Hadronic multiplicity at RHIC (and LHC)

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Abstract. We give a theoretical interpretation of experimental data on hadronic multiplicity, as a function of centrality and rapidity, in $Au - Au$ and $d - Au$ collisions at RHIC, in the framework of the Parton Saturation Model. We also present, in the same approach, our predictions for $p - Pb$ and $Pb - Pb$ collisions at LHC energies.

1. High energy strong interactions

Hadronic scattering at high energies allow us to explore the region of small Bjorken’s $x$, where the coupling constant $\alpha_s$ of the strong interactions is small but the density of partons (prevalently gluons) is so large that the conventional perturbative approach is not applicable because of strong non-linear effects. In fact, the one-gluon exchange approximation, the basis of normal perturbative QCD (pQCD) calculations, is valid only when the gluons interact independently from each other, and this cannot be true if their density is too high.

A more precise way of stating the problem is to note that the partons taking part in a high energy scattering have energies between $E_{hadron}$ (the hadron energy) and a minimum value of the order of $m_\pi$ (pion mass). The Bjorken’s $x$, i.e. the fraction of longitudinal momentum carried by the parton, is $x = E_{parton}/E_{hadron}$ and it ranges from $m_\pi/E_{hadron}$ to 1. It is evident that a higher incident energy $E_{hadron}$ provides experimental access to regions of smaller $x$.

On the other hand, the number of gluons per unit rapidity in a proton, $dN/dy = xG(x, Q^2)$, as measured at HERA [2] for various $x$ and $Q^2$, rises rapidly at small $x$. This rise becomes more and more dramatic with increasing $Q^2$, as can be seen in figure 1.

It is evident that at very high $Q^2$ and at very small $x$ the normal evolution equations, based on pQCD, are no longer valid: the parton distribution functions can not grow unlimitedly, because this would correspond to an uncontrolled growing of the cross-section (calculated as a convolution of the parton distribution functions) which would eventually violate the unitarity bound. We conclude, therefore, that the number of gluons $xG(x, Q^2)$ (the contribution from quarks and antiquarks in these small-$x$ regions is irrelevant) must saturate. We then see how by a simple, logical argument, the idea of parton saturation arises naturally in high energy strong interactions. I will give here a brief sketch of the theory underlying the parton saturation model, namely the Color Glass Condensate. A more satisfactory presentation can be found in [3] and references therein.

2. The Color Glass Condensate

The matter controlling high energy strong interactions is universal, independent of the hadron (including heavy nuclei) under consideration. It exists over sizes larger than $l \sim 1$ fm, $\tau 10^{-23}$ sec.

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1 Work done in collaboration with D. Kharzeev (BNL) and E. Levin (Tel Aviv Univ.) [1]
Its name, Color Glass Condensate (CGC), originates from:

**Color**: the gluons are colored.

**Glass**: the gluons at small $x$ are emitted by other partons at larger $x$. In the infinite momentum frame, these larger momentum partons travel very fast and their natural time scales are Lorentz dilated. As a consequence, the low $x$ gluons evolve very slowly compared to their own natural time scales. Thus they form a “glass”.

**Condensate**: the parton saturation occurs when the two competing processes, gluon emission and gluon recombination, compensate each other. The emission rate is proportional to the gluon density $\rho$, while the recombination rate depends on the interactions between gluons, namely it is proportional to $\rho^2$ times the coupling constant $\alpha_s$. Therefore we have saturation when the condition $\rho \sim \alpha_s \rho^2$ is fulfilled, giving $\rho \propto 1/\alpha_s$, characteristic of Bose condensation phenomena.

It follows that there is a critical momentum scale $Q_s$ which controls the occupation number through $1/\alpha_s(Q_s)$; this is the *saturation scale*. The transverse phase space density $n \equiv (1/\pi R^2)(dN/dy dp_T^2)$ ($R$ is the transverse size of the hadron or the nucleus under consideration; $n$ is the two-dimensional transverse density, the density in the longitudinal direction will be assumed uniform: this is a common assumption in heavy ion collisions phenomenology) is constant (i.e. “saturated”) for $p_T < Q_s$; for $p_T > Q_s$ the gluon density is smaller than the critical one, the gluons therefore interact independently and the perturbative result $n \propto \alpha_s(p_T)/(R^2 p_T^2)$ is recovered.

