INFINITE POPULATIONS OF MIGRANTS AS COMPLEX SYSTEMS: SELF-REGULATION

YURI KOZITSKY

Abstract. A model is proposed and studied describing an infinite population of point migrants arriving in and departing from \( \mathcal{X} \subseteq \mathbb{R}^d, \ d \geq 1 \). Both these acts occur at random with state-dependent rates. That is, depending on their geometry the existing migrants repel and attract the newcomers, which makes the population a complex system. Its states are probability measures on an appropriate configuration space, and their evolution \( \mu_0 \rightarrow \mu_t \) is obtained by solving the corresponding Fokker-Planck equation. The main result is the conclusion that this evolution of states preserves their sub-Poissonicity, and hence a local self-regulation (suppression of clustering) takes place due to the inter-particle repulsion – no matter of how small range. Further possibilities to study the proposed model with the help of this result are also discussed.

1. Introduction and Setup

1.1. Infinite populations in noncompact habitats. “A complex system is any system featuring a large number of interacting components (agents, processes, etc.) whose aggregate activity is nonlinear (not derivable from the summations of the activity of individual components) and typically exhibits hierarchical self-organization under selective pressures.” This typical description\(^1\) (definition) of a complex system clearly demonstrates that self-regulation belongs to its crucial properties arising from interactions between the constituents. Large systems of living entities are always named in this context. The development of the mathematical theory of such systems is a challenging task of modern applied mathematics \(2, 3, 23\).

This includes also modeling populations as configurations of point entities placed in some continuous habitat, see \(5, 6, 22, 24, 26\), in particular, those where the dynamics amounts to the appearance and disappearance of their members. For such models, a key question of their theory is how the appearance/disappearance of a given entity is related to its interaction with the rest of the population and what might be the outcome of such interaction.

In simple models, the considered populations are finite and have no structure. The state space of such a population is usually taken as the set of nonnegative integers \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). In such a case, in state \( n \in \mathbb{N}_0 \) the population consists of \( n \) entities, which is its complete characterization. Then the only observed result of the interplay between the appearance and disappearance is the evolution of the number of entities in the population. Of course, possible interactions in this case can be taken into account only indirectly. The theory of such models goes back to works by A. Kolmogorov and W. Feller, see \(9\) Chapter XVII] and, e.g.,

\(^1\)see [http://www.informatics.indiana.edu/rocha/publications/complex/csm.html](http://www.informatics.indiana.edu/rocha/publications/complex/csm.html)
for a more recent account of the related concepts and results. Therein, the time evolution of the probability of having \( n \) entities is obtained by solving the Kolmogorov equation with a tridiagonal infinite matrix on the right-hand side. Its entries are expressed in terms of birth and death rates, \( \lambda_n \) and \( \mu_n \), respectively. Possible influence of the entities on each other is reflected in the dependence of theses rates on \( n \). If the increase of \( \lambda_n \) and \( \mu_n \) is controlled by affine functions of \( n \), the solution of the Kolmogorov equation is given by a stochastic semigroup, see, e.g., [1] and the literature quoted in that work.

A population becomes a complex system if it is given an inner structure closely related to interactions of its constituents – now explicitly taken into consideration. An inner structure of a given population can be set by assigning a trait to each of its constituents. Usually, interactions are binary, dependent on the traits of the interacting entities. The set of all such traits, \( X \), is mostly a locally compact topological space. It is called habitat being interpreted as the set of spatial locations of the entities. Its topology is employed to determine the (finite) collection of entities interacting with a given one. Namely, it is supposed that the traits of the entities interacting with the entity with trait \( x \) lie in a compact neighborhood of \( x \). This determines the local structure of the population. In order to qualitatively distinguish between local and global aspects of its dynamics, one has to assume that \( X \) is noncompact. If the whole population is finite, it is contained in a compact subset \( Y \subset X \) – a finite sum of compact sets containing the trait of each single entity. If the population ‘stays’ in this \( Y \) forever, then \( Y \) is an actual habitat of the population, in contrast to our assumption. If in the course of evolution the population disperses beyond each compact subset of \( X \), it can be characterized as developing. In such systems, boundary effects play a major role, see [7], whereas inter-particle interactions in the bulk are not essential [19]. Thus, in order to understand mathematical mechanisms of interaction-induced self-regulation, one has to study the dynamics of an infinite population in a noncompact habitat.

1.2. Poisson states and self-regulation in infinite populations. In this article, we propose and study a model of an infinite population of point ‘particles’ (migrants), arriving in and departing from a habitat \( X \subset \mathbb{R}^d, d \geq 1 \). Both these acts of their dynamics occur at random and are described in a way that takes into account possible heterogeneity of the habitat and inter-particle attraction and repulsion (competition). It is clear, cf. [18 Sect. 2.3], that without interaction the distribution of migrants should eventually reflect the heterogeneity of the habitat and be eventually Poissonian [15] – in view of the randomness mentioned above and the lack of interactions. It might also be clear that if the already existing population attracts the newcomers, they can form dense clusters and thus their distribution will no longer be Poissonian. Then the question which we raise and answer here is whether mutual repulsion (competition) – which gives rise to the increase of emigration – can prevent the appearance of dense clusters and thus make the states nearly Poissonian. If this is the case, one can qualify it as a kind of self-regulation of the considered population of migrants – similarly as in the case of birth-and-death [17] and fission-death [21] systems.

