LONG-TERM REGULARITY OF THE PERIODIC EULER–POISSON SYSTEM FOR ELECTRONS IN 2D

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Abstract. We study a basic plasma physics model—the one-fluid Euler-Poisson system on the square torus, in which a compressible electron fluid flows under its own electrostatic field. In this paper we prove long-term regularity of periodic solutions of this system in 2 spatial dimensions.

Our main conclusion is that on a square torus of side length $R$, if the initial data is sufficiently close to a constant solution, then the solution is wellposed for a time at least $R/\epsilon^2 (\log R)^{O(1)}$, where $\epsilon$ is the size of the initial data.

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1. Introduction

1.1. Derivation of the equation. A plasma is a collection of charged particles interacting with each other via the coulomb forces. The plasma is the most ubiquitous form of matter in the universe, from heavenly bodies such as interstellar hydrogen and the interior of stars, to terrestrial objects like fluorescent tubes and neon signs. In addition, recent advances in controlled nuclear fusion requires a better understanding of the behavior of a plasma confined to a bounded region, for example, a tokamak fusion reactor, which resembles a torus in shape. We refer the interested reader to [4] and [7] for physics reference in book form.

The Euler–Poisson system describes the motion of a nonrelativistic warm adiabatic plasma consisting of electrons and ions. We assume the following in the derivation of the system.

- The plasma is nonrelativistic, so its dynamics follows Newton’s laws. Also, the main interaction between the ions is via the electrostatic field, which obeys the Poisson equation; the magnetic interaction is much smaller and can be neglected.
- The plasma is warm, so we need to consider the thermal pressure of the electrons, arising from the temperature of the plasma. The relation among the density, temperature, and pressure of the plasma satisfies the constitutive equation.
• The plasma is *adiabatic*. This means no heat flow within the plasma. The temperature of the plasma, though, will still vary with time. This is because the plasma is compressible, and when it is compressed, the mechanical work done on it is converted to thermal energy, so the plasma will heat up, and vice versa.

• The plasma consists of free electrons and ions. As ions are much heavier than electrons (for example, \(m_p = 1836 m_e\)) they move much more slowly than electrons, so we are mainly interested in the motion of the *electrons*. Hence we can model the motion of the plasma by a single fluid obeying the **compressible Euler equation**.

The above leads to the **Euler–Poisson one-fluid** model of the plasma. Let \(e\) denote the elementary charge, and \(m\) the mass of the electron. They are fundamental physical constants whose values are fixed throughout. The dynamical variables are the electron density \(n\), velocity \(v\), pressure \(p\), and the electrostatic field \(\phi\) the electrons produce.

By the conservation of charge, \(n\) satisfies the **continuity equation**

\[
  n_t + \nabla \cdot (nv) = 0.
\]  
(1.1)

The motion of the electrons satisfies Newton’s second law

\[
  nm D_t v = F.
\]  
(1.2)

On the left hand side of (1.2) is the material derivative of \(v\)

\[
  D_t v = v_t + (v \cdot \nabla) v,
\]

which equals the acceleration of an electron at a given point. On the right hand side of (1.2) is the net force acting on that electron

\[
  F = -\nabla p + (-ne)(-\nabla \phi) = -\nabla p + en\nabla \phi
\]

where \(p\) is the thermal pressure of the electrons, and \(\phi\) is the electrostatic potential they produce. Thus we obtain the **Euler equation** for the plasma:

\[
  nm(v_t + (v \cdot \nabla) v) = -\nabla p + en\nabla \phi.
\]  
(1.3)

The electrostatic potential \(\phi\) is related to \(n\) by the **Poisson equation**

\[
  \epsilon_0 \Delta \phi = e(n - n_0)
\]  
(1.4)

where \(\epsilon_0\) is the vacuum permittivity, and \(n_0\) is the charge density of the nuclei.

To close the system we need to find the pressure \(p\). The ideal gas law says

\[
  p = nk_B T
\]  
(1.5)

where \(k_B\) is the Boltzmann constant, and \(T\) is also a function of \(n\). The work required to compress the plasma is \(-pdV\). By conservation of energy, this work is converted to the thermal energy of the plasma. Hence

\[
  cnVdT = dQ = -pdV
\]  
(1.6)

where \(c\) is the heat capacity of an electron. By the equipartition theorem, \(c = dk_B/2\), where \(d\) counts the degree of freedom of the electrons. Putting this and (1.5) into (1.6) we get

\[
  \frac{d}{2} VdT = -TdV
\]
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which one can integrate to obtain \( \ln V + (d/2) \ln T = \text{const.} \)
\( \) or \( VT^{d/2} = \text{const.} \), so \( T \propto V^{-2/d} \propto n^{2/d}. \)
\( \) Again using (1.5) we get
\[
p \propto n^{1+d/2}.
\]
Electrons have no internal degrees of freedom, so \( d \) is simply the number of spatial dimensions. In our case \( d = 2 \), and the constitutive equation reads
\[
p = \theta n^2/2, \tag{1.7}
\]
where \( \theta = 2k_B T_0/n_0 \) depends on the initial condition of the plasma. We can use (1.7) to eliminate the pressure \( p \) in (1.3). For simplicity we assume \( \theta \) doesn’t depend on the position. Then we obtain
\[
n_t + \nabla \cdot (nv) = 0,
\]
\[
m(v_t + (v \cdot \nabla)v) + \theta \nabla n = e \nabla \phi,
\]
\[
\epsilon_0 \Delta \phi = e(n - n_0). \tag{1.8}
\]

1.2. Conservation laws. The continuity equation (1.1) implies the conservation of charge
\[
\frac{d}{dt} \int n = -\int -\nabla \cdot (nv) = 0. \tag{1.9}
\]
Also conserved is the energy
\[
E = \frac{1}{2} \int mn v^2 + \epsilon_0 (\nabla \phi)^2 + \theta n^2, \tag{1.10}
\]
where the three terms on the right correspond to the kinetic energy, electrostatic energy and thermal energy of the plasma respectively.

Next we look at the evolution for the vorticity \( \omega = \nabla \times v \). Taking the curl of the second equation in (1.8) we get its evolution equation
\[
\omega_t = -(v \cdot \nabla)\omega - \omega(\nabla \cdot v).
\]
If the flow is assumed to be irrotational (i.e., \( \omega = 0 \)) at the beginning, then it remains so forever.

Under the assumption of irrotationality, we have \((v \cdot \nabla)v = \nabla(|v|^2)/2\). Then the second equation in (1.8) shows that \( v_t \) is a gradient. Integrating over the whole torus gives the conservation of momentum
\[
\frac{d}{dt} \int v = 0. \tag{1.11}
\]

1.3. Normalization. We can rescale the variables to normalize all the constants in the system (1.8) to 1. To do so, we first list the dimensions of all the physical constants:

| Constant | \( m \) | \( e \) | \( n_0 \) | \( \theta \) | \( \epsilon_0 \) |
|----------|---------|--------|--------|--------|--------|
| Dimension | \( MN^{-1} \) | \( CN^{-1} \) | \( NL^{-3} \) | \( L^5T^{-2}MN^{-2} \) | \( L^{-3}T^2M^{-1}C^2 \) |

where \( L, T, N, M \) and \( C \) stand for the dimensions of length, time, number (of electrons), mass and charge, respectively. Then we list the dimensions of all the physical variables, and the
substitution that makes them dimensionless.

| Variable | x       | t       | n       | v       | φ       |
|----------|---------|---------|---------|---------|---------|
| Dimension| L       | T       | NL$^{-3}$| LT$^{-1}$| L$^2$T$^{-2}$MC$^{-1}$ |
| Substitution | $\sqrt{\frac{\epsilon_0 \theta}{e^2}} x'$ | $\sqrt{\frac{\epsilon_0 m}{e^2 n_0}} t'$ | $n_0(1 + \rho)$ | $\sqrt{\frac{n_0 \theta}{m}} v'$ | $\frac{n_0 \theta}{e} \phi'$ |

Now all the constants are normalized to 1, and the system (1.8) becomes

\begin{align*}
\rho_t + \nabla \cdot ((1 + \rho)v) &= 0, \\
v_t + (v \cdot \nabla)v + \nabla \rho &= \nabla \phi, \\
\Delta \phi &= \rho,
\end{align*}

(1.12)

**Remark 1.1.** There is no more scaling symmetry to be exploited. Hence $R$ is a genuine parameter of the system, and our results depend on $R$ explicitly.

The quantity $X_0 = \sqrt{\epsilon_0 \theta / e^2}$ is called the **Debye length**. It is the length scale beyond which local fluctuation in charge density (e.g., near the boundary of the container) does not have a significant effect. Hence we assume

$$R = \text{size of torus} / X_0 \gg 1.$$  

We also assume the plasma is **charge-neutral**, i.e.,

$$\int \rho = 0$$

so that the third equation in (1.12) is solvable on the torus. By a change of reference frame ($\bar{v} = v - v_0, \bar{\rho}(x, t) = \rho(x + v_0t, t)$), we can also assume the **zero momentum condition**

$$\int v = 0.$$  

By (1.11), this condition persists for all $t$.

The trivial solution $(\rho, v) = (0, 0)$ is an equilibrium of the system (1.12). Our main results in this paper are the long-term stability of this equilibrium.

Near this equilibrium the system (1.12) linearizes to

\begin{align*}
\rho_t &= -\nabla \cdot v, \\
v_t &= -\nabla \rho + \nabla \phi = -\nabla \rho - \nabla |\nabla|^{-2} \rho = -\nabla (1 + |\nabla|^2) |\nabla|^{-2} \rho,
\end{align*}

(1.13)

which can then be written in matrix form as

$$\frac{d}{dt} \begin{pmatrix} \rho \\ v \end{pmatrix} = \begin{pmatrix} 0 & -\nabla \cdot v \\ -\nabla (1 + |\nabla|^2) |\nabla|^{-2} & 0 \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}$$

and the eigenvalues of the matrix on the right hand side are (formally)

$$\pm \sqrt{\nabla \cdot \nabla (1 + |\nabla|^2) |\nabla|^{-2}} = \pm i \sqrt{1 + |\nabla|^2}.$$

**Definition 1.2.** We define the operator $\Lambda$ using the recipe

$$\mathcal{F}(\Lambda u)(\xi) = \sqrt{1 + |\xi|^2} \mathcal{F}(u),$$

where $\mathcal{F}$ denotes the Fourier transform.
1.4. **The main theorem.** To state our main results we need to introduce some function spaces. All functions and integrals are on the torus \((\mathbb{R}/\mathbb{Z})^2\) unless stated otherwise.

Let \(\varphi\) be a smooth cutoff function that is 1 on \(B(0,2/3)\) and vanishes outside \(B(0,3/2)\). Let \(\varphi_j = \varphi(x/2^j) - \varphi(x/2^{j-1})\), \(\varphi_{\leq j} = \varphi(x/2^j)\).

Let \(P_k\) be the Littlewood-Paley projection onto frequency \(2^k\), so that
\[
\mathcal{F}(P_k u) = \varphi_k \mathcal{F} u, \quad \mathcal{F}(P_{\leq k} u) = \varphi_{\leq k} \mathcal{F} u.
\]

For \(j \geq 1\) let \(Q_j\) be the physical localization at scale \(\approx 2^j\), that is, multiplication by \(\varphi_j\). Let \(Q_0 = id - \sum_{j \geq 1} Q_j\).

Now we can define the norms.

**Definition 1.3.** Define \(||x|| = d(x, (\mathbb{R}/\mathbb{Z})^2)\). Fix \(M \geq 10\) and define
\[
||u||_X = \sum_{k \in \mathbb{Z}} 2^{Mk} ||P_k u||_{L^\infty}, \quad ||u||_Z = ||(1+||x||)^{2/3} A^{M+2} u||_{L^2}.
\]

One can think of the \(X\) norm as \(W^{M,\infty}\), and the \(Z\) norm as \(W^{M+2,1,2+}\).

**Lemma 1.4.** (i) For \(k \in \mathbb{Z}\) we have
\[
||P_k u||_{W^{M+2,1,2+}} \lesssim ||u||_Z,
\]
where \(a^+\) denotes an exponent larger than \(a\) but can be arbitrarily close to \(a\).

(ii) \[
||u||_X \lesssim \sum_{k \in \mathbb{Z}} 2^{k^+Mk} ||P_k u||_{L^2} \lesssim ||u||_{H^{M+2}}, \quad ||u||_Z \lesssim R^{2/3} ||u||_{H^{M+2}}.
\]

(iii) If \(m \geq 0\) and \(T\) is a differential operator whose symbol is in the class \(S^0_{1,0}\), then
\[
||Tu||_{W^{M-m,\infty}} \lesssim ||u||_X.
\]

(iv) For \(k \in \mathbb{Z}\) we have
\[
||(1+||x||)^{2/3} P_k u||_{L^2} \lesssim 2^{-(M+2)k^+} ||u||_Z.
\]

(v) Calderon-Zygmund operators are bounded on \(H^N, X\) and \(Z\).

**Proof.** (i) follows from Hölder’s inequality and the fact that \((1+||x||)^{-2/3} \in L^{3^+}\). (ii) follows from the Bernstein inequality. (iii) follows from the bound \(||TP_k u||_{L^\infty} \lesssim ||P_k u||_{L^\infty}\). (iv) and (v) are due to the fact that the weight \((1+||x||)^{2/3} \in A_2\), see Theorem 2 and Corollary in Section V.4 of Stein. \(\square\)

Now we can state the main results about the 2D Euler–Poisson system (1.12) in this paper. Throughout the paper \(N\) and \(M\) are fixed integers, on which all implicit constants depends unless stated otherwise. We assume \(M \geq 10\), \(\epsilon\) is small enough and \(R\) is large enough.

**Theorem 1.5.** There is a constant \(c > 0\) such that if \(N \geq M + 5, 1 \leq R \leq \exp(ce^{-c})\) and
\[
||\nabla ||^{-1} \rho_0||_{H^{N+1}} + ||v_0||_{H^N} \leq \epsilon, \quad (1.14)
\]
then there is
\[
T_{R, \epsilon} \approx R/c^2 (\log R)^{O(1)} \quad (1.15)
\]
such that (1.12) with initial data \((\rho_0, v_0)\), has a unique solution with \((A \nabla ||^{-1} \rho, v) \in C([0, T_{R, \epsilon}], H^N)\).
Theorem 1.6. If $N \geq \max(3(M + 4), 106)$, $R \gg 1$ and
\[
\|\nabla^{-1} \rho_0\|_{H^{N+1}} + \|v_0\|_{H^N} + \|\Lambda \nabla^{-1} \rho_0\|_Z + \|v_0\|_Z \leq \epsilon,
\] (1.16)
then there is $T'_{R, \epsilon} \approx R^{10/9 - O(1/N)} \epsilon^{-2/3 + O(1/N)}$ (1.17)
such that (1.12) with initial data $(\rho_0, v_0)$, has a unique solution with $(\Lambda \nabla^{-1} \rho, v) \in C([0, T'_{R, \epsilon}], H^N)$.

Remark 1.7. (i) For specific values of the constants in exponents see Proposition 2.3 and 2.4. Technically the proof gives a constant $\epsilon$ depending on the choice of $N$ and $M$ in Theorem 1.5, but this dependence can be removed using persistence of regularity (see for details.)

(ii) If the assumption made on $R$ and $\epsilon$ in Theorem 1.5 does not hold, then we have $R \gtrsim \epsilon^{-100}$ (say), and Theorem 1.6 gives a better bound. Either lifespan is longer than $R/\epsilon$, which is the most one can hope for without using the normal form, for no decay can be expected of the $L^2$ norm of the solution, giving a lower bound of $\gtrsim \epsilon/R$ of its $L^\infty$ norm.

(iii) Theorem 1.6 implies global regularity in the Euclidean case, and gives a quantitative version of the theorem of Ionescu–Pausader [26] and Li–Wu [34]. A notable difference from those two works is that here the spatial weights on the Sobolev norms are equivalent to $x^{2/3}$ in the Euclidean case, which is smaller than $x$ previously. This is due to quartic energy estimates which allow for more flexibility in choosing the spatial weights. The choice of $x^{2/3}$ as spatial weights will be explained later when the $Z$-norm estimates are discussed.

