A magneto-viscoelasticity problem with a singular memory kernel

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\textbf{Abstract}

The existence of solutions to a one-dimensional problem arising in magneto-viscoelasticity is here considered. Specifically, a non-linear system of integro-differential equations is analyzed; it is obtained coupling an integro-differential equation modeling the viscoelastic behaviour, in which the kernel represents the relaxation function, with the non-linear partial differential equations modeling the presence of a magnetic field. The case under investigation generalizes a previous study since the relaxation function is allowed to be unbounded at the origin, provided it belongs to $L^1$; the magnetic model equation adopted, as in the previous results \cite{21, 22, 24, 25} is the penalized Ginzburg-Landau magnetic evolution equation.

\textbf{1 Introduction}

The study of magneto-viscoelastic materials is motivated by the interest on mechanical properties of innovative materials widely studied in a variety of applications. In particular, as far as the coupling between mechanical and magnetic effects is concerned, the interest is motivated by new materials such as Magneto Rheological Elastomers or, in general, magneto-sensitive polymeric composites (see \cite{14, 15} and references therein). A variational approach to study multiscale models, in this context, is given in \cite{8}. The results here presented are connected to a wide research project concerning the analytical study of differential and integro-differential models connected to mechanical properties of materials. Thus, in \cite{24, 25, 29} magneto-elasticity problems are considered, in \cite{21, 22} magneto-viscoelasticity problems are studied. Then, in turn, the case of a 1-dimensional, and of a 3-dimensional, body is investigated under the assumption of a regular kernel representing the relaxation modulus. Later, materials with memory characterized by a singular kernel integro-differential equations are studied in \cite{19, 20, 23}. Indeed, as pointed out therein, the case is of interest not only to model different physical behaviours but also under the analytical viewpoint. The interest in singular kernel problems goes back to Boltzmann \cite{5} and later, is testified, analytically, by the results of Berti \cite{4}, Giorgi and Morro \cite{9}, Grasselli and Lorenzi \cite{10} and Hanyga et al. \cite{11, 12, 13}. In addition, fractional derivative models, since the works of Rabotnov \cite{17} and Koeller \cite{16}, are

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employed in [3, 6, 7]. Here, a viscoelastic body is studied under the assumption of a relaxation modulus, modeled by a $L^1$ function, coupled with a magnetic field.

The problem to study is concerned with the behaviour of a viscoelastic body subject also to the presence of a magnetic field. The body is assumed to be one-dimensional. In particular the problem under investigation is motivated by a great interest in the realization of new materials which, on one side couple a viscoelastic behaviour with a magnetic one, see

$$u_t(t) - \int_0^t G(t - \tau)u_{xx}(\tau)d\tau - u_1 - \int_0^t \frac{\lambda}{2} (\Lambda(m) \cdot m)_x d\tau = \int_0^t f(\tau)d\tau$$

in $Q$

$$m_t + m \frac{|m|^2 - 1}{\delta} + \lambda \Lambda(m) u_x - m_{xx} = 0,$$

(1.1)

together with the initial and boundary conditions

$$u(\cdot, 0) = u_0 = 0, \quad m(\cdot, 0) = m_0, \quad |m_0| = 1 \text{ in } \Omega,$$

(1.2)

$$u = 0, \quad \frac{\partial m}{\partial \nu} = 0 \quad \text{on } \Sigma = \partial \Omega \times (0, T),$$

(1.3)

where $\Omega = (0, 1)$, $Q := \Omega \times (0, T)$ and $\mathcal{M} \equiv (0, m)$, letting $m = (m_1, m_2)$, is the magnetization vector, orthogonal to the conductor so that, since $u \equiv (u, 0, 0)$, when both quantities are written in $\mathbb{R}^3$; in addition, $\nu$ is the outer unit normal at the boundary $\partial \Omega$, $\Lambda$ is a linear operator defined by $\Lambda(m) = (m_2, m_1)$, the scalar function $u$ is the displacement in the direction of the conductor itself, here identified with the $x-$axis and $\lambda$ is a positive parameter. In addition, the term $f$ represents an external force which also includes the deformation history. Moreover we assume:

$$u_1 \in L^2(\Omega), \quad m_0 \in H^1(\Omega), \quad f \in L^2(Q).$$

(1.4)

