Abstract

Since its inception, the full Lagrangian meshless smoothed particle hydrodynamics (SPH) method has experienced a tremendous enhancement in methodology and impacted a range of multi-physics applications in science and engineering. The paper presents a concise review on latest developments and achievements of the SPH method, including (1) brief review of theory and fundamental with kernel corrections, (2) the Riemann-based SPH method with dissipation limiting and high-order data reconstruction by using MUSCL, WENO and MOOD schemes, (3) particle neighbor searching with particle sorting and efficient dual-criteria time stepping schemes, (4) total Lagrangian formulation with stabilized, dynamics relaxation and hourglass control schemes, (5) fluid-structure interaction scheme with interface treatments and multi-resolution discretizations, (6) novel applications of particle relaxation for mesh and particle generations. Last but not least, benchmark tests for validating computational accuracy, convergence, robustness and efficiency are also supplied accordingly.

Keywords: Multi-resolution SPH, Multi-phase flows, Complex interface, Fluid-structure interaction

1. Introduction

As a fully Lagrangian meshless method, whereby a set of particles are introduced to discretize the continuum media and their interactions determined by a Gaussian-like kernel function to approximate the mechanics, the smooth particle hydrodynamics (SPH) [1, 2] has been demonstrated to be a compromising alternative of mesh-based methods and received significant interest in the past decades [3, 4, 5, 6]. Thanks to its Lagrangian feature, the SPH method has shown peculiar advantages in handling free-surface flows [7, 8] involving violent impact and breaking events [9], structure analysis with crack propagation and large deformation [4, 10, 11] and multi-physics problems [12] including fluid-structure interactions (FSI) [13, 14, 15], multi-phase flows [16, 17, 18], additive manufacturing [19, 20] and cardiac modeling [21, 22, 23], and comprehensive reviews can be found in recent Refs. [3, 14, 24, 25, 26, 27, 28]. From the methodological point of view, tremendous efforts have been devoted to address the improvement of convergence, consistency and stability, the treatment of boundary conditions, the adaptive discretization and extension to multi-physics applications, as highlighted by Refs. [12, 29, 30].

This paper aims at providing a concise description on the state-of-the-art methodology development and achievement for the SPH method. In particular, attempts are devoted to address the perspectives of the Riemann-based SPH method with dissipation limiting and high-order data reconstruction, efficient particle-interaction configuration updating, stabilized scheme, dynamics relaxation and hourglass control for total Lagrangian formulation, novel applications for mesh and particle gen-
erations, and FSI schemes. This paper is organized as
follows. In Section 2, we will briefly summarize the
theory and fundamental of the SPH method. In Section
3, the traditional and Riemann-based SPH discretizations
of the fluid dynamics is presented with special attention
devoted to the Riemann solver with dissipation limiting,
high-order reconstruction and efficient update of particle-
interaction configuration. In Section 4, we will present the
total Lagrangian SPH formulation with stabilized scheme,
dynamics relaxation and hourglass control. Section 5 re-
ports the interface treatments and multi-resolution dis-
cretization for FSI. Section 6 focuses on the novel appli-
cations of the particle relaxation for high-quality particle
and unstructured mesh generations. Finally, concluding
remarks are given in Section 7.

2. SPH methodology

2.1. Theory and fundamental of SPH

In SPH method, a set of Lagrangian particles whose in-
teractions determined by a Gaussian-like kernel function
is introduced to discretize the continuum media. Then,
the particle-average based discretization of a variable field
\( f(\mathbf{r}) \) can be defined as

\[
f_i = \int \mathbf{f}(\mathbf{r}) W(\mathbf{r}_i - \mathbf{r}, h) d\mathbf{r}.
\]

Here, \( i \) is the particle index, \( f_i \) the discretized particle-
average variable and \( \mathbf{r}_i \) the particle position. For the
compact-support kernel function \( W(\mathbf{r}_j - \mathbf{r}, h) \), \( h \) is
the smoothing length determining a radially symmetric sup-
port domain with respect to \( \mathbf{r}_i \). As the mass of each par-
icle \( m_i \) is known and invariant (indicating mass conser-
vation), one has the particle volume \( V_i = m_i/\rho_i \) with \( \rho_i \)
denoting the particle-average density.

By introducing particle summation, Eq. (1) can be approx-
imated as

\[
f(\mathbf{r}) \approx \sum_j V_j f_j W(\mathbf{r} - \mathbf{r}_j, h) = \sum_j \frac{m_j}{\rho_j} f_j W(\mathbf{r} - \mathbf{r}_j, h).
\]

Here, the summation is over all the neighboring particles
\( j \) located in the support domain of the particle of particle
\( i \). Substituting the variable \( f(\mathbf{r}) \) with density, one gets the
approximation of the particle-average density

\[
\rho_i \approx \sum_j \frac{m_j}{\rho_j} W(\mathbf{r}_i - \mathbf{r}_j, h) \rho_j = \sum_j m_j W_{ij},
\]

which is an alternative way to write the continuity equa-
tion for updating the density.

Similarly, the approximation of the spatial derivative of
\( f(\mathbf{r}) \) can be derived as

\[
\nabla f_i = \nabla f_i \nabla \mathbf{r} + \nabla f_i \approx -\sum_j V_{ij} \nabla W_{ij},
\]

where the inter-particle difference value \( f_{ij} = f_i - f_j \), and
\( \nabla W_{ij} = e_{ij} \frac{\partial W}{\partial r} \), with \( r_{ij} \) and \( e_{ij} \) are the distance and unit
vector of the particle pair \((i, j)\), respectively. The strong-
form approximation of the derivative is used to determine
the local structure of a field. On the other hand, with a
slight different modification, Eq. (4) can be rewritten into
a weak form as

\[
\nabla f_i = \nabla f_i \nabla \mathbf{r} - 2 \sum_j V_{ij} \nabla W_{ij},
\]

where the inter-particle average value \( \bar{f}_{ij} = (f_i + f_j)/2 \).
The weak-form approximation of derivative is used to
compute the surface integration with respect to a vari-
able for solving its conservation law. Thanks to its anti-
symmetric property, i.e., \( \nabla W_{ij} = -\nabla W_{ji} \), the momentum
conservation of the particle system is implied.

2.2. Kernel correction

When particle is close to boundary or particle distri-
bution is irregular, the 0-order and 1st-order consistency of
particle approximation of Eqs. (2) and (4) are not satis-
fied, respectively. To remedy this issue, a number of
correction techniques has been proposed in the litera-
ture \cite{6, 31, 32, 33}. Here, we briefly review the two mostly
applied techniques, i.e., kernel correction and kernel gra-
dient correction which are also known as Shepard filter
and renormalization formulation \cite{4, 34}, respectively.

Kernel correction

Following Refs. \cite{4, 31, 33, 35, 36}, the improved partition
of unity, 0-order consistency, can be achieved by interpo-
lated with a correction kernel $\tilde{W}$ as
\[ f(r) = \sum_j V_j f_j \tilde{W}(r - r_j) = \sum_j V_j f_j \alpha(r) W_j(r), \quad (7) \]
where the parameter $\alpha_i$ is evaluated by enforcing that any constant distribution is exactly interpolated, that is
\[ C_0 = \sum_j V_j C_0 \tilde{W}(r - r_j). \quad (8) \]
Therefore, the following condition must be satisfied with the corrected kernel
\[ \sum_j V_j f_j \tilde{W}(r - r_j) = 1, \quad (9) \]
and this gives
\[ \alpha(r) = \frac{1}{\sum_j V_j W_j(r)}. \quad (10) \]
The scalar parameter of $\alpha(r)$ is also known as Shepard filter [4,31] and provides a much improved partition of unity, in particular for particles near the domain boundary [35,36].

**Kernel gradient correction**

to assess the consistency order of the particle approximation of Eq. (4), we can Taylor-expand $f_j$ around $r$
\[ f_j \approx f(r) + (r_j - r) \cdot \nabla f(r) + O(r^2), \quad (11) \]
and substitute it to Eq. (4)
\[ \nabla f(r) = -\sum_j V_j \{ f(r_j) + (r_j - r) \cdot \nabla f(r) \} \otimes \nabla W(r_j - r, h). \quad (12) \]
Accurate approximation of $\nabla f(r)$ requires that
\[ \sum_j V_j \nabla W(r_j - r, h) \approx 0, \quad (13) \]
and
\[ \sum_j V_j (r_j - r) \nabla W(r_j - r, h) \approx I, \quad (14) \]
where $I$ is the unit matrix. Eq. (14), a correction matrix $\mathbb{B}$ is introduced to modify the kernel gradient as
\[ \tilde{W}_i W_{ij} = \mathbb{B}_i \nabla W_{ij}, \quad (15) \]
and inserted to Eq. (14)
\[ \sum_j V_j (r_j - r) \mathbb{B}_j \nabla W_{ij} = I. \quad (16) \]
This gives
\[ \mathbb{B}_i = \left( \sum_j V_j (r_j - r) \nabla W_{ij} \right)^{-1}. \quad (17) \]
The use of the kernel gradient correction can improve the gradient approximation, in particular for irregular distributed particles [4,34,37]. However, extra computational efforts are induced due to the interpolation and matrix inverse for each particle at every time step. On the other hand, this kernel gradient correction is widely applied in the total Lagrangian formulation as it only be calculated once at the initial reference configuration [38,39,40] which will be presented in details in Section 4.

3. Fluid dynamics

3.1. Governing equations

For inviscid flow, the conservation of mass and momentum in the Lagrangian frame can be written as
\[ \begin{cases} \frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{v} \\ \frac{d\mathbf{v}}{dt} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \mathbf{f'}, \end{cases} \quad (18) \]
where $\mathbf{v}$ is the velocity, $\rho$ the density, $p$ the pressure, $\mathbf{g}$ the gravity, $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ stands for the material derivative and $\mathbf{f'}$ presents the force exerting on the fluid due to the existence of solid, which can be the solid wall or flexible structure.

To close the system of Eq. (18), the pressure can be calculated with an artificial isothermal equation of state (EoS) in the form
\[ p = \rho^\gamma \frac{c_s^2}{\gamma} \left( \frac{\rho}{\rho_0} \right)^\gamma - 1, \quad (19) \]
where $\gamma = 1$ or 7 is an empirically determined constant, $\rho^0$ the reference density and $c$ the speed of sound [41]. Following the weakly-compressible assumption [42], an artificial sound speed of $c = 10U_{\text{max}}$, where $U_{\text{max}}$ denoting the maximum anticipated flow speed, is employed for density fluctuation to be approximately 1%, implying the Mach number $M \approx 0.1$.

3.2. Traditional SPH method

Having Eqs. (3) and (6), the discretization of Eq. (18) in the traditional SPH formulation reads [6, 37, 43, 44]

$$\frac{d}{dt} \rho_i = \rho_i \sum_j m_j \frac{v_{ij} \cdot \nabla_i W_{ij}}{\rho_j} \cdot \nabla_i W_{ij}$$

$$\frac{d}{dt} v_i = - \sum_j m_j \left( \frac{\rho_i + \rho_j}{\rho_j} \right) \nabla_i W_{ij} + g + f^s,$$

where $v_{ij} = v_i - v_j$ is the relative velocity. To dampen the pressure oscillation and prevent instability in the particle motion, where single particle moves in a rather chaotic way, Monaghan and Gingold [43] introduced a Neumann–Richtmeyer type artificial viscosity term

$$\Pi_i = - \sum_j m_j \alpha \frac{\rho_i}{\rho_j} \frac{h_i c_i}{r_{ij}^2} \nabla_i W_{ij},$$

where $c = (c_i + c_j)/2$, $\bar{\rho} = (\rho_i + \rho_j)/2$ and $\alpha \leq 1.0$ is a tunable parameter. While a moderate artificial viscosity is able to stabilize the computation, it may lead to excessive dissipation which affects the physical flow characteristics [37, 46]. Another weakness is that the tunable parameter $\alpha$ requires careful numerical calibrations and its values usually are case dependent [5].

3.2.1. Density reinitialization

Implementing density reinitialization, i.e., the density is integrated by the continuity equation and periodically reinitialized by applying proper formulation, is an efficient way to address the high frequency density oscillations. The straight forward formulation reads [4, 16, 47]

$$\rho_i = \frac{\sum_j m_j W_{ij}}{\sum_j V_i W_{ij}},$$

by using the Shepard filter [4] of Eq. (10), resulting first-order accuracy. Colagrossi and Landrini [16] suggested to consider a mean-least-squares (MLS) kernel interpolation

$$\rho_i = \sum_j m_j W_{ij}^{\text{MLS}}(r_j),$$

where $W_{ij}^{\text{MLS}}$ is the MLS kernel [16] given by

$$W_{ij}^{\text{MLS}}(r_j) = \mathbf{M}_i^{-1} \mathbf{e}_i \cdot \mathbf{b}_{ij} W(r_j)$$

$$\mathbf{b}_{ij}^T = [1, (x_j - x_i), (y_j - y_i), (z_j - z_i)]; \mathbf{e}_i^T = [1, 0, 0].$$

This formulation achieves second-order accuracy and shows good results while is computationally rather expensive [48].

More recently, Zhang et al. [49] and Rezavand et al. [50] proposed new density reinitialization formulations read

$$\rho_i = \rho_i^0 \left( \frac{\sum W_{ij}^0}{\sum W_{ij}^0} + \max(0, \rho^s - \rho_i) \frac{\sum W_{ij}}{\sum W_{ij}^0} \rho_i^0 \right),$$

and

$$\rho_i = \rho_i^0 \frac{\sum W_{ij}}{\sum W_{ij}^0},$$

for free-surface and internal flow, respectively. Here, $\rho^s$ denotes the density before reinitialization and superscript 0 represents the initial reference value. Note that the density reinitialization is applied every $n$-time steps, for example $n = 20$ in Refs. [16, 47] and Zhang et al. [49] apply it every advection time step which consists of several acoustic time steps for particle relaxation.

3.2.2. Diffusive term in the continuity equation

Another approach to reduce the density oscillation is to introduce a diffusive term into the continuity equation of Eq. (20) resulting smooth pressure field and stable time integration. Inspired by the Riemann-based SPH method [27, 51], Ferrari et al. [8] modified the original SPH discretization of the continuity equation as

$$\frac{d \rho_i}{dt} = \rho_i \sum_j m_j \frac{v_{ij} \cdot \nabla_i W_{ij} + \mathcal{D}_i}{\rho_j},$$

by introducing a Rusanov diffusive term $\mathcal{D}_i$ defined as

$$\mathcal{D}_i = - \max\left( c_i, c_j \right) \sum_j \frac{m_j}{\rho_j} \frac{\partial W_{ij}}{\partial r_{ij}}.$$
As highlighted by Ref. [18], the Rusanov diffusive term \( \mathcal{D}_i \) is independent of any tunable parameter and no artificial viscosity of Eq. (21) is required in the momentum equation. However, this scheme is not compatible with the hydrostatic solution, exhibiting unphysical free-surface motion and expansion in long term simulations due to the inconsistency induced by the singularity of the density approximation at the surface [48, 52].

Molten and Colagrossi [53] pursued a similar idea of introducing diffusive term but it still suffers the incompatibility with the hydrostatic solution [54]. To decrease such artifacts, Antuono et al. [48] proposed an improvement and the diffusive term reads

\[
\mathcal{D}_i = -\delta h c \sum_j m_j \rho_j \left( \frac{\rho_i - \rho_j}{r_{ij}} \nabla_{ij} + \tilde{\nabla}_{ij} \right) \cdot \nabla W_{ij}, \quad (29)
\]

where \( \delta \) is a non-dimensional parameter, and \( \tilde{\nabla}_{ij} = \nabla \rho_i + \nabla \rho_j \) is a renormalized correction term to prevent the singularity at the surface [55]. This corrected \( \delta \)-term is compatible with the hydrostatic solution, whereas induces extra computational efforts due to the correction term of \( \tilde{\nabla}_{ij} \). Note that the parameter \( \delta \) is not freely tunable and usually set as 0.1 [55], and a small amount of artificial viscosity defined by Eq. (21) with parameter of \( \alpha = 0.02 \) is still applied for numerical stability.

### 3.3. Riemann-based SPH method

As a variant of the SPH method, the Riemann-based SPH method solves a one-dimensional Riemann problem along each particle pair to determine its interaction. Compared with the traditional SPH method presented in Section 3.2 the Riemann-based SPH method introduces implicit numerical dissipation other than use explicit artificial viscosity, and achieves the dissipation in a more accurate manner [37, 56, 57, 58]. Monaghan [56] pointed out that the artificial viscosity [45] is analogous to the dissipative terms of the Riemann solver, which scales with the wave speed and the velocity jump between interacting particles, whereas showed that using the exact, or well approximated, Riemann solution can obtain more accurate results in capturing shock wave. The pioneering work of developing Riemann-based SPH method can be tracked back to Vila [37], where a Riemann-based ALE-SPH scheme was proposed by discretizing the Euler equations in conservative form and calculating the fluxes between particles with a Riemann solver. It is worth noting that the particle in the Riemann-based ALE-SPH scheme represents a volume of the considered discretized fluid medium and it may move with the fluid velocity (Lagrangian description), remain still (Eulerian description) or move in any arbitrary way. Instead of following the ALE description, Parshikov et al. [59] and Parshikov and Stanislav [60] proposed a Riemann-based SPH formulation in purely Lagrangian framework by using a first-order Riemann solution to describe the contact interaction between particles. To improve the accuracy of capturing strong shocks, Inutsuka [61] has reformulated the Riemann-based SPH formulation with second-order Riemann solution. Then, Cha and Whitworth [62] derived different versions of Riemann-based SPH schemes, and performed a von Neumann stability analysis and concluded that the Riemann-based SPH is stable for all wavelengths, while the traditional SPH is unstable for certain wavelengths. Subsequently, enormous progress has been made toward accomplishing high-order data reconstruction [63, 64, 65, 66], dissipation limiting [67, 68] and solid boundary treatment [69, 70], which will be reviewed in the following parts.

To derive the Riemann-based SPH discretization of Eq. (18) we rewrite its traditional SPH discretization as

\[
\begin{align*}
\frac{d\rho_i}{dt} &= 2\rho_i \sum_j m_j \left( \frac{\rho_i - \rho_j}{r_{ij}} \right) \nabla W_{ij}, \\
\frac{d\mathbf{v}_i}{dt} &= -2 \sum_j m_j \left( \frac{\rho_i}{\rho_j} \right) \nabla W_{ij} + \mathbf{g} + \mathbf{f}',
\end{align*}
\]

by introducing the inter-particle average velocity \( \overline{\mathbf{v}}_{ij} = (\mathbf{v}_i + \mathbf{v}_j) / 2 \) and pressure \( \overline{p}_{ij} = (p_i + p_j) / 2 \). Then, the inter-particle average variables are replaced by the Riemann solution, i.e., \( U^* \) and \( P^* \), resulting

\[
\begin{align*}
\frac{d\rho_i}{dt} &= 2\rho_i \sum_j m_j \left( \frac{\mathbf{v}_i - \mathbf{v}^*}{r_{ij}} \right) \nabla W_{ij}, \\
\frac{d\mathbf{v}_i}{dt} &= -2 \sum_j m_j \left( \frac{P_i}{P^*} \right) \nabla W_{ij},
\end{align*}
\]

where \( \mathbf{v}^* = U^* \mathbf{e}_j + \left( \overline{\mathbf{v}}_{ij} - \overline{U}_{e_j} \right) \). In this case, the inter-particle average variables are replaced by solutions of the Riemann problem, implying that numerical dissipation, i.e., density regularization and numerical viscosity, is implicitly present.

In Riemann-based SPH the solution to the Riemann problem is reduced to a one-dimensional problem constructed along the interaction line of particles. Then, the
first step is to construct the $L$ and $R$ states between each pair of interacting particles. Following the Godunov-type method, which applies piece-wise constant assumption, i.e., first-order reconstruction, the $L$ and $R$ states are defined as

$$
\begin{align*}
\left( \rho_L, U_L, P_L \right) &= \left( \rho_i, v_i \cdot e_{ij}, p_i \right) \\
\left( \rho_R, U_R, P_R \right) &= \left( \rho_j, v_j \cdot e_{ij}, p_j \right)
\end{align*}
$$

(32)

In this case, the initial Riemann left and right states are on particles $i$ and $j$, respectively, and the discontinuity is at the middle point $r_{ij} = \frac{1}{2} (r_i + r_j)$, as shown in Figure 1.

Figure 1: Construction of Riemann problem along the interacting line of particles $i$ and $j$.