The CGC picture is based on a classical effective theory, originally proposed by McLerran and Venugopalan [4] to describe the gluon distribution in large nuclei: the valence quarks of the hadrons (fast partons) are treated as a source for a classical color field representing the small-$x$ (soft) gluons. The classical approximation is appropriate since the saturated gluons have large occupation numbers. The theory implies a non linear renormalization group equation [5]. The CGC approach is justified in the limit of $Q_s \gg \Lambda_{QCD}$: as we will see in the following, at RHIC energies this condition is only marginally satisfied but LHC will provide a better opportunity to test this theory.
In the recent years, a significant experimental evidence has been found in support of the hypothesis of the CGC \[3\], mainly from HERA and RHIC. In particular the RHIC data concern the multiplicity of produced hadrons and will be the focus of the rest of this paper.

3. QCD in the high density regime and heavy ion collisions

I will introduce here the basic ideas of parton saturation in an intuitive way. Consider the wave function of a nucleus boosted to a large momentum. The nucleus is Lorentz–contracted, and partons “live” on a thin sheet in the transverse plane. Each parton occupies the transverse area $\pi/Q^2$ determined, by uncertainty principle, by its transverse momentum $Q$, and can be probed with the cross section $\sigma \sim \alpha_s(Q^2)/Q^2$. On the other hand, the entire transverse area of the nucleus is $S_A \sim \pi R^2_A$. Therefore, if the number of partons exceeds

$$N_A \sim \frac{S_A}{\sigma} \sim \frac{\pi}{\alpha_s(Q^2)} Q^2 R^2_A,$$

(1)

they will begin to overlap in the transverse plane and start interacting with each other, which prevents further growth of parton densities. This happens when the transverse momenta of the partons are on the order of

$$Q_s^2 \sim \alpha_s(Q_s^2) \frac{N_A}{\pi R^2_A} \sim A^{1/3},$$

(2)

which is called the “saturation scale”. In the saturation regime, as is apparent from equations. (1) and (2), the multiplicity of the produced partons should be proportional to

$$N_s \equiv xG_A(x, Q^2_s) \sim \frac{\pi}{\alpha_s(Q_s^2)} Q_s^2 R^2_A \sim A.$$

(3)

($xG_A(x, Q^2_s)$ is the gluon distribution function for a nucleus of atomic number $A$). From equation (1) it is also evident that the saturation scale $Q_s$ depends on the nucleus: the heavier the nucleus, the larger the saturation scale. This is not in contrast with the fact that the CGC is an universal form of matter: the matter is the same, for all hadrons and nuclei, but the momentum scale at which it starts to reveal itself changes with the hadron and with the incident energy, as we will see in the following. This is also the reason why the experiments with high energy heavy ions are so interesting: they allow a larger $Q_s$ for the starting of saturation phenomena, they are more “efficient” because in order to see the same effect in hadron collisions, one would need much higher energies.

In the weak coupling regime, the density of partons ($\rho \propto 1/\alpha_s$) becomes very large. In the first approximation, the multiplicity in this high density regime scales with the number of participants. There is, however, an important logarithmic correction to this from the evolution of parton structure functions with the saturation scale $Q^2_s$, which we discuss below.

While this “derivation” has been very simplistic, the formulas (2),(3) can be reproduced by more sophisticated methods \[6, 7\], which also allow to reconstruct the coefficient of proportionality in (2):

$$Q_s^2 = \frac{8\pi^2 N_c}{N_c^2 - 1} \alpha_s(Q_s^2) \frac{xG(x, Q^2_s) n_{\text{part}}(b)}{2},$$

(4)

where $N_c = 3$ is the number of colors and $n_{\text{part}}$ is the density of participants in the transverse plane. We divide $n_{\text{part}}$ by 2 to get the density of those nucleons in a single nucleus which will participate in the collision at a given impact parameter. equation (4) implies that $Q_s^2$ depends also on the impact parameter $b$, i.e. on the centrality of the collision, through the density $n_{\text{part}}$, as we will see in the next section.
Table 1. $\sigma_{\text{in}}$ at various energies.

| $\sqrt{s}$ [GeV] | $\sigma_{\text{in}}$ [mb] |
|------------------|--------------------------|
| 20               | 30                       |
| 56               | 37                       |
| 130              | 41                       |
| 200              | 42                       |
| 5500             | 70                       |

4. Geometry of the collision

The most popular method of evaluating the relevant variables characterizing the geometry of the collision in a nucleus-nucleus interaction (number of participating nucleons, number of nucleon-nucleon collisions) is the Glauber model, based on eikonal approximation [8].

