DEGENERATIONS AND MIRROR CONTRACTIONS OF CALABI-YAU COMPLETE INTERSECTIONS
VIA BATYREV-BORISOV MIRROR SYMMETRY

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Abstract. We show that the dual of the Cayley cone, associated to a Minkowski sum decomposition of a reflexive polytope, contains a reflexive polytope admitting a nef-partition. This nef-partition corresponds to a Calabi-Yau complete intersection in a Gorenstein Fano toric variety degenerating to an ample Calabi-Yau hypersurface in another Fano toric variety. Using the Batyrev-Borisov mirror symmetry construction, we found the mirror contraction of a Calabi-Yau complete intersection to the mirror of the ample Calabi-Yau hypersurface.

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0. Introduction.

Reflexive polytopes were introduced by V. Batyrev in [B2]. These are lattice polytopes Δ in a real vector space \( \mathbb{R}^d \) with lattice points in \( \mathbb{Z}^d \) corresponding to monomials of the anticanonical degree on a Gorenstein Fano toric variety. Such polytopes are determined by the property that they have vertices at lattice points and have the origin in their interior with the dual polytope \( \Delta^* = \{ y \in \mathbb{R}^d \mid \langle \Delta, y \rangle \geq -1 \} \) satisfying the same property. This was the starting point for the Batyrev construction of a large class of mirror pairs of Calabi-Yau hypersurfaces in toric varieties in [B2].

A Gorenstein Fano toric variety associated to a reflexive polytope Δ can be defined as \( X_\Delta = \text{Proj}(\mathbb{C}[\sigma \cap \mathbb{Z}^{d+1}]) \), where \( \sigma = \mathbb{R}_{\geq 0} \cdot (\Delta, 1) \subset \mathbb{R}^{d+1} \). It contains an affine torus \( T = (\mathbb{C}^*)^d \) as a dense open subset which acts naturally on the toric

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A Calabi-Yau hypersurface $Y_{\Delta}$ in a Gorenstein Fano toric variety $X_{\Delta}$ can be viewed as the Zariski closure of a hypersurface

$$\sum_{m \in \Delta \cap \mathbb{Z}^d} a_m t^m = 0$$

in the affine torus $(\mathbb{C}^*)^d \subset X_{\Delta}$, where $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$, $a_m \in \mathbb{C}$ and $t^m = t_1^{m_1} \cdots t_d^{m_d}$ for the coordinates $t_1, \ldots, t_d$ on the torus.

More generally, a nef Calabi-Yau complete intersection in a Gorenstein Fano toric variety $X_{\Delta}$ corresponds to a Minkowski sum decomposition of the reflexive polytope $\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_k$ by lattice polytopes. The Calabi-Yau complete intersection $Y_{\Delta_0, \ldots, \Delta_k}$ is the closure of the affine complete intersection

$$\sum_{m \in \Delta_i \cap \mathbb{Z}^d} a_{i,m} t^m = 0, \quad i = 0, \ldots, k$$

in $(\mathbb{C}^*)^d \subset X_{\Delta}$ with generic coefficients $a_{i,m} \in \mathbb{C}$. A complete intersection in a toric variety is called nondegenerate if every intersection with a $T$-orbit is either transversal or empty. A generic nef Calabi-Yau complete intersection is nondegenerate by Lemma 4.3 in [M1] and Proposition 6.8 in [D].

The mirror construction of Batyrev is a pair of families of nondegenerate Calabi-Yau hypersurfaces obtained as maximal projective crepant partial resolutions of $Y_{\Delta}$ and $Y_{\Delta^*}$. Generalizing the polar duality of reflexive polytopes, L. Borisov in [Bo] introduced the notion of nef-partition, which is a Minkowski sum decomposition of the reflexive polytope $\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_k$ by lattice polytopes such that the origin $0 \in \Delta_i$ for all $i$. A nef-partition has a dual nef-partition defined as the Minkowski sum decomposition of the reflexive polytope $\nabla = \Delta_0 + \cdots + \Delta_k$ in the dual vector space with $\nabla_j$ determined by $\langle \Delta_i, \nabla_j \rangle \geq -\delta_{ij}$ for all $0 \leq i, j \leq k$, where $\delta_{ij}$ is the Kronecker symbol. One of the basic properties of the nef-partitions is that $\Delta^* = \text{Conv}(\Delta_0, \ldots, \Delta_k)$ and $\nabla^* = \text{Conv}(\nabla_0, \ldots, \nabla_k)$. The Batyrev-Borisov mirror symmetry construction is a pair of families of nondegenerate nef Calabi-Yau complete intersections obtained as maximal projective crepant partial resolutions of $Y_{\Delta_0, \ldots, \Delta_k}$ and $Y_{\nabla_0, \ldots, \nabla_k}$.

A topological mirror symmetry test for compact $n$-dimensional Calabi-Yau manifolds $V$ and $V^*$ is a symmetry of their Hodge numbers: $h^{p,q}(V) = h^{n-p,q}(V^*)$, $0 \leq p, q \leq n$. For singular varieties Hodge numbers must be replaced by the stringy Hodge numbers $h^{p,q}_{st}$ introduced by V. Batyrev in [B3]. The usual Hodge numbers coincide with the stringy Hodge numbers for nonsingular Calabi-Yau varieties. Moreover, all crepant partial resolutions $\tilde{V}$ of singular Calabi-Yau varieties $V$ have the same stringy Hodge numbers: $h^{p,q}_{st}(\tilde{V}) = h^{p,q}_{st}(V)$. In [BB03], Batyrev and Borisov show that the pair of Calabi-Yau complete intersections $V = Y_{\Delta_0, \ldots, \Delta_k}$ and $V^* = Y_{\nabla_0, \ldots, \nabla_k}$ pass the mirror symmetry test. One of the main ingredients of their proof was the use of the Cayley trick which associates to a Calabi-Yau complete intersection $Y_{\Delta_0, \ldots, \Delta_k}$ a generalized Calabi-Yau hypersurface in a higher dimensional Fano toric variety $\text{Proj}(\mathbb{C}[\tilde{\sigma} \cap \mathbb{Z}^{d+k+1}])$, where $\tilde{\sigma} = \{\sum_{i=0}^k t_i \Delta_i, t_0, \ldots, t_k | t_i \in \mathbb{R}_{\geq 0}\} \subset \mathbb{R}^{d+k+1}$, called the Cayley cone associated to the polytopes $\Delta_0, \ldots, \Delta_k$.

There are six different reflexive polytopes of dimension $d+k$ associated to a Minkowski sum decomposition of a $d$-dimensional reflexive polytope $\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_k$ into $k+1$ lattice polytopes. One of them, contained in the Cayley
cone $\sigma$ at an integral distance $k + 1$ from the origin, is isomorphic to

$$(k + 1)\text{Conv}(\Delta_0, \Delta_1 + e_1, \ldots, \Delta_k + e_k) - \sum_{i=1}^{k} e_i$$

in $\mathbb{R}^{d+k} \cong \mathbb{R}^d \oplus \mathbb{R}^k$ where $\{e_1, \ldots, e_k\}$ is the standard basis of $\mathbb{R}^k$. The dual $\sigma^*$ of the Cayley cone also contains a reflexive polytope isomorphic to

$$(k + 1)\text{Conv}\left(\left\{ u - \sum_{i=1}^{k} \min(\Delta_i, u)e_i^* \mid u \in \Delta^* \right\} \cup \{e_1^*, \ldots, e_k^*\} \right) - \sum_{i=1}^{k} e_i^*$$

in $\mathbb{R}^{d+k} \cong \mathbb{R}^d \oplus \mathbb{R}^k$, where $\{e_1^*, \ldots, e_k^*\}$ is the standard basis of $\mathbb{R}^k$ dual to $\{e_1, \ldots, e_k\}$. The reflexive polytopes (1) and (2) are not dual to each other, and their dual polytopes give another two reflexive polytopes. It turns out that while the polytope (1) may not admit a nef-partition (if $\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_k$ is not a nef-partition), the reflexive polytope (2) always admits one: $\nabla_0 = \text{Conv}(\{u - \sum_{i=1}^{k} \min(\Delta_i, u)e_i^* \mid u \in \Delta^* \} \cup \{e_1^*, \ldots, e_k^*\})$, $\nabla_i = \nabla_0 - e_i^*$, for $i = 1, \ldots, k$. The dual of this nef-partition is $\Delta_0 + \cdots + \Delta_k$, where $\Delta_0 = \text{Conv}(\Delta_1 + e_1, 0)$, for $i = 0, \ldots, k$, and $e_0 := -\sum_{i=1}^{k} e_i$. This reflexive polytope together with its dual $\text{Conv}(\nabla_0, \ldots, \nabla_k)$ are the other two reflexive polytopes associated to a Minkowski sum decomposition of the reflexive polytope $\Delta$.

The dual of $\nabla_0 + \cdots + \nabla_k$ is the reflexive polytope $\text{Conv}(\Delta_0, \ldots, \Delta_k)$. The Gorenstein Fano toric variety $X_{\Delta_0 + \cdots + \Delta_k}$, whose fan consists of the cones over the proper faces of $\text{Conv}(\Delta_0, \ldots, \Delta_k)$, is the ambient space of deformations of the Gorenstein Fano toric variety $X_{\Delta^*}$ in [M5]. In this paper, we show that the embedding $X_{\Delta^*} \hookrightarrow X_{\nabla_0 + \cdots + \nabla_k}$ realizes the ample Calabi-Yau hypersurface $Y_{\Delta^*}$ as a complete intersection in $X_{\nabla_0 + \cdots + \nabla_k}$, which deforms to a nondegenerate Calabi-Yau complete intersection $Y_{\nabla_0, \ldots, \nabla_k}$ corresponding to the nef-partition $\nabla_0 + \cdots + \nabla_k$. The degeneration $Y_{\nabla_0, \ldots, \nabla_k} \twoheadrightarrow Y_{\Delta^*}$ can be lifted to the degeneration of a maximal projective crepant partial resolution $Y'_{\nabla_0, \ldots, \nabla_k}$ of $Y_{\nabla_0, \ldots, \nabla_k}$ to a partial resolution $Y'_{\Delta^*}$ of $Y_{\Delta^*}$. Taking maximal projective crepant partial resolution $Y''_{\Delta^*}$ of $Y'_{\Delta^*}$, we obtain a geometric transition (a contraction followed by smoothing) from a minimal Calabi-Yau hypersurface to a minimal Calabi-Yau complete intersection: $Y''_{\nabla_0, \ldots, \nabla_k} \twoheadrightarrow Y'_{\Delta^*} \hookrightarrow Y'_{\Delta^*}$. According to a conjecture of D. Morrison in [Mo], every geometric transition between Calabi-Yau manifolds should correspond to a mirror geometric transition between the mirror partners of the original Calabi-Yau manifolds with the roles of degeneration and contraction reversed. In Section 5, we explicitly construct a natural contraction of a minimal Calabi-Yau complete intersection $Y''_{\Delta^*} \twoheadrightarrow Y''_{\Delta^*}$ to a degenerate Calabi-Yau hypersurface $Y''_{\Delta^*}$ in a maximal projective crepant partial resolution of $X_{\Delta}$. The smoothing of $Y''_{\Delta^*}$ to a nondegenerate Calabi-Yau hypersurface $Y''_{\Delta^*}$ gives a geometric transition $Y''_{\Delta^*} \twoheadrightarrow Y'_{\Delta^*} \hookrightarrow Y'_{\Delta^*}$, which should be the mirror of the above one. We use the method of Batyrev in [B1], [BvS] to support the mirror correspondence of geometric transitions by showing that the degeneration of the hypergeometric series arising from the main period of Calabi-Yau varieties coincides with the hypergeometric series of the maximal projective partial crepant resolution of the degenerate Calabi-Yau. These hypergeometric series determine the mirror map between the Kahler and complex moduli spaces (see [CK, Sec. 6.3.4]).
The construction of deformations of ample Calabi-Yau hypersurfaces and their partial resolutions are consistent with our conjecture in [M4] that all deformations of Calabi-Yau complete intersections (of dimension $\geq 3$) in toric varieties are Calabi-Yau complete intersections in higher dimensional toric varieties. An application of deformations of Gorenstein Fano toric varieties to deformations of nef Calabi-Yau complete intersections and a generalization of the above geometric transitions will appear in [M6]. These constructions together with the previously known geometric transitions between Calabi-Yau hypersurfaces in [BeKKI, Mo] give a strong evidence that the web of Calabi-Yau complete intersections in toric varieties can be connected by explicit geometric transitions.