For convenience, including for having translation invariance, we take the habitat \( X = \mathbb{R}^d \), equipped with the natural (Euclidean) distance and the corresponding mathematical structures: topology, Borel \( \sigma \)-field of subsets, etc. The (pure) states of the population are configurations – locally finite subsets \( \gamma \subset \mathbb{R}^d \). The latter
means that, for a given $\gamma$, the set $\gamma_\Lambda := \gamma \cap \Lambda$ is finite whenever $\Lambda \subset \mathbb{R}^d$ is compact. By $\Gamma$ we denote the collection of all such configurations $\gamma$. For each compact $\Lambda \subset \mathbb{R}^d$, one defines the counting map $\Gamma \ni \gamma \mapsto N_\Lambda(\gamma) := |\gamma_\Lambda|$, where $|\cdot|$ denotes cardinality. Thereby, one sets $\Gamma^{\Lambda,n} := \{\gamma \in \Gamma : |\gamma_\Lambda| = n\}$, $n \in \mathbb{N}_0$, and equips $\Gamma$ with the $\sigma$-field generated by all such $\Gamma^{\Lambda,n}$. This allows for considering probability measures on $\Gamma$ as states of the population – a natural way of doing this in view of the randomness of the basic evolution acts. Among such states are Poissonian ones. In these states, the particles are independently distributed over $\mathbb{R}^d$, see [15, Chapter 2]. They may serve as reference states, with which other states are compared.

The homogeneous Poisson measure $\pi_\kappa$ with density $\kappa > 0$ is defined by its values on $\Gamma^{\Lambda,n}$ with $n \in \mathbb{N}_0$ and compact $\Lambda$, given by the formula

$$\pi_\kappa(\Gamma^{\Lambda,n}) = (\kappa V(\Lambda))^{n} \exp \left(-\kappa V(\Lambda)\right) / n!,$$

where $V(\Lambda)$ denotes Lebesgue’s measure (volume) of $\Lambda$. The probabilistic interpretation of (1.1) might be as follows: $\Gamma^{\Lambda,n}$ is the event “there is $n$ particles in $\Lambda$", and $\pi_\kappa(\Gamma^{\Lambda,n})$ is its probability if the state is $\pi_\kappa$. The expected number of particles in $\Lambda$ then is $\kappa V(\Lambda) = \text{density} \times \text{volume}$. The independent character of the distribution of particles is reflected in the fast decay of the right-hand side of (1.1) with $n$. Then, for a state $\mu$, the appearance of dense clusters of particles in a given $\Lambda$ in this state can be established by the fact that the decay of $\mu(\Gamma^{\Lambda,n})$ is not as fast as in (1.1), i.e., the probability law of $N_\Lambda$ has a heavy tail. In this article, we employ sub-Poissonian states (cf. Definition 2.1 and Remark 2.1 below), for each of which and for every compact $\Lambda \subset \mathbb{R}^d$, the following holds

$$\forall n \in \mathbb{N} \quad \mu(\Gamma^{\Lambda,n}) \leq n! \left(\frac{\kappa}{n}\right)^n \pi_\kappa(\Gamma^{\Lambda,n}),$$

with some positive $\kappa$. In view of this bound, sub-Poissonian states are characterized by the lack of clustering or, equivalently, by the lack of heavy tails of the law of $N_\Lambda$ (following by (1.2) and Stirling’s formula). The particles in sub-Poissonian state are either independent in taking their positions or ‘prefer’ to stay away of each other.

The counting map $\Gamma \ni \gamma \mapsto N_\Lambda(\gamma)$ can also be defined for $\Lambda = \mathbb{R}^d$. Set $\Gamma^n = \{\gamma \in \Gamma : |\gamma| = n\}$. The set of finite configurations

$$\Gamma_0 := \bigcup_{n \in \mathbb{N}_0} \Gamma^n$$

is clearly a measurable subset of $\Gamma$. In a state, $\mu$, with the property $\mu(\Gamma_0) = 1$, the system is finite. Let us show that $\pi_\kappa(\Gamma_0) = 0$. To this end, we take an exhausting sequence $\{\Lambda_m\}_{m \in \mathbb{N}}$ of compact $\Lambda_m \subset X$. That is, $\Lambda_m \subset \Lambda_{m+1}$, $m \in \mathbb{N}$, and each $x \in X$ is eventually contained in its elements. It is clear that $V(\Lambda_m) \to +\infty$ in this case. Moreover, for each $n$, it is possible to pick $\{\Lambda_m\}_{m \in \mathbb{N}}$ in such a way that