At high energies the paths of the colliding nucleons can be approximated by straight lines, since in a typical interaction $t/s \ll 1$ and the typical scattering angle is small. This is the most important approximation underlying the Glauber approach to nuclear interactions. Other approximations which simplify calculations but are in principle unnecessary are the smallness of the nucleon–nucleon interaction radius compared to the typical nuclear size, and the neglect of the real part of the $NN$ scattering amplitude. A complete set of the relevant formulas can be found e.g. in [9].

It is customary and convenient to parameterize the centrality of the collision in terms of the “number of participants” $N_{\text{part}}$ – the number of nucleons which underwent at least one inelastic collision. This number can be directly measured experimentally (at least in principle) by detecting in the forward rapidity region the number of “spectator” nucleons $N_{\text{spect}}$ which did not take part in any inelastic collisions; obviously, for a nucleus with mass number $A$, $N_{\text{part}} = A - N_{\text{spect}}$.

The number of participating nucleons in a nucleus-A–nucleus-B interaction depends on the impact parameter $b$. In the eikonal approximation it can be evaluated as (see [8]):

$$N_{\text{part}}(b) = \int d^2 s \; n_{\text{part}}^A(b, s) = A \int d^2 s \; T_A(s) \left\{ 1 - [1 - \sigma_{\text{in}} T_B(b - s)]^B \right\} + B \int d^2 s \; T_B(b - s) \left\{ 1 - [1 - \sigma_{\text{in}} T_A(s)]^A \right\},$$ (5)

with the usual definition for the nuclear thickness function $T_A(s) = \int_{-\infty}^{\infty} dz \; \rho_A(z, s)$, normalized as $\int d^2 s \; T_A(s) = 1$; $\sigma_{\text{in}}$ is the proton-proton inelastic cross-section without diffractive component, with values given in table 1 [10, 11]. The nuclear density profile $\rho_A(z, s)$ is assumed to be a three-parameter Fermi distribution$^2$, with the parameters given in [12]. The decomposition of the coordinate vector $r$ into its longitudinal and transverse components $(z, s)$ is natural in this context.

$^2$ The only exception is the deuteron, for which $\rho_D$ is the square of the wave function integrated over the angular variables. In this case, however, a Monte Carlo calculation is more appropriate than the formulas given in this section (see [13, 1]). The reason is simply that a deuteron cannot be described as an homogeneous nucleus of radius about 2 fm and atomic number 2: this would imply, in the present approach, an unrealistically small density!
\( n_{part}^{AB}(b) \) the definition of the local density of participants \( n_{part}^{AB}(b, s) \) is evident; we will define its average over the transverse plane as

\[
\langle n_{part}^{AB}(b) \rangle = \frac{\int d^2 s \left| n_{part}^{AB}(b, s) \right|^2}{\int d^2 s n_{part}^{AB}(b, s)} .
\]

(6)

In the following we will need to use the average number of participants computed separately for nucleus-A and nucleus-B; it is given by

\[
\langle n_{part,A}^{AB}(b) \rangle = \frac{\int d^2 s n_{part,A}^{AB}(b, s) n_{part}^{AB}(b, s)}{\int d^2 s n_{part}^{AB}(b, s)} .
\]

(7)

\( n_{part,A}^{AB}(b) \) and \( n_{part,B}^{AB}(b) \) are the integrands of the first term and second term in the r.h.s of Eq. (5) respectively. Obviously, one has for their sum

\[
\langle n_{part,A}^{AB}(b) \rangle + \langle n_{part,B}^{AB}(b) \rangle = \langle n_{part}^{AB}(b) \rangle .
\]

The corresponding formulas for the proton–nucleus pA interaction can be deduced by setting \( B = 1 \) and using a delta-function for the proton thickness function (in the point-like approximation for the size of the proton). We get from Eq. (5):

\[
N_{part}^{pA}(b) = A \sigma_{in} T_A(b) + \left\{ 1 - [1 - \sigma_{in} T_A(b)]^A \right\} = A \sigma_{in} T_A(b) + \left\{ 1 - P_0^{pA}(b) \right\} .
\]

(8)

In the previous formula the function \( P_0^{pA}(b) \) is the probability of no interaction in a pA collision at impact parameter \( b \); the integration of \( [1 - P_0^{pA}(b)] \) over \( b \) gives the inelastic proton-nucleus cross section \( \sigma_{pA} \).

