Subgroup decomposition in $\text{Out}(F_n)$
Part III: Weak Attraction Theory

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Abstract

This is the third in a series of four papers, announced in [HM13a], that develop a decomposition theory for subgroups of $\text{Out}(F_n)$.

In this paper, given $\phi \in \text{Out}(F_n)$ and an attracting-repelling lamination pair for $\phi$, we study which lines and conjugacy classes in $F_n$ are weakly attracted to that lamination pair under forward and backward iteration of $\phi$ respectively. For conjugacy classes, we prove Theorem F from the research announcement, which exhibits a unique vertex group system called the “nonattracting subgroup system” having the property that the conjugacy classes it carries are characterized as those which are not weakly attracted to the attracting lamination under forward iteration, and also as those which are not weakly attracted to the repelling lamination under backward iteration. For lines in general, we prove Theorem G which characterizes exactly which lines are weakly attracted to the attracting lamination under forward iteration and which to the repelling lamination under backward iteration. We also prove Theorem H which gives a uniform version of weak attraction of lines.

Introduction

Many results about the groups $\text{MCG}(S)$ and $\text{Out}(F_n)$ are based on dynamical systems. The Tits alternative ([BFH00], [BFH05] for $\text{Out}(F_n)$; [McC85] and [Iva92] independently for $\text{MCG}(S)$) says that for any subgroup $H$, either $H$ is virtually abelian or $H$ contains a free subgroup of rank $\geq 2$, and these free subgroups are constructed by analogues of the classical “ping-pong argument” for group actions on topological spaces. Dynamical ping-pong arguments were also important in Ivanov’s classification of subgroups of $\text{MCG}(S)$ [Iva92]. And they will be important in Part IV [HM13d] where we prove our main theorem about subgroups of $\text{Out}(F_n)$, Theorem C stated in the Introduction [HM13a].

Ping-pong arguments are themselves based on understanding the dynamics of an individual group element $\phi$, particularly an analysis of attracting and repelling fixed sets of $\phi$, of their associated basins of attraction and repulsion, and of neutral sets which are neither attracted nor repelled. The proofs in [McC85, Iva92] use the action of $\text{MCG}(S)$ on Thurston’s space $\text{PM}(S)$ of projective measured laminations on $S$.

The proof of Theorem C in Part IV [HM13d] will employ ping-pong arguments for the action of $\text{Out}(F_n)$ on the space of lines $B = B(F_n)$, which is just the quotient of the action of $F_n$ on the space $\tilde{B}$ of two point subsets of $\partial F_n$. The basis of those ping-pong arguments
will be \textit{Weak Attraction Theory} which, given \( \phi \in \text{Out}(F_n) \) and a dual lamination pair \( \Lambda^\pm \in \mathcal{L}^\pm(\phi) \), addresses the following dynamical question regarding the action of \( \phi \) on \( \mathcal{B} \):

**General weak attraction question:** Which lines \( \ell \in \mathcal{B} \) are weakly attracted to \( \Lambda_+ \) under iteration of \( \phi \)? Which are weakly attracted to \( \Lambda_- \) under iteration of \( \phi^{-1} \)? And which are weakly attracted to neither \( \Lambda_+ \) nor \( \Lambda_- \)?

To say that \( \ell \) is weakly attracted to \( \Lambda_+ \) (under iteration of \( \phi \)) means that \( \ell \) is weakly attracted to a generic leaf \( \lambda \in \Lambda_+ \), that is, the sequence \( \phi^k(\ell) \) converges in the weak topology to \( \ell \) as \( k \to +\infty \). Note that this is independent of the choice of generic leaf of \( \Lambda_+ \), since all of them have the same weak closure, namely \( \Lambda_+ \).

Our answers to the above question are an adaptation and generalization of many of the ideas and constructions found in The Weak Attraction Theorem 6.0.1 of [BFH00], which answered a narrower version of the question above, obtained by restricting to a lamination \( \Lambda_+ \) which is topmost in \( \mathcal{L}(\phi) \) and to birecurrent lines. The answer was expressed in terms of the structure of an “improved relative train track representative” of \( \phi \).

In this paper we develop weak attraction theory to completely answer the general weak attraction question. Our theorems are expressed both in terms of the structure of a CT representative of \( \phi \), and in more invariant terms. The theory is summarized in Theorems F, G and H, versions of which were stated earlier in [HM13a]; the versions stated here are more expansive and precise. Theorem F focusses on periodic lines and on the nonattracting subgroup system; Theorems G and H are concerned with arbitrary lines. Each of these theorems has applications in Part IV [HM13d].

**The nonattracting subgroup system: Theorem F.** We first answer the general weak attraction question restricted to “periodic” lines in \( \mathcal{B} \), equivalently circuits in marked graphs, equivalently conjugacy classes in \( F_n \). The statement uses two concepts from Part I [HM13b]: geometricity of general EG strata (which was in turn based on geometricity of top EG strata as developed in [BFH00]), and vertex group systems.

**Theorem F** (Properties of the nonattracting subgroup system). For each rotationless \( \phi \in \text{Out}(F_n) \) and each \( \Lambda^\pm \in \mathcal{L}^\pm(\phi) \) there exists a subgroup system \( \mathcal{A}_{na}(\Lambda^\pm) \), called the nonattracting subgroup system, with the following properties:

1. (Proposition 1.4 (1)) \( \mathcal{A}_{na}(\Lambda^\pm) \) is a vertex group system.

2. (Proposition 1.4 (2)) \( \Lambda^\pm \) is geometric if and only if \( \mathcal{A}_{na}(\Lambda^\pm) \) is not a free factor system.

3. (Corollary 1.7) For each conjugacy class \( c \) in \( F_n \) the following are equivalent:
   - \( c \) is not weakly attracted to \( \Lambda_+^\phi \) under iteration of \( \phi \);
   - \( c \) is carried by \( \mathcal{A}_{na}(\Lambda^\pm) \).

4. (Corollary 1.9) \( \mathcal{A}_{na}(\Lambda^\pm) \) is uniquely determined by items (1) and (3).

5. (Corollary 1.10) For each conjugacy class \( c \) in \( F_n \), \( c \) is not weakly attracted to \( \Lambda^+ \) by iteration of \( \phi \) if and only if \( c \) is not weakly attracted to \( \Lambda^- \) by iteration of \( \phi^{-1} \).
Furthermore (Definition 1.2), choosing any CT $f: G \to G$ representing $\phi$ with EG-stratum $H_r$ corresponding to $\Lambda^\pm$, the nonattracting subgroup system $A_{na}(\Lambda^\pm)$ has a concrete description in terms of $f$ and the indivisible Nielsen paths of height $r$ (the latter are described in ([FH11] Corollary 4.19) or Fact I.1.40). The description given in Definition 1.2 is our first description in terms of $f$ independent of the choice of CT (item (4) above). Corollary 1.10 (item (5) above) shows moreover that $A_{na}(\Lambda^\pm)$ is indeed well-defined independent of the choice of nonzero power of $\phi$, depending only on the cyclic subgroup $\langle \phi \rangle$ and the lamination pair $\Lambda^\pm$.

**Notation:** The nonattracting subgroup system $A_{na}(\Lambda^\pm)$ depends not only on the lamination pair $\Lambda^\pm$ but also on the outer automorphism $\phi$ (up to nonzero powers). Often we emphasize this dependence by building $\phi$ into the notation for the lamination itself, writing $\Lambda^\pm_{\phi}$ and $A_{na}(\Lambda^\pm_{\phi})$.

**The set of nonattracted lines: Theorems G and H.** Theorem G, a vague statement of which was given in the introduction, is a detailed description of the set $B_{na}(\Lambda^+_{\phi}; \phi)$ of all lines $\gamma \in B$ that are not attracted to $\Lambda^+_{\phi}$ under iteration by $\phi$. Theorem H is a less technical and more easily applied distillation of Theorem G, and is applied several times in Part IV [HM13d].

As stated in Lemma 2.1, there are three somewhat obvious subsets of $B_{na}(\Lambda^+_{\phi}; \phi)$. One is the subset $B(A_{na}(\Lambda^+_{\phi}))$ of all lines supported by the nonattracting subgroup system $A_{na}(\Lambda^+_{\phi})$. Another is the subset $B_{gen}(\phi^{-1})$ of all generic leaves of attracting laminations for $\phi^{-1}$. The third is the subset $B_{sing}(\phi^{-1})$ of all singular lines for $\phi^{-1}$: by definition these lines are the images under the quotient map $\tilde{B} \mapsto B$ of those endpoint pairs $\{\xi, \eta\} \in \tilde{B}$ such that $\xi, \eta$ are each nonrepelling fixed points for the action of some automorphism representing $\phi^{-1}$.

In Definition 2.2 we shall define an operation of “ideal concatenation” of lines: given a pair of lines which are asymptotic in one direction, they define a third line by concatenating at their common ideal point and straightening, or what is the same thing by connecting their opposite ideal points by a unique line.

Theorem G should be thought of as stating that $B_{na}(\Lambda^+_{\phi}; \phi)$ is the smallest set of lines that contains $B(A_{na}(\Lambda^+_{\phi})) \cup B_{gen}(\phi^{-1}) \cup B_{sing}(\phi^{-1})$ and is closed under this operation of ideal concatenation. It turns out that only a limited amount of such concatenation is possible, namely, extending a line of $B(A_{na}(\Lambda^+_{\phi}))$ by concatenating on one or both ends with a line of $B_{sing}(\phi^{-1})$, producing a set of lines we denote $B_{ext}(\Lambda^+_{\phi}; \phi^{-1})$ (see Section 2.2).

**Theorem G (Theorem 2.6).** If $\phi, \phi^{-1} \in \text{Out}(F_n)$ are rotationless and if $\Lambda^\pm_{\phi} \in \mathcal{L}^\pm(\phi)$ then

$$B_{na}(\Lambda^+_{\phi}) = B_{ext}(\Lambda^+_{\phi}; \phi^{-1}) \cup B_{gen}(\phi^{-1}) \cup B_{sing}(\phi^{-1})$$

Note that the first of the three terms in the union is the only one that depends on the lamination pair $\Lambda^+_{\phi}$; the other two depend only on $\phi^{-1}$.

For certain purposes in Part IV [HM13d] the following corollary to Theorem G is useful in being easier to directly apply. In particular item (2) provides a topologically uniform version of weak attraction:
Theorem H (Corollary 2.17). Given rotationless \( \phi, \phi^{-1} \in \text{Out}(F_n) \) and a dual lamination pair \( \Lambda^\pm \in L^\pm(\phi) \), the following hold:

1. Any line \( \ell \in B \) that is not carried by \( A_{\text{na}}(\Lambda^\pm) \) is weakly attracted either to \( \Lambda^+ \) by iteration of \( \phi \) or to \( \Lambda^- \) by iteration by \( \phi^{-1} \).

2. For any neighborhoods \( V^+, V^- \subset B \) of \( \Lambda^+, \Lambda^- \), respectively, there exists an integer \( m \geq 1 \) such that for any line \( \ell \in B \) at least one of the following holds: \( \gamma \in V^-; \phi^m(\gamma) \in V^+; \) or \( \gamma \) is carried by \( A_{\text{na}}(\Lambda^\pm) \).

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1 The nonattracting subgroup system

Consider a rotationless \( \phi \in \text{Out}(F_n) \) and a dual lamination pair \( \Lambda^\pm \in L^\pm(\phi) \). Since \( \phi \) is rotationless its action on \( L(\phi) \) is the identity and therefore so is its action on \( L(\phi^{-1}) \); the laminations \( \Lambda^+_\phi \) and \( \Lambda^-_\phi \) are therefore fixed by \( \phi \) and by \( \phi^{-1} \). In this setting we shall define the nonattracting subgroup system \( A_{\text{na}}(\Lambda^\pm_\phi) \), an invariant of \( \phi \) and \( \Lambda^\pm_\phi \).

One can view the definition of \( A_{\text{na}}(\Lambda^\pm_\phi) \) in two ways. First, in Definition 1.2, we define \( A_{\text{na}}(\Lambda^\pm_\phi) \) with respect to a choice of a CT representing \( \phi \); this CT acts as a choice of “coordinate system” for \( \phi \), and with this choice the description of \( A_{\text{na}}(\Lambda^\pm_\phi) \) is very concrete. We derive properties of this definition in results to follow, from Proposition 1.4 to Corollary 1.8, including most importantly the proofs of items (1), (2) and (3) of Theorem F. Then, in Corollaries 1.9 and 1.10, we prove that \( A_{\text{na}}(\Lambda^\pm_\phi) \) is invariantly defined, independent of the choice of CT and furthermore independent of the choice of a positive or negative power of \( \phi \), in particular proving items (4) and (5) of Theorem F. The independence result is what allows us to regard the nonattracting subgroup system as an invariant of a dual lamination pair rather than of each lamination individually (but still with implicit dependence on \( \phi \) up to nonzero power).

Weak attraction. Recall (Section I.1.1.5)\(^1\) the notation \( B \) for the space of lines of \( F_n \) on which \( \text{Out}(F_n) \) acts naturally, and recall that that a line \( \ell \in B \) is said to be weakly attracted

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\(^1\)Cross references such as “Section I.X.Y.Z” refer to Section X.Y.Z of Part I [HM13b]. Cross references to the Introduction [HM13a], to Part I [HM13b], and to Part II [HM13c] are to the June 2013 versions.
to a generic leaf $\lambda \in \Lambda_\phi^+ \subset B$ under iteration by $\phi$ if the sequence $\phi^n(\ell)$ weakly converges to $\lambda$ as $n \to +\infty$, that is, for each neighborhood $U \subset B$ of $\lambda$ there exists an integer $N > 0$ such that if $n \geq N$ then $\phi^n(\ell) \in U$. Note that since any two generic leaves of $\Lambda_\phi^+$ have the same weak closure, namely $\Lambda_\phi^+$, this property is independent of the choice of $\lambda$; for that reason we often speak of $\ell$ being weakly attracted to $\Lambda_\phi^+$ by iteration of $\phi$.

This definition of weak attraction applies to $\phi^{-1}$ as well, and so we may speak of $\ell$ being weakly attracted to $\Lambda_\phi^-$ under iteration by $\phi^{-1}$. This definition also applies to iteration of a CT $f : G \to G$ representing $\phi$ on elements of the space $\hat{B}(G)$ (Section I.1.1.6), which contains the subspace $B(G)$ identified with $B$ by letting lines be realized in $G$, and which also contains all finite paths and rays in $G$. We may speak of such paths being weakly attracted to $\Lambda_\phi^+$ under iteration by $f$. Whenever $\phi$ and the $\pm$ sign are understood, as they are in the notations $\Lambda_\phi^+$ and $\Lambda_\phi^-$, we tend to drop the phrase “under iteration by . . .”.

**Remark 1.1.** Suppose that $\phi$ is a rotationless iterate of some possibly nonrotationless $\eta \in \Out(F_n)$ and that $\Lambda_\phi^\pm$ is $\eta$-invariant. Then $\gamma$ is weakly attracted to $\Lambda_\phi^+$ under iteration by $\eta$ if and only $\gamma$ is weakly attracted to $\Lambda_\phi^+$ under iteration by $\phi$. Our results therefore apply to $\eta$ as well as $\phi$.

### 1.1 The nonattracting subgroup system $A_{\text{na}}(\Lambda_\phi^\pm)$

The Weak Attraction Theorem 6.0.1 of [BFH00] answers the “general weak attraction question” posed above in the restricted setting of a lamination pair $\Lambda_\phi^\pm$ which is topmost with respect to inclusion, and under restriction to birecurrent lines only. The answer is expressed in terms of an “improved relative train track representative” $g : G \to G$, a “nonattracting subgraph” $Z \subset G$, a (possibly trivial) Nielsen path $\hat{\rho}_r$, and an associated set of paths denoted $\langle Z, \hat{\rho}_r \rangle$. The construction and properties of $Z$ and $\langle Z, \hat{\rho}_r \rangle$ are given in [BFH00, Proposition 6.0.4].

In Definition 1.2 and the lemmas that follow, we generalize $Z$, $\langle Z, \hat{\rho}_r \rangle$, and the nonattracting subgroup system beyond the topmost setting.

**Notation for the nonattracting subgroup system.** We use various notations in various contexts. Before Corollary 1.10 we will use the notation $A_{\text{na}}(\Lambda_\phi^\pm)$ for the subgroup system, presuming from the start what we shall eventually show in Corollary 1.7 regarding its independence from the choice of a CT representative, but leaving open for a while the issue of whether it depends on the choice of $\pm$ sign. After the latter independence is established in Corollary 1.10 we will switch over to the notation $A_{\text{na}}(\Lambda^\pm_\phi)$. When we wish to emphasize dependence on $\phi$ we sometimes use the notation $A_{\text{na}}(\Lambda^{\pm_\phi})$ or $A_{\text{na}}(\Lambda^{\pm_\phi})$; and when we wish to de-emphasize this dependence we sometimes use $A_{\text{na}}(\Lambda^\pm)$ or $A_{\text{na}}(\Lambda^\pm)$.

For the remainder of Section 1.1 we fix a rotationless $\phi \in \Out(F_n)$ and a lamination pair $\Lambda_\phi^\pm \in \mathcal{L}^\pm_\phi$.

For a review of CTs, completely split paths, and the terms of a complete splitting, we refer the reader to Section I.1.5.1, particularly Definition I.1.28.

**Definitions 1.2.** The graph $Z$, the path $\hat{\rho}_r$, the path set $\langle Z, \hat{\rho}_r \rangle$, and the subgroup system $A_{\text{na}}(\Lambda_\phi^\pm)$. 

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Consider a CT \( f : G \to G \) representing \( \phi \) such that \( \Lambda_\phi^+ \) corresponds to the EG stratum \( H_r \subset G \). We shall define the nonattracting subgraph \( Z \) of \( G \), and we shall define a path \( \hat{\rho}_r \), either a trivial path or a height \( r \) indivisible Nielsen path if one exists. Using these we shall define a graph \( K \) and an immersion \( K \to G \) by consistently gluing together the graph \( Z \) and the domain of \( \hat{\rho}_r \). We then define \( A_{\text{na}}(\Lambda_\phi^+) \) in terms of the induced \( \pi_1 \)-injection on each component of \( K \). We also define a groupoid of paths \( \langle Z, \hat{\rho}_r \rangle \) in \( G \), consisting of all concatenations whose terms are edges of \( Z \) and copies of the path \( \hat{\rho}_r \) or its inverse, equivalently all paths in \( G \) that are images under the immersion \( K \to G \) of paths in \( K \).

**Definition of the graph \( Z \).** The nonattracting subgraph \( Z \) of \( G \) is defined as a union of certain strata \( H_i \neq H_r \) of \( G \), as follows. If \( H_i \) is an irreducible stratum then \( H_i \subset Z \) if and only if no edge of \( H_i \) is weakly attracted to \( \Lambda \); equivalently, using Fact I.1.50 (1), we have \( H_i \subset G \setminus Z \) if and only if for some (every) edge \( E_i \) of \( H_i \), there exists \( k \geq 0 \) so that some term in the complete splitting of \( f_{\#}^k(E_i) \) is an edge in \( H_r \). If \( H_i \) is a zero stratum enveloped by an EG stratum \( H_s \) then \( H_i \subset Z \) if and only if \( H_s \subset Z \).

**Remark.** The nonattracting subgraph \( Z \) automatically contains every stratum \( H_i \) which is a fixed edge, an NEG-linear edge, or an EG stratum distinct from \( H_r \) for which there exists an indivisible Nielsen path of height \( i \). For a fixed edge this is obvious. If \( H_i \) is an NEG-linear edge \( E_i \) then this follows from (Linear Edges) which says that \( f(E_i) = E_i \cdot u \) where \( u \) is a closed Nielsen path, because for all \( k \geq 1 \) it follows that the path \( f_{\#}^k(E_i) \) completely splits as \( E_i \) followed by Nielsen paths of height \( < i \), and no edges of \( E_r \) occur in this splitting. For an EG stratum \( H_i \) with an indivisible Nielsen path of height \( i \) this follows from Fact I.1.41 (3) which says that for each edge \( E \subset H_i \) and each \( k \geq 1 \), the path \( f_{\#}^k(E) \) completely splits into edges of \( H_i \) and Nielsen paths of height \( < i \); again no edges of \( E_r \) occur in this splitting.

**Remark.** Suppose that \( H_i \) is a zero stratum enveloped by the EG stratum \( H_s \) and that \( H_i \subset Z \). Applying the definition of \( Z \) to \( H_i \) it follows that \( H_s \subset Z \). Applying the definition of \( Z \) to \( H_s \) it follows that no \( s \)-taken connecting path in \( H_i \) is weakly attracted to \( \Lambda_\phi^+ \). Applying (Zero Strata) it follows that no edge in \( Z \) is weakly attracted to \( \Lambda \).

**Definition of the path \( \hat{\rho}_r \).** If there is an indivisible Nielsen path \( \rho_r \) of height \( r \) then it is unique up to reversal by Fact I.1.40 and we define \( \hat{\rho}_r = \rho_r \). Otherwise, by convention we choose a vertex of \( H_r \) and define \( \hat{\rho}_r \) to be the trivial path at that vertex.

**Definition of the path set \( \langle Z, \hat{\rho}_r \rangle \).** Consider \( \widehat{\mathcal{B}}(G) \), the set of lines, rays, circuits, and finite paths in \( G \) (Definition I.1.1.5). Define the subset \( \langle Z, \hat{\rho}_r \rangle \subset \widehat{\mathcal{B}}(G) \) to consist of all elements which decompose into a concatenation of subpaths each of which is either an edge in \( Z \), the path \( \hat{\rho}_r \) or its inverse \( \hat{\rho}_r^{-1} \).

**Definition of the subgroup system \( A_{\text{na}}(\Lambda_\phi^+) \).** If \( \hat{\rho}_r \) is the trivial path, let \( K = Z \) and let \( h : K \to G \) be the inclusion. Otherwise, define \( K \) to be the graph obtained from the disjoint union of \( Z \) and an edge \( E_\rho \) representing the domain of the Nielsen path \( \rho_r : E_\rho \to G_r \), with identifications as follows. Given an endpoint \( x \in E(\rho) \), if \( \rho_r(x) \in Z \) then identify \( x \sim \rho_r(x) \). Given distinct endpoints \( x, y \in E(\rho) \), if \( \rho_r(x) = \rho_r(y) \notin Z \) then identify \( x \sim y \) (these points are already identified if \( \rho_r(x) = \rho_r(y) \in Z \)). Define \( h : K \to G \) to be the inclusion on \( Z \) and to be the map \( \rho_r \) on \( E_\rho \). By Fact I.1.40 the initial oriented edges of \( \rho_r \) and \( \hat{\rho}_r \) are distinct in \( H_r \), and since no edge of \( H_r \) is in \( Z \) it follows that the map \( h \) is an immersion. The restriction of \( h \) to each component of \( K \) therefore induces an injection.
Remark: The case of a top stratum. In the special case that $H_r$ is the top stratum of $G$, there is a useful formula for $A_{na}(\Lambda^+_{\phi})$ which is obtained by considering three subcases. First, when $\hat{\rho}_r$ is trivial we have $K = Z = G_{r-1}$. Second is the geometric case, where $\hat{\rho}_r$ is a closed Nielsen path whose endpoint is an interior point of $H_r$ (Fact I.1.42 (2a)), and so the graph $K$ is the disjoint union of $Z = G_{r-1}$ with a loop mapping to $\rho_r$. Third is the “parageometric” case, where $\hat{\rho}_r$ is a nonclosed Nielsen path having at least one endpoint which is an interior point of $H_r$ (Fact I.1.42 (1a)), and so $K$ is obtained by attaching an arc to $Z = G_{r-1}$ by identifying at most one endpoint of the arc to $G_{r-1}$; note in this case that union of noncontractible components of $K$ deformation retracts to the union of noncontractible components of $G_{r-1}$. From this we obtain the following formula:

$A_{na}(\Lambda^+_{\phi}) = \begin{cases} \{\pi_1 G_{r-1}\} & \text{if } \Lambda^+_{\phi} \text{ and } H_r \text{ are nongeometric} \\ \{\pi_1 G_{r-1}\} \cup \{\langle \rho_r \rangle\} & \text{if } \Lambda^+_{\phi} \text{ and } H_r \text{ are geometric} \end{cases}$

where in the geometric case $\{\langle \rho_r \rangle\}$ denotes the conjugacy class of the infinite cyclic subgroup generated by an element of $F_n$ represented by the closed Nielsen path $\rho_r$.

