Scaling of spin avalanches in growing networks

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Abstract
Growing networks decorated with antiferromagnetically coupled spins are archetypal examples of complex systems due to the frustration and the multivalley character of their energy landscapes. Here we use the damage spreading method (DS) to investigate the cohesion of spin avalanches in the exponential networks and the scale-free networks. On the contrary to the conventional methods, the results obtained from DS suggest that the avalanche spectra are characterized by the same statistics as the degree distribution in their home networks. Further, the obtained mean range $Z$ of an avalanche, i.e. the maximal distance reached by an avalanche from the damaged site, scales with the avalanche size $s$ as $Z/N^\beta = f(s/N^\alpha)$, where $\alpha = 0.5$ and $\beta = 0.33$. These values are true for both kinds of networks for the number $M$ of nodes to which new nodes are attached between 4 and 10; a check for $M = 25$ confirms these values as well.

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1 Introduction
In recent years, there is a growing interest in research on complexity science. Common features of physical, technical, biological, social and computational systems are qualified as constituting what we call complex systems. A final definition of a complex system would be premature. What is repeated in different formulations is 'a system built up from many interacting components'. Further, properties of a "complex system" are 'not fully explained by an understanding of its component parts' [1]. This notion allows to expect a formulation of phenomenological (as opposed to 'ab initio') laws which could be valid at intermediate levels of integration of system structure. The component parts can be reduced to simple mathematical objects, still the observed or derived laws can reveal new and unexpected aspects of a 'complex' system.

Archetypal examples of complex systems are growing networks [2]. They consist of nodes, and some of these nodes are linked together. This generalization of the Cartesian lattices became an object of research of computational scientists in 1998, when Watts and Strogatz published their seed paper [3]. Since then, the list of relevant references grown to thousands; for some recent monographs we refer to [1] [5] [6] [7] [8] [9]. A specific branch of complexity emerges around networks decorated with Ising spins $S_i = \pm 1$ at each node. There, the interaction between linked spins can favorize the same (ferromagnet) or different
(antiferromagnet, AF) signs. This generalization is known\cite{2} to enrich the list of problems and possible applications of complex networks.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{The avalanche spectrum $N_L(s)$ for the exponential networks, $M = 5$.}
\end{figure}

Here we are interested in statistics of avalanches in growing networks with AF interaction. The case of AF is more complex, because this interaction leads to the effect of frustration of spins in disordered structures\cite{10}. Frustration means in particular that once three nodes are connected to each other, spins at these nodes cannot have different signs; at least two spins must be of the same signs, what raises the energy. Further, there are many states with the minimal (ground state) energy, while for a ferromagnet there are only two ground states. In frustrated networks, the spectrum of ground states can be very rich and complex, and their number increases exponentially with the system size\cite{11}. Counting and classification of these states is then computationally unfeasible.

It seems that an information on the probability distribution of the number of states which lead to the same local minimum can be related with the spectrum of connected avalanches.

The numerical experiment is conducted as follows. We vary the external field what leads to flipping of groups of spins. Two runs are performed with using the same set of random numbers. In one run we keep one spin blocked in the direction opposite to its direction in equilibrium before the field is changed. The number of spins which behave in a different way is the size $s$ of a connected avalanche. The method is known as the damage spreading technique\cite{12, 13}. In previous numerical experiments\cite{14, 15} the size of an avalanches was calcu-
lated just as the number of flipped spins. That method did not differ between connected and disconnected avalanches.

The goal of this paper is twofold. First, the avalanche spectra in the growing networks are characterized by the same statistics, as the degree distributions in these networks. This means, that the probability distribution of avalanche size $s$ in the exponential networks is exponential, and the one in the scale-free networks is the power function. Our second finding is that the obtained mean range $Z$ of avalanches, i.e. the maximal distance reached by an avalanche from the damaged site, scales with the avalanche size $s$ as $Z/N^\beta = f(s/N^\alpha)$, where $N$ is the number of nodes in the network. The exponents $\alpha$ and $\beta$ increase from zero for $M = 1$ (trees) and they are approximately constant above some value of $M$. Here $M$ is the growing parameter, i.e. the number of nodes to which new nodes are attached. Above $M = 4$, the values of $\alpha$ and $\beta$ are almost the same for the exponential and the scale-free networks.

In next section, the calculations and results are described in details. Section 3 is devoted to discussion.

2 Calculations and results

The scheme of our calculation is basically the same as in [13, 15]: the difference is that here we apply the damage spreading technique. Avalanches are measured in the numerical experiment with the hysteresis loop. Energy of spin $S_i$ is

Figure 2: The avalanche spectrum $N_L(s)$ for the scale-free networks, $M = 5$. 
Figure 3: The average size $s$ of avalanche against the degree of the site where the avalanche was born, for the exponential networks, $M = 5$.

calculated as

$$E_i = S_i \left[ \sum_{j(i)} S_j - h \right]$$

(1)

where $h$ is the magnetic field energy. The energy units are $|J|$, where $J < 0$ is the antiferromagnetic exchange integral. The field is changed from $h_m$ to $-h_m$ by 1, where $h_m = k_{max} + 0.5$, and $k_{max}$ is the maximal degree in a given network. Each field change is performed $N$ times, where $N$ is the size of the network; in this way each spin is blocked once for the DS technique. Simulations are performed for four different orders of the sequence of nodes: (A) from the oldest to the newest nodes, (B) from the newest to the oldest nodes, (C) with random permutations, and (D) with one selected sequence. The obtained plots indicate that these variants do not differ qualitatively.

