Products of rough finite state machines

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Abstract

In this paper, we introduce the concept of several products of rough finite state machines. We establish their relationships through coverings and investigate some algebraic properties for these products.

Keywords: Rough finite state machine; Homomorphism; Covering; Direct product; Wreath product; Cascade product.

1 Introduction

The concept of finite state semiautomata (finite state machines) is well known (cf., e.g., [6, 7, 8, 9, 23]). A (deterministic) finite state machine is a triple \((Q, X, \delta)\), consisting of two (finite) sets \(Q\) (of states) and \(X\) (of inputs) and a map \(\delta: Q \times X \rightarrow Q\) (called the transition map). A nondeterministic version of a finite state machine, known as nondeterministic finite state machine, is also a triple \((Q, X, \delta)\), where \(Q\) and \(X\) are as above and \(\delta: Q \times X \rightarrow 2^Q\) is a map. The only difference between a deterministic and a nondeterministic finite state machine is in the value that the transition map returns. In case of previous the transition map returns a single state, while in case of later it returns a set of states.

An account of the fuzzy theoretic version of the notion of a finite state semiautomaton has been studied by Mordeson and Malik in [20], who called the resulting concept as a fuzzy finite state machine (see also [10]). This fuzzy theoretic version is obtained by allowing \(\delta(q, a)\), where \(q \in Q\) and \(a \in X\), to be not just a single state or even a subset of \(Q\), but a fuzzy (sub)set of \(Q\). Also, similar, or closely related, notions have been introduced and studied by Kim, Kim and Cho [12], Jun [11], and Li and Pedrycz [15]. In literature (cf., [4, 12, 14, 17, 18]), the crisp concepts of several types of products for finite state machines introduced and studied in [7] has been fuzzified by many researchers.

Pawlak’s rough set theory [20], like fuzzy set theory, is another mathematical approach to deal with imprecise, uncertain or incomplete information and knowledge. It has rapidly drawn attention of both mathematicians and computer scientists due to its ability to model many aspects of artificial intelligence and cognitive sciences, particularly in the areas of knowledge acquisition, decision analysis and expert systems. Following

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the advent of rough set theory, Basu [3] recently introduced the concept of a rough finite state (semi)automaton, by allowing a state, when given an input, to ‘transition’ to a rough set of the state set (rather than a subset or a fuzzy set) in a certain way and extended the idea further by designing a recognizer that accepts imprecise statements (cf., [3], for more details). Inspired from the work of Basu, Tiwari and Sharan [25] introduced the concepts of rough transformation semigroup associated with a rough finite state machine and coverings of rough finite state machines. Recently, Tiwari, Srivastava and Sharan [26] introduced and studied the algebraic concepts such as separatedness, connectedness and retrievability of such machines.

In this present work, our motive is to produce a new rough finite state machine by connecting the rough finite state machines. This we achieve by introducing different types of products between rough finite machines. Specifically, after providing a detail study of rough finite state machines, we introduce and study the notions of different types of products viz., direct product, cascade product and wreath product of rough finite state machines. We explore the relationship among such products through coverings, and investigate some algebraic properties of these products.

2 Preliminaries

In this section, we recall and study some concepts associated with rough sets, rough finite state machines and coverings, which we need in the subsequent sections.

2.1 Rough Sets

Over the past three decades, a number of definitions of a rough set have appeared in the literature (cf., e.g., [1, 10, 13, 19, 20, 21, 22, 27]). In [2], it has been shown that some of these are equivalent. In this paper, we follow the definition of a rough set as it is given in [27]. For completeness, we recall the following key notions.

Definition 2.1 [20] An approximation space is a pair \((X, R)\), where \(X\) is a nonempty set and \(R\) is an equivalence relation on \(X\).

If \(R\) is an equivalence relation on a nonempty set \(X\) and \(x \in X\), then let \([x]\) denote the set \(\{y \in X \mid xRy\}\), called an equivalence class or a block under \(R\). Let \(X/R = \{[x] \mid x \in X\}\).

Definition 2.2 [27] Given an approximation space \((X, R)\) and \(A \subseteq X\), the lower approximation \(\underline{A}\) of \(A\) and the upper approximation \(\overline{A}\) of \(A\) are defined as follows:

\[
\underline{A} = \bigcup \{[x] \in X/R \mid [x] \subseteq A\}, \\
\overline{A} = \bigcup \{[x] \in X/R \mid [x] \cap A \neq \emptyset\}.
\]

The pair \((\underline{A}, \overline{A})\) is called a rough set. We shall denote it by \(A\).

We now recall another concept of rough set from [19].

Definition 2.3 For an approximation space \((X, R)\) and \(A \subseteq X\), the pair \((\underline{A}, \overline{A})\) is called a rough set.

Let \((X, R)\) be an approximation space. Define a relation \(\equiv\) on \(2^X\) by \(A \equiv B \iff \overline{A} = \overline{B}\) and \(\underline{A} = \underline{B}\). Then \(\equiv\) is an equivalence relation on \(X\). The following is also a concept of rough set induced by the equivalence relation \(\equiv\).
Definition 2.4 A rough set in the approximation space \((X, R)\) is an equivalence class of \(P(X)/\equiv\).

Remark 2.1 In \cite{10}, the above definition of rough set is given in the generalized setup, precisely, \(R\) is a binary relation instead of an equivalence relation on a nonempty set \(X\).

Given an approximation space \((X, R)\) and \(A \subseteq X\), \(A\) and \(\overline{A}\) are interpreted as the collection of those objects of the domain \(X\) that definitely and possibly belongs to \(A\), respectively. Further, \(A\) is called definable (or exact) in \((X, R)\) iff \(A = \overline{A}\). Equivalently, a definable set is a union of blocks under \(R\). For any \(A \subseteq X\), \(A\), \(\overline{A}\) and \(\bar{B}nA\) are all definable sets in \((X, R)\).