3.3.1. Riemann solver with dissipation limiter

Since its inception, the Riemann-based SPH method has been applied to solve strong shocks problems [70, 71], solid mechanics problems [59, 60, 73], interface instability [74, 75] and magnetohydrodynamics (MHD) problems. However, it is generally too dissipative to reliably reproduce violent free-surface flows involving violent events such as impact and breaking [34, 77, 78]. To cope with the excessive dissipation introduced by directly applying a Riemann solver, Zhang et al. [67] proposed a simple low-dissipation limiter to the classic linearized Riemann solver [79, 80], ensuring no or decreased numerical dissipation for expansion or compression waves, respectively. This method is compatible with the hydrostatic solution and able to resolve violent wave breaking and impact events accurately, produces very small damping of mechanical energy, smooth pressure fields and predicts reasonable pressure peaks. Then, this method was extended by Rezavand et al. [18] for modeling multiphase flow with high density ratio and Zhang et al. [18] for FSI problems. Inspired by Ref. [67], Meng et al. [68] proposed a dissipation limiter to Roe’s approximated Riemann solver [81, 82] to develop a multiphase SPH model for complex interface flows. Furthermore, Meng et al. [68] derived the equivalent relation between the intrinsic dissipation of the Riemann solver and the Reynolds number for accurate modeling of viscous flow.

Linearized Riemann solver

According to Refs. [80, 83], the solution of a Riemann problem results in three waves emanating from the discontinuity, denoted by $(\rho_L^*, U_L^*, P_L^*)$ and $(\rho_R^*, U_R^*, P_R^*)$ as shown in Figure 2. Two waves, which can be shock or rarefaction wave, travel with the smallest or largest wave speed. The middle wave is always a contact discontinuity and separates two intermediate states. By assuming that the intermediate state satisfies $U_L^* = U_R^* = U^*$ and $P_L^* = P_R^* = P^*$, a linearized Riemann solver [83, 79] for smooth flows or with only moderately strong shocks can
be written as
\[
\begin{align*}
U^* &= \frac{\rho(\gamma+1)c_L^2 U_k + \rho_L P_k - P_R}{\rho(\gamma+1)c_L^2 + \rho_L c_L^2 + \rho_R c_R^2}, \\
p^* &= \frac{\rho(\gamma+1)c_L^2 P_k + \rho_L P_k + \rho_R P_R c_R^2 (U_L - U_R)}{\rho(\gamma+1)c_L^2 + \rho_L c_L^2 + \rho_R c_R^2},
\end{align*}
\]
(33)
where the limiter is defined as
\[
\beta = \min \left( \eta \max \left( \frac{U_L - U_R}{c_{LR}}, 0 \right), 1.0 \right),
\]
(34)
with \(c_{LR} = \frac{\rho_L c_L + \rho_R c_R}{\rho_L + \rho_R} \). Here, the dissipation limiter \(\beta\) ensures that there is no dissipation when the fluid is under the action of an expansion wave, i.e. \(U_L < U_R\), and that the parameter \(\eta\) is used to modulate dissipation when the fluid is under the action of a compression wave, i.e. \(U_L \geq U_R\). In Ref. [67], constant parameter \(\eta = 3\) is suggested according to numerical experiments. Meng et al. [68] presented that the relation between \(\eta\) and the parameter \(\alpha\) in artificial viscosity of Eq.(21) is given by
\[
\eta = \frac{\alpha h (c_L + c_R)}{2 (U_L - U_R)|\mathbf{r}_{ij}|},
\]
(35)
Having the equivalent relation of the artificial viscosity and the physical kinematics viscosity [59, 83], Meng et al. [68] derived the relation between the parameter \(\eta\) and the Reynolds number as
\[
\eta = \frac{2(d+2)U_c L_c}{Re(U_L - U_R)|\mathbf{r}_{ij}|},
\]
(36)
where \(d\) is the dimension, \(Re\) the Reynolds number, \(U_c\) and \(L_c\) are the characteristic velocity and length, respectively.

Figure 3 presents the validation of the linearized Riemann solver with dissipation limiter by studying the Taylor-Green vortex flow [67]. Without the dissipation limiter, the Riemann-based SPH is too dissipative to predict a reasonable kinetic energy decay. While with the limiter, low-dissipation feature and better agreement with the analytical solution are achieved in comparison with the traditional SPH method with the artificial viscosity (\(\alpha = 0.02\)) scheme. Also, the Riemann-based SPH achieves 2nd-order convergence for the total kinetic energy with increasing particle resolution. The performance of the Riemann-based SPH with dissipation limiter for modeling free-surface flows exhibiting violent events such as impact and breaking is addressed in Figure 4 where dam-break and sloshing flows are investigated by comparing with experimental data. Both tests demonstrate that the Riemann-based SPH can accurately predict the violent motion of free surface and meanwhile capture the impacting pressure reasonably.

**HLLC Riemann solver**

The HLLC Riemann solver proposed by Toro [83] has been applied in Riemann-based SPH method for capturing strong shock waves [70, 71] and energetic flows [58], exhibiting excessive numerical dissipation with the piecewise constant assumption [58]. Following Ref. [80], the wave speeds estimate respectively from Left (L) and Right (R) regions, \(s_L\) and \(s_R\), are
\[
s_L = U_L - c_L, \quad s_R = U_R + c_R,
\]
(37)
and then the intermediate wave speed \(s^*\) is then calculated as
\[
s^* = \frac{P_L - P_R + \rho_L U_L (s_L - U_L) - \rho_R U_R (s_R - U_R)}{\rho_L (s_L - U_L) - \rho_R (s_R - U_R)}.
\]
(38)
Subsequently, the intermediate states of pressure can be obtained accordingly by
\[
P_K^* = P_K + \rho_K (U_K - s_K) (U_K - s^*), \quad K = R, L.
\]
(39)
Finally, the HLLC solution to the Riemann problem is then expressed by
\[
F^{HLLC} = \begin{cases} 
F_L, & 0 \leq s_L \\
F_L^*, & s_L \leq 0 \leq s^* \\
F_R^*, & s^* \leq 0 \leq s_R \\
F_R, & 0 \geq s_R
\end{cases}, \quad K = L, R
\]
(40)

3.3.2. High-order reconstruction

Another key feature of the Riemann-based SPH is the possibility of implementing high-order data reconstruction [85, 86, 87, 88, 89, 90], which is widely applied in Eulerian Godunov-method [89, 90, 91] to decrease the dissipation and improve the accuracy [57, 34]. Vila [34] first introduced the MUSCL scheme for second-order data reconstruction into the Riemann-base ALE-SPH method. Similar approach was developed by Inutsuka et al. [61] for reformulating a second-order Riemann-based SPH method. Since then, more attempts have been aimed
at implementing different limiting functions for MUSCL scheme, e.g., van Leer limiter [76 92], SuperBee limiter [78 58 23] and Barth-Jespersen-type limiter [94], to reduce the numerical dissipation and increase spatial order of Riemann-based SPH method. More recently, increasing attentions are drawn to implement WENO scheme [89 91] and its variants [95 96 97]. Zhang et al. [98] have considered a fifth-order WENO reconstruction for computing one-dimensional problems, however, its multidimensional extension is not straightforward. The first WENO reconstruction for computing multi-dimensional problems is proposed by Avesani et al. [63], in which the directionally-biased multi-dimensional candidate stencils with high-order Moving-Least-Squares (MLS) reconstructions are combined with the WENO weighting strategy. Although this method achieves higher accuracy than those using linear reconstructions, it exhibits much lower computational efficiency due to a large number of multi-dimensional candidate-stencil evaluations. Nogueira et al. [99] proposed a SPH-MOOD-MLS method which uses a MLS-based approximation and a posteriori Multidimensional Optimal Order Detection (MOOD) approach for numerical stability. This method shows considerable improvement for modeling compressible flows with shock and blast waves. Different with Ref. [63], Zhang et al. [64] proposed an efficient one-dimensional 4-point stencil incremental stencil WENO (IS-WENO) reconstruction [65] along the interaction line of each particle pair, where the variable calculation of the missing points is based on SPH derivative approximation as that in MUSCL scheme, for the Riemann-based SPH method to increase accuracy by decreasing the numerical dissipation other than increasing the formal approximation order. This method preserves the capability of producing smooth and accurate pressure fields of the original method and now achieves also very small numerical dissipation. Similar with Ref. [64], Meng et al. [100] developed 5-point stencil WENO reconstruction [89 91] along the particle interacting line, while the missing variables are evaluated by firstly searching their nearest fluid particles and then adopting the first-order Taylor expansion. The tests showed that this method is robust and able to accurately capture shockwaves. Benefiting from the low-dissipation property, it also has a good performance in resolving small-scale structures in flows. Similar with Refs. [64 65], Meng et al. [100] implemented the TENO scheme [96] to capture the shocks and small-scale structures in some compressible flows, and obtain superior accuracy in some incompressible vortex flows and free surface flows.
Figure 4: Numerical modeling of free-surface flows by using the Riemann-based SPH method with linearized Riemann solver and dissipation limiter: Snapshots of particle distribution with pressure contour (left panel) and the time history of the numerical and experimental pressure signals (right panel) for dam-break (a) and sloshing (b) flows.
MUSCL reconstruction

The MUSCL scheme was developed by van Leer \cite{85, 86} to replace the piecewise constant approximation of Godunov’s scheme by reconstructing left and right states with piecewise linear approximations to calculate fluxes in the Eulerian methods \cite{80}. In MUSCL scheme, the spatial derivatives of field variable are used for data reconstruction, while direct usage results unstable scheme due to spurious oscillations near high gradients \cite{80}. To remedy the spurious oscillation, slope limiter or flux limiter, is applied to limit the approximated gradient near shocks or discontinuities.

Similar as in Eulerian method, the reconstructed, limited left and right states are used as input to the Riemann-based ALE-SPH method \cite{80}. With the piecewise linear approximation, the left and right states of the Riemann problem are reconstructed from

\[
\begin{align*}
\Phi_L &= \Phi_i + \frac{1}{2}(\Phi_i - \Phi_j) \nabla \Phi_j \cdot \mathbf{r}_{ij}, \\
\Phi_R &= \Phi_j - \frac{1}{2}(\Phi_i - \Phi_j) \nabla \Phi_i \cdot \mathbf{r}_{ij},
\end{align*}
\]  

where \( \phi \) is the limiting function and \( \lambda \) the ratio of successive gradients defined as \( \lambda_i = \frac{\Phi_i - \Phi_j}{\nabla \Phi_i \cdot \mathbf{r}_{ij}} \) and \( \lambda_j = \frac{\Phi_j - \Phi_i}{\nabla \Phi_j \cdot \mathbf{r}_{ij}} \), respectively. Here, the \( \nabla \Phi_i \) and \( \nabla \Phi_j \) are the corresponding gradients calculated from the SPH approximation as

\[
\nabla \Phi_i = \sum_j \frac{m_j}{p_j} (\Phi_j - \Phi_i) \nabla W_{ij},
\]

or with kernel grad correction defined in Eq. (16). Here, we briefly summarize several widely used slope limiters. In the Minmod limiter \cite{101}, the limiting function is defined as

\[
\phi_{mm}(\lambda) = \max[0, \min(1, \lambda)].
\]  

As for the SuperBee \cite{101} limiter, the limiting function is given by

\[
\phi_{sb}(\lambda) = \max[0, \min(2\lambda, 1), \min(\lambda, 2)].
\]  

As for the van Leer \cite{102} limiter, the limiting function is defined as

\[
\phi_{vl}(\lambda) = \frac{\lambda + |\lambda|}{1 + |\lambda|}.
\]

Base on the Barth-Jespersen-type limiter \cite{87}, Hopkins \cite{94} proposed a limiting function defined as

\[
\phi_{lj} = \min\left[1, \beta \min\left(1, \frac{\phi_{max}_{ij,ngh} - \Phi_j}{\Phi_j - \Phi_i, \Phi_i - \phi_{min}_{ij,ngh}}\right)\right],
\]  

where \( \beta \) is constant, \( \phi_{max}_{ij,ngh} \) and \( \phi_{min}_{ij,ngh} \) are the maximum and minimum values of \( \Phi_j \) among all neighbor particles \( i \) of particle \( j \), and \( \Phi_{max} \) and \( \Phi_{min} \) are the maximum and minimum values (over all pairs \( ij \) of the \( j \) neighbours of \( i \)) reconstructed on the ‘i side’ of the interface between particles \( i \) and \( j \) (i.e., \( \phi_{max}_{ij,mid} = \max(\Phi_i, \Phi_j, \Phi_{i,j,mid}) \)).

WENO reconstruction

Different with the classical point-wise one-dimensional WENO reconstruction from structured mesh data \cite{89, 91}, the reconstruction from scattered data, i.e., cell average data using unstructured mesh or meshless particle data, is numerically very critical challenging as it requires solving interpolation problems \cite{104, 105}, in particular when the reconstruction order is high, or when the scattered data are very unevenly distributed \cite{106, 107}. The mostly common procedure applied in the unstructured mesh method is to construct a set of reconstruction stencils for each element by dividing its neighbor elements to different groups \cite{108}. Concerning WENO date reconstruction in the Riemann-based SPH method, two types of stencil reconstruction, i.e., multi-dimensional \cite{63} and one-dimensional \cite{64} stencils, are developed.

Following Refs. \cite{106, 108}, Avesania et al. \cite{63} proposed a new class of MLS-ALE-SPH methods by first producing each particle a set of high-order MLS reconstructions based on multi-dimensional reconstructed stencils and then applying a nonlinear WENO technique to combine reconstructions with each other. For each particle \( i \), the reconstructed stencils are defined as

\[
\begin{align*}
S_{i,0} &= \bigcup_j P_j & |r_{ij}| \leq h \\
S_{i,k} &= \bigcup_j P_k & |r_{ij}| \leq 2h \quad \text{and} \quad \kappa \in [1, s] \\
S_{i,\kappa} &= \bigcup_j P_j & \theta \in [2(\kappa - 1)\pi/s, 2\kappa\pi/s]
\end{align*}
\]  

where \( S_{i,0} \) is the central stencil containing neighboring particle union \( P_0 \) with distance less than \( h \), and \( S_{i,k} \) with \( \kappa \in [1, s] \) are the one-sided stencils consisting of neighboring particle union \( P_\kappa \) with distance less than \( 2h \) and
located at specified Circular sector determined by the angle $\theta$ formed by the vector $\mathbf{r}_{ij}$ and the x-axis, as shown in Figure 5. Note that more one-sided stencils can be constructed by diving the cutoff region of particle $i$ into more Circular sectors. Having the definition of the constructed stencils, the Moving-Least-Squares interpolation is applied for each particle by assuming the reconstruction polynomials in the form

$$q_{i,k}(\mathbf{r}) = \Phi_i + \sum_{\chi=1}^{N} C^i_\chi B^i_\chi(\mathbf{r} - \mathbf{r}_i),$$  \hspace{1cm} (48)$$

for each stencil defined in Eq. (47). Here, $N$ is the size of the polynomial basis (that depends on the polynomial degree and on the space dimension) $B^i_\chi(\mathbf{r} - \mathbf{r}_i)$ are the associated basis functions, and $C^i_\chi$ are the (unknown) polynomial coefficients, more details are referred to Ref. [63]. With the constructed polynomials, the classical WENO scheme can be applied to obtain the final polynomial defined as

$$q_i(\mathbf{r}) = \sum_{\chi} w_{\chi} q_{i,\chi}(\mathbf{r}),$$  \hspace{1cm} (49)$$

with the normalized nonlinear weights given by

$$w_{\chi} = \frac{\alpha_{\chi}}{\sum_{\kappa} \alpha_{\kappa}}, \quad \alpha_{\chi} = \frac{\alpha_{c} \lambda_{\chi}}{(\beta_{\chi} + \epsilon)^4}. \hspace{1cm} (50)$$

Here, the constant parameter $\epsilon = 10^{-6}$, and $\lambda_0 = 1.0$ and $\lambda_{\chi} = 10^{-5}$ for central and sided stencils, respectively. For the calculation of the smoothness indicator, Avesania et al. [63] proposed the following equation

$$\beta_{\chi} = \sum_{\chi} \left( C^i_\chi \right)^2, \hspace{1cm} (51)$$

The MLS-ALE-SPH method was further improved by Nogueira et al. [99] using the MOOD paradigm to improve the accuracy and the robustness and Avesani et al. [66] adopting ADER approach (Arbitrary Derivative in space and time) to guarantee a high order space–time reconstruction. The MLS-ALE-SPH method and its improvements are able to capture the discontinuities and to maintain accuracy and low numerical dissipation in smooth regions, while they are generally excessive computational expensive due to particle search for all the stencils of each particle and the corresponding MLS interpolations.
To improve the computational efficiency of applying WENO reconstruction, Zhang et al. [64] developed a one-dimensional stencil reconstruction along the interacting line of each particle pair as shown in Figure 5. They first introduced an IS-WENO reconstruction, by which the full 4-point stencil as shown in Figure 5 is constructed following the concept of Refs. [96,97]. To construct the 4-point stencil for each interacting particle pair, such as particle $i$ and $j$, the values at the stencil points are calculated as

\[
\begin{align*}
q_{i-1} &= \Phi_i - \nabla \Phi_i \cdot r_{ij} \\
q_i &= \Phi_i \\
q_{i+1} &= \Phi_j \\
q_{i+2} &= \Phi_j + \nabla \Phi_j \cdot r_{ij}
\end{align*}
\] (52)

where $\Phi_i$ and $\Phi_j$ represent the primitive values, i.e., $\rho$, $P$ and $\mathbf{v} \cdot \mathbf{e}_{ij}$, at particle $i$ and $j$ respectively. In this 4-point stencil, two visual particles, namely $i-1$ and $i+1$, are constructed along the interacting line $r_{ij}$ with the gradients calculated from the SPH approximation of Eq. (42). Similar with Ref. [64], Wang et al. [65] introduced a 5-point stencil by constructing more visual particles along the interacting line $r_{ij}$ and whose values are calculated as

\[
q_m = \Phi_c + (i-m)\nabla \Phi_c \cdot r_{ij} \quad m \in \{i-2, i-1, i+1\} \] (53)

where $\Phi_c$ denotes the variable of particle $c$ which is the closed particle to the visual particle located at $r_i + (i-m) \cdot r_{ij}$. Compared with the 4-point stencil construction, the construction proposed by Wang et al. [65] provides the possibility of implementing the classical 5-order WENO scheme, while requires extra computational efforts for nearest particle search of each visual particles. This searching procedure increases the complexity of neighbor searching from $O(N)$ to $5O(N)$ [65].

Compared with the multi-dimension WENO reconstruction [63], the one-dimensional reconstruction [27,65] can no longer maintain the higher-order data reconstruction [63]. However, it is reasonable as the main objective of applying the WENO reconstruction aims to increase accuracy by decreasing the numerical dissipation other than increasing the formal approximation order of the SPH method [63], which depends on many factors and is quite difficult to achieve in practice. It is shown that a general SPH method applying Gaussian-like kernel achieves only 2nd-order convergence even when the integration error is sufficiently small [3,109].

\[
\begin{align*}
S_1 & \bullet \bullet \bullet \\
WENO & \bullet \bullet \bullet \bullet \bullet \\
S_2 & \bullet \bullet \bullet \\
IS - WENO & \bullet \bullet \bullet \bullet \bullet \\
S_3 & \bullet \bullet \\
S_4 & \bullet \\
S_5 & \bullet \\
S_6 & \bullet \\
S_7 & \bullet \\
S_8 & \bullet \\
S_9 & \bullet \\
S_{10} & \bullet \\
S_{11} & \bullet \\
S_{12} & \bullet \\
\end{align*}
\]

Figure 6: Full stencil and candidate stencils for classic 5-point WENO (upper panel) and 4-point IS-WENO [27,64] (bottom panel) for the data reconstruction at $r_{i+1/2}$.

Following the WENO reconstruction, the mid-point value, i.e., $q_{1/2}$ as shown in Figure 6 is predicted by the non-linear weighted average

\[
q_{1/2} = \sum_k w_k q_{1/2}^{(k)},
\] (54)

where $q_{1/2}^{(k)}$ and $w_k$, $k = 1, 2, 3$, are the reconstructed values from the candidate stencils and their non-linear weights.

