Gap probability of the circular unitary ensemble with a Fisher-Hartwig singularity and the coupled Painlevé V system

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Abstract We consider the circular unitary ensemble with a Fisher-Hartwig singularity of both jump type and root type at $z = 1$. A rescaling of the ensemble at the Fisher-Hartwig singularity leads to the confluent hypergeometric kernel. By studying the asymptotics of the Toeplitz determinants, we show that the probability of there being no eigenvalues in a symmetric arc about the singularity on the unit circle for a random matrix in the ensemble can be explicitly evaluated via an integral of the Hamiltonian of the coupled Painlevé V system in dimension four. This leads to a Painlevé-type representation of the confluent hypergeometric-kernel determinant. Moreover, the large gap asymptotics, including the constant terms, are derived by evaluating the total integral of the Hamiltonian. In particular, we reproduce the large gap asymptotics of the confluent hypergeometric-kernel determinant and the sine-kernel determinant, including the constant term conjectured earlier by Dyson.

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1 Introduction and statement of results

One of the most celebrated results in random matrix theory is the universality of local statistics of eigenvalues for large random matrices. Well-known universality classes arise in the unitary invariant random matrices in the bulk, and at the soft edges and hard edges [2, 32, 49]. Accordingly, for large unitary random matrices, the eigenvalues near a regular point in the bulk, near the soft edge and the hard edge, form determinantal point processes associated with the sine kernel, Airy kernel and Bessel kernel, respectively; see [49] and [2, Chapter 6]. These three determinantal point processes are now well understood. A significant feature is that the gap probabilities in these determinantal point processes can be expressed in terms of the Painlevé transcendents and their associated Hamiltonians; see [33, 34, 41, 51, 52]. The large gap asymptotics, including the non-trivial constant terms, are also worked out; see [17, 20, 21, 27, 28, 45].

Another basic universality class in the unitary random matrix ensembles is the confluent hypergeometric-kernel determinantal point process, having been used to describe the statistics of eigenvalues near a Fisher-Hartwig singularity in the spectrum; see [2, Chapter 6]. For example, consider the following Gaussian unitary ensemble with a Fisher-Hartwig singularity at the origin, defined by the joint probability density function

\[ p_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n} \prod_{j=1}^{n} e^{-\frac{\lambda_j^2}{2}} |\lambda_j|^{2n} \chi(\lambda_j) \prod_{j<k}(\lambda_j - \lambda_k)^2, \]  

(1.1)
where $Z_n$ is the normalization constant, the parameter $\alpha > -\frac{1}{2}$, and the function

$$\chi(x) = \begin{cases} e^{i\pi\beta}, & x < 0, \\ e^{-i\pi\beta}, & x > 0, \end{cases}$$

with $i\beta \in \mathbb{R}$. The simultaneous occurrence of a jump type singularity as $\chi(x)$, and a root type singularity $|x|^{2\alpha}$, indicates a Fisher-Hartwig singularity at the origin; cf. [18]. For $\alpha \in \mathbb{N}$ and $\beta = 0$, the ensemble is a degenerate GUE of size $n + \alpha$ with $\alpha$ eigenvalues located at the origin, a case considered by Chen and Feigin in [11]. If $\alpha = 0$ and $i\beta \in \mathbb{R}$, the ensembles can be realized as a conditional GUE. More precisely, one obtains a thinned process of eigenvalues by deleting each of the eigenvalues of the original GUE independently with probability $p = e^{-2i\beta}$ for $i\beta > 0$ (while $p = e^{2i\beta}$ if $i\beta < 0$); see [5, 6]. One may interpret the remaining and removed eigenvalues as observed and unobserved particles, respectively. Knowing the information that the particles observed are all negative when $i\beta > 0$ (or positive if $i\beta < 0$), the eigenvalue distribution of the conditional GUE is given by (1.1); see [4, 9]. The asymptotics of the partition function and the recurrence coefficients of the orthogonal polynomials associated with the Gaussian weight with a Fisher-Hartwig singularity at the origin have been obtained by Its and Krasovsky in [36]. The asymptotic analysis of the orthogonal polynomials therein implies that the scaling limit of the correlation kernel near the origin can be expressed in term of the confluent hypergeometric kernel. For $\beta = 0$, the kernel is reduced to the Bessel kernel of the first kind, obtained earlier by Kuijlaars and Vanlessen in [27].

The confluent hypergeometric kernel also arises in the Dyson circular unitary ensemble with a Fisher-Hartwig singularity

$$p_n(\theta_1, \ldots, \theta_n) = \frac{1}{Z_n} \prod_{j=1}^{n} |e^{i\theta_j} - 1|^{2\alpha} e^{i\beta(\theta_j - \pi)} \prod_{j<k}(e^{i\theta_j} - e^{i\theta_k})^2, \quad \theta_1, \ldots, \theta_n \in (0, 2\pi),$$

with real parameter $\alpha > -1/2$ and pure imaginary parameter $\beta$. Then, it is direct to see that the weight function $|e^{i\theta} - 1|^{2\alpha} e^{i\beta(\theta - \pi)}$ is positive for $\theta \in (0, 2\pi)$ and possesses a Fisher-Hartwig singularity of both root type and jump type at $\theta = 0$. With the normalization constant $Z_n', (1.2)$ is made a probability measure. We denote by $C_t$ an arc of the unit circle, oriented counterclockwise, that is,

$$C_t = \{e^{i\theta} : t \leq \theta \leq 2\pi - t\} \quad \text{for } 0 < t < \pi,$$

and define the following positive weight function $w(z; t)$ on $C_t$

$$w(z; t) = |z - 1|^{2\alpha} z^\beta e^{-\pi i\beta}, \quad z = e^{i\theta} \in C_t,$$

with real parameter $\alpha > -1/2$ and pure imaginary parameter $\beta$. Let $D_n(t)$ denote the Toeplitz determinant with the weight function (1.4), namely

$$D_n(t) = \det \left( \frac{1}{2\pi} \int_0^{2\pi} w(e^{i\theta}; t) e^{-(j-k)i\theta} d\theta \right)_{j,k=0}^{n-1} = \frac{1}{(2\pi)^n n!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{j=1}^{n} w(e^{i\theta_j}; t) \prod_{j<k} (e^{i\theta_j} - e^{i\theta_k})^2 \prod_{j=1}^{n} d\theta_j,$$
where the second expression is known as Heine’s multiple integral representation \[50\]. Then, the probability that all eigenvalue of the random matrix in the ensemble \[1.2\] are contained in the arc $C_t$ \[1.3\] can be expressed equally as

$$\text{Pro}(\{\theta_j \in (t, 2\pi - t) : 1 \leq j \leq n\}) = \frac{D_n(t)}{D_n(0)}. \quad (1.6)$$

The circular ensemble \[1.2\] was introduced by Deift, Krasovsky and Vasilevska in \[22\] to derive the large gap asymptotics for the Fredholm determinant of the confluent hypergeometric kernel. It was shown that the scaling limit of the correlation kernel near $z = 1$ is the confluent hypergeometric kernel

$$K^{(\alpha, \beta)}(u, v) = \frac{1}{2\pi i} \frac{\Gamma(1 + \alpha + \beta)\Gamma(1 + \alpha - \beta)}{\Gamma(1 + 2\alpha)^2} \frac{A(u)B(v) - A(v)B(u)}{u - v} \quad (1.7)$$

where

$$A(x) = \chi(x)^{1/2}|2x|^\alpha e^{-ix\psi(1 + \alpha + \beta, 1 + 2\alpha, 2ix)}, \quad B(x) = \overline{A(x)}, \quad (1.8)$$

the confluent hypergeometric function \[1\]

$$\psi(a, b, z) = 1 + \sum_{k=1}^{\infty} \frac{a(a + 1)\cdots(a + k - 1)}{b(b + 1)\cdots(b + k - 1)} \frac{z^k}{k!}, \quad (1.9)$$

and $\chi(x) = \begin{cases} e^{i\pi \beta}, & x < 0, \\ e^{-i\pi \beta}, & x > 0, \end{cases}$ as defined in \[1.11\]. If $\alpha = \beta = 0$, the kernel is reduced to the classical sine kernel

$$K^{(0,0)}(u, v) = K_{\sin}(u, v) = \frac{\sin(u - v)}{\pi(u - v)}. \quad (1.10)$$

Thus, for $n$ large, the gap probability \[1.6\] can be expressed in terms of the Fredholm determinant

$$\lim_{n \to \infty} \frac{D_n(2t_n)}{D_n(0)} = \text{det}(I - K_t^{(\alpha, \beta)}), \quad (1.11)$$

where $K_t^{(\alpha, \beta)}$ is the integral operator on $L^2(-t, t)$, with the confluent hypergeometric kernel \[1.7\]; see \[22\] Lemma 6. In \[22\], the large-$n$ asymptotics of the logarithmic derivative of the Toeplitz determinant $D_n(t_n)$ were derived, uniformly for the parameter $2t_0/n < t_n < \pi$ with $t_0$ sufficiently large. The boundary condition for $D_n(t)$ as $t \to \pi$ was obtain by comparing it with the Hankel determinant associated with the Legendre polynomials, of which the large-$n$ asymptotics was derived before in \[53\]. Therefore, asymptotic approximation of the Toeplitz determinant, uniformly for $2t_0/n < t_n < \pi$ with $t_0$ big enough, was obtained by integration. Then, recalling the relation \[1.11\], the large gap asymptotics of the determinant of confluent hypergeometric kernel as $t \to \infty$ was readily worked out. The constant term, namely the Widom-Dyson constant, was expressed in terms of the Barnes $G$-functions therein.

Recently, the studies of the asymptotics of Toeplitz determinant with Fisher-Hartwig singularities are of great interests. In \[18\], the asymptotics of Toeplitz determinant for a general weight function on the circle with several Fisher-Hartwig singularities were derived and the Barnes $G$-functions were involved. A nice review of the history of the asymptotics of Toeplitz determinants, with applications in the Ising model, can be found in \[19\]. In \[13\], Claeys, Its and Krasovsky studied the emergence of a Fisher-Hartwig singularity for the Toeplitz determinant.
They obtained transition type asymptotics of the Toeplitz determinant between the case with no singularity and the case with one Fisher-Hartwig singularity. Later, the transition asymptotics as two Fisher-Hartwig singularities merging into one Fisher-Hartwig singularity was considered by Claeys and Krasovsky in [14]. Both of the transition asymptotics are described by the Painlevé V transcendents with different boundary behaviors. Applications in two-dimensional Ising models and random matrix theory were also considered therein.

From the relation (1.11), the large-$n$ asymptotics of the Toeplitz determinant $D_n(\frac{2t}{n})$ defined in (1.5) for bounded $t$ will give the determinant of the confluent hypergeometric kernel on $(-t, t)$. Thus one obtains in the unitary ensembles (1.2) the probability that the symmetric interval around the Fisher-Hartwig singularity is free of eigenvalues. For $\beta = 0$, the kernel was reduced to the Bessel kernel of the first kind and the determinant can be expressed equally in terms of the Hamiltonian associated with the Painlevé V equation; see [54]. It was observed by Borodin and Deift [7] that the determinant of the confluent hypergeometric kernel, restricted to one side interval $(0, t)$, should be related to a solution to the Painlevé V equation. Since the confluent hypergeometric kernel is not symmetrical, thus the determinant on the interval $(-t, t)$ can not be obtained from that on the one side interval $(0, t)$. Moreover, it is interesting to note that as $\frac{2t}{n} \to 0$, in (1.4), the two Fisher-Hartwig singularities of jump type at $\theta = \frac{2\pi}{n}$ and $\theta = 2\pi - \frac{2\pi}{n}$ are merging together to the root type singularity $|e^{i\theta} - 1|^{2\alpha}$. The result of the process is a new Fisher-Hartwig singularity of both jump-type and the root-type singularity in the weight function in (1.2). The transition is different from the ones considered in [13, 14] as mentioned above. It is of interest to study the new transition asymptotics of the Toeplitz determinant from this point of view.

More precisely, this paper is devoted to the study of the determinant of the confluent hypergeometric kernel on $(-t, t)$ for both finite $t$ and the asymptotics as $t \to \infty$. Firstly, we derive the the large-$n$ asymptotics of the Toeplitz determinant $D_n(\frac{2t}{n})$ defined in (1.5) for bounded $t$. Using the relation (1.11) with the Toeplitz determinant, we obtain an integral representation of the determinant in term of the Hamiltonian of a coupled Painlevé V system. We further study the existence of solutions to the system and the asymptotics of the solutions. Secondly, we evaluate the total integral of the Hamiltonian, which then gives us the large gap asymptotics of the determinant, including the constant terms.

The Hamiltonian integral is closely related to the Jimbo-Miwa-Ueno tau-function, since the logarithmic derivative of the tau-function is the Hamiltonian; see [42]. The tau-functions of Painlevé equations play important roles in random matrix theory, two-dimensional quantum gravity, Ising models and many other areas of mathematical physics; see [29, 32, 37, 51]. Usually, it is a hard problem to determine the constant term in the asymptotics of the tau-functions. Recently, Its, Lisovyy, Prokhorov and their coauthors have developed a new method to evaluate the asymptotics of the tau-functions, including the constant terms; see [37, 38, 39]. A crucial observation is that the Hamiltonian is equal to the classical action up to a total differential. Thus, the evaluation of the integral of the Hamiltonian amounts to evaluate the integral of the classical action. A nice property of the action is that the differential of the action, with respect to the parameters of the equation or the Stokes multipliers, is again a total differential. Thus, the evaluation of the Hamiltonian integral is reduced to the calculation of an action integral for special parameters and Stokes multipliers. For certain parameters and Stokes multipliers, the Hamiltonian may admit solutions in terms of classical special functions, and its integral can be evaluated. Quite recently, the asymptotics of the tau-functions for classical Painlevé equations are successfully obtained by applying this method; see [37, 38, 39, 48]. More general Painlevé-type equations are also considered. For example, in [55], Xu and Dai obtained the
asymptotics of the Hamiltonian integral of the coupled Painlevé II system with an application in the large gap asymptotics of the Painlevé II and Painlevé XXXIV determinants. In the present paper, however, we find we are in a different position, to use the approach to evaluate the tau-function of non-classical Painlevé equations, namely the the coupled Painlevé V system in dimension four in the list of [43].

1.1 Coupled Painlevé V system

To state our main results, we need to introduce the the coupled Painlevé V system in dimension four, which is a special 4-dimensional Painlevé-type equation in the list of [43]. It is convenient to express the coupled Painlevé V system in the following Hamiltonian form

\[
\frac{dv_k}{ds} = \frac{\partial H}{\partial u_k}, \quad \frac{du_k}{ds} = -\frac{\partial H}{\partial v_k}, \quad k = 1, 2, \tag{1.12}
\]

where the Hamiltonian

\[
H = H(u_1, v_1, v_2; \alpha, \beta)
\]

is defined by

\[
sH = \frac{s}{2} H_V(u_1, v_1, s/2; \alpha, \beta) - \frac{s}{2} H_V(u_2, v_2, -s/2; \alpha, \beta) + u_1 u_2 (v_1 + v_2)(v_1 - 1)(v_2 - 1), \tag{1.13}
\]

and \(H_V\) is the classical Hamiltonian for the Painlevé V equation, such that

\[
sH_V(u, v; \alpha, \beta) = u^2 v(v - 1)^2 - suv - \alpha u(v^2 - 1) - \beta u(v - 1)^2; \tag{1.14}
\]

see [43, 40]. In terms of \(u_k\) and \(v_k\), we write (1.12) as the following system of equations

\[
\begin{align*}
\frac{dv_1}{ds} &= \frac{s}{2} u_1 - u_1^2 (v_1 - 1)(3v_1 - 1) - u_1 u_2 (v_2 - 1)(2v_1 + v_2 - 1) + 2(\alpha + \beta) u_1 v_1 - 2\beta u_1, \\
\frac{dv_2}{ds} &= -\frac{s}{2} u_2 - u_2^2 (v_2 - 1)(3v_2 - 1) - u_1 u_2 (v_1 - 1)(v_1 + 2v_2 - 1) + 2(\alpha + \beta) u_2 v_2 - 2\beta u_2, \\
\frac{du_1}{ds} &= -\frac{s}{2} v_1 + 2u_1 v_1 (v_1 - 1)^2 + u_2 (v_1 + v_2)(v_1 - 1)(v_2 - 1) - \alpha (v_1^2 - 1) - \beta (v_1 - 1)^2, \\
\frac{du_2}{ds} &= \frac{s}{2} v_2 + 2u_2 v_2 (v_2 - 1)^2 + u_1 (v_1 + v_2)(v_1 - 1)(v_2 - 1) - \alpha (v_2^2 - 1) - \beta (v_2 - 1)^2.
\end{align*}
\]

(1.15)

If the parameter \(\beta = 0\), we have the symmetry relations

\[
u_2 = -u_1 v_1^2 \quad \text{and} \quad v_2 = 1/v_1, \tag{1.16}
\]

as shown later in Section 2.4. Accordingly, the system of equations (1.15) is reduced to

\[
\begin{align*}
\frac{dv_1}{ds} &= \frac{1}{2} su_1 - 2u_1^2 v_1 (v_1^2 - 1) + 2\alpha u_1 v_1, \\
\frac{dv_1}{ds} &= -\frac{1}{2} sv_1 + u_1 (v_1^2 - 1)^2 - \alpha (v_1^2 - 1).
\end{align*}
\]

(1.17)

From the second equation in (1.17), we see that \(u_1\) can be expressed in terms of \(v_1\) and \(v_1'.\) Using this expression to delete \(u_1\) from the system, we find the second order nonlinear equation for \(v_1\)

\[
\frac{d^2 v_1}{ds^2} - \frac{2v_1}{v_1^2 - 1} \left(\frac{dv_1}{ds}\right)^2 + \frac{1}{s} \frac{dv_1}{ds} + \frac{v_1 (v_1^2 + 1)}{4 (v_1^2 - 1)} + \alpha \frac{v_1^2 + 1}{2s} = 0. \tag{1.18}
\]

Applying the Mobius transformation \(v = \frac{v_1 + 1}{v_1 - 1}\) to (1.18), we obtain the Painlevé III equation

\[
\frac{d^2 v}{ds^2} - \frac{1}{v} \left(\frac{dv}{ds}\right)^2 + \frac{1}{s} \frac{dv}{ds} - \frac{1}{16} \left(\alpha + \frac{1}{2}\right) v^2 + \left(\alpha - \frac{1}{2}\right) \frac{v^2}{16} = 0. \tag{1.19}
\]
Remark 1. If the parameters $\alpha = \beta = 0$, we obtain the Painlevé V equation from \eqref{1.18} after the transformation $q = v_1^2$, that is,
\[
\frac{d^2q}{ds^2} - \left( \frac{1}{q-1} + \frac{1}{2q} \right) \left( \frac{dq}{ds} \right)^2 + \frac{1}{s} \frac{dq}{ds} + \frac{q}{s} + \frac{q(q+1)}{2(q-1)} = 0.
\]  
(1.20)
Moreover, using the symmetry \eqref{1.16}, the Hamiltonian \eqref{1.13} is reduced to the classical Hamiltonian for the Painlevé V equation after elementary transformations,
\[
sH(u_1, v_1, u_2, v_2, s; 0, 0) = u_1^2(v_1^2 - 1)^2 - su_1v_1 = p^2q(q-1)^2 - spq,
\]
where $q = v_1^2$ and $p = u_1/v_1$. Denote
\[
\sigma(s) = sH(u_1, v_1, u_2, v_2, s; 0, 0),
\]
we get from the Hamilton equations $q' = \frac{\partial H}{\partial p}$ and $p' = -\frac{\partial H}{\partial q}$ the Jimbo-Miwa-Okamoto $\sigma$-form of the Painlevé V equation
\[
(s\sigma')^2 - (\sigma - s\sigma' + 4\sigma^2) (\sigma - s\sigma') = 0;
\]
(1.23)
see \cite{34,40}.

Quite recently, the coupled Painlevé systems have found interesting applications in random matrix theory. For example, a different coupled Painlevé V system was used in the studies of a singularly perturbed Laguerre unitary ensemble \cite{15,56}. The coupled Painlevé II system was applied in the studies of the Airy point process and the determinants of the Painlevé II and Painlevé XXXIV kernels \cite{12,55}. Moreover, the coupled Painlevé III system was involved in the Painlevé-type representation of the determinants of a Painlevé III kernel, which arose in the studies of a singularly perturbed Laguerre unitary ensemble \cite{13,56}.

We have the following existence and asymptotic results for the coupled Painlevé V system \eqref{1.15}. The solutions play a central role in our integral representation of the gap probability.

