INEQUALITIES FOR A NUMBER OF DIFFERENT TYPES OF CONVEXITY

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Abstract. Some inequalities for different types of convexity are established.

1. Introduction

In this section, some definitions of different types of convexity will be reminded.

In [1], Hudzik and Maligranda considered among others the class of functions which are s-convex in the second sense.

**Definition 1.** A function $f : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ = [0, \infty)$, is said to be s-convex in the second sense if

$$f(\alpha u + \beta v) \leq \alpha^s f(u) + \beta^s f(v)$$

for all $u, v \in [0, \infty)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $s$ fixed in $(0, 1]$. This class of s-convex functions in the second sense is usually denoted by $K^2_s$.

s-convexity reduces the ordinary convexity of functions defined on $[0, \infty)$ for $s = 1$.

For some information about convexity and s-convexity it is possible to refer to [1]-[7].

**Definition 2.** [8] A function $f : I \to \mathbb{R}$ is said to be quasi-convex if for all $x, y \in I$ and all $\alpha \in [0, 1], f(\alpha x + (1 - \alpha)y) \leq \max(f(x), f(y))$.

Godunova and Levin introduced the following concept in the paper [9].

**Definition 3.** A function $f : I \to \mathbb{R}$ is said to be belong to the class $Q(I)$ if it is nonnegative and for all $x, y \in I$ and $\lambda \in (0, 1)$, satisfies the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y).$$

**Definition 4.** [10] A function $f : I \to \mathbb{R}$ is said to be belong to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $\lambda \in [0, 1)$, satisfies the following inequality

$$f(\lambda x + (1 - \lambda)y) \leq f(x) + f(y).$$

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The results are obtained via following lemma.

**Lemma 1.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function on \([a, b]\) such that \( f \in L[a, b] \) with \( a < b \). For some fixed \( p, q > 0 \), followig equality holds:

\[
\int_a^b (x - a)^p (b - x)^q f(x) f(a + b - x) \, dx = (b - a)^{p+q+1} \int_0^1 (1 - t)^p t^q f(ta + (1 - t)b) f((1 - t)a + tb) \, dt.
\]

**Proof.** Changing the variable \( x = ta + (1 - t)b \) and simple calculations proceed the required result. \( \square \)

**Theorem 1.** Let \( f : [a, b] \subset [0, \infty) \rightarrow \mathbb{R}^+ \) be a continuous function on \([a, b]\) such that \( f \in L[a, b] \) with \( \theta \leq a < b < \infty \). If \( f \) is \( s \)-convex in the second sense, for some fixed \( p, q > 0 \) and \( s \in (0, 1] \) the following inequality holds

\[
\begin{align*}
\int_a^b (x - a)^p (b - x)^q f(x) f(a + b - x) \, dx & \leq \frac{(b - a)^{p+q+1}}{2} \left\{ (f^2(a) + f^2(b)) \left[ \beta(p + 1, 2s + q + 1) + \beta(q + 1, 2s + p + 1) \right] \right. \\
& \quad + 4f(a)f(b)\beta(p + s + 1, q + s + 1) \right\}
\end{align*}
\]

where \( \beta(m, n) = \int_0^1 t^{m-1}(1 - t)^{n-1} \, dt \), \( m, n > 0 \) is the Euler Beta function.

**Proof.** Using the inequality \( cd \leq \frac{1}{2}[c^2 + d^2] \) \( c, d \in \mathbb{R}^+ \) in the right hand side in Lemma II we obtain

\[
\begin{align*}
\int_a^b (x - a)^p (b - x)^q f(x) f(a + b - x) \, dx & \leq \frac{(b - a)^{p+q+1}}{2} \int_0^1 (1 - t)^p t^q \left\{ [f(ta + (1 - t) b)]^2 + [f((1 - t)a + tb)]^2 \right\} dt.
\end{align*}
\]

Since \( f \) is \( s \)-convex in the second sense, we can write

\[
\begin{align*}
\int_a^b (x - a)^p (b - x)^q f(x) f(a + b - x) \, dx & \leq \frac{(b - a)^{p+q+1}}{2} \int_0^1 (1 - t)^p t^q \left\{ [t^s f(a) + (1 - t)^s f(b)]^2 + [(1 - t)^s f(a) + t^s f(b)]^2 \right\} dt \\
& = \frac{(b - a)^{p+q+1}}{2} \int_0^1 (1 - t)^p t^q \left\{ (f^2(a) + f^2(b)) \left[ t^{2s} + (1 - t)^{2s} \right] + 4t^s(1 - t)^s f(a) f(b) \right\} \\
& = \frac{(b - a)^{p+q+1}}{2} \left\{ (f^2(a) + f^2(b)) \left[ \int_0^1 (1 - t)^p t^q + 2s dt + \int_0^1 (1 - t)^{p+2s} t^q dt \right] \\
& + 4f(a)f(b) \int_0^1 t^{s+s}(1 - t)^{p+s} dt \right\}.
\end{align*}
\]