Here is an organization of our paper. In Section 1, we study properties of reflexive Gorenstein cones and explicitly describe the Cayley cone and its dual together with the reflexive polytopes contained in these cones. Then we briefly overview some basic notation and facts of toric geometry. Section 3 explains the relation of the Cayley trick and deformations of Fano toric varieties constructed in [M5], and Section 4 constructs deformations of Calabi-Yau hypersurfaces. Finally, in Section 5, we construct two geometric transitions described above, and then Section 6 discusses degenerations of the main periods of Calabi-Yau complete intersections and Mirror Symmetry.

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1. Combinatorics of reflexive polytopes and Gorenstein cones.

In this section, we explicitly describe the reflexive polytopes arising from the construction of the Cayley cone associated to a Minkowski sum decomposition of a reflexive polytope in [BB01]. We show that the dual of the Cayley cone contains a reflexive polytope which admits a nef-partition introduced in [Bo].

Let $N$ be a lattice and $M$ be its dual lattice with a paring $\langle *, * \rangle : M \times N \to \mathbb{Z}$.

**Definition 1.1.** [B2] A lattice polytope $\Delta$ in $M_\mathbb{R} = M \otimes \mathbb{R}$ (i.e., its vertices are at the lattice points) is called a reflexive polytope if it contains 0 in its interior and the dual polytope

$$\Delta^* = \{ n \in N_\mathbb{R} \mid \langle m, n \rangle \geq -1 \forall m \in \Delta \}$$

in the dual vector space $N_\mathbb{R} = N \otimes \mathbb{R}$ is also a lattice polytope. The pair $\Delta$ and $\Delta^*$ is called a pair of dual reflexive polytopes and it satisfies $\Delta = (\Delta^*)^*$.

Reflexive polytopes are related to the notion of reflexive Gorenstein cones from [BB01]. Let $\tilde{M}$ and $\tilde{N}$ be lattices which are dual to each other. Let $\sigma \subset \tilde{M}_\mathbb{R}$ be a polyhedral cone with a vertex at 0. The dual cone of $\sigma$ is defined as

$$\sigma^\vee = \{ n \in \tilde{N}_\mathbb{R} \mid \langle m, n \rangle \geq 0 \forall m \in \sigma \}.$$

**Definition 1.2.** [BB01] A maximal dimensional polyhedral cone $\sigma$ is called Gorenstein, if it is generated by finitely many lattice points contained in the affine hyperplane $\{ x \in \tilde{M} \mid \langle x, n_\sigma \rangle = 1 \}$ for a unique $n_\sigma \in \tilde{N}$. A Gorenstein cone $\sigma$ is called reflexive if both $\sigma$ and $\sigma^\vee$ are Gorenstein cones, in which case they both have maximal dimension and uniquely determined $n_\sigma \in \tilde{N}$ and $m_\sigma, m_\sigma^\vee \in M$, which take value 1 at the primitive lattice generators of the respective cones. The positive integer $r = \langle m_\sigma^\vee, n_\sigma \rangle$ is called the index of the reflexive Gorenstein cones $\sigma$ and $\sigma^\vee$. 
Lemma 1.5. A third way to write the Cayley cone is given in the following lemma.

Proposition 1.3. [BBo1, Pr. 2.11] Let \( \sigma \) be a Gorenstein cone. Then \( \sigma \) is a reflexive Gorenstein cone of index \( r \) if and only if the polytope \( \sigma(r) - m_{\sigma^\vee} \) is a reflexive polytope with respect to the lattice \( \check{M} \cap n_{\sigma}^+ = \{ x \in M \mid \langle x, n_{\sigma} \rangle = 0 \} \).

As noted in Remark 1.13 in [BN], the reflexive polytopes \( \sigma(r) - m_{\sigma^\vee} \) and \( \sigma(r) - n_{\sigma} \) are combinatorially dual to each other, but not dual as lattice polytopes. In [BN, Proposition 1.15], the dual reflexive polytope \( (\sigma(r) - m_{\sigma^\vee})^* \) was obtained as \( \sigma(r)_i \) with respect to the refined affine lattice \( (\check{N} + \frac{1}{2} \mathbb{Z} n_{\sigma}) \cap \{ y \in \check{N}_{\mathbb{R}} \mid \langle m_{\sigma^\vee}, y \rangle = 1 \} \).

We will give an alternative description for the dual reflexive polytope.

Proposition 1.4. Let \( \sigma \) be a reflexive Gorenstein cone of index \( r \). Then the dual polytope \( (\sigma(r) - m_{\sigma^\vee})^* = \pi(\sigma(r)_i) \) with respect to the lattice \( \check{N}/\mathbb{Z} n_{\sigma} \simeq \text{Hom}(M \cap n_{\sigma}^+, \mathbb{Z}) \), where \( \pi : \check{N}_{\mathbb{R}} \rightarrow \check{N}_{\mathbb{R}}/\mathbb{R} n_{\sigma} \) is the quotient homomorphism.

Proof. Let the vertices \( v \) of \( \sigma(r)_i \) are in one-to-one correspondence with the facets \( F_v := \sigma(r)_i \cap v^\perp \) of the polytope \( \sigma(r) \) by the duality of the cones \( \sigma \) and \( \sigma^\vee \). Then \( \langle F_v - m_{\sigma^\vee}, v \rangle = -1 \) with respect to the pairing of \( \check{M} \) and \( \check{N} \). Consequently, \( \langle F_v - m_{\sigma^\vee}, \pi(v) \rangle = -1 \) with respect to the pairing of \( \check{M} \cap n_{\sigma}^+ \) and \( \check{N}/\mathbb{Z} n_{\sigma} \). Hence, all vertices of the dual polytope \( (\sigma(r) - m_{\sigma^\vee})^* \) in \( \check{N}_{\mathbb{R}}/\mathbb{R} n_{\sigma} \) are of the form \( \pi(v) \) for a vertex \( v \) of \( \sigma(r)_i \).

A special class of reflexive Gorenstein cones arises from a Calabi-Yau complete intersection in a Gorenstein Fano toric variety by a Cayley trick (see [BBo1]). Let \( \Delta \) be a reflexive polytope in \( M_\mathbb{R} \) and \( \Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_k \) be a Minkowski sum decomposition by lattice polytopes. By [BBo1, Proposition 3.6], the cone

\[
\sigma = \left\{ \sum_{i=0}^{k} t_i \Delta_i, t_0, \ldots, t_k \mid t_i \in \mathbb{R}_{\geq 0} \right\} \subset M_\mathbb{R} \oplus \mathbb{R}^{k+1}
\]
is reflexive Gorenstein of index \( k + 1 \). This cone is called the Cayley cone associated to the polytopes \( \Delta_0, \ldots, \Delta_k \). It can also be written as

\[
\bar{\sigma} = \mathbb{R}_{\geq 0} \cdot \text{Conv}(\Delta_0 + r_0, \Delta_1 + r_1, \ldots, \Delta_k + r_k),
\]
where \( \{r_0, \ldots, r_k\} \subset \mathbb{Z}^{k+1} \subset M \oplus \mathbb{Z}^{k+1} \) is the standard basis of the second summand. A third way to write the Cayley cone is given in the following lemma.

Lemma 1.5. There is equality of cones

\[
\bar{\sigma} = \left\{ (t \cdot \text{Conv}(\Delta_0, \Delta_1 + e_1, \ldots, \Delta_k + e_k), t) \mid t \in \mathbb{R}_{\geq 0} \right\},
\]
induced by the isomorphism

\[
M \oplus \mathbb{Z}^{k+1} \simeq M \oplus \mathbb{Z}^k \oplus \mathbb{Z}, \quad (m, \alpha_0, \ldots, \alpha_k) \mapsto (m, \alpha_1, \ldots, \alpha_k, \alpha_0 + \cdots + \alpha_k),
\]
where \( \{e_1, \ldots, e_k\} \) is the standard basis for the second summand \( \mathbb{Z}^k \).

The dual of the Cayley cone \( \bar{\sigma}^\vee \) can also be explicitly found.
Proposition 1.6. Let \( \bar{\sigma} \subset M_\mathbb{R} \oplus \mathbb{R}^{k+1} \) be the Cayley cone associated to \( \Delta_0, \ldots, \Delta_k \). Then
\[
\bar{\sigma}^\vee = \mathbb{R}_{\geq 0} \cdot \text{Conv}(\left\{ u - \sum_{i=0}^{k} \min(\Delta_i, u) r_i^* | u \in \Delta^* \right\} \cup \{ r_0^*, \ldots, r_k^* \}),
\]
where \( \{ r_0^*, \ldots, r_k^* \} \) is the basis of \( \mathbb{Z}^{k+1} \subset N \oplus \mathbb{Z}^{k+1} \) dual to \( \{ r_0, \ldots, r_k \} \).

Proof. We have \( u + \sum_{i=0}^{k} \alpha_i r_i^* \in \bar{\sigma}^\vee \) with \( u \in N_\mathbb{R} \) and \( \alpha_i \in \mathbb{R} \) if and only if \( \langle x_j + r_j, u + \sum_{i=0}^{k} \alpha_i r_i^* \rangle \geq 0 \) for all \( x_j \in \Delta_j \), \( j = 0, \ldots, k \). But the last inequality is equivalent to \( \alpha_j \geq -\langle x_j, u \rangle \) for all \( x_j \in \Delta_j \). Hence, \( \alpha_j \geq -\min(\Delta_j, u) \). Since 0 is in the interior of \( \Delta \), \( \min(\Delta, u) < 0 \) for \( u \neq 0 \), whence \( u + \sum_{i=0}^{k} \alpha_i r_i^* = -\min(\Delta, u) (u' - \sum_{i=0}^{k} \min(\Delta_i, u') r_i^*) + \sum_{i=0}^{k} \beta_i r_i^* \), where \( u' = \frac{u}{\min(\Delta, u)} \in \Delta^* \) and \( \beta_i = \alpha_i + \min(\Delta_i, u) \geq 0 \).

The following alternative view of the dual of the Cayley cone may also be useful.

Lemma 1.7. There is equality of cones
\[
\bar{\sigma}^\vee = \left\{ \left( t \cdot \text{Conv}(\left\{ u - \sum_{i=0}^{k} \min(\Delta_i, u) e_i^* | u \in \Delta^* \right\} \cup \{ e_1^*, \ldots, e_k^* \}), t \right) | t \in \mathbb{R}_{\geq 0} \right\}
\]
induced by the isomorphism
\[
N \oplus \mathbb{Z}^{k+1} \simeq N \oplus \mathbb{Z}^k \oplus \mathbb{Z}, \quad (n, \alpha_0, \ldots, \alpha_k) \mapsto (n, \alpha_1, \ldots, \alpha_k, \alpha_0 + \cdots + \alpha_k),
\]
where \( \{ e_1^*, \ldots, e_k^* \} \) is the standard basis for the second summand \( \mathbb{Z}^k \).

The above descriptions of the Cayley cone \( \bar{\sigma} \), associated to a Minkowski sum decomposition of the reflexive polytope \( \Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_k \), and of the dual cone \( \bar{\sigma}^\vee \) directly show that both cones are Gorenstein reflexive of index \( k + 1 \) with the unique lattice points \( n_{\bar{\sigma}} = r_0^* + r_1^* + \cdots + r_k^* \) and \( m_{\bar{\sigma}^\vee} = r_0 + r_1 + \cdots + r_k \). Next, we want to explicitly describe the reflexive polytopes \( \bar{\sigma}^{(k+1)} - m_{\bar{\sigma}^\vee} \) and \( \bar{\sigma}^{(k+1)} - n_{\bar{\sigma}} \), and their dual polytopes, arising from Proposition 1.4.

Consider the lattice \( \tilde{M} := M \oplus \mathbb{Z}^k \) and denote by \( \{ e_1, \ldots, e_k \} \) the standard basis for the second summand \( \mathbb{Z}^k \). Then \( \tilde{N} := N \oplus \mathbb{Z}^k \) is the dual to \( \tilde{M} \) lattice and set \( \{ e_1^*, \ldots, e_k^* \} \) be the dual to \( \{ e_1, \ldots, e_k \} \) basis in \( \mathbb{Z}^k \). The following statements follow trivially from Propositions 1.4 and 1.6.

