A new proof of Savin’s theorem on Allen-Cahn equations

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Abstract
In this paper we establish an improvement of tilt-excess decay estimate for the Allen-Cahn equation, and use this to give a new proof of Savin’s theorem on the uniform \( C^{1,\alpha} \) regularity of flat level sets, which then implies the one dimensional symmetry of minimizers in \( \mathbb{R}^n \) for \( n \leq 7 \). This generalizes Allard’s \( \varepsilon \)-regularity theorem for stationary varifolds to the setting of Allen-Cahn equations.

Keywords: Allen-Cahn equation, phase transition, improvement of tilt-excess decay, harmonic approximation, De Giorgi conjecture.

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1 Introduction

This paper is devoted to generalize the Allard’s regularity theory in Geometric Measure Theory to the setting of Allen-Cahn equations and discuss its application to the De Giorgi conjecture.

The Allen-Cahn equation

\[ \Delta u = u^3 - u, \]  

(1.1)

is a typical model of phase transition. By now, it has been studied from various aspects. One particular feature of this equation is its close relation with the minimal surface theory, through the singularly perturbed Allen-Cahn equation

\[ \Delta u_\varepsilon = \frac{1}{\varepsilon^2} \left( u_\varepsilon^3 - u_\varepsilon \right). \]
By this connection and in view of the Bernstein theorem for minimal hypersurfaces [23], De Giorgi made the following conjecture in [8]:

Let \( u \in C^2(\mathbb{R}^{n+1}) \) be a solution of (1.1) such that

\[
|u| \leq 1, \quad \frac{\partial u}{\partial x_{n+1}} > 0 \quad \text{in} \quad \mathbb{R}^{n+1}.
\]

Then \( u \) depends only on one variable, if \( n \leq 7 \).

This conjecture has been considered by many authors, including Ghoussoub and Gui [14], Ambrosio and Cabre [3] and Savin [19], and counterexamples are constructed by del Pino, Kowalczyk and Wei [10] when \( n \geq 8 \). In particular, Savin proved an improvement of flatness type result for minimizing solutions (i.e. a minimizer of the energy functional). This result says, if \( u \) is a minimizer, and if in a ball \( B_l \), the zero level set of \( u \) is trapped in a flat cylinder \( \{|x_{n+1}| < \theta\} \) which is sufficiently narrow (i.e. \( \theta l^{-1} \) sufficiently small), then by shrinking the radius of ball, possibly after a rotation of coordinates, the zero level set of \( u \) is trapped in a flatter cylinder.

By using this estimate, Savin proved

**Theorem 1.1.** Let \( u \) be a minimizer defined on the entire space \( \mathbb{R}^{n+1} \) where \( n \leq 6 \). Then \( u \) is one dimensional.

For \( n > 6 \), if we add some assumptions on the behavior of level sets of \( u \), it is still possible to prove the one dimensional symmetry of \( u \). This theorem also implies the original De Giorgi conjecture, with an addition assumption that

\[
\lim_{x_{n+1} \to \pm \infty} u(x, x_{n+1}) = \pm 1.
\]

This type of improvement of flatness result is a common theme in partial regularity theory for many elliptic problems, although sometimes in rather different forms. One main ingredient to establish this improvement of flatness is the blow up (or harmonic approximation) technique, which was first introduced in De Giorogii’s solution of the almost everywhere regularity problem for minimal hypersurfaces in [7].

Although the statement of Savin’s improvement of flatness result bears many similarities with De Giorgi’s theorem, the proof in [19] is rather different. In fact, it is based on Caffarelli-Cordoba’s proof of De Giorgi theorem in [5]. This approach uses the “viscosity” side of the problem, and relies heavily on a Krylov-Safanov’s type argument. In particular, Savin first obtains a Harnack inequality (hence some kind of uniform Hölder continuity) and then use this to prove that the blow up sequence converges uniformly to a harmonic function. Thus this argument can be understood as a kind of harmonic approximation in \( L^\infty \) norm.
Savin’s approach can be applied to many other problems, even without variational structure, see for example \[20, 21, 22\]. However, it seems that the maximum principle and Harnack inequality is crucial in this approach. At present it is not clear how to get this kind of improvement of flatness result for elliptic systems, where Harnack inequality may fail. Thus, in view of the connection between Allen-Cahn equations and the minimal surface theory, in this paper we intend to explore the variational side of improvement of flatness and establish some results which parallel the classical regularity theory in *Geometric Measure Theory*. In particular, as in Allard’s regularity theory \[1\] (see also \[16\, Section 6.5\] for an account), we use the following excess quantity

\[
\int [1 - (\nu_\varepsilon \cdot e)^2] \varepsilon |\nabla u_\varepsilon|^2,
\]

where \(\nu_\varepsilon = \nabla u_\varepsilon/|\nabla u_\varepsilon|\) is the unit normal vector to level sets of \(u_\varepsilon\). This quantity was first used by Hutchinson-Tonegawa \[15\] to derive the integer multiplicity of the limit varifold arising from general critical points in the Allen-Cahn problem.

This quantity can be seen as a measurement of the flatness of the level sets of \(u_\varepsilon\) (see Lemma 4.6 below). Similar to Allard’s \(\varepsilon\)-regularity theorem, if the excess in a ball is small, then after shrinking the radius of ball and possibly rotating the vector \(e\), the excess becomes smaller. This *improvement of tilt-excess* is the main step in the proof of Allard’s \(\varepsilon\)-regularity theorem, and also in our argument. In contrast to the quantity used in Savin’s improvement of flatness result, the excess is an energy type quantity. Indeed, if all the level sets \(\{u_\varepsilon = t\}\) can be represented by graphs along the \((n+1)\)-th direction, in the form \(\{x_{n+1} = h(x,t)\}\), then

\[
\int [1 - (\nu_\varepsilon \cdot e)^2] \varepsilon |\nabla u_\varepsilon|^2 = \int_{-1}^{1} \left( \int \frac{|\nabla h(x,t)|^2}{1 + |\nabla h(x,t)|^2} \varepsilon |\nabla u_\varepsilon(x,h(x,t))| dx \right) dt,
\]

which is almost a weighted Dirichlet energy.

From this, we see the problem can be approximated by harmonic functions (corresponding to critical points of the Dirichlet energy) if \(|\nabla h(x,t)|\) is small. This is exactly the content of *harmonic approximation* technique. Using the excess allows us to work in the Sobolev spaces and apply standard compactness Sobolev embeddings to get the blow up limit.

We also note that this type of tilt-excess decay result was known by Tonegawa, see \[24\], where he showed that this result implies the uniform \(C^{1,\alpha}\) regularity of intermediate transition layers in dimension 2.

However, in this tilt-excess decay we need to assume that

\[
\int [1 - (\nu_\varepsilon \cdot e)^2] \varepsilon |\nabla u_\varepsilon|^2 \gg \varepsilon^2. \tag{1.2}
\]
Compared to Allard’s ε-regularity theorem, this condition is rather unsatisfactory. It prevents us from applying this improvement of decay directly to deduce the uniform $C^{1,\alpha}$ regularity of intermediate transition layers, while Savin’s improvement of flatness does implies this directly. (This was also discussed by Tonegawa [24].)

In this paper we show this tilt-excess decay still implies the uniform $C^{1,\alpha}$ regularity, by exploiting some more properties implied by the fact that $u_\varepsilon$ is close to a one dimensional solution at $O(1)$ scale. We can show that $u_\varepsilon$ is close to a one dimensional solution at all scale up to $O(\varepsilon)$. This fact, together with the improvement of tilt-excess decay, gives a Morrey type bound. Here, once again due to the obstruction (1.2), this Morrey type bound does not imply the $C^{1,\alpha}$ regularity of $\{u_\varepsilon = 0\}$, but only a Lipschitz one. However, under the condition that $\{u_\varepsilon = 0\}$ is a Lipschitz graph, Caffarelli and Cordoba [6] has shown that transition layers are uniformly bounded in $C^{1,\alpha}$ for some $\alpha \in (0,1)$. Thus we get a full analogue of Allard’s ε-regularity theorem in the Allen-Cahn setting.

In this paper we do not fully avoid the use of maximum principle. For example, it seems that the Modica inequality is indispensable in our argument, because we need it to obtain a monotonicity formula. We also need to apply the moving plane (or sliding) method (as in Farina [12]) to deduce the one dimensional symmetry of some entire solutions. There we also need a function to control the behavior of $u$ at the place far away from the transition part. This function behaves like a distance function and this fact follows from the Modica inequality. However, we do avoid the use of any Harnack inequality. It may be possible to remove the above mentioned deficiency by strengthening the tilt-excess decay estimate (see Theorem 3.3 below), but the current version of Theorem 3.3 is already sufficient for proving Theorem 1.1, as discussed above.

This approach was used by the author in [25], where we consider a De Giorgi type conjecture for the elliptic system

$$\Delta u = uv^2, \quad \Delta v = vu^2, \quad u, v > 0 \text{ in } \mathbb{R}^n.$$ 

For the corresponding singularly perturbed system

$$\begin{cases}
\Delta u_\kappa = \kappa u_\kappa v_\kappa^2, \\
\Delta v_\kappa = \kappa v_\kappa u_\kappa^2,
\end{cases}$$

an improvement of flatness result was established by considering the quantity

$$\int |\nabla (u_\kappa - v_\kappa - e \cdot x)|^2.$$

Both proofs are same in the spirit. In particular, to prove that the blow up limit is a harmonic function, we mainly use the stationary condition associated to the equation but not the equation itself. This is more apparent in the current setting, because the
stationary condition for the singularly perturbed Allen-Cahn equations is directly linked to the corresponding one in their limit problem, the stationary condition for varifolds (in the sense of Allard [1], see [15]). Furthermore, since the excess is kind of $H^1$ norm, to prove the strong convergence in $H^1$ space, we implicitly use a Caccioppoli type inequality, which is again deduced from the stationary condition (see Remark 4.7 below).

Finally, although in our tilt-excess result (Theorem 3.3) and $\varepsilon$-regularity result (Theorem 9.1), we do not assume the solution to be a minimizer, we still need a kind of multiplicity one property of transition layers. (This is associated to the unit density property of the limit varifold.) Thus essentially this argument does not remove the restriction of no folding phenomena in Savin’s result.

The organization of this paper can be seen from the table of contents.

2 Settings and notations

Throughout this paper, we shall work in the following settings. We consider the Allen-Cahn equation in the general form as

$$\Delta u = W'(u),$$  \hspace{1cm} (2.1)

where $W$ is a double well potential. That is, $W \in C^3(\mathbb{R})$, satisfying

- $W \geq 0$, $W(\pm 1) = 0$ and $W > 0$ in $(-1,1)$;
- for some $\gamma \in (0,1)$, $W'' < 0$ on $(\gamma,1)$ and $W'' > 0$ on $(-1,-\gamma)$;
- $W'' \geq \kappa > 0$ for all $|x| \geq \gamma$.

A typical example is $W(u) = (1 - u^2)^2/4$.

Through a scaling $u_\varepsilon(X) := u(\varepsilon^{-1}X)$, we have the singularly perturbed version of the Allen-Cahn equation:

$$\varepsilon \Delta u_\varepsilon = \frac{1}{\varepsilon} W'(u_\varepsilon).$$  \hspace{1cm} (2.2)

Note that a solution $u_\varepsilon$ is a critical point of the functional

$$E_\varepsilon(u_\varepsilon) = \int \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon).$$  \hspace{1cm} (2.3)

We say $u_\varepsilon$ is a minimizer (or a minimizing solution), if for any ball $B$ in the definition domain of $u_\varepsilon$,

$$\int_B \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \leq \int_B \frac{\varepsilon}{2} |\nabla v|^2 + \frac{1}{\varepsilon} W(v),$$

for any $v \in H^1(B)$ such that $v = u$ on $\partial B$. 

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We will always assume that any solution $u_\varepsilon$ of (2.2) satisfies $|u_\varepsilon| \leq 1$. Then by standard elliptic estimates, there exists a universal constant $C(n)$ such that

$$
\varepsilon|\nabla u_\varepsilon| + \varepsilon^2|\nabla^2 u_\varepsilon| \leq C(n).
$$

(2.4)

Hence $u_\varepsilon$ is a classical solution.

For any smooth vector field $Y$ with compact support, by considering the domain variation in the form

$$u_\varepsilon^t(X) := u_\varepsilon(X + tY(X)), \quad \text{for } |t| \text{ small},$$

from the definition of critical points we get

$$
\frac{d}{dt} \int \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \bigg|_{t=0} = 0.
$$

Then after some integration by parts, we obtain the stationary condition satisfied by $u_\varepsilon$:

$$
\int \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) \text{div} Y - \varepsilon D Y(\nabla u_\varepsilon, \nabla u_\varepsilon) = 0.
$$

(2.5)

Finally, by the assumption on $W$, there exists a one dimensional solution $g(t)$ defined on $t \in (-\infty, +\infty)$,

$$g''(t) = W'(g(t)), \quad \lim_{t \to \pm\infty} g(t) = \pm 1,$n

(2.6)

where the convergence rate is exponential.

The first integral for $g$ can be written as

$$g'(t) = \sqrt{2W(g(t))} > 0.
$$

(2.7)

For any $\varepsilon > 0$, we denote $g_\varepsilon(t) = g(\varepsilon^{-1}t)$, which satisfies

$$g_\varepsilon''(t) = \frac{1}{\varepsilon^2} W'(g_\varepsilon(t)).
$$

(2.8)

Throughout this paper, $\sigma_0$ is the constant defined by

$$\sigma_0 := \int_{-\infty}^{+\infty} g'(t)^2 dt.
$$

(2.9)

In this paper we adopt the following notations.

- A point in $\mathbb{R}^{n+1}$ will be denoted by $X = (x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$.
- $B_r(X)$ denotes an open ball in $\mathbb{R}^{n+1}$ and $B_r(x)$ an open ball in $\mathbb{R}^n$. If the center is the origin 0, we write it as $B_r$ (or $B_r$).
\( C_r(x) = B_r(x) \times (-1, 1) \subset \mathbb{R}^{n+1} \) the finite cylinder over \( B_r(x) \subset \mathbb{R}^n \).

- \( e_i, 1 \leq i \leq n + 1 \) denote the standard basis in \( \mathbb{R}^{n+1} \).

- \( P \) denotes a hyperplane in \( \mathbb{R}^{n+1} \) and \( \Pi_P \) (or simply \( P \)) the orthogonal projection onto it. If \( P = \mathbb{R}^n \), we simply use \( \Pi \).

- \( G(n) \) denotes the Grassman manifold of unoriented \( n \)-dimensional planes in \( \mathbb{R}^{n+1} \).

- A varifold \( V \) is a Radon measure on \( \mathbb{R}^{n+1} \times G(n) \). We use \( \| V \| \) to denote the weighted measure of \( V \), that is, for any measurable set \( A \subset \mathbb{R}^{n+1} \),

\[
\| V \|(A) = V(A \times G(n)).
\]

- For a measure \( \mu \), \( \text{spt} \mu \) denotes its support.

- \( \nu_\varepsilon(X) = \frac{\nabla u_\varepsilon(X)}{\|\nabla u_\varepsilon(X)\|} \) if \( \nabla u_\varepsilon(X) \neq 0 \), otherwise we take it to be an arbitrary unit vector.

- \( \mu_\varepsilon := \varepsilon |\nabla u_\varepsilon|^2 dX \).

- \( \mathcal{H}^s \) denotes the \( s \)-dimensional Hausdorff measure.

- \( \omega_n \) denotes the volume of the unit ball \( B_1 \) in \( \mathbb{R}^n \).

- \( H^1 \) denote the Sobolev space with the norm \( (\int |\nabla u|^2 + |u|^2)^{1/2} \).

- \( \text{dist}_H \) is the Hausdorff distance between sets in \( \mathbb{R}^{n+1} \).

- Universal constants \( C, C_i \) and \( K_i \) (which are large) and \( c_i \) (which are small) depend only on the dimension \( n \) and the potential function \( W \).

Throughout this paper \( u_\varepsilon \) always denotes a solution of (2.2), satisfying the Modica inequality

\[
\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 \leq \frac{1}{\varepsilon} W(u_\varepsilon).
\]

This inequality may not be essential in the argument, but we prefer to assume it to make the arguments clear. We use \( \varepsilon \) (more precisely, it should be written as \( \varepsilon_i \)) to denote a sequence of parameters converging to 0.
Part I
Tilt-excess decay

3 Statement

The following quantity will play an important role in our analysis.

**Definition 3.1 (Excess).** Let $P$ be an $n$-dimensional hyperplane in $\mathbb{R}^{n+1}$ and $e$ one of its unit normal vector, $B_r(x) \subset P$ an open ball and $C_r(x) = B_r(x) \times (-1, 1)$ the cylinder over $B_r(x)$. The excess of $u_\varepsilon$ in $C_r(x)$ with respect to $P$ is

$$E(r; x, u_\varepsilon, P) := r^{-n} \int_{C_r(x)} \left[ 1 - (\nu_\varepsilon \cdot e)^2 \right] \varepsilon |\nabla u_\varepsilon|^2 dX. \quad (3.1)$$

If $P = \mathbb{R}^n$ and $e = e_{n+1}$, then

$$E(r; x, u_\varepsilon) = r^{-n} \int_{C_r(x)} \varepsilon \sum_{i=1}^n \left( \frac{\partial u_\varepsilon}{\partial x_i} \right)^2 dX.$$

**Remark 3.2.** For any unit vector $\nu$,

$$|\nu - e| |\nu + e| \geq \sqrt{2} \min\{|\nu - e|, |\nu + e|\}.$$

Thus

$$1 - (\nu \cdot e)^2 = \frac{1}{4} (1 - \nu \cdot e) (1 + \nu \cdot e)$$

$$= \frac{1}{4} |\nu - e|^2 |\nu + e|^2$$

$$\geq \frac{1}{2} \min\{|\nu - e|^2, |\nu + e|^2\}.$$

By projecting the unit sphere $\mathbb{S}^n$ to the real projective space $\mathbb{RP}^n$ (both with standard metric), we get

$$1 - (\nu \cdot e)^2 \geq c(n) \text{dist}_{\mathbb{RP}^n}(\nu, e)^2,$$

for some universal constant $c(n)$.

Our main objective in Part I is to prove the following decay estimate.
Theorem 3.3 (Tilt-excess decay). There exist five universal constants \( \delta_0, \tau_0, \varepsilon_0 > 0, \theta \in (0, 1/4) \) and \( K_0 \) large so that the following holds. Let \( u_\varepsilon \) be a solution of (2.2) with \( \varepsilon \leq \varepsilon_0 \) in \( B_4 \), satisfying the Modica inequality (2.10), \( |u_\varepsilon(0)| \leq \gamma \), and

\[
4^{-n} \int_{B_4} \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \leq (1 + \tau_0) \sigma_0 \omega_n. \tag{3.2}
\]

Suppose the excess with respect to \( \mathbb{R}^n \)

\[
\delta_\varepsilon^2 := E(2; 0, u_\varepsilon, \mathbb{R}^n) \leq \delta_0^2, \tag{3.3}
\]

where \( \delta_\varepsilon \geq K_0 \varepsilon \). Then there exists another plane \( P \), such that

\[
\theta^{-n} E(\theta; 0, u_\varepsilon, P) \leq \frac{\theta}{2} E(2; 0, u_\varepsilon, \mathbb{R}^n). \tag{3.4}
\]

Moreover, there exists a universal constant \( C \) such that

\[
\|e - e_{n+1}\| \leq C E(2; 0, u_\varepsilon, \mathbb{R}^n)^{1/2}, \tag{3.8}
\]

where \( e \) is the unit normal vector of \( P \) pointing to the above.

In the next section we shall see that (3.3) always holds, provided that we have (3.2) with \( \tau_0 \) sufficiently small (depending on \( \delta_0 \)). However, the assumption that \( \delta_\varepsilon \gg \varepsilon \) is crucial here, which is not so satisfactory compared to Allard’s and Savin’s version.

