On the collective curvature radiation

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ABSTRACT

In this paper, we study one possible mechanism of pulsar radio emission (i.e. with the collective curvature radiation of the relativistic particle stream moving along the curved magnetospheric magnetic field lines). We show that an electromagnetic wave that contains one cylindrical harmonic \( \exp \{i\phi\} \) cannot be radiated by the curvature radiation mechanism, which corresponds to the radiation of a charged particle moving along curved magnetic field lines. The point is that a particle in a vacuum radiates the triplex of harmonics \((s, s \pm 1)\) in which the polarization of the emitted wave changes from one point to another on a circle of constant radius, while for one \( s \)-harmonic the polarization remains constant. So, for the collective curvature radiation, the wave polarization is very important and cannot be fixed a priori. For this reason, the polarization of real unstable waves must be determined directly from the solution of wave equations for the media. Its electromagnetic properties should be described by the dielectric permittivity tensor \( \hat{\varepsilon}(\omega, k, r) \), which contains information on the reaction on all possible types of radiation.

Key words: radiation mechanisms: general – pulsars: general.

1 INTRODUCTION

Curvature radiation is a type of bremsstrahlung radiation when a radiated charged particle moves along a curved trajectory with a curvature radius \( \rho_0 \) and its acceleration is orthogonal to the velocity \( \mathbf{v} \). The cyclotron rotation of a charged particle in the external magnetic field \( B \) is an example of this motion when \( \rho_0 = v_\perp/\omega_c \). Here, \( \omega_c \) is the cyclotron frequency and \( \omega_c = eBm_c\gamma \), where \( e \) and \( m_c \) are the charge and mass of a particle, respectively, \( \gamma \) is the particle Lorentz factor and \( v_\perp \) is the component of the particle velocity, which is orthogonal to the magnetic field.

Moving along the circular trajectory, a particle radiates at harmonics of the cyclotron frequency: \( \omega = n\omega_c \). This radiation is called cyclotron radiation for a non-relativistic particle and synchrotron radiation for a relativistic particle \((\gamma \gg 1)\). For the synchrotron radiation, at large numbers of cyclotron harmonics, the maximum of the radiated power is \( I \propto \gamma^3 \). The total radiation power \( I \) also grows with the particle energy, \( I \propto \gamma^2 \). Therefore, the synchrotron radiation of relativistic particles is widely presented in space radiation (Ginzburg & Syrovatskii 1964).

It is important to stress that although the length of formation of the curvature radiation is larger than the wavelength \( \lambda \), it is much less than the curvature radius \( \rho_0 \). So, the properties of the curvature radiation do not differ from those of the synchrotron radiation in which the cyclotron radius is equal to the local curvature radius \( \rho_0 \). The frequency of the maximum of the spectral power, \( \omega \simeq \gamma^2/\rho_0 \), and radiation power, \( I = 4/3 \varepsilon c^2 \gamma^3/\rho_0^2 \), increases with the particle energy. Here, the dependence on \( \gamma \) is stronger than for the synchrotron radiation because the curvature is fixed, and does not fall with the energy, as for the motion in the constant magnetic field.

The curvature mechanism of radiation is believed to be connected with the mechanism of the coherent pulsar radio emission. Indeed, in the region of the open magnetic field lines in the pulsar magnetosphere, there is a relativistic electron–positron plasma moving with relativistic velocities along the curved magnetic field lines. For the typical values of curvature radius, \( \rho_0 \approx 10^8 \) cm, and the Lorentz factor of electrons and positrons, \( \gamma \approx 10^7 \), the characteristic frequency of the curvature radiation is in the radio band. We can expect to find collective curvature radiation in the magnetosphere, where the curvature frequency coincides with the plasma frequency, \( \omega_p/\gamma^{3/2} \). Here, \( \omega_p = (4\pi\varepsilon n_e/m_e)^{1/2} \) is the usual plasma frequency, where \( n_e \) is the plasma number density. In the strong magnetic field of the pulsar magnetosphere, when the
charged particles can move along magnetic field lines only, the frequency of the plasma oscillations is $\gamma^{-3/2}$ times less than the usual plasma frequency $\omega_p$.

It seems natural to continue the analogy between curvature radiation and cyclotron radiation for collective radiation. However there is an essential difference between them, which means that we cannot use the equations for cyclotron plasma radiation and rewrite them for curvature radiation by replacing the cyclotron radius by the curvature radius. The problem is that at each point of plasma in the magnetic field, the distribution of particles over transverse velocities is isotropic. All directions of particle transverse motion exist, so that the average velocity equals zero. This is not the case for curvature radiation, when all particles have only one direction of motion along the magnetic field.