(iv) The Klein-Gordon equation with mass $m = 1$ on a torus of size $R$ can be rescaled to the unit torus, but with mass $m = R$. When $R$ is of unit size, Theorem 1.5 gives a lifespan $\gtrsim \epsilon^{-2}$ and recovers the theorem of Delort–Szeftel [12]. In general, our bounds depend on $R$ in a uniform way, thus reinstating the exceptional set of measure zero that has to be excluded from the parameter space in Delort [8] and Fang–Zhang [17].

(v) Unlike Faou–Germain–Hani [16] and Buckmaster–Germain–Hani–Shatah [3], our proofs of Theorems 1.5 and 1.6 do not rely on the number-theoretic properties of the resonance set. Hence it is straightforward to generalize our results to nonsquare tori with bounded aspect ratio.

1.5. Previous work on long-term regularity. The Euler–Poisson system (1.12) is a symmetrizable quasilinear hyperbolic system, as already shown in [26]. Therefore local regularity of solutions with sufficiently smooth initial data follows from [29]. It is long-term regularity that is of interest here.

In the Euclidean case, global regularity in 3D was shown by Guo in [21], and in 2D shown independently by Ionescu–Pausader [26] and Li–Wu [34]. Extensions of this model whose global regularity is known include the nonneutral case [20], the Euler–Maxwell equation (19, 27 in 3D and 15 in 2D), the Euler–Poisson ion equation [24], two fluid models [22] for nonrelativistic models and [23] for relativistic models), general Klein-Gordon systems [11, 18, 27] for generic parameters and [14] for all parameters) and coupled wave-Klein-Gordon systems [28].

1.5.1. Periodic solutions. Because of the lack of dispersion on compact domains, global regularity is hard to come by in the periodic case. The study of periodic dispersive equations was initiated by Bourgain, who showed almost global regularity of the quadratic Klein-Gordon equation for almost every mass on the circle [2]. Using the normal form method, this result has since been generalized to semilinear [8, 17] and quasilinear [9, 12] Klein-Gordon equations on tori, spheres [10] and Zoll manifolds [13].
It should also be mentioned that when the size of the torus is large (known as the large box limit), Faou–Germain–Hani [16], and more recently Buckmaster–Germain–Hani–Shatah [3], were able to derive a continuous resonance equation that describes the long-term behavior of the solution of a cubic Schrödinger equation on the torus of any dimension.

1.6. Main ideas of the proof. Since the seminal work of Klainerman [30]–[33], Christodoulou [6], and Shatah [35], the proof of long-term regularity of such systems consists of the following two aspects:

(1) Energy estimates (high order Sobolev norms) to control high frequencies;

(2) Dispersive estimates of the $L^\infty$ norm of the solution to control low frequencies.

Starting from Shatah [35], Poincaré’s normal form method (see [1, 5] for book reference) has proved to be successful in the study of long-term solutions of nonlinear evolutions. Basically one transforms quadratic nonlinearities to cubic ones to gain better integrability of the decay of the solution. To adapt this general framework to the case of the torus, one needs to overcome the difficulty that the $1/t$ decay of the linear evolution of the Euclidean Klein-Gordon equation is only valid for time $t \lesssim R$. Beyond this time the solution wraps around and superimposes with itself. As a rough estimate, notice that the group velocity of the Klein-Gordon wave is $\nabla \Lambda(\xi) = O(1)$, so after time $t \gtrsim R$ the solution is able to wrap around the torus $O(t/R)$ times, both horizontally and vertically. Thus the Euclidean theory only gives a bound of the form

$$\|u\|_{L^\infty} \lesssim t^{-1}(t/R)^2 = t/R^2$$

on the torus of size $R$. Suppose the nonlinearity has degree $D$. Using Gronwall’s inequality one arrives at an energy estimate schematically of the form

$$E(t) \leq E(0) \exp \left(C \int_0^t \|u(s)\|_{L^\infty}^{D-1} \right) ds \leq E(0) \exp \left(C \frac{t^D \epsilon^{D-1}}{R^{2D-2}} \right).$$

Hence one is only able to close the estimate up to time

$$T \lesssim R^{2D-2} \epsilon^{D-1}. $$

For quadratic nonlinearity this gives a lifespan of $R/\sqrt{\epsilon}$. This is nontrivial only in the large box limit $R \to \infty$; when $R \approx 1$, this lifespan is even shorter than provided by local regularity.

To improve on (1.20), we will combine the following three ingredients:

• **Quartic energy estimates.** It has been observed in [33, 26, 34] that the Klein-Gordon equation has no time resonance, and the normal form transform effectively makes the nonlinearity cubic, and allows for quartic energy estimates. To overcome the loss of derivatives arising from quasilinearity, we make use of paradifferential calculus, which has already found application to similar quasilinear evolution equations in, for example, [15].

• **Z-norm estimates** in the large box limit. When $R$ is very large compared to $1/\epsilon$, the dispersive estimates are done using a bootstrap argument in a suitable $Z$-norm (spatially weighted Sobolev norm) of the profile. The argument is similar to the Euclidean case in [26]. Thanks to the normal form transform, the nonlinearity is now cubic, so we only need a decay better than $1/\sqrt{t}$. We will still optimize the Euclidean decay rate, which translates to longer lifespan in the large box case. This is done by adjusting the spatial weight.

More precisely, in the Euclidean case, a spatial weight of $x^\alpha$, where $\alpha \in (0, 1)$, leads to $t^{-\alpha}$ decay of the $L^\infty$ norm of the solution, and no decay of the $L^2$ norm. Localized initial data will spread a distance of $\approx t$ after time $t$. At such distances, the $Z$ norm of the cubic
nonlinearity is contributed mainly by two parts: one where all three factors are at distance $O(1)$ from the origin, and the other where all three factors are at distance $\approx t$. Using the $L^2 \times L^\infty \times L^\infty \to L^2$ trilinear estimate, the first part decays like $t^{-2}$, while the second part decays like $t^{-3\alpha}$ (note that we only need the unweighted $L^2$ norm of the first factor, which contributes another factor $t^{-\alpha}$ in decay.) Putting the weight back onto the nonlinearity gives a decay rate of $\max(t^{-2+\alpha}, t^{-2\alpha})$, which attains the minimum of $t^{-4/3}$ when $\alpha = 2/3$. This is well integrable, and translates to the fact that the exponent of $R$ in the lifespan is larger than 1 in the large box case.

- **Strichartz estimates** in the small box regime. When $R$ is close to 1, the decay (1.18) is no longer useful; in fact we don’t expect any decay of the $L^\infty$ norm of the linear evolution. Trivial integration will produce a factor of $t$ in the energy estimate. This factor can be saved using Strichartz estimates, which in $\mathbb{R}^2$ reads (see [37] for a textbook reference)

$$\|e^{it\Delta} u\|_{L^q_t L^r_x} \lesssim \|u\|_{L^2_x}, \quad 1/q + 1/r = 1/2, \quad (q,r) \neq (2,\infty).$$

(1.21)

We will show its analog for the Klein-Gordon equation on $\mathbb{T}^2$ at the endpoint $(q,r) = (2,\infty)$. This fits nicely into the energy estimate (1.19) thanks again to the cubic nonlinearity (so $D-1 = 2$). The logarithmic loss in time in the endpoint case of (1.21) is reflected in (1.15), and the loss of derivative in quasilinearity is recovered by the energy estimates.

1.7. **Organization.** The rest of the paper is organized as follows: In section 2 we establish local wellposedness of the Euler-Poisson system and state the main bootstrap propositions. In section 3 we introduce paradifferential calculus and derive the linear dispersive estimates and multilinear paraproduct estimates to be used in the rest of the paper. In section 4 we obtain the quartic energy estimates. In section 5 we prove Theorem 1.5, and in section 6 we prove Theorem 1.6.

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2. **Local wellposedness and bootstrap propositions**

By the irrotationality and zero momentum conditions, we have $v = \nabla h$ for some $h$. Let $g = |\nabla|^{-1}\rho$, and

$$U = \Lambda g + i|\nabla| h = X + iY.$$  

The charge-neutrality and zero momentum conditions imply that $\mathcal{F}U(0) = 0$, and the evolution equations for $X$ and $Y$ are

$$X_t = \Lambda Y - \Lambda R_j (\rho v_j), \quad Y_t = -\Lambda X - |\nabla|(v^2/2),$$

(2.1)

where $\rho = \Lambda^{-1}|\nabla| X$, $v_j = R_j Y$, and $R_j = |\nabla|^{-1}\partial_j$ is the Riesz transform. Note that the action of the Riesz transform on the zero frequency is assumed to be 0, consistent with the charge-neutral condition. Repeated indices imply Einstenian summation throughout the paper.
2.1. Local Wellposedness. The local wellposedness of the Euler-Poisson system (1.12) on $\mathbb{R}^2$ was worked out in Proposition 2.2 (i) of [26] using Bona-Smith approximation. The key point in the proof is an energy identity making use of the symmetrizability of the Euler-Poisson system, which is manifest after multiplying the second equation by $n$. Since the proof of the energy identity uses nothing more than integration by parts, it continues to hold on the torus, which then shows local wellposedness of the Euler-Poisson system on the torus.

Proposition 2.1. If $N \geq 3$, $U \in H^N$ and $\|U_0\|_{H^3}$ is sufficiently small, then there is $U \in C([0,1], H^N) \cap C^1([0,1], H^{N-1})$ solving (1.12) with initial data $U_0$.

It follows from Proposition 2.1 and Lemma 1.4 (ii) that

Proposition 2.2. If $N \geq M + 2$, $T > 0$ and $\sup_{t \in [0,T]} \|U(t)\|_{H^N}$ is sufficiently small, then $\|U(t)\|_X$ and $\|V(t)\|_Z$ are finite and continuous on $[0,T]$.

2.2. Bootstrap propositions. In this subsection we lay out the bootstrap propositions and use them to show Theorem 1.5 and Theorem 1.6. Throughout the paper we assume $R \gg 1$ and put

$$\mathcal{L} = \log(t + 1), \quad \mathcal{L}_R = \log R.$$ 

We will use $\mathcal{L}$ in Section 6 and $\mathcal{L}_R$ in Section 3 and Section 5.

Proposition 2.3. Fix $N \geq M + 5 \geq 15$. Assume (1.14) holds with $\epsilon$ small enough. Also assume

$$\|U\|_{L^\infty([0,t],H^N)} \leq \epsilon_1,$$

$$\|U\|_{L^2([0,t],X)} \leq \epsilon_2,$$

with $\epsilon_1, \epsilon_2$ small enough. Then

$$\|U\|_{L^\infty([0,t],H^N)} \leq \epsilon + \epsilon_1^{3/2} + \epsilon_1 \epsilon_2,$$

$$\|U\|_{L^2([0,t],X)} \leq \mathcal{L}_R \sqrt{1 + t/R \epsilon_1 (1 + \mathcal{L}_R^{3/2} \epsilon_2^2)}.$$ 

Proof of Theorem 1.5. We can choose $\epsilon_1 \approx \epsilon$, $T_0 \approx R/(\epsilon^2 (\log R)^{7/2})$ and $\epsilon_2 \approx (\log R)^{-3/4}$ such that for $t \leq T_0$, (2.3) and (2.4) give (2.2) with the strict inequality. Note that we need $R \leq \exp(\epsilon^{-3/7})$ to recover the $L^2 X$ norm. Now the result follows from local wellposedness (Proposition 2.1) and continuity of the $X$ norm (Proposition 2.2). See Section 2.1 of [26] for details.

Proposition 2.4. Fix $N \geq \max(3(M + 4), 106)$. Assume (1.16) holds with $\epsilon$ small enough. Define the profile $V(t) = e^{it\Lambda} U(t)$. Assume

$$\|U\|_{L^\infty([0,t],H^N)} \leq \epsilon_1,$$

$$\sup_{[0,t]} \|V\|_Z \leq \epsilon_1,$$

with $\epsilon_1$ small enough. Then

$$\|U\|_{L^\infty([0,t],H^N)} \leq \epsilon + \epsilon_1^{3/2} + t^{6/5} \epsilon_1^2/R^{4/3},$$

$$\sup_{[0,t]} \|V\|_Z \leq \epsilon + (t^{1+9/N}/R^{4/3} + 1) \epsilon_1^2 + (t^{3+33/N}/R^{10/3-2/N} + 1) \epsilon_1^3.$$ 

Proof of Theorem 1.6. We can choose \( \epsilon_1 \approx \epsilon \) such that all \( t \leq T_0 \) defined in the statement, (2.6) and (2.7) lead to (2.5) with the strict inequality. Then the proof is similar to that of Theorem 1.5. \( \square \)

3. Linear dispersive and multilinear paraproduct estimates

3.1. Linear dispersive estimates. The first ingredient in the proof of global existence is dispersive estimates. In the following we will use the fact that \( P_{< \ell} = 0 \) unless \( t \lesssim R \).

Lemma 3.1. For \( k \in \mathbb{Z} \), \( 1 \leq p \leq q \leq \infty \) with \( 1/p + 1/q = 1 \) we have

\[
\| P_k e^{-it\Lambda} u \|_{L^q} \lesssim [(1 + t)^{-1} (t/R + 1)^{2/3} t^{1/p - 1/q}] u_{L^p}.
\]

Proof. By interpolation and the conservation of the \( L^2 \) norm under \( e^{-it\Lambda} \) we can assume \( p = 1 \) and \( q = \infty \). Since \( P_k = P_k P_{\leq 0} \) for \( k < 0 \), and \( P_k \) is bounded on \( L^\infty \), we can further assume \( k \in \mathbb{N} \) or \( k = 0 \). By Poisson summation,

\[
P_k e^{-it\Lambda} u(x) = \frac{C}{R^2} \sum_{\xi \in (2\pi\mathbb{Z}/R)^2} e^{ix \cdot \xi - it\Lambda(\xi)} \varphi_k(\xi) \hat{u}(\xi) = \int G_k(x, y, t) u(y) dy,
\]

\[
G_k(x, y, t) = \sum_{z \in (2\pi\mathbb{Z})^2} K_k(x, y + z, t),
\]

\[
K_k(x, y, t) = \int e^{i(x-y) \cdot \xi - it\Lambda(\xi)} \varphi_k(\xi) d\xi.
\]

Trivially \( K_k(x, y + z, t) \lesssim 2^{2k} \). When \( t \geq 1 \), we can get a better bound using the method of stationary phase. The gradient of the phase is

\[
\nabla ((x - y - z) \cdot \xi - t\Lambda(\xi)) = x - y - z - t\xi/\Lambda(\xi).
\]

It vanishes only when \( |x - y - z| \leq 2t \), which happens for \( O((t/R + 1)^2) \) values of \( z \). For such \( z \), the Hessian of the phase is

\[
-t \begin{pmatrix} \Lambda''(\xi) & 0 \\ 0 & 1/\Lambda(\xi) \end{pmatrix} = -t \begin{pmatrix} 1/\Lambda(\xi)^3 & 0 \\ 0 & 1/\Lambda(\xi) \end{pmatrix},
\]

with nonvanishing determinant \( t^2/\Lambda(\xi)^4 \). Then by stationary phase,

\[
K_k(x, y + z, t) \lesssim (t^2/\Lambda(\xi)^4)^{-1/2} \lesssim t^{-1/2} 2^k.
\]

For other values of \( z \), the gradient of the phase is \( \gtrsim |x - y - z| \), so

\[
\sum_{z \in (2\pi\mathbb{Z})^2 \atop |x - y - z| > 2t} K_k(x, y + z, t) \lesssim 2^{2k} \sum_{z \in (2\pi\mathbb{Z})^2 \atop |x - y - z| > 2t} |x - y - z|^{-3} \lesssim t^{-1} 2^k.
\]

Combining the two bounds shows the claim. \( \square \)

Lemma 3.2. (i) For \( k \in \mathbb{Z} \) we have

\[
\| P_k e^{-it\Lambda} u \|_{L^\infty} \lesssim 2^{k/3 - (M+2/3)k^+} (1 + t)^{-\frac{2}{3} +} (t/R + 1)^{4/3} \| u \|_Z.
\]

