1 Introduction

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$ and denote $\rho(x) := d(x, \partial \Omega)$, $x \in \Omega$. Denote by $\phi_1$ and $\lambda_1$ respectively the first (positive) eigenfunction and the first eigenvalue of $-\Delta$ in the space $H^1_0(\Omega)$. Also, let $G$ denote the positive Green’s function for $-\Delta$ in $\Omega$.

Assume that $K \in C_{\text{loc}}^\nu(\Omega) \ (\nu \in (0, 1))$ is such that

$$\inf_{\Omega} K > 0, \quad \text{and} \quad K = O(\rho^{-\beta}) \quad \text{near} \quad \partial \Omega, \quad \text{for some} \quad \beta \geq 0.$$ 

Given $\alpha > 0$, we consider the following fourth order singular elliptic problem:

$$(P) \quad \begin{cases} 
\Delta^2 u = K(x) u^{-\alpha} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \quad u|_{\partial \Omega} = 0, \quad \Delta u|_{\partial \Omega} = 0. 
\end{cases}$$

There is a large literature concerning such singular problems (as well as the corresponding systems) for second order elliptic operators wherein questions of existence, uniqueness and multiplicity, regularity, asymptotic behaviour, symmetry, etc. have been investigated (see for instance [1], [3], [10], [14], [21], [23], [27], [28]). Similar results for the quasilinear case have been obtained in [22] and [23]. We refer the reader to the two excellent surveys [19] and [26] for more details.

There are very few results available which concern fourth order singular problems similar to $(P)$. In [20], the author studies the problem $\Delta^2 u = u^{-\alpha}$, $\alpha < 1$, but with Dirichlet boundary condition. Furthermore, the author assumes that the domain is a perturbation of the ball to ensure positivity of the associated Green’s function. Using the Schauder fixed point theorem.

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to a suitable integral formulation of the problem in an appropriate cone of positive continuous functions, the existence and the uniqueness of a solution in $C^2(\Omega) \cap C^0_0(\Omega)$ that behaves like $\rho^2$ near the boundary is shown in this work. Since such a boundary behaviour is expected, the restriction $\alpha < 1$ is necessary.

In contrast with [20], we consider the problem $(P)$ for a general smooth bounded domain $\Omega$ with Navier boundary conditions. We first clarify the notion of a solution to $(P)$:

**Definition 1.1.** A function $u \in C^2(\Omega)$ is a solution to $(P)$ if $u > 0$ in $\Omega$, $u = \Delta u = 0$ on $\partial \Omega$ and satisfies the following integral identity for any $\psi \in C^2(\Omega) \cap C^0_0(\Omega)$:

$$\int_{\Omega} \Delta u \Delta \psi \, dx = \int_{\Omega} K(x) u^{-\alpha} \psi \, dx. \quad (1.1)$$

**Remark 1.2.**

1. We require $C^2(\Omega)$ regularity to be able to define $\Delta u = 0$ on $\partial \Omega$.
2. A consequence of the above definition is that a solution $u$ to $(P)$ necessarily satisfies

$$\int_{\Omega} K(x) u^{-\alpha}(x) \rho(x) \, dx < \infty.$$ 

To see this, plug in the test function $\psi = \phi_1$ in (1.1), where $\phi_1$ is the first (normalized) positive eigenfunction of $-\Delta$ on $H^1_0(\Omega)$.
3. Definition 1.1 is similar to the concept of very weak solution given in [6] (see definition 0.2 there) for solving second order elliptic problems with $L^1$- (or measure) data. We adapt this notion here for fourth order elliptic equations.

The solution to $(P)$ can be defined equivalently using the Green’s representation formula (see proposition 4.1 in section 4). It is easy to see that the equation in $(P)$ is equivalent to the following second order elliptic system:

$$(PS) \begin{cases} 
-\Delta u = v & \text{in } \Omega, \ u > 0 \quad \text{in } \Omega, \\
-\Delta v = K(x) u^{-\alpha} & \text{in } \Omega, \\
u|_{\partial \Omega} = 0, \ v|_{\partial \Omega} = 0.
\end{cases}$$

Nevertheless, $(PS)$ is not a cooperative system and hence monotone methods can not be used to prove existence of solutions to $(PS)$, as is done in [11] for the single equation. Furthermore, for $\alpha \in (0, 1)$, the problem $(P)$ has a variational structure and the energy functional $J$ associated to $(P)$ is defined as follows:

$$(1.2) \quad J(w) \overset{\text{def}}{=} \frac{1}{2} \int_{\Omega} (\Delta w)^2 \, dx - \frac{1}{1-\alpha} \int_{\Omega} K(x) w^{1-\alpha} \, dx \quad \text{for} \quad w \in X \overset{\text{def}}{=} H^2(\Omega) \cap H^1_0(\Omega).$$

Clearly, $J$ is well defined in the cone of nonnegative functions in $X$ provided $K$ has moderate singularity near $\partial \Omega$. But the main difficulty is that truncation techniques (which work in case of second order elliptic equations) can not be used directly since we are in the $H^2$-framework. This makes it difficult to employ variational methods for studying $(P)$. Another difficulty is that the Schauder fixed point theorem (used in [20]) works only in the case $\alpha < 1$ where the invariance of the solution operator with respect to a cone of positive solutions can be ensured. For these reasons, our approach in this paper is slightly different. We first approximate the singular problem $(P)$ by a family of problems $(P_\epsilon)$ with regular terms as given below and use apriori estimates to show the existence of solution. We now state the results that we prove:
Theorem 1.3. Assume that $\alpha + \beta < 2$. Then there exists a unique solution $u$ to $(P)$. Furthermore, there exist $c_1, c_2 > 0$ such that

$$c_1 \rho(x) \leq u(x) \leq c_2 \rho(x).$$

The idea behind the proof (see section 3) is to approximate the problem in the following way:

$$(P_{\epsilon}) \quad \left\{ \begin{array}{ll}
\Delta^2 u = K_\epsilon(x)(u + \epsilon)^{-\alpha} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u|_{\partial\Omega} = 0, \Delta u|_{\partial\Omega} = 0.
\end{array} \right.$$  

The existence of the solution $u_\epsilon$ to $(P_{\epsilon})$ can be obtained by the Schauder fix point theorem. We then prove a priori estimates on $\{u_\epsilon\}_{\epsilon > 0}$ using crucially the restrictions on $\alpha, \beta$ and pass to the limit as $\epsilon \to 0^+$. The following nonexistence result proves that the restriction $\alpha + \beta < 2$ is sharp in the above results:

Theorem 1.4. Assume that $\alpha + \beta \geq 2$. Then, there is no solution to $(P)$. 

