Poincaré and $sl(2)$ algebras of order 3

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Abstract

In this paper we initiate a general classification for Lie algebras of order 3 and we give all Lie algebras of order 3 based on $sl(2,\mathbb{C})$ and $iso(1,3)$ the Poincaré algebra in four-dimensions. We then set the basis of the theory of the deformations (in the Gerstenhaber sense) and contractions for Lie algebras of order 3.

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1 Introduction and motivation

The concept of symmetry, and its associated algebraic structures, is central in the understanding of the properties of physical systems. This means, in particular, that a better comprehension of the laws of physics may be achieved by an identification of the possible mathematical structures as well as their classification. For instance, the properties of elementary particles and their interactions are very well understood within Lie algebras. Moreover, the discovery of supersymmetry gave rise to the concept of Lie superalgebras which becomes central in theoretical physics and mathematics. Of course not all the mathematical structures would be relevant in physics. For instance, they are constraint by the principle of quantum mechanics and relativity. This was synthesize in two no-go theorems which restrict drastically the possible Lie algebras [2] and Lie superalgebras [15] one is able to consider in physics.

But it turns out that Lie (super)algebras are not the only allowed structures one is able to consider. Several attempts to construct models based on different algebras were proposed. Here we focus on one of the possible extensions called fractional supersymmetry [3, 4, 5, 6, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29] together with its associated underlying algebraic structure named $F$–Lie algebra or Lie algebra of order $F$ [22, 27] (note that a different approach has been proposed in [17]). Lie algebras of order $F$ lead to new models of symmetry of space-time i.e. lead to some new non-trivial extensions of the Poincaré algebra, which involves $F$–ary ($F \geq 3$) relations instead of the usual quadratic ones [20, 21, 22, 29]. These new structures can be seen as a possible generalization of Lie (super)algebras. An $F$–Lie algebra admits a $Z_F$–gradation, the zero-graded part being a Lie algebra. An $F$–fold symmetric product (playing the role of the anticommutator in the case $F = 2$) expresses the zero graded part in terms of the non-zero graded part.

Subsequently, a detailed analysis when $F = 3$ and for a specific extension of the Poincaré algebra was undertaken together with its explicit implementation in quantum field theory [20, 21, 22, 29]. However, this
general study revealed some difficulties that are not already resolved. This means that in order to understand
the impact of these new structures, a general algebraic study should be undertaken. Thus, the aim of this
paper is two-fold. Firstly a general classification of Lie algebra of order 3 is initiated when the zero graded
part of the algebra is either (i) \(\mathfrak{sl}(2)\) or (ii) the Poincaré algebra in four dimensions \(\mathfrak{iso}(1, 3)\). It is then
shown that the structure of the algebra is relatively rigid and a few examples of Lie algebras of order 3
are possible in the former case (see Theorem 3.1). Although in the latter cases, since the generators of the
space-time translation commute, there are many possible extensions of the Poincaré algebra (see Theorem
4.8). Secondly, a theory of deformations and contractions is presented. This can be seen as a natural extension
of the theory of contraction/deformation of Lie (super)algebras (see for example [12, 13]) to Lie algebras
of order 3. Indeed, contraction/deformation are relevant in physics in the sense that they may provide a
relationship between two different theories. For instance, the Poincaré algebra of special relativity and the
Galilean algebra of non-relativistic physics are related through an İnönü-Wigner contraction. In the same
way it is known that the \(N\)-extended supersymmetric extension of the Poincaré algebra can be obtained
through a contraction of the superalgebra \(\mathfrak{osp}(4|N)\). Similarly the extension of the Poincaré algebra studied
in [20, 21, 22, 29] was obtained through a contraction of a certain Lie algebra of order 3 [26].

The content of the paper is organised as follow. In the next section the definition of Lie algebras of order
3 is recalled. Explicit examples are then given. In section three a classification of all Lie algebras of order 3
associated to \(\mathfrak{sl}(2)\) is given. Section four is devoted to a general study of Lie algebras of order 3 associated to
the Poincaré algebra in four dimensions \(\mathfrak{iso}(1, 3)\). In section five the general notion (in a topological sense) of
contractions is defined. This notion being too general more useful contractions (İnönü-Wigner contractions)
are introduced. Section six is devoted to the implementation of the theory of deformations of Lie algebra
of order 3 in the Gerstenhaber sense. Infinitesimal and isomorphic deformations are then introduced.

2 Lie algebras of order 3

In this section we recall the definition and some basic properties of Lie algebras of order \(F\) introduced in
[20] and [27] and we define the algebraic variety of these algebraic structures.

2.1 Definition and examples of elementary Lie algebras of order 3

**Definition 2.1** Let \(F \in \mathbb{N}^+\). A \(\mathbb{Z}_F\)-graded \(\mathbb{C}\)-vector space \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \cdots \oplus \mathfrak{g}_{F-1}\) is called a complex
Lie algebra of order \(F\) if

1. \(\mathfrak{g}_0\) is a complex Lie algebra.
2. For all \(i = 1, \ldots, F-1\), \(\mathfrak{g}_i\) is a representation of \(\mathfrak{g}_0\). If \(X \in \mathfrak{g}_0, Y \in \mathfrak{g}_i\) then \([X, Y]\) denotes the action
   of \(X \in \mathfrak{g}_0\) on \(Y \in \mathfrak{g}_i\) for all \(i = 1, \ldots, F-1\).
3. For all \(i = 1, \ldots, F-1\) there exists an \(F\)-linear, \(\mathfrak{g}_0\)-equivariant map
   \(\mu_i : S^F(\mathfrak{g}_i) \rightarrow \mathfrak{g}_0\),

   where \(S^F(\mathfrak{g}_i)\) denotes the \(F\)-fold symmetric product of \(\mathfrak{g}_i\), satisfying the following (Jacobi) identity

   \[
   \sum_{j=1}^{F+1} [Y_j, \mu_i(Y_1, \ldots, Y_j-1, Y_{j+1}, \ldots, Y_{F+1})] = 0,
   \]

   for all \(Y_j \in \mathfrak{g}_i, j = 1, \ldots, F+1\).

**Remark 2.2** If \(F = 1\), by definition \(\mathfrak{g} = \mathfrak{g}_0\) and a Lie algebra of order 1 is a Lie algebra. If \(F = 2\), then
\(\mathfrak{g}\) is a Lie superalgebra. Therefore, Lie algebras of order \(F\) appear as some kind of generalisations of Lie
algebras and superalgebras.

Note that by definition the following Jacobi identities are satisfied by a Lie algebra of order \(F\):
For any $X, X', X'' \in \mathfrak{g}_0$,

$$[[X, X'], X''] + [[X', X''], X] + [[X'', X], X'] = 0. \quad \text{J1}$$

For any $X, X' \in \mathfrak{g}_0$ and $Y \in \mathfrak{g}_i, i = 1, \ldots, F - 1$,

$$[[X, X'], Y] + [[X', Y], X] + [[Y, X], X'] = 0. \quad \text{J2}$$

For any $X \in \mathfrak{g}_0$ and $Y_j \in \mathfrak{g}_i, \ j = 1, \ldots, F, i = 1, \ldots, F - 1$,

$$[X, \mu_i(Y_1, \ldots, Y_F)] = \mu_i([X, Y_1], \ldots, Y_F) + \cdots + \mu_i(Y_1, \ldots, [X, Y_F]). \quad \text{J3}$$

For any $Y_j \in \mathfrak{g}_i, j = 1, \ldots, F + 1, i = 1, \ldots, F - 1$,

$$\sum_{j=1}^{F+1} [Y_j, \mu_i(Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_{F+1})] = 0. \quad \text{J4}$$

**Proposition 2.3** Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{F-1}$ be a Lie algebra of order $F$, with $F > 1$. For any $i = 1, \ldots, F - 1$, the $\mathbb{Z}_F$-graded vector spaces $\mathfrak{g}_0 \oplus \mathfrak{g}_i$ is a Lie algebra of order $F$. We call these type of algebras elementary Lie algebras of order $F$.

In [26] an inductive process for the construction of Lie algebras of order $F$ starting from a Lie algebra of order $F_1$ with $1 \leq F_1 < F$ is given. In this paper we are especially concerned by deformation and classification problems. Moreover, we restrict ourselves to elementary Lie algebras of order 3, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We also denote the 3-linear map $\mu_1$ by the 3-entries bracket $\{\ldots\}$ and we refer to it as a 3-bracket. Non-trivial examples of Lie algebras of order $F$ (finite and infinite-dimensional) are given in [26] and [27].

We now give some examples of finite-dimensional Lie algebras of order 3, which will be relevant in the sequel.

**Example 2.4** Let $\mathfrak{g}_0 = \mathfrak{so}(2, 3)$ and $\mathfrak{g}_1$ its adjoint representation. Let $\{J_a, a = 1, \ldots, 10\}$ be a basis of $\mathfrak{g}_0$ and $\{A_a = \text{ad}(J_a), a = 1, \ldots, 10\}$ be the corresponding basis of $\mathfrak{g}_1$. Thus, one has $[J_a, J_b] = \text{ad}([J_a, J_b])$. Let $g_{ab} = \text{Tr}(A_a A_b)$ be the Killing form. Then one can endow $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a Lie algebra of order 3 structure given by

$$\{A_a, A_b, A_c\} = g_{ab} J_c + g_{ac} J_b + g_{bc} J_a.$$

**Example 2.5** Let $\mathfrak{g}_0$ be the Poincaré algebra in four dimensions and $\{L_{mn}, P_m : L_{mn} = -L_{nm}, m < n, m, n, = 0, \ldots, 3\}$ be a basis of $\mathfrak{g}_0$ with the non-zero brackets

$$[L_{mn}, L_{pq}] = \eta_{mq} L_{pm} - \eta_{mq} L_{pm} + \eta_{np} L_{mq} - \eta_{np} L_{mq},$$

$$[L_{mn}, P_p] = \eta_{np} P_m - \eta_{mp} P_n. \quad \text{(2.1)}$$

Let now $\mathfrak{g}_1$ be the 4-dimensional vector representation of $\mathfrak{g}_0$, the action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is given by

$$[L_{mn}, V_p] = \eta_{mp} V_m - \eta_{mp} V_m, \quad [P_m, V_n] = 0$$

where $\{V_m : \ m = 0, \ldots, 3\}$ is a basis of $\mathfrak{g}_1$. The following brackets on $\mathfrak{g}_1$

$$\{V_m, V_n, V_r\} = \eta_{mn} P_r + \eta_{mr} P_n + \eta_{rn} P_m, \quad \text{(2.2)}$$

with the metric $\eta_{mn} = \text{diag}(1, -1, -1, -1)$ endow $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ with an elementary Lie algebra of order 3 structure.
2.2 The variety elementary $\mathcal{F}_{m,n}$ of Lie algebras of order 3

Let $g = g_0 \oplus g_1$ be an elementary Lie algebra of order 3 and let $A = (g_0 \otimes g_0) \oplus (g_0 \otimes g_1) \oplus S^{3}(g_1)$. The multiplication of Lie algebra of order 3 is given by the linear map

$$\varphi : A \to g$$

satisfying the conditions J1-J4. Let $\varphi_1, \varphi_2, \varphi_3$ be the restrictions of $\varphi$ to each of the terms of $A$

\begin{align*}
\varphi_1 : g_0 \otimes g_0 &\to g_0, \\
\varphi_2 : g_0 \otimes g_1 &\to g_1, \\
\varphi_3 : S^{3}(g_1) &\to g_0.
\end{align*}

We denote this by $\varphi = (\varphi_1, \varphi_2, \varphi_3)$.

Let $\{X_i : i = 1, \ldots, m\}$ and resp. $\{Y_a : a = 1, \ldots, n\}$ be a basis of $g_0$ and resp. $g_1$. The maps $\varphi_i$ $(i = 1, 2, 3)$ are defined by their structure constants with regard to this basis

$$\varphi_1(X_i, X_j) = C^k_{ij}X_k, \quad \varphi_2(X_i, Y_b) = D^c_{ib}Y_c \quad \text{and} \quad \varphi_3(Y_a, Y_b, Y_c) = E^i_{abc}X_i.$$  \hspace{1cm} (2.3)

The structure constants $(C^k_{ij}, D^c_{ib}, E^i_{abc})$ verify the following conditions:

\begin{align*}
C^k_{ij} &= -C^k_{ji}, \\
E^i_{abc} &= E^i_{acb} = E^i_{bac} = E^i_{caba} = E^i_{cab} = E^i_{cda}.
\end{align*}  \hspace{1cm} (2.4)

and the polynomial equations corresponding to the Jacobi conditions J1-J4 are:

\begin{align*}
C^k_{ij}C^n_{kl} + C^k_{jk}C^n_{li} + C^k_{ki}C^n_{lj} &= 0, \\
C^k_{ij}D^c_{ka} - D^c_{ja}D^i_{kb} + D^c_{ja}D^i_{kb} &= 0, \\
E^j_{abc}C^k_{ij} - D^d_{ja}E^k_{dbc} - D^d_{ja}E^c_{adk} - D^d_{ja}E^k_{adb} &= 0, \\
D^i_{jia}E^i_{abc} + D^i_{jia}E^i_{cda} + D^i_{jia}E^i_{cda} + D^i_{jia}E^i_{abc} &= 0.
\end{align*}  \hspace{1cm} (2.5)

Let $C^N$ be the vector space whose elements are the $N$-tuple $(C^k_{ij}, E^i_{abc}, D^i_{jia})$, with $N = mC^2_0 + mn^2 + mC^3_n$. The polynomial equations determine an algebraic variety $\mathcal{F}_{m,n}$ embedded in $C^N$. Each point of $\mathcal{F}_{m,n}$ correspond to an $(m + n)$-dimensional Lie algebra of order 3. Thus we identify any elementary Lie algebra of order 3 with the bracket $\varphi$ to a point of $\mathcal{F}_{m,n}$.

