Dark Matter–admixed Rotating White Dwarfs as Peculiar Compact Objects

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Abstract

Discoveries of anomalous compact objects challenge our understanding of the standard theory of stellar structures and evolution, so they serve as an excellent laboratory for searching for new physics. Earlier studies on spherically symmetric dark matter (DM)–admixed compact stars could explain a handful of anomalies. In this paper, we investigate the observational signatures of DM-admixed rotating white dwarfs, and make connections to observed peculiar compact objects. We compute the equilibrium structures of DM-admixed rotating white dwarfs using a self-consistent, two-fluid method, with the DM component being a nonrotating degenerate Fermi gas. We find that admixing DM to rotating white dwarfs could: (1) account for some peculiar white dwarfs that do not follow their usual mass–radius relation; (2) allow stable rapid-rotating white dwarfs that are free from thermonuclear runaway to exist, which could explain some soft gamma-ray repeaters/anomalous X-ray pulsars; and (3) produce universal I (moment of inertia)–Love (tidal Love number)–Q (quadrupole moment) relations that span bands above those without DM admixture, thus providing an indirect way of searching for DM in white dwarfs through gravitational-wave detection. To conclude, DM-admixed rotating white dwarfs can account for some peculiar compact objects. Our results suggest a systematic approach to accounting for the unusual compact objects that upcoming surveys could discover.

Unified Astronomy Thesaurus concepts: Gravitational waves (678); Stellar rotation (1629); White dwarf stars (1799); Dark matter (353)

1. Introduction

New telescopes, such as LISA and LSST, are expected to begin their operations in the upcoming decade. It is estimated that over 150 million white dwarfs (WDs) could be detected in the final phase of the 10 yr LSST survey (Fantin et al. 2020) and that more than 25,000 WD binaries could be observed through the LISA telescope (Korol et al. 2017). Furthermore, more than 10,000 supernovae Type Ia supernovae are expected to be observed by LSST each year (Villar et al. 2018), with the number for the total population of Type Ia supernovae being many more. We anticipate that the overwhelming volume of data for WDs and Type Ia supernovae resulting from astronomical surveys could contain a considerable number of anomalies, particularly thanks to the use of automatic and robust anomaly detection pipelines (Chan et al. 2022). Such anomalous objects could signify the presence of new physics. One example is dark matter (DM)–admixed stars. DM constitutes more than 95% of the mass in a typical galaxy (Freese 2017) and 26% of the mass–energy budget of the universe (Peter 2012). DM could be captured by normal matter (NM) in a region with a high concentration of DM particles (Casanellas & Lopes 2009; Sulistiyowati et al. 2014; Arun et al. 2019). Stars may then contain a DM component. Studies have been conducted on the effects of admixing DM to the structures and dynamics of compact and exotic objects, such as WDs (Leung et al. 2013), neutron stars (Sandin & Ciarcia-luti 2009; Leung et al. 2011; Rezaei 2018), and quark stars (Mukhopadhyay & Schaffner-Bielich 2016), for which the host object itself is an exotic object to be discovered.

These studies show that the effects of DM admixture on compact stars could be significant and observable. These DM-admixed compact stars would have unusual stellar properties, such as masses and radii, and the final fate of their evolution would deviate from normal expectations for compact stars without DM admixture. For instance, Rezaei (2017) discovered that the mass–radius relations of DM-admixed neutron stars agree with some observed anomalous compact objects, such as EXO 1745–248, 4U 1608–52, and 4U 1820–30. Bhat & Paul (2020) showed that the admixture of DM can explain the cooling rate of some pulsars/neutron stars, e.g., PSR B0656+14, PSR B1706-44, and PSR B2334+61, which could not be explained if the popular APR equation of state (EOS) is assumed. Leung et al. (2019) found that the accretion-induced collapse of a DM-admixed WD (DMWD) can explain the formations of some low-mass neutron stars that are incompatible with the conventional formation paths of core-collapse supernovae. Leung et al. (2015) and Chan et al. (2021) showed that DM-admixed Type Ia supernovae would produce light curves consistent with some unexplained peculiar supernovae. Bramante (2015) proposed that black hole–forming DM could implode and destroy pulsars, thus explaining the missing pulsar problem near the galactic center.

Furthermore, studies of DM-admixed compact stars may lead to constraints on the properties of DM. Graham et al. (2018) considered the thermalization of WD cores by DM particles, and constrained the DM properties by comparing the measured rate of Type Ia supernovae with those due to the DM heating effect. Fedderke et al. (2020) constrained the galactic abundance of charged massive relics by studying the old WD population. Kouvaris & Tinyakov (2011a), McDermott et al.
channel. However, the nonluminous nature of DM makes it difficult to detect through conventional telescopes. Its gravitational effects, however, can be indirectly observed through gravitational-wave signatures (Jung & Shin 2019). One important prediction waiting to be confirmed via gravitational-wave detection is the \( I \)-Love–\( Q \) relations. Yagi & Yunes (2013) discovered universal relations that are independent of the EOS among the (all rotationally induced) moment of inertia \( I \), the mass quadrupole moment \( Q \), and the tidal Love number \( \lambda \) of a neutron star. The existence of such universal relations has subsequently been found for WDs (Boshkayev & Quevedo 2018; Taylor et al. 2019; Roy et al. 2021). Such universal relations could be due to isodensity self-similarity, an approximate symmetry that emerges for compact stellar objects (Yagi & Yunes 2017). The \( I \)-Love–\( Q \) information of a compact star is imprinted in its gravitational-wave signature (Flanagan & Hinderer 2008; Hinderer et al. 2010; Lau et al. 2010). Even though the sensitivity of current gravitational-wave detectors does not allow for direct independent measurement of the individual \( I \)-Love–\( Q \) numbers, methods for obtaining and testing the universal relations have been proposed (Yagi & Yunes 2013; Samajdar & Dietrich 2020). Similarly, the \( I \)-Love–\( Q \) relations have been applied to constrain extra dimensions (Chakravarti et al. 2020) and gravitational theories beyond relativity (Doneva et al. 2015; Silva et al. 2021). Future gravitational-wave measurements of the \( I \)-Love–\( Q \) numbers could provide an excellent laboratory for testing and understanding fundamental physics, including astrophysical DM.

DM admixture adds an extra degree of freedom to the current stellar theory, allowing it to explain both ordinary and anomalous objects. Given that WDs constitute a considerable subset of observed stellar objects, and that the DM-admixed model has been successful in accounting for some peculiar compact stars, we believe that DMWDs deserve more attention. If DMWDs exist, they would also promise indirect channels of DM detection. Furthermore, rotation is essential in studying WD structures and evolutions (Yoon & Langer 2004, 2005). Rotating WDs have also been proposed as progenitors of neutron stars that are formed by accretion-induced collapses (Yoon & Langer 2005; Fryer & New 2011; Hachisu et al. 2012) and superluminous thermonuclear supernovae (Pfannes et al. 2010; Wang et al. 2014; Fink et al. 2018). However, the effect of DM admixture on rotating WDs has never been considered. Moreover, the universal \( I \)-Love–\( Q \) relations are useful for understanding compact object physics. They have been used to study exotic stars, such as dark stars (Maselli et al. 2017) and neutron stars admixed with quark matter (Bauswein et al. 2019). It will be interesting to see whether such universal relations continue to hold for DM-admixed rotating WDs (DMRWDs). Therefore, we extend previous studies of DMWDs to rotating WDs, to provide predictions in order to facilitate searches for such objects. The outline of the paper is as follows. Section 2 describes our method of constructing DMRWDs and obtaining the \( I \)-Love–\( Q \) numbers. Section 3 provides a summary and discussion of the results. Section 4 concludes our study. The connections between this work and previous work on DM-admixed Type Ia supernovae is discussed in Appendix A; and the stability of DMRWDs is analyzed in Appendix B; Equation (14) for DMRWDs is derived in Appendix C; and the formation of DMRWDs is discussed Appendix D.

