Abstract. In this paper, the synchronizability problem of dynamical networks is addressed, where better synchronizability means that the network synchronizes faster with lower-overshoot. The $L_2$ norm of the error vector $e$ is taken as a performance index to measure this kind of synchronizability. For the equilibrium synchronization case, it is shown that there is a close relationship between the $L_2$ norm of the error vector $e$ and the $H_2$ norm of the transfer function $G$ of the linearized network about the equilibrium point. Consequently, the effect of the network coupling topology on the $H_2$ norm of the transfer function $G$ is analyzed. Finally, an optimal controller is designed, according to the so-called $LQR$ problem in modern control theory, which can drive the whole network to its equilibrium point and meanwhile minimize the $L_2$ norm of the output of the linearized network.

Keywords. network, synchronizability, $L_2$ norm, $H_2$ norm, $LQR$ problem, optimal control.

1 Introduction

The topic of synchronizability of dynamical networks has attracted increasing interest recently (see [1, 3, 6, 10, 11, 12, 14] and references therein), which is mostly referred to as how easy it is for the network to synchronize. Technically, this is a problem whether there is a wide range of coupling strength in which the synchronization is stable, and the wider the range, the better the synchronizability of the network. Accordingly, for some cases when the synchronized region is unbounded, a better synchronizability means that the synchronization can be achieved with a smaller coupling strength [10]. Generally, the synchronizability of a network depends on the underlying coupling topology of the network. Studies show that the ability for a network to synchronize is related to the spectral properties of the outer coupling matrix of the network [10, 11] and thus influenced by the structural properties of the network, such as average distance [14], degree homogeneity [6], clustering coefficient [12], degree correlation [1], betweenness centrality [3], etc.

*This work is supported by both the National Science Foundation of China under grants 60674093, 60334030 and the City University of Hong Kong under the Research Enhancement Scheme and SRG grant 7002134.

† Corresponding author: chaoliu@pku.edu.cn
In this paper, the synchronizability problem of dynamical networks is considered from a different point of view: suppose that the coupling strength of a network belongs to the range in which the synchronization is stable; then, how fast the synchronization will be achieved? This problem is important for the reason that in practical engineering implementation (such as communications via chaotic synchronization), synchronization is expected to be achieved not only easily but also swiftly. In [7], it is qualitatively pointed out that networks with diagonalizable outer coupling matrices may synchronize faster than the ones with non-diagonalizable outer coupling matrices. Clearly, a measure is needed for a quantitative description of the swiftness of network synchronization. To meet this objective, the $L_2$ norm of the error vector $e$, denoted as $\|e\|_2$, is taken in this paper as a performance index of this kind of network synchronizability. As will be seen later, the quantity $\|e\|_2$ presents a suitable measure of both swiftness and overshoot (referring to the largest difference among various node dynamics before the synchronization is achieved): the smaller the quantity $\|e\|_2$, the faster with smaller overshoot the network synchronization.

Furthermore, as shown by the numerical examples given below, the quantity $\|e\|_2$ is influenced by the coupling topology of the network. Thus, the investigation on the relationship between $\|e\|_2$ and the network structure is of significance. In this paper, for the case that the synchronous state is an equilibrium point, it is pointed out that $\|e\|_2$ is upper-bounded by the product of the vector 2-norm of the initial error vector $e_0$ and the $H_2$ norm of the transfer function $G(s)$, denoted as $\|G(s)\|_2$ or simply $\|G\|_2$, of the linearized network about the equilibrium point. Thus, the smaller the $\|G\|_2$, the smaller the $\|e\|_2$ as well. As pointed out in [2], the relationship between $\|G(s)\|_2$ and the network structure is quite complicated. Under some assumptions, it is proved in this paper (see Theorem 1 and Example 4) that $\|G\|_2$ will not increase as the real eigenvalues of the symmetrical outer coupling matrix increase.

For a linear time-invariant system, the linear quadratic regulator problem, or simply the $LQR$ problem, is a classical problem in modern control theory. The objective of the $LQR$ problem is to find an optimal control law $u(t)$ such that the state $x(t)$ is driven into a (small) neighborhood of the origin while minimizing a quadratic performance ($L_2$ performance) index on $u$ and $x$. In fact, the $LQR$ problem is posed traditionally as the minimization problem of the $L_2$ norm of the regulator output of the system. In this paper, based on the techniques of the $LQR$ problem, an optimal controller design is developed so as to drive the network dynamics onto some homogenous stationary states while minimizing the $L_2$ norm of the output of the linearized network.

The rest of the paper is organized as follows. In Section 2, some preliminary definitions and lemmas necessary for successive development are presented. In Section 3, some numerical examples are provided to illustrate that the quantity $\|e\|_2$ presents a suitable measure of both swiftness and overshoot of the network synchronization. For the equilibrium synchronization case, the relationship between $\|e\|_2$ and the network structure is investigated in Section 4. Based on the results of the $LQR$ problem, an $LQR$ optimal controller is proposed in Section 5. The paper is concluded by the last Section.
2 Preliminaries

$L_2[a, b]$ is an infinite-dimensional Hilbert space, which consists of all square-integrable and Lebesgue measurable functions defined on an interval $[a, b]$ with the scalar inner product

$$\langle f, g \rangle = \int_a^b f(t) g(t) dt,$$

while if the functions are vector or matrix-valued, the inner product is defined as

$$\langle f, g \rangle = \int_a^b \text{trace}[f(t) \ast g(t)] dt,$$

where $\ast$ denotes complex conjugate transpose, and the induced norm is defined as

$$\|f\|_2 = \sqrt{\langle f, f \rangle},$$

for $f, g \in L_2[a, b]$.

