Integrable Systems and Factorization Problems

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The present lectures were prepared for the Faro International Summer School on Factorization and Integrable Systems in September 2000. They were intended for participants with the background in Analysis and Operator Theory but without special knowledge of Geometry and Lie Groups. The text below represents a sort of compromise: it is certainly impossible not to speak about Lie algebras and Lie groups at all; however, in order to make the main ideas reasonably clear, I tried to use only matrix algebras such as $\mathfrak{gl}(n)$ and its natural subalgebras; Lie groups used are either $GL(n)$ and its subgroups, or loop groups consisting of matrix-valued functions on the circle (possibly admitting an extension to parts of the Riemann sphere). I hope this makes the environment sufficiently easy to live in for an analyst. The main goal is to explain how the factorization problems (typically, the matrix Riemann problem) generate the entire small world of Integrable Systems along with the geometry of the phase space, Hamiltonian structure, Lax representations, integrals of motion and explicit solutions. The key tool will be the classical r-matrix (an object whose other guise is the well-known Hilbert transform). I do not give technical details, unless they may be exposed in a few lines; on the other hand, all motivations are given in full scale whenever possible. I hope that this choice agrees with the spirit of the Faro School and will help to bridge the gap between different branches of Mathematical Analysis.

1. Introduction

The story of the discovery of the modern theory of Integrable Systems is certainly too long (and too well-known), and I can hardly add anything new; so let me tell just a few words before addressing the bulk of the subject. The study of completely integrable systems goes back to the classical papers of Euler, Lagrange, Jacobi, Liouville and others on analytical mechanics. By the end of the XIX-th century all interesting examples seemed to have been exhausted, and the interest has shifted to the qualitative study of chaotic behaviour. The new age in the study of integrable systems has begun with the famous paper [1] on the KdV equation, which was the first example of an infinite dimensional dynamical system with nontrivial but highly regular behaviour and with a rich excitation spectrum. Amplifying the earlier results of Gardner, Greene, Kruskal and Miura
and of Peter Lax [2] Faddeev and Zakharov [3] have shown in 1971, exactly 30 years ago, that the KdV equation is in fact a completely integrable Hamiltonian system in a technical sense. Within a short time, many more examples have been discovered, notably, the sine–Gordon equation, the first ever example of a relativistic completely integrable system [4]. These discoveries were particularly exciting in view of the possible physical applications: while it was of course clear that “generic” nonlinear equations are non-integrable, it has been argued that Fundamental Physics always deals with highly non-generic equations which might be integrable in some sense or other. These initial hopes have been fulfilled only partially: one major obstacle is that the new technique does not apply to (natural) non-linear equations in realistic space-time dimension. On the other hand, the mathematics of complete integrability has proved to be extraordinarily rich, bringing together Functional Analysis, Algebraic Geometry, Lie Groups, Representation Theory, Symplectic Geometry and much more. The analytic machinery used in the initial papers was based on the Inverse Scattering Problem; the subsequent developments allowed to single out the basic geometric ideas of the theory and to provide a unified basis for different examples. One of the key ingredients of this geometric approach are Infinite Dimensional Lie Groups; in some loose sense, one can say that Integrable Systems always possess some rich hidden symmetry. One may recall that one of the original motivations of S.Lie has been the use of continuous transformation groups for the study of differential equations. With the modern methods at hand, we came closer to that goal; it has now become clear, however, that almost all classical examples of integrable mechanical systems from the XIX century textbooks, as well as the infinite dimensional systems associated with integrable PDE’s are related not to finite-dimensional Lie groups, but rather to their infinite-dimensional analogs.

The general geometric construction that we shall discuss below allows to unify the following characteristic features that are typical for all examples known so far:

(i) The equations of motion are compatibility conditions for a certain auxiliary system of linear equations.

(ii) They are Hamiltonian with respect to a natural Poisson bracket.

(iii) Integrals of motion are spectral invariants of the auxiliary linear operator. They are in involution with respect to the Poisson bracket referred to above.

(iv) The solution of the equations of motion reduces to some version of the Riemann-Hilbert problem.

Depending on the nature of the auxiliary linear problem, the associated nonlinear equations may be divided into the following three groups:

a) Finite-dimensional systems,

b) Infinite-dimensional systems with one or two spatial variables,

c) Integrable systems on one-dimensional lattices.
In case (a) the auxiliary linear problem is the eigenvalue problem for a finite-dimensional matrix (possibly depending on an additional parameter). In case (b) the associated linear operator is differential. In case (c) it is a difference operator.

As it happens, the key properties 1–4 referred to above are corollaries of a single general theorem which may be adapted to numerous concrete applications. The original idea of this theorem is due to B.Kostant and M.Adler; its important amplification brings in the notion of classical r-matrix which provides a link between abstract Riemann-Hilbert problems and the ideas of Quantum Group Theory. The statement and proof of this theorem are particularly simple for systems of types (a) and (b). (Lattice systems require a special treatment, since the associated Poisson brackets belong to a different and more sophisticated class. We shall discuss this case later on, but it is natural to begin with the simpler cases (a) and (b)).

One more word of caution: while it is aesthetically very attractive to deduce a large variety of examples together with their explicit solutions from a simple general construction, there is one important disadvantage: for a given dynamical system (even if it is known to be completely integrable!) it is very difficult to tell a priori, what is the underlying Lie group, or Lie algebra. The practical way around this difficulty is to look at various examples associated with different Lie algebras; with some skill, one manages to recognize among these examples both classical and new integrable systems which admit physical interpretation. The list of interesting Lie algebras includes:

1. Finite dimensional semisimple Lie algebras. Associated integrable systems include open Toda lattice and other finite dimensional Hamiltonian systems which may be integrated in elementary functions (rational functions of $\exp t$, or $t$, where $t$ is the time variable).

