DISTANCE DIFFERENCE FUNCTIONS ON NON-CONVEX BOUNDARIES OF RIEMANNIAN MANIFOLDS

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ABSTRACT. We show that a complete Riemannian manifold with boundary is uniquely determined, up to an isometry, by its distance difference representation on the boundary. Unlike previously known results, we do not impose any restrictions on the boundary.

1. Introduction

Let \( M = (M, g) \) be a complete, connected, \( C^\infty \) Riemannian manifold with boundary \( F = \partial M \neq \emptyset \). We study the following structure introduced by Lassas and Saksala [8]. For every \( x \in M \) consider the distance difference function

\[
D_x: F \times F \to \mathbb{R}
\]

defined by

\[
D_x(y, z) = d_M(x, y) - d_M(x, z), \quad y, z \in F,
\]

where \( d_M \) is the arclength distance in \( M \). Note that for every \( x_1, x_2 \in M \), the distance \( d_M(x_1, x_2) \) is realized by a shortest path which is a \( C^1 \) curve, see e.g. [1]. The boundary is not assumed convex, thus a shortest path can touch the boundary, bend along it, etc.

We regard the collection of functions \( D_x, x \in M \), as a map

\[
\mathcal{D}: M \to \mathcal{C}(F \times F)
\]

given by

\[
\mathcal{D}(x) = D_x, \quad x \in M.
\]

Here \( \mathcal{C}(F \times F) \) denotes the space of all continuous functions on \( F \times F \). The map \( \mathcal{D} \) is called the distance difference representation of \( M \), and the set \( \mathcal{D}(M) \subset \mathcal{C}(F \times F) \) is called the distance difference data [8] or travel time difference data [3]. In this paper we use the former term.

The main result of this paper tells that the distance difference data \( \mathcal{D}(M) \) determine \( M \) uniquely up to a Riemannian isometry. See Theorem 1 below for the precise statement. This improves results from [3] and [4] where similar theorems were obtained under additional assumptions on the boundary. Moreover in Theorem 2 we show that the geometry of an arbitrary open region \( U \subset M \) is determined by the partial distance difference data \( \mathcal{D}(U) \).

The intuition behind the problem is the following. Imagine that \( M \) is some material object of interest, for example the Earth, and \( g \) represents the speed...
of wave propagation in $M$. A point $x \in M$ can be a spontaneous spherical wave source (for example, think of microseismic events in the Earth’s crust). An observer measures arrival times of the wave at a dense set of points on the surface. Since the time of the event in the interior is unknown, the information obtained from the measurement is precisely the same as that provided by the function $D_x$. Knowing the set $D(M)$ means that such information is collected from a dense set of points in the interior, and the goal is to learn the geometry of $M$ from these data. See [8, 3] for more detailed discussion of applications and [5] for applications of ordinary distance representations. i.e., plain distance functions $d_M(x, \cdot)$ rather than the differences.

Previous results. Lassas and Saksala [8] proved the unique determination of $M$ in a different setting where $M$ is compact and has no boundary, and the “observation domain” $F$ is an open subset of $M$ rather than the boundary. In [4] this is generalized to the case of a complete but possibly non-compact manifold $M$ and partial distance difference data $D(U)$ where $U \subset M$ is an open set whose geometry is to be determined.

The case when $F = \partial M$ turns out to be more difficult. It is partly addressed in [3] and [4] where the unique determination of the manifold is proved under various additional assumptions on the boundary. The assumption in [4] is the nowhere concavity of the boundary and in [3] it is a less restrictive “visibility” condition. We emphasize that in this paper we do not make additional assumptions on the boundary.

The similar problem about boundary distance data (without the “difference” part) was solved earlier, see [5, 6]. See also [7] for similar problems in the Lorentzian setting.

