The role of axisymmetric flow configuration in the estimation of the analogue surface gravity and related Hawking like temperature

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Abstract
We investigate the salient features of the acoustic geometry corresponding to the axially symmetric accretion of dissipationless inhomogeneous fluid onto a non-rotating astrophysical black hole under the influence of a generalized pseudo-Schwarzschild gravitational potential. For a few chosen flow configurations, we determine the location of the acoustic horizon and calculate analytically the corresponding analogue surface gravity $\kappa$ and the associated analogue Hawking temperature $T_AH$. We study the dependence of $\kappa$ on various boundary conditions and geometry governing the dynamic and thermodynamic properties of the background flow.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Contemporary works in the field of analogue gravity phenomena have attracted significant attention in the community in theoretical front [1–4] as well as in the laboratory set up...
Proper equivalence has been established between the physics of the propagating acoustic (and acoustic type) perturbations embedded in an inhomogeneous dynamical fluid system and certain kinematic features of the general theory of relativity. Such formalism has opened up the possibility of simulating various important features of black hole space time within the laboratory set up.

Conventional works in this field, however, concentrate on the physical systems for which gravity like effects are realized as emergent phenomena. Such systems do not usually contain any source that produces active gravitational field in any form. In recent years, though, attempts have been made to study the analogue effects in strong gravity environment [10–18]. The uniqueness of such systems lies in the fact that those are the only analogue models studied so far that simultaneously contain both kind of horizons, the gravitational as well as the acoustic—allowing one to go for a close comparison between the general relativistic and the analogue Hawking effect.

Till date, analogue gravity effects in the axisymmetrically accreting black hole systems have been studied for the flow structure assumed to be in hydrostatic equilibrium in the vertical direction. Two other configurations for the axially symmetric black hole accretion are also possible. The accretion flow in conical equilibrium [19] and ‘flat disc’ kind of flow with constant flow thickness have also been studied in the literature (for a detailed review, see, e.g., [20]). As a compromise between the easy handling of the Newtonian framework of gravity and more rigorous and non-tractable complete general relativistic description of the strong gravity space time, four different ‘modified’ Newtonian ‘black hole potentials’ have been introduced in the literature which are commonly known as pseudo-Schwarzschild potentials (for further details about such potentials, see, e.g., [21] and [22]). The three aforementioned different flow structures in the pseudo-Schwarzschild gravitational potentials have recently been studied in great detail [23].

In our present work, we study the analogue gravity effects in accreting black hole systems under the influence of pseudo-Schwarzschild potentials for three different flow configurations—the flow in conical and in vertical equilibrium and the flow with constant thickness (height). The main motivation behind this work is to study the dependence of the salient features of the acoustic geometry on the background matter geometry realized through the aforementioned three different flow configurations. To accomplish such a task, we calculate the acoustic surface gravity for the same set of accretion parameters but for all possible flow configurations in a pseudo-Schwarzschild potentials.

2. Acoustic surface gravity for classical analogue systems

Classical analogue gravity systems (alternatively, the classical ‘black hole analogues’) are fluid dynamical analogue of black holes in general relativity. Such an analogue may occur when a small linear perturbation propagates through a dissipationless inhomogeneous barotropic transonic fluid at finite temperature. The corresponding acoustic metric, which specifies the geometry in which the perturbation propagates, may be constructed in terms of the flow variables defining the unperturbed background continuum. The transonic surface acts as acoustic horizon—a null hypersurface with acoustic null geodesics, the phonons, as its generators. The acoustic black hole horizon, which resembles the black hole event horizons in many ways and forms at the regular transonic point of the fluid, whereas an acoustic white hole horizon may be formed at the hypersurface where the fluid makes a discontinuous sonic transition, e.g., through a stationary shock [13, 24].
In his pioneering work, Unruh [25] introduced the concept of acoustic geometry inside a supersonic fluid and demonstrated that an analogue surface gravity $\kappa$ may be associated with an acoustic black hole type event horizon, and one of the most interesting aspects of the acoustic horizon is to emit the Hawking type radiation of thermal phonons. Such acoustic Hawking radiation may be characterized by an analogue Hawking temperature $T_{AH} = \frac{\hbar \kappa}{2\pi}$. In Unruh’s original approach, the acoustic surface gravity could be associated with the component of the bulk velocity of the flow normal to the acoustic horizon $u_{\perp}$ and the speed of propagation of the acoustic perturbation $c_s$ as

$$\kappa \propto \left( \frac{1}{c_s} \frac{\partial u_{\perp}}{\partial \eta} \right)_{r_h},$$

where $\frac{\partial}{\partial \eta} = \eta \mu \partial_{\mu}$ represents the space derivative taken along the normal to the acoustic horizon, and the quantities in brackets in equation (1) have been evaluated at the location of the acoustic horizon $r_h$.

In equation (1), however, the sound speed was assumed to be a position independent constant. Unruh’s work was followed by several other important contributions ([26–29], to name a few). The concept of the acoustic geometry in a transonic fluid was first realized by Moncrief [30] while studying the stability properties of the relativistic spherical accretion onto astrophysical black holes. Visser [28] implemented a contribution due to the position dependent sound speed and obtained a modified expression for the surface gravity

$$\kappa \propto \left[ c_s \frac{\partial}{\partial \eta} (c_s - u_{\perp}) \right]_{r_h}.$$  

The acoustic horizon is a surface defined by the equation [28]

$$u_{\perp}^2 - c_s^2 = 0,$$

for a stationary background flow configuration. Equation (3) basically states that the acoustic horizons are a transonic surface. The supersonic region of a transonic flow defines the acoustic ergo region.

The concept of acoustic geometry has been extended to a relativistic fluid flow in a general background space time [31]. For an ideal barotropic fluid, the relativistic Euler equation and the equation of continuity obtained from the energy momentum conservation can be linearized in order to obtain the wave equation for the propagating perturbation in analogue curved space time along with the corresponding acoustic metric. The generalized form of the acoustic surface gravity turns out to be

$$\kappa = \left| \frac{\sqrt{\chi^{\mu} \chi_{\mu}}}{(1 - c_s^2)} \frac{\partial}{\partial \eta} (u_{\perp} - c_s) \right|_{r_h},$$

where $\chi^{\mu}$ is the Killing field which is null on the corresponding acoustic horizon. The algebraic expression corresponding to the $\sqrt{\chi^{\mu} \chi_{\mu}}$ may thus be evaluated once the background stationary metric governing the fluid flow as well as the propagation of the perturbation in a specified geometry with well posed boundary conditions are realized. It is worth mentioning that the generalized form for $\kappa$ as defined in equation (4) can further be reduced to its Newtonian/semi-Newtonian counterpart depending on the nature of the gravitational potential describing the background fluid motion.

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6 Hereafter, the phrases ‘acoustic’ and ‘analogue’ will be used synonymously for the sake of brevity.
3. Acoustic surface gravity for axisymmetric black hole accretion

From (2) and (4), it is clear that, in order to find the acoustic surface gravity \( \kappa \) for the Newtonian as well as for the relativistic acoustic geometry, it is sufficient to calculate the location \( r_h \) of the acoustic horizon, the sound speed \( c_s \) of the small linear perturbation and its normal space gradient \( dc_s/d\eta \), as well as the normal (to the acoustic horizon) component of the flow velocity \( u_\perp \) and its normal space gradient \( du_\perp/d\eta \), evaluated on the acoustic horizon.

For a transonic accretion onto an astrophysical black hole, one thus needs to consider the Euler equation and the equation of continuity for a specific symmetry of the problem. The Euler and the continuity equation may then be linearized (for a general linearization scheme and its application to the study of axisymmetric black hole accretion for three different flow geometries, see \[23\]) in order to construct the corresponding acoustic metric and thus to specify the relevant acoustic geometry. The structure of the stationary background fluid flow may be provided by the stationary solution of the Euler and the continuity equations. For a certain set of values of the initial conditions describing the transonic accretion flow governed by certain barotropic equation of state, it may be possible to find the location of the saddle type transonic point, which is identical to the acoustic black hole horizon \[12, 13, 15, 31\]. The quantities \([u_\perp, c_s, du_\perp/dr, dc_s/dr]\) subject to the gravitational field which governs the accretion may then be evaluated at the acoustic horizon radius \( r_h \) to estimate the acoustic surface gravity using equation (4). The quantity \( \sqrt{\chi_{\mu\chi_{\mu}}} \) evaluated at the acoustic horizon is a function of \( r_h = r_{\text{transonic}} \) (as will be demonstrated in subsequent paragraphs) depending on initial conditions.

In general, we consider the equatorial slice of the axisymmetric gravitating accretion of hydrodynamic fluid onto non-rotating black holes. Accretion is assumed to possess finite radial velocity \( u \) commonly known as the ‘advective velocity’ in the accretion literature. Considering \( v \) to be the magnitude of the three velocity, \( u \) is the component of three velocity perpendicular to the set of timelike hypersurfaces \( \{\Sigma_v\} \) defined by \( v^2 = \text{constant} \). The advective velocity \( u \) is thus perpendicular to the acoustic horizon and hence \( u \) is identical with \( u_\perp \). Hereafter, we drop the subscript \( \perp \) in \( u_\perp \) and simply use \( u \) instead. The gravitational field of the black hole is assumed to be described by certain pseudo-Schwarzschild potentials.

As mentioned in section 1, several pseudo-Schwarzschild potentials exists in the literature to mimic the space time around a non-rotating black hole. In this work, we formulate the general equations in terms of a generalized pseudo-Schwarzschild potential \( \Phi \) and subsequently present the specific results for the following four pseudo-Schwarzschild black hole potentials

\[
\Phi_1 = -\frac{1}{2(r-1)}, \quad \Phi_2 = -\frac{1}{2r} \left[ 1 - \frac{3}{2r} + 12 \left( \frac{1}{2r} \right)^2 \right],
\]
\[
\Phi_3 = -1 + \left( 1 - \frac{1}{r} \right), \quad \Phi_4 = \frac{1}{2} \ln \left( 1 - \frac{1}{r} \right).
\]

The potential \( \Phi_1 \) was introduced in \[45\]. \( \Phi_1 \) correctly represents the location of the marginally stable and the marginally bound orbit, \( r_s \) and \( r_b \), respectively. For further detail about the marginally stable and the bound orbit, see, e.g., \[47\]. The angular momentum distribution of the accreting matter in the Schwarzschild metric is best represented using the potential \( \Phi_1 \).

The potential \( \Phi_2 \) had been proposed in \[46\], to analyse the normal mode of the acoustic oscillations within a thin accretion disc and can reproduce the correct general relativistic value of the angular velocity at the marginally stable orbit and the best approximate value of the radial epicycle frequency for radial distance greater than the marginally stable orbit.