For classic 5-point stencil WENO scheme, the reconstructed values are defined as

\[
\begin{align*}
{d_{1/2}^{(1)}} &= \frac{1}{6} (2q_{i-2} - 7q_{i-1} + 11q_i) \\
{d_{1/2}^{(2)}} &= \frac{1}{6} (-q_{i-1} + 5q_i + 2q_{i+1}) \\
{d_{1/2}^{(3)}} &= \frac{1}{6} (2q_i + 5q_{i+1} - q_{i+2})
\end{align*}
\] (55)

with the renormalized nonlinear weights given by

\[
w_k = \frac{\alpha_k}{\sum_{s=1}^{3} \alpha_s}, \quad \alpha_k = \frac{d_k}{(\beta_k + \epsilon)},
\] (56)

Also, the smoothing indicator is calculated from

\[
\begin{align*}
\beta_1 &= \frac{1}{6} (q_{i-2} - 4q_{i-1} + 3q_i)^2 + \frac{1}{12} (q_{i-2} - 2q_{i-1} + q_i)^2 \\
\beta_2 &= \frac{1}{6} (q_{i-1} - q_{i+1})^2 + \frac{1}{12} (q_{i-1} - 2q_i + q_{i+1})^2 \\
\beta_3 &= \frac{1}{6} (q_i - 4q_{i+1} + q_{i+2})^2 + \frac{1}{12} (q_i - 2q_{i+1} + q_{i+2})^2
\end{align*}
\] (57)
As for the 4-point stencil IS-WENO scheme, these reconstructed values are defined as [27, 97]
\[
\begin{cases}
q_{1/2}^{(1)} = \frac{1}{4} q_0 + \frac{1}{2} q_1 \\
q_{1/2}^{(2)} = -\frac{1}{2} q_{-1} + \frac{3}{2} q_0 \\
q_{1/2}^{(3)} = \frac{1}{2} q_0 + \frac{5}{6} q_1 - \frac{1}{6} q_2
\end{cases}
\]  
(58)

Also, the non-linear weights are defined as following Fu et al. [96], the cutoff and dissipation properties [95, 96, 97] and here,
\[
d_f = \sum_{i=1}^{3} \frac{d_i}{\alpha_i},
\]  
(59)

where the linear weights are determined as \(d_1 = 1/3\), \(d_2 = 1/6\) and \(d_3 = 1/2\). \(\beta_k\), \(k = 1, 2, 3\), and \(\beta_{12}\) are the smoothness indicators for the candidate stencils,
\[
\begin{align*}
\beta_1 &= (q_{i+1} - q_i)^2 \\
\beta_2 &= (q_i - q_{i-1})^2 \\
\beta_{12} &= \frac{1}{2} (q_{i+1} - q_i)^2 + \frac{1}{2} (3q_{i+1} - 2q_i + q_{i-1})^2 \\
\beta_3 &= \frac{1}{2} (q_i - 2q_{i+1} + q_{i+2})^2 + \frac{1}{2} (3q_i - 4q_{i+1} + q_{i+2})^2
\end{align*}
\]  
(60)

and \(\tau_4\) is a global reference smoothness indicator [96] given as
\[
\tau_4 = \left| q_{i-1}(547q_{i-1} - 2522q_i + 1922q_{i+1} - 494q_{i+2}) + q_i(3423q_{i-1} - 5966q_i + 1602q_{i+1}) + q_{i+1}(2843q_{i-1} - 1642q_i + 267q_{i+2}) \right| / 240.
\]  
(61)

Both the WENO and IS-WENO reconstruction can be further improved by the TENO scheme [99] with the non-linear weights are reformulated as
\[
w_k = \frac{d_k \delta_k}{\sum_{k'=0}^{2} d_k \delta_{k'}}.
\]  
(62)

Here, \(d_k\) are optimal weights with respect to their dispersion and dissipation properties [95, 96, 97] and \(\delta\) is a sharp cutoff function to determine the contribution of each stencil. Following Fu et al. [96], the cutoff function is defined as
\[
\delta_k = \begin{cases}
0, & \text{if } \chi_r < C_r \\
1, & \text{otherwise}
\end{cases}
\]  
(63)

with threshold \(C_r = 10^5\) and the parameter \(\chi_r\) is a normalized smoothness measure. For each sub-stencil, \(\chi_r\) is defined as
\[
\chi_r = \frac{\gamma_r}{\sum_{r=0}^{2} \gamma_r},
\]  
(64)

where \(\gamma\) is a scale separation to distinguish the discontinuity form the smooth region. For 5-point WENO scheme,
\[
\gamma_r = \left(1 + \frac{\tau_5}{\beta_r + \epsilon}\right)^{\frac{6}{5}}, \quad \tau_5 = |\beta_0 - \beta_2|,
\]  
(65)

and for 4-point IS-WENO
\[
\gamma_r = \left(1 + \frac{\tau_4}{\beta_r + \epsilon}\right)^{\frac{6}{5}}.
\]  
(66)

The convergence rate of different data reconstruction, i.e., piece-wise constant reconstruction termed as "Baseline", MUSCL and IS-WENO schemes, is presented in Figure [7] which gives the density error with increasing particle resolution for one-dimensional acoustic wave propagation [64]. Both MUSCL and IS-WENO reconstructions achieve second-order convergence, which is the formal accuracy of a general SPH approximation with Gaussian-like smoothing kernels when the particle integration error is negligible [109]. As expected, the Baseline achieves first-order convergence only, and MUSCL exhibits considerably larger errors due to numerical dissipation.

MOOD scheme

The aforementioned MUSCL and WENO scheme provide a priori limitation procedure, which is performed with data at time \(t^r\) to eliminate the spurious numerical oscillation in the vicinity of discontinuity at time \(t^{r+1}\) [63]. Recently, Clain et al. [110] proposed a posteriori limiting paradigm, multi-dimensional optimal order detection (MOOD), within the finite volume Eulerian framework on unstructured mesh. The MOOD paradigm consists of detecting problematic situations after each time update of the solution and of reducing the local polynomial degree before recomputing the solution. Nogueira et al. [99] implemented the MOOD paradigm in the MLS-WENO-SPH method [63, 66] to determine, a posteriori, the optimal order of the polynomial reconstruction of MLS interpolation for each particle that provides the best compromise between accuracy and stability. Then, Antona et al. [111] extended this method to the simulation of weakly-compressible viscous flow.

Different with Refs. [99, 111], where the posteriori limiting procedure is performed at the MLS interpolation process, we derive herein the exploitation of MOOD.
paradigm to determine the optimal data reconstruction of the Left and Right states in the Riemann-based SPH method. The key idea is to introduce a Data Reconstruction Degree decrementing process to replace the counterpart based on Particle Polynomial Degree (PPD) applied in the original MOOD paradigm [99, 110]. More precisely, the present MOOD paradigm consists of two ingredients, a DRD and a detector. The DRD indicates the optimal data reconstruction, Baseline, MUSCL or WENO scheme, for the Riemann problem to obtain the candidate Riemann solution of $\Phi^*$, i.e., $U^*$ and $P^*$. The detector controls the admissibility of the resulting Riemann solution and the particle DRD will be decremented from the WENO scheme to MUSCL scheme and further to Godunov scheme when a detector is activated. The Riemann solution with first-order, robust Godunov scheme, i.e., the reconstruction with piecewise constant assumption, is assumed to be always valid as the original MOOD paradigm in Refs. [110, 99]. Figure 8 sketches the Riemann-based SPH method with Godunov in the top panel, MUSCL or WENO reconstruction, while displays in the bottom panel the present posteriori MOOD procedure. Concerning the detector, the physical admissibility detection

$$0 < P^* \leq P_{\text{threshold}},$$  \hspace{1cm} (67)

or the discrete maximum principle

$$\min(\rho_i, \rho_j) \leq \rho^* \max(\rho_i, \rho_j),$$ \hspace{1cm} (68)

proposed by Ref. [99] can be applied.

![Figure 7: Numerical study of one-dimensional acoustic wave: the convergence of the density error as a function of particle resolution by using the Baseline scheme, MUSCL with the Sweby limiter and IS-WENO reconstructions.](image)

![Figure 8: Sketch of the Riemann-based SPH (top panel) and the one with MOOD posteriori limiting paradigm (bottom panel).](image)

3.4. Particle-interaction configuration

In the particle-base methods, the pairwise interaction between neighboring particles is determined through a Gaussian-like kernel function which has radial-symmetric compact support. Therefore, implementing the particle-interaction configuration, i.e., searching of neighbor particles and computing corresponding kernel weights and gradients, is a critical aspect of the high-performance particle-based solver. Concerning the searching of neighbor particles, two different approaches, i.e., cell-linked
list (CLL) [112] and Verlet list (VL) [113], are widely used in the particle-method community [114, 115, 116]. With different neighboring search technique, the particle-interaction configuration can be updated accordingly. For the CLL approach, the neighboring search procedure must be performed at each numerical iteration, indicating that the particle-interaction configuration is also updated accordingly. As for the VL approach, a VL containing all potential neighboring particles is created and stored for each particle. Therefore, a VL may be used for multiple times without executing neighbor search if the particle-interaction configuration can be obtained [114, 116]. Notwithstanding the wide implementation of the CLL and VL approach in the particle-based methods, they may become not sufficiently efficient when adaptive particle resolution with variable smoothing lengths is applied [117, 118], where tree-based neighboring search technique can be applied for address this issue [119, 120, 121]. With the CLL and VL approaches in hand, several schemes, dual-criteria time stepping [49] and multi-cell linked lists [125], are developed for further improvement of the computational efficiency.

### 3.4.1. Cell-linked and Verlet lists

In the linked list approach, i.e., CLL and VL, the whole computational domain is partitioned into equisized cells and each cell creates a list consisting of the references of all the particles located within it, as shown in Figure 9.

For the CLL approach, the cell size is equal to that of the support or cut-off radius of the kernel function, i.e., $2h$, and the searching of neighbor particles is restricted to the nearest neighboring cells, 9 cells in two dimensions as shown in Figure 9. After the neighbor-searching operation, the kernel function values and interaction forces can be calculated with respect to the particles belong to these neighboring cells only if they are found within the cut-off radius. As the CLL approach does not store the neighboring-particle identities and the corresponding kernel function values, the particle-interaction configuration must be updated multiple times during a single time step in the time integration, e.g., the particle-interaction configuration is updated twice when the kick-drift-kick time integration scheme is applied [67, 126, 127].

Different with the CLL approach, the VL increases the cell size to $2h + \Delta h$, creates and stores a Verlet list which contains the references to all potential neighboring particles for each particle by checking all particles within the adjacent cells [114], as shown in Figure 9. Without conducting neighbor particle search, a VL can be used for multiple times, twice for one single time step with the kick-drift-kick scheme, with non-vanishing kernel function values [114].

Dominguez et al. [114] conducted a comprehensive study of the CLL and VL approaches, and concluded that the VL approach has to use cells with considerably larger size than the cut-off radius to increase the reuse of the Verlet lists. They demonstrated that with a 50% increase of the cell size, only a slight performance gain of 8% is achieved when the Verlet lists are reused for 13 times in 7 time steps. Winkler et al. [128] also found that in general this approach is not able to substitute the CLL due to its poor performance. More recently, Fraga Filho et al. [129] evaluated the performance of the CLL and the VL approaches, and concluded that the VL approach is an optimisation proposal in which the neighbour list is not update at each numerical iteration through an appropriate choice of the cutoff radius ensuring no accuracy loss in the location of neighbor particles.

### 3.4.2. Particle sorting with space filling curve

Traditionally, particles are generated and stored following a given order, row or column order for Lattice distributed particles, and the corresponding memory allocation of particle data is unaltered during the simulation. This results poor temporal and spatial data access and insufficient usage of memory hierarchy [117, 114] due to the full Lagrangian feature of the particle-based methods. Therefore, sorting particle data to change their memory location for better data locality, which has positive effects on hardware caching [130], can improve the memory access and decrease the computational time, achieving scalability and efficiency for large scale particle simulations. To that end, particle date arrays including particle index and other physical variables are rearranged so that the neighbouring particles are close in computer memory space. This procedure can be realized by implementing sorting algorithm with proper space filling curve (SFC), for example the Morton SFC [131] and the Hilbert SFC [132], which traverses higher dimensional space in a continuous fashion [133]. Note that particle sorting does not change the particle-interaction configuration.
Springel [117] implemented an efficient Hilbert SFC in a cosmological N-body/SPH code to domain decomposition and particle sorting within each processor, exhibiting approximated speedup of 2 compared with random sorting. For pure SPH simulation, the CLL approach is introduced to avoid the naive $O(n^2)$ neighbor searching and particle sorting can be conducted with respect to the mapped cell index [114, 123]. Dominguez et al. [114] showed that implementing particle sorting with the Morton SFC [131] can increase the computational performance about 20% for weakly-compressible SPH simulations. Since then, particle sorting has been implemented in particle-based code with the coupling of MPI [134], GPU-acceleration [122, 135, 128] and share-memory high-performance computing strategy [123].

Following Refs. [114, 123], the cell index in the CLL will be defined by a SFC function [131, 132] $H(x_{cell}, y_{cell}, z_{cell}) = n_{cell}$ which produces one dimensional indices $n_{cell}$ where two cells $i$ and $j$ that are geometrically close will be ordinally close. As pointed out by Dominguez et al. [114] particle sorting provides a way to improve the data access pattern and neighboring particle search, while it increases the memory requirements.

### 3.4.3. Dual-criteria time stepping

Zhang et al. [49] proposed a dual-criteria time stepping scheme to optimize the computational efficiency of the WCSPH method by introducing two time-step criteria characterized by the particle advection and the acoustic speeds, respectively. In this scheme, the advection criterion determines the updating frequency of the particle-interaction configuration, i.e., the simplest VL approach with a cell size of $2h$ and the corresponding kernel weights and gradients, and the acoustic criterion controls the frequency of the pressure relaxation process, i.e., the time integration of the particle density, position and velocity due to the action of pressure gradient.

The time-step size determined by the advection criterion, termed $\Delta_{ad}$, has the following form

$$\Delta_{ad} = CFL_{ad} \min\left(\frac{h}{|v_{max}|}, \frac{h^2}{\nu}\right),$$

(69)

where $CFL_{ad} = 0.25$, $|v_{max}|$ is the maximum particle advection speed in the flow and $\nu$ the kinematic viscosity. The time-step size according to the acoustic criterion, termed $\Delta_{ac}$, has the form

$$\Delta_{ac} = CFL_{ac} \frac{h}{c + |v_{max}|},$$

(70)
where $CFL_{nc} = 0.6$. Therefore, the pressure relaxation process is carried out approximately $k \approx \frac{\Delta t}{\Delta x}$ times, for example $k$ is about 4 to 5 when considering inviscid flow [49], during one advection step. Also, the particle-interaction configuration is not altered in one advection time step, a large $CFL_{nc}$ value typically for a Eulerian method is allowable without introducing numerical instability.

As reported in Ref. [49], the dual-criteria time stepping scheme can achieve an speedup up to 2.80 with good robustness and accuracy, in comparison to the traditional counterpart where the CLL approach is applied.

4. Solid mechanics

In the SPH method, there generally two types of formulations, namely update Lagrangian (UL) and total Lagrangian (TL) formulations, have been developed for solid dynamics. The UL formulation, where the current configuration is used as the reference, suffers from several shortcomings, e.g. the presence of tensile instability [136, 137] and the reduced order of convergence for derived variable [10]. To address these problems, many modifications by correcting the kernel function [33, 38, 139], improving the interpolation integral [4, 15, 140, 141, 142] or introducing transport-velocity formulation [137] have been proposed. Compared with the UL formulation, the TL formulation shows promising potential in the simulation of finite deformation due to its attractive advantages in being free from tensile instability and ensuring 1st-order consistency when computing deformation gradient by introducing the kernel gradient correction. Since its inception, it has been applied for the problems of necking and fracture in thermomechanical deformations [143], fluid-structure interaction (FSI) [13, 144, 14, 145] and biomechanics [146, 22, 22], among many others. In this paper, we focus on the TL formulation with highlights on stabilized term, the steady state solution and the hourglass control scheme.

4.1. Governing equations

The kinematics of the finite deformations can be characterized by introducing a deformation map $\varphi$, where a material point $r^0$ can thus be mapped from the initial reference configuration $\Omega^0 \subset \mathbb{R}^d$ to the point $r = \varphi(r^0, t)$ in the deformed configuration $\Omega = \varphi(\Omega^0)$. Here, the superscript $(\bullet)^0$ denotes the quantities in the initial reference configuration. Accordingly, the deformation tensor $F$ can be defined by its derivative with respect to the initial reference configuration as

$$ F = \nabla^0 \varphi = \frac{\partial \varphi}{\partial r^0} = \frac{\partial r}{\partial r^0} \quad (71) $$

With the definition of the displacement $u = r - r^0$, the deformation tensor $F$ can also be calculated through

$$ F = \nabla^0 u + I, \quad (72) $$

where $I$ represents the unit matrix.

In total Lagrangian framework, the conservation of mass and the linear momentum corresponding to the solid mechanics can be expressed as

$$ \begin{cases} \rho = \rho^0 \frac{1}{J} \\ \rho^0 \frac{\partial u}{\partial t} = \nabla^0 \cdot F^T + \rho^0 g \end{cases} \quad (73) $$

where $\rho$ is the density, $J = \det(F)$ and $F$ the first Piola-Kirchhoff stress tensor and $F = FS$ with $S$ denoting the second Piola-Kirchhoff stress tensor. In particular, when the material is linear elastic and isotropic, the constitutive equation can be simply given by

$$ S = \lambda \text{tr}(\mathbb{E}) I + 2G (\mathbb{E} - \frac{1}{3} \text{tr}(\mathbb{E}) I) = \lambda \text{tr}(\mathbb{E}) I + 2\mu \mathbb{E}, \quad (74) $$

where $\lambda$ and $\mu$ are the Lamé parameters [147], $K = \lambda + (2\mu/3)$ the bulk modulus and $G = \mu$ the shear modulus. The relation between the two moduli reads

$$ E = 2G (1 + \nu) = 3K (1 - 2\nu), \quad (75) $$

with $E$ denoting the Young’s modulus and $\nu$ the Poisson’s ratio. Note that the sound speed of solid structure is defined as $c_s = \sqrt{K/\rho}$. The Neo-Hookean material model can be defined in a general form with the introduction of the strain-energy density function

$$ W = \mu \text{tr}(\mathbb{E}) - \mu \ln J + \frac{\lambda}{2} (\ln J)^2. \quad (76) $$

Then, the second Piola-Kirchhoff stress $S$ is derived as

$$ S = \frac{\partial W}{\partial \mathbb{E}}. \quad (77) $$
4.2. Total Lagrangian formulation

In the TL formulation, the correction matrix of Eq. (17) of the kernel gradient correction is calculated from the initial reference configuration as [39]

$$\Xi_a^0 = \left( \sum_b V_b (r_b^0 - r_a^0) \otimes \nabla_a^0 W_{ab} \right)^{-1},$$  \hspace{1cm} (78)

where

$$\nabla_a^0 W_{ab} = \frac{\partial W(r_{ab}, h)}{\partial r_{ab}} e_{ab}^0,$$  \hspace{1cm} (79)

denotes the gradient of the kernel function. Here, the subscript $a$ and $b$ are introduced to denote the solid particles. It is worth noting that the correction matrix is only calculated once before the simulation as it is evaluated at the initial reference configuration. Then, the discretization form of the mass and momentum conservation equations, Eq. (73), yields

\[
\begin{cases}
\rho_a = \rho^0 \frac{1}{\det(T)} \\
\frac{\partial}{\partial t} \epsilon_a^0 = \frac{1}{\rho^0} \sum_b V_b V_a \bar{P}_{ab} \nabla_a^0 W_{ab} + \mathbf{g} + \mathbf{f}',
\end{cases}
\hspace{1cm} (80)
\]

where $\mathbf{f}'$ denotes the force exerting on the solid particles due to the existence of the fluid particles and $\bar{P}_{ab}$ denotes the inter-particle averaged first Piola-Kirchhoff stress and is defined by

$$\bar{P}_{ab} = \frac{1}{2} (P_{ab}^0 + P_{ba}^0).$$  \hspace{1cm} (81)

Note that the first Piola-Kirchhoff stress tensor is computed from the constitutive law with the deformation tensor $F$ given by

$$F = \left( \sum_b V_b (r_b - r_a) \otimes \nabla_a^0 W_{ab} \right) \Xi_a^0 + I.$$

\hspace{1cm} (82)

4.3. Stabilized scheme

Without appropriate stabilization technique, the original TL formulation may exhibit spurious fluctuations especially in the vicinity of sharp spatial gradients. This deficiency can result in numerical instability and lead to wrongly predicted deformation for problems involving large strain. To rectify this deficiency, Lee et al. [148] proposed a Jameson-Schmidt-Turkel SPH (JST-SPH) method, which shows good performance of eliminating spurious pressure oscillations in the simulation of nearly incompressible solid. In JST-SPH methodology, the nodally conservative JST stabilization is additively decomposed into harmonic operator (2nd-order) and biharmonic operator (4th-order) which require excessive computational efforts [149]. In a more recent work, Lee et al. [150] further proposed a total Lagrangian upwind SPH (TLU-SPH) method by introducing a characteristic-based Riemann solver in conjunction with a linear reconstruction procedure to guarantee the consistency and conservation of the overall algorithm. This method also shows good performance in the simulation of nearly and truly incompressible explicit fast solid dynamics with large deformations.

