A characterization of boundary conditions yielding maximal monotone operators.

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MATH-AN-11-2013
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August 14, 2014

Abstract. We provide a characterization for maximal monotone realizations for a certain class of (nonlinear) operators in terms of their corresponding boundary data spaces. The operators under consideration naturally arise in the study of evolutionary problems in mathematical physics. We apply our abstract characterization result to Port-Hamiltonian systems and a class of frictional boundary conditions in the theory of contact problems in visco-elasticity.

Keywords and phrases: Maximal monotone operators, boundary data spaces, nonlinear boundary conditions, Port-Hamiltonian systems, frictional boundary conditions

Mathematics subject classification 2010: 47B44,47F05,47N20,46N20
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1 Introduction

As it was shown in several articles ([20, 21, 22, 23, 26, 32, 33, 34, 35]) evolutionary problems in classical mathematical physics can often be written as a differential equation of the form

$$(\partial_0 M + A) u = f,$$

where $u$ is the unknown, $f$ is a given source term, $\partial_0$ denotes the temporal derivative, $M$ is a suitable bounded operator acting in space-time and $A$ is a maximal monotone (possibly nonlinear) operator, which frequently is a suitable restriction of a block operator matrix of the form

$$
\begin{pmatrix}
0 & D \\
G & 0
\end{pmatrix},
$$

where $G$ and $D$ are densely defined closed linear operators satisfying $-G^* \subseteq D$.

The aim of this article is to provide a characterization of all maximal monotone restrictions of $(1)$. This characterization will be given in terms of the so-called boundary data spaces, introduced in [24, 27], associated with the operators $G$ and $D$. Moreover, we give a characterization of skew-selfadjoint (and hence maximal monotone) restrictions of $(1)$, which is a natural question arising for instance in the study of energy preserving evolutionary problems (see e.g. [25, 24]).

The question of maximal monotone (or m-accretive) realizations of certain operators or relations was studied in various papers. For instance in 1959, Phillips [19] provides a characterization of m-accretive realizations of linear operators using indefinite metrics on Hilbert spaces on the one hand and the Neumann-Cayley transform on the other hand. Later on these results were generalized to linear relations in [8]. More recently, in [1] we find a characterization result for m-accretive extensions of linear relations in Hilbert spaces using the theory of Friedrichs- and Neumann-extensions of symmetric relations [4]. Another strategy to study extensions of operators or relations uses the theory of boundary triplets or, more general, boundary relations (see e.g. [6, 2, 7]). So, for instance in [10, Chapter 3] the question of m-accretive extensions of sectorial operators is addressed and a characterization is given in terms of boundary triplets. To the authors best knowledge all these strategies are restricted to the case of linear operators or relations and we emphasize here that our approach also works for nonlinear realizations.

The article is structured as follows. In Section 2 we recall some well-known facts on maximal monotone relations and we refer the reader to [3, 11, 17] for a detailed study of this topic. Moreover, we recall the definition and basic properties of so-called boundary data spaces (see [24]). In the third section we prove our main theorem (Theorem 3.1): a characterization of all maximal monotone restrictions of operators of the form $(1)$ in terms of the associated boundary data spaces. One part of this statement was already proved by the author in [33] but for sake of completeness we state the proof once again. Moreover we give a characterization of all skew-selfadjoint restrictions (Corollary 3.8). Section 4 is devoted to the comparison of abstract boundary data spaces as they were introduced in Section 2 and the classical trace spaces $H^\frac{1}{2}(\partial \Omega)$ and $H^{-\frac{1}{2}}(\partial \Omega)$ for bounded Lipschitz-domains $\Omega \subseteq \mathbb{R}^n$ (see e.g. [18]). In particular, we show how classical boundary conditions of Dirichlet-, Neumann- or Robin-type can be formulated within the framework of boundary data spaces associated with the operators $\text{grad}$ and $\text{div}$. We conclude the article with two applications. In the first one we study so-called linear Port-Hamiltonian systems as they were introduced in [12, 13] and show, how
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these systems can be embedded into our abstract setting. In the second example we consider frictional boundary conditions of monotone type arising in the theory of contact problems in visco-elasticity ([14, 15, 31]). Moreover, by means of this example we illustrate, how to formulate boundary conditions on different parts of the boundary within our framework.

Throughout, all Hilbert spaces are assumed to be complex and the inner products are denoted by $\langle \cdot | \cdot \rangle$, which are assumed to be linear in the second and conjugate linear in the first argument. The induced norm is denoted by $|\cdot|$. Moreover, for a Hilbert space $H$ and a closed subspace $V \subseteq H$ we denote by $\pi_V : H \to V$ the orthogonal projection onto $V$. The adjoint $\pi_V^* : V \to H$ is then the canonical embedding and the projector on $V$ is given by $P_V := \pi_V^* \pi_V : H \to H$.

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2.1 Maximal monotone relations

In this subsection we recall some well-known facts about maximal monotone relations. We refer the reader to the monographs [3, 11, 17] and the references therein for a deeper study of maximal monotone relations and for the proofs of most of the statements in this subsection. We introduce some algebraic notions for binary relations in order to work with binary relations in a comfortable way.

**Definition.** Let $H_0, H_1$ be two Hilbert spaces and $A \subseteq H_0 \oplus H_1$. For $M \subseteq H_0$ we define the *post-set of $M$ under $A$* by

$$A[M] := \{ y \in H_1 | \exists x \in M : (x, y) \in A \}.$$  

Analogously, for $N \subseteq H_1$ we define the *pre-set of $N$ under $A$* by

$$[N]A := \{ x \in H_0 | \exists y \in N : (x, y) \in A \}.$$  

The *inverse relation* $A^{-1} \subseteq H_1 \oplus H_0$ is given by

$$A^{-1} := \{ (y, x) \in H_1 \oplus H_0 | (x, y) \in A \}.$$  

Moreover, for $B \subseteq H_0 \oplus H_1$ and $\lambda \in \mathbb{C}$ we define

$$\lambda A + B := \{ (x, \lambda y + z) \in H_0 \oplus H_1 | (x, y) \in A \land (x, z) \in B \}.$$  

The relation $A$ is called *bounded*, if for each bounded set $M \subseteq H_0$ the post-set $A[M]$ is also bounded.

In the literature, the notion of maximal monotonicity is frequently defined for set valued mappings, i.e. mappings of the form $A : D(A) \subseteq H \to \mathcal{P}(H)$. However, we prefer the notion of binary relations $A \subseteq H \oplus H$ instead of set-valued mappings, that is, we identify a set-valued mapping $A$ with the relation

$$\{(u, v) \in H \oplus H | u \in D(A), v \in A(u)\}.$$
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Finally we define the adjoint relation $A^*$ of $A$ by

$$A^* = \{(y, x) \in H_1 \oplus H_0 \mid (x, y) \in A\}^\perp \subseteq H_1 \oplus H_0.$$

Remark 2.1. We note that $A^*$ is always a closed linear relation. Moreover, a pair $(u, v) \in H_1 \oplus H_0$ belongs to $A^*$ if and only if

$$\langle y|u \rangle_{H_1} = \langle x|v \rangle_{H_0}$$

for all pairs $(x, y) \in A$, see e.g. [23, p.14].

We now give the definition of monotonicity and maximal monotonicity of binary relations.

**Definition.** Let $A \subseteq H \oplus H$. Then $A$ is called **monotone**, if for each $(u, v), (x, y) \in A$ the inequality

$$\Re \langle u - x|v - y \rangle \geq 0$$

holds. A monotone relation $A$ is called **maximal monotone**, if there exists no proper monotone extension, i.e. for each monotone $B \subseteq H \oplus H$ with $A \subseteq B$ it follows that $A = B$.

Remark 2.2. A maximal monotone relation $A \subseteq H \oplus H$ is **demi-closed**, i.e. for each sequence $((x_n, y_n))_{n \in N}$ in $A$, where $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ or $x_n \rightarrow x$ and $y_n \rightharpoonup y$ for some $x, y \in H$ as $n \rightarrow \infty$ we have $(x, y) \in A$ (see e.g. [17, Proposition 1.1]). Moreover, for each $x \in H$ the post-set $A[x]$ is closed and convex.

Classical examples of maximal monotone relations are skew-selfadjoint operators, non-negative selfadjoint operators and subgradients of lower semicontinuous, convex functions (see [30]). In 1962 G. Minty proves the following celebrated characterization for maximal monotonicity.

**Theorem 2.3** (Minty’s Theorem, [16]). Let $A \subseteq H \oplus H$ be monotone. Then the following statements are equivalent

(i) $A$ is maximal monotone,

(ii) For all $\lambda > 0$ the relation $1 + \lambda A$ is onto, i.e. $(1 + \lambda A)[H] = H$,

(iii) There exists $\lambda > 0$ such that $1 + \lambda A$ is onto.

Using this theorem, we can define the Yosida-approximation of maximal monotone relations.

**Definition.** Let $A \subseteq H \oplus H$ be maximal monotone and $\lambda > 0$. Then we define the mapping $A_\lambda : H \rightarrow H$ by

$$A_\lambda(x) = \lambda^{-1} \left( x - (1 + \lambda A)^{-1}(x) \right),$$

the so-called **Yosida-approximation of $A$**. Note that due to the monotonicity of $A$, the relation $(1 + \lambda A)^{-1}$ defines a Lipschitz-continuous mapping with smallest Lipschitz-constant less than or equal to 1 and by Theorem 2.3 this mapping is defined on the whole Hilbert space $H$. Consequently, $A_\lambda$ is also a Lipschitz-continuous mapping defined on the whole space $H$.

**Proposition 2.4** (see e.g. [17, Theorem 1.3]). Let $A \subseteq H \oplus H$ be maximal monotone and set

$$A^0(x) := P_{A[x]}(0) \quad (x \in [H|A])$$

Here, we denote by 1 the identity on $H$.

By $P_{A[x]}$ we denote the orthogonal projection on the closed convex set $A[x]$.
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the principal section of \( A \). Then:

(a) For each \( \lambda > 0 \) the mapping \( A_\lambda \) is maximal monotone,

(b) For each \( \lambda > 0 \) and \( x \in H \) we have \( \left((1 + \lambda A)^{-1}(x), A_\lambda(x)\right) \in A \),

(c) For each \( \lambda > 0 \) and \( x \in [H]A \) we have \( \lvert A_\lambda(x) \rvert \leq \lvert A^0(x) \rvert \).

We conclude this subsection with two statements about the construction of maximal monotone relations from given ones.

**Proposition 2.5.** Let \( H_0, H_1 \) be two Hilbert spaces and \( B_0 \subseteq H_0 \oplus H_0 \), \( B_1 \subseteq H_1 \oplus H_1 \) maximal monotone relations. Then

\[
B_0 \oplus B_1 := \left\{ ((u, x), (v, y)) \in (H_0 \oplus H_1)^2 \mid (u, v) \in B_0, (x, y) \in B_1 \right\}
\]

defines a maximal monotone relation on \( H_0 \oplus H_1 \).

**Proof.** The proof is straightforward and we therefore omit it. \( \square \)

**Proposition 2.6.** Let \( H_0, H_1 \) be two Hilbert spaces and \( B \subseteq H_0 \oplus H_0 \) be maximal monotone and bounded. Moreover, let \( T : H_1 \rightarrow H_0 \) be linear and bounded. If \( T[H_1] \cap [H_0]B \neq \emptyset \), then

\[
T^*BT := \{(x, T^*y) \in H_1 \oplus H_1 \mid (Tx, y) \in B\}
\]

is maximal monotone.