\( \Phi_3 \) and \( \Phi_4 \) have been formulated in \[21\]. While \( \Phi_3 \) produces the accurate value of the free fall acceleration of a test particle at any given radial distance, \( \Phi_4 \) gives the value of a free
fall acceleration that is equal to the value of the co-variant component of the three dimensional free fall acceleration vector of a test particle that is at rest in the Schwarzschild reference frame. A detailed comparative discussion of the effective (gravitational plus centrifugal) black hole potential for all the aforementioned pseudo potentials for the axisymmetric flow of non self-gravitating matter around non-rotating compact astrophysical objects with respect to the effective potentials corresponding to the actual Schwarzschild metric is available in the section 2 of [22].

The low angular momentum sub-Keplerian advective inviscid flow will be considered where the specific flow angular momentum $\lambda$ will be assumed to be a position independent constant. Viscous transport of angular momentum will not be taken into account since close to the black hole, the infall time scale for the highly supersonic flow is rather small compared to the corresponding viscous time scale (for further details, see, e.g., [22, 32, 33] and references therein). Also for advective accretion, large radial velocity at larger distances are the consequence of the small rotational energy of the flow [34–36].

The energy-momentum tensor of a relativistic ideal fluid can be expressed as

$$T^{\mu\nu} = (\epsilon + p)v^\mu v^\nu + pg^{\mu\nu},$$

$\epsilon$ being the energy density including the rest mass density and the internal energy, and $p$ is fluid pressure. Vanishing of the four divergence of the energy momentum tensor provides the general relativistic version of the Euler equation

$$T^{\mu\nu}_{;\nu} = 0.$$  \hspace{1cm} (5)

The continuity equation is obtained from

$$(\rho v^\mu)_{;\mu} = 0.$$ \hspace{1cm} (6)

We define two killing vectors $\xi^\mu = \delta^\mu_t$ and $\phi^\mu = \delta^\mu_\phi$ corresponding to stationarity and axisymmetry of the flow, respectively.

We contract equation (5) with $\phi^\mu$ to obtain,

$$\phi^\mu [(\epsilon + p)v^\mu v^\nu]_{;\nu} + \phi^\mu p_{;\nu} g^{\mu\nu} = 0.$$  \hspace{1cm} (7)

But $\phi^\nu p_{;\nu} = 0$ due to axisymmetry, hence

$$\phi^\mu [(\epsilon + p)v^\mu v^\nu]_{;\nu} = 0,$$

which further provides,

$$g_{\mu\phi} [\epsilon + p)v^\mu v^\nu]_{;\nu} = 0,$$

since $\phi^\mu = \delta^\mu_\phi$. Since $g_{\mu\nu,\nu} = 0$, equation (7) can be written as

$$[g_{\mu\phi} (\epsilon + p)v^\mu v^\nu]_{;\nu} = 0;$$

from where we obtain

$$[\phi^\mu h v^\nu]_{;\nu} = 0.$$ \hspace{1cm} (8)

From equation (8), one thus infers $\phi^\mu h v^\mu = hv_\phi$ and hence $hv_\phi$, the angular momentum per baryon for the axisymmetric flow, is conserved. We write $L = -hv_\phi$.

We now contract equation (5) with $\xi^\mu$$

$$\xi^\mu T^{\mu\nu}_{;\nu} = 0,$$

to obtain that $hv_\nu$ is conserved. $hv_\nu$ may be interpreted as the conserved specific energy of the flow and is denoted by $\mathcal{E}$. For adiabatic accretion, $\mathcal{E}$ is a first integral of motion along a the
flow streamline, and is identified with the relativistic Bernoulli’s constant. Hence, the quantity

\[ \frac{\mathcal{L}}{\mathcal{E}} = -\frac{v_\phi}{v_t} = \text{Constant} = \lambda \, \text{(say)}, \]

where \( \lambda \) is the specific constant angular momentum of the flow for an inviscid axisymmetric stationary accretion.

The angular velocity \( \Omega \) is defined as

\[ \Omega = \frac{v_\phi}{v_t}. \]

With the help of the following transformation properties of the metric elements

\[ g^{tt} = -\frac{g_{\phi\phi}}{g_{\phi\phi} - g_{tt} g_{\phi\phi}}, \]

\[ g^{t\phi} = \frac{g_{t\phi}}{g_{t\phi} - g_{tt} g_{\phi\phi}}, \]

\[ g^{\phi\phi} = -\frac{g_{\phi\phi}}{g_{\phi\phi} - g_{tt} g_{\phi\phi}}, \]

\[ g_{\phi\phi} = 1 \]

and

\[ g^{rr} = \frac{1}{g_{rr}}, \]

we obtain

\[ \Omega = \frac{v_\phi}{v_t} \]

\[ = \frac{g^{t\phi} v_t + g^{\phi\phi} v_\phi}{g^{t\phi} v_\phi + g^{t\phi} v_t} \]

\[ = \frac{g^{t\phi} - g^{\phi\phi} \lambda}{g^{tt} - g^{\phi\phi} \lambda} \]

\[ = \frac{g^{t\phi} + \lambda g_{\phi\phi} + \lambda g_{t\phi}}{g_{\phi\phi} + \lambda g_{t\phi}}. \]

Thus, the corresponding angular velocity \( \Omega \) may be expressed as,

\[ \Omega = -\frac{g_{t\phi} + \lambda g_{\phi\phi}}{g_{\phi\phi} + \lambda g_{t\phi}}, \tag{9} \]

where \( g_{ij} \) are the metric components.

The corresponding surface gravity \( \kappa \) can now be expressed as

\[ \kappa = \left[ \frac{\mathcal{K}^{\mu}_{\mu}}{-g_{rr}} \right]^{1/2} \left[ \frac{1}{1 - c_s^2} \left[ \frac{d}{dr} (u - c_s) \right] \right]_{r_1}, \tag{10} \]
where \( \chi^\mu = \xi^\mu + \Omega \zeta^\mu \) and the Killing vectors \( \xi^\mu \) and \( \zeta^\mu \) are the generators of the temporal and the axial symmetry group. The norm of the Killing vector \( \chi^\mu \) may be computed as

\[
\sqrt{\chi^\mu \chi^\mu} = \left( g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} \right)^{1/2} = \frac{\Sigma \Lambda}{\sqrt{\Sigma^2 + \Lambda^2}},
\]

where

\[
\Sigma^2 = g_{t\phi}^2 - g_{tt} g_{\phi\phi},
\]
\[
\Lambda^2 = (g_{tt} + \lambda^2 g_{t\phi} + \lambda^2 g_{\phi\phi}).
\]

In the Newtonian limit

\[
g_{tt} = 1 + 2\Phi, \quad g_{\phi\phi} = -r^2, \quad g_{rr} = -1, \quad g_{t\phi} = 0,
\]

\( \Phi \) being the corresponding pseudo potential. Hence, the acoustic surface gravity for axisymmetric black hole accretion under the influence of the pseudo-Schwarzschild potential becomes

\[
\kappa = \left| \frac{1}{(1 + \Phi) \left( 1 - \lambda^2 \frac{r^2}{r^2} - 2\Phi \lambda^2 \frac{r^2}{r^2} \right)} \left( \frac{1}{1 - c_s^2} \left[ \frac{dr}{dr} - \frac{d \omega}{dr} \right] \right) \right|_{r_h}.
\]

with the corresponding analogue Hawking temperature

\[
T_{AH} = \frac{\kappa}{2\pi}.
\]

Since the quantities \( r_h \) and \( c_s \), \( \frac{dr}{dr} \) and \( \frac{d \omega}{dr} \) (evaluated at \( r_h \)) are expressed in terms of the specific energy of the accretion flow \( \mathcal{E} \), the specific angular momentum of the accreting matter \( \lambda \) and the polytropic index \( \gamma \) for an axisymmetric accretion onto a non-rotating black hole, the temperature \( T_{AH} \) is a function of \( \mathcal{E}, \lambda, \gamma \). In the subsequent sections, we present further details about the parametrization scheme of the flow.

The general relativistic Hawking temperature \( T_H \) is given by

\[
T_H = \frac{\hbar c^3}{8\pi GM_{BH} k_B} \approx 6.17 \times 10^{-8} \frac{M_{\odot}}{M_{BH}} \text{ K},
\]

where \( M_{\odot} \) and \( M_{BH} \) stand for the solar and black-hole masses, respectively. From Eq. (16), it is obvious that the observation of the Hawking radiation is possible only for a black hole of relatively small mass (e.g., a primordial black hole) and the maximal Hawking temperature is obtained when the black-hole mass approaches zero.

For an analogue Hawking temperature, such a straightforward dependence on a mass parameter of the system is not available. Instead, the temperature \( T_{AH} \) nonlinearly depends on the set of parameters \( [\mathcal{E}, \lambda, \gamma] \). One thus needs to explore the three dimensional parameter space spanned by \( [\mathcal{E}, \lambda, \gamma] \) to find out what initial set of boundary conditions are needed to obtain a significant \( T_{AH} \) and hence the observable signature of the analogue Hawking radiation. For axisymmetric accretion, one also needs to understand which flow configuration out of the three discussed in subsequent sections, yields large analogue Hawking temperature.

The Hawking like effects in a dispersive medium can manifest non-conventional classical features which may, in principle, be observed within the laboratory set up \([37–42]\). The origin of such non-trivial features may be attributed to a modified dispersion relation close to the acoustic horizon. In the limit of the strong dispersion relation where the background flow has non-constant velocity gradient, the measure of the analogue temperature, unlike the general relativistic Hawking temperature, depends on the frequency of the propagating perturbation embedded in the background flow. The distinction between the analogue Hawking like effects in a dispersive medium and the standard general relativistic Hawking effect is in that the former is sensitive to the spatial velocity gradient of the stationary background flow solutions.
The dispersion relation near the acoustic horizon strongly influences the Hawking like spectra for a nonlinear velocity profile of the background fluid. Such phenomenon, is however not expected to take place for the gravitational horizon. Close to the gravitational horizon, the modification of the dispersion relation is known to appear at extremely short length scale—below the Planck’s scale to be more specific and no significant deviation from the aforementioned universality can be perceived thereof. Nevertheless, for non-gravitational cases, the phase integral method of determination of the associated analogue temperature ceases to be valid if the velocity gradient tends to diverge \[41, 42\]. The acoustic surface gravity as expressed in (14) is an analytical function of the space gradient of the steady state bulk velocity of the background fluid. Such velocity gradient influences the universality (as well as the departure from it) of the Hawking radiation, and various other properties of the anomalous scattering of the acoustic mode due to the modified dispersion relation at the acoustic horizon.

The acoustic surface gravity estimated in our present work can further be used to study the modified dispersion relation as well as the non universal features of the classical Hawking like process for a flat background flow subject to the gravitational force. In addition to the space velocity gradient, the surface gravity for the accreting black hole system is a function of the space gradient of the propagation speed of acoustic perturbations. Hence, for a non isothermal equation of state, the position dependent sound speed will contribute to the modified dispersion relation.