2. Methodology

2.1. Equations of Hydrostatic Equilibrium

In this work, we consider a fully degenerate light Fermionic DM model (Narain et al. 2006). Following the previous studies on DMWDs (Chan et al. 2021), we choose the DM particle mass to be 0.1 GeV. We assume that the DM component is inherited from a zero-age main-sequence star. The formation of DM-admixed main-sequence stars will be discussed in Appendix D.

We compute a series of DMRWDs by solving the Newtonian hydrostatic equations, including the centripetal force:

\[
\nabla P_i = -\rho_i \nabla \Phi + \rho_i \omega^2 \hat{s} \cdot \delta \hat{\tau}.
\]

(1)

Here, the subscript \( i = 1, 2 \) denotes the DM (NM) quantities, and \( \rho, P, \) and \( \omega \) are the density, pressure, and angular speed of the fluid element, respectively. \( s \) is the perpendicular distance from the rotation axis, and \( \hat{s} \) is the unit vector orthogonal to and pointing away from that axis. \( \omega \) is assumed to be a function of \( s \) only. \( \delta \) is the Kronecker delta function, indicating that only the NM is rotating. \( \Phi \) is the gravitational potential governed by the two-fluid Poisson equation:

\[
\nabla^2 \Phi = 4\pi G (\rho_1 + \rho_2).
\]

(2)

The use of the Newtonian framework is justified, since the rotation speeds and compactness of WDs are small. Following Erguchi & Mueller (1985), Hachisu (1986), and Aksenov & Blinnikov (1994), we can integrate the equation of equilibrium:

\[
\int \frac{dP_i}{\rho_i} = -\Phi + \delta_{i2} \int \omega(s)^2 ds + C_i,
\]

(3)

where \( C_i \) is an integration constant. In particular, following Hachisu (1986), we define:

\[
\int \frac{dP_i}{\rho_i} = H_i,
\]

(4)

\[
\int \omega(s)^2 ds = -h^2 \psi_i.
\]

(5)

where \( H \) is the enthalpy, \( \psi \) is the rotational potential, and \( h^2 \) is a constant to be determined. So, the equation of equilibrium can be written in the integral form:

\[
H_i + \Phi + \delta_{i2} h^2 \psi_i = C_i.
\]

(6)

We assume that the WD is rigidly rotating, so \( \psi_i = -s^2/2 \) and \( h_i = \omega \).
2.2. The Self-consistent Method

We follow Hachisu (1986) in adopting an axial symmetric spherical grid. We adopt dimensionless units, so we set the gravitational constant $G$, the maximum NM density $\rho_{\text{Max}2}$, and the NM equatorial radius $r_{\text{eq}2}$ to be 1. The computational domain is described by the radial coordinate $r$ and $\mu = \cos \theta$, where $\theta$ is the polar angle. We divide $\mu$ and $r$ into the equal portions $N_\mu$ and $N_r$, respectively. Therefore:

\[
\begin{align*}
\rho_j &= r_0 \frac{j - 1}{N_r - 1}, \quad (1 \leq j \leq N_r), \\
\mu_k &= \frac{k - 1}{N_\mu - 1}, \quad (1 \leq k \leq N_\mu).
\end{align*}
\]

Here, $r_0$ is the size of the computational domain. In this work, when there is no DM component, we choose $N_\mu = 257$, $r_0 = \frac{16}{21}$, and $N_r = 257$. When a DM admixture occurs, $r_0$ and $N_r$ are enlarged to accommodate the DM fluid. We compute the gravitational potential in spherical coordinates, using the multipole expansion method:

\[
\Phi(\mu, r) = -4\pi G \int_0^\infty dr' \int_0^1 d\mu' \\
\times \sum_{n=0}^{l_{\text{max}}} P_{2n}(\mu) P_{2n}(\mu') \rho(\mu', r'),
\]

where $P_{2n}(\mu)$ is the Legendre polynomial and $l_{\text{max}}$ is the maximum number of harmonics. We choose $l_{\text{max}} = 16$ and:

\[
\begin{align*}
P_{2n}(r', r) &= \begin{cases} r'^{2n+2}/r^{2n+1}, & r' < r, \\
r'^{2n}/r^{2n-1}, & r' > r. \end{cases}
\end{align*}
\]

To compute the equilibrium structure of a pure-NM rotating WD, we need to specify the boundary condition for which $\rho = 0$. The equatorial boundary is set at $r_{\text{eq}2} = 1$, $\theta = \frac{\pi}{2}$, while the axis boundary is set at $0 \leq r_{\text{eq}2} \leq 1$, $\theta = 0$. The axis ratio of the NM is defined as $r_{\text{eq}2} = r_{\text{eq}2}/r_{\text{eq}2}$. There are 3 degrees of freedom, and we need one more parameter to specify the system completely. We choose the radial position of the DM equatorial boundary $r_{\text{DM}1}$.

We obtain the equilibrium structures by first specifying $\rho_{\text{Max}2}$, $r_{\text{eq}2}$, and $r_{\text{DM}1}$, then solving Equation (6) iteratively. We first guess the initial density profiles for the DM and NM and compute $\Phi$. Then, we can get $C_i$ from the DM and NM boundaries. After that, we invert Equation (6) to obtain the new density profiles from the DM/NM enthalpy. We then update $\Phi$ and $C_i$ again. This procedure continues until the relative changes for $\rho$, $C_i$, and the maxima of $H_i$ for both fluids are less than $10^{-10}$. In general, the DM mass changes during iterations. Therefore, we use the bisection method to vary $r_{\text{DM}1}$ in order to obtain the targeted DM mass.

2.3. Extracting the $I$–Love–$Q$ Information

We can determine $I$, $\lambda_T$, and $Q$ by postprocessing the density profiles. The moment of inertia is given as:

\[
I = I_1 + I_2,
\]

\[
I_1 = \int \rho_1 r^2 d\tau,
\]

while the mass quadrupole moment is:

\[
Q = Q_1 + Q_2,
\]

\[
Q_i = \int \rho_i r^2 P_2(\cos \theta) d\tau,
\]

where $d\tau$ is the volume element and $P_2(\theta)$ is the $l=2$ Legendre polynomial. Following Boshkayev et al. (2017), we scale $I$, and $Q$ by:

\[
I \to \left( \frac{c^2}{G} \right) \frac{I}{M^2},
\]

\[
Q \to \left( \frac{6c^2 Q}{J^2} \right).
\]

Here, $M = M_1 + M_2$ is the total mass and $J$ is the total angular momentum of the DMRWD. On the other hand, the tidal deformability $\tilde{k}$ is defined as (Boshkayev et al. 2017):

\[
\tilde{k} = \frac{3 - \tilde{\eta}}{2(2 + \tilde{\eta})}.
\]

$\tilde{\eta}$ is obtained by solving Radau’s equation for the unperturbed spherically symmetric configuration (Poisson & Will 2014):

\[
\frac{d}{dr}(r \tilde{\eta}) = 6(1 - D(r)(\tilde{\eta} + 1)) - \tilde{\eta}(\tilde{\eta} - 2).
\]

Here, $D(r) = 4\pi^3 \rho(r)/3m(r)$, and $4\pi^3 /3m(r)$ is the average density of the enclosed mass at a radial distance $r$. The boundary condition is $\tilde{\eta}(0) = 0$. We show in Appendix C the derivation of the two-fluid Radau equation. To compute Equation (13), we would evaluate $\tilde{\eta}$ at the stellar radius $R$, which we take as the larger of the NM and DM radii. $\lambda_T$ is related to $\tilde{k}$ as:

\[
\lambda_T = \frac{2R^5}{3G} \tilde{k},
\]

We then scale $\lambda_T$ as (Boshkayev et al. 2017):

\[
\lambda_T \to \frac{10\lambda_T}{G^4M^2}.
\]

Previous studies on the $I$–Love–$Q$ relations have assumed slowly rotating WDs, the equilibrium structures of which have been computed based on the Hartle formalism. In this work, we reproduce the slowly rotating limit results by using a $r_{\text{eq}2}$ close to 1.

