cap V = \emptyset$, and then $\overline{A} \cap V = \emptyset$, a contradiction. \qed 

Similar to Theorem 2.5 the correspondence between $P$ and $\Gamma(P)$ has been studied (see [13, 14, 21, 11]). Here we use the adjunction in Proposition 3.10 to give a unified proof.

Lemma 3.16. ([13]) Let $P$ and $Q$ be complete lattices, and let $g : P \to Q$ and $d : Q \to P$ be maps which preserve arbitrary sups and $g \circ d = 1_Q$.

1. If $P$ is completely distributive, then so is $Q$.
2. If $P$ is hypercontinuous, then so is $Q$.

Theorem 3.17. Let $P$ be a poset,

1. $P$ is continuous iff the lattice $\Gamma(P)$ is a continuous lattice.
2. $P$ is quasicontinuous iff the lattice $\Gamma(P)$ is a quasicontinuous lattice.
3. $P$ is algebraic iff the lattice $\Gamma(P)$ is an algebraic lattice.
4. $P$ is quasialgebraic iff the lattice $\Gamma(P)$ is a quasialgebraic lattice.

Proof. (1) If $P$ is continuous, then $\sigma(P)$ is a completely distributive lattice by Theorem 2.5. Both $\sigma(P)$ and $\Gamma(P)$ are continuous lattices by Proposition 2.1. Conversely, if $\Gamma(P)$ is continuous, then $\sigma(\Gamma(P))$ is completely distributive, and so $\sigma(P)$ is a completely distributive lattice by Proposition 3.10 and Lemma 3.16. This shows that $P$ is a continuous poset.

(2) If $P$ is quasicontinuous, then $\sigma(P)$ is a hypercontinuous lattice by Theorem 2.5. It follows that $\Gamma(P)$ is a quasicontinuous lattice by Theorem 2.2. Conversely, if $\Gamma(P)$ is quasicontinuous, then $\sigma(\Gamma(P))$ is hypercontinuous, and so $\sigma(P)$ is a hypercontinuous lattice by Proposition 3.10 and Lemma 3.16. This shows that $P$ is a quasicontinuous poset.

(3) For each nonempty $B \in \Gamma(P)$, it is easy to see that $B \ll B$ in $\Gamma(P)$ if and only if there exists some $k \in K(P)$ such that $B = \downarrow k$. If $P$ is algebraic, then $A = \cl(\bigcup \downarrow k : k \in A \cap K(P))$. Thus $\Gamma(P)$ is an algebraic lattice. Conversely, if $\Gamma(P)$ is an algebraic lattice, $P$ is continuous by (1). Both $\sigma(P)$ and $\Gamma(P)$ are completely distributive. This implies that $P$ is algebraic.

(4) If $P$ is quasialgebraic, then by Theorem 2.5 $\sigma(P)$ is a hypercontinuous and algebraic lattice. Let $L = \sigma(P)$. Then by Theorem 2.5 and Theorem 2.2, $\omega(L^{op}) = v(L) = \sigma(L)$ is an algebraic lattice. This implies that $\Gamma(P)$ is a quasialgebraic lattice from Lemma 3.1. Conversely, if $\Gamma(P)$ is a quasialgebraic lattice, then by Lemma 3.1, $\omega(L^{op}) = v(L)$ is an algebraic lattice. By (2) and Theorem 2.5, $P$ is quasicontinuous and $L$ is a hypercontinuous lattice. By Theorem 2.2, $\sigma(L) = v(L)$ and is a completely distributive algebraic lattice. This together implies that $\sigma(P)$ is a hyperalgebraic lattice. Thus $P$ is a quasialgebraic poset, by Theorem 2.5. \qed 

4. Distributive continuous lattice

Now we return to continuous lattices to consider the condition that $\sigma(L^{op}) = v(L^{op})$. At first, we list some well-known results.

Fact 4.1. If $X$ and $Y$ are both sober spaces and the open set lattice $\mathcal{O}(X)$ is isomorphic to $\mathcal{O}(Y)$, then $X$ is homeomorphic to $Y$.

