QFS-space and its properties

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Abstract

In this paper, the concept of quasi-finitely separating map and quasiapproximate identity are introduced. Based on these concepts, QFS-spaces and quasicontinuous maps are defined. Properties and characterizations of QFS-spaces are explored. Main results are: (1) Each QFS-space is quasicontinuous space; (2) Closed subspaces, quasicontinuous projection spaces of QFS-spaces are QFS-spaces; (3) Continuous retracts of QFS-spaces are QFS-spaces and a kind of retracts of QFS-spaces are constructed; (4) Upper powerspaces of continuous QFS-spaces are FS-spaces.

Keywords: quasicontinuous-space; QFS-space; continuous retract; quasicontinuous projection; upper powerspace

2000 MSC: 54B20; 54D99; 06B35; 06F30

1. Introduction

Domain theory was first introduced by Dana Scott in the early 1970s, and the main purpose is to provide a mathematical tool for the semantics of functional programming languages, see[1, 2, 3, 20]. The most distinctive feature of domain theory is that it integrates order structures, topology structures and computer science[4, 6, 7, 15, 16, 17]. The main objects of domain theory are posets and domains, see[4, 5, 8]. In [16], Directed space is defined by Hui Kou. It is easy to see that directed spaces are equivalent to monotone determined spaces, which is defined in [17]. DTOP, the category of directed spaces was shown to be cartesian closed.

In [9], Chen, Kou and Lyu investigated two approximation relations on a T_0 topological space, the n-approximation and the d-approximation. Different kinds of continuous spaces are defined by the two approximations. They showed that a T_0 topological space is n-continuous iff it is a c-space iff it is d-continuous directed space. In order to find the maximal full cartesian closed subcategories of d-continuous directed space, Xie and Kou [13] introduced the concept of FS-space.

As a generalization of d-continuous spaces, the concept of quasicontinuous space was introduced in [21] by Feng and Kou. There are many examples of directed spaces which are quasicontinuous spaces but not continuous spaces. Therefore, a quasicontinuous space may not be a continuous space actually, let alone an FS-space. Now questions naturally arise: what is the counterpart of an FS-space in the realm of quasicontinuous spaces and to what extent does the counterpart share nice properties possessed by FS-spaces? In this paper we manage to give answers to these questions. First, we introduce the concepts of quasi-finitely separating map and quasi-approximate identity. Then QFS-spaces and quasicontinuous maps are defined accordingly, and properties as well as characterizations of QFS-domains are explored. Main results are: (1) Each QFS-space is quasicontinuous space; (2) Closed subspaces, quasicontinuous projection...
spaces of QFS-spaces are QFS-spaces; (3) Continuous retracts of QFS-spaces are QFS-spaces and a kind of retracts of QFS-spaces are constructed; (4) Upper powerspaces of continuous QFS-spaces are FS-spaces.

2. Preliminaries

Now, we introduce the concepts needed in this paper. The readers can also consult[8, 11, 12, 14]. A nonempty set $L$ endowed with a partial order $\leq$ is called poset. A subset $D \subseteq L$ is called directed set if for any $x, y \in D$, there exists $d \in D$ such that $x, y \leq d$. A poset $L$ is called a directed complete poset (dcpo, for short) if any directed subset of $L$ has a sup in $L$. For any $x, y \in L$, we say that $x$ is way below $y$ (denoted by $x \ll y$) if for any directed set $D$ of $L$, $y \leq \bigvee D$ implies that there is some $d \in D$ with $x \leq d$. A poset $L$ is called continuous if for any $x \in L$, $\{a \in L : a \ll x\}$ is directed set and has $x$ as its supremum. For a subset $A$ of $L$, let $\uparrow A = \{x \in L : \exists a \in A, a \leq x\}$, $\downarrow A = \{x \in L : \exists a \in A, x \leq a\}$. We use $\uparrow a$ (resp. $\downarrow a$) instead of $\uparrow \{a\}$ (resp. $\downarrow \{a\}$) when $A = \{a\}$. $A$ is called an upper (resp. a lower) set if $A = \uparrow A$ (resp. $A = \downarrow A$).

Let $L$ be a poset and $U \subseteq L$. Then $U$ is called Scott open if it satisfies: (1) $U = \uparrow U$; (2) For any directed sets $D \subseteq L$, $\forall D \subseteq U$ implies $D \cap U \neq \emptyset$. The collection of all Scott open subsets of $L$ is called the Scott topology of $L$ and denoted by $\sigma(L)$.

In this paper, topological spaces will always be supposed to be $T_0$ spaces. For a topological space $X$, its topology is denoted by $\tau$. The partial order $\leq$ is defined on $X$ by $x \leq y$ if and only if $x \in \overline{\{y\}}$ is called the specialization order$[11, 18, 19]$. From now on, all order-theoretical statements about $T_0$ spaces, such as upper sets, lower sets, directed sets, and so on, always refer to the specialization order.

A net of a topological space $X$ is a map $\xi : J \rightarrow X$, where $J$ is a directed set. Usually, we denote a net by $(x_j)_{j \in J}$. Let $x \in X$, saying $(x_j)_{j \in J}$ converges to $x$, denoted by $(x_j)_{j \in J} \rightarrow x$, if $(x_j)_{j \in J}$ is eventually in every open neighborhood of $x$, that is, for any open neighborhood $U$ of $x$, there exist $j_0 \in J$ such that for every $j \in J$, $j \geq j_0 \Rightarrow x_j \in U$.

Let $X$ be a $T_0$ topological space, then any directed subset of $X$ can be regarded as a net, and its index set is itself. We use $D \rightarrow r x$ to represent $D$ converges to $x$. Define notation $D(X) = \{(D, x) : x \in X, D$ is a directed subset of $X$ and $D \rightarrow r x\}$. It is easy to verify that, for any $x, y \in X$, $x \leq y$ if and only if $(y) \rightarrow r x$ and for directed set $D$, if $\exists d \in D, x \leq d$, then $D \rightarrow r x$. Therefore, if $x \leq y$, then $(\{y\}, x) \in D(X)$. Next, we give the concept of directed space. A subset $U$ of $X$ is called a directed open set if $\forall (D, x) \in D(X), x \in U \Rightarrow D \cap U \neq \emptyset$. Denote all directed open sets of $X$ by $d(X)$. Obviously, every open set of $X$ is directed open, that is, $\tau \subseteq d(X)$.

Let $X$ be a $T_0$ topological space. $X$ is called directed space if every directed open set of $X$ is an open set, that is, $d(X) = \tau[13, 16]$.

**Remark 2.1.** [13, 16] Let $X$ be a $T_0$ topological space.

1. The definition of directed space here is equivalent to the monotone determined space defined in [17].
2. Every poset equipped with the Scott topology is a directed space [16, 18], besides, each Alexandroff space is a directed space. Thus, the directed space extends the concept of the Scott topology.
3. If $U \in d(X)$, $U = \uparrow U$.
4. $X$ equipped with $d(X)$ is a $T_0$ topological space such that $\leq d = \leq$, where $\leq d$ is the specialization order relative to $d(X)$.
5. For a directed subset $D$ of $X$, $D \rightarrow x \iff D \rightarrow d(x)$ for all $x \in X$, where $D \rightarrow d(x)$ means that $D$ converges to $x$ with respect to the topology $d(X)$.
6. For each $x \in X$, $\downarrow x$ is directed closed.