2.4. EoSs

We focus on using the ideal degenerate Fermi gas (Shapiro & Teukolsky 1983; Camenzind 2007), assuming a carbon–

\footnote{We anticipate that the relativistic effect of a DMRWD will be small, provided that the WD is not rapidly rotating and $\rho_{\text{Max}2}$ is below $10^{10.5}$ g cm$^{-3}$. We thank the anonymous referee for pointing this out.}
massive WDs from Dufour et al. taken from Dufour et al. fractions
The Astrophysical Journal, lack of general relativistic effect and total mass approaches the Chandrasekhar limit; and this could be due to the rotating DMRWDs with different mass fractions lying outside the band of possible pure-NM models.

some peculiar WDs that are either too light or too massive, for the NM to explore the most of our studies, but we also consider several more EOSs the Harrison–Wheeler (HW) EOSs (Harrison et al. 1965) and several parameterized deleptonization formulae to describe the electron fraction at high density, in Arutyunyan et al. (1971; ASC), Liebendorfer (2005; G15, N13), Cabezón et al. (2018; S15), Dessart et al. (2006), and Abdikamalov et al. (2010; VUL).

3. Results
3.1. Mass–Radius Relations

The success in using the DM-admixed neutron star model to account for unusual compact objects that do not follow their expected mass–radius relations (Molla et al. 2020; Rahman et al. 2020; Das et al. 2021; Lee et al. 2021) inspires us to perform a similar analysis for peculiar WDs. We extract the observed mass–radius data for the WDs from Dufour et al. (2017) and Należyty & Madej (2004), and show them in Figure 1. We also append the mass–radius relations in the same figure for pure-NM rigidly rotating WDs. Most of the observed WDs agree with the pure-NM models. However, there exist some peculiar WDs that are either too light or too massive, lying outside the band of possible pure-NM models.

We aim to apply the DMRWD model to explain these anomalies. We compute the mass–radius relations for rigidly rotating DMRWDs with different mass fractions $\epsilon$ and show them in Figure 1. We find that the admixed DM affects the mass–radius relation in two ways. First, at a given large radius, the total mass is reduced. Second, the Chandrasekhar limit is increased. In particular, the increase in the Chandrasekhar limit is not observed in the results of Leung et al. (2013). They

assumed a heavier DM particle mass in the range of 1–100 GeV. In such a case, the DM component is very compact, and it affects the global structure of the star significantly. In this work, we have assumed a sub-gigaelectronvolt Fermionic DM model. In such a scenario, the DM component is more diffuse and extended (see Figure 2), thus producing a less significant effect on the structure of the NM component. We also observe that admixing DM changes the concavity of the mass–radius relation. For instance, at a radius of $10^8$ cm, the mass–radius relation changes from concaving upward to downward. The concavity change is present for the mass–radius relation of a pure-NM WD when one follows the relation from a larger to a smaller radius. Such a change indicates that the WD is approaching the Chandrasekhar limit. The DMRWD (assuming a 0.1 GeV DM particle mass) has a larger Chandrasekhar limit, and the limit is achieved at a smaller radius (higher density). Thus DMRWDs, say with $\epsilon = 0.2$ and a radius of $10^8$ cm, are not yet compact enough to reach the Chandrasekhar limit. This explains why the concavity of the mass–radius relation changes when DM is admixed.

Admixing sub-gigaelectronvolt Fermionic DM to a WD can make it exceptionally light/compact. Figure 1 shows that some of the observed anomalously low-mass/compact WDs lie within the bands of the possible models spanned by rigidly rotating DMRWDs with different $\epsilon$. Note that we sample only data with error bars of less than 10% for both masses and radii, and that there are no error bars for very massive WDs, which cast doubts on whether they are actually super-Chandrasekhar. We still include these suspicious data points, to show the capability of our model, but readers should interpret these WDs with caution. We magnify Figure 1 to show anomalous low-mass WDs with error bars in Figure 3. We observe that these anomalous WDs deviate from the pure-NM line (blue), which the error bars cannot cover. Such a result shows that alternative explanations are essential for understanding these anomalies. To conclude, the DMRWD model could account for WDs with anomalous mass–radius properties. The nature of the DM

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Masses against radii for rigidly rotating DMRWDs, with DM mass fractions $\epsilon = 0$, 0.1, and 0.2, indicated by the blue, green, and red bands, respectively, spanning possible models from the nonrotating (smaller mass) to the critically rotating (larger mass) limits. We terminate the mass–radius relations at $\rho = \rho_c$ (see Section 3.2). We also show the data for observed WDs taken from Dufour et al. (2017; orange circles) and Należyty & Madej (2004; purple triangles) as scattered points. We note that the most massive WD confirmed so far has a mass of 1.365 $M_\odot$ (Caiazzo et al. 2021), so some of the massive WDs from Dufour et al. (2017) should be interpreted with care.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Density profiles of DMRWDs that have different DM mass fractions $\epsilon$. Left panel: DM. Right panel: NM. In each panel, the upper (lower) plot shows the polar (equatorial) density profiles. These DMRWDs all have an NM central density of $10^9$ g cm$^{-3}$, and are rigidly rotating at their Keplerian limit.}
\end{figure}

8 However, the theoretical line starts to deviate from the majority when the total mass approaches the Chandrasekhar limit; and this could be due to the lack of general relativistic effect and/or the realistic treatment of the EoS. See, for example, Boshkayev et al. (2013) and Nunes et al. (2021).
admixed in these peculiar WDs could then be constrained by measuring their radii and rotational periods.

3.2. Soft Gamma-Ray Repeaters and Anomalous X-Ray Pulsars

Boshkayev et al. (2013) pointed out that massive, fast-rotating (rigidly), and highly magnetized WDs are candidates for soft gamma-ray repeaters (SGRs) and anomalous X-ray pulsars (AXPs; Malheiro et al. 2012). These high-energy objects have rotation periods of $2 \lesssim P \lesssim 12$ s. However, the fact that an ordinary WD rotating with a period of $2 \text{s}$ would be close to the Chandrasekhar limit casts doubt on whether these peculiar rotators could be stable rotating ordinary WDs.

To further investigate this issue, we follow Boshkayev et al. (2013) in computing the $\omega$--$J$ relations for a pure-NM rigidly rotating WD, and show them in the upper left panel of Figure 4. In the same figure, we show constant density boundaries for the catastrophic events that a WD could experience: (1) core ignition (SNe; $\rho_{\text{SNe}} = 2 \times 10^{9}$ g cm$^{-3}$); (2) core electron capture (EC; $\rho_{\text{EC}} = 10^{10}$ g cm$^{-3}$); and (3) inverse beta decay (IBD; $\rho_{\beta} = 1.37 \times 10^{11}$ g cm$^{-3}$ for a cold Fermi gas EOS), beyond which no stable WD would exist (Shapiro & Teukolsky 1983; Boshkayev et al. 2013). Fast rotators with $\omega > 1$ s$^{-1}$ are close to the SNe boundaries, which is consistent with our expectations. We define the critical rotation periods $P_{\text{Crit}}$ for SNe, EC, and IBD as the Keplerian rotation period for a rigidly rotating WD at the corresponding central density. We show these values for the pure-NM model in the first column of Table 1. We observe that $P_{\text{Crit}}^{\text{SNe}} = 2.307$ s. Therefore, any SGRs or AXPs that are deemed to be consistent with a pure-NM WD, with $2 \lesssim P \lesssim 2.307$ s, would be near- to super-Chandrasekhar WDs ($\sim 1.45$--$1.48$ $M_{\odot}$) lying in the SNe region. This is shown in the left panel of Figure 5 with the white band.

