HAMILTONIAN AND SYMPLECTIC SYMMETRIES: AN INTRODUCTION

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In memory of Professor J.J. Duistermaat (1942–2010)

Abstract. Classical mechanical systems are modeled by a symplectic manifold \((M, \omega)\), and their symmetries, encoded in the action of a Lie group \(G\) on \(M\) by diffeomorphisms that preserves \(\omega\). These actions, which are called “symplectic”, have been studied in the past forty years, following the works of Atiyah, Delzant, Duistermaat, Guillemin, Heckman, Kostant, Souriau, and Sternberg in the 1970s and 1980s on symplectic actions of compact abelian Lie groups that are, in addition, of “Hamiltonian” type, i.e. they also satisfy Hamilton’s equations. Since then a number of connections with combinatorics, finite dimensional integrable Hamiltonian systems, more general symplectic actions, and topology, have flourished. In this paper we review classical and recent results on Hamiltonian and non Hamiltonian symplectic group actions roughly starting from the results of these authors. The paper also serves as a quick introduction to the basics of symplectic geometry.

1. Introduction

Symplectic geometry is a geometry concerned with the study of a notion of signed area, rather than length or distance. It can be, as we will see, less intuitive than Euclidean or metric geometry and it is taking mathematicians many years to understand some of its intricacies (which is still work in progress).

The word “symplectic” goes back to Hermann Weyl’s (1885-1955) book [159] on Classical Groups (1946). It derives from a Greek word meaning “complex”. Since the word “complex” had already a precise meaning in mathematics, and was already used at the time of Weyl, he took the Latin roots of “complex” (which means “plaited together”) and replaced them by the Greek roots “symplectic”.

The origins of symplectic geometry are in classical mechanics, where the phase space of a mechanical system is modeled by a “symplectic manifold” \((M, \omega)\), that is, a smooth manifold \(M\) endowed with a non-degenerate closed 2-form \(\omega \in \Omega^2(M)\), called a “symplectic form”. At each point \(x \in M\), \(\omega_x : T_x M \times T_x M \to \mathbb{R}\) is an antisymmetric bilinear form on \(T_x M\), and given \(u, v \in T_x M\) the real number \(\omega_x(u, v)\) is called the “symplectic area” spanned by \(u\) and \(v\). Intuitively, \(\omega\) gives a way to measure area along 2-dimensional sections of \(M\), which itself can be of an arbitrarily large dimension.

The most typical example of a symplectic manifold is a cotangent bundle, the phase space of mechanics, which comes endowed with a canonical symplectic form. Initially it was the study of mechanical systems which motivated many of the developments in symplectic geometry.

Joseph-Louis Lagrange (1736-1813) gave the first example of a symplectic manifold in 1808, in his study of the motion of the planets under the influence of their mutual gravitational interaction [100, 101]. An explicit description of Lagrange’s construction and his derivation of what are known today as Hamilton’s equations is given by Weinstein in [156, Section 2].

The origins of the current viewpoint in symplectic geometry may be traced back to Carl Gustav Jacob Jacobi (1804-1851) and then William Rowan Hamilton’s (1805-1865) deep formulation of Lagrangian mechanics, around 1835. Hamilton was expanding on and reformulating ideas of Galileo
Galilei (1564-1642), Christiaan Huygens (1629-1695), Leonhard Euler (1707-1883), Lagrange, and Isaac Newton (1642-1727) about the structure and behavior of orbits of planetary systems.

At the time of Newton and Huygens the point of view in classical mechanics was geometric. Later Lagrange, Jacobi, and Hamilton approached the subject from an analytic viewpoint. Through their influence the more geometric viewpoint fell out of fashion. Further historical details and references are given by Weinstein in [156]. Several treatments about mechanical systems in the 1960s and 1970s, notably including [7, 9, 11, 2, 147], had an influence in the development of ideas in symplectic geometry.

The modern viewpoint in symplectic geometry starts with the important contributions of a number of authors in the early 1970s (some slightly before or slightly after) including the works of Ralph Abraham, Vladimir Arnold, Johannes J. Duistermaat, Victor Guillemin, Bertram Kostant, Paulette Libermann, George Mackey, Jerrold Marsden, Clark Robinson, Jean-Marie Souriau, Shlomo Sternberg, and Alan Weinstein. Even at these early stages, many other authors contributed to aspects of the subject so the list of developments is extensive and we do not make an attempt to cover it here.

Symplectic geometry went through a series of developments in the period 1970-1985 where connections with other areas flourished, including: (i) geometric, microlocal and semiclassical analysis, as in the works of Duistermaat, Heckman, and Hörmander [35, 32, 33]; Duistermaat played a leading role in establishing relations between the microlocal and symplectic communities in particular through his article on oscillatory integrals and Lagrange immersions [33]; (ii) completely integrable systems, of which Duistermaat’s article on global action-angle coordinates [34] may be considered to mark the beginning of the global theory of completely integrable systems; (iii) Poisson geometry, as in Weinstein's foundational article [155]; (iv) Lie theory and geometric quantization, as in Kostant and Souriau’s geometric quantization [96, 145] (in early 1960s the quantum viewpoint had already reached significant relevance in mathematics, see Mackey’s mathematical foundations of quantum mechanics [104], on which the works by Segal [143] and Kirillov [93] had an influence; and (v) symplectic and Hamiltonian group actions, as pioneered by Atiyah [10], Guillemin-Sternberg [73], Kostant [95], and Souriau [146]. It is precisely symplectic and Hamiltonian group actions that we are interested in this paper, and we will give abundant references later.

An influential precursor in the study of global aspects in symplectic geometry, the study of which is often referred to as “symplectic topology”, is Arnold’s conjecture [9, Appendix 9] (a particular case appeared in [8]; see Zehnder’s article [162] for an expository account). Arnold’s conjecture is a higher dimensional analogue of the classical fixed point theorem of Henri Poincaré (1854-1912) and George Birkhoff (1884-1944) which says that any area-preserving periodic twist of a closed annulus has at least two geometrically distinct fixed points. This fixed point result can be traced to the work of Poincaré in celestial mechanics [137], where he showed that the study of the dynamics of certain cases of the restricted 3-Body Problem may be reduced to investigating area-preserving maps, and led him to this result, which he stated in [138] in 1912. The complete proof was given by Birkhoff [19] in 1925. Arnold realized that the higher dimensional version of the result of Poincaré and Birkhoff should concern “symplectic maps”, that is, maps preserving a symplectic form, and not volume-preserving maps, and formulated his conjecture. Arnold’s conjecture has been responsible for many of the developments in symplectic geometry (as well as in other subjects like Hamiltonian dynamics and topology).

In 1985 Gromov [64] introduced pseudoholomorphic curve techniques into symplectic geometry and constructed the first so called “symplectic capacity”, a notion of monotonic symplectic invariant pioneered by Ekeland and Hofer [42, 78, 79] and developed by Hofer and his collaborators, as well as many others, from the angle of dynamical systems and Hamiltonian dynamics.
There have been many major developments since the early 1980s, and on many different fronts of the symplectic geometry and topology, and covering them (even very superficially) would be beyond the scope of this paper. In this article we study only on the topic of symplectic and Hamiltonian group actions, item (v) above, starting roughly with the work of Atiyah and Guillemin-Sternberg.

While the phase space of a mechanical system is mathematically modeled by a symplectic manifold, its symmetries are described by symplectic group actions. The study of such symmetries or actions fits into a large body of work by the name of “equivariant symplectic geometry”, which includes tools of high current interest also in algebraic geometry, such as equivariant cohomology on which we will (very) briefly touch.

Mathematically speaking, equivariant symplectic geometry is concerned with the study of smooth actions of Lie groups $G$ on symplectic manifolds $M$, by means of diffeomorphisms $\varphi \in \text{Diff}(M)$ which pull-back the symplectic form $\omega$ to itself: $\varphi^* \omega = \omega$. A map $\varphi$ satisfying this condition is called a “symplectomorphism” following Souriau, or a “canonical transformation”. Actions satisfying this natural condition are called “symplectic”. As a first example of a symplectic action consider $S^2 \times (\mathbb{R}/\mathbb{Z})^2$ with the product form (of any areas forms on $S^2$ and $(\mathbb{R}/\mathbb{Z})^2$). The action of the 2-torus $(\mathbb{R}/\mathbb{Z})^2$ by translations on the right factor is symplectic.

In this paper we treat primarily the case when $G$ is a compact, connected, abelian Lie group, that is, a torus: $T \simeq (S^1)^k$, $k \geq 1$. Let $t$ be the Lie algebra of $T$, and let $t^*$ be its dual Lie algebra. We think of $t$ as the tangent space at the identity $1 \in T$.

Equivalently, a $T$-action is symplectic if $L_{X_M} \omega = 0$ for every $X \in t$, where $L$ is the Lie derivative and $X_M$ is the vector field generated by the $T$-action from $X \in t$ via the exponential map. In view of the homotopy formula for the Lie derivative, this is equivalent to

$$d(\omega(X_M, \cdot)) = 0$$

for every $X \in t$.

A fundamental subclass of symplectic actions admit what is called a “momentum map” $\mu: M \to t^*$, which is a $t^*$-valued smooth function on $M$ which encodes information about $M$ itself, the symplectic form, and the $T$-action, and is characterized by the condition that for all $X \in t$:

$$-d(\mu, X) = \omega(X_M, \cdot).$$

Such very special symplectic actions are called “Hamiltonian” (the momentum map was introduced generally for any Lie group action by Kostant [95] and Souriau [145]). A simple example would be to take $M = S^2 \subset \mathbb{R}^3$ and $T = S^1$ acting by rotations about the $z = 0$ axis in $\mathbb{R}^3$. In this case $t^* \simeq \mathbb{R}^* \simeq \mathbb{R}$ and $\mu: (\theta, h) \mapsto h$.

The fundamental observation here is that the right hand side of equation (2) is always a closed 1-form by (1) and being Hamiltonian may be rephrased as the requirement that this form is moreover exact. Therefore, the obstruction for a symplectic action to being Hamiltonian lies in the first cohomology group $H^1(M; \mathbb{R})$.

In particular, any symplectic action on a simply connected manifold is Hamiltonian. Notice that is an extremely stringent condition, for instance by (2) it forces the action to have fixed points on a compact manifold (because $\mu$ always has critical points, and these correspond to the fixed points of the action).

Many symplectic actions of interest in complex algebraic geometry and Kähler geometry are symplectic but not Hamiltonian; one such case is the action of the 2-torus on the Kodaira variety, which appears in Kodaira’s description [94, Theorem 19] of the compact complex analytic surfaces that have a holomorphic $(2, 0)$-form that is nowhere vanishing, described later in this paper (Example 5.5). Other symplectic actions that do not admit a momentum map include examples of interest
in classical differential geometry (e.g. multiplicity free spaces), and topology (e.g. nilmanifolds over nilpotent Lie groups).

Recent work of Susan Tolman (Theorem 3.18) indicates that even for low dimensional Lie groups $G$ most symplectic actions are not Hamiltonian. The advantage of having the existence of a momentum map $\mu: M \to t^*$ for a symplectic action has led to a rich general theory, part of which is described in this article. One can often find out information about $(M, \omega)$ and the $T$-action through the study of $\mu$. For instance, if $\dim M = 2 \dim T$, Delzant proved that the image $\mu(M) \subset t^*$ completely characterizes $(M, \omega)$ and the $T$-action, up to symplectic and $T$-equivariant transformations. We will see a proof of this result later in the paper.

Hamiltonian actions have been extensively studied since the 1970s following the seminal works of Atiyah [10], Delzant [31], Duistermaat-Heckman [35], Guillemin-Sternberg [73], and Kostant [95], and have been a motivation to study more general symplectic actions.

The majority of proofs/results about general symplectic actions use in an essential way the Hamiltonian theory, but also include other ingredients. The fact that there is not necessarily a momentum map $\mu$ means that Morse theory for $\mu$ and Duistermaat-Heckman theory, often used in the Hamiltonian case, must be replaced by alternative techniques.

Research on symplectic actions is still at its infancy and there are many unsolved problems. A question of high interest has been whether there are symplectic non-Hamiltonian $S^1$-actions with some, but only finitely many fixed points on compact connected manifolds. The aforementioned result by Tolman provides an example [149] of such an action with thirty two fixed points. In [62] the authors give a general lower bound for the number of fixed points of any symplectic $S^1$-action, under a mild assumption.

Approximately the first half of the paper concerns the period from 1970 to approximately 2002, where the emphasis is on symplectic Hamiltonian actions, its applications, and its implications, including the (subsequent) interactions with completely integrable systems. The second half of the paper concerns symplectic actions which are not necessarily Hamiltonian, with a focus on the developments that took place in the approximate period from 2002 to 2015.

Of course this separation is somewhat artificial, because Hamiltonian actions play a fundamental role in the study of other types of symplectic actions.

The paper gives a succinct introduction to the basics of symplectic geometry, followed by an introduction to symplectic and Hamiltonian actions, and it is written for a general audience of mathematicians. It is not a survey, which would require, due to the volume of works, a much longer paper. We will cover a few representative proofs with the goal of giving readers a flavor of the subject. The background assumed is knowledge of geometry and topology for instance as covered in second year graduate courses.

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Outline of Topics. Section 2 give a succinct introduction to symplectic geometry for a general mathematical audience.

Sections 3 and 4 introduce the basics of symplectic and Hamiltonian Lie group actions.

Section 5 contains examples of Hamiltonian and symplectic non-Hamiltonian torus actions.

Section 6 includes classification results on symplectic Hamiltonian actions of Lie groups. In most cases, the Lie group is compact, connected, and abelian; but a certain case of noncompact groups which is pertinent to completely integrable systems is also included.

The material from Section 7 onwards is probably less well known to nonexperts; it focuses on developments on symplectic group actions, not necessarily Hamiltonian, in the past fifteen years, with an emphasis on classification results in terms of symplectic invariants.
2. Symplectic manifolds

Symplectic geometry is concerned with the study a notion of signed area, rather than length, distance, or volume. In this sense it is a peculiar type of geometry, which displays certain non-intuitive features, as we see in this section.

Symplectic geometry displays a degree of flexibility and rigidity at the same time which makes it a rich subject, the study of which is of interest well beyond its original connection to classical mechanics.

2.1. Basic properties. For the basics of symplectic geometry, we recommend the textbooks [81, 22, 112]. This section gives a quick overview of the subject, and develops the fundamental notions which we need for the following sections. Unless otherwise specified all manifolds are $C^\infty$-smooth and have no boundary.

**Definition 2.1.** A symplectic manifold is a pair $(M, \omega)$ consisting of a smooth $C^\infty$-manifold $M$, and a smooth 2-form $\omega$ on $M$ which is: (1) closed, i.e. $d\omega = 0$; (2) non-degenerate, i.e. for each $x \in M$ it holds that if $u \in T_x M$ is such that $\omega_x(u,v) = 0$ for all $v \in T_x M$ at $x$, then necessarily $u = 0$. The form $\omega$ is called a symplectic form.

Conditions (1) and (2) can be geometrically understood as we will see.

**Example 2.2.** In dimension 2 a symplectic form and an area form are the same object. Accordingly, the simplest example of a symplectic manifold is given by a surface endowed with an area form. A typical non-compact example is the Euclidean space $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n)$ equipped with the symplectic form $\sum_{i=1}^{n} dx_i \wedge dy_i$. Any open subset $U$ of $\mathbb{R}^{2n}$ endowed with the symplectic form given by this same formula is also a symplectic manifold.

Let $X$ be a smooth $n$-dimensional manifold and let $(V, x_1, \ldots, x_n)$ be a smooth chart for $X$. To this chart we can associate a cotangent chart $(T^*V, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ on which we can define, in coordinates, the smooth 2-form given by the formula $\omega_0 := \sum_{i=1}^{n} dx_i \wedge d\xi_i$.

**Proposition 2.3.** The expression for $\omega_0$ is coordinate independent, that is, it defines a canonical form on the cotangent bundle $T^*X$, which is moreover symplectic and exact.

**Proof.** It follows by observing that $\omega_0 = -d\alpha$ where $\alpha := \sum_{i=1}^{n} \xi_i dx_i$, which one can check that is intrinsically defined. □

A way to construct symplectic manifolds is by taking products, and endowing them with the product symplectic form. As such, $S^2 \times \mathbb{R}^{2n}$ or $T^*X \times T^*Y$ are naturally symplectic manifolds, where $X$ and $Y$ are manifolds of any finite dimension.

There is a geometric interpretation of the closedness of the symplectic form in terms of area of a surface, as follows. Define the symplectic area of a surface $S$ (with or without boundary) inside of a symplectic manifold $(M, \omega)$ to be the integral

$$\text{symplectic area of } S := \int_S \omega \in \mathbb{R}.\quad (3)$$

By Stokes’ theorem, the closedness condition $d\omega = 0$ implies:

**Proposition 2.4.** Every point $x \in M$ has an open neighborhood such that if the surface $S$ is contained in $U$ then the symplectic area of $S$ does not change when deforming $S$ inside of $U$ while keeping the boundary $\partial S$ of $S$ fixed under the deformation. If $\omega$ is exact then $U = M$. 
Proof. If $d\omega = 0$ then locally near every point $x \in M$, $\omega = d\sigma$ for some smooth 1-form $\sigma \in \Omega^1(M)$. Hence

$$\int_S \omega = \int_S d\sigma = \int_{\partial S} \sigma,$$

and the result follows.

In view of Proposition 2.3 and Proposition 2.4 we obtain the following.

**Corollary 2.5.** The symplectic area of any surface $S$ in a cotangent bundle $(T^*X, \omega_0)$ depends only on the boundary $\partial S$. Moreover, if $\partial S = \emptyset$ then the symplectic area of $S$ is zero.

The fact that $\omega$ is non-degenerate gives:

**Proposition 2.6.** Let $(M, \omega)$ be a symplectic manifold. There is an isomorphism between the tangent and the cotangent bundles $TM \to T^*M$ by means of the mapping $X \mapsto \omega(X, \cdot)$.

In other words, the symplectic form allows us to give a natural correspondence between one-forms and vector fields.

One can ask some basic questions about symplectic manifolds. For instance, one can wonder:

**Question 2.7.** Does the 3-dimensional sphere $S^3$ admit a symplectic form?

The answer is “no”, because symplectic manifolds are even-dimensional; for otherwise the non-degeneracy condition is violated, which follows from elementary linear algebra. Similarly:

**Question 2.8.** Does the Klein bottle admit a symplectic form?

The answer to this question is again “no” since symplectic manifolds must be orientable; indeed, since the symplectic form $\omega$ is non-degenerate, the wedge $\omega^n = \omega \wedge \ldots \wedge \omega$, where $2n$ is the dimension of the manifold, is a volume form giving an orientation to $M$.

To summarize:

**Proposition 2.9.** Symplectic manifolds are even-dimensional and orientable.

To continue the discussion with spheres:

**Question 2.10.** Does the 4-dimensional sphere $S^4$ admit a symplectic form?

The answer is given by the following.

**Proposition 2.11.** Let $(M, \omega)$ be a compact symplectic manifold of dimension $2n$. Then its even-dimensional cohomology groups are non trivial, that is, $H^{2k}(M, \mathbb{R}) \neq 0$ for $1 \leq k \leq n$.

**Proof.** The cohomology class $[\omega^k]$ is nontrivial. This is an exercise which follows from Stokes’ theorem, using $d\omega = 0$. □

One immediate consequence of these observations is the following.

**Proposition 2.12.** The 2-dimensional sphere $S^2$ is the only sphere $S^n, n \geq 1$, which may be endowed with a symplectic form.

However, the question whether an arbitrary manifold it admits or not a symplectic form is in general very difficult.

For instance if $N$ is a closed oriented 3-manifold, in Friedl-Vidussi [57] and Kutluhan-Taubes [99], the authors study when a closed 4-manifold of the form $S^1 \times N$ admits a symplectic form, which turns out to imply that $N$ must fiber over the circle $S^1$ (for details refer to the aforementioned papers).
2.2. Symplectomorphisms. The natural maps between symplectic manifolds are the diffeomorphisms which preserve the symplectic structure, they are called canonical transformations, symplectic diffeomorphisms, or following Souriau [146], symplectomorphisms.

Definition 2.13. A symplectomorphism \( \varphi: (M_1, \omega_1) \to (M_2, \omega_2) \) between symplectic manifolds is a diffeomorphism \( \varphi: M_1 \to M_2 \) which satisfies \( \varphi^* \omega_2 = \omega_1 \). In this case we say that \( (M_1, \omega_1) \) and \( (M_2, \omega_2) \) are symplectomorphic.

Recall that the expression \( \varphi^* \omega_2 = \omega_1 \) in Definition 2.13 means that \((\omega_2)_x(d_x \varphi(u), d_x \varphi(v)) = (\omega_1)_x(u, v)\) for every point \( x \in M_1 \) and for every pair of tangent vectors \( u, v \in T_x M_1 \). That is, the symplectic area spanned by \( u, v \) coincides with the symplectic area spanned by the images \( d_x \varphi(u), d_x \varphi(w) \), for every \( x \in M_1 \) and for every \( u, v \in T_x M \).

Remark 2.14. Roughly speaking one can view symplectomorphisms as diffeomorphisms preserving the area enclosed by loops, or rather, the sum of the areas enclosed by their projections onto a collection of 2-dimensional planes. For instance, if \((M, \omega) = (\mathbb{R}^6, dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3)\), with coordinates \((x_1, y_1, x_2, y_2, x_3, y_3)\), then you would want to preserve the area (counted by \( \omega \) with sign depending on the orientation of the region inside) of the projection of any loop in \( \mathbb{R}^6 \) onto the \((x_1, y_1), (x_2, y_2)\) and \((x_3, y_3)\) planes. I learned how to think about symplectomorphisms in this way from Helmut Hofer.

The symplectic volume (or Liouville volume) of a symplectic manifold of dimension \( 2n \) is

\[
\text{vol}(M, \omega) := \frac{1}{n!} \int_M \omega^n.
\]

Of course, since symplectomorphisms preserve \( \omega \), they preserve the symplectic volume but the converse is false (we discuss this in Section 2.6).

Since the late twentieth century it is known that symplectic manifolds have no local invariants except the dimension. This is a result due to Jean-Gaston Darboux (1842-1917).

Theorem 2.15 (Darboux [30], 1882). Let \( (M, \omega) \) be a symplectic manifold. Near each point \( p \in M \) one can find coordinates \((x_1, y_1, \ldots, x_n, y_n)\) in which the symplectic form \( \omega \) has the expression \( \omega = \sum_{i=1}^{n} dx_i \wedge dy_i \). That is, any two symplectic manifolds \((M_1, \omega_1)\) and \((M_2, \omega_2)\) of the same dimension are locally symplectomorphic near any choice of points \( p_1 \in M_1 \) and \( p_2 \in M_2 \).

Theorem 2.15 gives a striking difference with Riemannian geometry, where the curvature is a local invariant.

Remark 2.16. It is important to note, however, that there are local aspects of symplectic actions which have only been understood recently (certainly much later than Darboux’s result), but in these cases we are not concerned with the local normal form of the symplectic form itself, but instead with that a geometric object (a form, a vector field, an action, etc.) defined on the manifold. For instance in the case of symplectic group actions which we will discuss later the local normal form of symplectic Hamiltonian actions is due to Guillemin-Marle-Sternberg [74, 105], and in the general case of symplectic actions to Ortega-Ratiu [117] and Benoist [15], in a neighborhood of an orbit.