Remark 2.2 In \cite{2}, it has shown that the Definitions 2.2, 2.3 and 2.4 introduced by different researchers at different time, are essentially equivalent to each other for a given approximation space \((X, R)\). Even these equivalent definitions provide different algebras by taking different algebraic operations.

In our case, we will follow the concept of rough sets given in Definition 2.2.

2.2 Rough finite state machines

The notion of rough finite state machine has been firstly proposed by Basu \cite{3}. In this subsection, our aim is to discuss the concept of a rough finite state machine in details.

Throughout this section, \(X^*\) is the set of all words on \(X\) (i.e., finite strings of elements of \(X\), which form a monoid under concatenation of strings) including the empty word (which we shall denote by \(e\)).

We begin with the following concept of nondeterministic finite state machine.

Definition 2.5 A (nondeterministic) finite state machine is a triple \((Q, X, \delta)\), where \(Q\) is a nonempty finite set of states, \(X\) is a nonempty finite set of inputs and a map \(\delta : Q \times X \rightarrow 2^Q\), called the transition map (or more precisely, \(\delta\) is a map such that \(\delta(q, a)\), where \(q \in Q\) and \(a \in X\), is a subset of \(Q\)).

The transition map \(\delta : Q \times X \rightarrow 2^Q\) can be extended to the map \(\delta^* : Q \times X^* \rightarrow 2^Q\) such that

(i) \(\forall q \in Q, \delta(q, e) = \{q\}\), and

(ii) \(\forall q \in Q, \forall x \in X^*\) and \(\forall a \in X, \delta(q, xa) = \bigcup\{\delta(p, a) : p \in \delta^*(q, x)\}\).

A rough finite state machine is a natural generalization of above nondeterministic finite state machine. The difference is only that in case of a rough finite state machine the transition map returns a rough set of states instead of a set of states, as in the case of nondeterministic finite state machine. This roughness arises due to presence of an equivalence relation on its state-set. Formally, a rough finite state machine can be defined as follows:

Definition 2.6 A rough finite state machine (or RFSM) is a 4-tuple \(M = (Q, R, X, \delta)\), where \(Q\) is a nonempty finite set (the set of states of \(M\)), \(R\) is an equivalence relation on \(Q\), \(X\) is a nonempty finite set (the set
of inputs) and $\delta : Q \times X \to A$, where $A = \{(\overline{A}, \overline{A}) : A \subset Q\}$ is a map (called the rough transition map) such that for each $(q, a) \in Q \times X$, $\delta(q, a) = (\overline{A}, \overline{A})$ being a rough set in $(Q, R)$ for some $A \subset Q$.

We shall denote $\overline{A}$ and $\overline{A}$ as $\delta(q, a)$ and $\delta(q, a)$ respectively. Also, throughout, we will write the set of all rough sets $\{\overline{A}, \overline{A}\}$ in the approximation space $(Q, R)$ just as $\overline{A}$.

**Example 2.1** Consider a RFSM $(Q, R, X, \delta)$, where $Q = \{q_1, q_2, q_3, q_4, q_5\}$, $R$ is an equivalence relation on $Q$ with $Q/R = \{\{q_1, q_2\}, \{q_3, q_5\}, \{q_4\}\}$, $X = \{a, b\}$ and the rough transition map $\delta$ is given by the following table:

| $Q$     | $\delta(q, a)$ | $\delta(q, b)$ |
|---------|----------------|----------------|
| $q_1$   | $\{q_4\}$     | $\{q_3, \overline{q}_3, \overline{q}_4\}$ |
| $q_2$   | $\overline{q}_2, \{q_3, q_5\}$ | $\{q_3, q_5, \overline{q}_3, \overline{q}_4\}$ |
| $q_3$   | $\{q_1, q_2\} \cup \{q_4\}$ | $\{q_1, q_2\} \cup \{q_4\}$ |
| $q_4$   | $\{q_1, q_2\} \cup \{q_4\}$ | $\{q_1, q_2\} \cup \{q_4\}$ |
| $q_5$   | $\{q_1, q_2\} \cup \{q_4\}$ | $\{q_1, q_2\} \cup \{q_4\}$ |

**Table 2.1 State Transition Table**

**Remark 2.3** Let $(Q, X, \delta)$ be a nondeterministic finite state machine and $R$ be an equivalence relation on $Q$ such that for all $A \subset Q, A = \overline{A} = C$ (say). Then by identifying $\delta$ with the map $\delta : Q \times X \to A$ given by $\delta(q, a) = (C, C), \forall (q, a) \in Q \times X$, we see that every nondeterministic finite state machine can be viewed as a RFSM as defined in [9].

Let $(Q, R, X, \delta)$ be an RFSA and $D$ be the set of all definable sets generated by $R$ over $Q$. Then transition map $\delta$ of a RFSA $(Q, R, X, \delta)$ can be extended to a map $\delta^D : D \times X \to A$, as we proceed to explain next.

**Definition 2.7** Let $(Q, R, X, \delta)$ be an RFSM. Then the block transition map $\delta^D : D \times X \to A$ is defined as follows: $\forall B \in Q/R$ and $\forall a \in X$,

$$\delta^D(D, a) = (\overline{\delta^D(D, a)}, \overline{\delta^D(D, a)})$$

where

$$\delta^D(D, a) = \bigcup \{\delta(q, a) : q \in B \subset D, B \in Q/R\}$$

and

$$\delta^D(D, a) = \bigcup \{\delta(q, a) : q \in B \subset D, B \in Q/R\}.$$  

**Example 2.2** Consider the RFSM given in Example 2.1. Then the block transitions can be evaluated as:

$$\delta^D(\{q_1, q_2\} \cup \{q_3, q_5\}, a) = \bigcup \{\delta(q, a) : q \in B \subset \{q_1, q_2\} \cup \{q_3, q_5\} \}$$

$$\delta^D(\{q_1, q_2\} \cup \{q_3, q_5\}, a) = \bigcup \{\delta(q, a) : q \in B \subset \{q_1, q_2\} \cup \{q_3, q_5\} \}.$$
Let \( Q, R, X, \delta \) be an RFSM. Define \( \delta^* : Q \times X^* \to A \) as follows:

(i) \( \delta^*(q, e) = ([q], [q]), \forall q \in Q, \) and

(ii) \( \forall q \in Q, \forall x \in X^* and \forall a \in X, \delta^*(q, xa) = (\delta^*(q, x), \delta^* (q, xa)) \),

where \( \delta^*(q, xa) = \delta^D(D, x, a) \) and \( \delta^* (q, xa) = \delta^D(\delta^*(q, x), a). \)

A block transition map can also be extended, as explained next.