2. Loop algebras, or affine Lie algebras. Associated integrable systems are finite dimensional Hamiltonian systems which may be integrated in Abelian functions of time. Integrable tops, as well as almost all classical examples from the XIXth century Analytical Mechanics find their place here.

3. Double loop algebras and their central extensions. This class of algebras accounts for integrable PDE's admitting the zero curvature representation (such as the Nonlinear Schroedinger equation, the Sine-Gordon equation and many others).

4. Algebras of pseudodifferential operators. The KdV equation and its higher analogs come from this example, although it is more practical to derive them from double loop algebras.$^1$

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$^1$ A further generalization is possible: we can add one more spatial variable and consider the loop algebra based on the algebra of pseudodifferential operators; this yields the so called KP equation for functions of three variables.
5. Algebra of vector fields on the line. This algebra, or rather its central extension (the Virasoro algebra) and the associated loop algebra again are related to the KdV equation.

As we see from this list, the choice of the Lie algebra determines not only the possible kinematics (i.e., the structure of the phase space) of the dynamical systems which admit a realization based on this algebra, but also the functional class of the possible solutions. In all cases, it is very important to examine central extensions of the algebras in question (if any), as well as their non-trivial automorphisms: they usually lead to new examples. Non-trivial central extensions exist in cases 2, 3, 4, 5, each one leading to a non-trivial theory. As for outer automorphisms of these algebras, they serve, for instance, to define twisted loop algebras with important range of applications (see Section 7.1) and are also used in the study of nonlinear finite-difference equations.

It is virtually impossible to provide these lectures with a comprehensive list of references. The point of view adopted here is certainly rather subjective. I have included a few references to the old original papers as well as to several reviews.

2. A few Preliminaries: Poisson Brackets, Coadjoint Orbits, etc.

The first question that is prior to the study of the dynamics of a mechanical system is that of its kinematics, i.e. of the structure of its phase space. Typically, the phase space of an individual dynamical system should be a symplectic manifold. However, an important conclusion which may be drawn from the practical study of numerous examples is that integrable systems associated with auxiliary linear problems always arise in families. The appropriate geometrical setting for the study of such families is provided by the theory of Poisson manifolds. Let me recall that a Poisson bracket on a smooth manifold $M$ is the structure of a Lie algebra on the space $C^\infty(M)$; moreover, the Poisson bracket satisfies the Leibniz rule, i.e., it is a derivation with respect to each argument. In local coordinates, a Poisson bracket is written as

$$\{\varphi, \psi\}(x) = \sum_{i,j} \pi_{ij}(x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_j},$$

where $\pi_{ij}$ is an antisymmetric tensor (Poisson tensor) satisfying a quadratic differential constraint which assures the Jacobi identity. When the manifold $M$ is symplectic, i.e., admits a nondegenerate closed 2-form $\omega = \sum \omega^i dx_i \wedge dx_j$, the associated Poisson tensor $\pi_{ij} = (\omega^i)^{-1}$. Reciprocally, whenever the Poisson tensor is nondegenerate, its inverse is a symplectic form. In general, a Poisson manifold is not symplectic; the fundamental theorem which goes back to Lie asserts that it
always admits a stratification whose strata are already symplectic manifolds \((\text{the symplectic leaves of our Poisson manifold})\). The geometrical meaning of this decomposition is very simple. Any \(H \in C^\infty(M)\) defines on \(M\) a Hamiltonian vector field which acts on \(\varphi \in C^\infty(M)\) via
\[
X_H \varphi = \{H, \varphi\};
\]
for a given point \(x \in M\) the tangent vectors \(X_H (x)\) span a linear subspace in the tangent space \(T_x M\); this is precisely the tangent space to the symplectic leave passing through \(x\). By construction, Hamiltonian vector fields are tangent to symplectic leaves, and hence the Hamiltonian flows are preserving each leaf separately. A closely related property of general Poisson manifolds is the existence of Casimir functions. By definition, a function \(H \in C^\infty(M)\) is called a Casimir function if it lies in the center of the Poisson bracket; equivalently, Casimir functions define trivial Hamiltonian equations on \(M\). Restrictions of Casimir functions to symplectic leaves in \(M\) are constants; reciprocally, the common level surfaces of Casimir functions define a stratification of \(M\); typically, this stratification is more coarse than the stratification into symplectic leaves (i.e., symplectic leaves are not completely separated by the values of the Casimirs), but in many applications the knowledge of Casimir functions yields a sufficiently accurate description.

A very typical example of a Poisson manifold is the dual space of a Lie algebra.

I shall briefly recall the corresponding construction, since it proved to be very important for the study of integrable systems. Let \(\mathfrak{g}\) be a Lie algebra, \(\mathfrak{g}^*\) its dual space, \(P (\mathfrak{g}^*)\) the space of polynomial functions on \(\mathfrak{g}^*\). By the Leibniz rule, a Poisson bracket on \(P (\mathfrak{g}^*)\) is completely determined by its value on the subspace of linear functions \(\mathfrak{g} \subset P (\mathfrak{g}^*)\); for \(X, Y \in \mathfrak{g}\) let us set simply
(2.1)
\[
\{X, Y\}(L) = \langle L, [X, Y] \rangle, \ L \in \mathfrak{g}^*.
\]

The Jacobi identity for the bracket (2.1) follows from that for the Lie bracket in \(\mathfrak{g}\); since \(P (\mathfrak{g}^*)\) is dense in \(C^\infty (\mathfrak{g}^*)\), it canonically extends to all smooth functions. Explicitly, we have
(2.2)
\[
\{\varphi_1, \varphi_2\}(L) = \langle L, [d\varphi_1 (L), d\varphi_2 (L)] \rangle.
\]