The average number of participants in a pA collision can be obtained as:

\[
\langle N_{part}^{pA} \rangle = \frac{\int d^2 b N_{part}^{pA}(b)}{\int d^2 b [1 - P_0^{pA}(b)]} = A \sigma_{in} T_A(b) + 1 ;
\]

(9)

the first term in the r.h.s. gives the mean number of participants \( \langle N_{part,A}^{pA} \rangle \) in the nucleus. As in the case of nucleus-nucleus collision, we will need to compute the mean density of participants in nucleus A, defined as:

\[
\langle n_{part,A}^{pA} \rangle = \frac{\langle N_{part,A}^{pA} \rangle}{\sigma_{in}} = \frac{A}{\sigma_{pA}} .
\]

(10)

In practice, the information about the impact parameter dependence is extracted by analyzing the data in various centrality bins. The physical observable most frequently used to estimate the centrality of the collision is the multiplicity of charged particles \( N_{ch} \). We will assume that the average value of \( N_{ch} \) produced in a collision at impact parameter \( b \) is determined by the number of participating nucleons \( N_{part}(b) \). The actual multiplicity will fluctuate around its mean value according to:

\[
P(N_{ch}, \langle N_{ch}(b) \rangle) = \frac{1}{\sqrt{2\pi a(N_{ch}(b)} \} C(\langle N_{ch}(b) \rangle) \exp \left\{ - \frac{[N_{ch} - \langle N_{ch}(b) \rangle]^2}{2a(N_{ch}(b))} \right\} ,
\]

(11)

where the factor \( C(N) \equiv 2/[1 + erf(\sqrt{N/2a})] \) is introduced to ensure that the fluctuation function \( P(N_{ch}, N) \) satisfies \( \int_0^\infty dN_{ch} P(N_{ch}, N) = 1 \). The numerical value of \( C(N) \) is 1 with very good accuracy for almost all cases of practical interest (it can exceed 1 for very peripheral
collisions, where the number of participants and consequently $N_{ch}$ is small: in such a case it is important to include the factor $C(N)$ to have a correct normalization).

The parameter $a$ gives the width of the fluctuations and depends on the experimental apparatus: it can be fixed by fitting the minimum bias distribution of events as was done, for example, in [14]. We will also assume the proportionality between $N_{ch}$ and $N_{part}$ (this is valid only for the total hadron yields) when computing the differential inelastic cross section; this proportionality is not exact, but the shape of minimum bias distribution has been found insensitive to this assumption ([14]).

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The minimum bias differential cross section can be obtained as $\left( N(b) \equiv qN_{part}(b), \text{where } q \text{ is a constant} \right)$:

$$\frac{d\sigma_{mb}}{dN_{ch}} = \int d^2b P(N_{ch}, N(b)) \left[ 1 - P_0(b) \right]; \quad (12)$$

Here $P_0(b)$ is the probability of no interaction at the impact parameter $b$: for a nucleus-nucleus collision $P_0(b) = [1 - \sigma_{in} T_{AB}(b)]^{AB}$, where $T_{AB}$ is the overlap function: $T_{AB}(b) = \int d^2s T_A(s) T_B(b-s)$; in the case of $B=1$, $P_0(b)$ reduces to $P_{0}^{pA}(b)$ defined above. In the following, all of the formulas will refer to A-B collisions; with obvious modifications they are valid also in the p-A case.

The total nucleus-nucleus cross section is then obtained by integrating Eq. (12) over $dN_{ch}$:

$$\sigma_{AB} = \int dN_{ch} \frac{d\sigma_{mb}}{dN_{ch}} = \int d^2b \left[ 1 - P_0(b) \right]. \quad (13)$$

The mean value of any physical observable $O$ (known in terms of the impact parameter $b$) can be computed as:

$$\langle O \rangle = \frac{1}{\sigma_{AB}} \int dN_{ch} \frac{d\sigma_{mb}}{dN_{ch}} O(b) \quad (14)$$

To obtain the corresponding average for a given centrality cut we have to limit the integrations in the previous formula in the appropriate way, for instance the expression:

$$\langle O \rangle \bigg|_{N_{ch}>N_0} = \frac{\int_{N_0} dN_{ch} \frac{d\sigma_{mb}}{dN_{ch}} O(b)}{\int_{N_0} dN_{ch} \frac{d\sigma_{mb}}{dN_{ch}}} \quad (15)$$

gives the average value of the observable $O$ in the fraction of the total cross section defined by the limit $N_0$.

