DIRECT CONSTRUCTION OF SYMMETRY-BREAKING DIRECTIONS IN BIFURCATION PROBLEMS WITH SPHERICAL SYMMETRY

SANJAY DHARMAVARAM
Department of Mathematics
1 Dent Dr, Bucknell University
Lewisburg, PA 17837, USA

TIMOTHY J. HEALEY
Department of Mathematics
Cornell University
Ithaca, NY 14853, USA

ABSTRACT. We consider bifurcation problems in the presence of $O(3)$ symmetry. Well known group-theoretic techniques enable the classification of all maximal isotropy subgroups of $O(3)$, with associated mode numbers $\ell \in \mathbb{N}$, leading to 1-dimensional fixed-point subspaces of the $(2\ell+1)$-dimensional space of spherical harmonics. In each case the so-called equivariant branching lemma can then be used to establish the existence of a local branch of bifurcating solutions having the symmetry of the respective subgroup. To first-order, such a branch is a precise linear combination of the $2\ell+1$ spherical harmonics, which we call the bifurcation direction. Our work here is focused on the direct construction of these bifurcation directions, complementing the above-mentioned classification. The approach is an application of a general method for constructing families of symmetric spherical harmonics, based on differentiating the fundamental solution of Laplace’s equation in $\mathbb{R}^3$.

1. Introduction. Steady bifurcation problems posed on spherical domains typically involve nonlinear elliptic equations that are equivariant under a representation of $O(3)$. Examples include the buckling of thin elastic shells [8], convection patterns in spherical layers [1], and two-phase patterns in lipid-bilayer structures [4], [5]. Local bifurcation then entails the analysis of reduced bifurcation equations posed on the $(2\ell+1)$-dimensional space of surface harmonics of order $\ell \in \mathbb{N}$. It has long been recognized that group-theoretic methods can be used to simplify an otherwise formidable analysis, cf. [12]. In particular, upon classification of appropriate subgroups, the existence of a plethora of local, symmetry-breaking bifurcations is readily established via the so-called equivariant branching lemma, cf. [3].

These valuable results are incomplete, however, in the sense that the actual post-critical modes, corresponding to local solutions of the nonlinear problem, remain to be determined. More specifically, while the linearized problem admits an entire

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* Corresponding author: Sanjay Dharmavaram.
span of $2\ell + 1$ surface harmonics at criticality, only precise linear combination correspond to first-order solutions of the nonlinear system. The classification in [3] gives the appropriate subgroups of $O(3)$ for a given $\ell \in \mathbb{N}$, but not the post-critical modes. The latter are useful, e.g., in understanding/visualizing symmetry-breaking behavior, for numerical bifurcation purposes, cf. [1], [4], [15], etc.

The determination of a post-critical mode ostensibly involves the $(2\ell + 1) \times (2\ell + 1)$ irreducible matrix representation of $O(3)$ - not just the trace, sufficient for dimension calculations, cf. [3]. Even for rather moderate values such of $\ell$, the size of the square matrix involved is large, rendering the process cumbersome at best. To illustrate the idea, we revisit the problem presented in [4], arising from the equilibrium conditions for the van der Waals/Cahn-Hilliard model on the unit sphere $\mathbb{S}^2$. The nonlinear governing equations read

$$\begin{align*}
-\epsilon \Delta u + \sigma(\lambda + u) - \frac{1}{4\pi} \int_{\mathbb{S}^2} \sigma(\lambda + u) \, ds &= 0 \text{ on } \mathbb{S}^2, \\
\int_{\mathbb{S}^2} u \, ds &= 0.
\end{align*}$$

subject to the constraint (2)

Here $\epsilon > 0$ is a fixed small (material) parameter, $\sigma = W'$, where $W : \mathbb{R} \to \mathbb{R}$ is a smooth 2-well potential, $\lambda \in \mathbb{R}$ is the bifurcation parameter, and $\lambda + u$ represents the total value of the phase field on $\mathbb{S}^2$. Following the development in [4], we associate (1) with that of finding the zeros of a mapping $F : \mathbb{R} \times X \to Y$, viz.,

$$F(\lambda, u) = 0,$$

where the Banach spaces

$$X := C^{2,\alpha}(\mathbb{S}^2) \cap Y,$$

$$Y := \left\{ u \in C^{0,\alpha}(\mathbb{S}^2) : \int_{\mathbb{S}^2} u \, ds = 0 \right\},$$

are equipped with the usual H"{o}lder norms ($0 < \alpha \leq 1$).

Its easy to see that $u \equiv 0$ satisfies (1) for all values of $\lambda$, i.e., $F(\lambda, 0) \equiv 0$. The rigorous linearization of (1) along the trivial line of solutions is then given by

$$A(\lambda)[h] := D_u F(\lambda, 0) = -\epsilon \Delta h + \sigma'(\lambda)h = 0 \text{ on } \mathbb{S}^2,$$

for $h \in X$, where $A(\lambda) : X \to Y$ is a Fredholm operator operator of index zero. Equation (4) admits nontrivial solutions if and only if $\lambda \in \mathbb{R}$ satisfies the characteristic equation

$$-\sigma'(\lambda)/\epsilon = \ell(\ell + 1) \text{ for } \ell \in \mathbb{N}.$$  