(Note that \(d\varphi_i (L) \in (\mathfrak{g}^*)^* \simeq \mathfrak{g}\), and hence the Lie bracket is well defined). The bracket (2.2) is usually called the Lie–Poisson bracket. Its properties are closely related to the distinguished representation of the associated Lie group, the coadjoint representation. Let \(G\) be a Lie group with Lie algebra \(\mathfrak{g}\); \(\exp : \mathfrak{g} \to G\) the

\(^2\)The integrability condition which assures the local existence of the submanifold with the given tangent distribution immediately follows from the Jacobi identity; the subtle part of the proof consists in checking that the possible jumps of the rank of the Poisson tensor \(\pi\) do not lead to singularities of the leaves; one can check that the Lie derivative of \(\pi\) along any Hamiltonian vector field is zero, and hence its rank is constant along the leaves (but may jump in the transversal direction).
exponential map. The adjoint and coadjoint representations of $G$ acting in $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively, are defined by

$$Ad \, g \cdot X = \left( \frac{d}{dt} \right)_{t=0} g \cdot \exp t \cdot g^{-1}, \; X \in \mathfrak{g},$$

$$\langle Ad^* g \cdot L, X \rangle = \langle L, Ad \, g^{-1} \cdot X \rangle, \; X \in \mathfrak{g}, \; L \in \mathfrak{g}^*.$$

Set

$$ad \, X \cdot Y = \left( \frac{d}{dt} \right)_{t=0} Ad \, \exp t \cdot Y, \; ad^* \, X \cdot L = \left( \frac{d}{dt} \right)_{t=0} Ad^* \, \exp t \cdot L.$$

Clearly, one has $ad \, X \cdot Y = [X,Y]$, $ad^* \, X = - (ad \, X)^*$. The following fundamental theorem again goes back to Lie; it was rediscovered by Kirillov and Kostant in 1960’s:

**Theorem 2.1.** (i) Symplectic leaves of the Lie-Poisson bracket coincide with $G$-orbits in $\mathfrak{g}^*$ (coadjoint orbits). (ii) Casimir functions of the Lie-Poisson bracket are precisely the coadjoint invariant functions on $\mathfrak{g}^*$.

It is very easy to verify a somewhat weaker property.

**Proposition 2.2.** Let $\varphi \in C^\infty(\mathfrak{g}^*)$ be an arbitrary function; the Hamiltonian equation of motion defined by $\varphi$ with respect to the Lie-Poisson bracket may be written in the following form:

$$\frac{dL}{dt} = -ad^* \, d\varphi \, (L) \cdot L, \; L \in \mathfrak{g}^*; \tag{2.3}$$

in other words, the velocity vector, associated with any Hamiltonian equation on $\mathfrak{g}^*$ is automatically tangent to the coadjoint orbit passing through $L$.

In the context of integrable systems coadjoint orbits are of particular importance: in many applications, the phase spaces of integrable systems are coadjoint orbits for some appropriate Lie group$^3$. Classification of coadjoint orbits for particular Lie groups is a good exercise (which may be quite involved depending on the nature of the Lie group); let us just quote a few examples which will be useful in the sequel.

$^3$It’s probably worth making some precisions: when we deal with concrete equations, coadjoint orbits are almost always a good starting point, but it may be quite useful to enrich our tools. In some cases, an orbit is too big for our purpose and it is possible to cancel out some degrees of freedom by passing to the quotient space over some manifest symmetry group. On the other hand, in some cases, an orbit is too small and it’s more practical to use a bigger phase space which is mapped onto the orbit in a way which is compatible with its Poisson structure. Finally, there are classes of examples when the context of Lie algebras appears to be too restrictive and we have to deal with nonlinear Poisson brackets from the very beginning. We shall comment on these examples later on (see Section 9).
Example 2.3. Let \( g = \mathfrak{gl}(n) \) be the full matrix algebra; its dual space \( g^* \) may be canonically identified with \( g \) by means of the invariant inner product

\[
\langle X, Y \rangle = \text{tr} XY. \tag{2.4}
\]

Thus the adjoint and the coadjoint representations of the corresponding Lie group \( G = GL(n) \) are identical; we have \( \text{Ad}^* g \cdot L = gLg^{-1} \); the coadjoint orbits consist of conjugate (isospectral) matrices; their classification is given by the Jordan normal form. Casimir functions are spectral invariants of matrices; their level surfaces consist of a finite number of coadjoint orbits.

Example 2.4. Let \( \mathfrak{b}_+ \subset \mathfrak{g} \) be the subalgebra of upper triangular matrices; the pairing \( (2.4) \) allows to identify its dual with the space \( \mathfrak{b}_- \) of lower triangular matrices. The coadjoint representation of the corresponding Lie group \( B_+ \) of upper triangular invertible matrices is given by

\[
\text{Ad}^* b \cdot F = P_- \left( b \cdot F \cdot b^{-1} \right), \quad b \in B_+, \quad F \in \mathfrak{b}_-, \tag{2.5}
\]

where \( P_- : \mathfrak{g} \to \mathfrak{b}_- \) is the projection operator which replaces by zeros all matrix coefficients above the principal diagonal.

In this example the adjoint and the coadjoint representations are inequivalent.

After this brief discussion of Poisson geometry and coadjoint orbits let us return to the study of integrable systems. The use of linear Poisson brackets and of coadjoint orbits seems a good guess to get a proper kinematical description of our future examples; this suggestion is further supported by the following simple observation which specializes proposition 2.2 above.