5. Hadronic multiplicity

We will see now how to apply the saturation idea presented in section 3 to calculate the multiplicity production in heavy ion collisions.

First of all, we have to give a numerical estimate of the saturation scale: we refer, for this purpose, to the first RHIC data (central $Au - Au$ collision at $\sqrt{s} = 130$ GeV and pseudorapidity $|\eta| < 1$ [15]).

Equation (4) can be solved by iterations; a self–consistent solution can be found at $Q_s^2 \simeq 2 \text{ GeV}^2$ if we use $xG(x, Q_s^2) \simeq 2$ [16] at $x \simeq 2Q_s/\sqrt{s} \simeq 0.02$, with $\alpha_s(Q_s^2) \simeq 0.6$. We use an explicit expression [7] for the number of produced partons

$$\frac{d^2N}{d^2bd\eta} \bigg|_{|\eta|<1} = c \frac{N_c^2 - 1}{4\pi^2 N_c} \frac{1}{\alpha_s} Q_s^2,$$  \quad (16)
where $c$ is the “parton liberation” coefficient accounting for the transformation of virtual partons in the initial state to the on–shell partons in the final state. Integration over the transverse coordinate and the use of (4) yield simply

$$\frac{dN}{d\eta} \bigg|_{|\eta|<1} = c N_{\text{part}} x G(x, Q_s^2). \quad (17)$$

If we assume that $dN/d\eta \simeq 3/2 \frac{dn_{\text{exp}}}{d\eta}$, take $x G(x, Q_s^2) \simeq 2$ at $Q_s^2 \simeq 2 \text{ GeV}^2$ and $x \simeq 2Q_s/\sqrt{s} \simeq 0.02$ [16], and use $N_{\text{part}} \simeq 339$ (obtained as explained in section 4) for the 6% centrality cut, the experimental number [15] $dn/d\eta = 555 \pm 12(\text{stat}) \pm 35(\text{syst})$ translates into the following value of the “parton liberation” coefficient:

$$c = 1.23 \pm 0.20. \quad (18)$$

This number appears to be close to unity, as expected by Mueller [7], which implies a very direct correspondence between the number of the partons in the initial and final states. Moreover, this may imply that the number of particles is conserved through the parton–to–hadron transformation – a miraculous fact first noted in the context of the “local parton–hadron duality” hypothesis [17].

The value (18) can be compared to the lattice calculation [18] by Krasnitz and Venugopalan, which yields $c = 1.29 \pm 0.09$. An analytical calculation for $c$ has been presented by Kovchegov [19], with the result $c = 2 \ln 2 \simeq 1.39$.

To compute the centrality dependence, we still need to know the evolution of the gluon structure function with the density of partons, which is proportional to the mean density of participants in the transverse plane. We will assume that this evolution is governed by the DGLAP equation [20], and take

$$x G(x, Q_s^2) \sim \ln \left( \frac{Q_s^2}{\Lambda_{QCD}^2} \right). \quad (19)$$

The dependence (19) emerges when the radiation of gluons is treated classically, and so is consistent with equation (4). The equations (4) and (17) can now be used to evaluate the centrality dependence; with the parameters described above we get

$$\frac{2}{N_{\text{part}}} \frac{dN}{d\eta} \bigg|_{|\eta|<1} \simeq 0.82 \ln \left( \frac{Q_s^2}{\Lambda_{QCD}^2} \right), \quad (20)$$

where we take $\Lambda_{QCD} \simeq 200$ MeV and use the values of $Q_s$ obtained from equation (4).

From equation (20) it is possible, now, to calculate the centrality dependence of the hadron multiplicity at midrapidity (or, more precisely, for $|\eta| < 1$) and $\sqrt{s} = 130$ GeV. Let us see how to extend it at different energies and also how to take into account the rapidity dependence [25, 26, 13, 27].

