ENTROPY AND FINITESS OF GROUPS WITH ACYLINDRICAL SPLITTINGS

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Abstract. We prove that there exists a positive, explicit function $F(k, E)$ such that, for any group $G$ admitting a $k$-acylindrical splitting and any generating set $S$ of $G$ with $\text{Ent}(G, S) < E$, we have $|S| \leq F(k, E)$. We deduce corresponding finiteness results for classes of groups possessing acylindrical splittings and acting geometrically with bounded entropy: for instance, $D$-quasiconvex $k$-malnormal amalgamated products acting on $\delta$-hyperbolic spaces or on $\text{CAT}(0)$-spaces with entropy bounded by $E$. A number of finiteness results for interesting families of Riemannian or metric spaces with bounded entropy and diameter also follow: Riemannian 2-orbifolds, non-geometric 3-manifolds, higher dimensional graph manifolds and cusp-decomposable manifolds, ramified coverings and, more generally, $\text{CAT}(0)$-groups with negatively curved splittings.

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Introduction

In this paper we are interested in finitely generated groups $G$ admitting $k$-acylindrical splittings, that is isomorphic to the fundamental group of a graph of groups such that the action of $G$ on the corresponding Bass-Serre tree is (non elementary and) $k$-acylindrical. The notion of acylindricity is due to Sela in [Sel],
and arises naturally in the context of Bass-Serre theory. It is a geometric translation of the notion of malnormal amalgamated product, introduced by Karras and Solitar \cite{Ka-Sol} at the beginning of the seventies (see Section \S 1 for details). We recall that an action without inversions of a finitely generated group on a simplicial tree $T$ is said to be $k$-acylindrical if the fixed point set of any element $g \in G$ has diameter at most $k$.

The more elementary examples of groups possessing a $k$-acylindrical splitting are the free products and the fundamental groups of compact surfaces of negative Euler characteristic, but the class is considerably larger and encompasses several interesting classes of amalgamated groups which naturally arise in Riemannian and metric geometry, as we shall see. During the years, the existence of acylindrical actions on simplicial trees has been mainly used to prove some accessibility results \cite{Sel, Ka-We2}. More recently, there has been an increasing interest on groups acting acylindrically on Gromov hyperbolic spaces (see \cite{Osi, Mi-Os, Sis} and references therein for this more general notion of acylindricity). We shall instead focus on growth and finiteness properties of such groups and of some classes of spaces arising as quotients of geometric actions of these groups.

Recall that the entropy, or exponential growth rate, of a group $G$ with respect to a finite generating set $S$ is defined as

$$\text{Ent}(G, S) := \lim_{n \to +\infty} \frac{1}{n} \log |S^n|$$

where $|S^n|$ is the cardinality of the ball of radius $n$ centered at the identity element, with respect to the word metric relative to $S$. We will also deal with groups $G$ acting discretely by isometries on general (non-compact) proper metric spaces $Y$; in this case, the entropy of the action is defined as

$$\text{Ent}(G \bowtie Y) := \limsup_{R \to +\infty} \frac{1}{R} \log |B_Y(y_0, R) \cap G y_0|$$

where $B_Y(y_0, R)$ denotes the ball of radius $R$ centred at $y_0$ (the limit is clearly independent from the choice of the base point $y_0 \in Y$). When $X$ is a closed Riemannian manifold it is customary to call \textit{(volume-)entropy} of $X$ the entropy of $G = \pi_1(X)$ acting on its Riemannian universal covering space $Y = \tilde{X}$; this number coincides with the exponential growth rate of the volume of balls in $X$, and it is well known that it equals, in non-positive curvature, the \textit{topological entropy} of the geodesic flow on the unitary tangent bundle of $X$, cp. \cite{Man}. We extend this terminology to any compact metric space $X$ obtained as the quotient $X = G \backslash Y$ of a simply connected, geodesic space $Y$ by a discrete group of isometries (possibly, with fixed points). We will come back shortly to the analytic information encrypted in this asymptotic invariant for Riemannian manifolds.

The first main result of this paper, from a group-theoretic point of view, is the following:

\textbf{Theorem 1 (Entropy-Cardinality inequality).} \textit{Let $G$ be a group acting by automorphisms without inversions, non-elementarily and $k$-acylindrically on a tree. For any finite generating set $S$ of $G$ of cardinality $n$, we have}

$$\text{Ent}(G, S) \geq \frac{\log \left( \sqrt[3]{n} - 1 \right)}{80(k+3)}$$

\textit{(1)}
Remark. The above inequality is far from being optimal, and makes sense only for \( n \gg 0 \). The meaning is qualitative: the entropy \( \text{Ent}(G, S) \) diverges as \( |S| \) becomes larger and larger.

A first remarkable example of Entropy-Cardinality inequality was given by Arzhantseva-Lysenok [Ar-Ly]: for any given hyperbolic group \( G \) there exists a constant \( \alpha(G) \) such that for any non-elementary, finitely generated subgroup \( H \) and any finite generating set \( S \) of \( H \) the inequality \( \text{Ent}(H, S) \geq \log(\alpha(G) \cdot |S|) \) holds. One interest of similar inequalities is that they generally represent a step forward to prove (or disprove) the realizability of the algebraic entropy for a group \( G \). On the other hand, the Entropy-Cardinality inequality proved in this paper has a different theoretical meaning, since the cardinality of \( S \) is bounded in terms of a universal function, only depending on the entropy and the acylindricity constant \( k \), and not on the group \( G \) itself.

The idea of proof of (1) is relatively elementary and based on the construction of free subgroups of \( G \) of large rank. Any collection of hyperbolic elements of \( G \) admitting disjoint domains of attraction generates a free subgroup (namely, a Schottky subgroup) by a classical ping-pong argument. So, the strategy for Theorem 1 is to show that from any sufficiently large generating set \( S \) one can produce large collections (compared to \( |S| \)) of hyperbolic elements with suitable configurations of axes and uniformly bounded \( S \)-lengths. However, quantifying this idea turns out to be a rather complicate combinatorial problem, since we need to control the mutual positions of the axes as well as the translational lengths of all the hyperbolic elements under consideration. The first part of the paper (§1.1-§1.4), is entirely devoted to developing the combinatorial tools needed to prove this inequality.

In the second part of the paper we focus on algebraic and geometric applications of the Entropy-Cardinality inequality. As an immediate consequence, we get general finiteness results for abstract groups with \( k \)-acylindrical splittings and uniformly bounded entropy, provided we know that they possess a complete set of relators of uniformly bounded length (cp. Theorem 2.1 in Section §2). The following are particularly interesting cases:

**Corollary 2.** The number of isomorphism classes of marked, \( \delta \)-hyperbolic groups \((G, S)\) with a non-elementary \( k \)-acylindrical splitting and satisfying \( \text{Ent}(G, S) \leq E \) is finite, bounded by an explicit function \( M(k, \delta, E) \).

**Corollary 3.** The number of groups \( G \) admitting a non-elementary \( k \)-acylindrical splitting, with a \( D \)-quasiconvex action on:
(i) either some (proper, geodesic) \( \delta \)-hyperbolic space \((X, d)\),
(ii) or on some CAT(0)-space \((X, d)\),
and satisfying \( \text{Ent}(G \circlearrowleft X) \leq E \) is finite. Their number is bounded by a function of \( k, \delta, D, E \) in case (i), and of \( k, D, E \) in case (ii).

(We stress the fact that, in the above corollary, the hyperbolic or CAT(0)-spaces \( X \) the group \( G \) acts on are not supposed to be fixed).

\footnote{It is worth noticing here that given a finitely generated group \( G \) acting \( k \)-acylindrically on a simplicial tree \( T \), any finitely generated subgroup \( H < G \) also acts \( k \)-acylindrically on the minimal subtree \( T_H \), globally preserved by \( H \); therefore Theorem 1 also holds for all subgroups of \( G \), provided that their action on \( T \) is still non-elementary.}

\footnote{The algebraic entropy \( \text{Ent}_{alg}(G) \) of a finitely generated group \( G \) is defined as the infimum of the entropies \( \text{Ent}(G, S) \), when \( S \) varies among all possible, finite generating sets for \( G \).}
A typical case where a group $G$ admits an action on a $CAT(0)$-space and a $k$-acylindrical splitting occurs when $G$ is the fundamental group of a space $X = X_1 \sqcup_Z X_2$ which is the gluing of two locally $CAT(0)$-spaces $X_i$ along two isometric, locally convex subspaces $Z_i \cong Z$ (or $X = X_0 \sqcup_\phi$ is obtained by identifying two such subspaces $Z_i \subset X_0$ to each other by an isometry $\phi$), and the resulting space $X$ is locally, negatively curved around $Z$. Namely, we will say that $X$ has a negatively curved splitting if the subspace $Z$ obtained by identifying $Z_1$ to $Z_2$ has a neighbourhood $U(Z)$ in $X$ such that $U(Z) \setminus Z$ is a locally $CAT(-\kappa)$-space, for some $\kappa > 0$. The fact that $Z$ possesses such a neighborhood ensures that $\pi_1(Z)$ is a malnormal subgroup in each $\pi_1(X_i)$, and therefore $\pi_1(X)$ has a 1-acylindrical splitting: we refer to Section 2.2 and to the Appendix C for details. We then have:

**Theorem 4.** The number of homotopy types of compact, locally $CAT(0)$-spaces $X$ admitting a non-trivial negatively curved splitting, satisfying $\text{Ent}(X) < E$ and $\text{diam}(X) < D$ is finite.

**Corollary 5.** There exist only finitely many non-diffeomorphic closed, non-positively curved manifolds $X$ of dimension different from 4 admitting a non-trivial negatively curved splitting and satisfying $\text{Ent}(X) < E$ and $\text{diam}(X) < D$. In dimension 4, the same is true up to homotopy equivalence.

It is worth noticing that Corollary 5 holds more generally for manifolds $X$ with metrics of curvature of any possible sign, provided that $X$ also admits a non-trivial, negatively curved splitting (this follows from the combination of Theorem 4 with Lemma 2.4 as in the proof of Theorem 6 for non-triangular 2-orbifolds).

The question whether a given family of Riemannian manifolds contains only a finite number of topological types has a long history: the ancestor of all finiteness results is probably Weinstein’s theorem [Wei] on finiteness of the homotopy types of pinched, positively curved, even dimensional manifolds. A few years later, Cheeger’s celebrated Finiteness Theorem appeared, for closed Riemannian manifolds with bounded sectional curvature and, respectively, lower and upper bounds on volume and diameter [Che, Pe]. Several generalizations [Gr-Pe, GPW1] with relaxed assumptions on the curvature then followed, until, in the nineties, the attention of geometers turned to Riemannian manifolds satisfying a lower bound on the Ricci curvature, driven by Gromov’s Precompactness Theorem. Substantial progresses in understanding the diffeomorphism type of Gromov-Hausdorff limits and the local structure of manifolds under lower Ricci curvature bounds were then made –by no means trying to be exhaustive– by Anderson-Cheeger [An-Ch], Cheeger-Colding [Ch-Co] and, more recently by Kapovich-Wilking [Ka-Wi] (see also Breuillard-Greene-Tao’s work [BGT], for a more group-theoretical approach to the generalized Margulis’ Lemma under packing conditions –a macroscopic translation of a lower Ricci curvature bound).

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3 Notice that some form of strictly negative curvature must be assumed to deduce that $\pi_1(Z)$ is malnormal in $\pi_1(X)$, and it is not sufficient to require that $Z$ is a locally convex subset of a locally $CAT(0)$-space $X$ of rank 1. A counterexample is provided, for instance, by a 3-dimensional irreducible manifold $X$ with non trivial JSJ splitting and one component of hyperbolic type: by [Leeb], $X$ can be given a non-positively curved metric of rank 1 in the sense of [Bal], and possesses a totally geodesic, embedded torus $Z$ whose fundamental group is not malnormal in $\pi_1(X)$. 
Corollary 5 represent an attempt to get rid of lower curvature bounds, at least in non-positive curvature, replacing it only by a bound of an asymptotic invariant. Recall that, for a closed Riemannian manifold $X$, a lower bound of the Ricci curvature $\text{Ricci}_X \geq -(n-1)K^2$ implies a corresponding upper bound of the entropy $\text{Ent}(X) \leq (n-1)K$, by the classical volume-comparison theorems of Riemannian geometry. However the entropy, being an asymptotic invariant, only depends on the large-scale geometry of the universal covering $\tilde{X}$, and can be seen as an averaged version of the curvature (this can be given a precise formulation in negative curvature by integrating the Ricci curvature on the unitary tangent bundle of $X$ with respect to a suitable measure, cp. [Kn]). Therefore, the condition $\text{Ent}(X) < E$ is much weaker than a lower bound on the Ricci curvature. To get a glimpse of the difference, remark that the class of closed, Riemannian manifolds (of dimension $n \geq 3$) with uniformly bounded entropy and diameter is not precompact with respect to the Gromov-Hausdorff distance (see [Rev2], Remark 2), and neither is the family of Riemannian structures with uniformly bounded entropy and diameter on any given $n$-dimensional manifold $X$ (see [Rev1], Example 2.29).

The first results about families of Riemannian metric and metric-measured spaces with uniformly bounded entropy, such as lemmas à-la-Margulis, finiteness and compactness results etc., were given by Besson-Courtois-Gallot’s in [BCG3] (yet unpublished), and are the object of [BCGS]. Other local topological rigidity results under entropy bounds have recently appeared in the authors’ [Ce-Sa].

Under this perspective, Corollary 5 might be compared with the well-know finiteness result for negatively curved $n$-manifolds $X$ with uniformly bounded diameter and sectional curvature $K(X) \geq -k^2$ (which follows from a version of the Margulis’ Lemma in non-positive curvature, as stated for instance in [Bu-Ka], and from the aforementioned Cheeger’s finiteness theorem). It is a challenging open question to know whether the conclusion of Corollary 5 extends to all closed, negatively curved manifolds with uniformly bounded entropy and diameter.

The Entropy-Cardinality inequality becomes a powerful tool, when applied to families of Riemannian manifolds enjoying strong topological-rigidity properties. To illustrate this fact, we present now some basic examples of application of Theorem 1 to particular classes of spaces whose groups naturally possess acylindrical splittings and presentations with an uniform bound on the acylindricity constant and on the length of relators.

A. Two-dimensional orbifolds of negative orbifold characteristic.

Orbifolds were introduced by Satake [Sat] in the late fifties under the name of $V$-manifolds and later popularized by Thurston ([Thu]) who used them to show the existence of locally homogeneous metrics on Seifert fibered manifolds. Generally speaking, $n$-dimensional orbifolds are mild generalization of manifolds, whose points have neighborhoods modeled on the quotient of $\mathbb{R}^n$ (or on the upper half space $\mathbb{R}^n_+$) by the action of a finite group of transformations; we refer to section 2.3 for precise definitions and isomorphisms of 2-orbifolds. In the ’80s Fukaya introduced the equivariant Gromov-Hausdorff distance ([Fuk]), and used it to study Riemannian orbifolds. Since then, several authors gave attention to spectral and finiteness results on Riemannian orbifolds (see for instance [Bor], [Far], [Sta], [Pr-St], [Pro]), possibly because of their application to string theory ([ALR]). We show:
Theorem 6. Let $\mathcal{O}^2(E,D)$ be the class of Riemannian, compact, 2-orbifolds (with or without boundary) with conical singularities and $\chi_{\text{orb}}(O) \leq 0$, satisfying $\text{Ent}(O) \leq E$ and $\text{diam}(O) \leq D$. This class contains only a finite number of isomorphism types.

We stress the fact that the orbifold metrics in the class $\mathcal{O}^2(E,D)$ under consideration are not supposed to be negatively or nonpositively curved. Notice that the analogous result for compact surfaces easily follows from basic estimates of the algebraic entropy of a surface group, together with the aforementioned Gromov’s inequality

$$\text{Ent}(X) \text{diam}(X) \geq \frac{1}{2} \text{Ent}_{\text{alg}}(\pi_1(X))$$

Actually, it is well known that the algebraic entropy of a compact surface $X$ of genus $g$ with $h$ boundary component is bounded from below by $\log(4g + 3h - 3)$ (cp. [delH]), therefore $\text{Ent}(X)$ and $\text{diam}(X)$ bound $g$ and $h$. The orbifold case is significantly more tricky: we use the fact that non-triangular orbifolds of negative Euler characteristic always admit a 2-acylindrical splitting (a proof of this is given in the Appendix A, Proposition A.1), so we can apply the Entropy-Cardinality inequality to particular, well-behaved presentations of the orbifold groups. On the other hand, triangular orbifolds do not admit such splittings, and we are forced to a direct computation, using arguments from classical small cancellation theory.

Remark. The above finiteness theorem marks a substantial difference with the analogue question in geometric group theory: in fact, the number of 2-orbifold groups $G$ admitting a generating set $S$ such that $\text{Ent}(G,S) \leq E$ is not finite (at least, without any additional, uniform hyperbolicity assumption on the groups $G$). Actually, on any topological surface $S$ of genus $g$ with $k$ conical points of orders $p_1,\ldots,p_k$ there always exists a generating set $S$ of cardinality at most $2g + k$, such that $\text{Ent}(G,S)$ is smaller than the entropy of the free group on $S$, independently from the choice of the orders $p_1,\ldots,p_k$ (we thank R. Coulon for pointing out this fact to us). The reason for this difference is that, on Riemannian orbifolds, any torsion element $g \in G$ has a fixed point on $\tilde{X}$, which gives rise to arbitrarily small loops increasing substantially the entropy; this does not happen for the action of $G$ on its (non-simply connected) Cayley graph $G(G,S)$. Notice also that the existence of torsion elements with unbounded orders prevents $(G,S)$ to be $\delta$-hyperbolic, for any fixed $\delta$.