From the aforementioned discussions, one understands that the main task hereafter boils down to the identification of the location of the acoustic horizon and to the evaluation of \(c_s, \frac{dc_s}{dr}, \frac{du}{dr}\) as defined on the horizon, in terms of the initial boundary conditions governing the flow. This will enable us to conceive a ‘calibration space’ spanned by various astrophysically relevant parameters governing the flow, for which the extremization of \(\kappa\) as well as \(T_{AH}\) can be performed for the axisymmetric accretion of a particular geometric configuration and described by a specific equation of state under the influence of a generalized pseudo-Schwarzschild black hole potential. We then perform a similar operation for various other flow geometries described by different equations of state. This will give us a comprehensive idea about the influence of initial boundary conditions governing the flow and about the nature of the geometric configuration of the black hole accretion flow. Besides, this analysis will help us perform the extremization process of the analogue surface gravity and understand the reason for the departure of the associated Hawking like effects from the universal behaviour due to the modified dispersion relation in the background flow in a strong gravity environment.

In a related work \[43\], we plan to show how such parameter spaces and the flow geometry dependence of \(\kappa\) can be studied using the full general relativistic framework.

4. On flow geometries and equation of state

The governing equations describing the dynamics of axially symmetric pseudo-Schwarzschild inviscid hydrodynamics accretion are the equation for the conservation of linear momentum (the Euler equation):

\[
\frac{\partial}{\partial t} u(r, t) + u(r, t) \frac{\partial}{\partial r} u(r, t) + \frac{1}{\rho(r, t)} \frac{\partial p(r, t)}{\partial r} \frac{\lambda^2}{r^2} + \Phi' = 0, \tag{17}
\]

where \(u, \rho\) and \(p\), being the flow velocity, the fluid density and the pressure, respectively, are functions of both \(r\) and \(t\). \(\Phi'\) represents the space derivative \(\frac{\partial \Phi}{\partial r}\), where \(\Phi\) may be taken as any one of the pseudo-Schwarzschild potentials described in equation as mentioned in section 3.
The mass conservation equation (the continuity equation) can be written as:

$$\frac{\partial}{\partial t} \rho(r, t) + \frac{\partial}{\partial r} \left[ \rho(r, t) u(r, t) r H \right] = 0.$$  \hspace{1cm} (18)

The quantity $H$ is the flow thickness which is different in three different flow configurations described in detail below.

4.1. Three different geometrical configurations of background stationary axisymmetric fluid

In the standard literature of accretion astrophysics (see, e.g., [47]), the local flow thickness for an inviscid axisymmetric flow described by a adiabatic as well as an isothermal equation of state, can have three different geometric configurations as described below:

4.1.1. Flow with constant thickness. The thickness of the flow (the disc height) is taken to be constant—such height does not change with the radial distance (measured from the event horizon along the equatorial plane) for a stationary flow configuration. The geometric configuration resembles a right circular cylinder of constant height with a symmetry about the $z$ axis. In absence of convention currents along any non-equatorial direction (the assumption adopted in the present work), any circular plane orthogonal to the axis of symmetry may thus be considered as the equatorial plane on which the flow dynamics is studied. This is considered to be the simples possible flow geometry. Since in all our calculations the differential form of the accretion rate will be involved (see the subsequent sections), the disc height for a constant thickness flow may be normalized to unity as long as the calculation of the value of the acoustic surface gravity $\kappa$ is concerned.

4.1.2. Wedge shaped flow with conical geometry. In its next variant, we consider the quasi-spherical flow. The absolute spherical symmetry (Bondi [44] like flow) is destroyed due to the introduction of the angular momentum of the flow. However, the flow thickness remains proportional to the radial distance and the ratio of the local flow thickness to the local radial distance remains constant. The numerical value of the constant is determined by the solid angle ($< 4\pi$) subtended by the flow at the centre. Once again, as long as the computation of the value of $\kappa$ is concerned, the absolute value of the geometric constant measuring the ratio $H/r$, $H$ being the local flow thickness and $r$ being the local radius respectively, may be normalized to unity for the convenience of the calculation.

It is to be mentioned that the conical flow is the ideal-most candidate to model the inviscid flow since the absence of viscosity implies the existence of a weakly rotating flow which can be best modelled by a rotating quasi-spherical mass possessing a constant specific angular momentum. Incidentally, the first ever modelling of the multi-transonic flow under the influence of the pseudo-Schwarzschild black hole potential was performed for quasi-spherical flow in conical geometry [19].

4.1.3. Flow in hydrostatic equilibrium along the vertical direction. The background stationary flow is assumed to have a rather complex dependence of local flow thickness on the local radial distance (as well as on the speed of propagation of the acoustic perturbation embedded in the background fluid flow). The central plane is assumed to coincide with the equatorial plane of the black hole and the gravity is balanced by the component of the fluid pressure acting along the vertical direction. While the variation of the dynamical flow variables (bulk flow velocity, for example) are studied over the equatorial plane only, the corresponding thermodynamic variables are usually averaged over the flow thickness (local disc height), see, e.g.,
4.2. Equation of state

All the aforementioned geometric configurations will be studied for both an adiabatic as well as an isothermal equation of state, i.e., two different variants of a barotropic equation of state in general where the speed of the propagation of the acoustic perturbation $c_s$ can be expressed as

$$c_s = \left( \frac{\partial \rho}{\partial p} \right)^{\frac{1}{2}}.$$

An adiabatic equation of state of the form $p = K \rho^\gamma$ will be adopted where $\gamma = c_p/c_v$ is the ratio of the specific heats of the fluid at the constant pressure and at the constant volume, respectively. The entropy per particle $\sigma$ is related to $K$ and $\gamma$ as (Landau L. D., & Lifshitz E. M., 1994, Statistical Mechanics, Oxford: Pergamon, p. 125)

$$\sigma = \frac{1}{\gamma - 1} \log K + \frac{\gamma}{\gamma - 1} + \text{constant},$$

where the constant depends on the chemical composition of the accreting material. The above equation implies that $K$ is a measure of the specific entropy of the background matter flow. The polytropic (adiabatic) sound speed $c_{s,\text{adia}} = \frac{\gamma K}{\rho}$ depends on the radial distance $r$ since the bulk flow temperature varies with $r$ to keep the total specific energy of the flow to be invariant.

The isothermal flow will be described by the equation $p = \rho \kappa B T \mu m_H$. The quantities $K$, $\gamma$, $\kappa B$, $T$, $\mu$ and $m_H$ are the entropy per particle, the Boltzmann constant, the isothermal flow temperature, the reduced mass and the mass of the Hydrogen atom, respectively. The flow temperature for a stationary configuration remains invariant with respect to the radial distance for isothermal fluid, hence the isothermal sound speed $c_{s,\text{iso}} = \left( \frac{\rho}{\rho} \right)^{\frac{1}{2}}$ is position independent and is entirely described by the bulk flow temperature $T$ (which is used as parameter in our work, see subsequent sections).

With the help of the equation of state, and the specified radial dependence of the flow thickness, we can find stationary solutions of the Euler and the continuity equations and draw the Mach number versus the radial distance phase portrait for the integral flow solutions. With this, we obtain the detailed information about the location of the acoustic horizon as well as the horizon related quantities.

5. Adiabatic accretion

For an adiabatic equation of state of the form $p = K \rho^\gamma$, we integrate the time independent part of the Euler equation\footnote{The equation of state has been used to integrate the $\frac{1}{2} \rho \frac{d}{dr}$ term since an equation of state connecting $p$ and $\rho$, i.e., a barotropic equation of state, is necessary to integrate the aforementioned term contains $p$ and $\rho$, both of which are position dependent quantity.}. The integral solution of the stationary part of the Euler equation provides the energy first integral of motion of the following form:

$$\mathcal{E} = \frac{u^2}{2} + \frac{c_s^2}{\gamma - 1} + \frac{\lambda^2}{2 \rho^2} + \Phi. \quad (19)$$

The conserved specific energy $\mathcal{E}$ does not depend on the flow configuration for obvious reason. Since $r$ dependence of $\Phi$ varies for different flow geometries, the integral solution of
the continuity equation, which is another first integral of motion and is referred to as the mass accretion rate, will be different for three different accretion configurations. Expressions for the mass accretion rate can be obtained as

\[ \dot{M}_{\text{CH}} = \rho u r H_c, \]  
\[ \dot{M}_{\text{CM}} = \Theta \rho u r^2, \]  
\[ \dot{M}_{\text{VE}} = \sqrt{\frac{1}{\gamma} \mu s, \rho r^\frac{3}{2} \left( \Phi' \right)^{-\frac{1}{2}}}, \]

where the subscript CH, CM and VE stands for the flow with constant height (CH), in conical model (CM), and in vertical equilibrium (VE), and implies that the respective algebraic equations are to be solved to obtain the critical point for the corresponding flow geometries. The quantity \( H_c \) is the constant disc height and \( \Theta \) is the solid angle sustained by the flow. The mass accretion rate for the flow in vertical equilibrium not only depends on the matter geometry (through the radial dependence of \( H \)), but also the information about the space time geometry is encrypted in \( \dot{M}_{\text{VE}} \) through the explicit appearance of the derivative of the pseudo-Schwarzschild potential. One defines the entropy accretion rate as ([19, 48]):

\[ \dot{M} = \dot{M}_{\text{P}} \frac{1}{r^2} K^{\frac{1}{2}}. \]

Substitution of \( \dot{M} \) from equations (20a)–(20c) in the above equation provides the expression for \( \dot{M}\) in terms of the adiabatic sound speed, radial distance and the flow velocity. The space gradient of the sound speed and the flow velocity for various flow geometries can be obtained by differentiating equation (19) and equation (21):

\[ \left( \frac{d c_s}{dr} \right)_{\text{CH}} = (1 - \gamma) \frac{c_s}{u} \left( \frac{1}{2} \frac{du}{dr} + \frac{u}{2r} \right), \]
\[ \left( \frac{d c_s}{dr} \right)_{\text{CM}} = (1 - \gamma) \frac{c_s}{u} \left( \frac{1}{2} \frac{du}{dr} + \frac{u}{r} \right), \]
\[ \left( \frac{d c_s}{dr} \right)_{\text{VE}} = \left( 1 - \gamma \right) \frac{c_s}{u} \left[ \frac{du}{dr} + \frac{3}{2 r} - \frac{\Phi''(r)}{\Phi'(r)} \right], \]

\[ \left( \frac{du}{dr} \right)_{\text{CH}} = \frac{u \left( \frac{c_s^2}{2} + \frac{u^2}{2} - \Phi'(r) \right)}{\left( u^2 - c_s^2 \right)}, \]
\[ \left( \frac{du}{dr} \right)_{\text{CM}} = \frac{u \left( \frac{2c_s^2}{r} + \frac{u^2}{r^2} - \Phi'(r) \right)}{\left( u^2 - c_s^2 \right)}, \]
\[ \left( \frac{du}{dr} \right)_{\text{VE}} = \frac{u \left[ \frac{c_s^2}{(1 + \gamma)} \left( \frac{1}{2} - \frac{\Phi''(r)}{\Phi'(r)} \right) + \frac{u^2}{r^2} - \Phi'(r) \right]}{\left( u^2 - \frac{2}{1+\gamma} c_s^2 \right)}. \]

The critical point conditions may be obtained by simultaneously making the numerator and the denominator of equations (23a)–(23c) vanish, and the aforementioned critical point conditions may thus be expressed as:
where $r_c$ is the location of the critical point.