Theorem 1. For pure imaginary parameter $\beta$ and $\alpha > -1/2$, there exist solutions to the coupled Painlevé V system \eqref{1.15} with the following asymptotic behavior
\[
u_1(s) = 1 + O(s^{2\alpha+1}) + O(s), \quad is \to 0^+,
\]
(1.24)

\[
u_2(s) = 1 + O(s^{2\alpha+1}) + O(s), \quad is \to 0^+,
\]
(1.25)

\[
u_3(s) = 1 + O(s^{2\alpha+1}) + O(s), \quad is \to 0^+,
\]
(1.26)

\[
u_4(s) = 1 + O(s^{2\alpha+1}) + O(s), \quad is \to 0^+,
\]
(1.27)

\[
u_5(s) = 1 + O(s^{2\alpha+1}) + O(s), \quad is \to 0^+,
\]
(1.28)

\[
u_6(s) = 1 + O(s^{2\alpha+1}) + O(s), \quad is \to 0^+,
\]
(1.29)

\[
u_7(s) = 1 + O(s^{2\alpha+1}) + O(s), \quad is \to 0^+,
\]
(1.30)

\[
u_8(s) = 1 + O(s^{2\alpha+1}) + O(s), \quad is \to 0^+.
\]
(1.31)
Moreover, the Hamiltonian
\[ H(s; \alpha, \beta) = H(u_1, v_1, u_2, v_2, s; \alpha, \beta), \]
associated with these \( u_k(s) \) and \( v_k(s) \), is pole-free for \( s \in -i(0, +\infty) \) and has the asymptotic behavior
\[ H(s; \alpha, \beta) = \frac{\Gamma(1 + \alpha + \beta) \Gamma(1 + \alpha - \beta) \cos(\pi \beta)}{i\pi 2^{2\alpha + 1}(2\alpha + 1)\Gamma(1 + 2\alpha)^2} |s|^{2\alpha} + O\left(s^{2\alpha + 1}\right), \quad s \to 0^+, \quad (1.32) \]
and
\[ H(s; \alpha, \beta) = \frac{s}{16} + \frac{i}{2\alpha} - \left(\alpha^2 - \beta^2 + \frac{1}{4}\right) \frac{1}{s} + O\left(\frac{1}{s^2}\right), \quad s \to +\infty. \quad (1.33) \]

**Remark 2.** For the parameters \( \alpha = 1/2 \) and \( \beta = 0 \), the coupled Painlevé V system admits solutions in terms of classical special functions; see Section 2.4 for details. In particular, we have
\[ H(u_1, v_1, u_2, v_2, s; 1/2, 0) = \frac{s}{16} + \frac{i}{4} I_0((is/4)) I_0((is/4)), \quad (1.34) \]
where \( I_0(s) \) is the modified Bessel function of order zero. It is noted that for this special solution, the integral of the Hamiltonian can be easily evaluated.

### 1.2 Gap probability and Painlevé-type formulas

The gap probability for the eigenvalues of the circular unitary ensemble with a Fisher-Hartwig singularity can be expressed in terms of the Toeplitz determinant associated with the weight function \( (1.4) \). Via deriving the asymptotics of the Toeplitz determinant, we give explicit expressions for the gap probability as an integral of the Hamiltonian to the coupled Painlevé V system in dimension four.

**Theorem 2.** For pure imaginary parameter \( \beta \) and \( \alpha > -1/2 \), as \( n \to \infty \) and \( t \to 0^+ \) in a way such that \( nt \) is bounded, we have the asymptotic approximation of the Toeplitz determinant associated with the weight function \( (1.4) \)
\[ \frac{D_n(t)}{D_n(0)} = \exp\left(\int_0^{-2int} H(\tau; \alpha, \beta) d\tau + O\left(\frac{1}{n}\right)\right), \quad (1.35) \]
where \( H(s; \alpha, \beta) \) is the Hamiltonian for the coupled Painlevé V system with the properties specified in Theorem 1 and the error term is uniform for \( nt \) bounded.

It then follows from \( (1.6) \) and Theorem 2 that we have the Painlevé-type representation of the gap probability for the circular ensemble with a Fisher-Hartwig singularity \( (1.2) \) as \( n \to \infty \)
\[ \text{Pro}(\{\theta_j \in (t, 2\pi - t) : 1 \leq j \leq n\}) = \exp\left(\int_0^{-2int} H(\tau; \alpha, \beta) d\tau + O\left(\frac{1}{n}\right)\right). \]

The gap probability can also be expressed in terms of the Fredholm determinant of the confluent hypergeometric kernel, as given in \( (1.11) \), such that
\[ \lim_{n \to \infty} \frac{D_n(\frac{2\alpha}{n})}{D_n(0)} = \det \left(I - K_s^{(\alpha, \beta)}\right), \]
where \( K_s^{(\alpha, \beta)} \) is the operator with the confluent hypergeometric kernel acting on \( L^2(-s, s) \); see [22]. As a direct application of the above Theorem, we obtain an explicit representation of the determinant of the confluent hypergeometric kernel.
Theorem 3. For pure imaginary parameter $\beta$ and $\alpha > -1/2$, let $K_s^{(\alpha, \beta)}$ be the operator with the confluent hypergeometric kernel $K^{(\alpha, \beta)}(u, v)$ in (1.7) acting on $L^2(-s, s)$, $s > 0$, we have
\[ \det \left( I - K_s^{(\alpha, \beta)} \right) = \exp \left( \int_{-4is}^{4is} H(\tau; \alpha, \beta) d\tau \right), \] (1.36)
where $H(s; \alpha, \beta)$ is the Hamiltonian for the coupled Painlevé V system with the properties specified in Theorem 1.

For the special parameters $\alpha = \beta = 0$, the confluent hypergeometric kernel is reduced to the classical sine kernel (1.10). Applying Theorem 1, Theorem 3 and Remark 1, we recover the following Painlevé V representation of the determinant of the sine kernel, which was obtained earlier by Jimbo, Miwa, Mōri and Sato [41].

Corollary 1. Let $K_s^{(0, 0)}$ be the operator with sine kernel (1.10) acting on $L^2(-s, s)$, $s > 0$, we have
\[ \det \left( I - K_s^{(0, 0)} \right) = \exp \left( \int_0^s \sigma_V(\tau) d\tau \right), \] (1.37)
where $\sigma_V(s)$ is the solution of the equation
\[ (s\sigma_V'')^2 + 4(4\sigma_V - 4s\sigma_V' - \sigma_V')\left(\sigma_V - s\sigma_V'\right) = 0, \] (1.38)
characterized by the asymptotic behavior
\[ \sigma_V(s) \sim -\frac{2}{\pi} s, \quad s \to 0^+ \quad \text{and} \quad \sigma_V(s) \sim -s^2 - \frac{1}{4}, \quad s \to +\infty. \] (1.39)

1.3 Total integral of the Hamiltonian and large gap asymptotics

As is noted in Remark 2 that for special parameters, the coupled Painlevé V system (1.15) admits special function solutions. For these special solutions, the integral of the Hamiltonian associate can be evaluated explicitly. For general parameters, the derivation is divided into two steps. First we evaluate the derivative of the Hamiltonian integral by deriving and applying certain differential identities, with respect to the parameters, for the Hamiltonian. Then, after taking integration we obtain the following total integral of the Hamiltonian.

Theorem 4. For pure imaginary parameter $\beta$ and $\alpha > -1/2$, we have the total integral of the Hamiltonian after regularization at infinity
\[ \int_0^c H(\tau; \alpha, \beta) d\tau + \int_{-c}^{-i\infty} \left[ H(\tau; \alpha, \beta) - \frac{\tau}{16} - \frac{i c^2}{2} + \alpha^2 - \beta^2 + \frac{1}{4} \right] d\tau \]
\[ = \frac{c^2}{32} + \frac{i c}{2} \left( \alpha^2 - \beta^2 + \frac{1}{4} \right) \ln |c| + \ln \left( \frac{\sqrt{\pi} G(1/2) G(1 + 2\alpha)}{2^{2\alpha^2} G(1 + \alpha + \beta) G(1 + \alpha - \beta)} \right), \] (1.40)
where $ic > 0$ and $G$ is the Barnes $G$-function.

As an application of the above total integral of the Hamiltonian, we derive the large gap asymptotics of the the confluent hypergeometric-kernel determinant, including the explicitly given constant term. From the large-$s$ asymptotic formula (1.33), we see that the second integral in (1.40) is of order $O(1/c)$ as $ic \to +\infty$. Therefore, let $c = -4is$ and substitute (1.40) into (1.36), we obtain the following theorem as $s \to +\infty$. 

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Theorem 5. For pure imaginary parameter $\beta$ and $\alpha > -1/2$, let $K^{(\alpha,\beta)}_s$ be the operator with the confluent hypergeometric kernel $K^{(\alpha,\beta)}(u,v)$ in (1.7) acting on $L^2(-s,s)$, we have the asymptotics as $s \to +\infty$

$$\det(I - K^{(\alpha,\beta)}_s) = \frac{\sqrt{\pi}G(\frac{1}{2})^2 G(1 + 2\alpha)}{2^{2\alpha^2} G(1 + \alpha + \beta) G(1 + \alpha - \beta)} s^{-(\alpha^2 - \beta^2 + \frac{1}{4})} e^{-\frac{s^2}{2}} \left[ 1 + O\left(\frac{1}{s}\right) \right] , \quad (1.41)$$

where $G$ is the Barnes $G$-function.

We reproduce the large gap asymptotics of the determinant with a confluent hypergeometric kernel, obtained by Deift, Krasovsky and Vasilyevskaya [22]. Particularly, we recover the large gap asymptotics of the sine-kernel determinant by taking $\alpha = \beta = 0$ in (1.41) and using the relation between the Barnes $G$-function and the Riemann zeta-function that $\ln G(1/2) = \frac{3}{2} \zeta'(1) - \frac{1}{4} \ln \pi + \frac{1}{24} \ln 2$.

Corollary 2. Let $K^{(0,0)}_s$ be the operator with sine kernel (1.10) acting on $L^2(-s,s)$, we have the large gap asymptotics, as $s \to +\infty$,

$$\ln \det \left( I - K^{(0,0)}_s \right) = -s^2 \frac{2}{2} - \frac{\ln s}{4} + \frac{\ln 2}{12} + 3\zeta'(1) + O\left(\frac{1}{s}\right) , \quad (1.42)$$

where $\zeta'(s)$ is the derivative of the Riemann zeta-function.

Using the series expansion of $\sigma_V(s)$ as $s \to +\infty$, of which the first few terms can be found in (1.39), Dyson was able to obtain the asymptotics of the sine-kernel determinant of the form (1.42) in [26]. While the constant factor can not be derived from the series expansion of $\sigma_V(s)$, it was conjectured to be given by the Riemann zeta-function $c_0 = \frac{\ln 2}{12} + 3\zeta'(1)$ by Dyson in the same paper [26]. This conjecture has been proved rigorously by Krasovsky [45] and Ehrhardt [27] using different methods. In the present paper, our result provides another proof of the large gap asymptotics (1.42) including the constant conjectured by Dyson.

The rest of the paper is arranged as follows. In Section 2, we formulate the model Riemann-Hilbert problem (RH problem or RHP, for short) for $\Psi(\zeta; s)$. We derive a Lax pair corresponding to the model RH problem, of which the compatibility condition is expressed as a coupled Painlevé V system. Several differential identities of the Hamiltonian are also established, which are crucial in the evaluation of the total integral of the Hamiltonian. We then justify the solvability of $\Psi(\zeta; s)$ for the parameters $\alpha > -1/2$, $\beta \in i\mathbb{R}$ and $s \in -i(0, \infty)$ by proving a vanishing lemma. The solvability implies the pole-free of the Hamiltonian associated with the coupled Painlevé V system for $is > 0$. The special function solutions to the coupled Painlevé system are also derived for certain specific parameters in this section. In Section 3, we derive asymptotic formulas for the coupled Painlevé V system as $is \to 0^+$ by using Deift-Zhou nonlinear steepest descent method for the RH problems. While the large-$s$ analysis will be carried out in Section 4. These two sections together provide a proof of Theorem 4. In Section 5 we formulate a RH problem for $Y$, corresponding to the orthogonal polynomials associated with the weight function (1.4) on an arc, we also derive a differential identity connecting the Toeplitz determinant (1.5) with $Y$. In Section 6, we carry out, in full details, the Deift-Zhou nonlinear steepest descent analysis of the RH problem for $Y$. The asymptotics of the orthogonal polynomials and the differential identity then lead to a proof of Theorem 2 in this section. Section 7 will be devoted to the evaluation of the total integral of the Hamiltonian and the large gap asymptotic analysis of the model. Theorem 3 is also proved in this last section.
2 Model RH problem and coupled Painlevé V system

In this section, we introduce a RH problem for $\Psi(\zeta; s)$ for later use. We derive a Lax pair for the solution to the RH problem for $\Psi(\zeta; s)$, which turns out to be a Garnier system in the list of [43, 44], with three regular singularities and one irregular singularity of order two. The compatibility condition of the Lax pair is described by the coupled Painlevé V system in dimension four. We then prove the solvability of this model RH problem and the pole-free of the Hamiltonian associated with the coupled Painlevé V system for $is > 0$. The Hamiltonian plays a central role in the derivation of our main results on the asymptotics for the Toeplitz determinant. In this section, we will also show that for special parameters, the coupled Painlevé system admits special function solutions.

2.1 A Model RH problem

We formulate a model RH problem, which will pave the road to the steepest descent analysis in later sections. The model RH problem for $2 \times 2$ matrix function $\Psi(\zeta) = \Psi(\zeta; s)$ is the following:

(a) $\Psi(\zeta; s)$ is analytic in $\zeta \in \mathbb{C} \setminus \bigcup_{j=1}^{v} \Sigma_j$, where the oriented $\zeta$-contours

\[ \Sigma_1 = 1 + e^{\frac{2\pi i}{4}} \mathbb{R}^+, \Sigma_2 = -1 + e^{\frac{3\pi i}{4}} \mathbb{R}^+, \Sigma_3 = -1 + e^{-\frac{3\pi i}{4}} \mathbb{R}^+, \Sigma_4 = e^{-\frac{\pi i}{2}} \mathbb{R}^+, \Sigma_5 = 1 + e^{-\frac{\pi i}{4}} \mathbb{R}^+, \Sigma_6 = (0, 1) \text{ and } \Sigma_7 = (-1, 0), \]

as depicted in Figure 1, see also an illustration of the regions $\Omega_j$ for $j = 1, 2, \cdots, 5$, each having $\Sigma_j$ and $\Sigma_{j+1}$ as portions of its boundary.

![Figure 1: Contours and regions for the model RH problem](image)

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(b) $\Psi(\zeta; s)$ satisfies the jump conditions

$$\Psi_+(\zeta; s) = \Psi_-(\zeta; s) e^{2\pi i \sigma_3}, \quad \zeta \in \Sigma_4,$$

$$
\begin{cases}
\left( \begin{array}{cc}
1 & 0 \\
e^{-\pi i (\alpha - \beta)} & 1 \\
\end{array} \right), & \zeta \in \Sigma_1, \\
\left( \begin{array}{cc}
1 & 0 \\
e^{\pi i (\alpha - \beta)} & 1 \\
\end{array} \right), & \zeta \in \Sigma_2, \\
\left( \begin{array}{cc}
1 & -e^{-\pi i (\alpha - \beta)} \\
0 & 1 \\
\end{array} \right), & \zeta \in \Sigma_3, \\
r^{2\pi i \sigma_3}, & \zeta \in \Sigma_4, \\
\left( \begin{array}{cc}
1 & -e^{-\pi i (\alpha - \beta)} \\
0 & 1 \\
\end{array} \right), & \zeta \in \Sigma_5, \\
\left( \begin{array}{cc}
0 & -e^{\pi i (\alpha - \beta)} \\
e^{-\pi i (\alpha - \beta)} & 0 \\
\end{array} \right), & \zeta \in \Sigma_6, \\
\left( \begin{array}{cc}
0 & -e^{-\pi i (\alpha - \beta)} \\
e^{\pi i (\alpha - \beta)} & 0 \\
\end{array} \right), & \zeta \in \Sigma_7.
\end{cases}
$$

For later use, we denote the jump matrix on $\Sigma_k$ by $J_k$ for $k = 1, 2, \ldots, 7$.

(c) As $\zeta \to \infty$, we have

$$\Psi(\zeta; s) = \left( I + \frac{\Psi_1(s)}{\zeta} + \frac{\Psi_2(s)}{\zeta^2} + O \left( \frac{1}{\zeta^3} \right) \right) \zeta^{-\beta \sigma_3 e^{\pi i s \sigma_3}},$$

(2.2)

where the branch cut of the function $\zeta^\beta$ is taken along $(0, -i\infty)$ such that $\arg \zeta \in (-\pi/2, 3\pi/2)$, and $\sigma_3$ is the Pauli matrix. This and the other Pauli matrices are

$$\sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma_2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right) \quad \text{and} \quad \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$$

(2.3)

(d) As $\zeta \to 0$, we have

$$\Psi(\zeta; s) = \Psi^{(0)}(\zeta; s) e^{\alpha \sigma_3 E_j^{(0)}}, \quad \zeta \in \Omega_j, \ j = 1, 3, 4,$$

(2.4)

where $\Psi^{(0)}(\zeta; s) = \Psi_0^{(0)}(s) \left( I + \Psi_1^{(0)}(s) \zeta + O \left( \zeta^2 \right) \right)$ is analytic at $\zeta = 0$, and $\z^\beta$ takes the principal branch. The connection matrices are given below

$$E_1^{(0)} = I, \quad E_4^{(0)} = J_6^{-1}, \quad E_3^{(0)} = J_6^{-1} J_4^{-1},$$

where $J_k$ is the jump matrix for $\Psi$ on $\Sigma_k$ for $k = 1, 2, \ldots, 7$; cf. (2.1).
(e) As $\zeta \to 1$, we have

$$\Psi(\zeta; s) = \Psi^{(1)}(\zeta; s) \begin{pmatrix} 1 & c_1 \ln(\zeta - 1) \\ 0 & 1 \end{pmatrix} E_j^{(1)}, \quad \zeta \in \Omega_j, \; j = 1, 4, 5,$$

where $\Psi^{(1)}(\zeta; s) = \Psi_0^{(1)}(s) \left( I + \Psi_1^{(1)}(s)(\zeta - 1) + O((\zeta - 1)^2) \right)$ is analytic at $\zeta = 1$, the logarithmic function takes the principal branch, the constant $c_1 = -\frac{1}{2\pi i} e^{-i(\alpha - \beta)}$ and the connection matrices are given as

$$E_1^{(1)} = I, \quad E_5^{(1)} = J_1^{-1}, \quad E_4^{(1)} = J_1^{-1}J_5^{-1},$$

$J_k$ being the jump matrix for $\Psi$ on $\Sigma_k$ for $k = 1, 2, \ldots, 7$; see (2.1).

(f) As $\zeta \to -1$, we have

$$\Psi(\zeta; s) = \Psi^{(2)}(\zeta; s) \begin{pmatrix} 1 & c_2 \ln(\zeta + 1) \\ 0 & 1 \end{pmatrix} E_j^{(2)}, \quad \zeta \in \Omega_j, \; j = 1, 2, 3,$$

where $\Psi^{(2)}(\zeta; s) = \Psi_0^{(2)}(s) \left( I + \Psi_1^{(2)}(s)(\zeta + 1) + O((\zeta + 1)^2) \right)$ is analytic at $\zeta = -1$. The branch for the logarithm is taken so that $\arg(\zeta + 1) \in (0, 2\pi)$. The constant $c_2 = \frac{1}{2\pi i} e^{-i(\alpha - \beta)}$ and the connection matrices are given as follow

$$E_1^{(2)} = I, \quad E_2^{(2)} = J_2^{-1}, \quad E_3^{(2)} = J_2^{-1}J_3^{-1},$$

where $J_k$ is the jump matrix for $\Psi$ on $\Sigma_k$ for $k = 1, 2, \ldots, 7$; see (2.1).

In the above model RH problem, the connection matrices describing the sector-wise behaviors of $\Psi$ near the node points $\zeta = 0$ and $\pm 1$, are specified in a way such that they are in compliance with the jump conditions (2.1) in neighborhoods of the origin and $\zeta = \pm 1$.

### 2.2 Lax pair and the coupled Painlevé V system

In this section, we derive a Lax pair from the RH problem for $\Psi$. The Lax pair is associated with a special Garnier system, of which the compatibility condition is expressed as the coupled Painlevé V system in dimension four.