Consider a continuous-time linear system,

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

where $x$, $u$ and $y$ are the state, input and output of the system, respectively, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{l \times n}$ are given constant matrices. The transfer function from $u$ to $y$ is $G(s) = C(sI - A)^{-1}B$. If $A$ is stable, then the $H_2$ norm of system (2) is represented by the $H_2$ norm of the transfer function $G(s)$, which is defined by

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}\{G^*(j\omega)G(j\omega)\} d\omega}.$$

It can be proved that $\|G(s)\|_2^2$ equals the overall output energy of the system response to the impulse input. For computing $\|G(s)\|$, the following formula is convenient.

**Lemma 1** [13]: If $A$ is stable, then $\|G(s)\|_2^2 = \text{trace}(B^T Y B)$, where matrix $Y$ is the solution to the following Lyapunov equation:

$$YA + A^TY + CT C = 0.$$  

(3)

Equivalently,

$$\|G(s)\|_2^2 = \inf_{P>0} \{ \text{trace}(B^T P B) : PA + A^T P + CT C \leq 0 \}.$$  

(4)

The so-called linear quadratic regulator (LQR) problem is an optimal control problem with a quadratic performance ($H_2$ norm) criterion. For the linear time-invariant system

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0,$$

where $x_0$ is arbitrarily given, the regulator problem refers to finding a control function $u(t)$ defined on $[t_0, T]$, which can be a function of the state $x(t)$, such that the state $x(t)$ is driven into a (small) neighborhood of origin at time $T$, $T < \infty$. Since every physical system has energy limitation, and large control action (even if it is realizable) can easily drive the system out of its valid region, certain limitations have to be imposed on the control in practical engineering implementation. For these reasons, the regulator problem is usually posed as an optimal control problem with a certain combined performance index on $u$ and $x$. Focusing on the infinite time regulator problem (i.e., $T \to \infty$) and without loss of generality assuming $t_0 = 0$, the LQR problem is formulated as follows: Find a control $u(t)$ defined on $[0, \infty)$ such that the state $x(t)$ is driven to the origin at $t \to \infty$ and
the following performance index is minimized:

$$
\min_u \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt
$$

(5)

for some $Q = Q^*$, $S$ and $R = R^* > 0$. Here, $R > 0$ emphasizes that the control energy has to be finite, i.e., $u(t) \in L_2[0, \infty)$. Moreover, it is assumed that

$$
\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0, \quad R > 0.
$$

(6)

Then, (6) can be factored as

$$
\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} = \begin{bmatrix} C^* \\ D^* \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}
$$

and (5) can be written as

$$
\min_{u(t) \in L_2[0, \infty)} \|Cx + Du\|^2_2.
$$

Traditionally, the LQR problem is posed as the following minimization problem:

$$
\min_{u(t) \in L_2[0, \infty)} \|y\|^2_2
$$

subject to:

$$
\begin{align*}
\dot{x} &= Ax + Bu, \quad x(t_0) = x_0 \\
y &= Cx + Du.
\end{align*}
$$

(7)

For the above LQR problem, the following lemma is useful.

**Lemma 2** [13]: Suppose that in (7):

(A1) $(A, B)$ is stabilizable;

(A2) $D$ has full column rank with $[D \ D_\perp]$ being unitary;

(A3) $(C, A)$ is detectable;

(A4) $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ has full column rank for all $\omega$.

Then, there exists a unique optimal control $u = Fx$ for the LQR problem (7), where

$$
F = -(B^*X + DC)
$$

(8)

and $X$ is the stabilizing solution to the following Riccati equation:

$$
(A - BD^*C)^*X + X(A - BD^*C) - XBB^*X + C^*D_\perp D_\perp^*C = 0.
$$

(9)

Moreover, the minimized $L_2$ norm of the output $y(t)$ is given by

$$
\min_{u(t) \in L_2[0, \infty)} \|y\|^2_2 = \|G_c x_0\|^2_2,
$$

(10)

where $G_c$ is the transfer function of the system

$$
\begin{align*}
\dot{x} &= (A + BF)x + Ix_0 \delta(t), \quad x(0-) = 0 \\
y &= (C + DF)x,
\end{align*}
$$

with $I$ being the identity matrix and $\delta(t)$ the impulse function.
3 \textit{L}_2 \text{ norm performance index of network synchronization}

Consider a network of \( N \) identical dynamical nodes, described by

\[
\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} m_{ij} \Gamma x_j, \quad i = 1, 2, \cdots, N,
\]

where \( f(x) \) governs the dynamics of each individual node, \( \sigma \) is the coupling strength, \( \Gamma \) is the inner linking matrix, and \( M = [m_{ij}] \) is the outer coupling matrix.