Next, we use Theorem 1.3 to obtain the existence of a path-connected branch of solutions to the following bifurcation problem:

$$(P_{\lambda}) \quad \left\{ \begin{array}{ll}
\Delta^2 u = K(x)u^{-\alpha} + \lambda f(u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u|_{\partial\Omega} = 0, \Delta u|_{\partial\Omega} = 0
\end{array} \right.$$  

where $\lambda$ is the bifurcation parameter and $f$ a function satisfying the following assumptions:

1. $f: [0, \infty) \to [0, \infty)$ is a twice continuously differentiable map with $f(0) = 0$.
2. $f(t)$ is a finite product of functions of the form $g(t^p), p > 0$, where $g$ is a real entire function on $\mathbb{R}$.
3. $f' \geq 0$ and $\lim_{t \to \infty} \frac{f(t)}{t} > 0$.

Given a positive continuous function $\phi$ on $\Omega$, denote by

$$C_\phi(\Omega) := \left\{ u \in C(\overline{\Omega}) : \sup_{\Omega} \frac{|u|}{\phi} < +\infty \right\},$$

with the norm

$$\|u\|_{C_{\phi}(\Omega)} := \sup_{\Omega} \frac{|u|}{\phi}$$

and the “positive cone”

$$C_{\phi}^+(\Omega) := \left\{ u \in C_{\phi}(\Omega) : \inf_{\Omega} \frac{u}{\phi} > 0 \right\}.$$  

We define the inverse of the biharmonic operator denoted as $(\Delta^2)^{-1}$ as follows:

$$(\Delta^2)^{-1}h = u$$

where for $h$ in an appropriate space, $u$ solves the inhomogeneous problem:

$$(1.5) \quad \left\{ \begin{array}{ll}
\Delta^2 u = h & \text{in } \Omega, \\
u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0
\end{array} \right.$$
The bifurcation analysis is done in the space $\mathbb{R} \times C_{\phi_1}(\Omega)$. Therefore, we consider the following set of all solutions (in the sense of definition [1.1])

\begin{align}
S = \{ u \in C^4(\Omega) \cap C^2(\partial \Omega), u > 0 \text{ solves } (P_\lambda) \} \subset \mathbb{R} \times C_{\phi_1}(\Omega).
\end{align}

Consider the following solution operator associated to $(P_\lambda)$:

\begin{align}
F(\lambda, u) = u - (\Delta^2)^{-1}(K(x)u^{-\alpha} + \lambda f(u)), \ (\lambda, u) \in \mathbb{R} \times C_{\phi_1}^+(\Omega), \ 0 < \alpha + \beta < 2.
\end{align}

Using the framework of analytic bifurcation theory as developed in the works [8] and [9] (see also [4] and [7]), we obtain an analytic global unbounded path of solutions to $(P_\lambda)$:

**Theorem 1.5.** Let $f$ satisfy conditions $(f_0) - (f_2)$ and $\alpha + \beta < 2$. Then, $F : \mathbb{R} \times C_{\phi_1}^+(\Omega) \to C_{\phi_1}^1(\Omega)$ is an analytic map (see definition [5.1]). Furthermore, there exists $\Lambda \in (0, \infty)$ and an unbounded set $A \subset (-\infty, \Lambda] \times C_{\phi_1}^+(\Omega) \subset S$ of solutions to $(P_\lambda)$ which is globally parametrised by a continuous map :

\begin{align}
(-\infty, \infty) \ni s \mapsto (\lambda(s), u(s)) \in A.
\end{align}

Moreover, the following properties hold along this path $A$:

(i) $(\lambda(s), u(s)) \to (0, u_0)$ in $\mathbb{R} \times C_{\phi_1}(\Omega)$ as $s \to 0$, where $u_0$ is the unique solution to $(P)$.

(ii) $\|u(s)\|_{C_{\phi_1}(\Omega)} \to \infty$ as $s \to \infty$.

(iii) $A$ has at least one asymptotic bifurcation point $\Lambda_\alpha \in [0, \Lambda]$. That is, there exist sequences $\{s_n\}_{n \in \mathbb{N}} \subset (0, \infty)$, $\{(\lambda(s_n), u(s_n))\} \subset A$ such that $s_n \to \infty$, $\lambda(s_n) \to \Lambda_\alpha$ and $\|u(s_n)\|_{C_{\phi_1}(\Omega)} \to \infty$.

(iv) $\{s \geq 0 : \partial_s F(\lambda(s), u(s)) \text{ is not invertible} \}$ is a discrete set.

(v) (A is an “analytic” path) At each of its points $A$ has a local analytic re-parameterization in the following sense: For each $s^* \in \mathbb{R}$ there exists a continuous, injective map $\rho^* : (-1, 1) \to \mathbb{R}$ such that $\rho^*(0) = s^*$ and the re-parametrisation

\begin{align}
(-1, 1) \ni t \to (\lambda(\rho^*(t)), u(\rho^*(t))) \in A \text{ is analytic.}
\end{align}

Furthermore, the map $s \mapsto \lambda(s)$ is injective in a neighborhood of $s = 0$ and for each $s^* > 0$ there exists $\epsilon^* > 0$ such that $\lambda$ is injective on $[s^*, s^* + \epsilon^*]$ and on $[s^* - \epsilon^*, s^*]$.

(vi) For any $\lambda \leq 0$, there exists almost one solution to $(P_\lambda)$ and $A \cap (-\infty, 0) \times C_{\phi_1}(\Omega)$ is a single analytic curve which is a graph from the $\lambda$ axis consisting of non-degenerate solutions $u_\lambda$. In particular, we can take $\lambda(s) = s$ for $s < 0$.

The paper is organized as follows: In Section 3 we prove Theorem 1.3 using a version of Hopf principle recalled in proposition 2.1. In Section 4 we study the equivalence between the two definitions of a solution and prove Theorem 1.4. Finally, in Section 5 we prove Theorem 1.5.
2 Some preliminary results for Theorem 1.3

We first prove a version of Hopf principle.

