ELLIPITIC GENERA, TORUS ORBIFOLDS AND MULTI-FANS; II

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1. Introduction

This article is a continuation of [HM2].

Elliptic genera for manifolds introduced by Ochanine and other people has a remarkable feature called rigidity. If the circle group acts non-trivially on a closed almost complex (or more generally stably almost complex) manifold whose first Chern class is divisible by a positive integer $N$ greater than 1, then its equivariant elliptic genus of level $N$ is rigid, that is, it is a constant character of the circle group. It was conjectured by Witten [W] and proved by Taubes [T], Bott-Taubes [BT] and Hirzebruch [Hir]. Liu [L] found a simple proof using the modular property of elliptic genera.

For stably almost complex orbifolds an invariant called orbifold elliptic genus is defined. Though the rigidity property does not hold in this case in the same form as the case of manifolds, versions of rigidity theorem can be formulated and shown to hold [Hat]. The orbifold elliptic genus is also defined for multi-fans and is likewise expected to have nice properties [HM2]. The notion of multi-fan is a generalization of that of fan in the theory of toric varieties. It was first introduced in [M] and was further developed in [HM1]. In [HM2] we proved a rigidity theorem concerning the elliptic genus for complete simplicial multi-fans and gave an application to toric varieties to the effect that a non-singular complete toric variety of dimension $n$ whose canonical line bundle is linearly equivalent to a $T$-Cartier divisor of the form $nD$ is isomorphic to a certain projective space bundle over a projective line.

In [Hat] we proved rigidity theorems and vanishing theorems concerning orbifold elliptic genus for general almost complex orbifolds. In this paper we shall give analogues of those vanishing theorems for multi-fans (Theorem 3.2, Theorem 3.3 and theorem 3.4). As a special feature about multi-fans, vanishing of orbifold elliptic genus and modified orbifold elliptic genus holds under suitable assumptions.

As an application it will be shown that a $Q$-factorial complete toric variety of dimension $n$ whose canonical divisor is linearly equivalent to a $T$-Cartier divisor of the form $nD$ is either non-singular or isomorphic to a certain weighted projective space (Proposition 5.6 and Corollary 5.9). This result was already proved by O. Fujino [Fu] for projective toric varieties. The author is grateful to him for having communicated his result to the author.

The paper is organized as follows. In Section 2 we recall some basic facts about multi-fans from [HM1] and [HM2]. In Section 3 we define the elliptic genus, orbifold elliptic genus and modified orbifold elliptic genus of a pair of a multi-fan and a set of generating vectors and formulate main theorems. Properties of the $T_y$-genus, orbifold $T_y$-genus and modified orbifold $T_y$-genus of multi-fans are discussed. In Section 4 the proofs of the main results are given. Section 5 is devoted to applications to special types of multi-fans and, in particular, fans associated to toric varieties.
The author wishes to thank M. Masuda, coauthor of the joint papers [HM1] and [HN2]. The collaboration with him was much profitable to the author for this work as well. He declined to be coauthor of this paper which in fact grew up from our collaboration.

2. Multi-fans

The present paper depends heavily on its first part [HM2]. In this section we shall summarize materials we need in the sequel from [HM1] and [HM2].

Let $L$ be a lattice of rank $n$. An $n$-dimensional simplicial multi-fan in $L$ is a triple $\Delta = (\Sigma, C, w^\pm)$. We shall call it simply a multi-fan in this paper. Here $\Sigma$ is an augmented simplicial set, that is, $\Sigma$ is a simplicial set with empty set $\ast = \emptyset$ added as $(-1)$-dimensional simplex. $\Sigma^{(k)}$ denotes the $k - 1$ skeleton of $\Sigma$ so that $\ast \in \Sigma^{(0)}$. We assume that $\Sigma = \sum_{k=0}^\infty \Sigma^{(k)}$, and $\Sigma^{(n)} \neq \emptyset$. $C$ is a map from $\Sigma^{(k)}$ into the set of $k$-dimensional strongly convex rational polyhedral cones in the vector space $L_\mathbb{R} = L \otimes \mathbb{R}$ for each $k$ such that, if $J$ is a face of $I$, then $C(J)$ is a face of $C(I)$. $w^\pm$ are maps $\Sigma^{(n)} \to \mathbb{Z}_{\geq 0}$. We set $w(I) = w^+(I) - w^-(I)$. A vector $v \in L_\mathbb{R}$ will be called generic if $v$ does not lie on any linear subspace spanned by a cone in $C(\Sigma)$ of dimension less than $n$. For a generic vector $v$ we set $d_v = \sum_{v \in C(I)} w(I)$, where the sum is understood to be zero if there is no such $I$. We call a multi-fan $\Delta = (\Sigma, C, w^\pm)$ of dimension $n$ pre-complete if the integer $d_v$ is independent of the choice of generic vectors $v$. We call this integer the degree of $\Delta$ and denote it by $\deg(\Delta)$.

For each $K \in \Sigma$ we set

$$\Sigma_K = \{ J \in \Sigma \mid K \subset J \}.$$

It inherits the partial ordering from $\Sigma$ and becomes an augmented simplicial set where $K$ is the unique minimum element in $\Sigma_K$. Let $(L_K)_\mathbb{R}$ be the linear subspace of $L_\mathbb{R}$ generated by $C(K)$. Let $L^K_\mathbb{R}$ be the quotient space of $L_\mathbb{R}$ by $(L_K)_\mathbb{R}$ and $L^K$ the image of $L$ in $L^K_\mathbb{R}$. $L^K_\mathbb{R}$ is identified with $L^K \otimes \mathbb{R}$. For $J \in \Sigma_K$ we define $C^K(J)$ to be the cone $C(J)$ projected on $L^K_\mathbb{R}$. We define two functions

$$w^K_\pm: \Sigma_K^{\{n-|K|\}} \subset \Sigma^{(n)} \to \mathbb{Z}_{\geq 0}$$

to be the restrictions of $w^\pm$ to $\Sigma_K^{\{n-|K|\}}$. The triple $\Delta_K := (\Sigma_K, C_K, w^K_\pm)$ is a multi-fan in $L^K$ and is called the projected multi-fan with respect to $K \in \Sigma$. If $K = \emptyset$ then $\Delta_K = \Delta$.

A pre-complete multi-fan $\Delta = (\Sigma, C, w^\pm)$ is said to be complete if the projected multi-fan $\Delta_K$ is pre-complete for any $K \in \Sigma$. A multi-fan is complete if and only if the projected multi-fan $\Delta_J$ is pre-complete for any $J \in \Sigma^{(n-1)}$.

Let $M$ be an oriented closed manifold of dimension $2n$ with an effective action of an $n$-dimensional torus $T$. We assume further that the fixed point set $M^T$ is not empty. There is a finite number of subcircles of $T$ such that the fixed point set of each subcircle has codimension 2 components. Let $\{M_i\}_{i=1}^t$ be those components which have non-empty intersection with $M^T$. We call $M$ a torus manifold if a preferred orientation of each $M_i$ is given. The $M_i$ are called characteristic submanifolds. A complete multi-fan $\Delta(M) = (\Sigma(M), C(M), w^\pm(M))$ in the lattice $H_2(BT)$ is associated with $M$ in a canonical way, where $BT$ is the classifying space of $T$. If $K \in \Sigma(M)$ then the projected multi-fan $\Delta(M)_K$ is closely related to the multi-fan associated with $M_K = \cap_{i \in K} M_i$, where $M_K$ is regarded as a union of torus manifolds.

Let $\Delta = (\Sigma, C, w^\pm)$ be a multi-fan in $L$. If $T$ denotes the torus $L_\mathbb{R}/L$, then $L$ can be canonically identified with $H_2(BT)$. Then there is a unique primitive vector $v_i \in L = H_2(BT)$ which generates the cone $C(i)$ for each $i \in \Sigma(1)$. $\Delta$ is called non-singular.
if \( \{ v_i \ | i \in I \} \) is a basis of the lattice \( L = H_2(BT) \) for each \( I \in \Sigma^{(n)} \). The multi-fan associated with a torus manifold is a complete non-singular multi-fan.

It is sometimes more convenient to consider a set of vectors \( \mathcal{V} = \{ v_i \in L \}_{i \in \Sigma^{(1)}} \) such that each \( v_i \) generates the cone \( C(i) \) in \( L_\mathbb{R} \) but is not necessarily primitive. This is the case for multi-fans associated with torus orbifolds. A torus orbifold \( X \) is a closed oriented orbifold of even dimension with an effective action of a torus of half the dimension of the orbifold \( X \) with some additional condition. A set of codimension 2 suborbifolds \( X_i \) called characteristic suborbifolds is similarly defined as in the case of torus manifolds. To each subcircle \( S \) which fixes \( X \) pointwise there is a unique finite covering \( \tilde{S} \) and an effective action of \( \tilde{S} \) on the orbifold cover of each fiber of the normal bundle. This defines a vector \( v_i \) in \( \text{Hom}(S^1, T) = H_2(BT) = L \) as before. In this way a multi-fan \( \Delta(X) \) and a set of vectors \( \mathcal{V}(X) = \{ v_i \}_{i \in \Sigma^{(1)}} \) are associated to the torus orbifold \( X \).

Hereafter multi-fans are assumed to be complete and we shall always consider the pair of a multi-fan \( \Delta \) and a set of vectors \( \mathcal{V} = \{ v_i \in L \}_{i \in \Sigma^{(1)}} \) as above. In case \( \Delta \) is non-singular it is further assumed that all the \( v_i \) are primitive. If \( I \) is in \( \Sigma^{(n)} \), then \( \{ v_i \}_{i \in I} \) becomes a basis of vector space \( L_\mathbb{R} \). In case \( \Delta \) is non-singular it is a basis of the lattice \( L \). In general, for \( I \in \Sigma^{(n)} \), we define \( L_{I, \mathcal{V}} \) to be the sublattice of \( L \) generated by \( \{ v_i \}_{i \in I} \).

Let \( L^*_{I, \mathcal{V}} \) be the dual lattice of \( L_{I, \mathcal{V}} \) and \( \{ u_i^* \} \) the basis of \( L^*_{I, \mathcal{V}} \) dual to \( \{ v_i \}_{i \in I} \). We identify \( L^*_{I, \mathcal{V}} \) with the lattice in \( L^*_\mathbb{R} \) given by

\[
\{ u \in L^*_\mathbb{R} \mid \langle u, v \rangle \in \mathbb{Z}, \text{ for any } v \in L_{I, \mathcal{V}} \},
\]

where \( \langle u, v \rangle \) is the dual pairing. For \( h \in L/L_{I, \mathcal{V}} \) and \( u \in L^*_{I, \mathcal{V}} \) we define

\[
\chi_I(u, h) = e^{2\pi i \langle u, v(h) \rangle},
\]

where \( v(h) \in L \) is a representative of \( h \). If one fixes \( u, h \mapsto \chi_I(u, h) \) gives a character of the group \( L/L_{I, \mathcal{V}} \).

The dual lattice \( L^* = H^2(BT) \subset H^2(BT; \mathbb{R}) \) is canonically identified with \( \text{Hom}(T, S^1) \). The latter is embedded in the character ring \( R(T) \). In fact \( R(T) \) can be considered as the group ring \( \mathbb{Z}[L^*] \) of the group \( L^* = \text{Hom}(T, S^1) \). It is convenient to write the element in \( R(T) \) corresponding to \( u \in H^2(BT) \) by \( t^u \). The homomorphism \( v^*: R(T) \to R(S^1) = \mathbb{Z}[t, t^{-1}] \) induced by an element \( v \in H_2(BT) = \text{Hom}(S^1, T) \) can be written in the form

\[
v^*(t^u) = t^{(u, v)},
\]

where \( t^m \in R(S^1) \) is such that \( t^m(g) = g^m \) for \( g \in S^1 \).

More generally, set \( L_{\mathcal{V}} = \bigcap_{I \in \Sigma^{(1)}} L_{I, \mathcal{V}} \), and let \( L^*_{\mathcal{V}} \) be the dual lattice of \( L_{\mathcal{V}} \). \( L^*_{\mathcal{V}} \) contains all \( L^*_{I, \mathcal{V}} \) and is generated by all the \( u_i^* \)'s. The group ring \( \mathbb{Z}[L^*_{\mathcal{V}}] \) contains \( \mathbb{Z}[L^*] = R(T) \) and has a basis \( \{ t^u \mid u \in L^*_{\mathcal{V}} \} \) with multiplication determined by the addition in \( L^*_{\mathcal{V}} \):

\[
t^u t^v = t^{u+v}.
\]

If \( v \) is a vector in \( L_{\mathcal{V}} \), then \( v \) determines a homomorphism \( v^*: \mathbb{Z}[L^*_{\mathcal{V}}] \to R(S^1) = \mathbb{Z}[t, t^{-1}] \) sending \( t^u \) to \( t^{(u, v)} \). If we vary \( v \) then \( v^*(t^u) \) determines \( t^u \).

