Offensive $k$-alliances in graphs

Henning Fernau$^1$, Juan A. Rodríguez$^2$ and José M. Sigarreta$^3$

$^1$FB 4-Abteilung Informatik
Universität Trier,
54286 Trier, Germany.
e-mail:fernau@uni-trier.de

$^2$Departament d’Enginyeria Informàtica i Matemàtiques
Universitat Rovira i Virgili,
Av. Països Catalans 26, 43007 Tarragona, Spain.
e-mail:juanalberto.rodriguez@urv.cat

$^3$Departamento de Matemáticas
Universidad Carlos III de Madrid,
Avda. de la Universidad 30, 28911 Leganés (Madrid), Spain.
e-mail:josemaria.sigarreta@uc3m.es

Abstract

Let $G = (V,E)$ be a simple graph. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $\delta_X(v)$ denotes the number of neighbors $v$ has in $X$. A nonempty set $S \subseteq V$ is an offensive $r$-alliance in $G$ if $\delta_S(v) \geq \delta_S(v) + r$, $\forall v \in \partial(S)$, where $\partial(S)$ denotes the boundary of $S$. An offensive $r$-alliance $S$ is called global if it forms a dominating set. The global offensive $r$-alliance number of $G$, denoted by $\gamma_o^r(G)$, is the minimum cardinality of a global offensive $r$-alliance in $G$. We show that the problem of finding optimal (global) offensive $r$-alliances is NP-complete and we obtain several tight bounds on $\gamma_o^r(G)$.

Keywords: Computational complexity, offensive alliances, alliances in graphs, domination.

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1 Introduction

The mathematical properties of alliances in graphs were first studied by P. Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi [13]. They proposed different types of alliances: namely, defensive alliances [11, 12, 13, 21], offensive alliances [4, 5, 7, 17, 18] and dual alliances or powerful alliances [1]. A generalization of these alliances called $r$-alliances was presented by K. H. Shafique and R. D. Dutton [19, 20].

In this paper, we study the mathematical properties of offensive $r$-alliances. We begin by stating the terminology used. Throughout this article, $G = (V, E)$ denotes a simple graph of order $|V| = n$. We denote two adjacent vertices $u$ and $v$ by $u \sim v$. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors $v$ has in $X$: $N_X(v) := \{u \in X : u \sim v\}$, and the degree of $v$ in $X$ will be denoted by $\delta_X(v) = |N_X(v)|$. We denote the degree of a vertex $v \in V$ by $\delta(v)$ and the degree sequence of $G$ by $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. The complement of the vertex-set $S$ in $V$ is denoted by $\bar{S}$ and the boundary, $\partial(S)$, of $S$ is defined by

$$\partial(S) := \bigcup_{v \in S} N_{\bar{S}}(v).$$

For $r \in \{2 - \delta_1, \ldots, \delta_1\}$, a nonempty set $S \subset V$ is an offensive $r$-alliance in $G$ if for every $v \in \partial(S)$,

$$\delta_S(v) \geq \delta_S(v) + r. \quad (1)$$

or, equivalently,

$$\delta(v) \geq 2\delta_S(v) + r. \quad (2)$$

An offensive 1-alliance is an offensive alliance and an offensive 2-alliance is a strong offensive alliance as defined in [7, 17, 18].

The offensive $r$-alliance number of $G$, denoted by $a^o_r(G)$, is defined as the minimum cardinality of an offensive $r$-alliance in $G$. Notice that

$$a^o_{r+1}(G) \geq a^o_r(G). \quad (3)$$

The offensive 1-alliance number of $G$ is known as the offensive alliance number of $G$ and the offensive 2-alliance number is known as the strong offensive alliance number [7, 17, 18].

A set $S \subset V$ is a dominating set in $G = (V, E)$ if for every vertex $u \in \bar{S}$, $\delta_S(u) > 0$ (every vertex in $\bar{S}$ is adjacent to at least one vertex in $S$). The
domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in $G$.

