Submanifolds of warped product manifolds $I \times_f S^{m-1}(k)$ from a $p$-harmonic viewpoint

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Abstract

We study $p$-harmonic maps, $p$-harmonic morphisms, biharmonic maps, and quasiregular mappings into submanifolds of warped product Riemannian manifolds $I \times_f S^{m-1}(k)$ of an open interval and a complete simply-connecteded $(m - 1)$-dimensional Riemannian manifold of constant sectional curvature $k$. We establish an existence theorem for $p$-harmonic maps and give a classification of complete stable minimal surfaces in certain three dimensional warped product Riemannian manifolds $\mathbb{R} \times_f S^2(k)$, building on our previous work. When $f \equiv \text{Const.}$ and $k = 0$, we recapture a generalized Bernstein Theorem and hence the Classical Bernstein Theorem in $\mathbb{R}^3$. We then extend the classification to parabolic stable minimal hypersurfaces in higher dimensions.

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1 Introduction

In [8], we make the first general study of submanifolds in warped product Riemannian manifolds $I \times_f S^{m-1}(k)$ of nonconstant curvature from differential geometric viewpoint. Here $I \subset \mathbb{R}$ is an open interval, $S^{m-1}(k)$ is an $(m - 1)$-dimensional complete, simply-connected, Riemannian manifold of constant sectional curvature $k$, and $f$ is a warping function. This is in contrast to the study of submanifolds in (real, complex, Sasakian, ..., etc.) space forms due to the simplicity of their curvature tensors (see, for instance, [3, 4, 5]). The study is also in contrast to the recent work in treating Riemannian warped product manifolds as submanifolds from the viewpoint of isometric immersions (cf. [6, 9]).

The purpose of this paper is to study submanifolds in warped product Riemannian manifolds $\mathbb{R}^m(k, f) := I \times_f S^{m-1}(k)$ (or simply denoted by $I \times_f S$) of nonconstant curvature from a $p$-harmonic viewpoint. In particular, we study $p$-harmonic maps, $p$-harmonic morphisms, biharmonic maps, and

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quasiregular mappings into submanifolds of $R^m(k, f)$. Furthermore, building on our previous work, for concave warping function $f$ with bounded derivative $|f'| \leq \sqrt{k}$ on $R$, we give a classification Theorem of complete stable minimal surfaces in three dimensional warped product Riemannian manifolds $R \times f S^2(k)$. When $f \equiv$ Const. and $k = 0$, we recapture a generalized Bernstein Theorem ([12, 11, 22], cf. Theorem 7.2) and hence The Classical Bernstein Theorem ([1], cf. Theorem 7.3) in $R^3$. The techniques that we utilized, are sufficiently general to extend the classification theorem for surfaces to parabolic stable minimal hypersurfaces in higher dimensional warped product Riemannian manifolds (cf. Theorem 5.1).

This article is organized as follows: After this first introductory section, we recall some necessary formulas, notations and basic results on warped product manifolds $R^m(k, f)$, and our previous work on submanifolds of $R^m(k, f)$ in section 2, and then describe $p$-harmonic maps and $p$-harmonic morphisms into submanifolds of $R^m(k, f)$ in section 3, biharmonic maps into $R^m(k, f)$ or submanifolds of $R^m(k, f)$ in section 4, quasiregular mappings into $R \times f S$ in section 5, a link to manifolds with warped cylindrical ends in section 6, a classification theorem of complete stable minimal surfaces in three dimensional warped product Riemannian manifolds $R \times f S^2(k)$ in section 7, a classification of parabolic stable minimal hypersurfaces in $R \times f S^n(k)$ in section 8, and $p$-hyperbolic manifolds and stable minimal hypersurfaces in $R \times f S^n(k)$ in section 9.

2 Preliminaries

We recall some basic facts, notations, definitions, and inequalities for Riemannian submanifolds and warped product manifolds (cf. [3, 20]), and some known results about submanifolds of $R^m(k, f)$ and $R \times f S$ (see [8] for details).

2.1 Basic equations and inequalities

Let $M$ be a submanifold of dimension $n \geq 2$ in a Riemannian manifold $\tilde{M}$ with Levi-Civita connection $\tilde{\nabla}$. Denote by $\nabla$ and $\Gamma(TM)$, the Levi-Civita connection of $M$ and the (infinite dimensional) vector space of smooth sections of a smooth tangent bundle $TM$ of $M$ respectively. The formulas of Gauss is given by (cf. [3])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

(2.1)

for $X, Y \in \Gamma(TM)$, where $h$ is the second fundamental form of $M$ in $\tilde{M}$. The mean curvature vector field of a submanifold $M$ is defined by $H = \frac{1}{n} \text{trace} h$. A submanifold $M$ in $\tilde{M}$ is called totally geodesic (resp. minimal) if $h \equiv 0$ (resp. $H \equiv 0$). A minimal hypersurface $M$ in a Riemannian manifold $\tilde{M}$ is said to be stable minimal, if it is a local minimal of area functional. Thus, if $M$ is stable minimal in $\tilde{M}$, then for every $\phi \in C^0_0(M)$,

$$\int_M (\text{Ric} \tilde{M} (\nu) + |A|^2) \phi^2 dv \leq \int_M |\nabla \phi|^2 dv,$$

(2.2)
Lemma 2.1. The curvature tensor $\tilde{R}$ of $R^m(k, f)$ satisfies

$$
\tilde{R}(\partial_t, X)\partial_t = \frac{f''}{f} X,
$$

$$
\tilde{R}(X, \partial_t)Y = \langle X, Y \rangle \frac{f''}{f} \partial_t,
$$

$$
\tilde{R}(X, Y)\partial_t = 0,
$$

$$
\tilde{R}(X, Y)Z = \frac{k - f'^2}{f^2} \{ \langle Y, Z \rangle X - \langle X, Z \rangle Y \}
$$

where $\nu$ is a unit normal vector field to $\tilde{M}$, $|A|^2$ is the squared of the length of the second fundamental form of $M$ in $\tilde{M}$.

2.2 Warped products

Let $B$ and $F$ be two Riemannian manifolds of positive dimensions equipped with Riemannian metrics $g_B$ and $g_F$, respectively, and let $f$ be a positive function on $B$. Consider the product manifold $B \times F$ with its projection $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The warped product $\tilde{M} = B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

$$
||X||^2 = ||\pi_*(X)||^2 + f^2(\pi(x))||\eta_*(X)||^2
$$

for any tangent vector $X \in T_x\tilde{M}$. Thus, we have $g = \pi^*(g_B) + (f \circ \pi)^2\eta^*(g_F)$. The function $f$ is called the warping function of the warped product.

Let $\mathcal{L}(B)$ and $\mathcal{L}(F)$ denote the set of lifts of vector fields on $B$ and $F$ to $B \times_f F$ respectively. For each $q \in F$, the horizontal leaf $\eta^{-1}(q)$ is a totally geodesic submanifold of $B \times_f F$ isometric to $B$. For each $p \in B$, $\pi^{-1}(p)$ is an $(n - b)$-dimensional totally umbilical submanifold of $B \times_f F$ that is homothetically isomorphic to $F$ with scalar factor $\frac{1}{f(p)}$, where $b$ is the dimension of $B$. The submanifolds $\pi^{-1}(p) = \{ p \} \times F, p \in B$, and $\eta^{-1}(q) = B \times \{ q \}, q \in F$ are called fibers and leaves respectively. A vector field on $\tilde{M}$ is called vertical if it is always tangent to fibers; and horizontal if it is always orthogonal to fibers. We use the corresponding terminology for individual tangent vectors as well. A vector field on $\tilde{M}$ is called basic if $X$ is horizontal and $\pi$-related to a vector field $X_\pi$ on $B$.