$$\sum_{m=1}^{\infty} \pi_\kappa(\Gamma^{\Lambda_m,n}) < \infty,$$

see (1.1). Assume now that $\pi_\kappa(\Gamma_0) > 0$. By (1.3) it then follows that $\pi_\kappa(\Gamma^n) > 0$ for some $n \in \mathbb{N}$. Fix this $n$ and write $\Gamma^n = \Gamma^{\Lambda_m,n} \cup \Gamma^n_{\epsilon,m}$, $\Gamma^n_{\epsilon,m} := \Gamma^n \setminus \Gamma^{\Lambda_m,n}$, which makes sense for each $n$. For a given $\epsilon$, we pick $m_\epsilon$ such that $V(\Lambda_{m_\epsilon}) \kappa \geq n$.
and
\[
\max\left\{ \pi_\varepsilon(\Gamma^{A_m,n}); \sum_{k=m_\varepsilon+1}^{\infty} \pi_\varepsilon(\Gamma^{A_k,n}) \right\} < \frac{\varepsilon}{2},
\]
which is possible in view of (1.1) and (1.4). Clearly, the letter inequality remains true after replacing \( m_\varepsilon \) by any \( m > m_\varepsilon \). By our assumptions each \( \gamma \in \Gamma^n \) is contained in some \( \Lambda_m \), hence
\[
\forall m > m_\varepsilon \quad \Gamma^{A_m,n} \subset \bigcup_{k=m_\varepsilon+1}^{\infty} \Gamma^{A_k,n},
\]
by which and (1.5) we then conclude that \( \pi_\varepsilon(\Gamma^n) < \varepsilon \), which in fact yields \( \pi_\varepsilon(\Gamma_0) = 0 \). Hence, the system in state \( \pi_\varepsilon \) is infinite. A nonhomogeneous Poisson measure \( \pi_\varrho \), characterized by a density \( \varrho: \mathbb{R}^d \to [0, +\infty) \), satisfies (1.1) with \( \varphi V(\Lambda) \) replaced by \( \int_\Lambda \varrho(x)dx \). Then either \( \pi_\varrho(\Gamma_0) = 1 \) or \( \pi_\varrho(\Gamma_0) = 0 \), depending on whether or not \( \varrho \) is globally integrable. In this work, we consider infinite systems.

1.3. The model. To characterize states on \( \Gamma \) one employs observables – appropriate functions \( F: \Gamma \to \mathbb{R} \). Their evolution is obtained from the backward Kolmogorov equation
\[
\frac{d}{dt}F_t = LF_t, \quad F_t|_{t=0} = F_0, \quad t > 0,
\]
where the Kolmogorov operator \( L \) specifies the model. The states’ evolution is then obtained from the forward Kolmogorov (called also Fokker–Planck) equation
\[
\frac{d}{dt}\mu_t = L^*\mu_t, \quad \mu_t|_{t=0} = \mu_0,
\]
related to that in (1.6) by the duality \( \mu_t(F_0) = \mu_0(F_t) \), where
\[
\mu(F) := \int_\Gamma F(\gamma)\mu(d\gamma).
\]
Direct solving of (1.7) assumes putting it in a Banach space setting, which includes also defining \( L^* \) as a linear operator in the corresponding Banach space, see [19]. For infinite populations, this is usually impossible. Instead, one may solve the corresponding weak Fokker-Planck equation
\[
\mu_t(F) = \mu_0(F) + \int_0^t \mu_s(LF)ds,
\]
that has to hold for all \( F \) from a sufficiently big class of functions. We refer the reader to the monograph [4] for a general theory of Fokker-Planck equations.

The model proposed and studied in this work is specified by the following Kolmogorov operator
\[
(LF)(\gamma) = \int_{\mathbb{R}^d} E^+(x,\gamma) [F(\gamma \cup x) - F(\gamma)] dx
- \sum_{x \in \gamma} E^-(x,\gamma \setminus x) [F(\gamma) - F(\gamma \setminus x)].
\]
The first term in (1.9) describes immigration with rate
\[
E^+(x,\gamma) = b^+(x) + \sum_{y \in \gamma} a^+(x - y),
\]
where \( b^+ \) and \( a^+ \) are nonnegative. Here the first term corresponds to state-independent immigration, whereas the second one describes attraction of the arriving 'immigrants' by the existing population. The second term in (1.9) describes emigration. Similarly as in (1.10), we take it in the form

\[
E^-(x, \gamma) = b^-(x) + \sum_{y \in \gamma} a^-(x - y),
\]

(1.11)

where both \( b^- \) and \( a^- \) are nonnegative. Note that the second term in (1.11) describes repulsion of the particle located at \( x \) by the rest of the population, that can also be considered as competition. Note also that the dependence of \( b^\pm \) on \( x \in X \) reflects possible heterogeneity of the habitat. In the sequel, the functions \( a^\pm \) and \( b^\pm \) are called kernels. The probabilistic meaning of both terms in (1.9) is as follows. For a given \( \gamma \) and \( x \in \gamma \), the probability to find a particle at \( x \) after time \( t \) is \( \exp(-tE^-(x, \gamma \setminus x)) \). Likewise, the probability to find a new particle at \( y \in \mathbb{R}^d \) after time \( t \) is \( 1 - \exp(-tE^+(y, \gamma)) \), where the difference in these two formulas is related to the different signs of the corresponding terms of \( L \) and different initial conditions.

In this article, the model parameters are supposed to satisfy the following.

**Assumption 1.1.** The kernels \( a^\pm \) in (1.10) and (1.11) are continuous and belong to \( L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \). The kernels \( b^\pm \) are continuous and bounded.

According to this we set

\[
\|a^\pm\| = \sup_{x \in \mathbb{R}^d} a^\pm(x), \quad \|b^\pm\| = \sup_{x \in \mathbb{R}^d} b^\pm(x).
\]

(1.12)

The assumed continuity has rather technical origin, whereas the boundedness and integrability are essential.

**Remark 1.2.** Concerning the kernels \( a^\pm \), the following alternatives are possible:

(i) (long competition) there exists \( \theta > 0 \) such that \( a^-(x) \geq \theta a^+(x) \) for all \( x \in \mathbb{R}^d \);

(ii) (short competition) for each \( \theta > 0 \), there exists \( x \in \mathbb{R}^d \) such that \( a^-(x) < \theta a^+(x) \).