**Proposition 1.** We have the following Lax pair

$$\frac{d}{d\zeta} \Psi(\zeta; s) = L(\zeta; s)\Psi(\zeta; s), \quad \frac{d}{ds} \Psi(\zeta; s) = U(\zeta; s)\Psi(\zeta; s),$$

where

$$L(\zeta; s) = \frac{s}{4}\sigma_3 + \frac{A_0(s)}{\zeta} + \frac{A_1(s)}{\zeta - 1} + \frac{A_2(s)}{\zeta + 1},$$

$$U(\zeta; s) = \frac{1}{4}\zeta\sigma_3 + B(s),$$

In this section, we derive a Lax pair from the RH problem for $\Psi$. The Lax pair is associated with a special Garnier system, of which the compatibility condition is expressed as the coupled Painlevé V system in dimension four.
with the coefficients given below

\[
A_0(s) = \begin{pmatrix}
    u_1(s)v_1(s) + u_2(s)v_2(s) - \beta & - (u_1(s)v_1(s) + u_2(s)v_2(s) - \alpha - \beta) y(s) \\
    (u_1(s)v_1(s) + u_2(s)v_2(s) + \alpha - \beta) / y(s) & -u_1(s)v_1(s) - u_2(s)v_2(s) + \beta
\end{pmatrix},
\]

(2.10)

\[
A_k(s) = \begin{pmatrix}
    -u_k(s)v_k(s) & u_k(s)y(s) \\
    -u_k(s)v_k^2(s)/y(s) & u_k(s)v_k(s)
\end{pmatrix}, \quad k = 1, 2,
\]

(2.11)

and

\[
B(s) = \frac{1}{s} \begin{pmatrix}
    0 & b_1(s) y \\
    b_2(s)/y & 0
\end{pmatrix},
\]

(2.12)

where

\[
\begin{align*}
    b_1(s) &= -u_1(s)(v_1(s) - 1) - u_2(s)(v_2(s) - 1) + \alpha + \beta \\
    b_2(s) &= -u_1(s)v_1(s)(v_1(s) - 1) - u_2(s)v_2(s)(v_2(s) - 1) + \alpha - \beta.
\end{align*}
\]

(2.13)

The compatibility condition of the Lax pair is expressed as the coupled Painlevé V system

\[
\begin{align*}
    s \frac{du_1}{ds} &= \frac{3}{2} u_1 - u_1^2(v_1 - 1)(3v_1 - 1) - u_1 u_2(v_2 - 1)(2v_1 + v_2 - 1) + 2(\alpha + \beta) u_1 v_1 - 2 \beta u_1, \\
    s \frac{du_2}{ds} &= -\frac{3}{2} u_2 - u_2^2(v_2 - 1)(3v_2 - 1) - u_1 u_2(v_1 - 1)(v_1 + 2v_2 - 1) + 2(\alpha + \beta) u_2 v_2 - 2 \beta u_2, \\
    s \frac{dv_1}{ds} &= -\frac{3}{2} v_1 + 2u_1 v_1(v_1 - 1)^2 + u_2(v_1 + v_2)(v_1 - 1)(v_2 - 1) - \alpha(v_1^2 - 1) - \beta(v_1 - 1)^2, \\
    s \frac{dv_2}{ds} &= \frac{3}{2} v_2 + 2u_2 v_2(v_2 - 1)^2 + u_1(v_1 + v_2)(v_1 - 1)(v_2 - 1) - \alpha(v_2^2 - 1) - \beta(v_2 - 1)^2.
\end{align*}
\]

(2.14)

Let

\[
sH = sH(u_1, v_1, u_2, v_2, s; \alpha, \beta) = -\frac{s}{2} (\Psi_1)_{11} - (\alpha^2 - \beta^2),
\]

(2.15)

where \((\Psi_1)_{11}\) is the \((1, 1)\)-entry of the coefficient \(\Psi_1\) in the large-\(\zeta\) asymptotic approximation \((2.2)\) of \(\Psi(\zeta; s)\), then we have

\[
sH = \frac{s}{2} H_V(u_1, v_1, s/2; \alpha, \beta) - \frac{s}{2} H_V(u_2, v_2, -s/2; \alpha, \beta) + u_1 u_2(v_1 + v_2)(v_1 - 1)(v_2 - 1),
\]

(2.16)

as defined in \([13]\), where \(H_V(u, v, s; \alpha, \beta)\) is the Hamiltonian for the Painlevé V equation

\[
sH_V(u, v, s; \alpha, \beta) = u^2(v - 1)^2 - suv - \alpha u(v^2 - 1) - \beta u(v - 1)^2.
\]

(2.17)

The system \((2.14)\) is equivalent to the following Hamiltonian formulation

\[
\frac{dv_k}{ds} = \frac{\partial H}{\partial u_k}, \quad \frac{du_k}{ds} = -\frac{\partial H}{\partial v_k}, \quad k = 1, 2.
\]

(2.18)

**Proof.** Note that all the jump matrices in \((3.2)\) of the RH problem for \(\Psi(\zeta; s)\) are independent of the variables \(\zeta\) and \(s\). Then \(\frac{d}{ds} \Psi(\zeta; s)\), \(\frac{d}{ds} \Psi_s(\zeta; s)\) and \(\Psi(\zeta; s)\) satisfy the same jump conditions. Thus, the matrix-valued functions \(L(\zeta; s) = \frac{d}{ds} \Psi(\zeta; s)\Psi^{-1}(\zeta; s)\) and \(U(\zeta; s) = \frac{d}{ds} \Psi(\zeta; s)\Psi^{-1}(\zeta; s)\) are meromorphic for \(\zeta\) in the complex plane with only possible isolated singularities at \(\zeta = 0, \pm 1\). Then, it follows from the local behavior of \(\Psi(\zeta; s)\) as \(\zeta \to \infty, \zeta \to 0,\)
and $\zeta \to \pm 1$; cf. (2.2), (2.4), (2.5) and (2.6), that $L$ and $U$ are rational functions in $\zeta$ and take the form as given in (2.8) and (2.9), respectively.

Using the fact that $\det \Psi = 1$, we have $\text{tr} L = \text{tr} U = 0$ and thus all the coefficients $A_k$, $k = 0, 1, 2$ and $B$ are trace-zero. Substituting the behavior of $\Psi$ at infinity (2.2) into the first equation of the Lax pair (2.7), we find after comparing the coefficient of $\frac{1}{\zeta}$ that

$$A_0 + A_1 + A_2 = \begin{pmatrix} -\beta & -\frac{s}{2}(\Psi_1)_{12} \\ \frac{s}{2}(\Psi_1)_{21} & \beta \end{pmatrix},$$

(2.19)

where $\Psi_1$ is the coefficient of the large $\zeta$ asymptotic of $\Psi(\zeta; s)$ in (2.2). Moreover, combining the master equation in (2.7) with the local behavior (2.4) of $\Psi(\zeta; s)$ at $\zeta = 0$, and (2.5)-(2.6) at $\zeta = \pm 1$, we have

$$\det A_0 = -\alpha^2, \quad \det A_k = 0, \quad k = 1, 2.$$ 

Thus, the coefficients $A_k$, $k = 0, 1, 2$, can be taken in the form appeared in (2.10) and (2.11).

Similarly, substituting the behavior of $\Psi$ at infinity (2.2) into the second equation of the Lax pair (2.7) and equalizing the zero-order terms in $\zeta$, we find

$$B = \begin{pmatrix} 0 & -\frac{s}{2}(\Psi_1)_{12} \\ \frac{s}{2}(\Psi_1)_{21} & 0 \end{pmatrix},$$

(2.20)

where $\Psi_1$ is the coefficient of the large-$\zeta$ asymptotic approximation of $\Psi(\zeta; s)$ in (2.2). This, together with (2.19), implies that $B$ is given by (2.12).

The compatibility condition of the Lax pair (2.7) is readily written as

$$\begin{align*}
\frac{d}{ds} A_0 &= [B, A_0], \\
\frac{d}{ds} A_1 &= [B + \frac{1}{4}\sigma_3, A_1], \\
\frac{d}{ds} A_2 &= [B - \frac{1}{4}\sigma_3, A_2],
\end{align*}$$

(2.21)

where the commutator $[A, B] = AB - BA$. Denote $a(s)$ by

$$a(s) = (A_0)_{11}(s) = u_1(s)v_1(s) + u_2(s)v_2(s) - \beta.$$ 

(2.22)

Recalling the definition of $b_k$ in (2.13) and substituting (2.10), (2.11) and (2.12) into (2.21) yields

$$\begin{align*}
\frac{s}{ds}((a - \alpha)y) &= 2ab_1y, \\
\frac{s}{ds}((a + \alpha)y) &= 2ab_2/y, \\
\frac{s}{ds}(u_kv_k) &= b_1u_kv_k^2 + b_2u_k, \quad k = 1, 2, \\
\frac{s}{ds}(u_kv) &= (-1)^{k+1}s_{k}u_ky + 2b_1u_kv_ky, \quad k = 1, 2.
\end{align*}$$

(2.23)

Then, the first two equations imply that the gauge parameter $y(s)$ of the Lax pair satisfies the equation

$$sy'(s)/y(s) = b_1(s) - b_2(s) = u_1(s)(v_1(s) - 1)^2 + u_2(s)(v_2(s) - 1)^2 + 2\beta.$$ 

(2.24)

This equation, together with the last two equations of (2.23), gives us the system (2.14).
Substituting the large-ζ expansion (2.2) into the first equation of the Lax pair (2.7), and comparing the coefficients of $1/\zeta^2$, we find
\[
\frac{s}{4} [\sigma_3 \Psi_1, \Psi_1] + \frac{s}{4} [\Psi_2, \sigma_3] - \beta [\Psi_1, \sigma_3] - \Psi_1 = A_1 - A_2.
\] (2.25)
Hence, combining the definition (2.15) with (2.11) and (2.25), we obtain (2.16). By (2.16), it is readily verified that the Hamiltonian system (2.18) is equivalent to the system of equations (2.14). This completes the proof of the proposition.

We then derive several differential identities for later use.

**Proposition 2.** Let $\Psi^{(1)}$ and $\Psi^{(2)}$ be the matrix functions in the asymptotic behaviors of $\Psi(\zeta; s)$ near $\zeta = 1$ and $\zeta = -1$, defined respectively in (2.5) and (2.6), we have
\[
 u_k(s)v_k(s) = \frac{(-1)^k}{\pi} e^{(-1)^{k+1} \pi i (\alpha - \beta)} \frac{d}{ds} \left( \Psi_1^{(k)}(s) \right)_{21}, \quad k = 1, 2.
\] (2.26)
Moreover, we have
\[
y(s) = \left( \Psi_0^{(0)}(s) \right)_{11}^{11} \left( \Psi_0^{(0)}(s) \right)_{21}^{21}
\] (2.27)
and
\[
s \frac{d}{ds} \ln y = u_1(s) (v_1(s) - 1)^2 + u_2(s) (v_2(s) - 1)^2 + 2\beta.
\] (2.28)
Denoting
\[
d(s) = 2\alpha \left( \Psi_0^{(0)}(s) \right)_{11}^{11} \left( \Psi_0^{(0)}(s) \right)_{21}^{21},
\] (2.29)
we have
\[
s \frac{d}{ds} \ln d(s) = b_1(s) + b_2(s) = -u_1(s) (v_1^2(s) - 1) - u_2(s) (v_2^2(s) - 1) + 2\alpha.
\] (2.30)

**Proof.** Substituting (2.4), (2.5) and (2.6) into the first equation of (2.7), we find
\[
 A_0(s) = \alpha \Psi_0^{(0)}(s) \sigma_3 \left\{ \Psi_0^{(0)}(s) \right\}^{-1},
\] (2.31)
\[
 A_k(s) = c_k \Psi_0^{(k)}(s) \sigma_3 \left\{ \Psi_0^{(k)}(s) \right\}^{-1},
\] (2.32)
where $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and the constant $c_k = (-1)^k \frac{1}{2\pi} e^{(-1)^{k+1} \pi i (\alpha - \beta)}$ for $k = 1, 2$. Similarly, upon substitution of (2.4), (2.5) and (2.6) into the second equation of (2.7), it follows that
\[
 \frac{d}{ds} \Psi_0^{(0)}(s) = B \Psi_0^{(0)}(s),
\] (2.33)
\[
 \frac{d}{ds} \Psi_0^{(k)}(s) = \left( \frac{(-1)^{k+1}}{4} \sigma_3 + B \right) \Psi_0^{(k)}(s),
\] (2.34)
and
\[
 \frac{d}{ds} \Psi_1^{(k)}(s) = \frac{1}{4} \left\{ \Psi_0^{(k)}(s) \right\}^{-1} \sigma_3 \Psi_0^{(k)}(s)
\] (2.35)
for $k = 1, 2$. We then get from (2.32) and (2.35) that
\[
\frac{d}{ds} \left( \Psi^{(k)}(s) \right)_{21} = \frac{1}{2c_k} (A_k(s))_{11}
\]  
(2.36)
for $k = 1, 2$. This, together with (2.11), implies (2.26).

Equation (2.27) follows from a combination of (2.10) and (2.31). Indeed, we have
\[
y(s) = \left( \frac{A_0(s)}{s} \right)_{11} + \alpha \left( \frac{\Psi^{(0)}(s)}{s} \right)_{11} = \left( \Psi^{(0)}(s) \right)_{11}.
\]

Here use has been made of the fact that $\det \Psi^{(0)}(s) = 1$. In view of (2.13) and (2.27), and splitting (2.33) to entries, we derive the logarithmic derivative (2.30) for the gauge parameter $d(s)$. The equation (2.28) has already been derived before; see (2.24). This completes the proof.

The following differential identities of the Hamiltonian with respect to the parameters $\alpha$ and $\beta$ are crucial to the evaluation of the total integral of the Hamiltonian.

**Proposition 3.** The Hamiltonian $H = H(u_1, v_1, u_2, v_2, s; \alpha, \beta)$ is equivalent to the classical action up to a total differential
\[
H = \left( u_1 \frac{dv_1}{ds} + u_2 \frac{dv_2}{ds} - H \right) + \frac{d}{ds} \left( sH + \alpha \ln d(s) - \beta \ln y(s) - 2(\alpha^2 - \beta^2) \ln s \right),
\]  
(2.37)
where $y(s)$ and $d(s)$ satisfy the properties given in Proposition 2. The Hamiltonian system implies the following useful differential formulas for the classical action
\[
\frac{d}{d\alpha} \left( u_1 \frac{dv_1}{ds} + u_2 \frac{dv_2}{ds} - H \right) = \frac{d}{ds} \left( u_1 d \frac{d}{d\alpha} v_1 + u_2 \frac{d}{d\alpha} v_2 - \ln d + 2\alpha \ln s \right),
\]  
(2.38)
\[
\frac{d}{d\beta} \left( u_1 \frac{dv_1}{ds} + u_2 \frac{dv_2}{ds} - H \right) = \frac{d}{ds} \left( u_1 \frac{d}{d\beta} v_1 + u_2 \frac{d}{d\beta} v_2 + \ln y - 2\beta \ln s \right).
\]  
(2.39)

**Proof.** It follows from the Hamiltonian system (2.18) and (2.16) that
\[
\frac{d}{ds} (sH(s)) = -\frac{1}{2} \left( u_1(s)v_1(s) - u_2(s)v_2(s) \right).
\]  
(2.40)

Using this equation, together with (2.16), (2.28) and (2.30), we readily verify (2.37). The verification of (2.38) is straightforward, by substituting (2.18), (2.16) and (2.30) into it. While that of (2.39) follows from (2.18), (2.16) and (2.28). Here we regard $u_k$ and $v_k$, $k = 1, 2$, as functions of $\alpha$ and $\beta$, and $\alpha$, $\beta$ and $s$ are independent.

### 2.3 Vanishing lemma and existence of solution to the model RH problem

We will prove the existence of solution to the model RH problem for $\Psi(\zeta; s)$ if the parameters $\alpha > -1/2$, $\beta \in i\mathbb{R}$ and $s \in -i(0, +\infty)$. We start with the vanishing lemma which shows that the homogeneous RH problem has only zero solution.
Lemma 1. For $\alpha > -1/2$, $\beta \in i\mathbb{R}$ and $s \in (-i(0, +\infty)$, we suppose that $\hat{\Psi}(\zeta)$ satisfies the same jump conditions (2.1) and the same behaviors (2.4), (2.5), and (2.6) as $\Psi(\zeta)$, respectively as $\zeta$ tends to the origin and $\pm 1$. Further assume the behavior of $\hat{\Psi}(\zeta)$ to be

$$
\hat{\Psi}(\zeta) = O \left( \frac{1}{\zeta} \right) \zeta^{-\beta s_3} e^{\frac{1}{2} \zeta s_3}, \quad \text{as } \zeta \to \infty.
$$

(2.41)

Then, we have $\hat{\Psi}(\zeta) = 0$ for $\zeta \in \mathbb{C}$.

Proof. To normalize the behavior as $\zeta \to \infty$, we introduce the first transformation

$$
\hat{\Psi}^{(1)}(\zeta) = \hat{\Psi}(\zeta) e^{-\frac{1}{2} \pi i \beta s_3} \varphi(\zeta) \beta s_3 e^{-\frac{1}{2} \zeta s_3}, \quad \zeta \notin \Sigma_4 \cup \Sigma_6 \cup \Sigma_7,
$$

(2.42)

where the contours are illustrated in Figure 1, and

$$
\varphi(\zeta) = \zeta + \sqrt{\zeta^2 - 1}.
$$

(2.43)

The branch for $\sqrt{\zeta^2 - 1}$ is taken such that $\arg(\zeta \pm 1) \in (-\pi, \pi)$ and the branch for $\zeta^\beta$ is taken as in (2.49), such that $\arg \zeta \in (-\pi/2, 3\pi/2)$. Then, we have the normalized behavior at infinity

$$
\hat{\Psi}^{(1)}(\zeta) = O \left( \frac{1}{\zeta} \right),
$$

(2.44)

and the modified jumps

$$
\hat{\Psi}_+^{(1)}(\zeta) = \hat{\Psi}_-^{(1)}(\zeta) \begin{cases} 
\begin{pmatrix} 1 & 0 \\ e^{-\pi i \varphi(\zeta) 2 \beta e^{-\frac{1}{2} \zeta s_3}} & 1 \end{pmatrix}, & \zeta \in \Sigma_1, \\
\begin{pmatrix} 1 & 0 \\ e^{\pi i (\alpha - 2 \beta) \varphi(\zeta) 2 \beta e^{-\frac{1}{2} \zeta s_3}} & 1 \end{pmatrix}, & \zeta \in \Sigma_2, \\
\begin{pmatrix} 1 & 0 \\ e^{-\pi i (\alpha - 2 \beta) \varphi(\zeta) - 2 \beta e^{\frac{1}{2} \zeta s_3}} & 1 \end{pmatrix}, & \zeta \in \Sigma_3, \\
\begin{pmatrix} 1 & 0 \\ -e^{-\pi i \varphi(\zeta) - 2 \beta e^{\frac{1}{2} \zeta s_3}} & 1 \end{pmatrix}, & \zeta \in \Sigma_5, \\
\begin{pmatrix} 0 & -e^{\pi i \varphi(\zeta) 2 \beta e^{\frac{1}{2} \zeta s_3}} \\ e^{-\pi i \varphi(\zeta) - \frac{1}{2} s_3} & 0 \end{pmatrix}, & \zeta \in \Sigma_6, \\
\begin{pmatrix} 0 & -e^{-\pi i \varphi(\zeta) - \frac{1}{2} s_3} \\ e^{\pi i \varphi(\zeta) - \frac{1}{2} s_3} & 0 \end{pmatrix}, & \zeta \in \Sigma_7.
\end{cases}
$$

(2.45)
To deform the jump contours to the real axis, we take the second transformation

\[
\hat{\Psi}^{(2)}(\zeta) = \hat{\Psi}^{(1)}(\zeta) \left\{ \begin{array}{l}
\begin{pmatrix}
1 & 0 \\
e^{-\pi im_a(\varphi(\zeta) + \frac{1}{2} s\zeta)} & 1
\end{pmatrix}, & \text{Im} \zeta > 0, \ z \in \Omega_5, \\
1 & e^{\pi im_a(\varphi(\zeta) - \frac{1}{2} s\zeta)} \\
0 & 1
\end{array}
\right.
\]

\[
\begin{array}{l}
e^{-\pi im_a(\varphi(\zeta) - 2\beta e^{\frac{1}{2} s\zeta} e^{-\pi i(\alpha - 2\beta)\varphi(\zeta) - 2\beta e^{\frac{1}{2} s\zeta}})} \\
0 & 1
\end{array}
\]

\[
I,
\]

otherwise.

