Cardinality of a floor function set

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Abstract
Fix a positive integer $X$. We quantify the cardinality of the set $\{\lfloor X/n \rfloor\}_{n=1}^X$. We discuss restricting the set to those elements that are prime, semiprime or similar.

1 Introduction
Throughout we will restrict the variables $m$ and $n$ to positive integer values. For any real number $X$ we denote by $\lfloor X \rfloor$ its integer part, that is, the greatest integer that does not exceed $X$. The most straightforward sum of the floor function is related to the divisor summatory function since

$$
\sum_{n \leq X} \left\lfloor \frac{X}{n} \right\rfloor = \sum_{n \leq X} \tau(n),
$$

where $\tau(n)$ is the number of divisors of $n$. From [2, Theorem 2] we infer

$$
\sum_{n \leq X} \left\lfloor \frac{X}{n} \right\rfloor = X \log X + X(2\gamma - 1) + O\left(X^{517/1648+o(1)}\right),
$$

where $\gamma$ is the Euler–Mascheroni constant, in particular $\gamma \approx 0.57722$.

Recent results have generalised this sum to

$$
\sum_{n \leq X} f\left(\left\lfloor \frac{X}{n} \right\rfloor\right),
$$

where $f$ is an arithmetic function (see [1], [3] and [4]).

In this paper we take a different approach by examining the cardinality of the set

$$
S(X) := \left\{ m : m = \left\lfloor \frac{X}{n} \right\rfloor \text{ for some } n \leq X \right\}.
$$

Or main results are as follows.
**Theorem 1.1.** Let $X$ be a positive integer and let

$$b = \frac{-1 + \sqrt{4X + 1}}{2}.$$ 

We have

$$|S(X)| = [b] + \left\lfloor \frac{X}{[b+1]} \right\rfloor.$$ 

**Theorem 1.2.** We have

$$|S(X)| = 2\sqrt{X} + O(1).$$

## 2 Proof of Main Theorems

Throughout let

$$b = \frac{-1 + \sqrt{4X + 1}}{2}, \quad (2.1)$$

and note that

$$\frac{X}{b} = b + 1.$$ 

We define 2 sets:

$$S_1(X) = \left\{ m : m = \left\lfloor \frac{X}{n} \right\rfloor, m \leq b \right\}. \quad (2.2)$$

and

$$S_2(X) = \left\{ m : m = \left\lfloor \frac{X}{n} \right\rfloor, n(n-1) \leq X \right\}. \quad (2.3)$$

We will quantify $S_1(X)$ and then show that $S_1(X) \cup S_2(X) \subseteq S(X)$. This will allow us to use the inclusion-exclusion principle once we quantify $|S_2(X)|$ and $|S_1(X) \cap S_2(X)|.$

We start by calculating the number of elements of $S_1(X)$. Let $m$ be an arbitrary positive integer with

$$m \leq b = \frac{-1 + \sqrt{4X + 1}}{2}.$$ 

This means that

$$m^2 + m - X \leq 0$$

which implies that $m(m + 1) \leq X$. Thus

$$\frac{X}{m(m + 1)} \geq 1$$
and therefore
\[ \frac{X}{m} - \frac{X}{m+1} \geq 1. \]

Since the interval from \( \frac{X}{m} \) to \( \frac{X}{m+1} \) is at least 1 there must be an integer \( n \) such that
\[ \frac{X}{m+1} < n \leq \frac{X}{m}, \tag{2.4} \]
from which
\[ X - n < mn \leq X. \]

In turn this implies that
\[ \frac{X}{n} - 1 < m \leq \frac{X}{n}. \]

This means that \( m = \lfloor X/n \rfloor \) and so \( m \in S_1(X) \). From (2.2) there are \( \lfloor b \rfloor \) possible values of \( m \). From (2.4) we see that can always find an \( n \) to give us any of these values of \( m \). Therefore the numbers 1, 2, \ldots, \( \lfloor b \rfloor \) are the only elements of \( S_1(X) \) and so
\[ |S_1(X)| = \lfloor b \rfloor. \tag{2.5} \]