In case (i), \( a^+ \) decays faster than \( a^- \), and hence the effect of repulsion from the existing population prevails. If \( b^+(x) \equiv 0 \), new members appear only due to the existing population, which can also be interpreted as their birth. In this case, \( a^+ \) is usually referred to as dispersal kernel and (i) then corresponds to short dispersal. This particular case of (1.9) with nonzero \( b^- \) and \( a^- \) is the Bolker-Pacala model – introduced in [5, 6] and studied in [16, 17, 25]. Such models with short dispersal are employed to describe, e.g., the evolution of cell communities [10]. An instance of the long competition (short dispersal) is given by \( a^+ \) with finite range, i.e., \( a^+(x) \equiv 0 \) for all \( |x| \geq r \), and \( a^-(x) > 0 \) for such \( x \). In case (ii), \( a^- \) decays faster than \( a^+ \), and hence some of the newcomers can be out of the competitive influence of the existing population. Models of this kind can be adequate, e.g., in plant ecology with the long-range dispersal of seeds, cf. [24]. Notably, the equality \( a^+ = a^- \) – corresponding to case (i) – does not yet mean the lack of interaction. This is a typical example of ‘nonlinearity’ of a complex system mentioned above.

Particular cases of the model specified by (1.9) were studied in: (a) [12, 16, 17], case of \( b^+ \equiv 0 \); (b) [18], case of \( a^+ \equiv 0 \). In case (a), our result – formulated in
Theorem 2.9 below – yields an extension of the corresponding result of [16] as it holds for both cases, (i) and (ii), mentioned in Remark 1.2.

In Section 2 below, we introduce necessary technicalities and then formulate the result: Theorem 2.9 and Corollary 2.10. Thereafter, we make a number of comments to these statements and compare them with the facts known for similar models. We also discuss possible continuation of studying the proposed model based on the result stated in Theorem 2.9. A complete proof of the latter will be done in a separate publication.

2. The Result

By \( \mathcal{B}(\mathbb{R}^d) \) we denote the sets of all Borel subsets of \( \mathbb{R}^d \). The configuration space \( \Gamma \) is equipped with the vague (weak-hash) topology, see [8], and thus with the corresponding Borel \( \sigma \)-field \( \mathcal{B}(\Gamma) \), which makes it a standard Borel space. Note that \( \mathcal{B}(\Gamma) \) is exactly the \( \sigma \)-field generated by the sets \( \Gamma^\Lambda_n \) mentioned in Introduction. By \( \mathcal{P}(\Gamma) \) we denote the set of all probability measures on \( (\Gamma, \mathcal{B}(\Gamma)) \).

2.1. Sub-Poissonian measures. The space of finite configurations \( \Gamma_0 \) defined in (1.3) can be equipped with the topology induced by the vague topology of \( \Gamma \). It is precisely the weak topology determined by bounded continuous functions \( f \in \mathcal{C}_b(\mathbb{R}^d) \). Then the corresponding Borel \( \sigma \)-field \( \mathcal{B}(\Gamma_0) \) is a sub-field of \( \mathcal{B}(\Gamma) \).

One can show that a function \( G : \Gamma_0 \rightarrow \mathbb{R} \) is measurable if and only if there exists a family of symmetric Borel functions \( G(n) : (\mathbb{R}^d)^n \rightarrow \mathbb{R} \), \( n \in \mathbb{N} \) such that

\[
G(\{x_1, \ldots, x_n\}) = G(n)(x_1, \ldots, x_n). \tag{2.1}
\]

We also set \( G(0) = G(\emptyset) \) and consider the following class of functions.

Definition 2.1. A measurable function, \( G : \Gamma_0 \rightarrow \mathbb{R} \), is said to have bounded support if there exist \( N \in \mathbb{N} \) and a compact \( \Lambda \) such that: (a) \( G(n) \equiv 0 \) for all \( n > N \); (b) \( G(\eta) = 0 \) whenever \( \eta \) is not a subset of \( \Lambda \). By \( B_{bs} \) we will denote the set of all bounded functions with bounded support. For \( G \in B_{bs} \), \( N_G \) and \( \Lambda_G \) will denote the least \( N \) and \( \Lambda \) as in (a) and (b), respectively. We also set \( C_G = \sup_{\eta \in \Gamma_0} |G(\eta)| \).

For \( G \in B_{bs} \), set

\[
(KG)(\gamma) = \sum_{\eta \subset \gamma} G(\eta), \quad \gamma \in \Gamma. \tag{2.2}
\]

Remark 2.2. For each \( G \in B_{bs} \), \( KG \) is measurable and such that \( |(KG)(\gamma)| \leq C_G(1 + |\gamma \cap \Lambda_G|^{|N_G|}) \) with \( C_G, \Lambda_G \) and \( N_G \) as in Definition 2.1.

The Lebesgue-Poisson measure \( \lambda \) is defined on \( \Gamma_0 \) by the integrals

\[
\int_{\Gamma_0} G(\eta) \lambda(d\eta) = G(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G(n)(x_1, \ldots, x_n) dx_1 \cdots dx_n, \tag{2.3}
\]

holding for all \( G \in B_{bs} \), for which \( G(n)'s \) are defined as in (2.1).