Definition 2.9 For an RFSM \((Q, R, X, \delta)\), the block transition map \( \delta^D : D \times X^* \to A \) can be extended to a map \( \delta^{*D} : D \times X^* \to A \) as follows:

\[
\delta^{*D}(D, x) = (\delta^{*D}(D, x), \delta^{*D}(D, x)), \quad \delta^{*D}(D, x) = \bigcup \{\delta^*(q, x) : q \in B \subseteq D, B \in Q/R\} and
\]

\[
\delta^{*D}(D, x) = \bigcup \{\delta^*(q, x) : q \in B \subseteq D, B \in Q/R\}.
\]

Following is require to prove the extension of rough transition map.

Definition 2.10 Let \((Q, R, X, \delta)\) be an RFSM and \( D \) be a definable set generated by \( R \) over \( Q \). Then

\[
\delta^{*D}(D, xa) = (\delta^{*D}(D, xa), \delta^{*D}(D, xa)), \quad \delta^{*D}(D, xa) = \delta^D(\delta^{*D}(D, x), a) and
\]

\[
\delta^{*D}(D, xa) = \delta^D(\delta^{*D}(D, x), a), \forall x \in X^* and \forall a \in X.
\]

Lemma 2.1 Let \((Q, R, X, \delta)\) be an RFSM. Then

\[
\delta^*(q, xy) = (\delta^*(q, xy), \delta^*(q, xy)), \quad \delta^*(q, xy) = \delta^{*D}(\delta^*(q, x), y) and
\]

\[
\delta^*(q, xy) = \delta^{*D}(\delta^*(q, x), y), \forall q \in Q and \forall x, y \in X^*.
\]
Proof: Let $q \in Q$ and $x, y \in X^*$. We prove the result by induction on $|y| = n$. If $n = 1$, let $y = a$. Then from Definition 2.8,

\[
\delta^*(q, xa) = (\delta(q, x), \delta^*(q, xa)), \quad \text{where}
\]

\[
\delta^*(q, xa) = \delta^D(\delta^*(q, x), a) \quad \text{and}
\]

\[
\delta^*(q, xa) = \delta^D(\delta^*(q, x), a).
\]

Thus the result is true for $n = 1$. Now, suppose the result is true for all $x \in X^*$ and $y \in X^*$ such that $|y| = n$. Let $y = ua$, where $|u| = n$. Then

\[
\delta^*(q, xy) = \delta^*(q, xua) = \delta^*(q, za), \quad \text{where } z = xu
\]

\[
= \delta^D(\delta^*(q, z), a)
\]

\[
= \delta^D(\delta^*(q, x), u, a) \quad \text{(by induction)}.
\]

On the other hand

\[
\delta^D(\delta^*(q, x), y) = \delta^D(\delta^*(q, x), ua)
\]

\[
= \delta^D(D, ua), \quad \text{where } D = \delta^*(q, x)
\]

\[
= \delta^D(D^*, (D, x), a), \quad \text{where } D = \delta^*(q, x)
\]

\[
= \delta^D(D^*, (\delta^*(q, x), u), a).
\]

Thus

\[
\delta^*(q, xy) = \delta^D(\delta^*(q, x), y).
\]

Similarly, \(\delta^*(q, xy) = \delta^D(\delta^*(q, x), y)\).

Hence the result is true for $|y| = n + 1$.

Keeping the above in mind, it seems reasonable to accept the following also a definition of an RFSM. By the abuse of notation, we shall write $X$, $\delta$ and $\delta^D$ instead of $X^*$, $\delta^*$ and $\delta^D$ respectively.

Definition 2.11 A rough finite state machine (or RFSM) is a 4-tuple $M = (Q, R, X, \delta)$, where $Q$ is a nonempty finite set (the set of states of $M$), $R$ is a given equivalence relation on $Q$, $X$ is a monoid (whose elements are the input symbol) and $\delta : Q \times X \rightarrow A$ is a map (called the rough transition map) such that

(i) $\delta(q, e) = ([q], [q]), \forall q \in Q$, and

(ii) $\forall q \in Q, \forall x, y \in X$, $\delta(q, xy) = (\delta(q, x), \delta(q, y))$, where

\[
\delta(q, xy) = \delta^D(\delta(q, x), y) \quad \text{and} \quad \delta(q, xy) = \delta^D(\delta(q, x), y).
\]

Next, we introduce the concept of homomorphism between two rough finite state machines, which is a natural generalization of the same concept associated with finite state machines. In the case of finite state machines, recall that the homomorphism between two nondeterministic finite state machines $(Q, X, \delta)$ and $(R, Y, \mu)$ is a pair of maps $f : Q \rightarrow R$ and $g : X \rightarrow Y$ such that $f(\delta(q, x)) \subseteq \mu(f(q), g(x))$. 

6
Definition 2.12 A homomorphism from an RFSM $M_1 = (Q_1, R_1, X_1, \delta_1)$ to an RFSM $M_2 = (Q_2, R_2, X_2, \delta_2)$ is a pair of maps $f : Q_1 \rightarrow Q_2$ and $g : X_1 \rightarrow X_2$ such that

(i) $(p, q) \in R_1 \Rightarrow (f(p), f(q)) \in R_2$, $\forall p, q \in Q_1$, and

(ii) $f(\delta_1(q, x)) \subseteq \delta_2(f(q), g(x))$ or $(f(\delta_1(q, x)), f(\delta_1(q, x))) \subseteq (\delta_2(f(q), g(x)), \delta_2(f(q), g(x)))$, $\forall q \in Q_1$ and $\forall x \in X_1$.