Let

\[
e_i(t) = x_i(t) - x_1(t), \quad i = 1, \cdots, N,
\]

and

\[
e(t) = [e_1^T \ e_2^T \ \cdots \ e_N^T]^T
\]

denote the error vector of network (11). Then, network (11) is said to achieve (asymptotical) synchronization if

\[
e(t) \to 0, \text{ as } t \to \infty.
\]

Let \( \|e(t)\|_2([0, T]) \), or simply \( \|e(t)\|_2 \) when no confusion may be caused, denote the \( L_2 \) norm of \( e(t) \) on a given interval \([0, T]\). Then, according to (11),

\[
\|e(t)\|_2 = \sqrt{\langle e, e \rangle} = \sqrt{\int_0^T \text{trace}[e(t)^*e(t)]dt}.
\]

In fact, \( \|e(t)\|_2 \) represents the energy of the error signal \( e \) on the interval \([0, T]\), which is in proportion to the area between the error function \( e \) and the time axis. Hence, \( \|e(t)\|_2 \) can be used as a quantitative measure of the swiftness and overshoot of the network synchronization. In what follows, two examples are first given for illustration.

\textbf{Example 1:} Suppose that each single node in network (11) is a Chua’s oscillator. In the dimensionless form, Chua’s oscillator is described by

\[
\begin{align*}
\dot{x}_1 &= \alpha(-x_1 + x_2 - f(x_1)), \\
\dot{x}_2 &= x_1 - x_2 + x_3, \\
\dot{x}_3 &= -\beta x_2 - \gamma x_3,
\end{align*}
\]

where \( f(\cdot) \) is a piecewise linear function:

\[
f(x_1) = m_1 x_1 + \frac{1}{2} (m_2 - m_1) (|x_1 + 1| - |x_1 - 1|).
\]

Take parameters \( \alpha = 9, \beta = 14, \gamma = 0.01, m_1 = -0.714, \) and \( m_2 = -1.14 \), so that Chua’s oscillator (15) generates a double-scroll chaotic attractor. Set \( \Gamma = \text{diag}(1, 1, 1) \) in (11). Fig. 1 shows the different synchronization performances of network (11) with the same coupling strength \( \sigma (\sigma = 6) \) but different coupling configurations. The corresponding values of \( \|e(t)\|_2 \) are computed numerically as given in Table 1.
Fig. 1 Synchronization of the state variables of network (11) with 20 nodes and the same coupling strength in different coupling configurations.

![Nearest-neighbor coupling](image1)

![Star-shaped coupling](image2)

![Global coupling](image3)

| Coupling         | Nearest-neighbor | Star-shaped | Global |
|------------------|------------------|-------------|--------|
| $\|e\|_2$        | 9.2744           | 0.8030      | 0.4142 |

Table 1 Values of $\|e\|_2$.

As to the case of equilibrium synchronization of network (11), since the synchronous state $s$ is known, the error vector can be defined in the following way:

\[
e_i = x_i(t) - s, \quad i = 1, \cdots, N,
\]

\[
e(t) = [e_1^T \ e_2^T \cdots e_N^T]^T.
\]

**Example 2:** Consider a Lur'e system,

\[
\begin{aligned}
\dot{x}_1 &= (A_1 - 2B_1C_1)x_1 + B_1f_1(y_1), \\
y_1 &= C_1x_1,
\end{aligned}
\]

where $x_1$ is the state, $y_1$ is the measured output,

\[
A_1 = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}, \quad B_1 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

and the nonlinear function $f_1(y_1) = |y_1 + 1| - |y_1 - 1|.$

A network with system (18) as individual nodes is given as follows:

\[
\begin{aligned}
\dot{x} &= (I_N \otimes (A_1 - 2B_1C_1) - \sigma M \otimes \Gamma)x + (I_N \otimes B_1)f(y), \\
y &= (I_N \otimes C_1)x,
\end{aligned}
\]

(19)
where $x = (x_1^T, \ldots, x_N^T)^T$, $y = (y_1, \ldots, y_N)^T$ and $f(y) = (f_1(y_1), \ldots, f_N(y_N))^T$.

Network (19) can be viewed as a large-scale system with measured output and feedback. Assume that system (19) is observable. Then, the states of network nodes achieve synchronization if and only if all the outputs of network nodes achieve synchronization. Thus, one only needs to consider the outputs of network (19).

Replace $x_i$ with $y_i$ and set $s = 0$ in (17). Then, $e_i = y_i$ and $e = y$. Let $a = 10$, $b = 3$, $b_1 = 0$, $b_2 = 1$, $c_1 = c_2 = 1$, and $\sigma = 2$ in (18). Fig. 2 shows the different performances of outputs of network (19) with the same coupling strength $\sigma$ ($\sigma=2$) but different coupling configurations. The corresponding values of $\|y\|_2$ are listed in Table 2.

![Fig. 2](image)

### Table 2: Values of $\|y\|_2$.  

| Coupling          | Nearest-neighbor | Star-shaped | Global  |
|-------------------|------------------|-------------|---------|
| $\|y\|_2$         | 0.7296           | 0.5768      | 0.3990  |

**Remark 1:** As indicated by the above examples, in the same situation of stable synchronization, the performances of synchronization of networks can be quite different.