An offensive $r$-alliance $S$ is called global if it forms a dominating set, i.e., $\partial(S) = \overline{S}$. The global offensive $r$-alliance number of $G$, denoted by $\gamma^o_r(G)$, is the minimum cardinality of a global offensive $r$-alliance in $G$. Clearly,

$$\gamma^o_{r+1}(G) \geq \gamma^o_r(G) \geq \gamma(G) \quad \text{and} \quad \gamma^o_r(G) \geq a^o_r(G). \tag{4}$$

Notice that if every vertex of $G$ has even degree and $k$ is odd, $k = 2l - 1$, then every offensive $(2l - 1)^o$-alliance in $G$ is an offensive $(2l)$-alliance. Hence, in such a case, $a^o_{2l-1}(G) = a^o_{2l}(G)$ and $\gamma^o_{2l-1}(G) = \gamma^o_{2l}(G)$. Analogously, if every vertex of $G$ has odd degree and $k$ is even, $k = 2l$, then every offensive $(2l)$-alliance in $G$ is an offensive $(2l + 1)$-alliance. Hence, in such a case, $a^o_{2l}(G) = a^o_{2l+1}(G)$ and $\gamma^o_{2l}(G) = \gamma^o_{2l+1}(G)$.

2 On the complexity of finding optimal offensive $r$-alliances

For the class of complete graphs of order $n$, $G = K_n$, we have the exact value of $a^o_r(G)$. That is,

$$n - 1 = a^o_{n-1}(K_n) = a^o_{n-2}(K_n) \geq a^o_{n-3}(K_n) = a^o_{n-4}(K_n) = n - 2 \ldots \geq a^o_{5-n}(K_n) = a^o_{4-n}(K_n) = 2 \geq a^o_{3-n}(K_n) = 1.$$ 

Hence, for every $r \in \{3 - n, \ldots, n - 1\}$, $a^o_r(K_n) = \lceil \frac{n+r-1}{2} \rceil$. In this case, every offensive $r$-alliance is global and every vertex-set of cardinality $\lceil \frac{n+r-1}{2} \rceil$ is a (global) offensive $r$-alliance.

As we will see below, in general, the problem of finding optimal (global) offensive $r$-alliances is NP-complete. That is, we are interested in the computational complexity of the following optimization problems.

Offensive $r$-Alliance problem ($r$-OA):

Given: A graph $G = (V, E)$ and a positive integer $k \leq |V|$. 

Question: Is there an offensive $r$-alliance in $G$ of size $k$ or less?

**Global offensive $r$-Alliance problem ($r$-GOA):**

Given: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

Question: Is there a global offensive $r$-alliance in $G$ of size $k$ or less?

### 2.1 Offensive alliances

Our reasoning will use and generalize the following observation:

**Proposition 1.** [7] On cubic graphs, every vertex cover is a strong offensive alliance and vice versa.

With some gadgetry, this was used in [9] to show NP-hardness of finding small offensive alliances. We will generalize those results in the following.

**Theorem 2.** $\forall r$: $r$-OA is NP-complete.

*Proof.* It is clear that $r$-OA is in NP.

Consider first the case that $r \geq 3$. For any connected $r$-regular graph $G = (V, E)$, it can be seen that $C \subseteq V$ is a minimum vertex cover iff $C$ is a minimum $r$-offensive alliance. Clearly, any vertex cover is an $r$-OA. Let $S$ be an $r$-OA. By definition, $S \neq \emptyset$. Discuss $x \in S$. Any neighbor of $x$ must have $r$, i.e., all, neighbors in $S$, and we can continue the argument with those vertices taking the role of $x$, till the whole graph is exhausted (since it is connected by assumption). Hence, the complement of $S$ forms an independent set, which means that $S$ itself is a vertex cover. Since it is well-known that the vertex cover problem, restricted to $r$-regular graphs is NP-complete for any $r \geq 3$, see [8] for a recent account related to approximability results, the claim follows for $r \geq 3$.

Now, we show: if $r$-OA is NP-hard, then so is $(r-1)$-OA. By induction, the whole claim will follow.

Let $(G = (V, E), k)$ be an instance of $r$-OA, with $n = |V|$. We construct an instance of $(r-1)$-OA as follows: $G' = (V', E')$ with $V' = V \times \{1, 2, 3\} \cup \{c_1, \ldots, c_{n-r+2}\}$. In $E'$, we find the following edges (and only those):
- $\{(u, 1), (v, 1)\} \in E'$ iff $\{(u, 2), (v, 2)\} \in E'$ iff $\{u, v\} \in E$;
- $\{(u, 1), (u, 3)\} \in E'$ and $\{(u, 2), (u, 3)\} \in E'$ for any $u \in V$;
- $\{(u, 3), c_j\} \in E'$ for any $u \in V$ and any $1 \leq j \leq n - r + 2$;

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Clearly, $K \mid N$ of size at most $k$ for $G$ iff $S \times \{1, 2\}$ is a $(r - 1)$-OA of size at most $2k$ for $G'$, and that there is no other possibility to form smaller $(r - 1)$-OAs in $G'$ due to the attached clique.