**Proposition 2.1.** Let $h \in L^\infty(\Omega)$ be a nonnegative function. Let $u$ be the classical solution to (1.5). Then there exists a constant $C > 0$ (independent of $h$) such that the following inequality holds:

\[
(2.1) \quad u(x) \geq C \rho(x) \int_{\Omega} h(y) \rho(y) dy.
\]

**Proof.** Since $h \in L^\infty(\Omega)$, $u$ solves the following system:

\[
(2.2) \quad \begin{cases}
-\Delta u = v & \text{in } \Omega, \\
-\Delta v = h & \text{in } \Omega, \\
u|_{\partial \Omega} = v|_{\partial \Omega} = 0.
\end{cases}
\]

Recall from lemma 3.2 in [5] that for any nonnegative function $h \in L^\infty(\Omega)$, the unique solution $w$ to the problem

\[
\begin{cases}
-\Delta w = h & \text{in } \Omega, \\
w|_{\partial \Omega} = 0
\end{cases}
\]

satisfies the estimate:

\[
w(x) \geq C \rho(x) \int_{\Omega} h(y) \rho(y) dy \quad x \in \Omega,
\]

where the constant $C$ does not depend to $h$.

We apply the previous inequality to $u$ and $v$ to get

\[
u(x) \geq C \rho(x) \int_{\Omega} v(y) \rho(y) dy \geq C^2 \rho(x) \int_{\Omega} \rho(y)^2 dy \int_{\Omega} h(z) \rho(z) dz
\]

which completes the proof.

By a simple approximation argument and the maximum principle, we have the

**Corollary 2.2.** Let $h \rho \in L^1(\Omega)$ and nonnegative. Then any $u$ solving (1.5) (in the sense of definition 1.1) satisfies the inequality (2.1).

We next have the following regularity and uniform estimate result:

**Lemma 2.3.** Let $h \in C^0_{loca}(\Omega)$ be a nonnegative function such that $h \rho^\delta \in L^\infty(\Omega)$ for some $0 < \delta < 2$. Let $u \in C^2(\Omega)$ be the solution (in the sense of definition 1.1) to (1.5). Then there exist constants $C > 0$ (dependent on $\|h \rho^\delta\|_{L^\infty(\Omega)}$, $\nu$ and $\delta$) and $0 < \theta < 1$ (depending on $\nu$ and $\delta$) such that the following inequality holds:

\[
(2.3) \quad \|u\|_{C^{2,\theta}(\Omega)} \leq C.
\]

**Proof.** Since $h \rho \in L^1(\Omega)$, from the above corollary we obtain that $u \geq c \rho$ for some $c > 0$. Since $u \in C^2(\Omega) \cap C^0(\Omega)$, we obtain that $u \sim \rho$ near $\partial \Omega$. We note that $v := -\Delta u \in C_{loca}^{2,\nu}(\Omega)$ by elliptic regularity and is a nonnegative function by the maximum principle. Consider the equivalent system for $u,v$ as in (2.2). Then we have

\[
|\Delta v| \leq C_0 \rho^{-\delta}, \quad \text{where } C_0 := \|h \rho^\delta\|_{L^\infty(\Omega)}.
\]
Let \( w := w(\delta) \) denote the unique positive solution to
\[
\begin{aligned}
-\Delta w &= w^{-\delta} \text{ in } \Omega, \\
w &= 0 \text{ on } \partial\Omega.
\end{aligned}
\]

From [11], there exist positive constants \( c_1 < c_2 \) such that the following estimates hold:
\[
\begin{align*}
c_1 \rho &\leq w \leq c_2 \rho \quad \text{if } 0 < \delta < 1, \\
c_1 \rho \ln\left(\frac{D}{\rho}\right) &\leq w \leq c_2 \rho \ln\left(\frac{D}{\rho}\right) \quad \text{if } \delta = 1 \ (D := \text{diam}(\Omega)) \quad \text{and} \\
c_1 \rho^{\frac{\delta}{\delta+1}} &\leq w \leq c_2 \rho^{\frac{\delta}{\delta+1}} \quad \text{if } 1 < \delta < 2.
\end{align*}
\]

Choosing the constant \( M > 0 \) large enough (depending on \( C_0, c_1 \) and \( \delta \)) and using the weak comparison principle, we can conclude \( v - Mw \leq 0 \). Thus, we have
\[
0 \leq v \leq Mw \leq Mc_2 \rho^\mu \quad \text{for some } \mu > 0.
\]

By noting (2.4) and (2.5), appealing to Proposition 3.4 in [24] we obtain that \( v \in C^0(\overline{\Omega}) \) for some \( \theta \in (0, 1) \), \( \|v\|_{C^0,\theta}(\Omega) \leq C = C(C_0, \delta) \). We then apply the classical elliptic theory to get \( u \in C^{2,\theta}(\overline{\Omega}) \) and \( \|u\|_{C^{2,\theta}(\overline{\Omega})} \leq \tilde{C} = \tilde{C}(C_0, \delta, \nu) \).

We can now show the following result on existence of \( C^2(\overline{\Omega}) \) solution (as in definition [11]) by means of a simple approximation argument:

**Proposition 2.4.** Let \( h \) be a nonnegative function such that \( h \rho^\delta \in L^\infty(\Omega) \) for some \( \theta := \theta(\delta, \nu) \in (0, 1) \) and \( \|v\|_{C^0,\theta}(\overline{\Omega}) \leq C = C(C_0, \delta) \). We then apply the classical elliptic theory to get \( u \in C^{2,\theta}(\overline{\Omega}) \) and \( \|u\|_{C^{2,\theta}(\overline{\Omega})} \leq \tilde{C} = \tilde{C}(C_0, \delta, \nu) \). \( \square \)

We recall the Hardy Inequality for \( H^s \) spaces:

**Lemma 2.5** (see Lemma 3.2.6.1 in [24]). Let \( s \in [0, 2] \) such that \( s \neq \frac{1}{2} \) and \( s \neq \frac{3}{2} \). Then the following generalisation of Hardy’s inequality holds:
\[
\|\rho^{-s} g\|_{L^2(\Omega)} \leq C\|g\|_{H^s(\Omega)} \quad \text{for all } g \in H^s_0(\Omega).
\]

Finally, we state the following regularity result.

\[
(2.6) \quad \|\rho^{-s} g\|_{L^2(\Omega)} \leq C\|g\|_{H^s(\Omega)} \quad \text{for all } g \in H^s_0(\Omega).
\]
Lemma 2.6. Assume that $h$ is a nonnegative function such that $h\rho^{\delta} \in L^\infty(\Omega)$ for some $0 < \delta < 2$. Let $u \in C^2(\overline{\Omega}) \cap C_0(\overline{\Omega})$ be the solution of (1.5). Then $u \in W_{loc}^{4,p}(\Omega) \cap H^{4-s}(\Omega)$ for any $1 \leq p < \infty$ and $s \in \left(\delta - \frac{1}{2}, 2\right] \setminus \left\{\frac{1}{2}, \frac{3}{2}\right\}$.