Let $X$ be a $T_0$ topological space. The topology generated by the complements of all principal filters $\uparrow x$ is called the lower topology and denoted by $\omega(X)$. The common refinement $d(x) \vee \omega(X)$ of the directed topology and lower topology is called the Lawson topology, denoted by $\lambda(X)$. If $X$ be a directed space, then $\lambda(X) = d(x) \vee \omega(X) = \tau \vee \omega(X)$.

Let $X$ be a directed space and $x, y \in X$. We say that $x$ is $d$-way below $y$, denoted by $x \ll d y$, if for any directed subset $D$ of $X$, $D \rightarrow y$ implies $x \leq d$ for some $d \in D$. If $x \ll d y$, then $x$ is called $d$-compact element of $X$. For any directed space $X$ and $x \in X$, we denote $K_d(X) = \{x \in X : x \ll d x\}$, $\downarrow d x = \{y \in X : y \ll d x\}$.
and \( \uparrow_d x = \{ y \in X : x \ll_d y \} \). The d-way below relation is a natural extension of way-below relation. Let \( X \) be a directed space. \( X \) is called d-continuous if \( \downarrow_d x \) is directed and \( \downarrow_d x \to x \) for any \( x \in X \).

Let \( X \) be a \( T_0 \) topological space and \( \mathcal{P}^w(X) \) be the family of all nonempty finite subsets of \( X \). Given any two subsets \( G, H \) of \( X \), we define \( G \ll H \) iff \( H \subseteq \uparrow G \). A family of finite sets \( \mathcal{F} \subseteq \mathcal{P}^w(X) \) is said to be directed if given \( F_1, F_2 \) in the family, there exists \( F \in \mathcal{F} \) such that \( F \subseteq \uparrow F_1 \cap \uparrow F_2 \). Unless otherwise specified, directed family is always defined as above. Let \( \mathcal{F} \subseteq \mathcal{P}^w(X) \) be a directed family. We say that \( \mathcal{F} \to x \) if for any open neighbourhood \( U \) of \( x \), there exists some \( F \in \mathcal{F} \) such that \( \mathcal{F} \subseteq U \).

Let \( X \) be a directed space and \( G, H \subseteq X \). We say that \( G \) d-approximates \( H \), denoted by \( G \ll_d H \), if for any directed subset \( D \) of \( X \), \( D \to h \) for some \( h \in H \) implies \( D \cap \uparrow G \neq \emptyset \). We write \( G \ll_d x \) for \( G \subseteq \{ x \} \). \( G \) is said to be d-compact if \( G \ll_d G \). For any \( T_0 \) topological space \( X \) and \( F \subseteq X \), we denote \( \uparrow_d F = \{ x \in X : F \ll_d x \} \). A topological space \( X \) is called d-quasicontinuous if it is a directed space such that for any \( x \in X \), the family \( \text{fin}_d(x) = \{ F : F \in \mathcal{P}^w(X), F \ll_d x \} \) is a directed family and converges to \( x \).

**Theorem 2.2.** [21] Let \( X \) be a d-quasicontinuous space. The following statements hold.

1. Given any \( H \in \mathcal{P}^w(X) \) and \( y \in X \), \( H \ll_d y \) implies \( H \ll_d \mathcal{F} \subseteq \mathcal{P}^w(X) \).
2. Given any \( F \in \mathcal{P}^w(X) \), \( \uparrow_d F = (\uparrow F)^\circ \). Moreover, \( \{ \uparrow_d F : F \in \mathcal{P}^w(X) \} \) is a base of \( \tau \).

Let \( X \) be a d-quasicontinuous space and \( x \in X \), then \( \uparrow_d \{ x \} = \uparrow_d x \) is a open subset of \( X \). A \( T_0 \) topological space \( X \) is called locally hypercompact, if for any open subsets \( U \) of \( X \) and \( x \in U \), there exists some \( F \in \mathcal{P}^w(X) \) such that \( x \in (\uparrow F)^\circ \subseteq F \subseteq U \).

**Theorem 2.3.** [21] Let \( X \) be a directed space, Then the following three conditions are equivalent to each other.

1. \( X \) is a quasicontinuous space;
2. \( X \) is a locally hypercompact space;
3. There exists a directed family \( \mathcal{F} \subseteq \text{fin}(x) \) such that \( \mathcal{F} \to x \) for any \( x \in X \).

**Lemma 2.4.** [21] Let \( X \) be a \( T_0 \) topological space, \( \mathcal{F} \subseteq \mathcal{P}^w(X) \) be directed family, \( G, H \in \mathcal{P}^w(X), x \in X \). If \( G \ll_d H \) and \( \mathcal{F} \to x \in H \), then there exists \( F \in \mathcal{F} \) such that \( F \subseteq \uparrow G \).

Continuous dcpos and quasicontinuous dcpos endowed with the Scott topology can be viewed as special directed continuous and d-quasicontinuous spaces.

**Definition 2.5.** [21] Suppose \( X, Y \) are two \( T_0 \) topological spaces. A function \( f : X \to Y \) is called directed continuous if it is monotone and preserves all limits of directed subset of \( X \); that is, \( (D, x) \in D(X) \Rightarrow (f(D), f(x)) \in D(Y) \).

Here are some characterizations of the directed continuous functions.

**Proposition 2.6.** [21] Suppose \( X, Y \) are two \( T_0 \) topological spaces. \( f : X \to Y \) is a function between \( X \) and \( Y \). Then

1. \( f \) is directed continuous if and only if \( \forall U \in d(Y), f^{-1}(U) \in d(X) \).
2. If \( X, Y \) are directed spaces, then \( f \) is continuous if and only if it is directed continuous.

An approximate identity for a directed space \( X \) is a directed set \( \mathcal{D} \subseteq [X \to X] \) satisfying \( \mathcal{D} \to 1_X \), that is, \( \forall x \in X, \mathcal{D}(x) \to x \) (pointwise convergence), where \( [X \to X] \) is the family of all continuous functions on \( X \).

**Definition 2.7.** [13] A continuous function \( \delta : X \to X \) on a directed space \( X \) is finitely separating if there exists a finite set \( F_\delta \) such that for each \( x \in X \), there exists \( y \in F_\delta \) such that \( \delta(x) \leq y \leq x \). A directed space is finitely separated if there is an approximate identity for \( X \) consisting of finitely separating functions. A finitely separated directed space that is also a c-space will be called an FS-space.

In [13], Xie and Kou showed that a finitely separated space is already a c-space, hence an FS-space.
Definition 2.8. Let $X$ be a directed space, $B \subseteq \mathcal{P}^{w}(X)$.

(1) $B$ is called a quasibase of $X$ if $B \cap \text{fin}(x)$ is a directed family and $B \cap \text{fin}(x) \rightarrow_{\tau} x$. (2) A quasibase is called countable if $|B|$ is countable.

Let $X$ be a directed space, then $X$ is $d$-quasicontinuous spaces iff $X$ has a quasibase.

3. QFS-space

In this section, two technical notions of quasi-finitely separating map and quasi-approximate identity are introduced. With these notions, we define QFS-spaces and study basic properties and constructions of them.

Definition 3.1. Let $X$ be a directed space. A map $\delta : X \rightarrow \mathcal{P}^{w}(X)$ is said to be quasi-finitely separating if it satisfies the following three conditions:

(i) If $a \leq b$, then $\delta(b) \subseteq \uparrow \delta(a)$;

(ii) There exists a finite set $F_{\delta} \subseteq X$ (called a quasi-finitely separating set) such that for any $x \in X$, there exists $y \in F_{\delta}$ with $x \in \uparrow y \subseteq \uparrow \delta(x)$.