In 1981 Alan Weinstein referred to symplectic geometry in [156] as “the more flexible geometry of canonical (in particular, area preserving) transformations instead of the rigid geometry of Euclid; accordingly, the conclusions of the geometrical arguments are often qualitative rather than quantitative.”

Manifestations of “rigidity” in symplectic geometry were discovered in the early days of modern symplectic geometry by Mikhail Gromov, Yakov Eliashberg, and others.
Theorem 2.17 (Eliashberg-Gromov [43, 44, 65]). The group of symplectomorphisms of a compact symplectic manifold is $C^0$-closed in the group of symplectomorphisms.

Remark 2.18. The group of symplectomorphisms of a manifold often has a complicated (but extremely interesting) structure, and many basic questions about it remain open, see Leonid Polterovich’s book [139]. Interest in the behavior of symplectic matrices may be found in the early days of symplectic geometry, in important work of Clark Robinson [141, 140]. See also Arnold [6] for work in a related direction.

2.3. Moser stability. In 1965 Jürgen Moser published an influential article [114] in which he showed that:

Theorem 2.19 (Moser [114]). If $\omega$ and $\tau$ are volume forms on a compact connected oriented smooth manifold without boundary $M$ such that
\[
\int_M \omega = \int_M \tau
\]
then there exists a diffeomorphism $\psi: M \to M$ such that $\psi^* \tau = \omega$.

Remark 2.20. It follows that the total symplectic area of $S^2$ given by (3) completely determines the symplectic form on $S^2$ (and $S^2$ is the only symplectic sphere according to Proposition 2.12).

An extension to fiber bundles of this result appears in [90].

The proof of Moser’s theorem uses in an essential way the compactness of $M$, but the method of proof, known as Moser’s method and described below, may be generalized. The following is the generalization (1979) to noncompact manifolds.

Theorem 2.21 (Greene-Shiohama [66]). If $M$ is a noncompact connected smooth manifold and $\omega$ and $\tau$ are volume forms on $M$ such that
\[
\int_M \omega = \int_M \tau \leq \infty
\]
then there is a volume preserving diffeomorphism $\varphi: M \to M$ such that $\varphi^* \tau = \omega$ provided that every end $\epsilon$ of $M$ it holds that $\epsilon$ has finite volume with respect to $\omega$ if and only if has finite volume with respect to $\tau$.

Remark 2.22. If the assumption on the ends does not hold then they prove that their result does not necessarily hold [66, Example in page 406].

This result was recently extended to fiber bundles with noncompact fibers in [131]. The simplest incarnation of this result is the case of trivial fiber bundles, which is equivalent to considering, instead of two volume forms, two smooth families of volume forms $\omega_t, \tau_t$, indexed by some compact manifold without boundary which plays the role of parameter space $B$. The Greene-Shiohama theorem produces for each point $t$ a volume preserving diffeomorphism $\varphi_t$ with the required properties, but there is no information given about how the $\varphi_t$ change when $t$ changes in $B$.

In the same article where Moser proved Theorem 2.19 he also proved the following stability result for symplectic forms.
Theorem 2.23 (Moser [114], 1965). Suppose we have a smooth family of symplectic forms \( \{\omega_t\}_{t \in [0,1]} \) on a compact smooth manifold \( M \) with exact derivative \( \frac{d\omega_t}{dt} = d\sigma_t \) (or \( [\omega_t] \) constant in \( t \)) then there exists a smooth family \( \{\varphi_t\}_{t \in [0,1]} \) of diffeomorphisms of \( M \) such that \( \varphi_t^* \omega_t = \omega_0 \).

Moser’s article introduced a method, known as “Moser’s method” to prove this stability result. In addition to Moser’s article there are now many expositions of the result, see for instance [112].

2.4. The fixed point theorem of Poincaré and Birkhoff. In his work in celestial mechanics [137] Poincaré showed the study of the dynamics of certain cases of the restricted 3-Body Problem may be reduced to investigating area-preserving maps. He concluded that there is no reasonable way to solve the problem explicitly in the sense of finding formulae for the trajectories.

Instead of aiming at finding the trajectories, in dynamical systems one aims at describing their analytical and topological behavior. Of a particular interest are the constant ones, i.e., the fixed points.

The development of the modern field of dynamical systems was markedly influenced by Poincaré’s work in mechanics, which led him to state (1912) the Poincaré-Birkhoff Theorem [138, 19]. It was proved in full by Birkhoff in 1925.

We will formulate the result (equivalently) for the strip covering the annulus.

Definition 2.24. Let \( S = \mathbb{R} \times [-1, 1] \). A diffeomorphism \( F: S \to S, F(q, p) = (Q(q, p), P(q, p)) \), is an area-preserving periodic twist if: (1) area preservation: it preserves area; (2) boundary invariance: it preserves \( \ell_{\pm} := \mathbb{R} \times \{\pm 1\} \), i.e. \( P(q, \pm 1) := \pm 1 \); (3) boundary twisting: \( F \) is orientation preserving and \( \pm Q(q, \pm 1) > \pm q \) for all \( q \); (4) periodicity: \( F(q + 1, p) = (1, 0) + F(q, p) \) for all \( p, q \).

The following is the famous result of Poincaré and Birkhoff on area-preserving twist maps.

Theorem 2.25 (Poincaré-Birkhoff [138, 19]). An area-preserving periodic twist \( F: S \to S \) has at least two geometrically distinct fixed points.

Arnold formulated the higher dimensional analogue of this result, the Arnold Conjecture [9] (see also [14], [80], [81], [162]). The conjecture says that a “Hamiltonian map” on a compact symplectic manifold possesses at least as many fixed points as a function on the manifold has critical points, we refer to Zehnder [162] for details. A. Weinstein [157] observed that Arnold’s conjecture holds on compact manifolds when the Hamiltonian map belongs to the flow of a sufficiently small Hamiltonian vector field. The first breakthrough on the conjecture was by Charles Conley and Eduard Zehnder [28], who proved it for the 2\(n\)-torus (a proof using generating functions was later given by Chaperon [25]). According to their theorem, any smooth symplectic map \( F: \mathbb{T}^{2d} \to \mathbb{T}^{2d} \) that is isotopic to the identity has at least \( 2d + 1 \) many fixed points. The second breakthrough was by Floer [51, 52, 53, 54].

There have been many generalizations of this result, see for instance [56, 129].

2.5. Lagrangian submanifolds. In the years 1970–1975 Alan Weinstein proved a series of foundational theorems about what Maslov called Lagrangian submanifolds which were influential in the development of symplectic geometry.

Definition 2.26. A submanifold \( C \) of a symplectic manifold \( (M, \omega) \) is isotropic if the symplectic form vanishes along \( C \), that is, \( i^* \omega = 0 \) where \( i: C \to M \) is the inclusion mapping.

It is an exercise to verify that if \( C \) is isotropic then \( 2 \dim C \leq \dim M \).
For instance, in the symplectic plane $(\mathbb{R}^2, dx \wedge dy)$ any line $x = constant$ or $y = constant$ is an isotropic submanifold. More generally:

**Example 2.27.** For any constants $c_1, \ldots, c_n$, $X_{c_1 \ldots c_n} := \{ (x_1, y_1, \ldots, x_n, y_n) \mid x_i = c_i \ \forall i = 1, \ldots, n \}$ is an isotropic submanifold of $(\mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i)$ of dimension $n$.

The following is due to Maslov [108].

**Definition 2.28.** An isotropic submanifold $C$ of a symplectic manifold $(M, \omega)$ is Lagrangian if $2 \dim C = \dim M$.

Cotangent bundles provide a source of Lagrangian submanifolds.

**Lemma 2.29.** The image of a section $s: X \to T^*X$ of the contangent bundle $T^*X$ is Lagrangian if and only if $\text{ds} = 0$.

The following is one of the foundational results of symplectic geometry, it is known as the Langrangian neighborhood theorem.

**Theorem 2.30 (Weinstein [153]).** Let $M$ be a smooth $2n$-dimensional manifold and let $\omega_0$ and $\omega_1$ be symplectic forms on $M$. Let $X$ be a compact $n$-dimensional submanifold. Suppose that $X$ is a Lagrangian submanifold of both $(M, \omega_0)$ and $(M, \omega_1)$. Then there are neighborhoods $V_0$ and $V_1$ of $X$, and a symplectomorphism $\varphi: (V_0, \omega_0) \to (V_1, \omega_1)$ such that $i_1 = \varphi \circ i_0$ where $i_0: X \to V_0$ and $i_1: X \to V_1$ are the inclusion maps.

Using Theorem 2.30, Weinstein proved the following result.

**Theorem 2.31 (Weinstein [153]).** Let $(M, \omega)$ be a symplectic manifold. Let $X$ be a closed Lagrangian submanifold. Let $\omega_0$ be the standard cotangent symplectic form on $T^*X$. Let $i_0: X \to T^*X$ be the Lagrangian embedding given by the zero section and $i: X \to T^*X$ be the Lagrangian embedding given by the inclusion. Then there are neighborhoods $V_0$ of $X$ in $T^*X$ and $V$ of $X$ in $M$, and a symplectomorphism $\varphi: (V_0, \omega_0) \to (V, \omega)$ such that $i = \varphi \circ i_0$.

As we will see later in this paper, isotropic and Lagrangian submanifolds play a central role in the theory of symplectic group actions (as well as in other parts of symplectic geometry, for instance the study of intersections of Lagrangian submanifolds, see Arnold [5], Chaperon [24], and Hofer [77]).

2.6. **Monotonic symplectic invariants.** Let $B^{2n}(R)$ be the open ball of radius $R > 0$. If $U$ and $V$ are open subsets of $\mathbb{R}^{2n}$, with the standard symplectic form $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$.

For $n \geq 1$ and $r > 0$ let $B^{2n}(r) \subset \mathbb{C}^n$ be the $2n$-dimensional open symplectic ball of radius $r$ and let $Z^{2n}(r) = \{ (z_i)_{i=1}^{n} \in \mathbb{C}^n \mid \|z_i\| < r \}$ be the $2n$-dimensional open symplectic cylinder of radius $r$. Both inherit a symplectic structure from their embedding as a subset of $\mathbb{C}^n$ with symplectic form $\omega_0 = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$.

A symplectic embedding $f: U \to V$ is a smooth embedding such that $f^*\omega_0 = \omega_0$.

Similarly one defines symplectic embeddings $f: (M_1, \omega_1) \to (M_2, \omega_2)$ between general symplectic manifolds.

If there is a symplectic embedding $f: U \to V$ then $\text{vol}(U)$ is at most equal to $\text{vol}(V)$, that is, the volume provides an elementary embedding obstruction.

**Theorem 2.32 (Gromov [64]).** There is no symplectic embedding of $B^{2n}(1)$ into $Z^{2n}(r)$ for $r < 1$. 


This result shows a rigidity property that symplectic transformations exhibit, in contrast with their volume-preserving counterparts. It shows that in addition to the volume there are other obstructions which are more subtle and come from the symplectic form, they are called symplectic capacities.

Denote by \( \mathcal{E} \ell \) the category of ellipsoids in \( \mathbb{R}^{2n} \) with symplectic embeddings induced by global symplectomorphisms of \( \mathbb{R}^{2n} \) as morphisms, and by \( \text{Symp}^{2n} \) the category of symplectic manifolds of dimension \( 2n \), with symplectic embeddings as morphisms. A symplectic category is a subcategory \( C \) of \( \text{Symp}^{2n} \) containing \( \mathcal{E} \ell \) such that \( (M, \omega) \in C \) implies that \( (M, \lambda \omega) \in C \) for all \( \lambda > 0 \).

Let \( d \) be a fixed integer such that \( 1 \leq d \leq n \). Following Ekeland-Hofer and Hofer [42, 78] we make the following definition of monotonic symplectic invariant.

**Definition 2.33.** A symplectic \( d \)-capacity on a symplectic category \( C \) is a functor \( c \) from \( C \) to the category \( (\mathbb{R}_{\geq 0}, \leq) \) satisfying: i) Monotonicity: \( c(M_1, \omega_1) \leq c(M_2, \omega_2) \) if there is a morphism from \( (M_1, \omega_1) \) to \( (M_2, \omega_2) \); ii) Conformality: \( c(M, \lambda \omega) = \lambda c(M, \omega) \) for all \( \lambda > 0 \). iii) Non-triviality: \( c(B^{2n}(1)) > 0, c(B^{2n}(1) \times \mathbb{R}^{2(n-d)}) < \infty \), and \( c(B^{2(d-1)}(1) \times \mathbb{R}^{2(n-d+1)}) = \infty \). If \( d = 1 \), a symplectic \( d \)-capacity is called a symplectic capacity.

Symplectic capacities are a fundamental class of invariants of symplectic manifolds. They were introduced in Ekeland and Hofer’s work [42, 78].

The first symplectic capacity was the Gromov radius, constructed by Gromov in [64]: it is the radius of the largest ball (in the sense of taking a supremum) that can be symplectically embedded in a manifold:

\[
(M, \omega) \mapsto \sup \left\{ R > 0 \mid B^{2n}(R) \hookrightarrow M \right\},
\]

where \( \hookrightarrow \) denotes a symplectic embedding. The fact that the Gromov radius is a symplectic capacity is a deep result, it follows from Theorem 2.32.

The symplectic volume (4) is a symplectic \( n \)-capacity.

Today many constructions of symplectic capacities are known, see [26]; therein one can find for instance two of the best known capacities, the Hofer-Zehnder capacities and the Ekeland-Hofer capacities, but there are many more.

The following results [76, 136] clarify the existence of continuous symplectic \( d \)-capacities. In what follows a capacity satisfies the exhaustion property if value of the capacity on any open set equals the supremum of the values on its compact subsets.

**Theorem 2.34** (Guth [76]). Let \( n \geq 3 \). If \( 1 < d < n \), symplectic \( d \)-capacities satisfying the exhaustion property do not exist on any subcategory of the category of symplectic \( 2n \)-manifolds.

That is, other than the volume, the monotonic invariants of symplectic geometry only measure 2-dimensional information.

The assumption on the previous theorem was removed in [136] where the authors prove that symplectic \( d \)-capacities do not exist on any subcategory of the category of symplectic \( 2n \)-manifolds.

There are invariants of symplectic manifolds which do not fit Definition 2.33, see for instance [112]. An equivariant theory of symplectic capacities appears in [49], we will discuss it later.

3. Symplectic and Hamiltonian actions

3.1. Lie groups and Lie algebras. Lie groups are smooth manifolds that are simultaneously groups, and as such they can also act on smooth manifolds and describe their symmetries.
They are named after Sophus Lie (1842-1899), one of the most influential figures in differential geometry to whom many of the modern notions can be traced back, including that of a transformation (Lie) group, and some particular instances of the notion of a momentum map (already mentioned in the introduction, but which we will formally define shortly).

Recall that a Lie group is a pair \((G, \star)\) where \(G\) is smooth manifold and \(\star\) is an internal group operation \(\star: G \times G \to G\) which is smooth and such that \(G \to G, g \mapsto g^{-1}\) is also smooth.

**Example 3.1.** The most important example of Lie group for the purpose of this paper is the circle, which may be viewed in two isomorphic ways, either as a quotient of \(\mathbb{R}\) by its integral lattice \(\mathbb{Z}\), or as a subset of the complex numbers \((\mathbb{R}/\mathbb{Z}, +) \simeq (S^1 := \{z \in \mathbb{C} \mid |z| = 1\}, \cdot)\)

**Example 3.2.** A torus is a compact, connected, abelian Lie group, and one can prove that such group is isomorphic to a product of circles, that is, \(((\mathbb{R}/\mathbb{Z})^k, +) \simeq ((S^1)^k := \{z \in \mathbb{C} \mid |z| = 1\}^k, \cdot, \ldots, \cdot)\). The integer \(k\) is the dimension of the torus. Other well known Lie groups are the general linear group \(\text{GL}(n, \mathbb{R})\) and the orthogonal group \(\text{O}(n)\) endowed with matrix multiplication.

A subset \(H\) of \(G\) is a Lie subgroup if it is a subgroup of \(G\), a Lie group, and the inclusion \(H \hookrightarrow G\) is an immersion.

**Theorem 3.3** (Cartan [23]). A closed subgroup of a Lie group is a Lie subgroup.

Let \(G/H\) be the space of right cosets endowed with the quotient topology. One can prove that there exists a unique smooth structure on \(G/H\) for which the quotient map \(G \to G/H\) is smooth. The \(G\)-action on \(G\) descends to an action on \(G/H\): \(g \in G\) acts on \(G/H\) by sending \(g' H\) to \((g g') H\). In this case \(G/H\) is called a homogeneous space.

The transformations between Lie groups are the Lie group homomorphisms. Let \((G, \cdot), (G', \star)\) be Lie groups. A Lie group homomorphism is a smooth map \(f: G \to G'\) such that \(f(a \cdot b) = f(a) \star f(b)\) for all \(a, b \in G\). A Lie group isomorphism is a diffeomorphism \(f: G \to G'\) such that \(f(a \cdot b) = f(a) \star f(b)\) for all \(a, b \in G\).

A Lie algebra is a vector space \(V\), together with a bilinear map \([\cdot, \cdot]: V \times V \to V\), called a Lie bracket, which satisfies: \([\zeta, \eta] = -[\eta, \zeta]\) (antisymmetry) and \([\zeta, [\eta, \rho]] + [\eta, [\zeta, \rho]] + [\rho, [\zeta, \eta]] = 0\) for all \(\zeta, \eta, \rho \in V\) (Jacobi identity).

For instance, the space of \(n\)-dimensional matrices with real coefficients endowed with the commutator of matrices as bracket, is a Lie algebra.

For the properties of Lie algebras and for how from a given a Lie group one can define its associated Lie algebra, see for instance Duistermaat-Kolk [37].

For this paper, if \(T\) is an \(n\)-dimensional torus, a compact, connected, abelian Lie group and \(1\) is the identity of \(T\), the Lie algebra of \(T\) is the additive vector space \(t := T_1 T\) endowed with the trivial bracket.

3.2. Lie group actions. Let \((G, \star)\) be a Lie group, and let \(M\) be a smooth manifold. A smooth \(G\)-action on \(M\) is a smooth map \(G \times M \to M\), denoted by \((g, x) \mapsto g \cdot x\), such that \(e \cdot x = x\) and \(g \cdot (h \cdot x) = (g \star h) \cdot x\), for all \(g, h \in G\) and for all \(x \in M\).

For instance, we have the following smooth actions. The map \(S^1 \times \mathbb{C}^n \to \mathbb{C}^n\) on \(\mathbb{C}^n\) given by \((\theta, (z_1, z_2, \ldots, z_n)) \mapsto (\theta z_1, z_2, \ldots, z_n)\) is a smooth \(S^1\)-action on \(\mathbb{C}^n\). Also, any Lie group \(G\) acts on itself by left multiplication \(L(h): g \mapsto gh\) and analogously right multiplication, and also by the adjoint action \(\text{Ad}(h): g \mapsto hgh^{-1}\). If \(G\) is abelian, then \(\text{Ad}(h)\) is the identity map for every \(h \in G\).

We say that the \(G\)-action is effective if every element in \(T\) moves at least one point in \(M\), or equivalently \(\cap_{x \in M} G_x = \{e\}\), where \(G_x := \{t \in G \mid t \cdot x = x\}\) is the stabilizer subgroup of the
G-action at x. The action is free if \( G_x = \{ e \} \) for every \( x \in M \). The action is semi-free if for every \( x \in M \) either \( G_x = G \) or \( G_x = \{ e \} \). The action is proper if for any compact subset \( K \) of \( M \) the set of all \( (g, m) \in G \times M \) such that \( (m, g \cdot m) \in K \) is compact in \( G \times M \).

Remark 3.4. There are obstructions to the existence of effective smooth \( G \)-actions on compact and non-compact manifolds, even in the case that the \( G \)-action is only required to be smooth. For instance, in \([161]\) Corollary in page 242 it is proved that if \( N \) is an \( n \)-dimensional manifold on which a compact connected Lie group \( G \) acts effectively and there are \( \sigma_1, \ldots, \sigma_n \in H^1(M, \mathbb{Q}) \) such that \( \sigma_1 \cup \ldots \cup \sigma_n \neq 0 \) then \( G \) is a torus and the \( G \)-action is locally free. In \([161]\) Yau also proves several other results giving restrictions on \( G, M \), and the fixed point set \( M^G \). If the \( G \)-action is moreover assumed to be symplectic or \( \mathbb{K} \)ähler, there are even more non-trivial constraints. \( \Box \)

The set \( G \cdot x := \{ t \cdot x \mid t \in G \} \) is the \textit{G-orbit} that goes through the point \( x \).

**Proposition 3.5.** The stabilizer \( G_x \) is a Lie subgroup of \( G \).

*Proof.* By Theorem 3.3 it suffices to show if that \( G_x \) is closed in \( G \). Let \( A \colon G \times M \to M \) denote the \( G \)-action. For each \( x \in G \) let \( i_x \colon G \to G \times M \) be the mapping \( i_x(g) := (g, x) \). Then \( G_x = (i_x)^{-1}(A^{-1}(x)) \), and hence \( G_x \) is closed since \( \{ x \} \) is closed and \( i_x \) and \( \varphi \) are continuous mappings. \( \square \)

The following can be easily checked.

**Proposition 3.6.** The action of \( G \) is proper if \( G \) is compact.

For each closed subgroup \( H \) of \( T \) which can occur as a stabilizer subgroup, the orbit type \( M^H \) is defined as the set of all \( x \in M \) such that \( T_x \) is conjugate to \( H \), but because \( T \) is commutative this condition is equivalent to the equation \( T_x = H \).

Each connected component \( C \) of \( M^H \) is a smooth \( T \)-invariant submanifold of \( M \). The connected components of the orbit types in \( M \) form a finite partition of \( M \), which actually is a Whitney stratification. This is called the orbit type stratification of \( M \).

There is a unique open orbit type, called the principal orbit type, which is the orbit type of a subgroup \( H \) which is contained in every stabilizer subgroup \( T_x, x \in M \).

Because the effectiveness of the action means that the intersection of all the \( T_x, x \in M \) is equal to the identity element, this means that the principal orbit type consists of the points \( x \) where \( T_x = \{ 1 \} \), that is where the action is free. If the action is free at \( x \), then the linear mapping \( X \mapsto X_M(x) \) from \( t \) to \( T_x M \) is injective.

**Theorem 3.7** (Theorem 2.8.5 in Duistermaat-Kolk \([37]\)). The principal orbit type \( M_{\text{reg}} \) is a dense open subset of \( M \), and connected if \( G \) is connected.

The following notions will be essential later on.

**Definition 3.8.** The points \( x \in M \) at which the \( T \)-action is free are called the regular points of \( M \), and the principal orbit type, the set of all regular points in \( M \), is denoted by \( M_{\text{reg}} \). The principal orbits are the orbits in \( M_{\text{reg}} \).

