Entanglement entropy and Möbius transformations for critical fermionic chains.

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Entanglement entropy may display a striking new symmetry under Möbius transformations. This symmetry was analysed in our previous work for the case of a non-critical (gapped) free homogeneous fermionic chain invariant under parity and charge conjugation. In the present work we extend and analyse this new symmetry in several directions. First, we show that the above mentioned symmetry also holds when parity and charge conjugation invariance are broken. Second we extend this new symmetry to the case of critical (gapless) theories. Our results are further supported by numerical analysis. For some particular cases, analytical demonstrations show the validity of the extended symmetry. We finally discuss the intriguing parallelism of this new symmetry and space-time conformal transformations.
I. INTRODUCTION

Entanglement entropy is recognized as a most useful tool for the study of critical properties of quantum extended systems [1–10]. For instance, in one dimension, the entanglement entropy in the ground state of a critical system grows logarithmically with the size of the subsystem. The coefficient of the logarithmic term encodes the central charge of the corresponding conformal field theory. In fact, in the pioneer works [1], [3], the behaviour under space-time conformal transformations of the two-point function in a 1+1 dimensional critical theory, was used to compute the leading scaling behaviour of the entanglement entropy.

In the case of a general translation invariant free fermionic chain one can enlarge the group of symmetries of the entanglement entropy, including not only space-time symmetries (like the conformal transformations) but also a realization of the Möbius group that acts on the coupling constants of the theory.

The first glimmer of this new symmetry came from Ref. [11], where Franchini, Its, Jin and Korepin discovered that in the space of couplings of the XY spin chain there are ellipses and hyperbolas along which the von Neumann entanglement entropy of the ground state is constant. They unravelled this property by a direct analysis of the expressions for the entropy, obtained by Its, Jin and Korepin [12] and Peschel [13] in terms of elliptic integrals.

Recently, in Ref. [14], we noticed that this invariance occurs in more general spinless fermionic chains and it is not only valid for the von Neumann entropy but also for the Rényi entanglement entropy [15]. This fact implies something deeper: the full spectrum of the two-point correlation function remains invariant on these curves.

In the same paper [14], we traced back the origin of this symmetry. For any fermionic chain described by a quadratic Hamiltonian, the Rényi entanglement entropy of the ground state can be written in terms of the determinant of the two-point correlation matrix. In the case of periodic and homogeneous Hamiltonians, the correlation function of an interval of contiguous sites is a block Toeplitz matrix. The asymptotics of a block Toeplitz determinant can be obtained by solving a Riemann-Hilbert problem [16, 17]. For certain cases, where the model has a mass gap, the solution to this problem has been found [12, 18]. The determinant can be written in terms of the Riemann theta function.
of a compact Riemann surface, defined by a hyperelliptic curve whose genus is related to the range of the couplings. Then, it is easy to see that the determinant of the correlation matrix is invariant under the action of Möbius transformations on the hyperelliptic curve. Moreover, since it only depends on the couplings of the theory, we have families of gapped theories connected by a subgroup of these transformations with the same ground state entanglement entropy. From this point of view, the conics of constant entanglement entropy found in [11] correspond to flows of the Möbius transformations for genus one.

In this paper, we further analyse the behaviour of the entanglement entropy under these Möbius transformations. In particular, we shall consider critical theories as well as Hamiltonians that break parity (the same as reflection) and/or charge conjugation symmetries. These aspects are relevant when we compute the entanglement entropy [19, 20]. In this sense, recall that even if the Hamiltonian violates parity this does not necessarily imply that its vacuum breaks this symmetry. Actually, if the Hamiltonian is non critical, the ground state will preserve parity and the entanglement entropy will remain invariant under the above Möbius transformations. On the contrary, if the theory is massless, the vacuum can break the parity symmetry.

From the above geometrical perspective, critical theories with a parity invariant vacuum correspond to hyperelliptic curves where some pairs of branch points degenerate at the unit circle. Therefore, the associated Riemann surface is pinched and the behaviour of the entropy changes. In fact, we shall see that it is no longer invariant under Möbius transformations, but it changes like a product of homogeneous fields. A different case is that of a critical theory with a parity symmetry breaking vacuum. We shall show that in this situation the entanglement entropy transforms also like homogeneous fields, but of different dimension.

Another natural question is what happens when the subsystem is made out of several disjoint intervals. The difficulty here is that the correlation matrix is not block Toeplitz anymore, but its principal submatrix. However, using the expression proposed in [21] for the determinant of these kind of matrices, we are able to deduce the transformation law of the entanglement entropy for these disconnected subsystems. This reveals a striking similarity with its behaviour under conformal transformations on the real space when the theory is massless.

The paper is organized in the following way. In Section II, we introduce the basic nota-
tions and definitions as well as the models we shall study, discussing its critical properties and the symmetries of its ground state. In section III, we focus on non-critical theories with ground state being parity invariant. We shall show that the entanglement entropy of a single interval is invariant under Möbius transformations. The proof presented here is more general and simpler than that of [14]. Moreover, the new approach reveals in a transparent way that the Möbius symmetry of entanglement entropy is a consequence of the asymptotic invariance of the spectrum of the correlation matrix. Section IV is dedicated to critical theories. The first part of this section is concerned with theories with parity invariant ground state while in the second part we consider those theories with parity symmetry breaking ground state. In both cases we find that the transformation law of the entropy is analogous to that of a product of homogeneous fields inserted at the discontinuities of the symbol of the correlation matrix. Their scaling dimension is the ingredient that distinguishes between vacua that preserve parity and those that break it. In Section V, we shall extend this transformation law to the case of several disjoint intervals, comparing it with the behaviour under global conformal transformations in the real space. Finally, in Section VI we shall end with some conclusions. In the Appendices, we analyse in detail the case of Hamiltonians with parity and charge conjugation symmetries and study the degenerate limit of their associated hyperelliptic curves, both if they correspond to massive or massless theories.

II. ENTANGLEMENT ENTROPY IN THE FREE FERMIONIC CHAIN

Let us consider a unidimensional chain of $N$ spinless fermions described by the following periodic, quadratic, translation invariant Hamiltonian with long range couplings ($L < N/2$)

$$H = \frac{1}{2} \sum_{n=1}^{N} \sum_{l=-L}^{L} \left( 2A_l a_n^\dagger a_{n+l} + B_l a_n^\dagger a_{n+l}^\dagger - B_l a_n a_{n+l} \right).$$  \hspace{1cm} (1)

Fermionic annihilation and creation operators at site $n$, $a_n$ and $a_n^\dagger$, satisfy the canonical anticommutations relations

$$\{a_n, a_m\} = \{a_n^\dagger, a_m^\dagger\} = 0, \quad \{a_n^\dagger, a_m\} = \delta_{nm},$$  \hspace{1cm} (2)

and we fix periodic boundary conditions, $a_{n+N} = a_n$. The Hamiltonian is Hermitian when $A_{-l} = \overline{A}_l$. In addition, we shall impose, without loss of generality, that $B_{-l} = -B_l$. 


Since the Hamiltonian is translation invariant, we may introduce the Fourier modes
\[ b_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta_k} a_n, \quad \theta_k = \frac{2\pi k}{N}. \]

After performing the appropriate Bogoliubov transformation \( d_k = \cos \xi_k b_k + i \sin \xi_k b_k^\dagger \), we obtain the diagonalized Hamiltonian (see e.g. [20]),
\[ H = \mathcal{E} + \sum_{k=0}^{N-1} \Lambda(\theta_k) \left( d_k^\dagger d_k - \frac{1}{2} \right), \]
where
\[ \mathcal{E} = \frac{1}{2} \sum_{k=0}^{N-1} \Theta(e^{i\theta_k}) \]
is an irrelevant constant shift in the energy levels and, for \( \theta \in (-\pi, \pi] \),
\[ \Lambda(\theta) = \sqrt{\Theta^+(e^{i\theta})^2 + |\Xi(e^{i\theta})|^2 + \Theta^-(e^{i\theta})} \]
is the dispersion relation. Both these functions may be expressed in terms of the Laurent polynomials
\[ \Theta(z) = \sum_{l=-L}^{L} A_l z^l, \quad \Xi(z) = \sum_{l=-L}^{L} B_l z^l \quad \text{and} \quad \Theta^\pm(z) = \frac{\Theta(z) \pm \Theta(z^{-1})}{2}, \]
that map meromorphically the Riemann sphere \( \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) to itself. Due to the properties of the coupling constants we have \( \Theta^\pm(z) = \Theta^\pm(z^{-1}) = \pm \Theta^\pm(z) \) and \( \Xi(z^{-1}) = -\Xi(z) \).

A general stationary state is obtained by selecting a set of modes \( K \subset \{0, \ldots, N-1\} \) to which we associate
\[ |K\rangle = \left( \prod_{k \in K} d_k^\dagger \right) |0\rangle, \quad \text{with} \quad d_k |0\rangle = 0, \quad \forall k, \]
so that \( |0\rangle \) represents the vacuum of the Fock space for the Bogoliubov modes \( d_k \). The corresponding energy eigenvalue is
\[ E_K = \mathcal{E} + \frac{1}{2} \sum_{k \in K} \Lambda(\theta_k) - \frac{1}{2} \sum_{k \in K} \Lambda(\theta_k). \]
The ground state is obtained when we select the modes in the Dirac sea
\[ |\text{GS}\rangle = \left( \prod_{\Lambda(\theta_k) < 0} d_k^\dagger \right) |0\rangle, \]
and has an energy

\[ E_{\text{GS}} = \mathcal{E} - \frac{1}{2} \sum_{k=0}^{N-1} |\Lambda(\theta_k)|. \]

Notice that if

\[ \Theta^{-}(e^{i\theta})^2 \leq \Theta^{+}(e^{i\theta})^2 + |\Xi(e^{i\theta})|^2, \tag{7} \]

for all \( \theta \in (-\pi, \pi] \), then the ground state is the Fock space vacuum \( |0\rangle \); while if the inequality is not true for some values of \( \theta \) then the state of minimum energy is obtained by exciting the modes with negative energy (Dirac sea). In Fig. 1, we represent these different possibilities. In the same figure, we also show a third possibility. This case corresponds to a critical gapless Hamiltonian with non-negative dispersion relation but vanishing at some values of \( \theta \). It corresponds to situations in which (7) holds but the right hand side vanishes at some points. As we shall see later the theory has different properties in the three cases and it requires a separate analysis.