(iii) For each directed set $D \subseteq X$ and $D \rightarrow_{\tau} x$, $\delta(D) \Rightarrow_{\tau} \delta(x)$, that is, if $\delta(x) \subseteq U \in \tau$, $\delta(d) \subseteq U$ for some $d \in D$.

Denote all maps from $X$ to $\mathcal{P}^{w}(X)$ by $[X \rightarrow \mathcal{P}^{w}(X)]$. $D \subseteq [X \rightarrow \mathcal{P}^{w}(X)]$ is called directed if $\delta_{1}, \delta_{2} \in D$, $\delta(x) \subseteq \uparrow \delta_{1}(x) \cap \uparrow \delta_{2}(x)$ for any $x \in X$. We will use $1_{X}^{\delta}$ to denote the map $1_{X}^{\delta} : P \rightarrow \mathcal{P}^{w}(X)$ with $1_{X}^{\delta}(x) = \{x\}$ for any $x \in X$. A quasi-approximate identity for a directed space $X$ is a directed set $D \subseteq [X \rightarrow \mathcal{P}^{w}(X)]$ satisfying $D \rightarrow_{\tau} 1_{X}^{\delta}$, that is, $\forall x \in X$, $D(x) \rightarrow_{\tau} x$ (pointwise convergence).

Definition 3.2. A directed space $X$ is called QFS-domain if there is a quasi-approximate identity for $X$ consisting of quasi-finitely separating maps.

Let $X$ be a QFS-space and $\delta$ be any quasi-finitely separating map on $X$ with quasi-finitely separating set $F_{\delta}$. Since every $x \in X$ is above some $y \in F_{\delta}$ by Definition 3.1, $X = \uparrow F_{\delta}$. Since $F_{\delta}$ is finite, QFS-spaces is a finitely generated upper set.

Let $X$ be a finite set and $D = \{1_{X}^{\delta}\}$, then $1_{X}^{\delta}$ is quasi-finitely separating map, $D$ is directed and $D \rightarrow 1_{X}^{\delta}$. Then $X$ is a QFS-space.

Proposition 3.3. Let $X$ be a directed space. If $\delta$ is a quasi-finitely separating map on $X$, then for any $x \in X$, $\delta(x) \ll_{d} x$.

Proof: Suppose that $x \in X$, directed set $D \subseteq X$ and $D \rightarrow_{\tau} x$. Let $F_{D} = \{y \in F_{\delta} : \exists d \in D, d \in \uparrow y \subseteq \uparrow \delta(d)\}$. For any $y \in F_{D}$, we can choose a $d_{y} \in D$ such that $d_{y} \in \uparrow y \subseteq \uparrow \delta(d_{y})$. Let $D_{F} = \{d_{y} : y \in F_{D}\}$, then $D_{F}$ is finite. Since $D$ is directed set, $D_{F} \subseteq \downarrow d_{0}$ for some $d_{0} \in D$. Then $d_{0} \in \uparrow \delta(d)$ for any $d \in D$.

If $d_{0} \notin \uparrow \delta(x)$, then $\delta(x) \subseteq X \downarrow d_{0}$. Since $X \downarrow d_{0}$ is directed open set and $\delta(D) \Rightarrow_{\tau} \delta(x)$, $\delta(d) \subseteq X \downarrow d_{0}$ for some $d \in D$. Contradiction with $d_{0} \in \uparrow \delta(d)$. Hence $d_{0} \in \uparrow \delta(x)$ and $\delta(x) \ll_{d} x$.

Proposition 3.4. Let $X$ be a FS-space, then $X$ is QFS-space.

Proof: Let $X$ be a FS-space and $\mathcal{D}$ be an approximate identity for $X$ consisting of finitely separating functions, let $\mathcal{D}^{\ast} = \{\delta^{\ast} : \delta \in \mathcal{D}\}$, where $\delta^{\ast}(x) = \{\delta(x)\}$ for any $\delta \in \mathcal{D}$ and $x \in X$. Then $\mathcal{D}^{\ast}$ is a quasi-approximate identity for $X$ consisting of quasi-finitely separating maps.

Proposition 3.5. Let $X$ be a QFS-space and $\mathcal{D}$ be a quasi-approximate identity consisting of quasi-finitely separating maps for $X$, then $X$ is $d$-quasicontinuous space. In particular, $\{\delta(x) : \delta \in \mathcal{D}, x \in X\}$ is quasibase of $X$.

Proof: Let $X$ be a QFS-space and $\mathcal{D}$ be a quasi-approximate identity for $X$ consisting of quasi-finitely separating maps. Since $\mathcal{D}(x) = \{\delta(x)\}_{\delta \in \mathcal{D}}$ is directed and $\{\delta(x)\}_{\delta \in \mathcal{D}} \rightarrow_{\tau} x$, $X$ is $d$-quasicontinuous space by $\text{fin}_{d}(x) \rightarrow x$ and Theorem 2.3.

By Proposition 3.3, $\{\delta(x) : \delta \in \mathcal{D}, x \in X\}$ is quasibase of $X$. 

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Lemma 3.6. Let $X$ be a QFS-space and $D$ be a quasi-approximate identity consisting of quasi-finitely separating maps for $X$. For any open subset $U \subseteq X$, let $V_{\delta} = \{x \in X : \delta(x) \subseteq U\}$ for any $\delta \in D$. Then $V_{\delta} \subseteq d(X)$, $\bigcup_{\delta \in D} V_{\delta} = U$ and $\{V_{\delta} : \delta \in D\}$ is directed family with respect to inclusion order.

Proof: First we show that $\bigcup_{\delta \in D} V_{\delta} = U$. For any $x \in \bigcup_{\delta \in D} V_{\delta}$, there exists $\delta \in D$ such that $\uparrow \delta(x) \subseteq U$. Since $\delta$ is quasi-finitely separating map, $x \in \uparrow \delta(x) \subseteq U$. Conversely, for any $x \in U$, let $\mathcal{D}(x) = \{\delta \in D(x) : x \in \delta(x)\}$. Then there exists $\delta_0 \in \mathcal{D}$ such that $\uparrow \delta_0(X) \subseteq U$. Thus $x \in V_{\delta_0} \subseteq \bigcup_{\delta \in D} V_{\delta}$.

Then we show that $V_{\delta}$ is open set for any $\delta \in D$. Let $x \in V_{\delta}$ and $x \leq y$. Since $\delta$ is quasi-finitely separating, $\delta(y) \subseteq \uparrow \delta(x) \subseteq U$, $y \in V_{\delta}$. Thus $V_{\delta}$ is an upper set. For any directed subsets $D \subseteq L$ with $D \rightarrow x$, let $x \in V_{\delta}$. By $\delta(D) \Rightarrow x \subseteq U$, $\delta(d) \subseteq U$ for some $d \in D$. Then $\uparrow \delta(d) \subseteq U$, $d \in V_{\delta}$. Hence $V_{\delta}$ is open set.