The critical point conditions for the constant height flow, the conical model flow and the flow in vertical equilibrium are stated in equations (24a), (24b), (25a), (25b) and (26a), (26b) respectively. Linearizing the Euler and the continuity equation for accretion in hydrostatic equilibrium, one can show that the linear perturbation propagates with the speed $\sqrt{\frac{2}{\gamma+1}} c_s$ instead of $c_s$. We thus define $\sqrt{\frac{2}{\gamma+1}} c_s$ to be the ‘effective’ sound speed for such a flow. This happens because, for the flow in vertical equilibrium, the expression for the flow thickness contains the adiabatic sound speed $c_s = \gamma p/\rho$. Hence, the critical point $r_c$ actually is identical with the radial location of the event horizon.

For a set of fixed values of $[\mathcal{E}, \lambda, \gamma]$, the location of the critical point for a particular flow model can be obtained by substituting the corresponding critical point condition (as expressed in equations (24a)–(26b) in the energy first integral (19) for a particular pseudo-Schwarzschild black hole potential. Once these expressions are substituted, the energy first integral becomes an algebraic expression of $r_c$. Exact value of $r_c$ for the constant height flow, the flow in conical model and in hydrostatic equilibrium can thus be obtained by solving the following equations

$$\mathcal{E}_{\text{CH}} - \frac{1}{2} \left(\frac{\gamma+1}{\gamma-1}\right) \left[ r_c \Phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] - \Phi(r_c) - \frac{\lambda^2}{2r_c^2} = 0, \quad (27a)$$

$$\mathcal{E}_{\text{CM}} - \frac{1}{4} \left(\frac{\gamma+1}{\gamma-1}\right) \left[ r_c \Phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] - \Phi(r_c) - \frac{\lambda^2}{2r_c^2} = 0, \quad (27b)$$

$$\mathcal{E}_{\text{VE}} - \frac{2\gamma}{\gamma-1} \left[ r_c \Phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] \left[ 3 - r_c \left( \frac{d\Phi}{dr} \right)' \right]^{-1} - \Phi(r_c) - \frac{\lambda^2}{2r_c^2} = 0. \quad (27c)$$

The exact location of $r_c$ can be evaluated once the astrophysically relevant range of $[\mathcal{E}, \lambda, \gamma]$ can be realized. One can argue [18] that the relevant values in the parameter space $[\mathcal{E}, \lambda, \gamma]$ can be set as $[1 \lesssim \mathcal{E} \lesssim 2, 0 < \lambda \lesssim 2, 4/3 < \gamma \leq 5/3]$. A solution to equations (27a)–(27c) may exhibit either one (saddle type), or three (one centre type flanked by two saddle type) critical points depending on the chosen set of parameters $[\mathcal{E}, \lambda, \gamma]$. Certain $[\mathcal{E}, \lambda, \gamma]_{\text{mc}} \subset [\mathcal{E}, \lambda, \gamma]$ thus provides the multi criticality in accretion solutions, where the subscript ‘mc’ stands for ‘multi critical’. The acoustic horizon is thus
a collection of the ‘sonic’ points for which the radial Mach number becomes unity. Such a horizon is located on the combined integral solution of equations (22a)–(22c) and (23a)–(23c). For an inviscid flow, a physically acceptable transonic solution which passes through a saddle type sonic point can be realized. Such a solution would be an example which confirms the hypothesis that every saddle type critical point is accompanied by its sonic point but no centre type critical point has its sonic counterpart. For an axisymmetric configuration, in all three geometries discussed in this work, a multi-critical flow is thus a theoretical abstraction where three critical points (out of which one is always a centre type, through which the integral solution can never pass) are obtained as a mathematical solution of the energy conservation equation (through the critical point condition), whereas a multi-transonic flow is a realistic configuration where accretion solution passes through two different saddle type sonic points. One should, however, note that a smooth accretion solution can never encounter more than one regular sonic point, hence no continuous transonic solution exists which passes through two different acoustic horizons. The only way the multi-transonicity could be realized is a combination of two different otherwise smooth solutions passing through two different saddle type critical (and hence sonic) points and are connected to each other through a discontinuous shock transition. Such a shock has to be stationary and will be located in between two sonic points. For a specific [E, λ, γ]Shock ⊂ [E, λ, γ]inc, three critical points (two saddles embracing a centre one) are routinely obtained but no stationary shock forms. Hence, no multi-transonicity is observed even if the flow is multi-critical, and real physical accretion solution can have access only to the outer type saddle point out of the two. Thus, multi-critical accretion and multi-transonic accretion are not topologically isomorphic in general. A true multi-transonic flow can only be realized for [E, λ, γ]Shock ⊂ [E, λ, γ]inc, if the criteria for forming a standing shock are met (for details about such shock formation and related multi-transonic shocked flow topologies, see [22, 32, 49]). For a mono-transonic flow, one can have only one acoustic horizon on which the related surface gravity may be evaluated. For multi-transonic shocked flow solutions passing through two different saddle type critical points and are connected to each other through a discontinuous shock transition. Such a shock has to be stationary and will be located in between two sonic points. For a specific [E, λ, γ]Shock ⊂ [E, λ, γ]inc, three critical points (two saddles embracing a centre one) are routinely obtained but no stationary shock forms. Hence, no multi-transonicity is observed even if the flow is multi-critical, and real physical accretion solution can have access only to the outer type saddle point out of the two. Thus, multi-critical accretion and multi-transonic accretion are not topologically isomorphic in general. A true multi-transonic flow can only be realized for [E, λ, γ]Shock ⊂ [E, λ, γ]inc, if the criteria for forming a standing shock are met (for details about such shock formation and related multi-transonic shocked flow topologies, see [22, 32, 49]). For a mono-transonic flow, one can have only one acoustic horizon on which the related surface gravity may be evaluated. For multi-transonic shocked accretion, however, one can have two black hole type acoustic horizons (at the inner and the outer saddle type critical point) and can calculate the corresponding two different values of the acoustic surface gravity. We show this in subsequent sections.

Once the critical point is located, the critical derivatives of the sound speed \( \left( \frac{d^2 u}{dr^2} \right) \) and of the flow velocity \( \left( \frac{du}{dr} \right) \), evaluated at the critical point \( r_c \) (which coincides with the location of the acoustic horizon \( r_{ch} \)), can be obtained for various flow models by applying L’ Hospital’s rule to the numerator and the denominator of (23a), (23b) and (23c)

\[
\begin{align*}
\left| \frac{du}{dr} \right|_{CH,r_c} & = \frac{1}{r_c} \left( 1 - \frac{\gamma}{1 + \gamma} \right) \sqrt{r_c \Phi'(r_c) - \frac{\lambda^2}{r_c^2}} \pm \frac{1}{r_c} \left( 1 - \frac{\gamma}{1 + \gamma} \right)^2 \left( r_c \Phi'(r_c) - \frac{\lambda^2}{r_c^2} \right) \sqrt{r_c \Phi'(r_c) - \frac{\lambda^2}{r_c^2}} \left( \frac{2}{r_c^2} \left( \frac{r_c \Phi'(r_c)}{2} - \frac{\lambda^2}{2r_c^2} \right)^2 \frac{2(2\gamma-1)}{r_c} \frac{r_c \Phi'(r_c)}{2} - \frac{\lambda^2}{2r_c^2} + \Phi''(r_c) \right),
\end{align*}
\]

\[
\begin{align*}
\left| \frac{du}{dr} \right|_{CM,r_c} & = \frac{2}{r_c} \left( 1 - \frac{\gamma}{1 + \gamma} \right) \sqrt{r_c \Phi'(r_c) - \frac{\lambda^2}{2r_c^2}} \pm \frac{4}{r_c^2} \left( 1 - \frac{\gamma}{1 + \gamma} \right)^2 \left( \frac{r_c \Phi'(r_c)}{2} - \frac{\lambda^2}{2r_c^2} \right)^2 \frac{2(2\gamma-1)}{r_c} \frac{r_c \Phi'(r_c)}{2} - \frac{\lambda^2}{2r_c^2} + \Phi''(r_c) \right) \frac{2}{r_c^2} \left( \frac{r_c \Phi'(r_c)}{2} - \frac{\lambda^2}{2r_c^2} \right)^2 \frac{2(2\gamma-1)}{r_c} \frac{r_c \Phi'(r_c)}{2} - \frac{\lambda^2}{2r_c^2} + \Phi''(r_c) \right),
\end{align*}
\]

\[
\begin{align*}
\end{align*}
\]

(28a)

(28b)
the following first integral of motion of the Euler equation. The integral solution of the time independent Euler equation provides

\[ \left( \frac{du}{dr} \right)_{\mathrm{VE}} = 2u_c \left( \frac{\gamma - 1}{8\gamma} \right) \left( \frac{3}{r_c} + \frac{\Phi''(r_c)}{\Phi'(r_c)} \right) \]

\[ \pm \sqrt{\frac{\gamma + 1}{4\gamma}} \left[ u_c^2 \frac{\gamma - 1}{\gamma + 1} \left( \frac{3}{r_c} + \frac{\Phi''(r_c)}{\Phi'(r_c)} \right) \right]^2 - u_c^2 \frac{1 + \gamma}{2} \left( \frac{\Phi''(r_c)}{\Phi'(r_c)} \right) - \frac{2\gamma}{(1 + \gamma)^2} \left( \frac{\Phi''(r_c)}{\Phi'(r_c)} \right)^2 \]

\[ + \frac{6(\gamma - 1)}{\gamma(\gamma + 1)^2} \frac{\Phi'(r_c)}{\Phi'(r_c)} - \frac{6(2\gamma - 1)}{\gamma^2(\gamma + 1)^2} - \frac{3\lambda^2}{r_c^2} \right]^{1/2}. \quad \text{(28c)} \]

The quantity \( u_c \) in (28c) may be substituted from equations (26a) and (26b).

