Functional analytic insights into irreversibility of quantum resources

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We propose an approach to the study of quantum resource manipulation based on the basic observation that quantum channels which preserve certain sets of states are contractive with respect to the base norms induced by those sets. We forgo the usual physical assumptions on quantum dynamics: instead of enforcing complete positivity, trace preservation, or resource-theoretic considerations, we study transformation protocols as norm-contractive maps. This allows us to apply to this problem a technical toolset from functional analysis, unifying previous approaches and introducing new families of bounds for the distillable resources and the resource cost, both one-shot and asymptotic. Since our expressions lend themselves naturally to single-letter forms, they can often be calculated in practice; by doing so, we demonstrate with examples that they can yield the best known bounds on quantities such as the entanglement cost. As applications, we not only give an alternative derivation of the recent result of [arXiv:2111.02438] which showed that entanglement theory is asymptotically irreversible, but also provide the quantities introduced in that work with explicit operational meaning in the context of entanglement distillation through a novel generalised form of hypothesis testing relative entropy. Besides entanglement, we reveal a new irreversible quantum resource: through improved bounds for state transformations in the resource theory of magic-state quantum computation, we show that there exist qutrit magic states that cannot be reversibly interconverted under stabiliser protocols.

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1. INTRODUCTION

The practical use of various physical features of quantum mechanical systems to enhance tasks such as computation, communication, or information processing is underlain by our ability to transform these resources efficiently. To understand the limits of such transformations has therefore been one of the most pressing problems of quantum information science. The ultimate extent of resource conversion is characterised by asymptotic transformation rates — that is, the question of how well we can transform quantum systems when allowed to use an unbounded number of their copies. The study of these rates forms the foundation of quantum information theory, embodied by problems such as the calculation of quantum channel capacities [1–3] or the distillation [4] and dilution [5] of quantum entanglement. Unfortunately, an exact understanding of the best achievable rates is often intractable, and the available bounds are, in many cases, quite loose. The problem is exacerbated by the wide variety of different settings and quantum resources beyond entanglement, each seemingly requiring specialised and unique approaches to fully characterise.

The unified framework of quantum resource theories [6, 7] was conceived to understand the common features of the various resources encountered in the processing of quantum systems, and has already provided significant insights into both asymptotic [6, 8–12] and single-shot [12–17] state transformations. Its remarkable generality has allowed for the consolidated description of many physical phenomena in a broad, axiomatic manner under a comprehensive mathematical formalism [18–20]. However, this same generality can also be a drawback: in order to remain general, one can only make broadly applicable assumptions on the considered resources, which means that specific features of different resources might remain unaccounted for. This could suggest that it is not in our best interest to strive for such general approaches, and instead more tailored methods should be employed to obtain better results. We hereby put this premise into question by introducing a framework that is strictly more general than previous approaches, yet it yields an improvement over known bounds for asymptotic resource transformations and reveals features of quantum resource manipulation that prior methods were unable to uncover.

1.1. Quantum resources and their manipulation

A quantum resource theory is a mathematical model of operational constraints encountered in manipulating quantum systems, stemming from either practical or fundamental limitations. It is specified by a set $\mathcal{F}$ of free states, comprising certain density operators on a family of quantum systems, and a set $\mathcal{O}$ of free operations, i.e. state-to-state transformations that connect those systems. Operationally, $\mathcal{F}$ and $\mathcal{O}$ represent states and operations that are relatively inexpensive to construct or implement in the laboratory; together, they form the basic ingredients of a quantum resource theory. In order to be physically realisable, any transformation in $\mathcal{O}$ should be a quantum channel, i.e. a completely positive and trace preserving linear map. In the resource theory of quantum entanglement [21], for example, $\mathcal{F}$ is taken to be the set of separable states over bipartite quantum systems [22], while historically $\mathcal{O}$ is usually taken to be the set of transformation implementable via local operations assisted by classical communication (LOCC) [23], although other choices of operations have also found fruitful applications [24–28].

One of the central problems of quantum resource theories is to characterise the ultimate asymptotic transformation rates that are achievable by means of the free operations. In particular, one is interested in two opposite processes: on the one hand, that of distillation [4], which consists of taking many copies of a given state $\rho$ and transforming them, via free operations, into as many copies as possible of some standard unit of resource $\phi$ (typically a pure state); and on the other hand,
that of \textit{dilution} \cite{5}, in which we take as few copies as possible of $\phi$ and attempt to transform them into copies of $\rho$. In both cases, we allow for an asymptotically small error in the transformation, as quantified by the trace norm. The largest ratio between output copies of $\phi$ and input copies of $\rho$ in distillation is called the \textit{distillable resource} of $\rho$ under operations in $\mathcal{O}$, and denoted by $R_{d,\mathcal{O}}(\rho)$. Conversely, the smallest ratio between input copies of $\phi$ and output copies of $\rho$ in dilution is called the \textit{resource cost} of $\rho$ under operations in $\mathcal{O}$, and denoted by $R_{c,\mathcal{O}}(\rho)$.

For many sets $\mathcal{O}$ whose construction stems from physical considerations, the ‘no-free-lunch’ inequality $R_{c,\mathcal{O}}(\rho) \geq R_{d,\mathcal{O}}(\rho)$ can be shown to hold for all $\rho$. Notably, it can be strict, and this fact is at the heart of the phenomenon of resource irreversibility \cite{28–32}. In the context of entanglement theory, when $\phi$ is chosen to be the ‘entanglement bit’ (or \textit{ebit}), the above two measures turn into the distillable entanglement and the entanglement cost \cite{21}; irreversibility in the context of entanglement manipulation under LOCC is a long-known fact \cite{29, 30}, and even stronger variants of it were shown recently \cite{28, 31}.

More generally, computing either $R_{c,\mathcal{O}}(\rho)$ or $R_{d,\mathcal{O}}(\rho)$ for a given state $\rho$ is typically a challenging task. While upper bounds on $R_{c,\mathcal{O}}(\rho)$ or lower bounds on $R_{d,\mathcal{O}}(\rho)$ are relatively easy to obtain by the explicit construction of resource manipulation protocols, lower bounding $R_{d,\mathcal{O}}(\rho)$ or upper bounding $R_{c,\mathcal{O}}(\rho)$ requires a control over all possible protocols — in general, a highly non-trivial task.

Since the physical meaning and origin of $\mathcal{O}$ typically depends heavily on the specific type of quantum resource under examination, an approach often taken in the operational characterisation of quantum resources is to employ the natural axiom sometimes referred to as the ‘golden rule’ of quantum resource theories \cite{7}: a free operation should never generate the given resource, in the sense that if $\sigma \in \mathcal{F}$, then $\Lambda(\sigma) \in \mathcal{F}$. We denote by $\mathcal{O}_{\text{max}} \subseteq \text{CPTP}$ the maximal set of all channels that satisfy this condition. This assumption is intended to be a weak and non-restrictive condition that any physical class of free transformations ought to obey, allowing one to describe all such operations in a single framework.

Beyond mathematical convenience, is such an approach physically meaningful? Take, as an example, the resource theory of entanglement: the operations $\mathcal{O}_{\text{max}}$ — known as non-entangling or separability-preserving maps \cite{26, 33} — include all naturally ‘free’ classes of maps such as local operations and classical communication as well as all separable operations, but also transformations such as the swap channel, which can be considered costly to realise in practice because they require quantum communication. Thus, in practical scenarios, one is typically concerned with some physically motivated set of operations of interest $\mathcal{O} \subseteq \mathcal{O}_{\text{max}}$. It then follows immediately that — for any possible choice of $\mathcal{O}$ — the rate at which a resource can be distilled can only increase under $\mathcal{O}_{\text{max}}$, i.e. $R_{d,\mathcal{O}_{\text{max}}}(\rho) \geq R_{d,\mathcal{O}}(\rho)$, while the cost can only decrease, i.e. $R_{c,\mathcal{O}_{\text{max}}}(\rho) \leq R_{c,\mathcal{O}}(\rho)$. Importantly, this immediately gives

$$R_{c,\mathcal{O}}(\rho) \geq R_{c,\mathcal{O}_{\text{max}}}(\rho) \geq R_{d,\mathcal{O}_{\text{max}}}(\rho) \geq R_{d,\mathcal{O}}(\rho)$$

as long as the operations $\mathcal{O}_{\text{max}}$ satisfy the property that the resource cost does not exceed the distillable resource. The latter is a natural condition satisfied in virtually all settings of physical interest, and can be ensured generally under very mild assumptions \cite{10, 34}. We thus see immediately that studying the relation between resource distillation and dilution under $\mathcal{O}_{\text{max}}$ can immediately shed light on the corresponding relations under any physical class of operations $\mathcal{O}$. Notably, this is in spite of the fact that the operations in $\mathcal{O}_{\text{max}}$ may or may not be physical themselves.
1.2. Basic idea

The above discussion tells us that in order to lower bound \( R_{c,\mathcal{O}}(\rho) \) or upper bound \( R_{d,\mathcal{O}}(\rho) \), we can focus instead on doing the same for \( R_{c,\mathcal{O}_{\text{max}}}(\rho) \) and \( R_{d,\mathcal{O}_{\text{max}}}(\rho) \). But this does not look substantially easier than the original task — \( \mathcal{O}_{\text{max}} \) may be simpler than \( \mathcal{O} \) due to its axiomatic origin, but to lower bound \( R_{c,\mathcal{O}_{\text{max}}}(\rho) \), for instance, we still have to keep under control all possible resource dilution protocols in \( \mathcal{O}_{\text{max}} \). The situation would be radically different if we decided instead to resort to the no-free-lunch bound \( R_{c,\mathcal{O}_{\text{max}}}(\rho) \geq R_{d,\mathcal{O}_{\text{max}}}(\rho) \): indeed, now the right-hand side can be lower bounded by making educated guesses of possible distillation protocols. While this ‘shortcut strategy’ works in certain cases, the bounds obtained in this way are typically very loose — this is the case, for instance, for certain entangled states \cite{28}. In order to improve on such bounds, we notice a curious property that has emerged: although the operations \( \mathcal{O}_{\text{max}} \) are a relaxation of the physically relevant class \( \mathcal{O} \), distillation protocols under \( \mathcal{O}_{\text{max}} \) provide better bounds on \( R_{c,\mathcal{O}} \) than distillation protocols under \( \mathcal{O} \) would (cf. (1)). What if we took this insight further?

In this work, we thus take the unorthodox approach of improving on asymptotic transformation bounds by relaxing the constraints on quantum resource manipulation even more — in practice, by considering a strictly larger set of maps than \( \mathcal{O}_{\text{max}} \). But if \( \mathcal{O}_{\text{max}} \) was intended to be the largest possible class of quantum channels relevant in the given resource theory, then how could we hope to define even larger classes? Our main idea may seem quite radical: first, we give up the physicality of the allowed maps, replacing complete positivity and trace preservation with contractivity with respect to a suitably chosen ‘resource norm’ \( \| \cdot \|_{\mathcal{O}} \), expressed in formula as
\[
\|\Lambda(X)\|_{\mathcal{O}} \leq \|X\|_{\mathcal{O}}
\]
for all operators \( X \); and we renounce also the golden rule of resource non-generation, replacing it with contractivity with respect to a suitably chosen ‘resource norm’ \( \| \cdot \|_{\mu} \), expressed in formula as
\[
\|\Lambda(X)\|_{\mu} \leq \|X\|_{\mu}
\]
for all \( X \). The resulting enlarged set of free operations will be denoted by \( \mathcal{O}_{\mu} \supseteq \mathcal{O}_{\text{max}} \supseteq \mathcal{O} \).

Ideally, this set \( \mathcal{O}_{\mu} \) should obey the no-free-lunch inequality \( R_{c,\mathcal{O}_{\mu}}(\rho) \geq R_{d,\mathcal{O}_{\mu}}(\rho) \). If that is the case, then we can write \( R_{c,\mathcal{O}}(\rho) \geq R_{c,\mathcal{O}_{\mu}}(\rho) \geq R_{d,\mathcal{O}_{\mu}}(\rho) \geq R_{d,\mathcal{O}_{\text{max}}}(\rho) \); therefore, and this is our main point, regardless of the unphysicality of \( \mathcal{O}_{\mu} \), the quantity \( R_{d,\mathcal{O}_{\mu}}(\rho) \) potentially gives a better lower bound on the physically relevant quantity \( R_{c,\mathcal{O}}(\rho) \) than \( R_{d,\mathcal{O}_{\text{max}}}(\rho) \) did. Coupled with the fact that any ansatz of a distillation protocol in \( \mathcal{O}_{\mu} \) provides a lower bound for \( R_{d,\mathcal{O}_{\mu}} \), the latter can serve as a new, potentially stronger class of computable lower bounds for resource cost. An analogous statement holds for distillation: \( R_{c,\mathcal{O}_{\mu}}(\rho) \) leads to a better upper bound on \( R_{d,\mathcal{O}}(\rho) \) than \( R_{c,\mathcal{O}_{\text{max}}}(\rho) \), due to the chain of inequalities \( R_{d,\mathcal{O}}(\rho) \leq R_{d,\mathcal{O}_{\mu}}(\rho) \leq R_{c,\mathcal{O}_{\mu}}(\rho) \leq R_{c,\mathcal{O}_{\text{max}}}(\rho) \), and it is in turn straightforward to estimate from above.

In short, the starting point of this work is the realisation that going beyond physically allowed quantum channels and into the realm of maps that are not completely positive can yield better bounds on operational asymptotic resource quantifiers. This programme will prove successful in several ways: most notably, the asymptotic bounds we obtain can go beyond all previously known bounds, such as ones based on the quantum relative entropy \cite{6,35}, revealing features and limitations of quantum resources that remained completely out of reach of previous theoretical approaches. One of the reasons behind the success of our methods is that by forgoing difficult-to-handle constraints such as complete positivity and resource non-generation, and focusing instead on simpler norm-based constraints such as contractivity, we can employ a wealth of functional and convex analytic tools to tackle the problem. A by-product of this approach is that our results will hold for virtually every convex quantum resource theory.
1.3. Technical hurdles and main contributions

The above idealised picture, although appealing, contains some potential pitfalls in passing to the asymptotic limit.

First, for larger sets of operations such as $\mathcal{O}_\mu$ it is typically not at all easy to prove the no-free-lunch inequality $R_{c,\mathcal{O}_\mu}(\rho) \geq R_{d,\mathcal{O}_\mu}(\rho)$ that underpins this whole approach. The origin of this difficulty lies in the generic asymptotic inequivalence of the two norms employed, the trace norm $\|\cdot\|_1$ on the one hand and the resource norm $\|\cdot\|_\mu$ on the other. Namely, even when the Hilbert space $\mathcal{H}$ corresponding to a single quantum system is assumed to be finite dimensional so that the norms $\|\cdot\|_1$ and $\|\cdot\|_\mu$ are equivalent, in formula $c \|\cdot\|_1 \leq \|\cdot\|_\mu \leq C \|\cdot\|_1$ for some constants $C \geq c > 0$, such constants will in general depend on the dimension; as a consequence, while the output state $\omega_n = \Lambda_n(\phi^\otimes n)$ of a dilution process is guaranteed to be close in trace norm to $\rho^\otimes m$, or $\|\omega_n - \rho^\otimes m\|_1 \leq \epsilon_n$ with $\lim_{n \to \infty} \epsilon_n = 0$, this fact alone does not imply that $\|\omega_n\|_\mu$ is close to $\|\rho^\otimes m\|_\mu$. This consideration prevents us from using the $\|\cdot\|_\mu$-contractivity of $\Lambda_n$ to relate directly $\|\phi^\otimes n\|_\mu$ with $\|\rho^\otimes m\|_\mu$.

Second, even if we circumvent the above obstacle, we are left with the problem of lower bounding the asymptotic distillation rate $R_{d,\mathcal{O}_\mu}$ with a quantity that can actually be evaluated in practice. This is typically done by reducing it to a one-shot distillation protocol, where only a single copy of a state is used. On the one hand, this might not be as simple in our approach as in some previous frameworks, because the employed resource norm $\|\cdot\|_\mu$ is not guaranteed to be multiplicative, i.e. it may be the case that $\|\rho^\otimes n\|_\mu < \|\rho\|_\mu^n$. On the other hand, we have to be careful not to take a bound which is too loose — we want to benefit from working in the extended framework of ‘unphysical’ operations $\mathcal{O}_\mu$, and not merely recover previously known restrictions.

Overcoming these difficulties can be regarded as our main technical contribution. We will achieve this by identifying a novel type of hypothesis testing relative entropy, dubbed the emancipated hypothesis testing relative entropy (see Section 3.2), which plays a central role in characterising distillation under operations in $\mathcal{O}_\mu$. Our key insight is that this quantity obeys a particular type of continuity expressed by the ‘$\epsilon$-$\delta$ lemma’ below (Lemma 4). By using this observation, we are able to provide a variant of the no-free-lunch relation $R_{c,\mathcal{O}_\mu}(\rho) \geq R_{d,\mathcal{O}_\mu}(\rho)$ that allows for straightforward lower bounding of the involved rates through suitable choices of norms.

The results we will obtain in this way compete with state-of-the-art bounds on general transformation rates, recovering and improving on many of the leading asymptotic and non-asymptotic bounds for the manipulation of important quantum resources. We first demonstrate the effectiveness of our techniques by applying them to quantum entanglement [21], in which context we provide in particular a number of novel lower bounds on the entanglement cost of any state that are efficiently computable via semidefinite programs. Obtaining such bounds is a notoriously thorny problem, especially since Hastings’s counterexample [36] has disproved the additivity conjectures [37] that would have allowed for a significant simplification, leaving us with intractable regularised expressions for asymptotic quantities. We then apply our toolset to the theory of magic states [38, 39] (non-stabiliser quantum computation), where our results immediately yield a number of bounds for the rates of transformations both in the case of many-qudit as well as many-qubit systems. We show that our results can strictly improve on previously known bounds in this setting, including ones based on the regularised relative entropy of magic [38] and one-shot variants thereof [40].

Crucially, the bounds established in our work are strong enough to reveal the irreversibility of quantum resources, that is, the inability to reversibly interconvert two quantum states under the constraints of a given resource theory. This phenomenon — although well known in some restricted settings such as the theory of entanglement under local operations and classical com-
unication — has been famously difficult to characterise in general [8, 33, 41]. A pertinent aspect of the problem is that, if a given resource can be shown to be irreversible under all choices of free operations, then this entails that there cannot exist a single monotone that completely determines asymptotic convertibility of quantum states, which would mirror the role of entropy in thermodynamics [9, 42–44]. Only recently was it shown that such a strong irreversibility materialises for entanglement theory [28]: there exist no free operations that can make entanglement reversible, and so this theory cannot be governed by a single entropic quantity. Our results can be used not only to recover the main finding of [28] in a somewhat different manner, but also to endow the monotones introduced in [28] with a precise operational meaning as the distillable entanglement under linear maps that contract the entanglement negativity.

Beyond entanglement theory, our general approach uncovers a new irreversible resource: we show the theory of magic-state quantum computation to be irreversible under all stabiliser operations, i.e. under all operations that can be implemented by using only Clifford unitaries, Pauli measurements, and ancillary stabiliser states. We do so by exemplifying a pair of qutrit pure magic states that cannot be converted reversibly using general types of free operations that include all stabiliser protocols. These results showcase the breadth and generality of our approach, obtained without detriment of its effectiveness — indeed, the strength of our bounds compared to all previously known approaches allowed us to substantially advance our understanding of the problem of resource irreversibility, which no previous restrictions were able to do.