(ii)

\[
\| e^{-it\Lambda} u \|_X \lesssim (1 + t)^{-\frac{2}{3} +} (t/R + 1)^{4/3} \| u \|_Z.
\]
Proof. By the Bernstein inequality and Lemma 3.1 for any \( c \in (1/3, 1/2) \),
\[
\|P_k e^{-it\Lambda} u\|_{L^\infty} \lesssim 2^k \|P_k e^{-it\Lambda} u\|_{L^{2/c}} \\
\lesssim 2^{k+2(1-c)k'} (1 + t)^{-1+c} (t/R + 1)^{2(1-c)} \|P_{[k-1,k+1]} u\|_{L^{2/(2-c)}}.
\]
Then (i) follows from Lemma 1.4 (i). To get (ii) we sum (i) over \( k \in \mathbb{Z} \).
\[\square\]

Lemma 3.3.
\[
\|e^{-is\Lambda} u\|_{L^2([0,t])X} \lesssim \mathcal{L}_R \sqrt{1 + t/R} \|u\|_{H^{M+2}}.
\]
Proof. For \( k \in \mathbb{Z} \) let
\[
T_k : L^2_x \to L^2([0,t])L^\infty_x, \quad u \mapsto P_k e^{-is\Lambda} u.
\]
Then
\[
T_k^* : L^2([0,t])L^1_x \to L^2_x, \quad u \mapsto \int_0^t P_k e^{is\Lambda} u(s) ds.
\]
Then \( \|T_k\| = \|T_k^*\| = \|T_k T_k^*\|^{1/2} \), where
\[
T_k T_k^* : L^2([0,t])L^1_x \to L^2([0,t])L^\infty_x, \quad u \mapsto \int_0^t P_k^2 e^{i(s'-s)\Lambda} u(s') ds'.
\]
First we suppose \( t \leq R \). By Lemma 3.1 (note that \( t/R + 1 \leq 2 \)),
\[
\|P_k^2 e^{i(s'-s)\Lambda} u(s')\|_{L^\infty_x} \lesssim (1 + |s' - s|)^{-1} 2^{k'} \|u(s)\|_{L^2_x}.
\]
By Young’s inequality (note that now \( \mathcal{L}_R > \log(1 + t) \)),
\[
\|T_k T_k^* u\|_{L^2([0,t])L^\infty_x} \lesssim \|(1 + |\cdot|)^{-1}\|_{L^1([0,t])} 2^{k'} \|u\|_{L^2([0,t])L^1_x} \lesssim 2^{k'} \mathcal{L}_R \|u\|_{L^2([0,t])L^1_x}
\]
so \( \|T_k\| \lesssim 2^{k'} \sqrt{\mathcal{L}_R} \). For \( t > R \) we use the conservation of the \( L^2 \) norm to take an \( \ell^2 \) sum of time intervals of length \( R \) to get \( \|T_k\| \lesssim 2^{k'} \sqrt{\mathcal{L}_R (1 + t/R)} \).

Applying the above bound to \( P_{[k-1,k+1]} u \) and summing over \( k \geq -\mathcal{L}_R \), using the Cauchy-Schwarz inequality on the right hand side, we get the desired bound.
\[\square\]

3.2. Paradifferential calculus. We will use Weyl quantization of paradifferential operators on the torus.

Definition 3.4. Given a symbol \( a = a(x, \xi), (\mathbb{R}/R\mathbb{Z})^2 \times (\mathbb{R}^2 \setminus 0) \to \mathbb{C} \), define the operator \( T_a \) using the following recipe:
\[
\mathcal{F}(T_a f)(\xi) = \frac{C}{R^2} \sum_{\eta \in (2\pi R^2)^2} \varphi_{<-10} \left( \frac{\xi - \eta}{\xi + \eta} \right) \mathcal{F}_x a \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta),
\]
where \( C \) is a normalization constant (independent of \( R \)) such that \( T_1 = \text{id} \).

Remark 3.5. When \( \xi + \eta = 0 \), the \( \varphi_{<-10} \) factor is defined to be 0, so \( a(x, 0) \) will never be used. Also the case \( \xi = \eta = 0 \) is of no concern, for \( T_a \) will only be applied to \( U \), for which we have assumed \( \mathcal{F}U(0) = 0 \).

The next lemma follows directly from the definition.
Lemma 3.6. (i) If \( a \) is real valued, then \( T_a \) is self-adjoint.
(ii) If \( a(x, -\zeta) = a(x, \zeta) \) and \( f \) is real valued, so is \( T_a f \).
(iii) If \( a = P(\zeta) \), then \( T_a f = P(D) f \) is a Fourier multiplier.

The following symbol norm will be used.

Definition 3.7. For \( p \in [1, \infty] \) and \( m \in \mathbb{R} \) define

\[
|a|(x, \zeta) = \sum_{|I| \leq 8} |\xi^I| |\partial_\xi^I a(x, \zeta)|, \quad \|a\|_{L^p_m} = \sup_{\zeta \in \mathbb{R}^d} (1 + |\zeta|)^{-m} \|a(x, \zeta)\|_{L^p_0(\mathbb{R}/\mathbb{Z}_2^2)}.
\]

Here \( m \) is the order of the symbol, in the sense of Hörmander.

Lemma 3.8. A multiplier whose symbol is of class \( S^m_{1,0} \) has finite \( L^\infty_m \) norm.

The norm of the product of two operators can be bounded using Leibniz’s rule and Hӧlder’s inequality. The result is

Lemma 3.9. For fixed \( m \in \mathbb{R} \), \( p, q, r \in [1, \infty] \) with \( 1/p = 1/q + 1/r \) we have

\[
\|ab\|_{L^p_{m+n}} \lesssim \|a\|_{L^p_m} \|b\|_{L^r_n}.
\]

Paradifferential operators in \( L^p_m \) act like differential operators of order \( m \) with \( L^p \) coefficients.

Lemma 3.10. For fixed \( m \in \mathbb{R} \), \( p, q, r \in [1, \infty] \) with \( 1/p = 1/q + 1/r \) we have

\[
\|P_k T_a f\|_{L^r} \lesssim 2^{mk+} \|a\|_{L^p_m} \|P_{k-2,k+2} f\|_{L^r}.
\]

Proof. First we assume that \( k \geq -\log R - 1 \); otherwise \( P_k f = 0 \) for any function \( f \) on \( (\mathbb{R}/\mathbb{Z})^2 \) because \( \text{supp} \varphi_k \) and \( (\mathbb{Z}/\mathbb{R})^2 \) are disjoint.

The Schwartz kernel for \( T_a \) is

\[
I(x, y) = \int \frac{C'}{R^4} \sum_{\xi, \eta \in (2\pi\mathbb{Z}/R)^2} a(z, \frac{\xi + \eta}{2}) e^{i(\xi \cdot (x-z) + \eta \cdot (z-y))} \varphi \leq_{10} \left( \frac{|\xi - \eta|}{|\xi + \eta|} \right) \varphi_k(\xi) dz.
\]

\[
= \int \frac{C'}{R^4} \sum_{\xi, \eta \in (2\pi\mathbb{Z}/R)^2} a(z, \frac{\xi + \eta}{2}) \varphi \leq_{10} \left( \frac{|\xi - \eta|}{|\xi + \eta|} \right) \varphi_k(\xi)
\times \left( \prod_{j=1}^{|I|} \Delta^{\xi_j}_{1/R} \right) \left( \prod_{j=1}^{|I|} \Delta^{\eta_j}_{1/R} \right) e^{i(\xi \cdot (x-z) + \eta \cdot (z-y))}
\times \left( \prod_{j=1}^{|I|} \Delta^{\xi_j}_{1/R} \right) \left( \prod_{j=1}^{|I|} \Delta^{\eta_j}_{1/R} \right) e^{i(\xi \cdot (x-z) + \eta \cdot (z-y))}
\times R^{n+|I|} \prod_{j=1}^{|I|} (e^{2\pi i(x_j - z_j)/R} - 1) \prod_{j=1}^{|I|} (e^{2\pi i(y_j - z_j)/R} - 1) dz.
\]

Here

\[
\Delta^\xi_j a = \frac{a(x, \zeta + h \epsilon_j) - a(x, \zeta)}{h}
\]

is the finite difference quotient. By summation by parts in \( \xi \) and \( \eta \) it follows that

\[
I(x, y) = \int \frac{C'}{R^4} \sum_{\xi, \eta \in (2\pi\mathbb{Z}/R)^2} e^{i(\xi \cdot (x-z) + \eta \cdot (z-y))}
\times \left( \prod_{j=1}^{|I|} \Delta^{\xi_j}_{1/R} \right) \left( \prod_{j=1}^{|I|} \Delta^{\eta_j}_{1/R} \right) \left( a \left( z, \frac{\xi + \eta}{2} \right) \varphi \leq_{10} \left( \frac{|\xi - \eta|}{|\xi + \eta|} \right) \varphi_k(\xi) \right)
\times R^{n+|I|} \prod_{j=1}^{|I|} (e^{2\pi i(x_j - z_j)/R} - 1) \prod_{j=1}^{|I|} (e^{2\pi i(y_j - z_j)/R} - 1) dz.
\]
By the fundamental theorem of calculus, the Leibniz rule, the bounds $\nabla \varphi_k \lesssim 2^{-k}$, $\nabla \varphi_{\leq 10} \lesssim 1$ and the triangle inequality, the difference quotient in (3.2) can be bounded by $C_{|I|+|J|}$ times $g_{I,J}(z, \xi, \eta)$

$$g_{I,J}(z, \xi, \eta) = \int_{[0,1]|I|+|J|} 2^{-|I|+|J|-l} \left| \nabla_{\xi} a \left( z, \xi + \eta + \sum_{j=1}^{|I|+|J|} t_j e_j / R \right) \right| dt_1 \cdots dt_{|I|+|J|}.$$ 

When $k \geq - \log R + O(1)$, the second argument of $a$ is still $\approx 2^k$, it follows from the definition of $|a|$ that for $|I| + |J| \leq 5$,

$$g_{I,J}(z, \xi, \eta) \lesssim 2^{-|I|+|J|} \int_{[0,1]|I|+|J|} \left| a \left( z, \xi + \eta + \sum_{j=1}^{|I|+|J|} t_j e_j / R \right) \right| dt_1 \cdots dt_{|I|+|J|}.$$ 

Using $|e^{2\pi i x/R} - 1| \approx \||x||/R$ and $||x-z| + |y| \approx ||x-y|| + ||x-z||$, where $||x|| = d(x, RZ)$, we have

$$|I(x, y)| \lesssim \min_{|I|+|J| \leq 5} \int R^{-4} \sum_{\xi, \eta \in (2\pi / R)^2 \cap \{ |\xi|, |\eta| \leq 2^k \}} g_{I,J}(z, \xi, \eta) \prod_{j=1}^{|I|} \| x_I - y_I \| \prod_{j=1}^{|J|} \| z_I - y_I \| dz$$

$$\lesssim \min_{|I|+|J| \leq 5} \int R^{-4} \sum_{\xi, \eta \in (2\pi / R)^2 \cap \{ |\xi|, |\eta| \leq 2^k \}} g_{I,J}(z, \xi, \eta) \prod_{j=1}^{|I|} \| x_I - y_I \| \prod_{j=1}^{|J|} \| x_I - z_I \| dz.$$ 

Therefore

$$|Ta f(x)| \lesssim \min_{|I|+|J| \leq 5} \int K(y, z) f(x-y) g_{I,J}(x-z) dy dz,$$

where

$$K(y, z) = \frac{1}{(1 + 2^k(\|y\| + \|z\|))^5}, \quad \tilde{g}_{I,J}(z) = \sum_{\xi, \eta \in (2\pi / R)^2 \cap \{ |\xi|, |\eta| \leq 2^k \}} g_{I,J}(z, \xi, \eta).$$

Now we can pass the $L^p$ norm inside the integral and the sum, and then apply Hölder’s inequality. From the bound $\|K\|_{L^1} \lesssim 2^{-4k}$ and $\|g\|_{L^q} \lesssim 2^{4k-mk^+} \|a\|_{L^q_m}$ follows the lemma with $f$ in place of $P_{[k-2, k+2]} f$.

When $k = - \log R + O(1)$, we can take $|I| = |J| = 0$, in which case $g_{I,J}(z, \xi, \eta) = |a(z, \xi + \eta)| \lesssim |a|(|z, \xi + \eta|)$. The same conclusion follows from the bound $\|g\|_{L^q} \lesssim R^{m-4} \|a\|_{L^q_m}$.

To show the lemma itself, note in the definition (3.11), $|\xi|/2 < |\eta| < 2|\xi|$, so $\varphi_k(\xi) > 0$ implies $\varphi_k(2|\xi|/2) = 1$, and hence $P_k T_a f = P_k T_a P_{[k-2, k+2]} f$.

Paradifferential operators extract the “quasilinear” part of products, leaving “semilinear” remainders.

**Definition 3.11.** Given two functions $f$ and $g$, define

$$H(f, g) = fg - Tfg - Tg f.$$ 

From $P_k H(f, g) = P_k H(P_{> k-20} f, P_{> k-20} g)$ and Lemma 3.10 we get

**Lemma 3.12.** For fixed $p, q, r \in [1, \infty]$ with $1/p = 1/q + 1/r$ we have

$$\| P_k H(f, g) \|_{L^p} \lesssim \| P_{> k-20} f \|_{L^q} \| P_{> k-20} g \|_{L^r}.$$
Next we show the commutator estimates of paradifferential operators.

**Definition 3.13.** Given symbols $a_1, \ldots, a_n$, define the operator

$$E(a_1, \ldots, a_n) = T_{a_1} \cdots T_{a_n} - T_{a_1 \cdots a_n}.$$ 

**Lemma 3.14.** For fixed $m_j \in \mathbb{R}$, $p, q_j, r \in [1, \infty]$ ($j = 1, \ldots, n$) with $1/p = 1/q_1 + \cdots + 1/q_m + 1/r$ we have

$$\|P_k E(a_1, \ldots, a_n) f\|_{L^p} \lesssim 2^{(\sum_{j=1}^m m_j - 1)k^+} \prod_{j=1}^n (\|a_j\|_{\mathcal{L}^m_{1q_j}} + \|\nabla_a a_j\|_{\mathcal{L}^m_{1q_j}}) \|P_{[k-2n, k+2n]} f\|_{L^r}.$$ 

Roughly speaking, the operator $E(a_1, \ldots, a_n)$ is one order smoother than either term on the right, so it can be thought of an “error term”.

**Proof.** Lemma 3.9 and Lemma 3.10 allow us to use induction on $n$, so it suffices to show the case when $n = 2$. Decompose $a_j = a_j^L + a_j^H$ ($j = 1, 2$), where $a_j^L = P_{\leq k-20} a_j$, and put

$$E^L(a_1, a_2) = E(a_1^L, a_2^L),$$

$$E^H(a_1, a_2) = E(a_1, a_2) - E^L(a_1, a_2) = E(a_1^H, a_2) + E(a_1^L, a_2^H).$$

For $E^H(a_1, a_2)$, we use Lemma 3.9 and Lemma 3.10 to get

$$\|P_k E(a_1^H, a_2^H) f\|_{L^p} \lesssim 2^{(m_1 + m_2 - 1)k^+} (\|a_1^H\|_{\mathcal{L}^m_{1q_1}} + \|\nabla_a a_1\|_{\mathcal{L}^m_{1q_1}}) \|P_{[k-4, k+4]} f\|_{L^r}.$$

A similar bound, with $\nabla_x$ hitting $a_2$, holds for $P_k E(a_1^L, a_2^H) f$.