B. Non-geometric 3-manifolds.

A compact 3-manifold, possibly with boundary, is called non-geometric if its interior cannot be endowed with a complete Riemannian metric locally isometric to one of the eight 3-dimensional complete, maximal, homogeneous model geometries: $\mathbb{R}^3, S^3, H^3, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, H^2 \times \mathbb{R}, Nil$ and $Sol$. We restrict our attention to orientable manifolds, for the sake of simplicity without spherical boundary components (since, clearly, punctures cannot be detected by the fundamental group in dimension 3).
By classical results of 3-dimensional topology and by the solution of the Geometrization Conjecture, any such manifold \(X\) falls into one of the following mutually disjoint classes (as explained, for instance, in [AFW]):

(i) either \(X\) is not prime, and is different from \(P^3R \# P^3R\) (the only compact, non-prime manifold without spherical boundary components admitting a geometric structure, modelled on \(S^2 \times R\));

(ii) or \(X\) is irreducible, has a non-trivial JSJ decomposition and is not finitely covered by a torus bundle (in which case, it would admit a \(Sol\)-geometry).

In the first case, the fundamental group of \(X\) admits a non-elementary, 1-acylindrical splitting corresponding to the prime decomposition, while in case (ii) the JSJ-decomposition induces a (at most) 4-acylindrical splitting of \(\pi_1(X)\), by [Wi-Za] (cp. also Section \(\S\)4 in [Ce-Sa1], and [Cer2] for details about the degree of acylindricity of the splitting over the abelian subgroups corresponding to the JSJ tori, according to the different types of adjacent JSJ-components).

In [Ce-Sa1] the authors examined the local rigidity properties of Riemannian, non-geometric 3-manifolds with torsionless fundamental group, under uniformly bounded entropy and diameter. Here we consider, more generally, the class \(\mathcal{M}_{\eta_0}^\partial(E,D)\) of compact, orientable non-geometric 3-manifolds (with possibly empty, non-spherical boundary), possibly with torsion, endowed with Riemannian metrics with entropy and diameter bounded from above by two positive constants \(E\) and \(D\). Acylindricity of the splitting of their fundamental groups is the key to the following:

**Theorem 7.** The number of isomorphism classes of fundamental groups of manifolds in \(\mathcal{M}_{\eta_0}^\partial(E,D)\) is less than \((e^{1120ED} + 1)^{32}\).

Moreover, the homotopy type (and, in turns, the diffeomorphism type) of compact 3-manifolds without spherical boundary components is determined by their fundamental group up to a finite number of choices, by Johannson and Swarup works [Jo1],[Swa] (see the discussion in Section 2.4 for details, and Theorem B.1 in Appendix B). We therefore obtain:

**Corollary 8.** The number of diffeomorphism types in \(\mathcal{M}_{\eta_0}^\partial(E,D)\) is finite.

Notice that, while Theorem 7 gives an explicit (albeit ridiculously huge) estimate of the number of groups in \(\mathcal{M}_{\eta_0}^\partial(E,D)\), Corollary 8 does not provide any explicit estimate of the number of diffeomorphism types.

**Remark.** The bound we found in Theorem 7 is explicit, but far from being optimal. Once finiteness for this class is known, one might try to use more efficient, computer-assisted algorithms to find reasonable estimates of their number.

**C. Ramified coverings of hyperbolic manifolds**

Another interesting class of spaces whose fundamental groups admit acylindrical splittings is the one of cyclic ramified coverings of hyperbolic manifolds. The construction is due to Gromov-Thurston ([Gr-Th]) and represents an important source of examples of manifolds admitting pinched, negatively curved metrics but not hyperbolic ones. A degree \(k\) ramified cover \(X_k\) of a hyperbolic manifold \(X\) is obtained by excising a totally geodesic hypersurface with boundary \(Z\) in \(X\), and then glueing several copies of \(X - Z\) along a “\(k\)-paged open book”, whose leaves are copies \(Z_i\) of \(Z\) joined together at the ramification locus \(R = \partial Z\). Any such covering admits a singular, locally \(CAT(-1)\) metric, and its fundamental group splits as a free
product of CAT(-1) groups amalgamated over the fundamental group of the locally convex subspace $Z_1 \cup Z_k$ of $X_k$ (given by two pages of the book). We will recall in Section §2.5 this construction in more detail, and show that these manifolds naturally fall in the class of spaces with negatively curved splittings. However, we will be interested in metrics with curvature of any possible sign on such manifolds. Namely, let $\mathcal{R}^4(E, D)$ be the space of all 4-dimensional Riemannian, cyclic ramified coverings of compact, orientable hyperbolic manifolds, whose entropy and diameter are respectively bounded by $E$ and $D$; and let $\mathcal{R}^{\neq 4}(E, D)$ the corresponding space of ramified coverings in dimensions different from 4. Then:

**Corollary 9.** The class $\mathcal{R}^4(E, D)$ contains only finitely many different homotopy types, and $\mathcal{R}^{\neq 4}(E, D)$ only finitely many diffeomorphism types.

We stress the fact that the manifolds under consideration are (genuine) ramified coverings of any possible hyperbolic manifold, and not just of one fixed hyperbolic manifold.

**D. High dimensional graph and cusp decomposable manifolds.**

High dimensional graph manifolds have been introduced by Frigerio, Lafont and Sisto in [FLS], and cusp-decomposable manifolds by Nguyen Phan in [Ngu]. Roughly speaking, an (extended) $n$-dimensional graph manifold $X$ is obtained gluing together, via affine diffeomorphisms of their boundaries, several elementary, building blocks $X_i$ which are diffeomorphic to products $H^{k_i} \times T^{n-k_i}$, where $T^{n-k_i}$ is a $(n-k_i)$-dimensional torus (representing the local fibers of the graph manifold), and $H^{k_i}$ is a manifold of dimension $k_i \geq 2$ with toroidal boundary components, obtained from a hyperbolic manifold with cusps by truncating some cusps along (the quotient of) flat horospheres; a block of the form $H^2 \times T^{n-2}$ is also called a surface piece. A high dimensional graph manifold $X$ is called irreducible if none of the fibers of two adjacent blocks represent the same element in $\pi_1(X)$, and purely hyperbolic if there are no fibers at all (that is every piece $X_i$ is a truncated $n$-dimensional hyperbolic manifold).

Cusp decomposable manifolds are defined similarly to purely hyperbolic high dimensional graph manifolds, but starting from building blocks which are negatively curved, locally symmetric manifolds with cusps, and gluing, always via affine diffeomorphisms the boundary infra-nilmanifolds obtained by truncating the cusps. We say that a $n$-dimensional graph manifold, or a cusp decomposable manifold $X$, is non-elementary if it is obtained by identifying at least two boundaries (of one or more building blocks).

High dimensional graph enjoy strong topological rigidity properties, as they are aspherical spaces and satisfy the Borel Conjecture in dimension $n \geq 6$ (cf. [FLS], Thm.3.1 and §3.4, Rmk.3.7); moreover, the diffeomorphism type of (closed) cusp decomposable manifolds, or of all high dimensional graph manifolds whose boundary components do not belong to surface pieces, is determined by the fundamental

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6Notice that the above class does no cover the case of non-geometric 3-manifolds, since surface pieces are trivial products, so the blocks do not take into account Seifert fibrations.

7An affine diffeomorphism, in this context, is the composition of an isomorphism of the two nilpotent Lie groups composed with the left multiplication by an element of the group. In [Ngu] the author points out the necessity of realizing the gluings via affine diffeomorphisms to have strong differential rigidity: in fact, Aravinda-Farrell show in [Ar Fa] the existence of non-affine gluing maps, for the double of a hyperbolic cusped manifold $X$, giving rise to a manifold which is not diffeomorphic to the one obtained by gluing the two copies via the identity map of $\partial X$. 
group within the respective classes (cp. [FLS] Thm.0.7 and [Ngu]). The importance of considering affine gluings, in order to get rigidity, was pointed out by Aravinda-Farrell in [Ar-Fa], where they considered the so called twisted doubles — which also are included in the class defined in [FLS].

Non-elementary, irreducible high dimensional graph manifolds groups possess 2-acylindrical splittings, while cusp decomposable manifolds (or purely hyperbolic, high dimensional graph manifolds) groups have 1-acylindrical splittings. Therefore, we obtain:

**Corollary 10.** Let $\mathcal{G}^g(E, D)$ (resp. $\mathcal{G}(E, D)$) be the class of compact (resp. closed) Riemannian, non-elementary irreducible high dimensional graph manifolds with entropy and diameter bounded by $E$, $D$, and let $\mathcal{C}(E, D)$ be the class of closed Riemannian, non-elementary cusp decomposable manifolds satisfying the same bounds:
(i) $\mathcal{G}^g(E, D)$ contains finitely many homotopy types, and $\mathcal{G}(E, D)$ only finitely many diffeomorphism types;
(ii) $\mathcal{C}(E, D)$ contains a finite number of diffeomorphism types.

Notice that we do not bound a priori the dimension of the manifolds in these classes: this follows, in the aftermath, from bounding their entropy and diameter. Moreover, notice that, while the results on Riemannian 2-orbifolds and ramified coverings (though concerning metrics of any possible sign) still pertain to the framework of spaces of negative curvature, the classes of non-geometric 3-manifolds and of high dimensional graph or cusp decomposable manifolds escape from the realm of non-positive curvature. This is clear for non-prime 3-manifolds, and follows from Leeb’s work [Leeb] for irreducible 3-manifolds. Cusp decomposable manifolds obtained by gluing boundary infra-nilmanifolds which are not tori do not admit nonpositively curved metrics, by the Solvable Subgroup Theorem; also, in [FLS] there are examples of $n$-dimensional graph manifolds not supporting any locally $\text{CAT}(0)$-metric, for $n \geq 4$.

**Acknowledgements.** The first author acknowledges financial support by the Max-Planck Institut für Mathematik and praises the excellent working conditions provided by the Institut. Both the authors wish to thank R. Coulon for useful discussions during his stay in Rome in 2016, and S. Gallot for many precious hints.

1. Groups with acylindrical splittings

This first part is devoted to introducing some basic facts about groups with acylindrical splittings (Section 1.1), and to developing the combinatorial tools needed to prove the Entropy-Cardinality inequality (1) (Sections 1.2-1.3).

A finitely generated group $G$ possesses an acylindrical splitting if $G$ is a non-trivial amalgamated product or HNN-extension, thus isomorphic to the fundamental group of a non-trivial graph of groups $\mathcal{G}$, and the canonical action of $G$ on the Bass-Serre tree associated to $\mathcal{G}$ is acylindrical. We recall the notion of acylindrical action on a simplicial tree:

**Definition 1.1 (Acylnidrical actions on trees).**
Let $G$ be a discrete group acting by automorphisms without edge inversions (i.e. with no element swapping the vertices of some edge) on a simplicial tree $T$, endowed with its natural simplicial distance $d_T$ (i.e. with all edges of unit length).
We shall say that the action \( G \curvearrowright T \) is \textit{k-acylindrical} if the set of fixed points of every elliptic element \( g \in G \) has diameter less or equal to \( k \), and that the action is \textit{acylindrical} if it is \( k \)-acylindrical for some \( k \).

The notion of \( k \)-acylindrical action of a discrete group on a tree is a geometric reformulation of the notion of \textit{k-step malnormal amalgamated product}, cp. \cite{Ka-So}. We recall that an element \( g \) belonging to an amalgamated product \( G = A \ast_C B \) is written in \textit{normal form} when it is expressed as \( g = g_1g_2 \cdots g_n \), where \( g_0 \in C \) and, for \( i \geq 1 \), no \( g_i \) belongs to \( C \) and two successive \( g_i \) belong to different factors of the product; the integer \( n = \ell(g) \) is then called the \textit{syllable length} of \( g \) (then, the identity and the elements of \( C \) have zero syllable length by definition).

An amalgamated product \( G = A \ast_C B \) is called \textit{k-step malnormal} if \( gCg^{-1} \cap C = \{1\} \) for all \( g \in G \) with \( \ell(g) \geq k + 1 \) (in particular, \( k = 0 \) if and only if \( C \) is a malnormal subgroup of \( G \); and, by definition, free products are \((-1\text{-})\text{malnormal})

A similar definition can be given for a group \( A \ast_{\varphi} = \langle A, t \mid \text{rel}(A), t^{-1} ct = \varphi(c) \rangle \) which is a HNN-extension of \( A \) with respect to an isomorphism \( \varphi : C_- \to C_+ \) between subgroups \( C_-, C_+ \). Namely, by Britton’s Lemma every element \( g \in G^\ast \) can be written in a \textit{normal form} as

\[
g = g_0\varepsilon_1g_1 \cdots g_{m-1}\varepsilon_m g_m
\]

where \( g_0 \in A \), \( \varepsilon_i = \pm 1 \), and \( g_i \in A \setminus C_\varepsilon_i \) if \( \varepsilon_{i+1} = -\varepsilon_i \); the syllable length of \( g \) is defined in this case as \( \ell(g) = m \). Then, a HNN-extension \( G = A \ast_{\varphi} \) is called \textit{k-step malnormal} if \( gC_\varepsilon g^{-1} \cap C_{\varepsilon'} = \{1\} \) for any \( \varepsilon,\varepsilon' \in \{\pm\} \) and for all \( g \in G \) with \( \ell(g) \geq k + 2 \).

It is then easy to check that a group \( G \) which is a \((k-1)\)-step malnormal amalgamated product or HNN-extension admits a \( k \)-acylindrical action on his Bass-Serre tree. Conversely, if \( G \) admits a \( k \)-acylindrical splitting having a segment (resp. a loop) as underlying graph \( \mathcal{G} \), then \( G \) is a \((k-1)\)-step malnormal amalgamated product (resp. HNN-extension); see \cite{Cer2} for further details.

In what follows, we will be interested in \textit{non-elementary} acylindrical splittings: that is, splittings of \( G \) as a non-trivial amalgamated or HNN-extension for which the action on the corresponding Bass-Serre tree is non-elementary. Since non-trivial amalgamated products and HNN extensions act without global fixed points on their Bass-Serre tree, this will be equivalent to asking that the action of \( G \) is not \textit{linear} (see the next subsection for the basic terminology for acylindrical group actions on trees), i.e. \( G \) splits as a \textit{malnormal, non-trivial} amalgamated product \( A \ast_C B \) (resp. HNN-extension \( A \ast_{\varphi} \)) with \( C \) of index greater than 2 in \( A \) or \( B \) (resp. \([A : C_-] + [A : C_+] \geq 3\)).

1.1. Basic facts on acylindrical actions on trees.

In the following, we will always tacitly assume that the action of \( G \) on the simplicial tree \( T \) under consideration is by automorphisms and without edge inversions\footnote{This seems to contrast with the definition for amalgamated products, but it actually yields that \( G = A \ast_{\varphi} \) is a 0-malnormal HNN extension if and only if \( C_+, C_- \) are malnormal and conjugately separated in \( A \), and that \( G \) is \((-1\text{-})\text{malnormal} \) if \( C_\pm = \{1\} \). This is due to the relation \( t^{-1}C_- t = C_+ \); one might express the same condition by imposing \( gC_\varepsilon g^{-1} \cap C_{\varepsilon'} = \{1\} \) for all \( g \) with \( \ell(g) = k + 1 \) and whose normal form satisfies some additional (awkward to write) restrictions.}

\footnote{An action of a group \( G \) on a simplicial tree \( T \) is said to be without \textit{(edge inversions)} if for any oriented edge \( e \) and for any \( g \in G \) we have \( ge \neq \bar{e} \), where \( \bar{e} \) is the edge \( e \) with the opposite orientation.}
Then, the elements of $G$ can be divided into two classes, according to their action: elliptic and hyperbolic elements. They are distinguished by their translation length, which is defined, for $g \in G$, as

$$
\tau(g) = \inf_{v \in T} d_T(v, g \cdot v)
$$

where $d_T$ is the natural simplicial distance of $T$, i.e. with all edges of unit length. If $\tau(g) = 0$ the element $g$ is called elliptic, otherwise it is called hyperbolic.

We shall denote by $\text{Fix}(g)$ the set of fixed points of an elliptic element $g$. We recall that $\text{Fix}(g)$ is a (possibly empty) connected subtrees of $T$. If $h$ is a hyperbolic element then $\text{Fix}(h) = \emptyset$ and $h$ has a unique axis.

$$
\text{Ax}(h) = \{ v \in T \mid d_T(v, h(v)) = \tau(h) \}
$$

on which it acts by translation: each element on the axis of $h$ is translated at distance $\tau(h)$ along the axis, whereas elements at distance $\ell$ from the axis are translated of $\tau(h) + 2\ell$.

Let $T_G$ be the minimal subtree of $T$ which is $G$-invariant: the action of $G$ is said elliptic if $T_G$ is a point, and linear if $T_G$ a line; in both cases we shall say that the action of $G$ is elementary.