Proposition 2.5. Assume that \( g = \mathfrak{gl}(n) \) is identified with its dual space and equipped with the Lie–Poisson bracket. For any \( \varphi \in C^\infty(\mathfrak{g}) \) the Hamiltonian equation of motion is written in the form

\[
\frac{dL}{dt} = - [d\varphi(L), L]; \tag{2.5}
\]

hence all Hamiltonian flows on \( \mathfrak{g} \) preserve spectral invariants of matrices.

Equation (2.5) looks exciting; one is tempted to compare it with the famous Lax equations. A closer look on the picture reveals, however, that proposition 2.5 leads to a deception. Indeed, spectral invariants of matrices are Casimir functions for the Lie-Poisson bracket; their conservation is a trivial fact which has nothing to do with integrability of equation (2.5). There is very little chance that this equation with arbitrary Hamiltonian \( \varphi \) will be completely integrable; on the other hand, the spectral invariants themselves which seem to be natural candidates to

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4 Throughout these lectures inner product means a nondegenerate symmetric bilinear form (\( \mathbb{C} \)-bilinear in the case of complex algebras); over the reals we do not impose any positivity condition.
provide integrable systems, generate trivial flows, in view of the following simple fact:

**Proposition 2.6.** For any Lie algebra \( \mathfrak{g} \) a function \( \varphi \in C^\infty(\mathfrak{g}^*) \) is a Casimir function for the Lie–Poisson bracket on \( \mathfrak{g}^* \) if and only if

\[
\text{ad}^* d\varphi(L) \cdot L = 0
\]

for any \( L \in \mathfrak{g}^* \). (Note that \( d\varphi(L) \in (\mathfrak{g}^*)^* \simeq \mathfrak{g} \); when \( \mathfrak{g} \) and \( \mathfrak{g}^* \) are identified, this relation is reduced to \( [d\varphi(L), L] = 0 \).)

Despite this initial setback, the original idea to use Lie–Poisson brackets and coadjoint orbits can be saved. However, instead of the initial Lie-Poisson bracket which provides the set of spectral invariants but does not yield any nontrivial dynamics associated with them, we must find a different one. It’s at this point that the classical r-matrix is brought into play.

3. Classical r-matrices and Lax Equations

Let \( \mathfrak{g} \) be a Lie algebra. We shall say that \( r \in \text{End} \mathfrak{g} \) is a classical r-matrix if the bracket

\[
[X,Y]_r = \frac{1}{2} ([rX,Y] + [X,rY])
\]

is a Lie bracket, i.e. if it satisfies the Jacobi identity. The skew symmetry of (3.1) is obvious for any \( r \). We denote the Lie algebra with the bracket (3.1) by \( \mathfrak{g}_r \), and say that \( (\mathfrak{g}, \mathfrak{g}_r) \) is a double Lie algebra.

If \( \mathfrak{g} \) is a double Lie algebra, there are two different Poisson brackets in the space \( \mathfrak{g}^* \), namely, the Lie-Poisson brackets of \( \mathfrak{g} \) and \( \mathfrak{g}_r \). The latter bracket will be referred to as the r-bracket, for short.

A class of double Lie algebras which is of great importance for applications is constructed as follows. Assume that there is a vector space decomposition of \( \mathfrak{g} \) into a direct sum of two Lie subalgebras, \( \mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_- \). Let \( P_{\pm} \) be projection operators onto \( \mathfrak{g}_\pm \) parallel to the complementary subalgebra; set

\[
r = P_+ - P_-
\]

In this case, bracket (3.1) is given by

\[
[X,Y]_r = [X_+, Y_+] - [X_-, Y_-],
\]

where \( X_{\pm} = P_{\pm} X, Y_{\pm} = P_{\pm} Y \). In other words, the bracket (3.1) is the difference of Lie brackets in \( \mathfrak{g}_+ \) and \( \mathfrak{g}_- \). The Jacobi identity for \( \mathfrak{g}_r \) is obvious from (3.3).
As discussed in Section 6, in typical applications the Lie algebra \( g \) is a loop algebra, i.e., an algebra of matrix-valued functions on the circle, and \( g_\pm \) its subalgebra consisting of functions which are analytic inside (resp., outside) the circle. In that case, the classical r-matrix (3.1) is precisely the Hilbert transform. Of course, general classical r-matrices need not have this simple form (although (3.1) is by far the most important example of all). We shall discuss the general theory of r-matrices a little later; let us first state the key theorem which motivates the definition.

3.1. Involutivity Theorem

Let \( I(g^*) \) be the ring of Casimir functions on \( g^* \) (with respect to the original Lie-Poisson bracket); equivalently, \( I(g^*) \subset C^\infty(g^*) \) is the set of coadjoint invariants.

**Theorem 3.1.** (i) Functions in \( I(g^*) \) are in involution with respect to the r-bracket on \( g^* \). (ii) The equations of motion induced by \( h \in I(g^*) \) with respect to the r-bracket have the form

\[
\frac{dL}{dt} = -\text{ad}^*_g M \cdot L, \quad M = r(dh(L))
\]

If \( g \) admits a nondegenerate invariant bilinear form, so that \( \text{ad}^*_g \simeq \text{ad}_g \), equations (2.4) have the Lax form

\[
\frac{dL}{dt} = [L, M].
\]

**Proof.** (i) Let \( h_1, h_2 \in I(g^*) \); set \( dh_i(L) = X_i \) for brevity. By definition,

\[
\{h_1, h_2\}_r(L) = \langle L, [X_1, X_2]_r \rangle
\]

\[
= \frac{1}{2} \langle L, [rX_1, X_2] + [X_1, rX_2] \rangle
\]

\[
= \frac{1}{2} \langle \text{ad}^*_g X_2 \cdot L, rX_1 \rangle - \frac{1}{2} \langle \text{ad}^*_g X_1 \cdot L, rX_2 \rangle = 0,
\]

since, by proposition 2.6, \( \text{ad}^*_g X_2 \cdot L = \text{ad}^*_g X_1 \cdot L = 0 \). (ii) We have

\[
\frac{dL}{dt} = -\text{ad}^*_g dh(L) \cdot L;
\]

(3.1) implies that

\[
ad^*_g X \cdot L = \frac{1}{2} (\text{ad}^*_g r X \cdot L + r^* (\text{ad}^*_g X \cdot L));
\]
since $h \in I(g^*)$, the second term in (3.5) vanishes.