We shall prove this theorem by contradiction. Assume it does not hold. Thus there exists a sequence of \( \varepsilon_i \) (for simplicity we shall drop the subscript \( i \)) and corresponding solutions \( u_\varepsilon \) satisfying all of the assumptions in Theorem 3.3, in particular:

1. We have a sequence of \( \tau_\varepsilon \to 0 \) such that

\[
4^{-n} \int_{B_4} \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \leq (1 + \tau_\varepsilon) \sigma_0 \omega_n. \tag{3.5}
\]

2. The excess

\[
\delta_\varepsilon^2 := E(2; 0, u_\varepsilon, \mathbb{R}^n) \to 0, \tag{3.6}
\]

where

\[
\frac{\delta_\varepsilon}{\varepsilon} \to +\infty, \quad \text{as} \quad \varepsilon \to 0. \tag{3.7}
\]

But for any unit vector \( e \) with

\[
\|e - e_{n+1}\| \leq C E(2; 0, u_\varepsilon, \mathbb{R}^n)^{\frac{1}{2}}, \tag{3.8}
\]
where the constant $C$ will be determined below (by the constant in (8.1)), we have
\[ \theta^{-n} E(\theta; 0, u_\varepsilon, P) \geq \frac{\theta}{2} E(2; 0, u_\varepsilon, \mathbb{R}^n). \] (3.9)

Here $P$ is the hyperplane orthogonal to $e$ and $\theta$ is a constant to be determined later, too.

The remaining part will be devoted to derive a contradiction from these assumptions (3.5)-(3.9).

### 4 Compactness results

In this section, we study the convergence of various quantities associated to $u_\varepsilon$.

Recall that we have assumed the Modica inequality (2.10) holds for $u_\varepsilon$. An important consequence of this inequality is the

**Proposition 4.1 (Monotonicity formula).** For any $X \in B_3$, 
\[ r^{-n} \int_{B_r} \frac{\varepsilon}{2} \left| \nabla u_\varepsilon \right|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \] is non-decreasing in $r \in (0, 1)$.

By combining the monotonicity formula (4.1) with (3.8), we see

**Corollary 4.2.** For any $B_r(X) \subset B_3$, 
\[ \int_{B_r(X)} \frac{\varepsilon}{2} \left| \nabla u_\varepsilon \right|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \leq 8^n \sigma_0 \omega_n r^n. \] (4.1)

We can use the main result in Hutchinson and Tonegawa [15] to study the convergence of $u_\varepsilon$. Define the varifold $V_\varepsilon$ by
\[ <V_\varepsilon, \Phi(X, S)> = \int \Phi(x, I - \nu_\varepsilon \otimes \nu_\varepsilon) \varepsilon \left| \nabla u_\varepsilon \right|^2 dx, \quad \forall \Phi \in C_0^\infty(C_2 \times G(n)). \]

Hutchinson and Tonegawa proved that, as $\varepsilon \to 0$, $V_\varepsilon$ converges in the sense of varifolds to a stationary, rectifiable varifold $V$ with integer density (modulo division by the constant $\sigma_0$). From this, it can be directly checked that the measures $\mu_\varepsilon$ converge to $\|V\|$ weakly.

Moreover, for any $t \in (-1, 1)$ fixed, $\{u_\varepsilon = t\}$ converges to $\text{spt}\|V\|$ in the Hausdorff distance. Hence we must have $0 \in \text{spt}\|V\|$ because $0 \in \{|u_\varepsilon| \leq \gamma\}$.

In [15], using (11.5) and Proposition 4.1, Hutchinson and Tonegawa also proved that the discrepancy quantity
\[ \left( \frac{1}{\varepsilon} W(u_\varepsilon) - \frac{\varepsilon}{2} \left| \nabla u_\varepsilon \right|^2 \right) \to 0, \quad \text{in } L^1_{\text{loc}}(C_3). \] (4.2)

With the bound (3.5) we claim that
Proposition 4.3 (Limit varifold). The limit \( \|V\| = \sigma_0 H^n|_{\mathbb{R}^n} \).

Proof. This is because
\[
4^{-n}\|V\|(B_4) = \sigma_0 \omega_n,
\]
which is obtained by taking the limit in (3.5) and using the integer multiplicity of \( V \). Then by the monotonicity formula for stationary varifold (cf. [16, Theorem 6.3.2]), we deduce that \( V \) is a cone. To show that \( V \) is the standard varifold associated to this plane with unit density, we can apply Allard’s \( \varepsilon \)-regularity result to deduce that \( \text{spt}\|V\| \cap B_{1/2} \) is a smooth hypersurface and combine this fact with the cone property of \( V \).

Now we show that away from \( \mathbb{R}^n \), \( u_\varepsilon \) is close to \( \pm 1 \).

Proposition 4.4. For any \( h > 0 \), if \( \varepsilon \) sufficiently small,
\[
(1 - u_\varepsilon^2) + |\nabla u_\varepsilon| \leq C(h) e^{-\frac{|x_{n+1}|}{C_\varepsilon}} \quad \text{in} \quad C_2 \setminus \{|x_{n+1}| \leq h\}.
\]
In particular, \( \{u_\varepsilon = 0\} \cap C_2 \) lies in a \( h \)-neighborhood of \( \mathbb{R}^n \cap B_2 \).

Proof. By [15], \( u_\varepsilon^2 \) converges to 1 uniformly on any compact set outside \( \text{spt}\|V\| = \mathbb{R}^n \). In particular, for all \( \varepsilon \) small,
\[
u_\varepsilon^2 \geq \gamma \quad \text{in} \quad C_3 \setminus \{|x_{n+1}| \geq \frac{h}{2}\}.
\]
Then there exists a constant \( C \) depending only on the potential function \( W \) such that
\[
\Delta(1 - u_\varepsilon^2) \geq \frac{1}{C_\varepsilon^2}(1 - u_\varepsilon^2) \quad \text{in} \quad C_3 \setminus \{|x_{n+1}| \geq \frac{h}{2}\}.
\]
From this we can apply Lemma B.1 to get the exponential decay of \( 1 - u_\varepsilon^2 \). The estimate for \( |\nabla u_\varepsilon| \) follows from standard interior gradient estimate. \( \square \)

Remark 4.5. We can show that \( u_\varepsilon \) converges to 1 locally uniformly in \( C_2 \cap \{x_{n+1} > 0\} \), and to \(-1\) locally uniformly in \( C_2 \cap \{x_{n+1} < 0\} \). Together with the previous proposition, this implies that
\[
dist_H(\{u_\varepsilon = 0\} \cap C_1, \mathbb{R}^n \cap C_1) \to 0.
\]
Indeed, if \( u_\varepsilon \) converges to 1 (or \(-1\)) on both sides of \( \mathbb{R}^n \), the multiplicity of \( V \) will be greater than 2 (see [15, Theorem 1, (4)]). This contradicts Proposition 4.3.

The following lemma implies that if we have (3.5), then (3.6) always holds.

Lemma 4.6. Let \( u_\varepsilon \) be a sequence of solutions satisfying (3.5) and the Modica inequality (2.10) in \( B_4 \). Then the excess with respect to \( \mathbb{R}^n \),
\[
E(2; 0, u_\varepsilon) \to 0, \quad \text{as} \ \varepsilon \to 0.
\]
Proof. For any \( \eta \in C_0^\infty(C_2) \), take the vector field \( Y = (0, \cdots, 0, \eta x_{n+1}) \) and substitute it into the stationary condition. This leads to

\[
0 = \int_{C_2} \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) \left( \eta + \frac{\partial \eta}{\partial x_{n+1}} x_{n+1} \right) - \eta \nu_{\varepsilon,n+1}^2 |\nabla u_\varepsilon|^2 - x_{n+1} \sum_{i=1}^{n+1} \frac{\partial \eta}{\partial x_i} \nu_{\varepsilon,i} \nu_{\varepsilon,n+1} |\nabla u_\varepsilon|^2. \tag{4.3}
\]

By (4.2) and our assumption, the measures

\[
\left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) dx, \quad \nu_{\varepsilon,i} \nu_{\varepsilon,n+1} |\nabla u_\varepsilon|^2 dx
\]

all converge to some measures supported on \( \mathbb{R}^n \). Thus

\[
\lim_{\varepsilon \to 0} \int_{C_2} \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) \frac{\partial \eta}{\partial x_{n+1}} x_{n+1} - x_{n+1} \sum_{i=1}^{n+1} \frac{\partial \eta}{\partial x_i} \nu_{\varepsilon,i} \nu_{\varepsilon,n+1} |\nabla u_\varepsilon|^2 = 0.
\]

Substituting this into (4.3) and applying the Modica inequality (2.10), we can finish the proof. \( \square \)

Remark 4.7. Although we shall not use the Caccioppoli type inequality, here we show how to use the stationary condition to derive it.

Take a \( \psi \in C_0^\infty((-1,1)) \) satisfying \( 0 \leq \psi \leq 1, \psi \equiv 1 \) in \((-1/2, 1/2), |\psi'| \leq 3\). For any \( \phi \in C_0^\infty(B_1) \), take \( \eta(x,x_{n+1}) = \phi(x)^2 \psi(x_{n+1})^2 \) and replace \( x_{n+1} \) by \( x_{n+1} - \lambda \) in (4.3), where \( \lambda \in (-1,1) \) is an arbitrary constant. By this choice we get

\[
0 = \int_{C_1} \left[ \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right] \left[ \phi^2 \psi^2 + 2 \phi^2 \psi \psi' (x_{n+1} - \lambda) \right] - \phi^2 \psi^2 \nu_{\varepsilon,n+1}^2 |\nabla u_\varepsilon|^2 - (x_{n+1} - \lambda) \sum_{i=1}^{n} 2 \phi \psi \frac{\partial \phi}{\partial x_i} \nu_{\varepsilon,i} \nu_{\varepsilon,n+1} |\nabla u_\varepsilon|^2 \tag{4.4}
\]

\[
- (x_{n+1} - \lambda) 2 \phi^2 \psi \psi' \nu_{\varepsilon,n+1}^2 |\nabla u_\varepsilon|^2.
\]

First consider those terms containing \( \psi'(x_{n+1}) \). Since \( \psi' \equiv 0 \) in \( B_1 \times \{|x_{n+1}| < 1/2\} \), with the help of Proposition 4.4, we get

\[
\int_{C_1} \left[ \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right] 2 \phi^2 \psi \psi' (x_{n+1} - \lambda) - (x_{n+1} - \lambda) 2 \phi^2 \psi \psi' \nu_{\varepsilon,n+1}^2 |\nabla u_\varepsilon|^2 = O(e^{-\frac{1}{\varepsilon}}).
\]

Substituting this into (4.4) leads to

\[
\int_{C_1} \left[ \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 - \nu_{\varepsilon,n+1}^2 |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right] \phi^2 \psi^2 \tag{4.5}
\]
\[
\int_{C_1} (x_{n+1} - \lambda) \sum_{i=1}^{n} 2\phi\psi^2 \frac{\partial \phi}{\partial x_i} \nu_{e,i} \nu_{e,n+1} \varepsilon |\nabla u_\varepsilon|^2 + O(e^{-c/\varepsilon}).
\]

By the Cauchy inequality,
\[
\int_{C_1} (x_{n+1} - \lambda) \sum_{i=1}^{n} 2\phi\psi^2 \frac{\partial \phi}{\partial x_i} \nu_{e,i} \nu_{e,n+1} \varepsilon |\nabla u_\varepsilon|^2
\leq \frac{1}{4} \int_{C_1} \phi^2 \psi^2 \sum_{i=1}^{n} \nu_{e,i} \varepsilon |\nabla u_\varepsilon|^2 + 64 \int_{C_1} |\nabla \phi|^2 \psi^2 (x_{n+1} - \lambda)^2 \varepsilon |\nabla u_\varepsilon|^2.
\]

Substituting this into (4.5) and noting that
\[
\sum_{i=1}^{n} \nu_{e,i}^2 = 1 - (\nu_{e} \cdot e_{n+1})^2,
\]
and by the Modica inequality (2.10),
\[
\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 - \nu_{e,n+1} \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \geq [1 - (\nu_{e} \cdot e_{n+1})^2] \varepsilon |\nabla u_\varepsilon|^2,
\]
we obtain the following Caccioppoli type inequality
\[
\int_{C_1} \phi^2 \psi^2 [1 - (\nu_{e} \cdot e_{n+1})^2] \varepsilon |\nabla u_\varepsilon|^2 \leq 2^n \int_{C_1} |\nabla \phi|^2 \psi^2 (x_{n+1} - \lambda)^2 \varepsilon |\nabla u_\varepsilon|^2 + C e^{-\frac{c}{\varepsilon}}. \quad (4.6)
\]

**Remark 4.8.** Since we only have a control on \(1 - (\nu_{e} \cdot e_{n+1})^2\), in view of Remark 3.2, \(\nu_{e}\) may be close to \(e_{n+1}\) or \(-e_{n+1}\). To exclude one possibility, we need to use the unit density property. A subtle point here is that, without such assumptions, we cannot say that \(1 - \nu_{e} \cdot e_{n+1}\) (or \(1 + \nu_{e} \cdot e_{n+1}\)) is small everywhere. This is related to the possible interface foliation (and consequently the higher multiplicity of the limit varifold \(V\)), see the examples constructed by del Pino-Kowalczyk-Wei-Yang in [9].

### 5 Lipschitz approximation

Let
\[
f_\varepsilon(x) = \int_{-1}^{1} [1 - (\nu_{e}(x, x_{n+1}) \cdot e_{n+1})^2] \varepsilon |\nabla u_\varepsilon(x, x_{n+1})|^2 dx_{n+1}.
\]

By Lemma 4.6, \(f_\varepsilon \to 0\) in \(L^1(B_1)\). Consider the Hardy-Littlewood maximal function
\[
Mf_\varepsilon(x) := \sup_{r \in (0,1)} r^{-n} \int_{B_r(x)} f_\varepsilon(y) dy.
\]

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Given an $l > 0$, by the weak $L^1$ estimate, there exists a constant $C(n)$ depending only on the dimension $n$ such that
\[
\mathcal{H}^n \left( \{ Mf_\varepsilon > l \} \cap B_1 \right) \leq \frac{C(n)}{l} \| f_\varepsilon \|_{L^1(B_1)} = C(n) \frac{\delta_\varepsilon^2}{l}.
\]
(5.1)

Denote the set $B_1 \setminus \{ Mf_\varepsilon > l \}$ by $W_\varepsilon$. Note that since the integrand in (5.1) and hence $f_\varepsilon(x)$ are continuous functions, $W_\varepsilon$ is an open set.

Given $b \in (0, 1)$ and $l > 0$, we say a point $X \in \{|u_\varepsilon| < 1 - b\} \cap C_1$ is good, if
\[
\sup_{0 < r < 1} r^{-n} \int_{B_r(X)} \left[ 1 - (\nu_\varepsilon \cdot e_{n+1})^2 \right] \varepsilon |\nabla u_\varepsilon|^2 dX < l.
\]

Good points form a set $A_\varepsilon$ and we let $B_\varepsilon = (\{|u_\varepsilon| < 1 - b\} \cap C_1) \setminus A_\varepsilon$ be the set of bad points. Note that since $\left[ 1 - (\nu_\varepsilon(x, x_{n+1}) \cdot e_{n+1})^2 \right] \varepsilon |\nabla u_\varepsilon(x, x_{n+1})|^2$ is continuous, $A_\varepsilon$ is an open set and $B_\varepsilon$ is relatively closed in $\{|u_\varepsilon| < 1 - b\} \cap C_1$. Clearly we have $W_\varepsilon \subset \Pi(A_\varepsilon)$.

We first note that $B_\varepsilon$ is small in some sense.

**Lemma 5.1.** There exists a universal constant $C$ such that
\[
\mu_\varepsilon(B_\varepsilon) \leq C \frac{\delta_\varepsilon^2}{l}.
\]

**Proof.** For any $X \in B_\varepsilon$, by definition there exists an $r_X \in (0, 1)$ satisfying
\[
r_X^n \leq \frac{1}{l} \int_{B_{r_X}(X)} \left[ 1 - (\nu_\varepsilon \cdot e_{n+1})^2 \right] \varepsilon |\nabla u_\varepsilon|^2 dX.
\]

By Vitali covering lemma, we can choose a countable set of $X_i \in B_\varepsilon$ such that $B_{r_i}(X_i)$ (here $r_i := r_{X_i}$) are disjoint, and
\[
B_\varepsilon \subset \bigcup_i B_{5r_i}(X_i).
\]

Then
\[
\mu_\varepsilon(B_\varepsilon) \leq \sum_i \mu_\varepsilon(B_{5r_i}(X_i)) \leq C \sum_i r_i^n \quad \text{(by (4.1))}
\]
\[
\leq C l^{-1} \int_{B_{r_i}(X_i)} \left[ 1 - (\nu_\varepsilon \cdot e_{n+1})^2 \right] \varepsilon |\nabla u_\varepsilon|^2 dX \leq C l^{-1} \delta_\varepsilon^2. \quad \square
\]
Another fact about $B_{\varepsilon}$ is

**Lemma 5.2.** $\mathcal{H}^n(\Pi(B_{\varepsilon})) \leq C(n)l^{-1}\delta^2_{\varepsilon}$.

**Proof.** This is because $\Pi(B_{\varepsilon}) \subset B_1 \setminus W_{\varepsilon}$. Hence we can apply (5.1). □

Next we show that in $A_{\varepsilon}$, level sets of $u_{\varepsilon}$ are essentially Lipschitz graphs.

**Lemma 5.3.** Given a constant $b \in (0, 1)$, if $l$ is small enough, for any $t \in (-1 + b, 1 - b)$, $\{u_{\varepsilon} = t\} \cap A_{\varepsilon}$ can be represented by a Lipschitz graph $\{x_{n+1} = h_{\varepsilon}(x)\}$. The Lipschitz constant of $h_{\varepsilon}$ is controlled by $c_0(b, l)$, which satisfies $\lim_{l \to 0} c_0(b, l) = 0$.

**Proof.** Fix a point $X_0 \in A_{\varepsilon}$ with $u_{\varepsilon}(X_0) = t$. After a rescaling $v(X) = u_{\varepsilon}(X_0 + \varepsilon X)$, we are in the situation that

\[
\Delta v = W'(v), \quad \text{in } B_{\varepsilon-1},
\]

(5.2)

\[
\int_{B_R(0)} \frac{1}{2} |\nabla v|^2 + W(v) \leq CR^n, \quad \forall \, R \in (0, \varepsilon^{-1}),
\]

(5.3)

\[
\int_{B_1(0)} \sum_{i=1}^n \left( \frac{\partial v}{\partial x_i} \right)^2 \leq l.
\]

(5.4)

We claim that there exists an $l_0$ small such that for all $l \leq l_0$, we have two constants $c_1(b, l) \in (0, 1/2)$ and $c_2(b)$ so that

\[
\left| \frac{\partial v}{\partial x_n} \right| \geq (1 - c_1(b, l)) |\nabla v| \geq c_2(b) \quad \text{in } B_1.
\]

(5.5)

Assume by the contrary that we have a sequence of $v_i$ satisfying all of these conditions (5.2)-(5.4) with $l$ replaced by $l_i$, which goes to 0 as $i \to 0$. By standard elliptic estimates and Arzelà-Ascoli theorem, $v_i$ converges to a function $v$ in $C^2_{loc}(\mathbb{R}^{n+1})$. $v$ is still a solution of (5.2) in $\mathbb{R}^{n+1}$, and because

\[
|v(0)| = \lim_{i \to +\infty} |v_i(0)| \leq 1 - b,
\]

we must have $|v| < 1$ in $\mathbb{R}^{n+1}$. After passing to the limit in (5.4) (where $l$ is replaced by $l_i$) and (5.3), we see $v(X) \equiv g(x_{n+1} + t)$ for some $t \in \mathbb{R}$. Then by (2.7),

\[
\left| \frac{\partial v}{\partial x_{n+1}}(X) \right| = |\nabla v(X)| = \sqrt{2W(v(X))} \geq c(b) \quad \text{in } B_1.
\]

Thus for all $i$ large, $v_i$ satisfies (5.5). This also proves that $c_1(b, l)$ converges to 0 as $l \to 0$.