If we use the following equalities above we get the required result:

\[
\int_0^1 (1 - t)^p t^{q+2s} dt = \beta(p + 1, 2s + q + 1),
\]
\[
\int_0^1 (1-t)^{p+2s} t^q dt = \beta(q + 1, 2s + p + 1)
\]

and
\[
\int_0^1 t^{q+s} (1-t)^{p+s} dt = \beta(p + s + 1, q + s + 1).
\]

\[\square\]

**Corollary 1.** In Theorem 1, if we choose \( p = q \) following inequality holds:
\[
\int_a^b (x-a)^p (b-x)^p f(x) f(a + b - x) dx 
\leq \frac{(b - a)^{2p+1}}{2} \left\{ (f^2(a) + f^2(b)) \left[ \beta(p + 1, 2s + p + 1) + \beta(p + 1, 2s + p + 1) \right] + 4f(a)f(b)\beta(p + s + 1, p + s + 1) \right\}.
\]

**Corollary 2.** In Theorem 1, if \( f \) is symmetric function, \( f(x) = f(a + b - x) \), following inequality holds:
\[
\int_a^b (x-a)^p (b-x)^q f^2(x) dx 
\leq \frac{(b - a)^{p+q+1}}{2} \left\{ (f^2(a) + f^2(b)) \left[ \beta(p + 1, q + 3) + \beta(q + 1, p + 3) \right] + 4f(a)f(b)\beta(p + 2, q + 2) \right\}
\]

for convex functions.

**Corollary 3.** In Theorem 1, if we choose \( s = 1 \) following inequality holds
\[
\int_a^b (x-a)^p (b-x)^q f(x) f(a + b - x) dx 
\leq \frac{(b - a)^{p+q+1}}{2} \left\{ (f^2(a) + f^2(b)) \left[ \beta(p + 1, q + 3) + \beta(q + 1, p + 3) \right] + 2f(a)f(b)\beta(p + 1, p + 3) \right\}
\]

for convex functions.

**Corollary 4.** In Corollary 3, if we choose \( p = q \) and \( f(x) = f(a + b - x) \), we obtain the following inequalities respectively
\[
\int_a^b (x-a)^p (b-x)^p f(x) f(a + b - x) dx 
\leq (b - a)^{2p+1} \left\{ (f^2(a) + f^2(b)) \beta(p + 1, p + 3) + 2f(a)f(b)\beta(p + 1, p + 3) \right\}
\]

and
\[
\int_a^b (x-a)^p (b-x)^q f^2(x) dx 
\leq \frac{(b - a)^{p+q+1}}{2} \left\{ (f^2(a) + f^2(b)) \left[ \beta(p + 1, q + 3) + \beta(q + 1, p + 3) \right] + 4f(a)f(b)\beta(p + 2, q + 2) \right\}
\]

for convex functions.

Following results are about quasi-convex functions.
Theorem 2. Let \( f : [a, b] \subset [0, \infty) \to \mathbb{R}^+ \) be a continuous function on \([a, b]\) such that \( f \in L[a, b] \) with \( 0 \leq a < b < \infty \). If \( f \) is quasi-convex, for some fixed \( p, q > 0 \) the following inequality holds
\[
\int_a^b (x-a)^p (b-x)^q f(x)f(a+b-x)dx 
\leq (b-a)^{p+q+1} (\max \{ f(a), f(b) \})^2 \beta(p+1, q+1)
\]
where \( \beta \) is the Euler Beta function.

Proof. Using the inequality \( cd \leq \frac{1}{2}[c^2 + d^2] \) \( c, d \in \mathbb{R}^+ \) in the right hand side in Lemma 1 and quasi convexity of \( f \), we obtain
\[
\int_a^b (x-a)^p (b-x)^q f(x)f(a+b-x)dx 
\leq \frac{(b-a)^{p+q+1}}{2} \int_0^1 (1-t)^p t^q \left\{ |f((1-t)a + tb)|^{(1-t)b} + |f((1-t) \beta(p+1, q+1)\}ight\} dt.
\]
\[
\leq (b-a)^{p+q+1} (\max \{ f(a), f(b) \})^2 \int_0^1 (1-t)^p t^q dt.
\]
The proof is completed. \( \square \)

Corollary 5. In Theorem 2

- If \( f \) is increasing, the following inequality holds
  \[
  \int_a^b (x-a)^p (b-x)^q f(x)f(a+b-x)dx 
  \leq (b-a)^{p+q+1} f^2(b) \beta(p+1, q+1).
  \]

- If \( f \) is decreasing, the following inequality holds
  \[
  \int_a^b (x-a)^p (b-x)^q f(x)f(a+b-x)dx 
  \leq (b-a)^{p+q+1} f^2(a) \beta(p+1, q+1).
  \]

Following result is about \( P \)-convexity.

