A self-adjoint arrival time operator inspired by measurement models

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Abstract

We introduce an arrival time operator which is self-adjoint and, unlike previously proposed arrival time operators, has a close link to simple measurement models. Its spectrum leads to an arrival time distribution which is a variant of the Kijowski distribution (a re-ordering of the current) in the large momentum regime but is proportional to the kinetic energy density in the small momentum regime, in agreement with measurement models. A brief derivation of the latter distribution is given. We make some simple observations about the physical reasons for self-adjointness, or its absence, in arrival time operators and also compare our operator with the dwell time operator.

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I. INTRODUCTION

The arrival time problem in quantum mechanics is the question of determining the probability that an incoming wave packet, for a free particle, arrives at the origin in a given time interval \([1–5]\). Classically, for a particle with initial position \(x\) and momentum \(p\), the arrival time is given by the quantity

\[
\tau = -\frac{mx}{p}.
\] (1)

The quantum problem is most simply solved using the spectrum of an operator corresponding to this quantity, such as that first studied by Aharonov and Bohm \([6]\),

\[
\hat{T}_{AB} = -\frac{m}{2} \left( \hat{x} \frac{1}{\hat{p}} + \frac{1}{\hat{p}} \hat{x} \right).
\] (2)

A heuristic result due to Pauli \([7]\) (updated by Galapon \([8]\)) indicates that an object such as this, which is conjugate to a Hamiltonian with a semi-bounded spectrum, cannot be self-adjoint. Indeed we find that its eigenstates, which in the momentum representation (with \(\hat{x} \rightarrow i\hbar \partial/\partial p\)) are given by

\[
\phi_{\tau}(p) = \left( \frac{|p|}{2\pi m \hbar} \right)^{\frac{1}{4}} e^{i \frac{p^2}{2m\hbar} \tau},
\] (3)

are complete but not orthogonal. There is a POVM associated with these states from which an arrival time distribution can be constructed and it coincides with that postulated by Kijowski \([9]\), namely

\[
\Pi_K(\tau) = |\langle \psi | \phi_{\tau} \rangle|^2
\]

\[
= \frac{1}{m} \langle \psi_{\tau} | [\hat{p}, \frac{1}{2} \delta(\hat{x}) \delta(\hat{x})] \frac{1}{2} \hat{p} | \psi_{\tau} \rangle
\] (4)

(where \(\Pi(\tau)d\tau\) is the probability of arriving at the origin between \(\tau\) and \(\tau + d\tau\), for which there is some experimental evidence \([10]\). This is related by a simple operator re-ordering to the quantum-mechanical current at the origin, \(\langle \hat{J}(t) \rangle\), the expected result on classical grounds, where the current operator is given by

\[
\hat{J}(t) = \frac{1}{2m} (\hat{p}\delta(\hat{x}(t)) + \delta(\hat{x}(t))\hat{p}),
\] (5)

with \(\hat{x}(t) = \hat{x} + \hat{p}t/m\).
II. SELF-ADJOINT ARRIVAL TIME OPERATORS.

The lack of self-adjointness of Eq.(2) is not necessarily a problem but nevertheless a number of efforts have been made to restore it. To this end, we note that self-adjointness may be achieved by a number of simple modifications of the states Eq.(3). The states

$$\phi_\tau(p) = \left(\frac{|p|}{2\pi m \hbar}\right)^{\frac{1}{2}} e^{i\epsilon(p) \frac{p^2}{2m\hbar} \tau},$$

(6)

where $\epsilon(p)$ is the sign function, are orthogonal and complete and so are eigenstates of a self-adjoint operator. This operator, first considered by Kijowski [9] and subsequently explored at length by Delgado and Muga [11] may be written

$$\hat{T}_{KDM} = -\frac{m}{2} \left(\hat{x} \frac{1}{|p|} + \frac{1}{|p|} \hat{x}\right),$$