Lemma 1.8. Let \( \bar{\sigma} \subset M_\mathbb{R} \oplus \mathbb{R}^{k+1} \) be the Cayley cone associated to lattice polytopes \( \Delta_0, \ldots, \Delta_k \) in \( M_\mathbb{R} \) such that \( \Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_k \) is reflexive. Then
\[
\bar{\sigma}^{(k+1)} - m_{\bar{\sigma}^\vee} \simeq (k + 1) \text{Conv}(\Delta_0, \Delta_1 + e_1, \ldots, \Delta_k + e_k) - \sum_{i=1}^{k} e_i
\]
induced by the isomorphism \( \tilde{M} \cap n_{\tilde{\sigma}}^\perp \simeq \tilde{M}, \quad (m, \alpha_0, \ldots, \alpha_k) \mapsto (m, \alpha_1, \ldots, \alpha_k) \), where \( \tilde{M} := M \oplus \mathbb{Z}^{k+1} \) and \( n_{\tilde{\sigma}} = \sum_{i=0}^{k} r_i^* \). Also,\[
\pi(\bar{\sigma}^{(1)}) = \text{Conv}\left( \Delta_0 - \sum_{i=1}^{k} e_i, \Delta_1 + e_1, \ldots, \Delta_k + e_k \right),
\]
where \( \pi : \tilde{M} \to \tilde{M}/\mathbb{Z}m_{\bar{\sigma}^\vee} \simeq \tilde{M}, \quad (m, \alpha_0, \ldots, \alpha_k) \mapsto (m, \alpha_1 - \alpha_0, \ldots, \alpha_k - \alpha_0) \).
Lemma 1.9. Let $\tilde{\sigma} \subset M_\mathbb{R} \oplus \mathbb{R}^{k+1}$ be the Cayley cone associated to lattice polytopes $\Delta_0, \ldots, \Delta_k$ in $M_\mathbb{R}$ such that $\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_k$ is reflexive. Then

$$\tilde{\sigma}^\vee_{(k+1)-n_\sigma} \simeq (k+1)\text{Conv}\left(\left\{u - \sum_{i=1}^{k} \min \langle \Delta_i, u \rangle e_i^* \mid u \in \Delta^* \right\} \cup \{e_0^*, \ldots, e_k^*\}\right) - \sum_{i=1}^{k} e_i^*$$

under the isomorphism $\tilde{N} \cap m_{0^\vee}^+ \simeq \tilde{N}$, $(n, \alpha_0, \ldots, \alpha_k) \mapsto (n, \alpha_1, \ldots, \alpha_k)$, where $\tilde{N} := N \oplus \mathbb{Z}^{k+1}$, $m_{0^\vee}^+ = \sum_{i=0}^{k} r_i$. Also,

$$\pi(\tilde{\sigma}^\vee_{(1)}) = \text{Conv}\left(\left\{u - \sum_{i=0}^{k} \min \langle \Delta_i, u \rangle e_i^* \mid u \in \Delta^* \right\} \cup \{e_0^*, \ldots, e_k^*\}\right),$$

where $\pi : \tilde{N} \rightarrow \tilde{N}/\mathbb{Z}m_{0^\vee}^+ \simeq \tilde{N}$, $(n, \alpha_0, \ldots, \alpha_k) \mapsto (n, \alpha_1 - \alpha_0, \ldots, \alpha_k - \alpha_0)$, and $e_0^* := -\sum_{i=1}^{k} e_i^*$.

It is straightforward to check that the above natural isomorphisms $\tilde{M} \cap n_{0^\vee}^+ \simeq \tilde{M}$, $\tilde{N}/\mathbb{Z}m_{0^\vee}^+ \simeq \tilde{N}$, $\tilde{M}/\mathbb{Z}m_{0^\vee}^+ \simeq \tilde{M}$ respect pairings. Hence, by combining the above lemmas with Proposition 1.4 we get two pairs of reflexive polytopes.

Proposition 1.10. Let $\Delta$ be a reflexive polytope in $M_\mathbb{R}$ and $\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_k$ be a Minkowski sum decomposition by lattice polytopes in $M_\mathbb{R}$. Denote $\hat{\Delta}_0 = \text{Conv}(\Delta_0, \Delta_1 + e_1, \ldots, \Delta_k + e_k)$, $\hat{\Delta}_i = \hat{\Delta}_0 - e_i$ in $M_\mathbb{R}$, for $i = 1, \ldots, k$. Then $\hat{\Delta}_0 + \cdots + \hat{\Delta}_k = (k+1)\hat{\Delta}_0 + e_0$ is a reflexive polytope in $M_\mathbb{R} = M_\mathbb{R} \oplus \mathbb{R}^k$ with the dual reflexive polytope

$$(\hat{\Delta}_0 + \cdots + \hat{\Delta}_k)^* = \text{Conv}\left(\left\{u - \sum_{i=0}^{k} \min \langle \Delta_i, u \rangle e_i^* \mid u \in \Delta^* \right\} \cup \{e_0^*, \ldots, e_k^*\}\right)$$

in $\tilde{N}_\mathbb{R} = N_\mathbb{R} \oplus \mathbb{R}^k$, where $e_0 = -\sum_{i=1}^{k} e_i$, $e_0^* = -\sum_{i=1}^{k} e_i^*$.

Proposition 1.11. Let $\Delta$ be a reflexive polytope in $M_\mathbb{R}$ and $\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_k$ be a Minkowski sum decomposition by lattice polytopes in $M_\mathbb{R}$. Denote $\hat{\nabla}_0 = \text{Conv}(\{u - \sum_{i=0}^{k} \min \langle \Delta_i, u \rangle e_i^* \mid u \in \Delta^* \} \cup \{e_0^*, \ldots, e_k^*\})$, $\hat{\nabla}_i = \hat{\nabla}_0 - e_i^*$ in $\tilde{N}_\mathbb{R}$, for $i = 1, \ldots, k$. Then $\hat{\nabla}_0 + \cdots + \hat{\nabla}_k = (k+1)\hat{\nabla}_0 + e_0^*$ is a reflexive polytope in $\tilde{M}_\mathbb{R} = M_\mathbb{R} \oplus \mathbb{R}^k$ with the dual reflexive polytope

$$(\hat{\nabla}_0 + \cdots + \hat{\nabla}_k)^* = \text{Conv}(\Delta_0 + e_0, \Delta_1 + e_1, \ldots, \Delta_k + e_k)$$

in $\hat{M}_\mathbb{R} = M_\mathbb{R} \oplus \mathbb{R}^k$, where $e_0 = -\sum_{i=1}^{k} e_i$, $e_0^* = -\sum_{i=1}^{k} e_i^*$.

It turns out that one of the reflexive polytopes in the above propositions always admits a nef-partition introduced in [Bo]. A nef-partition of a reflexive polytope $\Delta$ is a Minkowski sum decomposition $\Delta = \Delta_0 + \cdots + \Delta_k$ by lattice polytopes such that the origin $0 \in \Delta_j$ for all $i$. If one defines the polytopes

$$\nabla_j = \{y \in \mathbb{R}^d \mid \langle \delta_{ij}, x \rangle \geq 0 \forall x \in \Delta_i, i = 0, \ldots, k\}$$

for $j = 0, \ldots, k$, then $\nabla = \nabla_0 + \cdots + \nabla_k$ is a reflexive polytope and $\nabla_j$ are lattice polytopes with $0 \in \nabla_j$ for all $j$. The nef-partitions $\Delta = \Delta_0 + \cdots + \Delta_k$ and $\nabla = \nabla_0 + \cdots + \nabla_k$ are called dual to each other.

For convenience, we will introduce the following notation. For any two subsets $P$ and $Q$ of a real vector space we denote by $P \uplus Q := \text{Conv}(P \cup Q)$, the convex hull of the union of $P$ and $Q$. The operation $\uplus$ is clearly associative and commutative.
By Propositions 3.1 and 3.2 in [Bo], we have the following dualities.

**Proposition 1.12.** [Bo] Let $\Delta$ be a reflexive polytope in $M_\mathbb{R}$ and let $\Delta = \Delta_0 + \cdots + \Delta_k$ be a nef-partition and $\nabla = \nabla_0 + \cdots + \nabla_k$ be the dual nef-partition in $N_\mathbb{R}$. Then $(\Delta_0 \cup \cdots \cup \Delta_k)^* = \nabla_0 + \cdots + \nabla_k$ and $(\nabla_0 \cup \cdots \cup \nabla_k)^* = \Delta_0 + \cdots + \Delta_k$.

Now, suppose $\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_k$ is a Minkowski sum decomposition of a reflexive polytope by lattice polytopes in $M_\mathbb{R}$. By construction, we see that $\nabla_0 + \cdots + \nabla_k$ is a nef-partition even if $\Delta = \Delta_0 + \cdots + \Delta_k$ is not a nef-partition, since $0 \in \nabla_i$, for $i = 0, \ldots, k$, and the polytopes $\nabla_i$ are convex hulls of lattice points. It is not difficult to determine that the dual nef-partition is $\Delta_0 + \cdots + \Delta_k$, where $\Delta_i := \text{Conv}(\Delta_i + e_i, 0)$. In particular, we have

**Proposition 1.13.** Let $\Delta$ be a reflexive polytope in $M_\mathbb{R}$ and $\Delta = \Delta_0 + \cdots + \Delta_k$ be a Minkowski sum decomposition by lattice polytopes in $M_\mathbb{R}$. Then $\nabla_0 + \cdots + \nabla_k$ is a reflexive polytope in $N_\mathbb{R} = N_\mathbb{R} \oplus \mathbb{R}^k$. Moreover, $\nabla_0 + \cdots + \nabla_k$ is a nef-partition dual to $\Delta_0 + \cdots + \Delta_k$ and the following identities hold:

$$(\nabla_0 \cup \cdots \cup \nabla_k)^* = \Delta_0 + \cdots + \Delta_k, \quad (\Delta_0 \cup \cdots \cup \Delta_k)^* = \nabla_0 + \cdots + \nabla_k.$$ \hfill (1.13)

One can easily find the lattice points in $\Delta_0 \cup \cdots \cup \Delta_k$ and $\nabla_0 \cup \cdots \cup \nabla_k$. Denote by $L(P)$ and $l(P)$ the set and the number of lattice points in a polytope $P$ in a real vector space.

**Proposition 1.14.** Let $\Delta$ be a reflexive polytope in $M_\mathbb{R}$ and $\Delta = \Delta_0 + \cdots + \Delta_k$ be a Minkowski sum decomposition by lattice polytopes in $M_\mathbb{R}$. Then $L(\Delta_0 \cup \cdots \cup \Delta_k) = \{0\} \cup \bigcup_{i=0}^k L(\Delta_i + e_i)$, $l(\Delta_0 \cup \cdots \cup \Delta_k) = 1 + \sum_{i=0}^k l(\Delta_i)$, $L(\nabla_0) = \{n - \sum_{i=1}^k \min(\Delta_i, n)e_i^* \mid n \in \Delta^* \cap N\} \cup \{e_1^*, \ldots, e_k^*\}$, $l(\nabla_0) = l(\Delta^*) + k$, $l(\nabla_0 \cup \cdots \cup \nabla_k) = (k + 1)l(\Delta^*) + k^2$.

The reflexive polytope $\Delta_0 \cup \cdots \cup \Delta_k$ arises from the construction of deformations of Gorenstein Fano toric varieties $X_\Delta^*$ associated to the fan generated by the faces of $\Delta = \Delta_0 + \cdots + \Delta_k$ in [M5]. The deformations are realized by complete intersections in a higher dimensional Fano toric variety whose fan is generated by the faces of $\Delta_0 \cup \cdots \cup \Delta_k$. On the other hand, the reflexive polytope $\Delta_0 + \cdots + \Delta_k$ arises from the Cayley trick: it corresponds to the dual of the canonical line bundle (or the anticanonical degree) on the projective space bundle associated to a Calabi-Yau complete intersection in the Fano toric variety $X_\Delta$ whose fan is generated by the faces of $\Delta^*$. We will review these constructions in detail in Section 3.