For $E^L(a_1, a_2)$, since $a_j^L = P_{\leq k-20} a_j$, we have

$$\mathcal{F}(P_k E(a_1^L, a_2^L) f)(\xi) = \frac{C^2}{R^4} \sum_{0, \xi, \zeta \in (2\pi/2R)^2} A(\xi, \eta, \zeta) \hat{f}(\eta, \zeta),$$

where

$$A = \mathcal{F}_x a_1^L \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \mathcal{F}_x a_2^L \left( \eta - \zeta, \frac{\eta + \zeta}{2} \right) - \mathcal{F}_x a_1^L \left( \xi - \eta, \frac{\xi + \zeta}{2} \right) \mathcal{F}_x a_2^L \left( \eta - \zeta, \frac{\xi + \zeta}{2} \right),$$

$$A_1 = \int_0^1 (\nabla_\zeta \mathcal{F}_x a_1^L) \left( \xi - \eta, \frac{\xi + \zeta + t(\eta - \xi)}{2} \right) \cdot \mathcal{F}_x a_2^L \left( \eta - \zeta, \frac{\xi + \zeta + t(\eta - \xi)}{2} \right) \frac{dt}{2},$$

$$A_2 = \int_0^1 \mathcal{F}_x a_1^L \left( \xi - \eta, \frac{\xi + \zeta + t(\eta - \xi)}{2} \right) \cdot (\nabla_\zeta \mathcal{F}_x a_2^L) \left( \eta - \zeta, \frac{\xi + \zeta + t(\eta - \xi)}{2} \right) \frac{dt}{2},$$

$$A_1 = \frac{1}{2i} \mathcal{F}_x (\nabla_\zeta a_1) \left( \xi - \eta, \frac{\xi + \zeta + t(\eta - \xi)}{2} \right) \cdot \mathcal{F}_x (\nabla_\zeta a_2) \left( \eta - \zeta, \frac{\xi + \zeta + t(\eta - \xi)}{2} \right) \frac{dt}{2},$$

$$A_2 = \frac{1}{2i} \mathcal{F}_x (\nabla_\zeta a_1) \left( \xi - \eta, \frac{\xi + \zeta + t(\eta - \xi)}{2} \right) \cdot \mathcal{F}_x (\nabla_\zeta a_2) \left( \eta - \zeta, \frac{\xi + \zeta + t(\eta - \xi)}{2} \right) \frac{dt}{2}. $$
The Schwartz kernel for $P_k E^L(a_1, a_2)$ is then

$$I(x, y) = R^{-6} \sum_{\xi, \eta, \zeta \in (\mathbb{Z}/R)^2} \int A(\xi, \eta, \zeta) e^{i(\xi(x-z_1) + \eta(y-z_2) + \zeta(z_2-y))} \varphi_k(\zeta)dz_1dz_2.$$ 

The same argument as in Lemma 3.10 (applied to $P_{[k-4,k+4]} f$ instead of $f$) shows the claim. □

### 3.3. Multilinear paraproduct estimates

We also need to bound multilinear paraproducts on the torus.

**Definition 3.15.** If $m$ is a Schwartz function on $(\mathbb{R}^2)^n$, define

$$\|m\|_{S^\infty} = \|Fm\|_{L^1},$$

$$\|m\|_{S^\infty_{k_1, \ldots, k_n}} = \|\varphi_k(\xi_1 + \cdots + \xi_n)m(\xi_1, \ldots, \xi_n)\varphi_{k_1}(\xi_1) \cdots \varphi_{k_n}(\xi_n)\|_{S^\infty}.$$ 

**Definition 3.16.** Throughout the paper we let $K$ be the largest of $k, k_1, \ldots, k_n$.

The next lemma allows us to estimate the $S^\infty$ norm of various symbols.

**Lemma 3.17.** (i) $\|m_1 m_2\|_{S^\infty} \leq \|m_1\|_{S^\infty} \|m_2\|_{S^\infty}$.

(ii) $\|m \cdot \otimes^m_{j=1} \varphi_{k_j}\|_{S^\infty} \leq \sum_{l=0}^{n+1} \sum_{j=0}^{n} 2^{lk_j} \|\varphi_{[k_j-1, k_j+1]} \nabla^{l}_{j} m\|_{L^\infty}$.

(iii) $\left\| m \sum_{\xi \in (2\pi \mathbb{Z}^2)^n} \delta_\xi \right\|_{S^\infty(\mathbb{R}/\mathbb{Z}^2)^n}) \leq \|m\|_{S^\infty((\mathbb{R}^2)^n)}$.

That is, we can bound the $S^\infty$ norm of a multiplier supported on the lattice $(2\pi \mathbb{Z}^2)^n$ by the $S^\infty$ norm of (one of) its extension to $(\mathbb{R}^2)^n$. By scaling symmetry, this also applies to multipliers supported on a rescaled lattice.

**Proof.** (i) follows directly from the definition. (ii) is [15], Lemma 3.3. (iii) follows from the definition, the Poisson summation formula

$$\mathcal{F}^{-1} \left( f \sum_{\xi \in (2\pi \mathbb{Z}^2)^n} \delta_\xi \right)(x) = \sum_{z \in (\mathbb{Z}^2)^n} \mathcal{F}^{-1} f(x + z)$$

and the triangle inequality. □

The $L^p$ boundedness of a paraproduct of functions is well known.

**Lemma 3.18.** Fix $p, p_j \in [1, \infty]$ ($j = 1, \ldots, n$) and $1/p = 1/p_1 + \cdots + 1/p_n$. Let

$$\mathcal{F} f(\xi) = \sum_{\xi_j \in (2\pi \mathbb{Z}/R)^2 \atop \xi_1 + \cdots + \xi_n = \xi} m(\xi_1, \ldots, \xi_n) \prod_{j=1}^{n} \mathcal{F} f_j(\xi_j).$$

Then

$$\|f\|_{L^p} \lesssim \|m\|_{S^\infty} \prod_{j=1}^{n} \|f_j\|_{L^{p_j}},$$

where the $L^p$ norms are taken on $(\mathbb{R}/R\mathbb{Z})^2$. 

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**THE PERIODIC EULER–POISSON SYSTEM FOR ELECTRONS IN 2D**

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Definition 4.1. For an integer \( N \geq 2 \) define
\[
\mathcal{U} = X + iT_{\sqrt{1 + \rho}} Y, \quad \mathcal{E} = \|P_{\geq 0} \mathcal{U}\|_{H^N}^2 = \|P_{\geq 0} \Lambda^N \mathcal{U}\|_{L^2}^2.
\]

We first show that \( \mathcal{E} \) is close to the usual \( H^N \) norm, up to a cubic error.

Proposition 4.2. If \( \|\mathcal{U}\|_{H^2} \) is sufficiently small then
\[
|\mathcal{E} - \|P_{\geq 0} \mathcal{U}\|_{H^N}^2| \lesssim \|\mathcal{U}\|_{H^N}^3.
\]

Proof. From Lemma 3.10 and Sobolev embedding it follows that
\[
\|P_{\geq 0}(\mathcal{U} - U)\|_{H^N} \lesssim \|\rho\|_{L^\infty} \|\mathcal{U}\|_{H^N} \lesssim \|\mathcal{U}\|_{H^2}.
\]
If \( \|\mathcal{U}\|_{H^2} \) is sufficiently small, the above also gives \( \|P_{\geq 0} \mathcal{U}\|_{H^N} \lesssim \|\mathcal{U}\|_{H^N} \). Combining the two bounds shows the claim.

The rest of this section is devoted to estimating \( d\mathcal{E}/dt \). To begin with, the evolution equation for \( \mathcal{U} \) is
\[
\partial_t \mathcal{U} = \Lambda(Y - iT_{\sqrt{1 + \rho}} X) - \Lambda R_j(T_{\rho} v_j + T_{v_j} \rho + H(\rho, v_j))
- iT_{\sqrt{1 + \rho}} |\nabla|(T_{v_j} v_j + H(v_j, v_j)/2) + iT_{b_k \sqrt{1 + \rho}} Y.
\]

Using the definition of \( E, T_{\Lambda(\zeta)} = \Lambda \) (by Lemma 3.6(ii)) and \( \zeta_j \zeta_j = |\zeta|^2 \) we get
\[
(\partial_t + iT_{\sqrt{1 + \rho} \Lambda(\zeta)}) \mathcal{U} = -iE(\Lambda(\zeta), T_{\sqrt{1 + \rho}}) X - E(\sqrt{1 + \rho} \Lambda(\zeta), \sqrt{1 + \rho}) Y
+ E(\Lambda(\zeta) \zeta_j / |\zeta|, v_j, \rho, \zeta_j / |\zeta|) Y - \Lambda R_j T_{\rho} v_j - iT_{\sqrt{1 + \rho}} |\nabla| T_{v_j} v_j
- \Lambda R_j H(\rho, v_j) - iT_{\sqrt{1 + \rho}} |\nabla| H(v_j, v_j)/2 + iT_{b_k \sqrt{1 + \rho}} Y.
\]

Using \( E(a, 1) = E(1, b) = 0, [T_a, T_b] = E(a, b) - E(b, a) \) and the bilinearity of \( E \) we get
\[
(\partial_t + iT_{\sqrt{1 + \rho} \Lambda(\zeta)} + iT_{v_j \zeta}) \mathcal{U} = \mathcal{Q} + \mathcal{S} + \mathcal{C}, \tag{4.1}
\]
where
\[
\mathcal{Q} = -iE(\Lambda(\zeta), T_{\sqrt{1 + \rho}}) X - E(\Lambda(\zeta), \sqrt{1 + \rho} - 1) Y + E(\Lambda(\zeta) \zeta_j / |\zeta|, \rho, \zeta_j / |\zeta|) Y
- iE(\Lambda(\zeta) \zeta_j / |\zeta|, v_j, \zeta_j / |\zeta|) X + E(\zeta_j / |\zeta|, v_j, \zeta_j / |\zeta|) Y + iT_{|\nabla| Y}/2,
\]
\[
\mathcal{S} = -\Lambda R_j H(\rho, v_j) - iT_{\sqrt{1 + \rho}} |\nabla| H(v_j, v_j)/2,
\]
\[
\mathcal{C} = -E((\sqrt{1 + \rho} - 1) \Lambda(\zeta), \sqrt{1 + \rho} - 1) Y + T_{\sqrt{1 + \rho} - 1} E(\zeta_j / |\zeta|, v_j, \zeta_j / |\zeta|) Y
+ iT_{v_j \zeta} T_{\sqrt{1 + \rho} - 1} |\nabla| H(v_j, v_j)/2 + iT_{b_k \sqrt{1 + \rho} - |\nabla| Y}/2 Y
\]
are quasilinear quadratic, semilinear quadratic and cubic terms, respectively.
By Lemma 3.6 (i), \( T_{\sqrt{1+\rho}A(\zeta) + v_\zeta} \) is self-adjoint, so \( \langle T_{\sqrt{1+\rho}A(\zeta) + v_\zeta} f, f \rangle \in \mathbb{R} \), where the inner product is taken on the torus. Now we can decompose \( dE/dt \) accordingly:

\[
\frac{d}{dt} E = 2\Re((\partial_t + iT_{\sqrt{1+\rho}A(\zeta)} + iT_{v_\zeta})P_{\geq 0}\Lambda^N U, P_{\geq 0}\Lambda^N U) = 2(E_Q + E_S + E_4),
\]

where (note that \( [T_{\Lambda(\zeta)}, P_{\geq 0}\Lambda^N] = \Lambda, P_{\geq 0}\Lambda^N = 0) \)

\[
E_Q = \Re([iT_{\sqrt{1+\rho}A(\zeta)} + v_\zeta, P_{\geq 0}\Lambda^N U + P_{\geq 0}\Lambda^N Q, P_{\geq 0}\Lambda^N U],
E_S = \Re(P_{\geq 0}\Lambda^N S, P_{\geq 0}\Lambda^N U),
E_4 = \Re(P_{\geq 0}\Lambda^N S, iP_{\geq 0}\Lambda^N T_{\sqrt{1+\rho} - 1}Y) + \Re(P_{\geq 0}\Lambda^N C, P_{\geq 0}\Lambda^N U)
\]

are quasilinear cubic, semilinear cubic and quartic energies, respectively.

4.2. Bounding the quartic energy.

**Proposition 4.3.** If \( \|U\|_{H^2} \) is sufficiently small then

\[
E_4 \lesssim \|U\|^{2}_X \|U\|^{2}_{H^N}.
\]

**Proof.** By Lemma 3.12 Lemma 3.10 and Lemma 1.4 (iii), for \( k \geq -2 \) we have

\[
\|P_kS\|_{H^N} \lesssim 2^{(N+1)k} \|P_{k-20}(\rho, v)\|_{H^\infty} \|P_{k-20}v\|_{L^2} \lesssim 2^{Nk}\|U\|\|H_k - 20U\|_{L^2}, \quad (4.2)
\]

\[
\|P_kT_{\sqrt{1+\rho} - 1}Y\|_{H^N} \lesssim \|\rho\|_{L^\infty} \|P_{k-20}U\|_{H^N} \lesssim \|\rho\|_{L^\infty} \|P_{k-20}U\|_{H^N} \lesssim \|\rho\|_{L^\infty} \|U\|\|H_k - 20U\|_{H^N}. \quad (4.3)
\]

The desired bound for the first term follows after taking the \( \ell^2 \) sum in \( k \).

For the second term, by Proposition 4.2 and Lemma 1.4 (iii) it suffices to show

\[
\|P_{\geq 0}\Lambda^N C\|_{H^N} \lesssim \|(\rho, v)\|^{2}_{W^{1,\infty}} \|U\|_{H^N}. \quad (4.4)
\]

The desired bounds for the first five terms of \( P_{\geq 0}C \) follow from Lemma 3.8 Lemma 3.9 Lemma 3.10 and Lemma 3.14. The desired bound for the sixth term follows from Lemma 3.10 and (4.2). The desired bound for the last term follows from Lemma 3.10 using the identity

\[
\partial_t \sqrt{1+\rho} = \frac{-\nabla \cdot ((1 + \rho)v)}{2\sqrt{1+\rho}} = \frac{\nabla Y - (\sqrt{1+\rho} - 1)\nabla \cdot v}{2\sqrt{1+\rho}} - \frac{v \cdot \nabla \rho}{2\sqrt{1+\rho}}.
\]

\( \square \)

4.3. Bounding the semilinear energy.

**Proposition 4.4.** If \( N \geq 9 \) then

\[
\int_0^t E_S(s)ds \lesssim \|U\|^{2}_{L^\infty([0,t])H^N} + \|U\|^{2}_{L^2([0,t])X}\|U\|^{2}_{L^\infty([0,t])H^N}.
\]

**Proof.** Let \( U_+ = U \) and \( U_- = \tilde{U} \). Then \( E_S \) is a linear combination of terms of the form \( \Re \mathcal{E}_{\mu\nu} \), where

\[
\mathcal{E}_{\mu\nu} = \langle P_{\geq 0}\Lambda^{N+1}T_3H(T_1U_\mu, T_2U_\nu), P_{\geq 0}\Lambda^N U \rangle,
\]
$T_1, T_2$ and $T_3$ are Calderon-Zygmund operators, and $\mu, \nu \in \{+, -\}$. Let

$\Phi_{\mu\nu}(\xi_1, \xi_2) = \Lambda(\xi_1 + \xi_2) - \mu \Lambda(\xi_1) - \nu \Lambda(\xi_2), \tag{4.5}$

$I_{S\mu\nu}[f_1, f_2, f_3] = \frac{C^2}{R^4} \sum_{\xi_j \in (2\pi^2/R)^2} \frac{1 - \varphi \leq -10 \left( \frac{|\xi_1|}{|\xi_1 + 2\xi_2|} \right) - \varphi \leq -10 \left( \frac{|\xi_2|}{|\xi_2 + 2\xi_1|} \right)}{\Phi_{\mu\nu}(\xi_1, \xi_2)} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_1 + \xi_2),$

$I_{S\mu\nu} = I_{S\mu\nu}[T_1 U_\mu, T_2 U_\nu, P_{\geq 0}^2 \Lambda^{2N+1} T_3^s U].$

By (2.1), the evolution equation for $U$ is

$$U_t = -i \Lambda U - N, \quad N = \Lambda R_f(\rho v_j) + i|\nabla|(v^2/2). \tag{4.6}$$

Let $N_+ = N$ and $N_- = \tilde{N}$. Then

$$\frac{dI_{S\mu\nu}^0}{dt} = I_{S\mu\nu}^0 [T_1(U_\mu)_t, T_2 U_\nu, P_{\geq 0}^2 \Lambda^{2N+1} T_3^s U] + I_{S\mu\nu}^0 [T_1 U_\mu, T_2(U_\nu)_t, P_{\geq 0}^2 \Lambda^{2N+1} T_3^s U]$$

$$+ I_{S\mu\nu}^0 [T_1 U_\mu, T_2 U_\nu, P_{\geq 0}^2 \Lambda^{2N+1} T_3^s U_t]$$

$$= \mathcal{E}_{\mu\nu}^0 - I_{S\mu\nu}^0 [T_1 N_\mu, T_2 U_\nu, P_{\geq 0}^2 \Lambda^{2N+1} T_3^s U] - I_{S\mu\nu}^0 [T_1 U_\mu, T_2 N_\nu, P_{\geq 0}^2 \Lambda^{2N+1} T_3^s U]$$

$$- I_{S\mu\nu}^0 [T_1 U_\mu, T_2 U_\nu, P_{\geq 0}^2 \Lambda^{2N+1} T_3^s N].$$

Integration by parts in time gives

$$\int_0^t \mathcal{E}_{\mu\nu}^0 (s) ds = I_{S\mu\nu}^0 (t) - I_{S\mu\nu}^0 (0) \tag{4.7}$$

$$+ \int_0^t (I_{S\mu\nu}^0 [T_1 N_\mu, T_2 U_\nu, P_{\geq 0}^2 \Lambda^{2N+1} T_3^s U] + I_{S\mu\nu}^0 [T_1 U_\mu, T_2 N_\nu, P_{\geq 0}^2 \Lambda^{2N+1} T_3^s U]) (s) ds \tag{4.8}$$

$$+ \int_0^t I_{S\mu\nu}^0 [T_1 U_\mu, T_2 U_\nu, P_{\geq 0}^2 \Lambda^{2N+1} T_3^s N] (s) ds \tag{4.9}.$$
so $|\xi_2^L|\nabla_{\xi_2}^L \Phi_{\mu^-} | \lesssim_L |\xi_1|$. By \eqref{4.10}, $|\xi_2^L|\nabla_{\xi_2}^L (\Phi_{\mu^-})^{-1} | \lesssim_L |\xi_1|^{2L+1}$, and the bound follows Lemma 3.17 (ii).