**Remark 2:** It is also clear that swiftness and overshoot are important indexes for describing the synchronous behaviors. The $L_2$ norm of the error vector $e$, i.e., $\|e\|_2$ as defined in (13) or (17) for the equilibrium synchronization case, can properly measure the swiftness and overshoot of the network synchronization: the smaller the $\|e\|_2$, the faster with lower overshoot the network synchronization. Thus, the quantity $\|e\|_2$ can be taken as a performance index of network synchronizability.

**Remark 3:** The quantity $\|e\|_2$ is influenced by many factors of the network, such as

1. network structure, particularly the coupling strength $\sigma$ and the outer coupling matrix $M$;
2. dynamical components, particularly the individual dynamics determined by $f$, the synchronous state $s(t)$, and the inner linking matrix $\Gamma$.  

□
4 Local synchronization to network equilibrium point

In the literature where the eigenvalues of the outer coupling matrix are used to measure the network synchronizability, it is a topic of great interest to investigate the relationship between the eigenvalues and the network structure thereby finding suitable ways to enhance the synchronizability. In this section, for the case of equilibrium synchronization, the relationship between the quantity \( \|e\|_2 \) and some network parameters is explored.

4.1 \( \|G\|_2 \) as a constraint of \( \|e\|_2 \)

Suppose that each single node in network (11) has a measured output \( y = Cx \). Then, the equations of network (11) can be written as follows:

\[
\begin{align*}
\dot{x}_i &= f(x_i) - \sigma \sum_{j=1}^{N} m_{ij} \Gamma x_j, \quad x_i(0) = x_{i0}, \quad i = 1, \ldots, N, \\
y_i &= Cx_i,
\end{align*}
\]

(20)

If \( C = I_n \), then \( y_i = x_i \) is just the state of the \( i \)th node.

Let \( s \) denote an equilibrium point of the individual node, satisfying \( f(s) = 0 \). Then, the linearized equations of (20) about the synchronous solutions \( x_i = s \) are as follows:

\[
\begin{align*}
\dot{\eta}_i &= Df(s)\eta_i - \sigma \sum_{j=1}^{N} m_{ij} \Gamma \eta_j, \quad \eta_i(0) = x_{i0} - s = \eta_{i0}, \quad i = 1, \ldots, N, \\
e_i &= C\eta_i,
\end{align*}
\]

(21)

where \( \eta_i (\eta_i = x_i - s) \) and \( e_i \) are the state error vector and the output error vector to the \( i \)th node, respectively. Viewing the impulse function \( \eta_{i0}\delta(t) \) as an input to system (21), system (21) can be equivalently written as

\[
\begin{align*}
\dot{\eta}_i &= Df(s)\eta_i - \sigma \sum_{j=1}^{N} m_{ij} \Gamma \eta_j + \eta_{i0}\delta(t), \quad \eta_i(0_-) = 0, \quad i = 1, \ldots, N, \\
e_i &= C\eta_i,
\end{align*}
\]

(22)

Using the Kronecker product, the error system (22) can be rewritten as

\[
\begin{align*}
\dot{\eta} &= (I_N \otimes Df(s) - \sigma \sum_{i=1}^{N} m_{ij} \Gamma \eta_j + \eta_{i0}\delta(t), \quad \eta_i(0_-) = 0, \\
e &= \eta_{i0}\delta(t).
\end{align*}
\]

(23)

As in Example 2, suppose that system (23) is observable. Then, \( \|e\|_2 \) can be taken as a measure of the swiftness and overshoot of the synchronization of network (20). Let \( G(s) \) denote the transfer function of the error equation (23) from \( u \) to \( e \). Then

\[
G(s) = (I_N \otimes C)[s(I_N \otimes I_n) - (I_N \otimes Df(s) - \sigma \sum_{i=1}^{N} m_{ij} \Gamma \eta_j + \eta_{i0}\delta(t))]^{-1}(I_N \otimes I_n).
\]

(24)

Let the transfer function \( G(s) \) be given in (24), then Lemma 3: The inequality

\[
\|e\|_2 \leq \|G\|_2 \|\eta_{i0}\|_2
\]

holds, where \( \|e\|_2 \) is the \( L_2 \) norm of the error vector \( e(t) \) on the interval \([0, \infty)\), \( \|G\|_2 \) is the \( H_2 \) norm of the transfer function \( G(s) \), and \( \|\eta_{i0}\|_2 \) is the vector 2-norm of the initial error vector \( \eta_{i0} \).

**Proof:** Since \( u(t) = \eta_{i0}\delta(t) \), \( e(t) = g(t)\eta_{i0} \), where \( g(t) \) denotes the corresponding bilateral Laplace
transform of $G(s)$. Then, by Parseval’s identity,

$$\|e(t)\|_2 = \|g(t)\eta_0\|_2 = \|G\eta_0\|_2 \leq \|G\|_2 \|\eta_0\|_2.$$  

Thus, the quantity $\|e\|_2$ is upper-bounded by the product $\|G\|_2 \|\eta_0\|_2$. Since $\eta_0$ is the given initial error vector, $\|G\|_2$ can be taken as a constraint of $\|e\|_2$. In fact, as introduced in Sec.1, $\|G\|_2^2$ equals the overall output energy of the system response to the impulse input.

**Remark 4:** The advantages of using the quantity $\|G\|_2$ include:

1. $\|G\|_2$ can be numerically computed;
2. the synchronizability is affected by many factors of a network, while $\|G\|_2$ can be seen as an overall reflection of these network factors consisting of both structural and dynamical ones.