The structure and main findings of our paper are as follows.

• In Section 2, we set up the setting of our work and introduce all of the relevant concepts in the description of quantum resource theories and their manipulation.

• In Section 3, we introduce tight bounds on resource conversion in the one-shot setting, characterising exactly the restrictions on achievable regimes in resource transformations (Theorems 1 and 2). We broadly divide our results between bounds that apply to the task of distillation (purification) as well as more general bounds, applicable in particular to the task of dilution (reverse of distillation). Our description of distillation protocols in Theorem 2 is enabled by a new generalisation of the hypothesis testing relative entropy that we introduce in Section 3.2. The key insights here are precise trade-offs between the performance of resource manipulation protocols and the values of related norm-based resource quantifiers (Lemma 4), which will later form the foundation of our asymptotic bounds.

• Section 4 is concerned with the establishment of our main results: a variety of bounds on the asymptotic rates of transformations between quantum states. First, in Section 4.1 we introduce our strongest bounds that have potential to improve on previously known limitations for resource conversion (Theorems 5 and 6). These bounds require the evaluation of asymptotic regularised quantities, which may be very difficult to compute in full generality. In Section 4.2 we therefore propose a method to bound them by single-letter quantities, which yields a number of efficiently computable bounds on transformation rates that can be evaluated as convex or even semidefinite programs (notably, Corollary 9). Section 4.3 is then concerned with showing that one can, nevertheless, estimate the regularised quantities of Section 4.1 in some cases, allowing for the evaluation of our bounds when the single-letter approaches are not good enough. Finally, Section 4.4 discusses how appropriate choices of norms in our expressions can be used to recover the best known efficiently computable bounds on resource distillation in the literature.

• Aiming to give more practical insight into the abstract resource-theoretic methods of the previous sections, Section 5 provides general pointers as to how our results can be applied to specific theories in practice and what such applications require.
• In Section 6, we explicitly apply our results to the theory of entanglement manipulation. We discuss the relation between our bounds and known entanglement measures. We provide a class of new lower bounds on entanglement cost based on the reshuffling criterion for separability (Corollary 12). We finally show how the irreversibility of entanglement manipulation can be retrieved with a somewhat different technique using our new results (Proposition 13).

• Finally, Section 7 is concerned with the theory of magic-state quantum computation. We show how to apply our results in detail and provide several new bounds on the asymptotic transformation rates. We establish, in particular, the irreversibility of the theory of magic for many-qudit systems (Theorem 14), and discuss a conjecture related to an analogous irreversibility also in the case of many-qubit magic.

2. PRELIMINARIES

Our considerations will take place in the space of self-adjoint operators acting on a separable Hilbert space. Specifically, let $\mathcal{H}$ be a separable Hilbert space, finite or infinite dimensional. We use $\mathcal{T}(\mathcal{H})$ to denote the Banach space of all trace-class operators on $\mathcal{H}$, and $\mathcal{B}(\mathcal{H})$ the dual space of all bounded operators on $\mathcal{H}$. For any $Y \in \mathcal{B}(\mathcal{H})$ and $X \in \mathcal{T}(\mathcal{H})$, we use the notation $\langle Y, X \rangle := \text{Tr} Y^\dagger X$, which corresponds to the Hilbert–Schmidt inner product when $Y$ is also a Hilbert–Schmidt operator. The restrictions of $\mathcal{T}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ to self-adjoint operators are denoted as $\mathcal{T}_{sa}(\mathcal{H})$ and $\mathcal{B}_{sa}(\mathcal{H})$, respectively, but note that — unless otherwise specified — all of the operators we consider in this work will be self-adjoint, and we will sometimes not remark this explicitly. The sets $\mathcal{T}_+(\mathcal{H})$ and $\mathcal{B}_+(\mathcal{H})$ are the cones of positive self-adjoint operators in the respective space. Finally, $\mathcal{D}(\mathcal{H})$ stands for the set of all density operators, that is, elements of $\mathcal{T}_{sa}(\mathcal{H})$ normalised to have unit trace. For any $|\psi\rangle \in \mathcal{H}$, we sometimes use the shorthand $\psi := |\psi\rangle\langle\psi| \in \mathcal{T}_+(\mathcal{H})$.

Given two Hilbert spaces $\mathcal{H}$, $\mathcal{H}'$, we use $\text{CP}(\mathcal{H} \rightarrow \mathcal{H}')$ to denote the set of all completely positive linear maps from $\mathcal{T}_{sa}(\mathcal{H})$ to $\mathcal{T}_{sa}(\mathcal{H}')$; for simplicity, we will often omit the spaces and simply write $\text{CP}$. The set of quantum channels (completely positive and trace preserving maps) will be denoted by $\text{CPTP}$. All logarithms are to base 2.

2.1. Asymptotic and non-asymptotic resource conversion

The general approach to quantum resource theories begins with the identification of a particular set $F \subseteq \mathcal{D}(\mathcal{H})$ of so-called free states. We will only assume that this is some convex and closed (in the topology induced by the trace norm $\|\cdot\|_1$) subset of density matrices.

The problem that we will focus on in particular concerns the asymptotic conversion of quantum systems. Let us define the rate of transforming a state $\rho \in \mathcal{D}(\mathcal{H}_{in})$ into another state $\sigma \in \mathcal{D}(\mathcal{H}_{out})$, with $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ not necessarily the same, by means of some class of free operations $\mathbb{O}$ as

$$r(\rho \rightarrow \sigma) := \sup \left\{ r \left| \lim_{n \to \infty} \inf_{\Lambda_n \in \mathbb{O}} \| \Lambda_n (\rho^\otimes n) - \sigma^\otimes [r_n] \|_1 = 0 \right. \right\}.$$  \hspace{1cm} (2)

Here, each operation $\Lambda_n$ is a map $\mathcal{T}_{sa}(\mathcal{H}_{in}^\otimes n) \rightarrow \mathcal{T}_{sa}(\mathcal{H}_{out}^\otimes [r_n])$. Throughout the manuscript, we simply use the notation $\mathbb{O}$ without an explicit reference to the underlying spaces. We further assume that each space $\mathcal{T}_{sa}(\mathcal{H}^\otimes n)$ has its own corresponding set of free states $F$. In practical manipulation of quantum resources, one often identifies a reference state $\phi$ — often assumed to be a highly resourceful state, or some state that allows for the efficient use of
a given resource — and treats it as a standard ‘unit’ of the given resource. The most important tasks to study are then resource distillation (asymptotic conversion into \(\phi\)) and resource dilution (asymptotic conversion from \(\phi\)). Having fixed a choice of \(\phi\), we define the \textit{distillable resource} \(R_{d,\mathcal{O}}\) and the \textit{resource cost} \(R_{c,\mathcal{O}}\) as

\[
R_{d,\mathcal{O}}(\rho) := r(\rho \rightarrow_{\Delta} \phi), \quad R_{c,\mathcal{O}}(\rho) := r(\phi \rightarrow_{\Delta} \rho)^{-1}.
\]

To understand such conversion more precisely, we can first consider non-asymptotic transformations, where the number of copies of an input state is taken to be finite and we fix some allowed error in the conversion. We thus define the \textit{one-shot} \(\varepsilon\)-error distillable resource \(R^{(1),\varepsilon}_{d,\mathcal{O}}\) and \textit{one-shot} \(\varepsilon\)-error resource cost \(R^{(1),\varepsilon}_{c,\mathcal{O}}\) as

\[
R^{(1),\varepsilon}_{d,\mathcal{O}}(\rho) := \sup \left\{ m \in \mathbb{N} \mid \inf_{\Lambda \in \mathcal{O}} \frac{1}{2} \| \Lambda(\rho) - \phi^\otimes m \|_1 \leq \varepsilon \right\},
\]

\[
R^{(1),\varepsilon}_{c,\mathcal{O}}(\rho) := \inf \left\{ m \in \mathbb{N} \mid \inf_{\Lambda \in \mathcal{O}} \frac{1}{2} \| \Lambda \left( \phi^\otimes m \right) - \rho \|_1 \leq \varepsilon \right\}.
\]

We then have

\[
R_{d,\mathcal{O}}(\rho) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} R^{(1),\varepsilon}_{d,\mathcal{O}}(\rho^\otimes n),
\]

\[
R_{c,\mathcal{O}}(\rho) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} R^{(1),\varepsilon}_{c,\mathcal{O}}(\rho^\otimes n).
\]

The operation that involves the limit in \(n\) in the above expressions is called a \textit{regularisation}. From the technical standpoint, it constitutes the main hurdle to the calculation of the distillable resource and of the resource cost. Since the disproof of the additivity conjectures [37] by Hastings [36], we know that in general it cannot be omitted.

We will also consider zero-error resource manipulation, where no error whatsoever is allowed in the transformations, even at the one-shot level; specifically,

\[
R^{(1),\text{exact}}_{d,\mathcal{O}}(\rho) := \sup \left\{ m \in \mathbb{N} \mid \inf_{\Lambda \in \mathcal{O}} \| \Lambda(\rho) - \phi^\otimes m \|_1 = 0 \right\},
\]

\[
R^{(1),\text{exact}}_{c,\mathcal{O}}(\rho) := \inf \left\{ m \in \mathbb{N} \mid \inf_{\Lambda \in \mathcal{O}} \| \Lambda \left( \phi^\otimes m \right) - \rho \|_1 = 0 \right\},
\]

and

\[
R_{d,\mathcal{O}}(\rho) = \liminf_{n \to \infty} \frac{1}{n} R^{(1),\text{exact}}_{d,\mathcal{O}}(\rho^\otimes n),
\]

\[
R_{c,\mathcal{O}}(\rho) = \limsup_{n \to \infty} \frac{1}{n} R^{(1),\text{exact}}_{c,\mathcal{O}}(\rho^\otimes n).
\]

\textbf{2.2. Norm-based free operations}

\textit{Resource monotones} (or \textit{resource measures}) are functions \(M_{\mathcal{F}} : \mathcal{D}(\mathcal{H}) \to \mathbb{R}_+ \cup \{+\infty\}\) that satisfy \(M_{\mathcal{F}}(\rho) \geq M_{\mathcal{F}}(\Lambda(\rho))\) for any free operation \(\Lambda \in \mathcal{O}\). This includes monotones based on the relative entropy and other distance measures, or functions such as the base norm induced by the set \(\mathcal{F}\). Any such monotone identifies a class of operations which includes all maps \(\mathcal{O}\), but may be strictly larger: namely, the quantum channels contracting the given measure, that is,

\[
\mathcal{O} \subseteq \left\{ \Lambda : \mathcal{T}_{\text{sa}}(\mathcal{H}_\text{in}) \to \mathcal{T}_{\text{sa}}(\mathcal{H}_\text{out}) \mid \Lambda \in \text{CPTP}, \ M_{\mathcal{F}}(\Lambda(\rho)) \leq M_{\mathcal{F}}(\rho) \ \forall \rho \in \mathcal{D}(\mathcal{H}_\text{in}) \right\}.
\]
We will in particular specialise to the case when \( M_{\varepsilon}(\rho) = \|\rho\|_\mu \) is given by the value of some norm on \( T_{sa}(\mathcal{H}) \) or, more generally, a ‘norm-like’ function — we will only need to assume that \( \|\cdot\|_\mu \) is absolutely homogeneous (\( \|tX\|_\mu = |t|\|X\|_\mu \)) and satisfies the triangle inequality, from which it also follows that \( \|\cdot\|_\mu \) is convex. In order to ensure a proper normalisation of this quantity, we will assume that, for any state \( \rho \), the function satisfies \( \|\rho\|_\mu \geq 1 \). We refer to any such \( \|\cdot\|_\mu \) as a norm for simplicity.

An intuitive and often encountered example of a norm-like resource monotone is the entanglement negativity \([45, 46]\): for any bipartite quantum state, the value of \( \Gamma_1 \) where \( \Gamma \) denotes partial transposition (see Section 6) is a useful gauge of how entangled a given state is. Such norm-based measures can be defined in any resource theory \([47]\), as we will describe shortly, and the restriction to these functions will make it easier to treat also non-positive operators in the same formalism.

Let us follow the line of reasoning based on contractivity further. When acting on a general operator \( X \in T_{sa}(\mathcal{H}) \), any map \( \Lambda \in \text{CPTP} \) acts as a contraction in trace norm: \( \|\Lambda(X)\|_1 \leq \|X\|_1 \).

Relaxing the assumption of complete positivity and assuming only this contractivity property, we end up with a general set of maps defined as

\[
\mathcal{O}_\mu := \left\{ \Lambda : T_{sa}(\mathcal{H}_{\text{in}}) \to T_{sa}(\mathcal{H}_{\text{out}}) \mid \|\Lambda(X)\|_1 \leq \|X\|_1, \|\Lambda(X)\|_\mu \leq \|X\|_\mu \ \forall X \right\} .
\]  

(9)

When \( \|\cdot\|_\mu \) is chosen to be a norm which contracts under all free operations \( \mathcal{O}_{\text{max}} \), we necessarily have that \( \mathcal{O}_\mu \supset \mathcal{O}_{\text{max}} \) as desired. Different choices of the norm \( \|\cdot\|_\mu \) will lead to operations with different properties, with the basic underlying idea being the contractivity of the chosen norm.

For any choice of \( \|\cdot\|_\mu \), we define the function \( \|\cdot\|_\mu^\circ : \mathcal{B}_{sa}(\mathcal{H}) \to \mathbb{R}_+ \cup \{+\infty\} \) as

\[
\|\cdot\|_\mu^\circ = \sup \left\{ \langle \cdot, W \rangle \mid \|W\|_\mu \leq 1 \right\} .
\]  

(10)

This is the norm dual to \( \|\cdot\|_\mu \) or, more generally, the support function of the set \( \left\{ W \mid \|W\|_\mu \leq 1 \right\} \). The crucial property is the Cauchy–Schwarz inequality: \( \langle Q, X \rangle \leq \|Q\|_\mu^\circ \|W\|_\mu \) for any \( Q \in \mathcal{B}_{sa}(\mathcal{H}) \), \( W \in T_{sa}(\mathcal{H}) \), which follows directly from the definition.

3. ONE-SHOT RESOURCE MANIPULATION

We first derive general conditions on the possibility of transforming a given state into, or from, a given reference state in the one-shot setting.

Given a state \( \rho \), we define its \( \varepsilon \)-ball in trace norm as \( B_\varepsilon(\rho) := \{ X \mid \|X\|_1 \leq 1, \|X - \rho\|_1 \leq \varepsilon \} \).

We re-emphasise that the operators \( X \) allowed here are not assumed to be positive, but we always take them to be self-adjoint. Note that the trace distance in quantum information theory is typically defined as \( \frac{1}{2}\|X - \rho\|_1 \); such a definition is motivated by the Helstrom–Holevo theorem \([48, 49]\), which however applies only when both \( X \) and \( \rho \) have unit trace. Since \( X \) is not a state, we employ here the more relevant distance \( \|X - \rho\|_1 \), although we do explicitly account for the factor of \( \frac{1}{2} \) in the definitions of all operational quantities (cf. Section 2) for consistency with other results.
3.1. One-shot dilution

**Theorem 1.** If there exists a map $\Lambda \in \mathcal{O}_\mu$ such that $\|\Lambda(\phi) - \rho\|_1 \leq \varepsilon$ for some pure state $\phi$, then

$$\|\phi\|_\mu \geq \inf_{X \in \mathcal{B}_c(\rho)} \|X\|_\mu. \quad (11)$$

Conversely, if

$$\frac{1}{\|\phi\|_\mu^\sigma} \geq \inf_{X \in \mathcal{B}_c(\rho)} \|X\|_\mu,$$ \quad (12)

then there exists a map $\Lambda \in \mathcal{O}_\mu$ such that $\|\Lambda(\phi) - \rho\|_1 \leq \varepsilon$.

Before proving the Theorem, let us discuss its immediate consequences. The inequality in (11) will be of most interest to us, as it establishes a general bound that applies to any manipulation protocol; this includes, in particular, any subset of the operations $\mathcal{O}_\mu$, and thus also physically relevant classes of quantum channels. Comparing (11) and (12) we see that, in order to give necessary and sufficient conditions for resource dilution under $\mathcal{O}_\mu$, we need to understand precisely the relation between the norm $\|\phi\|_\mu$ and its dual $1/\|\phi\|_\mu^\sigma$. Indeed, we will see that this relation forms the foundation of many of the results of this work, and it will be crucial to understand it precisely.

In particular, if a given pure state satisfies $\|\phi\|_\mu = 1/\|\phi\|_\mu^\sigma$, we obtain a necessary and sufficient condition for approximate one-shot conversion from this state into any other state. If we further assume that

$$\|\phi^{\otimes m}\|_\mu = \frac{1}{\|\phi^{\otimes m}\|_\mu^\sigma} = \left(\frac{1}{\|\phi\|_\mu^\sigma}\right)^m = \|\phi\|_\mu^m, \quad (13)$$

then we can in fact give an exact expression for one-shot $\varepsilon$-error resource cost of any state:

$$R^{(1),\varepsilon}_{c,\mathcal{O}_\mu}(\rho) = \left[ \inf_{X \in \mathcal{B}_c(\rho)} \frac{\log \|X\|_\mu}{\log \|\phi\|_\mu} \right]. \quad (14)$$

Is an assumption such as (13) physically justifiable? Indeed, there are cases of resource theories in which it holds true, notably the theory where we choose $\|X\|_\mu = \|X^T\|_1$ (resource theory of non-positive partial transpose) and $\phi$ is a maximally entangled pure state. Of particular importance is the fact that, for $\varepsilon = 0$, we get

$$R^{(1),\text{exact}}_{c,\mathcal{O}_\mu}(\rho) = \left[ \frac{\log \|\rho\|_\mu}{\log \|\phi\|_\mu} \right], \quad (15)$$

which gives an exact operational meaning to the quantity $\|\rho\|_\mu$ — for non-positive partial transpose, this endows the well-known logarithmic negativity \cite{vidal02,horodecki01} with a new operational meaning as the one-shot zero-error entanglement cost under maps which contract the negativity.
More generally, we might not get a strict equality between the norm $\|\cdot\|_\mu$ and its dual; nevertheless, as long as we can ensure that

$$\|\phi^\otimes m\|_\mu = \Theta\left(\frac{1}{\|\phi^\otimes m\|_\sigma}\right)$$

(as functions of $m$), then this will suffice to allow us to understand the asymptotic properties of resource dilution exactly. Generally speaking, this latter requirement is more realistic and easier to meet in applications — it holds, for instance, in the resource theory of entanglement, where $\|X\|_\mu$ denotes the base norm with respect to separable states. We refer the reader to the forthcoming Section 6 for a detailed discussion on the applications to entanglement theory.