It is interesting to discuss how the discrete transformations of parity (P) and charge conjugation (C) act on this system. In terms of the creation-annihilation operators they are defined by

\[ P : a_n \mapsto ia_{N-n}, \quad C : a_n \mapsto a^\dagger_n. \tag{8} \]

The Hamiltonian is \( P \) invariant if the couplings \( A_l \) are real, while \( PC \) is a symmetry if \( B_l \in \mathbb{R} \). In the latter case, \( H - \mathcal{E} \) goes to \( \mathcal{E} - H \).

The action of \( P \) and \( C \) on the Fourier and Bogoliubov modes are

\[ P : b_k \mapsto ib_{-k}, \quad C : b_k \mapsto b_{-k}^\dagger, \tag{9} \]

\[ P : d_k \mapsto id_{-k}, \quad C : d_k \mapsto d_{-k}^\dagger, \tag{10} \]

where \( \overline{d}_k = \cos \xi_k b_k - i \sin \xi_k b_{-k}^\dagger \) represents the Bogoliubov mode for the Hamiltonian transformed under \( PC \), i.e. replacing \( B_l \) by its complex conjugate \( \overline{B}_l \). Note that the Bogoliubov modes transform covariantly under parity even if the Hamiltonian is not invariant, while they are covariant under charge conjugation only if the Hamiltonian is \( PC \) symmetric.

The above observations have important consequences for determining the symmetries of the ground state. In particular, one can notice that if the Hamiltonian has a gap, the ground state is always \( P \) invariant, irrespectively of the symmetries of the Hamiltonian. This corresponds to plot a in Fig. 1. On the contrary, if the dispersion relation attains
FIG. 1: Dispersion relation $\Lambda(\theta)$ for three representative Hamiltonians that violate parity. In the case $a$ the theory has a gap, hence its ground state is $|0\rangle$, which preserves parity. The dispersion relation in $b$ is negative for a set of modes (shadowed interval). Therefore the theory is gapless and the ground state is obtained by filling these modes (the Dirac sea), breaking the parity of the state. In the panel $c$, $\Lambda$ is non-negative but it has zeros. Then the model is gapless but the ground state is $|0\rangle$ and, hence, invariant under parity.

negative values, the ground state is given by the excitation of the modes in the Dirac sea, and then it is not parity invariant. This situation is represented in the plot $b$ in Fig. 1. There is a third possibility plotted in Fig. 1 $c$, in which the Hamiltonian is gapless and the ground state is parity invariant. The three scenarios will be analysed separately in the following sections. As we will show, their ground state entanglement entropy behaves under Möbius transformation in three different ways.
In the following paragraphs we will introduce the basic definitions and notation to discuss the Rényi entanglement entropy for this model.

We divide the chain into two subsystems $X \cup Y = \{1, \ldots, N\}$, so that the Hilbert space factorizes as $H = H_X \otimes H_Y$. We are interested in the Rényi entanglement entropy $S_{\alpha,X}$ of the subsystem $X$ for the ground state $|GS\rangle$ of \([1]\). If we introduce the partition function \([3]\)

$$Z_{\alpha,X} = \text{Tr}(\rho_X^\alpha), \quad (11)$$

the entropy is defined as

$$S_{\alpha,X} = \frac{1}{1-\alpha} \log Z_{\alpha,X}. \quad (12)$$

Due to the properties of the ground state of a Gaussian theory, the Wick decomposition holds for the $n$-point correlation functions; and therefore $Z_{\alpha,X}$ and the entropy can be recast in terms of the two-point correlations \([2, 22]\),

$$V_{nm} = \left\langle \begin{pmatrix} a_n \\ a_n^\dagger \end{pmatrix} \begin{pmatrix} a_m \n, m = 1, \ldots, N. \quad (13)\right.\right.

In fact,

$$Z_{\alpha,X} = \det F_{\alpha}(V_X), \quad (14)$$

where

$$F_{\alpha}(x) = \left(\frac{1+x}{2}\right)^\alpha + \left(\frac{1-x}{2}\right)^\alpha \quad (15)$$

and $V_X$ is the $2|X|$ by $2|X|$ dimensional block matrix $V_X = (V_{nm})$, $n, m \in X$ with $2 \times 2$ submatrices being

$$V_{nm} = \frac{1}{N} \sum_{k=0}^{N-1} G(\theta_k)e^{i\theta_k(n-m)}, \quad (16)$$

with

$$G(\theta) = \begin{cases} 
I, & \text{if } \Lambda(\theta) < 0 \text{ and } \Lambda(-\theta) > 0, \\
M(\theta), & \text{if } \Lambda(\theta) \geq 0 \text{ and } \Lambda(-\theta) \geq 0, \\
-I, & \text{if } \Lambda(\theta) > 0 \text{ and } \Lambda(-\theta) < 0, 
\end{cases} \quad (17)$$

and

$$M(\theta) = \frac{1}{\sqrt{\Theta^+(e^{i\theta})^2 + |\Xi(e^{i\theta})|^2}} \begin{pmatrix} \Theta^+(e^{i\theta}) & \Xi(e^{i\theta}) \\
\Xi(e^{i\theta}) & -\Theta^+(e^{i\theta}) \end{pmatrix}. \quad (17)$$
In the following sections it will be crucial the analytic continuation $\mathcal{M}(z)$ of $M(\theta)$ from the unit circle $\gamma = \{ z = e^{i\theta} \}$ to the Riemann sphere $\overline{\mathbb{C}}$, namely

$$
\mathcal{M}(z) = \frac{1}{\sqrt{\Theta'(z)^2 - \Xi(z)\Xi(\overline{z})}} \begin{pmatrix}
\Theta'(z) & \Xi(z) \\
-\Xi(\overline{z}) & -\Theta'(z)
\end{pmatrix}.
$$

(18)

Notice that we have used $\Xi(z^{-1}) = -\Xi(z)$.

To understand the analytic structure of $\mathcal{M}(z)$ we introduce the polynomial

$$
P(z) = z^{2L} \left( \Theta'(z)^2 - \Xi(z)\Xi(\overline{z}) \right).
$$

(19)

Then, one immediately sees that $\mathcal{M}(z)$ is bivalued in $\overline{\mathbb{C}}$ and actually is meromorphic in the Riemann surface represented by the complex curve

$$
w^2 = P(z).
$$

(20)

This curve represents a double covering of the Riemann sphere with branch points at the zeros of $P(z)$. The genus of the Riemann surface is related to the range of the couplings of the Hamiltonian, $g = 2L - 1$. Due to the properties of $\Theta$ and $\Xi$, one deduces that $P(z)$ has real coefficients and verifies $z^{4L}P(z^{-1}) = P(z)$. This implies that the zeros of $P(z)$ come in quartets related by inversion and complex conjugation, except for the real ones that come in pairs related by inversion. In Fig. 2 we present an example corresponding to $L = 2$.

The non-critical theories, i. e. those which have a mass gap or equivalently $\Lambda(\theta) > 0$, can be easily characterized in terms of the roots of $P(z)$. To see it, simply consider the relation

$$
(\Lambda(\theta) + \Lambda(-\theta))^2 = 4|P(e^{i\theta})|.
$$

Hence if $\Lambda(\theta) > 0$ for any $\theta$, $P(e^{i\theta}) \neq 0$ and there are no roots at the unit circle.

In the other case, that is, if $P(z)$ has roots at the unit circle, they are necessarily degenerate, the dispersion relation vanishes at some points and it corresponds to a critical massless theory.

In the next sections we shall take $X$ to be a single interval in the chain composed of $|X|$ consecutive sites, namely $X = \{1, \ldots, |X|\}$. In this case $V_X$ is a block Toeplitz matrix of the type that has been studied in [16, 18]. The more general case of $X$ composed of several intervals will be discussed later in section [VI].
FIG. 2: Possible arrangement of the branch points and cuts of $w = \sqrt{P(z)}$ for genus $g = 3$ ($L = 2$).

Half of them must be inside the unit circle and the rest outside it. Since the branch points must be related by inversion and complex conjugation, $z_1 = z_8^{-1}$, $z_2 = z_7^{-1}$, $z_3 = z_4 = z_5^{-1} = z_6^{-1}$.

III. MÖBIUS INVARIANCE IN NON CRITICAL CHAINS

We will consider in first place the case of non critical Hamiltonians for which $\Lambda(\theta) > 0$ for any $\theta$. Therefore, $G(\theta) = M(\theta)$. In addition, we are interested in the thermodynamic limit $N \to \infty$. Hence the sum in (16) is transformed into the integral

$$V_{nm} = \frac{1}{2\pi i} \oint_{\gamma} z^{n-m} M(z) \frac{dz}{z},$$

(21)

where $M(z)$ is the analytic continuation of $M(\theta)$ from the unit circle $\gamma$ to the Riemann sphere defined in (18).

In [14] we introduced a new symmetry of the Rényi entanglement entropy using the connection of the latter with the solution of a Wiener-Hopf problem. Here we present a more direct approach to the subject based on the existence of a similarity transformation for the correlation matrix in the asymptotic limit.