Let $\mathcal{H}$ and $\mathcal{V}$ denote the projections of tangent spaces of $\tilde{M}$ onto the subspaces of horizontal and vertical vectors, respectively. We use the same letters to denote the horizontal and vertical distributions.

On the warped product $R \times_f S$, let $t$ be an arclength parameter of $R$. Let us denote by $\partial_t$, the lift to $R \times_f S$ of the standard vector filed $\frac{d}{dt}$ on $R$. Thus, $\partial_t \in \mathcal{L}(R)$.

For each vector field $V$ on $R \times_f S$, we decompose $V$ into a sum

$$
V = \varphi_V \partial_t + \hat{V},
$$

where $\varphi_V = \langle V, \partial_t \rangle$ and $\hat{V}$ is the vertical component of $V$ that is perpendicular to $\partial_t$, or the projection of $V_{(p, q)}$ onto its vertical subspace $T_{(p, q)}(p \times S)$.

Lemma 2.1. The curvature tensor $\tilde{R}$ of $R^m(k, f)$ satisfies

$$
\tilde{R}(\partial_t, X)\partial_t = \frac{f''}{f} X,
$$

$$
\tilde{R}(X, \partial_t)Y = \langle X, Y \rangle \frac{f''}{f} \partial_t,
$$

$$
\tilde{R}(X, Y)\partial_t = 0,
$$

$$
\tilde{R}(X, Y)Z = \frac{k - f'^2}{f^2} \{ \langle Y, Z \rangle X - \langle X, Z \rangle Y \}
$$

for any tangent vector $X \in T_x\tilde{M}$. Thus, we have $g = \pi^*(g_B) + (f \circ \pi)^2\eta^*(g_F)$. The function $f$ is called the warping function of the warped product.
for $X, Y, Z \in \mathcal{L}(S)$.

Proof. Follows from [20, p. 210].

2.3 Notions of transverse and $\mathcal{H}$-submanifolds

By a slice of $\mathbb{R} \times f S$ we mean a hypersurface of $R^m(k, f)$ given by $S(t_0) := \{t_0\} \times S$ for some $t_0 \in \mathbb{R}$. A submanifold $M$ of $\mathbb{R} \times f S$ is called a transverse submanifold if it is contained in a slice $S(t_0) := \{t_0\} \times S$ (with the metric: $f^2(t_0)g_k$) for some $t_0 \in \mathbb{R}$, i.e. $\partial_t^T = 0$, where $\partial_t^T$ is the tangential projection of $\partial_t$ onto $M$.

For simplicity, we call a submanifold $M$ in $\mathbb{R} \times f S$ an $\mathcal{H}$-submanifold if the horizontal vector field $\partial_t$ is tangent to $M$ at each point on $M$, i.e. $\partial_t^\perp = 0$, where $\partial_t^\perp$ denotes the normal component of $\partial_t$ to $M$.

2.4 Submanifolds of $\mathbb{R} \times f S$

Let $M$ an $n$-dimensional submanifold of $R^m(k, f)$ and $e_1, \ldots, e_n$ an orthonormal frame on $M$. Then

$$\Phi = \sum_{j=1}^n \varphi_j^2$$

(2.6)

is called the total scalar projection of $TM$ onto $\partial_t$, where $\varphi_j = \langle e_j, \partial_t \rangle$.

For a submanifold $M$ in $R^m(k, f)$, the total scalar projection $\Phi$ satisfies $0 \leq \Phi \leq 1$, with $\Phi = 0$ (respectively, $\Phi = 1$) holding at each point if and only if $M$ is a transverse submanifold (respectively, $M$ is an $\mathcal{H}$-submanifold).

2.5 Minimal submanifolds of $\mathbb{R} \times f S$

**Proposition 2.1.** [8] Let $M$ be a minimal submanifold of $\mathbb{R} \times f S$.

(a) If $(\ln f)^n + k/f^2 = 0$ on $\mathbb{R}$, then the Ricci curvature of $M$ satisfies

$$\text{Ric}(X) \leq \frac{(n-1)(k-f^2)}{f^2} \langle X, X \rangle,$$

(2.7)

for $X \in \Gamma(TM)$. The equality sign of (2.7) holds identically if and only if $M$ is a totally geodesic submanifold of constant curvature $(k - f^2)/f^2$.

(b) If $(\ln f)^n + k/f^2 \neq 0$ on $\mathbb{R}$, then the Ricci curvature of $M$ satisfies

$$\text{Ric}(X) \leq \left\{ ((2-n)\varphi_X^2 - \Phi) \left( \frac{k}{f^2} + (\ln f)^n \right) + \frac{(n-1)(k-f^2)}{f^2} \right\} \langle X, X \rangle.$$

(2.8)

The equality sign of (2.8) holds identically if and only if one of the following two cases occurs:

(b.1) $M$ a transverse submanifold which lies in a slice $S(t_0)$ with $f'(t_0) = 0$ as a totally geodesic submanifold;

(b.2) $M$ is an $\mathcal{H}$-submanifold which is locally the warped product $I \times f N^{n-1}$ of $I$ and a totally geodesic submanifold $N^{n-1}$ of $S$.

Furthermore, if case (b.1) occurs, then $\text{Ric}(X) = \frac{(n-1)(k-f^2)}{f^2} \langle X, X \rangle, X \in TM$. 

Proposition 2.2. Let $M$ be a Riemannian $n$-manifold whose scalar curvature satisfies
\[ \tau \geq \frac{n(n-1)(\tilde{k} - f'^2)}{2f^2} \]  
for some real number $\tilde{k} \geq k$, where $f$ be a positive function satisfying $f' \neq 0$ and $(\ln f)'' > -k/f^2$ on $I$. Then $M$ cannot be isometrically immersed in $R^m(k, f)$ as a minimal submanifold.

Proposition 2.3. Let $M$ be a Riemannian $n$-manifold whose scalar curvature satisfies
\[ \tau > \frac{n(n-1)(k - f'^2)}{2f^2} \]  
at one point, where $f$ is a positive function satisfying $(\ln f)'' \geq -k/f^2$ on $I$. Then $M$ cannot be isometrically minimally immersed in $R^m(k, f)$.

2.6 Classification of parallel submanifolds in $R^m(k, f)$

Theorem 2.1. If $R^m(k, f)$ contains no open subsets of constant curvature, then a submanifold $M$ of $R^m(k, f)$ is a parallel submanifold if and only if one of the following statements holds:

1. $M$ is a transverse submanifold which lies in a slice $S(t_0)$ of $R^m(k, f)$ as a parallel submanifold.
2. $M$ is an $H$-submanifold which is locally the warped product $I \times_f N^{n-1}$, where $N^{n-1}$ is a submanifold of $S$. Furthermore, we have
   (2.1) if $f' \neq 0$ on $I$, then $M$ is totally geodesic in $R^m(k, f)$;
   (2.2) if $f' = 0$ on $I$, then $N^{n-1}$ is a parallel submanifold of $S$.