**Proof of Theorem 1.** As $\|\cdot\|_\mu$ is a monotone under $O_\mu$ by definition, for any protocol $\Lambda \in O_\mu$ such that $\Lambda(\phi) = X$ with $X \in B_\varepsilon(\rho)$, we have

$$\|\phi\|_\mu \geq \|X\|_\mu \geq \inf_{X \in B_\varepsilon(\rho)} \|X\|_\mu.$$ (17)

On the other hand, take any operator $X \in B_\varepsilon(\rho)$ and define the map

$$\Lambda(Z) = \langle Z, \phi \rangle X.$$ (18)

For any $Z \in T_{sa}(\mathcal{H})$, by the Cauchy–Schwarz inequality for $\|\cdot\|_\mu$ it holds that

$$\|\Lambda(Z)\|_\mu = |\langle Z, \phi \rangle| \|X\|_\mu \leq \|Z\|_\mu \|\phi^\otimes \|X\|_\mu$$ (19)

and similarly

$$\|\Lambda(Z)\|_1 \leq \|Z\|_1 \|\phi\|_\infty \|X\|_1 \leq \|Z\|_1.$$ (20)

What this means is that, as long $\|X\|_\mu \leq 1/\|\phi\|_\mu^\otimes$, the operation satisfies $\Lambda \in O_\mu$ and performs the desired transformation $\Lambda(\phi) = X$. Optimising over $X \in B_\varepsilon(\rho)$ shows the achievability of the stated bound. □

### 3.2. One-shot distillation

In order to characterise one-shot resource distillation, we will use a quantity based on the hypothesis testing relative entropy $D_H^\varepsilon$. Recall that, for two states $\rho$ and $\sigma$, their hypothesis testing relative entropy is given by [51, 52]

$$D_H^\varepsilon(\rho\|\sigma) = -\log \inf \left\{ \langle Q, \sigma \rangle \mid 0 \leq Q \leq 1, \langle Q, \rho \rangle \geq 1 - \varepsilon \right\}.$$ (21)

We define a generalisation of this quantity, dubbed the **emancipated hypothesis testing relative entropy**, as follows:\footnote{The symbol $\mathcal{B}$ is the letter ‘h’ in the Etruscan alphabet. The Etruscans were an ancient and somewhat mysterious pre-Roman people inhabiting today’s Tuscany — which is named after them. Their language is lost, except for the alphabet and a few words. Emperor Claudius, perhaps one of its last speakers, had written a treatise on Etruscan history, which has unfortunately not survived through the centuries.}

$$D_B^\varepsilon(\rho\|X) := -\log \inf \left\{ \langle Q, X \rangle \mid -1 \leq Q \leq 1, \langle Q, \rho \rangle \geq 1 - \varepsilon \right\},$$ (22)
where we take \( \log(x) = -\infty \) when \( x \leq 0 \) for consistency. Here we do not stipulate any physical meaning of this quantity a priori — it is difficult to interpret it as ‘hypothesis testing’ since we are no longer assuming that \( Q \) forms a part of a valid POVM, but it will nevertheless be crucial in the description of resource distillation. We also define the non-logarithmic variant

\[
3 h\left(\| - \| B_2\right) = \frac{2}{2} h\left(\| - \| B_2\right) - \text{Tr}X,
\]

for convenience.

We note that \( D^c_\mathcal{B} \) can alternatively be expressed as a function of the standard hypothesis testing relative entropy \( D^c_\mathcal{H} \): re-parametrising \( Q := 2Q' - 1 \) with \( Q' \in [0, 1] \) in the definition of \( D^c_\mathcal{H} \) we have that

\[
d^c_\mathcal{H}(\rho\|X)^{-1} = 2d^{c/2}_\mathcal{H}(\rho\|X)^{-1} - \text{Tr} X,
\]

where \( d^c_\mathcal{H}(\rho\|X) := 2D^c_\mathcal{H}(\rho\|X) \).

We now use this quantity to study the conditions for the existence of one-shot distillation protocols.

**Theorem 2.** If there exists a map \( \Lambda \in \mathcal{O}_\mu \) such that \( \|\Lambda(\rho) - \phi\|_1 \leq \varepsilon \) for some pure state \( \phi \), then

\[
\frac{1}{\|\phi\|_\mu} \leq \inf_{\|Z\|_\mu \leq 1} d^c_\mathcal{B}(\rho\|Z).
\]

Conversely, if

\[
\|\phi\|_\mu \leq \inf_{\|Z\|_\mu \leq 1} d^c_\mathcal{B}(\rho\|Z),
\]

then there exists a map \( \Lambda \in \mathcal{O}_\mu \) such that \( \|\Lambda(\rho) - \phi\|_1 \leq \varepsilon \).

As before, let us discuss what the result immediately tells us. First, under the assumption \( \|\phi\|_\mu = 1/\|\phi\|_\mu \), we have a necessary and sufficient condition for approximate one-shot conversion from any state into the pure state \( \phi \). If one additionally assumes that

\[
\|\phi^{\otimes m}\|_\mu = \left(\frac{1}{\|\phi\|_\mu}\right)^m = \|\phi\|_\mu^m,
\]

then Theorem 2 gives an exact expression for one-shot distillable resources:

\[
R^{(1)}_{d,\mathcal{O}}(\rho) = \left[ \frac{\inf_{\|Z\|_\mu \leq 1} D^c_\mathcal{B}(\rho\|Z)}{\log \|\phi\|_\mu} \right].
\]

For the case of \( \varepsilon = 0 \), we obtain in particular that

\[
R^{(1), \text{exact}}_{d,\mathcal{O}}(\rho) = \left[ \frac{\inf_{\|Z\|_\mu \leq 1} D^c_\mathcal{B}(\rho\|Z)}{\log \|\phi\|_\mu} \right].
\]

As mentioned in Section 3.1, the assumption of (27) might not be exactly satisfied in all theories of interest, although it is often asymptotically tight. In the study of partial transposition, where
norm and the weak* topologies, a straightforward consequence of the definitions and of the fact and continuity of the map \( \text{weak* compact} \). Finally, for dimensional spaces, these steps require some elaboration in the infinite-dimensional case. The and in the third line we used Sion’s minimax theorem \([53]\). Although straightforward in finite-

weak* compact and a weak* closed set, the set \( \{ Q \in \mathcal{B} \mid \| Q \|_\infty \leq 1, \langle Q, \rho \rangle \geq 1 - \varepsilon \} \) will form a key technical insight of many of our bounds.

**Lemma 3.** For any choice of \( \| \cdot \|_\mu \), it holds that

\[
\inf_{\| Z \|_{\mu} \leq 1} d_\mathcal{B}^\varepsilon(\rho \| Z) = \max \left\{ \frac{1}{\| Q \|_\mu^\circ} \mid \| Q \|_\infty \leq 1, \langle Q, \rho \rangle \geq 1 - \varepsilon \right\}.
\]

For \( \varepsilon = 0 \), this can also be expressed as

\[
\inf_{\| Z \|_{\mu} \leq 1} d_\mathcal{B}^0(\rho \| Z) = \max \left\{ \langle W, \rho \rangle \mid \| W \|_\mu^\circ \leq 1, \| W \|_\infty \leq \langle W, \rho \rangle \right\}.
\]

**Proof.** Follows since

\[
\inf_{\| Z \|_{\mu} \leq 1} d_\mathcal{B}^\varepsilon(\rho \| Z) = \inf_{\| Z \|_{\mu} \leq 1} \left( \min \left\{ \langle Q, Z \rangle \mid \| Q \|_\infty \leq 1, \langle Q, \rho \rangle \geq 1 - \varepsilon \right\} \right)^{-1}
\]

\[
= \left( \sup_{\| Z \|_{\mu} \leq 1} \min \left\{ \langle Q, Z \rangle \mid \| Q \|_\infty \leq 1, \langle Q, \rho \rangle \geq 1 - \varepsilon \right\} \right)^{-1}
\]

\[
= \left( \min \left\{ \| Q \|_\mu^\circ \mid \| Q \|_\infty \leq 1, \langle Q, \rho \rangle \geq 1 - \varepsilon \right\} \right)^{-1}
\]

\[
= \max \left\{ \frac{1}{\| Q \|_\mu^\circ} \mid \| Q \|_\infty \leq 1, \langle Q, \rho \rangle \geq 1 - \varepsilon \right\}.
\]

Here, in the first line we used compactness of the set \( \{ Q \in \mathcal{B}_\text{sa}(\mathcal{H}) \mid \| Q \|_\infty \leq 1, \langle Q, \rho \rangle \geq 1 - \varepsilon \} \), and in the third line we used Sion’s minimax theorem \([53]\). Although straightforward in finite-dimensional spaces, these steps require some elaboration in the infinite-dimensional case. The set \( B_\infty := \{ Q \in \mathcal{B}_\text{sa}(\mathcal{H}) \mid \mathcal{T}_\text{sa}(\mathcal{H}) \mid \| Q \|_\infty \leq 1 \} \) is compact in the weak* topology induced on \( \mathcal{B}_\text{sa}(\mathcal{H}) \) by its pre-dual \( \mathcal{T}_\text{sa}(\mathcal{H}) \) thanks to the Banach–Alaoglu theorem \([54, \text{Theorem 2.6.18}]\); since for \( \rho \in \mathcal{T}_\text{sa}(\mathcal{H}) \) the function \( \mathcal{B}_\text{sa}(\mathcal{H}) \ni Q \mapsto \langle Q, \rho \rangle \) is weak* continuous by definition, the set \( \{ Q \in \mathcal{B}_\text{sa}(\mathcal{H}) \mid \langle Q, \rho \rangle \geq 1 - \varepsilon \} \) is immediately seen to be weak* closed; being the intersection of a weak* compact and a weak* closed set, the set \( \{ Q \in \mathcal{B}_\text{sa}(\mathcal{H}) \mid \| Q \|_\infty \leq 1, \langle Q, \rho \rangle \geq 1 - \varepsilon \} \) is also weak* compact. Finally, \( B_\mu := \{ Z \in \mathcal{T}_\text{sa}(\mathcal{H}) \mid \| Z \|_{\mu} \leq 1 \} \) is convex. Together with the bilinearity and continuity of the map \( (B_\mu, B_\infty) \ni (Z, Q) \mapsto \langle Q, Z \rangle \) with respect to the product of the trace norm and the weak* topologies, a straightforward consequence of the definitions and of the fact
that $B_\infty$ is bounded in operator norm, this ensures that the conditions of the minimax theorem are satisfied.

In the case $\epsilon = 0$, we have that

$$\inf_{\|Z\|_\mu \leq 1} d_\Theta^\epsilon (\rho \|Z\|) = \max \left\{ \frac{1}{\|Q\|_\mu^\circ} \left| \|Q\|_\infty = 1, \langle Q, \rho \rangle = 1 \right\} \right.$$ 

$$= \max \left\{ \langle W, \rho \rangle \mid \|W\|_\mu^\circ = 1, \|W\|_\infty = \langle W, \rho \rangle \right\} \right.$$ 

$$= \max \left\{ \langle W, \rho \rangle \mid \|W\|_\mu^\circ \leq 1, \|W\|_\infty \leq \langle W, \rho \rangle \right\} \right.$$ 

through a change of variables $Q = \|Q\|_\mu^\circ W$. □

The statement of the Theorem can then be shown as follows.

**Proof of Theorem 2.** On the one hand, the fact that any $\Lambda \in O_\mu$ is a contraction with respect to the norm $\|\cdot\|_\mu$ means that its adjoint map, $\Lambda^\dagger$, will necessarily contract the dual norm $\|\cdot\|_\mu^\circ$. Specifically,

$$\|\Lambda^\dagger (\phi)\|_\mu^\circ = \sup_{\|Z\|_\mu \leq 1} \langle \Lambda(Z), \phi \rangle$$ 

$$\leq \sup_{\|Z\|_\mu \leq 1} \langle Z', \phi \rangle$$ 

$$= \|\phi\|_\mu^\circ. \quad (34)$$

In an analogous way, we see that $\Lambda^\dagger$ is a contraction with respect to the operator norm $\|\cdot\|_\infty$. Additionally, we have that

$$\|\phi - \Lambda (\rho)\|_1 = \sup \{ \langle W, \phi - \Lambda (\rho) \rangle \mid \|W\|_\infty \leq 1 \}$$ 

$$\geq \langle \phi, \phi - \Lambda (\rho) \rangle$$ 

$$= 1 - \langle \Lambda^\dagger (\phi), \rho \rangle. \quad (35)$$

This altogether means that any $\Lambda \in O_\mu$ such that $\|\Lambda (\rho) - \phi\|_1 \leq \epsilon$ gives a feasible solution to Eq. (30) as $Q = \Lambda^\dagger (\phi)$, yielding

$$\inf_{\|Z\|_\mu \leq 1} d_\Theta^\epsilon (\rho \|Z\|) \geq \frac{1}{\|\phi\|_\mu^\circ}. \quad (36)$$

On the other hand, taking an optimal $Q$ in (30) with $1/\|Q\|_\mu^\circ = \inf_{\|Z\|_\mu \leq 1} d_\Theta^\epsilon (\rho \|Z\|)$, the map $\Lambda$ defined as

$$\Lambda (Z) = \langle Q, Z \rangle \phi \quad (37)$$

can be seen to satisfy

$$\|\Lambda (Z)\|_1 \leq \|Z\|_1 \|Q\|_\infty \|\phi\|_1 \leq \|Z\|_1, \quad (38)$$

$$\|\Lambda (Z)\|_\mu \leq \|Z\|_\mu \|Q\|_\mu^\circ \|\phi\|_\mu \leq \|Z\|_\mu \quad (39)$$

using that $1/\|Q\|_\mu^\circ \geq \|\phi\|_\mu$ by assumption. This implies that $\Lambda \in O_\mu$, and verifying that any map $\Lambda$ in (37) satisfies

$$\|\Lambda (\rho) - \phi\|_1 = |\langle Q, \rho \rangle - 1| \leq \epsilon \quad (40)$$

because $\langle Q, \rho \rangle \in [1 - \epsilon, 1]$ concludes the proof. □
Remark. The fact that
\[ \inf_{\|Z\|_\rho \leq 1} d_B^0(\rho\|Z) = \max \left\{ \left\langle W, \rho \right\rangle \mid \|W\|_\mu = 1, \|W\|_\infty = \left\langle W, \rho \right\rangle \right\}, \]  
(41)
as shown in Lemma 3, makes explicit the equality between this problem and the class of tempered resource monotones introduced in [28], which are defined exactly by the right-hand side of (41).

We can also note that, due to the self-adjointness of $Q$, the constraints $\|Q\|_\infty \leq 1$ and $\left\langle Q, \rho \right\rangle = 1$ are equivalent to $1 \geq Q \geq 2\Pi - 1$, where $\Pi_\rho$ is the projection onto the support of $\rho$. This means that we can equivalently write
\[ \inf_{\|Z\|_\rho \leq 1} d_B^0(\rho\|Z) = \max \left\{ \frac{1}{\|Q\|_\mu} \mid 2\Pi_\rho - 1 \leq Q \leq 1 \right\}, \]  
(42)
which explicitly shows that computing $d_B^0$ depends only on $\text{supp} \rho$.

3.3. Relation between distillation and dilution: the $\varepsilon$-$\delta$ lemma

Our approach to describing asymptotic resource manipulation will rely on a precise understanding of the relations and trade-offs between the achievable performances of one-shot distillation and dilution.

Previously, the question of one-shot manipulation received significant attention particularly in the resource theory of thermodynamics [55, 56], where it was shown that the task of one-shot work extraction (distillation) is governed by the hypothesis testing relative entropy
\[ \max_{\|\tilde{\rho}\|_\mu \leq 1} \log \inf \left\{ \lambda \mid \tau_{\tilde{\rho} - 1} \leq 1 \right\} \]  
(43)
The relations between these two fundamental quantities have been studied in great detail in a number of works [58–62] (owing also to the applications of the two entropies in information theory and quantum cryptography [63]), which allowed for an exact understanding of the operational aspects of thermodynamics in both the one-shot and asymptotic setting.

Although the beginnings of the applications of $D^c_H$ and $D^c_{\max}$ to more general quantum resource theories can be traced back to the seminal works by Brandão and Plenio [8, 26], the precise quantitative study of the non-asymptotic interrelations in the manipulation of quantum resources has only been undertaken very recently [34, 64]. In these works, conceptually similar ideas based on trade-offs between $D^c_{\max}$ and $D^c_H$ were employed. However, it is already known that $D^c_{\max}$ is not the right quantity to consider in the study of resources such as quantum entanglement [15, 65], where resource dilution is not governed by $D^c_{\max}$, but rather by a different quantity based on the standard robustness $R^c$ — or, equivalently, the base norm. The results of previous works thus failed to establish the tightest possible bounds, in particular in the asymptotic setting, because they relaxed the problem to the study of $D^c_{\max}$, which can only provide a looser bound.

Here, we establish a bound which constrains the trade-off between smoothed norms $\|\cdot\|_\mu$ and the function $D^\delta_B$. We can regard the following result in at least two possible ways. On the one hand, it as a natural generalisation of the so-called `$\varepsilon$-lemma' that played a key role in our proof of the fundamental irreversibility of entanglement theory [28, Lemma S6] (see also Section 6.2 below). On the other, since we have seen that these two quantities govern one-shot resource dilution and distillation, respectively, we can consider them as more appropriate objects of study.
than the max-relative entropy and the hypothesis testing relative entropy in our framework. The result below can then be understood as an extension of the one shot yield–cost trade-off relations of Refs. [34, 64] to our setting, where the free operations are contractions with respect to certain norms.

**Lemma 4** ($\varepsilon$-$\delta$-lemma). Choose any $\varepsilon, \delta \in [0, 1)$ such that $\delta + \varepsilon < 1$. Then

\[
\inf_{X \in B_\varepsilon(\rho)} \log \|X\|_\mu \geq \inf_{\|Z\|_\mu \leq 1} D_\delta^\varepsilon(\rho; Z) + \log (1 - \delta - \varepsilon). \tag{44}
\]

**Proof.** Recall from Lemma 3 that

\[
\inf_{\|Z\|_\mu \leq 1} D_\delta^\varepsilon(\rho; Z) = \log \max \left\{ \frac{1}{\|Q\|_\mu^\varepsilon} \left| \langle Q, \rho \rangle \right| \geq 1 - \delta, \|Q\|_\infty \leq 1 \right\}. \tag{45}
\]

Take $Q$ as an operator optimal for the above, and take $X$ as any operator such that $\|X - \rho\|_1 \leq \varepsilon$. Then

\[
\|X\|_\mu \geq \sup \left\{ \langle X, W \rangle \mid \|W\|_\mu^\varepsilon \leq 1 \right\}
\geq \left\langle X, \frac{Q}{\|Q\|_\mu^\varepsilon} \right\rangle
\geq \left\langle \rho - X, \frac{Q}{\|Q\|_\mu^\varepsilon} \right\rangle
\geq (1 - \delta) \frac{1}{\|Q\|_\mu^\varepsilon} - \|\rho - X\|_1 \|Q\|_\mu^\varepsilon
\geq (1 - \delta - \varepsilon) \frac{1}{\|Q\|_\mu^\varepsilon}, \tag{46}
\]

where the first line is by weak Lagrange duality, and in the fourth line we used the Cauchy–Schwarz inequality (or, specifically, the Hölder inequality for $\|\cdot\|_1$). Optimising over all feasible $X$, we get the desired inequality.

\[\square\]

### 4. ASYMPTOTIC RESOURCE MANIPULATION

The purpose of this section is to employ the technical results of the above section to derive explicit bounds on the asymptotic rates of state transformations, focusing in particular on the distillation and dilution of a given resource. Our strategy follows the conceptual scheme summarised in Figure 1: to obtain lower bounds on the efficiency of resource dilution, i.e. on the resource cost, we will connect it with the (smoothed) norm $\|\cdot\|_\mu^\varepsilon$ (Theorem 1), then use the $\varepsilon$-$\delta$-lemma (Lemma 4) to relate this with the emancipated hypothesis testing relative entropy $D_\delta^\varepsilon$ (22), and finally use this latter quantity to lower bound the cost. This is precisely the technique underlying the proof of Theorem 5 below. An analogous strategy is used to prove Theorem 6. The purpose of doing so, naturally, is that the smoothed emancipated hypothesis testing relative entropy, initially defined as an infimum, can alternatively be expressed as a supremum (see Lemma 3). By means of Theorem 5, therefore, we can generate and explicitly compute lower bounds to the resource cost in a systematic fashion.
4.1. Bounds on distillable resource and resource cost

Let us begin with an explicit investigation of how the one-shot results obtained in Section 3 can help us shed light on the asymptotic properties of resource manipulation. Although the bounds in this section are defined through asymptotic regularisations which do not necessarily have known computable forms, they provide important conceptual generalisations of other bounds that have appeared previously. We will shortly see how to use the results presented here to obtain also computable, single-letter bounds, as well as how the regularised quantities themselves can be bounded in some cases.