The next lemma resumes some facts about centralizers and normalizers of hyperbolic elements of acylindrical actions:

**Lemma 1.2.** Let $G \acts T$ be a $k$-acylindrical action on a tree. Let $h \in G$ be hyperbolic, and let $h_0$ be a hyperbolic element with $\text{Ax}(h_0) = \text{Ax}(h)$ and $\tau(h_0)$ minimal:

(i) any hyperbolic element $g$ with $\text{Ax}(g) = \text{Ax}(h)$ is a multiple of $h_0$;

(ii) if $s\text{Ax}(h) = \text{Ax}(h)$ then either $shs^{-1} = h$ or $shs^{-1} = h^{-1}$, depending on whether $s$ preserves the orientation of $\text{Ax}(h)$ or not;

(iii) the centralizer $Z(h)$ is the infinite cyclic subgroup generated by $h_0$;

(iv) the normalizer $N(h)$ is either equal to $Z(h)$, or to an infinite dihedral group generated by $\{h_0, \sigma\}$, where $\sigma$ is an elliptic element swapping $\text{Ax}(h_0)$;

(v) if the action is linear then $G$ is virtually cyclic.

**Proof.** Since $h_0$ has minimum translation length among hyperbolic elements with the same axis, any hyperbolic $g$ with $\text{Ax}(g) = \text{Ax}(h)$ acts on $\text{Ax}(h_0)$ as $h_0^n$, for some $n \in \mathbb{Z}$; hence $g = h_0^n$ necessarily, by acylindricity, which shows (i).

To see (ii), notice that if $\text{Ax}(h) = s\text{Ax}(h) = \text{Ax}(shs^{-1})$ then $shs^{-1} = h^\pm 1$ by (i), since $\tau(shs^{-1}) = \tau(h)$. Moreover, as $s$ stabilizes $\text{Ax}(h)$, if $s$ is hyperbolic then it has the same axis as $h$, preserves the orientation of the axis and commutes with $h$; that is $shs^{-1} = h$. On the other hand, if $s$ is elliptic, by acylindricity it can only swap $\text{Ax}(h)$ by a reflection through a vertex (because $G$ acts without edge inversions), so $shs^{-1}$ acts on $\text{Ax}(h)$ as $h^{-1}$; this implies that $shs^{-1} = h^{-1}$, again by acylindricity.

Assertion (i) implies that the subset of hyperbolic elements of $G$ whose axes are equal to $\text{Ax}(h)$ is included in $Z(h)$. Reciprocally, if $s \in Z(h)$ then, by assumption, $shs^{-1} = h$ and $s\text{Ax}(h) = \text{Ax}(h)$; therefore, by the above discussion, $s$ is a hyperbolic element which has the same axis as $h$. This proves (iii).

To see (iv), let $s \in N(h)$. As $s(h_0)s^{-1} = \langle h_0 \rangle$ and conjugation is an automorphism orientation. Notice that if a group $G$ acts on a simplicial tree $T$ transitively on the (un)oriented edges, then $G$ acts without inversions on the barycentric subdivision of $T$. 

---

**Note:** The above text is a continuation of the discussion on the classification of elements in a group $G$ with respect to their translation length and their actions on simplicial trees. The focus is on distinguishing between elliptic and hyperbolic elements, and the implications of these classifications on the structure of subgroups and actions.

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**Reference:**

The text references the use of simplicial trees, which are fundamental in the study of geometric group theory, particularly in the context of hyperbolic groups and their actions. The notation $\text{Ax}(h)$ refers to the axis of an element $h$, which is a key concept in understanding the geometry of these groups.

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**Further Reading:**

For a deeper understanding, one might consult texts on geometric group theory, such as "Geometric Group Theory" by Daniel T. Wise, or "Hyperbolic Groups" by Michael R. Bridson and André Haefliger, which provide comprehensive insights into the subjects discussed in the context of simplicial trees and their applications.

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**Additional Information:**

The natural simplicial distance $d_T$ is a measure of the length of the shortest path between two points in the tree $T$. This distance is fundamental in defining the translation length of elements, which is a key invariant in the study of group actions on simplicial trees.
of $G$, we deduce that $s h_0 s^{-1} = h_0^{±1}$, so $s$ stabilizes $\text{Ax}(h)$. If $s$ is hyperbolic, we know by (ii) that it belongs to the infinite cyclic subgroup $\langle h_0 \rangle$. On the other hand, we have seen that if $s$ is elliptic then it acts on $\text{Ax}(h)$ by a reflection through a vertex $v$, so $s^2 = 1$ and $s h_0 s^{-1} = h_0^{−1}$ by acylindricity. If $s'$ is another element of $N(h) \setminus Z(h)$, then also $s'$ acts by swapping $\text{Ax}(h)$ and fixing some $v' \in \text{Ax}(h)$. Then, $s' s^{-1} = s' s$ acts on $\text{Ax}(h)$ as $h_0^s$, and by acylindricity $s' s = h_0^s$. This shows that, if $N(h) \neq Z(h)$ then $N(h) \cong D_{\infty}$.

Finally, let us show (v). Assume that the minimal $G$-invariant subtree of $T$ is a line. Since the action is non-elliptic, there exists a hyperbolic element $h$ preserving this line (as $G$ acts on $T$ without global fixed points, its elements cannot be all elliptic, or there would exist $s_1, s_2$ with $\text{Fix}(s_1) \cap \text{Fix}(s_2) = \emptyset$ and this would produce a hyperbolic element $h = s_1 s_2$). Any other element $s \in G$ leaves $\text{Ax}(h)$ invariant, so it is in $N(h)$ by (ii), and then $G$ is cyclic or dihedral by (iii) and (iv).

The next lemma bounds the size of the intersection of the axes of two hyperbolic elements in terms of the acylindricity constant $k$ and of the translation lengths:

**Lemma 1.3.** Let $G \curvearrowright T$ be a non-elementary, $k$-acylindrical action on a tree. Let $h_1, h_2 \in G$ be two hyperbolic elements with distinct axes: then,

$$\text{diam}(\text{Ax}(h_1) \cap \text{Ax}(h_2)) \leq \tau(h_1) + \tau(h_2) + k$$

**Proof.** If $\text{Ax}(h_1) \cap \text{Ax}(h_2) = \emptyset$ the assertion is trivially verified. Up to taking inverses, we may assume that the elements $h_1, h_2$ translate $I = \text{Ax}(h_1) \cap \text{Ax}(h_2) \neq \emptyset$ in the same direction, and we let $I$ be oriented by this direction. We argue by contradiction: assume that $\text{diam}(I) \geq \tau(h_1) + \tau(h_2) + k + 1$. Then, the element $[h_1, h_2]$ would fix pointwise the initial subsegment of length $k + 1$ of $I$ and this, by $k$-acylindricity, implies that $[h_1, h_2] = 1$. This contradicts the assumption that $h_1$ and $h_2$ have distinct axes. 

### 1.2. Schottky and pairwise-Schottky subgroups.

Recall that the subgroup $\langle g_1, ..., g_n \rangle$ of $G$ generated by $g_1, ..., g_n$ is a **Schottky subgroup** (of rank $n$), if for any $i = 1, ..., n$ it is possible to find subsets $X_i \subset T$ for $i = 1, ..., n$ such that:

(i) $X_i \cap X_j = \emptyset$ for $i \neq j$;

(ii) $g_i^{±1}(T \setminus X_i) \subseteq X_i$ for all $i$.

The $X_i$'s are called **joint domains of attraction** of the $g_i$'s.

**Lemma 1.4.** Let $G$ be a group acting on a simplicial tree $T$. Let $h_1, h_2$ be two hyperbolic elements such that:

$$\text{diam}(\text{Ax}(h_1) \cap \text{Ax}(h_2)) < \min\{\tau(h_1); \tau(h_2)\}$$

Then the group $\langle h_1, h_2 \rangle$ is a rank 2 free, Schottky subgroup of $G$.

This lemma is folklore (see for instance [Ka-We1]). However let us describe the domains of attraction $X_1, X_2$ for future reference. Let $p_i : T \rightarrow \text{Ax}(h_i)$ be the projection and define $J_1 = p_1(\text{Ax}(h_2))$ and $J_2 = p_2(\text{Ax}(h_1))$: then one can choose $X_i = \{v \in T \mid p_i(v) \notin J_i\}$.
Definition 1.5 (pS-family). Let $G$ act on a simplicial tree $T$ and let $\mathcal{H} = \{h_1, ..., h_m\}$ be a collection of hyperbolic elements such that:
\[
\text{diam}(Ax(h_i) \cap Ax(h_j)) < \min\{\tau(h_i), \tau(h_j)\}, \quad \text{for any } i \neq j
\]
We shall call $\mathcal{H}$ a pairwise Schottky family (shortly, pS-family) and let $X_{i,j}$, $X_{j,i}$ be the domains of the pair $\{h_i, h_j\}$, $1 \leq i < j \leq m$ (namely, where $X_{i,j}$ is the domain of attraction of $h_i$, with respect to the pair $\{h_i, h_j\}$).

Remark 1.6. It is worth to stress the difference between a Schottky subgroup and the group $H$ generated by a pS-family $\mathcal{H} = \{h_1, ..., h_m\}$. By definition, for any pair of elements $h_i, h_j \in \mathcal{H}$ of a pS-family it is possible to find domains of attraction $X_{i,j}$ and $X_{j,i}$, so that $\{h_i, h_j\}$ are in Schottky position and generate a rank 2 free subgroup of $G$. However the whole collection $\mathcal{H}$ generally is not in Schottky position, as the domains $X_{i,j}$, $X_{j,i}$ are not joint domains of attraction for all the $h_i$’s, as they depend on the pair $\{h_i, h_j\}$ (notice that the intersection $X_i := \bigcap_{j \neq i} X_{i,j}$ might be empty for some index $i$). In particular $H$ generally is not free.

1.3. Large pS-families with universally bounded $S$-lengths.

We start now considering finitely generated marked groups $(G, S)$ acting on trees, where $S$ is any finite generating set for $G$. We will denote by $|\cdot|_S$ the associated word metric, and by $S^n$ the relative ball of radius $n$, centered at the identity element. Let us also denote by $S^\text{hyp}_{\text{hyp}}$ the subset of hyperbolic elements in $G$ of $S$-length smaller than or equal to $n$.

We start noticing that, when the action is non-elementary, there always exist hyperbolic elements of $S$-length at most 2 (see for instance [Ce-Sa1], §2):

Lemma 1.7. Let $G \acts T$ be an action on a simplicial tree, and let $S$ be any finite generating set for $G$:

(i) if the action is non-elliptic, then there exists a hyperbolic element $h \in G$ such that $|h|_S \leq 2$. Namely, either $h \in S$, or $h$ is the product of two elliptic elements $s_1, s_2 \in S$ with $\text{Fix}(s_1) \cap \text{Fix}(s_2) = \emptyset$;
(ii) if the action is non-elementary, then for any hyperbolic element $h \in G$ there exists $s \in S$ which does not belong to the normalizer $N(h)$.

The next two Propositions are preparatory to the Entropy-Cardinality inequality. They show that, from any generating set $S$ of a group $G$ acting acylindrically and non-elementarily on a tree, one can always produce a large (compared to $|S|$) pS-family $\mathcal{H}$ of hyperbolic elements with universally bounded $S$-length, whose axes are, moreover, in one of two basic configurations.

Proposition 1.8 (Large conjugacy classes of hyperbolic with bounded $S$-lengths). Let $G \acts T$ be a non-elementary, $k$-acylindrical action on a tree, and let $S$ be any finite generating set for $G$. There exists a subset $\mathcal{H} \subset S^\text{hyp}_{\text{hyp}}$ of hyperbolic elements, with $|\mathcal{H}| \geq \sqrt[4]{|S|}$ such that:

(i) the elements of $\mathcal{H}$ have all different axes;
(ii) every element of $\mathcal{H}$ has at least $\sqrt[4]{|S|}/2$ conjugates with distinct axes in $S^\text{hyp}_{\text{hyp}}$. 
Proposition 1.9 (Large pS-families with universally bounded S-lengths).

Let \( G \trianglelefteq \mathcal{T} \) be a non-elementary, \( k \)-acylindrical action on a tree. For any finite generating set \( S \) of \( G \) there exists a pairwise Schottky family \( \mathcal{S} \) with the following properties:

(i) all the elements of \( S \) have the same translation length;
(ii) \( |h|_S \leq 20(k + 3) \) for all \( h \in S \);
(iii) \( |S| \geq \sqrt[4]{|S|/2} \).
(iv) either there exists \( h_1 \in S \) such that \( \text{Ax}(h) \cap \text{Ax}(h_1) \neq \emptyset \) for all \( h \in S \), or all the axes of the elements in \( S \) have pairwise empty intersection.

In order to prove the above Propositions, we preliminary show:

Lemma 1.10. Under the above assumptions, we have \(|S_{hyp}^4| \geq \sqrt{|S|}\). Moreover, in \( S_{hyp}^4 \) there are at least two hyperbolic elements having distinct axes.

Proof of Lemma 1.10. Let \( h \in G \) be a hyperbolic element such that \(|h|_S \leq 2\). Consider the set \( C_S(h) \subset S^4 \) of conjugates \( s h s^{-1} \) of \( h \) by all the elements of \( S \). Observe that either \(|C_S(h)| \geq \sqrt{|S|}\), or there exists a subset \( S_h \subset S \) of cardinality at least \( \sqrt{|S|} \) such that \( s h s^{-1} = s' h s'^{-1} \) for all \( s, s' \in S_h \). In the first case we deduce that \(|S_{hyp}^4| \geq \sqrt{|S|}\). In the second case, choose \( s_0 \in S_h \) and consider the map \( F : S_h \to S^2 \) defined by \( F(s) = s^{-1} s_0 \). By definition, \( F \) is an injective map. Moreover, since \( s_0 h s_0^{-1} = h s_0^{-1} \) for all \( s \in S_h \), we have \( s^{-1} s_0 h = h s_0^{-1} \), so \( \text{Im}(F) \subset Z(h) \). By Lemma 1.2 we know that \( Z(h) \) is infinite cyclic, generated by a hyperbolic element \( h_0 \); we deduce that \( \text{Im}(F) \subset S_{hyp}^4 \), so \(|S_{hyp}^4| \geq |S_{hyp}^2| \geq \sqrt{|S|}\).

Finally, notice that, in any case, \( C_S(h) \) contains at least two hyperbolic elements with distinct axes: otherwise, every \( s \in S \) would be in the normalizer \( N(h) \) and the action of \( G \) on \( \mathcal{T} \) would be elementary.

Proof of Proposition 1.9. Consider the equivalence relation between hyperbolic elements \( h, h' \in G \) defined by \( h \sim h' \) if and only if \( \text{Ax}(h) = \text{Ax}(h') \), and let \( C_1, \dots, C_m \) be the equivalence classes in \( S_{hyp}^4 \) with respect to \( \sim \).

By Lemma 1.10 we know that \(|S_{hyp}^4| \geq \sqrt{|S|}\), so:

(a) either there exists an equivalence class, say \( C_1 \), such that \(|C_1| \geq \sqrt{|S|}\);
(b) or \( m \geq \sqrt{|S|}\).

In the first case, notice that we have \( m \geq 2 \), always by Lemma 1.10. So, in case (a), let \( C_1 = \{h_1, \dots, h_n\} \) with \( n \geq \sqrt{|S|}\), and let \( h \in C_2 \); then, the set \( \mathcal{H} = \{h_i h h_i^{-1} \mid h_i \in C_1\} \) is a collection of \( n \) hyperbolic elements in \( S_{hyp}^{12} \) with distinct axes. Indeed, if \( \text{Ax}(h_i h h_i^{-1}) = \text{Ax}(h_j h h_j^{-1}) \) then \( h_j^{-1} h_i \) would preserve both \( \text{Ax}(h_i) = \text{Ax}(h_j) \) and \( \text{Ax}(h) \); this would imply \( \text{Ax}(h) = \text{Ax}(h_i) = \text{Ax}(h_j) \), a contradiction. Moreover, in this case there are also at least \( n \) distinct conjugates (with distinct axes) of \( h \) and of each \( h_i h h_i^{-1} \) in \( S_{hyp}^{12} \), since the collection of elements \( h_j h h_j^{-1} = h_j h h_j^{-1} \) is also a collection of conjugates of \( h_i h h_i^{-1} \).

Therefore, in this case we conclude that \(|\mathcal{H}| \geq n \geq \sqrt{|S|}\).

In case (b), let \( \mathcal{R} = \{h_1, \dots, h_m\} \) be a set of representatives for the classes \( C_i \): we will now prove that every \( h_i \) has at least \( \sqrt{m} \) distinct conjugates in \( S_{hyp}^{20} \).