2.2 Global offensive alliances

Cami et al. [2] showed NP-completeness for $r = 1$. We are going to modify their construction to show NP-completeness for any fixed $r$. Since we are dealing with the degree of vertices both in $G$ and within the new graph $G'$ as constructed below, we are going to attach $G$ and $G'$ to $\delta$ to avoid confusion in our notation.

**Theorem 3.** $\forall r: r$-GOA is NP-complete.

**Proof.** Membership in NP is clear.

The construction in [2] can be modified to work for any case $r \leq 1$. Let $(G, k)$ be an instance of Dominating Set with minimum degree $|r| + 1$, with $G = (V, E)$. To any $v \in V$, attach $\delta_G(v) + r - 1 \geq 0$ copies of $K_2$ with one edge per $K_2$-copy, this way yielding a new graph $G' = (V', E')$ with $G$ as a subgraph; call the new neighbors of vertices from $V$ $A$-vertices and collect them into set $A$, and call $N(A) \setminus V$ $B$-vertices.

If $D \subseteq V$ is a dominating set in $G$, then $S = D \cup A$ is a $r$-GOA. Clearly, $S$ is a dominating set in $G'$. Now, consider a $B$-vertex $v$. Obviously, $N(v) \subseteq A$, and therefore $|N_{G'}(v) \cap S| \geq |N_G(v) \cap \bar{S}| + r$. Any vertex $v \in V \setminus D$ has a neighbor $d \in D$. Hence, $|N_G(v) \cap \bar{S}| \leq \delta_G(v) - 1$, while $|N_{G'}(v) \cap \bar{S}| \geq \delta_G(v) + (r - 1) + 1 = \delta_G(v) + r$. Therefore, $S$ is a valid $r$-GOA.

Conversely, let $S$ be a $r$-GOA of $G'$. Since $S$ is a dominating set, for each $K_2$-copy attached to $G$, either the corresponding $A$- or the corresponding $B$-vertex is in $S$. Consider some $v \in V \setminus S$. $v$ must be dominated. If no neighbor of $v$ in $V$ is in $S$, then $|N_G(v) \cap S| \leq \delta_G(v) + r - 1$, while $|N_G \cap \bar{S}| \geq \delta_G(v)$, which leads to a contradiction. Hence, $S \cap V$ is a dominating set in $G$.

Combining the arguments, we obtain: $G = (V, E)$ has a dominating set of size at most $k$ iff $G' = (V', E')$ has a $r$-GOA of size $k + \sum_v (\delta_G(v) + r - 1) = k + (r - 1)|V| + 2|E|$.

Now, we consider the case $r \geq 2$. Let $(G, k)$ be an instance of Dominating Set with minimum degree 1, with $G = (V, E)$. To any $v \in V$, attach $\delta_G(v) + r - 1 \geq 1$ so-called $A$-vertices. All $A$-vertices together form an independent
set. Let \( A(v) = \{(v, 1), \ldots, (v, \delta_G(v) + r - 1)\} \) denote the set of \( A \)-vertices attached to \( v \in V \). We denote the \( B \)-vertices attached to the \( A \)-vertices in \( A(v) \) by \( B(v) \) and can describe them as \( B(v) = \left( \frac{A(v)}{r} \right) \), i.e., the \( r \)-element subsets of \( A(v) \). Each \( X \in B(v) \) has as neighbors exactly the \( A \)-vertices listed in \( X \). This describes the graph \( G' = (V', E') \) as obtained from \( G \).

If \( D \subseteq V \) is a dominating set in \( G \), then \( S = D \cup A \) is a \( r \)-GOA in \( G' \). Clearly, \( S \) is a dominating set in \( G' \). Now, consider a \( B \)-vertex \( v \). Obviously, \( N(v) \subseteq A(v) \), and therefore \( |N_G(v) \cap S| = r \geq |N_G(v) \cap \bar{S}| + r \). Any vertex \( v \in V \setminus D \) has a neighbor \( d \in D \). Hence, \( |N_G(v) \cap \bar{S}| \leq \delta_G(v) - 1 \), while \( |N_G(v) \cap S| \geq \delta_G(v) + (r - 1) + 1 = \delta_G(v) + r \). Therefore, \( S \) is a valid \( r \)-GOA.