Statement of the results. We begin with a global determination result:

**Theorem 1.** Let $M = (M, g)$ and $M' = (M', g')$ be complete, connected Riemannian $n$-manifolds $(n \geq 2)$ with a common boundary $F = \partial M = \partial M' \neq \emptyset$. Assume that the distance difference data of $M$ and $M'$ coincide as subsets of $C(F \times F)$, i.e. $D(M) = D'(M')$ where $D$ and $D'$ are their respective distance difference representations, see (1.1) and (1.2).

Then $M$ and $M'$ are isometric via a Riemannian isometry that fixes $F$.

The words “common boundary” in the theorem require clarification. Their precise meaning is the following: $F$ is a topological manifold and it is identified with $\partial M$ and $\partial M'$ by means of some homeomorphisms. In other words, we assume that $M$ and $M'$ induce the same topology on $F$ but do not assume that they induce the same differential structure and metric.

Like in [4], we actually prove a more general result on unique determination of local geometry from the corresponding partial data:

**Theorem 2.** Let $M = (M, g)$ and $M' = (M', g')$ be complete, connected Riemannian $n$-manifolds $(n \geq 2)$ with a common boundary $F = \partial M = \partial M' \neq \emptyset$. Let $D$ and $D'$ denote the distance difference representations of $M$ and $M'$ in $C(F \times F)$, see (1.1) and (1.2). Let $U \subset M$ and $U' \subset M'$ be open sets such that

(1.3) $D(U) = D'(U').$

Then there exists a Riemannian isometry $\phi: U \to U'$ (with respect to the metrics $g$ and $g'$) such that $\phi|_{U \cap F}$ is the identity.
Note that Theorem 1 is a special case of Theorem 2 for \( U = M \) and \( U' = M' \). The proof of Theorem 2 occupies the rest of the paper. It builds upon results from [4] and [5]. In particular one of the important ingredients of the proof is Proposition 2.4 borrowed from [4].

The paper is organized as follows. In Section 2 we fix notation, collect preliminaries and construct a candidate for the isometry \( \phi \). In Section 3 we find a region on the boundary where a given distance function is regular, see Lemma 3.3. This step of the proof is essentially borrowed from [5]. In Section 4 we show that certain minimizing geodesics in \( M \) are mapped by \( \phi \) to geodesics in \( M' \) (but not necessarily preserving the arc length). Similar proof steps can be found in [8, 3, 4] but the details in each case are different. Finally in Section 5 we prove the key Lemma 5.1, which in a sense reconstructs the metric tensor at a point, and deduce the theorems. The arguments in Section 5 are similar to those in [4] with some modifications.

**Remarks on regularity.** The Riemannian manifolds in this paper are \( C^\infty \). This is different from [4] where the proof of the main result requires only sectional curvature bounds and works for Alexandrov spaces as well. More regularity is needed in Proposition 2.4 whose proof in [4] depends on smooth extension of the metric beyond the boundary, and in Lemma 3.3 where we rely on properties of cut loci.

It is plausible that the result holds under weaker regularity assumptions such as uniform bounds on the sectional curvature and the second fundamental form of the boundary. Such an improvement would imply stability of manifold determination with respect to Gromov-Hausdorff topology, cf. [4, Proposition 6.4].

### 2. Preliminaries and notation

Let \( M, g, \mathcal{D}, U, M', g', \mathcal{D}', U' \) be as in Theorem 2. We denote by \( D_x \) and \( D'_x \) the distance difference functions defined by (1.1) for \( M \) and \( M' \), resp. In the sequel a number of lemmas are stated only for \( M \) but they apply to both \( M \) and \( M' \).

**Notation.** For \( x \in M \), we denote by \( T_x M \) the tangent space of \( M \) at \( x \) and by \( S_x M \) the unit sphere of \( T_x M \) (with respect to \( g \)). That is,
\[
S_x M = \{ v \in T_x M : \|v\|_g = 1 \}.
\]
As usual we write \( \langle u, v \rangle \) instead of \( g(u, v) \) for \( u, v \in T_x M \).