To this purpose, let

\[ C_{\mathcal{R}}(h_i) = \{h_j h_i h_j^{-1} \mid h_j \in \mathcal{R}\} \subset S_{hyp}^{12} \]
Again, either \( C_\mathcal{R}(h_i) \) contains at least \( \sqrt{m} \) distinct conjugates of \( h_i \) and we are done, or there exists a subset \( \mathcal{H}_i \subset \mathcal{R} \) with cardinality \( |\mathcal{H}_i| \geq \sqrt{m} \) such that
\[
h h_i h^{-1} = h' h_i h'^{-1} =: g_i \quad \text{for all} \ h, h' \in \mathcal{H}_i
\]
(2)
In this case, consider the map \( c_{h_i} : \mathcal{H}_i \rightarrow S_{hyp}^{20} \), defined by \( h \mapsto c_{h_i}(h) := h^2 h_i h^{-2} \).
To show that there are at least \( \sqrt{m} \) distinct conjugates of \( h_i \), it is enough to show that this map is injective. Actually, assume that \( c_{h_i}(h) = c_{h_i}(h') \), for some \( h, h' \in \mathcal{H}_i \); as, by (2) we have that \( h^{-1} h' =: \gamma_i \in Z(h_i) \), we would deduce
\[
g_i = h h_i h^{-1} = h^{-1} (h'^2 h_i h'^{-2}) h = \gamma_i g_i \gamma_i^{-1}
\]
that is \( [g_i, \gamma_i] = 0 \). Therefore, \( \gamma_i \in Z(h_i) \cap Z(g_i) = Z(h_i) \cap Z(h h_i h^{-1}) \) which is possible if and only if \( Ax(h_i) = h Ax(h_i) \). But then \( h \) and \( h_i \) would share the same axis, which contradicts the fact that they are distinct elements of \( \mathcal{R} \).
We have therefore showed that, in case (b), every \( h_i \in \mathcal{R} \) has at least \( \sqrt{m} \) distinct conjugates in \( S_{hyp}^{20} \). Now, if we have \( Ax(s h_i s^{-1}) = Ax(s' h_i s'^{-1}) \) for \( s, s' \in S \), then \( (s^{-1} s') \cdot Ax(h_i) = Ax(h_i) \), so \( (s^{-1} s') h_i (s^{-1} s')^{-1} = h_i' \) for some \( \epsilon \in \{ \pm 1 \} \), by Lemma 1.2 that is, either \( s' h_i s'^{-1} = s h_i s^{-1} \) or \( s' h_i s'^{-1} = s h_i s^{-1} = (s h_i s^{-1})^{-1} \). It readily follows that \( h_i \) has at least \( \sqrt{m}/2 \) conjugates in \( S_{hyp}^{20} \).

**Proof of Proposition 1.9.** By Proposition 1.8 we know that there is a collection \( \mathcal{H} \subset S_{hyp}^{20} \) of hyperbolic elements which are all conjugates of one \( h_0 \in \mathcal{H} \), have distinct axes, and with cardinality \( |\mathcal{H}| = n \geq \sqrt{|S|} \). Moreover, for any pair of elements \( h, h' \in \mathcal{H} \) we know by Lemma 1.3 that
\[
\text{diam}(Ax(h) \cap Ax(h')) \leq \tau(h) + \tau(h') + k = 2 \cdot \tau(h_0) + k
\]
It follows that, for all \( p > 2 + \frac{k}{\tau(h_0)} \):
\[
\text{diam}(Ax(h^p) \cap Ax(h'^p)) < \tau(h^p) = \tau(h'^p)
\]
for all \( h, h' \in \mathcal{H} \). Therefore, the collection \( \mathcal{H}_1 = \{ h^{k+3} \mid h \in \mathcal{H} \} \) provides a pairwise Schottky family satisfying (i), (ii) and (iii). We shall now prove that we can extract from \( \mathcal{H}_1 \) a subset \( S \) with \( |S| \geq \sqrt{n} \) satisfying one of the two conditions in (iv). Actually, let \( m(h) \) be the number of elements of \( \mathcal{H}_1 \) whose axis intersects \( Ax(h) \), and let \( m = \max_{h \in \mathcal{H}_1} m(h) \). Then either there exists an element of \( \mathcal{H}_1 \), say \( h_1 \), with \( m(h_1) \geq \sqrt{n} \), or \( m(h) < \sqrt{n} \) for all \( h \in \mathcal{H}_1 \). In the first case, we define \( S = \{ h \in \mathcal{H}_1 \mid Ax(h) \cap Ax(h_1) \neq \emptyset \} \) and we are done. Otherwise, let \( \mathcal{S} = \{ h_1, ..., h_m \} \subset \mathcal{H}_1 \) be a maximal subset of elements with pairwise disjoint axes, and let \( \mathcal{S}_i \subset \mathcal{H}_1 \) be the subset of elements whose axes intersect \( Ax(h_i) \). As \( \mathcal{S} \) is maximal, we have
\[
\mathcal{H}_1 = \bigcup_{i=1}^{m} \mathcal{S}_i
\]
so \( n \leq m \cdot |\mathcal{S}_i| < m \sqrt{n} \). Hence, \( |\mathcal{S}| = m > \sqrt{n} \).

1.4. **Extraction of Schottky subgroups of large rank.**
In this section we consider a pS-family \( \mathcal{S} = \{ h_1, ..., h_m \} \) with \( \tau(h_i) = \tau \), \( \forall i = 1, ..., m \) and whose axes satisfy one of the two configurations explained in Proposition 1.9.
We will explain how to extract from the group \( H = \langle \mathcal{S} \rangle \) generated by \( \mathcal{S} \) a genuine Schottky subgroup of rank \( r \geq \sqrt{\frac{n-1}{2}} \), whose generators have \( \mathcal{S} \)-length \( \leq 4 \).
1.4.1. \textit{pS-families distributed along one axis.} Here we study the case where all axes of the elements in $\mathcal{S}$ intersect an axis, say $\text{Ax}(h_1)$. We shall in particular examine two opposite (but not exhaustive) subcases:

(a) the case where \textit{the intersection between the axes with $\text{Ax}(h_1)$ agglomerate together}, i.e. there exists a segment $I \subset \text{Ax}(h_1)$ with diam$(I) \leq 2\tau$ containing all the intersections $J_{1,i} := \text{Ax}(h_1) \cap \text{Ax}(h_i)$; 

(b) the case where \textit{the intersections with $\text{Ax}(h_1)$ are sparse along $\text{Ax}(h_1)$}, i.e. $J_{1,i} = \text{Ax}(h_1) \cap \text{Ax}(h_i) \neq \emptyset$, $\forall i = 2, \ldots, M$ but $J_{i,j} := \text{Ax}(h_i) \cap \text{Ax}(h_j) = \emptyset$ for every $i, j \geq 2$ with $i \neq j$.

(Notice that $J_{1,i} = p_1(\text{Ax}(h_i))$, so the notation is consistent with the one in §1.2).

For easy future reference we will call such families \textit{agglomerated} (in case (a)) or \textit{sparse} (in case (b)) with respect to $h_1$.

The next Proposition shows that these configurations are actually more frequent than one can expect:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{agglomerated_family.png}
\caption{pS-family agglomerated with respect to $h_1$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{sparse_family.png}
\caption{pS-family sparse with respect to $h_1$}
\end{figure}

\textbf{Proposition 1.11.} Let $\mathcal{S} = \{h_1, \ldots, h_m\}$ be a pS-family with $\text{Ax}(h_1) \cap \text{Ax}(h_i) \neq \emptyset$ for all $i$. Then, there exists a subset $\mathcal{S}' \subset \mathcal{S}$, containing $h_1$, which either is agglomerated with respect to $h_1$ and has cardinality $m' \geq \sqrt{m}$, or is sparse with respect to $h_1$ and has cardinality $m' \geq \sqrt{m} + 1$.

\textit{Proof.} For any $k = 2, \ldots, m$ let us denote as before $J_{1,k} = \text{Ax}(h_1) \cap \text{Ax}(h_k)$ and let $\mathcal{S}^* = \mathcal{S} \setminus \{h_1\}$. Since $\mathcal{S}$ is a pS-family such that $\tau(h_i) = \tau$, we know that diam$(J_{1,k}) < \tau$. Orient $\text{Ax}(h_1)$ by the translation direction of $h_1$. Then, to each $J_{1,k}$ we associate $v_k^-, v_k^+$, respectively the initial and final vertex of $J_{1,k}$ with respect to the orientation of $\text{Ax}(h_1)$. We then consider the following order relation on $\{h_k\}_{k=2}^m$: we shall say that $h_k \leq h_j$ if
Let \( \text{Lemma 1.12.} \)

The relation \( \leq \) defines a total order on \( S^* \). Up to re-indexing the elements in \( S^* \) we may assume that \( h_i < h_j \) if and only if \( i < j \). Set \( i_1 = 2 \) and let \( I_1 \) be the interval on \( \text{Ax}(h_{i_1}) \) starting at \( v_{i_1}^- \) and having length \( 2\tau \), and define \( C_{I_1} \) be the set of those elements in \( S^* \) such that \( J_{I_1} \subseteq I_1 \) and \( m_1 = |C_{I_1}| \). By definition \( C_{I_1} \) is non-empty since \( h_2 \in C_{I_1} \). If \( m_1 \geq \sqrt{m} - 1 \), then we define \( S' = \{ h_1 \} \cup C_{I_1} \) and we are done, as \( S' \) is an agglomerated family with respect to \( h_1 \). Otherwise set \( i_2 = m_1 + 2 \), consider the segment \( I_2 \) starting at \( v_{i_2}^- \) of length \( 2\tau \) and define \( C_{I_2} = \{ h_k \in S^* \setminus C_{I_1} \mid J_{I_2} \subseteq I_2 \} \) and \( m_2 = |C_{I_2}| \). Then either \( m_2 \geq \sqrt{m} - 1 \), and we conclude taking \( S' = \{ h_1 \} \cup C_{I_2} \), or we set \( i_3 = m_1 + m_2 + 2 \). We proceed in this way until either we find a \( C_{I_k} \) whose cardinality is greater or equal to \( \sqrt{m} - 1 \), or we exhaust the set \( S^* \). Assume that the process ends after \( K \) steps and that all of the \( C_{I_k} \) have cardinality strictly smaller than \( \sqrt{m} - 1 \): then, we consider the set \( \{ h_{i_1}, \ldots, h_{i_K} \} \). Notice that, since \( \text{diam}(I_k) = 2\tau \) and \( \text{diam}(J_{i_1}) < \tau \) for all \( i = 2, \ldots, m \) we deduce that \( d_{\tau}(\text{Ax}(h_{i_k}), \text{Ax}(h_{i_j})) > 0 \) if \( k \neq j \). Therefore, the collection \( \{ \text{Ax}(h_{i_1}), \ldots, \text{Ax}(h_{i_K}) \} \) is a collection of pairwise disjoint axes, such that each element in the collection has a non-empty intersection with \( \text{Ax}(h_{i_1}) \); so \( S' = \{ h_1 \} \cup \{ h_{i_1}, \ldots, h_{i_K} \} \) is a \( \text{pS-family} \) sparse with respect to \( h_1 \) and such that \( \tau(h') = \tau \) for any \( h' \in S' \). Observe that \( K > \frac{m-1}{\sqrt{m}-1} \geq \sqrt{m} \), so \( |S'| \geq \sqrt{m} + 1 \). \( \square \)

We now show how to extract a free subgroup of rank \( r \geq m' - 1 \) from a \( \text{pS-family} \) \( S' = \{ h_1, \ldots, h_{m'} \} \) whose axes are in configuration (a) or (b).

**Lemma 1.12.** Let \( S' = \{ h_1, \ldots, h_{m'} \} \subseteq G \) be a \( \text{pS-family} \) with \( \tau(h_i) = \tau \) for all \( i \), agglomerated with respect to \( h_1 \). Then, \( \langle h_1^4, \ldots, h_{m'}^4 \rangle \) is a Schottky subgroup of \( G \).

**Proof.** Let \( I \subseteq \text{Ax}(h_{i_1}) \) be a segment with \( \text{diam}(I) \leq 2\tau \) containing all the intersections \( J_{I,i} = \text{Ax}(h_{i_1}) \cap \text{Ax}(h_i) \), let \( J_{i,j} = \text{Ax}(h_i) \cap \text{Ax}(h_j) \) and let us denote by \( T' \subseteq T \) the subset:

\[
T' = I \cup \left( \bigcup_{i,j \geq 2, i 
eq j} J_{i,j} \right)
\]

Since \( J_{i,j} \neq \emptyset \) and is contained into \( I \) for all \( i \neq 1 \), we conclude that \( T' \) is a connected subtree of \( T \). Moreover, since \( \tau(h_i) = \tau \) for all \( i \), by definition of agglomerated, \( \text{pS-family} \), we have \( \text{diam}(T') \leq \text{diam}(I) + 2 \max_{i,j} \text{diam}(J_{i,j}) < 4\tau \). It follows that the sets \( X_i = \{ v \mid p_i(v) \notin T' \} \) are joint domains of attraction for the collection \( \{ h_i^4 \}_{i=1,\ldots,m'} \). Actually, \( X_i \cap X_j = \emptyset \) for any \( i \neq j \) and if \( v \in X \setminus X_i \) the projection of \( v \) on \( \text{Ax}(h_i) \) is contained in \( T' \), as \( \tau(h_i^\pm) = 4\tau > \text{diam}(T') \); hence the projection of \( h_i^\pm(v) \) on \( \text{Ax}(h_i) \) lies in \( \text{Ax}(h_i) \setminus T' \), i.e. \( h_i^\pm(v) \in X_i \). \( \square \)

**Lemma 1.13.** Let \( S' = \{ h_1, \ldots, h_{m'} \} \subseteq G \) be a \( \text{pS-family} \) with \( \tau(h_i) = \tau \) for all \( i \), which is sparse with respect to \( h_1 \). Then, \( \langle h_2, \ldots, h_{m'} \rangle \) is a Schottky subgroup of \( G \).

**Proof.** For each \( i = 2, \ldots, m' \) define \( X_i = \{ v \in T \mid p_i(v) \notin J_{1,i} \} \). By definition the projections of the \( X_i \)'s on \( \text{Ax}(h_{i_1}) \) are disjoint; so, if \( v \in X \setminus X_i \), then its projection on \( \text{Ax}(h_i) \) lies in \( J_{1,i} \). As \( \tau(h_i) > \text{diam}(J_{1,i}) \) it follows that \( h_i^\pm(v) \in X_i \). \( \square \)
1.4.2. \textit{pS-families with disjoint axes.}

We now study the case where the axes of the family $\mathcal{S} = \{h_1, \ldots, h_m\}$ are all disjoint. To properly describe this case, let us introduce some notation. Let $p_i : T \to \text{Ax}(h_i)$ be the projection, let $\mathcal{A} = \bigcup_{i=1}^{m} \text{Ax}(h_i)$ and let $\overline{\mathcal{T}}_S$ be the convex hull of $\mathcal{A}$.

We shall call the \textit{nerves} of $\mathcal{S}$ the connected components $N_S(j)$ of $\overline{\mathcal{T}}_S \setminus \mathcal{A}$. We remark that, if $\text{Ax}(h_i) \cap N_S(j) \neq \emptyset$, then the intersection consists of a single vertex $n_i(j)$, which coincides with the projection $p_i(\text{Ax}(h_k))$ for each $\text{Ax}(h_k)$ whose intersection with $N_S(j)$ is non-empty; we shall call it the \textit{i-nervertex} of $N_S(j)$.

We will focus on 2 particular subcases:

(c) the case where for each $i = 1, \ldots, m$ the subset $J_i = \bigcup_{j \neq i} \{p_i(\text{Ax}(h_j)) \mid j \neq i\}$ of the projections on $\text{Ax}(h_i)$ of all the other axes is included in a subsegment $I_i \subset \text{Ax}(h_i)$ of length strictly less than $\tau$ (see Fig. 4);

(d) the case where there are precisely $m - 1$ nerves $N_A(1), \ldots, N_A(m - 1)$ and for any $j = 1, \ldots, m - 1$ the nerve $N_A(j)$ coincides with the geodesic segment connecting $\text{Ax}(h_j)$ to $\text{Ax}(h_{j+1})$ (see Figure 5, disregarding the dotted lines and the $\text{Ax}(h_i h_i^{-1})$'s).

We shall refer to such configurations as a \textit{disjoint family with small projections} (in case (c)) and a \textit{disjoint sequential family} (in case (d)).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure3.png}
\caption{Joint domains of attraction as in Lemma 1.12}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure4.png}
\caption{Disjoint pS-family with small projections}
\end{figure}
Again, the next Proposition shows that the configurations (c) and (d) are frequent.

**Proposition 1.14.** Let $S = \{h_1, \ldots, h_m\}$ be a $pS$-family with pairwise disjoint axes. Then, there exists a subset $S' \subseteq S$ of cardinality $m' \geq \sqrt{m}$ which is either a disjoint $pS$-family with small projections or a disjoint sequential $pS$-family.

To better handle in the proof of Proposition 1.14 the possible configurations of the axes of a general $pS$-family $S = \{h_1, \ldots, h_m\}$ with pairwise disjoint axes, let us define the graph $\Gamma_S$ of the nerves of $S$, as follows: its vertex set $V(\Gamma_S)$ is given by the nerves $N_S(j)$ and by the axes $Ax(h_i)$ of the elements of $S$, and there is an edge between $N_S(j)$ and $Ax(h_i)$ if and only if $N_S(j) \cap Ax(h_i) \neq \emptyset$.

Notice that, as it is build from the tree $T_S$, the graph $\Gamma_S$ is a tree. The leaves of $\Gamma_S$ are all vertices of type $Ax(h_i)$, for suitable $i$'s, and there are no edges connecting two vertices of type $N_S(j)$ or two vertices of type $Ax(h_i)$. Moreover, we clearly have $m + 1 \leq |V(\Gamma_S)| \leq 2m - 1$ (3)

with $|V(\Gamma_S)| = m + 1$ implying that $S$ is a disjoint family with small projections (namely, a single point $n_i(j)$ on each axis $Ax(h_i)$) and $|V(\Gamma_S)| = 2m - 1$ if and only if the degree of any vertex of type $N_S(j)$ is equal to 2.
Lemma 1.15. Let $\Gamma$ be a connected tree with $n$ vertices, and let $\ell$, $d$ denote respectively the number of leaves and the diameter: if $d$ is even, then $\ell \cdot d \geq 2(n - 1)$.

Proof of Lemma 1.15. Let $\gamma$ be a diameter of $\Gamma$, with central point $p$ (which is a vertex, since $d$ is even). Summing the number of vertices different from $p$ belonging to all segments $p\gamma_i$, where $\gamma_i$ runs over the leaves of $\Gamma$, we find $\ell \cdot \frac{d}{2} \geq n - 1$, which is the announced inequality.