The avalanche spectra, i.e. the number $N_L$ of avalanches of size $s$ for the exponential and the scale-free growing networks are shown in Fig. 1 and 2, respectively. These calculations are performed for the network size $N = 2 \times 10^3$ nodes, and the spectra are averaged over $K = 10$ networks. The obtained relations for the exponential networks can be parametrized as $N_L(s) \propto \exp(-\phi s)$, where $\phi = 0.45$ for $M = 1$, $\phi = 0.21$ for $M = 2$, and $\phi = 0.11$ for $M = 5$. For the scale-free networks, the relation $N_L(s) \propto s^{-\gamma}$ applies. There, $\gamma = 2.71$ for $M = 1$, $\gamma = 2.14$ for $M = 2$, and $\gamma = 1.86$ for $M = 5$. 

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These results indicate that the avalanche spectrum is determined by degree distribution of the network. We checked how the avalanche size depends on the degree of a node where the damage was started. The obtained plots are shown in Figs. 3 and 4, for the exponential and the scale-free networks, respectively.

We investigate also the range $Z$ of avalanches, i.e. the maximal distance reached by an avalanche from the damaged site, against the avalanche size $s$. Due to computational limitations, these calculations could be performed for the network size not larger than $N = 300$. However, here we got the finite-size scaling relations

$$
\frac{Z}{N^\beta} = f\left(\frac{s}{N^\alpha}\right)
$$

with the shapes of the functions $f(s)$ shown in Figs. 5-8 for $M = 1$ and $M = 5$, the exponential and the scale-free networks. For $M = 1$ results are averaged over $K = 10^3$ networks ($N = 300$) and over $K = 10^4$ (for others $N$). For $M = 5$ number of networks are $K = 10^3$ except $N = 300$ where $K = 10^2$. The results on the obtained exponents $\alpha$ and $\beta$ against the growing parameter $M$ are shown in Fig. 9. These results indicate that $M = 1$ is a special case, where the scaling relation does not depend on the network size $N$. On the contrary, for $M$ between 4 and 10 the plots do not depend qualitatively on $M$, and the values of the exponents $\alpha$ and $\beta$ seem to reach their asymptotic values. This is confirmed by the calculation for $M = 25$, where we get $\alpha = 0.52(0.51)$ and $\beta = 0.34(0.33)$ for the scale-free (exponential) networks. Most important result
is that the exponents $\alpha$ and $\beta$ are the same for the exponential networks and the scale-free networks.

Figure 5: The scaling relation between the range $Z$ of avalanches and their size $s$ for the exponential trees ($M = 1$).

3 Discussion

Our numerical results indicate that the spectrum of avalanches is described by the same function as the degree distribution in the growing network. A most simple explanation of this result could be that an avalanche contains all nodes in the direct neighborhood of a damaged node; the range of such avalanches is just one. We see that this is not the case; in general, the range of avalanche exceeds 1. Still, the observed coincidence indicates that there is a monotoneous relation between the size of avalanche and the degree of the node at the avalanche origin. We checked numerically that indeed such a relation does appear. In other words, an avalanche born in a damaged site increases with the degree of this site. However, the obtained exponent $\gamma$ clearly decreases with the growing parameter $M$. This is in contradiction to the node degree distribution in the scale-free networks, where the appropriate exponent does not vary with $M$. Perhaps the finite size effect is enhanced here by the deviation of the relation $\langle s(k) \rangle$ from linearity, as observed in Fig. 4. Note that this argument is not related to the exponential networks, where the deviation from linearity (Fig. 3) is not observed, but the exponent $\phi$ is known to decrease with $M$ [15].

The second result independent on the network topology is the scaling relation
between the range and size of the avalanches, i.e. \( Z/N^\beta = f(s/N^\alpha) \). For trees \((M = 1)\), the exponents \( \alpha \) and \( \beta \) vanish, what reflect the fact that in linear chains the range and the size of avalanches is the same. For \( M = 4 \) and larger, \( \alpha \) is close to \( 1/2 \) and \( \beta \) is close to \( 1/3 \) with the numerical accuracy. This is true for the exponential and the scale-free networks. The plateau of the function \( f(s) \), shown in Fig. 3, reveals that the range of avalanches is limited: The scaling relation found here suggests that the origin of this limitation is the size of the network. We are not aware of any analytical calculation of the range of avalanches. The topological disorder of the network, combined with the frustration, can produce a kind of magnetic disorder, analogous to the Random Field Model [16, 17]; perhaps some results of this model could be applied also to the antiferromagnetic growing networks.

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Figure 9: The exponents $\alpha$ and $\beta$ against the growing parameter $M$ for the exponential (EX) and the scale-free (SC) networks.