Let us now consider the action of the group $GL(m, n) \cong GL(m) \times GL(n)$ on $\mathcal{F}_{m,n}$. For any $(h_0, h_1) \in GL(m, n)$, this action is defined by

$$(h_0, h_1) \cdot (\varphi_1, \varphi_2, \varphi_3) \to (\varphi'_1, \varphi'_2, \varphi'_3)$$

where

\begin{align*}
\varphi'_1(X_1, X_2) &= h_0^{-1}\varphi_1(h_0(X_1), h_0(X_2)), \\
\varphi'_2(X_1, Y_2) &= h_1^{-1}\varphi_2(h_0(X_1), h_1(Y_2)), \\
\varphi'_3(Y_1, Y_2, Y_3) &= h_0^{-1}\varphi_3(h_1(Y_1), h_1(Y_2), h_1(Y_3)).
\end{align*}  \hspace{1cm} (2.6)

where $(Y_1, Y_2, Y_3)$ represents an element of $S^{3}(g_1)$. The group $GL(m, n)$ can be embedded in $GL(n + m)$. It corresponds to the subgroup of $GL(m + n)$ which let invariant the subspaces $g_0$ and $g_1$ of $g$. Denote by $O_\varphi$ the orbit of $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ with respect to this action. Then the algebraic variety $\mathcal{F}_{m,n}$ is fibered by isemis orbits. The quotient set is the set of isomorphism classes of $(m + n)$-dimensional elementary Lie algebras of order 3.
3 \( \mathfrak{sl}(2) \)-algebras of order 3

In this section we study complex Lie algebras of order 3, \( \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) for which \( \mathfrak{g}_0 \cong \mathfrak{sl}(2) \) and \( \mathfrak{g}_1 \) is an arbitrary representation of \( \mathfrak{sl}(2) \). We denote by \( X_+, X_-, X_0 \) a standard basis of \( \mathfrak{g}_0 \)

\[
[X_0, X_+] = 2X_+, \quad [X_0, X_-] = -2X_-, \quad [X_+, X_-] = X_0, \quad (3.1)
\]

and \( D_\ell (\ell \in \mathbb{N}) \) an irreducible representation of dimension \( \ell + 1 \).

**Theorem 3.1** The graded complex vector space \( \mathfrak{g} \cong \mathfrak{sl}(2) \oplus \mathfrak{g}_1 \), with \( \mathfrak{g}_1 \) a representation of \( \mathfrak{sl}(2) \) is provided with a non-trivial Lie algebra structure of order 3 if and only if:

1. \( \mathfrak{g}_1 \cong D_2 (D_2 = \langle Y_2, Y_0, Y_{-2} \rangle) \), with the non-zero three-brackets

\[
\{Y_2, Y_{-2}, Y_0\} = X_0, \quad \{Y_0, Y_0, Y_0\} = 6X_0, \quad \{Y_2, Y_2, Y_{-2}\} = 2X_+, \quad \{Y_{-2}, Y_0, Y_0\} = 2X_-.
\]

2. \( \mathfrak{g}_1 \cong D_2 \oplus D_0^{(1)} \oplus \cdots \oplus D_0^{(k)} (D_2 = \langle Y_2, Y_0, Y_{-2} \rangle, D_0^{(k)} = \langle \lambda_k \rangle) \), with the non-zero three-brackets

\[
\{\lambda_i, \lambda_j, Y_2\} = \alpha_{ij} X_+, \quad \{\lambda_i, \lambda_j, Y_0\} = \alpha_{ij} X_0, \quad \{\lambda_i, \lambda_j, Y_{-2}\} = \alpha_{ij} X_-.
\]

and \( \alpha_{ij} \in \mathbb{C} \).

3. \( \mathfrak{g}_1 \cong D_1 \oplus D_0 (D_1 = \langle Y_1, Y_{-1} \rangle, D_0 = \langle \lambda \rangle) \), with the non-zero three-brackets

\[
\{\lambda, Y_1, Y_1\} = -2X_+, \quad \{\lambda, Y_1, Y_{-1}\} = X_0, \quad \{\lambda, Y_{-1}, Y_{-1}\} = 2X_-.
\]

**Proof.** Since \( \mathfrak{g}_1 = \bigoplus_{\ell_k} D_{\ell_k} \), with \( D_{\ell_k} \) an irreducible representation of dimension \( \ell_k + 1 \), the 3–brackets \( \{\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1\} \) contain terms like (i) \( \{D_{\ell_1}, D_{\ell_1}, D_{\ell_1}\} \), (ii) \( \{D_{\ell_1}, D_{\ell_1}, D_{\ell_2}\} \), (iii) \( \{D_{\ell_1}, D_{\ell_2}, D_{\ell_3}\} \).

1. Consider firstly the case \( \{D_1, D_0, D_1\} \). A simple weight argument shows that \( \ell \) is even, furthermore the non-vanishing three brackets are

\[
\{Y_i, Y_j, Y_k\} = \begin{cases} 
\alpha_{ijk} X_+ & i + j + k = 2, \\
\beta_{ijk} X_0 & i + j + k = 0, \\
\gamma_{ijk} X_- & i + j + k = -2.
\end{cases}
\]

Suppose firstly that \( \ell = 2 \). The action of \( \mathfrak{sl}(2) \) on \( D_2 \) is

\[
[X_0, Y_{-1}] = -2Y_{-1}, \quad [X_{-}, Y_0] = 2Y_{-1}, \quad [X_{-}, Y_1] = -Y_0, \quad [X_+, Y_0] = -2Y_1, \quad [X_0, Y_1] = 2Y_1.
\]

From symmetry considerations one has \( \alpha_{ijk} = \gamma_{i-j-k} \), and the Jacobi identity J3 gives \( \alpha_{1,-1} = \alpha_{1,0,0} = \gamma_{-1,1,1} = \gamma_{1,-1,0} = 2t, \beta_{i-j-k} = \beta_{0,0,0} = 6t \), with \( t \in \mathbb{C} \). Furthermore, a direct calculus shows that the Jacobi identities J4 are satisfied for any \( t \). If \( t = 0 \), the Lie algebra of order 3 is trivial. If \( t \neq 0 \) all the algebras are equivalent. Since for \( \mathfrak{sl}(2) \) the Casimir operator is given by \( Q = \frac{1}{2}H^2 + X_+X_- + X_-X_+ \) we have \( \text{Tr}(X_+X_-) = g_+ = g_- = 1, \text{Tr}(X_0X_0) = g_{00} = 2 \) and the three-brackets (3.2) can be rewritten [26]

\[
\{Y_i, Y_i, Y_k\} = g_{ij} X_k + g_{jk} X_i + g_{ki} X_j,
\]

(here \( X_+, X_0, X_- \) are denoted \( X_2, X_0, X_{-2} \)).

Now we assume that \( \ell > 2 \) and we prove that \( \{D_\ell, D_\ell, D_\ell\} = 0 \).
1. The bracket \( \{Y_0, Y_0, Y_0\} = 0 \) as a consequence of J4 applied to \((Y_0, Y_0, Y_0)\).

2. The bracket \( \{Y_0, Y_0, Y_i\} = 0 \) as a consequence of J4 applied to \((Y_0, Y_0, Y_i)\).

3. The bracket \( \{Y_0, Y_i, Y_j\} = 0, \ i + j \neq 0 \) as a consequence of J4 applied to \((Y_0, Y_0, Y_i, Y_j)\).

4. The bracket \( \{Y_0, Y_i, Y_{i-1}\} = 0 \). If \( i \neq 2 \) this is a consequence of J4 applied to \((Y_0, Y_i, Y_{i-1})\) and if \( i = 2 \) this is a consequence of J4 applied to \((Y_2, Y_0, Y_{-2})\).

5. The bracket \( \{Y_i, Y_j, Y_k\} = 0, \ i + j + k \neq 0 \) as a consequence of J4 applied to \((Y_0, Y_i, Y_j, Y_k)\).

6. The bracket \( \{Y_i, Y_j, Y_{i-j}\} = 0 \) as a consequence of J4 applied to \((Y_k, Y_i, Y_j, Y_{i-j})\) with \( k \neq i, j, -i - j, 0 \). Such a \( k \) always exists since the dimension of \( \mathcal{D}_\ell \) is bigger than 5.

II. Consider now the case \( \{\mathcal{D}_{\ell_1}, \mathcal{D}_{\ell_1}, \mathcal{D}_{\ell_2}\} \).

- We prove now that \( \{\mathcal{D}_\ell, \mathcal{D}_\ell, \mathcal{D}_0\} = 0 \) for all \( \ell \) but \( \ell = 1 \).

  1. It is easy to see that \( \ell = 0 \) leads to a trivial Lie algebra of order 3.

  2. If \( \ell = 1 \) we denote \( \mathcal{D}_0 = < \lambda > \) and \( \mathcal{D}_1 = < Y_1, Y_{-1} > \). The action of \( \mathfrak{sl}(2) \) on \( \mathcal{D}_1 \) is

\[
\begin{align*}
[X_0, Y_1] &= Y_1, & [X_+, Y_{-1}] &= Y_1, \\
[X_-, Y_1] &= Y_{-1}, & [X_0, Y_{-1}] &= -Y_{-1},
\end{align*}
\]

and a simple weight argument gives

\[
\{\lambda, Y_1, Y_1\} = \alpha X_+, \quad \{\lambda, Y_1, Y_{-1}\} = \beta X_0, \quad \{\lambda, Y_{-1}, Y_{-1}\} = \gamma X_-.
\]

The Jacobi identities J3 and J4 give \( \alpha = -2t, \beta = t, \gamma = 2t, \ t \in \mathbb{C} \).

3. If \( \ell = 2 \) using the Clebsch-Gordan decomposition \( \mathcal{D}_2 \otimes \mathcal{D}_2 = D_4 \oplus D_2^c \oplus \mathcal{D}_0 \), since the representation \( D_2^c \) is antisymmetric in the permutation of the two factors \( \mathcal{D}_2 \), we have \( \{D_2, D_2, D_0\} = 0 \).

4. If \( \ell > 2 \), we denote \( \mathcal{D}_\ell = < Y_\ell, Y_{\ell-2}, \ldots, Y_{-\ell} > \). A simple weight arguments shows that the possible non-vanishing 3-brackets are: \( \{\lambda, Y_\ell, Y_{\ell+2}\}, \{\lambda, Y_{\ell-2}, Y_{\ell+2}\}, \{\lambda, Y_{\ell-2}, Y_{\ell-4}\} \).