(7)

and is a quantization of the classical expression $-mx/|p|$. A second modification is to superpose opposite values of $\tau$ in Eq.(3) and then note that the subsequent states, which are proportional to $|p|^{\frac{1}{2}} \sin(p^2 \tau/2m\hbar)$ are orthogonal and are the eigenstates of the self-adjoint operator

$$\hat{T}_{MI} = \sqrt{\hat{T}_{AB}^2},$$

(8)

considered by de la Madrid and Isidro [12]. This is a quantization of $m|x|/|p|$. A third modification is to note that the orthogonality of the states $|p|^{\frac{1}{2}} \sin(p^2 \tau/2m\hbar)$ is not affected by restriction to positive or negative momenta so we may consider these two sectors separately and as a consequence the operator

$$\hat{T}_3 = \theta(\hat{p})\hat{T}_{MI}\theta(\hat{p}) - \theta(-\hat{p})\hat{T}_{MI}\theta(-\hat{p}),$$

(9)

is self-adjoint. This operator is a quantization of the classical expression $m|x|/p$. These three examples all side-step the Pauli theorem since they do not have canonical commutation with the Hamiltonian. Furthermore, they all give probability distributions which are simple variants on the Kijowski distribution, the expected result [13]. From these three examples, we see the following: self-adjoint modifications of the Aharonov-Bohm operator are easily obtained by relinquishing just one or two bits of information, namely the signs of $x$, or $p$, or both.

A closely analogous situation arises with the momentum operator on the half-line $x > 0$ [14]. There, the momentum operator cannot be made self-adjoint since it generates translations into negative $x$. However, $\hat{p}^2$ can be made self-adjoint, with suitable boundary
conditions, and hence, by the spectral theorem, $|\hat{p}|$ can be made self-adjoint. Hence just by relinquishing information about the sign of $\hat{p}$ a self-adjoint operator is obtained. Differently put, the obstruction to self-adjointness on the half-line lies in the sign function of $\hat{p}$. The issue is that the operator $\theta(\hat{x})\theta(\hat{p})\theta(\hat{x})$ cannot be made self-adjoint. A simple measurement model of momentum on the half-line using sequential position measurements indicates why this is to be expected on physical grounds: reflection off the origin makes it impossible to fully distinguish between left and right moving states. (We argue elsewhere, with simple measurement models, that this is also the underlying physical reason why the Aharonov-Bohm operator is not self-adjoint – the comparison with the momentum operator on the half-line is not just an analogy [13]).

Aside from the self-adjointness issue, all four of the above operators suffer from a number of problems. Firstly, their relation to actual measurements, or at least to simple measurement models is not obvious since there is no physical system that couples to the Aharonov-Bohm operator or any of the above variants of it. Secondly all the above arrival time operators yield an arrival time distribution which is the Kijowski distribution or variants thereof for all ranges of momenta. Simple measurement models (such as those based on a complex potential [10, 15] or stopwatch [16, 17]) agree with this for large momenta. But crucially, for small momenta, and more specifically for arrival time measurements more precise than $\hbar m/p^2$ (the energy time [18]), reflection off the detector becomes significant and measurement models typically yield a distribution proportional to the kinetic energy density

$$\Pi(\tau) = N \langle \psi_\tau | \hat{p} \delta(\hat{x}) \hat{p} | \psi_\tau \rangle,$$

(10)

where $N$ is a model-dependent normalization factor [10, 15, 17, 19].

Thirdly, even after normalizable states are constructed (by superposing over narrow ranges of $\tau$) from the eigenstates of $\hat{T}_{AB}$ and its three modifications, they all go like $p^{1/2}$ for small $p$, which means that they have infinite $(\Delta x)^2$, so have poor spatial localization properties, contrary to the intuitive notion of what an arrival time state should look like. The last two problems are issues around small momentum behaviour and are solved by an arrival time operator whose eigenstates go like $p$ for small $p$. (See Ref. [20] for related criticisms of traditional arrival time operators). The main result of this paper, and the solution to these three problems, is the construction of an arrival time operator directly inspired by simple measurement models.
We mention, tangentially, the work of Grot, Rovelli and Tate, who removed the singularity at $p = 0$ in $\hat{T}_{AB}$ by a somewhat artificial regularization procedure [21]. This produced a self-adjoint operator but the low momentum behaviour produced a spatial spread even more severe than the examples above [17], reiterating the need for a physically motivated handling of the low momentum regime.

III. MEASUREMENT MODELS

To motivate our proposed new arrival time operator and also to substantiate the arrival time distribution in the low momentum regime Eq.(10) we consider two measurement models for arrival time.