The acoustic surface gravity \( \kappa \) as defined in equation (14) may now be evaluated for various space time geometries for adiabatic accretion. The location of the acoustic horizon (the critical point \( r_c \)) and \( [u, c_s, d\rho/dr, du/dr] \) can be evaluated as a function of the initial boundary conditions as defined by the parameters \( [\mathcal{E}, \lambda, \gamma] \) for the adiabatic flow and \( [u, c_s, du/dr] \) can be calculated as a function of \( [T, \lambda] \) for the isothermal flow for a fixed flow geometry in all four pseudo potentials as well as under the influence of a particular pseudo potentials in all three different flow geometries.

### 6. Isothermal accretion

For an isothermal equation of state of the form \( p \propto \rho \), we integrate the time independent part of the Euler equation. The integral solution of the time independent Euler equation provides the following first integral of motion

\[ \frac{u^2}{2} + c_s^2 \ln \rho + \frac{\lambda^2}{2r^2} + \Phi(r) = \text{Constant}. \quad \text{(29)} \]

Obviously, this constant of motion cannot be identified with the specific energy of the flow. The isothermal sound speed is proportional to \( T^{\frac{1}{2}} \). The mass accretion rate, another first integral of motion of the accreting system of aforementioned kind, may be obtained for three different flow geometries as

\[ M_{\mathrm{CH}}^{\mathrm{iso}} = \rho urH_c, \quad \text{(30a)} \]

\[ M_{\mathrm{CM}}^{\mathrm{iso}} = \Theta \rho ur^2, \quad \text{(30b)} \]

\[ M_{\mathrm{VE}}^{\mathrm{iso}} = c_s \rho ur^\frac{3}{2} (\Phi')^{-\frac{3}{2}}. \quad \text{(30c)} \]

The space gradients of the velocities for these three models become

\[ \left( \frac{du}{dr} \right)_{\mathrm{CH}}^{\mathrm{iso}} = u \left( \frac{c_s^2}{7} - \Phi'(r) + \frac{\lambda^2}{r^2} \right), \quad \text{(31a)} \]

\[ \left( \frac{du}{dr} \right)_{\mathrm{CM}}^{\mathrm{iso}} = u \left( \frac{2c_s^2}{\Phi'(r)} - \Phi'(r) + \frac{\lambda^2}{r^2} \right), \quad \text{(31b)} \]

\[ \left( \frac{du}{dr} \right)_{\mathrm{VE}}^{\mathrm{iso}} = u \left[ \frac{c_s^2}{7} \left( \frac{3}{7} - \frac{\Phi''(r)}{\Phi'(r)} \right) - \Phi'(r) + \frac{\lambda^2}{r^2} \right], \quad \text{(31c)} \]
which provides the following critical point conditions

\[
(u)_{rc} = (c_s)_{rc} = \sqrt{\frac{\kappa_B}{\mu m_H}} T^2 = \sqrt{r_c \left[ \Phi'_{rc} - \frac{\lambda^2}{r_c^2} \right]},
\]

(32)

\[
(u)_{rc} = (c_s)_{rc} = \sqrt{\frac{\kappa_B}{\mu m_H}} T^2 = \sqrt{\frac{1}{2} \left( r_c \left[ \Phi'_{rc} - \frac{\lambda^2}{r_c^2} \right] \left[ \Phi''_{rc} - 2 r_c \left[ \Phi''_{rc} \right] \right] \right)^{-\frac{1}{2}}},
\]

(33)

and

\[
(u)_{rc} = (c_s)_{rc} = \sqrt{\frac{\kappa_B}{\mu m_H}} T^2 = \sqrt{\frac{\sqrt{2}}{2} \left( r_c \left[ \Phi'_{rc} - \frac{\lambda^2}{r_c^2} \right] \left[ \Phi''_{rc} - 2 r_c \left[ \Phi''_{rc} \right] \right] \right)^{-\frac{1}{2}}},
\]

(34)

for the flows with constant thickness (32), in conical equilibrium (33) and in hydrostatic equilibrium in the vertical direction (34), respectively. A two parameter input \([T, \lambda]\) (\(T\) being the isothermal flow temperature), can solve equations (32)–(34) to obtain the location of the acoustic horizon for three different flow configurations as mentioned above. The critical space gradients of the flow velocities as evaluated on the acoustic horizon are given by

\[
\left| \frac{du}{dr} \right|_{iso}^{\text{CH}}_{r_c} = \pm \frac{1}{\sqrt{2}} \sqrt{-\Phi''(r_c) + \left( \frac{c_s^2}{r_c^2} + \frac{3\lambda^2}{r_c^2} \right)^2},
\]

(35a)

\[
\left| \frac{du}{dr} \right|_{iso}^{\text{CM}}_{r_c} = \pm \frac{1}{\sqrt{2}} \sqrt{-\Phi''(r_c) - \left( \frac{2c_s^2}{r_c^2} + \frac{3\lambda^2}{r_c^2} \right)^2},
\]

(35b)

and

\[
\left| \frac{du}{dr} \right|_{iso}^{\text{VE}}_{r_c} = \pm \frac{1}{\sqrt{2}} \sqrt{c_s^2 \left[ \left( \Phi''(r_c) \right)^2 - \left( \Phi'''(r_c) \right) \right] - \left( \Phi''(r_c) + \frac{3c_s^2}{2r_c^2} \right)}.
\]

(35c)

Hence, the acoustic surface gravity for isothermal accretion can be evaluated for three different flow models as a function of only two parameters, namely, the flow angular momentum \(\lambda\) and the isothermal flow temperature \(T\).

7. Analytical calculation of the acoustic surface gravity and the corresponding analogue Hawking temperature

In this section, we calculate the surface gravity \(\kappa\) for a fluid gravitating in the Paczyński and Wiita [45] pseudo-Schwarzschild potential \(\Phi_1 = -\frac{1}{2(r - 1)}\) for the flows with constant height, in conical shape and in hydrostatic equilibrium in the vertical direction, for both the adiabatic as well as the isothermal accretion, and will study the variation of \(\kappa\) for three different flow geometries used. In addition, we calculate \(\kappa\) for the same set of initial boundary conditions for both the adiabatic and the isothermal accretion under the influence of all four pseudo-Schwarzschild potentials under consideration in the present work, for a flow in any of the three geometries mentioned above. Studying the acoustic surface gravity \(\kappa\) as a function of \(\Phi\) provides information about the dependence of \(\kappa\) on the background space time geometry.

7.1. Adiabatic accretion

In subsequent sections, we will calculate \(\kappa\) for three different models using Paczyński and Wiita potential [45] for adiabatic accretion.
7.1.1. Accretion flow with constant thickness. We start with the simplest flow configuration—axisymmetric flow with constant thickness. The space gradient of the speed of sound and the flow velocity can be computed as:

\[
\frac{dc_s}{dr} = \frac{c_s(1 - \gamma)}{2} \left[ \frac{1}{r} + \frac{1}{u} \frac{du}{dr} \right],
\]

(36a)

\[
\frac{du}{dr} = \frac{u}{r} \left[ \frac{c_s^2}{r^2} + \frac{u^2}{r^2} - \frac{1}{2(r^2 - 1)r^2} \right].
\]

(36b)

The corresponding critical point conditions can thus be obtained as:

\[
(c_s)_{rc} = (u)_{rc} = \sqrt{\frac{\frac{r_c^2}{2(r_c^2 - 1)^2}}{\lambda^2 \frac{r_c}{r_c^2}}},
\]

(37)

By substituting the above condition into the equation for the energy first integral (19), a fourth degree polynomial in \(r_c\) can be obtained in the form

\[
r_c^4 + \Gamma_1 r_c^3 + \Gamma_2 r_c^2 + \Gamma_3 r_c + \Gamma_4 = 0,
\]

(38)

where

\[
\Gamma_1 = \frac{(\gamma - 3) - 8\varepsilon(\gamma - 1)}{4\varepsilon(\gamma - 1)}; \quad \Gamma_2 = \frac{(2\varepsilon - 1)(\gamma - 1) + 2\lambda^2}{2\varepsilon(\gamma - 1)},
\]

\[
\Gamma_3 = \frac{-2\lambda^2}{\varepsilon(\gamma - 1)}; \quad \Gamma_4 = \frac{\lambda^2}{\varepsilon(\gamma - 1)}.
\]

The location of the acoustic horizon in terms of \([\varepsilon, \lambda, \gamma]\) can be obtained analytically by solving the algebraic equation (38) for \(r_c\) using the Ferrari’s method (for the details of the Ferrari’s method and its use in classical algebra, see, e.g., [50]). Then, from equation (37), one finds the flow velocity and the sound speed for each solution \(r_c\). The critical space gradient of the flow velocity and the sound speed evaluated on the acoustic horizon can then be obtained by applying l’Hospital’s rule on the numerator and the denominator of \(du/dr\) in (36a), and then by substituting the value of \((du/dr)_{rc}\) in the expression of \(dc_s/dr\) in (36b) on the acoustic horizon:

\[
\left( \frac{dc_s}{dr} \right)_{rc} = \frac{u_c}{r_c} \left[ \frac{1}{r_c} - \frac{1}{u_c} \right] - \frac{u_c^2(1 - 3\gamma)}{2(1 + \gamma)} \frac{1}{r_c^2} + \frac{1}{4(1 + \gamma)^2} \left[ \frac{3\lambda^2}{4r_c^4} \right].
\]

\[
\left( \frac{du}{dr} \right)_{rc} = \frac{u_c}{r_c} \left[ \frac{1}{r_c} - \frac{1}{u_c} \right] - \frac{u_c^2(1 - 3\gamma)}{2(1 + \gamma)^2} + \frac{1}{(1 + \gamma)(r_c^2 - 1)^3} - \frac{3\lambda^2}{r_c^4(1 + \gamma)^2}.
\]

(39)

Note that the quantities \(r_c, c_s, dc_s/dr\) and \(du/dr\) evaluated at the acoustic horizon are expressed in terms of elementary functions of \(\varepsilon, \lambda, \gamma\). Hence, the surface gravity \(\kappa\) can be calculated analytically as a function of \([\varepsilon, \lambda, \gamma]\) since

\[
\kappa_{CH} = \zeta_{CH} \left( r_c, c_s, \frac{dc_s}{dr}, \frac{du}{dr} \right)_{rc},
\]

(40)

as is obvious from equation (14).

