Massive photons in particle and laser physics

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Abstract

This article applies the theory of massive electrodynamics to the Dirac equation with the aim to find the generalized Volkov solution with massive photon field. The resulting equation is the Riccati equation which cannot be solved in general. We use the approximative Volkov function for massive photons and consider an electron in the periodic field and in the laser pulse of the $\delta$-function form. We derive the modified Compton formulas for the interaction of the multiphoton object with an electron for both cases.

KEY WORDS: Volkov solution, Riccati equation, massive photons, Compton effect.
1 Introduction

The introduction of the massive photon into field theory is elementary from the mathematical point of view. However, the physical reasons for such generalization require serious motivation.

We know from the special theory of relativity, that the relativistic mass formula

\[ m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \]  

where \( m_0 \) is the rest mass, has physical meaning for \( v = c \), only if \( m_0 = 0 \). Since the velocity of photon in vacuum is \( v = c \), it follows from the view point of the special theory of relativity that the rest mass of photon is zero.

Nevertheless, massless photon has a momentum

\[ p = \frac{E}{c} = \frac{\hbar \omega}{c}, \]  

as it follows from the Einstein relativistic mass formula \( E = \sqrt{c^2 p^2 + m^2 c^4} \) in which we put the zero rest mass of photon. Only moving photon has mass as follows from the Einstein formula \( E = mc^2 \). Mass of the moving photon is \( m_\gamma = \hbar \omega/c^2 \). A non-zero photon mass would have several implications, such as a frequency-dependent speed of light and the existence of longitudinal electromagnetic waves. Photon with the nonzero rest mass is evidently in contradiction with special relativity. Arnold Sommerfeld (1954), who first considered superluminal velocities and theoretically discovered the Čerenkov effect, wrote no remark on the massive photons in his famous Optics.

If we suppose that the momentum of massive photon is \( p = \hbar \omega/c \), then from the Einstein formula follows that the energy of massive photon is \( E = \sqrt{\hbar^2 \omega^2 + m^2 c^4} \).

The corresponding Planck formula for the density \( P(\omega) \) of the black body radiation is as follows:

\[ P(\omega) = \left( \frac{\omega^2}{\pi^2 c^3} \right) \frac{E}{e^{E/kT} - 1}; \quad E = \sqrt{\hbar^2 \omega^2 + m^2 c^4}, \]

where we used the frequency \( \omega \) instead of the momentum of photon, because the frequency is used in experiment and not momentum of photon. The massless limit of the formula (3) is the original Planck law. We show later that \( E \) given by eq. (3) is in harmony with the quantum definition of massive photon. The quantity \( c' \) is the velocity of photons inside the black body and it must be involved in the number of electromagnetic modes inside the blackbody. We can put approximately \( c' \approx c \). To our knowledge there is no experimental evidence that the modified Planck law is correct. It means that massive photons cannot be involved into the theory of the black body radiation. To our knowledge the precise measurement of the the anomalous magnetic moment of electron and Lamb shift agree with QED formulas with zero photon rest mass. On the other hand, if photons are moving in electromagnetic field, then they have nonzero rest mass (Ritus, 1969). This mass is complex quantity, while we will consider here only real quantity. It follows from the polarization operator in external fields. This operator substantially differs from the operator in the dielectric medium (Pardy, 1994), where the fundamental role plays the
index of refraction. Polarization of vacuum can be determined also by the source theory methods (Dittrich, 1978). The photon mass following from the vacuum polarization is not generated by the Higgs mechanism, or by the Schwinger mechanism. This mass is of the dynamical origin corresponding to the radiative corrections. To our knowledge, the experiments with the black body radiation in magnetic or electric field was never performed.

The formal introduction of the rest mass of photon exist in quantum electrodynamics, where for instance the processes with soft photons are calculated. In these calculations the photon mass is introduced in order to avoid the infrared divergences (Berestetzkii et al. 1989).

We shall see later that introducing the nonzero photon mass modifies Coulomb law. Such modification is discussed in literature (Okun, 1981). It is evident that massive photons play crucial role in gravity. However, this problem was not discussed in the prestige articles (Okun, 2002).

On the other hand, the possibility that photon may be massive particle has been treated by many physicists. The discussion is also devoted to the existence of the mass of neutrino and its oscillations, which can form some analogue with the photons with the same importance. The established fact is that the massive electrodynamics is a perfectly consistent classical and quantum field theory (Feldmann et al., 1963; Minkowski et al., 1971; Goldhaber et al., 1971). In all respect the quantum version has the same status as the standard QED. In this article we do not solve the radiative problems in sense of article by Nieuwenhuizen (1973). Our goal is to determine the Volkov solution of the Dirac equation with massive photons. The resulting equation is the Riccati equation which cannot be solved in general. So, we derive only some approximative formulas.

In particle physics and quantum field theory (Ryder, 1985; de Wit et al., 1986; Commins et al., 1983) photon is defined as a massless particle with spin 1. Its spin is along or in opposite direction to its motion. The massive photon as a neutral massive particle is usually called vector boson. The equation for vector boson was derived in the unified theory of the electro-weak interaction. There are other well known examples of massive spin-1 particles. For instance neutral $\rho$-meson, $\varphi$-meson and $J/\psi$ particle, bosons $W^\pm$ and $Z^0$ in particle physics.