Similarly if \( v_1 \) and \( v_2 \) are vectors in \( L \) they define a homomorphism from a 2-dimensional torus \( T^2 \) into \( T \) and induce a homomorphism \( (v_1, v_2)^*: \mathbb{Z}[L^*] \to R(T^2) = \mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}] \) defined by

\[
(v_1, v_2)^*(t^u) = t_1^{(u, v_1)} t_2^{(u, v_2)}.
\]

If \( v_1 \) and \( v_2 \) belong to \( L_{\mathcal{V}} \), then \( (v_1, v_2)^* \) extends to a homomorphism \( \mathbb{Z}[L^*_{\mathcal{V}}] \to R(T^2) \). We define the equivariant cohomology \( H^*_T(\Delta) \) of a complete multi-fan \( \Delta \) as the face ring
of the simplicial complex $\Sigma$. Namely let $\{x_i\}$ be indeterminates indexed by $\Sigma^{(1)}$, and let $R$ be the polynomial ring over the integers generated by $\{x_i\}$. We denote by $\mathcal{I}$ the ideal in $R$ generated by monomials $\prod_{i \in J} x_i$ such that $J \notin \Sigma$. $H^2_T(\Delta)$ is by definition the quotient $R/\mathcal{I}$. We regard $H^2(\mathbb{T})$ as a submodule of $H^2_T(\Delta)$ by the formula

$$u = \sum_{i \in \Sigma^{(1)}} \langle u, v_i \rangle x_i.$$  

This determines an $H^*(\mathbb{T})$-module structure of $H^2_T(\Delta)$. It should be noticed that this module structure depends on the choice of $\Sigma'$ as above.

For each $I \in \Sigma^{(n)}$ we define the restriction homomorphism $\iota^*_I : H^2_T(\Delta) \to L^*_\Sigma$ by

$$\iota^*_I(x_i) = \begin{cases} u^I_i & \text{for } i \in I \\ 0 & \text{for } i \notin I. \end{cases}$$

$\iota^*_I|H^2(\mathbb{T})$ is the identity map for any $I$, and $\sum_{I \in \Sigma^{(n)}} \iota^*_I$ is injective. Note that, if $\Delta$ is non-singular, then $\iota^*_I$ maps $H^2_T(\Delta)$ into $H^2(\mathbb{T})$.

**Lemma 2.1.** For any $x = \sum_{i \in \Sigma^{(1)}} c_i x_i \in H^2_T(\Delta)$, $c_i \in \mathbb{Z}$, the element

$$\sum_{I \in \Sigma^{(n)}} \frac{w(I)}{|L/L_I, y|} \sum_{h \in L/L_I, y} \chi_I(\iota^*_I(x), h) t^{\iota^*_I(x)}$$

in $\mathbb{C}[L^*_\Sigma]$ actually belongs to $R(\mathbb{T})$.

See Lemma 2.1 in [HM2].

We also use an extended version of this Lemma. Let $K \in \Sigma^{(k)}$ and let $\Delta_K = (\Sigma_K, C_K, w_K)$ be the projected multi-fan. If $I \in \Sigma^{(l)}$ contains $K$, then $I$ is considered as lying in $\Sigma^{(l-k)}$. In order to avoid some notational confusions we introduce the link $\Sigma_K'$ of $K$ in $\Sigma$. It is a simplicial set consisting simplices $J$ such that $K \cup J \in \Sigma$ and $K \cap J = \emptyset$. There is an isomorphism from $\Sigma_K'$ to $\Sigma_K$ sending $J \in \Sigma_K'$ to $K \cup J \in \Sigma_K$. Let $K * \Sigma_K'$ be the join of $K$ (regarded as a simplicial set) and $\Sigma_K'$. Its simplices are of the form $J_1 \cup J_2$ with $J_1 \subset K$ and $J_2 \subset \Sigma_K'$. The torus $T^K$ corresponding to $\Delta_K$ is a quotient of $T$. We consider the polynomial ring $R_K$ generated by $\{x_i \mid i \in K \cup \Sigma_K^{(1)}\}$ and the ideal $\mathcal{J}_K$ generated by monomials $\prod_{i \notin J} x_i$ such that $J \notin K * \Sigma_K'$. We define the equivariant cohomology $H^*_T(\Delta_K)$ of $\Delta_K$ with respect to the torus $T$ as the quotient ring $R_K/\mathcal{J}_K$.

Note that $H^*_T(\Delta_K)$ is different from $H^*_T(K)$.

$H^2(\mathbb{T})$ is regarded as a submodule of $H^2_T(\Delta_K)$ by a formula similar to (11). This defines an $H^*(\mathbb{T})$-module structure on $H^*_T(\Delta_K)$. The projection $H^2_T(\Delta) \to H^2_T(\Delta_K)$ is defined by sending $x_i$ to $x_i$ for $i \in K \cup \Sigma_K^{(1)}$ and putting $x_i = 0$ for $i \notin K \cup \Sigma_K^{(1)}$. The restriction homomorphism $\iota^*_I : H^2_T(\Delta_K) \to L^*_\Sigma$ is also defined for $I \in \Sigma_K^{(n-k)}$ by $\iota^*_I(x_i) = u^I_i$. If $X$ is a torus orbifold, then $H^*_T(\Sigma(X)_K)$ is related to the equivariant cohomology $H^*_T(X_K)$ with respect to the group $T$ (not with respect to $T_K$).

**Theorem 2.2.** Let $\Delta$ be a complete simplicial multi-fan. Let $x = \sum_{i \in K \cup \Sigma_K^{(1)}} c_i x_i \in H^2_T(\Delta_K)$ be as above with all $c_i$ integers. Then the expression

$$\sum_{I \in \Sigma_K^{(n-k)}} \frac{w(I)}{|L/L_I, y|} \sum_{h \in L/L_I, y} \chi_I(\iota^*_I(x), h) t^{\iota^*_I(x)}$$

belongs to $R(\mathbb{T})$. 
Corollary 2.3. Let \( v_1 \) and \( v_2 \) be generic vectors in \( L \) such that \( \langle u^I, v_1 \rangle \) and \( \langle u^I, v_2 \rangle \) are integers for all \( I \in \Sigma^{(n-k)} \), and let \( x = \sum_{i \in \Sigma^{(n-k)}(1)} c_i x_i \in H_T^2(\Delta_K) \) with all \( c_i \) integers. Then

\[
\sum_{I \in \Sigma^{(n-k)}} w(I) \frac{[L/L_{I,Y}]}{[L/L_{I,Y}]} \sum_{h \in L/L_{I,Y}} \chi_I(t^*_I(x), h) t_1^{(i_1^*(x), v_1)} t_2^{(i_2^*(x), v_2)} \prod_{i \in I \setminus K} (1 - \chi_I(u^I_i, h) - 1 - t_1^{(-u^I_i, v_1)} t_2^{(-u^I_i, v_2)})
\]

belongs to \( R(T^2) = \mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}] \).

See Corollary 2.4 of [HM2]. Note that if \( v_1 \) and \( v_2 \) belong to \( L_{I,Y} \) then they satisfy the condition in Corollary 2.3.

Let \( \Delta \) be a complete multi-fan in a lattice \( L \) and \( \mathcal{V} = \{ v_i \}_{i \in \Sigma^{(1)}} \) a set of prescribed edge vectors as before. An \( H^*(BT) \)-module structure of \( H^2(\Delta) \) is defined by (1). The class

\[
\sum_{i \in \Sigma^{(1)}} x_i \in H_T^2(\Delta)
\]

will be called the \textit{equivariant first Chern class} of the pair \( (\Delta, \mathcal{V}) \), and will be denoted by \( c^*_1(\Delta, \mathcal{V}) \). When \( \Delta \) is non-singular, \( \mathcal{V} \) consists of primitive vectors which is determined by \( \Delta \) by our convention. In this case we simply write \( c^*_1(\Delta) \) and call it the \textit{equivariant first Chern class} of the non-singular multi-fan \( \Delta \).

The image of \( c^*_1(\Delta, \mathcal{V}) \) in \( H^2(\Delta, \mathcal{V}) := H_T^2(\Delta)/H^2(BT) \) is called the first Chern class of \( (\Delta, \mathcal{V}) \) and is denoted by \( c_1(\Delta, \mathcal{V}) \). Let \( N > 1 \) be an integer. The first Chern class \( c_1(\Delta, \mathcal{V}) \) is divisible by \( N \) in \( H^2(\Delta, \mathcal{V}) \) if and only if \( c^*_1(\Delta, \mathcal{V}) \) is of the form

\[
c_1^*(\Delta, \mathcal{V}) = Nx + u, \ x \in H^2_T(\Delta), \ u \in H^2(BT).
\]

We set \( u^I = V_1(c^*_1(\Delta, \mathcal{V})) = \sum_{i \in I} u_i \in L^*_I, \mathcal{V} \). Note that \( u^I \) does not belong to \( L^* = H^2(BT) \) in general.

Lemma 2.4. The following three conditions are equivalent:

(i) the first Chern class \( c_1(\Delta, \mathcal{V}) \) is divisible by \( N \),

(ii) \( u^I \mod N \) regarded as an element of \( L^*_I/NL^*_I \) is independent of \( I \in \Sigma^{(n)} \) and belongs to the image of \( L^* = H^2(BT) \),

(iii) there is an element \( u \in H^2(BT) \) such that \( \langle u, v_i \rangle = 1 \mod N \) for all \( i \in \Sigma^{(1)} \).

See Lemma 4.1 in [HM2].

Remark 2.5. Let \( M \) be a torus manifold and \( \Delta(M) \) its associated multi-fan. Put \( \check{H}_T^2(M) = H_T^2(M)/S\)-torsion, where \( S \) is the subset of \( H^*(BT) \) multiplicatively generated by nonzero elements in \( H^2(BT) \). In [M] it was shown that there is a canonical embedding of \( \check{H}_T^2(\Delta(M)) \) in \( H_T^2(M) \), and, in case \( M \) is a stably almost complex torus manifold, \( c^*_1(M) \in H_T^2(M) \) descends to \( c^*_1(\Delta(M)) \in H_T^2(\Delta(M)) \). It follows that, if \( M \) is a stably almost complex torus manifold and \( c_1(M) \) is divisible by \( N \), then \( c_1(\Delta(M)) \) is also divisible by \( N \). Even if \( X \) is a stably almost complex orbifold it can be shown that \( c^*_1(X) \in H_T^2(X; \mathbb{R}) \) descends to \( c^*_1(\Delta(X), \mathcal{V}(X)) \in H_T^2(\Delta(X)) \otimes \mathbb{R} \). But the divisibility of the first Chern class has no meaning with real coefficients. We have to work with orbifold cohomology theory with integer coefficients, cf. Remark 3.2 in [Hat].

An element \( x = \sum_{i \in \Sigma^{(1)}} c_i x_i \) is called \textit{T-Cartier} if \( t^*_I(x) = \sum_{i \in \Sigma^{(1)}} c_i u^I_i \) belongs to \( L^* \) for each \( I \in \Sigma^{(n)} \). Note that this definition depends on the choice of \( \mathcal{V} \). The equivariant
The first Chern class $c^T_1(\Delta, \mathcal{V})$ is $T$-Cartier if and only if $u^I = \sum_{i \in I} u^I_i$ belongs to $L^*$ for each $I \in \Sigma(n)$. The Chern class $c_1(\Delta, \mathcal{V})$ is said to be $T$-Cartier divisible by $N$ if $x$ is $T$-Cartier in $\mathbb{C}$. In this case $c^T_1(\Delta, \mathcal{V})$ is also $T$-Cartier.

Remark 2.6. Let $X$ be a complete $\mathbb{Q}$-factorial toric variety. Its fan can be considered as a simplicial multi-fan $\Delta(X) = (\Sigma(X), C(X), w(\pm)(X))$ in the lattice $L = \text{Hom}(S^1, T) = H_2(BT)$ with $w^+(X) = 1$, $w^-(X) = 0$ for any $I \in \Sigma(n)(X)$. We take the primitive vector $v_i$ of the 1-dimensional cone $C(i)$ for each $i \in \Sigma(1)$ and set $\mathcal{V} = \{v_i\}$. Let $D_i$ be the divisor corresponding to the cone $C(i)$. In our language it is the divisor corresponding to the characteristic suborbifold $X_i$. Every $T$-Weil divisor $D$ is written uniquely in the form

$$D = \sum_{i \in \Sigma(1)} c_i D_i, \ c_i \in \mathbb{Z},$$

and $D$ is $T$-Cartier if and only if $\sum_{i \in \Sigma(1)} c_i u^I_i \in L^*$ for each $I \in \Sigma(n)$, cf. [HM2]. This suggests the above definition of $T$-Cartier elements in $H^*_T(\Delta)$. Each $u \in L^*$ determines a rational function on $X$. The corresponding Cartier divisor is denoted by $[t^u]$. Let $\text{Div}_T(X)$ be the group of $T$-Cartier divisors, $\text{Pic}(X)$ the group of line bundles on $X$ and $A_{n-1}(X)$ the group of all Weil divisors. Then there is a morphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^* & \longrightarrow & \text{Div}_T(X) & \longrightarrow & \text{Pic}(X) & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^* & \longrightarrow & \bigoplus_{i \in \Sigma(1)} \mathbb{Z} \cdot D_i & \longrightarrow & A_{n-1}(X) & \longrightarrow & 0
\end{array}$$

where the map $L^* = \text{Hom}(T, S^1) \rightarrow \text{Div}_T(X)$ is given by

$$u \mapsto [\text{div}(t^u)] = \sum_i (u, v_i) D_i.$$ 

See [Ful]. Comparing (1) and (3) we see that the group of all $T$-Weil divisors $\bigoplus_{i \in \Sigma(1)} \mathbb{Z} \cdot D_i$ can be identified with $H^*_T(\Delta)$ and $T$-Cartier divisors with $T$-Cartier elements.