**Proof.** The monotonicity of \( T^*BT \) is obvious. For showing the maximal monotonicity we use Minty’s Theorem (Theorem 2.3). For that purpose, let \( f \in H_1 \). Since for each \( \lambda > 0 \) the mapping \( T^*B_\lambda T : H_1 \rightarrow H_1 \) is monotone and Lipschitz-continuous (and hence, maximal monotone cf. [35, Corollary 2.8]), we find \( x_\lambda \in H_1 \) such that

\[
x_\lambda + T^*B_\lambda(Tx_\lambda) = f.
\]

We show that the family \( (x_\lambda)_{\lambda > 0} \) is bounded. For doing so let \( x^* \in H_1 \) such that \( Tx^* \in [H_0]B \). Then

\[
\Re \langle x_\lambda - x^* | f - (x^* + T^*B_\lambda(Tx^*)) \rangle = |x_\lambda - x^*|^2 + \Re \langle x_\lambda - x^* | T^*B_\lambda(Tx_\lambda) - T^*B_\lambda(Tx^*) \rangle
\]

\[
\geq |x_\lambda - x^*|^2,
\]

due to the monotonicity of \( T^*B_\lambda T \). The latter implies

\[
|x_\lambda| \leq |x_\lambda - x^*| + |x^*| \\
\leq |f| + 2|x^*| + \lVert T^* \rVert \lVert B_\lambda(Tx^*) \rVert \\
\leq |f| + 2|x^*| + \lVert T^* \rVert \lVert B^0(Tx^*) \rVert
\]

for all \( \lambda > 0 \), where we have used Proposition 2.4 (c). From the boundedness of \( (x_\lambda)_{\lambda > 0} \) we
derive the boundedness of \((1 + \lambda B)^{-1} (Tx)\) \(\lambda \in [0, 1]\). Indeed, we estimate for all \(\lambda \in [0, 1]\)
\[
|\begin{pmatrix} 1 + \lambda B \end{pmatrix}^{-1} (Tx) | \leq |(1 + \lambda B)^{-1} (Tx) - (1 + \lambda B)^{-1} (Tx^*)| + |(1 + \lambda B)^{-1} (Tx^*)| \\
\leq \|T\| |(|\lambda| + |x^*|)| + |\lambda B (Tx^*) + Tx^*| \\
\leq \|T\| |(|\lambda| + 2|x^*|) + |B^0 (Tx^*)|.
\]
Since \((1 + \lambda B)^{-1} (Tx), B(Tx)\) \(\in B\) (see Proposition 2.4 (b)) and since \(B\) is bounded, we obtain
\[
C := \sup_{\lambda \in [0, 1]} |B(\lambda Tx)| < \infty.
\]
The latter gives that \(B_\lambda (Tx)_{\lambda > 0}\) has a weak convergent subsequence \(B_{\lambda_n} (Tx_{\lambda_n})_{n \in \mathbb{N}}\) with \(\lambda_n \to 0\) and we denote its weak limit by \(y\). Let now \(\lambda, \mu \in [0, 1]\). Then we compute
\[
|x_\lambda - x_\mu|^2 \\
= \Re(x_\lambda - x_\mu|f - T^* B_\lambda (Tx) - (f - T^* B_\mu (Tx_\mu)) \\
= \Re(x_\lambda - x_\mu|B_\mu (Tx_\mu) - B_\lambda (Tx_\mu)) \\
= \Re(\lambda B_\lambda (Tx) + (1 - \lambda B)^{-1} (Tx) - \mu B_\mu (Tx_\mu) - (1 - \mu B)^{-1} (Tx_\mu)|B_\mu (Tx_\mu) - B_\lambda (Tx_\lambda)) \\
\leq \Re(\lambda B_\lambda (Tx) - \mu B_\mu (Tx_\mu)|B_\mu (Tx_\mu) - B_\lambda (Tx_\lambda)) \\
\leq 2C^2 (\lambda + \mu),
\]
where we have again used Proposition 2.4 (b). Thus \((x_{\lambda_n})_{n \in \mathbb{N}}\) is a Cauchy-sequence and hence, it converges and we denote its limit by \(x\). By the continuity of \(T\) we have \(Tx_{\lambda_n} \to Tx\) as \(n \to \infty\) and
\[
|(1 + \lambda_n B)^{-1} (Tx_{\lambda_n}) - Tx| \leq \lambda_n |B_{\lambda_n} (Tx_{\lambda_n})| + |Tx_{\lambda_n} - Tx| \to 0 \quad (n \to \infty).
\]
By the demi-closedness of \(B\) (see Remark 2.2) we get that \((Tx, y) \in B\), which implies \((x, T^* y) \in T^* BT\). Moreover
\[
f = \lim_{n \to \infty} (x_{\lambda_n} + T^* B_{\lambda_n} (Tx_{\lambda_n})) = x + T^* y,
\]
or in other words \((x, f) \in 1 + T^* BT\). □

**Remark 2.7.** A similar result was shown by Robinson 29 without imposing boundedness of \(B\), but with an additional compatibility assumption on \(T\) and \(B\) and assuming the closedness of the ranges of \(T\) and \(T^*\).

### 2.2 Boundary data spaces

In this subsection we recall the notion and some basic properties of boundary data spaces as they were introduced in 23. Throughout, let \(H_0, H_1\) be two Hilbert spaces and \(G_\circ : \mathcal{D}(G_\circ) \subseteq H_0 \to H_1\), \(D_\circ : \mathcal{D}(D_\circ) \subseteq H_1 \to H_0\) be two densely defined, closed linear operators with \(G_\circ \subseteq -D_\circ^*\) and consequently \(D_\circ \subseteq -G_\circ^*\). We set \(G := -D_\circ^*\) and \(D := -G_\circ^*\).

**Example 2.8.** As a guiding example for the situation above we set \(H_0 := L_2(\Omega)\) and \(H_1 := L_2(\Omega)^n\) for some open \(\Omega \subseteq \mathbb{R}^n\). We define the gradient \(\text{grad}_c\) with “vanishing trace” as the
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closure of
\[
\text{grad}_c \mid_{C^\infty_c(\Omega)} : C^\infty_c(\Omega) \subseteq L_2(\Omega) \to L_2(\Omega)^n
\]
\[
\phi \mapsto (\partial_i \phi)_{i \in \{1, \ldots, n\}},
\]
where we denote by $C^\infty_c(\Omega)$ the space of arbitrarily differentiable functions with compact support in $\Omega$. The domain of grad$_c$ then coincides with the classical Sobolev space $W^{1,2}_{0,0}(\Omega)$. Analogously, we define the operator div$_c$ as the closure of
\[
div_c \mid_{C^\infty_c(\Omega)^n} : C^\infty_c(\Omega)^n \subseteq L_2(\Omega)^n \to L_2(\Omega)
\]
\[
(\psi_i)_{i \in \{1, \ldots, n\}} \mapsto \sum_{i=1}^n \partial_i \psi_i.
\]
The domain of div$_c$ then consists of those $L_2$-vector fields whose distributional divergence is an $L_2$-function and which satisfy an abstract Neumann-boundary condition.\footnote{Indeed, if $\partial \Omega$ is regular enough then $\psi \in \mathcal{D}(\text{div}_c)$ satisfies $N \cdot \psi = 0$, where $N$ denotes the unit outward normal vector field at $\partial \Omega$, see Section 4.}

We set grad := $-\text{div}_c^*$ and div := $-\text{grad}_c^*$ and get grad$_c \subseteq$ grad as well as div$_c \subseteq$ div. The domains of grad and div are then the maximal sets of $L_2$-functions or -vector fields such that the distributional gradient or divergence is again an $L_2$-vector field or -function, respectively.

We recall the notion of short Sobolev-chains (see [23] Section 2.1]).

**Definition.** Let $C : \mathcal{D}(C) \subseteq H \to H$ be a closed, densely defined linear operator with $0 \in \rho(C)$. Then we denote by $H^1(C)$ the Hilbert space given by the domain $\mathcal{D}(C)$ equipped with the inner product $\langle \cdot , \cdot \rangle_{H^1(C)} := \langle C \cdot , C \cdot \rangle_H$. Moreover, we set $H^{-1}(C)$ as the completion of $H$ with respect to the norm induced by the inner product $\langle \cdot , \cdot \rangle_{H^{-1}(C)} := \langle C^{-1} \cdot , C^{-1} \cdot \rangle_H$. Then $H^1(C) \hookrightarrow H \hookrightarrow H^{-1}(C)$ with continuous and dense embeddings and we call the triple $(H^1(C), H, H^{-1}(C))$ the short Sobolev-chain associated with $C$.

**Remark 2.9.** It is easy to see that the operator $C : H^1(C) \to H$ is unitary. Moreover, the operator $C : \mathcal{D}(C) \subseteq H \to H^{-1}(C)$ has a unitary extension which we also denote by $C$.

**Proposition 2.10** ([23] Lemma 2.1.16]). Let $G : \mathcal{D}(G) \subseteq H_0 \to H_1$ be a closed densely defined linear operator. Then\footnote{Recall that $|G| := \sqrt{G^*G}$ is a self-adjoint operator and thus, $0 \in \rho(|G| + i)$.} $G : H^1(|G| + i) \to H_1$ is bounded. Moreover, the operator $G : \mathcal{D}(G) \subseteq H_0 \to H^{-1}(|G^*| + i)$ has a unique bounded extension.

**Definition** ([23] Section 5.2]). Let $G_c, D_c, G$ and $D$ as above. We define
\[
\mathcal{B}D(G) := \mathcal{D}(G_c)^{-1}H^1(|G| + i) = \mathcal{N}(1 - DG),
\]
\[
\mathcal{B}D(D) := \mathcal{D}(D_c)^{-1}H^1(|D| + i) = \mathcal{N}(1 - GD),
\]
the so-called boundary data spaces, where the orthogonal complements are taken with respect to the inner products in $H^1(|G| + i)$ and $H^1(|D| + i)$, respectively.
Remark 2.11. According to the projection theorem we have

\[ H^1(|G| + i) = H^1(|G_c| + i) \oplus BD(G), \]
\[ H^1(|D| + i) = H^1(|D_c| + i) \oplus BD(D). \]

This could be interpreted as a decomposition result for elements in \( H^1(|G| + i) \) and \( H^1(|D| + i) \) into one part with “vanishing trace” (in \( H^1(|G_c| + i) \) or \( H^1(|D_c| + i) \), respectively) and one part carrying the whole information about the behaviour at the boundary.

Finally, we recall the following result from [24].

Proposition 2.12 ([24, Theorem 5.2]). Let \( G_c, G, D_c, D \) as above. Then \( G[BD(G)] \subseteq BD(D) \) and \( D[BD(D)] \subseteq BD(G) \). Moreover, the operators

\[ \overset{\bullet}{G} : BD(G) \to BD(D) \]
\[ \overset{\bullet}{D} : BD(D) \to BD(G), \]

defined as the restrictions\(^{6}\) of \( G \) and \( D \), respectively, are unitary with \( (\overset{\bullet}{G})^* = \overset{\bullet}{D} \).

3 A characterization of maximal monotone realizations

In this section we give a characterization for maximal monotone realizations for a certain class of operators in terms of the corresponding boundary data spaces. As in Subsection 2.2, let \( H_0, H_1 \) be two Hilbert spaces and \( G_c : D(G_c) \subseteq H_0 \to H_1, D_c : D(D_c) \subseteq H_1 \to H_0 \) be two densely defined, closed linear operators with \( D_c \subseteq -G^*_c \). We set \( D := -G^*_c \) and \( G := -D^*_c \), which in particular yields \( D_c \subseteq D \) as well as \( G_c \subseteq G \). Let \( A : D(A) \subseteq H_0 \oplus H_1 \to H_0 \oplus H_1 \) a possibly nonlinear operator with

\[ A \subseteq \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix}. \]

Recall that we have the following orthogonal decompositions

\[ H^1(|G| + i) = H^1(|G_c| + i) \oplus BD(G), \]
\[ H^1(|D| + i) = H^1(|D_c| + i) \oplus BD(D). \]

The corresponding projections will be denoted by

\[ \pi_{G_c} : H^1(|G| + i) \to H^1(|G_c| + i), \]
\[ \pi_{BD(G)} : H^1(|G| + i) \to BD(G). \]

\(^{6}\)In other words we have

\[ \overset{\bullet}{G} = \pi_{BD(D)} G \pi_{BD(G)} \]

and analogously

\[ \overset{\bullet}{D} = \pi_{BD(G)} D \pi_{BD(D)} \].
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and

\[ \pi_{D_c} : H^1(|D| + i) \to H^1(|D_c| + i), \]

\[ \pi_{BD(D)} : H^1(|D| + i) \to BD(D). \]

Our main theorem reads as follows.