While massless photon is described by the Maxwell Lagrangian, the massive photon is described by the Proca Lagrangian from which the field equations follow. The massive electrodynamics can be considered as a generalization of massless electrodynamics. The well known area where the massive photon or boson plays substantial role is the theory of superconductivity (Ryder, 1985), plasma physics (Anderson, 1963), waveguides and so on. Of course the mass of photon is not the relativistic vacuum rest mass, but effective mass which is generated by the physical properties of medium, or by some mechanism such as the Higgs mechanism, Schwinger mechanism and so on. In this sense, the physics of massive photon is meaningful, the generalized Volkov solution of the Dirac equation with massive photon field is physically meaningful too and it is worthwhile to investigate problems with the massive photons.

In order to be pedagogically clear, following author’s article (Pardy, 2002), we treat, in section 2, the massive spin 1 quantum field theory and in section 3, we derive the Volkov solution of the Dirac equation for massless photon field. In section 4, we find
the Riccati equation which involves mass of photon. Then we discuss the emission of massive photons by electron in the periodic and δ-form electromagnetic field. We derive generalized Compton formulas for interaction of the multiphotonic object with electron.

2 Massive fields with spin 1

If spin zero particles and fields are described by the scalar source, then a vector source denoted here as $J^\mu(x)$ can be considered as a candidate for the description of the spin 1 fields and particles. However, there exist some obstacles because source $J^\mu(x)$ has four components and spin one particles have only three spin possibilities.

It was shown by Schwinger (1970) that the action of the spin 1 massive particles is as follows:

$$W(J) = \frac{1}{2} \int (dx)(dx') \left\{ J_\mu(x) \Delta_+(x - x') J^\mu(x') + \frac{1}{m^2} \partial_\mu J^\mu(x) \Delta_+(x - x') \partial'_\nu J^\nu(x') \right\}. \quad (4)$$

The field of spin one particles can be defined using the definition of the test source $\delta J^\mu(x)$ by the relation

$$\delta W(J) = \int (dx) \delta J^\mu(x) \varphi_\mu(x), \quad (5)$$

where $\varphi_\mu$ is the field of particles with spin 1. After performing variation of the formula (4) and comparison with eq. (5) we get the equation for field of spin 1 in the following form:

$$\varphi_\mu(x) = \int (dx') \Delta_+(x - x') J_\mu(x') - \frac{1}{m^2} \partial_\mu \int (dx') \Delta_+(x - x') \partial'_\nu J^\nu(x'). \quad (6)$$

The divergence of the vector field $\varphi_\mu(x)$ is given by the relation

$$\partial_\mu \varphi^\mu(x) = \int (dx') \Delta_+(x - x') \partial'_\mu J^\nu(x') - \frac{1}{m^2} \partial^2 \int (dx') \Delta_+(x - x') \partial'_\nu J^\nu(x') =$$

$$\frac{1}{m^2} \partial_\mu J^\mu(x), \quad (7)$$

as a consequence of scalar equation

$$-\partial^2 \Delta_+ = \delta(x - x') - m^2 \Delta_+. \quad (8)$$

Further, we have after applying operator $(-\partial^2 + m^2)$ on the equation (6) the following equations:

$$(-\partial^2 + m^2) \varphi_\mu(x) = J_\mu(x) - \frac{1}{m^2} \partial_\mu \partial'_\nu J^\nu(x), \quad (9)$$

$$(-\partial^2 + m^2) \varphi_\mu(x) + \partial_\mu \partial'_\nu \varphi^\nu(x) = J_\mu(x), \quad (10)$$
as a consequence of eq. (7).

It may be easy to cast the last equation into the following form

\[ \partial^\nu G_{\mu\nu} + m^2 \varphi_\mu = J_\mu, \]  

(11)

where

\[ G_{\mu\nu}(x) = -G_{\nu\mu}(x) = \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu. \]

(12)

Identifying \( G_{\mu\nu} \) with \( F_{\mu\nu} \) of the electromagnetic field we get instead of eq. (10) and eq. (11) so-called the Proca equation for the electromagnetic field with the massive photon.

\[ (-\partial^2 + m^2) A_\mu(x) + \partial_\mu \partial_\nu A^\nu(x) = J_\mu(x), \]

(13)

\[ \partial^\nu F_{\mu\nu} + m^2 A_\mu = J_\mu, \]

(14)

\[ F_{\mu\nu}(x) = -F_{\nu\mu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu. \]

(15)

In case \( m^2 \neq 0 \), we can put \( \partial_\mu A_\mu = 0 \) in order to get:

\[ (-\partial^2 + m^2) A_\mu(x) = 0, \quad \partial_\mu A_\mu = 0. \]

(16)

The solution of the system (16) is the plane wave

\[ A_\mu = \varepsilon_\mu(k)e^{ikx}, \quad k^2 = -m^2 \]

(17)

with \( k\varepsilon(k) = 0 \), which is precisely the correct definition of a massive particle with spin 1. Later, we will see how to generalize this procedure to the situation of the massive electrodynamics in dielectric and magnetic media.

The equation (11) can be derived also from the action

\[ W = \int (dx) (J^\mu(x)\varphi_\mu(x) + \mathcal{L}(\varphi(x))), \]

(18)

where

\[ \mathcal{L} = -\frac{1}{2} \left( \frac{1}{2} \left( \partial^\nu \varphi^\mu - \partial^\mu \varphi^\nu \right) (\partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu) + m^2 \varphi^\mu \varphi_\mu \right), \]

(19)

where we have used the arrangement

\[ \int (dx) \varphi^\mu (-\partial^2) \varphi_\mu = \int (dx) \partial^\nu \varphi^\mu \partial_\nu \varphi_\mu \]

(20)

and

\[ \int (dx) \varphi^\mu \partial_\mu \partial^\nu \varphi_\nu = - \int (dx) \varphi^\nu \partial^\mu \partial_\mu \varphi_\nu = - \int (dx) \varphi_\mu \partial^\mu \partial^\nu \varphi_\nu. \]