Let $V_{\delta_1}, V_{\delta_2} \in \{V_{\delta} : \delta \in D\}$. Since $\mathcal{D}$ is directed, $\delta(x) \subseteq \uparrow \delta_1(x) \cap \delta_2(x)$ for any $x \in X$. If $x \in V_{\delta_1}$, then $\uparrow \delta(x) \subseteq \uparrow \delta_1(x) \subseteq U$ and $x \in V_{\delta_2}$. Hence $V_{\delta_1} \subseteq V_{\delta_2}$. Similarly, $V_{\delta_2} \subseteq V_{\delta_1}$. $\{V_{\delta} : \delta \in D\}$ is directed family with respect to inclusion order. □

Lemma 3.7. Let $X$ be a QFS-space and $D$ be a quasi-approximate identity consisting of quasi-finitely separating maps for $X$. For any $F \in \mathcal{P}^w(X)$, $\uparrow F = \bigcap_{\delta \in D} \bigcup_{x \in F} \uparrow \delta(x)$ and $\{\bigcup_{x \in F} \uparrow \delta(x) : \delta \in D\}$ is directed family.

Proof: Obviously, $\uparrow F \subseteq \bigcap_{\delta \in D} \bigcup_{x \in F} \uparrow \delta(x)$. Conversely, if there exists $y \in \bigcap_{\delta \in D} \bigcup_{x \in F} \uparrow \delta(x)$ but $y \notin \uparrow F$, then $\uparrow F \notin \bigcap_{\delta \in D} \bigcup_{x \in F} \uparrow \delta(x)$. Then $\bigcup_{x \in F} \uparrow \delta(x) \notin \bigcap_{\delta \in D} \bigcup_{x \in F} \uparrow \delta(x)$ by Lemma 3.6. Since $\uparrow F$ is compact, there exists $\delta_i \subseteq D$, $i = 1, \ldots, n$, such that $\uparrow F \subseteq \bigcup_{i=1}^n \{x \in X : \uparrow \delta_i(x) \subseteq \bigcup_{x \in F} \uparrow \delta(x)\}$. Since $\{V_{\delta} : \delta \in D\}$ is directed family, there exists $\delta_0 \in \mathcal{D}$ such that $\uparrow F \subseteq \bigcup_{X \in F} \uparrow \delta_0(x) \subseteq \bigcup_{x \in F} \uparrow \delta(x)$. Hence $y \notin \bigcup_{x \in F} \uparrow \delta(x)$. Thus $\uparrow F = \bigcap_{\delta \in D} \bigcup_{x \in F} \uparrow \delta(x)$. Obviously, $\{\bigcup_{x \in F} \uparrow \delta(x) : \delta \in D\}$ is directed family. □

Definition 3.8. A family $F$ of maps from $X$ to $\mathcal{P}^w(X)$ is said to satisfy property $\mathcal{P}$ if given any $F_i \ll_d x_i$, $i = 1, 2$, there is a $F \in \mathcal{F}$ such that $F_i \ll_d F(x_i) \ll_d x_i$.

Proposition 3.9. Let $X$ be a QFS-space and $D$ be a quasi-approximate identity consisting of quasi-finitely separating maps for $X$. Then $X$ is a d-quasicontinuous space and $D$ satisfies property $\mathcal{P}$.

Proof: Suppose $X$ is a QFS-space, then $X$ is d-quasicontinuous space by Proposition 3.5. Given any pairs $F_i \ll_d x_i$, $i = 1, 2$, let $D$ be a quasi-approximate identity for $X$ consisting of quasi-finitely separating maps. For any $i$, $1 \leq i \leq 2$, $D(x_i) \rightarrow x_i \in \uparrow F_i \in \tau$. Then there exists $\delta \in \mathcal{D}$ such that $\delta_0(x_i) \subseteq \uparrow F_i$. Thus $F_i \ll_d \delta_0(x_i) \ll_d x_i$ by Proposition 3.3. $D$ satisfies property $\mathcal{P}$. □

4. Properties of QFS-spaces

In this section, we show that closed subspaces ,quasicontinuous projection spaces and continuous retracts of QFS-spaces are QFS-spaces. Further more, a kind of continuous retracts of QFS-space are constructed.

Let $X = (X, \tau)$ be a topological space, $A \subseteq X$, then $A = (A, \tau|_A)$ is a subspace of $X$. For $x, y \in A$, it is easy to see that $x \leq y$ iff $x \leq \tau|_A y$, where $\leq \tau|_A$ is the specialization order relative to $\tau|_A$.

Theorem 4.1. Let $X$ be a QFS-space and $A \subseteq X$ be a directed closed set. Then $A$ is a QFS-space.

Proof: Since $X$ is a QFS-domain, there exists a quasi-approximate identity $D$ for $X$ consisting of quasi-finitely separating maps. Let $D^* = \{\delta_\alpha : \delta \in D\}$, where $\delta_\alpha(x) = \delta(x) \cap A$ for any $x \in A$. Now we prove it in three steps.

(1) Let $\Gamma(X)$ be set of all directed closed subsets of $X$ and $\Gamma(A)$ be set of all directed closed subsets of $A$. Since $A$ is directed set, $\Gamma(A) = \{G \in \Gamma(A) : G \subseteq A\}$. Hence $d(A) = \{A \setminus G : G \in \Gamma(A)\} = \{A \setminus G : G \subseteq \Gamma(X)\} = \{A \setminus G : G \subseteq \Gamma(X)\} = \{A \setminus \Gamma(X) \setminus G : G \subseteq \Gamma(X)\} = \{A \setminus \Gamma(X) \setminus G : G \subseteq \Gamma(X)\} = \{A \setminus U : U \subseteq d(X)\} = d(X)|_A$. $A$ is directed space.
Proof: verify that \( f \) is quasicontinuous, if \( f \) is monotone, that is, if \( H \subseteq \uparrow G \), then \( f(H) \subseteq \uparrow f(G) \) and if \( F \subseteq \mathcal{P}^w(X) \) and \( f \rightarrow \tau \), then \( \{ f(F) : F \in F \} \rightarrow \tau \). Hence we can show that \( \delta(D) \Rightarrow \tau|_{\uparrow a} \delta(a) \).

By Definition 3.1, \( \mathcal{D}^* \) is a set consisting of quasi-finitely separating maps on \( A \).

(3) Then we show that \( \mathcal{D}^* \) is a quasi-approximate identity for \( A \). It is easy to see that \( \mathcal{D}^* \) is directed. For each \( a \in U \in \uparrow \mathfrak{A}_A \), there exists \( V \in \tau \) such that \( U = A \cap V \). Since \( a \in V \) and \( \mathcal{D}(a) = \{ \delta(a) : \delta \in \mathcal{D} \} \rightarrow \tau \), \( \delta^0(a) \subseteq V \) for some \( \delta^0 \in \mathcal{D} \). Then \( \delta^0_a \) is a set consisting of quasi-finitely separating maps on \( A \).

This shows that \( \mathcal{D}^* \) is a quasi-approximate identity for \( A \).

By (1), (2), and (3), \( A \) is a QFS-space. □

**Definition 4.2.** A function \( f : X \rightarrow Y \) between directed space is quasicontinuous if \( f \) is monotone, that is, if \( H \subseteq \uparrow G \), then \( f(H) \subseteq \uparrow f(G) \) and if \( F \subseteq \mathcal{P}^w(X) \) and \( f \rightarrow \tau \), then \( \{ f(F) : F \in F \} \rightarrow \tau \).

If \( f : X \rightarrow Y \) is a quasicontinuous, then \( f \) is a continuous.

**Proposition 4.3.** If \( A \) is a function \( f : X \rightarrow Y \) between directed space is quasicontinuous, then

1. \( f \) is a continuous;
2. If \( F \rightarrow \mathcal{D} \), then \( \{ f(F) : F \in F \} \rightarrow \tau \).

