A TQFT associated to the LMO invariant of three-dimensional manifolds∗†‡

Dorin Cheptea and Thang T Q Le

Abstract

We construct a Topological Quantum Field Theory (in the sense of Atiyah [1]) associated to the universal finite-type invariant of 3-dimensional manifolds, as a functor from the category of 3-dimensional manifolds with parametrized boundary, satisfying some additional conditions, to an algebraic-combinatorial category. It is built together with its truncations with respect to a natural grading, and we prove that these TQFTs are non-degenerate and anomaly-free. The TQFT(s) induce(s) a (series of) representation(s) of a subgroup $\mathcal{L}_g$ of the Mapping Class Group that contains the Torelli group. The $N = 1$ truncation produces a TQFT for the Casson-Walker-Lescop invariant.

In [18] the Kontsevich integral $Z(L)$, i.e. the universal finite-type invariant for links, has been extended to an invariant $Z^{LMO}(M)$ of 3-dimensional manifolds. The later is universal for rational homology 3-spheres [15]. The task of putting $Z^{LMO}$ in the structural framework of TQFT was partially accomplished in [22], however that construction uses a twisted gluing of cobordisms and the resulting anomaly is complicated. The construction is however important to establish, since TQFT naturally connects a manifold invariant (in this case LMO) to the Mapping Class Group. It also aims to shed some light to the question of topological interpretation of quantum invariants of manifolds.

Our construction allows us to associate to the LMO invariant an infinite-dimensional linear representation of the Torelli group, in fact of a larger Lagrangian subgroup of the Mapping Class Group. The new results of this paper are:

- proving an isomorphism (Proposition 2.3) reducing the study of the LMO invariant of 3-dimensional manifolds with parametrized boundary to that of finite-type invariants of string-links
- the construction of the composition of chord diagrams from truncations (Theorem 2.12) and establishing an important limit property (Lemma 3.5)
- proving the non-degeneracy of the TQFT (Theorem 3.2), which means it is possible to calculate the induced representation in purely combinatorial terms
- a combinatorial description of the map $Z(M) \mapsto Z(\hat{M})$ (Proposition 4.3), where $M$ denotes the cobordism, and $\hat{M}$ the closed 3-manifold obtained by "caping" its boundary

The natural truncation induces a TQFT for the Walker-Lescop extension of the Casson invariant, and we can identify Morita’s representation as its first non-trivial part.

The essential difference between this TQFT and the Reshetikhin-Turaev TQFT for quantum invariants is that the present is tailored for integer and rational homology spheres, because $Z^{LMO}$ is strong for them, and weaker if the rank of homology is bigger. Hence we consider connected cobordisms between connected surfaces. When gluing, we discriminate between the domain and the range of a cobordism. In particular, while we regard the standard surface $\Sigma_g$ of genus $g$ in the domain as the boundary of the standard handlebody $N_g$, we regard $\Sigma_g$ in the range as the boundary of the complement of $N_g$ in $S^3$. Thus, gluing identically in our TQFT produces

∗ 2000 Mathematics Subject Classification: 57M27.
† Key words: 3-dimensional manifolds, homology spheres, knots, Kontsevich integral, LMO invariant, quantum invariants, TQFT, Torelli group, Casson invariant
‡ The results of this article were obtained when the authors were at the Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260-2900, USA
$S^3$ as opposed to $\#_g(S^2 \times S^1)$ in the Reshetikhin-Turaev TQFT. It remains, however, to interpret our TQFT as a "perturbative expansion around 0" of the Reshetikhin-Turaev TQFT \cite{ReshetikhinTuraev1991}.

This paper is organized as follows. In Section 1 we recall the topological categories $\mathcal{O}, \mathcal{Z}$ introduced in \cite{ChepteaLe2008}, and the pertaining results that we will need subsequently in this paper. The categories $\mathcal{A}$ and $\mathcal{A}^{\le N}$ of chord diagrams are explained in Section 2. We use a simpler definition of $Z$ on elementary pseudo-quasi-tangles\footnote{"pseudo" stands for the presence of 3-valent vertices}, and an even associator, while the one in \cite{ChepteaLe2008} is based on the Klužník-Zamoždchikov associator. Important results here are Proposition 2.3 and Theorem 2.12. We also recall for comparison the multiplication \footnote{"pseudo" stands for the presence of 3-valent vertices} of graphs $\Gamma$ let us consider:

\[ \bullet \text{ the chain graph}\footnote{this terminology we borrow from \cite{ChepteaLe2008}} \text{ suggestively denoted } \begin{tikzpicture} [baseline=-.5ex,scale=0.75] \draw (0,0) circle (0.25) node {\tiny \textbf{1}}; \draw (0,0) -- (1,1) node [midway,above] {\tiny \textbf{1}}; \draw (1,1) -- (2,0) node [midway,below] {\tiny \textbf{1}}; \draw (2,0) -- (1,-1) node [midway,below] {\tiny \textbf{1}}; \draw (1,-1) -- (0,0) node [midway,above] {\tiny \textbf{1}}; \end{tikzpicture} \text{ times } g. \]

\[ \bullet \text{ the oriented manifold which is the union of } \gamma \in \mathbb{N}^* \text{ copies of } [0,1], \text{ each copy labeled (colored) by a distinct element of a finite abstract ordered set } X \text{ of asterisks. This special graph will be denoted } \begin{tikzpicture} [baseline=-.5ex,scale=0.75] \draw (0,0) -- (0,1) node [midway,above] {\tiny \textbf{1}}; \draw (0,1) -- (1,1) node [midway,above] {\tiny \textbf{1}}; \draw (1,1) -- (1,0) node [midway,below] {\tiny \textbf{1}}; \draw (1,0) -- (0,0) node [midway,below] {\tiny \textbf{1}}; \end{tikzpicture} \text{ times } g. \]

\[ \bullet \text{ it is convenient to set } \Gamma^0 = \text{ one point } \text{ as a chain graph. Chord diagrams on } \Gamma^0 \text{ automatically can have only internal vertices.} \]
1 The topological category

1.1 Definitions (see also [9]). Two triplets \((K, G_1, G_2)\) and \((L, H_1, H_2)\), consisting of a framed oriented link \(K\) (respectively \(L\)) in \(S^3\), and two disjoint (and disjoint from the corresponding link) embedded framed chain graphs are equivalent (notation \(\cong\)) if there is a PL-homeomorphism \(\phi : S^3 \rightarrow S^3\) which preserves the link and the embedded framed graphs, i.e. \(\phi\) sends \(K\) to \(L\), the first embedded framed graph \(G_1 = (\Gamma_1, R_1)\) to the first embedded framed graph \(H_1 = (\Delta_1, S_1)\), and the second \(G_2 = (\Gamma_2, R_2)\) to the second one \(H_2 = (\Delta_2, S_2)\). Here \(\emptyset\) is also considered a framed oriented link in \(S^3\). Call \((\Gamma_1, R_1)\) the bottom, and \((\Gamma_2, R_2)\) the top of the triplet.

Let \(M\) be a compact oriented 3-manifold with boundary \(\partial M = (-S_1) \cup S_2\), suppose that parametrizations \(f_i : \Sigma_{g_i} \rightarrow S_i\) are fixed. Such \((M, f_1, f_2)\) is called a (parametrized) \((2+1)\)-cobordism. \(S_1\) is the bottom, \(S_2\) - the top of the cobordism. The cobordisms \((M, f_1, f_2)\) and \((N, h_1, h_2)\) are equivalent (homeomorphic) if there is a PL-homeomorphism \(F : M \rightarrow N\) sending bottom to bottom and top to top preserving the parametrizations, i.e. \(F \circ f_i = h_i, i = 1, 2\).

Fix \(N_g\) - a standard neighbourhood of \(\Gamma^g\) in \(S^3\). \(\Sigma_g = \partial N_g \subset S^3\) is the standard oriented surface of genus \(g\). Let \(\overline{N_g}\) be the handlebody complement of \(N_g\) in \(S^3\). We also denote by \(\overline{\Gamma^g}\) the core of \(\overline{N_g}\). Clearly \(\partial \overline{N_g} = -\Sigma_g\). When \(g = 0\) we assume that \(\Gamma^g\) is a point, and \(N_g\) is a ball. Using the parametrizations, we can glue the standard handlebody \(N_{g_1}\) to the bottom and the standard anti-handlebody \(\overline{N_{g_2}}\) to the top of \(M\). The result \(\overline{M} := M \cup_{f_1} N_{g_1} \cup_{-f_2} (\overline{N_{g_2}})\) is called the filling of \((M, f_1, f_2)\).

1.2 Surgery description of gluing cobordisms. Let \(\mathfrak{G}\) denote set of equivalence classes of triplets \((L, G_1, G_2)\) in \(S^3\). Let \(\mathfrak{C}\) denote the set of equivalence classes of \(3\)-cobordisms, with connected non-empty bottom and top.

Proposition. 1) The map \(\kappa : \mathfrak{G} \rightarrow \mathfrak{C}\) that associates to every equivalence class of triplets \((L, G_1, G_2)\) the equivalence class of cobordisms \((M, f_1, f_2)\), obtained by doing surgery on \(L \subset S^3\), removing tubular neighbourhoods \(N_1, N_2\) of each \(G_1, G_2\), and recording the parametrizations of the two obtained boundary components, is well-defined and surjective. If one glues according to these parametrizations a standard handlebody to \(-\partial N_1\) and a standard anti-handlebody to \(-\partial N_2\), then one obtains \(S^4\).

2) Let a first Kirby move on a triplet be the cancellation / insertion of a \(0^\pm 1\) separated by an \(S^2\) from anything else, and an extended (generalized) second Kirby move be a slide over a link component of an arc, either from another link component or from a chain graph. Then if one factors \(\mathfrak{G}\) by the extended Kirby moves and changes of orientations of link components, the induced map \(\pi\) is a bijection. \(\square\)

For example, to represent the \(3\)-cobordism \((\Sigma_g \times [0, 1], p_1, p_2)\), where \(p_i : \Sigma_g \rightarrow \Sigma_g \times I \subset S^3\) are two copies of the standard embedding of \(\Sigma_g\) in \(S^3\), we can choose the framed graphs \(R_1, R_2\) as in Figure 1. Let \(\Gamma\), respectively \(\Gamma'\) generically denote the bottom, respectively the top of a triplet. Call the union of the lower half-circles and the horizontal segments of \(\Gamma\), the horizontal line of \(\Gamma\). Similarly, call the union of the upper half-circles and the horizontal segments of \(\Gamma'\), the horizontal line of \(\Gamma'\). See Figure 2a.

![Figure 1: The preferred choice of ribbons R_i, i = 1, 2 for (Sigma_g x I, p_1, p_2).](image)

1.3 Proposition. Let \((M_1, f_1, f'_1)\) and \((M_2, f_2, f'_2)\) be two \(3\)-cobordisms with connected non-empty bottoms and tops, represented by triplets \((L_1, G_1, G'_1)\) and \((L_2, G_2, G'_2)\). Remove a 3-ball neighbourhood of the horizontal
1.5 Proposition. Suppose $M$ is a connected compact oriented 3-manifolds with two distinguished (not necessarily connected) boundary components $\partial M = (-S_1) \cup S_2$, let $f_1$, $f_2$ be parametrizations of these surfaces, let $N_1$ and $N_2$ be corresponding-to-the-genera disjoint unions of standard handlebodies, respectively anti-handlebodies, and let $i = (i_1, i_2) : \partial M \hookrightarrow M$ be the inclusion. The following conditions are equivalent:

1. $H_1(M, M) = 0$
2. $H_i(M, Z) = i_*(H_1(\partial M, Z)/(f_1, -f_2)_* H_1(N_1 \cup -N_2, Z))$

They imply:

3. $2 \cdot \text{rank } H_1(M; Z) = \text{rank } H_1(\partial M; Z)$

1.5 Proposition. Suppose $M$ is a connected compact oriented 3-manifolds with two distinguished (not necessarily connected) boundary components $\partial M = (-S_1) \cup S_2$, let $f_1$, $f_2$ be parametrizations of these surfaces, let $N_1$ and $N_2$ be corresponding-to-the-genera disjoint unions of standard handlebodies, respectively anti-handlebodies, and let $i = (i_1, i_2) : \partial M \hookrightarrow M$ be the inclusion. The following conditions are equivalent:

1. $H_1(M, M) = 0$
2. $H_i(M, Q) = i_*(H_1(\partial M, Q)/(f_1, -f_2)_* H_1(N_1 \cup -N_2, Q))$

They imply:

3. $2 \cdot \text{rank } H_1(M; Q) = \text{rank } H_1(\partial M; Q)$
1.6 Description of the categories $\Omega \supset 3$. Objects in each of these are natural numbers. The morphisms between $g_1$ and $g_2$ are equivalence (homeomorphism) classes of connected 3-cobordisms with bottom $S_1$ of genus $g_1$ and top $S_2$ of genus $g_2$, satisfying the $F$-semi-Lagrangian conditions:

$$f_1 \ast L^a \supseteq f_2 \ast L^b$$

(1.2)

where $L^a = \ker (\text{incl}_*: H_1(\Sigma_g, F) \to H_1(N, F))$, and $L^b = \ker (\text{incl}_*: H_1(\Sigma_g, F) \to H_1(\overline{N}, F))$, and $F = \mathbb{Z}$ or $\mathbb{Q}$. The composition-morphism of two cobordisms $(M_1, f_1)$ and $(M_2, f_2, f'_2)$ is the equivalence class of the 3-cobordism $(M_2 \cup_{f_2=0} f'_2 \cdot M_1, f_1, f'_2)$.

In general condition 1.2 over $\mathbb{Z}$ is stronger than 1.2 over $\mathbb{Q}$. It also may hold with strict inclusion. 6

1.7 Proposition 6. The composition of two morphisms (say, class of $M$ and class of $N$) in category $\Omega$ (respectively 3) is again a morphism in the category $\Omega$ (respectively 3). □

Let us restrict to 3-cobordisms $M$ of the form $(\Sigma_g \times [0, 1], f \times 0, f' \times 1)$, with $f, f' \in \text{Aut}(\Sigma_g)$, i.e. the parametrization of the top differs by that of the bottom by the automorphism $w = (f')^{-1} \circ f$. The equivalence classes of this cobordism depends only on the isotopy class of $w$ (i.e. we don’t need to specify both $f, f'$). The equivalence class of $M = (\Sigma_g \times [0, 1], f \times 0, f' \times 1)$ is a $\mathbb{Z}$-semi-Lagrangian cobordism iff it is a $\mathbb{Q}$-semi-Lagrangian cobordism if it satisfies $L^a = w_*(L^b)$ and $L^b = w_*(L^a)$, i.e. is a Lagrangian cobordism. Then $\hat{M}$ is always a $\mathbb{Z}$-homology sphere. 8

The composition of two cobordisms $(\Sigma_g \times I, f_1 \times 0, f'_1 \times 1) \cong (\Sigma_g \times I, w_1 \times 0, id \times 1)$ and $(\Sigma_g \times I, f_2 \times 0, f'_2 \times 1) \cong (\Sigma_g \times I, w_2 \times 0, id \times 1)$ along $(f_2 \times 0) \circ ((f'_1)^{-1} \times 1)$ (respectively $(w_2 \times 0) \circ (id \times 1)$) is the 3-cobordism $(\Sigma_g \times I, f_2 \circ (f'_1)^{-1} \circ f'_1 \times 0, f'_2 \times 1(\Sigma_g \times I, (f'_2)^{-1} \circ f_2\circ (f'_1)^{-1} \circ f'_1 \times 0, id \times 1) \cong (\Sigma_g \times I, (w_2 \circ w_1) \times 0, id \times 1)$. In particular, the composition of two morphisms of category $\Sigma$ is again a morphism in the same category.

1.8 Definition 6. Denote by $L_g$ the subgroup of the Mapping Class Group, consisting of isotopy classes of elements $w \in \text{Aut}(\Sigma_g)$ such that $w_*(L^a) = L^a$ and $w_*(L^b) = L^b$ (over $\mathbb{Q}$ or over $\mathbb{Z}$, is equivalent by the above), and call it the Lagrangian subgroup of the MCG.

The TQFT of the LMO invariant induces a representation of $L_g$. This subgroup of $MCG(g)$ is big enough to be interesting, it contains the Torelli group. Its image under the action on homology is the group (not normal as subgroup of $Sp(2g, \mathbb{Z})$) of matrices of the form $\begin{pmatrix} A & 0
0 & (A^T)^{-1} \end{pmatrix}$, where $A \in GL(g, \mathbb{Z})$.

Remark. Let $\lambda$ denote the Casson invariant of homology 3-spheres. By fixing the standard handlebody of genus $g$ in $\mathbb{R}^3 \subset S^3$ we fixed a Heegaard homeomorphism that Morita 21 calls $\tau_g$, and by taking the filling $(\Sigma_g \times I, \varphi, id)$ we obtain a manifold denoted by Morita $W_\varphi$.

2 The algebraic-combinatorial category

In 22, an essential part of constructing a TQFT associated to $Z^{LMO}$ has been completed. Namely, first the Kontsevich integral was extended to an invariant $Z(G)$ of oriented framed trivalent graphs $G$ in $S^3$ (see 22 theorem 1.4)). A framed graph $G \subset S^3$ is represented as a plane projection (with implicit blackboard framing), then decomposed into elementary pseudo-quasi-tangles, and $Z$ is defined for each piece (see 22 figure 2 for the exact definition of $Z$). It is easy to observe that in order to verify the independence of $Z$ of the decomposition into pseudo-quasi-tangles and the invariance under extended (generalized) Reidemeister moves for trivalent graphs, one is forced to introduce relations that “move” (in the sense of Proposition 2.2) a box-diagram over a trivalent vertex to a box-diagram. This implies the branching relations (figure 5 here, figure 1 in 22) These relations are necessary to impose regardless of the definition of $Z$ for the neighbourhood of a trivalent vertex.