Following [12] we introduce the operator

$$K_X v(z) = v(z) - \frac{1}{2\pi i} \oint_{\gamma} \frac{(z/y)^{|X|} - 1}{z-y} (I - M(y)) v(y) \, dy, \quad v \in L^2(\gamma) \otimes \mathbb{C}^2,$$

(22)

defined to act on the Hilbert space $L^2(\gamma) \otimes \mathbb{C}^2$ with scalar product

$$(v_1, v_2) = \frac{1}{2\pi i} \oint_{\gamma} v_1(y)^\dagger v_2(y) \frac{dy}{y}.$$
In the topology induced by this inner product, $K_X$ is a continuous operator for any $\mathcal{M}$ bounded in $\gamma$ and, as we shall show now, it fulfills the important property

$$\det F_\alpha(V_X) = \det F_\alpha(K_X).$$

(23)

In the following, we prove this fact.

Consider an orthonormal basis in $L^2(\gamma) \otimes \mathbb{C}^2$ of the form $\{z^n e_\nu | n \in \mathbb{Z}, \nu = 1, 2\}$, where the vectors $e_\nu$ form the standard basis in $\mathbb{C}^2$. One can immediately compute the matrix representation of $K_X$ in this basis to get

$$\begin{pmatrix}
I & 0 & 0 \\
(V_{na}) & (V_{nm}) & (V_{nb}) \\
0 & 0 & I
\end{pmatrix},$$

where the different indices are meant to run through the following ranges: $a \leq 0$, $n, m = 1, \ldots, |X|$ and $b > |X|$. Now due to the block form of $K_X$ one has

$$\det F_\alpha(K_X) = \det F_\alpha(I)^2 \cdot \det F_\alpha(V_X),$$

and looking at the definition (15) of $F_\alpha$ one immediately obtains $F_\alpha(I) = I$, from which the result follows.

The relation between $K_X$ and $V_X$ can be extended to the $|X| \to \infty$ limit. We may define

$$Kv(z) = v(z) + \lim_{\mu \to 1} \int_{\gamma} \frac{1}{2\pi i} \oint_{\mu z - y} \frac{I - \mathcal{M}(y)}{\mu} v(y) \, dy,$$

where we take the lateral limit for real values of $\mu$ smaller than 1. The matrix representation of $K$ in the basis introduced before is simply

$$\begin{pmatrix}
I & 0 \\
(V_{na}) & (V_{nm})
\end{pmatrix},$$

with $a \leq 0$ and $n, m > 0$.

If we call $V$ the inductive limit of $V_X$ when $|X| \to \infty$, we can immediately extend the relation (23) to this limit and thus we have

$$\det F_\alpha(V) = \det F_\alpha(K).$$

(24)

We shall denote by $Z_\alpha$ and $S_\alpha$ the partition function and the entanglement entropy in this limit $|X| \to \infty$,

$$Z_\alpha = \det F_\alpha(K),$$
\[ S_\alpha = \frac{1}{1-\alpha} \log Z_\alpha. \]

The computation of the determinant of \( K \) for real couplings \( A_l, B_l \) has been the subject of references \[12, 18\]. There the authors transform this into a Wiener-Hopf factorization problem. Later on we will use their results. For the moment, however, we are not interested into the computation of the partition function or the Rényi entanglement entropy, but rather in its invariance properties. Hence a simpler approach is enough.

We shall proceed in two steps. First we consider general Möbius transformations and check under which circumstances the determinant \( Z_\alpha = \det F_\alpha(K) \) is left invariant. In the next step we will ask which of these transformations are physical. By physical we mean those transformations that can be implemented as a change in the coupling constants of the theory.

Let us consider a Möbius transformation on \( \mathbb{C} \) defined by

\[
z' = \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}). \tag{25}\]

This induces a transformation on the symbol \( \mathcal{M}(z) \) such that \( \mathcal{M}'(z') = \mathcal{M}(z) \) and consequently on the operator \( K \), so that

\[
K'v(z') = v(z') + \lim_{\mu \to 1} \frac{1}{2\pi i} \oint_{\gamma} \frac{I - \mathcal{M}'(y')}{\mu z' - y'} v(y') \, dy',
\]

where for convenience we use primed integration variable. Now we perform the change of variables \( y'(y) \) induced by \( (25) \),

\[
K'v(z') = v(z') + \lim_{\mu \to 1} \frac{1}{2\pi i} \oint_{\gamma'} \frac{I - \mathcal{M}(y)}{\mu z' - y'(y)} v(y'(y)) \frac{\partial y'}{\partial y} \, dy,
\tag{26}

where the relation \( \mathcal{M}'(y'(y)) = \mathcal{M}(y) \) was used and \( \gamma' = \{ y, |y'(y)| = 1 \} \). The crucial property is that for any Möbius transformation

\[
z'(z) - y'(y) = \left( \frac{\partial z'}{\partial z} \right)^{1/2} \left( \frac{\partial y'}{\partial y} \right)^{1/2} (z - y).
\]

Plugging this into \( (26) \) we have

\[
K'v(z'(z)) = v(z'(z)) + \left( \frac{\partial z'}{\partial z} \right)^{-1/2} \lim_{\mu \to 1} \frac{1}{2\pi i} \oint_{\gamma} \frac{I - \mathcal{M}(y)}{\mu z - y} \left( \frac{\partial y'}{\partial y} \right)^{1/2} v(y'(y)) \, dy. \tag{27}
\]
Here we have assumed that we can safely apply the Cauchy’s integral theorem in order to deform $\gamma'$ into $\gamma$, which in particular demands that $M$ is analytic in a region in with both curves are homotopic. Defining

$$T v(z) = \left( \frac{\partial z'}{\partial z} \right)^{1/2} v(z'(z)),$$

we finally obtain

$$TK'v = KT v.$$

Therefore $K$ and $K'$ are related by a similarity transformation and all its spectral invariants coincide. In particular $\det F_\alpha(K) = \det F_\alpha(K')$.

To close the previous discussion, some words of caution are in order. On the one hand, in order to be able to apply the Cauchy theorem we need not only that $M(y)$ is analytic, as we actually demanded, but also $v(y'(y))$ should have the same property. On the other hand, in the definition of $T$ we implicitly use analytic continuation for determining $Tv(z)$ for $z \in \gamma$. This implies that we should restrict ourselves to situations in which $v$ and $K'v$ are analytic in a region such that $\gamma$ and its Möbius transformed are homotopic. Finally, note that the analyticity condition on $M(z)$ can be simply stated by asking that for any $z_i$ root of $P(z)$ inside (outside) the unit circle, its Möbius transformed $z'_i$ has to be also inside (outside).

The previous difficulties do not arise if the Möbius transformation preserves the unit circle. In this case it is of the form

$$z' = \frac{az + b}{bz + a}, \quad |a|^2 - |b|^2 = 1. \tag{28}$$

Then $T$ is a bounded operator and the similarity relation between $K$ and $K'$ holds for any $v \in L^2(\gamma) \otimes \mathbb{C}^2$.

This is also the case of physical interest. The reason is that the infinitesimal Möbius transformation associated with changes of the coupling constants of the theory must preserve the unit circle.

Indeed, as it is discussed in [14], the Möbius transformation \[25\] acts on the couplings $A = (A_L, A_{L-1}, \ldots, A_{-L})$ and $B = (B_L, B_{L-1}, \ldots, B_{-L})$ like the spin $L$ representation of $SL(2, \mathbb{C})$. One immediately sees that only a subset of Möbius transformations map the original set of couplings $A, B$ to another set $A', B'$ that fulfils the conditions for the hermiticity of the Hamiltonian and the antisymmetry of $\Xi'$: $A'_L = \overline{A'}_L$ and $B'_{-l} = -B'_l$. 


An equivalent, and more appropriate way of imposing these conditions is to demand that the roots of the new polynomial

\[ P'(z') = (cz + d)^{-4L}P(z) \]

come also in quartets: if \( z'_i \) is a root of \( P' \) then \( \bar{z}'_i \) and \( 1/z'_i \) are also roots. A way of guaranteeing this is by selecting those Möbius transformations that commute with complex conjugation and inversion. These correspond in particular to the group \( SO(1,1) \) with transformations like

\[ z' = \frac{z \cosh \zeta + \sinh \zeta}{z \sinh \zeta + \cosh \zeta}. \]  

(29)

Therefore, due to the physical meaning of our symmetry we are led to consider the 1+1 Lorentz group which has the important property of preserving the unit circle.

**To summarize:** Möbius transformations in the \( z \) plane induce a change in the coupling constants of the free fermionic chain. They are physically admissible if they are like those in (29). Moreover, if we deal with a non critical theory, then

\[ Z'_\alpha = Z_\alpha \quad \text{and} \quad S'_\alpha = S_\alpha, \]  

(30)

where these are respectively the partition function of (11) and the Rényi entanglement entropy (12) for an interval in the infinite size limit. This is easily derived from the relations \( K' = T^{-1}KT \), \( Z_\alpha = \det F_\alpha(K) \) and \( S_\alpha = \log Z_\alpha/(1-\alpha) \), that hold for physically admissible coupling constants.

**IV. MÖBIUS TRANSFORMATIONS IN CRITICAL THEORIES**

Our next goal is to extend the considerations in the previous section to the case of massless critical theories. In this case, the symbol is not holomorphic in the unit circle and the previous results do not apply. In addition, we must distinguish between ground states that are parity invariant and those that break this symmetry.

**Parity invariant Ground States**

First, let us consider massless theories with a ground state invariant under parity symmetry. An example is depicted in Fig. 1c. In the previous section we associated a
Riemann surface to the fermionic chain. The critical point is attained when two branch points collide at the unit circle. This corresponds to the pinching of the corresponding Riemann surface. Hence $\mathcal{M}(z)$ is not holomorphic in the unit circle and the entanglement entropy scales logarithmically with the size of the subsystem, with a coefficient proportional to the number of pinchings.