A bijective homomorphism $(f, g)$ from an RFSM $M_1$ to an RFSM $M_2$ is called an isomorphism. If there is an isomorphism from RFSM $M_1$ to RFSM $M_2$, then $M_1$ is said to be isomorphic to $M_2$, and is denoted by $M_1 \cong M_2$.

Example 2.3 Let $M_1 = (Q_1, R_1, X_1, \delta_1)$ and $M_2 = (Q_2, R_2, X_2, \delta_2)$ be two rough finite state machines, where $Q_1 = \{q_1, q_2, q_3, q_4\}$, $Q_1/R_1 = \{(q_1), (q_2, q_4), (q_3)\}$, $X_1 = \{a, b\}$, $Q_2 = \{q'_1, q'_2, q'_3, q'_4\}$, $Q_2/R_2 = \{(q'_1), (q'_2, q'_4), (q'_3)\}$, $X_2 = \{a', b'\}$ and the rough transition functions $\delta_1$ and $\delta_2$ are respectively given as follows:

| $Q$ | $\delta_1(q, a)$ | $\delta_1(q, b)$ |
|-----|-----------------|-----------------|
| $q_1$ | $\{q_1\} \cup \{q_3\}$ | $\{q_2, q_4\}$ |
| $q_2$ | $\{q_1, q_2, q_4\} \cup \{q_3\}$ | $\{q_2, q_4\}, \{q_1\} \cup \{q_2, q_4\} \cup \{q_3\}$ |
| $q_3$ | $\{q_1\}$ | $\{q_1\}, \{q_1\} \cup \{q_3\}$ |
| $q_4$ | $\{q_2, q_4\}, \{q_1\} \cup \{q_2, q_4\}$ | $\{q_1\}, \{q_1\} \cup \{q_3\}$ |

| $Q$ | $\delta_2(q', a')$ | $\delta_2(q', b')$ |
|-----|-----------------|-----------------|
| $q'_1$ | $\{q'_1\} \cup \{q'_4\}$ | $\{q'_2, q'_4\}$ |
| $q'_2$ | $\{q'_1, q'_4\} \cup \{q'_2, q'_4\}$ | $\{q'_1\}, \{q'_1\} \cup \{q'_4\}$ |
| $q'_3$ | $\{q'_1\}$ | $\{q'_1\}, \{q'_1\} \cup \{q'_4\}$ |
| $q'_4$ | $\{q'_1, q'_2, q'_4\} \cup \{q'_3\}$ | $\{q'_2, q'_4\}, \{q'_1\} \cup \{q'_2, q'_4\} \cup \{q'_3\}$ |

Table 2.3 State Transition Table

A pair of maps $f : Q_1 \rightarrow Q_2$ and $g : X_1 \rightarrow X_2$, where $f(q_1) = q'_1, f(q_2) = q'_4, f(q_3) = q'_3, f(q_4) = q'_2$ and $g(a) = a', g(b) = b'$ is clearly a homomorphism from $M_1$ to $M_2$.

Remark 2.4 From the Definition 2.12 one can easily see that how in a simple way we are introducing the concept of homomorphism in the case of rough finite state machines from the concept of homomorphism of finite state machines. Contrary to it, it is easy to see that if the same concept is known in the case of rough finite state machines, one can easily guess the similar concept in the case of finite state machines. So, now onward we will introduce the concepts for rough finite state machines without recalling the similar concepts for finite state machines.

The concept of coverings of finite state machines has been introduced and studied in [7]. We close this subsection by recalling the concept of covering of rough finite state machines, recently introduced in [26].

Definition 2.13 Let $M_1 = (Q_1, R_1, X_1, \delta_1)$ and $M_2 = (Q_2, R_2, X_2, \delta_2)$ be rough finite state machines. Then a pair of maps $\eta : Q_2 \rightarrow Q_1$ (onto) and $\xi : X_1 \rightarrow X_2$ is called a covering of $M_1$ by $M_2$, if
(i) \( (p, q) \in R_2 \Rightarrow (\eta(p), \eta(q)) \in R_1, \forall p, q \in Q_2 \), and
(ii) \( \forall q_2 \in Q_2 \) and \( \forall x \in X_1, \delta_1(\eta(q_2), x) \subseteq \eta(\delta_2(q_2, \xi(x))) \) or \( (\bar{\delta}_1(\eta(q_2), x)) \subseteq \eta(\bar{\delta}_2(q_2, \xi(x))) \), where \( \xi : X_1 \to X_2 \) is a map such that \( \xi(e_1) = e_2 \) and \( \xi(x) = \xi(x_1)\xi(x_2)\ldots\xi(x_n) \), \( \forall x = x_1x_2\ldots x_n \in X_1 \).

We shall denote by \( M_1 \leq M_2 \), the covering of \( M_1 \) by \( M_2 \).

## 3 Products of rough finite state machines

In this section, we introduce several products for rough finite state machines. We explore the notions of coverings for these products and also examine some algebraic properties. For the terminology in (crisp) automata theory, we refer to [7].

Let \( (p_1, p_2), (q_1, q_2) \in R \) if \( (p_1, q_1) \in R_1 \) and \( (p_2, q_2) \in R_2 \). It is easy to see that \( R \) turns out to be an equivalence relation on \( Q_1 \times Q_2 \), as \( R_1 \) and \( R_2 \) are equivalence relations on \( Q_1 \) and \( Q_2 \) respectively. It is easy to see that the relation \( R \) on \( Q_1 \times Q_2 \) is nothing but \( R_1 \times R_2 \).

We begin with the following concept of (full) direct product of two rough finite state machines from [20]. In case of finite state machines, this product may be interpreted as the ‘parallel composition’ of two finite state machines (cf., e.g., Döfler [4]).