**Case 2.2:** $\nu = -$. Then for $L \geq 1$ we have $|\xi_2^L|\nabla_{\xi_2}^L \Phi_{\mu^+} | \lesssim_L |\xi_2|^{L(1 + |\xi_2|)}$. By (4.6) and the Sobolev multiplication theorem, $|\xi_2| \lesssim \Phi_{\mu^+}$, so by \eqref{4.10} and Lemma 3.17, the bound can be improved to $\Phi_{\mu^+}^{-1} \lesssim 2^{-k_3^+}$.

Now we bound \eqref{4.7}, \eqref{4.8} and \eqref{4.9}. By Lemma 3.18 and Lemma 4.5

$$I_S^{\mu'}[T_1 P_{k_1} f_1, T_2 P_{k_2} f_2, P_{k_3}^2 \Lambda^{2N+1} T_3^+ f_3] \lesssim \sup_{k_1 \in \mathbb{Z}} \|P_{k_1} f_1\|_{W^8+m, \infty} \|P_{k_2} f_2\|_{H^{N-m}} \|P_{k_3} f_3\|_{H^{N-m}}.$$  \hfill (4.12)

A similar bound with $f_1$ and $f_2$ swapped holds. The additive restriction of frequencies implies $|k_1 - k_2| = O(1)$, so summing over $k_1$, $k_2$ and $k_3$ using the Cauchy-Schwarz inequality gives

$$I_S^{\mu'}[T_1 f_1, T_2 f_2, P_{k_3}^2 \Lambda^{2N+1} T_3^+ f_3] \lesssim \sup_{k_1 \in \mathbb{Z}} \|P_{k_1} f_1\|_{W^8+m, \infty} \sum_{k,l \in \mathbb{Z}} 2^{-N|m|} \|P_{k+l} f_2\|_{H^N} \|P_{k_3} f_3\|_{H^{N-m}}$$

$$\lesssim \sup_{k \in \mathbb{Z}} \|P_{k} f_1\|_{W^8+m, \infty} \|f_2\|_{H^N} \|f_3\|_{H^{N-m}}.$$  \hfill (4.12)

By \eqref{4.12} with $f_1 = U_{\mu}$, $f_2 = U_{\nu}$, $f_3 = U$, $m = 0$ and $N \geq 9$,  \hfill (4.7)

$$\lesssim \|U(t)\|^3_{H^N} + \|U(0)\|^3_{H^N} \lesssim \|U\|^3_{L^\infty([0,t])H^N}.$$  \hfill (4.7)

By \eqref{4.6} and Lemma 1.4(iii), $\|P_{k} N\|_{W^8, \infty} \lesssim \|\rho, v\|_{W^8, \infty} \lesssim \|U\|^2_X$, so by \eqref{4.12} with $f_1 = N_{\mu}$, $f_2 = U_{\nu}$, $f_3 = U$, $m = 0$,  \hfill (4.8)

$$\lesssim \|U\|^2_{L^2([0,t])X} \|U\|^2_{L^\infty([0,t])H^N}.$$  \hfill (4.8)

By \eqref{4.6} and the Sobolev multiplication theorem, $\|N\|_{H^{N-1}} \lesssim \|U\|_X \|U\|_{H^N}$, so by \eqref{4.12} with $f_1 = U_{\mu}$, $f_2 = U_{\nu}$, $f_3 = N$, $m = 1$, the same holds for \eqref{4.9}.

Combining the three bounds shows the proposition.  \hfill $\square$

### 4.4. Bounding the quasilinear energy.

**Proposition 4.6.** If $N \geq 10$ and $\|U\|_{H^3}$ is sufficiently small then

$$\int_0^t \mathcal{E}_Q(s) ds \lesssim \|U\|^3_{L^\infty([0,t])H^N} + \|U\|^2_{L^2([0,t])X} \|U\|^2_{L^\infty([0,t])H^N}.$$  \hfill (4.13)

**Proof.** Up to a quartic error like the second term in $\mathcal{E}_4$, which can be bounded using \eqref{4.13}, Lemma 3.9 Lemma 3.10 Lemma 3.14 and the bound

$$\|\sqrt{1 + \rho - 1 - \rho/2}\|_{W^{1, \infty}} \lesssim \|\rho\|_{W^{1, \infty}}.$$
Lemma 4.7. \[ E^\mu_\nu = \text{Re} \frac{C^2}{R^3} \sum_{\xi_j \in (2\pi Z/R)^2} q(\xi_1, \xi_2) \bar{U}_\mu(\xi_1) \bar{U}_\nu(\xi_2) \mathcal{U}(\xi_1 + \xi_2), \]

\[ q(\xi_1, \xi_2) = n_1(\xi_1)n_2(\xi_2)n_3(\xi_1 + \xi_2) \left[ n_4(\xi_1 + \xi_2)n_5(\xi_2) - (n_4n_5) \left( \frac{\xi_1 + 2\xi_2}{2} \right) \right] \]

\[ \times \varphi_{\leq -10} \left( \frac{\left| \xi_1 \right|}{\left| \xi_1 + 2\xi_2 \right|} \right). \]

Let \( n_j \in S_{1,0}^{m_j} \), \( m_j = 2N + 1 \) and \( n_2, n_3 \leq \varphi_{\geq -1}. \) Let

\[ I^\mu_\nu[f_1, f_2, f_3] = \text{Re} \frac{C^2}{R^3} \sum_{\xi_j \in (2\pi Z/R)^2} \frac{q(\xi_1, \xi_2)}{\Phi^\mu_{\mu\nu}(\xi_1, \xi_2)} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_1 + \xi_2), \]

\[ I^\mu_\nu_Q = I^\mu_\nu[U_\mu, U_\nu, \mathcal{U}]. \]

Similarly integration by parts in time gives

\[ \int_0^t E^\mu_\nu(s) ds = I^\mu_\nu_Q(t) - I^\mu_\nu_Q(0) \]  \hspace{1cm} (4.13)

\[ + \int_0^t I^\mu_\nu_Q[N_\mu, U_\nu, \mathcal{U}](s) ds \]  \hspace{1cm} (4.14)

\[ + \int_0^t (I^\mu_\nu_Q[U_\mu, (U_\nu)_t + i\Delta U_\nu, \mathcal{U} + i\Delta \mathcal{U}](s) ds. \]  \hspace{1cm} (4.15)

The bound then follows from the corresponding bounds for (4.13), (4.14) and (4.15), to be shown in Proposition 4.8 and Proposition 4.9 below. \( \square \)

To estimate \( I^\mu_\nu_Q \), we need to bound the \( S^\infty \) norm of the \( q \) multiplier.

**Lemma 4.7.**

\[ ||q||_{S^\infty_{k_1, k_2, k_3}} \lesssim 2^{2Nk_3^3 + k_1} 1_{k_1 \leq k_3 - 6}, \quad ||\nabla q||_{S^\infty_{k_1, k_2, k_3}} \lesssim 2^{(2N - 1)k_3^3 + k_1} 1_{k_1 \leq k_3 - 6}. \]

**Proof.** The support part comes from the \( \varphi_{\leq -10} \) factor. The bound follows from the identity

\[ n_4(\xi_1 + \xi_2)n_5(\xi_2) - (n_4n_5) \left( \frac{\xi_1 + 2\xi_2}{2} \right) = \frac{1}{2} \int_0^1 \xi_1 \cdot \nabla (n_4(\xi_1)n_5(\eta_1)) dt, \]

where \( \xi_t = ((1 + t)\xi_1 + 2\xi_2)/2 \) and \( \eta_t = ((1 - t)\xi_1 + 2\xi_2)/2 \). Then we use Lemma 3.17 (i), (ii) to bound the \( S^\infty \) norm of the integrand. The bound on \( \nabla q \) follows in a similar way. \( \square \)

**Proposition 4.8.** If \( N \geq 10 \) and \( ||U||_{H^2} \) is sufficiently small then

\[ (4.13) \lesssim ||U||_{L^\infty([0,t])H^N}, \quad (4.14) \lesssim ||U||_{L^2([0,t])X} ||U||_{L^\infty([0,t])H^N}. \]

**Proof.** By Lemma 4.5 and Lemma 4.7 \( ||q/\Phi^\mu_{\mu\nu}||_{S^\infty_{k_1, k_2, k_3}} \lesssim 2^{2Nk_3^3 + 7k_1^3 + k_1} \), so by Lemma 3.18

\[ I^\mu_\nu_Q[P_{k_1}f_1, P_{k_2}f_2, P_{k_3}f_3] \lesssim 2^{2Nk_3^3 + 7k_1^3 + k_1} ||P_{k_1}f_1||_{L^\infty} ||P_{k_2}f_2||_{L^2} ||P_{k_3}f_3||_{L^2}. \]
Summing over $k_1 \in \mathbb{Z}$ and $k_2 - k_3 = O(1)$ and using the Cauchy-Schwarz inequality gives
\[
I_Q^{\mu \nu}[f_1, f_2, f_3] \lesssim \sum_{k \in \mathbb{Z}} 2^{k(k+1)} \|P_k f_1\|_{L^\infty} \|f_2\|_{H^N} \|f_3\|_{H^N}.
\] (4.16)

Now put $f_1 = U_\mu$, $f_2 = U_\nu$ and $f_3 = U$. By Lemma 1.4 (ii), Proposition 4.2 and $N \geq 10$,
\[
(4.13) \lesssim \|U(t)\|_{H^N}^3 + \|U(0)\|_{H^N}^3 \lesssim \|U\|_{L^\infty([0,t])}^3 \|U\|_{H^N}.
\]

The integrand in (4.14) can be bounded by ($f_1 = N_\mu$, $f_2 = U_\nu$, $f_3 = U$)
\[
I_Q^{\mu \nu}[N_\mu, U_\nu, U] \lesssim \sum_{k \in \mathbb{Z}} 2^{k(k+1)} \|P_k (\rho v, v^2)\|_{L^\infty} \|U\|_{H^N}^2 \lesssim \|(\rho v, v^2)\|_{B^0_{\infty,1}} \|U\|_{H^N}^2.
\]

Since $M \geq 9$, by [Tr], Theorem 2 (i) and Lemma 1.4 (iii) we have
\[
\|(\rho v, v^2)\|_{B^0_{\infty,1}} \lesssim \|(\rho, v)\|_{B^0_{\infty,1}}^2 \lesssim \|U\|_{X}^2.
\]

Integrating in time gives
\[
(4.14) \leq \int_0^t I_Q^{\mu \nu}[N_\mu, U_\nu, U](s)ds \lesssim \|U\|_{L^2([0,t])}^2 \|U\|_{L^\infty([0,t])}^2 \|U\|_{H^N}.
\]

**Proposition 4.9.** If $\|U\|_{H^3}$ is sufficiently small then
\[
(4.15) \lesssim \|U\|_{L^2([0,t])}^2 \|U\|_{L^\infty([0,t])}^2 \|U\|_{H^N}.
\]

**Proof.** By (4.4), the integrand of (4.15) becomes
\[
I_Q^{\mu \nu}[U_\mu, (Q + S + C)_\nu, U] + I_Q^{\mu \nu}[U_\mu, U_\nu, Q + S + C] - I_Q^{\mu \nu}[U_\mu, U_\nu, iT_{v_\nu} U] - I_Q^{\mu \nu}[U_\mu, iT_{v_\nu} U, U] - I_Q^{\mu \nu}[U_\mu, U_\nu, iT_{(\sqrt{1-p} - 1)\Lambda(\xi)} U].
\] (4.17)

By (4.16), (4.2), (4.4), Lemma 3.8, Lemma 3.9, Lemma 3.10, Lemma 3.14 and the fact that $\|(\rho, v)\|_{W^{1,\infty}} \lesssim \|(\rho, v)\|_{H^3} \lesssim \|U\|_{H^3}$ is sufficiently small,
\[
(4.17) \lesssim \|U\|_{X} \|U\|_{H^N} \|Q + S + C\|_{H^N} \lesssim \|U\|_{X}^2 \|U\|_{H^N}^2.
\]

For (4.18), by Lemma 3.9 (ii), the operator $iT_{v_\xi}$ maps real valued functions to real valued functions, so $(iT_{v_\xi} U)_{\nu} = iT_{v_\xi} U_{\nu}$. Taking the complex conjugation on the third slot of $I_Q^{\mu \nu}$ into account we have
\[
(4.18) = \Re \sum_{\xi, \eta, \theta \in (2\pi \mathbb{Z})^2} r_{\mu \nu}[\xi, \eta, \theta] \hat{U}_\mu(\xi - \eta - \theta) \hat{f}(\eta) \hat{\varphi}(\xi) \hat{v}(\theta),
\]
where
\[
r_{\mu \nu}[\xi, \eta, \theta] = \frac{2\eta_1 + \theta_1}{2} \times \frac{q_{k_1, k_2, k_3}(\xi - \eta - \theta, \eta + \theta)}{\Phi_{\mu+}(\xi - \eta - \theta, \eta + \theta)} \varphi_{\mu+} \left( \left| \frac{\theta}{2\eta + \theta} \right| \right) \varphi_{\mu+} \left( \left| \frac{|\theta|}{2\eta - \theta} \right| \right),
\]
\[
q_{k_1, k_2, k_3}(\xi_1, \xi_2) = n(\xi_1, \xi_2) \varphi_{k_1}(\xi_1) \varphi_{k_2}(\xi_2) \varphi_{k_3}(\xi_1 + \xi_2).
\]
Since
\[
(\partial_2 \Phi^{-1}_{\mu+})(\xi - \eta - \theta, \eta + t\theta) = \frac{\nabla \Lambda(\xi - \theta + t\theta) - \nabla \Lambda(\eta + t\theta)}{\Phi(\xi - \theta, \eta + t\theta)^2} = \int_0^1 (\xi - \eta - \theta) \cdot \nabla^2 \Lambda(\eta + s(\xi - \eta - \theta) + t\theta) ds
\]

Thus by Lemma 3.17 for \(k_1 \leq k_2 - 6\) and \(k_3 = k_2 + O(1)\) we have
\[
\|\partial_2 (\Phi^{-1}_{\mu+})(\xi - \eta - \theta, \eta + t\theta)\|_{S^\infty_{k_1,k_2,k_3}} \lesssim 2^{8k_1^+ + k_1^+}.
\]

Using the fundamental theorem of calculus as in Lemma 4.7 we obtain
\[
\|r_{\mu,j}(\xi, \eta, \theta)\varphi_{k_4}(\theta)\|_{S^\infty} \lesssim 2^{2Nk_4^+ + 10k_4^+ + k_4^+} 1_{k_1,k_4 \leq k_3 - 5}.
\]

Using Lemma 3.18 and Lemma 1.3 (iii) and summing over \(k_1, k_4 \leq k_3 - 5\) and \(k_2 = k_3 + O(1)\) gives
\[
(4.18) \lesssim \|U\|_X^2 \|U\|_{H^N}^2.
\]

For (4.19) we distinguish two cases.