We start with a general estimate of the efficiency of resource dilution.

**Theorem 5.** For any state $\rho$, any pure state $\phi$, and any $\delta \in [0, 1)$, the rate of transformation from $\phi$ to $\rho$ under $O_\mu$ satisfies

$$r(\phi \xrightarrow{O_\mu} \rho)^{-1} \geq \limsup_{n \to \infty} \frac{1}{n} \frac{\inf_{\|Z\|_\mu \leq 1} D_\delta^\phi (\rho^\otimes n \| Z)}{L_{\mu, \infty}(\phi)}$$

$$= \limsup_{n \to \infty} \frac{1}{n} \log \sup \left\{ \frac{1}{\|Q\|_\mu} : \|Q\|_{\infty} \leq 1, \langle Q, \rho^\otimes n \rangle \geq 1 - \delta \right\}^{\geq 1},$$

(47)

where

$$L_{\mu, \infty}(\phi) := \limsup_{n \to \infty} \frac{1}{n} \log \|\phi^\otimes n\|_\mu.$$  

(48)

This bound has the potential to improve on known lower bounds on resource cost $R_{c,O_\mu}(\rho) = r_{O_\mu}(\phi \to \rho)^{-1}$, and in particular it majorises the class of bounds which was used recently in [28] to show the irreversibility of entanglement theory. The quantity $L_{\mu, \infty}(\phi)$ represents here a parameter that characterises the resource properties of the given target state $\phi$, and we will see that, in relevant cases, it can be straightforwardly computed or bounded.

However, a major question is: can the regularisation of the emancipated hypothesis testing relative entropy $D_\delta^\phi$ be evaluated in practice? Although we are unable to provide an answer to this problem at this stage, we do remark the conceptual similarity of this bound to the asymptotic
bound of Brandão and Plenio [26]; indeed, the quantity encountered in their work,

$$\limsup_{n \to \infty} \frac{1}{n} \inf_{\sigma \in \mathcal{F}} D_{\mathcal{H}}^\delta(\rho^{\otimes n} \| \sigma),$$

(49)
can also be used as a lower bound for resource cost, but it can never be better than our result in Theorem 5, since $D_{\mathcal{H}}^\delta(\rho^{\otimes n} \| \sigma) \geq D_{\mathcal{H}}^{\overline{\delta}}(\rho^{\otimes n} \| \sigma)$ by definition. Interestingly, the quantity in (49) was conjectured to reduce to a well-known resource monotone: the regularised relative entropy of the given resource [8, 26]. Although some issues have recently emerged [66] with the claimed proof of this result in Refs. [8, 26], assuming that the main claims of [8, 26] are in fact correct, our Theorem 5 then has the potential to outperform relative-entropy–based bounds in general. As we will shortly see, there are indeed examples in the resource theories of entanglement and magic where the emancipated hypothesis testing relative entropy yields a strict improvement over all such previously known bounds.

It is therefore a very interesting open problem to determine whether an exact form of the asymptotic regularisation for the quantity $h$ in (47) can be established, and what the equivalent of relative entropy would be in this setting. This would lead to a new entropic-like lower bound on resource dilution cost that can be strictly better than the regularised relative entropy.

**Proof of Theorem 5.** Assume that $C$ is any achievable rate of dilution, and consider a sequence of operations $(\Lambda_n)_n \in \mathcal{O}_\mu$ such that $\frac{1}{n} \| \Lambda_n(\sigma^{[Cn]}) - \rho^{\otimes n} \|_1 \leq \epsilon_n$, with the error $\epsilon_n$ asymptotically vanishing. From Theorem 1 we have that

$$\| \sigma^{[Cn]} \|_\mu \geq \inf_{X \in \mathcal{B}_n(\rho^{\otimes n})} \| X \|_\mu,$$

(50)

Using the $\epsilon$-$\delta$-lemma (Lemma 4), we get that

$$\| \sigma^{[Cn]} \|_\mu \geq \inf_{\| Z \|_\mu \leq 1} D_{\mathcal{H}}^\delta(\rho^{\otimes n} \| Z) (1 - \delta - 2\epsilon_n)$$

(51)

for $n$ sufficiently large and $\delta$ sufficiently small so that $\delta + 2\epsilon_n < 1$. Thus

$$\frac{1}{n} \log \| \sigma^{[Cn]} \|_\mu \geq \frac{1}{n} \inf_{\| Z \|_\mu \leq 1} D_{\mathcal{H}}^\delta(\rho^{\otimes n} \| Z) + \frac{\log(1 - \delta - 2\epsilon_n)}{n}.$$

(52)

We can rewrite this as

$$\frac{\log \| \sigma^{[Cn]} \|_\mu}{n} \geq \frac{1}{n} \inf_{\| Z \|_\mu \leq 1} D_{\mathcal{H}}^\delta(\rho^{\otimes n} \| Z) + \frac{\log(1 - \delta - 2\epsilon_n)}{n}.$$

(53)

Taking $\lim_{\delta \to 0} \limsup_{n \to \infty}$ of both sides yields

$$C \limsup_{n \to \infty} \frac{1}{n} \log \| \sigma^{[Cn]} \|_\mu \geq \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \inf_{\| Z \|_\mu \leq 1} D_{\mathcal{H}}^\delta(\rho^{\otimes n} \| Z).$$

(54)

Upper bounding $\limsup_{n \to \infty} \frac{1}{n} \log \| \sigma^{[Cn]} \|_\mu \leq L_{\mu, \infty}(\sigma)$ concludes the proof of the first line of (47). The equality on the second line follows from the dual expression derived in Lemma 3. □

**Remark.** We note that our definition of the rate $r(\sigma \xrightarrow{\mu} \rho)$ assumes that the transformation error $\epsilon_n$ vanishes asymptotically. In the proof of the Theorem, this is not strictly required: all we need here is for the error to eventually become sufficiently small so that $\delta + 2\epsilon_n < 1$, allowing for the
crucial $\varepsilon$-$\delta$-lemma (Lemma 4) to be applied. If we instead required $\delta$ to go to 0, this would then give us the bound

$$r(\phi \xrightarrow{O_n} \rho)^{-1} \geq \tilde{r}(\phi \xrightarrow{O_n} \rho)^{-1} \geq \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{\frac{1}{n} \inf_{\|Z\|_\mu \leq 1} D_{\|\cdot\|_{\mu}}^\delta (\rho^{\otimes n} | Z)}{L_{\mu,\infty}(\phi)},$$

(55)

which applies even to a modified notion of rate $\tilde{r}(\phi \xrightarrow{O_n} \rho)$, defined analogously to (2) but where the trace norm error $\varepsilon_n = \frac{1}{2} \|\Lambda(\rho^{\otimes n}) - \rho^{\otimes (r_n)}\|_1$ satisfies only $\limsup_{n \to \infty} \varepsilon_n < 1/2$ instead of $\lim_{n \to \infty} \varepsilon_n = 0$ as in (2). Such a bound is known as a ‘pretty strong converse’ bound [67] (cf. [28] in the context of distillation and dilution of quantum resources). For simplicity, here we only explicitly consider rates with vanishing error.

We also note that taking the limit $\delta \to 0$ could be expected to make the regularisation in Eq. (55) easier to compute than the $\delta$-dependent form.

A similar type of bound can be established for distillable resource yield.

**Theorem 6.** For any state $\rho$, any pure state $\phi$, and any $\varepsilon \in [0, 1)$, the rate of transformation from $\rho$ to $\phi$ under $O_\mu$ satisfies

$$r(\rho \xrightarrow{O_\mu} \phi) \leq \frac{\liminf_{n \to \infty} \frac{1}{n} \inf_{X \in B_n(\rho^{\otimes n})} \log \|X\|_{\mu}}{-L_{\mu,\infty}(\phi)},$$

(56)

where

$$L_{\mu,\infty}(\phi) := \limsup_{n \to \infty} \frac{1}{n} \log \|\phi^{\otimes n}\|_{\mu}^\circ.$$

(57)

For consistency, if $\|\phi^{\otimes n}\|_{\mu}^\circ = 1$, we understand the bound as the trivial restriction $r(\rho \xrightarrow{O_\mu} \phi) \leq \infty$.

**Proof.** Let $D$ be an achievable rate for distillation, and consider a sequence of protocols $(\Gamma_n)_{n \in \mathbb{N}} \in O_\mu$ such that $\frac{1}{2} \|\Gamma_n(\rho^{\otimes n}) - \phi^{\otimes n}(D_n)\|_1 := \delta_n$, with $\delta_n$ denoting the asymptotically vanishing distillation error. Theorem 2 then gives

$$\frac{1}{\|\phi^{\otimes[D_n]}\|_{\mu}^\circ} \leq \inf_{\|Z\|_{\mu} \leq 1} d_{\|\cdot\|_{\mu}}^{2\delta_n} (\rho^{\otimes n} | Z).$$

(58)

For $n$ sufficiently large and $\varepsilon$ sufficiently small so that $2\delta_n + \varepsilon < 1$, the $\varepsilon$-$\delta$-lemma (Lemma 4) allows us to estimate

$$\frac{|D_n|}{|D_n|} \frac{1}{|D_n|} \left( -\log \|\phi^{\otimes[D_n]}\|_{\mu}^\circ \right) \leq \frac{1}{n} \inf_{X \in B_n(\rho^{\otimes n})} \log \|X\|_{\mu} \leq \frac{\log(1 - 2\delta_n - \varepsilon)}{n}. $$

(59)

Taking $\lim_{\varepsilon \to 0} \liminf_{n \to \infty}$ of both sides and bounding

$$\liminf_{n \to \infty} \frac{1}{|D_n|} \left( -\log \|\phi^{\otimes[D_n]}\|_{\mu}^\circ \right) \geq \liminf_{n \to \infty} \frac{1}{n} \left( -\log \|\phi^{\otimes n}\|_{\mu}^\circ \right) = -L_{\mu,\infty}(\phi)$$

(60)

gives the stated result.
Remark. Again, here we do not actually need the transformation error \( \delta_n \) or \( \varepsilon_n \) to go to 0 asymptotically; if we required that \( \varepsilon \to 0 \) instead, this would yield the pretty strong converse bound
\[
\liminf_{n \to \infty} \inf_{X \in \mathcal{B}_n} \frac{1}{n} \log \|X\|_\mu \geq -L_{\mu,\infty}(\phi),
\]
which would apply for a modified definition of rate \( \tilde{r}(\rho \to \phi) \) where we only require the transformation error to satisfy \( \limsup_{n \to \infty} \delta_n < 1/2 \) instead of \( \lim_{n \to \infty} \delta_n = 0 \). As before, this could potentially constitute a more easily computable formula.

Remark. One may wonder whether our bounds in Theorems 5 and 6 are simply a statement of the ‘no free lunch’ relation \( R_{\mathcal{C},\mathcal{O}_\mu}(\rho) \geq R_{\mathcal{D},\mathcal{O}_\mu}(\rho) \), that is, the fact that one cannot extract more resources from a given state than it costs to produce that same state. As expounded in Sec. 1.2, this is certainly the conceptual intuition behind our approach: we use the quantity \( D_{\mathcal{B}_n}'(\cdot) \) to upper bound distillable resources. However, our method does not directly go through \( R_{\mathcal{D},\mathcal{O}_\mu} \) to lower bound distillable resources.

In fact, it is a noteworthy observation that the inequality \( R_{\mathcal{C},\mathcal{O}_\mu}(\rho) \geq R_{\mathcal{D},\mathcal{O}_\mu}(\rho) \) cannot be proved directly with the same methods. As a matter of fact, we do not yet know whether such inequality holds in full generality. Let us examine more closely the reason why this is so. Given a state \( \rho \), consider a pair of achievable rates \( C, D \) for dilution and distillation, respectively. According to (2), this means that there exist operations \( \Lambda_n, \Lambda'_n \in \mathcal{O}_\mu \) with the property that
\[
\|\Lambda_n(\rho^{\otimes n}) - \rho^{\otimes [Cn]}\|_1 = \varepsilon_n \text{ and } \|\Lambda'_n(\rho^{\otimes n}) - \rho^{\otimes [Dn]}\|_1 = \delta_n
\]
satisfy \( \lim_{n \to \infty} \varepsilon_n = 0 = \lim_{n \to \infty} \delta_n \). A standard way of proceeding would be to concatenate these operations and conclude that if \( CD > 1 \) then we would be able to distil more copies of \( \phi \) than we started with, something that should be impossible with free operations. We would therefore conclude that \( CD \leq 1 \), and upon taking the supremum over achievable rate pairs this would yield that \( R_{\mathcal{D},\mathcal{O}_\mu}(\rho) \leq R_{\mathcal{C},\mathcal{O}_\mu}(\rho) \), as desired.

Let us try to make this reasoning rigorous. Defining \( \Gamma_n := \Lambda'_n^{\otimes [Cn]} \Lambda_n \in \mathcal{O}_{\mu} \), we have immediately that
\[
\|\Gamma_n(\phi^{\otimes n}) - \phi^{\otimes [Dn]}\|_1 \leq \|\Gamma_n(\phi^{\otimes n}) - \Lambda'_n^{\otimes [Cn]}(\rho^{\otimes [Cn]})\|_1 + \|\Lambda'_n^{\otimes [Cn]}(\rho^{\otimes [Cn]}) - \phi^{\otimes [Dn]}\|_1
\]
\[
= \|\Lambda'_n^{\otimes [Cn]}(\Lambda_n(\phi^{\otimes n}) - \rho^{\otimes [Cn]})\|_1 + \|\Lambda'_n^{\otimes [Cn]}(\rho^{\otimes [Cn]}) - \phi^{\otimes [Dn]}\|_1 \geq \varepsilon_n + \delta_n
\]
where in the last line we used contractivity of \( \|\cdot\|_1 \) under operations in \( \mathcal{O}_{\mu} \). Leveraging also contractivity of \( \|\cdot\|_\mu \), we obtain that \( \|\Gamma_n(\phi^{\otimes n})\|_\mu \leq \|\phi^{\otimes n}\|_\mu \). Therefore, \( \phi^{\otimes [Dn]} \) lies to within a vanishingly small trace distance from a state whose \( \mu \)-norm is upper bounded by \( \|\phi^{\otimes n}\|_\mu \).

Importantly, however, this does not allow us to draw any conclusions concerning the \( \mu \)-norm of \( \phi^{\otimes [Dn]} \) itself. Indeed, to do so one could be tempted to apply the triangle inequality
\[
\|\phi^{\otimes [Dn]}\|_\mu \leq \|\phi^{\otimes n}\|_\mu + \|\Gamma_n(\phi^{\otimes n}) - \phi^{\otimes [Dn]}\|_\mu\]
\]
however, in order to further upper bound \( \|\Gamma_n(\phi^{\otimes n}) - \phi^{\otimes [Dn]}\|_\mu \), we would need to somehow turn the \( \mu \)-norm into the trace norm, as we know the latter to be vanishingly small. This is in
principle certainly possible, because we are working in finite-dimensional spaces and hence all norms are equivalent. Doing so, one obtains a relation of the form

\[
\|\Gamma_n(\phi^\otimes n) - \phi^\otimes D[C_n]\|_\mu \leq N_n \|\Gamma_n(\phi^\otimes n) - \phi^\otimes D[C_n]\|_1 \leq N_n (\epsilon_n + \delta[C_n])
\]

(64)

for some constant \(N_n > 0\). We would be able to complete the proof if we could show that the right-hand side of the above inequality converges to zero. The key issue that prevents us from doing so is that the constant \(N_n\) can diverge as \(n \to \infty\) — and, typically, it will do so exponentially fast.

The above reasoning should convince the reader that the naïve way of proving the no free lunch relation fails in this case. Although we do not expect this inequality to hold exactly in all resource theories, it turns out that we can recover a variant of such a relation by employing our key technical tool, the \(\epsilon\)-\(\delta\) lemma (Lemma 4). Indeed, combining Theorems 5 and 6 (which ultimately rely on that lemma) gives us

\[
\frac{R_{C,\mathcal{O}_\mu}(\rho)}{-L_{\mu,\infty}(\phi)} \geq \frac{R_{d,\mathcal{O}_\mu}(\rho)}{L_{\mu,\infty}(\phi)}.
\]

(65)

In this context, the values of \(L_{\mu,\infty}(\phi)\) and \(-L_{\mu,\infty}^o(\phi)\) can be thought of as ‘renormalisation factors’ that account for the specific properties of the norm \(\|\cdot\|_\mu\) and the choice of the target state \(\phi\). Since by the Cauchy–Schwartz inequality \(\|\phi^\otimes n\|_\mu \|\phi^\otimes n\|^o_\mu \geq \langle \phi^\otimes n, \phi^\otimes n \rangle = 1\), we see immediately that \(-L_{\mu,\infty}^o(\phi) / L_{\mu,\infty}(\phi) \leq 1\); thus, the above relation (65) would be implied by the no free lunch inequality \(R_{C,\mathcal{O}_\mu}(\rho) \geq R_{d,\mathcal{O}_\mu}(\rho)\). This resembles analogous results in the description of quantum resource transformations under completely positive and trace-preserving maps, where similar renormalisation coefficients are needed [10, 34].

### 4.2. Single-letter bounds

As a first step towards simplifying the bounds obtain in the previous section, let us notice that, instead of using the parameters \(\delta\) (in Theorem 5) or \(\epsilon\) (in Theorem 6), we could have simply fixed them as zero to begin with. This has the intuitive interpretation of using the zero-error distillation rate to lower bound the resource cost, or vice versa. We formalise this as follows.