3. Elliptic genera of multi-fans

Let $\Delta$ be a complete simplicial multi-fan in a lattice $L$ and $\mathcal{V} = \{v_i\} \in \Sigma(1)$ a set of prescribed edge vectors as in Section 2. We first recall the definition of (equivariant) elliptic genus $\varphi(\Delta, \mathcal{V})$ and (equivariant) orbifold elliptic genus $\hat{\varphi}(\Delta, \mathcal{V})$ of the pair $(\Delta, \mathcal{V})$ after [HM2]. They are defined in such a way that, for an almost complex torus orbifold $X$, $\varphi(\Delta(X), \mathcal{V}(X))$ and $\hat{\varphi}(\Delta(X), \mathcal{V}(X))$ coincide with the elliptic genus $\varphi(X)$ and the orbifold elliptic genus $\hat{\varphi}(X)$ of $X$ respectively.

We consider the function $\phi(z, \tau, \sigma)$ of $z, \sigma$ in $\mathbb{C}$ and $\tau$ in the upper half plane $\mathcal{H}$ given by the following formula

$$\phi(z, \tau, \sigma) = \zeta^{-\frac{1}{2}} \frac{1 - \zeta t}{1 - t} \prod_{k=1}^{\infty} \frac{(1 - \zeta t q^n)(1 - \zeta^{-1} t^{-1} q^n)}{(1 - t q^n)(1 - t^{-1} q^n)},$$

where $t = e^{2\pi \sqrt{-1} \tau}$, $q = e^{2\pi \sqrt{-1} \sigma}$ and $\zeta = e^{2\pi \sqrt{-1} \sigma}$. Note that $|q| < 1$ and $\phi$ is a meromorphic function of $z, \tau$ and $\sigma$. We let act the group $SL_2(\mathbb{Z})$ on $\mathbb{C} \times \mathcal{H}$ by

$$A(z, \tau) = \left( \begin{array}{cc}
\frac{z}{c \tau + d} \\
\frac{a \tau + b}{c \tau + d}
\end{array} \right) A \tau = \left( \begin{array}{cc}
\frac{a \tau + b}{c \tau + d}
\end{array} \right), \quad A = \left( \begin{array}{cc}
a & b \\
c & d
\end{array} \right) \in SL_2(\mathbb{Z}).$$
emphasize the variables. 

\[ \phi(A(z, \tau), \sigma) = e^{\pi \sqrt{-1}c(2z \sigma + (c \tau + d) \sigma)} \phi(z, \tau, (c \tau + d) \sigma), \]

(4)

\[ \phi(z + m \tau + n, \tau, \sigma) = e^{-2 \pi \sqrt{-1}m \sigma} \phi(z, \tau, \sigma) = \zeta^{-m} \phi(z, \tau, \sigma). \]

In the sequel we fix the set \( \mathcal{V} \) and put \( H_I = L/L_{I, \mathcal{V}} \). Let \( v \in L_{\mathcal{V}} \) be a generic vector. The (equivariant) elliptic genus \( \varphi^v(\Delta, \mathcal{V}) \) along \( v \) and the (equivariant) orbifold elliptic genus \( \hat{\varphi}^v(\Delta, \mathcal{V}) \) along \( v \) of the pair \( (\Delta, \mathcal{V}) \) are defined by

\[ \varphi^v(\Delta, \mathcal{V}) = \sum_{I \in \Sigma(n)} \frac{w(I)}{|H_I|} \sum_{h \in H_I} \prod_{i \in I} \phi(-\langle u_i^I, zv + v(h) \rangle, \tau, \sigma), \]

and

\[ \hat{\varphi}^v(\Delta, \mathcal{V}) = \sum_{I \in \Sigma(n)} \frac{w(I)}{|H_I|} \sum_{(h_1, h_2) \in H_I \times H_I} \prod_{i \in I} \zeta^{\langle u_i^I, v(h_1) \rangle} \phi(-\langle u_i^I, zv - \tau v(h_1) + v(h_2) \rangle, \tau, \sigma). \]

(5)

where \( v(h), v(h_1), v(h_2) \in L \) are representatives of \( h, h_1, h_2 \in H_I \) respectively. The above expressions give well-defined functions independent of the choice of representatives \( v(h), v(h_1), v(h_2) \) as is easily seen from (4). They are meromorphic functions in the variables \( z, \tau, \sigma \) and sometimes written as \( \varphi^v(\Delta, \mathcal{V}; z, \tau, \sigma) \) and \( \hat{\varphi}^v(\Delta, \mathcal{V}; z, \tau, \sigma) \) to emphasize the variables.

For each \( K \in \Sigma(k) \) with \( k > 0 \) let \( L_K \) be the kernel of the projection map \( L \to L_K^* \) and let \( L_{K, \mathcal{V}} \) be the sublattice of \( L_K \) generated by \( v_i \in K \). We set \( H_K = L_K/L_{K, \mathcal{V}} \). If \( J \subset K \) then we have \( L_J \cap L_{K, \mathcal{V}} = L_{I, \mathcal{V}} \), and hence \( H_J \) is canonically embedded in \( H_K \). We set

\[ \hat{H}_K = H_K \setminus \bigcup_{J \not\subset K} H_J. \]

The subset \( \hat{H}_K \) is characterized by

(6)

\[ \hat{H}_K = \{ h \in H_K \mid \langle u_i^K, v(h) \rangle \not\in \mathbb{Z} \quad \text{for any} \ i \in K \}, \]

where \( \{ u_i^K \} \) is the basis of \( L_{K, \mathcal{V}}^* \) dual to the basis \( \{ v_i \}_{i \in K} \) of \( L_{K, \mathcal{V}} \) and \( v(h) \in L_K \) is a representative of \( h \in H_K \). For the minimum element \(* = 0 \in \Sigma(0)\) we set \( \hat{H}_* = H_* = 0 \).

If \( K \) is contained in \( I \in \Sigma(n) \), then the canonical map \( L_{I, \mathcal{V}}^* \to L_{K, \mathcal{V}}^* \) sends \( u_i^I \) to \( u_i^K \) for \( i \in K \) and to 0 for \( i \in I \setminus K \). Therefore, if \( h \) is in \( H_K \), then \( \langle u_i^I, v(h) \rangle = 0 \) for \( i \in I \setminus K \), and \( \langle u_i^I, v(h) \rangle = \langle u_i^K, v(h) \rangle \) for \( i \in K \). Here \( v(h) \in L_K \) is regarded as lying in \( L \). Then \( \hat{\varphi}^v(\Delta, \mathcal{V}) \) can also be written in the following form which is sometimes useful.

\[ \hat{\varphi}^v(\Delta, \mathcal{V}) = \sum_{k=0}^{n} \sum_{K \in \Sigma(k)} \sum_{h_i \in \hat{H}_K} \zeta^{\langle u^K, v(h_1) \rangle}. \]

(7)

\[ \sum_{I \in \Sigma(n-k)} \frac{w(I)}{|H_I|} \sum_{h_2 \in H_I} \prod_{i \in I} \phi(-\langle u_i^I, zv - \tau v(h_1) + v(h_2) \rangle, \tau, \sigma), \]

where \( u^K = \sum_{i \in K} u_i^K \).
Note. In the sum above with respect to \( K \in \Sigma^{(k)} \) and \( h_1 \in \hat{H}_K \), the term corresponding to \( K = * \in \Sigma^{(0)} \) and \( h_1 = 0 \in \hat{H}_* = 0 \) is equal to \( \varphi^v(\Delta, \mathcal{Y}) \).

It is also sometimes useful to take a representative \( v(h) \) of \( h \in H_I \) such that

\[
0 \leq \langle u^I_i, v(h) \rangle < 1 \quad \text{for all } i \in I.
\]

Such a representative is unique. We denote the value \( \langle u^I_i, v(h) \rangle \) by \( f_{I,h,i} \) for such a representative \( v(h) \). If \( h \) lies in \( H_K \) for \( K \in \Sigma^{(k)} \) contained in \( I \), then \( f_{I,h,i} = 0 \) for \( i \notin K \), and \( f_{I,h,i} \) depends only on \( K \) for \( i \in K \) which we shall denote by \( f_{K,h,i} \). The sum \( \sum_{i \in K} f_{K,h,i} \) will be denoted by \( f_{K,h} \). Note that (6) can be rewritten as

\[
\hat{H}_K = \{ h \in H_K \mid f_{K,h,i} \neq 0 \quad \text{for any } i \in K \}.
\]

If we choose representatives \( v(h) \) for \( h \in H_I \) satisfying (8), then (7) can be put in the form

\[
\varphi^v(\Delta, \mathcal{Y}) = \sum_{k=0}^{n} \sum_{K \in \Sigma^{(k)}, h_1 \in \hat{H}_K} \zeta^{f_{K,h}}.
\]

(9)

\[
\sum_{I \in \Sigma^{(n-k)}} \frac{w(I)}{|H_I|} \sum_{h_2 \in H_I} \prod_{i \in I} \phi(-\langle u^I_i, zv - \tau v(h_1) + v(h_2) \rangle, \tau, \sigma).
\]

Let \( N > 1 \) be an integer. For a rational number \( f = \frac{a}{r} \) with \( r \) relatively prime to \( N \), we take an integer \( d \) such that \( dr \equiv 1 \mod N \) and define

\[
\hat{f} = ds.
\]

The integer \( \hat{f} \) is defined modulo \( N \).

Assume that \( N \) is relatively prime to \( |H_I| \) for all \( I \in \Sigma^{(n)} \). We put \( \sigma = \frac{k}{N} \) with \( 0 < k < N \), and define the modified orbifold elliptic genus \( \check{\varphi}^v(\Delta, \mathcal{Y}) = \check{\varphi}^v(\Delta, \mathcal{Y}; z, \tau, \sigma) \) of level \( N \) by

\[
\check{\varphi}^v(\Delta, \mathcal{Y}) = \sum_{I \in \Sigma^{(n)}} \frac{w(I)}{|H_I|} \sum_{(h_1, h_2) \in \hat{H}_I \times \hat{H}_I} \prod_{i \in I} \xi^{\langle u^I_i, v(h_1) \rangle} \phi(-\langle u^I_i, zv - \tau v(h_1) + v(h_2) \rangle, \tau, \sigma).
\]

It has also the following expression:

\[
\check{\varphi}^v(\Delta, \mathcal{Y}) = \sum_{k=0}^{n} \sum_{K \in \Sigma^{(k)}, h_1 \in \hat{H}_K} \zeta^{\hat{f}_{K,h}}.
\]

(10)

\[
\sum_{I \in \Sigma^{(n-k)}} \frac{w(I)}{|H_I|} \sum_{h_2 \in H_I} \prod_{i \in I} \phi(-\langle u^I_i, zv - \tau v(h_1) + v(h_2) \rangle, \tau, \sigma)
\]

If we choose representatives \( v(h) \) satisfying (8), then

\[
\check{\varphi}^v(\Delta, \mathcal{Y}) = \sum_{k=0}^{n} \sum_{K \in \Sigma^{(k)}, h_1 \in \hat{H}_K} \zeta^{\hat{f}_{K,h}}.
\]

\[
\sum_{I \in \Sigma^{(n-k)}} \frac{w(I)}{|H_I|} \sum_{h_2 \in H_I} \prod_{i \in I} \phi(-\langle u^I_i, zv - \tau v(h_1) + v(h_2) \rangle, \tau, \sigma).
\]
**Proposition 3.1.** Let \( \varphi^v(\Delta, \mathcal{V}) = \sum_{s=0}^{\infty} \varphi_s(z)q^s \) be the expansion into power series, then \( \zeta^2 \varphi_s(z) \) belongs to \( R(S^1) \otimes \mathbb{Z}[\zeta, \zeta^{-1}] \), where \( R(S^1) \) is identified with \( \mathbb{Z}[t, t^{-1}] \). Let \( r \) be the least common multiple of \( \{ |H_I| \}_{I \in \Sigma(n)} \). Then \( \hat{\varphi}^v(\Delta, \mathcal{V}) \) can be expanded in the form \( \hat{\varphi}^v(\Delta, \mathcal{V}) = \sum_{s=0}^{\infty} \hat{\varphi}_s(z)q^s \), where \( \zeta^2 \hat{\varphi}_s(z) \) belongs to \( R(S^1) \otimes \mathbb{Z}[\zeta^2, \zeta^{-2}] \). Similarly \( \tilde{\varphi}^v(\Delta, \mathcal{V}) \) can be expanded in the form \( \tilde{\varphi}^v(\Delta, \mathcal{V}) = \sum_{s=0}^{\infty} \tilde{\varphi}_s(z)q^s \), where \( \zeta^2 \tilde{\varphi}_s(z) \) belongs to \( R(S^1) \otimes \mathbb{Z}[\zeta, \zeta^{-1}]((\zeta^N)) \).