The model adopted here to describe the magneto-elastic interaction is introduced in [28, 24, 25, 29] and the case of magneto-viscoelastic regular behaviour is given in [21, 22]. In fact, the kernel in the linear integro-differential equation, which represents the relaxation function $G$, is assumed here to satisfy weaker functional requirements with respect to the classical regularity requirements. In particular, the relaxation function $G(t)$ is assumed to be such that

$$G \in L^1(0, T) \cap C^2(0, T), \quad \forall T \in \mathbb{R}^+;$$

(1.5)

the relaxation function $G(t)$ is assumed to satisfy the further requirements, which follow from the physics of the model,

$$G(t) > 0, \quad \dot{G}(t) \leq 0, \quad \ddot{G}(t) \geq 0, \quad t \in (0, \infty).$$

(1.6)

Note that, in the classical model the relaxation function, further to satisfy conditions (1.6), is assumed to be $C^2[0, T]$, $\forall T \in \mathbb{R}^+$. To this aim, in the following Section 2, a suitable sequence of approximated classical problems is constructed. In the same Section also some apriori estimates are obtained. Crucial in our analysis is the assumption $u_0 = 0$.

The subsequent Section 3, is devoted to prove the existence of a weak solution to the problem (1.1) with the initial and boundary conditions (1.2) - (1.3).
2 Approximated problems and a priori estimates

In this Section, the approximation strategy is devised and, then, some estimates which are needed to prove the existence results are given.

First of all, observe that, the reason why equation (1.1) is written under the form of an evolution equation is that the classical model [26, 27] is not defined since it depends on $G(0)$ and on the integral of $\dot{G}$ which is not assumed to be in $L^1$ at the origin. However, in our case, even if the the kernel $G$ of the integral equation is singular at the origin, the regularity requirements it is supposed to satisfy (Cf. (1.5)), guarantee that the classical problem can be adopted to model the magneto-viscoelastic behaviour of the material as soon as we consider a time $t > 0$. Hence, here a sequence of time-translated approximated problems is constructed. Specifically, let $\varepsilon$ denote a small parameter $0 < \varepsilon \ll 1$ and consider an approximated problem corresponding to each value of the parameter $\varepsilon$ defined via a $\varepsilon$ time translation, that is, let us introduce, corresponding to each $\varepsilon > 0$, the translated relaxation function $G_\varepsilon(\cdot) := G(\varepsilon + \cdot)$. Furthermore, it is coupled with a penalized version of the magnetization equation with penalization parameter $0 < \delta \ll 1$, i.e. adopting the same model in [21] where, now, the magnetization problem is coupled with a translated viscoelasticity equation.

Then, we can introduce the problem $P_\varepsilon$ given by:

$$
\begin{align*}
\left\{ \begin{array}{l}
\ddot{u}_\varepsilon - G_\varepsilon(0)u_{\varepsilon xx} - \int_0^t \dot{G}_\varepsilon(t-\tau)u_{\varepsilon xx}(\tau) d\tau - \frac{\lambda}{2}(\Lambda(m_\varepsilon) \cdot m_\varepsilon)_x = f \\
m_\varepsilon + m_\varepsilon|\frac{m_\varepsilon}{\delta}|^2 - 1 + \lambda \Lambda(m_\varepsilon)_x = 0,
\end{array} \right. \\
\text{in } Q \\
\text{together with the initial and boundary conditions}
\end{align*}
$$

$$
\begin{align*}
u_\varepsilon(\cdot,0) = u_0 = 0, \quad \dot{u}_\varepsilon(\cdot,0) = u_1, \quad m_\varepsilon(\cdot,0) = m_0, \quad \text{in } \Omega,
\end{align*}
$$

$$
\begin{align*}
u_\varepsilon = 0, \quad \frac{\partial m_\varepsilon}{\partial \nu} = 0 \quad \text{on } \Sigma = \partial \Omega \times (0,T),
\end{align*}
$$

where $G_\varepsilon(t) \in C^2[0,T]$ and, hence, the non-linear integro-differential problem $P_\varepsilon$ is well defined. Specifically, according to [21], such a problem admits a unique strong solution.

**Lemma 2.1** Let $\bar{u}$ denote a solution to the problem

$$
\begin{align*}
\ddot{u}_\varepsilon - G_\varepsilon(0)\bar{u}_{\varepsilon xx} - \int_0^t \dot{G}_\varepsilon(t-\tau)\bar{u}_{\varepsilon xx}(\tau) d\tau = F \quad \text{in } Q,
\end{align*}
$$

where in the r.h.s. $F \in L^2(Q)$. The initial and boundary conditions, in turn, are

$$
\begin{align*}
u_\varepsilon(\cdot,0) = u_0 = 0, \quad \bar{u}_\varepsilon(\cdot,0) = u_1, \quad \text{in } \Omega \\
\bar{u} = 0, \quad \text{on } \Sigma = \partial \Omega \times (0,T),
\end{align*}
$$

it follows

$$
\begin{align*}
\frac{1}{2} \int_{\Omega} G(t+\varepsilon)|\bar{u}_\varepsilon|^2 \, dx + \frac{1}{2} \int_{\Omega} |\bar{u}_1|^2 \, dx \leq \frac{1}{2} \int_{\Omega} |u_1|^2 \, dx ds + \frac{1}{2} \int_{\Omega} G(\varepsilon)|u_\varepsilon(0)|^2 + \\
+ \int_{\Omega} \int_{0}^{t} F \bar{u}_\varepsilon \, dx \, ds.
\end{align*}
$$