3 $p$-harmonic maps and $p$-harmonic morphisms into submanifolds in $R^m(k, f)$

A smooth map $u : M \rightarrow N$ is said to be $p$-harmonic, $p > 1$, if it is a critical point of the $p$-energy functional:
\[ E_p(u) = \frac{1}{p} \int_M |du|^p dv \]
with respect to any compactly supported smooth variation, where $|du|$ is the Hilbert-Schmidt norm of the differential $du$ of $u$. It follows from the first variational formula for the $p$-energy functional, $u$ is $p$-harmonic if and only if $u$ is a weak solution of the Euler-Lagrange equation $\text{div}(|du|^{p-2} du) = 0$. Examples of $p$-harmonic maps include geodesics, minimal submanifolds, conformal maps between manifolds of the same dimensions, and harmonic maps (when $p = 2$) (cf. [19, 23, 25, 27]). A $C^2$ map $u : N_1 \rightarrow M$ is called a $p$-harmonic morphism if for any $p$-harmonic function $f$ defined on an open set $V$ of $M$, the composition $f \circ u$ is $p$-harmonic on $u^{-1}(V)$. Examples of $p$-harmonic morphisms include the Hopf fibrations.
3.1 Existence of $p$-Harmonic Maps

**Theorem 3.1.** Let $M$ be a complete Riemannian manifold and $N$ be a compact Riemannian manifold with the universal cover $\tilde{N} = R \times f \, S^{m-1}(k)$, where $f$ is a positive convex function on $R$, and $k \leq 0$. Then any continuous map from $M$ into $N$ of finite $p$-energy can be deformed to a $C^{1,\alpha}$ $p$-harmonic map minimizing $p$-energy in its homotopy class, where $1 < p < \infty$.

**Proof.** In view of Lemma 2.1, $\tilde{N}$ is a complete simply-connected Riemannian manifold of nonpositive curvature. Then the assertion follows from [28, Theorem 2.1 or Corollary 2.4].

3.2 Maps of compact manifolds

**Theorem 3.2.** Every $p$-harmonic map from a compact manifold into $M$ or $R^m(k,f)$ is constant, provided $k \leq 0$, $f$ is a positive convex function on an open interval $I$, and $M$ is a totally geodesic submanifold of $R^m(k,f)$. In particular, if $M$ is a non-transversal parallel submanifold of $R^m(k,f)$ where $k \leq 0$, and $f$ is a positive convex function satisfying $f' \neq 0$ and $(\ln f)' \neq -k/f^2$ on an open interval $I$, then every $p$-harmonic map from a compact manifold into $M$ is constant.

**Proof.** By Lemma 2.1 $I \times f \, S$ is nonpositively curved. Thus, the Gauss curvature equation implies that $\text{Sec}_{M} = \text{Sec}_{R^m(k,f)} \leq 0$. It follows that the image under any $p$-harmonic map of a compact manifold lies in a domain of a strictly convex function, e.g. the squared distance function (cf [2]). But this is impossible unless it is constant by [32] or [25, Theorem 8.1]. This proves the first assertion. Now the second assertion follows from the Classification Theorem 2.1 of parallel submanifolds, that $M$ has to be totally geodesic in $R^m(k,f)$.

**Corollary 3.1.** Let $M$ be a submanifold of $R^m(k,f)$ as in Theorem 3.2. Then there are no compact geodesics (without boundary), and no compact minimal submanifolds in $M$ or in $R^m(k,f)$.

**Proof.** This follows from the previous Theorem 3.2 and [25, Theorems 1.10 (i) and 1.14 (i) p.635,637] which state that a curve parametrized proportionally to the arc length is $p$-harmonic for any $p \geq 1$ if and only if it is a geodesic, and an isometric immersion of $M$ is minimal if and only if it is $p$-harmonic for every $1 < p < \infty$.

3.3 Maps of complete noncompact manifolds

In the following, we assume that $N_1$ is a complete noncompact Riemannian manifold and $B(x_0;r)$ is the geodesic ball of radius $r$ centered at $x_0 \in N_1$. We recall some notions from [29].
**Definition 3.1.** A function $f$ on $N_1$ is said to have $p$-finite growth (or, simply, is $p$-finite) if there exists $x_0 \in N_1$ such that

$$\liminf_{r \to \infty} \frac{1}{r^p} \int_{B(x_0;r)} |f|^q dv < \infty; \quad (3.1)$$

it has $p$-infinite growth (or, simply, is $p$-infinite) otherwise.

**Definition 3.2.** A function $f$ has $p$-mild growth (or, simply, is $p$-mild) if there exist $x_0 \in N_1$, and a strictly increasing sequence of $\{r_j\}_{j=0}^\infty$ going to infinity, such that for every $l_0 > 0$, we have

$$\sum_{j=l_0}^{\infty} \left( \frac{(r_{j+1} - r_j)^p}{\int_{B(x_0;r_{j+1}) \setminus B(x_0;r_j)} |f|^q dv} \right)^{\frac{1}{p-1}} = \infty; \quad (3.2)$$

and has $p$-severe growth (or, simply, is $p$-severe) otherwise.

**Definition 3.3.** A function $f$ has $p$-obtuse growth (or, simply, is $p$-obtuse) if there exists $x_0 \in N_1$ such that for every $a > 0$, we have

$$\int_a^\infty \left( \frac{1}{\int_{\partial B(x_0;r)} |f|^q dv} \right)^{\frac{1}{p-1}} dr = \infty; \quad (3.3)$$

and has $p$-acute growth (or, simply, is $p$-acute) otherwise.

**Definition 3.4.** A function $f$ has $p$-moderate growth (or, simply, is $p$-moderate) if there exist $x_0 \in N_1$, and $F(r) \in \mathcal{F}$, such that

$$\limsup_{r \to \infty} \frac{1}{r^p F^{-1}(r)} \int_{B(x_0;r)} |f|^q dv < \infty. \quad (3.4)$$

And it has $p$-immoderate growth (or, simply, is $p$-immoderate) otherwise, where

$$\mathcal{F} = \{ F : [a, \infty) \to (0, \infty) | \int_a^\infty \frac{dr}{r F(r)} = +\infty \text{ for some } a \geq 0 \}. \quad (3.5)$$

(Notice that the functions in $\mathcal{F}$ are not necessarily monotone.)

**Definition 3.5.** A function $f$ has $p$-small growth (or, simply, is $p$-small) if there exists $x_0 \in N_1$, such that for every $a > 0$, we have

$$\int_a^\infty \left( \frac{r}{\int_{B(x_0;r)} |f|^q dv} \right)^{\frac{1}{p-1}} dr = \infty; \quad (3.6)$$

and has $p$-large growth (or, simply, is $p$-large) otherwise.

We introduce the following notion in [27]:

**Definition 3.6.** A function $f$ has $p$-balanced growth (or, simply, is $p$-balanced) if $f$ has one of the following: $p$-finite, $p$-mild, $p$-obtuse, $p$-moderate, or $p$-small growth, and has $p$-imbalanced growth (or, simply, is $p$-imbalanced) otherwise.
The above definitions depend on $q$, and $q$ will be specified in the context in which the definition is used.