**Case 1:** \(\nu = +\). Since \((\sqrt{1 + \rho} - 1)\Lambda(\xi)\) is real valued, we get the same cancellation as in (4.18) from the complex conjugation in the definition of \(I_Q^{\mu+}\), so by Lemma 3.18
\[
(4.19) \lesssim \|U\|_X^2 \|U\|_{H^N}^2.
\]

**Case 2:** \(\nu = -\). On the support of \(q\) we have \(k_1 \leq k_2 - 6\) and \(k_3 \geq 0\), so Case 2.2 of Lemma 4.5 gives
\[
\|\Phi^{-1}_{\mu+}\|_{S^\infty_{k_1,k_2,k_3}} \lesssim 2^{-k_3}, \text{ so } \|q/\Phi_{\mu+}\|_{S^\infty_{k_1,k_2,k_3}} \lesssim 2^{(2N-1)k_3 + k_1}.
\]

This can be used to obtain the desired bound by recovering the loss of derivative in \(T_{(\sqrt{1 + \rho} - 1)\Lambda(\xi)}\):
\[
\|T_{(\sqrt{1 + \rho} - 1)\Lambda(\xi)}U\|_{H^{N-1}} \lesssim \|U\|_X \|U\|_{H^N}.
\]

Then Lemma 3.18 gives
\[
(4.19) \lesssim \|U\|_X^2 \|U\|_{H^N}^2.
\]

Combining the three bounds and integrating in time show the claim.

4.5. Quartic energy estimates.

**Proof of (2.5).** By (1.14) and conservation of the energy \(E(\theta)\) we know that \(\sup_{[0,t]} E \lesssim \epsilon^2\). By Proposition 4.2 (1.14) and (2.2) we know that \(E(0) \lesssim \epsilon^2\) and \(\|P_{\geq 0} U(t)\|_{H^N}^2 \leq E(t) + \epsilon^2\).

By Proposition 4.3, Proposition 4.4 and Proposition 4.6 we have
\[
E(t) = E(0) + O(\epsilon^3 + \epsilon_1^2 \epsilon_2^2) \lesssim \epsilon^2 + \epsilon_1^3 + \epsilon_1^2 \epsilon_2^2.
\]

Then the same bound holds for \(\|P_{\geq 0} U(t)\|_{H^N}^2\), and
\[
\|U(t)\|_{H^N}^2 \lesssim \|P_{\geq 0} U(t)\|_{H^N}^2 + E(t) \lesssim \epsilon^2 + \epsilon_1^3 + \epsilon_1^2 \epsilon_2^2.
\]

Taking the square root gives (2.3).

**Proof of (2.6).** Integrate Lemma 3.2 (ii) in \(t\) and put it in (2.3).
5. Strichartz estimates for small $R$

5.1. Definition of the profile. The evolution equation (4.6) for $U$ can be rewritten as

\[ U_t + iAU = \sum_{\mu, \nu = \pm} N_{\mu\nu}[U_{\mu}, U_{\nu}]. \]

Define the profile $V_\pm(t) = e^{\pm i t \Lambda} U_\pm(t)$. Then the evolution equation for $V$ is

\[ \dot{V}_t(\xi) = \frac{C}{R^2} \sum_{\mu, \nu = \pm} \sum_{\xi_1, \xi_2 \in (2\pi R)^2} e^{it\Phi_{\mu\nu}(\xi_1, \xi_2)} m_{\mu\nu}(\xi_1, \xi_2) \dot{V}_\mu(\xi_1) \dot{V}_\nu(\xi_2), \tag{5.1} \]

where $\Phi_{\mu\nu}$ is defined in (4.5), and $m_{\mu\nu}$ are sums of terms of the form $a_0 a_1$,

\[ a_0 \in \{ \Lambda(\xi_1 + \xi_2)|\xi_1|/\Lambda(\xi_1), \Lambda(\xi_1 + \xi_2)|\xi_2|/\Lambda(\xi_2), |\xi_1 + \xi_2| \} \tag{5.2} \]

and $a_1$ is a Calderon-Zygmund operator on $(\mathbb{R}^2)^2$. By Lemma 4.4 (v) we can assume $a_1 = 1$ is the identity.

Thanks to Lemma 4.5 we can integrate (5.1) by parts in $t$ to get

\[ V(t) - V(0) = \sum_{\mu, \nu = \pm} \left( W_{\mu\nu}(t) - W_{\mu\nu}(0) - \int_{t_0}^t H_{\mu\sigma}(s) ds \right), \tag{5.3} \]

\[ \mathcal{F}W_{\mu\nu}(\xi, t) = \frac{C}{R^2} \sum_{\xi_1, \xi_2 \in (2\pi R)^2} e^{it\Phi_{\mu\nu}(\xi_1, \xi_2)} m_{\mu\nu}(\xi_1, \xi_2) \dot{V}_\mu(\xi_1, t) \dot{V}_\nu(\xi_2, t), \tag{5.4} \]

\[ \mathcal{F}H_{\mu\rho}(\xi, t) = \frac{C^2}{R^4} \sum_{\xi_1, \xi_2, \xi_3 \in (2\pi R)^2} e^{it\Phi_{\mu\rho}(\xi_1, \xi_2, \xi_3)} m_{\mu\rho}(\xi_1, \xi_2, \xi_3) \dot{V}_\mu(\xi_1, t) \dot{V}_\nu(\xi_2, t) \dot{V}_\rho(\xi_3, t), \tag{5.5} \]

where $m_{\mu\rho}$ is a linear combination of multipliers of the form $m_{\mu\sigma}m_{\nu\rho}/\Phi_{\mu\sigma}$ or $m_{\sigma \rho}m_{\mu\nu}/\Phi_{\sigma \rho}$, see (5.2). We need to bound the $S^\infty$ norms of such multipliers.

**Lemma 5.1.** For $k_j \in \mathbb{Z}$ we have

\[ \|m_{\mu\nu}/\Phi_{\mu\nu}\|_{S^\infty_{k_1, k_2, k_3}} \lesssim 2^{\max k_j + 7 \min k_j}. \tag{5.6} \]

If $m_{\mu\rho} = m_{\mu\sigma}m_{\nu\rho}/\Phi_{\mu\sigma}$, then for $k, k_j$ and $l \in \mathbb{Z}$ we have

\[ \|\varphi_l(\xi_2 + \xi_3)m_{\mu\rho}(\xi_1, \xi_2, \xi_3)\|_{S^\infty_{k_1, k_2, k_3:k}} \lesssim 2^{\max k_j + 7 \min(k_1, l)} \lesssim 2^{2\max k_j + 7 \text{med} k_j}. \tag{5.7} \]

Similar bounds hold if $m_{\mu\rho} = m_{\sigma \rho}m_{\mu\nu}/\Phi_{\sigma \rho}$.

**Proof.** (5.6) and the first bound of (5.7) follow from (5.2), Lemma 3.17 and Lemma 4.5. The second bound of (5.7) follows from the fact that at least two of $k_j \geq \min(k_1, l) - O(1)$. \hfill $\square$

5.2. Strichartz Estimates. In this subsection we show the Strichartz estimate (2.3).

**Proposition 5.2.** Assume $N \geq M + 5$ and (2.2). Then

\[ \|U\|_{L^2([0, t]) \times \mathcal{S}^2_1} \lesssim \mathcal{L}_R \sqrt{1 + t/R\epsilon_1(1 + \mathcal{L}_R^{3/2}\epsilon_2^2)}. \]
Proof. From (5.3) it follows that

$$U(t) = e^{-it\Lambda} \left( U(0) + \sum_{\mu, \nu = \pm} (W_{\mu\nu}(t) - W_{\mu\nu}(0)) + \int_0^t \sum_{\mu\nu\rho} H_{\mu\nu\rho}(s) ds \right).$$

**Part 1:** The linear term. By Lemma 3.3 and (2.2),

$$\|e^{-is\Lambda}U(0)\|_{L^2([0,t])^X} \lesssim \mathcal{L}_R \sqrt{1 + t/R\epsilon_1}.$$

**Part 2:** The quadratic boundary terms. We rewrite (5.4) as

$$e^{-it\Lambda(\xi)}\hat{W}_{\mu\nu}(\xi, t) = \frac{C}{R^2} \sum_{\xi_1, \xi_2 \in (2\pi R)^2} m_{\mu\nu}(\xi_1, \xi_2) \hat{U}_\mu(\xi_1, t)\hat{U}_\nu(\xi_2, t).$$

We view $W_{\mu\nu} = W_{\mu\nu}[V_\mu, V_\nu]$ as a bilinear form and decompose

$$W_{\mu\nu} = \sum_{k_1, k_2 \in \mathbb{Z}} W_{\mu\nu}^{k_1, k_2}, \quad W_{\mu\nu}^{k_1, k_2} = W_{\mu\nu}[P_{k_1}V_\mu, P_{k_2}V_\nu].$$

By symmetry we can assume $k_1 \leq k_2$. Since by (5.2) and (4.10),

$$(m_{\mu\nu}/\Phi_{\mu\nu})(\xi_1, \xi_2) \lesssim (|\xi_1| + |\xi_2|)(1 + \min(|\xi_1|, |\xi_2|, |\xi_1 + \xi_2|)),$$

we have

$$\|W_{\mu\nu}^{k_1, k_2}\|_{L^2} \lesssim 2^{k_2 + k_1 + |k_2|} \|\phi_{k_1}\mathcal{F}U\|_{L^1}\|\phi_{k_2}\mathcal{F}U\|_{L^2}$$

$$\lesssim 2^{-Nk_2 + k_1 + k_2 + k_1}\|P_{k_1}U\|_{L^2}\|P_{k_2}U\|_{H^N}, \quad (5.10)$$

$$\|P_k W_{\mu\nu}^{k_1, k_2}\|_{L^2} \lesssim 2^{k} \|\mathcal{F}W_{\mu\nu}^{k_1, k_2}\|_{L^\infty} \lesssim 2^{k_2 + k_1 + k}\|P_{k_1}U\|_{L^2}\|P_{k_2}U\|_{L^2}$$

$$\lesssim 2^{-Nk_2 + k_1 + k_2 + k}\|P_{k_1}U\|_{L^2}\|P_{k_2}U\|_{H^N}. \quad (5.11)$$

Since $\mathcal{F}W_{\mu\nu}^{k_1, k_2}$ is supported on the ball $B(0, O(2^{k_2}))$, by (2.2) and (5.10) we have

$$\|W_{\mu\nu}^{k_1, k_2}\|_{H^{M+2}} \lesssim 2^{(M+2-N)k_2 + k_1 - |k_1|}.$$
view it as a trilinear form $H_{\mu \nu \rho} = H_{\mu \nu \rho}(V_{\mu}, V_{\nu}, V_{\rho})$ and decompose

$$H_{\mu \nu \rho} = \sum_{l \in \mathbb{Z}, \sigma = \pm} H_l^{\mu \nu \rho},$$

$$H_l^{\mu \nu \rho}[V_{\mu}, V_{\nu}, V_{\rho}](t) = W_{\mu \sigma}[V_{\mu}, P_l e^{it \Lambda} N_{\nu \rho}^\sigma [e^{-it \Lambda} V_{\nu}, e^{-it \Lambda} V_{\rho}]]$$

$$+ W_{\sigma \rho}[P_l e^{it \Lambda} N_{\mu \nu}^\sigma [e^{-it \Lambda} V_{\mu}, e^{-it \Lambda} V_{\nu}], V_{\rho}],$$

(5.12)

Assume $k_1 \leq k_2 \leq k_3$. By Lemma 3.18 ($p_1 = p_2 = \infty$, $p_3 = 2$) and (5.7),

$$\|P_l H_{k_1 k_2 k_3, l}\|_{L^2} \lesssim 2^{k_3 + 7k_2^2} \|P_l U\|_{L^\infty} \|P_k U\|_{L^\infty} \|P_k U\|_{L^2}.$$

Summing over $k_1, k_2 \in \mathbb{Z}$, $-\mathcal{L}_R \leq l \leq k_3 + O(1)$ and $k_3 \geq k - O(1)$ we get

$$\|P_k H_{\mu \nu \rho}(t)\|_{L^2} \lesssim (\mathcal{L}_R + k^+ + 1) 2^{(2-N)k^+} \|U(t)\|_{H^N}^3 \|U(t)\|_{H^N}.$$

Taking an $\ell^2$ sum in $k \geq -\mathcal{L}_R$ and using (2.2) and $N \geq M + 5$ we get

$$\|H_{\mu \nu \rho}(t)\|_{H^{M+2}} \lesssim \mathcal{L}_R^{3/2} \|U(t)\|_X^{2/3} \epsilon_1.$$

By Lemma 3.3 we then have

$$\left\| e^{-is \Lambda} \int_0^s H_{\mu \nu \rho}(s') ds' \right\|_{L^2([0,t], X)} \lesssim \int_0^t \left\| e^{-is \Lambda} H_{\mu \nu \rho}(s') \right\|_{L^2([s', t], X)} ds'$$

$$\lesssim \mathcal{L}_R \sqrt{1 + t/R} \|H_{\mu \nu \rho}\|_{L^1([0,t], H^{M+2})}$$

$$\lesssim \mathcal{L}_R^{5/2} \sqrt{1 + t/R} \epsilon_1^2.$$

Combining Part 1 through Part 3 shows the claim. Note that Part 2 is dominated by Part 1 and Part 3. □

For future use we also need to bound $H_l^{\mu \nu \rho}$ for small $l$.

**Proposition 5.3.** Assume $N \geq M + 3$ and (2.2). Then for $l \leq 0$ we have

$$\|H_l^{\mu \nu \rho}\|_{H^{M+2}} \lesssim 2^{3l} \epsilon_1^3.$$

**Proof.** We use $m_{\mu \nu}(|\xi_1, \xi_2| \lesssim |\xi_1| + |\xi_2|$ to get

$$\|F N_{\nu \rho}^\sigma [V_{\nu}, V_{\sigma}]\|_{L^\infty} \lesssim \|V\|_{H^1}^2 \lesssim \epsilon_1^2.$$

When $|\xi_2| \lesssim 1$, by (5.9) we also have $(m_{\mu \nu}/\Phi_{\mu \nu})(|\xi_1, \xi_2| \lesssim \Lambda(\xi_1)$, so

$$\|W_{\mu \sigma}[V_{\mu}, P_l e^{it \Lambda} N_{\nu \rho}^\sigma [V_{\nu}, V_{\rho}]\|_{H^{M+2}} \lesssim \|F A^{M+3} V\| + \|F P_l N_{\nu \rho}^\sigma [V_{\nu}, V_{\sigma}]\|_{L^2}$$

$$\lesssim \|V\|_{H^{M+3}} \|F P_l N_{\nu \rho}^\sigma [V_{\nu}, V_{\sigma}]\|_{L^1}$$

$$\lesssim 2^{3l} \epsilon_1^3$$

and a similar bound holds for $\|P_l W_{\sigma \rho}[P_l N_{\nu \rho}^\sigma [V_{\nu}, V_{\sigma}], V_{\rho}]\|_{H^{M+2}}$. □
6. Z-norm estimates for large $R$

This section is devoted to the proof of the Z norm estimate (2.7), which is contained in Proposition 6.3 and Proposition 6.4 below.