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**Proof.** Here, for the reader convenience, we give the proof which follows the original one by Dafermos \[20\, 27\]. First of all, equation \((2.14)\), when we change the integration variable \(\tau\) into \(s = t - \tau\) add and subtract the term

\[
\int_0^t G^\tau(s) u_{xx}(t) ds = [G^\tau(t) - G^\tau(0)] u_{xx}(t),
\]

can be written in the following equivalent form

\[
\bar{u}_{tt} - G(t + \varepsilon) \bar{u}_{xx} + \int_0^t \dot{G}(s + \varepsilon) [\bar{u}_{xx}(t) - \bar{u}_{xx}(t - s)] ds = F. \tag{2.14}
\]

Then multiplication of equation \((2.14)\) by \(\bar{u}_t\), and integration over \(\Omega\) gives

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\bar{u}_t|^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Omega G(t + \varepsilon) \bar{u}_x \bar{u}_{xt} dx + \int_\Omega \bar{u}_t(t) dx \int_0^t \dot{G}(s + \varepsilon) [\bar{u}_{xx}(t) - \bar{u}_{xx}(t - s)] ds = \int_\Omega F \bar{u}_t dx \tag{2.15}
\]

that is, since \(\bar{u}_t\) is independent of \(s\)

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\bar{u}_t|^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Omega G(t + \varepsilon) |\bar{u}_x|^2 dx - \frac{1}{2} \int_\Omega \dot{G}(t + \varepsilon) |\bar{u}_x|^2 dx - \int_\Omega dx \int_0^t \dot{G}(s + \varepsilon) \bar{u}_{xt} [\bar{u}_x(t) - \bar{u}_x(t - s)] ds = \int_\Omega F \bar{u}_t dx. \tag{2.16}
\]

Now, observe that

\[
- \int_\Omega \int_0^t \dot{G}(s + \varepsilon) \bar{u}_{xt}[\bar{u}_x(t) - \bar{u}_x(t - s)] dx ds
\]

\[
= -\frac{1}{2} \frac{d}{dt} \int_0^t ds \int_\Omega \dot{G}(s + \varepsilon) |\bar{u}_x(t) - \bar{u}_x(t - s)|^2 dx
\]

\[
+ \frac{1}{2} \int_\Omega \dot{G}(t + \varepsilon) |\bar{u}_x(t) - \bar{u}_x(0)|^2 dx
\]

\[
- \int_\Omega \int_0^t \dot{G}(s + \varepsilon) \bar{u}_{xt}(t - s)[\bar{u}_x(t) - \bar{u}_x(t - s)] dx ds
\]

\[
= -\frac{1}{2} \frac{d}{dt} \int_0^t ds \int_\Omega \dot{G}(s + \varepsilon) |\bar{u}_x(t) - \bar{u}_x(t - s)|^2 dx
\]

\[
+ \frac{1}{2} \int_\Omega \dot{G}(t + \varepsilon) |\bar{u}_x(t) - \bar{u}_x(0)|^2 dx
\]

\[
+ \int_\Omega \int_0^t \dot{G}(s + \varepsilon) \frac{d}{ds} \bar{u}_x(t - s)[\bar{u}_x(t) - \bar{u}_x(t - s)] dx ds
\]

\[
= -\frac{1}{2} \frac{d}{dt} \int_0^t ds \int_\Omega \dot{G}(s + \varepsilon) |\bar{u}_x(t) - \bar{u}_x(t - s)|^2 dx
\]

\[
+ \frac{1}{2} \int_\Omega \dot{G}(t + \varepsilon) |\bar{u}_x(t) - \bar{u}_x(0)|^2 dx
\]

\[
- \frac{1}{2} \int_\Omega \int_0^t \dot{G}(s + \varepsilon) \frac{d}{ds} |\bar{u}_x(t) - \bar{u}_x(t - s)|^2 dx ds
\]
\begin{align*}
= -\frac{1}{2} \frac{d}{dt} \int_0^t ds \int_\Omega \hat{G}(s+\varepsilon)|\vec{u}_x(t) - \vec{u}_x(t-s)|^2 dx \\
+ \frac{1}{2} \int_\Omega \int_0^t \ddot{G}(s+\varepsilon)|\vec{u}_x(t) - \vec{u}_x(t-s)|^2 dx ds.
\end{align*}

Substitution of the latter in (2.16), combined with integration over the time interval \((0, t)\), taking into account the sign conditions (1.6), implies (2.13) and, hence, in the case of homogeneous initial displacement condition, completes the proof. \(\Box\)

This last estimate, later on, is combined with the following one.