**Theorem 3.3.** Let $M$ and $R^m(k, f)$ be as in Theorem 3.2, and $u$ from $N_1$ into $M$ or $R^m(k, f)$ be a smooth (a) harmonic map where $p = 2$ or (b) $p$-harmonic morphism with $p > 2$. Then $u$ is either a constant or the function $\text{dist}^2(u(x), y_0)$ on $N_1$ has $p$-imbalanced growth for all $q > p - 1$. Here $\text{dist}^2(u(x), y_0)$ is the square of the distance between $u(x)$ and a fixed point $y_0$ in $M$ or $R^m(k, f)$.

**Proof.** This follows immediately from [29, Theorem 5.4.(i).], based on a composition formula and estimates on $p$-subharmonic functions [29, Theorems 2.1-2.5]. For $p > 2$, the composition $\text{dist}^2(u(x), y_0)$ of a $p$-harmonic morphism $x \mapsto u(x)$ and a convex function $y \mapsto \text{dist}^2(y, y_0)$ is $p$-subharmonic on $N_1$ (cf. [29, Theorem 5.2.]). For $p = 2$, see also [9].

**4 Biharmonic maps into $R^m(k, f)$ or submanifolds of $R^m(k, f)$**

We apply our results from previous sections to study biharmonic, conformal, $p$-harmonic maps into $R^m(k, f)$ or into submanifolds of $R^m(k, f)$. We also study isometric minimal immersions in $R^m(k, f)$ and its submanifolds.

A smooth map $u : M \to N$ between two Riemannian manifolds is said to be biharmonic if $u$ is a critical point of bi-energy:

$$E_2(u) = \int_M |(d + d^*)^2 u|^2 dx = \int_M |\tau(u)|^2 dx$$

with respect to any compactly supported variation, and polyharmonic of order $k$ if $u$ is a critical point of

$$E_k(u) = \int_M |(d + d^*)^k u|^2 dx$$

with respect to any compactly supported variation, where $\tau(u)$ is the tension field of $u$, and $d^*$ is the adjoint of the exterior differential operator $d$.

A $C^\infty$ section of a bundle over a Riemannian manifold has $p$-imbalanced growth if its norm is so, and $p$-balanced growth otherwise.

**Theorem 4.1.** Let $M$ be a submanifold of $R^m(k, f)$ as in Theorem 3.2. Let $u$ be a smooth biharmonic isometric immersion of a complete manifold $N_1$ into $M$ or $R^m(k, f)$. If for some $q > 2$, $\tau(u)$ has $2$-balanced growth, then we have:

1. $\tau(u)$ is parallel. Further, if $u$ is not harmonic, then $N_1$ is parabolic.

2. $u$ is either the unique harmonic map unless it is a constant or maps $N_1$ onto a closed geodesic $\gamma$ in $M$ or $R^m(k, f)$ (in the latter case, we have uniqueness up to rotations of $\gamma$); or $u$ is of rank one in which case $\tau(u)$ at each point is tangent to the image curve of $u$. 

(3) If, for some \( x_0 \in M \) and \( q = 2 \), \( du \) has 2-balanced growth, then \( u \) is a harmonic map minimizing energy in its homotopy class.

(4) Under the assumption of (3), if for some \( q > 2 \), and \( y_0 \in u(N_1) \), its distance function, defined by \( \text{dist}(u(x), y_0) \) for \( x \in N_1 \), has 2-balanced growth, then \( u \) is constant.

Proof. We note that both \( M \) and \( R^m(k, f) \) are simply-connected manifolds of nonpositive curvature, but are not necessarily complete. However, the conclusions follow by proceeding exactly as in the proof of [29, Theorem 9.1](cf. [30]).

Corollary 4.1. Let \( M \) and \( R^m(k, f) \) be as in Theorem 4.1. Let \( u \) be a smooth biharmonic isometric immersion of a complete manifold \( N_1 \) into \( R^m(k, f) \) (resp. \( M \)) with mean curvature vector field \( H \) which has 2-balanced growth. Then we have:

(i) \( N_1 \) is a minimal submanifold of \( R^m(k, f) \) (resp. of \( M \)) for \( u : N_1 \to R^m(k, f) \) (resp. \( u : N_1 \to M \)).

(ii) \( u \) is \( p \)-harmonic for every \( 1 < p < \infty \).

(iii) \( u \) is polyharmonic of order \( j \) for every \( j \in \{1, 2, \cdots \} \).

Proof. We first assume the case \( u(N_1) \subset M \), and note that if \( u \) is an isometric immersion, then its tension field \( \tau(u) \) agrees with its mean curvature \( H \) and \( |du| = \sqrt{\dim N_1} \). It follows from Theorem 4.1 (1) that either \( H \equiv 0 \), then we have proved (i), or \( |H| \equiv C \), a nonzero constant, i.e. \( u \) is not harmonic. But if \( |H| \equiv C \neq 0 \), then the growth assumption on \( H \) implies the same growth condition on \( |du| = \sqrt{\dim N_1} \). Thus, by Theorem 4.1 (3), \( u \) would be harmonic and hence \( H \equiv 0 \), a contradiction. This proves that \( N_1 \) is a minimal submanifold of \( M \), and hence a minimal submanifold of \( R^m(k, f) \) by Theorem 2.1 that \( M \) is a totally geodesic submanifold of \( R^m(k, f) \). The same technique also proves the case \( u(N_1) \subset R^m(k, f) \), and the assertion (i) follows. Now assertions (ii) and (iii) follow from [25, Theorem 1.14, p.637].

Theorem 4.2. Let \( M \) and \( f \) be as in Proposition 2.2 or 2.3. Then there are neither \( p \)-harmonic conformal immersions \( u : M \to R^m(k, f), p = \dim N_1 \), nor \( p \)-harmonic isometric immersions \( u : M \to R^m(k, f), p > 1 \).

Proof. Suppose on the contrary, then by a Theorem of Takeuchi [23] or [25, Theorem 1.14], \( u(M) \) would be minimal in \( R^m(k, f) \), contradicting Proposition 2.2 or 2.3.

5 Quasiregular mappings into \( \mathbb{R} \times_f S \)

Our previous ideas can be naturally applied to the study of quasiregular mappings. These mappings are generalizations of complex analytic func-
tions on the plane, to higher dimensional Euclidean spaces; even more gen-
erally to Riemannian \( n \)-manifolds. While analytic functions pull back har-
monic (resp. superharmonic) functions on an open subset of \( \mathbb{R}^2 \) to harmonic
(resp. superharmonic) functions, quasiregular mappings pull back \( n \)-harmonic (resp. \( n \)-superharmonic) functions on manifolds to \( A \)-harmonic (resp. \( A \)-superharmonic) functions (of type \( n \)) (cf [15] and [16]).