**Corollary 7.** For any state \(\rho\) and any pure state \(\phi\), the rate of transformation from \(\phi\) to \(\rho\) satisfies

\[
r(\phi \rightarrow \rho) \geq \frac{\limsup_{n \to \infty} \frac{1}{n} \log \sup \left\{ \langle W, \rho \otimes Z \rangle \mid \|W\|_\mu \leq 1, \|W\|_\infty = \langle W, \rho \otimes Z \rangle \right\}}{L_{\mu,\infty}(\phi)}.
\]

(66)

and analogously the reverse transformation satisfies

\[
r(\rho \rightarrow \phi) \leq \frac{\liminf_{n \to \infty} \frac{1}{n} \log \|\rho \otimes n\|_\mu}{-L_{\mu,\infty}^o(\phi)}.
\]

(67)

The evaluation of such bounds is generally non-trivial. It certainly can, however, simplify under additional assumptions. Assume, for instance, that \(\|X^\otimes n\|_\mu \leq \|X\|_{C_n}^o\) which in practice will
be satisfied in virtually all cases of interest; we then immediately get
\[ R_{d,\mathcal{O}_\mu}(\rho) = r(\rho \xrightarrow{\mathcal{O}_\mu} \phi) \leq \frac{\log \|\rho\|_\mu}{-L^0_{\mu,\infty}(\phi)}. \] (68)

In entanglement theory, choosing \( \|X\|_\mu = \|X^\top\|_1 \) gives \( L^0_{\mu,\infty}(\phi) = 1 \) and the above recovers one of the oldest known computable upper bounds on distillable entanglement: the logarithmic negativity [50].

We have thus obtained what could be considered an unsurprising bound: the rate of distillation is upper bounded by the quantity \( \log \|\rho\|_\mu \), which characterises one-shot, zero-error dilution, and thus clearly should upper bound the asymptotic rate of dilution (as the cost can only increase in the one-shot setting). This natural correspondence does not immediately extend to the bound in the other direction: it is not clear whether one-shot, zero-error distillation gives a better bound than asymptotic zero-error distillation. This can be considered as a consequence of the fact that at no point do we assume that the set of operations \( \mathcal{O}_\mu \) is closed under tensor product — it might be the case that \( \Lambda \in \mathcal{O}_\mu \), but the \( \Lambda \otimes - \) copy protocol \( \Lambda \otimes \rho \) is no longer in \( \mathcal{O}_\mu \).

We will thus need more assumptions in order to bound \( 1 - h^0_\mu \). To see this, recall that
\[ \inf_{\|Z\|_\mu \leq 1} D^0_{\mathcal{B}}(\rho^\otimes n\|Z) = \log \sup \left\{ \langle W, \rho^\otimes n \rangle \mid \|W\|_\mu \leq 1, \|W\|_\infty = \langle W, \rho^\otimes n \rangle \right\}. \] (69)

We would now like to lower bound \( \inf_{\|Z\|_\mu \leq 1} D^0_{\mathcal{B}}(\rho^\otimes n\|Z) \) with \( n \inf_{\|Z\|_\mu \leq 1} D^0_{\mathcal{B}}(\rho\|Z) \) so that the problem reduces to a single-letter quantity. As the operator norm \( \|\cdot\|_\infty \) is multiplicative, such a bound would indeed be possible provided that
\[ \|W^\otimes n\|_\mu^0 \leq \left( \|W\|_\mu^0 \right)^n, \] (70)

so that we can take \( W^\otimes n \) as a feasible solution. However, for many resource theories of interest the opposite inequality is satisfied here: since \( \|\cdot\|_\mu \) is typically a submultiplicative norm, this implies the supermultiplicativity of \( \|\cdot\|_\mu^0 \) under tensor product. The only way to satisfy (70) in such cases is for equality to hold there; luckily, such multiplicative norms are available in many theories of interest.

We formalise all of the above as follows.

**Corollary 8.** If the given norm satisfies \( \|X^\otimes n\|_\mu \leq \|X\|_\mu^n \) for all \( X \), then
\[ r(\rho \xrightarrow{\mathcal{O}_\mu} \phi) \leq \frac{\log \|\rho\|_\mu}{-L^0_{\mu,\infty}(\phi)}. \] (71)

If the given norm satisfies \( \|Y^\otimes n\|_\mu^0 \leq \left( \|Y\|_\mu^0 \right)^n \) for all \( Y \), then
\[ r(\phi \xrightarrow{\mathcal{O}_\mu} \rho)^{-1} \geq \frac{\inf_{\|Z\|_\mu \leq 1} D^0_{\mathcal{B}}(\rho\|Z)}{L^0_{\mu,\infty}(\phi)} \log \sup \left\{ \langle W, \rho \rangle \mid \|W\|_\mu \leq 1, \|W\|_\infty = \langle W, \rho \rangle \right\}. \] (72)

Both of the above are simultaneously true if and only if \( \|X^\otimes n\|_\mu = \|X\|_\mu^n \) and \( \|Y^\otimes n\|_\mu^0 = \left( \|Y\|_\mu^0 \right)^n \) for all \( X \) and \( Y \).
In practice, we will often obtain a single-letter lower bound for resource cost in a slightly
different manner. Say that a given resource theory is characterised by a norm \( \| \cdot \|_\mu \) which is not
necessarily multiplicative itself, but there exists a multiplicative norm \( \| \cdot \|_\gamma \) such that \( \| X \|_\gamma \leq \| X \|_\mu \)
for all \( X \). We can then bound
\[
\inf_{\| Z \|_\mu \leq 1} D_0^0 (\rho^\otimes n \| Z) \geq \inf_{\| Z \|_\gamma \leq 1} D_0^0 (\rho^\otimes n \| Z) \geq n \inf_{\| Z \|_\gamma \leq 1} D_0^1 (\rho \| Z)
\]
for any \( \rho \). We state the resulting lower bound as a separate result, as it will find important
applications in several theories of interest.

**Corollary 9 (Multiplicative norm bound).** Let \( \| \cdot \|_\gamma \) be a norm such that \( \| X \|_\gamma \leq \| X \|_\mu \) \( \forall X \) and
\[\| Y^\otimes n \|_\gamma^o = \left(\| Y \|_\gamma^o\right)^n \forall Y.\] Then
\[
r(\phi \xrightarrow{\otimes} \rho)^{-1} \geq \frac{\inf_{\| Z \|_\gamma \leq 1} D_0^0 (\rho \| Z)}{L_{\mu,\infty}(\phi)} \log \sup \left\{ \langle W, \rho \rangle \mid \| W \|_\gamma^o \leq 1, \| W \|_\infty = \langle W, \rho \rangle \right\}. \]

Note the necessity to consider two different norms, \( \| \cdot \|_\gamma \) and \( \| \cdot \|_\mu \), in this bound.

### 4.3. Lower bounds on resource cost of mixtures

One potential drawback of the single-letter bounds of the previous section is that they rely
on the quantity \( \inf_{\| Z \|_\mu \leq 1} D_0^0 (\rho \| Z) \), which will trivialise when the state \( \rho \) is full rank. Below, we
present an alternative approach to bounding the rate \( r(\phi \rightarrow \rho) \) that avoids this problem, and can
be applied also to full-rank states in finite dimensions. This provides a more direct way of lower
bounding the smoothed and regularised quantity encountered in Theorem 5, without going to the
inherently single-shot quantity \( D_0^0 \).

**Theorem 10.** Let \( \phi \) be pure, and let \( \rho \) be a state decomposable as \( \rho = \sum_x p_x \Pi_x \), where the range of \( x \)
is finite, for each \( x \) the operator \( \Pi_x \) is a projector onto a subspace of dimension \( d_x \), and \( \Pi_x \Pi_{x'} = 0 \) for \( x \neq x' \). Consider a norm \( \| \cdot \|_\gamma \), such that \( \| X \|_\gamma \leq \| X \|_\mu \) \( \forall X \) and \( \| Y \otimes Z \|^o_\gamma \leq \| Y \|^o_\gamma, \| Z \|^o_\gamma \) \( \forall Y, Z. \) Then
\[
r(\phi \xrightarrow{\otimes} \rho)^{-1} \geq \frac{\sum_x p_x \log \frac{1}{\| \Pi_x \|_\gamma} - H(p)}{L_{\mu,\infty}(\phi)}, \]
where \( H(p) := -\sum_x p_x \log p_x \) is the entropy of \( p \).

The assumption that the range of \( x \) is finite is necessary to apply the typicality arguments in
what follows.

**Proof.** Since we intend to apply Theorem 5, the first step is to lower bound
\[
\sup \left\{ \frac{1}{\| Q \|^o_\mu} \left| \langle Q, \rho^\otimes n \rangle \right| \geq 1 - \delta, \| Q \|_\infty \leq 1 \right\}
\]
for large values of $n$. For some fixed (small) parameter $\eta > 0$, define the strongly typical subspace of sequences $x^n = x_1 \ldots x_n$ by setting [68, Definition 14.7.2]

$$T_\eta := \left\{ x^n \mid \frac{1}{n} N(x|x^n) - p_x \leq \eta \left( 1 - \delta_{0,p_x} \right) \forall x \right\},$$

(77)

where $\delta_{0,q} = 1$ if $q = 0$ and $\delta_{0,q} = 0$ otherwise, and $N(x|x^n)$ denotes the number of times the symbol $x$ appears in the sequence $x^n$. It is well known that [68, Eq. (14.72) and (14.73)]

$$\lim_{n \to \infty} \sum_{x^n \in T_\eta} p_{x^n} = 1,$$

(78)

$$|T_\eta| \leq 2^n (H(p) + c\eta),$$

(79)

where $p_{x^n} := \prod_{i=1}^n p_{x_i}$, and $c := - \sum_{x: p_x > 0} \log p_x > 0$ is a positive constant. Denoting analogously $\Pi_{x^n} := \bigotimes_{i=1}^n \Pi_{x_i}$ and $d_{x^n} := \prod_{i=1}^n d_{x_i}$, let us now define

$$Q_n := \sum_{x^n \in T_\eta} \Pi_{x^n}.$$  

(80)

On the one hand, since $Q_n$ is an orthogonal projector we have that $\|Q_n\|_\infty = 1$; on the other, for every $\delta > 0$ and for sufficiently large $n$ it holds that

$$\langle Q_n, \rho^{\otimes n} \rangle = \sum_{x^n} p_{x^n} \left( Q_n, \frac{\Pi_{x^n}}{d_{x^n}} \right) = \sum_{x^n \in T_\eta} p_{x^n} \geq 1 - \delta,$$

(81)

Now, we estimate

$$\|Q_n\|_\mu^0 \leq \|Q_n\|_\gamma^0 \leq \sum_{x^n \in T_\eta} \|\Pi_{x^n}\|_\gamma^0 \leq \sum_{x^n \in T_\eta} \prod_{x} \left( \|\Pi_{x}\|_\gamma^0 \right)^{N(x|x^n)} \leq \sum_{x^n \in T_\eta} \prod_{x} \left( \|\Pi_{x}\|_\gamma^0 \right)^{n(p_x + \eta)} \leq |T_\eta| \prod_{x} \left( \|\Pi_{x}\|_\gamma^0 \right)^{n(p_x + \eta)} \leq 2^n (H(p) + c\eta + \sum_{x} (p_x \log \|\Pi_{x}\|_\gamma^0 + \eta \log \|\Pi_{x}\|_\gamma^0)),$$

(82)

Here, (i) follows from the inequality $\|\cdot\|_\gamma \leq \|\cdot\|_\mu$, which implies by duality that $\|\cdot\|_\gamma^0 \geq \|\cdot\|_\mu^0$; (ii) comes from the sub-multiplicativity of $\|\cdot\|_\gamma^0$; (iii) descends from the definition of strongly typical set; and (iv) is just an application of (79).
Thanks to Theorem 5, from the above chain of inequalities we deduce that

\[
\frac{1}{L_{\mu,\infty}(\phi)} \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{Q, \rho^{\otimes n}} \left\{ \frac{1}{\|Q\|_{\mu}^n} \langle Q, \rho^{\otimes n} \rangle \geq 1 - \delta, \|Q\|_{\infty} \leq 1 \right\}
\geq \limsup_{n \to \infty} \frac{1}{n} \log \frac{1}{\|Q_n\|_{\mu}^n}
\geq - \sum_x p_x \log \|\Pi_x\|_{\gamma} - H(p) - \eta \left( c + \sum_x \log \|\Pi_x\|_{\gamma} \right).
\]

(83)

Since \(\eta > 0\) was arbitrary, we can take \(\eta \to 0^+\) and complete the proof.

\[\square\]

**Remark.** A closer look to the proof of Theorem 10 shows that we in fact proved the slightly stronger bound

\[
\frac{1}{L_{\mu,\infty}(\phi)} \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{Q, \rho^{\otimes n}} \left\{ \frac{1}{\|Q\|_{\mu}^n} \langle Q, \rho^{\otimes n} \rangle \geq 1 - \delta, \|Q\|_{\infty} \leq 1 \right\}
\geq \limsup_{n \to \infty} \frac{1}{n} \log \frac{1}{\|Q_n\|_{\mu}^n}
\geq - \sum_x p_x \log \|\Pi_x\|_{\gamma} - H(p) - \eta \left( c + \sum_x \log \|\Pi_x\|_{\gamma} \right).
\]

(84)

from which one deduces (75) by using the triangle inequality and replacing \(\|\cdot\|_{\mu}\) with \(\|\cdot\|_{\gamma}\).

### 4.4. On positivity, and comparison with other norm-based bounds

Here we take a short intermezzo to highlight the importance of making an appropriate choice of the norm \(\|\cdot\|_{\mu}\) to be used in the bounds of this work. We do this through a comparison with a different approach, which traces its conceptual beginnings to works in entanglement theory by Rains [69], and which found use recently in the establishment of several norm-based bounds in the theory of entanglement [27, 31, 70] and magic [40].

Although previous works did not explicitly formalise it in this way, their approach can be thought of as studying resource transformations under a class of completely positive maps defined as

\[
\mathcal{O}_{\mu+} := \{ \Lambda \mid \Lambda \in \text{CP}, \|\Lambda(X)\|_1 \leq \|X\|, \|\Lambda(X)\|_{\mu} \leq \|X\|_{\mu} \}.
\]

(85)

This means that \(\mathcal{O}_{\mu+} \subset \mathcal{O}_{\mu}\), and hence it appears that this choice of operations should never provide better bounds through the approach that we have employed in our work (cf. the discussion in Section 1.2). In particular, our lower bounds on resource cost (Theorem 5 and Corollary 8) provide a strict improvement over lower bounds on entanglement cost which were established in [31, 70].

It might then come as a surprise that a naïve application of the upper bound for distillable resource that we have found (Theorem 6 and Corollary 8) can actually perform worse than corresponding bounds derived in [27, 40]. The reason for this is that the bounds in the latter works make an explicit use of the positivity of operations \(\mathcal{O}_{\mu+}\), which in this case results in tighter limitations. But the same result can be recovered in our approach, as we now show.

Define the function \(p_{\mu}(X)\) as a ‘positive’ variant of the norm \(\|\cdot\|_{\mu}\):

\[
p_{\mu}(X) := \inf \left\{ \|Z\|_{\mu} \mid X \leq Z \right\}.
\]

(86)

Note that \(p_{\mu}(X) \leq \|X\|_{\mu}\). Although useful, this quantity is not a norm per se. We can, however, define a norm by simply symmetrising the above, that is,

\[
\|X\|_{\mu+} := \sup \{p_{\mu}(X), p_{\mu}(-X)\}.
\]

(87)
Since every norm can be written as \( \|Z\|_\mu = \inf \left\{ \lambda \mid Z = \lambda U, \|U\|_\mu \leq 1 \right\} \), we can rewrite \( \|X\|_{\mu^+} \) as
\[
\|X\|_{\mu^+} = \inf \left\{ \sup \{\lambda_+, \lambda_-\} \mid -\lambda_- Q \leq X \leq \lambda_+ Z, \|Q\|_\mu, \|Z\|_\mu \leq 1 \right\} = \inf \left\{ \lambda \mid -\lambda Q \leq X \leq \lambda Z, \|Q\|_\mu, \|Z\|_\mu \leq 1 \right\}.
\] (88)

It is not difficult to see that this is a norm with the unit ball \((B_\mu - T_\mu(\mathcal{H})) \cap (-B_\mu + T_\mu(\mathcal{H}))\), where \(B_\mu := \left\{Z \mid \|Z\|_\mu \leq 1\right\}\). It follows by construction that \(\|\cdot\|_{\mu^+}\) is in fact a monotone under \(O_\mu\), allowing us to immediately apply our results.

Importantly, as long as \(X \geq 0\), it holds that \(\|X\|_{\mu^+} = p_\mu(X)\). Applying the bounds of Theorem 6 and Corollary 8 and making use of the fact that \(O_\mu\) are positive maps — meaning that, for any \(X \geq 0\), \(\Lambda(X)\) will always be a positive semidefinite operator — we obtain the following result.

**Corollary 11.** For any state \(\rho\) and any pure state \(\phi\), the rate of transformation from \(\rho\) to \(\phi\) under completely positive operations in \(O_\mu\) satisfies
\[
r(\rho \xrightarrow{O_{\mu^+}} \phi) \leq \lim_{\epsilon \to 0} \lim \inf_{n \to \infty} \frac{1}{n} \inf_{X \in B_\mu(\rho^\otimes n) \cap T_\mu(\mathcal{H})} \log p_\mu(X) - L_{\mu,\infty}(\phi) \leq \lim \inf_{n \to \infty} \frac{1}{n} \log p_\mu(\rho^\otimes n) - L_{\mu,\infty}(\phi).
\] (89)

As long as the norm \(\|\cdot\|_\mu\) is sub-multiplicative, i.e. \(\|X \otimes Y\|_\mu \leq \|X\|_\mu \|Y\|_\mu\), this gives
\[
r(\rho \xrightarrow{O_{\mu^+}} \phi) \leq \frac{\log p_\mu(\rho) - L_{\mu,\infty}(\phi)}{-L_{\mu,\infty}(\phi)} = \frac{\log \|\rho\|_{\mu^+}}{-L_{\mu,\infty}(\phi)}.
\] (90)

This in fact recovers the upper bound on distillable entanglement \(E_W\) found in [27] and an upper bound on magic state distillation \(\theta_{\max}\) from [40] as special cases. Since these are the best known computable upper bounds in their corresponding resource theories, we thus see that our results can recover the leading constraints on resource distillation, as long as care is taken to choose a suitable norm in the application of our bounds.

It is rather interesting to note that a similar ‘positivity’ assumption does not appear to be helpful in improving our bounds on the reverse transformation \(\phi \to \rho\). On the one hand, replacing a norm \(\|\cdot\|_\mu\) with \(\|\cdot\|_{\mu^+}\) will, in general, sacrifice its multiplicativity, which we need to apply the single-letter bounds in Corollary 8 or 9; on the other hand, since the quantity \(D^\phi\) that we employ is a maximisation, explicitly adding a positivity assumption there would only lead to a worse bound (and would, in fact, reduce to the previous bounds of [31, 70]). This can be considered as the reason why our approach allowed us to improve on previous bounds: they all took positivity for granted, while it appears that the characterisation of resource dilution benefits from going beyond positive operators.