From the extended $Z$ Murakami and Ohtsuki 22 derived an invariant of oriented 3-manifolds with boundary, along the same lines the $Z^{LMO}$ is constructed 18 from the Kontsevich integral of framed links. But in 22 a twisted gluing is used for composing cobordisms, and the resulting TQFT has complicated anomaly.

2.1 The modules of chord diagrams. Let $\Gamma$ be a graph, we will be mainly interested in the cases $\Gamma = 1$-manifold and $\Gamma = a$ chain graph. Let $\mathcal{A}(\Gamma)$ be the formal series completion with respect to the degree of the $\mathbb{Q}$-vector space freely generated by the set of homeomorphism classes of chord diagrams with support $\Gamma$, without
self-loops and univalent vertices, modulo AS, IHX, STU and branching relations (which are homogeneous with respect to the degree).\footnote{There are essentially two conventions in defining STU and AS relations, and drawing certain elements of $A(\Gamma)$, as shown in figure 24. Note that in convention 1, which is the one that we use (as well as 22, 23), AS relations refer only to internal vertices, and no cyclic order of edges adjacent to external trivalent vertices is defined. In this convention, as a consequence, the LHS of STU-left is equal to minus the RHS of STU-left. Using the second convention, the definition of a chord diagram has to be changed as to account for the cyclic order of edges adjacent to external trivalent vertices. The two $A(\Gamma)$, from the two conventions, are canonically isomorphic; in fact only the meaning of some diagrams as elements of $A(\Gamma)$ is changed by adding a $-$ sign.}

We will use the following box-diagram notation for the formal sum of chord diagrams, as shown in figure 2.\footnote{For convention 2 all coefficients $c_i = 1$. Then the box-diagrams in the two conventions correspond precisely one to the other via the canonical isomorphism between the conventions.}

There outside the drawn part the diagrams are identical, the vertical edge is dashed, the horizontal edges are arbitrary. If the horizontal edge $i$ is dashed, then $c_i = 1$, if it is bold, then $c_i$ is as shown in figure 2.\footnote{One can check (e.g. by induction on the number of internal vertices of chord diagrams) that this comultiplication is well-defined (remember the presence of STU relations).}

The branching relations, introduced in $\Box$ figure 1] are shown in figure 5 using this box-notation.

Similarly, let $A(\emptyset)$ denote the formal series completion with respect to the degree of the $Q$-module freely generated by the set of homeomorphism classes of open chord diagrams without self-loops and univalent vertices, modulo AS and IHX relations. For a chord diagram $D$, denote $[D]$ the corresponding element of $A(\Gamma)$. $A(\Gamma)$ and $A(\emptyset)$ are co-algebras with respect to the decomposition of the dashed part of a diagram in connected components (the elements represented by diagrams that have non-empty connected dashed part are defined to be primitive)\footnote{One can reformulate this statement: Every IHX (respectively AS) relation on-the-left-of-the-trivalent-vertex is a consequence of branching relations and IHX (respectively AS) relations on-the-right-of-the-trivalent-vertex.}

$A(\emptyset)$ is an algebra with respect to disjoint union, and together with (completed) comultiplication $\Delta$ forms a Hopf algebra. Note that $A(\Gamma)$ is an $A(\emptyset)$-module with respect to the disjoint union.

2.2 Proposition. (a) The box-STU and box-AS relations, schematically shown in figure 4 hold in $A(\Gamma)$.

(b) The three relations in figure 7 hold in $A(\Gamma)$.

(c) The box-STU and box-AS relations can be “moved” over any trivalent vertex of $\Gamma$, using only branching relations (see figure 2b for an example).\footnote{The reason we choose $\sum_i$ instead of $\prod_i$ is Lemma 3.5. See the remark after it.}

Proof. (a) Let $x_i$ denote the horizontal edges. Let $[D^1]$, $[D^H]$, $[D^X]$ denote the three terms of the box-STU relation. Note that the brackets are also part of the notation. $D^V$ means the box-diagram, which is not a chord diagram. Let $[D^V_{x_i}]$ denote the element of $A(\Gamma)$ corresponding to the chord diagram obtained from $D^V$ by replacing the box with a prolongation of the vertical edge until the the edge $x_i$. With similar notations $[D^H_{x_i}]$ and $[D^X_{x_i}]$, note that for $i \neq j$, $[D^H_{x_i,x_j}] = [D^X_{x_i,x_j}]$. Hence:

\[
RHS = \sum_{x_i} \sum_{x_j} c_i c_j [D^H_{x_i,x_j}] - \sum_{x_i} \sum_{x_j} c_j c_i [D^X_{x_i,x_j}] = \sum_{i=j} c_i^2 [D^H_{x_i,x_i}] + \sum_{i \neq j} c_i c_j [D^H_{x_i,x_j}] - \sum_{i \neq j} c_j c_i [D^X_{x_i,x_j}]
\]

where the equality before the last we have used an IHX, STU, or convention-1 form of STU-left for each $x_i$. The proof of the box-AS relation is elementary, using AS relations and the definition of coefficients $c_i$.

(b) Consider all dashed/bold possibilities for the edges. The relations then follow from the AS, IHX, STU and branching relations.

(c) Every box-diagram is a sum of box-diagrams with small boxes. For the later follow the calculation shown in figure 5 for the box-STU case. The box-AS case is obvious. \fbox{\sffamily{\small $\Box$}}
Convention 1:  

\[
\begin{align*}
\text{STU} & = \begin{array}{c}
\text{STU-left} \\
\text{STU-right}
\end{array} \\
\text{and} & =
\end{align*}
\]

(Consequence: LHS (STU-left) = - RHS (STU-left))

Convention 2:  

\[
\begin{align*}
\text{STU-left} & = \begin{array}{c}
\text{STU-right}
\end{array} \\
\text{and} & =
\end{align*}
\]

Figure 3: The two conventions for chord diagrams

\[
\begin{align*}
a: \quad & \sum_i \begin{array}{c}
\text{i}
\end{array} = c_1 + c_2 + \ldots + c_n \\
b: \quad & i \begin{array}{c}
\text{i}
\end{array} = -
\end{align*}
\]

Figure 4: The box-diagram

Figure 5: The 8 branching relations (all but the vertical edge are bold): one for each possible orientations of the 3 bold edges

Figure 6: a: The box-STU relation (each term in the RHS contains a double sum over the horizontal edges), b: The box-AS relation
Figure 7: Invariance over “elementary pseudo-tangles” (the dashed/bold type of horizontal edges is arbitrary, the vertical edge is dashed)

Figure 8: “Moving” a box-STU relation over a trivalent vertex

Figure 9: Relation 9: the ambiguity of “moving” a dashed end off the horizontal line.
\[ \prod_{i \in \mathbb{N}} D^i / \prod_{i \in \mathbb{N}} \mathcal{R}^i = D/\mathcal{R}, \text{ i.e. factoring and taking completion commute. We will not use anywhere below the next proposition that } A \cong D/\mathcal{R}, \text{ our object is always } A. \]

### 2.3 Proposition

Denote \([g] \not\in \{1, \ldots, g\}. \) Let \( \phi : \{g\} \to \bigcirc \cdots \bigcirc \) be the embedding of \([g]\) onto the upper half-circles of \( \bigcirc \cdots \bigcirc \), sending the arrow labeled \( i \) to the \( i^{th} \) upper half-circle of \( \Gamma^g \), preserving orientation. Then it extends to an isomorphism of \( \mathbb{Q} \)-vector spaces \( \tilde{\alpha} : A(\{g\}) \to A(\Gamma^g) \).

**Proof.** Fix an arbitrary degree \( i \) of chord diagrams. Then \( \phi \) induces a homomorphism of vector spaces \( \phi_* : D^i(\{g\}) \to D^i(\Gamma^g) \), under which \( R(\{g\}) \) is sent exactly to the set of AS, IHX and STU relations in \( \Gamma^g \) that involve only diagrams with support in \( \phi(\{g\}) \). For simplicity of notation, let us denote \( \phi_* D^i(\{g\}) \) by \( D^i(\{g\}) \), and \( \phi_* R(\{g\}) \) by \( R(\{g\}) \).

Replace each external trivalent vertex in \( \Gamma^g - \phi(\{g\}) \) of a chord diagram by a small box (and add a sign to it, the coefficient \( c_i \)), then “move”, using the branching relations, one by one all boxes off \( \Gamma^g - \phi(\{g\}) \). This assigns to an arbitrary chord diagram with support in \( \Gamma^g \) a diagram with boxes (with a \( \pm \) sign) with support in \( \phi(\{g\}) \). It depends of the choice of the sequence of trivalent vertices over which boxes are “moved” in \( \Gamma^g \). Observe, however, that different such choices result in diagrams with boxes, representing elements of \( D^i(\{g\}) \) that differ one from the other by a sum (with coefficients \( \pm 1 \)) of relations depicted in figure 2. Let us call them Relations 9 as reference to figure 9 in [22]. By linearity, this defines a homomorphism of \( \mathbb{Q} \)-vector spaces \( \alpha : D^i(\Gamma^g) \to D^i(\{g\}) \), which when restricted to \( D^i(\{g\}) \to \mathcal{D}^i(\{g\}) \) is the canonical quotient map. Here \( \mathcal{D} \) is the \( \mathbb{Q} \)-vector subspace of \( D^i(\{g\}) \) generated by the set of Relations 9.

Proposition 2.2(b) implies that Relations 9 are true in \( A^i(\{g\}), \) i.e. \( \mathcal{D}^i \subset \mathcal{R}^i(\{g\}) \). Let \( \beta : \mathcal{D}^i(\{g\})/\mathcal{D}^i(\{g\}) = \mathcal{D}^i(\{g\}) \) be the canonical projection. Let us observe that every branching relation \( R, \alpha(R) = 0. \) Therefore \( (\beta \circ \alpha)(R) = 0, \) so if we denote by \( \mathcal{S}^i \) the \( \mathbb{Q} \)-vector subspace of \( D^i(\Gamma^g) \) generated by the set of branching relations, then \( \mathcal{S}^i \subset \mathcal{D}^i \subset \mathcal{R}^i(\{g\}) \).

On the other hand, any IHX, AS, STU or branching relation on \( \Gamma^g \) is, by Proposition 2.2, a sum of IHX, AS and STU relations on \( \phi(\{g\}) \), plus a sum of branching relations. Indeed, an IHX or AS relation refers only to a neighbourhood outside \( \Gamma^g - \phi(\{g\}) \), hence the “moving” procedure can be applied simultaneously to all terms of the relation; while a STU relation is, up to sign, a box-STU relation, therefore using Proposition 2.2(c) can be “moved” to a box-STU relation with support in \( \Gamma^g - \phi(\{g\}) \), the later being a consequence of \( \mathcal{R}^i(\{g\}) \) by Proposition 2.2(a). The difference between the start and the end of each step of a “moving” procedure is, of cause, an element of \( \mathcal{S}^i \). Hence \( \mathcal{R}^i(\Gamma^g) = \mathcal{R}^i(\{g\}) + \mathcal{S}^i \).

The two established relations imply \( \mathcal{R}^i(\Gamma^g) \cap D^i(\{g\}) \subset \mathcal{R}^i(\{g\}) \). Since the opposite inclusion is obvious, \( \mathcal{R}^i(\Gamma^g) \cap D^i(\{g\}) = \mathcal{R}^i(\{g\}) \). Then, by the second isomorphism theorem for vector spaces, \( \mathcal{D}^i(\{g\})/\mathcal{R}^i(\{g\}) \cong \mathcal{D}^i(\Gamma^g) \). Composing with \( \phi_* \) from the first paragraph, we obtain \( \phi_* : A^i(\{g\}) \to A^i(\Gamma^g) \), for every \( i \geq 0 \). Moreover the induced \( \tilde{\alpha} : A(\{g\}) \to A(\Gamma^g) \) preserves the topology.

**Remark.** This Proposition still holds if \( \Gamma \) has two or more connected components, but we can ”eliminate” the horizontal line of only one component. If we ”eliminate” more than one horizontal line, the corresponding \( \phi_* \) is still well-defined and surjective.

### 2.4 The algebra structure of \( A(\{g\}) \) and the set \( \mathcal{C}_g. \)

Let \( A_c(\{g\}) \) be the \( \mathbb{Q} \)-vector subspace of \( A(\{g\}) \) generated by formal series of diagrams on \( \{g\} \) with no components of the dashed graph disconnected from the support. Viewing each chord diagram as a union of the connected components of the dashed graph that do not meet the support with the part that meets the support, we get \( A_c(\{g\}) = A(\emptyset) \otimes C_\ast \). Let a(\{g\}) be the \( \mathbb{Q} \)-vector subspace of \( A_c(\{g\}) \) generated by formal series of diagrams on \( \{g\} \) with non-empty and connected dashed graph (and connected to the support). a(\{g\}) is precisely the set of primitive elements of \( A_c(\{g\}) \). A similar notation a(\Gamma) for any abstract graph \( \Gamma \) is self-evident. \( A_c(\{g\}) \) is an algebra with respect to juxtaposition of the bold vertical arrows. Denote this associative, generally (if \( g > 1 \)) non-commutative operation \( \bullet \). In fact \( A_c(\{g\}) \) is a co-commutative Hopf algebra [25, Proposition 1.5]. The following is apparently ”common knowledge”:

**Proposition** 1) a(\{g\}) is a Lie algebra over \( \mathbb{Q} \) with respect to the operation \( (x, y) \mapsto x \bullet y - y \bullet x \).
2) Let \( \hat{I} \) be the topological ideal of \( \mathcal{A}_c(\{I[g]\}) \) generated by \( \mathfrak{a}(\{I[g]\}) \). Then \( \exp : \hat{I} \to 1 + \hat{I} \) and \( \log : 1 + \hat{I} \to \hat{I} \), defined by \( \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) and \( \log(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \), where the product is the operation \( \bullet \), satisfy \( \exp \circ \log = \text{id}_{1+\hat{I}} \) and \( \log \circ \exp = \text{id}_{\hat{I}} \). In particular, \( \exp \) and \( \log \) are bijections.

3) \( \exp \) is a bijection from \( \mathfrak{a}(\{I[g]\}) \subset \hat{I} \) to the set of group-like elements in \( 1 + \hat{I} \).

4) If \( \alpha, \beta \in \mathfrak{a}(\{I[g]\}) \), then \( \exp(\alpha) \bullet \exp(\beta) = \exp(\gamma) \) for some \( \gamma \in \mathfrak{a}(\{I[g]\}) \). Moreover, \( \gamma \) is given by the Campbell-Hausdorff formula.

5) \( \hat{I} \) coincides with the set of formal series of chord diagrams of degree \( \geq 1 \).

**Proof.** 1) The statement is sufficient to prove for \( x, y = \) diagrams with connected dashed graph. Using STU relations, as shown in Figure 10, we can interchange two consecutive external vertices, one from \( x \), the other from \( y \), on any bold arrow, up to \( \pm \) a diagram with connected dashed graph. Therefore iteratively we can interchange all external vertices of \( x \) with all external vertices of \( y \), obtaining \( x \bullet y - y \bullet x = \) a sum of diagrams with connected dashed graph.

2), 3) and 4) are classical statements. The proofs in [23, Theorem 7.2, Corollary 7.3 and Theorem 7.4] apply môt-a-môt. For 3) and 4) note that if \( \gamma \) is primitive, then \( \gamma \in \mathfrak{a}(\{I[g]\}) \).

5) Since the set of formal series of chord diagrams of degree \( \geq 1 \) is an ideal containing \( \mathfrak{a}(\{I[g]\}) \), and is closed topologically, \( \hat{I} \) certainly belongs to it. Conversely, pick an arbitrary connected component \( y' \) of the dashed graph of a chord diagram. Observe that using the “trick” in Figure 10, up to \( \pm \) a diagram with the number of connected components of the dashed graph less by 1, \( y' \) can be assumed to have all external vertices below all the other external vertices of the diagram. Hence an induction on the number of connected components of the dashed graph shows that any chord diagram of degree \( \geq 1 \) is a sum of terms of type \( \pm z_1 \bullet z_2 \bullet \ldots \bullet z_k \), \( k \geq 1 \), with \( z_j \) a diagram in \( \mathfrak{a}(\{I[g]\}) \). We conclude that the set of finite sums of chord diagrams of degree \( \geq 1 \) is contained in \( \hat{I} \). Hence so is its completion.

\[ \includegraphics[width=0.8\textwidth]{Figure10.png} \]

Figure 10: Two consecutive external vertices from connected components \( x' \) and \( y' \) can be interchanged up to \( \pm \) a diagram with dashed graph having one component less.

This Proposition holds if we replace \( \mathcal{A}_c(\{I[g]\}) \) by \( \mathcal{A}(\{I[g]\}) \) and \( \mathbb{Q} \) by \( \mathcal{A}(\emptyset) \). It suggests us to consider an operation on \( \mathcal{C}_\emptyset \), the set of connected 3-cobordisms with empty bottom and connected top, to correspond to the multiplication in \( \mathcal{A}(\{I[g]\}) \). Let \( (M_1, \emptyset, f_1), (M_2, \emptyset, f_2) \in \mathcal{C}_\emptyset \), and let \( (L_1, G_1) \subset S^3 \), \( (L_2, G_2) \subset S^3 \) such that \( \kappa(L_1, G_1) = (M_1, \emptyset, f_1) \), \( \kappa(L_2, G_2) = (M_2, \emptyset, f_2) \). Remove a tubular neighbourhood of the horizontal line of \( G_1 \) (= a ball in \( S^3 \)), and similarly for \( G_2 \). Up to isotopy we can assume each looks as the box in Figure 11. Glue the two boxes from left to right, and fill back in the standard way a horizontal line. Denote the result by \( (L_1 \cup L_2, G_1 \bullet G_2) \), and define:

\[ (M_1, \emptyset, f_1) \bullet (M_2, \emptyset, f_2) = \kappa(L_1 \cup L_2, G_1 \bullet G_2) \]  \hspace{1cm} (2.1)

Observe that the new 3-cobordism does not depend on the choice of pairs \( (L_i, G_i) \), such that \( \kappa(L_i, G_i) = (M_i, \emptyset, f_i) \), since \( (L_1 \cup L_2, G_1 \bullet G_2) \) needs to be determined only up to extended KI, KII relations, and change of orientation of link components. In the case of \( g = 0 \), \( \bullet \) is the connected sum. Hence this operation is an alternative (to composition of cobordisms) way of generalizing connected sum. Note that \( (M_1, \emptyset, f_1) \bullet (M_2, \emptyset, f_2) = \)

\[ \text{The notion of a pair } (L, G) \text{ is defined in the same way as that of a triplet.} \]
of graded modules. It is isomorphism in degree \( \leq \) invariant under the second Kirby move \([18, 25]\).