**Theorem 4.4.** Let \( X \) be a QFS-space, \( f : X \rightarrow X \) be a quasicontinuous projection and \( f(X) = (f(X), \tau|_{f(x)}) \) be directed space. Then \( f(X) \) is a QFS-space.

Proof: Since \( X \) is a QFS-domain, there exists a quasi-approximate identity \( \mathcal{D} \) for \( X \) consisting of quasi-finitely separating maps. Let \( \mathcal{D}^* = \{ f \circ \delta : \delta \in \mathcal{D} \} \), where \( f \circ \delta(x) = f(\delta(x)) \) for any \( x \in X \). Firstly, we verify that \( \mathcal{D}^* \) satisfies the conditions in Definition 3.1. Let \( X = (X, \tau), f(X) = (f(X), \tau|_{f(x)}) \).

(i) For any \( a, b \in f(X) \) with \( a \leq b \), we have \( f \circ \delta(b) = f(\delta(b)) \subseteq \uparrow f(\delta(a)) = \uparrow f \circ \delta(a) \) for any \( \delta \in \mathcal{D} \).

(ii) For any \( \delta \in \mathcal{D} \), there exists a finite set \( F_\delta \subseteq X \) such that for any \( x \in X \), there exists \( y \in F_\delta \) with \( x \in \uparrow \delta(y) \). Noticing that \( f \) is a projection, then for every \( x \in X \), we have \( f(x) = x \). By the quasiuniformity of \( f \) and \( f \in f(F_\delta) \), we have that \( x \in \uparrow f(x) \), \( f(\uparrow y) \cap f(\uparrow \delta(x)) \subseteq \uparrow f(\delta(x)) \cap f(x) = \uparrow f(x) \). Hence \( f \circ \delta \) is quasi-finitely separating set of \( f \).

(iii) Let \( D \) be a directed set of \( f(X) \) and \( D \rightarrow \tau|_{\uparrow a} \), then \( D \) is directed set of \( X \) and \( D \rightarrow \tau \). Hence \( \delta(D) \Rightarrow \tau|_{\uparrow a} \delta(a) \).

By Definition 3.1, \( \mathcal{D}^* \) is a set consisting of quasi-finitely separating maps on \( f(X) \).

Then we show that \( \mathcal{D}^* \) is a quasi-approximate identity for \( f(X) \). It is easy to see that \( \mathcal{D}^* \) is directed. For each \( x \in U \in \uparrow \mathfrak{A}(X) \), there exists \( V \in \tau \) such that \( U = f(X) \cap V \). Since \( x \in V \), \( \mathcal{D}(x) = \{ \delta(x) : \delta \in \mathcal{D} \} \rightarrow \tau \). Since \( f \) is quasicontinuous, \( \{ f(\delta(x)) : \delta \in \mathcal{D} \} \rightarrow \tau(\mathcal{D}) \). Hence \( \{ f \circ \delta(x) : \delta \in \mathcal{D} \} \rightarrow \tau|_{\uparrow a} \), \( x \) by \( f(x) = x \).

This shows that \( \mathcal{D}^* \) is a quasi-approximate identity for \( f(X) \). So, \( f(X) \) is a QFS-space. □

**Definition 4.5.** Let \( X, Y \) be two topological space. \( Y \) is called a continuous retract of \( X \) if there exist continuous functions \( f : X \rightarrow Y \) and \( g : Y \rightarrow X \) such that \( f \circ g = 1_Y \).

**Theorem 4.6.** Let \( X \) be a QFS-space and \( Y \) be a directed space. If \( Y \) is a continuous retract of \( X \), then \( Y \) is a QFS-space.

Proof: Let \( X \) be a QFS-space and \( Y \) be a retract of \( X \). Then there exist continuous functions \( f : X \rightarrow Y \) and \( g : Y \rightarrow X \) such that \( f \circ g = 1_Y \). We show that \( Y \) is a QFS-space. Suppose \( \mathcal{D} \) is a quasi-approximate identity for \( X \) consisting of quasi-finitely separating maps. Let \( \varepsilon : Y \rightarrow \mathcal{P}^w(Y) \) be \( \varepsilon(y) = \{ f(x) : x \in \delta(g(y)) \} \) for
any \( y \in Y \). We show that \( D^* = \{ \varepsilon : \delta \in D \} \) is a quasi-approximate identity for \( Y \) consisting of quasi-finitely separating maps.

1. \( \varepsilon \) is quasi-finitely separating map.
   
   (i) For any \( y_1, y_2 \in Y \) and \( y_1 \leq y_2 \). Since \( g \) is monotone, \( g(y_1) \leq g(y_2) \). Hence \( \varepsilon(y_2) = \{ f(x) : x \in \delta(g(y_2)) \} \subseteq \{ f(x) : x \in \delta(g(y_1)) \} = \varepsilon(y_1) \). Thus \( \delta(g(y_2)) \subseteq \delta(g(y_1)) \). Hence \( \delta(g(y_2)) \subseteq \delta(g(y_1)) \). Then there exists \( z \in F_X \) such that \( y \in \delta(z) \subseteq \delta(g(y_1)) \). Then \( y \in \delta(z) \subseteq \delta(g(y_1)) \).

   (ii) Since \( \delta \) is quasi-finitely separating, there exists a finite set \( F_\delta \subseteq X \) such that for any \( x \in X \), there exists \( y \in F_\delta \) with \( x \in \delta(z) \subseteq \delta(g(y)) \). Let \( F_\delta = f(F_\delta) \). For any \( x \in X \), we have \( f(g(y)) = y \). Since \( g(y) \in X \), there exists \( z \in F_X \) such that \( g(y) \in \delta(z) \subseteq \delta(g(y_1)) \). Then \( y \in \delta(z) \subseteq \delta(g(y_1)) \).

   (iii) For any directed set \( D \subseteq Y \) and \( D \to \tau_\delta, y \), then \( g(D) \subseteq X \) is directed set and \( g(D) \to \tau_\delta, g(y) \). Then \( \delta(g(D)) = \delta(g(y)) \). Let \( \varepsilon(y) = \{ f(x) : x \in \delta(g(y)) \} \subseteq V \in V \), then \( \delta(g(y)) \subseteq f^{-1}(V) \). By Definition 3.1 and \( \delta(g(D)) = \delta(g(y)) \), \( \delta(g(D)) \subseteq f^{-1}(V) \). That is \( \varepsilon(d) = \{ f(x) : x \in \delta(g(D)) \} \subseteq V \), then \( \varepsilon(D) \to \tau_\delta, \varepsilon(y) \).

By Definition 3.1, \( D^\ast \) is a set consisting of quasi-finitely separating maps on \( Y \).

(2) Then we show that \( D^\ast \) is a quasi-approximate identity for \( f(X) \). \( D^\ast \) is directed since \( D \) is directed.

\[ \forall y \in Y, \ D^\ast(y) = \{ \varepsilon(y) : \varepsilon \in D^\ast \} = \{ \{ f(x) : x \in \delta(g(y)) \} : \delta \in D \} \subseteq \{ \{ f(x) : x \in \delta(g(y)) \} : \delta \in D \} \]}

Proposition 4.7. Let \( X \) be a QFS-space, \( A \subseteq X \) be a lower bound of \( \uparrow b \) for some \( b \in X \). Then \{\( a \)\} \cup \( \uparrow b \) is a continuous retract of \( X \), that is, \{\( a \)\} \cup \( \uparrow b \) is QFS-space.