### 5. APPLYING THE RESULTS TO SPECIFIC RESOURCES

The basic ingredient of any resource theory of quantum states is the set \(\mathbb{F}\) of states which are designated as free. This set immediately gives rise to several different norms that can be employed in the framework of this work.
Entanglement Base norm $\|X\|_{\mu}$ No $|\phi\rangle$ $2^{n+1} - 1$ $2^n$ $1$ $1$

[21] Negativity $\|X^T\|_1$ Yes $|\phi_2\rangle$ $2^n$ $2^n$ $1$ $1$

Reshuffled negativity $\|X^R\|_1$ Yes $|\phi_2\rangle$ $2^n$ $2^n$ $1$ $1$

Qudit magic Base norm $\|X\|_{\mathbb{F}_w}$ No $|S\rangle$ $\leq 3^n$ $2^n$ $\leq \log 3$ $1$

Wigner negativity $\|X\|_{W}$ Yes $|S\rangle$ $\left(\frac{5}{3}\right)^n$ $1$ $\log \frac{5}{3}$ $0$

Qubit magic Base norm $\|X\|_{\mathbb{F}_{STAB}}$ No $|T\rangle$ $\leq \sqrt{2}^n$ $4 - 2\sqrt{2}$ $\leq \log 1.29$ $\log (4 - 2\sqrt{2})$

Stabiliser norm $\|\cdot\|_\phi$ Yes $|\text{Hog}\rangle$ $\leq \left(\frac{19}{5}\right)^n$ $\left(\frac{12}{5}\right)^n$ $\leq \log \frac{19}{5}$ $\log \frac{12}{5}$

TABLE I. **Survey of resources, norms-based monotones, and choices of reference states $|\phi\rangle$.** We overview the formalism of this work applied to the two considered resource theories, namely, quantum entanglement and magic-state quantum computation. For each resource, we give a selection of norm-based measures that can be used in our framework. We specify whether a given norm has a multiplicative dual (so that Corollary 9 can be applied). We then give estimates on, when known, exact values of the parameters $L_{\mu,\infty}(\phi)$ and $L_{\mu,\infty}(\phi)$ that are required for the application of Theorems 5 and 6. For the theory of magic, the upper bounds on $\|\phi^{\otimes n}\|_\mu$ are based on the evaluation of the single-letter quantity $\|\phi\|_\mu$ (see [34, 38, 39, 71]) and using the sub-multiplicativity of the norm; for $L_{\mathbb{F}_{STAB},\infty}(|N\rangle\langle N|)$, we also give improved estimates obtained in Section 7, while for $L_{\mathbb{F}_{STAB},\infty}(|T\rangle\langle T|)$, we use a many-copy estimate that was obtained in [72]. The evaluation of $\|\phi^{\otimes n}\|_\mu$ proceeds by first computing $\|\phi\|_\mu$ (see [40, 71]), and then using the fact that, for the considered cases, the dual norms are multiplicative [40, 71]. See Sections 6 and 7 for more details on the choices of target states and on the computation of the quantities.

A basic assumption we will make is that $\mathbb{F}$ is a closed and convex set. In finite-dimensional theories, as long as $\text{span}(\mathbb{F}) = \mathcal{T}_{sa}(\mathcal{H})$, i.e. the free states are a full-measure subset of all quantum states, then a natural norm that can be defined is the **base norm** [73, 74]:

$$\|X\|_{\mathbb{F}} := \inf \left\{ \lambda_+ + \lambda_- \mid X = \lambda_+ \sigma_+ - \lambda_- \sigma_- , \, \sigma_\pm \in \mathbb{F} \right\}. \quad (91)$$

The condition on the span of $\mathbb{F}$ is satisfied in resource theories such as quantum entanglement or magic. Even when $\text{span}(\mathbb{F}) \neq \mathcal{T}_{sa}(\mathcal{H})$, which happens in particular for several infinite-dimensional resource theories of operational interest, the quantity in (91) can still be defined, although care needs to be taken as it will be infinite for some states [75, 76]. We use the term ‘base norm’ regardless of whether $\|X\|_{\mathbb{F}}$ defines a proper norm. This quantity is directly related to a commonly used resource monotone called the **standard robustness**, defined as $R_\mathbb{F}(\rho) := \inf \left\{ \lambda \mid \rho + \lambda \sigma_- = (1 + \lambda) \sigma_+ , \, \sigma_\pm \in \mathbb{F} \right\}$. It is not difficult to notice that, for a quantum state, we have $\|\rho\|_{\mathbb{F}} = 1 + 2R_\mathbb{F}(\rho)$. 


Another type of norm can be defined if the free states satisfy that $\mathcal{F} = \text{conv} \{ |\psi\rangle \langle \psi| \mid |\psi\rangle \in \mathcal{V} \}$ for some set $\mathcal{V}$ defined in the underlying Hilbert space $\mathcal{H}$, such that $\text{span} (\mathcal{V}) = \mathcal{H}$. This is the case for resource theories such as entanglement, magic, or quantum coherence. We can then define

$$
\|X\|_\mathcal{V} := \inf \left\{ \sum_i |\lambda_i| \mid X = \sum_i \lambda_i |v_i\rangle \langle w_i|, \lambda_i \in \mathbb{C}, |v_i\rangle, |w_i\rangle \in \mathcal{V} \right\}.
$$

This gives rise to the projective tensor norm in entanglement theory [77, 78], the $\ell_1$ norm in the resource theory of coherence [79], and the so-called dyadic negativity in magic theory [80]. Note that this norm can be defined even for linear operators which are not self-adjoint.

Yet another form of a norm can be defined based on another resource measure, the generalised robustness $R^\mathcal{F}_\mathcal{F}(\rho) := \inf \{ \lambda \mid \rho + \lambda \omega = (1 + \lambda) \sigma, \sigma \in \mathcal{F}, \omega \in \mathcal{D}(\mathcal{H}) \}$, as follows. The function

$$
\|X\|_{\mathcal{F},\mathcal{F}} := \inf \{ \lambda \mid -\lambda \sigma_- \leq X \leq \lambda \sigma_+, \sigma_\pm \in \mathcal{F} \}
$$

(93)

can be seen to be a norm in the case of finite-dimensional theories as soon as $\mathcal{F}$ contains at least one state of full rank, which is satisfied in virtually all cases of interest. For any state, it holds that $\|\rho\|_{\mathcal{F},\mathcal{F}} = 1 + R^\mathcal{F}_\mathcal{F}(\rho)$.

A useful property of the above norms is that, for a pure state $\phi$, their dual norms all coincide [47]:

$$
\|\phi\|_{\mathcal{F}}^\circ = \|\phi\|_{\mathcal{V}}^\circ = \|\phi\|_{\mathcal{F},\mathcal{F}}^\circ = \sup \{ \langle \phi, \sigma \rangle \mid \sigma \in \mathcal{F} \},
$$

(94)

where we assumed that each of the norms is well defined. Since we have already seen that evaluating the dual norm on the target state $\phi$ in distillation and dilution protocols is a crucial ingredient of many of our bounds, this insight can help in the applications of the different norms.

Any of the above norms can be directly used in all of our results. The crucial point to consider in any resource theory is how to obtain single-letter bounds on asymptotic transformations rates, leading to efficiently computable bounds. To clarify the applicability of our single-letter results (Corollaries 8, 9, and 11), we now summarise how they may be applied in a given resource theory.

Assume that $\sigma \in \mathcal{F} \Rightarrow \sigma^\otimes n \in \mathcal{F} \forall n \in \mathbb{N}$, which is a weak assumption satisfied in virtually all theories of interest. Consider then any class of quantum channels $\mathcal{O}$ under which the base norm $\|\cdot\|_\mathcal{F}$ is contractive — this includes, in particular, the set $\mathcal{O}_{\text{max}}$ of resource non-generating quantum channels, and all subsets thereof. Then:

(i) In order to upper bound the distillation rate $r(\rho \xrightarrow{\mathcal{O}} \phi)$, one needs to compute or upper bound the regularised quantity $L^0_{\mathcal{F},\infty}(\phi)$. Then, Corollary 8 or 11 can be used to establish that

$$
r(\rho \xrightarrow{\mathcal{O}} \phi) \leq \frac{\log \|\rho\|_{\mathcal{F},\mathcal{F}}}{-L^0_{\mathcal{F},\infty}(\phi)} \leq \frac{\log \|\rho\|_{\mathcal{F}}}{-L^0_{\mathcal{F},\infty}(\phi)}.
$$

(95)

(ii) In order to upper bound the dilution rate $r(\phi \xrightarrow{\mathcal{O}} \rho)$, one needs to compute or upper bound the regularised quantity $L_{\mathcal{F},\infty}(\phi)$, as well as find a multiplicative norm $\|\cdot\|_\gamma$ such that $\|\cdot\|_\gamma \leq \|\cdot\|_\mathcal{F}$. Corollary 9 then gives

$$
r(\phi \xrightarrow{\mathcal{O}} \rho)^{-1} \geq \frac{\inf_{\|Z\|_\gamma \leq 1} D^0_{\mathcal{F}}(\rho\|Z)}{L_{\mathcal{F},\infty}(\phi)}.
$$

(96)

An alternative, but more tailored approach that might only be applicable to specific quantum states is provided in Theorem 10.
We discuss these problems in specific resource theories. We also present an overview of different norm choices for representative resource theories in Table I.

6. APPLICATION: QUANTUM ENTANGLEMENT

In the manipulation of quantum entanglement, the relevant set of free states is separable states, which can be defined on a bipartite quantum system $AB$ with Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ by

$$\mathcal{F} = \mathcal{F}_{\text{SEP}} := \text{conv}\{ |\psi\rangle\langle\psi|_A \otimes |\phi\rangle\langle\phi|_B \},$$

where the convex hull ranges over all normalised pure states $|\psi\rangle_A \in \mathcal{H}_A$ and $|\phi\rangle_B \in \mathcal{H}_B$. Every state $\sigma_{AB} \in \mathcal{F}_{\text{SEP}}$ can thus be written as [81]

$$\sigma_{AB} = \int |\psi\rangle\langle\psi|_A \otimes |\phi\rangle\langle\phi|_B \, d\mu(\psi, \phi),$$

where $\mu$ is a Borel probability measure on the product of the sets of local (normalised) pure states; conversely, every state of the form (98) is clearly separable.

The relevant target state $\phi$ here is the two-qubit maximally entangled state $\phi = \Phi_2$, where $|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$; we will refer to the corresponding rates of transformations as distillable entanglement $E_{d,\mathcal{O}}(\rho) := r(\rho \rightarrow \Phi_2)$ and entanglement cost $E_{c,\mathcal{O}}(\rho) := r(\Phi_2 \rightarrow \rho)^{-1}$.

As for the class $\mathcal{O}$, perhaps the best motivated from an operational standpoint is that of local operations and classical communication, denoted by $\mathcal{O}_{\text{LOCC}}$ [23]. Other useful classes are $\mathcal{O}_{\text{max}} := \mathcal{O}_{\text{NE}}$ (non-entangling, a.k.a. separability-preserving operations [28, 33]), $\mathcal{O}_{\text{SEP}}$ (separable channels) [82], and $\mathcal{O}_{\text{PPT}}$ (channels $\Lambda : AB \rightarrow A'B'$ such that $\Gamma_A \circ \Lambda \circ \Gamma_A$ is CPTP, where $\Gamma$ denotes the partial transpose) [69].

6.1. Basic bounds on entanglement cost and distillable entanglement

The set $\mathcal{F}_{\text{SEP}}$ immediately gives rise to the base norm $\|X\|_{\mathcal{F}_{\text{SEP}'}}$, which we recall as

$$\|X\|_{\mathcal{F}_{\text{SEP}'}} := \inf \left\{ \lambda_+ + \lambda_- \mid X = \lambda_+ \sigma_+ - \lambda_- \sigma_- , \sigma_{\pm} \in \mathcal{F}_{\text{SEP}} \right\}.$$

As we mentioned before, for normalised quantum states this norm is equivalent to the measure known as the (standard) robustness of entanglement [83], in the sense that $\|\rho\|_{\mathcal{F}_{\text{SEP}'}} = 1 + 2R^e_{\mathcal{F}_{\text{SEP}}} (\rho)$.

Let us now discuss the asymptotic properties of this norm. Using the fact that $\sigma, \tau \in \mathcal{F}_{\text{SEP}} \Rightarrow \sigma \otimes \tau \in \mathcal{F}_{\text{SEP}}$, it can be shown that $\|X^{\otimes n}\|_{\mathcal{F}_{\text{SEP}}} \leq \|X\|^n_{\mathcal{F}_{\text{SEP}'}}$, which immediately gives the bound

$$L_{\mathcal{F}_{\text{SEP}}^{\infty}}(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \|\phi^{\otimes n}\|_{\mathcal{F}_{\text{SEP}}} \leq \log \|\phi\|_{\mathcal{F}_{\text{SEP}'}}$$

for any pure state $\phi$. However, a much tighter bound can be obtained for pure states, where it was shown [75, 83] that $1 + R^e_{\mathcal{F}_{\text{SEP}}} (\phi) = \sum_{i=1}^n \alpha_i$, where $\{\alpha_i\} \in \mathbb{R}_+$ are the Schmidt coefficients of a given state, i.e. $|\phi\rangle = \sum_{i=1}^n \alpha_i |e_i f_i\rangle$ for some orthonormal bases $\{|e_i\rangle\}_i, \{|f_i\rangle\}_j$. This means that

$$1 + R^e_{\mathcal{F}_{\text{SEP}}} (\phi^{\otimes n}) = \left(1 + R^e_{\mathcal{F}_{\text{SEP}}} (\phi)\right)^n$$

for any pure $\phi$, and hence

$$L_{\mathcal{F}_{\text{SEP}}^{\infty}}(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \left(1 + 2R^e_{\mathcal{F}_{\text{SEP}}} (\phi^{\otimes n})\right) = \lim_{n \to \infty} \frac{1}{n} \log \left(1 + R^e_{\mathcal{F}_{\text{SEP}}} (\phi^{\otimes n})\right)$$

(101)
For the relevant choice of the two-qubit maximally entangled state as the target, this gives $L_{F_{\text{SEP}},\infty}(\Phi_2) = 1$.

The dual norm here is the overlap $\|X\|_{F_{\text{SEP}}} = \sup_{\sigma \in F_{\text{SEP}}} |\langle X, \sigma \rangle|$. It is known [84] that $\|\phi\|_{F_{\text{SEP}}}^\circ = (\sup_{f} \alpha_i)^{-1}$ for any pure state $\phi$, which in particular means that $\|\phi^\otimes n\|_{F_{\text{SEP}}}^\circ = \left(\|\phi\|_{F_{\text{SEP}}}^\circ\right)^n$, yielding

$$L_{F_{\text{SEP}},\infty}^\circ(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \|\phi^\otimes n\|_{F_{\text{SEP}}}^\circ = \log \|\phi\|_{F_{\text{SEP}}}^\circ.$$  \hspace{1cm} (102)

For the two-qubit maximally entangled state, we once again have that $L_{F_{\text{SEP}},\infty}^\circ(\Phi_2) = 1$.

We now focus on evaluating the asymptotic bounds established in our work. The physically relevant sets of free operations in this theory all form subsets of the maximal set of free channels, $\mathcal{O}_{\text{max}}$, which comprises all non-entangling maps: $\mathcal{O}_{\text{max}} = \{ \Lambda \in \text{CPTP} \mid \Lambda(\sigma) \in F_{\text{SEP}} \ \forall \sigma \in F_{\text{SEP}} \}$. By choosing any norm which contracts under those maps — such as the base norm $\|\cdot\|_{F_{\text{SEP}}}$ or the norm $\|\cdot\|_{\gamma, F_{\text{SEP}}}$ based on the generalised robustness — we can use the properties of the norms to constrain the operational properties of entanglement.

Although it is not known what the regularised bound of Theorem 6 evaluates to, the single-letter bound in Corollary 8,

$$E_{d, \mathcal{O}_{F_{\text{SEP}}}}(\rho) \leq \log \left( 1 + 2R_{F_{\text{SEP}}}^\tau(\rho) \right),$$  \hspace{1cm} (103)

is known [50]. As for the lower bound on entanglement cost, the generalised hypothesis testing relative entropy corresponds to a quantity introduced in [28] as the ‘tempered robustness’ $R_{F_{\text{SEP}}}^\tau$:

$$\inf_{\|Z\|_{F_{\text{SEP}}} \leq 1} D_{\text{B}}^0(\rho\|Z) = \log \sup \left\{ \langle W, \rho \rangle \mid \|W\|_{F_{\text{SEP}}}^\circ \leq 1, \|W\|_\infty = \langle W, \rho \rangle \right\}$$

$$= 1 + 2R_{F_{\text{SEP}}}^\tau(\rho).$$  \hspace{1cm} (104)

Our zero-error bound in Corollary 7 then recovers a result of [28], although the regularised bound in Theorem 5 has a strong potential to improve on that result, provided that it can be computed exactly.

### 6.2. Bounds from partial transposition

The issue we encounter is that the norm $\|\cdot\|_{F_{\text{SEP}}}$ is not multiplicative, preventing an efficient evaluation of the lower bounds on $E_{c, \mathcal{O}_{F_{\text{SEP}}}}$, and in particular the computation of the single-letter results. We therefore consider a relaxation: instead of the set $\mathbb{F} = F_{\text{SEP}}$, we consider as our set of interest the set of all states with a positive partial transpose, or in fact an even larger set: the set $\mathbb{F}_{\text{PPT}} = \{ X \mid X^T \geq 0, \text{Tr} X = 1 \}$. The base norm $\|\cdot\|_{\mathbb{F}_{\text{PPT}}}$ is then simply the entanglement negativity, $\|X\|_{\mathbb{F}_{\text{PPT}}} = \|X^T\|_1$, which is straightforwardly verified to be multiplicative. By the inclusion $F_{\text{SEP}} \subseteq F_{\text{PPT}}$ it holds that $\|X^T\|_1 \leq \|X\|_{F_{\text{SEP}}}$, which allows us to apply Corollary 9 to give

$$E_{c, \mathcal{O}_{F_{\text{SEP}}}}(\rho) \geq \inf_{\|Z^T\|_1 \leq 1} D_{\text{B}}^0(\rho\|Z)$$

$$= \log \max \left\{ \langle W, \rho \rangle \mid \|W\|_\infty \leq 1, \|W\|_\infty = \langle W, \rho \rangle \right\}.$$  \hspace{1cm} (105)

This quantity exactly equals the tempered logarithmic negativity $E_{\tau}$ of [28]. We have thus not only recovered this single-letter bound on entanglement cost, but in fact endowed it with an operational meaning: using Theorem 2, we have that

$$E_{d, \mathcal{O}_{\mathbb{F}_{\text{PPT}}}}^{(1), \text{exact}}(\rho) = \left[ E_{\tau}(\rho) \right].$$  \hspace{1cm} (106)
in other words, $E_T(\rho)$ quantifies exactly the distillable entanglement under linear maps which contract the trace norm and the negativity.