Next we define equivariant maps, following on Definition 2.13. Suppose that \( G \) acts smoothly on \((M_1, \omega_1)\) and \((M_2, \omega_2)\).

**Definition 3.9.** A \( G \)-equivariant diffeomorphism (resp. \( G \)-equivariant embedding) \( \varphi \colon (M_1, \omega_1) \to (M_2, \omega_2) \) is a symplectomorphism (resp. embedding) \( \varphi \colon M_1 \to M_2 \) such that \( \varphi(g \cdot x) = g \star \varphi(x) \quad \forall g \in G, \ x \in M_1 \), where \( \cdot \) denotes the \( G \)-action on \( M_1 \) and \( \star \) denotes the \( G \)-action on \( M_2 \). In this case we say that \((M_1, \omega_1)\) and \((M_2, \omega_2)\) are \( G \)-equivariantly diffeomorphic (resp. that \((M_1, \omega_1)\) is symplectically and \( G \)-equivariantly embedded in \((M_2, \omega_2)) \).
It is useful to work with a notion of equivariant map up to reparametrizations of the acting group. If \( G_1 \) and \( G_2 \) are isomorphic Lie groups (possibly equal) acting symplectically on \((M_1, \omega_1)\) and \((M_2, \omega_2)\) respectively, \( \varphi: (M_1, \omega_1) \to (M, \omega_2) \) is an equivariant diffeomorphism (resp. equivariant embedding) if it is a diffeomorphism (resp. an embedding) for which there exists an isomorphism \( f: G_1 \to G_2 \) such that \( \varphi(g \cdot x) = f(g) \cdot \varphi(x) \) \( \forall g \in G_1 \), \( \forall x \in M_1 \), where \( \cdot \) denotes the \( G_1 \)-action on \( M_1 \) and \( \ast \) denotes the \( G_2 \)-action on \( M_2 \). In this case we say that \((M_1, \omega_1)\) and \((M_2, \omega_2)\) are equivariantly diffeomorphic (resp. that \((M_1, \omega_1)\) is symplectically and equivariantly embedded in \((M_2, \omega_2)\)). This more general notion is particularly important when working on equivariant symplectic packing problems \([123, 124, 130, 50, 49]\), because if \( f \) is only allowed to be the identity these problems are too rigid to be of interest.

**Proposition 3.10.** For \( x \in M \), the stabilizer \( G_x \) of a proper \( G \)-action is compact and \( A_x: g \mapsto g \cdot x: G \to M \) induces a smooth \( G \)-equivariant embedding \( \alpha_x: G \times M \to M \) with closed image equal to \( G \cdot x \).

In this paper we will be concerned with actions on symplectic manifolds which preserve the symplectic form (defined in Section 3.3). We will later describe a result of Benoist and Ortega-Ratiu which gives a symplectic normal form for proper actions in the neighborhood of any \( G \)-orbit \( G \cdot x \) of a symplectic manifold \((M, \omega), x \in M \) (this is Theorem 7.3).

Let \( T \) be an \( n \)-dimensional torus with identity 1 and let \( t := T \cdot 1 \) be its Lie algebra. Let \( X \in t \). There exists a unique homomorphism \( \alpha_X: \mathbb{R} \to T \) with \( \alpha_X(0) = 1, \alpha'_X(0) = X \). Define the so called exponential mapping \( \exp: t \to T \) by

\[
\exp(X) := \alpha_X(1)
\]

The exponential mapping \( \exp: t \to T \) is a surjective homomorphism from the additive Lie group \((t, +)\) onto \( T \). Furthermore, \( t_{\mathbb{Z}} := \ker(\exp) \) is a discrete subgroup of \((t, +)\) and \( \exp \) induces an isomorphism from \( t/t_{\mathbb{Z}} \) onto \( T \), which we also denote by \( \exp \).

Because \( t/t_{\mathbb{Z}} \) is compact, \( t_{\mathbb{Z}} \) has a \( \mathbb{Z} \)-basis which at the same time is an \( \mathbb{R} \)-basis of \( t \), and each \( \mathbb{Z} \)-basis of \( t_{\mathbb{Z}} \) is an \( \mathbb{R} \)-basis of \( t \).

Using coordinates with respect to an ordered \( \mathbb{Z} \)-basis of \( t_{\mathbb{Z}} \), we obtain a linear isomorphism from \( t \) onto \( \mathbb{R}^n \) which maps \( t_{\mathbb{Z}} \) onto \( \mathbb{Z}^n \), and therefore induces an isomorphism from \( T \) onto \( \mathbb{R}^n/\mathbb{Z}^n \). The set \( t_{\mathbb{Z}} \) is called the integral lattice in \( t \).

However, because we do not have a preferred \( \mathbb{Z} \)-basis of \( t_{\mathbb{Z}} \), we do not write \( T = \mathbb{R}^n/\mathbb{Z}^n \).

Using (5) one can generate vector fields on a smooth manifold from a given action.

**Definition 3.11.** For each \( X \in t \), the vector field **infinitesimal action** \( X_M \) of \( X \) on \( M \) is defined by

\[
X_M(x) := \text{tangent vector to } \left. t \mapsto \exp(tX) \cdot x \right|_{t=0}
\]

i.e. \( X_M(x) = d/dt|_{t=0} \exp(tX) \cdot x \).

In the sequel, \( \mathcal{X}^\infty(M) \) denotes the Lie algebra of all smooth vector fields on \( M \), provided with the **Lie brackets** \([X, Y] \) of \( \mathcal{X} \): \( \mathcal{X}, Y \in \mathcal{X}^\infty(M): [X, Y]f := X(Y(f)) - Y(X(f)), \forall f \in C^\infty(M) \). The Lie brackets vanish when the flows of the vector fields \( X \) and \( Y \) commute. Therefore

\[
[X_M, Y_M] = 0
\]

for all \( X, Y \in t \).
3.3. Symplectic and Hamiltonian actions. Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold. Let $G$ be a Lie group.

Definition 3.12. The action $\phi: G \times M \to M$ is symplectic if $G$ acts by symplectomorphisms, i.e. for each $t \in G$ the diffeomorphism $\varphi_t: M \to M$ given by $\varphi_t(x) := t \cdot x$ is such that $(\varphi_t)^* \omega = \omega$. The triple $(M, \omega, \phi)$ is called a symplectic $G$-manifold.

Let $L_X$ denote the Lie derivative with respect to the vector field $X$, and $i_X \omega$ the inner product of $\omega$ with $X$, obtained by inserting $X$ in the first slot of $\omega$. The fact that the $T$-action is symplectic says that for every $X \in t$

\begin{equation}
(8) \quad d(i_X \omega) = L_X \omega = 0,
\end{equation}

so the 1-form

\begin{equation}
(9) \quad i_X \omega = \omega(X_M, \cdot)
\end{equation}

is closed. The first identity in (8) follows from the homotopy identity

\begin{equation}
(10) \quad L_v = d \circ i_v + i_v \circ d
\end{equation}

combined with $d\omega = 0$. The case when (9) is moreover an exact form, for each $X \in t$, has been thoroughly studied in the literature.

Indeed, there is a special type of symplectic actions which appear often in classical mechanics, and which enjoy a number of very interesting properties: they are called Hamiltonian actions, named after William Hamilton (1805-1865).

Let $T$ be an $n$-dimensional torus, a compact, connected, abelian Lie group, with Lie algebra $t$. Let $t^*$ be the dual of $t$.

Definition 3.13. A symplectic action $T \times M \to M$ is Hamiltonian if there is a smooth map $\mu: M \to t^*$ such that Hamilton’s equation

\begin{equation}
(11) \quad -d\langle \mu(\cdot), X \rangle = i_X \omega := \omega(X_M, \cdot), \quad \forall X \in t,
\end{equation}

holds, where $X_M$ is the vector field infinitesimal action of $X$ on $M$, and the left hand-side of equation (11) is the differential of the real valued function $\langle \mu(\cdot), X \rangle$ obtained by evaluating elements of $t^*$ on $t$.

There is a natural notion of symplectic and Hamiltonian vector field. Given a smooth function $f: M \to \mathbb{R}$, let $\mathcal{H}_f$ be the vector field defined by Hamilton’s equation $\omega(\mathcal{H}_f, \cdot) = -df$.

Definition 3.14. We say that a smooth vector field $\mathcal{Y}$ on a symplectic manifold $(M, \omega)$ is symplectic if its flow preserves the symplectic form $\omega$. We say that $\mathcal{Y}$ is Hamiltonian if there exists a smooth function $f: M \to \mathbb{R}$ such that $\mathcal{Y} = \mathcal{H}_f$.

It follows that a $T$-action on $(M, \omega)$ is symplectic if and only if all the vector fields that it generates through (6) are symplectic. A symplectic $T$-action is Hamiltonian if all the vector fields $X_M$ that it generates are Hamiltonian, i.e. for each $X \in t$ there exists a smooth solution $\mu_X: M \to \mathbb{R}$, called a Hamiltonian or energy function, to $-d\mu_X = \omega(X_M, \cdot)$.

Proposition 3.15. A any symplectic $T$-action on a simply connected manifold $(M, \omega)$ is Hamiltonian.

Proof. The obstruction for $\omega(X_M, \cdot)$ to being exact lies in the first cohomology group of the manifold $H^1(M, \mathbb{R}) = 0$. If the manifold is simply connected then $\pi_1(M) = 0$, and hence $H^1(M, \mathbb{R}) = 0$. □
The natural transformations between symplectic manifolds \((M_1, \omega_1)\) and \((M_2, \omega_2)\) endowed with symplectic \(T\)-actions are the \(T\)-equivariant diffeomorphisms which preserve the symplectic form, they are called \(T\)-equivariant symplectomorphisms.

Remark 3.16. Kostant [95] and Souriau [145] gave the general notion of momentum map (we refer to Marsden-Ratiu [106] Pages 369, 370 for the history). The momentum map may be defined generally for a Hamiltonian action of a Lie group. It was a key tool in Kostant [96] and Souriau discussed it at length in [146]. In this paper we only deal with the momentum map for a Hamiltonian action of a torus.

3.4. Conditions for a symplectic action to be Hamiltonian. A Hamiltonian \(S^1\)-action on a compact symplectic manifold \((M, \omega)\) has at least \(\frac{1}{2} \dim M + 1\) fixed points. This follows from the fact that, if the fixed set is discrete, then the momentum map \(\mu: M \to \mathbb{R}\) is a perfect Morse function whose critical set is the fixed set. Therefore, the number of fixed points is equal to the rank of \(\sum_{i=1}^{\frac{1}{2}\dim M} H_i(M; \mathbb{R})\). Finally, this sum is at least \(\frac{1}{2} \dim M + 1\) because \([1], [\omega], [\omega^2], \ldots, [\omega^{\frac{1}{2}\dim M}]\) are distinct cohomology classes.

We are not aware of general criteria to detect when a symplectic action is Hamiltonian, other than in a few specific situations. In fact, one striking question is:

Question 3.17. Are there non-Hamiltonian symplectic \(S^1\)-actions on compact connected symplectic manifolds with non-empty discrete fixed point set?

In recent years there has been a flurry of activity related to this question, see for instance Godinho [60, 61], Jang [83, 84], Pelayo-Tolman [132], and Tolman-Weitsman [150]. Recently Tolman constructed an example [149] answering Question 3.17 in the positive.

Theorem 3.18 (Tolman [149]). There exists a symplectic non-Hamiltonian \(S^1\)-action on a compact connected manifold with exactly 32 fixed points.

Tolman and Weitsman proved the answer to the question is no for semifree symplectic actions (they used equivariant cohomological methods, briefly covered here in Section 4.4).

Theorem 3.19 (Tolman-Weitsman [150]). Let \(M\) be a compact, connected symplectic manifold, equipped with a semifree, symplectic \(S^1\)-action with isolated fixed points. Then if there is at least one fixed point, the circle action is Hamiltonian.

In the Kähler case the answer to the question is a classical theorem of Frankel (which started much of the activity on the question).

Theorem 3.20 (Frankel [55]). Let \((M, \omega)\) be a compact connected Kähler manifold admitting an \(S^1\)-action preserving the Kähler structure. If the \(S^1\)-action has fixed points, then the action is Hamiltonian.

Ono [116] proved the analogue of Theorem 3.20 for compact Lefschetz manifolds and McDuff [111] Proposition 2 proved the a symplectic version of Frankel’s theorem (later generalized by Kim [91] to arbitrary dimensions).

Theorem 3.21 (McDuff [111]). A symplectic \(S^1\)-action on a compact connected symplectic 4-manifold with fixed points must be Hamiltonian.

On the other hand, McDuff [111] Proposition 1] constructed a compact connected symplectic 6-manifold with a non-Hamiltonian symplectic \(S^1\)-action which has fixed point set equal to a union of tori.

Less is known for higher dimensional Lie groups; the following corresponds to [58 Theorem 3.13].
Theorem 3.22 (Giacобе [53]). A symplectic action of a n-torus on a compact connected symplectic 2n-manifold with fixed points must be Hamiltonian.

Theorem 3.22 appears as [33 Corollary 3.9]. If \( n = 2 \) this is deduced from the classification of symplectic 4-manifolds with symplectic 2-torus actions in [125, Theorem 8.2.1] (Theorem 8.26 covered later in this paper) in view of [10, Theorem 1.1].

There are also related results by Ginzburg describing the obstruction to the existence of a momentum map for a symplectic action, see [59], where he showed that a symplectic action can be decomposed as cohomologically free action, and a Hamiltonian action.

In the present paper we will focus on symplectic actions of tori of dimension \( k \) on manifolds of dimension \( 2n \) where \( k \geq n \). For these there exist recent complete classifications of certain classes of symplectic actions, which are described in the following sections, in which a complete answer to the following more general question can be given in terms of the vanishing of certain invariants.

**Question 3.23.** When is a symplectic torus action on a compact connected symplectic manifold Hamiltonian? Describe precisely the obstruction to being Hamiltonian.

When in addition to being symplectic the action is Hamiltonian, then necessarily \( n = k \), but there are many non-Hamiltonian symplectic actions when \( n = k \), and also when \( n \geq k + 1 \).

### 3.5 Monotonic symplectic G-invariants

Let \( G \) be a Lie group. We denote the collection of all \( 2n \)-dimensional symplectic \( G \) manifolds by \( \text{Symp}^{2n,G} \). The set \( \text{Symp}^{2n,G} \) is a category with morphisms given by \( G \)-equivariant symplectic embeddings. We call a subcategory \( \mathcal{C}_G \) of \( \text{Symp}^{2n,G} \) a symplectic \( G \)-category if \( (M, \omega, \phi) \in \mathcal{C}_G \) implies \( (M, \lambda \omega, \phi) \in \mathcal{C}_G \) for any \( \lambda \in \mathbb{R} \setminus \{0\} \).

**Definition 3.24.** Let \( \mathcal{C}_G \) be a symplectic \( G \)-category. A generalized symplectic \( G \)-capacity on \( \mathcal{C}_G \) is a map \( c: \mathcal{C}_G \to [0, \infty] \) satisfying:

i) **Monotonicity:** if \( (M, \omega, \phi), (M', \omega', \phi') \in \mathcal{C}_G \) and there exists a \( G \)-equivariant symplectic embedding \( M \xrightarrow{\mathcal{G}} M' \) then \( c(M, \omega, \phi) \leq c(M', \omega', \phi') \);

ii) **Conformality:** if \( \lambda \in \mathbb{R} \setminus \{0\} \) and \( (M, \omega, \phi) \in \mathcal{C}_G \) then \( c(M, \lambda \omega, \phi) = |\lambda| c(M, \omega, \phi) \).

**Definition 3.25.** For \( (N, \omega_N, \phi_N) \in \mathcal{C}_G \) we say that \( c \) satisfies \( N \)-non-triviality if \( 0 < c(N) < \infty \).

**Definition 3.26.** We say that \( c \) is tamed by \( (N, \omega_N, \phi_N) \in \text{Symp}^{2n,G} \) if there exists some \( a \in (0, \infty) \) such that the following two properties hold:

1. if \( M \in \mathcal{C}_G \) and there exists a \( G \)-equivariant symplectic embedding \( M \xrightarrow{\mathcal{G}} N \) then \( c(M) \leq a \);
2. if \( P \in \mathcal{C}_G \) and there exists a \( G \)-equivariant symplectic embedding \( N \xrightarrow{\mathcal{G}} P \) then \( a \leq c(P) \).

For any integer \( 1 \leq d \leq n \) the standard action of the \( d \)-dimensional torus \( \mathbb{T}^d \) on \( \mathbb{C}^n \) is given by \( \phi_{\mathbb{C}^n}((z_i)_{i=1}^n) = (\alpha_1 z_1, \ldots, \alpha_d z_d, z_{d+1}, \ldots, z_n) \). It induces actions of \( \mathbb{T}^d = \mathbb{T}^k \times \mathbb{T}^{d-k} \) on \( \mathbb{B}^{2n}(1) \) and \( \mathbb{Z}^{2n}(1) \), which in turn induce the actions of \( \mathbb{T}^k \times \mathbb{R}^{d-k} \) on \( \mathbb{B}^{2n}(1) \) and \( \mathbb{Z}^{2n}(1) \) for \( k \leq d \). The action of an element of \( \mathbb{T}^k \times \mathbb{R}^{d-k} \) is the action of its image under the quotient map \( \mathbb{T}^k \times \mathbb{R}^{d-k} \to \mathbb{T}^d \). We endow \( \mathbb{B}^{2n}(1) \) and \( \mathbb{Z}^{2n}(1) \) with these actions.

**Definition 3.27.** A generalized symplectic \((\mathbb{T}^k \times \mathbb{R}^{d-k})\)-capacity is a symplectic \((\mathbb{T}^k \times \mathbb{R}^{d-k})\)-capacity if it is tamed by \( \mathbb{B}^{2n}(1) \) and \( \mathbb{Z}^{2n}(1) \).

Given integers \( 0 \leq k \leq m \leq n \) we define the \((m, k)\)-equivariant Gromov radius \( c_{B}^{m,k} : \text{Symp}^{2n,\mathbb{R}^k} \to [0, \infty], (M, \omega, \phi) \mapsto \sup \{ r > 0 \mid \mathbb{B}^{2m}(r) \xrightarrow{\mathbb{R}^k} M \} \), where \( \xrightarrow{\mathbb{R}^k} \) denotes a symplectic \( \mathbb{R}^k \)-embedding and \( \mathbb{B}^{2m}(r) \subset \mathbb{C}^m \) has the \( \mathbb{R}^k \)-action given by rotation of the first \( k \) coordinates.

**Proposition 3.28 (49).** If \( k \geq 1 \), \( c_{B}^{m,k} \) is a symplectic \( \mathbb{R}^k \)-capacity.
Let $\text{vol}(E)$ denote the symplectic volume of a subset $E$ of a symplectic manifold and $\text{Symp}^{2n, T^n}$ the category of $2n$-dimensional symplectic toric manifolds. A toric ball packing $P$ of $M$ is given by a disjoint collection of symplectically and $T^n$-equivariantly embedded balls. As an application of symplectic $G$-capacities to Hamiltonian $T^n$-actions we define the toric packing capacity $\mathcal{T} : \text{Symp}^{2n, T^n} \to [0, \infty]$,

$$(M, \omega, \phi) \mapsto \left( \sup \{ \text{vol}(P) \mid P \text{ is a toric ball packing of } M \} / \text{vol}(B^{2n}(1)) \right)^{\frac{1}{2n}},$$

**Proposition 3.29** ([49]). $\mathcal{T} : \text{Symp}^{2n, T^n} \to [0, \infty]$ is a symplectic $T^n$-capacity.

In general, symplectic $G$-capacities provide a general setting to define monotonic invariants of integrable systems. There should be many such invariants but so far few are known beyond the toric case, and the semitoric case also discussed in [49].

### 4. Properties of Hamiltonian actions

#### 4.1. Marsden-Weinstein symplectic reduction

Even though one cannot in general take quotients of symplectic manifolds by group actions and get again a symplectic manifold, for Hamiltonian actions we have the following notion of “symplectic quotient”.

**Theorem 4.1** (Marsden-Weinstein [107], Meyer [109]). Let $(M, \omega)$ be a symplectic manifold and let $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$ acting on $(M, \omega)$ in a Hamiltonian fashion with momentum map $\mu : M \to \mathfrak{g}^*$. Let $i : \mu^{-1}(\lambda) \to M$ be the inclusion map and suppose that $G$ acts freely on $\mu^{-1}(\lambda)$. Then the orbit space $M_{\text{red}, \lambda} := \mu^{-1}(\lambda)/G$ is a smooth manifold, the projection $\pi : \mu^{-1}(\lambda) \to M_{\text{red}, \lambda}$ is a principal $G$-bundle, and there is a symplectic form $\omega_{\text{red}, \lambda}$ on $M_{\text{red}, \lambda}$ such that $\pi^* \omega_{\text{red}, \lambda} = i^* \omega$.

The “symplectic quotient” $(M_{\text{red}}, \omega_{\text{red}})$ is called the Marsden-Weinstein reduction of $(M, \omega)$ for the $G$-action at $\lambda$. Symplectic reduction has numerous applications in mechanics and geometry, see for instance [106]. Here we will give one in the proof of the upcoming result Theorem 6.6.

#### 4.2. Atiyah-Guillemin-Sternberg convexity

It follows from equation (11) that Hamiltonian $T$-actions on compact connected manifolds have fixed points because zeros of the vector field $X_M$ correspond to critical points of $\langle \mu(\cdot), X \rangle$ and $\langle \mu(\cdot), X \rangle$ always has critical points in a compact manifold. The Atiyah-Guillemin-Sternberg Convexity Theorem (1982, [10, 73]) says that $\mu(M)$ is a convex polytope.

**Theorem 4.2** (Atiyah [10], Guillemin-Sternberg [73]). If an $m$-dimensional torus $T$ acts on a compact, connected $2n$-dimensional symplectic manifold $(M, \omega)$ in a Hamiltonian fashion, the image $\mu(M)$ under the momentum map $\mu : M \to t^*$ is the convex hull of the image under $\mu$ of the fixed point set of the $T$-action. In particular, $\mu(M)$ is a convex polytope in $t^*$.

The fixed point set in this theorem is given by a collection of symplectic submanifolds of $M$.

The polytope $\mu(M)$ is called the momentum polytope of $M$. One precedent of this result appears in Kostant’s article [97].

Other convexity theorems were proven later by Birtea-Ortega-Ratiu [20], Kirwan [93] (in the case of compact, non-abelian group actions), Benoist [15], and Giacobbe [58], to name a few. Convexity in the case of Poisson actions has been studied by Alekseev, Flaschka-Ratiu, Ortega-Ratiu and Weinstein [4, 47, 118, 158] among others.

Given a point $x \in \mu(M)$, its preimage $\mu^{-1}(x)$ is connected (this is known as Atiyah’s connectivity Theorem). Moreover, it is diffeomorphic to a torus of dimension $\ell$, where $\ell$ is the dimension of the lowest dimensional face $F$ of $\mu(M)$ such that $x \in F$.