By (5.5), the level set $\{v = v(0)\} \cap B_1(0)$ is a Lipschitz graph of the form $\{x_{n+1} = h(x)\}$ with the Lipschitz constant $c_0(b, l) \leq 2c_1(b, l)$. □
From this proof, we see for any $X = (x, x_{n+1}) \in A_\varepsilon$, $\Pi^{-1}(x) \cap \{u_\varepsilon = u_\varepsilon(X)\} \cap B_{L\varepsilon}(X) = \{X\}$ if we fix $L$ and then choosing $l$ sufficiently small. In fact, this is the context of the following lemma, which is essentially [15, Proposition 5.6].

**Lemma 5.4.** Given $L, M > 0$, $b \in (0, 1)$ and $t \in (-1 + b, 1 - b)$, there exists an $l_1$ small, if $u$ is a solution of (2.1) in $B_{2L}$ satisfying $u(0) = t$, the energy bound

$$\left(2L\right)^{-n} \int_{B_{2L}} \frac{1}{2} |\nabla u|^2 + W(u) \leq M,$$

and

$$\left(2L\right)^{-n} \int_{B_{2L}} \left[1 - (\nu \cdot e_{n+1})^2\right] |\nabla u|^2 \leq l_1,$$

(5.6)

then $\{u = t\} \cap B_L \cap \Pi^{-1}(0) = \{0\}$.

**Proof.** In contrast to [15, Proposition 5.6], here we do not assume the smallness of the discrepancy quantity

$$\left(\frac{3L}{2}\right)^{-n} \int_{B_{3L}} \left|W(u) - \frac{1}{2} |\nabla u|^2\right|.$$

However, this can be made arbitrarily small if we have chosen $l_1$ sufficiently small in (5.6). This claim can be proved by contradiction using a compactness argument. With this we can proceed as in [15] to prove that $g^{-1} \circ u$ is close to $x_{n+1} + t$ in $C^1(B_L)$ for some $t \in \mathbb{R}$. $\square$

The above results only give a clear picture of $\{u_\varepsilon = t\}$ at $O(\varepsilon)$ scale. Now since we also assume that $u_\varepsilon$ has a kind of unit density property, we can further claim that

**Lemma 5.5.** Given $b \in (0, 1)$, for every $t \in (-1 + b, 1 - b)$ and $x \in \Pi(A_\varepsilon)$, there exists exactly one point in $\Pi^{-1}(x) \cap \{u_\varepsilon = t\}$.

This lemma is a consequence of the following fact if we have chosen $R_0$ large in the following lemma and $l$ sufficiently small in the definition of $A_\varepsilon$. Note that the following result can be seen as a quantitative version of the multiplicity one property for the limit varifold $V$.

**Lemma 5.6.** For any $\delta > 0$, there exist three constants $R_0$ large and $\tau_1, l_2$ small so that the following holds. Suppose that $u_\varepsilon$ is a solution of (2.2) in $B_{R_0}$, where $\varepsilon \leq 1$, satisfying $|u_\varepsilon(0)| \leq \gamma$, the Modica inequality (2.10) and

$$R_0^{-n} \int_{B_{R_0}} \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \leq (1 + \tau_1) \sigma_0 \omega_n,$$

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and
\[ R_0^{-n} \int_{B_{R_0}} \frac{\varepsilon_k}{2} |\nabla u_k|^2 + \frac{1}{\varepsilon_k} W(u_{\varepsilon_k}) \leq (1 + \tau_k) \sigma_0 \omega_n, \tag{5.7} \]
then
\[ \{ u_\varepsilon = u_\varepsilon(0) \} \cap B_1 \subset \delta - \text{neighborhood of } \mathbb{R}^n \cap B_1. \]

**Proof.** Assume that we have a sequence of solutions \( u_k \) satisfying the assumptions in this lemma, with \( \varepsilon \) replaced by \( \varepsilon_k \in (0, 1] \), and
\[ R_0^{-n} \int_{B_{R_0}} \varepsilon_k |\nabla u_k|^2 + \frac{1}{\varepsilon_k} W(u_{\varepsilon_k}) \leq (1 + \tau_k) \sigma_0 \omega_n, \tag{5.7} \]
where \( \tau_k \to 0 \), and
\[ R_0^{-n} \int_{B_{R_0}} \varepsilon_k \sum_{i=1}^n \frac{\partial u_k}{\partial x_i}^2 \, dX \to 0, \tag{5.8} \]
but there exists \( X_k = (x_k, x_{n+1,k}) \in \{ u_\varepsilon = 0 \} \cap B_1 \) with \( |x_{n+1,k}| \geq \delta \). The constant \( R_0 \) will be determined below. Without loss of generality, assume that \( X_k \) converges to some point \( X_\infty = (x_\infty, x_{n+1,\infty}) \in B_1 \) with \( |x_{n+1,\infty}| \geq \delta \).

We divide the proof into two cases.

**Case 1.** \( \varepsilon_k \) converges to some \( \varepsilon_0 > 0 \) (after subtracting a subsequence).

By standard elliptic estimates and Arzelà-Ascoli theorem, \( u_k \) converges to a function \( u_\infty \) in \( C^2(B_{R_0-1}) \). Note that we always have \( |u_k| < 1 \) and hence \( |u_\infty| \leq 1 \) in \( B_{R_0-1} \). \( u_\infty \) is a solution of \( (2.2) \) with \( \varepsilon \) replaced by \( \varepsilon_0 \). Since \( |u_\infty(0)| \leq \gamma < 1 \), by the strong maximum principle, \( |u_\infty| < 1 \) strictly in \( B_{R_0-1} \). It can be directly checked that \( u_\infty \) is a solution of \( (2.2) \) with \( \varepsilon \) replaced by \( \varepsilon_0 \). We still have
\[ \int_{B_{R_0-1}} \frac{\varepsilon_0}{2} |\nabla u_\infty|^2 + \frac{1}{\varepsilon_0} W(u_\infty) \leq \sigma_0 \omega_n (R_0 - 1)^n. \tag{5.9} \]
Since \( \varepsilon_0 \leq 1 \), we cannot have \( u_\infty \equiv u_\infty(0) \) because otherwise
\[ \int_{B_{R_0-1}} \frac{\varepsilon_0}{2} |\nabla u_\infty|^2 + \frac{1}{\varepsilon_0} W(u_\infty) \geq \frac{1}{\varepsilon_0} W(u_\infty(0)) \omega_n (R_0 - 1)^{n+1} > \sigma_0 \omega_n (R_0 - 1)^n, \]
if we choose \( R_0 \) large to satisfy the last inequality. (It depends only the dimension \( n \) and \( W \).)

Passing to the limit in \( (5.8) \) we see
\[ R_0^{-n} \int_{B_{R_0}} \varepsilon_0 \sum_{i=1}^n \left( \frac{\partial u_\infty}{\partial x_i} \right)^2 \, dX = 0. \]
Thus \( u_\infty(X) \equiv u(x_{n+1}) \). Moreover, if \( R_0 \) is sufficiently large in (5.9),

\[
\frac{\partial u_\infty}{\partial x_{n+1}}(X) > 0, \quad \text{in } B_1.
\]

In particular, \( u_\infty \neq u_\infty(0) \) in \( B_1 \setminus \mathbb{R}^n \). However, by the convergence of \( X_k \) and uniform convergence of \( u_k \), \( u_\infty(X_\infty) = u_\infty(0) \). Because \( X_\infty \in B_1 \setminus \mathbb{R}^n \), this is a contradiction.

**Case 2.** \( \varepsilon_k \to 0 \).

Let \( V_k \) be the varifold associated to \( u_k \) as defined in Section 4. For any \( \eta \in C_0^\infty(B_{R_0}) \), let

\[
\Phi(X, S) = \eta(X) < S e_{n+1}, e_{n+1} > \in C_0^\infty(B_{R_0} \times G(n)).
\]

By (5.8),

\[
<V_k, \Phi> = \int_{B_{R_0}} \eta(X) \varepsilon_k \sum_{i=1}^{n} \left( \frac{\partial u_k}{\partial x_i} \right)^2 dX \to 0.
\]

Let \( V_\infty \) be the limit varifold of \( V_k \), which is stationary rectifiable with unit density. Then

\[
0 = <V, \Phi> = \int \eta(X) < T e_{n+1}, e_{n+1} > d\|V_\infty\|,
\]

where \( T \) is the weak tangent plane of \( V \) at \( X \). This implies that \( T = \mathbb{R}^n \|V_\infty\| \) a.e.. Thus \( V_\infty = \sigma_0 \sum_j i(T_j) \) in \( B_{R_0/2} \times (-R_0/2, R_0/2) \), where \( T_j = \mathbb{R}^n \times \{(0, t_j)\} \) for some \( j \). By our assumptions, there are at least two components, say \( T_0 \) and \( T_1 \), associated to 0 and \( X_\infty \) respectively.

However, by passing to the limit in (5.7), we obtain

\[
\|V\|(B_{R_0}) \leq \sigma_0 \omega_n R_0^n = \sigma_0 \|T_0\|(B_{R_0}).
\]

Thus we cannot have any more components other than \( T_0 \), which is a contradiction.

**Remark 5.7.** *It will be useful to write the dependence of \( \delta \) and \( l_2 \) reversely as \( \delta = c_2(l_2) \). This function is a modulus of continuity, i.e. a non-decreasing function satisfying \( \lim_{l_2 \to 0} c_2(l_2) = 0 \).*

For any \( X_0 = (x_0, x_{0,n+1}) \in A_\varepsilon \) and \( r \in (\varepsilon, 1/R_0) \), by considering

\[
\bar{u}_{\varepsilon,r}(X) = u_\varepsilon(X_0 + rX),
\]

the previous lemma implies that

\[
\{u_\varepsilon = u_\varepsilon(X_0)\} \cap (B_{1/2}(X_0) \setminus B_\varepsilon(X_0)) \subset \{ |x_{n+1} - x_{0,n+1}| \leq c_2(l) |x - x_0| \}.
\]

(5.10)
Together with Lemma 5.4, this implies that for every \( t \in (-1 + b, 1 - b) \) and \( x \in \Pi(A_\varepsilon) \), there exists at most one point in \( \Pi^{-1}(x) \cap \{u_\varepsilon = t\} \).

On the other hand, by Remark 4.5, for each \( x \in B_1 \),

\[
u_\varepsilon(x, 1) > 1 - b, \quad u_\varepsilon(x, -1) < -1 + b.
\]

Thus there must exist one \( x_{n+1} \in (-1, 1) \) such that \( u_\varepsilon(x, x_{n+1}) = t \).

In conclusion, for any \( x \in \Pi(A_\varepsilon) \), there exists a unique point \( (x, x_{n+1}) \in \Pi^{-1}(x) \cap \{u_\varepsilon = t\} \). This completes the proof of Lemma 5.5.

Note that we have assumed that \( u_\varepsilon > 0 \) in \( C_1 \cap \{x_{n+1} > h\} \) and \( u_\varepsilon < 0 \) in \( C_1 \cap \{x_{n+1} < -h\} \) for some \( h > 0 \). (This \( h \) can be made arbitrarily small as \( \varepsilon \to 0 \).) Then by continuity, for any \( r \in (\varepsilon, 1/R_0) \), (5.10) can be improved to

\[
\{x_{n+1} - x_{0, n+1} > c_2(l)|x - x_0|\} \cap (B_{1/2}(X_0) \setminus B_{r}(X_0)) \subset \{u_\varepsilon > u_\varepsilon(X_0)\},
\]

(5.11)

When \( r = \varepsilon \), we can combine this with Lemma 5.4 to obtain that,

\[
\frac{\partial u_\varepsilon}{\partial x_{n+1}}(X) \geq (1 - c_1(b, l))|\nabla u_\varepsilon(X)| \geq \frac{c(b)}{\varepsilon}, \quad \forall X \in A_\varepsilon.
\]

(5.12)

In the following we denote the Lipschitz functions by \( h_\varepsilon^t \) for \( t \in (-1 + b, 1 - b) \). Note that Lemma 5.4 implies that the definition domains of \( h_\varepsilon^t \) can be made to be a common one, \( \Pi(A_\varepsilon) \). By (5.12), \( h_\varepsilon^t \) is strictly increasing in \( t \in (-1 + b, 1 - b) \).

In the above construction, \( h_\varepsilon^t \) is only defined on a subset of \( B_1 \), but we can extend it to \( B_1 \) without increasing the Lipschitz constant by letting (see for example [16, Theorem 3.1.3])

\[
h_\varepsilon^t(x) := \inf_{y \in \Pi(A_\varepsilon)} h_\varepsilon^t(y) + c_3(b, l)|y - x|, \quad \forall x \in B_1.
\]

(5.13)

Here \( c_3(b, l) = \max\{c_0(b, l), c_2(l)\} \). This extension preserves the monotonicity of \( h_\varepsilon^t \) in \( t \).

In Section 7 and 8, \( b \) and hence \( l \) may be decreased further. Thus it is worthy to note the dependence of these Lipschitz functions on \( b \) and \( l \).

**Remark 5.8.** If we decrease \( l \), the definition domain of \( h_\varepsilon^t \) also decreases. But on the common part, these two constructions coincide. If we define two families by choosing two \( 0 < b_1 < b_2 < 1 \), these two families also coincide on \( (-1 + b_2, 1 - b_2) \).

Notation: \( D_\varepsilon = \Pi(A_\varepsilon) \).

In the following it will be useful to keep in mind that, on \( \{u_\varepsilon = t\} \cap A_\varepsilon \),

\[
\frac{\partial u_\varepsilon}{\partial x_{n+1}} = \left(\frac{\partial h_\varepsilon^t}{\partial t}\right)^{-1}, \quad \frac{\partial u_\varepsilon}{\partial x_i} = \left(\frac{\partial h_\varepsilon^t}{\partial t}\right)^{-1} \frac{\partial h_\varepsilon^t}{\partial x_i}, \quad 1 \leq i \leq n.
\]

(5.14)

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6 Estimates on $h^t_\varepsilon$

First we give an $H^1$ bound.

**Lemma 6.1.** There exists a constant $C$ independent of $\varepsilon$ such that,

$$\int_{-1+b}^{1-b} \int_{B_1} |\nabla h^t_\varepsilon|^2 dx dt \leq C(b) \delta^2_\varepsilon.$$

**Proof.** First since $H^n(B_1 \setminus D_\varepsilon) \leq C \delta^2_\varepsilon$, and $|\nabla h^t_\varepsilon| \leq c_3(b, l)$ in $B_1$ for all $t \in (-1+b, 1-b)$, we have

$$\int_{-1+b}^{1-b} \int_{B_1 \setminus D_\varepsilon} |\nabla h^t_\varepsilon|^2 dx dt \leq C \delta^2_\varepsilon \quad (6.1)$$

Next, by noting that on $A_\varepsilon$, $\varepsilon|\nabla u_\varepsilon| \geq c(b)$, we have

$$\begin{align*}
\delta^2_\varepsilon & \geq \int_{A_\varepsilon} \left[1 - (\nu_\varepsilon \cdot e_{n+1})^2\right] \varepsilon |\nabla u_\varepsilon|^2 dX \\
& = \int_{-1+b}^{1-b} \left( \int_{\{u_\varepsilon = t\}} \left(1 - (\nu_\varepsilon \cdot e_{n+1})^2\right) \varepsilon |\nabla u_\varepsilon| dA \right) dt \\
& \geq c(b) \int_{-1+b}^{1-b} \left( \int_{A_\varepsilon} \left[1 - \frac{1}{1 + |\nabla h^t_\varepsilon|^2}\right] \sqrt{1 + |\nabla h^t_\varepsilon|^2} dx \right) dt \\
& \geq c(b) \int_{-1+b}^{1-b} \left( \int_{A_\varepsilon} |\nabla h^t_\varepsilon|^2 dx \right) dt,
\end{align*}$$

if we have chosen $l$ so small that the Lipschitz constants of $h^t_\varepsilon$, $c_3(b, l) \leq 1/2$. \hfill \Box

With this lemma in hand, we first choose a $t_\varepsilon \in (-1+b, 1-b)$ such that

$$\int_{B_1} |\nabla h^{t_\varepsilon}_\varepsilon|^2 dx \leq C(b) \delta^2_\varepsilon,$$

and then take a $\lambda_\varepsilon$ so that the function defined by

$$\bar{h}_\varepsilon := \frac{1}{\delta_\varepsilon} h^{t_\varepsilon}_\varepsilon - \lambda_\varepsilon, \quad (6.2)$$

satisfies $\int_{B_1} \bar{h}_\varepsilon = 0$. Then by the Poincare inequality

$$\int_{B_1} \bar{h}_\varepsilon(x)^2 dx \leq C(n) \int_{B_1} |\nabla h_\varepsilon(x)|^2 dx \leq C(b). \quad (6.3)$$
Thus we can assume that (after passing to a subsequence of $\varepsilon \to 0$), $\bar{h}_\varepsilon$ converges weakly to an $\bar{h}$ in $H^1(B_1)$ and strongly in $L^2(B_1)$.

Let

$$\bar{h}_\varepsilon^t := \frac{1}{\delta_\varepsilon} h_\varepsilon^t - \lambda_\varepsilon.$$ 

In $D_\varepsilon$,

$$0 \leq \frac{\partial h_\varepsilon^t}{\partial t} = \left( \frac{\partial u_\varepsilon}{\partial x_{n+1}} \right)^{-1} \leq C(b)\varepsilon;$$

with a constant $C(b)$ depending only on $b$. Hence

$$0 \leq h_\varepsilon^{1-b} - h_\varepsilon^{-1+b} \leq C(b)\varepsilon, \quad \text{in } D_\varepsilon. \quad (6.5)$$

This also holds for $x \in B_1 \setminus D_\varepsilon$ by (5.13).

Then for any $-1 + b < t_1 < t_2 < 1 - b$,

$$\int_{B_1} (h_\varepsilon^{t_1} - h_\varepsilon^{t_2})^2 \leq C(b)\varepsilon^2. \quad (6.6)$$

Because $\delta_\varepsilon \gg \varepsilon$, for any sequence of $\tilde{t}_\varepsilon \in (-1 + b, 1 - b)$, $\bar{h}_\varepsilon^{\tilde{t}_\varepsilon}$ still converges to $\bar{h}$ in $L^2(B_1)$.

Since $\delta_\varepsilon^{-1} \nabla h_\varepsilon^t$ are uniformly bounded in $L^2(B_1 \times (-1 + b, 1 - b), \mathbb{R}^n)$, we can assume that it converges weakly to some limit in $L^2(B_1 \times (-1 + b, 1 - b), \mathbb{R}^n)$. By the above discussion, this limit must be $\nabla \bar{h}$.

By Remark 5.8, $\bar{h}$ is independent of the choice of $b$. Hence we have a universal constant $C$, which is independent of $b$ and $l$, such that

$$\int_{B_1} |\nabla \bar{h}|^2 + \bar{h}^2 \leq C. \quad (6.7)$$

Concerning the size of $\lambda_\varepsilon$, we have

**Lemma 6.2.** $\lim_{\varepsilon \to 0} |\lambda_\varepsilon \delta_\varepsilon| = 0$.

**Proof.** Note that

$$\lambda_\varepsilon \delta_\varepsilon = \int_{B_1} h_\varepsilon^t. \quad (6.8)$$

By Proposition 4.4,

$$\lim_{\varepsilon \to 0} \sup_{C_1 \cap \{|u_\varepsilon| \leq 1-b\}} |x_{n+1}| = 0.$$

Thus

$$\lim_{\varepsilon \to 0} \sup_{t \in (-1+b,1-b)} \sup_{x \in D_\varepsilon} |h_\varepsilon^t(x)| = 0. \quad (6.9)$$
For any \( x \in B_1 \setminus D_\varepsilon \), by Lemma 5.2,
\[
\text{dist}(x, D_\varepsilon) \leq C(n) l^{-\frac{1}{2}} \delta _\varepsilon ^{\frac{3}{2}}.
\]
Thus since the Lipschitz constant of \( h_\varepsilon ^t \) is smaller than \( c_3(b, l) \leq 1, \)
\[
\sup_{t \in (-1 + b, 1-b)} \sup_{x \in B_1 \setminus D_\varepsilon} |h_\varepsilon ^t(x)| \leq \sup_{t \in (-1 + b, 1-b)} \sup_{x \in D_\varepsilon} |h_\varepsilon ^t(x)| + C(n, l) \delta _\varepsilon ^{\frac{3}{2}}.
\]
Combining this with (6.9) we see
\[
\lim_{\varepsilon \to 0} \sup_{t \in (-1 + b, 1-b)} \sup_{x \in B_1} |h_\varepsilon ^t(x)| = 0.
\]
Substituting this into (6.8) we can finish the proof. \( \square \)

Next we establish an \( L^2 \) bound. This can be viewed as a Poincare inequality on the varifold \( V_\varepsilon \) (while the Caccioppoli type inequality is a reverse Poincare inequality).