Lemma 4.2. ([7] Lemma III-5.7) Let $L$ be a quasicontinuous domain. If $A = \uparrow A$ is compact in the Scott topology, then every Scott open neighborhood $U$ of $A$ contains a finite set $F$ such that $A \subseteq \uparrow F \subseteq \uparrow F \subseteq U$. Furthermore, $A$ is a directed intersection of all finitely generated upper sets that contain $A$ in their Scott interior.

Lemma 4.3. ([5]) Let $L$ be a complete lattice.

1. The Lawson topology $(L, \lambda(L))$ is compact and $T_1$.
2. $L$ is quasicontinuous if and only if $(L, \lambda(L))$ is Hausdorff.
3. A subset $M$ of $L$ is closed in the lower topology if and only if $M = \uparrow M$ and if for every ultrafilter $\mathcal{F}$ with $M \in \mathcal{F}$, $\liminf \mathcal{F} = \bigvee \{F : F \in \mathcal{F}\} \in M$.
4. A subset $M$ of $L$ is closed in the lower topology if and only if $M = \uparrow M$ and $M$ is closed in the Lawson topology.
Proposition 4.4. Let \( L \) be a continuous lattice, the following conditions are equivalent:

1. \( \sigma(L^{op}) = v(L^{op}) \).
2. The bi-Scott topology \( \sigma_B(L) = \sigma(L) \lor \sigma(L^{op}) \) is compact and Hausdorff.
3. Let \( Q(L) \) be the set of compact saturated sets of \((L, \sigma(L))\), ordered by reverse inclusion, then \( \sigma(L^{op}) \cong Q(L) \).

Proof. (1) \( \Rightarrow \) (2). The bi-Scott topology \( \sigma_B(L) = \sigma(L) \lor \sigma(L^{op}) = \sigma(L) \lor v(L^{op}) = \sigma(L) \lor \omega(L) = \lambda(L) \), and is compact and Hausdorff by Lemma 4.3.

(2) \( \Rightarrow \) (3). The Lawson topology of \( L \) is compact and Hausdorff by the above proposition. If the bi-Scott topology of \( L \) is also compact and Hausdorff, then it is equal to the Lawson topology, since \( \lambda(L) \subseteq \sigma_B(L) \).

(3) \( \Rightarrow \) (1). From the assumption and the proof above, we can see that \( Q(L) \cong v(L^{op}) \cong \sigma(L^{op}) \).

Proposition 4.5. Let \( L \) be a distributive continuous lattice such that \( \sigma(L^{op}) = v(L^{op}) \).

1. Define \( I : L \to \sigma(L^{op}) \) by \( I(x) = \langle L \setminus \uparrow x \rangle \) for each \( x \in L \), then \( I \) preserves arbitrary sups.

2. For each \( U \in \sigma(L^{op}) \), \( U = \langle \{ L \setminus \uparrow F : F \subseteq_{L} L, \uparrow F \subseteq U \} \rangle = \langle \{ \uparrow F : F \subseteq_{L} L, \uparrow F \subseteq U \} \rangle = \langle \{ F : F \subseteq_{L} L, \uparrow F \subseteq U \} \rangle \).

Remark 4.6. There is a one-to-one correspondence between distributive continuous lattices (that is, continuous frames and locally compact sober spaces in the sense of a duality of categories, namely Hofmann-Lawson duality (cf. [13])). Theorem 3.13 shows that if \( L \) is a distributive continuous lattice such that the hull-kernel topology of \( SpecL \) is just the Scott topology, then \( \sigma(L^{op}) = v(L^{op}) \) and \( \Sigma(L^{op}) \) is sober and locally compact. There exists a complete lattice \( W \) such that \( \sigma(W) \) is a continuous lattice, but \( W \) is not quasicontinuous (see [7] Theorem 3.13).
There exists $D W L, \sigma$

**Proof.** (Theorem II-4.10) Let $E L$ be represented as the intersection of the family of finitely generated lower sets in $\Gamma (P)$. Proposition 4.10.

