Short Proof of the Gallai-Edmonds Structure Theorem

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Abstract
We derive the Gallai-Edmonds Structure Theorem from Hall’s Theorem.

Introduction and Definitions
In 1971, Anderson [1] gave a derivation of Tutte’s Theorem from Hall’s Theorem. In this note, we give a concise proof of the Gallai-Edmonds Structure Theorem; our proof-method seems new but similar to that of Anderson. For completeness, we proceed with the basic definitions and relevant formulations; the reader familiar with the Gallai-Edmonds Structure Theorem (see, for example, [2] pp.93–95) is asked to skip to the next section.

A set $M$ of edges in a graph $G$ is a matching if no two edges of $M$ share a vertex. A vertex of $G$ incident with no edge in $M$ is $M$-exposed. A matching is perfect if it has no exposed vertex, and near-perfect if it has exactly one exposed vertex. A graph $G$ is factor-critical if for every vertex $v \in G$ the graph $G-v$ has a perfect matching.

In what follows, $S$ and $T$ are subsets of the vertex set of $G$; we set $s := |S|$ and $t := |T|$.

The [odd, even] $S$-components are the connected components of the graph $G-S$ [of odd, even cardinality, respectively]. The number of odd $S$-components is denoted by $\text{od}(S)$, the total number of vertices in those $S$-components which have no perfect matching is denoted by $\text{df}(S)$, while the quantity $\text{df}(S) := \text{od}(S) - s$ is the deficiency of $S$. It is important to notice that (1) every matching in $G$ leaves at least $\text{df}(S)$ nodes exposed, and (2) if a matching $M$ leaves exactly $\text{df}(S)$ nodes exposed then $M$ contains a perfect matching of each even $S$-component, a near-perfect matching of each odd $S$-component, and matches all nodes of $S$ to nodes in distinct $S$-components. If (2) is the case for some $S$ and $M$, then $S$ is called Tutte-Berge, while $M$ is maximum by (1). The Tutte-Berge Formula asserts that a Tutte-Berge set $S$ always exists.

If $s + \text{od}(S) \geq 2$, we denote by $\langle G, S \rangle$ the bipartite minor of $G$ obtained from $G$ by deleting the vertices of the even $S$-components, contracting each odd $S$-component to a single node, and deleting the edges spanned by $S$. A monochromatic set $S$ of vertices in a bipartite graph $H$ satisfies Hall’s condition [resp., with surplus $k \in \mathbb{N}$] if for every non-empty subset $T$ of $S$, the size of the neighborhood of $T$ in $H$ is at least $t$ [resp., $t+k$]. In these terms, Hall’s Marriage Theorem asserts that $S$ is covered in $H$ by some matching if and only if $S$ satisfies Hall’s condition. Finally, $S$ is Gallai-Edmonds [with respect to $G$] if:

(a) the even $S$-components, if any, have a perfect matching;
(b) the odd $S$-components, if any, are factor critical;
(c) if $S$ is non-empty, then $S$ satisfies Hall’s condition with surplus one in $\langle G, S \rangle$.

The Gallai-Edmonds Structure Theorem
(i) For every graph $G$, there exists a Gallai-Edmonds set, $S$.

(ii) $S$ is Tutte-Berge. Consequently, every maximum matching of $G$ contains a near-perfect matching of each odd $S$-component, a perfect matching of each even $S$-component, and matches all nodes of $S$ to nodes in distinct $S$-components.

(iii) The underlying vertex set of the odd $S$-components is the set $D(G)$ of the vertices left exposed by at least one maximum matching of $G$, while $S$ is the neighborhood of $D(G)$. In particular, $G$ has a unique Gallai-Edmonds set.

**Proof:** Among the subsets of the vertex set of $G$ with maximum deficiency, let $S$ have minimum $\text{df}(S)$. We show, by induction on $|G|$, that $S$ is Gallai-Edmonds.

Suppose that $C$ is an even $S$-component with no perfect matching. Fix $v \in C$ and set $S' := S \cup \{v\}$. Then $\text{df}(S') \geq \text{df}(S)$ while $\text{df}(S') < \text{df}(S)$. This contradiction shows that $S$ satisfies (a).

Suppose that $v$ is a vertex in an odd $S$-component $C$ such that $C - v$ has no perfect matching. By induction, the graph $H := C - v$ has a Gallai-Edmonds set, $T$. In particular, $\text{df}_H(T) \geq 2$. Set $S' := S \cup T \cup \{v\}$. Then $\text{df}(S') = \text{od}(S') - |S'| = [\text{od}(S) + \text{od}_H(T) - 1] - [s + t + 1] = \text{df}(S) + \text{df}_H(T) - 2 \geq \text{df}(S)$ while $\text{df}(S') < \text{df}(S)$. Thus, $S$ satisfies (b).

Suppose that $T$ is a smallest non-empty subset of $S$ violating Hall’s condition with surplus one in $(G, S)$. Set $S' := S - T$. If $T$ consists of a single vertex with no neighbors in $(G, S)$ then $\text{df}(S') > \text{df}(S)$ which is a contradiction. Else, $T$ satisfies Hall’s condition in $(G, S)$. By (b) and Hall’s Theorem, $T$ is contained in an even $S'$-component with a perfect matching (this even component is spanned by $T$ and the $t$ odd $S$-components “neighboring” with $T$ in $G$). Consequently, $\text{df}(S') = \text{df}(S)$ and $\text{df}(S') < \text{df}(S)$. Thus, $S$ satisfies (c) whence (i).

Let now $S$ be Gallai-Edmonds, and let $v$ be a fixed vertex in an odd $S$-component $C$. By (c) and Hall’s Theorem, $S$ can be matched to nodes in $s$ distinct odd $S$-components different from $C$. By (a) and (b), this matching can be extended by (near-)perfect matchings of the $S$-components to obtain a matching $M$ avoiding $v$. Since $M$ leaves $\text{df}(S)$ nodes exposed, it is maximum, and (ii) and (iii) follow immediately. \qed

**Concluding Remarks**

1. The above proof can be formulated as an exercise, as follows: (A) Among the sets with maximum deficiency, consider a set $S$ which minimizes the total number of vertices in the $S$-components with no perfect matching; (B) prove, by contradiction, that $S$ is Gallai-Edmonds; (C) deduce the Gallai-Edmonds Structure Theorem.

2. Among the sets $T$ with minimum number of even $T$-components with no perfect matching, we could choose $S$ to maximize the difference between $\text{df}(S)$ and the total number of vertices in the odd $S$-components. The above proof would then be repeated almost verbatim.
3. The proof is further shortened if the Tutte-Berge Formula is taken for granted. In fact, 
$S$ can be chosen among the Tutte-Berge sets to minimize the total number of vertices 
in the odd $S$-components. Then (a) and (ii) become superfluous for $S$, while (b) and 
(c) can be shown as above but simpler.

4. By incorporating the proof of Hall’s Theorem into the above inductive argument, one 
can derive the Gallai-Edmonds Structure Theorem “from scratch.”

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**References**

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[2] L. Lovász and M.D. Plummer: Matching Theory, North-Holland Mathematics Studies 121 
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