**Example 3:** Consider Example 2 again. The linearized equation of network (19) about the equilibrium point $s = [0 \ 0 \ 0]$ is given as follows:

$$\begin{align*}
\dot{\eta} &= (I_N \otimes A_1 - \sigma M \otimes \Gamma) \eta + (I_N \otimes I_n) \eta_0 \delta(t), \quad \eta(0) = 0, \\
\epsilon &= (I_N \otimes C) \eta.
\end{align*}$$  

(26)

The corresponding values of $\|G\|_2$ with the three different network configurations are listed in Table 3.

| Coupling | Nearest-neighbor | Star-shaped | Global |
|----------|------------------|-------------|--------|
| $\|G\|_2$ | 2.5091           | 2.4383      | 1.2565 |
| $\|e\|_2$ | 0.7296           | 0.5768      | 0.3990 |

Table 3 Values of $\|G\|_2$ of network (19) with different network configurations.

### 4.2 Relationship between $\|G\|_2$ and network structure

Example 3 shows that $\|G\|_2$ is influenced by the network configurations. Thus, the relationship between $\|G\|_2$ and the network structure is a problem deserving further investigation.

In this section, it is always assumed that the outer coupling matrix $M$ is symmetrical. Then, there exists a unitary matrix $U \in \mathbb{R}^{N \times N}$ such that $M = U \Delta U^{-1}$, where $\Delta = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ is a diagonal matrix with the diagonal entries $\lambda_i$, $i = 1, \ldots, N$, being the eigenvalues of matrix $M$. Let $\eta = (U \otimes I_n) \xi$, $u = (U \otimes I_n) \omega$, and $z = (U^{-1} \otimes I_l) e$. Then, the error equation (23) can be rewritten as follows:

$$\begin{align*}
\dot{\xi} &= (I_N \otimes Df(s) - \sigma \Delta \otimes \Gamma) \xi + (I_N \otimes I_n) \omega, \\
z &= (I_N \otimes C) \xi.
\end{align*}$$  

(27)

Note that system (27) is composed of $N$ uncoupled subsystems:

$$\begin{align*}
\dot{\xi}_i &= (Df(s) - \sigma \lambda_i \Gamma) \xi_i + I_n \omega_i, \\
z_i &= C \xi_i, \\
i &= 1, \ldots, N,
\end{align*}$$  

(28)

where $\omega_i$ is the $i$th component of the input vector $\omega = (U^{-1} \otimes I_n) u = (U^{-1} \otimes I_n) \eta_0 \delta(t)$. Let

$$E_i = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0),$$

be a matrix with the $i$th diagonal entry being 1 and all the other entries being zero. Then, $\omega_i = (E_i \otimes I_n) (U^{-1} \otimes I_n) \omega = (E_i U^{-1} \otimes I_n) \eta_0 \delta(t)$ and $\omega_{i0} = (E_i U^{-1} \otimes I_n) \eta_0$. 
From (28), the condition for ensuring the stable synchronous state
\[ \begin{align*}
    x_1 &= x_2 = \cdots = x_N = s
\end{align*} \]
is that the \( N \) matrices
\[ \begin{align*}
    Df(s) - \sigma \lambda_i \Gamma, \quad i = 1, \cdots, N,
\end{align*} \] (29)
are all stable.

Let \( T(s) \) denote the transfer function of (27) from \( \omega \) to \( z \). Then
\[ \begin{align*}
    T(s) &= (I_N \otimes C)[s(I_N \otimes I_n) - (I_N \otimes Df(s) - \sigma \Delta \otimes \Gamma)]^{-1}(I_N \otimes I_n) \\
        &= (U^{-1} \otimes I)G(s)(U \otimes I_m).
\end{align*} \]
Since both \( (U^{-1} \otimes I_l) \) and \( (U \otimes I_m) \) are unitary transformations, one has
\[ \|T(s)\|_2 = \|G(s)\|_2. \] (30)
Let \( T_i(s) = C[sI_n - (Df(s) - \sigma \lambda_i \Gamma)]^{-1}I_n \) denote the transfer function of the \( i \)th subsystem in (28). Then
\[ T(s) = \begin{bmatrix}
    T_1(s) \\
    \vdots \\
    T_N(s)
\end{bmatrix} \]
and
\[ \|T(s)\|_2^2 = \sum_{i=1}^{N} \|T_i(s)\|_2^2. \] (31)

**Assumption 1:** Suppose that the outer coupling matrix \( M \) is symmetrical, diffusive and irreducible, with the off-diagonal entries \( m_{ij} \leq 0 \) and the diagonal entries \( m_{ii} = -\sum_{j=1,j\neq i}^{N} m_{ij} \), for \( i, j = 1, 2, \cdots, N \).

The outer coupling matrix \( M \) satisfying Assumption 1 has a zero eigenvalue of multiplicity 1, and its other eigenvalues are all positive.