(21)

Using the last equation (21) we get the Lagrange function in the following standard form:

\[ \mathcal{L} = -\frac{1}{2} \left( \partial^\nu \varphi^\mu \partial_\nu \varphi_\mu - (\partial_\mu \varphi^\mu)^2 + m^2 \varphi^\mu \varphi_\mu \right). \]

(22)
If we use the A- and F-symbols, we receive from eq. (19) the Proca Lagrangian

$$\mathcal{L} = -\frac{1}{2} \left( \frac{1}{2} F^{\mu \nu} F_{\mu \nu} + m^2 A^\mu A_\mu \right),$$  \hspace{1cm} (23)

or,

$$\mathcal{L} = -\frac{1}{2} \left( \partial^\nu A^\mu \partial_\nu A_\mu - (\partial_\mu A^\mu)^2 + m^2 A^\mu A_\mu \right).$$  \hspace{1cm} (24)

By variation of the corresponding Lagrangian for the massive field with spin 1 we get evidently the massive Maxwell equations.

It is evident that the zero mass limit does not exist for $\partial_\mu J^\mu(x) \neq 0$. In such a way we are forced to redefine action $W(J)$. One of the possibilities is to put

$$\partial_\mu J^\mu(x) = mK(x)$$  \hspace{1cm} (25)

and identify $K(x)$ in the limit $m \to 0$ with the source of massless spin zero particles. Since the zero mass particles with zero spin are experimentally unknown in any event, we take $K(x) = 0$ and we write

$$W_{[m=0]}(J) = \frac{1}{2} \int (dx)(dx') J_\mu(x) D_+(x - x') J^\mu(x'),$$  \hspace{1cm} (26)

where

$$\partial_\mu J^\mu(x) = 0$$  \hspace{1cm} (27)

and

$$D_+(x - x') = \Delta_+(x - x'; m = 0).$$  \hspace{1cm} (28)

In case we want to work with electrodynamics in medium it is necessary to involve such parameters as velocity of light $c$, magnetic permeability $\mu$ and the dielectric constant $\varepsilon$. Then the corresponding equations for electromagnetic potentials which are compatible with the Maxwell equations are as follows (Schwinger et al., 1976):

$$\left( \Delta - \frac{\mu \varepsilon}{c^2} \frac{\partial^2}{\partial t^2} + \frac{n^2 c^2}{\hbar^2} \right) A^\mu = \frac{\mu}{c} \left( g^{\mu \nu} + \frac{n^2 - 1}{n^2} \eta^\mu \eta^\nu \right) J_\nu,$$  \hspace{1cm} (29)

where the corresponding Lorentz gauge is defined in the Schwinger et al. (1976) article in the following form

$$\partial_\mu A^\mu - (\mu \varepsilon - 1)(\eta \partial)(\eta A) = 0,$$  \hspace{1cm} (30)

where $\eta^\mu = (1, 0)$ is the unit time-like vector in the rest frame of the medium. The four-potentials are $A^\mu(\phi, \mathbf{A})$ and the four-current $J^\mu(c\varphi, \mathbf{J})$, $n$ is the index of refraction of this medium.

So, the massive electrodynamics in medium can be constructed by generalization of massless electrodynamics to the case with massive photon.

In superconductivity, photon is a massive spin 1 particle as a consequence of a broken symmetry of the Landau-Ginzburg Lagrangian. The Meissner effect can be used as an experimental demonstration that photon in a superconductor is a massive particle. In particle physics the situation is analogous to the situation in superconductivity. The
masses of particles are also generated by the broken symmetry or in other words by the Higgs mechanism. Massive particles with spin 1 form the analogue of the massive photon.

Kirzhnitz and Linde (1972) proposed a qualitative analysis wherein they indicated that, as in the Landau-Ginzburg theory of superconductivity, the Meissner effect can also be realized in the Weinberg model. Later, it was shown that the Meissner effect is realizable in renormalizable gauge fields and also in the Weinberg model (Yildiz, 1977).

While the photon propagator $D(k)$ in the momentum representation in the massless electrodynamics is

$$D(k) = \frac{1}{|k|^2 - n^2(k^0)^2 - i\epsilon}, \quad (31)$$

the massive photon propagator is of the form (here we introduce $\hbar$ and $c$):

$$D(k, m^2) = \frac{1}{|k|^2 - n^2(k^0)^2 + \frac{m^2 c^2}{\hbar^2} - i\epsilon}, \quad (32)$$

where this propagator is derived from an assumption that the photon energetic equation is

$$|k|^2 - n^2(k^0)^2 = -\frac{m^2 c^2}{\hbar^2}, \quad (33)$$

where $n$ is the parameter of the medium and $m$ is mass of photon in this medium.

From eq. (33) the dispersion law for the massive photons follows:

$$\omega = \frac{c}{n} \sqrt{k^2 + \frac{m^2 c^2}{\hbar^2}}. \quad (34)$$

Let us remark here that such dispersion law is valid not only for the massive photon but also for electromagnetic field in waveguides and electromagnetic field in ionosphere. It means that the corresponding photons are also massive and the theory of massive photons is physically meaningful.