For \( x \neq y \in M \), we denote by \([xy]\) a shortest path from \( x \) to \( y \). In general, a shortest path is not unique; we assume that some choice of \([xy]\) is fixed for every pair \( x, y \). By \( \overrightarrow{xy} \) we denote the unit tangent vector of \([xy]\) at \( x \). That is, \( \overrightarrow{xy} \in S_x M \) is the initial velocity of the unit-speed parametrization of \([xy]\).

We need the following standard implication of the Gauss Lemma: If \( x \neq y \in M \) and the distance function \( d_M(\cdot, y) \) is differentiable at \( x \) then the Riemannian gradient of this function at \( x \) is given by
\[
(2.1) \quad \text{grad}_g d_M(\cdot, y)|_x = -\overrightarrow{xy}.
\]
Equivalently, the differential of \( d_M(\cdot, y) \) at \( x \) is given by
\[
(2.2) \quad d_x d_M(\cdot, y) = -\langle \cdot, \overrightarrow{xy} \rangle_g
\]
for all \( v \in T_x M \).

To see why (2.1) holds (even in manifolds with boundary), observe that \( d_M(\cdot, y) \) is a 1-Lipschitz function and it decays with speed 1 along \([xy]\). Hence \(-\overrightarrow{xy}\) is the
direction of maximum growth of this function and the growth rate is 1, therefore it is the gradient.

We regard the target space $C(F \times F)$ of $D$ with the sup-norm distance,

$$\|u - v\| = \sup_{x,y \in F} |u(x, y) - v(x, y)|, \quad u, v \in C(F \times F).$$

If $F$ is not compact then this distance can attain infinite values. However the distance between functions from $D(M)$ is always finite. Indeed, the triangle inequality for $d_M$ implies that

$$\|D(x) - D(y)\| \leq 2d_M(x, y) < \infty$$

for all $x, y \in M$. This inequality also shows that $D$ is a 2-Lipschitz map.

We need the following result from [4].

Proposition 2.1 ([4, Proposition 7.1]). The map $D: M \to C(F \times F)$ is a locally bi-Lipschitz homeomorphism between $M$ and $D(M)$.

Now, applying Proposition 2.1 to $M$ and $M'$ and using the assumption (1.3) of Theorem 2, we can define a locally bi-Lipschitz homeomorphism $\phi: U \to U'$ by

$$\phi = (D')^{-1} \circ D|_U.$$ 

The definition of $\phi$ implies that

$$(2.3) \quad D_x = D'_{\phi(x)}$$

for all $x \in U$. Our ultimate goal is to prove that $\phi$ is a Riemannian isometry.

3. Nearest and almost nearest boundary points

Since $M$ is complete, all closed balls of the metric $d_M$ are compact, see e.g. Proposition 2.5.22. Therefore for every $x \in M$ there exists at least one nearest boundary point, i.e., a point $y \in F$ realizing the minimum of the function $d_M(x, \cdot)|_F$.

Lemma 3.1. Let $x \in U$ and let $y \in F$ be a nearest boundary point to $x$ in $M$. Then $y$ in a nearest boundary point to $\phi(x)$ in $M'$.

Proof. A point $y \in F$ is a nearest boundary point to $x$ if and only if

$$\forall z \in F \quad D_x(y, z) \leq 0,$$

see (1.1). By (2.3), this relation implies the same one for $D'_{\phi(x)}$ in place of $D_x$. Hence $y$ in a nearest boundary point to $\phi(x)$ in $M'$.

Now we can prove that $\phi$ satisfies the last requirement of Theorem 2

Lemma 3.2. $U \cap F = U' \cap F$ and $\phi|_{U \cap F}$ is the identity map.