The difficulty in this case is the following. The logarithmic term is well known and does not change under Möbius transformations. However we do not know in general how to compute the non-universal constant which differs from one theory to another. In order to deal with this problem and determine how the entropy behaves under Möbius transformations we follow two different strategies. The first strategy is to consider special cases in which we have an explicit expression for the entropy. In this case we can easily deduce its behaviour under the transformations. The second strategy is to study the limit to criticality starting from a non-critical theory like those considered in the previous section. From the results obtained with the two strategies we will be able to conjecture a simple transformation law for the entropy. We cannot prove this conjecture in general but numerical checks leave no doubt that it is correct.

As we mentioned before, in some special situations we have a completely explicit expression for the asymptotic behaviour of the entanglement entropy. For example, when $A_l \in \mathbb{R}$ and $B_l = 0$. In this case one immediately sees that the symbol $\mathcal{M}(z)$ at the unit circle is $\pm \sigma_z$. The discontinuities are the roots of $z^L \Theta(z)$ at the unit circle, which we will denote by $u_\kappa = \exp(i\theta_\kappa)$, $\kappa = 1, \ldots, R$ in anticlockwise order. Due to the parity invariance, $\Theta(z^{-1}) = \Theta(z)$, the roots $u_\kappa$ come in pairs related by inversion. We shall assume that all of them are simple roots. This implies that they are different from $\pm 1$.

From this particular form of the symbol, it is clear that the entropy $S_{\alpha,X}$ and the partition function $Z_{\alpha,X}$ may depend solely on the set of roots $u = (u_1, \ldots, u_R)$. Based on [21] (see also [24]), we can write the asymptotic behavior in the following convenient way

$$S_{\alpha,X}(u) = \frac{(\alpha + 1)R}{12\alpha} \log |X| - \frac{\alpha + 1}{12\alpha} \sum_{1 \leq \kappa \neq \nu \leq R} (-1)^{\kappa + \nu} \log(u_\kappa - u_\nu) + RI_\alpha + \ldots, \quad (31)$$

where

$$I_\alpha = \frac{1}{2\pi i(1 - \alpha)} \int_{-1}^{1} \frac{d \log F_\alpha(\lambda)}{d \lambda} \log \left[ \frac{\Gamma(1/2 - i\omega(\lambda))}{\Gamma(1/2 + i\omega(\lambda))} \right] d\lambda, \quad (32)$$
with $\Gamma(z)$ the Gamma function,

$$\omega(\lambda) = \frac{1}{2\pi} \log \left| \frac{\lambda - 1}{\lambda + 1} \right|,$$

and the dots represent terms that vanish in the large $|X|$ limit.

Now applying any Möbius transformation

$$u'_\kappa - u'_\nu = \left( \frac{\partial u'_\kappa}{\partial u_\kappa} \frac{\partial u'_\nu}{\partial u_\nu} \right)^{1/2} (u_\kappa - u_\nu),$$

(34)

to the entropy (31), we obtain in the asymptotic limit that

$$S'_{\alpha,X}(u') = S_{\alpha,X}(u) + \frac{\alpha + 1}{12\alpha} \sum_{\kappa=1}^{R} \log \left| \frac{z'_\kappa - z'_{-1}\kappa}{z_\kappa - z_{-1}\kappa} \right| + K'_\alpha(u') + \ldots,$$

(35)

We conjecture that this transformation also applies for a general critical Hamiltonian with parity preserving vacuum. To further motivate the conjecture and gain a better understanding of its origin, we will study the limit to criticality of a symmetric theory.

Let us consider a general non critical Hamiltonian with parity and charge conjugation symmetry, i.e. $A_l, B_l \in \mathbb{R}$ $\forall l$. Criticality in this case is achieved when pairs of roots of $P(z)$, say $z_{j\kappa}$ and $z_{-1\kappa}$ approach $u_\kappa$ at the unit circle. Geometrically this corresponds to the pinching of some cycles in the associated Riemann surface. As the roots approach its limit the entropy diverges. In the Appendix C we compute its behaviour. The result can be written as

$$S_{\alpha} = -\frac{\alpha + 1}{12\alpha} \sum_{\kappa=1}^{R} \log |z_{j\kappa} - z_{-1\kappa}^{-1}| + K_{\alpha}(u) + \ldots,$$

(36)

where the dots stand for contributions that vanish in the limit $z_{j\kappa} \to u_\kappa, \kappa = 1, \ldots, R$. We have also assumed that $u_\kappa \neq u_\xi, \kappa \neq \xi$ and we have omitted the explicit dependence on the non degenerate branch points.

We are interested in studying the behaviour of $K_{\alpha}$ under an admissible Möbius transformation. For that purpose we can use the invariance of $S_{\alpha}$. Indeed, one has

$$S'_{\alpha} = -\frac{\alpha + 1}{12\alpha} \sum_{\kappa=1}^{R} \log |z'_j - z'_{-1\kappa}| + K'_{\alpha}(u') + \ldots,$$

but

$$z'_j - z'^{-1}_j = \frac{z'_j}{z_j} \left| \frac{\partial z'_j}{\partial z_j} \right| (z_j - z^{-1}_j).$$
Here we have used that Möbius transformations in $SO(1, 1)$ commute with complex conjugation. As these transformations also commute with inversion and the points $u_\kappa$ lie in the unit circle, we have

$$S'_\alpha = -\frac{\alpha + 1}{12\alpha} \sum_{\kappa=1}^{R} \left( \log |z_{j_\kappa} - \bar{z}_{j_\kappa}^{-1}| + \log \frac{\partial u'_\kappa}{\partial u_\kappa} \right) + K'_\alpha(u') + \ldots .$$

And from the invariance of $S_\alpha$ one has

$$K'_\alpha(u') = K_\alpha(u) + \frac{\alpha + 1}{12\alpha} \sum_{\kappa=1}^{R} \log \frac{\partial u'_\kappa}{\partial u_\kappa} . \quad (37)$$

It may seem strange that the entropy diverges at the critical points ($z_{j_\kappa} = \bar{z}_{j_\kappa}^{-1}$). The reason for that is simple. We took the asymptotic $|X| \to \infty$ limit in the non critical theory. This renders a finite entropy. At criticality due to the logarithmic scaling with the size of the subsystem, it diverges. In fact, we may have a finite entropy in the critical theory by restoring the finite size interval $X$. In this case the entropy of the critical theory up to terms that vanish in the asymptotic limit is [20]

$$S_{\alpha,X}(u) = \frac{(\alpha + 1)R}{12\alpha} \log |X| + C_\alpha(u) + \cdots , \quad (38)$$

where we have reintroduced the dependence on the degeneration points $u_\kappa$. The expression for the constant term $C_\alpha(u)$ is not known in general.

It is very suggestive to compare the formula above with (36). One sees that, up to constant terms, (38) is obtained from (36) by simply replacing $|z_{j_\kappa} - \bar{z}_{j_\kappa}^{-1}|$ with $|X|^{-1}$. If we assume that the finite corrections are invariant under Möbius transformations, or in other words

$$K'_\alpha(u') - C'_\alpha(u') = K_\alpha(u) - C_\alpha(u) ,$$

then one immediately obtains a result identical to (35), that is,

$$S'_{\alpha,X}(u') = S_{\alpha,X}(u) + \frac{\alpha + 1}{12\alpha} \sum_{\kappa=1}^{R} \log \frac{\partial u'_\kappa}{\partial u_\kappa} . \quad (39)$$

For the partition function, we have

$$Z'_{\alpha,X}(u') = \prod_{\kappa=1}^{R} \left( \frac{\partial u'_\kappa}{\partial u_\kappa} \right)^{2\Delta_\alpha} Z_{\alpha,X}(u) , \quad (40)$$

with $\Delta_\alpha = (\alpha^{-1} - \alpha)/24$. Therefore $Z_{\alpha,X}$ transforms like the product of homogeneous fields of dimension $2\Delta_\alpha$ inserted at the pinchings $u_\kappa$. 
We conjecture the same behaviour for the entanglement entropy (or the partition function) of the free fermionic chain for any finite range critical Hamiltonian with parity symmetric vacuum. This has been checked numerically. The results shown in Fig. 3 are quite convincing.

FIG. 3: Numerical check of transformation law (39) for a critical theory with $L = 2$ and pinchings at $\pm \theta_F$, as we depict in the inset. We have computed numerically for several sizes of $X$ the change of the Rényi entaglement entropy for $\alpha = 2$ under $SO(1,1)$, (29). The solid line is our conjecture (39) expressed in terms of the pinching angle $\theta_F$. The initial value of the entropy is set at $\theta_F = 3\pi/4$ (filled dot). Observe that the finite size effects are relevant when $\theta_F$ approaches zero. In this case, all the roots of $P(z)$ are close to $z = 1$, the stable fixed point of (29).

**Ground states breaking parity symmetry**

The conjecture stated above was motivated by considering the critical theory as the limit of non critical ones. There are cases, however, in which this strategy is not possible. Consider a theory with a non parity invariant vacuum like the one plotted in Fig. b. It is evident that this dispersion relation can not be considered as the limit of a non-critical
theory. We shall therefore adopt a different strategy.

In the previous discussion we emphasized the close relation between the logarithmic scaling of the entropy and the scaling dimension of the partition function under Möbius transformations. This scaling is associated to discontinuities of the symbol due to the degeneration of branch points at the unit circle. The idea is to apply the same relation with full generality.

As it was discussed in [20] (it is also easy to see from (17)), the discontinuities in the symbol for critical theories with vacuum preserving parity symmetry correspond to a global change of sign. For cases in which the vacuum breaks parity invariance the discontinuities are of a different type. The symbol changes from $M$ to $-I$ at the Fermi point, $\theta_F$, or from $M$ to $I$ at its opposite value, $-\theta_F$.