- We now consider the brackets of type \( \{\mathcal{D}_\ell, \mathcal{D}_0, \mathcal{D}_0\} \). By a weight argument and identity J4 one has \( \ell = 2 \) and the non-trivial 3-brackets are:

\[
\begin{align*}
\{\lambda, Y_2\} &= \alpha X_+,
\{\lambda, Y_0\} &= \alpha X_0,
\{\lambda, Y_{-2}\} &= \alpha X_-.
\end{align*}
\]

- We now prove that \( \{\mathcal{D}_{\ell_1}, \mathcal{D}_{\ell_2}, \mathcal{D}_{\ell_2}\} = 0 \) with \( \ell_1, \ell_2 \neq 0 \).
1. Let $\ell_1 = \ell_2 = 2$, we denote $\mathcal{D}_2 = \langle Y_2, Y_0, Y_{-2} \rangle$, $\mathcal{D}'_2 = \langle Y'_2, Y'_0, Y'_{-2} \rangle$ the two three-dimensional representations. In this case, we have four types of brackets: $\{\mathcal{D}_2, \mathcal{D}_2, \mathcal{D}_2\}$, $\{\mathcal{D}_2, \mathcal{D}'_2, \mathcal{D}'_2\}$, $\{\mathcal{D}_2, \mathcal{D}_2, \mathcal{D}_2\}$, $\{\mathcal{D}_2, \mathcal{D}'_2, \mathcal{D}'_2\}$. The possible non-vanishing three-brackets are

\[
\begin{align*}
\{Y_i, Y_j, Y_k\} &= \alpha g_{ij}X_k + \alpha g_{jk}X_i + \alpha g_{ki}X_j, \\
\{Y'_i, Y'_j, Y'_k\} &= \alpha' g_{ij}X_k + \alpha' g_{jk}X_i + \alpha' g_{ki}X_j \\
\{Y_2, Y_2, Y'_2\} &= \alpha_1 X_+ \\
\{Y'_2, Y'_2, Y'_2\} &= \alpha_2 X_+ \\
\{Y'_2, Y'_2, Y'_2\} &= \alpha_3 X_+ \\
\{Y'_2, Y'_2, Y'_2\} &= \alpha_4 X_+
\end{align*}
\]

(plus similar terms with $\{Y', Y', Y\}$ and coefficients $\alpha'_1, \ldots, \beta'_4$, as $\{Y'_2, Y'_2, Y_{-2}\} = \alpha'_1 X_+$, etc.). Working out the Jacobi identity $J4$, one gets that the coefficients $\alpha, \alpha_1, \ldots, \beta_4, \alpha', \alpha'_1, \ldots, \beta'_4$ must be zero. In fact one has

(a) $(Y_2, Y_2, Y'_2)$ gives $\alpha_1 = \alpha'_1 = 0$;
(b) $(Y_{-2}, Y_{-2}, Y_2)$ gives $\alpha = \alpha_2 = 0$ (resp. $\alpha' = \alpha'_2 = 0$);
(c) $(Y_0, Y'_0, Y_0)$ gives $\alpha_3 = 0$ (resp. $\alpha'_3 = 0$);
(d) $(Y_0, Y_0, Y'_2)$ gives $\alpha_4 = 0$ (resp. $\alpha'_4 = 0$);
(e) $(Y_2, Y_2, Y_{-2})$ gives $\beta_1 = 0$ (resp. $\beta'_1 = 0$);
(f) $(Y'_2, Y_0, Y'_2)$ gives $\beta_2 = 0$ (resp. $\beta'_2 = 0$);
(g) $(Y_{-2}, Y'_0, Y'_2)$ gives $\beta_3 = 0$ (resp. $\beta'_3 = 0$);
(h) $(Y_2, Y_0, Y'_0)$ gives $\beta_4 = 0$ (resp. $\beta'_4 = 0$).

Thus all the brackets vanishes.

2. $\ell_1 \neq 2, \ell_2 \neq 2$ arbitrary $\mathcal{D}_{\ell_1} = \langle Y_{i_1}, Y_{i_2}, \ldots, Y_{i_{\ell_1}} \rangle$ and $\mathcal{D}_{\ell_2} = \langle Y'_{i_1}, Y'_{i_2}, \ldots, Y'_{i_{\ell_2}} \rangle$. We consider the bracket $\{Y_i, Y_j, Y'_k\}$. The identity $J4$ applied to $(Y_i, Y_j, Y'_k)$ leads to the vanishing of the 3 brackets except when $2i + k \neq 0, \pm 2$ or $(i \neq \ell$ and $i + j + k \neq 2)$. The identity $J4$ applied to $(Y_i, Y_j, Y'_k)$ leads to the vanishing of the 3 brackets except when $2j + k \neq 0, \pm 2$ or $(j \neq \ell$ and $i + j + k \neq 2)$. The cases that remains to be studied are:

(a) If $k = -2j + 2 = -2i + 2$ then $i = j$ and the bracket $\{Y_i, Y_i, Y'_{2i+2}\}$ vanishes.
(b) If $k = -2j + 2 = -2i - 2$ then $i = j - 1$ which is not possible.
(c) If $k = -2j + 2 = -2i - 2$ then $j = i + 2$ and the bracket reduces to $\{Y_i, Y_{i+2}, Y'_{2i+2}\}$. Then identity $J4$ applied to $(Y_i, Y_i, Y_{i+2}, Y'_{2i+2})$ and to $(Y_{i+2}, Y_i, Y_{i+2}, Y'_{2i+2})$ leads to $\{Y_i, Y_{i+2}, Y'_{2i+2}\} = 0$.
(d) If $k = -2j = -2i$ then $i = j$ and the bracket vanishes as before. If $k = -2j = -2i - 2$ then $j = i + 1$ which is also excluded. If $k = -2j - 2 = -2i - 2$ then $i = j$ and the bracket vanishes as before.

3. Let $\ell_1 = 2, \ell_2 \neq 2$ and consider the brackets of the type $\{\mathcal{D}_2, \mathcal{D}_2, \mathcal{D}'_{\ell_2}\}$. The Jacobi identity $J4$ applied on $(Y_{i_1}, Y_{i_2}, Y'_{i_3})$ with $Y_{i_1}, Y_{i_2} \in \mathcal{D}_2$ and $Y_{i_3} \in \mathcal{D}_3$ leads to $\{\mathcal{D}_2, \mathcal{D}_2, \mathcal{D}'_{\ell_2}\} = 0$.

III. Consider now the case $\{\mathcal{D}_{\ell_1}, \mathcal{D}_{\ell_2}, \mathcal{D}_{\ell_3}\}$.

If $\ell_1 = \ell_2 = 0$, by weight arguments we have $\ell_3 = 2$ and the possible non-vanishing three brackets are

\[
\begin{align*}
\{\lambda_1, \lambda_2, Y_2\} &= \alpha_{12}X_+ \\
\{\lambda_1, \lambda_2, Y_0\} &= \alpha_{12}X_0 \\
\{\lambda_1, \lambda_2, Y_{-2}\} &= \alpha_{12}X_-
\end{align*}
\]

where $\mathcal{D}'_0 = \langle \lambda_1 \rangle$, $\mathcal{D}_2 = \langle \lambda_2 \rangle$ and $\mathcal{D}_2 = \langle Y_2, Y_0, Y_{-2} \rangle$.
The Jacobi identity \( J_4 \) with \( \{,\} \) brackets are
\[
\{Y_1, Y_1, \lambda\} = -2\alpha X_+, \quad \{Y_1, Y_-, \lambda\} = \alpha X_0, \quad \{Y_-, Y_-, \lambda\} = 2\alpha X_-
\]
\[
\{Y'_1, Y'_1, \lambda\} = -2\alpha' X_+, \quad \{Y'_1, Y'_-, \lambda\} = \alpha' X_0, \quad \{Y'_-, Y'_-, \lambda\} = 2\alpha' X_-
\]
\[
\{Y_1, Y'_1, \lambda\} = \beta_1 X_+, \quad \{Y_1, Y'_-, \lambda\} = \beta_2 X_0, \quad \{Y'_1, Y_-, \lambda\} = \beta_3 X_0, \quad \{Y'_-, Y_-, \lambda\} = \beta_4 X_-
\]
The Jacobi identity \( J_4 \) with \( \{,\} \) implies \( \alpha = \beta_2 = 0 \), with \( \{,\} \) implies \( \alpha' = \beta'_2 = 0 \), with \( \{,\} \) implies \( \beta_1 = 0 \) and with \( \{,\} \) implies \( \beta_4 = 0 \).

- If \( \ell_1, \ell_2, \ell_3 \neq 0 \), let \( Y \in D_{\ell_1}, Y' \in D_{\ell_2}, Y'' \in D_{\ell_3} \). Then \( J_4 \) applied to \( (Y, Y', Y'') \) leads to \( (Y, Y', Y'') = 0 \).

Taking all the cases obtained above, the only Lie algebras of order 3 associated to \( \{,\} \) are isomorphic to the Poincaré algebra and \( \{,\} \) is isomorphic to the Poincaré algebra. Such algebras are called Poincaré algebras of order 3.

### Remark 3.2
In [26] two families of algebras associated to \( \mathfrak{sp}(n) \) were constructed. We have however check that they coincide when \( n = 1 \) i.e. for \( \mathfrak{sl}(2) \). This algebra is the one given in Eq. [3.2]. The algebra [3.3] was also obtained in [26] and the algebra [3.4] in [26] and [1].

### 4 Poincaré algebras of order 3

In this section we study and provide a systematic method to obtain all elementary Lie algebras of order 3, \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) where \( \mathfrak{g}_0 \) is isomorphic to the Poincaré algebra and \( \mathfrak{g}_1 \) is an arbitrary finite dimensional representation of the Poincaré algebra. Such algebras are called Poincaré algebras of order 3.

To construct these algebras, we proceed in several steps. First, we extend the action of \( \mathfrak{so}(1, 3, \mathbb{C}) \) on \( \mathfrak{g}_1 \), with \( \mathfrak{g}_1 \) a finite dimensional representation of \( \mathfrak{so}(1, 3, \mathbb{C}) \), to the action of the complexified Poincaré algebra on \( \mathfrak{g}_1 \). Then, we construct the \( \mathfrak{so}(1, 3, \mathbb{C}) \)-equivariant mappings from \( \mathcal{S}(\mathfrak{g}_1) \) into \( D_{1,1} \), with \( D_{1,1} \) the vector representation of \( \mathfrak{so}(1, 3, \mathbb{C}) \). Finally, we obtain all Lie algebras of order 3, \( \mathfrak{g} = (\mathfrak{so}(1, 3, \mathbb{C}) \oplus D_{1,1}) \oplus \mathfrak{g}_1 \).

### 4.1 Finite dimensional representations of the Poincaré algebra

The Poincaré algebra in \((1+3)\)dimensions \( \mathfrak{iso}(1, 3) \) (see Example 2.4 for notations) is given by
\[
[L_{mn}, L_{pq}] = \eta_{mq} L_{pm} - \eta_{mp} L_{qm} + \eta_{mp} L_{mq}, \quad [L_{mn}, P^p] = \eta_{mp} P_m - \eta_{mp} P_n, \quad [P_m, P_n] = 0.
\]

where \( \eta_{mn} \) is the Minkowski metric. Let \( \mathfrak{iso}(1, 3, \mathbb{C}) = \mathfrak{iso}(1, 3) \otimes \mathbb{C} \) be the complexified of \( \mathfrak{iso}(1, 3) \). Its Levi part is isomorphic to \( \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \). Consider in (4.1) the following change of basis
\[
U_0 = i L_{12} - L_{03}; \quad V_0 = i L_{12} + L_{03};
\]
\[
U_+ = \frac{1}{2} (i L_{23} - L_{31} - L_{10} + i L_{02}); \quad V_+ = \frac{1}{2} (i L_{23} - L_{31} + L_{10} + i L_{02});
\]
\[
U_- = \frac{1}{2} (i L_{23} + L_{31} - L_{10} + i L_{02}); \quad V_- = \frac{1}{2} (i L_{23} + L_{31} + L_{10} - i L_{02});
\]
\[
\begin{pmatrix} p_{+-} & p_{-+} \\ p_{++} & p_{-} \end{pmatrix} = P_m \sigma^m = \begin{pmatrix} P_0 + P_3 & P_1 - i P_2 \\ P_1 + i P_3 & P_0 - P_3 \end{pmatrix},
\]

(with \( \sigma^0 \) the identity matrix and \( \sigma^i, i = 1, 2, 3 \) the Pauli matrices). In this basis the \( \mathfrak{iso}(1, 3, \mathbb{C}) \) brackets are given by
\[ U_0, U_\pm = \pm 2U_{\pm}, \quad V_0, V_\pm = \pm 2V_{\pm}, \]
\[ U_+, U_- = U_0, \quad V_+, V_- = V_0, \]
\[ U_+, p_{\pm \epsilon} = p_{\pm \epsilon}, \quad V_+, p_{\pm \epsilon} = -p_{\pm \epsilon}, \]
\[ U_-, p_{\pm \epsilon} = p_{\pm \epsilon}, \quad V_-, p_{\pm \epsilon} = -p_{\pm \epsilon}, \]
\[ U_0, p_{\pm \epsilon'} = \epsilon' p_{\pm \epsilon'}, \quad V_0, p_{\pm \epsilon'} = \epsilon' p_{\pm \epsilon'}, \] (4.3)

(with \( \epsilon, \epsilon' = \pm \)).

Let \( \mathcal{D}_i \) be the irreducible \((i+1)\)-dimensional representation of \( \mathfrak{sl}(2) \). We note by \( \mathcal{D}_{i,j} = \mathcal{D}_i \otimes \mathcal{D}_j \), \( i,j \in \mathbb{N} \) the irreducible representation (since there is a factor 2 in the first line on equation (4.3) \( i \) belongs to \( \mathbb{N} \) and not to \( \frac{1}{2} \mathbb{N} \) of dimension \((i+1)(j+1)\) of \( \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \) defined from

\[ \rho(U_0 + V_0)(x \otimes y) = [U_0, x] \otimes y + x \otimes [V_0, y]. \]

Let \( g_1 = \bigoplus D_{i_k, j_k} \) be an arbitrary reducible representation of \( \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \).

