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The cohomology of the Steenrod algebra and the mod $p$ Lannes-Zarati homomorphism

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**Abstract**

For any pointed space $X$, the mod $p$ Lannes-Zarati homomorphism $\varphi^X_p$ is considered as a graded associated version of the mod $p$ Hurewicz map $h_* : \pi_* (QX) \rightarrow H_* QX$ in the $E_2$-term of the Adams spectral sequence. In this paper, we investigate the behavior of $\varphi^p_{3}$ and $\varphi^*(B\mathbb{Z}/p)$ ($s \leq 1$) for an odd prime $p$.

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**Keywords:**
spherical classes
Hurewicz map
Lannes-Zarati homomorphism
Adams spectral sequence
cohomology of the Steenrod algebra

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* This research is funded by the National Foundation for Science and Technology Development (NAFOSTED) of Viet Nam under grant number 101.04-2019.10.

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1. Introduction

The paper is a continuation of our previous one [6], which will refer to as Part I. For any unstable $A$-module, let us recall that the mod $p$ Lannes-Zarati homomorphism, which was first introduced by Lannes and Zarati in [23], for each $s \geq 0$, is defined as follows

$$\varphi_s^M : \text{Ext}^{s, s+t}_A(M, \mathbb{F}_p) \rightarrow (\mathbb{F}_p \otimes_A \mathcal{R}_s M)^\#.$$  \hspace{1cm} (1.1)

Here $A$ denote the mod $p$ Steenrod algebra and $\mathcal{R}_s M$ denote the Singer construction (see Singer [32], [33], Lannes-Zarati [23], Zarati [35], see also Hái [11], Powell [30] and citations therein for a detail description). For an $A$-module $N$, we denote by $N^\#$ the linear dual of $N$.

Our interest in the map (1.1) (or its dual) lies in the fact that it is closely related to the mod $p$ Hurewicz map. Indeed, if $M$ is the reduced mod $p$ (singular) cohomology of a pointed space $X$, then $\varphi_s^M$ is considered as a graded associated version of the mod $p$ Hurewicz map

$$h_s : \pi_*^S X = \pi_* QX \rightarrow H_* QX$$

of the infinite loop space $QX := \lim\Omega^n \Sigma^n X$ in the $E_2$-term of the Adams spectral sequence (see Lannes and Zarati [21], [22] for $p = 2$ and Kuhn [20] for an odd prime $p$). Hence, the study of the behavior of the mod $p$ Lannes-Zarati homomorphism actually corresponds to the description of the image of the mod $p$ Hurewicz map, and therefore, it also closely corresponds to the famous conjectures on spherical classes of Curtis and Wellington for $S^0$ (see Curtis [10], Wellington [34]) and of Eccles for any pointed CW-complex $X$ (which is mentioned in [36]).

We refer to the introduction of Part I for a detail survey of known facts about the mod $p$ Lannes-Zarati homomorphism.

In Part I, we initiated the use of the lambda algebra to study the image and the kernel of the map (1.1) for $M = \mathbb{F}_p$. The advantages of our method is that we can avoid using the knowledge of the so-called “hit problem” for $\mathcal{R}_s \mathbb{F}_p$ as in [14], [12], [13], [15]. Therefore, it can help us to not only recover previous known results with little computation involved (however, we have not pointed out this fact in Part I), but also obtain new results about the behavior of $\varphi_s^p$ for $s \leq 2$ with $p$ odd. However, for $s$ higher, the computation remains difficult because the Adem relations of the mod $p$ Dyer-Lashof algebra, considered as the dual of $\mathcal{R}_s \mathbb{F}_p$, in general, is hard to exploit (see the proof of Theorem 4.2 in [6]).

To overcome this difficulty, in this paper, we develop the power operation $P^0$ acting on $\text{Ext}^s_A(\mathbb{F}_p, \mathbb{F}_p)$ (see Lurie-icus [25] or May [7]). For $M = \mathbb{F}_p$ and $M = P := \tilde{H}^*(B\mathbb{Z}/p)$, we show that there exist the power operations $P^0$ acting on $\text{Ext}^s_A(M, \mathbb{F}_p)$ and on $(\mathbb{F}_p \otimes_A \mathcal{R}_s M)^\#$. Moreover, these actions are compatible with each other through the mod $p$ Lannes-Zarati homomorphism $\varphi_s^M$, which is given by the following proposition.
**Proposition 4.3.** The following diagram is commutative

\[
\begin{array}{ccc}
\text{Ext}^{s,s+t}_A(M, \mathbb{F}_p) & \xrightarrow{p^0} & \text{Ext}^{s,p(s+t)}_A(M, \mathbb{F}_p) \\
\varphi_s^M & \downarrow & \varphi_s^M \\
(\mathbb{F}_p \otimes_A \mathcal{R}_s M)^t & \xrightarrow{p^0} & (\mathbb{F}_p \otimes_A \mathcal{R}_s M)^{p(s+t)-s},
\end{array}
\]

for \(M = \mathbb{F}_p\) and \(M = P\).

Recall that, for \(M = \mathbb{F}_p\) and \(M = P\), a family \(\{a_i : i \geq i_0\} \subset \text{Ext}^{s,*}_A(M, \mathbb{F}_p)\) is called a \(\mathcal{P}^0\)-family if \(a_{i+1} = \mathcal{P}^0(a_i)\) for \(i \geq i_0\). The above result allows us to determine \(\varphi_s^M(a_i)\) through \(\varphi_s^M(a_{i_0})\), this makes reduce significantly the computation.

Using the Proposition 4.3 combining with the method of Part I, we have the following theorem, which is the first of our main results.

**Theorem 5.1.** The third Lannes-Zarati homomorphism

\[
\varphi_{3,F}^p : \text{Ext}_A^{3,3+t}(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow (\mathbb{F}_p \otimes_A \mathcal{R}_3 \mathbb{F}_p)^t
\]

is a monomorphism for \(t = 0\) and vanishing at all positive stems \(t\).

From the results in [6] and Theorem 5.1, we observe that, for an odd prime \(p\), the behavior of the mod \(p\) Lannes-Zarati \(\varphi_s^M, s > 2\), is similar to the case \(p = 2\). By the conjecture on spherical classes due to Wellington [34] and Conjecture 1.2 in [17], it leads us to a conjecture.

**Conjecture 1.1 (Cf. [17, Conjecture 1.2]).** Given an unstable \(A\)-module \(M\), the mod \(p\) Lannes-Zarati homomorphism

\[
\varphi_s^M : \text{Ext}_A^{s,s+t}(M, \mathbb{F}_p) \longrightarrow (\mathbb{F}_p \otimes_A \mathcal{R}_s M)^t
\]

is trivial at all positive stems \(t\), for \(s > 2\).

For \(p = 2\), the conjecture is verified for \(M = \mathbb{F}_2\) with \(3 \leq s \leq 5\) by Hưng et al. (see [14], [12], [13], [15]), and for \(M = \mathcal{H}^*(B\mathbb{Z}/2)\) with \(3 \leq s \leq 4\) by Hưng-Tuấn [17]. For \(p\) odd and \(M = \mathbb{F}_p\), Theorem 5.1 shows that this conjecture is true for \(s = 3\). The previous results motivated us to study the behavior of \(\varphi_s^{p,F}\).

In order to investigate the behavior of \(\varphi_s^{p,F}\) we, on the one hand, construct a spectral sequence, which is a generalized version of the one used in Cohen-Lin-Mahowld [8], Lin [24] and Chen [4]. Using this spectral sequence to compute \(\text{Ext}_A^*(P, \mathbb{F}_p)\), we obtain the following theorems, which are the second of our main results.

**Theorem 5.3 (Cf. [9, Theorem 1.1]).** The Ext group \(\text{Ext}_A^{0,t}(P, \mathbb{F}_p)\) has an \(\mathbb{F}_p\)-basis consisting of all elements.
1. $\hat{h}_i \in \Ext_A^{0,2(p-1)p^{i-1}}(P, F_p)$, $i \geq 0$;
2. $\hat{h}_i(k) \in \Ext_A^{0,2kp^{i-1}}(P, F_p)$, $i \geq 0, 1 \leq k < p - 1$.

The Ext group $\Ext_A^{1,1+t}(P, F_p)$ is given by the following theorem.

**Theorem 5.4.** The Ext group $\Ext_A^{1,1+t}(P, F_p)$ has an $F_p$-basis consisting of all elements given by the following list

1. $\alpha_0 \hat{h}_i \in \Ext_A^{1,2(p-1)p^{i}}(P, F_p)$, $i \geq 1$;
2. $\alpha_0 \hat{h}_i(k) \in \Ext_A^{1,2kp^{i}}(P, F_p)$, $i, 1 \leq k < p - 1$;
3. $\hat{\alpha}(\ell) \in \Ext_A^{1,2(p+\ell)+2}(P, F_p)$, $0 \leq \ell < p - 2$;
4. $h_i \hat{h}_i(1) \in \Ext_A^{1,2(p-1)p^{i+2p^{i-1}}}(P, F_p)$, $i \geq 0$;
5. $h_i \hat{h}_j \in \Ext_A^{1,2(p-1)(p^{i}+p^{j})^{-1}}(P, F_p)$, $i, j \geq 0, j \neq i, i + 1$;
6. $h_i \hat{h}_j(k) \in \Ext_A^{1,2(p-1)p^{i}+2kp^{j-1}}(P, F_p)$, $i, j \geq 0, j \neq i, i + 1, 1 \leq k < p - 1$;
7. $\hat{d}_i(k) \in \Ext_A^{1,2(p-1)(p^{i}+p^{j+1})+2kp^{i-1}}(P, F_p)$, $i \geq 1, 1 \leq k \leq p - 1$;
8. $\hat{k}_i \in \Ext_A^{1,2(k+1)p^{i+1}-1}(P, F_p)$, $i \geq 0, 1 \leq k < p - 1$;
9. $\hat{p}_i(k) \in \Ext_A^{1,2(p-1)(p^{i}+p^{j+1})+2(k+1)p^{j-1}}(P, F_p)$, $i \geq 0, 1 \leq k < p - 1$.

On the other hand, we describe the dual of $\mathcal{R}_s M$ in term of the mod $p$ Dyer-Lashof algebra $R$. We show that $(\mathcal{R}_s M)^\#$ is considered as a quotient right $A$-module of $R_s \otimes M^\#$. Here $R_s$ denote the subspace of $R$ spanned by all monomials of length $s$, and the right $A$-action on $R$ is given via the Nishida relations [7]. Based upon the description, we obtain a chain-level representation of $\varphi_s^M$ for any unstable $A$-module $M$ in term of the lambda algebra.

**Proposition 3.7.** For any unstable $A$-module $M$, the projection

$$\tilde{\varphi}_s^M : \Lambda_s \otimes M^\# \longrightarrow (\mathcal{R}_s M)^\#$$

given by

$$\lambda_I \otimes \ell \longrightarrow (-1)^{(s-1)(s-2)} [Q^I \otimes \ell]$$

is a chain-level representation of the mod $p$ Lannes-Zarati homomorphism $\varphi_s^M$.

Here, we denote by $Q^i, \beta Q^i$ the generators of the mod $p$ Dyer-Lashof algebra. The proposition is a generalized version of Proposition 3.4 in [6] to the case of an arbitrary unstable $A$-module. Moreover, it is also one of the main tools to study the behavior of $\varphi_s^P$.

Using the knowledge above to determine the image and the kernel of $\varphi_s^P$, we also obtain the following results, which are the third of our main results.
Theorem 5.6. The Lannes-Zarati homomorphism \( \varphi_0^P : \text{Ext}_{A}^{0,t}(P, \mathbb{F}_p) \rightarrow (\mathbb{F}_p \otimes_{A} \mathcal{R}_0 P)^\# \) is an isomorphism.

This result is similar to the case \( p = 2 \) due to Hùng and Tuán [17].

Theorem 5.7. The Lannes-Zarati homomorphism \( \varphi_1^P : \text{Ext}_{A}^{1,1+t}(P, \mathbb{F}_p) \rightarrow (\mathbb{F}_p \otimes_{A} \mathcal{R}_1 P)^\# \) sends

(i) \( h_i \hat{h}_i(1) \) to \( \left[ \beta Q^j ab^{(p^i-1)} \right] \), for \( i \geq 0 \);
(ii) \( h_i \hat{h}_j \) to \( \left[ \beta Q^j ab^{(p^j-1)} \right] \) for \( 0 \leq j < i \);
(iii) \( h_i \hat{h}_j(k) \) to \( \left[ \beta Q^j ab^{(p^j-1)} \right] \) for \( 0 \leq j < i \), \( 1 \leq k < p - 1 \);
(iv) \( \hat{k}_i(k) \) to \( (P^0)^j \left( \left[ \beta Q^{k+1} ab^{[k]} \right] \right) \), \( i \geq 0, 1 \leq k < p - 1 \); and
(v) others to zero.

Here, we denote by \( a^ib^c \) the \( \mathbb{F}_p \)-generator of \( \tilde{H}_{2s+t}(B\mathbb{Z}/p) \).

From Corollary 5.8, \( \varphi_1^P \) is not an epimorphism. This fact is similar to the case \( p = 2 \) (see Remark 6.6).

It should be noted that our strategy in term of the lambda algebra is also valid for the case \( p = 2 \) with suitable modifications. Therefore, using this method, we can recover the results of Lannes-Zarati [23] and Hùng et al. [14], [12], [13], [15] [17] with a little computation (see section 6).

The paper is organized as follows. Section 2 is a preliminary on the Singer-Hùng-Sum chain complex, the lambda algebra as well as the Dyer-Lashof algebra that are required for other sections. In section 3, we recall the mod \( p \) Lannes-Zarati homomorphism and its chain-level representation in Singer-Hùng-Sum chain complex, which is presented in [6]. In addition, we also describe therein the dual of the Singer construction \( \mathcal{R}_s M \) in term of the Dyer-Lashof algebra. The section 4 provides a development of the power operations. The behavior of the mod \( p \) Lannes-Zarati homomorphism is presented in section 5. We recover the results of Lannes-Zarati [23] and Hùng et al. [14], [12], [13], [15] [17] in section 6. In appendix, we construct a spectral sequence to compute \( \text{Ext}_A^s(P, \mathbb{F}_p) \). Using this spectral sequence, we give proofs for Theorem 5.3 and Theorem 5.4.

2. Preliminaries

Unless stated otherwise, we will be working over the prime field \( \mathbb{F}_p \), where \( p \) is an odd prime. In addition, we will use the standard notations as in Part I.

2.1. The Singer-Hùng-Sum chain complex

Let \( E_n \) be an \( s \)-dimensional \( \mathbb{F}_p \)-vector space. It is well-known that the mod \( p \) cohomology of the classifying space \( BE_n \) is given by
\[ P_s := H^*BE_s = E(x_1, \ldots, x_s) \otimes \mathbb{F}_p[y_1, \ldots, y_s], \]

where \((x_1, \ldots, x_s)\) is a basis of \(H^1BE_s = \text{Hom}(E_s, \mathbb{F}_p)\) and \(y_i = \beta(x_i)\) for \(1 \leq i \leq s\) where \(\beta\) denote the Bockstein homomorphism. Here \(E(\ldots)\) and \(\mathbb{F}_p[\ldots]\) are standard notations for the exterior algebra and the polynomial algebra respectively over \(\mathbb{F}_p\) generated by the indicated variables.

Let \(GL_s\) denote the general linear group \(GL_s := GL(E_s)\). The group \(GL_s\) acts on \(E_s\) and then on \(H^*BE_s\).

For any \(s\)-tuple of non-negative integers \((r_1, \ldots, r_s)\), put \([r_1, \ldots, r_s] := \det(y_1^{p^j} \ldots y_s^{p^j})\), and define \(L_{s,i} := \left[0, \ldots, \hat{i}, \ldots, s\right]; \quad L_s := L_{s,s}; \quad q_{s,i} := L_{s,i}/L_s\), for any \(1 \leq i \leq s\).

In particular, \(q_{s,s} = 1\) and by convention, set \(q_{s,i} = 0\) for \(i < 0\). The degree of \(q_{s,i}\) is \(2(p^s - p^i)\). Define \(V_s := V_s(y_1, \ldots, y_s) := \prod_{\lambda_j \in \mathbb{F}_p} (\lambda_1 y_1 + \cdots + \lambda_{s-1} y_{s-1} + y_s)\).

For non-negative integers \(k, r_k+1, \ldots, r_s\), set

\[
[k; r_k+1, \ldots, r_s] := \frac{1}{k!} \begin{vmatrix}
  x_1 & \cdots & x_s \\
  \cdot & \cdots & \cdot \\
  x_1 & \cdots & x_s \\
  y_1^{p^k+1} & \cdots & y_s^{p^k+1} \\
  \cdot & \cdots & \cdot \\
  y_1^{p^s} & \cdots & y_s^{p^s}
\end{vmatrix}.
\]

For \(0 \leq i_1 < \cdots < i_k \leq s - 1\), we define

\[
M_{s;i_1,\ldots,i_k} := [k; 0, \ldots, \hat{i}_1, \ldots, \hat{i}_k, \ldots, s - 1],
\]

\[
R_{s;i_1,\ldots,i_k} := M_{s;i_1,\ldots,i_k} L_{s}^{p-2}.
\]

Let \(\Phi_s := H^*BE_s[L_s^{-1}]\) be the localization of \(H^*BE_s\) obtained by inverting \(L_s\). The action of \(GL_s\) on \(H^*BE_s\) extends an action of it on \(\Phi_s\). Set

\[
\Delta_s := \Phi_s^{T_s}, \quad \Gamma_s := \Phi_s^{GL_s},
\]

where \(T_s\) is the subgroup of \(GL_s\) consisting of all upper triangular matrices with 1’s on the main diagonal.
Put $u_i := M_{i;i-1}/L_{i-1}$ and $v_i := V_i/q_{i-1,0}$. From [16], we have
\[
\begin{align*}
\Delta_s &= E(u_1, \ldots, u_s) \otimes \mathbb{F}_p[v_s^{\pm 1}, \ldots, v_s^{\pm 1}], \\
\Gamma_s &= E(R_{s;0}, \ldots, R_{s;s-1}) \otimes \mathbb{F}_p[q_s, q_{s,1}, \ldots, q_{s,s-1}].
\end{align*}
\]
Let $\Delta^+_s$ be the subspace of $\Delta_s$ spanned by all monomials of the form
\[
u_1^{(p-1)i_1-\epsilon_1} \cdots u_s^{(p-1)i_s-\epsilon_s}, \epsilon_i \in \{0, 1\}, 1 \leq i \leq s, i_1 \geq \epsilon_1,
\]
and let $\Gamma^+_s := \Gamma_s \cap \Delta^+_s$.