For plasma physics, the problem of collective curvature radiation is complicated because it demands the consideration of an essentially non-uniform plasma. Collective curvature radiation does not result from the change of parameters of the magnetic field and plasma in space. These effects can be taken into account in the local approximation because the wavelength of the radiation is much less than the scales of inhomogeneities. In order not to lose the curvature radiation, we also need to take into consideration the turn of the vector of anisotropy of the particle distribution function, $f(p) \propto \delta(p - p_l B/B)$, in space. Here, $p_l$ is the longitudinal particle momentum.

Two parameters of the curvature radiation (i.e. the length of formation $l_i = \rho_0 v_\parallel \approx \lambda y^2 \gg \lambda$ and the width of the radiation directivity $\delta \phi \approx \gamma^{-1}$) are connected by the relation $l_i/\rho_0 \approx \delta \phi$. Thus, the particle is in synchronism with the wave (i.e. the particle sees the constant wave phase) along the path on which the wave intensity changes essentially. The value of $\lambda$ is the wavelength of the curvature radiation, $\lambda \approx \rho_0 v_\parallel^2$.

Beskin, Gurevich & Istomin (1993, hereafter BGI) have solved the problem regarding the calculation of the dielectric permittivity in the geometrical optics approximation for a non-uniform anisotropic plasma particle distribution. They have also described the collective curvature–plasma interaction, when the electromagnetic waves that are connected with the curvature radiation are amplified simultaneously by the Cherenkov mechanism. This effect is absent in the vacuum. However, this procedure is complicated and demands a clear understanding. Because of this, there have been some incorrect statements in the literature (see, for example, Nambu 1989; Machabeli 1991, 1995).

Over many years, the problem of collective curvature radiation has been investigated in another way (Asseo, Pellat & Sol 1983; Larroche & Pellat 1987; Lyutikov, Machabeli & Blandford 1999; Kaganovich & Lyubarsky 2010). These authors considered a more simple task connected with the pure cylindrical geometry, which can be solved ‘exactly’. In such a statement, the magnetic field lines are considered to be concentric, with the relativistic plasma moving (i.e. rotating) along the magnetic field lines because of the centrifugal drift directed parallel to the cylindrical axis (z-coordinate) with the velocity $u = c \rho_0/\rho \ll c$. Here, again, $\rho_0 = c \omega_0 e$. However, this approach cannot be used when analysing curvature radiation (Beskin, Gurevich & Istomin 1988a).

Let us choose the electromagnetic fields of the wave, as was done by all the authors mentioned above, in the form

$$(E, B) = [E(\rho), B(\rho)] \times \exp(-i\omega t + i \phi + ikz).$$

(1)

Here, $\omega$, $s$ is an integer number defining the azimuthal wave vector $k_\phi$ and $k_z$ is the longitudinal wave vector along the cylinder. In this approach, the wave amplitudes $E(\rho)$ and $B(\rho)$ are considered to be functions of the radial distance $\rho$ only. Moreover, it is not the vectors $E$ and $B$, but their cylindrical components $(E, B)_\phi$, $(E, B)_x$ and $(E, B)$, that depend on the coordinate $\rho$ only. This means that the wave polarization follows the magnetic field, turning from one point to another. This is also the case if we have the definite boundary condition (e.g. putting the system into the metallic coat). With such suggestions, we come to the one-dimensional problem, which can be easily solved. Here, we show that such a wave does not have any relation to the curvature radiation.

Let us consider a particle moving exactly along the circle of radius $\rho_0$ with a constant velocity $v$; this motion corresponds to the infinite magnetic field. Then, the radiated power is equal to the work of the wave electric field under the particle electric current. The electric current is

$$j = ev\delta(\phi - \Omega t) \delta(z) \frac{\delta(\rho - \rho_0)}{\rho} e_\phi,$$

(2)

where $\Omega = v/\rho_0$. For a selected polarization, we obtain

$$\int j E dr = ev E_\phi(\rho_0) \exp(-i\omega t + i s\Omega t).$$

(3)

As we can see, the radiation is possible only if $\omega - s\Omega = 0$ (i.e. $\omega = k_\phi v$). This is just the condition of Cherenkov not curvature radiation. The point is that a wave with such polarization cannot be radiated by the curvature mechanism. The difference between the curvature wave and the Cherenkov wave is in the finite interaction time of the bremsstrahlung radiation with a radiated particle. The freely propagating wave with almost constant polarization deflects from the direction of a particle motion. As a result, the non-zero projection of the wave electric field on the particle velocity (i.e. on the direction of the electric current) occurs, and the wave takes away the energy from the particle. This continues the finite time $\tau = l_i/v$, which can be determined from the relation $\tau(\omega - k_\parallel v) \approx 1$. For the relativistic particle ($v \approx c$), $\tau = (\rho_0^2/\omega c^2)^{1/3} \approx \rho_0/c\gamma$. Below, we find the real polarization of the curvature radiation.