From Lemma 1.8, we can see that the reflexive polytopes $\Delta_0 \cup \cdots \cup \Delta_k$ and $\Delta_0 + \cdots + \Delta_k$ must be the same up to a linear transformation and a change of the lattice. The same should hold for the dual reflexive polytopes. These transformations can be explicitly described as follows.

**Lemma 1.15.** The homomorphism of lattices

$$\varphi : M \oplus \mathbb{Z}^k \to M \oplus \mathbb{Z}^k, \quad m + \sum_{i=1}^k \alpha_i e_i \mapsto (k + 1)m + \sum_{i=1}^k \alpha_i ((k + 1)e_i + e_0),$$

maps $\Delta_0 \cup \cdots \cup \Delta_k$ onto $\Delta_0 + \cdots + \Delta_k$. 

Lemma 1.16. The homomorphism of lattices

$$\varphi^* : \mathbb{Z}^k \to \mathbb{Z}^k, \quad n + \sum_{i=1}^k \alpha_i e_i^* \mapsto (k+1)n + \sum_{i=1}^k \alpha_i ((k+1)e_i^* + e_0),$$

maps $$(\Delta_0 + \cdots + \Delta_k)^*$$ onto $$(\Delta_0 \cup \cdots \cup \Delta_k)^*$$.  

Finishing this section, we will look at what happens if $$\Delta = \Delta_0 + \cdots + \Delta_k$$ is a nef-partition. In this case the dual nef-partition $$\nabla = \nabla_0 + \cdots + \nabla_k$$ satisfies $$\langle \Delta_i, \nabla_j \rangle \geq -\delta_{ij}$$ for all $$0 \leq i, j \leq k$$, and the dual to $$\Delta$$ reflexive polytope is $$\Delta^* = \text{Conv}(\nabla_0, \ldots, \nabla_k)$$ with $$\nabla_i \cap \nabla_j = \{0\}$$ for all $$i, j$$. Now if $$0 \neq u \in \nabla_i \cap N$$, then $$-1 = \min \langle \Delta, u \rangle = \min \langle \Delta_0, u \rangle + \cdots + \min \langle \Delta_k, u \rangle$$. Since $$\langle \Delta, u \rangle \geq 0$$ for $$j \neq i$$ and $$\langle \Delta_i, u \rangle \geq -1$$, we conclude that $$\min \langle \Delta_j, u \rangle = 0$$ for $$j \neq i$$ and $$\min \langle \Delta_i, u \rangle = -1$$. Hence, by Proposition 1.6, we get

$$\hat{\sigma}^* = \mathbb{R}_{\geq 0} \cdot \text{Conv}(\nabla_0 + r_0^*, \ldots, \nabla_k + r_k^*).$$

Similarly, $$\hat{\tilde{\sigma}}_0 = \text{Conv}(\nabla_0, \nabla_1 + e_1^*, \ldots, \nabla_k + e_k^*) = \hat{\Delta}_0$$. Applying Proposition 1.12, we get the following dualities for eight reflexive polytopes of dimension $$d + k$$ corresponding to a dual pair of nef-partitions of dimension $$d$$.

Proposition 1.17. Let $$\Delta$$ be a reflexive polytope in $$\mathbb{R}_d$$ and let $$\Delta = \Delta_0 + \cdots + \Delta_k$$ be a nef-partition and $$\nabla = \nabla_0 + \cdots + \nabla_k$$ be the dual nef-partition in $$\mathbb{R}_k$$. Then

$$(\Delta_0 \cup \cdots \cup \Delta_k)^* = \hat{\nabla}_0 + \cdots + \hat{\nabla}_k$$

and $$(\nabla_0 \cup \cdots \cup \nabla_k)^* = \Delta_0 + \cdots + \Delta_k$$ are nef-partitions respectively dual to

$$(\nabla_0 \cup \cdots \cup \nabla_k)^* = \Delta_0 + \cdots + \Delta_k$$

and $$(\Delta_0 \cup \cdots \cup \Delta_k)^* = \hat{\nabla}_0 + \cdots + \hat{\nabla}_k$$.

2. Some basics from toric geometry.

This section will review some basic facts from [C1], [C2], [F] on toric geometry. See [D], [O] for additional references.

Let $$X_{\Sigma}$$ be a $$d$$-dimensional toric variety associated with a finite rational polyhedral fan $$\Sigma$$ in $$N_{\mathbb{R}}$$. Denote by $$\Sigma(1)$$ the finite set of the 1-dimensional cones $$\rho$$ in $$\Sigma$$, which correspond to the torus invariant divisors $$D_{\rho}$$ in $$X_{\Sigma}$$. From the work of David Cox (see [C1]), every toric variety can be described as a categorical quotient of a Zariski open subset of an affine space by a subgroup of a torus. For simplicity, assume that the 1-dimensional cones $$\Sigma(1)$$ span $$N_{\mathbb{R}}$$. Consider the polynomial ring $$S(\Sigma) := \mathbb{C}[x_{\rho} \mid \rho \in \Sigma(1)]$$, called the homogeneous coordinate ring of the toric variety $$X_{\Sigma}$$, and the corresponding affine space $$\mathbb{C}^{\Sigma(1)} = \text{Spec}(\mathbb{C}[x_{\rho} \mid \rho \in \Sigma(1)])$$. Let $$B = \{ \prod_{\rho \in \sigma} x_{\rho} \mid \sigma \in \Sigma \}$$ be the ideal in $$S(\Sigma)$$. This ideal determines a Zariski closed set $$V(B)$$ in $$\mathbb{C}^{\Sigma(1)}$$, which is invariant under the diagonal group action of the subgroup

$$G = \left\{ (\mu_{\rho}) \in (\mathbb{C}^*)^{\Sigma(1)} \mid \prod_{\rho \in \Sigma(1)} \mu_{\rho}^{(u,v_{\rho})} = 1 \forall u \in M \right\}$$

of the torus $$(\mathbb{C}^*)^{\Sigma(1)}$$ on the affine space $$\mathbb{C}^{\Sigma(1)}$$, where $$v_{\rho}$$ denotes the primitive lattice generator of the 1-dimensional cone $$\rho$$. Then by Theorem 2.1 in [C1], the toric variety $$X_{\Sigma}$$ is the categorical quotient $$(\mathbb{C}^{\Sigma(1)} \setminus V(B))/G$$. This presentation is important because it allows us to work with closed subvarieties of the toric variety. In particular, a torus invariant divisor $$D_{\rho}$$ is given by the equation $$x_{\rho} = 0$$. 


The ring $S(\Sigma)$ is graded by the Chow group $A_{d-1}(X_\Sigma) \simeq \text{Hom}(G, \mathbb{C}^*)$, and $\deg(\prod_{\rho \in \Sigma(1)} x_{\rho}^{b_{\rho} m}) = [\sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho}] \in A_{d-1}(X_\Sigma)$. For a torus invariant Weil divisor $D = \sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho}$, there is a one-to-one correspondence between the monomials of $C[x_{\rho} : \rho \in \Sigma(1)]$ in the degree $[\sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho}] \in A_{d-1}(X_\Sigma)$ and the lattice points inside the polytope

$$\Delta_D = \{ m \in M_R \mid \langle m, v_{\rho} \rangle \geq -b_{\rho} \forall \rho \in \Sigma(1) \}$$

by associating to $m \in \Delta_D$ the monomial $\prod_{\rho \in \Sigma(1)} x_{\rho}^{b_{\rho} (m, v_{\rho})} = x_m \prod_{\rho \in \Sigma(1)} x_{\rho}^{b_{\rho}}$ where $x_m$ will denote $\prod_{\rho \in \Sigma(1)} x_{\rho}^{(m, v_{\rho})}$. If we denote the homogeneous degree of $S(\Sigma)$ corresponding to $\beta = [D] \in A_{d-1}(X_\Sigma)$ by $S(\Sigma)_\beta$, then by Proposition 1.1 in [C1], we also have a natural isomorphism

$$H^0(X_\Sigma; O_{X_\Sigma}(D)) \simeq S(\Sigma)_\beta.$$ 

In particular, every hypersurface in $X_\Sigma$ of degree $\beta = \sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho}$ corresponds to a polynomial

$$\sum_{m \in \Delta_D \cap M} a_m \prod_{\rho \in \Sigma(1)} x_{\rho}^{b_{\rho} + (m, v_{\rho})}$$

with the coefficients $a_m \in \mathbb{C}$.

Every lattice polytope $\Delta$ in $M_R$ determines the Weil divisor $D_\Delta = \sum_{\rho \in \Sigma(1)} -\min(\Delta, v_{\rho}) D_{\rho}$ on $X_\Sigma$. By Theorem 1.6 in [M2] we know that if $D$ is a Cartier divisor on a compact toric variety $X_\Sigma$, then $O_{X_\Sigma}(D)$ is generated by global sections if $D$ is numerically effective (nef). In this case, by [F, p. 68], we get $D = D_\Delta$. Also, if for a lattice polytope $\Delta$ the divisor $D_\Delta$ is nef, then $\Delta D_\Delta = \Delta$. Additionally, this correspondence preserves sums: if $D_{\Delta_1}$ and $D_{\Delta_2}$ are nef then $D_{\Delta_1 + \Delta_2} = D_{\Delta_1} + D_{\Delta_2}$. Moreover, the following holds.

**Lemma 2.1.** Let $X_\Sigma$ be a compact toric variety associated to a fan $\Sigma$ in $N_R$. Suppose $\Delta_1$ and $\Delta_2$ are lattice polytopes in $M_R$ then $D_{\Delta_1 + \Delta_2}$ is a nef divisor on $X_\Sigma$ if $D_{\Delta_1}$ and $D_{\Delta_2}$ are nef on $X_\Sigma$.

**Proof.** If $D_{\Delta_1 + \Delta_2}$ is a nef divisor on $X_\Sigma$ then $\Sigma$ is a refinement of the normal fan of $\Delta_1 + \Delta_2$. But since $\Delta_i$ is a Minkowski summand of $\Delta_1 + \Delta_2$, the normal fan of $\Delta_1 + \Delta_2$ is a refinement of the normal fans of $\Delta_i$, for $i = 1, 2$. Hence, $\Sigma$ is a refinement of the normal fans of $\Delta_1$ and $\Delta_2$. This implies that $D_{\Delta_1}$ and $D_{\Delta_2}$ are nef on $X_\Sigma$, if $D_{\Delta_1 + \Delta_2}$ is a nef divisor on $X_\Sigma$. The other direction follows from the fact that the sum of nef divisors is nef. \qed

From Mori’s theory we know that nef divisors correspond to contractions and for toric varieties this correspondence can be formulated as in Theorem 1.2 in [M3].

**Theorem 2.2.** Let $[D] \in A_{d-1}(X_\Sigma)$ be a nef divisor class on a compact toric variety $X_\Sigma$ of dimension $d$. Then, there exists a unique compact toric variety $X_{\Sigma_D}$ with a surjective toric morphism $\pi : X_{\Sigma_D} \to X_{\Sigma_D}$ such that $\pi^*[Y] = [D]$ for some ample divisor $Y$ on $X_{\Sigma_D}$. Moreover, $\dim X_{\Sigma_D} = \dim D$, and the fan $\Sigma_D = \Sigma_D$, the normal fan of polytope $\Delta_D$, for a torus invariant $D$. 