The validity of eq. (33) can be verified using very simple idea that for $n = 1$ the Einstein equation for mass and energy has to follow. Putting $p = \hbar k$, $\hbar k^0 = \hbar (\omega/c) = (E/c)$, we get the Einstein energetic equation

$$E^2 = p^2 c^2 + m^2 c^4. \quad (35)$$

The propagator for the massive photon is then derived as

$$D_+(x - x', m^2) = \frac{i}{c 4\pi^2} \int_0^\infty d\omega \frac{\sin^2[n^2\omega^2 / c^2 - m^2 c^2 / \hbar^2]^{1/2} |x - x'| e^{-i\omega|t - t'|}}{|x - x'|}. \quad (36)$$

The function (36) differs from the the original function $D_+$ by the factor

$$\left(\frac{\omega^2 n^2}{c^2} - \frac{m^2 c^2}{\hbar^2}\right)^{1/2}. \quad (37)$$

From eq. (36) the potentials generated by the massless or massive photons respectively follow. In case of the massless photon, the potential is according to Schwinger defined by the formula ($m = 0$):
\[ V(x - x') = \int_{-\infty}^{\infty} d\tau D_+(x - x', \tau) = \int_{-\infty}^{\infty} d\tau \left\{ \frac{i}{c} \frac{1}{4\pi^2} \int_{0}^{\infty} d\omega \frac{\sin \frac{\omega}{c} |x - x'|}{|x - x'|} e^{-i\omega |\tau|} \right\}. \] (38)

The \( \tau \)-integral can be evaluated using the mathematical formula

\[ \int_{-\infty}^{\infty} d\tau e^{-i\omega |\tau|} = \frac{2}{i\omega} \] (39)

and the \( \omega \)-integral can be evaluated using the formula

\[ \int_{0}^{\infty} \sin ax \frac{dx}{x} = \frac{\pi}{2}, \quad \text{for} \quad a > 0. \] (40)

After using eqs. (39) and (40), we get

\[ V(x - x') = \frac{1}{c} \frac{1}{4\pi} \frac{1}{|x - x'|}. \] (41)

In case of the massive photon, the mathematical determination of potential is the analogical to the massless situation only with the difference we use the propagator (36) and the table integral (Gradshteyn et al., 1965)

\[ \int_{0}^{\infty} \frac{dx}{x} \sin \left( \frac{p\sqrt{x^2 - u^2}}{c} \right) = \frac{\pi}{2} e^{-pu}. \] (42)

Using this integral we get that the potential generated by the massive photons is

\[ V(x - x', m^2) = \frac{1}{c} \frac{1}{4\pi} \frac{1}{|x - x'|} \exp \left\{ -\frac{mc}{\hbar} |x - x'| \right\}. \] (43)

So, we see that in case of the massive photons, the potential generated by the massive particle is of the Yukawa form.

### 3 Volkov solution of the Dirac equation with massless photons

Let us remember the derivation of the Volkov (1935) solution of the Dirac equation in vacuum (we use here the method of derivation and metric convention of Berestetzki et al. (1989)): \[
(\gamma(p - eA) - m)\Psi = 0. \] (44)

where

\[ A^\mu = A^\mu(\phi); \quad \phi = kx. \] (45)

We suppose that the four-potential satisfies the Lorentz gauge condition

\[ \partial_\mu A^\mu = k_\mu (A^\mu)' = (k_\mu A^\mu)' = 0, \] (46)

where the prime denotes derivation with regard to \( \phi \). From the last equation follows
\[ kA = \text{const} = 0, \]  
(47)
because we can put the constant to zero. The tensor of electromagnetic field is

\[ F_{\mu\nu} = k_{\mu}A^{\prime}_{\nu} - k_{\nu}A^{\prime}_{\mu}. \]  
(48)

Instead of the linear Dirac equation (44) we consider the quadratical equation, which we get by multiplication of the linear equation by operator \( (\gamma(p - eA) + m) \), (Berestetzkii et al., 1989). We get:

\[
\left[ (p - eA)^2 - m^2 - \frac{i}{2} eF_{\mu\nu}\sigma^{\mu\nu} \right] \psi = 0. 
\]  
(49)

Using \( \partial_{\mu}(A^\mu\psi) = A^\mu\partial_{\mu}\psi \), which follows from eq. (46), and \( \partial_{\mu}\partial^{\mu} = \partial^2 = -p^2 \), with \( p_{\mu} = i(\partial/\partial x^\mu) = i\partial_{\mu} \), we get the quadratical Dirac equation for the four potential of the plane wave:

\[
[ -\partial^2 - 2i(A\partial) + e^2A^2 - m^2 - ie(\gamma k)(\gamma A^\prime) ] \psi = 0. 
\]  
(50)

We are looking the solution of the last equation in the form:

\[ \psi = e^{-ipx}F(\varphi). \]  
(51)

After insertion of this equation into (50), we get with \( (k^2 = 0) \)

\[
\partial^{\mu}F = k^{\mu}F', \quad \partial_{\mu}\partial^{\mu}F = k^2F'' = 0, 
\]  
(52)
the following equation for \( F(\varphi) \)

\[
2i(kp)F' + [ -2e(pA) + e^2A^2 - ie(\gamma k)(\gamma A^\prime) ] F = 0. 
\]  
(53)

The integral of the last equation is of the form:

\[
F = \exp \left\{ -i \int_0^{kx} \left[ \frac{e(pA)}{(kp)} - \frac{e^2}{2(kp)}A^2 \right] d\varphi + \frac{e(\gamma k)(\gamma A)}{2(kp)} \right\} \frac{u}{\sqrt{2p_0}}, 
\]  
(54)
where \( u/\sqrt{2p_0} \) is the arbitrary constant bispinor.