**Lemma 6.3.** There exists a universal constant \( C \) such that
\[
\int _{C_{3/4}} (x_{n+1} - \lambda _\varepsilon \delta _\varepsilon )^2 \varepsilon |\nabla u_\varepsilon |^2 \leq C \delta _\varepsilon ^{2}.
\]  
(6.10)

**Proof.** We divide the proof into two steps. In the following we shall fix two numbers \( 0 < b_2 < b_1 < 1 \) so that \( W'' \geq \kappa \) in \((-1, -1 + b_1) \cup (1 - b_1, 1)\).

**Step 1.** Here we give an estimate in the part \( \{|u_\varepsilon| < 1 - b_2\} \cap C_1\), i.e.

\[
\int _{\{ |u_\varepsilon| < 1 - b_2 \} \cap C_1} (x_{n+1} - \lambda _\varepsilon \delta _\varepsilon )^2 \varepsilon |\nabla u_\varepsilon |^2 \leq C \delta _\varepsilon ^{2}.
\]  
(6.11)

First, by (6.2) and (6.6),
\[
\int _{-1+b_2}^{1-b_2} \int _{B_1} (h_\varepsilon ^t - \lambda _\varepsilon \delta _\varepsilon )^2 dxdt \leq C \delta _\varepsilon ^{2}.
\]  
(6.12)

Then by a change of variables, the gradient bound (2.4) and the Lipschitz bound on \( h_\varepsilon ^t \), we obtain
\[
\int _{A_\varepsilon} (x_{n+1} - \lambda _\varepsilon \delta _\varepsilon )^2 |\nabla u_\varepsilon |^2 = \int _{-1+b_2}^{1-b_2} \int _{D_\varepsilon} (h_\varepsilon ^t - \lambda _\varepsilon \delta _\varepsilon )^2 (1 + |\nabla h_\varepsilon ^t|^2) \varepsilon \frac{\partial u_\varepsilon}{\partial x_{n+1}} dxdt \leq C \int _{-1+b_2}^{1-b_2} \int _{D_\varepsilon} (h_\varepsilon ^t - \lambda _\varepsilon \delta _\varepsilon )^2 dxdt.
\]  
(6.13)
\[ \leq C\delta^2. \]

In \( B_\varepsilon \), by Lemma 5.1 and Lemma 6.2,
\[ \int_{B_\varepsilon} (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon)^2 \varepsilon |\nabla u_\varepsilon|^2 \leq C\mu_\varepsilon (B_\varepsilon) \leq C\delta^2. \tag{6.14} \]

Combining (6.13) and (6.14) we get (6.11).

**Step 2.** We claim that in the part \{\( |u_\varepsilon| > 1 - b_2 \) \( \cap \mathcal{C}_{3/4} \),
\[ \int_{\{|u_\varepsilon|>1-b_2\}\cap\mathcal{C}_{3/4}} (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon)^2 \varepsilon |\nabla u_\varepsilon|^2 \leq C\delta^2. \tag{6.15} \]

Choose a function \( \zeta \in C^\infty(\mathbb{R}) \), satisfying
\[
\begin{align*}
\zeta(t) &\equiv 1, \text{ in } \{ |t| > 1 - b_1 \}, \\
\zeta(t) &\equiv 0, \text{ in } \{ |t| < 1 - b_2 \}, \\
|\zeta'| + |\zeta''| &\leq C \text{ in } \{ 1 - b_1 \leq |t| \leq 1 - b_2 \}.
\end{align*}
\]

We also fix an function \( \eta \in C^\infty_0(B_1 \times \{|x_{n+1}| < 4/3\}) \) so that \( 0 \leq \eta \leq 1, \eta \equiv 1 \) in \( \mathcal{C}_{3/4} \).

It can be checked directly that
\[ \Delta (\varepsilon |\nabla u_\varepsilon|^2) \geq \frac{\kappa}{\varepsilon^2} (\varepsilon |\nabla u_\varepsilon|^2), \text{ in } \{ |u_\varepsilon| > 1 - b_1 \}. \tag{6.16} \]

Multiplying this equation by \( (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon)^2 \eta \zeta(u_\varepsilon) \) and integrating by parts, we obtain
\[
\begin{align*}
\int_{B_2} & (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon)^2 \eta \zeta(u_\varepsilon) \varepsilon |\nabla u_\varepsilon|^2 \\
&\leq \frac{\varepsilon^2}{\kappa} \int_{B_2} \Delta \left[ (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon)^2 \eta \right] \zeta(u_\varepsilon) \varepsilon |\nabla u_\varepsilon|^2 \\
&+ \frac{\varepsilon^2}{\kappa} \int_{B_2} 4 (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_{n+1}} \eta \zeta'(u_\varepsilon) \varepsilon |\nabla u_\varepsilon|^2 \\
&+ \frac{\varepsilon^2}{\kappa} \int_{B_2} 2 (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon)^2 (\nabla \eta \cdot \nabla u_\varepsilon) \zeta'(u_\varepsilon) \varepsilon |\nabla u_\varepsilon|^2 \\
&+ \frac{\varepsilon^2}{\kappa} \int_{B_2} (\zeta''(u_\varepsilon) |\nabla u_\varepsilon|^2 + \zeta'(u_\varepsilon) \Delta u_\varepsilon) (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon)^2 \eta \varepsilon |\nabla u_\varepsilon|^2.
\end{align*}
\]

In the right hand side, the first term is bounded by
\[ \frac{\varepsilon^2}{\kappa} \int_{B_2} \Delta \left[ (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon)^2 \eta \right] \zeta(u_\varepsilon) \varepsilon |\nabla u_\varepsilon|^2 \leq C\varepsilon^2, \tag{6.18} \]
because both $\Delta \left[(x_{n+1} - \lambda \varepsilon \delta \varepsilon)^2 \eta \right]$ and $\zeta(u\varepsilon)$ are bounded by a universal constant.

Note that the supports of $\zeta'(u\varepsilon)$ and $\zeta''(u\varepsilon)$ belong to $\{|u\varepsilon| < 1 - b\}$. By the Cauchy inequality, the second term is bounded by

$$\frac{\varepsilon^2}{\kappa} \int_{B_2} 4 \left(x_{n+1} - \lambda \varepsilon \delta \varepsilon \right) \frac{\partial u\varepsilon}{\partial x_{n+1}} \eta \zeta'(u\varepsilon) \varepsilon |\nabla u\varepsilon|^2$$

(6.19)

$$\leq C \varepsilon \left[ \int_{\{|u\varepsilon| < 1 - b\} \cap B_2} \left(x_{n+1} - \lambda \varepsilon \delta \varepsilon \right)^2 \eta^2 \varepsilon |\nabla u\varepsilon|^2 \right]^{\frac{1}{2}} \left[ \int_{\{|u\varepsilon| < 1 - b\} \cap B_2} \left(\varepsilon \frac{\partial u\varepsilon}{\partial x_{n+1}}\right)^2 \varepsilon |\nabla u\varepsilon|^2 \right]^{\frac{1}{2}}$$

$$\leq C \varepsilon \left( \delta \varepsilon + e^{-\frac{\varepsilon}{2}} \right),$$

by Proposition 4.4, (6.11) and the fact that $\varepsilon |\frac{\partial u\varepsilon}{\partial x_{n+1}}| \leq C(n)$.

Similarly, the third term

$$\frac{\varepsilon^2}{\kappa} \int_{B_2} (x_{n+1} - \lambda \varepsilon \delta \varepsilon)^2 (\nabla \eta \cdot \nabla u\varepsilon) \zeta'(u\varepsilon) \varepsilon |\nabla u\varepsilon|^2$$

$$\leq C \varepsilon \left[ \int_{\{|u\varepsilon| < 1 - b\} \cap B_2} (x_{n+1} - \lambda \varepsilon \delta \varepsilon)^2 \eta^2 \varepsilon |\nabla u\varepsilon|^2 \right]^{\frac{1}{2}} \left[ \int_{\{|u\varepsilon| < 1 - b\} \cap B_2} \left(\varepsilon \frac{\partial u\varepsilon}{\partial x_{n+1}}\right)^2 \varepsilon |\nabla u\varepsilon|^2 \right]^{\frac{1}{2}}$$

(6.20)

$$\leq C \varepsilon \left( \delta \varepsilon + e^{-\frac{\varepsilon}{2}} \right).$$

Finally, in the last term, by employing (2.4), we have

$$\frac{\varepsilon^2}{\kappa} \int_{B_2} (\zeta''(u\varepsilon)|\nabla u\varepsilon|^2 + \zeta'(u\varepsilon)\Delta u\varepsilon) (x_{n+1} - \lambda \varepsilon \delta \varepsilon)^2 \eta \varepsilon |\nabla u\varepsilon|^2$$

$$\leq C \int_{\{|u\varepsilon| < 1 - b\} \cap B_2} (x_{n+1} - \lambda \varepsilon \delta \varepsilon)^2 \eta \varepsilon |\nabla u\varepsilon|^2$$

(6.21)

$$\leq C \delta \varepsilon^2.$$

Substituting (6.18)-(6.21) into (6.17), and noting the fact that $\delta \varepsilon \gg \varepsilon$, we obtain (6.15). Combining (6.11) and (6.15) we finish the proof. \qed

Once we have this bound, we can further show

**Corollary 6.4.** For any $\sigma > 0$, there exists a $b > 0$ such that

$$\int_{\{|u\varepsilon| > 1 - b\} \cap C_{3/4}} (x_{n+1} - \lambda \varepsilon \delta \varepsilon)^2 \varepsilon |\nabla u\varepsilon|^2 \leq \sigma \varepsilon^2 + \varepsilon^2.$$ 

**Proof.** We only need to note that, in deriving (6.21), instead of using the bound (2.4), we can use

$$\varepsilon^2 \left(|\nabla u\varepsilon|^2 + |\Delta u\varepsilon| \right) \leq 2W(u\varepsilon) + |W'(u\varepsilon)|,$$

which is small in $\{1 - 2b < |u\varepsilon| < 1 - b\}$, if we have chosen $b$ small. \qed

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Finally we give an estimate on $\frac{\partial h}{\partial t}$.

**Lemma 6.5.** There exists a set $E_\varepsilon \subset D_\varepsilon$ with $\mathcal{H}^n(D_\varepsilon \setminus E_\varepsilon) \leq C\delta_\varepsilon$, such that, for any $X_\varepsilon \in \Pi^{-1}(E_\varepsilon) \cap A_\varepsilon$ with $u_\varepsilon(X_\varepsilon) = t$,

$$\lim_{\varepsilon \to 0} \varepsilon \frac{\partial h^\varepsilon}{\partial t}(X_\varepsilon) = g'(g^{-1}(t)).$$

**Proof.** Let $E_\varepsilon = D_\varepsilon \cap \{Mf_\varepsilon < \delta_\varepsilon\}$. By (5.1),

$$\mathcal{H}^n(D_\varepsilon \setminus E_\varepsilon) \leq \mathcal{H}^n(B_1 \setminus \{Mf_\varepsilon > \delta_\varepsilon\}) \leq C\delta_\varepsilon \to 0.$$

For any $X_\varepsilon \in \Pi^{-1}(E_\varepsilon) \cap A_\varepsilon$, consider

$$v_\varepsilon(X) := u_\varepsilon(X_\varepsilon + \varepsilon X), \quad \text{for } X \in B_{\varepsilon^{-1/2}}.$$

$v_\varepsilon$ is a solution of (2.1). By definition, $v_\varepsilon(0) = u_\varepsilon(X_\varepsilon) = t \in (-1+b, 1-b)$ because $X_\varepsilon \in A_\varepsilon$. As usual we can assume that $v_\varepsilon$ converges to a $v_\infty$ in $C^2_{\text{loc}}(\mathbb{R}^{n+1})$, which is a solution of (2.1) on $\mathbb{R}^{n+1}$.

By the definition of Hardy-Littlewood maximal function and our choice of $E_\varepsilon$,

$$\sup_{0 < r < \varepsilon^{-1/2}} r^{-n} \int_{B_r} \sum_{i=1}^n \left( \frac{\partial v_\varepsilon}{\partial x_i} \right)^2 \leq \delta_\varepsilon \to 0.$$

After passing to the limit, we see $v_\infty$ depends only on $x_{n+1}$. Then by (4.1), we have the energy bound

$$\int_{B_r} \frac{1}{2} |\nabla v_\infty|^2 + W(v_\infty) \leq 8^n \sigma_0 \omega_n r^n, \quad \forall \ r > 1.$$

From this we deduce that $v_\infty(X) \equiv g(x_{n+1} + g^{-1}(t))$.

This then implies that

$$\varepsilon \frac{\partial u_\varepsilon}{\partial x_{n+1}}(X_\varepsilon) = \frac{\partial v_\varepsilon}{\partial x_{n+1}}(0) \to \frac{\partial v_\infty}{\partial x_{n+1}}(0) = g'(g^{-1}(t)).$$

The claim then follows from (5.14). \qed

In fact, this proof shows the convergence is uniform.
7 The blow up limit

In this section we prove that

**Proposition 7.1.** \( \bar{h} \) is harmonic in \( B_1 \).

Fix an \( \eta \in C_0^\infty((-1,1)) \), such that \( 0 \leq \eta \leq 1 \), \( \eta \equiv 1 \) in \( (-1/2,1/2) \) and \( |\eta'| \leq 2 \). For any \( \varphi \in C_0^\infty(B_1) \), let \( X(x,x_{n+1}) = \varphi(x)\eta(x_{n+1})e_{n+1} \), which is a smooth vector field with compact support in \( C_1 \).

We want to substitute this \( X \) into the stationary condition (2.5). Note that

\[
DX(x,x_{n+1}) = \eta(x_{n+1})\nabla \varphi(x) \otimes e_{n+1} + \varphi(x)\eta'(x_{n+1})e_{n+1} \otimes e_{n+1},
\]

\[
\text{div}X(x,x_{n+1}) = \varphi(x)\eta'(x_{n+1}).
\]

Since \( \text{div}X \) vanishes in \( B_1 \times (-1/2,1/2) \), by Proposition 4.4,

\[
\int_{C_1} \left[ \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right] \text{div}X = O(e^{-\frac{\varepsilon}{2}}). \tag{7.1}
\]

Similarly,

\[
\int_{C_1} \varphi(x)\eta'(x_{n+1}) \varepsilon \left( \frac{\partial u_\varepsilon}{\partial x_{n+1}} \right)^2 = O(e^{-\frac{\varepsilon}{2}}). \tag{7.2}
\]

Thus from the stationary condition we obtain

\[
\int_{C_1} \varepsilon \eta \left( \sum_{i=1}^n \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \right) \frac{\partial u_\varepsilon}{\partial x_{n+1}} = O(e^{-\frac{\varepsilon}{2}}) = o(\delta_\varepsilon), \tag{7.3}
\]

thanks to our assumption (3.7).

First note that

\[
\int_{C_1 \cap \{|u_\varepsilon| \geq 1-b\}} \varepsilon \eta \left( \sum_{i=1}^n \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \right) \frac{\partial u_\varepsilon}{\partial x_{n+1}} dx dx_{n+1}
\]

\[
\leq C(n) \sup_{B_1} |\nabla \phi| \left[ \int_{C_1 \cap \{|u_\varepsilon| \geq 1-b\}} \varepsilon \left( \sum_{i=1}^n \frac{\partial u_\varepsilon}{\partial x_i} \right)^2 \right]^{1/2} \left[ \int_{C_1 \cap \{|u_\varepsilon| \geq 1-b\}} \varepsilon \left( \frac{\partial u_\varepsilon}{\partial x_{n+1}} \right) \right]^{1/2} \tag{7.4}
\]

\[
\leq C(\varphi) o_b(1) \delta_\varepsilon,
\]

where \( o_b(1) \) converges to 0 as \( b \to 0 \) by Lemma B.3.
Next in $B_\varepsilon$,

$$
\int_{B_\varepsilon} \varepsilon \eta \left( \sum_{i=1}^{n} \frac{\partial u_{\varepsilon}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \right) \frac{\partial u_{\varepsilon}}{\partial x_{n+1}} dx_{n+1}
\leq C(n) \sup_{B_1} |\nabla \phi| \left[ \int_{B_\varepsilon} \varepsilon \left( \sum_{i=1}^{n} \frac{\partial u_{\varepsilon}}{\partial x_i} \right)^2 \right]^{1/2} \left[ \int_{B_\varepsilon} \varepsilon \left( \frac{\partial u_{\varepsilon}}{\partial x_{n+1}} \right)^2 \right]^{1/2}
\leq C(\varphi) \delta_{\varepsilon} \mu_{\varepsilon}(B_\varepsilon)^{1/2}
\leq C(\varphi) \delta_{\varepsilon}^2.
$$

(7.5)

Substituting these into (7.3) and noting that $\eta \equiv 1$ on $A_\varepsilon$ because $A_\varepsilon \subset B_1 \times \{ |x_{n+1}| < 1/2 \}$, we see

$$
\int_{A_\varepsilon} \varepsilon \varphi(x) \left( \sum_{i=1}^{n} \frac{\partial u_{\varepsilon}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \right) \frac{\partial u_{\varepsilon}}{\partial x_{n+1}} dx_{n+1} = o(\delta_{\varepsilon}) + o_b(1) \delta_{\varepsilon}.
$$

(7.6)

In $A_\varepsilon$, by using the transformation $(x, x_{n+1}) = (x, h^t_{\varepsilon}(x))$ and (5.14), this integral can be transformed into

$$
\int_{-1+b}^{1-b} \int_{D_\varepsilon \setminus E_\varepsilon} \varepsilon \left( \frac{\partial h^t_{\varepsilon}}{\partial x_{n+1}} \right)^{-1} \sum_{i=1}^{n} \frac{\partial h^t_{\varepsilon}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx dt = o(\delta_{\varepsilon}) + o_b(1) \delta_{\varepsilon}.
$$

(7.7)

We need to further divide $A_\varepsilon$ into two parts, using the set $E_\varepsilon$ introduced in Lemma 6.5.

In the first part $D_\varepsilon \setminus E_\varepsilon$,

$$
\int_{-1+b}^{1-b} \int_{D_\varepsilon \setminus E_\varepsilon} \varepsilon \left( \frac{\partial h^t_{\varepsilon}}{\partial x_{n+1}} \right)^{-1} \sum_{i=1}^{n} \frac{\partial h^t_{\varepsilon}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx dt
\leq \left( \sup_{A_\varepsilon} \left| \varepsilon \left( \frac{\partial h^t_{\varepsilon}}{\partial x_{n+1}} \right)^{-1} \right| \right) \left( \sup_{B_1} |\nabla \varphi| \right) \left[ \int_{-1+b}^{1-b} \int_{D_\varepsilon \setminus E_\varepsilon} \sum_{i=1}^{n} \left( \frac{\partial h^t_{\varepsilon}}{\partial x_i} \right)^2 dx dt \right]^{1/2} \mathcal{H}^{n}(D_\varepsilon \setminus E_\varepsilon)^{1/2}
\leq C \delta_{\varepsilon}^3 = o(\delta_{\varepsilon}),
$$

by (2.4), (5.14), Lemma 6.1 and Lemma 6.5.