The scaling behaviour of block Toeplitz determinants associated to matrix symbols with jump discontinuities, has been studied for the first time in [20]. In that paper we propose an expression for the leading logarithmic scaling of the determinant when the two lateral limits commute.

The result is that the new discontinuities (with lateral limits $M$ and $\pm I$) give a contribution to the coefficient of the logarithmic term of the entropy that is half of that associated to the pinchings at the unit circle (in which the two lateral limits differ by a sign). Then, using the connection between the contribution to the logarithmic term and the scaling dimension under admissible Möbius transformations discussed above, we propose that these new insertions have half the dimension of those considered in the previous section.

If we add to the pinchings $u_\kappa$ discussed previously the points of the unit circle where the new jump discontinuities (from $M$ to $\pm I$) take place and call these by $v_\sigma = \exp(i\theta_\sigma)$, $\sigma = 1, \ldots, Q$, the behaviour under admissible Möbius transformations of the partition function reads

$$Z'_{a,X}(u', v') = \prod_{\kappa=1}^{R} \left( \frac{\partial u'_\kappa}{\partial u_\kappa} \right)^{2\Delta_a} \prod_{\sigma=1}^{Q} \left( \frac{\partial v'_\sigma}{\partial v_\sigma} \right)^{\Delta_a} Z_{a,X}(u, v).$$

(41)

This expression covers the most general form for the behaviour under admissible Möbius transformations of the entanglement entropy of an interval in the ground state of a free, homogeneous fermionic chain. Its numeric accuracy is tested in Fig. [4].
FIG. 4: Numerical check of the transformation (41) for a critical theory with $L = 2$ and a Dirac sea depicted in the inset. We plot the Rényi entanglement entropy with $\alpha = 2$ for different lengths of $X$ as a function of the Fermi point $\theta_F$ under the $SO(1,1)$ group, (29). The solid lines represent our conjectured transformation (41) expressed in terms of $\theta_F$. The initial value of the entropy is set at $\theta_F = 3\pi/4$ and $\tilde{\theta}_F = \pi/2$ (filled dots). Observe that the finite size effects are relevant when $\theta_F$ approaches zero. In this case, all the roots of $P(z)$ are close to $z = 1$, the stable fixed point of (29).

We may generalize this discussion to excited states and to subsystems composed of several intervals. The latter will be discussed in a subsequent section. But before doing that we shall apply the transformations to the simplest fermionic chain with range $L = 1$, that via Jordan-Wigner transform is mapped into the XY spin chain.

V. APPLICATION TO THE XY-MODEL WITH A DZYALOSHINSKI-MORIYA COUPLING

In this section we shall apply the previous results to the XY spin chain with a Dzyaloshinski-Moriya (DM) coupling that breaks the parity symmetry of the model. In particular we will show how to compute the scaling behaviour of the entanglement entropy
for the critical Ising universality class.

As we mentioned before, a Jordan-Wigner transform maps the model into a free fermionic chain with nearest neighbours couplings. The coupling constants with the standard parametrization of the theory are $A_1 = A_{-1} = 1 + is$, $A_0 = -h$ and $B_1 = -B_{-1} = \gamma$; $h$ represents the transversal magnetic field, the DM coupling $s$ is a drift term that breaks spatial parity invariance and $\gamma$ is the spin space anisotropy parameter.

The dispersion relation is

$$\Lambda(\theta) = \sqrt{(h - 2 \cos \theta)^2 + 4\gamma^2 \sin^2 \theta + 2s \sin \theta}. \quad (42)$$

The theory is critical when

$$s^2 - \gamma^2 > 0, \quad \text{and} \quad (h/2)^2 - s^2 + \gamma^2 < 1; \quad \text{(Region A)},$$

or when

$$s^2 - \gamma^2 < 0, \quad \text{and} \quad h = 2; \quad \text{(Region B)}.$$

In Fig. [5] we show a plot of the critical regions. The ground state of the theories in Region A (for $s \neq 0$) breaks parity invariance. Region B corresponds to the critical Ising universality line and the ground state is invariant under parity.

**Parity preserving theories: $s = 0$**

In the first part of this section we shall consider the parity invariant Hamiltonian $s = 0$, i.e. the pure XY spin chain. Moreover, $\gamma = 0$ (no anisotropy) corresponds to the XX model and $\gamma = 1$ to the quantum Ising model.

An interesting feature of the critical XX model ($s = \gamma = 0, |h| < 2$) is that it is possible to compute analytically the full scaling behaviour of the critical theory, including the finite term. This is usually not known for a generic fermionic chain. Using well know results based on the Fisher-Hartwig conjecture for Toeplitz determinants we have $[21, 23, 24]$

$$S_{\alpha,X}(u_1, u_2) = \frac{\alpha + 1}{6\alpha} \log |X| + \frac{\alpha + 1}{12\alpha} \log |u_1 - u_2| + 2I_\alpha + ..., \quad (43)$$

where $u_1 = h/2 + i\sqrt{1 - (h/2)^2}$ and $u_2 = \bar{u}_1$ are the two points at the unit circle in which the roots of $P(z)$ degenerate, and $I_\alpha$ defined in Eq. [32] is the universal constant contribution.
Observe that the predicted transformation properties of the entropy under admissible Möbius transformations \[ [10] \] are fulfilled, i.e.

\[
S'_{\alpha,X}(u'_1, u'_2) = S_{\alpha,X}(u_1, u_2) + \frac{\alpha + 1}{12\alpha} \log \left( \frac{\partial u'_1}{\partial u_1} \frac{\partial u'_2}{\partial u_2} \right).
\]

Contrary to the case of the critical XX line, the scaling behaviour for the entanglement entropy in the critical Ising universality class \((h = 2)\) is only partially known. Until very short ago we were able to determine it only for the special point that corresponds to the Ising model \((\gamma = 1, h = 2)\). This was based on the relation between the correlations of the Ising model and those of the XY model without magnetic field after specializing it to their respective critical points \([25]\), as well as by employing field theory methods \([26]\).
The result is
\[
S_{\alpha,X}^{\gamma=1,h=2} = \frac{1}{2} S_{\alpha,2X}(i, -i) = \frac{\alpha + 1}{12\alpha} \log |X| + \frac{\alpha + 1}{6\alpha} \log 2 + I_\alpha + \ldots, \tag{44}
\]
where, for convenience, we have changed the notation to make explicit the value of the couplings of the theory. Recently an extension of this relation was proposed in [14]. It allows to determine the entanglement entropy for part of the critical line (|\gamma| \leq 1). Here we present an alternative derivation of the same result, based on the Möbius transformation studied in the previous section. An advantage of the new approach is that it can be applied to the whole critical line.

We first consider the action of the Möbius transformation [29] on the coupling constants h and \gamma [14], that is,
\[
\gamma' = \frac{\gamma}{h/2 \sinh 2\zeta + \cosh 2\zeta}, \quad h'/2 = \frac{h/2 \cosh 2\zeta + \sinh 2\zeta}{h/2 \sinh 2\zeta + \cosh 2\zeta}.
\]
Observe also that in the critical Ising model two real roots of \(P(z)\) degenerate at \(u = 1\), hence
\[
\frac{\partial u'}{\partial u} \bigg|_{u=1} = e^{-2\zeta}.
\]
If we finally take \(\gamma = \exp(-2\zeta)\) and apply (35), we have
\[
S_{\alpha,X}^{\gamma,h=2} = S_{\alpha,X}^{\gamma=1,h=2} + \frac{\alpha + 1}{12\alpha} \log \gamma. \tag{45}
\]
Alternatively, by using (44), we obtain the simple result
\[
S_{\alpha,X}^{\gamma,h=2} = \alpha + 1 \frac{1}{12\alpha} \log 4\gamma |X| + I_\alpha + \ldots \tag{46}
\]
for the entanglement entropy along the critical Ising line \(h = 2\) and large \(|X|\).

**Parity breaking theories: s \neq 0**

In the second part of this section we consider \(s \neq 0\). This implies the breaking of parity invariance in the Hamiltonian.

As already discussed, if the theory has a mass gap the vacuum is still parity invariant and the new coupling \(s\) does not affect the entanglement entropy. The same happens in the critical Ising universality line. However in Region A \((s^2 - \gamma^2 > 0, \ (h/2)^2 - s^2 + \gamma^2 < 1)\)
the vacuum breaks parity invariance and the entanglement entropy does depend on \( s \). The Fermi points \( \theta_j, j = 1, 2 \) at which the dispersion relation changes sign, satisfy

\[
\cos \theta_j = \frac{-h/2 \pm \sqrt{(s^2 - \gamma^2)(s^2 - \gamma^2 + 1 - (h/2)^2)}}{s^2 - \gamma^2 + 1},
\]

with \( \theta_j \in (-\pi, 0] \), for \( s > 0 \). We have 4 discontinuities for the symbol at the unit circle \( v_j = e^{i\theta_j} \) and \( v_{4-j} = e^{-i\theta_j}, j = 1, 2 \).

In this region, an analytic expression for the finite terms of the entropy is not known except for some particular cases. Those will be discussed below. However, following the general discussion of the previous section we can actually determine the behaviour of the finite terms under an admissible Möbius transformation. First, notice that the new coupling constant transforms like

\[
s' = \frac{s}{h/2 \sinh 2\zeta + \cosh 2\zeta}.
\]

Since the discontinuities of the symbol \( \theta_j \) do not correspond to degenerate roots of \( P(z) \), we have \( R = 0, Q = 4 \), and according to (41),

\[
S^{\gamma',s',h'}_{a,X} = S^{\gamma,s,h}_{a,X} + \frac{\alpha + 1}{24\alpha} \sum_{\sigma=1}^{4} \log \frac{\partial v'_\sigma}{\partial v_\sigma}.
\]

For the particular case \( \gamma = 0 \) we can indeed compute the full asymptotic expression for the entanglement entropy and check the conjecture. In this situation the symbol is either \( \pm I \) or \( \pm \sigma_z \). We can then reduce the problem to a scalar symbol where much more is known (see e. g. [24]). Adapting the results from [24] to our situation we obtain

\[
S_{a,X} = \frac{\alpha + 1}{6\alpha} \log |X| + \frac{\alpha + 1}{12\alpha} \log \left( \frac{s^2 - (h/2)^2 + 1}{s^2 + 1} \right) + 2I_a + \ldots .
\]

This makes sense for \( s^2 - (h/2)^2 + 1 > 0 \), i.e. when we are in the critical region A with \( \gamma = 0 \). The case \( h = 2 \) was already obtained in [20].