**Remark 3.2.** The matrix $M$ in (3.4) is not uniquely defined: one can always add to it something which commutes with $L$. Here is a useful option: set

(3.6) \[ M_\pm = \pm \frac{1}{2} (r \pm \text{Id}) (dh (L)); \]

equation (3.4) holds with *any* of these two operators. Below, we shall see that this form of $M$-operator appears naturally from the global formula for the solutions.

Theorem 3.1 has a transparent geometrical meaning: it shows that the trajectories of the dynamical systems with Hamiltonians $h \in I(g^*)$ lie in the intersection of *two families of orbits* in $g^*$, the coadjoint orbits of $g$ and $g_r$. Indeed, the coadjoint orbits of $g_r$ are preserved by all Hamiltonian flows in $g_r$. On the other hand, because of (2.4), the flow is always tangent to the $g$-orbits in $g^*$.

In many applications the intersections of orbits are precisely the “Liouville tori” for our dynamical systems.

### 3.2. Factorization Theorem

The scheme outlined so far incorporates only two of the three main features of the inverse scattering method: the Poisson brackets and the Lax form of the equations of motion. As it happens, the most important feature of this method, the reduction of the equations of motion to the Riemann problem, is already implicit in our scheme. An abstract version of the Riemann problem is provided by the *factorization problem* in Lie groups.

We shall state a factorization theorem, which is the global version of Theorem 3.1, for the simplest r-matrices of the form (3.2). Let $G$ be a connected Lie group with Lie algebra $g$, and let $G_\pm$ be its subgroups corresponding to $g_\pm$.

**Theorem 3.3.** Let $h \in I(g^*)$, $X = dh (L)$. Let $g_\pm (t)$ be the smooth curves in $G_\pm$ which solve the factorization problem

(3.7) \[ \exp tX = g_+ (t) \cdot g_- (t)^{-1}, \quad g_+ (0) = e. \]

Then the integral curve $L(t)$ of equation (2.4) with $L(0) = L$, is given by any of the two formulae,

(3.8) \[ L(t) = \text{Ad}_{\text{g}_r}^* g_+(t)^{-1} \cdot L = \text{Ad}_{\text{g}_r}^* g_-(t)^{-1} \cdot L. \]

**Proof.** Differentiating (3.8) with respect to $t$ we get

\[ \frac{dL}{dt} = -\text{ad}^* (g_+^{-1} \dot{g}_+) \cdot L = -\text{ad}^* (g_-^{-1} \dot{g}_-) \cdot L. \]
We shall check that $g^{-1}_\pm \dot{g}_\pm = M_\pm$, where $M_\pm$ are the M-operators from (3.6). Due to our special choice of $r$ we have
\[
M_\pm(t) = \pm P_\pm X(t),
\]
where $X(t) = dh(L(t))$. The $Ad^*G$-invariance of $h$ implies that
\[
X(t) = Ad_g(t)^{-1} \cdot X.
\]

**Exercise 3.4.** Check this formula in matrix case (“gradients of invariant functions are covariant”).

Writing (3.7) in the form $g_+(t) \exp tX = g_-(t)$ and differentiating with respect to $t$, we get
\[
g^{-1}_+ \dot{g}_+ + Ad_g(t)^{-1} \cdot X = \dot{g}_- g_-^{-1}
\]
Since $g^{-1}_\pm \dot{g}_\pm \in g_\pm$, this implies $g^{-1}_\pm \dot{g}_\pm = \pm P_\pm X(t)$, as desired. ☐

Note that the two possible choices of sign in (3.8) are equivalent precisely because $X(t)$ belongs to the centralizer of $L(t)$, i.e., $adX(t) \cdot L(t) = 0$ (in fact, this is the characteristic property of Casimir functions).

By the implicit function theorem, the factorization (3.7) exists for $t$ sufficiently small; note that in our proof we need not assume that this factorization exists globally for all $t$. Geometrically, this means that the solution of the Lax equation exists as long as the curve $\exp tX$ remains in the “big cell” $G_+ \times G_- \subset G$; in other words, the flow associated with the Lax equation is not necessarily complete. One can show in typical examples that the curve intersects “complementary” cells of positive codimension transversally and returns back to the big cell; for the exceptional values of $t$ the solution “escapes to infinity”, i.e., displays a pole in $t$.

### 3.3. Factorization Theorem and Hamiltonian Reduction

A more geometric proof of Theorem 3.3 is based on Hamiltonian reduction. Recall that the Hamiltonian reduction applies to Hamiltonian dynamical systems with high degree of symmetry; it allows to exclude certain redundant degrees of freedom. Classically, the use of reduction is to simplify multidimensional systems getting quotient, or reduced, systems of lower dimension. However, as pointed out in [9], one can reverse this reasoning and use Hamiltonian reduction in the opposite direction, starting with a simple multidimensional system with high symmetry (“free dynamics”) and getting a complicated reduced system as an output. In order to apply this idea, one has to answer the following questions:

1. Find a “big” phase space and suggest the “free dynamics” which will yield Lax equations as the quotient system.
2. Make sure there is an expected high degree of symmetry for the free system.
3. Perform the reduction.