Proof. Since $U$ and $U'$ are open subsets of $M$ and $M'$, they are topological manifolds, possibly with boundaries $\partial U = U \cap F$ and $\partial U' = U' \cap F$. Since $\phi$ is a homeomorphism between $U$ and $U'$, it sends boundary to boundary. Thus

$$(3.1) \quad \phi(U \cap F) = U' \cap F.$$

Any point $x \in U \cap F$ is a unique nearest boundary point to itself. This and Lemma 3.1 imply that $x$ is a unique nearest boundary point to $\phi(x)$. Since $\phi(x) \in F$ by (3.1), it follows that $\phi(x) = x$. Thus $U \cap F = U' \cap F$ and $\phi|_{U \cap F}$ is the identity.
Let $p \in M \setminus F$ and let $q \in F$ be a nearest boundary point to $p$. Consider a shortest path $[pq]$. It meets $F$ only at $q$, therefore it is a Riemannian geodesic. Furthermore, the first variation formula implies that $[pq]$ meets $F$ orthogonally at $q$. These properties imply that $[pq]$ is a unique shortest path between $p$ and $q$.

As shown in [5, Lemma 2.13], $p$ and $q$ are not conjugate along $[pq]$, hence $p$ is not a cut point of $q$ and therefore the distance function $d_M(\cdot, q)$ is smooth at $p$. Since the cut-point relation is closed, similar properties hold for all boundary points sufficiently close to $q$. Namely we have the following lemma.

**Lemma 3.3.** Let $p \in M \setminus F$ and let $q \in F$ be a nearest boundary point to $x$. Then there exists an neighborhood $V \subset F$ of $q$ such that for every $z \in V$ the following holds:

1. There is a unique shortest paths $[pz]$ in $M$;
2. $[pz] \cap F = \{z\}$ and $[pz]$ meets $F$ transversally;
3. the function $d_M(p, \cdot)$ is differentiable at $z$;
4. the function $d_M(\cdot, z)$ is differentiable at $p$.

**Proof.** The proof is similar to that of Lemma 2.14 in [5] and its essence is explained above. Here are the formal details.

In order to use standard properties of Riemannian cut loci, we extend $M$ beyond the boundary to obtain a complete boundaryless Riemannian $n$-manifold $\hat{M}$ such that $F$ is a smooth hypersurface in $\hat{M}$ separating $M$ from its complement. Since $q$ is a nearest to $p$ point of $F$, $[pq]$ is a unique shortest path between $p$ and $q$ in both $M$ and $\hat{M}$. By [5, Lemma 2.13], $p$ and $q$ are not conjugate along $[pq]$.

Therefore $q$ is not a cut point of $p$ in $\hat{M}$. Hence there is a neighborhood $W$ of $q$ in $\hat{M}$ such $d_{\hat{M}}(p, \cdot)$ is smooth on $W$ and every point $z \in W$ is connected to $p$ by a unique $\hat{M}$-minimizing geodesic whose direction at $z$ depends smoothly on $z$. Since $[pq]$ meets $F$ orthogonally at $q$ and has no other points on $F$, one can choose a neighborhood $W_0 \subset W$ of $q$ such that, for every $z \in W_0$ the $\hat{M}$-minimizing geodesic $[pz]$ intersects $F$ at most once and transversally. For $z$ from the “half-neighborhood” $W_0 \cap M$, this property implies that $[pz] \subset M$ and therefore $[pz]$ is a shortest path in both $\hat{M}$ and $M$. Hence $d_M(p, z) = d_{\hat{M}}(p, z)$ for all $z \in W_0 \cap M$.

Thus for all $z \in V := W_0 \cap F$ the requirements (1)–(3) of the lemma are satisfied. To prove (4), recall that the cut-point relation is symmetric. Hence, for every $z \in V$, $p$ is not a cut point of $z$ and a similar argument shows that $d_M(\cdot, z) = d_{\hat{M}}(\cdot, z)$ in a neighborhood of $p$. These properties imply that $d_M(\cdot, z)$ is differentiable at $p$. \qed

**Remark 3.4.** By Lemma 3.1, a nearest boundary point $y$ to $x \in U$ is also a nearest boundary point to $\phi(x)$ in $M'$. Applying Lemma 3.3 to both manifolds and taking the intersection of the respective neighborhoods, we obtain a neighborhood $V \subset F$ satisfying the requirements (1)–(4) of the lemma for both $x \in M$ and $\phi(x) \in M'$.