More recently, Zhang et al. [151] proposed an efficient artificial damping method by introducing a Kelvin-Voigt (KV) type damper for total Lagrangian formulation by introducing appropriate damping terms into the constitutive equation. Besides, many stabilization strategies for update Lagrangian formulation, e.g. artificial viscous fluxes [4, 152], conservative strain smoothing regularization [153] and Riemann-based scheme [59], have also been proposed. In KV model, a viscous damper and a purely elastic spring connected in parallel are involved. Following the same idea, an elastic solid undergoing large strains can also be modeled with the mechanical components of springs and dashpots. Thus, the total stress $\sigma_{total}$ can be decomposed into two parts, i.e., the elastic stress $\sigma_S$ and the damper stress $\sigma_D$ as

$$\sigma_{total} = \sigma_S + \sigma_D,$$  \hspace{1cm} (83)

where the damper stress is defined by

$$\sigma_D = \frac{\eta}{\rho} \frac{\partial \epsilon(t)}{\partial t}.$$  \hspace{1cm} (84)

Here, $\partial \epsilon(t)/\partial t$ denotes the strain rate and $\eta$ the physical viscosity. Applying the KV model to TL-SPH formulation, the second Piola-Kirchhoff stress $S$ can be rewritten as

$$S = S_S + S_D,$$  \hspace{1cm} (85)

where $S_S$ is given by the constitutive equation of Eq. (74) or Eq. (76), and the damper $S_D$ is defined as

$$S_D = \frac{\pi}{\rho} \frac{\partial \epsilon(t)}{\partial t} = \frac{\pi}{2} \left( \frac{\partial F}{\partial t} \right)^T F + \frac{1}{2} \left( \frac{\partial F}{\partial t} \right) \left( \frac{\partial F}{\partial t} \right)^T.$$  \hspace{1cm} (86)
By introducing a von Neumann-Richtmyer type scaling factor with the speed of sound \( c \), the artificial viscosity \( \pi \) in Eq. (86) is defined by

\[
\pi = \alpha \rho c h,
\]

where \( \alpha = 0.5 \) is a constant parameter and \( h \) denotes the smoothing length.

Figure 10 shows the validations of the total Lagrangian formulation and the KV-type damper for solid mechanics and its applications in bio-mechanics. Figure 10a presents the time histories of velocity and displacement in the length direction at the right tip end of an elastic cable which experiences a wave propagation initialized by imposing a velocity \( v = 5 \) m/s along the length direction on the right quarter [154]. For the original TL formulation without stabilized schemes, excessive oscillation and similar overshoots in the velocity and displacement profiles are exhibited. With both JST [148] and KV-type [154] stabilized scheme, correct velocity and displacement are predicted as expected, whereas small overshoots are observed with the JST scheme [148]. Figure 10b reports the deformed configuration with von Mises stress contour and the displacement of the free end for three-dimensional bending rubber-like cantilever whose bottom face is clamped to the ground and its body is allowed to bend freely by imposing an initial uniform velocity [40]. Compared with the numerical data in literature obtained by mesh-based method [155], the nonlinear deformation of the structure is accurately predicted by the TL formulation with the KV-type damper. The robustness and versatility in biomedical applications of the TL formulation is portrayed in Figure 10c where the C-shaped stent is considered by imposing initial velocity at its top and bottom [154]. To the best knowledge of the authors, this is first time that an SPH-based method is successfully extended to the simulation of realistic cardiovascular stent and this will open up interesting possibilities for modeling bio-mechanical applications.

4.4. Steady state solution

Coupling a mechanical system to converge to a static equilibrium state plays a key role in static and dynamic analyses. For SPH-based simulations, where an explicit time integration scheme is mainly applied, fast achieving the static equilibrium state is a very critical numerical challenge. Also, traditional implicit methods solving the entire system with iterative solvers are not applicable for large scale applications especially in nonlinear cases. Instead, dynamic relaxation, which was originally proposed for finite element method (FEM) [156][157][158], is attractive because its explicit iterative algorithm is simpler and more efficient.

Generally, there are two groups of dynamic relaxation technique, i.e., viscous dynamic relaxation (VDR) and kinetic dynamic relaxation (KDR) [159][160][161], with respect to the manner of applying damping. In VDR, an artificial viscous damping term is added into the equation of motion to reduce the number of iterations. While VDR is able to obtain system equilibrium without the loss of momentum conservation, it has one difficulty that the solution for nonlinear problems may be path dependent and vary with different damping ratio as noted in Ref. [162]. Also, the relaxation process is slow if insufficient damping ratio is applied, on the other hand, excessive damping not only can lead to numerical stability issue for explicit damping scheme but also can hinder the system from achieving the correct final steady state as the damped velocity can be very small. The inefficiency of the damping leads low efficiency to achieve the final solution, which is the main drawback of VDR [163]. As an alternative approach, the KDR was first proposed by Cundall [164] for static structure analysis. In the KDR, the damping is introduced to a dynamic system by resetting all current velocities to zero when kinetic energy peak is detected [158]. After performing this relaxation procedure several times for successive local peak kinetic energy, the final state can be achieved and the static equilibrium solution is obtained. The procedure is simple with no damping coefficient required and has been applied to a wide range of engineering problems [163][165]. However, it also exhibits two obvious drawbacks. One is that it violates the momentum conservation. Therefore, it cannot be applied to moving systems for their equilibrium state analysis due to the variation of the entire mechanical energy. The other is that the additional process for kinetic energy peak detection brings extra computational efforts [160][166]. Concerning the development of dynamic relaxation in SPH method, the first effort is credited to Lin et al. [162] where a dynamic relaxation scheme is developed for shell-based SPH by exploiting the formulations in finite element analysis. However, the damping matrix
Figure 10: Total Lagrangian SPH formulation for solid mechanics: (a) Velocity (left panel) and displacement (right panel) profiles for wave propagation in an elastic cable with different stabilized scheme. (b) Deformed configurations with von Mises stress contour (left panel) and the displacement profile at the free end (right panel) for bending rubber-like cantilever. (c) The deformed configuration with von Mises stress contour of C-shaped stent.
as well as the damping ratio have to be chosen suitably, which is not an easy task.

Recently, Zhu et al. [167] proposed a VDR-type dynamic relaxation method for SPH by introducing efficient momentum-conservative damping. Specifically, an artificial viscous force is first introduced into the momentum conservation equation which is rewritten as

$$\frac{dv}{dt} = \frac{1}{\rho^0} \nabla^0 \cdot \overrightarrow{P}^T + g + f^G + f^V,$$ (88)

where \( f^V \) denotes the added damping term. Following the TL formulation, this viscous damping term can be discretized as

$$G^0 = \frac{\eta}{\rho^0} (\nabla^0)^2 v = \frac{2\eta}{m_n} \sum_a V_a V_b \frac{\partial W_{ab}}{\partial r_{ab}} |^0,$$ (89)

where \( \eta \) is the dynamic viscosity.

Note that along with the acoustic and body-force time-step size criteria, the time-step size during the simulation would be also constrained by

$$\Delta t \leq \frac{h^2}{\nu_D},$$ (90)

where \( D = \{1, 2, 3\} \) for one-, two- or three-dimensional cases, respectively. This limitation may lead excessive computational efforts especially when large damping ratio and high resolution ratio are applied. To release this limitation, an operator splitting scheme [168, 22] is first applied to decouple the momentum conservation equation Eq. (88) into the original momentum part and the damping part. Then, two operators \( S_m \) and \( S_d \), which yield

\[
S_m : \ v(t + \Delta t) = v(t) + \left( \frac{1}{\rho^0} \nabla^0 \cdot \overrightarrow{P}^T + g \right) \Delta t, \quad (91)
\]

and

\[
S_d : \ v(t + \Delta t) = v(t) + G^V \Delta t, \quad (92)
\]

are introduced as forward Euler scheme is adopted for time integration. Subsequently, the first order Lie-Trotter splitting scheme [169] is applied to approximate the solution from time \( t \) to \( t + \Delta t \) by

$$v(t + \Delta t) = S_d^{(\Delta t)} \circ S_m^{(\Delta t)} v(t),$$ (93)

where the symbol \( \circ \) denotes the separation of each operator and indicates that \( S_d^{(\Delta t)} \) is applied after \( S_m^{(\Delta t)} \). As demonstrated in Refs. [170, 171], a larger time-step size is allowed when suitable implicit formulations are constructed to solve the viscous term.

To avoid large scale matrix operations for traditional implicit formulations, the entire-domain-related damping step is sequentially split into particle-by-particle operators, e.g. by second-order Strang splitting [172], as

$$S_d^{(\Delta t)} = D_1^{\frac{\Delta t}{2}} \circ D_2^{\frac{\Delta t}{2}} \circ \ldots \circ D_{N_p-1}^{\frac{\Delta t}{2}} \circ D_{N_p}^{\frac{\Delta t}{2}} \circ D_{N_p-1}^{\frac{\Delta t}{2}} \circ \ldots \circ D_2^{\frac{\Delta t}{2}} \circ D_1^{\frac{\Delta t}{2}},$$ (94)

where \( N_p \) denotes the total number of particles and \( D_a \) the split damping operator corresponding to particle \( a \). Two efficient schemes, the particle-by-particle splitting scheme and the pairwise splitting scheme, are then proposed for the local damping operator [167]. The new time-step velocity updating for the entire field can be thus achieved by carrying out the local split operator to all particles for half a time step and then performing the operator to these particles in a reverse sequence for another half time step [173] as shown in Eq. (94).

### 4.4.1. Particle-by-particle splitting scheme

In an implicit formulation, the local damping term in Eq. (89) can be rewritten as

$$f^V = \left( \frac{dv_a}{dt} \right)^V = \frac{2\eta}{m_a} \sum_b V_a V_b \frac{\partial W_{ab}}{\partial r_{ab}} |^0,$$ (95)

where \( v_{ab}^{n+1} = v_{ab}^n + dv_{ab} - dv_{ab} \) with \( dv_{ab} \) and \( dv_{ab} \) representing the incremental change of velocity for particle \( a \) and its neighboring particles \( b \) induced by viscous acceleration. After denoting

$$B_{ab} = 2\eta V_a V_b \frac{1}{r_{ab}} \frac{\partial W_{ab}}{\partial r_{ab}} |^0,$$ (96)

and

$$E_{ab} = -2\eta \sum_b V_a V_b \frac{\partial W_{ab}}{\partial r_{ab}} |^0,$$ (97)

the implicit formulation Eq. (95) can be simplified to

$$E_{ab} = \left( \sum_{b} B_{ab} - m_a \right) dv_{ab} - \sum_{b} B_{ab} dv_{ab}.$$ (98)

A gradient descent method [174] is then adopted to evaluate \( dv_a \) and \( dv_{ab} \). In Eq. (98), the gradient \( \nabla E_{ab} \) with respect
to variables \((d\mathbf{v}_a, d\mathbf{v}_1, d\mathbf{v}_2, \ldots, d\mathbf{v}_N)^T\) gives

\[
\nabla \mathbf{E}_a = \left( \sum_b B_b - m_a, -B_1, -B_2, \ldots, -B_N \right)^T.
\]

(99)

Let

\[
(d\mathbf{v}_a, d\mathbf{v}_1, d\mathbf{v}_2, \ldots, d\mathbf{v}_N)^T = k \nabla \mathbf{E}_a,
\]

(100)

where \(k\) is known as the learning rate \[174\]. By substituting Eqs. (99) and (100) into Eq. (98), the learning rate can be obtained, i.e.

\[
k = \left( \sum_b B_b - m_a \right)^2 + \sum_b (B_b)^2 \right)^{-1} E_a.
\]

(101)

According to Eqs. (99) and (100), the incremental change of velocity by viscous damping can be thus achieved. In order to ensure momentum conservation, the velocities of neighboring particles are then modified by the above predicted incremental change. In summary, the local update of velocities includes two steps as follows. The first step calculates the incremental change for velocity by gradient descent method, i.e.,

\[
\begin{aligned}
\mathbf{v}^{a+1}_a &= \mathbf{v}^a_a + d\mathbf{v}_a = \mathbf{v}^a_a + (\sum_b B_b - m_a) k \\
\mathbf{v}^p_1 &= \mathbf{v}^1_1 + d\mathbf{v}_1 = \mathbf{v}^1_1 - B_1 k \\
\mathbf{v}^p_2 &= \mathbf{v}^2_2 + d\mathbf{v}_2 = \mathbf{v}^2_2 - B_2 k \\
\mathbf{v}^p_N &= \mathbf{v}^N_N + d\mathbf{v}_N = \mathbf{v}^N_N - B_N k \\
\end{aligned}
\]

where the superscript \(p\) denotes the predicted value.

The second step ensures momentum conservation, which yields

\[
\begin{aligned}
\mathbf{v}^{p+1}_1 &= \mathbf{v}^1_1 - B_1 \left( \mathbf{v}_i^{p+1} - \mathbf{v}_i^p \right) / m_1 \\
\mathbf{v}^{p+1}_2 &= \mathbf{v}^2_2 - B_2 \left( \mathbf{v}_i^{p+1} - \mathbf{v}_i^p \right) / m_2 \\
\vdots \\
\mathbf{v}^{p+1}_N &= \mathbf{v}^N_N - B_N \left( \mathbf{v}_i^{p+1} - \mathbf{v}_i^p \right) / m_N \\
\end{aligned}
\]

(102)

As the velocities are updated implicitly, much larger time-step size is allowed and the following viscous criterion

\[
\Delta t \leq 50 \frac{b^2}{vD},
\]

(104)

which is about 100 times larger than the corresponding explicit method as presented in Eq. (90), is adopted. For solid, the artificial dynamic viscosity \(\eta = \rho v\) as shown in Eq. (88) is defined by

\[
\eta = \frac{1}{4} \beta \sqrt{\frac{E}{\rho} L} = \frac{\beta}{4 \sqrt{\rho E L}},
\]

(105)

where \(E\) is the Young’s modulus, \(L\) the characteristic length scale of the problem and \(\beta\) denotes a parameter relating to the body shape. Note that choosing different value for the parameter \(\beta\) may alter, though not much, the speed to final state. For fluid, the viscosity is defined by

\[
\eta = \rho U_{\text{max}} L.
\]

(106)

4.4.2. Pairwise splitting scheme

The pairwise splitting scheme is inspired by the work of Ref. [170], where particle velocity is updated implicitly and locally in a pairwise fashion. By adopting the second-order Strang splitting [172], the damping operator corresponding to each particle \(i\) as given in Eq. (94) is further split based on its neighbors, i.e.,

\[
D^{(N)}_a = D^{(N)}_{a,b} \circ D^{(N)}_{a,b} \cdots D^{(N)}_{a,b} \circ D^{(N)}_{a,b} \circ D^{(N)}_{a,b} \cdots D^{(N)}_{a,b} \circ D^{(N)}_{a,b} \circ D^{(N)}_{a,b},
\]

(107)

where \(D^{(N)}_{a,b}\) denotes the interaction between particle \(a\) and its neighbors. Specifically, the incremental changes for velocity of a specific particle pair induced by viscosity can be written in implicit form as

\[
\begin{aligned}
m_a d\mathbf{v}_a &= B_b (\mathbf{v}_{ab} + d\mathbf{v}_a - d\mathbf{v}_b) \\
m_b d\mathbf{v}_b &= -B_a (\mathbf{v}_{ab} + d\mathbf{v}_a - d\mathbf{v}_b).
\end{aligned}
\]

(108)

Here, \(B_b\) is defined in Eq. (96) and it is obvious that this process does not change the conservation of momentum. Then, \(d\mathbf{v}_a\) and \(d\mathbf{v}_b\) can be obtained straightforwardly by solving Eq. (108), which yields

\[
\begin{aligned}
d\mathbf{v}_a &= m_b B_{ab} \frac{B_{b,vb} - B_{ab,v} - m_b v_b I_b}{B_{b,vb} - B_{ab,v} - m_b v_b I_b} - m_a \frac{B_{ab,v} - m_a v_a I_a}{B_{ab,v} - m_a v_a I_a} \mathbf{b} \\
d\mathbf{v}_b &= -m_a \frac{B_{ab,v} - m_a v_a I_a}{B_{ab,v} - m_a v_a I_a} \mathbf{b}
\end{aligned}
\]

(109)

By sweeping over all neighboring particle pairs for half a time step and then over these particles in a reverse sequence for another half time step, the incremental changes for velocity of particle \(a\) and all its neighbors can be thus achieved. Compared to the particle-by-particle splitting method, this scheme leads more errors in solving viscosity due to the further splitting in pairwise fashion. However, it is unconditional stable and thus more suitable for
problems with high spatial resolution and high damping ratio.

4.4.3. Random-choice strategy

It is worth noting that the added viscous force would hinder the system achieving correct steady state especially when the damping ratio is large, which may lead to the different solutions of the nonlinear problems with different damping ratio [162]. Thus, a suitable damping ratio has to be selected [162, 175] for faster reaching to final state of the system with the aforementioned viscous damping methods.

To avoid this damping-ratio-related problem and relax the limitation on the choice of large damping ratio, a random-choice strategy is presented, in which the viscosity term is imposed randomly rather than at every time step. To achieve this, the artificial dynamic viscosity \( \eta \) is modified as

\[
\tilde{\eta} = \begin{cases} 
\frac{\eta}{\alpha} & \text{if } \alpha > \phi \\
0 & \text{otherwise}
\end{cases},
\]

where \( \phi \) is a random number uniformly distributed between 0 to 1, and \( \alpha = 0.2 \) a parameter determining the probability. Therefore, the resistance on displacement induced by the large artificial viscosity can be released randomly, which eliminates the damping-ratio-related issue and accelerates the achievement to the final state. Note that this strategy also helps to save much computational cost since the computation of damping is only carried out at a small fraction of time steps.

The performance of the VDR-type dynamics relaxation scheme is validated by considering an elastic block sliding along a smooth slope accelerated by the gravity [167]. Figure 11 presents the time histories of the distance between the block center and the slope, and the displacements of the block center in \( x \)- and \( y \)-direction. With the dynamics relaxation scheme, the steady state is quickly achieved by surpassing oscillations.

4.5. Hourglass control scheme

In the FEM method, hourglass modes represent zero-energy modes in the sense that the element deforms without an associated increase of the elastic energy when reduced-integration elements are employed. Insufficient integral points lead to rank deficiency of FEM stiffness matrix, which further causes the non-uniqueness of governing equations. Similarly, the SPH method is also susceptible to hourglass modes as all field variables and their derivatives are evaluated at the same position [141, 176]. These modes cannot be detected and can be developing over time [177] if the number of integration points is reduced, and therefore the solution is polluted with arbitrary amounts of strain energy and entirely dominated by these modes [139].

To address the hourglass modes in SPH method, Sweigle et al. [178] proposed to replace the strain measure by a non-local approximation based on gradient approach. Beissel and Belytschko [179] treated these singular modes by the addition to the potential energy functional of a stabilization term which contains the square of the residual of the equilibrium equation. A more straightforward idea is to introduce additional integral points to calculate derivatives away from particles with zero derivatives of the kernel function [152, 180]. Two sets of points, velocity points and stress points, are applied to discretize the calculation domain, one carrying the velocity and the other carrying the stress. While the velocity gradient and stress are computed on stress points, the divergence of stress is calculated using stress points as neighbor particles and then sampled at velocity points. The enhanced stability of additional stress points was confirmed [39, 181, 182, 183, 184]. However, estimating the stresses induces extra computational efforts and how to place the stress points is till not fully addressed.