As shown in [4], (5) has roots for each $\ell \in \mathbb{N}$, denoted $\lambda = \lambda_\ell$, provided that $\epsilon > 0$ is sufficiently small. Corresponding to such a root, equation (4) admits a family of surface harmonics as nontrivial solutions:

$$h = Y_{\ell,m}(x), \quad m = -\ell, \ldots, \ell.$$  

In terms of spherical coordinates $x = (x, y, z) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$, these are defined by

$$\begin{align*}
Y_{\ell,0}(x) &= P_\ell(\cos \theta), \\
Y_{\ell,m}(x) &= N_{\ell,m} P_{\ell,m}(\cos \theta) \cos m\varphi, \quad m = 1, \ldots, \ell, \\
Y_{\ell,m}(x) &= N_{\ell,m} P_{\ell,-m}(\cos \theta) \sin m\varphi, \quad m = -\ell, \ldots, -1,
\end{align*}$$

where $N_{\ell,m}$ are normalization constants.
where $P_\ell$ denotes the Legendre polynomials and $P_{\ell,m}$ the associated Legendre polynomials, with $N_{\ell,m} = \sqrt{2(l+|m|)!/(l+|m|)}$, a normalization constant such that\(^1\)

(7) form an orthogonal set with respect to the $L^2$ inner-product on $S^2$. We define the space of surface harmonics of order $\ell \in \mathbb{N}$ via

$$V_\ell := \text{span}\{Y_{\ell,m} : -\ell \leq m \leq \ell\}. \quad (8)$$

In view of (4), observe that $A(\lambda_h)[h] = 0$ for all $h \in V_\ell$.

The natural action of $O(3)$ on functions in $X$ (or $Y$) is given by

$$\gamma u(x) := u(\Gamma^T x) \forall \Gamma \in O(3), \quad (9)$$

where $\Gamma$ is a $3 \times 3$ orthogonal matrix, and $x \in \mathbb{R}^3$ with $|x| = 1$. It easily follows from (1) that (3) is equivariant with respect to the natural action, viz.,

$$F(\lambda, \gamma u) = \gamma F(\lambda, u) \forall \Gamma \in O(3). \quad (10)$$

Let $G \subset O(3)$ be a subgroup, and define the linear subspace

$$X_G := \{u \in X : \gamma u = u \forall \Gamma \in G\}. \quad (11)$$

Then (10) implies that $F(\lambda, \cdot) : X_G \rightarrow Y_G$, where $Y_G$ is defined as in (11).

From the classification in ([3]; Thm. XIII.9.9), we are given the subgroups $G \subset O(3)$ and corresponding mode numbers $\ell \in \mathbb{N}$ such that (3) restricted to $\mathbb{R} \times X_G$ yields a standard 1-dimensional bifurcation problem, i.e., $\dim(\text{Null}(A_\ell)|_{X_G}) = 1$, cf.(4). In this work we are concerned with finding the precise linear combination,

$$Y^*_\ell := \sum_{-\ell \leq m \leq \ell} c_m Y_{\ell,m}, \quad (12)$$

such that $Y^*_\ell \in X_G$, where the coefficients, $c_m, -\ell \leq m \leq \ell$, are to be determined.

Let $T_\ell(\Gamma)$ denote the $(2\ell + 1) \times (2\ell + 1)$ irreducible matrix representation of $O(3)$, defined by

$$\sum_{-\ell \leq m \leq \ell} \sum_{-\ell \leq j \leq \ell} [T_\ell(\Gamma)]_{mj} c_j Y_{\ell,m}(x) : = \sum_{-\ell \leq m \leq \ell} c_m Y_{\ell,m}(\Gamma^T x) \forall \Gamma \in O(3), \quad (13)$$

cf. [14]. Then (9), (11), (12) and (13) yield,

$$Y^*_\ell \in X_G \iff T_\ell(\Gamma)c = c \forall \Gamma \in G, \quad (14)$$

where $c = (c_{-\ell}, c_{-\ell+1}, \cdots, c_{\ell}) \in \mathbb{R}^{2\ell+1}$.

Of course it is enough to enforce the right side of (4) for the generators of $G$, suggesting a rather clumsy approach to “searching” for $c \in \mathbb{R}^{2\ell+1}$. More systematically, the average of $T_\ell$ over the subgroup $G$ yields a projection onto a 1-dimensional subspace of $\mathbb{R}^{2\ell+1}$, any non-zero element of which serves as $c$. However, this requires assembling the representation $T_\ell$ explicitly, which is an inconvenient if not formidable task for even moderate values of $\ell \in \mathbb{N}$. In this work we present a direct and elegant alternative to the right side of (14), based on an old observation of Maxwell, i.e., that any spherical harmonic can be realized as a sequence of directional derivatives acting on the fundamental solution of Laplace’s equation in $\mathbb{R}^3$, viz., $1/r$, where $r^2 = x^2 + y^2 + z^2$. It was first recognized by Poole [11], and later by Hodgkinson [7], that complete families of spherical harmonics invariant with respect to the symmetry of a Platonic solid can be constructed by such a procedure. Later Meyer

\(^1\)We choose the inner-product $\int_{S^2} Y_{\ell,m} Y^*_{\ell',m'} \, da = \frac{4\pi}{2\ell' + 1} \delta_{\ell\ell'} \delta_{m m'}$, for convenience.
presented a rigorous, systematic approach to that construction accounting for all possible subgroups of $O(3)$. Here we merely specialize the approach [10] to the subspaces $V_t$, cf. (8) corresponding to any subgroup $G \subset O(3)$. In particular, for the subgroups catalogued in [3], we can obtain any bifurcation mode $Y^\gamma_t \in X_G$ by successive differentiations. For subgroups outside of those in [3] the same method can be used to construct bases for $\text{Null } A(\lambda_t)|_{X_G}$.

The outline of this work is as follows: In Section 2 we consider the tetrahedral subgroup $T$ and provide the details for constructing $T$-invariant spherical harmonics via directional derivatives of $1/r$. In addition we provide a recipe for expressing these directional derivatives in terms of standard spherical harmonics. In Section 3 we consider all other finite subgroups of $O(3)$. In Table 1, based on the work of Meyer [10], we summarizes the invariant spherical harmonic basis functions tabulated by their respective subgroups. We provide several examples in Section 4, mostly illustrating the utility of the method for constructing bifurcation directions, as described above. We pick one or two examples from each of the discrete subgroups listed in Theorem 9.9 of [3], constructing the bifurcation direction and illustrating its nodal pattern in each case.