All powers of \( (\gamma k)(\gamma A) \) above the first are equal to zero, since

\[
(\gamma k)(\gamma A)(\gamma k)(\gamma A) = -\gamma k(\gamma k)(\gamma A)(\gamma A) + 2(kA)(\gamma k)(\gamma A) = -k^2A^2 = 0. \]  
(55)

Then we can write:

\[
\exp \left\{ \frac{e(\gamma k)(\gamma A)}{2(kp)} \right\} = 1 + \frac{e(\gamma k)(\gamma A)}{2(kp)}. \]  
(56)

So, the solution is of the form:

\[
\Psi_p = R \frac{u}{\sqrt{2p_0}} e^{iS} = \left[ 1 + \frac{e}{2kp}(\gamma k)(\gamma A) \right] \frac{u}{\sqrt{2p_0}} e^{iS}, \]  
(57)
where \( u \) is an electron bispinor of the corresponding Dirac equation

\[
(\gamma p - m)u = 0. \quad (58)
\]

The mathematical object \( S \) is the classical Hamilton-Jacobi function, which was determined in the form:

\[
S = -px - \int_0^{kx} \frac{e}{kp} \left[ (pA) - \frac{e}{2} (A^2) \right] d\varphi. \quad (59)
\]

The current density is

\[
\dot{j} = \bar{\Psi} p \gamma^\mu \Psi_p, \quad (60)
\]

where \( \bar{\Psi} \) is defined as the transposition of (57), or,

\[
\bar{\Psi}_p = \frac{\bar{u}}{\sqrt{2p_0}} \left[ 1 + \frac{e}{2kp} (\gamma A)(\gamma k) \right] e^{-iS}. \quad (61)
\]

After insertion of \( \Psi_p \) and \( \bar{\Psi}_p \) into the current density, we have:

\[
\dot{j} = \frac{1}{p_0} \left\{ p^\mu - eA^\mu + k^\mu \left( \frac{e(pA)}{(kp)} - \frac{e^2 A^2}{2(kp)} \right) \right\} + k^\mu i e \frac{F_{\alpha\beta}}{8(kp)p_0} F_{\alpha\beta}(u^* \sigma^{\alpha\beta} u), \quad (62)
\]

which is in agreement with formula in the Meyer article (1971).

The so called kinetic momentum corresponding to \( \dot{j} \) is as follows:

\[
J^\mu = \bar{\Psi}_p(p^\mu - eA^\mu) \Psi_p = \bar{\Psi}_p \gamma^0 (p^\mu - eA^\mu) \Psi_p =
\]

\[
\left\{ p^\mu - eA^\mu + k^\mu \left( \frac{e(pA)}{(kp)} - \frac{e^2 A^2}{2(kp)} \right) \right\} + k^\mu \frac{ie}{8(kp)p_0} F_{\alpha\beta}(u^* \sigma^{\alpha\beta} u), \quad (63)
\]

where

\[
\sigma^{\alpha\beta} = \frac{1}{2} (\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha). \quad (64)
\]

4 Volkov solution of the Dirac equation for massive photons

The original Volkov solution is based on the assumption that photon has zero rest mass, or, \( k^2 = 0 \). Our goal is to consider the solution of the Dirac equation in case that \( k^2 = M^2 \), where \( M \) is the rest mass of photon. We use here the metrical notation of Berestetzkii et al. (1989).

We apply the procedure of the preceding section for the case of the massive photon, and we write:

\[
\psi = e^{-ipx} F(\varphi), \quad (65)
\]

where for \( F \) we get the following equation
\[ M^2 F'' - 2i(kp)F' + G(\varphi)F = 0 \]  
with

\[ G(\varphi) = 2e(pA) - e^2 A^2 + ie(\gamma k)(\gamma A'). \]  

The equation (66) differs from the original Volkov equation only by means of the massive term. However, the equation is substantially new, because of the second derivative of the function \( F \). The solution of the last equation can be easily obtained in the approximative form in case that \( M \to 0 \). However, let us try to find the exact solution, which was not described, to our in physical or mathematical journals.

In order to find such solution, we transcribe this equation in the form:

\[ F'' + aF' + bF = 0, \]

where

\[ a = -\frac{2i(kp)}{M^2}, \quad b(\varphi) = \frac{G(\varphi)}{M^2}. \]

Using the substitution

\[ F = v(\varphi)e^{-\frac{1}{2}a\varphi}, \]

we get simple equation for \( v(\varphi) \):

\[ v'' + P(\varphi)v = 0, \]

where

\[ P(\varphi) = \left(-\frac{a^2}{4} + b\right). \]

Using the substitution

\[ v(\varphi) = e^{\int_0^\varphi T(\varphi)d\varphi}, \]

we get from eq. (71)

\[ T' + T^2 + P(\varphi) = 0. \]

Equation (74) is so called Riccati equation. The mass term is hidden in \( P(\varphi) \). It is well know that there is no general form of solution of this equation. There is only some solution expressed in the elementary functions for some specific functions \( P(\varphi) \). Nevertheless, there is interesting literature concerning the Riccati equation. For instance, Riccati equation is applied in the supersymmetric quantum mechanics (Cooper et al., 1995), in variational calculus (Zelekin, 1998), nonlinear physics, (Matveev et al., 1991), in renormalization group theory (Buchbinder et al., 1992; Milton et al., 2001) and in thermodynamics (Rosu et al., 2001).

With regard to circumstances, we are forced to find some approximative solution with form similar to the original Volkov solution. Let us show the derivation of such the
approximative solution. We hope it will play the same role in quantum electrodynamics with the massive photon as in the case with the massless photon.

There are many approximative methods for solution of this problem. We choose the elementary method which was also applied to the Schrödinger equation and which is described for instance in the monograph of Mathews et al. (1964).