Proof. That $u \in W_{loc}^{4,p}(\Omega)$ for $1 \leq p < \infty$ follows from standard elliptic regularity result. Fix any $s \in \left(\delta - \frac{1}{2}, 2\right] \setminus \left\{\frac{1}{2}, \frac{3}{2}\right\}$. We claim that $h \in H^{-s}(\Omega)$. Indeed, using lemma 2.5, for any $\xi \in H^{s}_0(\Omega)$,

$$\int_{\Omega} h\xi = \int_{\Omega} (h\rho^s)(\xi\rho^{-s})$$

$$\leq C \int_{\Omega} (\rho^{s-\delta})(\rho^{-s}|\xi|)$$

$$\leq C\|\rho^{s-\delta}\|_{L^2(\Omega)}\|\rho^{-s}\xi\|_{L^2(\Omega)}$$

$$\leq C\|\xi\|_{H^{s}_0(\Omega)}.$$ 

Hence by elliptic regularity used successively to $v$ and $u$ we obtain that $u \in H^{4-s}(\Omega)$. \qed

3 Proof of Theorem 1.3

We first show that the solution is unique. Let $u_1$ and $u_2$ be two solutions to $(P)$. Then,

$$(3.1) \quad \int_{\Omega} (\Delta(u_1 - u_2))^2dx = \int_{\Omega} K(x)(u_1 - u_2)(u_1 - u_2)dx \leq 0.$$ 

Therefore, since $u_1, u_2 \in H^1_0(\Omega)$, we obtain $u_1 \equiv u_2$.

Fix $\epsilon > 0$. We next prove the existence of a unique solution to $(P_{\epsilon})$. Let $W$ be the positive cone of $C_0(\overline{\Omega})$, i.e.

$$W \defeq \{u \in C_0(\overline{\Omega}) \mid u \geq 0 \text{ in } \Omega\}.$$ 

We define the functional $\Phi : W \to W$ as the solution to the following problem:

$$\left\{ \begin{array}{l}
\Delta^2\Phi(u) = K_\epsilon(x)(u + \epsilon)^{-\alpha} \quad \text{in } \Omega, \\
\Phi(u)|_{\partial \Omega} = 0, \quad \Delta\Phi(u)|_{\partial \Omega} = 0.
\end{array} \right.$$ 

By the elliptic regularity theory, $\Phi$ is a compact linear operator on $C_0(\overline{\Omega})$, and by the weak comparison principle, leaves the closed convex set $W$ invariant. Hence, by the Schauder fixed point theorem, there exists $u_\epsilon \in W$ solution to $(P_{\epsilon})$. Using a similar argument as in (3.1), $u_\epsilon$ is the unique solution to $(P_{\epsilon})$. By elliptic regularity, we also have $u_\epsilon \in C^2(\overline{\Omega})$.

Multiplying the equation satisfied by $u_\epsilon$ by $\phi_1$, we obtain that

$$(3.2) \quad \lambda_1^2 \int_{\Omega} u_\epsilon \phi_1 dx = \int_{\Omega} \Delta^2 u_\epsilon \phi_1 dx = \int_{\Omega} K_\epsilon(x)\phi_1(u_\epsilon(x) + \epsilon)^{-\alpha} dx.$$ 

First we show a uniform lower bound:

Proposition 3.1. There exists a constant $C > 0$ independent of $\epsilon$ such that $u_\epsilon \geq C\rho$ in $\Omega$. 

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Proof. We first show the following fact:

\[ (3.3) \inf_{\epsilon > 0} \int_{\Omega} K_\epsilon \phi_1 (u_\epsilon + \epsilon)^{-\alpha} \, dx > 0. \]

We argue by contradiction. Suppose, up to a subsequence,

\[ \int_{\Omega} K_\epsilon \phi_1 (u_\epsilon + \epsilon)^{-\alpha} \, dx \to 0 \text{ as } \epsilon \to 0^+. \]

Using (3.2) this implies that \( \int_{\Omega} u_\epsilon \phi_1 \, dx \to 0 \) and hence \( u_\epsilon \to 0 \) in \( L^1_{\text{loc}}(\Omega) \) as \( \epsilon \to 0^+ \).

Again up to a subsequence, we deduce that

\[ u_\epsilon \to 0 \quad \text{and} \quad K_\epsilon (u_\epsilon + \epsilon)^{-\alpha} \to 0 \text{ a.e. in } \Omega \text{ as } \epsilon \to 0^+ \]

which is a contradiction. This proves (3.3) above.

By elliptic regularity theory, \( u_\epsilon \in C^{2,\gamma}(\Omega) \) for any \( \gamma \in (0,1) \) and from Proposition 2.1 the estimate

\[ u_\epsilon(x) \geq C \rho(x) \int_{\Omega} K_\epsilon (u_\epsilon + \epsilon)^{-\alpha} \rho \, dy \quad (3.5) \]

holds. The conclusion follows from (3.3). \( \square \)

**Proposition 3.2.** There exists \( \theta \in (0,1) \) independent of \( \epsilon > 0 \) such that

\[ \sup_{\epsilon > 0} \| u_\epsilon \|_{C^{2,\theta}(\Omega)} < +\infty. \]

Proof. From the last proposition, it follows that

\[ K_\epsilon (u_\epsilon + \epsilon)^{-\alpha} \rho^{\alpha + \beta} \in L^\infty(\Omega). \quad (3.6) \]

Noting that \( 0 < \alpha + \beta < 2 \) and invoking lemma 2.3 the conclusion follows. \( \square \)

Let \( u_\epsilon \to u \) in \( C^2(\Omega) \) as \( \epsilon \to 0 \). From (3.6), we note that given \( \psi \in C^2(\Omega) \cap C_0(\Omega) \) there exists \( p > 1 \) such that

\[ \left\{ K_\epsilon (u_\epsilon + \epsilon)^{-\alpha} \psi \right\}_{\epsilon > 0} \text{ is a bounded family in } L^p(\Omega). \]

We can now use Vitali’s convergence theorem to directly pass to the limit as \( \epsilon \to 0 \) in (\( P_\epsilon \)) to conclude that \( u \) solves (\( P \)).