2. Constructing symmetric spherical harmonics.

2.1. Spherical harmonics as directional derivatives. Let us first recall that if $y(x)$ is a harmonic function, i.e., $\Delta y(x) = 0$ on $\mathbb{R}^3$, where $\Delta$ is the Laplacian on $\mathbb{R}^3$, then $y(x)|_{r=1}$ is a surface harmonic. Maxwell’s insight on the connection between spherical harmonics and directional derivatives follows from the simple observation that $1/r$ is a harmonic function on $\mathbb{R}^3 - \{0\}$,

$$\Delta(1/r) = 0. \quad (15)$$

By taking the directional derivative of (15) in the direction $a := (a_x, a_y, a_z) \in \mathbb{R}^3$, we obtain

$$\Delta [D_a(1/r)] = 0, \quad (16)$$

where $D_a(\cdot) := [a_x \partial_x + a_y \partial_y + a_z \partial_z](\cdot)$ is the directional derivative along $a$. Therefore, $D_a(1/r)|_{r=1}$ is a spherical harmonic. Similarly, by taking higher directional derivatives of (15) along $a_1, \ldots, a_\ell \in \mathbb{R}^3, \ell \in \mathbb{N}$, we obtain

$$\Delta [D_{a_1} D_{a_2} \cdots D_{a_\ell}(1/r)] = 0. \quad (17)$$

Thus, $D_{a_1} D_{a_2} \cdots D_{a_\ell}(1/r)|_{r=1}$ is a spherical harmonic of order $\ell$. The converse, that every spherical harmonic of order $\ell$ can be uniquely written as a product of directional derivatives of $1/r$ (up to reordering) is called Sylvester’s theorem [13]. This relation between directional derivatives and spherical harmonics can be used to construct spherical harmonics that are invariant under the action of a subgroup of $G \subset O(3)$, e.g., $Y^\gamma_t \in X_G$, cf. (12).

The three dimensional representation corresponding to $\ell = 1$ is precisely $O(3)$, i.e., $T_1 \equiv O(3)$. Let $\Gamma \in G$, where $G \subset O(3)$ is a subgroup. In view of (9), observe that $\gamma(1/r) = (1/r)$, and since $D_a(1/r) = -a \cdot x/r^3$, it follows that $\gamma(D_a(1/r)) = -(a \cdot \Gamma^T x)/r^3 = D_{\Gamma a}(1/r)$. More generally, we have

$$\gamma \left[ D_{a_1} \cdots D_{a_\ell} \left( \frac{1}{r} \right) \right] = \left[ D_{\Gamma a_1} \cdots D_{\Gamma a_\ell} \left( \frac{1}{r} \right) \right]. \quad (18)$$
It follows from (18) that by choosing directions \( a_1, \ldots, a_\ell \) such that
\[
D_{\Gamma a_1} \cdots D_{\Gamma a_\ell} \left( \frac{1}{r} \right) = D_{a_1} \cdots D_{a_\ell} \left( \frac{1}{r} \right),
\]
we can construct spherical harmonics that are invariant under the action of \( G \). For axi-symmetric spherical harmonics, the obvious choice for directional derivatives is along the axis of symmetry. For polyhedral subgroups of \( O(3) \) the vectors may be chosen to lie along symmetry directions of the corresponding regular polyhedron [10], [11].

2.2. Constructing invariant spherical harmonics: Tetrahedral case. As an illustration of the utility of (19), we first consider the tetrahedral group \( T \subset O(3) \), viz., the proper-rotational symmetry group of a regular tetrahedron, to obtain \( T \)-invariant spherical harmonics. For a tetrahedron oriented as depicted in Figure 1, the matrices
\[
S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
\]
generate the group. Consider the operator, \( \mathcal{T}_3 : C^\infty(\mathbb{R}^3 - \{0\}) \to C^\infty(\mathbb{R}^3 - \{0\}) \), defined by
\[
\mathcal{T}_3(\cdot) := D_k D_j D_i (\cdot),
\]
where \( i, j, k \) are the cartesian basis vectors of \( \mathbb{R}^3 \) (pointing along \( x, y \) and \( z \) axes, resp.). Using (18) and \( Si = -i, Sj = -j, Sk = k, Ti = k, Tj = k, Tk = j \), it follows that
\[
S \left[ \mathcal{T}_3(1/r) \right] = (D_k)(-D_j)(-D_i)(1/r) = \mathcal{T}_3(1/r),
\]
\[
T \left[ \mathcal{T}_3(1/r) \right] = (D_i)(D_k)(D_j)(1/r) = \mathcal{T}_3(1/r).
\]
Therefore, \( \mathcal{T}_3(1/r)|_{r=1} \) is a \( T \)-invariant spherical harmonic. Straightforward computation yields
\[
\mathcal{T}_3(1/r) = -15(xyz)/(x^2 + y^2 + z^2)^{7/2},
\]
which, when evaluated at \( r = 1 \) and expressed in spherical coordinates results in

\[
\begin{align*}
\mathcal{T}_3(1/r)|_{r=1} &= -\sqrt{15}Y_{3,-2}(\theta, \varphi), \\
\end{align*}
\]

where the numerical prefactor of \( Y_{3,-2} \) in (25) results form the normalization constant, cf. (7). The function (25) up to an arbitrary amplitude is plotted in the Fig. (2).