Let \( \mathcal{P}_{lm}(\Gamma) \) be the set of all probability measures on \( \Gamma \) that have all local moments. This means that such measures satisfy the condition

\[
\int_{\Gamma} |\gamma\Lambda|^{|n|} \mu(d\gamma) < \infty,
\]
holding for all \( n \in \mathbb{N} \) and compact \( \Lambda \subset \mathbb{R}^d \). Let \( \Theta \) be the set of all continuous compactly supported functions \( \theta : \mathbb{R}^d \to (-1,0] \). Then, for each \( \mu \in \mathcal{P}(\Gamma) \), the function

\[
F^\theta(\gamma) := \prod_{x \in \gamma} (1 + \theta(x)) = \exp \left( \sum_{x \in \gamma} \log(1 + \theta(x)) \right), \quad \gamma \in \Gamma,
\]

is \( \mu \)-integrable. For \( \theta \in \Theta \), the map \( \gamma \mapsto \sum_{x \in \gamma} \log(1 + \theta(x)) \) – and hence the function \( F^\theta \) – are continuous in the vague topology. It is clear that each \( \theta \in \Theta \) is absolutely integrable on \( X \); hence, one can write

\[
\| \theta \| = \int_X |\theta(x)| \, dx.
\]

(2.4)

For \( \theta \in \Theta \), we define functions \( \Gamma_0 \ni \eta \mapsto e(\theta; \eta), \Gamma_0 \ni \eta \mapsto e_n(\theta; \eta), n \in \mathbb{N}_0 \), by setting

\[
e(\theta; \eta) = \prod_{x \in \eta} \theta(x), \quad e_n(\theta; \eta) = e(\theta; \eta) \mathbb{I}_{\Gamma^n}(\eta),
\]

(2.5)

where \( \mathbb{I}_{\Gamma^n} \) is the corresponding indicator. For \( n = 0 \), the second term in (2.5) has to be understood by taking into account that \( \Gamma_0 = \{\emptyset\} \) and \( \prod_{x \in \emptyset} \theta(x) = 1 \).

Clearly, \( e_n(\theta; \cdot) \in B_b \); hence, \( K e_n(\theta; \cdot) \) satisfies the corresponding estimate as in Remark 2.2, which implies that \( K e_n(\theta; \cdot) \) is \( \mu \)-integrable for all \( \mu \in \mathcal{P}_{lm}(\Gamma) \).

**Definition 2.3.** The set of states \( \mathcal{P}_{exp}(\Gamma) \) contains all those \( \mu \in \mathcal{P}_{lm}(\Gamma) \), for each of which, the map

\[
\Theta \ni \theta \mapsto \mu(K e_n(\theta; \cdot)) = \int_{\Gamma} (K e_n(\eta; \cdot))(\gamma) \mu(d\gamma)
\]

(2.6)

can be extended to a continuous monomial of order \( n \) on the real Banach space \( L^1(X) \) satisfying the estimate

\[
|\mu(K e_n(\theta; \cdot))| \leq \frac{1}{n!} \varkappa^n \| \theta \|^n,
\]

(2.7)

for a certain \( \varkappa > 0 \) and \( \| \cdot \| \) as in (2.4).

It can be shown that, for each \( \mu \in \mathcal{P}_{exp}(\Gamma) \), the right-hand side of (2.6) might be written in the form

\[
\mu(K e_n(\theta; \cdot)) = \frac{1}{n!} \int_{X^n} k_{\mu}^{(n)}(x_1, \ldots, x_n) \theta(x_1) \cdots \theta(x_n) \, dx_1 \cdots dx_n,
\]

(2.8)

where \( k_{\mu}^{(n)} \) – \( n \)-th order correlation function of \( \mu \) – is a symmetric elements of \( L^\infty(X^n) \) satisfying the following

\[
0 \leq k_{\mu}^{(n)}(x_1, \ldots, x_n) \leq \varkappa^n,
\]

(2.9)

with \( \varkappa \) being the same as in (2.4). The right-hand inequality in (2.4) is known as Ruelle’s bound. Note that \( k_{\mu}^{(0)}(x_1, \ldots, x_n) \equiv 1 \) since \( \mu(\Gamma) = 1 \). For the homogeneous Poisson measure, it follows that

\[
k_{\pi_n}^{(n)}(x_1, \ldots, x_n) = \varkappa^n, \quad n \in \mathbb{N}_0.
\]

(2.10)
For a compact $\Lambda \subset X$, let $\mathbb{1}_\Lambda$ be its indicator. Then one can write

$$N_\Lambda(\gamma) = |\gamma \cap \Lambda| = \sum_{x \in \gamma} \mathbb{1}_\Lambda(x),$$

which can be generalized to the following, cf. (2.2),

$$N^n_\Lambda(\gamma) = \sum_{l=1}^{n} l! S(n, l) \sum_{\{x_1, \ldots, x_l\} \subset \gamma} \mathbb{1}_\Lambda(x_l) \cdot \cdot \cdot \mathbb{1}_\Lambda(x_1)$$

(2.11)

where $S(n, l)$ is Stirling’s number of second kind – the number of ways to divide $n$ labeled items into $l$ unlabeled groups, see [28, Chapter 2]. Now we apply (2.8) and (2.9) in (2.11) and obtain

$$\mu(N^n_\Lambda) = \sum_{l=1}^{n} l! S(n, l) \int_{\Lambda^l} f(\gamma) \rho_{n, l}(\gamma) d\gamma$$

(2.12)

By the latter equality it follows that

$$\pi_{\mu}(N^n_\Lambda) = T_n(\kappa V(\Lambda)),$$

where $T_n$ is Touchard’s polynomial, cf. [28, Chapter 2]. Then

$$\int_{\Gamma} \exp(\beta N_\Lambda(\gamma)) \pi_{\mu}(d\gamma) = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} T_n(\kappa V(\Lambda))$$