The paper is organized as follows. In Section 2, we find that the polarization of the curvature wave does not correspond to one cylindrical harmonic. In Section 3, we show that the non-linear wave interaction can lead to significant changes in the propagation of cylindrical modes. In Section 4, the BGI permittivity tensor is derived from the permittivity corresponding to one cylindrical mode. Finally, in Section 5, we discuss our main results.
2 POLARIZATION OF THE CURVATURE WAVE

The radiation field of the electric current density $j$ and the electric charge density $\rho_e$ of a moving particle with charge $e$ is described by the retarded potentials (Landau & Lifshitz 1975):

$$ A = \frac{1}{c} \int \frac{J(t')}{R} \, dt'; $$

$$ \Phi = \int \frac{\rho_e(t')}{R} \, dt'. $$

Here, $t' = t - R/c$ is the retarded time and $R$ is the distance from the charge location at time $t'$ to the observer, which has the cylindrical coordinates $(\rho, \phi, z)$:

$$ R = \left(\rho^2 + z^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi' - \phi)\right)^{1/2}. $$

After the Fourier transformation of potentials (4) and (5) over time, we obtain

$$ A_\omega = \frac{1}{2\pi} \int A(t) \exp(i\omega t) \, dt, $$

$$ \Phi_\omega = \frac{1}{2\pi} \int \Phi(t) \exp(i\omega t) \, dt. $$

It is now convenient to replace the integration over time $t$ by the integration over the retarded time $t'$, and then over the angle $(\phi' - \phi)$. As a result, for the Cartesian components $(x, y, z)$ of the vector potential $A$ and the scalar potential $\Phi$, we can obtain

$$ [A_\omega; \Phi_\omega] = \frac{e\rho_0}{2\pi c} \exp(i\omega \phi/\Omega) \left[ -K_s; K_s; 0; \frac{c}{v} K_0 \right]. $$

Here, the quantities $K_0, K_s,$ and $K_c$ are the functions of coordinates $\rho$ and $z$ only, and they are equal to

$$ K_0 = \int \frac{\exp[i\omega(R/c + \Omega^{-1}\alpha)]}{R + v\rho \sin \alpha/c} \, d\alpha, $$

$$ K_s = \int \frac{\exp[i\omega(R/c + \Omega^{-1}\alpha)] \sin \alpha}{R + v\rho \sin \alpha/c} \, d\alpha, $$

$$ K_c = \int \frac{\exp[i\omega(R/c + \Omega^{-1}\alpha)] \cos \alpha}{R + v\rho \sin \alpha/c} \, d\alpha, $$

$$ R = \left(\rho^2 + z^2 + \rho_0^2 - 2\rho\rho_0 \cos \alpha\right)^{1/2}. $$

Equation (10) is valid at any point $r$ (i.e. not only in the wave zone). The dependence over the angle $\phi$ is given by the exponent $\exp(i\omega \phi/\Omega)$. From the periodicity over $\phi$, we have $\omega = s\Omega$.

The key point of the above expansion (10) is that the radiated wave is the superposition of three harmonics: $s, s - 1$ and $s + 1$. For example, the azimuthal electric field $E_{\phi\omega}$ is equal to

$$ E_{\phi\omega} = \frac{ie\omega}{v} \left[ -\frac{\rho_0}{\rho} \Phi_\omega + \frac{v}{c} A_{\phi\omega} \right] = -\frac{e\rho_0 \omega}{2\pi c v^2} \exp[i\omega \phi] \left[ \frac{\rho_0}{\rho} K_0 - \frac{v^2}{c^2} (K_s \sin \phi + K_c \cos \phi) \right]. $$

The first term in equation (12), which is proportional to the scalar potential $\Phi$, is not important in the wave zone, $\rho \gg \rho_0$, but it is significant in the near zone on the particle trajectory $\rho = \rho_0$. Because of this term, the particle, which is in resonance with one of the three harmonics, that is, with $s (\omega = s\Omega)$, is beaten out of the synchronism by the neighbouring harmonics $s \pm 1$. The electric field $E_{\phi\omega}$ changes its sign during the time $\tau$. The synchronism condition (i.e. $1 - \cos \Omega \tau \simeq 1 - v^2/c^2 = \gamma^{-2}$) defines the time $\tau$

$$ \tau \simeq 1/\Omega \gamma = \rho_0/c\gamma, $$

which coincides with the time of formation of the curvature radiation.