**Definition 3.1** [20] Let \( M_1 = (Q_1, R_1, X_1, \delta_1) \) and \( M_2 = (Q_2, R_2, X_2, \delta_2) \) be rough finite state machines. Then the RFSM \( M_1 \times M_2 = (Q_1 \times Q_2, R, X_1 \times X_2, \delta_1 \times \delta_2) \) is called (full) direct product of \( M_1 \) and \( M_2 \), where \( \delta_1 \times \delta_2 : (Q_1 \times Q_2) \times (X_1 \times X_2) \to A \) is a map such that \( \delta_1 \times \delta_2((q_1, q_2), (x_1, x_2)) = \left((\delta_1(q_1, x_1), \delta_2(q_2, x_2)), (\delta_1(q_1, x_1), \delta_2(q_2, x_2))\right) \), \( \forall (q_1, q_2) \in Q_1 \times Q_2 \) and \( \forall (x_1, x_2) \in X_1 \times X_2 \).

Inspired from [7], we now introduce more direct products of two rough finite state machines.

**Definition 3.2** Let \( M_1 = (Q_1, R_1, X, \delta_1) \) and \( M_2 = (Q_2, R_2, X, \delta_2) \) be rough finite state machines. Then the RFSM \( M_1 \wedge M_2 = (Q_1 \times Q_2, R, X, \delta_1 \wedge \delta_2) \) is called the restricted direct product of \( M_1 \) and \( M_2 \), where \( \delta_1 \wedge \delta_2 : (Q_1 \times Q_2) \times X \to A \) is a map such that \( \delta_1 \wedge \delta_2((q_1, q_2), x) = ((\delta_1(q_1, x), \delta_2(q_2, x)), (\delta_1(q_1, x), \delta_2(q_2, x))) \), \( \forall (q_1, q_2) \in Q_1 \times Q_2 \) and \( \forall x \in X \).

Let \( \overline{X} \) be a finite set and \( f : \overline{X} \to X_1 \times X_2 \) be a map. Also, let \( p_1 \) and \( p_2 \) be the projection mappings of \( X_1 \times X_2 \) onto \( X_1 \) and \( X_2 \) respectively, i.e., \( p_1 : X_1 \times X_2 \to X_1 \) and \( p_2 : X_1 \times X_2 \to X_2 \). Then the following is the concept of generalized direct product of rough finite state machines.

**Definition 3.3** Let \( M_1 = (Q_1, R_1, X_1, \delta_1) \) and \( M_2 = (Q_2, R_2, X_2, \delta_2) \) be rough finite state machines. Then the RFSM \( M_1 \ast M_2 = (Q_1 \times Q_2, R, \overline{X}, \delta_1 \ast \delta_2) \) is called general direct product of \( M_1 \) and \( M_2 \), where \( \delta_1 \ast \delta_2 : (Q_1 \times Q_2) \times \overline{X} \to A \) is a map such that \( \delta_1 \ast \delta_2((q_1, q_2), x) = ((\delta_1(q_1, p_1(x))), (\delta_2(q_2, p_2(f(x)))) \), \( \forall (q_1, q_2) \in Q_1 \times Q_2 \) and \( \forall x \in \overline{X} \).

**Remark 3.1** (i) If \( \overline{X} = X_1 \times X_2 \) and \( f \) is the identity map, then the general direct product \( M_1 \ast M_2 \) reduces to full direct product.
(ii) If $\delta_1 = X_1 = X_2$ and $f$ is the identity map, then the general direct product $M_1 \ast M_2$ reduces to restricted direct product.

The following proposition shows the relation between (full) direct product and restricted direct product through covering.

**Proposition 3.1** Let $M_1 = (Q_1, R_1, X_1, \delta_1)$ and $M_2 = (Q_2, R_2, X_2, \delta_2)$ be rough finite state machines. Then $M_1 \cap M_2 \preceq M_1 \times M_2$.

**Proof.** Let $\eta$ be an identity map on $Q_1 \times Q_2$. Then $\forall ((q_1, q_2), (q'_1, q'_2)) \in Q_1 \times Q_2$, $((q_1, q_2), (q'_1, q'_2)) \in R \Rightarrow (\eta(q_1, q_2), \eta(q'_1, q'_2)) \in R$. Define a map $\xi : X \rightarrow X \times X$ by $\xi(x) = (x, x)$, $\forall x \in X$. Now, for $(q_1, q_2) \in Q_1 \times Q_2$, and $x \in X$, $(\delta_1 \times \delta_2)(\eta(q_1, q_2), x) = (\delta_1 \times \delta_2)((q_1, q_2), x) = ((\delta_1(q_1, x), \delta_2(q_2, x)), (\delta_1(q_1, x), \delta_2(q_2, x)) = (\delta_1 \times \delta_2)((q_1, q_2), (x, x)) = (\delta_1 \times \delta_2)((q_1, q_2), (x, x))$. Also, $\xi(e) = (e, e)$, $e$ being the identity of $X$ and $\xi(x) = 0(x_2)\cdots 0(x_{n-1})0(x_n), \forall x = x_1x_2\ldots x_n \in X$. Thus $M_1 \cap M_2 \preceq M_1 \times M_2$.

The following lemma is useful to introduce the wreath product of rough finite state machines.

**Lemma 3.1** Let $S_1$ and $S_2$ be semigroups. Then $(S_1^{S_2} \times S_2, \ast)$ is a semigroup, where $S_1^{S_2} = \{ f \mid f : Q_2 \rightarrow S_1 \}$, $(f, s) \ast (g, t) = (fg, st)$ and $(f, g)(q_2) = f(g)(q_2), \forall f, g \in S_1^{S_2}, s, t \in S_2$ and $q_2 \in Q_2$.