Proof: Let \( A = \{ a \} \cup \uparrow b \), then \( \varepsilon : \{ a \} \cup \uparrow b \to X \), defined by \( g(x) = x \). Let \( f : X \to \{ a \} \cup \uparrow b \), defined by \( f(x) = x \), \( x \in A \) or \( f(x) = a \), \( x \notin A \). It is easy to see that \( g \) is continuous.

Obviously, \( f \) is monotone and \( f = f^\downarrow \). Let \( D \subseteq X \) be directed set and \( D \to \tau_\delta \). (1) If \( f(x) = x \in \uparrow b \), then there exists \( d_1 \in D \cap \uparrow b \) such that \( d_1 \leq d_2 \). If \( x \in A \cap U \) for some \( U \in \tau_\delta \), then there exists \( d_2 \in D \cap U \) such that \( d_1 \leq d_2 \). Since \( D \cap A \subseteq Y \), \( D \cap A \subseteq Y \). Hence \( D \cap A \subseteq Y \). Hence \( D \cap A \subseteq Y \).

By (1) and (2), \( A \) is a directed space. Hence \( A \) is QFS-space by Theorem 4.6.

Suppose \( X, Y \) are two directed spaces. Let \( X \times Y \) represents the Cartesian product of \( X \) and \( Y \), then we have a natural partial order on it: \( \forall (x_1, y_1), (x_2, y_2) \in X \times Y, (x_1, y_1) \leq (x_2, y_2) \iff x_1 \leq x_2, y_1 \leq y_2, \) which is called the pointwise order on \( X \times Y \). Now, we define a topological space \( X \times Y \) as follows[13]:

(1) The underlying set of \( X \times Y \) is \( X \times Y \);

(2) The topology on \( X \times Y \) is defined as follows: for each given directed set \( D \subseteq X \times Y \) and \( (x, y) \in X \times Y \), \( D \to \tau_{X \times Y} (x, y) \in X \times Y \iff \pi_1(D) \to \tau_X x \in X, \pi_2(D) \to \tau_Y y \in Y \), that is, a subset \( U \subseteq X \times Y \) is open if and only if for every directed limit defined as above \( D \to \tau_{X \times Y} (x, y) \in U \Rightarrow U \cap D \neq \emptyset \).

Proposition 4.8. [13] Suppose \( X \) and \( Y \) are two directed spaces.

(1) The topological space \( X \times Y \) defined above is a directed space and satisfies the following properties:

The specialization order on \( X \times Y \) equals to the pointwise order on \( X \times Y \).

(2) Suppose \( Z \) is another directed space, then \( f : X \times Y \to Z \) is continuous if and only if it is continuous in each variable separately.

Theorem 4.9. Let \( X, Y \) be QFS-space and \( X \) be core-compact, then \( X \times Y \) is QFS-space.
Proof: Let $X, Y$ be QFS-space and. Suppose $\mathcal{D}$, $\mathcal{E}$ be quasi-approximate identity for $X$, $Y$ consisting of quasi-finitely separating maps. Let $\mathcal{D}^* = \{ (\delta, \varepsilon) : \delta \in \mathcal{D}, \varepsilon \in \mathcal{E} \}$. Since $X$ is core-compact, $X \otimes Y = X \times Y$ by Theorem 4.2 in [13].

It is easy to verify that $\mathcal{D}^*$ is quasi-approximate identity for $X \otimes Y$ consisting of quasi-finitely separating maps. \hfill \Box

5. QFS-space and Convex Powerspace of directed space

Definition 5.1. [25] Let $X$ be a directed space.

(1) Denote $\mathcal{Q}_f^{(X)} = \{ \uparrow F : F \in \mathbb{P}^w(X) \}$. Define an order $\leq_{\mathcal{Q}}$ on $\mathcal{Q}_f^{(X)}$: $\uparrow F_1 \leq_{\mathcal{Q}} \uparrow F_2 \Leftrightarrow \uparrow F_2 \subseteq \uparrow F_1$.

(2) Let $\mathcal{F} = \{ \uparrow F : F \in \mathbb{P}^w(X) \} \subseteq \mathcal{Q}_f^{(X)}$ be a directed set (respect to order $\leq_{\mathcal{Q}}$) and $\uparrow H \in \mathcal{Q}_f^{(X)}$.

Define a convergence notation $\mathcal{F} \Rightarrow_{\mathcal{Q}} \uparrow H \Leftrightarrow$ there exists finite directed sets $D_1, \ldots, D_n \subseteq X$ such that

1. $H \cap \{ x : D_i \to x \} \neq \emptyset$ for any $i = 1, \ldots, n$;
2. $H \subseteq \{ x : D_i \to x, i = 1, \ldots, n \}$;
3. $\forall (d_1, \ldots, d_n) \in \prod_{i=1}^n D_i$, there exists some $\uparrow F \in \mathcal{F}$ such that $\uparrow F \subseteq \uparrow d_i$.

(3) A subset $\mathcal{U} \subseteq \mathcal{Q}_f^{(X)}$ is called a $\Rightarrow_{\mathcal{Q}}$ convergence open set of $\mathcal{Q}_f^{(X)}$ if and only if for any directed subset $\mathcal{F}$ of $\mathcal{Q}_f^{(X)}$ and $\uparrow H \in \mathcal{Q}_f^{(X)}$, $\mathcal{F} \Rightarrow_{\mathcal{Q}} \uparrow H \in \mathcal{U}$ implies $\mathcal{F} \cap \mathcal{U} \neq \emptyset$. Denote all $\Rightarrow_{\mathcal{Q}}$ convergence open set of $\mathcal{Q}_f^{(X)}$ by $\mathcal{O}_{\Rightarrow_{\mathcal{Q}}}(\mathcal{Q}_f^{(X)})$.

Proposition 5.2. [25] Let $X$ be a directed space, then

(1) $(\mathcal{Q}_f^{(X)}), \mathcal{O}_{\Rightarrow_{\mathcal{Q}}}(\mathcal{Q}_f^{(X)}))$ is a directed space, that is, $\mathcal{O}_{\Rightarrow_{\mathcal{Q}}}(\mathcal{Q}_f^{(X)})) = d(\mathcal{Q}_f^{(X)}))$ and the specialization order $\leq_{\mathcal{Q}}$ of $\mathcal{O}_{\Rightarrow_{\mathcal{Q}}}(\mathcal{Q}_f^{(X)}))$ equals to $\leq_{\mathcal{Q}}$;

(2) $(\mathcal{Q}_f^{(X)}, \mathcal{O}_{\Rightarrow_{\mathcal{Q}}}(\mathcal{Q}_f^{(X)}))$ respect to the set union operation $\cup$ is a directed deflationary semilattice;

(3) Suppose $X$ is a directed space, then $(\mathcal{Q}_f^{(X)}), \mathcal{O}_{\Rightarrow_{\mathcal{Q}}}(\mathcal{Q}_f^{(X)}))$ is the upper powerspace of $X$, that is, endowed topology $\mathcal{O}_{\Rightarrow_{\mathcal{Q}}}(\mathcal{Q}_f^{(X)}), (\mathcal{Q}_f^{(X)}, \cup) \cong \mathcal{P}(U)$.