Figure 11: \((L, G)\) less a tubular neighbourhood of the horizontal line of \(G\).

\((M_1, \emptyset, f_1) \# (M_2, \emptyset, f_2)\). In particular the sets \(\mathcal{G} \cap \{Z\text{-cobordism}\}\) and \(\mathcal{G} \cap \{Q\text{-cobordism}\}\) are closed under \(\ast\).

2.5 The LMO invariant for closed manifolds and extending the maps \(\iota_n\). In \([18]\) from the Kontsevich integral an invariant of oriented framed links \(L\) was constructed, which does not change under Kirby-1,2 moves and change of orientation of components of \(L\). We recall it here, together with the maps \(\iota_n\), necessary to extend it to an invariant of unions of embedded framed chain graphs in \(S^3\).

Let \(\mathcal{A}^{\emptyset}\) be the formal series completion of the \(Q\)-vector space generated by the homeomorphism classes of open chord diagrams without univalent vertices (but allowing dashed self-loops - these are set of degree 0) modulo AS and IHX relations. Let \(\mathcal{B}(X)\) be the formal power series completion of the \(Q\)-vector space generated by the homeomorphism classes of open chord diagrams without dashed self-loops, with the univalent vertices colored by elements of \(X\), modulo AS and IHX relations.

Denote \([m]^{\not=} = \{1, \ldots, m\}\), and let \(\Gamma = \sqcup_m S^1\), where each component is colored by a different element of \([m]\). Let \(\mathcal{B}([m])\) be the subspace of \(\mathcal{B}(m) \cup \{\ast\}\), generated by the diagrams with one \(\ast\)-colored vertex, and \(f_i : \mathcal{B}([m]) \to \mathcal{B}([m]), f_i := \text{average of the diagrams obtained by attaching the \(\ast\)-vertex near all }i\text{-vertices.}

We can define a map \(\varphi : \mathcal{C}(m) := \frac{\mathcal{B}([m])}{\sum_i f_i | \mathcal{C}(m)|} \to \mathcal{A}(\Gamma), \varphi := \text{average of the diagrams obtained by attaching }i\text{-coloured vertices to the }i\text{th copy of }S^1\text{ in }\Gamma, \forall i\). (One checks that the definition on diagrams extends over relations to a map between formal series completions.) This map is in fact an isomorphism of \(Q\)-modules (and co-algebras). For details, please consult \([18, 25]\). (If \(\Gamma = S^1, \mathcal{C}(1)\) and \(\mathcal{A}(S^1)\) are algebras, but \(\varphi\) is not an algebra homomorphism.)

For every \(n \geq 0\) define a map \(\kappa_n : \mathcal{C}(m) \to \mathcal{A}^{\emptyset}\); \(\kappa_n(K) = 0\), if \(\exists i\) such that the number of \(i\)-colored vertices is not \(2n\), \(\kappa_n(K) = \text{sum of all ways of attaching }i\text{-coloured vertices in pairs, }\forall i\), otherwise. Let \(\mathcal{O}_n\) be the ideal of \(\mathcal{A}^{\emptyset}\) generated by \(\bigcirc + 2n\). \((\mathcal{A}^{\emptyset})\) is an algebra with respect to disjoint union. It can be shown that as modules (and even as algebras) \(\mathcal{A}^{\emptyset}/\mathcal{O}_n \cong \mathcal{A}(\emptyset)\). Now, let \(\iota_n = q_n \circ \kappa_n \circ \varphi^{-1} : \mathcal{A}(\Gamma) \to \mathcal{C}(m) \to \mathcal{A}^{\emptyset} \to \mathcal{A}(\emptyset)/\mathcal{O}_n \cong \mathcal{A}(\emptyset),\) where \(q_n\) is the quotient map.

Let \(\kappa_n^* : \mathcal{B}(X \sqcup \{\ast\}) \to \mathcal{B}(X)\) be defined as \(\kappa_n\), but only involving \(\ast\)-colored vertices (see \([18, 25]\) for details).

Let \(P_n = \text{Im}(\kappa_n^*)\). The map \(\kappa_n^*\) passes to the quotient from the definition on \(\mathcal{C}(m)\), and hence we get a submodule \(P_n\) of \(\mathcal{C}(m)\). The relations \(P_n\) also commute with \(\varphi\). Define the quotient map \(j_n : \mathcal{A}^{\emptyset}/\mathcal{O}_n \to \mathcal{A}^{\emptyset}/\mathcal{O}_n, P_n, P_{n+1}\) of graded modules. It is isomorphism in degree \(\leq n\), and is the main ingredient in showing that \(\iota_n(\hat{Z}(L)) \leq n\) is invariant under the second Kirby move \([18, 25]\).

This construction can be extended for \(\Gamma = \Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2}\), disjoint union of two chain graphs and \(m\).
copies of $S^1$, i.e. $\iota_n : A(\sqcup_m S^1) \to A(\emptyset) / \mathcal{O}_n \cong A(\emptyset)$ can be extended (meaning that for $g_1 = g_2 = 0$, $\iota_n$ acts exactly as $\iota_n$) to a map:

$$\tilde{\iota}_n = \tilde{q}_n \circ \kappa_n \circ \tilde{\varphi}^{-1} : A(\Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2}) \to A(\Gamma^{g_1} \sqcup \Gamma^{g_2}) / \mathcal{O}_n \cong A(\Gamma^{g_1} \sqcup \Gamma^{g_2})$$

(2.2)

where the corresponding homomorphism $\tilde{\varphi}^{-1}$ refers only to all present circle components of $\Gamma$. Here, to define the preimage of $\tilde{\varphi} : C(\Gamma^{g_1}, [m], \Gamma^{g_2}) \to A(\Gamma^{g_1}, \sqcup_m S^1, \Gamma^{g_2})$ we consider absolutely analogous chord diagrams with support the disjoint union of two chain graphs $\Gamma^{g_1}, \Gamma^{g_2}$, and points indexed by elements of $[m]$ (it is convenient NOT to call these points vertices), $\kappa_n$ is extended in the same manner, and $\tilde{q}_n$ is just the quotient map. $\tilde{\varphi}$ is an isomorphism with the proof of Section 2 of [18]. Moreover, the similarly constructed map $\tilde{j}_n$ is an isomorphism in degree $\leq n$. Namely, and this is exactly the statement of Lemma 2.3 of [22], if $\Gamma = \Gamma^{g_1} \sqcup (\sqcup_m S^1) \sqcup \Gamma^{g_2}$, then $\tilde{j}_n : (A(\Gamma^{g_1} \sqcup \Gamma^{g_2}) \cong A(\Gamma^{g_1} \sqcup \Gamma^{g_2}) / \mathcal{O}_n)_{\leq n} \cong (A(\Gamma^{g_1} \sqcup \Gamma^{g_2}) / \mathcal{O}_n, P_{n+1})_{\leq n}$. To check this fact it is enough to follow the proof of lemma 3.3 in [18] or proposition 4.4 in [25].

Let $Z(L)$ be the usual Kontsevich integral of the (oriented) framed link $L, \nu$ be $Z$ of the zero-framed unknot. Denote $\tilde{Z}(L) := Z(L) \otimes \nu(L)$, meaning we take the “connected sum” of $Z(L)$ on each its component with $\nu$. Like $Z(L), \tilde{Z}(L)$ is also group-like of the form $1 + (terms \ of \ degree \ \geq 1)$. Let $\sigma_{\pm}$ be the number of positive, resp. negative eigenvalues of the linking matrix of $L$. Denote $O^{+1}$, resp. $O^{-1}$ the unknot with +1, resp. −1 framing, and $S^3$ the 3-manifold obtained by surgery on the framed link $L$ in $S^3$. Recall the definition of the LMO invariant for oriented closed 3-manifolds $M \equiv S^3$:

$$\Omega_n(S^3) := \left( \frac{\iota_n(\tilde{Z}(L))}{\iota_n(\tilde{Z}(O^{+1}))^{\sigma_{+}} \cdot \iota_n(\tilde{Z}(O^{-1}))^{\sigma_{-}}} \right)_{\leq n}$$

(2.3)

and:

$$Z^{\text{LMO}}(M) := \sum_{n \geq 0} \Omega_n(M)_n$$

and for $\mathbb{Q}$-homology spheres also:

$$Z^{\text{LMO}}(M) := \sum_{n \geq 0} d(M)^{-n} \Omega_n(M)_n$$

where $d(M) = |\text{det}(lk(L))|$, which is 0 if $H_1(S^3, \mathbb{Q}) \neq 0$ and $|H_1(M, \mathbb{Z})|$ otherwise. We use the convention $|\text{det}(lk(\emptyset))| = 1$. Then we have $\Omega_{n+1}(S^3)_{\leq n} = d(M) \cdot \Omega_n(S^3)$, hence we can write $Z^{\text{LMO}}(M)_{\leq n} = d(M)^{-n} \Omega_n(M)$. More precisely, the following holds [25, Proposition 4.5]:

$$[\iota_{n+1}(\tilde{Z}(L))]_{\leq n} = (-1)^{|L|} |\text{det}(lk(L))| [\iota_n(\tilde{Z}(L))]_{\leq n}$$

(2.4)

and therefore we can define:

$$c_{\_} = \lim_{n \to \infty} (-1)^n [\iota_n(\tilde{Z}(O^{+1}))]_{\leq n}$$

(2.5)

$$c_{\_} = \lim_{n \to \infty} [\iota_n(\tilde{Z}(O^{-1}))]_{\leq n}$$

(2.6)

These elements of $A(\emptyset)$ are canonical constants in the theory of LMO invariant. [23] implies

$$Z^{\text{LMO}}_{\leq N}(M) = \left( \frac{(-1)^{N \sigma_{\pm}}}{d(M)^{N}} \cdot \left( \frac{\tilde{Z}(L)}{c_{\_}^{\sigma_{\_}}} \right) \right)_{\leq N}$$

(2.7)

where the notation $[\leq N]$ means the minimal internal degree in the sense of 2.8.

We restrict to the case of $\mathbb{Q}$-homology spheres. Since $\tilde{Z}(L \sqcup L') = \tilde{Z}(L)\tilde{Z}(L')$, we have $\Omega_n(M \# M') = \Omega_n(M)\Omega_n(M')$, therefore:

\[\text{It can be shown by induction that then for } |L| = 1 \text{ the formal graded series } \log(\text{element}) \text{ is a primitive element of } A(\sqcup S^1), \text{ and has no part of degree } 0, \text{ hence it is a formal power series of chord diagrams with connected dashed part. More precisely, a statement similar to Proposition 2.4 holds.}\]
\[ Z^{LMO}(M \# M') = Z^{LMO}(M)Z^{LMO}(M') \]  

And since \( \overline{S_L} = S_L \), where \( \overline{L} \) is the mirror image of \( L \), we have \( \text{[13]} \):  

\[ Z^{LMO}(-M)_n = (-1)^{n(b_1+1)}Z^{LMO}(M)_n \]  

where \( b_1 = \text{rank } H_1(M, \mathbb{Z}) \). Let \( \omega^{LMO}(M) = \log(Z^{LMO}(M)) \). Then the two formulas above can be re-written:

\[ \omega^{LMO}(M \# M')_n = \omega^{LMO}(M)_n + \omega^{LMO}(M')_n \]

\[ \omega^{LMO}(-M) = \sum_{n=1}^{\infty} (-1)^{n(b_1+1)}\omega^{LMO}(M)_n \]

\[ \omega^{LMO}(-M) = \begin{cases} \omega^{LMO}(M), & \text{if } b_1 = \text{even} \\ \omega^{LMO}(M), & \text{if } b_1 = \text{odd} \end{cases} \]

where the conjugation in \( \mathcal{A}(\emptyset) \) is defined as identity on chord diagrams of even degree, and multiplication by \(-1\) on chord diagrams of odd degree.

Also define as in \( \text{[22]} \) \( \hat{Z}(L \cup G) = Z(L \cup G) \otimes (\nu^{\otimes |L|}) \), i.e. add \( \nu \) to \( Z \) of each component of \( L \).

### 2.6 The definition of \( Z \) on elementary pseudo-quasi-tangles

To extend \( \Omega_n(S^n_L) \in \mathcal{A}(\emptyset) \) to invariants \( \Omega_n(L, G) \in \mathcal{A}(\Gamma) \), where \( \Gamma \) is \( G \) as abstract graph, we will extend now \( \hat{Z}(L) \) to \( \hat{Z}(L \cup G) \). However we shall do this differently from Murakami and Ohnuki \( \text{[22]} \) see Figure 2 there], who use Knizhnik-Zamolodchikov associator. We will use the even associator.

Let \( G \) be an embedded framed graph in \( S^3 \). Fix a plane projection such that \( G \) is given the blackboard framing. This projection of \( G \) can be decomposed into elementary tangles\(^{10}\):  

\[ \begin{array}{cc}
\includegraphics[width=0.1\textwidth]{tangle1} & \includegraphics[width=0.1\textwidth]{tangle2} \\
\includegraphics[width=0.1\textwidth]{tangle3} & \includegraphics[width=0.1\textwidth]{tangle4} \\
\includegraphics[width=0.1\textwidth]{tangle5} & \includegraphics[width=0.1\textwidth]{tangle6} \\
\includegraphics[width=0.1\textwidth]{tangle7} & \includegraphics[width=0.1\textwidth]{tangle8} \\
\end{array} \]

We need only to specify the definition of \( Z \) on the first two, since on the others we know it from the link case.

Let \( \Gamma \) be an abstract (disjoint union of) chain graph(s), and \( \epsilon, \Gamma = \Gamma \) with edge \( e \) erased. Suppose \( \epsilon, \Gamma \) is also a chain graph. A similar notation \( \epsilon, G \) for a framed graph \( G \) is self-evident. Define the map \( \epsilon_{(e)} : \mathcal{A}(\Gamma) \to \mathcal{A}(\epsilon, \Gamma) \), \( \epsilon_{(e)}(D) = 0 \), if \( D \) has an external vertex on the removed edge, and \( \epsilon_{(e)}(D) = D \), otherwise. To verify well-defineness of \( \epsilon_{(e)} \) it is enough to check its invariance under branching relations of diagrams on \( \Gamma \). There are 3 diagrams involved in a branching relation. Suppose \( v \) is a trivalent vertex of \( \Gamma \), and \( e_1, e_2, e_3 \) the edges adjacent to \( v \). Edge \( e \) cannot be repeated twice among \( e_1, e_2, e_3 \), since then \( \epsilon, \Gamma \) would not be a (union of) chain graph(s). Therefore we can assume \( e = e_1, e \neq e_2, e \neq e_3 \). It is easy to check that then one of the three diagrams in the relation is sent to 0 by \( \epsilon_{(e)} \), while the other two are sent to diagrams that form an AS relation in \( \mathcal{A}(\epsilon, \Gamma) \).

If \( e \) is an edge of \( G \), denote by \( S_e G \) the graph obtained from \( G \) by reversing the orientation of the edge \( e \) (without changing the framing). If \( \Gamma \) is the underlying abstract graph of \( G \), denote by \( S_e \Gamma \) the underlying abstract graph of \( S_e G \). Let \( \epsilon_{(e)} : \mathcal{A}(\Gamma) \to \mathcal{A}(S_e \Gamma) \) be the linear map which sends every diagram \( D \) in \( \mathcal{A}(\Gamma) \) to the diagram obtained from \( D \) by reversing the orientation of \( e \), multiplied by \((-1)^m \), where \( m \) is the number of vertices of \( D \) on the edge \( e \).

We define \( Z \) for the elementary tangles \( \begin{array}{cc}
\includegraphics[width=0.1\textwidth]{tangle1} & \includegraphics[width=0.1\textwidth]{tangle2} \\
\end{array} \) to satisfy the following two conditions (compare with \( \text{[22]} \) Proposition 1.5):  

1. \( Z(S_e G) = \epsilon_{(e)} Z(G) \), for any embedded framed graph \( G \) and edge \( e \)
2. \( Z(\epsilon, G) = \epsilon_{(e)} Z(G) \), for any (disjoint union of) embedded chain graph(s) \( G \) and edge \( e \), such that \( \epsilon, G \) is still a (disjoint union of) embedded chain graph(s)

Moreover, we seek to define \( Z(\begin{array}{c}
\includegraphics[width=0.1\textwidth]{tangle3} \\
\end{array}) \) to be of the form \( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{tangle9} \\
\end{array} \) (for all possible 8 orientation). By condition \( (2) \) above we must have \( a = b = c^{-1} \), and hence also \( Z(\begin{array}{c}
\includegraphics[width=0.1\textwidth]{tangle10} \\
\end{array}) \) = \( \nu^{1/2} \). But \( Z(\begin{array}{c}
\includegraphics[width=0.1\textwidth]{tangle11} \\
\end{array}) \) = \( \nu^{1/2} \), therefore we must
require $a = b = c^{-1} = \nu^{1/4}$, i.e. $Z(\begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,.5) .. controls (0,.7) and (.25,.7) .. ++(0,.25);
\draw (.25,.5) .. controls (0,.7) and (-.25,.7) .. ++(0,.25);
\end{tikzpicture}) = \begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,1) -- ++(0,.5);
\draw (.25,1) -- ++(0,.5);
\end{tikzpicture}$. Similarly $Z(\begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,.5) .. controls (0,.7) and (.25,.7) .. ++(0,.25);
\draw (.25,.5) .. controls (0,.7) and (-.25,.7) .. ++(0,.25);
\end{tikzpicture}) = \begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,1) -- ++(0,.5);
\draw (.25,1) -- ++(0,.5);
\end{tikzpicture}$.