(2.13)

$$= \exp((\kappa V(\Lambda)) (e^\beta - 1)), \quad \beta \in \mathbb{R}.$$

At the same time, for each $\mu \in \mathcal{P}_{\exp}(\Gamma)$, we have

$$\int_{\Gamma} \exp(\beta N_\Lambda(\gamma)) \mu(d\gamma) = \sum_{n=0}^{\infty} e^{\beta n} \mu(\Gamma^{\Lambda, n}),$$

(2.14)

which for $\pi_{\mu}$ reads

$$\int_{\Gamma} \exp(\beta N_\Lambda(\gamma)) \pi_{\mu}(d\gamma) = \sum_{n=0}^{\infty} e^{\beta n} \pi_{\mu}(\Gamma^{\Lambda, n})$$

(2.15)

Now we take into account (2.12), set in (2.14) and (2.15) $t = e^\beta$, and obtain from (2.13) the following

$$\sum_{n=0}^{\infty} t^n \mu(\Gamma^{\Lambda, n}) \leq \sum_{n=0}^{\infty} t^n \pi_{\mu}(\Gamma^{\Lambda, n}) = e^{-\kappa V(\Lambda)} \exp(t \kappa V(\Lambda)).$$

Then (1.2) is obtained from the latter by a standard Cauchy-like estimation.
Remark 2.4. Each $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ is sub-Poissonian in the sense of (1.2) and (2.12). Furthermore, let $G \in B_{\text{bs}}$ be positive, i.e., $G(\eta) \geq 0$ for $\lambda$-almost all $\eta \in \Gamma_0$, see (2.7). Then

$$\mu(KG) \leq \pi_\mu(KG) = G(\varnothing) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{X^n} G(n)(x_1, \ldots, x_n)dx_1 \cdots dx_n.$$  

The proof of the latter estimate is based on (2.9) and the fact that $\mu(KG) = G(\varnothing) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{X^n} G(\varnothing) \lambda(d\eta)$, holding for all $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$. In the second line of (2.16) we use the correlation function of $\mu$, related to $k^{(n)}_{\mu}$, $n \in \mathbb{N}_0$ in the sense of (2.1).

By (2.5) we have $e(\theta; \eta) = \sum_{n=0}^{\infty} e_n(\theta; \eta)$, which implies, see also (2.7), that

$$\mu(Ke(\theta; \cdot)) = \mu(F^\theta) = \int_{\Gamma} \prod_{x \in \gamma} (1 + \theta(x)) \mu(d\gamma). \quad (2.17)$$

This means that $\Theta \ni \mu(F^\theta)$ can be extended to a real exponential type entire function of $\theta \in L^1(X)$. For the homogeneous Poisson measure, the integral in (2.17) can be calculated explicitly, cf. (2.10),

$$\pi_\mu(F^\theta) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{X^n} k^{(n)}_{\pi_\mu}(x_1, \ldots, x_n) \theta(x_1) \cdots \theta(x_n)dx_1 \cdots dx_n \quad (2.18)$$

2.2. The statement. Before formulating the result we define the set of functions $F: \Gamma \rightarrow \mathbb{R}$ which we then use in (1.3).

Definition 2.5. The set $\mathcal{G}$ of functions $G: \Gamma_0 \rightarrow \mathbb{R}$ is defined as that containing all those $G$ which satisfy

$$\forall C > 0 \quad \int_{\Gamma_0} C^{[\theta]} |G(\eta)| \lambda(d\eta) < \infty.$$  

Note that

$$\int_{\Gamma_0} C^{[\theta]} |e(\theta; \eta)| \lambda(d\eta) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (C[\theta])^n < \infty.$$  

Hence, $e(\theta; \cdot) \in \mathcal{G}$ if $\theta$ is integrable.

Proposition 2.6. For each $G \in \mathcal{G}$, the function $\Gamma \ni \gamma \mapsto (KG)(\gamma)$ is $\mu$-integrable whenever $\mu$ is in $\mathcal{P}_{\text{exp}}(\Gamma)$. 

Proof. For \( G \in \mathcal{G} \), compact \( \Lambda \) and \( N \in \mathbb{N} \), set
\[
G^{\Lambda,N}(\eta) = \begin{cases} G(\eta), & \eta \subset \Lambda \text{ and } |\eta| \leq N; \\ 0, & \text{otherwise.} \end{cases}
\]
Then \( G^{\Lambda,N} \in B_{bs} \) and hence, see (2.10) and (2.11),
\[
\mu(KG^{\Lambda,N}) = G(\emptyset) + \sum_{n=1}^{N} \int_{\Lambda^n} k_\mu^{(n)}(x_1, \ldots, x_n)G^{(n)}(x_1, \ldots, x_n)dx_1 \cdots dx_n
\]
\[
\leq |G(\emptyset)| + \sum_{n=1}^{\infty} \frac{x_n}{n!} \int_{X^n} |G^{(n)}(x_1, \ldots, x_n)|dx_1 \cdots dx_n < \infty.
\]
Then the stated property follows by Lebesgue’s dominated convergence theorem.
\[\square\]

By Proposition 2.6 it follows that, for \( \mu \in \mathcal{P}_\exp(\Gamma) \) and \( G \in \mathcal{G} \), the following holds
\[
\mu(KG) = \int_{\Gamma_0} k_\mu(\eta)G(\eta)\lambda(d\eta),
\]
see (2.10). Now we set
\[
\mathcal{F} = \{ F = KG : G \in \mathcal{G} \}.
\]

Proposition 2.7. For each \( F \in \mathcal{F} \), it follows that \( LF \in \mathcal{F} \).