(2.46)

Then, \(\hat{\Psi}^{(2)}(\zeta)\) is analytic in \(\mathbb{C} \setminus \mathbb{R}\), with jump condition on the real axis as follows:

\[
\hat{\Psi}^{(2)}_+(\zeta) = \hat{\Psi}^{(2)}_-(\zeta) \hat{J}(\zeta), \quad \zeta \in \mathbb{R},
\]

where the orientation is from left to right, and the jump matrices are

\[
\hat{J}(\zeta) = \left\{ \begin{array}{l}
\begin{pmatrix}
0 & -e^{-\pi im_a(\varphi(\zeta) - 2\beta e^{\frac{1}{2} s\zeta})} \\
e^{-\pi im_a(\varphi(\zeta) + \frac{1}{2} s\zeta)} e^{\pi i(\alpha - 2\beta)\varphi(\zeta) - 2\beta e^{\frac{1}{2} s\zeta}} & 1
\end{pmatrix}, & \zeta > 1, \\
\begin{pmatrix}
0 & -e^{-\pi im_a(\varphi(\zeta) - \frac{1}{2} s\zeta)} \\
e^{-\pi im_a(\varphi(\zeta) + \frac{1}{2} s\zeta)} & 1
\end{pmatrix}, & 0 < \zeta < 1,
\end{array}
\right.
\]

\[
\begin{array}{l}
e^{-\pi im_a(\varphi(\zeta) - \frac{1}{2} s\zeta)} e^{\pi i(\alpha - 2\beta)\varphi(\zeta) - 2\beta e^{\frac{1}{2} s\zeta}} \\
0 & 1
\end{array}
\]

where \((\hat{\Psi}^{(2)}(\zeta))^*\) denotes the Hermitian conjugate of \(\hat{\Psi}^{(2)}(\zeta)\). We see that \(Q(\zeta)\) is analytic for \(\zeta \in \mathbb{C} \setminus \mathbb{R}\). For \(\alpha > -1/2\), we have

\[
\int_{\mathbb{R}} Q_+(x) dx = \int_{\mathbb{R}} \hat{\Psi}^{(2)}_+(x) (\hat{J}^{-1}(x))^* (\hat{\Psi}^{(2)}_+(x))^* dx = 0,
\]

(2.51)
where the jump $\hat{J}(x)$ is introduced in (2.47). Adding to (2.51) its Hermitian conjugate, for purely imaginary $s$ and $\beta$, we have

$$\int_{-\infty}^{-1} \hat{\Psi}_+^{(2)}(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (\hat{\Psi}_+^{(2)}(x))^* \, dx + \int_{1}^{\infty} \hat{\Psi}_+^{(2)}(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (\hat{\Psi}_+^{(2)}(x))^* \, dx = 0. \tag{2.52}$$

Thus, the first column of $\hat{\Psi}_+^{(2)}(x)$ vanishes for $x > 1$ and $x < -1$. From (2.47) one sees that $\hat{\Psi}^{(2)}(\zeta)$ can be analytically extended from the upper-half complex plane to the lower-half, acrossing $(1, \infty)$. Thus, the first column of $\hat{\Psi}^{(2)}(\zeta)$ vanishes in the upper-half $\zeta$-plane. This and the jump condition (2.47) then imply the vanishing of the second column of $\hat{\Psi}^{(2)}(\zeta)$ in the lower-half plane.

To show the vanishing of the remaining column of $\hat{\Psi}^{(2)}(\zeta)$ in the whole plane, we consider two scalar functions defined as

$$g_k(\zeta) = \begin{cases} -\left(\hat{\Psi}^{(2)}(\zeta)\right)_{k2} e^{-\frac{i\pi}{4}} \varphi(\zeta)^{2\beta} e^{-\pi i \alpha}, & \text{Im} \zeta > 0, \\ \left(\hat{\Psi}^{(2)}(\zeta)\right)_{k1} e^{\frac{i\pi}{4}}, & \text{Im} \zeta < 0, \end{cases} \tag{2.53}$$

for $k = 1, 2$. Then, it follows from (2.47) and (2.49) that $g_k(\zeta)$ is analytic and bounded for $\zeta \notin (-\infty, 1]$ and $g_k(\zeta) = O(e^{-|s|/4})$ for purely imaginary $\zeta$ and $is > 0$. Applying Carlson’s theorem, we have $g_k(\zeta) = 0$ for $k = 1, 2$. This then implies that the second column of $\hat{\Psi}^{(2)}(\zeta)$ vanishes in the upper-half $\zeta$-plane, and the first column of $\hat{\Psi}^{(2)}(\zeta)$ also vanishes in the lower-half plane. This completes the proof the lemma. \hfill \Box

By a standard analysis [23, 30, 31, 58], the vanishing lemma implies the existence of unique solution to the RH problem for $\Psi(\zeta; s)$ for the parameters $\alpha > -1/2, \beta \in i\mathbb{R}$ and $s \in -i(0, +\infty)$.

**Proposition 4.** For $\alpha > -1/2$, $\beta \in i\mathbb{R}$ and $s \in -i(0, +\infty)$, there exists unique solution to the RH problem for $\Psi(\zeta; s)$. Then, $\Psi(\zeta; s)$ is a global function of $s$ on $-i(0, +\infty)$, that is, it is free of poles for $s \in -i(0, +\infty)$. Particularly, the Hamiltonian $H(s)$ (2.15) is free of poles for $s \in -i(0, +\infty)$.

### 2.4 Special function solutions

Now we show that the coupled Painlevé V system (1.15) admits special function solutions for some particular parameters. The special solutions are useful in our derivation of the total integral of the Hamiltonian.

For $\beta = 0$, the jump on $\Sigma_1$ vanishes; cf. (2.1). It is readily verified that $\sigma_1 \Psi(-\zeta) \sigma_1$ also solves the RH problem for $\Psi(\zeta)$, formulated in Section 2.1. Then, it follows from the uniqueness of solution to the RH problem for $\Psi(\zeta)$ that

$$\sigma_1 \Psi(-\zeta) \sigma_1 = \Psi(\zeta), \tag{2.54}$$

where $\sigma_1$ is the Pauli matrix; cf. (2.3). Substituting (2.54) into (2.4), we have the constraint

$$\Psi_0^{(0)}(s) = \sigma_1 \Psi_0^{(0)}(s) \sigma_3. \tag{2.54}$$

Noting that $\det \Psi_0^{(0)}(s) = 1$, we can write

$$\Psi_0^{(0)}(s) = \frac{I - i\sigma_2}{\sqrt{2}} l(s) \sigma_3 \tag{2.55}$$

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for a certain scalar function \( l(s) \). We define
\[
P(\zeta) = e^{-\frac{1}{4}\pi i \sigma_3} \zeta^{-\frac{1}{4}\sigma_3} \frac{(I + i\sigma_3)}{\sqrt{2}} \psi(\sqrt{\zeta}) e^{\frac{1}{4}\pi i \sigma_3} \begin{pmatrix} I, & \quad 0 < \arg \zeta < \pi, \\ 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad -\pi < \arg \zeta < 0.
\]
Then \( P(\zeta) \) satisfies the following model RH problem.

\[\begin{align*}
\Sigma_{P,1} \\
\Sigma_{P,2} \\
\Sigma_{P,3} \\
\Sigma_{P,4}
\end{align*}\]

Figure 2: Contours for the RH problem of \( P(\zeta) \)

(a) \( P(\zeta) \) is analytic in \( \mathbb{C} \setminus \bigcup_{k=1}^{4} \Sigma_{P,k} \), where \( \Sigma_{P,1} = 1 + e^{\frac{\pi i}{3}} \mathbb{R}^+, \Sigma_{P,2} = (0, 1), \Sigma_{P,3} = 1 + e^{-\frac{\pi i}{3}} \mathbb{R}^+ \) and \( \Sigma_{P,4} = (1, +\infty) \), as illustrated in Figure 2.

(b) \( P(\zeta) \) satisfies the jump condition
\[P_+(\zeta) = P_-(\zeta) J_{P,k}(\zeta), \quad \zeta \in \Sigma_{P,k}, \quad k = 1, 2, 3, 4,\]
where
\[
J_{P,1} = \begin{pmatrix} 1 & 0 \\ e^{-\pi i (\alpha - \frac{1}{2})} & 1 \end{pmatrix}, \quad J_{P,2} = e^{\pi i (\frac{1}{2} - \alpha) \sigma_3}, \quad J_{P,3} = \begin{pmatrix} 1 & 0 \\ e^{\pi i (\alpha - \frac{1}{2})} & 1 \end{pmatrix}, \quad J_{P,4} = i\sigma_2.
\]

(c) \( P(\zeta) \) has the asymptotic behavior as \( \zeta \to \infty \)
\[
P(\zeta) = \zeta^{-\frac{1}{4}\sigma_3} \frac{I - i\sigma_1}{\sqrt{2}} \left( I + O\left(\frac{1}{\sqrt{\zeta}}\right)\right) e^{\frac{1}{4}\sqrt{\zeta} \sigma_3} \psi(\sqrt{\zeta}), \quad 0 < \arg \zeta < \pi,
\]
and
\[
P(\zeta) = \zeta^{-\frac{1}{4}\sigma_3} \frac{I - i\sigma_1}{\sqrt{2}} \left( I + O\left(\frac{1}{\sqrt{\zeta}}\right)\right) e^{\frac{1}{4}\sqrt{\zeta} \sigma_3} (-i\sigma_2)
\]
\[= \zeta^{-\frac{1}{4}\sigma_3} e^{-\frac{\pi i}{4} \sigma_3} \frac{I - i\sigma_1}{\sqrt{2}} \left( I + O\left(\frac{1}{\sqrt{\zeta}}\right)\right) e^{-\frac{1}{4}\sqrt{\zeta} \sigma_3}, \quad -\pi < \arg \zeta < 0.
\]
(d) $P(\zeta)$ satisfies the asymptotic behavior near the origin

$$P(\zeta) = (I + c(s)\sigma_+) \left( I + O(\sqrt{\zeta}) \right) l(s)^{\sigma_3} \zeta^{(\frac{\pi}{4} - \frac{1}{2})\sigma_3}, \quad (2.60)$$

where $c(s)$ is a scalar function depending only on $s$, $l(s)$ is introduced in (2.55), and the branch of the power function is chosen such that $\arg \zeta \in (0, 2\pi)$.

(e) $P(\zeta)$ satisfies the asymptotic behavior as $\zeta \to 1$

$$P(\zeta) = O(\ln(\zeta - 1)). \quad (2.61)$$

The RH problem is equivalent to the model RH problem considered by the authors in [57]. We can write

$$P(\zeta; s) = ie^{-\frac{\pi}{2} \sigma_3} 2^{-\frac{1}{2} \sigma_3} \sigma_1 \Psi_{0}^{XZ} \left( \frac{1 - \zeta}{4}, is \right) \sigma_3, \quad (2.62)$$

where $\Psi_{0}^{XZ}(\zeta, s)$ is the solution to the model problem for $\Psi_{0}(\zeta, s)$, formulated in [57, Section 1.2], with parameters $\gamma = -1/2$ and $\Theta = 1/2 - \alpha$ therein.

If we further assume that the parameter $\alpha = 1/2$, the RH problem for $P(\zeta)$ can be solved explicitly in terms of the modified Bessel functions

$$P(\zeta) = \left( I - \frac{\Gamma_{v}(|s|/4)}{\Gamma_{v}(|s|/4)} \sigma_- \right) e^{-\frac{\pi}{2} \sigma_3} \left( \frac{|s|}{4} \right)^{\frac{1}{2} \sigma_3} \sigma_3 \Phi_{B} \left( \frac{s^2}{16} (\zeta - 1) \right) \sigma_3, \quad (2.63)$$

where $\Phi_{B}(\zeta)$ is defined in (4.20) with order zero, and $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The lower triangular matrix is chosen to ensure the fulfilment of the asymptotic behavior (2.60). From (4.19), we have the large-$\zeta$ behavior

$$P(\zeta) = \zeta^{-\frac{1}{2} \sigma_3} \frac{1 - i \sigma_1}{\sqrt{2} s} \left\{ I + \frac{1}{\sqrt{s}} \left( \begin{array}{cc} 1 & 2i \\ -2i & 1 \end{array} \right) - \frac{1}{8} \sigma_3 \right\}^{\frac{s}{2} \sqrt{\zeta \sigma_3}} e^{\frac{s}{4} \sqrt{\zeta \sigma_3}} + O \left( \frac{1}{\zeta} \right) \quad (2.64)$$

for $0 < \arg \zeta < \pi$. Moreover, it follows from (4.21) that $P(\zeta)$ satisfies the differential equation

$$\frac{dP(\zeta)}{d\zeta} = \left( \begin{array}{cc} 0 & 0 \\ \frac{s}{8} & 0 \end{array} \right) + \frac{is}{8(\zeta - 1)} \left( \begin{array}{cc} \frac{\Gamma_{v}(|s|/4)}{\Gamma_{v}(|s|/4)} & 1 \\ -\frac{\Gamma_{v}(|s|/4)^2}{\Gamma_{v}(|s|/4)^2} & -\frac{\Gamma_{v}(|s|/4)}{\Gamma_{v}(|s|/4)} \end{array} \right) P(\zeta). \quad (2.65)$$

From the symmetry of $\Psi(\zeta)$ in (2.54) for $\beta = 0$, we have

$$u_1 v_1 = -u_2 v_2, \quad u_2 = -u_1 v_1^2, \quad v_2 = 1/v_1, \quad y(s) = 1. \quad (2.66)$$

Substituting (2.56) into (2.7), we see that $P(\zeta)$ solves the differential equation with coefficients involving $u_1(s)$ and $v_1(s)$

$$\frac{dP(\zeta)}{d\zeta} = \left( \begin{array}{cc} 0 & 0 \\ \frac{s}{8} & 0 \end{array} \right) + \frac{u_1}{2(\zeta - 1)} \left( \begin{array}{cc} 1 - v_1^2 & -i(v_1 + 1)^2 \\ -i(v_1 - 1)^2 & v_1^2 - 1 \end{array} \right) P(\zeta). \quad (2.67)$$
Now we are in a position to derive the special function solutions to the the coupled Painlevé system of equations (1.15) for $\alpha = 1/2$ and $\beta = 0$. The integral of the Hamiltonian associated with the special function solutions is explicitly calculated, and furnishes as boundary condition in our evaluation of the total integral of the Hamiltonian for general parameters later.

**Proposition 5.** For $\alpha = 1/2$ and $\beta = 0$, the coupled Painlevé system of equations (1.15) admits the following special function solutions

\[
\begin{align*}
u_1(s) &= \frac{s}{16} \left( i + \frac{I_0(|s|/4)}{I_0(|s|/4)} \right)^2, \\
u_2(s) &= -\frac{s}{16} \left( i - \frac{I_0(|s|/4)}{I_0(|s|/4)} \right)^2, \\
u_1(s) &= \frac{i - \frac{I_0(|s|/4)}{I_0(|s|/4)}}{i + \frac{I_0(|s|/4)}{I_0(|s|/4)}}, \\
u_2(s) &= \frac{i + \frac{I_0(|s|/4)}{I_0(|s|/4)}}{i - \frac{I_0(|s|/4)}{I_0(|s|/4)}}
\end{align*}
\]

and the Hamiltonian associated with $u_k(s)$ and $v_k(s)$ takes the form

\[
H (u_1, v_1, u_2, v_2, s; 1/2, 0) = \frac{s}{16} + \frac{i}{4} \frac{I_0(|s|/4)}{I_0(|s|/4)},
\]

where

\[
I_0(s) = \sum_{n=0}^{\infty} \frac{1}{n!^2} \left( \frac{s^2}{4} \right)^n
\]

is the modified Bessel function of order zero.

**Proof.** In this specific case when $\alpha = 1/2$ and $\beta = 0$, the special function solutions $u_1(s)$ and $v_1(s)$ in (2.68) and (2.69) are readily derived by comparing (2.65) with (2.67). We further obtain $u_2(s)$ and $v_2(s)$ in (2.68) and (2.69) by using the relation (2.66). The special Hamiltonian of the form (2.70) follows from (2.16), (2.2), (2.58) and (2.64). This completes the proof of the proposition. \qed

3 Asymptotics of the coupled Painlevé V system as $is \to 0^+$

In this section, we derive the asymptotic approximations of the solutions to the coupled Painlevé V system as $is \to 0^+$ by performing Deift-Zhou nonlinear steepest descent analysis [16, 24, 25] of the RH problems for $\Psi(\zeta; s)$.

3.1 Outer parametrix

Let

\[
X(\xi; s) = \left( \frac{|s|}{2} \right)^{-\beta\sigma_3} \Psi(2\xi/|s|; s).
\]

Then, $X(\xi)$ satisfies a re-scaled version of the RH problem for $\Psi(\zeta)$ in the complex $\xi$-plane with $\xi = |s|\zeta/2$. Note that, the jump contour $(-1, 1)$ in the $\zeta$-plane is mapped on to $(-|s|/2, |s|/2)$ in the $\xi$-plane. As $is \to 0^+$, we consider the following approximate RH problem $\Phi(\xi)$ by ignoring the jump conditions for $X(\xi)$ along the shrinking line segment $(-|s|/2, |s|/2)$. 

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RH problem for $\Phi$

(a) $\Phi(\xi)$ is analytic in $\mathbb{C} \setminus \bigcup_{j=1}^{5} \Sigma_{\Phi,j}$, where the oriented contours

$$
\Sigma_{\Phi,1} = e^{\frac{\pi i}{4}} \mathbb{R}^+, \quad \Sigma_{\Phi,2} = e^{\frac{3\pi i}{4}} \mathbb{R}^+, \quad \Sigma_{\Phi,3} = e^{-\frac{3\pi i}{4}} \mathbb{R}^+, \quad \Sigma_{\Phi,4} = e^{-\frac{\pi i}{4}} \mathbb{R}^+, \quad \Sigma_{\Phi,5} = e^{-\frac{\pi i}{4}} \mathbb{R}^+,
$$

as illustrated in Figure 3.

(b) $\Phi(\xi)$ satisfies the jump conditions

$$
\Phi_+(\xi) = \Phi_-(\xi) \begin{cases}
\begin{pmatrix} 1 & 0 \\ e^{-\pi i(\alpha-\beta)} & 1 \end{pmatrix}, & \xi \in \Sigma_{\Phi,1}, \\
\begin{pmatrix} 1 & 0 \\ e^{\pi i(\alpha-\beta)} & 1 \end{pmatrix}, & \xi \in \Sigma_{\Phi,2}, \\
\begin{pmatrix} 1 & -e^{-\pi i(\alpha-\beta)} \\ 0 & 1 \end{pmatrix}, & \xi \in \Sigma_{\Phi,3}, \\
e^{2\pi i \beta \sigma_3}, & \xi \in \Sigma_{\Phi,4}, \\
\begin{pmatrix} 1 & -e^{\pi i(\alpha-\beta)} \\ 0 & 1 \end{pmatrix}, & \xi \in \Sigma_{\Phi,5}.
\end{cases}
$$

(c) As $\xi \to \infty$, we have

$$
\Phi(\xi) = \left( I + \frac{\Phi_1}{\xi} + O\left(\frac{1}{\xi^2}\right) \right) \xi^{-\beta \sigma_3} e^{-\frac{1}{2} \xi \sigma_3}, \tag{3.3}
$$

where $\Phi_1 = \begin{pmatrix} -e^{-\pi i\beta} & -e^{-\pi i\beta} \\ e^{\pi i\beta} & e^{\pi i\beta} \end{pmatrix}$ and arg $\xi \in (-\pi/2, 3\pi/2)$. 

Figure 3: The jump contours and regions of the RH problem for $\Phi(\xi)$
(d) The description of the asymptotic behavior of $\Phi(\xi)$ as $\xi \to 0$ is divided into two cases, namely $2\alpha \notin \mathbb{N}$ and $2\alpha \in \mathbb{N}$.

In the first case, if $2\alpha \not\in \mathbb{N}$, there exists a function $\Phi^{(0)}(\xi)$, analytic at $\xi = 0$, such that

$$\Phi(\xi) = \Phi^{(0)}(\xi)\xi^{\alpha_3}C_j, \quad \xi \to 0, \; \xi \in \Omega_{\Phi,j}, \; j = 1, 2, 3, 4, 5, \tag{3.4}$$

where $\Omega_{\Phi,j}$ are the sectors illustrated in Figure 3, the branch of $\xi^{\alpha_3}$ is chosen such that $\arg \xi \in (-\pi/2, 3\pi/2)$, the constant matrix

$$C_1 = \begin{pmatrix} 1 & \frac{\sin(\pi(\alpha+\beta))}{\sin(2\pi\alpha)} \\ 0 & 1 \end{pmatrix}, \quad 2\alpha \not\in \mathbb{N}, \; \alpha > \frac{1}{2},$$

and the other constant matrices are determined by $C_1$ and the jump conditions.