**Lemma 4.1** Let \( \mathcal{D} \) be a (finite dimensional) representation of \( \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \). The action of \( \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \) on \( \mathcal{D} \) extends to an action of \( \mathfrak{iso}(1,3,\mathbb{C}) \) on \( \mathcal{D} \) such that:

1. The operators \( \rho(P_m) \) \((m = 0, \cdots, 3)\), are nilpotent.

2. Let

\[ A_p = \bigcap_{p_0 + p_1 + p_2 + p_3 = p} \ker \left( (\rho(P_0))^{p_0} (\rho(P_1))^{p_1} (\rho(P_2))^{p_2} (\rho(P_3))^{p_3} \right), \]

then, there exists an \( N \) such that we have the filtration

\[ A_1 \subset A_2 \subset \cdots \subset A_N = \mathcal{D}, \]

and for every \( 0 \leq p \leq N, \ A_p \) is an \( \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \) module.

3. \( \mathcal{D} \) is indecomposable (i.e. one can find an irreducible representation \( \mathcal{D}' \subset \mathcal{D} \) of \( \mathfrak{iso}(1,3) \) such that it is impossible to have \( \mathcal{D} = \mathcal{D}' \oplus \mathcal{D}'' \) where \( \mathcal{D}'' \) is stable under the action \( \rho(P_0) - \) see Examples [4.2 below –]),

\[ 0 \xleftarrow{\rho(P_m)_{A_1}} A_1 \xrightarrow{\rho(P_m)_{A_2}} A_2 \xrightarrow{\rho(P_m)} \cdots \xrightarrow{\rho(P_m)} A_N = \mathcal{D}. \]

Let \( B_p^m = \rho(P_m)(A_p) \). Then \( B_p^0 = B_p^i \) for \( i = 1, 2, 3 \). We denote by \( B_p \) this space and we have

\[ B_p \subset A_p \otimes \mathcal{D}_{1,1} \subset A_{p-1}. \]

**Proof.** 1. Let \( \lambda_0 \) be an arbitrary eigenvalue of \( \rho(P_0) \) and \( E_0 = \ker (\lambda_0 - \rho(P_0)) \subset \ker (\lambda_0 - \rho(P_0))^{n_0} \subset \mathcal{D} \) (with \( \ker (\lambda_0 - \rho(P_0))^{n_0} \) the generalised eigenspace). Denote \( \lambda_i \) \((i = 1, 2, 3)\) an arbitrary eigenvalue of \( \rho(P_i)|_{E_0} \) and let \( V = \ker (\lambda_i - \rho(P_i))^{n_i} \subset \ker (\lambda_0 - \rho(P_0)) \subset E_0 \). Since \([\rho(P_i), \rho(P_0)] = 0\) then \( E_0 \) is invariant by the action of \( P_i \). In \( V \) we have \( \rho(P_0)|_V = \lambda_0 \)Id, \( \rho(P_i)|_V = \lambda_i \Id \), since \([L_{\lambda_0}, P_i] = 0 \) we have \( \lambda_0 = \lambda_i = 0 \). And \( \rho(P_0), \rho(P_i) \) are nilpotent operators.

2. Since the operators \( \rho(P_m) \) are nilpotent (we denote \( n_m \) the index of nilpotency of \( \rho(P_m) \)) and are commuting operators, it is obvious that there exists an \( N \geq \max\{n_0, n_1, n_2, n_3\} \) such that \( A_N = \mathcal{D} \). Furthermore, since for all \( L \in \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \), there exists a \( P \in \mathcal{D}_{1,1} \) such that \([L, P] = P_0 \) we have \([L, P^p] = pP_0 P^{p-1} \). This means that if \( v \in A_p, [L, v] \in A_p \). Thus, \( A_p \) is an \( \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \) module.

3. Let \( w \in B_{p_i}^0 \), then there exists \( \nu \in A_p \) such that \( w = [P_0, \nu] \). Since \( A_p \) is an \( \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \) module we have \([L_{\lambda_0}, \nu] = \nu \) \((i = 1, 2, 3)\) with \( \nu \in A_p \). Using the Jacobi identity of Lie algebras, \( w = [P_0, \nu] = [P_0, [L_{\lambda_0}, \nu]] \) leads to \([P_0, \nu] = w = [L_{\lambda_0}, [P_0, \nu]] \) and \( B_{p_i}^0 \subset B_{p_i}^1 \). The converse goes along the same lines and we have \( B_{p_i}^0 = B_{p_i}^1 \) for \( i = 1, 2, 3 \). Finally, the \( \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \) equivariance of the mapping guarantees that, \( B_p \subset A_p \otimes \mathcal{D}_{1,1} \subset A_{p-1} \) and \( \mathcal{D} \) is indecomposable. QED.
Example 4.2 (1) If $g_1 = D_{1,1} \oplus D_{0,0}$ the vector plus the scalar representations of $sl(2) \oplus sl(2)$, $P_m$ can be represented by $5 \times 5$ nilpotent matrices. If we denote $(v_m, m = 0, \cdots, 3)$ (resp. $(w_0)$) a basis of $D_{1,1}$ (resp. $D_{0,0}$) we can define

$$
\rho(P_m)v_0 = v_m, \quad \rho(P_m)v_n = 0.
$$

(2) If $g_1 = D_{1,0} \oplus D_{0,1}$, since $D_{1,0} \otimes D_{1,1} = D_{2,1} \oplus D_{0,1} \otimes D_{0,1}$, $P_m$ can be represented by the $4 \times 4$ matrices $\rho(P_m) = \begin{pmatrix} 0 & \sigma_m \\ 0 & 0 \end{pmatrix}$ such that for $\psi \in D_{1,0}, \tilde{\chi} \in D_{0,1}$ we have

$$
\rho(P_m)\psi = 0, \quad \rho(P_m)\tilde{\chi} = \sigma_m\psi.
$$

(3) The example above can be even refined. Let $g_1 = D_{1,0} \oplus D_{0,1} \oplus D_{0,1}'$. The action of the $P$'s can be defined as follow:

$$
\rho(P_m)\psi = 0, \quad \rho(P_m)\psi' = 0, \quad \rho(P_m)\tilde{\chi} = \sigma_m\psi,
$$

with $\psi \in D_{1,0}, \psi' \in D_{1,0}', \tilde{\chi} \in D_{0,1}$. Here, $\text{Ker} (\text{ad} \ P_m) = D_{1,0} \oplus D_{1,0}'$. Here, $\text{Ker} (\text{ad} \ P_m)^2 = D_{1,0} \oplus D_{0,1} \oplus D_{0,1}'$ $(m = 0, \cdots, 3)$ and $B_2 = D_{1,0}$.

Remark 4.3 If $g_1$ is an irreducible representation of $sl(2) \oplus sl(2)$, then the action of $\text{ad} \ P$ on $g_1$ is trivial. Indeed, since $g_1$ is irreducible, $\text{Ker} (\text{ad} \ P_m)$ is equal either to $g_1$ or $\{0\}$. But since $\text{ad} \ P_0, \cdots, \text{ad} \ P_3$ commute they can be simultaneously diagonalised this means that $\text{Ker} (\text{ad} \ P_m) \neq \{0\}$ and the action of $\text{ad} \ P$ on $g_1$ is trivial.

4.2 $sl(2) \oplus sl(2)$–equivariant mappings

Now, we construct the possible $sl(2) \oplus sl(2)$–equivariant mappings from $S^3(g_1)$ into $D_{1,1}$, with $g_1$ an arbitrary representation of $sl(2) \oplus sl(2)$. We recall the following isomorphisms of representations of $GL(A) \times GL(B)$ [S]?

$$
S^p(A \oplus B) = \bigoplus_{k=0}^{p} S^k(A) \otimes S^{p-k}(B),
$$

$$
S^p(A \otimes B) = \bigoplus_{\Gamma} S^{\Gamma}(A) \otimes S^{\Gamma}(B),
$$

where the second sum is taken over all Young diagrams $\Gamma$ of length $p$ and $S^{\Gamma}(A)$ denotes the irreducible representation of $GL(A)$ corresponding to the Young symmetriser of $\Gamma$. In particular this gives

$$
S^3(A \oplus B \oplus C) = S^3(A) \oplus S^3(B) \oplus S^3(C) \oplus A \otimes B \otimes C \oplus 
$$

$$
S^2(A) \otimes B \oplus S^2(A) \otimes C \oplus S^2(B) \otimes A \oplus S^2(B) \otimes C \oplus S^2(C) \otimes A \oplus S^2(C) \otimes B \quad (4.5)
$$

$$
S^2(A \otimes B) = S^2(A) \otimes S^2(B) \oplus A^2(A) \otimes A^2(B) \oplus
$$

$$
S^3(A \otimes B) = S^3(A) \otimes S^3(B) \oplus S^3(A) \otimes S^3(B) \oplus A^3(A) \otimes A^3(B) \oplus A^3(A) \otimes A^3(B).
$$

Let $g_1$ be a representation of $sl(2) \oplus sl(2)$ and let $D_{1,1}$ be the vector representation of $sl(2) \oplus sl(2)$. Using the first equation given in [1.3], since $g_1$ is a reducible representation of $sl(2) \oplus sl(2)$, $S^3(g_1)$ reduces to three types of terms (i) $S^3(D)$, (ii) $S^2(D) \otimes D'$ and (iii) $D \otimes D' \otimes D''$ with $D, D', D''$ three irreducible representations. Thus all possible $sl(2) \oplus sl(2)$–equivariant mappings are of the type (i) $S^3(D) \rightarrow D_{1,1}$, (ii) $S^2(D) \otimes D' \rightarrow D_{1,1}$ and (iii) $D \otimes D' \otimes D'' \rightarrow D_{1,1}$. We now characterise more precisely these mappings.

Explicit description of the $sl(2) \oplus sl(2)$ equivariant mappings

(i) Type I. $sl(2) \oplus sl(2)$–equivariant mappings: $S^3(D) \rightarrow D_{1,1}$

Let $D = D_a \otimes D_b$ with $a, b \in \mathbb{N}$. $D_{1,1} \subseteq D_{a,b} \otimes D_{a,b} \otimes D_{a,b}$ if $a$ and $b$ odd. From the third equation of (4.5) $D_{1,1} \subseteq S^3(D_{a,b})$ if either
Let $\sigma = \chi, \Gamma$.

We call these $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$--equivariant mappings, mappings of type $I_S$ (symmetric), $I_A$ (antisymmetric) and $I_M$ (mixed) respectively. In particular, when $D = D_{a,a}$ and $a$ odd, the mapping $S^3(D_{a,a}) \to D_{1,1}$ is always $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$--equivariant and is called type $I_{0S}, I_{0A}, I_{0M}$ respectively. The extension of the Poincaré algebra given in Example 2.5 is of type $I_{0M}$ with $D = D_{1,1}$.

(ii) Type $II$. $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$--equivariant mappings: $S^2(D) \otimes D' \to D_{1,1}$.

Let $D = D_{a,b}$ and $D' = D_{c,d}$, $D_{1,1} \subseteq D_{a,b} \otimes D_{a,b} \otimes D_{c,d}$ if $c,d$ odd, and there exists an $n$ such that $2a - 2n - c = 1$ or $c - 2a + 2n = 1$ (and similar relations for $b,d$). From the second equation of (4.5) $D_{1,1} \subseteq S^2(D_{a,b}) \otimes D_{c,d}$ if either

\[ II_S : D_{1} \subseteq S^2(D_{a}) \otimes D_{c} \text{ and } D_{1} \subseteq S^2(D_{b}) \otimes D_{d}; \]

\[ II_A : D_{1} \subseteq \Lambda^2(D_{a}) \otimes D_{c} \text{ and } D_{1} \subseteq \Lambda^2(D_{b}) \otimes D_{d}. \]

We call these $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$--equivariant mappings, mappings of type $II_S$ and $II_A$ respectively. In particular, when $a,b$ even $D_{0,0} \subseteq S^2(D_{a,b})$ and $D_{1,1} \subseteq S^2(D_{a,b}) \otimes D_{1,1}$ (type $II_{0S}$). The extension of the Poincaré algebra given in Example 2.6 is of type $II_{0S}$ with $D' = D_{2,0} \oplus D_{0,2}, D = D_{1,1}$.

(iii) Type $III$. $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$--equivariant mappings: $D \otimes D' \otimes D'' \to D_{1,1}$.