In order to verify the behaviour of (49) under Möbius transformations, we need to compute the product of the complex Jacobians at the insertions \( v_i \),

\[
\prod_{\sigma=1}^{4} \frac{\partial v'_\sigma}{\partial v_\sigma} = \left( \frac{s^2 + 1}{s^2 + (h/2 \sinh 2\zeta + \cosh 2\zeta)^2} \right)^2
\]

\[
= \left( \frac{s^2 + 1}{s^2 + 1} \cdot \frac{s^2 - (h'/2)^2 + 1}{s^2 - (h/2)^2 + 1} \right)^2.
\]

From the second line one notice that (49) transforms according to (48).
VI. SEVERAL DISJOINT INTERVALS AND THE RELATION WITH CONFORMAL INVARIANCE

A different symmetry of the critical models is the conformal invariance. This symmetry has been extensively studied in the context of 1+1 dimensional quantum field theory since it was recognized in Ref. [27] as a powerful tool to determine the correlation functions of non trivial massless theories. These techniques have been applied in the context of entanglement entropy by different authors [1, 3, 28].

In order to compare the behaviour of the entropy under Möbius and conformal transformations, it is convenient to extend the results of section IV to more general subsystems. As such we are going to consider here the case in which $X$ is composed of $P$ disjoint intervals, i.e.

$$X = \bigcup_{i=1}^{P} [x_{2i-1}, x_{2i}].$$ (51)

We also introduce a new notation for the partition function, i.e.

$$Z_{\alpha}(u, \nu; \xi) = \text{Tr}(\rho_{X}^{\alpha}),$$ (52)

where as before $\rho_{X}$ is the reduced density matrix of the subsystem $X$ of the ground state of a critical theory, $u = (u_1, \ldots, u_R)$ are the positions of the pinchings at the unit circle and $\nu = (v_1, \ldots, v_Q)$ with $v_{\sigma} = \exp(\theta_{\sigma})$ are the points in the complex plane associated to the Fermi points or their opposite.

This case can not be analyzed with the tools used in the previous sections, because the correlation matrix $V_{X}$ is no longer of the block Toeplitz type. Rather, it is a principal submatrix of a block-Toeplitz matrix.

The correlation matrices associated to multi-intervals were studied in [21]. From the results stated there, one derives in the asymptotic limit that

$$Z_{\alpha}(u, \nu; \xi) = \prod_{1 \leq \tau < \tau' \leq 2P} Z_{\alpha}(u, \nu; x_{\tau}, x_{\tau'})^{-\sigma_{\tau} \sigma_{\tau}'},$$

where $\sigma_{\tau} = (-1)^{\tau}$. Observe that the partition functions in the product of the right hand side correspond, all of them, to those of a single interval $[x_{\tau}, x_{\tau'}]$ for different values of $\tau$ and $\tau'$. Therefore, the results of section IV apply. Using (41) and taking into account that

$$\sum_{1 \leq \tau < \tau' \leq 2P} \sigma_{\tau} \sigma_{\tau'} = -P,$$
as can be easily deduced, one finally obtains
\[
Z'_\alpha(u', v'; x') = \prod_{\kappa=1}^{R} \left( \frac{\partial u'_\kappa}{\partial u_\kappa} \right)^{2P\Delta_\alpha} \prod_{\sigma=1}^{Q} \left( \frac{\partial v'_\sigma}{\partial v_\sigma} \right)^{P\Delta_\alpha} Z_\alpha(u, v; x) \tag{53}
\]
in the large $|x_\tau - x_{\tau'}|$ limit.

On the other hand, a global conformal transformation in space is given by
\[
x' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}),
\]
with $x, x' \in \mathbb{R}$. Under this transformation the partition function scales as
\[
Z_\alpha(u, v; x) \mapsto \left( a x + b \right)^{2P\Delta_\alpha} \left( c x + d \right)^{2C\Delta_\alpha} Z_\alpha(u, v; x'),
\]
where the central charge $C$ depends on the number of pinchings $C = \frac{R}{2} + \frac{Q}{4}$.

It is manifest the striking similarity of this expression and (53) with the rôle of the endpoints of the intervals $x_\tau$ replaced by the pinchings $u_\kappa, v_\sigma$.

Actually, it is possible to obtain a unified expression if one consider simultaneously Möbius and conformal transformations. Consider the map induced by an element of the direct product $SO(1, 1) \times SL(2, \mathbb{R})$ on the space of couplings, pinchings and endpoints
\[
(A, B, u, v, x) \mapsto (A', B', u', v', x')
\]
where the element in the first group factor acts as a Möbius transformation in $A, B, u$ and $v$ while the conformal transformations in the second factor act on $x$. The complex Jacobian determinant in $(u_\kappa, x_\tau)$ or $(v_\sigma, x_\tau)$ is given respectively by
\[
J_{\kappa\tau} = \frac{\partial u'_\kappa}{\partial u_\kappa} \frac{\partial x'_\tau}{\partial x_\tau},
\]
and
\[
K_{\sigma\tau} = \frac{\partial v'_\sigma}{\partial v_\sigma} \frac{\partial x'_\tau}{\partial x_\tau},
\]
because the transformation of $u$ and $v$ do not depend on $x$ and viceversa.

Therefore, under a general transformation in $SO(1, 1) \times SL(2, \mathbb{R})$ eqs. (53) and (54) can be combined to give the following transformation law
\[
Z'_\alpha(u', v'; x') = \prod_{\kappa, \tau} J_{\kappa\tau}^{\Delta_\alpha} \prod_{\sigma, \tau} K_{\sigma\tau}^{\Delta_\alpha/2} Z_\alpha(u, v; x). \tag{55}
\]
Note that this expression can be interpreted as the transformation of the expectation value, in a covariant theory in $S^1 \times \mathbb{R}$ of homogeneous fields of dimension $\Delta_\alpha$ at insertion points $(u_\kappa, x_\tau)$ and of dimension $\Delta_\alpha/2$ at $(v_\sigma, x_\tau)$. A particular example with $R = 2$, $Q = 0$, $P = 1$ is depicted in Fig. 6.

![Diagram](fig6.png)

**FIG. 6:** The partition function $Z_\alpha$ of an interval $X = [x_1, x_2]$ and pinchings at $u_1$, and $u_2$ behaves under $SO(1, 1) \times SL(2, \mathbb{R})$ as the expectation value of a product of homogeneous fields inserted at the points $(u_1, x_1), (u_1, x_2), (u_2, x_1)$ and $(u_2, x_2)$.

The natural question, whose answer we do not know yet, is if there are more general transformations and/or more general configurations for the insertion points that lead to an expression similar to (55).

**VII. CONCLUSIONS**

In this paper we have extended the results of Ref. [14] on the invariance of the entanglement entropy under Möbius transformations. In that work we restricted ourselves to non-critical quadratic spinless fermionic chains with parity and charge conjugation symmetries. In addition, we only considered subsystems formed by a single interval of contiguous sites. Here we have improved these results in several ways.

First, we present a more general and simple proof of the above mentioned invariance.
This allowed us to extend the domain of application of the previous results to include parity broken and/or charge symmetry broken Hamiltonians.

We then moved to the case of critical theories discussing how Möbius transformations act on the corresponding partition function. Critical theories are characterized by the existence of jump discontinuities in the symbol, that is, the Fourier transformation of the 2-point correlation function. We showed that in this case the partition function transforms like the product of homogeneous fields inserted at those discontinuities. The dimensions of these homogeneous fields depend on whether the discontinuity is associated to the existence or not of a Dirac sea. If there is a Dirac sea, then the ground state breaks parity symmetry.

One striking aspect of this behaviour of the partition function is its parallelism with the conformal transformations of primary fields in CFT. Notice though that the latter act on space-time itself. Actually, under conformal transformation the partition function behaves also as the product of homogeneous fields but now inserted at the boundaries of the subsystem. This relation is still more evident if we consider subsystems composed of several intervals. In this case, under Möbius transformations, the insertions take place at the discontinuities of the symbol. Their dimensions depend on the number of end points of the intervals that form the subsystem. Conversely, under conformal transformations, the homogeneous fields are inserted at the end points of the intervals. Their dimensions depend on the number of discontinuities in the symbol.

This close similarity suggests that it is possible to consider both transformations together. This amounts to the introduction of a larger group that includes Möbius and conformal transformations. This larger group acts on a phase space obtained from the cartesian product of the space of discontinuities of the symbol (momentum space) and of the end points of the intervals (real space). In this scenario the partition function transforms like the product of homogeneous fields inserted at the points consisting of combinations of discontinuities and end points.

The previous picture is very attractive. It immediately suggests a generalization of our results to larger groups containing the direct product of Möbius and conformal. Even more general configurations in phase space can be envisaged.

Whether this project can be carried out, its meaning and further applications remains an open question.
Acknowledgments: Research partially supported by grants 2016-E24/2, DGIID-DGA and FPA2015-65745-P, MINECO (Spain). FA is supported by FPI Grant No. C070/2014, DGIID-DGA/European Social Fund, and thanks the hospitality and warm atmosphere of Instituto de Física, Universidade de Brasília where some of this research was carried out. ARQ is supported by CNPQ under process number 307124/2016-9 and thanks Departamento de Física Teórica, Universidad de Zaragoza for the hospitality and nice atmosphere where part of this work was conducted. FA and ARQ gratefully acknowledge support from the Simons Center for Geometry and Physics, Stony Brook University at which some of the research for this paper was performed during the Program ”Entanglement and Dynamical Systems”. In particular, we thank Prof. V. Korepin for discussions during our stay in SCGP.