4 Proof of Theorem 1.4

We first prove the following equivalent way of defining a solution to (\( P \)):

**Proposition 4.1.** \( u \in C^2(\Omega) \cap C_0(\Omega) \) is a solution to (\( P \)) (in the sense of definition 1.1) if \( u > 0 \) in \( \Omega \) and verifies

\[ u(x) = \int_{\Omega} G(x,y) \left( \int_{\Omega} G(y,z) K(z) u^{-\alpha}(z) \, dz \right) \, dy. \quad (4.1) \]
Proof. Assume first that $u$ satisfies Definition 1.1. From the estimates in Proposition 4.13 in [18] and noting $\int_{\Omega} K(z) \rho(z) u^{-\alpha}(z)dz < \infty$ (see Remark 1.2), we obtain by Fubini’s theorem that for any $x \in \Omega$,

$$\int_{\Omega} G(x,y) \left( \int_{\Omega} G(y,z) K(z) u^{-\alpha}(z)dz \right) dy = \int_{\Omega} K(z) u^{-\alpha}(z)dz \int_{\Omega} G(x,y) G(y,z) dy < \infty.$$ 

Therefore, by classical arguments, $u$ satisfies (4.1).

Now assume that $u \in C^2(\Omega) \cap C_0(\Omega)$, $u > 0$ in $\Omega$ and verifies (4.1). Let us show that $u$ satisfies Definition 1.1. For that, observe that for $\eta > 0$ small enough,

$$u(x) - \frac{\eta}{\lambda_1} \phi_1(x) = \int_{\Omega} G(x,y) \left( \int_{\Omega} G(y,z) (K(z) u^{-\alpha}(z) - \eta \phi_1(z))dz \right) \geq 0. \quad (4.2)$$

Thus $u \geq c \rho$ for some $c > 0$. From the $C^2$-regularity of $u$, we also have that $u \leq C \rho$ for $C > 0$. Therefore, by the assumptions on $K$, for any $\psi \in C^2(\Omega) \cap C_0(\Omega)$,

$$\int_{\Omega} K(x) u^{-\alpha}(x) \psi(x) dx < \infty$$

and hence $u$ satisfies (4.1).

Now we prove Theorem 1.4.

Proof. Let $\alpha + \beta \geq 2$ and $u$ be a solution to $(\text{P})$. From Proposition 4.1, the inequality (4.2) holds and noting that $u \in C^1(\Omega)$, we obtain that $u \sim \rho$ in $\Omega$. Since $\alpha + \beta \geq 2$, from Theorem 2.4 in [21], we get the required contradiction.

5 Bifurcation results

In this section we prove Theorem 1.5. We consider the following bifurcation framework (see Chapter 9 in [7] or Theorem 1.13 in [4] for more details):

Let $X, Y$ be real Banach spaces, $U \subset \mathbb{R}^+ \times X$ an open set. Let $\Psi : U \rightarrow Y$ be a map.

Definition 5.1. $\Psi$ is said to be real analytic on $U$ if for each $x \in U$ there is an $\varepsilon > 0$ and continuous $k$-homogeneous polynomials $P_k : U \rightarrow Y$ such that $\Psi(x + h) = \sum_{k=0}^{\infty} P_k(h)$ if $\|h\| < \varepsilon$.

Define the solution set

$$S = \{ (\lambda, x) \in U : \Psi(\lambda, x) = 0 \}$$

and the non-singular solution set

$$N = \{ (\lambda, x) \in S : \text{Ker}(\partial_x \Psi(\lambda, x)) = \{0\} \}.$$ 

Definition 5.2. A distinguished arc is a maximal connected subset of $N$.

Suppose that

(G1) Bounded closed subsets of $S$ are compact in $\mathbb{R} \times X$.

(G2) $\partial_x \Psi(\lambda, x)$ is a Fredholm operator of index zero for all $(\lambda, x) \in S$. 

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There exists an analytic function \((\lambda, u) : (-\epsilon, \epsilon) \to \mathbb{S}\) such that \(\partial_u \Psi(\lambda(s), u(s))\) is invertible for all \(s \in (-\epsilon, \epsilon)\) and \(\lim_{s \to 0^+} \Psi(\lambda(s), u(s)) = (0, u_0)\) where \(u_0 \in \mathbb{X}\) is the unique solution to \(\Psi(0, u_0) = 0\).

Let

\[ A_0 = \{(\lambda(s), u(s)) : s \in (-\epsilon, \epsilon)\} \neq \emptyset. \]

Obviously, \(A_0 \subset \mathbb{S}\). The following result gives a global extension of the function \((\lambda, u)\) from \((\epsilon, \epsilon)\) to \((-\infty, \infty)\) in the real analytic case.

**Theorem 5.3.** Suppose \((G1)-(G3)\) hold. Then, \((\lambda, u)\) can be extended as a continuous map (still called) \((\lambda, u) : (-\infty, \infty) \to \mathbb{S}\) with the following properties:

(a) Let \(A \overset{\text{def}}{=} \{(\lambda(s), u(s)) : s \in \mathbb{R}\}\). Then, \(A \cap N\) is an almost countable union of distinct distinguished arcs \(\bigcup_{i=1}^{n} A_i, n \leq \infty\).

(b) \(A_0 \subset A_1\).

(c) \(\{s \in \mathbb{R} : \ker(\partial_u \Psi(\lambda(s), u(s))) \neq \{0\}\}\) is a discrete set.

(d) At each of its points \(A\) has a local analytic re-parameterization in the following sense:

For each \(s^* \in \mathbb{R}\) there exists a continuous, injective map \(\rho^* : (-1, 1) \to \mathbb{R}\) such that \(\rho^*(0) = s^*\) and the re-parametrisation

\((-1, 1) \ni t \mapsto (\lambda(\rho^*(t)), u(\rho^*(t))) = \mathcal{A}\) is analytic.