### 6.3. Bounds from reshuffling criterion

We will now employ Corollary 9 to deduce a novel lower bound on the entanglement cost. To construct it, let us consider the *reshuffling* operation on $T(\mathcal{H}_{AB})$, the Banach space of trace class operators acting on the bipartite Hilbert space $\mathcal{H}_{AB}$ [78, 85–89]. A quick and painless way to define it in a rigorous manner while encompassing the case of infinite-dimensional Hilbert spaces is to think of it as an isometry on the Hilbert–Schmidt class $\mathcal{HS}(\mathcal{H}_{AB}) \supseteq T(\mathcal{H}_{AB})$, defined on an arbitrary

$$X^R := \sum_{i,j,k,l} X_{ij,kl} |i\rangle \langle j| \otimes |k\rangle \langle l|, \quad \text{(107)}$$

where the sum on the right-hand side converges in Hilbert–Schmidt norm because $\sum_{i,j,k,l} |X_{ij,kl}|^2 = \|X\|_2^2 < \infty$. A little thought reveals that $X \mapsto X^R$ is in fact an isometry, so that

$$\text{Tr} \left[ (X^R)^\dagger Y^R \right] = \langle X^R, Y^R \rangle = \langle X, Y \rangle = \text{Tr} \left[ X^\dagger Y \right] \quad \forall X, Y \in \mathcal{HS}(\mathcal{H}_{AB}). \quad \text{(108)}$$

In what follows, we will most often consider its restriction to the trace class $T(\mathcal{H}_{AB})$. We note that this approach requires us to go beyond self-adjoint operators, so in this section we drop the assumption of self-adjointness that we have taken throughout this paper.

The importance of the reshuffling operation stems from the fact that it maps separable states to trace class (rather than simply Hilbert–Schmidt) operators, and moreover [78, 85–89]

$$\|\sigma^R\|_1 \leq 1 \quad \forall \sigma \in \mathbb{P}_{\text{SEP}}. \quad \text{(109)}$$

To verify this, we start by observing that for every two pure states $|\psi\rangle = \sum_{i} \psi_i |i\rangle \in \mathcal{H}_A$ and $|\phi\rangle = \sum_{j} \phi_j |j\rangle \in \mathcal{H}_B$ it holds that

$$\langle |\psi\rangle \langle \psi|_A \otimes |\phi\rangle \langle \phi|_B \rangle^R = \left( \sum_{i,j,k,l} \psi_i^* \phi_k^* \phi_j^* \phi_l^* \langle k| \otimes | j\rangle \langle l| \right)^R$$

$$= \sum_{i,j,k,l} \psi_i^* \phi_k^* \phi_j^* \phi_l^* \langle i| \otimes | k\rangle \langle j| \langle l|$$

$$= |\psi\rangle \langle \phi^*|_A \otimes |\psi^*\rangle \langle \phi|_B. \quad \text{(110)}$$

Thus

$$\sigma^R = \left( \int |\psi\rangle \langle \psi|_A \otimes |\phi\rangle \langle \phi|_B \ d\mu(\psi, \phi) \right)^R$$

$$= \int \langle |\psi\rangle \langle \psi|_A \otimes |\phi\rangle \langle \phi|_B \rangle^R d\mu(\psi, \phi) \quad \text{(111)}$$
where the identity in the second line follows from the fact that the Bochner integral commutes with any continuous linear operator — the reshuffling, being an isometry, is automatically continuous. Finally,

$$\|\sigma^R\|_1 = \left\| \int |\psi\rangle \langle \phi^*| \otimes |\psi\rangle \langle \phi| \, d\mu(\psi, \phi) \right\|_1 \leq \sup_{|\psi\rangle, |\phi\rangle} \left\| |\psi\rangle \langle \phi^*| \otimes |\psi\rangle \langle \phi| \right\|_1 = 1. \quad (112)$$

We are now ready to state the following:

**Corollary 12** (Reshuffling lower bound). Given an arbitrary bipartite quantum state \( \rho_{AB} \), we have that

$$E_{c, O_{\text{LOCC}}} (\rho_{AB}) \geq E_{c, O_{\text{NE}}} (\rho_{AB}) \geq \log N_1^R (\rho_{AB}), \quad (113)$$

where the **tempered reshuffled negativity** is defined by

$$N_1^R (\rho_{AB}) := \max \left\{ \langle W, \rho \rangle \mid \|W^R\|_\infty \leq 1, \langle W, \rho \rangle = \|W\|_\infty \right\}. \quad (114)$$

**Remark.** The tempered reshuffled negativity, exactly as the tempered negativity introduced in [28], can be computed efficiently via a semidefinite program. To see that this is the case, it suffices to reformulate slightly (114) as

$$N_1^R (\rho_{AB}) = \max \left\{ \langle W, \rho \rangle \mid \left( \frac{1}{(W^R)^*} W^R \right) \geq 0, \quad -\langle W, \rho \rangle \mathbb{1} \leq W \leq \langle W, \rho \rangle \mathbb{1} \right\}. \quad (115)$$

Note that unlike the partial transposition, the reshuffling operation is not Hermiticity-preserving.

**Proof of Corollary 12.** We intend to apply Corollary 9 with the choices \( \|\cdot\|_\mu = \|\cdot\|_{\text{SEP}} \) and \( \|\cdot\|_Y = \|\cdot\|_{\text{SEP}} \). First of all, let us verify that \( \|\cdot\|_1 \leq \|\cdot\|_{\text{SEP}} \). To this end, for any \( \delta > 0 \) pick some operator \( X \in \mathcal{F}_{\text{sa}} (\mathcal{H}) \) and a decomposition \( X = a \sigma_+ - b \sigma_- \) with \( \sigma_\pm \in \mathcal{F}_{\text{SEP}} \) and \( a + b \leq \|X\|_{\text{SEP}} + \delta \); we have that

$$\|X^R\|_1 = \|a \sigma_+ - b \sigma_-\|_1 \leq a \|\sigma_+\|_1 + b \|\sigma_-\|_1 \leq a + b \leq \|X\|_{\text{SEP}} + \delta. \quad (116)$$

Since this holds for arbitrary \( \delta > 0 \), we conclude that indeed \( \|X^R\|_1 \leq \|X\|_{\text{SEP}} \), as claimed.

We now compute the dual norm to \( \|\cdot\|_Y = \|\cdot\|_{\text{SEP}} \). Using (108) yields that

$$\|Y\|_Y^\circ = \sup \left\{ \langle X, Y \rangle \mid \|X\|_Y \leq 1 \right\} = \sup \left\{ \langle X^R, Y^R \rangle \mid \|X^R\|_1 \leq 1 \right\} = \sup \left\{ \langle Z, Y^R \rangle \mid \|Z\|_1 \leq 1 \right\} = \|Y^R\|_\infty. \quad (117)$$

Fortunately, this is indeed a multiplicative norm, i.e.

$$\|Y^{\otimes n}\|_Y^\circ = \|(Y^{\otimes n})^R\|_{\text{SEP}} = \|(Y^R)^{\otimes n}\|_{\infty} = \|Y^R\|_\infty^n = \left( \|Y\|_Y^\circ \right)^n \quad (118)$$

for all \( Y \). We are thus ready to apply Corollary 9, which yields

$$E_{c, O_{\mu}} (\rho_{AB}) = r (\Phi_2 \circ_{\mu} \rho)^{-1} \geq \frac{1}{L_{\text{SEP}, \circ}(\Phi_2)} \log \sup \left\{ \langle W, \rho \rangle \mid \|W^R\|_\infty \leq 1, \langle W, \rho \rangle = \|W\|_\infty \right\} \quad (119)$$

where we recalled that \( L_{\text{SEP}, \circ}(\Phi_2) = 1 \).

To complete the proof it suffices to note that non-entangling operations are always contractive with respect to the norm \( \|\cdot\|_{\text{SEP}} \), so that \( E_{c, O_{\text{LOCC}}} (\rho_{AB}) \geq E_{c, O_{\text{NE}}} (\rho_{AB}) \geq E_{c, O_{\text{SEP}}} (\rho_{AB}) \). \( \square \)
It is an open problem to find examples of states for which the reshuffled tempered negativity yields a lower bound on the entanglement cost that is both non-trivial and better than other bounds known so far, but we will shortly see that it can match the bound obtained from partial transposition, allowing us to provide an alternative derivation of the irreversibility of entanglement theory.

6.4. Recovering the irreversibility of entanglement theory

To establish the asymptotic irreversibility of a theory, it suffices to exemplify states \( \rho, \rho' \) such that \( r(\rho \rightarrow \rho') r(\rho' \rightarrow \rho) < 1 \). We will first show how this can be done in the theory of entanglement, recovering the recent result of [28] by means of a more general class of examples.

For an integer \( d \geq 3 \), define the \( d \times d \)-dimensional state \( \omega_d \) as

\[
\omega_d := \frac{1}{d(d-1)} \sum_{i,j=1}^d (|ii\rangle\langle ii| - |ii\rangle\langle jj|) = \frac{1}{d-1} (P_d - \Phi_d),
\]

(120)

where \( \Phi_d := \frac{1}{d} \sum_{i,j=1}^d |ii\rangle\langle jj| \) and \( P_d := \frac{1}{d} \sum_{i=1}^d |ii\rangle\langle ii| \). Note that for \( d = 3 \) the above state reproduces precisely the state \( \omega_3 \) from [28].

We proceed to establish a slight generalisation of the irreversibility of entanglement shown in [28].

**Proposition 13.** For any \( d \geq 3 \), the manipulation of the state \( \omega_d \) in the resource theory of entanglement is irreversible under all non-entangling operations. Specifically,

\[
E_{\text{c, O}_{\text{NE}}}(\omega_d) > E_{d, \text{O}_{\text{NE}}}(\omega_d).
\]

(121)

**Proof.** Applying Theorem 8, we obtain

\[
r(\omega_d \rightarrow \Phi_2) \leq \frac{\log \|\omega_d\|_{\text{FSEP}}}{\log \|\Phi_2\|_{\text{FSEP}}} = \log \frac{d}{d-1},
\]

(122)

where we observed that since \( \omega_d + \frac{1}{d-1} \Phi_d = \frac{1}{d-1} P_d \), it follows that \( R_{\text{FSEP}}^d(\omega_d) \leq \frac{1}{d-1} \). Incidentally, equality holds in this latter estimate, although we shall not need this observation here. For the lower bound, we use the base norm \( \|\cdot\|_{\text{FSEP}} \) and aim to bound the transformation rate \( r(\Phi_2 \rightarrow \omega_d) \) through Corollary 9. To this end, set

\[
W_d := \alpha_d P_d - \beta_d \Phi_d,
\]

(123)

where

\[
\alpha_d := \begin{cases} 2 & d = 3, \\ d & d \geq 4 \end{cases}, \quad \beta_d := \begin{cases} 3 & d = 3, \\ \frac{2d}{d-2} & d \geq 4. \end{cases}
\]

(124)

We obtain that

\[
\|W_d\|_{\infty} = \|\alpha_d P_d - \beta_d \Phi_d\|_{\infty} = \|\alpha_d P_d - \frac{\beta_d}{d} \Phi_d\|_{\infty} = \max \left\{ \frac{\alpha_d - \beta_d}{d}, \frac{\beta_d}{d} \right\} = 1,
\]

(125)

and moreover

\[
\langle W_d, \omega_d \rangle = \text{Tr} \left( (\alpha_d P_d - \beta_d \Phi_d) \frac{P_d - \Phi_d}{d-1} \right) = \alpha_d = \max \{ |\alpha_d|, |\alpha_d - \beta_d| \} = \|W_d\|_{\infty};
\]

(126)
putting all together, we see that
\[
E_{c, \text{ONE}}(\omega_d) \geq r(\Phi_2 \xrightarrow{\otimes_{\text{SEP}}} \omega_d)^{-1} \geq \log (W_d, \omega_d) = \log \alpha_d
\]

\[
> \log \frac{d}{d-1} \geq r(\omega_d \xrightarrow{\otimes_{\text{SEP}}} \Phi_2) \geq E_{d, \text{ONE}}(\omega_d),
\] (127)
as claimed. □

**Remark.** The proof technique in [28] (cf. (105)) uses in a similar way the norm \(\|(-)^T\|_1\) based on the partial transpose, instead of that based on the reshuffling criterion employed here. For the special state \(\omega_d\), both of these choices lead to equally tight (and optimal, as can be seen by following [28]) bounds.

7. APPLICATION: RESOURCE THEORY OF MAGIC

The free states in the resource theory of magic are the stabiliser states \(F := F_{\text{STAB}}\), composed of convex mixtures of pure states \(|\psi_U\rangle\langle\psi_U|\) generated as \(|\psi_U\rangle = U |0\rangle^\otimes n\) through the application of all Clifford unitaries \(U\). Any state which is not in \(F_{\text{STAB}}\) is called a magic state, and in particular the distillation of highly resourceful, pure magic states is a cornerstone of many fault-tolerant quantum computation subroutines [90]. The resource theory of magic was formalised in [38], where transformation rates were studied for the first time. The free operations of this theory are typically taken to be the stabiliser operations \(O_{\text{STAB}}\), built through Clifford gates, Pauli measurements, and preparations of ancillary states in the computational basis. The motivation for such a choice can be understood as the fact that the application of any operation in \(O_{\text{STAB}}\) can be efficiently simulated on a classical computer [91], while operations beyond \(O_{\text{STAB}}\) may require the costly magic states for their implementation. However, recent results showed that even larger classes of operations admit efficient classical simulation algorithms [80, 92–97], which could suggest that characterising the manipulation of magic states under larger types of operations may be of interest.

7.1. The case of qudits

The case of \(d\)-dimensional magic theory (i.e. defined for systems composed of \(n\) qudits, where \(d\) is an odd prime) is amenable to a particularly convenient characterisation, owing to the fact that the discrete Wigner function \([98]\) can be defined (see \([38, 40]\) for an overview). The set of states with a positive Wigner representation, denoted \(F_W\), then forms a useful approximation to the stabiliser states \(F_{\text{STAB}}\), and crucially it holds that any stabiliser protocol \(\Lambda\) satisfies \(\Lambda[F_W] \subseteq F_W [38]\), meaning that such protocols are also free operations with respect to the set \(F_W\) of states with a positive Wigner function. We then use this choice of free states to study the asymptotic properties of magic state transformations.

Let us hereafter focus on the case \(d = 3\). For any \((a_1, a_2) \in \mathbb{Z}_3 \times \mathbb{Z}_3\), the Heisenberg–Weyl operators are defined as
\[
T_{(a_1, a_2)} := \omega^{-2a_1a_2} Z^{a_1} X^{a_2},
\] (128)
where \(\omega := e^{2\pi i/3}\), and \(X\) and \(Z\) are the clock and shift operators, respectively. These operators are used to define the phase space point operators
\[
A_{(0,0)} := \frac{1}{3} \sum_{a_1, a_2 = 0}^{2} T_{(a_1, a_2)},
\]
\[
A_{(a_1, a_2)} := T_{(a_1, a_2)} A_{(0,0)} = A_{(a_1, a_2)}^{T},
\] (129)
which then allow us to define the discrete Wigner function $W_{a_1,a_2}$ as

$$W_{a_1,a_2}(Y) := \frac{1}{3} \langle A(a_1,a_2), Y \rangle.$$  \hfill (130)

For any state $\rho$, the Wigner representation $\{W_{a_1,a_2}(\rho)\}_{(a_1,a_2)\in\mathbb{Z}_3\times\mathbb{Z}_3}$ forms a quasi-probability distribution over $\mathbb{Z}_3 \times \mathbb{Z}_3$. The corresponding Wigner trace norm is given by

$$\|Y\|_W := \sum_{a_1,a_2=0}^2 |W_{a_1,a_2}(Y)|,$$  \hfill (131)

and the quantity $\log \|\rho\|_W$ has been dubbed the mana of a quantum state $\rho$ [38]. The free states are then defined as

$$\mathcal{F}_W := \left\{ \rho \in \mathcal{D}(\mathcal{H}) \mid W_{a_1,a_2}(\rho) \geq 0 \quad \forall (a_1,a_2) \in \mathbb{Z}_3 \times \mathbb{Z}_3 \right\} = \left\{ \rho \in \mathcal{D}(\mathcal{H}) \mid \|\rho\|_W = 1 \right\}.$$  \hfill (132)

The generalisation to many copies is straightforward: one defines the Heisenberg–Weyl operators as $T_{(a_1,1,a_1,2)\oplus\cdots\oplus(a_{n_1,1,a_{n_2,2})} := T_{(a_1,1,a_1,2)} \otimes \cdots \otimes T_{(a_{n_1,1,a_{n_2,2})}$, and the definitions of the phase space point operators and Wigner function are extended analogously, e.g.

$$W_{a_1,1,a_1,2,\ldots,a_n,1,a_n,2}(Y) = \frac{1}{3^n} \langle A(a_1,1,a_1,2)\oplus\cdots\oplus(a_n,1,a_n,2), Y \rangle.$$  \hfill (133)

Crucially, the Wigner trace norm is multiplicative: $\|X \otimes Y\|_W = \|X\|_W \|Y\|_W$ for any $X, Y$ [38]. Coupled with the fact that $\|X\|_W \leq \|X\|_{\mathcal{F}_{\text{STAB}}}$, this will allow us to employ the Wigner trace norm in the multiplicative norm bound on resource cost (Corollary 9).

In this resource theory, states such as the Strange state $|S\rangle = (|1\rangle - |2\rangle)/\sqrt{2}$ and the Norell state $|N\rangle = (-|0\rangle + 2|1\rangle - |2\rangle)/\sqrt{6}$ have received attention as ‘maximally magical’ states. In particular, Ref. [38] noticed that both of these states maximise the Wigner trace norm $\|\cdot\|_W$ among all qutrit states, and raised the question of whether the states are asymptotically equivalent, that is, whether $r(|S\rangle \otimes |N\rangle \langle N|) = r(|N\rangle \langle N| \rightarrow |S\rangle \langle S|) = 1$. Ref. [40] showed that this is not the case by proving that $r(|N\rangle \rightarrow |S\rangle) < 1$. However, this did not rule out the possibility that $|S\rangle$ can be transformed to $|N\rangle$ at a rate which makes the two state interconvertible in the asymptotic limit. An important question therefore remained open: is this resource theory asymptotically reversible?

7.2. Irreversibility of magic manipulation

Instead of the conversion between $|S\rangle$ and $|N\rangle$, we consider the transformation between the Norrel state $|N\rangle$ and the Hadamard ‘+’ state $|H_+\rangle$, defined as the $+1$ eigenstate of the Hadamard gate

$$H = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$  \hfill (134)

To upper bound the rate $r(|H_+\rangle \langle H_+| \rightarrow |N\rangle \langle N|)$ we use (95) with the choice of norm $\|\cdot\|_{g,\mathcal{F}_W}$ defined as in (93). In this case the bound reduces to that found in Ref. [40]:

$$r(|H_+\rangle \langle H_+| \rightarrow |N\rangle \langle N|) \leq \frac{\log (R_{\mathcal{F}_W}^g (|H_+\rangle \langle H_+|) + 1)}{-\log \|N\rangle \langle N\|_{\mathcal{F}_W}^g} = \frac{\log (3 - \sqrt{3})}{\log \frac{3}{2}} \approx 0.59,$$  \hfill (135)
where we used the known value $\| |N\rangle \langle N|^{\otimes n}\|_{F_W} = (\frac{7}{3})^n$ [40], whose multiplicativity implies that $L_{F_W,\infty}^\circ (|N\rangle \langle N|) = \log \| |N\rangle \langle N|\|_{F_W}^\circ$, and the known value of $R_{F_W}^\circ (|H_+\rangle \langle H_+|) = 2 - \sqrt{3}$ [40]. For the other direction, we would like to use our Corollary 9 (see also (96)). Introducing the tempered mana

$$W_t(\rho) := \inf_{\|Z\|_W \leq 1} D_0^\circ(\rho\|Z) = \log \max \left\{ \langle \rho, X \rangle \mid \|X\|_W^2 \leq 1, \|X\|_\infty = \text{Tr} \rho X \right\},$$

(136)

where $\|X\|_W^2 = \max_{a_1, a_2} d |W_{a_1, a_2}(X)|$ is known as the Wigner spectral norm, we can compute $W_t(|H_+\rangle \langle H_+|) = \log \frac{1}{2}(1 + 2\sqrt{3})$ (see Appendix A for details). However, the naive bound $L_{F_W,\infty}(\rho) \leq \log \|\rho\|_{F_W}$ now only gives

$$r(|N\rangle \langle N| \xrightarrow{O_{F_W}} |H_+\rangle \langle H_+|) \leq \log \frac{11}{3} \approx 1.74,$$

(137)

which altogether only lets us bound the rates as

$$r(|H_+\rangle \langle H_+| \xrightarrow{O_{F_W}} |N\rangle \langle N|) \ r(|N\rangle \langle N| \xrightarrow{O_{F_W}} |H_+\rangle \langle H_+|) \leq 1.02.$$  

(138)

We can see that the bound is almost good enough, but not quite sufficient to show irreversibility — we will need a slightly stronger bound here.