This completes Definitions 1.2.

Remark 1.3. In the special case that the stratum $H_r$ is geometric, the 1-complex $K$ lives naturally as an embedded subcomplex of the geometric model $X$ for $H_r$ (Definition I.2.4), as follows. By item (4a) of that definition, we may identify $K$ with the subcomplex $Z \cup j(\partial_0 S) \subset G \cup j(\partial_0 S) \subset X$ in such a way that the immersion $K \to G$ is identified with the restriction to $K$ of the deformation retraction $d: X \to G$. The subgroup system $A_{na}(\Lambda^+_{\phi}) = \{\pi_1 K\}$ may therefore be described as the conjugacy classes of the images of the inclusion induced injections $\pi_1 K_i \to \pi_1 X \approx F_n$, over all noncontractible components $K_i \subset X$. Noting that $j: S \to X$ maps each boundary component $\partial_1 S, \ldots, \partial_m S$ to $G_{r-1} \subset Z \subset K$ and maps $\partial_0 S$ to $j(\partial_0 S) \subset K$, we have $j(\partial S) \subset K$. It follows in the geometric case that Proposition I.3.3 applies to $\{\pi_1 K\}$, the conclusion of which will be used in the proof of the following proposition.

Recall the characterization of geometricity of $\Lambda^+_{\phi}$ given in Proposition I.2.18, expressed in terms of the free factor support of the boundary components of $S$. Our next result, among other things, gives a different characterization of geometricity of a lamination $\Lambda^+_{\phi} \in \mathcal{L}(\phi)$, expressed in terms of the nonattracting subgroup system $A_{na}(\Lambda^+_{\phi})$.

**Proposition 1.4** (Properties of the nonattracting subgroup system). Given a CT $f: G \to G$ representing $\phi$ with EG stratum $H_r$ corresponding to $\Lambda^+_{\phi}$, the subgroup system $A_{na}(\Lambda^+_{\phi})$ satisfies the following:

1. $A_{na}(\Lambda^+_{\phi})$ is a vertex group system.
2. $A_{na}(\Lambda^+_{\phi})$ is a free factor system if and only if the stratum $H_r$ is not geometric.
3. $A_{na}(\Lambda^+_{\phi})$ is malnormal, with one component for each noncontractible component of $K$.  

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Proof. First we show that any subgroup \( A \) for which \( [A] \in \mathcal{A}_{na}(\Lambda^+_\phi) \) is nontrivial and proper, as required for a vertex group system. Nontriviality follows because only noncontractible components of \( K \) are used. To prove properness: if \( \hat{\rho}_r \) is trivial then any circuit containing an edge of \( H_r \) is not carried by \( \mathcal{A}_{na}(\Lambda^+_\phi) \); if \( \hat{\rho} = \rho_r \) is nontrivial then any circuit containing an edge of \( H_r \) but not containing \( \rho_r \) is not carried by \( \mathcal{A}_{na}(\Lambda^+_\phi) \).

We adopt the notation of Definition 1.2. By applying Fact I.1.42 and Proposition I.2.18, when \( \Lambda^+_\phi \) is not geometric then \( \hat{\rho}_r \) is either trivial or a nonclosed Nielsen path, and when \( \Lambda^+_\phi \) is geometric then \( \hat{\rho}_r \) is a closed Nielsen path. We prove (1)—(3) by considering these three cases of \( \hat{\rho}_r \) separately.

**Case 1:** \( \hat{\rho}_r \) is trivial. In this case \( K = Z \), and \( \mathcal{A}_{na}(\Lambda^+_\phi) \) is the free factor system associated to the subgraph \( Z \subset G \). Item (3) follows immediately.

**Case 2:** \( \hat{\rho}_r = \rho_r \) is a nonclosed Nielsen path. We prove that \( \mathcal{A}_{na}(\Lambda^+_\phi) \) is a free factor system following an argument of [BFH00] Lemma 5.1.7. By Fact I.1.42 (1) there is an edge \( E \subset H_r \) that is crossed exactly once by \( \rho_r \). We may decompose \( \rho_r \) into a concatenation of subpaths \( \rho_r = \sigma E \tau \) where \( \sigma, \tau \) are paths in \( G_r \setminus \text{int}(E) \). Let \( \tilde{G} \) be the graph obtained from \( G \setminus \text{int}(E) \) by attaching an edge \( J \), letting the initial and terminal endpoints of \( J \) be equal to the initial and terminal endpoints of \( \rho_r \), respectively. The identity map on \( G \setminus \text{int}(E) \) extends to a map \( h: \tilde{G} \to G \) that takes the edge \( J \) to the path \( \hat{\rho}_r \), and to a homotopy inverse \( \tilde{h}: G \to \tilde{G} \) that takes the edge \( E \) to the path \( \tilde{\sigma}J \tilde{\tau} \). We may therefore view \( \tilde{G} \) as a marked graph, pulling the marking on \( G \) back via \( h \). Notice that \( K \) may be identified with the subgraph \( Z \cup J \subset \tilde{G} \), in such a way that the map \( h: \tilde{G} \to G \) is an extension of the map \( h: K \to G \) as originally defined. It follows that \( \mathcal{A}_{na}(\Lambda^+_\phi) \) is the free factor system associated to the subgraph \( Z \cup J \).

In this case, as in Case 1, item (3) follows immediately because of the identification of \( K \) with a subgraph of the marked graph \( \tilde{G} \).

**Case 3:** \( \hat{\rho}_r = \rho_r \) is a closed Nielsen path. In this case \( H_r \) is geometric. Adopting the notation of the geometric model \( X \) for \( H_r \), Definition I.2.4, by Remark 1.3 we have \( \mathcal{A}_{na}(\Lambda^+_\phi) = [\pi_1 K] \) for a subgraph \( K \subset L \) containing \( j(\partial S) \). Applying Proposition I.3.3 it follows that \( [\pi_1 K] \) is a vertex group system.

If \( \mathcal{A}_{na}\Lambda^+_\phi = [\pi_1 K] \) were a free factor system then, since each of the conjugacy classes \( [\partial_0 S], \ldots, [\partial_m S] \) is supported by \( [\pi_1 K] \), it would follow by Proposition I.2.18 (5) that \( [\pi_1 S] \subset [\pi_1 K] \). However, since \( S \) supports a pseudo-Anosov mapping class, it follows that \( S \) contains a simple closed curve \( c \) not homotopic to a curve in \( \partial S \). By Lemma I.2.7 (2) we have \( [j(c)] \not\in [\pi_1 K] \) while \( [j(c)] \in [\pi_1 S] \). This is a contradiction and so \( \mathcal{A}_{na}\Lambda^+_\phi \) is not a free factor system.

In this case, item (3) is a consequence of Lemma I.2.7 (1). \( \square \)

Item (2) in the next lemma states that \( \langle Z, \hat{\rho}_r \rangle \) is a groupoid, by which we mean that the tightened concatenation of any two paths in \( \langle Z, \hat{\rho}_r \rangle \) is also a path in \( \langle Z, \hat{\rho}_r \rangle \) as long as that concatenation is defined. For example, the concatenation of two distinct rays in \( \langle Z, \hat{\rho}_r \rangle \) with the same base point tightens to a line in \( \langle Z, \hat{\rho}_r \rangle \).

**Lemma 1.5.** Assuming the notation of Definitions 1.2,

(1) The map \( h \) induces a bijection between \( \hat{B}(K) \) and \( \langle Z, \hat{\rho}_r \rangle \).
(2) \( \langle Z, \hat{\rho}_r \rangle \) is a groupoid.

(3) The set of lines carried by \( \langle Z, \hat{\rho}_r \rangle \) is the same as the set of lines carried by \( A_{na}(\Lambda^+_{\phi}) \).

(4) The set of circuits carried by \( \langle Z, \hat{\rho}_r \rangle \) is the same as the set of circuits carried by \( A_{na}(\Lambda^+_{\phi}) \).

(5) The set of lines carried by \( \langle Z, \hat{\rho}_r \rangle \) is closed in the weak topology.

(6) If \([A_1], [A_2] \in A_{na}(\Lambda^+_{\phi})\) and if \(A_1 \neq A_2\) then \(A_1 \cap A_2 = \{1\}\) and \(\partial A_1 \cap \partial A_2 = \emptyset\).

Proof. We make use of four evident properties of the immersion \( h : K \rightarrow G \). The first is that every path in \( K \) with endpoints, if any, at vertices is mapped by \( h \) to an element of \( \langle Z, \hat{\rho}_r \rangle \). The second is that \( h \) induces a bijection between the vertex sets of \( K \) and of \( Z \cup \partial \hat{\rho}_r \). The third is that for each edge \( E \) of \( Z \), there is a unique edge of \( K \) that projects to \( E \) and that no other subpath of \( K \) has at least one endpoint at a vertex and projects to \( E \). The last is that if \( \hat{\rho}_r \) is non-trivial then it has a unique lift to \( K \) (because its unique illegal turn of height \( r \) does). Together these imply (1) which implies (2) and (3). Item (4) follows from (3) using the natural bijection between periodic lines and circuits. Item (5) follows from (1) and Fact I.1.8. Item (6) follows from Proposition 3 and Fact I.1.2.

The following lemma is based on Proposition 6.0.4 and Corollary 6.0.7 of [BFH00].

Lemma 1.6. Assuming the notation of Definitions 1.2, we have:

(1) If \( E \) is an edge of \( Z \) then \( f_{\#}(E) \in \langle Z, \hat{\rho}_r \rangle \).

(2) \( \langle Z, \hat{\rho}_r \rangle \) is \( f_{\#} \)-invariant.

(3) If \( \sigma \in \langle Z, \hat{\rho}_r \rangle \) then \( \sigma \) is not weakly attracted to \( \Lambda^+_{\phi} \).

(4) For any finite path \( \sigma \in G \) with endpoints at fixed vertices, the converse to (3) holds: if \( \sigma \) is not weakly attracted to \( \Lambda \) then \( \sigma \in \langle Z, \hat{\rho}_r \rangle \).

(5) \( f_{\#} \) restricts to bijections of the the following sets: lines in \( \langle Z, \hat{\rho}_r \rangle \); finite paths in \( \langle Z, \hat{\rho}_r \rangle \) whose endpoints are fixed by \( f \); and circuits in \( \langle Z, \hat{\rho}_r \rangle \).

Proof. In this proof we shall freely use that \( \langle Z, \hat{\rho}_r \rangle \) is a groupoid, Lemma 1.5 (2).

\( \langle Z, \hat{\rho}_r \rangle \) contains each fixed or linear edge by construction. Given an indivisible Nielsen path \( \rho_i \) of height \( i \), we prove by induction on \( i \) that \( \rho_i \) is in \( \langle Z, \hat{\rho}_r \rangle \). If \( H_i \) is NEG this follows from (NEG Nielsen Paths) and the induction hypothesis. If \( H_i \) is EG then Fact I.1.41 (3) applies to show that \( H_i \subset Z \); combining this with Fact I.1.41 (3) again and with the induction hypothesis we conclude that \( \rho_i \in \langle Z, \hat{\rho}_r \rangle \).

Since all indivisible Nielsen paths and all fixed edges are contained in \( \langle Z, \hat{\rho}_r \rangle \), it follows that all Nielsen path are contained in \( \langle Z, \hat{\rho}_r \rangle \), which immediately implies that \( \langle Z, \hat{\rho}_r \rangle \) contains all exceptional paths.

Suppose that \( \tau = \tau_1 \ldots \tau_m \) is a complete splitting of a finite path that is not contained in a zero stratum. Each \( \tau_i \) is either an edge in an irreducible stratum, a taken connecting path in a zero stratum, or, by the previous paragraph, a term which is not weakly attracted to \( \Lambda \) and which is contained in \( \langle Z, \hat{\rho}_r \rangle \). If \( \tau_i \) is a taken connecting path in a zero stratum
$H_t$ that is enveloped by an EG stratum $H_s$ then, by definition of complete splitting, $\tau_i$ is a maximal subpath of $\tau$ in $H_t$; since $\tau \not\subset H_t$ it follows that $m \geq 2$, and by applying (Zero Strata) it follows that at least one other term $\tau_j$ is an edge in $H_s$. In conjunction with the second Remark in Definitions 1.2, this proves that $\tau$ is contained in $\langle Z, \hat{\rho}_r \rangle$ if and only if each $\tau_i$ that is an edge in an irreducible stratum is contained in $Z$ if and only if $\tau$ is not weakly attracted to $\Lambda^+_\phi$.

We apply this in two ways. First, this proves item (4) in the case that $\sigma$ is completely split. Second, applying this to $\tau = f_\#(E)$ where $E$ is an edge in $Z$, item (1) follows in the case that $f_\#(E)$ is not contained in any zero stratum. Consider the remaining case that $\tau = f_\#(E)$ is contained in a zero stratum $H_t$ enveloped by the EG stratum $H_s$. By definition of complete splitting, $\tau = \tau_1$ is a taken connecting path. By Fact I.1.45 the edge $E$ is contained in some zero stratum $H_{t'}$ enveloped by the same EG stratum $H_s$. Since $E \subset Z$, it follows that $H_s \subset Z$, and so $H^*_s \subset Z$, and so $\tau \subset Z$, proving (1).

Item (2) follows from item (1), the fact that $f_\#(\hat{\rho}_r) = \hat{\rho}_r$ and the fact that $\langle Z, \hat{\rho}_r \rangle$ is a groupoid.

Every generic leaf of $\Lambda^+_{\phi}$ contains subpaths in $H_r$ that are not subpaths of $\hat{\rho}_r$ or $\hat{\rho}^{-1}_r$ and hence not subpaths in any element of $\langle Z, \hat{\rho}_r \rangle$. Item (3) therefore follows from item (2).

To prove (5), for lines and finite paths the implication (2) $\Rightarrow$ (5) follows from Corollary 6.0.7 of [BFH00]. For circuits, use the natural bijection between circuits and periodic lines, noting that this bijection preserves membership in $\langle Z, \hat{\rho}_r \rangle$.

It remains to prove (4). By (5), there is no loss of generality in replacing $\sigma$ with $f^k_\#(\sigma)$ for any $k \geq 1$. By Fact I.1.35 this reduces (4) to the case that $\sigma$ is completely split which we have already proved.

\[\square\]

### 1.2 Applications and properties of the nonattracting subgroup system.

We now show that the nonattracting subgroup system $\mathcal{A}_{na}(\Lambda^+_{\phi})$ deserves its name.

**Corollary 1.7.** For any rotationless $\phi \in \text{Out}(F_n)$ and $\Lambda^+_{\phi} \in \mathcal{L}(\phi)$, a conjugacy class $[a]$ in $F_n$ is not weakly attracted to $\Lambda^+_{\phi}$ if and only if it is carried by $\mathcal{A}_{na}(\Lambda^+_{\phi})$.

**Proof.** Let $f: G \to G$ be a CT representing a rotationless $\phi \in \text{Out}(F_n)$ and assume the notation of Definitions 1.2. By Lemma 1.5 (4), it suffices to show that a circuit in $G$ is not weakly attracted to $\Lambda^+_{\phi}$ under iteration by $f_\#$ if and only if it is carried by $\langle Z, \hat{\rho}_r \rangle$. Both the set of circuits in $\langle Z, \hat{\rho}_r \rangle$ and the set of circuits that are not weakly attracted to $\Lambda^+_{\phi}$ are $f_\#$-invariant. We may therefore replace $\sigma$ with any $f^k_\#(\sigma)$ and hence may assume that $\sigma$ is completely split. After taking a further iterate, we may assume that some coarsening of the complete splitting of $\sigma$ is a splitting into subpaths whose endpoints are fixed by $f$. Lemma 1.6 (4) completes the proof.

\[\square\]

**Corollary 1.8.** For any rotationless $\phi \in \text{Out}(F_n)$ and $\Lambda^+_{\phi} \in \mathcal{L}(\phi)$, and for any finite rank subgroup $B < F_n$, if each conjugacy class in $B$ is not weakly attracted to $\Lambda^+_{\phi}$ then there exists a subgroup $A < F_n$ such that $B < A$ and $[A] \in \mathcal{A}_{na}(\Lambda^+_{\phi})$.

**Proof.** By Corollary 1.7 the conjugacy class of every nontrivial element of $B$ is carried by the subgroup system $\mathcal{A}_{na}(\Lambda^+_{\phi})$ which, by Proposition 1.4 (1), is a vertex group system. Applying Lemma I.3.1, the conclusion follows.

\[\square\]
Using Corollary 1.8 we can now prove some useful invariance properties of $A_{na}(\Lambda^+_\phi)$, for instance that $A_{na}(\Lambda^+_\phi)$ is an invariant of $\phi$ and $\Lambda^+_\phi$ alone, independent of the choice of CT representing $\phi$.

**Corollary 1.9.** For any rotationless $\phi \in \text{Out}(F_n)$ and any lamination $\Lambda^+_{\phi} \in \mathcal{L}(\phi)$ we have:

1. The nonattracting subgroup system $A_{na}(\Lambda^+_\phi)$ is the unique vertex group system such that the conjugacy classes it carries are precisely those which are not weakly attracted to $\Lambda^+_\phi$ under iteration of $\phi$.

2. $A_{na}(\Lambda^+_\phi)$ depends only on $\phi$ and $\Lambda^+_\phi$, not on the choice of a CT representing $\phi$.

3. The dependence in (2) is natural in the sense that if $\theta \in \text{Out}(F_n)$ then $\theta(A_{na}(\Lambda^+_\phi)) = A_{na}(\Lambda^+_{\theta \phi \theta^{-1}})$ where $\Lambda^+_{\theta \phi \theta^{-1}}$ is the image of $\Lambda^+_\phi$ under the bijection $\mathcal{L}(\phi) \mapsto \mathcal{L}(\theta \phi \theta^{-1})$ induced by $\theta$.

**Proof.** By Proposition 1.4 (1), $A_{na}(\Lambda^+_\phi)$ is a vertex group system. By Lemma I.3.1, $A_{na}(\Lambda^+_\phi)$ is determined by the set of conjugacy classes of elements of $F_n$ that are carried by $A_{na}(\Lambda^+_\phi)$, and by Corollary 1.7 these conjugacy classes are determined by $\phi$ and $\Lambda^+_\phi$ alone, independent of choice of a CT representing $\phi$, namely they are the conjugacy classes weakly attracted to $\Lambda^+_\phi$ under iteration of $\phi$. This proves (1) and (2). Item (3) follows by choosing any CT $f : G \to G$ representing $\phi$ and changing the marking on $G$ by the conjugator $\theta$ to get a CT representing $\theta \phi \theta^{-1}$. \hfill $\square$

The following shows that not only is $A_{na}(\Lambda^+_\phi)$ is invariant under change of CT, but it is invariant under inversion of $\phi$ and replacement of $\Lambda^+_\phi$ with its dual lamination.

**Corollary 1.10.** Given $\phi, \phi^{-1} \in \text{Out}(F_n)$ both rotationless, and given a dual lamination pair $\Lambda^+ \in \mathcal{L}(\phi), \Lambda^- \in \mathcal{L}(\phi^{-1})$, we have $A_{na}(\Lambda^+; \phi) = A_{na}(\Lambda^-; \phi^{-1})$.

**Notational remark.** Based on Corollary 1.10, we introduce the notation $A_{na}\Lambda^+_\phi$ for the vertex group system $A_{na}(\Lambda^+; \phi) = A_{na}(\Lambda^-; \phi^{-1})$.

**Proof.** For each nontrivial conjugacy class $[a]$ in $F_n$, we must prove that $[a]$ is weakly attracted to $\Lambda^+$ under iteration by $\phi$ if and only if $[a]$ is weakly attracted to $\Lambda^-$ under iteration by $\phi^{-1}$. Replacing $\phi$ with $\phi^{-1}$ it suffices to prove the “if” direction. Applying Theorem I.1.30, choose a CT $f : G \to G$ representing $\phi$ having a core filtration element $G_r$ such that $[G_r] = F_{\text{supp}}(\Lambda^+)$, and so $H_r \subset G$ is the EG stratum corresponding to $\Lambda^+$. We adopt the notation of Definitions 1.2.

Suppose that $[a]$ is not weakly attracted to $\Lambda^+$ under iteration by $\phi$. Then the same is true for all $\phi^{-k}([a])$ and so $\phi^{-k}([a]) \in \langle Z, \hat{\rho}_r \rangle$ for all $k \geq 0$ by Corollary 1.7.

Arguing by contradiction, suppose in addition that $[a]$ is weakly attracted to $\Lambda^-$ under iteration by $\phi^{-1}$. Applying Corollary 1.5 (5) it follows that a generic line $\gamma$ of $\Lambda^-$ is contained in $\langle Z, \hat{\rho}_r \rangle$. However, since $F_{\text{supp}}(\gamma) = F_{\text{supp}}(\Lambda^-) = [G_r]$, it follows that $\gamma$ has height $r$. If $\hat{\rho}_r$ is trivial then $\gamma$ is a concatenation of edges of $Z$ none of which has height $r$, a contradiction. If $\hat{\rho}_r = \rho_r$ is nontrivial then all occurrences of edges of $H_r$ in $\gamma$ are contained
in a pairwise disjoint collection of subpaths each of which is an iterate of \( \rho_r \) or its inverse. By Fact I.1.42, at least one endpoint of \( \rho_r \) is disjoint from \( G_{r-1} \). If \( \rho_r \) is not closed then we obtain an immediate contradiction. If \( \rho_r \) is closed then \( \gamma \) is a bi-infinite iterate of \( \rho_r \), but this contradicts [BFH00] Lemma 3.1.16 which says that no generic leaf of \( \Lambda_{\psi}^+ \) is periodic. \( \square \)

We conclude this section with the following result, needed for the proof of Corollary 2.17, which generalizes Fact I.1.12 from free factor systems to nonattracting subgroup systems. Given an end \( \mathcal{E} \), we say that \( \mathcal{E} \) is carried by \( \langle Z, \hat{\rho}_r \rangle \) if some ray representing \( \mathcal{E} \) is in \( \langle Z, \hat{\rho}_r \rangle \).