Furthermore, the map \(s \mapsto \lambda(s)\) is injective in a neighborhood of \(s = 0\) and for each \(s^* \neq 0\) there exists \(\epsilon^* > 0\) such that \(\lambda\) is injective on \([s^*, s^* + \epsilon^*]\) and on \([s^* - \epsilon^*, s^*]\).

(e) Only one of the following alternatives occurs:

(i) \(|(\lambda(s), u(s))|_{\mathbb{R} \times \mathbb{X}} \to \infty\) as \(s \to +\infty\) (resp. \(s \to -\infty\)).

(ii) a subsequence \(\{(\lambda(s_n), u(s_n))\}\) approaches the boundary of \(\mathbb{U}\) as \(s_n \to +\infty\) (resp. \(s_n \to -\infty\)).

(iii) \(\mathcal{A}\) is the closed loop:

\[ \mathcal{A} = \{(\lambda(s), u(s)) : -T \leq s \leq T, (\lambda(T), u(T)) = (\lambda(-T), u(-T))\} \text{ for some } T > 0. \]

In this case, choosing the smallest such \(T > 0\) we have

\[ (\lambda(s + 2T), u(s + 2T)) = (\lambda(s), u(s)) \text{ for all } s \in \mathbb{R}. \]

(f) Suppose \(\partial_u \Psi(\lambda(s_1), u(s_1))\) is invertible for some \(s_1 \in \mathbb{R}\). If for some \(s_2 \neq s_1\), we have

\[ (\lambda(s_1), u(s_1)) = (\lambda(s_2), u(s_2)) \]

then (e)/(iii) occurs and \(|s_1 - s_2|\) is an integer multiple of \(2T\). In particular, the map \(s \mapsto (\lambda(s), u(s))\) is injective on \([-T, T]\).

**Remark 5.4.** We remark that theorem 9.1.1 in [7] deals with “bifurcation from the first eigenvalue” type of situation whereas Theorem 1.13 in [4] concerns the bifurcation from origin. The conditions \((G1) - (G3)\) assumed there are required only to ensure that the starting analytic path corresponding to \(A_0\) is available for global extension. In our case, we make this as an assumption \((G3)\) above. Hence the proof given in [7] and in [4] holds good in our case as well.
We recall the following result from [4] (proposition 2.1).

**Proposition 5.5.** Let \( g : \mathbb{R} \to \mathbb{R} \) be an entire function with \( g(0) = 0 \). Define \( M_k(a) = \max_{-a, a} g^{(k)}(z) \), \( k = 1, 2, 3, ... \) Assume that for any \( a \geq 0 \), there exists \( \mu > 0 \) such that the series \( \sum_{k=0}^\infty \frac{M_k(a)}{\mu^k} \) converges. Then, for any \( \phi \in C_0(\Omega) \), \( \phi > 0 \) in \( \Omega \), we have \( C_\phi(\Omega) \ni u \mapsto g(u) \in C_\phi(\Omega) \) is an analytic map. Furthermore, if \( \inf_{[0, \infty)} g' > 0 \), then \( g \) maps \( C^+_\phi(\Omega) \) into itself.

Consider now the solution operator \( F \) associated to \((P_\lambda)\) defined in [1.7].

**Proposition 5.6.** The map \( F \) takes \( \mathbb{R} \times C^+_\phi(\Omega) \) into \( C^+_\phi(\Omega) \) and is analytic. Furthermore, if \( \lambda \geq 0 \), then \( F(\lambda, \cdot) \) maps \( C^+_\phi(\Omega) \) into \( C^+_\phi(\Omega) \).

**Proof.** Step 1: The map \( C^+_\phi(\Omega) \ni u \mapsto K(x)u^{-\alpha} + \lambda f(u) \in C^\alpha_{\phi_1}(\Omega) \) is analytic.

Given \( u \in C^+_\phi(\Omega) \), it follows that \( K(x)u^{-\alpha} \in C^\alpha_{\phi_1}(\Omega) \). Then following the arguments in Step 1 of prop.2.3 in [4], we obtain the analyticity of the map.

Step 2: The map \( C^\alpha_{\phi_1}(\Omega) \ni u \mapsto (\Delta^2)^{-1}u \in C^\phi(\Omega) \) is a linear continuous map (and hence analytic). Furthermore, this map takes \( C^+_\phi(\Omega) \) into \( C^+_\phi(\Omega) \).

We observe that \((\Delta^2)^{-1}\) is well defined on \( C^\alpha(\Omega) \). Indeed, since \( \alpha + \beta < 2 \), from lemma 2.3 there exists a unique solution \( w \in C^{2, \theta}(\Omega), 0 < \theta < 1 \), solving

\[
\begin{aligned}
\Delta^2 w &= u \text{ in } \Omega, \quad u \in C^\alpha_{\phi_1}(\Omega), \\
\lambda w &= 0 \text{ on } \partial \Omega.
\end{aligned}
\]

Clearly, \( w := (\Delta^2)^{-1}u \in C^\phi(\Omega) \). If additionally \( u \in C^+_\phi(\Omega) \), from the Hopf principle in corollary 2.3 we also have that \( w \in C^+_\phi(\Omega) \). The proof of the proposition follows by combining steps 1 and 2. \( \square \)

We now prove the existence of \( A_0 \):

**Proposition 5.7.** Let \( 0 < \alpha + \beta < 2 \). There exists a \( \lambda_0 > 0 \) such that for all \( \lambda \in (-\lambda_0, \lambda_0) \), there exists a non degenerate solution \( u_\lambda \in C^+_\phi(\Omega) \) to \((P_\lambda)\). Furthermore, the map

\[
(-\lambda_0, \lambda_0) \ni \lambda \mapsto u_\lambda \in C^+_\phi(\Omega)
\]

is analytic and \( \|u_\lambda - u_0\|_{C^\phi(\Omega)} \to 0 \) as \( \lambda \to 0 \) where \( u_0 \) is the unique solution to \((P)\).