For the details of proof we refer to [HM2]. We introduce an auxiliary variable \( \tau \) with \( \Im(\tau) > 0 \) and put \( q_1 = e^{2\pi \sqrt{-1}\tau} \). The proof amounts to showing that

\[
\sum_{I \in \Sigma(n)} \frac{w(I)}{|H_I|} \sum_{h_2 \in H_I} \prod_{i \in I} \phi(-2u_i, zv - \tau_1v(h_1) + v(h_2)), 
\]

is expanded in a power series

\[
\hat{\varphi}_{K, h_1}(z, \tau_1, \tau, \sigma) = \sum_{s_1, s_2 \in \mathbb{Z}_{\geq 0}} \hat{\varphi}_{s_1, s_2}(z, q_1)\zeta^{s_1}q^{s_2}.
\]

where \( \zeta^2 \hat{\varphi}_{s_1, s_2}(z, q_1) \) a finite sum of the expressions of the form

\[ (11) \sum_{I \in \Sigma(n-k)} \frac{w(I)}{|H_I|} \sum_{h_1 \in H_I} \prod_{i \in I \setminus K} (1 - \chi_I(u_i, h_2)q^{(u_i, h_1)}) \]

The expression (11) belongs to \( \mathbb{Z}[t, t^{-1}, q_1, q_1^{-1}] \) by Corollary 2.3. From this fact the statement for \( \hat{\varphi}^v(\Delta, \mathcal{V}) \) follows. Note that \( \hat{\varphi}^v(\Delta, \mathcal{V}) \) does not have negative power of \( q \) by [HM2]. The cases of \( \varphi^v(\Delta, \mathcal{V}) \) and \( \tilde{\varphi}^v(\Delta, \mathcal{V}) \) are similar.

The elliptic genus \( \varphi(\Delta, \mathcal{V}) = \varphi(\Delta, \mathcal{V} ; \tau, \sigma) \in (R(T) \otimes \mathbb{Z}[\zeta, \zeta^{-1}])[[q]] \), the orbifold elliptic genus \( \hat{\varphi}(\Delta, \mathcal{V}) = \hat{\varphi}(\Delta, \mathcal{V} ; \tau, \sigma) \in (R(T) \otimes \mathbb{Z}[\zeta^2, \zeta^{-2}])[[q]] \) and the modified orbifold elliptic genus \( \tilde{\varphi}(\Delta, \mathcal{V}) = \tilde{\varphi}(\Delta, \mathcal{V} ; \tau, \sigma) \in (R(T) \otimes \mathbb{Z}[\zeta, \zeta^{-1}]((\zeta^N)))[[q]] \) of level \( N \) are defined by

\[
\varphi^v(\Delta, \mathcal{V}) = \varphi^v(\Delta, \mathcal{V} ; \tau, \sigma), \quad \varphi^v(\Delta, \mathcal{V}) = \hat{\varphi}^v(\Delta, \mathcal{V} ; \tau, \sigma) \quad \text{and} \quad \varphi^v(\Delta, \mathcal{V}) = \tilde{\varphi}^v(\Delta, \mathcal{V} ; \tau, \sigma)
\]

where one varies generic vectors \( v \) in \( L_T \).

**Note.** Once \( \varphi(\Delta, \mathcal{V}) \), \( \hat{\varphi}(\Delta, \mathcal{V}) \) and \( \tilde{\varphi}(\Delta, \mathcal{V}) \) are defined as above we can define \( \varphi^v(\Delta, \mathcal{V}), \hat{\varphi}^v(\Delta, \mathcal{V}) \) and \( \tilde{\varphi}^v(\Delta, \mathcal{V}) \) by using the above formulas for any \( v \in L \).

Let \( N > 1 \) be an integer. When \( \sigma = \frac{k}{N}, 0 < k < N \), the genera \( \varphi(\Delta, \mathcal{V}) \) and \( \hat{\varphi}(\Delta, \mathcal{V}) \) will be also called of level \( N \).

**Theorem 3.2.** Let \( (\Delta, \mathcal{V}) \) be a pair of complete simplicial multi-fan in a lattice \( L \) of rank \( n \) and a set of generating edge vectors. Let \( N > 1 \) be an integer relatively prime to \( |H_I| \) for every \( I \in \Sigma(n) \). If \( c_1(\Delta, \mathcal{V}) \) is divisible by \( N \), then the modified orbifold elliptic genus \( \hat{\varphi}(\Delta, \mathcal{V}; \tau, \sigma) \) of level \( N \) constantly vanishes.

**Theorem 3.3.** Let \( (\Delta, \mathcal{V}) \) be a pair of complete simplicial multi-fan in a lattice \( L \) of rank \( n \) and a set of generating edge vectors. Let \( N > 1 \) be an integer. If \( c_1(\Delta, \mathcal{V}) \) is \( T \)-Cartier divisible by \( N \), then the orbifold elliptic genus \( \tilde{\varphi}(\Delta, \mathcal{V}; \tau, \sigma) \) of level \( N \) constantly vanishes.

**Theorem 3.4.** Let \( (\Delta, \mathcal{V}) \) be a pair of complete simplicial multi-fan in a lattice \( L \) of rank \( n \) and a set of generating edge vectors. If \( c_1(\Delta, \mathcal{V}) = 0 \), then the orbifold elliptic genus \( \tilde{\varphi}(\Delta, \mathcal{V}; \tau, \sigma) \) constantly vanishes.
The following examples are obtained by using Theorem 3.4 in [HM2]. Consider the multi-fan $\Delta = (\Sigma, C, w^\pm)$ where

$$\Sigma^{(1)} = \{1, 2, 3\}, \Sigma^{(2)} = \{\{1\}, \{2\}, \{3\}\},$$

and $C(1), C(2), C(3)$ are half lines in $\mathbb{R}^2$ generated by $e_1 = (1, 0), e_2 = (0, 1), -e_1 - be_2$ respectively, where $b > 0$ is an integer. We set $w^+(I) = 1$ and $w^-(I) = 0$ so that $w(I) = 1$ for all $I \in \Sigma^{(2)}$.

**Example 1.** Define $\mathcal{V} = \{v_1, v_2, v_3\}$ by

$$v_1 = e_1, v_2 = e_2, v_3 = -e_1 - be_2.$$

Then

$$\hat{\phi}(\Delta, \mathcal{V}) =$$

$$\sum_{(m_1, m_2) \in \mathbb{Z}^2} t_1^{-m_1} t_2^{-m_2} \frac{(1 - \zeta^{b+2})(1 - \zeta^{2m_2})}{(1 - \zeta^{m_1})(1 - \zeta^{m_2})(1 - \zeta^{m_1 - bm_2})(1 - \zeta^{2m_2})} \Phi(\sigma, \tau)^2.$$

If $b$ is odd, then $c_1(\Delta, \mathcal{V})$ is divisible by $b + 2$, and we have

$$\hat{\phi}(\Delta, \mathcal{V}) =$$

$$\sum_{(m_1, m_2) \in \mathbb{Z}^2} t_1^{-m_1} t_2^{-m_2} \frac{(1 - \zeta^{b+2})(1 - \zeta^{2m_2})}{(1 - \zeta^{m_1})(1 - \zeta^{m_2})(1 - \zeta^{m_1 - bm_2})(1 - \zeta^{2m_2})} \Phi(\sigma, \tau)^2 = 0,$$

where $b = 2l - 1$ and $\sigma = \frac{k}{b+2}, \ 0 < k < b + 2$.

**Example 2.** Define $\mathcal{V} = \{v_1, v_2, v_3\}$ by

$$v_1 = e_1, v_2 = be_2, v_3 = -e_1 - be_2$$

so that $v_1 + v_2 + v_3 = 0$. Then

$$\hat{\phi}(\Delta, \mathcal{V}) =$$

$$\sum_{(m_1, m_2) \in \mathbb{Z}^2} t_1^{-m_1} t_2^{-m_2} \frac{(1 - \zeta^{b+2})(1 - \zeta^{2m_2})}{(1 - \zeta^{m_1})(1 - \zeta^{m_2})(1 - \zeta^{m_1 - bm_2})(1 - \zeta^{2m_2})} \Phi(\sigma, \tau)^2.$$

If $b$ is relatively prime to 3, then $c_1(\Delta, \mathcal{V})$ is divisible by 3, and we have

$$\hat{\phi}(\Delta, \mathcal{V}) =$$

$$\sum_{(m_1, m_2) \in \mathbb{Z}^2} t_1^{-m_1} t_2^{-m_2} \frac{(1 - \zeta^{3d})(1 - \zeta^{2m_2})}{(1 - \zeta^{m_1})(1 - \zeta^{m_2})(1 - \zeta^{m_1 - bm_2})(1 - \zeta^{2m_2})} \Phi(\sigma, \tau)^2 = 0,$$

where $d$ is such that $db \equiv 1 \mod 3$ and $\sigma = \frac{k}{3}, \ k = 1, 2$.

**Example 3.** If $b = 2$ in Example 1, then $c_1(\Delta, \mathcal{V})$ is $T$-Cartier divisible by 2. This corresponds to the toric variety $\mathbb{P}^2(2, 1, 1)$, see Corollary 3.9. In this case

$$\hat{\phi}(\Delta, \mathcal{V}) =$$

$$\sum_{(m_1, m_2) \in \mathbb{Z}^2} t_1^{-m_1} t_2^{-m_2} \frac{(1 - \zeta^2)(1 + \zeta^{m_2})}{(1 - \zeta^{m_1})(1 - \zeta^{m_2})(1 - \zeta^{m_1 - 2m_2})} \Phi(\sigma, \tau)^2.$$
Remark 3.5. For ordinary fans $c_1(\Delta, \mathcal{V})$ has infinite order when the vectors $v_i$ are taken primitive. But there are examples of general multi-fans with vanishing $c_1(\Delta, \mathcal{V})$.

Proofs of Theorem 3.2, Theorem 3.3 and Theorem 3.4 will be given in Section 4.

The elliptic genus $\varphi(\Delta, \mathcal{V})$ reduces to the so-called $T_y$-genus for $q = 0$ if it is multiplied by $\zeta^{n/2}$ and if $\zeta$ is substituted by $-y$. Namely

$$T_y(\Delta, \mathcal{V}) = \sum_{I \in \Sigma(n)} \frac{w(I)}{|H_I|} \sum_{h \in H_I} \prod_{i \in I} \frac{1 + y \chi_I(u_i', h)^{-1} t^{-u_i'}}{1 - \chi_I(u_i', h)^{-1} t^{-u_i'}}.$$  

In [HM2] it was shown that $T_y$-genus $T_y(\Delta, \mathcal{V})$ had the following expression.

(12) 

$$T_y(\Delta, \mathcal{V}) = \sum_{k=0}^{n} h_k(\Delta)(-y)^k,$$

where $h_k(\Delta)$ is defined by

$$h_k(\Delta) = \sum_{I \in \Sigma(n), \mu(I)=k} w(I) \quad \text{with} \quad \mu(I) = \# \{i \in I \mid \langle u_i', v \rangle > 0\}.$$  

Here $v$ is a generic vector but $h_k(\Delta)$ does not depend on the choice of $v$. The equality (12) shows that $T_y(\Delta, \mathcal{V})$ is rigid and is independent of $\mathcal{V}$. So we write it simply $T_y(\Delta)$.

In [HM2] it was also shown that $T_y(\Delta)$ had the following expression.

(13) 

$$T_y(\Delta) = \sum_{k=0}^{n} e_k(\Delta)(-1 - y)^{n-k}$$

where $e_k(\Delta) = \sum_{J \in \Sigma(k)} \deg(\Delta_J)$.

Note. We have $h_k(\Delta) = h_{n-k}(\Delta)$. $h_0(\Delta) = T_0(\Delta)$ is the Todd genus of $\Delta$, and $h_n(\Delta) = \deg(\Delta)$ by definition of the latter. Hence $T_0[\Delta]$ equals $\deg(\Delta)$, cf. [HM2].

We define orbifold $T_y$-genus of $(\Delta, \mathcal{V})$ by

$$\hat{T}_y(\Delta, \mathcal{V}) = \sum_{k=0}^{n} \sum_{K \in \Sigma(k)} \sum_{h \in \tilde{H}_K} (-y)^{f_{K,h}} T_y(\Delta_K).$$

It is equal to the degree zero term of $\zeta^{n/2}\hat{\varphi}(\Delta, \mathcal{V})$ with $\zeta$ substituted by $-y$.