Conversely, let \( S \) be a \( r \)-GOA of \( G' \) of size \( k + |A| \). Notice that this bound is met if \( S \cap V \) is a dominating set in \( G \) and all \( A \)-vertices go into \( S \). Consider an \( A(v) \)-vertex \( x \) and assume \( x \notin S \). Then, either there is a \( y \in S \cap N(x) \cap B(v) \), or \( v \in S \), since otherwise \( x \) would not be dominated.

Altogether, \( x \) has \( \left( \frac{\delta_G(v) + r - 1}{r} \right) + 1 \) many neighbors. Since \( S \) is an \( r \)-GOA, more than \( |A(v)| = \delta_G(v) + r - 1 \) vertices from the gadget attached to \( v \) would be in \( S \), this way violating the bound on the size of \( S \). Consider some \( v \in V \setminus S \). \( v \) must be dominated. If no neighbor of \( v \) in \( V \) is in \( S \), then \( |N_G(v) \cap S| \leq \delta_G(v) + r - 1 \), while \( |N_G(v) \cap \bar{S}| \geq \delta_G(v) \), which leads to a contradiction. Hence, \( S \cap V \) is a dominating set in \( G \).

Combining the arguments, we obtain: \( G = (V, E) \) has a dominating set of size at most \( k \) iff \( G' = (V', E') \) has a \( r \)-GOA of size \( k + \sum_v (\delta_G(v) + r - 1) = k + (r - 1)|V| + 2|E| \).

\[ \square \]

### 3 Bounding the offensive \( r \)-alliance number

**Theorem 4.** For any graph \( G \) of order \( n \) and minimum degree \( \delta \), and for every \( r \in \{2 - \delta, \ldots, \delta\} \),

\[
\left\lceil \frac{\delta + r}{2} \right\rceil \leq a^o_r(G) \leq \gamma^o_r(G) \leq n - \left\lfloor \frac{\delta - r + 2}{2} \right\rfloor.
\]

**Proof.** Let \( v \) be a vertex of minimum degree in \( G \) and let \( Y \subset N_V(v) \) such that \( |Y| = \left\lceil \frac{\delta + r}{2} \right\rceil \). Let \( S = \{v\} \cup N_V(v) - Y \). Hence, \( S \) is a dominating set and

\[
\delta_S(v) = \left\lceil \frac{\delta + r}{2} \right\rceil \geq \left\lceil \frac{\delta + r}{2} \right\rceil = \delta - \left\lfloor \frac{\delta + r}{2} \right\rfloor + r = \delta_S(v) + r.
\]
Thus,
\[ \delta_S(u) \geq \delta_S(v) \geq \delta_S(v) + r \geq \delta_S(u) + r, \quad \forall u \in S. \]

Therefore, \( \bar{S} \) is a global offensive \( r \)-alliance in \( G \) and, as a consequence, the upper bound follows.

On the other hand, let \( X \subset V \) be an offensive \( r \)-alliance in \( G \). For every \( v \in \partial(X) \) we have
\[
\delta(v) = \delta_X(v) + \delta_{\bar{X}}(v) \\
\delta(v) \leq \delta_X(v) + \frac{\delta(v) - r}{2} \\
\frac{\delta(v) + r}{2} \leq \delta_X(v) \leq |X| \\
\delta + r \leq |X|.
\]

Therefore, the lower bound follows.

The bounds are attained for every \( r \) in the case of the complete graph \( G = K_n \).

A set \( S \subset V \) is a \( k \)-dominating set if for every \( v \in \bar{S} \), \( \delta_S(v) \geq k \). The \( k \)-domination number of \( G \), \( \gamma_k(G) \), is the minimum cardinality of a \( k \)-dominating set in \( G \). The following result generalizes, to \( r \) alliances, some previous results obtained for \( r = 1 \) and \( r = 2 \) \([15, 18]\).