We denote by \( W_{loc}^{1,p}(M) \) the Sobolev space whose real-valued functions on
\( M \) are locally \( p \)-integrable and have locally \( p \)-integrable partial distributional
first derivatives. A continuous mapping \( u : M \rightarrow N \) between two Riemannian
\( n \)-manifolds is said to be quasiregular if \( u \) is in \( W_{loc}^{1,n}(M) \), and there exists a
constant \( 1 \leq K < \infty \) such that the differential \( du_x \) and the Jacobian \( J_u(x) \)
satisfy
\[
|du_x|^n \leq K J_u(x)
\]
for (a.e.) almost every \( x \in M \), where the operator norm of differential
\[
|du_x| = \max \{ du_x(\xi) : \xi \in T_x(M), |\xi| = 1 \}.
\]
A quasiregular mapping is said to be quasiconformal if it is a homemorphism. A
continuous mapping \( u : M \rightarrow N \) is said to be a quasi-isometry if \( u \) is in \( W_{loc}^{1,1}(M) \), if \( J_u(x) \geq 0 \) a.e., and if there exists a constant \( 1 \leq L < \infty \) such
that the differential \( du_x \) satisfies
\[
\frac{1}{L} |\xi| \leq |du_x(\xi)| \leq L |\xi|
\]
for (a.e.) almost every \( x \in M \), and \( \xi \in T_x(M) \). Examples of quasiregular mapping include isometries, quasi-isometries (with \( K = L^{2(n−1)} \)), Möbius maps, and holomorphic maps from the complex plane to a Riemann surface.

We denote by \( A \) a measurable cross section in the bundle whose fiber at
a.e. \( x \) in \( M \) is a continuous map \( A_x \) on the tangent space \( T_x(M) \) into \( T_x(M) \).
We assume further that there are constants \( 1 < p < \infty \) and \( 0 < \alpha \leq \beta < \infty \)
such that for a.e. \( x \) in \( M \) and all \( h \in T_x(M) \), we have
\[
\langle A_x(h), h \rangle_M \geq \alpha |h|^p,
\]
\[
|A_x(h)| \leq \beta |h|^{p−1},
\]
\[
\langle A_x(h_1) - A_x(h_2), h_1 - h_2 \rangle_M > 0, \quad h_1 \neq h_2,
\]
and
\[
A_x(\lambda h) \equiv |\lambda|^{p−2} \lambda A_x(h), \quad \lambda \in \mathbb{R} \setminus \{0\}.
\]
A function \( f \in W_{loc}^{1,p}(M) \) is a weak solution (resp. supersolution, subsolution) of the equation
\[
\text{div} A_x(\nabla f) = 0 \ (\text{resp.} \leq 0, \geq 0),
\]
if for all nonnegative \( \varphi \in C_0^\infty(M) \)
\[
\int_M \langle A_x(\nabla f), \nabla \varphi \rangle dv = 0 \ (\text{resp.} \geq 0, \leq 0)
\]
The equation (5.7) is called $\mathcal{A}$--harmonic equation, and continuous solutions of (5.7) are called $\mathcal{A}$--harmonic (of type $p$). In the case $\mathcal{A}_p(h) \equiv |h|^{p-2}h$, $\mathcal{A}$--harmonic functions are $p$--harmonic. A lower (resp. upper) semicontinuous function $f : M \to R \cup \{\infty\}$ (resp. $\{-\infty\} \cup R$) is $\mathcal{A}$--superharmonic (resp. $\mathcal{A}$-subharmonic) (of type $p$) if it is not identically infinite, and it satisfies the $\mathcal{A}$--comparison principle: i.e., for each domain $D \subset M$ and for each function $g \in C(D)$ which is $\mathcal{A}$--harmonic in $D$, $g \leq f$ (resp. $g \geq f$) in $\partial D$ implies $g \leq f$ (resp. $g \geq f$) in $D$. An $\mathcal{A}$--superharmonic (resp. $\mathcal{A}$--subharmonic) function $f$ is called $p$--superharmonic (resp. $p$--subharmonic) if $\mathcal{A}_p(h) \equiv |h|^{p-2}h$.

$\mathcal{A}$--superharmonic and $\mathcal{A}$-subharmonic functions are closely related to subsolutions and supersolutions of (5.7). For a discussion of the $\mathcal{A}$-harmonic equation, we refer the reader to J. Heinonen, T. Kipeläinen and O. Martio’s book [17].

A complete noncompact manifold $M$ is said to be $\mathcal{A}$--parabolic (resp. $p$--parabolic) (of type $p$) if every nonnegative measurable $\mathcal{A}$-superharmonic (resp. $p$-superharmonic) (of type $p$) function is constant, and $\mathcal{A}$--hyperbolic (resp. $p$--hyperbolic) (of type $p$) otherwise.

Throughout this section, we assume $S$ is a Riemannian $(m-1)$-manifold of constant sectional curvature $k$. We begin with a general

**Theorem 5.1.** Let $N_1$ be an $\mathcal{A}_1$-parabolic manifold (of type $m$) and $N_2$ be an $\mathcal{A}_2$-hyperbolic manifold (of type $m$). Then there does not exist any quasiregular mapping $u$ from $N_1$ into $N_2$, unless it is a constant.

The case $\mathcal{A}_1(\nabla \varphi) = \mathcal{A}_2(\nabla \varphi) = |\nabla \varphi|^{m-2}\nabla \varphi$ is due to T. Coulhon, I. Holopainen and L. Saloff-Coste [10]:

**Proposition 5.1.** Every quasiregular mapping $u$ from an $m$-parabolic manifold into an $m$-hyperbolic manifold is constant.

Theorem 5.1 recaptures classical Picard’s Theorem, which states that every analytic function $u$ on the complex plane $C$ omits at least two different values must be constant. This is the case for its lift $\tilde{u} : C \to D$, where $m = 2$, $K = 1$ in (5.1), $A_{1x}(h) = A_{2x}(h) = h$, $N_1 = C$ is parabolic, and $N_2 = D$, the universal cover of $C\backslash\{z_1, z_2, \cdots\}$, is hyperbolic (cf. also [27]).

**Proof.** Suppose on the contrary. Let $f$ be a nonconstant positive $\mathcal{A}_2$-superharmonic function (of type $m$) on $N_2$, and $f_j$ be a nonconstant supersolution of $\mathcal{A}_2$-harmonic equations, where $f_j = \min\{f, j\}$ and $j$ is a positive integer (cf. [17], 7.2, 7.20). Then $u$ would pull back $f_j$ on $N_2$ to a nonconstant positive supersolution $f_j \circ u$ of $\mathcal{A}_3$-harmonic equations on $N_1$, where $\mathcal{A}_3$ is the pull-back of $\mathcal{A}_2$ under $f_j \circ u$ satisfying (5.3)-(5.6) (cf. [16], (2.9a), (2.9b)). It follows that there would exist a compact set $C \subset N_1$ such that $\inf_\varphi \int_{N_1} |\nabla \varphi|^m dV > 0$, where the infimum is taken over all $\varphi \geq 1$ on $C$ and $\varphi \in C_0^\infty(N_1)$ (cf. [16], 5.2). In view of (5.3), we would have

$$\inf_\varphi \int_{N_1} \langle A_1(\nabla \varphi), \nabla \varphi \rangle dV \geq \inf_\varphi \alpha \int_{N_1} |\nabla \varphi|^m dV > 0$$