When the FEM using the one-point reduced finite element, a mean deformation gradient is obtained, resulting mean strain and stress over a single element. The SPH method also evaluates a mean strain and stress at the center of a particle through the weighted averaging over the neighboring particles. Recently, Ganzenmüller [139] recognized the analogy between SPH collocation and FEM using the one-point reduced element. Inspired by Flanagan and Belytschko [185], Ganzenmüller pointed out that the mean stress-strain description can only represent a fully linear velocity field, which implies node or particle displacement should be exactly described by the deformation gradient, and node or particle displacement incompatible with the linear deformation field are identified as the hourglass modes. The distance of particle \( a \)
and $b$ can be estimated by the deformation gradient $F_a$ as
\[
\langle r_{ab} \rangle^a = F_a r_{ab}.
\] (111)
When the hourglass mode exists, the estimated distance between the two particles is inconsistent with the actual distance. This difference defines the error vector
\[
\Phi_{ab} = \langle r_{ab} \rangle^a - r_{ab}.
\] (112)
A correction force, proportional to the error vector, is introduced, i.e., an artificial stiffness is added to counteract hourglass modes. Then, the scalar $\delta_{ab}^a$ can be defined as
\[
\delta_{ab}^a = \Phi_{ab}^a / |r_{ab}|.
\] (113)
$\delta_{ab}^a$ is the projection of error vector onto current particle distance vector. The hourglass correction force per unit volume is derived as
\[
\hat{f}_{HG}^{ab} = -E \delta_{ab}^a r_{ab} / |r_{ab}|^2.
\] (114)

The stiffness is linear in $\delta_{ab}^a$, and described using the Young’s modulus $E$ of the material. A normalized smoothing kernel is used to be consistent with SPH collocation, and a explicit systematization via the arithmetic mean is applied as the hourglass forces, $\hat{f}_{ab}^a$ and $\hat{f}_{ba}^b$, are unsymmetrical. Therefore, the smoothed and symmetric correction force between particle $a$ and $b$ can be expressed as
\[
\hat{f}_{HG}^{ab} = -\frac{1}{2} \delta_{ab}^a V_a V_b W_{ab} \left( E_a \delta_{ab}^a + E_b \delta_{ba}^b \right) r_{ab} / |r_{ab}|.
\] (115)
where $\alpha$ is a dimensionless constant which determines the amplitude of the hourglass control. Finally, the total hourglass correction force of particle $a$ over all neighboring particles is
\[
\hat{f}_{HG}^{ab} = -V_a \sum_b V_b W_{ab} \hat{f}_{HG}^{ab} = \sum_b -\frac{1}{2} \alpha V_a V_b W_{ab} \left( E_a \delta_{ab}^a + E_b \delta_{ba}^b \right) r_{ab} / |r_{ab}|^2.
\] (116)

Figure 11: Numerical investigation of the block sliding along slope. The configuration (upper panel) and the time histories of the distance between the block center and the slope (a) and displacement of the block center in $x$– and $y$–directions (b).
This hourglass control algorithm has successfully applied in geomaterials [186], FSI with GPU acceleration [187, 188], etc.

5. Fluid-structure interaction

Fluid-structure interaction, where the structure represents either movable rigid or flexible structures is ubiquitous in natural phenomena, e.g. aerial animal flying, aquatic animal swimming and blood circulation, and also plays a crucial role in the design of many engineering systems, e.g. automobile, aircraft, spacecraft, engines and energy harvesting device. This phenomenon is characterized by the multiphysics coupling between the laws that describe fluid dynamics and structural mechanics. Due to the intrinsic complexity of the interaction between a movable or flexible structure and a surrounding or internal fluid flow, computational study of this type multiphysics problems is highly challenging.

The conventional FSI algorithms are based on mesh-based methods, i.e., the finite difference method (FDM) [189], the FEM [190] and the finite volume method (FVM) [191], by implementing monolithic or partitioned approach. The monolithic approach treats the coupled problem as a whole with proper combination of the sub-system, applying a single solver to simultaneously solve the governing equations of the fluid and solid dynamics in FSI. One typical example is the arbitrary Lagrangian-Eulerian (ALE) description of the FEM method [192] where moving mesh is introduced to fluid discretization for addressing the issues of unacceptable mesh distortion near the structure undergoing large deformations. This approach encounters difficulties of the convective terms treatment and the challenging of complex mesh regeneration, in particular when large structure deformation is evolved [193]. The partitioned approach strives to solve each sub-problem separately, applying computational fluid and computational solid solves for the fluid and solid, respectively, with communication of FSI interface data. Typical example are the immersed-boundary method (IBM) [194] which utilizes two overlapped Lagrangian and Eulerian meshes. In IBM method, the fluid equation is solved on the Eulerian mesh and the effects of solid structure are taken into account by distributing the forces computed on the deformed Lagrangian mesh to the Eulerian counterpart using proper kernel function. Compared with the monolithic one, the partitioned approach suffers from Lagrangian-Eulerian mismatches on the kinematics and the distribution of solid structure forces due to the fairly weak coupling formulation.

As an alternative for tackling FSI problems, the meshless methods, i.e., the SPH [1, 2, 67], the MPS [195] and the DEM [196] provide unified monolithic approach with pure Lagrangian discretization of both the fluid and solid equations. In recent years, the meshless methods have attracted significant attention in studying FSI problems, owing to their peculiar advantages in handling material interfaces [18][137] and the capability of capturing violent events such as wave impact and breaking [64]. Promising results have been obtained by the weakly-compressible SPH method [13, 14, 27, 64, 188, 197, 198, 199, 200, 201], incompressible SPH method [144, 202, 203], MPS method [204], SPH-DEM or MPS-DEM methods [205, 206] and their combinations with FEM by using partition approach [207, 208, 209, 210, 211].

In this part, we focus on the recent developments of unified SPH method for FSI problems and special attention are devoted to the FSI interface treatment, multi-resolution discretization and time stepping schemes, which are key components of the accurate and efficient FSI algorithm. Concerning the applications of FSI algorithm in engineering, comprehensive reviews can be found in Refs. [9, 14, 15, 26, 28, 212].

5.1. Treatments of FSI interface

In unified SPH-FSI computation, the fluid-structure coupling is resolved by treating the surrounding movable or flexible structure as moving solid boundary for fluid with imposing free- or no-slip boundary condition at the fluid-structure interface.

Concerning the treatment of movable or flexible boundary, several methods have been proposed and they are generally categorized into three schemes, i.e., boundary force particle, dummy particle and one-sided Riemann scheme. In the first scheme, the structure particle is behaving as the moving boundary force particle [213] and a repulsive force is introduced to prevent particle penetration. This scheme was first proposed by Monaghan and Kajtar [213], where one layer boundary particles provide Lennard-Jones potential repulsive force for fluid particle, and was improved by Liu et al. [214] and further by
Zhang et al. [213] with introducing a new numerical approximation scheme for estimating field functions of solid particles. In the second scheme, the structure is presented by the dummy particle whose velocity and pressure are interpolated from fluid particles [55, 84] to solve governing equations. Notwithstanding its wide application, this approach exhibits excessive computational efforts which are inherently expensive in three-dimensional simulations due to the data interpolation. In the third scheme, the one-sided Riemann problem is constructed along the structure norm and solved to determine the FSI coupling [67]. This scheme has demonstrated its robustness, accuracy and efficiency [49, 67, 145].

Note that there are other schemes, for example ghost particles [5] and semi-analytical approach [216, 217, 218], have been applied for handing the FSI interface treatment in the separation of boundary particles. The CD-SBT has shown its accuracy and robustness in the simulation of multi-phase flow [221], free-surface flows interacting with movable rigid objects [222] and hydro-elastic FSI [198].

5.1.1. Boundary force particle

The boundary force particle scheme is firstly proposed by Monaghan and Kajtar [213] for approximating arbitrarily shaped boundaries. The movable boundary is modeled by one layer boundary particles with particle spacing as a factor of 3 less than the fluid particle spacing, as shown in the left panel of Figure 12 and these particles interact with the fluid particles by forces depending on the separation of the particles and pre-determined parameters. Then, the force $f_i^b$ in Eq. (15) acting on a fluid particle $i$, due to the presence of the neighboring solid particle $a$, is given by [213, 219]

$$f_i^b = \beta \sum_a \phi_{ia} \frac{r_{ia}}{r_{ia}(r_{ia} - \Delta)}.$$  

where

$$\phi_{ia} = \begin{cases} \frac{1}{32}(1 + \frac{2}{3}q + 2q^2)(1.5 - q)^3 & q < 1.5 \\ 0 & q \leq 1.5 \end{cases}$$

(118)

Here, $\beta = v_{\text{max}} / (\rho a d p_0^2)$ with $v = \frac{1}{2} a d c$ and $\Delta = d p_0 / 3$ is the separation of boundary particles.

This solid boundary treatment is widely applied in SPH simulations for dealing with FSI interface treatment involving complex solid shape in single- and multi-phase flows [213, 220], and also extended for SPH-FEM coupling scheme [207]. However, using the artificial repulsive forces violates the kernel truncation in the immediate vicinity of the solid boundaries as only a single layer of particles are required to mimic the boundaries. To address this issue, Liu et al. [214] developed a coupled dynamic SBT scheme, termed as CD-SBT, by introducing ghost particles along with the repulsive force particles, as shown in the right panel of Figure 12. In the CD-SBT scheme, the repulsive particles are similar to that of Ref. [213] with identical particle spacing of fluid particle. Ghost particles are located outside the repulsive ones and initially generated in a regular or irregular distribution [214]. An improved repulsive force in the form of

$$f_{ia}^r = 0.01c^2 \sum_a \chi f(\eta) \frac{r_{ia}}{r_{ia}^2},$$

(119)

with

$$\eta = r_{ia} / 0.75h,$$

$$\chi = 1 - \frac{r_{ia}}{dp_0}, 0 < r_{ia} < dp_0,$$

$$f_\eta = \begin{cases} 2/3 & 0 < \eta < 2/3 \\ (2\eta - 1.5\eta^2) & 2/3 < \eta < 1 \\ 0.5(2 - \eta^2) & 1 < \eta < 2 \\ 0 & \text{otherwise} \end{cases}$$

(120)

is proposed. Also, both the repulsive particles and ghost particles are dynamically evolved in the SPH approximation of the governing equations, and their density and velocity can be interpolated from the fluid particles by

$$\{ \rho_i = \sum_m m_i W_{ia}^{\text{new}} \}
\{ \mathbf{v}_i = - \sum_m \mathbf{v}_m W_{ia}^{\text{new}} \mathbf{v}_i \},$$

(121)

where $W_{ia}^{\text{new}}$ represents the corrected kernel function with Shepard filter or MLS method. Its straightforward to note that the repulsive force particle provides a penetration force and its combination with ghost particle restores consistency by extending the full support domain of fluid particles. The CD-SBT has shown its accuracy and robustness in the simulation of multi-phase flow [221], free-surface flows interacting with movable rigid objects [222] and hydro-elastic FSI [198].

5.1.2. Dummy particle

In dummy particle scheme, the solid is discretized with dummy particles in a layer of width $r_c$, the cutoff radius of
the kernel function, along the interface or the particles of the flexible structure to represent dummy particles when interacting with fluid particles. As shown in the left panel of Figure 13, the fluid particles (in blue) near the wall interact with the dummy particles (in black) which lie within the support radius of the smoothing kernel function. Then, the force $f_s$ acting on the fluid is decomposed into the pressure force $f_s^p$ and viscous force $f_s^v$ which are defined as

$$f_s^p = -\frac{2}{m_i} \sum_a V_i V_a p_a \nabla W_{ia},$$
$$f_s^v = \frac{2}{m_i} \sum_a \eta V_i V_a v_i - \bar{v}_a \frac{\partial W_{ia}}{\partial r_{ia}},$$

Figure 12: Sketch of multi-phase particles interacting with solid particles along the normal vector through the one-side Riemann problem.

Figure 13: Sketch of multi-phase particles interacting with solid particles along the normal vector through the one-side Riemann problem.

(122)

to impose the no-slip boundary condition. Note that the $v_a$ of Eq. (124) represents the velocity of the movable solid or flexible structure. And the dummy particle pressure $p_a$ of Eq. (123) can be calculated with a summation over all contribution of the neighboring fluid particles by [55]

$$p_a = \sum_i p_i W_{MLS}(r_i) dV_i + 2d\rho a \cdot g,$$

(125)

or [84]

$$p_a = \sum_i p_i W_{ai} + (g - a_a) \cdot \sum_i r_{ai} W_{ai}.$$  

(126)

Here, $a_a$ is solid acceleration. With the interpolated pressure $p_a$, the dummy particle density $\rho_a$ can be derived from the EoS of Eq. (19). Compared with the interpolation of Eq. (125), Eq. (126) is written in the more general formulation with moving solid present, making it a suitable choice for FSI involving the moving or flexible structure.

With the dummy particle, the kernel truncation of fluid particles close to the structure is avoided, ensuring ap-
forces right panel of Figure 13. Solved to realize the FSI coupling [67] as shown in the man problem is constructed along the solid normal and structure with dummy particles, while a one-sided Riemann scheme also represents the movable or flexible phase FSI [203]. Notwithstanding its wide application, the dummy particle results excessive computational efforts which are inherently expensive in three-dimensional simulations due to the introduction of physical variable interpolation.

5.1.3. One-sided Riemann-based scheme

Similar to the dummy particle scheme, the one-sided Riemann scheme also represents the movable or flexible structure with dummy particles, while a one-sided Riemann problem is constructed along the solid normal and solved to realize the FSI coupling [67] as shown in the right panel of Figure 13.

In the one-sided Riemann scheme [67], the pressure forces \( f^p_i \) are rewritten as

\[
f^p_i = -\frac{2}{m_i} \sum_a V_i V_a p^* \nabla_i W_{ia},
\]

where \( p^* \) is the Riemann solution of the one-sided Riemann problem whose left and right states are defined as

\[
\begin{align*}
(\rho_L, U_L, p_L) &= (\rho_f, -\mathbf{n}_a \cdot \mathbf{v}_i, p_i) \\
(\rho_R, U_R, p_R) &= (\rho_0, -\mathbf{n}_a \cdot (2\mathbf{v}_i - \mathbf{v}_a), p_a)
\end{align*}
\]

Here, \( \mathbf{n}_a \) is the local normal vector pointing from solid to fluid, and the pressure \( p_a = p_f + \rho_0 g \cdot \mathbf{r}_a \). With the dummy particle pressure, its density \( \rho_0 \) is also calculated through the EoS presented in Eq. [19]. Compared with the aforementioned dummy particle scheme, the present one is more simple and efficient due to the fact that the one-sided Riemann problem is solved in a particle-by-particle fashion and no interpolation of states for the solid particles is required.

For each solid particles, the normal vector in the reference configuration can be calculated by [67]

\[
\mathbf{n}_a^0 = \frac{\Phi(\mathbf{r}_a^0)}{|\Phi(\mathbf{r}_a^0)|}, \quad \Phi(\mathbf{r}_a^0) = -\sum_{b \neq a} \nabla_{ab} W_{ab} \nabla_{ab},
\]

where the summation is over wall particles only. For static solid structure, the normal vector is not altered during the computation. For flexible structure, the normal vector should be updated accordingly when deformation occurs. To void of frequent summation computation in Eq. [129], the updated normal vector for flexible structure can be obtained by

\[
\mathbf{n}_a = \mathbf{Q} \mathbf{n}_a^0,
\]

where \( \mathbf{Q} \) is the orthogonal matrix of the deformation tensor \( \mathbf{F} \) and can be calculated with polar decomposition.

Figure 14 presents the numerical investigation of wave interaction with an oscillating wave surge converter (OWSC) [223] with SPHinXsys library [12] by using Riemann-based WCSPH method with one-sided Riemann-based fluid-solid interface treatment. It is observed that smooth velocity fields are produced even when complex interactions between the wave and the flap are involved. Also, wave-structure interaction and wave loading are well predicted in comparison with the experimental data [224], numerical results obtained with commercial software FLUENT [224] and SPH results in the literature [225, 226].

5.2. Multi-resolution scheme

When applying the particle-based solver for modeling FSI problems, single spatial-temporal resolution, where a uniform particle spacing is used for discretizing the entire computational domain and a single time step being the smallest one of these required by the fluid and solid structure is used for time integration [13], is commonly employed. In such a case, the single-resolution approach is computationally expensive and high memory consumption in the application where the structure requires locally refined resolution or the global computational domain is quite larger with respect to the critical sub-domain. Therefore, developing a multi-resolution scheme for particle-based FSI solver is desirable in the computational efficiency point of view.

For particle-based simulation, different accurate, stable and consistent multi-resolution schemes have been developed for discretizing the fluid equations and they are generally classified into four classes: adaptive particle refinement (APR) with or without particle splitting/merging [117, 227, 228, 229, 230], non-spherical particle scheme [231, 232], domain-decomposition based scheme [233, 234] or the hybrid scheme [235, 236, 237]. Notwithstanding these progress, the development of multi-resolution
scheme for particle-base FSI solver has merged in the very recent years. The pioneering work is credited to Khayyer et al. [204] where a multi-resolution scheme is developed for MPS-FSI solver. Then, Sun et al. [203] extended the multi-resolution scheme developed by Barcarolo et al. [235] to the simulation of multi-phase hydroelastic FSI problems. More recently, Zhang et al. [23] proposed a multi-resolution SPH method for fluid-flexible structure interaction with special attention in dealing with enforcing the momentum conservation and force matching at the fluid-structure interface.

It is worth noting that consistent particle resolution is applied through FSI interface in Ref. [203] where the APR scheme is applied in predefined area interface located in the fluid domain, with proper particle splitting/merging [235]. On the other hand, different particle resolution is adopted in the FSI interface of Refs. [204, 23, 238] with proper handling of momentum conservation and force matching. In this work, we focus on the second approach as the first one involves splitting/merging scheme which is not the main objective of the survey.

5.2.1. Multi-resolution discretization

In the multi-resolution framework of Ref. [23], the fluid and solid equations are discretized by different spatial-temporal resolutions. In this case, the solid structure can be resolved at a higher spatial resolution, and the computational efficiency is enhanced when a lower resolution discretization for the fluid is sufficient. This strategy is suitable for applications where the structure is relatively thin, i.e. the structure has a considerable small spatial scale compared with fluid, or when the structure has a high Poisson ratio which results smaller time step size than that required by the fluid.

With different spatial resolutions are applied across the FSI interface, the interaction pressure force \( f_{i}^{p} \) and viscous force \( f_{i}^{v} \) are rewritten as

\[
\begin{align*}
    f_{i}^{p} (h') &= -\frac{2}{m_i} \sum_a V_i V_a \tilde{p}_{ia} V_i W(r_{ia}, h') , \\
    f_{i}^{v} (h') &= \frac{2}{m_i} \sum_a \eta V_i V_a \frac{\nabla W(r_{ia}, h')}{\nabla r_{ia}} , \\
\end{align*}
\]

where \( h' \) denotes the smoothing length used for fluid with the assumption of \( h' \geq h \), ensuring that a fluid particle \( i \) can be searched and tagged as a neighboring particle of a solid particle \( a \) which is located in the neighborhood of particle \( i \). Having Eq. (131) in hand, the fluid forces...
exerting on the solid structure can be derived straightforwardly.

5.2.2. Multi-time stepping with position-based Verlet scheme

Following Ref. [22][145], the time-step criterion for the solid integration is given as

$$\Delta^s = 0.6 \min \left( \frac{h^s}{c^s + |v_{\text{max}}|}, \sqrt{\frac{h^s}{|\alpha_{\text{diff}}|_{\text{max}}}} \right).$$

(132)

With the advection criterion $\Delta^f_{ac}$ of Eq. (69) and the acoustic criterion $\Delta^f_{ac}$ of Eq. (70) in hand, three different time step sizes are introduced. Generally, $\Delta^s < \Delta^f_{ac}$, due to the fact that $c^s > c^f$. Other than choosing $\Delta^s$ as the single time step for both fluid and structure, one can carry out the structure time integration $\kappa = \lfloor \frac{\Delta^s}{\Delta^f_{ac}} \rfloor + 1$ times, where $\lfloor \cdot \rfloor$ represents the integer operation, during one acoustic time step of fluid integration. As different time steps are applied in the integration of fluid and solid equations, the issue of force mismatch in the fluid-structure interaction may be encountered. That is, in the imaginary pressure and velocity calculation, the velocity and acceleration of solid particles in Eq. (131) may present several different values updated after each $\Delta^s$. Another issue is that the momentum conservation in the fluid and structure coupling may be violated. To address the force-calculation mismatch, Zhang et al. [23] proposed to calculate the imaginary pressure $p^d$ and velocity $v^d$ as

$$\begin{cases} p^d_a = p_i + \rho_{\text{max}}(0, (g - \frac{\bar{\alpha}_a}{\alpha_{\text{diff}}}) \cdot n^s)(r_a \cdot n^s) \\ v^d_a = 2v_i - \bar{v}_a \end{cases},$$

(133)

where $\bar{v}_a$ and $\frac{\bar{\alpha}_a}{\alpha_{\text{diff}}}$ represents the single averaged velocity and acceleration of solid particles during a fluid acoustic time step.

Also, to address the momentum conservation issue, Zhang et al. [23] developed a position-based Verlet scheme. Instead of starting with a half step for velocity followed by a full step for position and another half step for velocity as in the velocity-based Verlet scheme [84], the position-based Verlet does the opposite: a half step for position followed by a full step for velocity and another half step for position. Figure 15 depicts the velocity- and position-based Verlet schemes assuming that $\kappa = 4$ for the integration of fluid and solid equations. Note that since the position is updated twice and the velocity once with the acceleration at the half step, using the Taylor expansion one can find that the position-based scheme has the same 2nd-order accuracy as the original one.