The energy dependence of the saturation scale is assumed to be the same observed at HERA [21]; at $y = 0$:

$$Q_s^2(x) = Q_{s0}^2 \left( \frac{x}{x_0} \right)^{-\lambda} = Q_{s0}^2 \left( \frac{\sqrt{s}}{\sqrt{s_0}} \right)^{\frac{\lambda}{1+\lambda/2}} \quad (21)$$

(the r.h.s. is obtained with $x = Q_s/\sqrt{s}$) with $\lambda = 0.288$. equation 21 has been derived analytically in [22], and the estimate given for $\lambda$ is in excellent agreement with the experimental fit.
Figure 2. The centrality dependence (measured in terms of the number of participants) of the hadronic multiplicity at RHIC energies in the interval of pseudorapidity $|\eta| < 1$ (from [23, 24]).

Figure 2 shows the experimental data on the hadronic multiplicity measured by PHOBOS in $Au - Au$ collisions at different incident energies as a function of the number of participants (related to the impact parameter $b$). The left plot shows the data at $\sqrt{s} = 130$ GeV [23] compared to the theoretical predictions of a few models. The saturation model (labeled “Kharzeev/Nardi”) is indicated with the solid line. The normalization factor in equation (20) was chosen to fit the preliminary data published in [15] (solid square); the $N_{part}$ dependence was then a genuine prediction of the model [14] and was later confirmed by the experimental data. The right plot shows the analogous results at $\sqrt{s} = 200$ and 19.6 GeV. The same excellent agreement can be seen in the comparison with PHENIX data [1].

To describe the rapidity distribution, we have to note first that the Bjorken variables $x_1$ and $x_2$ for the interacting partons in the two colliding nuclei are related to $y$ by:

$$x_1 = \frac{Q}{\sqrt{s}} e^{-y} \quad \text{and} \quad x_2 = \frac{Q}{\sqrt{s}} e^{y}.$$  \hspace{1cm} (22)

This implies that two different saturation scales must be considered: $Q_s(x_1)$ and $Q_s(x_2)$, with

$$Q^2_s(x_1, 2) = Q^2_{s0}\left(\frac{x_1}{x_0}\right)^{-\lambda} = Q^2_{s0}\left(\frac{\sqrt{s}}{\sqrt{s_0}}\right)^{\frac{\lambda}{1+\lambda/2}} \exp\left\{\mp \frac{\lambda y}{1+\lambda/2}\right\}. \hspace{1cm} (23)$$

Different values of $y$ correspond to different regions of $x$ for the two nuclei: it is even possible that one of the two nuclei is in the saturation region and the other is not (this happens when $y$ is close to $\pm y_{beam}$).

Finally, we must introduce the unintegrated gluon distribution function $\varphi(x, k_t)$, related to $xG(x, Q^2)$ by:

$$xG(x, Q^2) = \int Q^2 \, dk_t^2 \, \varphi(x, k_t)$$  \hspace{1cm} (24)

and giving the probability of having a gluon, inside a nucleon, with longitudinal momentum fraction $x$ and transverse momentum $k_t$. The lower limit $\Lambda_{QCD}$ in the integral in equation (24) (and in the following equations) is omitted for simplicity. The corresponding function for a nucleus will be denoted by $\varphi_A(x, k_t)$. 
According to the above considerations, we assume that \( \varphi(x, k_t) \) saturates at small \( p_T \); naively, we expect that it behaves like in figure 3. For simplicity, we will assume the following (crude!) analytical expression for \( \varphi \):

\[
\varphi_A(x, k_T^2) = \begin{cases} 
\kappa' \frac{S_A}{\alpha_s} (1 - x)^4 & \text{for } k_t < Q_s(x) \\
\kappa \frac{\alpha_s}{k_T^2} (1 - x)^4 & \text{for } k_t > Q_s(x) 
\end{cases}
\]  

(25)

In the region \( k_t > Q_s \) there is no saturation, the unintegrated gluon distribution function shows normal bremsstrahlung radiation. However, in the saturation region, \( k_t < Q_s \), instead of growing, as pQCD would predict, \( \varphi_A \) saturates (the integral of (25), in this region, gives equation (3)). The factor \( S_A \) is the overlap area of the colliding nuclei (proportional to \( T_{AB}(b) \), see section 4). The constant \( \kappa \) is simply an overall normalization factor: it is included in the constant \( c \) of equation (17) and it is, therefore, irrelevant (\( \kappa' \) guarantees the continuity of \( \varphi \) at \( k_t = Q_s \)).

