THE SIGN PROBLEM IS THE SOLUTION

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Abstract
The unquenched spectral density of the Dirac operator at $\mu \neq 0$ is complex and has oscillations with a period inversely proportional to the volume and an amplitude that grows exponentially with the volume. Here we show how the oscillations lead to the discontinuity of the chiral condensate.

1 Lessgo!

The sign problem in QCD at non-zero baryon chemical potential, $\mu$, has been an obstacle for more than two decades. New innovations have given exiting results \cite{1} even though the core of the sign problem in QCD remains. Here we address a closely related and equally long-standing problem (see e.g. \cite{2}):

How does chiral symmetry breaking at $\mu \neq 0$ manifest itself in the spectrum of the Dirac operator?

At $\mu = 0$ the Dirac spectrum is located on the imaginary axis and the discontinuity of the chiral condensate at zero quark mass, $m$, is proportional to the eigenvalue density at the origin (the Banks Casher relation). While this is quite intuitive the situation for $\mu \neq 0$ is quite puzzling. The chemical potential sends the eigenvalues, $z_k$, of $D + \mu \gamma_0$ off into the complex plane while the discontinuity of the chiral condensate remains. The solution \cite{3} of this (Silver Blaze \cite{4}) puzzle is in a sense the sign problem: Due to the sign problem the eigenvalue density becomes a complex and strongly oscillating function and the oscillations lead to the discontinuity of the chiral condensate. To show this we start from the exact solution \cite{5} of the eigenvalue density for $\mu \ll \Lambda_{\text{QCD}}$ and $m^2 \ll 1/\sqrt{\langle \ldots \rangle}$ (\ldots is the quenched average)

$$\rho_{N_f}(x, y; m; \mu) = \frac{\langle \sum_k \delta^2(x + iy - z_k) \det^{N_f}(D + \mu \gamma_0 + m) \rangle}{\langle \det^{N_f}(D + \mu \gamma_0 + m) \rangle}$$

and compute the chiral condensate $\Sigma = \lim_{m \to 0, V \to \infty} \Sigma_{N_f}(m)$ using

$$\Sigma_{N_f}(m) = \frac{1}{V} \int dx \, dy \, \rho_{N_f}(x, y; m; \mu) \frac{\det^{N_f}(D + \mu \gamma_0 + m)}{\det^{N_f}(D + \mu \gamma_0 + m)}$$

Given the nature of the sign problem it is perhaps not surprising that the oscillations of the eigenvalue density \cite{1} have a period inversely proportional to the volume and an amplitude growing exponentially large with the volume \cite{6}. The results \cite{7,5,6} for the eigenvalue density are ideally suited to resolve these oscillations since they describe eigenvalues $|z| \sim \mathcal{O}(1/(\Sigma V))$. In \cite{5} the eigenvalue density was derived using non hermitian random matrix theory \cite{8} while in \cite{6} it was derived from the chiral Lagrangian using the replica method \cite{9}.

On the next pages we explain the original \cite{3} direct computation of $\Sigma_{N_f}(m)$ through several figures.

2 The phase quenched way

Before going to the real problem we take a short aside and look at the way chiral symmetry breaking affects the spectral density of the Dirac operator in phase quenched QCD (tantamount to QCD at non-zero isospin chemical potential). This case serves to show how the perhaps more familiar mean field results are a special limit of the exact results. The real and positive eigenvalue density

$$\rho_{2n}(x, y; m; \mu) = \frac{\langle \sum_k \delta^2(x + iy - z_k) \det(D + \mu \gamma_0 + m)^{2n} \rangle}{\langle \det(D + \mu \gamma_0 + m)^{2n} \rangle}$$
is plotted in the top panel of figure 1 (the explicit expression is given in (76) of [6]). As expected the eigenvalues have spread out away from the imaginary axis. Notice the absence of eigenvalues at \( z = \pm m \) and at \( z = 0 \). Using (2) the corresponding chiral condensate follows, see the lower panel of figure 1. The drop of the chiral condensate as the quark mass enters the eigenvalue distribution is easy to understand by an electrostatic analogy and is consistent with lattice measurements [10]. Furthermore, taking \( m\Sigma V \gg 1 \) and \( \mu^2 F^2 V \gg 1 \) the mean field result (\( \Sigma_{2n}^{MF} = m\Sigma^2/(2\mu^2 F^2) \) for \( m\pi/(2\mu) < 1 \) and otherwise) is reproduced by the exact result. (\( F \) is the pion decay constant.)