Appendix A

For completeness we shall review in this appendix the computation of the entanglement entropy for a general parity and charge conjugation preserving free fermionic Hamiltonian. We follow [12, 14, 18].

It will be convenient to introduce the characteristic polynomial of $V_X$, $D_X(\lambda) = \det(\lambda-V_X)$ in terms of which the entropy reads

\[
S_{\alpha,X} = \lim_{\delta \to 1} \frac{1}{4\pi i} \oint_C f_\alpha(\lambda/\delta) \frac{d \log D_X(\lambda)}{d \lambda} d\lambda. \tag{A1}
\]

where $f_\alpha(x) = \log F_\alpha(x)/(1-\alpha)$ (see [15]) and $C$ is the contour depicted in Fig. 7 surrounding the eigenvalues $v_l$ of $V_X$, all of them lying in the real interval $[-1,1]$.

![Fig. 7: Contour of integration, cuts and poles for the computation of $S_{\alpha,X}$. The cuts for the function $f_\alpha$ extend to $\pm \infty$.](image)
So far we have only transformed the problem of computing the entropy into that of the characteristic polynomial of $V_X$. But, it happens that an explicit expression for $D_X(\lambda)$ is available if charge conjugation is preserved. In this case $A_l, B_l \in \mathbb{R}$ and the symbol reads

$$M(z) = U \begin{pmatrix} 0 & g(z) \\ g(z)^{-1} & 0 \end{pmatrix} U^{-1},$$  \hspace{1cm} (A2)

with

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

and

$$g(z) = \sqrt{\frac{\Theta(z) + \Xi(z)}{\Theta(z) - \Xi(z)}}.$$  \hspace{1cm} (A3)

The roots of $P(z) = z^{2L}(\Theta(z) + \Xi(z))(\Theta(z) - \Xi(z))$ denoted by $z_j$ are either zeros or poles of the rational function $g^2(z)$. According to this we assign an index $\epsilon_j$ to every root, which is +1 if $z_j$ is a zero and −1 if it is a pole of $g^2(z)$.

Assume for the moment that all the roots of $P(z)$, denoted by $z_1, z_2, \ldots, z_{4L}$, are simple. The curve $w^2 = P(z)$ is a double covering of the Riemann sphere $\overline{\mathbb{C}}$ determining a genus $g = 2L - 1$ Riemann surface. Notice that we have $g^2(z) = g^2(\bar{z}) = 1/g^2(z^{-1})$. Hence the roots are related by inversion and conjugation, so that half of them are inside the unit circle and the other half outside. We order the roots such that $z_1, \ldots, z_{2L}$ are inside the unit circle and the rest lie outside. The cuts $\Sigma_\rho, \rho = 0, \ldots, g$, in the complex plane are chosen to join $z_{2\rho+1}$ and $z_{2\rho+2}$ so that they do not intersect the unit circle. Adapted to our choice of cuts, we consider as a basis of the homology the cycles $a_r, b_r$ ($r = 1, \ldots, g$) so that $a_r$ encloses the cut $\Sigma_r$ anticlockwise and the dual cycle $b_r$ surrounds the branch points $z_2, z_3, \ldots, z_{2r+1}$ clockwise. The canonical basis of holomorphic forms

$$d\omega_r = \frac{\varphi_r(z)}{\sqrt{P(z)}} \, dz,$$  \hspace{1cm} (A4)

with $\varphi_r(z)$ a polynomial of degree smaller than $g$. It is chosen so that $\oint_{a_r} d\omega_r' = \delta_{rr'}$. The $g \times g$ symmetric matrix of periods $\Pi = (\Pi_{rr'})$ is defined by

$$\Pi_{rr'} = \oint_{b_r} d\omega_{r'}.$$  \hspace{1cm} (A5)

We introduce the $\vartheta$ function with characteristics $\vartheta \left[ \begin{pmatrix} \bar{p} \\ \bar{q} \end{pmatrix} \right] : \mathbb{C}^g \rightarrow \mathbb{C}$ associated to $\Pi$ as

$$\vartheta \left[ \begin{pmatrix} \bar{p} \\ \bar{q} \end{pmatrix} \right] (s|\Pi) = \sum_{\bar{n} \in \mathbb{Z}^g} e^{\pi i (\bar{n} + \bar{\rho}) \Pi \cdot (\bar{n} + \bar{\rho}) + 2\pi i (s + \bar{q}) \cdot (\bar{n} + \bar{\rho})}, \quad \bar{p}, \bar{q} \in \mathbb{R}^g.$$  \hspace{1cm} (A6)
The normalized version of the $\vartheta$ function is

$$\hat{\vartheta}\left[\vec{p} \mid \vec{q}\right](\vec{s}) = \hat{\vartheta}\left[\vec{p} \mid \vec{q}\right](\vec{s}|\Pi) = \vartheta\left[\vec{p} \mid \vec{q}\right](\vec{s}|\Pi).$$  \hspace{1cm} (A7)

For large $|X|$ expansion, one may find that

$$\log D_X(\lambda) = |X| \log(\lambda^2 - 1) + \log \left(\hat{\vartheta}\left[\vec{p} \mid \vec{q}\right](\beta(\lambda)\vec{e}) \cdot \hat{\vartheta}\left[\vec{p} \mid \vec{q}\right](-\beta(\lambda)\vec{e})\right) + \ldots, \hspace{1cm} (A8)$$

where

$$\beta(\lambda) = \frac{1}{2\pi i} \log \frac{\lambda + 1}{\lambda - 1}$$

and $\vec{e} \in \mathbb{Z}^g$ with its first $L - 1$ entries equal to 0 and the last $L$ equal to 1. The characteristics $\vec{\mu}, \vec{\nu} \in (\mathbb{Z}/2)g$ depend on the indices assigned to the branch points and are defined by

$$\mu_r = \frac{1}{4}(\epsilon_{2r+1} + \epsilon_{2r+2}); \quad \nu_r = \frac{1}{4} \sum_{j=2}^{2r+1} \epsilon_j; \quad r = 1, \ldots, g.$$ 

Recall that $\epsilon_j = 1$ if $z_j$ is a root of $g^2(z)$ (see (A3)) or $-1$ if it is a pole.

Of course, the expression for $D_X(\lambda)$ cannot depend on the particular choice of cuts, provided none of them crosses the unit circle. This has been explicitly shown in [14]. We shall make use of this fact further below.

It is also interesting to remark that the expression in (A8) is explicitly invariant under Möbius transformations. Moreover, a Möbius transformation move the branch points and cuts. Thus the holomorphic forms (A4) change, but not the matrix of periods (A5). Therefore, the $\vartheta$ functions do not change and then the entropy depending on them remains equal.

This property implies that $D_X(\lambda)$ only depends on the set of independent cross ratios that can be obtained with the branch points of the hyperelliptic curve. In addition, this symmetry is very useful to find dualities and other relations between different Hamiltonians in terms of the entanglement entropy. This has been studied in detail in Ref. [14].

**Appendix B**

In this and the next Appendixes we shall study the coalescence of pairs of roots of $P(z)$ degenerating into a single double root. This produces a pinching of the associated
complex curve $w^2 = P(z)$.

We will treat two separate cases. In one of them the roots degenerate at a point at the unit circle. In the other one they degenerate outside of the unit circle. In the first case the entropy diverges logarithmically while it has a finite limit in the second. In this Appendix we will study the latter case. The first one will be treated in the next Appendix.

The analysis of a pinching of the hyperelliptic curve $w^2 = P(z)$ requires the choice of a homology basis $(a, b)$. This will be conveniently chosen such that each pair of branch points being degenerated is surrounded by one of the $a$-cycles. An example of such basis is given in the Appendix A. In this basis, it becomes easy to deal with the divergences appearing in the matrix of periods. So, fix an order for the roots, compatible with the requirements of Appendix A. Let us focus on the situation when the roots $z_j$ and $z_{j+1}$, with $j = 2\hat{r} + 1$, approach each other. In this case, the cycle $a_{\hat{r}}$ is enclosing them. Notice that this cannot be done if the two roots were in opposite sides of the unit circle.

If we compute the period matrix we immediately see that the only entry that diverges when $z_j \to z_{j+1}$ is $\Pi_{\hat{r}\hat{r}} \sim -i/\pi \log |z_j - z_{j+1}|$. The remaining terms of this matrix, i.e. $\Pi_{lm}, l, m \neq \hat{r}$, are finite in this limit. Moreover, they define a new $g - 1 \times g - 1$ period matrix denoted by $\Pi_\circ$. This new period matrix is associated to the Riemann surface resulting from the removal of the two merging branch points. The entries $\Pi_{l\hat{r}}, l \neq \hat{r}$ define a $g - 1$ dimensional vector denoted $\vec{\Delta}_\circ$.

There are two different cases for the resulting theta function after the limit. The simplest one is when the two degenerating points are of different character. This means that one is a zero and the other is a pole of $g^2(z)$. As a result $\mu_{\hat{r}} = 0$. Due to the divergent behaviour of $\Pi_{\hat{r}\hat{r}}$, the only surviving terms in the sum of (A6) are those with $n_{\hat{r}} = 0$. Hence, we have

$$\lim_{z_j \to z_{j+1}} \vartheta \left[ \vec{\mu} \right] (s | \Pi) = \vartheta \left[ \vec{\mu}_\circ \right] (s_\circ | \Pi_\circ),$$

(B1)

where $\vec{\mu}_\circ, \vec{\nu}_\circ$ and $s_\circ$ stand for the $g - 1$ dimensional vectors obtained from $\vec{\mu}, \vec{\nu}$ and $s$, respectively, by removing the $\hat{r}$ entry. Therefore, after the limit, we get a theta function associated to the Riemann surface of genus $g - 1$ obtained after the removal of the two colliding branch points.