Proof of Proposition 1.14. Let $n$ be the number of vertices of the tree $\Gamma_S$. Notice that, by the very definition of $\Gamma_S$, the diameter $d$ is realized by a geodesic $c$ connecting two leaves and is even. Hence, by the above lemma, either the number of leaves of $\Gamma_S$ is $\ell \geq \sqrt{(n - 1)}$ or its diameter is $d \geq 2\sqrt{(n - 1)}$. Assume that the first case holds: each leaf of $\Gamma_S$ corresponds to an axis $Ax(h_i)$ so let $S' \subset S$ be the subset of those elements whose axes correspond to the leaves of $\Gamma_S$. For any $h, h_i \in S'$ the projection of $Ax(h)$ on $Ax(h_i)$ is one point, namely the nerve $n_i(j)$ of the only nerve $N_S(j)$ connected to $Ax(h_i)$. Thus, the collection $S'$ is a disjoint pS-subfamily of $S$ with small projections and $|S'| = \sqrt{(n - 1)} \geq \sqrt{m}$, by (3). Assume now that we are in the second case. Let $Ax(h_{j_1}), \ldots, Ax(h_{j_{\ell + 1}})$ be the axes corresponding to the odd vertices of the geodesic $c$: then, the set $S' = \{h_{j_1}, \ldots, h_{j_{\ell + 1}}\}$ is a sequential disjoint pS-subfamily of $S$ with $|S'| = p + 1 > \frac{d}{2} \geq \sqrt{(n - 1)} \geq \sqrt{m}$, by (3). □

We now show how to extract free subgroups of rank $r \geq m'$ from any disjoint pS-family $S' = \{h_1, \ldots, h_{m'}\}$ whose axes are in one of the configurations (c) or (d).

Lemma 1.16. Let $S' = \{h_1, \ldots, h_{m'}\}$ be a disjoint pS-family with small projections and such that $\tau(h_i) = \tau$ for all $i$. Then $S'$ generates a Schottky group.

Proof. We set $X_i = \{v \in \mathcal{T} | p_i(v) \notin I_i\}$ for any $i = 1, \ldots, m'$. By assumption $p_i(Ax(h_k)) \in I_i$ for any $k \neq i$ and thus we conclude that $X_i \cap X_k = \emptyset$ (otherwise $\mathcal{T}$ would contain a loop). Now observe that if $v \in X \setminus X_i$, then $p_i(v)$ is in $I_i$. Since $\tau(h_i) = \tau > \text{diam}(I_i)$ and $p_i(h_i(v)) = h_i(p_i(v))$ we conclude that $h_i^{\pm 1}(v) \in X_i$. Thus $\{h_1, \ldots, h_{m'}\}$ is a Schottky subgroup. □

Lemma 1.17. Let $S' = \{h_1, \ldots, h_{m'}\}$ be a disjoint sequential pS-family such that $\tau(h_i) = \tau$ for all $i$. Then there exist $\varepsilon_2, \ldots, \varepsilon_{m' - 1} \in \{\pm 1\}$, such that the family $S'' = \{h_1, \ldots, h_k^{\pm \varepsilon_k}h_{k+1}^{-\varepsilon_k}, \ldots, h_{m'}\}$ for $k = 2, \ldots, m' - 1$, generates a Schottky subgroup.

Proof. For each $i = 2, \ldots, m' - 1$ choose $\varepsilon_i$ so that the segment $[n_i(i - 1), n_i(i)]$ is coherently oriented with the translation direction of $h_i^{\varepsilon_i}$. Now consider the set $S'' = \{h_1, \ldots, h_k^{\varepsilon_k}h_{k+1}^{-\varepsilon_k}, \ldots, h_{m'}\}$, and notice that it has a single nerve which is given by

$$[n_1(1), h_2^{\varepsilon_2}n_2(2)] \bigcup_{k=2}^{m'-1} [n_k(k-1), h_k^{\varepsilon_k}n_{k+1}(k)] \bigcup [h_{m'}^{\varepsilon_{m'-1}}n_{m'}(m'-1), n_{m'}(m'-1)]$$

(see Fig. [5]). We can then apply Lemma 1.14 to conclude that the group generated by $S''$ is a Schottky group. □
1.5. Proof of the Entropy-cardinality inequality, Theorem 1

By Proposition 1.9, for any generating set $S$ with $|S| = n$ we can find a pS-family $S$ of cardinality $m \geq \sqrt[3]{n}/\sqrt{2}$ such that $\tau(h) = \tau$ and $|h|_S \leq 20(k + 3)$ for all $h \in S$. Moreover, by the same theorem, either there exists $h_1 \in S$ such that $Ax(h_i) \cap Ax(h_1) \neq \emptyset$ for all $i$, or the axes of elements in $S$ are pairwise disjoint. If the axes of all the elements in $S$ intersect $Ax(h_1)$, we know by Proposition 1.11 that there exists a pS-subfamily $S' \subset S$ which either is agglomerated with cardinality $m' \geq \sqrt{m}$, or is sparse with respect to $h_1$ and has cardinality $m' \geq \sqrt{m} + 1$. In both cases, we deduce from Lemmas 1.12 and 1.13 a free Schottky subgroup $H$ of rank $m' \geq 32\sqrt{n}/4\sqrt{2}$, generated by elements having $S'$-length at most 4.

On the other hand, if the axes of $S$ are pairwise disjoint, by Proposition 1.14 there exists a disjoint, pS-subfamily $S' \subset S$ of cardinality $m' \geq \sqrt{m}/\sqrt{2}$, generated by elements having $S'$-length at most 4. Hence,

$$\text{Ent}(G, S) \geq \text{Ent}(H, d_S) \geq \frac{1}{80(k + 3)} \text{Ent}(H, S') \geq \frac{1}{80(k + 3)} \log (2m' - 1)$$

which gives the announced inequality.

□

2. Applications

Recall that, given a marked group $(G, S)$, a complete set of relators for $G$ is a finite subset $R$ of the free group on $S$ such that $G \cong \mathbb{F}(S)/\langle\langle R \rangle\rangle$, where $\langle\langle R \rangle\rangle$ denotes the normal subgroup generated by the elements in $R$.

A first consequence of the Entropy-Cardinality inequality is a general, finiteness theorem for abstract groups admitting acylindrical splittings, which has an interest in its own:

**Theorem 2.1.** The number of isomorphism classes of marked groups $(G, S)$ admitting a non-elementary, $k$-acylindrical splitting and a complete set of relators of length less than $\ell$, satisfying $\text{Ent}(G, S) \leq E$ is finite, bounded by an explicit function $N(k, \ell, E)$.

**Proof of Theorem 2.1.** The Entropy-Cardinality inequality yields a corresponding bound on the cardinality of $S$ in terms of the acylindricity constant and of $E$:

$$|S| \leq \left(1 + e^{E(80k+3)}\right)^{32} = n(k, E)$$

Now, the number of possible presentations by relators of length smaller than $\ell$ on an alphabet $A$ of at most $n$ letters is roughly bounded by $2^{n^2 + n\ell} \times$ the number of subsets $S$ of $A$, times the number of subsets $R$ of $A':$ this gives the announced bound of the number of marked groups $(G, S)$ by the function $N(k, \ell, E) = 2^{n(k, E)+n(k, E)^\ell}$. □

In the following, we will make use of these basic facts about amalgamated products and HNN-extensions over malnormal subgroups (the proof of Lemma 2.2(i) and (ii) can be found, with the original terminology, in [Ka-S1], Section §5; the case of HNN-extensions is analogous, cp. [Cer2] for more details):
Lemma 2.2. Let $A \ast_C B$ be the amalgamated product of two groups $A, B$. Assume that $C$ is a proper subgroup having index greater than $2$ in $A$ or in $B$, and let $\iota_A : C \rightarrow A$ and $\iota_B : C \rightarrow B$ be the natural inclusions:

(i) if one between $\iota_A(C), \iota_B(C)$ is malnormal in the respective group, then $A \ast_C B$ is a $1$-step malnormal amalgamated product, and has a non-elementary $2$-acylindrical splitting;

(ii) if both $\iota_A(C)$ and $\iota_B(C)$ are malnormal in the respective groups, then $A \ast_C B$ is a $0$-step malnormal amalgamated product, and has a non-elementary $1$-acylindrical splitting.

Lemma 2.3. Let $A \ast_\varphi = \langle A, t \mid t^{-1}ct = \varphi(c) \rangle$ be the HNN-extension of $A$ with respect to an isomorphism $\varphi : C_+ \rightarrow C_-$ between two subgroups one of which has index at least two in $A$, and let $\iota : C_+ \rightarrow A \ast_\varphi$ be the natural inclusion:

(i) if $\iota(C_-)$ is malnormal in $A \ast_\varphi$, then $C_+$ and $C_-$ are malnormal and conjugately separated in $A$;

(ii) if the subgroups $C_+, C_-$ are malnormal and conjugately separated in $A$, then $A \ast_\varphi$ is a $0$-malnormal HNN-extension, hence it, admits a non-elementary $1$-acylindrical splitting.

2.1. Quasiconvex groups of $\delta$-hyperbolic and $CAT(0)$-spaces.

Recall that an action of a group $G$ on a $\delta$-hyperbolic or $CAT(0)$-space $X$ is called quasiconvex if there exists an orbit $S = Gx_0$ which is a $D$-quasi-convex subset of $X$ (i.e. all the geodesics joining two points $x_1, x_2 \in S$ are included in the closed $D$-neighborhood of $S$), for some $D > 0$. One also says that $G$ is a quasi-convex group of $X$.

Proof of Corollary 2. Any marked $\delta$-hyperbolic group $(G, S)$ possesses a complete set of relators of length $\ell \leq (4\delta + 6)$ (see for instance [Br-Ha], Chapter III.$\Gamma$, Proposition 2.2): the conclusion then immediately follows from Theorem 2.1 \(\square\)

Proof of Corollary 3. Let $G$ act on a proper, geodesic space $(X, d)$, and let $x_0 \in X$ a point with $D$-quasi-convex orbit. By a classical argument, the set $S = \{g \in G \mid d(x_0, g.x_0) \leq 2D + 1\}$ generates $G$. Actually, for every $g \in G$, consider a geodesic $c : [0, \ell] \rightarrow X$ from $x_0$ to $g.x_0$, and orbit points $g(k).x_0$ such that $g(0) = 1$, $g([\ell]) = g$ and $d(c(k), g(k).x_0) \leq D$, given by the condition of $D$-quasiconvexity. Then, setting $\gamma(1) = g(1)$ and $\gamma(k) = g(k-1)^{-1}g(k)$, one has that the $\gamma(k)$’s are in $S$ and $g = \gamma(1) \cdot \gamma(2) \cdots \gamma(\ell)$. By construction, we also have

$$\frac{1}{2D + 1} d(x_0, g.x_0) \leq |g|_S \leq d(x_0, g.x_0) + 1$$

(4)

therefore the marked group $(G, S)$ is $(2D + 1, 1)$-quasi-isometric to the orbit $G.x_0$, endowed with the distance $d$ induced by $X$; it follows that, in case (i), $(G, S)$ is $\delta'$-hyperbolic, for some $\delta' = \delta(D, \delta)$. On the other hand, from the left-hand side of (4) we deduce that

$$Ent(G, S) \leq (2D + 1) \cdot Ent(G \cap X) \leq (2D + 1)E$$

The conclusion in case (i) then follows from Corollary 2.

In case (ii), we proceed similarly to [Br-Ha] Chapter III.$\Gamma$, Proposition 2.2, by
where the words $s$ are conjugates of words $r_i$ of $S$-length at most $8D+6$. Actually, choose again geodesics $c_i : [0, \ell_i] \to X$ from $x_0$ to $\sigma_i x_0$, and then orbit points $g_i(k) x_0$ with $g_i(0) = 1$, $g_i([\ell_i]) = \sigma_i$ for all $i$ and $d(c_i(k), g_i(k) x_0) \leq D$, provided by the $D$-quasiconvexity. We then consider the elements $\gamma_i(k) := g_i(k) r_i(k) g_i(k)^{-1}$ and $\mu_i(k) := g_i(k) r_i(k) g_i(k)^{-1}$ of $G$, and notice that
\[ d(x_0, \gamma_i(k) x_0) \leq 2D + d(c_i(k), c_i(k+1)) = 2D + 1. \]

and that, by the convexity of the metric of $X$,
\[ d(x_0, \mu_i(k) x_0) \leq 2D + d(c_i(k), c_{i+1}(k)) \leq 2D + d(x_0, \sigma_{i+1}(x_0) = 4D + 1 \]

Therefore, $\gamma_i(k) \in S$ and $|\mu_i(k)| \leq 4D + 2$, by \cite{[4]}; so, $\mu_i(k)$ can be represented by a word $\tilde{\mu}_i(k)$ on $S$ of length $\leq 4D + 2$. Accordingly, all the relations $\sigma_i s_{i+1} \sigma_i^{-1}$ can be decomposed as products of conjugates
\[ \sigma_i s_{i+1} \sigma_i^{-1} = g_i([\ell_i]) r_i([\ell_i]) g_i([\ell_i])^{-1} \cdots g_i(k) r_i(k) g_i(k)^{-1} \cdots g_i(1) r_i(1) g_i(1)^{-1} \]

where the words $r_i(k) := \tilde{\mu}_i(k+1) \gamma_{i+1}^{-1} \tilde{\mu}_i(k) \gamma_i(k)$ represent relations on $S$ whose $S$-lengths do not exceed $8D + 6$. The conclusion for case (ii) then follows from Theorem 2.1.

2.2. CAT(0)-spaces with negatively curved splittings.

We say that a locally CAT(0)-space $X$ \textbf{admits a splitting} if $X$ is isometric to the gluing $Y_1 \sqcup \phi Y_2$ of two locally CAT(0)-spaces $Y_1, Y_2$ along compact, locally convex, isometric subspaces $Z_i \hookrightarrow X_i$ via an isometry $\phi : Z_1 \to Z_2$; or if $X$ is isometric to the space $Y \sqcup \phi$ obtained by identifying two such subspaces $Z_i \subset Y$ to each other by an isometry $\phi$. The splitting is \textbf{non-trivial} if the corresponding splitting of $\pi_1(X)$ as an amalgamated product or HNN-extension is non-trivial. Notice that the space obtained by such gluings is always locally CAT(0) (cp. \cite{[Br-Ha], Prop. 11.6}). We will say that $X$ has a \textbf{negatively curved splitting} if the subspace $Z$ of $X$ obtained by identifying $Z_1$ to $Z_2$ has a neighbourhood $U(Z)$ in $X$ such that $U(Z) \setminus Z$ is a locally CAT(0) space for some $k > 0$.

The following fact is crucial to prove acylindricity for negatively curved splittings of CAT(0)-spaces, and we believe it is folklore; we will give a proof of it in Appendix \cite{[C]}. by completeness.

\textbf{Proposition C.1} Let $Z$ be a compact, locally convex subspace of a compact, complete locally CAT(0)-space $X$. Assume that $X$ is negatively curved around $Z$: then, $H = \pi_1(Z)$ is malnormal in $G = \pi_1(X)$.

\textbf{Proof of Theorem 4.} If $X$ admits a non-trivial, negatively curved splitting $X = X_1 \sqcup_{\phi} X_2$ or $X = X_0 \sqcup \phi$, along two isometric, locally convex subspaces $Z_i \cong Z$ identified to each other via an isometry $\phi$, then $\pi_1(X)$ splits as a non-trivial amalgamated product of the groups $G_i = \pi_1(X_i)$ over $H = \pi_1(Z)$, or as a non-trivial
HNN-extension of $G_0 = \pi_1(X_0)$ along subgroups $H_i = \pi_1(Z_i)$, via the isomorphism $\phi_* : H_1 \rightarrow H_2$. The subgroups $H_i$ and $H$, are malnormal respectively in each $G_i$ and in $G$, by Proposition \text{[C.1]} Moreover, each $H_i$ does not have index 2 in $G_i$, or it would be normal and malnormal, hence trivial (since the splitting is supposed to be non-trivial). Therefore, $\pi_1(X)$ admits a non-elementary 1-acylindrical splitting.

By Corollary \text{3}, we deduce that $\pi_1(X)$ belongs to a finite class of groups, whose number is bounded by a universal function of $E$ and $D$. Since the locally $\text{CAT}(0)$-spaces are aspherical, we can conclude by Whitehead’s theorem the finiteness of the homotopy types.

Proof of Corollary \text{5}. It follows from the fact that, in dimension greater than 4, the homeomorphism type of closed, non-positively curved manifolds is determined by their homotopy type, from the solution of the Borel Conjecture by Bartels-Lück for $\text{CAT}(0)$-manifolds \text{[Ba-Lü]} Moreover, the works of Kirby-Siebenmann \text{[Ki-Si]} and of Hirsch-Mazur \text{[Hi-Ma]} on PL structures and their smoothings imply the finiteness of smooth structures in dimension $n \geq 5$ (cp. also \text{[Lan]}, Thm. 7.2).

The fact that the fundamental group determines the diffeomorphism type is well-known in dimension 2, and follows in dimension 3 from Perelman’s solution of the geometrization conjecture (we now know that any closed, negatively curved 3-manifold also admits a hyperbolic metric; so Mostow’s rigidity applies).