Finishing this section, we will recall an alternative way to describe projective toric varieties using the language of Gorenstein cones. Suppose that $\Delta$ is a lattice polytope in $M_{\mathbb{R}}$ such that its support function $\psi_{\Delta} = -\min(\Delta \ominus \cdot)$ is strictly convex with respect to the fan $\Sigma$. In this case, the divisor $D_{\Delta}$ is ample and $\Sigma = \Sigma_{\Delta}$ is the normal fan of $\Delta$. Consider the Gorenstein cone

$$K = \{(t\Delta, t) \mid t \in \mathbb{R}_{\geq 0}\} \subset M_{\mathbb{R}} \oplus \mathbb{R}.$$  

The projective toric variety $X_{\Delta} := X_{\Sigma_{\Delta}}$ can be represented as $\text{Proj}(\mathbb{C}[K \cap (M \oplus \mathbb{Z})])$. Moreover, if $\beta \in A_{d-1}(X_{\Delta})$ is the class of the ample divisor $D_{\Delta} = \sum_{\rho \in \Sigma_{\Delta}} b_{\rho} D_{\rho}$, then there is a natural isomorphism of graded rings

$$\mathbb{C}[K \cap (M \oplus \mathbb{Z})] \cong \bigoplus_{i=0}^{\infty} S(\Sigma_{\Delta}) \beta,$$

sending $\chi^{(m,i)} \in \mathbb{C}[K \cap (M \oplus \mathbb{Z})]$, to $\prod_{\rho \in \Sigma_{\Delta}} x^{i_{\rho} + (m,\nu_{\rho})} = x^{m} \prod_{\rho \in \Sigma_{\Delta}} x^{i_{\rho}}$. This correspondence allows to translate an equation of a hypersurface given by a polynomial in homogeneous coordinates

$$\sum_{m \in \Delta \cap M} a_{m} \prod_{\rho \in \Sigma_{\Delta}} x^{i_{\rho} + (m,\nu_{\rho})} = \sum_{m \in \Delta' \cap M} a_{m} x^{m} \prod_{\rho \in \Sigma_{\Delta}} x^{i_{\rho}}$$

into the homogeneous element $\sum_{m \in \Delta \cap M} a_{m} \chi^{(m,i)}$ of the Gorenstein ring.

3. Cayley trick and deformations of Fano toric varieties.

To describe the Cayley trick used in mirror symmetry by [BBo1] we start with a Gorenstein Fano toric variety $X_{\Delta} := X_{\Sigma_{\Delta}}$, whose (normal) fan $\Sigma_{\Delta}$ of the reflexive polytope $\Delta$ consists of the cones generated by the proper faces of the dual reflexive polytope $\Delta^{\ast}$ in $N_{\mathbb{R}}$. Consider a Minkowski sum decomposition $\Delta = \Delta_{0} + \cdots + \Delta_{k}$ by lattice polytopes. The anticanonical divisor $D_{\Delta} = \sum_{\rho \in \Sigma_{\Delta}} D_{\rho}$ on the Fano toric variety $X_{\Delta}$ is ample, and, in particular, nef. Applying Lemma 2.1, we get the nef divisors $D_{\Delta_{0}}, \ldots, D_{\Delta_{k}}$ on $X_{\Delta}$. Given a collection of line bundles on a variety, the Cayley trick associates to it the projective space bundle. In our case we get the $\mathbb{P}^{k}$-bundle $\mathbb{P}(E_{\Delta_{0}, \ldots, \Delta_{k}}) \rightarrow X_{\Delta}$, where

$$E_{\Delta_{0}, \ldots, \Delta_{k}} = \mathcal{O}_{X_{\Delta}}(D_{\Delta_{0}}) \oplus \cdots \oplus \mathcal{O}_{X_{\Delta}}(D_{\Delta_{k}}).$$

By [O, p. 58], we know that this bundle is a toric variety with its fan in $N_{\mathbb{R}} \oplus \mathbb{R}^{k}$.

**Proposition 3.1.** The torus invariant anticanonical divisor on $\mathbb{P}(E_{\Delta_{0}, \ldots, \Delta_{k}})$ is big and nef and equals $D_{\Delta_{0} + \cdots + \Delta_{k}}$.

**Proof.** We only need to check that for the torus invariant anticanonical divisor $Y$ of the toric variety $\mathbb{P}(E_{\Delta_{0}, \ldots, \Delta_{k}})$ there is equality of polytopes: $\Delta_{Y} = \Delta_{0} + \cdots + \Delta_{k}$ in $M_{\mathbb{R}} \oplus \mathbb{R}^{k}$. But this follows immediately from the fan description in [O, p. 58] and Proposition 1.10. \qed

By Theorem 2.2, for the nef divisor $D_{\Delta_{0} + \cdots + \Delta_{k}}$ we get the contraction

$$\mathbb{P}(E_{\Delta_{0}, \ldots, \Delta_{k}}) \rightarrow X_{\Delta_{0} + \cdots + \Delta_{k}}$$

which relates the projective bundle to the Fano toric variety. In the case, when the anticanonical divisor on $\mathbb{P}(E_{\Delta_{0}, \ldots, \Delta_{k}})$ is ample (i.e., the vector bundle $E_{\Delta_{0}, \ldots, \Delta_{k}}$
is ample) we get the equality \( \mathbb{P}(E_{\Delta_0, \ldots, \Delta_k}) = X_{\tilde{\Delta}_0 + \cdots + \tilde{\Delta}_k} \). The projective bundle \( \mathbb{P}(E_{\Delta_0, \ldots, \Delta_k}) \to X_\Delta \) can also be viewed as a contraction corresponding to the polytope \( \Delta \) in \( M_\mathbb{R} \subset M_\mathbb{R} \oplus \mathbb{R}^k \) and its nef divisor on \( \mathbb{P}(E_{\Delta_0, \ldots, \Delta_k}) \).

The Fano toric varieties \( X_\Delta \) and \( X_{\tilde{\Delta}_0 + \cdots + \tilde{\Delta}_k} \) can also be described in the language of Gorenstein cones from [BB01]. Let \( \sigma = \{(t_\Delta, t) \mid t \in \mathbb{R}_{\geq 0}\} \subset M_\mathbb{R} \oplus \mathbb{R} \) and

\[
\tilde{\sigma} = \left\{ \left( \sum_{i=0}^{k} t_i \Delta_i, t_0, \ldots, t_k \right) \mid t_i \in \mathbb{R}_{\geq 0} \right\} \subset M_\mathbb{R} \oplus \mathbb{R}^{k+1}.
\]

Then, by the correspondence at the end of Section 2 and Lemma 1.5, we have \( X_\Delta = \text{Proj}(\mathbb{C}[\sigma \cap (M \oplus \mathbb{Z})]) \) and \( X_{\tilde{\Delta}_0 + \cdots + \tilde{\Delta}_k} = \text{Proj}(\mathbb{C}[\tilde{\sigma} \cap \tilde{M}]) \), where \( \tilde{M} = M \oplus \mathbb{Z}^{k+1} \). Inclusion of cones \( \sigma \subset \tilde{\sigma} \) via

\[
M_\mathbb{R} \oplus \mathbb{R} \hookrightarrow M_\mathbb{R} \oplus \mathbb{R}^{k+1}, \quad (m, r) \mapsto (m, r, \ldots, r),
\]

induces an injective homomorphism \( \mathbb{C}[\sigma \cap (M \oplus \mathbb{Z})] \hookrightarrow \mathbb{C}[\tilde{\sigma} \cap \tilde{M}] \) and a surjective morphism \( \text{Spec}(\mathbb{C}[\sigma \cap M]) \to \text{Spec}(\mathbb{C}[\sigma \cap (M \oplus \mathbb{Z})]) \) of affine toric varieties. It also induces a rational map \( \text{Proj}(\mathbb{C}[\sigma \cap M]) \dasharrow \text{Proj}(\mathbb{C}[\sigma \cap (M \oplus \mathbb{Z})]) \) of projective toric varieties. This map coincides with the morphism \( \mathbb{P}(E_{\Delta_0, \ldots, \Delta_k}) \to X_\Delta \), if \( E_{\Delta_0, \ldots, \Delta_k} \) is an ample vector bundle.

There is more story to the Cayley trick in associating a semiample hypersurface in the projective bundle to the nef Calabi-Yau complete intersection on \( X_\Delta \) given by global sections of \( \mathcal{O}_{X_\Delta}(D_{\Delta_0}), \ldots, \mathcal{O}_{X_\Delta}(D_{\Delta_k}) \), but we will not need this here.

Now, let us show how the Cayley trick is related to deformations of Fano toric varieties. Consider the Fano toric variety \( X_{\Delta^*} \), whose fan \( \Sigma_{\Delta^*} \) in \( M_\mathbb{R} \) consists of the cones generated by the proper faces of the reflexive polytope \( \Delta = (\Delta^*)^* \). Take the same Minkowski sum decomposition \( \Delta = \Delta_0 + \cdots + \Delta_k \) as above. We have a natural inclusion of spaces \( M_\mathbb{R} \subset M_\mathbb{R} \oplus \mathbb{R}^k \) which induces the inclusion of polytopes \( \Delta \subset (k+1)(\Delta_0 \cup \cdots \cup \Delta_k) \) and the map of fans over the proper faces of these polytopes.

**Theorem 3.2.** [M5] Associated to the map of fan \( \Sigma_{\Delta^*} \to \Sigma_{(\tilde{\Delta}_0, \ldots, \tilde{\Delta}_k)}^* \), the toric morphism \( X_{\Delta^*} \to X_{(\tilde{\Delta}_0, \ldots, \tilde{\Delta}_k)}^* \) is an embedding, whose image is a complete intersection given by the equations

\[
\prod_{v_\rho \in \Delta_i + e_i} x_\rho = \prod_{v_\rho \in \Delta_0 + e_0} x_\rho = 0,
\]

for \( i = 1, \ldots, k \), where \( x_\rho \) are the homogeneous coordinates of the toric variety \( X_{(\tilde{\Delta}_0, \ldots, \tilde{\Delta}_k)}^* \) corresponding to the vertices \( v_\rho \) of the polytope \( \tilde{\Delta}_0 \cup \cdots \cup \tilde{\Delta}_k \).

Let \( l(\Delta^*) \) denote the number of lattice points in the reflexive polytope \( \Delta^* \). By [M5], we have \( (k!l(\Delta^*) - k) \)-parameter embedded deformation family of \( X_{\Delta^*} \) in \( X_{(\tilde{\Delta}_0, \ldots, \tilde{\Delta}_k)}^* \) given by the equations:

\[
\left( x_i^{e_i} - 1 + \sum_{n \in (\Delta^* \cap \mathbb{N}) \setminus \{0\}} \lambda_{i,n} x^{n \sum_{j=1}^{k} \min(\Delta_j, n)e_j} \right) \prod_{v_\rho \in \Delta_0 + e_0} x_\rho = 0
\]

for \( i = 1, \ldots, k \).

The embedding \( X_{\Delta^*} \hookrightarrow X_{(\tilde{\Delta}_0, \ldots, \tilde{\Delta}_k)}^* \) can also be described in the language of Gorenstein cones. Let \( \sigma \) and \( \tilde{\sigma} \) be the same cones as above. Associated to the
inclusion of cones $\sigma \subset \tilde{\sigma}$, there is a projection $\tilde{\sigma}^\vee \to \sigma^\vee$ induced by

$$\tilde{N} := N \oplus \mathbb{Z}^{k+1} \to N \oplus \mathbb{Z}, \quad (n, \alpha_0, \ldots, \alpha_k) \mapsto (n, \alpha_0 + \cdots + \alpha_k)$$

and the corresponding ring homomorphism

$$\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{N}] \longrightarrow \mathbb{C}[\sigma^\vee \cap (N \oplus \mathbb{Z})],$$

which is surjective by Lemma 2.2 in [M5]. Hence, we get the embedding

$$\text{Spec}(\mathbb{C}[\tilde{\sigma}^\vee \cap (N \oplus \mathbb{Z})]) \hookrightarrow \text{Spec}(\mathbb{C}[\tilde{\tau}^\vee \cap \tilde{N}])$$

of affine toric varieties. By (3) and Lemma 1.7, if $\beta = \deg(\prod_{v_\rho \in \Delta_0+e_0} x_\rho) = [D_{\tilde{\psi}_0}]$, then

$$\mathbb{C}[\tilde{\tau}^\vee \cap \tilde{N}] \simeq \bigoplus_{i=0}^{\infty} S(\tilde{\psi}_0)_{i\beta}, \quad x^{u+\sum_{j=0}^{k} a_j r_j^*} \mapsto x^{u+\sum_{j=1}^{k} \alpha_j e_j^*} \prod_{v_\rho \in \Delta_0+e_0} x^{\alpha_0+\cdots+\alpha_k}.$$

Since $\Sigma_{\tilde{\psi}_0} = \Sigma_{(\Delta_0\oplus \cdots \oplus \Delta_k)^*}$ by Proposition 1.13, we also get the embedding of projective toric varieties

$$X_{\Delta^*} = \text{Proj}(\mathbb{C}[\tilde{\tau}^\vee \cap (N \oplus \mathbb{Z})]) \hookrightarrow X_{(\Delta_0\oplus \cdots \oplus \Delta_k)^*} = \text{Proj}(\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{N}]),$$

where the image is a complete intersection given by $\chi_{r_i^*}^* - \chi_{\tilde{r}_0}^*$, for $i = 1, \ldots, k$.