4. **Shortest paths to boundary points**

The main result of this section is Lemma 4.2 about $\phi$-images of certain geodesics (compare with [8, Lemma 2.9] and [4, Lemma 6.3]). The following preparation lemma characterizes these geodesics in terms of distance difference functions.

**Lemma 4.1.** Let $p \in M \setminus F$ and let $z \in F$ be a point satisfying conditions (1)–(4) from Lemma 3.3. Then for every $x \in M$ the following holds: $x \in [pz]$ if and only
if the function \( \Phi: F \to \mathbb{R} \) given by
\[
\Phi(y) = D_p(y, z) - D_x(y, z), \quad y \in F,
\]
where \( D_p = D(y) \) and \( D_x = D(x) \) (see (1.1) and (1.2)) attains its maximum at \( z \).

Proof. Substituting (1.1) into (4.1) yields that \( \Phi(y) = \Psi(y) + C \) where
\[
\Psi(y) = d_M(p, y) - d_M(x, y)
\]
and \( C = d_M(x, z) - d_M(p, z) \) does not depend on \( y \). We prove the statement of lemma for \( \Psi \) instead of \( \Phi \). The statements for \( \Phi \), \( \Psi \) are equivalent since the two functions have the same points of maxima.

To prove the “only if” part, consider \( x \in [pz] \). For all \( y \in F \) we have
\[
\Psi(y) = d_M(p, y) - d_M(x, y) \leq d_M(p, x)
\]
by the triangle inequality. Since \( x \in [pz] \), this inequality turns into equality for \( y = z \). Hence
\[
\Psi(z) = d_M(p, x) = \max_{y \in F} \Psi(y).
\]
Thus \( z \) is a point of maximum of \( \Psi \).

To prove the “if” part, consider \( x \in M \) and assume that \( \Psi \) attains its maximum at \( z \). First we show that \( \overrightarrow{zx} = 0 \). Suppose the contrary. Let \( v \in T_zF \) be the orthogonal projection of the vector \( \overrightarrow{zx} \) to \( T_zF \). Since \( \overrightarrow{zx} \) and \( \overrightarrow{zp} \) belong to the hemisphere of \( S_zM \) bounded by the hyperplane \( T_zF \subset T_zM \), these two vectors have different projections to \( T_zF \). Hence \( v \neq 0 \) and moreover
\[
\langle v, \overrightarrow{zx} - \overrightarrow{zp} \rangle > 0.
\]
Let \( \gamma: [0, \varepsilon) \to F \) be a smooth curve with \( \gamma(0) = z \) and \( \dot{\gamma}(0) = v \). By our assumptions the function \( d_M(p, \gamma) \) is differentiable at \( z \), hence by (2.2),
\[
\frac{d}{dt}d_M(p, \gamma(t))|_{t=0} = -\langle v, \overrightarrow{zp} \rangle.
\]
Construct a smooth variation of curves \( \{\sigma_t\}, t \in [0, \varepsilon), \) where \( \sigma_0 = [xz] \) and \( \sigma_t \) connects \( x \) to \( \gamma(t) \) for every \( t \). By the first variation formula,
\[
\frac{d}{dt} \text{length}(\sigma_t) = -\langle v, \overrightarrow{zx} \rangle.
\]
Hence
\[
d_M(x, \gamma(t)) \leq \text{length}(\sigma_t) \leq d_M(x, z) - t\langle v, \overrightarrow{zx} \rangle + o(t), \quad t \to 0.
\]
This and (4.3) imply that
\[
\Psi(\gamma(t)) = d_M(p, \gamma(t)) - d_M(x, \gamma(t)) \geq \Psi(z) + t\langle v, \overrightarrow{zx} - \overrightarrow{zp} \rangle + o(t), \quad t \to 0.
\]
By (4.2), this implies that \( \Psi(\gamma(t)) > \Psi(z) \) for a sufficiently small \( t > 0 \). Hence \( \Psi(z) \) is not a maximum of \( \Psi \), a contradiction.