These formulas are each for the 8 possible orientations.

2.7 Theorem. $Z(G)$ is an isotopy invariant of embedded framed chain graphs.

Proof. In a big part, this is mostly a repetition of the proofs of the statements in [22, Section 1], hence we only sketch here the details that are not identical. First, one shows that $Z(G)$ is invariant under isotopies of the plane. If such isotopies fix a neighbourhood of each trivalent vertex, the result is known from the link case. If isotopies move such neighbourhoods “as a whole”, the result follows using branching relations [22, Lemma 1.2].

Finally, it is sufficient to show $Z(\begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,.5) .. controls (0,.7) and (.25,.7) .. ++(0,.25);
\draw (.25,.5) .. controls (0,.7) and (-.25,.7) .. ++(0,.25);
\end{tikzpicture}) = Z(\begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,1) -- ++(0,.5);
\draw (.25,1) -- ++(0,.5);
\end{tikzpicture})$ and $Z(\begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,1) -- ++(0,.5);
\draw (.25,1) -- ++(0,.5);
\end{tikzpicture}) = Z(\begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,1) -- ++(0,.5);
\draw (.25,1) -- ++(0,.5);
\end{tikzpicture})$.

Secondly, one shows that $Z(G)$ is invariant under extended Reidemeister moves. This is also easily achieved from results known from the link case and the branching relations [22, Lemma 1.4].

To prove the two remaining relations, note that in [14, page 8] it is proved (using an even associator) that

$$Z(\begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,.5) .. controls (0,.7) and (.25,.7) .. ++(0,.25);
\draw (.25,.5) .. controls (0,.7) and (-.25,.7) .. ++(0,.25);
\end{tikzpicture}) = \Delta(\nu^4) \quad \text{and} \quad Z(\begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,.5) .. controls (0,.7) and (.25,.7) .. ++(0,.25);
\draw (.25,.5) .. controls (0,.7) and (-.25,.7) .. ++(0,.25);
\end{tikzpicture}) = \Delta(\nu^4) \quad \text{branching relations}.$$

Therefore $Z(\begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,.5) .. controls (0,.7) and (.25,.7) .. ++(0,.25);
\draw (.25,.5) .. controls (0,.7) and (-.25,.7) .. ++(0,.25);
\end{tikzpicture}) = \begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,1) -- ++(0,.5);
\draw (.25,1) -- ++(0,.5);
\end{tikzpicture}$. Similarly $Z(\begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,1) -- ++(0,.5);
\draw (.25,1) -- ++(0,.5);
\end{tikzpicture}) = Z(\begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,1) -- ++(0,.5);
\draw (.25,1) -- ++(0,.5);
\end{tikzpicture})$. □

The above properties (1) and (2) we have now for granted (compare to [22, Proposition 1.5]). It then follows directly from their definitions in section 3.1 that $\tau^{\leq N}$ and $\tau$ also enjoy properties (1) and (2).

It seems that statements in this section (using even associator) have been known to different people, but a complete proof was missing from the literature.

Conjecture A. If $G$ is a chain graph, then this definition of $Z(G)$ using even associator coincides with the definition in [22], which uses KZ associator.

We have been able to obtain only partial results toward the proof of this conjecture. The results of this paper are equally true for any associator for which Theorem 2.7 holds. Also, note that branching relations must be introduced (in addition to IHX, AS and STU) regardless how one defines $Z(G)$.

Remark. If we use even associator it is easy to see that $Z(\begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,1) -- ++(0,.5);
\draw (.25,1) -- ++(0,.5);
\end{tikzpicture}) = \phi_*(\begin{tikzpicture}[baseline=-.25em]
\draw (-.25,0) -- ++(0,1);
\draw (-.25,1) -- ++(0,.5);
\draw (.25,1) -- ++(0,.5);
\end{tikzpicture})$, where $\phi_*$ is the isomorphism from Proposition 2.3.

2.8 The composition of chord diagrams. Let $(\Gamma^{g_1}, \Gamma^{g_2})$ be an ordered pair of chain graphs. Every time we consider such a pair, $\Gamma^{g_1}$ is the union of its horizontal line and the upper half-circles, and $\Gamma^{g_2}$ is the union of its horizontal line and lower half-circles. Denote $\mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2}) = \mathcal{A}(\Gamma^{g_1} \sqcup \Gamma^{g_2})$, where the order $(\Gamma^{g_1}, \Gamma^{g_2})$ has been specified. As we have remarked after Proposition 2.3, every element of $\mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})$ has a representative, a formal series (with rational coefficients) of chord diagrams, whose external vertices don’t meet one horizontal line. We will “remove” a horizontal line when needed using this isomorphism.
Besides (total) degree, which is half the number of all vertices, a chord diagram has internal degree, half the number of internal vertices. For arbitrary $\alpha \in \mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})$, choose a formal sum of chord diagrams (an element of $D(\Gamma^{g_1}, \Gamma^{g_2})$) representing $\alpha$, express each chord diagram $d$ as a product $a \cdot \beta$, where $a$ is an open chord diagram (i.e., an element of $D(\emptyset)$), say of internal degree $n$, and all connected components of $\beta$ intersect the support $\Gamma$ non-empty. Using STU relations, rewrite $\beta$ (in $\mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})$) as a finite formal sum of chord diagrams without internal vertices, hence of internal degree zero. Group the terms of $\alpha$ with the same internal degree to obtain $\alpha^{[n]} \in \mathcal{A}(\Gamma^{g_1}, \Gamma^{g_2})$, $\forall n$. (Note that $\alpha^{[n]}$ is in general a formal series, not just a finite sum.) Since STU are the only internal degree non-homogeneous relations, $\alpha^{[n]}$, $\forall n$ are uniquely determined by $\alpha$. Call $\alpha = \sum_{n \geq 0} \alpha^{[n]}$ the internal degree decomposition of $\alpha$, and $\alpha^{[n]}$ its internal degree $n$ part.

For any non-negative integer $N$, define $A^{\leq N}(\emptyset) = A(\emptyset)/D^{[>N]}$, where $D^{[>N]}$ is the subspace spanned by diagrams of degree $> N$. The sequence of natural $\mathbb{Q}$-linear projection maps $A^{\leq N+1}(\emptyset) \to A^{\leq N}(\emptyset)$, that forget the degree $N+1$ part, i.e.:

$$\ldots \to A^{\leq N+1}(\emptyset) \to A^{\leq N}(\emptyset) \to \ldots \to A^{\leq 1}(\emptyset) \to A^{\leq 0}(\emptyset) = \mathbb{Q}$$

has inverse limit the direct product of homogeneous degree parts, i.e. $A(\emptyset)$. (This has already been used in Proposition 2.4.) Absolutely similarly, with the only observation that the degree is the minimal internal degree of diagrams, we define $A^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) = A(\Gamma^{g_1}, \Gamma^{g_2})/D^{[>N]}$ and present $A(\Gamma^{g_1}, \Gamma^{g_2})$ as the inverse limit of a similar sequence:

$$\ldots \to A^{\leq N+1}(\Gamma^{g_1}, \Gamma^{g_2}) \to A^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) \to \ldots \to A^{\leq 1}(\Gamma^{g_1}, \Gamma^{g_2}) \to A^{\leq 0}(\Gamma^{g_1}, \Gamma^{g_2}) = A_{\emptyset}((\Gamma^{g_1}, \Gamma^{g_2}))$$

Of cause, $A(\emptyset)$ and $A(\Gamma^{g_1}, \Gamma^{g_2})$ are also inverse limits of infinite subsequences of the above, e.g. if we only consider $\mathbb{N}$ even. Note also the natural isomorphism $A^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) \cong A^{\leq N}(\emptyset) \otimes \mathbb{Q} A_{\emptyset}((\Gamma^{g_1}, \Gamma^{g_2}))$.

The $\mathbb{Q}$-vector spaces $A^{\leq N}(\Gamma)$ have a $A^{\leq N}(\emptyset)$-module structure given by looking at the multiplication by $A(\emptyset)$ in $A(\Gamma)$ in the quotient: the product of $\alpha \in A^{\leq N}(\emptyset)$ with $\beta \in A^{\leq N}(\Gamma)$ is $(\alpha \beta)^{[\leq N]} \in A^{\leq N}(\Gamma)$, where the multiplication $\alpha \beta$ is induced by the disjoint union of chord diagrams. Similarly the multiplication $\bullet$ in $A(\Gamma)$ induces one on $A^{\leq N}(\Gamma)$, making it an algebra.

However for the comultiplication we don’t have the trouble of quotienting through $[>N]$. Indeed, $\Delta : A(\Gamma) \to A(\Gamma) \otimes A(\Gamma)$ preserves both the degree and the internal degree parts, as it is easy to observe from its definition. Hence it induces $\Delta^{\leq N} : A^{\leq N}(\Gamma) \to (A^{\leq N}(\emptyset) \otimes A^{\leq N}(\emptyset))^{\leq N} \subset A^{\leq N}(\emptyset) \otimes A^{\leq N}(\emptyset)$. We can drop the index in $\Delta^{\leq N}$ since the quotient $A(\Gamma) \to A^{\leq N}(\Gamma)$ admits a canonical section $A^{\leq N}(\Gamma) \to A(\Gamma)$, which is to view every formal series of diagrams of minimal internal degree $\leq N$ as a formal series of diagrams, i.e. $\Delta^{\leq N}$ can be equally viewed as the restriction of $\Delta$ to a submodule. An element of $\beta \in A^{\leq N}(\Gamma)$ will be called primitive if $\Delta^{\leq N} \beta = \beta \otimes 1 + 1 \otimes \beta$. (In fact $\beta$ is primitive in $A^{\leq N}(\Gamma)$ iff it is primitive in $A(\Gamma)$.) $\alpha \in A^{\leq N}(\Gamma)$ will be called group-like if $\Delta^{\leq N} \alpha = (\alpha \otimes \alpha)^{[\leq N]}$. Moreover, even though $\Delta^{\leq N}$ are not homomorphisms, Proposition 2.4 holds true if we replace $A(\Gamma)$, $A(\Gamma)$ and $\Gamma$ by their internal degree $\leq N$ truncations $A^{\leq N}(\Gamma)$, $A^{\leq N}(\Gamma)$, and $\tilde{\Gamma}^{\leq N}$, provided we use the above definitions of primitive and group-like. It follows from the fact that $A^{\leq N}(\Gamma) = A^{\leq N}(\emptyset) \otimes A_{\emptyset}(\Gamma)$. This is a simple, yet important observation. $A^{\leq N}(\Gamma)$, and via Proposition 2.3 also $A^{\leq N}(\Gamma)$, have all the properties of non-commutative co-commutative Hopf algebras, except $\Delta$ being homomorphisms.\(^{11}\)

Let $\Gamma^{g_1}, \Gamma^{g_2}$, $\Gamma^{g_3}$ be two fixed ordered pairs of chain graphs. For $\alpha \in A(\Gamma^{g_1}, \Gamma^{g_2})$, $\beta \in A(\Gamma^{g_2}, \Gamma^{g_3})$, represented by single chord diagrams $x$, respectively $y$, let $\alpha \ast \beta$ denote the element of $A(\Gamma^{g_1}, \Gamma^{g_3})$, represented by the diagram obtained by attaching $\phi_{\ast}^{-1}y$ (the horizontal line of $\Gamma^{g_2}$ removed) on top of $\phi_{\ast}^{-1}x$ (the horizontal line of $\Gamma^{g_2}$ removed). For $g = 0$ set $\ast$ to be the formal multiplication. Extend $\ast$ by linearity to formal power series of chord diagrams. Note that $\ast$ is associative.

Let $z_g \in A(\Gamma_g, \Gamma_g)$ and $z_g^N \in A^{\leq N}(\Gamma_g, \Gamma_g)$ be defined by:

$$z_g = \frac{Z(T_g) \otimes (y^{'1/2})^{\otimes 2g}}{c_g^1 + c_g^2} \quad (2.10)$$

\(^{11}\)Alternatively, we can redefine the tensor products $A^{\leq N}(\Gamma) \otimes A^{\leq N}(\Gamma) := A^{\leq N}(\Gamma) \otimes A^{\leq N}(\Gamma) / (A^{\leq N}(\Gamma) \otimes A^{\leq N}(\Gamma))^{>N}$, i.e. enlarge the notion of graded Hopf algebras with many interesting examples. But apparently this approach is disliked by algebraists.
where $T_g$ is the $q$-tangle from figure 7, with the non-associative structure $(\ldots ((**)((**)))(**)) \ldots$, $\nu = Z(O) \in \mathcal{A}(\emptyset)$ is the Kontsevich integral of the zero-framed unknot, and $\otimes$ means taking the connected sum of chord diagrams on each of the $2g$ components, $c_+, c_-$ have been defined in 2.5. Note that $Z(T_g) \otimes (\nu^{1/2})^\otimes 2g$ has internal degree 0. For $g = 0$ define $z_0 = z_0^\emptyset = 1$.

2.9 Proposition. 1) Let $*$ be the gluing operation defined above, let $\tilde{i}_N : \mathcal{A}(\Gamma^{g_1} \sqcup (L_2 \sqcup L_3)^1 \sqcup \Gamma^{g_3}) \to \mathcal{A}(\Gamma^{g_1} \sqcup \Gamma^{g_3})$ be the $\mathcal{A}(\emptyset)$-linear map defined by $\bullet\bullet\bullet$, which refers exactly to all present circle components (in this case $2g_2$). Then

\[
\ell^{\leq N} (\alpha, \beta) = (-1)^{g_2} (\tilde{i}_N(\alpha * z^{\leq N}_{g_2} \beta)) \quad (2.12)
\]
defines a $\mathcal{A}^{\leq N}(\emptyset)$-bilinear form $\mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) \otimes \mathcal{A}^{\leq N}(\Gamma^{g_2}, \Gamma^{g_3}) \to \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_3})$.

2) For any $N$ and any respective elements $\alpha, \beta, \gamma$ we have $\ell^{\leq N}(\ell^{\leq N}(\alpha, \beta), \gamma) = \ell^{\leq N}(\alpha, \ell^{\leq N}(\beta, \gamma))$. 

Proof. 1) Let $\alpha \in \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}), \beta \in \mathcal{A}^{\leq N}(\Gamma^{g_2}, \Gamma^{g_3}), z^{\leq N}_{g_2} \in \mathcal{A}^{\leq N}(\Gamma^{g_2}, \Gamma^{g_2})$. Then, since $\tilde{\varphi}$ is an isomorphism, $\tilde{\varphi}^{\otimes 1}(\alpha * z_{g_2}^\otimes \beta)^{\leq N} = 0$, i.e. if we factor in $\mathcal{C}(\Gamma^{g_1}, [m], \Gamma^{g_2})$ by the subspace spanned by diagrams with internal degree $> N$, hence $\tilde{i}_N(\alpha * z_{g_2}^\otimes \beta)^{\leq N} = 0 \in \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2})$. Similarly for $\beta$. $\ell^{\leq N}(\emptyset)$-bilinearity of $(\alpha, \beta) \mapsto \tilde{i}_N(\alpha * z_{g_2}^\otimes \beta)^{\leq N}$ is obvious.

2) follows from the fact that $*$ is associative, and $\tilde{i}_N(\tilde{i}_N(\alpha * z_{g_2}^\otimes \beta) * z_{g_2}^\otimes \gamma) = \tilde{i}_N(\alpha * z_{g_1}^\otimes \tilde{i}_N(\beta * z_{g_2}^\otimes \gamma)) = \tilde{i}_N(\alpha * z_{g_1}^\otimes \beta * z_{g_2}^\otimes \gamma)$. □

Note that when $g_1 = 0$, $\ell^{\leq N}$ becomes a $\mathcal{A}^{\leq N}(\emptyset)$-linear map:

\[
\ell^{\leq N} : \mathcal{A}^{\leq N}(\Gamma^{g_2}) \otimes \mathcal{A}^{\leq N}(\Gamma^{g_2}, \Gamma^{g_3}) \to \mathcal{A}^{\leq N}(\Gamma^{g_3})
\]

Hence every element in $\mathcal{A}^{\leq N}(\Gamma^{g_2}, \Gamma^{g_3})$ defines a $\mathcal{A}^{\leq N}(\emptyset)$-linear map from $\mathcal{A}^{\leq N}(\Gamma^{g_2})$ to $\mathcal{A}^{\leq N}(\Gamma^{g_3})$, and the induced map $\tilde{\ell}^{\leq N} : \mathcal{A}^{\leq N}(\Gamma^{g_2}) \to \mathcal{A}^{\leq N}(\Gamma^{g_3})$ is $\mathcal{A}^{\leq N}(\emptyset)$-linear. Composing with the isomorphism $\tilde{\phi}_n : \mathcal{A}^{\leq N}(\Gamma^{g_2}) \to \mathcal{A}(\Gamma^{g_2})$ we obtain an $\mathcal{A}^{\leq N}(\emptyset)$-linear map also denoted $\tilde{\ell}^{\leq N} : \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) \to \mathcal{A}^{\leq N}(\Gamma^{g_1}) \otimes \mathcal{A}^{\leq N}(\Gamma^{g_2})$.