**Lemma 2.2** Let \((u^\varepsilon, m^\varepsilon)\) denote a solution to the problem (2.7)–(2.9), then it follows that we have

\[
\begin{align*}
\frac{1}{2} \int_\Omega G^\varepsilon(t)|u^\varepsilon_x|^2 dx + \frac{1}{2} \int_\Omega |u^\varepsilon|^2 dx + \int_0^t \int_\Omega |m^\varepsilon|^2 dx + \frac{1}{2} \int_\Omega |m^\varepsilon_x|^2 dx + \\
\lambda \int_\Omega \Lambda(m^\varepsilon) \cdot m^\varepsilon u^\varepsilon_x dx + \frac{1}{4} \int_\Omega \frac{(|m^\varepsilon|^2 - 1)^2}{\delta} dx \leq \int_0^t \int_\Omega f u^\varepsilon dx + \frac{1}{2} \int_\Omega |m^\varepsilon_0|^2 dx + \frac{1}{2} \int_\Omega |u^\varepsilon_1|^2 dx.
\end{align*}
\]  

**Proof** For sake of simplicity, all the superscripts are omitted. Consider the second equation in (2.7)

\[
m_t + m \frac{|m|^2 - 1}{\delta} + \lambda \Lambda(m) u_x - m_{xx} = 0.
\]  

Taking the scalar product with \(m_t\), after integration over \(\Omega\), it follows

\[
\int_\Omega |m_t|^2 dx + \frac{d}{dt} \int_\Omega \frac{|m|^2 - 1}{\delta} dx + \frac{1}{2} \frac{d}{dt} \int_\Omega |m_x|^2 dx + \lambda \int_\Omega (\Lambda(m) \cdot m_t) u_x dx = 0
\]  

and hence, after integration over \((0, t)\), recall (1.2), i.e. \(|m_0| = 1\),

\[
\int_0^t \int_\Omega |m_t|^2 dx + \frac{1}{4} \int_\Omega \frac{(|m|^2 - 1)^2}{\delta} dx + \frac{1}{2} \int_\Omega |m_x|^2 dx + \lambda \int_0^t \int_\Omega (\Lambda(m) \cdot m_t) u_x dx = 0
\]  

Now, multiplying the first equation in (2.7) by \(u_t\) and integrating over \(\Omega\), recalling Lemma 2.1, it follows

\[
\frac{1}{2} \int_\Omega G(t+\varepsilon)|u_x|^2 dx + \frac{1}{2} \int_\Omega |u_t|^2 dx \leq \int_\Omega \int_0^t \left[ \frac{f + \lambda}{2} (\Lambda(m) \cdot m)_x \right] u_t dx ds + \frac{1}{2} \int_\Omega G(\varepsilon)|u_x(0)|^2 dx + \frac{1}{2} \int_\Omega |u_1|^2 dx
\]  

\[
(2.21)
\]
where, since the initial homogeneous datum \((u_0 = 0)\) is assigned, the term where \(G(\varepsilon)\) appears cancels. Since 
\[
(\Lambda(m) \cdot m_t) = \frac{1}{2} \frac{d}{dt} (\Lambda(m) \cdot m) \quad \text{and} \quad (\Lambda(m) \cdot m_x) = \frac{1}{2} \frac{d}{dx} (\Lambda(m) \cdot m),
\]
the combination of (2.20) and (2.21) allows to write 
\[
\frac{1}{2} \int_{\Omega} G(t + \varepsilon) |u_x|^2 \, dx + \frac{1}{2} \int_{\Omega} |u_t|^2 \, dx + \frac{\lambda}{2} \int_0^t \int_{\Omega} [(\Lambda(m) \cdot m) u_x]_t \, dx + \int_0^t \int_{\Omega} |m_t|^2 \, dx
\]
\[
+ \frac{1}{4} \int_{\Omega} (|m|^2 - 1)^2 \, dx + \frac{1}{2} \int_{\Omega} |m_x|^2 \, dx \leq \int_0^t \int_{\Omega} f u_t \, dx \, ds + \frac{1}{2} \int_{\Omega} |m_0|^2 \, dx + \frac{1}{2} \int_{\Omega} |u_1|^2 \, dx;
\]
which completes the proof. \(\square\)

The following estimates can then be proved.