Let $D = D_{a,b}, D' = D_{c,d}$ and $D'' = D_{e,f}$ with $a \geq c \geq e, D_{1,1} \subseteq D_{a,b} \otimes D_{c,d} \otimes D_{e,f}$ if $a + c + e$ and $b + d + f$ are odd and if there exists an $n$ such that $a + c + e - 2n = 1$ or $e - a + c - 2n = 1$ (plus similar relations for $b,d,f$). There are many $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$--equivariant mappings of these types.

We now give explicit examples of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$--equivariant mappings of type $I, II$ and $III$.

**Example 4.4** Let $D = D_{1,0} \otimes D_{1,1} = D_{4,0} \otimes D_{1,0}$. Using conventional notations for spinors, let $D_{1,0} = \{ \psi_\alpha, \alpha = 1, 2 \}$ and $D_{1,1} = \{ \chi^\dot{\alpha}, \dot{\alpha} = 1, 2 \}$ be the spinor representations of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Introduce the Dirac $\Gamma$--matrices ($\Gamma_m, \Gamma_n = \Gamma_m \Gamma_n + \Gamma_n \Gamma_m = 2 \eta_{mn} I_4$, with $I_4$ the four dimensional identity matrix)

\[ \Gamma_m = \begin{pmatrix} 0 & \sigma_m \\ \bar{\sigma}_m & 0 \end{pmatrix}, \]

where $\sigma_0 = \bar{\sigma}_0$ is the identity matrix and $\bar{\sigma}_i = -\sigma_i, i = 1, 2, 3$ with $\sigma_i$ the Pauli matrices. The index structure of the $\sigma_m$--matrices is as follow $\sigma_m \to \bar{\sigma}_m \to \sigma_m \to \bar{\sigma}_m$. We also define $\psi_\alpha = \varepsilon_{\alpha \beta} \psi^\beta, \psi^\alpha = \varepsilon^{\alpha \beta} \psi_\beta$, $\chi^\dot{\alpha} = \bar{\varepsilon}^{\dot{\alpha} \dot{\beta}} \chi^\dot{\beta}, \bar{\chi}^\dot{\alpha} = \varepsilon^{\dot{\alpha} \dot{\beta}} \bar{\chi}^\dot{\beta}$ with the invariant antisymmetric $\mathfrak{sl}(2)$--matrices $\varepsilon, \bar{\varepsilon}$ given by $\varepsilon_{12} = \bar{\varepsilon}_{12} = -1$, $\varepsilon_{12} = \varepsilon_{12} = 1$. A direct calculation gives $\bar{\sigma}_m \dot{\beta} \beta = \varepsilon^{\beta \alpha} \bar{\varepsilon}^{\dot{\alpha} \dot{\beta}} \sigma_{maa}$. Furthermore since the Dirac $\Gamma$--matrices are representations of the Clifford algebra, we have the relations $\sigma_m \bar{\sigma}_n + \sigma_n \bar{\sigma}_m = \eta_{mn} \sigma_0$, and thus $\sigma_{m\dot{a}a} \bar{\sigma}_{na} \dot{\alpha} = T_{mn} \bar{\sigma}_{na} = 2 \eta_{mn}$.

Now, we consider the representation

\[ D'_{1,0} = S^2(D_{1,0}). \]

We introduce the projector (Young symmetrizer)

\[ P_{1,2,3} = \frac{1}{3} (1 - (12) + (13) - (123)) \]
with
\[(a \ b) = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, (a \ b \ c) = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}\]
two cycles of \(S_3\) the group of permutation with three elements. A direct calculation gives
\[P_{\frac{1}{2}} \left( \psi_\alpha \otimes \psi_\beta \otimes \psi_\gamma \right) = \varepsilon_{\alpha\beta\lambda} \lambda,\]
and \(D_{1,0}' = \langle \lambda_\alpha, \alpha = 1, 2 \rangle\) (the same result can be obtained using the usual calculus of the Clebsch-Gordan coefficients). Proceeding along the same lines with \(D_{0,1}\) and introducing
\[D_{0,1}' \cong S \circ (D_{0,1}) = \langle \bar{\rho}^\alpha, \bar{\alpha} = 1, 2 \rangle\]
we obtain
\[P_{\frac{1}{2}} \left( \psi_\alpha \otimes \psi_\beta \otimes \psi_\gamma \right) \otimes P_{\frac{1}{2}} \left( \bar{\chi}_\alpha \otimes \bar{\chi}_\beta \otimes \bar{\chi}_\gamma \right) = \varepsilon_{\alpha\beta\gamma} \bar{\varepsilon}_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \lambda_\gamma \otimes \bar{\rho}_\gamma.\]
Symmetrising the R.H.S. we then get
\[S^3 \left( \psi_\alpha \otimes \bar{\chi}_\bar{\alpha} \otimes \psi_\beta \otimes \bar{\chi}_\bar{\beta} \otimes \psi_\gamma \otimes \bar{\chi}_\bar{\gamma} \right) = \varepsilon_{\alpha\beta\gamma} \varepsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \lambda_\gamma \bar{\rho}_\gamma + \varepsilon_{\gamma\alpha\bar{\beta}} \varepsilon_{\bar{\gamma}\bar{\alpha}\beta} \lambda_\beta \bar{\rho}_\beta + \varepsilon_{\bar{\beta}\gamma\alpha} \varepsilon_{\beta\bar{\gamma}\bar{\alpha}} \lambda_\bar{\alpha} \bar{\rho}_\bar{\alpha}. \tag{4.6}\]

Now, from the isomorphism of \(D_{1,0} \otimes D_{0,1}\) with the vector representation, and using the relation \(\sigma_{m\alpha\bar{\alpha}} \bar{\sigma}_n \bar{\alpha} = 2 \eta_{mn}\) we have the correspondence
\[V_m = \bar{\sigma}_m \bar{\alpha} \psi_\alpha \otimes \bar{\chi}_\bar{\alpha}, \quad \psi_\alpha \otimes \bar{\chi}_\bar{\alpha} = \frac{1}{2} \sigma_m \alpha \bar{\alpha} V_m, \]
\[P_m = \bar{\sigma}_m \alpha \lambda_\alpha \otimes \bar{\rho}_\bar{\alpha}, \quad \lambda_\alpha \otimes \bar{\rho}_\bar{\alpha} = \frac{1}{2} \sigma_m \alpha \bar{\alpha} V_m, \tag{4.7}\]
(thus \((P_m, m = 0, \cdots , 3) \sim D_{1,1}, (V_m, m = 0, \cdots , 3) \sim D_{1,1}'\)) and equations (4.6) reduce to
\[S^3 (V_m \otimes V_n \otimes V_p) = \eta_{mn} P_p + \eta_{np} P_m + \eta_{pm} P_n.\]
We denote \(3 \text{iso}(1, 3, \mathbb{C}) = \text{iso}(1, 3, \mathbb{C}) \oplus D_{1,1}\) the corresponding Lie algebra of order 3. If we now take the real form of \(\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)\) corresponding to \(\mathfrak{sl}(2, \mathbb{C})\) (the universal covering group of \(SO(1, 3)\) being \(SL(2, \mathbb{C})\)), the representation \(D_{1,0}\) and \(D_{0,1}\) become complex conjugate. Thus if we take \(\bar{\chi}_\bar{\alpha} = (\psi_\alpha)^*\) (the complex conjugate of \(\psi_\alpha\)), and similarly \(\bar{\rho}_\bar{\alpha} = (\lambda_\alpha)^*\), \(V_m\) and \(P_m\) become real vectors of \(\mathfrak{so}(1, 3)\).

**Example 4.5** Let \(D = D_{3,3}\). Using spinor notations
\[D_{3,0} = \langle \psi_{\alpha\beta\gamma}, \alpha, \beta, \gamma = 1, 2 \rangle, \quad D_{0,3} = \langle \bar{\chi}_{\alpha\bar{\beta}\bar{\gamma}}, \bar{\alpha}, \bar{\beta}, \bar{\gamma} = 1, 2 \rangle,\]
with \(\psi_{\alpha\beta\gamma}, \bar{\chi}_{\alpha\bar{\beta}\bar{\gamma}}\) symmetric spinor-tensors. This case is more involved than the previous one, because
\[S \circ (D_{3,0})\]
is a reducible representation. However, using the correspondence (4.7) elements of \(D_{3,3}\) are symmetric traceless tensors of order three
\[T_{mnp} = \bar{\sigma}_m \bar{\alpha} \bar{\sigma}_n \beta \bar{\sigma}_p \gamma \psi_{\alpha\beta\gamma} \bar{\chi}_{\alpha\beta\gamma}\]
(the symmetry of \(T\) comes from the symmetry of \(\psi\) and \(\bar{\chi}\) and \(T_{mnp} \eta_{nm} = 0\) from \(\sigma_{m\alpha\bar{\alpha}} \sigma_{n\beta\bar{\beta}} = 2 \varepsilon_{\alpha\beta} \bar{\varepsilon}_{\bar{\alpha}\bar{\beta}}\)).

Now it is easy to see that the mapping
\[T_{m1n1p1} \otimes T_{m2n2p2} \otimes T_{m3n3p3} \rightarrow \eta_{m1m2} \eta_{n1n2} \eta_{p1p3} \eta_{p2p3} P_{p3}\]
is \(\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)\)-equivariant. Thus, symmetrising the R.H.S we have

Let us denote the characterisation of $sl_2$. There exist an $sl_2$, $D_5$. Suppose there exists $D_2$. Suppose $D_1$. Suppose $D_5$. There exist a representation $D_2$. Suppose there exist $a,b,c,d,e,f$ such that $V \subseteq S^2(D_{a,b}) \otimes D_{c,d}$ together with $S^2(D_{a,b}) \otimes D_{c,d} = 0$ gives a contradiction and $[V, D_{c,d}] = 0$.

Type II : Assume $V \subseteq S^2(D_{a,b}) \otimes D_{c,d}$ and $S^3(D_{a,b}) = S^3(D_{c,d}) = 0$ (not of type I). In this case $c,d$ are odd.

1. Suppose $D_{c,d} \subseteq [V, D_{a,b}]$. The Jacobi identity J4 with $Y_1 = Y_2 = Y_3 = Y_4 \in D_{a,b}$ leads to a contradiction, thus $[V, D_{a,b}] = 0$.

2. Suppose there exist a representation $D_{a,b} \subseteq D$ such that $D_{a,b} \subseteq [V, D_{c,d}]$. Since $a,b$ are odd, $c,d$ are even and thus $V \not\subseteq S^2(D_{a,b}) \otimes D_{c,d}$. The Jacobi identity J4 with $Y_1, Y_2, Y_3 \in D_{a,b}, Y'_4 \in D_{c,d}$ together with $S^2(D_{a,b}) \otimes D_{c,d} = 0$ gives a contradiction and $[V, D_{c,d}] = 0$.

3. Suppose there exists $D_{c,d} \subseteq D$, with $D_{c,f} \neq D_{c,d}$, such that $D_{c,f} \subseteq [V, D_{a,b}]$. The same argument as in the point 1. above gives $[V, D_{a,b}] = 0$.

4. Suppose there exists $D_{c,f} \subseteq D$, with $D_{c,f} \neq D_{c,d}$, such that $D_{c,f} \subseteq [V, D_{c,d}]$. The Jacobi identity J4 with $Y_1 = Y_2 = Y \in D_{a,b}$ and $Y'_3 = Y'_4 = Y' \in D_{c,d}$ gives $[V, Y, Y', Y''] + [Y', Y, Y''] = 0$. If we suppose that $\{Y, Y', Y''\} = P \in V$ we know from the points 1. and 3. above that $[P, Y] = 0$. Thus the previous identity becomes $[Y', \{Y, Y', Y''\}] = [Y', P] = 0$ and $[V, D_{c,d}] = 0$.

5. Suppose there exists $D_{c,f} \subseteq D$, with $D_{c,f} \neq D_{a,b}$ and $D_{c,f} \neq D_{c,d}$ such that either $D_{a,b} \subseteq [V, D_{c,f}]$ or $D_{c,d} \subseteq [V, D_{c,f}]$. The Jacobi identity J4 with $Y_1 = Y_2 = Y \in D_{a,b}, Y' \in D_{c,d}, Y'' \in D_{e,f}$ gives $2[Y, \{Y, Y', Y''\}] + [Y', \{Y, Y, Y''\}] + [Y'', \{Y, Y, Y''\}] = 0$. If $\{Y, Y', Y''\} \in V$ or $\{Y, Y, Y''\} \in V$ since $[V, D_{a,b}] = [V, D_{c,d}] = 0$ (see 1., 2., 3. and 4. above), the previous identity reduces to $[Y'', \{Y, Y, Y''\}] = [Y'', P] = 0$ and thus $[V, D_{e,f}] = 0$. 