Such a result poses two potential problems for explaining SGRs/AXPs using the WD model. First, a pure-NM WD that rotates at $2 \lesssim P \lesssim 2.307$ would lie in the SNe region. Thus, for the pure-NM WD to avoid SNe, it should have a low density and temperature. In particular, we notice one example of an

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**Figure 3.** The same as Figure 1, but for a magnified plot with error bars included. The left (right) panel shows the data points extracted from Dufour et al. (2017) (Należyty & Madej 2004). Here, we plot only the $\epsilon = 0$ model, to facilitate visualization.

**Table 1**

| $\epsilon$ | $P_{\text{Crit}}^{\text{SNe}}$ | $P_{\text{Crit}}^{\text{EC}}$ | $P_{\text{Crit}}^{\text{IBD}}$ |
|------------|-------------------------------|-------------------------------|-------------------------------|
| 0.00       | 2.307                         | 1.175                         | 0.361                         |
| 0.05       | 2.303                         | 1.142                         | 0.341                         |
| 0.1        | 2.181                         | 1.070                         | 0.324                         |
| 0.2        | 1.916                         | 0.943                         | 0.303                         |

---

We note that $\rho_{\text{SNe}}$, $\rho_{\text{EC}}$, and $\rho_{\beta}$ might vary according to the microphysical model assumed. Nonetheless, we quote their canonical values to qualitatively understand the evolution of massive WDs.
DMRWDs using (Koester 1976; Nunes et al. 2021):

\[ \frac{T_{\text{eff}}}{g} = 2.05 \times 10^{-10} \epsilon^{2.48}. \]  

(17)

Here, \( T_{\text{eff}} \) is the effective surface temperature,\(^9\) which we set to \( 10^5 \) K, and \( g \) is the surface gravity. We show the estimation in Figure 6.

In the same figure, we append the carbon ignition curve (gray) and the carbon–oxygen fusion curve (brown), taken from Nomoto et al. (1984) and Yakovlev et al. (2006), respectively. It is believed that to the right of the gray line, carbon ignition would lead to a thermonuclear runaway. We observe that all massive super-Chandrasekhar models are too hot to exist stably as rotating WDs. The understanding of thermonuclear supernova progenitors and explosions is still highly uncertain. Given the large diversity of Type Ia supernova light curves, it is natural to expect a large variety of supernova progenitors, and hence a large range of central \( \rho \) and \( T \). In other words, the carbon ignition curve might shift. In addition, pycnocnuclear burning occurs over a long timescale, so less massive models with \( M \sim 1.45 \, M_\odot \) might be able to escape being supernovae. However, the second issue is that such a value might be too large. The most massive WD discovered so far has a mass of \( 1.365 \, M_\odot \) (Caiazzo et al. 2021).

As we discussed earlier, the super-Chandrasekhar WDs with masses \( \gtrsim 1.46 \, M_\odot \) discovered in Dufour et al. (2017) are highly suspicious, due to the lack of error bars. Therefore, the expected mass of a WD being a rapidly rotating AXPs/SGR would challenge the well-established data. Theoretically, it is also difficult to form a WD with \( M \sim 1.45 \, M_\odot \). For instance, by running a series of accreting WD models, with different mass accretion rates, using the MESA stellar evolution code (Paxton et al. 2011, 2013, 2015, 2018, 2019), we found that central ignition occurs for \( M \sim 1.4 \, M_\odot \). Therefore, even if 1E 1547-54 is an \( M \sim 1.45 \, M_\odot \) WD, it is unlikely to be formed by mass accretion. It is also unlikely that 1E 1547-54 is a super-Chandrasekhar WD formed by a binary WD merger, because Schwab et al. (2016) have shown that massive, super-Chandrasekhar WD mergers would end up as either (1) supernovae; (2) sub-Chandrasekhar WDs; or (3) neutron stars. Therefore, 1E 1547-54 should not be a normal WD.

Although a WD model is preferred (Malheiro et al. 2012), the true identity of 1E 1547-54 is still unknown. We show that this object might be a rigidly rotating DMRWD. To confirm our hypothesis, we compute the \( \omega\)-\( J \) relations for rigidly rotating DMRWDs with different \( \epsilon \), and show them in Figure 4. We also compute their critical rotation periods and list them in Table 1. We observe that the constant density boundaries for catastrophic events (SNe, EC, and IBD) shift upward when \( \epsilon \) increases—admixing DM makes the NM component at a fixed \( \rho_c \) rotate faster. Thus, \( \rho_{\text{SNe}}^{\text{Crit}}, \rho_{\text{EC}}^{\text{Crit}} \), and \( \rho_{\text{Crit}}^\beta \) (see Table 1 for their definitions) all decrease as \( \epsilon \) increases, allowing for more DMRWDs to rotate faster without reaching \( \rho_{\text{SNe}} \). We show in the right panel of Figure 5 that if we assumed \( \epsilon = 0.2 \), then a DMRWD rotating with \( P = 2.07 \) s is indeed possible. We use Equation (17) to estimate the core temperatures of these models and find that they lie outside the forbidden region in Figure 6. These models are also less massive (\( \sim 1.4 \, M_\odot \)). We remark that the effective temperatures of pulsars span from \( 10^5 \) to \( 10^7 \) K (Ng & Kaspi 2011). We find that \( T_e \) reaches \( 10^9 \) K if \( T_{\text{eff}} \sim 10^{35} \) K. Carbon ignition would initiate at such a high temperature. Therefore, 1E 1547-54 could be a rigidly rotating DMRWD if its \( T_{\text{eff}} \) is \( 10^7 \) K or less, which is at the lower end of the typical range of pulsar surface temperatures.

The observed spinning down of 1E 1547-54 provides additional evidence for its nonordinary structure. As shown by Boshkayev et al. (2013), under the constant-mass assumption, some super-Chandrasekhar WDs would spin up (increase in \( \omega \)) as they lose \( J \). Such a phenomenon has been observed by Shapiro et al. (1990) and Gerosa & Papasotiriou (2000).

Here, we reproduce their results in Figure 5. Following the isomass contours on the figure, we observe a handful of super-Chandrasekhar WDs spinning up as they gradually lose \( J \). These models are all super-Chandrasekhar WDs so massive that degenerate pressure alone cannot support their structures. So they rotate faster to provide stronger centrifugal force as they lose \( J \). However, several follow-up studies on 1E 1547-54 have revealed that it is spinning down (Mukhopadhyay & Rao 2016),\(^{10}\) which is inconsistent with the fact that pure-NM WDs with periods of \( 2.07 \) s should be spinning up, as they lose \( J \).

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\(^{10}\) See also http://www.physics.mcgill.ca/~pulsar/magnetar/main.html.  

\(^9\) The surface temperature of 1E 1547-54 is still unknown; therefore, our discussion only serves as a rough estimation.
et al. 2017), and the possibility that 1E 1547-54 is a WD-like object should not be ruled out.