The symplectic form $\omega$ must vanish along each $\mu^{-1}(x)$, that is, $\mu^{-1}(x)$ is isotropic.
4.3. Duistermaat-Heckman theorems. Roughly at the same time as Atiyah, Guillemin-Sternberg proved the convexity theorem, Duistermaat and Heckman proved an influential result which we will describe next. Let \((M, \omega)\) be a 2n-dimensional symplectic manifold. Suppose that \(T\) is a torus acting on a symplectic manifold \((M, \omega)\) in a symplectic Hamiltonian fashion with momentum map \(\mu: M \to \mathfrak{t}^*\). Assume moreover that \(\mu\) is proper, that is, for every compact \(K \subseteq \mathfrak{t}^*\), the preimage \(\mu^{-1}(K)\) is compact.

Definition 4.3. The Liouville measure of a Borel subset \(U\) of \(M\) is \(m_\omega(U) := \int_U \frac{\omega^n}{m}\). The Duistermaat-Heckman measure \(m_{\text{DH}}\) on \(\mathfrak{t}^*\) is the push-forward measure \(\mu_* m_\omega\) of \(m_\omega\) by \(\mu: M \to \mathfrak{t}^*\).

Let \(\lambda\) be the Lebesgue measure in \(\mathfrak{t}^* \simeq \mathbb{R}^m\).

Theorem 4.4 (Duistermaat-Heckman [35, 36]). There is a function \(f: \mathfrak{t}^* \to \mathbb{R}\) such that \(f\) is a polynomial of degree at most \(n - m\) on each component of regular values of \(\mu\), and \(m_\omega(U) = \int_U f \, d\lambda\).

Definition 4.5. The function \(f\) is called the Duistermaat-Heckman polynomial.

In the case of the symplectic-toric manifold \((S^2, \omega, S^1)\), the Liouville measure is \(m_\omega([a, b]) = 2\pi (b - a)\) and the Duistermaat-Heckman measure is the characteristic function \(2\pi \chi_{[a,b]}\).

If \(T\) acts freely on \(\mu^{-1}(0)\), then it acts freely on fibers \(\mu^{-1}(t)\) for which \(t \in \mathfrak{t}^*\) is closed to 0. Consider the Marsden-Weinstein reduced space \(M_t = \mu^{-1}(t)/T\) (see [2 Section 4.3]) with the reduced symplectic form \(\omega_t\) as in the proof of Theorem 6.6.

Theorem 4.6 (Duistermaat-Heckman [35, 36]). The cohomology class \([\omega_t]\) varies linearly in \(t\).

Theorem 4.6 does not hold for non-proper momentum maps, see [128, Remark 4.5].

4.4. Atiyah-Bott-Berline-Vergne localization. A useful tool in the study of properties of symplectic \(S^1\)-actions has been equivariant cohomology, because it encodes well the fixed point set information. Within symplectic geometry, equivariant cohomology is an active area, and in this paper we do not touch on it but only give the very basic definition, a foundational result, and an application to symplectic \(S^1\)-actions. Although equivariant cohomology may be defined generally we concentrate on the case of \(S^1\)-equivariant cohomology.

Definition 4.7. Let \(S^1\) act on a smooth manifold \(M\). The equivariant cohomology of \(M\) is \(H^*_S^1(M) := H^*(M \times_{S^1} S^\infty)\).

Example 4.8. If \(x\) is a point then \(H^*_S^1(x, \mathbb{Z}) = H^*(\mathbb{C}P^\infty, \mathbb{Z}) = \mathbb{Z}[t]\).

If \(V\) is an equivariant vector bundle over \(M\), then the equivariant Euler class of \(V\) is the Euler class of the vector bundle \(V \times_{S^1} S^\infty\) over \(M \times_{S^1} S^\infty\). The equivariant Chern classes of equivariant complex vector bundles are defined analogously.

If \(M\) is oriented and compact then the projection map \(\pi: M \times_{S^1} S^\infty \to \mathbb{C}P^\infty\) induces a natural push-forward map, denoted by \(\int_M: \pi_*: H^i_{S^1}(M, \mathbb{Z}) \to H^{i-\dim M}(\mathbb{C}P^\infty, \mathbb{Z})\). In particular \(\pi_*(\alpha) = 0\) for all \(\alpha \in H^i_{S^1}(M, \mathbb{Z})\) when \(i < \dim M\). For a component \(F\) of the fixed point set \(M^{S^1}\) we denote by \(e_{S^1}(\mathbb{N}_F)\) the equivariant Euler class of the normal bundle to \(F\).

Theorem 4.9 (Atiyah-Bott [11], Berline-Vergne [18]). Fix \(\alpha \in H^*_S^1(M, \mathbb{Q})\). As elements of \(\mathbb{Q}(t)\),

\[\int_M \alpha = \sum_{F \subseteq M^{S^1}} \int_F \frac{\alpha|_F}{e_{S^1}(\mathbb{N}_F)},\]

where the sum is over all fixed components \(F\).
Let $S^1$ act symplectically on a symplectic manifold $(M, \omega)$, and let $J: TM \to TM$ be a compatible almost complex structure. If $p \in M^{S^1}$ is an isolated fixed point, then there are well-defined non-zero integer weights $\xi_1, \ldots, \xi_n$ in the isotropy representation $T_pM$ (repeated with multiplicity). Indeed, there exists an identification of $T_pM$ with $\mathbb{C}^n$, where the $S^1$ action on $\mathbb{C}^n$ is given by $\lambda \cdot (z_1, \ldots, z_n) = (\lambda^{\xi_1} z_1, \ldots, \lambda^{\xi_n} z_n)$; the integers $\xi_1, \ldots, \xi_n$ are determined, up to permutation, by the $S^1$-action and the symplectic form. The restriction of the $i$th-equivariant Chern class $p$ is given by $c_i(M)|_p = \sigma_i(\xi_1, \ldots, \xi_n) t^i$ where $\sigma_i$ is the $i$th elementary symmetric polynomial and $t$ is the generator of $H^2_{S^1}(p, \mathbb{Z})$. For example, $c_1(M)|_p = \sum_{i=1}^n \xi_i t$ and the equivariant Euler class of the tangent bundle at $p$ is given by $e_{S^1}(N_p) = c_n(M)|_p = \left(\prod_{j=1}^n \xi_j\right) t^n$. Hence,

$$\int_p c_1(M)|_p e_{S^1}(N_p) = \frac{\sigma_i(\xi_1, \ldots, \xi_n)}{\prod_{j=1}^n \xi_j} t^{i-n}.$$ 

We can naturally identify $c_1(M)|_p$ with an integer $c_1(M)(p)$: the sum of the weights at $p$. Let $\Lambda_p$ be the product of the weights (with multiplicity) in the isotropy representation $T_pM$ for all $p \in M^{S^1}$. 

**Definition 4.10.** The mapping $c_1(M): M^{S^1} \to \mathbb{Z}$ defined by $p \mapsto c_1(M)(p) \in \mathbb{Z}$ is called the *Chern class map* of $M$.

**Proposition 4.11 ([132]).** Let $S^1$ act symplectically on compact symplectic $2n$-manifold $(M, \omega)$ with isolated fixed points. If the range of $c_1(M): M \to \mathbb{Z}$ contains at most $n$ elements, then

$$\sum_{p \in M^{S^1}} \frac{1}{\Lambda_p} = 0.$$ 

for every $k \in \mathbb{Z}$.

**Proof.** Let \{\{c_1(M)(p) \mid p \in M^{S^1}\} = \{k_1, \ldots, k_\ell\}$ and define for $i \in \{1, \ldots, \ell\}$, $A_i := \sum_{p \in M^{S^1}} 1_{c_1(M)(p) = k_i} \frac{1}{\Lambda_p}$. 

Consider the $\ell \times \ell$ matrix $B$ given by $B_{ij} := (k_i)_{j-1}$, where $1 \leq i, j \leq \ell$. Since $\ell \leq n$ by assumption, $\int_M c_1(M)^j = 0$ for all $j < \ell$. Applying Theorem 4.9 to the elements $1, c_1(M), \ldots, c_1(M)^{\ell-1}$ gives a homogenous system of linear equations $B \cdot (A_1, \ldots, A_\ell) = (0, \ldots, 0)$. Since $B$ is a Vandermonde matrix, we have that $\det(B(\ell)) \neq 0$. Thus, it follows that $A_1 = \cdots = A_\ell = 0$. 

Let $X$ and $Y$ be sets and let $f: X \to Y$ be a map. We recall that $f$ is *somewhere injective* if there is a point $y \in Y$ such that $f^{-1}\{y\}$ is the singleton.

**Theorem 4.12 ([132]).** Let $S^1$ act symplectically on compact symplectic $2n$-manifold $(M, \omega)$ with isolated fixed points. If the Chern class map is somewhere injective, then the circle action has at least $n + 1$ fixed points.

**Proof.** Since the Chern class map is somewhere injective there is $k \in \mathbb{Z}$ such that $\sum_{p \in M^{S^1}} \frac{1}{\Lambda_p} \neq 0$. 

By Proposition 1.11, this implies that the range of the Chern class map contains at least $n + 1$ elements; a fortiori, the action has at least $n + 1$ fixed points.
4.5. Further Topics. There exists an extensive theory of Hamiltonian actions and related topics, see for instance the books by Guillemin [68], Guillemin-Sjamaar [72], and Ortega-Ratiu [118].

There are many influential works which we do not describe here for two reasons, brevity being the main one, but also because they are more advanced and more suitable for a survey than a succinct invitation to the subject. These works include: Sjamaar-Lerman [144] work on stratifications; Kirwan’s convexity theorem [93] (which generalizes the Atiyah-Guillemin-Sternberg theorem to the non abelian case); and Lerman’s symplectic cutting procedure [102] (a procedure to “cut” a symplectic manifolds that has many applications in equivariant symplectic geometry and completely integrable systems).

5. Examples

5.1. Symplectic Hamiltonian actions. The following is an example of a Hamiltonian symplectic action.

Example 5.1. The first example of a Hamiltonian torus action is \((S^2, \omega = d\theta \wedge dh)\) equipped with the rotational circle action \(R/\mathbb{Z}\) about the vertical axis of \(S^2\) (depicted in Figure 3). This action has momentum map \(\mu : S^2 \to \text{Lie}(S^1) = T_1(S^1) \simeq \mathbb{R}\) equal to the height function \(\mu(\theta, h) = h\), and in this case the momentum polytope image is the interval \(\Delta = [-1, 1]\). Another example (which generalizes this one) is the \(n\)-dimensional complex projective space equipped with a \(\lambda\)-multiple, \(\lambda > 0\), of the Fubini-Study form \((\mathbb{C}P^n, \lambda \cdot \omega_{FS})\) and the rotational \(\mathbb{T}^n\)-action induced from the rotational \(\mathbb{T}^n\)-action on the \((2n+1)\)-dimensional complex plane. This action is Hamiltonian, with momentum map

\[
\mu_{\mathbb{C}P^n, \lambda} : z = [z_0 : z_1 : \ldots, z_n] \mapsto \left(\frac{\lambda|z_1|^2}{\sum_{i=0}^n |z_i|^2}, \ldots, \frac{\lambda|z_n|^2}{\sum_{i=0}^n |z_i|^2}\right).
\]

The associated momentum polytope is \(\mu_{\mathbb{C}P^n, \lambda}(\mathbb{C}P^n) = \Delta = \text{convex hull} \{0, \lambda e_1, \ldots, \lambda e_n\}\), where \(e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)\) are the canonical basis vectors of \(\mathbb{R}^n\). 


The category of Hamiltonian actions, while large, does not include some simple examples of symplectic actions, for instance free symplectic actions on compact manifolds, because Hamiltonian actions on compact manifolds always have fixed points.

5.2. Symplectic non Hamiltonian actions. In the following three examples, there does not exist a momentum map; they are examples of what we later call “maximal symplectic actions” (discussed in Section 7).

Example 5.2. The 4-torus \((\mathbb{R}/\mathbb{Z})^2 \times (\mathbb{R}/\mathbb{Z})^2\) endowed with the standard symplectic form, on which the 2-dimensional torus \(T := (\mathbb{R}/\mathbb{Z})^2\) acts by multiplications on two of the copies of \(\mathbb{R}/\mathbb{Z}\) inside of \((\mathbb{R}/\mathbb{Z})^4\), is symplectic manifold with symplectic orbits which are 2-tori.

Example 5.3. Let \(M := S^2 \times (\mathbb{R}/\mathbb{Z})^2\) be endowed with the product symplectic form of the standard area form on \((\mathbb{R}/\mathbb{Z})^2\) and the standard area form on \(S^2\). Let \(T := (\mathbb{R}/\mathbb{Z})^2\) act on \(M\) by translations on the right factor. This is a free symplectic action the orbits of which are symplectic 2-tori.

Example 5.4. Let \(P := S^2 \times (\mathbb{R}/\mathbb{Z})^2\) equipped with the product symplectic form of the standard symplectic (area) form on \(S^2\) and the standard area form on the sphere \((\mathbb{R}/\mathbb{Z})^2\). The 2-torus \(T := (\mathbb{R}/\mathbb{Z})^2\) acts freely by translations on the right factor of \(P\). Let the finite group \(\mathbb{Z}/2\mathbb{Z}\) act on \(S^2\) by rotating each point horizontally by 180 degrees, and let \(\mathbb{Z}/2\mathbb{Z}\) act on \((\mathbb{R}/\mathbb{Z})^2\) by the antipodal action on the first circle \(\mathbb{R}/\mathbb{Z}\). The diagonal action of \(\mathbb{Z}/2\mathbb{Z}\) on \(P\) is free. Therefore, the quotient
space \( S^2 \times \mathbb{Z}/2\mathbb{Z} (\mathbb{R}/\mathbb{Z})^2 \) is a smooth manifold. Let \( M := S^2 \times \mathbb{Z}/2\mathbb{Z} (\mathbb{R}/\mathbb{Z})^2 \) be endowed with the symplectic form \( \omega \) and \( T \)-action inherited from the ones given in the product \( S^2 \times (\mathbb{R}/\mathbb{Z})^2 \), where \( T = (\mathbb{R}/\mathbb{Z})^2 \). The action of \( T \) on \( M \) is not free, and the \( T \)-orbits are symplectic 2-dimensional tori. Notice that the orbit space \( M/T \) is \( S^2/(\mathbb{Z}/2\mathbb{Z}) \), which is a smooth orbifold with two singular points of order 2, the South and North poles of \( S^2 \) (this orbifold will play an important role in the classification of maximal symplectic actions).

The following two are examples of what we later call coisotropic actions (discussed in Section 7).

**Example 5.5.** (Kodaira [94] and Thurston [148]) The first example of a (non-Hamiltonian) symplectic torus action with coisotropic (in fact, Lagrangian) principal orbits is the Kodaira variety [94] (also known as the Kodaira-Thurston manifold [148]), which is a torus bundle over a torus constructed as follows. Consider the product symplectic manifold \((\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})^2, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)\), where \((x_1, y_1) \in \mathbb{R}^2\) and \((x_2, y_2) \in (\mathbb{R}/\mathbb{Z})^2\). Consider the action of \((j_1, j_2) \in \mathbb{Z}^2\) on \((\mathbb{R}/\mathbb{Z})^2\) by the matrix group consisting of \(
abla = \begin{pmatrix} 1 & j_2 \\ 0 & 1 \end{pmatrix}\), where \(j_2 \in \mathbb{Z}\) (notice that \(j_1\) does not appear intentionally in the matrix). The quotient of this symplectic manifold by the diagonal action of \(\mathbb{Z}^2\) gives rise to a compact, connected, symplectic 4-manifold \((\text{KT}, \omega) := (\mathbb{R}^2 \times \mathbb{Z}/2\mathbb{Z} (\mathbb{R}/\mathbb{Z})^2, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)\) on which the 2-torus \(T := \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}\) acts symplectically and freely, where the first circle acts on the \(x_1\)-component, and the second circle acts on the \(y_2\)-component (one can check that this action is well defined). Because the \(T\)-action is free, all the orbits are principal, and because the orbits are obtained by keeping the \(x_2\)-component and the \(y_1\)-component fixed, \(dx_2 = dy_1 = 0\), so the orbits are Lagrangian.

The symplectic manifold in \((\text{KT})\) fits in the third case in Kodaira [94, Theorem 19]. Thurston rediscovered it [148], and observed that there exists no Kähler structure on KT which is compatible with the symplectic form (by noticing that the first Betti number \(b_1(\text{KT})\) is 3). It follows that no other symplectic 2-torus action on KT is Hamiltonian because in that case it would be toric and \(b_1(\text{KT})\) would vanish since toric varieties are simply connected (Proposition 6.7).

![Figure 1. A symplectic 2-torus action on \(S^2 \times (\mathbb{R}/\mathbb{Z})^2\).](image)

**Example 5.6.** This is an example of a non-Hamiltonian, non-free symplectic 2-torus action on a compact, connected, symplectic 4-manifold. Consider the compact symplectic 4-manifold \((M, \omega) := ((\mathbb{R}/\mathbb{Z})^2 \times S^2, dx \wedge dy + d\theta \wedge dh)\). There is a natural action of the 2-torus \(T := \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}\) on expression \((M, \omega)\), where the first circle of \(T\) acts on the first circle \(\mathbb{R}/\mathbb{Z}\) of the left factor of \(M\), and the right circle acts on \(S^2\) by rotations (about the vertical axis); see Figure 1. This \(T\)-action is symplectic. However, it is not a Hamiltonian action because it does not have fixed points. It is also not free, because the stabilizer subgroup of a point \((p, q)\), where \(q\) is the North or South pole of \(S^2\), is a circle subgroup. In this case the principal orbits are the products of the circle orbits.
of the left factor \((\mathbb{R}/\mathbb{Z})^2\), and the circle orbits of the right factor (all orbits of the right factor are circles but the North and South poles, which are fixed points). Because these orbits are obtained by keeping the \(y\)-coordinate on the left factor constant, and the height on the right factor constant, \(dy = dh = 0\), which implies that the product form vanishes along the principal orbits, which are Lagrangian, and hence coisotropic.

6. Classifications of Hamiltonian actions

6.1. Classification of symplectic-toric manifolds.

6.1.1. Delzant polytopes. Let \(\Delta\) be an \(n\)-dimensional convex polytope in \(t^*\). Let \(F\) be the set of all codimension one faces of \(\Delta\). Let \(V\) be the set of vertices of \(\Delta\). For every \(v \in V\), we write \(F_v = \{f \in F \mid v \in f\}\), that is, \(F_v\) is the set of faces of \(\Delta\) which contain the vertex \(v\). Following Guillemin [68, page 8] we define a very special type of polytope.

**Definition 6.1.** We say that \(\Delta\) is a Delzant polytope if: i) for each \(f \in F\) there are \(X_f \in t_\mathbb{Z}\) and \(\lambda_f \in \mathbb{R}\) such that the hyperplane which contains \(f\) has defining equation \(\langle X_f, \xi \rangle + \lambda_f = 0, \xi \in t^*\), and \(\Delta\) is contained in the set of \(\xi \in t^*\) such that \(\langle X_f, \xi \rangle + \lambda_f \geq 0\); ii) for every \(v \in V\), \(\{X_f \mid f \in F_v\}\) is a \(\mathbb{Z}\)-basis of \(t_\mathbb{Z}\).

The definition of a Delzant polytope implies the following.

**Lemma 6.2.** Let \(\Delta\) be a Delzant polytope in \(t^*\). Then for each \(f \in F\) there exists \(X_f \in t_\mathbb{Z}\) and \(\lambda_f \in \mathbb{R}\) such that \(\Delta = \{\xi \in t^* \mid \langle X_f, \xi \rangle + \lambda_f \geq 0 \ \forall f \in F\}\).

**Corollary 6.3.** For every \(v \in V\), \(#(F_v) = n\).

![Figure 2](image)

**Figure 2.** The two left most polygons are Delzant. The right 3-polytope is not Delzant (there are four vertices meeting at the frontal vertex).

Delzant (1988, [31]) proved that if the Hamiltonian action in the convexity theorem (Theorem 4.2) is effective and \(m = n\), then \(\mu(M)\) is a Delzant polytope.

For any \(z \in \mathbb{C}^F\) and \(f \in F\) we write \(z(f) := z_f\), which we view as the coordinate of the vector \(z\) with the index \(f\). Let \(\pi: \mathbb{R}^F \to t\) be the linear map \(\pi(t) := \sum_{f \in F} t_f X_f\). Because, for any vertex \(v\) of the Delzant polytope \(\Delta\), the \(X_f\) with \(f \in F_v\) form a \(\mathbb{Z}\)-basis of \(t_\mathbb{Z}\) which is also an \(\mathbb{R}\)-basis of \(t\), we have \(\pi(\mathbb{Z}^F) = t_\mathbb{Z}\) and \(\pi(\mathbb{R}^F) = t\). It follows that:

**Proposition 6.4.** The map \(\pi\) induces a surjective Lie group homomorphism \(\pi': \mathbb{R}^F/\mathbb{Z}^F = (\mathbb{R}/\mathbb{Z})^F \to t/t_\mathbb{Z}\), and hence a surjective homomorphism \(\exp \circ \pi': \mathbb{R}^F/\mathbb{Z}^F \to T\).

Write \(n := \ker \pi\) and

\[(13) \quad N = \ker(\exp \circ \pi'),\]

which is a compact abelian subgroup of \(\mathbb{R}^F/\mathbb{Z}^F\). Actually, \(N\) is connected (see [39] Lemma 3.1), and isomorphic to \(n/n_\mathbb{Z}\), where \(n_\mathbb{Z} := n \cap \mathbb{Z}^F\) is the integral lattice in \(n\) of the torus \(N\).
6.1.2. Symplectic-toric manifolds. The following notion has been a source of inspiration to many authors working on symplectic and Hamiltonian group actions, as well as finite dimensional integrable Hamiltonian systems.

**Definition 6.5.** A symplectic toric manifold is a compact connected symplectic manifold \((M, \omega)\) of dimension \(2n\) endowed with an effective Hamiltonian action of a torus \(T\) of dimension \(n\).

For instance, the effective \(S^1\)-action by rotations about \(z = 0\) of \(S^2\) (Figure 3) is symplectic and Hamiltonian, and hence \((S^2, \omega, S^1)\) is a symplectic-toric manifold, where \(\omega = d\theta \wedge dh\) is the standard area form in spherical coordinates. If \((x, y, z) \notin \{(0, 0, -1), (0, 0, 1)\}\), then \(T_{(x, y, z)} = \{e\}\), and \(T_{(0, 0, -1)} = T_{(0, 0, 1)} = S^1\). Identifying the dual Lie algebra of \(S^1\) with \(\mathbb{R}\) (by choosing a basis), the momentum map is, in spherical coordinates, given by \(\mu(\theta, z) = z\). The image of \(\mu\) is the closed interval \([-1, 1]\), which is convex hull of the image of the fixed point set \(\{(0, 0, -1), (0, 0, 1)\}\) as in Theorem 6.6.

According to the following result by Thomas Delzant, the interval \([-1, 1]\) completely characterizes the symplectic geometry of \((S^2, \omega, S^1)\) (see Figure 3).