**Lemma 2.3** Let \((u^\varepsilon, m^\varepsilon)\) denote a solution to the problem (2.7)–(2.9), then the following estimates hold

\[
\int_{\Omega} |u_x^\varepsilon|^2 \, dx \leq C_1
\]
\[
\int_{\Omega} |u_t^\varepsilon|^2 \, dx \leq C_2
\]
\[
\int_{\Omega} |m_x^\varepsilon|^2 \, dx \leq C_3
\]
\[
\int_{Q} |m_t^\varepsilon|^2 \, dx \, dt \leq C_4
\]
\[
\int_{\Omega} \frac{(|m|^2 - 1)^2}{\delta} \, dx \leq C_5
\]

where \(C_k, k = 1, 2, 3, 4, 5,\) depend on \(T, m_0, f, u_1,\) but do not depend on \(\varepsilon\) nor on \(\delta.\)

**Proof** Consider the inequality (2.17) proved in Lemma 2.2, where for simplicity, all the subscripts are omitted and where the initial data are included within the constant \(C(m_0, u_1),\)

\[
\frac{1}{2} \int_{\Omega} G(t + \varepsilon) |u_x|^2 \, dx + \frac{1}{2} \int_{\Omega} |u_t|^2 \, dx + \frac{1}{4} \int_{\Omega} \frac{|m|^2 - 1}{\delta} \, dx + \frac{1}{2} \int_{\Omega} |m_x|^2 \, dx +
\]
\[
\int_0^t \int_{\Omega} |m_t|^2 \, dx \leq -\frac{\lambda}{2} \int_{\Omega} (\Lambda(m) \cdot m) u_x \, dx + \int_0^t \int_{\Omega} f u_t \, dx \, ds + C(m_0, u_1).
\]

Now we have

\[
\left| -\frac{\lambda}{2} \int_{\Omega} (\Lambda(m) \cdot m) u_x \, dx \right| \leq \frac{\lambda}{2} \int_{\Omega} |m|^2 |u_x| \, dx.
\]
Furthermore, observe that

\[
\int_\Omega |m|^2 |u_x| \, dx = \sqrt{\delta} \int_\Omega \frac{|m|^2 - 1}{\sqrt{\delta}} |u_x| \, dx + \int_\Omega |u_x| \, dx \leq (2.26)
\]

\[
\frac{\sqrt{\delta}}{2} \int_\Omega \frac{(|m|^2 - 1)^2}{\delta} \, dx + \sqrt{\frac{2}{\delta}} \int_\Omega |u_x|^2 \, dx + \frac{\sigma}{2} \int_\Omega |u_x|^2 \, dx + \frac{1}{2\sigma} |\Omega| ,
\]

where both \( \sigma < 1 \) and \( \delta < 1 \). Since \( G(t + \varepsilon) > G(T + 1), \forall t \in (0, T) \), we can choose \( \sigma \) and \( \delta \) so that

\[
\lambda \sqrt{\delta} < \frac{1}{2}, \quad \lambda (\sqrt{\delta} + \sigma) < G(T + 1) \quad (2.27)
\]

and hence we obtain:

\[
\int_0^t \int_\Omega |m|^2 |u_x| \, dx \, dt + \frac{1}{2} \int_\Omega |u_t|^2 \, dx + \int_0^t \int_\Omega |m|^2 \, dx \, dt + \frac{1}{2} \int_\Omega |m_x|^2 \, dx + \frac{1}{8} \int_\Omega \frac{(|m|^2 - 1)^2}{\delta} \, dx \leq \int_0^t \int_\Omega f \, dx \, dt + \frac{1}{2} \int_\Omega |m_0|^2 \, dx \, dt + \frac{1}{2} \int_0^t |u|^2 \, dx \, dt + \frac{1}{2\sigma} |\Omega| \quad (2.28)
\]

Hence, if we set

\[
E(t) := \frac{1}{4} \int_\Omega G(t) |u_x|^2 \, dx + \frac{1}{2} \int_\Omega |u_t|^2 \, dx + \frac{1}{2} \int_\Omega |m|^2 \, dx + \frac{1}{2} \int_\Omega |m_x|^2 \, dx + \frac{1}{8} \int_\Omega \frac{(|m|^2 - 1)^2}{\delta} \, dx ,
\]

noting that

\[
\left| \int_0^t \int_\Omega f \, dx \, ds \right| \leq \frac{1}{2} \int_0^t \int_\Omega |u_t|^2 \, dx \, ds + \frac{1}{2} \int_0^t \int_\Omega |f|^2 \, dx \, ds \quad (2.30)
\]

we obtain

\[
E(t) - \int_0^t E(\tau) \, d\tau \leq C(T, m_0, f, u_1) .
\]