Although the proof based on this approach is much longer than the elementary computation presented above, it is more transparent and explains the origin of the result. A simple candidate for the big phase space is the cotangent bundle $T^*G$ equipped with the canonical symplectic structure. Let us first of all describe the “free dynamics” on $T^*G$. The group $G$ acts on itself by left and right translations; these actions naturally lift to $T^*G$; both actions are Hamiltonian with respect to the canonical symplectic structure. Let us identify $T^*G$ with $G \times g^*$ by means of left translations.

**Exercise 3.5.** In left trivialization the action of $G$ on $T^*G \simeq G \times g^*$ by left (right) translations is given by

\begin{align}
\lambda(g) : (x, L) &\mapsto (gx, L), \\
\rho(g) : (x, L) &\mapsto (xg^{-1}, Ad^*g \cdot L).
\end{align}

(In left trivialization the action of $G$ by right translations induces a nontrivial action in the fiber $g^*$; it is easy to check that it is precisely the coadjoint action.) Left-invariant functions on $T^*G$ are identified with functions on $g^*$. Since the canonical Poisson bracket of left-invariant functions is also left-invariant, this induces a Poisson structure on $g^*$; it is easy to check that it coincides with the Lie-Poisson bracket. Casimir functions on $g^*$ canonically lift to smooth functions on $T^*G$ which are $G$-biinvariant. For $h \in I(g^*)$ let us denote by $\hat{h} \in C^\infty(T^*G)$ the corresponding biinvariant Hamiltonian on $T^*G$.

**Lemma 3.6.** The Hamiltonian flow on $T^*G$ defined by $\hat{h}$ is given (in left trivialization) by

\begin{align}
F_t : (x, L) &\mapsto (x \cdot \exp t\hat{h}(L), L), \quad x \in G, \quad L \in g^*.
\end{align}

In other words, integral curves of $\hat{h}$ project to left translates of one-parameter subgroups in $G$; the choice of $\hat{h}$ determines the (constant) velocity vector $\hat{h}(L)$ which depends only on the initial data.

Since the “free Hamiltonian” $\hat{h}$ is $G$-biinvariant, the flow $F_t$ admits reduction with respect to any subgroup $U \subset G \times G$. There is, at this stage, a very large freedom in the choice of such a subgroup which all lead to different but meaningful quotient systems. The particular choice which is imposed by the choice the r-matrix (3.2) is $U = G_+ \times G_-$. By (3.9), with our choice of trivialization, the action of $G_+ \times G_-$ on $T^*G \simeq G \times g^*$ is given by

\begin{align}
(g_+, g_-) : (x, L) &\mapsto (g_+xg_-^{-1}, Ad^*_G g_- \cdot L).
\end{align}

We now turn to the reduction procedure. In textbooks the reduction is usually described in a rather complicated way (which involves the moment map and a good deal of symplectic geometry) (see, e.g., [7]). Here is an elementary substitute. Let
us ask what is bad about the naive suggestion: consider a group action \( G \times M \rightarrow M \) on a symplectic manifold and take the projection \( \pi : M \rightarrow M/G \) onto the quotient space? The answer is: the quotient space is no longer symplectic, since symplectic forms transform by pullback, and there is no natural symplectic form on \( M/G \) (it in not even in general even dimensional!) But on the other hand, the quotient space does carry a Poisson bracket (which transforms by push-forward!). The difficult part of reduction consists in the description of the particular symplectic leaves of this quotient Poisson bracket; it's at this stage that one needs the moment map and all other machinery. If we do not want a too detailed description, or if we manage to guess the symplectic leaves in some other way, everything becomes simple! In the present case, we can display a map \( \pi \) which is constant on the orbits of \( G_+ \times G_- \) in \( M \) and hence its image yields a model of the quotient space; the Poisson structure on this quotient is easy to recognize.

For \( x \in G \) we denote by \( x_\pm \) the solution of the factorization problem

\[
(3.12) \quad x = x_+ \cdot x_-^{-1}, \ x_+ \in G_+, \ x_- \in G_-.
\]

**Lemma 3.7.** (i) The map \( \pi : T^*G \rightarrow \mathfrak{g}^* : (x,L) \mapsto -\cdot \text{Ad}_G^* x_-^{-1} \cdot L \) is constant on the orbits of \( G_+ \times G_- \) in \( T^*G \). (ii) If \( G \) is globally diffeomorphic to \( G_+ \times G_- \), i.e., the factorization problem (3.12) is always solvable, \( \pi \) is a global cross-section of this action. (iii) The induced Poisson structure on \( T^*G/G_+ \times G_- \approx \mathfrak{g}^* \) coincides with the Lie-Poisson bracket for \( \mathfrak{g}_r \approx \mathfrak{g}_+ \oplus \mathfrak{g}_- \).

The check of (i) and (ii) is immediate; the proof of (iii) requires a little knowledge of symplectic geometry (or else a three-line computation); we shall not present it here (see [8]).

**Lemma 3.8.** The flow (3.10) factorizes over \( \mathfrak{g}_r^* \); in other words, there exists a natural flow \( \bar{F}_t : \mathfrak{g}_r^* \rightarrow \mathfrak{g}_r^* \) (called the quotient flow) which makes the following diagram commutative; the quotient flow \( \bar{F}_t : \mathfrak{g}_r^* \rightarrow \mathfrak{g}_r^* \) is given by

\[
(3.13) \quad \bar{F}_t : L \mapsto \text{Ad}_G^* g_- ((t)^{-1} \cdot L,
\]
where, as in (3.7), \( g_+(t) g_-(t)^{-1} = \exp t \, db(L) \).