In the second case, if $2\alpha \in \mathbb{N}$, there exists a function $\tilde{\Phi}^{(0)}(\xi)$, analytic at $\xi = 0$, such that

$$\Phi(\xi) = \tilde{\Phi}^{(0)}(\xi)\xi^{\alpha_3} \begin{pmatrix} 1 & -(1)^{2\alpha}\frac{\sin(\pi(\alpha+\beta))}{\pi} \ln \xi \\ 0 & 1 \end{pmatrix} \tilde{C}_j, \quad \xi \to 0, \; \xi \in \Omega_{\Phi,j}, \; j = 1, 2, 3, 4, 5, \tag{3.5}$$

where $\Omega_{\Phi,j}$ are the sectors illustrated in Figure 3, the branches of $\xi^{\alpha_3}$ and $\ln \xi$ are chosen such that $\arg \xi \in (-\pi/2, 3\pi/2)$, the constant matrix $\tilde{C}_1$ is the identity matrix and the other constant matrices are determined by $\tilde{C}_1$ and the jump conditions.

The connection matrices $C_j$ are specified so that $\Phi$ the asymptotic behavior near the origin be in accordance with the jump conditions, that is, for $\xi \in \Sigma_{\Phi,4}$, we have the constraint

$$\Phi^{-}(\xi)^{-1}\Phi^{+}(\xi) = J_3J_2C_1^{-1}e^{-2\pi i\alpha_3}C_j1^{-1}J_5^{-1} = e^{2\pi i\beta_3}, \tag{3.6}$$

where $J_j$ denote the jumps on $\Sigma_{\Phi,j}$, given in (3.2). These are the same as the jumps on $\Sigma_j$; see (2.1). Similarly, we readily verify that the jump conditions are also fulfilled by (3.5) in the case $2\alpha \in \mathbb{N}$.

The solution to the above RH problem can be constructed explicitly in terms of the confluent hypergeometric function; see [15] and [13]. The parametrix is equivalent to the one, namely $M(\xi)$, used in [13] Section 4.2.1 after some elementary transfromation

$$\Phi(\xi) = e^{\frac{\pi i \beta_3}{2}}\sigma_3M(e^{\frac{\pi i}{2}}\xi)\sigma_3, \quad \arg \xi \in (-\pi/2, 3\pi/2).$$

The analytic function $\Phi^{(0)}(\xi)$ brought in (3.4) now takes

$$\Phi^{(0)}(\xi) = e^{-\frac{\pi i}{2}}\begin{pmatrix} e^{-\frac{\pi i(\alpha+\beta)}{2}}\frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+2\alpha)}\psi(\alpha+\beta, 1 + 2\alpha, i\xi) & e^{-\frac{\pi i(\alpha-\beta)}{2}}\frac{\Gamma(2\alpha)}{\Gamma(\alpha+\beta)}\psi(-\alpha+\beta, 1 - 2\alpha, i\xi) \\ e^{-\frac{\pi i(\alpha-\beta)}{2}}\frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+2\alpha)}\psi(1 + \alpha+\beta, 1 + 2\alpha, i\xi) & e^{-\frac{\pi i(\alpha+\beta)}{2}}\frac{\Gamma(2\alpha)}{\Gamma(\alpha-\beta)}\psi(-1 - \alpha+\beta, 1 - 2\alpha, i\xi) \end{pmatrix}, \tag{3.7}$$

where $\psi$ is the confluent hypergeometric function given in (1.9). Also, the first column of $\Phi^{(0)}(\xi)$ in (3.5) is the same as that of $\Phi^{(0)}(\xi)$; see [13] Section 4.2.1.

It is straightforward to see that $\Phi(\xi)$ shares the same jump conditions with $X(\xi)$ for $|\xi| > |s|/2$. For some small constant radius $\epsilon > 0$, we intend to construct a local parametrix $L(\xi)$ in a neighborhood $U(0; \epsilon)$ of the origin, which shares the same jump conditions as $X(\xi)$ and matches with $\Phi(\xi)$ on the boundary of $U(0; \epsilon)$, namely, on $|\xi| = \epsilon$.  

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3.2 Local parametrix

RH problem for $L$

(a) $L(\xi)$ is analytic in $U(0; \epsilon) \setminus \Sigma^X$, where $\Sigma^X$ denotes the jump contours for $X(\xi)$, a re-scaled version as illustrated in Figure [1].

(b) $L(\xi)$ satisfies the same jump conditions on $U(0; \epsilon) \cap \Sigma^X$ as $X(\xi)$.

(c) As $\epsilon \to 0^+$, we have the matching condition

$$L(\xi) = (I + O(s^{2\alpha+1})) \Phi(\xi),$$

for $|\xi| = \epsilon$ with a constant $\epsilon > 0$.

From the asymptotic behavior (3.4) and (3.5) of $\Phi(\xi)$ near the origin, we may seek a solution to the above RH problem of the form

$$L(\xi) = \Phi^{(0)}(\xi) \left( \begin{array}{cc} 1 & |s|^{2\alpha} k(\xi/|s|) \\ 0 & 1 \end{array} \right) \xi^{\alpha \sigma_3} C_j \quad \text{if} \quad 2\alpha \notin \mathbb{N}, \quad \alpha > \frac{1}{2},$$

or

$$L(\xi) = \tilde{\Phi}^{(0)}(\xi) \left( \begin{array}{cc} 1 & |s|^{2\alpha} k(\xi/|s|) \\ 0 & 1 \end{array} \right) \xi^{\alpha \sigma_3} \left( \begin{array}{cc} 1 & (-1)^{2\alpha} \sin(\pi(\alpha/2)) \ln \xi \\ 0 & 1 \end{array} \right) \tilde{C}_j \quad \text{if} \quad 2\alpha \in \mathbb{N},$$

for $\xi/|s| \in \Omega_j$, $j = 1, 2, 3, 4, 5$; cf. Figure [1] where $\Phi^{(0)}(\xi)$ and $C_j$ are given in (3.7) and (3.4), $\tilde{\Phi}^{(0)}(\xi)$ and $\tilde{C}_j$ are given in (3.5). The branches of $\xi^\alpha$ and $\ln \xi$ are chosen so that arg $\xi \in (-\pi/2, 3\pi/2)$.

In view of (3.9) and (3.6), we see that $L(\xi)$ has the same jumps as $X(\xi)$ on $\Sigma^X \setminus [-|s|/2, |s|/2]$. To ensure the fulfilment by $L(\xi)$ of the jump condition on $(-|s|/2, |s|/2)$ as $X(\xi)$, and the matching condition (3.8), we look for the function $k(\xi)$, as a solution of the following scalar RH problem for $\alpha > -1/2$.

RH problem for $k$

(a) $k(\xi)$ is analytic in $\mathbb{C} \setminus [-1/2, 1/2]$.

(b) $k(\xi)$ satisfies the same jump relations

$$k_+(x) - k_-(x) = \begin{cases} -e^{\pi i (\alpha-\beta)} x^{2\alpha}, & x \in (0, 1/2), \\ -e^{\pi i (\alpha+\beta)} |x|^{2\alpha}, & x \in (-1/2, 0). \end{cases}$$

(c) As $\xi \to \infty$, $k(\xi) = O(1/\xi)$.

Applying the Sokhotski-Plemelj formula, we obtain an explicit representation

$$k(\xi) = -\frac{e^{\pi i (\alpha-\beta)}}{2\pi i} \int_0^{1/2} x^{2\alpha} \frac{dx}{x-\xi} - \frac{e^{\pi i (\alpha+\beta)}}{2\pi i} \int_{-1/2}^0 |x|^{2\alpha} \frac{dx}{x-\xi} \quad \text{for} \quad \xi \in \mathbb{C} \setminus [-1/2, 1/2].$$
Having had \( k(\xi) \) as in (3.12), we readily verify that \( L(\xi) \) in (3.9) shares the same jumps as \( X(\xi) \) on real intervals. Indeed, for \( \xi \in (0, |s|/2) \), we have

\[
L_-(\xi)^{-1}L_+(\xi) = J_5J_1(I - e^{\pi i(\alpha-\beta)}\sigma_+) = J_6, \tag{3.13}
\]

where \( J_6 \) denotes the constant jump of \( \Psi(\zeta) \) on the contour \( \Sigma_\zeta \); cf. (2.1) and (3.2). Now we turn to \( \xi \in (-|s|/2, 0) \). First, with the branch chosen such that \( \arg \xi \in (-\pi/2, 3\pi/2) \), we have

\[
\xi^\alpha = \xi^\alpha = |\xi|^{\alpha}e^{\pi i \alpha} \quad \text{and} \quad (\ln \xi)_+ = (\ln \xi)_-
\]

for \( \xi \) on the negative real axis. Therefore, we have

\[
L_-(\xi)^{-1}L_+(\xi) = J_3J_2(I - e^{\pi i(\alpha-\beta)}\sigma_+) = J_7. \tag{3.14}
\]

Thus, \( L(\xi) \) has the same jump as \( X(\xi) \) for \( \xi \in (-|s|/2, 0) \cup (0, |s|/2) \), and hence for \( \xi \in U(0, \epsilon) \cap \Sigma X \). Accordingly, we see that \( X(\xi)L(\xi)^{-1} \) is analytic in \( 0 < |\xi| < \epsilon \).

To show the analyticity of \( X(\xi)L(\xi)^{-1} \) at \( \xi = 0 \), we study the behavior of \( L(\xi) \) near the origin. From (3.12), we have for \( 2\alpha \notin \mathbb{N} \)

\[
k(\xi) = k_r(\xi) + \begin{cases} -\frac{\sin(\pi(\alpha + \beta))}{\sin(2\pi \alpha)}\xi^{2\alpha}, & \text{arg} \xi \in (0, \pi), \\ \frac{\sin(\pi(\alpha - \beta))}{\sin(2\pi \alpha)}(\xi e^{\pi i})^{2\alpha}, & \text{arg} \xi \in (-\pi, 0), \end{cases} \tag{3.15}
\]

where \( k_r(\xi) \) is an analytic function in \( |\xi| < 1/2 \). Similarly, for \( 2\alpha \in \mathbb{N} \), we obtain from (3.12) that

\[
k(\xi) = \hat{k}_r(\xi) + \begin{cases} -\frac{(1)^{2\alpha+1}\sin(\pi(\alpha + \beta))}{\pi}\xi^{2\alpha} \ln \xi, & \text{arg} \xi \in (0, \pi), \\ -\frac{(1)^{2\alpha+1}\sin(\pi(\alpha - \beta))}{\pi}\xi^{2\alpha} \ln \xi + e^{\pi i(\alpha-\beta)}\xi^{2\alpha}, & \text{arg} \xi \in (-\pi, 0), \end{cases} \tag{3.16}
\]

where \( \hat{k}_r(\xi) \) is an analytic function in \( |\xi| < 1/2 \).

For \( 2\alpha \notin \mathbb{N} \) and \( \alpha > -1/2 \), a combination of (2.4), (3.1), (3.4), (3.9) and (3.15) shows that \( X(\xi)L(\xi)^{-1} \) is bounded in a neighborhood of \( \xi = 0 \). While for \( 2\alpha \in \mathbb{N} \), it follows from (2.4), (3.1), (3.5), (3.10) and (3.16) that \( X(\xi)L(\xi)^{-1} \) is also bounded in a neighborhood of \( \xi = 0 \). Thus, the singularity at the origin is removable. Therefore, \( X(\xi)L(\xi)^{-1} \) is analytic for \( |\xi| < \epsilon \).

For later use, we also derive from (3.12) that

\[
k(\xi) \sim \frac{e^{\pi i(\alpha+\beta)}}{2\pi i} - 2^{-2\alpha} \ln \left( \xi + \frac{1}{2} \right) \quad \text{as} \quad \xi \to -\frac{1}{2}, \tag{3.17}
\]

and

\[
k(\xi) \sim -\frac{e^{\pi i(\alpha-\beta)}}{2\pi i} - 2^{-2\alpha} \ln \left( \xi - \frac{1}{2} \right) \quad \text{as} \quad \xi \to \frac{1}{2}. \tag{3.18}
\]

### 3.3 Final transformation

Next, we define

\[
Z(\xi) = \begin{cases} X(\xi)\Phi(\xi)^{-1}, & |\xi| > \epsilon, \\ X(\xi)L(\xi)^{-1}, & |\xi| < \epsilon. \end{cases} \tag{3.19}
\]
As shown earlier in this section, \(Z(\xi)\) is analytic for \(|\xi| \neq \epsilon\). From (3.4), (3.9), (3.12) and (3.19), we have the order estimate for the jump of \(Z(\xi)\)

\[
J_Z(\xi) = I + c_{\alpha, \beta} \frac{|s|^{2\alpha+1}}{\xi} \Phi(0)(\xi)\sigma_+ \Phi(0)(\xi)^{-1} + O\left(s^{2\alpha+2}\right) \quad \text{as} \quad is \to 0^+,
\]

where the constant \(c_{\alpha, \beta} = \frac{e^{\pi i \alpha} \cos(\beta \pi)}{i \pi 2^{2\alpha+1}(2\alpha+1)}\), and the error term is uniform for \(\xi\) on the clockwise jump contour \(|\xi| = \epsilon\). Therefore, the matching condition (3.8) is fulfilled. Hence, the RH problem for \(Z(\xi)\) is a small-norm RH problem. By a standard analysis [23], we have the piecewise approximation as \(is \to 0^+\)

\[
Z(\xi) = \begin{cases} 
I + c_{\alpha, \beta} \frac{|s|^{2\alpha+1}}{\xi} \Phi(0)(\xi)\sigma_+ \left\{ \Phi(0)(\xi) \right\}^{-1} + \varepsilon_Z(\xi), & |\xi| > \epsilon, \\
I + c_{\alpha, \beta} \frac{|s|^{2\alpha+1}}{\xi} \left[ \Phi(0)(\xi)\sigma_+ \left\{ \Phi(0)(\xi) \right\}^{-1} - \Phi(0)(\xi)\sigma_+ \left\{ \Phi(0)(\xi) \right\}^{-1} \right] + \varepsilon_Z(\xi), & |\xi| < \epsilon,
\end{cases}
\]

where the error term \(\varepsilon_Z(\xi) = O\left(s^{2\alpha+2}\right)\) as \(is \to 0^+\), uniformly in all \(\xi\) in the complex plane.

### 3.4 Proof of Theorem 1: Asymptotics of \(sH\) as \(is \to 0^+\)

Now we have the following asymptotic approximation of \(X(\xi) = X(\xi; s)\), introduced in (3.1):

\[
X(\xi) = \begin{cases} 
Z(\xi)\Phi(\xi), & |\xi| > \epsilon, \\
Z(\xi)L(\xi), & |\xi| < \epsilon;
\end{cases}
\]

as \(\xi \to \infty\), and taking (2.2) and (3.1) into account, we have

\[
(\Psi_1)_{11} = 2(X_1)_1/|s|.
\]

Using this together with (2.16), (3.21), (3.22) and (3.3), we find

\[
H = -(\Psi_1)_{11}/2 = \frac{2}{s}(\alpha^2 - \beta^2)
\]

\[
= -\frac{1}{is}((Z_1)_{11} + (\Phi_1)_{11}) - \frac{\alpha^2 - \beta^2}{s}
\]

\[
= \frac{\Gamma(1 + \alpha + \beta)|\Gamma(1 + \alpha - \beta)\cos(\pi \beta)|}{i \pi 2^{2\alpha+1}(2\alpha+1)|s|^{2\alpha} + O\left(s^{2\alpha+1}\right)}.
\]

This gives (1.32).

Next, we derive the small-\(s\) behavior of \(y(s)\), \(d(s)\), and other functions. It follows from (3.1), (3.21) and (3.22) that

\[
\Psi(2\xi/|s|) = \left(\frac{|s|}{2}\right)^{\beta_2} X(\xi) = \left(\frac{|s|}{2}\right)^{\beta_2} \left[I + O\left(s^{2\alpha+1}\right)\right] L(\xi) \quad \text{as} \quad is \to 0^+.
\]

for \(|\xi| < \epsilon\). Substituting (3.9) and (3.15) into (3.24), and using the relations (2.27) and (2.29), we obtain

\[
y(s) = \frac{\Gamma(1 + \alpha - \beta)}{\Gamma(1 + \alpha + \beta)} e^{-\pi i \beta} \left(\frac{|s|}{2}\right)^{2\beta} \left[1 + O\left(s^{2\alpha+1}\right)\right], \quad is \to 0^+.
\]
and
\[ d(s) = 2\alpha \frac{\Gamma(1 + \alpha - \beta)\Gamma(1 + \alpha + \beta)}{\Gamma(1 + 2\alpha)^2} e^{-\pi i \alpha} \left( \frac{|s|}{2} \right)^{2\alpha} [1 + O(s^{2\alpha+1})], \quad is \to 0^+. \] (3.26)

Similarly, the relation (2.32), together with (3.9), (3.17), (3.18) and (3.24), gives us
\[ u_1(s) = -\frac{\Gamma(1 + \alpha + \beta)\Gamma(1 + \alpha - \beta)}{2\pi i \Gamma(1 + 2\alpha)^2} e^{-\pi i \beta} \left( \frac{|s|}{2} \right)^{2\alpha} [1 + O(s^{2\alpha+1}) + O(s)], \quad is \to 0^+, \]
\[ v_1(s) = 1 + O(s^{2\alpha+1}) + O(s), \quad is \to 0^+, \]
\[ u_2(s) = \frac{\Gamma(1 + \alpha + \beta)\Gamma(1 + \alpha - \beta)}{2\pi i \Gamma(1 + 2\alpha)^2} e^{\pi i \beta} \left( \frac{|s|}{2} \right)^{2\alpha} [1 + O(s^{2\alpha+1}) + O(s)], \quad is \to 0^+, \]
and
\[ v_2(s) = 1 + O(s^{2\alpha+1}) + O(s), \quad is \to 0^+. \]

We mention that the error term \( O(s) \) in the above formulas has its roots back in the approximation of \( \Phi(0) \) in (3.7) as \( is \to 0^+ \)
\[ \Phi^{(0)}(\pm |s|/2) = \Phi^{(0)}(0) + O(s). \]

These are formulas (1.24), (1.25), (1.26) and (1.27), respectively. Thus, we have proved the small-\( s \) part of Theorem 1.

4 Asymptotics of the coupled Painlevé V system as \( is \to +\infty \)

In this section, we derive the asymptotics of the coupled Painlevé V system as \( is \to +\infty \) by using Deift-Zhou nonlinear steepest descent analysis of the RH problem for \( \Psi(\zeta; s) \).

4.1 Normalization

We introduce a normalization of \( \Psi(\zeta; s) \) at infinity of the form
\[ A(\zeta) = \Psi(\zeta; s) \exp(-sg(\zeta)\sigma_3), \] (4.1)

where
\[ g(\zeta) = \frac{1}{4} \sqrt{\zeta^2 - 1} \] (4.2)

is analytic in \( \zeta \in \mathbb{C} \setminus [-1, 1] \), such that \( \arg(\zeta \pm 1) \in (-\pi, \pi) \).
RH problem for $A$

(a) $A(\zeta)$ is analytic in $\mathbb{C} \setminus \{\cup_{j=1}^{7}\Sigma_j\}$, where the contours are shown in Figure 1.

(b) $A(\zeta)$ satisfies the jump conditions

$$A_+(\zeta) = A_-(\zeta) e^{2\pi i \sigma_3},$$

$$\begin{cases}
1 & 0 \\
0 & 1
\end{cases}, \quad \zeta \in \Sigma_1,$$

$$\begin{cases}
1 & 0 \\
0 & 1
\end{cases}, \quad \zeta \in \Sigma_2,$$

$$\begin{cases}
1 & -e^{-\pi i(\alpha-\beta)} e^{2sg(\zeta)} \\
0 & 1
\end{cases}, \quad \zeta \in \Sigma_3,$$

$$\begin{cases}
e^{-\pi i(\alpha-\beta)} e^{-2sg(\zeta)} \\
1 & 0
\end{cases}, \quad \zeta \in \Sigma_4,$$

$$\begin{cases}
e^{-\pi i(\alpha-\beta)} e^{-2sg(\zeta)} \\
1 & 0
\end{cases}, \quad \zeta \in \Sigma_5,$$

$$\begin{cases}
e^{-\pi i(\alpha-\beta)} e^{-2sg(\zeta)} \\
1 & 0
\end{cases}, \quad \zeta \in \Sigma_6,$$

$$\begin{cases}
e^{-\pi i(\alpha-\beta)} e^{-2sg(\zeta)} \\
1 & 0
\end{cases}, \quad \zeta \in \Sigma_7.$$

(c) As $\zeta \to \infty$, we have

$$A(\zeta) = \left(I + O\left(\frac{1}{\zeta}\right)\right) \zeta^{-\beta \sigma_3},$$

where $\text{arg } \zeta \in (-\pi/2, 3\pi/2)$.