Note that the choice of derivative directions in (21) can be visualized as the vectors pointing to the mid point of three edges of the tetrahedron, shown in Figure (1). Other invariant spherical harmonics can be produced by different choices. Two specific choices are particularly relevant for the discussion below: (i) vectors pointing to the four vertices of the tetrahedron, and (ii) six vectors pointing to the mid point of the edges of the inscribing cube cf. Fig. (1). The corresponding operators, denoted \( \mathcal{T}_4, \mathcal{T}_6 : C^\infty(\mathbb{R}^3 - \{0\}) \to C^\infty(\mathbb{R}^3 - \{0\}) \) are given by

\[
\begin{align*}
\mathcal{T}_4 &:= (D_i + D_j + D_k)(-D_i + D_j + D_k)(D_i - D_j + D_k)(D_i + D_j - D_k), \\
\mathcal{T}_6 &:= (D_i^2 - D_j^2)(D_j^2 - D_k^2)(D_k^2 - D_i^2).
\end{align*}
\]

It is easy to verify (following (22) and (23)) that \( \mathcal{T}_4(1/r)|_{r=1} \) and \( \mathcal{T}_6(1/r)|_{r=1} \) are \( \mathbb{T} \)-invariant spherical harmonics. The subscripts in (21), (26) and (27) remind the reader the order of derivatives, which is equivalent to the order of spherical harmonics.

Note that the operators (21), (26) and (27) can be mapped to polynomials by the formal identification:

\[
D_i \leftrightarrow x, \quad D_j \leftrightarrow y, \quad D_k \leftrightarrow z.
\]

This connection between operators and polynomials is convenient, and we henceforth employ the following notation:

\[
D_i := \hat{x}, \quad D_j := \hat{y}, \quad D_k := \hat{z}.
\]

Accordingly, (21), (26) and (27) now read

\[
\begin{align*}
\mathcal{T}_3 &= \hat{z}\hat{y}\hat{x}, \\
\mathcal{T}_4 &= (\hat{x} + \hat{y} + \hat{z})(-\hat{x} + \hat{y} + \hat{z})(\hat{x} - \hat{y} + \hat{z})(\hat{x} + \hat{y} - \hat{z}),
\end{align*}
\]
\[ T_6 = (x^2 - y^2)(y^2 - z^2)(z^2 - x^2). \quad (32) \]

Since the choice of directional derivatives leading to a given spherical harmonic is not necessarily unique, multiple copies of the same set of vectors can be used to produce higher order spherical harmonics. For instance, \( T_6^4(1/r)|_{r=1}, T_4^2(1/r)|_{r=1}, T_6(1/r)|_{r=1} \) \( (p, q \text{ and } s \text{ are non-negative integers, with superscripts denoting the power of the operator}) \) are spherical harmonics of order \( 3p, 4q \text{ and } 6s \) respectively. More generally, these operators may be combined as linear combination of \( T_6^s T_4^q T_6^p(1/r)|_{r=1} \) such that \( 6s + 4q + 3p = \ell \) to obtain \( T \)-invariant spherical harmonic of order \( \ell \). The operator corresponding to this linear combination can be interpreted as a homogeneous polynomial of degree \( \ell \) in \( T_3, T_4 \) and \( T_6 \), formally interpreted as an operator. Thus, for every homogeneous polynomial \( P \in H_\ell \) (where \( H_\ell \) denotes the module of homogeneous polynomial of degree \( \ell \) in \( T_3, T_4 \) and \( T_6 \)) a polynomial operator \( P \mapsto \mathcal{P} \) can be defined via \( (28) \). We say \( \mathcal{P} \) is \emph{homogeneous polynomial operator} when \( P \) is a homogeneous polynomial. It follows from the discussion above that the function \( \mathcal{P}(1/r)|_{r=1} \) is a \( T \)-invariant spherical harmonic. We now discuss the converse result \cite{10}.

Let us first note that if \( P_1 \) and \( P_2 \) are operators constructed from homogeneous polynomials \( P_1, P_2 \in H_\ell \), respectively such that \( P_1 \neq P_2 \), then it does not necessarily follow that \( P_1(1/r)|_{r=1} \neq P_2(1/r)|_{r=1} \). This is because some polynomial operators are trivially zero, since \( (x^2 + y^2 + z^2)(1/r) = \Delta(1/r) \equiv 0 \). However, if \( P_1 \neq P_2(\text{mod } x^2 + y^2 + z^2) \), then indeed \( P_1(1/r)|_{r=1} \neq P_2(1/r)|_{r=1} \). Therefore, only homogeneous polynomials modulo \( x^2 + y^2 + z^2 \) in \( H_\ell \) lead to distinct spherical harmonics. For polynomial operators \( (30)-(32) \), it can be verified (computer algebra may be used for this purpose) that

\[ T_6^2 = \frac{1}{16} T_4^4 - 27 T_4^3 \mod (x^2 + y^2 + z^2). \quad (33) \]

Thus, the powers of operator \( T_6 \) beyond 1 can be rewritten in terms of \( T_3 \) and \( T_4 \). Based on this observation it can be shown (cf. [10] for details) that

\[ \left\{ T_6^s T_4^q T_6^p(1/r)|_{r=1}, 3p + 4q + 6s = \ell, \ p, q \in \mathbb{N}, \ s \in \{0, 1\} \right\} \]

forms a basis for \( T \)-invariant spherical harmonic of order \( \ell \).