For any $A$-module $M$, define the stable total power $S_s(x_1, y_1, \ldots, x_s, y_s; m) \in \Delta_s \otimes M$, for $m \in M$, as follows (see Hưng-Sum [16])
\[
S_s(x_1, y_1, \ldots, x_s, y_s; m) := \sum_{\epsilon_i = 0, 1, i, j \geq 0} (-1)^{\epsilon_1+i+\cdots+i_s} u_s^{\epsilon_s} \cdots u_1^{(p-1)i_1-\epsilon_1} \cdots u_s^{(p-1)i_s-\epsilon_s} \otimes (\beta^1 P^{\epsilon_1} \cdots \beta^{\epsilon_s} P^i_m)(m). \tag{2.1}
\]
For convenience, we put $S_s(m) := S_s(x_1, y_1, \ldots, x_s, y_s; m)$, and $S_s(M) := \{S_s(m) : m \in M\}$.

Then $\Gamma^+_M := \{(\Gamma^+ M)_s\}_{s \geq 0}$, where $(\Gamma^+ M)_0 := M$ and $(\Gamma^+ M)_s := \Gamma^+_s S_s(M) = \{vS_s(m) : v \in \Gamma^+_s, m \in M\}$, is a differential $\mathbb{F}_p$-module.

For $v = \sum_{\epsilon, \ell} v_{\epsilon, \ell} u_s^{(p-1)\ell-\epsilon} \in \Gamma^+_s$ and $m \in M$, where $v_{\epsilon, \ell} \in \Gamma^+_{s-1}$, the differential in $\Gamma^+_M$ is given by
\[
\partial(v S_s(m)) = (-1)^{\deg v + 1} \sum_{\epsilon, \ell} (-1)^{\ell} v_{\epsilon, \ell} S_{s-1}(\beta^{1-\epsilon} P^\ell m). \tag{2.2}
\]

In [16], Hưng and Sum showed that $H_s(\Gamma^+ M) \cong \text{Tor}^4_s(\mathbb{F}_p, M)$ for any $A$-module $M$. Therefore, $\Gamma^+ M$ is a suitable complex to compute $\text{Tor}^4_s(\mathbb{F}_p, M)$.

2.2. The lambda algebra and the Dyer-Lashof algebra

Let $\Lambda$ denote the lambda algebra, which is isomorphic to the co-Koszul resolution of the Steenrod algebra (see Priddy [31]). For any $A$-module $M$, $\Lambda \otimes M^\#$ is a suitable complex to compute $\text{Ext}^4_s(M, \mathbb{F}_p)$.

Recall that $\Lambda$ is the unital, graded, associative differential algebra over $\mathbb{F}_p$ generated by $\lambda_{i-1}$ $(i > 0)$ of degree $2i(p-1)-1$ and $\mu_{j-1}$ $(j \geq 0)$ of degree $2j(p-1)$ satisfying the Adem relations (see [2] [3], [34] and [31])
\[
\sum_{i+j=n} \binom{i+j}{i} \lambda_{i-1+pm} \lambda_{j-1+m} = 0,
\]
\[
\sum_{i+j=n} \binom{i+j}{i} (\lambda_{i-1+pm\mu_{j-1}+m} - \mu_{i-1+pm\lambda_{j-1}+m}) = 0,
\]
for all \(m \geq 1\) and \(n \geq 0\); and
\[
\sum_{i+j=n} \binom{i+j}{i} \lambda_{i+pm\mu_{j-1}+m} = 0,
\]
\[
\sum_{i+j=n} \binom{i+j}{i} \mu_{i+pm\mu_{j-1}+m} = 0,
\]
for all \(m \geq 0\) and \(n \geq 0\).

The differential is given by
\[
d(\lambda_{n-1}) = \sum_{i+j=n} \binom{i+j}{i} \lambda_{i-1} \lambda_{j-1},
\]
\[
d(\mu_{n-1}) = \sum_{i+j=n} \binom{i+j}{i} (\lambda_{i-1} \mu_{j-1} - \mu_{i-1} \lambda_{j-1}),
\]
\[
d(\sigma \tau) = (-1)^{\deg \sigma \deg \tau} d(\tau) + d(\sigma) \tau.
\]

For convenience, we denote \(\lambda^1_{i-1} = \lambda_{i-1}\) and \(\lambda^0_{i-1} = \mu_{i-1}\). Let \(\Lambda_s\) denote the subspace of \(\Lambda\) spanned by all monomial \(\lambda^{e_1}_{i_1-1} \cdots \lambda^{e_s}_{i_s-1}\) of length \(s\). By the Adem relations, \(\Lambda_s\) has an additive basis consisting of all admissible monomials (which are monomials of the form \(\lambda_i = \lambda^{e_1}_{i_1-1} \cdots \lambda^{e_s}_{i_s-1} \in \Lambda_s\) satisfying \(pi_k - \epsilon_k \geq i_{k-1}\) for \(2 \leq k \leq s\)).

Given an \(A\)-module \(M\), the differential of complex \(\Lambda \otimes M^\#\) is given by, for \(\lambda \in \Lambda\) and \(h \in M^\#\),
\[
d(\lambda \otimes h) = d(\lambda) \otimes h + \sum_{i-\epsilon \geq 0} (-1)^{\deg \lambda + (1-\epsilon) \deg h} \lambda^{e_i}_{i-1} \otimes h \beta^{1-\epsilon} P^i. \tag{2.3}
\]

Based upon the results of Hưng-Sum [16] and Priddy [31], for any \(A\)-module \(M\), there exists an isomorphism of differential \(\mathbb{F}_p\)-modules \(\nu^A := \{\nu^A_s\}_{s \geq 0} : \Gamma^+ M \to \Lambda^\# \otimes M\) given by
\[
\nu^A_s(u_1^{\epsilon_1} v_1^{(p-1)i_1-\epsilon_1} \cdots u_s^{\epsilon_s} v_s^{(p-1)i_s-\epsilon_s} S_s(m)) = (-1)^{i_1+\cdots+i_s+\sum_{\ell<k} \epsilon_\ell \epsilon_k} (\lambda^{e_1}_{i_1-1} \cdots \lambda^{e_s}_{i_s-1})^* \otimes m,
\]
where \((\lambda^{e_1}_{i_1-1} \cdots \lambda^{e_s}_{i_s-1})^*\) is the dual of \(\lambda^{e_1}_{i_1-1} \cdots \lambda^{e_s}_{i_s-1}\) respect to the admissible basis.

An important quotient algebra of \(\Lambda\) is the mod \(p\) Dyer-Lashof algebra \(R\), which is also well-known as the algebra of homology operations acting on the homology of infinite loop spaces.
For any monomial \( \lambda_I = \lambda_{i_1}^{e_1} \cdots \lambda_{i_s}^{e_s} \in \Lambda_s \), we define the excess of \( \lambda_I \) or of \( I \) to be
\[
\text{exc}(\lambda_I) = \text{exc}(I) = 2i_1 - \epsilon_1 - \sum_{k=2}^{s} 2(p - 1)i_k + \sum_{k=2}^{s} \epsilon_s.
\]

Then, the mod \( p \) Dyer-Lashof algebra is the quotient algebra of \( \Lambda \) over the (two-sided) ideal generated by all monomials of negative excess (see Curtis [10], Wellington [34]).

Let \( Q^I = \beta^e Q^{i_1} \cdots \beta^s Q^{i_s} \) denote the image of \( \lambda_I \) under the canonical projection, and let \( R_s \) denote the subspace of \( R \) spanned by all monomials of length \( s \), then \( R_s \) is isomorphic to \( \mathcal{B}[s]^\# \) as \( A \)-coalgebras (see the description of \( \mathcal{B}[s] \) in section 3 below), where the \( A \)-action on \( R \) is given by the Nishida relations (see May [7]).

From the above result, we observe that the restriction of \( \nu_s := \nu_s^\mathbb{F}_p \) on \( \mathcal{B}[s] \) is isomorphism between \( \mathcal{B}[s] \) and \( R_s^\# \).

3. The mod \( p \) Lannes-Zarati homomorphism

In the section, we review some main results about the mod \( p \) Lannes-Zarati homomorphism in [6]. In addition, we also construct an additive basis of the Singer construction as well as an additive basis of the dual of the Singer construction, these results are new. By the results, we get a chain-level representation of the mod \( p \) Lannes-Zarati homomorphism in term of the lambda algebra, this is one of main tool for section 5.

Recall that \( \mathcal{M} \) is the category of \( A \)-modules of finite type and \( \mathcal{U} \) is the full subcategory of \( \mathcal{M} \) consisting of all unstable \( A \)-modules.

The destabilization functor \( \mathcal{D} : \mathcal{M} \to \mathcal{U} \) is the left adjoint to the inclusion \( \mathcal{U} \hookrightarrow \mathcal{M} \).
It can be described more explicitly as follows:
\[
\mathcal{D}(M) := M/EM,
\]
where \( EM := \text{Span}_{\mathbb{F}_p} \{ \beta^e P^i x : \epsilon + 2i > \deg(x), x \in M \} \). That \( EM \) is an \( A \)-submodule of \( M \) is a consequence of the Adem relations.

For any \( A \)-module \( M \), there exists a natural \( A \)-homomorphism \( \mathcal{D}(M) \rightarrow \mathcal{F}_p \otimes_A M \) (see [6, section 2.1] for a detail description). This in turns induces maps between corresponding derived functors:
\[
i_s^M : \mathcal{D}_s(M) \rightarrow \text{Tor}_s^A(\mathbb{F}_p, M).
\]

On the other hand, for an unstable \( A \)-module \( M \), the Singer construction \( \mathcal{R}_s \) (see the definition below) provides a functorial \( A \)-submodule \( \mathcal{R}_s M \) of \( P_s \otimes M \). Lannes and Zarati [23] for \( p = 2 \) and Zarati [35] for \( p \) odd showed that there is an isomorphism
\[
\alpha_s(\Sigma M) : \mathcal{D}_s(\Sigma^{1-s} M) \rightarrow \Sigma \mathcal{R}_s M \subset \mathcal{D}_0(P_s \otimes \Sigma M) \cong P_s \otimes SM.
\]

By the results, for any unstable \( A \)-module \( M \) and for \( s \geq 0 \), there exists a homomorphism \( (\mathcal{R}_s^M)^\# \) such that the following diagram commutes:
\[ \mathcal{D}_s(\Sigma^{1-s}M) \xrightarrow{\alpha_s(\Sigma M)} \Sigma \mathcal{R}_s M \xrightarrow{} \Sigma P_s \otimes M \]
\[ \mathcal{I} \mathcal{R}_s \mathcal{M} \]
\[ \text{Tor}_s^4(\mathbb{F}_p, \Sigma^{1-s}M). \]

Because the Steenrod algebra \( A \) acts trivially on the target, \( (\varphi_s^M)^\# \) factors through \( \mathbb{F}_p \otimes_A \Sigma \mathcal{R}_s M \). Therefore, after desuspending, we obtain the dual of the mod \( p \) Lannes-Zarati homomorphism
\[ (\varphi_s^M)^\#: (\mathbb{F}_p \otimes_A \mathcal{R}_s M)^t \rightarrow \text{Tor}_s^4(\mathbb{F}_p, \Sigma^{-s}M) \cong \text{Tor}_s^4(\mathbb{F}_p, M). \]

The linear dual
\[ \varphi_s^M: \text{Ext}_A^{s,s+t}(M, \mathbb{F}_p) \rightarrow (\mathbb{F}_p \otimes_A \mathcal{R}_s M)^t, \]
is called the mod \( p \) Lannes-Zarati homomorphism.

In order to construct a chain-level representation of \( \varphi_s^M \), we need to recall some main points of the Singer construction.

Let \( \Sigma_{p^s} \) be the symmetric group acting on the underlying set of the group \( E_s := (\mathbb{Z}/p)^s \) and \( r_s: E_s \hookrightarrow \Sigma_{p^s} \) be the inclusion via the action by translations. Denote by \( \mathbb{Z}/p \) the trivial \( \Sigma_{p^s} \)-module of \( \mathbb{Z}/p \) and \( \mathbb{Z}/p \) the module \( \mathbb{Z}/p \) with \( \Sigma_{p^s} \) acting via the signature. Put
\[ \mathcal{B}[s] := \text{im} \left( H^*(B\Sigma_{p^s}; \mathbb{Z}/p) \xrightarrow{r_s^*} H^*(BE_s; \mathbb{Z}/p) \right); \]
\[ \mathcal{B}[s] := \text{im} \left( H^*(B\Sigma_{p^s}; \mathbb{Z}/p) \xrightarrow{r_s^*} H^*(BE_s; r_s^*\mathbb{Z}/p) \right). \]

Let \( D[s] := \mathbb{F}_p[y_1, \ldots, y_s]^{GL_s} \). The structure of \( \mathcal{B}[s] \) and \( \mathcal{B}[s] \) are given by the following proposition.

**Proposition 3.1 (Mùi [29], Zarati [35]).**

1. \( \mathcal{B}[s] \) is a free \( D[s] \)-module generated by
\[ \left\{ 1, M_{s;i_1, \ldots, i_k} L_s^{p-2+(p-1)[\frac{i_k-1}{2}]} \right\} \]
for \( 0 \leq i_1 < \cdots < i_k \leq s-1 \).
2. \( \mathcal{B}[s] \) is a free \( D[s] \)-module generated by
\[ \left\{ L_s^{\frac{p-1}{2}}, M_{s;i_1, \ldots, i_k} L_s^{\frac{p-3}{2}+(p-1)[\frac{i_k}{2}]} \right\} \]
for \( 0 \leq i_1 < \cdots < i_k \leq s-1 \).
Here denote \([x]\) the largest integer number that is not greater than \(x\).

For any unstable \(A\)-module \(M\), the (unstable) total power \(St_s(x_1, y_1, \ldots, x_s, y_s; m)\), for \(m \in M\), is defined as follows (see Zarati [35])

\[
St_s(x_1, y_1, \ldots, x_s, y_s; m) := (-1)^s \left[ \frac{[m]}{[s]} \right] L_s^{p-1|m|} S_s(m) \in P_s \otimes M.
\]

For convenience, we put \(St_s(m) := St_s(x_1, y_1, \ldots, x_s, y_s; m)\) and \(St_s(M) := \{St_s(m) : m \in M\}\).

Given an unstable \(A\)-module \(M\), the module \(\mathcal{R}_sM\) is defined by (see Zarati [35])

\[
\mathcal{R}_sM = \mathcal{B}[s] \cdot St_s(M^+) \oplus \mathcal{B}[s] \cdot St_s(M^-),
\]

where \(M^+\) (resp. \(M^-\)) is the subspace consisting of all elements of even degree (resp. odd degree) of \(M\). Then, for each \(s \geq 0\), the assignment \(M \mapsto \mathcal{R}_sM\) provides an exact functor from \(U\) to itself.

A chain-level representation of \((\varphi_s^M)^\#\) in the Singer-Hmg-Sum chain complex is given in [6] by the following theorem.

**Theorem 3.2 (Chön-Nhut [6, Theorem 3.1]).** For any unstable \(A\)-module \(M\), the inclusion \((\varphi_s^M)^\# : \mathcal{R}_sM \rightarrow (\Gamma^+M)_s\) given by

\[
\gamma \mapsto (-1)^{(s-2)(s-1)} \gamma
\]

is a chain-level representation of the dual of the mod \(p\) Lannes-Zarati homomorphism \((\varphi_s^M)^\#\).

In order to construct a chain-level representation of \(\varphi_s^M\), we need to investigate more carefully the structure of the dual of the Singer construction. It should be noted that, in [19], Kuhn constructed the dual of the Singer construction by geometrical approach. Here we use the algebraic one.

**Proposition 3.3.** Given \(M\) an unstable \(A\)-module, \(\mathcal{R}_sM\) has an \(\mathbb{F}_p\)-basis given by

\[
\mathcal{C} := \left\{ R^\sigma_{s,0} q^j_{s,0} \cdots R^\sigma_{s,s-1} q^j_{s,s-1} S_s(m) \right\}
\]

for \(m\) running through a homogeneous basis of \(M\), \(\sigma_k \in \{0, 1\}, j_1 \in \mathbb{Z}, j_k \geq 0, 2 \leq k \leq s\) and \(2j_1 + \sigma_1 + \cdots + \sigma_s \geq |m|\).

**Proof.** First, using [29, Theorem 4.17] (see also [6, Theorem 2.2]), it is easy to show that \(\mathcal{C}\) is contained in \(\mathcal{R}_sM\).

From the definition, for any \(x \in \mathcal{R}_sM\), it can be written by \(x = \lambda St_s(m)\) for some \(m\), where \(\lambda \in \mathcal{B}[s]\) if \(|m| = 2n\) and \(\lambda \in \mathcal{B}[s]\) if \(|m| = 2n + 1\).
If \(|m| = 2n\), then \(x\) can be written by \(x = (-1)^{sn} \lambda q^s_n S_s(m)\), where \(\lambda \in \mathcal{A}[s]\).

On the other hand, by result of Chôn [5, Proposition 3.4], \(\mathcal{A}[s]\) has an \(\mathbb{F}_p\)-basis consisting of all elements

\[
R^{s_1}_{\delta_1}g_{j_1}^{j_1} \cdots R^{s_j}_{\delta_j}g_{j_j}^{j_j},
\]

for \(\sigma_k \in \{0, 1\}, j_1, \cdots, j_s \in \mathbb{Z}, j_k \geq 0, 2 \leq k \leq s\) and \(2j_1 + \sigma_1 + \cdots + \sigma_s \geq 0\). Therefore, \(x\) can be written as a linear combination of elements of \(\mathcal{G}\).

Otherwise, if \(|m| = 2n + 1\), then \(x\) can be written by \(x = (-1)^{sn} \lambda L_s^{p-1/2}(2n+1) S_s(m)\), where \(\lambda\) is a sum of \(f_i g_i\) with \(f_i \in D[s]\) and \(g_i = L_s^{p-1} \) or \(g_i = M_{s;i_1 \cdots i_k} L_s^{p-1/2 + (p-1)[\frac{k}{2}] - 1}\).

If \(g_i = L_s^{p-1}\), then \(f_i g_i L_s^{p-1/2}(2n+1) = f_i q^{n+1}_s\). Therefore, \(f_i g_i S_s(m)\) can be expressed as a linear combination of the needed form.

If \(g_i = M_{s;i_1 \cdots i_k} L_s^{p-1/2 + (p-1)[\frac{k}{2}] - 1}\), then

\[
f_i g_i L_s^{p-1/2}(2n+1) = f_i R_{s;i_1 \cdots i_k} q^{n+1}_s.
\]

Hence, by [5, Proposition 3.7], \(f_i g_i S_s(m)\) can be also expressed as a linear combination of the elements in \(\mathcal{G}\).

Thus, \(\mathcal{G}\) is a set of generators of \(\mathcal{A}_s M\) as an \(\mathbb{F}_p\)-vector space.

Also by the Chôn’s result [5, Proposition 3.7], it is easy to verify that the set \(\mathcal{G}\) is linear independent.

The proof is complete. \(\square\)

Hence, give an unstable \(A\)-module \(M\), the Singer functor \(\mathcal{A}_s M\) is an \(A\)-submodule of \(\mathcal{A}[s] \cdot S_s(M)\). Therefore, in dual, \(\mathcal{A}_s M\) is isomorphic to a quotient of \(R_s \otimes M^\#\). In order to define the structure of \(\mathcal{A}_s M\), we need the following results.