6.1. Integration by parts in phase space. We need a lemma to integrate by parts in phase space.

**Lemma 6.1** ([22], Lemma A.2 or [27], Lemma 5.4). Let $0 < \epsilon \leq 1/\epsilon \leq K$. Suppose $f, g : \mathbb{R}^d \to \mathbb{R}$ satisfies $|\nabla f| \geq 1_{\text{supp}g}$, and for all $L \geq 2$, $\nabla^L f \lesssim L^{1-L}$ on $\text{supp} g$.

Then

$$\int e^{iKf} g \lesssim_{d,L} (K\epsilon)^{-L} \sum_{l=0}^L \epsilon^l \|\nabla^l g\|_{L^1}.$$ 

We will only use the case when $\epsilon = 1$, for which the bound reads

$$\int e^{iKf} g \lesssim_{d,L} K^{-L} \|g\|_{W^{L,1}}.$$ 

**Lemma 6.2.** Let $r \geq 2$. Suppose $f, g : \mathbb{R}^d \to \mathbb{R}$ satisfies $|\nabla f| \leq r/2$ on $\text{supp} g$, and for all $L \geq 2$, we have $\nabla^L f \lesssim_L r$ on $\text{supp} g$. Let

$$K(x) = \int e^{i(x \cdot \xi + f(\xi))} g(\xi) d\xi.$$ 

Then for all $L \geq 1$ we have

$$\|K\|_{L^1(\mathbb{R}^d \setminus B(0,r))} \lesssim_L r^{-L} \|g\|_{W^{L+1,d,1}}.$$ 

**Proof.** We have

$$x \cdot \xi + f(\xi) = \frac{|x|}{2} \left( \frac{2x}{|x|} \cdot \xi + \frac{2f(\xi)}{|x|} \right).$$

Suppose $|x| \geq r$. Let $K = |x|/2 \geq r/2 \geq 1$, and

$$F(\xi) = \left( \frac{2x}{|x|} \cdot \xi + \frac{2f(\xi)}{|x|} \right).$$

Then $|\nabla F| \geq 1$ and $\nabla^L F \lesssim_L 1$. By Lemma 6.1 for $|x| \geq r$ we have

$$K(x) \lesssim_L |x|^{-L} \|g\|_{W^{L,1}}.$$ 

The result follows from integrating this bound with $L + d$ in place of $L$. 

6.2. Bounding the quadratic boundary terms. In this section we bound the Z norm of the quadratic boundary terms $W_{\mu\nu}$.

**Proposition 6.3.** Assume $N \geq 3(M+4)$ and (2.7). Then

$$\|W_{\mu\nu}(t)\|_Z \lesssim (t^{1+9/N}/R^{4/3} + 1)\epsilon_1^2.$$
Proof. We use the decomposition (5.8) and assume by symmetry $k_1 \leq k_2$ (except in Case 4.1 below). We distinguish several cases to estimate

$$
\|W_{\mu\nu}\|_Z \approx \left\|2^{j/3}Q_j \Lambda^{M+2} W_{\mu\nu}\right\|_{L^2}^2 \lesssim \sum_{j \geq 0} 2^{-j/N}\|U\|^2_{H^N} \lesssim \epsilon_1^2.
$$

**Case 1:** $k_2 \geq j/N$. We sum (5.10) over $k_1 \in \mathbb{Z}$ and $k_2 \geq j/N$, and use $(M + 3 - N)k_2 \leq -2Nk_2/3 - k_2 \leq -2j/3 - j/N$ to get

$$
\left\|\sum_{k_1 \in \mathbb{Z}, k_2 \geq j/N} 2^{2j/3}\|Q_j \Lambda^{M+2} W_{k_1,k_2}\|_{L^2}^2\right\|_{L^2}^2 \lesssim \sum_{j \geq 0} 2^{-j/N}\|U\|^2_{H^N} \lesssim \epsilon_1^2.
$$

**Case 2:** $k_1 \leq -3j/4$. We sum (5.10) with $k_1 \leq -3j/4$ and $k_2 \in \mathbb{Z}$ and use $N \geq M + 4$ to get the same bound as **Case 1**.

**Case 3:** $P_{k_3} W_{\mu\nu}$ for $k_3 \leq -3j/4 < k_1$. We sum (5.11) with $k_1 \in [k_3, k_2]$, $k_3 \leq -3j/4$ and $k_2 \in \mathbb{Z}$ to get the same bound as **Case 1**.

**Case 4:** $-3j/4 < k_3 \leq j/N + O(1)$, $1 \leq i \leq 3$.

**Case 4.1:** $j \leq \mathcal{L} + 5$. We decompose

$$
W_{k_1,k_2} = \sum_{j_1,j_2 \geq 0} W_{j_1,j_1,j_2,k_2}^{\mu\nu},
$$

$$
W_{j_1,j_1,j_2,k_2}^{\mu\nu} = W_{j_1}^{\mu\nu}[P_{[k_1-1,k_1+1]}Q_{j_1} P_{k_1} V_{\mu}, P_{[k_2-1,k_2+1]}Q_{j_2} P_{k_2} V_{\nu}].
$$

We now assume $j_1 \leq j_2$ instead of $k_1 \leq k_2$. By Lemma 3.18, 5.6, Lemma 3.1 conservation of the $L^2$ norm, Hölder’s inequality and Lemma 1.4 (iv) we have

$$
\|\Lambda^{M+2} P_{k_3} W_{j_1,j_1,j_2,k_2}^{\mu\nu}(t)\|_{L^2} \lesssim 2^{(M+3)K+7k_2^+} \left|\epsilon - i t \mu \Lambda\right| Q_{j_1} P_{k_1} V_{\mu}(t) \|_{L^\infty} \|e^{-it\mu\Lambda} Q_{j_2} P_{k_2} V_{\nu}(t)\|_{L^2}
$$

$$
\lesssim 2^{(M+3)K+7k_2^+ + 2k_1^+} \left|\epsilon - i t \mu \Lambda\right| Q_{j_1} P_{k_1} V_{\mu}(t) \|_{L^1} \|Q_{j_2} P_{k_2} V_{\nu}(t)\|_{L^2}
$$

$$
\lesssim 2^{8K + j_1/3 - 2j_2/3 - t/R + 1/2} \epsilon_1^2,
$$

where we recall $K = \max k_i$. We sum over $j_1, j_2 \in \mathbb{Z}$, $j_1 \leq j_2$ and $-\mathcal{L} - 5 < -3j/4 < k_i \leq j/N + O(1) \leq \mathcal{L}/N + O(1)$ to get

$$
\sum_{-2j/3 < k_i \leq j/N + O(1)} \|\Lambda^{M+2} P_{k_3} W_{k_1,k_2}^{\mu\nu}(t)\|_{L^2} \lesssim \frac{(t/R + 1)^2}{(1 + t)} \epsilon_1^2.
$$

Then we sum over $0 \leq j \leq \min(\mathcal{L}, \log R) + O(1)$ and use $N \geq 27$ to get

$$
\left\|\sum_{-3j/4 < k_i \leq j/N + O(1)} 2^{2j/3}\|Q_j \Lambda^{M+2} P_{k_3} W_{k_1,k_2}^{\mu\nu}(t)\|_{L^2}\right\|_{L^2}^2 \lesssim (t^{1+9/N}/R^{4/3} + 1) \epsilon_1^2.
$$
Case 4.2: $j > \mathcal{L} + 5$. In this case $t < 2^{j-5}$. We decompose
\[
W_{\mu
u}^{k_1,k_2} = A_{k_1,k_2}^{\mu\nu} + B_{k_1,k_2}^{\mu\nu}, \\
A_{k_1,k_2}^{\mu\nu} = W_{\mu\nu}[P[k_{-1},k_{+1}]Q_{\leq j-5}P_k V_\mu P_k V_\nu], \\
B_{k_1,k_2}^{\mu\nu} = W_{\mu\nu}[P[k_{-1},k_{+1}]Q_{\geq j-5}P_k V_\mu P_k V_\nu],
\]
where we have used $P[k_{-1},k_{+1}]P_k = P_k$.

For $A$ we have, by (5.10) and the conservation of the $L^2$ norm,
\[
\|A_{k_1,k_2}^{\mu\nu} \|_{L^2} \lesssim 2^{(M+2-N)k_1^2 + k_2 + k_3} \|Q_{\geq j-5}P_k V_\mu P_k V_\nu \|_{L^2} \|P_k U\|_{H^N}.
\]

We sum over $k_1, k_2 \in \mathbb{Z}$, $j \geq 0$ and use $N \geq M + 4$ and Lemma 1.4 (iv) to get
\[
\left\| \sum_{k_1,k_2 \in \mathbb{Z}} 2^{2j/3} \|Q_j A_{k_1,k_2}^{\mu\nu} \|_{L^2} \right\|_{L^2_{t \geq 0}} \lesssim \|V\|_{H^N} \lesssim \varepsilon_1^2.
\]

To bound $B$, we can assume that the support of $Q_j$ intersects the torus. This implies that $2^{j-1} < R/\sqrt{2}$, or $R > 2^{j-1/2}$. We write
\[
A_{k_1,k_2}^{\mu\nu} = \sum_{\xi_1,\xi_2 \in (2\pi \mathbb{Z}/R)^2} e^{i\xi_1 y + i\xi_2 z} A_{\xi_1,\xi_2}^{\mu\nu} \Lambda(\xi_1,\xi_2) M_{\mu\nu}^\Lambda(\xi_1,\xi_2) M_{\xi_1,\xi_2}^{\mu\nu}
\]
where
\[
K(x,y,z,t) = \sum_{y',z' \in (\mathbb{R}Z)^2} K(x,y+y',z+z',t),
\]
with
\[
K(x,y,z,t) = \int e^{i\phi_{\mu\nu}(\xi_1,\xi_2)} \Lambda(\xi_1,\xi_2) M_{\mu\nu}^{\Lambda}(\xi_1,\xi_2) M_{\xi_1,\xi_2}^{\mu\nu}\Lambda(\xi_1,\xi_2) M_{\mu\nu}^{\Lambda}(\xi_1,\xi_2) M_{\xi_1,\xi_2}^{\mu\nu} d\xi_1 d\xi_2.
\]

Since $\Lambda_{\mu\nu} = \Lambda_{0}$ is of the form (5.2), for $L \geq 0$ we have
\[
\nabla^L m_{\mu\nu}(\xi_1,\xi_2) \lesssim_L 2^{Mk_1 + Lk_2} (\|\xi_1\| + \|\xi_2\|)(1 + \min(\|\xi_1\|,\|\xi_2\|,\|\xi_1 + \xi_2\|) - L).
\]

From (6.2), $\nabla^L (\phi_1 \Lambda^M) \lesssim_L 2^{Mk_+ - Lk}$ and (4.11) it follows that
\[
\nabla^L \left( \Lambda(\xi_1 + \xi_2) M_{\mu\nu}^{\Lambda}(\xi_1,\xi_2) M_{\xi_1,\xi_2}^{\mu\nu}\Lambda(\xi_1,\xi_2) M_{\mu\nu}^{\Lambda}(\xi_1,\xi_2) M_{\xi_1,\xi_2}^{\mu\nu} \right) \lesssim L 2^{(M+2)k_1^2 + k_2} 2^{(M+2)\min k_1 + 2^{-L} \min k_i}.
\]

Using $-3j/4 < k_i \leq j/N + O(1)$ we have
\[
\left\| \Lambda(\xi_1 + \xi_2) M_{\mu\nu}^{\Lambda}(\xi_1,\xi_2) M_{\mu\nu}^{\Lambda}(\xi_1,\xi_2) \right\|_{L^2_{x,y,z} \leq 2^{(M+2)k_1^2 + 3k_2 + 2k_3 + 18j}.
\]

Since $|\nabla \Lambda| \leq 1$, we have $|t \nabla \phi_{\mu\nu}(\xi_1,\xi_2)| < 4t < 2^{j-3}$, so by Lemma 6.2 (with $L = 20$ and $d = 4$),
\[
\|1_{|x-y| \leq 2^{-j}2} K(x,y,z,t) \|_{L^1_{x,y,z} \leq 2^{(M+2)k_1^2 + 3k_2 + 2k_3 - 2j}.
\]
When $\varphi_j(x)\varphi_{\leq j-5}(y) > 0$, we have $|x-y| > 2^{j-2}$, and for all $y' \in (R\mathbb{Z})^2\setminus\{(0,0)\}$, $|x-y-y'| > R/2 - 2^{j-1} > 2^{j-2}$. Then by (6.11),
\[
\|\varphi_j(x)\varphi_{\leq j-5}(y)G(x,y,z,t)\|_{L^1_{y,z}((R/R\mathbb{Z})^2)} \lesssim \text{right hand side of (6.3)}.
\]
Combining this with Bernstein's inequality $\|P_\ell V\|_{L^\infty} \lesssim 2^k \|P_\ell V\|_{L^2} = 2^k \|P_\ell U\|_{L^2}$ we get
\[
\|Q_j \Lambda^{M+2} P_k \mathbb{B}_{k_1,k_2}^{\mu\nu}\|_{L^2} \lesssim 2^j \|Q_j \Lambda^{M+2} P_k \mathbb{B}_{k_1,k_2}^{\mu\nu}\|_{L^\infty} \lesssim 2^{(M+2)k_2^+ + 2k_3 + k_1 - j} \|P_k U\|_{L^2}^2 \|P_{k_2} U\|_{L^2}.
\]
We sum over $k_i \leq j/N + O(1)$, $j > \mathcal{L} + 5$ and use $N \geq M + 9$ to get
\[
\left| \sum_{-3j/4 < k_j \leq j/N + O(1)} 2^{2j/3} \|Q_j \Lambda^{M+2} P_k \mathbb{B}_{k_1,k_2}^{\mu\nu}\|_{L^2} \right|_{\ell^2_{j > \mathcal{L} + 5}} \lesssim \|U\|_{H^N}^2 \leq \epsilon_1^2.
\]
Combining Case 1 through Case 4 above shows Proposition 6.3. □

6.3. Bounding the cubic bulk terms. In this section we bound the $Z$ norm of the cubic bulk terms $H^{\mu\nu\rho}$.

**Proposition 6.4.** Assume $N \geq \max(3(M + 4), 106)$ and (2.7). Then
\[
\int_0^t \|H^{\mu\nu\rho}(s)\|_{Z} ds \lesssim (t^{3+33/N}/R^{10/3-2/N} + 1)^{\epsilon_1^3}.
\]

**Proof.** Recall (5.12). We assume by symmetry $k_1 \leq k_2 \leq k_3$ (except in Case 5.1 below). From (5.2), (5.9) and (4.10) it follows in the same way as (5.10) that
\[
\|\Lambda^{M+2} H^{\mu\nu\rho}_{k_1,k_2,k_3}\|_{L^2} + \|\Lambda^{M+2} H^{\mu\nu\rho}_{k_1,k_2,k_3,l}\|_{L^2} \lesssim 2^{k_1 + k_2 + (M+2)k_2^+ + 2k_3 + k_1} \|P_{k_1} U\|_{L^2} \|P_{k_2} U\|_{L^2} \|P_{k_3} U\|_{L^2}.
\] (6.4)

We distinguish several cases.