Apart from quantifying \( S_1(X) \) we also note that the fact that 1, 2, \ldots, \( \lfloor b \rfloor \) \( \in \) \( S_1(X) \) implies that \( S_1(X) \subseteq S(X) \). By reference to the definitions of \( S_2(X) \) and \( S(X) \) we see that \( S_2(X) \subseteq S(X) \). Thus
\[ S_1(X) \cup S_2(X) \subseteq S(X) \]
and so, using the inclusion-exclusion principle,
\[ |S(X)| = |S_1(X)| + |S_2(X)| - |S_1(X) \cap S_2(X)|. \tag{2.6} \]

We now consider the cardinality of \( S_2(X) \). We show that \( n(n-1) < X \) implies \( \lfloor X/n \rfloor \) and \( \lfloor X/(n-1) \rfloor \) are distinct. We have
\[ \left| \frac{X}{n-1} - \frac{X}{n} \right| = \frac{X}{n-1} - \frac{X}{n} - \left\{ \frac{X}{n-1} \right\} + \left\{ \frac{X}{n} \right\}, \]
where \( \{ \} \) represents, as usual, the fractional part of the real number. So
\[ \left| \frac{X}{n-1} - \frac{X}{n} \right| = \frac{X}{n(n-1)} + t, \tag{2.7} \]
where \( t \in (-1, 1) \). Recalling that \( n(n-1) \leq X \) we have
\[ \frac{X}{n(n-1)} \geq 1. \]

Substituting into (2.7) we see that
\[ \left| \frac{X}{n-1} - \frac{X}{n} \right| > 0 \]
which implies that $\lfloor X/n \rfloor$ and $\lfloor X/(n-1) \rfloor$ are distinct. Since $n(n-1) \leq X$ we have, solving the quadratic equation,
\[ n \leq \frac{X}{b} \tag{2.8} \]
and so
\[ |S_2(X)| = \left\lfloor \frac{X}{b} \right\rfloor = [b] + 1. \tag{2.9} \]

To finish the proof it only remains to consider $|S_1(X) \cap S_2(X)|$. We have seen that $S_1(X) = \{1, 2, \cdots, [b]\}$. From (2.8) we see that
\[ n \leq \frac{X}{b} = b + 1. \]
So the values of $n$ in $S_2(X)$ are $1, 2, \cdots \lfloor b+1 \rfloor$ and therefore
\[ S_2(X) = \left\{ \left\lfloor \frac{X}{[b+1]} \right\rfloor, \left\lfloor \frac{X}{[b]} \right\rfloor, \cdots, X \right\}. \]

The set $S_1(X) \cap S_2(X)$ will be non empty if
\[ \left\lfloor \frac{X}{[b+1] - c} \right\rfloor = [b - d], \]
for some $c, d \geq 0$. From this we deduce that
\[ \frac{X}{b+1 - c} < b + 1 - d, \]
and so
\[ X < ((b+1) - d)(b+1 - c). \]
Recalling that $X = b(b+1)$ we have
\[ b + 1 - c(b + d + 1) - d(b + 1) > 0, \]
which is only possible if $c = d = 0$. Thus there will be at most one element of $S_1(X) \cap S_2(X)$ and this one element will occur if, and only if,
\[ \left\lfloor \frac{X}{[b+1]} \right\rfloor = [b]. \]
In fact,
\[ |S_1(X) \cap S_2(X)| = 1 - \left( \left\lfloor \frac{X}{[b+1]} \right\rfloor - [b] \right). \]
Combining this equation with (2.6), (2.5) and (2.9) and simplifying completes the proof of Theorem 1.1.