In $A_\varepsilon \cap \Pi^{-1}(E_\varepsilon)$,

$$
\int_{-1+b}^{1-b} \int_{E_\varepsilon} \varepsilon \left( \frac{\partial h^t_{\varepsilon}}{\partial x_{n+1}} \right)^{-1} \sum_{i=1}^{n} \frac{\partial h^t_{\varepsilon}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx dt = \int_{-1+b}^{1-b} \int_{E_\varepsilon} g'(g^{-1}(t)) \sum_{i=1}^{n} \frac{\partial h^t_{\varepsilon}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx dt + o(\delta_{\varepsilon}),
$$

where in the last equality we have used

$$
\sup_{E_\varepsilon} \left| \varepsilon \left( \frac{\partial h^t_{\varepsilon}}{\partial x_{n+1}} \right)^{-1} - g'(g^{-1}(t)) \right| = o_{\varepsilon}(1) \rightarrow 0,
$$

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and the bound from Lemma 6.1,
\[
\int_{-1+b}^{1-b} \int_{E_\varepsilon} \sum_{i=1}^n \frac{\partial h^t_{\varepsilon}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx \, dt \leq \left[ \int_{-1+b}^{1-b} \int_{B_1} |\nabla h^t_{\varepsilon}|^2 \, dx \, dt \right]^{1/2} \left[ \int_{-1+b}^{1-b} \int_{B_1} |\nabla \varphi|^2 \, dx \, dt \right]^{1/2} \leq C \delta_{\varepsilon}.
\]

By Cauchy inequality we also have
\[
\int_{-1+b}^{1-b} \int_{B_1 \setminus E_\varepsilon} g'(g^{-1}(t)) \sum_{i=1}^n \frac{\partial h^t_{\varepsilon}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx \, dt \leq \sup_{B_1} |\nabla \varphi| \left[ \int_{-1+b}^{1-b} \int_{B_1} |\nabla h^t_{\varepsilon}|^2 \, dx \, dt \right]^{1/2} H^n(B_1 \setminus E_\varepsilon)^{1/2} \leq C \delta_{\varepsilon}^{3/2} = o(\delta_{\varepsilon}),
\]
by Lemma 6.1, (5.1) and Lemma 6.5.

In conclusion, we get
\[
\int_{-1+b}^{1-b} \int_{B_1} g'(g^{-1}(t)) \sum_{i=1}^n \frac{\partial h^t_{\varepsilon}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx \, dt = o_b(1) \delta_{\varepsilon} + o(\delta_{\varepsilon}).
\]

By the weak convergence of \(\delta_{\varepsilon}^{-1} \nabla h^t_{\varepsilon}\) to \(\nabla \bar{h}\) in \(L^2(B_1 \times (-1 + b, 1 - b))\), we can let \(\varepsilon \to 0\) to obtain
\[
\left[ \int_{-1+b}^{1-b} g'(g^{-1}(t)) dt \right] \left[ \int_{B_1} \sum_{i=1}^n \frac{\partial \bar{h}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx \right] = o_b(1). \tag{7.8}
\]

For \(b \in (0, 1/2)\),
\[
\int_{-1+b}^{1-b} g'(g^{-1}(t)) dt = \int_{g^{-1}(1-b)}^{g^{-1}(1+b)} g'(s)^2 \, ds \geq \frac{\sigma_0}{1000}.
\]

At the first step, we can choose a smaller \(\tilde{b}\) and get another family \(\tilde{h}^t_{\varepsilon}\) for \(t \in (-1 + \tilde{b}, 1 - \tilde{b})\). Assume that its limit is \(\tilde{h}\). By Remark 5.8, \(\tilde{h}^t_{\varepsilon} = h^t_{\varepsilon}\) for \(t \in (-1 + b, 1 - b)\). Then by (6.6), \(\tilde{h} = \tilde{h}\). In other words, the limit \(\tilde{h}\) does not depend on \(b\).

After taking \(b \to 0\) in (7.8), we get
\[
\int_{B_1} \sum_{i=1}^n \frac{\partial \tilde{h}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx = 0.
\]

Since \(\tilde{h} \in H^1(B_1)\) and \(\varphi \in C^\infty_0(B_1)\) can be arbitrary, standard harmonic function theory implies that \(\tilde{h}\) is harmonic in \(B_1\).
8 Proof of the tilt-excess decay

Recall that $\bar{h}$ is a harmonic function satisfying (see (6.7))

$$\int_{B_1} |\nabla \bar{h}|^2 + \bar{h}^2 \leq C,$$

standard interior gradient estimates imply that

$$|\nabla \bar{h}(0)| \leq C, \quad \sup_{B_r} |\nabla \bar{h} - \nabla \bar{h}(0)| \leq Cr, \quad \forall r \in (0, 1/2). \quad (8.1)$$

Thus

$$\int_{B_r} |\nabla \bar{h} - \nabla \bar{h}(0)|^2 \leq Cr^{n+2}, \quad \forall r \in (0, 1/2). \quad (8.2)$$

We first consider the special case when $\nabla \bar{h}(0) = 0$, and then reduce the general case to this one.

8.1 The case $\nabla \bar{h}(0) = 0$

Take a $\psi \in C^\infty_0((-1, 1))$ satisfying $0 \leq \psi \leq 1$, $\psi \equiv 1$ in $(-1/2, 1/2)$, $|\psi'| \leq 3$. For any $r \in (0, 1/4)$, choose an $\phi \in C^\infty_0(B_{2r})$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ in $B_r$. In the stationary condition (2.5), take the vector field

$$Y = \phi(x)^2 \psi(x_{n+1})^2 (x_{n+1} - \lambda_\varepsilon \delta_{\varepsilon}) e_{n+1},$$

where $\lambda_\varepsilon$ is the constant appearing in (6.2).

By this choice of $Y$ we get

$$0 = \int_{C_1} \left[ \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right] \left[ \phi^2 \psi^2 + 2 \phi^2 \psi \psi' (x_{n+1} - \lambda_\varepsilon \delta_{\varepsilon}) \right]$$

$$-\phi^2 \psi^2 \nu_{x_{n+1}} \varepsilon |\nabla u_\varepsilon|^2 - (x_{n+1} - \lambda_\varepsilon \delta_{\varepsilon}) \sum_{i=1}^n 2\phi \psi^2 \frac{\partial \phi}{\partial x_i} \nu_{x_{n+1}} \varepsilon |\nabla u_\varepsilon|^2 \quad (8.3)$$

As in the proof of Caccioppoli inequality (4.6), those terms containing $\psi'$ can be bounded by

$$\int_{C_1} \left[ \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right] 2\phi^2 \psi \psi' (x_{n+1} - \lambda_\varepsilon \delta_{\varepsilon}) - (x_{n+1} - \lambda_\varepsilon \delta_{\varepsilon}) 2\phi^2 \psi \psi' \nu_{x_{n+1}} \varepsilon |\nabla u_\varepsilon|^2 = O(e^{-\frac{1}{C\varepsilon}}).$$
By the Modica inequality (2.10), (8.3) can be transformed to
\[
\int_{C_1} \phi^2 \psi^2 [1 - (\nu_n \cdot e_{n+1})^2] \varepsilon |\nabla u_\varepsilon|^2 \\
\leq \int_{C_1} -2\phi \psi^2 (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon) \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \nu_{\varepsilon,i} \nu_{\varepsilon,n+1} \varepsilon |\nabla u_\varepsilon|^2 + O(e^{-\frac{1}{C_\varepsilon}}).
\]

Since \(1 - \psi^2 \equiv 0\) in \(|x_{n+1}| \leq 1/2\), as before we have
\[
\int_{C_1} \phi^2 (1 - \psi^2) [1 - (\nu_n \cdot e_{n+1})^2] \varepsilon |\nabla u_\varepsilon|^2 = O(e^{-\frac{1}{C_\varepsilon}}).
\]

Thus we obtain
\[
\int_{C_1} \phi^2 [1 - (\nu_n \cdot e_{n+1})^2] \varepsilon |\nabla u_\varepsilon|^2 \leq \int_{C_1} -2\phi \psi^2 (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon) \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \nu_{\varepsilon,i} \nu_{\varepsilon,n+1} \varepsilon |\nabla u_\varepsilon|^2 + O(e^{-\frac{1}{C_\varepsilon}}) .
\] (8.4)

Now we consider the convergence of the integral in the right-hand side of (8.4).

**Lemma 8.1.**
\[
\lim_{\varepsilon \to 0} \delta_\varepsilon^{-2} \int_{C_1} 2\phi \psi^2 (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon) \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \nu_{\varepsilon,i} \nu_{\varepsilon,n+1} \varepsilon |\nabla u_\varepsilon|^2 \\
= \int_{-1}^{1} g'(g^{-1}(t))dt \left[ \int_{B_1} \phi^2 |\nabla \bar{h}(x)|^2 dx \right].
\]

**Proof.** In \(|u_\varepsilon| \geq 1 - b\),
\[
\left| \int_{|u_\varepsilon| \geq 1 - b} 2\phi \psi^2 (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon) \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \nu_{\varepsilon,i} \nu_{\varepsilon,n+1} \varepsilon |\nabla u_\varepsilon|^2 \right| \\
\leq C \sup_{B_1} \left| \phi \psi^2 \nabla \phi \right| \left[ \int_{|u_\varepsilon| \geq 1 - b} \sum_{i=1}^{n} \nu_{\varepsilon,i}^2 \varepsilon |\nabla u_\varepsilon|^2 \right]^{\frac{1}{2}} \left[ \int_{|u_\varepsilon| \geq 1 - b} (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon)^2 \varepsilon |\nabla u_\varepsilon|^2 \right]^{\frac{1}{2}} \\
\leq o_b(1) \delta_\varepsilon^2,
\]
by the definition of \(\delta_\varepsilon\) and Corollary 6.4.
\[ \leq C \sup_{\{u_\varepsilon \leq 1 - b\}} \left| x_{n+1} - \lambda_\varepsilon \delta_\varepsilon \right| \sup_{B_1} \left| \phi \psi^2 \nabla \phi \right| \left[ \int_{B_\varepsilon} \sum_{i=1}^{n} \nu_{\varepsilon,i} \varepsilon \left| \nabla u_\varepsilon \right|^2 \right]^{1/2} \left[ \int_{B_\varepsilon} \varepsilon \left| \nabla u_\varepsilon \right|^2 \right]^{1/2} \leq o(\delta_\varepsilon^2), \]

where we have used the definition of excess, Lemma 5.1 and the fact that \( \{u_\varepsilon \leq 1 - b\} \) belongs to a small neighborhood of \( \{x_{n+1} = 0\} \), and thus by combining this fact with Lemma 6.2 we obtain

\[ \lim_{\varepsilon \to 0} \sup_{\{u_\varepsilon \leq 1 - b\}} \left| x_{n+1} - \lambda_\varepsilon \delta_\varepsilon \right| = 0. \] (8.5)

In \( A_\varepsilon \), first due to \( A_\varepsilon \subset \{\left| x_{n+1} \right| \leq 1/2\} \), \( \psi(x_{n+1}) \equiv 1 \) in \( A_\varepsilon \). Thus if writing in the \((x, t)\) coordinates, we have

\[ \int_{A_\varepsilon} 2\phi \psi^2 (x_{n+1} - \lambda_\varepsilon \delta_\varepsilon) \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \nu_{\varepsilon,i} \varepsilon \left| \nabla u_\varepsilon \right|^2 = \int_{-1+b}^{1-b} \int_{D_\varepsilon} 2\phi \left( \nabla \phi \cdot \nabla h_\varepsilon^t \right) \left( h_\varepsilon^t - \lambda_\varepsilon \delta_\varepsilon \right) \varepsilon \left( \frac{\partial h_\varepsilon^t}{\partial t} \right)^{-1} dxdt. \] (8.6)

Note that in \( A_\varepsilon \),

\[ \varepsilon \left( \frac{\partial h_\varepsilon^t}{\partial t} \right)^{-1} = \varepsilon \frac{\partial u_\varepsilon}{\partial x_{n+1}} \leq C(n). \]

Let \( E_\varepsilon \) be the set defined as in Lemma 6.5. By the Cauchy inequality, Lemma 6.1, (6.2), (6.6) and Sobolev inequality,

\[ \int_{-1+b}^{1-b} \int_{D_\varepsilon \setminus E_\varepsilon} 2\phi \left( \nabla \phi \cdot \nabla h_\varepsilon^t \right) \left( h_\varepsilon^t - \lambda_\varepsilon \delta_\varepsilon \right) \varepsilon \left( \frac{\partial h_\varepsilon^t}{\partial t} \right)^{-1} dxdt \leq C \left[ \int_{-1+b}^{1-b} \int_{D_\varepsilon \setminus E_\varepsilon} \left( \nabla \phi \cdot \nabla h_\varepsilon^t \right)^2 dxdt \right]^{1/2} \left[ \int_{-1+b}^{1-b} \int_{D_\varepsilon \setminus E_\varepsilon} \left( h_\varepsilon^t - \lambda_\varepsilon \delta_\varepsilon \right)^2 \phi^2 dxdt \right]^{1/2} \leq C \delta_\varepsilon H^n(D_\varepsilon \setminus E_\varepsilon)^{\frac{p-1}{4p}} \left[ \int_{-1+b}^{1-b} \int_{B_1} \lambda_{\varepsilon,\delta_\varepsilon}^2 \phi \right]^{1/2} dt \leq C H^n(D_\varepsilon \setminus E_\varepsilon)^{\frac{p-1}{4p}} \delta_\varepsilon^2 = o(\delta_\varepsilon^2). \]

In the above \( p > 1 \) is a constant depending only on the dimension \( n \). This estimate gives

\[ \int_{-1+b}^{1-b} \int_{D_\varepsilon \setminus E_\varepsilon} 2\phi \left( \nabla \phi \cdot \nabla h_\varepsilon^t \right) \left( h_\varepsilon^t - \lambda_\varepsilon \delta_\varepsilon \right) \varepsilon \left( \frac{\partial h_\varepsilon^t}{\partial t} \right)^{-1} dxdt = o(\delta_\varepsilon^2). \] (8.7)
This implies that
\[ \delta_{\varepsilon}^{-2} \int_{C_1} 2 \phi_0^2 (x_{n+1} - \lambda_{\varepsilon} \delta_{\varepsilon}) \sum_{i=1}^{n} \frac{\partial \phi_0}{\partial x_i} \nu_{\varepsilon,n+1 \varepsilon} (\nabla u_{\varepsilon})^2 \]
\[ = \delta_{\varepsilon}^{-2} \int_{-1+b}^{1-b} \int_{E_{\varepsilon}} 2 \phi (\nabla \phi \cdot \nabla h_{\varepsilon}^t) (h_{\varepsilon}^t - \lambda_{\varepsilon} \delta_{\varepsilon}) \varepsilon \left( \frac{\partial h_{\varepsilon}^t}{\partial t} \right)^{-1} dxdt + o_b(1) + o_{\varepsilon}(1). \]

In $E_{\varepsilon}$, by Lemma 6.1, (6.2)-(6.6) and the Cauchy inequality, we have
\[ \left| \int_{-1+b}^{1-b} \int_{E_{\varepsilon}} 2 \phi (\nabla \phi \cdot \nabla h_{\varepsilon}^t) (h_{\varepsilon}^t - \lambda_{\varepsilon} \delta_{\varepsilon}) dxdt \right| \leq C \delta_{\varepsilon}^2. \]

Then by Lemma 6.5, we obtain
\[ \int_{-1+b}^{1-b} \int_{E_{\varepsilon}} 2 \phi (\nabla \phi \cdot \nabla h_{\varepsilon}^t) (h_{\varepsilon}^t - \lambda_{\varepsilon} \delta_{\varepsilon}) \varepsilon \left( \frac{\partial h_{\varepsilon}^t}{\partial t} \right)^{-1} dxdt \]
\[ = \int_{-1+b}^{1-b} \int_{E_{\varepsilon}} 2 \phi (\nabla \phi \cdot \nabla h_{\varepsilon}^t) (h_{\varepsilon}^t - \lambda_{\varepsilon} \delta_{\varepsilon}) g'(g^{-1}(t)) dxdt + o(\delta_{\varepsilon}^2). \]

Finally, similar to (8.7), we have
\[ \int_{-1+b}^{1-b} \int_{B_1 \setminus E_{\varepsilon}} 2 \phi (\nabla \phi \cdot \nabla h_{\varepsilon}^t) (h_{\varepsilon}^t - \lambda_{\varepsilon} \delta_{\varepsilon}) g'(g^{-1}(t)) dxdt = o(\delta_{\varepsilon}^2). \]

This, combined with Lemma 6.5, then implies that
\[ \delta_{\varepsilon}^{-2} \int_{C_1} 2 \phi_0^2 (x_{n+1} - \lambda_{\varepsilon} \delta_{\varepsilon}) (\nabla \phi \cdot \nu_{\varepsilon,n+1 \varepsilon}) (\nabla u_{\varepsilon})^2 \]
\[ = \delta_{\varepsilon}^{-2} \int_{-1+b}^{1-b} \int_{B_1} 2 \phi (\nabla \phi \cdot \nabla h_{\varepsilon}^t) [h_{\varepsilon}^t - \lambda_{\varepsilon} \delta_{\varepsilon}] g'(g^{-1}(t)) dxdt + o_b(1) + o_{\varepsilon}(1). \]

By the Rellich compactness theorem and Lemma 6.1 and (6.2)-(6.6), it can be directly checked that
\[ \lim_{\varepsilon \to 0} \delta_{\varepsilon}^{-2} \int_{-1+b}^{1-b} \int_{B_1} 2 \phi (\nabla \phi \cdot \nabla h_{\varepsilon}^t) [h_{\varepsilon}^t - \lambda_{\varepsilon} \delta_{\varepsilon}] g'(g^{-1}(t)) dxdt \]
\[ = \left[ \int_{-1+b}^{1-b} g'(g^{-1}(t)) dt \right] \left[ \int_{B_1} 2 \phi (\nabla \phi \cdot \nabla \bar{h}) \bar{h} dx \right]. \]

Since $\bar{h}$ is a harmonic function, an integration by parts gives
\[ \int_{B_1} 2 \phi (\nabla \phi \cdot \nabla \bar{h}) \bar{h} dx = - \int_{B_1} \phi^2 \nabla \bar{h}^2 dx. \]
Now we have proved that
\[
\lim_{\varepsilon \to 0} \delta_{\varepsilon}^{-2} \int_{C_1} 2\phi \psi^2 (x_{n+1} - \lambda_{\varepsilon} \delta_{\varepsilon}) \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \nu_{\varepsilon, i} \nu_{\varepsilon, n+1} \varepsilon |\nabla u_{\varepsilon}|^2
\]
\[
= - \left[ \int_{-1+b}^{1-b} g'(g^{-1}(t))dt \right] \left[ \int_{B_1} \phi^2 |\nabla \bar{h}|^2 dx \right] + o_b(1).
\]
As in the proof of Proposition 7.1, we can let \(b \to 0\) to finish the proof.

Note that
\[
\int_{-1}^{1} g'(g^{-1}(t))dt = \int_{-\infty}^{+\infty} g'(s)^2 ds = \sigma_0.
\]

By (8.2), we can choose an \(\theta \in (0, 1/2)\) so that
\[
\theta^{-n} \int_{B_{2\theta}} |\nabla \bar{h}|^2 \leq C(n) \theta^2 \leq \frac{\theta}{4\sigma_0}.
\]

Then by choosing \(r = 2\theta\) in the definition of \(\phi\), (8.4) and Lemma 8.1 give, for all \(\varepsilon\) small,
\[
\theta^{-n} \int_{C_{\theta}} \left[ 1 - (\nu_{\varepsilon} \cdot e_{n+1}) \right] \varepsilon |\nabla u_{\varepsilon}|^2 \leq \frac{\theta}{3} \delta_{\varepsilon}^2,
\]
which contradicts the initial assumption (3.9). This completes the proof of improvement of tilt-excess decay in the special case \(\nabla \bar{h}(0) = 0\).