7.1.2. Conical model. For a conical flow, the space gradient of \(c_s\) and \(u\) can be obtained as

\[
\frac{dc_s}{dr} = \frac{c_s(1 - \gamma)}{2} \left[ \frac{1}{r} + \frac{1}{u} \frac{du}{dr} + \frac{2}{r} \right],
\]

\[
\frac{du}{dr} = \frac{u}{r} \left[ \frac{2c_s^2}{r^2} + \frac{u^2}{r^2} - \frac{1}{2(r^2 - 1)r^2} \right].
\]

(41)
Hence, the critical point condition becomes
\[
(c_v)_{rc} = (u)_{rc} = \sqrt{\frac{r_c}{4(r_c - 1)^2} - \frac{\lambda^2}{2r_c^2}}.
\]  
(42)

The corresponding fourth degree polynomial in \( r_c \) can be expressed as
\[
r_c^4 + \Gamma_1 r_c^3 + \Gamma_2 r_c^2 + \Gamma_3 r_c + \Gamma_4 = 0,
\]  
(43)

where
\[
\Gamma_1 = \frac{(3\gamma - 5) - 16\epsilon(\gamma - 1)}{8\epsilon(\gamma - 1)}, \quad \Gamma_2 = \frac{2(\gamma - 1)(2\epsilon - 1) - \lambda^2(\gamma - 3)}{4\epsilon(\gamma - 1)},
\]
\[
\Gamma_3 = \frac{\lambda^2(\gamma - 3)}{2\epsilon(\gamma - 1)}, \quad \Gamma_4 = \frac{\lambda^2(3 - \gamma)}{4\epsilon(\gamma - 1)}.
\]

The critical gradient of the sound speed and the flow velocity can be obtained as,
\[
\left( \frac{dc}{dr} \right)_{rc} = \frac{2u_c}{r_c} \left( \frac{1 - \gamma}{1 + \gamma} \right) - \frac{u_c(1 - \gamma)^2}{r_c^2(1 + \gamma)^2} \left( \frac{\lambda^2(2 - r_c)(1 - \gamma)}{2r_c^2(1 + \gamma)} \right)
\]
\[
+ \frac{(1 - \gamma)^2[r_c(3 - 2\gamma) + (2\gamma - 1)]^2}{8r_c(1 + \gamma)(r_c - 1)^3}
\]  
(44a)

\[
\left( \frac{du}{dr} \right)_{rc} = \frac{2u_c}{r_c} \left( \frac{1 - \gamma}{1 + \gamma} \right) - \frac{4u_c^2}{r_c^2} \left( \frac{1 - \gamma}{1 + \gamma} \right)^2 \left( \frac{2\lambda^2(2 - r_c)}{(1 + \gamma)r_c^2} + \frac{r_c(3 - 2\gamma) + (2\gamma - 1)}{2r_c(1 + \gamma)(r_c - 1)^3} \right)^2.
\]  
(44b)

Using equation (42)–(44b), the acoustic surface gravity for the conical flow
\[
\kappa_{CM} = \zeta_{CM} \left( r, c_s, \frac{dc}{dr}, \frac{du}{dr} \right)_{rc}.
\]  
(45)

can thus be calculated analytically as a function of \([\epsilon, \lambda, \gamma]\).

### 7.1.3. Flow in hydrostatic equilibrium in vertical direction.

For a flow in hydrostatic equilibrium in the vertical direction, the velocity gradients and the corresponding critical point conditions become
\[
\frac{dc}{dr} = c_s \left( \frac{\gamma - 1}{\gamma + 1} \right) \left[ \frac{-1}{u} \frac{du}{dr} - \frac{5r - 3}{2r(r - 1)} \right],
\]
\[
\frac{du}{dr} = u(\gamma + 1) \left[ \frac{\lambda^2}{r^2} - \frac{1}{2(\gamma - 1)^2} + \frac{c_s^2(5\gamma - 3)}{r(r - 1)(\gamma + 1)} \right],
\]  
(46)

\[
\sqrt{\frac{2}{1 + \gamma}(c_v)_{rc} = (u)_{rc} = \sqrt{2 \left[ \Phi'(r_c) - \frac{\lambda^2}{r^2} \right] \left[ \frac{1}{r_c} - \frac{\Phi''(r_c)}{\Phi'(r_c)} \right]}.
\]  
(47)

The corresponding fourth degree polynomial in \( r_c \) can be expressed as
\[
r_c^4 + \Gamma_1 r_c^3 + \Gamma_2 r_c^2 + \Gamma_3 r_c + \Gamma_4 = 0,
\]  
(48)
where
\[
\Gamma_1 = \frac{5 - 16\epsilon - \frac{2\gamma}{\gamma - 1}}{10\epsilon}, \quad \Gamma_2 = \frac{6\epsilon - 3 + \frac{\gamma - 5}{\gamma - 1}\lambda^2}{10\epsilon},
\]
\[
\Gamma_3 = \frac{8\lambda^2}{10(\gamma - 1)\epsilon}, \quad \Gamma_4 = \frac{(\gamma + \lambda^2)}{10(\gamma - 1)\epsilon}.
\]

The critical gradient of the flow velocity can be found as
\[
\left( \frac{du}{dr} \right)_{rc} = -\beta - \sqrt{\beta^2 - 4\alpha\delta},
\]
where,
\[
\alpha = 4\gamma,
\]
\[
\beta = \frac{2(\gamma - 1)(5r_c - 3)u_c}{r_c(r_c - 1)},
\]
\[
\delta = \frac{\lambda^2(\gamma + 1)(r_c - 2)}{2r_c^2(r_c - 1)} - \frac{\gamma + 1}{2r_c(r_c - 1)^3} + \frac{(5r_c - 3)(\gamma - 1)}{2r_c(r_c - 1)^3} - \frac{\lambda^2(5r_c - 3)(\gamma - 1)}{r_c^2(r_c - 1)}
\]
\[
- \frac{5(\gamma + 1)}{2(r_c - 1)^2(5r_c - 3)} + \frac{5\lambda^2(\gamma + 1)}{r_c^2(5r_c - 3)}.
\]

Hence, the critical gradient of the speed of sound can be found as
\[
\left( \frac{dc_s}{dr} \right)_{rc} = \left( \sqrt{\frac{(\gamma + 1)(r_c^2 - 2(r_c - 1)^2\lambda^2)}{2r_c^2(r_c - 1)(5r_c - 3)}} \right) \left( \frac{\gamma - 1}{\gamma + 1} \right) \left[ -\frac{1}{u_c} \left( \frac{du}{dr} \right)_{rc} - \frac{5r_c - 3}{2r_c(r_c - 1)} \right],
\]
where \( \left( \frac{du}{dr} \right)_{rc} \) is to be substituted from equation (49).

### 7.2. Isothermal accretion

For an isothermal flow under the influence of the Paczyński and Wiita (1980) [45] potential, specific energy does not remain one of the first integrals of motion any more. The mass accretion rate, however, still remains a constant of motion. Since the temperature is constant, the value of the isothermal sound speed \( c_s = \sqrt{\kappa_B/(\mu m_H)T^2} \) is position independent and hence \( dc_s/dr = 0 \) identically. Mach number profile for the isothermal accretion is thus found to be a scaled down version of the dynamical velocity profile. The stationary solution is completely characterized by two parameters \([T, \lambda]\), \( T \) being the isothermal flow temperature.

#### 7.2.1. Accretion flow with constant thickness

For a constant thickness flow, the velocity gradient
\[
\frac{du}{dr} = u \left[ \frac{c_s^2}{\tau} + \frac{c_s^2}{\tau} - \frac{1}{2(\alpha - 1)^2} \right]
\]
provides the critical point condition as
\[
(u)_{rc} = (c_s)_{rc} = \sqrt{\kappa_B/(\mu m_H)T^2} = \sqrt{\left[ \frac{r_c}{2(r_c - 1)^2} - \frac{\lambda^2}{r_c^2} \right]},
\]
The corresponding fourth degree polynomial is
\[
r_c^4 + \Gamma_1 r_c^3 + \Gamma_2 r_c^2 + \Gamma_3 r_c + \Gamma_4 = 0,
\]

\(\)
where
\[
\Gamma_1 = -2 - \frac{1}{2c_s^2}, \quad \Gamma_2 = 1 + \frac{\lambda^2}{c_s^2}, \quad \Gamma_3 = \frac{-2\lambda^2}{c_s^2}, \quad \Gamma_4 = \frac{\lambda^2}{c_s^2}.
\]

(54)

The critical flow velocity gradient becomes
\[
\left(\frac{du}{dr}\right)_{rc} = \sqrt{r_c + 1 - \frac{\lambda^2}{r_c^3}}.
\]

(55)

Although both \(c_s\) and \(du/dr\) evaluated at the acoustic horizon depend only on the angular momentum of the flow and not on the flow temperature, the location of the acoustic horizon itself (the critical point \(r_c\)) is a function of both \(T\) and \(\lambda\), hence
\[
\kappa_{\text{iso}}^{\text{CH}} = \frac{\kappa_{B}}{\mu m_{\text{H}}} T^2 = \sqrt{\frac{r_c - 1}{4(r_c - 1)^2} - \frac{\lambda^2}{r_c^3}}.
\]

(56)

and the critical gradient of the flow velocity is thus
\[
\left(\frac{du}{dr}\right)_{rc} = \sqrt{r_c + 1 - \frac{\lambda^2}{r_c^3}}.
\]

(61)

which is identical to that obtained for a flow with constant thickness, (eq. (55)). The corresponding acoustic surface gravity
\[
\kappa_{\text{iso}}^{\text{CM}} = \frac{\kappa_{B}}{\mu m_{\text{H}}} \left(\frac{du}{dr}\right)_{rc}
\]

(62)

can thus be calculated analytically as a function of \([T, \lambda]\).

7.2.2. Conical model. For a conical flow, the velocity gradient
\[
\frac{du}{dr} = u \left[ \frac{2c_s^2}{r} + \frac{\lambda^2}{r} - \frac{1}{2(r-1)^2} \right] \left( u^2 - c_s^2 \right)
\]

(57)

provides the critical point condition as
\[
(u)_{rc} = (cs)_{rc} = \sqrt{\frac{\kappa_B}{\mu m_{\text{H}}} T^2} = \sqrt{\frac{r_c - 1}{4(r_c - 1)^2} - \frac{\lambda^2}{r_c^3}}.
\]

(58)

The corresponding polynomial in \(r_c\) becomes
\[
r_c^4 + \Gamma_1 r_c^3 + \Gamma_2 r_c^2 + \Gamma_3 r_c + \Gamma_4 = 0.
\]

(59)

where
\[
\Gamma_1 = -2 - \frac{1}{2c_s^2}, \quad \Gamma_2 = 1 + \frac{\lambda^2}{2c_s^2}, \quad \Gamma_3 = \frac{-2\lambda^2}{c_s^2}, \quad \Gamma_4 = \frac{\lambda^2}{2c_s^2}
\]

(60)

and the critical gradient of the flow velocity is thus
\[
\left(\frac{du}{dr}\right)_{rc} = \sqrt{r_c + 1 - \frac{\lambda^2}{r_c^3}}.
\]

(61)

which is identical to that obtained for a flow with constant thickness, (eq. (55)). The corresponding acoustic surface gravity
\[
\kappa_{\text{iso}}^{\text{CM}} = \frac{\kappa_{B}}{\mu m_{\text{H}}} \left(\frac{du}{dr}\right)_{rc}
\]

(62)

can thus be calculated analytically as a function of \([T, \lambda]\).