The approximation consists at the application of the following inequalities:

$$|F''(\varphi)| \ll |F'(\varphi)|; \quad |F'''(\varphi)| \ll |F(\varphi)|. \quad (75)$$

Then, we get the original Volkov solution with the difference that the existence of the nonzero photon mass will be involved only in the exponential expansion. Or, with $U = e(\gamma k)(\gamma A)/2(kp)$, we perform the expansion:

$$e^U = \left\{ 1 + \frac{1}{1!} U + \frac{1}{2!} U^2 + \frac{1}{3!} U^3 + ... \right\} =$$

$$\left\{ 1 + \frac{e}{2(kp)}(\gamma k)(\gamma A) + \frac{1}{2!} \left( \frac{e}{2(kp)} \right)^2 (-M^2A^2) + \frac{1}{3!} \left( \frac{e}{2(kp)} \right)^3 (\gamma k)(\gamma A)(-M^2A^2) + \frac{1}{4!} \left( \frac{e}{2(kp)} \right)^4 (M^4A^4) + ... \right\}, \quad (76)$$

where we have used equation (55) in the modified form

$$(\gamma k)(\gamma A)(\gamma k)(\gamma A) = -(\gamma k)(\gamma k)(\gamma A)(\gamma A) + 2(kA)(\gamma k)(\gamma A) = -k^2A^2 = -M^2A^2. \quad (77)$$

with $k^2 = M^2$ for massive photons. We see that in this method of approximation the Massive solution involves the Volkov solution as the basic term and then the additional terms containing photon mass.

After performing some algebraic operations, we get the first approximation of the Volkov solution with the massive photon in the following form

$$\Psi_p = R(A, M^2) \frac{u}{\sqrt{2p_0}} e^{iS} = \left[ 1 + \frac{e}{2kp} (\gamma k)(\gamma A) - \left( \frac{e}{2(kp)} \right)^2 M^2A^2 + ... \right] \frac{u}{\sqrt{2p_0}} e^{iS}. \quad (78)$$

Now, we are prepared to solve some physical problems with the Volkov solution with massive photons.

5 Emission of massive photons by electron moving in the periodic field

Let us consider the monochromatic circularly polarized electromagnetic wave with the four potential

$$A = a_1 \cos \varphi + a_2 \sin \varphi; \quad a_3 = 0; \quad \varphi = kx \quad (79)$$
with \( k^\mu = (\omega, \mathbf{k}) \) being a wave 4-vector and \( k^2 = M^2 \), the 4-amplitudes \( a_1 \) and \( a_2 \) are the same and one another perpendicular, or

\[
a_1^2 = a_2^2 = a^2; \quad a_1 a_2 = 0. \tag{80}
\]

We shall also use the Lorentz gauge condition, which gives \( a_1 k = a_2 k = 0 \).

The wave function is then of the form:

\[
\psi_p = \left\{ 1 + \frac{e}{2(kp)} \left[ (\gamma k)(\gamma a_1) \cos \varphi + (\gamma k)(\gamma a_2) \sin \varphi - \frac{e}{2(kp)} a^2 M^2 + \ldots \right] \right\} \frac{u(p)}{\sqrt{2q_0}} \times \exp \left\{ -ie \frac{a_1 p}{kp} \sin \varphi + ie \frac{a_2 p}{kp} \cos \varphi - iq \right\}, \tag{81}
\]

where

\[
q^\mu = p^\mu - e^2 \frac{a^2}{2(kp)} (k^\mu) \tag{82}
\]
is the time-averaged value of the eq. (62).

The corresponding matrix element is of the obligate form (Berestetzii et al., 1989).

After performing the appropriate mathematical operation we get the \( \delta \)-function in the matrix element, from which the conservations laws follow in the form

\[
sk + q = q' + k' \tag{83}
\]

The interpretation of this formula is as follows: \( s \) massive photons with momentum \( k \) are absorbed by electron with momentum \( q \) and only one massive photon is emitted with the 4-vector \( k' \), and the final momentum of electron is \( q' \). So, we see that the Volkov solution gives the multiphoton processes, which are intensively studied in the modern physics (Delone et al., 2000).

For the periodic wave it is

\[
q^2 = q'^2 = m^2_\ast; \quad m_\ast = m \sqrt{1 + \frac{e^4 a^4 M^2}{(2kp)^2 m^2} - \frac{e^2}{m^2 a^2}} \tag{84}
\]

which can be interpreted as a mass shift of electron in the periodic field, or, the mass renormalization.

If we consider an electron at a rest (\( q = 0, q_0 = m_\ast \)), then from the formula (82), (83) and (84) follows

\[
(s^2 + 1) \frac{M^2}{2m_\ast \omega} \frac{1}{\omega'} + s \frac{1}{\omega} - \frac{1}{\omega} = \frac{s}{m_\ast} (1 - \cos \Theta); \quad s = 1, 2, 3, \ldots n. \tag{85}
\]

The massless limit of the last formula is the well known Compton formula (with \( M = 0 \))

\[
\omega' = \frac{s \omega}{1 + \frac{s \omega}{m_\ast} (1 - \cos \Theta)}, \tag{86}
\]

13
where $\theta$ is an angle between $k$ and $k'$. So we see that frequencies $\omega'$ are harmonic frequencies of $\omega$.