(5.9)
where the infimum is taken over all \( \varphi \ge 1 \) on \( C \) and \( \varphi \in C^\infty_0(N_1) \). By an exhaustion argument (cf. e.g. [29], 5), based on Harnack’s principle, Hölder continuity estimates, and Arzela-Ascoli Theorem, there would exist a nonconstant positive \( A_1 \)-superharmonic function (of type \( m \)) on \( N_1 \) (called Green function on \( M \) for the operator \( A_1 \)) (cf. [16], 3.27), contradicting the hypothesis that \( N_1 \) is an \( A_1 \)-parabolic manifold (of type \( m \)).

\[ \text{Corollary 5.1. Every } m \text{-harmonic morphism } u \text{ from an } m \text{-parabolic manifold into an } m \text{-hyperbolic manifold is constant}. \]

\[ \text{Proof. It follows from the fact that every } m \text{-harmonic morphism } u \text{ between } m \text{-manifolds is conformal (cf. [21]), and hence quasiregular (in which } K = 1 \text{ in (5.1))}. \]

\[ \text{Theorem 5.2. Let } f \text{ be a positive function on the Euclidean line } \mathbb{R} \text{ satisfying } \min\{\frac{f''}{f}, \frac{f'^2-k}{f^2}\} \ge a^2 \text{ with } a > 0. \text{ Then there does not exist any quasiregular mapping } u \text{ from any } A \text{-parabolic manifold } N \text{ (of type } m \text{) into } \mathbb{R} \times fS, \text{ unless it is a constant.} \]

\[ \text{Proof. By virtue of Lemma [21], } \mathbb{R} \times fS \text{ is a complete simply-connected manifold with sectional curvature } K \le -a^2. \text{ Then, for any domain } \Omega \text{ relatively compact in } \mathbb{R} \times fS \text{ with smooth boundary } \partial \Omega, x_0 \in \Omega \text{ and } r(x) = \text{dist}(x_0, x), \text{ we have via Gauss’ lemma, Stokes’ Theorem and the Hessian Comparison Theorem that} \]

\[ \int_{\partial \Omega} 1 \, dS \ge \int_{\partial \Omega} \langle \nabla r, \nu \rangle \, dS = \int_{\Omega} \Delta r \, dV \ge (m - 1) \int_{\Omega} \frac{\cosh ar}{a \sinh ar} \, dV \ge (m - 1) \int_{\Omega} a \, dV. \]  

\[ (5.10) \]

Hence,

\[ \text{Area}(\partial \Omega) \ge (m - 1)a \text{ vol}(\Omega). \]  

\[ (5.11) \]

Now set

\[ \Psi(t) = \inf \{ \text{Area}(\partial \Omega) : \Omega \subset \subset R \times fS, \partial \Omega \in C^\infty, \text{vol}(\Omega) \ge t \}. \]

Then, for any \( p \in (1, \infty) \), \( \Psi(t) \) satisfies

\[ \int_{t_0}^{\infty} \frac{1}{\Psi(t)^{p-1}} \, dt < \infty \]  

\[ (5.12) \]

where \( t_0 > 0 \) is a constant. It follows from a Theorem of Troyanov [24] that there exists a nonconstant positive supersolution of \( p \)-harmonic equation defined on \( \mathbb{R} \times fS \), or \( \mathbb{R} \times fS \) is \( p \)-hyperbolic for every \( p > 1 \). Choose \( p = m \), and the assertion follows from Theorem 5.1 (in which \( A_1 = A \), and \( A_2(h) \equiv |h|^{m-2}h \)).
Corollary 5.2. Let $f$ and $N$ be as in Theorem 5.2 and $M$ be a totally geodesic $n$-submanifold of $R^m(k, f)$. Then (i) Every quasiregular mapping $u$ from $N$ into $R^m(k, f)$ is constant. (ii) Every quasiregular mapping from an $A$–parabolic $n$-manifold (of type $n$) into $M$ is constant.

Proof. (i) In view of (5.1), $u : N \to R^m(k, f)$ is quasiregular as a mapping into $R \times_f S$, and hence a constant by Theorem 5.2. (ii) By the totally geodesic assumption, $M$ is an $n$-manifold with sectional curvature bounded above by $-a^2$, and hence $M$ is $n$-hyperbolic. The assertion follows from Theorem 5.1.

Corollary 5.3. Let $f$ be as in Theorem 5.2. If $N$ is a Riemannian $n$-manifold of nonnegative Ricci curvature, then there does not exist any quasiregular mapping $u$ from $N$ into $R \times_f S$, unless it is a constant.

Proof. By virtue of Bishop’s Volume Comparison Theorem and $A$–superharmonic estimates, $N$ is $A$–parabolic (cf. [29, Corollary 3.3]). The assertion follows from Theorem 5.2.

Corollary 5.4. Let $f$ be as in Theorem 5.2 and let $f_1$ be a positive concave function on the Euclidean line $R$ satisfying $f_1'' \leq k$. Then there is neither any nonconstant quasiregular mapping $u$ from $R \times_f S$ into $R \times_f S$, nor nonconstant quasiregular mapping $u_1 : M_1 \to M$ between complete totally geodesic $n$-submanifolds $M_1(\subset R \times_f S)$ and $M(\subset R \times_f S)$.

Proof. By assumption and Lemma 2.1 $R \times_f S$ has nonnegative sectional curvature. The assertion follows from Corollary 5.3.

As noted above, quasiconformal mappings and quasi-isometries are special cases of quasiregular mappings (in which $K = L^{2(n-1)}$), we have

Corollary 5.5. Let $f$ be as in Theorem 5.2. Then every quasiregular map from $E^m$ into $R \times_f S$ is constant. In particular, there is neither a quasi-isometry from $E^m$ into $R \times_f S$ whose Jacobian is positive almost everywhere, nor a quasiconformal map from $E^m$ into $R \times_f S$.

6 A link to manifolds with warped cylindrical ends

Our previous study can also be linked to manifolds with warped cylindrical ends. A manifold $N_1$ is said to have a warped cylindrical end if there exists a compact domain $D \subset N_1$ and a compact Riemannian manifold $(K, g_K)$ such that $N_1 \setminus D = (1, \infty) \times_f K$, the warped product of $(1, \infty)$ and $K$. An obvious example is the Euclidean plane with warping function $f_1(t) = t$. As a second example, the warped product $I \times_f S$, where $I = (1, \infty)$ and $S = S^{m-1}(1)$ is an $(m - 1)$-manifold with a warped cylindrical end, in which $D$ is the empty set, and $f_1 = f$. 
Theorem 6.1. Let $N$ be as in Theorem 5.2, and $N_2$ be an $m$-manifold with a warped cylindrical end such that the warping function $f_2$ satisfies $\int_1^\infty \frac{1}{f_2(t)} dt < \infty$, then there does not exist any nonconstant quasiregular mapping $u$ from $N$ into $N_2$. In particular, there is no nonconstant $m$-harmonic morphism from $N$ into $N_2$.

Proof. According to a Theorem of M. Troyanov [24], an $m$-manifold $N_2$ with a warped cylindrical end is $p$-parabolic if and only if its warping function $f_2$ satisfies $\int_1^\infty f_2(t)^{\frac{m_2-1}{1-p}} dt = \infty$. Hence, $N_2$ is $m$-hyperbolic, and $N$ is $m$-parabolic. Now the assertion follows from Theorem 5.2 and Corollary 5.1. 