As we discussed earlier, understanding the structures of rotating WDs is essential to predicting their final fates in qualitative terms (Yoon & Langer 2005). Because some studies (Leung et al. 2015; Chan et al. 2021) have used the DM-admixed Type Ia supernova model to account for peculiar transients, it would be interesting to see if admixing DM would alter the evolutionary path of WDs toward thermonuclear runaway. We have therefore presented some preliminary analysis in Appendix A. Furthermore, it is important to know whether the DMRWD that we consider here is stable against secular instability. We have examined the stabilities of DMRWDs, and these are presented in Appendix B.

### 3.3. Applications of the I-Love–Q Relations

This section focuses on the effects of $\epsilon > 0$ on the well-known I-Love–Q universal relations for WDs.

#### 3.3.1. The Deviation of the Universal Relations

We have computed the I-Love–Q relations for rigidly rotating DMRWDs using several of the EOSs mentioned in Section 2.4. We construct a few sequences of DMRWD models by varying the maximum NM density $\rho_{\text{Max}}$ from $10^6$ to $\sim 10^9$ g cm$^{-3}$, for fixed $\epsilon$, ranging from 0 to 0.3.

We show the scatter plots obtained by assuming different EOSs in Figure 7. In particular, the upper left panels in Figures 7 (a) and (b) show that, when there is no DM admixture, the I-Love–Q relations for WDs computed using different EOSs lie approximately on the same line. In particular, we find that the maximum deviations from the best-fit line are within $\sim 1\%$ in the log$_{10}$ scale. These results also agree with previous studies on the I-Love–Q relations of slowly rotating WDs, particularly for the universality lines and the corresponding maximum deviations. We also observe that a small mass fraction (e.g., $\epsilon \sim 0.01$; not shown in Figure 7) of DM admixture makes no change to the I-Love–Q universality. Most of the deviations due to DM admixture are within the average fitting uncertainties of the pure-NM model. However, we observe two features when $\epsilon$ increases beyond 0.01. First, Figure 7 shows that the I-Love–Q relations remain independent of the EOS assumed. Second, the universal lines are all shifted to higher $Q$ values relative to those with $\epsilon = 0$, which is shown in the upper panels of Figures 8 (a) and (b). The relative deviations between the universal Love–Q ($I-Q$) relations for DMRWDs and those for pure-NM models become larger than the corresponding average fitting uncertainty $\Delta = 0.16 (0.08)$ of the latter, and the deviation increases with $\epsilon$. We fit the I-Love–Q relations for DMRWDs with different $\epsilon$ using a log-linear function:

$$\log_{10} y = a + b \log_{10} x,$$

with $y$ being $\lambda_T$ or $I$ and $x = Q$. We show the results in Figure 9 and Table 2. The parameter $a$ is strongly correlated with $\epsilon$, while $b$ shows a much weaker correlation, reflecting the negligible changes in the slopes of the I-Love–Q relations for different $\epsilon$.

#### 3.3.2. Detection Prospects

Gravitational waves are emitted in binary WD merger events (Gianninas et al. 2014; McNeill et al. 2019; Yoshida 2021). Although the signal is below the detection thresholds of the current gravitational-wave detectors, it has been estimated that a binary WD merger event could be resolvable for the next generation of detectors (Yu & Jeffery 2010; Korol et al. 2020). In fact, a method for estimating WD masses by extracting the tidal information from the gravitational-wave signatures has been proposed (Wolz et al. 2020). Hence, we anticipate that measurements of the I-Love–Q relations for WDs could help to reveal the existence of DM in rotating WDs. In reality, each WD may acquire a different amount of DM. Therefore, the WDs would scatter within a band of I-Love–Q relations spanned by different $\epsilon$. We show these bands, together with the uncertainties of the pure-NM models, in the lower panels of Figures 8 (a) and (b). They are computed first by taking the differences between the best-fit lines for the DM-admixed models and those for the pure-NM models, then plotting the differences versus $Q$ as solid lines. Finally, we fill...
Figure 8. (a) Upper panel: best-fit linear lines of the log10$\lambda_T$ vs. log10$Q$ relation for slowly, rigidly rotating DMRWDs with different DM mass fractions $\epsilon$. The best-fit linear line is generated by fitting the scatter points in Figure 7. Lower panel: $\Delta \lambda_T/\langle \lambda_T \rangle = (\lambda_T^{\text{DM}} - \lambda_T^{\text{NM}})/\langle \lambda_T^{\text{NM}} \rangle$ vs. log10$Q$ for different $\epsilon$. $\Delta \lambda_T$ is computed by taking the difference between the best-fit line of a DM-admixed model and the pure-NM model. The solid gray line indicates the average fitting uncertainty $\delta = 0.16$ of the Love–Q relations for the pure-NM model. The solid blue line represents the estimated measurement uncertainty $\bar{\delta} = 0.08$, as taken from Yagi & Yunes (2013) and Yunes & Siemens (2013). We fill the areas between solid lines to form different bands. In the areas between the solid lines, there are overlapping bands. We interpret this figure in the following way: the WDs lying in the green strip have 0.1 < $\epsilon < 0.2$, for instance, and the WDs lying in the blue (gray) strip are within the measurement (fitting) uncertainty. (b) The same as (a), but for log10$I$ vs. log10$Q$, with $\delta = 0.08$ and $\beta = 0.1$.

Figure 9. Correlations between the parameters $a = (a' - a)/a$ and $b$ with the DM mass fraction $\epsilon$ for the $\lambda$–$Q$ relation (left panel) and the $I$–$Q$ relation (right panel). The blue lines are the fitting functions presented in Equation (20), while the orange lines are just discrete linear lines connecting the data points.

Table 2

| $\epsilon$ | $a_{\lambda-Q}$ | $b_{\lambda-Q}$ | $a_{I-Q}$ | $b_{I-Q}$ |
|------------|-----------------|-----------------|------------|------------|
| 0.00       | −8.77           | 5.23            | −4.36      | 2.11       |
| 0.01       | −8.95           | 5.24            | −4.45      | 2.12       |
| 0.05       | −9.48           | 5.28            | −4.65      | 2.13       |
| 0.1        | −9.90           | 5.30            | −4.78      | 2.13       |
| 0.2        | −10.67          | 5.32            | −4.97      | 2.13       |
| 0.3        | −11.54          | 5.34            | −5.29      | 2.13       |

in the areas between the solid lines to form different bands. In particular, since the uncertainties (shown as the gray band) of the pure-NM models are small, any significant deviations that lie above the pure-NM version of the $I$–Love–Q relations is possibly a sign of a subsolar mass scale of DM admixture.

Are DMRWDs detectable by measuring the deviations from the $I$–Love–Q relations? As mentioned in Yagi & Yunes (2013) and Yunes & Siemens (2013), $\lambda_T$ and $I$ could be measured by gravitational-wave and/or electromagnetic observations. We take their estimated uncertainties as 60% and 10%, respectively. We compare these values to the deviations in $\lambda_T$ and $I$ in the lower panels of Figures 8 (a) and (b), respectively. We find it possible to detect DMRWDs with $\epsilon \gtrsim 0.1$. Future space-based detectors should have improved accuracy to detect or constrain DMRWDs with $\epsilon < 0.1$.