The advantage of the Wigner-based approach over an optimisation with respect to stabiliser states is that we can more easily evaluate $\| \cdot \|_{F_W}$ for many copies of states. For example, we compute (see Appendix A)

$$\| |N\rangle \langle N|^{\otimes 2}\|_{F_W} = \frac{11}{3} < 4 = \| |N\rangle \langle N|\|_{F_W}^2.$$

(139)

Then, we get a better bound on the regularisation of $\| \cdot \|_{F_W}$ as

$$L_{F_W,\infty}(\rho) \leq \frac{1}{2} \log \|\rho^{\otimes 2}\|_{F_W},$$

(140)

which yields

$$r(|N\rangle \langle N| \xrightarrow{O_{F_W}} |H_+\rangle \langle H_+|) \leq \frac{\frac{1}{2} \log \| |N\rangle \langle N|^{\otimes 2}\|_{F_W}}{W_t(|H_+\rangle \langle H_+|)} = \frac{\frac{1}{2} \log \frac{11}{3}}{\log(1 + 2\sqrt{3}) - \log 3} \approx 1.63.$$  

(141)

Combining with the reverse bound as above, we obtain

$$r(|H_+\rangle \langle H_+| \xrightarrow{O_{F_W}} |N\rangle \langle N|) \ r(|N\rangle \langle N| \xrightarrow{O_{F_W}} |H_+\rangle \langle H_+|) \leq \frac{\log \frac{11}{3} \log \left(3 - \sqrt{3}\right)}{\log \frac{9}{4} \log \frac{1 + 2\sqrt{3}}{3}} \leq 0.96.$$  

(142)

Asymptotic irreversibility is therefore established. Summing up, what we have shown is as follows.

**Theorem 14.** The resource theory of multi-qudit magic is asymptotically irreversible under any class of operations $O$ that preserves the set of states with positive Wigner function. Specifically, if $\sigma \in F_W \Rightarrow \Lambda(\sigma) \in F_W$ for all $\Lambda \in O$, then

$$r(|H_+\rangle \langle H_+| \xrightarrow{O} |N\rangle \langle N|) < r(|N\rangle \langle N| \xrightarrow{O} |H_+\rangle \langle H_+|)^{-1}.$$  

(143)
The result applies not only to stabiliser protocols, but also to the more general classes of completely stabiliser-preserving [94] and completely positive-Wigner-preserving maps [95], both of which are strictly larger than stabiliser operations [97].

We note that the irreversibility shown here is stronger than the irreversibility found in entanglement theory: the manipulation of pure entangled states is known to be reversible [4], and only for a class of rank-2 states could the irreversibility under non-entangling transformations be established [28]. Here in the theory of magic, on the contrary, we see that not even pure states can be reversibly manipulated.

We also stress that previously known bounds, including those based on the regularised relative entropy of magic $D^\infty_{\text{STAB}}$ [38], are not strong enough to show the irreversibility revealed by our results. To see this, note that [40]

$$D^\infty_{\text{STAB}}(|H_+\rangle\langle H_+|) = \log \left(3 - \sqrt{3}\right), \quad D^\infty_{\text{STAB}}(|N\rangle\langle N|) = \log \frac{3}{2},$$

from which we only get that [38]

$$r(|N\rangle\langle N| \xrightarrow{O_{\text{max}}} |H_+\rangle\langle H_+|) \leq \frac{\log \frac{3}{2}}{\log(3 - \sqrt{3})} \approx 1.71.$$  (145)

This is weaker than our bound in (141) and, in particular, insufficient to establish Theorem 14 — one can notice that (145) is just the inverse of Eq. (135).

7.3. The case of qubits

We have been unable to decisively show that irreversibility occurs also in the resource theory of magic for qubits. However, we present below some partial results and a conjecture whose validity would indeed rule out the reversibility of many-qubit magic.

The discrete Wigner function does not allow for a straightforward application to the qubit case that would recover the nice properties of the qudit resource theory based on $F_W$. However, a function conceptually similar to the Wigner negativity is the so-called stabiliser norm, defined for an $n$-qubit operator $X$ as [99]

$$\|X\|_{\mathcal{P}} := \frac{1}{2^n} \sum_{P \in \mathcal{P}} |\text{Tr}(X P)|,$$  (146)

where $\mathcal{P}$ denotes all $n$-qubit Pauli operators. Its dual norm can be obtained as

$$\|X\|_{\mathcal{P}^*} = \max_{P \in \mathcal{P}} |\text{Tr}(X P)|.$$  (147)

Both of the norms are not difficult to see to be multiplicative on tensor products.

Importantly, $\sigma \in F_{\text{STAB}} \Rightarrow \|\sigma\|_{\mathcal{P}} \leq 1$, which means that $\|X\|_{\mathcal{P}} \leq \|X\|_{F_{\text{STAB}}}$. The stabiliser norm approach does not always yield a good approximation for the set of stabiliser states: the set of $n$-qubit states with $\|\sigma\|_{\mathcal{P}} \leq 1$ is much larger than the set $F_{\text{STAB}}$ when $n > 1$ [96]. Nevertheless, we can use it to define the tempered stabiliser norm

$$P_\tau(\rho) := \inf_{\|Z\|_{\mathcal{P}} \leq 1} D^0_{\mathcal{P}}(\rho\|Z) = \log \max \left\{ \langle \rho, X \rangle \mid \|X\|_{\mathcal{P}} \leq 1, \|X\|_{\infty} = \langle \rho, X \rangle \right\}$$  (148)

and use the multiplicativity of $\|\cdot\|_{\mathcal{P}}$ to apply Corollary 9 and establish $P_\tau$ as a single-letter bound on transformation rates.
An important class of states in the characterisation of multi-qubit magic are the Hoggar states |Hog⟩ (see e.g. [39, 100, 101]). A Hoggar state is any 3-qubit state defined by taking a fiducial state, e.g.

|Hog⟩₀ \propto (-1 + 2i, 1, 1, 1, 1, 1, 1, 1)ᵀ

(149)

and defining |Hog⟩ as belonging to the orbit of |Hog⟩₀ under the Pauli group, i.e. |Hog⟩ = P |Hog⟩₀ for some P ∈ P. These states enjoy a very strong symmetry: it holds that

|⟨Hog|P|Hog⟩⟩ = \frac{1}{3}

(150)

for all non-trivial Paulis P ∈ P \{ 1 \}. Importantly, each Hoggar state satisfies the property that its standard robustness equals its generalised robustness [34, 39]:

R^s_{\text{stab}} (|Hog⟩⟨Hog|) + 1 = R^g_{\text{stab}} (|Hog⟩⟨Hog|) + 1 = \frac{1}{\||Hog⟩⟨Hog||^o_{\text{stab}}} = \frac{12}{5}.

(151)

Since the type of symmetry in Eq. (150) imposes a similar, rather symmetric structure on the tensor products of Hoggar states, one could reasonably expect that this will also lead to an equality between the robustness measures for many copies of this state.

**Conjecture 15.** The standard robustness equals the generalised robustness for many copies of Hoggar states:

R^s_{\text{stab}} (|Hog⟩⟨Hog|^o^n) = R^g_{\text{stab}} (|Hog⟩⟨Hog|^o^n) ∀n.

(152)

Establishing the conjecture would then directly lead to the following.

**Corollary 16.** If Conjecture 15 is true, then the resource theory of many-qubit magic is asymptotically irreversible under stabiliser-preserving operations. Specifically, for any class of operations σ such that σ ∈ P_{\text{stab}} ⇒ Λ(σ) ∈ P_{\text{stab}} ∀Λ ∈ σ, the single-qubit |T⟩ state satisfies

r(|T⟩⟨T|) < r(|Hog⟩⟨Hog|). (153)

**Proof.** The assumption that R^s_{\text{stab}} (|Hog⟩⟨Hog|^o^n) = R^g_{\text{stab}} (|Hog⟩⟨Hog|^o^n) tells us that R^s_{\text{stab}} + 1 is sub-multiplicative for the Hoggar state. This gives

L_{P_{\text{stab}},σ} (|Hog⟩⟨Hog|) ≤ \log(1 + R^s_{\text{stab}} (|Hog⟩⟨Hog|))) = \log \frac{12}{5}

(154)

where we used the known value of R^s_{\text{stab}} (|Hog⟩⟨Hog|)) [39]. Computing P, (|T⟩⟨T|) = \frac{1 + \sqrt{2}}{2} and applying Corollary 9 then gives

r(|Hog⟩⟨Hog|) ≤ \frac{\log \frac{12}{5}}{\log(1 + \sqrt{2}) - 1} ≈ 4.65.

(155)

On the other hand, using the fact that for any pure state of up to three qubits it holds that

\|Φ^o^n\|_{P_{\text{stab}}} = (\|Φ^o\|_{P_{\text{stab}}}^n) \leq [71], we can upper bound the reverse transformation rate using the norm \|\cdot\|_{\text{stab}} (which equals a bound previously shown in Ref. [80]):

r(|T⟩⟨T|) ≤ \frac{\log (R^g_{\text{stab}} (|T⟩⟨T|) + 1)}{-\log |||Hog⟩⟨Hog||^o_{\text{stab}}} = \frac{1 + \log(2 - \sqrt{2})}{\log \frac{12}{5}} \approx 0.18,

(156)

We note that in [39], the name ‘robustness of magic’ was used to refer to the base norm \|\cdot\|_{P_{\text{stab}}} itself; here it stands for R^g_{\text{stab}} (ρ) = \frac{1}{2} (\|ρ\|^2_{P_{\text{stab}}} - 1), for consistency with the terminology used in entanglement theory and other resources.
where we used that $R_{\text{STAB}}^\forall (|T\rangle\langle T|) + 1 = 2(2 - \sqrt{2})$ [71, 80] and $\|\text{Hog}\rangle\langle \text{Hog}\|_\text{F_{STAB}} = \frac{5}{12}$ [34]. Altogether,

$$r(|T\rangle\langle T| \xrightarrow{\text{O_{STAB}}} |\text{Hog}\rangle\langle \text{Hog}|) r(|\text{Hog}\rangle\langle \text{Hog}| \xrightarrow{\text{O_{sup}}} |T\rangle\langle T|) \leq \frac{1 + \log(2 - \sqrt{2})}{\log(1 + \sqrt{2}) - 1} \approx 0.84.$$ (157)

This concludes the proof. □

Of course, Conjecture 15 does not appear to be a necessary requirement for the irreversibility of many-qubit magic, and one could also expect that a transformation other than $|\text{Hog}\rangle\langle \text{Hog}| \leftrightarrow |T\rangle\langle T|$ could be used to establish the irreversibility of the theory. We only present Conjecture 15 as one possible way to approach this problem, motivated by the fact that the tempered stabiliser norm $P_t$ provides a strong bound for Hoggar state manipulation.

Even if the Conjecture itself is not found to be true, we hope that our discussion in this section motivates further research into the asymptotic manipulation of non-stabiliser states, as well as, more generally, into what is perhaps the most intriguing question prompted by our framework: what is it that makes a particular resource theory irreversible?

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Appendix A: Details of Wigner function computations

We first show Eq. (139), namely

\[ \|\langle N\rangle\langle N\rangle\|_{F_W}^2 = \frac{11}{3} < 4 = \|\langle N\rangle\langle N\rangle\|^2_{F_W}. \]  

(A1)

For the single copy of the state \( |N\rangle\langle N| \), we can bound \( \|\langle N\rangle\langle N\rangle\|_{F_W} \leq \|\langle N\rangle\langle N\rangle\|_{F_{STAB}} = 2 \), where the latter value was obtained in [34]. The opposite inequality follows from the fact that \( \|\langle N\rangle\langle N\rangle\|_{F_W} = 1 + 2R_{F_W}^s (\|\langle N\rangle\langle N\rangle\|) \geq 1 + 2R_{F_W}^s (\|\langle N\rangle\langle N\rangle\|) = 2 \), where the last equality was shown in [40].

Recall that, given the Heisenberg-Weyl operators \( T_{(a_1,a_2)} \) of a single-qutrit Hilbert space, the bipartite operators take the form \( T_{(a_1,a_2)b(b_1,b_2)} = T_{(a_1,a_2)} \otimes T_{(b_1,b_2)} \). The bipartite Wigner representation is then defined analogously as \( W_{(a_1,a_2),b(b_1,b_2)}(\rho) = \frac{1}{2} \text{Tr} \rho A_{(a_1,a_2)b(b_1,b_2)} \), where \( A_{(a_1,a_2)b(b_1,b_2)} = T_{(a_1,a_2)b(b_1,b_2)} A_{(0,0)b(0,0)} T_{(a_1,a_2)b(b_1,b_2)}^\dagger \) with \( A_{(0,0)b(0,0)} = \frac{1}{9} \sum_{a_1,a_2,b_1,b_2=0} A_{(a_1,a_2)b(b_1,b_2)}. \)

Consider now the state \( |N\rangle\langle N| \otimes |N\rangle\langle N| \), whose Wigner representation \( W_{a_1,a_2,b_1,b_2}(|N\rangle\langle N| \otimes |N\rangle\langle N|) \) takes the following values:
Define $X_+$ as the operator with the following Wigner representation:

| $(b_1, b_2)$ | $(a_1, a_2)$ | (0, 0) | (0, 1) | (0, 2) | (1, 0) | (1, 1) | (1, 2) | (2, 0) | (2, 1) | (2, 2) |
|--------------|--------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| (0, 0)       | 0            | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| (0, 1)       | 0.1          | 0.05   | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    |
| (0, 2)       | 0.1          | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    |
| (1, 0)       | 0.1          | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    |
| (1, 1)       | 0.1          | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    |
| (1, 2)       | 0.1          | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    |
| (2, 0)       | 0.1          | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    |
| (2, 1)       | 0.1          | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    |
| (2, 2)       | 0.1          | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    | 0.1    |

and the operator $X_-$ with the following Wigner representation:

| $(b_1, b_2)$ | $(a_1, a_2)$ | (0, 0) | (0, 1) | (0, 2) | (1, 0) | (1, 1) | (1, 2) | (2, 0) | (2, 1) | (2, 2) |
|--------------|--------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| (0, 0)       | 0            | 0.18   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   |
| (0, 1)       | 0.18         | 0.18   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   |
| (0, 2)       | 0.36         | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   |
| (1, 0)       | 0.36         | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   |
| (1, 1)       | 0.36         | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   |
| (1, 2)       | 0.36         | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   |
| (2, 0)       | 0.36         | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   |
| (2, 1)       | 0.36         | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   |
| (2, 2)       | 0.36         | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   | 0.36   |
It then holds that $|N\rangle\langle N|_{\otimes 2} = X_+ - X_-$ by construction. Furthermore, defining

$$|v_1\rangle := \frac{1}{\sqrt{10}}(1, 0, 1, -1, -2, -1, 1, 0, 1)^T$$

$$|v_2\rangle := \frac{1}{2}(0, 1, 0, -1, 0, 1, 0)^T$$

$$|v_3\rangle := \frac{1}{3}(1, 1, 1, 1, 1, 1, 1, 1, 1)^T$$

$$|v_4\rangle := \frac{1}{\sqrt{6}}(-1, 0, 1, -1, 0, 1, 0)^T$$

$$|v_5\rangle := \frac{1}{2\sqrt{3}}(0, -1, -2, 1, 0, -1, 2, 1, 0)^T$$

one can verify that

$$X_+ = |N\rangle\langle N|_{\otimes 2} + \frac{5}{12} |v_1\rangle\langle v_1| + \frac{5}{12} |v_2\rangle\langle v_2| + \frac{1}{3} |v_3\rangle\langle v_3| + \frac{1}{2} |v_4\rangle\langle v_4| + \frac{1}{12} |v_5\rangle\langle v_5|$$

$$X_- = \frac{5}{12} |v_1\rangle\langle v_1| + \frac{5}{12} |v_2\rangle\langle v_2| + \frac{1}{3} |v_3\rangle\langle v_3| + \frac{1}{2} |v_4\rangle\langle v_4| + \frac{1}{12} |v_5\rangle\langle v_5|,$$

which shows that $X_\pm$ are both positive semidefinite. As $X_\pm$ have a positive Wigner representation by construction, we get

$$\| |N\rangle\langle N|_{\otimes 2} \|_{W} \leq \mathrm{Tr} X_+ + \mathrm{Tr} X_- = \frac{11}{3}. \quad (A4)$$

This value can be verified to be optimal by employing the dual form of $\| \cdot \|_{W}$, but we do not need this for our argument.

We now show how to bound the value of $\mathcal{W}_t(|H_+\rangle\langle H_+|)$, which we used in Eq. (141). Recall that

$$\mathcal{W}_t(x) = \log \max \left\{ \langle \rho, X \rangle \mid \| X \|_W \leq 1, \|X\|_\infty = \mathrm{Tr} \rho X \right\}, \quad (A5)$$

where

$$\|X\|_W = \max_{a_1, a_2} 3 |W_{a_1, a_2}(X)|. \quad (A6)$$

Consider then the ansatz

$$X = \frac{1 + 2\sqrt{3}}{3} |H_+\rangle\langle H_+| - \frac{2\sqrt{3} - 1}{3} |H_-\rangle\langle H_-| - \frac{1}{3} |H_i\rangle\langle H_i|, \quad (A7)$$

where $|H_-\rangle$ and $|H_i\rangle$ are the eigenvectors of the Hadamard gate corresponding to the eigenvalues $-1$ and $+i$, respectively. Clearly, $\|X\|_\infty = \langle H_+|X|H_+\rangle = \frac{1 + 2\sqrt{3}}{3}$. Computing the Wigner representation of $X$, we can see that it takes the form:

$$\begin{array}{c|ccc}
 a_2 \\
 a_1 \hline
 0 & 1 & 2 \\
 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
 2 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\end{array}$$

Thus $\|X\|_W = 1$, and we conclude that

$$\mathcal{W}_t(|H_+\rangle\langle H_+|) \geq \log \langle H_+|X|H_+\rangle = \log \frac{1 + 2\sqrt{3}}{3}. \quad (A8)$$

A numerical evaluation can be used to confirm that this value is in fact optimal.