$\sigma (L^0) = \nu (L^0)$ is necessary and whether it implies that $\text{Spec} L$ is a Scott space. In the case of distributive algebraic lattices, we have the following result.

**Proposition 4.7.** Let $L$ be a distributive algebraic lattice, then the following conditions are equivalent:

1. $L$ is a hyperalgebraic lattice,
2. $\sigma (L^0) = \nu (L^0)$,
3. The spectrum of $L$ is a quasialgebraic domain equipped with the Scott topology,
4. $L^0$ is a quasialgebraic lattice.

**Proof.** (1) $\iff$ (3) $\iff$ (4). (cf. [21])

(1) implies (2). Since (1) $\Rightarrow$ (3), $L^0$ is isomorphic to the lattice of Scott closed subsets of a quasialgebraic domain. Then we have $\sigma (L^0) = \nu (L^0)$ by Proposition 3.7 since any quasialgebraic domain is locally compact with respect to the Scott topology.

(2) implies (1). $\sigma (L^0) = \nu (L^0) = \omega (L)$, where $\omega (L)$ is an algebraic lattice by Lemma 3.1. So we have that $L^0$ is a quasialgebraic lattice from Theorem 2.5 and that $\sigma (L^0)$ is a hyperalgebraic lattice by Lemma 2.3. There exists an adjunction between $\sigma (L^0)$ and both preserve arbitrary sups by Proposition 4.5. Thus we get the conclusion that $L$ is a hyperalgebraic lattice.

Next, we construct a complete lattice $P$ such that $(P, \nu (P))$ is sober and locally compact, but the Scott topology on the lattice of closed subsets is not equal to the lower Vietoris topology. $P$ endowed with the Scott topology is neither locally compact nor first-countable still we have $\sigma (\Gamma (P)) = \nu (\Gamma (P))$.

**Example 4.8.** Let $P = (\mathbb{N} \times \mathbb{N}) \cup \{\top\}$ with a partial order defined as follows:

(i) $\forall (m, n) \in \mathbb{N} \times \mathbb{N}, \bot \leq (m, n) \leq \top$;

(ii) $\forall (m_1, n_1), (m_2, n_2) \in \mathbb{N} \times \mathbb{N}, (m_1, n_1) \leq (m_2, n_2)$ iff $m_1 = m_2$ and $n_1 \leq n_2$.

(1) When considering $P$ with the upper topology and let $L = \nu (P)$, it is easy to see that $(P, \nu (P))$ is a sober space. We also have that $(P, \nu (P))$ is locally compact, indeed $L$ is a distributive algebraic lattice. Obviously, $P$ is isolated in $L$. For any finite subset $F$ of $P$, we will show that $P \setminus F \not\subseteq P \setminus F$ in $\nu (P)$ by the Alexander's Lemma. Let $\{x_i : i \in I\}$ be a subset of $P$ such that $P \setminus F \subseteq \bigcup_{i \in I} P \setminus x_i$, i.e., $\bigcap_{i \in I} x_i \subseteq F$. If $\bigcap_{i \in I} x_i = \{\bot\}$, then there exists $x_i, x_j, i, j \in I$ such that $x_i \cap x_j = \{\bot\}$. If $\bigcap_{i \in I} x_i \neq \{\bot\}$, then $\{x_i : i \in I\}$ has a minimal element $x_i = \neq \bot$. In the first case $P \setminus F \subseteq P \setminus x_i \cup P \setminus x_j$ and in the second case $P \setminus F \subseteq P \setminus x_i$, which together imply that $L$ is an algebraic lattice. Thus $\sigma (L^0) \neq \nu (L^0)$ by Proposition 4.7 since $P$ is not quasialgebraic. It is not hard to verify that $\Sigma (L^0)$ is a sober but not locally compact space.

(2) When considering $P$ with the Scott topology, we have $\sigma (\Gamma (P)) = \nu (\Gamma (P))$.