Extend $\mathcal{C}(\emptyset, \Gamma^{g_2})$ to $\mathcal{A}^{\leq N}(\emptyset) \to \mathcal{A}^{\leq N}(\emptyset, \Gamma^{g_2})$, and compose with $\tilde{\ell}^{\leq N}$ to obtain a $\mathcal{A}^{\leq N}(\emptyset)$-linear map $(\ell^{\leq N})^\star : \mathcal{A}^{\leq N}(\emptyset) \to \mathcal{A}^{\leq N}(\emptyset)^\star$. Namely $(\ell^{\leq N})^\star(\beta)(\alpha) = \ell^{\leq N}(\alpha, \beta), \forall \alpha, \beta \in \mathcal{A}^{\leq N}(\emptyset)$. Similarly, there is a map $(\ell^{\leq N})^\star : \mathcal{A}^{\leq N}(\Gamma^{g_1}) \to \mathcal{A}^{\leq N}(\Gamma^{g_1})^\star$.

The second part of the above proposition shows that $\tilde{\ell}^{\leq N}(\ell^{\leq N}(\beta, \gamma)) = \tilde{\ell}^{\leq N}(\gamma) \circ \ell^{\leq N}(\beta)$ for any corresponding $\beta, \gamma$, i.e. the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) \otimes \mathcal{A}^{\leq N}(\Gamma^{g_2}, \Gamma^{g_3}) & \xrightarrow{\ell^{\leq N}} & \mathcal{A}^{\leq N}(\Gamma^{g_1}, \Gamma^{g_3}) \\
\downarrow \tilde{\ell}^{\leq N} \otimes \tilde{\ell}^{\leq N} & & \downarrow \tilde{\ell}^{\leq N} \\
\mathcal{A}^{\leq N}(\Gamma^{g_1})^\star \otimes \mathcal{A}^{\leq N}(\Gamma^{g_2})^\star \otimes \mathcal{A}^{\leq N}(\Gamma^{g_3})^\star & \xrightarrow{\text{evaluation}} & \mathcal{A}^{\leq N}(\Gamma^{g_1})^\star \otimes \mathcal{A}^{\leq N}(\Gamma^{g_3})^\star
\end{array}
\]

Remark. If we were to use Knizhnik-Zamolodchikov or any other associator, in the definition of $\ell$, between $\alpha, z_g$ and $\beta$ we would have to insert an element $A$ in the space of chord diagrams on $2g$ arrows alternatively oriented downward and upward, and its horizontal reflection, such that $(\phi_+^{-1}Z(\underbrace{\bullet\bullet\dotsc\bullet\bullet}_{2g})) * A = \nu^{1/2}g$; and similarly for $\beta$. For the even associator, $A$ can be taken 1, i.e. it can be omitted. Conjecture A above claims that $A = 1$ for any associator.

2.10 The categories $\mathcal{A}^{\leq N}$ and $\mathcal{A}$. Let $\mathcal{A}^{\leq N}$ be the category with objects $\mathcal{A}^{\leq N}(\Gamma^{g}) \equiv \mathcal{A}^{\leq N}(\Gamma^{g}, \Gamma^{g}), g \geq 0$, and morphisms the set of $\mathcal{A}^{\leq N}(\emptyset)$-homomorphisms between these modules. Similarly define the category $\mathcal{A}$. Note that via the isomorphism $\phi_*$ in Proposition 2.3 we can identify $\mathcal{A}(\Gamma_g)$ and $\mathcal{A}(\Gamma^*_g)$. 

\footnote{If one of elements belongs to the space of chord diagrams on certain $2g$ arrows that connect $2g$ points on a "bottom line" with $2g$ points on a "top line", as is $z_g$, we still can define $\star$. It extends by linearity and is associative.}
2.11 Proposition. Let $\ell \leq N$ be the bilinear form defined by the previous proposition, and $\tilde{\ell} \leq N$ be the induced map $\mathcal{A}^{\leq N}(\Gamma^g, \Gamma^g) \to \mathcal{A}^{\leq N}(\mathcal{I}_{[g_1]}, \mathcal{I}_{[g_2]}) = \text{Hom}(\mathcal{A}^{\leq N}(\mathcal{I}_{[g_1]}), \mathcal{A}^{\leq N}(\mathcal{I}_{[g_2]}))$. Denote $w_g = Z(W_g) = Z(W_g)^{\leq N} \in \mathcal{A}^{\leq N}(\Gamma^g, \Gamma^g)$, where $W_g$ is the embedded framed graph in figure 12, where the first $\Gamma^g$ corresponds to the lower of the two chain graphs in the picture. (For $g = 0$, $W_g = 1$.) Then $\hat{\ell} \leq N(w_g)$ is the identity operator on $\mathcal{A}^{\leq N}(\mathcal{I}_{[g]})$.

Note that $w_g$ has zero as internal degree $\geq 1$ parts.

Figure 12: The embedded framed graph $W_g$

2.12 Theorem. 1) There is a (unique) $\mathcal{A}(\emptyset)$-bilinear form $\ell : \mathcal{A}(\Gamma^g, \Gamma^g) \otimes \mathcal{A}(\Gamma^g, \Gamma^g) \to \mathcal{A}(\Gamma^g, \Gamma^g)$, such that its $\leq N$ internal degree part coincides with $\tilde{\ell} \leq N$.

2) Let $\ell$ be the induced map $\mathcal{A}(\Gamma^g, \Gamma^g) \to \mathcal{A}(\mathcal{I}_{[g_1]}, \mathcal{I}_{[g_2]}) = \text{Hom}_{\mathcal{A}(\emptyset)}(\mathcal{A}(\mathcal{I}_{[g_1]}), \mathcal{A}(\mathcal{I}_{[g_2]}))$, and denote as before $w_g = Z(W_g) \in \mathcal{A}(\Gamma^g, \Gamma^g)$, where $W_g$ is shown in figure 12. Then $\hat{\ell}(w_g)$ is the identity operator on $\mathcal{A}(\mathcal{I}_{[g]})$.

We will prove these statements in Section 3 (after 3.6, resp. 3.7).

3 The TQFT

Now we can formally construct the truncated and full TQFTs. We will show that they are non-degenerate and anomaly-free. The truncated TQFTs are with respect to the internal degree. Since the map $\hat{i}_N$, which we had to introduce if we want to have invariance under Kirby moves for chain graphs, decreases the total degree of a diagram by $2gN$, and since $\hat{i}_N$ must be applied every time we glue two cobordisms, one should not expect the theory to truncate with respect to the total degree of chord diagrams.

For our purposes a TQFT $(T, \tau)$ based on the cobordism category $\mathfrak{C}$ (or a subcategory of it) is 1) a covariant functor $T$ from the category those objects are the objects of $\mathfrak{C}$ (i.e. natural numbers) and morphisms are the homeomorphisms of parametrized surfaces to a subcategory $\mathfrak{C}_K$ of the full category of $\mathfrak{C}$-modules, such that $T(0) = K$, where $K$ is a commutative module; and 2) a map $\tau$ that associates to each 3-cobordism $(M, f_1, f_2)$ a $K$-homomorphism $\tau(M) : T(\Sigma_1) \to T(\Sigma_2)$, satisfying the following axioms:

(A1) (Naturality) If $(\Sigma_1, \Sigma_1', (M_1, \Sigma_2, \Sigma_2')$ are two 3-cobordisms, and $f : M_1 \to M_2$ is a homeomorphism of 3-cobordisms, preserving the parametrizations, then the following diagram is commutative:

$$T(\Sigma_1) \xrightarrow{\tau(M_1)} T(\Sigma_1')$$

$$\downarrow T(f|_{\Sigma_1}) \downarrow T(f|_{\Sigma_2})$$

$$T(\Sigma_2) \xrightarrow{\tau(M_2)} T(\Sigma_2')$$

(A2) (Functoriality) If $M_1, M_2$ are 3-cobordisms, $f = f_2 \circ (f_1)^{-1} : \partial_{\text{top}}(M_1) \to \partial_{\text{bottom}}(M_2)$ is the gluing homeomorphism, and denote $M = M_2 \cup_f M_1$, then $\tau(M) = k \cdot \tau(M_2) \circ \tau(M_1)$. $k \in K$ is called the anomaly.

(A3) (Normalization) Let $(\Sigma \times [0, 1], (\Sigma \times 0, p_1), (\Sigma \times 1, p_2))$ be the 3-cobordism mentioned in 1.2, then

$$\tau(\Sigma_g \times [0, 1], (\Sigma_g \times 0, p_1), (\Sigma_g \times 1, p_2)) = id_{T(\Sigma_g)}$$
(A4) (pseudo-Hermitian structure) There is a superstructure on each element $V$ of $\mathfrak W_K$, i.e. it admits an antimorphism $\tau : V \to V$ (a map linear in 0-supergrading and antilinear in 1-supergrading), that commutes with ($\approx$ is natural with respect to) surface homeomorphisms. There is a canonical map $V \to V^*$, which composed with the above antimorphism extends (from the particular case when $T(\Sigma) = K$) to an antimorphism $\tau : \text{Mor}(T(\Sigma_1), T(\Sigma_2)) \to \text{Mor}(T(-\Sigma_2), T(-\Sigma_1))$, that commutes with homeomorphisms of 3-cobordisms, such that

$$\tau(-M) = \tau(M)$$

We can not require multiplicativity or self-duality since in the category $\Omega$ all cobordisms are connected. Conditions (A1-A3) say that $\tau : \Omega \to A$ is a pseudo-functor. $\tau$ would be a true functor when there is no anomaly. If the set of $\tau(M, S^2, \Sigma)$’s spans (in the closure for infinite-dimensional modules) $T(\Sigma)$, the TQFT is called non-degenerate.

### 3.1 $\tau^\leq N$ and $\tau$

The definition of $\Omega_n(S^3_2)$ and the proof of its invariance under Kirby moves have been extended [22 proposition 2.4] to an invariant of embeddings $L \sqcup G \to S^3$ and extended (generalized) Kirby moves. Again, as long as $Z(L \sqcup G)$ has been shown well-defined, it does not matter which associator we use.

**Proposition 1** Let $n \in \mathbb N$, and let $L \sqcup G \to S^3$ be an arbitrary embedding of a link and a (union of) chain graph(s) in $S^3$, $\sigma_+, \sigma_-$ be the number of positive, respectively negative eigenvalues of $\text{lk}(L)$, the linking matrix of $L$, $g$ the total number of circle components of $G$. Define:

$$\Omega_n(L, G) := \left( \frac{i_n(Z(L \sqcup G))}{i_n(Z(O^{+i})^{\sigma_+} \cdot i_n(Z(O^{-i})^{\sigma_-})} \right)^{[\leq n]} \in A^{\leq n}(\Gamma) \subset A(\Gamma)$$

where $G$ is an abstract graph. Then for every $m \leq n$ the internal degree part $\Omega_n(L, G)^{[m]}$ is invariant under extended (generalized) KI and KII moves, and under orientation change of components of $L$.

2) With the above notations, and denoting $d_L = |\det(\text{lk}(L))|$, the following relation holds in $A(\Gamma)$ for any (not necessarily connected) chain graph $G$ and link $L$:

$$\left( i_{n+1}Z(L, G) \right)^{[\leq n]} = (-1)^{|L|\det(\text{lk}(L))} \left( i_nZ(L, G) \right)^{[\leq n]}$$

(3.1)

and therefore:

$$\Omega_{n+1}(L, G)^{[\leq n]} = d_L \cdot \Omega_n(L, G)$$

3) $\frac{1}{d_L}\Omega_n(L, G)$ is a group-like element of $A^{\leq n}(\Gamma) \subset A(\Gamma)$ of the form $1 +$ higher order terms.

**Proof.** 1) By the well definiteness of the internal degree parts (see 2.8), it is enough to show that $\Omega_n(L, \Gamma)$ stays invariant under the moves, which is precisely the statement in Proposition 2.4 in [22] and the remarks after it. There the proof is similar to the case of links (see Section 3 in [18], or Section 4 in [25]).

2) Follow the proof of Proposition 4.5 in [25].

3) First note that $Z$ of any elementary pseudo- quasi-tangle is group-like of the desired form. Indeed, if one uses KZ associator, the elements $a, b$ in [22 page 503] used in the definition of $Z$ for the vicinity of trivalent vertices are clearly so. If one uses even associator, then (even simpler) it follows from the fact that $\Delta \nu = \nu \otimes \nu$ and $\nu = 1 + h.o.t.$ Hence $Z(L \sqcup G)$ is group-like of the form $1 + h.o.t.$ for any $L \sqcup G \to S^3$ (compare with [18 subsection 1.4]). That $\Delta$ commutes with $i_n$ follows from the fact that $\hat{\Delta}$ commutes with $\hat{\phi}$, and an explicit calculation of $\Delta \circ (\hat{q}_n \circ \hat{k}_n)$ and $(\hat{q}_n \circ \hat{k}_n) \otimes (\hat{q}_n \circ \hat{k}_n) \circ \Delta$ for any diagram with $2n$ legs of each colour $1, \ldots, |L|$, just as in the case $G = \emptyset$ [25] [18]. Similarly it follows that $\frac{1}{d_L}\Omega_n(L, G)$ has the form $1 + h.o.t.$ (compare with [18] Lemma 4.7). □

Let $M$ be a morphism in $\Omega$ between $g_1$ and $g_2$. Let $(L, G_1, G_2)$ be such that $\kappa(L, G_1, G_2) = (M, f_1, f_2)$. By Proposition 2.1 in [22], the ambiguity in this choice is a finite sequence of extended KI and KII moves, and change of orientation of link components. In Theorem 1.4 of [22] using KZ associator, or alternatively in 2.6 here using even associator, the Kontsevich integral is extended to an isotopy invariant of chain graphs in $S^3$, and hence of embeddings $L \sqcup G_1 \sqcup G_2$ in $S^3$. Suppose that $G_1, G_2$ regarded as abstract graphs are $\Gamma^{g_1}, \Gamma^{g_2}$. Then let us define:
\[\tau(M, f_1, f_2) = \frac{1}{\det(\kappa(L))^n} \left( \frac{\iota_n(\tilde{Z}(L \cup G_1 \cup G_2))}{\iota_n(\tilde{Z}(O^{-1}))} \right)^n \in A(\Gamma^{g_1}, \Gamma^{g_2}) \] (3.2)

where \( \det(\kappa(L)) \) represents the internal degree part, \( \sigma_+ , \sigma_- \) are the number of positive and negative eigenvalues of \( \kappa(L) \), and \( \iota_n \) refers to the circle components of chord diagrams, all coming here from the components of the link \( L \). As before (see the proof of Proposition 2.3) we can assume that the vertices of chord diagrams are off the horizontal lines. \( c_+, c_- \) have been defined in 2.5. We use the convention \( \det(\kappa(\emptyset)) = 1 \). Also, let:

\[\tau^{\leq N}(M, f_1, f_2) = \frac{(-1)^{\sigma_+ N}}{d(M)^N} \left( \frac{\iota_N(\tilde{Z}(L \cup G_1 \cup G_2))}{c_+^{\sigma_+} \cdot c_-^{\sigma_-}} \right)^{\leq N} \in A^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) \] (3.3)

3.2 Non-degeneracy of the TQFT(s). We can define now similarly \[\phi^*_{\xi}\] and \( A^*_{\xi}(\{g\}) \) for cobordisms \((M, \emptyset, f)\) with only one connected and parametrized boundary component, as long as \( \tilde{M} \) is a \( Z \)-homology sphere. We think of the boundary as the top of the 3-cobordism. Then \( \tau(M, \emptyset, f) \in A(\Gamma^0) \), and hence via the isomorphism \( \phi_*^{-1} \) we obtain an element \((M, \emptyset, f)\) in \( A(\{g\}) \). (Similarly add \( \leq N \).)

**Theorem.** Let \( A^{\leq N}_{\tau}(\{g\}) \), respectively \( A_{\tau}(\{g\}) \), be the \( \mathbb{Q} \)-vector subspace of \( A^{\leq N}(\{g\}) \), respectively \( A(\{g\}) \), generated by all \( \phi_*^{-1}(\tau^{\leq N}(M, \emptyset, f)) \), respectively all \( \phi_*^{-1}(\tau(M, \emptyset, f)) \), such that \( \tilde{M} \) is a \( Z \)-homology sphere. Then the completion of \( A^{\leq N}_{\tau}(\{g\}) \) is \( A^{\leq N}(\{g\}) \), and the completion of \( A_{\tau}(\{g\}) \) is \( A(\{g\}) \).

It is known [12, Proposition 13.1] that for every chord diagram \( \xi \in A(\{g\}) \) of degree \( m \), with connected dashed graph, there exist string links \( L^\pm \) such that \( Z(L^\pm) = 1 + \xi + o(m+1) \). For a very intuitive geometric realization of \( L^\pm \) see also [13, 17].

3.3 Lemma. For every \( n \geq 0 \) and every chord diagram \( \xi \in A(\{g\})_{\leq n} \), there exist string links \( L_1, \ldots, L_k \) and positive integers \( a_1, \ldots, a_k \) such that \( \sum_{i=1}^k a_i Z(L_i) = \xi + o(n+1) \).