Then deformations of the Fano toric variety $X_{\Delta^*}$ are $\text{Proj}(\mathbb{C}[\tilde{\tau}^\vee \cap \tilde{N}]/I)$, where the ideal $I \subset \mathbb{C}[\tilde{\tau}^\vee \cap \tilde{N}]$ is generated by

$$\chi_{r_i^*}^* - \chi_{\tilde{r}_0}^* + \sum_{n \in (\Delta^* \cap N) \setminus \{0\}} \lambda_{i,n} x^{n-\sum_{j=0}^{k} \min(\Delta_j,n) r_j^*},$$

for $i = 1, \ldots, k$, where $\{r_j^* \cup \tilde{r}_0^*\}$ is the basis of $\mathbb{Z}^{k+1} \subset N \oplus \mathbb{Z}^{k+1}$.

The ambient toric variety $X_{(\Delta_0\oplus \cdots \oplus \Delta_k)^*} = \text{Proj}(\mathbb{C}[\tilde{\tau}^\vee \cap \tilde{N}])$ of the deformation of $X_{\Delta^*}$ is related to the Fano toric variety $X_{\Delta_0\oplus \cdots \oplus \Delta_k} = \text{Proj}(\mathbb{C}[\tilde{\sigma} \cap \tilde{M}])$ from the Cayley trick by the duality of the Gorenstein cones. Note that the reflexive polytopes associated to these toric varieties are not dual to each other, but a precise relation between them is described in Lemmas 1.15 and 1.16.

We will conclude this section by considering the case when $\Delta = \Delta_0 + \cdots + \Delta_k$ is a nef-partition in $M_\mathbb{R}$ and $\nabla = \nabla_0 + \cdots + \nabla_k$ is the dual nef-partition in $N_\mathbb{R}$. In this case, by Proposition 1.17, we get $X_{(\Delta_0\oplus \cdots \oplus \Delta_k)^*} = X_{\nabla^\vee_0 + \cdots + \nabla^\vee_k}$ and $X_{(\tilde{\psi}_0\oplus \cdots \oplus \tilde{\psi}_k)^*} = X_{\tilde{\nabla}_0 + \cdots + \tilde{\nabla}_k}$. The fan of the projective space bundle $\mathbb{P}(\mathcal{E}_{\nabla_0}, \ldots, \mathcal{E}_{\nabla_k})$ is a refinement of the normal fan of $\nabla_0 + \cdots + \nabla_k$, which is obtained by a subdivision of the faces of the reflexive polytope $\Delta_0 \cup \cdots \cup \Delta_k = (\nabla_0 + \cdots + \nabla_k)^*$. Intersection of this fan with the linear subspace $M_\mathbb{R} \subset \tilde{M}_\mathbb{R} \oplus \mathbb{R}^k$ gives a subdivision $\Sigma_{\Delta^*}'$ of the normal fan $\Sigma_{\Delta^*}$ of the polytope $\Delta^*$. The

By the the embeddings of toric varieties from [M5, Section 7] we have a commutative diagram:

$$\begin{array}{ccc}
X_{\Sigma_{\Delta^*}'} & \hookrightarrow & \mathbb{P}(\mathcal{E}_{\nabla_0}, \ldots, \mathcal{E}_{\nabla_k}) \\
\downarrow & & \downarrow \\
X_{\Delta^*} & \hookrightarrow & X_{(\Delta_0\oplus \cdots \oplus \Delta_k)^*} = X_{\tilde{\psi}_0 + \cdots + \tilde{\psi}_k}
\end{array}$$
Similarly, if $\Sigma'_{\chi}$ is obtained by intersecting the fan of $\mathbb{P}(E_{\Delta_0,\ldots,\Delta_k})$ with the sub-
space $N_R \subset \tilde{N}_R \oplus \mathbb{R}^k$, then

$$X_{\Sigma'_{\chi}} \hookrightarrow \mathbb{P}(E_{\Delta_0,\ldots,\Delta_k}) \rightarrow X_\Delta$$

$$\Downarrow \Downarrow$$

$$X_{\Sigma_{\chi}} \hookrightarrow X_{(\Sigma_0 \cup \ldots \cup \Sigma_k)^*} = X_{\tilde{\Delta}_0 + \cdots + \tilde{\Delta}_k}.$$

4. DEFORMATIONS OF CALABI-YAU HYPERSURFACES IN FANO TORIC VARIETIES.

In this section we show that deformations of Fano toric varieties induce deformations of Calabi-Yau hypersurfaces. The embedding of the ambient Fano toric variety realizes a Calabi-Yau hypersurface as a Calabi-Yau complete intersection in a higher dimensional Fano toric variety. The deformations of the resulting complete intersections are “polynomial”, corresponding to changing the coefficients at the monomials. As before, we assume for the rest that $\Delta$ is a reflexive polytope and $\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_k$ is a Minkowski sum decomposition by lattice polytopes in $M_R$.

**Theorem 4.1.** Let $Y_{\Delta'} \subset X_{\Delta'} = \text{Proj}(\mathbb{C}[\bar{\sigma}' \cap (N \oplus \mathbb{Z})])$ be an ample Calabi-Yau hypersurface given by the equation

$$\sum_{n \in \Delta' \cap N} a_n \chi^{(n,1)} = 0,$$

where $a_n \in \mathbb{C}$. Then the image of $Y_{\Delta'}$ by the embedding $X_{\Delta'} \hookrightarrow X_{(\tilde{\Delta}_0 \cup \ldots \cup \tilde{\Delta}_k)^*} = \text{Proj}(\mathbb{C}[\bar{\sigma}' \cap \tilde{N}])$ is a nef Calabi-Yau complete intersection given by the equations

$$a_0 \chi^0 + \sum_{n \in (\Delta' \cap N) \setminus \{0\}} a_n \chi^{n - \sum_{j=0}^k \min(\Delta_j,n) r_j^*} = 0, \quad \chi_i^0 - \chi_i^* = 0, \quad i = 1, \ldots, k.$$

**Proof.** We need to show that the kernel of the surjective $\mathbb{Z}$-graded ring homomorphism

$$\mathbb{C}[\bar{\sigma}' \cap \tilde{N}] \longrightarrow \mathbb{C}[\bar{\sigma}' \cap (N \oplus \mathbb{Z})]/(f),$$

where $f = \sum_{n \in \Delta' \cap N} a_n \chi^{(n,1)}$, is generated by

$$\tilde{f} = a_0 \chi^0 + \sum_{n \in (\Delta' \cap N) \setminus \{0\}} a_n \chi^{n - \sum_{j=0}^k \min(\Delta_j,n) r_j^*}$$

and $\chi_i^0 - \chi_i^*$, for $i = 1, \ldots, k$. By [A, pp. 162-163] or Lemma 2.2 in [M5], we already know that the kernel of the surjective ring homomorphism

$$\mathbb{C}[\bar{\sigma}' \cap \tilde{N}] \longrightarrow \mathbb{C}[\bar{\sigma}' \cap (N \oplus \mathbb{Z})]$$

is an ideal generated by the regular sequence $\chi_i^0 - \chi_i^*$, for $i = 1, \ldots, k$. Therefore, it suffices to show that any preimage of $f$ by this homomorphism is in the ideal generated by $\tilde{f}$ and $\chi_i^0 - \chi_i^*$, for $i = 1, \ldots, k$.

For $n \in (\Delta' \cap N) \setminus \{0\}$, the preimage of $\chi^{(n,1)}$ by the ring homomorphism induced by $N \oplus \mathbb{Z}^{k+1} \rightarrow N \oplus \mathbb{Z}$, $(n, \alpha_0, \ldots, \alpha_k) \mapsto (n, \alpha_0 + \cdots + \alpha_k)$, is a linear combination of $\chi^{n + \sum_{j=0}^k \alpha_j r_j^*}$ with $\sum_{j=0}^k \alpha_j = 1$ and $n + \sum_{j=0}^k \alpha_j r_j^* \in \bar{\sigma}'$. But the last condition means $\min(n + \sum_{j=0}^k \alpha_j r_j^*, \Delta_l + r_l) \geq 0$, whence $\alpha_l \geq -\min(n, \Delta_l)$ for all $l$. Since

$$1 = \sum_{j=0}^k \alpha_j \geq -\sum_{j=0}^k \min(n, \Delta_j) = -\min(n, \sum_{j=0}^k \Delta_j) = -\min(n, \Delta) = 1,$$
we get $\alpha_j = -\min\{n, \Delta_j\}$. It is also clear that a preimage of $\chi^{(0,1)}$ coincides with $\chi^{\delta_i}$ modulo $\chi^{\delta_i} - \chi^{\delta_0}$, for $i = 1, \ldots, k$. \hfill \Box

Translating the above statement by the correspondence (3) into homogeneous coordinates we get:

**Theorem 4.2.** Let $Y_{\Delta^*} \subset X_{\Delta^*}$ be a Calabi-Yau hypersurface given by the equation

$$
\sum_{n \in \Delta^* \cap N} a_n x^n \prod_{\rho \in \Sigma_{\Delta^*}(1)} x_{\rho} = 0,
$$

where $a_n \in \mathbb{C}$. Then the image of $Y_{\Delta^*}$ under the embedding $X_{\Delta^*} \hookrightarrow X_{(\Delta_0|\cdots|\Delta_k)^*}$ is a nef Calabi-Yau complete intersection given by the equations

$$
\sum_{n \in \Delta^* \cap N} a_n x^{n - \sum_{j=1}^{k} \min\{\Delta_j, n\} e_j^*} \prod_{\nu_{\rho} \in \Delta_0 + e_0} x_{\rho} = 0, \prod_{\nu_{\rho} \in \Delta_i + e_i} x_{\rho} - \prod_{\nu_{\rho} \in \Delta_0 + e_0} x_{\rho} = 0,
$$

for $i = 1, \ldots, k$.

The ample Calabi-Yau hypersurface $Y_{\Delta^*} \subset X_{\Delta^*}$ deforms to a generic nef Calabi-Yau complete intersection $Y_{\nabla_0 + \cdots + \nabla_k}$ in the Fano toric variety $X_{(\Delta_0|\cdots|\Delta_k)^*} = X_{\nabla_0 + \cdots + \nabla_k}$ corresponding to the nef-partition $\nabla_0 + \cdots + \nabla_k$:

$$
\left(\sum_{j=1}^{k} a_{i,j} x^{e_j^* - \delta_i e_i^*} + \sum_{n \in \Delta^* \cap N} a_n x^{n - \delta_i e_i^* - \sum_{j=1}^{k} \min\{\Delta_j, n\} e_j^*}\right) \prod_{\nu_{\rho} \in \Delta_i + e_i} x_{\rho} = 0
$$

for $i = 0, \ldots, k$, where $a_{i,j}, a_{i,n} \in \mathbb{C}$ are the coefficients, and $\delta_i = 1$, if $i \neq 0$, $\delta_0 = 0$. (Note that the lattice points corresponding to the monomials are precisely the lattice points of the polytope $\nabla_i$ in Proposition 1.14.)