Theorem 1.2, that is
\[ |S(X)| = 2\sqrt{X} + O(1), \]
follows immediately from Theorem 1.1.
3 Discussion

We can generalise $S(X)$ by considering elements of $S(X)$ that are divisible by some positive integer $d \leq X$. This is interesting in its own right but could also form the basis for calculating something much more interesting: the number of primes, semi primes or similar in $S(X)$.

Let

$$S_d(X) = \left\{ m : m = \left\lfloor \frac{X}{n} \right\rfloor \text{ for some } n \leq X, d \mid \left\lfloor \frac{X}{n} \right\rfloor \right\}.$$

A standard approach to express $S_d(X)$ would be to follow a path involving an indicator function, differences of floor functions, the $\psi$ function and exponential sums, hoping that we can bound the exponential sums (here $\psi(y) = y - \lfloor y \rfloor - 1/2$). Unfortunately this is not the case here. The process yields

Lemma 3.1.

$$|S_d(X)| = \frac{4X^{1/2}}{3d} + \sum_{r=1}^{\lfloor X/a \rfloor} \sum_{j=\lfloor \frac{X}{dj} \rfloor} \left( \psi \left( \frac{X}{dj} + 1 \right) - \psi \left( \frac{X}{dj} \right) \right) + O(1),$$

where $a = X/b$.

A proof is given in Section 5. Calculating various sums using Maple suggests that the double sum cannot successfully be bound. In fact Maple suggests that the double sum is asymptotically equivalent to $2X^{1/2}/3d$. If this argument is correct then

$$S_d(X) \sim \frac{2X^{1/2}}{d},$$

as one would expect heuristically.

4 Trivial bounds

In the absence of a better approach we outline some trivial bounds on $|S_d(X)|$. The interested reader may wish to improve these bounds.

**Theorem 4.1.** For a real positive $X$ and a positive integer $d \leq X$ with $d \neq 1$ we have

$$\frac{X^{1/2}}{d} + O(1) \leq |S_d(X)| \leq \frac{X}{2} + O(1).$$

**Proof.** The lower bound follows from the fact that $1, 2, \ldots, \lfloor b \rfloor \in S(X)$ (see Section 2). Of these $\lfloor b \rfloor / d$ will be divisible by $d$. Recalling that $b = X^{1/2} + O(1)$ the result follows. The upper bound flows from the fact that of the $X$ numbers in the sequence $(\lfloor X/n \rfloor)_{n=1}^{X}$ the number 1 appears $X/2$ times if $X$ is even and $(X + 1)/2$ times if $X$ is odd. \qed
5 Proof of Lemma 3.1

To simplify notation we will let

\[ a = \frac{X}{b}. \]

It is clear that

\[ S_d(X) = S^{-}_d(X) \cup S^+_d(X), \tag{5.1} \]

where

\[ S^{-}_d(X) = \left\{ m : m = \left\lfloor \frac{X}{n} \right\rfloor \text{ for some } n < a, d | \left\lfloor \frac{X}{n} \right\rfloor \right\} \]

and

\[ S^+_d(X) = \left\{ m : m = \left\lfloor \frac{X}{n} \right\rfloor \text{ for some } n \geq a, d | \left\lfloor \frac{X}{n} \right\rfloor \right\}. \]

From Section 2 it is clear that the numbers 1, 2, \ldots, \lfloor b \rfloor will be elements of \( S^+(x) \). Of these exactly \( \lfloor \lfloor b \rfloor /d \rfloor \) will be divisible by \( d \) and so

\[ |S^+_d(x)| = X^{1/2} + O(1). \tag{5.2} \]

We now quantify \( S^{-}_d(X) \). We observe that if

\[ \frac{X}{dj} < n < \frac{X}{dj+1} \]

then

\[ dj < \frac{X}{n} \leq dj + 1 \]

and so

\[ \left\lfloor \frac{X}{n} \right\rfloor = dj \text{ for some } j, \]

which implies that

\[ m = \left\lfloor \frac{X}{n} \right\rfloor \in S^{-}_d(X). \]