**Theorem 3.1.** The operator \( A \) is maximal monotone if and only if there exists a maximal monotone relation \( h \subseteq BD(G) \oplus BD(G) \) such that

\[ D(A) = \left\{ (u, v) \in D(G) \times D(D) \mid \left( \pi_{BD(G)}u, \bullet \pi_{BD(D)}v \right) \in h \right\}. \quad (2) \]

In [33] it was proved that \( A \) is maximal monotone if \( D(A) \) is given by (2) for a maximal monotone relation \( h \). However, for sake of completeness we will recall this result below. First we start with the following observation.

**Lemma 3.2.** Let \( (u, v) \in D(G) \times D(D) \). Then

\[ \Re \left\langle \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{H_0 \oplus H_1} = \Re \langle \pi_{BD(G)}u \mid \bullet \pi_{BD(D)}v \rangle_{BD(G)}. \]

**Proof.** We compute

\[ \Re \left\langle \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{H_0 \oplus H_1} = \Re \langle Dv \mid u \rangle_{H_0} + \Re \langle Gu \mid v \rangle_{H_1} \]

\[ = \Re \langle Dc_\pi_{D_c} \pi_{D_c}v \mid u \rangle_{H_0} + \Re \langle D\pi_{BD(D)}^* \pi_{BD(D)}v \mid u \rangle_{H_0} \]

\[ + \Re \langle Gu \mid \pi_{BD(G)}^* \pi_{BD(D)}v \rangle_{H_1} + \Re \langle Gu \mid \pi_{BD(G)}^* \pi_{BD(D)}v \rangle_{H_1} \]

\[ = \Re \langle D\pi_{BD(D)}^* \pi_{BD(D)}v \mid u \rangle_{H_0} + \Re \langle Gu \mid \pi_{BD(G)}^* \pi_{BD(D)}v \rangle_{H_1} \]

\[ = \Re \langle D\pi_{BD(D)}^* \pi_{BD(D)}v \mid \pi_{BG} \pi_{BG} Gv \rangle_{H_0} + \Re \langle D\pi_{BD(D)}^* \pi_{BD(D)}v \mid \pi_{BG} \pi_{BG} Gv \rangle_{H_0} \]

\[ + \Re \langle Gu \mid \pi_{BD(G)}^* \pi_{BD(D)}v \rangle_{H_1} + \Re \langle Gu \mid \pi_{BD(G)}^* \pi_{BD(D)}v \rangle_{H_1} \]

\[ = \Re \langle D\pi_{BD(D)}^* \pi_{BD(D)}v \mid \pi_{BG} \pi_{BG} Gv \rangle_{H_0} + \Re \langle Gu \mid \pi_{BD(G)}^* \pi_{BD(D)}v \rangle_{H_1} \]

\[ = \Re \langle \pi_{BG} \pi_{BG} Gv \mid u \rangle_{H_1} + \Re \langle Gu \mid \pi_{BD(G)}^* \pi_{BD(D)}v \rangle_{H_1} \]

\[ = \Re \langle \pi_{BG} \pi_{BG} Gv \mid u \rangle_{H_1} + \Re \langle Gu \mid \pi_{BD(G)}^* \pi_{BD(D)}v \rangle_{H_1}. \]

\[ \square \]

**Lemma 3.3.** Let \( A \) be maximal monotone and \( (u, v) \in D(G) \times D(D) \). Then \( (u, v) \in D(A) \) if and only if \( (\pi_{BD(G)}^* \pi_{BD(D)}Gv, \pi_{BD(G)}^* \pi_{BD(D)}v) \in D(A) \).
Proof. We compute for every \((x, y) \in \mathcal{D}(A)\) using Lemma 3.2:

\[
\text{Re} \left( \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} \begin{pmatrix} \pi_{BD(G)}^* \pi_{BD(G)}^* u \\ \pi_{BD(D)}^* \pi_{BD(D)}^* v \end{pmatrix} - A \begin{pmatrix} x \\ y \end{pmatrix} \right) - \begin{pmatrix} \pi_{BD(G)}^* \pi_{BD(G)}^* u \\ \pi_{BD(D)}^* \pi_{BD(D)}^* v \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} \right)_{H_0 \oplus H_1}
\]

\[
= \text{Re} \left( \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} \begin{pmatrix} \pi_{BD(G)}^* \pi_{BD(G)}^* u - x \\ \pi_{BD(D)}^* \pi_{BD(D)}^* v - y \end{pmatrix} \right)_{H_0 \oplus H_1}
\]

\[
= \text{Re}(\pi_{BD(G)}(\pi_{BD(G)}^* \pi_{BD(G)}^* u - x) \cdot \pi_{BD(D)}(\pi_{BD(D)}^* \pi_{BD(D)}^* v - y))_{BD(G)}
\]

\[
= \text{Re}(\pi_{BD(G)}(u - x) \cdot \pi_{BD(D)}(v - y))_{BD(G)}
\]

\[
= \text{Re} \left( \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} \begin{pmatrix} u - x \\ v - y \end{pmatrix} \right)_{H_0 \oplus H_1}
\]

\[
= \text{Re} \left( \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} - A \begin{pmatrix} x \\ y \end{pmatrix} \right)_{H_0 \oplus H_1}
\]

If \((u, v) \in \mathcal{D}(A)\) the last term is non-negative and thus

\[
(\pi_{BD(G)}^* \pi_{BD(G)}^* u, \pi_{BD(D)}^* \pi_{BD(D)}^* v) \in \mathcal{D}(A),
\]

according to the maximality of \(A\). If on the other hand

\[
(\pi_{BD(G)}^* \pi_{BD(G)}^* u, \pi_{BD(D)}^* \pi_{BD(D)}^* v) \in \mathcal{D}(A),
\]

then the first term in the latter equalities is non-negative and hence, again by the maximality of \(A\) we deduce \((u, v) \in \mathcal{D}(A)\). \qed

Proposition 3.4. Let \(A\) be maximal monotone. Then there exists a relation \(h \subseteq BD(G) \oplus BD(G)\) such that

\[
\mathcal{D}(A) = \left\{ (u, v) \in \mathcal{D}(G) \times \mathcal{D}(D) \mid (\pi_{BD(G)}^* u, \pi_{BD(D)}^* v) \in \mathcal{D}(A) \right\}.
\]

Proof. We define

\[
h := \left\{ (x, y) \in BD(G) \oplus BD(G) \mid (\pi_{BD(G)}^* x, \pi_{BD(D)}^* y) \in \mathcal{D}(A) \right\}.
\]

Let \((u, v) \in \mathcal{D}(G) \times \mathcal{D}(D)\). Then, using Lemma 3.3 we get that

\[
(\pi_{BD(G)}^* u, \pi_{BD(D)}^* v) \in \mathcal{D}(A) \iff (\pi_{BD(G)}^* u, \pi_{BD(D)}^* y) \in \mathcal{D}(A) \iff (u, v) \in \mathcal{D}(A).
\]

Proposition 3.5. Let \(A\) be maximal monotone and \(h \subseteq BD(G) \oplus BD(G)\) such that

\[
\mathcal{D}(A) = \left\{ (u, v) \in \mathcal{D}(G) \times \mathcal{D}(D) \mid (\pi_{BD(G)}^* u, \pi_{BD(D)}^* v) \in \mathcal{D}(A) \right\}.
\]

Then \(h\) is maximal monotone.
3 A characterization of maximal monotone realizations

Proof. Let \((x, y), (w, z) \in h\). Then \((\pi_{BD(G)}^*, \pi_{BD(D)}^*) \, G \, y\), \((\pi_{BD(G)}^*, \pi_{BD(D)}^*) \, G \, z) \in D(A)\). By Lemma [3.2] we obtain

\[
\Re(x - w | y - z)_{BD(G)} = \Re \left( \pi_{BD(G)}^* \left( \pi_{BD(G)}^* x - \pi_{BD(G)}^* w \right) \right. \\
\left. \left. D \pi_{BD(D)}^* \left( \pi_{BD(D)}^* y - \pi_{BD(D)}^* z \right) \right) \right)_{BD(G)}
\]

\[
= \Re \left( \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} \begin{pmatrix} \pi_{BD(G)}^* x - \pi_{BD(G)}^* w \\ \pi_{BD(D)}^* G y - \pi_{BD(D)}^* G z \end{pmatrix} \right.
\left. \begin{pmatrix} \pi_{BD(G)}^* x - \pi_{BD(G)}^* w \\ \pi_{BD(D)}^* G y - \pi_{BD(D)}^* G z \end{pmatrix} \right)_{H_0 \oplus H_1}
\]

\[
= \Re \left( A \left( \pi_{BD(G)}^* x - \pi_{BD(D)}^* G y \right) \right. \\
\left. \left. - A \left( \pi_{BD(D)}^* G z \right) \right) \geq 0, \right)
\]

which proves the monotonicity of \(h\). For showing the maximal monotonicity we use Minty’s Theorem. Let \(f \in BD(G)\). Then there exists \((u, v) \in D(A)\) such that

\[
\begin{pmatrix} u \\ v \end{pmatrix} + A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \pi_{BD(G)}^* f \\ \pi_{BD(D)}^* G f \end{pmatrix},
\]

i.e.

\[
\begin{align*}
    u + Dv &= \pi_{BD(G)}^* f, \\
v + Gu &= \pi_{BD(D)}^* G f.
\end{align*}
\]

The latter especially yields that \(Dv \in D(G)\) and \(Gu \in D(D)\). Moreover, we get that

\[
\begin{align*}
v - GDv &= v - G(\pi_{BD(G)}^* f - u) \\
&= v - (\pi_{BD(D)}^* G f - Gu) \\
&= v - v \\
&= 0
\end{align*}
\]

yielding that \(v = \pi_{BD(D)}^* \pi_{BD(D)} v\). Thus, the first equality in (3) gives

\[
\pi_{BD(G)}^* u + D \pi_{BD(D)} v = f.
\]

Moreover, by definition \((\pi_{BD(G)}^* u, D \pi_{BD(D)} v) \in h\), which yields

\[
(\pi_{BD(G)}^* u, f) \in 1 + h.
\]

Hence, we have found out that \((1 + h)\, BD(G) = BD(G)\), which implies the maximal monotonicity of \(h\).

Together with Proposition [3.3] the latter proposition shows one implication in Theorem [3.1]. For the missing implication we recall the result and the proof of [33, Theorem 4.1].
Proposition 3.6. Let $h \subseteq BD(G) \oplus BD(G)$ be maximal monotone and

$$D(A) = \left\{ (u,v) \in D(G) \times D(D) \mid (\pi_{BD(G)}u, \pi_{BD(D)}v) \in h \right\}.$$ 

Then $A$ is maximal monotone.