**Proof.** Induction on \( n \). For \( n = 0 \), \( Z(\text{trivial string link}) = 1 \in A(\{g\}) \). For \( n = 1 \), \( \xi \) must have connected dashed graph, hence the claim follows from the mentioned result of Habegger and Masbaum, because \( Z(\text{trivial string link}) = 1 \). For general \( n \), suppose \( \xi \) has degree \( m \). We prove the statement first for \( m = n \), then for \( m = n - 1, \ldots, 1 \) \((m = 0 \) is obvious). For arbitrary \( m \), by the same argument as in the proof of Proposition 2.4.5) we can assume \( \xi = \sum_{i=1}^k \pm \xi_1 \cdots \pm \xi_k \), \( \xi_i \) connected dashed graph and degree \( \geq 1 \). If \( k = 1 \), by the induction hypothesis there exist \( a_1^i \in \mathbb{Z} \) and string links \( L_1^i \) such that \( \sum a_1^i Z(L_1^i) = \xi + o(n) \). Therefore \( \sum_{i=1}^k a_1^i \cdots a_k^i \cdot Z(L_1^i \cdots L_k^i) = \sum_{i=1}^k a_1^i Z(L_1^i) \cdots \sum_{i=1}^k a_k^i Z(L_k^i) = (\xi + o(n)) \cdots (\xi_k + o(n)) = \xi_1 \cdots \xi_k + o(n+1) \). If \( k = 1 \), by Habegger-Masbaum result, there is a string link \( L \) such that \( Z(L) - Z(\text{trivial string link}) = \xi + o(m+1) \). Therefore the statement for \( m \) follows from the fact that it holds for \( m + 1 \) (express in the latter formula the degree \( m + 1 \) terms of \( o(m+1) \)). If \( k = 1 \) and \( m = n \), it is precisely the Habegger-Masbaum result. Note that all coefficients \( a_i \) appearing throughout the proof can be arranged positive or negative as we wish [15, 17], hence the ones in the statement can be ensured positive.

It is known [17] Theorem 4.5; see also [13] that for any connected trivalent graph \( D \) of degree \( n \) there exist \( \mathbb{Z} \)-homology 3-spheres \( M^\pm \) such that \( Z^{LMO}(M^+) = 1 + D + o(n+1) \in A(\emptyset) \). (This is proved there for \( Z^{lmo}, \)
but it is obviously then true for \( Z^{LMO} \). Since \( Z^{LMO}(S^3) = 1 \in \mathcal{A}(\emptyset) \), with a proof absolutely similar to the one above, we have:

**3.4 Lemma.** For any \( n \geq 0 \) and any chord diagram \( \xi \in \mathcal{A}(\emptyset)_{\leq n} \), there exist \( \mathbb{Z} \)-homology spheres \( M_1, \ldots, M_k \) and positive integers \( b_1, \ldots, b_k \) such that \( \sum_{i=1}^k b_i Z^{LMO}(M_i) = \xi + o(n+1) \). In particular the set \{ \( \sum_i b_i Z^{LMO}(M_i) \mid M_i \mathbb{Z} \)-homology sphere, \( b_i \in \mathbb{N}^* \) \} is dense in \( \mathcal{A}(\emptyset) \).

**Proof of Theorem 3.2.** Let \( \overline{H}_{\{g\}} \) denote the graph with \( 2g \) edges oriented alternatively down- and upward. Lemma 3.3 is clearly true for \( \xi \in \mathcal{A}(\overline{H}_{\{g\}}) \) as well. Therefore for every \( n \geq 0 \) and every \( \beta \in \mathcal{A}(\overline{H}_{\{g\}}) \) there exist string links \( L_i \) and \( a_i \in \mathbb{Q} \) such that \( \sum a_i \cdot Z(L_i) = \beta \cdot \left( \bigotimes_{i=1}^n \nu^{-1} \right) + o(n+1) \). Using the operation \( \ast \) defined in 2.8 attach \( Z(\underbrace{\cdot \cdot \cdot \cdot \cdot}_{k}) \) on top and \( Z(\underbrace{\cdot \cdot \cdot \cdot \cdot}_{n}) \) below each side of this equality to obtain the existence of embedded framed graphs \( G_i \) and \( a_i \in \mathbb{Q} \) such that \( \sum a_i \cdot Z(G_i) = \beta \cdot \left( \bigotimes_{i=1}^n \nu^{-1} \right) \ast \left( \bigotimes_{i=1}^n \nu^{-1} \right) + o(n+1) \). Using the operation \( \ast \) below each side of this equality to obtain the existence of embedded framed graphs \( G_i \) and \( a_i \in \mathbb{Q} \) such that \( \sum a_i \cdot Z(G_i) = \beta \cdot \left( \bigotimes_{i=1}^n \nu^{-1} \right) \ast \left( \bigotimes_{i=1}^n \nu^{-1} \right) + o(n+1) \).

**Remark.** The statement of Theorem 3.2 remains true for \( \overline{H}_{\{g\}} \) replaced by \( \overline{\Gamma} \) = union of chain graphs, since in the above proof we only used that the ”capping map” \( \beta \mapsto \tilde{\beta} \) is linear and surjective, which is clear by the remark after Proposition 2.3.

**3.5 Lemma.** For any \( \beta \in \mathcal{A}(\overline{\Gamma}^{g_2}, \overline{\Gamma}^{g_3}) \) and any sequence of elements \( \alpha^n \in \mathcal{A}(\overline{\Gamma}^{g_1}, \overline{\Gamma}^{g_2}) \), such that for every \( n \), \( (\alpha^n)_{\leq n} = (\alpha^{n+1})_{\leq n} = \ldots \), both sides of the following equalities are well-defined and the equalities holds:

\[
\ell^{\leq n} (\lim_{n} \alpha^n, \beta) = \lim_{n} \ell^{\leq n} (\alpha^n, \beta) \tag{3.4}
\]

A similar property holds for the rôle of two arguments of \( \ell^{\leq n} \) reversed.
A TQFT associated to the LMO invariant of three-dimensional manifolds

Proof. The existence of \( \alpha := \lim n \alpha^n \in A(\Gamma^g, \Gamma^{g^2}) \) follows directly from the fact that we defined the topology on \( A(\Gamma^g, \Gamma^{g^2}) \) such that \( \text{dist}(p, q) < \frac{1}{m^k} \) if and only if \( p - q \) has degree \( n \). Then, since \( \alpha_{>n+gN} \) does not contribute to \( \ell^{\leq N}(\alpha, \beta) \leq n \), we have:

\[
\lim_n \ell^{\leq N}(\alpha, \beta) \leq n = \lim_n \ell^{\leq N}(\alpha_{\leq n+gN}, \beta) \leq n = \lim_m \ell^{\leq N}(\alpha_{\leq m}, \beta) = \ell^{\leq N}(\lim m \alpha_{\leq m}, \beta) = \ell^{\leq N}(\alpha, \beta)
\]

The existence of the third limit and the second equality follow from a standard Cauchy-sequences argument. The fourth equality is true since \( \lim m \alpha_{\leq m} \) commutes with \( \star, \iota_N \) and \( \llbracket \leq N \rrbracket \). On the other hand \( \ell^{\leq N}(\alpha^n, \beta) \) and \( \ell^{\leq N}(\alpha, \beta) \) agree in degree \( \leq n - 2gN \). Hence \( \ell^{\leq N}(\alpha^n, \beta) = \ell^{\leq N}(\alpha, \beta) \leq n \). Putting the two together we obtain \( \text{Lemma 3.5} \). □

Remark. If in the statement of this Lemma we assume that \( \lim n \alpha^n \) exists, then we can relax the topology: \( \text{distance}(p, q) \leq \frac{1}{n} \iff p - q \) has no terms of degree \( < n \).

3.6 The functors \( \Omega \rightarrow A^{\leq N} \) and \( 3 \rightarrow A^{\leq N} \): gluing formula and normalization. Set \( T^{\leq N}(g) = A^{\leq N}(\Gamma^g) \), if \( g > 0 \), \( T^{\leq N}(\emptyset) = A^{\leq N}(\emptyset) \). In this case \( K = A^{\leq N}(\emptyset) \). Set \( T(g) = A(\Gamma^g) \), if \( g > 0 \), and \( T(0) = A(\emptyset) \). In this case \( K = A(\emptyset) \). Now, let us start verifying the axioms of TQFT. Set \( T(f|_{\Sigma}) = id_{A(\Gamma^g)} \) for any homeomorphism \( f \) of the parametrized surfaces. Then \( T \) is a covariant functor, and the naturality axiom (A1) is obvious. The same is true for \( T^{\leq N} \). We will derive now a gluing formula.

Theorem. 1) Let \( (M_1, f_1, f_1') \) and \( (M_2, f_2, f_2') \) be two 3-cobordisms. Suppose \( (M_1, f_1, f_1') = \kappa(L_1, G_1, G_1'), \) \( (M_2, f_2, f_2') = \kappa(L_2, G_2, G_2'), \) and \( (M_2 \cup f_2, f_2') = \kappa(L_1 \cup L_0 \cup L_2, G_1, G_2') \), the later triplet obtained from the previous two by the construction described in Proposition 1.3. Denote \( \sigma_1^+ = \text{sign}_+(l_k(L_1)), \sigma_2^+ = \text{sign}_+(l_k(L_2)), \sigma_+ = \text{sign}_+(l_k(L_1) \cup L_0 \cup L_2) \), and let \( g \) be the genus of the connected closed surface along which is this splitting. Then the integer \( s(M_1, M_2) = \sigma_1^+ + \sigma_2^+ + g - \sigma_+ \) is an invariant of the decomposition \( M = M_2 \cup f_2, f_2' \), i.e. it does not depend on the choice of triplets representing the 3-cobordisms \( M_1 \) and \( M_2 \).

2) Let \( (M_1, f_1, f_1') \) and \( (M_2, f_2, f_2') \) be two \( \emptyset \text{-HH} \). Denote \( d = |H_1(M_2 \cup f_2, f_2')|, d_1 = |H_1(M_1, Z)|, \) \( d_2 = |H_1(M_2, Z)| \). Suppose that these cobordisms are glued along a surface of genus \( g \). Then:

\[
\tau^{\leq N}(M_2 \cup_{f_2, f_2'} M_1, f_1, f_1') = \left( -1 \right)^N \left( \frac{c_+}{c_-} \right)^{\llbracket \leq N \rrbracket} \left( \frac{d_1 d_2}{d} \right)^N \cdot \ell^{\leq N}(\tau^{\leq N}(M_1, f_1, f_1'), \tau^{\leq N}(M_2, f_2, f_2'))
\]

where \( (-1)^N \cdot \left( \frac{c_+}{c_-} \right)^{\llbracket \leq N \rrbracket} \in A^{\leq N}(\emptyset) \), the multiplication by scalars is thought in the category \( A^{\leq N} \), and \( \sigma_1^+ + \sigma_2^+ + g - \sigma_+ \) is an integer.

Proof. 1) Proposition 11 of [1].

2) Let \( (L_1, G_1, G_1') \), \( (L_2, G_2, G_2'), \) and \( (L_1 \cup L_0 \cup L_2, G_1, G_2') \) be as above, and let \( (\sigma_+, \sigma_-) \), \( (\sigma_1^+, \sigma_2^+) \), respectively \( (\sigma_2^+, \sigma_2^-) \) be the signatures of \( l_k(L_1 \cup L_0 \cup L_2) \), \( l_k(L_1) \), resp. \( l_k(L_2) \). Then, temporarily abbreviating \( c^{\llbracket \leq N \rrbracket} \) and \( c^{\llbracket \leq N \rrbracket} \) to \( c_+ \) and \( c_- \):

\[
\tau^{\leq N}(M_2 \cup_{f_2, f_2'} M_1, f_1, f_1') = \left( -1 \right)^{\sigma_+ + N} \left( \frac{c_+}{c_-} \right)^{\llbracket \leq N \rrbracket} \cdot \left( \frac{d_1 d_2}{d} \right)^N \cdot \ell^{\leq N}(\tau^{\leq N}(M_1, f_1, f_1'), \tau^{\leq N}(M_2, f_2, f_2'))
\]
where we have used that \( \sigma_+ + \sigma_- = \sigma_1^+ + \sigma_2^- + \sigma_2^+ + 2 \cdot g \). Observe that in the second equality, when "braking" \( \bar{Z} \) into three, on each component of \( L_0 \) a \( \nu^{1/2} \) "goes" to \( Z \) of \( G'_1 \) or \( G_2 \), and another \( \nu^{1/2} \) goes to \( z_g \). In fact, the two middle expressions are written for the even associator. For any other associator we would insert between the *'s the element \( A \) mentioned in the remark at the end of 2.7.

Let \( (L, G, G') \) be a triplet and \( (M, f, f') = \kappa(L, G, G') \). We can talk about linking number between a link component \( K \) and a circle \( U \) of a chain graph, as well as between two circles \( U \) and \( V \) of chain graphs: \( \text{lk}(K, U) = \text{lk}(U, K) \) is defined to be the linking number between \( K \) and the knot obtained from the graph by deleting all but the circle component \( U \), and similarly for \( \text{lk}(U, V) \). The linking matrix of a triplet is then:

\[
\text{lk}(L, G, G') = \begin{pmatrix}
\text{lk}(L) & \text{lk}(G, L) & \text{lk}(G, G') \\
\text{lk}(G, L) & \text{lk}(G, G) & \text{lk}(G, G') \\
\text{lk}(G', L) & \text{lk}(G', G) & \text{lk}(G', G')
\end{pmatrix}
= \begin{pmatrix}
A & B^T & C^T \\
B & D & E^T \\
C & E & F
\end{pmatrix}
\] (3.6)

where \( A, D, F \) are symmetric matrices. In \( \mathbb{R} \) it has been shown that the semi-Lagrangian condition can be expressed:

\[
D = BA^{-1}B^T, \quad F = CA^{-1}C^T
\] (3.7)

(for \( \mathbb{Q} \)-cobordisms this in particular means that the entries on the left-hand side, a priori in \( \mathbb{Z} \left[ \frac{1}{2 \cdot \text{gcd}(g)} \right] \), must be in \( \mathbb{Z} \)), and for the case \( F = \mathbb{Q} \) additionally \( BA^{-1}C^T \in M_{g_1 \times g_2}(\mathbb{Z}) \). We will need the following elementary

Remark. The signature of a symmetric \( 2g \times 2g \)-matrix \(
\begin{pmatrix}
A & -I \\
-I & 0
\end{pmatrix}
\) with integer, respectively real entries is \((g, g)\). The determinant of such a matrix is \((-1)^g\).

Proof of Proposition 2.11 Note that \( w_g = \tau((\Sigma_g \times [0, 1]), (\Sigma_g \times 0, p_1), (\Sigma_g \times 1, p_2)) \). Using the gluing formula (3.5), for any QHC \( (M, f_1, f_2), \tau^{\leq N}((\Sigma_g \times [0, 1]) \cup_{p_1 \circ (f_2)^{-1}} M, f_1, p_2) = \left( \begin{pmatrix}
\left( \frac{c_1}{c_2} \right)^{\leq N} \\
(\text{lk}(L)) \end{pmatrix} \right)^{(d, d_a)^{N}} \),

\[
\ell^{\leq N}((M, f_1, f_2), w_g). \quad \text{If } \tilde{M} = S^2_g, \text{ and the linking matrix of } L \text{ is } \text{lk}(L), \text{ then the linking matrix of the link } L \cup L_0 \text{ is } \begin{pmatrix}
\text{lk}(L) & 0 & 0 \\
0 & -I & 0 \\
0 & 0 & -I
\end{pmatrix} \sim \begin{pmatrix}
0 & -I & 0 \\
0 & 0 & -I \\
0 & 0 & -I
\end{pmatrix}.
\]

Using the above remark, \( \sigma_+ = \sigma_1^+ + g, \sigma_- = \sigma_1^- + g \), \( \sigma_1^2 = \sigma_2^2 = 0, \) \( d_2 = \text{lk}(\emptyset) = 1, \) \( d_1 = d \). Observe that \((\Sigma_g \times [0, 1]) \cup_{p_1 \circ (f_2)^{-1}} M, f_1, p_2) \cong (M, f_1, f_2) \), we have \( \ell^{\leq N}((M, f_1, f_2), w_g) = \tau^{\leq N}(M, f_1, f_2) \). In particular, this holds if \((M, f_1, f_2) \) is a ZHC with bottom \( S^2 \), and hence also for any \((M, \emptyset, f) \) such that \( \tilde{M} \) is a Z-homology sphere. The statement now follows from Theorem 3.2 and Lemma 3.5. □

Note that Proposition 2.11 verifies Axiom (A3) for the truncated TQFTs \( \Omega \to A^{\leq N} \) and \( \bar{Z} \to A^{\leq N} \).