This contradiction shows that \( \overrightarrow{zx} = 0 \), hence either \( x \in [zp] \) or \( p \subset [zx] \). It remains to rule out the latter case. Suppose that \( p \subset [zx] \). Then we can repeat the argument of the “only if” part with \( p \) and \( z \) swapped. Namely, for all \( y \in F \),
\[
\Psi(y) = d_M(p, y) - d_M(x, y) \geq -d_M(p, x)
\]
by the triangle inequality. This inequality turns into equality only for \( y = z \) since \( [pz] \) intersects \( F \) only at \( z \) and this intersection is transversal. Thus \( \Psi(z) \) is a strict minimum of \( \Psi \) rather than the maximum, a contradiction. This finishes the proof of the “if” part and of the lemma. \( \square \)
Now we are in a position to prove the main result of this section.

**Lemma 4.2.** Let \( p \in U \setminus F, \ p' = \phi(p) \) and let \( V \subset F \) be a neighborhood constructed in Remark 3.4. Then for every \( z \in V, \)
\[
\phi([pz] \cap U) = [p'z] \cap U'
\]
where the shortest paths in the left- and right-hand side are in \( M \) and \( M' \), resp.

**Proof.** Since \( \phi \) is a bijection between \( U \) and \( U' \), we can reformulate the lemma as follows: a point \( x \in U \) belongs to \([pz]\) if and only if \( \phi(x) \) belongs to \([p'z]\). By Lemma 3.1 \( x \in [pz] \) if and only if \( z \) is a point of maximum of the function \( \Phi \).

5. DERIVATIVE OF \( \phi \) AND PROOF OF THE THEOREMS

Recall that \( \phi : U \to U' \) is a locally bi-Lipschitz homeomorphism. By Rademacher’s theorem, every locally Lipschitz is differentiable almost everywhere. Applying this to \( \phi \) and \( \phi^{-1} \) yields that \( \phi \) is differentiable a.e. and its differential \( d\phi \) at any differentiability point \( x \in U \) is a non-degenerate linear map from \( T_xM \) to \( T_{\phi(x)}M' \).

In the next key lemma we show that this differential is an isometry.

**Lemma 5.1.** Let \( p \in U \) be a point where \( \phi \) is differentiable and \( p' = \phi(p) \). Then the differential \( d\phi : T_pM \to T_{p'}M' \) is a linear isometry with respect to \( g \) and \( g' \).

**Proof.** Let \( V \subset F \) be a neighborhood constructed in Remark 3.4. Then every point \( z \in V \) satisfies conditions (1)–(4) of Lemma 3.3 for both \( p \) and \( p' \) in \( V \).

These conditions imply that for every \( z \in V \), there is a unique shortest path \([pz]\) and it initial direction \( \overrightarrow{pz} \) depends continuously on \( z \). Hence the map \( z \mapsto \overrightarrow{pz} \) is a homeomorphism from \( V \) onto an open subset \( \Sigma \) of the sphere \( S_\rho M \).

Pick two different vectors \( v_1, v_2 \in \Sigma \) and let \( z_1, z_2 \in V \) be such that \( v_i = \overrightarrow{pz_i}, \ i = 1, 2 \). Let \( \gamma_i, i = 1, 2 \), denote the unit-speed parametrization of \([pz_i] \) with \( \gamma_i(0) = p \). By the choice of \( V \) (see Lemma 3.3(4)) the function
\[
t \mapsto D_{\gamma_1(t)}(z_1, z_2) = d_M(\gamma_1(t), z_1) - d_M(\gamma_1(t), z_2)
\]
is differentiable at \( t = 0 \), and by (2.2) its derivative is given by
\[
\frac{d}{dt} D_{\gamma_1(t)}(z_1, z_2)\big|_{t=0} = -1 + \langle v_1, v_2 \rangle.
\]

Similarly (swapping \( v_1 \) and \( v_2 \)),
\[
\frac{d}{dt} D_{\gamma_2(t)}(z_2, z_1)\big|_{t=0} = -1 + \langle v_1, v_2 \rangle.
\]
The scalar products above are \( g \)-products in \( T_pM \).