Thus, the radiated curvature wave consists of three harmonics $s$ and $s \pm 1$ with a fixed relation between their amplitudes. We can see from equation (12) that the polarization of the wave for its cylindrical components in the triplex changes from one point to another on a circle of constant radius, whereas it is constant for one $s$-harmonic. Namely, this circumstance provides the curvature mechanism of the radiation. The appearance of harmonics $s \pm 1$, except for the resonant one $s = \omega/\Omega$, is a result of the additional modulation of the radiation field induced by a modulation of the particle electric current, having the harmonic $s = 1$. Now, we can understand why the simple problem of the collective curvature radiation in the cylindrical geometry with only one azimuthal harmonic $\exp(is\phi)$ does not reveal any significant amplification of waves (Asseo et al. 1983; Lyutikov et al. 1999; Kaganovich & Lyubarsky 2010). In this case, the chosen wave polarization does not contain primordially the curvature mechanism. When considering the collective curvature radiation of a stream of particles moving on a circle, the unperturbed state is such that the electric current does not depend on the azimuthal angle $\phi$. Therefore, we can consider the excitation of only one cylindrical $s$-harmonic. However, here we want to show that even a weak non-linear interaction of $s$-harmonics leads to the excitation of different values of $s$, in particular the three harmonics $s$ and $s \pm 1$. 

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3 COLLECTIVE TRIPLE RADIATION

In Section 2, we have shown that the curvature radiation of one charged particle cannot be described in the pure cylindrical geometry by one azimuthal harmonic exp (isφ). In a collective radiation, the modulation of the particle electric current appears together with the electromagnetic field excitation. Because of this, the resonant azimuthal harmonic s = ωp/νp mixes with the harmonics of the electric current modulation and produces all possible values of s. In Section 4, we see that all azimuthal harmonics s give a contribution to the response of a media on an electromagnetic field. However, in this section, we demonstrate that the collective curvature radiation of only the triplex of azimuthal harmonics (s, s ± 1) differs significantly from that of one harmonic s, as is usually considered in the literature.

Let us consider the simple cylindrical one-dimensional problem of radiation of the cold stream of plasma particles with charge e and mass m∗, moving along the infinite azimuthal magnetic field B0 = Bφ. In this case, the particles can move only in the φ-direction with the velocity vφ at different cylindrical radius ρ. The unperturbed number density n(0) and velocity vφ(0) are constants (i.e. they do not depend on ρ). The electric current j has only a φ-component as well as a B⊥-component of the wave magnetic field (Bρ = Bφ = 0). Accordingly, the wave electric field has two components: Eρ and Eφ (Ei = 0).

The dependence of the wave fields over time and coordinates is as follows:

\[ [Eρ; Eφ; B⊥] = [Eρ(ρ); Eφ(ρ); B⊥(ρ)] \exp(-iωt + isφ). \]  

Then, from the Maxwell equations, we obtain

\[ \frac{dEρ(ρ)}{dρ} = \frac{iσ}{ρ} Eσ(ρ) - \frac{iρω^2}{c^2} Eρ(ρ) - \frac{Eφ(ρ)}{ρ}, \]  

and

\[ \frac{dEφ(ρ)}{dρ} = \frac{-iσ}{ρ} Eσ(ρ) + \frac{4πσ}{ω} jφ(ρ) - \frac{Eφ(ρ)}{ρ}. \]  

Here, the index σ corresponds to one of the three harmonics s or s ± 1. For simplicity, we use the dimensionless variable r defined as r = ρωc, as well as quantities \( \Lambda = \omega_p^2/(ω^2γ^2) \) and \( J_0 = 4πjφ(0)/(Λω). \) Here, \( ω_p = (4πne^2/m^*_e)^{1/2} \) is the plasma frequency, and \( γ \) is the Lorentz factor of the particle motion: \( γ = (1 - v^2_φ/c^2)^{-1/2}. \) Following these definitions, equations (15) and (16) take the following forms:

\[ \frac{dEρ(r)}{dr} = \frac{iσ}{r} Eσ(r) - \frac{iω^2}{r} Eρ(r) - \frac{Eφ(r)}{r}, \]

\[ \frac{dEφ(r)}{dr} = \frac{-iσ}{r} Eσ(r) + \frac{4πσ}{r} J_0 - \frac{Eφ(r)}{r}. \]

As has already been stressed, here we consider the interaction of three waves s and s ± 1. It is important that these are not independent and that their interaction is realized by the static electric field \([Eρ(ρ); Eφ(ρ)] \exp (iφ) \) having the first azimuthal harmonic s = 1. This electrostatic field is the result of the non-linear interactions of high-frequency neighbouring harmonics s and s ± 1. The equations for the mode s = 1 under the same definitions are

\[ \frac{dn}{dt} + ∇(nv) = 0; \]

\[ \left( \frac{∂}{∂t} + v∇ \right) p = e \left( E + \left[ \frac{v}{c}, B \right] \right). \]