**Theorem 1:** Suppose that the inner linking matrix \( \Gamma = kI \), where \( k > 0 \) is a constant and \( I \) the identity matrix. Suppose that the outer coupling matrix \( M \) satisfies Assumption 1 with the eigenvalues given as follows:
\[ 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N. \] (32)
Then
\[ \|T_i(s)\|_2^2 \leq \|T_j(s)\|_2^2, \] (33)
where \( 1 \leq j < i \leq N \), and
\[ \begin{align*}
    \|T(s)\|_2^2 \geq \|T_1(s)\|_2^2 + (N - 1)\|T_N(s)\|_2^2 \geq N\|T_N(s)\|_2^2, \\
    \|T(s)\|_2^2 \leq \|T_1(s)\|_2^2 + (N - 1)\|T_2(s)\|_2^2 \leq N\|T_1(s)\|_2^2. \] (34)

**Proof:** Let \( P \) be an arbitrary positive definite matrix such that
\[ P(Df(s) - \sigma \lambda_j \Gamma) + (Df(s) - \sigma \lambda_j \Gamma)^T P + C^T C < 0. \]
By the assumption that \( \Gamma = kI \), one has
\[ P\Gamma + \Gamma^T P > 0. \]
Then
\[ P(Df(s) - \sigma \lambda_i \Gamma) + (Df(s) - \sigma \lambda_i \Gamma)^T P + C^T C \]
\[ = P(Df(s) - \sigma \lambda_i \Gamma) + (Df(s) - \sigma \lambda_i \Gamma)^T P + C^T C - \sigma(\lambda_i - \lambda_j)(P \Gamma + \Gamma^T P) \]
\[ < -\sigma(\lambda_i - \lambda_j)(P \Gamma + \Gamma^T P) \leq 0. \]
The above inequality and Lemma 1 together leads to the assertions of the theorem. \qed

Remark 5: Since \( \lambda_1 = 0 \), \( \|T_1(s)\|_2 \) in (31) represents the \( H_2 \) norm of the transfer function of each individual node in the network.

Remark 6: Theorem 1 provides a simple way for comparing \( \|G\|_2 \) of a network with its different possible structures. It can be deduced from Theorem 1 that \( \|G\|_2 \) will not increase as the eigenvalues of the outer coupling matrix increase.

Example 4: Consider Example 2 again. The eigenvalues of the global coupling matrix \( M_{glo} \) are \( \lambda_1 = 0 \) and \( \lambda_2 = \cdots = \lambda_N = N \), while the eigenvalues of the star-shaped coupling matrix \( M_{sta} \) are \( \lambda_1 = 0, \lambda_2 = \cdots = \lambda_{N-1} = 1 \) and \( \lambda_N = N \). Let \( G_{glo} \) and \( G_{sta} \) denote the transfer functions of the global coupling and the star-shaped coupling networks, respectively. Then, by Theorem 1,
\[ \|G_{glo}\|_2^2 = \|T_{glo}\|_2^2 = \|T_1(s)\|_2^2 + (N - 1)\|T_{\lambda=N}\|_2^2 \leq \|T_1(s)\|_2^2 + \|T_{\lambda=N}\|_2^2 + (N - 2)\|T_{\lambda=1}\|_2^2 = \|T_{sta}\|_2^2 = \|G_{sta}\|_2^2. \]
It is consistent with the numerical results given in Example 3.

For Example 2, through integral computations, an analytical expression of \( \|T_i\|_2^2 \) can be obtained as follows:
\[ \|T_i\|_2^2 = \frac{2(c_1^2 + c_2^2)\sigma^2 \lambda_i^2 + \frac{(3c_1^2 + c_2^2)b + 2c_1c_2(1 - a)}{4\sigma^3 \lambda_i^3} + \frac{(c_1 b - c_2 a)^2}{6b \sigma^2 \lambda_i^2} + \frac{a c_1^2 + c_2^2 a + c_1^2}{4a + 2b^2} \sigma \lambda_i + 2ab}{6b} \]
By (35) and (31), Table 4 gives the different values of \( \|G_{glo}\|_2 \) and \( \|G_{sta}\|_2 \) of network (19) as the node number \( N \) increases.

| \( N \) | 1   | 10  | 100 | 1000 | \( \infty \) |
|------|-----|-----|-----|------|----------|
| \( \|G_{glo}\|_2 \) | 1.0801 | 1.3190 | 1.4492 | 1.4696 | 1.2910 |
| \( \|G_{sta}\|_2 \) | 1.0801 | 2.2619 | 6.9840 | 22.0434 | \( \infty \) |

Table 4 Values of \( \|G\|_2 \) of network (19) with different network configurations.

5 Optimal controller design

So far, the pinning control strategy is extensively used for achieving synchronization of dynamical networks [5, 8]. The main advantage of the pinning control strategy is that only a few network nodes are needed to be controlled. AS far as the control effects (referring to the swiftness and overshoot of the synchronization) and the control cost (represented by the \( L_2 \) norm of the control input) are concerned, however, pinning control may not be the best choice. It is revealed by the \( LQR \) problem, as illuminated in [7], when matrices \( C \) and \( D \) are properly selected, both the control effects and the control cost can be simultaneously evaluated by the \( L_2 \) norm of the measured output \( y \) of the
controlled network. The smaller the $\|y\|$, the better the control effects and the lower the control cost. In this section, based on the optimal solution to the LQR problem, an LQR optimal controller is designed for network \((11)\), which can drive the network onto some homogenous stationary states while minimizing the quantity $\|y\|$.