6 Emission of massive photons by electron moving in the impulsive force

We use the $\delta$-function form of the ultrashort laser pulse (Pardy, 2003)

$$A_\mu = a_\mu \eta(\varphi),$$  \hspace{1cm} (87)

where $\eta(\varphi)$ is the Heaviside unit step function defined as follows: $\eta(\varphi) = 0; \varphi < 0$, and $\eta(\varphi) = 1; \varphi \geq 0$. Then, the function $S$ and $R$ in the Volkov solution $\Psi_p$ are as follows (Pardy, 2003):

$$S = -px - \left[ e \frac{a p}{k p} - \frac{e^2}{2 k p} a^2 \right] \varphi, \quad R = \left[ 1 + \frac{e}{2 k p} (\gamma k)(\gamma a) \eta(\varphi) + \ldots \right].$$  \hspace{1cm} (88)

So, we get the matrix element in the form:

$$M = g \int d^4 x \bar{\Psi}_{p'} O \Psi_p e^{ik'x} \sqrt{2\omega'},$$  \hspace{1cm} (89)

where $O = \gamma e^{\mu}$, $g = -ie^2$ in case of the electromagnetic interaction and

$$\bar{\Psi}_{p'} = \frac{\bar{u}(p')}{\sqrt{2p'_0}} \bar{R}(p') e^{-iS(p')}.$$  \hspace{1cm} (90)

In such a way, using above definitions, we write the matrix element in the form:

$$M = \frac{g}{\sqrt{2\omega'}} \frac{1}{\sqrt{2p'_0 2p_0}} \int d^4 x \bar{R}(p') OR(p)e^{-iS(p') + iS(p)} e^{ik'x}.$$  \hspace{1cm} (91)

The quantity $\bar{R}(p')$ follows immediately from eq. (87), namely:

$$\bar{R}' = \left[ 1 + \frac{e}{2 k p'} (\gamma k)(\gamma a) \eta(\varphi) + \ldots \right] = \left[ 1 + \frac{e}{2 k p'} (\gamma a)(\gamma k) \eta(\varphi) + \ldots \right].$$  \hspace{1cm} (92)

Using

$$-iS(p') + iS(p) = i(p' - p) + i(\alpha' - \alpha) \varphi,$$  \hspace{1cm} (93)

where

$$\alpha = \left( e \frac{a p}{k p} - \frac{e^2}{2 k p} a^2 \right), \quad \alpha' = \left( e \frac{a p'}{k p'} - \frac{e^2}{2 k p'} a^2 \right).$$  \hspace{1cm} (94)

we get:

$$M = \frac{g}{\sqrt{2\omega'}} \frac{1}{\sqrt{2p'_0 2p_0}} \int d^4 x \bar{u}(p') \bar{R}(p') OR(p)u(p)e^{i(p' - p)x} e^{i(\alpha' - \alpha)\varphi} e^{ik'x}.$$  \hspace{1cm} (95)
We get after $x$-integration:

$$M = \frac{g}{\sqrt{2\omega'}} \frac{1}{\sqrt{2p_0'2p_0}} \bar{u}(p') R(p') OR(p) u(p) \delta^{(4)}(kl + p - k' - p'). \quad (96)$$

We see from the presence of the $\delta$-function in eq. (96) that during the process of the interaction of electron with the laser pulse the energy-momentum conservation law holds good:

$$lk + p = k' + p'; \quad l = \alpha - \alpha'. \quad (97)$$

The last equation describes the so called multiphoton process, which can be also described using Feynman diagrams and which are studied in the different form intensively in the modern physics of multiphoton ionization of atoms (Delone et al, 2000; Pardy, 2003).

If we introduce the angle $\Theta$ between $k$ and $k'$, then, with $|k| = \omega$ and $|k'| = \omega'$, we get from the squared equation (97) in the rest system of electron, where $p = (m_*, 0)$, the following equation $k = (\omega, k)$:

$$(l^2 + 1) \frac{M^2}{2m_\omega \omega'} + l \frac{1}{\omega'} - \frac{1}{\omega} = \frac{l}{m_\omega} (1 - \cos \Theta); \quad l = \alpha - \alpha', \quad (98)$$

which is modification of the original equation for the Compton process

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m} (1 - \cos \Theta). \quad (99)$$

We see that the substantial difference between single photon interaction and $\delta$-pulse interaction is the factor $l = \alpha - \alpha'$.

We know that the last formula of the original Compton effect can be written in the form suitable for the experimental verification, namely:

$$\Delta \lambda = 4\pi \frac{\hbar}{mc} \sin^2 \frac{\Theta}{2}. \quad (100)$$

which was used by Compton for the verification of the quantum nature of light (Rohlf, 1994).

Let us remark, the equation $lk + p = k' + p'$ is the symbolic expression of the nonlinear Compton effect and it concerns only the situation where $l$ photons are absorbed at a single point, and it does not describe the process where electron scatters twice, or more, as it traverses the laser focus. The nonlinear Compton process was experimentally confirmed (Bulla et al., 1996).

### 7 Discussion

The present article is continuation of the author discussion on laser interaction with electrons (Pardy, 1998, 2001), where the Compton model of laser acceleration was proposed and author article (Pardy, 2003), where the $\delta$-form laser pulse was considered.

The $\delta$-form laser pulses are the idealization of the experimental situation in laser physics. It was demonstrated theoretically that at present time the zeptosecond and
subzeptosecond laser pulses of duration $10^{-21} - 10^{-22}$ s can be realized by the petawatt lasers (Kaplan et al., 2002). It means that the generation of the ultrashort laser pulses is the keen interest in development of laser physics.

We have derived modified Compton formulas which involve multiphoton interaction of laser beam with electron. In case of the periodic field, the multiplicity is formed by the natural numbers and in case of the $\delta$-pulse, by number $l = \alpha - \alpha'$. This effect can be interpreted in such a way that the photonic object with s or l photons interacts simultaneously with one electron. We do not think that the photonic object is consequence of the Bose-Einstein condensation of photons in laser beam. It behaves as photonic elementary object and probably it can be used in the experiments in particle physics.

The Volkov solution of the Dirac equation for electromagnetic potential with massive photons concerns not only the superconductive medium but also the electron-positron plasma, ionosphere medium, photons in waveguides, or massive photons generated hypothetically during inflation (Prokopec et al., 2003).