Suppose there is an anomalous detection of $\lambda_T$ or $I$ with respect to a fixed $Q$. If such a deviation is a result of DM admixture, we can infer the value of $a$ from $\lambda_T$ or $I$:

$$a' \approx \log_{10} x' - b_0 \log_{10} Q'. \quad (19)$$

Here, the subscripts refer to anomalous detections, and those with the subscript zero refer to the pure-NM model with $x = \lambda_T$ or $I$. We also assume that the best-fit parameter $b_0$ of the DM-admixed models does not vary significantly from that of the pure-NM model, so we can substitute their values with those of the pure-NM model. We fit the relation between $a$ and $\epsilon$ for the $\lambda_T$–$Q$ and $I$–$Q$ universal curves and obtain:

$$\epsilon (\bar{a}_{\lambda_T-Q}) = 0.468 \tanh (2.756 \bar{a}_{\lambda_T-Q})^{3.461}$$
$$-0.012 \tanh (\bar{a}_{\lambda_T-Q})^{0.239},$$

$$\epsilon (\bar{a}_{I-Q}) = 0.433 \tanh (2.984 \bar{a}_{I-Q})^{1.296}$$
$$+0.084 \tanh (14.832 \bar{a}_{I-Q})^{16.31}. \quad (20)$$

For $0 \leq \epsilon \leq 0.3$, here, $\bar{a} = (a' - a)/a$ is the relative change in $a$ with respect to the pure-NM model. Using the above equation, one can directly infer the mass fractions $\epsilon$ of DM admixture in a rotating WD.
Taylor et al. (2019) demonstrated that the $I$–Love–$Q$ relations for WDs depend on the degree of differential rotation (i.e., the angular velocity profile). However, we have computed the $I$–Love–$Q$ relations for DMRWDs, assuming different rotation profiles. These include the $j$-const, $v$-const, and “Kepler” profiles (see Hachisu 1986 and Yoshida 2019). We still observe that the $I$–Love–$Q$ relations for DMRWD are universal, but that they differ from those of the pure-NM rotating WD. Assuming that the rotation rules that we assumed are representative enough, we conclude that the deviations of the DM-admixed $I$–Love–$Q$ relations from those of the pure-NM model are insensitive to whether the WD is rotating rigidly or differentially. We note that the universality of the $I$–Love–$Q$ relations is violated for hot WDs (Boshkayev & Quevedo 2018). However, our DMRWD models produce deviations from the NM version of the universal relations that are rather different from those for hot WDs.

### 3.4. Heavier/Lighter DM Particles?

Chan et al. (2021) and Leung et al. (2022) have shown that the properties of Fermionic DM-admixed compact stars change sharply around a DM particle mass of 0.1 GeV. Therefore, our results should be sensitive to the choice of such a value. We reproduce the analysis presented in Sections 3.2 and 3.3 and give qualitative descriptions of how our results would change according to the DM particle mass. In particular, we choose 0.2 GeV (0.08 GeV) to represent the heavy (light) DM limit.

If we assume a particle mass of 0.2 GeV, then the DM acts as a compact core. We find that the WD total mass and critical rotation velocity are reduced, so there will be fewer DMRWDs that can rotate at 2.307 s and be free from the thermonuclear runaway. We also find that the $I$–Love–$Q$ relations deviate more than those for the 0.1 GeV case. Also, the $I$–Love–$Q$ relations break down when the log quadrupole moment is small; the results when assuming a particle mass of 0.08 GeV are the other way round—the DM component usually extends to a larger radius than the NM component. We find that the critical rotation velocity increases, so there are more available DMRWDs that can rotate at a period of 2.307 s and be free from the thermonuclear runaway. However, these models have a huge DM mass. It is doubtful whether such massive WDs exist. Also, the $I$–Love–$Q$ relations are still universal, showing few differences from the 0.1 GeV case. However, when the DM component is much larger than the NM, it would be difficult to correctly reproduce the composite $I$–Love–$Q$ numbers.

### 4. Conclusion

Earlier studies on DM-admixed neutron stars and supernovae could explain a handful of peculiar astrophysical objects. In this paper, we investigate the observational signatures of DMRWDs.

We first compute the mass–radius relations for rigidly rotating DMRWDs. Our DMRWD models could successfully account for some of the anomalously low- or high-mass WDs that have been observed. We then show that admixing DM increases the rotational velocity of a WD, thus allowing the existence of stable rapidly rotating WDs that could be free from the thermonuclear runaway. These DMRWDs have a near-Chandrasekhar mass and are spinning down, which is consistent with some fast-rotating SGRs/AXPs (such as 1E 1547-54). We further discover that DMRWDs follow universal $I$–Love–$Q$ relations that deviate significantly from those of the pure-NM models, with a larger deviation for a larger $\epsilon$. Since each WD may have a different DM fraction, the WD $I$–Love–$Q$ relations span bands above the universal lines for the pure-NM models, and the deviations for $\epsilon \gtrsim 0.1$ are large enough to be observable. Finally, we give an empirical formula for extracting the DM mass fractions in DMRWDs from the deviations of the $I$–Love–$Q$ relations, with respect to those of the pure-NM model.

Several issues remain to be resolved. First, we have assumed that the DM is nonrotating. Even though accretion can change the collective motion of the DM component, it was pointed out by Iorio (2010) that a neutron star in the galaxy could accrete DM at a rate of $10^7$ kg s$^{-1}$. This value is so small that there could be no considerable amount of DM buildup at the outer envelope on the timescale of the age of the universe. Thus, the nonrotation approximation shall remain valid throughout the evolution, from the zero-age main sequence to the formation of a DMRWD. However, we remark that degenerate pressure is an effective quantum self-interaction. Thus, the DM component may have inherited collective motion during the molecular cloud collapse. The gravitational force from the NM component could also drag the DM component. Nonetheless, we assume the DM component to be nonrotating for this exploratory study.

Second, our analysis is solely based on the Newtonian framework. As the compactness of WDs continues to increase when one considers $\rho \rightarrow 3 \times 10^{10} \text{ g cm}^{-3}$, general relativistic effects become more important. We expect our results, when computed beyond such a value, to alter when general relativity is considered. Still, the results computed around $\rho \sim \rho_{\text{SN}}$ shall also be mildly affected. To extend our work to the general relativistic regime, one would need to include a second component of non-self-interacting DM into the matter source term of the Hartle–Thorne metric (Hartle 1967), or one could solve a set of self-consistent two-fluid elliptical equations to compute the structures of relativistic DMRWDs. These could also be used to examine DM-admixed rotating neutron stars.

In conclusion, we predict the observational signatures of DM admixtures in rotating WDs and connect our results to some peculiar compact objects. Our results could be applied to studying the potentially large number of unusual compact objects that could be discovered by next-generation LSST-like surveys.

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### Appendix A

#### Connections to the DM-admixed Supernova Model

Yoon & Langer (2005) analyzed presupernova evolution by considering the $J$–$M$ trajectory. In their study, a series of constant-NM central density models are fitted and plotted in the
smaller angular momentum barrier would mean a shorter total mass. We include rigid and differential rotation models in constructing the colored boundaries. The dashed-dotted lines represent the boundaries computed for the pure-NM model.

Figure 10. The same as Figure 4, but for the angular momenta against total mass. We choose $\alpha$ as a free parameter, $\epsilon$ is large could be understood by the following quantitative parameter, $M$ would behave as:

$$M = M_{\epsilon}(M_1) + M_1. \quad (A2)$$

The hydrostatic equations implied that $M_2$ depends on $M_1$, which is a decreasing function bounded from below by 0. When $M_1$ is large, $M_2$ will be close to 0. As such, we have:

$$M \approx M_1. \quad (A3)$$

So $M$ scales linearly with $M_1$ for large $M_1$ in the Newtonian limit, with no maximum $M_1$. This argument could also be applied to the case of admixing DM with a particle mass of more than 1 GeV, but the DM mass required to be admixed to increase the total mass would be large, and therefore these trends have not been observed by Leung et al. (2013).