Let $X$ be a directed space and $\uparrow G, \uparrow H \in \mathcal{Q}_f^{(X)}$, then $\uparrow G \ll_{\mathcal{Q}} \uparrow H \Leftrightarrow \forall F \Rightarrow_{\mathcal{Q}} \uparrow H, \uparrow F \subseteq \uparrow G$ for some $\uparrow F \in \mathcal{F}$ by the definition of way below. In this paper, the way below relation with respect to $\leq_{\mathcal{Q}}$ and $\Rightarrow_{\mathcal{Q}}$ is denoted by $\ll_{\mathcal{Q}}$. Let $\uparrow G \ll_{\mathcal{Q}} \uparrow H$. Since $\uparrow H \leq_{\mathcal{Q}} \uparrow x$ for any $x \in \mathcal{X}$, $\uparrow G \ll_{\mathcal{Q}} \uparrow x$.

The following lemma shows that the way below relation on the poset $\mathcal{Q}_f^{(X)}$ agrees with the way-below relation defined for finite subsets of directed space.

Proposition 5.3. Let $X$ be a directed space and $\uparrow G, \uparrow H \in \mathcal{Q}_f^{(X)}$, then $\uparrow G \ll_{\mathcal{Q}} \uparrow H$ if and only if $G \ll H$.

Proof: If $\uparrow G \ll_{\mathcal{Q}} \uparrow H$, $D$ is a directed subset of $X$ and $D \rightarrow x \in H$. Let $\mathcal{F} = \{ \{d\} : d \in D \}$, then

1. $\{ x \cap \{ x \cap \{ x : D \to x \} \neq \emptyset \}$;
2. $\{ x : D \to x \}$;
3. $\forall (d_1, \ldots, d_n) \in \prod_{i=1}^n D_i$, there exists some $\uparrow F \in \mathcal{F}$ such that $\uparrow F \subseteq \uparrow d_i$.

Let $G \ll H$ and $\mathcal{F} \Rightarrow_{\mathcal{Q}} \uparrow H$. Then there exists finite directed sets $D_1, \ldots, D_n \subseteq X$ such that

1. $H \cap \{ x : D_i \to x \} \neq \emptyset$ for any $i = 1, \ldots, n$;
2. $H \subseteq \{ x : D_i \to x, i = 1, \ldots, n \}$;
3. $\forall (d_1, \ldots, d_n) \in \prod_{i=1}^n D_i$, there exists some $\uparrow F \in \mathcal{F}$ such that $\uparrow F \subseteq \uparrow d_i$.

For any $i$, let $x_i \in H \cap \{ x : D_i \to x \}$, then $D_i \cap \uparrow x_i \neq \emptyset$. Thus $D_i \cap \uparrow x_i \neq \emptyset$. Let $d_i \in D_i \cap \uparrow x_i$, then there exists some $\uparrow F \in \mathcal{F}$ such that $\uparrow F \subseteq \uparrow d_i \subseteq \uparrow G$. So, $G \ll_{\mathcal{Q}} \uparrow H$.

Proposition 5.4. Let $X$ be a continuous space, $\mathcal{F} \subseteq \mathcal{Q}_f^{(X)}$ and $H \in \mathbb{P}^w(X)$. Then $\mathcal{F} \Rightarrow_{\mathcal{Q}} \uparrow H$ if and only if $\mathcal{F} \subseteq \mathcal{U}$ for any $\mathcal{U} \in d(X)$.

Proof: Let $\mathcal{F} \Rightarrow_{\mathcal{Q}} \uparrow H$ and $H \subseteq \mathcal{U} \in d(X)$, then there exists finite directed sets $D_1, \ldots, D_n \subseteq X$ such that

1. $H \cap \{ x : D_i \to x \} \neq \emptyset$ for any $i = 1, \ldots, n$;
2. $H \subseteq \{ x : D_i \to x, i = 1, \ldots, n \}$;
3. $\forall (d_1, \ldots, d_n) \in \prod_{i=1}^n D_i$, there exists some $\uparrow F \in \mathcal{F}$ such that $\uparrow F \subseteq \uparrow d_i$.

Hence for any $i$, $D_i \to h$ for some $h \in H$. Since $h \in U$, $D_i \cap U \neq \emptyset$. Let $d_i \in D_i \cap U$, then there exists some $\uparrow F \in \mathcal{F}$ such that $\uparrow F \subseteq \uparrow d_i \subseteq \uparrow U$, that is, $F \subseteq \mathcal{F} \subseteq \mathcal{U}$. \hfill \Box
If $H \subseteq U$ implies there exists $\uparrow F \in F$ such that $F \subseteq \uparrow F \subseteq U$ for any $U \in d(X)$. For any $h \in H$, since $X$ is continuous space, $\downarrow U$ is directed set and converges to $x$. Let $H = \{h_1, \cdots \, h_n\}$ and $D_i = \downarrow h_i$. Obviously, $H \cap \{x : D_i \to x\} \neq \emptyset$ for any $i = 1, \cdots , n$ and $H \subseteq \{x : D_i \to x, i = 1, \cdots , n\} \cap \{d_1, \cdots , d_n\} \in \prod_i d_i$, $d_i \ll h_i$. Since $H \subseteq \bigcup_i d_i \in d(x)$, then there exists some $F \in F$ such that $\uparrow F \in \bigcup_i d_i \subseteq \bigcup_i d_i$. Thus $F \Rightarrow \omega \uparrow F$ be Definition 5.1.

Proposition 5.5. Let $X$ be a continuous QS-space, then $Q_{fin}(X)$ is a FS-space.

Proof: Let $X$ be a QS-space and $D$ be a quasi-approximate identity for $X$ consisting of quasi-finitely separating maps. Let $\varepsilon : Q_{fin}(X) \to P^\omega (Q_{fin}(X))$ be $\varepsilon(\uparrow F) = \uparrow \bigcup_{x \in F} \delta(x)$. Now we show that $D^* = \{\varepsilon : \varepsilon \in D\}$ is a quasi-approximate identity for $Q_{fin}(X)$ consisting of quasi-finitely separating maps.

(1) $\varepsilon$ is quasi-finitely separating map.

(i) For any $\uparrow F_1, \uparrow F_2 \in Y$ and $\uparrow F_2 \subseteq \uparrow F_1$. $\varepsilon(\uparrow F_2) = \uparrow \bigcup_{x \in F_2} \delta(x) = \uparrow \bigcup_{x \in \uparrow F_2} \delta(x) \subseteq \uparrow \bigcup_{x \in \uparrow F_1} \delta(x) = \varepsilon(\uparrow F_1)$.

(ii) Since $\delta$ is quasi-finitely separating, there exists a finite set $F_3 \subseteq X$ such that for any $x \in X$, there exists $y \in F_3$ with $x \in y \subseteq \delta(x)$. Let $F_3 = \{\uparrow \bigcup_{x \in G} \delta(x) : G \neq \emptyset, G \subseteq F_3\}$. For any $\uparrow F \in Q_{fin}(X)$, let $D_F = \{y \in F_3 : x \in y \subseteq \delta(x)\} : y \subseteq \delta(x)$, then $D_F \subseteq F_3$. Since $F \subseteq D_F \subseteq \bigcup_{x \in D_F} \delta(x).$

(iii) Let $F = \{\uparrow F : F \in \mathcal{P}^\omega (X)\} \subseteq Q_{fin}(X)$ be a directed set(respect to order $\subseteq \omega$) and $\uparrow H \subseteq Q_{fin}(X)$. If $F \Rightarrow \omega \uparrow H \Rightarrow$ there exists finite directed sets $D_1, \cdots , D_n \subseteq X$ such that

1. $H \cap \{x : D_i \to x\} \neq \emptyset$ for any $i = 1, \cdots , n$;
2. $H \subseteq \{x : D_i \to x, i = 1, \cdots , n\};$
3. $\forall (d_1, \cdots , d_n) \in \prod_i D_i$, there exists some $F \in F$ such that $\uparrow F \in \bigcup_i \uparrow d_i$.