2.3. **Expressing in terms of standard spherical harmonics.** We now discuss how spherical harmonics written using directional derivatives, discussed above, can be expressed in terms of the standard spherical harmonics, cf. \cite{7}. Let us define new operators,

\[ \xi := \left( \hat{x} - i\hat{y} \right), \quad (35a) \]

\[ \eta := \left( \hat{x} + i\hat{y} \right), \quad (35b) \]

where \( i^2 = -1 \). These are motivated by the following identities that relate directional derivatives to standard spherical harmonics \cite{6}:

\[ \hat{z}^\ell \xi^m (\xi + \eta^n) \left( \frac{1}{r} \right) \bigg|_{r=1} = (-1)^{\ell-m} \sqrt{2(\ell-m)!} \ell! |Y_{\ell,m}|, \quad (36a) \]

\[ \hat{z}^\ell \left( \frac{1}{r} \right) \bigg|_{r=1} = (-1)^\ell \ell! |Y_{\ell,0}|, \quad (36b) \]

\[ i\hat{z}^\ell \xi^m (\xi - \eta^n) \left( \frac{1}{r} \right) \bigg|_{r=1} = (-1)^{\ell-m} \sqrt{2(\ell-m)!} \ell! |Y_{\ell,-|m|}|, \quad (36c) \]

Using (35) and the identity \((\hat{x}^2 + \hat{y}^2 + \hat{z}^2) = 0\), we obtain the relation

\[ \hat{x}\hat{y} = -\hat{z}^2. \]  

(37)

We now show that every \(T\)-invariant spherical harmonic, previously expressed in (34) as a linear combination of directional derivatives, can be expressed as a linear combination of standard spherical harmonics. First note that since \(T_3, T_4\) and \(T_6\) are each homogeous polynomial operators in \(\hat{x}, \hat{y}\) and \(\hat{z}\), any operator that generates a \(T\)-invariant spherical harmonic of order \(\ell\), as in (34), can be written as

\[ \mathcal{P} = \sum_{m,n,p} a_{mnp}\hat{x}^m\hat{y}^n\hat{z}^p, \]  

(38)

where \(a_{mnp} \in \mathbb{Z}\) and \(m, n, p \in \mathbb{N}\) such that \(m + n + p = \ell\). Substituting \(\hat{x} = (\hat{\xi} + \hat{\eta})/2, \hat{y} = i(\hat{\xi} - \hat{\eta})/2\), we obtain

\[ \mathcal{P} = \sum_{m,n,p} a_{mnp}\frac{(i)^n}{2^{m+n}}\hat{z}^p(\hat{\xi} + \hat{\eta})^m(\hat{\xi} - \hat{\eta})^n. \]  

(39)

Now observe that every term in \((\hat{\xi} + \hat{\eta})^m(\hat{\xi} - \hat{\eta})^n\) can be expressed as \(\hat{\xi}^j\hat{\eta}^k\), such that \(j + k = m + n\). Also, if \(\alpha_{jk} \in \mathbb{Z}\) is the coefficient of term \(\hat{\xi}^j\hat{\eta}^k\), then it is clear that the coefficient of \(\hat{\xi}^j\hat{\eta}^k\) is \((-1)^n\alpha_{jk}\). We can then rewrite (39) as

\[ \mathcal{P} = \sum_{m,n,p,j,k} a_{mnp}\frac{(i)^n}{2^{m+n}}\hat{z}^p\alpha_{jk} \left[\hat{\xi}^j\hat{\eta}^k + (-1)^n\hat{\xi}^j\hat{\eta}^k\right], \]  

(40)

where the summation involving \(j\) and \(k\) are such that \(j, k \in \mathbb{N}\), \(j + k = m + n\). Define \(j' = \min\{j, k\}\). By factoring \((\hat{\xi}\hat{\eta})^{j'}\) from the term in brackets of (40) and using (37), we obtain

\[ \mathcal{P} = \sum_{m,n,p,j,k} a'_{mnp,j,k}(i)^n(-1)^{j'}\hat{z}^{j'+2j'} \left[\hat{\xi}^{m+n-2j'} + (-1)^n\hat{\eta}^{m+n-2j'}\right], \]  

(41)

where \(a'_{mnp,j,k} := a_{mnp}\alpha_{jk}/2^{m+n}\). Note that every term in (41) has the form \(\hat{z}^u[\hat{\xi}^v \pm \hat{\eta}^v]\) for some \(u, v \in \mathbb{N}\). Thus, it follows from (36) that any \(T\)-invariant spherical harmonic constructed using operator \(\mathcal{P}\) can be written in terms of standard spherical harmonics (7).

We now apply this to operators \(T_3, T_4\) and \(T_6\), (cf. (30)-(32)). Straightforward calculations using (35) and (37) yield

\[ T_3 = \frac{i}{4}\hat{z}(\hat{\xi}^2 - \hat{\eta}^2), \]  

(42)

\[ T_4 = \frac{1}{4}[14\hat{\xi}^4 + (\hat{\xi}^4 + \hat{\eta}^4)], \]  

(43)

\[ T_6 = \frac{1}{32}\left[(\hat{\xi}^6 + \hat{\eta}^6) - 33\hat{\xi}^4(\hat{\xi}^2 + \hat{\eta}^2)\right]. \]  

(44)

Indeed, using (36) we verify (cf. (25)) that

\[ T_3(1/r)|_{r=1} = -\sqrt{15}Y_{3,-2}, \]  

(45)

Similarly, applying (36) to (43) and (44), we obtain

\[ T_4(1/r)|_{r=1} = 84Y_{4,0} + 12\sqrt{35}Y_{4,4}, \]  

(46)

\[ T_6(1/r)|_{r=1} = 45\sqrt{462}Y_{6,6} - 99\sqrt{210}Y_{6,2}. \]  

(47)
3. Operators for finite subgroups of $O(3)$. The procedure discussed in section (2.2) can be similarly applied to other subgroups of $O(3)$ to obtain homogeneous polynomial operators which generate corresponding invariant spherical harmonics. These operators can be obtained by choosing a set of vectors in $\mathbb{R}^3$ satisfying property (19) with respect to the corresponding subgroup. Conversely, it can be shown (cf. [10]) that for every subgroup $G$ of $O(3)$, there are a finite number of homogeneous polynomial operators $G_i$, where $i \in I$, (were $I$ is a finite index set; $i$ denotes the degree of the operator) such that every $G$-invariant spherical harmonic can be written as a linear combination of basis functions of the form $[\prod_{i \in I} G_i](1/r)_{|r|=1}$. Here, $I^N = I \times \cdots \times I$ ($N$ times).