One can easily check that $(P, \sigma (P))$ is neither locally compact nor first-countable. For any $U \in \sigma (\Gamma (P))$, let $A = \Gamma (P) \setminus U$. We will show that $A$ is a closed subset of $(\Gamma (P), \sigma (\Gamma (P)))$. Let $A^* = \bigcup \{A : A \in \text{Max} A\}$, then $A^* \subseteq A$ since $A^* = \eta^{-1} (A)$. If $A \subseteq A^*$, then $A = \Gamma (P)$. Otherwise, $(i) \times \mathbb{N}$ is a finite subset for each $i \in \mathbb{N}$. $A^*$ with the order inherited form $P$ is an algebraic dcpo. We see that $A$ is a closed subset of $(\Gamma (A^*), \sigma (\Gamma (A^*)))$, and is also a closed subset of $(\Gamma (A^*), \nu (\Gamma (A^*)))$ since $\sigma (\Gamma (A^*)) = \nu (\Gamma (A^*))$. This means that $A$ can be represented as the intersection of a family of finitely generated lower sets in $\Gamma (A^*)$. Thus $A$ can be represented as the intersection of the family of finitely generated lower sets in $\Gamma (P)$, which implies that $A$ is a closed subset of $\Gamma (P)$ with the lower Vietoris topology.

Hertling [10] constructs a complete lattice that is not jointly Scott continuous. Actually, the construction of Hertling works for any complete lattice that is not core-compact with respect to the Scott topology.

**Lemma 4.9.** (Theorem II-4.10) Let $X$ be a $T_0$ space, the set $E = \{(x, U) \in X \times \mathcal{O} (X) : x \in U\}$ is open in $X \times \Sigma \mathcal{O} (X)$ iff $X$ is core-compact.

**Proposition 4.10.** Let $P$ be a complete lattice and $L = P \times \sigma (P)$. If $(P, \sigma (P))$ is not core-compact, then $L$ is not jointly Scott continuous and then $\Sigma (L \times L) \neq \Sigma L \times \Sigma L$.

**Proof.** By the above lemma, the set $E = \{(x, U) \in P \times \sigma (P) : x \in U\}$ is not open in $\Sigma P \times \Sigma (\sigma (P))$. Suppose that the map sup: $(L, \sigma (L)) \times (L, \sigma (L)) \rightarrow (L, \sigma (L)), ((x, U), (y, V)) \mapsto (x \vee y, U \cup V)$ is continuous. Since $E$ is open in $\Sigma_L, \sup^{-1} (E)$ is open in $(L, \sigma (L)) \times (L, \sigma (L))$. For any $(a, W) \in E$, we have $(a, \emptyset) \subseteq (\bot, W) = (a, W) \in E$. There exists $D_1, D_2 \in \sigma (L)$ such that $(a, \emptyset) \subseteq (\bot, W) \in D_1 \times D_2 \subseteq \sup^{-1} (E)$.

Let $E_1 = \{x \in P : (x, \emptyset) \in D_1\}$ and $E_2 = \{x \in \sigma (P) : (\bot, P) \in D_2\}$, then $a \in E_1, E_1 \subseteq \sigma (P)$ and $W \in E_2, E_2 \subseteq \sigma (\sigma (P))$. For each $b, V \in E_1 \times E_2, (b, V) \subseteq (\bot, P) \subseteq (b, V) \in \sup (D_1 \times D_2) \subseteq E$. Contradiction. Thus $L$ is not jointly Scott continuous and then $\Sigma (L \times L) \neq \Sigma L \times \Sigma L$.

In the following, we use the complete lattice $P$ in Example 4.8 to construct a complete lattice $L$ such that $(L, \sigma (L))$ is sober but not jointly Scott continuous.
Example 4.11. Let $P$ be the complete lattice in Example 4.8. Let $M = \{\bot, \top\} \cup (\mathbb{N} \to \mathbb{N})$ with a partial order defined as follows:

(i) $\forall f \in (\mathbb{N} \to \mathbb{N}), \bot \preceq f \preceq \top$;
(ii) $\forall f, g \in (\mathbb{N} \to \mathbb{N}), f \preceq g$ if $\forall i \in \mathbb{N}, f(i) \geq g(i)$.