Now consider the images of these curves and vectors under \( \phi \) and \( d\phi \). For \( i = 1, 2 \), define
\[
\lambda_i = \|d\phi(v_i)\|_{g'}
\]
and

\[ w_i = \frac{d_p \phi(v_i)}{\lambda_i}. \]

Let \( \varepsilon > 0 \) be such that \( \gamma_i([0, \varepsilon]) \subset U \) for \( i = 1, 2 \). Then by Lemma 5.1, \( \phi \circ \gamma_i \mid [0, \varepsilon) \) parametrizes an initial interval of \( [p'z_i] \). The velocity of \( \phi \circ \gamma_i \mid [0, \varepsilon) \) at 0 equals

\[ d_p \phi(v_i) = \lambda_i w_i, \]

hence \( w_i = p'z_i, \ i = 1, 2. \)

Now similarly to (5.1) we calculate the derivative

\[ \frac{d}{dt} D_{\phi(\gamma_i(t))}(z_1, z_2) \bigg|_{t=0} = \lambda_1(-1 + \langle w_1, w_2 \rangle). \]

By (2.3), the functions differentiated in (6.1) and (5.3) are the same, hence

\[ -1 + \langle v_1, v_2 \rangle = \lambda_1(-1 + \langle w_1, w_2 \rangle) \]

where \( \langle w_1, w_2 \rangle \) is the scalar product with respect to \( g' \). Similarly from (5.2) we obtain that

\[ -1 + \langle v_1, v_2 \rangle = \lambda_2(-1 + \langle w_1, w_2 \rangle). \]

By (5.2) and (5.5),

\[ \lambda_1(-1 + \langle w_1, w_2 \rangle) = \lambda_2(-1 + \langle w_1, w_2 \rangle), \]

therefore \( \lambda_1 = \lambda_2 \) (note that \( \langle w_1, w_2 \rangle \neq 1 \) since \( w_1 \) and \( w_2 \) are different unit vectors). Substituting the definitions of \( \lambda_1 \) and \( \lambda_2 \) we obtain that

\[ ||d_p \phi(v_1)||_{g'} = ||d_p \phi(v_2)||_{g'} \]

Since \( v_1 \) and \( v_2 \) are arbitrary vectors from \( \Sigma \), this identity implies that the function \( v \mapsto ||d_p \phi(v)||_{g'} \) is constant on \( \Sigma \). We denote this constant by \( \lambda \). Since \( \Sigma \) is an open subset of the sphere \( S_p M \), it follows that \( d_p \phi \) is a \( \lambda \)-homothetic linear map:

\[ ||d_p \phi(v)||_{g'} = \lambda ||v||_g \]

for all \( v \in T_p M \). Hence \( d_p \phi \) preserves the angles, in particular \( \langle v_1, v_2 \rangle = \langle w_1, w_2 \rangle \).

Proof of the theorems 1 and 2. As shown in Lemma 6.1, the derivative of our bi-Lipschitz homeomorphism \( \phi \) is an isometry almost everywhere. Hence \( \phi \) is a 1-Lipschitz map, i.e. it does not increase arclength distances. The same holds for \( \phi^{-1} \), therefore \( \phi \) is a distance isometry. By the Myers-Steenrod theorem (9, see also 10, Ch. 5, Theorem 18) every distance isometry between Riemannian manifolds is a smooth Riemannian isometry. Thus \( \phi \) is a Riemannian isometry. Lemma 3.2 implies the last claim of Theorem 2 and this finishes the proof of Theorem 2. As explained in the introduction, Theorem 2 implies Theorem 1.

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