The gluon distribution function \( xG_A(x, p_T^2) \) is given by:

\[
xG_A(x, p_T^2) = \begin{cases} 
\kappa' \frac{S_A p_T^2}{\alpha_s} (1 - x)^4 & \text{for } p_t < Q_s(x) \\
\kappa \frac{\alpha_s}{p_T^2} S_A Q_s^2(x) (1 - x)^4 & \text{for } p_t > Q_s(x) 
\end{cases}
\]  

(26)

where a term proportional to \( \alpha_s \) has been neglected (its contribution is numerically small as compared to the one coming from the region below \( Q_s \)).

With these ingredients it is easy to write the formula for the inclusive production [28, 25]:

\[
E \frac{d\sigma}{d^3p} = \frac{4\pi N_c}{N_c^2 - 1} \frac{1}{p_T^2} \int p_t d^2 k_t \alpha_s \varphi_{A_1}(x_1, k_t^2) \varphi_{A_2}(x_2, (p - k_t)^2),
\]  

(27)

We can compute the multiplicity distribution by integrating Eq. (27) over \( p_t \), namely,

\[
\frac{dN}{dy} = \frac{1}{S} \int d^2 p_t E \frac{d\sigma}{d^3p};
\]  

(28)
\( S \) is either the inelastic cross section for the minimum bias multiplicity, or a fraction of it corresponding to a specific centrality cut.

Equation (28) (with (27)) can be integrated numerically to get the results on hadronic multiplicity that can be compared to experimental data. However we want to present here a simplified version of this formula to extract some physical idea. First, we note that in equation (27) the largest contribution to the integral comes from the two regions \( k_t \ll p_t \) and \( |k_t - p_t| \ll p_t \).

In the first case one can write \( \varphi_{A_1}(x_2, (p - k)_t^2) \sim \varphi_{A_2}(x_2, p_t^2) \), in the second case, by changing the integration variable \( k_t \to k_t - p_t \) one has \( \varphi_{A_1}(x_1, k_t^2) \to \varphi_{A_1}(x_1, (p - k)_t^2) \sim \varphi_{A_1}(x_1, p_t^2) \).

Therefore equation (28) becomes:

\[
\frac{dN}{dy} = \frac{4\pi N_c \alpha_s}{S N_c^2 - 1} \times \\
\times \int \frac{d^2p_t}{p_t^2} \left( \varphi_{A_1}(x_1, p_t^2) \int d^2k_t^2 \varphi_{A_2}(x_2, k_t^2) + \varphi_{A_2}(x_2, p_t^2) \int d^2k_t^2 \varphi_{A_1}(x_1, k_t^2) \right) = \\
= \frac{4\pi N_c \alpha_s}{S N_c^2 - 1} \int \frac{d^2p_t}{p_t^2} \left( \varphi_{A_1}(x_1, p_t^2) x_2 G_{A_1}(x_2, p_t^2) + \varphi_{A_2}(x_2, p_t^2) x_1 G_{A_1}(x_1, p_t^2) \right) = \\
= \frac{4\pi N_c \alpha_s}{S N_c^2 - 1} \int \frac{d^2p_t}{p_t^2} \frac{d^2}{d^2p_t} \left( x_1 G_{A_1}(x_1, p_t^2) x_2 G_{A_2}(x_2, p_t^2) \right). \tag{29}
\]

With the substitution of (26) one observes that the hadronic multiplicity for any \( y \) is always proportional to \( SQ_{s,\text{min}} (Q_{s,\text{min}} \) is the minimum between \( Q_s(x_1) \) and \( Q_s(x_2) \) according to (23)), i.e. it is proportional to the number of participating nucleons of the saturated nucleus [13, 27]. The property of the scaling according to the number of participants is peculiar of this model and it is confirmed by experiment.

Usually the experimental results are given in terms of the pseudorapidity rather than \( y \); the conversion between the two variables is straightforward, see for instance [25].

In the formulas of this section the impact-parameter dependence has been omitted: one should however remember that the saturation scales always depends on \( \beta \) according to (4), therefore \( dN/dy \) also depends on \( \beta \). The comparison with experimental data requires also the restriction to limited intervals of centrality, according to the procedure shown by equation (15).