**Lemma 1.11.** Assume the notation of Definitions 1.2.

1. Every sequence \( \gamma_i \) of lines in \( G \) not carried by \( \langle Z, \hat{\rho}_r \rangle \) has a subsequence that weakly converges to a line \( \gamma \) not carried by \( \langle Z, \hat{\rho}_r \rangle \).

2. The weak accumulation set of every end not carried by \( \langle Z, \hat{\rho}_r \rangle \) contains a line not carried by \( \langle Z, \hat{\rho}_r \rangle \).

**Proof.** If \( H_r \) is non-geometric then \( A_{na}(\Lambda^\pm) \) is a free factor system and the lemma follows from Fact I.1.12 and Lemma III.1.5 (3).

Suppose that \( H_r \) is geometric and so \( \langle Z, \hat{\rho}_r \rangle = \langle Z, \rho_r \rangle \). Let \( \rho_r = \alpha \ast \beta \) be concatenated at the unique illegal turn \( t = \{d_\alpha, d_\beta\} \) in \( H_r \), where \( d_\alpha \) is the terminal direction of \( \alpha \) and \( d_\beta \) is the initial direction of \( \beta \). Let

\[
L = \max\{\text{Length}(\alpha), \text{Length}(\beta)\}
\]

Let \( \Sigma \) be the set of all paths of length \( \leq 2L \) that occur as subpaths of an element of \( \langle Z, \rho_r \rangle \). Note that for any path of the form \( \alpha' \ast \beta' \in \Sigma \) such that \( \text{Length}(\alpha') = \text{Length}(\alpha) \) and \( \text{Length}(\beta') = \text{Length}(\beta) \) and such that \( d_\alpha \) is the terminal direction of \( \alpha' \) and \( d_\beta \) is the initial direction of \( \beta' \), we have \( \alpha' = \alpha \) and \( \beta' = \beta \) so \( \alpha' \ast \beta' = \rho_r \).

We claim that if \( \gamma \in \hat{B}(G) \) and if every subpath of \( \gamma \) of length \( \leq 2L \) is contained in \( \Sigma \) then there is a subpath \( \gamma' \) of \( \gamma \) that is contained in \( \langle Z, \rho_r \rangle \) and that contains all of \( \gamma \) with the possible exception of initial and terminal subpaths of length \( \leq L \).

To prove the claim, let \( \hat{\gamma} \) be a lift of \( \gamma \) to the universal cover of \( G \). Given \( \tilde{t} = \{\tilde{d}_\alpha, \tilde{d}_\beta\} \) an illegal turn in \( \hat{\gamma} \) that projects to \( t = \{d_\alpha, d_\beta\} \), we say that \( \tilde{t} \) is buffered if \( \hat{\gamma} \) contains at least \( L \) edges on each side of the turn \( \tilde{t} \); if this is the case then \( \hat{\gamma} \) has a subpath which contains the turn \( \tilde{t} \) and is a lift of \( \rho_r \). Suppose that \( \tilde{t}_1 \) and \( \tilde{t}_2 \) are buffered illegal turns in \( \hat{\gamma} \) that project to \( t \), and that there are no other illegal turns that project to \( t \) between \( \tilde{t}_1 \) and \( \tilde{t}_2 \). Let \( \hat{\gamma}_i \subset \hat{\gamma} \) be the lift of \( \rho_r \) or \( \rho_r^{-1} \) that contains \( \tilde{t}_i \), and let \( \sigma \) be the projection of the subpath of \( \gamma \) which starts at the turn \( \tilde{t}_1 \) and ends at the turn \( \tilde{t}_2 \). If \( \sigma \) has length \( \leq 2L \) then \( \sigma \) is a subpath of a path contained in \( \langle Z, \rho_r \rangle \), in which case it follows that \( \hat{\gamma}_1 \) and \( \hat{\gamma}_2 \) intersect in at most an endpoint; and the same evidently follows if \( \sigma \) has length \( > 2L \).

We may therefore decompose \( \hat{\gamma} \) into subpaths \( \hat{\gamma}_i \) as follows: any subpath of \( \hat{\gamma} \) that projects to either \( \rho_r \) or \( \rho_r^{-1} \) is a \( \hat{\gamma}_i \); each remaining edge is a \( \hat{\gamma}_i \). If \( \hat{\gamma} \) has an initial vertex, then remove the initial segment preceding the first subpath that projects to \( \rho_r \) or \( \rho_r^{-1} \) if this segment has length \( < L \) and remove the initial segment of length \( L \) otherwise. Treat the terminal vertex, if there is one, similarly. The resulting subpath of \( \hat{\gamma} \) projects to the desired subpath \( \gamma' \) of \( \gamma \). This completes the proof of the claim.
If \( \gamma_i \in B(G) \) is a sequence of lines not in \( \langle Z, \rho_r \rangle \) then it follows from the claim that each \( \gamma_i \) has a subpath \( \alpha_i \) of edge length \( \leq 2L \) that is not in \( \Sigma \). There are only finitely many paths of edge length \( \leq 2L \), so by passing to a subsequence we may assume that \( \alpha = \alpha_i \) is independent of \( i \). It follows that this subsequence has a weak limit which is a line containing the path \( \alpha \), and this line is not in \( \langle Z, \rho_r \rangle \).

If \( r \) is a ray in \( G \) no subray of which is contained in \( \langle Z, \rho_r \rangle \) then, focusing on subrays obtained by deleting initial paths of length \( > L \), it follows from the claim that \( r \) has infinitely many pairwise nonoverlapping subpaths \( \alpha_i \) of edge length \( \leq 2L \) that are not in \( \Sigma \). Again, passing to a subsequence, we may assume that \( \alpha = \alpha_i \) is independent of \( i \). It follows that \( r \) has a weak limit which is a line containing the path \( \alpha \), and this line is not in \( \langle Z, \rho_r \rangle \). □

2 Nonattracted lines

In the previous section, given a rotationless \( \phi \in \text{Out}(F_n) \) and \( \Lambda^+_{\phi} \in \mathcal{L}(\phi) \), we described the set of conjugacy classes that are not weakly attracted to \( \Lambda^+_{\phi} \) under iteration by \( \phi \) — they are precisely the conjugacy classes carried by the nonattracting subgroup system \( \mathcal{A}_{na}(\Lambda^+_{\phi}) \).

In this section we state Theorem 2.6 that characterizes those lines that are not weakly attracted to \( \Lambda^+_{\phi} \) under iteration by \( \phi \). Our characterization starts with Lemma 2.1 that lays out three particular types of such lines: lines carried by \( \mathcal{A}_{na}(\Lambda^\pm_{\phi}) \); singular lines of \( \phi^{-1} \); and generic leaves of laminations in \( \mathcal{L}(\phi^{-1}) \). Theorem 2.6 will say that, in addition to these three subsets, by concatenating elements of these subsets in a very particular manner one obtains the entire set of lines not weakly attracted to \( \Lambda^+_{\phi} \). The proof of this theorem will occupy the remaining subsections of Section 1.

2.1 Theorem G — Characterizing nonattracted lines

From here up through Section 2.4 we adopt the following:

Notational conventions: Assume \( \phi, \psi = \phi^{-1} \in \text{Out}(F_n) \) are rotationless and that \( \Lambda^\pm_{\phi} \in \mathcal{L}^\pm(\phi) \) is a laminating pair. We also denote

\[
\Lambda^+_{\phi} = \Lambda^-_{\psi} \in \mathcal{L}(\psi) \quad \text{and} \quad \Lambda^-_{\phi} = \Lambda^+_{\psi} \in \mathcal{L}(\phi)
\]

Applying [FH11] Theorem 4.28 (or see Theorem I.1.30), choose \( f: G \to G \) and \( f': G' \to G' \) to be CTs representing \( \phi \) and \( \psi \), respectively, the first with EG stratum \( H_r \subset G \) associated to \( \Lambda^+_{\phi} \), and the second with EG stratum \( H'_u \subset G' \) associated to \( \Lambda^+_{\psi} \), so that

\[
[G_r] = F_{\text{supp}}(\Lambda^+_{\phi}) = F_{\text{supp}}(\Lambda^-_{\psi}) = [G'_u] \\
[G_{r-1}] = [G'_{u-1}]
\]

To check that this is possible, after choosing \( f: G \to G \) to satisfy the one condition \([G_r] = F_{\text{supp}}(\Lambda^+_{\phi}) \) we may then choose \( f' \) to satisfy the two conditions \([G'_u] = [G_r] \) and \([G'_t] = [G_{r-1}] \) for some \( t < u \), but then by (Filtration) in Definition I.1.29 it follows that \([G'_t] = [G'_{u-1}] \). For other laminations in the set \( \mathcal{L}(\psi) \), or strata or filtration elements of \( G' \) that occur in the course of our presentation, we use notation like \( \Lambda^-_{t} \), or \( H'_{t} \) or \( G'_{t} \) with the subscript \( t \), as in the previous paragraph.
The reader may refer to Section I.1.4 for a refresher on basic concepts regarding the set $P(\phi)$ of principal automorphisms representing $\phi \in \text{Out}(F_n)$, and on the set $\text{Fix}(\bar{\Phi}) \subset \partial F_n$ of points at infinity fixed by the continuous extension $\bar{\Phi} : \partial F_n \to \partial F_n$ of an automorphism $\Phi \in \text{Out}(F_n)$.

We also recall/introduce some notations and definitions related to a rotationless outer automorphism $\psi \in \text{Out}(F_n)$.

- $B_{na}(\Lambda^+_\phi) = B_{na}(\Lambda^+_\phi; \phi)$ denotes set of all lines in $B$ that are not weakly attracted to $\Lambda^+_\phi$ under iteration by $\phi$.
- $B_{sing}(\psi)$ denotes the set of singular lines of $\psi$: by definition, $\ell \in B$ is a singular line for $\psi$ if there exists $\Psi \in P(\psi)$ (= the set of principal automorphisms representing $\psi$) such that $\partial \ell \subset \text{Fix}_N(\Psi)$.
- $B_{gen}(\psi)$ denotes the set of all generic leaves of all elements of $L(\psi)$.

**Lemma 2.1.** Given rotationless $\phi, \psi = \phi^{-1} \in \text{Out}(F_n)$ and $\Lambda^+_\phi \in L(\phi)$, if $\gamma \in B_{na}(\Lambda^+_\phi)$, then any of the following three conditions then $\gamma \in B_{na}(\Lambda^+_\phi)$:

1. $\gamma$ is carried by $A_{na}(\Lambda^+_\phi)$.
2. $\gamma \in B_{sing}(\psi)$
3. $\gamma \in B_{gen}(\psi)$.

**Proof.** Case (1) is a consequence of the following: no conjugacy class carried by $A_{na}(\Lambda^+_\phi)$ is weakly attracted to $\Lambda^+_\phi$ (Corollary 1.7); axes of conjugacy classes are dense in the set of all lines carried by $A_{na}(\Lambda^+_\phi)$ (as is true for any subgroup system); and $B_{na}(\Lambda^+_\phi)$ is a weakly closed subset of $B$, which follows from the fact that being weakly attracted to $\Lambda^+_\phi$ is a weakly open condition on $B$, an evident consequence of the definition of an attracting lamination.

For Case (3), suppose that $\gamma$, and hence each $\phi^i(\gamma)$, is a generic leaf of some $\Lambda^-_i \in L(\psi)$. Choose $[a]$ to be a conjugacy class represented by a completely split circuit in $G'$ such that some term of its complete splitting is an edge of $H'_1$. By Fact I.1.59 (1), $[a]$ is weakly attracted to $\gamma$ under iteration by $\psi$. If $\gamma$ were weakly attracted to $\Lambda^+_\phi$ under iteration by $\phi$ then, since the $\phi^i(\gamma)$'s all have the same neighborhoods in $B$, the lamination $\Lambda^+_\phi$ would be in the closure of $\gamma$, and so $[a]$ would be weakly attracted to $\Lambda^+_\phi$ under iteration by $\psi = \phi^{-1}$, contradicting Fact I.1.59 (2).

For Case (2), choose $\Psi \in P(\psi)$ and a lift $\hat{\gamma}$ of $\gamma$ with endpoints in $\text{Fix}_N(\hat{\Psi}) = \partial \text{Fix}(\Psi) \cup \text{Fix}_+(\Psi)$. Assuming that $\gamma$ is weakly attracted to $\Lambda^+_\phi$ under iteration by $\phi$, we argue to a contradiction. Since $\gamma$ is $\phi_\#$-invariant, $\Lambda^+_\phi$ is contained in the weak closure of $\gamma$. Let $\ell$ be a generic leaf of $\Lambda^+_\phi$. Since $\ell$ is birecurrent, $\ell$ is contained in the accumulation set of at least one of the endpoints, say $P$, of $\hat{\gamma}$. If $P \in \partial \text{Fix}(\Psi)$ then $\ell$ is carried by $A_{na}(\Lambda^+_\phi)$ in contradiction to Case (1) and the obvious fact that $\ell$ is weakly attracted to $\Lambda^+_\phi$. Thus $P \in \text{Fix}_+(\hat{\Psi})$. By Lemma I.1.52, there is a conjugacy class $[a]$ that is weakly attracted to every line in the weak accumulation set of $P$ under iteration by $\psi$, and so is weakly attracted to $\Lambda^+_\phi$ under iteration by $\psi$. As in Case (3), this contradicts Fact I.1.59 (2). □
As shown in Example 2.4 below, by using concepts of concatenation one can sometimes construct lines in $\mathcal{B}_{\text{na}}(\Lambda)$ not accounted for in the statement of Lemma 2.1. In the next definition we extend the usual concept of concatenation points to allow points at infinity.

**Definition 2.2.** Given any marked graph $K$ and oriented paths $\gamma_1, \gamma_2 \in \hat{\mathcal{B}}(K)$, we that $\gamma_1, \gamma_2$ are *concatenable* if there exist lifts $\bar{\gamma}_1, \bar{\gamma}_2 \subset \hat{K}$ with initial endpoints $P_n^{-} \in \bar{K} \cup \partial F_n$ and terminal endpoints $P_n^{+} \in \bar{K} \cup \partial F_n$ satisfying $P_n^{+} = P_2^{+}$ and $P_n^{-} \neq P_2^{+}$. The concatenation of $\bar{\gamma}_1 \cup \bar{\gamma}_2$ is the oriented path with endpoints $P_1^{-}, P_2^{+}$, denoted $\bar{\gamma}_1 \circ \bar{\gamma}_2$. Its projection to $K$, denoted $\gamma_1 \circ \gamma_2$, is called a *concatenation of $\gamma_1, \gamma_2$*. This operation is clearly associative and so we can define multiple concatenations. This operation is also invertible, in particular any concatenation of the form $\gamma = \alpha \circ \nu \circ \beta$ can be rewritten as $\nu = \alpha^{-1} \circ \gamma \circ \beta^{-1}$.

Notice that “the” upstairs concatenation $\hat{\gamma}_1 \circ \hat{\gamma}_2$ is well-defined, but “the” downstairs concatenation $\gamma_1 \circ \gamma_2$ is not generally well-defined: this fails precisely when $P_1^{+} = P_2^{+}$ is an endpoint of the axis of some element $\gamma$ of $F_n$ and neither $P_1^{-}$ nor $P_2^{+}$ is the opposite endpoint, in which case one can replace either of $\hat{\gamma}_1, \hat{\gamma}_2$ by a translate under $\gamma$ to get a different concatenation downstairs. This is a mild failure, however, and it is usually safe to ignore.

A subset of $\hat{\mathcal{B}}(K)$ is *closed under concatenation* if for any oriented paths $\gamma_1, \gamma_2 \in \hat{\mathcal{B}}(K)$, any of their concatenations $\gamma_1 \circ \gamma_2$ is an element of $\hat{\mathcal{B}}(K)$.

**Lemma 2.3.** Continuing with the Notational Convention above, the set of elements of $\hat{\mathcal{B}}(G)$ that are not weakly attracted to $\Lambda_\phi^+$ under iteration by $\phi$ is closed under concatenation. In particular, $\mathcal{B}_{\text{na}}(\Lambda_\phi^+)$ is closed under concatenation.

**Proof.** Consider a concatenation $\gamma_1 \circ \gamma_2$ with accompanying notation as in Definition 2.2. For each $m \geq 0$, the path $f_m^\#(\gamma_1 \circ \gamma_2)$ is the concatenation of a subpath of $f_m^\#(\gamma_1)$ and a subpath of $f_m^\#(\gamma_2)$. Letting $\ell$ be a generic leaf of $\Lambda_\phi^+$, by Fact I.1.57 we may write $\ell$ as an increasing union of nested tiles $\alpha_1 \subset \alpha_2 \subset \cdots$ so that each $\alpha_j$ contains at least two disjoint copies of $\alpha_{j-1}$. By assumption $\gamma_1$ has the property that there exists an integer $J$ so that if $\alpha_j$ occurs in $f_m^\#(\gamma_1)$ for arbitrarily large $m$ then $j \leq J$, and $\gamma_2$ satisfies the same property. This property is therefore also satisfied by $\gamma_1 \circ \gamma_2$ (with a possibly larger bound $J$) and so $\gamma_1 \circ \gamma_2$ is not weakly attracted to $\Lambda_\phi^+$. \hfill \square

**Example 2.4.** The set of lines satisfying (1), (2) and (3) of Lemma 2.1 is generally not closed under concatenation. For example, suppose that for $i = 1, 2$ we have singular lines for $\psi$ of the form $\gamma_i = \alpha_i' \beta_i' \subset G'$ where $\alpha_i' \subset G'_u$ is a principal ray of $\Lambda_\psi^+$ (Definition I.1.50) and $\beta_i' \subset G'_{u-1}$. Let $\mu' \subset G'_{u-1}$ be any line that is asymptotic in the backward direction to $\beta_i'$ and in the forward direction to $\beta_2'$. Then $\mu'$ is carried by $\mathcal{A}_\text{na}(\Lambda_\psi^+) = 2$ and $\gamma_3' = \gamma_1' \circ \mu' \circ \gamma_2'$ is not weakly attracted to $\Lambda_\phi^+$. However, $\gamma_3'$ does not in general satisfy any of (1), (2) and (3) of Lemma 2.1.

We account for these kinds of examples as follows. (See also Propositions 2.18 and 2.19.)

**Definition 2.5.** Given a subgroup $A < F_n$ such that $[A] \in \mathcal{A}_{\text{na}}(\Lambda_\phi^+)$ and given $\Psi \in P(\psi)$, we say that $\Psi$ is $A$-related if $\text{Fix}_N(\hat{\Psi}) \cap \partial A \neq \emptyset$. Define the *extended boundary* of $A$ to be

$$
\partial_{\text{ext}}(A, \psi) = \partial A \cup \left( \bigcup_{\Psi} \text{Fix}_N(\hat{\Psi}) \right)
$$
where the union is taken over all $A$-related $\Psi \in P(\psi)$. Let $B_{\text{ext}}(A, \psi)$ denote the set of lines that have lifts with endpoints in $\partial_{\text{ext}}(A, \psi)$; this set is independent of the choice of $A$ in its conjugacy class. Define

$$B_{\text{ext}}(\Lambda_\phi^\pm; \psi) = \bigcup_{A \in \mathcal{A}_{na}(\Lambda_\phi^\pm)} B_{\text{ext}}(A, \psi)$$

Basic properties of $B_{\text{ext}}(\Lambda_\phi^\pm; \psi)$ are established in the next section.

We conclude this section with the statement of our main weak attraction result. The proof is given in Section 2.5.

**Theorem 2.6 (Theorem G).** If $\phi, \psi = \phi^{-1} \in \text{Out}(F_n)$ are rotationless and $\Lambda_\phi^\pm \in \mathcal{L}^\pm(\phi)$ then

$$B_{na}(\Lambda_\phi^+, \phi) = B_{\text{ext}}(\Lambda_\phi^+, \psi) \cup B_{\text{sing}}(\psi) \cup B_{\text{gen}}(\psi)$$

**Remark 2.7.** The sets $B_{\text{ext}}(\Lambda_\phi^+, \psi)$, $B_{\text{sing}}(\psi)$, and $B_{\text{gen}}(\psi)$ need not be pairwise disjoint. For example, every line carried by $G_{u-1}$ is in $B_{\text{ext}}(\Lambda_\phi^+, \psi)$ and some of these can be in $B_{\text{sing}}(\psi)$ or in $B_{\text{gen}}(\psi)$.

**Remark 2.8.** It is not hard to show that if $\gamma \in B_{na}(\Lambda_\phi^+)$ is birecurrent then $\gamma$ is either carried by $A_{na}(\Lambda_\phi^\pm)$ or is a generic leaf of some element of $\mathcal{L}(\psi)$. This shows that Theorem 2.6 contains the Weak Attraction Theorem (Theorem 6.0.1 of [BFH00]) as a special case.

### 2.2 $B_{\text{ext}}(\Lambda_\phi^\pm; \psi) \cup B_{\text{sing}}(\psi) \cup B_{\text{gen}}(\psi)$ is closed under concatenation

We continue with the notation for an inverse pair of rotationless outer automorphisms $\phi, \psi = \phi^{-1} \in \text{Out}(F_n)$ established at the beginning of Section 2.1.

Much of the work in this section is devoted to revealing details of the structure of $B_{\text{ext}}(\Lambda_\phi^\pm; \psi)$. After a few such lemmas/corollaries, the main result of this section is that the union of the three subsets of $B_{na}(\Lambda_\phi^\pm)$ occurring in Theorem 2.6 is closed under concatenation; see Proposition 2.14.

We shall abuse notation for elements of the set $\mathcal{A}_{na}(\Lambda_\phi^\pm)$ as described in Section I.1.1.2, writing $A \in \mathcal{A}_{na}(\Lambda_\phi^\pm)$ to mean that $A$ is a subgroup of $F_n$ whose conjugacy class $[A]$ is an element of the set $\mathcal{A}_{na}(\Lambda_\phi^\pm)$. Since $\mathcal{A}_{na}(\Lambda_\phi^\pm)$ is a malnormal subgroup system (Proposition 1.4), this notational abuse should not cause any confusion.

**Lemma 2.9.** If $\Psi_1 \neq \Psi_2 \in \text{Aut}(F_n)$ are representatives of $\psi$ and $P \in \text{Fix}(\hat{\Psi}_1) \cap \text{Fix}(\hat{\Psi}_2)$ then there exists a nontrivial $a \in \text{Fix}(\Psi_1) \cap \text{Fix}(\Psi_2)$ determining an inner automorphism $i_a$, and there exists $A \in \mathcal{A}_{na}(\Lambda_\phi^\pm)$, such that $a \in A$ and $P \in \text{Fix}(i_a) \subset \partial A$.

**Proof.** Choosing $a$ so that $\Psi_1 = i_a \Psi_2$ it follows that $i_a = \Psi_1 \Psi_2^{-1}$ fixes $P$, and so $a \in \text{Fix}(\Psi_1) \cap \text{Fix}(\Psi_2)$ (Fact I.1.19), implying that $[a]$ is not weakly attracted to $\Lambda_\phi^+$. Applying Corollary 1.7, the conjugacy class $[a]$ is carried by $\mathcal{A}_{na}(\Lambda_\phi^\pm)$, and so there exists $A \in \mathcal{A}_{na}(\Lambda_\phi^\pm)$ such that $a \in A$, which implies that $\text{Fix}(i_a) \subset \partial A$.