**Proof.** We would like to apply the analytic version of the implicit function theorem. Given \( u \in C^+_\phi(\Omega) \), we can check that the linearised operator \( \partial_u F(\lambda, u) : C^\phi(\Omega) \to C^\phi(\Omega) \) is given by

\[
\partial_u F(\lambda, u) \phi = \phi + (\Delta^2)^{-1} \left( [\alpha K(x)u^{-\alpha-1} - \lambda f'(u)] \phi \right).
\]

Note that \( K(x)u^{-\alpha-1} \phi \in C^\alpha_{\phi_1}(\Omega) \). Indeed, for some \( C > 0 \),

\[
\|K(x)u^{-\alpha-1} \phi\|_{C^\alpha_{\phi_1}(\Omega)} \leq C \|\phi\|_{C^\phi(\Omega)}.
\]

Therefore from lemma 2.3 we obtain a constant \( \theta \in (0, 1) \) (depending only on \( \nu, \alpha \) and \( \beta \)) such that

\[
\left\{ K(x)u^{-\alpha-1} \phi \right\} \text{ bounded in } C^\alpha_{\phi_1}(\Omega) \implies \left\{ (\Delta^2)^{-1}(K(x)u^{-\alpha-1} \phi) \right\} \text{ bounded in } C^{2, \theta}(\Omega).
\]
We infer then that \( \partial_u F(\lambda, u) \) is a compact perturbation of the identity and hence is a Fredholm operator of 0-index. Next, we show that \( \partial_u F(\lambda, u) \) is invertible for \( \lambda \leq 0 \). If \( \phi \) belongs to the kernel of \( \partial_u F(\lambda, u) \), denoted by \( N(\partial_u F(\lambda, u)) \), we will have

\[
\int_\Omega (\Delta \phi)^2 dx + \alpha \int_\Omega K(x) u^{-\alpha-1} \phi^2 dx - \lambda \int_\Omega f'(u) \phi^2 dx = 0.
\]

Using \((f_2)\) and non positivity of \( \lambda \) we get \( \phi \equiv 0 \) from the above identity. Therefore, if \( \lambda \leq 0 \), we have \( N(\partial_u F(\lambda, u)) = \{0\} \) which implies that \( \partial_u F(\lambda, u) \) is invertible.

Appealing to the real analytic version of implicit function theorem (see [7]), we obtain a

**Proof.** Using the assumption on \( \partial \) kernel of \( \partial \) operator of 0-index. Next, we show that

\[
\text{Proposition 5.8. There exists } \Lambda > 0 \text{ such that } (P_\lambda) \text{ admits no solution for } \lambda > \Lambda.
\]

**Proof.** Using the assumption on \( K \) and \((f_2)\) we note that for some positive constants \( c_1, c_2 \)

\[
K(x) t^{-\alpha} + \lambda f(t) \geq c_1 + c_2 \lambda t \quad \text{for all } x \in \Omega, t > 0.
\]

Let \( \lambda > 0 \). We multiply the equation in \((P_\lambda)\) by \( \phi_1 \) and use the above inequality to get for any solution \( u \) to \((P_\lambda)\):

\[
\lambda_1^2 \int_\Omega \phi_1 dx = \int_\Omega K(x) \phi_1 u^{-\alpha} dx + \lambda \int_\Omega f(u) \phi_1 dx \geq c_1 \int_\Omega \phi_1 dx + c_2 \lambda \int_\Omega u \phi_1 dx.
\]

Hence, necessarily, \( \lambda \leq \frac{\lambda_1^2}{c_2} \). We finally prove Theorem 5.3.

**Proof.** Let \( \mathcal{U} := \mathbb{R} \times \mathcal{C}_{\phi_\lambda}^+(\Omega) \). We first check that \( F \) satisfies the conditions \((G1) - (G3)\) in order to apply Theorem 5.3. From the regularity estimate in lemma 2.3 we deduce that any bounded subset of \( \mathcal{S} \) is relatively compact in \( \mathbb{R} \times \mathcal{C}_{\phi_\lambda} \), i.e. \((G1)\) holds. \((G2) - (G3)\) follow from the above proposition. Hence theorem 5.3 asserts the existence of \( \mathcal{A} \subset \mathcal{S} \) satisfying \((a) - (e)\). (i) follows from proposition 5.7.

(ii) We first prove that assertion in the alternative \((e)(i)\) of theorem 5.3 occurs. We do this by ruling out the possibilities \((e)(ii)\) and \((e)(iii)\). The case \((e)(ii)\) can be ruled out as follows. Suppose there exists a sequence \( \{ (\lambda(s_n), u(s_n)) \} \subset \mathcal{A} \) such that \( (\lambda(s_n), u(s_n)) \to (\lambda, \tilde{u}) \in \partial \mathcal{U} \) in \( \mathbb{R} \times \mathcal{C}_{\phi_\lambda}(\Omega) \) as \( s_n \to \infty \). In particular, \( \tilde{u} \notin \mathcal{C}_{\phi_\lambda}^+(\Omega) \). Applying corollary 2.2 we get for some \( C > 0 \) independent of \( n \),

\[
(5.3) \quad u(s_n)(x) \geq C \rho(x) \int_\Omega (K(y) u(s_n)^{-\alpha} + \lambda(s_n) f(u(s_n))) \rho(y) dy \quad \forall x \in \Omega.
\]

Passing to the limit as \( n \to \infty \) and using Fatou’s Lemma, we get

\[
(5.4) \quad \tilde{u}(x) \geq C \rho(x) \int_\Omega (K(y) \tilde{u}^{-\alpha} + \tilde{\lambda} f(\tilde{u})) \rho(y) dy
\]

which contradicts the assumption \( \tilde{u} \notin \mathcal{C}_{\phi_\lambda}^+(\Omega) \) if \( \tilde{\lambda} \geq 0 \).

Next we rule out alternative \((e)(iii)\). For that, we observe that \( u_0 \) is the unique solution to \((P_0)\) and from the implicit function theorem, \( A_0 \) is the unique branch of solutions emanating from \((0, u_0)\). Therefore, \( \mathcal{A} \) can not bend back to join the point \((0, u_0)\).
Hence alternative (e)(i) of theorem 5.3 holds. From proposition 5.8 the conclusion (ii) of theorem follows.

(iii) follows in view of (ii) and the fact that there is no solution for all large $\lambda$ (prop. 5.8).

(iv) and (v) of theorem 1.5 follow directly from (c) and (d) of theorem 5.3.