### 3 The unquenched way

In full QCD the eigenvalues of the Dirac operator for a given gauge configuration are indistinguishable from what we would have obtained in the phase quenched case. However, due to the phase of the fermion determinant, the average spectral density is entirely different. To show this we have plotted the unquenched eigenvalue density in the top panel of figure 2. Its explicit expression is given in (73) of [6]. Here it is sufficient to note that this expression naturally separates into the quenched spectral density and a remainder

\[
\rho_{N_f \Sigma V}(x, y, m; \mu) = \rho_Q(x, y; \mu) + \rho_U(x, y, m; \mu).
\]  

(4)

These two parts are shown in the two lower panels of figure 2. The quenched eigenvalue density is real and positive and behaves as the phase quenched eigenvalue density (except for the dip at the quark mass). The unquenched part contains the complex oscillations of which we have only displayed the real
The eigenvalue density of the full QCD Dirac operator for one flavor (real part only) and $\mu F \sqrt{V} = 8$ and $m \Sigma V = 80$. The bottom figure shows the difference between the full (top) and quenched (middle) eigenvalue density. Notice the similarity between quenched and the phase quenched spectral density shown in figure 1. For $y = 0$ the oscillations start $|x| = m$.

For the values of the parameters $m \Sigma V$ and $\mu^2 F^2 V$ given in figure 2 the maximum value of the amplitude of the oscillations is 400 times larger than the plateau of the quenched spectral density. For better illustration the oscillation have been clipped. Inserting the density (4) into (2) leads to two terms

$$\Sigma_{N_f}(m) = \Sigma_Q(m, \mu) + \Sigma_U(m, \mu).$$

(5)
The quenched part, $\Sigma_Q$, drops to zero as $m$ comes inside the support the eigenvalue density. However, the unquenched oscillating part exactly makes up for this and leaves the full chiral condensate $\Sigma_{N_f}(m)$ independent of $\mu$ and therefore equal to the result for $\mu = 0$ [11] (see figure 3). While $\Sigma_Q$ is built up by the eigenvalue density inside the quark mass (the contributions from $|x| > m$ cancel each other), the unquenched part $\Sigma_U$ is built up by the density $\rho_U$ outside the quark mass (not easy to understand by an electrostatic analogy). This is possible since the oscillations have a period of order $1/V$ and an amplitude that grows exponentially with $V$ [3]. Moreover, the contribution from the unquenched part comes from the entire oscillating region and not from the boundary of the support of $\rho_U$. 

Fig. 3: The chiral condensate as a function of quark mass obtained from the three densities in figure 2 using (2).

Top: The chiral condensate in full QCD with the discontinuity at $m = 0$ (independent of $\mu$) is the sum of the two below.

Middle: The quenched chiral condensate (dependent on $\mu$) [13]. No discontinuity at $m = 0$.

Bottom: The contribution from the oscillating part of the eigenvalue density (dependent on $\mu$).
4 Conclusions

While the discontinuity of the chiral condensate is usually associated with a dense spectrum of Dirac eigenvalues on the imaginary axis near zero, the sign problem allows for an alternative mechanism: The discontinuity of the chiral condensate arises from strong oscillations of the eigenvalue density. Having a period of order $1/V$ and an amplitude that grows exponentially large with $V$ the oscillations directly reflect the sign problem. The oscillations are present as soon as the quark mass hits the support of the eigenvalue density, i.e. when the phase quenched theory enters the BEC phase. A lattice simulation in this region therefore must be able to deal with the oscillations in order to study the correct mechanism for spontaneous chiral symmetry breaking in QCD [12].

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