It is easy to see that $M \cong \sigma(P)$. Let $L = P \times M$, $L$ is not jointly Scott continuous by Proposition 4.10. Let $A$ be a non-empty Scott closed subset of $L$, then $A = \downarrow \mathrm{Max}A$, where $\mathrm{Max}A$ denotes the set of all maximal elements of $A$. We will show that $\Sigma L$ is a sober space by proving that $A$ can not be an irreducible closed set when $|MA| \geq 2$.

- Case 1. $|\pi_M(\mathrm{Max}A)| = 1$. $\pi_P(A)$ must be a Scott closed subset of $P$ with $|\pi_P(A)| \geq 2$. Since $\Sigma P$ is a sober space, there exists $B, C \in \Gamma(P)$ such that $\pi_P(A) \subseteq B \cup C$ but $\pi_P(A) \nsubseteq B$ and $\pi_P(A) \nsubseteq C$. It follows that $A \subseteq B \times \pi_M(A) \cup C \times \pi_M(A)$ but $A$ does not contained in any of them.

- Case 2. $|\pi_M(\mathrm{Max}A)| \geq 2$ and $\top \in \pi_M(\mathrm{Max}A)$. Let $A_0 = \{(x, y) \in \mathrm{Max}A : y = \top\}$ and $B_0 = \{(x, y) \in \mathrm{Max}A : y \neq \top\}$, then $A_0 \neq \emptyset$, $B_0 \neq \emptyset$ and $A \subseteq A_0 \cup \downarrow B_0$. We claim that $\downarrow A_0$ is a Scott closed subset of $L$. Suppose not, there exists a directed subset $D \subseteq \downarrow A_0$ such that $\bigvee D \nsubseteq A_0$. Let $D' = \{(\pi_P(d), \top) : d \in D\}$, then $D'$ is a directed subset in $\downarrow A_0$. We have that $\bigvee D' \nsubseteq A_0$ since $\bigvee D' \geq \bigvee D$, and that $\bigvee D' \not\subseteq \downarrow B_0$ since $\pi_M(\bigvee B_0) < \pi_M(\bigvee D') = \top$. Contradiction. Obviously, $A \nsubseteq A_0$ and $A \nsubseteq \downarrow B_0$.

- Case 3. $|\pi_M(\mathrm{Max}A)| \geq 2$ and $\top \in \pi_M(\mathrm{Max}A)$. Let $g = \pi_M(\bigvee A)$, then $g \in (\mathbb{N} \to \mathbb{N})$.

Case 3.1. $\bot \nsubseteq \pi_M(\mathrm{Max}A)$. By assumption, there exists some $i_0 \in \mathbb{N}$ such that $B_0 = \{(x, y) \in \mathrm{Max}A : y(i_0) > g(i_0)\} \neq \emptyset$. Let $A_0 = \{(x, y) \in \mathrm{Max}A : y = g(i_0)\}$, then $A_0 \neq \emptyset$ and $A \subseteq A_0 \cup \downarrow B_0$. We claim that $\downarrow A_0$ is a Scott closed subset of $L$. Suppose not, there exists some directed subset $D \subseteq A_0$ such that $\bigvee D \nsubseteq A_0$. For each $d \in D$, define $f_d : \mathbb{N} \to \mathbb{N}, f_d(n) = \pi_M(d)(n)$ if $n \neq i_0; f_d(n) = g(i_0)$ if $n = i_0$. Let $D' = \{(\pi_P(d), f_d) : d \in D\}$, then $D'$ is a directed subset in $\downarrow A_0$ with $\bigvee D' \geq \bigvee D$. We have that $\bigvee D' \nsubseteq A_0$ since $\pi_M(\bigvee B_0) < \pi_M(\bigvee D')$. Contradiction. Obviously, $A \nsubseteq A_0$ and $A \nsubseteq \downarrow B_0$. Thus $A$ is not irreducible.