Note, on application of Gronwall’s Lemma, it follows that

\[
E(t), \int_0^t E(\tau) \, d\tau \leq \tilde{C}(T, m_0, f, u_1) ,
\]

and the proof is completed since all the inequalities (2.23) are implied.

\[\square\]

3 Existence result for the limit problem

This Section is devoted to prove the existence of weak solutions to the non-linear integro-differential problem (1.1) – (1.3). The key tools are the estimates which are independent of \( \varepsilon \). Here the limit when the parameter \( \varepsilon \to 0 \) is studied. This allows us to establish the existence result in the generalized case of singular kernel, as far as the viscoelastic behaviour is concerned: this result generalizes the previous one in [21].

**Theorem 3.1** For all \( T > 0 \), there exists a weak solution \((u, m)\) to the problem (1.1)-(1.2)-(1.3), that is a vector function \((u, m)\) s.t.
• $u \in L^\infty(0, T; H^1_0(\Omega))$;
• $u_t \in L^\infty(0, T; L^2(\Omega))$;
• $m \in L^\infty(0, T; H^1(\Omega))$;
• $m_t \in L^2(Q)$.

which satisfies

\[ \begin{align*}
- \int_Q \phi_t u^\varepsilon(t)dxdt + \int_Q \int_0^t G^\varepsilon(t - \tau)u^\varepsilon_x(\tau)\phi_x d\tau dxdt &+ \int_Q \int_0^t \frac{\lambda}{2} \Lambda(m^\varepsilon) \cdot m^\varepsilon \phi_x d\tau dxdt \\
- \int_Q \left[ u_1 + \int_0^t f(\tau)d\tau \right] \phi dxdt &+ \int_Q \psi_t \cdot m^\varepsilon dxdt + \int_Q m_0 \cdot \psi(. , 0)dxdt + \\
\int_Q \left( \frac{|m^\varepsilon|^2 - 1}{\delta} \right) \psi \cdot m^\varepsilon dxdt &- \int_Q \int_0^t \lambda u^\varepsilon_x \Lambda(m^\varepsilon) \cdot \psi dxdt - \int_Q m^\varepsilon \cdot \psi dxdt = 0 .
\end{align*} \]

(3.32)

\[ \forall \phi \text{ smooth s.t. } \phi(0, t) = \phi(1, t) = 0, \phi(. , T) = 0, \text{ and } \forall \psi \equiv (\psi_1, \psi_2) \text{ s.t. } \psi(x, T) = 0. \]

**Proof** By a weak solution to

\[ \begin{align*}
&u_t^\varepsilon(t) - \int_0^t G^\varepsilon(t - \tau)u^\varepsilon_x(\tau)d\tau - u_1 - \int_0^t \frac{\lambda}{2} (\Lambda(m^\varepsilon) \cdot m^\varepsilon)_x d\tau = \int_0^t f(\tau)d\tau \\
&u^\varepsilon(\cdot , 0) = u_0 = 0, \ u^\varepsilon(x, t) = 0 \text{ on } \partial\Omega \times (0, T)
\end{align*} \]

we mean a function $u(x, t)$ such that

\[ \begin{align*}
&- \int_Q \phi_t u + \int_Q \int_0^t G(t - \tau)\phi x u(\tau)d\tau + \int_Q \int_0^t \frac{\lambda}{2} (\Lambda(m) \cdot m)\phi x d\tau = (3.34) \\
&\int_Q \phi \left[ u_1 + \int_0^t f(\tau)d\tau \right]
\end{align*} \]

\[ \forall \phi \text{ smooth s.t. } \phi(0, t) = \phi(1, t) = 0, \phi(\cdot , T) = 0, \text{ where } Q = \Omega \times (0, T) \text{ (we dropped the measure of integration } dxdt). \]

By a weak solution to

\[ \begin{align*}
m_t^\varepsilon + m^\varepsilon \frac{|m|^2 - 1}{\delta} + \lambda \Lambda(m)u_x - m_{xx} = 0 \text{ in } Q
\end{align*} \]