In the position-based Verlet scheme, as the velocity field is updated only once in the current fluid acoustic time step criterion, time marching of the momentum equations for fluid and solid are exactly consistent as the velocity marching interval $(\Delta t_{ac})_n = \sum_{x=0}^{n-1}(\Delta t^s)_x$, as shown Figure 15. Therefore, the position-based Verlet algorithm achieves strict momentum conservation in fluid-structure coupling, when multiple time steps is employed. In contrast, the velocity-based Verlet scheme does not guarantee momentum conservation as $0.5 \left| (\Delta t_{ac})_n + (\Delta t^s)_n \right| \neq 0.5 \sum_{x=0}^{n-1} \left| (\Delta t^s)_{x-1} + (\Delta t^s)_x \right|$, as also shown in Figure 15.

Figure 16 reports the validation of the multi-resolution scheme implemented in the open-source library SPHinXs [12][146] by simulating two FSI benchmark tests, i.e., flow-induced vibration of a beam attached to a cylinder and dam-break flow with elastic gate [145]. For both tests, the deformation of the flexible structure induced by the fluid-structure interaction is accurately captured in comparison with experimental data [13] and numerical data in the literature [13][144][200][239][240]. Concerning the computational efficiency, approximated set up in the order of $10^2$ is achieved compared with the simulation in single-resolution scenario as reported in Ref. [23].

6. Particle and mesh generation

Generating high-quality unstructured mesh or particle distributions is essentially important to mesh-based or particle-based methods in scientific computing [5][222]. However, complex geometries are always involved in industrial applications bringing a critical challenge for generating high-quality particle distributions or mesh for arbitrarily complex geometry.

For particle-based methods, there are generally two approaches to generate the initial particle distributions, i.e., (a) initiating particles on a lattice structure and (b) generating particles on a volume element mesh. The first
approach, positioning particles on a cubic lattice structure, is widely used in particle-based methods community. For example, Dominguez et al. [243] proposed a pre-processing tool for the DualSPHysics library where particles are generated on lattice structure and three-dimensional object is represented by particle model with excluding the outside particles. In this approach, the particles are equally distributed, however, a very fine spatial resolution is needed to correctly portray the complex geometry. The second approach generating particles at the center of tetra- or hexahedron volume elements has been widely used in application of bird strike [244, 245] and provided by state-of-art commercial pre-processing tools. This approach can accurately portray the complex surface, however, comprise drawbacks of non-uniform particle spacing and volume which may reduce the interpolation accuracy. Recently, the weighted Voronoi tessellation (WVT) method has been applied for generating initial particle distribution by Diehl et al. [246] for SPH astrophysical simulation and by Siemann and Ritt [247] for SPH modeling of bird-strike. Also, Vela et al. [248] proposed an algorithm for constructing complex initial density distributions with low noise.

Concerning high-quality mesh generation, various mesh generation techniques have also been developed, e.g. advancing front/layer methods [249, 250] initiating meshing from the boundary to domain interior, refinement based Delaunay triangulation [251, 252] inserting new Steiner points into a Delaunay mesh, centroidal Voronoi tessellations (CVT) [253, 254, 255, 256] and particle-based method [257, 258, 259, 260, 261] in which a relaxation strategy basing on the physical analogy between a simple mesh and a truss structure is applied. Among them, the particle-based mesh generation method has been widely studied due to the efficiency and versatility feature.

More recently, Fu et al. [120] and Zhu et al. [262] presented a novel application of the particle-relaxation in SPH methodology for high-quality unstructured mesh and body-fitted particle distribution, respectively, for arbitrarily complex geometry. Ji et al. [263, 264] further improved the particle-relaxation for mesh generation by exploiting multi-phase algorithm and introducing feature boundary correction term. The general procedure consists of three steps. First, the geometry surface is represented by zero level-set function by parsing corresponding computer-aided design (CAD) file [262]. Second, several steps of particle-relaxation is conducted by solving a set of physically-motivated model equations in the SPH methodology [120]. In this step, proper surface bonding techniques were developed for achieving body-fitted feature of particle distribution. Third, a set of neighboring particles generates a locally valid Voronoi diagram at the interior of the domain by using Delaunay triangulation.

Figure 15: Sketch of velocity- and position-based Verlet schemes with assumption that $\kappa = 4$. 
Figure 16: Numerical investigation of FSI problems with SPHinXsys library in multi-resolution scenario. (a) Flow-induced vibration of an elastic beam attached to a cylinder: snapshots of vorticity field and deformed beam configuration with von Mises contour (left panel) and the comparisons of the amplitude and the frequency of the beam oscillation with data in the literature. (b) Dam-break flow through an elastic gate: snapshots compared against experimental frames [13] (upper panel), vertical (bottom left panel) and horizontal (bottom right panel) displacements of the free end of the plate and their comparisons with experimental [13] and numerical [134] data.
method \[120\]. Note that the third step is only for mesh
generation.

6.1. Surface representation

By parsing a CAD file, the geometry surface can be
constructed and represented by the zero level-set of the
signed-distance function
\[ \Gamma = \{(x, y, z) \mid \phi (x, y, z, t) = 0\} . \]
(134)

Then, the normal direction \( \mathbf{N} = (n_x, n_y, n_z)^T \) of the surface
can be evaluated from
\[ \mathbf{N} = \frac{\nabla \phi}{|\nabla \phi|} . \]
(135)

To discretize the level-set function, a Cartesian back-
ground mesh is generated in the whole computational do-
main. The level-set value \( \phi \) is equal to the distance from
the cell center to the geometry surface. Besides, the nega-
tive phase with \( \phi < 0 \) is defined if the cell center is inside
the geometry and positive phase with \( \phi > 0 \) otherwise.

6.2. Particle relaxation

Starting from a preconditioned Lattice or random parti-
cle distribution generated inside the domain of the geom-
etry, a physics-driven relaxation process is introduced to
define particles evolution. The relaxation is governed by
the momentum conservation equation
\[ \frac{d \mathbf{v}_i}{d t} = -\frac{2}{m_i} \sum_j V_i V_j \mathbf{p}_{ij} \nabla \mathbf{W}_{ij} \]
\[ + \frac{2}{m_i} \sum_j V_i \frac{\eta_j \mathbf{v}_j}{\eta_i + \eta_j} \frac{\partial \mathbf{W}_{ij}}{\partial r_{ij}} , \]
(136)

where \( \mathbf{v} \) is the advection velocity and \( \eta = 0.2h\rho|\mathbf{v}| \). In Fu et al. \[120\], \( \mathbf{p}_{ij} = \frac{1}{2}(p_i + p_j) \) and the particle pressure
is defined by an equation of state
\[ p = p_0 \frac{\rho_i^2}{\rho_f^2} , \]
(137)

which incorporates a target density field \( \rho_i \) and \( p_0 \) is the
reference pressure \[120\]. In Zhu et al. \[262\], a constant
pressure \( p_{ij} = p_0 \) is applied as a homogeneous particle
distribution can be obtained by applying the transport-
velocity formulation \[109, 127, 137\] with a constant den-
sity and background pressure. Then, the particle evolution
is defined by
\[ \mathbf{r}^{n+1} = \mathbf{r}^n + \Delta t \frac{d \mathbf{v}^n}{d t} , \]
(138)

where \( a^n = F^n \). Note that only the instant acceleration
is considered for the evolution and particle velocity is set
to zero at the beginning of each time step to achieve a
fully stationary state following Ref. \[120, 127, 137\]. For
numerical stability, the time-step size \( \Delta t \) is constrained by
the body force criterion
\[ \Delta t \leq 0.25 \frac{h}{|\mathbf{v}(d)|} . \]
(139)

6.3. Surface bounding method

To achieve the body-fitted feature, a suitable boundary
condition treatment is required. In Ref. \[120\], a dynamic
ghost-particle method enforcing symmetry conditions at
all domain boundaries is adopted. However, it is chal-
lenge to construct ghost particles for complex geometries.
Zhu et al. \[262\] proposed a simple surface particle bound-
ing method. Specifically, the level-set value \( \phi \) and normal
direction \( \mathbf{N}_i \) of each particle are first surface particle bound-
ing method. Then, particles position is updated according to
\[ \mathbf{r}_i = \begin{cases} \mathbf{r}_i - \left( \phi_i + \frac{1}{2} \Delta x \right) \mathbf{N}_i & \phi_i \geq -\frac{1}{2} \Delta x , \\ \mathbf{r}_i & \text{otherwise}, \end{cases} \]
(140)

where \( \Delta x \) denotes the initial particle spacing. Figure
\[17\] presents the illustration of surface particles bound-
ing. When the particle locates outside of the geometry,
it will be enforced back on the surface following the nor-
mal direction and a body-fitted particle distribution can be
achieved accordingly. Note that, the surface particles are
relocated at \( \phi = -\frac{1}{2} \Delta x \) instead of \( \phi = 0 \) implying that the
material interface assumed to be located at \( \phi = 0 \).

Figure \[18\] portrays the generated unstructured mesh
for tyra and gear body, and particle model for an-
atomy heart and propeller. It is obvious that the
particle-relaxation generates high-quality globally opti-
mized adaptive isotropic meshes and well-regularized par-
ticle distribution for three-dimensional body with high ge-
ometric complexity.
7. Conclusion

In this paper, we present a concise review of SPH method on methodology development and recent achievement with highlights of aspects including numerical algorithms for fluid dynamics, solid mechanics and FSI, and novel applications in mesh and particle generations. Fundamentals and theory of the SPH method are first summarized. Recent developments of Riemann-based SPH method are presented with key aspects of Riemann-solver with dissipation limiter and high-order data reconstructions with MUSCL, WENO and MOOD schemes. Techniques for particle neighbor searching and efficient update of particle configuration are recalled. Concerning the total Lagrangian formulations, stabilized schemes, steady state solution and hourglass control algorithms are reported. For FSI coupling, treatments of FSI interface and discretization schemes in multi-resolution scenario are surveyed. Last but not least, recent novel SPH applications in mesh and particle generations are reviewed. Abundant validations and benchmark test for demonstrating the computational accuracy, convergence, efficiency and stability are also supplied in this survey.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

C. Zhang and X.Y. Hu would like to express their gratitude to Deutsche Forschungsgemeinschaft (DFG) for their sponsorship of this research under grant number DFG HU1527/12-4.

References

[1] L. B. Lucy, A numerical approach to the testing of the fission hypothesis, The Astronomical Journal 82 (1977) 1013–1024.
[2] R. A. Gingold, J. J. Monaghan, Smoothed particle hydrodynamics: theory and application to non-spherical stars, Mon. Not. R. Astron. Soc. 181 (3) (1977) 375–389.
[3] J. J. Monaghan, Smoothed particle hydrodynamics, Annual review of astronomy and astrophysics 30 (1) (1992) 543–574.
[4] P. Randles, L. Libersky, Smoothed particle hydrodynamics: Some recent improvements and applications, Comput. Methods Appl. Mech. Eng. 139 (1-4) (1996) 375–408.
[5] G.-R. Liu, M. B. Liu, Smoothed particle hydrodynamics: A meshfree particle method, World scientific, 2003.
[6] M. B. Liu, G. R. Liu, Smoothed particle hydrodynamics (SPH): an overview and recent developments, Arch. Comput. Methods Eng. 17 (1) (2010) 25–76.
[7] J. J. Monaghan, Simulating free surface flows with SPH, J. Comput. Phys. 110 (2) (1994) 399–406.
[8] A. Ferrari, M. Dumbser, E. F. Toro, A. Armanini, A new 3d parallel SPH scheme for free surface flows, Computers & Fluids 38 (6) (2009) 1203–1217.
[9] M. Luo, A. Khayyer, P. Lin, Particle methods in ocean and coastal engineering, Applied Ocean Research 114 (2021) 102734.
Figure 18: Mesh and particle generations for three-dimensional body with complex geometry. (a) Delaunay triangulation of the unstructure for tyra and tear body [264]. (b) Particle distribution of anatomy heart and propeller [262].
[10] J. Bonet, A. Burton, A simple average nodal pressure tetrahedral element for incompressible and nearly incompressible dynamic explicit applications, Communications in Numerical Methods in Engineering 14 (5) (1998) 437–449.

[11] J. J. Monaghan, SPH without a tensile instability, Journal of Computational Physics 159 (2) (2000) 290–311.

[12] C. Zhang, M. Rezavand, Y. Zhu, Y. Yu, D. Wu, W. Zhang, J. Wang, X. Hu, SPHinXsys: An open-source multi-physics and multi-resolution library based on smoothed particle hydrodynamics, Computer Physics Communications (2021) 108066.

[13] C. Antoci, M. Gallati, S. Sibilla, Numerical simulation of fluid–structure interaction by SPH, Computers & Structures 85 (11-14) (2007) 879–890.

[14] M. Liu, Z. Zhang, Smoothed particle hydrodynamics (SPH) for modeling fluid-structure interactions, SCIENCE CHINA Physics, Mechanics & Astronomy 62 (8) (2019) 984701.

[15] A.-m. Zhang, P.-n. Sun, F.-r. Ming, A. Colagrossi, Smoothed particle hydrodynamics and its applications in fluid-structure interactions, Journal of Hydrodynamics, Ser. B 29 (2) (2017) 187–216.

[16] A. Colagrossi, M. Landrini, Numerical simulation of interfacial flows by smoothed particle hydrodynamics, Journal of Computational Physics 191 (2) (2003) 448–475.

[17] Z.-B. Wang, R. Chen, H. Wang, Q. Liao, X. Zhu, S.-Z. Li, An overview of smoothed particle hydrodynamics for simulating multiphase flow, Applied Mathematical Modelling 40 (23-24) (2016) 9625–9655.

[18] M. Rezavand, C. Zhang, X. Hu, A weakly compressible SPH method for violent multi-phase flows with high density ratio, Journal of Computational Physics 402 (2020) 109092.

[19] M. Russell, A. Souto-Iglesias, T. Zohdi, Numerical simulation of laser fusion additive manufacturing processes using the SPH method, Computer Methods in Applied Mechanics and Engineering 341 (2018) 163–187.

[20] C. Meier, S. L. Fuchs, A. J. Hart, W. A. Wall, A novel smoothed particle hydrodynamics formulation for thermo-capillary phase change problems with focus on metal additive manufacturing melt pool modeling, Computer Methods in Applied Mechanics and Engineering 381 (2021) 113812.

[21] É. Lluch, M. De Craene, B. Bijnens, M. Sermesant, J. Noailly, O. Camara, H. G. Morales, Breaking the state of the heart: Meshless model for cardiac mechanics, Biomechanics and modeling in mechanobiology 18 (6) (2019) 1549–1561.

[22] C. Zhang, J. Wang, M. Rezavand, D. Wu, X. Hu, An integrative smoothed particle hydrodynamics method for modeling cardiac function, Computer Methods in Applied Mechanics and Engineering 381 (2021) 113847.

[23] C. Zhang, M. Rezavand, X. Hu, A multi-resolution SPH method for fluid-structure interactions, Journal of Computational Physics 429 (2021) 110028.

[24] D. Violeau, B. D. Rogers, Smoothed particle hydrodynamics (SPH) for free-surface flows: past, present and future, Journal of Hydraulic Research 54 (1) (2016) 1–26.

[25] M. S. Shadloo, G. Oger, D. Le Touzé, Smoothed particle hydrodynamics method for fluid flows, towards industrial applications: Motivations, current state, and challenges, Computers & Fluids 136 (2016) 11–34.

[26] T. Ye, D. Pan, C. Huang, M. Liu, Smoothed particle hydrodynamics (sph) for complex fluid flows: Recent developments in methodology and applications, Physics of Fluids 31 (1) (2019) 011301.

[27] Z. Zhang, T. Long, J. Chang, M. Liu, A smoothed particle element method (SPEM) for modeling fluid-structure interaction problems with large fluid deformations, Computer Methods in Applied Mechanics and Engineering 356 (2019) 261–293.
[28] H. Gotoh, A. Khayyer, Y. Shimizu, Entirely lagrangian meshfree computational methods for hydroelastic fluid-structure interactions in ocean engineering—reliability, adaptivity and generality, Applied Ocean Research 115 (2021) 102822.

[29] S. J. Lind, B. D. Rogers, P. K. Stansby, Review of smoothed particle hydrodynamics: Towards converged lagrangian flow modelling, Proceedings of the Royal Society A 476 (2241) (2020) 20190801.

[30] R. Vacondio, C. Altomare, M. De Leffe, X. Hu, D. Le Touzé, S. Lind, J.-C. Marongiu, S. Marrone, B. D. Rogers, A. Souto-Iglesias, Grand challenges for smoothed particle hydrodynamics numerical schemes, Computational Particle Mechanics 8 (3) (2021) 575–588.

[31] H. Takeda, S. M. Miyama, M. Sekiya, Numerical simulation of viscous flow by smoothed particle hydrodynamics, Progress of theoretical physics 92 (5) (1994) 939–960.

[32] G. R. Johnson, S. R. Beissel, Normalized smoothing functions for SPH impact computations, International Journal for Numerical Methods in Engineering 39 (16) (1996) 2725–2741.

[33] W. K. Liu, S. Jun, Y. F. Zhang, Reproducing kernel particle methods, International journal for numerical methods in fluids 20 (8-9) (1995) 1081–1106.

[34] J. P. Vila, SPH renormalized hybrid methods for conservation laws: applications to free surface flows, in: Meshfree methods for partial differential equations II, Springer, 2005, pp. 207–229.

[35] J. Bonet, T.-S. Lok, Variational and momentum preservation aspects of smooth particle hydrodynamic formulations, Computer Methods in applied mechanics and engineering 180 (1-2) (1999) 97–115.

[36] Y. Zhu, C. Zhang, X. Hu, A consistency-driven particle-advection formulation for weakly-compressible smoothed particle hydrodynamics, Computers & Fluids 230 (2021) 105140.

[37] J. P. Vila, On particle weighted methods and smooth particle hydrodynamics, Mathematical models and methods in applied sciences 9 (02) (1999) 161–209.

[38] J. Bonet, S. Kulasegaram, Correction and stabilization of smooth particle hydrodynamics methods with applications in metal forming simulations, International journal for numerical methods in engineering 47 (6) (2000) 1189–1214.

[39] R. Vignjevic, J. R. Reveles, J. Campbell, SPH in a total lagrangian formalism, CMC-Tech Science Press- 4 (3) (2006) 181.

[40] C. Zhang, J. Wang, M. Rezavand, D. Wu, X. Hu, An integrative smoothed particle hydrodynamics method for modeling cardiac function, Computer Methods in Applied Mechanics and Engineering 381 (2021) 113847.

[41] J. R. Macdonald, Some simple isothermal equations of state, Reviews of Modern Physics 38 (4) (1966) 669.

[42] J. P. Morris, P. J. Fox, Y. Zhu, Modeling low reynolds number incompressible flows using SPH, Journal of Computational Physics 136 (1) (1997) 214–226.

[43] J. J. Monaghan, Smoothed particle hydrodynamics and its diverse applications, Annu. Rev. Fluid Mech. 44 (2012) 323–346.

[44] X. Hu, N. Adams, A multi-phase SPH method for macroscopic and mesoscopic flows, J. Comput. Phys. 213 (2006) 844–861.

[45] J. J. Monaghan, R. A. Gingold, Shock simulation by the particle method SPH, Journal of computational physics 52 (2) (1983) 374–389.

[46] A. Ferrari, M. Dumbser, E. Toro, A. Armanini, A new stable version of the SPH method in lagrangian coordinates, Communications in Computational Physics 4 (2) (2008) 378–404.

[47] B. Ren, M. He, P. Dong, H. Wen, Nonlinear simulations of wave-induced motions of a freely floating body using WCSPH method, Applied Ocean Research 50 (2015) 1–12.
[48] M. Antuono, A. Colagrossi, S. Marrone, Numerical diffusive terms in weakly-compressible SPH schemes, Computer Physics Communications 183 (12) (2012) 2570–2580.

[49] C. Zhang, M. Rezavand, X. Hu, Dual-criteria time stepping for weakly compressible smoothed particle hydrodynamics, Journal of Computational Physics 404 (2020) 109135.

[50] M. Rezavand, C. Zhang, X. Hu, Generalised and efficient wall boundary condition treatment in gpu-accelerated smoothed particle hydrodynamics (2021). [arXiv:2110.02821].

[51] B. Ben Moussa, J. Vila, Convergence of SPH method for scalar nonlinear conservation laws, SIAM Journal on Numerical Analysis 37 (3) (2000) 863–887.

[52] J. Cercos-Pita, R. Dalrymple, A. Herault, Diffusive terms for the conservation of mass equation in SPH, Applied Mathematical Modelling 40 (19-20) (2016) 8722–8736.