It is easy to understand that only the φ-component of the Euler equation is needed, while the radial component only provides us with the equilibrium configuration across the infinite magnetic field. We represent the plasma number density and the plasma velocity as the expansion over powers of the wave amplitude:

\[ v_φ = v_φ^{(0)} + δv_φ^{(1)} + δv_φ^{(2)} + \ldots, \]

\[ n = n^{(0)} + δn^{(1)} + δn^{(2)} + \ldots. \]

The linear response can easily be found as

\[ n^{(1)} = n^{(0)} \frac{kν_φ^{(1)}}{ω - kν_φ^{(0)}}, \]

\[ v_φ^{(1)} = \frac{eEφ}{m^*_eγ^2γ} \left( \frac{1}{ω - kν_φ^{(0)}} \right). \]

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where $k = s/l$. However, for the non-linear current, the non-linear relation between $\delta v_\phi$ and $\delta p_\phi$ should be taken into account:

$$\delta p_\phi = m_c y^3 \delta v_\phi - \frac{3}{2} m_c y^{(0)} v_\phi^2 \frac{\delta v_\phi^2}{c^2}. \quad (27)$$

The result of cumbersome but straightforward calculations is

$$J_s = \frac{1}{1 - s v^{(0)}_\phi / r} \left\{ - \frac{E_\phi'}{1 - s v^{(0)}_\phi / r} + \alpha - \frac{r A_{s,s}}{v^{(0)}_\phi} \right\}, \quad (28)$$

$$J_{s-1} = \frac{1}{1 - (s-1) v^{(0)}_\phi / r} \left\{ - \frac{E_\phi'}{1 - (s-1) v^{(0)}_\phi / r} + \alpha A_{s-1,s} \right\}, \quad (29)$$

$$J_{s+1} = \frac{1}{1 - (s+1) v^{(0)}_\phi / r} \left\{ - \frac{E_\phi'}{1 - (s+1) v^{(0)}_\phi / r} + \alpha A_{s,s+1} \right\}, \quad (30)$$

$$Z = \frac{1}{(v^{(0)}_\phi)^2} \left\{ \frac{E_\phi'}{1/r} + \alpha \left( \frac{E_\phi^{(s+1)} E_\phi^{(s)}}{1 - s v^{(0)}_\phi / r} \right) \right\}, \quad (31)$$

$$A_{s,s} = \frac{1}{1 - s v^{(0)}_\phi / r} + \frac{1}{1 - j v^{(0)}_\phi / r} - 3 y^2. \quad (32)$$

Here, $\alpha = c/(m_c sv^{(0)}_\phi)$ is the particle velocity divided over the velocity of light. The same quantities for plane waves can be found in BGI. The above equations (17–20 and 28–31) are evaluated with the vacuum initial condition for the normal mode, which can be presented analytically as $E_\phi^{(0)} = -J_s'(r)$ and $E_\phi^{(0)} = i \phi J_s'(r)$. Here, $J_s(r)$ is the Bessel function. It should be noted that the singularity in equation (17) is passed smoothly by the additional small term $+i\phi$ in the resonance denominators in equations (25) and (26).

In numerical calculations, equations (17)–(20) for $s = s$ and $s = s \pm 1$ were solved with two different values for the quantities $J_s$ and $Z$. In the first case, we neglect non-linear terms in equations (28)–(31), while the second case corresponds to the full non-linear problem. Fig. 1 shows the results obtained for these two cases. For a better representation of the influence of the non-linear current, we choose amplitudes of the $s = 1$ and $s + 1$ modes 20 times higher than the amplitude of the $s$ mode. In reality, the $s$ mode interacts with the whole continuum of modes, so this model assumption is reasonable. Fig. 1 shows that, in this case, the intensity of the wave $|E|^2$ is approximately 2.5 times larger than in the case when the non-linear current is neglected. Hence, we can conclude that a three-wave interaction is effective.

Thus, we have shown that the triplex of cylindrical harmonics, which corresponds better to the curvature mechanism, is amplified more effectively than the separate harmonic having the single value of the azimuthal number. In fact, the real polarization of the collective curvature mode can be obtained only by calculating the permittivity tensor of the streaming plasma in the strong curved magnetic field. The solution of the wave equations produces not only the dispersive equation for normal waves, $\omega = \omega(k)$, but also defines their polarization. A priori, it is unclear what polarization corresponds to unstable modes.

At first sight, the problem considered above is essentially non-linear and has no direct connection with the question of the linear wave amplification. We have included non-linearity only in order to connect harmonics $s$ and $s \pm 1$ self-consistently. Also, the appearance of neighbouring harmonics $s \pm 1$, even for small non-linearity, strongly changes the $s$-mode amplification. It is also clear that the interaction of $s \pm 1$ waves with the field $s = 1$ will result in all azimuthal harmonics.