Fix a non-negative integer \(s\), for any non-negative integer \(n\), let \(\mathcal{J}_n\) be the set of all admissible string \(I = (\epsilon_1, i_1, \ldots, \epsilon_s, i_s)\) satisfying \(\text{exc}(I) \geq n\); and let \(\mathcal{J}_n\) be the set of all string \(J = (\sigma_1, j_1, \ldots, \sigma_s, j_s)\) satisfying the condition \(\sigma_k \in \{0, 1\}, 1 \leq k \leq s, j_1, \cdots, j_s \in \mathbb{Z}, j_k \geq 0, 2 \leq k \leq s\) and \(2j_1 + \sigma_1 + \cdots + \sigma_s \geq n\).

Obviously, we obtain that

**Lemma 3.4.** The map \(\phi_n : \mathcal{J}_n \to \mathcal{J}_n\) given by \(\phi_n(\sigma_1, j_1, \ldots, \sigma_s, j_s) = (\epsilon_1, i_1, \ldots, \epsilon_s, i_s)\) where \(\epsilon_k = \sigma_k\) and

\[
i_k = p^{s-k}(j_1 + \sigma_1 + \cdots + j_k + \sigma_k) + \sum_{t=0}^{s-k-1} (p^{s-k} - p^t)(j_{k+t+1} + \sigma_{k+t+1}),
\]

for \(1 \leq k \leq s\), is a bijection.

**Proof.** By Hung-Sum [16, Lemma 3.19], it is sufficient to prove the excess of \(I\) is not smaller than \(n\). This is immediate since \(\text{exc}(I) = 2j + \sigma_1 + \cdots + \sigma_s\). \(\square\)
Given an unstable $A$-module $M$, for a fixed homogeneous element $m \in M$, we order the set of monomials
\[
\{u_1^{\epsilon_1} v_1^{(p-1)i_1-\epsilon_1} \cdots u_s^{\epsilon_s} v_s^{(p-1)i_s-\epsilon_s} S_s(m) : (\epsilon_1, i_1, \ldots, \epsilon_s, i_s) \in \mathcal{J}_{m}\}
\]
by agreeing that
\[
u_1^{\epsilon_1} v_1^{(p-1)i_1-\epsilon_1} \cdots u_s^{\epsilon_s} v_s^{(p-1)i_s-\epsilon_s} S_s(m) > \nu_1^{\epsilon_1'} v_1^{(p-1)i_1'-\epsilon_1'} \cdots u_s^{\epsilon_s'} v_s^{(p-1)i_s'-\epsilon_s'} S_s(m)
\]
if and only if $(\epsilon_1, i_1, \ldots, \epsilon_s, i_s) > (\epsilon_1', i_1', \ldots, \epsilon_s', i_s')$ in the lexicographical order from the left.

Based upon the result of Hung-Sum [16, Lemma 3.11], we get that

**Lemma 3.5.** Given an unstable $A$-module $M$, for any homogeneous element $m \in M$, let $(\sigma_1, j_1, \ldots, \sigma_s, j_s) \in \mathcal{J}_{m}$ and $(\epsilon_1, i_1, \ldots, \epsilon_s, i_s) = \phi_{|m|}(\sigma_1, j_1, \ldots, \sigma_s, j_s)$. Then,

\[
R_{s:0}^{\sigma_1 j_1} \cdots R_{s:s-1}^{\sigma_s j_s} q_{s-1}^{i_s} S_s(m) = \nu_1^{\epsilon_1} v_1^{(p-1)i_1-\epsilon_1} \cdots u_s^{\epsilon_s} v_s^{(p-1)i_s-\epsilon_s} S_s(m) + \text{smaller monomials.}
\]

We identify $R_sM$ with its image in $R_s^\# \otimes M$ via the map $\nu_s^M$ (see subsection 2.2 for the definition of $\nu_s^M$). By the identification, the dual $(R_sM)^\#$ can be considered as a quotient right $A$-module of $R_s \otimes M^\#$.

**Proposition 3.6.** Given an unstable $A$-module $M$, the set of all elements $Q^I \otimes \ell = \beta^{\epsilon_s} Q_1^{i_1} \cdots \beta^{\epsilon_s} Q_s^{i_s} \otimes \ell$, for $I \in \mathcal{J}_{|\ell|}$ and $\ell$ running through a homogeneous basis of $M^\#$, represents an $\mathbb{F}_p$-basis of $(R_sM)^\#$.

**Proof.** Let $B$ be a homogeneous basis of $M$ and $B^\#$ be the dual basis of $B$. For any $\ell \in B^\#$ such that $\ell = m^\ast$, $m \in B$, by Lemma 3.5,

\[
\left\langle R_{s:0}^{\sigma_1 j_1} \cdots R_{s:s-1}^{\sigma_s j_s} q_{s-1}^{i_s} S_s(m), \beta^{\epsilon_s} Q_1^{i_1} \cdots \beta^{\epsilon_s} Q_s^{i_s} \otimes \ell \right\rangle = \begin{cases} 
\pm 1, & \phi_{|m|}(\sigma_1, j_1, \ldots, \sigma_s, j_s) = (\epsilon_1, i_1, \ldots, \epsilon_s, i_s); \\
0, & \phi_{|m|}(\sigma_1, j_1, \ldots, \sigma_s, j_s) < (\epsilon_1, i_1, \ldots, \epsilon_s, i_s).
\end{cases}
\]

Therefore, the set of all elements $Q^I \otimes \ell = \beta^{\epsilon_s} Q_1^{i_1} \cdots \beta^{\epsilon_s} Q_s^{i_s} \otimes \ell$ for $I \in \mathcal{J}_{|\ell|}$ represents a linear independent set of $(R_sM)^\#$. It implies that the set of all elements $Q^I \otimes \ell$ for $\ell \in B^\#$ and $I \in \mathcal{J}_{|\ell|}$ represents a linear independent set.

From Lemma 3.4, the number of elements of this set is equal to the dimension of $R_sM$.

The proof is complete. ∎
Observe that the elements $Q^I \otimes \ell \in R_s \otimes M^\#$ of $\text{exc}(I) < |\ell|$ represents a trivial element in $(R_s M)^\#$. Indeed, from [6, Corollary 2.8], any homogeneous element $\gamma \in R_s M$ can be expressed in a linear combination of monomials $u_1^{\epsilon_1} v_1^{(p-1)i_1 - \epsilon_1} \cdots u_s^{\epsilon_s} v_s^{(p-1)i_s - \epsilon_s} S_s(m)$ for $(\epsilon_1, i_1, \ldots, \epsilon_s, i_s)$ admissible and $\text{exc}(\epsilon_1, i_1, \ldots, \epsilon_s, i_s) \geq |m|$. Therefore, $\langle \gamma, Q^I \otimes \ell \rangle = 0$ for all $\gamma \in R_s M$.

Taking dual Theorem 3.2, we have the following result.

**Proposition 3.7.** For any unstable $A$-module $M$, the projection

$$\tilde{\varphi}_s^M : \Lambda_s \otimes M^\# \longrightarrow (R_s M)^\#$$

given by

$$\lambda_I \otimes \ell \longrightarrow (-1)^{(s-1)(s-2)} [Q^I \otimes \ell]$$

is a chain-level representation of the mod $p$ Lannes-Zarati homomorphism $\varphi_s^M$.

This proposition is a generalization of [6, Proposition 3.4] to the case of an arbitrary unstable $A$-module.

4. The power operations

This section is devoted to develop the power operations, these are useful tools to study the behavior of the Lannes-Zarati homomorphism in the next section.

From Liulevicius [25], [26] and May [27], there exists a power operation

$$\mathcal{P}^0 : \text{Ext}_{A}^{s,s+t}(F_p, F_p) \longrightarrow \text{Ext}_{A}^{s,p(s+t)}(F_p, F_p).$$

Its chain-level representation in $\Lambda$ is given by

$$\tilde{\mathcal{P}}^0 (\lambda_{i-1}^{\epsilon_1} \cdots \lambda_{i-1}^{\epsilon_s}) = \begin{cases} \lambda_{pi_1-1}^{\epsilon_1} \cdots \lambda_{pi_s-1}^{\epsilon_s}, & \epsilon_1 = \cdots = \epsilon_s = 1, \\ 0, & \text{otherwise}. \end{cases}$$

**Lemma 4.1.** The operation $\tilde{\mathcal{P}}^0$ does not make increasing the excess of elements in $\Lambda$, therefore, the exists an operation, which is also denoted by $\tilde{\mathcal{P}}^0$, acting on the Dyer-Lashof algebra $R$ given by

$$\tilde{\mathcal{P}}^0 (\beta_{\epsilon_1} Q^{i_1} \cdots \beta_{\epsilon_s} Q^{i_s}) = \begin{cases} \beta_{\epsilon_1} Q^{pi_1} \cdots \beta_{\epsilon_s} Q^{pi_s}, & \epsilon_1 = \cdots = \epsilon_s = 1, \\ 0, & \text{otherwise}. \end{cases}$$

**Proof.** It is sufficient to show that if $\lambda_{i-1}^{1} \cdots \lambda_{i-1}^{1}$ has negative excess then so does $\lambda_{pi_1-1}^{1} \cdots \lambda_{pi_s-1}^{1}$ for $s \geq 2$. 


By inspection, one gets
\[
\text{exc}(\lambda_{p_{i_1}-1}^1 \cdots \lambda_{p_{i_s}-1}^1) = 2p_{i_1} - \sum_{k=2}^{s} 2p(p-1)i_k + (s-2)
\]
\[
= p \cdot \text{exc}(\lambda_{i_1-1}^1 \cdots \lambda_{i_s-1}^1) - (p-1)(s-2).
\]
Therefore, if \(\text{exc}(\lambda_{i_1-1}^1 \cdots \lambda_{i_s-1}^1) < 0\) then \(\text{exc}(\lambda_{p_{i_1}-1}^1 \cdots \lambda_{p_{i_s}-1}^1) < 0\). □

**Lemma 4.2.** The operation \(\tilde{\mathcal{P}}^0\) is compatible with the action of \(A\). In particular,
\[
\tilde{\mathcal{P}}^0((\beta^{\epsilon_1}Q^{i_1} \cdots \beta^{\epsilon_s}Q^{i_s})P^k) = (\tilde{\mathcal{P}}^0(\beta^{\epsilon_1}Q^{i_1} \cdots \beta^{\epsilon_s}Q^{i_s}))P^{pk}.
\] (4.1)

**Proof.** It is sufficient to show the assertion of lemma in the case \(\epsilon_1 = \cdots = \epsilon_s = 1\).

We will prove by induction on \(s\).

For \(s = 1\), it is easy to see that
\[
\tilde{\mathcal{P}}^0((\beta Q^i)P^k) = \tilde{\mathcal{P}}^0((-1)^{k}\left((p-1)(i-k)-1\right)\beta Q^{i-k})
\]
\[
= (-1)^{k}\left((p-1)(i-k)-1\right)\beta Q^{pi-pk},
\]
and
\[
(\tilde{\mathcal{P}}^0(\beta Q^i))P^{pk} = \beta Q^{pi}P^{pk} = (-1)^{pk}\left((p-1)(pi-pk)-1\right)\beta Q^{pi-pk}.
\]

Since \((-1)^{pk}\left((p-1)(pi-pk)-1\right) \equiv (-1)^{k}\left((p-1)(i-k)-1\right) \mod p\), we have the assertion.

For \(s > 1\), by the inductive hypothesis,
\[
\tilde{\mathcal{P}}^0((\beta Q^{i_1} \cdots \beta Q^{i_s})P^k)
\]
\[
= \tilde{\mathcal{P}}^0\left(\sum_{t}(-1)^{k+t}\left((p-1)(i_1-k)-1\right)\beta Q^{i_1-k+t}(\beta Q^{i_2} \cdots \beta Q^{i_s})P^t\right)
\]
\[
+ \tilde{\mathcal{P}}^0\left(\sum_{t}(-1)^{k+t}\left((p-1)(i_1-k)-1\right)Q^{i_1-k+t}(\beta Q^{i_2} \cdots \beta Q^{i_s})\beta P^t\right)
\]
\[
= \sum_{t}(-1)^{k+t}\left((p-1)(i_1-k)-1\right)\beta Q^{p(i_1-k+t)}(\beta Q^{pi_2} \cdots \beta Q^{pi_s})P^{pt}.
\]

On the other hand,
\[
(\tilde{\mathcal{P}}^0(\beta Q^{i_1} \cdots \beta Q^{i_s}))P^{pk} = (\beta Q^{pi_1} \cdots \beta Q^{pi_s})P^{pk}
\]
\[
\begin{align*}
&= \sum_j (-1)^{pk+j} \left( (p-1)(pi_1 - pk) - 1 \right) \beta Q^{pi_1 - pk + j} (\beta Q^{pi_2} \ldots \beta Q^{pi_s}) P^j \\
&+ \sum_j (-1)^{pk+j} \left( (p-1)(pi_1 - pk) - 1 \right) Q^{pi_1 - pk + j} (\beta Q^{pi_2} \ldots \beta Q^{pi_s}) \beta P^j \\
&= \sum_j (-1)^{k+j} \left( (p-1)(i_1 - k) - 1 \right) \beta Q^{pi_1 - pk + j} (\beta Q^{pi_2} \ldots \beta Q^{pi_s}) P^j.
\end{align*}
\]

If \( j \) is not divisible by \( p \) then \( (p-1)(pi_2 - j) - 1 \equiv j - 1 \mod p \), while \( j - p\ell \equiv j \mod p. \) Therefore,

\[
(\beta Q^{pi_2} \ldots \beta Q^{pi_s}) P^j \\
= \sum_j (-1)^{j+\ell} \left( (p-1)(pi_2 - j) - 1 \right) \beta Q^{pi_2 - j + \ell} (\beta Q^{pi_3} \ldots \beta Q^{pi_s}) P^\ell \\
+ \sum_j (-1)^{j+\ell} \left( (p-1)(pi_2 - j) - 1 \right) Q^{pi_2 - j + \ell} (\beta Q^{pi_3} \ldots \beta Q^{pi_s}) \beta P^j \\
= \sum_j (-1)^{j+\ell} \left( (p-1)(pi_2 - j) - 1 \right) \beta Q^{pi_2 - j + \ell} (\beta Q^{pi_3} \ldots \beta Q^{pi_s}) P^\ell = 0.
\]

Thus,

\[
(\overline{P}^0 (\beta Q^{i_1} \ldots \beta Q^{i_s})) P^{pk} \\
= \sum_j (-1)^{k+\ell} \left( (p-1)(i_1 - k) - 1 \right) \beta Q^{p(i_1 - k + \ell)} (\beta Q^{pi_2} \ldots \beta Q^{pi_s}) P^{pt}.
\]

The lemma is proved. \( \Box \)

It is easy to see that if \( (Q^I)\beta = 0 \) then \( (\overline{P}^0 (Q^I))\beta = 0 \) for \( Q^I \in R. \) This fact together with Lemma 4.2 show that the operation \( \overline{P}^0 \) induces a power operation on \((\mathbb{F}_p \otimes \mathcal{A}_p \mathbb{F}_p)^\#;\)

which is also denoted by \( P^0. \)

Let \( H := \hat{H}_*(BZ/p) = P^\#. \) It is known that \( H \) has an \( \mathbb{F}_p \)-basis consisting of all elements

\[ \{a^\epsilon b^{[t]} : \epsilon = 0, 1, t \geq 0, t + \epsilon > 0\}, \]

where \( a b^{[t]} \) is the dual of \( xy^t \) in \( P. \) In addition, \( H \) admits a right \( A \)-module structure, with \( A \)-action given by

\[
b^{[t]} \beta^p P^k = \binom{t - (p-1)k - \epsilon}{k} a^\epsilon b^{[t-(p-1)k-\epsilon]}, \tag{4.2}
\]
and $\theta$ acting trivially on $a$ for $\theta \in \tilde{A}$, where $\tilde{A}$ is the augmentation ideal of $A$.

Recall that $\Lambda$ denote the lambda algebra. Then $\Lambda \otimes H$ is a suitable complex to compute $\text{Ext}^s_{\Lambda}(P, F_p)$. The differential of $\Lambda \otimes H$ is given by, for $\lambda \in \Lambda$,

$$d(\lambda \otimes a^i b^j) = d(\lambda) \otimes a^i b^j + \sum_{i, \delta \geq 0} (-1)^{\deg \lambda + (1 - \delta)}e$$

$$\times \left( t - (p - 1)i - 1 + \delta \right) \lambda^{i-1} \otimes a^{\epsilon + 1 - \delta} b^{[i-(p-1)i-1+\delta]}. \tag{4.3}$$

Define the operation $\psi : H_t :\rightarrow H_{p(t+1)-1}$ given by

$$\psi(a^i b^j) = \begin{cases} ab^[p(t+1)-1], & \epsilon = 1, \\ 0, & \epsilon = 0, \end{cases}$$

which is the dual of the so-called Kameko operation [18] (see also Minami [28] for an odd prime $p$). Since, $\psi(ab^i P^i) = (\psi(ab^i))P^i$, there exists an operation, which is also denoted by $\tilde{P}^0$, acting on the chain complex $\Lambda \otimes H$, given as follows

$$\lambda_{t_1 \cdots t_s}^{\epsilon_1} \cdots \lambda_{s_i \cdots s_j}^{\epsilon_s} \otimes a^i b^j \rightarrow \begin{cases} \lambda_{p_i \cdots p_j}^{\epsilon_1} \cdots \lambda_{p_i \cdots p_j}^{\epsilon_s} \otimes \epsilon_1 a^{[p(t+1)-1]}, & \epsilon_1 \cdots \epsilon_s = \epsilon = 1, \\ 0, & \text{otherwise}. \end{cases}$$

The latter commutes with the differential given in (4.3). Therefore, it induces an operation also denoted by $\tilde{P}^0$ acting on $\text{Ext}^s_{\Lambda} P, F_p)$.

Similarly, the power operation $\tilde{P}^0$ acting on $\Lambda \otimes H$ also induces a power operation on $(F_p \otimes_A \mathcal{R}_s P)^\#$ which is also denoted by $\tilde{P}^0$.

**Proposition 4.3.** The following diagram is commutative

$$\begin{array}{ccc}
\text{Ext}^s_{\Lambda} (M, F_p) & \xrightarrow{\varphi^M_s} & \text{Ext}^s_{\Lambda} (M, F_p) \\
\downarrow & & \downarrow \varphi^M_s \\
(F_p \otimes_A \mathcal{R}_s M)^\#_t & \xrightarrow{\varphi^M_s} & (F_p \otimes_A \mathcal{R}_s M)^\#_{(p(s+1)-1)^t}
\end{array}$$

for $M = F_p$ and $M = P$.

**Proof.** We prove the proposition for the case $M = P$, other case is proved similarly.

It is sufficient to prove $\tilde{P}^0_s$ and $\tilde{P}^P_s$ commute with each other. Moreover, by the definition of $\tilde{P}^0$ s, we need only to verify for the case $\lambda_{t_1 \cdots t_s}^{\epsilon_1} \cdots \lambda_{s_i \cdots s_j}^{\epsilon_s} \otimes ab^i$.