Table 1. Subgroups of $O(3)$ and their invariant spherical harmonic basis. Here $s \in \{0, 1\}, p, q \in \mathbb{N} \cup \{0\}$.

| Group       | Invariant Spherical Harmonic Basis | Order |
|-------------|-----------------------------------|-------|
| $T$         | $T \times T \times T \times (1/r)|_{|r|=1}$ | $6s + 4p + 3q$ |
| $O$         | $C_2 T \times T \times (1/r)|_{|r|=1}$ | $9s + 6p + 4q$ |
| $I$         | $T \times T \times T \times (1/r)|_{|r|=1}$ | $15s + 10p + 6q$ |
| $T \times Z_2$ | $T \times T \times (1/r)|_{|r|=1}$ | $6s + 6p + 4q$ |
| $O \times Z_2$ | $C_2 T \times (1/r)|_{|r|=1}$ | $6p + 4q$ |
| $I \times Z_2$ | $T \times T \times (1/r)|_{|r|=1}$ | $10p + 6q$ |
| $O^{-}$     | $T \times T \times (1/r)|_{|r|=1}$ | $4p + 3q$ |
| $Z_n$       | $\mathcal{Z}^n C_{(1/r)|_{|r|=1}}, \mathcal{Z}^n S_{(1/r)|_{|r|=1}}$ | $p + qn$ |
| $D_n$       | $\mathcal{Z}^n C_{(1/r)|_{|r|=1}}, \mathcal{Z}^n S_{(1/r)|_{|r|=1}}$ | $2p + qn, 2p + 1 + qn$ (resp.) |
| $D^e_n$     | $\mathcal{Z}^n C_{(1/r)|_{|r|=1}}, \mathcal{Z}^n S_{(1/r)|_{|r|=1}}$ | $p + qn$ |
| $Z_{n+1} \oplus Z_{n+2}$ (odd $n$) | $\mathcal{Z}^{n+1} C_{(1/r)|_{|r|=1}}, \mathcal{Z}^{n+1} S_{(1/r)|_{|r|=1}}$ | $2p + j + qn$ |
| $Z_{n+1} \oplus Z_{n+2}$ (even $n$) | $\mathcal{Z}^{n} C_{(1/r)|_{|r|=1}}, \mathcal{Z}^{n} S_{(1/r)|_{|r|=1}}$ | $2p + qn$ |
| $D^e_{n+1}$ (even $n$), $D^o_{n+1}$ (odd $n$) | $\mathcal{Z}^{n+1} C_{(1/r)|_{|r|=1}}, \mathcal{Z}^{n+1} S_{(1/r)|_{|r|=1}}$ | $2p + 2q, 2p + 1 + (2q + 1)n$ (resp.) |
| $D^e_{n+1}$ (odd $n$), $D^o_{n+1}$ (even $n$) | $\mathcal{Z}^{n+1} C_{(1/r)|_{|r|=1}}, \mathcal{Z}^{n+1} S_{(1/r)|_{|r|=1}}$ | $2p + qn$ |

Table 1 shows the finite subgroups of $O(3)$ (employing the notation of [3]) along with their respective functions that span the basis of the corresponding invariant spherical harmonic subspace, where $p, q, j \in \mathbb{N} \cup \{0\}$ and $s \in \{0, 1\}$. The order of the spherical harmonic basis function shown in the second column is given in the last column. Note that some subgroups (rows 8, 9, 11, 12, 13 of Table 1) have invariant basis functions generated by both $C$ and $S$ operators, as seen in the second column. For these cases, the two entries on the third column represent the order of spherical harmonic functions for the two respective cases. Also, some subgroups (last four rows of Table 1) have the same basis functions, seen in the second column, depending on if $n$ is even or odd. The operators that appear in the table are given by

\[
\mathcal{O}_4 := 14\hat{z}^4 + (\hat{\xi}^4 + \hat{\eta}^4), \\
\mathcal{O}_6 := \hat{z}^2(\hat{\xi}^4 + \hat{\eta}^4) - 2\hat{z}^6, \\
\mathcal{O}_9 := i\left[\hat{z}(\hat{\xi}^8 - \hat{\eta}^8) - 74\hat{z}^5(\hat{\xi}^4 - \hat{\eta}^4)\right], \\
\mathcal{I}_6 := 11\hat{z}^6 + \hat{z}(\hat{\xi}^6 + \hat{\eta}^6), \\
\mathcal{I}_{10} := 494\hat{z}^{10} - 228\hat{z}^5(\hat{\xi}^5 + \hat{\eta}^5) + (\hat{\xi}^{10} + \hat{\eta}^{10}).
\]
Figure 3. $O \oplus Z_2^c$-invariant spherical harmonic for (a) $\ell = 4$ and (b) $\ell = 6$.

\begin{align*}
I_{15} := i \left( \tilde{\xi}^{15} - \tilde{\eta}^{15} + 522 \tilde{z}^{10}(\tilde{\xi}^{10} - \tilde{\eta}^{10}) - 10005 \tilde{z}^{5}(\tilde{\xi}^{5} - \tilde{\eta}^{5}) \right) \\
C_n := (\tilde{\xi}^n + \tilde{\eta}^n) \\
S_n := i(\tilde{\xi}^n - \tilde{\eta}^n)
\end{align*}

These operators were obtained by expressing the ones considered in [10] in terms of (35). It is clear from (36) and the discussion in Section 2.3 that the basis functions of Table 1 can be expanded in terms of standard surface spherical harmonics.