Furthermore,

\[ |S^{-}_d(X)| = \sum_{\frac{X}{d+1} < n \leq \frac{X}{dj}} 1, \]

if the elements of \( \{\lfloor X/n \rfloor\}_{1}^{[a]} \) are distinct. To see that this condition is true we note that \( n < a \) implies that \( n(n+1) < X \) which means that \( X/n(n+1) > 1 \). Thus \( \lfloor X/n \rfloor - \lfloor X/(n+1) \rfloor > 0 \) which proves distinctiveness.

Next, since \( n < a \) we also have that

\[ \frac{X}{dj} - \frac{X}{dj+1} = X \frac{1}{m(m+1)} < X \frac{1}{m^2} < 1, \]

where

\[ m = \left\lfloor \frac{X}{n} \right\rfloor \in S^{-}_d(X). \]

Thus\( |S^{-}_d(x)| = X^{1/2} + O(1). \)
the last inequality being justified by the fact that \( n < a \) implies that \( m > \frac{X}{d^{\frac{1}{2}}} \). This means that there can only be one value of \( n \) between
\[
\frac{X}{d^{j}} \text{ and } \frac{X}{d^{j} + 1}.
\]
Therefore
\[
|S^{-}_{d}(X)| = \sum_{r=1}^{[a]} \sum_{j = \frac{X}{rd}}^{\frac{X}{rd}} 1(j),
\]
where
\[
1(j) = \begin{cases} 
1 & \text{if } X/(dj + 1) < n \leq X/dj \text{ for some } n \in \mathbb{N}, \\
0 & \text{otherwise.}
\end{cases}
\]
We can replace the indicator function with floor functions as follows:
\[
|S^{-}_{d}(X)| = \sum_{r=1}^{[a]} \sum_{j = \frac{X}{rd}}^{\frac{X}{rd}} \left| \frac{X}{jd} \right| - \left| \frac{X}{d^{j} + 1} \right|. 
\tag{5.3}
\]
For any real \( t \in \mathbb{R} \) we denote
\[
\psi(t) = t - [t] - \frac{1}{2}.
\]
Replacing the floor functions in (5.3) with the \( \psi \) function we obtain
\[
|S^{-}_{d}(X)| = \sum_{r=1}^{[a]} \sum_{j = \frac{X}{rd}}^{\frac{X}{rd}} \left( X/dj - X/dj + 1 + \psi \left( \frac{X}{dj} \right) - \psi \left( \frac{X}{dj + 1} \right) \right) 
\tag{5.4}
= S_{1} + S_{2},
\]
where
\[
S_{1} = \sum_{r=1}^{[a]} \sum_{j = \frac{X}{rd}}^{\frac{X}{rd}} \left( X/dj - X/dj + 1 \right)
\]
and
\[
S_{2} = \sum_{r=1}^{[a]} \sum_{j = \frac{X}{rd}}^{\frac{X}{rd}} \left( \psi \left( \frac{X}{dj + 1} \right) - \psi \left( \frac{X}{dj} \right) \right).
\]
Estimating $S_1$ we have
\[
S_1 = \sum_{r=1}^{[a]} \sum_{j=-\frac{X}{rd}}^{\frac{X}{rd}} \frac{X}{dj(dj+1)}
\]
\[
= \frac{X}{d} \sum_{r=1}^{[a]} \sum_{j=-\frac{X}{rd}}^{\frac{X}{rd}} \frac{1}{dj^2} - \frac{X}{d} \sum_{r=1}^{[a]} \sum_{j=-\frac{X}{rd}}^{\frac{X}{rd}} \frac{1}{dj^2(dj+1)}
\]
\[
= \frac{X}{d^2} \sum_{r=1}^{[a]} \sum_{j=-\frac{X}{rd}}^{\frac{X}{rd}} \frac{1}{j^2} + O \left( X \sum_{r=1}^{[a]} \sum_{j=-\frac{X}{rd}}^{\frac{X}{rd}} \frac{1}{j^3} \right). \tag{5.5}
\]