From Proposition 3.7, it is easy to check that

$$\tilde{\varphi}^P_s (\tilde{P}^0 (\lambda_{t_1 \cdots t_s}^{\epsilon_1} \cdots \lambda_{s_i \cdots s_j}^{\epsilon_s} \otimes ab^i)) = \tilde{\varphi}^P_s (\lambda_{p_i \cdots p_j}^{\epsilon_1} \cdots \lambda_{p_i \cdots p_j}^{\epsilon_s} \otimes ab^i(p(t+1)-1))$$
\[
\begin{align*}
&= (-1)^{(s-1)(s-2)/2} [\beta Q^{p_1} \cdots \beta Q^{p_s} \otimes a_b]^{p(t+1)-1]} \\
&= \tilde{P}^0(\tilde{\varphi}_s^P)(\lambda_{i-1}^1 \cdots \lambda_{s-1}^1 \otimes ab[t]).
\end{align*}
\]

The proof is complete. \(\square\)

5. The behavior of the mod \(p\) Lannes-Zarati homomorphism

In this section, we use the chain-level representation map of the \(\varphi_s^M\) constructed in the previous section to investigate its behavior.

5.1. The behavior of \(\varphi_3^F\)

**Theorem 5.1.** The third Lannes-Zarati homomorphism

\[
\varphi_3^F : \text{Ext}_A^{3,3+t}(F_p, F_p) \longrightarrow (F_p \otimes_A \mathcal{R}_3 F_p)_t^\infty
\]

is a monomorphism for \(t = 0\) and vanishing for all \(t > 0\).

From Proposition 3.7, we get the following lemma. It allows us to reduce the computation in the proof of the above theorem.

**Lemma 5.2.** If \(\lambda_I \in \Lambda_s\) and \(\lambda_J \in \Lambda_t\) such that \(\varphi_s^F(\lambda_I) = 0\) or \(\varphi_t^F(\lambda_J) = 0\) then \(\tilde{\varphi}_s^F(\lambda_I \lambda_J) = 0\).

**Proof of Theorem 5.1.** By the results of Liulevicius [26] and Aikawa [1], \(\text{Ext}_A^{3,3+t}(F_p, F_p)\) is spanned by following elements (for convenience we will write \(\text{Ext}_A^{s,s+t}\) for \(\text{Ext}_A^{s,s+t}(F_p, F_p)\))

1. \(h_i h_j h_k = [\lambda_{p_i-1}^0 \lambda_{p_j-1}^1 \lambda_{p_k-1}^1] \in \text{Ext}_A^{3,2(p-1)(p_i+p_j)+k}, 0 \leq i \leq j \leq k \leq 4;\)
2. \(\alpha_0 h_i h_j = [\lambda_{p_i-1}^1 \lambda_{p_j-1}^1 \lambda_{p_k-1}^0] \in \text{Ext}_A^{3,2(p-1)(p_i+p_j)+1}, 1 \leq i \leq j \leq 2;\)
3. \(\alpha_0^2 h_i = [\lambda_{p_i-1}^1 (\lambda_{p_j+1}^0)^2] \in \text{Ext}_A^{3,2(p-1)p_i+2}, i \geq 1;\)
4. \(\alpha_0^3 = [L_i \lambda_{i-1}^0] \in \text{Ext}_A^{3,3};\)
5. \(\tilde{\lambda}_i h_j = [L_i \lambda_{p_j}^1] \in \text{Ext}_A^{3,2(p-1)(p_i+1)+p_j}, i, j \geq 0, j \neq i+2;\)
6. \(\tilde{\lambda}_i \alpha_0 = [L_i \lambda_{i-1}^0]; i \geq 0;\)
7. \(h_{i;1} h_j = [\lambda_{p_i+1-1}^0 \lambda_{p_j+1}^1 \lambda_{p_i-1}^1] \in \text{Ext}_A^{3,2(p-1)(p_i+2+p_j)+p_j}, i, j \geq 0, j \neq i+2, i-1;\)
8. \(h_{i;1} \alpha_0 = [\lambda_{p_i+1}^1 \lambda_{p_j+1}^0 \lambda_{p_i-1}^1] \in \text{Ext}_A^{3,2(p-1)(p_i+2+p_j)+1}, i \geq 1;\)
9. \(h_{i;2} h_j = [\lambda_{2p_i+1} \lambda_{p_j-1}^1 \lambda_{2p_i-1}^1] \in \text{Ext}_A^{2(p-1)(p_i+1)+p_i+p_i+1}, i, j \geq 0, j \neq i+2, i \pm 1, i;\)
10. \(h_{i;2} \alpha_0 = [\lambda_{2p_i+1} \lambda_{p_j-1}^1 \lambda_{2p_i-1}^0] \in \text{Ext}_A^{3,2(p-1)(2p_i+1)+p_i+1}, i \geq 1;\)
11. \(h_j \rho = [\lambda_{p_i-1}^1 \lambda_{p_j}^1 \lambda_{p_i-1}^0] \in \text{Ext}_A^{3,2(p-1)(p_i+2)+1}, j \geq 2;\)
12. \(h_{i;3} = (\mathcal{P}^0)^t[\lambda_{3p_i-1}^1 \lambda_{3p_i-1}^1 \lambda_{2p_i-1}^0] \in \text{Ext}_A^{3,2(p-1)(3p_i+2+2p_i+1)+p}, p \neq 3, i \geq 0;\)
13. \( h'_{3,2,1} = [\lambda^1_{3p-1} \lambda^1_{2p-1} \lambda^0_{1}] \in \text{Ext}^3_A(\lambda^2_{p-1}(3p+2), p \neq 3); \)
14. \( h_{i;2,2,1} = (P^0)^i[\lambda^1_{2p^2-1} \lambda^1_{2p-1} \lambda^0_{1}] \in \text{Ext}^3_A(\lambda^2_{p-1}(2p^2+2), p = 3, i \geq 0); \)
15. \( h''_{2,2,1} = [\lambda^1_{2p^2-1} \lambda^1_{2p-1} \lambda^0_{1}] \in \text{Ext}^3_A(\lambda^2_{p-1}(3p+2), p = 3); \)
16. \( h_{i;1,3,1} = (P^0)^i[\lambda^1_{3p-1} \lambda^1_{3p-1} \lambda^0_{1}] \in \text{Ext}^3_A(\lambda^2_{p-1}(2p^2+2), p \neq 3, i \geq 0); \)
17. \( h'_{1,3,1} = [\lambda^1_{p-1} \lambda^1_{2p-1} \lambda^0_{1}] \in \text{Ext}^3_A(\lambda^2_{p-1}(3p+2), p \neq 3); \)
18. \( h_{i;2,1,2} = (P^0)^i[\lambda^1_{2p^2-1} \lambda^1_{2p-1} \lambda^0_{1}] \in \text{Ext}^3_A(\lambda^2_{p-1}(2p^2+2), p \neq 3, i \geq 0); \)
19. \( h_{i;1,2,3} = (P^0)^i[\lambda^1_{3p-1} \lambda^1_{3p-1} \lambda^0_{1}] \in \text{Ext}^3_A(\lambda^2_{p-1}(2p^2+2), p \neq 3, i \geq 0); \)
20. \( g_3 = [\lambda^1_{p-1} \lambda^1_{2p-1} \lambda^0_{1}] \in \text{Ext}^3_A(\lambda^2_{p-1}(3p+2), p \neq 3); \)
21. \( g'_3 = [\lambda^1_{p-1} \lambda^1_{2p-1} \lambda^0_{1}] \in \text{Ext}^3_A(\lambda^2_{p-1}(3p+2), p \neq 3); \)
22. \( f_i = (P^0)^{i-1}[M] \in \text{Ext}^3_A(\lambda^2_{p-1}(3p+2), i \geq 1); \)
23. \( g_i = (P^0)^{i-1}[N] \in \text{Ext}^3_A(\lambda^2_{p-1}(3p+2), i \geq 1); \)

where

\[ L_i = (P^0)^i \left( \sum_{j=1}^{(p-1)} \frac{(-1)^j}{j} \lambda^1_{p-j-1} \lambda^1_{j-1} \right), i \geq 0; \]

\[ M = \sum_{j=1}^{p-1} \frac{(-1)^j}{j} \left( \lambda^1_{p-j-1} \lambda^1_{p-j} - 2 \lambda^1_{p^2-1} \lambda^1_{j-1} \lambda^1_{2p-j-1} \right); \]

\[ N = \sum_{j=1}^{p-1} \frac{(-1)^j}{j} \left( \lambda^1_{p-j-1} \lambda^1_{2p-j} - 2 \lambda^1_{p^2-1} \lambda^1_{j-1} \lambda^1_{2p-j-1} \right). \]

Observe that the elements \( h_i h_j h_k \) (0 \( \leq i \leq j - 2 \leq k - 4 \), \( o_0 h_i h_j \) (1 \( \leq i \leq j - 2 \), \( h_{i;1,2} \), \( h_{i;1,3,1} \), \( h'_{1,3,1} \), \( h_{0;1,2,3} \), and \( f_0 \) are represented by cycles of negative excess. Since the images of the cycles under \( \varphi^F_{3,p} \) are trivial in \( R_3 \), then the images of these elements under \( \varphi^F_{3,p} \) are trivial.

By Proposition 4.3,

\[ \varphi^F_{3,p}(h_{i;1,3,1}) = \varphi^F_{3,p}(h_{0;1,3,1}) = (P^0)^i(\varphi^F_{3,p}(h_{0;1,3,1})) = 0, i \geq 0. \]

By the same argument, we get \( \varphi^F_{3,p}(h_{i;1,2,3}) = 0 \) for \( i \geq 0 \) and \( \varphi^F_{3,p}(f_i) = 0 \) for \( i \geq 1 \).

From the proof of [6, Theorem 4.2], we have that \( \varphi^F_{2,p}(\lambda^1_{2p+1-1} \lambda^1_{p-1}) = 0 \) for \( i \geq 0 \) and \( \varphi^F_{2,p}(\lambda^1_{2p^2-1} \lambda^1_{p^2-1}) = 0 \). Therefore, by Lemma 5.2, we obtain that

- \( \varphi^F_{3,p}(\lambda^1_{2p^2+1-1} \lambda^1_{p^2-1} \lambda^1_{p-1}) = 0; \)
- \( \varphi^F_{3,p}(\lambda^1_{2p^2+1-1} \lambda^1_{p^2-1} \lambda^1_{p^2-1}) = 0; \)
\[
\begin{align*}
\tilde{\varphi}_3^p (\lambda_{p'-1}^1 \lambda_{0-1}^0) &= 0, \\
\tilde{\varphi}_3^p (\lambda_{3p-1}^1 \lambda_{0-1}^0) &= 0, \text{ and} \\
\tilde{\varphi}_3^p (\lambda_{2p^2-1}^1 \lambda_{1-1}^0) &= 0.
\end{align*}
\]

It follows that the images of elements \( h_{i;2,1}h_j, h_{i;2,1}\alpha_0, h_j\rho, h_{3,2,1} \) and \( h'_{2,2,1} \) are trivial under the map \( \varphi_3^p \).

By [6, Theorem 4.2], one gets \( \tilde{\varphi}_2^p (L_i) = \beta Q^{(p-1)p^i} \beta Q^{p^i}. \) Therefore, we obtain

\[
\begin{align*}
\tilde{\varphi}_3^p (L_i \lambda_{p'}) &= -\beta Q^{(p-1)p^i} \beta Q^{p^i}, \\
\tilde{\varphi}_3^p (L_i \lambda_{0-1}^0) &= -Q^{(p-1)p^i} \beta Q^{p^i} Q^0.
\end{align*}
\]

Since the right hand side of the first formula is of negative excess for \( i, j \geq 0 \), it implies that \( \varphi_3^p (\tilde{\lambda}_i h_j) = 0 \).

By the proof of [6, Theorem 4.2], \( \beta Q^i Q^0 = 0 \in R_2 \), so that the right hand side of the second formula is trivial in \( R_3 \). It follows \( \varphi_3^p (\alpha_0 \tilde{\lambda}_i) = 0 \). By the same method, we also get \( \varphi_3^p (\alpha_0^2 h_i) = 0 \).

From above, \( \tilde{\varphi}_2^p (\lambda_{2p-1}^1 \lambda_{0}^0) = 0 \) and \( \tilde{\varphi}_2^p (\lambda_{p-1}^1 \lambda_{1}^0) = 0 \), therefore,

\[
\begin{align*}
\tilde{\varphi}_3^p (\lambda_{3p^2-1}^1 \lambda_{2p-1}^0) &= 0, \\
\tilde{\varphi}_3^p (\lambda_{2p^3-1}^1 \lambda_{2p-1}^0) &= 0, \text{ and} \\
\tilde{\varphi}_3^p (\lambda_{2p^2-1}^1 \lambda_{p-1}^0) &= 0.
\end{align*}
\]

It follows that \( \varphi_3^p (h_{0;3,2,1}) = 0, \varphi_3^p (h_{0;2,2,1}) = 0 \) and \( \varphi_3^p (h_{0;2,1,2}) = 0 \), therefore, using Proposition 4.3, \( \varphi_3^p (h_{i;3,2,1}) = 0, \varphi_3^p (h_{i;2,2,1}) = 0 \) and \( \varphi_3^p (h_{i;2,1,2}) = 0 \) for \( i \geq 0 \).

Applying the Adem relation, in \( R_2 \), one gets \( \beta Q^3 Q^0 = 0 \) and \( \beta Q^6 Q^0 = 0 \), therefore,

\[
\begin{align*}
\tilde{\varphi}_3^p (\lambda_{0}^1 (\lambda_{-1}^0)^2) &= -\beta Q^3 Q^0 Q^0 = 0, \text{ and} \\
\tilde{\varphi}_3^p (\lambda_{1}^1 (\lambda_{0}^0)^2) &= -\beta Q^6 Q^0 Q^0 = 0.
\end{align*}
\]

Hence, \( \varphi_3^p (\phi_3) = 0 \) and \( \varphi_3^p (\phi_3') = 0 \).

By inspection, using Proposition 3.7, one gets

\[
\tilde{\varphi}_3(N) = -\sum_{j=1}^{p-1} \frac{(-1)^j}{j} \left( 2\beta Q^{ip} \beta Q^{2p^2-jp} \beta Q^p + 2\beta Q^{p^2+jp} \beta Q^{p^2-jp} \beta Q^p - \beta Q^{2p^2} \beta Q^j \beta Q^{p-j} \right).
\]

Since two first terms of the right hand side of the formula are of negative excess, then

\[
\tilde{\varphi}_3(N) = \sum_{j=1}^{p-1} \frac{(-1)^j}{j} \beta Q^{2p^2} \beta Q^j \beta Q^{p-j}.
\]
It is easy to verify that \(\beta Q^j \beta Q^{p-j} = 0\) for \(j < p - 1\), therefore,
\[
\bar{\varphi}_3(N) = -\beta Q^{2p^2} \beta Q^{p-1} \beta Q^1.
\]
Applying the Adem relation, we get
\[
\beta Q^{2p^2} \beta Q^{p-1} = \sum_i (-1)^{2p^2+i+1} \frac{(p-1)(i+1-p) - 1}{pi-2p^2 - 1} \beta Q^{2p+p-1-i} \beta Q^i.
\]
Since \(pi \geq 2p^2 + 1\), then \(pi = 2p^2 + pa\) for some \(a \geq 1\). In this case, we get
\[
\text{exc}(\beta Q^{2p+p-1-i} \beta Q^i) = 2(2p^2 + p - 1 - i) - 2(p - 1)i = 4p^2 + 2p - 2 - 2pi = 4p^2 + 2p - 2 - 4p^2 - 2pa = 2p - 2pa - 2 < 0.
\]
Therefore, \(\beta Q^{2p^2} \beta Q^{p-1} = 0\), and then \(\bar{\varphi}_3(N) = 0\). It implies that \(\varphi^F_3(g_1) = 0\) and, hence, \(\varphi^F_3(g_i) = 0\) for \(i \geq 1\).

Finally, it is easy to check that \(\varphi_3(\alpha_3^p) = -Q^0Q^0Q^0 \neq 0 \in R_3\).
The proof is complete. \(\Box\)

5.2. The behavior of \(\varphi_s^P\) for \(s \leq 1\)

First, we need to determine the Ext groups \(\text{Ext}_s^*(P,F_p)\) for \(s \leq 1\). By direct computation on the spectral sequence constructed in Appendix, we get

**Theorem 5.3** (Cf. [9, Theorem 1.1]). The Ext group \(\text{Ext}_s^{1,t}(P,F_p)\) has an \(F_p\)-basis consisting of all elements

1. \(\hat{h}_i := [ab^{(p-1)p^{i-1}}] \in \text{Ext}_s^{0,2(p-1)p^{i-1}}(P,F_p), i \geq 0;\)
2. \(\hat{h}_i(k) := [ab^{kp^{i-1}}] \in \text{Ext}_s^{0,2kp^{i-1}}(P,F_p), i \geq 0, 1 \leq k < p - 1.\)

**Theorem 5.4.** The Ext group \(\text{Ext}_s^{1,1+t}(P,F_p)\) has an \(F_p\)-basis consisting of all elements given by the following list

1. \(\alpha_0\hat{h}_i = \left[\lambda_{-1}ab^{(p-1)p^{i-1}}\right], i \geq 1;\)
2. \(\alpha_0\hat{h}_i(k) = \left[\lambda_{-1}ab^{kp^{i-1}}\right], i \geq 1, 1 \leq k < p - 1;\)
3. \(\tilde{\alpha}(\ell) = \left[\lambda_{-1}ab^{(p+\ell)p^{i-1}} + (\ell + 1)\lambda_{0}ab^{[\ell+1]}\right], 0 \leq \ell < p - 2;\)
4. \(\hat{h}_i\hat{h}_i(1) = \left[\lambda_{p^{i-1}}ab^{[p^{i-1}]}\right], i \geq 0;\)
5. \(\hat{h}_ih_j = \left[\lambda_{p^{i-1}}ab^{(p-1)p^{j-1}}\right], i,j \geq 0, j \neq i, i+1;\)
6. \(\hat{h}_ih_j(k) = \left[\lambda_{p^{i-1}}ab^{kp^{j-1}}\right], i,j \geq 0, j \neq i, i+1, 1 \leq k < p - 1;\)
7. \(\tilde{d}_i(k) = (\mathcal{P}_0)^{i-1}\left(\lambda_{p^{i-1}}ab^{[kp+p-2]}\right), i \geq 1, 1 \leq k \leq p - 1;\)
8. \( \hat{k}_i(k) = (P^0)^i \left( \sum_{j=0}^k \frac{1}{j+1} \lambda_j^{ab(k-j)p+j} \right), i \geq 0, 1 \leq k < p-1; \)

9. \( \hat{p}_i(k) = (P^0)^i \left( \sum_{j=0}^{p-1-k} \frac{1}{j+1} \lambda_j^{ab(p-j-1)p+k+j} \right), i \geq 0, 1 \leq k < p-1. \)

**Proposition 5.5.** The \( \text{Ext}^s_A(\mathbb{F}_p, \mathbb{F}_p) \)-module \( \text{Ext}^s_A(P, \mathbb{F}_p) \), for \( s \leq 1 \), is generated by \( \hat{h}_i \) (i \( \geq 0 \)), \( \hat{h}_i(k) \) (i \( \geq 0, 1 \leq k < p-1 \), \( \lambda \alpha(\ell) \) (0 \( \leq \ell < p-2 \)), \( \lambda \alpha_1(k) \) (i \( \geq 1, 1 \leq k < p-1 \)), \( \hat{k}_i(k) \) (i \( \geq 0, 1 \leq k < p-1 \)) and \( \hat{p}_i(k) \) (i \( \geq 0, 1 \leq k < p-1 \)) subject only to the following relations:

- \( \lambda \lambda_i \lambda_i+1 = 0, i \geq 0; \)
- \( \lambda \lambda_i \lambda_i+1(k) = 0, i \geq 0, 1 \leq k < p-1; \)
- \( \lambda \lambda_i \lambda_i = 0, i \geq 0; \)
- \( \lambda \lambda_i \lambda_i(k) = 0, i \geq 0, 2 \leq k < p-1; \)
- \( \lambda \alpha_0 \lambda_0 = 0; \)
- \( \lambda \alpha_0 \lambda_0(k) = 0, 1 \leq k < p-1. \)

These results will be proved in the Appendix.