Case 3.2. $\bot \in \pi_M(\mathrm{Max}A)$ and $|\pi_M(\mathrm{Max}A) \cap (\mathbb{N} \to \mathbb{N})| = 1$. Let $(x, g) \in \mathrm{Max}A$, then $A \subseteq \downarrow (x, g) \cup \pi_P(A) \times \{\bot\}$. Obviously, $A \nsubseteq (x, g)$ and $A \nsubseteq \pi_P(A) \times \{\bot\}$.

Case 3.3. $\bot \in \pi_M(\mathrm{Max}A)$ and $|\pi_M(\mathrm{Max}A) \cap (\mathbb{N} \to \mathbb{N})| = 1$. There exists some $i_0 \in \mathbb{N}$ such that $B_0 = \{(x, y) \in \mathrm{Max}A : y(i_0) > g(i_0)\} \neq \emptyset$. Let $A_0 = \{(x, y) \in \mathrm{Max}A : y(i_0) = g(i_0)\}$, then $A_0 \neq \emptyset$. Let $A_1 = \downarrow A_0 \cup \pi_P(A) \times \{\bot\}$ and $B_1 = \downarrow B_0 \cup \pi_P(A) \times \{\bot\}$. Similar to case 3.1, we have that $A_1$ and $B_1$ are both Scott closed subsets of $L$ with $A \subseteq A_1 \cup B_1$. But $A$ is contained in non of them.

So we prove that $\Sigma L$ is a sober space while $L$ is not jointly Scott continuous. And we can see from the proof that for any depo $Q$, if $\Sigma Q$ is sober that $\Sigma (Q \times M)$ is also a sober space.

5. Consonance of the lower powerspace

Given a topological space $X$, we compared the Scott topology on $C(X)$ with the lower Vietoris topology on $C(X)$. Where the latter is often called the lower powerspace over $X$ (also called the Hoare powerspace) and is denoted by $P_H(X)$. Using $Q(X)$ to denote the set of all compact saturated subsets of $X$, the upper powerspace over $X$ is $Q(X)$ with the upper Vietoris topology, which is generated by $\square U = \{K \in Q(X) : K \subseteq U\}$ as a basis, $U \in \mathcal{O}(X)$, $Q(X)$ with the upper Vietoris topology is also called the Smyth powerspace over $X$ and is denoted by $P_S(X)$. For each $K \in Q(X)$, let $\Phi(K) = \{U \in \mathcal{O}(X) : K \subseteq U\}$. Then $\Phi(K)$ is a Scott open filter of $\mathcal{O}(X)$ and $K = \bigcap \Phi(K)$. The concept of consonance is as follows.
Definition 5.1. A topological space $X$ is consonant if and only if for every $\mathcal{H} \in \sigma(\mathcal{O}(X))$ and every $U \in \mathcal{H}$ there exists $K \in Q(X)$ such that $U \in \Phi(K) \subseteq \mathcal{H}$.

For a sober space $X$, by the Hoffmann-Mislove theorem (see [1] Theorem II-1.20]), $X$ is consonant if and only if the Scott topology on $\mathcal{O}(X)$ has a basis consisting of Scott open filters. Recently, Brecht and Kawai [2] proved that for a topological space $X$, the consonance of $X$ is equivalent to the commutativity of the upper and lower powerspaces in the sense that $P_H (P_S(X)) \cong P_S(P_H(X))$ under a naturally defined homeomorphism.

In that paper, they asked the following question: if $X$ is consonant, is $P_H(X)$ also consonant? We will give a partial answer below.

Theorem 5.2. Let $X$ be a consonant topological space. If $\Sigma(\prod^n \mathcal{O}(X)) = \prod^n (\Sigma \mathcal{O}(X))$ for each $n \in \mathbb{N}$, then $P_H(X)$ is consonant.