**Proof.** Let $f, g, h \in S_1^{S_2}$ and $s, t, u \in Q_2$. Then $((f, s) \ast (g, t)) \ast (h, u) = (fg, st) \ast (h, u) = ((fg)h, stu) = (fg, stu) = (f, s) \ast (gh, tu) = (f, s) \ast ((h, t) \ast (h, u))$. Now, let $I \in S_1^{S_2}$ such that $I(q_2) = e_1, e_1$ being the identity of $S_1$. Then it can be seen that $(I, e_2)$ is an identity of $S_1^{S_2} \times S_2$, where $e_2$ is the identity of $S_2$. Thus $(S_1^{S_2} \times S_2, \ast)$ is a semigroup with identity $(I, e_2)$.

Now, we introduce the wreath product of rough finite state machines, which is a generalization of the same concept for finite state machines (cf., [2]).

**Definition 3.4** Let $M_1 = (Q_1, R_1, X_1, \delta_1)$, $M_2 = (Q_2, R_2, X_2, \delta_2)$ be rough finite state machines. Then the RFSM $M_1 \circ M_2 = (Q_1 \times Q_2, R, X_1^{Q_2} \times X_2, \delta_1 \circ \delta_2)$ is called the **wreath product** of $M_1$ and $M_2$, where $\delta_1 \circ \delta_2 : (Q_1 \times Q_2) \times (X_1^{Q_2} \times X_2) \rightarrow A$, is a map such that $\delta_1 \circ \delta_2)((q_1, q_2), (f, x)) = (\delta_1(q_1, f(q_2)), \delta_2(q_2, x)) = (\delta_1(q_1, f(q_2)), \delta_2(q_2, x)), \forall (q_1, q_2) \in Q_1 \times Q_2$ and $\forall (f, x) \in X_1^{Q_2} \times X_2$.

Let $M_n = (Q_n, R_n, X_n, \delta_n), n = 1, 2, 3, 4$ be rough finite state machines. Then $\delta_1 \times \delta_2$ and $\delta_1 \circ \delta_2$, $i = 1, 2, 3, 4, j = 1, 2, 3, 4$, appearing below associated with rough finite state machines $M_1 \times M_4$ and $M_1 \circ M_4$, $i = 1, 2, 3, 4, j = 1, 2, 3, 4$, respectively have their usual meaning.

Also, $R_1 \times R_2$ appearing below is a relation on $Q_1 \times Q_2$, defined as $((p_1, p_2), (q_1, q_2)) \in R_1 \times R_2$ iff $(p_1, q_1) \in R_1$ and $(p_2, q_2) \in R_2$, $i = 1, 2, 3, 4, j = 1, 2, 3, 4$.

Now, we have the following.

**Proposition 3.2** Let $M_i = (Q_i, R_i, X_i, \delta_i)$, where $i = 1, 2, 3, 4$ be rough finite state machines. Then $(M_1 \circ M_4) \times (M_3 \circ M_4) \preceq (M_1 \times M_4) \circ (M_2 \times M_4)$.

**Proof.** Let $M_1 \circ M_4 = (Q_1 \times Q_2, R_1 \times R_2, (X_1^{Q_2} \times X_2, \delta_1 \circ \delta_2)$ and $M_2 \circ M_4 = (Q_3 \times Q_4, R_3 \times R_4, (X_3^{Q_4} \times X_4, \delta_3 \circ \delta_4)$. Then $(M_1 \circ M_4) \times (M_2 \circ M_4) = ((Q_1 \times Q_2) \times (Q_3 \times Q_4), (R_1 \times R_2) \times (R_3 \times R_4), (X_1^{Q_2} \times X_2) \times (X_3^{Q_4} \times X_4, \delta_1 \circ \delta_2 \circ \delta_3 \circ \delta_4)$. Therefore, $(M_1 \circ M_4) \times (M_2 \circ M_4) \preceq (M_1 \times M_4) \circ (M_2 \times M_4)$.
Definition 3.5 Let $M_1 = (Q_1, R_1, X_1, \delta_1)$, $M_2 = (Q_2, R_2, X_2, \delta_2)$ be rough finite state machines and $\omega : Q_2 \times X_2 \to X_1$ be a map. Then the RFMSM $M_1 \omega M_2 = (Q_1 \times Q_2, R, X_2, \delta_1 \omega \delta_2)$ is called the cascade product of $M_1$ and $M_2$, where $\delta_1 \omega \delta_2 : (Q_1 \times Q_2) \times X_2 \to A$, is a map such that $(\delta_1 \omega \delta_2)((q_1, q_2), x_2) = ((\delta_2(q_2, x_2)), (\delta_1(q_1, \omega(q_2, x_2)), \delta_2(q_2, x_2)))$. Then for $(q_1, q_2) \in Q_1 \times Q_2$ and $x_2 \in X_2$, $(\delta(q_1, q_2, x_2), 1) = (\delta_1(q_1, \omega(q_2, x_2)), \delta_2(q_2, x_2))$.

Remark 3.2 Let $M_1 \omega M_2 = (Q_1 \times Q_2, R, X_2, \delta_1 \omega \delta_2)$ be the cascade product of rough finite state machines $M_1$ and $M_2$ such that $X_1 = X_2 = X(say)$ and $\omega : Q_2 \times X \to X$ be the map, then the restricted direct products of $M_1$ and $M_2$ is a special case of their cascade products.

Finally, we introduce the following concept of cascade product of rough finite state machines.
\[(f_1, x_n) \Rightarrow \forall f = f_1 \circ \cdots \circ f_n \in X_1^{Q_2} \text{ and } \forall x = x_{1 \times 2 \times \cdots \times n} \in X_2. \text{ Thus } M_1 \circ M_2 \leq M_1 \circ M_2.\]

The following propositions are direct consequences of the associativity of products of rough finite state machines.