3.7 Absence of anomaly. In \( \mathbb{R} \) it has been shown that with the above notations the linking matrix

\[
\text{lk}(L_1 \cup L_0 \cup L_2) = \begin{pmatrix}
A & B^T & 0 & 0 \\
B & BA^{-1}B^T & -I & 0 \\
0 & -I & DC^{-1}DT & D \\
0 & 0 & DT & C
\end{pmatrix}
\] (3.8)

where \( A = \text{lk}(L_1) \in M_{|L_1| \times |L_1|}(\mathbb{Z}), \) \( C = \text{lk}(L_2) \in M_{|L_2| \times |L_2|}(\mathbb{Z}), \) \( B = \text{lk}(G'_1, L_1) \in M_{g_1 \times |L_1|}(\mathbb{Z}), \) \( D = \text{lk}(G_2, L_2) \in M_{g \times |L_2|}(\mathbb{Z}), \) \( BA^{-1}B^T, DC^{-1}DT \in M_{g \times g}(\mathbb{Z}). \) There it has been proven the following
Proposition 3.9. The signature of the matrix \( \sigma_1^1 + \sigma_2^2 + g, \sigma_1^2 + \sigma_2^1 + g \), where \((\sigma_1^1, \sigma_1^2)\), respectively \((\sigma_2^1, \sigma_2^2)\), is the signature of \( \text{lk}(L_1) \), respectively \( \text{lk}(L_2) \). Also the following holds:

\[
\det(\text{lk}(L_1 \cup L_0 \cup L_2)) = (-1)^g \cdot \det(\text{lk}(L_1)) \cdot \det(\text{lk}(L_2))
\]  

(3.9)

Therefore, we can re-write the gluing formula (3.9):

\[
\tau \leq N (M_2 \cup f_2 \circ (f_1)_{-1}, M_1, f_1, f'_2) = \ell \leq N (\tau \leq N (M_1, f_1, f'_1), \tau \leq N (M_2, f_2, f'_2))
\]  

(3.10)

Proof of Theorem 2.12. 1) By construction, the inverse limits \( \lim_{\infty \rightarrow N} A^{\leq N}(\emptyset) = A(\emptyset) \) and \( \lim_{\infty \rightarrow N} A^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) = A(\Gamma^{g_1}, \Gamma^{g_2}) \). Let us show that the following diagram is commutative for every \( N \in \mathbb{N} \):

\[
\begin{array}{ccc}
A^{\leq N+1}(\Gamma^{g_1}, \Gamma^{g_2}) \otimes A^{\leq N+1}(\Gamma^{g_2}, \Gamma^{g_3}) & \rightarrow & A^{\leq N}(\Gamma^{g_1}, \Gamma^{g_2}) \otimes A^{\leq N}(\Gamma^{g_2}, \Gamma^{g_3}) \\
\downarrow \ell \leq N+1 & & \downarrow \ell \leq N \\
A^{\leq N+1}(\Gamma^{g_1}, \Gamma^{g_3}) & \rightarrow & A^{\leq N}(\Gamma^{g_1}, \Gamma^{g_3})
\end{array}
\]  

(3.11)

where the horizontal arrows are the maps that forget the degrees \( N + 1 \) parts. Let \( \alpha = \tau \leq N+1(M_1) \), \( \beta = \tau \leq N+1(M_2) \) for some QHC \( M_1 \) and \( M_2 \). Then as previously observed (\( \tau \leq N+1(M_i) \))\( \leq N \) = \( \tau \leq N(M_i) \), \( i = 1, 2 \), i.e. \( \alpha \leq N = \tau \leq N(M_1) \), \( \beta \leq N = \tau \leq N(M_2) \). By the gluing formula (3.10) we then have \( \tau \leq N+1(M_2 \cup M_1) = \ell \leq N+1(\alpha, \beta) \) and \( \tau \leq N(M_2 \cup M_1) = \ell \leq N(\alpha \leq N, \beta \leq N) \). Again, using now \( \tau \leq N+1(M_2 \cup M_1) \)\( \leq N \) = \( \tau \leq N(M_2 \cup M_1) \), we get \( (\ell \leq N+1(\alpha, \beta)) \leq N = \ell \leq N(\alpha \leq N, \beta \leq N) \). Hence the diagram (3.11) is commutative for \( \alpha, \beta \) as above. By the remark after the proof of Theorem 3.2, and by Lemma 3.5, the diagram is then commutative for arbitrary \( \alpha, \beta \).

Therefore there exists a well-defined \( A(\emptyset) \)-bilinear map \( \ell : A(\Gamma^{g_1}, \Gamma^{g_2}) \otimes A(\Gamma^{g_2}, \Gamma^{g_3}) \rightarrow A(\Gamma^{g_1}, \Gamma^{g_2}) \otimes A(\Gamma^{g_2}, \Gamma^{g_3}) \rightarrow A(\Gamma^{g_1}, \Gamma^{g_3}) \), such that when restricting to the internal degree \( \leq N \) parts one obtains the map \( \ell \leq N \).

2) By the proof of 1), \( \ell(w_g) = \lim_{\infty \rightarrow N} \ell \leq N(w_g^N) \). By Proposition 2.11 the operators \( \ell \leq N(w_g^N) \) are identities, hence so is the limit. \( \lhd \)

3.8 The functors \( \Omega \rightarrow A \) and \( \mathfrak{3} \rightarrow A \). Theorem 2.12.1 shows that \( \ell \leq N \) are the \( \leq N \)-truncations of \( \ell \). With \( \mathfrak{3} \rightarrow A \), this implies Axiom (A2) for the non-truncated TQFTs \( \mathfrak{3} \rightarrow A \) and \( \Omega \rightarrow A \):

\[
\tau(M_2 \cup f_2 \circ (f_1)_{-1}, M_1, f_1, f'_2) = \ell(\tau(M_1, f_1, f'_1), \tau(M_2, f_2, f'_2))
\]  

(3.12)

Theorem 2.12.2) in particular implies Axiom (A3) for the non-truncated TQFTs.

3.9 Lemma. For any \( \beta \in A(\Gamma^{g_1}, \Gamma^{g_2}) \) and any sequence of elements \( \alpha^n \in A(\Gamma^{g_1}, \Gamma^{g_2}) \), such that for every \( n \) \( (\alpha^n)_{\leq N} = (\alpha^{n+1})_{\leq N} = \ldots \), both sides of the following equalities are well-defined and the equalities holds:

\[
\ell(\lim_{n} \alpha^n, \beta) = \lim_{n} \ell(\alpha^n, \beta)
\]  

(3.13)

A similar property holds for the role of two arguments of \( \ell \) reversed.

Proof. \( \ell \leq N \) are the \( \leq N \)-truncations of \( \ell \). Apply \( \mathfrak{3} \rightarrow A \) and pass to the limit (keeping, for example \( n = (2g+1)N \)). \( \lhd \)

Theorem 3.2 shows that our TQFTs are non-degenerate, and Lemmas 3.5 and 3.9 show that \( \ell \leq N \) and \( \ell \) are continuous maps.

3.10 Conjugation. The operation of conjugation in \( A(\emptyset) \) can be extended as follows. Grade the modules \( A(\emptyset) \), \( A(\Gamma^g) \) and \( A(\Gamma^{g_1}, \Gamma^{g_2}) \) by the internal degree, and define for an arbitrary chord diagram \( D \), and an arbitrary natural number \( k \):

\[
\frac{D^{[2k]}}{D^{[2k+1]}} = \frac{D^{[2k]}}{D^{[2k+1]}}
\]
Note that \( \mathcal{T}(\Sigma_g) = \mathcal{T}(-\Sigma_g) = \mathcal{A}(\Gamma^g) \), and \( \mathcal{T}(f) = \text{id}, \forall f \in \text{Homeo}(\Sigma_g) \), hence the naturality of this antimorphism is obvious. \( \tau : \mathcal{A}(\Gamma^g, \Gamma^{g_2}) \to \mathcal{A}(\Gamma^{g_2}, \Gamma^g) \) satisfies the requirement of axiom (A4) to commute with homeomorphisms of 3-cobordisms, because already \( \tau \) is defined for homeomorphism classes. The same can be repeated with added \( \leq N \).

Also \( \tilde{\epsilon}_N \mathcal{Z}(L, G) = (-1)^{[L|N]} \mathcal{Z}(L, G) \) (compare with [18 Proposition 5.2]). This is true for the Murakami-Ohtsuki extension of \( Z \) because \( a \) and \( b \) from \( \mathbb{Z} \), and hence \( Z \) (vicinity of a trivalent vertex) are “mirrors” of themselves, which is easy to check. For the extension of \( Z \) from 2.6 this property is obvious. From the proof of Proposition 5.2 in [18] it also follows that \( c_- = \tau c_+ \). Hence for any \( N \):

\[
\tau^{\leq N}(-M) = (-1)^{\tau^{\leq N}} \left( \frac{\tilde{\epsilon}_N(\mathcal{Z}(L, G))}{(\sigma_+^{\leq N})^\tau, (\sigma_-^{\leq N})^{\tau\sigma_-}} \right)^{[\leq N]} = (-1)^{\tau^{\leq N}} \left( \frac{\tilde{\epsilon}_N(\mathcal{Z}(L, G))}{(\sigma_+^{\leq N})^\tau, (\sigma_-^{\leq N})^{\tau\sigma_-}} \right)^{[\leq N]} = \tau^{\leq N}(M)
\]

(3.14)

Therefore also \( \tau(-M) = \tau(M) \). Using this formula and (3.12), it follows that for \( \alpha = \tau(M_1), \beta = \tau(M_2) \), where \( M_i \) are 3-cobordisms in our category, we have \( \ell(\alpha, \beta) = \ell(\tau(M_1), \tau(M_2)) = \ell(M_2 \cup M_1) = \tau((-M_1) \cup (-M_2)) = \ell(\tau(-M_1), \tau(-M_2)) = \ell(\tau(M_2), \tau(M_1)) = \ell(\beta, \pi) \). By Theorem 3.2 and (3.14), the same relation holds for arbitrary \( \alpha, \beta \). In particular, it remains true with \( \leq N \) added. Axiom (A4) is therefore verified for truncated and non-truncated TQFTs.

3.11 Conclusions and consequences. The full and truncated TQFTs are now completely constructed. The full TQFT induces a linear representation \( L_g \to GL_{\mathcal{A}(\emptyset)}(\mathcal{A}(\Gamma^g)) \). The truncated TQFTs induce linear representations \( L_g \to GL_{\mathcal{A}(\emptyset)}(\mathcal{A}(\Gamma^g)) \). It is known \( [9] \) that any ZHS can be obtained as filling of a parametrized 3-cobordism \( (\Sigma_g \times I, w, \text{id}) \) for some \( g \geq 0 \) and some \( w \in \mathcal{T}_g \), the Torelli group of genus \( g \). Furthermore \( [24] \) it even suffices to consider only \( w \in \mathcal{K}_g \), the kernel of the Johnson homomorphism, or topologically the subgroup of \( \mathcal{T}_g \) generated by Dehn twists on bounding simple closed curves. Our TQFTs, of course, induce linear representations of both these subgroups of \( L_g \). The group \( L_g \) has not been studied before, no explicit set of generators, less so one of relations, is known.

Note, that theorem 3.2 and Lemmas 3.5 and 3.9 not only allow a well-defined non-truncated TQFT (Theorem 2.12) and prove the non-degeneracy, but also solve the realization problem for links, string links, three-dimensional manifolds and chain graphs, by showing (see Lemmas 3.3 and 3.4) that \( Z(\text{links}), \tau(\text{3-manifolds}), Z(\text{string links}) \) and \( \tau(\text{3-manifolds with boundary}) \) in the closure generate the corresponding spaces of chord diagrams: \( \mathcal{A}(\bigcup \ldots \bigcup), \mathcal{A}(\emptyset) \) and \( \mathcal{A}(\bigcup \{g\}) \). (For links, the correspondent for Habegger-Masbaum result follows easily from Habiro’s calculus of claspers \([15]\).) Without proving Theorem 3.2 even partial results of this sort were hard to obtain, as we can exemplify by the following

**Proposition.** For every \( N \geq 0 \) and every \( \mathbb{Q} \)-homology handlebody \( (M, f_1, f_2) \), \( \tilde{\epsilon}^{\leq N}(\tau^{\leq N}(M, f_1, f_2)) \) sends the \( \mathcal{A}^{\leq N}(\emptyset) \)-submodule of \( \mathcal{A}^{\leq N}(\{g\}) \) generated by \( \exp(\alpha), \alpha \in a(\{g\})^{\leq N} \) to itself.

**Proof.** By Proposition 3.1.3 \( \tau^{\leq N}(M, f_1, f_2) \) is group-like. Observe that \( \Delta \) commutes with *, and repeating the argument from the proof of 3.1.3 for \( \tilde{\epsilon}_N \) in the definition of \( \ell^{\leq N} \), we can see that \( \tilde{\epsilon}^{\leq N}(\tau^{\leq N}(M, f_1, f_2)) \) takes a group-like element of \( \mathcal{A}^{\leq N}(\Gamma^g) \) of the form \( 1 + \text{h.o.t.} \) to a group-like element of \( \mathcal{A}^{\leq N}(\Gamma^g) \) of the form \( 1 + \text{h.o.t.} \) Now apply Proposition 2.4.5 and 3) for the truncated case. Hence it sends the \( \mathcal{A}^{\leq N}(\emptyset) \)-submodule of \( \mathcal{A}^{\leq N}(\{g\}) \) generated by \( \exp(\alpha), \alpha \in a(\{g\})^{\leq N} \) to itself.

**Remark.** This construction of TQFT can be done also in the language of the Aarhus integral \( \mathfrak{B} \).

3.12 Chord-handle canceling. In [18] Le, Murakami and Ohtsuki have introduced the chord-KII move to mirror the second Kirby move for links, which then allowed them to define \( Z^{\text{LMO}} \). However, it is well-known that handle canceling can not be obtained solely by Kirby-2, and would require in addition Kirby-1. But no corresponding chord-KI move exists, the invariance of \( Z^{\text{LMO}} \) under Kirby-1 is achieved via normalization. Therefore there is no a priori reason to suspect that a chord-canceling-handle relation is true for arbitrary chord diagrams. But, the result obtained here above allow us to prove:

**Proposition.** The chord-canceling-handle relation, schematically depicted in figure [18] holds for arbitrary \( \beta \in \mathcal{A}(\{g\}) \). (The upper part of each \( F_i \) should be read as \( Z(\text{drawn tangle}) \).)

**Proof.** For arbitrary \( \beta, F_1 \) differs from \( F_2 \) by a chord-KII move. (An argument similar to the one in [18] Proposition 3.2| works.) But now \( F_2 = \ell(\beta, w_g) = \beta = F_3 \). \( \square \)
4 A TQFT for the Casson-Walker-Lescop invariant

The term of degree one of \( Z^{LMO} \) of a 3-manifold is \((-1)^{b_1(M)} \lambda(M) \theta \), where \( b_1(M) \) is the first Betti number, \( \lambda(M) \) is the Casson invariant (in Walker-Lescop extension) and \( \theta \) is the (only) open chord diagram of degree 1, which looks like \( \theta \). Let us recall the definition and basic properties of Casson invariant. Let \( K \) be a knot in an oriented \( \mathbb{Z} \)-homology 3-sphere \( M \), and \( \Delta_K(t) = a_0 + a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) + \ldots \) be its Alexander polynomial normalized such that \( \Delta_K(1) = 1 \). Denote \( \lambda'(K) = \frac{1}{2} \Delta''(1) = \sum_n n^2 a_n \).

4.1 Theorem (Casson). There is an integer-valued invariant \( \lambda \) for oriented integer homology 3-spheres such that:

1. \( \lambda \mod 2 \) is the Rohlin invariant
2. \( \lambda(M) = 0 \) for any homotopy 3-sphere
3. \( \lambda(-M) = -\lambda(M) \)
4. \( \lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2) \)
5. If \( K \) is a knot in an oriented integer homology 3-sphere \( M \), and \( M(K, \frac{1}{2}) \) denotes the integer homology 3-sphere obtained from \( M \) by a \( \frac{1}{2} \)-surgery on \( K \), then \( \lambda(M(K, \frac{1}{2})) = \lambda(M) + n\lambda'(K) \).

Property (5) from this theorem for \( n = \pm 1 \) with \( \lambda(S^3) = 0 \) determine \( \lambda \) uniquely, since any integer homology 3-sphere can be obtained from \( S^3 \) by a succession of \( \pm 1 \)-surgeries on knots. \( \lambda \) was extended to rational homology 3-spheres by Walker, and corresponding properties (4) and (5) were given by Lescop [19].

\[ (4') \quad \lambda(M_1 \# M_2) = |H_1(M_2, \mathbb{Z})| \lambda(M_1) + |H_1(M_1, \mathbb{Z})| \lambda(M_2) \]

\[ (5') \quad \lambda(M(L, \frac{p_1}{q_1}, \ldots, \frac{p_d}{q_d})) = \left|H_1(M, \mathbb{Z})\right| \lambda(M) + F_M(L, \frac{p_1}{q_1}, \ldots, \frac{p_d}{q_d}) \]

where \( M(L, \frac{p_1}{q_1}, \ldots, \frac{p_d}{q_d}) \) is the manifold obtained from \( M \) by performing rational surgery with indicated coefficients on the components of the link \( L \), and \( F_M(L, \frac{p_1}{q_1}, \ldots, \frac{p_d}{q_d}) \) is a certain function on the set of surgery presentations in \( M \), which is essentially a function of the linking matrix, homology and Alexander polynomial [19].