7.2.3. Accretion in hydrostatic equilibrium in vertical direction. For a flow in hydrostatic equilibrium in the vertical direction, the corresponding quantities are
\[
\frac{du}{dr} = u \left[ \frac{c_s^2(5r-3)}{2(r-1)^3} + \frac{\lambda^2}{r^2} - \frac{1}{2(r-1)^2} \right] \left( u^2 - c_s^2 \right).
\]

(63)
Figure 1. Acoustic surface gravity versus specific angular momentum of the flow for a mono-transonic adiabatic accretion characterized by $\dot{E} = 0.06$ and $\gamma = 1.333$ for three different flow geometries: a flow in hydrostatic equilibrium in the vertical direction (solid green line), a constant thickness flow (long dashed black line) and the conical model (dotted red line).

\begin{equation}
\left(\frac{d\zeta}{dr}\right)_{rc} = \sqrt{\frac{\kappa B}{(\mu m_H)T}} \frac{1}{2} \left[ \frac{1}{2(r_c - 1)^2} - \frac{\lambda^2}{r_c^2} \right], \end{equation}

\begin{equation}
r_c^4 + \Gamma_1 r_c^3 + \Gamma_2 r_c^2 + \Gamma_3 r_c + \Gamma_4 = 0, \end{equation}

where

\begin{align*}
\Gamma_1 &= -\frac{1 + 8c_s^2}{5c_s^2}, & \Gamma_2 &= \frac{3c_s^2 + 2\lambda^2}{5c_s^2}, & \Gamma_3 &= -\frac{4\lambda^2}{5c_s^2}, & \Gamma_4 &= \frac{2\lambda^2}{5c_s^2}, \\
\left(\frac{d\zeta}{dr}\right)_{rc} &= \sqrt{\frac{\kappa_{\text{iso}} V_{\text{E}}}{2c_s^2(r_c - 1)(5r_c - 3)}} \left[ \frac{5r_c^2 - 3}{2(r_c - 1)^2} - \frac{2\lambda^2(5r_c^2 - 9r_c + 3)}{r_c^3} \right]^{\frac{1}{2}}, \end{align*}

and the corresponding acoustic surface gravity

\begin{equation}
\kappa_{\text{iso}} V_{\text{E}} = \kappa_{\text{iso}} V_{\text{E}}(r, c_s, \frac{d\zeta}{dr})_{rc}, \end{equation}

can be evaluated accordingly.

8. Dependence of the acoustic surface gravity on the flow geometry and initial boundary conditions

8.1. Adiabatic flow

8.1.1. Mono-transonic accretion. Figure 1 shows the acoustic surface gravity $\kappa$ as a function of the specific angular momentum $\lambda$ for a mono-transonic adiabatic accretion. The specific energy and the polytropic index have been kept constant at the fixed values $\dot{E} = 0.06$ and $\gamma = 1.333$, respectively. The range of $\lambda$ for which $\kappa$ has been calculated for a particular flow model constructs a subset in the $[\dot{E}, \lambda, \gamma]$ parameter space for which the representative flow
models produces mono-transonic accretion for a fixed set of values of \([E]\). Alternative ranges for \(\lambda\) for other similar subsets of \([E, \lambda, \gamma]\) parameter space may also be considered to study the \(\kappa - \lambda\) profile for other fixed values of \([E]\).

The solid green line represents the \(\kappa - \lambda\) profile for mono-transonic accretion in vertical equilibrium (VE), whereas the dotted red and the long dashed black lines represent such dependence for a flow with constant height (CH) and for a conical flow (CM), respectively.

It is usually observed that \(\kappa\) varies with \(\lambda\) nonlinearly and non monotonically. For relatively lower values of the specific angular momentum, the acoustic surface gravity correlates with the specific angular momentum and attains a peak characterized by an unique value of the specific angular momentum denoted by \(\lambda_{\text{max}}\), and subsequently falls of with \(\lambda\). \(\lambda_{\text{max}}\) is different for different flow models and one notes that

\[
\lambda_{\text{CM}} > \lambda_{\text{VE}} > \lambda_{\text{CH}}. 
\]

(69)

For a set of fixed values of \([E, \lambda]\), for a particular flow model, \(\lambda_{\text{max}}\) can be calculated completely analytically. We illustrate such procedure for a constant thickness flow. A similar procedure may be applied to calculate \(\lambda_{\text{max}}\) for the other two flow geometries.

For a constant thickness flow, equation (40) provides the dependence of \(\kappa\) on the critical points, on the sound speed and its space gradient, and on the space gradient of the flow velocity itself, everything evaluated at the acoustic horizon (the critical point). The quantities \((c_s)_{r_c}\), \((dc_s/dr)_{r_c}\) and \((du/dr)_{r_c}\) can be expressed as a function of \(r_c\) and \([E, \lambda, \gamma]\) using equations (37) and (40), respectively. The critical point \(r_c\) itself can be computed in terms of \([E, \lambda, \gamma]\) by solving the polynomial in \(r_c\) as represented through equation (38). Hence, the acoustic surface gravity \(\kappa\) for a constant thickness flow can be specified in terms of \([E, \lambda, \gamma]\). Analytical expression for \(\kappa_{\text{CH}}\equiv\kappa_{\text{CH}}[E, \lambda, \gamma]\) can be maximized with respect to the specific angular momentum and the corresponding \(\lambda_{\text{max}}\) can thus be obtained.

From figure 1, one should note that for the common range of the specific angular momentum for which all three flow models will produce a mono-transonic flow for a fixed set of value of \([E, \lambda]\), does not allow to explore the complete non monotonic \(\kappa - \lambda\) profile for all three flow models simultaneously. For example, for the common range of \(\lambda\) for which all three flow geometries produce a mono-transonic accretion, for both the constant height flow and the conical model, the acoustic surface gravity \(\kappa\) will anti correlate with \(\lambda\), whereas for the full range of allowed \(\lambda\) for which a mono-transonic accretion forms at individual level, the \(\kappa - \lambda\) profile for both the aforementioned flows exhibits a maximum.

The \(\kappa - E\) profile depicted in figure 2 shows a similar behaviour. For a fixed set of \([\lambda, \gamma]\) for a constant height flow, \(\kappa_{\text{CH}}\) apparently correlates with \(E\), whereas \(\kappa_{\text{CM}}\) and \(\kappa_{\text{VE}}\) anti-correlate with \(E\). Such a trend, however, does not provide the complete information about the \(\kappa - E\) profile in general since the range of \(E\) for which the figure is drawn is taken from the common region of \([E, \lambda, \gamma]\) space for which all the three flow geometries produce a mono-transonic accretion for a fixed value of \([\lambda, \gamma]\). If one allows \(\lambda\) to vary with \(E\) for the entire range of the specific energy for which a particular flow model provides the mono-transonic accretion for a fixed value of \([\lambda, \gamma]\), \(\kappa - E\) profile would have a non monotonic behaviour with a corresponding \(E_{\text{max}}\) separately for every flow configuration. The corresponding \(E_{\text{max}}\) for every flow model could then be estimated by maximizing the expression for the acoustic surface gravity with respect to \(E\) by keeping \([\lambda, \gamma]\) constant. One thus understands that the ‘anti correlating’ \(\kappa - E\) profiles for a conical flow (represented by the dotted red curve) and for a flow in hydrostatic equilibrium along the vertical direction (solid green curve) are the post-peak \((E > E_{\text{max}})\) descending parts of the complete non monotonic \(\kappa - E\) profiles for the corresponding flow configurations. Similarly, the ‘correlating’ \(\kappa - E\) profile for a constant height flow (represented by the long dashed black curve) is the pre-peak \((E < E_{\text{max}})\) ascending part of the complete
Figure 2. Acoustic surface gravity versus specific energy of the flow for a mono-transonic adiabatic accretion characterized by $\lambda = 1.835$ and $\gamma = 1.333$ for three different flow geometries: a flow in hydrostatic equilibrium in the vertical direction (solid green line), a constant thickness flow (long dashed black line) and the conical model (dotted red line). Only the common range of specific energy for which all the flow configurations produce an adiabatic mono-transonic flow is presented. See text for details.

Figure 3. Acoustic surface gravity versus specific energy of the flow for a mono-transonic adiabatic accretion characterized by $\lambda = 1.85$ and $\mathcal{E} = 0.06$ for three different flow geometries: the flow in hydrostatic equilibrium in the vertical direction (solid green line), the constant thickness flow (long dashed black line) and the conical model (dotted red line).

non monotonic $\kappa - \mathcal{E}$ profile for the corresponding flow geometry. Hence, $\mathcal{E}^{\text{CH}}_{\text{max}}$ is the largest among all the values of $\mathcal{E}_{\text{max}}$ corresponding to all three different flow configurations.

Figure 3 shows the $\kappa - \gamma$ profile (for a fixed set of $[\mathcal{E}, \lambda]$). It is obvious from the figure that $\gamma^{\text{CH}}_{\text{max}}$ has the largest value among all three values of $\gamma_{\text{max}}$ corresponding to three different flow
configurations. $E_{\text{max}}$ and $\gamma_{\text{max}}$ for any particular model can also be estimated by maximizing $\kappa$ with respect to the respective parameters.

One thus understands that for a mono-transonic adiabatic accretion characterized by a fixed set of values of $[E, \lambda, \gamma]$, the analogue surface gravity for three different flow configurations exhibit the following trend

$$\kappa_{\text{CM}}^{\text{ad}} > \kappa_{\text{VE}}^{\text{ad}} > \kappa_{\text{CH}}^{\text{ad}}.$$  (70)

Hence for the adiabatic mono-transonic accretion, the flow in conical shape produces the largest whereas the flow with constant thickness the lowest value $T_{\text{AH}}$ of the analogue Hawking like temperature for the same set of initial boundary conditions.