**Theorem 5.** For any simple graph \( G \) of order \( n \), minimum degree \( \delta \), and Laplacian spectral radius \( \mu_* \),
\[
\left\lfloor \frac{n}{\mu_*} \left[ \frac{\delta + r}{2} \right] \right\rfloor \leq \gamma^o_r(G) \leq \left\lfloor \frac{\gamma_r(G) + n}{2} \right\rfloor.
\]

**Proof.** Let \( H \subset V \) be an \( r \)-dominating set of \( G \) of minimum cardinality. If \( |\bar{H}| = 1 \), then \( \gamma_r(G) = n - 1 \) and \( \gamma^o_r(G) \leq n - 1 \). If \( |\bar{H}| \neq 1 \), let \( \bar{H} = X \cup Y \) be a partition of \( \bar{H} \) such that the edge-cut between \( X \) and \( Y \) has the maximum cardinality. Suppose \( |X| \leq |Y| \). For every \( v \in Y \), \( \delta_H(v) \geq r \) and \( \delta_X(v) \geq \delta_Y(v) \). Therefore, the set \( W = H \cup X \) is a global offensive \( r \)-alliance in \( G \), i.e., for every \( v \in Y \), \( \delta_W(v) \geq \delta_Y(v) + r \). Then we have,
\[
2|X| + \gamma^o_r(G) \leq n
\]
\[\text{(5)}\]

\(^1\)i.e., the largest Laplacian eigenvalue of \( G \). The reader is referred to \([6, 14]\) for a detailed study and survey on the Laplacian matrix of a graph and its eigenvalues.
and
\[ \gamma^o_r(G) \leq |X| + \gamma_r(G). \tag{6} \]

Thus, by (5) and (6), we obtain the upper bound.

It was shown in [10] that the Laplacian spectral radius of \( G, \mu_* \), satisfies
\[
\mu_* = 2n \max \left\{ \frac{\sum_{v_i \sim v_j} (w_i - w_j)^2}{\sum_{v_i \in V \sum_{v_j \in V} (w_i - w_j)^2} : w \neq \alpha j \text{ for } \alpha \in \mathbb{R}} \right\}, \tag{7}
\]
where \( V = \{v_1, v_2, ..., v_n\} \), \( j = (1, 1, ..., 1) \) and \( w \in \mathbb{R}^n \). Let \( S \subset V \). From (7), taking \( w \in \mathbb{R}^n \) defined as
\[
w_i = \begin{cases} 1 & \text{if } v_i \in S; \\ 0 & \text{otherwise} \end{cases}
\]
we obtain
\[
\mu_* \geq n \sum_{v \in S} \delta_S(v) \tag{8} \]
Moreover, if \( S \) is a global offensive \( r \)-alliance in \( G \),
\[
\delta_S(v) \geq \left\lceil \frac{\delta(v) + r}{2} \right\rceil, \quad \forall v \in \bar{S}. \tag{9}
\]
Thus, (8) and (9) lead to
\[
\mu_* \geq n \frac{\delta + r}{|S|} \tag{10} \]
Therefore, solving (10) for \( |S| \) we obtain the lower bound.

The above-mentioned bounds are attained, for instance, in the case of the complete graph of order \( n \).

**Corollary 6.** For any simple graph \( G \) of order \( n \), minimum degree \( \delta \), and for every \( r \in \{1, ..., \delta\} \),
\[
\gamma^o_r(G) \leq \left\lfloor \frac{n(2r + 1)}{2r + 2} \right\rfloor.
\]
Proof. The bound immediately follows from the following bound on $\gamma_r(G)$ [3]:
\[
\delta \geq r \Rightarrow \gamma_r(G) \leq \frac{rn}{r+1}.
\] (11)

Corollary 7. Let $L(G)$ be the line graph of a $\delta$-regular graph $G$ of order $n$. Then
\[
\gamma_o^L(L(G)) \geq \frac{n}{4} \left[ \frac{2(\delta-1) + r}{2} \right].
\]

Proof. We denote by $A$ the adjacency matrix of $L(G)$ and by $2(\delta-1) = \lambda_0 > \lambda_1 > \cdots > \lambda_b = -2$ its distinct eigenvalues. We denote by $L$ the Laplacian matrix of $L(G)$ and by $\mu_0 = 0 < \mu_1 < \cdots < \mu_b$ its distinct Laplacian eigenvalues. Then, since $L = 2(\delta-1)I_n - A$, the eigenvalues of both matrices, $A$ and $L$, are related by
\[
\mu_l = 2(\delta-1) - \lambda_l, \quad l = 0, \ldots, b.
\] (12)

Thus, the Laplacian spectral radius of $L(G)$ is $\mu_b = 2\delta$. Therefore, the result immediately follows. \qed

There are some immediate bounds on $\gamma_o^r(G)$ derived from the following remarks.