Proof. First we prove that $A$ is monotone. For doing so let $(u,v),(x,y) \in D(A)$. Then by Lemma 3.2 we obtain

$$\Re \left< A \begin{pmatrix} u \\ v \end{pmatrix} - A \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} \right>_{H_0 \oplus H_1} = \Re \left< \pi_{BD(G)}u - \pi_{BD(G)}x, \dot{\pi}_{BD(D)}v - \dot{\pi}_{BD(D)}y \right>_{BD(G)} \geq 0,$$

which shows the monotonicity of $A$. Next, we prove that $A$ is closed. For that purpose let $((u_n,v_n))_{n \in \mathbb{N}}$ be a sequence in $D(A)$ such that $(u_n,v_n) \to (u,v)$ as $n \to \infty$ and $\left( A \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right)_{n \in \mathbb{N}}$ is convergent. By the closedness of $G$ and $D$ we obtain $(u,v) \in D(G) \times D(D)$ and $u_n \to u$ as well as $v_n \to v$ in $H^1(|G| + i)$ and $H^1(|D| + i)$, respectively. Thus $\pi_{BD(G)}u_n \to \pi_{BD(G)}u$ and $\dot{\pi}_{BD(D)}v_n \to \dot{\pi}_{BD(D)}v$. The closedness of $h$ now yields the assertion. Finally, we prove the maximality of $A$ by using Minty’s Theorem. We note that since $A$ is monotone and closed, it suffices to prove that $1 + A$ has dense range. So let $(f,g) \in D(G_c) \times D(D_c)$ and define\footnote{Recall that $-DG_c = G_c^*G_c$ and $-G_cD = D^*D$ are non-negative selfadjoint operators and hence, $1 - DG_c$ and $1 - G_cD$ are boundedly invertible.}

$$\tilde{u} := (1 - DG_c)^{-1} (f - D_c g) \in D(DG_c),$$
$$\tilde{v} := (1 - G_cD)^{-1} (g - G_c f) \in D(G_cD).$$

Then

$$\tilde{u} + D\tilde{v} = (1 - DG_c)^{-1} (f - D_c g) + D(1 - G_cD)^{-1} (g - G_c f)$$
$$= (1 - DG_c)^{-1} ((1 - DG_c)D(1 - G_cD)^{-1}(g - G_c f))$$
$$= (1 - DG_c)^{-1} (f - D_c g + (1 - DG_c)D(1 - G_cD)^{-1}(g - G_c f))$$
$$= (1 - DG_c)^{-1} (f - G_c f)$$
$$= f$$

and analogously

$$\tilde{v} + G\tilde{u} = (1 - G_cD)^{-1} (g - G_c f) + G(1 - DG_c)^{-1} (f - D_c g) = g.$$ 

Moreover, we define

$$u := \tilde{u} + \pi^{\text{\#}}_{BD(G)} (1 + h)^{-1} ( - \dot{\pi}_{BD(D)} G_c \tilde{u} ) \in D(G),$$
$$v := \tilde{v} - \pi^{\text{\#}}_{BD(D)} G_c (1 + h)^{-1} ( - \dot{\pi}_{BD(D)} G_c \tilde{u} ) \in D(D).$$
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Then clearly

\[ u + Dv = \tilde{u} + D\tilde{v} = f, \]
\[ v + Gu = \tilde{v} + G\tilde{u} = g. \]

Moreover, since \( \tilde{u} \in D(G_e) \) we have that

\[ \pi_{BD(G)}u = (1 + h)^{-1} \left( -D \pi_{BD(D)}G_e\tilde{u} \right). \]

Using that

\[ \pi_{BD(D)}G_e\tilde{u} = \pi_{BD(D)}G_e(1 - DG_e)^{-1} (f - DcG) \]
\[ = \pi_{BD(D)}(1 - G_eD)^{-1}G_ef + \pi_{BD(D)}g - \pi_{BD(D)}(1 - G_eD)^{-1}g \]
\[ = \pi_{BD(D)}(1 - G_eD)^{-1} (G_ef - g) \]
\[ = -\pi_{BD(D)}\tilde{v} \]

we obtain

\[ \dot{D} \pi_{BD(D)}v + \pi_{BD(D)}u = \dot{D} \pi_{BD(D)}\tilde{v} - (1 + h)^{-1} \left( -D \pi_{BD(D)}G_e\tilde{u} \right) + (1 + h)^{-1} \left( -D \pi_{BD(D)}G_e\tilde{u} \right) \]
\[ = \dot{D} \pi_{BD(D)}\tilde{v} \]
\[ = -\dot{D} \pi_{BD(D)}G_e\tilde{u}. \]

Thus,

\[ \pi_{BD(D)}u = (1 + h)^{-1} \left( -D \pi_{BD(D)}G_e\tilde{u} \right) = (1 + h)^{-1} \left( \dot{D} \pi_{BD(D)}v + \pi_{BD(D)}u \right), \]

which is equivalent to \((\pi_{BD(G)}u, \dot{D} \pi_{BD(D)}v) \in h\), which yields \((u, v) \in D(A)\).

We conclude this section with a characterization of all skew-selfadjoint realizations of \( A \).

**Proposition 3.7.** Let \( A \) be linear, where \( D(A) \) is given by \( h \) for some linear relation \( h \subseteq BD(G) \oplus BD(G) \). Then \( A \) is densely defined and \( A^* \subseteq -\left( \begin{array}{cc} 0 & D \\ G_e & 0 \end{array} \right) \) with

\[ D(A^*) = \left\{ (x, y) \in D(G) \times D(D) \mid (\pi_{BD(G)}x, \dot{D} \pi_{BD(D)}y) \in -h^* \right\}. \]

**Proof.** Note that due to the linearity of \( h \) we have \((0, 0) \in h \) and thus,

\[ \left( \begin{array}{cc} 0 & D_e \\ G_e & 0 \end{array} \right) \subseteq A. \]

This shows that \( A \) is densely defined. Moreover, we deduce that

\[ A^* \subseteq -\left( \begin{array}{cc} 0 & D \\ G & 0 \end{array} \right). \]
Let \((x, y) \in \mathcal{D}(A^*)\). Then, for all \((u, v) \in \mathcal{D}(A)\) we have

\[
\left\langle A \begin{pmatrix} u \\ v \end{pmatrix} \bigg| \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_{H_0 \oplus H_1} = \left\langle \begin{pmatrix} u \\ v \end{pmatrix} \bigg| \begin{pmatrix} -Dy \\ -Gx \end{pmatrix} \right\rangle_{H_0 \oplus H_1}.
\]

The left hand side of the latter equation gives

\[
\langle Dv|x \rangle_{H_0} + \langle Gu|y \rangle_{H_1} = \langle D_x \pi^*_D, \pi_D, v|x \rangle_{H_0} + \langle \pi^*_BD(G) \cdot D \pi BD(D)v|x \rangle_{H_0} + \langle \pi^*_BD(G) \cdot G \pi BD(G)u|y \rangle_{H_1} = -\langle \pi^*_D, \pi_D, v|Gx \rangle_{H_1} + \langle \pi^*_BD(G) \cdot D \pi BD(D)v|x \rangle_{H_0} - \langle \pi^*_G, \pi_G, u|Dy \rangle_{H_0} + \langle \pi^*_BD(G) \cdot G \pi BD(G)u|y \rangle_{H_1}.
\]

On the other hand

\[
\langle u| -Dy \rangle_{H_0} + \langle v| -Gx \rangle_{H_1} = -\langle \pi^*_G, \pi_G, u|Dy \rangle_{H_0} - \langle \pi^*_BD(G) \cdot \pi BD(D)u|Dy \rangle_{H_0} - \langle \pi^*_D, \pi_D, v|Gx \rangle_{H_1} - \langle \pi^*_BD(G) \cdot \pi BD(D)u|Gx \rangle_{H_1}.
\]

Thus, we end up with

\[
\langle \pi^*_BD(G) \cdot D \pi BD(D)v|x \rangle_{H_0} + \langle \pi^*_BD(G) \cdot G \pi BD(G)u|y \rangle_{H_1} = -\left( \langle \pi^*_BD(G) \cdot \pi BD(D)v|Gx \rangle_{H_1} + \langle \pi^*_BD(G) \cdot \pi BD(D)u|Dy \rangle_{H_0} \right) = -\left( \langle \pi^*_BD(G) \cdot D \pi BD(D)v|Gx \rangle_{H_1} + \langle D \pi BD(D)v|Gx \rangle_{H_1} \right),
\]

which yields

\[
\langle \pi^*_BD(G) \cdot D \pi BD(D)v|x \rangle_{H_1} = -\langle \pi^*_BD(D) \cdot G \pi BD(G)u|y \rangle_{H_1} - \langle \pi^*_BD(D) \cdot \pi BD(D)u|Dy \rangle_{H_0}.
\]

Hence,

\[
\langle D \pi BD(D)v|\pi BD(G)x \rangle_{BD(G)} = \langle \pi BD(G)u| - D \pi BD(D)y \rangle_{BD(G)},
\]

which yields, using Remark 2.1

\[
(\pi BD(G)x, -D \pi BD(D)y) \in h^*.
\]

Assume now that \((x, y) \in \mathcal{D}(G) \times \mathcal{D}(D)\) with \((\pi BD(G)x, -D \pi BD(D)y) \in h^*\). Then, for each \((u, v) \in \mathcal{D}(A)\) we compute

\[
\left\langle A \begin{pmatrix} u \\ v \end{pmatrix} \bigg| \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_{H_0 \oplus H_1} = \langle Dv|x \rangle_{H_0} + \langle Gu|y \rangle_{H_1} = -\langle \pi^*_D, \pi_D, v|Gx \rangle_{H_1} + \langle \pi^*_BD(G) \cdot D \pi BD(D)v|x \rangle_{H_0} - \langle \pi^*_G, \pi_G, u|Dy \rangle_{H_0} + \langle \pi^*_BD(G) \cdot G \pi BD(G)u|y \rangle_{H_1} = -\langle u|Gx \rangle_{H_1} + \langle \pi^*_BD(G) \cdot \pi BD(D)v|Gx \rangle_{H_1} + \langle \pi^*_BD(G) \cdot D \pi BD(D)v|x \rangle_{H_0}.
\]
4 Classical trace spaces

\[-(u|Dy)_{H_0} + \langle \pi_{BD(G)} \pi_{BD(G)} u|Dy \rangle_{H_0} + \langle \pi_{BD(D)}^* G \pi_{BD(G)} u|y \rangle_{H_1}\]

\[= -(v|Gx)_{H_1} + (\dot{D} \pi_{BD(D)} u|\pi_{BD(G)} x)_{BD(G)}\]

\[-(u|Dy)_{H_0} - \langle \pi_{BD(G)} u| - \dot{D} \pi_{BD(D)} y \rangle_{BD(G)}\]

\[\left( \begin{pmatrix} u \\ v \end{pmatrix} \right) - \left( \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right)_{H_0 \oplus H_1},\]

where we again have used Remark 2.1. This completes the proof. \[\square\]

As a consequence of our considerations above, we obtain the following corollary.

**Corollary 3.8.** The operator $A$ is skew-selfadjoint if and only if there exists a skew-selfadjoint relation $h \subseteq BD(G) \oplus BD(G)$ such that the domain of $A$ is given by (2).

### 4 Classical trace spaces

In this section we compare the classical trace spaces $H^{\pm \frac{1}{2}}(\partial\Omega)$ with the abstract boundary data spaces $BD(\text{grad})$ and $BD(\text{div})$, where grad and div are defined as in Section 2. Throughout this section we assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded Lipschitz domain.

We recall the definition of the classical trace spaces $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$.

**Proposition 4.1** ([IS Theorem 1.2]). The operator

$$\gamma_D : C^\infty(\overline{\Omega}) \subseteq H^1(|\text{grad}| + i) \to L_2(\partial\Omega)$$

$$u \mapsto u|_{\partial\Omega}$$

is bounded and thus, it has a unique bounded extension to $H^1(|\text{grad}| + i)$, which will be again denoted by $\gamma_D$.

**Definition.** We set $H^{\frac{1}{2}}(\partial\Omega) := \gamma_D[H^1(|\text{grad}| + i)]$ and equip this space with the norm

$$|u|_{H^{\frac{1}{2}}(\partial\Omega)} := \sqrt{|u|_{L_2(\partial\Omega)}^2 + \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^n} \, dy \, dx.}$$

Moreover, we set $H^{-\frac{1}{2}}(\partial\Omega) := H^\ast(\partial\Omega)$, the dual space of $H^{\frac{1}{2}}(\partial\Omega)$.

**Remark 4.2.** One can show that

$$\iota : H^{\frac{1}{2}}(\partial\Omega) \to L_2(\partial\Omega)$$

$$u \mapsto u$$

is bounded and has dense range, see [IS Theorem 4.9] (indeed, one can even show the compactness of this embedding, [IS Theorem 6.2]). Consequently, one obtains that

$$\iota' : L_2(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$

$$f \mapsto \left( H^{\frac{1}{2}}(\partial\Omega) \ni u \mapsto \langle f|u \rangle_{L_2(\partial\Omega)} \right)$$
is also bounded with dense range (and even compact).