We now estimate
\[
\frac{X}{d^2} \sum_{r=1}^{[a]} \sum_{j=-\frac{X}{rd}}^{\frac{X}{rd}} \frac{1}{j^2}.
\]

Using Abel summation we have
\[
\sum_{j=-\frac{X}{rd}}^{\frac{X}{rd}} \frac{1}{j^2} = \frac{(X-r)/rd}{((X-r)/rd)^2} - \frac{X/rd}{(X/rd)^2} - \int_{(X-r)/rd}^{X/rd} t \left( \frac{-2}{t^3} \right) dt
\]
\[
= \frac{1}{(X-r)/rd} - \frac{1}{X/rd} + 2 \int_{(X-r)/rd}^{X/rd} \frac{1}{t^2} dt
\]
\[
= \frac{-r^2d}{X(X-r)} + 2 \frac{r^2d}{X(X-r)}
\]
\[
= \frac{r^2d}{X(X-r)}.
\]

So
\[
\frac{X}{d^2} \sum_{r=1}^{[a]} \sum_{j=-\frac{X}{rd}}^{\frac{X}{rd}} \frac{1}{j^2} = \frac{X}{d^2} \sum_{r=1}^{[a]} \frac{r^2d}{X(X-r)}
\]
\[
= \frac{1}{d} \sum_{r=1}^{[a]} \frac{r^2}{X-r}.
\]

Using Abel summation again we have
\[
\frac{X}{d^2} \sum_{r=1}^{[a]} \sum_{j=-\frac{X}{rd}}^{\frac{X}{rd}} \frac{1}{j^2} = \frac{1}{d} \left[ \frac{|a|((|a|+1)(2|a|+1)}}{6(X-|a|)} - \int_{1}^{[a]} \frac{u(u+1)(2u+1)}{6(X-u)^2} du \right].
\]
Observe that $[a] = X^{1/2} + O(1)$. Thus

$$
\frac{X}{d^2} \sum_{r=1}^{X^{1/2}} \sum_{j=\lfloor \frac{X}{rd} \rfloor}^{X} \frac{1}{j^2} = \frac{1}{6d} \left[ \frac{2X^{3/2} + O(X)}{X - X^{1/2} + O(1)} + O(1) \right]
$$

$$
= \frac{X^{1/2}}{3d} + O(1). \tag{5.6}
$$

Using a similar analysis we have

$$
O \left( X \sum_{r=1}^{X^{1/2}} \sum_{j=\lfloor \frac{X}{rd} \rfloor}^{X} \frac{1}{j^3} \right) = O(X^{-1/2}). \tag{5.7}
$$

Substituting (5.6) and (5.7) into (5.5) we conclude that

$$
S_1 = \frac{X^{1/2}}{3d} + O(1). \tag{5.8}
$$

substituting this expression for $S_1$ into (5.4) and then (5.4) and (5.2) into (5.1) completes the proof.

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References

[1] O. Bordellès, L. Dai, R. Heyman, H. Pan, I. E. Shparlinski, ‘On a sum involving the Euler function’, Journal of Number Theory, accepted manuscript, available at https://doi-org.wwproxy1.library.unsw.edu.au/10.1016/j.jnt.2019.01.006

[2] J. Bourgain and N. Watt, ‘Mean square of zeta function, circle problem and divisor problem revisited’, Preprint (2017) available at arXiv:1709.04340 [math.NT]

[3] S. Chern ‘Notes on sums involving the Euler function’, Preprint (2018) available at arXiv:1812.04657[math.NT]

[4] A. Goswami, ‘On a partial sum related to the Euler function’, Preprint (2018) available at arXiv:1812.07556 [math.NT]