### 8.2 The general case

In general \(\nabla \bar{h}(0)\) may not be 0, and we only have an estimate as in (8.1). Here we show how to reduce this problem to the special case treated in the previous subsection.

For each \(\varepsilon\), take an rotation \(T_{\varepsilon} \in SO(n + 1)\) so that
\[
T_{\varepsilon} e_{n+1} = e_{\varepsilon} := \frac{e_{n+1} + \delta_{\varepsilon} \nabla \bar{h}(0)}{(1 + \delta_{\varepsilon}^2 |\nabla \bar{h}(0)|^2)^{1/2}}.
\]

Then we define
\[
\tilde{u}_{\varepsilon}(X) := u_{\varepsilon}(T_{\varepsilon}X),
\]
which is still a solution of (2.2) in \(B_4\).

By (8.1),
\[
|e_{\varepsilon} - e_{n+1}| \leq C(n) \delta_{\varepsilon}.
\]

We can also make \(T_{\varepsilon}\) satisfy the following estimates.
Lemma 8.2.
\[ \|T_\varepsilon - I\| \leq C(n)\delta_\varepsilon, \quad \|\Pi \circ T_\varepsilon - I_{\mathbb{R}^n}\| \leq C(n)\delta_\varepsilon^2. \] (8.11)

Proof. By choosing a basis in \( \mathbb{R}^n \) so that \( \nabla \overline{h}(0) = |\nabla \overline{h}(0)| e_n \). We have defined \( T_\varepsilon e_{n+1} \). Then we take
\[ T_\varepsilon e_i = e_i, \quad \text{for } 1 \leq i \leq n-1, \]
\[ T_\varepsilon e_n = \frac{e_n - \delta_\varepsilon |\nabla \overline{h}(0)| e_{n+1}}{1 + \delta_\varepsilon^2 |\nabla \overline{h}(0)|^2}^{1/2}. \]
In particular, \( T_\varepsilon \) is only a rotation in the \((e_n, e_{n+1})\)-plane.

Since \( \delta_\varepsilon |\nabla \overline{h}(0)| \leq 1/2 \), the first inequality in (8.11) can be directly verified. For the second one, we only need to check that
\[ |\Pi \circ T_\varepsilon e_n - e_n| = \left| \frac{e_n}{(1 + \delta_\varepsilon^2 |\nabla \overline{h}(0)|^2)^{1/2}} - e_n \right| = 1 - \frac{1}{(1 + \delta_\varepsilon^2 |\nabla \overline{h}(0)|^2)^{1/2}} \leq C\delta_\varepsilon^2 |\nabla \overline{h}(0)|^2. \]

For \( 1 \leq i \leq n-1 \), we have \( \Pi \circ T_\varepsilon e_i = e_i \). \( \square \)

Similar to \( \nu_\varepsilon \), define the unit normal vector \( \tilde{\nu}_\varepsilon \) associated to \( \tilde{u}_\varepsilon \) as before. Then we claim that

Lemma 8.3. There exists a constant \( C(n) \) depending only on \( n \) such that
\[ \int_{C_{3/4}} \left[ 1 - (\tilde{\nu}_\varepsilon \cdot e_{n+1})^2 \right] \varepsilon|\nabla \tilde{u}_\varepsilon|^2 \leq C(n)\delta_\varepsilon^2. \]

Proof. First by change of variables and noting (8.11), we have
\[ \int_{C_{3/4}} \left[ 1 - (\tilde{\nu}_\varepsilon \cdot e_{n+1})^2 \right] \varepsilon|\nabla \tilde{u}_\varepsilon|^2 = \int_{T^{-1}_\varepsilon(B_{3/4} \times \{|x_{n+1}| < 1/2\})} \left[ 1 - (\nu_\varepsilon \cdot e_\varepsilon)^2 \right] \varepsilon|\nabla u_\varepsilon|^2 + O(e^{-c/\varepsilon}) \] (8.12)
\[ \leq \int_{C_1} \left[ 1 - (\nu_\varepsilon \cdot e_\varepsilon)^2 \right] \varepsilon|\nabla u_\varepsilon|^2 + O(e^{-c/\varepsilon}), \]
where \( O(e^{-c/\varepsilon}) \) denotes the contribution from the part near \( B_1 \times \{\pm 1\} \) (by applying Proposition 4.4).

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Note that
\[
1 - (\nu_\varepsilon \cdot e_{n+1})^2 \leq 1 - (\nu_\varepsilon \cdot e_{n+1})^2 + 2 (\nu_\varepsilon \cdot e_{n+1})^2 \left(1 - \frac{1}{(1 + \delta_\varepsilon^2|\nabla \tilde{h}(0)|^2)^{1/2}}\right) + 2 \delta_\varepsilon |\nu_\varepsilon \cdot e_{n+1}| |\nu_\varepsilon \cdot \nabla \tilde{h}(0)|.
\]
By definition
\[
\int_{C_1} [1 - (\nu_\varepsilon \cdot e_{n+1})^2] \varepsilon |\nabla u_\varepsilon|^2 = \delta_\varepsilon^2.
\]
Next,
\[
2 (\nu_\varepsilon \cdot e_{n+1})^2 \left(1 - \frac{1}{(1 + \delta_\varepsilon^2|\nabla \tilde{h}(0)|^2)^{1/2}}\right) \leq C(n) \delta_\varepsilon^2.
\]
Finally, by noting that
\[
|\nu_\varepsilon \cdot \nabla \tilde{h}(0)| \leq |\nabla \tilde{h}(0)| \left(\sum_{i=1}^n \nu_{\varepsilon,i}^2\right)^{1/2} \leq C(n) \left[1 - (\nu_\varepsilon \cdot e_{n+1})^2\right]^{1/2},
\]
we can use the Cauchy inequality to derive that
\[
\int_{C_1} \delta_\varepsilon |\nu_\varepsilon \cdot e_{n+1}| |\nu_\varepsilon \cdot \nabla \tilde{h}(0)| \varepsilon |\nabla u_\varepsilon|^2
\leq C\delta_\varepsilon \left(\int_{C_1} |\nu_\varepsilon \cdot e_{n+1}|^2 \varepsilon |\nabla u_\varepsilon|^2\right)^{1/2} \left(\int_{C_1} [1 - (\nu_\varepsilon \cdot e_{n+1})^2] \varepsilon |\nabla u_\varepsilon|^2\right)^{1/2}
\leq C\delta_\varepsilon^2.
\]
Putting these together we get
\[
\int_{C_1} [1 - (\nu_\varepsilon \cdot e_\varepsilon)^2] \varepsilon |\nabla u_\varepsilon|^2 \leq C \delta_\varepsilon^2.
\]
Substituting this into (8.12) we can finish the proof. \qed

With this lemma in hand, we can proceed as before to construct the Lipschitz functions \(\tilde{h}_\varepsilon^t\), and prove that \(\delta_\varepsilon^{-1} (\tilde{h}_\varepsilon^t - \lambda_\varepsilon \delta_\varepsilon)\) converge to a harmonic function \(\tilde{h}\), weakly in \(H^1(B_{3/4})\) and strongly in \(L^2(B_{3/4})\).

However by the definition of \(\tilde{u}_\varepsilon\), the graph of \(\tilde{h}_\varepsilon^t\) is only a rotation of the one of \(h_\varepsilon^t\). More precisely, for any \(x \in B_{3/4}\) and \(t \in (-1 + b, 1 - b)\),
\[
\tilde{h}_\varepsilon^t(x) + \delta_\varepsilon \nabla \tilde{h}(0) \cdot x \overline{1 + \delta_\varepsilon^2|\nabla \tilde{h}(0)|^2}^{1/2} = h_\varepsilon^t(\Pi \circ T_\varepsilon(x, \tilde{h}_\varepsilon^t(x))].
\]

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Since the Lipschitz constant of $h^t_\varepsilon$ is bounded by $1/2$, by (8.11) and a direct expansion, we obtain
\[
\tilde{h}_\varepsilon^t(x) = h^t_\varepsilon(x) - \delta_\varepsilon \nabla h(0) \cdot x + O(\delta_\varepsilon^2).
\]
Hence
\[
\tilde{h}(x) = \lim_{\varepsilon \to 0} \left[ \tilde{h}_\varepsilon^t - \lambda_\varepsilon \right] = \lim_{\varepsilon \to 0} \left[ h^t_\varepsilon - \lambda_\varepsilon - \nabla \tilde{h}(0) \cdot x \right] = \tilde{h}(x) - \nabla \tilde{h}(0) \cdot x.
\]
Thus $\tilde{h}$ is a harmonic function in $B_{3/4}$ satisfying $\nabla \tilde{h}(0) = 0$. Then we can proceed as in the previous subsection. By choosing a smaller $\theta$ to incorporate the constance $C$ appearing in Lemma 8.3, for all $\varepsilon$ small we have
\[
\theta^{-n} \int_{C_\theta} \left[ 1 - (\tilde{\nu}_\varepsilon \cdot e_{n+1})^2 \right] \varepsilon |\nabla \tilde{u}_\varepsilon|^2 \leq \frac{\theta}{2C(n)} \delta_\varepsilon^2.
\]
Here $C(n)$ is the constant appearing in Lemma 8.3, due to change variable associated to the rotation $T_\varepsilon$. After rotating back, this contradicts (3.9) and finishes the proof of Theorem 3.3.

**Part II**

**Uniform $C^{1,\alpha}$ regularity of intermediate layers**

**9 Statement**

In this part we prove the following local uniform $C^{1,\alpha}$ bound for intermediate layers. This parallels Allard’s $\varepsilon$-regularity theorem for stationary varifold.

**Theorem 9.1.** For any $b \in (0,1)$, there exist five universal constants $\varepsilon_A, \tau_A, \alpha_A \in (0,1)$ and $R_A, K_A$ so that the following holds. Let $u_\varepsilon$ be a solution of (2.2) with $\varepsilon \leq \varepsilon_A$, satisfying $|u_\varepsilon(0)| \leq \gamma$ and
\[
R_A^{-n} \int_{B_{R_A}} \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \leq (1 + \tau_A) \omega_n \sigma_0.
\]
Then there exists a hyperplane, say $\mathbb{R}^n$ (after a suitable rotation), such that for any $t \in (-1 + b, 1 - b)$, $\{u_\varepsilon = t\} \cap C_1$ can be represented by the graph of the function $x_{n+1} = h^t_\varepsilon(x)$, where
\[
\|h^t_\varepsilon\|_{C^{1,\alpha}(B_1)} \leq K_A.
\]
10 A Morrey type bound

In this section $u_\varepsilon$ denotes a fixed solution satisfying all of the assumptions in Theorem 9.1. Here we prove

**Lemma 10.1.** There exist two universal constants $K_1$ and $K_2$, such that for any $X_0 \in \{|u_\varepsilon| \leq \gamma\} \cap B_1$ and ball $B_r(X_0)$ with $r \in (K_1\varepsilon, \theta)$, we can find a unit vector $e_r(X_0)$ to satisfy

$$r^{-n} \int_{B_r(X_0)} \left[1 - (\nu_\varepsilon \cdot e_r(X_0))^2\right] \varepsilon |\nabla u_\varepsilon|^2 \leq \max\{K_2\varepsilon^2 r^{-2}, K_2\tau_4^2 r^\alpha\}. \tag{10.1}$$

Here $\alpha = \frac{\log 2}{\log \theta} \in (0, 1)$.

For convenience, we shall replace the cylinders $C_2$ and $C_\theta$ in Theorem 3.3 by balls $B_1$ and $B_\theta$ respectively. This may change the constants in that theorem by a factor, which however only depends on the dimension $n$ and does not affect our argument too much.

By the monotonicity formula (Proposition 4.1) and (9.1), if $R_A$ is sufficiently large, for any $X \in B_1$ and $r \in (0, R_A - 1)$,

$$r^{-n} \int_{B_r(X)} \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \leq (1 + 2\tau_4)\omega_n \sigma_0. \tag{10.2}$$

By applying Proposition 4.4 to $u_\varepsilon(rX)$, we obtain

**Lemma 10.2.** For any $\delta > 0$, there exists a $K(\delta)$ so that the following holds. For any $X \in \{|u_\varepsilon| \leq \gamma\} \cap B_1$ and $r \in (K(\delta)\varepsilon, 1)$, there exists a hyperplane $P_r(X)$ such that

$$\text{dist}_H(\{u_\varepsilon = u_\varepsilon(X)\} \cap B_r(X), P_r(X) \cap B_r(X)) \leq \delta r.$$

By Lemma 4.6, if $r \geq K_1\varepsilon$ ($K_1$ a constant determined by Lemma 4.6), the excess with respect to $P_r(X)$ (with unit normal vector $e_r(X)$)

$$E(r; X, u_\varepsilon, P_r(X)) \leq \delta_0^2. \tag{10.3}$$

Then Theorem 3.3 can be stated as

**Lemma 10.3.** If $E(r; X, u_\varepsilon, P_r(X)) \geq K_0 r^{-1} \varepsilon$, we can find another hyperplane $\tilde{P}_r(X)$ such that

$$\theta^{-n} E(\theta r; X, u_\varepsilon, \tilde{P}_r(X)) \leq \frac{\theta}{2} E(r; X, u_\varepsilon, P_r(X)).$$

Here one of the unit normal vector of $\tilde{P}_r(X)$, $\tilde{e}_r(X)$ satisfies

$$\|\tilde{e}_r(X) - e_r(X)\| \leq CE(r; X, u_\varepsilon, P_r(X))^{\frac{1}{2}}.$$
With this lemma in hand we can prove Lemma 10.1.

**Proof of Lemma 10.1.** Assume $X_0 = 0$ is the origin. For $k \geq 0$, let $r_k = \theta^k$. Define

$$E_k := \min_{e \in S^n} \varepsilon^{-2} r_k^{-2-n} \int_{B_{r_k}} [1 - (\nu_{\varepsilon} \cdot e)^2] \varepsilon |\nabla u_{\varepsilon}|^2.$$ 

Assume that $\bar{e}_k$ is a unit vector attaining this minima.

As in (10.3), for all $r_k \geq K_1 \varepsilon$,

$$E_k \leq \delta_0^2 \varepsilon^{-2} r_k^2. \quad (10.4)$$

Lemma 10.3 implies that, if $E_k \geq K_0$, then

$$E_{k+1} \leq \frac{\theta^2}{2} E_k. \quad (10.5)$$

Moreover, by the definition of $E_k$, we always have

$$E_{k+1} \leq \theta^{2-n} E_k. \quad (10.6)$$

Let $k_1$ be the unique number satisfying $\theta^{k_1} \in [K_1 \varepsilon, K_1 \theta^{-1} \varepsilon]$.

Now we derive the claimed bound on $E_k$ from (10.4)-(10.6), for $k \leq k_1$. Let $k_0$ be the smallest number such that, for all $k > k_0$,

$$E_k \leq K_0 \theta^{2-n}. \quad (10.7)$$

If $k_0 = 0$, then for any $k > 0$ and $r_k \geq K_1 \varepsilon$,

$$\int_{B_{r_k}} [1 - (\nu_{\varepsilon} \cdot \bar{e}_k)^2] \varepsilon |\nabla u_{\varepsilon}| \leq K_0 \theta^{2-n} \varepsilon^2 r_k^{n-2}.$$ 

This can also be extended to those $r \in [K_1 \varepsilon, \theta]$ by choosing a (unique) $k$ so that $r \in [r_{k+1}, r_k]$.

Next we assume $k_0 > 0$. By definition, $E_{k_0} > K_0 \theta^{2-n}$. By (10.6), $E_{k_0-1} \geq K_0$. Then we can apply (10.5) to deduce that

$$E_{k_0-1} \geq \frac{2}{\theta^2} E_{k_0}.$$ 

In particular,

$$E_{k_0-1} \geq E_{k_0} \geq K_0 \theta^{2-n}.$$ 

With this estimate we can repeat the above procedure to obtain that, for all $i \in [0, k_0)$,

$$E_i \geq \frac{2}{\theta^2} E_{i+1} \geq K_0 \theta^{2-n}.$$
This then implies that for all \( i \leq k_0 \),
\[
E_i \leq \left( \frac{\theta^2}{2} \right)^i E_0,
\]
which implies that for all \( i \leq k_0 \),
\[
\int_{B_{r_i}} \left[ 1 - (\nu \cdot \bar{e}_i)^2 \right] \varepsilon |\nabla u_\varepsilon| \leq \delta_0^2 r_i^{n+\alpha}.
\]
(10.8)

This estimate can also be extended to those \( r \in [r_{k_0}, \theta) \) by choosing an \( i \) so that \( r \in [r_{i+1}, r_i) \).

In conclusion, for \( r \in [r_{k_0}, \theta) \), we have the estimate (10.8), and for \( r \in (K_1 \varepsilon, r_{k_0}) \) we can apply (10.7). These two give (10.1).

Next we show that we can choose \( e_r(X_0) \) independent of \( r \).

**Lemma 10.4.** For any \( \sigma > 0 \), there exist two universal constants \( K_3 \) and \( K_4 \) (\( K_4 \) independent of \( \sigma \)) so that the following holds. For any \( X_0 \in \{|u_\varepsilon| \leq \gamma \} \cap B_1 \) and ball \( B_r(X_0) \) with \( r \in (K_3 \varepsilon, \theta) \), there exists a unit vector \( e(X_0) \) such that
\[
r^{-n} \int_{B_r(X_0)} \left[ 1 - (\nu \cdot e(X_0))^2 \right] \varepsilon |\nabla u_\varepsilon|^2 \leq \sigma^2 + K_4 r^2 \tau^2 \alpha.
\]
(10.9)

Here \( e(X_0) \) is independent of \( r \in (K_3 \varepsilon, \theta) \).

**Proof.** We keep notations as in the proof of Lemma 10.1.

For any \( r \in (K_1 \varepsilon, \theta) \), combining Remark 3.2 and Lemma B.5, we have
\[
\int_{B_{2r}} \left[ 1 - (\nu \cdot e_{2r}(X_0))^2 \right] \varepsilon |\nabla u_\varepsilon|^2 + \int_{B_r} \left[ 1 - (\nu \cdot e_r(X_0))^2 \right] \varepsilon |\nabla u_\varepsilon|^2
\]
\[
\geq c(n) \int_{B_r} [\text{dist}_{\mathbb{R}^n}(\nu, e_r(X_0))^2 + \text{dist}_{\mathbb{R}^n}(\nu, e_{2r}(X_0))^2] \varepsilon |\nabla u_\varepsilon|^2
\]
\[
\geq c(n) \text{dist}_{\mathbb{R}^n}(e_{2r}(X_0), e_r(X_0))^2 \int_{B_r} \varepsilon |\nabla u_\varepsilon|^2
\]
\[
\geq c(n) \text{dist}_{\mathbb{R}^n}(e_{2r}(X_0), e_r(X_0))^2 r^n.
\]

For \( k < k_0 \), this gives
\[
\text{dist}_{\mathbb{R}^n}(e_{k+1}(X_0), e_k(X_0)) \leq C \tau_A r_k^{\frac{\alpha}{2}} = C \tau_A \theta^{\frac{\alpha}{2}} k.
\]

Summing in \( i \) from \( k \) to \( k_0 \), we see
\[
\text{dist}_{\mathbb{R}^n}(e_k(X_0), e_k(X_0)) \leq C \tau_A \theta^{\frac{\alpha}{2}} k = C \tau_A r_k^{\frac{\alpha}{2}}, \quad \forall k < k_0.
\]
(10.10)
For $k \in [k_0, k_1)$, we have
\[
\text{dist}_{\mathbb{R}^n}(e_{k+1}(X_0), e_k(X_0)) \leq C \varepsilon r_k^{-1} = C \varepsilon \theta^{-k}.
\] (10.11)

Let $k_2 \leq k_1$ be the largest number satisfying
\[
\varepsilon C \theta - 1 \leq K_3 := C(\sigma^{-1} - 1)\theta^{-k_2}.
\]

Summing (10.11) from $k$ to $k_2$, we get
\[
\text{dist}_{\mathbb{R}^n}(e_{k_2}(X_0), e_k(X_0)) \leq C \varepsilon \theta^{-k_2} \leq \sigma,
\] (10.12)

In particular,
\[
\text{dist}_{\mathbb{R}^n}(e_{k_2}(X_0), e_{k_0}(X_0)) \leq \sigma.
\] (10.13)

Let $e(X_0) = e_{k_2}(X_0)$, by combining (10.10)-(10.13) we obtain, for any $k \in (0, k_2)$,
\[
\text{dist}_{\mathbb{R}^n}(e_k, e(X_0)) \leq \sigma + C \tau_A r_k^{\alpha_2}.
\]

Note that for any $k \geq 0$,
\[
1 - (\nu \cdot e(X_0))^2 \leq [1 - (\nu \cdot e_k)^2] + C(n)\text{dist}_{\mathbb{R}^n}(e_k, e(X_0))^2.
\]

Together with (10.1), this gives
\[
\int_{B_{r_k}} [1 - (\nu \cdot e(X_0))^2] \varepsilon |\nabla u_\varepsilon|^2 \leq C \sigma^2 + C \delta_0^2 r_k^{n+\alpha}.
\]

For any $r \in (K_3 \varepsilon, \theta)$, by choosing a $k$ so that $r \in (r_k, r_{k+1}]$, we then get
\[
\int_{B_r} [1 - (\nu \cdot e(X_0))^2] \varepsilon |\nabla u_\varepsilon|^2 \leq C \left( \sigma^2 + \delta_0^2 r_k^{n+\alpha} \right),
\] (10.14)

for some universal constant $C$.