### 4 TENSOR DERIVATION

In this section, we show that the asymptotic behaviour of the BGI dielectric tensor for a large enough curvature radius $\rho_0$ can be found directly from the plasma response on one cylindrical mode. For the infinite toroidal magnetic field, only the response to the toroidal component of the wave electric field $E_\phi$ is to be taken into consideration (Beskin 1999). Here, we consider the stationary medium only, so the time dependence can be chosen as $\exp(-i \omega t)$. Making summation over all cylindrical modes, we can write

$$D_\phi (\rho, \phi) = E_\phi (\rho, \phi) - \sum_{s=-\infty}^{\infty} E_\phi (\rho, s) K (\rho, s) \exp(is \phi), \quad (32)$$

where

$$K (\rho, s) = \frac{4 \pi e^2}{\omega} \int \frac{v_\phi}{\omega - sv_\phi / \rho} \frac{\partial f^{(0)}}{\rho} \, d\rho. \quad (33)$$

Here, $f^{(0)}(p_\phi)$ is the unperturbed distribution function. By making the Fourier transformation

$$E_\phi (\rho, s) = \frac{1}{2 \pi} \int_0^{2 \pi} E_\phi (\rho, \phi') \exp(-i s \phi') \, d\phi'. \quad (34)$$
and the transition to the Cartesian coordinate system, we can obtain

\[ D_x = E_x + \frac{1}{2\pi} \int \frac{\rho' \, d\rho' \, d\phi'}{\rho'} \sum_{s=-\infty}^{\infty} E_\phi(\rho', \phi') \delta(\rho - \rho') \exp[is(\phi - \phi')] \sin \phi, \]

\[ D_y = E_y - \frac{1}{2\pi} \int \frac{\rho' \, d\rho' \, d\phi'}{\rho'} \sum_{s=-\infty}^{\infty} E_\phi(\rho', \phi') \delta(\rho - \rho') \exp[is(\phi - \phi')] \cos \phi. \]

We choose the local coordinate system with the \( y \)-axis directed along the magnetic field and the \( x \)-axis orthogonal to it. From the above equations (35 and 36), we can obtain the permittivity kernel components:

\[ \varepsilon_{yy}(\mathbf{r}, \mathbf{r}') = 1 - \frac{1}{2\pi} \frac{1}{\rho} \sum_{s=-\infty}^{\infty} \delta(\rho - \rho') \exp[is(\phi - \phi')] \cos \phi \cos \phi'; \]

\[ \varepsilon_{yx}(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \frac{1}{\rho} \sum_{s=-\infty}^{\infty} \delta(\rho - \rho') \exp[is(\phi - \phi')] \cos \phi \sin \phi'; \]

\[ \varepsilon_{xy}(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \frac{1}{\rho} \sum_{s=-\infty}^{\infty} \delta(\rho - \rho') \exp[is(\phi - \phi')] \sin \phi \cos \phi'; \]

\[ \varepsilon_{xx}(\mathbf{r}, \mathbf{r}') = 1 - \frac{1}{2\pi} \frac{1}{\rho} \sum_{s=-\infty}^{\infty} \delta(\rho - \rho') \exp[is(\phi - \phi')] \sin \phi \sin \phi'. \]

This provides the material relationship

\[ D_i(\mathbf{r}) = \int \varepsilon_{ij}(\mathbf{r}, \mathbf{r}') E_j(\mathbf{r}') \, d\mathbf{r}'. \]

It should be noted that the operator presented above satisfies the needed symmetry condition

\[ \varepsilon_{ij}(\mathbf{r}, \mathbf{r}', \omega) = \varepsilon_{ji}(\mathbf{r}', \mathbf{r}, -\omega) \]
Ya. N. Istomin, A. A. Philippov and V. S. Beskin

[provided by the condition \( K(r, s, \omega) = K(r, -s, -\omega) \). As is well known (Kadomtsev 1965; Bornatici & Kravtsov 2000), it is this symmetrical form of permittivity tensor that is used for the calculation of the components of the permittivity tensor \( \varepsilon_{ij}(\omega, k, r) \)

\[
\varepsilon_{ij}(\omega, k, \eta \to r) = \int \varepsilon_{ij}(\omega, \xi, \eta) \exp(-ik\xi) \, d\xi.
\]

Here, \( \eta = (r + r')/2 \) and \( \xi = r - r' \). It is important that the above tensor (equation (43)) only describes correctly wave–particle interaction in inhomogeneous media with slowly varying parameters (Bernstein & Friedland 1984).