**Lemma 2.10.** For each $\Psi \in P(\psi)$ there exists at most one $A \in \mathcal{A}_{na}(\Lambda_\phi^\pm)$ such that $\Psi$ is $A$-related.
Proof. Suppose that for \( j = 1, 2 \) there exist \( A_j \in A_{na}(\Lambda^+_\phi) \) and \( P_j \in \text{Fix}_N(\hat{\psi}) \cap \partial A_j \). The line \( \hat{\gamma} \) connecting \( P_1 \) to \( P_2 \) projects to a line \( \gamma \in B_{\text{sing}}(\psi) \) that by Lemma 2.1 is not weakly attracted to \( A^+ \). Since \( P_j \in \partial A_j \), and since by Lemma 1.5 (3) each line that is carried by \( A_j \) is contained in \( (Z, \rho) \), the ends of \( \gamma \) are contained in \( (Z, \hat{\rho}) \), and so we may assume that \( \gamma = \bar{p} \circ \gamma_0 \circ \rho_+ \) where the rays \( \rho_-, \rho_+ \) are in \( (Z, \hat{\rho}) \). After replacing \( \gamma \) with a \( \phi_{\#} \)-iterate we may also assume that the central subpath \( \gamma_0 \) has endpoints at fixed vertices. Since none of \( \gamma, \rho_- \), \( \rho_+ \) are weakly attracted to \( A^+_\phi \), by Lemma 2.3 neither is \( \gamma_0 = \rho_- \circ \gamma \circ \bar{p}_+ \).

Lemma 1.6 (4) implies that \( \gamma_0 \) is contained in \( (Z, \hat{\rho}) \) and Lemma 1.5 (2) then shows that \( \gamma \) is contained in \( (Z, \hat{\rho}) \). By Lemma 1.5 (3) it follows that \( \gamma \) is carried by \( A_{na}(\Lambda^+_\phi) \), which means that \( \partial \tilde{\gamma} = \{ P_1, P_2 \} \subset \partial A_3 \) for some \( A_3 \in A_{na}(\Lambda^+_\phi) \). By Proposition 1.4 (3) and Fact 1.1.2 it follows that \( A_1 = A_3 = A_2 \).

Corollary 2.11. If \( \Psi \in P(\psi), A \in A_{na}(\Lambda^+_\phi), \) and \( \Psi \) is \( A \)-related, then \( \text{Fix}(\Psi) < A \) and each point of \( \text{Fix}_N(\hat{\psi}) \setminus \partial A \) is an isolated attractor for \( \hat{\psi} \).

Proof. If \( \text{Fix}(\Psi) \) is trivial then by Lemma 1.1.20 each point of \( \text{Fix}_N(\hat{\psi}) \) is an isolated attractor and we are done. Otherwise, noting that the conjugacy class of each nontrivial element of \( \text{Fix}(\Psi) \) is carried by \( A_{na}(\Lambda^+_\phi) \), applying Corollary 1.8 we have \( \text{Fix}(\Psi) < A' \) for some \( A' \in A_{na}(\Lambda^+_\phi) \), and so \( \partial \text{Fix}(\Psi) \subset \partial A' \). It follows that \( \Psi \) is \( A' \)-related. By Lemma 2.10 we have \( A' = A \), and applying Lemma 1.1.20 completes the proof.

Corollary 2.12. If \( A_1 \neq A_2 \in A_{na}(\Lambda^+_\phi) \) then \( \partial \text{ext}(A_1, \psi) \cap \partial \text{ext}(A_2, \psi) = \emptyset \).

Proof. We assume that \( Q \in \partial \text{ext}(A_1, \psi) \cap \partial \text{ext}(A_2, \psi) \) and argue to a contradiction. After interchanging \( A_1 \) and \( A_2 \) if necessary, we may assume by Proposition 1.4 (3) and Fact 1.1.2 that \( Q \not\in \partial A_1 \) and hence that \( Q \in \text{Fix}_N(\hat{\psi}) \) for some \( A_1 \)-related \( \Psi_1 \in P(\psi) \). Lemma 2.10 implies that \( \Psi_1 \) is not \( A_2 \)-related and so \( Q \not\in \partial A_2 \). The only remaining possibility is that \( Q \in \text{Fix}_N(\hat{\psi}_2) \) for some \( A_2 \)-related \( \Psi_2 \in P(\psi) \). But then Lemma 2.9 implies that \( Q \in \partial A_3 \) for some \( A_3 \in A_{na}(\Lambda^+_\phi) \), and then Lemma 2.10 implies that \( A_1 = A_3 = A_2 \).

Corollary 2.13. If \( \Psi \in P(\psi), A \in A_{na}(\Lambda^+_\phi), \) and \( \text{Fix}_N(\hat{\psi}) \cap \partial \text{ext}(A, \psi) \neq \emptyset \) then \( \Psi \) is \( A \)-related; in particular, \( \text{Fix}_N(\hat{\psi}) \subset \partial \text{ext}(A, \psi) \).

Proof. Choose \( Q \in \text{Fix}_N(\hat{\psi}) \cap \partial \text{ext}(A, \psi) \). If \( Q \in \partial A \) we’re done so we may assume that \( Q \in \text{Fix}_N(\hat{\psi}) \) for some \( A \)-related \( \Psi' \). If \( \Psi = \Psi' \) we are done. Otherwise, \( \partial \text{Fix}(\hat{\psi}) \) implies that \( Q \in \partial A' \) for some \( A' \in A_{na}(\Lambda^+_\phi) \) and Corollary 2.12 implies that \( A' = A \) so again we are done.

Proposition 2.14. If the oriented lines \( \gamma_1, \gamma_2 \) are in the set \( B_{ext}(\Lambda^+_\phi; \psi) \cup B_{\text{sing}}(\psi) \cup B_{\text{gen}}(\psi) \) and are concatable then any concatenation \( \gamma_1 \circ \gamma_2 \) is also in that set.

More precisely, given lifts \( \tilde{\gamma}_j \) with initial and terminal endpoints \( P_j^- \) and \( P_j^+ \) respectively, if \( P_j^+ = P_j^- \) and \( P_j^- \neq P_j^+ \) then either there exists \( \Psi \in P(\psi) \) such that the three points \( P_j^-, P_j^+, P_j^- \) are in \( \text{Fix}_N(\hat{\psi}) \) or there exists \( A \in A_{na}(\Lambda^+_\phi) \) such that those three points are in \( \partial \text{ext}(A, \psi) \).
Proof. The first sentence is an immediate consequence of the second, to whose proof we now turn.

Case 1: $\gamma_1 \in B_{\text{sing}}(\psi)$. We have $P_1^-, P_1^+ \in \text{Fix}_N(\hat{\Psi})$ for some $\Psi \in P(\psi)$. There are three subcases. First, if $P_2^-, P_2^+ \in \partial_{\text{ext}}(A, \psi)$ for some $A \in A_{\text{na}}(\Lambda^-_\phi)$ then $P_1^-, P_1^+ \in \partial_{\text{ext}}(A, \psi)$ by Corollary 2.13 and we are done. The second subcase is that $P_2^-, P_2^+ \in \text{Fix}_N(\hat{\Psi}')$ for some $\Psi' \in P(\psi)$; if $\Psi' = \Psi$ then we are done; if $\Psi' \neq \Psi$ then $P_1^+ = P_2^- \in \partial_{\text{ext}}(A, \psi)$ for some $A \in A_{\text{na}}(\Lambda^+_\phi)$ by Lemma 2.9, and so $P_2^-, P_2^+ \in \partial_{\text{ext}}(A, \psi)$ by Corollary 2.13 so we are reduced to the first subcase. The final subcase is that $\gamma_2$ is a generic line of some $\Lambda^-_\gamma \in \mathcal{L}(\psi)$. Assuming without loss that $P_2^+ \notin \text{Fix}_N(\hat{\Psi})$, the projection of $\Psi_\#(\hat{\gamma}_2)$ is a generic leaf of $\Lambda^-_\gamma$ that is asymptotic to $\gamma_2$ but not equal to $\gamma_2$. The next lemma says that this puts us in the second subcase and so we are done.

Lemma 2.15. Assume that $\theta \in \text{Out}(F_n)$ is rotationless. If $\ell', \ell''$ are each generic lines of elements of $\mathcal{L}(\theta)$, and if some end of $\ell'$ is asymptotic to some end of $\ell''$, then $\ell', \ell'' \in B_{\text{sing}}(\psi)$.

This lemma extends Lemma 3.3 of [HM11] in which it is assumed that $\phi$ is irreducible.

Putting off the proof of Lemma 2.15 for a bit, we continue with the proof of Proposition 2.14. Having finished Case 1, by symmetry we may now assume that $\gamma_j \notin B_{\text{sing}}(\psi)$ for $j = 1, 2$.

Case 2: $P_1^-, P_1^+ \in \partial_{\text{ext}}(A, \psi)$ for some $A \in A_{\text{na}}(\Lambda^+_\phi)$. If $P_2^+ \in \partial_{\text{ext}}(A, \psi)$ we are done. By Corollary 2.12, the only remaining possibility is that $\gamma_2$ is a generic leaf of an element of $\mathcal{L}(\psi)$. If $P_1^+ \in \text{Fix}_N(\hat{\Psi})$ for some $\Psi \in P(\psi)$ then, as shown above, Lemma 2.15 implies that $P_2^+ \in \text{Fix}_N(\hat{\Psi})$ and hence that $\gamma_2 \in B_{\text{sing}}(\psi)$ which is a contradiction. Thus $P_1^+ \in \partial A$. Since $\gamma_2$ is birecurrent, Fact I.1.8 implies that $\gamma_2$ is carried by $A_{\text{na}}(\Lambda^+_\phi)$.

Applying Lemma 1.5 (6) it follows that $P_2^+ \in \partial A$ and we are done.

By symmetry of $\gamma_1$ and $\gamma_2$, the only remaining case is:

Case 3: $\gamma_1$ and $\gamma_2$ are generic leaves of elements of $\mathcal{L}(\psi)$. Since they have asymptotic ends, they are leaves of the same element of $L(\psi)$ so are singular lines by Lemma 2.15. As we have already considered this case, the proof is complete.

It remains to prove Lemma 2.15, but we first prove the following, which is similar to Lemma 5.11 of [BH92], Lemma 4.2.6 of [BFH00] and Lemma 2.7 of [HM11].

Lemma 2.16. Suppose that $f : G \to G$ is a CT, that $H_\tau$ is an EG stratum, that $\gamma \subset G$ is line of height $r$ with exactly one illegal turn of height $r$ and that $f^{K_\gamma}(\gamma)$ is $r$-legal for some minimal $K_\gamma$. Then $K_\gamma \leq K$ for some $K$ that is independent of $\gamma$.

Proof. If the lemma fails there exists a sequence $\gamma_i$ such that $K_i = K_{\gamma_i} \to \infty$. Write $\gamma_i = \sigma_i \tau_i$ where the turn $(\sigma_i, \tau_i)$ is the illegal turn of height $r$. After passing to a subsequence we may assume that $\sigma_i \to \sigma$ and $\tau_i \to \tau$ for some rays $\sigma$ and $\tau$. The line $\gamma = \sigma \tau$ has height $r$ and $f^k(\gamma)$ has exactly one illegal turn of height $r$ for all $k \geq 0$. Lemma 4.2.6 of [BFH00] implies that there exists $m > 0$ and a splitting $f^m(\gamma) = \hat{R} - \rho \cdot \hat{R}^+$ where $\rho$ is the unique indivisible Nielsen path of height $r$. It follows that for all sufficiently large $i$, $f^i(\gamma_i)$ has a decomposition into subpaths $f^i(\gamma_i) = \hat{R} - \rho \hat{R}^+$ where the height $r$ illegal turn in $\rho$ is the only height $r$ illegal turn in $\gamma_i$. Since any such decomposition is a splitting, $f^i(\gamma_i)$ has an illegal turn of height $r$ for all $k$ in contradiction to our choice of $\gamma_i$. 

\end{proof}
Proof of Lemma 2.15. By symmetry we need prove only that $\ell' \in B_{\text{sing}}(\theta)$. By Fact I.1.61, each end of each generic leaf of an element of $\mathcal{L}(\theta)$ has the same free factor support as the whole leaf, and so $\ell'$ and $\ell''$ must be generic leaves of the same $\Lambda \in \mathcal{L}(\theta)$.

Let $f : G \to G$ be a CT representing $\theta$ and let $H_r$ be the EG stratum corresponding to $\Lambda$. For each $j \geq 0$, there are generic leaves $\ell_j'$ and $\ell_j''$ of $\Lambda$ such that $f_j^\#(\ell_j') = \ell'$ and $f_j^\#(\ell_j'') = \ell''$. Fixing a common end of $\ell'$ and $\ell''$, the corresponding common ends of $\ell_j'$ and $\ell_j''$ determine a maximal common subray $R_j$ of $\ell_j'$ and $\ell_j''$. Denote the rays in $\ell_j'$ and $\ell_j''$ that are complementary to $R_j$ by $R_j'^\ell$ and $R_j''$ respectively. Let $\gamma_j = R_j'^\ell R_j''$. Suppose at first that each $\gamma_j$ is $r$-legal. Lemma 5.8 of [BH92] implies that no height $r$ edges of $\gamma_j$ are cancelled when $f_j^\#(\gamma_j)$ is tightened to $\gamma_0$. Let $E_j$, $E_j'$ and $E_j''$ be the first height $r$ edges of $R_j, R_j'$ and $R_j''$ respectively, let $w_j, w_j'$ and $w_j''$ be their initial vertices and let $d_j, d_j'$ and $d_j''$ be their initial directions. Let $\mu_j'$ be the finite subpath of $\ell_j'$ connecting $w_j'$ to $w_j$. To complete the proof in this case we will show that $\mu_0'$ is a Nielsen path, that $R_0$ is the principal ray determined by iterating $d_0$ (see Definition I.1.50) and that $R_0'$ is the principal ray determined by iterating $d_0'$. If $w_j = w_j'$ then $d_j, d_j'$ determine distinct gates, and otherwise $w_j, w_j'$ are each incident to an edge of height $< r$. A similar statement holds for $w_j, w_j''$. In all cases it follows that $w_j, w_j'$ and $w_j''$ are principal vertices of $f$. Moreover, the following hold for all $i, j \geq 0$:

$$f^i(w_{j+i}) = w_j \quad f^i(w'_{j+i}) = w'_j \quad f^i(d_{j+i}) = d_j \quad f^i(d'_{j+i}) = d'_j \quad f^i(\mu_{j+i}') = \mu_j'$$

The first two of these equalities imply that $w = w_j, w' = w_j' \in \text{Fix}(f)$ are independent of $j$; the third and fourth imply that $E = E_j$ and $E' = E_j'$ are independent of $j$; in conjunction with Lemma I.1.57 (2), the last equality implies that $\mu = \mu_j'$ is a Nielsen path that is independent of $j$. It follows that $\ell'$ is the increasing union of the subpaths $f_\#(E')\mu f_\#(E)$ and so $\ell'$ is a pair of principal rays connected by a Nielsen path. Applying Fact I.1.47 completes the proof that $\ell' \in B_{\text{sing}}(\theta)$ when each $\gamma_j$ is $r$-legal.

It remains to consider the case that some $\gamma_j$ is not $r$-legal. Assuming without loss that $\gamma_0$ is not $r$-legal, each $\gamma_j$ is not $r$-legal. Lemma 2.16 implies that $f_j^k(\gamma_j)$ has an illegal turn of height $r$ for all $k \geq 0$ and Lemma 4.2.6 of [BFH00] implies that there is a splitting $\gamma_j = \tau_j' \cdot \rho_j \cdot \tau_j''$ where some $f_j^\#$-iterate of $\rho_j$ is the unique indivisible Nielsen path $\rho$ with height $r$. Since $f_j^i(\rho_{j+i}) = \rho_j$ for all $i, j \geq 0$, $\rho_j = \rho$ for all $j$. Let $E'$ be the first edge of height $r$ in the ray $\tau_0$ and let $E$ be the initial edge of $\rho_0$. Both of these edges are contained in $\ell'$ and we let $\mu$ be the subpath of $\ell'$ that connects their initial vertices. Arguing as in the previous case, $\mu$ is a Nielsen path and $\ell'$ is the increasing union of the subpaths $f_\#(E')\mu f_\#(E)$ which proves that $\ell' \in B_{\text{sing}}(\theta)$.

\[\square\]

2.3 Application — Proof of Theorem H

Before turning in later sections to the proof of Theorem 2.6 (Theorem G), we use it to prove the following:

Corollary 2.17 (Theorem H). Given rotationless $\phi, \psi = \phi^{-1} \in \text{Out}(F_n)$, a dual laminating pair $\Lambda_\phi^\pm \in \mathcal{L}^\pm(\phi)$, and a line $\gamma \in \mathcal{B}$, the following hold:

1. If $\gamma$ is not carried by $\mathcal{A}_{\phi}(\Lambda_\phi^\pm)$ then it is either weakly attracted to $\Lambda_\phi^+$ under iteration by $\phi$ or to $\Lambda_\phi^-$ under iteration by $\psi$.

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(2) For any weak neighborhoods $V^+$ and $V^-$ of generic leaves of $\Lambda_\phi^+$ and $\Lambda_\phi^-$, respectively, there exists an integer $m \geq 1$ (independent of $\gamma$) such that at least one of the following holds: $\gamma \in V^-; \phi^m(\gamma) \in V^+$; or $\gamma$ is carried by $A_{na}(\Lambda_\phi^\pm)$.

Proof. For (1) we assume that $\gamma$ is not weakly attracted to $\Lambda_\phi^+$ under iteration of $\phi$ and that $\gamma$ is not weakly attracted to $\Lambda_\psi^\pm = \Lambda_\phi^\pm$ under iteration of $\psi = \phi^{-1}$, and we prove that $\gamma$ is carried by $A_{na}(\Lambda_\phi^\pm)$. Applying Theorem 2.6 to both $\psi$ and $\phi$, we have

$$\gamma \in B_{na}(\Lambda_\phi^+, \phi) \cap B_{na}(\Lambda_\psi^+, \psi)$$

$$= \left( B_{ext}(\Lambda_\phi^+, \psi) \cup B_{sing}(\psi) \cup B_{gen}(\psi) \right) \cap \left( B_{ext}(\Lambda_\phi^-, \phi) \cup B_{sing}(\phi) \cup B_{gen}(\phi) \right) \quad (*)$$

and using this we proceed by cases.

**Case 1:** $\gamma \in B_{gen}(\phi) \cup B_{gen}(\psi).$ By symmetry we may assume $\gamma \in B_{gen}(\phi)$ and so $\gamma$ is a generic leaf of some $\Lambda_t^+ \in L(\phi)$. Since $\gamma \in B_{na}(\Lambda_\phi^+, \phi)$ it follows that $\Lambda_\phi^+ \nsubseteq \Lambda_t^+$, and so the stratum $H_t$ associated to $\Lambda_t^+$ is contained in $Z$, by Fact I.1.58 and the definition of $Z$. Since $(Z, \tilde{\rho}_t)$ is $f_{g_t}$-invariant (Lemma 1.6(2)) and the set of lines that it carries is closed (Lemma 1.5(5)), $(Z, \tilde{\rho}_t)$ carries $\Lambda_t^+$ by (Fact I.1.58). Lemma 1.5(4) then implies that $A_{na}(\Lambda_\phi^\pm)$ carries $\Lambda_t^+$ and hence $\gamma$.

Having settled Case 1 we may assume that

$$\gamma \in \left( B_{ext}(\Lambda_\phi^+, \psi) \cup B_{sing}(\psi) \right) \cap \left( B_{ext}(\Lambda_\phi^-, \phi) \cup B_{sing}(\phi) \right)$$

**Case 2:** $\gamma \in B_{ext}(\Lambda_\phi^+; \psi) \cup B_{ext}(\Lambda_\phi^-; \phi)$. By symmetry we may assume $\gamma \in B_{ext}(\Lambda_\phi^+; \psi)$, so there is a lift $\tilde{\gamma}$ with endpoints $P, Q \in \partial_{ext}(A, \psi)$ for some $A \in A_{na}(\Lambda_\phi^\pm)$.

**Case 2a:** At least one of $P$ or $Q$ is not in $\partial A$, say $P \not\in \partial A$. It follows that $P \in \text{Fix}_N(\tilde{\Psi}) \setminus \partial A$ for some $A$-related $\Psi \in P(\psi)$. Applying Corollary 2.11 it follows that $P$ is an isolated attracting point of $\text{Fix}_N(\tilde{\Psi})$. Since $P \in \partial_{ext}(A, \psi)$, by Corollary 2.12 and Lemma 2.9 it follows that $\phi = \Psi^{-1}$ is the only automorphism representing $\phi$ with $P \in \text{Fix}(\tilde{\Phi}) = \text{Fix}(\tilde{\Psi})$. Since $P$ is a repeller for the action of $\tilde{\Phi}$, we have $P \not\in \text{Fix}_N(\tilde{\Phi})$, and so $\gamma \not\in B_{sing}(\phi)$ and $\gamma \not\in B_{ext}(\Lambda_\phi^+; \phi)$. By (*) it follows that $\gamma \in B_{gen}(\phi)$, reducing to Case 1.

**Case 2b:** $P, Q \in \partial A$, and so $\gamma$ is carried by $A_{na}(\Lambda_\phi^\pm)$ and we are done.

Having settled Cases 1 and 2, we are reduced to the following:

**Case 3:** $\gamma \in B_{sing}(\phi) \cup B_{sing}(\psi)$. By symmetry we may assume $\gamma \in B_{sing}(\phi)$, and so there exists $\Phi \in P(\phi)$ and a lift $\tilde{\gamma}$ with endpoints $P, Q \in \text{Fix}_N(\tilde{\Phi})$.

**Case 3a:** $\text{Fix}(\Phi)$ is nontrivial. By Corollary 2.11 there exists $A \in A_{na}(\Lambda_\phi^\pm)$ such that $\text{Fix}(\Phi) < A$, and so $\Phi$ is $A$-related and $\gamma \in B_{ext}(A, \phi) \subset B_{ext}(\Lambda_\phi^\pm; \phi)$, reducing to Case 2.

**Case 3b:** $\text{Fix}(\Phi)$ is trivial. It follows that $P, Q$ are isolated attractors in $\text{Fix}_N(\tilde{\Phi})$. Lemma 2.9 combined with the assumption of Case 3b implies that $\Psi = \Phi^{-1}$ is the only automorphism representing $\psi$ with $P \in \text{Fix}(\tilde{\Psi})$. As in Case 2a, using (*) we conclude that $\gamma \in B_{gen}(\psi)$, reducing to Case 1.

This completes the proof of (1).
We prove (2) by contradiction. If (2) fails then there are neighborhoods $V^+, V^-$ of generic leaves of $\Lambda^+, \Lambda^-\,$ respectively, a sequence of lines $\gamma_i \in B$ and a sequence of positive integers $m_i \to \infty$, such that for all $i$ we have: $\gamma_i \notin V^-; \phi^{2m_i}(\gamma_i) \notin V^+$; and $\gamma_i$ is not carried by $A_{na}(\Lambda^\pm)$. We may assume that $V_+\,$ has the property $\phi(V_+) \subset V_+$, because generic leaves of $\Lambda^+$ have a neighborhood basis of such sets. Similarly, we may assume that $V_- \subset \phi(V_-)$. By Lemma 1.11, some subsequence of $\phi^{m_i}(\gamma_i)$ has a weak limit $\gamma$ that is not carried by $A_{na}(\Lambda^\pm)$.