2.3. Two-dimensional orbifolds.

We recall shortly some basic facts about orbifolds; for a primer concerning 2-dimensional orbifolds we refer to \text{Sco} and \text{Thu}. Following Thurston \text{Thu}, a $n$-dimensional orbifold $O$ (without boundary) is a Hausdorff, paracompact space which is locally homeomorphic either to $\mathbb{R}^n$, or to the quotient of $\mathbb{R}^n$ by a finite group action; similarly, $n$-orbifolds with boundary also have points whose neighbourhood is homeomorphic to the quotient of the half-space $\mathbb{R}^n_+$ by a finite group action. For the sake of simplicity we shall consider uniquely compact 2-dimensional orbifolds with \text{conical singular points}, that is points which have a neighborhood modelled on the quotient of $D^2$ by a finite cyclic group. Nevertheless, it follows from the description of singularities in \text{Sco} and \text{Thu} that given a general compact 2-orbifold there exists a canonically constructed double cover which has only conical singularities; this cover is obtained by doubling the underlying space along the \text{reflector lines}, duplicating the conical singular points and trasforming the so called \text{corner reflectors} into conical singular points.

We shall denote by $O = O(g, h; p_1, \ldots, p_k)$ the compact 2-orbifold having as underlying topological space a compact surface $|O|$ of genus $g \in \mathbb{Z}$ (using negative values for the genus of non-orientable surfaces), $h$ boundary components and $k$ singular points $x_1, \ldots, x_k$ of orders $p_1, \ldots, p_k$. By the classification of compact 2-orbifolds given by Thurston in \text{Thu}, Ch. 13, an orbifold with conical singular points is completely determined by its underlying topological space together with the number and the orders of its singular points. In view of this fact, we shall say that two smooth compact 2-orbifolds with conical singularities are \text{isomorphic} if they have the same underlying surface, the same number of singular points, and the same order at each singular point, up to permutations.

For a formal definition of the orbifold fundamental group, we refer to \text{BMP}; to our purposes, it will be sufficient to recall that the orbifold fundamental group of $O = O(g, h; p_1, \ldots, p_k)$ admits one of the following presentations:
good orbifolds (usually referred as distinguish between those orbifolds which are finitely covered by a compact surface from [Thu] (Theorem 13.3.6) that if depending on the sign of the genus characteristic and $\text{LCM}$ over, the Euler characteristic a good orbifold $O$ is endowed with any them strictly greater than 3.

orbifolds of hyperbolic type Riemannian metric. We shall accordingly call compact 2-orbifolds with negative simply connected surface– admits a $\pi$ structure $O$ in the introduction, as Ent($O$) with constant curvature). The entropy of $O$ form $O$ by such groups, and we will need an ad-hoc computation to conclude. We will denote cardinality inequality. In the second case, acylindrical splittings are not available for will show in detail in Proposition A.1 of Appendix A, and we can use the Entropy-In fact, in the first case, the orbifold groups admit a 2-acylindrical splitting, as we

$(a)$ non-triangular, Riemannian 2-orbifolds of hyperbolic type; $(b)$ triangular 2-orbifolds of hyperbolic type.

In fact, in the first case, the orbifold groups admit a 2-acylindrical splitting, as we will show in detail in Proposition [A.1] of Appendix [A] and we can use the Entropy-cardinality inequality. In the second case, acylindrical splittings are not available for such groups, and we will need an ad-hoc computation to conclude. We will denote by $\mathcal{O}_h^{b,n}(E, D)$ and $\mathcal{O}_h^{b,n}(E, D)$ the classes of compact, Riemannian 2-orbifolds of negative orbifold Euler characteristic with entropy and diameter bounded by $E$ and $D$, which fall, respectively, in cases $(a)$ and $(b)$.

$$
\pi_1^{orb}(O) = \left\{ a_1, b_1, \ldots, b_r, c_1, \ldots, c_k, d_1, \ldots, d_h \mid \prod_{i=1}^{g}[a_i, b_i] \cdot \prod_{j=1}^{p} c_j \cdot \prod_{\ell=1}^{h} d_\ell = 1 \right\}
$$

$$
\pi_1^{orb}(O) = \left\{ a_1, \ldots, a_{|g|}, c_1, \ldots, c_k, d_1, \ldots, d_h \mid \prod_{i=1}^{g|a_i|^2} a_i \cdot \prod_{j=1}^{k} c_j \cdot \prod_{\ell=1}^{h} d_\ell = 1 \right\}
$$
depending on whether the genus is positive or negative. The generators $a_i$'s, $b_i$'s, $d_i$'s are the fundamental system of generators of $|O|$, with $d_\ell$ corresponding to the $l$-th boundary compoent, whereas the $c_i$'s represent the generators of the isotropy groups associated to the singular points of $O$.

The usual Euler characteristic can be generalized in a natural way to the case of compact 2-orbifolds ($\text{Sc}$), and in the case we are considering the formula reads:

$$
\chi_{orb}(O(g, h; p_1, \ldots, p_k)) = \chi(|O|) - \sum_{i=1}^{k} \left( 1 - \frac{1}{p_i} \right) = \begin{cases} 2 - 2g - h - \sum_{i=1}^{k} \left( 1 - \frac{1}{p_i} \right) & g < 0 \\ 2 - |g| - h - \sum_{i=1}^{k} \left( 1 - \frac{1}{p_i} \right) & g > 0 \end{cases}
$$
depending on the sign of the genus $g$. The orbifold Euler characteristic is useful to distinguish between those orbifolds which are finitely covered by a compact surface (usually referred as good orbifolds) and those who are not (the bad ones): it follows from [Thu] (Theorem 13.3.6) that if $O(g, h; p_1, \ldots, p_k)$ has positive orbifold Euler characteristic and $\text{LCM}(p_1, \ldots, p_k) \cdot \chi_{orb}(O) \neq 1, 2$ then it is a bad orbifold. Moreover, the Euler characteristic a good orbifold $O$ determines the kind of geometric structure that can be given to $O$: that is, whether its universal cover $\tilde{O}$—which is a simply connected surface—admits a $\pi_1^{orb}(O)$-invariant spherical, flat or hyperbolic Riemannian metric. We shall accordingly call compact 2-orbifolds with negative Euler characteristic orbifolds of hyperbolic type.

Among compact 2-orbifolds of hyperbolic type there is a particular family whose groups are generated by symmetries of hyperbolic triangles along their edges: these groups are called triangle groups, and the corresponding orbifolds are those of the form $O(0, 0; p, q, r)$, with $p = 2$, $q = 3$, $r \geq 5$ or with $p, q, r \geq 3$ and at least one of them strictly greater than 3.

In Theorem [6] we consider general Riemannian 2-orbifolds: that is, good compact 2-orbifolds $O$ (with conical singularities) whose orbifold universal cover $\tilde{O}$ is endowed with any $\pi_1^{orb}(O)$-invariant Riemannian metric (that is, not necessarily with constant curvature). The entropy of $O$ is correspondingly defined, as explained in the introduction, as $\text{Ent}(O) = \text{Ent}(\pi_1^{orb}(O) \rhd \tilde{O})$.

Notice that compact, two-dimensional orbifolds with zero orbifold Euler characteristic yield only finitely many isomorphism classes (cp. [Thu], Theorem 13.3.6), as it can be checked directly from the Euler characteristic formula, without any assumption on their entropy and diameter.

The proof Theorem [6] will then split in two separate cases:

(a) non-triangular, Riemannian 2-orbifolds of hyperbolic type;
(b) triangular 2-orbifolds of hyperbolic type.
Proof of finiteness of isomorphism types in $\Theta^2_{h,nt}(E,D)$.

The orbifold groups in this class admit a 2-acylindrical splitting, cf. Proposition A.1 of Appendix A. Then, for any orbifold $\mathcal{O} \in \Theta^2_{h,nt}(E,D)$ we choose $\tilde{x}_0 \in \mathcal{O}$, and apply to $G = \pi_1^{orb}(\mathcal{O})$ and to the open ball $U = B_\delta(\tilde{x}_0, 2D + \epsilon)$ of $\mathcal{O}$, for arbitrary $\epsilon > 0$, the following, classical result (see, for instance, Serre, p.30):

Lemma 2.4. Let $G$ act by homeomorphisms on a path-connected, simply connected topological space $X$, and let $U$ be a path-connected open set such that $G.U = X$. Let $S = \{ s \in G \mid sU \cap U \neq \emptyset \}$ and $T = \{ (s_1, s_2) \in S \times S \mid U \cap (s_1 U) \cap (s_1 s_2 U) \neq \emptyset \}$. Then $S$ generates $G$, and $G \cong F(\Sigma)/\langle \langle \Theta \rangle \rangle$ where $\Sigma$ is the set of symbols $\{ x_s \mid s \in S \}$, and $\Theta$ is the set of words on $S \cup S^{-1}$ given by $\{ x_{s_1}x_{s_2}x_{s_1^{-1}} \mid (s_1, s_2) \in T \}$.

(Notice that if $(s_1, s_2) \in T$, then $s_1 s_2 \in S$ so $x_{s_1 s_2}$ makes sense).

For $M = 2D + \epsilon$, the lemma yields a finite generating set $S_M$ of $G$, such that $d(x_0, g \cdot x_0) < M$ for all $g \in S_M$, which we call $M$-short generators of $G$ at $x_0$, and a triangular presentation of the group: that is, such that the group of relations is generated, as a normal subgroup of $F(S_M)$, by relators of length at most 3.

Since $d(x_0, g \cdot x_0) \leq M \cdot |g|_{S_M}$, we have $\text{Ent}(\pi_1(\mathcal{O}), S_M) \leq M \cdot \text{Ent}(\mathcal{O}) \leq ME$. Letting $\epsilon \to 0$, we deduce from Theorem 1 that $|S_M| \leq (e^{600DE} + 1)^{32} = N(E,D)$. As in the proof of Theorem 2.1 observe now that the number of possible triangular presentations that can be build with letters from some subset $S$ of an alphabet of cardinality $N$ does not exceed $2^{N+3}$; therefore, the number of such orbifold groups is bounded from above by $2^{N(E,D)} + 2^{N(E,D)^3}$. To conclude, remark that $\pi_1^{orb}(\mathcal{O})$ determines the isomorphism class of $\mathcal{O}$ for closed orbifolds, while for orbifolds with boundary the isomorphism class is determined by the orbifold group and the number of boundary components; in any case, there are a finite number of non-isomorphic 2-orbifolds for any given group $\pi_1^{orb}(\mathcal{O})$. \hfill \Box

Proof of finiteness of isomorphism types in $\Theta^2_{h,ml}(E,D)$.

In this case, we call $G_{p,q,r} = \pi_1^{orb}(\mathcal{O}(0,0;p,q,r))$ the triangle orbifold group and consider the fixed point $\tilde{x}_0 \in \mathcal{O}(0,0;p,q,r)$ of the torsion element $c$ of order $r \geq 4$. To evaluate the Poincaré series of $G_{p,q,r}$ at $\tilde{x}_0$, we need the following

Lemma 2.5. Let $G_{p,q,r} = \langle a, b, c \mid abc = a^p = b^q = c^r = 1 \rangle$ where $3 \leq p \leq q \leq r$ and $r \geq 4$ be the fundamental group of a triangular orbifold of hyperbolic type. The following set of elements of $Z_p * Z_r$ naturally injects into $G_{p,q,r}$:

$$\mathcal{W}_r = \left\{ c^{j_1}a c^{j_2}a \cdots c^{j_m}a \mid m \in \mathbb{N}, 2 \leq j_i \leq \frac{r}{2} \right\}$$

(By a slight abuse of notation, we will use $\mathcal{W}_r$ and the reduced forms $c^{j_1}a \cdots c^{j_m}a$ to denote both the elements of $Z_p * Z_r$ and their images in $G_{p,q,r}$).

Assuming this lemma for a moment, we can compute:

$$P_s(G_{p,q,r}, \tilde{x}_0) \geq \sum_{g \in \mathcal{W}_r} e^{-s d(\tilde{x}_0, g \cdot \tilde{x}_0)} = \sum_{n} \sum_{2 \leq j_i \leq \frac{r}{2}} e^{-s d(\tilde{x}_0, c^{j_1}a \cdots c^{j_m}a \cdot \tilde{x}_0)} \quad (5)$$

Moreover, since we are considering triangular orbifolds of diameter bounded by $D$ and $\tilde{x}_0$ is the fixed point of $c$, we have:

$$d(\tilde{x}_0, c^{j_1}a \cdots c^{j_m}a \cdot \tilde{x}_0) \leq nd(a,a,o) + \sum_{i=1}^{n} d(a, c^{j_i}a) \leq 2nD$$
which plugged into (5) yields:

\[ P_s(G_{p,q,r}, \tilde{x}_0) \geq \sum_{n} \sum_{2 \leq j \leq \ell} e^{-2sD} \geq \sum_{n>0} \left( \frac{T}{2} - 1 \right) e^{-s2D} \]

Since \( P_s(G_{p,q,r}, \tilde{x}_0) \) converges for all \( s > E \), this shows that \([\frac{T}{2} - 1] \leq e^{2ED} \).

As \( r \geq q \geq p \) this proves the finiteness of the fundamental groups and of isomorphisms classes of triangular orbifolds with bounded entropy and diameter. □

**Proof of Lemma 2.5** By the canonical presentations of the compact, 2-orbifold groups recalled before, we know that \( G_{p,q,r} = \mathbb{Z}_p \ast \mathbb{Z}_r / \langle \langle (ca)^q \rangle \rangle \), since \( b^{-1} = ca \).

We start with the case where \( q \geq 4 \). Following [Ly-Sch], we call a word on \( S = \{a, a^{-1}, c, c^{-1}\} \) with normal form \( w = a^{p_1}c^{q_1} \cdots a^{p_n}c^{q_n} \), possibly with \( p_1 = 0 \) or \( q_n = 0 \), *weakly cyclically reduced* if the last syllable of \( w \) (that is, \( c^{q_n} \), if \( q_n \neq 0 \)) is different from the first one. If \( R \) denotes the set of weakly cyclically reduced conjugates of \( (ca)^q \), then the quotient \( G_{p,q,r} = \mathbb{Z}_p \ast \mathbb{Z}_r / \langle \langle R \rangle \rangle \) satisfies the \( C'(\frac{1}{6}) \) condition for free products (every prefix of an element \( r \in R \) which is a piece has syllable length \( \frac{1}{2} \) smaller than \( \frac{1}{3} \ell(r) \)). It follows from small cancellation theory that any element \( w \in \mathbb{Z}_p \ast \mathbb{Z}_r \) belonging to \( \langle \langle R \rangle \rangle \) has a normal form which contains as a subword a prefix \( r_0 \) of some element in \( R \) of syllable length \( \ell(r_0) \geq 5 \). It is straightforward to check that none of the elements of \( \mathcal{W}_r \mathcal{W}_r^{-1} \) contains such a subword, hence \( \mathcal{W}_r \) injects into \( G_{p,q,r} \).

The argument when \( p = q = 3 \) and \( r \geq 4 \) is the similar, with the difference that in this case the set \( R \) of weakly cyclically reduced conjugates of \( (ca)^q \) does not satisfy the \( C'(\frac{1}{6}) \) condition, but conditions \( C'(\frac{1}{4}) \) and \( T(4) \) (given \( r_1, r_2, r_3 \in R \), at least one among \( r_1r_2, r_2r_3, r_3r_1 \) is semi-reduced, cf. [Ly-Sch]). Nevertheless, if \( R \) satisfies \( C'(\frac{1}{4}) \) and \( T(4) \), it is still true that any reduced word \( w \in \langle \langle R \rangle \rangle \) contains a prefix \( r_0 \) of an element in \( R \) as subword, with syllable length \( \ell(r_0) \geq 4 \) ([Ly-Sch], Ch.5, Thm.4.4). We then proceed as before and deduce that \( \mathcal{W}_r \) injects into \( G_{p,q,r} \). □

### 2.4. Non-geometric 3-manifolds.

In this section we will prove the finiteness results Theorem 7 and Corollary 8 for non-geometric Riemannian 3-manifolds with bounded entropy and diameter. We will first show the finiteness of fundamental groups of manifolds in the class \( \mathcal{M}^{\text{ng}}(E, D) \), and then explain how to deduce Corollary 8 from Theorem 7.

The proof of the finiteness of the isomorphism classes of fundamental groups in \( \mathcal{M}^{\text{ng}}(E, D) \) relies on the fact that the fundamental group of every non-geometric 3-manifold, closed or compact with non-spherical boundary components, admits a non-elementary 4-acylindrical splitting; this fact was proved by [Wi-Za] (see also [Ce2] for further details). The splitting is relative either to the decomposition of \( \pi_1(X) \) as a free product given by prime decomposition, or to the decomposition of \( \pi_1(X) \) as an amalgamated product over rank 2, abelian subgroups provided by the JSJ-decomposition, for irreducible manifolds, cf. [Ce-Sa1]. Section 4.4. We can then apply the Entropy-Ccardinality inequality to the classical triangular presentation of \( \pi_1(X) \) given by the Lemma 2.4 and proceed as in the proof of Theorem 6 case (a).