Observe that for a given $\ell \in \mathbb{N}$, the dimension of the invariant subspace of spherical harmonics (with respect to a given group) is determined by the number of solutions to linear Diophantine equation obtained by setting the third column equal to $\ell$.

4. Examples. We now present several examples, mostly illustrating the utility of the method for constructing bifurcation directions. In each case we use (36) to translate the operator representation for invariant spherical harmonics that appear in Table 1 to their corresponding standard spherical harmonic representation. We pick one or two examples from each of the discrete subgroups listed in Theorem 9.9 of [3]. In each case, we give the determining equation from Table 1, the explicit bifurcation direction (12), and a figure illustrating its nodal character.

Example 1. $O \oplus Z_2^c$, $\ell = 4$:

Table 1, row 5: $6p + 4q = 4 \Rightarrow p = 0, q = 1$.

Invariant bifurcation direction:

\[ Y^*_4 = O_4(1/r)|_{r=1} = \left[ 14 \tilde{z}^4(1/r) + (\tilde{\xi}^4 + \tilde{\eta}^4) \right](1/r)|_{r=1} = 336Y_{4,0} + 48\sqrt{35}Y_{4,4}, \]

cf. Figure 3(a).

Example 2. $O \oplus Z_2^c$, $\ell = 6$:

Table 1, row 5: $6p + 4q = 6 \Rightarrow p = 1, q = 0$.

Invariant bifurcation direction:

\[ Y^*_6 = O_6(1/r)|_{r=1} = \left[ \tilde{z}^2(\tilde{\xi}^4 + \tilde{\eta}^4)(1/r) - 2\tilde{z}^6(1/r) \right]|_{r=1} \]
\[
\sqrt{2 \cdot 2! \cdot 10! Y_{6,4}} - 2 \cdot 6! Y_{6,0},
\]
cf. Figure 3(b).

**Figure 4.** (a) \(\mathcal{O}\)-invariant spherical harmonic for \(\ell = 9\); (b) \(\mathcal{O}^-\)-invariant spherical harmonic for \(\ell = 9\).

**Example 3.** \(\mathcal{O}\), \(\ell = 9\):
Table 1, row 2: \(9 + 6p + 4q = 9 \Rightarrow p = 0, q = 0\).
Invariant bifurcation direction:
\[
Y_g^* = \mathcal{O}_9(1/r)|_{r=1} = i \left[ \hat{z}(\hat{\xi}^8 - \hat{\eta}^8)(1/r) - 34 \hat{z}^3(\hat{\xi}^4 - \hat{\eta}^4)(1/r) \right] |_{r=1} 
= -\sqrt{2 \cdot 17!} Y_{9,-8} + 34\sqrt{2 \cdot 5! \cdot 13!} Y_{9,-4},
\]
cf. Figure 4(a).

**Example 4.** \(\mathcal{O}^-, \ell = 3\):
Table 1, row 7: \(4p + 3q = 3 \Rightarrow p = 0, q = 1\).
Invariant bifurcation direction:
\[
Y_3^* = T_3(1/r)|_{r=1} = -\sqrt{15} Y_{3,-2},
\]
which is the same as (25), i.e., it is also invariant under \(T\), cf. Figure 2.

**Example 5.** \(\mathcal{O}^-, \ell = 9\):
Table 1, row 7: \(4p + 3q = 9 \Rightarrow p = 0, q = 3\).
Invariant bifurcation direction:
\[
Y_9^* = T_3^2(1/r)|_{r=1} = -\frac{1}{64}\left[ \sqrt{2 \cdot 3! \cdot 15!} Y_{9,-6} - 3\sqrt{2 \cdot 7! \cdot 11!} Y_{9,-2} \right],
\]
cf. Figure 4(b).
Figure 5. $I \oplus Z_2^c$-invariant basis functions for (a) $\ell = 6$ and (b) $\ell = 10$.

Example 6. $I \oplus Z_2^c$, $\ell = 6$:
Table 1, row 6: $10p + 6q = 6 \Rightarrow p = 0$, $q = 1$.
Invariant bifurcation direction:
\[
Y_6^* = \mathcal{I}_6(1/r)|_{r=1} = \left[11\hat{z}^6(1/r) + \hat{z}(\hat{\xi}^5 + \hat{\eta}^5)(1/r)\right]|_{r=1}
= 11 \cdot 6! Y_{6,0} - \sqrt{2} \cdot 11! Y_{6,5},
\]

Example 7. $I \oplus Z_2^c$, $\ell = 10$:
Table 1, row 6: $10p + 6q = 10 \Rightarrow p = 1$, $q = 0$.
Invariant bifurcation direction:
\[
Y_{10}^* = \mathcal{I}_{10}(1/r)|_{r=1} = 494 \cdot 10! Y_{10,0} + 228 \cdot \sqrt{2} \cdot 5! \cdot 15! Y_{10,5} + \sqrt{2} \cdot 20! Y_{10,10},
\]

Figure 6. $I$-invariant spherical harmonic for $\ell = 15$. 
Example 8. \( I, \ell = 15: \)
Table 1, row 3: \( 15 + 10p + 6q = 15 \Rightarrow p = q = 0. \)
Invariant bifurcation direction:
\[
Y^{*}_{15} = \mathcal{Z}_{15}(1/r)|_{r=1} = \sqrt{2 \cdot 30!} Y_{15,-15} - 522 \sqrt{2 \cdot 5! \cdot 25!} Y_{15,-10} - 10005 \sqrt{2 \cdot 10! \cdot 20!} Y_{15,-5},
\]
cf. Figure 6.

\[\text{(a)} \hspace{1cm} \text{(b)}\]

Figure 7. \( D_6^d \)-invariant spherical harmonic of order \( \ell = 3 \): (a) front view and (b) top view.