**Theorem 6.6 (Delzant [31]).** Given any Delzant polytope \(\Delta \subset t^*\), there exists a compact connected symplectic manifold \((M_\Delta, \omega_\Delta)\) with an effective Hamiltonian \(T\)-action with momentum map \(\mu_\Delta: M_\Delta \to t^*\) such that \(\mu_\Delta(M_\Delta) = \Delta\). Any symplectic toric manifold \((M, \omega)\) is \(T\)-equivariantly symplectomorphic to \((M_{\mu(M)}, \omega_{\mu(M)})\). Two symplectic-toric manifolds are \(T\)-equivariantly symplectomorphic if and only if they have the same momentum map image, up to translations.

**Proof.** We will prove the existence part [31] pages 328, 329 following [39]: we are going to prove that for any Delzant polytope \(\Delta\) there exists a symplectic-toric manifold \(M_\Delta\) such that \(\mu_\Delta(M_\Delta) = \Delta\), and which is obtained as the reduced phase space for a linear Hamiltonian action of the torus \(N\) in \((13)\) on a symplectic vector space \(E\), at a value \(\lambda_N\) of the momentum mapping of the Hamiltonian \(N\)-action, where \(E\), \(N\) and \(\lambda_N\) are determined by \(\Delta\).

On the complex vector space \(\mathbb{C}^F\) of all complex-valued functions on \(F\) we have the action of \(\mathbb{R}^F/\mathbb{Z}^F\), where \(t \in \mathbb{R}^F/\mathbb{Z}^F\) maps \(z \in \mathbb{C}^F\) to the element \(t \cdot z \in \mathbb{C}^F\) defined for \(f \in F\) by \((t \cdot z)(f) = e^{2\pi i t f} z_f\). The infinitesimal action of \(Y \in \mathbb{R}^F = \text{Lie}(\mathbb{R}^F/\mathbb{Z}^F)\) is given by \((Y \cdot z)(f) = 2\pi i Y_f z_f\), which is a Hamiltonian vector field defined by the function \(z \mapsto (Y, \mu(z)) = \sum_{f \in F} Y_f \frac{|z_f|^2}{2} = \sum_{f \in F} Y_f \frac{x_f^2 + y_f^2}{2},\) and with respect to the symplectic form \(\omega_{\mathbb{C}^F} := \frac{1}{4\pi} \sum_{f \in F} dz_f \wedge d\overline{z}_f = \frac{1}{2\pi} \sum_{f \in F} dx_f \wedge dy_f\), if

![Figure 3. The simplest symplectic-toric manifold: \(S^2\) endowed with the standard area form and rotational action of \(S^1\) about the vertical axis.](image-url)
\[ z_f = x_f + iy_f, \text{ with } x_f, y_f \in \mathbb{R}. \] Since the right hand side of (6.1.2) depends linearly on \( Y \), we view \( \mu(z) \) as an element of \((\mathbb{R}^F)^* \simeq \mathbb{R}^F\), with the coordinates

\begin{equation}
\mu(z)_f = |z_f|^2/2 = (x_f^2 + y_f^2)/2, \quad f \in F.
\end{equation}

In other words, the action of \( \mathbb{R}^F/\mathbb{Z}^F \) on \( C^F \) is Hamiltonian with respect to \( \omega^{C^F} \) and with momentum map \( \mu : C^F \to (\text{Lie}(\mathbb{R}^F/\mathbb{Z}^F))^* \) given by (14). It follows that the action of \( N \) on \( C^F \) is Hamiltonian with momentum map \( \mu_N := \iota_n^* \circ \mu : C^F \to n^* \), where \( \iota_n : n \to \mathbb{R}^F \) denotes the identity viewed as a linear mapping from \( n \subset \mathbb{R}^F \) to \( \mathbb{R}^F \), and its transposed \( \iota_n^* : (\mathbb{R}^F)^* \to n^* \) assigns to a linear form on \( \mathbb{R}^F \) its restriction to \( n \).

Let \( \lambda \) denote the element of \((\mathbb{R}^F)^* \simeq \mathbb{R}^F\) with the coordinates \( \lambda_f, f \in F \). Write \( \lambda_N = \iota_n^*(\lambda) \). It follows from Guillemin [68, Theorem 1.6 and Theorem 1.4] that \( \lambda_N \) is a regular value of \( \mu_N \). Hence \( Z := \mu_N^{-1}(\{\lambda_N\}) \) is a smooth submanifold of \( C^F \), and that the action of \( N \) on \( Z \) is proper and free. As a consequence the \( N \)-orbit space \( M_\Delta := Z/N \) is a smooth 2\( n \)-dimensional manifold such that the projection \( p : Z \to M_\Delta \) exhibits \( Z \) as a principal \( N \)-bundle over \( M_\Delta \) for the \( N \)-action at \( \lambda_N \), see Theorem 1.1. Moreover, there is a unique symplectic form \( \omega_\Delta \) on \( M_\Delta \) such that \( p^* \omega_\Delta = t_Z^* \omega^{C^F} \), where \( t_Z \) is the identity viewed as a smooth mapping from \( Z \) to \( C^F \). \( M_\Delta \) is the Marsden-Weinstein reduction [2, Section 4.3] of \((C^F, \omega^{C^F})\). On the \( N \)-orbit space \( M_\Delta \), we still have the action of the torus \((\mathbb{R}^F/\mathbb{Z}^F)/N \simeq T \), with momentum mapping \( \mu_\Delta : M \to t^* \) determined by \( \pi^* \circ \mu_\Delta \circ p = (\mu - \lambda)|_Z \). The torus \( T \) acts effectively on \( M \) and \( \mu_\Delta(M) = \Delta \), see Guillemin [68, Theorem 1.7], and therefore we have constructed the symplectic-toric manifold \((M_\Delta, \omega_\Delta)\) with \( T \)-action and momentum map \( \mu_\Delta : M_\Delta \to t^* \) such that \( \mu_\Delta(M) = \Delta \), from \( \Delta \).

There have been generalizations of Theorem 6.6 for instance to multiplicity-free group actions by Woodward [160], and to symplectic-toric orbifolds by Lerman and Tolman [103]. An extension to noncompact symplectic manifolds was recently given by Karshon and Lerman [88].

Because any symplectic-toric manifold is obtained by symplectic reduction of \( C^1 \), it admits a compatible \( T \)-invariant Kähler metric. Delzant [31, Section 5] observed that \( \Delta \) gives rise to a fan, and that the symplectic toric manifold with Delzant polytope \( \Delta \) is \( T \)-equivariantly diffeomorphic to the toric variety \( M^{\text{toric}} \) defined by the fan. Here \( M^{\text{toric}} \) is a complex \( n \)-dimensional complex analytic manifold, and the action of \( T \) on \( M^{\text{toric}} \) has an extension to a complex analytic action on \( M^{\text{toric}} \) of the complexification \( T_C \) of \( T \).

A detailed study of the relation between the symplectic-toric manifold and \( M^{\text{toric}} \) appears in [39]. The following proof illustrates the interplay between the symplectic and algebraic view points in toric geometry.

**Proposition 6.7.** Every symplectic-toric manifold is simply connected.

**Proof.** Every symplectic toric manifold may be provided with the structure of a toric variety defined by a complete fan, cf. Delzant [31, Section 5] and Guillemin [68, Appendix 1]. On the other hand, Danilov [29, Theorem 9.1] observed that such a toric variety is simply connected. The proof goes as follows: the toric variety has an open cell which is isomorphic to the complex space \( \mathbb{C}^n \), whose complement is a complex subvariety of complex codimension one. Hence all loops may be deformed into the cell and contracted within the cell to a point. \( \square \)

### 6.2. Log symplectic-toric manifolds

Recently there has been a generalization of symplectic-toric geometry to a class of Poisson manifolds, called log-symplectic manifolds. Log-symplectic manifolds are generically symplectic but degenerate along a normal crossing configuration of smooth hypersurfaces.
Guillemin, Miranda, Pires and Scott initiated the study of log symplectic-toric manifolds in their article [69]. They proved the analogue of Delzant’s theorem (Theorem 6.6) in the case where the degeneracy locus of the associated Poisson structure is a smooth hypersurface.

Degeneracy loci for Poisson structures are often singular. In [67] the authors consider the mildest possible singularities (normal crossing hypersurfaces) and gave an analogue of Theorem 6.6. Next we informally state this result to give a flavor of the ingredients involved (being precise would be beyond the scope of this paper).

The notion of isomorphism below generalizes the classical notion taking into account the log symplectic structure.

**Theorem 6.8 ([67]).** There is a one-to-one correspondence between isomorphism classes of oriented compact connected log symplectic-toric $2n$-manifolds and equivalence classes of pairs $(\Delta, M)$, where $\Delta$ is a compact convex log affine polytope of dimension $n$ satisfying the Delzant condition and $M \to \Delta$ is a principal $n$-torus bundle over $\Delta$ with vanishing toric log obstruction class.

Log-symplectic geometry and its toric version are an active area of research which is related to tropical geometry and the extended tropicalizations of toric varieties defined by Kajiwara [85] and Payne [122].

Convexity properties of Hamiltonian torus actions on log-symplectic manifolds were studied in [70], where the authors prove a generalization of Theorem 4.2.

### 6.3. Classification of Hamiltonian $S^1$-spaces

In addition to Delzant’s classification (Theorem 6.6) there have been other classifications of Hamiltonian $G$-actions on compact symplectic $2n$-manifolds. In this section we outline the classification when $G = S^1$ and $n = 2$ due to Karshon.

**Definition 6.9.** A Hamiltonian $S^1$-space is a compact connected symplectic 4-manifold equipped with an effective Hamiltonian $S^1$-action

Let $(M, \omega, S^1)$ be a Hamiltonian $S^1$-space. We associate it a labelled graph as follows. Let $\mu: M \to \mathbb{R}$ be the momentum map of the $S^1$-action. For each component $\Sigma$ of the set of fixed points of the $S^1$-action there is one vertex in the graph, labelled by $\mu(\Sigma) \in \mathbb{R}$.

If $\Sigma$ is a surface, the corresponding vertex has two additional labels, one is the symplectic area of $\Sigma$ and the other one is the genus of $\Sigma$.

Let $F_k$ be a subgroup of $k$ elements of $S^1$. For every connected component $C$ of the set of points fixed by $F_k$ there is an edge in the graph, labeled by the integer $k > 1$. The component $C$ is a 2-sphere, which we call a $F_k$-sphere. The quotient circle $S^1/F_k$ rotates it while fixing two points, and the two vertices in the graph corresponding to the two fixed points are connected in the graph by the edge corresponding to $C$.

**Theorem 6.10 (Audin, Ahara, Hattori [3, 12, 13]).** Every Hamiltonian $S^1$-space is $S^1$-equivariantly diffeomorphic to a complex surface with a holomorphic $S^1$-action which is obtained from $\mathbb{CP}^2$, a Hirzebruch surface, or a $\mathbb{CP}^1$-bundle over a Riemann surface (with appropriate circle actions), by a sequence of blow-ups at the fixed points.

Let $A$ and $B$ be connected components of the set of fixed points. The $S^1$-action extends to a holomorphic action of the group $\mathbb{C}^\times$ of non-zero complex numbers. Consider the time flow given by the action of the subgroup $\{\exp(t) \mid t \in \mathbb{R}\}$.

**Definition 6.11.** We say that $A$ is greater than $B$ if there is an orbit of the $\mathbb{C}^\times$-action which at time $t = \infty$ approaches a point in $A$ and at time $t = -\infty$ approaches a point in $B$.

Take any of the complex surfaces with $S^1$-actions considered by Audin, Ahara and Hattori, and assign a real parameter to every connected component of the fixed point set such that these
parameters are monotonic with respect to the partial ordering we have just described. If the manifold contains two fixed surfaces we assign a positive real number to each of them so that the difference between the numbers is given by a formula involving the previously chosen parameters.

Karshon proved that for every such a choice of parameters there exists an invariant symplectic form and a momentum map on the complex surface such that the values of the momentum map at the fixed points and the symplectic areas of the fixed surfaces are equal to the chosen parameters. Moreover, every two symplectic forms with this property differ by an $S^1$-equivariant diffeomorphism.

**Theorem 6.12** (Karshon [87]). If two Hamiltonian $S^1$-spaces have the same graph, then they are $S^1$-equivariantly symplectomorphic. Moreover, every compact 4-dimensional Hamiltonian $S^1$-space is $S^1$-equivariantly symplectomorphic to one of the spaces listed in the paragraph above.

A generalization of this classification result to higher dimensions has been recently obtained by Karshon-Tolman [89]. The authors construct all possible Hamiltonian symplectic torus actions for which all the nonempty reduced spaces are two dimensional (and not single points), the manifold is connected and the momentum map is proper as a map to a convex set.

The study of symplectic and Hamiltonian circle actions has been an active topic of current research, see for instance McDuff-Tolman [113], where they show many interesting properties, for instance that if the weights of a Hamiltonian $S^1$-action on a compact symplectic symplectic manifold $(M, \omega)$ at the points at which the momentum map is a maximum are sufficiently small, then the circle represents a nonzero element of $\pi_1(\text{Ham}(M, \omega))$, where $\text{Ham}(M, \omega)$ is the group of Hamiltonian symplectomorphisms of $(M, \omega)$.

In [63] Godinho and Sabatini construct an algorithm to obtain linear relations among the weights at the fixed points which under certain conditions determines a family of vector spaces which contain the admissible lattices of weights.

Concerning symplectic $S^1$-actions (not necessarily Hamiltonian), see Godinho’s articles [60] [61].

### 6.4. Hamiltonian $(S^1 \times \mathbb{R})$-actions and classification of symplectic-semitoric manifolds.

Semitoric systems, also called symplectic semitoric manifolds, are a rich class of integrable systems which, in the case of compact phase space, take place on the Hamiltonian $S^1$-spaces of the previous section.

Let $(M, \omega)$ be a symplectic 4-dimensional manifold. The Poisson brackets of two real valued smooth functions $f$ and $g$ on $M$ are defined by $\{f, g\} := \omega(\mathcal{H}_f, \mathcal{H}_g)$.

**Definition 6.13.** An integrable system with two degrees of freedom is a smooth map $F = (f_1, f_2): M \to \mathbb{R}^2$ such that $\{f_1, f_2\} = 0$ and the Hamiltonian vector fields $\mathcal{H}_{f_1}, \mathcal{H}_{f_2}$ are linearly independent almost everywhere.

A theorem of Eliasson characterizes the so called “non-degenerate” (the term “non-degenerate” is a generalization of “Morse non-degenerate” which is more involved to define [135] here). The following is a particular instance of Eliasson’s general theorem, of interest to us here.

**Theorem 6.14** (Eliasson [45] [46]). Let $F := (f_1, f_2): (M, \omega) \to \mathbb{R}^2$ be an integrable system with two degrees of freedom all of the singularities of which are non-degenerate, and with no hyperbolic blocks. There exist local symplectic coordinates $(x_1, x_2, \xi_1, \xi_2)$ about every non-degenerate critical point $m$, in which $m = (0, 0, 0, 0)$, and $(F - F(m)) \circ \varphi = g \circ (q_1, q_2)$, where $\varphi = (x_1, x_2, \xi_1, \xi_2)^{-1}$ and $g$ is a diffeomorphism from a small neighborhood of the origin in $\mathbb{R}^4$ into another such neighborhood, such that $g(0, 0, 0, 0) = (0, 0, 0, 0)$ and $(q_1, q_2)$ are, depending on the rank of the critical point, as follows. If $m$ is a critical point of $F$ of rank zero, then $q_j$ is one of

(i) $q_1 = (x_1^2 + \xi_1^2)/2$ and $q_2 = (x_2^2 + \xi_2^2)/2$. 

(ii) \( q_1 = x_1 \xi_2 - x_2 \xi_1 \) and \( q_2 = x_1 \xi_1 + x_2 \xi_2 \);

If \( m \) is a critical point of \( F \) of rank one, then
\[
q_1 = \frac{(x_1^2 + \xi_1^2)}{2} \quad \text{and} \quad q_2 = \xi_2.
\]

The assumption of not having hyperbolic blocks is simply to reduce the complexity of the statement of the theorem, but is not really needed to understand the discussion which follows.

**Remark 6.15.** The analytic case of Theorem 6.14 is due to Rüßmann [132] for two degrees of freedom systems and to Vey [151] in any dimension. 

**Definition 6.16.** A semitoric system \( F := (f_1, f_2) : M \to \mathbb{R}^2 \) on a connected symplectic 4-manifold \((M, \omega)\) is an integrable system with two degrees of freedom such that \( f_1 \) is the momentum map of a Hamiltonian \( S^1 \)-action, \( f_1 \) is a proper map, and the singularities of \( F \) are non-degenerate, without hyperbolic blocks, and hence they are of the form given in Theorem 6.14.

**Remark 6.17.** In the definition above, \( f_1 \) gives rise to a Hamiltonian \( S^1 \)-action on \( M \), and \( f_2 \) gives rise to a Hamiltonian \( \mathbb{R} \)-action on \( M \); conceptually, their flows, one after the other, produce a Hamiltonian \( (S^1 \times \mathbb{R}) \)-action; details of the precise relation between \((S^1 \times \mathbb{R})\)-actions and semitoric systems are spelled out in [49] Section 3.

**Remark 6.18.** If \( M \) is compact then \((M, \omega)\) endowed with the Hamiltonian \( S^1 \)-action with momentum map \( f_1 \) is a Hamiltonian \( S^1 \)-space.

**Definition 6.19.** Suppose that \( F_1 = (f_1^1, f_1^2) : (M_1, \omega_1) \to \mathbb{R}^2 \) and \( F_2 = (f_2^1, f_2^2) : (M_2, \omega_2) \to \mathbb{R}^2 \) are semitoric systems. We say that they are isomorphic if there exists a symplectomorphism \( \varphi : (M_1, \omega_1) \to (M_2, \omega_2) \), and a smooth map \( \phi : F_1(M_1) \to \mathbb{R} \) with \( \partial_2 \phi \neq 0 \), such that \( \varphi^* f_1^1 = f_1^1 \) and \( \varphi^* f_2^1 = \phi(f_1^1, f_1^2) \).

Semitoric systems are classified under the assumption that each singular fiber contains at most one singular point of type (1.ii); these points are called focus-focus (or nodal, in algebraic geometry). The singular fiber containing a focus-focus point is a 2-torus pinched precisely at the focus-focus point (i.e. topologically a 2-sphere with the north and south poles identified). Semitoric systems satisfying this condition are called simple.

**Theorem 6.20** ([133] [134]). Simple semitoric systems \((M, \omega, F := (f_1, f_2))\) are determined, up to isomorphisms, by a convex polygon endowed with a collection of interior points, each of which is labelled by a tuple \((k \in \mathbb{Z}, \sum_{i,j=1}^\infty a_{ij}X^iY^j)\). Here \( \Delta \) is obtained from \( F(M) \) by appropriately unfolding the singular affine structure induced by \( F \), \( k \) encodes how twisted the singular Lagrangian fibration \( F \) is between consecutive focus-focus points arranged according to the first component of their image in \( \mathbb{R}^2 \), and the Taylor series \( \sum_{i,j=1}^\infty a_{ij}X^iY^j \) encodes the singular dynamics of the vector fields \( \mathcal{H}_{f_1}, \mathcal{H}_{f_2} \). Conversely, given a polygon with interior points \( p_1, \ldots, p_n \), and for each \( p_k \) a label \((k \in \mathbb{Z}, \sum_{i,j=1}^\infty a_{ij}X^iY^j)\), one can construct \((M, \omega)\) and a semitoric system \( F : M \to \mathbb{R}^2 \) having this data as invariants.

In [120] Palmer defined the moduli space of semitoric systems, which is an incomplete metric space, and constructed its completion. In [36] the connectivity properties of this space were studied using \( \text{SL}_2(\mathbb{Z}) \) equations.

Four-dimensional symplectic-toric manifolds are a very particular case of compact semitoric systems (in which the manifold is closed, and the only invariant is the convex polygon). Every semitoric

\[\text{this is really not any polygon, but a polygon of so called semitoric type, which generalize the notion of Delzant polygon (which was applicable to toric systems) to this more general context.}\]
system takes place on a Hamiltonian $S^1$-space, and the relation has been made completely explicitly recently. We call Karshon graph the labelled directed graph in Theorem 6.12.

**Theorem 6.21** (Hohloch-Sabatini-Sepe [82]). Let $F := (f_1, f_2): (M, \omega) \to \mathbb{R}^2$ be a simple semitoric system on a compact manifold with $m_f$ focus-focus critical points and underlying Hamiltonian $S^1$-space $(M, \omega, f_1)$ with momentum map $f_1$. Then the associated convex polygon in Theorem 6.20 and $m_f$ determines the Karshon graph, thus classifying $(M, \omega, S^1)$ up to $f_1$-equivariant symplectomorphisms.

There has been recent work generalizing the convex polygon in Theorem 6.20 to higher dimensional semitoric systems by Wacheux [152].

The Fomenko school has powerful and far reaching methods to study the topology of singularities of integrable systems [21].

7. Properties of symplectic actions

7.1. Fundamental form of a symplectic action. Let $T$ be a torus with Lie algebra $\mathfrak{t}$. Suppose that $T$ acts symplectically on a connected symplectic manifold $(M, \omega)$.

**Proposition 7.1.** There is a unique antisymmetric bilinear form $\omega^t$ on $\mathfrak{t}$, which we call the “fundamental form”, such that

$$\omega^t(X, Y) = \omega_x(X_M(x), Y_M(x))$$

for every $X, Y \in \mathfrak{t}$ and every $x \in M$.

**Proof.** Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth vector fields on $M$ satisfying $L_\mathcal{X}\omega = 0$ and $L_\mathcal{Y}\omega = 0$. Then by the homotopy identity (10)

$$i_{[X,Y]}\omega = L_\mathcal{X}(i_\mathcal{Y}\omega) = i_\mathcal{X}(d(i_\mathcal{Y}\omega)) + d(i_\mathcal{X}(i_\mathcal{Y}\omega)) = -d(\omega(\mathcal{X}, \mathcal{Y})).$$

Here we have used that $L_\mathcal{X}\omega = 0$ in the first equality, the homotopy formula for the Lie derivative in the second equality, and the closedness of $\omega$, the homotopy identity and $L_\mathcal{X}\omega = 0$ in the third equality.

Now take $\mathcal{X} = X_M, \mathcal{Y} = Y_M$ where $X, Y \in \mathfrak{t}$. Then from (7) and (16) we have $\mathcal{H}(X_M, Y_M) = 0$ and hence $d(\omega(X_M, Y_M)) = 0$, and the connectedness of $M$ implies that $x \mapsto \omega_x(X_M(x), Y_M(x))$ is constant. $\square$

7.2. Benoist-Ortega-Ratiu symplectic normal form. Let $(M, \omega)$ be a symplectic manifold endowed with a proper symplectic action of a Lie group $G$.