---

\(^{10}\) Notice that the small cancellation theory on free products differs from general cancellation theory, the relevant length and notion of *piece* being given by the syllable length and by the subdivision in syllables provided by the normal form.
Proof of Theorem 7. For any $X \in \mathcal{M}^0_n(E, D)$ pick $x \in X$ and let $M = 2D + \epsilon$. Then, consider the set $S_M$ of $M$-short generators at $x$. As $\pi_1(X)$ has a 4-acylindrical splitting, it follows from from Theorem 1 that
\[
E \geq \text{Ent}(X) \geq \frac{1}{M} \text{Ent}(\pi_1(X), S_M) \geq \frac{1}{560M} \log \left( \frac{\sqrt{\# S_M}}{4} - 1 \right)
\]
and letting $\epsilon \to 0$ we obtain $|S_M| \leq (e^{120ED} + 1)^{32} = N(E, D)$. Therefore, $X$ admits a triangular presentation on a generating set of cardinality at most $N(E, D)$. Since the number of possible triangular presentations that can be build with letters from some subset $S$ of an alphabet $A$ of $N$ letters does not exceed $2^{N+N^3}$, this concludes the proof. \hfill \Box

Now, the following statement is consequence of several results of 3-dimensional geometry and topology. Since it relies on facts which are now folklore (and are sometimes only sketched in literature), we will provide a full proof in Appendix B, together with all the references and the 3-dimensional topology tools needed for it.

**Theorem 2.6.** There exist only finitely many pairwise non-diffeomorphic, compact orientable 3-manifolds without spherical boundary components with given fundamental group $G$.

Corollary 8 then follows from the fact that the fundamental groups of Riemannian manifolds in the class $\mathcal{M}^0_n(E, D)$ belong to a finite collection.

2.5. Ramified coverings.

We briefly recall the construction of a cyclic ramified covering of a hyperbolic manifold, according to Gromov-Thurston [Gr-Th].

Let $Z_0$ be a two-sided hypersurface with boundary in some closed, orientable $n$-manifold $X_0$, and call $R_0 = \partial Z_0$ the (possibly disconnected) boundary. Cut $X_0$ along $Z_0$, thus obtaining a topological, compact manifold $\hat{X}_0$ with boundary; the boundary is given by two copies $Z_0^-$, $Z_0^+$ of $Z_0$, with $Z_0^- \cap Z_0^+ = R_0$.

Then, consider the topological manifold $X_k$ obtained by taking $k$ copies $X_k$ of $\hat{X}_0$, for $i = 1, \ldots, k - 1$, and gluing $X_i$ to $\hat{X}_{i+1}$ by identifying the boundaries $Z_i^+$, $Z_{i+1}^-$. Finally, let $X_k$ be the closed manifold obtained by identifying $Z_k^+ \to Z_1^-$, and call $Z_i$ the boundaries so identified inside $X_k$.

The resulting manifold $X_k$ can be given a smooth structure with a smooth projection onto the initial manifold $p : X_k \to X_0$ which is a smooth $k$-sheeted covering outside the ramification locus $R = p^{-1}(R_0)$; the ramification locus $R$ is the boundary of each $Z_i$, and around $R$ the projection writes as $(x, z) \mapsto (x, z^k)$, with respect to suitable coordinates for $X_k$ and for $X_0$, identifying the (trivial) normal bundles of $R$ and $R_0$ to $R_0 \times D^2$.

Moreover, choosing $X_0$ hyperbolic and the submanifolds $Z_0$, $R_0$ totally geodesic in $X_0$, the new manifold $X_k$ can be given a singular, locally $\text{CAT}(-1)$-metric $g_k$ which makes of $\bigcup_i Z_i$ a totally geodesic (singular) hypersurface of $X_k$, with totally geodesic boundary $R$, and such that the restriction $p_{|X_k-R}$ is a Riemannian covering (cp. [Gr-Th]). Namely, $\bigcup_i Z_i$ looks like a $k$-paged book, consisting of $k$ copies of $Z$ joined together at $R$, each pair of consecutive pages forming an angle $2\pi$ and a locally convex subset of $X_k$. The singular metric, around the ramification submanifold $R$, can be written as
where \( r \) represents the distance to \( R \), and \( g_0 \) the hyperbolic metric of \( Z_0 \). (As shown in [Gr-Th], this metric can then be smoothed to obtain a true Riemannian metric \( g_0 \) of strictly negative curvature \( K(X_k) \leq -1 \), and even pinched around \(-1\), provided that the normal injectivity radius of \( R \) is sufficiently large).

We will call a Riemannian manifold obtained by choosing any Riemannian metric on such \( X_k \) (possibly with variable sectional curvature, of any possible sign, and not necessarily locally isometric to the base hyperbolic manifold \( X_0 \setminus R_0 \)), for any \( k \geq 2 \), a Riemannian ramified covering of \( X_0 \).

**Proof of Corollary 10** Let \( \hat{X}_{k-1} \subset X_k \) be the union of the \( \hat{X}_i \)’s, for \( 2 \leq i \leq k \). By Van Kampen theorem, \( G_k = \pi_1(X_k) \) can be written as the amalgamated product \( G_1 *_{E,D} G_{k-1} \) of \( G_1 = \pi_1(\hat{X}_1) \) and \( G_{k-1} = \pi_1(\hat{X}_{k-1}) \) along \( H = \pi_1(Z_k^- \cup Z_k^+) \) (which is immersed in \( G_{k-1} \) via the isomorphisms induced by the identification diffeomorphisms \( Z_k^- \simeq Z_k^1 \) and \( Z_k^+ \simeq Z_k^2 \)). As the image \( Z_1 \cup Z_k \) of \( Z_k^- \cup Z_k^+ \) in \( X_k \) is locally convex with respect to the singular, \( CAT(-1) \)-metric of \( X_k \) described above, the subgroup \( H \) is a malnormal subgroup of \( G \) by Proposition C.1 of Appendix C.

We conclude that \( G \) admits a 0-acylindrical splitting; moreover, as \( H \) has not index two in \( G \) (being malnormal) the splitting is non-elementary.

We can then apply the Entropy-Cardinality inequality to a triangular presentation of \( \pi_1(X) \), as in the proof of Theorems 6 and 7. Namely, we choose some point \( x \) for every \( X \in \mathcal{R}(E,D) \), we consider a triangular presentation of \( \pi_1(X) \) by the \( M \)-short generating sets \( S_M \) for \( M = 2D + \epsilon \), and as \( d(x, g \cdot x) \leq M \cdot \|g\|_{S_M} \) for all \( g \in S_M \), we deduce by Theorem 1 that \( |S_M| \leq N(E,D) \). Therefore, the number of possible fundamental groups in \( \mathcal{R}(E,D) \) is explicitly bounded in terms of \( E, D \). Since every \( X_k \) is an aspherical manifold admitting a \( CAT(-1) \)-metric, we infer the finiteness of homotopy types (and of diffeomorphisms types in dimension different from 4) as explained in the proof of Corollary 5.

\[ \square \]

2.6. **Higher dimensional graphs and cusp decomposable manifolds.**

**Proof of Corollary 16** By [FLS], Proposition 6.4, we know that the fundamental groups of irreducible higher graph manifolds admit 2-acylindrical splittings. On the other hand, the fundamental groups of cusp decomposable manifolds possess non-elementary, 1-acylindrical splittings. Actually, the decomposition of \( \pi_1(X) \) corresponding to the cusp decomposition is obtained by identifying the cusp subgroups, and these subgroups are malnormal in the fundamental group of each bounded cusp manifold with horoboundary they belong to, and conjugately separated if they belong to the same group (cf. delH-AWe, or just apply Proposition C.1 of Appendix C to the whole, convex, cusp neighbourhoods). As a consequence, by Lemmas 2.2 and 2.3, the fundamental group of a cusp decomposable manifold can be presented as a 0-step malnormal amalgamated product or HNN-extension, and admits a 1-acylindrical splitting. Moreover, the Bass-Serre tree of the splittings corresponding to the decompositions of non-elementary higher graph or cusp decomposable manifolds is, by definition, neither a vertex nor a line. Therefore, the number of fundamental groups of manifolds in the classes \( \mathcal{R}^0(E,D) \) and \( \mathcal{C}(E,D) \) is finite, by the same argument used for Theorems 6 and 7. Since higher graph and cusp decomposable manifolds are aspherical (cf. [FLS], Corollary 3.3 and [Ngu]) we
immediately infer the finiteness of the homotopy types in $G^\partial(E,D)$ and $C(E,D)$. By the topological and differential rigidity properties of higher graph and cusp decomposable manifolds recalled in the introduction, we also deduce the finiteness of diffeomorphism types in $G(E,D)$ and $C(E,D)$. □

Remark 2.7. The finiteness result holds, more generally, also for the diffeomorphism types of non-irreducible high dimensional graph manifold with boundary, admitting at least one internal walls with transverse fibers, and whose boundary components do not belong to surface pieces and (see [FLS], Sections §5).

Appendix A. Acylindrical splittings of hyperbolic 2-orbifolds

Proposition A.1. Let $\mathcal{O}$ be a compact 2-orbifold of hyperbolic type with conical singularities. If $\mathcal{O}$ is not a hyperbolic triangular orbifold, then $\pi_1^{orb}(\mathcal{O})$ admits a 2-acylindrical splitting.

Proof. First notice that all compact 2-orbifolds of hyperbolic type with non-empty boundary have orbifold fundamental group which is a non-trivial free product of finite and infinite cyclic groups, hence $\pi_1^{orb}(\mathcal{O})$ has a 0-acylindrical, splitting in this case. Moreover, the splitting is necessarily non-elementary (otherwise the orbifold would be a disc with two singular points of order two, and it would not have negative orbifold Euler characteristic).

Assuming then that $\mathcal{O}$ is a compact, 2-orbifold of genus $g$ of hyperbolic type without boundary, which is not a hyperbolic triangular orbifold. By the formula for the orbifold Euler characteristic one of the following holds:

- $g > 1$;
- $g = 1$ and $\mathcal{O}$ has at least one singular point;
- $g = 0$ and $\mathcal{O}$ has $m \geq 4$ singular points, at least one of which has order greater than 2;
- $g = -1$ and $\mathcal{O}$ has 2 singular points, one of which has order greater than 2;
- $g = -1$ and $\mathcal{O}$ has $m \geq 3$ singular points;
- $g = -2$ and $\mathcal{O}$ has at least one singular point;
- $g < -2$.

The proof then is obtained by cutting any such orbifold $\mathcal{O}$ into two 2-orbifolds with boundary, and using repeatedly the following

Lemma A.2. If $\mathcal{O} = O(g, h; p_1, ..., p_k)$ is a compact 2-orbifold with boundary of hyperbolic type, the infinite cyclic subgroups $\langle d_i \rangle$, corresponding to the boundary curves form a collection of malnormal, conjugately separated subgroups of $\pi_1^{orb}(\mathcal{O})$.

The lemma can be checked directly by looking at the aforementioned presentations of the orbifold fundamental group: it is sufficient to notice that the boundary curves are represented by primitive elements of infinite order in a non-trivial free product of cyclic groups, different from $\mathbb{Z}_2 \ast \mathbb{Z}_2$, which do not belong to the same conjugacy class. A more geometric justification to malnormality is that $\mathcal{O}$ can be given a geometric structure of a hyperbolic 2-orbifold with cusps, with the boundary subgroups $\langle d_i \rangle$ becoming the parabolic subgroups associated to the cusps.

Now, if $g \geq 1$, choose a simple closed curve $\delta$ which does not disconnect $|\mathcal{O}|$; after possibly modify the curve $\delta$ in order to avoid the singular points, cut $|\mathcal{O}|$ along
that curve. We obtain a new orbifold $\mathcal{O}'$ with genus $g - 1$ and two new boundary components $\delta_1, \delta_2$; clearly, $\chi_{orb}(\mathcal{O}') = \chi_{orb}(\mathcal{O}) < 0$. Since $\mathcal{O}'$ is an orbifold of hyperbolic type, the classes $d_1, d_2$, represented by $\delta_1$ and $\delta_2$ in the fundamental group of $\mathcal{O}'$, generate two conjugately separated, malnormal subgroups in $\pi_1^{orb}(\mathcal{O}')$, by Lemma A.1. Then, by Lemma 2.3 we know that $\pi_1^{orb}(\mathcal{O})$ is the HNN-extension $\pi_1^{orb}(\mathcal{O})*_{\phi}$ defined by the isomorphism $\varphi : \langle d_1 \rangle \to \langle d_2 \rangle$, $\varphi(d_1) = d_2$; this yields a 2-acylindrical splitting of $\pi_1^{orb}(\mathcal{O})$. By construction, the Bass-Serre tree of this splitting is neither a point nor a line, so the splitting is non-elementary.

Assume now that $g = 0$ and that $\mathcal{O}$ has at least $m \geq 4$ singular points, one of which of order $r \geq 3$. Consider a simple closed curve $\delta$ which separates $|\mathcal{O}|$ into two disc orbifolds $\mathcal{O}_1, \mathcal{O}_2$, each containing at least 2 singular points, with, let's say, $\mathcal{O}_1$ containing the singular point of order $r \geq 3$. Denoting by $d$ the classes represented by the boundary curve in each $\mathcal{O}_i$, the orbifold fundamental groups have presentations:

$$\pi_1^{orb}(\mathcal{O}_1) = \langle c_1, \ldots, c_k, d \mid c_1 \cdots c_k d = 1, c_1^{p_1} = \cdots = c_k^{p_k} = 1 \rangle$$

$$\pi_1^{orb}(\mathcal{O}_2) = \langle c_{k+1}, \ldots, c_m, d \mid c_{k+1} \cdots c_m d = 1, c_{k+1}^{p_{k+1}} = \cdots = c_m^{p_m} = 1 \rangle$$

Notice that $\chi(\mathcal{O}_1) = \chi(\mathcal{O}_2) - 1 - \sum_{i=1}^{k+1}(1 - \frac{1}{p_i}) < 0$, therefore the infinite cyclic subgroup $\langle d \rangle$ is a malnormal subgroup of $\pi_1^{orb}(\mathcal{O}_1)$, by Lemma A.1. Moreover, $\pi_1^{orb}(\mathcal{O})$ splits non-trivially as $\pi_1^{orb}(\mathcal{O}_1)*_{\langle d \rangle} \pi_1^{orb}(\mathcal{O}_2)$, and Lemma 2.2 implies that this is a non-elementary, 2-acylindrical splitting.

If $g = -1$, consider a closed loop $\delta$ enclosing the singular points $x_1, \ldots, x_m$ and cut $\mathcal{O}$ along this loop. We obtain two orbifolds with boundary: a disc with $m$ singular points $\mathcal{O}_1$, and a Möbius strip $\mathcal{O}_2$. Observe that $\chi(\mathcal{O}_1) = \chi(\mathcal{O}) < 0$ and, calling again $d$ the classes represented by the boundary loops $\delta$ in each $\pi_1^{orb}(\mathcal{O}_i)$, the subgroup $\langle d \rangle$ is malnormal in $\pi_1^{orb}(\mathcal{O}_1)$ by Lemma A.2 while $\langle d \rangle$ is a subgroup of index two in $\pi_1^{orb}(\mathcal{O}_2) = \langle a, d \mid a^2 d = 1 \rangle$. As $\pi_1^{orb}(\mathcal{O}) = \pi_1^{orb}(\mathcal{O}_1)*_{\langle d \rangle} \pi_1^{orb}(\mathcal{O}_2)$, we have again by Lemma 2.2 that $\pi_1^{orb}(\mathcal{O})$ is a non-trivial, 1-step malnormal amalgamated product, and possesses a non-elementary, 2-acylindrical splitting.

Finally, if $g \leq -2$ then $\mathcal{O}$ can be cut along a boundary loop $\delta$ in two orbifolds $\mathcal{O}_i$, and we can assume that either $\mathcal{O}_1$ has genus 1 and at least one singular point, or $\mathcal{O}_1$ has genus greater than 1 and no singular points. In the first case, one has $\pi_1^{orb}(\mathcal{O}_1) = \langle a, c, d \mid a^2 c d = c^k = 1 \rangle \cong \mathbb{Z} * \mathbb{Z}_p$, whereas in the second one $\pi_1^{orb}(\mathcal{O}_1) = \langle a_1, a_2, d \rangle \cong \mathbb{Z} * \mathbb{Z}$. In both cases, $\pi_1^{orb}(\mathcal{O})$ splits as a non-trivial amalgamated product $\pi_1^{orb}(\mathcal{O}_1)*_{\langle d \rangle} \pi_1^{orb}(\mathcal{O}_2)$ with $\langle d \rangle$ malnormal in $\pi_1^{orb}(\mathcal{O}_1)$, which gives again a non-elementary 2-acylindrical splitting.

**Appendix B. 3-manifolds with prescribed fundamental group**

The following statement is consequence of a number of classical results, which we will recall hereafter for the convenience of the reader:

**Theorem B.1.** There exist only finitely many pairwise non-diffeomorphic, compact orientable 3-manifolds without spherical boundary components with given fundamental group $G$.

To begin with, recall that in dimension 3 the homeomorphism type determines the diffeomorphism type, by the celebrated works of Moise, Munkres and Whitehead [Mo], [Mun1], [Mun2], [Whi].
Now, Theorem B.1 is well-known for closed 3-manifolds. Actually, if $X$ and $X'$ are prime, closed, orientable, 3-manifolds with isomorphic fundamental groups, then $X$ and $X'$ are homeomorphic, unless $X$ and $X'$ are lens spaces; this follows from basic facts of 3-dimensional topology and from the solution of the Geometrization Conjecture (see, for instance, \cite{AFW}, chapters 1&2). Moreover, by the classification of lens spaces, for every fixed $p \in \mathbb{N}$ there exists only a finite number of lens spaces $L(p,q)$ having $\mathbb{Z}_p$ as fundamental group (see for example \cite{AFW}, pp. 27–28).

On the other hand, for non-prime, closed 3-manifolds, the statement follows by Kneser’s theorem and the fact that the homeomorphism type of a connected sum is determined by the prime factors up to a finite number of choices, the indeterminacy being given by the orientations of the summands.