The flow \( \bar{F}_t \) is precisely the result of the reduction procedure. Note that (3.13) involves only \( g_- \); this is due to our choice of trivialization of \( T^*G \); trivialization by right translations yields the equivalent formula for the quotient flow \( \bar{F}_t : L \mapsto Ad_{g_+}^* g_+(t)^{-1} \cdot L \).

In general, of course, \( G \) need not be diffeomorphic to \( G_+ \times G_- \); still, \( g^* \) provides a model for a “big cell” in the quotient space \( T^*G/G_+ \times G_- \); one can show that under very mild restrictions the action (3.11) is proper and hence the quotient space is a well-defined manifold; the quotient flow induced on this manifold is also well defined and may be regarded as the natural completion of the incomplete flow associated with Lax equations.

The choice of the subgroup \( G_+ \times G_- \subset G \times G \) as the symmetry group for the “free system” may seem arbitrary; indeed, there are many other possible choices leading to meaningful examples. (Among the dynamical systems which may be obtained in this way, there are, e.g., the Calogero-Moser systems, cf. [9].) The key property which characterizes our special situation is the simple description of the quotient space: as we see from Lemma 3.8, in this case the quotient space is simply the dual of a Lie algebra with its Lie-Poisson bracket, and its symplectic leaves (which are the symplectic quotients of \( T^*G \)) are the coadjoint orbits of \( G_r \); for other choices of the symmetry group the description of the quotient space will be more complicated and it may no longer be a homogeneous space.

4. Classical Yang-Baxter Identity

We have already mentioned that the most interesting \( r \)-matrices are associated with decompositions of the Lie algebra into complementary Lie subalgebras. However, it is worth examining the general conditions on \( r \) which follow from the Jacobi identity for the \( r \)-bracket. These conditions are known as the classical Yang-Baxter equations; they were first derived as a semi-classical approximation to the quantum Yang-Baxter equations which arise in the study of quantum completely integrable systems.

The restrictions on \( r \) which follow from the Jacobi identity are quite easy to establish. For \( r \in \text{End} \, g \) set

\[
B_r(X,Y) = [rX, rY] - r([X,Y] + [X, rY]).
\]

**Proposition 4.1.** The \( r \)-bracket (3.1) satisfies the Jacobi identity if and only if, for any \( X, Y, Z \in g \),

\[
[B_r(X,Y),Z] + [B_r(Y,Z),X] + [B_r(Z,X),Y] = 0.
\]

The proof is straightforward: just substitute (3.1) into the Jacobi identity and regroup the terms. The necessary and sufficient condition (4.2) is usually replaced
by sufficient conditions which are bilinear rather than trilinear. The simplest sufficient condition is the so-called classical Yang-Baxter equation (CYBE)

\[(4.3)\quad B_r(X,Y) = 0.\]

Another important sufficient condition is the modified classical Yang-Baxter equation (mCYBE)

\[(4.4)\quad B_r(X,Y) = -c [X,Y], \quad c = \text{const}.\]

By rescaling, we may always assume that \(c = 1\). Note that the r-matrices (3.2) satisfy mCYBE with \(c = 1\).

The reason for the study of classical r-matrices satisfying the modified classical Yang-Baxter identity is that although they do not in general have the simple form (3.2), one can still associate with them a factorization problem. By contrast, the ordinary classical Yang-Baxter identity (4.3) represents a degenerate case and does not lead to a factorization problem.

Let us briefly describe the corresponding construction. Given an r-matrix which satisfies mCYBE, set

\[(4.5)\quad r_{\pm} = \frac{1}{2} (r \pm \text{Id}).\]

**Proposition 4.2.** For each \(X,Y \in \mathfrak{g}\)

\[ [r_{\pm} X, R_{\pm} Y] = r_{\pm} ([X,Y]_r), \]

i.e., \(r_{\pm} : \mathfrak{g}_R \to \mathfrak{g}\) are Lie algebra homomorphisms.

Set \(\mathfrak{g}_{\pm} = \text{Im} r_{\pm}\). Clearly, \(\mathfrak{g}_{\pm}\) is a Lie subalgebra of \(\mathfrak{g}\). If \(r\) has the form (3.2), then \(r_{\pm} = \pm P_{\pm}\) and the subalgebras \(\mathfrak{g}_{\pm} = P_{\pm}(\mathfrak{g})\) are complementary. In general case this is no longer true. However, this difficulty may be resolved by passing to the double of \(\mathfrak{g}\). By definition,

\(\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}\)

is the direct sum of two copies of \(\mathfrak{g}\).

**Proposition 4.3.** The mapping \(i_r : \mathfrak{g}_r \to \mathfrak{g} \oplus \mathfrak{g} : X \longmapsto (r_+ X, r_- X)\) is a Lie algebra embedding, and each \(Y \in \mathfrak{g}\) has a unique decomposition, \(Y = Y_+ - Y_-\), where \((Y_+, Y_-) \in \text{Im} i_r\).

The last assertion follows immediately from the obvious identity \(r_+ - r_- = \text{Id}\).

Now, let \(G, G_r\) be (local) Lie groups which correspond to \(\mathfrak{g}, \mathfrak{g}_r\). The homomorphisms \(r_{\pm}\) give rise to the Lie group homomorphisms which we denote by the same letters. Put \(G_{\pm} = r_{\pm}(G_R)\). The composition of maps

\(i_r : G_r \longrightarrow G \times G : x \longmapsto (r_+ x, r_- x)\)

is a Lie group embedding. Consider the map

\(m : G \times G \longrightarrow G : (u, v) \longmapsto uv^{-1} \).
Then \( f = m \circ i_r : G_r \to G \) is a local homeomorphism, and therefore an arbitrary element \( y \in G \) which is sufficiently close to unity admits a unique representation \( y = y_+ y_-^{-1} \) with \((y_+, y_-) \in \text{Im} i_r\).