4.3 Lie algebras of order 3 associated to the Poincaré algebra

Let us denote $V$ the vector space isomorphic to $D_{1,1}$ generated by the vectors $P_m, m = 0, \cdots, 3$. Now, from the characterisation of $sl_2 \oplus sl_2$–equivariant mappings from $S^3(D)$ into $D_{1,1}$ and lemma 4.1, we construct Lie algebras of order 3 whose zero graded part is isomorphic to the Poincaré algebra.

Theorem 4.8 Let $D$ be a reducible representation of $sl_2 \oplus sl_2$ such that:

1. the action of $sl_2 \oplus sl_2$ on $D$ extends to an action of $iso(1, 3, \mathbb{C})$ on $D$ as in Lemma 2.3.

2. there exist an $sl_2 \oplus sl_2$–equivariant mapping from $S^3(D) \rightarrow V$.

Then if $g = iso(1, 3, \mathbb{C}) \oplus D$ is a Lie algebra of order 3, the action of $V$ on $D$ is trivial.

Proof.

Type I : Assume $S^3(D_{a,b}) \rightarrow V$. This means that $a, b$ are odd.

1. Suppose there exists a representation $D' \subseteq D$ (not necessarily irreducible) such that $D' \subseteq [V, D_{a,b}]$. The Jacobi identity J4 with $Y_1 = Y_2 = Y_3 = Y_4 \in D_{a,b}$ leads to a contradiction, thus $[V, D_{a,b}] = 0$.

2. Suppose there exist a representation $D_{c,d} \subseteq D$ such that $D_{a,b} \subseteq [V, D_{c,d}]$. Since $a, b$ are odd, $c, d$ are even and thus $V \not\subseteq S^2(D_{a,b}) \otimes D_{c,d}$. The Jacobi identity J4 with $Y_1, Y_2, Y_3 \in D_{a,b}, Y'_4 \in D_{c,d}$ together with $S^2(D_{a,b}) \otimes D_{c,d} = 0$ gives a contradiction and $[V, D_{c,d}] = 0$.
Type III: Assume $V \subseteq D_{a,b} \otimes D_{c,d} \otimes P$, $S^3(D_{a,b}) = S^3(D_{c,d}) = 0$ (not of type I) and $S^2(D_{a,b}) \otimes D_{c,d} = S^2(D_{c,d}) \otimes D_{a,b} = S^2(D_{a,b}) \otimes D_{c,d} = S^2(D_{c,d}) \otimes D_{a,b} = S^2(D_{a,b}) \otimes D_{c,d} = 0$ (not of type II).

1. If we assume $[V, D_{a,b}] \subseteq D_{g,h}$ with $D_{g,h} = D_{c,d}$ or $D_{g,h} \neq D_{c,d}$, $D_{g,h} \neq D_{c,f}$, the Jacobi identity $J_4$ with $Y_1, Y_2 \in D_{a,b}$, $Y_3 \in D_{c,d}$, $Y_4 \in D_{c,f}$ leads to a contradiction and $[V, D_{a,b}] = 0$.

2. Suppose there exists $D_{g,h}$ such that $[V, D_{g,h}] \not\subseteq D_{a,b}$, the Jacobi identity $J_4$ with $Y_1 \in D_{a,b}$, $Y_2 \in D_{c,d}$, $Y_3 \in D_{c,f}$ and $Y_4 \in D_{g,h}$ leads to a contradiction and thus $[V, D_{g,h}] = 0$.

This means that the action of $V$ on $D$ is trivial and thus $[V, D] = 0$. The remaining Jacobi identities are easy to be checked. Which ends the proof. QED.

**Corollary 4.9** With the hypothesis of theorem 4.8 the action of $V$ on $D$ is trivial i.e. $[V, D] = 0$.

**Remark 4.10** Differently as in theorem 4.8 if $g = \langle \text{sl}(2) \oplus \text{sl}(2) \oplus V \rangle \otimes D$ is a Lie algebra of order 3 satisfying $[V, D] = 0$ then $S^3(D) \rightarrow V$. Indeed if we suppose for contradiction that $S^3(D) \rightarrow \text{sl}(2) \oplus \text{sl}(2)$, and let $Y_1, Y_2, Y_3 \in D$ such that $\{Y_1, Y_2, Y_3\} = aL$ with $L \in \text{sl}(2) \oplus \text{sl}(2), a \in \mathbb{C}$. The Jacobi identity $J_3$ with $P \in V$ and $Y_1, Y_2, Y_3$ as above leads to $a = 0$ since the elements of $D$ commute with the elements of $V$ and $L$ do not commute with $P$.

The following property classify all Lie algebras of order 3 based on the Poincaré algebra such that the representation $D$ is of dimension 4.

**Proposition 4.11** Let $g = \text{iso}(1,3,\mathbb{C}) \oplus D$ be an elementary Lie algebra of order 3, with $D$ a representation of dimension 4. Then,

1. $D \cong D_{1,1}$;
2. $[V, D_{1,1}] = 0$;
3. $g \cong 3\text{iso}(1,3,\mathbb{C})$ (the complexified of the Lie algebra of order three of Example 2.9).

Proof. 1. Since the representation $D_{\ell}$, $\ell \in \mathbb{N}$ of $\text{sl}(2)$ is of dimension $\ell + 1$, the four dimensional representations of $\text{sl}(2) \oplus \text{sl}(2)$ are (up a permutation of the action $\text{sl}(2) \oplus \text{sl}(2))$:

$D_{0,0}, D_{1,0} \oplus D_{0,0}, D_{1,1}, D_{1,0} \oplus D_{0,1}, D_{1,0} \oplus D_{0,0} \oplus D_{0,0}, D_{0,0} \oplus D_{0,0} \oplus D_{0,0} \oplus D_{0,0}$.

Since $S^3(D) \rightarrow \text{iso}(1,3,\mathbb{C})$, a simple weights argument shows that the only possibilities are (i) $D = D_{1,1}$ with $V \subseteq S^3(D)$ (in agreement with the first description of $\text{sl}(2) \oplus \text{sl}(2)$—equivariant mappings) or (ii) by Example 2.4 $\text{sl}(2) \cong D_{2,0} \subseteq S^3(D_{2,0})$. In the second case we have by Remark 4.3 $[V, D_{2,0}] = 0$ and thus the Jacobi identity $J_3$ with $P \in V, Y_1, Y_2, Y_3 \in D_{2,0}$ leads to a contradiction since $P$ commute with $Y_i$ and not commute with $U$ (see Eq. 4.3). Thus, the only non-trivial Lie algebra of order 3 is then constructed with $D_{1,1}$.

2. Since $D$ is an irreducible representation of $\text{sl}(2) \oplus \text{sl}(2)$, by Remark 4.3, the action of $V$ on $D$ is trivial and $[V, D] = 0$.

3. Since $[V, D] = 0$, by Corollary 4.10 we have $V \subseteq S^3(D)$ (a simple weights argument, as we have seen in 1. above, also show that $\text{sl}(2) \oplus \text{sl}(2) \subsetneq S^3(D)$). This means, introducing $v_{++}, v_{+-}, v_{-+}, v_{--}$ a basis of $D$ (with notations similar to 1.2), and using a simple weights argument, that the only non-trivial trilinear brackets are:

$$\begin{align*}
\{v_{++}, v_{++, v_{--}}\} & = \alpha_1 P_{++}, & \{v_{++}, v_{--}, v_{++}\} & = \alpha_2 P_{--}, \\
\{v_{++}, v_{+-}, v_{-+}\} & = \alpha_3 P_{+-}, & \{v_{+-}, v_{--}, v_{++}\} & = \alpha_4 P_{+--}, \\
\{v_{+-}, v_{-+}, v_{--}\} & = \alpha_5 P_{-+}, & \{v_{-+}, v_{+-}, v_{++}\} & = \alpha_6 P_{-++}, \\
\{v_{-+}, v_{-+}, v_{--}\} & = \alpha_7 P_{-+-}, & \{v_{--}, v_{-+}, v_{++}\} & = \alpha_8 P_{---}.
\end{align*}$$

The action of $\text{sl}(2) \oplus \text{sl}(2)$ on $D$ is given by (see 4.3)}
\[ [U_+, v_-] = v_{+\varepsilon}, \quad [U_-, v_+] = v_{-\varepsilon}, \quad [V_+, v_{-\varepsilon}] = -v_{+\varepsilon}, \quad [V_-, v_{+\varepsilon}] = -v_{-\varepsilon}, \quad [U_0, v_{\varepsilon'}] = \varepsilon v_{\varepsilon'}, \quad [V_0, v_{\varepsilon'}] = \varepsilon' v_{\varepsilon'}, \]

with \( \varepsilon, \varepsilon' = \pm \). The Jacobi identity J3 gives

\[ \alpha_2 = -\frac{1}{2} \alpha_1, \beta_1 = \alpha_1, \beta_2 = -\frac{1}{2} \alpha_1, \gamma_1 = \frac{1}{2} \alpha_1, \gamma_2 = -\alpha_1, \delta_1 = \frac{1}{2} \alpha_1, \delta_2 = -\alpha_1. \]

If \( \alpha_1 \neq 0 \), we set

\[ v_0 = -\sqrt{\frac{1}{2\alpha_1}} (v_{+-} + v_{-+}), \quad v_3 = -\sqrt{\frac{1}{2\alpha_1}} (v_{+-} - v_{-+}), \quad v_1 = -\sqrt{\frac{1}{2\alpha_1}} (v_{++} + v_{--}), \quad v_2 = i \sqrt{\frac{1}{2\alpha_1}} (v_{++} - v_{--}), \]

and we get

\[ \{v_\mu, v_\nu, v_\rho\} = \eta_{\mu\nu} P_\rho + \eta_{\mu\rho} P_\nu + \eta_{\nu\rho} P_\mu. \]

So, the algebra is isomorphic to the complexified elementary Lie algebra of order 3 of Example 2.5. This results remains true if we consider its real form corresponding to the Lie algebra of order 3 of example 2.5. If we consider the complex case, every contraction of \( \varphi \), \( \varepsilon, \varepsilon = \text{regular} \) and \( \varepsilon = \text{infinitesimal} \). A further particularisation inspired from the Weimar-Woods construction [13] is given by

\[ h_\varepsilon = \text{diag}(\varepsilon^{a_1}, \ldots, \varepsilon^{a_m}, \varepsilon^{b_1}, \ldots, \varepsilon^{b_n}) \]

with \( a_i, b_j \in \mathbb{Z} \), \( i = 1, \ldots, m, \quad j = 1, \ldots, n \). Hence \( h_{0, \varepsilon}(X_i) = \varepsilon^{a_i} X_i \) and \( h_{0, \varepsilon}(Y_j) = \varepsilon^{a_j} Y_j \) and \( \{5.1\} \) become

\[ \varphi_{1, \varepsilon}(X_i, X_j) = \varepsilon^{a_i+a_j-a_k} C_{ij}^k X_k, \quad \varphi_{2, \varepsilon}(X_i, Y_j) = \varepsilon^{a_i+b_j-b_k} D_{ij}^k Y_k, \quad \varphi_{3, \varepsilon}(Y_i, Y_j, Y_k) = \varepsilon^{h_i+h_j+h_k-a_\ell} E_{ijk} Y_\ell. \]

5 Contractions of the Poincaré-algebra of order 3

5.1 Contractions of elementary Lie algebras of order 3

The variety \( F_{m, n} \) being an algebraic variety, one can naturally endow it with the Zariski topology.

**Definition 5.1** A contraction of \( \varphi \) is a point \( \varphi' \in F_{m, n} \) such that \( \varphi' \in \overline{\mathcal{O}_\varphi} \), the closure in the Zariski sense.

In the complex case, the notion of contraction is equivalent to the following. Let \( \varphi = (\varphi_1, \varphi_2, \varphi_3) \) be a given multiplication of elementary Lie algebras of order 3, \( g = g_0 \oplus g_1 \) and let \( (h_p)_{p \in \mathbb{N}} \) (with \( h_p = (h_{0, p}, h_{1, p}) \in GL(m, n) \)) be a sequence of isomorphisms. Define \( \varphi_p = (\varphi_{1, p}, \varphi_{2, p}, \varphi_{3, p}) \) by

\[
\begin{align*}
\varphi_{1, p}(X_1, X_2) &= h_{0, p}^{-1} \varphi_1(h_{0, p}(X_1), h_{0, p}(X_2)), \\
\varphi_{2, p}(X_1, Y_2) &= h_{1, p}^{-1} \varphi_2(h_{0, p}(X_1), h_{1, p}(Y_2)), \\
\varphi_{3, p}(Y_1, Y_2, Y_3) &= h_{0, p}^{-1} \varphi_3(h_{1, p}(Y_1), h_{1, p}(Y_2), h_{1, p}(Y_3)).
\end{align*}
\]

If the limit \( \lim_{p \to +\infty} \varphi_p \) exists, this limit is in the closure of \( \mathcal{O}_\varphi \). Then it is a contraction of \( \varphi \). Note that, in the complex case, every contraction of \( \varphi \) is obtained by this way [11].