A simple measurement model for the probability of a particle crossing the origin in a given time interval $[0, \tau]$ involves spatial measurements onto $x > 0$ and $x < 0$ described by projectors $P = \theta(\hat{x})$ and $\bar{P} = \theta(-\hat{x})$, and we simply check to see if the particle is on opposite sides of the origin at the initial and final times. The probability for crossing then is

$$p(0, \tau) = \langle \psi | \bar{P} P(\tau) \bar{P} | \psi \rangle + \langle \psi | P \bar{P}(\tau) P | \psi \rangle. \quad (11)$$

Since $dP(t)/dt = \hat{J}(t)$, where $\hat{J}(t)$ is the current operator Eq.(5), this may be rewritten

$$p(0, \tau) = \int_0^\tau dt \langle \psi | \bar{P} \hat{J}(t) \bar{P} | \psi \rangle - \int_0^\tau dt \langle \psi | P \hat{J}(t) P | \psi \rangle, \quad (12)$$

thus indicating the appearance of the current operator in arrival time probabilities derived from a measurement model (like many other models, e.g. Ref.[16]).

We may use this formula to see the origin of the low momentum regime formula Eq.(10) as follows. We suppose that prior to $t = 0$, the system has been subjected to frequent projections onto $x < 0$. If these projections are sufficiently frequent (in comparison to the energy time $\hbar m/p^2$) the incoming state will be significantly reflected. At $t = 0$ it will therefore have the approximate form $\psi(x) = \theta(-x)(\phi(x) - \phi(-x))$, for some $\phi(x)$, which is zero at $x = 0$ but has non-zero derivative. The arrival time distribution is then

$$\Pi(\tau) = \langle \psi | \hat{J}(\tau) | \psi \rangle = -\frac{i\hbar}{2m} [\psi^*(0, \tau) \psi'(0, \tau) - \psi(0, \tau) \psi''(0, \tau)], \quad (13)$$
where the dash denotes spatial derivative. This is zero at $\tau = 0$ so we have to expand the wave function for small $\tau$. We make use of the free particle propagator and write

$$\psi(0, \tau) = \int_{-\infty}^{0} dy \left( \frac{m}{2\pi i \hbar \tau} \right)^{\frac{1}{2}} e^{i \frac{m y^2}{2\hbar}} \psi(y, 0),$$

(14)

where we have used the fact that $\psi(x > 0, 0) = 0$. Making the change of variables $y = z\tau^\frac{1}{2}$, expanding $\psi(z\tau^\frac{1}{2}, 0)$, and recalling that $\psi(0, 0) = 0$, we thus obtain

$$\psi(0, \tau) \approx \left( \frac{m\tau}{2\pi i \hbar} \right)^{\frac{1}{2}} \psi'(0, 0) \int_{-\infty}^{0} dz \ z e^{i \frac{m z^2}{2\hbar}}.$$ 

(15)

Evaluating the integral we then find

$$\langle J(\tau) \rangle \approx \frac{1}{2\pi^{\frac{1}{2}}} \left( \frac{\hbar}{m} \right)^{\frac{3}{2}} \tau^\frac{1}{2} |\psi'(0, 0)|^2.$$ 

(16)

This is of the desired form, Eq.(10). (A similar derivation was given in Ref.[22] with different aims). Hence the low momentum (strong measurement) regime result arises simply because the strong measurement causes the wave function to vanish at the origin so it is necessary to expand around this for small times.

A second simple model for measuring the arrival time, considered by numerous authors[16, 17], consists of a stopwatch – a system with coordinate $y$ and zero Hamiltonian which couples to the particle through the interaction $p_y \theta(-x)$. Classically, for a particle approaching from the left it therefore causes a shift in the stopwatch variable $y$ for the entire time the particle is in $x < 0$, stopping when the particle reaches the origin, with final value

$$y(T) - y(0) = \int_{0}^{T} dt \ \theta \left( -x - \frac{p}{m} t \right),$$

(17)

where $T$ is taken to be very large. This is easily seen to be equal classically to $-mx/p$ for $p > 0$. The right-hand side of Eq.(17) provides an alternative starting point for quantization which, however, turns out to be difficult to solve in the quantum case so we do not pursue it here.