Similarly, we have the following Liouville-type results for $p$-harmonic morphisms between manifolds with warped cylindrical ends:

Theorem 6.2. Let $N_i$ ($i = 1, 2$) be an $m_i$-manifold with a warped cylindrical end such that the warping functions $f_1$ and $f_2$ satisfy $\int_1^\infty f_1(t)^{\frac{m_1-1}{1-p}} dt = \infty$ and $\int_1^\infty f_2(t)^{\frac{m_2-1}{1-p}} dt < \infty$. Then every $p$-harmonic morphism from $N_1$ into $N_2$ is constant.

As an obvious example of Theorem 6.2, there does not exist a nonconstant $p$-harmonic morphism from the Euclidean space $E^{m_1}$ into $E^{m_2}$ for $m_1 \leq p < m_2$.

In view of the above second example of an $m$-manifold with a warped cylindrical end, Theorems 5.1, 6.1 and 6.2 yield the following two results.

Corollary 6.1. Let $N_i$ ($i = 1, 2$) be an $m$-manifold with a warped cylindrical end such that the warping functions $f_1$, $f_2$ satisfy $\int_1^\infty \frac{dt}{f_1(t)} = \infty$ and $\int_1^\infty \frac{dt}{f_2(t)} < \infty$. Let $I = (1, \infty)$ and $S = S^{m-1}$. Then we have:

1. Every quasiregular mapping (in particular, every $m$-harmonic morphism) from $N_1$ to $I \times f S$ is constant, whenever $f$ satisfies $\int_1^\infty \frac{dt}{f(t)} < \infty$.
2. Every quasiregular mapping (in particular, every $m$-harmonic morphism) from $I \times f S$ to $N_2$ is constant, whenever $f$ satisfies $\int_1^\infty \frac{dt}{f(t)} = \infty$.
3. Every quasiregular mapping from $N_1$ to $N_2$ is constant. In particular, there is no nonconstant $m$-harmonic morphism from $N_1$ to $N_2$.

Corollary 6.2. Let $N_i$, and $f_i$ be as in Theorem 6.2 for $i = 1, 2$. Let $I = (1, \infty)$ and $S = S^{m-1}$. Then we have:

1. Every $p$-harmonic morphism from $N_1$ to $I \times f S$ is constant, whenever $f$ satisfies $\int_1^\infty f(t)^{\frac{m_1-1}{1-p}} dt < \infty$.
2. Every $p$-harmonic morphism from $I \times f S$ to $N_2$ is constant, whenever $f$ satisfies $\int_1^\infty f(t)^{\frac{m_2-1}{1-p}} dt = \infty$.

7 Classification of complete stable minimal surfaces in $R \times_f S^2(k)$

Theorem 7.1. Let $M$ be a stable minimal surface of $R \times_f S$, where $f$ is a positive $C^2$ concave function with bounded derivative $|f'| \leq \sqrt{k}$ on $R$. Then
M is totally geodesic. Furthermore,
(a) If \((\ln f)'' + k/f^2 = 0\) on \(R\), then \(M\) is a plane.
(b) If \((\ln f)'' + k/f^2 \neq 0\) on \(R\), then one of the following two cases occur
\(b.1\) \(M\) is a transverse submanifold which is a slice \(S(t_0)\) with \(f'(t_0) = 0\)
as a totally geodesic submanifold of \(R \times f S\); or
\(b.2\) \(M\) is an \(H\)-submanifold which is locally the warped product \(I \times f N^1\)
of \(I\) and a geodesic \(N^1\) of \(S\).

Moreover, if case \((b.1)\) occurs, then \(\text{Sec}(X) = \frac{(k-f'^2)}{f^2} \langle X, X \rangle, X \in \Gamma(TM)\).

To be self-contained, we provide the following complete

**Proof.** By virtue of the assumption \(f'' \leq 0, |f'| \leq \sqrt{K}\) on \(R\), and Lemma 2.1, \(\tilde{M} = R \times f S\) is a complete simply-connected manifold with sectional curvature \(\tilde{K} \geq 0\), and \(\text{Ric} \tilde{M} \geq 0\).

Since \(M\) is a minimal surface with Guass curvature \(K\) in \(\tilde{M}\), it follows from the Guass curvature equation that

\[
0 \leq \tilde{K} = K - h_{11} h_{22} + h_{12}^2 = K + h_{11}^2 + h_{12}^2 = K + \frac{1}{2} |A|^2. \tag{7.1}
\]

Hence, the stability inequality \([2,2]\) and \((7.1)\) imply that for every \(\varphi \in C^0_0(M)\),

\[
-2 \int_M K \varphi^2 dv \leq \int_M |A|^2 \varphi^2 dv \leq \int_M (\text{Ric} \tilde{M}(\nu) + |A|^2) \varphi^2 dv \tag{7.2}
\]

\(\leq \int_M |\nabla \varphi|^2 dv,\)

(Following \([26]\)) Firstly, we claim if \(M\) is conformally equivalent to the plane or equivalent if \(M\) is parabolic (i.e. there does not exist a positive superharmonic function unless it is a constant), then \(M\) is totally geodesic: Proceed as in \([26\] p.152-153)(in which \(b = |A|^2 \varphi^2\), and \(c_1 = 1\)).

For any fixed compact set \(K\) in \(M\), choose a sufficiently large \(r > 0\) so that the ball \(B_r\) of radius \(r\) covers \(K\) and pick

\[
\varphi_r = \begin{cases} 
1 & \text{on } K \\
0 & \text{on } \partial B_r \\
\text{harmonic} & \text{in } B_r \setminus K.
\end{cases}
\]

Set \(\phi = \varphi_r\) in \((7.2)\). Since \(\Delta \varphi_r = 0\) in \(M \setminus K\), \(\varphi_r = 0\) on \(\partial B_r\) and by divergence
theorem
\[ \int_K |A|^2 \phi^2 \, dv \leq c_1 \int_{B_r \setminus K} |A| \phi^2 \, dv \]
\[ = c_1 \int_{B_r \setminus K} \text{div}(\phi_r \nabla \phi_r) \, dv \]
\[ = c_1 \int_{\partial K} \frac{\partial \phi_r}{\partial n} \, ds \]
\[ = c_1 \int_{\Sigma} \frac{\partial \phi_r}{\partial n} \, ds \]

where \( \Sigma \) is a hypersurface between \( \partial K \) and \( B_r \) and \( n \) is the unit outer normal vector.

The last step follows from the harmonicity of \( \phi_r \) between \( \partial K \) and \( \Sigma \). By the maximum principle \( 0 \leq \phi_r \leq 1 \) and \( \phi_r \) increases as \( r \) increases. Then \( \phi_r \) converges to a constant function \( \phi_\infty \equiv 1 \). Otherwise \( \phi_\infty \) would be a nonconstant positive superharmonic function on \( M \), a contradiction to the parabolicity.

By an interior elliptic estimate [13], \( \nabla \phi_r \rightarrow 0 \) uniformly on compact subsets of \( M \setminus K \) as \( r \) tends to \( \infty \). It follows from (7.3) that \( A \equiv 0 \) and hence \( M \) is totally geodesic.