When $|H_I|$ is relatively prime to an integer $N > 1$ for every $I \in \Sigma(n)$, the modified orbifold $T_y$-genus of level $N$ of $(\Delta, \mathcal{V})$ is defined by

$$\tilde{T}_y(\Delta, V) = \sum_{k=0}^{n} \sum_{K \in \Sigma(k)} \sum_{h \in \tilde{H}_K} (-y)^{f_{K,h}} T_y(\Delta_K),$$

where $-y = e^{2\pi \sqrt{-1} I}$, $0 < l < N$. It is equal to the degree zero term of $\zeta^{n/2}\hat{\varphi}(\Delta, \mathcal{V})$ with $\zeta$ substituted by $-y$.

Suppose that $c_1(\Delta, \mathcal{V})$ is $T$-Cartier. Then $\langle u_i', v(h) \rangle = \langle c_1(\Delta, \mathcal{V}), v(h) \rangle$ is an integer because $c_1(\Delta, \mathcal{V}) \in \mathbb{Z}^r$. Since $f_{K,h} \equiv \langle c_1(\Delta, \mathcal{V}), v(h) \rangle$ mod $\mathbb{Z}$ for $I \supset K$, $f_{K,h}$ is an integer for any $K \in \Sigma(k)$ and $h \in \tilde{H}_K$. It follows that $\tilde{T}_y(\Delta, \mathcal{V})$ is a polynomial in $-y$. This is the case in particular when $c_1(\Delta, \mathcal{V})$ is $T$-Cartier divisible by $N$. 

Lemma 4.1. Assume that $v \langle (14)$ such a vector $v$ as above can be taken in $1$ is an integer relatively prime to $N$. Since the index of $H$ is relatively prime to $N$ is divisible by $c$ as before. For any $I$, $A \in SL_2(\mathbb{Z})$ we set

$$(\phi^v)^A(\Delta, \mathcal{V}; z, \tau, \sigma) = \phi^v(\Delta, \mathcal{V}; A(z, \tau), \sigma).$$

Similarly if $N > 1$ is an integer relatively prime to $|H_I|$ for all $I \in \Sigma(n)$, then we set

$$(\phi^v)^A(\Delta, \mathcal{V}; z, \tau, \sigma) = \phi^v(\Delta, \mathcal{V}; A(z, \tau), \sigma), \sigma = \frac{k}{N}, 0 < k < N.$$

If $c_1(\Delta, \mathcal{V})$ is divisible by $N$, then it follows from (2) that

$$\langle u^I, v \rangle = \langle \iota^*_I(c_1^T(\Delta, \mathcal{V})), v \rangle = N\langle \iota^*_I(x), v \rangle + \langle u, v \rangle.$$

for any $I \in \Sigma(n)$. In particular the mod $N$ value of the integer $\langle u^I, v \rangle$ is equal to the mod $N$ value of $\langle u, v \rangle$ and is independent of $I$. It will be denoted by $h(v)$.

When $x$ is $T$-Cartier in (2) $\langle u^I, v \rangle$ and $\langle \iota^*_I(x), v \rangle$ are integers for any vector $v \in L$.

Lemma 4.1. Assume that $c_1^T(\Delta, \mathcal{V})$ is divisible by $N$. If one of the following conditions is satisfied, then there exists a generic vector $v \in L$ such that $h(v)$ is defined and relatively prime to $N$.

(a) $N$ is relatively prime to $|H_I|$ for all $I \in \Sigma(n)$.
(b) $c_1(\Delta, \mathcal{V})$ is $T$-Cartier divisible by $N$.
(c) $L_{\mathcal{V}} = L_{I, \mathcal{V}}$ for all $I \in \Sigma(n)$.

Such a vector $v$ is taken in $L_{\mathcal{V}}$ for the cases (a) and (c).

Proof. Fix an element $I \in \Sigma(n)$. Since the $u^I$ form a basis of $L_{I, \mathcal{V}}$, there is a $v \in L_{I, \mathcal{V}}$ such that $\langle u^I, v \rangle$ takes a given integer value. In particular there is a $v \in L_{I, \mathcal{V}}$ such that $\langle u^I, v \rangle$ is relatively prime to $N$.

In the case (a) the index of $L_{\mathcal{V}}$ in $L_{I, \mathcal{V}}$ is relatively prime to $N$ as is easily seen. Hence $v$ as above can be taken in $L_{\mathcal{V}}$. In the case (c) $v$ lies in $L_{\mathcal{V}} = L_{I, \mathcal{V}}$.

Since $c_1^T(\Delta, \mathcal{V})$ is divisible by $N$, the value $\langle u^I, v \rangle \mod N$ is independent of $I$ and equal to $h(v)$. \qed

Proposition 3.6. Under the situation of Theorem 3.3 the orbifold $T_\mathcal{V}$-genus $\hat{T}_\mathcal{V}(\Delta, \mathcal{V})$ is a polynomial with integer coefficients divisible by

$$\sum_{k=0}^{N-1} (-y)^k.$$

In fact, by Theorem 3.3 $\hat{\phi}(\Delta, \mathcal{V})$ vanishes for $-y = e^{2\pi \sqrt{-1} \frac{k}{N}}$, $0 < k < N$, under the assumption, and hence its degree 0 term $\hat{T}_\mathcal{V}(\Delta, \mathcal{V})$ does so. Thus it is a polynomial in $-y$ which vanishes for $-y = e^{2\pi \sqrt{-1} \frac{k}{N}}$, $0 < k < N$. Hence $\hat{T}_\mathcal{V}(\Delta, \mathcal{V})$ must be divisible by $\sum_{k=0}^{N-1} (-y)^k$.

The following two propositions are corollaries of Theorem 3.2 and Theorem 3.4.

Proposition 3.7. Under the situation of Theorem 3.2 the modified orbifold $T_\mathcal{V}$-genus $\hat{T}_\mathcal{V}(\Delta, \mathcal{V})$ of level $N$ vanishes.

Proposition 3.8. Under the situation of Theorem 3.4 the orbifold $T_\mathcal{V}$-genus $\hat{T}_\mathcal{V}(\Delta, \mathcal{V})$ vanishes.
Lemma 4.2. Assume that \( N \) is relatively prime to \( |H_1| \) for all \( I \in \Sigma^{(a)} \) and \( c_1(\Delta, \mathcal{V}) \) is divisible by \( N \). Then \( (\tilde{\varphi}^\nu)^A(\Delta, \mathcal{V}; z, \tau, \sigma) \) with \( \sigma = \frac{k}{N}, 0 < k < N \) has the following expression.

\[
(\tilde{\varphi}^\nu)^A(\Delta, \mathcal{V}; z, \tau, \sigma) = e^{\pi \sqrt{-1}(n(c\tau+d)\sigma^2 - 2c(u,v)z\sigma)} \sum_{I \in \Sigma^{(a)}} \frac{w(I)}{|H_I|} \sum_{(h_1,h_2) \in H_I \times H_I} e^{-2\pi \sqrt{-1}(\xi(x)_I, d\tau v(h_1))} e^{-2\pi \sqrt{-1}(\xi(x)_I, c(zv+2v(h_2)))} \prod_{i \in I} e^{2\pi \sqrt{-1}(u_i'(-c\tau+d)\sigma v(h_1))} \phi(-\langle u_i', zv - \tau v(h_1) + v(h_2) \rangle, \tau, (c\tau + d)\sigma).
\]

(15)

Lemma 4.3. Assume that \( c_1(\Delta, \mathcal{V}) \) is \( T \)-Cartier divisible by \( N \). Then \( (\tilde{\varphi}^\nu)^A(\Delta, \mathcal{V}; z, \tau, \sigma) \) with \( \sigma = \frac{k}{N}, 0 < k < N \), has the following expression.

\[
(\tilde{\varphi}^\nu)^A(\Delta, \mathcal{V}; z, \tau, \sigma) = e^{\pi \sqrt{-1}(n(c\tau+d)\sigma^2 - 2c(u,v)z\sigma)} \sum_{I \in \Sigma^{(a)}} \frac{w(I)}{|H_I|} \sum_{(h_1,h_2) \in H_I \times H_I} e^{-2\pi \sqrt{-1}(\xi(x)_I, c(zv))} \prod_{i \in I} e^{2\pi \sqrt{-1}(u_i'(-c\tau+d)\sigma v(h_1))} \phi(-\langle u_i', zv - \tau v(h_1) + v(h_2) \rangle, \tau, (c\tau + d)\sigma).
\]

(16)

**Proof.** We first prove Lemma 4.2. By definition we have

\[
(\tilde{\varphi}^\nu)^A(\Delta, \mathcal{V}; z, \tau, \sigma) = \sum_{I \in \Sigma^{(a)}} \frac{w(I)}{|H_I|} \prod_{(h_1,h_2) \in H_I \times H_I} \zeta^{(u_i', v(h_1))} \phi(-\langle u_i', zv - (a\tau + b)v(h_1) + (c\tau + d)v(h_2) \rangle, A\tau, \sigma).
\]

Using (14) we get

\[
\prod_{i \in I} \zeta^{(u_i', v(h_1))} \phi(-\langle u_i', zv - (a\tau + b)v(h_1) + (c\tau + d)v(h_2) \rangle, A\tau, \sigma) = \zeta^{\sum_i (u_i', v(h_1))} e^{\pi \sqrt{-1}(n(c\tau+d)\sigma^2 + 2c(u',-zv+(a\tau+b)v(h_1)-(c\tau+d)v(h_2))\sigma)} \prod_{i \in I} \phi(-\langle u_i', zv - (a\tau + b)v(h_1) + (c\tau + d)v(h_2) \rangle, \tau, (c\tau + d)\sigma).
\]

(17)

We have

\[
c((a\tau + b)v(h_1) - (c\tau + d)v(h_2)) = -v(h_1) + (c\tau + d)(av(h_1) - cv(h_2)).
\]

Noting that \( \tilde{f}_1 + \tilde{f}_2 \equiv (f_1 + f_2) \mod N \) we see that

\[
\zeta^{\sum_i (u_i', v(h_1))} = \zeta^{(u', v(h_1))}.
\]

Since \( c_1(\Delta, \mathcal{V}) \) is divisible by \( N \), we get from (22)

\[
\langle u_i', v(h_1) \rangle = N\langle \xi(x)_I, v(h_1) \rangle + \langle u, v(h_1) \rangle,
\]

and hence \( \langle u_i', v(h_1) \rangle \equiv \langle u, v(h_1) \rangle \mod N \) because \( u \in L^* \) and \( \langle u, v(h_1) \rangle \in \mathbb{Z} \). Therefore

\[
\zeta^{\sum_i (u_i', v(h_1))} = \zeta^{(u, v(h_1))} = e^{2\pi \sqrt{-1}(u, v(h_1))\sigma},
\]

and

\[
\zeta^{\sum_i (u_i', v(h_1))} e^{-2\pi \sqrt{-1}(u_i', v(h_1))\sigma} = e^{-2\pi \sqrt{-1}(k \xi(x), v(h_1))}.
\]

(18)
Lemma 4.4. Instead of (18), the rest of the proof is entirely similar to that of Lemma 4.2.

\[ e^{2\pi \sqrt{-1}(u', zv)} = e^{2\pi \sqrt{-1}(u, zv)} e^{2\pi \sqrt{-1}k(x, zv)}. \]

Let \( \rho : H_I \times H_I \to H_I \times H_I \) be the map defined by

\[ \rho(h_1, h_2) = (\tilde{h}_1, \tilde{h}_2) = (ah_1 - ch_2, -bh_1 + dh_2). \]

\( \rho \) is bijective and its inverse is given by

\[ \rho^{-1}(\tilde{h}_1, \tilde{h}_2) = (d\tilde{h}_1 + c\tilde{h}_2, b\tilde{h}_1 + a\tilde{h}_2). \]

Then \( av(h_1) - cv(h_2) \) and \( -bv(h_1) + dv(h_2) \) are representatives of \( \tilde{h}_1 \) and \( \tilde{h}_2 \) which we shall denote by \( v(\tilde{h}_1) \) and \( v(\tilde{h}_2) \) respectively. We then have

\[ v(h_1) = dv(\tilde{h}_1) + cv(\tilde{h}_2). \]

In view of (18), (19) and (20) the right hand side of (17) is equal to

\[ e^{2\pi \sqrt{-1}(u', (c\tau + d)\sigma)v(h_1))} \prod_{i \in I} \phi(-\langle u_i', zv - v(\tilde{h}_1)\tau + v(\tilde{h}_2)\rangle, \tau, (c\tau + d)\sigma). \]

Summing up over \((h_1, h_2)\) is the same as summing up over \((\tilde{h}_1, \tilde{h}_2)\). Hence from (21) we get (15) with \( h_i \) replaced by \( \tilde{h}_i \) for \( i = 1, 2 \). This proves Lemma 4.2.