The behavior of the Lannes-Zarati homomorphism \( \varphi^P_s \) for \( s \leq 1 \) is given by the following theorems.

**Theorem 5.6.** The Lannes-Zarati homomorphism \( \varphi^P_0 : \text{Ext}^0_A(P, \mathbb{F}_p) \to (\mathbb{F}_p \otimes_A \mathcal{R}_0 P)_t^\# \) is an isomorphism.

**Proof.** It is easy to see that \( (\mathbb{F}_p \otimes_A \mathcal{R}_0 P)_t^\# \) is spanned by \( ab^{kp'-1} \) for \( i \geq 0 \) and \( 1 \leq k \leq p-1 \). Therefore, the assertion of the theorem is followed from Theorem 5.3 and Proposition 3.7. \( \square \)

**Theorem 5.7.** The Lannes-Zarati homomorphism \( \varphi^P_1 : \text{Ext}^1_A(1+p)(P, \mathbb{F}_p) \to (\mathbb{F}_p \otimes_A \mathcal{R}_1 P)_t^\# \) sends

(i) \( \lambda \lambda_i \lambda_i(1) \) to \( [\beta Q^p ab^{p-1-1} \right], i \geq 0; \)

(ii) \( \lambda \lambda_i \lambda_i \) to \( [\beta Q^p ab^{p-1-1} \right] \) for \( 0 \leq j < i; \)

(iii) \( \lambda \lambda_i \lambda_i \) to \( [\beta Q^p ab^{p-1-1} \right] \) for \( 0 \leq j < i, 1 \leq k < p-1; \)

(iv) \( \hat{k}_i(k) \) to \( (P^0)^i \left( [\beta Q^k ab^{k]} \right), i \geq 0, 1 \leq k < p-1; \) and

(v) others to zero.

**Proof.** Using Proposition 3.7, it is easy to verify that the images of the following elements

- \( \lambda \alpha_0 \lambda_i \) for \( i \geq 1; \)
- \( \lambda \alpha_0 \lambda_0(k) \) for \( i \geq 1, 1 \leq k < p-1; \)
- \( \lambda \alpha_0 \lambda_0 \) for \( 0 \leq \ell < p-2; \)
• $h_i h_j$ for $0 \leq i < j - 1$; and  
• $h_i h_j(k)$ for $0 \leq i < j - 1, 1 \leq k < p - 1$

are trivial.

By inspection, using Proposition 3.7, we get

$$\varphi_1^P(\lambda_p^{-1}ab^{[kp+p-2]}) = [\beta Q^p ab^{[kp+p-2]}].$$

Since $2p - (2(kp + p - 2) + 1) < 0$ for all $k \geq 1$, it follows that $\varphi_1^P(\hat{d}_1(k)) = 0$ for all $1 \leq k \leq p - 1$. Using Proposition 4.3, we obtain

$$\varphi_1^P(\hat{d}_1(k)) = (P^0)^{i-1} \varphi_1^P(\hat{d}_1(k)) = 0.$$

By the same argument, since $2(j + 1) - 2((p - 1 - j)p + k + j) + 1) < 0$ for all $0 \leq j \leq p - 1 - r$ and $k \geq 1$, it implies $\varphi_1^P(\hat{p}_0(k)) = 0$. In addition, using Proposition 4.3, we get $\varphi_1^P(\hat{p}_i(k)) = 0$.

Finally, using Proposition 3.7, it is easy verify that, in $(\mathcal{R}_1 P)^\#$, 

• $\varphi_1^P(h_i \hat{h}_1(1)) = [\beta Q^{p^i} ab^{[p^i-1]}] \neq 0$, for $i \geq 0$;  
• $\varphi_1^P(h_i \hat{h}_j) = [\beta Q^{p^i} ab^{[(p-1)p^i-1]}] \neq 0$ for $0 \leq j < i$;  
• $\varphi_1^P(h_i \hat{h}_j(1)) = [\beta Q^{p^i} ab^{[kp^i-1]}] \neq 0$ for $0 \leq j < i$, $1 \leq k < p - 1$; and  
• $\varphi_1^P(\hat{k}_i(k)) = (P^0)^{i} ([\beta Q^{k+1} ab^{[k]}]) \neq 0$, $i \geq 0, 1 \leq k < p - 1$.

The proof is complete. \(\square\)

**Corollary 5.8.** The Lannes-Zarati homomorphism $\varphi_1^P$ is not an epimorphism.

**Proof.** It is easy to see that $[\beta Q^{p-1} b^{[1]} + Q^{p-1} a]$ is non-trivial in $(\mathbb{F}_p \otimes_A \mathcal{R}_1 P)^\#$. It follows that, by Theorem 5.7, $\varphi_1^P$ is not an epimorphism. \(\square\)

### 6. The case $p = 2$

Our framework is also valid for $p = 2$ with suitable modifications.

For $p = 2$, the lambda algebra is generated by $\lambda_i$ of degree $i$ for $i \geq 0$ satisfying the Adem relations:

$$\lambda_i \lambda_j = \sum_t \left( \frac{t - j - 1}{2t - i} \right) \lambda_{i + j - t} \lambda_t, \quad i > 2j.$$ 

Therefore, a monomial $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_s} \in \Lambda_s$ is called admissible if $i_k \leq 2i_{k+1}$ for $1 \leq k < s$. The excess of $\lambda_I$ or $I$ is defined by
exc(λ_I) = exc(I) = i_1 - i_2 - \cdots - i_s.

The Dyer-Lashof algebra is the quotient algebra of Λ by the (two sided) ideal of Λ generated by all monomials of negative excess. We also denote Q^I the image of λ_i under the canonical projection.

Hence, Proposition 3.6 and Proposition 3.7 become respectively as follows.

**Proposition 6.1.** Given an unstable A-module M, the set of all elements Q^I ⊗ ℓ for ℓ running through a homogeneous basis of M^#, I admissible and exc(I) ≥ |ℓ|, represents an F_2-basis of (R_s M)^#.

**Proposition 6.2.** For an unstable A-module M, the projection \( \tilde{\varphi}^M_s : \Lambda_s \otimes M^# \to (R_s M)^# \) given by

\[
\lambda_I \otimes \ell \mapsto [Q^I \otimes \ell]
\]

is a chain-level representation of the mod 2 Lannes-Zarati homomorphism \( \varphi_s^M \).

For \( M = F_2 \) and \( M = \tilde{H}^*(B\mathbb{Z}/2) \), the squaring operation \( \tilde{Sq}^0 \) acting on \( \Lambda \otimes M^# \) induce squaring the operations acting on \( (R_s M)^# \). Moreover, these squaring operations induce squaring operations \( Sq^0 \)'s acting on the domain and the range of the mod 2 Lannes-Zarati homomorphism.

**Proposition 6.3.** The following diagram is commutative

\[
\begin{array}{ccc}
\text{Ext}_A^{s,s+t}(M,F_2) & \xrightarrow{Sq^0} & \text{Ext}_A^{s,2(s+t)}(M,F_2) \\
\varphi_s^M & \downarrow & \varphi_s^M \\
(\mathbb{F}_2 \otimes_A R_s M)^#_t & \xrightarrow{Sq^0} & (\mathbb{F}_2 \otimes_A R_s M)^#_{2t+s},
\end{array}
\]

for \( M = F_2 \) and \( M = \tilde{H}^*(B\mathbb{Z}/2) \).

In the rest of the section, we recover all known results for \( p = 2 \).

**Proposition 6.4** ([23], [14], [12], [13], [15]).

(i) The first Lannes-Zarati homomorphism \( \varphi_{F_2}^1 \) is an isomorphism.
(ii) The second Lannes-Zarati homomorphism \( \varphi_{F_2}^2 \) is an epimorphism.
(iii) The s-th Lannes-Zarati homomorphism \( \varphi_{F_2}^s \) vanishes at all positive stems in \( \text{Ext}_A^s(F_2,F_2) \) for \( 3 \leq s \leq 5 \).
Proof. The statements (i) and (ii) are easily proved by using Proposition 6.2 and the representations of \( h_i \) and \( h_i h_j \) on lambda algebra (see Lin [24] for example).

We only give an illustrated example for our method, the detail proof of (iii) is computed by the same argument.

We prove \( \varphi_5^F(U_i) = 0 \). By Chen [4], the element \( U_i \in \text{Ext}_A^{5,2i+8+2i+3+2i}(F_2, F_2) \) is represented in lambda algebra by the cycle

\[
\tilde{U}_i = (\tilde{S} q^i)^{i}(\lambda_{191}(\lambda_{15}^2 \lambda_{39} + \lambda_{39} \lambda_{15}^2) \lambda_0 + \lambda_{63}^2 \lambda_{47} \lambda_{87} \lambda_0 + \lambda_{127} \lambda_{31} \lambda_{63} \lambda_{39} \lambda_0), i \geq 0.
\]

Since \( \tilde{\varphi}_4^F(\lambda_{31} \lambda_{63} \lambda_{39} \lambda_0) = 0 \), \( \tilde{\varphi}_3^F(\lambda_{47} \lambda_{87} \lambda_0) = 0 \) and \( \tilde{\varphi}_4^F(\lambda_{15}^2 \lambda_{39} \lambda_0) = 0 \), then

\[
\tilde{\varphi}_5^F(\tilde{U}_0) = Q^{191} Q^{39} Q^{15} Q^{15} Q^0.
\]

Applying the Adem relation, we get \( Q^{15} Q^0 = 0 \in R_2 \), it implies that \( \tilde{\varphi}_5^F(\tilde{U}_0) = 0 \).

Hence, \( \varphi_5^F(U_0) = 0 \) and then \( \varphi_5^F(U_i) = \varphi_5^F((S q^0)^i(U_0)) = (S q^0)^i(\varphi_5^F(U_0)) = 0. \)

Proposition 6.5 ([17]).

(i) The zero-th Lannes-Zarati homomorphism \( \varphi_0^P \) is an isomorphism on \( \text{Ext}_A^0(P, F_2) \).

(ii) The first Lannes-Zarati homomorphism \( \varphi_1^P \) is a monomorphism on \( \text{Span}\{h_i h_j : i \geq j\} \) and vanishes on \( \text{Span}\{h_i h_j : i < j\} \).

(iii) The s-th Lannes-Zarati homomorphism \( \varphi_s^P \) vanishes in all positive stems in \( \text{Ext}_A^s(P, F_2) \) for \( 2 \leq s \leq 4 \).

Proof. The statements (i) and (ii) are easily proved by using Proposition 6.2 and the representations of \( \hat{h}_i \) and \( h_i h_j \) on \( \Lambda \otimes P^\# \) (see Lin [24] for example).

Similar to above proposition, here we only give some illustrated examples, the detail proof of (iii) is followed by the same argument.

First, we prove \( \varphi_2^P(\tilde{c}_i) = 0 \) for \( i \geq 0 \). From Lin [24], \( \tilde{c}_i \in \text{Ext}_A^{2,2i+3+2i+1+2i-1}(P, F_2) \) is represented in \( \Lambda \otimes P^\# \) by the cycle

\[
\tilde{c}_i = (\tilde{S} q^0)^i(\lambda_2^2 b[2]), i \geq 0.
\]

Since \( exc(\lambda_2^2) = 0 < 2 \), then, using Proposition 6.2 and Proposition 6.1, it follows that \( \tilde{\varphi}_2^P(\tilde{c}_0) = 0 \), hence, \( \varphi_2^P(c_0) = 0 \). Therefore, \( \varphi_2^P(c_i) = (S q^0)^i(\varphi_2^P(c_0)) = 0 \) for \( i \geq 0 \).

Second, we show that \( \varphi_3^P(\alpha_{16}(i)) = 0 \) for \( i \geq 0 \). From Lin [24], the element \( \alpha_{16}(i) \in \text{Ext}_A^{3,2i+3+2i+2-2}(P, F_2) \) is represented in \( \Lambda \otimes P^\# \) by the cycle

\[
\tilde{\alpha}_{16}(i) = (\tilde{S} q^0)^i(\lambda_7 \lambda_0 b[2] + \lambda_3 \lambda_9 + \lambda_7 \lambda_5 \lambda_3 b[1]), i \geq 0.
\]

By Proposition 6.2 and Proposition 6.1, it is easy to verify that \( \tilde{\varphi}_3^P(\tilde{\alpha}_{16}(0)) = Q^7 Q^7 Q^0 b[2] \).
Since $\text{exc}(Q^7Q^7\ell^0) = 0 < 2$, it implies that $\bar{\varphi}_4^P(\tilde{\alpha}_{16}(0)) = 0$. Hence, $\varphi_3^P(\alpha_{16}(0)) = 0$ and then $\varphi_3^P(\alpha_{16}(i)) = 0$ for $i \geq 0$.

Finally, we verify that $\varphi_4^P(\gamma_{63}(i)) = 0$ for $i \geq 0$. From Lin [24], the element $\gamma_{63}(i) \in \text{Ext}_{A}^{4,2^{i+1}+2^{i+1}+2^{i-1}}(P,\mathbb{F}_2)$ is represented in $\Lambda \otimes P^\#$ by the cycle

$$
\bar{\gamma}_{63}(i) = (\tilde{S}q)^0(i)\lambda_3\lambda_7\lambda_23\lambda_0b^{[2]} + \lambda_{47}(\lambda_2^2\lambda_9 + \lambda_9\lambda_3^2)b^{[1]}
$$

$$
+ (\lambda_7^2\lambda_{121} + \lambda_31\lambda_7\lambda_15\lambda_9 + \lambda_{15}\lambda_{47}\lambda_2^2)b^{[1]}), i \geq 0.
$$

By the same method, it is easy to verify that $\bar{\varphi}_4^P(\gamma_{63}(0)) = Q^{47}Q^9Q^3b^{[1]}$.

Applying the Adem relation, we get that $Q^9Q^3b^{[1]} = 0 \in R_3$, then $\tilde{\varphi}_4^P(\gamma_{63}(0)) = 0$. It implies that $\varphi_3^P(\gamma_{63}(0))$ and then $\varphi_3^P(\gamma_{63}(i)) = 0$ for $i \geq 0$. \qed

**Remark 6.6.** By Proposition 6.1, it is easy to verify that $(\mathbb{F}_2 \otimes A \otimes \mathcal{P}^\#)$ is spanned by

$$\left\{ \left[ Q^{2^{i-1}}b^{[2^{j-1}]} \right] : i \geq j \right\} \cup \left\{ (S^q)^t \left( \left[ Q^{2^{j-1}b^{[1]}} \right] + \left[ Q^{2^{j+1-1}b^{[2]}} \right] \right) : i \geq 0, j \geq 1 \right\}.$$

Therefore, the first Lannes-Zarati homomorphism $\varphi_1^P$ is not an epimorphism.

**Acknowledgment**

The authors would like to thank Lê Minh Hà, Jean Lannes and Geoffrey Powell for many fruitful discussions. The authors are grateful to the referee for helpful comments and corrections. The paper was completed while the first author was visiting the Viet Nam Institute for Advanced Study in Mathematics (VIASM) and the second author was a PhD student at Quy Nhon University (QNU). They thank the VIASM and QNU for support and hospitality.

**Appendix A. The proofs of Theorem 5.3 and Theorem 5.4**

In order to prove these theorems, we need to construct a spectral sequence to compute $\text{Ext}_A^n(P,\mathbb{F}_p)$. Note that the spectral sequence is defined for all $A$-module of finite type. However, here we only treat for $M = P$.

We filter the complex $\Lambda \otimes H$ (see section 4 for the description of this complex) by the filtration $F^n := F^n(\Lambda \otimes H), n \geq 0$ where $F^0(\Lambda \otimes H) := 0$ and for $n > 0$,

$$F^n(\Lambda \otimes H) := \{ \lambda \otimes h \in \Lambda \otimes H : |h| \leq n \}.$$ 

Since (4.3), it is clear that $F^n(\Lambda \otimes H)$ is a subcomplex of $\Lambda \otimes H$ satisfying $\cup_i F^n(\Lambda \otimes H) = \Lambda \otimes H$ and $\cap_i F^n(\Lambda \otimes H) = 0$. Therefore, the filtration gives rise to a spectral sequence converging to $\text{Ext}_A^n(P,\mathbb{F}_p)$. The differential $d_r : E_r^{n,s,t} \longrightarrow E_r^{n-r,s+1,t-1}$ is an $\mathbb{F}_p$-linear map.
In the spectral sequence, \( n \) is the filtration degree, \( s \) is the homological degree, \( t \) is the internal degree and \( s + t \) is the total degree.