4.2 The degree \( N \leq 1 \) truncation of our TQFT can be thought of as a TQFT for the Casson-Walker-Lescop invariant. Note that the ring \( A^{\leq 1}(\theta) = \langle \{r + s\theta \mid r, s \in \mathbb{Q}, |r|, |s| \leq 1 \} \rangle \cdot \mathbb{Q} \) \((\theta^2) \cong \langle \mathbb{Q}[\theta]/(\theta^2) \rangle \cong \{ \begin{pmatrix} r & s \\ 0 & r \end{pmatrix} \mid r, s \in \mathbb{Q} \} \)

Denote it by \( R \). Observe [19] that \( c_+ = 1 - \frac{1}{10} + \text{h.o.t.}, \quad c_- = 1 + \frac{1}{10} + \text{h.o.t.} \), hence \( c_+ c_- = 1 + \text{terms of degree } \geq 2 \), and therefore \( z_g = (-1)^{g-1} Z(T_g) \otimes (\nu^{1/2})^\otimes 2g \). If \( \alpha \in A^{\leq 1}(\Gamma^9, \Gamma^{9g}) \) and \( \beta \in A^{\leq 1}(\Gamma^{9g}, \Gamma^{9g-3}) \), then:

\[ \ell_{\text{Casson}}(\alpha, \beta) = (\ell_1(\alpha \ast z_g^{-1} \ast \beta))^{[\leq 1]} \]

(4.1)

If \( g = 0 \), \( \ell_{\text{Casson}} \) is the disjoint union (multiplication). A formula for \( Z(T_1) \) and \( Z(W_1) \) is given for example in [14]. Using the even associator it is easy to write down \( z_g^{-1} \) and \( w_1 \) explicitly, at least the degree \( \leq 3 \) terms.
Let \( \kappa(L, G_1, G_2) = (M, f_1, f_2) \). Then, keeping in mind that \( c_+ c_- = 1 \) terms of degree \( \geq 2 \), denote
\[
c(M, f_1, f_2) := \tau^{\leq 1}(M, f_1, f_2) = \frac{(-1)^{\sigma_+}}{d(M)} \bar{a}(\tilde{Z}(L, G_1, G_2))[\leq 1]
\]
where \( d(M) = |H_1(\tilde{M}, \mathbb{Z})| \) and \( \sigma_+ = \text{sign}_+(L) \), is an invariant of 3-cobordisms of category \( \Omega \). In particular, for cobordisms between \( S^2 \) and \( S^2 \), \( c(M, \text{id}_{S^2}, \text{id}_{S^2}) = \frac{(-1)^{\sigma_+}}{d(M)} \bar{a}(\tilde{Z}(L))[\leq 1] = Z^{\text{LMO}}(M)[\leq 1] = 1 + \frac{\lambda(M)}{2} \theta \), where we have identified \( \mathcal{A}^{\leq 1}(\Gamma^0, \Gamma^0) \equiv \mathbb{R} = \mathcal{A}^{\leq 1}(0) \). The filling of the composition of two 3-cobordisms between \( S^2 \) and \( S^2 \) is clearly the connected sum of the fillings. Hence \( c(M_2 \cup M_1, S^2, S^2) = c(M_1, S^2, S^2)c(M_2, S^2, S^2) \) implies, as it is easy to check, property (4) of the Casson invariant (the generalized version for \( \mathbb{Q} \)-HS). As we have shown the following axioms of TQFT hold:
\[
c(M_2 \cup f_2 c(f_1)^{-1} M_1, f_1, f_2') = \ell_{\text{Casson}}(c(M_1, f_1, f_1'), c(M_2, f_2, f_2')) \\
c(\Sigma_g \times [0, 1], p_1, p_2) = \text{id}_{\mathcal{A}^{\leq 1}(\Gamma^g)} \\
c(-M, -f_2, -f_1) = \frac{c(M_1, f_1, f_2)}{c(M_2, f_2, f_2')}
\]
where the notations are obvious. \( R, \mathcal{A}^{\leq 1}(\Gamma^0, \Gamma^g) \) and \( \mathcal{A}^{\leq 1}(\Gamma^g, \Gamma^{g_2}) \) are \( \mathbb{Z}_2 \)-graded by the internal degree; the conjugation changes the sign of the internal degree 1 part. In particular (4.3) implies property (3) of the CWL invariant. It is natural to try now to obtain property (5) of the CWL invariant as a consequence of the rational surgery formula of Bar-Natan and Lawrence.

Unfortunately, explicit calculations for \( c(M, f_1, f_2) \), as expected, are rather hard to do. Now we would like to show that the induced representation \( L_g \to GL(A^{\leq 1}(\Gamma^g)) \) descends to Morita’s homomorphism \( \lambda^* : K_g \to \mathbb{Z} \). (\( \lambda^* \) extends to \( L_g \), but fails to be a homomorphism there.)

4.3 Proposition. 1) Let \( \mathcal{B} \) be the completion of the \( \mathbb{Q} \)-vector subspace of \( \mathcal{A}(\Gamma^g, \Gamma^{g_1} \cup (\bigcup_m S^1) \cup \Gamma^{g_2}) \) generated by finite sums of chord diagrams which intersect \( \Gamma^{g_1} \cup \Gamma^{g_2} \). Then \( p : \mathcal{A}(\Gamma^g, \Gamma^{g_1} \cup (\bigcup_m S^1) \cup \Gamma^{g_2}) \to \mathcal{A}(\bigcup_m S^1), \) the natural map "erase \( \Gamma^g \) and \( \Gamma^{g_2} \) from a chord diagram", if it does not intersect \( \Gamma^{g_1} \cup \Gamma^{g_2} \), and set \( \text{set} = 0 \), otherwise, is well-defined and the following sequence is short exact:
\[
0 \to \mathcal{B} \to \mathcal{A}(\Gamma^g, \Gamma^{g_1} \cup (\bigcup_m S^1) \cup \Gamma^{g_2}) \xrightarrow{p} \mathcal{A}(\bigcup_m S^1) \to 0
\]
We will denote also by \( p \) the induced maps on minimal internal degree \( \leq N \) parts. They have the same property.

2) Denote by \( r \) the maps similar to \( p \) from 1) corresponding to the case \( m = 0 \). Then the following diagram is commutative:
\[
\begin{array}{ccc}
\mathcal{A}(\Gamma^g, \Gamma^{g_1} \cup (\bigcup_m S^1) \cup \Gamma^{g_2}) & \xrightarrow{\text{N}} & \mathcal{A}(\Gamma^g, \Gamma^{g_1} \cup \Gamma^{g_2}) \\
\downarrow p & & \downarrow r \\
\mathcal{A}(\bigcup_m S^1) & \xrightarrow{\text{N}} & \mathcal{A}(0)
\end{array}
\]
3) For every embedding \( L \cup G \hookrightarrow S^3 \), such that \( G \) as an abstract graph is \( \Gamma^{g_1} \cup \Gamma^{g_2} \), \( p \bar{Z}(L \cup G) = \bar{Z}(L) \).
4) For every embedding \( L \cup G \hookrightarrow S^3 \), such that \( G \) as an abstract graph is \( \Gamma^{g_1} \cup \Gamma^{g_2} \), and every \( N \geq 1 \), \( p(\tau^{\leq N}(\kappa(L \cup G))) = Z^{\text{LMO}}(\kappa(L \cup G))[\leq N] \). In particular (if \( N = 1 \), \( p(c(M, f_1, f_2)) = 1 + \frac{\lambda(M)}{2} \theta \).
5) If \( \varphi_1, \varphi_2 \in K_g \), then \( p(c(\Sigma_g \times I, \varphi_2 \circ \varphi_1, id)) = p(c(\Sigma_g \times I, \varphi_1, id))p(c(\Sigma_g \times I, \varphi_2, id)) \).

Proof. 1) The following argument can be worked for every fixed degree, and since all relations are homogeneous, we can use the universality property of the direct product as mentioned in 2.2 to obtain the desired statement. Consider the corresponding diagram before introducing relations:
\[
0 \to \mathcal{B} \to \mathcal{D}(\Gamma^g, \Gamma^{g_1} \cup (\bigcup_m S^1) \cup \Gamma^{g_2}) \xrightarrow{p} \mathcal{D}(\bigcup_m S^1) \to 0
\]
The terms of any relation for diagrams on \( \Gamma^{g_1} \cup (\bigcup_m S^1) \cup \Gamma^{g_2} \), either all intersect \( \Gamma^{g_1} \cup \Gamma^{g_2} \), or none does. Hence, if we denote by \( R_1 \) the \( \mathbb{Q} \)-vector space generated by relations of the first type, by \( R_2 \) - the space generated by relation of the second type, and by \( R \) - the one generated by all relations, then \( R/R_1 \cong R_2 \). All in all we get a diagram:
where all columns and the first two rows are short exact. The arrows $i$ and $p$ in the third row are then induced and make the diagram commutative. They clearly are the maps described in the statement. The exactness in the third row follows from the exactness in the second.

2) Let $\alpha \in \mathcal{A}(\Gamma^{g_1} \sqcup (\sqcup_m \mathcal{S}^1) \sqcup \Gamma^{g_2})$ and $\beta$ be such that $\tilde{\varphi}(\beta) = \alpha$. (Recall that $\tilde{\varphi}^{-1} = q_N \circ \kappa_N \circ \tilde{\varphi}^{-1}$.) A chord diagram $x$ from the expression of $\beta$ connects to $\Gamma^{g_1} \sqcup \Gamma^{g_2}$ if and only if its image via $\varphi$ is a sum $y$ of chord diagrams expressing $\alpha$, all connected to $\Gamma^{g_1} \sqcup \Gamma^{g_2}$. Again using the fact that the terms in any relation either all connect or all do not, $p(y) = 0$ implies that (in fact, if and only if) $q_N \circ \kappa_N(x)$ connects to $\Gamma^{g_1} \sqcup \Gamma^{g_2}$, i.e. $r(q_N \circ \kappa_N \circ \tilde{\varphi}^{-1}(y)) = r(q_N \circ \kappa_N(x)) = 0$.

Now, if we decompose $\beta = \beta_1 + \beta_2$ such that all terms in $\beta_1$ connect to $\Gamma^{g_1} \sqcup \Gamma^{g_2}$ and all terms in $\beta_2$ do not, the result follows: $(r \circ \tilde{\varphi}(\alpha)) = (r(\varphi(\beta_1) + \varphi(\beta_2))) = r(q_N \circ \kappa_N(x_1) + q_N \circ \kappa_N(x_2)) = q_N \circ \kappa_N(x_2) = \tilde{\varphi}(\beta_2) = \tilde{\varphi}(p(\alpha))$.

3) Decompose $L \cup G$ into elementary pseudo-quasi-tangles. Observe that for everyone, except $\big/\big\slash$’s and $\big\backslash\big/\big\backslash$’s (possibly with multiple strands), $Z$ either returns diagrams either all in $\mathcal{B}$, or all having no intersection between the dashed graph and $\Gamma^{g_1} \sqcup \Gamma^{g_2}$. Thus, suppressing $G$ for these elementary tangles corresponds precisely to applying $p$.

The remaining cases. Observe, first, that one can “lift $L$ above $G$”, leaving only some ”fingers” from $L$ attached to $G$. To see this, from a generic plane projection of $L \cup G$ on $\mathbb{R}^2 \subset \mathbb{R}^3$ obtain an isotopic embedding of $G \cup L$ in $\mathbb{R}^3$, such that $G$ is in an $\varepsilon$-neighbourhood of the plane $\{z = 0\} \subset \mathbb{R}^3$, and $L$, except for some fingers that correspond to intersections between $G$ and $L$ in the original plane projection, lies in an $\varepsilon$-neighbourhood of the plane $\{z = 1\} \subset \mathbb{R}^3$. Hence, by ”opening the two-page book”, we can find such a tangle decomposition that all occurring associator-tangles are of one of the following three types:

(A) refer only to $G$ or only to $L$.

(B) a single middle strand, which comes from $L$, the left-most strand (with ”big” multiplicity) comes from $G$, the right-most strand (also with ”big” multiplicity) comes from $L$. Moreover, if such an associator-tangle occurs, its inverse (on the same strands) will occur ”soon”;

(C) one of the left-most two strands is a single strand coming from $L$, all other strands come from $G$.

We will assume that he associator $\Phi$ is horizontal, i.e. it is a formal series in two non-commuting variables $r_{12}, r_{23}$, which correspond to a dashed line joining these indicated strands.

(A) If all strands are from $G$ or none are from $G$, then all terms of $\Phi_{\pm 1} = Z(\text{tangle})$, connect, respectively do not connect to $\Gamma^{g_1} \sqcup \Gamma^{g_2}$.

(C) If two of the three strands come from $G$, $\Phi_{\pm 1} = Z(\text{tangle})$ will have all terms connected to $\Gamma^{g_1} \sqcup \Gamma^{g_2}$, then, eliminating $G$ corresponds precisely to replacing this tangle-associator by the single strand from $L$, i.e. corresponds to applying $p$ in this case.

(B) If exactly one (multiple) strand comes from $G$, this corresponds to setting one of the two non-commutative variables $r_{12}, r_{23}$ zero. But, as we mentioned above, such tangles occur in pairs with their opposite. Then, both $\Phi$ and $\Phi_{\pm 2} = \Phi^{-1}$ occur. Setting one of $r_{12}, r_{23}$ zero, still leaves a series and its inverse (elementary exercise). Thus, eliminating $G$ corresponds again to applying $p$.

4) Recall the definitions of $r \leq N$ and $Z_{\text{LMO}}$. Apply $p$ and use the result of part 3. Then, use the commutativity of the diagram from part 2) to obtain the desired relation.

5) Applying $p$ to $\Phi$, $p(c(\Sigma_g \times I, \varphi_2 \cdot \varphi_1, \text{id})) = p(\tilde{\varphi}(c(\Sigma_g \times I, \varphi_1, \text{id})))$. Using part 4), $p(c(\Sigma_g \times I, \varphi_2 \cdot \varphi_1, \text{id})) = 1 + \frac{\lambda(W_{\varphi_2} + \varphi_1)}{2} \theta$, $p(c(\Sigma_g \times I, \varphi_1, \text{id})) = 1 + \frac{\lambda(W_{\varphi_1})}{2} \theta$, and $p(c(\Sigma_g \times I, \varphi_2, \text{id})) = 1 + \frac{\lambda(W_{\varphi_2})}{2} \theta$.

where $W_{\varphi_1} = (\Sigma_g \times I, \varphi_1, \text{id})$. But $1 + \frac{\lambda(W_{\varphi_2} + \varphi_1)}{2} \theta = \left(1 + \frac{\lambda(W_{\varphi_2})}{2} \theta\right) \left(1 + \frac{\lambda(W_{\varphi_1})}{2} \theta\right)$ in $R$, because $\lambda^*: \mathcal{K}_g \to \mathbb{Z}$, $\lambda^*(\varphi) := \lambda(W_{\varphi})$ is a satisfactory $\lambda^*(\varphi_2 \cdot \varphi_1) = \lambda(\varphi_1) + \lambda(\varphi_2)$ by 24.

$\square$
Remark. Using expression (4.11) for ℓ and observing that p commutes with i₁ and with taking [≤ 1] by Proposition 4.3.2, we can re-write \( p(\ell_{\text{Casson}}(c(\Sigma_g \times I, \phi_1, id)) \circ \ell_{\text{Casson}}(c(\Sigma_g \times I, \phi_2, id))) \) as \( \iota_1(p(c(\Sigma_g \times I, \phi_1, id) \ast z_1^g \ast c(\Sigma_g \times I, \phi_2, id))) \). Expressing \( c(\Sigma_g \times I, \phi, id) \) by \( c(\Sigma_g \times I, \phi, id) \), and keeping in mind the definition of \( \ell \) and properties of \( \iota_1 \), we can get to having to apply \( p \) on \( (\tilde{Z}(L_1, G_1, G_1') \ast z_1^g \ast \tilde{Z}(L_2, G_2, G_2')) \), respectively to apply \( p \) on \( (\tilde{Z}(L_1, G_1, G_1') \ast z_1^g \ast \tilde{Z}(L_2, G_2, G_2')) = p(\tilde{Z}(L_1, G_1, G_1') \ast z_1^g \ast \tilde{Z}(L_2, G_2, G_2')) \), for suitably chosen \( L_i \) in the triplets. This gives another proof of Proposition 4.3.5). We, thus, can obtain a proof of the fact that \( \lambda^* : K_g \rightarrow \mathbb{Z} \) is a homomorphism, using the Kontsevich integral.

References

[1] M. Atiyah, Topological Quantum Field Theories, Publications Mathématiques IHES 68, 175-186 (1988)
[2] D. Bar-Natan, On the Vassiliev knot invariants, Topology 34, 423-472 (1995)
[3] D. Bar-Natan, R. Lawrence, A rational surgery formula for the LMO invariant, arXiv math.GT/0007045 (2000)
[4] D. Bar-Natan, T. Le, D. Thurston, Two applications of elementary knot theory to Lie algebras and Vassiliev invariants, Topology 37, 1-31 (2003)
[5] C. Blanchet, H. Habegger, G. Masbaum, P. Vogel, Topological quantum field theories derived from the Kauffman bracket, Topology 34, no.4, 883-927 (1995)
[6] D. Cheptea, T. Le, 3-cobordisms with their rational homology on the boundary, preprint
[7] D. Cheptea, G. Massuyeau, Tangles, cobordisms, and their LMO-type invariants, in preparation
[8] D. Cheptea, Universal quantum invariants and the induced representation of the Torelli group, in preparation
[9] A. Fomenko, S. Matveev, Algorithmic and computer methods for three-manifolds, Kluwer Academic Publishers (1997)
[10] C. Gille, On the Le-Murakami-Ohtsuki invariant in degree 2 for several classes of 3-manifolds, J Knot Theory Ramifications 12 (1), 17-45 (2003)
[11] R. E. Gompf, A. I. Stipsicz, 4-manifolds and Kirby calculus, Graduate Studies in Mathematics 20, AMS (1999)
[12] N. Habegger, G. Masbaum, The Kontsevich integral and Milnor’s invariants, Topology 39, 1253-1289 (2000)
[13] C. Blanchet, H. Habegger, K. Orr, Finite type three manifold invariants -realization and vanishing, J Knot Theory Ramifications 8 (8), 1001-1007 (1999)
[14] C. Blanchet, Milnor link invariants and quantum 3-manifold invariants, Comment. Math. Helv. 74, no.2, 322-344 (1999)
[15] K. Habiro, Claspers and finite-type invariants of links, Geometry and Topology 4, 1-83 (2000)
[16] T. T. Q. Le, An invariant of integral 3-spheres which is universal for all finite type invariants, AMS Translation series 2, 179, 75-100 (1997)
[17] T. T. Q. Le, The LMO invariant, “Invariants de noeuds et de variétés de dimension 3”, École d’été de Mathématiques, Institut Fourier, Grenoble (1999)
[18] T. T. Q. Le, J. Murakami, T. Ohtsuki, On a universal perturbative invariant of 3-manifolds, Topology 37, no.3, 539-574 (1998)
[19] C. Lescop, Global surgery formula for the Casson-Walker invariant, Princeton University Press (1996)
[20] V. Turaev, Quantum Invariants of Knots and 3-Manifolds, Walter de Gruyter (1994)
[21] P. Vogel, Invariants de type fini, en “Nouveaux Invariants en Géométrie et en Topologie”, publié par D. Bennequin, M. Audin, J. Morgan, P. Vogel, Panoramas et Synthèses 11, Société Mathématique de France, 99-128 (2001)