As to Lemma 4.3, we have

\[ \zeta \sum_{I} (u_i', v(h_1)) e^{-2\pi \sqrt{-1}(u_i', v(h_1))\sigma} = 1 \]

instead of (18). The rest of the proof is entirely similar to that of Lemma 4.2. □

Lemma 4.4. Assume that \( N \) is relatively prime to \( |H_I| \) for all \( I \in \Sigma^{(n)} \) and \( c_1(\Delta, \mathcal{V}) \) is divisible by \( N \). Then the meromorphic function \((\tilde{\phi}^v)^A(\Delta, \mathcal{V}; z, \tau, \sigma)\) in \( z \) and \( \tau \) with \( \sigma = \frac{k}{N}, 0 < k < N, \) has no pole at \( z \in \mathbb{R} \).

Lemma 4.5. Assume that \( c_1(\Delta, \mathcal{V}) \) is \( T \)-Cartier divisible by \( N \). Then the meromorphic function \((\tilde{\phi}^v)^A(\Delta, \mathcal{V}; z, \tau, \sigma)\) in \( z \) and \( \tau \) with \( \sigma = \frac{k}{N}, 0 < k < N, \) has no pole at \( z \in \mathbb{R} \).

Proof. The expression (15) in Lemma 4.2 of the function \((\tilde{\phi}^v)^A(\Delta, \mathcal{V}; z, \tau, \sigma)\) can be rewritten in the following form as can be seen in a similar way to (10).

\[ (\tilde{\phi}^v)^A(\Delta, \mathcal{V}; z, \tau, \sigma) \]

\[ = e^{\pi \sqrt{-1}(nc(\tau + d)\sigma - 2c(u, v)z\sigma)} \sum_{I \in \Sigma^{(n-k)}} \sum_{h_1 \in H_I} e^{-2\pi \sqrt{-1}(u^K_h, (\tau + d)\sigma v(h_1))}. \]

\[ \sum_{I \in \Sigma^{(n-k)}} \frac{w(I)}{|H_I|} e^{-2\pi \sqrt{-1}(u^K_h, (\tau + d)\sigma v(h_1))} \prod_{h_2 \in H_I} \phi(-\langle u_i', zv - v(h_1) + v(h_2)\rangle, \tau, (\tau + d)\sigma). \]

Hence, in order to prove Lemma 4.4, it is sufficient to prove that

\[ \sum_{I \in \Sigma^{(n-k)}} \frac{w(I)}{|H_I|} e^{-2\pi \sqrt{-1}(u^K_h, (\tau + d)\sigma v(h_1))} \prod_{h_2 \in H_I} \phi(-\langle u_i', zv - v(h_1) + v(h_2)\rangle, \tau, (\tau + d)\sigma), \]

or, replacing \( ck \) by \( m \) and \( (\tau + d)\sigma \) by \( \sigma \),

\[ \sum_{I \in \Sigma^{(n-k)}} \frac{w(I)}{|H_I|} e^{-2\pi \sqrt{-1}(u^K_h, m(zv + v(h_2)))} \prod_{h_2 \in H_I} \phi(-\langle u_i', zv - v(h_1) + v(h_2)\rangle, \tau, \sigma). \]
has no pole at \( z \in \mathbb{R} \) for any fixed \( K \in \Sigma^{(k)} \) and \( h_1 \in \tilde{H}_K \). Note that
\[
e^{-2\pi \sqrt{-1}(\zeta_{\tau}(x), m(zv+v(h_2)))} = e^{-2\pi \sqrt{-1}(uK, mrv(h_1))} e^{-2\pi \sqrt{-1}(\zeta_{\tau}(x), m(zv-rv(h_1)+v(h_2)))}.
\]
By a similar argument to the proof of Proposition 3.1, we see that (22) can be expanded in the form
\[
e^{-2\pi \sqrt{-1}(uK, mrv(h_1))} \sum_{s=0}^{\infty} (\tilde{\varphi})^A_{K,h_1,s}(z) q^s,
\]
where \( \zeta_{\tau}(\tilde{\varphi})^A_{K,h_1,s}(z) \) belongs to \( R(S^1) \otimes \mathbb{Z}[\zeta^{\tau/2}, \zeta^{\tau-1}] / (\zeta^N) \). From this we can conclude that (22) has no pole at \( z \in \mathbb{R} \). We refer to Lemma in Section 7 of [Hit]. See also Section 5 of [HM2]. This finishes the proof of Lemma 4.5.

The proof of Lemma 4.5 is similar. We use Lemma 4.3 instead of Lemma 4.2. Then we have only to note that
\[
e^{2\pi \sqrt{-1}(\zeta_{\tau}(x),ckv)} = e^{2\pi \sqrt{-1}(\zeta_{\tau}(x),ck(zv+v(h_2)))}
\]
which holds because \( \zeta_{\tau}(x) \in L \) and hence \( \langle \zeta_{\tau}(x), v(h_i) \rangle \in \mathbb{Z}, i = 1, 2. \)

We now proceed to the proof of Theorem 3.2. We follow [L] for the idea of proof. We first show that \( \tilde{\varphi}^v(\Delta, \mathcal{Y}; z, \tau, \sigma) \) is a constant function as a function of \( z \).

We regard \( \tilde{\varphi}^v(\Delta, \mathcal{Y}; z, \tau, \sigma) \) as a meromorphic function of \( z \). By the transformation law \( \phi(z, \tau, \sigma) \) is an elliptic function in \( z \) with respect to the lattice \( \mathbb{Z} \cdot N\tau \oplus \mathbb{Z} \) for \( \sigma = \frac{k}{N} \) with \( 0 < k < N \). Hence \( \tilde{\varphi}^v(\Delta, \mathcal{Y}; z, \tau, \sigma) \) of level \( N \) is also an elliptic function in \( z \). Thus, in order to show that \( \tilde{\varphi}^v(\Delta, \mathcal{Y}; z, \tau, \sigma) \) is a constant function it suffices to show that it does not have poles.

Assume that \( z \) is a pole. Then \( 1 - t^m q^\alpha = 0 \) for some integer \( m \neq 0 \), some rational number \( r \) and a root of unity \( \alpha \). Consequently there are integers \( m_1 \neq 0 \) and \( k_1 \) such that \( m_1 z + k_1 \tau \in \mathbb{R} \). Then there is an element \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \) such that
\[
\frac{z}{ct+d} \in \mathbb{R}.
\]
Since
\[\tilde{\varphi}^v(\Delta, \mathcal{Y}; z, \tau, \sigma) = \tilde{\varphi}^v(\Delta, \mathcal{Y}; A^{-1}(\frac{z}{ct+d}, A\tau), \sigma) = (\tilde{\varphi}^v)^{A^{-1}}(\Delta, \mathcal{Y}; \frac{z}{ct+d}, A\tau, \sigma),\]
\( (\tilde{\varphi}^v)^{A^{-1}}(\Delta, \mathcal{Y}; \frac{z}{ct+d}, A\tau, \sigma) \) must have a pole at \( \frac{z}{ct+d} \in \mathbb{R} \). But this contradicts Lemma 4.4. This contradiction proves that \( \tilde{\varphi}^v(\Delta, \mathcal{Y}; z, \tau, \sigma) \) can not have a pole.

Since \( \tilde{\varphi}^v(\Delta, \mathcal{Y}; z, \tau, \sigma) \) is a constant function in \( z \) for every generic vector \( v \in L \), the equivariant modified orbifold genus \( \tilde{\varphi}(\Delta, \mathcal{Y}; \tau, \sigma) \) is constant as a function on \( T \). That constant is equal to \( \tilde{\varphi}^v(\Delta, \mathcal{Y}; z, \tau, \sigma) \) for any \( v \). On the other hand, using (14) and the fact that \( h(v) \mod N \) is independent of \( I \in \Sigma^{(n)} \), we have
\[
(\tilde{\varphi}^v)^{A^{-1}}(\Delta, \mathcal{Y}; z, \tau, \sigma) = \tilde{\varphi}^v(\Delta, \mathcal{Y}; z + \tau, \tau, \sigma) = \zeta^{ch(v)}\tilde{\varphi}^v(\Delta, \mathcal{Y}; z, \tau, \sigma) = \tilde{\varphi}^v(\Delta, \mathcal{Y}; \tau, \sigma).
\]
We choose a generic vector \( v \) such that \( h(v) \) is relatively prime to \( N \). Such a \( v \) exists by Lemma 4.4. Then \( \zeta^{ch(v)} \) is not equal to 1. Hence (23) implies that the constant \( \tilde{\varphi}(\Delta, \mathcal{Y}; \tau, \sigma) \) must vanish. This finishes the proof of Theorem 3.2.

The proof of Theorem 3.3 is similar. We have only to use Lemma 4.5 instead of Lemma 4.4.
For the proof of Theorem 3.4 we first note that the condition \( c_1(\Delta, \mathcal{V}) = 0 \) is equivalent to \( u = c_1^T(\Delta, \mathcal{V}) \in L^* \). In particular \( \langle u^I, v \rangle = \langle u, v \rangle \in \mathbb{Z} \) for all \( I \in \Sigma^{(n)} \). Then we have the equality

\[
(\hat{\phi}^v)^A(\Delta, \mathcal{V}; z, \tau, \sigma) = e^{\pi \sqrt{-1}(nc(\tau + d)\sigma^2 - 2c(\tau + d)\sigma z)} \sum_{I \in \Sigma^{(n)}} \frac{w(I)}{|H_I|} \sum_{(h_1, h_2) \in H_I \times H_{I'}} \prod_{i \in I} e^{2\pi \sqrt{-1}(u_i^I, (\tau + d)\sigma z h_1)} \phi(-\langle u_i^I, zv - \tau v(h_1) + v(h_2) \rangle, \tau, (\tau + d)\sigma),
\]

which can be proved in a similar way to (16). From this equality we can conclude that \( (\hat{\phi}^v)^A(\Delta, \mathcal{V}; z, \tau, \sigma) \) has no pole at \( z \in \mathbb{R} \) as in the proof of Lemma 4.5. Let \( N \) be an integer greater than 1. Then one sees, as in the proof of Theorem 3.2, that \( \hat{\phi}^v(\Delta, \mathcal{V}; z, \tau, \sigma) \) is a constant for \( \sigma = \frac{k}{N}, 0 < k < N \). Since this is true for any integer \( N > 1 \) for any generic \( v \in L_{\mathcal{V}}, \hat{\phi}^v(\Delta, \mathcal{V}; z, \tau, \sigma) \) must be a constant equal to \( \hat{\phi}(\Delta, \mathcal{V}; \tau, \sigma) \). Moreover we have

\[
\hat{\phi}^v(\Delta, \mathcal{V}; z + \tau, \tau, \sigma) = \zeta^{(u, v)} \hat{\phi}^v(\Delta, \mathcal{V}; z, \tau, \sigma).
\]

Then choosing \( v \in L_{\mathcal{V}} \) such that \( h(v) = \langle u, v \rangle \neq 0 \) we see that \( \hat{\phi}(\Delta, \mathcal{V}; \tau, \sigma) \) must be equal to 0.

5. Applications

Let \( \Delta = (\Sigma, C, \pm) \) be a complete simplicial multi-fan in a lattice \( L \) of rank \( n \). In this section a vector \( v_i \in L \) generating the cone \( C(i) \) for each \( i \in \Sigma^{(1)} \) will always be taken primitive so that \( \mathcal{V} = \{v_i\} \) is determined by \( \Delta \). Thus \( T_y(\Delta, \mathcal{V}) \) and \( \hat{T}_y(\Delta, \mathcal{V}) \) will be simply written \( T_y(\Delta) \) and \( \hat{T}_y(\Delta) \). Similarly \( c_1(\Delta, \mathcal{V}) \) is written \( c_1(\Delta) \).

Also the following condition will be assumed throughout this section.

(24) \textit{The Todd genus} \( T_0(\Delta) \) \textit{is equal to 1 and} \( w(I) = 1 \) \textit{for all} \( I \in \Sigma^{(n)} \).

Note. The condition (24) is always satisfied by complete simplicial ordinary fans. See e.g. [Ful].

Lemma 5.1. Under the condition (24) we have \( T_0(\Delta_K) = 1 \) for all \( K \in \Sigma^{(k)} \) and \( w_K(I) = 1 \) for all \( I \in \Sigma_K^{(n-k)} \) and \( K \in \Sigma^{(k)} \).

Proof. Take \( I \in \Sigma_K^{n-k} \). \( I \) is an element in \( \Sigma^{(n)} \) such that \( K \subset I \) and \( w_K(I) = w(I) = 1 \) by definition and by assumption.

As was remarked in Note after (13), \( T_0(\Delta) \) is equal to \( \text{deg}(\Delta) \). We shall show that \( \text{deg}(\Delta_K) = 1 \) for each \( K \in \Sigma^{(k)} \) which will prove Lemma. Take a generic vector \( \bar{v} \) in \( L_K^R \) and a generic vector \( v \) in \( L_{\mathcal{V}} \) which projects into \( \bar{v} \). Since \( \text{deg}(\Delta) = 1 \) and \( w(I) = 1 \) for all \( I \in \Sigma^{(n)} \), there is a unique \( I \in \Sigma^{(n)} \) with \( K \subset I \) such that \( v \) is contained in \( C(I) \). Then \( C_K(I) \) contains \( \bar{v} \).