5. Degenerations and Mirror Contractions of Calabi-Yau Complete Intersections.

In the previous section, we obtained a deformation of an ample Calabi-Yau hypersurface $Y_{\Delta^*} \subset X_{\Delta^*}$ in a Fano toric variety to a generic Calabi-Yau complete intersection in the Fano toric variety $X_{\nabla_0 + \cdots + \nabla_k}$. Equivalently, we have a degeneration of a generic Calabi-Yau complete intersection in $X_{\nabla_0 + \cdots + \nabla_k}$ to a generic Calabi-Yau hypersurface $Y_{\Delta^*} \subset X_{\Delta^*}$. Let $\Sigma'_{\nabla_0 + \cdots + \nabla_k}$ be a maximal projective subdivision of the normal fan of the reflexive polytope $\nabla_0 + \cdots + \nabla_k$, and let $\Sigma'_{\Delta^*}$ be the fan obtained by intersecting the cones of $\Sigma'_{\nabla_0 + \cdots + \nabla_k}$ with the linear subspace $M_R \subset \tilde{M}_R \oplus \mathbb{R}^k$. Then we have a commutative diagram:

$$
\begin{array}{ccc}
Y'_{\Delta^*} & \subset & X_{\Sigma'_{\Delta^*}} \hookrightarrow X_{\Sigma'_{\nabla_0 + \cdots + \nabla_k}} \\
\downarrow & & \downarrow \\
Y_{\Delta^*} & \subset & X_{\Delta^*} \hookrightarrow X_{\nabla_0 + \cdots + \nabla_k},
\end{array}
$$

where $Y'_{\Delta^*}$ is a crepant partial resolution of the ample Calabi-Yau hypersurface $Y_{\Delta^*}$. The hypersurface $Y'_{\Delta^*}$ deform to a Calabi-Yau complete intersection in $X_{\Sigma'_{\nabla_0 + \cdots + \nabla_k}}$. Correspondingly, a generic Calabi-Yau complete intersection $Y'_{\nabla_0 \cdots \nabla_k}$ in $X_{\Sigma'_{\nabla_0 + \cdots + \nabla_k}}$ degenerates to a generic Calabi-Yau hypersurface $Y'_{\Delta^*}$ in $X_{\Sigma'_{\Delta^*}}$.

Now, if $\Sigma''_{\Delta^*}$ is a maximal projective subdivision of the normal fan of the reflexive
polytope $\Delta^*$, which refines the fan $\Sigma^\prime_{\Delta^*}$, then we obtain a geometric transition from a minimal Calabi-Yau hypersurface to a minimal Calabi-Yau complete intersection:

$$Y'_{\Delta^*} \to Y_{\Delta^*} \leftrightarrow Y''_{\Delta^*},$$

where $Y''_{\Delta^*}$ is a maximal projective crepant partial resolution of the ample Calabi-Yau hypersurface $Y_{\Delta^*} \subset X_{\Delta^*}$.

By Morrison’s conjecture in [Mo], every geometric transition between Calabi-Yau manifolds should correspond to a mirror geometric transition between the mirror partners of the original Calabi-Yau manifolds with the roles of degeneration and contraction reversed. By the Batyrev-Borisov mirror symmetry construction in [B2] and [BB10] we know that the mirror of the Calabi-Yau hypersurface $Y'_{\Delta^*}$ is a nondegenerate Calabi-Yau hypersurface in a maximal projective crepant partial resolution of the Fano toric variety $X$ and the mirror of the Calabi-Yau complete intersection $Y'_{\Delta^*}$ is a nondegenerate Calabi-Yau complete intersection in a maximal projective crepant partial resolution of the Fano toric variety $X$. Theorem 5.1. Let $Y'_{\Delta_0,\ldots,\Delta_k} \subset X_{\Delta_0,\ldots,\Delta_k}$ be a Calabi-Yau complete intersection given by the equations

$$\left(1 - \sum_{m \in \Delta_i \cap M} a_m x^{m+m_i} \right) \prod_{v_j \in \mathcal{V}_i} x_{v_j} = 0, \quad i = 0, \ldots, k,$$

where $a_m \in \mathbb{C}$. Then the image of $Y'_{\Delta_0,\ldots,\Delta_k}$ under the contraction $X_{\Delta_0,\ldots,\Delta_k} \to X_{\Delta^*}$ is a nef Calabi-Yau hypersurface given by the equation

$$\left(1 - \prod_{i=0}^k \left( \sum_{m \in \Delta_i \cap M} a_m x^m \right) \right) \prod_{v_j \in \Delta^*} x_{v_j} = 0. \quad (5)$$

Proof. Note that the intersection of the Calabi-Yau complete intersection $Y'_{\Delta_0,\ldots,\Delta_k}$ with the dense affine torus $T = \text{Spec}(\mathbb{C}[M \oplus \mathbb{Z}^k])$ is a complete intersection given by the equations $1 - \sum_{m \in \Delta_i \cap M} a_m x^{m+m_i} = 0$, for $i = 0, \ldots, k$. We can find the image of $Y'_{\Delta_0,\ldots,\Delta_k}$ as the closure of the image of the affine complete intersection by the projection of tori $\text{Spec}(\mathbb{C}[M \oplus \mathbb{Z}^k]) \to \text{Spec}(\mathbb{C}[M])$ induced by the injective
lattice homomorphism $M \subset M \oplus \mathbb{Z}^k$. By eliminating $\chi^e_i$, for $i = 1, \ldots, k$, and $\chi^e_0 = \prod_{i=1}^k \chi^{-e_i}$ from the above equations we get the equation

$$1 - \prod_{i=0}^k \left( \sum_{m \in \Delta_i \cap M} a_m \chi^m \right) = 0$$

of the image in the affine torus $\text{Spec}(\mathbb{C}[M])$. The Zariski closure of this affine hypersurface is the nef Calabi-Yau hypersurface given by the equation (5) in homogeneous coordinates of $X_{\Sigma'_Y}$.

Denote by $Y'_\Delta$ the nef Calabi-Yau hypersurface in $X_{\Sigma'_Y}$, given by the equation (5). Notice that such a hypersurface is not generic. The geometric transition from a generic Calabi-Yau complete intersection $Y'_\Delta_{v_0,\ldots,v_k} \subset X_{\Sigma'_Y+\Delta_0+\ldots+\Delta_k}$ is completed by a smoothing of $Y'_\Delta$ to a nondegenerate Calabi-Yau hypersurface $Y''_\Delta$ in $X_{\Sigma'_Y}$:

$$Y''_\Delta \hookrightarrow Y'_\Delta \leftarrow Y'_\Delta_{v_0,\ldots,v_k}.$$  

6. DEGENERATIONS AND MIRROR CONTRACTIONS

We will support the mirror correspondence of the geometric transitions by showing that the degeneration of the main periods (determining the mirror map) for Calabi-Yau complete intersection $Y'_{\Delta_{v_0,\ldots,v_k}}$ and the hypersurface $Y''_\Delta$ coincide with the main periods of the minimal Calabi-Yau $Y''_{\Delta^*}$ and $Y'_{\Delta_{0,\ldots,\Delta_k}}$, respectively.

First, we recall the definition of the main period for the nondegenerate Calabi-Yau hypersurface $Y'_\Delta$ from [B1] (also, see [CK, Sec. 6.3.4]). Fix an integer basis $u_1, \ldots, u_d$ for the lattice $M$. Then $t_j = \prod_{\rho \in \Sigma'_Y} x_{\rho}^{u_{j,\rho}}$, for $j = 1, \ldots, d$, are the coordinates on the dense torus $T_N = \text{Spec}(\mathbb{C}[M]) = N \otimes \mathbb{C}^* \subset X_{\Sigma'_Y}$. Let $f_\Delta = 1 - \sum_{m \in \partial \Delta \cap M} b_m t^m$ be the Laurent polynomial determining the hypersurface $Y''_\Delta \cap T_N$, and let $\gamma \subset T_N$ be the cycle defined by $|t_1| = \cdots = |t_d| = 1$, then the main period for $Y''_\Delta$ equals the Euler integral

$$\Phi_{Y''_\Delta}(\beta) = \frac{1}{(2\pi i)^d} \int_\gamma \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d},$$

where $\beta = (b_m) \in \mathbb{C}^{d(\Delta)}$. The function $\Phi_{Y''_\Delta}(\beta)$ is called hypergeometric since it satisfies the GKZ hypergeometric system of differential equations (see [B1, Theorem 14.2]). It can be found as a power series expansion in the variables $b_m$:

$$\Phi_{Y''_\Delta}(\beta) = \sum_{l \in L_\Delta} \left( \sum_{m \in \partial \Delta \cap M} l_m! \prod_{m' \in \partial \Delta \cap M} \frac{b_{m'}}{l_{m'}} \right),$$

where $L_\Delta = \{(l_m)_{m \in \partial \Delta \cap M} | \sum_{m \in \partial \Delta \cap M} l_m m = 0, l_m \in \mathbb{Z}_{\geq 0} \forall m\}$. It can also be written in terms of the local coordinates on the complex moduli of $Y''_\Delta$ at a
maximally unipotent boundary point (see [BvS, CK]). As \( Y''_X \) degenerates to \( Y'_X \), the hypergeometric function \( \Phi_{Y''_X}(\beta) \) will degenerate to the Euler integral

\[
\Phi_{Y'_X}(\alpha) = \frac{1}{(2\pi \sqrt{-1})^d} \int \frac{1}{g_{\Delta}} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d},
\]

where \( g_{\Delta} = 1 - \prod_{i=0}^{k} (\sum_{m \in \Delta_i \cap M} a_m t^m) \) determines the hypersurface \( Y'_X \cap T_N \), and \( \alpha = (a_m) \in \mathbb{C}(\Delta_0) \times \cdots \times \mathbb{C}(\Delta_k) \). Similar to [B1, Ex. 14.5], substituting

\[
\frac{1}{g_{\Delta}} = \sum_{j=0}^{\infty} \prod_{i=0}^{k} \left( \sum_{m \in \Delta_i \cap M} a_m t^m \right)^j
\]

into the above integral and applying the Cauchy residue theorem gives

\[
\Phi_{Y'_X}(\alpha) = \sum_{l \in L_{\Delta_0, \ldots, \Delta_k}} \prod_{i=0}^{k} \left( \sum_{m \in \Delta_i \cap M} l_m \right)! \prod_{m \in \Delta_i \cap M} \frac{a_m^l}{l_m!}, \tag{7}
\]

where

\[
L_{\Delta_0, \ldots, \Delta_k} = \left\{ (l_m) \mid \sum_{i=0}^{k} \sum_{m \in \Delta_i \cap M} l_mm = 0, \sum_{m \in \Delta_i \cap M} l_m = \sum_{m \in \Delta_i \cap M} l_m \forall i \right\}
\]

is a subsemigroup of \( \mathbb{Z}_{\geq 0}^{\Delta_0} \oplus \cdots \oplus \mathbb{Z}_{\geq 0}^{\Delta_k} \). But (7) is precisely the hypergeometric series in [BvS, Def. 6.1.1, Pr. 6.1.4] for the Calabi-Yau complete intersection \( Y'_{\Delta_0, \ldots, \Delta_k} \) given by the Euler integral

\[
\Phi_{Y'_{\Delta_0, \ldots, \Delta_k}}(\alpha) = \frac{1}{(2\pi \sqrt{-1})^{d+k}} \int_{K'} \frac{1}{f_{\Delta_0} \cdots f_{\Delta_k}} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_{d+k}}{t_{d+k}},
\]

where \( t_1, \ldots, t_d, t_{d+1}, \ldots, t_{d+k} \) are the coordinates on the torus \( T_K = \text{Spec}(\mathbb{C}[\hat{M}]) \) corresponding to the lattice basis \( \{ u_1, \ldots, u_d, e_1, \ldots, e_k \} \) of \( \hat{M} = M \oplus \mathbb{Z}^k \), the cycle \( \hat{\gamma} \) is given by \( |t_j| = 1 \) for \( j = 1, \ldots, d + k \), and

\[
f_{\Delta_0} = 1 - \sum_{m \in \Delta_0 \cap M} a_m t^m, \quad f_{\Delta_i} = 1 - \sum_{m \in \Delta_i \cap M} a_m t^m t_{d+i}, \quad i = 1, \ldots, k,
\]

determine the affine complete intersection \( Y'_{\Delta_0, \ldots, \Delta_k} \cap T_K \). The series \( \Phi_{Y'_{\Delta_0, \ldots, \Delta_k}}(\beta) \) and \( \Phi_{Y'_X}(\alpha) \) are invariant under the natural torus action \( T_N \) on the space of Laurent polynomials in the variables \( t_j \) and can be expressed in the local coordinates on the complex moduli of \( Y''_X \) as in [CK, Sec. 6.3.4]. Therefore, the main period \( \Phi_{Y''_X}(\beta) \) of a minimal Calabi-Yau hypersurface \( Y''_X \) degenerates to the main period \( \Phi_{Y'_{\Delta_0, \ldots, \Delta_k}}(\alpha) \) of a minimal Calabi-Yau complete intersection \( Y'_{\Delta_0, \ldots, \Delta_k} \) as \( Y''_X \rightarrow Y'_X \). The main period \( \Phi_{Y'_{\Delta_0, \ldots, \Delta_k}}(\alpha) \) determines the mirror map between the complex moduli of \( Y'_{\Delta_0, \ldots, \Delta_k} \) and the Kähler moduli of the mirror partner \( Y'_{\check{\Delta}_0, \ldots, \check{\Delta}_k} \), which allows to compute the instanton numbers of rational curves in the latest Calabi-Yau as explained in [CK].
Similarly to the above, let us consider the main period for the Calabi-Yau complete intersection $Y'_{\tilde{\psi}_0,...,\tilde{\psi}_k}$:

$$
\Phi_{\tilde{\psi}_0,...,\tilde{\psi}_k}(\alpha) = \frac{1}{(2\pi \sqrt{-1})^{d+k}} \int \prod_{i=0}^{k} \frac{a_{i,i} \, dt_1}{t_1} \wedge \ldots \wedge \frac{dt_{d+k}}{t_{d+k}},
$$

where the coordinates on the torus $T_{\tilde{M}} = \text{Spec}(\mathbb{C}[N])$ corresponding to the dual basis $\{u_1^*,...,u_d^*,e_1^*,...,e_k^*\}$ of $N$ are denoted by $t_1,...,t_{d+k}$ again (abusing notation), and

$$
f_{\tilde{\psi}_i} = t_{d+i}^{-1} \left( a_{i,0} + \sum_{j=1}^{k} a_{i,j} t_{d+j} + \sum_{n \in \partial \Delta^* \cap N} a_{i,n} n^k \prod_{j=1}^{k} t_{d+j}^{-\min(\Delta_j,n)} \right), \quad i = 0,...,k,
$$