In addition, when $\epsilon$ is small, the $J$ for a given $M$ of a DMRWD right at $\rho \sim \rho_{SN}$ is larger than that of $\epsilon = 0$. Figure 2 illustrates the DM and NM density profiles for different $\epsilon$. We observe that the DM component with a sub-gigaelectronvolt particle mass is more extended and diffuse. Its central density is also lower than the NM, by one order of magnitude. Therefore, a small amount of DM admixture would not significantly affect the structure of the NM component. For instance, the NM density profiles only change mildly with small $\epsilon$, except the NM component rotates faster to balance the increase in gravitational attraction. However, as $\epsilon$ increases, the mass and radius of the NM component are reduced significantly, so that $J \sim \omega M_2 R_2^2$ is reduced.

We observe that a small amount (say $\epsilon \lesssim 0.1$) of admixed DM favors the thermonuclear explosion scenario in two ways—it allows a larger range of masses and reduces the angular momentum barrier for a WD to reach the SNe density $\rho_{SN}$.

Here, $\alpha$ varies according to the different rules assumed. There is a free parameter $\epsilon$, which measures how differential the rotation is. We choose $d$ to be the numerical value of the equatorial radius, where $\rho_\epsilon = 0.01$ is that of the (NM) central value. It is chosen to match the results of Yoon & Langer (2005). When a small amount of DM admixture is present (say, $\epsilon \lesssim 0.1$), the angular momentum barrier (shown as the red line) for SNe is shifted to a lower total mass. This leads to three consequences: (1) there is a larger range of $M$ available to initiate SNe; (2) DMRWDs need to lose less $J$ to reach $\rho \sim \rho_{SN}$; and (3) it is easier for accreting WDs evolving from the low-mass region in the diagram to end up having $\rho \sim \rho_{SN}$.

The shift of the angular momentum barrier to lower total mass is due to two reasons. First, the critical mass for SNe $M^{SN}_{\epsilon}\text{Crit}$, defined as the mass of a nonrotating DMWD that has a central density of $\rho_{SN}$, is reduced. We show $M^{SN}_{\epsilon}\text{Crit}$, $M^{EC}_{\epsilon}\text{Crit}$, and $M^{\beta}\text{Crit}_{\epsilon}$ for different $\epsilon$ in Table 3. Here, $M^{EC}_{\epsilon}\text{Crit}$ is the critical mass for EC, and $M^{\beta}\text{Crit}_{\epsilon}$ is the critical mass for IBD instability. As with $M^{SN}_{\epsilon}\text{Crit}$, $M^{EC}_{\epsilon}\text{Crit}$ and $M^{\beta}\text{Crit}_{\epsilon}$ are evaluated at $\rho = \rho_{EC}$ and $\rho_{\beta}$ respectively. We observe that $M^{EC}_{\epsilon}\text{Crit}$ and $M^{\beta}\text{Crit}_{\epsilon}$ first decrease and then increase as $\epsilon$ is increasing. In contrast, $M^{SN}_{\epsilon}\text{Crit}$ is an increasing function of $\epsilon$. These trends have been observed by Chan et al. (2021). Whether admixing a small fraction of DM would lead to an increase or a reduction of $M$ depends on the DM particle mass assumed. However, the increase of $M$ when $\epsilon$ is large could be understood by the following quantitative argument (in the Newtonian framework). At a fixed central density, and we vary the amount of DM admixture $M_1$ as a free parameter, $M$ would behave as:

$$M = M_{\epsilon}(M_1) + M_1. \quad (A2)$$

The hydrostatic equations implied that $M_2$ depends on $M_1$, which is a decreasing function bounded from below by 0. When $M_1$ is large, $M_2$ will be close to 0. As such, we have:

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In addition, when $\epsilon$ is small, the $J$ for a given $M$ of a DMRWD right at $\rho \sim \rho_{SN}$ is larger than that of $\epsilon = 0$. Figure 2 illustrates the DM and NM density profiles for different $\epsilon$. We observe that the DM component with a sub-gigaelectronvolt particle mass is more extended and diffuse. Its central density is also lower than the NM, by one order of magnitude. Therefore, a small amount of DM admixture would not significantly affect the structure of the NM component. For instance, the NM density profiles only change mildly with small $\epsilon$, except the NM component rotates faster to balance the increase in gravitational attraction. However, as $\epsilon$ increases, the mass and radius of the NM component are reduced significantly, so that $J \sim \omega M_2 R_2^2$ is reduced.

We observe that a small amount (say $\epsilon \lesssim 0.1$) of admixed DM favors the thermonuclear explosion scenario in two ways—it allows a larger range of masses and reduces the angular momentum barrier for a WD to reach the SNe density $\rho_{SN}$.

These results might connect with the DM-admixed supernova models proposed by Leung et al. (2015) and Chan et al. (2021). In particular, DMWDs with 0.1 GeV DM particle mass were used by Chan et al. (2021) as progenitors for studying supernova explosions. They showed that these exotic progenitors would produce dimmer and broader light curves compared with ordinary supernova models. This is because the photosphere evolves more slowly and loses thermal energy through pressure work done when expanding against an external DM...
potential. Their models match some peculiar supernovae with low-luminosity and slowly evolving light curves, with a range of $\epsilon$ from 0.05 to $\sim0.33$. Other than stability, another challenging issue of having DMWDs as supernova progenitors is the likelihood of them exploding. This work shows that from the angular momentum perspective, DMWDs are more likely to explode than ordinary WDs, given that the DM admixture is not too large. While $\epsilon \approx 0.2$ is the marginal case, when $\epsilon$ reaches about 0.3, such as in model DM3 presented by Chan et al. (2021), the angular momentum barrier becomes larger when compared with the pure-NM model, which disfavors explosion.

### Appendix B

**Stability Analysis**

An issue of concern is whether DMRWDs are stable against secular instability. When secular instability sets in, the star evolves until it reaches the point for dynamical instability, where it is unstable against gravitational collapse (Stergioulas 2003). As pointed out by Friedman et al. (1988) and Boshkayev et al. (2013), a turning-point method could be used to identify the onset of secular instability:

$$
\left( \frac{\partial M}{\partial \rho_c} \right)_J = 0. \quad (B1)
$$

Here, the NM central density is $\rho_c$. It has been shown that rotating WDs in the Newtonian regime are stable against secular instability (Ostriker & Bodenheimer 1968; Boshkayev et al. 2013). We show the total mass against $\rho_c$ for DMRWDs for different $\epsilon$ in Figure 11, together with constant $J$ contours lines. We do not observe any local extremum for constant $J$ lines, and thus we can conclude that DMRWDs are also stable against secular instability in the Newtonian framework. We also arrive at the same conclusion for differentially rotating models. We note that general relativistic effects would destabilize static and rotating WDs (Boshkayev et al. 2013). Therefore, changes to this set of results are expected in the general relativistic framework. In particular, it would be interesting to investigate whether admixing DM would postpone or advance the critical point of instability, or even create more than one instability point.