The following result will be used in the proof of Lipschitz regularity of $\{u_\varepsilon = 0\}$.

**Corollary 10.5.** For any $X_0 \in \{u_\varepsilon = 0\} \cap B_1$,
\[
|e(X_0) - e_{n+1}| \leq C (\tau_A + \sigma).
\]
Proof. By taking \( r = \theta \) in (10.9), we have
\[
\int_{B_{\theta}(X_0)} \left[ 1 - (\nu_\varepsilon \cdot e(X_0))^2 \right] \varepsilon |\nabla u_\varepsilon|^2 \leq C \left( \sigma^2 + \tau^2_\Lambda \right).
\]
On the other hand, by (9.1) and Lemma 4.6, we also have
\[
\int_{B_{\theta}(X_0)} \left[ 1 - (\nu_\varepsilon \cdot e_{n+1})^2 \right] \varepsilon |\nabla u_\varepsilon|^2 \leq C \tau^2_\Lambda.
\]
Similar to the proof of the previous lemma, combining these two and using Lemma B.5, we get
\[
\text{dist}_{\mathbb{R}P^n}(e(X_0), e_{n+1})^2 \leq \theta^{-n} \int_{B_{1/4}(X_0)} \left[ 1 - (\nu_\varepsilon \cdot e(X_0))^2 \right] \varepsilon |\nabla u_\varepsilon|^2 + \theta^{-n} \int_{B_{1/4}(X_0)} \left[ 1 - (\nu_\varepsilon \cdot e_{n+1})^2 \right] \varepsilon |\nabla u_\varepsilon|^2 \leq C \left( \sigma^2 + \tau^2_\Lambda \right).
\]
Finally, we can fix \( e(X_0) \) so that it points above. Thus the estimate on the distance in \( \mathbb{R}P^n \) can be lifted to an estimate in \( \mathbb{S}^n \).

We shall use these results to show that the level sets of \( u_\varepsilon \) are Lipschitz graphs in the form of \( x_{n+1} = h_\varepsilon(x) \). However, before proving this we first sketch a direct proof of Theorem 1.1.

11 A direct proof of Theorem 1.1

This section is devoted to a direct proof of Theorem 1.1. In fact, we prove something more.

Theorem 11.1. Suppose that \( u \) is a smooth solution of (2.1) on \( \mathbb{R}^{n+1} \), satisfying
\[
\lim_{R \to +\infty} R^{-n} \int_{B_R} \frac{1}{2} |\nabla u|^2 + W(u) \leq (1 + \tau_\Lambda) \omega_n \sigma_0. \tag{11.1}
\]
Then there exists a unit vector \( e \) such that \( u(X) \equiv u(e \cdot X) \).

In the following we will show that if \( u \) is a minimizing solution of (2.1) on \( \mathbb{R}^{n+1} \), where \( n \leq 6 \), then (11.1) is satisfied. Thus Theorem 1.1 is a corollary of this theorem.

Since \( u \) is an entire solution, by the main result of [17], \( u \) satisfies the Modica inequality and hence the monotonicity formula. This monotonicity ensures the existence of the limit in (11.1). It also implies that, for any ball \( B_R(X) \),
\[
R^{-n} \int_{B_R(X)} \frac{1}{2} |\nabla u|^2 + W(u) \leq (1 + \tau_\Lambda) \omega_n \sigma_0.
\]
With this bound, we can study the asymptotic behavior of $u$ through the scalings

$$u_\varepsilon(X) := u(\varepsilon^{-1}X).$$

As before, by Hutchinson-Tonegawa’s theorem, the varifolds $V_\varepsilon$ associated to $u_\varepsilon$ converge to a stationary varifold $V$ with integer multiplicity.

Furthermore, we claim that

**Proposition 11.2.** $V$ is a cone with respect to the origin $0$.

**Proof.** This is because for any $R > 0$, by the convergence of $\|V_\varepsilon\|$ and (4.2),

$$R^{-n}\|V\|(B_R) = \lim_{\varepsilon \to 0} R^{-n}\|V_\varepsilon\|(B_R)$$

$$= \lim_{\varepsilon \to 0} R^{-n} \int_{B_R} \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)$$

$$= \lim_{\varepsilon \to 0} (\varepsilon^{-1}R)^{-n} \int_{B_{\varepsilon^{-1}R}} \frac{1}{2} |\nabla u|^2 + W(u).$$

(11.2)

However, by (11.5) and Proposition 4.1, there exists a constant $\Theta$ such that

$$\lim_{\varepsilon \to 0} \varepsilon^n \int_{B_{\varepsilon^{-1}R}} \frac{1}{2} |\nabla u|^2 + W(u) = \Theta R^n, \; \forall R > 0.$$ 

Then by the monotonicity formula for stationary varifold (cf. [16, Theorem 6.3.2]), we deduce that $V$ is a cone.

By (11.1) and (11.2),

$$\|V\|(B_1) \leq (1 + \tau_A) \omega_n \sigma_0.$$ 

Hence we can apply Allard’s $\varepsilon$-regularity result to deduce that $\text{spt}\|V\| \cap B_{1/2}$ is a smooth hypersurface. Then by the previous proposition, $\text{spt}\|V\|$ must be a hyperplane and $V$ is the standard varifold associated to this plane with unit density.

This blowing down analysis also implies that, the distance function $\Phi_\varepsilon := g_\varepsilon^{-1} \circ u_\varepsilon$ (see Appendix A) converges to (up to a subsequence of $\varepsilon \to 0$) a linear function in the form $e \cdot X$ in $C_{loc}(\mathbb{R}^{n+1})$, where $e$ is a unit vector.

However, this argument does not show the uniqueness of this limit. Different subsequences of $\varepsilon \to 0$ may lead to different limits. To obtain the uniqueness of the blowing down limit, we use the following lemma.

**Lemma 11.3.** There exists a universal constant $C(n)$, such that for any ball $B_R$ with $R \geq 1$, we can find a unit vector $e_R$ to satisfy

$$\int_{B_R} \left[1 - (\nu \cdot e_R)^2\right] |\nabla u|^2 \leq C(n) R^{n-2}.$$ 

(11.3)
The proof is similar to the one of Lemma 10.1, see also the proof of [25, Theorem 2.3].
Note that \( e_R \) in this theorem may not be unique. In the following we assume that for each \( R > 1 \), we have fixed such a vector \( e_R \).
If \( n = 1 \), as \( R \to +\infty \), since \( e_R \) are unit vectors, we can take a subsequence of \( R_i \to +\infty \) so that \( e_{R_i} \to e_\infty \in S^1 \). Assume \( e_\infty = e_2 \). Then by taking limit in (11.3), we get for any \( R > 0 \),
\[
\int_{B_R} \left( \frac{\partial u}{\partial x_1} \right)^2 = 0.
\]
Thus \( u(x_1, x_2) \equiv u(x_2) \).
Now consider the case \( n \geq 2 \). Similar to Lemma 10.4, we also have

**Lemma 11.4.** There exists a unit vector \( e_\infty \) and a universal constant \( C(n) \) such that
\[
\int_{B_R} \left[ 1 - (\nu \cdot e_\infty)^2 \right] |\nabla u|^2 \leq C(n) R^{n-2}, \quad \forall R > 1. \tag{11.4}
\]

For the blowing down sequence \( u_\varepsilon \), (11.4) implies that
\[
\int_{B_1} \left[ 1 - (\nu_\varepsilon \cdot e_\infty)^2 \right] \varepsilon |\nabla u_\varepsilon|^2 \leq C(n) \varepsilon^2.
\]
For any \( \eta \in C^\infty_0(\mathbb{R}^{n+1}) \), let \( \Phi(X,S) = \eta(X)^2 < Se_\infty, e_\infty > \in C^\infty_0(\mathbb{R}^{n+1} \times G(n)) \). Then we have
\[
0 = \lim_{\varepsilon \to 0} < V_\varepsilon, \Phi > = < V, \Phi >.
\]
Thus for \( \|V\| \) a.a. \( X \), the tangent plane of \( V \) at \( X \) is the hyperplane orthogonal to \( e_\infty \). It can be directly checked that \( V \) must be the standard varifold associated to this hyperplane. (This can also be seen by noting that we have proved that spt\( \|V\| \) is a hyperplane.)
This also implies that, for all \( \varepsilon \to 0 \),
\[
\Phi_\varepsilon \to e_\infty \cdot X, \quad \text{in } C^\infty_{locc}(\mathbb{R}^{n+1}).
\]
Without loss of generality, assume \( e_\infty = e_{n+1} \).
Then by Theorem A.4, for any \( \delta > 0 \),
\[
\nabla \Phi_\varepsilon \to e_{n+1}, \quad \text{uniformly on } B_1 \cap \{|x_{n+1}| > \delta\}.
\]
By compactness, this still holds true if we replace the base point by any point \( X_0 \in \{u = 0\} \). Thus we arrive at

**Lemma 11.5.** For any \( \delta > 0 \), there exists an \( L \) such that, for any \( X \in \{ |\Phi| \geq L \} \),
\[
|\nabla \Phi(X) - e_{n+1}| \leq \delta.
\]
In particular, in \(|\Phi| > L\), \(u\) is increasing along directions in the cone
\[
\{e \cdot e_{n+1} \geq \delta\}.
\]

Then we can proceed as in [12] to deduce that \(u\) is increasing along directions in this cone everywhere in \(\mathbb{R}^{n+1}\). After letting \(L \to +\infty\), we deduce that for any unit vector \(e\) orthogonal to \(e_{n+1}\),
\[
e \cdot \nabla u \geq 0, \quad -e \cdot \nabla u \geq 0, \quad \text{in} \ \mathbb{R}^{n+1}.
\]
Thus \(\frac{\partial u}{\partial x_i} \equiv 0\) in \(\mathbb{R}^{n+1}\), for all \(1 \leq i \leq n\). This then implies that \(u\) depends only on \(x_{n+1}\).

Finally, by using (11.1), it can be checked directly that we must have \(u(X) \equiv g(x_{n+1} + t)\) for some \(t \in \mathbb{R}\).

Now assume that \(u\) is a minimizing solution of (2.1) on \(\mathbb{R}^{n+1}\), where \(n \leq 6\). First we can use standard comparison functions to deduce an energy bound.

**Lemma 11.6.** There exists a constant \(M\), which depends only on the dimension \(n\), such that
\[
\int_{B_R(X)} \frac{1}{2} |\nabla u|^2 + W(u) \leq MR^n, \quad (11.5)
\]
for any ball \(B_R(X)\).

As before we can consider the blowing down sequence \(u_\varepsilon\) and the associated varifold \(V_\varepsilon\). By [15, Theorem 2], its limit varifold \(V\) has unit density. In fact, in this case we can prove that \(\text{spt} \|V\| = \partial \Omega\), where \(\Omega\) has minimizing perimeter, see [18].

Moreover, \(\partial \Omega\) is still a cone. Now by the assumption that the dimension \(n \leq 6\), \(\partial \Omega\) must be a hyperplane, see Simons [23]. Then (11.2) gives
\[
\lim_{R \to +\infty} R^{-n} \int_{B_R} \frac{1}{2} |\nabla u|^2 + W(u) = \|V\|(B_1) = \omega_n \sigma_0.
\]
That is, \(u\) satisfies all of the assumptions in Theorem 11.1. Thus we get Theorem 1.1 by applying Theorem 11.1.

## 12 The Lipschitz nature of intermediate layers

Now we continue the proof of Theorem 9.1. In this section we first prove that \(\{u_\varepsilon = t\}\) can be represented by a Lipschitz graph in the \(x_{n+1}\) direction. This is the consequence of Corollary 10.5 and the following Lemma 12.2.

Before coming to Lemma 12.2, we need the following
Lemma 12.1. Let \( v \) be a solution of (2.1) in \( \mathbb{R}^{n+1} \). Assume there exists a constant \( \sigma \) small so that for all \( r \) large,
\[
\int_{B_r} \left[ 1 - (v \cdot e_{n+1})^2 \right] |\nabla v|^2 \leq \sigma^2 r^n,
\]
and
\[
\lim_{r \to +\infty} r^{-n} \int_{B_r} \frac{1}{2} |\nabla v|^2 + W(v) \leq (1 + \tau_A) \sigma_0 \omega_n.
\]
Then there exists a unit vector \( e \) satisfying
\[
|e - e_{n+1}| \leq C(n) \sigma,
\]
so that \( v(X) \equiv v(e \cdot X) \).

Proof. The only thing we need to check is that (12.1) implies (12.3). This can be directly verified by substituting \( u(X) \equiv g(e \cdot X + t) \) (\( t \) a constant) into (12.1). \( \square \)

Lemma 12.2. For any \( R > 1 \) and \( \sigma > 0 \) small, there exists \( \bar{R} > R \) so that the following holds. Let \( v \) be a solution of (2.1) in \( B_{\bar{R}} \), satisfying \( |v(0)| \leq \gamma \), the Modica inequality (2.10) and
\[
\bar{R}^{-n} \int_{B_{\bar{R}}} \frac{1}{2} |\nabla v|^2 + W(v) \leq (1 + \tau_A) \sigma_0 \omega_n.
\]
Suppose that for any \( r \in (R, \bar{R}) \),
\[
\int_{B_r} \left[ 1 - (v \cdot e_{n+1})^2 \right] |\nabla v|^2 \leq \sigma^2 r^n.
\]
Then by denoting \( \Phi := g^{-1} \circ v \) and assuming that \( v > 0 \) when \( x_{n+1} \gg 0 \),
\[
\sup_{B_{\bar{R}}} |\nabla \Phi - e_{n+1}| \leq \frac{1}{4}.
\]

Proof. Assume by the contrary, there exists an \( R > 0 \), a sequence of \( R_i \to +\infty \) and a sequence of solutions \( v_i \) to (2.1) defined on \( B_{R_i} \), satisfying \( |v_i(0)| \leq \gamma \), the Modica inequality (2.10) and
\[
R_i^{-n} \int_{B_{R_i}} \frac{1}{2} |\nabla v_i|^2 + W(v_i) \leq (1 + \tau_A) \sigma_0 \omega_n,
\]
and
\[
\int_{B_r} \left[ 1 - (v_i \cdot e_{n+1})^2 \right] |\nabla v_i|^2 \leq \sigma^2 r^n, \quad \forall r \in (R, R_i).
\]
But
\[
\sup_{B_{\bar{R}}} |\nabla \Phi_i - e_{n+1}| > \frac{1}{4}.
\]
Then we can assume $v_i$ converges to a smooth solution $v_\infty$ on any compact set of $\mathbb{R}^{n+1}$. By the monotonicity formula and (12.4), for any $r > 0$,

$$r^{-n} \int_{B_r} \frac{1}{2} |\nabla v_\infty|^2 + W(v_\infty) \leq (1 + \tau_A) \sigma_0 \omega_n.$$  

Passing to the limit in (12.5) we also have

$$\int_{B_r} \left[ 1 - (v_\infty \cdot e_{n+1})^2 \right] |\nabla v_\infty|^2 \leq \sigma^2 r^n, \quad \forall r > R.$$  

Then by the previous lemma (noting that $v_\infty(0) = \lim_{i \to +\infty} v_i(0)$ and $v_\infty > \gamma$ above $\mathbb{R}^n$), $v_\infty(X) \equiv g(e \cdot X + g^{-1}(v_\infty(0)))$ for some unit vector $e$ satisfying

$$|e - e_{n+1}| \leq \frac{1}{8}.$$  

Then

$$\Phi_i(X) := g^{-1} \circ v_i(X) \to e \cdot X \quad \text{in } C^1(B_R).$$  

In particular, for all $i$ large,

$$\sup_{B_R} |\nabla \Phi_i - e| \leq \frac{1}{4}.$$  

However, this contradicts (12.6).  

We can apply this lemma to $v(X) := u_\varepsilon(X_0 + \varepsilon X)$, where $u_\varepsilon$ is as in Theorem 9.1 and $X_0 \in \{u_\varepsilon = u_\varepsilon(0)\} \cap B_1$. Combined with Corollary 10.5 (provided $\varepsilon$ and $\sigma$ are sufficiently small), this results in

**Lemma 12.3.** For any $X_0 \in \{u_\varepsilon = u_\varepsilon(0)\} \cap B_1$, $\nabla u_\varepsilon \neq 0$ in $B_{K_4 \varepsilon}(X_0)$ and

$$|\nu_\varepsilon - e_{n+1}| \leq \frac{1}{2}.$$  

Here we only need to note that, at the beginning we have assumed that $u_\varepsilon > u_\varepsilon(0)$ in $\{x_{n+1} > 1/2\} \cap B_1$. Then by Lemma 10.2, $u_\varepsilon < u_\varepsilon(0)$ in $\{x_{n+1} < -1/2\} \cap B_1$.

Next by combining Lemma 10.2 and Lemma 10.4, for any $r \geq K_4 \varepsilon$,

$$\{(X - X_0) \cdot e(X_0) \geq \frac{r}{2}\} \cap B_r(X_0) \subset \{u_\varepsilon > u_\varepsilon(0)\}, \quad (12.7)$$  

thanks to the continuous dependence on $r$.

(12.7) also implies that for any $x \in B_1$, there exists a unique $x_{n+1} \in (-1, 1)$ so that $(x, x_{n+1}) \in \{u_\varepsilon = u_\varepsilon(0)\}$. Combined with the previous lemma, this then implies that

$$\{u_\varepsilon = u_\varepsilon(0)\} \cap B_1 = \{x_{n+1} = h_\varepsilon(x)\}, \quad x \in B_1.$$  

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Here $h_\varepsilon$ is a function with its Lipschitz constant bounded by 4. (This constant can be made as small as possible by decreasing $\varepsilon$, $\tau_\Lambda$ and $\sigma$.)

To complete the proof of Theorem 9.1, we directly apply the main result in [6]. Note that instead of the minimizing condition assumed in that paper, with our assumption (9.1) the argument still goes through.

## A A distance type function

In this appendix $u_\varepsilon$ always denotes a solution of (2.3), satisfying the Modica inequality (2.10). Here we introduce a distance type function associated to $u_\varepsilon$ and study its properties and convergence as $\varepsilon \to 0$. This is perhaps well known (see for example [11] for the parabolic Allen-Cahn case). However we do not find an exact reference, so we include some details here.

Recall that $g_\varepsilon(t) := g(\varepsilon^{-1}t)$ is a one dimensional solution of (2.2). Define

$$\Phi_\varepsilon(X) = g_\varepsilon^{-1}(u_\varepsilon(X)).$$

It satisfies

$$-\varepsilon \Delta \Phi_\varepsilon = f(\varepsilon^{-1}\Phi_\varepsilon)(1 - |\nabla \Phi_\varepsilon|^2),$$

where $f(t) := \frac{W'(g(t))}{\sqrt{2W(g(t))}}$. Note that $f(t) < 0$ is $C^2$ in a neighborhood of 1 (and $f(t) > 0$, $C^2$ in a neighborhood of $-1$).