(d) The behaviors of $A(\zeta)$ near $\pm 1$ and the origin are the same as $\Psi(\zeta; s)$; cf. (2.4), (2.5) and (2.6).

4.2 Outer parametrix

As $is \to +\infty$, the jump matrices (4.3) on the contours $\Sigma_k$, $k = 1, 2, 3, 5$ tend to the identity matrix exponentially fast. Thus, we consider the following approximate RH problem with jumps on the remaining contours.

RH problem for $A^{(\infty)}$

(a) $A^{(\infty)}(\zeta)$ is analytic in $\mathbb{C} \setminus \{\Sigma_4 \cup \Sigma_6 \cup \Sigma_7\}$; see Figure 1 for the contours.
(b) $A^{(\infty)}(\zeta)$ satisfies the jump conditions

$$
A_+^{(\infty)}(\zeta) = A_-^{(\infty)}(\zeta) \begin{cases} 
e^{2\pi i\sigma_3}, & \zeta \in \Sigma_4, \\
0 & -e^{\pi i(\alpha - \beta)} \\
e^{-\pi i(\alpha - \beta)} & 0 \\
0 & -e^{-\pi i(\alpha - \beta)} \\ne^{\pi i(\alpha - \beta)} & 0, & \zeta \in \Sigma_7.
\end{cases}
$$

(4.5)

(c) As $\zeta \to \infty$, we have

$$
A^{(\infty)}(\zeta) = \left(I + O\left(\frac{1}{\zeta}\right)\right)\zeta^{-\beta\sigma_3},
$$

(4.6)

where arg $\zeta \in (-\pi/2, 3\pi/2)$.

A solution can be constructed explicitly, that is,

$$
A^{(\infty)}(\zeta) = 2\beta\sigma_3 e^{-\frac{\beta}{2}\pi i\sigma_3} \left(\frac{I - i\sigma_1}{\sqrt{2}}\right) \left(\frac{\zeta - 1}{\zeta + 1}\right)^{\frac{1}{4}\sigma_3} \left(\frac{I + i\sigma_1}{\sqrt{2}}\right) \varphi(\zeta)^{-\beta\sigma_3} e^{\frac{\beta}{2}\pi i\sigma_3} D(\zeta)^{\sigma_3},
$$

(4.7)

where $\varphi(\zeta) = \zeta + \sqrt{\zeta^2 - 1}$ is defined as before in (2.43). The branches for $\sqrt{\zeta^2 - 1}$ and $\left(\frac{\zeta - 1}{\zeta + 1}\right)^{1/4}$ are taken such that arg $(\zeta \pm 1) \in (-\pi, \pi)$ and the branch for $\zeta^{\beta}$ is taken such that $\arg \zeta \in (-\pi/2, 3\pi/2)$. The function $D(\zeta)$ is the Szegő function which solves the following scalar RH problem:

(a) $D(\zeta)$ is analytic in $\mathbb{C} \setminus [-1, 1]$.

(b) $D(\zeta)$ satisfies the relation

$$
D_+(x)D_-(x) = \begin{cases} e^{\pi i\alpha}, & -1 < x < 0, \\
e^{-\pi i\alpha}, & 0 < x < 1.
\end{cases}
$$

(4.8)

Applying the Sokhotski-Plemelj formula, we have

$$
D(\zeta) = \exp \left\{ \frac{\alpha\sqrt{\zeta^2 - 1}}{2i} \left( -\int_{-1}^{0} \frac{1}{\sqrt{1-x^2}} \frac{dx}{\zeta - x} + \int_{0}^{1} \frac{1}{\sqrt{1-x^2}} \frac{dx}{\zeta - x} \right) \right\} = \left( -i + \sqrt{\zeta^2 - 1} \right)^{\alpha}\zeta, \quad \zeta \to \infty,
$$

(4.9)

where the branches for $\sqrt{\zeta^2 - 1}$ and $\zeta^{\alpha}$ are taken such that arg $(\zeta \pm 1)$, arg $\zeta \in (-\pi, \pi)$; see [36]. Straightforward calculation from (4.9) gives

$$
D(\zeta) = 1 - i\alpha/\zeta + O(1/\zeta^2), \quad \zeta \to \infty,
$$

(4.10)

$$
D(\zeta) = e^{-i\alpha\pi/2}(\zeta/2)^{\alpha}(1 + O(\zeta)), \quad \zeta \to 0, \quad \text{Im}\zeta > 0,
$$

(4.11)

and

$$
D(\zeta) = e^{i\alpha\pi/2} \left(1 + O(\sqrt{\zeta^2 - 1})\right), \quad \zeta \to \pm 1, \quad \text{Im}\zeta > 0.
$$

(4.12)
Substituting (4.11) into (4.7), we have
\[ A^{(\infty)}(\zeta) = 2^{2\beta_3} e^{-\frac{\beta}{2}\pi i \sigma_3} \left( I - i \sigma_2 \frac{1}{\sqrt{2}} \right) 2^{-\alpha_3} e^{-\frac{1}{2}\pi i \alpha_3} (I + O(\zeta)) \zeta^{\alpha_3}, \quad \zeta \to 0, \quad \text{Im} \zeta > 0, \] (4.13)
where the Pauli matrix \( \sigma_3 \) is defined in (2.3). The behavior of \( A^{(\infty)}(\zeta) \) as \( \zeta \to 0 \) from the lower-half plane can be obtained by combining (4.5) with (4.13). Similarly, we get by substituting (4.10) into (4.7)
\[ A^{(\infty)}(\zeta) = 2^{2\beta_3} e^{-\frac{\beta}{2}\pi i \sigma_3} \left( I + \frac{-i \alpha_3 + \frac{1}{2} \sigma_2}{\zeta} + O \left( \frac{1}{\zeta^2} \right) \right) 2^{-\beta_3} e^{\frac{1}{2}\pi i \beta_3} \zeta^{-\beta_3}, \quad \zeta \to \infty, \] (4.14)
for \( \arg \zeta \in (-\pi/2, 3\pi/2) \).

### 4.3 Local parametrices

We need the local parametrices \( A^{(+1)}(\zeta) \) and \( A^{(-1)}(\zeta) \) near \( \zeta = 1 \) and \( \zeta = -1 \), respectively.

(a) \( A^{(+1)}(\zeta) \) is analytic respectively in \( U(\pm 1) \backslash \{ \cup_{j=1}^{\infty} \Sigma_j \} \), where \( \Sigma_j \) are illustrated in Figure 4 and \( U(\pm 1) \) are small disks centered at \( \pm 1 \), respectively.

(b) \( A^{(+1)}(\zeta) \) satisfy the same jump conditions as \( A(\zeta) \) in \( U(\pm 1) \cap \{ \cup_{j=1}^{\infty} \Sigma_j \} \).

(c) \( A^{(+1)}(\zeta) \) satisfy the matching condition
\[ A^{(+1)}(\zeta) = (I + O(1/s)) A^{(\infty)}(\zeta) \] (4.15)
for \( \zeta \) on the boundaries of \( U(\pm 1) \), as \( is \to +\infty \).

We focus on the construction of the local parametrix near \( \zeta = -1 \). The local parametrix near \( \zeta = 1 \) can be constructed similarly. In the neighborhood \( U(-1) \), we seek a local parametrix of the form
\[ A^{(-1)}(\zeta) = E_{-1}(\zeta) \Phi_B (-|s|^2 g^2(\zeta)) \begin{cases} e^{\frac{1}{2}\pi i (\alpha - \beta) \sigma_3} e^{-sg(\zeta)\sigma_3}, & \text{Im} \zeta > 0, \\ i \sigma_3 e^{\frac{1}{2}\pi i (\alpha - \beta) \sigma_3} e^{-sg(\zeta)\sigma_3}, & \text{Im} \zeta < 0, \end{cases} \] (4.16)
where \( E_{-1}(\zeta) \) is defined and analytic in \( U(-1) \). From the definition (4.2) it is readily seen that \(-g^2(\zeta)\) serves as a conformal mapping in \( U(-1) \) and \( U(1) \). For later use, we assign
\[ \sqrt{-g^2(\zeta)} = \begin{cases} -ig(\zeta), & \text{Im} \zeta > 0, \\ ig(\zeta), & \text{Im} \zeta < 0, \end{cases} \] (4.17)
cut along \( (-\infty, -1] \cup [1, +\infty) \), taking principal branch, and being real positive for \(-1 < \zeta < 1\). The function \( \Phi_B(\zeta) \) in (4.16) solves the following model RH problem.

(a) \( \Phi_B(\zeta) \) is analytic in \( \mathbb{C} \backslash \Sigma_B \), where the contour \( \Sigma_B \) is illustrated in Figure 4

(b) \( \Phi_B(\zeta) \) satisfies the jump condition
\[ \Phi_B(\zeta) = \Phi_B(\zeta) J_B(\zeta), \quad \zeta \in \Sigma_B, \] (4.18)
where the jump \( J_B(\zeta) \) is also indicated in Figure 4.

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Figure 4: Contours $\Sigma_B$, sectors, and jumps for the Bessel model RH problem $\Phi_B$

(c) The asymptotic behavior of $\Phi_B(\zeta)$ at infinity is

$$\Phi_B(\zeta) = \zeta^{-\frac{1}{2} \sigma_3} \frac{I + i \sigma_1}{\sqrt{2}} \left( I + \frac{1}{8\sqrt{\zeta}} \begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix} \right) + O\left( \frac{1}{\zeta} \right) e^{\sqrt{\zeta} \sigma_3} \text{ as } \zeta \to \infty. \quad (4.19)$$

The solution to the above RH problem can be constructed in terms of Bessel functions

$$\Phi_B(\zeta) = \pi^{\frac{1}{2} \sigma_3} \begin{cases} 
I_0(\sqrt{\zeta}) & \frac{1}{2} \sqrt{\zeta} K_0(\sqrt{\zeta}) \\
\pi i \sqrt{\zeta} I_0(\sqrt{\zeta}) & -\sqrt{\zeta} K_0(\sqrt{\zeta}) \\
\pi i \sqrt{\zeta} I_0(\sqrt{\zeta}) & -\sqrt{\zeta} K_0(\sqrt{\zeta})
\end{cases}$$

for $\zeta \in I$, $\pi(\frac{1}{2} \sqrt{\zeta}) K_0(\sqrt{\zeta}) I(\sqrt{\zeta})$, for $\zeta \in II$, $\pi i \sqrt{\zeta} I_0(\sqrt{\zeta}) - \sqrt{\zeta} K_0(\sqrt{\zeta})$, for $\zeta \in III$, (4.20)

where $\arg \zeta \in (-\pi, \pi)$; see [46]. For later use, we mention that $\Phi_B$ satisfies the differential equation

$$\frac{d\Phi_B(\zeta)}{d\zeta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \Phi_B(\zeta). \quad (4.21)$$

To satisfy the matching condition, for $\zeta \in U(-1)$, we define $E_{-1}(\zeta)$ as

$$E_{-1}(\zeta) = A^{(\infty)}(\zeta) \begin{cases} 
e^{-\frac{1}{2} \pi i (\alpha - \beta) \sigma_3 \frac{I - i \sigma_3}{\sqrt{2}} \left( -|s|^2 g^2(\zeta) \right)^{\frac{1}{2}} \sigma_3}, & \text{Im } \zeta > 0, \\
e^{-\frac{1}{2} \pi i (\alpha + \beta) \sigma_3 \left( -i \sigma_2 \frac{I - i \sigma_3}{\sqrt{2}} \left( -|s|^2 g^2(\zeta) \right)^{\frac{1}{2}} \sigma_3}, & \text{Im } \zeta < 0, \end{cases} \quad (4.22)$$
where $g(\zeta)$ and $A^{(\infty)}(\zeta)$ are defined in (4.2) and (4.7), respectively. From (4.5) and (4.17), we see that $E_{-1}(\zeta)$ is analytic for $\zeta \in U(-1)$. Particularly, in view of (4.7) and (4.12), we have

$$E_{-1}(-1) = 2^{\beta_3} \sigma_3 e^{-\frac{1}{2} \beta_3 i \sigma_3} \frac{I - i \sigma_1}{\sqrt{2}} |s|^{\frac{1}{2} \sigma_3} 2^{\frac{1}{2} \sigma_3} e^{\frac{1}{2} \pi i \sigma_3}. \quad (4.23)$$

The other parametrix $A^{(1)}(\zeta)$ at $\zeta = 1$, in a $\zeta$-neighborhood $U(1)$, can be constructed similarly, also in terms of the Bessel-type model RH problem $\Phi_B$. More precisely, we define

$$A^{(1)}(\zeta) = E_1(\zeta) \sigma_3 \Phi_B(-|s|^2 g^2(\zeta)) \sigma_3 \left\{ \begin{array}{ll}
\frac{1}{2} \pi i (\alpha - \beta) \sigma_3 \frac{I + i \sigma_1}{\sqrt{2}} (-|s|^2 g^2(\zeta)) \frac{1}{2} \sigma_3, & \text{Im} \zeta > 0, \\
\frac{1}{2} \pi i (\alpha - \beta) \sigma_3 \frac{I - i \sigma_1}{\sqrt{2}} (-|s|^2 g^2(\zeta)) \frac{1}{2} \sigma_3, & \text{Im} \zeta < 0,
\end{array} \right. \quad (4.24)$$

where

$$E_1(\zeta) = A^{(\infty)}(\zeta) \left\{ \begin{array}{ll}
\frac{1}{2} \pi i (\alpha - \beta) \sigma_3 \frac{I + i \sigma_1}{\sqrt{2}} (-|s|^2 g^2(\zeta)) \frac{1}{2} \sigma_3, & \text{Im} \zeta > 0, \\
\frac{1}{2} \pi i (\alpha - \beta) \sigma_3 \frac{I - i \sigma_1}{\sqrt{2}} (-|s|^2 g^2(\zeta)) \frac{1}{2} \sigma_3, & \text{Im} \zeta < 0,
\end{array} \right. \quad (4.25)$$

and $A^{(\infty)}(\zeta)$ is given in (4.7). Using (4.5), (4.7), (4.12), and (4.17), we see that $E_1(\zeta)$ is analytic for $\zeta \in U(1)$. Moreover, we have

$$E_1(1) = 2^{\beta_3} \sigma_3 e^{-\frac{1}{2} \beta_3 i \sigma_3} \frac{I + i \sigma_1}{\sqrt{2}} |s|^{\frac{1}{2} \sigma_3} 2^{\frac{1}{2} \sigma_3} e^{-\frac{1}{2} \pi i \sigma_3}. \quad (4.26)$$

### 4.4 Final transformation

Next, we define

$$\hat{B}(\zeta) = \left\{ \begin{array}{ll}
A(\zeta) A^{(\infty)}(\zeta)^{-1}, & \zeta \in \{U(1) \cup U(-1)\}, \\
A(\zeta) A^{(\pm 1)}(\zeta)^{-1}, & \zeta \in U(\pm 1).
\end{array} \right. \quad (4.27)$$

For $\zeta \in \partial U(\pm 1)$, clockwise-oriented, and $\text{Im} \zeta > 0$, we have jumps

$$J_{B}(\zeta) = A^{(\pm 1)}(\zeta) A^{(\infty)}(\zeta)^{-1}. \quad (4.28)$$

As $is \to +\infty$, by a standard analysis of small-norm RH problems [23], we have

$$\hat{B}(\zeta) = I + \hat{B}_1(\zeta) \frac{s}{s^2} + O \left( \frac{1}{s^2} \right), \quad is \to +\infty, \quad (4.29)$$

where

$$\hat{B}_1(\zeta) = \frac{1}{2 \pi i} \oint_{\partial U(-1)} \frac{J_{B,1}(x) dx}{x - \zeta} + \frac{1}{2 \pi i} \oint_{\partial U(1)} \frac{J_{B,1}(x) dx}{x - \zeta}. \quad (4.30)$$

for $\zeta \in \mathbb{C} \setminus \{ \partial U(-1) \cup \partial U(1) \}$, and the integration paths in (4.30) are small clockwise circles encircling $-1$ and $1$, respectively. Here, $J_{B,1}$ is the coefficient taken from

$$J_B(\zeta) = I + \frac{J_{B,1}(\zeta)}{s} + O \left( \frac{1}{s^2} \right), \quad \zeta \in \partial U(-1) \cup \partial U(1), \quad is \to +\infty, \quad (4.31)$$

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where, for \( \zeta \in \partial U(-1) \),

\[
J_{B,1}(\zeta) = \frac{1}{2 \sqrt{\zeta^2 - 1}} A^{(\infty)}(\zeta) e^{-\frac{1}{2} \pi i (\alpha - \beta) \sigma_3 \left( \begin{array}{cc} -1 & -2i \\ 2i & 1 \end{array} \right)} \left( e^{\frac{1}{2} \pi i (\alpha - \beta) \sigma_3 A^{(\infty)}(\zeta)^{-1}} \right),
\]  

(4.32)

as follows from (4.16), (4.22) and (4.28), and a combination of (4.24), (4.25) and (4.28) gives

\[
J_{B,1}(\zeta) = \frac{1}{2 \sqrt{\zeta^2 - 1}} A^{(\infty)}(\zeta) e^{-\frac{1}{2} \pi i (\alpha - \beta) \sigma_3 \left( \begin{array}{cc} -1 & 2i \\ 2i & 1 \end{array} \right)} \left( e^{\frac{1}{2} \pi i (\alpha - \beta) \sigma_3 A^{(\infty)}(\zeta)^{-1}} \right)
\]  

(4.33)

for \( \zeta \notin \partial U(1) \), with the principal branches taken such that \( \arg(\zeta \pm 1) \in (-\pi, \pi) \). Putting in use (4.7) and (4.12), we obtain that \( J_{B,1}(\zeta) \) has simple poles \( \zeta = \pm 1 \) with the following leading term expansions

\[
J_{B,1}(\zeta) \sim \frac{1}{4(\zeta + 1)} 2^{\beta_3} e^{-\frac{1}{2} \pi i \sigma_3} (\sigma_3 - i \sigma_1) 2^{-\beta_3} e^{\frac{1}{2} \pi i \sigma_3} \quad \text{as} \quad \zeta \to -1,
\]  

(4.34)

and

\[
J_{B,1}(\zeta) \sim \frac{1}{4(\zeta - 1)} 2^{\beta_3} e^{-\frac{1}{2} \pi i \sigma_3} (\sigma_3 + i \sigma_1) 2^{-\beta_3} e^{\frac{1}{2} \pi i \sigma_3} \quad \text{as} \quad \zeta \to 1.
\]  

(4.35)

Now the residue theorem applies. From (4.34) and (4.35) we have

\[
\frac{1}{2 \pi i} \oint_{\partial U(-1)} \frac{J_{B,1}(x) dx}{x - \zeta} = \frac{1}{4} (\sigma_3 - i 2^{\beta_3} e^{-\frac{1}{2} \pi i \sigma_3} \sigma_1 2^{-\beta_3} e^{\frac{1}{2} \pi i \sigma_3} \left( \frac{1}{\zeta + 1} \right)
\]  

(4.36)

for \( \zeta \notin U(-1) \), and

\[
\frac{1}{2 \pi i} \oint_{\partial U(1)} \frac{J_{B,1}(x) dx}{x - \zeta} = \frac{1}{4} (\sigma_3 + i 2^{\beta_3} e^{-\frac{1}{2} \pi i \sigma_3} \sigma_1 2^{-\beta_3} e^{\frac{1}{2} \pi i \sigma_3} \left( \frac{1}{\zeta - 1} \right)
\]  

(4.37)

for \( \zeta \notin U(1) \). Then, substituting (4.36) and (4.37) into (4.30) gives the approximation as \( \zeta \to \infty \)

\[
\hat{B}_1(\zeta) = \frac{\sigma_3}{2\zeta} + O\left( \frac{1}{\zeta^2} \right).
\]  

(4.38)

### 4.5 Proof of Theorem 1: Asymptotics of \( sH \) as \( is \to +\infty \)

In the outer region \( |\zeta - 1| > \delta \) and \( |\zeta + 1| > \delta \) for some small positive constant \( \delta \), we get from the transformations (4.1) and (4.27)

\[
\Psi(\zeta) = B(\zeta) A^{(\infty)}(\zeta) \exp\left( \frac{s}{4} \sqrt{\zeta^2 - 1} \sigma_3 \right).
\]  

(4.39)

Expanding the last term as \( \zeta \to \infty \), we obtain

\[
\exp\left( \frac{s}{4} \sqrt{\zeta^2 - 1} \sigma_3 \right) = \exp\left( \frac{1}{4} s \zeta \sigma_3 \right) \left( I - \frac{s}{8 \zeta} \sigma_3 + O\left( \frac{1}{\zeta^2} \right) \right).
\]

Substituting this and the expansions (4.14) and (4.38) into (4.39), and recalling (2.15), we have

\[
H(s) = -\frac{(\Psi_{1,11})}{2} - \frac{(\alpha^2 - \beta^2)}{s} = \frac{s}{16} + \frac{i \alpha}{2} - \left( \frac{\alpha^2 - \beta^2 + 1}{4} \right) \frac{1}{s} + O\left( \frac{1}{s^2} \right).
\]  

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Thus, we have

and

where \( E \) is independent of \( s \) and analytic for \(|\zeta+1|<\delta\). This completes the proof of (1.33).