The proof of Theorem B.1 for general compact, orientable 3-manifolds with boundary is more tricky and due to Johannson (\cite{Jo1}, Corollary 29.3) in the case of irreducible manifolds with incompressible boundary. Recall that a compact 3-manifold $X$ is irreducible if any embedded 2-sphere bounds a 3-ball. The same result was proved, independently, by Swarup (\cite{Swa}, Theorem A), without the incompressibility assumptions. However, the part of Swarup’s proof dealing with possibly compressible boundary components invokes a proposition from \cite{Jo2} (namely, Proposition 3.9) that we were not able to track; since we noticed that this result, in more recent references like \cite{AFW}, is stated only for irreducible compact 3-manifolds with incompressible boundary, we find worth filling the details of the proof for general, compact manifolds with boundary without spherical boundary components, assuming Johannson’s statement. We will closely follow Swarup’s ideas, so no claim of originality is made.

Let us recall some basic terminology about 3-manifolds $X$ with boundary. A closed, properly embedded 2-disk $D \subset X$ (that is, such that $\partial D \subset \partial X$) is called essential if the loop $\partial D$ does not bound any embedded disk in $\partial X$. Two such disks $D, D'$ are said to be parallel if there is an ambient isotopy sending $D$ into $D'$; and one says that $X$ has incompressible boundary if there are no essential disks.

The surgery procedure for irreducible manifolds with compressible boundary. Let $X$ be a compact, irreducible 3-manifold: a disk system for $X$ is a collection $C$ of pairwise disjoint and non-parallel essential disks; the system is maximal if any collection $C'$ of essential disks properly containing $C$ contains a pair of parallel disks. Assume that $X$ has compressible boundary: we can then choose a non-empty, maximal disk system $C = \{D_1, \ldots, D_t\}$ and remove these disks from $X$. This procedure chops our irreducible manifold $X$ into a finite collection $\Gamma(X,C)$ of irreducible 3-manifolds with incompressible boundary $X_1, \ldots, X_n$ and finitely many 3-dimensional balls $B_1, \ldots, B_m$. Moreover, the collection $\Gamma(X,C) = \{X_1, \ldots, X_n, B_1, \ldots, B_m\}$ can be given a graph structure: the edges $d_i$ of $\Gamma(X,C)$ are in bijection with the disks $D_i$ of the maximal disk system $C$, and two vertices $v, v'$ of $\Gamma(X,C)$ (possibly with $v = v'$) are connected by $d_i$ if the disk $D_i$ bounds the corresponding manifolds or balls. The irreducible components $X_1, \ldots, X_n$ with incompressible boundary are uniquely determined up to diffeomorphism and do not depend on the particular maximal disk system $C$ (see \cite{Mat} pp. 167–168, or \cite{Mar}); on the other hand, the number $k$ of balls arising from the surgery procedure may depend on the choice of $C$.

This procedure can be inverted: we can reconstruct the manifold $X$ from $\Gamma(X,C)$.

\footnote{Johannson’s statement is more general and requires the manifolds to be Haken.}
by gluing back a 1-handle, i.e. a copy of $D^2 \times [0,1]$, for every edge of the graph; roughly speaking, $X$ appears as a “solid graph” whose vertices are the manifolds in the collection $\{X_1, \ldots, X_n, B_1, \ldots, B_m\}$ and whose edges are 1-handles connecting two (possibly equal) boundary components of the vertices. Using Van Kampen’s theorem we see that the fundamental group of $X$ is isomorphic to a free product $\pi_1(X) \cong \pi_1(X_1) \ast \cdots \ast \pi_1(X_n) \ast \mathbb{F}_k$ where $k$ is the number (possibly zero) of cycles in the graph $\Gamma(X, C)$.

Clearly, the number $n$ of compact, irreducible 3-manifolds $X_i$ with incompressible boundary components obtained by the surgery procedure is bounded, by Grushko’s theorem, by $N = n + k$. The next Lemma gives a bound of the numbers $r$ and $m$ of, respectively, 1-handles and balls appearing from the surgery procedure:

**Lemma B.2.** Let $N$ be the number of irreducible factors of $G = \pi_1(X)$ as a free product: then, $m \leq 2N$ and $r \leq 3N$.

**Proof.** We associate to $\Gamma(X, C)$ a graph of groups $\mathcal{G}(X, C)$, by assigning the group $G_{X_i} = \pi_1(X_i)$ to each vertex $X_i$, and the trivial groups to the vertices $B_j$ and to every edge $d_i$. Then, $\pi_1(\mathcal{G}) \cong G_{X_1} \ast \cdots \ast G_{X_n} \ast \mathbb{F}_k \cong G$ exactly. Notice that, from the non-parallelism condition, the degree of the vertices of $\Gamma(X, C)$ corresponding to the 3-balls is at least 3, unless the initial manifold was a solid torus, in which case the graph is just a loop and the collection of manifolds obtained after the surgery consists of a single 3-ball; therefore, we may assume that $\deg(B_i) \geq 3$ for $i = 1, \ldots, m$. On the other hand, since the initial manifold has compressible boundary, we know as well that $\deg(X_i) \geq 1$ for each $i = 1, \ldots, n$. Now, consider a maximal tree $T$ in $\Gamma(X, C)$: the maximal tree will have $n + m$ vertices and $n + m - 1$ edges. Let $E' = E(\Gamma(X, C)) \setminus E(T)$. Observe that, by construction, the adjunction of each edge of $E'$ to $T$ corresponds to add a free factor isomorphic to an infinite cyclic group, so $\#E' = k$. Then,

$$2 \cdot k + 2 \cdot (n + m - 1) \geq 2 \cdot \#E(\Gamma(X, C)) = \sum_{i=1}^{n} \deg(X_i) + \sum_{i=1}^{m} \deg(B_i) \geq n + 3m$$

hence $m \leq (2k + n - 2)$ and $r = \#E(\Gamma(X, C)) \leq k + n + m - 1 \leq 3k + 2n - 3$ and we conclude that $m$ and $r$ are (roughly) bounded respectively by $2N$ and $3N$. □

**Proof of Theorem B.1** for orientable manifolds with boundary. Let $G$ be a (compact) 3-manifold group, whose decomposition as a free product has $N$ indecomposable factors. If $X$ is an orientable, compact manifold with boundary without spherical boundary components and fundamental group $G$, it has a prime decomposition as a connected sum of irreducible manifolds and copies of $S^2 \times S^1$ (the only prime, non irreducible manifold without spherical boundary components), with at most $N$ factors. The homeomorphism type of a connected sum being uniquely determined by its factors and their orientations, it will then be enough to prove the theorem for irreducible manifolds. Now, by Lemma B.2 any compact, irreducible 3-manifold $X$ with fundamental group isomorphic to $G$ can be split using the surgery procedure in $n \leq N$ irreducible 3-manifolds $X_i$ with incompressible boundary and fundamental group $G_i$, plus a number $m \leq 2N$ of 3-balls; and $X$ is obtained as a solid graph on these pieces, attaching at most $r \leq 3N$ 1-handles. Moreover, notice that, by Kneser’s Theorem (holding for irreducible 3-manifolds with incompressible boundary components), the fundamental group of each $X_i$ is indecomposable, hence isomorphic to one indecomposable factor of the free product
decomposition of $G$. Now, by Johannson’s theorem, for each indecomposable factor $G_i$ of $G$ there exist only finitely many non-homeomorphic irreducible 3-manifolds $X_{i,a}$ with incompressible boundary with fundamental group $G_i$. Moreover, any two disks $D, D'$ in one of these $X_{i,a}$ are isotopic, and there are only two isotopy classes of diffeomorphisms $D^2 \to D^2$ (corresponding to the identity and to a reflection with respect to one axis); hence, once fixed two such pieces $X_{i,a}$ and $X_{j,b}$, there are essentially two inequivalent ways of attaching a 1-handle to them. Therefore, there are only finitely many manifolds which can be obtained as a solid graph on the (finitely many) pieces $X_{i,a}$, which concludes the proof. 

\section*{Appendix C. Malnormal subgroups of CAT(0)-groups}

We start giving a method to detect malnormal subgroups in fundamental group of locally $CAT(0)$-spaces. 

\textbf{Proposition C.1.} Let $Z$ be a compact, locally convex subspace of a compact, complete locally $CAT(0)$-space $X$. Assume that $X$ is negatively curved around $Z$; then, $H = \pi_1(Z)$ is malnormal in $G = \pi_1(X)$.

By \textit{negatively curved around $Z$} we mean that $Z$ has a neighbourhood $U(Z)$ in $X$ such that $U(Z) \setminus Z$ is a locally $CAT(-k)$-space, for some $k > 0$. Notice that this covers the case where $X$ is a complete Riemannian with sectional curvature $k_X < 0$, with no a-priori negative upper bound on the curvature.

\textit{Proof.} Let $\tilde{X} \to X$ the universal covering map: $\tilde{X}$ it is a $CAT(0)$-space. Let $\tilde{z}_0 \in p^{-1}(Z) \subset \tilde{X}$ be a point projecting to $z_0 \in Z$, and $H = \pi_1(Z, z_0)$. Finally, let $C_{\tilde{z}_0}p^{-1}(Z)$ denote the connected component of $p^{-1}(Z)$ containing the point $\tilde{z}$, and $\tilde{Z}_{\tilde{z}_0}$ the subset of $\tilde{X}$ obtained by lifting from $\tilde{z}_0$ any curve $\gamma$ of $Z$ based at $z_0$, and taking the endpoint $\tilde{\gamma}(1)$ of the lift $\tilde{\gamma}$. Then:

1. $C_{\tilde{z}_0}p^{-1}(Z) = \tilde{Z}_{\tilde{z}_0}$, and is a covering of $Z$. The fact that $C_{\tilde{z}_0}p^{-1}(Z)$ is a covering follows from ordinary theory of coverings, and the inclusion $\tilde{Z}_{\tilde{z}_0} \subset C_{\tilde{z}_0}p^{-1}(Z)$ is trivial. Conversely, if $\tilde{z} \in C_{\tilde{z}_0}p^{-1}(Z)$, it can be joined to $\tilde{z}_0$ by a curve $\tilde{\gamma}$ whose projection $\gamma$ stays in $Z$; hence $\tilde{z} = \tilde{\gamma}(1) \in \tilde{Z}_{\tilde{z}_0}$ by definition.

2. $\tilde{Z}_{\tilde{z}_0}$ is the universal covering of $Z$, and $H$ injects in $G$. Actually, since locally $CAT(0)$ spaces are locally convex, every class in $\pi_1(X, z_0)$ can be realized by a locally geodesic loop. Now, every locally geodesic loop $\gamma$ representing a class of $\pi_1(Z, z_0)$ lifts to a local geodesic $\tilde{\gamma}$ of $\tilde{X}$ from $\tilde{z}_0$ (the covering map being locally isometric). But every local geodesic in a $CAT(0)$ space is a true geodesic, hence $\tilde{\gamma}$ is not closed: this shows that $\tilde{Z}_{\tilde{z}_0}$ is simply connected, and that $\gamma$ does not represent the trivial element of $G$.

3. $\tilde{Z}_{\tilde{z}_0}$, endowed with the length structure induced by $Z$, is isometrically embedded in $\tilde{X}$; therefore, it is a convex subset of $\tilde{X}$. In fact, since $Z$ is locally convex in $X$, the inclusion $\tilde{Z}_{\tilde{z}_0} \subset \tilde{X}$ is a local isometry; but, $\tilde{X}$ being $CAT(0)$, geodesics in $\tilde{X}$ are unique, which implies that $\tilde{Z}_{\tilde{z}_0}$ is convex in $\tilde{X}$ and that the inclusion is a true isometric embedding.

\footnote{\textit{CAT(0)} spaces are assumed to be locally geodesic spaces (though non necessarily geodesic spaces), by definition.}
(4) \( C_{g\tilde{z}_0}p^{-1}(Z) = g \cdot \tilde{Z}_{\tilde{z}_0} \).

As in (1) one sees that \( C_{g\tilde{z}_0}p^{-1}(Z) = \tilde{Z}_{g\tilde{z}_0} \) (the subset obtained by lifting from \( g\tilde{z}_0 \) any curve \( \gamma \) of \( Z \) with base point \( z_0 \)), which clearly equals \( g \cdot \tilde{Z}_{\tilde{z}_0} \).

(5) \( \text{Stab}_G(\tilde{Z}_{\tilde{z}_0}) = H \).

The elements of \( H \) clearly stabilize \( \tilde{Z}_{\tilde{z}_0} \) (recall that \( h \in H \) acts on \( \tilde{x} \in \tilde{X} \) by lifting from \( z_0 \) the composition of a geodesic \( c \) from \( z_0 \) to \( x = p(\tilde{x}) \) with a loop \( \gamma \) at \( z_0 \) representing \( h \); so, the final point of the lift \( \tilde{\gamma}c \) belongs to \( \tilde{Z}_{\tilde{z}_0} \) by definition of \( \tilde{Z}_{\tilde{z}_0} \)).

Conversely: if \( g \in \text{Stab}_G(\tilde{Z}_{\tilde{z}_0}) \), then \( g\tilde{z}_0 \in \tilde{Z}_{\tilde{z}_0} \), and then the geodesic \( \tilde{\gamma} \) joining \( \tilde{z}_0 \) to \( g\tilde{z}_0 \) stays in \( \tilde{Z}_{\tilde{z}_0} \) (since this is a convex subset of \( \tilde{X} \)). As \( g \) is represented by the projection \( \gamma \) of \( \tilde{\gamma} \) in \( X \), which is included in \( Z \), then \( g \in \pi_1(Z) = H \).

(6) \( \text{Stab}_G(g \cdot \tilde{Z}_{\tilde{z}_0}) = gHg^{-1} \), and the number of connected components of \( p^{-1}(Z) \) is in bijection with the cosets space \( G/H \).

Both assertions follow from (4) and (5).

(7) Every \( h \in H \) acts on \( \tilde{X} \) by hyperbolic isometries, and the subset \( \text{Min}(h) \) where the displacement function \( d(\tilde{x}, h\tilde{x}) \) attains its minimum is included in \( \tilde{Z}_{\tilde{z}_0} \).

Since the action of \( G = \pi_1(X, z_0) \) on \( \tilde{X} \) is cocompact and without fixed points, then every element of \( G \) acts on \( \tilde{X} \) by hyperbolic isometries. We shall now prove that \( \text{Min}(h) \) is entirely included in \( \tilde{Z}_{\tilde{z}_0} \), for every \( h \in H \). Actually, let \( \tilde{x}_0 \) be an arbitrary point of minimum for the displacement function \( s_h(x) = d(\tilde{x}, h\tilde{x}) \), and consider the projection \( p : \tilde{X} \to \tilde{Z}_{\tilde{z}_0} \) (this is well defined, since \( \tilde{Z}_{\tilde{z}_0} \) is a convex subset). As \( \tilde{Z}_{\tilde{z}_0} \) is invariant under \( h \) by (5), and since \( p \) is a projection, we have \( h \cdot p(\tilde{x}_0) = p(h \cdot \tilde{x}_0) \). Therefore

\[
    d(p(\tilde{x}_0), h \cdot p(\tilde{x}_0)) = d(p(\tilde{x}_0), p(h \cdot \tilde{x}_0)) \leq d(\tilde{x}_0, h \cdot \tilde{x}_0)
\]

This shows that the point \( p(\tilde{x}_0) \in \tilde{Z}_{\tilde{z}_0} \) also realizes the minimum of \( s_h(x) \). By the \( h \)-invariance and the convexity of \( \text{Min}(h) \), we deduce that the orbits \( \{ h^n \cdot \tilde{x}_0 \} \) and \( \{ h^n \cdot \tilde{x}_0 \} \) lie on two parallel geodesics \( \gamma \) and \( p(\gamma) \), entirely included in \( \text{Min}(h) \); moreover, \( p(\gamma) \subset \tilde{Z}_{\tilde{z}_0} \), as \( \tilde{Z}_{\tilde{z}_0} \) is convex. So, \( \text{Min}(h) \) contains a flat band bounding \( \gamma \) and \( p(\gamma) \); the lifted neighbourhood \( \tilde{U}(\tilde{Z}_{\tilde{z}_0}) \) of \( \tilde{Z}_{\tilde{z}_0} \), being strictly negative curved outside \( \tilde{Z}_{\tilde{z}_0} \), this shows that \( \gamma \) and \( \tilde{x}_0 \) are necessarily included in \( \tilde{Z}_{\tilde{z}_0} \).

(8) \( H \) is malnormal in \( G \).

Assume that there exists \( h \in H^* \) and \( g \in G \) such that \( h' = ghg^{-1} \in H \). By (7), \( \text{Min}(h) \) is included in \( \tilde{Z}_{\tilde{z}_0} = C_{\tilde{z}_0}p^{-1}(Z) \); but as \( h' = ghg^{-1} \) is in \( H \), we also have \( \text{Min}(h') \subset C_{\tilde{z}_0}p^{-1}(Z) \). However, \( \text{Min}(h') = \text{Min}(ghg^{-1}) = g \cdot \text{Min}(h) \) is included in \( g \cdot \tilde{Z}_{\tilde{z}_0} = C_{g\tilde{z}_0}p^{-1}(Z) \), which is disjoint from \( C_{\tilde{z}_0}p^{-1}(Z) \) if \( g \notin H \), by (6). This shows that \( g \in H \) and that \( H \) is malnormal in \( G \). \( \square \)

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