**Proposition 4.3** ([18 Theorems 4.10, 5.5, 5.7]). The operator \( \gamma_D : H^1(\| \cdot \| + i) \to H^\frac{1}{2}(\partial \Omega) \) is bounded and \( N(\gamma_D) = H^1(\| \cdot \| + i) \). Moreover, there exists a bounded right inverse, i.e. there exists a bounded linear operator
\[
E : H^\frac{1}{2}(\partial \Omega) \to H^1(\| \cdot \| + i)
\]
such that \( \gamma_D \circ E = 1_{H^\frac{1}{2}(\partial \Omega)} \) the identity on \( H^\frac{1}{2}(\partial \Omega) \).

With the help of the last proposition we can show that \( BD(\text{grad}) \) and \( H^\frac{1}{2}(\partial \Omega) \) are isomorphic.

**Corollary 4.4.** The operator \( \gamma := \gamma_D \circ \pi^*_{BD(\text{grad})} : BD(\text{grad}) \to H^\frac{1}{2}(\partial \Omega) \) is a Banach space isomorphism.

**Proof.** That \( \gamma \) is one-to-one and bounded follows from Proposition 4.3. To see that \( \gamma \) is onto, let \( u \in H^\frac{1}{2}(\partial \Omega) \). Then we set \( v := \pi_{BD(\text{grad})} Eu \) and obtain
\[
u = \gamma_D(Eu) = \gamma_D(\pi^*_{\text{grad}_c}\pi_{\text{grad}_c} Eu + \pi^*_\text{BD(grad)}\pi_{\text{BD(grad)}} Eu) = \gamma_D(\pi^*_\text{BD(grad)}\pi_{\text{BD(grad)}} Eu) = \gamma(v).
\]
Moreover,
\[
|v|_{BD(\text{grad})} = |\pi^*_\text{BD(grad)} v|_{H^1(\| \text{grad} \| + i)} \leq \|E\| \|u\|_{H^\frac{1}{2}(\partial \Omega)},
\]
which shows the continuity of \( \gamma^{-1} \).

Using this observation, we may define an alternative, but equivalent, norm on \( H^\frac{1}{2}(\partial \Omega) \) by
\[
|u| := |\gamma^{-1} u|_{BD(\text{grad})} \quad (u \in H^\frac{1}{2}(\partial \Omega)).
\]
Using this norm, the operator \( \gamma : BD(\text{grad}) \to H^\frac{1}{2}(\partial \Omega) \) becomes unitary. In the subsequent part we will always assume that \( H^\frac{1}{2}(\partial \Omega) \) is equipped with this equivalent norm. In order to deal with normal derivatives, we need the following representation of the dual of \( BD(\text{grad}) \).

**Lemma 4.5.** Let \( T R(\text{div}) := (\text{div} - \text{div}_c) [H^1(\| \text{div} \| + i)] \subseteq H^{-1}(\| \cdot \| + i) \). Then, the mapping \( \Phi : T R(\text{div}) \to BD(\text{grad})' \) given by
\[
\Phi ((\text{div} - \text{div}_c) v)(u) := \left< (\text{div} - \text{div}_c) v | \pi^*_\text{BD(grad)} u \right>_{H^{-1}(\| \cdot \| + i), H^1(\| \cdot \| + i)}
\]
is unitary.

\(^8\)Recall that according to Proposition 2.10 the operator \( \text{div}_c \) is bounded as an operator from \( L^2(\Omega)^n \) to \( H^{-1}(\| \cdot \| + i) = H^{-1}(\| \cdot \| + i). \)
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Proof. Since the functional $(\text{div} - \text{div}_c)v$ vanishes on $B\mathcal{D}(\text{grad})^\perp = H^1(|\text{grad}_c| + i)$ we easily see that $\Phi$ is isometric. To show the surjectivity of $\Phi$ we take $\varphi \in B\mathcal{D}(\text{grad})'$. Then there exists $\tilde{u} \in B\mathcal{D}(\text{grad})$ such that
\[
\varphi(u) = \langle \tilde{u}|u \rangle_{B\mathcal{D}(\text{grad})} = \langle \text{grad} \pi_{\mathcal{B}(\text{grad})}^* \tilde{u}|\text{grad} \pi_{\mathcal{B}(\text{grad})}^* u \rangle_{L^2(\Omega)} + \langle \text{grad} \pi_{\mathcal{B}(\text{grad})}^* \tilde{u}|\text{grad} \pi_{\mathcal{B}(\text{grad})}^* u \rangle_{L^2(\Omega)}
\]
for every $u \in B\mathcal{D}(\text{grad})$, which proves that $\Phi$ is onto. \hfill \Box

Corollary 4.6. The spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $\mathcal{T}\mathcal{R}(\text{div})$ are isomorphic via the mapping $\Phi^* \circ \gamma'$, where $\gamma'$ denotes the dual mapping of $\gamma$, given in Corollary 4.4.

For a function $v \in H^1(|\text{div}| + i)$ one can define the boundary term $v \cdot N \in H^{-\frac{1}{2}}(\partial\Omega)$, where $N$ denotes the unit outward normal vector field on $\partial\Omega$ (which exists according to [18, Lemma 4.2]) via Green’s formula 9:
\[
\langle v \cdot N|u \rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} = \langle \text{div} v|E(u) \rangle_{L^2(\Omega)} + \langle \text{grad} E(u)|\text{grad}^* u \rangle_{L^2(\Omega)}
\]
for every $u \in B\mathcal{D}(\text{grad})$, which proves that $\Phi$ is onto.

Finally, we recall a result from [27].

Proposition 4.7 ([27, Theorem 4.5]). The operator
\[
\tilde{\gamma} := (\text{div} - \text{div}_c)\pi_{\mathcal{B}(\text{div})}^* : B\mathcal{D}(\text{div}) \to \mathcal{T}\mathcal{R}(\text{div})
\]
is unitary.

Some classical boundary conditions

This subsection is devoted to the study of classical boundary conditions within the framework of abstract boundary data spaces. Moreover, we discuss which boundary conditions yield a maximal monotone realization of the operator
\[
A \subseteq \begin{pmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{pmatrix}.
\]

To avoid technicalities, we only treat the most simple cases of such boundary conditions and refer to the next section for more advanced examples. We start with inhomogeneous Dirichlet and Neumann boundary conditions.

9Note that $\gamma^* = \gamma^{-1} : H^\frac{1}{2}(\partial\Omega) \to B\mathcal{D}(\text{grad})$ according to our choice of the norm on $H^\frac{1}{2}(\partial\Omega)$. 

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Dirichlet and Neumann boundary conditions

Throughout let \((u, v) \in \mathcal{D}(\text{grad}) \times \mathcal{D}(\text{div})\). The inhomogeneous Dirichlet boundary condition reads as
\[
\gamma_D u = f,
\]
for some \(f \in H^{\frac{3}{2}}(\partial \Omega)\). The latter is equivalent to the fact that
\[
\pi_{\mathcal{B}D(\text{grad})} u = \gamma^* f,
\]
by Corollary 4.4 and thus, the boundary relation \(h \subseteq \mathcal{B}D(\text{grad}) \oplus \mathcal{B}D(\text{grad})\) may be given by
\[
h := \{ (\gamma^* f, y) \mid y \in \mathcal{B}D(\text{grad}) \}.
\]
Obviously, this relation is maximal monotone and hence, the operator \(A\) with domain
\[
\mathcal{D}(A) = \left\{ (u, v) \in \mathcal{D}(\text{grad}) \times \mathcal{D}(\text{div}) \mid (\pi_{\mathcal{B}D(\text{grad})} u, \text{div } \pi_{\mathcal{B}D(\text{div})} v) \in h \right\}
\]
is maximal monotone. Note that only in the case of \(f = 0\), the operator is skew-selfadjoint.

In the same way one might deal with Neumann boundary conditions, given by
\[
v \cdot N = g
\]
for some \(g \in H^{-\frac{3}{2}}(\partial \Omega)\). Using (4) the latter means that for all \(u \in H^{\frac{3}{2}}(\partial \Omega)\) we have that
\[
\langle g | u \rangle_{H^{-\frac{3}{2}}(\partial \Omega), H^{\frac{3}{2}}(\partial \Omega)} = \Phi ((\text{div } - \text{div}_c) v) (\gamma^* u)
\]
or equivalently for all \(w \in \mathcal{B}D(\text{grad})\)
\[
(\gamma' g) (w) = \langle g | \gamma w \rangle_{H^{-\frac{3}{2}}(\partial \Omega), H^{\frac{3}{2}}(\partial \Omega)} = \Phi ((\text{div } - \text{div}_c) v) (w).
\]
Since \((\text{div } - \text{div}_c) v = (\text{div } - \text{div}_c) \pi_{\mathcal{B}D(\text{div})} \pi_{\mathcal{BD}(\text{grad})} v\), the latter means
\[
\gamma' g = \Phi \gamma \pi_{\mathcal{BD}(\text{div})} v
\]
or equivalently
\[
\pi_{\mathcal{BD}(\text{div})} v = \tilde{\gamma} \Phi^* \left( \gamma' g \right).
\]
using Lemma 4.3 and Proposition 4.7. Thus, the boundary relation \(h\) is given by
\[
h := \left\{ \left( x, \text{div } \tilde{\gamma} \Phi^* \left( \gamma' g \right) \right) \mid x \in \mathcal{B}D(\text{grad}) \right\},
\]
which is again maximal monotone and thus, the operator \(A\) with domain
\[
\mathcal{D}(A) = \left\{ (u, v) \in \mathcal{D}(\text{grad}) \times \mathcal{D}(\text{div}) \mid (\pi_{\mathcal{B}D(\text{grad})} u, \text{div } \pi_{\mathcal{BD}(\text{div})} v) \in h \right\}
\]
This gives
\[
L \text{ is maximal monotone if and only if } \gamma w = \langle \gamma \Phi^* \rangle g
\]
is maximal monotone. Note again that the operator gets skew-selfadjoint if and only if \( g = 0 \).

### Robin type boundary conditions

Let \((u, v) \in D(\text{grad}) \times D(\text{div})\). A Robin-type boundary condition may be written as
\[
\ell' \ell \gamma Du - v \cdot N = g,
\]
for some \( g \in H^{-\frac{1}{2}}(\partial \Omega) \), where \( \ell \) denotes the embedding \( H^{\frac{1}{2}}(\partial \Omega) \hookrightarrow L_2(\partial \Omega) \) and \( \ell' \) denotes the embedding \( L_2(\partial \Omega) \hookrightarrow H^{-\frac{1}{2}}(\partial \Omega) \) as in Remark 4.2. The boundary condition (6) means that for each \( w \in BD(\text{grad}) \) we have that
\[
\langle \ell' \ell \gamma Du | \ell \gamma w \rangle_{L_2(\partial \Omega)} - \Phi \left( \gamma \pi_{BD(\text{div})} v \right)(w) = \langle g | \gamma w \rangle_{H^{-\frac{1}{2}}(\partial \Omega), H^{\frac{1}{2}}(\partial \Omega)},
\]
or equivalently
\[
\langle \gamma^* \ell' \ell \gamma BD(\text{grad}) u | w \rangle_{BD(\text{grad})} = \langle \ell \gamma \pi_{BD(\text{div})} u | \ell \gamma w \rangle_{L_2(\partial \Omega)} - \Phi \left( \gamma \pi_{BD(\text{div})} v \right)(w) = \Phi \left( \gamma^* \Phi^* \gamma' g + \pi_{BD(\text{div})} v \right) \pi_{BD(\text{grad})} w
\]
\[
= \left( \langle \text{div} \langle \gamma^* \Phi^* \gamma' g + \pi_{BD(\text{div})} v \rangle \pi_{BD(\text{grad})} w \rangle_{H^{-1}(\text{grad} | + i), H^1(\text{grad} | + i)}
\]
\[
+ \left( \text{div} \langle \gamma^* \Phi^* \gamma' g + \pi_{BD(\text{div})} v \rangle \pi_{BD(\text{grad})} w \rangle_{L_2(\Omega)}
\]
\[
+ \left( \text{div} \langle \gamma^* \Phi^* \gamma' g + \pi_{BD(\text{div})} v \rangle \pi_{BD(\text{grad})} w \rangle_{L_2(\Omega)}
\]
\[
= \langle \text{div} \langle \gamma^* \Phi^* \gamma' g + \pi_{BD(\text{div})} v \rangle \pi_{BD(\text{grad})} w \rangle_{BD(\text{grad})}.
\]
This gives
\[
\gamma^* \ell' \ell \gamma BD(\text{grad}) u = \text{div} \left( \gamma^* \Phi^* \gamma' g + \pi_{BD(\text{div})} v \right),
\]
which leads to the following definition of the boundary relation \( h \)
\[
h := \left\{ (x, y) \mid \gamma^* \ell' \ell \gamma x = \text{div} \left( \gamma^* \Phi^* \gamma' g + y \right) \right\}.
\]
To see that this relation is maximal monotone, we state the following trivial observation.