(with $t_{d+i} = 1$ if $i = 0$) determine the affine complete intersection $Y'_{\tilde{\psi}_0,...,\tilde{\psi}_k} \cap T_{\tilde{M}}$, and $\alpha = (a_{i,*}) \in \mathbb{C}^{(k+1)((\Delta^*)^*+k)}$. Rewriting $a_{i,i}/f_{\tilde{\psi}_i}$ as a series

$$
\sum_{s=0}^{\infty} \left( \sum_{j \neq i} a_{i,j} t_{d+i}^{-1} t_{d+j} + \sum_{n \in \partial \Delta^* \cap N} a_{i,n} t_{d+i}^{-1} n^k \prod_{j=1}^{k} t_{d+j}^{-\min(\Delta_j,n)} \right)^s
$$

(here $j \neq i$ means $j \in \{0,...,k\} \setminus \{i\}$) and applying the Cauchy residue formula similar to [B1, Ex. 14.5] gives

$$
\Phi_{\tilde{\psi}_0,...,\tilde{\psi}_k}(\alpha) = \sum_{l \in L_{\tilde{\psi}_0,...,\tilde{\psi}_k}} \prod_{i=0}^{k} \frac{l_i!}{(-a_{i,i}) t_i} \Pi_{j \neq i} \frac{a_{i,j} l_{i,j}}{l_{j,i}} \Pi_{n \in \partial \Delta^* \cap N} \frac{a_{i,n} l_{i,n}}{l_{i,n}},
$$

where $l_i = \sum_{j \neq i} l_{i,j} + \sum_{n \in \partial \Delta^* \cap N} l_{i,n}$, and $L_{\tilde{\psi}_0,...,\tilde{\psi}_k}$ consists of vectors $(l_{i,*}) \in \mathbb{Z}_{\geq 0}^{(k+1)((\Delta^*)^*+k-1)}$ such that

$$
\sum_{s=0}^{k} \left( \sum_{j \neq i} l_{i,j} (\delta_j e_j^* - \delta_i e_i^*) + \sum_{n \in \partial \Delta^*} l_{i,n} \left( n - \delta_i e_i^* - \sum_{j=1}^{k} \min(\Delta_j,n) e_j^* \right) \right) = 0,
$$

where $\delta_j = 1$ if $j \neq 0$ and $\delta_0 = 0$. From Theorem 4.2, degeneration $Y'_{\tilde{\psi}_0,...,\tilde{\psi}_k} \Rightarrow Y'_{\Delta^*}$ corresponds to setting $a_{0,0} = a_0$, $a_{0,j} = 0$, $a_{i,n} = a_n$, $a_{i,i} = 1$, $a_{i,0} = -1$, $a_{i,0} = -1$, $a_{i,j} = 0$, $a_{i,n} = 0$ for $1 \leq i, j \leq k$, $i \neq j$, and $n \in \partial \Delta^* \cap N$. Coefficients $a_{i,*} = 0$ force the vanishing of the terms in the series $\Phi_{\tilde{\psi}_0,...,\tilde{\psi}_k}(\alpha)$ unless the corresponding $l_{i,*}$ equals zero. Hence, the nonvanishing terms in the series correspond to $l_0 = \sum_{n \in \partial \Delta^* \cap N} l_{0,n}$, $l_i = l_{i,0}$, for $i = 1,...,k$, and

$$
\sum_{n \in \partial \Delta^* \cap N} l_{0,n} \left( n - \sum_{j=1}^{k} \min(\Delta_j,n) e_j^* \right) + \sum_{i=1}^{k} l_{i,0} (-e_i^*) = 0,
$$

and the series $\Phi_{\tilde{\psi}_0,...,\tilde{\psi}_k}(\alpha)$ degenerates to

$$
\sum_{l} \left( \prod_{i=1}^{k} \frac{l_i!}{(-a_0) t_i} \frac{(-1)^{l_{i,0}}}{l_{i,0}!} \right) \frac{l_0!}{t_0!} \Pi_{n \in \partial \Delta^* \cap N} \frac{a_{0,n}^{l_{0,n}}}{l_{0,n}!} = \sum_{l \in L_{\Delta^*}} \frac{l_0!}{(-a_0)^{l_0}} \Pi_{n \in \partial \Delta^* \cap N} \frac{a_{0,n}^{l_{0,n}}}{l_{0,n}!},
$$

where

$$
L_{\Delta^*} = \left\{ (l_n)_{n \in \partial \Delta^* \cap N} \mid \sum_{n \in \partial \Delta^* \cap N} l_n n = 0 \right\} \subset \mathbb{Z}_{\geq 0}^{((\Delta^*)^*+k-1)}, \quad l_0 = \sum_{n \in \partial \Delta^* \cap N} l_n.$$
But the last series is the hypergeometric series corresponding to the main period of Calabi-Yau hypersurface $Y_\Delta^\ast$, at the maximally unipotent boundary point (see [B1, CK]). Thus, we showed that the main period $\Phi_{\varphi_0,\ldots,\varphi_k}(\alpha)$ of a minimal Calabi-Yau complete intersection $Y_{\varphi_0,\ldots,\varphi_k}$ degenerates to the main period $\Phi_{Y_{\Delta}^\ast}(\alpha_0)$ of a minimal Calabi-Yau hypersurface $Y_{\Delta}^\ast$, as $Y_{\varphi_0,\ldots,\varphi_k} \to Y_{\Delta}^\ast$.

There is more work to be done in computing the mirror map itself from the main periods of Calabi-Yau, but we will finish this paper by explaining the expected relationship of the complex and Kahler moduli of the Calabi-Yau varieties involved in our geometric transitions. For definitions and notation we refer to the book [CK].

A minimal Calabi-Yau variety $V$ has a Kahler cone $K(V)$, a complexified Kahler space

$$K_C(V) = \{ \omega \in H^2(V, \mathbb{C}) \mid \text{Im}(\omega) \in K(V) \}/\text{im}H^2(V, \mathbb{Z}),$$

and the complexified Kahler moduli space $K(V) = K_C(V)/\text{Aut}(V)$. For a non-degenerate Calabi-Yau complete intersection $V$ in a maximal projective partial crepant resolution $X_\Sigma$ of a Gorenstein Fano toric variety, one considers a toric part of the Kahler cone $K(V)_{\text{toric}} = K(V) \cap H^2_{\text{toric}}(V)$, where $H^2_{\text{toric}}(V)$ is the image of the restriction map $H^2(X_\Sigma) \to H^2(V)$, and the corresponding toric Kahler moduli space $K(V)_{\text{toric}}$.

On the complex side, one also considers a part of the complex moduli space of Calabi-Yau complete intersection $V \subset X_\Sigma$. Let $V$ be the closure of the affine complete intersection

$$\sum_{m \in \Delta_i \cap \mathbb{Z}^d} a_{i,m} t^m = 0, \quad i = 0, \ldots, k$$

in $(\mathbb{C}^*)^d \subset X_\Sigma$, where $\Delta = \Delta_0 + \cdots + \Delta_k$ is a Minkowski sum decomposition of a reflexive polytope. Then the polynomial moduli space of the complete intersection $V$ can be constructed similar to [BC, Sect. 13] as a geometric quotient $\mathcal{M}(V)_{\text{poly}} = U/\text{Aut}(X_\Sigma)$, where $U$ is an open subset in $\mathbb{P}(L(\Delta_i \cap \mathbb{Z}^d)) \times \cdots \times \mathbb{P}(L(\Delta_k \cap \mathbb{Z}^d))$ corresponding to a subset of the set of quasismooth complete intersections (see [M1]) with $L(\Delta_i \cap \mathbb{Z}^d)$ denoting the vector space of Laurent polynomials $\sum_{m \in \Delta_i \cap \mathbb{Z}^d} a_{i,m} t^m$.

In practice, one replaces the toric Kahler moduli space and the polynomial moduli space of a Calabi-Yau intersection $V$ with suitable compactifications realized as toric varieties associated with a secondary fan (see [CK]). We will denote them by $\mathcal{K}(V)_{\text{toric}}$ and $\mathcal{M}(V)_{\text{poly}}$, respectively.

The geometric transitions $Y_{\varphi_0,\ldots,\varphi_k}^\ast \leftrightarrow Y_{\Delta}^\ast$, and $Y_{\Delta}^\ast \leftrightarrow Y_{\Delta_0,\ldots,\Delta_k}^\ast$ induce inclusions

$$\mathcal{M}(Y_{\Delta}^\ast)_{\text{poly}} \subset \mathcal{M}(Y_{\varphi_0,\ldots,\varphi_k}^\ast)_{\text{poly}}, \quad \mathcal{M}(Y_{\Delta_0,\ldots,\Delta_k}^\ast)_{\text{poly}} \subset \mathcal{M}(Y_{\Delta}^\ast)_{\text{poly}},$$

corresponding to degenerations on the complex side, and inclusions

$$\mathcal{K}(Y_{\varphi_0,\ldots,\varphi_k}^\ast)_{\text{toric}} \subset \mathcal{K}(Y_{\Delta}^\ast)_{\text{toric}}, \quad \mathcal{K}(Y_{\Delta_0,\ldots,\Delta_k}^\ast)_{\text{toric}} \subset \mathcal{K}(Y_{\Delta}^\ast)_{\text{toric}},$$

corresponding to contractions on the Kahler side. These should fit into the following commutative diagrams:

$$\begin{align*}
\mathcal{M}(Y_{\Delta}^\ast)_{\text{poly}} &\subset \mathcal{M}(Y_{\varphi_0,\ldots,\varphi_k}^\ast)_{\text{poly}} & \mathcal{M}(Y_{\Delta_0,\ldots,\Delta_k}^\ast)_{\text{poly}} &\subset \mathcal{M}(Y_{\Delta}^\ast)_{\text{poly}} \\
\mathcal{K}(Y_{\varphi_0,\ldots,\varphi_k}^\ast)_{\text{toric}} &\subset \mathcal{K}(Y_{\Delta}^\ast)_{\text{toric}} & \mathcal{K}(Y_{\Delta_0,\ldots,\Delta_k}^\ast)_{\text{toric}} &\subset \mathcal{K}(Y_{\Delta}^\ast)_{\text{toric}}
\end{align*}$$
where the vertical arrows are the mirror morphisms between the complex and Kähler moduli spaces. Moreover, the degenerations of the main periods of Calabi-Yau varieties (calculated above) should induce degenerations of the mirror morphisms between the ambient moduli to the mirror morphisms of the enclosed moduli. The inclusions of the moduli can be described explicitly in terms of the inclusions of the respective secondary fans.

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