The following result is a consequence of the Modica inequality.

**Proposition A.1.** $|\nabla \Phi_\varepsilon| \leq 1$.

**Proof.** Since $u_\varepsilon = g_\varepsilon(\Phi_\varepsilon)$,

$$|\nabla u_\varepsilon| = g'_\varepsilon(\Phi_\varepsilon)|\nabla \Phi_\varepsilon|.$$

Thus $|\nabla \Phi_\varepsilon| \leq 1$ is equivalent to

$$|\nabla u_\varepsilon|^2 \leq g'_\varepsilon(\Phi_\varepsilon)^2.$$

The first integral for $g_\varepsilon$ is

$$\frac{\varepsilon}{2} (g'_\varepsilon)^2 = \frac{1}{\varepsilon} W(g_\varepsilon).$$

Then the final equivalent statement is exactly the Modica inequality for $u_\varepsilon$. $\square$

By using (A.1), the limit of $\Phi_\varepsilon$, $\Phi_0$ can be characterized as a viscosity solution of the eikonal equation: In $\{\Phi_0 > 0\}$, $\Phi_0$ is a viscosity solution of

$$|\nabla \Phi_0|^2 - 1 = 0.$$
In $\{\Phi_0 < 0\}$, $\Phi_0$ is a viscosity solution of

$$1 - |\nabla \Phi_0|^2 = 0.$$

This is similar to the vanishing viscosity method (see for example Fleming-Souganidis [13]). However, here we would like to give a direct proof.

**Proposition A.2.** For any $\delta > 0$, there exists $\varepsilon_2, \tau_2 > 0$, if $u_\varepsilon$ is a solution of (2.2) in $B_2$ with $\varepsilon \leq \varepsilon_2$, satisfying $|u_\varepsilon(0)| \leq \gamma$ and (9.1) with $\tau_\varepsilon$ replaced by $\tau_2$, then there exists a set $\Omega \subset B_1$, with $0 \in \partial \Omega$ and $\partial \Omega$ being a smooth minimal hypersurface, such that

$$\sup_{B_1} |\Phi_\varepsilon - d_{\partial \Omega}| \leq \delta. \quad (A.2)$$

Here $d_{\partial \Omega}$ is the signed distance function to $\partial \Omega$, which is positive in $\Omega$.

**Proof.** We fix $\tau_\varepsilon$ so that it fulfills the assumption in Allard’s $\varepsilon$-regularity theorem. By Hutchinson-Tonegawa [15], if we have a sequence of $u_\varepsilon$ with $\varepsilon \to 0$, satisfying all of the assumptions in this proposition, then the associated varifold $V_\varepsilon$ converges to a stationary rectifiable varifold $V$ with integer multiplicity. Moreover, (9.1) and the monotonicity formula implies that

$$2^{-n} \|V\|(B_2) \leq (1 + \tau_\varepsilon) \sigma_0 \omega_n.$$

Then Allard’s regularity theorem (cf. [1] and [16, Theorem 6.5.1]) implies that $\text{spt} \|V\| \cap B_{3/2}$ is a smooth hypersurface and $V \cap B_{3/2}$ is the standard varifold associated to this hypersurface with density one. This hypersurface also divides $B_1$ into two parts, say $\Omega$ and $B_1 \setminus \Omega$. Moreover, as in Remark 4.5, $u_\varepsilon$ converges to 1 uniformly in any compact set of $\Omega$, and to $-1$ uniformly in any compact set of $\Omega^c$.

Thus if $\varepsilon_2$ is small enough, we can assume that there exists a set $\Omega$ with $0 \in \partial \Omega$ and $\partial \Omega$ being a smooth minimal hypersurface, such that

$$\text{dist}_H (\{u_\varepsilon > 0\} \cap B_1, \Omega \cap B_1) \leq \frac{\delta}{8}.$$

In particular,

$$\sup_{B_1} \left| \text{dist}_{\{u_\varepsilon = 0\}} - d_{\partial \Omega} \right| \leq \frac{\delta}{8}.$$

By Proposition A.1, in $\{u_\varepsilon > 0\} \cap B_1$,

$$\Phi_\varepsilon(X) \leq \text{dist}_{\{u_\varepsilon = 0\}}(X) \leq \text{dist}_{\partial \Omega}(X) + \delta.$$

Similarly, in $\{X : |\text{dist}_{\partial \Omega}(X)| \leq \frac{\delta}{4}\}$, $|\Phi_\varepsilon| \leq \delta/2$. Thus in this part,

$$|\Phi_\varepsilon - d_{\partial \Omega}| \leq |\Phi_\varepsilon| + |d_{\partial \Omega}| \leq \delta.$$
Thus to prove (A.2), we only need to show that if \( X \in \Omega \cap \{ X : \text{dist}_{\partial \Omega}(X) \geq \frac{\delta}{4} \} \), where \( \text{dist}_{\{u_{\varepsilon}=0\}} \geq \delta/8 \), then
\[
\Phi_{\varepsilon}(X) \geq \text{dist}_{\{u_{\varepsilon}=0\}}(X) - \frac{\delta}{16}.
\]
However, this can be proved directly by constructing a comparison function in \( B_{\text{dist}_{\{u_{\varepsilon}=0\}}(X) - \frac{\delta}{16}}(X) \), by noting that \( u_{\varepsilon} \) is close to 1 in this ball, if we have chosen \( \varepsilon \) sufficiently small (compared to \( \delta \)).

In the following we assume that as \( \varepsilon \to 0 \), \( \Phi_{\varepsilon} \) converges to a distance function \( \Phi_0 \) uniformly. Now we present a fact about the \( C^1 \) convergence of \( \Phi_{\varepsilon} \) near a \( C^1 \) point of \( \Phi_0 \).

First we establish the uniform semi-concavity of \( \Phi_{\varepsilon} \).

**Lemma A.3.** Let \( \Phi_{\varepsilon} > 1/2, |\nabla \Phi| \leq 1 \), satisfy (A.1) in \( B_1 \). Then
\[
\nabla^2 \Phi_{\varepsilon}(0) \leq C,
\]
where \( C \) is a constant depending only on the dimension \( n \).

The constant \( 1/2 \) is not essential here. It can be replaced by any positive constant.

**Proof.** We shall work in the setting where \( \varepsilon = 1 \) and the ball is \( B_R \). Here \( \varepsilon = R^{-1} \). For simplicity, all subscripts will be dropped.

Take a unit vector \( \xi \). By directly differentiating (A.1) in the direction \( \xi \), we get
\[
-\Delta \Phi_{\xi} = f'(\Phi)(1 - |\nabla \Phi|^2)\Phi_{\xi} - 2f(\Phi) \sum_{k=1}^{n+1} \Phi_{k}\Phi_{\xi k},
\]
\[
-\Delta \Phi_{\xi \xi} = f'(\Phi)(1 - |\nabla \Phi|^2)\Phi_{\xi \xi} + f''(\Phi)(1 - |\nabla \Phi|^2)\Phi_{\xi}^2
-4f'(\Phi) \sum_{k=1}^{n+1} \Phi_{k}\Phi_{\xi k}\Phi_{\xi} - 2f(\Phi) \sum_{k=1}^{n+1} \Phi_{k}\Phi_{\xi k}^2.
\]

Take an \( \eta \in C_0^\infty(B_{R/2}) \) such that \( \eta \equiv 1 \) in \( B_{R/4} \), \( 0 \leq \eta \leq 1 \), \( |\nabla \eta| \leq 8R^{-1} \) and \( \eta^{-1}|\nabla \eta|^2 + |\Delta \eta| \leq 100R^{-2} \). Denote \( w := \eta\Phi_{\xi \xi} \). Since \( w = 0 \) on \( \partial B_{R/2} \), it attains its maxima at an interior point \( X_0 \), where
\[
\nabla w = \eta \nabla \Phi_{\xi \xi} + \Phi_{\xi \xi} \nabla \eta = 0,
\]
(A.3)

\[
0 \geq \Delta w = \Delta \Phi_{\xi \xi} \eta + 2\nabla \Phi_{\xi \xi} \nabla \eta + \Phi_{\xi \xi} \Delta \eta
\geq -f'(\Phi)(1 - |\nabla \Phi|^2)w - f''(\Phi)(1 - |\nabla \Phi|^2)\Phi_{\xi}^2 \eta
\]

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\[ +4f'(\Phi) \sum_{k=1}^{n+1} \Phi_k \Phi_{\xi_k} \Phi_{\xi} \eta + 2f(\Phi) \sum_{k=1}^{n+1} \Phi_k \Phi_{\xi_k \xi} \eta + 2f(\Phi) \sum_{k=1}^{n+1} \Phi_k^2 \xi_k \\
+ 2\nabla \Phi_{\xi \xi} \nabla \eta + w\eta^{-1} \Delta \eta. \]

Substituting (A.3) into this, and applying the Cauchy inequality to the third term, we obtain
\[
4 \frac{f'(\Phi)^2}{f(\Phi)} |\nabla \Phi|^2 \xi^2 \eta + f''(\Phi)(1 - |\nabla \Phi|^2) \xi^2 \eta \\
\geq -f'(\Phi)(1 - |\nabla \Phi|^2)w - 2f(\Phi)\nabla \Phi \nabla \eta \eta^{-1}w + f(\Phi)\eta^{-1}w^2 + w\eta^{-1} \Delta \eta - 2w\eta^{-2}|\nabla \eta|^2.
\]

This implies that
\[
w(x_0) \leq \frac{f'(\Phi)}{f(\Phi)} (1 - |\nabla \Phi|^2)\eta + 2|\nabla \Phi| |\nabla \eta| + \frac{1}{f(\Phi)} |\Delta \eta| + \frac{2}{f(\Phi)} \eta^{-1}|\nabla \eta|^2 \\ + \frac{1}{f(\Phi)} \left( 4 \frac{f'(\Phi)^2}{f(\Phi)} |\nabla \Phi|^2 \xi^2 \eta + f''(\Phi)(1 - |\nabla \Phi|^2) \xi^2 \eta \right)^{\frac{1}{2}} \tag{A.4}
\]

Since \( \Phi \geq R/2 \) in \( B_{R/2} \), by the definition of \( f \) and some standard estimates on \( g(t) \),
\[
f(\Phi) > c \quad \text{in} \quad B_{R/2},
\]
\[
|f'(\Phi)| + |f''(\Phi)| \leq Ce^{-cR} \quad \text{in} \quad B_{R/2}.
\]

Substituting these into (A.4) and noting that \( |\nabla \Phi| \leq 1 \) and the condition on \( \eta \), we get
\[
\sup_{B_{R/4}} \Phi_{\xi \xi} \leq w(x_0) \leq CR^{-1}.
\]

Rescaling back we get the claimed estimate. \( \square \)

**Theorem A.4.** Assume that \( \Phi_\varepsilon \) converges to \( \Phi_0 \) in \( C^0(\Omega) \), where \( \Omega \subset \mathbb{R}^{n+1} \) is an open set and \( \Phi_0 > 0 \) in \( \Omega \). If \( \Phi_0 \in C^1(\Omega) \), then \( \Phi_\varepsilon \) converges to \( \Phi_0 \) in \( C^1_{\text{loc}}(\Omega) \).

**Proof.** Fix an open set \( \Omega_0 \subset \subset \Omega \). Take an arbitrary sequence \( X_\varepsilon \in \Omega_0 \) such that \( X_\varepsilon \to X_0 \in \Omega_0 \) as \( \varepsilon \to 0 \). By the uniform semi-concavity of \( \Phi_\varepsilon \) in \( \Omega_0 \), there exists a constant \( C(\Omega_0) \) such that, for all \( \varepsilon > 0 \)
\[
\bar{\Phi}_\varepsilon(X) := \Phi_\varepsilon(X) - C(\Omega_0) |X - X_\varepsilon|^2
\]
are concave in \( \Omega_0 \). In particular, for any unit vector \( e \) and \( h < \text{dist}(\Omega_0, \partial \Omega) \),
\[
\bar{\Phi}_\varepsilon(X_\varepsilon + he) \leq \bar{\Phi}_\varepsilon(X_\varepsilon) + h \nabla \bar{\Phi}_\varepsilon(X_\varepsilon)e.
\]

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Because $|\nabla \Phi_\varepsilon(X_\varepsilon)| \leq 1$, we can assume that $\nabla \Phi_\varepsilon(X_\varepsilon)$ converge to a vector $\xi$. By the uniform convergence of $\Phi_\varepsilon$ in $\Omega$, in the limit we have

$$\Phi_0(X_0 + he) \leq \Phi_0(X_0) + h(\xi \cdot e) + \frac{C(\Omega_0)h^2}{2}, \quad \forall h > 0.$$ 

Since $\Phi_0$ is differentiable at $X_0$, letting $h \to 0$, we see $\xi = \nabla \Phi_0(X_0)$. 

\[\square\]

B Several technical results

Here we collect some technical results used in this paper.

The first one is an exponential decay estimate. This has been used in many places and can be proved by various methods, so here we only state the result.

**Lemma B.1.** If in the ball $B_{2R}(0)$, $u \in C^2$ satisfies

$$\begin{cases}
    \Delta u \geq Mu, \\
    0 \leq u \leq 1,
\end{cases} \quad \text{(B.1)}$$

then

$$\sup_{B_R(0)} u \leq C_1(n)e^{-C_2(n)RM^\frac{1}{2}},$$

where $C_1(n)$ and $C_2(n)$ depend on the dimension $n$ only.

The next one gives a control of the discrepancy using the excess.

**Lemma B.2.** Given $M > 0$, for any $\tau > 0$ there exists a $\delta > 0$ so that the following holds. Suppose that $u$ is a solution of (2.1) in $B_2$, satisfying

$$\int_{B_2} \frac{1}{2} |\nabla u|^2 + W(u) \leq M,$$

$$\int_{B_2} [1 - (\nu \cdot e_{n+1})^2] |\nabla u|^2 \leq \delta.$$ 

Then

$$\int_{B_1} \left| W(u) - \frac{1}{2} |\nabla u|^2 \right| \leq \tau.$$ 

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Proof. This can be proved by a contradiction argument. We need to use the fact that for any one dimensional solution, i.e. a solution \( u \) satisfying
\[
\int_{B_2} \left[ 1 - \left( \nu \cdot e_{n+1} \right)^2 \right] |\nabla u|^2 = 0,
\]
we have the first integral (2.7), that is,
\[
W(u) = \frac{1}{2} |\nabla u|^2.
\]

The following result says the energy \( \epsilon |\nabla u_\epsilon|^2 \) is mostly concentrated on the transition part \( \{ |u_\epsilon| \leq 1 - b \} \).

**Lemma B.3.** Let \( u_\epsilon \) be a solution of (2.2) defined in \( B_2 \), satisfying
\[
\int_{B_2} \frac{\epsilon}{2} |\nabla u_\epsilon|^2 + \frac{1}{\epsilon} W(u_\epsilon) \leq M.
\]
For any \( \delta > 0 \), there exists a \( b > 0 \) such that
\[
\int_{\{ |u_\epsilon| > 1 - b \} \cap B_1} \epsilon |\nabla u_\epsilon|^2 \leq \delta.
\]

This is essentially [15, Proposition 5.1]. We just need to note that by the Modica inequality, we can bound \( \epsilon |\nabla u_\epsilon|^2 \) by \( \epsilon^{-1} W(u_\epsilon) \).

In fact we can give a different proof, by using the method in the proof of Corollary 6.4. More precisely, we have

**Lemma B.4.** Let \( u_\epsilon \) be as in the previous lemma. For any \( 1 \leq i \leq n + 1 \) and \( \delta > 0 \), there exists a \( b > 0 \) such that,
\[
\int_{\{ |u_\epsilon| > 1 - b \} \cap B_1} \epsilon \left( \frac{\partial u_\epsilon}{\partial x_i} \right)^2 \leq \delta \int_{\{ |u_\epsilon| < 1 - b \} \cap B_2} \epsilon \left( \frac{\partial u_\epsilon}{\partial x_i} \right)^2 + C \epsilon^2.
\]
Here \( C \) is a universal constant.

Finally, we give a result concerning the lower bound of energy in a ball.

**Lemma B.5.** For any \( M > 0 \), there exist two constants \( c(M) \) and \( R(M) \) so that the following holds. Assume \( u \) is a solution of (2.2) in \( B_R \) where \( R \geq 2R(M) \), satisfying \( |u(0)| \leq \gamma \), the Modica inequality and the energy bound
\[
\int_{B_R} \frac{1}{2} |\nabla u|^2 + W(u) \leq MR^n.
\]
Then for any \( r \in [1, R/2] \),
\[
\int_{B_r} |\nabla u|^2 \geq c(M)r^n. \tag{B.2}
\]
Proof. We first prove that, under the assumptions of this lemma (with a different constant $R_1(M)$),
\[ \int_{B_{R/2}} |\nabla u|^2 \geq c(M) R^n, \] (B.3)
if $R \geq R_1(M)$.

Assume this is not true, that is, the claimed $R_1(M)$ does not exist. Then there exists an $M > 0$ and a sequence of $u_i$, which are solutions of (2.2) in $B_{R_i}$ where $R_i \to +\infty$, satisfying $|u_i(0)| \leq \gamma$, the Modica inequality and the energy bound
\[ \int_{B_{R_i}} \frac{1}{2} |\nabla u_i|^2 + W(u_i) \leq M R_i^n, \]
but
\[ R_i^{-n} \int_{B_{R_i}} |\nabla u|^2 \to 0. \] (B.4)

Let $\varepsilon_i = R_i^{-1}$ and $u_{\varepsilon_i}(X) := u_i(R_i X)$. Then
\[ \int_{B_1} \frac{\varepsilon_i}{2} |\nabla u_{\varepsilon_i}|^2 + \frac{1}{\varepsilon_i} W(u_{\varepsilon_i}) \leq M, \]
\[ \int_{B_{1/2}} \varepsilon_i |\nabla u_{\varepsilon_i}|^2 \to 0. \] (B.5)

By the main result in [15], $\varepsilon_i |\nabla u_i|^2 dX \rightharpoonup \mu$ as measures, where $\mu$ is a positive Radon measure (in fact, the weight measure associated to the limit varifold, as in Section 4). Moreover, $u_{\varepsilon_i} \to \pm 1$ locally uniformly outside spt$\mu$. However, (B.5) obviously implies that $\mu(B_{1/2}) = 0$. Thus for all $i$ large, $|u_{\varepsilon_i}| > \gamma$ in $B_{1/2}$. This contradicts our assumption that $|u_{\varepsilon_i}(0)| \leq \gamma$ and proves (B.3).

By the monotonicity formula (recall that we have assumed the validation of the Modica inequality), for any $r \in (1, R)$
\[ \int_{B_r} \frac{1}{2} |\nabla u|^2 + W(u) \leq M r^n. \] (B.6)

Thus the above discussion covers the case $r \in [R_1(M), R]$ in (B.2), that is, (B.3) holds for every $r \in [R_1(M), R]$.

For the remaining case, we only need to note that it is impossible to have $u \equiv u(0)$, because otherwise
\[ \int_{B_r} \frac{1}{2} |\nabla u|^2 + W(u) = \omega_n W(u(0)) r^{n+1} \geq M r^n, \]
provided \( r \geq \frac{M}{\omega_{n+1}(\inf_{s \in [-\gamma,\gamma]} W(s))} \). Then it can be directly verified that

\[
\int_{B_1} |\nabla u|^2 \geq c(M),
\]

by using (B.6) with \( r = \frac{M}{\omega_{n+1}(\inf_{s \in [-\gamma,\gamma]} W(s))} \).

Choosing \( R(M) = \max\{R_1(M), \frac{M}{\omega_{n+1}(\inf_{s \in [-\gamma,\gamma]} W(s))}\} \) we can finish the proof. \( \square \)

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