**Lemma 4.8.** Let \( H \) be a Hilbert space and \( L \subseteq H \oplus H \). Moreover let \((x, y) \in H \times H\). Then \( L \) is maximal monotone if and only if
\[
L + \{(x, y)\} = \{(u + x, v + y) \mid (u, v) \in L\}.
\]
is maximal monotone.

The latter gives, that $h$ is maximal monotone if and only if
\[ h - \left\{ (0, \text{div } \tilde{\varphi} \Phi^* \gamma') \right\} \]
is maximal monotone. The latter relation is nothing but the non-negative, selfadjoint operator $\gamma^* \iota^* \iota \gamma$ and thus, maximal monotone. We note here that a Robin boundary condition of the form
\[ \iota^* \iota \gamma D u + v \cdot N = g \] (7)
does not lead to a maximal monotone relation $h$ and hence not to a maximal monotone realization of (5). Indeed, a boundary condition of the form (7) would yield a relation of the form
\[ h = \left\{ (x, y) \mid -\gamma^* \iota^* \gamma x = \text{div } \tilde{\varphi} \Phi^* \gamma' g + y \right\}, \]
which is not even monotone.

**Remark 4.9.** In applications it turns out that different boundary conditions are imposed on different parts of the boundary. We refer the reader to the next section, where in the concrete case of a contact problem in visco-elasticity such boundary conditions are studied.

### 5 Examples

#### 5.1 Port-Hamiltonian systems

In this section we study so-called linear Port-Hamiltonian systems. Originally, these systems were defined in the context of differential forms in [36]. However, we follow the notion given in [13, Chapter 7]. Throughout, let $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$ with $a < b$. Moreover, let $P_1 \in \mathbb{C}^{n \times n}$ an invertible, selfadjoint matrix, $P_0 \in \mathbb{C}^{n \times n}$ be skew-selfadjoint and $H \in L^\infty([a, b]; \mathbb{C}^{n \times n})$ such that $H(t)$ is selfadjoint and there exists $c > 0$ with $H(t) \geq c$ for almost every $t \in [a, b]$. The differential operator under consideration is a suitable restriction $A$ of
\[ P_1 \partial H + P_0 H, \]
with maximal domain, where $\partial : H^1([a, b]; \mathbb{C}^n) \subseteq L_2([a, b]; \mathbb{C}^n) \to L_2([a, b]; \mathbb{C}^n)$ denotes the usual weak derivative on $L_2$. In particular we want to characterize those restrictions $A$, which yield a maximal monotone operator in a suitable Hilbert space. In case of a linear operator $A$, a class of maximal monotone realizations was given in [13, Section 4], [12, Theorem 7.2.3] (see also Theorem 5.6 below).

**Lemma 5.1.** Let $H$ be a Hilbert space and $P \in L(H)$ selfadjoint with $P \geq c > 0$. Moreover, let $A \subseteq H \oplus H$ be a maximal monotone relation. We denote by $H_P$ the Hilbert space $H$ equipped with the weighted inner product
\[ (x|y)_H := (Px|y)_H \quad (x, y \in H). \]
Then $AP := \{(x, y) \in H_P \oplus H_P \mid (Px, y) \in A\}$ is maximal monotone in $H_P$. 

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Proof. The monotonicity of $AP$ in $H_P$ is clear. Moreover, if $(u, v) \in H_P \oplus H_P$ satisfies

$$\Re \langle x - u | y - v \rangle_{H_P} \geq 0$$

for each $(x, y) \in AP$, then

$$\Re \langle \tilde{x} - Pu | y - v \rangle_H \geq 0$$

for each $(\tilde{x}, y) \in A$ and thus, $(Pu, v) \in A$. The latter yields $(u, v) \in AP$ and thus, $AP$ is maximal monotone.

The last lemma shows, that we can assume without loss of generality that $H(t) = 1$ for each $t \in [a, b]$. For linear maximal monotone operators $A$, the last lemma was also shown in [12, Lemma 7.2.2] with an alternative proof. Moreover, since $P_0$ is skew-selfadjoint, it suffices to treat the case $P_0 = 0$, since if $P_1 \partial + P_0$ is maximal monotone then so is $P_1 \partial$ and vice versa.

Thus, we are led to consider maximal monotone realizations of the operator $P_1 \partial$. Although, this operator seems not to be of the form discussed in Section 3, a simple trick adopted from [22] will allow us to write the operator as a block operator matrix of the form

$$\begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix}.$$  

Without loss of generality we assume that the interval $[a, b]$ is symmetric around 0, i.e. $a = -b$. We consider the following operators.

Definition. Let $L_{2,e}([-b, b]; \mathbb{C}^n) := \{ f \in L_2([-b, b]; \mathbb{C}^n) \mid f(x) = f(-x) \quad x \in [-b, b] \text{ a.e.}\}$ and $L_{2,o}([-b, b]; \mathbb{C}^n) := \{ f \in L_2([-b, b]; \mathbb{C}^n) \mid f(x) = -f(-x) \quad x \in [-b, b] \text{ a.e.}\}$. Then clearly, $L_{2,e}$ and $L_{2,o}$ are orthogonal closed subspaces of $L_2$ such that

$$L_2([-b, b]; \mathbb{C}^n) = L_{2,e}([-b, b]; \mathbb{C}^n) \oplus L_{2,o}([-b, b]; \mathbb{C}^n).$$

We denote the corresponding projectors by $\pi_e$ and $\pi_o$, respectively. Moreover, we define

$$\partial_e : H^1([-b, b]; \mathbb{C}^n) \cap L_{2,e}([-b, b]; \mathbb{C}^n) \subseteq L_{2,e}([-b, b]; \mathbb{C}^n) \to L_{2,o}([-b, b]; \mathbb{C}^n)$$

and

$$\partial_o : H^1([-b, b]; \mathbb{C}^n) \cap L_{2,o}([-b, b]; \mathbb{C}^n) \subseteq L_{2,o}([-b, b]; \mathbb{C}^n) \to L_{2,e}([-b, b]; \mathbb{C}^n)$$

as the usual weak derivative restricted to the even and odd functions, respectively. Consequently,

$$\partial = \begin{pmatrix} \pi_e & \pi_o^* \\ \pi_e^* & \pi_o \end{pmatrix} \begin{pmatrix} 0 & \partial_0 \\ \partial_e & 0 \end{pmatrix},$$

which shows that $\partial$ and $\begin{pmatrix} 0 & \partial_0 \\ \partial_e & 0 \end{pmatrix}$ are unitarily equivalent.

Lemma 5.2. We set $\partial_{e,c} := -\partial_e^*$ and $\partial_{o,c} := -\partial_o^*$. Then $\partial_{o,c} \subseteq \partial_o$ and $\partial_{e,c} \subseteq \partial_e$ with $D(\partial_{o,c}) = \{ u \in D(\partial_o) \mid u(-b) = u(b) = 0 \}$ and $D(\partial_{o,c}) = \{ u \in D(\partial_e) \mid u(-b) = u(b) = 0 \}$. Moreover,

$$BD(\partial_o) = \text{span} \{ \sinh e_i \mid i \in \{1, \ldots, n\} \} \quad \text{and} \quad BD(\partial_e) = \text{span} \{ \cosh e_i \mid i \in \{1, \ldots, n\} \},$$

where $e_i$ denotes the $i$-th canonical basis vector in $\mathbb{C}^n$.

Proof. The proof is straightforward and we therefore omit it. 

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5.1 Port-Hamiltonian systems

We come back to the operator $P_1 \partial$, which can be written as

$$\begin{pmatrix} \pi_e^* & \pi_o^* \\ \pi_o P_1 \pi_e^* & 0 \end{pmatrix} \begin{pmatrix} \partial_c & 0 \\ 0 & \partial_c \end{pmatrix} \begin{pmatrix} \pi_o^* \\ \pi_e^* P_1 \pi_o^* \partial_c \end{pmatrix} = \begin{pmatrix} \pi_e^* & \pi_o^* \\ \pi_o P_1 \pi_e^* \partial_c & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \pi_o^* \pi_e^* \partial_c \end{pmatrix}$$

and the operator $\begin{pmatrix} 0 & \pi_e P_1 \pi_e^* \partial_c \\ \partial_c \pi_o P_1 \pi_e^* & 0 \end{pmatrix}$ now fits into our abstract framework with $D := \pi_e P_1 \pi_e^* \partial_c$, $D_c := \pi_e P_1 \pi_e^* \partial_{b,c}$ and $G := \partial_c \pi_o P_1 \pi_e^*$, $G_e := \partial_{c,e} \pi_o P_1 \pi_e^*$.

**Remark 5.3.** We note that $BD(G)$ and $BD(D)$ are both $n$-dimensional spaces. More precisely, let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of the symmetric and invertible matrix $P_1$ counted with multiplicity. Moreover, we denote by $b_1, \ldots, b_n$ the corresponding pairwise orthonormal eigenvectors. Then

$$BD(G) = \text{span}\{ (x \mapsto \cosh (\lambda_i^{-1} x) b_i) \mid i \in \{1, \ldots, n\}\},$$

$$BD(D) = \text{span}\{ (x \mapsto \sinh (\lambda_i^{-1} x) b_i) \mid i \in \{1, \ldots, n\}\}.$$  

**Theorem 5.4.** Let $A \subseteq P_1 \partial \mathcal{H} + P_0 \mathcal{H}$ an arbitrary (possibly nonlinear) restriction and let $G, G_c, D, D_c$ as above. Then $A$ is maximal monotone with respect to the weighted inner product

$$\langle u|v \rangle_{\mathcal{H}} := \langle Hu|v \rangle_{L^2([-b,b];\mathbb{C}^n)} \quad (u, v \in L^2([-b,b];\mathbb{C}^n))$$

if and only if there exists a maximal monotone relation $h \subseteq BD(G) \oplus BD(G)$ such that

$$D(A) = \left\{ u \in L^2([-b,b];\mathbb{C}^n) \mid \mathcal{H} u \in H^1([-b,b];\mathbb{C}^n), \left( \pi_{BD(G)} \pi_e \mathcal{H} u, D_{\pi_{BD(D)} \pi_e \mathcal{H} u} \right) \in h \right\}.$$  

**Proof.** This is a direct consequence of Theorem 3.1 and the considerations above. \(\square\)

In \[12\] we find a characterization for a class of maximal monotone realizations of $P_1 \partial \mathcal{H} + P_0 \mathcal{H}$ in terms of the so-called boundary flow and boundary effort, defined as

$$f_\partial(u) := \frac{1}{\sqrt{2}} (-P_1 \mathcal{H} u(b) + P_1 \mathcal{H} u(-b))$$

and

$$e_\partial(u) := \frac{1}{\sqrt{2}} (\mathcal{H} u(b) + \mathcal{H} u(-b)),$$

respectively. In the next lemma we provide a formulation of these terms within our framework.