IV. NEW OPERATOR AND ITS PROPERTIES

Motivated by the above observations, we look for a new time operator defined in terms of the $\hat{J}(t)$. We begin by noting that classically,

$$\frac{-m x}{|p|} = \int_{-\infty}^{\infty} dt \ J(t),$$

(18)
where $J(t) = (p/m)\delta(x + pt/m)$ is the classical current. Quantization of the left-hand side yields the KDM operator Eq. (7). However, we may equally take the right-hand side as the starting point for quantization and this leads to a different operator, namely

$$\hat{T} = \int_{-\infty}^{\infty} dt \, \hat{J}(t), \quad (19)$$

whose properties we now study. By sandwiching between momentum states and evaluating the integral, our arrival time operator becomes

$$\hat{T} = -\frac{m}{2} \left( \hat{x} \frac{1}{|\hat{p}|} (1 + \hat{R}) + \frac{1}{|\hat{p}|} (1 + \hat{R}) \hat{x} \right), \quad (20)$$

where $\hat{R}$ is the reflection operator which is defined by $\hat{R}|x\rangle = |-x\rangle$. It is self-adjoint, as we shall see. (See also Ref. [13]).

The extra terms, in comparison to Eq. (7), involving reflection, deserve further comment and to this end we note it bears a close comparison with the dwell time operator. This is the operator describing the time spent by a particle in a spatial region $[0, L]$, defined by

$$\hat{T}_D = \int_{-\infty}^{\infty} dt \, e^{i\hat{p}Ht} \, P_L e^{-i\hat{p}Ht}, \quad (21)$$

where $H = \hat{p}^2/2m$ and $P_L = \int_{0}^{L} dx |x\rangle\langle x|$ is a projector on the region $[0, L]$. The dwell time operator commutes with the Hamiltonian so is unaffected by the Pauli theorem and indeed it is self-adjoint. The dwell time operator may also be written more explicitly as

$$\hat{T}_D = \frac{mL}{|\hat{p}|} \left( 1 + e^{-\frac{i\hat{p}L}{\hbar}} \sin \left( \frac{\hat{p}L}{\hbar} \right) \hat{R} \right), \quad (22)$$

in which one can clearly see reflection effects. Classically, the dwell time can be written as the difference between two arrival times, at $x = L$ and $x = 0$ but not in general in the quantum case Eq. (22). However, it is true for low momenta, $pL \ll \hbar$, where one can see that our new arrival time operator satisfies

$$\hat{T}_D = e^{\frac{i\hat{p}L}{\hbar}} \hat{T} e^{-\frac{i\hat{p}L}{\hbar}} - \hat{T}. \quad (23)$$

Consider now the spectrum of the arrival time operator Eq. (20). The eigenvalue equation in momentum space is

$$-i \frac{m\hbar}{2} \left[ \frac{\partial}{\partial \hat{p}} \frac{1}{|\hat{p}|} \left( 1 + \hat{R} \right) + \frac{1}{|\hat{p}|} \left( 1 + \hat{R} \right) \frac{\partial}{\partial \hat{p}} \right] \phi_{\tau}(\hat{p}) = \tau \phi_{\tau}(\hat{p}). \quad (24)$$
The eigenstates can be written as the sum of their symmetric and anti-symmetric parts

\[ \phi_\tau(p) = \phi^S_\tau(p) + \phi^A_\tau(p) \]

and the eigenvalue equation reduces to a coupled system of first order equations

\[ \frac{\partial}{\partial p} \left( \frac{1}{|p|}\phi^S_\tau(p) \right) = \frac{i\tau}{m\hbar} \phi^A_\tau(p), \quad (25) \]

and

\[ \frac{\partial}{\partial p} \phi^A_\tau(p) = \frac{i\tau|p|}{m\hbar} \phi^S_\tau(p). \quad (26) \]

Solving for the antisymmetric part of the state we obtain the second order differential equation

\[ \frac{\partial^2}{\partial p^2} \phi^A_\tau(p) - \frac{2}{p} \frac{\partial}{\partial p} \phi^A_\tau(p) + \left( \frac{\tau}{m\hbar} \right)^2 p^2 \phi^A_\tau(p) = 0. \quad (27) \]