Let $\uparrow \bigcup_{x \in U} \delta(x) \subseteq U \in \tau$. For any $h_i \in H$, there exists directed subset $D_i$ of $X$ such that $D_i \to x$. Then $\delta(D_i) \Rightarrow \tau h_i$. Since $X$ is a continuous space. Then $\uparrow \delta(d_i) \subseteq U$ for some $d_i \in D_i$. Then there exists some $\uparrow F \in F$ such that $\uparrow F \in \bigcup_i \uparrow d_i$. It is easy to see that $\uparrow \bigcup_{x \in \uparrow F} \delta(x) \subseteq \bigcup_{x \in \{d_i : h_i \in H\}} \delta(x) \subseteq U$. This is show that $\varepsilon(\mathcal{F}) \Rightarrow \omega \varepsilon(\uparrow H)$ by Proposition5.4. That is $\varepsilon$ is a continuous functions.

By Definition 3.1, $D^*$ is a set consisting of quasi-finitely separating maps on $Y$.

(2) Then we show that $D^*$ is a quasi-approximate identity for $f(X)$. $D^*$ is directed since $D$ is directed. $\uparrow F \in Q_{fin}(X), D^*(\uparrow F) = \{\varepsilon(\uparrow F) : \varepsilon \in D^*\} = \{\uparrow \bigcup_{x \in F} \delta(x) : \delta \in D\}$. Let $F \subseteq U \in \tau$. For any $f_i \in F$, $i = 1, \cdots , n$, $D(f_i) \to x_i$, then $\delta(u_i) \subseteq U$. Since $D$ is directed, there exist $\delta \in D$ such that $\delta(f_i) \subseteq \delta(u_i)$, $i = 1, \cdots , n$. Since $\uparrow \delta(f_i) \subseteq U$ and $\bigcup_{x \in \uparrow F} \delta(x) \subseteq U$, $D^*(\uparrow F) \Rightarrow \omega \uparrow F$ by Proposition5.4. This shows that $D^*$ is a quasi-approximate identity for $f(X)$. So, $Q_{fin}(X)$ is a FS-space by Definition 2.7.

Acknowledgments

This work is supported by the NSFY of China (Nos. 11401435). The authors are grateful to the referees for their valuable comments which led to the improvement of this paper.

Conflict of interest

The authors declare that there is no conflict of interest in this paper.

References

References

[1] X. Q. Xu, J. B. Yang, Topological representation of distributive hypercontinuous lattices, Chinese Annals of Mathematics, 2009, 30B: 199-206.
[2] R. M. Amadio, P. L. Curien, Domains and Lambda-Calculi, Cambridge:Cambridge University Press, 1998.
[3] J. B. Yang, M. K. Luo, Priestley spaces, quasi-hyperalgebraic lattices and Smyth powerdomains, Acta mathematics Sinica, 22(2006), 951-958.
[4] J. Goubault-Larrecq, Non-Hausdorff topology and domain theory: Selected topics in point-set topology, Cambridge University Press, (2013).
[5] L. Zhou, Q. Li, Convergence On Quasi-continuous Domain, Journal of Computational Analysis and Applications, 2013, 15(2).
[6] Q. Y. He, L. S. XU, X. Y. Xi, Consistent Smyth Powerdomains of Topological Spaces and Quasicontinuous Domains, Topology and its Applications,2017,228: 327-340. doi: 10.1016/j.topol.2017.06.016.
[7] M. Ern´e, The ABC Order and Topology, Berlin: Hedermann, 1991.
[8] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott, Continuous Lattices and Domains, Cambridge University Press, Cambridge, 2003.
[9] Y. X. chen, H. Kou, Z. C.lyu, Continuity and core compactness of topological spaces, arXiv:2203.06344. doi: 10.48550/arXiv.2203.06344.
[10] X. L. Xie, H. Kou, Lower power structures of directed spaces, J. Sichuan Univ., 57 (2020), 211C 217. doi: 10.3969/j.issn.0490-6756.2020.002.
[11] S. Luo, X. Xu, On Monotone Determined Spaces, Electronic Notes in Theoretical Computer Science, 2017, 333: 63-72. doi: 10.1016/j.entcs.2017.08.006.
[12] H. Miao, Q. Li, D. Zhao, On two problems about sobriety of topological spaces, Topology and its Applications, 2021, 295: 107667. doi: 10.1016/j.topol.2021.107667.
[13] X. L. Xie, H. Kou, The Cartesian closedness of c-spaces,AIMS Mathematics 2022, 7(9): 16315-16327. doi: 10.3934/math.2022891.
[14] G. Gierz, J. Lawson, Generalized continuous and hypercontinuous lattices, The Rocky Mountain Journal of Mathematics, 1981, 11(2): 271-296.
[15] M. Ern´e, Scott convergence and Scott topology in partially ordered sets II, Continuous Lattices, Springer, Berlin, Heidelberg, 1981, 61-96.
[16] Y. Yu, H. Kou, Directed spaces defined through $T_0$ spaces with specialization order (in Chinese), Journal of Sichuan University (Natural Science Edition), 2015, 52(2): 217-222. doi: 10.3969/j.issn.0490-6756.2015.02.001.
[17] M. Ern´e, Infinite distributive laws versus local connectedness and compactness properties, Topology and its Applications, 156, 2054-2069 (2009). doi: 10.1016/j.topol.2009.03.029.
[18] H. Kou, Directed spaces: An extended framework for domain theory, 1th Pan Pacific International Conference on Topology and Applications Min Nan Normal University, Zhangzhou City(2015), 11, 25-30.
[19] M. Che, H. Kou, A cartesian closed full subcategory of c-spaces (in Chinese), Journal of Sichuan Normal University(Natural Science), 2020, 43(06): 756-762. doi: 10.3969/j.issn.1001-8395.2020.06.005.
[20] K. KEIMEL. Topological cone: functional analysis in $T_0$-setting, Semigroup Forum, 2008, 77(1):109-142. doi: 10.1007/s00233-008-9087-0.
[21] H. Feng, H. Kou, Quasicontinuity and meet-continuity of $T_0$ spaces(inChinese), Journal of Sichuan University (Natural Science Edition), 2017, 54(05): 905-910. doi: 10.3969/j.issn.0490-6756.2017.05.002.
[22] X. Y. Zhang, Y. W. Hou, W. F. Zhang. The properties of continuous space(in Chinese), Fuzzy Systems and Mathematics, 2017, 31(2): 57-61.
[23] W. Wu, H. L. Zhang, Y. Wang, The Way Below Bases of Directed Spaces(In chinese), Journal of Systems Science and Mathematical Sciences, 2022, 42(04):1060-1066. doi: 10.12341/jssms21243.
[24] R. Heckmann, K. Keimel, Quasicontinuous Domains and the Smyth Powerdomain, Electronic Notes in Theoretical Computer Science 298 (2013) 215C232. doi: 10.1016/j.entcs.2013.09.015.
[25] X. L. Xie, Y. X. Chen, H. Kou, Power structures of directed spaces, arXiv:2203.04506. doi: 10.48550/arXiv.2204.09926.