Example 9. \( D_6^d, \ell = 3: \)
Table 1, row 14: \( 2p + 3q = 3 \Rightarrow p = 0, q = 1. \)
Invariant bifurcation direction:
\[
Y^{*}_{3} = C_{3}(1/r)|_{r=1} = \sqrt{2 \cdot 6!} Y_{3,3},
\]
cf. Figure 7.

Example 10. \( D_4^d, \ell = 5: \)
Table 1, penultimate row: \( n = 2, \ 2p + 1 + 2(2q + 1) = 5 \Rightarrow p = 1, q = 0. \)
Invariant bifurcation direction:
\[
Y^{*}_{5} = \hat{z} S_{2}(1/r)|_{r=1} = i \hat{z}^{3}(\hat{\xi}^2 - \hat{\eta}^2)(1/r)|_{r=1} = -\sqrt{2 \cdot 3! \cdot 7!} Y_{5,-2},
\]
cf. Figure 8.

We finish this section with two subgroups \( G \) and accompanying values of \( \ell \) such that \( \text{Null} A(\lambda_\ell)|_G \) is 2-dimensional; we generate two basis functions in each case.

Example 11. \( D_4 \oplus \mathbb{Z}_2, \ell = 4: \)
Table 1, last row: \( 2p + 4q = 4 \Rightarrow (p,q) = (2,0) \) or \( (0,1) \). The first choice generates the axisymmetric spherical harmonic \( Y_{4,0} \) which trivially is also invariant under \( D_4 \oplus \mathbb{Z}_2 \). The second choice generates
\[
C_{4}(1/r)|_{r=1} = \sqrt{2 \cdot 8!} Y_{4,4},
\]
which is shown in Figure 9.
Figure 8. $D_4^c$-invariant spherical harmonic of order $\ell = 5$: (a) Front view and (b) top View.

Figure 9. One of the basis function that generate the two dimensional subspace of $\mathbb{D}_4 \oplus \mathbb{Z}_2^c$-invariant spherical harmonic of order $\ell = 4$: (a) Front view and (b) top view.

Example 12. $\mathbb{O} \oplus \mathbb{Z}_2^c, \ell = 12$:
Table 1, row 5: $6p + 4q = 12 \Rightarrow (p, q) = (2, 0)$ or $(0, 3)$.

The first choice gives

$$Y_{12}^{*a} = O_6^2(1/r)|_{r=1} = \left[ \hat{z}^4(\hat{\xi}^8 + \hat{\eta}^8) - 4\hat{z}^8(\hat{\xi}^4 + \hat{\eta}^4) + 6\hat{z}^{12} \right](1/r)|_{r=1}$$

$$= \sqrt{2} \cdot 4! \cdot 20! Y_{12,8} - 4\sqrt{2} \cdot 8! \cdot 16! Y_{12,4} + 6 \cdot 12! Y_{12,0},$$

while the second produces

$$Y_{12}^{*b} = O_4^3(1/r)|_{r=1} = \left[ 14\hat{z}^4 + (\hat{\xi}^4 + \hat{\eta}^4) \right]^3(1/r)|_{r=1}$$
Figure 10. The two basis functions (a) and (b) that span the subspace of $O$-invariant spherical harmonics of order $\ell = 12$.

$$\left(\hat{\xi}^{12} + \hat{\eta}^{12}\right) + 42\hat{z}^4(\hat{\xi}^8 + \hat{\eta}^8) + 591\hat{z}^8(\hat{\xi}^4 + \hat{\eta}^4) + 2828\hat{z}^{12}\right)(1/r)|_{r=1} = \sqrt{2} \cdot 24!Y_{12,12} + 42 \cdot \sqrt{2} \cdot 4! \cdot 20!Y_{12,8} + 591 \cdot \sqrt{2} \cdot 8! \cdot 16!Y_{12,4} + 2828 \cdot 12!Y_{12,0},$$

where we have used the identity $\hat{\xi}\hat{\eta} = -\hat{z}^2$ (cf. (37)) to rewrite terms involving products of $\hat{\xi}$ and $\hat{\eta}$. Both of these are shown in Figure 10.

5. Concluding remarks. Together with the equivariant branching lemma, the results of Theorem 9.9 in [3] insure, for the given subgroup $G \subset O(3)$ and $\ell \in \mathbb{N}$, the existence of a nontrivial, local bifurcating branch of solutions of the form

$$\lambda = \tilde{\lambda}(\epsilon), \quad u = \bar{u}(\epsilon) = \epsilon Y^{\ast}_\ell + o(\epsilon), \quad \text{as } \epsilon \to 0,$$

where $\bar{u}(\epsilon) \in X_G$, for all sufficiently small $\epsilon$, and where $Y^{\ast}_\ell$ is the bifurcation direction, cf. (12). Our complementary results presented herewith provide an efficient and systematic approach to the precise determination of $Y^{\ast}_\ell$. We also mention that in problems like [4], [5], such local symmetric branches (57) are only the start of global bifurcating solution branches inheriting the same symmetry.

Finally, we note the existence of local, $O(3)$-symmetry-breaking solutions exhibiting sub-maximal isotropy, cf. [2], [9]. These arise in the context of isotropy subgroups $G \subset O(3)$ for which the invariant bifurcation problem on $\mathbb{R} \times X_G$ is not 1-dimensional. In contrast to the applications presented here based on the equivariant branching lemma, the existence of such solutions depends on a detailed analysis of the bifurcation equations. Although we do not carry out such analyses here, Examples 11 and 12 illustrate the construction of basis functions in the context of 2-dimensional local bifurcation equations. Finally we note that invariant spherical-harmonic bases for the planar subgroups of $O(3)$, as well as the dimension of fixed-point subspaces for all subgroups and representations are provided in [2].

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E-mail address: sd282@cornell.edu

E-mail address: tjh10@cornell.edu