Suppose that the controlled network is given as follows:

\[
\begin{cases}
  \dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} m_{ij} y_j + Bu_i, \\
y_i = Cx_i + Du_i,
\end{cases}
\tag{36}
\]

where $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{l \times n}$, $D \in \mathbb{R}^{l \times m}$ are given constant matrices, $u = (u_1^T, \ldots, u_N^T)^T$ is the controller to be designed, and $y = (y_1^T, \ldots, y_N^T)^T$ is the measured output of the LQR problem. As in Sec. 4, let $s$ denote an equilibrium point of the individual node and let $\eta = (\eta_1^T, \ldots, \eta_N^T)^T$, where $\eta_i = x_i - s$, $i = 1, \ldots, N$, be the state error vector.

Suppose that the controller is a feedback law of the state error, i.e., $u = F\eta$, where $F$ is the feedback gain matrix to be determined. Then, by the Kroneck product, the equation of the linearized system of \((36)\) about the synchronous solution $x_1 = \cdots = x_N = s$ is given as follows:

\[
\begin{cases}
  \dot{\eta} = A_c\eta + B_cu, \\
y = C_c\eta + D_cu,
\end{cases}
\tag{37}
\]

where $A_c = I_N \otimes Df(s) - \sigma M \otimes \Gamma$, $B_c = I_N \otimes B$, $C_c = I_N \otimes C$ and $D_c = I_N \otimes D$. Furthermore, suppose that the matrices $A_c$, $B_c$, $C_c$ and $D_c$ satisfy the assumptions (A1-A4) as given in Lemma 2. Then, by Lemma 2, the feedback gain $F$ for the LQR problem of the linearized system \((37)\) is obtained as

\[
F = -(B_c^T X + D_c C_c),
\tag{38}
\]

where $X$ is the stabilizing solution to the following Riccati equation:

\[
(A_c - B_c D_c C_c)^* X + X(A_c - B_c D_c C_c) - X B_c B_c^* X + C_c^* (D_c)^\perp (D_c)^\perp C_c = 0.
\tag{39}
\]

**Remark 7:** The optimal controller $u = F\eta$ with $F$ given in \((38)\) has the property that it controls network \((36)\) to the synchronous state $x_1 = \cdots = x_N = s$ and meanwhile minimizes the $L_2$ norm of the output $y$ of the linearized system \((37)\).

As the node number $N$ of network \((36)\) increases, the direct computation of the feedback gain $F$ as given in \((38)\) will become harder. For the case when the outer matrix $M$ is symmetrical, the feedback gain $F$ can be alternatively given in terms of the $N$ uncoupled subsystems. The main advantage of this approach is that the decentralized feedback gain $F_i$ will be given only based on the information of the $i$th subsystem, $i = 1, 2, \cdots, N$.

By using the unitary transformation, as used in Sec. 4, system \((37)\) becomes

\[
\begin{cases}
  \dot{\xi} = (I_N \otimes Df(s) - \sigma \Delta \otimes \Gamma)\xi + (I_N \otimes B)\eta, \\
z = (I_N \otimes C)\xi + (I_N \otimes D)\omega,
\end{cases}
\tag{40}
\]

where $\eta = (U \otimes I_n)\xi$, $u = (U \otimes I_n)\omega$, and $z = (U^{-1} \otimes I_l)y$. The $N$ uncoupled controlled subsystems are as follows:

\[
\begin{cases}
  \dot{\xi}_i = A_i\xi_i + B\omega_i, \\
  z_i = C\xi_i + D\omega_i, \quad i = 1, \cdots, N,
\end{cases}
\tag{41}
\]
where \( A_i = Df(s) - \sigma \lambda_i \Gamma \). By the analysis in Sec. 4, one has
\[
\|y\|_2^2 = \|z\|_2^2 = \sum_{i=1}^{N} \|z_i\|_2^2.
\]

Suppose that the constant matrices \( A_i, B, C \) and \( D \) in (41) satisfy the assumptions (A1-A4) as given in Lemma 2, for \( i = 1, \cdots, N \). Then, the controller \( F_i \) to the \( i \)th subsystem can be given as follows:
\[
F_i := -(B^*X_i + DC),
\]
where \( X_i \) is the stabilizing solution to the following Riccati equation:
\[
(Df(s) - \lambda_i \Gamma - BD^*C)^*X_i + X_i(A - \lambda_i \Gamma - BD^*C) - X_iBB^*X_i + C^*D_\perp D^*_\perp C = 0,
\]
for \( i = 1, 2, \cdots, N \). Moreover, the feedback gain matrix \( F \) is given by
\[
F = (U \otimes I_m) \begin{bmatrix} F_1 & \cdots & F_N \end{bmatrix} (U^* \otimes I_n). \tag{44}
\]

**Example 5:** Consider Example 1 again. Suppose that the network is with the global coupling configuration. The objective is to design a controller such that network (11) with Chua’s oscillators can be driven to the synchronous solution \( x_1 = \cdots = x_N = s = [0 \ 0 \ 0]^T \).