Substituting the kernel components, we can find

\[
\varepsilon_{xx}(\omega, k, \eta) = 1 - \frac{1}{2\pi} \int d\xi \exp(-ik\xi) \frac{1}{|\eta - \xi/2|} \sum_{n=-\infty}^{\infty} \delta(|\eta + \xi/2| - |\eta - \xi/2|) K(|\eta + \xi/2|, s)
\]

\[
\times \exp[is(\phi - \phi')] \sin \phi \sin \phi',
\]

\[
\varepsilon_{yy}(\omega, k, \eta) = 1 - \frac{1}{2\pi} \int d\xi \exp(-ik\xi) \frac{1}{|\eta - \xi/2|} \sum_{n=-\infty}^{\infty} \delta(|\eta + \xi/2| - |\eta - \xi/2|) K(|\eta + \xi/2|, s)
\]

\[
\times \exp[is(\phi - \phi')] \cos \phi \cos \phi',
\]

\[
\varepsilon_{zz}(\omega, k, \eta) = 1 - \frac{1}{2\pi} \int d\xi \exp(-ik\xi) \frac{1}{|\eta - \xi/2|} \sum_{n=-\infty}^{\infty} \delta(|\eta + \xi/2| - |\eta - \xi/2|) K(|\eta + \xi/2|, s)
\]

\[
\times \exp[is(\phi - \phi')] \cos \phi \cos \phi'.
\]

As a result, the integrals above (equations 44–47) are reduced to integration over \( \xi \), which is perpendicular to \( \eta \). However, the expression for the delta-functions in equations (44)–(47) is

\[
\delta(\ldots) = \frac{\delta(\theta - \pi/2)}{(\eta + \xi/2) - |\eta - \xi/2|} + \frac{\delta(\theta + \pi/2)}{(\eta + \xi/2) - |\eta - \xi/2|},
\]

where \( \theta \) is the angle between vectors \( \eta \) and \( \xi \). So, the integration over angles can be done easily. Finally, from the transition \( \eta \to r \), we can obtain \( \cos \alpha_\eta \to \cos \alpha = 1 \). Hence, according to equation (50), \( (k\xi) = k_\parallel |\xi| \), where \( k_\parallel \) is the component of the wave vector parallel to the external magnetic field.

The property of the absence of \( k_\perp \) is very important. It provides the same symmetry as in the case of a homogeneous medium:

\[
\varepsilon_{ij}(-\omega, -k, -B, r) = \varepsilon_{ij}(\omega, k, B, r) \quad \text{(Istomin 1994)}.
\]

This result differs from that obtained by Lyutikov et al. (1999). In this work, the importance of the transformation (43) was neglected.

Finally, using the Taylor expansion over \( |\xi| \) and the reduction of the resonant denominator to the delta-function, we can obtain

\[
\sum_{n=-\infty}^{\infty} \frac{1}{\omega|\eta + \xi/2|/v_\phi} \to \pi \int \delta \left[ s - \omega(|\eta|^2 + |\xi|^2)^{1/2} \right] \, ds.
\]

As a result, we can write

\[
\varepsilon_{xx} = -\frac{8\pi^2 e^2}{\omega} \int F''(\kappa) \frac{v_\phi}{\omega} \frac{\partial f^{(0)}}{\partial \rho_0} d\rho_0,
\]

\[
\varepsilon_{yy} = -\varepsilon_{xx},
\]

and

\[
\varepsilon_{zz} = -\frac{8\pi^2 e^2}{\omega} \int F''(\kappa) \frac{\rho_0^{1/3} v_\phi^{1/3}}{\omega^{2/3}} \frac{\partial f^{(0)}}{\partial \rho_0} d\rho_0.
\]

Here,

\[
F(\kappa) = \frac{1}{4\pi} \int_0^{+\infty} \exp(\text{i}(\kappa t + \text{i} r^3/3)) \, dt.
\]
On the collective curvature radiation

and

\[ \kappa = \frac{2(\omega - k_1 v_s)}{\omega^{1/3} v_s^{2/3} \rho_0^{1/3}}, \]  

(56)

where the prime denotes the derivative and \( \rho_0 \) is the curvature radius of the magnetic field.

If there is a high enough curvature radius of the field lines in the pulsar magnetospheres, we can use the asymptotic behaviour of \( F(\kappa) \) for \( \kappa \gg 1 \):

\[ F(\kappa) \approx \frac{1}{\pi \kappa} + \frac{2i}{\pi \kappa^2} + \ldots. \]  

(57)

After integration by parts, the final result is

\[ \varepsilon_{ij} = \left( 1 - \frac{3}{2} \frac{\omega^2 v_i}{\gamma^3 \rho_0 \omega_0^3} - i \frac{\omega^2 v_i}{\gamma^3 \rho_0 \omega_0^3} \right) \left( 1 - \frac{\omega^2 v_i}{\gamma^3 \rho_0 \omega_0^3} \right) \]  

(58)