To contradict (1) we show that $\gamma$ is not weakly attracted to $\Lambda^+_\phi$ or to $\Lambda^-\phi$. By symmetry we need show only the first, namely, that the sequence $\phi^m(\gamma)$ does not weakly converge to $\Lambda^+_\phi$. If it does then $\phi^m(\gamma) \in V^+$ for some $M$. Since $V^+$ is open there exists $I$ such that $\phi^{m_i+M}(\gamma_i) \in V^+$ for all $i \geq I$. Since $\phi(V^+) \subset V^+$, it follows that $\phi^m(\gamma_i) \in V^+$ for all $m \geq m_i + M$ and $i \geq I$. We can choose $i \geq I$ so that $m_i \geq M$, and it follows that $\phi^{2m_i}(\gamma_i) \in V^+$, a contradiction. \qed

2.4 Nonattracted lines of EG height.

We continue the Notational Conventions established at the beginning of Section 2.1. By combining Proposition I.2.18 with the following equations of free factor systems

$$[G_r] = \mathcal{F}_{\text{supp}}(\Lambda^\pm_\phi) = [G'_u], \quad [G_{r-1}] = [G'_{u-1}]$$

we may conclude that the stratum $H_u$ is geometric if and only if the stratum $H'_u$ is geometric. The realizations of a line $\gamma \in B$ in the marked graphs $G, G'$ will be denoted $\gamma_G, \gamma_{G'}$ respectively, or just as $\gamma, \gamma'$ when we wish to abbreviate the notation, or even both just as $\gamma$ when we wish for further abbreviation.

In this section we focus on the special case of case of Theorem 2.6 (Theorem G) concerned with those lines $\gamma$ such that $\gamma_G$ has height $r$, equivalently $\gamma_{G'}$ has height $u$. We give necessary and sufficient conditions for $\gamma$ to be weakly attracted to $\Lambda^+_\phi$ under iteration of $\phi$, expressed in terms of the form of $\gamma_{G'}$. This is the analog of Proposition 6.0.8 of [BFH00] which has the additional hypothesis that $\gamma$ is birecurrent, and the proof of which is separated into geometric and non-geometric cases. In our present setting we drop the birecurrence hypothesis, and we also separate the proof into the non-geometric case in Lemma 2.18 and the geometric case in Lemma 2.19. The conclusions of two lemmas describe $\gamma_{G'}$ in explicit detail which, while more than we need for our applications, is included because it is needed for the proof and it helps clarify the picture. Although in this section we do not yet derive the conclusions of Theorem 2.6 (Theorem G) for the height $r$ case, that will be done as part of the general derivation of those conclusions in Section 2.5.

**Lemma 2.18** (Height $r$ lines in the nongeometric case). Assuming that the strata $H_r, H'_u$ are not geometric, and with the notation above, if $\gamma \in B$ has realization $\gamma_G$ in $G$ of height $r$ and $\gamma_{G'}$ in $G'$ of height $u$, and if $\gamma$ is not weakly attracted to $\Lambda^+_\phi$, then its realization $\gamma_{G'}$, satisfies at least one of the following:

1. $\gamma_{G'}$ is a generic leaf of $\Lambda^-_\phi$.

2. $\gamma_{G'}$ decomposes as $R_1\mu R_2$ where $R_1$ and $R_2$ are principal rays for $\Lambda^-_\phi$ and $\mu$ is either the trivial path or a nontrivial path of one of the forms $\alpha, \beta, \alpha\beta, \alpha\beta\alpha$, such that $\beta$ is a nontrivial path of height $< s$ and $\alpha$ is a height $s$ indivisible Nielsen path.
(3) \(\gamma_{G^r} \text{ or } \gamma_{G^r}^{-1}\) decomposes as \(R_1^1 \mu R_2\) where \(R_1\) is a principal ray for \(\Lambda^-\), \(R_2\) is a ray of height \(< s\), and \(\mu\) is either trivial or a height \(s\) Nielsen path.

**Proof.** By Proposition I.2.18, since \(H_r\) is not a geometric stratum, neither is \(H'_u\). Let \(\alpha\) denote the unique (up to reversal) indivisible Nielsen path of height \(s\) in \(G'_u\), if it exists; by Fact I.2.3 and Fact I.1.42 (1) it follows that \(\alpha\) is not closed and that we may orient \(\alpha\) so that its initial endpoint \(v\) is an interior point of \(H'_u\).

We adopt the abbreviated notation \(\gamma\) for \(\gamma_G\) and \(\gamma'\) for \(\gamma_{G^r}\).

We first show that \(\gamma\) contains infinitely many edges in \(H_r\) by proving that if a line \(\gamma\) has height \(r\) and only finitely many edges in \(H_r\) then \(\gamma\) is weakly attracted to \(\Lambda^+\). To prove this, write \(\gamma = \gamma_0 \gamma_+\) where \(\gamma_-, \gamma_+ \subset G_{r-1}\) and \(\gamma_0\) is a finite path whose first and last edges are in \(H_r\). Since \(f_\#\) restricts to a bijection on lines of height \(r-1\), it follows that \(f_\#^k(\gamma)\) has height \(r\) for all \(k\), and so \(f_\#^k(\gamma_0)\) is a nontrivial path of height \(r\) for all \(k\). By Fact I.1.35 there exists \(K \geq 0\) such that \(f_\#^K(\gamma_0)\) completely splits into terms each of which is either an edge or Nielsen path of height \(r\) or a path in \(G_{r-1}\), with at least one term of height \(r\). Let \(f_\#^K(\gamma_0) = \gamma'_+, \gamma'_0, \gamma'_-\) where \(\gamma'_-, \gamma'_+\) are in \(G_{r-1}\) and \(\gamma'_0\) is the maximal subpath of \(f_\#^K(\gamma_0)\) whose first and last edges are in \(H_r\), so \(\gamma'_0\) is nontrivial and completely split. Since \(f_\#^K(\gamma) = [f_\#^K(\gamma_0) \cdot \gamma'_0 \cdot \gamma'_+]'\) then, using RTT-(i), it follows that there is a splitting \(f_\#^K(\gamma) = \gamma'_0 \cdot \gamma'_+\) where \(\gamma'_0 = [f_\#^K(\gamma_0) \cdot \gamma'_-]\) and \(\gamma'_+ = [\gamma'_+ f_\#^K(\gamma_0)]\) are in \(G_{r-1}\). By Fact I.1.42 (1), each term in this splitting of \(f_\#^K(\gamma)\) which is an indivisible Nielsen path of height \(r\) is adjacent to a term that is an edge in \(H_r\). It follows that at least one term in the splitting of \(f_\#^K(\gamma)\) is an edge in \(H_r\), implying that \(\gamma\) is weakly attracted to \(\Lambda^+\).

Since \(\gamma\) contains infinitely many edges in \(H_r\), and since \([G_r] = [G'_u]\) and the graphs \(G_r, G'_u\) are both core subgraphs, the line \(\gamma_{G^r}\) contains infinitely many edges in \(H'_u\).

In the part of the proof of the nongeometric case of Proposition 6.0.8 of [BFH00] that does not use birecurrence and so is true in our context, it is shown that there exists \(M' > 0\) so that for every finite subpath \(\gamma'_i\) of \(\gamma'\) there exists a line or circuit \(\tau'_i\) in \(G'\) that contains at most \(M'\) edges of \(H'_u\) such that \(\gamma'_i\) is a subpath of \(g_\#^k(\tau'_i)\) for some \(k_i \geq 0\). If \(G_{r-1} = \emptyset\) then \(\tau'_i\) is a circuit; otherwise \(\tau'_i\) is a line. (This is proved in two parts. First, in what is called step 2 of that proof, an analogous result is proved in \(G\). Then the bounded cancellation lemma is used to transfer this result to \(G'\); the case that \(G_{r-1} = \emptyset\) is considered after the case that \(G_{r-1} \neq \emptyset\).)

Choose a sequence of finite subpaths \(\gamma'_i\) of \(\gamma'\) that exhaust \(\gamma'\) and let \(\tau'_i\) and \(k_i\) be as above so that \(\gamma'_i\) is a subpath of \(g_\#^k(\tau'_i)\) and so that \(\tau'_i\) contains at most \(M'\) edges of \(H'_u\). Since \(\gamma'\) contains infinitely many \(H'_u\) edges, we have \(k_i \to +\infty\) as \(i \to +\infty\).

By Lemma I.1.54 there exists \(d > 0\) depending only on the bound \(M'\) such that \(g_\#^d(\tau'_i)\) has a splitting into terms each of which is either an edge or indivisible Nielsen path of height \(s\) or a path in \(G'_{u-1}\). By taking \(i\) so large that \(k_i \geq d\) we may replace each \(\tau'_i\) by \(g_\#^d(\tau'_i)\) and each \(k_i\) by \(k_i - d\), and hence we may assume that \(\tau'_i\) has a splitting

\[
\tau'_i = \tau'_{i,1} \cdots \tau'_{i,l_i}
\]

each of whose terms is an edge or Nielsen path of height \(s\) or a path in \(G'_{u-1}\). The number of edges that \(\tau'_i\) has in \(H'_u\) is still uniformly bounded, and so \(l_i\) is uniformly bounded. Passing to a subsequence, we may assume that \(l_i\) and the ordered sequence of height \(s\) terms in \(\tau'_i\)
are independent of $i$. We may also assume that $l = l_i$ is minimal among all such choices of $\gamma_i$ and $\tau^i_i$.

**Case A:** $l = 1$. In this case $\tau^i_1 = E$ is a single edge of $H'_u$ and so, by I.1.58, $\gamma'$ is a leaf of $\Lambda^\phi_\psi$. If both ends of $\gamma'$ have height $s$ then $\gamma'$ is generic by Fact I.1.61 and case (1) is satisfied.

Suppose one end of $\gamma'$, say the positive end, has height $\leq s - 1$. We have a concatenation $\gamma' = \overline{R}_1 R_2$ where the ray $R_1$ starts with an edge of $H'_u$ and the ray $R_2$ is contained in $G'_{u-1}$. By Fact I.1.37, the concatenation point is a principal vertex. By Lemma I.1.57 (2), for each $m$ there is an $m$-tile in $\gamma'$ which is an initial segment of $R_1$. By Corollary I.1.60 it follows that $R_1$ is a principal ray. This shows that (3) is satisfied with trivial $\mu$.

**Case B:** $l \geq 2$. Choose a subpath $\nu'_i \subset \tau^i_1$, with endpoints not necessarily at vertices, such that $g_{\#}^{k_i}(\nu'_i) = \gamma'_i$. Let $\tau''_i$ be the subpath obtained from $\tau'_i$ by removing the initial segment $\tau'_{i,1}$ and the terminal segment $\tau'_{i,l}$ of the splitting ($*$), so either $\tau''_i = \tau'_{i,2} \cdots \tau'_{i,l-1}$ or, when $l = 2$, $\tau''_i$ is the trivial path at the common vertex along which $\tau'_{i,1}$ and $\tau'_{i,2}$ are concatenated. After passing to a subsequence, we may assume that $\tau''_i \subset \nu'_i$; if no such subsequence existed then we could reduce $l$ by removing either $\tau'_{i,1}$ or $\tau'_{i,l}$ from $\tau'_i$. For the same reason, we may assume that $\gamma'$ has a finite subpath that contains $g_{\#}^{k_i}(\tau''_i)$ for all $i$.

After passing to a subsequence, we may assume that $\mu = g_{\#}^{k_i}(\tau''_i)$ is independent of $i$. Since the sequence of height $s$ terms in $\tau''_i$ is independent of $i$, it follows that $\mu$ is either trivial or has a splitting into terms each of which is either $\alpha$, or $\bar{\alpha}$, or a path in $G'_{u-1}$. Since the endpoints of $\alpha$ are distinct, no two adjacent terms in this splitting can both be $\alpha$ or $\bar{\alpha}$, and so each subdivision point of the splitting is in $G'_{u-1}$. Since $v$ is an interior point of $H'_u$, for any occurrence of $\alpha$ or $\bar{\alpha}$ as a term of $\mu$ the endpoint $v$ must be an endpoint of $\eta$. It follows that $\mu$ can be written in one of the forms given in item (2), after possibly inverting $\gamma'$.

Write $\gamma'$ as $\overline{R}_1 \mu R_2$. If $\tau'_{i,1}$ is an edge $E$ in $H'_u$ then $E = \tau'_{i,1}$ for all $i$, and the ray $R_1$ is the increasing union of $g_{\#}^{k_i}(E) \subset \overline{g_{\#}^{k_i}(E)} \subset \cdots$, so $R_1$ is a principal ray for $\Lambda^\phi_\psi$. Otherwise $\tau'_{i,1}$ is a path in $G'_{u-1}$ for all $i$ and $R_1$ is a ray in $G'_{u-1}$. Using $\tau'_{i,1}$ similarly in place of $\tau'_{i,1}$, $R_2$ is either a principal ray for $\Lambda^\phi_\psi$ or a ray in $G'_{u-1}$. At least one of $R_1$ and $R_2$ is a principal ray. If they are both principal rays then item (2) holds, otherwise (3) holds. \[\square\]

For stating the geometric case, Lemma 2.19, we continue with the notation laid out in the opening paragraph of Section 2.4. Assuming the strata $H_r$, $H'_u$ are geometric, we let $\rho_r$, $\rho'_u$ be the closed indivisible Nielsen paths in $G_r$, $G'_u$ of heights $r, u$, respectively; by applying Proposition I.2.18, up to reorienting these Nielsen paths we have $[\rho_r] = [\rho'_u]$.

**Lemma 2.19** (Height $r$ lines in the geometric case). Assuming that $H_r$, $H'_u$ are geometric, and with notation as above, if $\gamma \in \mathcal{B}$ is a line of height $r$ that is not weakly attracted to $\Lambda^\phi_\psi$ then its realization $\gamma_{G'_u}$ has at least one of the following forms:

1. $\gamma_{G'_u}$ or $\bar{\gamma}_{G'_u}$ is the bi-infinite iterate of $\rho'_u$.
2. $\gamma_{G'_u}$ is a generic leaf of $\Lambda^\phi_\psi$.
(3) $\gamma_G$ decomposes as $\overline{R_1 \mu R_2}$ where $R_1$ and $R_2$ are principal rays for $\Lambda^-_b$ and $\mu$ is either the trivial path, a finite iterate of $\rho'_u$ or its inverse, or a nontrivial path of height $< u$. 

(4) $\gamma_G'$ or $\tilde{\gamma}_G'$ decomposes as $\overline{R_1 R_2}$ where $R_1$ is a principal ray for $\Lambda^-_b$ and the ray $R_2$ either has height $< u$ or is the singly infinite iterate or $\rho'_u$ or its inverse.

Proof. By restricting $f: G \to G$ to the component of $G_r$ that contains $H_r$, restricting $f': G' \to G'$ to the component of $G'_u$ that contains $H'_u$, and replacing $\phi, \psi$ with their restrictions to Out$(\pi_1 G) = \text{Out}(\pi_1 G')$ (Fact I.1.4), we may assume that $H_r, H'_u$ are the top strata.

Throughout this proof, to simplify notation we shall identify a downstairs abstract line $\gamma \in \mathcal{B}$ with its realization in $G'$, and an upstairs abstract line $\tilde{\gamma} \in \mathcal{B}$ with its realization in the universal cover $G'$, and we shall elide the subscript $G'$ from $\gamma_{G'}$ and $\tilde{\gamma}_{G'}$.

Here is a rough outline of the proof. We use a geometric model for $f'$ and $H'_u$, in particular the surface $S$ and the pseudo-Anosov mapping class $\theta \in \mathcal{MG}(S)$ which are part of the geometric model. We also use Proposition I.2.15 which identifies the unstable and stable laminations $\Lambda^u, \Lambda^s$ of $\theta$ with $\Lambda^+_b, \Lambda^-_b$ respectively. For each line $\gamma$ of height $u$ in $G'$, we shall define a canonical decomposition of $\gamma$ as an alternating concatenation of “overpaths” and “underpaths”. Roughly speaking an overpath is a subpath that begins and ends with edges of $H'_u$, that has a homotopy pullback to the surface $S$, and that is maximal with respect to these properties. An overpath can be a finite subpath of $\gamma$, a subray, or the whole line. We show that overpaths have disjoint interiors, and we define underpaths to be the components of the complements of the interiors of the overpaths. We show that this “over–under” decomposition is natural with respect to the action of $f'$, which is captured by the slogan “the iterates of an overpath of $\gamma$ are overpaths of the iterates of $\gamma$”. Combining this with the hypothesis that $\gamma$ is not weakly attracted to $\Lambda^+_b$ under iteration of $\phi$, we shall conclude that the homotopy pullbacks to $S$ of the overpaths of $\gamma$ are not weakly attracted to $\Lambda^u$ under iteration of $\theta$. Combining this with an application of Nielsen-Thurston theory (Proposition I.2.14) we will by a case analysis derive strong consequences on the structure of the overpath–underpath decomposition of $\gamma$, showing that this structure directly yields one of conclusions (1)–(4).

In order to formalize overpath–underpath decompositions we use the peripheral Bass-Serre tree $T \varsubsetneq T$ associated to a geometric model as a bookkeeping device. We begin with a review of these topics from Part I [HM13b].

**Geometric model.** In our present context where $H'_u$ is the top stratum, a geometric model for $f'$ and $H'_u$ (given in Definition I.2.4) is the same thing as a weak geometric model (Definitions I.2.1).

To simplify notation we denote $L = G'_{u-1}$ and its total lift to $\tilde{G}'$ as $\tilde{L}$.

We recall the following elements of the data comprising a (weak) geometric model for $f'$ and $H'_u$, separated into static data and dynamic data. The static data includes: a compact surface $S$ with a distinguished upper boundary component $\partial_b S$ and with lower boundary $B = \partial S - \partial_b S$; a quotient complex $Y$ obtained by gluing $S$ and $L$ using an attaching map $\alpha: B \to L$ which restricts to a homotopically nontrivial closed edge path on each component of $B$; an embedding $G' \hookrightarrow Y$ extending the embedding $G'_{u-1} = L \hookrightarrow Y$; and a deformation retraction $d: Y \to G'$ which takes $\partial_b S$, regarded as a closed curve based at the unique point
associated to the geometric model $Y$ is a straightforward consequence of Bass-Serre theory ([SW79]) and Definition I.2.10. Justifying this description is a straightforward consequence of Bass-Serre theory ([SW79]) and Definition I.2.10.

Consider the following pushout diagram:

\[
\begin{array}{ccc}
\hat{S} \sqcup \hat{L} & \xrightarrow{j} & \hat{Y} \\
\downarrow{\hat{q}} & & \downarrow{\hat{q}} \\
S \sqcup L & \xrightarrow{j} & Y
\end{array}
\]

where $q$ is the universal covering map, $\hat{S} \sqcup \hat{L} = \hat{Y}$ is the subspace of the Cartesian product $(S \sqcup L) \times Y$ consisting of all pairs $(x, y)$ such that $j(x) = q(y)$, and the upper and left arrows are restrictions of the two projection maps of the Cartesian product. The group $F_n$ acts on $\hat{Y}$ by deck transformations and on $S \sqcup L$ trivially, inducing a diagonal action on $\hat{S} \sqcup \hat{L} = \hat{Y}$, such that $\hat{j}$ is $F_n$-equivariant and $\hat{q}$ is a covering map with deck transformation group $F_n$. The components of $\hat{S}$ are called $S$-vertex spaces, the components of $\hat{L}$ are $L$-vertex spaces, and the components of $B = \partial \hat{S}$ are edge spaces. These components are indexed as follows together with their respective images in $\hat{Y}$ and stabilizer groups:

\[
\begin{align*}
\hat{S} &= \cup_s \hat{S}_s, & \hat{S}_s &= \hat{j}(\hat{S}_s) \subset \hat{Y}, & \Gamma_s &= Stab(\hat{S}_s) = Stab(\hat{S}_s) < F_n \\
\hat{L} &= \cup_l \hat{L}_l, & \hat{L}_l &= \hat{j}(\hat{L}_l) \subset \hat{Y}, & \Gamma_l &= Stab(\hat{L}_l) = Stab(\hat{L}_l) < F_n \\
\hat{B} &= \cup_b \hat{B}_b, & \hat{B}_b &= \hat{j}(\hat{B}_b) \subset \hat{Y}, & \Gamma_b &= Stab(\hat{B}_b) = Stab(\hat{B}_b) < F_n
\end{align*}
\]

Note that the $\hat{S}_s$ is connected and it is cocompact under the action of $\Gamma_s$, so the same is true of $\hat{S}_s$; similar statements hold for the actions of $\Gamma_l$ and $\Gamma_b$. Also by restricting $\hat{q}$ we get universal covering maps of each $\hat{S}_s$ over $S$ with deck transformation group $\Gamma_s$, and of each $\hat{L}_l$ over some component of $L$ with deck group $\Gamma_l$, and of each $\hat{B}_b$ over some component of $B$ with deck group $\Gamma_b$.

The domain and range restrictions of $\hat{j}$ are also denoted with subscripts, e.g. $\hat{j}_s : \hat{S}_s \to \hat{S}_s$. Although $\hat{j}_l$ is a homeomorphism, $\hat{j}_s$ and $\hat{j}_b$ need not be injective. However $\hat{j}_s$ restricts to a homeomorphism from the manifold interior $\text{int}(\hat{S}_s)$ onto an open subset of $\hat{Y}$ contained in $\hat{S}_s$ denoted by convention $\text{int}(\hat{S}_s)$, and $\hat{j}_b$ restricts to a bijection of components between $\partial \hat{S}_s$ and $\hat{S}_s - \text{int}(\hat{S}_s)$.

The Bass-Serre tree $T$ is a bipartite tree with: one $S$-vertex denoted $V_s$ for each $\hat{S}_s$; one $L$-vertex denoted $V_l$ for each $\hat{L}_l$; and one edge denoted $E_b$ for each $\hat{B}_b$ that is attached
to the unique $V_s$ and $V_l$ having the properties $\hat{B}_b \subset \hat{S}_s$ and $\hat{B}_b \subset \hat{L}_l$. The action of $F_n \curvearrowright \tilde{Y}$ induces the action $F_n \curvearrowright T$. Note that $T$ can be characterized algebraically: the conjugacy class $[\pi_1 S]$ equals the set $\{\Gamma_s\}$ of $S$-vertex stabilizers, and the latter corresponds bijectively to $\{V_s\}$ since $\pi_1 S$ is its own normalizer in $F_n$ (Lemma I.2.7 (2)). Also, the union of the conjugacy classes constituting the subgroup system $[\pi_1 L]$ equals the set $\{\Gamma_l\}$ which corresponds bijectively to $\{V_l\}$ since $[\pi_1 L]$ is malnormal (Lemma I.2.7 (1)). The tree $T$ thus has one $S$-vertex $V_s$ for each $\Gamma_s$, one $L$-vertex $V_l$ for each $\Gamma_l$, with an edge connecting $V_s$ to $V_l$ if and only if $\Gamma_s \cap \Gamma_l$ is a nontrivial subgroup of $F_n$.