It is easy to deduce that
\[
Df(s) = \begin{bmatrix} -\alpha(m_2 + 1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{bmatrix}.
\]

Consequently, the linearized equation of the controlled network is given by
\[
\begin{cases}
\dot{\eta} = (I_N \otimes Df(s) - \sigma M \otimes \Gamma)\eta + (I_N \otimes B)u, \\
y = (I_N \otimes C)\eta + (I_N \otimes D)u,
\end{cases} \tag{45}
\]
where the constant matrices \( B = C = D = I_3 \) in this example.

By the above analysis, the LQR optimal controller can be designed by solving the Riccati equation (39). For comparison, a pinning control strategy is also considered, where 6 nodes are randomly selected to be pinned by controllers \( u_i = -\sigma \cdot d \cdot I_3\eta_i \) with \( \sigma = 1 \) and \( d = 20 \).

Fig. 3 shows the different performances of the synchronous behavior of the two controlled networks, by the LQR optimal controller and the pinning controller, respectively. The corresponding values of \( \|y\|_2 \) are listed in Table 5.
LQR optimal control          pinning control

Fig. 3  Different effects of the LQR and pinning controllers.

| Controller | LQR optimal control | Pinning control |
|------------|---------------------|-----------------|
| $\|y\|_2$  | 0.1309              | 4.9799          |

Table 5  Values of $\|y\|_2$.

6  Conclusion

In this paper, synchronizability of dynamical networks is considered based on some new measures: the swiftness and overshoot of the network synchronization. The quantity $\|e\|_2$, which represents the $L_2$ norm of the synchronization error vector $e(t)$, is taken as the performance index of this kind of synchronizability. It has been shown by several numerical examples, $\|e\|_2$ presents a suitable measure of both swiftness and overshoot of network synchronization: the smaller the values of $\|e\|_2$, the faster with smaller overshoot the network synchronization. For the case when the synchronous state is an equilibrium point, $\|e\|_2$ is upper-bounded by the product of the vector 2-norm of the initial error vector $e_0$ and the $H_2$ norm of the transfer function $G(s)$, denoted as $\|G(s)\|_2$, of the linearized network about the equilibrium point. The relationship between $\|G(s)\|_2$ and the network structure has also been discussed. Under some assumptions, it has been proved that $\|G(s)\|_2$ will not increase as the real eigenvalues of the outer coupling matrix increase. Finally, based on the techniques of the LQR control theory, an optimal controller has been suggested to drive the network onto some homogenous stationary states, which, in the mean time, can minimize the $L_2$ norm of the output of the linearized network. Further research along this direction seems to be quite promising as long as the network energy and performance are concerned, therefore deserves further efforts.

References

[1] M. di Bernardo, F. Garofalo and F. Sorrentino, Synchronizability of degree correlated networks, arXiv: cond-mat/0504335.
[2] Z. S. Duan, J. Z. Whang, G. R. Chen and L. Huang, Complexity in linearly coupled dynamical networks: some unusual phenomena in energy accumulation, 2007, submitted.
[3] H. Hong, B. J. Kim, M. Y. Choi and H. Park, Factors that predict better synchronizability on complex networks, Phys. Rev. E, vol. 69, 2004, 067105.
[4] G. Hu, J. Z. Yang and W. J. Liu, Instability and controllability of linearly coupled oscillators: eigenvalues analysis, Phys. Rev. E, vol. 58, no. 4, 1998, pp. 4440-4453.
[5] X. Li, X. F. Wang and G. R. Chen, Pinning a complex dynamical network to its equilibrium, IEEE Trans. Circuits Syst.-I , vol. 51, no. 48, 2004, pp. 2074-2087.
[6] T. Nishikawa, A. E. Motter, Y. C. Duan and F. C. Hoppensteadt, Heterogeneity in oscillator networks: are smaller worlds easier to synchronize? *Phys. Rev. Lett.*, vol. 91, no. 1, 2003, 014101.

[7] T. Nishikawa and A. E. Motter, Synchronization is optimal in nondiagonalizable networks, *Phys. Rev. Lett.*, vol. 91, no. 1, 2003, 014101.

[8] X. F. Wang and G. R. Chen, Pinning control of scale-free dynamical networks, *Physica A*, vol. 310, 2002, pp. 521-531.

[9] C. W. Wu and L. O. Chua, A unified framework for synchronization and control of dynamical systems, *Int. J. Bifurc. Chaos*, vol. 4, no. 4, 1994, pp. 979-989.

[10] C. W. Wu, Perturbation of coupling matrices and its effect on the synchronizability in arrays of coupled chaotic systems, *Phys. Lett. A*, vol. 319, 2003, pp. 495-503.

[11] C. W. Wu, Synchronizability of networks of chaotic systems coupled via a graph with a prescribed degree sequence, *Phys. Lett. A*, vol. 346, 2005, pp. 281-287.

[12] X. Wu, B. Wang, T. Zhou, W. Wang, M. Zhao and H. Yang, Synchronizability of highly clustered scale-free networks, *Chinese Phys. Lett.*, vol. 23, no. 4, 2006, pp. 1046-1049.

[13] K. M. Zhou, J. C. Doyle and K. Glover, Robust and optimal control, *Englewood Cliffs: Prentice-Hall*, 1996.

[14] T. Zhou, M. Zhao, and B. H. Wang, Better synchronizability predicted by crossed double cycle, *Phys. Rev. E*, vol. 73, 2006, 037101.