**Proposition 3.4** Let \(M_1, M_2\) and \(M_3\) be rough finite state machines. Then

(i) \((M_1 \times M_2) \times M_3 \cong M_1 \times (M_2 \times M_3),\)

(ii) \((M_1 \land M_2) \land M_3 \cong M_1 \land (M_2 \land M_3),\)

(iii) \((M_1 \circ M_2) \circ M_3 \cong M_1 \circ (M_2 \circ M_3),\) and

(iv) \((M_1^\delta_3 M_2) \omega_2 M_3 \cong M_1 \omega_3(M_2 \omega_3 M_3),\) where \(\omega_3\) and \(\omega_4\) are determined by \(\omega_1\) and \(\omega_2\) in a natural way.

Let \(M_n = (Q_n, R_n, X_n, \delta_n), n = 1, 2, 3\) be rough finite state machines. Then \(\delta_1 \times \delta_2, \delta_1 \land \delta_2, \delta_1 \circ \delta_1\) and \(\delta_1 \omega_1\), where \(\omega_1\) and \(\omega_2\) appearing below are rough transition maps associated with rough finite state machines \(M_1 \times M_2, M_1 \land M_1, M_1 \circ M_1\) and \(M_1 \omega_1 M_1, i = 1, 2, j = 3,\) respectively.

**Proposition 3.5** Let \(M_1 = (Q_1, R_1, X_1, \delta_1), M_2 = (Q_2, R_2, X_2, \delta_2)\) and \(M_3 = (Q_3, R_3, X_3, \delta_3)\) be rough finite state machines such that \(M_1 \leq M_2\). Then

(i) \((a) M_1 \times M_3 \leq M_2 \times M_3 \text{ and } (b) M_1 \times M_3 \leq M_3 \times M_2,\)

(ii) if \(X = X_1 = X_2\), then \((a) M_1 \times M_3 \leq M_2 \times M_3 \text{ and } (b) M_1 \times M_3 \leq M_3 \times M_2,\)

(iii) \((a) M_1 \circ M_3 \leq M_2 \circ M_3 \text{ and } (b) M_3 \circ M_3 \leq M_2 \circ M_3,\)

(iv) \((a) \text{ given } \omega_1 : Q_1 \times X_1 \to X_1, \text{ there exists } \omega_2 : Q_2 \times X_2 \to X_2, \text{ such that } M_1 \omega_1 M_3 \leq M_2 \omega_2 M_3 \text{ and } (b) \text{ if } (\eta, \xi) \text{ is a covering of } M_1 \text{ by } M_2, \text{ then for each } \omega_1 : Q_1 \times X_1 \to X_1 \text{ there exists } \omega_2 : Q_2 \times X_2 \to X_2 \text{ such that } \omega_1 M_2 \omega_2 M_3 \leq M_2 \omega_2 M_2.\)

**Proof.** As \(M_1 \leq M_2\), there exist \(\text{an onto map } \eta : Q_2 \to Q_1\) and a map \(\xi : X_1 \to X_1\) such that \((q_2, q'_2) \in R_2 \Rightarrow (\eta(q_2), \eta(q'_2)) \in R_1, q_2, q'_2 \in Q_2,\) and \(\delta_1(\eta(q_2), x) \leq (\delta_1(\eta(q_2), x), q_1) \leq (\delta_1(\eta(q_2, q_1, x)), q_1) \leq (\delta_1(\eta(q_2, q_1), x), q_1) \leq \eta(\delta_1(\eta(q_2, q_1), x), q_1) \leq \eta(\delta_1(\eta(q_2, q_1), x), q_1), \forall x \in X_1, \text{ and } \xi(x) = \xi(x_1) \xi(x_2) \cdots \xi(x_n), \forall x = x_1 x_2 \cdots x_n \in X_1.\)

(i) \((a) \text{ Let } M_1 \times M_3 = (Q_1 \times Q_3, R_1 \times R_3, X_1 \times X_3, \delta_1 \times \delta_3) \text{ and } M_2 \times M_3 = (Q_2 \times Q_3, R_2 \times R_3, X_2 \times X_3, \delta_2 \times \delta_3). \text{ Define an onto map } \eta_\xi : Q_2 \times Q_3 \to Q_1 \times Q_3 \text{ by } \eta_\xi(q_2, q_3) = (\eta(q_2, q_3)\), \(q_2, q_3) \) and a map \(\xi_\xi : X_1 \times X_3 \to X_2 \times X_3 \text{ by } \xi_\xi(x_1, x_3) = (\xi(x_1), x_3). \text{ Then } ((q_2, q_1, q_3)) \in R_2 \times R_3 \Rightarrow \{(\eta(q_2, q_1), (q_2, q_3)) \in R_1 \times R_3, \forall((q_2, q_3), (q_2, q_3)) \in Q_2 \times Q_3, \text{ Let } (q_2, q_3) \in Q_2 \times Q_3 \text{ and } (x_1, x_3) \in X_1 \times X_3, \text{ then } \delta_1(\delta_2(\eta(q_2, q_3), (x_1, x_3))) \leq \delta_1(\delta_2(\eta(q_2, q_3), (x_1, x_3))) \leq \delta_1(\delta_2(\xi_\xi(x_1), x_3)) \leq (\delta_1(\delta_2(\eta(q_2, q_3), (x_1, x_3))) \leq (\delta_1(\delta_2(\xi_\xi(x_1), x_3))) \leq (\delta_1(\delta_2(\xi_\xi(x_1), x_3))) \leq \eta_\xi(\delta_1(\delta_2(\xi_\xi(x_1), x_3))) \leq \xi_\xi(\delta_1(\delta_2(\xi_\xi(x_1), x_3))) \leq \xi_\xi(\delta_1(\delta_2(\xi_\xi(x_1), x_3))) \leq \eta(\delta_1(\delta_2(\xi_\xi(x_1), x_3))) \leq \eta(\delta_1(\delta_2(\xi_\xi(x_1), x_3))) \leq \eta(\delta_1(\delta_2(\xi_\xi(x_1), x_3))) \leq \eta(\delta_1(\delta_2(\xi_\xi(x_1), x_3))) \leq \eta(\delta_1(\delta_2(\xi_\xi(x_1), x_3))). \text{ Now, } \xi_\xi(e_1, e_3) = (\xi_1(e_1), e_3) = (e_1, e_3), \text{ where } e_1, e_3 \text{ being the identity elements of } X_1, X_3 \text{ respectively and } \xi_\xi(x, y) = (\xi(x), y) = (\xi(x) x_2 \cdots x_n), y_1 y_2 \cdots y_n = (\xi(x_1) x_2 \cdots x_n), y_1 y_2 \cdots y_n) = ((\xi(x_1), y_1) (\xi(x_2), y_2) \cdots (\xi(x_n), y_n)), \forall x = x_1 x_2 \cdots x_n \in X_1 \text{ and } \forall y = y_1 y_2 \cdots y_n \in X_3. \text{ Thus } M_1 \times M_3 \leq M_2 \times M_3.\)

(b) The proof is similar to that of Proposition 3.4 (i) (a).