(3.35)

\[ \begin{align*}
m(\cdot , 0) = m_0, \ \frac{\partial m}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0, T)
\end{align*} \]

we mean a function $m \equiv (m_1, m_2)$ such that

\[ \begin{align*}
- \int_Q \psi_1 \cdot m + \int_Q \psi \cdot m^\varepsilon \frac{|m|^2 - 1}{\delta} + \int_Q \psi \cdot \Lambda(m)u_x + \int_Q \psi_x \cdot m_x - \int_\Omega m_0 \cdot \psi(x, 0)dx &= 0 .
\end{align*} \]

(3.36)
∀ψ s.t. ψ(x, T) = 0 where ψ ≡ (ψ_1, ψ_2). We start from \((u^\varepsilon, m^\varepsilon)\) satisfying

\[
\begin{aligned}
- \int_\Omega \phi_t u^\varepsilon(t) + \int_\Omega \int_0^t G^\varepsilon(t - \tau) u_x^\varepsilon(\tau) \phi_x d\tau + \int_\Omega \frac{1}{2} \Lambda(m^\varepsilon) \cdot m^\varepsilon \phi_x d\tau &= \\
&= \int_\Omega \left[u_1 + \int_0^t f(\tau) d\tau\right] \phi \tag{3.37}
\end{aligned}
\]

- \int_\Omega \psi_t \cdot m^\varepsilon - \int_\Omega m_0 \cdot \psi(\cdot, 0) - \int_\Omega \left(\frac{|m^\varepsilon|^2}{\delta} - 1\right) \psi \cdot m^\varepsilon + \int_\Omega \lambda u_x^\varepsilon \Lambda(m^\varepsilon) \cdot \psi + \int_\Omega m_x^\varepsilon \psi_x = 0.

From the estimates we have

\[u^\varepsilon\] is bounded in \(L^\infty(0, T; H^1_0(\Omega))\), \quad u^\varepsilon(\cdot, t)\] is bounded in \(H^1_0(\Omega)\) if a.e. \(t \in (0, T)\). \quad (3.38) \quad (3.39)

Then, there exists \(u \in L^\infty(0, T; H^1_0(\Omega))\) such that

\[u^\varepsilon \rightharpoonup u\] in \(L^\infty(0, T; H^1_0(\Omega))\) and also in \(L^2(0, T; H^1_0(\Omega))\). \quad (3.40)

Moreover for a.e. \(t\) there exists \(v(\cdot, t) \in H^1_0(\Omega)\) such that

\[u^\varepsilon(\cdot, t) \rightharpoonup v(\cdot, t)\] in \(H^1_0(\Omega), u^\varepsilon(\cdot, t) \longrightarrow v(\cdot, t)\] in \(L^2(\Omega)\). \quad (3.41)

We suppose \(L^2(0, T; H^1_0(\Omega))\) equipped with the scalar product

\[
\int_0^T \int_\Omega u_x v_x dx dt.
\]

Let \(\varphi \in \mathcal{D}(0, T), \chi \in \mathcal{D}(\Omega),\) from (3.41) we derive

\[
\int_0^T \int_\Omega u_x^\varepsilon(\varphi \chi)_x \longrightarrow \int_0^T \int_\Omega u_x(\varphi \chi)_x = \int_0^T \varphi \int_\Omega u_x \chi_x
\]

One has also for a.e. \(t\) by (3.41)

\[
\int_\Omega u_x^\varepsilon \chi_x \longrightarrow \int_\Omega v_x \chi_x.
\]

Moreover due to the estimates that we have \(u_x^\varepsilon, \int_\Omega u_x^\varepsilon \chi_x\) are uniformly bounded in \(t\) and by the Lebesque theorem one derive that

\[
\int_0^T \int_\Omega u_x^\varepsilon(\varphi \chi)_x = \int_0^T \varphi \int_\Omega u_x^\varepsilon \chi_x \longrightarrow \int_0^T \varphi \int_\Omega v_x \chi_x.
\]

Comparing (3.42), (3.45) we derive that for a.e. \(t\)

\[
\int_\Omega u_x^\varepsilon \chi_x = \int_\Omega v_x \chi_x \quad \forall \chi \in \mathcal{D}(\Omega).
\]

This implies that

\[u(\cdot, t) = v(\cdot, t)\] a.e. \(t \in (0, T)\). \quad (3.46)

Since \(m^\varepsilon\) is such that

\[m^\varepsilon\] is bounded in \(L^\infty(0, T; H^1(\Omega))\), \quad (3.47) \quad (3.48)
arguing as above one derives the existence of \( m \in L^\infty(0,T;H^1(\Omega)) \) s.t. up to a subsequence

\[
\begin{align*}
\mathbf{m}^\varepsilon(\cdot,t) \quad & \text{is bounded in} \quad H^1(\Omega) \subset C^{1/2}(\bar{\Omega}) \quad \text{a.e.} \quad t \in (0,T). \\
\mathbf{m}^\varepsilon & \to \mathbf{m} \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)), \\
\mathbf{m}^\varepsilon(\cdot,t) & \to \mathbf{m}(\cdot,t) \quad \text{in} \quad C^0(\bar{\Omega}), \quad \text{a.e.} \quad t.
\end{align*}
\]