**Further remarks.** We are abusing the terminology and notation of Definition I.2.10 by stripping away $\partial_h S$ from the graph $L$, with the effect (see Remark I.2.11) that valence 1 vertices of the Bass-Serre tree of Definition I.2.10 have been stripped away in forming $T$. Other valence 1 vertices may remain in $T$, namely those associated to “free lower boundary circles” of $S$.

The failure of the maps $\tilde{j}_s: \tilde{S}_s \to \tilde{S}_s$ and $\tilde{j}_b: \tilde{B}_b \to \tilde{B}_b$ to be homeomorphisms stems from non local injectivity of the attaching map $\alpha: B \to L$. The $\alpha$ factors on each component of $B$ as a finite sequence of Stallings folds followed by a local injection, which lifts to equivariant factorizations of each $\tilde{j}_s$ and each $\tilde{j}_b$, each term of which is a homotopy equivalence, and so each $\tilde{j}_s$ and $\tilde{j}_b$ is a homotopy equivalence. In particular each $\tilde{S}_s$ is contractible.

**Action of the subgroup** $\text{Aut}_\psi(F_n) = \text{Aut}_\phi(F_n)$ on $T$. This subgroup of $\text{Aut}(F_n)$ consists of all automorphisms representing all outer automorphisms $\psi^i = \phi^{-i}$, $i \in \mathbb{Z}$. The *exponent* of $\Psi \in \text{Aut}_\psi(F_n)$ is the integer $i \in \mathbb{Z}$ for which $\Psi$ represents $\psi^i$; this defines an epimorphism $\text{Aut}_\psi(F_n) \to \mathbb{Z}$ with kernel $\text{Inn}(F_n)$. For $\Psi$ of non-negative exponent $i \geq 0$ the lift of the CT $(f')^i_\Psi: G' \to G'$ that corresponds to $\Psi$ is denoted $f'_\Psi: G' \to \tilde{G}'$.

We extend the action $\text{Inn}(F_n) \approx F_n \curvearrowright T$ to an action $\text{Aut}_\psi(F_n) \curvearrowright T$ as follows. First, under the action of $\text{Out}(F_n)$ on conjugacy classes of subgroups of $F_n$, the group $\text{Aut}_\psi(F_n)$ preserves the conjugacy class $[\pi_1 S] = \{\Gamma_s\}$ inducing the $S$-vertex action $\text{Aut}_\psi(F_n) \curvearrowright \{\Gamma_s\} \leftarrow \{V_s\}$; we denote $\Psi(V_s) = V_{\Psi(s)}$. Similarly, the group $\text{Aut}_\psi(F_n)$ preserves each component of $[\pi_1 L]$ inducing the $L$-vertex action $\text{Aut}_\psi(F_n) \curvearrowright \{\Gamma_l\} \leftarrow \{V_l\}$ which we denote $\Psi(V_l) = V_{\Psi(l)}$. Finally, for each $\Psi \in \text{Aut}_\psi(F_n)$, if $V_s, V_l$ are connected by an edge $E_b$ then $\Gamma_s \cap \Gamma_l$ is nontrivial, so $\Psi(\Gamma_s) \cap \Psi(\Gamma_l)$ is nontrivial, and so $V_{\Psi(s)}, V_{\Psi(l)}$ are connected by an edge $E_{\Psi(b)} = \Psi(E_b)$.

We need a topological characterization of the action $\text{Aut}_\psi(F_n) \curvearrowright T$ expressed in terms of lifts $f'_\Psi$ defined for non-negative exponent. For this purpose we need some additional notation.

The embedding $G' \subset Y$ and deformation retraction $d: Y \to G'$ lift to an $F_n$-equivariant embedding $\tilde{G}' \subset \tilde{Y}$ and deformation retraction $\tilde{d}: \tilde{Y} \to \tilde{G}'$. Let $\tilde{L} \subset \tilde{G}'$ be the full lift of $L$. Let $\tilde{H}'_s = \tilde{G}' \setminus \tilde{L}$ which is the full lift of $H'_s = G' \setminus L$. Let $\tilde{G}'_s = \tilde{G}' \cap \tilde{S}_s = \tilde{d}(\tilde{S}_s) = \tilde{d}(\tilde{\gamma}(\tilde{S}))$; note that $\tilde{G}'_s$ is connected since $\tilde{S}$ is connected, and $\Gamma_s$ acts cocompactly on $\tilde{G}'_s$ since it acts cocompactly on $\tilde{S}$; it follows that we may naturally identify $\partial \Gamma_s = \partial \tilde{S}_s = \partial \tilde{G}'_s \subset \partial F_n$, from which it follows in turn that each line in $\tilde{G}'$ with idea endpoints in $\partial \Gamma_s$ is contained in the subgraph $\tilde{G}'_s$. Let $\tilde{H}'_{u,s} \subset \tilde{H}'_s$ be the subgraph of all edges of $\tilde{H}'_s \cap \tilde{G}'_s$ (the latter intersection may contain some isolated vertices which we avoid by defining $\tilde{H}'_{u,s}$ in this manner). Note that the components of the subgraph $\tilde{G}'_s \setminus \tilde{H}'_{u,s}$ are precisely the components of $\tilde{S}_s - \text{int}(\tilde{S}_s)$,
each a component of $\hat{B}_b$ corresponding bijectively to edges $E_b$ incident to $V_s$.

Given an topological representative of an outer automorphism of $F_n$, recall the usual correspondence between lifts of that topological representative to the universal cover and automorphisms representing that outer automorphism, defined by inducing the same continuous extension on $\partial F_n$ (see e.g. Section I.1.5.3). If $\Psi \in \text{Aut}_\psi(F_n)$ represents $\psi^i$ with $i \geq 0$ then the CT $(f')_# : G' \rightarrow G'$ is a topological representative of $\psi^i$, and the lift of $(f')_#$ that corresponds to $\Psi$ is denoted $\tilde{f}_\Psi : \tilde{G}' \rightarrow \tilde{G}'$. We use the same symbol for the continuous extension to the Gromov compactification $\tilde{f}_\Psi : \tilde{G}' \cup \partial F_n \rightarrow \tilde{G}' \cup \partial F_n$.

**Full height lines, realization in $T$, and over–under decompositions.** Consider a line $\tilde{\gamma}$ in $\tilde{G}'$ of full height, meaning that $\tilde{\gamma}$ contains an edge of $\tilde{H}'_\psi$. We define the realization $\gamma_T$ of $\tilde{\gamma}$ in $T$, and in parallel we define the over–under decomposition of $\tilde{\gamma}$ in $G'$.

To start, we define the $S$-vertices in $\gamma_T$ and their associated overpaths in $\tilde{\gamma}$. Given an $S$-vertex $V_s \in T$, we put $V_s$ in $\gamma_T$ if and only if $\tilde{\gamma} \cap \text{int}(\tilde{S}_s) \neq \emptyset$, equivalently $\tilde{\gamma} \cap \tilde{H}'_{u,s}$ contains an edge. For any $S$-vertex $V_s \in \gamma_T$ its associated overpath $\gamma_s \subset \tilde{\gamma}$ is defined to be the longest subpath of $\tilde{\gamma}$ having the property that each ideal endpoint of $\tilde{\gamma}_s$ is in $\partial F_s$ and the edge of $\tilde{\gamma}_s$ incident to each finite endpoint of $\tilde{\gamma}_s$ is in $\tilde{H}'_{u,s}$. We note that $\tilde{\gamma}_s \subset \tilde{G}'_s$. We note also that distinct overpaths have disjoint interiors, that is, given overpaths $\gamma_s, \gamma_{s'} \subset \tilde{\gamma}$ with $s \neq s'$ their intersection $\tilde{\gamma}_s \cap \tilde{\gamma}_{s'}$, which is a path in $\tilde{G}'_s \cap \tilde{G}'_{s'} \subset \tilde{L}$, is either empty or a common endpoint, for if this were not true then: if $\tilde{\gamma}_s = \tilde{\gamma}_{s'}$ we would contradict the fact that $\tilde{L}$ contains no edges of $\tilde{H}'_\psi$, whereas if $\tilde{\gamma}_s \neq \tilde{\gamma}_{s'}$ then the intersection $\tilde{\gamma}_s \cap \tilde{\gamma}_{s'}$ would have a finite endpoint $x$ with incident edge $E$ such that $x$ is also a finite endpoint of one of $\tilde{\gamma}_s$ or $\tilde{\gamma}_{s'}$ with incident edge $E$ and hence $E \subset \tilde{H}'_\psi$, also a contradiction.

Next we define the underpaths of $\tilde{\gamma}$ and their associated $T$-vertices. The underpaths are the components of $\tilde{\gamma} - \bigcup_s \text{int}(\tilde{\gamma}_s)$, a disjoint union of possibly degenerate subintervals of $\tilde{\gamma}$, each contained in $\tilde{L}$. Given an $L$-vertex $V_l \in T$, we put $V_l$ in $\gamma_T$ if and only if one of the underpaths, denoted $\gamma_{l,u}$, is contained in the $L$-vertex space $\tilde{L}_l$. Finally, given an edge $B_b \subset T$ with endpoints $V_s, V_l$, we put $B_b$ in $\gamma_T$ if and only if $\gamma_s \cap \gamma_l$ is nonempty, in which case that intersection is a point that we denote $p_b$.

This completes the definition of $\gamma_T$, although we must still check that it is indeed a path in the tree $T$. Choosing an orientation of $\tilde{\gamma}$, by construction we have decomposed $\tilde{\gamma}$ into an alternating concatenation of overpaths and underpaths, what we call the over–under decomposition of $\tilde{\gamma}$, and associated to this decomposition we have an expression of $\gamma_T$ as a concatenation of edges of $T$. To prove that $\gamma_T$ is a path it suffices to show that this concatenation is locally injective at each vertex. Supposing that in $\gamma_T$ the $S$-vertex $V_s$ is preceded by an edge $E_b$ and followed by an edge $E_{b'}$, it follows that the overpath $\gamma_s$ is a finite path with endpoints $p_b \in \hat{B}_b$ and $p_{b'} \in \hat{B}_{b'}$; the desired inequality $E_b \neq E_{b'}$ follows from the inequality $\hat{B}_b \neq \hat{B}_{b'}$ which is true because, otherwise, it would follow that $\gamma_s \subset \hat{B}_b = \hat{B}_{b'}$ contradicting that $\gamma_s$ contains in edge of $\tilde{H}'_\psi$. And supposing that in $\gamma_T$ the $L$-vertex $V_l$ is preceded by an edge $E_b$ with opposite $S$-vertex $V_s$ and followed by an edge $E_{b'}$ with opposite $S$-vertex $V_{s'}$, by construction the over–under decomposition has three successive terms $\gamma_s \gamma_{s'} \gamma_l$, and so by construction $\gamma_s$ and $\gamma_{s'}$ have disjoint interiors; but $\gamma_s$ contains every $\tilde{H}'_\psi$ edge in $\tilde{\gamma}$ including at least one such edge, and $\gamma_{s'}$ contains every $\tilde{H}'_{s'}$ edge in $\tilde{\gamma}$ including at least one such edge, and it follows that $V_s \neq V_{s'}$ and so $E_b \neq E_{b'}$.

**$\text{Aut}_\psi(F_n)$-equivariance.** We show that realization of lines in $T$ is equivariant under the
actions of $\text{Aut}_\psi(F_n)$ on $\widehat{B}$ and on $T$, in the sense that the following equation holds for each $\Psi \in \text{Aut}_\psi(F_n)$ and each top height line $\widehat{\gamma} \in \widehat{B}$:

$$(\Psi(\widehat{\gamma}))_T = \Psi(\widehat{\gamma}_T) \quad (*)$$

Since this equation is obviously true for $\Psi \in \text{Inn}(F_n)$ of exponent zero, it suffices to consider the case of positive exponent 1, meaning $\Psi$ corresponds to $\tilde{f}_\psi' : \tilde{G}' \to \tilde{G}'$ which is a lift of $f'_\psi$. In this case the line on the left hand side of this equation is the realization in $T$ of the line $(f'_\psi)(\widehat{\gamma})$, which by the Bounded Cancellation Lemma (see Fact I.1.5) is characterized as the unique line in $\tilde{G}'$ at finite Hausdorff distance from the image $\tilde{f}_\psi'(\widehat{\gamma})$, and so $(f'_\psi)_{\#}$ is the unique line in $\tilde{G}'$ contained in the image $\tilde{f}_\psi'(\widehat{\gamma})$. We may therefore prove the above equation by examining directly how to obtain the over–under decomposition of the line $(f'_\psi)_{\#}(\widehat{\gamma})$ from the over–under decomposition of $\widehat{\gamma}$.

If $\widehat{\gamma}_T = V_s$ degenerates to a single $S$-vertex then we have implications

$$\widehat{\gamma} = \widehat{\gamma}_s \subset \tilde{S}_s \implies \partial \widehat{\gamma} \subset \partial \Gamma_s \implies \Psi(\partial \widehat{\gamma}) \subset \partial \Gamma_{\Psi(s)} \implies \Psi(\widehat{\gamma}) = \Psi(\widehat{\gamma}_s) \subset \tilde{S}_{\Psi(s)}$$

$$\widehat{\gamma} \not\subset \widehat{L} \implies \Psi(\widehat{\gamma}) \not\subset \widehat{L}$$

which implies that $\Psi(\widehat{\gamma})_T = V_{\Psi(s)} = \Psi(\widehat{\gamma}_T)$ and we are done.

We turn to the remaining case that $\widehat{\gamma}_T$ does not degenerate to a single $S$-vertex. Write the over–under decomposition of $\widehat{\gamma}_T$ as the concatenation of paths $\alpha_j, j \in J$, where $J \subset \mathbb{Z}$ is some subinterval, so that if $j$ is even then $\alpha_j = \overline{\gamma}_{s(j)}$ is an overpath and if $j$ is odd then $\alpha_j = \overline{\gamma}_{s(j)}$ is an underpath. For each $j \in J$ let $p_j, p_{j+1} \in \tilde{G}' \cup \partial F_n$ be the initial and terminal endpoints of $\alpha_j$. We shall define for each $j \in J$ a path $\delta_j \subset \tilde{G}'$ which will turn out to be the overpath or underpath in $(f'_\psi)_{\#}(\widehat{\gamma})$ corresponding to $\alpha_j$.

We first define $\delta_j$ when $j \in J$ is even. Note the following chain of equivalences:

$$j - 1 \in J \iff p_j \in \tilde{G}' \iff V_{l(j-1)} \in \widehat{\gamma}_T \text{ is defined } \iff \tilde{L}_{l(j-1)} \text{ is defined}$$

$$j + 1 \in J \iff p_{j+1} \in \tilde{G}' \iff V_{l(j+1)} \in \widehat{\gamma}_T \text{ is defined } \iff \tilde{L}_{l(j+1)} \text{ is defined}$$

Consider $\beta_j = (f'_\psi)_{\#}(\alpha_j)$, the path in $\tilde{G}'$ with initial and terminal endpoints $q_j = f'_\psi(p_j), q_{j+1} = f'_\psi(p_{j+1})$. Note the implications

$$j - 1 \in J \implies p_j \in \tilde{L}_{l(j-1)} \implies q_j \in \Psi(\tilde{L}_{l(j-1)}) = \tilde{L}_{\Psi(l(j-1))}$$

and

$$j - 1 \not\in J \implies p_j \in \partial \Gamma_{s(j)} \implies q_j \in \Psi(\partial \Gamma_{s(j)}) = \partial \Gamma_{\Psi(s(j))}$$

and similarly for $q_{j+1}$ depending on whether or not $j + 1 \in J$. It follows that $\beta_j$ is contained in the connected subgraph $\text{St}(\tilde{S}_{\Psi(s(j))}) \subset \tilde{G}'$ consisting of the union of $\tilde{S}_{s(j)}$ with whichever of the $L$-vertex spaces $\tilde{L}_{l(j-1)}, \tilde{L}_{l(j+1)}$ is defined. Note that the edges of $\tilde{H}'_{\psi(s)}$ in $\text{St}(\tilde{S}_{\Psi(s(j))})$ are precisely those in $\tilde{H}'_{\psi(s)}$. The path $\beta_j$ must contain at least one of those edges, because $q_j, q_{j+1}$ are separated by at least one edge of $\tilde{H}'_{\psi(s)}$. To see why: if $j - 1, j + 1$ are both defined then $\tilde{L}_{l(j-1)}, \tilde{L}_{l(j+1)}$ are distinct components of $\tilde{L}$ and so $\tilde{L}_{\Psi(l(j-1))}, \tilde{L}_{\Psi(l(j+1))}$
are distinct components and are separated by at least one edge of \( \hat{H}'_u \); whereas if only one of \( j-1, j+1 \) is defined, say \( j-1 \) is defined, then the ideal point \( p_{j+1} \in \partial \Gamma_s(j) \) is not contained in the Gromov boundary of the component \( \hat{L}_{(j-1)} \) that contains \( p_{j-1} \), and so the ideal point \( q_{j+1} \in \partial \Gamma_{\psi(s(j))} \) is not contained in the Gromov boundary of the component \( \hat{L}_{\psi((j-1))} \) that contains \( q_{j-1} \), and so \( \hat{L}_{\psi((j-1))} \) is separated from \( q_{j+1} \in \partial \hat{G}' \) by at least one edge of \( \hat{H}'_u \). It therefore makes sense to define \( \delta_j \) to be the longest subpath of \( \beta_j \) having the property that each ideal endpoint of \( \delta_j \) is contained in \( \partial \Gamma_{\psi(s(j))} \) and each finite endpoint of \( \delta_j \) is incident to an edge of \( \delta_j \) in \( \hat{H}'_{u, \psi(s(j))} \). Let \( r_j, r_{j+1} \) be the initial and terminal endpoints of \( \delta_j \). Note that whichever of \( r_j, r_{j+1} \) is infinite, it is equal to the corresponding \( q_j, q_{j+1} \), and in particular is contained in \( \partial \Gamma_{\psi(s(j))} = \partial \hat{S}_{\psi(s(j))} \). Note also that whichever of the points \( r_j, r_{j+1} \) is finite, it is an endpoint of an edge of \( \hat{H}'_{u, \psi(s(j))} \) and so is contained in \( \hat{S}_{\psi(s(j))} \). It follows that \( \delta_j \subset \hat{S}_{\psi(s(j))} \).

We next define \( \delta_j \subset \hat{L}_{\psi((j))} \) when \( j \in J \) is odd, for which we must define its endpoints. If \( j-1, j+1 \in J \) then the overpaths \( \delta_{j-1}, \delta_{j+1} \) have already been defined, with \( \delta_{j-1} \) having terminal finite endpoint \( r(j) \in \hat{L}_{\psi((j))} \) and \( \delta_{j+1} \) having initial finite endpoint \( r(j+1) \in \hat{L}_{\psi((j))} \). Otherwise, exactly one of \( j-1, j+1 \) is in \( J \), say \( j-1 \in J \), so the overpath \( \delta_{j-1} \) has already been defined with terminal endpoint \( r_j \in \hat{L}_{\psi((j))} \); we also have ideal points \( p_{j+1} \in \partial \hat{L}_{(j)} = \partial \Gamma_{\psi((j))} \) and \( q_{j+1} = \psi(p_{j+1}) \in \partial \Gamma_{\psi((j))} = \partial \hat{L}_{\psi((j))} \) and we set \( r_{j+1} = q_{j+1} \). In either case we have defined points \( r_j, r_{j+1} \in \hat{L}_{\psi((j))} \cup \partial \Gamma_{\psi((j))} \) and we let \( \delta_j \subset \hat{L}_{\psi((j))} \) to be the path with initial endpoint \( r_j \) and terminal endpoint \( r_{j+1} \).

By construction, the paths \( \delta_j \) concatenate without cancellation and form the over–under decomposition of some line. Also by construction that line is contained in \( \hat{f}'(\gamma_T) \), and so that line is equal to \( (\hat{f}'_{\beta}(\hat{\gamma})) \) whose realization in \( T \) equals \( (\psi(\hat{\gamma}))_T \). Also by construction, the path in \( T \) corresponding to the concatenation of the \( \delta_j \) is precisely \( \psi(\hat{\gamma}_T) \). This completes the proof of the equivariance equation \((\ast)\).

**Proper geodesic overpaths do not cross the stable lamination.** Fix a hyperbolic structure on \( S \) with totally geodesic boundary, and let \( \Lambda^u, \Lambda^s \subset \text{int}(S) \) be the unstable and stable laminations associated to the pseudo-Anosov mapping class \( \theta \). To avoid confusion with the label “\( s \)” we shall speak of “stable principal regions” rather than “principal regions of \( \Lambda^s \)”.

For each vertex \( V_s \in T \) lift to a hyperbolic structure with totally geodesic boundary on the universal covering space \( \hat{S}_s \) with deck transformation group \( \Gamma_s \) acting by isometries. We will use the notions of proper geodesics (lines, rays, and arcs) and proper equivalence of proper geodesics, in \( S \) and in each \( \hat{S}_s \), as given in Definition I.2.13. Choose once and for all a homeomorphism \( \Theta: S \to S \) representing the mapping class \( \theta \), such that \( \Theta \) preserves \( \Lambda^s \) and \( \Lambda^u \) and their principal regions; since \( \phi \) is rotationless it follows that \( \Theta \) preserves each individual principal region and fixes its cusps ([Mil82], Theorem 9).

We have the following composition of quasi-isometries with uniform constants independent of \( s \) and the associated composition of continuous extensions to Gromov compactifications:

\[
\hat{S}_s \xrightarrow{j_s} \hat{S}_s \xrightarrow{d_s} \hat{G}'_s \\
\hat{S}_s \cup \partial \Gamma_s \xrightarrow{j_s} \hat{S}_s \cup \partial \Gamma_s \xrightarrow{d_s} \hat{G}'_s \cup \partial \Gamma_s
\]
Let \( \tilde{\Lambda}_u^s \subset \text{int}(\tilde{S}_s) \) be the total lift of \( \Lambda^u \), and let \( \tilde{\Lambda}_\phi^+ \) be the total lift to \( \tilde{G}' \) of \( \Lambda_\phi^+ \). It follows by Proposition I.2.15 that for each leaf \( \ell \subset \tilde{\Lambda}_u^s \) its image \( \tilde{d}_s \circ \tilde{\gamma}_s(\ell) \) is uniformly Hausdorff close in \( G' \) to a unique leaf of \( \tilde{\Lambda}_\phi^+ \).