(ii) \((a) \text{ Let } X = X_1 = X_2. \text{ Then } M_1 \land M_3 = (Q_1 \times Q_3, R_1 \times R_3, X, \delta_1 \land \delta_3) \)
and \( M_2 \land M_3 = (Q_2 \times Q_3, R_2 \times R_3, X, \delta_2 \land \delta_3) \). Define an onto map 
\( \eta_\Lambda : Q_2 \times Q_3 \to Q_1 \times Q_4 \) by \( \eta_\Lambda(q_2, q_3) = (\eta(q_2), q_3) \) and \( \xi_\Lambda = \xi \) be the map on \( X \). Then \( (\eta_\Lambda, \xi_\Lambda) \) is the required covering.

(b) Follows as above.

(iii) \((a)\) Let \( M_1 \circ M_3 = (Q_1 \times Q_3, R_1 \times R_3, X_{Q_1}^{Q_3} \times X_3, \delta_1 \circ \delta_3) \) and \( M_2 \circ M_3 = (Q_2 \times Q_3, R_2 \times R_3, X_{Q_2}^{Q_3} \times X_3, \delta_2 \circ \delta_3) \). Define an onto map 
\( \eta_\alpha : Q_2 \times Q_3 \to Q_1 \times Q_3 \) by \( \eta_\alpha(q_2, q_3) = (\eta(q_2), q_3) \) and a map \( \xi_\alpha : X_{Q_2}^{Q_3} \times X_3 \to X_1 \times X_3 \) by \( \xi_\alpha(f, x_3) = (\xi(f), x_3) \). Then for \((q_2, q_4, q_4', q_3') \in R_2 \land R_3 \Rightarrow ((\eta(q_2), q_3), (\eta(q_4), q_4')) \in R_1 \land R_3 \land \forall ((q_2, q_4)(q_4', q_3')) \in Q_2 \times Q_1 \). Also, let \((q_2, q_3) \in Q_2 \times Q_3 \) and \((f, x_3) \in X_{Q_2}^{Q_3} \times X_3 \). Then \( (\delta_1 \circ \delta_3)((\eta(q_2, q_3), (f, x_3)) = (\delta_1 \circ \delta_3)((\eta(q_2), q_3), (f, x_3)) = ((\delta_1(\eta(q_2)), f(q_2)), \delta_3(q_3, x_3)), (\delta_3(q_3, x_3))) \subseteq ((\eta(\delta_1(q_2, \xi(f(q_2)))), \delta_3(q_3, x_3))) = ((\delta_1 \circ \delta_3)(\eta(q_2, q_3), x_3)) \). Now, for \( I \in X_{Q_2}^{Q_3}, I(q_3) = e_1, e_1 \) being the identity of \( X_1 \), \( I(e_3) = X_1 \). Also, \( \xi(I, e_3) = (\xi \circ I, e_3) \), which is an identity element of \( X_{Q_1}^{Q_3} \). Again, \( \xi(f, x) = (\xi \circ f, x) = (\xi(f_1, f_2, f_3, f_4, \ldots, f_n), x_1(x_2, x_3, x_4, \ldots, x_n) = ((\xi \circ f_1, x_1), (\xi \circ f_2, x_2), (\xi \circ f_3, x_3), (\xi \circ f_4, x_4), \ldots, (\xi \circ f_n, x_n)) \). Again, \( \xi_I(f, x) = (\xi_I(f_1, f_2, f_3, f_4, \ldots, f_n), x_1(x_2, x_3, x_4, \ldots, x_n) = ((\xi_I \circ f_1, x_1), (\xi_I \circ f_2, x_2), (\xi_I \circ f_3, x_3), (\xi_I \circ f_4, x_4), \ldots, (\xi_I \circ f_n, x_n)) \). Hence the covering exist.

(b) Given \( \omega_1 : Q_1 \times X_1 \to X_3 \), let \( \omega_2 : Q_2 \times X_2 \to X_3 \) such that \( \omega_1(q_1, X_1) = \omega_2(q_1, X_2) = \omega_1(\eta(q_2), x_1) \). Define an onto map \( \eta_\omega : Q_1 \times Q_2 \to Q_2 \times Q_1 \) by \( \eta_\omega(q_1, q_2) = (q_1, \eta(q_2)) \) and set \( \xi_\omega = \xi \). Then for \((q_1, q_2), (q_4, q_3) \in R_3 \land R_3 \Rightarrow ((q_1, \eta(q_2)), (q_4, q_3)) \in R_3 \land R_3 \land \forall ((q_1, \eta(q_2)), (q_4, q_3)) \in Q_3 \times Q_3 \). Thus \( (\eta_\omega, \xi_\omega) \) is the required covering.

4 Conclusion

Chieﬂy inspired from [14], we have introduced and studied here the concept of rough ﬁnite state machine and several products viz., direct product, cascade product and wreath product of rough ﬁnite state machines. Also, we studied the relationship between these different products through coverings as well as examined some algebraic properties. We hope that, like fuzzy ﬁnite state machines, rough ﬁnite state machines, which is another dimension of application of rough set theory, will attract the researchers and the work carried out here will help in finding some successful applications of rough ﬁnite state machines.
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