From (2.27), (2.29), (4.13), (4.29) and (4.39), we obtain

\[
y(s) = 2^{2\beta} e^{-\beta \pi i} (1 + O(1/s)) , \quad is \to +\infty ,
\]

and

\[
d(s) = 2^{-2\alpha} e^{-\alpha \pi i} e^{is/2} (1 + O(1/s)) , \quad is \to +\infty .
\]

Now we derive the large-\( s \) behavior of \( u_k \) and \( v_k \) introduced in the Hamiltonian (1.13). In the region \(|\zeta + 1| < \delta\), we get by tracing back the invertible transformations (4.1) and (4.27)

\[
\Psi(\zeta) = \hat{B}(\zeta) E_{-1}(\zeta) \Phi_B (-|s|^2 g^2(\zeta)) e^{\frac{i}{2} \pi i (\alpha - \beta) \sigma_3} , \quad \text{Im} \zeta > 0 ,
\]

where \( E_{-1}(\zeta) \) is independent of \( s \) and analytic for \(|\zeta + 1| < \delta\), \( \hat{B}(\zeta) \) is also analytic for \(|\zeta + 1| < \delta\). Thus, we have

\[
A_2(s) = \hat{B}(-1) E_{-1}(-1) \left( \lim_{\zeta \to 0} \zeta \Phi_B(\zeta) \Phi_B^{-1}(\zeta) \right) \hat{B}(-1)^{-1} E_{-1}(-1) \hat{B}(1) E_{-1}(1) \hat{B}(1)^{-1}
\]

\[
= \frac{i}{2} s \hat{B}(-1) E_{-1}(-1) \sigma_+ E_{-1}(-1)^{-1} \hat{B}(-1)^{-1}
\]

\[
= \frac{i}{2} s \hat{B}(-1) 2^{3\beta \sigma_3} e^{-i \beta \pi i \sigma_3} (i \sigma_3 + a_1) e^{i \beta \pi i \sigma_3} 2^{-3\beta \sigma_3} \hat{B}(-1)^{-1}
\]

where use has been made of the differential equation (4.21) and formula (4.23). It follows from (4.29) that

\[
\hat{B}(-1) = I + O(1/s) , \quad is \to +\infty .
\]

Thus, from (2.11), (4.43) and (4.40)- (4.41), we obtain the asymptotic approximations of \( u_2(s) \) and \( v_2(s) \) as \( is \to +\infty \), as stated in (1.30) and (1.31).

Using the approximation of \( \Psi(\zeta; s) \) for \(|\zeta - 1| < \delta\), a similar argument justifies (1.28) and (1.29), namely, the asymptotic approximations of \( u_1(s) \) and \( v_1(s) \) as \( is \to +\infty \).

Thus, we have obtained the large-\( s \) asymptotics in Theorem 1. This, together with the small-\( s \) asymptotic formulas obtained in Section 3.4 and the existence results given in Proposition 4, completes the proof of Theorem I.

It is worth noting that with long but direct calculation the later terms in the asymptotic expansions of \( u_k(s) \) and \( v_k(s) \) can also be determined. For example, we have from (4.29) that

\[
\hat{B}(\pm 1) = I + \frac{\hat{B}_1(\pm 1)}{s} + O \left( \frac{1}{s^2} \right) , \quad is \to +\infty .
\]

Thus, the asymptotics (1.29) and (1.31) can be refined respectively to

\[
v_1(s) = i + \frac{\gamma_1}{s} + O(1/s^2) , \quad is \to +\infty ,
\]

and

\[
v_2(s) = -i + \frac{\gamma_2}{s} + O(1/s^2) , \quad is \to +\infty ,
\]

where the coefficients \( \gamma_1 \) and \( \gamma_2 \) can be determined by using \( \hat{B}_1(\pm 1) \).
5 RH problem for orthogonal polynomials and the differential identity

We consider the orthogonal polynomials associated with the weight function (1.4). Note that the weight function (1.4) can be analytically extended to the complex $z$-plane with a cut $[0, \infty)$

$w(z) = (z - 1)^{2\alpha}z^{\beta-\alpha}e^{-\pi i(\alpha+\beta)}, \quad \text{(5.1)}$

where the branch cut of $(z - 1)^{2\alpha}$ is taken along $[1, \infty)$ so that $\arg(z - 1) \in (0, 2\pi)$ and the branch of $z^{\beta-\alpha}$ is chosen so that $\arg z \in (0, 2\pi)$. Let $\pi_n(z; t) = z^n + \cdots$ be the monic orthogonal polynomial of degree $n$ with respect to the weight (1.4), satisfying the orthogonality relation

$\frac{1}{2\pi} \int_C \pi_n(e^{i\theta}; t)\bar{\pi}_m(e^{i\theta}; t)w(e^{i\theta}; t)d\theta = \chi^{-2}(t)\delta_{n,m}, \quad \text{(5.2)}$

where $\chi_n(t) > 0$ and $n, m = 0, 1, \cdots$. Denote the reverse polynomials associated with $\pi_n(z)$ by $\pi_n^*(z) = z^n\pi_n(1/z) = z^n\pi_n(1/z) = z^n + \cdots$,

where $\pi_n^*$ denotes the polynomial whose coefficients are complex conjugates of that of $\pi_n$. Let $Y(z)$ be the $2 \times 2$ matrix-valued function

$Y(z) = \begin{pmatrix} \pi_n(z) & \frac{1}{2\pi} \int_{C_t} \pi_n(x)w(x)dx \\ \frac{2}{\pi} \int z^n \pi_{n-1}(z) & \frac{\chi_n^2}{\pi_n^*} \int_{C_t} \pi_n^*(x)w(x)dx \end{pmatrix}, \quad \text{(5.3)}$

Then, $Y(z)$ is the unique solution to the RH problem below.

RH problem for $Y$

(a) $Y(z)$ is analytic in $\mathbb{C} \setminus C_t$.

(b) $Y(z)$ satisfies the jump condition

$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n}w(z; t) \\ 0 & 1 \end{pmatrix}, \quad z \in C_t, \quad \text{(5.4)}$

where the arc $C_t$ is defined in (1.3). The weight function $w(z; t)$, as given in (1.4), is the restriction of $w(z)$ (5.1) on the arc $C_t$.

(c) As $z \to \infty$, we have

$Y(z) = \begin{pmatrix} I + O(1) & O(\ln(z)) \\ O(\ln(z)) & I + O(1) \end{pmatrix} \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad \text{(5.5)}$

(d) Denoting $z_1 = e^{it}$ and $z_2 = e^{i(2\pi-t)}$, we have, respectively for $z_1 = z_1$ and $z_2 = z_2$, that

$Y(z) = \begin{pmatrix} O(1) & O(\ln(z-z_1)) \\ O(\ln(z-z_1)) & O(1) \end{pmatrix} \text{ as } z \to z_1. \quad \text{(5.6)}$
We have the differential identities with respect to $t$ in the following lemma, involving $Y(z)$ and the corresponding Toeplitz determinant. The lemma was first proved in [22].

**Lemma 2.** Let $D_n(t)$ be the Toeplitz determinant associated with the weight function (1.4), then we have
\[
\frac{d}{dt} \ln D_n(t) = -\frac{1}{2\pi} \sum_{j=1}^{2} z^{-n+1} w(z_j; t) \lim_{z \to z_j} \left( \frac{Y^{-1}(z)}{z} \frac{d}{dz} Y(z) \right)_{21},
\]
where $z_1 = e^{it}$ and $z_2 = e^{i(2\pi - t)}$.

**Proof.** We start with the expression of the Toeplitz determinant in terms of the leading coefficients of the orthonormal polynomials
\[
D_n(t) = \prod_{k=0}^{n-1} \chi_k^2(t).
\]
Taking logarithmic derivative on both side of (5.8) yields
\[
\frac{d}{dt} \ln D_n(t) = -2 \sum_{k=0}^{n-1} \sum_{j=1}^{2} \left( \frac{\chi_k^2(t)}{2} \sum_{j=1}^{2} \pi_k(z_j) \pi_k(z_j^{-1}) w(z_j; t) \right),
\]
where $z_1 = e^{it}$, $z_2 = e^{i(2\pi - t)}$ and $k = 0, 1, \ldots, n - 1$. Substituting (5.10) into (5.9) gives
\[
\frac{d}{dt} \ln D_n(t) = -\frac{1}{2\pi} \sum_{j=1}^{2} \left( \sum_{k=0}^{n-1} \sum_{j=1}^{2} \chi_k^2(t) \pi_k(z_j) \pi_k(z_j^{-1}) w(z_j; t) \right).
\]

From the Christoffel-Darboux identity [50]
\[
\sum_{k=0}^{n-1} \chi_k^2(t) \pi_k(z) \pi_k(z^{-1}) = z \chi_n^2(t) \left[ \pi_n(z^{-1}) \frac{d}{dz} \pi_n(z) - \pi_n(z) \frac{d}{dz} \pi_n(z^{-1}) \right] - n \chi_n^2(t) \pi_n(z) \pi_n(z^{-1}),
\]
and the recurrence relation [50]
\[
z \pi_n(1/z) = \chi_{n-1}(1/z)^2 \pi_{n-1}(1/z) + z^{1-n} \pi_n(0),
\]
we have
\[
\sum_{k=0}^{n-1} \chi_k^2(t) \pi_k(z) \pi_k(z^{-1})
\]
\[
= -\chi_{n-1}^2(t) \left[ \pi_n(z) \frac{d}{dz} \pi_{n-1}(z^{-1}) - \pi_{n-1}(z) \frac{d}{dz} \pi_n(z) + (n-1)z^{-1} \pi_n(z) \pi_{n-1}(z^{-1}) \right]
\]
\[
= -\chi_{n-1}^2(t) z^{1-n} \left[ \pi_n(z) \frac{d}{dz} \pi_{n-1}(z^{-1}) - \pi_{n-1}(z^{-1}) \frac{d}{dz} \pi_n(z) \right].
\]
Then, the differential identity (5.7) follows from (5.3), (5.11) and (5.14).
6 Asymptotics of the RH problem for $Y$

We carry out the Deift-Zhou nonlinear steepest descent analysis of the RH problem for $Y(z)$ as $n \to \infty$ and $t \to 0^+$ in a way such that $nt$ is bounded.

6.1 Normalization of the RH problem: $Y \to T$

We introduce the transformation $Y \to T$ to normalize the large-$z$ behavior of $Y(z)$,

$$T(z) = \begin{cases} Y(z) z^{-n \sigma_3}, & |z| > 1, \\ Y(z), & |z| < 1, \end{cases} \quad (6.1)$$

Then, it is readily seen that $T(z)$ satisfies the following RH problem.

**RH problem for $T$**

(a) $T(z)$ is analytic in $C \setminus C_t$.

(b) $T(z)$ satisfies the jump condition

$$T_+(z) = T_-(z) \begin{pmatrix} z^n & w(z) \\ 0 & z^{-n} \end{pmatrix}, \quad z \in C_t, \quad (6.2)$$

and

$$T_+(z) = T_-(z) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \in C \setminus C_t, \quad (6.3)$$

where the weight function $w(z)$ is defined in (5.1), $C = \{ z : |z| = 1 \}$ is the unit circle and the arc $C_t$ is defined in (1.3). Both the arcs in (6.2) and (6.3) are oriented counterclockwise.

(c) As $z \to \infty$, we have

$$T(z) = I + O \left( \frac{1}{z} \right). \quad (6.4)$$

(d) As $z \to z_1$ and $z \to z_2$, where $z_1 = e^{it}$ and $z_2 = e^{i(2\pi - t)}$, $T(z)$ shares the same behavior as $Y(z)$; cf. (5.6).

6.2 Opening of the lens: $T \to S$

The diagonal entries of the jump matrix for $T$ in (6.2) are highly oscillating for $n$ large. To turn the oscillation to exponential decays on certain contours, we deform the contours $C_t$ and introduce the second transformation $T \to S$. The transformation is based on the following factorization of the jump matrix (6.2)

$$\begin{pmatrix} z^n & w(z) \\ 0 & z^{-n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^{-n} w(z)^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & w(z) \\ -w(z)^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^n w(z)^{-1} & 1 \end{pmatrix}. \quad (6.5)$$
Define the transformation

\[
S(z) = \begin{cases} 
T(z) \begin{pmatrix} 1 & 0 \\ z^{-n}w(z)^{-1} & 1 \end{pmatrix}, & \text{for } z \in \Omega_E, \\
T(z) \begin{pmatrix} 1 & 0 \\ -z^{-n}w(z)^{-1} & 1 \end{pmatrix}, & \text{for } z \in \Omega_I, \\
T(z), & \text{otherwise,}
\end{cases}
\]

(6.6)

with the regions shown in Figure 5. Then, \(S(z)\) satisfies the following RH problem.

![Figure 5: Opening of the lens, contours and regions for the RH problem for \(S(z)\).](image)

**RH problem for \(S\)**

(a) \(S(z)\) is analytic in \(\mathbb{C} \setminus \Sigma_E \cup C \cup \Sigma_I\), where the jump curves are illustrated in Figure 5 and \(C = \{z : |z| = 1\}\) is the unit circle.
(b) $S(z)$ satisfies the jump condition
\[
S_+(z) = S_-(z) \left\{ \begin{array}{ll}
1 & \text{for } z \in \Sigma_E, \\
0 & \text{for } z \in C_t, \\
1 & \text{for } z \in \Sigma_I, \\
z^n & \text{for } z \in C \setminus C_t,
\end{array} \right.
\]
for $z \in \Sigma$.

(c) As $z \to \infty$, we have
\[
S(z) = I + O \left( \frac{1}{z} \right).
\]
(d) At $z_1 = e^{it}$ and $z_2 = e^{i(2\pi - t)}$, we have
\[
S(z) = O(\ln(z - z_1)) \text{ as } z \to z_1 \quad \text{and} \quad S(z) = O(\ln(z - z_2)) \text{ as } z \to z_2.
\]

6.3 Global parametrix

As $n \to \infty$, the jump matrix in (6.7) tends to the identity matrix for $z$ bounded away from the unit circle $C$. The arc $C_t$ also tends to the unit circle in the double scaling case as $n \to \infty$. Thus, we consider the approximate RH problem below, with jump occurs on the unit circle alone.

**RH problem for $N$**

(a) $N(z)$ is analytic in $\mathbb{C} \setminus C$, where $C$ is the unit circle.

(b) $N(z)$ satisfies the jump condition on the unit circle, counterclockwise oriented,
\[
N_+(z) = N_-(z) \begin{pmatrix} 0 & w(z) \\ -w(z)^{-1} & 0 \end{pmatrix},
\]
where the weight $w(z)$ is given in (5.1).

(c) As $z \to \infty$, we have
\[
N(z) = I + O \left( \frac{1}{z} \right).
\]

The solution to the above RH problem is given explicitly by
\[
N(z) = \begin{cases}
(z^{\frac{\beta - \alpha}{2}})^{\sigma_3}, & \text{if } |z| > 1, \\
(z - 1)^{i(\alpha + \beta)\sigma_3}e^{-\pi i(\alpha + \beta)\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{if } |z| < 1,
\end{cases}
\]
\[
N(z) = \begin{cases}
(z^{\frac{\beta - \alpha}{2}})^{\sigma_3}, & |z| > 1, \\
(z - 1)^{i(\alpha + \beta)\sigma_3}e^{-\pi i(\alpha + \beta)\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & |z| < 1,
\end{cases}
\]
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where the branch of $z^{\alpha-\beta}$ is chosen so that $\arg z \in (0, 2\pi)$ and both branch cuts of $(z - 1)^{\alpha + \beta}$ and $(z - 1)^{\beta - \alpha}$ are taken along $[1, \infty)$, with $\arg(z - 1) \in (0, 2\pi)$.

### 6.4 Local parametrix

The global parametrix $N(z)$ fails to approximate $S(z)$ near the endpoints $z_1$ and $z_2$, where the weight has jump discontinuities. For $n$ large, both the endpoints are close to 1 and belong to a neighborhood $U(1, r) = \{ z : |z - 1| < r \}$ of $z = 1$, with $r$ small but fixed. We intend to construct a local parametrix $P^{(1)}(z)$ in $U(1, r)$, which satisfies the same jump condition as $S(z)$ and matches with $N(z)$ on the boundary of $U(1, r)$.

**RH problem for $P^{(1)}$**

(a) $P^{(1)}(z)$ is analytic in $U(1, r) \setminus \{ \Sigma_E \cup C \cup \Sigma_I \}$; see Figure 5 for the contours.

(b) $P^{(1)}(z)$ satisfies the same jump condition as $S(z)$ on $U(1, r) \cap \{ \Sigma_E \cup C \cup \Sigma_I \}$; cf. (6.7).

(c) As $n \to \infty$, we have the matching condition

$$P^{(1)}(z) = \left( I + O\left( \frac{1}{n} \right) \right) N(z).$$  \hfill (6.13)

Let

$$M(z) = \begin{cases} z^{\frac{\alpha}{2}} w(z)^{-\frac{1}{2}} \sigma_3, & |z| < 1, z \in U(1, r), \\ z^{\frac{\alpha}{2}} w(z)^{\frac{1}{2}} \sigma_3 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & |z| > 1, z \in U(1, r). \end{cases}$$  \hfill (6.14)

where the principal branches are chosen for the power functions. We also define

$$f(z) = f(z; t) = e^{-\frac{2i}{t}} t^{-1} \ln z,$$  \hfill (6.15)

where the branch of $\ln z$ is chosen such that $\arg z \in (-\pi, \pi)$. Note that $f(z)$ is a conformal mapping from $U(1, r)$ to a neighborhood of the origin so long as $0 < r < 1$, and $f(z_1) = 1$ and $f(z_2) = -1$. We seek a solution to the above RH problem of the following form:

$$P^{(1)}(z) = E(z) \Psi(f(z); -2int) e^{\pm \frac{\pi i}{2}(\alpha - \beta)\sigma_3} M(z),$$ \hfill (6.16)

where $\Psi(\zeta; s)$ is the solution to the model RH problem discussed in Section 2.1. $E(z)$ is a certain matrix-valued analytic function defined in $U(1, r)$, to be determined. It then follows from the jumps (2.1) for $\Psi$ and the definition of $M(z)$ in (6.14) that $P^{(1)}(z)$ fulfills all jump conditions as $S(z)$ for $z \in U(1, r) \cap \{ \Sigma_E \cup C \cup \Sigma_I \}$. We then choose $E(z)$ so that the matching condition (6.13) holds.

**Proposition 6.** Let

$$E(z) = e^{\pm \frac{\pi i}{2} \zeta \sigma_3} e^{-\frac{\pi i}{2} \sigma_3 z^{\alpha - \beta} \sigma_3 \zeta} \left( \frac{f(z)}{z - 1} \right)^{-\beta \sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$ \hfill (6.17)

where $r \in (0, 1)$ is a small constant, the function $\zeta^{-\beta}$ takes principal branch, and the branch of $z^{(\alpha - \beta)/2}$ is chosen such that $\arg z \in (0, 2\pi)$. Then, $E(z)$ is analytic for $z \in U(1, r)$ and the matching condition (6.13) is satisfied for $nt$ bounded.
Proof. It follows from the definition of \( f(z) \) in (6.15) that \( \left( \frac{f(z)}{z-1} \right)^{-\beta} = e^{\frac{\pi i \beta}{2}} t^\beta + O(z - 1) \) is analytic for \( z \in U(1, r) \). With the branch chosen, it is readily seen that \( e^{\pm \pi i (\alpha - \beta)/2} z^{(\alpha - \beta)/2} \) is also analytic for \( z \in U(1, r) \) with \( r < 1 \). Thus, \( E(z) \) is analytic in \( U(1, r) \).