It may be shown that this equation has two linearly independent solutions in terms of Bessel functions, an antisymmetric one, \( p|p|^{\frac{1}{2}} J_{\frac{3}{4}} \left( \frac{p^2\tau}{2m\hbar} \right) \) and a symmetric one \( |p|^{\frac{3}{2}} J_{-\frac{1}{4}} \left( \frac{p^2\tau}{2m\hbar} \right) \) which is irrelevant so is dropped. Inserting this antisymmetric solution into Eq.(26) and using properties of the Bessel functions \([24]\) we obtain \( \phi^S_\tau(p) \) and the full normalized solution is then found to be

\[ \phi^A_\tau(p) = \frac{-i}{\sqrt{8m\hbar}} \left( p|p|^{\frac{1}{2}} J_{\frac{3}{4}} \left( \frac{p^2\tau}{2m\hbar} \right) - i|p|^{\frac{3}{2}} J_{-\frac{1}{4}} \left( \frac{p^2\tau}{2m\hbar} \right) \right). \quad (28) \]

These eigenstates may be shown, at some length, to be orthonormal and complete and thus \( \hat{T} \) is a self-adjoint operator \([13]\). Using the asymptotic forms of the Bessel functions, we obtain approximations for the eigenstates in the large and low momentum regimes respectively,

\[ \phi_\tau(p) \sim \begin{cases} -i e^{-\frac{\pi}{4}\epsilon(p)} \left( \frac{|p|}{2m\hbar} \right)^{\frac{1}{4}} e^{i\epsilon(p)} \frac{p^2\tau}{2m\hbar} & \frac{p^2\tau}{2m\hbar} \gg 1 \\ -i \frac{\tau}{2\Gamma(\frac{1}{4})(\hbar)^{\frac{1}{2}}}|p| & \frac{p^2\tau}{2m\hbar} \ll 1. \end{cases} \quad (29) \]

The arrival time probability for a time interval \([\tau_1, \tau_2]\) is given by

\[ p(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} d\tau \left| \langle \psi | \phi_\tau \rangle \right|^2. \quad (30) \]

For large time intervals, compared to the energy time, the large momentum regime applies, the eigenstates are of the form \([6]\) and thus the probability distribution in the large momentum regime is similar to that for \( \hat{T}_{KDM} \) (a variant of the Kijowski distribution\([25]\)). (Note, however, that the phase depending on \( \epsilon(p) \) means that the terms in the distribution representing interference between positive and negative momenta will have a different phase to that in the distribution derived from \( \hat{T}_{KDM} \). For time intervals short compared to the
energy time in Eq. (30), the low momentum regime applies. The eigenstates are of the form 
\[ \phi_\tau(p) \approx C|p|, \]
where \( C \) is read off from Eq. (29) and we easily obtain
\[
|\langle \psi | \phi_\tau \rangle|^2 \approx \frac{\pi}{2 (\Gamma (\frac{3}{2}))^2} \frac{\tau}{m^2 \hbar^2} \langle \psi | \hat{p} \delta(\hat{x}) | \hat{p} | \psi \rangle,
\]
which is the kinetic energy density, Eqs. (10), (16), the physically expected result in this
regime. The numerical precfactors are not exactly the same as in Eq. (16) - they differ by
about 20 percent but the overall factors in Eq. (31) are fixed by the completeness relation of
the eigenstates \( \phi_\tau(p) \). There is no reason to expect perfect agreement since the two formulae
have different origins. Note also that because the eigenstates go like \(|p|\) for small \( p \) so their
associated normalizable states, obtained by superposing over small ranges of \( \tau \), have finite
\((\Delta x)^2\), as desired.

V. SUMMARY AND CONCLUSION

In the construction of arrival time operators, different classical starting points lead to
inequivalent quantum operators with different physical predictions. The Aharonov-Bohm
operator and its variants do not capture the expected physical behaviour in the low momen-
tum regime. In this paper we constructed an arrival time operator taking a different starting
point, inspired by measurement models and classically equivalent to Eq. (1), thus obtaining
an arrival time probability giving the expected physical behaviour in both large and small
momentum regimes. These results and a number of associated results will be described at
greater length in another paper [13].

VI. ACKNOWLEDGEMENTS

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