Let $x \in M$, let $H := G_x$, and let $\mathfrak{l}$ be the kernel of the fundamental form $\omega^t$ (Proposition 7.1), let $\mathfrak{g}_M(x) := T_x(G \cdot x)$, and let $\alpha_x$ is as in Proposition 3.10. In addition, let $\omega^{G/H}$ be the $G$-invariant closed 2-form $(\alpha_x)^*\omega$ on $G/H$, and let $\omega^W$ be the symplectic form on $W := \mathfrak{g}_M(x)_{\omega_x}/(\mathfrak{g}_M(x)_{\omega_x} \cap \mathfrak{g}_M(x))$, defined as the restriction to $\mathfrak{g}_M(x)_{\omega_x}$ of $\omega_x$. The map $X + h \mapsto X_M(x)$ is a linear isomorphism from $\mathfrak{l}/\mathfrak{h}$ to $\mathfrak{g}_M(x)_{\omega_x} \cap \mathfrak{g}_M(x)$. The linearized action of $H$ on $T_x M$ is symplectic and leaves $\mathfrak{g}_M(x)_{\omega_x} \simeq \mathfrak{g}/\mathfrak{h}$ invariant, acting on it via the adjoint representation. It also leaves $\mathfrak{g}_M(x)_{\omega_x}$ invariant and induces an action of $H$ on $(W, \omega^W)$ by symplectic linear transformations. Let $E := (\mathfrak{l}/\mathfrak{h})^* \times W$, on which $h \in H$ acts by sending $(\lambda, w)$ to $(((\text{Ad}(h))^{-1})^\lambda(h, w)$. Choose $\text{Ad}H$-invariant linear complements $\mathfrak{e}$ and $\mathfrak{c}$ of $\mathfrak{h}$ and $\mathfrak{l}$ in $\mathfrak{g}$, respectively. Let $X \mapsto X_1 : \mathfrak{g} \mapsto \mathfrak{l}$ and $X \mapsto X_\mathfrak{h} : \mathfrak{g} \mapsto \mathfrak{h}$ denote the linear projection from $\mathfrak{g}$ onto $\mathfrak{l}$ and $\mathfrak{h}$ with kernel equal to $\mathfrak{c}$ and $\mathfrak{e}$, respectively. These projections are $\text{Ad}H$-equivariant. Define the smooth one-form $\eta^\#$ on $G \times E$ by $\eta^\#_{(g, \lambda, w)}((d_1 L_g)(X), (\delta \lambda, \delta w)) := \lambda(X_1) + \frac{1}{2}(\lambda, w) \omega^W(w, \delta w + X_\mathfrak{h} \cdot w)$ for all $g \in G, \lambda \in (\mathfrak{l}/\mathfrak{h})^*, w \in W$, and all $X \in \mathfrak{g}, \delta \lambda \in (\mathfrak{l}/\mathfrak{h})^*, \delta w \in W$. 

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Let \( G \times_H E \) denote the orbit space of \( G \times E \) for the proper and free action of \( H \) on \( G \times E \), where \( h \in H \) acts on \( G \times E \) by sending \((g, e)\) to \((gh^{-1}, h \cdot e)\). The action of \( G \) on \( G \times_H E \) is induced by the translational action of \( G \) on \( G \times E \).

Let \( \pi: G \times_H E \to G/H \) be induced by \((g, e) \mapsto g: G \times E \to G \). Because \( H \) acts on \( E \) by means of linear transformations, this projection exhibits \( G \times_H E \) as a \( G \)-homogeneous vector bundle over the homogeneous space \( G/H \), which fiber \( E \) and structure group \( H \).

**Proposition 7.2.** If \( \pi_H: G \times E \to G \times_H E \) denotes \( H \)-orbit mapping, then there is a unique smooth one-form \( \eta \) on \( G \times_H E \), such that \( \eta^\# = \pi_H^* \eta \).

Endow \( G \times_H E \) with the 2-form \( \pi^* \omega^{G/H} + d\eta \). This 2-form is symplectic.

The following is the local normal form of Benoist \[15\] Prop. 1.9 and Ortega and Ratiu \[117\] for a general proper symplectic Lie group action.

**Theorem 7.3** (Benoist \[15\], Ortega-Ratiu \[117\]). There is an open \( H \)-invariant neighborhood \( E_0 \) of the origin in \( E \), an open \( G \)-invariant neighborhood \( U \) of \( x \) in \( M \), and a \( G \)-equivariant symplectomorphism \( \Phi: (G \times_H E, \pi^* \omega^{G/H} + d\eta) \to (U, \omega) \) such that \( \Phi(H \cdot (1,0)) = x \).

For Hamiltonian symplectic actions, these local models had been obtained before by Marle \[105\] and Guillemin and Sternberg \[74, Section 41\].

The fundamental form \( \omega^t \) in Proposition 7.1 is an essential ingredient in the study of symplectic actions. In the case of Hamiltonian actions, it takes a very particular form as we see from the next result (but it will also turn out to be essential in the case of more general symplectic actions).

**Theorem 7.4.** Let \((M, \omega)\) be a compact connected symplectic manifold endowed with an effective symplectic action of an \( n \)-dimensional torus \( T \) on \((M, \omega)\). Then the following are equivalent: i) The action of \( T \) has a fixed point in \( M \); ii) \( \omega^t = 0 \) and \( M/T \) is homeomorphic to a convex polytope; iii) \( \omega^t = 0 \) and \( \text{H}^1(M/T, \mathbb{R}) = 0 \); iv) The action of \( T \) is Hamiltonian.

The proof of Theorem 7.3 uses Theorem 4.2, Proposition 7.1, Theorem 7.3, and the theory of \( V \)-parallel spaces (\[38\] Section 10), see \[38\] Corollary 3.9.

### 7.3. Symplectic orbit types.

While Hamiltonian actions of maximal dimension appear as symmetries in many integrable systems in mechanics, non-Hamiltonian actions also occur in physics, see eg. Novikov \[115\].

In the remaining of this paper we will give classifications of symplectic actions of tori (in the spirit of the Delzant classification Theorem 6.6) in two cases: maximal symplectic actions, and coisotropic actions.

We already described Hamiltonian actions in the previous sections in the case of \( \dim M = 2 \dim T \), and these are a special case of coisotropic actions, as we will see.

**Definition 7.5.** A submanifold \( C \) of the symplectic manifold \((M, \omega)\) is *symplectic* if the restriction \( \omega|_C := i^* \omega \) of the symplectic form \( \omega \) to \( C \), where \( i: C \to M \) is the inclusion map, is a symplectic form.

**Definition 7.6.** A *maximal symplectic action* is a symplectic action on compact symplectic manifold endowed with an effective symplectic action of a torus \( T \) with some symplectic \( T \)-orbit of maximal dimension \( \dim T \).

**Lemma 7.7.** Let \((M, \omega)\) be a symplectic manifold endowed with a symplectic \( T \)-action. If there is a symplectic \( \dim T \)-orbit then every \( T \)-orbit is symplectic and \( \dim T \)-dimensional.
**Proof.** Assume that there is a symplectic $\dim T$-orbit. Since the fundamental form $\omega^t$ (Proposition 7.1) is point independent, it is non-degenerate. Hence $\ker(\omega^t) = 0$. □

**Proposition 7.8.** A maximal symplectic action does not admit a momentum map, and hence it is not Hamiltonian.

**Proof.** The $T$-orbits of a Hamiltonian action are isotropic submanifolds, and hence not symplectic as it is the case for maximal symplectic actions. □

Recall that if $V$ is a subspace of a symplectic vector space $(W, \sigma)$, its **symplectic orthogonal complement** $V^\sigma$ consists of the vectors $w \in W$ such that $\sigma(w, v) = 0$ for all $v \in V$.

**Definition 7.9.** A submanifold $C$ of a symplectic manifold $(M, \omega)$ is **coisotropic** if for every $x \in C$ we have that $(T_x C)^\omega = T_x C$.

**Proposition 7.10.** If $C$ is a coisotropic $k$-dimensional submanifold of a $2n$-dimensional symplectic manifold $(M, \omega)$ then $k \geq n$.

**Proof.** If $C$ is a coisotropic submanifold of dimension $k$, then $2n - k = \dim (T_x C)^\omega \leq \dim (T_x C) = k$ shows that $k \geq n$. □

The submanifold $C$ has the minimal dimension $n$ if and only if $(T_x C)^\omega = T_x C$, if and only if $C$ is a Lagrangian submanifold of $M$ (Definition 2.28).

A Lagrangian submanifold is simultaneously a special case of coisotropic and isotropic submanifold. In fact, a submanifold is Lagrangian if and only if it is isotropic and coisotropic. Hence a symplectic torus action for which one can show that it has a Lagrangian orbit falls into the category of coisotropic actions. Therefore it also includes Hamiltonian actions of $n$-dimensional tori which are described in Section 3, because the preimage of any point in the interior of the momentum polytope is a Lagrangian orbit.

**Definition 7.11.** Let $T$ be a torus. A symplectic $T$-action on a compact connected symplectic manifold $(M, \omega)$ endowed with a symplectic $T$-action with coisotropic orbits is called a **coisotropic action**.

The following is a consequence of Theorem 7.3. In the following recall that $T$ denotes the kernel of the fundamental form $\omega^t$ (Proposition 7.1).

**Lemma 7.12.** Let $(M, \omega)$ be a compact connected symplectic manifold, and $T$ a torus which acts effectively and symplectically on $(M, \omega)$. Suppose that there exists a coisotropic principal $T$-orbit. Then every coisotropic $T$-orbit is a principal orbit and $\dim M = \dim T + \dim T$.

**Proof.** We use Theorem 7.3 with $G = T$. Since $T$ is abelian, the adjoint action of $H = T_x$ on $t$ is trivial, which implies that the coadjoint action of $H$ on the component $(l/h)^*$ is trivial. Let $T \cdot x$ be a coisotropic orbit. Then $W$ is zero. This implies that the action of $H$ on $E = (l/h)^*$ is trivial, and $T \times_H E = T \times_H (l/h)^*$ is $T$-equivariantly isomorphic to $(T/H) \times (l/h)^*$, where $T$ acts by left multiplications on the first factor. It follows that in the model all stabilizer subgroups are equal to $H$, and therefore $T_y = H$ for all $y$ in the $T$-invariant open neighborhood $U$ of $x$ in $M$. Since $M_{reg}$ is dense in $M$, there are $y \in U$ such that $T_y = \{1\}$, and therefore $T_x = H = \{1\}$, so the orbit $T \cdot x$ is principal. The statement $\dim M = \dim T + \dim T$ also follows. □

A similar argument to that in the proof of Lemma 7.12 using Theorem 7.3 (this time using the formula for the symplectic form in Theorem 7.3) shows:

**Proposition 7.13.** There exists a coisotropic orbit if and only if every principal orbit is coisotropic.
Proposition 7.14. The following hold:

1. The fundamental form $\omega^t$ vanishes identically if and only if $I := \ker(\omega^t) = t$ if and only if some $T$-orbit is isotropic if and only if every $T$-orbit is isotropic.

2. Every principal orbit is Lagrangian if and only if some principal orbit is Lagrangian if and only if $\dim M = 2 \dim T$ and $\omega^t = 0$.

Proof. The equivalence of $\omega^t = 0$ and $I = t$ is immediate. The equivalence between $\omega^t = 0$ and the isotropy of some $T$-orbit follows from Proposition 7.1.

If $x \in M_{reg}$ and $T \cdot x$ is a Lagrangian submanifold of $(M, \omega)$, then $\dim M = 2 \dim (T \cdot x) = 2 \dim T$, and $\omega^t = 0$ follows in view of the first statement in the proposition. Conversely, if $\dim M = 2 \dim T$ and $\omega^t = 0$, then every orbit is isotropic and for every $x \in M_{reg}$ we have $\dim M = 2 \dim T = 2 \dim (T \cdot x)$, which implies that $T \cdot x$ is a Lagrangian submanifold of $(M, \omega)$. □

In Guillemin and Sternberg [75], we find the following notion.

Definition 7.15. A symplectic manifold with a Hamiltonian action of an arbitrary compact Lie group is called a multiplicity-free space if the Poisson brackets of any pair of invariant smooth functions vanish.

There is a relationship between coisotropic actions and multiplicity-free spaces.

Proposition 7.16. For a torus $T$ acting on a closed connected symplectic manifold $(M, \omega)$, the principal orbits are coisotropic if and only if $(M, \omega)$ is a multiplicity-free space.

Proof. Let $f, g$ be in the set of $T$-invariant smooth functions and let $x \in M_{reg}$. We will use the notation $t_M(x) := T_x(T \cdot x)$. Since $M_{reg}$ is fibered by the $T$-orbits, $t_M(x)$ is the common kernel of the $T$-orbits, $t_M(x)$ is the set of expressions $\text{ker}(\omega^t) = t$ if and only if some $T$-orbit is isotropic if and only if every $T$-orbit is isotropic.

If $x \in M_{reg}$ and $T \cdot x$ is a Lagrangian submanifold of $(M, \omega)$, then $\dim M = 2 \dim (T \cdot x) = 2 \dim T$, and $\omega^t = 0$ follows in view of the first statement in the proposition. Conversely, if $\dim M = 2 \dim T$ and $\omega^t = 0$, then every orbit is isotropic and for every $x \in M_{reg}$ we have $\dim M = 2 \dim T = 2 \dim (T \cdot x)$, which implies that $T \cdot x$ is a Lagrangian submanifold of $(M, \omega)$. □

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Proposition 7.16. For a torus $T$ acting on a closed connected symplectic manifold $(M, \omega)$, the principal orbits are coisotropic if and only if $(M, \omega)$ is a multiplicity-free space.

Proof. Let $f, g$ be in the set of $T$-invariant smooth functions and let $x \in M_{reg}$. We will use use the notation $t_M(x) := T_x(T \cdot x)$. Since $M_{reg}$ is fibered by the $T$-orbits, $t_M(x)$ is the common kernel of the $T$-orbits, $t_M(x)$ is the set of all $x \in M_{reg}$. So if the principal $T$-orbits are coisotropic, we have that $\mathcal{H}_f(x), \mathcal{H}_g(x) \in t_M(x)^{\omega_x} \cap t_M(x)$. It follows that

$$\left\{ f, g \right\}(x) := \omega_x \left( \mathcal{H}_f(x), \mathcal{H}_g(x) \right) = 0 \quad \forall x \in t_M(x)^{\omega_x} \cap t_M(x)$$

Since the principal orbit type $M_{reg}$ is dense in $M$, we have that $\left\{ f, g \right\} = 0$ for all $f, g \in C^\infty(M)^T$ if the principal orbits are coisotropic. Conversely, if we have that $\left\{ f, g \right\} = 0$ for all $f, g \in C^\infty(M)^T$, then $t_M(x)^{\omega_x} \subset (t_M(x)^{\omega_x})^{\omega_x} = t_M(x)$ for every $x \in M_{reg}$, i.e. $T \cdot x$ is coisotropic. □

7.4. Stabilizer subgroups. The general problem we want to treat is:

Problem 7.17. Let $(M, \omega)$ be a closed connected symplectic manifold endowed with an effective symplectic $T$-action. Let $x \in M$. Characterize when $T_x$ is connected and when it is discrete.

We will prove the following result:

Theorem 7.18. Let $(M, \omega)$ be a symplectic manifold endowed with an effective symplectic $T$-action and let $x \in M$.

(i) If the $T$-action is coisotropic, then $T_x$ is connected.

(ii) If the $T$-action is maximal symplectic, then $T_x$ is finite.

Let $t_x$ denote the Lie algebra of the stabilizer subgroup $T_x$ of the $T$-action at $x$, which consists of of the $X \in t$ such that $X_M(x) = 0$. That is, $t_x$ is the kernel of the linear mapping $\alpha_x : X \to X_M(x)$ from $t$ to $T_x M$. 
Lemma 7.19 ([125]). Let $(M, \omega)$ be a symplectic manifold endowed with a maximal symplectic $T$-action. The stabilizer $T_x$ is a finite abelian group for every $x \in M$.

Proof. Since $t_x \subset \ker \omega^j$, by Lemma 7.7 we have that $t_x$ is trivial. □

Since every $T_x$ is finite, it follows from the tube theorem of Koszul (cf. [98] or [37] Theorem 2.4.1) and the compactness of $M$ that there exists only finitely many different stabilizer subgroups.

The following is statement (1) (a) in [15, Lemma 6.7]. It is a consequence of Theorem 7.3. We use Theorem 7.3 with $G = T$, with $H$ acting trivially on the factor $(1/h)^* \times W$. Recall that $t \in T$ acts on $T \times_H E$ by sending $H \cdot (t', e)$ to $H \cdot (tt', e)$. When $t = h \in H$, $E_{\mathbf{h}} \cdot (h t', e) = H \cdot (h t' h^{-1}, h \cdot e) = H \cdot (t', h \cdot e)$ since $T$ is abelian, and the action of $H$ on $T \times_H E$ is represented by the linear symplectic action of $H$ on $W$.

Lemma 7.20 (Benoist [15]). Let $(M, \omega)$ be a compact symplectic manifold endowed with a coisotropic $T$-action. For every $x \in M$, the stabilizer $T_x$ is connected.

Proof. Since $\dim M = (\dim T + \dim (1/h) + \dim W) - \dim H$ and because the assumption that the principal orbits are coisotropic implies that $\dim M = \dim T + \dim l$, see Lemma 7.12, it follows that $\dim W = 2 \dim H$.

Write $m = \dim H$. The action of $H$ by means of symplectic linear transformations on $(W, \omega^W)$ leads to a direct sum decomposition of $W$ into $m$ pairwise $\omega^W$-orthogonal two-dimensional $H$-invariant linear subspaces $E_j$, $1 \leq j \leq m$. For $h \in H$ and every $1 \leq j \leq m$, let $t_j(h)$ denote the restriction to $E_j \subset W \simeq \{0\} \times W \subset (1/h)^* \times W$ of the action of $h$ on $E$. Note that $\det t_j(h) = 1$, because $t_j(h)$ preserves the restriction to $E_j \times E_j$ of $\omega^W$, which is an area form on $E_j$.

Averaging any inner product in each $E_j$ over $H$, we obtain an $H$-invariant inner product $\beta_j$ on $E_j$, and $t_j$ is a homomorphism of Lie groups from $H$ to $\text{SO}(E_j, \beta_j)$, the group of linear transformations of $E_j$ which preserve both $\beta_j$ and the orientation. If $h \in H$ and $w \in W_{\text{reg}}$, then $h \cdot w = \sum_{j=1}^m t_j(h) w_j$ where $w = \sum_{j=1}^m w_j$, $w_j \in E_j$. Therefore $t_j(h) w_j = w_j$ for all $1 \leq j \leq m$ implies that $h = 1$. Hence the map $i : H \to \prod_{j=1}^m \text{SO}(E_j, \beta_j)$ defined by $i(h) = (t_1(h), \ldots, t_m(h))$ is a Lie group isomorphism, so $H$ is connected. □

It follows from Lemma 7.20 that $T_x$ is a subtorus of $T$.

8. Classifications of symplectic actions

8.1. Maximal symplectic actions. This section describes the invariants of maximal symplectic actions. Using these invariants we will construct a model to which $(M, \omega)$ is $T$-equivariantly symplectomorphic. Let $(M, \omega)$ be a compact connected symplectic manifold endowed with a maximal symplectic $T$-action of a torus $T$.

8.1.1. Orbit space. We denote by $\pi : M \to M/T$ the canonical projection $\pi(x) := T \cdot x$. The orbit space $M/T$ is endowed with the maximal topology for which $\pi$ is continuous (which is a Hausdorff topology). Since $M$ is compact and $M/T$ is compact and connected. By the tube theorem (see for instance [37] Theorem 2.4.1) if $x \in M$ there is a $T$-invariant open neighborhood $U_x$ of $T \cdot x$ and a $T$-equivariant diffeomorphism $\Phi_x : U_x \to T \times_{T_x} D_x$, where $D_x$ is an open disk centered at the origin in $\mathbb{C}^{k/2}$, $k := \dim M - \dim T$. In order to form the quotient, $h \in T_x$ acts on $T \times D_x$ by sending $(g, x)$ to $(gh^{-1}, h \cdot x)$, where $T_x$ acts by linear transformations on $D_x$. The action of $T$ on $T \times_{T_x} D_x$ is induced by the translational action of $T$ on the left factor of $T \times D_x$. The $T$-equivariant diffeomorphism $\Phi_x$ induces a homeomorphism $D_x/T_x \to \pi(U_x)$, which we compose with the projection $D_x \to D_x/T_x$ to get a map $\phi_x : D_x \to \pi(U_x)$. The proof of the following is routine.
Proposition 8.1. The collection \( \{ (\pi(U_x), D_x, \phi_x, T_x) \}_{x \in M} \) is an atlas for \( M/T \).

8.1.2. Flat connection. Consider the symplectic form on \( \mathbb{C}^m \omega^m := \frac{1}{2i} \sum_{j=1}^m d\overline{z}^j \wedge dz^j \). We identify each tangent space to \( T \) with \( t \) and each tangent space of a vector space with the vector space itself. The translational action of \( T \) on \( T \times \mathbb{C}^m \) descends to an action of \( T \) on \( T \times E \mathbb{C}^m \). Since the fundamental form (Proposition 7.1) \( \omega^T : t \times t \rightarrow \mathbb{R} \) is non-degenerate, it determines a unique symplectic form \( \omega^T \) on \( T \). The product symplectic form on \( T \times \mathbb{C}^m \), denoted by \( \omega^T \oplus \omega^m \), is defined pointwise at \((t, z)\) and a pair of vectors \(( (X, u), (X', u') \) by \( \omega^T(X, X') + \omega^m (u, u') \). The form \( \omega^T \oplus \omega^m \) descends to a symplectic form on \( T \times E_\mathbb{C}^m \). Theorem 7.3 gives us the following in the maximal symplectic case.

Lemma 8.2. There is an open \( \mathbb{T}^m \)-invariant neighborhood \( E \) of \( 0 \) in \( \mathbb{C}^m \), an open \( T \)-invariant neighborhood \( V_x \) of \( x \) in \( M \), and a \( T \)-equivariant symplectomorphism \( \Lambda_x : (T \times_T E, \omega^T \oplus \omega^m) \rightarrow (V_x, \omega) \) such that \( \Lambda_x([1, 0]_{T_x}) = x \).

Lemma 8.2 implies the following essential result.

Proposition 8.3. The collection \( \Omega := \{ \Omega_x \}_{x \in M} \) where \( \Omega_x := (T_x(T \cdot x))^\omega_T \), is a smooth distribution on \( M \) and \( \pi : M \rightarrow M/T \) is a smooth principal \( T \)-bundle of which \( \Omega \) is a \( T \)-invariant flat connection.

Let \( \psi : \widetilde{M/T} \rightarrow M/T \) be the universal cover of \( M/T \) based at \( p_0 = \pi(x_0) \). Let \( I_x \) be the maximal integral manifold of \( \Omega \). The inclusion \( i_x : I_x \rightarrow M \) is an injective immersion and \( \pi \circ i_x : I_x \rightarrow M/T \) is an orbifold covering. Since \( \widetilde{M/T} \) covers any covering of \( M/T \), it covers \( I_x \), which is a manifold. Because a covering of a smooth manifold is a smooth manifold, \( \widetilde{M/T} \) is a smooth manifold. Readers unfamiliar with orbifolds may consult [125] Section 9.