Next, we show that \( P^{(1)}(z) \) satisfies the matching condition (6.13). Since \( nt \) is bounded, we see that \( f(z) \) is large for large-\( n \). Hence, for \( z \in \partial U(1, r) \), in view of (6.16) and the large-\( \zeta \) behavior (2.2) of \( \Psi(\zeta; s) \), we obtain

\[
P^{(1)}(z)N(z)^{-1} = E(z) [I + O(1/n)] f(z)^{-\beta \sigma_3} z^{-\frac{3}{2} \sigma_3 e^{\pm \pi i (\alpha - \beta)/2} \sigma_3} M(z) N(z)^{-1}, \quad \pm \text{Im} z > 0.
\]  

(6.18)

Recalling the definitions of \( N(z) \) and \( M(z) \), given in (6.12) and (6.14), respectively, we have

\[
M(z) N(z)^{-1} = z^{\frac{3}{2} \sigma_3} e^{-\frac{3}{2} (\alpha + \beta) \sigma_3} z^{\frac{1}{2} (\alpha - \beta) \sigma_3} (z-1)^{\beta \sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]  

(6.19)

where the branch for \( z^{n/2} \) is chosen such that \( \text{arg} z \in (-\pi, \pi) \), the branch cut for \( z^{(\beta - \alpha)/2} \) is taken along the positive real axis so that \( \text{arg} z \in (0, 2\pi) \) and the branch cut for \( (z-1)^{-\beta} \) is chosen along \( [1, \infty) \) so that \( \text{arg}(z-1) \in (0, 2\pi) \). Note that \( |f(z)| = e^{-\frac{3}{2} \beta} e^{\beta \text{arg} z} \) is independent of \( t \), bounded and bounded away from zero for pure imaginary parameter \( \beta \) and \( z \in \partial U(1, r) \). Substituting (6.17) and (6.19) into (6.18) gives us the matching condition (6.13). This completes the proof of the proposition. \( \square \)

6.5 Final transformation: \( S \to R \)

Define

\[
R(z) = \begin{cases} 
S(z)N(z)^{-1}, & |z - 1| > r, \\
S(z)P^{(1)}(z)^{-1}, & |z - 1| < r.
\end{cases}
\]  

(6.20)

Then, \( R(z) \) satisfies the RH problem.

RH problem for \( R \)

(a) \( R(z) \) is analytic for \( C \setminus \Sigma_R \), where the remaining contour \( \Sigma_R \) is illustrated in Figure 6.

(b) \( R(z) \) satisfies the jump condition

\[
R_+(z) = R_-(z) J_R(z), \quad z \in \Sigma_R,
\]  

(6.21)

with

\[
J_R(z) = \begin{cases} 
P^{(1)}(z)N(z)^{-1}, & |z - 1| = r, \text{ clockwise}, \\
N(z)J_E(z)N(z)^{-1}, & z \in \Sigma_E, \ |z - 1| > r, \\
N(z)J_I(z)N(z)^{-1}, & z \in \Sigma_I, \ |z - 1| > r,
\end{cases}
\]  

(6.22)

where \( J_E(z) \) and \( J_I(z) \) denote the jumps of \( S(z) \) on \( \Sigma_E \) and \( \Sigma_I \), respectively; see (6.7) for the jump matrices and Figure 6 for the contours.

(c) As \( z \to \infty \), we have

\[
R(z) = I + O(1/z). \quad (6.23)
\]
It is readily seen that $J_R(z) - I$ is uniformly exponentially small for $z$ on the portions of $\Sigma_E$ and $\Sigma_I$, described in (6.22), as $n \to \infty$. Therefore, for large-$n$ and bounded $nt$, we have from the matching condition (6.13) that
\[
J_R(z) = I + O(1/n),
\]
uniformly for $z \in \Sigma_R$. As before, after a standard analysis of the small-norm RH problems [23], a combination of (6.21) and (6.24) gives
\[
R(z) = I + O(1/n), \quad \frac{d}{dz}R(z) = O(1/n),
\]
uniformly in the whole complex $z$-plane.

6.6 Proof of Theorem 2

As a consequence of the nonlinear steepest descent analysis of the RH problem for $Y$ performed in Section 6 and applying the differential identity (5.7), we derive the asymptotic approximation of the Toeplitz determinant as stated in Theorem 2.

Tracing back the series of transformations $Y \to T \to S \to R$, we find that
\[
Y(z) = R(z)E(z)\Psi(f(z); -2int)e^{\pm \frac{\pi i (\alpha - \beta)}{2} \sigma_3 z^2 \sigma_3 w(z) - \frac{1}{2} \sigma_3}, \quad \pm \text{Im } z > 0,
\]
for $|z| < 1$ and $|z - 1| < r$. From Proposition 6 we see that $E(z)$ is analytic and bounded for $|z - 1| < r$. We also have the analyticity of $R(z)$ in $|z - 1| < r$ and the approximation (6.25).

Thus, we have
\[
z^{-n}w(z) \left( Y(z)^{-1} \frac{d}{dz} Y(z) \right) = f'(z)e^{\pm i \pi (\alpha - \beta)} \left\{ \Psi^{-1}(f(z); -2int)\Psi_c(f(z); -2int) \right\} + O\left( \frac{1}{n} \right)
\]
respectively for \( \text{Im} \pm z > 0 \), with \( |z| < 1 \) and \( |z - 1| < r \). The definition of \( f'(z) = \frac{1}{i \pi z} \); cf. (6.15). Substituting (6.27) into the differential identity (5.7), we obtain

\[
\frac{d}{dt} \ln D_n(t) = -\frac{1}{2\pi i} e^{\pi i(\alpha - \beta)} \lim_{\zeta \to 1} \left( \Psi_+^\prime(\zeta; -2int)^{-1} \frac{d}{d\zeta} \Psi_+^\prime(\zeta; -2int) \right)_{21} - \frac{1}{2\pi i} e^{-\pi i(\alpha - \beta)} \lim_{\zeta \to -1} \left( \Psi_+^\prime(\zeta; -2int)^{-1} \frac{d}{d\zeta} \Psi_+^\prime(\zeta; -2int) \right)_{21} + O\left( \frac{1}{n} \right)
\]

\[
= -\frac{1}{2\pi i} e^{\pi i(\alpha - \beta)} \Psi_1^\prime(\zeta; -2int) + e^{-\pi i(\alpha - \beta)} \Psi_1^\prime(\zeta; -2int) + O\left( \frac{1}{n} \right), \quad (6.28)
\]

where \( \Psi_1^\prime(s) \) and \( \Psi_1^\prime(\zeta) \) are defined in (2.5) and (2.6), respectively. Comparing (2.26) with (2.40), we have

\[
-\frac{1}{\pi i} (e^{\pi i(\alpha - \beta)} \Psi_1^\prime(s) + e^{-\pi i(\alpha - \beta)} \Psi_1^\prime(\zeta))_{21} = 2sH(s) + c, \quad (6.29)
\]

where \( H(s) \) is the Hamiltonian for the coupled Painlevé V system and \( c \) is a constant independent of the variable \( s \). Using the asymptotics as \( is \to 0^+ \) in (1.32) and (3.27), we have the constant \( c = 0 \). Substituting (6.29) into (6.28), we then obtain by taking integration

\[
\frac{D_n(t)}{D_n(0)} = \exp \left( \int_0^{2int} H(\tau) d\tau + O(1/n) \right),
\]

where the error bound is uniform for \( nt \) bounded. Thus we obtain (1.35), and complete the proof of Theorem 2.

7 Proof of Theorem 4

Integrating the large-\( s \) asymptotic approximation of \( H(s; \alpha, \beta) \) in Theorem 1, we have as \( is \to +\infty \)

\[
\int_0^s H(\tau; \alpha, \beta) d\tau = \frac{s^2}{32} + \frac{i\alpha}{2}s - \left( \alpha^2 - \beta^2 + \frac{1}{4} \right) \ln |s| + C_1(\alpha, \beta) + O\left( \frac{1}{s} \right), \quad (7.1)
\]

where the path of integration is a line segment of the negative imaginary axis. Since \( H(s) \) is free of poles for \( s \in -i(0, +\infty) \) with at most weak singularity as \( is \to 0^+ \), as indicated in the asymptotic behavior (1.32), the integral (7.1) is well-defined.

To determine the constant \( C_1(\alpha, \beta) \), which contains the global information of the Hamiltonian, the strategy is the following. First, we express the integral alternatively in terms of an integral of the classical action by using the differential identity (2.37). The evaluation of the constant \( C_1(\alpha, \beta) \) amounts to a calculation of the constant term \( C_2(\alpha, \beta) \) in the asymptotic expansion of the action integral. Next, it is much more convenient to evaluate \( C_2(\alpha, \beta) \). Actually, by using the differential identities (2.38) and (2.39), we see that each derivative of the action integral, with respect to the parameters \( \alpha \) or \( \beta \), is a simple combination of the functions \( u_k(s) \) and \( v_k(s) \), not involving any integrals. Thus, the derivatives of the constant term \( C_2(\alpha, \beta) \) with respect to the parameters \( \alpha \) and \( \beta \) are then readily determined. Finally, it is known that for special parameters \( \alpha = 1/2 \) and \( \beta = 0 \), the Hamiltonian admits a special function solution (2.70) and its integral is explicitly calculated. This special case actually serves as an initial condition. The constant terms are thus determined, in terms of the Barnes \( G \)-function.
To integrate on both sides of the differential differential identity (2.37), we first consider the following limit as $is \to 0^+$,

$$
\lim_{is \to 0^+} \left( sH(s; \alpha, \beta) + \alpha \ln d(s) - \beta \ln y(s) - 2(\alpha^2 - \beta^2) \ln s \right)
$$

$$
= \alpha \ln \left( \frac{\Gamma(1 + \alpha + \beta) \Gamma(1 + \alpha - \beta)}{\Gamma(1 + 2\alpha)^2} \right) - \beta \ln \left( \frac{\Gamma(1 + \alpha - \beta)}{\Gamma(1 + \alpha + \beta)} \right) + \alpha \ln(2\alpha) - 2(\alpha^2 - \beta^2) \ln 2,
$$

(7.2)

by using (3.25) and (3.26), and the asymptotics of $H(s; \alpha, \beta)$ given in Theorem 1. Recalling (2.37), the convergence of the integral (7.1) and the boundedness near $0^+$ of the function in (7.2) imply that the action defined by

$$
\int_0^s H(\tau; \alpha, \beta) d\tau = \int_0^s \left[ u_1(\tau) \frac{dv_1(\tau)}{d\tau} + u_2(\tau) \frac{dv_2(\tau)}{d\tau} - H(\tau; \alpha, \beta) \right] d\tau + sH(s; \alpha, \beta) + \alpha \ln d(s) - \beta \ln y(s) - 2(\alpha^2 - \beta^2) \ln s - C_0,
$$

(7.3)

where $C_0$ is the limit in (7.2), the path of integration of the first integral is an interval of the negative imaginary axis as described in (7.1). The integration contour of the second integral consists of the intervals $(0, -i\delta_1], [-iM_1, s]$ and a curve $\Gamma$ connecting $\delta_1$ and $M_1$ in the right half complex plane $\text{Re} \tau > 0$ to avoid the poles of the action and the Hamiltonian, and such that the region inside the closed curve $-i(\delta_1, M_1) \cup \Gamma^{-1}$ contains no poles of the action and the Hamiltonian. From the large-$s$ asymptotic formulas (1.28), (1.30), (1.33), (4.44) and (4.45), we obtain

$$
\int_0^s \left[ u_1(\tau) \frac{dv_1(\tau)}{d\tau} + u_2(\tau) \frac{dv_2(\tau)}{d\tau} - H(\tau; \alpha, \beta) \right] d\tau = -\frac{s^2}{32} - \frac{ias}{2} + \left( \alpha^2 - \beta^2 + \frac{1}{4} - \frac{i(\gamma_1 + \gamma_2)}{8} \right) \ln |s| + C_2(\alpha, \beta) + O\left( \frac{1}{s} \right)
$$

(7.4)

and

$$
\frac{s^2}{16} + ias - 2(\alpha^2 - \beta^2) \ln |s| - 2(\alpha^2 + \beta^2) \ln 2 - \left( \alpha^2 - \beta^2 + \frac{1}{4} \right) + \alpha \ln \alpha + O\left( \frac{1}{s} \right)
$$

(7.5)

as $is \to +\infty$, where the logarithms take the principal branches, the constant $C_2(\alpha, \beta)$ depends only on the parameters $\alpha$ and $\beta$, and the constants $\gamma_1$ and $\gamma_2$ are the coefficients in the asymptotic expansions (4.44) and (4.45), respectively. In deriving (7.5), use has also been made of the large-$s$ asymptotic formulas (4.40) and (4.41). Substituting (7.1), (7.4) and (7.5) into (7.3), we
obtain
\[ C_1(\alpha, \beta) = C_2(\alpha, \beta) - \left(\alpha^2 - \beta^2 + \frac{1}{4}\right) - \alpha \ln 2 - 4\beta^2 \ln 2 \]
\[ - \alpha \ln \left(\frac{\Gamma(1 + \alpha + \beta)\Gamma(1 + \alpha - \beta)}{\Gamma(1 + 2\alpha)^2}\right) + \beta \ln \left(\frac{\Gamma(1 + \alpha - \beta)}{\Gamma(1 + \alpha + \beta)}\right), \] (7.6)

and the relation
\[ \gamma_1 + \gamma_2 = -4i. \] (7.7)

Next, we evaluate the constant \( C_2(\alpha, \beta) \) by using the differential identities (2.38) and (2.39). From the relation (7.7) and the large-s asymptotics of \( d(s) \), \( u_k(s) \) and \( v_k(s) \), given in (1.28), (1.30), (4.41), (4.44) and (4.45), we have
\[ u_1(s) \frac{dv_1(s)}{d\alpha} + u_2(s) \frac{dv_2(s)}{d\alpha} - \ln d(s) + 2\alpha \ln s = -\frac{i}{2} s^2 + 2\alpha \ln |s| + 2\alpha \ln 2 - \ln \alpha + O(1/s) \] (7.8)

and
\[ u_1(s) \frac{dv_1(s)}{d\beta} + u_2(s) \frac{dv_2(s)}{d\beta} + \ln y(s) - 2\beta \ln s = -2\beta \ln |s| + 2\beta \ln 2 + O(1/s) \] (7.9)
as \( s \to +\infty \). On the other hand, from the asymptotics of \( d(s) \), \( y(s) \), \( u_k(s) \) and \( v_k(s) \) as \( s \to 0^+ \), given in (3.25), (3.26) and Theorem 1, we have
\[ \lim_{is \to 0^+} \left( u_1(s) \frac{dv_1(s)}{d\alpha} + u_2(s) \frac{dv_2(s)}{d\alpha} - \ln d(s) + 2\alpha \ln s \right) = 2\alpha \ln 2 - \ln \frac{2\alpha \Gamma(1 + \alpha + \beta)\Gamma(1 + \alpha - \beta)}{\Gamma(1 + 2\alpha)^2} \] (7.10)

and
\[ \lim_{is \to 0^+} \left( u_1(s) \frac{dv_1(s)}{d\beta} + u_2(s) \frac{dv_2(s)}{d\beta} + \ln y(s) - 2\beta \ln s \right) = -2\beta \ln 2 + \ln \frac{\Gamma(1 + \alpha - \beta)}{\Gamma(1 + \alpha + \beta)}. \] (7.11)

Integrating both sides of (2.38) and (2.39) on the interval \([0, s]\) (the integration contour deformed if necessary, similar to (7.3), to keep away from the poles of the integrand), and using the approximations (7.4) and (7.8)-(7.11), we obtain the first-order differential equations
\[ \frac{d}{d\alpha} C_2(\alpha, \beta) = \ln 2 + \ln \left(\frac{\Gamma(1 + \alpha + \beta)\Gamma(1 + \alpha - \beta)}{\Gamma(1 + 2\alpha)^2}\right), \] (7.12)

and
\[ \frac{d}{d\beta} C_2(\alpha, \beta) = 4\beta \ln 2 - \ln \left(\frac{\Gamma(1 + \alpha - \beta)}{\Gamma(1 + \alpha + \beta)}\right). \] (7.13)

It follows from (7.12) and (7.13) that
\[ C_2(\alpha, \beta) = C_2(0, \beta) + \alpha \ln 2 + \int_0^{\alpha + \beta} \ln(\Gamma(1 + \tau))d\tau + \int_{-\beta}^{\alpha - \beta} \ln(\Gamma(1 + \tau))d\tau - \int_0^{2\alpha} \ln(\Gamma(1 + \tau))d\tau, \]
(7.14)
and
\[ C_2(0, \beta) = C_2(0, 0) + 2\beta^2 \ln 2 + \int_0^\beta \ln(\Gamma(1 + \tau))d\tau + \int_0^{\beta} \ln(\Gamma(1 + \tau))d\tau. \] (7.15)
Recalling the integral expression of the Barnes G-function

$$\ln G(1 + z) = \frac{z}{2} \ln(2\pi) - \frac{z(z + 1)}{2} + z \ln \Gamma(z + 1) - \int_0^z \ln \Gamma(1 + t) dt, \quad \text{Re } z > -1; \quad (7.16)$$

cf., e.g., [18 (2.37)], we obtain from (7.6) and (7.14) that

$$C_1(\alpha, \beta) = C_2(0, 0) - \ln \left( \frac{G(1 + \alpha + \beta) G(1 + \alpha - \beta)}{G(1 + 2\alpha)} \right) - 4\beta^2 \ln 2 - \frac{1}{4} - \int_0^\beta \ln(\Gamma(1 + \tau)) d\tau - \int_0^{-\beta} \ln(\Gamma(1 + \tau)) d\tau. \quad (7.17)$$

This equation, together with (7.15), implies that

$$C_1(\alpha, \beta) = C_2(0, 0) - \ln \left( \frac{G(3/2)^2}{G(2)} \right) - \frac{1}{4} - \ln \left( \frac{\pi G(1/2)^2}{G(1 + 2\alpha)} G(1 + \alpha + \beta) G(1 + \alpha - \beta) \right) - 2\beta^2 \ln 2 - \frac{1}{4}. \quad (7.18)$$

In particular, we have

$$C_1(1/2, 0) = C_2(0, 0) - \ln \left( \frac{G(3/2)^2}{G(2)} \right) - \frac{1}{4} = C_2(0, 0) - \ln \left( \pi G(1/2)^2 \right) - \frac{1}{4} \quad (7.19)$$

by using the recurrence relation

$$G(z + 1) = \Gamma(z) G(z), \quad G(1) = 1.$$

Comparing (7.18) and (7.19), we obtain

$$C_1(\alpha, \beta) = C_1(1/2, 0) + \ln \left( \frac{\pi G(1/2)^2 G(1 + 2\alpha)}{G(1 + \alpha + \beta) G(1 + \alpha - \beta)} \right) - 2\beta^2 \ln 2. \quad (7.20)$$

Substituting (7.20) into (7.1), we obtain the large-s asymptotic approximation of the integral of the Hamiltonian \( H(s) \)

$$\int_0^s H(\tau; 1/2, 0) d\tau = \frac{s^2}{32} + \frac{is}{2} - \left( \alpha^2 - \beta^2 + \frac{1}{4} \right) \ln |s| + C_1(1/2, 0) + \ln \left( \frac{\pi G(1/2)^2 G(1 + 2\alpha)}{G(1 + \alpha + \beta) G(1 + \alpha - \beta)} \right) - 2\beta^2 \ln 2 + O \left( \frac{1}{s} \right), \quad (7.21)$$

as \( is \to +\infty \).

Finally, we evaluate \( C_1(1/2, 0) \) by using the special function solution to the Hamiltonian for \( \alpha = 1/2 \) and \( \beta = 0 \). From (2.70), we have

$$\int_0^s H(\tau; 1/2, 0) d\tau = \frac{s^2}{32} + \frac{is}{4} - \frac{\ln |s|}{2} - \frac{\ln(\pi/2)}{2} + O \left( \frac{1}{s} \right), \quad is \to +\infty. \quad (7.22)$$

Here we have used the asymptotic formula \( I_0(x) = \frac{e^x}{\sqrt{2\pi x}} (1 + O(1/x)) \) as \( x \to +\infty \); see [1 (9.7.1)]. Comparing (7.21) and (7.22) yields

$$C_1(1/2, 0) = \frac{1}{2} \ln 2 - \frac{1}{2} \ln \pi.$$
Thus, we obtain
\[
\int_0^s H(\tau; \alpha, \beta) d\tau = \frac{s^2}{32} + \frac{i\alpha s}{2} - \left( \alpha^2 - \beta^2 + \frac{1}{4} \right) \ln(|s|/4)
+ \ln \left( \frac{\sqrt{\pi} G(1/2)^2 G(1 + 2\alpha)}{2^{2\alpha^2} G(1 + \alpha + \beta) G(1 + \alpha - \beta)} \right) + O \left( \frac{1}{s} \right),
\]
(7.23)
as \imath s \to +\infty. The equation can be written equivalently in the form of (1.40). This completes
the proof of Theorem 4.

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