Given a full height line \( \gamma \) in \( G' \) we define its proper geodesic overpaths in \( S \). Choose a lift \( \tilde{\gamma} \subset G' \) of \( \gamma \). For each of the overpaths \( \tilde{\gamma}_s \subset G'_s \subset \tilde{S}_s \), choose \( \tilde{\gamma}_s \subset \tilde{S}_s \) to be a geodesic whose endpoints in \( \partial \tilde{S}_s \cup \partial \tilde{\Gamma}_s \) map to the endpoints of \( \tilde{\gamma}_s \) under the map \( \tilde{d}_s \circ \tilde{\gamma}_s \) (we shall address below the non uniqueness of the choice of \( \tilde{\gamma}_s \), which may occur due to noninjectivity of the restriction of \( \tilde{\gamma}_s \) to lower boundary components). It follows that any finite endpoints of \( \tilde{S}_s \) are contained in the lower boundary \( \partial \tilde{S}_s \cap \tilde{B} \). We may immediately rule out the possibility that \( \tilde{\gamma}_s \subset \partial \tilde{S}_s \) as follows. If \( \tilde{\gamma}_s \) were contained in the lower boundary \( \partial \tilde{S}_s \cap \tilde{B} \) then it would follow that \( \tilde{\gamma}_s \subset \tilde{L} \cap \tilde{G}'_s \), and so \( \gamma \) is a line in \( L = G'_{u-1} \), contradicting that \( \gamma \) has full height. If \( \tilde{\gamma}_s \) is contained in the upper boundary \( \partial \tilde{S}_s \cap \tilde{B} \) then it has no finite endpoints and so must be an entire upper boundary component, from which it follows that \( \gamma \) is a bi-infinite iterate of \( \rho'_u \) or \( \rho''_u \), verifying conclusion (1) of Lemma 2.18.

Having reduced to the case \( \tilde{\gamma}_s \not\subset \partial \tilde{S}_s \), and recalling Definition I.2.13, it follows that:

- Each \( \tilde{\gamma}_s \) is a proper geodesic, with finite endpoints in \( \partial \tilde{S}_s \) and interior in \( \text{int}(\tilde{S}_s) \).

Projecting \( \tilde{\gamma}_s \) down to \( \tilde{S} \) we obtain a downstairs proper geodesic \( \gamma_s \), which we shall call a proper geodesic overpath of \( \gamma \). Note that, once an orientation of \( \gamma \) is fixed, the linearly ordered sequence of proper geodesic overpaths of \( \gamma \) is well-defined independent of the choice of a lift \( \tilde{\gamma} \). Note also that \( \tilde{\gamma}_s \), although not well-defined, is well-defined up to proper equivalence (Definition I.2.13), meaning that its infinite endpoints are well-defined and its finite endpoints are contained in well-defined lower boundary components.

Assuming now the hypothesis that the top height line \( \gamma \in B \) is not weakly attracted to \( \Lambda_\phi^+ \) under iteration of \( \phi \), we prove:

- No proper geodesic overpath \( \gamma_s \) of \( \gamma \) crosses the stable lamination.

We shall directly prove the contrapositive: if \( \gamma_s \) crosses the stable lamination then \( \gamma \) is weakly attracted to \( \Lambda_\phi^+ \).

Choose a lift \( \tilde{\gamma} \subset G' \) of \( \gamma \) with overpath \( \tilde{\gamma}_s \) projecting to \( \gamma_s \). Choose \( \Phi \in \text{Aut}_\psi(F_n) \) representing \( \phi \) so that \( \Phi(\Gamma_s) = \Gamma_s \), so \( \Psi(s) = s \). Denote \( \Psi = \Phi^{-1} \) representing \( \psi \), and so \( \Psi(s) = s \). Consider the sequence of lines \( \Phi^i(\tilde{\gamma}) \), \( i \geq 0 \), represented by lines in \( G' \) denoted \( \tilde{\gamma}_i \), with realization in \( T \) denoted \( \tilde{\gamma}_{i,T} \). It follows by equation (*) that \( V_s \in \tilde{\gamma}_{i,T} \) for all \( i \), with associated overpaths denoted \( \tilde{\gamma}_{i,s} \subset \tilde{G}'_s \), associated proper geodesic overpaths upstairs denoted \( \tilde{\gamma}_{i,s} \subset \tilde{S}_s \), and associate proper geodesic overpaths downstairs denoted \( \gamma_{i,s} \). We may choose a lift \( \tilde{\Theta} : \tilde{S}_s \to \tilde{S}_s \) of \( \Theta : S \to S \) whose action on \( \partial \Gamma_s \) agrees with the action of \( \Phi \); it follows that \( \Theta \) and \( \Phi \) also act compatibly on boundary components, in that \( \Theta(\tilde{B}_h) = \tilde{B}_\phi(h) \).

Combining this with equation (*) it follows that for each \( i \) the proper geodesics \( \Phi(\gamma_{i,s}) \) and \( \gamma_{i+1,s} \) are properly equivalent in the sense of Definition I.2.13, meaning that they have the same ideal endpoints and their finite endpoints are in the same boundary components. Using the notation \([\cdot]\) for proper equivalence, it follows that the sequence of proper geodesics \( \gamma_{i+1,s} \) represent the iterated sequence of proper equivalence classes \( \theta^i[\gamma_s] \).

Assuming that \( \gamma_s \) crosses the stable lamination, it follows by Nielsen–Thurston theory, Proposition I.2.14, that \( \gamma_s \) is weakly attracted to the unstable lamination under iteration.
of $\theta$, as stated in item (3) of that proposition: for each $\epsilon > 0$ and $M > 0$ there exists $I$ such that if $i \geq I$ then $\tilde{\gamma}_s,i$ has a subpath of length $\geq M$ at Hausdorff distance $< \epsilon$ from some subsegment of some leaf of the stable lamination in $\tilde{S}_s$. From this it follows that as $i \to \infty$ the image $\tilde{d}_s \circ \tilde{\gamma}_s(\tilde{\gamma}_s,i)$ contains longer and longer segments in common with leaves of $\Lambda^+_\phi$, and so $\tilde{\gamma}_s$ itself contains longer and longer such segments, and so $\tilde{\gamma}_s$ contains longer and longer segments of leaves of $\Lambda^+_\phi$. Since all leaves of $\Lambda^+_\phi$ are generic (Proposition I.3) it follows that $\gamma_\phi$ is weakly attracted to $\Lambda^+_\phi$.

**Completion of the proof.** Consider a line $\gamma \in \mathcal{B}$ of top height in $G'$ which is not weakly attracted to $\Lambda^+_\phi$, and choose a lift $\tilde{\gamma} \subset G'$. Consider an $S$-vertex $V_s \in \tilde{\gamma}_T$ with corresponding proper geodesic overpaths $\tilde{\gamma}_s \subset G'_s$ upstairs and $\gamma_s$ downstairs. Since $\gamma_s$ does not cross $\Lambda^u$ it follows that $\gamma_s$ and $\tilde{\gamma}_s$ are not proper geodesic arcs with endpoints in the boundary, and so $\tilde{\gamma}_s$ is not a finite arc. It follows that $V_s$ is not an interior point of the geodesic $\tilde{\gamma}_T$. This being true for all $S$-vertices in $\tilde{\gamma}_T$, and $\tilde{\gamma}_s$ having at least one $S$-vertex by virtue of $\gamma$ being of top height, up to choice of orientation the path $\tilde{\gamma}_T$ in $T$ has one of the following forms, with additional descriptions to follow:

**Degenerate $S$-point:** $\tilde{\gamma}_T = V_s$. In this case $\tilde{\gamma} = \tilde{\gamma}_s$ is a geodesic line in $\text{int}(\tilde{S}_s)$ and is either a leaf of $\tilde{\Lambda}^{\phi}$ or contained in some stable principal region $\tilde{P}$.

**One $S$-endpoint:** $\tilde{\gamma}_T = \left( V_s - \frac{E_{b}}{E_{l}} V_l \right)$. In this case $\tilde{\gamma}_s$ is a proper geodesic ray contained in some stable principal region $\tilde{P}$ projecting to a stable crown principal region $P$.

**Two $S$-endpoints:** $\tilde{\gamma}_T = \left( V_{s-} - \frac{E_{b-}}{E_{b+}} V_l - \frac{E_{b+}}{V_{s+}} \right)$. In this case, for each choice of $\pm$, each of $\tilde{\gamma}_{s\pm}$ is a proper geodesic ray contained in some stable principal region $\tilde{P}_{\pm}$ projecting to some stable crown principal region $P_{\pm}$.

We justify the additional descriptions in each case. In the case “Degenerate $S$-point” it follows immediately that $\gamma_s$ is a proper geodesic line in $S_s$ (reps.), and since it does not cross $\Lambda^{\phi}$ it must be either a leaf of $\Lambda^{\phi}$ or contained in a principal region of $\Lambda^{\phi}$; the rest follows. In the cases “One $S$-endpoint”, “Two $S$-endpoints” it follows immediately that $\gamma_s, \gamma_{s\pm}$ (resp.) is a proper geodesic ray, and since it does not cross $\Lambda^{\phi}$ and has an endpoint in $\partial S$ it follows that this ray is contained in some crown principal region; the rest follows.

We now prove in each case that one of conclusions (2), (3), or (4) of Lemma 2.19 holds (the case leading to conclusion (1) has already been handled above). In each case of the proof, we assert that certain rays are principal rays, and this assertion is justified by application of Fact I.1.49 (3b).

We consider first the case “Degenerate $S$-point” as it is denoted above. If $\gamma_s$ is a leaf of $\Lambda^u$ then by Proposition I.2.15 the line $\gamma$ is a leaf of $\Lambda^+_\phi$ which is conclusion (2). Suppose then that $\gamma_s$ is contained in a principal region $\tilde{P}$ of $\tilde{\Lambda}^{\phi}$ lifting $P$. Since $\psi$ is rotationless, we may choose a representative $\Psi \in \text{Aut}_\psi(F_n)$ representing $\psi$ so that $\Psi(\Gamma_s) = \Gamma_s$ and so that $\Psi$ fixes the points of $\partial F_n$ corresponding to points at infinity of $\tilde{P}$, namely its cusps and, in the case $P$ is a crown, the points of $\partial F_n$ corresponding to the ideal endpoints of the component of $\partial \tilde{S}_s$ contained in $\tilde{P}$. Corresponding to $\Psi$ there is a lift $\tilde{\Theta}^{-1} : \tilde{S}_s \to S_s$ of

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\( \Theta^{-1} \) that preserves the principal region \( \bar{P} \), fixing each of its points at infinity. Let \( \xi_1, \xi_2 \) be the ideal endpoints of \( \gamma_s \). If \( \xi_i \) is not a cusp then \( P \) is a crown and \( \xi_i \) is one of the ideal endpoints of the component of \( \partial S_s \) contained in \( \bar{P} \); it follows that at least one of \( \xi_1, \xi_2 \) is a cusp. If \( \xi_i \) is a cusp of \( P \) then the following hold: \( \xi_i \) is an attracting point for the action of \( \Theta^{-1} \) on \( \partial \Gamma_s \) (Proposition I.2.12), and so \( \xi_i \) is an attracting point for the action of \( \Psi \) on \( \partial \Gamma_s \); \( \xi_i \) is an attracting point for the action of \( \Psi \) on all of \( \partial F_n \) (Fact I.1.20); \( \xi_i \) is represented by a principal ray \( \bar{R}_i \) generated by an oriented edge \( \bar{E}_i \) with fixed initial direction and fixed initial vertex \( \bar{v}_i \) (Fact I.1.49), and furthermore \( \bar{E}_i \subset H'_{u,s} \) for otherwise either \( \xi_i \notin \partial \Gamma_s \) or \( \xi_i \in \partial \bar{L}_l \) for some \( l \).

If \( \xi_1, \xi_2 \) are both cusps, choose such principal rays \( \bar{R}_1, \bar{R}_2 \), and note that \( \bar{\mu} = [\bar{v}_1, \bar{v}_2] \) is either a trivial path or a Nielsen path of \( f_{\bar{\Psi}}' \). The path \( \bar{\mu} \), if not trivial, decomposes uniquely into fixed edges and indivisible Nielsen paths of \( f_{\bar{\Psi}}' \). Choose \( \bar{R}_1, \bar{R}_2 \) so as to minimize the number of copies of lifts of \( \rho'_u \) or \( \rho'_u \) in \( \bar{\mu} \). We claim that the interior of \( \bar{\mu} \) is disjoint from the interiors of \( \bar{R}_1 \) and \( \bar{R}_2 \), implying that \( \bar{\gamma} = \bar{R}_1 \bar{R}_2 \) where the ray \( \bar{R}_2 \) is the maximal subpath of \( \bar{\gamma} \) contained in \( \bar{\gamma}_s (\bar{\ell}) = \bar{B}_b \). The ray \( \bar{R}_1 = \bar{\gamma} - \bar{R}_2 \) is the minimal subpath containing every \( \bar{H}'_{u,s} \) edge of \( \bar{\gamma} \). Since \( f_{\bar{\Psi}}' \) is a principal lift fixing \( \xi_1, \xi_2 \) it follows that \( \bar{f}_{\bar{\Psi}}' \) fixes the common base point of \( \bar{R}_1 \) and \( \bar{R}_2 \) and fixes the initial direction of \( \bar{R}_1 \), and therefore \( \bar{R}_1 \) is a principal ray representing \( \xi_1 \), proving conclusion (4). If \( \bar{\ell} \) is an upper boundary component the same analysis works except that the line \( \bar{\gamma}_s (\bar{\ell}) \) is a lift of the bi-infinite iterate of \( \rho'_u \) or \( \rho'_u \), the subray \( \bar{R}_2 \subset \bar{\gamma} \) is the maximal subray that is a lift of a singly infinite iterate of \( \rho'_u \) or \( \rho'_u \), and \( \bar{R}_1 = \bar{\gamma} - \bar{R}_2 \); it still holds that \( \bar{f}_{\bar{\Psi}}' \) fixes the common base point of \( \bar{R}_1 \) and \( \bar{R}_2 \) and the initial direction of \( \bar{R}_1 \), and that \( \bar{R}_1 \) is a principal ray representing \( \xi_1 \), also verifying conclusion (4).

The remaining cases “One \( S \)-endpoint” and “Two \( S \)-endpoints” are very similar to each other, the first leading to conclusion (4) and the second to (3).

Consider the case “One \( S \)-endpoint” as it is denoted above. We have \( \bar{\gamma} = \bar{R}_1 \bar{R}_2 \) where \( \bar{R}_1 = \gamma_s \) and \( \bar{R}_2 = \gamma_l \). We know that \( \gamma_s \) is contained in a stable principal region \( \bar{P} \subset \bar{S} \) covering a crown principal region \( P \subset S \). We may choose \( \Psi \in \text{Aut}_\bar{P}(F_n) \) and \( \Theta : \bar{S}_s \to \bar{S}_s \) as in the case “Degenerate \( S \)-point”, fixing points at infinity of \( \bar{P} \). Note also that \( \Theta \) preserves the component \( \bar{B}_b \) of \( \partial \bar{S}_s \) corresponding to \( E_b \), that \( f_{\bar{\Psi}}' \) is a principal lift preserving \( \bar{B}_b \), and that the set \( \text{Fix}(f_{\bar{\Psi}}') \cap \bar{B}_b \) is nonempty and invariant under \( \Gamma_b \). From the hypothesis of the case “One \( S \)-endpoint” the rays \( \bar{R}_1 = \bar{\gamma}_s \subset \bar{S}_s \) and \( \bar{R}_2 = \bar{\gamma}_l \subset \bar{L}_l \) have a common finite endpoint \( x \in \bar{B}_b \), and initial edge of \( \bar{\gamma}_s \) is in \( \bar{H}'_{u,s} \). It follows that \( \bar{R}_1 \) is the principal ray representing \( \xi_1 \), proving conclusion (4).

In the remaining case “Two \( S \)-endpoints” one carries out the analysis of the previous paragraph on each of the rays \( \bar{R}_1 = \bar{\gamma}_{s-} \) and \( \bar{R}_2 = \bar{\gamma}_{s+} \), proving that they are principal rays,
2.5 General nonattracted lines and the Proof of Theorem G

We are given rotationless $\phi$, $\psi = \phi^{-1} \in \text{Out}(F_n)$, a lamination pair $\Lambda_{s}^{\pm} \in \mathcal{L}^\pm(\phi)$, and a CT $f : G \to G$ representing $\phi$ with EG stratum $H_{r}$ corresponding to $\Lambda_{s}^{+}$ (note that we are abandoning the notational conventions of Section 2.1).

From Definition 1.2 we have the path set $\langle Z, \hat{\rho}_{r} \rangle \subset \hat{\mathcal{B}}$. From Lemma 1.5 this path set is a groupoid and each line $\gamma \in \langle Z, \hat{\rho}_{r} \rangle$ is carried by $\mathcal{A}_{na}(\Lambda_{s}^{\pm})$. Recall also Lemma 2.1 which says that $\gamma \in \mathcal{B}_{na}(\Lambda_{s}^{\pm})$ as long as it satisfies at least one of the following conditions.

1. $\gamma$ is carried by $\mathcal{A}_{na}(\Lambda_{s}^{+})$.
2. $\gamma \in \mathcal{B}_{sing}(\psi)$.
3. $\gamma \in \mathcal{B}_{gen}(\psi)$.

Since each line carried by $\mathcal{A}_{na}(\Lambda_{s}^{+})$ is in $\mathcal{B}_{ext}(\Lambda_{s}^{\pm}; \psi)$ it follows that for each line $\gamma \in \langle Z, \hat{\rho}_{r} \rangle$ we have $\gamma \in \mathcal{B}_{ext}(\Lambda)$; we use this repeatedly in this section.

To simplify the notation of the proof we define the set of good lines in $\mathcal{B}$ to be

$$\mathcal{B}_{good}(\Lambda_{s}^{\pm}; \psi) = \mathcal{B}_{ext}(\Lambda_{s}^{\pm}; \psi) \cup \mathcal{B}_{sing}(\psi) \cup \mathcal{B}_{gen}(\psi)$$

and we repeatedly use Proposition 2.14 which with this notation says that $\mathcal{B}_{good}(\Lambda_{s}^{\pm}; \psi)$ is closed under concatenation.

The conclusion of Theorem G says that $\mathcal{B}_{na}(\Lambda_{s}^{+}) = \mathcal{B}_{good}(\Lambda_{s}^{\pm}; \psi)$. One direction of inclusion, namely $\mathcal{B}_{na}(\Lambda_{s}^{+}) \supset \mathcal{B}_{good}(\Lambda_{s}^{\pm}; \psi)$, follows from Lemma 2.1 and the fact that each line in $\mathcal{B}_{ext}(\Lambda_{s}^{\pm}; \psi)$ is a concatenation of lines in $\mathcal{B}_{sing}(\psi)$ and lines carried by $\mathcal{A}_{na}(\Lambda_{s}^{+})$.

We turn now to proof of the opposite inclusion $\mathcal{B}_{na}(\Lambda_{s}^{+}) \subset \mathcal{B}_{good}(\Lambda_{s}^{\pm}; \psi)$. Given $\gamma \in \mathcal{B}_{na}(\Lambda_{s}^{+})$, if the height of $\gamma$ is less than $r$ then $\gamma \in \langle Z, \hat{\rho}_{r} \rangle$ and we are done. Henceforth we proceed by induction on height. Define an inductive concatenation of $\gamma$ to be an expression of $\gamma$ as a concatenation of finitely many lines in $\mathcal{B}_{good}(\Lambda_{s}^{\pm}; \psi)$ and at most one line $\nu$ of height lower than $\gamma$. If we can show that $\gamma$ has an inductive concatenation, we prove that $\gamma$ is good as follows. In some cases $\nu$ does not occur in the concatenation and so $\gamma$ is good by Proposition 2.14. Otherwise, using invertibility of concatenation, it follows that $\nu$ is expressed as a concatenation of good lines plus the line $\gamma$, all of which are known to be in $\mathcal{B}_{na}(\Lambda_{s}^{+})$. Applying Lemma 2.3 we therefore have $\nu \in \mathcal{B}_{na}(\Lambda_{s}^{+})$. Applying induction on height it follows that $\nu$ is good, and so again $\gamma$ is good by Proposition 2.14.

The induction step breaks into two major cases, depending on whether or not the stratum of the same height as $\gamma$ is NEG or EG. For the case of an NEG stratum we will use the following:

**Lemma 2.20.** Suppose that $\phi, \psi = \phi^{-1} \in \text{Out}(F_n)$ are rotationless, that $f : G \to G$ is a CT representing $\phi$, that $E_{s}$ is the unique edge in an NEG stratum $H_{s}$, and that both endpoints of $E_{s}$ are contained in $G_{s-1}$. Let $E_{s}$ be a lift of $E_{s}$, let $f : G \to G$ be the lift of $f$ that
Since $\psi$ projected ray the highest edge splitting vertex determined by term is finite. Since $\gamma$ by $\gamma$ splitting. Lemma 1.6 (4) implies that each term is contained in $P$ and $\Psi$ is $P$-related. By Lemma 2.20, there exist $(\hat{\psi}^m)$ so assume otherwise. The ray $\tilde{f}$ and so each of $B_1, B_2$ is $\Psi$-invariant. By Fact I.1.21 applied to $\Psi \mid B_j,$ there exists $m > 0$ and points $P_j \in \text{Fix}_N(\tilde{\psi}^m) \cap \partial \tilde{C}_j$ for $j = 1, 2.$ Since the line $\tilde{\sigma}$ connecting $P_1$ to $P_2$ is not birecurrent it does not project to either an axis or a generic leaf of some element of $L(\phi^{-1}).$ Thus $\Psi^m \in P(\psi).$ Since $\psi$ is rotationless, $\Psi \in P(\psi)$ and $\sigma \in B_{\text{sing}}(\psi).$

Proof. By Fact I.1.44 and Definition I.1.29 (6), no component of $G_{s-1}$ is contractible. Letting $C_1, C_2 \subset G$ be the components of the full pre-image of $G_{s-1}$ that contain the initial and terminal endpoints of $\tilde{E}_s$ respectively, there are nontrivial free factors $B_1, B_2$ that satisfy $\partial B_j = \partial C_j.$ Each of $C_1, C_2$ is preserved by $\tilde{f}$ and so each of $B_1, B_2$ is $\Psi$-invariant. By Fact I.1.21 applied to $\Psi \mid B_j,$ there exists $m > 0$ and points $P_j \in \text{Fix}_N(\tilde{\psi}^m) \cap \partial \tilde{C}_j$ for $j = 1, 2.$ Since the line $\tilde{\sigma}$ connecting $P_1$ to $P_2$ is not birecurrent it does not project to either an axis or a generic leaf of some element of $L(\phi^{-1}).$ Thus $\Psi^m \in P(\psi).$ Since $\psi$ is rotationless, $\Psi \in P(\psi)$ and $\sigma \in B_{\text{sing}}(\psi).$

Fix now $s \geq r$ and assume as an induction hypothesis that all lines in $B_{\text{na}}(A_\phi^+) \cap B_{\text{good}}(A_\phi^+; \psi).$ Fix a line $\gamma \in B_{\text{na}}(A_\phi^+)$ of height $s \leq s.$ Let $\tilde{\gamma}$ be a lift of $\gamma$ and let $P$ and $Q$ be its initial and terminal endpoints respectively.

**Case 1:** $H_s$ is NEG. Let $E_s$ be the unique edge in $H_s.$ If $E_s$ is closed and $\gamma$ is a bi-infinite iterate of $E_s$ then $E_s \subset \sigma$ and $\gamma \in (\sigma, \hat{\rho}_r)$ so $\gamma \in B_{\text{ext}}(A_\phi^+; \psi).$ We may therefore assume that both endpoints of $E_s$ belong to $G_{s-1}.$

Orient $E_s$ so that its initial direction is fixed. Recall (Lemma 4.1.4 of [BFH00]) that for each occurrence of $E_s$ or $\overline{E}_s$ in the representation of $\gamma$ as an edge path, the line $\gamma$ splits at the initial vertex of $E_s,$ and we refer to this as the highest edge splitting vertex determined by the occurrence of $E_s.$ We also use this terminology for lifts of $E_s$ in the universal cover. By Fact I.1.37, highest edge splitting vertices are principal.