It is easy to see that
\[
E^{n,s,t}_0 = (F^n(\Lambda \otimes H)/F^{n-1}(\Lambda \otimes H))^t \cong \Sigma^n \Lambda_s,
\]
therefore,
\[
E^{n,s,t}_1 = H^s(E^{0,s,t}_0) \cong \Sigma^n E^{s,s+t-n}(\mathbb{F}_p, \mathbb{F}_p),
\]
and
\[
E^{n,s,t}_\infty \cong (F^nH^s(\Lambda \otimes H)/F^{n-1}H^s(\Lambda \otimes H))^t,
\]
where
\[
F^nH^s(\Lambda \otimes H) := \text{im}(H^s(F^n(\Lambda \otimes H)) \longrightarrow H^s(\Lambda \otimes H)).
\]
Therefore, \( \oplus_{n\geq 1} E^{n,s,t}_\infty \cong \text{Ext}^{s,s+t}_A(P, \mathbb{F}_p). \)

We denote \( E^{*,*,*}_\infty = \oplus_{n,t} E^{n,s,t}_\infty \) and \( E^{*,*,*}_\infty = \oplus_{n,t} E^{n,s,t}_\infty. \)

It is easy to check that the \( E^{*,0,*}_1 \) has a \( \mathbb{F}_p \)-basis consisting of all elements

- \( \alpha^b t, t + \epsilon \geq 1. \)

As the results of Liulevicius [26] (see also Aikawa [1]), we get the \( E^{*,1,*}_1 \) has a \( \mathbb{F}_p \)-basis consisting of all elements

- \( \alpha_0 a^b t = [\lambda^0_{p^0-1}a^b t], t + \epsilon \geq 1, \)
- \( h_i a^b t = [\lambda^1_{p^0-1}a^b t], i \geq 0, t + \epsilon \geq 1; \)

and the \( E^{*,2,*}_1 \) has an \( \mathbb{F}_p \)-basis consisting of all elements

- \( h_i h_j a^b t = [\lambda^1_{p^0-1}\lambda^1_{p^0-1}a^b t], 0 \leq i < j - 1, t + \epsilon \geq 1; \)
- \( h_{i,j} a^b t = [\lambda^1_{p^0-1}\lambda^0_{p^0-1}a^b t], i \geq 1, t + \epsilon \geq 1; \)
- \( \alpha_0 a^b t = [\lambda^0_{p^0-1}a^b t], t + \epsilon \geq 1; \)
- \( h_{i,2j+1} a^b t = [\lambda^1_{2p^0-1}\lambda^1_{2p^0-1}a^b t], i \geq 0, t + \epsilon \geq 1; \)
- \( h_{i,1} a^b t = [\lambda^1_{p^0+1-1}\lambda^1_{2p^0-1}a^b t], i \geq 0, t + \epsilon \geq 1; \)
- \( \rho a^b t = [\lambda^1_{1}\lambda^0_{1}a^b t], t + \epsilon \geq 1; \)
- \( \lambda^i a^b t = \left[ \sum_{j=1}^{(p-1)} \frac{(-1)^{j+1}}{j} \lambda^1_{p-1}\lambda^1_{jp^0-1}a^b t \right], i \geq 0, t + \epsilon \geq 1. \)

We will write \( \alpha \longrightarrow \beta \) to indicate that \( \alpha \) and \( \beta \) survive to \( E_r \) and \( d_r(\alpha) = \beta \) for some \( r \), therefore, both \( \alpha \) and \( \beta \) do not survive to \( E^{*,*,*}_\infty \). In such case, the element \( \beta \) is a boundary of a differential supported by \( \alpha \).

For example, we write
\[
ab^{(mp+k)p^0-1} \longrightarrow (k + 1)h_i ab^{((m-1)p^0+k+1)p^0-1}, \text{ for } i \geq 0, 1 \leq k \leq p - 1, m \geq 1,
\]
means
\[
d_{p^0(p-1)}(ab^{(mp+k)p^0-1}) = -(k + 1)h_i ab^{((m-1)p^0+k+1)p^0-1}.
\]
We can include $r$ by comparing the filtration degree of both sides of the formula.

In addition, for any non-negative integer $n$, we can write $n = mp + k$ for some $m \geq 0$ and $0 \leq k \leq p - 1$.

The differential $E_r^{*, 0, *} \xrightarrow{d_r} E_r^{*, 1, *}$ is given by the following lemma.

**Lemma A.1.** The non-trivial differentials $E_r^{*, 0, *} \xrightarrow{d_r} E_r^{*, 1, *}$ are listed as follows:

(A.1.1) $b^{[t]} \xrightarrow{\alpha_0} \alpha_0 b^{[t-1]}$, for $t \geq 1$;
(A.1.2) $ab^{[(mp+k)p^i-1]} \xrightarrow{(k+1)h_iab^{[(m-1)p+k+1)p^i-1]}}$, for $i \geq 0, 1 \leq k \leq p - 1$, $m \geq 1$.

**Proof of Theorem 5.3.** The formula (A.1.2) implies that the element $ab^{[kp^i-1]}$ is an infinite cycle and it survives to $E_{\infty}^{*, 0, *}$ for $i \geq 0$ and $1 \leq k \leq p - 1$.

The proof is complete. $\square$

The differential $E_r^{*, 1, *} \xrightarrow{d_r} E_r^{*, 2, *}$ is given by the following lemma.

**Lemma A.2.** The non-trivial differentials $E_r^{*, 1, *} \xrightarrow{d_r} E_r^{*, 2, *}$ are listed as follows:

(A.2.1) $\alpha_0 b^{[t]} \xrightarrow{\alpha_0^2} \alpha_0^2 b^{[t-1]}$, for $t \geq 1$;
(A.2.2) $\alpha_0 ab^{[mp+k]} \xrightarrow{(k+2)\rho ab^{[(m-2)p+k+2]}}$, for $0 \leq k < p - 2$, $m \geq 2$;
(A.2.3) $\alpha_0 ab^{[(mp+k)p^i-1]} \xrightarrow{(k+1)\alpha_0 h_iab^{[(m-1)p+k+1)p^i-1]}}$, for $i \geq 1, m \geq 1$;
(A.2.4) $\alpha_0 ab^{[(mp+k)p^i-p^i-2]} \xrightarrow{(k+1)\alpha_0 h_iab^{[(m-1)p+k+1)p^i-2]}}$, for $i \geq 1, m \geq 1$;
(A.2.5) $h_i b^{[t]} \xrightarrow{h_i \alpha_0} h_i \alpha_0 b^{[t-1]}$, for $i \geq 1, t \geq 1$;
(A.2.6) $h_0 b^{[mp+\ell]} \xrightarrow{(\ell - 1)\rho ab^{[(m-1)p+\ell]}}$, for $m \geq 1$ and $\ell \neq 1$;
(A.2.7) $h_0 b^{[(mp+1)p^i-p^i+2]} \xrightarrow{(e+1)h_0 h_jab^{[(m-1)p+e+1)p^i-p^i+2]}}$, for $i \geq 2, m \geq 0$;
(A.2.8) $h_iab^{[(mp+k)p^i-1]} \xrightarrow{(k+1)h_ih_jab^{[(m-1)p+k+1)p^i-1]}}$, for $i \geq 1, m \geq 1, 0 \leq j < i$;
(A.2.9) $h_iab^{[(mp+k)p^i+p^i-1]} \xrightarrow{(k+2)h_iab^{[(m-2)p+k+2)p^i-1+p^i-1]}}$, for $i \geq 1, m \geq 1$;
(A.2.10) $h_iab^{[(mp+k)p^i+(p-1)p^i-1]} \xrightarrow{-\frac{1}{2}(k-1)h_iab^{[(m-1)p+k)p^i+p^i-1]}}$, for $i \geq 1, m \geq 1$;
(A.2.11) $h_iab^{[(mp+k)p^i+2+p^i+1+p^i+u]} \xrightarrow{(i+2)h_iab^{[(m-2)p^i+2+(r+2)p^i+1+p^i}]}$, for $i \geq 1, m \geq 0, u = (p-1)p^i-1$;
(A.2.12) $h_iab^{[(mp+k)p^i-p^i+2+v]} \xrightarrow{(k+1)h_ih_jab^{[(m-1)p+k+1)p^i-p^i+2+v]}}$, for $j \geq i \geq 1, m \geq 1, v = (p-2)p^i+1+p^i+(p-1)p^i-1$;
(A.2.13) $h_iab^{[(mp+k)p^i-p^i+1+p^i+u]} \xrightarrow{(k+1)h_ih_jab^{[(m-1)p+k+1)p^i-p^i+1+p^i+w]}}$, for $j \geq i \geq 1, m \geq 0, u = (p-1)p^i-1$;
(A.2.14) $h_iab^{[(mp+k)p^i+1+p^i-1]} \xrightarrow{\lambda_iab^{[(m-1)p+k+2)p^i-1]}}$, for $i \geq 0, m \geq 1$;
(A.2.15) $h_iab^{[mp^i+2+kp^i+1+p^i+2+p^i]} \xrightarrow{(k+2)h_iab^{[(m-1)p+k+1)p^i+1+p^i+2]}}$, for $i \geq 0, m \geq 0$;
(A.2.16) $h_iab^{[(mp+\ell)p^i-p^i+2+w]} \xrightarrow{(\ell+1)h_ih_jab^{[(m-1)p+\ell+1)p^i-p^i+2+w]}}$, for $j \geq i \geq 0, m \geq 1, w = (p-2)p^i+1+ep^i+p^i-1$.
\begin{align*}
(A.2.17) \quad & h_iab^{(mp+\ell)p^i-p^{i+2}+kp^{i+1}+x} \rightarrow (\ell+1)h_\ell h_jab^{((m-1)p^i+\ell+1)p^i-p^{i+2}+kp^{i+1}+x}, \text{ for } j - 2 \geq i \geq 0, m \geq 1, k \neq p - 2, x = p^{i+1} - 1.

\textbf{Proof.} \quad \text{The lemma is proved by direct computation in the chain complex } \Lambda \otimes H \text{ and in the spectral sequence.}

\textbf{Proof of (A.2.1).} \quad \text{It is clear that, in } \Lambda \otimes H,

\begin{align*}
d(\lambda_{i-1}^0 b^t) &= \lambda_{i-1}^0 \lambda_{i-1}^0 ab^{[t-1]} \mod F^{2(t-1)},
\end{align*}

\text{for } t \geq 1. \text{ Therefore, in the spectral sequence, we get the formula.}

\textbf{Proof of (A.2.2).} \quad \text{By inspection, we have, in } \Lambda \otimes H,

\begin{align*}
d \left( \lambda_{i-1}^0 ab^{[mp+k]} + (k+1) \lambda_{i-1}^0 ab^{[(m-1)p^i+k+1]} + \left( \frac{k+2}{2} \right) \lambda_{i-1}^0 ab^{[(m-2)p^i+k+2]} \right) &= \left( \frac{k+2}{2} \right) \lambda_{i-1}^0 ab^{[(m-2)p^i+k+2]} \mod F^{2((m-2)p^i+k+2)},
\end{align*}

\text{for } 0 \leq k < p - 2, m \geq 2. \text{ Therefore, in the spectral sequence, we obtain the formula.}

\textbf{Proof of (A.2.3).} \quad \text{By inspection, in } \Lambda \otimes H, \text{ we have}

\begin{align*}
d(\lambda_{i-1}^0 ab^{[(mp+k)p^i-1]}) &= (k+1) \lambda_{i-1}^0 \lambda_{i-1}^0 ab^{[(m-1)p^i+k+1)p^i-1]} \mod F^{2k_1},
\end{align*}

\text{for } i \geq 1, \text{ where } k_1 = ((m-1)p+k+1)p^i - 1. \text{ Therefore, we get the formula in the spectral sequence.}

\textbf{Proof of (A.2.4).} \quad \text{It is easy to check that, in } \Lambda \otimes H,

\begin{align*}
d(\lambda_{i-1}^0 ab^{[(mp+k)p^i-p+p-2]} + \lambda_{i-1}^0 ab^{[(mp+k)p^i-p-1]} &= (k+1) \lambda_{i-1}^0 \lambda_{i-1}^0 ab^{[(m-1)p^i+k+1)p^i-p+p-2]} \mod F^{2k_2},
\end{align*}

\text{for } i \geq 1, \text{ where } k_2 = ((m-1)p+k+1)p^i - p + p - 2. \text{ Therefore, in the spectral sequence, we obtain (A.2.4).}

\textbf{Proof of (A.2.5).} \quad \text{Similar to the proof of (A.2.1).}

\textbf{Proof of (A.2.6).} \quad \text{In } \Lambda \otimes H, \text{ we have}

\begin{align*}
d \left( \lambda_{i-1}^0 b^{[kp+\ell]} + \frac{1}{2} (\ell+1) \lambda_{i-1}^0 b^{[(k-1)p+\ell+1]} \right) &= \frac{1}{2} (\ell-1) \lambda_{i-1}^0 \lambda_{i-1}^0 ab^{[(k-1)p^i+\ell]} \mod F^{2((k-1)p^i+\ell)},
\end{align*}

\text{for } k \geq 1 \text{ and } \ell \neq 1. \text{ Hence, in the spectral sequence, one gets the formula.}

\textbf{Proof of (A.2.7).} \quad \text{Put } s = (mp + e)p^i \text{ for } m \geq 0 \text{ and } 0 \leq e \leq p - 1. \text{ By inspection, in } \Lambda \otimes H, \text{ one gets}
\[
d d \left( \sum_{j=0}^{p-1} \lambda_j^1 b^{[s-p^2+(k-j)p+j+1]} + C_1 + \cdots + C_{p-k-1} \right) = \\
\quad -(e + 1) \lambda_0^1 \lambda_{p^i-1}^1 b^{[(m-1)p+e+1)p^i-p^2+kp+1]} \\
\qquad \mod F^{2 \cdot ((m-1)p+e+1)p^i-p^2+kp+1)-1},
\]
where
\[
C_n = \sum_{\ell=0}^{p-1} \binom{p + k + n - \ell}{n} \lambda_{mp+\ell}^1 b^{[s-(n+1)p^2+(k+n-\ell)p+\ell+1]},
\]
for \(i \geq 2\) and \(m \geq 1\). Thus, in the spectral sequence, we obtain the formula.

**Proof of (A.2.8).** It is immediate.

**Proof of (A.2.9).** By inspection, in \(\Lambda \otimes H\), one gets
\[
d (\lambda_{p^i-1}^1 ab^{((mp+k)p^i-1+p^i-1-1)}) = -(k + 1) \lambda_{p^i-1}^1 \lambda_{p^i-1-1}^1 ab^{[(m-1)p+k+1)p^i-1+p^i-1-1]} \\
\quad - \binom{k + 2}{2} \lambda_{p^i-1}^1 \lambda_{2p^i-1-1}^1 ab^{[(m-2)p+k+2)p^i-1+p^i-1-1]} \\
\qquad \mod F^{2 \cdot ((m-2)p+k+2)p^i-1+p^i-1-1)},
\]
for \(i \geq 1\) and \(m \geq 1\). Since \(\lambda_{p^i-1}^1 \lambda_{p^i-1-1}^1 = 0\) in \(\Lambda\) and the second term of the formula represents \(h_{i-1,1,2}ab^{((m-2)p+k+2)p^i-1+p^i-1-1]}\), in the spectral sequence, we obtain (A.2.9).

**Proof of (A.2.10).** In \(\Lambda \otimes H\), we have, letting \(s = (m-1)p+k\) for \(m \geq 1\) and \(0 \leq k \leq p-1\),
\[
d \left( 2\lambda_{p^i-1}^1 ab^{((mp+k)p^i+(p-1)p^i-1-1)} + (k + 1) \lambda_{2p^i-1}^1 ab^{[(s+1)p^i+(p-1)p^i-1-1]} \right) = \\
\quad (k+1) \lambda_{2p^i-1}^1 \lambda_{p^i-1-1}^1 + 2k \lambda_{p^i-1}^1 \lambda_{p^i+p^i-1-1}^1 ab^{[sp^i+p^i-1]} \mod F^{2(s+p^i-1)},
\]
for \(i \geq 1\) and \(m \geq 1\). Since, by the Adem relation, in \(\Lambda\), \(\lambda_{2p^i-1}^1 \lambda_{p^i-1-1}^1 + \lambda_{p^i-1}^1 \lambda_{p^i+p^i-1-1}^1 = 0\) and the element \(\lambda_{2p^i-1}^1 \lambda_{p^i-1-1}^1 ab^{[sp^i+p^i-1]}\) represents the element \(h_{i-1,2,1}ab^{[sp^i+p^i-1]}\), in the spectral sequence, we obtain the formula.

**Proof of (A.2.11).** Put \(u = (p-1)p^{i-1} - 1\). In \(\Lambda \otimes H\), using the Adem relation, one gets that the differential of the element
\[
\lambda_{p^i-1}^1 ab^{(mp+k)p^{i+2}+rp^{i+1}+p^i+u] + \sum_{j=2}^{p-1} \lambda_{jp^i-1}^1 ab^{[(mp+k)p^{j+2}+(r-j+1)p^{j+1}+jp^i+u]} \\
\quad + (r + 1) \lambda_{p^i+1+p^i-1}^1 ab^{[(mp+k-1)p^{i+2}+(r+1)p^{i+1}+p^i+u]}
\]
\[
+ \binom{r + 2}{2} \lambda_{2p^{i+1}+1}^1 \lambda_{p^i-1}^1 ab^{[(mp+k-2)p^{i+2}+(r+2)p^{i+1}+p^i+u]}
\]

is equal to

\[
\binom{r + 2}{2} \lambda_{2p^{i+1}+1}^1 \lambda_{p^i-1}^1 ab^{[(mp+k-2)p^{i+2}+(r+2)p^{i+1}+p^i+u]} \mod F^2((mp+k-2)p^{i+2}+(r+2)p^{i+1}+p^i+u),
\]

for \( i \geq 1 \) and \( m \geq 0 \).

Since, in the spectral sequence, \( \lambda_{2p^{i+1}+1}^1 \lambda_{p^i-1}^1 ab^{[(mp+k-2)p^{i+2}+(r+2)p^{i+1}+p^i+u]} \)
represents \( h_{i;2,1} ab^{[(mp+k-2)p^{i+2}+(r+2)p^{i+1}+p^i+u]} \), we obtain the formula.

**Proof of (A.2.12).** In \( \Lambda \otimes H \), one gets, for \( u = (p-1)p^{i-1} - 1 \) and \( v' = (mp+k)p^j \), that the differential of the element

\[
\lambda_{p^i-1}^1 ab^{[v'-p^{i+2}+(2p)p^{i+1}+p^i+u]} + \sum_{\ell=2}^{p-1} \lambda_{\ell p^i-1}^1 ab^{[v'-\ell p^{i+1}+p^i+u]}
\]

is equal to

\[
-(k+1) \lambda_{p^i-1}^1 \lambda_{p^i-1}^1 ab^{[(m-1)p+k+1)p^j-p^{i+2}+(2p)p^{i+1}+p^i+u]} \mod F^{2u_1},
\]

for \( 1 \leq i \leq j-2 \) and \( m \geq 1 \), where \( u_1 = ((m-1)p+k+1)p^j-p^{i+2}+(2p)p^{i+1}+p^i+u \).

Therefore, in the spectral sequence, one obtains the formula.

**Proof of (A.2.13).** In \( \Lambda \otimes H \), one gets, for \( j-2 \geq i \geq 1 \) and \( u = (p-1)p^{i-1} - 1 \),

\[
d \left( \lambda_{p^i-1}^1 ab^{[v'-p^{i+1}+p^i+u]} + \sum_{\ell=2}^{p-1} \lambda_{\ell p^i-1}^1 ab^{[v'-\ell p^{i+1}+p^i+u]} \right)
\]

\[- (k+1) \lambda_{p^i-1}^1 \lambda_{p^i-1}^1 ab^{[(m-1)p+k+1)p^j-p^{i+1}+p^i+u]} \mod F^{2u_2},
\]

for \( m \geq 1 \), where \( u_2 = ((m-1)p+k+1)p^j-p^{i+1}+p^i+u \). Hence, in the spectral sequence, we obtain the formula.