Figure 4 shows the results obtained in this model compared to the experimental data of \( Au - Au \) collisions at different energies (by PHOBOS and BRAHMS Collaborations). Again it should be stressed that the overall normalization is guaranteed by fitting the first PHOBOS data [15] at midrapidity for the 6\% most central collisions. The shapes of the curves are the predictions of the model. We can conclude that the present model reproduces very well all the data for the highest energies in the pseudorapidity interval \( |\eta| < 4 \). At the lowest energy, the data for the less central collisions are underestimated. This is not a surprise, since when the saturation scale becomes too small (this is the case in the peripheral collisions, but also in the region near \( \pm \eta_{beam} \) at higher energies) the saturation effects are no longer dominant in the production of particles.

The case of \( d - Au \) collisions requires some care, since the deuteron can not be treated as a heavy nucleus. The Glauber approximation used here to compute the number of participating nucleons is, therefore, not accurate. In this case, just to test our saturation model we “borrowed” the number of participants calculated by the experimental collaborations with Monte Carlo methods. The results are then in good agreement with the experimental data as, for instance, one can see in figure 5, for \( \eta > -2 \). The region \( \eta < -2 \) is the fragmentation region of the \( Au \) nucleus, where the present approach is not justified. In the right plot, the dashed line is obtained assuming \( dN/d\eta = N_{\text{part}}^{Au} \times dN_{pp}/d\eta \) in the \( Au \) fragmentation region (see [13]).
Figure 4. Rapidity distribution of charged particles in Au – Au collisions at RHIC for various centrality intervals. Left: PHOBOS data at energies $\sqrt{s} = 19.6$, 130 and 200 GeV [29]. Right: BRAHMS data at $\sqrt{s} = 200$ GeV [30]. The solid lines are the predictions of the saturation model.

Figure 5. Rapidity distribution of charged particle in $d – Au$ collisions at $\sqrt{s} = 200$ GeV for various centrality intervals (data from [31, 32]).
6. Predictions for LHC
Since the results for RHIC energies are encouraging, it is interesting to see what the saturation model predicts for $Pb - Pb$ and $p - Pb$ collisions at $\sqrt{s} = 5.5$ TeV (LHC) [27].

The main source of uncertainty is the parameter $\lambda$ in equation (21) governing the energy dependence: it is known with an experimental error, which can make the extrapolation to such high energy problematic. Moreover, equation (21) was obtained with the assumption of constant $\alpha_s$: if one takes into account the running of the coupling constant, instead of equation (21) one gets a more complicated expression for the energy dependence of $Q_s$ (see [27] and references therein). Therefore, we prefer to give a band of uncertainty to our prediction: in the following figures the solid and dashed lines indicate, respectively, the results obtained with equation (21) and those with running $\alpha_s$ (assuming the same initial condition at $\sqrt{s} = 130$ GeV).

![Figure 6. Centrality dependence of hadron production at LHC in $Pb - Pb$ interactions.](image)

Figure 6 shows the centrality dependence in $Pb - Pb$ interactions of hadronic production for different values of pseudorapidity. The pseudorapidity dependence is given in figure 7 for $Pb - Pb$ (left) and $p - Pb$ (right) collisions. As it is evident, the uncertainty in the extrapolation of the saturation scale gives a difference of about $10 - 15\%$ in the number of produced particles at midrapidity in central $Pb - Pb$ collisions, however, when compared with the spreading of results coming from other models in the literature (see [33]) ranging from 1500 to more than 6000, it indicates that the CGC gives a sufficiently precise prediction that should allow the future experiment to test it.

7. Conclusions
The results presented here and in many other recent works (see [3]) support the idea that the dense matter produced in the early stage of heavy ion collisions at RHIC can be described as a Color Glass Condensate. The CGC is interesting by itself, because it represents the universal form of high energy QCD wavefunctions at small $x$, not only of hadrons but heavy nuclei as well. It allows to study the QCD in the regime of high density and small coupling constant. It is important also for its relation with the study of deconfined matter which is the main goal of heavy ion collisions at high energy: it provides a theoretical description, based on QCD, of the initial state of the nucleus-nucleus system, from which the subsequent quark-gluon-plasma evolves.
The CGC hypothesis can be tested in a wide range of experimental environments not restricted to heavy ion collisions; further data, at RHIC and at LHC but also in $e−p$ interactions (HERA, eRHIC) are still needed both to corroborate this conclusion and to study with higher accuracy the properties of the CGC.

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