We may assume that \( v \) is chosen in such a way that \( \langle u_i^I, v \rangle \) is a sufficiently large positive number for each \( i \in K \). Assume that there is another \( I' \in \Sigma_K^{n-k} \) such that \( C_K(I') \) contains \( \bar{v} \). Then, \( \langle u_i^{I'}, v_i \rangle > 0 \) for all \( i \in I' \setminus K \). From the fact that \( \langle u_i^I, v \rangle \) is sufficiently large for every \( i \in K \), it follows that \( \langle u_i^{I'}, v \rangle > 0 \) also for all \( i \in K \) and hence \( v \in C(I') \). This contradicts the fact that \( I \) is the unique element in \( \Sigma^{(n)} \) such that \( I \supset K \) and \( v \in C(I) \). Hence \( I \in \Sigma_K^{n-k} \) is the unique element which contains \( \bar{v} \). Since \( w_K(I) = 1 \) we have \( \text{deg}(\Delta_K) = 1 \). \( \square \)
Lemma 5.2. Under the condition (24) the following equalities hold.

\[ h_k(\Delta) = \# \{ I \in \Sigma^{(n)} | \mu(I) = k \}, \]

in (12) and

\[ e_k(\Delta) = \# \Sigma^{(k)} \]

in (13).

**Proof.** (25) is immediate. \( \deg(\Delta_K) = 1 \) for all \( K \in \Sigma \) by Lemma 5.1. Then (26) follows. \( \square \)

Lemma 5.3. \( h_k(\Delta) > 0 \) for \( 0 \leq k \leq n \).

**Proof.** Let \( v \) be a generic vector. For \( I \in \Sigma^{(n)} \) we put

\[ \mu_v(I) = \# \{ i \in I | \langle u_i^I, v \rangle > 0 \}. \]

Then \( h_k(\Delta) = \# \{ I \in \Sigma^{(n)} | \mu_v(I) = k \} \) by (25). Fixing \( I \) we put

\[ \sigma_J = \{ v \in \mathbb{L}_\mathbb{R} | \langle u_j^I, v \rangle > 0 \text{ for } j \in J, \langle u_i^I, v \rangle < 0 \text{ for } i \not\in J \} \]

for \( J \subset I \). Then the collection \( \{ \sigma_J \}_{J \subset I} \) decomposes \( \mathbb{L}_\mathbb{R} \setminus \{ v | \langle u_i^I, v \rangle = 0 \text{ for some } i \in I \} \) into connected components. If we take \( J \subset I \) in \( \Sigma^{(k)} \) and \( v \in \sigma_J \), then \( \mu_v(I) = k \). This proves that \( h_k(\Delta) > 0 \). \( \square \)

**Note.** It is known that \( h_k(\Delta) \geq h_{k-1}(\Delta) \) for \( 0 \leq k \leq \left[ \frac{n}{2} \right] \) for the fan associated to a complete \( \mathbb{Q} \)-factorial projective toric variety, cf. [Ful].

We shall also use the following fact.

\[ 0 < f_{K,h} < k \text{ for } K \in \Sigma^{(k)}, \ k > 0, \text{ and } h \in \hat{H}_K. \]

In fact \( \langle u^K_i, v(h) \rangle \notin \mathbb{Z} \) for \( h \in \hat{H}_K \) and \( i \in K \) by (6). Hence \( 0 < f_{K,h,i} < 1 \) and \( 0 < f_{K,h} = \sum_{i \in K} f_{K,h,i} < k \).

Proposition 5.4. Let \( \Delta \) be a complete multi-fan of dimension \( n \) satisfying condition (24). If \( c_1(\Delta) \) is \( T \)-Cartier divisible by an integer \( N > 1 \), then \( N \) is equal to or less than \( n + 1 \). In the extremal case \( N = n + 1 \) the multi-fan \( \Delta \) is non-singular and the \( T_y \)-genus must be of the form

\[ T_y(\Delta) = \sum_{k=0}^{n} (-y)^k. \]

**Proof.** Suppose that \( c_1(\Delta) \) is \( T \)-Cartier divisible by \( N \). Then, by Proposition 3.6 \( \hat{T}_y(\Delta) \) is divisible by \( \sum_{k=0}^{N-1} (-y)^k \). On the other hand it is a polynomial of degree \( n \) with constant term \( T_0(\Delta) = 1 \). Therefore we must have \( N - 1 \leq n \).

Suppose that \( N = n + 1 \). Then the same reasoning as above shows that

\[ \hat{T}_y(\Delta) = \sum_{k=0}^{n} (-y)^k. \]

On the other hand we have

\[ T_y(\Delta) = \sum_{k=0}^{n} h_k(-y)^k \]
with $h_k > 0$ by Lemma 5.3. Hence we must have $T_y(\Delta) = \hat{T}_y(\Delta)$. It also follows that $\Delta$ is non-singular. For otherwise there would be an extra term $(-y)^{f_{K,h}} T_y(\Delta_K)$ where $T_y(\Delta_K) \neq 0$ since $T_0(\Delta_K) = 1$ by Lemma 5.1. □

Remark 5.5. In [HN2] it was shown that a complete non-singular simplicial multi-fan satisfying (28) is unique up to isomorphisms and is isomorphic to the $n$-dimensional projective space $\mathbb{P}^n$. There are exactly $n + 1$ primitive generating vectors $\{v_i\}_{i=1}^{n+1}$ and they satisfy the relation

$$v_1 + v_2 + \cdots + v_{n+1} = 0.$$  

Proposition 5.6. Let $\Delta$ be a complete multi-fan of dimension $n$ satisfying condition (24). If $c_1(\Delta)$ is $T$-Cartier divisible by $n$, then the following two possibilities occur.

(a) $\Delta$ is non-singular and

$$T_y(\Delta) = (1 - y) \sum_{k=0}^{n-1} (-y)^k.$$  

(b) $n \geq 2$ and $\Delta$ has a unique $K \in \Sigma$ such that $\hat{H}_K \neq \emptyset$. In this case

$$T_y(\Delta) = \sum_{k=0}^{n} (-y)^k,$$

$$T_y(\Delta_K) = \sum_{k=0}^{n-2} (-y)^k,$$

and

$$\hat{T}_y(\Delta) = T_y(\Delta) + (-y)T_y(\Delta_K) = (1 - y) \sum_{k=0}^{n-1} (-y)^k.$$  

Remark 5.7. In [HN2] it was shown that in the case (a) there are exactly $n + 2$ elements in $\Sigma^{(1)}$ and the corresponding primitive generating vectors satisfy the relations (under a suitable numbering)

$$v_0 + v_{n+1} + \sum_{i=2}^{n} k_i v_i = 0, \quad v_1 + v_2 + \cdots + v_n = 0,$$

where $(\sum_{i=2}^{n} k_i) + 2$ is divisible by $n$. It was also shown that $c_1(\Delta)$ of a complete non-singular simplicial multi-fan $\Delta$ satisfying such relations is divisible by $n$.

In the case (b) it will be shown in the proof that there are exactly $n + 1$ elements in $\Sigma^{(1)}$, and the corresponding primitive generating vectors satisfy the relations (under a suitable numbering)

$$2 \sum_{i=1}^{n-1} v_i + v_n + v_{n+1} = 0.$$  

Proof. We first show that the orbifold $T_y$-genus must be of the form

$$\hat{T}_y(\Delta) = (1 - y) \sum_{k=0}^{n-1} (-y)^k.$$  

$\hat{T}_y(\Delta)$ is a polynomial in $-y$ of degree $n$ divisible by $\sum_{k=0}^{n-1} (-y)^k$ by Proposition 3.6. We shall show that the constant term and the coefficient of the highest term are equal to
1. Then it would prove that $\tilde{T}_y(\Delta)$ must be of the form (29). The constant term of $T_y(\Delta)$ is $h_0(\Delta) = T_0(\Delta) = 1$ and its coefficient of the highest term is $h_n(\Delta) = h_0(\Delta) = 1$. So it suffices to show that $(-y)^{f_{K,h}} T_y(\Delta_K)$ has no constant term and its highest degree is less than $n$ for any $K \in \Sigma^{(1)}$ with $k > 0$ and $h \in H_K$.

As $f_{K,h} > 0$ by (24) there is no constant term. On the other hand $T_y(\Delta_K)$ is a polynomial of degree $n - k$ so that the highest degree of $(-y)^{f_{K,h}} T_y(\Delta_K)$ is $f_{K,h} + n - k$ and it is less than $n$ by (27). This finishes the proof of (29).

If $\tilde{T}_y(\Delta)$ is of the form (29), then $h_{n-1}(\Delta) = 1$ or 2 and

$$ e_1(\Delta) = nh_n(\Delta) + h_{n-1}(\Delta) = \begin{cases} n + 1, & h_{n-1}(\Delta) = 1, \\ n + 2, & h_{n-1}(\Delta) = 2. \end{cases} $$

Claim. Case 1. $h_{n-1}(\Delta) = 2$, $e_1(\Delta) = n + 2$. We have

$$ e_n(\Delta) = 2n, \quad h_k(\Delta) = \begin{cases} 1, & k = 0, n, \\ 2, & 1 \leq k \leq n - 1. \end{cases} $$

We consider the link $\text{Lk}\{i\}$ of $i \in \Sigma^{(1)}$ in $\Sigma$. The number $\#\text{Lk}\{i\}$ of $\text{Lk}\{i\}$ is at least $n$ in general by the completeness of $n$-dimensional multi-fan $\Delta$. In the present case it is equal to $n$ or $n + 1$.

We show that there is a vertex of $\Sigma$, i.e., an element $i \in \Sigma^{(1)}$ with $\#\text{Lk}\{i\} = n$. In fact if $\#\text{Lk}\{i\} = n + 1$ for all $i \in \Sigma^{(1)}$, then the set $\{i \in \Sigma^{(1)}\}$ would form an $n$-dimensional simplex in $\Sigma$. This is a contradiction because the highest dimension of simplices in $\Sigma$ is $n - 1$, since the dimension as a simplex of $I \in \Sigma^{(n)}$ is $n - 1$.

If $\#\text{Lk}\{i\} = n$, then the $n$ vertices of $\text{Lk}\{i\}$ do not form an $(n-1)$-dimensional simplex. For otherwise they together with $i$ would form an $n$-dimensional simplex. Hence the star $\text{St}\{i\}$ of $i$, i.e., the join of $i$ with $\text{Lk}\{i\}$ consists of $n$ simplices in $\Sigma^{(n)}$.

Assume now that $\#\text{Lk}\{i\} = n$. Then there is a unique $i' \in \Sigma^{(1)}$ such that $\text{Lk}\{i'\} = \text{Lk}\{i\}$. Each of the stars of $i$ and $i'$ consists of $n$ simplices in $\Sigma^{(n)}$. Hence

$$ e_n(\Delta) = 2n. $$

We also have

$$ e_k(\Delta) = \begin{cases} \binom{n}{k} + 2\binom{n}{k-1}, & 1 \leq k \leq n - 1, \\ 2n, & k = n, \\ 1, & k = 0. \end{cases} \quad (30) $$

On the other hand we have the relations

$$ e_k(\Delta) = \sum_{i=0}^{k} \binom{n-i}{n-k} h_{n-i}(\Delta). $$

This follows from

$$ T_y(\Delta) = \sum_{k=0}^{n} h_k(\Delta)(-y)^k = \sum_{k=0}^{n} e_k(\Delta)(-y - 1)^{n-k}. $$

If we put

$$ h_k(\Delta) = \begin{cases} 1, & k = 0, n, \\ 2, & 1 \leq k \leq n - 1 \end{cases} \quad (31) $$
in the above relations, then we have exactly the same values \((30)\) for \(e_k(\Delta)\). It follows that \(h_k(\Delta)\) is given by \((31)\). This proves Claim.

Since \(h_k(\Delta)\) is given by \((31)\), we have

\[
T_y(\Delta) = 1 + 2 \sum_{k=1}^{n-1} (-y)^k + (-y)^n = (1 - y) \sum_{k=0}^{n-1} (-y)^k = \hat{T}_y(\Delta).
\]

We conclude that \(\Delta\) is non-singular.

Claim. Case 2. \(h_{n-1}(\Delta) = 1, \ e_1(\Delta) = n + 1\). We have

\[
e_n(\Delta) = n + 1, \quad h_k(\Delta) = 1, \quad 0 \leq k \leq n.
\]

The fact that \(e_1(\Delta) = n + 1\) implies that \(Lk\{i\}\) consists of \(n\) vertices for any \(i \in \Sigma^{(1)}\). This in turn implies that \(e_n(\Delta) = n + 1\). Then \(h_k(\Delta)\) equals 1 for all \(0 \leq k \leq n\). In fact

\[
n + 1 = e_n(\Delta) = \sum_k h_k(\Delta) \geq n + 1
\]

since \(h_k(\Delta) > 0\). Hence \(h_k(\Delta)\) must be equal to 1 for all \(k\). This proves Claim.