[53] D. Molteni, A. Colagrossi, A simple procedure to improve the pressure evaluation in hydrodynamic context using the SPH, Computer Physics Communications 180 (6) (2009) 861–872.

[54] M. Antuono, A. Colagrossi, S. Marrone, D. Molteni, Free-surface flows solved by means of SPH schemes with numerical diffusive terms, Computer Physics Communications 181 (3) (2010) 532–549.

[55] S. Marrone, M. Antuono, A. Colagrossi, G. Colicchio, D. Le Touzé, G. Graziani, δ-SPH model for simulating violent impact flows, Computer Methods in Applied Mechanics and Engineering 200 (13) (2011) 1526–1542.

[56] J. J. Monaghan, SPH and riemann solvers, Journal of Computational Physics 136 (2) (1997) 298–307.

[57] B. B. Moussa, On the convergence of SPH method for scalar conservation laws with boundary conditions, Methods and applications of analysis 13 (1) (2006) 29–62.

[58] A. Rafiee, S. Cummins, M. Rudman, K. Thiagarajan, Comparative study on the accuracy and stability of SPH schemes in simulating energetic free-surface flows, European Journal of Mechanics-B/Fluids 36 (2012) 1–16.

[59] A. N. Parshikov, S. A. Medin, I. I. Loukashenko, V. A. Milekhin, Improvements in SPH method by means of interparticle contact algorithm and analysis of perforation tests at moderate projectile velocities, International Journal of Impact Engineering 24 (8) (2000) 779–796.

[60] A. N. Parshikov, S. A. Medin, Smoothed particle hydrodynamics using interparticle contact algorithms, Journal of computational physics 180 (1) (2002) 358–382.

[61] S. I. Inutsuka, Reformulation of smoothed particle hydrodynamics with riemann solver, Journal of Computational Physics 179 (1) (2002) 238–267.

[62] S.-H. Cha, A. P. Whitworth, Implementations and tests of godunov-type particle hydrodynamics, Monthly Notices of the Royal Astronomical Society 340 (1) (2003) 73–90.

[63] D. Avesani, M. Dumbser, A. Bellin, A new class of moving-least-squares weno–sph schemes, Journal of Computational Physics 270 (2014) 278–299.

[64] C. Zhang, G. Xiang, B. Wang, X. Hu, N. Adams, A weakly compressible SPH method with WENO reconstruction, Journal of Computational Physics 392 (2019) 1–18.

[65] P.-P. Wang, A.-M. Zhang, Z.-F. Meng, F.-R. Ming, X.-L. Fang, A new type of WENO scheme in SPH for compressible flows with discontinuities, Computer Methods in Applied Mechanics and Engineering 381 (2021) 113770.

[66] D. Avesani, M. Dumbser, R. Vancidio, M. Righetti, An alternative SPH formulation: ADER-WENO-SPH, Computer Methods in Applied Mechanics and Engineering 382 (2021) 113871.
[67] C. Zhang, X. Hu, N. A. Adams, A weakly compressible SPH method based on a low-dissipation riemann solver, J. Comput. Phys. 335 (2017) 605–620.

[68] Z.-F. Meng, P.-P. Wang, A.-M. Zhang, F.-R. Ming, P.-N. Sun, A multiphase SPH model based on roe’s approximate riemann solver for hydraulic flows with complex interface, Computer Methods in Applied Mechanics and Engineering 365 (2020) 112999.

[69] J.-C. Marongiu, F. Leboeuf, J. Caro, E. Parkinson, Free surface flows simulations in pelton turbines using an hybrid SPH-ALE method, Journal of Hydraulic Research 48 (sup1) (2010) 40–49.

[70] K. Puri, P. Ramachandran, A comparison of SPH schemes for the compressible euler equations, Journal of Computational Physics 256 (2014) 308–333.

[71] K. Puri, P. Ramachandran, Approximate riemann solvers for the godunov SPH (GSPH), Journal of Computational Physics 270 (2014) 432–458.

[72] F. V. Sirotkin, J. J. Yoh, A smoothed particle hydrodynamics method with approximate riemann solvers for simulation of strong explosions, Computers & Fluids 88 (2013) 418–429.

[73] V. Mehra, S. Chaturvedi, High velocity impact of metal sphere on thin metallic plates: a comparative smooth particle hydrodynamics study, Journal of Computational Physics 212 (1) (2006) 318–337.

[74] S.-H. Cha, S.-I. Inutsuka, S. Nayakshin, Kelvin–hilmoltz instabilities with godunov smoothed particle hydrodynamics, Monthly Notices of the Royal Astronomical Society 403 (3) (2010) 1165–1174.

[75] S. Borgani, G. Murante, R. Brunino, S.-H. Cha, Hydrodynamic simulations with the godunov SPH, in: Advances in Computational Astrophysics: Methods, Tools, and Outcome, Vol. 453, 2012, p. 259.

[76] K. Iwasaki, S.-i. Inutsuka, Smoothed particle magnetohydrodynamics with a riemann solver and the method of characteristics, Monthly Notices of the Royal Astronomical Society 418 (3) (2011) 1668–1688.

[77] V. Roubtsova, R. Kahawita, The sph technique applied to free surface flows, Computers & Fluids 35 (10) (2006) 1359–1371.

[78] P. K. Koukouvinis, J. S. Anagnostopoulos, D. E. Papantonis, An improved MUSCL treatment for the SPH-ALE method: comparison with the standard SPH method for the jet impingement case, International Journal for Numerical Methods in Fluids 71 (9) (2013) 1152–1177.

[79] E. Toro, A linearized riemann solver for the time-dependent euler equations of gas dynamics, Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences 434 (1892) (1991) 683–693.

[80] E. F. Toro, Riemann solvers and numerical methods for fluid dynamics: a practical introduction, Springer Science & Business Media, 2013.

[81] P. L. Roe, Approximate riemann solvers, parameter vectors, and difference schemes, Journal of computational physics 43 (2) (1981) 357–372.

[82] W. J. Rider, A review of approximate riemann solvers with godunov’s method in lagrangian coordinates, Computers & fluids 23 (2) (1994) 397–413.

[83] E. F. Toro, Riemann solvers and numerical methods for fluid dynamics: a practical introduction, Springer Science & Business Media, 2009.

[84] S. Adami, X. Hu, N. Adams, A generalized wall boundary condition for smoothed particle hydrodynamics, Journal of Computational Physics 231 (21) (2012) 7057–7075.

[85] B. Van Leer, Towards the ultimate conservative difference scheme III. upstream-centered finite-difference schemes for ideal compressible flow, Journal of Computational Physics 23 (3) (1977) 263–275.
[86] B. Van Leer, Towards the ultimate conservative difference scheme. v. a second-order sequel to godunov’s method, Journal of computational Physics 32 (1) (1979) 101–136.

[87] T. Barth, D. Jespersen, The design and application of upwind schemes on unstructured meshes, in: 27th Aerospace sciences meeting, 1989, p. 366.

[88] A. Harten, B. Engquist, S. Osher, S. R. Chakravarthy, Uniformly high order accurate essentially non-oscillatory schemes, III, in: Upwind and high-resolution schemes, Springer, 1987, pp. 218–290.

[89] C.-W. Shu, Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws, in: Advanced numerical approximation of nonlinear hyperbolic equations, Springer, 1998, pp. 325–432.

[90] S. Pirozzoli, Numerical methods for high-speed flows, Annual review of fluid mechanics 43 (2011) 163–194.

[91] C.-W. Shu, Essentially non-oscillatory and weighted essentially non-oscillatory schemes, Acta Numerica 29 (2020) 701–762.

[92] G. Murante, S. Borgani, R. Brunino, S.-H. Cha, Hydrodynamic simulations with the godunov smoothed particle hydrodynamics, Monthly Notices of the Royal Astronomical Society 417 (1) (2011) 136–153.

[93] B. D. Rogers, R. A. Dalrymple, P. K. Stansby, Simulation of caisson breakwater movement using 2-d SPH, Journal of Hydraulic Research 48 (sup1) (2010) 135–141.

[94] P. F. Hopkins, A new class of accurate, mesh-free hydrodynamic simulation methods, Monthly Notices of the Royal Astronomical Society 450 (1) (2015) 53–110.

[95] X. Hu, Q. Wang, N. A. Adams, An adaptive central-upwind weighted essentially non-oscillatory scheme, Journal of Computational Physics 229 (23) (2010) 8952–8965.

[96] L. Fu, X. Y. Hu, N. A. Adams, A family of high-order targeted ENO schemes for compressible-fluid simulations, Journal of Computational Physics 305 (2016) 333–359.

[97] B. Wang, G. Xiang, X. Hu, An incremental-stencil weno reconstruction for simulation of compressible two-phase flows, International Journal of Multiphase Flow 104 (2018) 20–31.

[98] X. Zhang, H. Tian, L. Kuo, W. Chen, A contact sph method with high-order limiters for simulation of inviscid compressible flows, Communications in Computational Physics 14 (02) (2013) 425–442.

[99] X. Nogueira, L. Ramírez, S. Clain, R. Loubère, L. Cueto-Felgueroso, I. Colominas, High-accurate SPH method with multidimensional optimal order detection limiting, Computer Methods in Applied Mechanics and Engineering 310 (2016) 134–155.

[100] Z.-F. Meng, A.-M. Zhang, P.-P. Wang, F.-R. Ming, B. C. Khoo, A targeted essentially non-oscillatory (TENO) SPH method and its applications in hydrodynamics, Ocean Engineering (2021) 110100.

[101] P. L. Roe, Characteristic-based schemes for the euler equations, Annual review of fluid mechanics 18 (1) (1986) 337–365.

[102] B. Van Leer, Towards the ultimate conservative difference scheme. II. monotonicity and conservation combined in a second-order scheme, Journal of computational physics 14 (4) (1974) 361–370.

[103] G.-S. Jiang, C.-W. Shu, Efficient implementation of weighted eno schemes, J. Comput. Phys. 126 (1) (1996) 202–228.

[104] R. Abgrall, On essentially non-oscillatory schemes on unstructured meshes: analysis and implementation, Journal of Computational Physics 114 (1) (1994) 45–58.

[105] O. Friedrich, Weighted essentially non-oscillatory schemes for the interpolation of mean values on unstructured grids, Journal of computational physics 144 (1) (1998) 194–212.
[106] M. Käser, A. Iske, ADER schemes on adaptive triangular meshes for scalar conservation laws, Journal of Computational Physics 205 (2) (2005) 486–508.

[107] M. Dumbser, M. Käser, V. A. Titarev, E. F. Toro, Quadrature-free non-oscillatory finite volume schemes on unstructured meshes for nonlinear hyperbolic systems, Journal of Computational Physics 226 (1) (2007) 204–243.

[108] M. Dumbser, M. Käser, Arbitrary high order non-oscillatory finite volume schemes on unstructured meshes for linear hyperbolic systems, Journal of Computational Physics 221 (2) (2007) 693–723.

[109] S. Litvinov, X. Hu, N. Adams, Towards consistence and convergence of conservative sph approximations, Journal of Computational Physics 301 (2015) 394–401.

[110] S. Clain, S. Diot, R. Loubère, A high-order finite volume method for systems of conservation laws—multi-dimensional optimal order detection (MOOD), Journal of computational Physics 230 (10) (2011) 4028–4050.

[111] R. Antona, R. Vacondio, D. Avesani, M. Righetti, M. Renzi, Towards a high order convergent ALE-SPH scheme with efficient WENO spatial reconstruction, Water 13 (17) (2021) 2432.

[112] W. Mattson, B. M. Rice, Near-neighbor calculations using a modified cell-linked list method, Computer Physics Communications 119 (2-3) (1999) 135–148.

[113] L. Verlet, Computer’”experiments” on classical fluids. i. thermodynamical properties of lennard-jones molecules, Physics Review 159 (1) (1967) 98.

[114] J. M. Domínguez, A. J. Crespo, M. Gómez-Gesteira, J. C. Marongiu, Neighbour lists in smoothed particle hydrodynamics, Int. J. Numer. Methods Fluids 67 (12) (2011) 2026–2042.

[115] J. M. Domínguez, G. Fourtakas, C. Altomare, R. B. Canelas, A. Tafuni, O. García-Feal, I. Martínez-Estévez, A. Mokos, R. Vacondio, A. J. Crespo, et al., DualSPHysics: from fluid dynamics to multiphysics problems, Computational Particle Mechanics (2021) 1–29.

[116] G. Viccione, V. Bovolin, E. P. Carratelli, Defining and optimizing algorithms for neighbouring particle identification in SPH fluid simulations, International Journal for Numerical Methods in Fluids 58 (6) (2008) 625–638.

[117] V. Springel, The cosmological simulation code GADGET-2, Monthly Notices of the Royal Astronomical Society 364 (4) (2005) 1105–1134.

[118] V. Springel, Smoothed particle hydrodynamics in astrophysics, Annual Review of Astronomy and Astrophysics 48 (2010) 391–430.

[119] L. Arge, M. D. Berg, H. Haverkort, K. Yi, The priority r-tree: A practically efficient and worst-case optimal r-tree, ACM Transactions on Algorithms 4 (1) (2008) 1–30.

[120] L. Fu, L. Han, X. Y. Hu, N. A. Adams, An isotropic unstructured mesh generation method based on a fluid relaxation analogy, Computer Methods in Applied Mechanics and Engineering 350 (2019) 396–431.

[121] S. Khorasanizade, J. Sousa, Improving linked-lists using tree search algorithms for neighbor finding in variable-resolution smoothed particle hydrodynamics, Computer Physics Communications 26 (1) (2019) 57–86.

[122] J. M. Domínguez, A. J. Crespo, M. Gómez-Gesteira, Optimization strategies for CPU and GPU implementations of a smoothed particle hydrodynamics method, Computer Physics Communications 184 (3) (2013) 617–627.

[123] D. Winkler, M. Rezavand, M. Meister, W. Rauch, GpuSPHASE—a shared memory caching implementation for 2d SPH using CUDA, Computer Physics Communications 235 (2019) 514–516.

[124] R. Fair, X. Guo, T. Cui, Particle sorting for the projection based particle method, Engineering Analysis with Boundary Elements 109 (2019) 199–208.
[125] Z.-X. Zhao, H. Liu, Z.-X. Gong, A high-efficiency smoothed particle hydrodynamics model with multi-cell linked list and adaptive particle refinement for two-phase flows, Physics of Fluids 33 (6) (2021) 064102.

[126] J. J. Monaghan, Smoothed particle hydrodynamics, Reports on progress in physics 68 (8) (2005) 1703.

[127] S. Adami, X. Hu, N. Adams, A transport-velocity formulation for smooth particle hydrodynamics, Journal of Computational Physics 241 (2013) 292–307.

[128] D. Winkler, M. Rezavand, W. Rauch, Neighbour lists for smoothed particle hydrodynamics on GPUs, Computer Physics Communications 225 (2018) 140–148.

[129] C. Fraga Filho, L. Schuina, B. Porto, An investigation into neighbouring search techniques in mesh-free particle methods: An evaluation of the neighbour lists and the direct search, Archives of Computational Methods in Engineering 27 (4) (2020) 1093–1107.

[130] R. W. Hockney, J. W. Eastwood, Computer simulation using particles, CRC Press, 2021.

[131] G. M. Morton, A computer oriented geodetic database and a new technique in file sequencing.

[132] D. Hilbert, Äijber die stetige abbildung einer linie auf ein fläddchenstäljck, Mathematische Annalen 38.

[133] B. Moon, H. Jagadish, C. Faloutsos, J. Saltz, Analysis of the clustering properties of the hilbert space-filling curve, IEEE Transactions on Knowledge and Data Engineering 13 (1) (2001) 124–141.

[134] M. Hofmann, G. Rünger, P. Gibbon, R. Speck, Parallel sorting algorithms for optimizing particle simulations, in: 2010 IEEE International Conference On Cluster Computing Workshops and Posters (CLUSTER WORKSHOPS), IEEE, 2010, pp. 1–8.

[135] M. Ihmsen, N. Akinici, M. Becker, M. Teschner, A parallel SPH implementation on multi-core CPUs, in: Computer Graphics Forum, Vol. 30, Wiley Online Library, 2011, pp. 99–112.

[136] J. P. Gray, J. J. Monaghan, R. P. Swift, SPH elastic dynamics, Computer methods in applied mechanics and engineering 190 (49) (2001) 6641–6662.

[137] C. Zhang, X. Y. Hu, N. A. Adams, A generalized transport-velocity formulation for smoothed particle hydrodynamics, Journal of Computational Physics 337 (2017) 216–232.

[138] M. Puso, J. Chen, E. Zywicz, W. Elmer, Meshfree and finite element nodal integration methods, International Journal for Numerical Methods in Engineering 74 (3) (2008) 416–446.

[139] G. C. Ganzenmüller, An hourglass control algorithm for lagrangian smooth particle hydrodynamics, Computer Methods in Applied Mechanics and Engineering 286 (2015) 87–106.

[140] L. D. Libersky, A. G. Petschek, Smooth particle hydrodynamics with strength of materials, in: Advances in the free-Lagrange method including contributions on adaptive gridding and the smooth particle hydrodynamics method, Springer, 1991, pp. 248–257.

[141] R. Vignjevic, J. Campbell, L. Libersky, A treatment of zero-energy modes in the smoothed particle hydrodynamics method, Computer methods in Applied mechanics and Engineering 184 (1) (2000) 67–85.

[142] J. Bonet, S. Kulasegaram, A simplified approach to enhance the performance of smooth particle hydrodynamics methods, Applied Mathematics and Computation 126 (2-3) (2002) 133–155.

[143] K. Ba, A. Gakwaya, Thermomechanical total lagrangian SPH formulation for solid mechanics in large deformation problems, Computer Methods in Applied Mechanics and Engineering 342 (2018) 458–473.
A. Khayyer, H. Gotoh, H. Falahaty, Y. Shimizu, An enhanced ISPH-SPH coupled method for simulation of incompressible fluid–elastic structure interactions, Computer Physics Communications 232 (2018) 139–164.

C. Zhang, M. Rezavand, X. Hu, A multi-resolution SPH method for fluid-structure interactions, Journal of Computational Physics (2020) 110028.

C. Zhang, M. Rezavand, Y. Zhu, Y. Yu, D. Wu, W. Zhang, S. Zhang, J. Wang, X. Hu, SPHinXsys: An open-source meshless, multi-resolution and multi-physics library, Software Impacts 6 (2020) 100033.

I. S. Sokolnikoff, R. D. Specht, et al., Mathematical theory of elasticity, Vol. 83, McGraw-Hill New York, 1956.

C. H. Lee, A. J. Gil, G. Greto, S. Kulasegaram, J. Bonet, A new jameson–schmidt–turkel smooth particle hydrodynamics algorithm for large strain explicit fast dynamics, Computer Methods in Applied Mechanics and Engineering 311 (2016) 71–111.

C. H. Lee, A. J. Gil, O. I. Hassan, J. Bonet, S. Kulasegaram, A variationally consistent streamline upwind petrov–galerkin smooth particle hydrodynamics algorithm for large strain solid dynamics, Computer Methods in Applied Mechanics and Engineering 318 (2017) 514–536.

C. H. Lee, A. J. Gil, A. Ghavamian, J. Bonet, A total lagrangian upwind smooth particle hydrodynamics algorithm for large strain explicit solid dynamics, Computer Methods in Applied Mechanics and Engineering 344 (2019) 209–250.

C. Zhang, Y. Zhu, Y. Yu, M. Rezavand, X. Hu, A simple artificial damping method for total lagrangian smoothed particle hydrodynamics, arXiv preprint arXiv:2102.04898.

P. Randles, L. Libersky, Normalized sph with stress points, International Journal for Numerical Methods in Engineering 48 (10) (2000) 1445–1462.

J.-S. Chen, C.-T. Wu, S. Yoon, Y. You, A stabilized conforming nodal integration for galerkin mesh-free methods, International journal for numerical methods in engineering 50 (2) (2001) 435–466.

C. Zhang, Y. Zhu, Y. Yu, M. Rezavand, X. Hu, A simple artificial damping method for total lagrangian smoothed particle hydrodynamics, arXiv preprint arXiv:2102.04898.

M. Aguirre, A. J. Gil, J. Bonet, A. A. Carreño, A vertex centred finite volume jameson–schmidt–turkel (jst) algorithm for a mixed conservation formulation in solid dynamics, Journal of Computational Physics 259 (2014) 672–699.

J. R. H. Otter, A. C. Cassell, R. E. Hobbs, POISSON, Dynamic relaxation, Proceedings of the Institution of Civil Engineers 35 (4) (1966) 633–656.

T. Belytschko, W. K. Liu, B. Moran, K. Elkhodary, Nonlinear finite elements for continua and structures, John wiley & sons, 2013.