Proof. For \( G \in B_{bs} \), se define, cf. (1.9),
\[
(\hat{L}G)(\eta) = \int_{X} E^+(x, \eta)G(\eta \cup x)dx + \sum_{x \in \eta} \int_{X} a^+(x-y)G(\eta \setminus x \cup y)dy
\]
\[
- \left( \sum_{x \in \eta} E^-(x, \eta \setminus x) \right) G(\eta) - \sum_{x \in \eta} \left( \sum_{y \in \eta \setminus x} a^-(x-y) \right) G(\eta \setminus x)
\]
\[
= : (A_1G)(\eta) + \cdots + (A_4G)(\eta),
\]
that ought to hold for \( \lambda \)-almost all \( \eta \in \Gamma_0 \). Here \( E^\pm \) are given in (1.10) and (1.11). It is clear that \( |(\hat{L}G)(\eta)| < \infty \) as the sums in (2.21) are finite and the integrals are taken over a compact subset of \( X \). To prove that \( LF \in \mathcal{F} \) we have to show that: (a) \( \hat{L} \) can be extended to a self-map of \( \mathcal{G} \); (b) this extension and the operator defined in (1.9) satisfy \( LKG = \hat{L}G \) holding for all \( G \in \mathcal{G} \). To get (a) we employ the following evident property of the integrals defined in (2.23)
\[
\int_{\Gamma_0} \left( \int_{X} H(x, \eta)dx \right) \lambda(d\eta) = \int_{\Gamma_0} \left( \sum_{x \in \eta} H(x, \eta \setminus x) \right) \lambda(d\eta),
\]
holding for all appropriate functions \( H \). Along with (2.22) we use the estimate, see (1.12).
\[
\left| \sum_{x \in \eta} E^\pm(x, \eta) \right| \leq |\eta| \left( \|b^\pm\| + \frac{1}{2} \|a^\pm\|(|\eta| - 1) \right) = : c^\pm(|\eta|), \quad \eta \in \Gamma_0.
\]
For \( C > 0 \), set \( \mathcal{G}_C = L^1(\Gamma_0; C^1(d\lambda)) \), and let \( \| \cdot \|_C \) stand for the corresponding norm. Then (a) will be done if we show that each \( A_i \) defined in the last line of (2.21) acts
as a bounded linear operator from \(G_C\) to \(G_{C+\varepsilon}\), that holds for each positive \(C\) and \(\varepsilon\). By (2.23) we get

\[
\sup_{n\in\mathbb{N}} c^+(n) \left( \frac{C}{C + \varepsilon} \right)^n =: \delta^+(C, \varepsilon) < \infty.
\]

Then by means of (2.22) it follows that

\[
\|A_1 G\|_C \leq \int_{\Gamma_0} \left( \int_{\mathcal{X}} |E^+(x, \eta)| |G(\eta \cup x)| dx \right) C^{[\eta]} \lambda(d\eta)
\]

\[
= \int_{\Gamma_0} \left( \sum_{x \in \eta} E^+(x, \eta \setminus x) \right) |G(\eta)| C^{[\eta]-1} \lambda(d\eta)
\]

\[
\leq C^{-1} \delta^+(C, \varepsilon) \|G\|_{C+\varepsilon}.
\]

In a similar way, we get

\[
\|A_2 G\|_C \leq \delta^+(C, \varepsilon) \|G\|_{C+\varepsilon},
\]

\[
\|A_3 G\|_C \leq \delta^{-}(C, \varepsilon) \|G\|_{C+\varepsilon},
\]

\[
\|A_4 G\|_C \leq C\delta^{-}(C, \varepsilon) \|G\|_{C+\varepsilon}.
\]

In combination with (2.24) this implies that \(\hat{L}\) acts as a bounded linear operator from each \(G_C\) to \(G_{C+\varepsilon}\), which yields (a) since

\[
\mathcal{G} = \bigcap_{C>0} G_C.
\]

For \(G \in B_{bs}\), the equality \(LKG = K\hat{L}G\) follows by [13, Proposition 3.1, page 209]. Its extension to \(G \in \mathcal{G}\) follows as in (a). \(\square\)

Now we are at a position to formulate our main statement.

**Definition 2.8.** By a solution of the Fokker-Planck equation (1.8) we understand a map \([0, +\infty) \ni t \mapsto \mu_t \in \mathcal{P}(\Gamma)\) such that: (a) each \(F \in \mathcal{F}\), see (2.20), is \(\mu_t\)-absolutely integrable for all \(t \geq 0\); (b) for each \(F \in \mathcal{F}\), the map \([0, +\infty) \ni t \mapsto \mu_t(LF) \in \mathbb{R}\) is integrable on each \([0, T], T > 0\) and (1.8) is satisfied.

According to this definition, if \(t \mapsto \mu_t\) is a solution, then \(t \mapsto \mu_t(F) \in \mathbb{R}\) is absolutely continuous, and hence \(\mu_t(F) \to \mu_0(F)\) as \(t \to 0\), where \(\mu_0\) is considered as the initial condition for (1.8).

**Theorem 2.9.** For each \(\mu_0 \in \mathcal{P}_{exp}(\Gamma)\), there exists a unique map \([0, +\infty) \ni t \mapsto \mu_t \in \mathcal{P}_{exp}(\Gamma)\) that solves (1.8).