8.1.3. Monodromy. The universal cover \( \widetilde{M/T} \) is a smooth manifold. Let \( \pi_1^{\text{orb}}(M/T, p_0) \) be the orbifold fundamental group, based at the same point as the universal cover. The mapping \( \pi_1^{\text{orb}}(M/T, p_0) \times \widetilde{M/T} \rightarrow \widetilde{M/T} \) given by \( ([\lambda], [\gamma]) \mapsto [\gamma \lambda] \) is a smooth action of \( \pi_1^{\text{orb}}(M/T, p_0) \) on \( \widetilde{M/T} \), which is transitive on each fiber \( \widetilde{M/T}_p \) of \( \psi : \widetilde{M/T} \rightarrow M/T \). For any loop \( \gamma : [0, 1] \rightarrow M/T \) in \( M/T \) such that \( \gamma(0) = p_0 \), denote by \( \lambda_\gamma : [0, 1] \rightarrow M \) its unique horizontal lift with respect to the connection \( \Omega \) such that \( \lambda_\gamma(0) = x_0 \), where by horizontal we mean that \( d\lambda_\gamma(t)/dt \in \Omega_{\lambda_\gamma(t)} \) for every \( t \in [0, 1] \). \( \Omega \) is an orbifold flat connection, which means that \( \lambda_\gamma(t) = \lambda_\delta(t) \) if \( \delta \) is homotopy equivalent to \( \gamma \) in the space of all orbifold paths in the orbit space \( M/T \) which start at \( p_0 \) and end at the given end point \( p = \gamma(1) \). Therefore there exists unique group homomorphism \( \mu : \pi_1^{\text{orb}}(M/T, p_0) \rightarrow T \) such that \( \lambda_\gamma(1) = \mu([\gamma]) \cdot x_0 \).

Definition 8.4. The homomorphism \( \mu \) does not depend on the choice of the base point \( x_0 \in M \); we call it the monodromy homomorphism of \( \Omega \).

8.1.4. Case \( \dim T = \dim M - 2 \). The case \( \dim T = \dim M - 2 \) is the only case in which can give a complete classification, thanks to the following.

Lemma 8.5 (Thurston). Given a positive integer \( g \) and an \( n \)-tuple \( (a_k)_{k=1}^n \), \( a_i \leq a_{i+1} \) of positive integers, there exists a compact, connected, boundaryless, orientable smooth orbisurface \( O \) with underlying topological space a compact, connected surface of genus \( g \) and \( n \) cone points of respective orders \( o_1, \ldots, o_n \). Secondly, let \( O, O' \) be compact, connected, boundaryless, orientable smooth orbisurfaces. Then \( O \) is diffeomorphic to \( O' \) if and only if the genera of their underlying surfaces are the same, and their associated increasingly ordered \( n \)-tuples of orders of cone points are equal.
According to Lemma \[8.5\] \(\text{sig}(O) := (g; \bar{\sigma})\) topologically classifies \(O\). We call this tuple the Fuchsian signature of \(O\). If \((O, \omega)\) is a symplectic orbisurface, \(\int_O \omega\) is the total symplectic area of \((O, \sigma)\).

It follows from Lemma \[8.5\] and the orbifold Moser’s theorem [110] Theorem 3.3, that in the maximal symplectic case with \(\dim T = \dim M - 2\) the Fuchsian signature and the symplectic area of \(M/T\) determine \(M/T\) up to symplectomorphisms.

Let \((\gamma, \bar{\sigma}) \in \mathbb{Z}^{1+m}\) be the Fuchsian signature of \(M/T\); let \(\{\gamma_k\}_{k=1}^m\) be a basis of small loops around the cone points \(p_1, \ldots, p_n\) of \(M/T\), viewed as an orbifold; let \(\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}\) be a symplectic basis of a free subgroup \(F\) of \(H^1_{\text{orb}}(M/T, \mathbb{Z}) = \langle \{\alpha_i, \beta_i\}_{i=1}^g, \{\gamma_k\}_{k=1}^m | \sum_{k=1}^m \gamma_k = 0, o_k \gamma_k = 0, 1 \leq k \leq m \rangle\) whose direct sum with the torsion subgroup of \(H^1_{\text{orb}}(M/T, \mathbb{Z})\) is \(H^1_{\text{orb}}(M/T, \mathbb{Z})\).

Let \(\mu_h\) be the homomorphism induced on homology by \(\mu\).

**Definition 8.6.** The monodromy invariant of the triple \((M, \omega, T)\) is the \(G_{(g; \bar{\sigma})}\)-orbit given by \(G_{(g; \bar{\sigma})} : ((\mu_h(\alpha_i), \mu_h(\beta_i))_{i=1}^2, (\mu_h(\gamma_k))_{k=1}^m) \in T_{(g; \bar{\sigma})}/G_{(g; \bar{\sigma})}\).

Even though the monodromy invariant depends on choices, one can show that it is well-defined.

**8.1.5. Classification.** The \(T\)-action on \(\tilde{M}/T \times_\Gamma T\) is the \(T\)-action inherited from the action of \(T\) by translations on the right factor of \(\tilde{M}/T \times T\).

One can show that there exists a unique 2-form \(\nu\) on \(M/T\) such that \(\pi^* \nu |_{\Omega_+} = \omega |_{\Omega_+}\) for every \(x \in M\). Moreover, \(\nu\) is a symplectic form. Therefore \((M/T, \nu)\) is a compact, connected, symplectic orbifold. The symplectic form on \(\tilde{M}/T\) is the pullback by the covering map \(\tilde{M}/T \to M/T\) of \(\nu\) and the symplectic form on \(T\) is the unique \(T\)-invariant symplectic form determined by \(\omega^\gamma\). The symplectic form on \(M/T \times T\) is the product symplectic form.

Let \(\pi_1^\text{orb}(M/T)\) act on \(\tilde{M}/T \times T\) by the diagonal action \(x(y, t) = (x \ast y^{-1}, \mu(x) \cdot t)\), where \(\ast : \pi_1^\text{orb}(M/T) \times \tilde{M}/T \to \tilde{M}/T\) denotes the natural action of \(\pi_1^\text{orb}(M/T)\) on \(\tilde{M}/T\).

The symplectic form on \(M/T \times \pi_1^\text{orb}(M/T)\) \(T\) is induced on the quotient by the product form. The following is the model of maximal symplectic actions. The \(T\)-action on \(\tilde{M}/T \times \pi_1^\text{orb}(M/T)\) \(T\) is inherited from the \(T\)-action on the right factor of \(\tilde{M}/T \times T\).

**Theorem 8.7** [125]. Let \((M, \omega)\) be a compact connected symplectic manifold endowed with a maximal symplectic \(T\)-action. Then \(M\) is \(T\)-equivariantly symplectomorphic to \(\tilde{M}/T \times \pi_1^\text{orb}(M/T)\) \(T\).

**Proof.** For any homotopy class \([\gamma]\) \(\in \tilde{M}/T\) and \(t \in T\), define \(\Phi([\gamma], t) := t \cdot \lambda_\gamma(1) \in M\). The assignment \(([\gamma], t) \mapsto \Phi([\gamma], t)\) defines a smooth covering \(\Phi : M/T \times T \to M\) between smooth manifolds. Let \([\delta]\) \(\in \pi_1^\text{orb}(M/T, p_0)\) act on \(\tilde{M}/T \times T\) by sending the pair \(([\gamma], t)\) to \(([\gamma \delta^{-1}], \mu([\delta]) t)\). One can show that this action is free, and hence the associated bundle \(M/T \times \pi_1^\text{orb}(M/T, p_0)\) \(T\) is a smooth manifold. The mapping \(\Phi\) induces a diffeomorphism \(\phi : \tilde{M}/T \times \pi_1^\text{orb}(M/T, p_0)\) \(T\) onto \(M\). By definition, \(\phi\) intertwines the action of \(T\) by translations on the right factor of \(M/T \times \pi_1^\text{orb}(M/T, p_0)\) \(T\) with the action of \(T\) on \(M\). It follows from the definition of the symplectic form on \(M/T \times \pi_1^\text{orb}(M/T, p_0)\) \(T\) that \(\phi\) is a \(T\)-equivariant symplectomorphism.

**Theorem 8.8** [125]. Compact connected symplectic 2n-dimensional manifolds \((M, \omega)\) endowed with a maximal symplectic \(T\)-action with \(\dim T = \dim M - 2\) are classified up to \(T\)-equivariant
symplectomorphisms by: 1) fundamental form $\omega^1$; $t \times t \to \mathbb{R}$; 2) Fuchsian signature $(g; \partial)$ of $M/T$; 3) symplectic area $\lambda$ of $M/T$; 4) monodromy of the connection $\Omega$ of orthocomplements to the $T$-orbits.

Moreover, for any list 1)-4) there exists a compact, connected symplectic manifold with an effective symplectic $T$-action of a torus $T$ of dimension $2n - 2$ with $(2n - 2)$-dimensional symplectic $T$-orbits whose list of invariants is precisely this one.

Theorem 8.8 is extension of Theorem 6.6 to a class of symplectic actions which are never Hamiltonian. The first part of Theorem 8.8 is uniqueness. The last part is an existence result for which we shall say, however, that any antisymmetric bilinear form can appear, essentially all tuples as in 2) (with very few exceptions), and any $\lambda > 0$ can appear in 3). Similarly for 4). Readers may consult [125] for the precise list.

8.2. Coisotropic actions. This section gives invariants of coisotropic actions. Using these invariants we construct a model of $(M, \omega)$ and the $T$-action. Let $(M, \omega)$ be a compact connected symplectic manifold endowed with a coisotropic $T$-action of a torus $T$.

8.2.1. Hamiltonian subaction. We construct the maximal subtorus $T_h$ of $T$ which acts in a Hamiltonian fashion on $(M, \omega)$, and for which accordingly there is an associated momentum map $\mu$ and an associated polytope $\Delta$ as described in Section 3.

The group $T_h$ is the product of all the different stabilizer subgroups of $T$, which is a subtorus of $T$.

Let $x \in M$ and let $m = \dim T_x$. Let $K$ be a complementary subtorus of the subtorus $H := T_x$ in $T$. For $t \in T$, let $t_x$ and $t_K$ be the unique elements in $T_x$ and $K$, respectively, such that $t = t_x t_K$. Recall that $I = \ker(\omega^1)$, where $\omega^1$ is the fundamental form in Proposition 7.1. Let $X \mapsto X_1$ be a linear projection from $t$ onto $I$.

The $H$-invariant inner product $\beta_j$ on $E_j$, introduced in the proof of Lemma 7.20, is unique, if we also require that the symplectic inner product of any orthonormal basis with respect to $\omega^W$ is equal to $\pm 1$. In turn this leads to the existence of a unique complex structure on $E_j$ such that, for any unit vector $e_j$ in $(E_j, \beta_j)$, we have that $e_j, i e_j$ is an orthonormal basis in $(E_j, \beta_j)$ and $\omega^W(e_j, i e_j) = 1$. This leads to an identification of $E_j$ with $\mathbb{C}$, which is unique up to multiplication by an element of $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.

In turn this leads to an identification of $W$ with $\mathbb{C}^m$, with the symplectic form $\omega^W$ defined by $\omega^C^m = \frac{1}{m!} \sum_{j=1}^m d\overline{z}^j \wedge dz^j$. The element $c \in T^m$ acts on $\mathbb{C}^m$ component wise by $(c \cdot z)^j = c^j z^j$. There is a unique Lie group isomorphism $\iota : H \to T^m$ such that $h \in H$ acts on $W = \mathbb{C}^m$ by sending $z \in \mathbb{C}^m$ to $\iota(h) \cdot z$. The identification of $W$ with $\mathbb{C}^m$ is unique up to a permutation of the coordinates and the action of an element of $T^m$.

Let $T$ act on $K \times (1/t_x)^* \times \mathbb{C}^m$ by $t \cdot (k, \lambda, z) = (t_K k, \lambda, \iota(t_x) \cdot z)$, endowed with the symplectic form given at a point $(k, \lambda, z)$ and pair of vectors $((X, \delta \lambda, \delta z), (X', \delta' \lambda, \delta' z))$ by $\omega^1(X, X') + \delta \lambda(X'_1) - \delta' \lambda(X_1) + \omega^C^m(\delta z, \delta' z)$.

Lemma 8.9. There is an open $T^m$-invariant neighborhood $V$ of $(0, 0)$ in $(1/t_x)^* \times \mathbb{C}^m$, an open $T$-invariant neighborhood $U_x$ of $x$ in $M$, and a $T$-equivariant symplectomorphism

$$\Phi : K \times V \to U_x$$

such that $\Phi(1, 0) = x$. 

Proposition 8.10. The product of all stabilizers is a subtorus of \( T \), denoted by \( T_h \), and it acts on \( M \) in a Hamiltonian fashion. Furthermore, any complementary subtorus \( T_I \) to \( T_h \) in \( T \) act freely on \( M \).

Proof. As a consequence of Lemma 8.9 the stabilizer subgroup of the \( T \)-action on \( K \times (1/t_x)^* \times \mathbb{C}^m \) at \((k, \lambda, z)\) is \( \{t_x \in T_x \mid \iota(t_x)^j = 1 \forall j \text{ such that } z^j \neq 0\} \). In view of (17) there are \( 2^m \) different stabilizer subgroups \( T_y, y \in U \). Since \( M \) is compact there are only finitely many different stabilizer subgroups of \( T \). The product of all the different stabilizer subgroups is a subtorus of \( T \) because the product of finitely many subtori is a compact and connected subgroup of \( T \), and therefore it is a subtorus of \( T \).

See [38, Corollary 3.11] for the second claim.

The torus \( T_I \) acts freely on \( M \), because if \( x \in M \), then \( T_x \subset T_h \), hence \( T_x \cap T_I \subset T_h \cap T_I = \{1\} \), which proves (iii).

The tube theorem of Koszul [98] or [37, Theorem 2.4.1] implies that there is a finite number of stabilizer subgroups, and hence (i) in Proposition 8.10, but we wanted to derive this from Lemma 8.9 because it is an essential result in the study of coisotropic actions.

Proposition 8.11. \( T_h \) is the unique maximal stabilizer subgroup of \( T \).

Proof. Since any Hamiltonian torus action has fixed points, it follows from Proposition 8.10 that there exist \( x \in M \) such that \( T_h \subset T_x \), hence \( T_h = T_x \) because the definition of \( T_h \) implies that \( T_x \subset T_h \) for every \( x \in M \).

Let
\[
\mu : M \to \Delta \subset t_h^*
\]
be the momentum map of \( T_h \)-action. Its fixed points are the \( x \in M \) such that \( \mu(x) \) is a vertex of the Delzant polytope \( \Delta \).

8.2.2. Orbit space. By Leibniz identity for the Lie derivative one can show that for each \( X \in I \), \( \hat{\omega}(X) := -i_{X_M} \omega \) is a closed basic one-form on \( M \) and \( X \mapsto \hat{\omega}(X)_x \) is an \( I^* \)-valued linear form on \( T_x M \), which we denote by \( \hat{\omega}_x \). Hence \( x \mapsto \hat{\omega}_x \) is a basic closed \( I^* \)-valued one-form on \( M \), which we denote by \( \hat{\omega} \) and \( \hat{\omega}_x(v)(X) = \omega_x(v, X_M(x)) \), \( x \in M, v \in T_x M, X \in I \).

In Lemma 8.9 with \( x \in M_{reg} \), where \( t_x = \{0\} \) and \( m = 0 \), at each point \( \hat{\omega} \) is \( (\delta t, \delta \lambda) \mapsto \delta \lambda : \{0\} \times I^* \to I^* \).

Proposition 8.12. For every \( p \in (M/T)_{reg} \) the induced map \( \hat{\omega}_p : T_p(M/T)_{reg} \to I^* \) is a linear isomorphism.

Therefore \( \zeta \in I^* \) acts on \( p \in (M/T)_{reg} \) by traveling for time 1 from \( p \) in the direction that \( \zeta \) points to. We denote the arrival point by \( p + \zeta \). This action is not defined on \((M/T) \setminus (M/T)_{reg}\), it is only defined in the directions of vectors which as linear forms vanish on the stabilizer subgroup of the preimage under \( \pi : M \to M/T \).

Since \( T_h \) is the maximal stabilizer subgroup (Proposition 8.11), and for each \( x, T_x \subset T_h \), the additive subgroup
\[
N := (1/t_h)^*
\]
viewed as the set of linear forms on \( I \) which vanish on the Lie algebra \( t_h \) of \( T_h \), is the maximal subgroup of \( I^* \) which acts on \( M/T \). This turns \( M/T \) into a \( I^*-parallel space \), intuitively a space modeled on \( I^* \). In [38, Section 11] it is proved that they are isomorphic to the product of a closed
convex set and a torus. In the case of $M/T$, the convex polytope is Delzant (Definition 6.1), and equal to $\Delta$ in (18).

**Proposition 8.13.** If $P$ is the period lattice of a $N$-action on $M/T$, the quotient Lie group $N/P$ is a torus, and $M/T$ is isomorphic to $\Delta \times (N/P)$.

An analysis of the singularities of $M/T$ allows one to define the structure of $\mathfrak{l}^*$-parallel space. Any $\xi \in \mathfrak{l}^*$ may be viewed as a constant vector field on $(M/T)_{reg}$.

8.2.3. *Singular connection.* If $\mathfrak{t}_1$ denotes the Lie algebra of $T_1$ in Proposition 8.10 then $t = \mathfrak{t}_h \oplus \mathfrak{t}_1$. Each linear form on $\mathfrak{t}_h^*$ has a unique extension to a linear form on $\mathfrak{l}$ which is zero on $\mathfrak{t}_1$. This leads to an isomorphism of $\mathfrak{t}_h^*$ with the subspace $(\mathfrak{l}/\mathfrak{l} \cap \mathfrak{t}_1)^*$ of $\mathfrak{l}^*$. This isomorphism depends on the choice of $T_1$. Since $l = \mathfrak{t}_h \oplus (\mathfrak{l} \cap \mathfrak{t}_1)$, $\mathfrak{l}^* = ((\mathfrak{l}/\mathfrak{l} \cap \mathfrak{t}_1)^* \oplus (\mathfrak{l}/\mathfrak{t}_h)^*$. Let

$$\mu : M \to \Delta \subset (\mathfrak{l}/\mathfrak{l} \cap \mathfrak{t}_1)^* \simeq \mathfrak{t}_h^*$$

be $\pi : M \to M/T$ followed by the projection $M/T \simeq \Delta \times (N/P) \to \Delta$. The map $\mu : M \to \mathfrak{t}_h^*$ is a momentum mapping for the Hamiltonian $T_1$-action on $M$ in Proposition 8.10 and coincides with (18). The composite of $\pi : M \to M/T$ with the projection from $M/T \simeq \Delta \times (N/P) \to N/P$ is torus-valued generalization of the $S^1$-momentum map of McDuff [111].

**Definition 8.14.** $L_\zeta \in \mathcal{X}^\infty(M_{reg})$ is a lift of $\zeta$ if $d_x \pi(L_\zeta(x)) = \zeta$ for all $x \in M_{reg}$.

The word “lift” is used above the sense $\zeta \in \mathfrak{l}^*$ is as a constant vector field on $(M/T)_{reg}$ of which $L_\zeta$ is a lift. Linear assignments of lifts $\zeta \in \mathfrak{l}^* \mapsto L_\zeta$ depending linearly on $\zeta$ and connections for the principal torus bundle $M_{reg} \to M_{reg}/T$ are equivalent. The essential ingredient for construction of the model of $(M, \omega)$ with $T$-action is the existence of the following connection:

**Theorem 8.15.** There is an antisymmetric bilinear map $c : N \times N \to \mathfrak{l}$ satisfying $\zeta(c(\zeta', \zeta'')) + \zeta'(c(\zeta', \zeta)) + \zeta''(c(\zeta, \zeta')) = 0$ for every $\zeta, \zeta', \zeta'' \in N$, and a connection

$$\zeta \in \mathfrak{l}^* \mapsto L_\zeta \in \mathcal{X}^\infty(M_{reg}),$$

whose Lie brackets satisfy

$$[L_\zeta, L_\eta] = c(\zeta, \eta)_M, \quad \forall \zeta, \eta \in N,$$

and $[L_\zeta, L_\eta] = 0$ otherwise, as well as $\omega_x(L_\zeta(x), L_\eta(x)) = -\mu(x)(c_h(\zeta, \eta))$ $\forall \zeta, \eta \in N$, $\forall x \in M$, where $\mu$ is the momentum map of the $T_h$-action in (20). $c_h(\zeta, \eta)$ denote the $\mathfrak{t}_h$-component of $c(\zeta, \eta)$ in $l = \mathfrak{t}_h \oplus (\mathfrak{l} \cap \mathfrak{t}_1)$, and $\omega(L_\zeta, L_\eta) = 0$ otherwise.

The construction of (21) is the most involved part of [35] (Proposition 5.5 therein). It is a singular connection which blows up at $M \setminus M_{reg}$. The map $c$ has a geometric interpretation, which we discuss next.

8.2.4. *Chern class.* There is an isomorphism $(M/T)_{reg} \simeq \Delta^{int} \times (N/P)$, induced by the isomorphism $M/T \simeq \Delta \times (N/P)$. Any connection for $T$-bundle $M_{reg} \to M_{reg}/T$ has a curvature form, a smooth $t$-valued two-form on $M_{reg}/T$. Its cohomology class of this curvature form is an element of $H^2(M_{reg}/T, t)$, which is independent of the choice of the connection. The $N$-action on $M/T$ leaves $M_{reg}/T \simeq (M/T)_{reg}$ invariant, with orbits isomorphic to $N/P$.

The pull-back to the $N$-orbits defines an isomorphism $H^2(M_{reg}/T, t) \to H^2(N/P, t)$, which is identified with $(\Lambda^2 N^*) \otimes t$ (this observation goes back to Élie Cartan).

It follows from the construction of the connection (21) that $c : N \times N \to \mathfrak{l}$, viewed as an element in $c \in (\Lambda^2 N^*) \otimes 1 \subset (\Lambda^2 N^*) \otimes t$ equals the negative of the pull-back to an $N$-orbit of the cohomology class of the curvature form. Hence $c : N \times N \to \mathfrak{l}$ in (22) is independent of $T_1$. The Chern class $C$ of
the principal $T$-bundle $\pi : M_{\text{reg}} \to M_{\text{reg}}/T$ is an element of $H^2(M_{\text{reg}}/T, \mathbb{T}_2)$. It is known that the image of $C$ in $H^2(M_{\text{reg}}/T, t)$ under the coefficient homomorphism $H^2(M_{\text{reg}}/T, \mathbb{T}_2) \to H^2(M_{\text{reg}}/T, t)$ is equal to the negative of the cohomology class of the curvature form of any connection in the principal $T$-bundle, and hence we have the following.

**Proposition 8.16.** The map $e : N \times N \to \mathbb{I}$ represents the Chern class $C$.

8.2.5. **Toric foliation.** Next we describe a foliation of $M$ by symplectic-toric manifolds as in Section 6.1.1. Let $D_x := \text{span}\{ L_\eta(x), Y_M(x) \mid Y \in \mathfrak{h}_{\text{h}}, \eta \in C \}$, $x \in M_{\text{reg}}$, where $C \oplus N = \mathfrak{t}'$ and let $\mathcal{D} := \{ D_x \mid x \in M \}$.