As has been mentioned in section 5, a multi-transonic accretion with stationary shock may be realized for all three different flow configurations for adiabatic as well as for isothermal accretion, studied here in this work. For such flow topologies, two black hole type acoustic horizons form at the inner and the outer saddle type sonic points. The corresponding acoustic surface gravity $\kappa_{\text{in}}$ and $\kappa_{\text{out}}$ can be evaluated at the inner and the outer sonic points, respectively. It has been observed that the overall $\kappa_{\text{in}} - [E, \lambda, \gamma]$ as well as the $\kappa_{\text{out}} - [E, \lambda, \gamma]$ profiles are qualitatively similar to the $\kappa - [E, \lambda, \gamma]$ profile for all three flow configurations, where $\kappa$ is the acoustic surface gravity evaluated for the mono-transonic accretion. For all three flow geometries considered in this work, the value of $\kappa_{\text{out}}$ is, however, several orders of magnitude less than the corresponding value of $\kappa_{\text{in}}$ for the same set of initial boundary conditions. This is because the outer acoustic horizons form at a much larger distance away from the black hole event horizon than the distance of the inner acoustic horizon. For a typical set of initial boundary conditions, the inner acoustic horizon may form at a distance 1.5–5 Schwarzschild radii away from the black hole event horizon whereas the outer acoustic horizon may be located at $10^{3}$–$10^{6}$ Schwarzschild radii away, or even more. The weakness of gravity at such a large distance where the outer sonic horizons form restrict the acoustic surface gravity to take a small numerical value. The analogue Hawking temperatures evaluated at the inner and the outer acoustic horizons take on the correspondingly large and small values, respectively.

### 8.1.2. Multi-transonic accretion with Shock

In figure 4, we plot $\kappa_{\text{in}}/\kappa_{\text{out}}$ as a function of $\lambda$ for a constant thickness flow (uppermost panel), a flow in hydrostatic equilibrium along the vertical direction (mid panel) and for a conical flow (lowermost panel). The range of $\lambda$ in this figure corresponds to the common value of the specific angular momentum for which shock forms for multi-transonic accretion in all three flow geometries for a fixed value of $[E, \lambda]$. The common range of $\lambda$ is chosen so that the inner acoustic horizon forms at a distance larger than two Schwarzschild radii from the black hole event horizon. Since we use pseudo-Schwarzschild potentials which are relatively less reliable in simulating the general relativistic space time close to the black hole event horizon, we prefer to restrict attention to the mono-transonic and the multi-transonic flow topologies for which the acoustic horizon does not form at a very close proximity to the black hole event horizon. From the figure, it is evident that

$$\lambda_{\text{CM}}^{\text{max}} > \lambda_{\text{VE}}^{\text{max}} > \lambda_{\text{CH}}^{\text{max}},$$  (71)

where $\lambda_{\text{max}}$ is the value of $\lambda$ at which the non monotonic $\kappa_{\text{in}}$ reaches its maximum. Interestingly enough, equation (69) is identical with equation (71), indicating the fact that the dependence of the acoustic surface gravity on initial boundary conditions is similar for both the mono- and the multi-transonic shocked accretion in all three flow geometries considered in this work. Once again, the $\kappa_{\text{in}}/\kappa_{\text{out}} - \lambda$ dependence does not exhibit the peaked non monotonic profile since the common range corresponding to the shock forming $\lambda$ is not sufficient to provide the required span for the $\kappa_{\text{in}}/\kappa_{\text{out}} - \lambda$ variation for any particular flow geometry for the entire range.
of $\lambda$ for which the corresponding flow model produces a shocked multi-transonic accretion for a fixed value of $[\mathcal{E}, \lambda]$.

8.2. Isothermal flow

Figure 5 demonstrates the dependence of the acoustic surface gravity on specific angular momentum for a mono-transonic isothermal accretion for three different flow models. We note that
Figure 5. Acoustic surface gravity versus specific angular momentum of the flow for a mono-transonic isothermal accretion characterized by the isothermal flow temperature $T_{10} = 22$ for three different flow geometries: the flow in hydrostatic equilibrium in the vertical direction (solid green line), the constant thickness flow (long dashed black line) and the conical model (dotted red line).

Figure 6. Acoustic surface gravity versus isothermal flow temperature for a mono-transonic isothermal accretion characterized by $\lambda = 1.8$ for three different flow geometries: the flow in hydrostatic equilibrium in the vertical direction (solid green line), the constant thickness flow (long dashed black line), and the conical model (dotted red line).

$$\lambda_{\text{CM}} \max > \lambda_{\text{VE}} \max > \lambda_{\text{CH}} \max.$$  \hfill (72)

In figure 6, we plot the dependence of $\kappa$ on the isothermal flow temperature $T$ (scaled by a factor of $10^{10}$ degree Kelvin - $T_{10} = T \times 10^{-10}$). We obtain

$$T_{\text{CM}} \max > T_{\text{VE}} \max > T_{\text{CH}} \max.$$  \hfill (73)
The value of $\lambda_{\text{max}}$ and $T_{\text{max}}$ for the mono-transonic isothermal flow may be estimated for all three flow geometries in a way similar to what has been accomplished for an adiabatic flow.

9. Discussion

Axisymmetric accretion onto a non-rotating astrophysical black hole under the influence of pseudo-Schwarzschild potentials is a natural example of classical analogue systems found in the universe. The corresponding acoustic geometry may be studied for three different flow configurations, viz., the accretion with constant flow thickness, the conical flow and the accretion disc in hydrostatic equilibrium in the vertical direction. For each background flow geometry, eight different configurations of acoustic geometry may be studied: adiabatic and isothermal accretion in four different pseudo potentials. For any specific pseudo potential, six different configurations of acoustic flow geometry may be studied: adiabatic and isothermal accretion in three different flow geometries. For any geometric configuration of the adiabatic flow under the influence of a particular potential, three initial parameters, viz., the specific energy $E$, the specific angular momentum $\lambda$, and the adiabatic index of the flow $\gamma$, completely specify the corresponding acoustic geometry. Similarly, for an isothermal flow in any potential and with any flow geometry, two initial parameters, viz., the constant flow temperature $T$ and the specific angular momentum $\lambda$, completely specify the corresponding acoustic geometry.

Among six possible flow configurations for any particular pseudo potential used, only the adiabatic flow in vertical equilibrium exhibits an ‘effective’ sound speed which is a scaled version of the adiabatic sound speed with a $\gamma$ dependent scaling constant. The scaling constant becomes unity for the isothermal flow. The reason is that for the accretion in vertical equilibrium the flow thickness is a function of the adiabatic sound speed, as well as a function of the space derivative of the pseudo potential used, since the expression for the flow thickness is obtained by balancing the pressure gradient with the relevant component of the gravitational force. One should, however, bear in mind that the corresponding expressions for the flow thickness in all three flow geometries used are derived using a set of idealized assumptions. In principle, a more realistic derivation of the flow thickness may be worked out by employing the non-LTE radiative transfer [51, 52] or by taking recourse to the Grad–Shafranov equations for the MHD flow [53–55].

For a multi-transonic flow, two acoustic black holes are formed at two regular saddle type sonic points, whereas the acoustic white hole forms at the shock location, in agreement with the results obtained by [24] and [13]. The acoustic surface gravity is formally infinite for the acoustic white hole since the flow velocity as well as the sound speed changes discontinuously at the shock location, in agreement with [56].

The surface gravity $\kappa$ (or $T_{\text{AH}}$) profile obtained for a multi-transonic flow at the inner acoustic horizon is similar to the $\kappa$ profile for a mono-transonic flow. This indicates that irrespective of the phase topology, the surface gravity is basically determined by the physical proximity of the acoustic horizon to the black hole event horizon. This is further supported by the fact that irrespective of the flow geometry, pseudo potentials or the equation of state used to describe the accretion flow, the value of $\kappa$ at the outer acoustic horizon (for a multi-transonic flow) is much less than that evaluated at the inner acoustic horizon.

For a fixed set of $[E, \lambda, \gamma]$ describing an adiabatic accretion as well as for a fixed set of $[T, \lambda]$ describing an isothermal accretion, the conical flow produces the largest whereas the flow with constant thickness the smallest surface gravity and the analogue Hawking temperature. The accretion flow in hydrostatic equilibrium in the vertical direction provides the value of $\kappa$ and $T_{\text{AH}}$ in between the respective values of $\kappa$ and $T_{\text{AH}}$ for the conical and the constant
thickness flow. The influence of the flow geometry in determining the acoustic surface gravity has thus been successfully investigated in this work.

One of our major achievements is the analytical computation of the acoustic surface gravity and the investigation of its dependence on various flow geometries and on various accretion parameters. However, one should note that this has been possible owing to a specific feature of the chosen pseudo potential. With the Paczyński & Wiita (1980) [45] potential \( \Phi_1 = -\frac{1}{2(r-1)} \), the energy first integral for the adiabatic accretion (19) as well as the critical point condition for the isothermal flow (equations (40)–(42)) can be recast into a fourth degree polynomial in \( r_c \) (equivalently in \( r_h \)) in an exactly solvable form. This has not been possible with other pseudo potentials, for which such an exactly solvable polynomial in \( r_h \) cannot be constructed. With other potentials, however, the surface gravity \( \kappa \) can still be studied as a function of various flow parameters and flow geometries, with the help of numerical methods. Remarkably, it has been demonstrated in the literature ([21, 22] and references therein) that out of the four pseudo-Schwarzschild potentials as shown in equation (43), the potential \( \Phi_1 \) mimics the Schwarzschild space time most efficiently in constructing the integral flow solution for transonic accretion.

However, even if one cannot employ a complete analytical calculation and even if \( \Phi_1 \) is the most suitable potential to mimic a Schwarzschild space time, our generalized formalism for the evaluation of \([u, c_s, \Delta c_s/dr, du/dr]_\Lambda \) and the corresponding values of \( \kappa \) and \( T_{\Lambda H} \) in terms of a general \( \Phi \) is still important in the following sense. There exists a possibility that a new form of \( \Phi \) more effective than \( \Phi_1 \) will be suggested in the future in order to approximate the Schwarzschild space time in constructing the integral accretion solutions for a transonic flow. In such a case, if a general model is capable of computing the acoustic surface gravity (and hence the analogue Hawking temperature) in a way presented in this work, then this model will be able to readily accommodate that novel form of the pseudo-Schwarzschild potential with no need to significantly change the fundamental structure of the formulation and the solution scheme. In this case, one need not worry about providing any new unique scheme valid exclusively for a particular form of pseudo potential.

The methodology developed in this paper can also be used to construct the relevant acoustic geometry for the equatorial slice of the accretion flow under the influence of various pseudo-Kerr potentials and to study the dependence of the surface gravity on black hole spin. Such work is in progress and will be reported elsewhere.

However, one should be cautious when using the pseudo potentials because none of the potentials discussed here can be directly derived from the Einstein equations. These potentials are used to obtain more accurate correction terms over and above the pure Newtonian results. Hence, any ‘radically new’ result obtained using these potentials should be cross checked very carefully against general relativity. Besides, one should bear in mind that these potentials are not too reliable for modeling the space time in the very close neighborhood of the event horizon since strong gravity effects dominate in this region. Hence, our formalism may not be quite realistic if the acoustic horizon forms very close to the black hole event horizon. We thus consider only those initial boundary conditions for which \( r_h > 2 \) ensuring our findings to be trustworthy.

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