(vi) We also note that (see Proposition 5.7) since $\partial_u F(\lambda, u_\lambda)$ is an invertible operator for $\lambda < 0$, the negative portion of $\mathcal{A}$ i.e., $\mathcal{A} \cap (-\infty, 0) \times C_{\mathcal{A}}(\Omega)$, is a single analytic curve (indeed a graph from the $\lambda$ axis) consisting of non-degenerate solutions $u_\lambda$. In particular, this curve does not undergo any bifurcations.

This completes the proof of the theorem.

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References

[1] Adimurthi and J. Giacomoni, Multiplicity of positive solutions for singular and critical elliptic problem in $\mathbb{R}^2$, Commun. Contemp. Math., 8 (2006), 621–656.

[2] G. Arioli, F. Gazzola, H. C. Grunau and E. Mitidieri, A semilinear fourth order elliptic problem with exponential nonlinearity, SIAM J. Anal., 36(4) 2005, 1226–1258.

[3] K. Bal and J. Giacomoni, A remark on symmetry of solutions to singular equations and applications, Eleventh International Conference Zaragoza-Pau on Applied Mathematics and Statistics, Monogr. Mat. Garcia Galdeano, 37 (2012), 25–35.

[4] B. Bougherara, J. Giacomoni and S. Prashanth, Analytic global bifurcation and infinite turning points for very singular problems, Calc. Var. Partial Differential Equations, 52(3-4), 829–856.

[5] H. Brezis and X. Cabré, Some simple nonlinear PDE’s without solutions, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat., 1(2) (1998), 223-262.

[6] H. Brezis, M. Marcus and A. C. Ponce, Nonlinear elliptic equations with measures revisited, Mathematical aspects of nonlinear dispersive equations, 55–109, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.

[7] B. Buffoni and J. F. Toland, Analytic theory of global bifurcation. An introduction, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2003.

[8] B. Buffoni, E. N. Dancer and J.F. Toland, The regularity and local bifurcation of steady periodic water waves, Arch. Rational Mech. Anal., 152 (2000), 207–240.

[9] B. Buffoni, E. N. Dancer and J.F. Toland, The sub-harmonic bifurcation of Stokes waves, Arch. Rational Mech. Anal., 152 (2000), 241–271.

[10] A. Canino, M. Degiovanni, A variational approach to a class of singular semilinear elliptic equations, J. Convex Anal. 11(1) (2004), 147–162.

[11] M. G. Crandall, P. H. Rabinowitz, and L. Tartar, On a Dirichlet problem with a singular nonlinearity, Comm. Partial Differential Equations, 2 (1977), 193–222.
[12] J. Dávila, L. Dupaigne, I. Guerra and M. Montenegro, Stable solutions for the bilaplacian with exponential nonlinearity, SIAM J. Math. Anal., 39(2) (2007), 565–592.

[13] J. Dávila, I. Flores and I. Guerra, Multiplicity of solutions for a fourth order problem with exponential nonlinearity, J. Differential Equations, 247 (2009), 3136–3162.

[14] J. Dávila, I. Flores and I. Guerra, Multiplicity of solutions for a fourth order equation with power-type nonlinearity, Math. Ann., 348 (2010), 143–193.

[15] R. Dhanya, J. Giacomoni, S. Prashanth and K. Saoudi, Global bifurcation and local multiplicity results for elliptic equations with singular nonlinearity of super exponential growth in $\mathbb{R}^2$, Adv. Differential Equations, 17 (3-4) (2012), 369–400.

[16] L. Dupaigne, M. Ghergu, O. Goubet and G. Warnault, The Gel’fand problem for the biharmonic operator, Arch. rational Mech. Anal., 208 (2013), 725–752.

[17] F. Gazzola, H. C. Grunau and M. Squassina, Existence and nonexistence results for critical growth biharmonic elliptic equations, Calc. Var. Partial Differential Equations, 18(2) (2003), 117–143.

[18] F. Gazzola, H. C. Grunau and G. Sweers, Polyharmonic boundary value problems. Positivity preserving and nonlinear higher elliptic equations in bounded domains, Lecture Notes in Mathematics, 1991. Springer-Verlag, Berlin, 2010.

[19] M. Ghergu and V. D. Radulescu, Singular Elliptic Problems: Bifurcation and Asymptotic Analysis, Oxford University Press, 2008.

[20] M. Ghergu, A biharmonic equation with singular nonlinearity, Proc. Edinb. Math. Soc., 55(2) (2012), 155–166.

[21] M. Ghergu, Lane-Emden systems with nonnegative exponents, J. Funct. Anal., 258(10) (2010), 3295–3318.

[22] J. Giacomoni, I. Schindler, and P. Takáč, Sobolev versus Hölder minimizers and existence of multiple solutions for a singular quasilinear equation, Annali della Scuol. Norm. Sup. di Pisa, Serie V, Vol VI, Fasc 1 (2007), 544–572.

[23] J. Giacomoni, I. Schindler, and P. Takáč, Singular quasilinear elliptic systems and Hölder regularity, Adv. Differential Equations, 20(3–4) (2015), 259–298.

[24] C. Gui and F.-H. Lin, Regularity of an elliptic problem with a singular nonlinearity, Proc. Roy. Soc. Edinburgh Sect. A, 123 (1993), 1021–1029.

[25] Y. Haitao, Multiplicity and asymptotic behaviour of positive solutions for a singular semilinear elliptic problem, Jour. Diff. equations, 189(2003),487-512.

[26] J. Hernández and F. J. Mancebo, Singular elliptic and parabolic equations, Handbook of Differential Equations, 3 (2006), 317–400.

[27] J. Hernández, F. Mancebo, J. M. Vega, On the linearization of some singular, nonlinear elliptic problems Ann. Inst. H. Poincaré Anal. Non Linéaire, 19 (2002), 777–813.
[28] N. Hirano, C. Saccon and N. Shioji, Brezis-Nirenberg type theorems and multiplicity of positive solutions for a singular elliptic problem, Jour. Diff. Equations, 245(2008), 1997-2037.
J.A. Leach and D. J. Needham, Matched Asymptotic Expansions in Reaction-diffusion Theory, Springer-Verlag, Berlin (2004).

[29] H. Triebel, Interpolation theory, function spaces, differential operators. Johann Ambrosius Barth, heidelberg, second edition, 1995.