**Proof of (A.2.14).** By inspection, in \( \Lambda \otimes H \), we have

\[
d \left( \lambda_{p^i-1}^1 ab^{[(mp+k)p^{i+1}+p^i-1]} + \sum_{j=2}^{p-1} \lambda_{j p^i-1}^1 ab^{[(mp+k-j+1)p^{i+1}+(j-1)p^i+p^i-1]} \right)
\]

\[= \sum_{j=1}^{p-1} \frac{(-1)^{j+1}}{j} \lambda_{(p-j)p^i-1}^1 \lambda_{j p^i-1}^1 ab^{[((m-1)p+k+2)p^{i+1}-1]} \mod F^{2(((m-1)p+k+2)p^{i+1}-1)},
\]
for \( i \geq 0 \) and \( m \geq 1 \).

Since the right hand side of the formula represents \( \hat{\lambda}_i ab^{[(m-1)p+k+2)p^{i+1}-1]} \), we obtain the formula.

**Proof of (A.2.15).** We have, in \( \Lambda \otimes H \), the differential of

\[
\lambda_{p^i-1}^1 ab^{[mp^{i+2}+kp^{i+1}+ep^i+p^{i}-1]} + \sum_{j=2}^{p-e} \frac{(e+j-1)}{j} \lambda_{jp^i-1}^1 ab^{[mp^{i+2}+(k-j+1)p^{i+1}+(e+j)p^{i}-1]}
\]

\[
+ (k + 1) \lambda_{p^{i+1}+p^i-1}^1 ab^{[(m-1)p^{i+2}+(k+1)p^{i+1}+ep^i+p^{i}-1]}
\]

\[
+ \frac{1}{2} k(e + 1) \lambda_{p^{i+1}+2p^i-1}^1 ab^{[(m-1)p^{i+2}+kp^{i+1}+(e+1)p^i+p^{i}-1]}
\]

is equal to

\[
\frac{1}{2} k(e + 1) \lambda_{p^{i+1}-1}^1 \lambda_{2p^i-1}^1 ab^{[(m-1)p^{i+2}+kp^{i+1}+(e+1)p^i+p^{i}-1]}
\]

\[
- (e + 1) \lambda_{p^{i+1}+p^i-1}^1 \lambda_{p^i-1}^1 ab^{[(m-1)p^{i+2}+kp^{i+1}+(e+1)p^i+p^{i}-1]}
\]

\[
\mod F^2((m-1)p^{i+2}+kp^{i+1}+(e+1)p^i+p^{i}-1),
\]

for \( i \geq 0 \) and \( m \geq 1 \).

Because of the Adem relation, we get \( \lambda_{p^{i+1}-1}^1 \lambda_{2p^i-1}^1 + \lambda_{p^{i+1}+p^i-1}^1 \lambda_{p^i-1}^1 = 0 \). Therefore, in the spectral sequence, we get the formula.

**Proof of (A.2.16).** In \( \Lambda \otimes H \), for \( j - 2 \geq i \geq 0, r > 0, m \geq 1 \) and \( u = p^i - 1 \), the differential of

\[
\lambda_{p^i-1}^1 ab^{[(mp+\ell)p^i-p^{i+2}+(p-2)p^i+1+ep^i+u]+}
\]

\[
\sum_{j=2}^{p-e} \frac{(e+j-1)}{j} \lambda_{jp^i-1}^1 ab^{[(mp+\ell)p^i-p^{i+2}+(p-1-j)p^{i+1}+(e+j-1)p^i]+}
\]

\[
- \lambda_{p^{i+1}+p^i-1}^1 ab^{[(mp+\ell)p^i-p^{i+2}+(p-1)p^i+1+ep^i]+}
\]

\[
- (e + 1) \lambda_{p^{i+1}+2p^i-1}^1 ab^{[(mp+\ell)p^i-p^{i+2}+(p-2)p^i+1+(e+1)p^i+u]}
\]

is equal to

\[
-(\ell + 1) \lambda_{p^{i-1}}^1 \lambda_{p^i-1}^1 ab^{[(m-1)p+\ell+1)p^i-p^{i+2}+(p-2)p^i+1+ep^i+u]}
\]

\[
\mod F^2((m-1)p^{i+2}+(p-2)p^{i+1}+ep^i+u),
\]

Therefore, in the spectral sequence, one gets the formula.

**Proof of (A.2.17).** Similarly, in \( \Lambda \otimes H \), we also have that, for \( j - 2 \geq i \geq 0, m \geq 1 \) and \( k \neq p-2 \), the differential of
\[ \lambda^1_{p^i-1}ab[(mp+\ell)p^j-p^{i+2} + kp^{i+1} + p^{i+1} - 1] + \\
(k + 1)\lambda^1_{p^{i+1}+p^j-1}ab[(mp+\ell)p^j-2p^{i+2} + (k+1)p^{i+1} + p^{i+1} - 1] \]

is equal to

\[- (\ell + 1)\lambda^1_{p^i-1} \lambda^1_{p^j-1}ab[(m-1)p+\ell+1)p^j-p^{i+2} + kp^{i+1} + p^{i+1} - 1] \mod F^2((m-1)p+p+1)p^j-p^{i+2} + kp^{i+1} + p^{i+1} - 1). \]

Hence, in the spectral sequence, we get the formula. \(\square\)

Since \(E^{*,1,*}_{\infty} \cong \text{Ext}^{1,1+*}_A(P,\mathbb{F}_p)\) as \(\mathbb{F}_p\)-vector spaces, in order to prove Theorem 5.4, first, we compute the \(\mathbb{F}_p\)-basis of \(E^{*,1,*}_{\infty}\).

**Proposition A.3.** The infinite term \(E^{*,1,*}_{\infty}\) has a \(\mathbb{F}_p\)-basis consisting of all elements given in Table A.1.

| Elements | Represented by | \(t\) | Range of indexes |
|----------|----------------|------|-----------------|
| \(\alpha_0 h_i\) | \(\alpha_0 ab[(p-1)p^{i-1}]\) | \(2(p-1)p^i - 1\) | \(i \geq 1\) |
| \(\alpha_0 h_i(k)\) | \(\alpha_0 ab[kp^{i-1}]\) | \(2kp^i - 1\) | \(i \geq 1, 1 \leq k \leq p - 1\) |
| \(\hat{\alpha}(\ell)\) | \(\alpha_0 ab[\ell+1]\) | \(2(p + \ell) + 1\) | \(0 \leq \ell < p - 2\) |
| \(h_i \hat{h}_i(1)\) | \(h_i ab[p^{i-1}]\) | \(2(p - 1)p^i + 2p^i - 2\) | \(i \geq 0\) |
| \(h_i \hat{h}_j\) | \(h_i ab[(p-1)p^{j-1}]\) | \(2(p - 1)(p^i + p^j) - 2\) | \(0 \leq j, i; j \neq i, i + 1\) |
| \(h_i \hat{h}_j(k)\) | \(h_i ab[kp^{i-1}]\) | \(2(p - 1)p^i + 2kp^i - 2\) | \(0 \leq j, i; j \neq i, i + 1\) |
| \(\hat{d}_i(k)\) | \(h_i ab[kp^{i}+(p-1)p^{i-1}]\) | \(2(p - 1)(p^i + p^{i-1}) + 2kp^i - 2\) | \(i \geq 1, 1 \leq k \leq p - 1\) |
| \(\tilde{h}_i(k)\) | \(h_i ab[kp^{i+1}+p^{i-1}]\) | \(2(k + 1)p^{i+1} - 2\) | \(i \geq 0, 1 \leq k < p - 1\) |
| \(\tilde{p}_i(k)\) | \(h_i ab[(p-1)p^{i+1}+(k+1)p^{i-1}]\) | \(2(p - 1)(p^i + p^{i+1}) + 2(k + 1)p^i - 2\) | \(i \geq 0, 1 \leq k < p - 1\) |

**Proof.** Since \((A.2.1)\), the element \(\alpha_0 b^{[t]}\) is not an infinite cycle. Thus, we need only to consider the elements \(\alpha_0 ab^{[t]}\) for \(t \geq 0, h_i b^{[t]}, i \geq 0, t > 0\) and \(h_i ab^{[t]}, i \geq 0, t \geq 0\).

**The case \(\alpha_0 ab^{[t]}\) for \(t \geq 0\)**

From \((A.2.2)\), it is easy to see that \(\alpha_0 ab^{[k]}\), for \(0 \leq k \leq p - 1\), and \(\alpha_0 ab^{[p+k]}\), for \(1 \leq k \leq p - 1\), are infinite cycles. The first element is a boundary of a differential supported by \(b^{[k+1]}\). Therefore, we have the following relations:

\[ \alpha_0 \hat{h}_0 = 0; \quad \alpha_0 \hat{h}_0(k) = 0, 1 \leq k < p - 1. \]  

(A.1)
The second element is also a boundary of a differential supported by $b^{[b+k+1]}$. However, in $\Lambda \otimes H$, we get
\[ d(b^{[p+k+1]}) = \lambda^0_{-1} ab^{[p+k]} + (k+2) \lambda^1_0 b^{[k+2]} \mod F^{2(k+2)-1}. \quad (A.2) \]

Since, in $\Lambda$, $\lambda^1_{0} \lambda^0_{-1} = 0$, it follows that $\lambda^0_0 b^{[k+2]}$ is a cycle for $k < p - 2$. Therefore, $h_0 b^{[k+2]}$ is an infinite cycle in the spectral sequence, and then, $\alpha_0 ab^{[p+k]}$ survives to $E_{\infty}^{*,1,*}$ represented by $(k+2)h_0 b^{[k+2]}$ of lower filtration degree.

It is easy to see that, from (A.2), $\alpha_0 ab^{[2p-2]}$ does not survive to $E_{\infty}^{*,1,*}$.

For $k = p - 1$, since $\lambda^0_0 ab^{[2p-1]}$ is a cycle in $\Lambda \otimes H$, from (A.2), it implies that $h_i b^{[p+1]}= [\lambda^1_0 b^{[p+1]}]$ is an infinite cycle in the spectral sequence. Hence, $\alpha_0 ab^{[2p-1]}$ survives to $E_{\infty}^{*,1,*}$.

From (A.2), it is sufficient to consider $\alpha_0 ab^{[mp+p-1]}$ and $\alpha_0 ab^{[mp+p-2]}$ for $n \geq 2$. The first element can be written in the form $\alpha_0 ab^{[(mp+k)p-1]}$ for some $i \geq 1$ and the second element can be written in the form $\alpha_0 ab^{[(mp+k)p-1]+[p-2]}$ for some $i \geq 1$.

In addition, by the formulas (A.2.3) and (A.2.4), we obtain that these elements are infinite cycles if and only if $m = 0$. That means the elements $\alpha_0 ab^{[kp-1]}$ and $\alpha_0 ab^{[kp+p-2]}$, for $1 \leq k \leq p - 1$, are only infinite cycles of these forms.

From (A.1.1), it is easy to see that $\alpha_0 ab^{[kp-1]}$ and $\alpha_0 ab^{[kp+p-2]}$ are boundaries and they are respectively supported by $b^{[kp]}$ and $b^{[kp+1]}$. However, in $\Lambda \otimes H$, we have
\[ d(b^{[kp+1]}) = \lambda^0_{-1} ab^{[kp+1]} + \lambda^1_0 b^{[kp+p-1]} \mod F^{2(kp+p-1)-1}; \quad (A.3) \]
and
\[ d(b^{[kp+1]} - 1) = \lambda^0_{-1} ab^{[kp+p-2]}. \]

The second formula implies that the element $\alpha_0 ab^{[kp+p-2]}$ does not survive to $E_{\infty}^{*,1,*}$. Since $\lambda^0_{1} b^{[kp+p-1]}$ represents $h_0 b^{[kp+p-1]}$, and $\lambda^0_{-1} ab^{[kp-1]}$ is a cycle in $\Lambda \otimes H$, it follows that $h_0 b^{[kp+p-1]}$ is an infinite cycle in the spectral sequence. Therefore, $\alpha_0 ab^{[kp-1]}$ survives and represents the elements $\alpha_0 \hat{h}_i \neq 0$ and $\alpha_0 \hat{h}_i (k) \neq 0$ for $i \geq 1$ and $1 \leq k < p - 1$.

Basing above computation, we obtain the first three generators in Table A.1.

The case $h_i b^{[t]}$ for $t > 0$

From (A.2.5), it is sufficient to consider $h_0 b^{[t]}$ for $t \geq 1$. We put $t = mp + \ell$. By (A.2.6), when $m = 0$, the element $h_0 b^{[t]}$, $1 \leq \ell \leq p - 1$, is an infinite cycle. It is clear that it is in a boundary supported by $b^{[p+\ell-1]}$. However, in $\Lambda \otimes H$, one gets
\[ d(b^{[p+\ell-1]}) = \lambda^0_{-1} ab^{[p+\ell-1]} + \ell \lambda^1_0 b^{[t]} \mod F^{2\ell-1}. \]

It follows that $h_0 b^{[t]}$ survives to $E_{\infty}^{*,1,*}$ and represents the element $\frac{1}{\ell} \alpha_0 ab^{[\ell-2]}$. 
For $m \geq 1$, by (A.2.6), it is sufficient to consider the case $\ell = 1$. That means we need only to consider the element of the form $h_0 b^{(mp+e)p'−p^2+kp+1}$.

By (A.2.7), when $m = 0$, the element $h_0 b^{[ep'−p^2+kp+1]}$ is an infinite cycle in the spectral sequence.

For $k = p−1$, by (A.3), we obtain that $h_0 b^{[ep'−p+1]}$ represents the element $−\alpha_0 \widehat{h}_i(e) \neq 0$.

For $k < p−1$, it is easy to see that $h_0 b^{[ep'−p^2+kp+1]}$ is a boundary of a differential supported by $b^{[ep'−p^2+(k+1)p]}$. Furthermore, in $\Lambda \otimes H$, one gets

$$d \left( b^{[ep'−p^2+(k+1)p]} \right) = \lambda_{0−1}^0 ab^{[ep'−p^2+(k+1)p−1]}
\begin{align*}
&\quad + \sum_{j=1}^{p−1} \lambda_{j−1}^1 b^{[ep'−p^2+(k−j+1)p+j]} \\
&\quad + (k+2)\lambda_{p−1}^1 b^{[ep'−2p^2+(k+2)p]} \mod F^{2(ep'−2p^2+(k+2)p)−1}.
\end{align*}$$

Since the first term of the right hand side of the formula represents $\alpha_0 ab^{[ep'−p^2+(k+1)p−1]}$, which does not survive to $E^{*,1*}_{\infty}$ (see the first case), it implies that $h_0 b^{[ep'−p^2+kp+1]}$ does not survive to $E^{*,1*}_{\infty}$.

Thus, this case does not give us any new generator.

The case $h_i ab^{[t]}$ for $t \geq 0$

First, we consider the case $i \geq 1$. From (A.2.8), it follows that $h_i ab^{[kp^i−1]}$, $1 \leq k \leq p−1$, $0 \leq j < i$, is an infinite cycle. It is easy to see that $h_i ab^{[kp^i−1]}$ is not a boundary; then, it represents the elements $h_i \widehat{h}_j \neq 0$ for $0 \leq j < i$ and $h_i \widehat{h}_j(k) \neq 0$ for $0 \leq j < i$, $1 \leq k < p−1$.

Since $\binom{k+2}{2} = 0$ if and only if $k = p−2$ or $k = p−1$, from (A.2.9), it is sufficient to consider three cases:

(i) $h_i ab^{[mp^i+(p−1)p^i−1]}$ for $i \geq 1$, $m \geq 1$;
(ii) $h_i ab^{[mp^i+p^i−1]}$ for $i \geq 1$, $m \geq 1$; and
(iii) $h_i ab^{[p^i−1]}$.

It should be noted that when $m = 0$ and $k < p−1$ the element $h_i ab^{[(mp+k)p^i−1+p^i−1−1]}$ reduces to the case considered above.

Now, we consider the elements of the form (i). By (A.2.10), $h_i ab^{[kp^i+(p−1)p^i−1−1]}$, for $1 \leq k \leq p−1$, is an infinite cycle.

In addition, it is in a boundary supported by $ab^{[p^i+1+(k−1)p^i+(p−1)p^i−1−1]}$. However, in $\Lambda \otimes H$, one gets

$$d \left( ab^{[p^i+1+(k−1)p^i+(p−1)p^i−1−1]} \right) = \lambda_{p^i−1−1}^1 ab^{[p^i+1+(k−2)p^i+p^i−1]}$$
\[-k\lambda_{p_i - 1}^1 ab^{[kp^i + (p-1)p^{i-1} - 1]} \mod F^{2(kp^i + (p-1)p^{i-1} - 1)}.

Since the second term of the right hand side of the formula is cycles in \(\Lambda \otimes H\) and the
first term represents the element \(h_{i-1} ab^{[kp^i + (k-2)p^{i-1} - 1]}\), then it is an infinite cycle.
Hence, \(h_i ab^{[kp^i + (p-1)p^{i-1} - 1]}\) survives and represents a non-trivial in \(E^{*,1,*}_\infty\).

For \(m \geq p\), the elements of the form (i) can be rewritten by \(h_i ab^{[(mp+k)p^i + (p-1)p^{i-1} - 1]}\)
for \(i \geq 1\).

Also by (A.2.10), for \(m \geq 1\), such element reduces to the case \(k = 1\), namely,
\(h_i ab^{[mp^i + (p-1)p^{i-1} - 1]}\).

It is clear that if \(1 \leq m \leq p - 1\), then \(h_i ab^{[mp^i + (p-1)p^{i-1} - 1]}\) is also a infinite cycle
and it is in a boundary supported by \(ab^{[(m+1)p^i + (p-1)p^{i-1} - 1]}\).

In \(\Lambda \otimes H\), we get
\[
d \left(ab^{[(m+1)p^i + (p-1)p^{i-1} - 1]}\right) = \lambda_{p_i - 1}^1 ab^{[(m+1)p^i + (p-1)p^{i-1} - 1]}
- \lambda_{p_i - 1}^1 ab^{[mp^i + (p-1)p^{i-1} - 1]} \mod F^{2(mp^i + (p-1)p^{i-1} - 1)}.
\]

It follows that \(h_i ab^{[mp^i + (p-1)p^{i-1} - 1]}\) survives to \(E^{*,1,*}_\infty\), and it represents the element
\(h_{i-1}\hat{h}_{i+1}(m+1) \neq 0\) for \(i \geq 1\). It should be noted that when \(m = p - 2\) and \(m = p - 1\),
the element \(\hat{h}_{i+1}(m+1)\) is respectively equal to \(\hat{h}_{i+1}1\).

When \(m \geq p\), put \(u = (p-1)p^{i-1} - 1\). From (A.2.11), the case (i) reduces to two cases:

(i.1) \(h_i ab^{[(mp+k)p^i + (p-2)p^{i+1} + p^i + u]}\); and

(i.2) \(h_i ab^{[(mp+k)p^i + (p-1)p^{i+1} + p^i + u]}\).