The proof of this statement is quite technical and will be done in a separate publication. It consists in constructing a map \([0, +\infty) \ni t \mapsto k_t\) such that: (a) each \(k_t\) is the correlation function of a unique \(\mu_t \in \mathcal{P}_{exp}(\Gamma)\), i.e., \(k_t = k_{\mu_t}\), see (2.8), (2.17); (b) the map \([0, +\infty) \ni t \mapsto \mu_t \in \mathcal{P}_{exp}(\Gamma)\) obtained in this way solves (1.8).

Note that, for each \(F \in \mathcal{F}\), the mentioned map \(t \mapsto k_t\) is such that the map, see (2.19),

\[
t \mapsto \mu_t(LF) = \mu_t(LKG) = \mu_t(K\hat{L}G) = \int_{\Gamma_0} k_t(\eta)(\hat{L}G)(\eta) \lambda(d\eta)
\]

is continuous and integrable on each \([0, T], T > 0\).
Corollary 2.10. If the initial state of the population is sub-Poissonian, i.e., \( \mu_0 \in \mathcal{P}_{\text{exp}}(\Gamma) \), then the evolution of its states \( \mu_0 \to \mu_t \) preserves this property. This is true for both long and short competition cases mentioned in Remark 1.2.

Since the solution described in Theorem 2.9 is unique, the evolution mentioned in Corollary 2.10 is the only possible, and hence the dynamics of the population manifests competition-caused self-regulation.

2.3. Further comments and comparison. For \( b^+ \equiv 0 \) and \( a^- \equiv 0 \), the model considered in this work gets exactly soluble. It is known as the continuum contact model, see [11], for which there is no self-regulation. Namely, under the following quite natural condition

\[
\inf_{x,y \in B} a^+(x - y) \geq \alpha > 0,
\]

satisfied in some ball \( B \), it was proved [11] Eq. (3.5), page 303] that

\[
k_{\mu_t}^{(n)}(x_1, \ldots, x_n) \geq \omega^n n!,
\]

holding for some \( \omega > 0 \), all \( n \geq 2 \) and \( t > 0 \), and almost all \( x_1, \ldots, x_n \in B \). Hence, for \( \theta(x) > 0 \), \( x \in B \), by (2.8) and (2.6) the latter implies

\[
\mu_t(Ke_n(\theta; \cdot)) \geq \omega^t,
\]

that clearly contradicts (2.7). It is possible to show that a similar bound holds true also in the model described by \( L \) as in (1.9) with \( a^- \equiv 0 \) and nonzero \( b^+ \). This means that the inter-particle competition represented in \( L \) by \( a^- \) – that gives rise to the increase of emigration – is the sole factor responsible for the effect mentioned in Corollary 2.10. Likewise, by comparing with the Bolker-Pacala model obtained from the letter by setting \( b^+ \equiv 0 \), one shows that if the following holds

\[
\inf_{x \in X} b^-(x) \geq \int_X a^+(x) dx,
\]

then the correlation functions \( k_{\mu_t} \) remain bounded in time, see [16]. That is, the global regulation may be achieved at the expense of large emigration. Moreover, the system eventually gets empty in this case.

As mentioned above, Poissonian states are completely characterized by their densities. That is, for \( \theta \in L^1(X) \) and a Poisson state \( \pi_\theta, \theta \in L^\infty(X) \), one has, cf. (2.18),

\[
\pi_\theta(F^\theta) = \exp \left( \int_X \theta(x) dx \right),
\]

which means that the corresponding correlation functions are

\[
k_{\pi_\theta}^{(n)}(x_1, \ldots, x_n) = \theta(x_1) \cdots \theta(x_n), \quad n \in \mathbb{N},
\]

and hence \( \theta(x) = k_{\pi_\theta}^{(1)}(x) \). At the mesoscopic level obtained by a scaling procedure, see [12] [20], the evolution of states is described as the evolution of their densities \( k_0^{(1)} \to k_t^{(1)} \) obtained by solving corresponding kinetic equations. Without interactions, the description of the evolution of Poisson states \( \pi_\theta \to \pi_{\theta_t} \) by equations like (1.8) is equivalent to the that obtained from the corresponding kinetic equations. Possible interactions in the system are taken into account in kinetic equations indirectly, and hence the mesoscopic description is less accurate. At the same time, by kinetic equations – and their more sophisticated versions [25] – it is possible to get much richer information as to the evolution of a given system, see [25] for a numerical study of the Bolker-Pacala model. It is believed that, for sub-Poissonian states,
passing from micro- to meso-scale – and hence from equations like (1.8) to kinetic equations – produces ‘minor errors’, and hence is acceptable. This means that the sub-Poissonicity established in Theorem 2.9 ‘justifies’ passing to the description of the evolution of the considered system based on the kinetic equation

\[
\frac{d}{dt} \varrho_t(x) = \left( b^+(x) - b^-(x) \right) \varrho_t(x) + \int_X a^+(x-y) \varrho_t(y) dy - \varrho_t(x) \int_X a^-(x-y) \varrho_t(y) dy,
\]

which can be derived from (1.6) and (1.9) in the same way as it was done for the Bolker-Pacala model in [12], and for a similar migration model in [20]. We believe that the study of (2.25) can yield further details of the evolution of the model proposed here – similarly as it was in the case for the models studied in [12, 20]. In particular, we expect to clarify the peculiarities of the cases mentioned in Remark 1.2. Note that the self-regulation described in Theorem 2.9 – a global effect – occurs even if the competition kernel is very short. We plan to study (2.25), also numerically, in a separate publication.

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