**Case 1:** $k_3 \geq 3 \max(j, \mathcal{L})/N$. We sum (6.4) over $k_1, k_2 \in \mathbb{Z}$ and $k_3 \geq 3 \max(j, \mathcal{L})/N$, and use $(M + 4 - N)k_3 \leq -2Nk_3/3 \leq -2 \max(j, \mathcal{L})$ to get
\[
\left| \sum_{k_1, k_2, k_3} 2^{2j/3} \|Q_j \Lambda^{M+2} H^{\mu\nu\rho}_{k_1,k_2,k_3}(t)\|_{L^2} \right|_{\ell^2_{j > 0}} \lesssim \sum_{j > 0} 2^{2j/3 - 2 \max(j, \mathcal{L})} \|U\|_{H^N}^3 \lesssim (1 + t)^{-4/3} \epsilon_1^3.
\]

**Case 2:** $l \leq -6 \max(j, \mathcal{L})/7$. By Proposition 5.3 we have
\[
\left| \sum_{l \leq -6 \max(j, \mathcal{L})/7} 2^{2j/3} \|Q_j \Lambda^{M+2} H^{\mu\nu\rho\sigma}_{l}(t)\|_{L^2} \right|_{\ell^2_{j > 0}} \lesssim \sum_{j > 0} 2^{2j/3 - 12 \max(j, \mathcal{L})/7} \epsilon_1^3 \lesssim (1 + t)^{-1.04} \epsilon_1^3.
\]

**Case 3:** $k_1 \leq -3 \max(j, \mathcal{L})/4$, $k_3 < 3 \max(j, \mathcal{L})/N$ and $l > -6 \max(j, \mathcal{L})/7$ (so $l \geq j + \mathcal{L} + k_3 \leq |k_1|$). By Bernstein’s inequality, Lemma 1.1 (i) and (2.2),
\[
\|P_k U\|_{L^2} = \|P_k V\|_{L^2} \lesssim 2^{3k/5} \|P_k V\|_{L^{5/4}} \lesssim 2^{3k/5} \epsilon_1.
\]
Using moreover Lemma 3.18 (5.7), Lemma 3.2 (i) and $M \geq 7$ we get
\[
\|\Lambda^{M+2} P_k h^{\mu\nu\rho}_{k_1,k_2,k_3,l}(t)\|_{L^2} \lesssim 2^{(M+2)k+2k/20} \|P_k h^{\mu\nu\rho}_{k_1,k_2,k_3,l}(t)\|_{L^{10/21}} \\
\lesssim 2^{(M+2)k+7k/20} \|P_{k_1} u\|_{L^\infty} \|P_{k_2} u\|_{L^\infty} \|P_{k_3} u\|_{L^2} \\
\lesssim 2^{(M+2-N)k_1+2k_3+k/20} \|P_{k_1} u\|_{L^\infty} \|P_{k_2} u\|_{L^\infty} \|P_{k_3} u\|_{L^2} 
\]
We sum over $k_2 \in [k_1,k_3] \cap \mathbb{Z}$, $k \leq k_3 + O(1)$, $k \in \mathbb{Z}$, $|l| \lesssim |k_1|$, $k_1 \leq -3 \max(j,\mathcal{L})/4$ and $j \geq 0$, and use $N \geq M + 5$ to get
\[
\| k_{j \geq 0} \left( \sum_{k_1 \leq -3 \max(j,\mathcal{L})/4} \sum_{k_2 \geq 0} \sum_{k_3 \geq 0} 2^{2j/3} \|Q_j \Lambda^{M+2} h^{\mu\nu\rho}_{k_1,k_2,k_3,l}(t)\|_{L^2} \right) \|_{L^2} 
\]
\[
\lesssim \sum_{j \geq 0} 2^{2j/3-7 \max(j,\mathcal{L})/6} \frac{(t/R + 1)^{4/3}}{(1+t)^{19/30}} \Xi_3 \approx (t/R + 1)^{4/3} \Xi_3 
\]
\[
\leq \sum_{j \geq 0} 2^{2j/3-7 \max(j,\mathcal{L})/6} \frac{(t/R + 1)^{4/3}}{(1+t)^{19/30}} \Xi_3 \approx (t/R + 1)^{4/3} \Xi_3 
\]
\[
\Xi_3 \lesssim (t/R + 1)^{4/3} \Xi_3 
\]

**Case 4:** $k \leq -6 \max(j,\mathcal{L})/7$, and $m_{\mu\rho}$ or $m_{\sigma\rho} = |\xi_1 + \xi_2 + \xi_3|$. Then from (5.9) it follows in the same way as (5.11) that
\[
\|\Lambda^{M+2} P_k h^{\mu\nu\rho}_{k_1,k_2,k_3,l}\|_{L^2} \lesssim 2^{2k_1+k_3+k/3} \|P_{k_1} u\|_{L^2} \|P_{k_2} u\|_{L^2} \|P_{k_3} u\|_{L^2} 
\]
We sum over $k \in \mathbb{Z}$, $k \leq -6 \max(j,\mathcal{L})/7$ to get the same bound as Case 2.

**Case 5:** $-6 \max(j,\mathcal{L})/7 < k_1$, $l < 3 \max(j,\mathcal{L})/N + O(1)$, $1 \leq i \leq 3$ and if $m_{\mu\rho}$ or $m_{\sigma\rho} = |\xi_1 + \xi_2 + \xi_3|$, then $k > -6 \max(j,\mathcal{L})/7$.

**Case 5.1:** $j \leq \mathcal{L} + 5$. We decompose
\[
h^{\mu\nu\rho}_{k_1,k_2,k_3,l} = \sum_{j_1,j_2,j_3} H^{\mu\nu\rho}_{j_1,j_2,j_3,k_1,k_2,k_3,l} 
\]
as in Case 4.1 in the proof of Proposition 5.3. We now assume $j_1 \leq j_2 \leq j_3$ instead of $k_1 \leq k_2 \leq k_3$. By Lemma 3.18 (5.7), Lemma 3.1 the conservation of the $L^2$ norm, Hölder’s inequality and Lemma 1.14 (iv) (we have (recall $K = \max(k_1,k_2,k_3)$)
\[
\|\Lambda^{M+2} P_k h^{\mu\nu\rho}_{j_1,j_2,j_3,k_1,k_2,k_3,l}(t)\|_{L^2} \lesssim 2^{k/N} \|\Lambda^{M+2} P_k h^{\mu\nu\rho}_{j_1,j_2,j_3,k_1,k_2,k_3,l}(t)\|_{L^{2N/(N+1)}} 
\]
\[
\lesssim 2^{k/N+(M+11)K^+} \|e^{-it\mu\Lambda} Q_{j_1} P_{k_1} V(t)\|_{L^{2N}} \|e^{-it\nu\Lambda} Q_{j_2} P_{k_2} V(t)\|_{L^{\infty}} \|e^{-it\rho\Lambda} Q_{j_3} P_{k_3} V(t)\|_{L^2} 
\]
\[
\lesssim 2^{k/N+(M+11)K^+} 2^{k_1+k_2+k_3} \frac{(t/R + 1)^{4-2/N}}{(1+t)^{12-2/N}} \|Q_{j_1} P_{k_1} V(t)\|_{L^{2N/(2N-1)}} \|Q_{j_2} P_{k_2} V(t)\|_{L^1} \|Q_{j_3} P_{k_3} V(t)\|_{L^2} 
\]
\[
\lesssim 2^{k/N+11K^+} (j_1+j_2-2j_3)/3 \frac{(t/R + 1)^{4-2/N}}{(1+t)^{2-2/N}} v_{j_1,k_1}(t) v_{j_2,k_2}(t) v_{j_3,k_3}(t) 
\]
where $v_{j,k} = 2^{(M+2)k+2j/3} \|Q_j P_k V\|_{L^2}$. By the AM-GM inequality and Lemma 1.14 (iii),
\[
v_{j_1,k_1} v_{j_2,k_2} v_{j_3,k_3} \lesssim v_{j_1,k_1}^3 + v_{j_2,k_2}^3 + v_{j_3,k_3}^3, 
\]
\[
\|v_{j,k}\|_{L^3} \lesssim \|v_{j,k}\|_{L^2} \lesssim \|V\| \lesssim \epsilon_1. 
\]
We sum over $j_1, j_2, j_3 \geq 0$ to get
\[
\|A^{M+2}P_k H_{k_1, k_2, k_3, l}^{\mu \nu \rho \sigma}(t)\|_{L^2} \lesssim 2^{k/N+11K^+}(t/R + 1)^{4-2/N}(1+t)^{2-1/N}e_1^3.
\]

We then sum over $-L - 5 < k_i, l \leq 3L/N + O(1)$ and $k \leq 3L/N + O(1)$ to get
\[
\sum_{-L - 5 < k_i, l \leq 3L/N + O(1)} \|A^{M+2}P_k H_{k_1, k_2, k_3, l}^{\mu \nu \rho \sigma}(t)\|_{L^2} \lesssim (t/R + 1)^{4-2/N}(1+t)^{2-35/N}e_1^3.
\]

Then we sum over $j \leq \min(L, \log R) + O(1)$ and use $N \geq 106$ to get
\[
\left\| \sum_{-L - 5 < k_i, l \leq 3L/N + O(1)} 2^{2j/3}\|Q_j A^{M+2}P_k H_{k_1, k_2, k_3, l}^{\mu \nu \rho \sigma}(t)\|_{L^2}\right\|_{L^2_{0 \leq j \leq L+O(1)}}^2 \lesssim (t^{2+33/N}/R^{10/3-2/N} + (1+t)^{-1.003})e_1^3.
\]

**Case 5.2: $j > L+5$.** We decompose
\[
H_{k_1, k_2, k_3, l}^{\mu \nu \rho \sigma} = A_{k_1, k_2, k_3, l}^{\mu \nu \rho \sigma} + B_{k_1, k_2, k_3, l}^{\mu \nu \rho \sigma},
\]
\[
A_{k_1, k_2, k_3, l}^{\mu \nu \rho \sigma} = H_l^{\mu \nu \rho \sigma}[P[k_1-1, k_1+1]Q_{j-4}P_{k_1}V_\mu, P[k_2-1, k_2+1]Q_{j-4}P_{k_2}V_\nu, P[k_3-1, k_3+1]Q_{j-4}P_{k_3}V_\rho],
\]
\[
B_{k_1, k_2, k_3, l}^{\mu \nu \rho \sigma} = \sum_{l_1, l_2, l_3 \geq l \in \{j-4, j-5\}} 2^{3l}P_{l_1, 1, l_2, 2, l_3}Q_{j-4}P_{l_2}V_{l_1}P_{l_3}V_{l_2}, P[k_3-1, k_3+1]Q_{j-4}P_{k_3}V_\rho].
\]

For $A$ we have, by (6.4) and Lemma 1.4(iv),
\[
\|A^{M+2}A_{k_1, k_2, k_3, l}^{\mu \nu \rho \sigma}\|_{L^2} \lesssim 2^{k_1+k_2+(M+3)k_3^+ + 2k_3} \|Q_{j-3}P_{k_1}V\|_{L^2} \|Q_{j-3}P_{k_2}V\|_{L^2} \|Q_{j-3}P_{k_3}V\|_{L^2} \lesssim 2^{-(M+2)(k_1^+ + k_2^+ + k_3^+ + k_1 + k_2 + 2k_3 - 2j^3/3)}e_1^3.
\]

We sum over $-6j/7 < k_i, l \leq 3j/N + O(1)$ and $j > L + 5$, and use $N \geq 28$ to get
\[
\left\| \sum_{-6j/7 < k_i, l \leq 3j/N + O(1)} 2^{2j/3}\|Q_j A^{M+2}A_{k_1, k_2, k_3, l}^{\mu \nu \rho \sigma}(t)\|_{L^2}\right\|_{L^2_{j>L+5}} \lesssim \sum_{j>L+5} j2^{(9/4-3)j}e_1^3 \lesssim (1+t)^{-1.01}e_1^3.
\]
To bound $B$, we assume $R > 2^{j-1/2}$ as before, and write
\[
\sum_{\mu,\nu,\rho,\sigma = \pm} \Lambda^{M+2} B_{k_1, k_2, k_3, k_4, l}(x, t) = \sum_{\mu,\nu,\rho,\sigma = \pm} \int G(x, y, z, w, t) Q_{l} P_{k_1} V_{\mu}(y, t) Q_{l} P_{k_2} V_{\nu}(z, t) Q_{l} P_{k_3} V_{\rho}(w, t) dydw,
\]
\[
G(x, y, z, w, t) = \sum_{y', z', w' \in (R\mathbb{E})^2} K(x, y + y', z + z', w + w', t),
\]
\[
K(x, y, z, w, t) = \int e^{i \phi_{\mu\rho}(\xi_1, \xi_2, \xi_3)} \phi_{\mu\rho}(\xi_1, \xi_2, \xi_3) = \Lambda(\xi_1 + \xi_2 + \xi_3) - \mu \Lambda(\xi_1) - \nu \Lambda(\xi_2) - \rho \Lambda(\xi_3),
\]
\[
\phi_{\mu\rho}(\xi_1, \xi_2, \xi_3) = (x - y) \cdot \xi_1 + (x - z) \cdot \xi_2 + (x - w) \cdot \xi_3 + t \phi_{\mu\rho}(\xi_1, \xi_2, \xi_3),
\]
\[
m_{k_1, k_2, k_3}(\xi_1, \xi_2, \xi_3) = \varphi_l(\xi_2 + \xi_3) m_{\mu\rho}(\xi_1, \xi_2, \xi_3) \varphi_{k_1}(\xi_1) \varphi_{k_2}(\xi_2) \varphi_{k_3}(\xi_3)
\]
or
\[
\text{or } \varphi_l(\xi_1 + \xi_2) m_{\mu\rho}(\xi_1, \xi_2, \xi_3) \varphi_{k_1}(\xi_1) \varphi_{k_2}(\xi_2) \varphi_{k_3}(\xi_3).
\]

From (6.2), $\nabla^L(\varphi_k \Lambda^M) \lesssim_L 2^{Mk^+ - Lk}$ and (4.11) it follows that if $m_{\mu\sigma}$ or $m_{\sigma\rho} \neq |\xi_1 + \xi_2 + \xi_3|$ then
\[
\nabla^L \left( \sum_{x, y} \Lambda^M \mu_{k_1, k_2, k_3} \varphi_{k_1}(\xi_1) \varphi_{k_2}(\xi_2) \varphi_{k_3}(\xi_3) \right) \lesssim_L 2^{(M+3)k^+_3 + 2k_3 (1 + 2^{-L \min(k,l)})}.
\]

Since $-6j/7 < k_i, l \leq 3j/N + O(1)$,
\[
\| \Lambda(\xi_1 + \xi_2 + \xi_3) \|_{W^{d, 1}} \lesssim_L 2^{(M+3)k_3^+ + 4k_3 + 2k_2 + 2k_1 + 54j}.
\]

Again $|\nabla^6 \phi_{\mu\rho}(\xi_1, \xi_2, \xi_3)| < 2^{-j}$, so by Lemma 6.2 ($L = 57$ and $d = 6$),
\[
\|1_{\max(|x-y|, |x-z|, |x-w|) > 2j/2} K(x, y, z, w, t) \|_{L^1_{x, y, z, w}} \lesssim_L 2^{(M+3)k_3^+ + 4k_3 + 2k_2 + 2k_1 - 54j}.
\]

Using $N \geq M + 9$ we argue as Case 4.2 of Proposition 6.3 to get (note an extra factor of $j$ coming from summation in $l$)
\[
\left\| \sum_{-6j/7 < k_i, l \leq 3j/N + O(1)} 2^{j/3} \| Q_j \Lambda^M + 2 B_{k_1, k_2, k_3, l}(t) \|_{L^2} \right\|_{L^\infty_{j > L + 5}} \lesssim (1 + t)^{-5/4 \epsilon_1^3}.
\]

If $m_{\mu\sigma}$ or $m_{\sigma\rho} = |\xi_1 + \xi_2 + \xi_3|$ then a similar bound to (5.5) holds for $\varphi_k(\xi_1 + \xi_2 + \xi_3) \Lambda(\xi_1 + \xi_2 + \xi_3) M^{\mu\rho}(\xi_1, \xi_2, \xi_3)$, with $\min(k_i, l)$ replaced by $\min(k_i, k_i, l)$, but the additional assumption $k > -6j/7$ shows that (6.6) holds for $P_k K(x, y, z, w, t)$, and the desired bound also follows from summing over $|k|, |l|$ and $|k_i| \lesssim j$.

Combining Case 1 through Case 5 above gives
\[
\| H_{\mu\rho}(t) \|_{L^\infty} \lesssim (t^{1/5} / R^{4/3} + t^{2 + 33/3} / R^{10/3 - 2/N} + (1 + t)^{-1.003} \epsilon_1^3).
\]

Integration in $t$ shows Proposition 6.4. □
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