Proof. Firstly, for each $n \in \mathbb{N}$, we define a map $\varphi_n : \prod^n \mathcal{O}(X) \to v(C(X))$ as follows:

$$\forall (U_1, U_2, \ldots, U_n) \in \prod^n \mathcal{O}(X), \varphi_n(U_1, U_2, \ldots, U_n) = \bigcap_{k=1}^n \diamond U_k.$$ 

$\varphi_n$ preserves arbitrary sups, since $\diamond (\bigcup_{i \in I} U_i) = \bigcup_{i \in I} \diamond U_i$ for any family $\{U_i \in \mathcal{O}(X) : i \in I\}$. Thus $\varphi_n$ is Scott continuous and is a continuous map from $\prod^n (\Sigma \mathcal{O}(X))$ to $\Sigma(v(C(X)))$, since $\Sigma(\prod^n \mathcal{O}(X)) = \prod^n (\Sigma \mathcal{O}(X))$ by assumption.

Generally, $\diamond U \cap \diamond V \neq \diamond (U \cap V)$ for $U, V \in \mathcal{O}(X)$. Let $\mathcal{B} = \{\bigcap_{U \in F} \diamond U : F \subseteq_f \mathcal{O}(X)\}$, which serves as a base for $P_H(X)$. It is easy to see that $\mathcal{B}$ is closed for finite union.

For any $\mathcal{A} \in \mathcal{O}(v(C(X)))$ and $A \in \mathcal{A}$. Then $A$ is equal to the union of a family of elements in $\mathcal{B}$. Without loss of generality, we assume that $A \neq \emptyset$. Here we use a trick that is frequently used. Let $\text{fin}(A)$ be the set of all finite unions of elements of the family, then $\text{fin}(A)$ is a directed subset of $\mathcal{B}$, since $\mathcal{B}$ is closed for finite union.

And then $\mathcal{A} = \bigcup \text{fin}(A) = \bigvee_{\text{fin}(A) \in \mathcal{A}'} \text{fin}(A) \in \mathcal{A}'$, where $\mathcal{A}'$ is Scott open. There exists some $\{U_j : 1 \leq j \leq n\} \subseteq \mathcal{O}(X) \setminus \{\emptyset\}$ such that $\bigcap_{j=1}^n \diamond U_j \in \text{fin}(A) \cap \mathcal{A}'$, i.e., $\varphi_n(U_1, U_2, \ldots, U_n) \in \mathcal{A}'$. Since $\varphi_n$ is a continuous map from $\prod^n (\Sigma \mathcal{O}(X))$ to $\Sigma(v(C(X)))$ by previous proof, there exists $\mathcal{H}_j \in \sigma(\mathcal{O}(X)), j = 1, 2, \ldots, n$ such that

$$\bigcup_{j=1}^n \diamond U_j \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_n \subseteq \varphi_n^{-1}(\mathcal{A}).$$

By assumption, $X$ is consonant. For each $1 \leq j \leq n$, there is $K_j \in \mathcal{O}(X)$ such that $U_j \in \Phi(K_j) \subseteq \mathcal{H}_j$. Obviously, the function $\eta : x \mapsto \downarrow x, X \to P_H(X)$ is a topological embedding.

Claim 1. $\bigcap_{j=1}^n \uparrow \mathcal{O}(X)(K_j)$ is compact and saturated in $P_H(X)$.

Claim 2. $\bigcap_{j=1}^n \uparrow \mathcal{O}(X)(K_j)$ is a Scott open filter by definition. According to the previous proof, in order to prove $\Phi(\bigcap_{j=1}^n \uparrow \mathcal{O}(X)(K_j)) \subseteq \mathcal{A}'$, we only need to prove $\Phi(\bigcap_{j=1}^n \uparrow \mathcal{O}(X)(K_j)) \subseteq \mathcal{A}'$. Suppose that $\bigcap_{j=1}^n \uparrow \mathcal{O}(X)(K_j) \subseteq \bigcap_{V \in \mathcal{F}} \diamond V$ for some $\mathcal{F} \subseteq_f \mathcal{O}(X)$. Then for each $V \in \mathcal{F}$, there is some $1 \leq j \leq n$ such that $K_j \subseteq V$. Otherwise, for any $1 \leq j \leq n$, there is $y_j \in K_j \setminus V$. Then $\downarrow y_1 \cup \downarrow y_2 \cup \cdots \cup \downarrow y_n \in \bigcap_{j=1}^n \uparrow \mathcal{O}(X)(K_j)$ but $\downarrow y_1 \cup \downarrow y_2 \cup \cdots \cup \downarrow y_n \notin \diamond V$, which contradicts the assumption. For each $1 \leq j \leq n$, let $F_j = \{V \in \mathcal{F} : K_j \subseteq V\}$ and let $V_j = \bigcup_{V \in F_j} V$. Then $V_j \in \Phi(K_j) \subseteq \mathcal{H}_j$ for each $j$, which implies that