Before we continue with the other case, we may discuss the applications of this to the entanglement entropy obtained from (A8). We first recall that the branch points are
related by inversion and conjugation. It means that if a pair of real roots merge there will be another merging pair related by inversion with the former. In the complex case there will be three other coalescing pairs, related by inversion and conjugation. In both cases, all pairs are composed of roots of different character. Therefore, by successive application of the coalescence limit we obtain a \( \vartheta \)-function of genus either \( g - 2 \) for real roots or \( g - 4 \) for complex ones.

The result of the previous analysis is that the entanglement entropy corresponds to that of a theory with range of couplings either \( L - 1 \) for the real case or \( L - 2 \) for the complex one. It is interesting to observe that this equivalence applies for the ground state entanglement entropy but not for the dynamics: the respective Hamiltonians do not coincide.

Returning to the general discussion, the second case corresponds to two merging roots that have the same character. Therefore, \( \mu_{\tilde{r}} = \pm 1/2 \). In the coalescence limit, \( \vartheta[\vec{\mu}/\vec{\nu}] \) vanishes. However, we are interested in the normalized theta function which does not vanish in this limit. We can compute its limit from

\[
\lim_{\bar{z}_j \to \bar{z}_{j+1}} \vartheta[\vec{\mu}/\vec{\nu}] (\vec{s}/\Pi) e^{-\pi \Pi_{\tilde{r}} i/4} = e^{2\pi i (s_{\tilde{r}} + u_{\tilde{r}}) \mu_{\tilde{r}} \vartheta[\vec{\mu}/\vec{\nu}]} (\vec{s} + \mu_{\tilde{r}} \Delta_{s}/\Pi_{\tilde{r}})
\]

\[
+ e^{-2\pi i (s_{\tilde{r}} + u_{\tilde{r}}) \mu_{\tilde{r}} \vartheta[\vec{\mu}/\vec{\nu}]} (\vec{s} - \mu_{\tilde{r}} \Delta_{s}/\Pi_{\tilde{r}}),
\]

obtained from the non vanishing terms in (A6), i.e. those with \( n_{\tilde{r}} = 0, -2 \mu_{\tilde{r}} \).

When applying this scenario to the entanglement entropy we observe that either two pairs of branch points, for the real case, or four for the complex one, should merge simultaneously. The resulting theta functions, after the merging, correspond to Riemann surfaces of genus either \( g - 2 \), for the real case, or \( g - 4 \). In contrast to the previous case of different character roots, in this limit the entanglement entropy is no longer equal to that of a theory with a smaller range of couplings.

**Appendix C**

Equipped with the previous results we can now study the behaviour of the entropy for critical theories with parity symmetric vacuum, i.e. those in which two branch points come together at the unit circle. This corresponds to the pinching of a nontrivial cycle of the Riemann surface determined by the symbol of the correlation matrix.
Here we are interested in the case of degenerating points of different type, i.e., one is a zero of \( g^2(z) \) and the other is a pole. In this case, as it is discussed in Appendix B, the limit of the \( \vartheta \) function is easily computed and coincides with that of a theory where the merging branch points are removed.

There is a problem, however, because the limiting procedure in the previous Appendix was carried out assuming that one of the basic \( a \) cycles of the homology encircles the merging points. However, the expression (A8) for the characteristic polynomial of \( V_X \) is valid when no \( a \) cycle intersects the unit circle. In this case, in which the branch points degenerate at the unit circle, the two prescriptions are not compatible.

The way to overcome this difficulty is by performing a modular transformation to a new basis \((a', b')\) where some \( a' \)-cycles cross the unit circle and enclose the pairs of degenerating branch points. We shall initially order the roots so that \( z_{2L} \) degenerates with \( z_{2L+2} = \overline{z}_{2L}^{-1} \) at the unit circle. If the previous roots are real we do not impose other conditions to the ordering. If they are not real we shall take \( z_{2L-1} = \overline{z}_{2L} \) and \( z_{2L+1} = \overline{z}_{2L+2} \) that also degenerate at the unit circle. In this ordering, where the first roots are inside and the last ones outside, a simple transposition of \( z_{2L} \) and \( z_{2L+1} \) induces the desired transformation of the basic cycles, see Fig. 8.

FIG. 8: Representation of the change of homology basis used to extract the divergent behaviour under a pinching. In the left panel the usual homology basis described in Appendix A, in which the \( a \) cycles do not cross the unit circle, is depicted. In the right panel we represent the alternative homology basis in which \( a' \) cycles enclose pairs of approaching branch points. This modular transformation is equivalent to a permutation in the labelling of the branch points: \( z_{2L} = z_{2L+1}' \) and \( z_{2L+1} = z_{2L}' \).

By recalling the results of [14] we obtain the following relation between the original
\( \vartheta \)-function and the new one,

\[
\hat{\vartheta} \left[ \vec{\nu} \right] (\beta(\lambda) \vec{e} | \Pi) = e^{\pi i \beta(\lambda)^2 (\Pi'_{L-1,L-1} + \Pi'_{L,L} - 2 \Pi'_{L,L-1}^{-1})} \hat{\vartheta} \left[ \vec{\mu} \right] (\beta(\lambda) \vec{e} | \Pi'),
\]

where \( \Pi' \) is the period matrix for the new basic cycles, \( e'_r = e_r - (\Pi'_{L,L} - \Pi'_{L-1,L}) \) and \( \vec{\mu}' \) differs from \( \vec{\mu} \) only in the \( L-1 \) and \( L \) entries:

\[
\mu'_{L-1} = \mu_{L-1} + \nu_L - \nu_{L-1} + 1/2, \quad \mu'_L = \mu_L - \nu_L + \nu_{L-1} + 1/2.
\]

We emphasize again that the advantage of using this basis of cycles is that the divergences of \( \Pi' \) are very simple to analyze. In fact, for the real pinching (when \( z_{2L} \) and \( z_{2L+2} \) degenerate), only \( \Pi'_{L,L} \) diverges, so that

\[
\Pi'_{L,L} \sim -i/\pi \log |z_{2L+2} - z_{2L}|,
\]

while the rest of the entries of \( \Pi' \) have a finite limit. In the complex pinching (when also \( z_{2L-1} = \bar{z}_{2L} \) and \( z_{2L+1} = \bar{z}_{2L+2} \) degenerate), both \( \Pi'_{L-1,L-1} \) and \( \Pi'_{L,L} \) have a divergent behaviour,

\[
\Pi'_{L-1,L-1} + \Pi'_{L,L} \sim -2i/\pi \log |z_{2L+2} - z_{2L}|.
\]

Moreover, we should have \( \epsilon_{2L} = -\epsilon_{2L+2} \) implying \( \mu'_L = 0 \). In the complex case we have also \( \epsilon_{2L-1} = -\epsilon_{2L+1} \), so that \( \mu'_{L-1} = \mu'_L = 0 \). These observations are crucial to obtain the following finite limit when the roots degenerate,

\[
\hat{\vartheta} \left[ \vec{\nu} \right] (\beta(\lambda) \vec{e} | \Pi) \to \hat{\vartheta} \left[ \vec{\mu}_o \right] (\beta(\lambda) \vec{e}_o | \Pi'_o).
\]

Here on the right hand side we have the theta function for a genus \( g - 1 \) Riemann surface in the real pinching or \( g - 2 \) in the complex one. The period matrix \( \Pi'_o \) is obtained in the real case by removing from \( \Pi' \) the \( L \) row and column. For the complex case, the removal are though the \( L-1 \) and \( L \) rows and columns. Likewise, \( \vec{\mu}_o, \vec{\nu}_o \) and \( \vec{e}_o \) stand for the vectors resulting after the removal of the \( L \) component if \( z_{2L} \in \mathbb{R} \) or both the \( L-1 \) and \( L \) ones if it is complex.

Taking into account the logarithmic divergence of \( \Pi'_{L,L} \) and eventually that of \( \Pi'_{L-1,L-1} \) we finally obtain the following asymptotic behaviour

\[
\log \hat{\vartheta} \left[ \vec{\mu} \right] (\beta(\lambda) \vec{e} | \Pi) = -\frac{c}{2\pi^2} \left( \log \frac{\lambda + 1}{\lambda - 1} \right)^2 \log |z_{2L} - z_{2L+2}| + \ldots,
\]
where \( c = 1/2 \) for the real pinching, \( c = 1 \) for the complex one and the dots stand for contributions which are finite in the limit \( z_{2L} \to z_{2L+2} \).

Finally, we plug \((C3)\) into \((A8)\) and use \((A1)\) to obtain

\[
S_\alpha \sim -c \frac{\alpha + 1}{6\alpha} \log |z_{2L} - z_{2L+2}|. \tag{C4}
\]

The above reasoning can be straightforwardly extended to situations where different couples of branch points degenerate at the unit circle. For instance, \( z_{j\kappa}, z_{j\kappa}^{-1} \to u_\kappa \) with \( u_\kappa = \exp(i\theta_\kappa) \), and \( u_\kappa \neq u_\xi, \kappa \neq \xi \). In this case,

\[
S_\alpha = -\frac{\alpha + 1}{12\alpha} \sum_{\kappa=1}^{R} \log |z_{j\kappa} - z_{j\kappa}^{-1}| + K_\alpha(u) + \ldots \tag{C5}
\]

where the dots stand for contributions that vanish in the limit \( z_{j\kappa} \to u_\kappa, \kappa = 1, \ldots, R \) and \( u = (u_1, \ldots, u_R) \). This is precisely the expression \((36)\).

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