Here, by definition, \( \omega = \omega - k v_s \), and the brackets \( \langle \rangle \) denote both the averaging over the particle distribution function \( f_{e^+e^-}(p_\phi) \) and the summation over the types of particles:

\[ \langle \ldots \rangle = \sum_{e^+e^-} \int \langle \ldots \rangle f_{e^+e^-}(p_\phi) \, dp_\phi. \]  

(59)

We see that the tensor above (equation 58) is just the BGI tensor. In the limit \( \rho_0 = \infty \), as expected, this tensor tends to the dielectric permittivity of a homogeneous plasma. The non-zero components \( \varepsilon_{xx}, \varepsilon_{ys} \) and \( \delta\varepsilon_{xx} = \varepsilon_{xx} - 1 \) in the tensor (58) for the finite curvature are a result of the non-local properties of the plasma response on the electromagnetic wave in the curved magnetic field. The parameter of non-locality \( (v_i/\omega)/\rho_0 \) is the ratio of the formation length of radiation to the curvature radius. For a vacuum, \( \omega_c \approx c/\gamma^2 \), and the length \( v_i/\omega \) coincides with the length of formation of the curvature radiation \( l_c \).

It is important that the components \( \varepsilon_{xx} = -\varepsilon_{ys} \) and \( \delta\varepsilon_{xx} \) essentially change the wave polarization. Following from the tensor of the dielectric permittivity (58), the relation between \( E_\phi \) and \( E_\rho \) of the wave electric field is

\[ (\varepsilon_{xx} + n_\rho n_\phi) E_\phi + (\delta\varepsilon_{xx} + 1 - n_\rho^2) E_\rho = 0, \]  

(60)

where \( n_\rho \) and \( n_\phi \) are components of the dimensionless wave vector: \( n = k c/\omega \). For the tangential wave propagation (i.e. for \( n_\rho = 0 \)), we have \( E_\phi \simeq (\delta\varepsilon_{xx}/n_{xx}) E_\rho \simeq (c/\rho_0 \omega_0) E_\rho \). As a result, a wave can produce the positive work under the electric particle current \( j_\phi \) (i.e. it can be excited). This is not the case if \( \delta\varepsilon_{xx} = \varepsilon_{xx} = 0 \) when \( E_\phi = 0 \). Note that for the low plasma density when \( \delta\varepsilon_{xx}, \delta\varepsilon_{ys}, \delta\varepsilon_{yy} \ll 1 \) when propagation does not differ practically from the vacuum case, there is no collective excitation. This has been shown by the Einstein coefficient method (Blandford 1975; Melrose 1978; Chugunov & Shaposhnikov 1988) and also by the dispersion relation solution with BGI dielectric tensor (Beskin, Gurevich & Istomin 1988b). However, as shown in Beskin et al. (1988b), in a dense plasma, when \( \delta\varepsilon_{xx} \gg 1 \), there exist modes for which the increment becomes positive. This is because two conditions must be fulfilled for the effective amplification of the collective curvature wave. First, the deviation of propagation of the electromagnetic wave from the plasma flow is needed for the wave electric field to produce the work under the electric current. Secondly, a ‘capture’ of the wave is required for the wave–particle interaction, which lasts much longer than it does in a vacuum.

5 DISCUSSION

Thus, as shown above, the wave polarization \( [E_\phi(p); E_\rho(p)] \exp(\pm i\phi) \) containing one cylindrical harmonic \( s \) suggests only the Cherenkov mechanism of radiation. In the curvature radiation mechanism of one particle in a vacuum, the generated wave consists of three harmonics \( s \) and \( s \pm 1 \). This property provides the exit from the phase synchronism of the wave with the particle motion, which is inherent in the bremsstrahlung radiation. For the collective curvature radiation, it is shown that the hydrodynamical model of plasma motion along the infinite magnetic field gives different results of the wave amplification, depending on the wave polarization. So, there is no other way to find the polarization of exiting waves apart from calculating the response of the medium on an electromagnetic field (i.e. using the dielectric permittivity). The correct procedure for calculating the dielectric permittivity using the expansion over cylindrical modes has also been shown above. We have demonstrated that the tensor obtained by this procedure coincides with the BGI tensor calculated previously by another method.

In conclusion, it is worth noting that unsuccessful attempts to find the collective curvature radiation have given rise to the term ‘curvature-drift instability’ (Lyutikov et al. 1999). As shown, the chosen simple wave polarization (i.e. one \( s \)-harmonic) means that only the Cherenkov mechanism of wave generation takes place. In this case, the centrifugal particle drift has a significant role. Practically, all curvature effects come only to this drift and the Cherenkov resonance on the drift motion produces small wave amplification in a better case (Kaganovich & Lyubarsky 2010). A stronger magnetic field produces less drift velocity and less Cherenkov effect, although the curvature of a particle motion does not depend on the magnetic field strength at all.
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