Secondly, we claim if \( M \) is conformally equivalent to the unit disk \( D \) endowed with the complete metric \( \frac{1}{(1+|z|^2)^2} |dz|^2 \), or equivalent if \( M \) is hyperbolic (i.e., \( M \) is not parabolic), then \( M \) is not a stable minimal submanifold of \( \tilde{M} \):

Proceed as in [26, p.154-155] in which \( c_1 = c_2 = 1 \) (cf. [12]). Suppose on the contrary, then by a well-known formula, its Gaussian curvature \( K \) is given by

\[ K = \frac{\Delta f}{f} - \frac{\left( \frac{\Delta f}{f} \right)^2}{f^2} \]

where \( \Delta \) is the Beltrami-Laplace operator on \( M \). And (7.2) implies that for every \( \phi \in C_0^\infty(M) \)

\[ -2 \int_M \phi^2 \left( \frac{\Delta f}{f} - \frac{\left( \frac{\Delta f}{f} \right)^2}{f^2} \right) \, dv \leq \int_M |\nabla \phi|^2 \, dv . \]  

(7.5)

Substituting \( \phi = f \phi \) into (7.5) for every \( \phi \in C_0^\infty(M) \), we have

\[ -2 \int_M f \phi^2 \Delta f - |\nabla f|^2 \phi^2 \, dv \leq \int_M |\nabla f|^2 \phi^2 + |\nabla \phi|^2 f^2 + 2 f \phi \nabla f \nabla \phi \, dv . \]  

(7.6)

Integration by parts and Cauchy-Schwarz inequality yield

\[ 3 \int_M |\nabla f|^2 \phi^2 \, dv \leq \int_M |\nabla \phi|^2 f^2 \, dv - \int_M 2 f \phi \nabla f \nabla \phi \, dv \]
\[ \leq 2 \int_M |\nabla \phi|^2 f^2 \, dv + \int_M \phi^2 |\nabla f|^2 \, dv . \]  

(7.7)

Then (7.7) implies

\[ \int_M |\nabla f|^2 \phi^2 \, dv \leq \int_M f^2 |\nabla \phi|^2 \, dv . \]  

(7.8)
Choose a cut-off function \( \varphi \) with compact support in \( B_r \) with \( |\nabla \varphi| \leq \frac{c}{r} \) and \( \varphi \equiv 1 \) on \( B_{\frac{3}{2}r} \) and use the fact that \( dv = \frac{1}{f^2} dx dy \)

\[
\int_{B_{\frac{3}{2}r}} |\nabla f|^2 dv \leq \frac{c^2}{r^2} \int_M f^2 dv \\
= \frac{c^2}{r^2} \int_D dx dy \\
= \frac{c^2 \pi}{r^2}
\]

which tends to 0 as \( r \) tends to \( \infty \). Hence \( f \) is a constant on \( M \), contradicting the completeness of the metric \( \frac{1}{f(z)} |dz|^2 \).

It follows from the Uniformization Theorem that \( M \) is totally geodesic. In view of Proposition 2.1 that the assertion (b) follows if \( (\ln f)'' + k/f^2 \neq 0 \) on \( R \); and \( M \) is a totally geodesic submanifold of nonnegative constant curvature \( (k - f^2)/f^2 \), if \( (\ln f)'' + k/f^2 = 0 \) on \( R \). In the latter case, \( M \) has to be flat or \( M \) is a plane, since a totally geodesic submanifold \( M \) of a sphere (which has positive constant curvature) is not stable. This completes the proof.  

When \( f = \text{Const} \), and \( k = 0 \), this result recaptures the following theorem of Colbrie-Fisher - Schoen [12], do Carmo - Peng [11], and Pogorelov [22]:

**Theorem 7.2.** Every complete stable minimal surface in \( R^3 \) is a plane.

Which implies the Classical Bernsten Theorem [1]:

**Theorem 7.3.** Every entire solution to the Minimal Surface Equation

\[
\text{div} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = 0
\]

on \( R^2 \) is an affine function.

# 8 Classification of parabolic stable minimal hypersurfaces in \( R \times_f S^n(k) \)

Utilizing the same technique in the last section, we obtain

**Theorem 8.1.** Let \( M \) be a parabolic stable minimal hypersurface in warped product Riemannian manifolds \( R \times_f S^n(k) \), where \( f \) is as in Theorem 7.1. Then \( M \) is totally geodesic. Furthermore,

(a) If \( (\ln f)'' + k/f^2 = 0 \) on \( R \), then \( M \) is a hyperplane.

(b) If \( (\ln f)'' + k/f^2 \neq 0 \) on \( R \), then one of the following two cases occur

(b.1) \( M \) a transverse submanifold which lies in a slice \( S(t_0) \) with \( f'(t_0) = 0 \) as a totally geodesic submanifold;

(b.2) \( M \) is an \( \mathcal{H} \)-submanifold which is locally the warped product \( I \times_f N^{n-1} \) of \( I \) and a totally geodesic submanifold \( N^{n-1} \) of \( S \).

Furthermore, if case (b.1) occurs, then \( \text{Ric}(X) = \frac{(n-1)(k-f^2)}{f^2} \langle X, X \rangle , X \in \Gamma(TM) \).
Proof. Proceed as in the first part of the proof of Theorem 7.1, \( M \) is totally geodesic, and Proposition 2.1 completes the proof. \( \square \)

9 \( p \)-hyperbolic manifolds and stable minimal hypersurfaces in \( \mathbb{R} \times_f S^n(k) \)

In the course of proving Theorem 5.2, one has shown the case \( p > 1 \) for the following

**Proposition 9.1.** Every complete, simply-connected, manifold with sectional curvature bounded above by a negative constant is \( p \)-hyperbolic for all \( p \geq 1 \). In particular, every \( n \)-dimensional hyperbolic space \( \mathbb{H}^n \) is \( p \)-hyperbolic for all \( p \geq 1 \).

Proof. For the case \( p = 1 \), this follows from [24, p.139]. \( \square \)

Let \( \mathbb{B}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 < 1\} \) be the unit \( n \)-ball. Assume that the hyperbolic space \( \mathbb{H}^n \) is modeled on the Euclidean unit \( n \)-ball \( (\mathbb{B}^n, \frac{4}{(1-|x|^2)}\ dx^n) \) where \( dx^n \) is Euclidean metric and \( x = (x_1, \ldots, x_n) \). By proposition 9.1, \( \mathbb{H}^n \) is \( p \)-hyperbolic for all \( p \geq 1 \). We remark that by proceeding exactly as in the proof of [18, Theorem 1.3], one can prove that every complete manifold \( M \) that is conformally equivalent to the unit \( n \)-ball \( \mathbb{B}^n \) cannot be stably minimally immersed in \( \mathbb{R} \times_f S^n(k) \), where \( f(x) = \sqrt{k}x + b \), for any constants \( k \geq 0 \) and \( b \). This is precisely the nonexistence theorem in \( \mathbb{R}^{n+1} \) [18, Theorem 1.3], since by Lemma 2.1 and Cartan-Ambrose-Hicks Theorem, such \( \mathbb{R} \times_f S^n(k) \) is isometric to \( \mathbb{R}^{n+1} \).

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