**Lemma 5.5.** Let $G$ be as above and denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of $P_1$ (counted with multiplicity) and by $b_1, \ldots, b_n \in \mathbb{C}^n$ the corresponding pairwise orthonormal eigenvectors. Define

$$Q : BD(G) \to \mathbb{C}^n$$

$$v \mapsto \sqrt{2} v(b)$$
and $S \in \mathbb{C}^{n \times n}$ via $S_{bi} := \lambda_i \tanh(\lambda_i^{-1} b_i)$ for $i \in \{1, \ldots, n\}$. Then $S$ is selfadjoint and strictly positive definite and $\sqrt{SQ}$ is unitary. Moreover, for $\mathcal{H}u \in H^1([-b, b]; \mathbb{C}^n)$ we have that
\[
e_{\partial}(u) = Q \left( \pi_{BD(G)} \pi_e \mathcal{H}u \right)
\]
\[
f_{\partial}(u) = -SQ \left( \dot{D} \pi_{BD(D)} \pi_o \mathcal{H}u \right).
\]

**Proof.** The fact that $S$ is selfadjoint and strictly positive definite holds, since
\[
S = U^* \begin{pmatrix}
\lambda_1 \tanh(\lambda_1^{-1} b) & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \\
\vdots & \ddots & 0 & \\
0 & \cdots & 0 & \lambda_n \tanh(\lambda_n^{-1} b)
\end{pmatrix} U,
\]
where $U$ is the unitary matrix defined via $Ue_i = b_i$ for each $i \in \{1, \ldots, n\}$. Moreover, for $v \in BD(G)$ we find a representation $v = \sum_{i=1}^n c_i \cosh(\lambda_i^{-1} \cdot b_i)$ for suitable constants $c_1, \ldots, c_n \in \mathbb{C}$. Thus, using $G v = \sum_{i=1}^n c_i \sinh(\lambda_i^{-1} \cdot b_i)$ and integration by parts we obtain
\[
|v|_{BD(G)}^2 = \sum_{i=1}^n |c_i|^2 \left( \int_{-b}^{b} \sinh(\lambda_i^{-1} x)^2 \, dx + \int_{-b}^{b} \cosh(\lambda_i^{-1} x)^2 \, dx \right)
\]
\[
= 2 \sum_{i=1}^n |c_i|^2 \lambda_i \sinh(\lambda_i^{-1} b) \cosh(\lambda_i^{-1} b)
\]
\[
= 2 \sum_{i=1}^n c_i \lambda_i \sinh(\lambda_i^{-1} b) b_i \sum_{i=1}^n c_i \cosh(\lambda_i^{-1} b) b_i \in \mathbb{C}^n
\]
\[
= (SQv|Qv)_{\mathbb{C}^n}
\]
\[
= |\sqrt{S}Qv|_{\mathbb{C}^n}^2.
\]

Finally, for $\mathcal{H}u \in H^1([-b, b]; \mathbb{C}^n)$ we compute
\[
e_{\partial}(u) = \frac{1}{\sqrt{2}} \left( \mathcal{H}u(b) + \mathcal{H}u(-b) \right)
\]
\[
= \sqrt{2} \left( \pi_e \mathcal{H}u \right)(b)
\]
\[
= \sqrt{2} \left( \pi_{BD(G)} \pi_e \mathcal{H}u \right)(b)
\]
\[
= Q \pi_{BD(G)} \pi_e \mathcal{H}u,
\]
as well as
\[
f_{\partial}(u) = \frac{1}{\sqrt{2}} \left( -P_1 \mathcal{H}u(b) + P_1 \mathcal{H}u(-b) \right)
\]
\[
= -\sqrt{2} P_1 \left( \pi_o \mathcal{H}u \right)(b)
\]
\[
= -\sqrt{2} P_1 \left( \pi_{BD(D)} \pi_o \mathcal{H}u \right)(b).
\]
Using that there exist \( c_1, \ldots, c_n \in \mathbb{C} \) such that \( \pi_{BD(D)} \pi_o \mathcal{H} u = \sum_{i=1}^{n} c_i \sinh(\lambda_i^{-1} \cdot b_i) \) we get that
\[
f_o(u) = -\sqrt{2} \sum_{i=1}^{n} c_i \lambda_i \sinh(\lambda_i^{-1} b_i).
\]

On the other hand we have
\[
\mathcal{D} \pi_{BD(D)} \pi_o \mathcal{H} u = \sum_{i=1}^{n} c_i \cosh(\lambda_i^{-1} \cdot b_i)
\]
from which we derive that
\[
f_o(u) = -SQ \left( \mathcal{D} \pi_{BD(D)} \pi_o \mathcal{H} u \right). \quad \Box
\]

We now recall a part of [12, Theorem 7.2.3] and provide an alternative proof within our framework.

**Theorem 5.6.** Let \( A \subseteq P_1 \partial \mathcal{H} + P_0 \mathcal{H} \) a linear restriction and let \( G, G_e, D, D_e \) as above. If there exists a matrix \( V \in \mathbb{C}^{n \times n} \) such that \( V^* V \leq 1 \) and
\[
\mathcal{D}(A) = \left\{ u \in L_2([-b, b]; \mathbb{C}^n) \left| \mathcal{H} u \in H^1([-b, b]; \mathbb{C}^n), \ (1 + V^* \cdot V) \left( f_o(u) \right. \right) = 0 \right\},
\]
then \( A \) defines a maximal monotone operator with respect to the weighted inner product
\[
\langle u|v \rangle_H := \langle \mathcal{H} u|v \rangle_{L_2([-b, b]; \mathbb{C}^n)} \quad (u, v \in L_2([-b, b]; \mathbb{C}^n)).
\]

**Proof.** According to Lemma 5.5 we may rewrite the domain of \( A \) as
\[
\mathcal{D}(A) = \left\{ u \in L_2([-b, b]; \mathbb{C}^n) \left| \mathcal{H} u \in H^1([-b, b]; \mathbb{C}^n), \ (1 + V^* \cdot V) \left( \begin{array}{c} 0 -S \\ 1 0 \end{array} \right) \left( \begin{array}{c} Q \pi_{BD(G)} \pi_o \mathcal{H} u \\ Q \mathcal{D} \pi_{BD(D)} \pi_o \mathcal{H} u \end{array} \right) = 0 \right\}.
\]

By Theorem 5.4 it suffices to check whether
\[
h := \left\{ (x, y) \in BD(G) \oplus BD(G) \left| \ (1 + V \cdot V) \left( \begin{array}{c} 0 -S \\ 1 0 \end{array} \right) \left( \begin{array}{c} Qx \\ Qy \end{array} \right) = 0 \right\} \quad (8)
\]
defines a maximal monotone relation. According to [12, Lemma 7.3.2] the kernel of \( (1 + V \cdot V) \) equals the set
\[
\{(1 - V \ell), (-1 - V \ell) \ | \ell \in \mathbb{C}^n\}.
\]
Let \((x, y) \in h\). Then there exists \( \ell \in \mathbb{C}^n \) such that \(-SQy = \ell - V \ell\) and \(Qx = -\ell - V \ell\). The latter implies, using Lemma 5.5 and \( V^* V \leq 1 \)
\[
\mathcal{R}(x|y)_{BD(G)} = \mathcal{R}(Q^{-1} (-\ell - V \ell) | (-SQ)^{-1} (\ell - V \ell))_{BD(G)}
\]
\[
= -\mathcal{R}(Q^{-1} (-\ell - V \ell) | (\sqrt{S}Q)^{-1} \sqrt{S^{-1}} (\ell - V \ell))_{BD(G)}
\]

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\[ = -2\Re(\sqrt{S}(-\ell - V\ell)|\sqrt{S^{-1}}(\ell - V\ell))C_{n} \]

\[ = -2\Re((-\ell - V\ell)|(-\ell - V\ell))C_{n} \]

\[ = -2\Re((V^*V - 1)\ell|\ell)C_{n} \geq 0. \]

Since \( h \) is linear, this gives the monotonicity of \( h \). For showing the maximality of \( h \) we take \( f \in BD(G) \). We have to find an element \( \ell \in C_{n} \) such that

\[ Q^{-1}(-\ell - V\ell) - Q^{-1}S^{-1}(\ell - V\ell) = f, \]

which is equivalent to the fact that

\[ (-S(1 + V) - (1 - V))\ell = SQf. \]

To show the existence of such a vector \( \ell \in C_{n} \) it suffices to prove that

\[ -S(1 + V) - (1 - V) \]

is injective. Take \( \ell \in C_{n} \) such that \(-S(1 + V)\ell - (1 - V)\ell = 0\). Then we estimate

\[ 0 \leq \langle S(1 + V)\ell|(1 + V)\ell\rangle_{C_{n}} \]

\[ = -\langle (1 - V)\ell|(1 + V)\ell\rangle_{C_{n}} \]

\[ = -\langle (1 - V^*V)\ell|\ell\rangle_{C_{n}} \leq 0 \]

and hence, \((1 + V)\ell = 0\). Therefore \((1 - V)\ell = -S(1 + V)\ell = 0\) and hence, we conclude that \( \ell = \frac{1}{2}((1 + V)\ell + (1 - V)\ell) = 0 \).

\[ \square \]

5.2 Frictional boundary conditions

In the context of contact problems in visco-elasticity we find so-called frictional boundary conditions, which should hold on the part of the boundary where the contact occurs. Examples for such boundary conditions can be found for instance in [14, Section 5], [31, p. 171 ff.] and [9, 134 ff.].

We follow the model presented in [14], which was already discussed by the author in [33] for the case where the frictional boundary condition holds on the whole boundary. The equations read as follows

\[ \partial_{0}^{2}u - \text{Div} T = f, \]

\[ T = C \text{Grad} u + D \text{Grad} \partial_{0}u, \]

where \( \partial_{0} \) stands for the temporal derivative, \( \text{Grad} \) denotes the symmetrized gradient and \( \text{Div} \) the row-wise divergence of a matrix with respect to the spatial variables (the precise definition will be given below). Following [33], the latter system can be reformulated as an equation of the form

\[ \left( \partial_{0} \mathcal{M} + \begin{pmatrix} 0 & \text{Div} \\ \text{Grad} & 0 \end{pmatrix} \right) \left( \begin{pmatrix} \partial_{0}u \\ -T \end{pmatrix} \right) = \begin{pmatrix} f \\ 0 \end{pmatrix} \]

for a suitable operator \( \mathcal{M} \). Throughout, we assume that \( \Omega \subseteq \mathbb{R}^{3} \) is a bounded Lipschitz-domain. Let \( \Gamma_{1}, \Gamma_{2}, \Gamma_{3} \subseteq \partial \Omega \) be three pairwise disjoint, measurable sets such that \( \Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{3} = \)
5.2 Frictional boundary conditions

Following [14] we impose the following boundary conditions

\[ \partial_0 u = f_1 \text{ on } \Gamma_1, \] (9)

\[ -T \cdot N = f_2 \text{ on } \Gamma_2, \] (10)

\[ (\partial_0 u, -T \cdot N) \in g \text{ on } \Gamma_3, \] (11)

for given functions \( f_1 \in L_2(\Gamma_1)^3, f_2 \in L_2(\Gamma_2)^3 \) and a binary relation \( g \subseteq L_2(\Gamma_3)^3 \oplus L_2(\Gamma_3)^3 \).

Since we are interested in maximal monotone realizations of the block operator matrix \( \begin{pmatrix} 0 & \text{Grad} \\ \text{Div} & 0 \end{pmatrix} \),

we restrict ourselves to maximal monotone relations \( g \) (in [14] also a class of non-monotone relations was discussed).