(3.49)

One can then pass to the limit in (3.37) to get (3.34), (3.36). Let us first derive

\[
\begin{align*}
\text{From (3.40) } & \quad \text{due to our assumptions for } t \text{ fixed we have when } \varepsilon \to 0 \\
& \quad \int_\Omega \phi_t u^\varepsilon \to \int_\Omega \phi_t u.
\end{align*}
\]

(3.50)

(3.51)

(3.52)

(3.53)

(3.54)

(3.55)

(3.56)

• From (3.40) \( u^\varepsilon \to u \) in \( L^2(0,T;H^1_0(\Omega)) = L^2(\mathcal{Q}) \) and thus, when \( \varepsilon \to 0 \)

\[
\begin{align*}
G^\varepsilon(t-t') & \to G(t-t') \quad \text{in} \quad L^1(0,t) \\
\text{and by (3.44), (3.47) } & \quad \int_\Omega \phi_x(x,t) u^\varepsilon_x(x,\tau)dx \to \int_\Omega \phi_x(x,t) u_x(x,\tau)dx \quad \text{a.e.} \quad \tau \in (0,t)
\end{align*}
\]

(3.57)

(3.58)

(3.59)

(3.60)

and these integrals are uniformly bounded independently of \( \varepsilon \). It follows that

\[
\begin{align*}
\int_0^t G^\varepsilon(t-t') \int_\Omega \phi_x(x,t) u^\varepsilon_x(x,\tau) d\tau dx & \to \int_0^t G(t-t') \int_\Omega \phi_x(x,t) u_x(x,\tau) d\tau dx.
\end{align*}
\]

(3.61)

Since these integrals are uniformly bounded independently it follows that

\[
\begin{align*}
\int_0^T \int_0^T G^\varepsilon(t-t') \int_\Omega \phi_x(x,t) u^\varepsilon_x(x,\tau) d\tau dx dt & \to \int_0^T \int_0^T G(t-t') \int_\Omega \phi_x(x,t) u_x(x,\tau) d\tau dx dt
\end{align*}
\]

i.e.

\[
\begin{align*}
\int_0^T \int_0^T G^\varepsilon(t-t') u^\varepsilon_x \phi_x & \to \int_0^T \int_0^T G(t-t') u_x \phi_x.
\end{align*}
\]

(3.62)

(3.63)

• Since \( \mathbf{m}^\varepsilon(\cdot,\tau) \to \mathbf{m}(\cdot,\tau) \) in \( C^0(\bar{\Omega}) \) a.e. \( \tau \) one has

\[
\begin{align*}
\int_0^T \int_0^t \frac{\lambda}{2} \Lambda(\mathbf{m}^\varepsilon) \cdot \mathbf{m}^\varepsilon \phi_x d\tau & \to \int_0^T \int_0^t \frac{\lambda}{2} \Lambda(\mathbf{m}) \cdot \mathbf{m} \phi_x d\tau
\end{align*}
\]

(3.64)

which completes the existence of solution to (3.34). To pass to the limit in the second equation in (3.37) due to (3.50)-(3.51) only perhaps the fourth integral is not clear. But from (3.51), (3.51)

\[
\begin{align*}
\Lambda(\mathbf{m}^\varepsilon) \cdot \psi(\cdot,t) & \to \Lambda(\mathbf{m}) \cdot \psi(\cdot,t) \quad \text{in} \quad L^2(\Omega), \quad \text{a.e.} \quad t.
\end{align*}
\]

(3.65)

This implies that

\[
\begin{align*}
\int_\Omega \lambda u^\varepsilon_x \Lambda(\mathbf{m}^\varepsilon) \cdot \psi(\cdot,t) & \to \int_\Omega \lambda u_x \Lambda(\mathbf{m}) \cdot \psi(\cdot,t) \quad \text{a.e.} \quad t.
\end{align*}
\]

(3.66)

Since all these integrals are uniformly bounded one deduces that

\[
\int_\mathcal{Q} \lambda u^\varepsilon_x \Lambda(\mathbf{m}^\varepsilon) \cdot \psi & \to \int_\mathcal{Q} \lambda u_x \Lambda(\mathbf{m}) \cdot \psi
\]

which completes the proof of the existence of a solution to (3.36).
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