First, we treat the element of the form (i.1).

From (A.2.12), it implies that \(h_i ab^{[kp^j - p^{i+2} + (p-2)p^{i+1} + p^i + u]}\) is an infinite cycle. It is easy to check that, in the spectral sequence, it is in a boundary of a differential supported by the element \(ab^{[kp^j - p^{i+2} + (p-1)p^{i-1} - 1]}\). In addition, in \(\Lambda \otimes H\),
\[
d \left(ab^{[kp^j - p^{i+1} + (p-1)p^{i-1} - 1]}\right) = -\lambda_{p_j - 1}^1 ab^{[kp^j - p^{i+1} - 1]}
- \lambda_{p_j - 1}^1 ab^{[kp^j - p^{i+2} + (p-2)p^{i+1} + p^i + u]} \mod F^{2(kp^j - p^{i+2} + (p-2)p^{i+1} + p^i + u)}.
\]

Since the first term of the right hand side of the formula represents an element which
does not survive to \(E^{*,1,*}_\infty\), it follows that \(h_i ab^{[kp^j - p^{i+2} + (p-2)p^{i+1} + p^i + u]}\) is a boundary,
and then, it does also not survive to \(E^{*,1,*}_\infty\).

Second, we treat the element of the form (i.2).

From (A.2.13), it follows that \(h_i ab^{[kp^j - p^{i+1} + p^i + (p-1)p^{i-1} - 1]}\) is an infinite cycle. It is easy to see that \(h_i ab^{[kp^j - p^{i+1} + p^i + (p-1)p^{i-1} - 1]}\) is in a boundary of a differential supported by the element \(ab^{[kp^j + (p-1)p^{i-1} - 1]}\). However, in \(\Lambda \otimes H\), one gets
\[ d \left( ab^{[kp'+(p-1)p^{i-1}-1]} \right) = -\lambda_{p'-1}^1 ab^{[kp'-1]} + \lambda_{p'-1}^1 ab^{[kp'-p^{i+1}+p^{i'}+(p-1)p^{i-1}-1]} \mod F^{2(kp'-p^{i+1}+p^{i'}+(p-1)p^{i-1}-1)}. \]

It follows that, in the spectral sequence, \( h_i ab^{[kp'-p^{i+1}+p^{i'}+(p-1)p^{i-1}-1]} \) survives to \( E_{\infty,1,*}^i \) and represents the element \( h_{i-1} \hat{h}_j(k) \neq 0 \), for \( j \geq i + 2, 1 \leq k \leq p - 1 \), in \( E_{\infty,1,*}^i \).

Next, we consider the element of the form (ii). It is easy to check that, for \( m \leq p - 1 \), the element \( h_i ab^{[(m+1)p^{i}-1]} \) represents the element \( h_i \hat{h}_i(m+1) \). However, in \( \Lambda \otimes H \), one gets

\[ d \left( ab^{[p^{i+1}+mp^{i'-1}]} \right) = m\lambda_{p'-1}^1 ab^{[(m+1)p^{i'}-1]} \mod F^{2((m+1)p^{i'}-1)}. \quad \text{(A.4)} \]

Therefore, we obtain relations \( h_i \hat{h}_i = 0 \), \( h_i \hat{h}_i(k) = 0 \) for \( i \geq 1, 2 \leq k < p - 1 \) for \( i \geq 1 \).

From (A.2.14), it follows that the element \( h_i ab^{[(mp+k)p^{i+1}+ep^{i'}+p^{i'-1}]} \) is not an infinite cycle, for \( m \geq 0, e = 0 \) and \( k = p - 1 \).

Hence, the element of the form (ii) reduces to the following cases:

(ii.1) \( h_i ab^{[kp^{i}+1+ep^{i'}+p^{i'}-1]} \) for \( k \leq p - 2 \);
(ii.2) \( h_i ab^{[(p-1)p^{i+1}+ep^{i'}+p^{i'-1}]} \) for \( e > 0 \);
(ii.3) \( h_i ab^{[(mp+k)p^{i+1}+ep^{i'}+p^{i'-1}]} \) for \( m > 0 \) and \( e > 0 \).

It should be noted that the two first cases are infinite cycles in the spectral sequence.

By inspection, it is easy to verify that, in \( \Lambda \otimes H \),

\[ d \left( ab^{[(k+1)p^{i+1}+(e-1)p^{i'}+p^{i'-1}]} \right) = e\lambda_{p'-1}^1 ab^{[kp^{i+1}+ep^{i'}+p^{i'}-1]} \mod F^{2(kp^{i+1}+ep^{i'}+p^{i'}-1)}. \quad \text{(A.5)} \]

Therefore, the element of the form (ii.1) is boundaries if \( e > 0 \). For \( e = p - 1 \), one gets the relations: \( h_i \hat{h}_{i+1} = 0 \) and \( h_i \hat{h}_{i+1}(k) = 0 \) for \( 1 \leq k \leq p - 1 \). However, for \( e = 0 \), the element \( h_i ab^{[kp^{i+1}+p^{i'}-1]} \) survives to \( E_{\infty,1,*}^i \).

By the same argument, in \( \Lambda \otimes H \), one gets

\[ d \left( ab^{[p^{i+2}+(e-1)p^{i'}+p^{i'-1}]} \right) = \sum_{j=1}^{p-1} (e+j-1) \lambda_{p'-1}^1 ab^{[(p-1)p^{i+1}+(e-1+j)p^{i'}+p^{i'}-1]} \mod F^{2(p^{i+1}+(e-1)p^{i'}+p^{i'}-1)}. \]

Since, for \( e > 0 \), the first sum of the right hand side of the formula is a cycle in \( \Lambda \otimes H \) and the last term represents \( h_{i+1} ab^{[p^{i+1}+(e-1)p^{i'}+p^{i'}-1]} \), then \( h_{i+1} ab^{[p^{i+1}+(e-1)p^{i'}+p^{i'}-1]} \) is an infinite cycle. Therefore, the element of the form (ii.2) survives to \( E_{\infty,1,*}^i \). For \( e = p - 1 \), it represents the element \( h_i \hat{h}_{i+2}(1) \neq 0 \).
Finally, by (A.2.15), the element of the form (ii.3) reduces two cases:

(iii.1) \( h_iab^{[mp^i+2+(p-2)p^{i+1}+ep^i+p^i-1]} \) for \( e > 0 \); and

(iii.2) \( h_iab^{[mp^i+2+kp^i+1+p^{i+1}-1]} \) for \( k \neq p-2 \).

From (A.2.16), the element \( h_iab^{[fp^i-p^{i+2}+(p-2)p^{i+1}+ep^i+u]} \) for \( e > 0, 1 \leq \ell \leq p - 1 \) and \( u = p^i - 1 \), is an infinite cycle. In addition, it is easy to check that, in \( \Lambda \otimes H \),

\[
d\left(ab^{[fp^i-p^{i+2}+(p-2)p^{i+1}+ep^i+u]}\right) = e\lambda^1_{p-1}ab^{[fp^i-p^{i+2}+(p-2)p^{i+1}+ep^i+u]} \mod F^2(fp^i-p^{i+2}+(p-2)p^{i+1}+ep^i+u).
\]

Since \( e > 0 \), it implies that \( h_iab^{[fp^i-p^{i+2}+(p-2)p^{i+1}+ep^i+u]} \) does not survive to \( E^{*,1,*}_{\infty} \).

Similarly, from (A.2.17), the element \( h_iab^{[fp^i-p^{i+2}+kp^i+1+p^{i+1}-1]} \) is an infinite cycle. In addition, we also have, in \( \Lambda \otimes H \), that

\[
d\left(ab^{[fp^i-p^{i+2}+(k+1)p^{i+1}+(p-1)p^i-1]}\right) = -\lambda^1_{p-1}ab^{[fp^i-p^{i+2}+kp^i+1+p^{i+1}-1]} + (k+2)\lambda^1_{p^i+1-1}ab^{[fp^i-2p^{i+2}+(k+2)p^{i+1}+(p-1)p^i-1]} \mod F^{2u_3},
\]

where \( u_3 = \ell p^j - 2p^{i+2} + (k+2)p^{i+1} + (p-1)p^i - 1 \).

By above computation, \( h_i+1ab^{[fp^j-2p^{i+2}+(k+2)p^{i+1}+(p-1)p^i-1]} \) survives to \( E^{*,1,*}_{\infty} \) if and only if \( k = p-1 \). Therefore, \( h_iab^{[fp^i-p^{i+2}+kp^i+1+p^{i+1}-1]} \) survives to \( E^{*,1,*}_{\infty} \) if and only if \( k = p-1 \). In this case \( h_iab^{[fp^i-1]} \) represents \( h_i\hat{h}_j(\ell) \neq 0 \) for \( 0 < i < j-1 \) and \( 1 \leq \ell \leq p-1 \).

It is easy to see that the element of the form (iii), \( h_iab^{[p^i-1]} \) for \( i \geq 1 \), represents the element \( h_ih_i(1) \) in the spectral sequence. From (A.4), it implies that this element survives to \( E^{*,1,*}_{\infty} \).

The element \( h_0ab^{[\ell]} \) can be considered as a special case of the element of the form (ii) for \( i = 0 \), therefore, it can be treated by the same method.

The proof is complete. \( \square \)

**Proof of Theorem 5.4.** Since \( E^{*,1,*}_{\infty} \simeq \text{Ext}^{1,1,*}_{\Lambda}(P, \mathbb{F}_p) \) as \( \mathbb{F}_p \)-vector spaces, \( \text{Ext}^{1,1,*}_{\Lambda}(P, \mathbb{F}_p) \) is spanned by the elements listed in Table A.1.

It is easy to check that the elements \( \alpha_i\hat{h}_i(i \geq 1) \), \( \alpha_i\hat{h}_i(k) \) \( i \geq 1, 1 \leq k < p-1 \), \( h_i\hat{h}_i(1) \) \( i \geq 0 \), \( h_i\hat{h}_j(i, j \geq 0, j \neq i, i+1) \) and \( h_i\hat{h}_j(k) \) \( i, j \geq 0, j \neq i, i+1, 1 \leq k < p-1 \) are represented by cycles in \( \Lambda \otimes H \) as in the statement of the theorem.

By inspection, we get the element \( \lambda^0_{-1}ab^{[p-\ell]} + (\ell+1)\lambda^0_{-1}ab^{[\ell+1]} \) is a cycle in \( \Lambda \otimes H \). Since the first term of the element represents the element \( \alpha_0ab^{[p+\ell]} \) in the spectral sequence, this cycle is a representative of \( \hat{\alpha}(\ell) \).

By the same argument, we get that the cycle \( \lambda^1_{-1}ab^{[kp+p-2]} \) represents the element \( \hat{\alpha}_1(k) \). Since \( \hat{\mathcal{P}}^0 \) commutes with the differential of \( \Lambda \otimes H \), then \( \hat{\mathcal{P}}^0(\lambda^1_{-1}ab^{[kp+p-2]}) \) is also
a cycle and it represents the element $h_2ab^{[kp^2+(p-1)p-1]}$. Therefore, $\tilde{\mathcal{P}}^0(\lambda_{p-1}^1ab^{[kp+p-2]})$ represents the element $\hat{d}_2(k)$. Similarly, we get $\hat{d}_i(k)$ is represented, in $\Lambda \otimes H$, by the cycle $(\mathcal{P}^0)^{i-1}(\lambda_{p-1}^1ab^{[kp+p-2]})$. In other words,

$$\hat{d}_i(k) = (\mathcal{P}^0)^{i-1}\left(\lambda_{p-1}^1ab^{[kp+p-2]}\right).$$

Other elements are treated similarly. □

**Proof of Proposition 5.5.** The proof of the proposition is based upon the computation in the proof of Proposition A.3.

From (A.1), we have the two last relations.

From (A.4), we obtain the relations $h_i\hat{h}_i = 0$, $h_i\hat{h}_i(k) = 0$ for $i \geq 1$, $2 \leq k < p-1$ for $i \geq 1$.

Finally, from (A.5), for $e = p-1$, we obtain the other relations. □

**References**

[1] T. Aikawa, 3-dimensional cohomology of the mod $p$ Steenrod algebra, Math. Scand. 47 (1) (1980) 91–115. MR 60080 (82g:55023).

[2] A.K. Bousfield, E.B. Curtis, D.M. Kan, D.G. Quillen, D.L. Rector, J.W. Schlesinger, The mod-$p$ lower central series and the Adams spectral sequence, Topology 5 (1966) 331–342.

[3] A.K. Bousfield, D.M. Kan, The homotopy spectral sequence of a space with coefficients in a ring, Topology 11 (1972) 79–106. MR 0283801.

[4] T.W. Chen, Determination of $Ext^1_\mathbb{Z}(\mathbb{Z}/2, \mathbb{Z}/2)$, Topol. Appl. 158 (5) (2011) 660–689.

[5] P.H. Chön, Modular coinvariants and the mod $p$ homology of $Q$S$^k$, Proc. Lond. Math. Soc. (3) 112 (2) (2016) 351–374. MR 3471252.

[6] P.H. Chön, P.B. Nhüt, On the mod $p$ Lannes-Zarati homomorphism, J. Algebra 537 (2019) 316–342. MR 3990046.

[7] F.R. Cohen, T.J. Lada, J.P. May, The Homology of Iterated Loop Spaces, Lecture Notes in Mathematics, vol. 533, Springer-Verlag, Berlin, 1976. MR 0436146 (55 #9096).

[8] R.L. Cohen, W.H. Lin, M.E. Mahowald, The Adams spectral sequence of the real projective spaces, Pac. J. Math. 134 (1) (1988) 27–55. MR 953499.

[9] M.D. Crossley, $A(p)$-annihilated elements in $H_*(C\mathbb{P}^\infty \times C\mathbb{P}^\infty)$, Math. Proc. Camb. Philos. Soc. 120 (3) (1996) 441–453. MR 1388199 (97d:55029).

[10] E.B. Curtis, The Dyer-Lashof algebra and the $A$-algebra, Ill. J. Math. 19 (1975) 231–246. MR 0377885 (51 #14054).

[11] N.D.H. Hái, Résolution injective instable de la cohomologie modulo $p$ d’un spectre de Thom et application, PhD thesis, Université Paris 13, 2010.

[12] N.H.V. Hưng, Spherical classes and the algebraic transfer, Trans. Am. Math. Soc. 349 (10) (1997) 3893–3910. MR 1433119 (98e:55020).

[13] N.H.V. Hưng, On triviality of Dickson invariants in the homology of the Steenrod algebra, Math. Proc. Camb. Philos. Soc. 134 (1) (2003) 103–113.

[14] N.H.V. Hưng, F.P. Peterson, $A$-generators for the Dickson algebra, Trans. Am. Math. Soc. 347 (12) (1995) 4687–4728. MR 1316852 (96c:55022).

[15] N.H.V. Hưng, V.T.N. Quỳnh, N.A. Tuân, On the vanishing of the Lannes-Zarati homomorphism, C. R. Math. Acad. Sci. Paris 352 (3) (2014) 251–254. MR 3167575.

[16] N.H.V. Hưng, N. Sum, On Singer’s invariant-theoretic description of the lambda algebra: a mod $p$ analogue, J. Pure Appl. Algebra 99 (3) (1995) 297–329. MR 1332903 (96c:55024).

[17] N.H.V. Hưng, N.A. Tuân, The generalized algebraic conjecture on spherical classes, Manuscr. Math. 162 (2020) 133–157.

[18] M. Kameko, Products of projective spaces as Steenrod modules, ProQuest LLC, Thesis (PhD), The Johns Hopkins University, Ann Arbor, MI, 1990. MR 2638633.
[19] N.J. Kuhn, The Whitehead conjecture, the tower of $S^1$ conjecture, and Hecke algebras of type $A$, J. Topol. 8 (1) (2015) 118–146. MR 3335250.
[20] N.J. Kuhn, Adams filtration and generalized Hurewicz maps for infinite loopspaces, Invent. Math. 214 (2) (2018) 957–998.
[21] J. Lannes, Sur le $n$-dual du $n$-ème spectre de Brown-Gitler, Math. Z. 199 (1) (1988) 29–42 (in French).
[22] J. Lannes, S. Zarati, Invariants of Hopf d’ordre supérieur et suite spectrale d’Adams, C. R. Acad. Sci., Sér. 1 Math. 296 (15) (1983) 695–698. MR 705694.
[23] J. Lannes, S. Zarati, Sur les foncteurs dérivés de la déstabilisation, Math. Z. 194 (1) (1987) 25–59. MR 871217 (88j:55014).
[24] W.H. Lin, $\text{Ext}_A^{4,\ast}(\mathbb{Z}/2,\mathbb{Z}/2)$ and $\text{Ext}_A^{5,\ast}(\mathbb{Z}/2,\mathbb{Z}/2)$, Topol. Appl. 155 (5) (2008) 459–496. MR 2380930 (2008j:55020).
[25] A. Liulevicius, The factorization of cyclic reduced powers by secondary cohomology operations, Proc. Natl. Acad. Sci. USA 46 (7) (1960) 978–981.
[26] A. Liulevicius, The factorization of cyclic reduced powers by secondary cohomology operations, Mem. Am. Math. Soc. 42 (1962) 112. MR 0182001 (31 #6226).
[27] J.P. May, A general algebraic approach to Steenrod operations, in: The Steenrod Algebra and its Applications. Proc. Conf. to Celebrate N. E. Steenrod’s Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970, in: Lecture Notes in Mathematics, vol. 168, Springer, Berlin, 1970, pp. 153–231. MR 0281196 (43 #6915).
[28] N. Minami, The iterated transfer analogue of the new doomsday conjecture, Trans. Am. Math. Soc. 351 (6) (1999) 2325–2351. MR 1443884 (99i:55023).
[29] H. Mùi, Modular invariant theory and cohomology algebras of symmetric groups, J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math. 22 (3) (1975) 319–369. MR 0422451 (54 #10440).
[30] G.M.L. Powell, On the derived functors of destabilization at odd primes, Acta Math. Vietnam. 39 (2) (2014) 205–236. MR 3212661.
[31] S.B. Priddy, Koszul resolutions, Trans. Am. Math. Soc. 152 (1) (1970) 39–60.
[32] W.M. Singer, The construction of certain algebras over the Steenrod algebra, J. Pure Appl. Algebra 11 (1–3) (1977–1978) 53–59. MR 0467746.
[33] W.M. Singer, Invariant theory and the Lambda algebra, Trans. Am. Math. Soc. 280 (2) (1983) 673–693.
[34] R.J. Wellington, The unstable Adams spectral sequence for free iterated loop spaces, Mem. Am. Math. Soc. 36 (258) (1982), viii+225. MR 646741 (83c:55028).
[35] S. Zarati, Dérivés du foncteur de déstabilisation en caractéristiques impaire et application, PhD thesis, Université Paris-Sud (Orsay), 1984.
[36] H. Zare, On spherical classes in $H_n QS^n$, PhD thesis, The University of Manchester, 2009.