$$\bigcap_{j=1}^n \diamond V_j = \varphi_n(V_1, V_2, \ldots, V_n) \in \mathcal{A}'$$

and it is obvious that $\bigcap_{j=1}^n \diamond V_j \subseteq \bigcap_{V \in \mathcal{F}} \diamond V$. Thus $\bigcap_{V \in \mathcal{F}} \diamond V \in \mathcal{A}'$, since $\mathcal{A}'$ is an upper set in the order of set inclusion.

Since $\bigcap_{j=1}^n \uparrow \mathcal{O}(X)(K_j) \subseteq \bigcap_{j=1}^n \mathcal{U}_j \subseteq \mathcal{A}$, we are done. \(\square\)

Corollary 5.3. Let $X$ be a consonant topological space. If $\mathcal{O}(X)$ is core-compact or first-countable, then $P_H(X)$ is consonant.

A natural question is whether consonance of $P_H(X)$ implies that $X$ is consonant. We answer this question negatively by a counterexample. First, we show the following result.
Proposition 5.4. Let $X$ be a topological space, $X$ is locally compact if and only if $X$ is core-compact and consonant.

Proof. ($\Rightarrow$) If $X$ is a locally compact space, then $X$ is core-compact. For any $H \in \sigma(O(X))$ and any $U \in H$. Suppose that $U$ is not empty, then for each $x \in U$ there is some $K_x \in Q(X)$ such that $x \in K_x \subseteq \subseteq U$. \( \bigcup_{x \in F} K_x^2 : F \subseteq f \ U \) is a directed subset of $O(X)$ with \( V \{ \bigcup_{x \in F} K_x^2 : F \subseteq f \ U \} = \bigcup_{x \in F} K_x^2 \subseteq \subseteq U \). There exists some $F \subseteq f \ U$ such that $\bigcup_{x \in F} K_x \in H$, since $H$ is Scott open. Then $\bigcup_{x \in F} K_x \in Q(X)$ and $U \in \Phi(\bigcup_{x \in F} K_x) \subseteq H$.

($\Leftarrow$) Let $X$ be a core-compact and consonant space. For any $X$ and any open set $U$ containing $x$, there exists some open set $V$ such that $x \in V \ll U$. We employ the interpolation property to find a sequence \( \{ V_i \}_{i \in N} \) of open sets such that $x \in V_i \ll \cdots \ll V_n \ll V_{n-1} \ll \cdots \ll V_1 \ll U$ (see [7]). Let $H = \{ W \in O(X) : W \supseteq V_i \}$ for some $i \in N$. It is easy to see that $H$ is a Scott open filter with $U \in H$. There exists some $K \in Q(X)$ such that $U \in \Phi(K) \subseteq H$. Then $x \in V \subseteq \bigcap H \subseteq K = \bigcap \{ W \in O(X) : K \subseteq W \} \subseteq U$. \[ \square \]

Example 5.5. Exercise V-5.25 of [7] gives an example of a $T_0$ space $X$ such that $O(X)$ is a continuous lattice but $X$ itself is not locally compact. By Proposition 5.4, $X$ is not consonant. Let $L = O(X)$, then $C(X) \equiv L^{op}$ and $v(C(X)) \equiv v(L^{op}) = \omega(L)$. By Lemma 3.1, $v(C(X))$ is a continuous lattice. According to Lemma 3.2 and 3.3, $P_H(X)$ is a sober and locally compact space. We can see from Proposition 5.4 that $P_H(X)$ is consonant while $X$ is not.

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