Theorem 3. Let \( f : [a, b] \subset [0, \infty) \to \mathbb{R} \) be a continuous function on \([a, b]\) such that \( f \in L[a, b] \) with \( 0 \leq a < b < \infty \). If \( f \) is \( P \)-convex, for some fixed \( p, q > 0 \) the following inequality holds
\[
\int_a^b (x-a)^p (b-x)^q f(x)f(a+b-x)dx 
\leq (b-a)^{p+q+1} (f(a) + f(b))^2 \beta(p+1, q+1)
\]
where \( \beta \) is the Euler Beta function.

Proof. Using the inequality \( cd \leq \frac{1}{2}[c^2 + d^2] \) \( c, d \in \mathbb{R}^+ \) in the right hand side in Lemma 1 \( P \)-convexity of \( f \) and the definition of \( \beta \) function, we get the desired result. \( \square \)

The last theorem is for \( Q(I) \) class functions.
**Theorem 4.** Let \( f : [a, b] \subset [0, \infty) \to \mathbb{R} \) be a continuous function on \([a, b]\) such that \( f \in L[a, b] \) with \( 0 \leq a < b < \infty \). If \( f \) belongs to \( Q(I) \) class, the following inequality holds

\[
\int_a^b (x-a)^p (b-x)^q f(x)f(a+b-x)dx \\
\leq \frac{(b-a)^{p+q+1}}{2} \big\{ \left[ f^2(a) + f^2(b) \right] (\beta(p+1, q-1) + \beta(p-1, q+1)) \\
+ 4f(a)f(b)\beta(p, q) \big\}
\]

for some fixed \( p, q > 1 \) and \( t \in (0, 1) \).

**Proof.** Using the inequality \( cd \leq \frac{1}{2} [c^2 + d^2] \) \( c, d \in \mathbb{R}^+ \) in the right hand side in Lemma 1, we obtain

\[
\int_a^b (x-a)^p (b-x)^q f(x)f(a+b-x)dx \\
\leq \frac{(b-a)^{p+q+1}}{2} \int_0^1 (1-t)^p t^q \left\{ \left[ f(ta+(1-t)b) \right]^2 + \left[ f((1-t)a+tb) \right]^2 \right\} dt.
\]

Since \( f \) belongs to \( Q(I) \), we can write

\[
\int_a^b (x-a)^p (b-x)^q f(x)f(a+b-x)dx \\
\leq \frac{(b-a)^{p+q+1}}{2} \int_0^1 (1-t)^p t^q \left\{ \left[ \frac{f(a)}{t} + \frac{f(b)}{1-t} \right]^2 + \left[ \frac{f(a)}{1-t} + \frac{f(b)}{t} \right]^2 \right\} dt
\]

\[
= \frac{(b-a)^{p+q+1}}{2} \left\{ \left[ f^2(a) + f^2(b) \right] \left( \int_0^1 (1-t)^p t^{q-2} dt + \int_0^1 (1-t)^{p-2} t^q dt \right) + 4f(a)f(b) \int_0^1 (1-t)^{p-1} t^{q-1} dt \right\}.
\]

If we use the following equalities above we get the required result:

\[
\int_0^1 (1-t)^p t^{q-2} dt = \beta(p+1, q-1),
\]

\[
\int_0^1 (1-t)^{p-2} t^q dt = \beta(p-1, q+1)
\]

and

\[
\int_0^1 (1-t)^{p-1} t^{q-1} dt = \beta(p, q).
\]

\( \square \)

**Corollary 6.** In Theorem 4 if \( f(a) = f(b) \) the following inequality holds

\[
\int_a^b (x-a)^p (b-x)^q f(x)f(a+b-x)dx \\
\leq \frac{(b-a)^{p+q+1}}{2} \left\{ \beta(p+1, q-1) + 2\beta(p, q) + \beta(p-1, q+1) \right\}
\]

where \( p, q > 1 \) and \( \beta \) is Euler beta function.

**Remark 1.** One may get some results for other types of convexity via Lemma 1.
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