Remark 8. If $S$ is an independent set in $G$, then $\bar{S}$ is a global offensive $r$-alliance in $G$ ($r \leq \delta$).

Remark 9. All global offensive $r$-alliance in $G$ is a $\left\lceil \frac{\delta + r}{2} \right\rceil$-dominating set in $G$ ($r \geq 2 - \delta$).

Therefore, the following bounds follow.
\[
\gamma_{\left\lceil \frac{\delta + r}{2} \right\rceil}(G) \leq \gamma_o^r(G) \leq n - \alpha(G),
\] (13)

where $\alpha(G)$ denotes the independence number of $G$.

The reader is referred to our previous works [15, 16, 17, 18] for a more detailed study on offensive 1-alliances and offensive 2-alliances.
References

[1] R. Brigham, R. Dutton and S. Hedetniemi, A sharp lower bound on the powerful alliance number of $C_m \times C_n$. Congr. Numer. 167 (2004) 57–63.

[2] A. Cami, H. Balakrishnan, N. Deo and R. Dutton, On the complexity of finding optimal global alliances. J. Combin. Math. Combin. Comput. 58 (2006).

[3] E. J. Cockayne, B. Gamble and B. Shepherd, An upper bound for the $k$-domination number of a graph. J. Graph Theory 9 (4) (1985) 533-534.

[4] M. Chellali and T. W. Haynes, Global alliances and independence in trees. Discussiones Mathematicae Graph Theory. In press.

[5] M. Chellali, Offensive alliances in trees. Submitted.

[6] D. M. Cvetković, M. Doob and H. Sachs, Spectra of Graphs - Theory and Application. Academic Press. New York, 1980.

[7] O. Favaron, G. Fricke, W. Goddard, S. Hedetniemi, S. T. Hedetniemi, P. Kristiansen, R. C. Laskar and R. D. Skaggs, Offensive alliances in graphs. Discuss. Math. Graph Theory 24 (2) (2004) 263–275.

[8] U. Feige, Vertex cover is hardest to approximate on regular graphs, Technical Report MCS 03-15 of the Weizmann Institute, 2003.

[9] H. Fernau and D. Raible, Alliances in graphs: a complexity-theoretic study, Software Seminar SOFSEM 2007, Student Research Forum, Proceedings Vol. II, 61–70.

[10] M. Fiedler, A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory. Czechoslovak Math. J. 25 (100) (1975) 619–633.

[11] G. H. Fricke, L. M. Lawson, T. W. Haynes, S. M. Hedetniemi and S. T. Hedetniemi, A note on defensive alliances in Graphs. Bull. Inst. Combin. Appl. 38 (2003) 37–41.

[12] T. W. Haynes, S. T. Hedetniemi and M. A. Henning, Global defensive alliances in graphs. Electron. J. Combin. 10 (2003) 139–146.
[13] P. Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi, Alliances in graphs. *J. Combin. Math. Combin. Comput.* **48** (2004) 157–177.

[14] B. Mohar, The Laplacian spectrum of graphs. In Y. Alavi, G. Chartrand, O. Ollermann, and A. Schwenk, editors, *Graph Theory, Combinatorics, and Applications*, pages 871–898. John Wiley and Sons, Inc., New York, 1991.

[15] J. A. Rodríguez and J. M. Sigarreta, Spectral study of alliances in graphs. *Discussiones Mathematicae Graph Theory* **27** (1) (2007) 143-157.

[16] J. A. Rodríguez and J. M. Sigarreta, Global alliances in planar graphs. *AKCE–International Journal of Graphs and Combinatorics*. In press.

[17] J. A. Rodríguez and J. M. Sigarreta, Offensive alliances in cubic graphs. *International Mathematical Forum* **1** (36) (2006) 1773–1782.

[18] J. A. Rodríguez-Velázquez and J. M. Sigarreta, Global offensive alliances in graphs. *Electronic Notes in Discrete Mathematics* **25** (2006) 157–164.

[19] K. H. Shafique and R. D. Dutton, Maximum alliance-free and minimum alliance-cover sets. *Congr. Numer.* **162** (2003) 139–146.

[20] K. H. Shafique and R. Dutton, A tight bound on the cardinalities of maximum alliance-free and minimum alliance-cover sets. *J. Combin. Math. Combin. Comput.* **56** (2006), 139–145.

[21] J. M. Sigarreta and J. A. Rodríguez, On defensive alliance and line graphs. *Applied Mathematics Letters* **19** (12) (2006) 1345–1350.