S. Jung, T.-Y. Kim, W.-S. Yoo, Dynamic relaxation using continuous kinetic damping—part i: Basic algorithm, Journal of Computational and Nonlinear Dynamics 13 (8).

J. Rodriguez, Numerical study of dynamic relaxation methods and contribution to the modelling of inflatable lifejackets, Ph.D. thesis, Ph. D. thesis, Université de Bretagne Sud (2011).

J. Alamatian, A new formulation for fictitious mass of the dynamic relaxation method with kinetic damping, Computers & Structures 90 (2012) 42–54.

M. Rezaiee-Pajand, H. Rezaee, Fictitious time step for the kinetic dynamic relaxation method, Mechanics of Advanced Materials and Structures 21 (8) (2014) 631–644.

J. Lin, H. Naceur, D. Coutellier, A. Laksimi, Efficient meshless SPH method for the numerical modeling of thick shell structures undergoing large deformations, International Journal of Non-Linear Mechanics 65 (2014) 1–13.
[163] K. S. Lee, S. E. Han, T. Park, A simple explicit arc-length method using the dynamic relaxation method with kinetic damping, Computers & Structures 89 (1-2) (2011) 216–233.

[164] P. Cundall, Explicit finite difference method in geomechanics, in: Second Int. Conf. Numerical Methods in Geomechanics, Blacksburg, 1976, Vol. 1, 1976, pp. 132–150.

[165] C. Douthe, O. Baverel, Design of nexorades or reciprocal frame systems with the dynamic relaxation method, Computers & Structures 87 (21-22) (2009) 1296–1307.

[166] I. Zardi, J. Alamatian, A new formulation for fictitious mass of viscous dynamic relaxation method, Mechanics Based Design of Structures and Machines 48 (5) (2020) 542–567.

[167] Y. Zhu, C. Zhang, X. Hu, A splitting random-choice dynamic relaxation method for smoothed particle hydrodynamics, arXiv preprint arXiv:2103.08932.

[168] J.-H. Wang, S. Pan, X. Y. Hu, N. A. Adams, A split random time-stepping method for stiff and nonstiff detonation capturing, Combustion and Flame 204 (2019) 397–413.

[169] R. I. McLachlan, G. R. W. Quispel, Splitting methods, Acta Numerica 11 (2002) 341.

[170] S. Litvinov, M. Ellero, X. Y. Hu, N. A. Adams, A splitting scheme for highly dissipative smoothed particle dynamics, J. Comput. Phys. 229 (2010) 5457–5464.

[171] J. J. Monaghan, On the integration of the sph equations for a highly viscous fluid, Journal of Computational Physics 394 (2019) 166–176.

[172] G. Strang, On the construction and comparison of difference schemes, SIAM journal on numerical analysis 5 (3) (1968) 506–517.

[173] K. Nguyen, A. Caboussat, D. Dabbdub, Mass conservative, positive definite integrator for atmospheric chemical dynamics, Atmospheric Environment 43 (40) (2009) 6287–6295.

[174] M. A. Nielsen, Neural networks and deep learning, Vol. 25, Determination press USA, 2015.

[175] M. Crisfield, Non-linear finite element analysis of solids and structures: Advanced topics (1997).

[176] C. Dyka, P. Randles, R. Ingel, Stress points for tension instability in SPH, International Journal for Numerical Methods in Engineering 40 (13) (1997) 2325–2341.

[177] T. Belytschko, Y. Guo, W. Kam Liu, S. Ping Xiao, A unified stability analysis of meshless particle methods, International Journal for Numerical Methods in Engineering 48 (9) (2000) 1359–1400.

[178] J. Swegle, S. Attaway, M. Heinstein, F. Mello, D. Hicks, An analysis of smoothed particle hydrodynamics, Tech. rep., Sandia National Labs., Albuquerque, NM (United States) (1994).

[179] S. Beissel, T. Belytschko, Nodal integration of the element-free galerkin method, Computer methods in applied mechanics and engineering 139 (1-4) (1996) 49–74.

[180] P. Randles, L. Libersky, A. Petschek, On neighborhoods, derivatives, and viscosity in particle codes, Tech. rep., Los Alamos National Lab., NM (US) (1999).

[181] T. Rabczuk, T. Belytschko, S. Xiao, Stable particle methods based on lagrangian kernels, Computer methods in applied mechanics and engineering 193 (12-14) (2004) 1035–1063.

[182] S. Xiao, T. Belytschko, Material stability analysis of particle methods, Advances in Computational Mathematics 23 (1) (2005) 171–190.

[183] B. Maurel, A. Combescure, An sph shell formulation for plasticity and fracture analysis in explicit dynamics, International journal for numerical methods in engineering 76 (7) (2008) 949–971.

[184] F. Ming, A. Zhang, S. Wang, Smoothed particle hydrodynamics for the linear and nonlinear analyses of elastoplastic damage and fracture of shell, International Journal of Applied Mechanics 7 (02) (2015) 1550032.
[185] D. Flanagan, T. Belytschko, A uniform strain hexahedron and quadrilateral with orthogonal hourglass control, International journal for numerical methods in engineering 17 (5) (1981) 679–706.

[186] M. R. I. Islam, C. Peng, A stabilized total-lagrangian sph method for large deformation and failure in geomaterials, arXiv preprint arXiv:1907.06990.

[187] J. O’Connor, B. D. Rogers, A fluid–structure interaction model for free-surface flows and flexible structures using smoothed particle hydrodynamics on a GPU, Journal of Fluids and Structures 104 (2021) 103312.

[188] L. Zhan, C. Peng, B. Zhang, W. Wu, A stabilized TL-WC SPH approach with GPU acceleration for three-dimensional fluid–structure interaction, Journal of Fluids and Structures 86 (2019) 329–353.

[189] G. E. Forsythe, W. R. Wasow, et al., Finite-difference methods for partial differential equations.

[190] T. E. Tezduyar, M. Behr, S. Mittal, J. Liou, A new strategy for finite element computations involving moving boundaries and interfaces, Computer methods in applied mechanics and engineering 94 (3) (1992) 353–371.

[191] H. K. Versteeg, W. Malalasekera, An introduction to computational fluid dynamics: the finite volume method, Pearson education, 2007.

[192] M. Souli, A. Ouahsine, L. Lewin, ALE formulation for fluid–structure interaction problems, Computer methods in applied mechanics and engineering 190 (5-7) (2000) 659–675.

[193] E. Oñate, S. Idelsohn, O. Zienkiewicz, R. Taylor, A finite point method in computational mechanics. applications to convective transport and fluid flow, International journal for numerical methods in engineering 39 (22) (1996) 3839–3866.

[194] C. S. Peskin, The immersed boundary method, Acta numerica 11 (2002) 479–517.

[195] S. Koshizuka, Y. Oka, Moving-particle semi-implicit method for fragmentation of incompressible fluid, Nuclear science and engineering 123 (3) (1996) 421–434.

[196] B. Mishra, R. K. Rajamani, The discrete element method for the simulation of ball mills, Applied Mathematical Modelling 16 (11) (1992) 598–604.

[197] G. Oger, P. M. Guilcher, E. Jaquin, L. Brosset, J. B. Deuff, D. Le Touzé, et al., Simulations of hydro-elastic impacts using a parallel SPH model, in: The Nineteenth International Offshore and Polar Engineering Conference, International Society of Offshore and Polar Engineers, 2009.

[198] M.-b. Liu, J.-r. Shao, H.-q. Li, Numerical simulation of hydro-elastic problems with smoothed particle hydrodynamics method, Journal of Hydraulics 25 (5) (2013) 673–682.

[199] L. Wang, F. Xu, Y. Yang, SPH scheme for simulating the water entry of an elastomer, Ocean Engineering 178 (2019) 233–245.

[200] L. Han, X. Hu, SPH modeling of fluid-structure interaction, Journal of Hydraulics 30 (1) (2018) 62–69.

[201] P.-N. Sun, D. Le Touzé, G. Oger, A.-M. Zhang, An accurate FSI-SPH modeling of challenging fluid-structure interaction problems in two and three dimensions, Ocean Engineering 221 (2021) 108552.

[202] A. Rafiee, K. P. Thiagarajan, An SPH projection method for simulating fluid-hypoelastic structure interaction, Computer Methods in Applied Mechanics and Engineering 198 (33-36) (2009) 2785–2795.

[203] P. Sun, D. Le Touzé, A.-M. Zhang, Study of a complex fluid-structure dam-breaking benchmark problem using a multi-phase sph method with apr, Engineering Analysis with Boundary Elements 104 (2019) 240–258.

[204] A. Khayyer, N. Tsuruta, Y. Shimizu, H. Gotoh, Multi-resolution mps for incompressible fluid-elastic structure interactions in ocean engineering, Applied Ocean Research 82 (2019) 397–414.
B. Ren, Z. Jin, R. Gao, Y. Wang, Z. Xu, SPH-DEM modeling of the hydraulic stability of 2d blocks on a slope, Journal of Waterway, Port, Coastal, and Ocean Engineering 140 (6) (2013) 04014022.

F. Xie, W. Zhao, D. Wan, Numerical simulations of liquid-solid flows with free surface by coupling IMPS and DEM, Applied Ocean Research 114 (2021) 102771.

Q. Yang, V. Jones, L. McCue, Free-surface flow interactions with deformable structures using an SPH-FEM model, Ocean Engineering 55 (2012) 136–147.

Y. Zhang, D. Wan, MPS-FEM coupled method for fluid–structure interaction in 3d dam-break flows, International Journal of Computational Methods 16 (02) (2019) 1846009.

X. Chen, Y. Zhang, D. Wan, Numerical study of 3-d liquid sloshing in an elastic tank by MPS-FEM coupled method, Journal of Ship Research.

C. Hermange, G. Oger, Y. Le Chenadec, D. Le Touzé, A 3d SPH–FE coupling for FSI problems and its application to tire hydroplaning simulations on rough ground, Computer Methods in Applied Mechanics and Engineering 355 (2019) 558–590.

G. Zhang, W. Zhao, D. Wan, Partitioned MPS-FEM method for free-surface flows interacting with deformable structures, Applied Ocean Research 114 (2021) 102775.

H.-G. Lyu, P.-N. Sun, X.-T. Huang, S.-Y. Zhong, Y.-X. Peng, T. Jiang, C.-N. Ji, A review of SPH techniques for hydrodynamic simulations of ocean energy devices, Energies 15 (2) (2022) 502.

J. J. Monaghan, J. B. Kajtar, SPH particle boundary forces for arbitrary boundaries, Computer Physics Communications 180 (2009) 1811–1820.

M. Liu, J. Shao, J. Chang, On the treatment of solid boundary in smoothed particle hydrodynamics, Science China Technological Sciences 55 (1) (2012) 244–254.

Z. Zhang, K. Walayat, J. Chang, M. Liu, Meshfree modeling of a fluid-particle two-phase flow with an improved SPH method, International Journal for Numerical Methods in Engineering 116 (8) (2018) 530–569.

M. Ferrand, D. R. Laurence, B. D. Rogers, D. Viroleau, C. Kassiotis, Unified semi-analytical wall boundary conditions for inviscid, laminar or turbulent flows in the meshless SPH method, International Journal for Numerical Methods in Fluids 71 (4) (2013) 446–472.

A. Mayrhofer, M. Ferrand, C. Kassiotis, D. Viroleau, F.-X. Morel, Unified semi-analytical wall boundary conditions in sph: analytical extension to 3-d, Numerical Algorithms 68 (1) (2015) 15–34.

L. Chiron, M. De Lefè, G. Oger, D. Le Touzé, Fast and accurate SPH modelling of 3d complex wall boundaries in viscous and non viscous flows, Computer Physics Communications 234 (2019) 93–111.

A. Valizadeh, J. J. Monaghan, A study of solid wall models for weakly compressible SPH, Journal of Computational Physics 300 (2015) 5–19.

J. J. Monaghan, A. Rafiee, A simple SPH algorithm for multi-fluid flow with high density ratios, International Journal for Numerical Methods in Fluids 71 (5) (2013) 537–561.

Z. Chen, Z. Zong, M. Liu, L. Zou, H. Li, C. Shu, An SPH model for multiphase flows with complex interfaces and large density differences, Journal of Computational Physics 283 (2015) 169–188.

M. Liu, J. Shao, H. Li, An sph model for free surface flows with moving rigid objects, International Journal for Numerical Methods in Fluids 74 (9) (2014) 684–697.

C. Zhang, Y. Wei, F. Dias, X. Hu, An efficient fully lagrangian solver for modeling wave interaction with oscillating wave surge converter, Ocean Engineering 236 (2021) 109540.
[224] Y. Wei, A. Rafiee, A. Henry, F. Dias, Wave interaction with an oscillating wave surge converter, part i: Viscous effects, Ocean Engineering 104 (2015) 185–203.

[225] A. Rafiee, B. Elsaesser, F. Dias, Numerical simulation of wave interaction with an oscillating wave surge converter, in: International Conference on Offshore Mechanics and Arctic Engineering, Vol. 55393, American Society of Mechanical Engineers, 2013, p. V005T06A013.

[226] M. Brito, R. Canelas, O. García-Feal, J. Domínguez, A. Crespo, R. Ferreira, M. Neves, L. Teixeira, A numerical tool for modelling oscillating wave surge converter with nonlinear mechanical constraints, Renewable Energy 146 (2020) 2024–2043.

[227] M. Lastiwka, N. Quinlan, M. Basa, Adaptive particle distribution for smoothed particle hydrodynamics, International Journal for Numerical Methods in Fluids 47 (10-11) (2005) 1403–1409.

[228] R. Vacondio, B. D. Rogers, P. K. Stansby, P. Mignosa, Variable resolution for SPH in three dimensions: Towards optimal splitting and coalescing for dynamic adaptivity, Computer Methods in Applied Mechanics and Engineering 300 (2016) 442–460.

[229] S. Khorasanizade, J. Sousa, Dynamic flow-based particle splitting in smoothed particle hydrodynamics, International Journal for Numerical Methods in Engineering 106 (5) (2016) 397–410.

[230] W. Hu, G. Guo, X. Hu, D. Negrut, Z. Xu, W. Pan, A consistent spatially adaptive smoothed particle hydrodynamics method for fluid–structure interactions, Computer Methods in Applied Mechanics and Engineering 347 (2019) 402–424.

[231] M. Liu, G. Liu, K. Lam, Adaptive smoothed particle hydrodynamics for high strain hydrodynamics with material strength, Shock Waves 15 (1) (2006) 21–29.

[232] J. M. Owen, J. V. Villumsen, P. R. Shapiro, H. Martel, Adaptive smoothed particle hydrodynamics: Methodology. II., The Astrophysical Journal Supplement Series 116 (2) (1998) 155.

[233] X. Bian, Z. Li, G. E. Karniadakis, Multi-resolution flow simulations by smoothed particle hydrodynamics via domain decomposition, Journal of Computational Physics 297 (2015) 132–155.

[234] K. Shibata, S. Koshizuka, T. Matsunaga, I. Masaie, The overlapping particle technique for multi-resolution simulation of particle methods, Computer Methods in Applied Mechanics and Engineering 325 (2017) 434–462.

[235] D. A. Barcarolo, D. Le Touzé, G. Oger, F. De Vuyst, Adaptive particle refinement and derefinement applied to the smoothed particle hydrodynamics method, Journal of Computational Physics 273 (2014) 640–657.

[236] M. Tanaka, R. Cardoso, H. Bahai, Multi-resolution MPS method, Journal of Computational Physics 359 (2018) 106–136.

[237] P. Omidvar, P. K. Stansby, B. D. Rogers, Wave body interaction in 2d using smoothed particle hydrodynamics (SPH) with variable particle mass, International Journal for Numerical Methods in Fluids 68 (6) (2012) 686–705.

[238] A. Khayyer, Y. Shimizu, H. Gotoh, S. Hattori, Multi-resolution ISPH-SPH for accurate and efficient simulation of hydroelastic fluid-structure interactions in ocean engineering, Ocean Engineering 226 (2021) 108652.

[239] S. Turek, J. Hron, Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow, in: Fluid-Structure Interaction, Springer, 2006, pp. 371–385.

[240] R. Bhardwaj, R. Mittal, Benchmarking a coupled immersed-boundary-finite-element solver for large-scale flow-induced deformation, AIAA Journal 50 (7) (2012) 1638–1642.

[241] F.-B. Tian, H. Dai, H. Luo, J. F. Doyle, B. Rousseau, Fluid–structure interaction involving
large deformations: 3d simulations and applications to biological systems, Journal of Computational Physics 258 (2014) 451–469.

[242] B. Fabritius, G. Tabor, Improving the quality of finite volume meshes through genetic optimisation, Engineering with Computers 32 (3) (2016) 425–440.

[243] J. Dominguez, A. Crespo, A. Barreiro, M. Gomez-Gesteira, A. Mayrhofer, Development of a new pre-processing tool for sph models with complex geometries, in: 6th International SPHERIC workshop, 2011, pp. 117–124.

[244] R. Vignjevic, M. Orłowski, T. De Vuyst, J. C. Campbell, A parametric study of bird strike on engine blades, International Journal of Impact Engineering 60 (2013) 44–57.

[245] S. Heimbs, Computational methods for bird strike simulations: A review, Computers & Structures 89 (23-24) (2011) 2093–2112.

[246] S. Diehl, G. Rockefeller, C. L. Fryer, D. Riemmiller, T. S. Statler, Generating optimal initial conditions for smoothed particle hydrodynamics simulations, Publications of the Astronomical Society of Australia 32.

[247] M. Siemann, S. A. Ritt, Novel particle distributions for sph bird-strike simulations, Computer Methods in Applied Mechanics and Engineering 343 (2019) 746–766.

[248] L. V. Vela, R. Sanchez, J. Geiger, Alaric: An algorithm for constructing arbitrarily complex initial density distributions with low particle noise for sph/spmhd applications, Computer Physics Communications 224 (2018) 186–197.

[249] R. L. Hner, Progress in grid generation via the advancing front technique, Engineering with Computers 12 (3-4) (1996) 186–210.

[250] P. J. Frey, H. Borouchaki, P. L. George, 3d delaunay mesh generation coupled with an advancing-front approach, Computer Methods in Applied Mechanics and Engineering 157 (1–2) (1998) 115–131.

[251] Jonathan, Richard, Shewchuk, Delaunay refinement algorithms for triangular mesh generation, Computational Geometry.

[252] L. P. Chew, Guaranteed-quality triangular meshes, cornell university.

[253] Q. Du, F. M. Gunzburger, Centroidal voronoi tessellations: Applications and algorithms, Siam Review 41 (4) (1999) 637–676.

[254] D. Qiang, M. Gunzburger, Grid generation and optimization based on centroidal voronoi tessellations, Applied Mathematics Computation 133 (2-3) (2002) 591–607.

[255] Y. Liu, W. Wang, V. Y. BRUNO LE, F. Sun, D. M. Yan, L. U. Lin, C. Yang, On centroidal voronoi tessellation energy smoothness and fast computation, ACM Transactions on Graphics (TOG) 29 (2009) 101.1–101.17.

[256] S. Valette, J. M. Chassery, R. Prost, Generic remeshing of 3d triangular meshes with metric-dependent discrete voronoi diagrams, IEEE Transactions on Visualization Computer Graphics 14 (2) (2008) 369–381.

[257] A. P. Witkin, P. S. Heckbert, Using particles to sample and control implicit surfaces, ACM.

[258] J. R. Bronson, J. A. Levine, R. T. Whitaker, Particle systems for adaptive, isotropic meshing of cad models, in: International Meshing Roundtable, 2010.

[259] P. O. Persson, Mesh generation for implicit geometries, massachusetts institute of technology.

[260] Z. Zhong, X. Guo, W. Wangy, B. Levyz, F. Suny, Y. Liux, W. Mao, Particle-based anisotropic surface meshing, ACM Transactions on Graphics 32 (4CD) (2013) 99.1–99.14.

[261] M. Meyer, P. Georgel, R. T. Whitaker, Robust particle systems for curvature dependent sampling of implicit surfaces, in: International Conference on Shape Modeling & Applications, 2005, pp. 124–133.
[262] Y. Zhu, C. Zhang, Y. Yu, X. Hu, A CAD-compatible body-fitted particle generator for arbitrarily complex geometry and its application to wave-structure interaction, Journal of Hydrodynamics 33 (2) (2021) 195–206.

[263] Z. Ji, L. Fu, X. Hu, N. Adams, A consistent parallel isotropic unstructured mesh generation method based on multi-phase SPH, Computer Methods in Applied Mechanics and Engineering 363 (2020) 112881.

[264] Z. Ji, L. Fu, X. Hu, N. Adams, A feature-aware sph for isotropic unstructured mesh generation, Computer Methods in Applied Mechanics and Engineering 375 (2021) 113634.