**Proposition 8.17.** The distribution $\mathcal{D} := \{ D_x \}_{x \in M}$ is smooth, integrable and $T$-invariant and the integral manifolds of $\mathcal{D}$ are $(2 \dim \mathfrak{t}_h)$-dimensional symplectic manifolds and $\mathfrak{t}_h$-equivariantly symplectomorphic to each other.

We pick an integral manifold of Proposition 8.17 and call it $M_{\text{h}}$. Then $\omega$ restricts to a symplectic form $\omega_{\text{h}}$ on $M_{\text{h}}$ and $T_{\text{h}}$ acts Hamiltonianly on it.

8.2.6. **Group extensions.** $N$ in [19] is the maximal subgroup of $\mathfrak{t}'$ which acts on $M/T$. Denote the flow after time $t \in \mathbb{R}$ of $v \in \mathcal{X}(M)$ by $e^t v$. This defines a map $v \mapsto e^v$, $\mathcal{X}(M) \to \text{Diff}^\infty(M)$, analogous to (5).

**Definition 8.18.** The extension of $N$ by $T$ is the Lie group $G := T \times N$ with operation

$$
(t, \zeta)(t', \eta) = (tt'e^{-(\zeta, \eta)/2}, \zeta + \eta).
$$

**Proposition 8.19.** The Lie group $G$ acts smoothly on $M$ by $(t, \zeta) \mapsto t_M \circ e^{t_c}$, where we are using the identification $G \simeq (t/\mathbb{T}_2) \times N$.

The projection $\pi : M \to M/T$ intertwines the action of $G$ on $M$ with the action of $N$ on $M/T$ and there is an exact sequence $1 \to T \to G \to N \to 1$, where $G \to N$ corresponds to passing from the action of $G$ on $M$ to the action of $N$ on $M/T$, on which the action of $T$ is trivial.

**Proposition 8.20.** The Lie algebra of $G$ with (23) is the two-step nilpotent Lie algebra $\mathfrak{g} = t \times N$ with $[(X, \zeta), (X', \eta)] = -c(\zeta, \eta), 0)$. The product $t \times N$ endowed with the operation $(X, \zeta)(X', \eta) = (X + X' - c(\zeta, \eta)/2, \zeta + \eta)$ is a two-step nilpotent Lie group with Lie algebra $\mathfrak{g}$, and the identity as the exponential map.

8.2.7. **Holonomy.** For $\zeta \in P$ and $p \in M/T$ consider the loop $\gamma_{\zeta}(t) := p + t \zeta$. If $p = \pi(x)$, then $\delta(t) = e^{t \zeta}(x)$, $0 \leq t \leq 1$ is the horizontal lift of $\gamma_{\zeta}$ which starts at $x$ because $\delta(0) = x$ and $\delta'(t) = L_{\zeta}(\delta(t))$ is a horizontal tangent vector mapped by $d_{\delta(t)}\pi$ to $\zeta$. Hence $\pi(\delta(t)) = \gamma_{\zeta}(t)$ for all $0 \leq t \leq 1$.

**Definition 8.21.** The element of $T$ which maps $\delta(0) = x$ to $\delta(1)$ is called the holonomy $\tau_{\zeta}(x)$ of the loop $\gamma_{\zeta}$ at $x$ with respect to the connection (21).

Because $\delta(1) = e^{L_{\zeta}}(x)$, we have that $\tau_{\zeta}(x) \cdot x = e^{L_{\zeta}}(x)$. The element $\tau_{\zeta}(x)$ does depend on the point $x \in M$, on the period $\zeta \in P$, and on the choice of connection (21). Let $H = \{ (t, \zeta) \in G \mid \zeta \in P, t \tau_{\zeta} \in \mathfrak{t}_h \}$. The elements $\tau_{\zeta} \in T$, $\zeta \in P$, encode the holonomy of (21). So the holonomy is an element of the set $\text{Hom}_c(P, T)$ of maps $\tau : P \to T$, denoted by $\zeta \mapsto \tau_{\zeta}$, such that $\tau_{\zeta} \tau_{\eta} = \tau_{\zeta + \eta}e^{c(\tau_{\zeta}, \eta)/2}$. There is a Lie subgroup $B \leq \text{Hom}_c(P, T)$ which eliminates the dependance on the choice of connection and base point, so the true holonomy invariant of $(M, \omega)$ is an element of $\text{Hom}_c(P, T)/B$. The precise definition of $B$ is technical and appeared in [38].
8.2.8. Nilmanifolds. The quotient $G/H$ is with respect to the non standard group structure in expression (23). On $G/H$ we still have the free action of the torus $T/T_h$, which exhibits $G/H$ as a principal $T/T_h$-bundle over the torus $(G/H)/T \simeq N/P$. Palais and Stewart [121] showed that every principal torus bundle over a torus is diffeomorphic to a nilmanifold for a two-step nilpotent Lie group. When the nilpotent Lie group is not abelian, then the manifold $M$ does not admit a Kähler structure, cf. Benson and Gordon [17]. Next we give a description of $G/H$.

**Proposition 8.22.** The $G$-space $G/H$ is isomorphic to the quotient of the simply connected two-step nilpotent Lie group $(t/t_h) \times N$ by the discrete subgroup of elements $(Z, \zeta)$ such that $e^Z \tau_\zeta \in T_h$.

**Proof.** The identity component $H^0 = T_h \times \{0\}$ of $H$ is a closed normal Lie subgroup of both $G$ and $H$. The mapping $(G/H^0)/(H/H^0) \to G/H$ given by $(g H^0) (H/H^0) \mapsto g H$ is a $G$-equivariant diffeomorphism. The structure in $G/H^0 = (T/T_h) \times N$ is $(t, \zeta) (t', \eta) = (tt' e^{-c_{t/h}(\zeta, \eta)/2}, \zeta + \eta)$, $t, t' \in T/T_h$, $\zeta, \eta \in N$, where $c_{t/h} : N \times N \to \mathbb{I}$ and the projection $I \to I/t_h$. Hence $G/H^0$ is a two-step nilpotent Lie group with universal covering $(1/t_h) \times N$ and covering group $(T/T_h) \times \mathbb{Z} \simeq (T_h/\mathbb{Z}) \times \mathbb{Z}$. Also $P \to H/H^0$ given by $\iota : \zeta \mapsto (\tau_\zeta^{-1}, \zeta) H^0$ is an isomorphism.

8.2.9. Classification. Let $h \in H$ act on $G \times M_h$ by $(g, x) \mapsto (gh^{-1}, h \cdot x)$ and consider $G \times_H M_h$.

The $T$-action by translations on the left factor of $G$ passes to an action on $G \times_H M_h$. Each of the fibers of $G \times_H M_h$ is identified with the symplectic-toric manifold $(M_h, \omega_h, T_h)$. Any complementary subtorus $T_l$ permutes the fibers of $G \times_H M_h \to G/H$, each of which is identified with $(M_h, \omega_h, T_h)$.

Next let us explicitly construct a symplectic form on $G \times_H M_h$. This construction uses in an essential way Lemma [8.9] but for simplicity here we skip the details as the general formula may be given directly. Let $\delta a = ((\delta t, \delta \zeta), \delta x)$, and $\delta' a = ((\delta' t, \delta' \zeta), \delta' x)$ be tangent vectors to the product $G \times M_h$ at the point $a = ((t, \zeta), x)$, where we identify each tangent space of $T$ with $t$. Write $X = \delta t + c(\delta \zeta, \zeta)/2$ and $X' = \delta' t + c(\delta' \zeta, \zeta)/2$. Let $X_h$ be $t_h$-component of $X \in t$ in $t_h \oplus t_f$, and similarly for $X_f$ and define

$$\Omega_a(\delta a, \delta' a) = \omega_l(\delta t, \delta' t) + \delta \zeta (X'_l) - \delta' \zeta (X_l) - \mu(x) (c_{t/h}(\delta \zeta, \delta' \zeta))$$

$$+ (\omega_h)_x (\delta x, (X'_h)_{M_h}(x)) - (\omega_h)_x (\delta' x, (X_h)_{M_h}(x)) + (\omega_h)_x (\delta x, \delta' x).$$

(24)

If $\pi_M$ is the projection $G \times M_h \to G \times_H M_h$, the $T$-invariant symplectic form on $G \times_H M_h$ is the unique two-form $\beta$ on $G \times_H M_h$ such that $\Omega = \pi_M^* \beta$.

We are ready to state the model theorem.

**Theorem 8.23.** Let $(M, \omega)$ be a compact connected symplectic manifold endowed with a coisotropic $T$-action. Then $(M, \omega)$ is $T$-equivariantly symplectomorphic to the total space $G \times_H M_h$ of the symplectic fibration $(M_h, \omega_h, T_h) \to (G \times H M_h, \Omega, T) \to G/H$ with base $G/H$ being a torus bundle over a torus, and symplectic toric manifolds $(M_h, \omega_h, T_h)$ as fibers. The $T$-action on $G \times_H M_h$ is the symplectic action by translations on the $T$-factor of $G$, and the symplectic form $\Omega$ is given pointwise by formula (24).

**Sketch of proof.** The map from $F : G \times_H M_h$ to $M$ given by $((t, \xi), x) \mapsto t \cdot e^{L_\xi}(x)$ is a $T$-equivariant symplectomorphism.

In view of Lemma [8.9], Theorem 8.15 and Proposition 8.17, it is not difficult to verify that $F$ is a $T$-equivariant diffeomorphism and $F^* \omega = \Omega$ since the way we have arrived at the model $(G \times_M M_h, \Omega, T)$ of $(M, \omega, T)$ is constructive. However careful checking is still fairly technical and not necessarily illuminating on a first reading; we refer interested readers to [83] for a full proof.

Notice that:
(i) If the $T$-action is free, then the Hamiltonian subtorus $T_h$ is trivial, and hence $M$ is itself a torus bundle over a torus. Concretely, $M$ is of the form $G/H$. The Kodaira variety (Example 5.5) is one of these spaces. Since $M$ is a principal torus bundle over a torus, it is a nilmanifold for a two-step nilpotent Lie group as explained in Palais-Stewart [121]. In the case when this nilpotent Lie group is not abelian, $M$ does not admit a Kähler structure, see Benson-Gordon [17].

(ii) In the case of 4-dimensional manifolds $M$, item (i) corresponds to the third case in Kodaira’s description [94, Theorem 19] of the compact complex analytic surfaces which have a holomorphic $(2, 0)$-form that is nowhere vanishing, see [40]. As mentioned, these were rediscovered by Thurston [148] as the first examples of compact connected symplectic manifolds without Kähler structure.

(iii) If on the other hand the $T$-action is Hamiltonian, then $T_h = T$, and in this case $M$ is itself a symplectic toric manifold an hence a toric variety (see [31, 68, 39] for the relations between symplectic toric manifolds and toric varieties).

(iv) Henceforth, we may view the coisotropic orbit case as a twisted mixture of the Hamiltonian case, and of the free symplectic case.

**Theorem 8.24 (35).** Compact connected symplectic manifolds $(M, \omega)$ with a coisotropic $T$-action are determined up to $T$-equivariant symplectomorphisms by: 1) fundamental form $\omega^t : t \times t \to \mathbb{R}$; 2) Hamiltonian torus $T_h$ and its associated polytope $\Delta$; 3) period lattice $P$ of $N = (1/t_h)^*$; 4) Chern class $c : N \times N \to \text{Hom}_{\text{c}}(P, T)$; 5) holonomy $[\tau : P \to T]_B \in \text{Hom}_{\text{c}}(P, T)/B$.

Moreover, for any list 1)-5) there exists a compact connected symplectic manifold with a coisotropic $T$-action whose list of invariants is precisely this one.

**Theorem 8.24** is analogous to Theorem 6.6.

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$^2$In [40] the authors show that a compact connected symplectic 4-manifold with a symplectic 2-torus action admits an invariant complex structure and give an identification of those that do not admit a Kähler structure with Kodaira’s class of complex surfaces which admit a nowhere vanishing holomorphic $(2, 0)$-form, but are not a torus or a K3 surface.
The first part of Theorem 8.24 is a uniqueness theorem. The last part is an existence theorem for which we have not provided details for simplicity. Nonetheless we shall say that, for example, any antisymmetric bilinear form can appear as invariant 1), and any subtorus $S \subset T$ and Delzant polytope can appear as ingredient 2) etc. This is explained in [38].

Example 8.25. In the case of the Kodaira variety $M = \mathbb{R}^2 \times_{\mathbb{Z}/2} (\mathbb{R}/\mathbb{Z})^2$ in Example 5.5, $T = (\mathbb{R}/\mathbb{Z})^2$, $t \simeq \mathbb{R}^2$ and its invariants are: 1) fundamental form: $\omega^t = 0$; 2) Hamiltonian torus: $T_h = \{(0, 0)\}$; Delzant polytope: $\Delta = \{(0, 0)\}$; 3) period lattice is $P = \mathbb{Z}^2$; 4) Chern class $c: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, defined by $c(e_1, e_2) = e_1$; 5) The holonomy is the class of $\tau$ given by $\tau e_1 = \tau e_2 = [0, 0]$. In this case $G = (\mathbb{R}/\mathbb{Z})^2 \times \mathbb{R}^2$, $M_h = \{p\}$, and $H = \{(0, 0)\} \times \mathbb{Z}^2$. The model of $M$ is $G \times_H M_h \simeq G/H \simeq \mathbb{R}^2 \times_{\mathbb{Z}^2} (\mathbb{R}/\mathbb{Z})^2$.

8.3. Symplectic 2-torus actions on 4-manifolds. Consider on $(\mathbb{R}/\mathbb{Z})^2 \times S^2$ the product symplectic form. The action of the 2-torus is: one circle acts on the first circle of $(\mathbb{R}/\mathbb{Z})^2$ by translations, while the other circle acts on $S^2$ by rotations about the vertical axis.

If $T$ is a 2-dimensional torus, consider the product $T \times t^*$ with the standard cotangent bundle form and the standard $T$-action on left factor of $T \times t^*$. If the symplectic-toric manifold $M_h$ is trivial, then the model for a symplectic $T$-action with coisotropic orbits simplifies greatly, and it splits into cases (1), (2) and (3) below.

The following is a simplified version of the main result of [125]; readers may consult [125, Theorem 8.2.1] for the complete version of the statement.

Theorem 8.26 ([125]). Let $(M, \omega)$ be a compact, connected, symplectic 4-manifold equipped with an effective symplectic action of a 2-torus $T$. If the symplectic $T$-action is Hamiltonian, then (1) $(M, \omega)$ is a symplectic toric 4-manifold, and hence classified up to $T$-equivariant symplectomorphisms by the image $\Delta$ of the momentum map $\mu: M \to t^*$ of the $T$-action.

If the symplectic $T$-action is not Hamiltonian, then one and only one of the following cases occurs:

(2) $(M, \omega)$ is equivariantly symplectomorphic to $(\mathbb{R}/\mathbb{Z})^2 \times S^2$.

(3) $(M, \omega)$ is equivariantly symplectomorphic to $(T \times t^*)/Q$ with the induced form and $T$-action, where $Q \leq T \times t^*$ is a discrete cocompact subgroup for the group structure $[23]$ on $T \times t^*$.

(4) $(M, \omega)$ is equivalently symplectomorphic to a symplectic orbifold bundle $\tilde{\Sigma} \times_{\pi^\text{orb}_1(\Sigma, p_0)} T$ over a good orbisurface $\Sigma$, where the symplectic form and $T$-action are induced by the product ones, and $\pi^\text{orb}_1(\Sigma, p_0)$ acts on $\tilde{\Sigma} \times T$ diagonally, and on $T$ is by means of any homomorphism $\mu: \pi^\text{orb}_1(\Sigma) \to T$.

The proof of Theorem 8.26 uses as stepping stones the maximal symplectic and coisotropic cases.

Idea of proof. The fundamental observation to use the results in order to prove Theorem 8.26 is that under the assumptions of Theorem 8.26 there are (by linear algebra of $\omega$) precisely two possibilities: (a) the $T$-orbits are symplectic 2-tori, so the fundamental form $\omega^t$ is non-degenerate and hence $t$ is trivial. This corresponds to case 4); (b) the 2-dimensional $T$-orbits are Lagrangian 2-tori, and hence $t = t$. This corresponds to cases 1), 2), and 3). Case 3) is derived from Theorems 8.23 and 8.24. Case 4) is derived from Theorems 8.7 and 8.8.

A significant part of the proof of Theorem 8.26 consists of unfolding item b) above into items 1), 2), 3) in the statement of Theorem 8.26.

Notice that item 1) is classified in terms of the Delzant polytope in view of Theorem 6.6 which is the only invariant in the Hamiltonian case.
Case 3) corresponds to the third case in the description of Kodaira [44, Th. 19] of the compact complex analytic surfaces which carry a nowhere vanishing holomorphic $(2, 0)$-form. These were rediscovered as the first examples of compact symplectic manifolds without Kähler structure by Thurston [148].

The article [41] shows that the first Betti number of $M/T$ is equal to the first Betti number of $M$ minus the dimension of $T$.

**Example 8.27.** The invariants of $M = S^2 \times_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{R}/\mathbb{Z})$ are: the non-degenerate antisymmetric bilinear form $\omega_{\mathbb{R}^2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; 2) The Fuchsian signature $(g; \sigma) = (0; 2, 2)$ of the orbit space $M/T^2$; 3) The symplectic area of $S^2/(\mathbb{Z}/2\mathbb{Z})$: 1 (half of the area of $S^2$); 4) The monodromy invariant: $g_{(0, 2, 2)} : (\mu_h(\gamma_1), \mu_h(\gamma_2)) = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \}$ · $[(1/2, 0), (1/2, 0)]$. Here the $\gamma_1, \gamma_2$ are small loops around the poles of $S^2$. Then $M/T = S^2/(\mathbb{Z}/2\mathbb{Z}), \pi^\text{orb}_1(M/T, p_0) = (\gamma_1 | \gamma_2^2 = 1) \simeq \mathbb{Z}/2\mathbb{Z}$, and $\mu : (\gamma_1 | \gamma_2^2 = 1) \to T = (\mathbb{R}/\mathbb{Z})^2$ is $\mu(\gamma_1) = [1/2, 0]$. We have a $T$-equivariant symplectomorphism $\tilde{M}/\tilde{T} \times _{\pi^\text{orb}_1(M/T, p_0)} T = S^2/(\mathbb{Z}/2\mathbb{Z}) \times _{\pi^\text{orb}_1(S^2/(\mathbb{Z}/2\mathbb{Z}), p_0)} (\mathbb{R}/\mathbb{Z})^2 \simeq M$.

9. **Final remarks**

In this paper we have covered symplectic Hamiltonian actions as contained in the works of Audin, Ahara, Hattori, Delzant, Duistermaat, Heckman Kostant, Atiyah, Guillemin, Karshon, Sternberg, Tolman, Weitsman [3, 12, 13, 33, 35, 71, 10, 73, 31] among others, and more general symplectic actions as in the works of Benoist, Duistermaat, Frankel, McDuff, Ortega, Ratiu, and the author [15, 16, 38, 117, 125] among others.

We have described classifications (on compact manifolds) in four cases: (i) **maximal Hamiltonian case**: Hamiltonian $T$-action, $\text{dim } M = 2 \text{dim } T$; (ii) **$S^1$-Hamiltonian case**: Hamiltonian $T$-action, $\text{dim } M = 4, \text{dim } T = 1$; (iii) **four-dimensional case**: $\text{dim } M = 4$ and $\text{dim } T = 2$; (iv) **maximal symplectic case**: there is a $T$-orbit symplectic orbit; (v) **coisotropic case**: there is a coisotropic orbit. We have outlined the connections of these works with complex algebraic geometry, in particular Kodaira’s classification of complex analytic surfaces [94], the theory of toric varieties and Kähler manifolds [39], and toric log symplectic-geometry [67]; geometric topology, in particular the work of Palais-Stewart [121] and Benson-Gordon [17] on torus bundles over tori and nilpotent Lie groups; also with orbifold theory (for instance Thurston’s classification of compact 2-dimensional orbifolds); and integrable systems, in particular the work of Guillemin-Sternberg on multiplicity-free spaces [75] and semitoric systems [82, 133, 134].

Some of the techniques to study Hamiltonian torus actions (see for instance the books by Guillemin [68], Guillemin-Sjamaar [72], and Ortega-Ratiu [118]) are useful in the study of non-Hamiltonian symplectic torus actions (since many non-Hamiltonian actions exhibit proper subgroups which act Hamiltonianally).

In the study of Hamiltonian actions, one tool that is often used is Morse theory for the (components of the) momentum map of the action. Since there is no momentum map in the classical sense for a general symplectic action, Morse theory does not appear as a natural tool in the non-Hamiltonian case.

There is an analogue, however, “circle valued-Morse theory” (since any symplectic circle action admits a circle-valued momentum map, see McDuff [111] and [127], which is also Morse in a sense) but it is less immediately useful in our setting; for instance a more complicated form of the Morse inequalities holds (see Pajitnov [119], Chapter 11, Proposition 2.4 and Farber [48, Theorem 2.4]).
and the theory appears more difficult to apply, at least in the context of non-Hamiltonian symplectic actions; see [127, Remark 6] for further discussion in this direction. This could be one reason that non-Hamiltonian symplectic actions have been studied less in the literature than their Hamiltonian counterparts.

The moduli space of coisotropic actions includes as a particular case Hamiltonian actions of maximal dimension (see [126] for the description of the moduli space of Hamiltonian actions of maximal dimension on 4-manifolds), classified in Delzant’s article [31].

We conclude with a general problem for further research:

Problem 9.1. Let $T$ be an $m$-dimensional torus (or even more generally, a compact Lie group). Give a classification of effective symplectic $T$-actions on compact connected symplectic $2n$-dimensional manifolds $(M,\omega)$. For instance, where $2n = 4$ or $2n = 6$.

In this paper we have given an answer to this question under the additional assumptions in the cases (i)-(v) above. Theorem 8.26 give the complete answer when $m = 2$ and $2n = 4$ (using Theorem 6.6 for case 1 therein), under no additional assumptions.

Current techniques are dependent on the additional assumptions (i.e. being Hamiltonian, having some orbit of a certain type etc.) and solving Problem 9.1 in further cases poses a challenge.

**Dedication**

This paper is dedicated to J.J. Duistermaat (1942–2010).

The memorial article [71] edited by V. Guillemin, Á. Pelayo, S. Vũ Ngòc, and A. Weinstein outlines some of Duistermaat’s most influential contributions (see also [135, Section 2.4]). Here is a brief extraction from that article: “We are honored to pay tribute to Johannes (Hans) J. Duistermaat (1942-2010), a world leading figure in geometric analysis and one of the foremost Dutch mathematicians of the XX century, by presenting a collection of contributions by some of Hans’ colleagues, collaborators and students. Duistermaat’s first striking contribution was his article “Fourier integral operators II’ with Hörmander (published in Acta Mathematica), a work which he did after his doctoral dissertation. Several influential results in analysis and geometry have the name Duistermaat attached to them, for instance the Duistermaat-Guillemin trace formula (1975), Duistermaat’s global action-angle Theorem (1980), the Duistermaat-Heckman Theorem (1982) and the Duistermaat-Grunbaum Bi-spectral Theorem (1986). Duistermaat’s papers offer an unusual display of originality and technical mastery.”

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