Moreover, Inönü-Wigner contractions [10] turn out to be a relevant subclass of contractions. We consider the automorphisms \( h_\varepsilon = (h_{0, \varepsilon}, h_{1, \varepsilon}) \) of the form \( h_{0, \varepsilon} = h_0^{(1)} + \varepsilon h_0^{(2)} \) and \( h_{1, \varepsilon} = h_1^{(1)} + \varepsilon h_1^{(2)} \) with \( h_0^{(1)}, h_1^{(1)} \) singular, \( h_0^{(2)}, h_1^{(2)} \) regular and \( \varepsilon \) infinitesimal. A further particularisation inspired from the Weimar-Woods construction [13] is given by

\[ h_\varepsilon = \text{diag}(\varepsilon^{a_1}, \ldots, \varepsilon^{a_m}, \varepsilon^{b_1}, \ldots, \varepsilon^{b_n}) \]

with \( a_i, b_j \in \mathbb{Z} \) (i = 1, ..., m, j = 1, ..., n). Hence \( h_{0, \varepsilon}(X_i) = \varepsilon^{a_i} X_i \) and \( h_{0, \varepsilon}(Y_j) = \varepsilon^{a_j} Y_j \) and \( \{5.1\} \) become

\[ \begin{align*}
\varphi_{1, \varepsilon}(X_i, X_j) &= \varepsilon^{a_i+a_j-a_k} C_{ij}^k X_k, \\
\varphi_{2, \varepsilon}(X_i, Y_j) &= \varepsilon^{a_i+b_j-b_k} D_{ij}^k Y_k, \\
\varphi_{3, \varepsilon}(Y_i, Y_j, Y_k) &= \varepsilon^{b_i+b_j+b_k-a_\ell} E_{ijk} Y_\ell.
\end{align*} \]
As already stated, one can define a contraction if the limit \( \varepsilon \to 0 \) exists, \( i.e. \) if
\[
a_i + a_j - a_k \geq 0, \quad a_i + b_j - b_k \geq 0
\]
and
\[
b_i + b_j + b_k - a_\ell \geq 0
\]
for any \( a \) and \( b \).

Examples.

Let \( \mathcal{F}_{1,1} \) be the algebraic variety of \( 2 = (1 + 1) \)-dimensional elementary Lie algebras of order 3. We consider a basis \( \{X,Y\} \) of \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) adapted for this decomposition.

**Proposition 5.2** Any two-dimensional Lie algebra of order 3 \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) is isomorphic to one of the following Lie algebras of order 3

1. \( \mathfrak{g}_1^3 \): \( \{Y,Y,Y\} = X, \{X,Y\} = 0 \);
2. \( \mathfrak{g}_2^3 \): \( \{X,Y\} = Y, \{Y,Y,Y\} = 0 \);
3. \( \mathfrak{g}_3^3 \): the trivial Lie algebra of order 3.

**Proof.** We consider the most general possibility for the structure constants of a two-dimensional elementary Lie algebra of order 3:

\[
[X,Y] = \alpha_1 Y, \{Y,Y,Y\} = \alpha_2 X.
\]

The Jacobi identities J1-J4 imply \( \alpha_1 \alpha_2 = 0 \), and we obtain

\[
\mathfrak{g}_1^3, \alpha_1 = 0, \alpha_2 = 1;
\]
\[
\mathfrak{g}_2^3, \alpha_1 = 1, \alpha_2 = 0;
\]
\[
\mathfrak{g}_3^3, \alpha_1 = 0, \alpha_2 = 0.
\]

QED

**Corollary 5.3** The variety \( \mathcal{F}_{1,1} \) of \( 2 \)-dimensional elementary Lie algebras of order 3, is the union of two irreducible algebraic components \( U_1 \) and \( U_2 \) with

\[
U_1 = \overline{\mathfrak{g}_1^3} \text{ and } U_2 = \overline{\mathfrak{g}_2^3}.
\]

**Proof.** One has the following contraction scheme

where by \( A \to B \) we denote a contraction of the algebra \( A \) to the algebra \( B \) (\( B \) is a contraction of \( A \)). QED

**Remark 5.4** The algebra \( \mathfrak{g}_1^3 \) has been considered in [3, 4, 5, 6, 18, 19].

5.2 Contraction which leads to a non-trivial extension of the Poincaré algebra

Let \( \mathfrak{g} = \mathfrak{so}(2,3) \oplus \text{ad } \mathfrak{so}(2,3) \). Using vector indices of \( \mathfrak{so}(1,3) \) coming from the inclusion \( \mathfrak{so}(1,3) \subset \mathfrak{so}(2,3) \) we introduce \( \{M_{mn} = -M_{nm}, M_{m4} = -M_{4m}, m,n = 0,\ldots,3, m < n\} \) a basis of \( \mathfrak{so}(2,3) \) and \( \{J_{mn} = -J_{nm}, J_{m4} = -J_{4m}, m,n = 0,\ldots,3, m < n\} \) the corresponding basis of \( \text{ad } \mathfrak{so}(2,3) \). The multiplication law \( \varphi \) of the elementary Lie algebra of order 3 \( \mathfrak{so}(2,3) \oplus \text{ad } \mathfrak{so}(2,3) \) writes
In this context, a deformation $\lambda$ will assume that

\[ \varphi_1(M_{mn}, M_{pq}) = -\eta_{mq}M_{np} - \eta_{mp}M_{nq} + \eta_{mq}M_{np} + \eta_{mp}M_{nq}, \]
\[ \varphi_1(M_{mn}, P_p) = -\eta_{mp}P_n + \eta_{np}P_m, \]
\[ \varphi_1(P_m, P_p) = 0, \]
\[ \varphi_2(L_{mn}, V_{pq}) = -\eta_{mq}V_{np} - \eta_{mp}V_{nq} + \eta_{mq}V_{np} + \eta_{mp}V_{nq}, \]
\[ \varphi_2(P_m, V_{pq}) = 0, \]
\[ \varphi_2(P_m, V_p) = 0, \]
\[ \varphi_3(V_{mn}, V_{pq}, V_{rs}) = 0, \]
\[ \varphi_3(V_{mn}, V_{pq}, V_r) = (\eta_{mp}V_{nq} - \eta_{mq}V_{np})P_r, \]
\[ \varphi_3(V_{mn}, V_p, V_r) = 0, \]
\[ \varphi_3(V_m, V_p, V_r) = \eta_{mp}P_r + \eta_{mr}P_p + \eta_{pr}P_m. \]

Thus $L_{mn}, P_m$ generate the Poincaré algebra and $V_{mn}$ and $V_m$ are in the adjoint and vector representations of $\mathfrak{so}(1,3)$.

**Remark 5.5** The subalgebra generated by $L_{mn}, P_m$ and $V_m$ is the algebra of Example 2.5.

### 6 Deformations of Lie algebras of order 3

#### 6.1 Definition

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie algebra of order 3 on the complex field (or on a field of characteristic zero). Let $A$ be a commutative local $\mathbb{K}$-algebra, $\mathfrak{g}$ its maximal ideal. We assume that the residual field is isomorphic to $\mathbb{K}$. A $A$-Lie algebra of order $F$ is a Lie algebra of order $F$ whose coefficients in a $\mathbb{K}$-basis belongs to $A$. We will assume that $A$ admits an augmentation $\epsilon : A \rightarrow \mathbb{K}$. The ideal $m_\epsilon := \text{Ker} \epsilon$ is the maximal ideal of $A$.

In this context, a deformation $\lambda$ of $\mathfrak{g}$ with the base $(A, m)$ or simply with the base $A$, is a $A$-Lie algebra of order $F$ on the tensor product $A \otimes_\mathbb{K} \mathbb{K}^n$ with brackets $[\cdot, \cdot]_\lambda$ and $\{\cdot, \cdot\}_\lambda$ satisfying the Jacobi identities J1-J4.
Sometimes, we add hypothesis on $A$, for example $A$ is finitely generated or $A$ is noetherian. In this paper, following [13], we assume that $A$ is a valuation algebra. It is a local algebra with the following property: if $x$ belongs to the field of fractions of $A$ but not to $A$, then the converse $x^{-1}$ belongs to $g$. The most interesting examples of deformations (the formal deformations of Gerstenhaber, the nonstandard perturbations) satisfy this hypothesis. Moreover, a deformation in a valued algebra always admits a finite decomposition, that is defined by a finite number of ideals of $A$. Then every deformation has a finite Krull dimension and this means that an hypothesis such as the noetherian property is superfluous.

6.2 The Gerstenhaber products

The Gerstenhaber products have been introduced in [9] and [10] to study the deformations of associative algebras and the structure of Hochschild cohomology groups. Let $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ be a given multiplication of elementary Lie algebras of order 3, $g = g_0 \oplus g_1$. The identities J1-J4 are equivalent to

$$
\begin{align*}
\varphi_1(\varphi_1(X_1, X_2), X_3) + \varphi_1(\varphi_1(X_3, X_1), X_2) + \varphi_1(\varphi_1(X_2, X_3), X_1) &= 0, \\
\varphi_2(\varphi_1(X_1, X_2), Y) + \varphi_2(\varphi_2(X_2, Y), X_1) + \varphi_2(\varphi_2(Y, X_1), X_2) &= 0, \\
\varphi_1(X, \varphi_2(Y_1, Y_2, Y_3)) - \varphi_3(\varphi_2(X, Y_1), Y_2, Y_3) - \varphi_3(\varphi_1(Y_1, \varphi_2(X, Y_2), Y_3)) &= 0, \\
\varphi_2(Y_1, \varphi_3(Y_2, Y_3, Y_4)) + \varphi_2(Y_2, \varphi_3(Y_1, Y_3, Y_4)) + \varphi_2(Y_3, \varphi_3(Y_1, Y_2, Y_4)) + \varphi_2(Y_4, \varphi_3(Y_1, Y_2, Y_3)) &= 0.
\end{align*}
$$

If $\varphi$ and $\varphi'$ are two products of elementary Lie algebras of order 3, we can define

$$
\begin{align*}
\varphi \circ_1 \varphi' : (g_0 \otimes g_0 \otimes g_0) 
&\to (g_0 \otimes g_0 \otimes g_0) \\
X_1 \otimes X_2 \otimes X_3 
&\to \varphi_1(\varphi'_1(X_1, X_2), X_3) + \varphi_1(\varphi'_1(X_3, X_1), X_2) + \varphi_1(\varphi'_1(X_2, X_3), X_1), \\
\varphi \circ_2 \varphi' : (g_0 \otimes g_0 \otimes g_1) 
&\to (g_1) \\
X_1 \otimes X_2 \otimes Y 
&\to \varphi_2(\varphi'_1(X_1, X_2), Y) + \varphi_2(\varphi'_2(X_2, Y), X_1) + \varphi_2(\varphi'_2(Y, X_1), X_2), \\
\varphi \circ_3 \varphi' : (g_0 \otimes S^3(g_1)) 
&\to (g_0) \\
X \otimes (Y_1, Y_2, Y_3) 
&\to \varphi_1(X, \varphi'_2(Y_1, Y_2, Y_3)) - \varphi_3(\varphi'_2(X, Y_1), Y_2, Y_3) - \varphi_3(\varphi_1(Y_1, \varphi'_2(X, Y_2), Y_3)), \\
\varphi \circ_4 \varphi' : (g_1 \otimes S^3(g_1)) 
&\to (g_1) \\
Y_1 \otimes (Y_2, Y_3, Y_4) 
&\to \varphi_2(Y_1, \varphi'_3(Y_2, Y_3, Y_4)) + \varphi_2(Y_2, \varphi'_3(Y_1, Y_3, Y_4)) + \varphi_2(Y_3, \varphi'_3(Y_1, Y_2, Y_4)) + \varphi_2(Y_4, \varphi'_3(Y_1, Y_2, Y_3)).
\end{align*}
$$

Proposition 6.1 The map $\varphi$ endows $g$ with a structure of elementary Lie algebra of order 3 iff

$$
\varphi \circ_1 \varphi = 0 \text{ for } i = 1, \ldots, 4.
$$

6.3 Gerstenhaber deformations

In this section we assume that the algebra of valuation $A$ is the algebra $\mathbb{C}[[t]]$ of formal sequence. Its law is written $\varphi_t$

$$
\varphi_t : A \to (g_0 \oplus g_1) \otimes \mathbb{C}[[t]]
$$

with

$$
\varphi_t = \varphi + t^1\psi^{(1)} + t^2\psi^{(2)} + \cdots + t^n\psi^{(n)} + \cdots,
$$

where the $\psi^{(i)}$'s are linear applications from $A$ to $g$, satisfying (6.3).