Then we see that

\[
(32) \quad T_y(\Delta) = \sum_{k=0}^{n} h_k(\Delta)(-y)^k = \sum_{k=0}^{n} (-y)^k.
\]

Since \(T_y(\Delta) \neq \hat{T}_y(\Delta)\), \(\Delta\) can not be non-singular.

Hereafter we assume that \(\Sigma^{(1)} = \{1, 2, \ldots, n+1\}\). We shall show that there is a unique \(K \in \Sigma\) such that \(\hat{H}_K \neq \emptyset\). Note that the generating vectors \(\{v_i\}_{i=1}^{n+1}\) satisfy a relation of the form:

\[
(33) \quad \sum_i a_i v_i = 0 \quad \text{with} \quad a_i \in \mathbb{Z}_{>0}.
\]

To see this, we write \(v_{n+1}\) in the form

\[
v_{n+1} = \sum_{i=1}^{n} b_i v_i, \quad b_i \in \mathbb{Q}.
\]

The convexity of rational \(n\)-dimensional cones \(C(I)\), \(I \in \Sigma^{(n)}\), implies that \(b_i\) must be negative for all \(i\). Hence \((33)\) follows. We assume that the greatest common divisor of \(\{a_i\}\) is equal to 1.

We may assume that the lattice \(L\) is \(\mathbb{Z}^n\). We write

\[
(v_1, v_2, \ldots, v_n, v_{n+1}) = (e_1, e_2, \ldots, e_n)A,
\]

with \((n, n+1)\) matrix \(A\) where \(e_i\) is the standard unit vector for \(1 \leq i \leq n\). If \(A_i\) denotes the \((n, n)\) matrix obtained from \(A\) by deleting the \(i\)-th column, then we see easily that

\[
|\det A_i| = d a_i
\]

for some positive integer \(d\). We shall show later that \(d\) is equal to 1.

We put \(\{i\}^* = \{1, 2, \ldots, n, n+1\} \setminus \{i\} \in \Sigma^{(n)}\). If \(L_{\{i\}^*,y}\) denotes the lattice generated by \(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1}\), then \(H_{\{i\}^*} = L/L_{\{i\}^*,y}\). Since

\[
(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1}) = (e_1, \ldots, e_n)A_i,
\]

we have

\[
(34) \quad |H_{\{i\}^*}| = |\det A_i| = d a_i.
\]
Claim.

\[ d = 1, \quad \sum_{i=1}^{n+1} a_i = 2n. \]

Proof of Claim.

Step 1. We show \( d = 1 \) and \( \sum_{i=1}^{n+1} a_i \leq 2n. \)

We put \(-y = 1\) in the polynomial \( \hat{T}_y(\Delta) \) in \(-y\). By (29) we have

\[ 2n = \hat{T}_y(\Delta)|_{-y=1} = \sum_{k=0}^{n} \sum_{K \in \Sigma(k)} \sum_{h \in H_K} T_y(\Delta_K)|_{-y=1}. \]

If we put \( T_y(\Delta_K) = \sum_{i=0}^{n-k} h_i(\Delta_K)(-y)^i \) for \( K \in \Sigma(k) \), then Lemma 5.1 and Lemma 5.3 yield \( h_i(\Delta_K) > 0 \). Therefore \( T_y(\Delta_K)|_{-y=1} \geq n + 1 - k \). Hence

\[ \sum_{k=0}^{n} \sum_{K \in \Sigma(k)} (n + 1 - k)|\hat{H}_K| \leq 2n. \]

On the other hand we have

\[ \sum_{k=0}^{n} \sum_{K \in \Sigma(k)} (n + 1 - k)|\hat{H}_K| = \sum_{I \in \Sigma(n)} |H_I|. \]

In fact \( H_I = \bigcup_{K \subset I} \hat{H}_K \), and the number of \( I = \{i\}^* \in \Sigma(n) \) containing a fixed \( K \in \Sigma(k) \) is equal to \( n + 1 - k \). Combining (36) with (34) and (35) we obtain

\[ d \sum_{i=1}^{n+1} a_i = \sum_{I \in \Sigma(1)} |H_{(i)}| \leq 2n. \]

Since all \( a_i \) are positive, \( d \) must be equal to 1, and we get \( \sum_{i=1}^{n+1} a_i \leq 2n. \)

Step 2. We show \( \sum_{i=1}^{n+1} a_i = 2n. \)

Since \( c_1(\Delta) \) is \( T \)-Cartier divisible by \( n \), the equivariant first Chern class is of the form

\[ c_1^T(\Delta) = \sum_{i \in \Sigma(1)} x_i = nx + u \]

with \( u \in H^2(BT) \) and \( \iota^*_I(x) \in L^* \) for all \( I \in \Sigma(n) \). In particular \( u^I = \iota^*(c_1^T(\Delta)) \) lies in \( L^* \). Since \( \iota^*_I(x) \in L^* \), we have

\[ \langle u^I, v_i \rangle \equiv \langle u, v_i \rangle \mod n \]

for all \( I \in \Sigma(n) \).

We may suppose that \( a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \) without loss of generality. Since \( \sum_{i=1}^{n+1} a_i = 2n \), we must have \( a_n = a_{n+1} = 1 \). Put \( I = \{n+1\}^* \) and \( I' = \{1\}^*. \) Then

\[- \sum_{i=1}^{n} a_i = \langle u^I, v_{n+1} \rangle = \langle u, v_{n+1} \rangle \equiv \langle u^{I'}, v_{n+1} \rangle = 1 = a_{n+1} \mod n. \]

Therefore \( \sum_{i=1}^{n+1} a_i \) is divisible by \( n \). But \( \sum_{i=1}^{n+1} a_i \leq 2n \) and \( a_i \geq 1 \) for all \( i \). Hence \( \sum_{i=1}^{n+1} a_i = 2n. \)

Step 3. We show that \( 2x_{n+1} \) is \( T \)-Cartier.
We first show that
\[ c^I_1(\Delta) = 2nx_{n+1} + u' \]
for \( I = \{n+1\}^* \). For that purpose put \( x' = c^I_1(\Delta) - u' \). Then \( \iota^*_I(x') = 0 \). Since the kernel of \( \iota^*_I \) is generated by \( x_{n+1} \) we see that \( x' = mx_{n+1} \) for some \( m \in \mathbb{Z} \). Hence \( \langle \iota^*_I(x'), v_{n+1} \rangle = m \) for \( I' = \{1\}^* \). On the other hand \( \langle \iota^*_I(c^I_1(\Delta)), v_{n+1} \rangle = 1 \), and \( \langle u', v_{n+1} \rangle = -\sum_{i=1}^{n} a_i = -2n + 1 \). Hence we get \( m = \langle \iota^*_I(x'), v_{n+1} \rangle = 2n \).

Comparing (38) with (37) we see that
\[ \langle u' - u', v_i \rangle \equiv 0 \mod n \]
for all \( 1 \leq i \leq n \). Since \( |A_{n+1}| = a_{n+1} = 1 \) the collection \( \{v_i\}_{i=1}^{n} \) generates the lattice \( L \). It follows that \( u - u' = nu_1 \) for some \( u_1 \in L^* \). Then
\[ n(2x_{n+1}) = nx + u - u' = n(x + u_1), \]
and \( 2x_{n+1} = x + u_1 \). Since \( x \) is \( T \)-Cartier by assumption, \( 2x_{n+1} \) is \( T \)-Cartier.

Step 4. We shall show that \( (a_1, \ldots, a_{n-1}, a_n, a_{n+1}) = (2, \ldots, 2, 1, 1) \) which will prove (5.7).

\( v_1, \ldots, v_n \) form a basis of \( L \) since \( |A_{n+1}| = 1 \). Therefore we may assume that \( L \) is \( \mathbb{Z}^n \) and \( v_i = e_i \), the standard unit vector. The fact that \( 2x_{n+1} \) is \( T \)-Cartier is equivalent to
\[ 2u_{n+1}' = \iota^*_I(2x_{n+1}) \in L^* \] for \( I = \{i\}^* \), \( 1 \leq i \leq n \).

If \( \{e_i^*\} \) is the basis of \( L^* \) dual to \( \{e_i\} \), then
\[ u_{n+1}' = -\frac{1}{a_i}e_i^* \]
for \( I = \{i\}^* \) since \( v_{n+1} = -\sum_{i=1}^{n} a_i e_i \). Therefore from the above condition \( 2u_{n+1}' \in L^* \) it follows that \( a_i \) must be equal to 1 or 2. But \( \sum_{i=1}^{n+1} a_i = 2n \). Hence
\[ (a_1, \ldots, a_{n-1}, a_n, a_{n+1}) = (2, \ldots, 2, 1, 1). \]

Now we put \( K = \{n, n+1\} \in \Sigma^{(2)} \). If \( J \in \Sigma \) does not contain \( K \), then \( J \) is a subset of \( \{1, \ldots, n-1, n\} \) or \( \{1, \ldots, n-1, n+1\} \). Since \( v_1, \ldots, v_{n-1}, v_n \) and \( v_1, \ldots, v_{n-1}, v_{n+1} \) are bases of \( L \), \( H_J \) is a trivial group.

On the other hand \( L_K \) is generated by \( v_n \) and \( (v_n + v_{n+1})/2 \), and \( L_{K, \{v_n, v_{n+1}\}} \) is generated by \( v_n, v_{n+1} \). This implies that \( H_K = L_K/L_{K, \{v_n, v_{n+1}\}} \cong \mathbb{Z}/2 \). As a representative \( v(h) \) of its generator \( h \) we can take \( (v_n + v_{n+1})/2 \). Similarly for \( I_i = \{1, \ldots, n, n+1\} \setminus \{i\} \) with \( 1 \leq i \leq n-1 \), we have \( H_{I_i} = L_{I_i}/L_{I_i} \cong \mathbb{Z}/2 \). This implies that \( H_K = \{h\} \), and \( K \) is the unique element in \( \Sigma \) such that \( H_K \neq \emptyset \). Moreover if we identify \( L^* \) with \( \mathbb{Z}^n \), then
\[ u_{n}^{K} = -\sum_{i=1}^{n-1} \frac{1}{2(n-1)} e_i + e_n \] and \( u_{n+1}^{K} = -\sum_{i=1}^{n-1} \frac{1}{2(n-1)} e_i \).

It follows that \( f_{K, h, i} = \langle u_{i}^{K}, v(h) \rangle = \frac{1}{2} \) for \( i = n, n+1 \), and \( f_{K, h} = 1 \). Therefore
\[ \hat{T}_y(\Delta) = T_y(\Delta) + (-y)T_y(\Delta_K). \]

From this and (32) we see also that
\[ T_y(\Delta_K) = \sum_{k=1}^{n-2} (-y)^k. \]
\[ \square \]
Remark 5.8. Let \( \Delta \) be a fan of the form (a) in Proposition 5.6. The corresponding toric variety is a projective space bundle over a projective line, c.f. \([HM2]\) and \([Fuj]\). In the notation of \([Fuj]\) it is written \( \mathbb{P}(\mathcal{O}(k_1) \oplus \mathcal{O}(k_2) \oplus \cdots \mathcal{O}(k_n)) \) with \( \sum_{i=1}^{n} k_i = 2 \).

The toric variety corresponding to the fan of the form (b) in Proposition 5.6 is the weighted projective space \( \mathbb{P}^{n}(2, \ldots, 2, 1, 1) \). Its orbifold structure is the one as the quotient \((\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*\) where the action of \( \mathbb{C}^* \) is given by

\[
(z_1, \ldots, z_{n-1}, z_n, z_{n+1}) \rightarrow (z^2 z_1, \ldots, z^2 z_{n-1}, zz_n z z_{n+1}).
\]

See \([Fuj]\). It should be noticed that the action of the finite group \((\mathbb{Z}/2)^{n-1}\) on \( \mathbb{P}^{n}(2, \ldots, 2, 1, 1) \) gives the same algebraic variety \( \mathbb{P}^{n}(2, \ldots, 2, 1, 1) \) but a different orbifold structure.

Since a toric variety is determined by its fan, we obtain

**Corollary 5.9.** Let \( X \) be a \( \mathbb{Q} \)-factorial complete toric variety of dimension \( n \) and \( K_X \) denote the canonical divisor of \( X \). If there exists a \( T \)-Cartier divisor \( D \) such that \( K_X \) is linearly equivalent to \( nD \), then \( X \) is isomorphic to a projective space bundle \( \mathbb{P}(\mathcal{O}(k_1) \oplus \mathcal{O}(k_2) \oplus \cdots \mathcal{O}(k_n)) \) with \( \sum_{i=1}^{n} k_i = 2 \) or to \( \mathbb{P}^{n}(2, \ldots, 2, 1, 1) \) as a toric variety.

Fujino \([Fuj]\) classified the \( n \)-dimensional projective toric varieties \( X \) such that \( K_X \) is \( \mathbb{Q} \)-Cartier and numerically equivalent to \( nD \) for some Cartier divisor \( D \). When \( X \) is \( \mathbb{Q} \)-factorial the conclusion is the same as Corollary 5.9.

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