The bosons $W^\pm$ and $Z^0$ are also massive and it means that the generalization of our approach to the situation in the standard model is evidently feasible. The vector mesons $\rho, \phi, J/\psi$ are generated during the nuclear collisions and probably, the Volkov solution for these massive vector particles will play substantial role in the nuclear physics.

References

Anderson, P. W. (1963). Plasmons, gauge invariance, and mass, Physical Review 130(1), 439.

Berestetzkii, V. B., Lifshitz, E. M. and Pitaevskii, L. P. (1989). Quantum Electrodynamics, Moscow, Nauka. (in Russian).

Bulla, C. et al. (1996). Observation of nonlinear effects in Compton scattering, Physical Review Letter 76, 3116.

Buchbinder, I. L., Odintsov, S. D. and Shapiro, I. L. (1992). Effective Action in Quantum Gravity, IOP Publishing Ltd.

Commins, E. D. and Bucksbaum, P. H. (1983). Weak Interactions of Leptons and Quarks, Cambridge University Press, Cambridge.

Cooper, F., Khare, A. and Sukhatme, U. (1995). Supersymmetry and quantum mechanics, Physics Reports 251, 267.

Delone, N. B. and Krainov, V. P. (2000). Multiphoton Processes in Atoms, 2nd ed., Springer-Verlag, Berlin, Heidelberg, New York.

de Wit B. and Smith, J. (1986). Field Theory in Particle Physics, Vol. I, Elsevier.

Dittrich, W. (1978). Source methods in quantum field theory, Fortschrritte der Physik 26, 289.

Feldman, G. and Mathews, P. T. (1963). Massive electrodynamics, Physical Review 130, 1633.

Goldhaber, A. S. and Nieto, M. M. (1971). Terrestrial and extraterrestrial limits on the
photon mass, Review of Modern Physics 43, 277.

Gradshteyn, I. S. and Ryzhik, I. M. (1965). Table of Integrals, Series and Products Academic Press, New York.

Kaplan, A. E. and Shkolnikov, P. L. (2002). Lasetron: A proposed source of powerful nuclear-time-scale electromagnetic bursts, Physical Review Letter 88(7), 074801.

Kirzhnitz, A. D. and Linde, A. D. (1972). Macroscopic consequences of the Weinberg model, Physics Letters 42B, 471.

Matveev, V. B. and Salle, M. A. (1991). Darboux Transformations and Solitons Springer, Berlin.

Mathews, J. and Walker, R. L. (1964). Mathematical Methods of Physics W. B. Benjamin, Inc., New York – Amsterdam.

Meyer, J. W. (1971). Covariant classical motion of electron in a laser beam, Physical Review D: Particles and Fields 3(2), 621.

Milton, K., Odintsov, S. D. and Zerbini, S. (2001). Bulk versus brane running couplings, e-print hep-th/0110051

Minkowski, P. and Seiler, R. (1971). Massive vector meson in external fields, Physical Review D: Particles and Fields 4, 359.

Okun, L. B. (1981). Leptons and Quarks, Nauka, Moscow. (in Russian).

Okun, L. B. (2002). Photons, clocks, gravity and the concept of mass, Nuclear Physics B (Proc. Supl.) 110, 151.

Okun, L. B., Selivanov, K. and Telegdi, V. (1999). Gravitation, photons, clocks, Physics Uspekhi 42, 1045.

Pardy, M. (1994). The Čerenkov effect with radiative corrections, Physics Letters B 325, 517.

Pardy, M. (1998). The quantum field theory of laser acceleration, Physics Letters A 243, 223.

Pardy, M. (2001). The quantum electrodynamics of laser acceleration, Radiation Physics and Chemistry 61, 391.

Pardy, M. (2002). Čerenkov effect with massive photons, International Journal of Theoretical Physics 41(5), 887.

Pardy, M. (2003). Electron in the ultrashort laser pulse, International Journal of Theoretical Physics 42(1), 99.

Prokopec, T. and Woodard, D. (2003). Vacuum polarization and photon mass in inflation, e-print astro-ph/0303358; CERN-TH/2003-065.

Ritus, V. I. (1969). Radiative effects and their enhancement in intensive electromagnetic field, Journal of Experimental and Theoretical Physics 57(6), 2176. (in Russian).

Rohlf, J. W. (1994). Modern Physics from $\alpha$ to $Z^0$, John Wiley & Sons, Inc. New York.

Rosu, A. C. and Aceves de la Crus. (2001). One-parameter Darboux-transformed quan-
tum action in thermodynamics, e-print quant-ph/0107043

Ryder, L. H. (1985). *Quantum Field Theory*, Cambridge University Press, Cambridge.

Schwinger, J. (1970). *Particles, Sources and Fields*, Vol. I Addison-Wesley, Reading, Massachusetts.

Schwinger, J., Tsai W. Y. and Erber T. (1976). Classical and quantum theory of synergetic synchrotron-Cerenkov radiation, *Annals of Physics (New York)* 96, 303.

Sommerfeld, A. (1954). *Optics*, Academic Press, New York 10, N. Y. USA.

van Nieuwenhuizen, P. (1973). Radiation of massive gravitation, *Physical Review D: Particles and Fields* 7(8), 2300.

Volkov, D. M., (1935). Über eine Klasse von Lösungen der Diracschen Gleichung, *Zeitschrift für Physik*, 94, 250

Yildiz, A. (1977). Meissner effect in gauge fields, *Physical Review D: Particles and Fields* 16(12), 3450.

Zelekin, M. I. (1998). *Homogenous Spaces and Riccati Equation in Variational Calculus*, Factorial, Moscow. (in Russian).