Proposition 6.2 Considering a deformation $\varphi_t$ of $\varphi$, the maps $\psi^{(p)}$ (with $p \in \mathbb{N}$) satisfy the equations

$$
\sum_{p+q=r} \psi^{(p)} \circ_i \psi^{(q)} = 0, \text{ for any } i = 1, \ldots, 4, \ r \in \mathbb{N}
$$

where $\psi^{(0)} = \varphi$. 18
Proof. As $\varphi_t$ is a deformation of $\varphi$ it satisfies $\varphi_t \circ \varphi_t = 0$. For $i = 1$, equation (6.6) is just the condition of the deformations of Gerstenhaber for Lie algebras. We explicitly prove (6.6) for $i = 2$ the two remaining cases being similar. If one checks only the terms in $t^2$, only the terms $\varphi + t\psi^{(1)} + t^2\psi^{(2)}$ will matter. Inserting

\[
\begin{align*}
\varphi_{t1} &= \varphi_1 + t\psi_1^{(1)} + t^2\psi_1^{(2)} \\
\varphi_{t2} &= \varphi_2 + t\psi_2^{(1)} + t^2\psi_2^{(2)} \\
\varphi_{t3} &= \varphi_3 + t\psi_3^{(1)} + t^2\psi_3^{(2)}
\end{align*}
\] (6.7)

in (6.1), the coefficient of degree 1 leads to

\[
\varphi_2(\psi_1^{(1)}(X_1, X_2), Y) + \varphi_2(\psi_1^{(1)}(X_1, X_2), Y) + \varphi_2(\psi_2^{(1)}(X_2, Y), X_1)
+ \psi_2^{(1)}(\varphi_2(X_2, Y), X_1) + \varphi_2(\psi_2^{(1)}(Y, X_1), X_2) + \varphi_2(\varphi_2(Y, X_1), X_2) = 0
\] (6.8)

and the coefficient of degree 2 gives

\[
\begin{align*}
\varphi_2(\psi_2^{(2)}(X_1, X_2), Y) + \psi_2^{(1)}(\varphi_2(X_1, X_2), Y) + \psi_2^{(1)}(\psi_1^{(1)}(X_2, X_2), Y) \\
+ \varphi_2(\psi_2^{(2)}(X_2, Y), X_1) + \varphi_2(\psi_2^{(2)}(X_2, Y), X_1) + \psi_2^{(1)}(\psi_2^{(1)}(X_2, Y), (X_1)
+ \varphi_2(\psi_2^{(2)}(Y, X_1), X_2) + \psi_2^{(2)}(\varphi_2(Y, X_1), X_2) + \psi_2^{(2)}(\psi_2^{(1)}(Y, X_1), X_2) = 0.
\end{align*}
\] (6.9)

Then $\sum_{p+q=1} \psi^{(p)} \circ_2 \psi^{(r)} = 0$ and $\sum_{p+q=2} \psi^{(p)} \circ_2 \psi^{(r)} = 0$. Similarly, one proves (6.6) for any $r \in \mathbb{N}^*$. QED

Definition 6.3 An infinitesimal deformation of $\varphi$ is a deformation $\varphi_t$ of the form

$\varphi_t = \varphi + t\psi^{(1)}$.

Let $\varphi_t = (\varphi_1 + t\psi_1^{(1)}, \varphi_2 + t\psi_2^{(1)}, \varphi_3 + t\psi_3^{(1)})$. Identities (6.3) for the coefficient of $t$ lead to

\[
\begin{align*}
\varphi_1(\psi_1^{(1)}(X_1, X_2), X_3) + \varphi_1^{(1)}(\varphi_1(X_1, X_2), X_3) + \varphi_1(\psi_1^{(1)}(X_3, X_1, X_2) \\
+ \psi_1^{(1)}(\varphi_1(X_3, X_1, X_2) + \varphi_1(\psi_1^{(1)}(X_2, X_3, X_1) + \psi_1^{(1)}(\varphi_1(X_2, X_3, X_1) = 0,
\end{align*}
\] (6.10)

\[
\begin{align*}
\varphi_2(\psi_1^{(1)}(X_1, X_2), Y) + \varphi_2^{(1)}(\varphi_1(X_1, X_2), Y) + \varphi_2(\psi_2^{(1)}(X_2, Y), X_1)
+ \psi_2^{(1)}(\varphi_2(X_2, Y), X_1) + \varphi_2(\psi_2^{(1)}(Y, X_1), X_2) + \varphi_2(\varphi_2(Y, X_1), X_2) = 0,
\end{align*}
\] (6.11)

Using (6.2) these equations write

$\varphi \circ_i \psi + \psi \circ_i \varphi = 0$, with $i = 1, \ldots, 4$, which is just equation (6.6) for $r = 1$. 

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Furthermore, the coefficient of \( t^2 \) obtained from (6.3) gives

\[
\psi_1^{(1)}(X_1, X_2) + \psi_1^{(1)}(X_3, X_1, X_2) + \psi_1^{(1)}(X_2, X_3, X_1) = 0,
\]

\[
\psi_2^{(1)}(X_1, X_2) + \psi_2^{(1)}(X_2, X_1, X_2) + \psi_2^{(1)}(X_2, X_3, X_1) = 0,
\]

\[
\psi_3^{(1)}(X, X_2) - \psi_3^{(1)}(X_1, X_2) = 0,
\]

(6.12)

which writes

\[
\psi^{(1)} \circ \psi^{(1)} = 0, \quad \text{with } i = 1, \ldots, 4.
\]

(6.13)

**Definition 6.4** Denote by

\[
Z(A) = \{(\psi_1, \psi_2, \psi_3) : A \rightarrow \mathfrak{g}\},
\]

where \( \psi_i \) (\( i = 1, 2, 3 \)) satisfy (6.10) and (6.12). The vector space \( Z(A) \) is called the infinitesimal deformation space of \( A \).

### 6.4 Isomorphic deformations

**Proposition 6.5** Let \( (\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \varphi) \in \mathcal{F}_{m,n} \) be an elementary Lie algebra of order 3. We consider a formal change of basis given by \( \text{Id} + tf_0 \in GL(\mathfrak{g}_0 \otimes \mathbb{C}[t]), \text{Id} + tf_1 \in GL(\mathfrak{g}_1 \otimes \mathbb{C}[t]) \). The isomorphic multiplication \( \varphi_t \) writes as the deformation

\[
\varphi_t = \varphi + t\psi + O(t^2),
\]

where \( \psi = (\psi_1, \psi_2, \psi_3) \) is given by

\[
\varphi_1(X_1, X_2) = \varphi_1(f_0(X_1), X_2) + \varphi_1(X_1, f_0(X_2)) - f_0(\varphi_1(X_1, X_2)),
\]

\[
\varphi_2(X, Y) = \varphi_2(f_0(X), Y) + \varphi_2(X, f_1(Y)) - f_1(\varphi_2(X, Y)),
\]

(6.14)

\[
\varphi_3(Y_1, Y_2, Y_3) = \varphi_3(f_1(Y_1), Y_2, Y_3) + \varphi_3(Y_1, f_1(Y_2), Y_3) + \varphi_3(Y_1, Y_2, f_0(Y_3)) - f_0(\varphi_3(Y_1, Y_2, Y_3)).
\]

**Proof.** We put

\[
\tilde{X}_i = (\text{Id} + tf_0)(X_i) = X_i + tf_0(X_i),
\]

\[
\tilde{Y}_j = (\text{Id} + tf_1)(Y_j) = Y_j + tf_1(Y_j),
\]

(6.15)

and we have

\[
\varphi_1(\tilde{X}_1, \tilde{X}_2) = h_0^{-1}\varphi_1(h_0(X_1), h_0(X_2)),
\]

\[
\varphi_2(\tilde{X}_1, \tilde{Y}_2) = h_1^{-1}\varphi_2(h_0(X_1), h_1(Y_2)),
\]

\[
\varphi_3(\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3) = h_0^{-1}\varphi_3(g_1(Y_1), g_1(Y_2), g_1(Y_3)).
\]

(6.16)

This can be written as

\[
\varphi_1(\tilde{X}_1, \tilde{X}_2) = \varphi_1(X_1, X_2) + t\psi_1(X_1, X_2) + O(t^2),
\]

\[
\varphi_2(\tilde{X}, \tilde{Y}) = \varphi_2(X, Y) + t\psi_2(X, Y) + O(t^2),
\]

\[
\varphi_3(\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3) = \varphi_3(Y_1, Y_2, Y_3) + t\psi_3(Y_1, Y_2, Y_3) + O(t^2),
\]

(6.17)

where, by a tedious but straightforward calculation, one has (6.14). QED
Definition 6.6 An elementary Lie algebra of order 3 \( g = g_0 \oplus g_1 \) is called rigid if all deformations of \( g \) are isomorphic to \( g \).

If \( g \) is rigid then \( g_0 \) is a rigid Lie algebra and the representation \( g_1 \) of \( g_0 \) is also rigid.

As an example of rigid Lie algebra of order 3 one has \( \mathfrak{sl}(2) \oplus \mathfrak{ad}(\mathfrak{sl}(2)) \) with \( \alpha = 1 \) (see Theorem 5.1 for notations). Other examples are given in subsections 5.1 and 5.2. An example of non-rigid Lie algebra of order 3 is also exhibited in subsection 5.2. Finally, note that some rigidity properties of representations of \( \mathfrak{sl}(2) \) can be found in [7].

Remark. Usually, rigidity is computed using cohomological methods with the Nijenhuis-Richardson theorem. This implies that we are able to define cohomology of algebra of order 3 with values in a \( g \)-module. For instance, it is easy to define 1 and 2 cochains and the corresponding cocycles and coboundaries space. For degree greater than 2, this is an open problem. For degree 2, we can quickly summarize the construction.

One denotes \( \psi \) (see Proposition 6.5) by \( \delta \varphi f \):

\[
\begin{align*}
\psi_1(X_1, X_2) &= (\delta \varphi f)(X_1, X_2), \\
\psi_2(X_1, Y_2) &= (\delta \varphi f)(X_1, Y_2), \\
\psi_3(Y_1, Y_2, Y_3) &= (\delta \varphi f)(Y_1, Y_2, Y_3)
\end{align*}
\]

(6.18)

for any \( X_i \in g_0 \) and \( Y_j \in g_1 \). Let \( Z(A) = Z^2(A) \) be the space of infinitesimal deformations (see Definition 6.3). Let \( B^2(A) \) be the subspace of \( Z^2(A) \) defined by

\[
B^2(A) = \{ \psi \in Z^2(A) : \psi = \delta \varphi f \},
\]

we obviously have \( B^2(A) \subseteq Z^2(A) \). The theorem of Nijenhuis Richardson is written in this frame:

If \( H^2 = Z^2/B^2 = \{0\} \), then elementary Lie algebra of order 3 \( g \) is rigid.

6.5 Some deformations on the Poincaré algebra of order 3

Let us consider the algebra (5.6). It leads to an explicit example of deformation. Let \( \varphi = (\varphi_1, \varphi_2, \varphi_3) \) be the law defined by (5.6). The deformation \( \varphi_t = (\varphi_{t1}, \varphi_{t2}, \varphi_{t3}) \) is given by

\[
\begin{align*}
\varphi_{t1}(L_{mn}, L_{pq}) &= \varphi_1(L_{mn}, L_{pq}), \\
\varphi_{t1}(L_{mn}, P_p) &= \varphi_1(L_{mn}, P_p), \\
\varphi_{t1}(P_m, P_p) &= -t^2 L_{mp}, \\
\varphi_{t2}(L_{mn}, V_{pq}) &= \varphi_2(L_{mn}, V_{pq}), \\
\varphi_{t2}(L_{mn}, V_p) &= \varphi_2(L_{mn}, V_p), \\
\varphi_{t2}(P_m, V_{pq}) &= t(\eta_{mp} V_q - \eta_{mq} V_p), \\
\varphi_{t2}(P_m, V_p) &= t V_{mp}, \\
\varphi_{t3}(V_{mn}, V_{pq}, V_{rs}) &= t((\eta_{mp} \eta_{rq} - \eta_{mq} \eta_{rp}) L_{rs} + (\eta_{mr} \eta_{qs} - \eta_{ms} \eta_{qr}) L_{pq} + (\eta_{pr} \eta_{qs} - \eta_{ps} \eta_{qr}) L_{mn}), \\
\varphi_{t3}(V_{mn}, V_{pq}, V_r) &= \varphi_3(V_{mn}, V_{pq}, V_r), \\
\varphi_{t3}(J_{mn}, P_p, P_r) &= t \eta_{pr} L_{mn}, \\
\varphi_{t3}(V_m, P_p, P_r) &= \varphi_3(V_m, P_p, P_r).
\end{align*}
\]

and we obtain (6.4).

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