### Appendix C

**Generalizing the Tidal Equation to the Two-fluid Case**

Radau’s equation, which gives the tidal Love number, can be obtained in an alternative way by considering the Poisson equation (Hinderer 2008):

$$
H''(r) + \frac{2}{r}H'(r) - \left[ \frac{6}{r^2} - 4\pi G\rho(r) \frac{d\rho(r)}{dP(r)} \right] H(r) = 0. \quad (C1)
$$

Here, $H(r)$ represents the Eulerian change of the Newtonian gravitational potential (Yip & Leung 2017). A change of the variable $Y(r) = \frac{H''(r)}{H' (r)}$ transforms the second-order equation to a first-order equation (Yip & Leung 2017):

$$
Y'(r) + \frac{Y(r)}{r} + \frac{Y(r)^2}{r} = \frac{6}{r} + \frac{4\pi^3}{m^2} \frac{d\rho(r)}{dP(r)}, \quad (C2)
$$

where we have made use of the chain rule $\frac{d\rho(r)}{dP(r)} = \frac{d\rho(r)}{dr} \frac{dr}{dP(r)}$, and the hydrostatic equation $\frac{d\rho(r)}{dP(r)} = -\frac{Gm(r)\rho(r)}{r^2}$. We can define $y(r) = Y(r) - \frac{4\pi^3}{m^2}$ to obtain (Yip & Leung 2017):

$$
y'(r) + \left[ \frac{1}{r} + \frac{8\pi^2\rho(r)}{m^2} \right] y(r) + \frac{y(r)^2}{r} = \frac{6}{r} - \frac{16\pi^2\rho(r)}{m^2}, \quad (C3)
$$

Rearranging terms, and denoting $D(r) = \frac{4\pi^2\rho(r)}{3m^2}$, we have:

$$
ry'(r) + y(r)^2 + y + 6D(r)(y(r) + 2) - 6 = 0. \quad (C4)
$$

This is to be solved with the initial condition $y(0) = -1$, and the tidal deformability is given as $\tilde{k} = \frac{2 - y(R)}{2[3 + y(R)]}$, where $R$ is the stellar radius. By inspection, we make a substitution $y = \eta - 1$ to transform the equation into:

$$
\eta(r)'^2 + \eta(r)\eta(r - 1) + 6D(r)(\eta(r) + 1) - 6 = 0, \quad (C5)
$$

for which we recover Radau’s equation with $\eta = \tilde{\eta}$. We notice that there are two-fluid generalizations for calculating the tidal Love numbers of hybrid stars under the general relativistic framework (Zhang et al. 2020; Leung et al. 2022). Therefore, the key to justifying our naive generalization of Equation (14) to the two-fluid situation is to take the Newtonian limit of the relativistic version of Equation (C1) that has been established for the two-fluid system. The relativistic version of the equation is (Hinderer 2008):

$$
rY'(r) + Y(r)^2 + Y(r)e^{\lambda(r)}[1 + 4\pi r^2(p(r) - \rho(r))] + r^2Q(r) = 0,
$$

$$
Q(r) = 4\pi e^{\lambda(r)} \left[ 5\rho(r) + 9p(r) + \frac{\rho(r) + p(r)}{dp/d\rho} \right] - 6\frac{\rho(r)}{r^2} - (\nu'(r))^2, \quad (C6)
$$

where we have adopted the geometric units $c = G = 1$. Here, $\lambda$, $\nu$, $\lambda$, and $\nu$ are related to the metric elements. The term...
\[
\frac{\rho(r) + p(r)}{dp/d\rho} \text{ should be treated carefully in the two-fluid case.}
\]

Fortunately, it can be decomposed into the contributions from NM and DM, if they do not interact with each other (Zhang et al. 2020; Leung et al. 2022):

\[
\frac{\rho(r) + p(r)}{dp/d\rho} = \frac{\rho_1(r) + p_1(r)}{dp_1/d\rho_1} + \frac{\rho_2(r) + p_2(r)}{dp_2/d\rho_2}.
\]  

(C7)

In the Newtonian limit \(e^{\lambda(r)} \approx 1\), \(v'(r) \approx 0\), and \(p \ll \rho\), we have:

\[
Q(r) \approx 4\pi \left[ \frac{\rho_1(r) dp_1(r)}{dr} + \frac{\rho_2(r) dp_2(r)}{dr} \right] - \frac{6}{r^2}.
\]  

(C8)

We make use of the hydrostatic equation \(dp(r)/dr = -Gm(r)/r^2\) to obtain:

\[
Q(r) \approx -4\pi \frac{r^2}{m(r)} \left[ \frac{dp_1(r)}{dr} + \frac{dp_2(r)}{dr} \right] - \frac{6}{r^2}.
\]  

(C9)

Since the densities of DM and NM can be added as scalars, we substitute this expression into Equation (C6) and use \(4\pi r^2 p(r) \approx 0\) to get:

\[
rY'(r) + Y(r)^2 + Y(r) - \left(4\pi r^4 \frac{dp_1(r)}{m(r) dr} + 6 \right) = 0,
\]  

(C10)

which is just Equation (C1), with \(\rho = \rho_1 + \rho_2\) and \(m(r)\) being the total enclosed mass.

Appendix D

The Formation of a DMRWD

We consider a scenario similar to that presented in Leung et al. (2013), where the star is born with an inherent admixture of DM, contributing an extra gravitational force to the zero-age main-sequence star. We assume a spherically symmetric cloud of NM and DM, having the constant densities \(\rho_1\) and \(\rho_2\), respectively. Their individual radii could be computed by \(R = (3M/4\pi \rho)^{1/3}\). In particular, we consider the situation with the DM radius \(R_1\) being larger than that of the NM, \(R_2\). The total energy \(E\) (gravitational + kinetic) is:

\[
E = \left( \frac{3}{5} G M_2^{5} \right. + \frac{3}{5} G M_2^{5} \left. + \frac{3}{2} \frac{GM_1 M_2}{R} - \frac{3}{10} \frac{G M_2 R_2^3}{R_1^3} \right) + \frac{3}{2} NKT + \frac{1}{2} M_1 v_1^2.
\]  

(D1)

Here, \(v_1\) is the DM thermal velocity, \(N = M_2/m_H\) is the total number of NM nuclei, and \(m_H\) is the molecular mass of hydrogen. Furthermore, since we have widely analyzed the model with \(\varepsilon = 0.2\) in the text, we assume \(M_1 \sim 1.4 \times 0.2 \sim 3 M_\odot\), and \(M_2 \sim 10.0 M_\odot\). For a typical collapsing molecular cloud, we have \(T \sim 150 K\) and \(\rho_2 \sim 10^6 m_{H} cm^{-3}\), and hence \(R_2 \sim 3.05 \times 10^{16} cm\) smaller than the Jeans radius. The maximum velocity DM \(v_{1,max}\), for it to be bounded by the combined gravitational force, is obtained by solving \(E(R_2) = 0\).

Here, \(v_{1,max} \sim 7.81 \times 10^5 cm s^{-1}\). For a given \(v_1 < v_{1,max}\), we would fix \(R_2\) and vary \(R_1\) to look for a solution where \(E < 0\). However, the most probable DM speed (assuming a Maxwell distribution) is \(v_{p1} \sim 10^{7} cm s^{-1}\). To take this into account, the bounded DM fraction is given by:

\[
f = \int_0^{v_{p1}} \frac{u}{u^2 exp(-u^2)} du.
\]  

(D2)

Here, \(u = v/v_{p1}\) and \(u_1 = v_1/v_{p1}\). We take a particular \(v_1 = 7.10 \times 10^5 cm s^{-1}\), and give two sets of solutions in terms of \(R_1, \rho_1\) to show that the requirement of \(E < 0\) could be satisfied: \(4.38 \times 10^{16} cm, 3539 GeV cm^{-3}\) and \(3.05 \times 10^{16} cm, 1.05 \times 10^{10} GeV cm^{-3}\). The required DM density in the first set of solutions is based on the state-of-the-art simulations, which have shown that the DM density at the galactic bulge could be \(\sim 3600 GeV cm^{-3}\) (Piffl et al. 2014). The required DM density in the other set of solutions is much larger. However, such a value is possible near the galactic center, and values with a similar order of magnitude have been adopted in studying the effects of DM annihilation on main-sequence stars (Moskalenko & Wai 2006; Iocco 2008). In conclusion, our estimations considering the DM velocity dispersions show that it is possible to trap a DM of \(0.3 M_\odot\) during the star-forming phase, provided that the molecular cloud is in the vicinity of the galactic center. Note that the DM and NM have different ambient densities, implying that they have different freefall times, and, in principle, the DM would not follow the trajectory of the NM.

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