**Case 1A:** Both ends of $\gamma$ have height $s.$ In this case $\gamma$ has a splitting in which each term is finite. Since $\gamma$ is not weakly attracted to $A^+,$ neither is any of the terms in the splitting. Lemma 1.6 (4) implies that each term is contained in $(\sigma, \hat{\rho}_r)$ and so $\gamma$ is contained in $(\sigma, \hat{\rho}_r)$ and we are done.

**Case 1B:** Exactly one end of $\gamma$ has height $s.$ We assume without loss that the initial end of $\gamma$ has height $s.$ Pick a lift $\tilde{\gamma},$ let $\tilde{E}_s$ be the last lift of $E_s$ crossed by $\tilde{\gamma},$ let $\tilde{x} \in \tilde{\gamma}$ be the highest edge splitting vertex determined by $\tilde{E}_s,$ and let $\tilde{\gamma} = \tilde{R}^{-1} \cdot \tilde{R}_+$ be the splitting at $\tilde{x}.$ The ray $\tilde{R}_-$ has height $s$ and crosses lifts of $E_s$ infinitely often, and as in case 1A the projected ray $R_-$ is contained in $(\sigma, \hat{\rho}_r).$ It follows that there exists a subgroup $A \in A_{\text{na}}(A_\phi^+)$ such that $P \in \partial A.$ Let $\tilde{f}$ be the lift of $f$ that fixes $\tilde{x}$ and let $\Phi$ be the corresponding element of $P(\phi).$ Lemma 2.20 implies that $\Psi = \Phi^{-1} \in P(\psi).$

We claim that $A$ is $\Phi$-invariant. By Lemma 1.5 (6) it suffices to show that $\hat{\Phi}(\partial A) \cap \partial A \neq \emptyset.$ This is obvious if $P \in \text{Fix}(\hat{\Phi})$ so assume otherwise. The ray $\hat{f}_\#(R_-)$ is contained in $(\sigma, \hat{\rho}_r)$ by Lemma 1.6 (2) so $P$ and $\hat{\Phi}(P)$ bound a line that projects into $(\sigma, \hat{\rho}_r)$ and so is carried by $A_{\text{na}}(\sigma).$ Another application of Lemma 1.5 (6) implies that $\Phi(P) \in \partial A$ as desired.

By Fact I.1.21 applied to $\Psi \mid A,$ the set $\text{Fix}_N(\tilde{\psi}^m) \cap \partial A$ is nonempty for some $m > 0.$ Since $\psi$ is rotationless and $\Psi \in P(\psi), \psi$, we may take $m = 1,$ from which it follows that $\Psi$ is $A$-related. By Lemma 2.20, there exist $P', Q' \in \text{Fix}_N(\Psi)$ so that the line $\tilde{\sigma} = P'Q'$
crosses $\tilde{E}_s$ in the same direction as $\tilde{\gamma}$ and crosses no other edge of height $\geq s$, and $\tilde{\sigma}$ projects to $\sigma \in B_{\text{sing}}(\psi)$. Let $\tilde{\sigma} = \tilde{R}^-_{1} \cdot \tilde{R}^+_1$ be the highest edge splitting determined by $\tilde{x}$. Assuming that $P \neq P'$, the line $\tilde{\mu} = \tilde{P}P'$ has endpoints in $\partial A \cup \text{Fix}_N(\Psi)$ and so projects to $\mu \in B_{\text{ext}}(\Lambda^+_{\phi})$. If $\gamma$ crosses $\tilde{E}_s$ in the backwards direction then $\tilde{E}_s$ is the last edge of both $\tilde{R}^-_-$ and $\tilde{R}^-_+$ and each of $\tilde{R}^+_-$ and $\tilde{R}^+_+$ have height $\leq s - 1$; otherwise each of $\tilde{R}^-_-$ and $\tilde{R}^+_+$ is a concatenation of $\tilde{E}_s$ followed by a ray of height $\leq s - 1$. In either case, assuming that $Q \neq Q'$, it follows that the line $\tilde{\nu} = \tilde{Q}Q'$ has height $\leq s - 1$. We therefore have an inductive concatenation $\gamma = \mu \circ \sigma \circ \nu$, with $\mu$ omitted when $P = P'$ and $\nu$ omitted when $Q = Q'$, and Case 1B is completed.

**Case 1C: Neither end of $\gamma$ has height $s$.** We induct on the number $m$ of height $s$ edges in $\gamma$. The base case, where $m = 0$, follows from induction on $s$. Let $\tilde{\gamma} = \tilde{R}^{-1} \tilde{R}^+_+$ be the splitting determined by the last highest edge splitting vertex $\tilde{x}$ in $\tilde{\gamma}$, let $\tilde{f}$ be the lift of $f$ that fixes $\tilde{x}$, and let $\Phi \in P(\phi)$ correspond to $\tilde{f}$. As in Case 1B, from Lemma 2.20 it follows that $\Psi = \Phi^{-1} \in P(\psi)$ and that there exist $P', Q' \in \text{Fix}_N(\Psi)$ so that the line $\tilde{\sigma}$ connecting $P'$ to $Q'$ crosses the last height $s$ edge of $\tilde{\gamma}$ in the same direction as $\tilde{\gamma}$ and crosses no other edge of height $\geq s$. Let $\tilde{\sigma} = \tilde{R}^-_{1-1} \tilde{R}^+_1$ be the highest edge splitting determined by $\tilde{x}$. The line $\mu_1 = \tilde{P}P'$ is obtained by tightening $\tilde{R}^{-1} \tilde{R}^+_1$, and the line $\mu_2 = Q'Q$ is obtained by tightening and $\tilde{R}^+_1 \cdot \tilde{R}^+_1$. These lines have height $\leq s$, cross fewer than $m$ edges of height $s$, and are not weakly attracted to $\Lambda^+_{\phi}$ by Lemma 2.3, because the rays $\tilde{R}^-_-, \tilde{R}^+_-, \tilde{R}^+_1$ and $\tilde{R}^+_1$ are not weakly attracted to $\Lambda^+_{\phi}$. By induction on $m$ we have $\mu_1, \mu_2 \in B_{\text{good}}(\Lambda^+_{\phi} ; \psi)$. Since $\sigma \in B_{\text{sing}}(\psi)$, it follows that $\gamma = \mu_1 \circ \sigma \circ \mu_2 \in B_{\text{good}}(\Lambda^+_{\phi} ; \psi)$, completing Case 1C.

**Case 2: $H_s$ is EG.** Let $\Lambda^+_{\phi} \in \mathcal{L}(\phi)$ be the lamination associated to $H_s$ with dual lamination denoted $\Lambda^-_{\phi} \in \mathcal{L}(\psi)$. Applying Theorem 1.1.30 with $C$ being $[G_r] \subset [G_s]$, let $f' : G' \to G'$ be a CT representing $\psi$ with EG stratum $H'_s$ associated to $\Lambda^-_{\phi}$ and EG stratum $H'_r$ associated to $\Lambda^-_{s}$ so that $[G_r] = [G'_r]$ and $[G_s] = [G'_s]$. Let $\gamma'$ be the realization of $\gamma$ in $G'$, a line of height $s'$. Using the $F_n$-equivariant identification $\partial G_s \approx \partial G'_s$, there is a lift $\tilde{\gamma}'$ of $\gamma'$ with endpoints $P, Q$.

**Case 2A: $\gamma$ is not weakly attracted to $\Lambda^+_{\phi}$.** This is the case where we apply Lemmas 2.18 and 2.19. In the situation where $\gamma'$ is a singular line of $\psi$ or a generic leaf of $\Lambda^+_{\phi}$, or in the geometric situation where $\gamma'$ is a bi-infinite iterate of the height $s'$ closed indivisible Nielsen path $\rho_{s'}$, we have $\gamma' \in B_{\text{good}}(\Lambda_{\phi}; \psi)$ and we are done. The situation where $\gamma'$ is a singular line of $\psi$ includes all cases of Lemmas 2.18 and 2.19 where $\gamma' = R_{-1} \mu R_{+}$, each of $R_{-}, R_{+}$ is either a height $s'$ principal ray or a singly infinite iterate of a height $s'$ closed indivisible Nielsen path, and $\mu$ is either trivial or a height $s'$ Nielsen path. We may therefore assume that none of these situations occurs. In all remaining situations, we divide into two subcases depending on whether one or two ends of $\gamma'$ have height $s'$.

Consider first the subcase where only one end of $\gamma'$, say the initial end, has height $s'$. By applying Lemma 2.18 (3) or Lemma 2.19 (4) we obtain a decomposition $\gamma' = R_{-1} \mu R_{+}$ where $R_{-}$ is a height $s'$ principal ray, $\mu$ is either a trivial path or a height $s'$ Nielsen path,
and the ray \( R_+ \) has height \(< s' \). Lifting the decomposition of \( \gamma' \) we obtain a decomposition \( \tilde{\gamma}' = \tilde{R}_+^{-1} \mu \tilde{R}_+ \) where \( \tilde{R}_+, \tilde{R}_- \) have endpoints \( P, Q \). Let \( x \) be the initial point of \( R_+ \), lifting to the initial point \( \tilde{x} \) of \( \tilde{R}_+ \). The component \( \Gamma \) of the full pre-image of \( G_{s'}^{-1} \) that contains \( \tilde{x} \) is \( \tilde{f}' \)-invariant and infinite and so there is a free factor \( B \) such that \( Q \in \partial B = \partial \Gamma \). Since \( \Psi \) is principal, Lemma I.1.21 implies the existence of \( Q' \in \Fix_N(\tilde{\Psi}) \cap \partial \Gamma \). The line \( \tilde{\tau}' \) connecting \( P \) to \( Q' \) projects to \( \tau' \in B_{\text{sing}}(\psi) \); let \( \tau \) be the realization of \( \tau' \) in \( G \). The line \( \tilde{\nu}' = \tilde{\tau}'^{-1} \circ \tilde{\gamma}' \) is contained in \( \Gamma \) and so projects to a line \( \nu' = \tau'^{-1} \circ \gamma' \) of height \(< s' \) whose realization in \( G \) is a line \( \nu \) of height \(< s \). We obtain an inductive concatenation \( \gamma = \tau \circ \nu \), completing the first subcase of Case 2A.

Consider next the subcase where both ends of \( \gamma' \) have height \( s' \). Applying Lemma 2.18 (2) or Lemma 2.19 (3), and keeping in mind the situations that we have assumed not to occur, there is a decomposition \( \gamma' = R_1^{-1} \mu R_2 \) where \( R_1, R_2 \) are both height \( s' \) principal rays, and \( \mu \) has one of the forms \( \beta, \alpha \beta, \beta \alpha, \alpha \beta \alpha \) where \( \beta \) is a nontrivial path of height \(< s' \) and \( \alpha \) (if it occurs) is a height \( s' \) nonclosed indivisible Nielsen path oriented to have initial vertex in the interior of \( H'_{s'} \) and terminal vertex in \( G'_{s'}^{-1} \). Absorbing occurrences of \( \alpha \) into the incident principal rays \( R_1, R_2 \), we obtain rays \( \tilde{R}, \tilde{R}_+ \) containing \( R_1, R_2 \) respectively, and a decomposition \( \gamma' = R_1^{-1} \beta R_+ \) which lifts to a decomposition \( \tilde{\gamma}' = \tilde{R}_1^{-1} \beta \tilde{R}_+ \). Let \( x \) be the initial point of \( \tilde{R}_- \). There is a principal lift \( \tilde{f} : G' \to \tilde{G}' \) with associated \( \Psi \in P(\psi) \) such that \( \tilde{R}_1 \) is a principal ray for \( \tilde{f} \) fixing the initial point \( \tilde{y} \) of \( \tilde{R}_1 \). Since either \( \tilde{x} = \tilde{y} \) or the segment \([\tilde{x}, \tilde{y}]\) is a lift of \( \alpha, \) it follows that \( \tilde{f} \) fixes \( \tilde{x} \) and that \( \tilde{f}(\tilde{R}_-) = \tilde{R}_- \). As in the previous subcase there is a ray based at \( \tilde{x} \) with height \(< s' \) and terminating at some \( Q' \in \Fix_N(\tilde{\Psi}) \). The line \( \tilde{\tau}' \) connecting \( P \) to \( Q' \) projects to \( \tau' \in B_{\text{sing}}(\psi) \) which is good, and the line \( \tilde{\tau}' = \tilde{\tau}'^{-1} \circ \gamma' \) has only one end with height \( s' \). By the previous subcase, the realization \( \sigma \) of \( \tilde{\tau} \circ \gamma \) in \( G \) is good and hence \( \gamma = \tau \circ \sigma \) is good.

**Case 2B:** \( \gamma \) is weakly attracted to \( \Lambda_+^\circ \). In this case \( H_s \subset Z \), for otherwise \( \gamma \) is weakly attracted to \( \Lambda_+^\circ \) as well, contrary to hypothesis.

**Special case:** We first consider the special case that \( \gamma \) decomposes at a fixed vertex \( v \) into two rays \( \gamma = \gamma_1 \gamma_2 \) so that \( \gamma_1 \) has height \(< s \) and \( \gamma_2 \in \langle Z, \hat{\rho}_v \rangle \). In \( G \) there is a corresponding decomposition \( \tilde{\gamma} = \tilde{\gamma}_1 \tilde{\gamma}_2 \) at a vertex \( \tilde{v} \), and there is a lift \( \tilde{f} \) fixing \( \tilde{v} \) with corresponding \( \Phi \in \text{Aut}(F_n) \) representing \( \phi \). Let \( \Psi = \Phi^{-1} \).

Recall the notation established in Definition 1.2 of the graph immersion \( h : K \to G \) used to define \( A_{\text{na}}(\Lambda_+^\circ) \). Since the ray \( \gamma_2 \) is an element of the path set \( \langle Z, \hat{\rho}_v \rangle \), it follows from Definition 1.2 that \( \gamma_2 \) lifts via the immersion \( h : K \to G \) to a ray in the finite graph \( K \). The image of this lifted ray must therefore be contained in a noncontractible component \( K_0 \) of \( K \). There is a lift of universal covers \( \tilde{h} : \tilde{K}_0 \to \tilde{G} \) such that \( \tilde{h}(\tilde{K}_0) \) contains \( \tilde{\gamma}_2 \) and such that the stabilizer of \( \tilde{h}(\tilde{K}_0) \) is a subgroup \( A \in A_{\text{na}}(\Lambda_+^\circ) \) whose conjugacy class is the one determined by the immersion \( h : K_0 \to G \). By construction we have \( Q \in \partial A \). If \( \tilde{\Phi}(Q) \neq Q \) then \( Q \) and \( \tilde{\Phi}(Q) \) bound a line that projects into \( \langle Z, \hat{\rho}_v \rangle \) and so is carried by \( A_{\text{na}}(\Lambda_+^\circ) \), and by applying Lemma 1.5 (6) it follows that \( \tilde{Q} \in \partial A ; \) this is also true if \( \tilde{\Phi}(Q) = Q \). In particular \( \tilde{\Phi} \) and therefore also \( \Psi \), preserves \( A \). By Fact 1.1.21 applied to \( \Psi | A \) there exists an integer \( q \geq 1 \) so that \( \Fix_N(\tilde{\Psi}) \cap \partial A \neq \emptyset \); we choose \( q \) to be the minimal such integer
and then we choose $Q' \in \text{Fix}_N(\tilde{\Psi}^q) \cap \partial A$. If $Q \neq Q'$ then the line $\beta$ connecting $Q$ to $Q'$ is carried by $A$ and so $\beta \in \mathcal{B}_{\text{good}}(A^\pm; \psi)$.

The component $C$ of $G_{t-1}$ that contains the ray $\gamma_1$ is noncontractible, and letting $\tilde{C}$ be the component of the full pre-image of $C$ that contains $\gamma_1$, the stabilizer of $\tilde{C}$ is a nontrivial free factor $B$ such that $\partial B = \partial \tilde{C}$. By construction we have $P \in \partial B$. Also $\tilde{C}$ is invariant under $\tilde{f}$ and so $B$ is invariant under $\Psi$. By Fact I.1.21 applied to $\Psi \mid B$ there exists an integer $p \geq 1$ so that $\text{Fix}_N(\tilde{\Psi}^p) \cap \partial B \neq \emptyset$; we choose $p$ to be the minimal such integer and then we choose $P' \in \text{Fix}_N(\tilde{\Psi}^p) \cap \partial B$. If $P \neq P'$ then the line $\nu$ connecting $P$ to $P'$ has height $< s$.

For some least integer $m > 0$ we have $P', Q' \in \text{Fix}_N(\Psi^m)$. If $P' \neq Q'$, consider the line $\mu$ connecting $P'$ to $Q'$. By hypothesis $\psi$ is rotationless and so $\Psi$ is principal if and only if $\Psi^m$ is principal. It follows that if $\Psi$ is principal then $m = 1$ and $\mu \in \mathcal{B}_{\text{sing}}(\psi)$, whereas if $\Psi$ is not principal then $\text{Fix}_N(\Psi^m) = \{P', Q'\}$ so $m = p = q = 1$ or 2 and either $\mu \in \mathcal{B}_{\text{gen}}(\psi)$ or $\mu$ is a periodic line corresponding to a conjugacy class that is invariant under $\phi^2$. In all cases, $\mu \in \mathcal{B}_{\text{good}}(A^\pm; \psi)$.

We therefore have an inductive concatenation of the form $\gamma = \nu \circ \mu \circ \tilde{\beta}$, where $\nu$ is omitted if $P = P'$, $\mu$ is omitted if $P' = Q'$, and $\tilde{\beta}$ is omitted if $Q' = Q$, but at least one of them is not omitted because $P \neq Q$. This completes the proof in the special case.

**General case.** First we reduce to the subcase that $\gamma$ has a subray of height $s$ in $\langle Z, \hat{\rho}_r \rangle$.

To carry out this reduction, after replacing $\gamma$ with some $\phi^k_\#(\gamma)$ we may assume that $\gamma$ contains a long piece of $\Lambda^+_\phi$ and so has a splitting $\gamma = R_- \cdot E \cdot R_+$ where $E$ is an edge of $H_s$ whose initial vertex and initial direction are principal. Lifting this splitting we have $\tilde{\gamma} = \tilde{R}_- \cdot \tilde{E} \cdot \tilde{R}_+$. Let $\tilde{f}$ be the principal lift that fixes the initial vertex of $\tilde{E}$ and let $\tilde{R}'$ be the principal ray determined by the initial direction of $\tilde{E}$. Neither the line $R_-R'$ nor the line obtained by tightening $\tilde{R}_+\tilde{R}'$ is weakly attracted to $\Lambda^+_\phi$, because $\tilde{R}_-$ and $\tilde{R}_+$ are not weakly attracted and the ray $R'$ is contained in $\langle Z, \hat{\rho}_r \rangle$. Each of these lines contains a subray of $R'$, and any subray of $R'$ contains a further subray of height $s$ in $\langle Z, \hat{\rho}_r \rangle$, and so it suffices to show that each of these lines is contained in $\mathcal{B}_{\text{good}}(\phi)$, which completes the reduction.

Let $t$ be the highest integer in $\{t, \ldots, s-1\}$ for which $H_t$ is not contained in $Z$. Using that $\gamma$ has a subray of height $s$ in $\langle Z, \hat{\rho}_r \rangle$, after making it a terminal subray by possibly inverting $\gamma$, there is a decomposition $\gamma = \ldots \nu_2 \mu_1 \nu_1 \mu_0$ into an alternating concatenation where the $\mu_i$’s are the maximal subpaths of $\gamma$ of height $> t$ that are in $\langle Z, \hat{\rho}_r \rangle$, and the $\nu_i$’s are the subpaths of $\gamma$ that are complementary to the $\mu_i$’s. Each subpath $\nu_i$ has fixed endpoints, is contained in $G_t$, and is not an element of $\langle Z, \hat{\rho}_r \rangle$. Further, $\nu_i$ is finite unless the decomposition of $\gamma$ is finite and $\nu_i$ is the leftmost term of the decomposition. Since $H_t$ is not a zero stratum, each component of $G_t$ is non-contractible and hence $f$-invariant. We prove that the above decomposition of $\gamma$ is finite by assuming that it is not and arguing to a contradiction.

We claim that for all $l$ and all $m \geq 1$ the following hold:

(1) If $\nu_l$ is finite, not all of $f^m_\#(\nu_l)$ is cancelled when $f^m_\#(\nu_l) f^m_\#(\nu_1) f^m_\#(\mu_{t-1})$ is tightened to $f^m_\#(\nu_l \nu_1 \mu_{t-1})$. Moreover, as $m \to \infty$ the part of $f^m_\#(\nu_l)$ that is not cancelled contains subpaths of $\Lambda^+_\phi$ which cross arbitrarily many edges of $H_r$. 37
(2) Not all of $f^m_{#}(\mu_1)$ is cancelled when $f^m_{#}(\nu_{l+1})f^m_{#}(\mu_1)f^m_{#}(\nu_l)$ is tightened to $f^m_{#}(\nu_{l+1}\mu_1\nu_l)$. Assuming without loss of generality that $m$ is large, (1) follows from finiteness of the path $\nu_l$ by applying Lemma 1.6 (4) which implies that the path $f^m_{#}(\nu_l)$ contains subpaths of $\Lambda_\phi^+$ that cross arbitrarily many edges of $H_r$, whereas $f^m_{#}(\mu_1)$ and $f^m_{#}(\mu_{l-1})$ contain no such subpaths. Item (2) follows from the fact that each component of $G_t$ is $f$-invariant which implies that $f^m_{#}(\nu_{l+1}\mu_1\nu_l) \not\subset G_t$.

Items (1) and (2) together imply that if $\nu_l$ is finite, the only cancellation that occurs to $f^m_{#}(\nu_l)$ when the concatenation $...f^m_{#}(\nu_2)f^m_{#}(\mu_1)f^m_{#}(\nu_1)f^m_{#}(\mu_0)$ is tightened to $f^m_{#}(\gamma)$ is that which occurs when the subpath $f^m_{#}(\mu_1)f^m_{#}(\nu_l)f^m_{#}(\mu_{l-1})$ is tightened to $f^m_{#}(\mu_1\nu_l\mu_{l-1})$. But then $f^m_{#}(\gamma)$ contains subpaths of a generic leaf of $\Lambda_\phi^+$ that cross arbitrarily many edges of $H_r$, in contradiction to the assumption that $\gamma$ is not weakly attracted to $\Lambda_\phi^+$.

Not only have we shown that the decomposition of $\gamma$ is finite, we have shown that no $\nu_l$ term of the decomposition can be finite, and so either $\gamma = \mu_0$ or $\gamma = \nu_1\mu_0$. If $\gamma = \mu_0$ then $\gamma \in \langle Z, \hat{\rho}_r \rangle$ and we are done. If $\gamma = \nu_1\mu_0$ then $\gamma$ falls into the special case and we are also done. This completes the proof of Theorem 2.6.

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