We define the operators involved:

**Definition.** We denote by \( L_2(\Omega)^{3\times3} \) the space of \( 3\times3 \)-matrix-valued \( L_2(\Omega) \) functions, equipped with the inner product

\[ \langle T|S \rangle_{L_2(\Omega)^{3\times3}} := \int_{\Omega} \text{trace}(T(t)^*S(t)) \, dt. \]

Moreover, we denote by \( L_{2,\text{sym}}(\Omega)^{3\times3} \) the closed subspace of symmetric-matrix-valued functions. We define the operator \( \text{Grad}_c \) as the closure of

\[ C_\infty^c(\Omega)^3 \subseteq L_2(\Omega)^3 \to L_{2,\text{sym}}(\Omega)^{3\times3} \]

\[ (\phi_i)_{i \in \{1,2,3\}} \mapsto \left( \frac{\partial_i \phi_j + \partial_j \phi_i}{2} \right)_{i,j \in \{1,2,3\}} \]

and the operator \( \text{Div}_c \) as the closure of

\[ C_\infty^c(\Omega)^{3\times3} \cap L_{2,\text{sym}}(\Omega)^{3\times3} \subseteq L_{2,\text{sym}}(\Omega)^{3\times3} \to L_2(\Omega)^3 \]

\[ (T_{ij})_{i,j \in \{1,2,3\}} \mapsto \left( \sum_{j=1}^{3} \partial_j T_{ij} \right)_{i \in \{1,2,3\}}. \]

Furthermore, we define the operator \( \text{Grad} := - (\text{Div}_c)^* \) and \( \text{Div} := - (\text{Grad}_c)^* \).

The operator matrix \( \begin{pmatrix} 0 & \text{Div} \\ \text{Grad} & 0 \end{pmatrix} \) is of the form discussed in Section 3, and thus, it suffices to show that the boundary conditions \( [12]-[14] \) can be realized by a maximal monotone relation on \( \mathcal{B}D(\text{Grad}) \). This is the aim of the rest of this subsection. We first note that Korn’s inequality holds for Lipschitz-domains (see e.g. [11]), which states that \( H^1(|\text{Grad}| + i) \) and \( H^1(|\text{grad}| + i)^3 \) are isomorphic via the identity-mapping. Following the reasoning of Section 4, we obtain that \( \mathcal{B}D(\text{Grad}) \) is isomorphic to \( H^+(\partial \Omega)^3 \) and consequently, there exists a continuous injection \( \kappa : \mathcal{B}D(\text{Grad}) \to L_2(\partial \Omega)^3 \) with dense range. In this sense, the boundary condition \([13]\) can be formulated as

\[ \pi_{L_2(\Gamma_1)^3}\kappa \pi_{\mathcal{B}D(\text{Grad})} \partial_0 u = f_1, \] (12)

which in particular implies \( f_1 \in \mathcal{R} \left( \pi_{L_2(\Gamma_1)^3}\kappa \right) \). We now define a maximal monotone relation
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on \((L_2(\Gamma_1)^3)^\perp = L_2(\Gamma_2)^3 \oplus L_2(\Gamma_3)^3\), which will represent the boundary conditions (10) and (11).

**Definition.** We denote by \(\pi_{\perp L_2(\Gamma_2)^3} : (L_2(\Gamma_1)^3)^\perp \to L_2(\Gamma_2)^3\) and by \(\pi_{\perp L_2(\Gamma_3)^3} : (L_2(\Gamma_1)^3)^\perp \to L_2(\Gamma_3)^3\) the orthogonal projections onto \(L_2(\Gamma_2)^3\) and \(L_2(\Gamma_3)^3\), respectively. We define the relation \(\tilde{g} \subseteq (L_2(\Gamma_1)^3)^\perp \oplus (L_2(\Gamma_1)^3)^\perp\) by

\[
\tilde{g} := \{ (x, y) | \pi_{\perp L_2(\Gamma_2)^3} y = f_2, (\pi_{\perp L_2(\Gamma_3)^3} x, \pi_{\perp L_2(\Gamma_3)^3} y) \in g \}.
\]

**Lemma 5.7.** The relation \(\tilde{g}\) is maximal monotone. Moreover, if \(g\) is bounded then so is \(\tilde{g}\).

**Proof.** The maximal monotonicity follows by Proposition 2.5 and the boundedness of \(\tilde{g}\) in case of a bounded relation \(g\) is obvious.

In [14] we find an assumption on \(g\), which in particular implies the boundedness of \(g\). So, henceforth, we will assume that \(g\) is bounded. Moreover, we define the closed subspace \(V\) of \(BD(\text{Grad})\) by

\[
V := \mathcal{N}(\pi_{\perp L_2(\Gamma_1)^3} \kappa),
\]

which consists of those elements in \(BD(\text{Grad})\), whose trace is supported on \(\Gamma_2 \cup \Gamma_3\).

**Lemma 5.8.** If \(g\) is bounded, then \(\tilde{h} := \pi_V \kappa^* \tilde{g} \pi_V^* \subseteq V \oplus V\) is maximal monotone.

**Proof.** This follows by Proposition 2.6 and Lemma 5.7.

We are now able to define the realization \(A \subseteq \begin{pmatrix} 0 & \text{Div} \\ \text{Grad} & 0 \end{pmatrix}\) subject to the boundary conditions (9)-(11).

**Theorem 5.9.** The nonlinear operator \(A \subseteq \begin{pmatrix} 0 & \text{Div} \\ \text{Grad} & 0 \end{pmatrix}\) with

\[
\mathcal{D}(A) := \left\{ (v, -T) \in \mathcal{D}(\text{Grad}) \times \mathcal{D}(\text{Div}) \mid \pi_{L_2(\Gamma_1)^3} \kappa \pi_{BD(\text{Grad})} v = f_1,
\left( \pi_V \pi_{BD(\text{Grad})} v, \pi_V \text{Div} \pi_{BD(\text{Div})} (-T) \right) \in \tilde{h} \right\}
\]

is maximal monotone.

**Proof.** We first note that

\[
\pi_{L_2(\Gamma_1)^3} \kappa \pi_{V^\perp} : V^\perp \to \mathcal{R}(\kappa \pi_{L_2(\Gamma_1)^3})
\]

is bijective according to the definition of \(V\). Hence, \(\pi_{L_2(\Gamma_1)^3} \kappa \pi_{BD(\text{Grad})} v = f_1\) is equivalent to

\[
\pi_{V^\perp} \pi_{BD(\text{Grad})} v = \tilde{f}_1 := (\pi_{L_2(\Gamma_1)^3} \kappa \pi_{V^\perp})^{-1} (f_1).
\]

Thus, defining

\[
h := \left\{ (x, y) \in BD(\text{Grad}) \oplus BD(\text{Grad}) \mid \pi_{V^\perp} x = \tilde{f}_1, (\pi_V x, \pi_V y) \in \tilde{h} \right\},
\]

(13)
we can write the domain of $A$ as

$$D(A) = \left\{ (v, -T) \in D(\text{Grad}) \times D(\text{Div}) \left| \left( \pi_{BD(\text{Grad})}v, \text{Div} \pi_{BD(\text{Div})} (-T) \right) \in h \right. \right\}.$$  

Hence, $A$ is maximal monotone if $h$ is maximal monotone according to Theorem 5.1. The latter follows by the maximal monotonicity of $\tilde{h}$ (see Lemma 5.8) and Proposition 2.5.

In the remaining part of this subsection we discuss, in which sense elements in the domain of $A$ satisfy the boundary conditions (9)-(11). Following the rationale of Section 4, (10) should hold in the sense that

$$((\text{Div} - \text{Div})\pi_{BD(\text{Div})}^* \pi_{BD(\text{Div})}(-T)|\pi_{BD(\text{Grad})}^* \pi_{BD(\text{Grad})} u)_{H^{-1}(\text{Div}^+ \cup \text{Grad}^+)} = \langle f_2|\kappa u \rangle_{L_2(\partial \Omega)^3}$$

for each $u \in BD(\text{Grad})$ with $\kappa u \in L_2(\Gamma)^3$ or, in other words, for each $u \in \mathcal{N}(\pi_{L_2(\Gamma)^3}^+ \kappa) =: V_2$. The latter gives

$$\langle \pi_{V_2} \pi_{V_2} \text{Div} \pi_{BD(\text{Div})}(-T) | u \rangle_{BD(\text{Grad})} = \langle \text{Div} \pi_{BD(\text{Div})}(-T) | \pi_{V_2}^* \pi_{V_2} u \rangle_{BD(\text{Grad})}$$

$$= \langle (\text{Div} - \text{Div})\pi_{BD(\text{Div})}^* \pi_{BD(\text{Div})}(-T)|\pi_{BD(\text{Grad})}^* \pi_{BD(\text{Grad})} u \rangle_{H^{-1}(\text{Div}^+ \cup \text{Grad}^+)}$$

$$= \langle f_2|\kappa \pi_{V_2}^* \pi_{V_2} u \rangle_{L_2(\partial \Omega)^3}$$

$$= \langle \pi_{V_2}^* \pi_{V_2} \kappa f_2 | u \rangle_{BD(\text{Grad})}$$

for each $u \in BD(\text{Grad})$. Thus, the appropriate formulation for (10) in our setting is

$$\pi_{V_2} \text{Div} \pi_{BD(\text{Div})}(-T) = \pi_{V_2} \kappa f_2.$$  (14)

Analogously (11) should hold in the sense that there exists a function $f_3 \in L_2(\Gamma)^3$ such that

$$(\pi_{L_2(\Gamma)^3}^+ \kappa \pi_{BD(\text{Grad})} \partial_0 u, f_3) \in g$$

and

$$\pi_{V_2} \text{Div} \pi_{BD(\text{Div})}(-T) \in V_3$$

where $V_3 := \mathcal{N}(\pi_{L_2(\Gamma)^3}^+ \kappa)$.

Let now $\left( \pi_{BD(\text{Grad})} \partial_0 u, \text{Div} \pi_{BD(\text{Div})}(-T) \right) \in h$, where $h$ is given by (13). Then

$$\pi_{V_2} \pi_{BD(\text{Grad})} \partial_0 u = \tilde{f}_1,$$

which yields (12). Moreover, $\left( \pi_{V} \pi_{BD(\text{Grad})} \partial_0 u, \pi_{V} \text{Div} \pi_{BD(\text{Div})}(-T) \right) \in \tilde{h}$, which implies the existence of an element $w \in (L_2(\Gamma)^3)^\perp$ such that

$$(\kappa \pi_{V_2}^* \pi_{V} \pi_{BD(\text{Grad})} \partial_0 u, w) \in \tilde{g}.$$
According to the definition of \( \tilde{g} \), we get that

\[
\pi_{L_2(\Gamma_2)^3} w = f_2 \quad \text{and} \quad (\pi_{L_2(\Gamma_3)^3} \kappa \pi_V \pi_{\mathcal{B}D(\mathcal{G}rad)} \partial_0 u, \pi_{L_2(\Gamma_3)^3} w) \in g.
\]

Hence, we get

\[
\pi_{V_2} \mathbf{\nabla} \pi_{\mathcal{B}D(\mathcal{G}rad)} (-T) = \pi_{V_2} \pi_V \pi_{\mathcal{B}D(\mathcal{G}rad)} (-T) = \pi_{V_2} \pi_V \pi_{\mathcal{B}D(\mathcal{G}rad)} (-T) = \pi_{V_2} \kappa^* \pi_{L_2(\Gamma_2)^3} w
\]

where we have used that \( \pi_{V_2} = \pi_{V_2} \pi_V \pi_{V_2} \), since \( V_2 \subseteq V \), and \( \pi_{V_2} \kappa^* \pi_{L_2(\Gamma_2)^3} \perp = 0 \) by the definition of \( V_2 \). This shows (14). Analogously, one obtains

\[
\pi_{V_3} \mathbf{\nabla} \pi_{\mathcal{B}D(\mathcal{G}rad)} (-T) = \pi_{V_3} \kappa^* \pi_{L_3(\Gamma_3)^3} w,
\]

which yields (15) for \( f_3 := \pi_{L_2(\Gamma_3)^3} w \).

Acknowledgement

The author thanks Birgit Jacob, who has initiated this study by a question on a conference and Marcus Waurick for careful reading.

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