The proof of Theorem 3.3 extends to the present setting with only minor changes.

It is probably worth giving examples of \( r \)-matrices satisfying (4.4) which are not of the form (3.2). Let \( g = gl(n) \) be the matrix algebra; let us consider its decomposition \( g = n_+ + h + n_- \) into direct sum of upper triangular, diagonal, and lower triangular matrices. Let \( P_\pm, P_0 \) be the corresponding projection operators. Let \( r_0 \) be the partially defined linear operator on \( g \) with domain \( \text{Dom}(r_0) = n_+ + n_- \) given by

\[
r_0 X = \begin{cases} 
X, & \text{if } X \in n_+, \\
-X, & \text{if } X \in n_-.
\end{cases}
\]

We want to extend \( r_0 \) to the entire linear space \( g \) in such a way that it satisfies the modified classical Yang-Baxter identity. Let us first drop the latter condition and consider all linear operators \( r \supset r_0 \), \( \text{Dom}(r) = g \). Let us set again \( r_\pm = \frac{1}{2} (r \pm \text{Id}), i_r = r_+ \oplus r_- \) and consider the subspace \( g_r = \text{Im} i_r \). It is easy to see that this subspace is transversal to the diagonal subalgebra \( g_d = \{(X, X) \in g \oplus g\} \); conversely, all extensions \( r \supset r_0 \) are in bijective correspondence with linear subspaces \( g_r \subset g \oplus g \) which contain \( n_+ \oplus n_- \) and are transversal to the diagonal. All such subspaces may be parametrized in the following way:

\[
g_r = \{(X_+ + X_0, X_- + \theta X_0) ; X \in g\},
\]

where \( X_\pm = \pm P_\pm X, X_0 = P_0 X \) and \( \theta \in \text{End} h \) is a linear operator; equivalently, the extensions \( r \supset r_0 \) are described by the von Neumann formulae,

\[
\left\{ \begin{array}{l}
X = X_+ - X_- + (\text{Id} - \theta) X_0, \\
r_0 X = X_+ + X_- + (\text{Id} + \theta) X_0.
\end{array} \right.
\]

The transversality condition is equivalent to the non-degeneracy of \( \text{Id} - \theta \). Since \( h \subset g \) is an abelian subalgebra which normalizes both \( n_+ \) and \( n_- \), it is easy to see that \( g_r \subset g \oplus g \) is a Lie subalgebra (and not merely a linear subspace) for any \( \theta \), and hence all \( r_\theta \supset r_0 \) satisfy mCYBE. In more general examples, when the relevant subalgebra is not abelian, there are additional algebraic constraints on \( \theta \) which make the construction partially rigid.

The moral of the story: One can construct new classical \( r \)-matrices from the standard ones of the form (3.2) by assuming that the simple formula (3.2) applies not globally, but only on some subspace of positive codimension and then using the extension theory of linear operators. The same trick works for loop algebras; in that case, the relevant codimensions are finite. Let us set, for example, \( \mathcal{S} = Lg, g = gl(n) \); by definition, \( \mathcal{S} \) is the algebra of Laurent polynomials with matrix coefficients. Set \( \mathcal{N}_+ = \{ X(z) = X_0 + X_1 z + X_2 z^2 + \ldots ; X_0 \in n_+ \}, \mathcal{N}_- = \{ X(\lambda) = X_0 + X_1 \lambda^{-1} + X_2 \lambda^{-2} + \ldots ; X_0 \in n_- \}. \) Then \( \mathcal{S} = \mathcal{N}_+ + h + \mathcal{N}_- \), where \( h \) is the subalgebra of constant diagonal matrices. Hence we may apply the previous
construction without any modification. A classification theorem, due to Belavin and Drinfeld, assures that all r-matrices on semisimple Lie algebras and their loop algebras satisfying mCYBE and certain additional conditions arise in a similar way; of course, the most difficult part of the construction is the case when \( \mathfrak{h} \) is replaced by a non-abelian subalgebra. (Cf. [10], [11].)

5. A Finite-dimensional Example

Let again \( \mathfrak{g} = \mathfrak{gl}(n) \) be the full matrix algebra. There are several natural decompositions of \( \mathfrak{g} \) into direct sum of complementary subalgebras, e.g.,

\[
\mathfrak{g} = \mathfrak{b}_+ + \mathfrak{n}_-, \tag{5.1}
\]

where \( \mathfrak{n}_- \) is the subalgebra of lower triangular nilpotent matrices and \( \mathfrak{b}_+ = \mathfrak{h} + \mathfrak{n}_+ \) the complementary subalgebra of upper triangular matrices. Another decomposition is

\[
\mathfrak{g} = \mathfrak{b}_+ + \mathfrak{k}, \tag{5.2}
\]

where \( \mathfrak{k} = \mathfrak{so}(n) \) is the Lie algebra of skew symmetric matrices. We can associate two standard classical r-matrices with these decompositions:

\[
r_{\text{Gauss}} = \mathring{P}_{\mathfrak{b}_+} - P_{\mathfrak{n}_-},
\]

\[
r_{\text{Iwasawa}} = \tilde{P}_{\mathfrak{b}_+} - P_{\mathfrak{k}},
\]

where \( P_{\mathfrak{b}_+}, P_{\mathfrak{n}_-}, \mathring{P}_{\mathfrak{b}_+}, P_{\mathfrak{k}} \) are the respective projection operators (mind that of course \( P_{\mathfrak{b}_+} \neq \tilde{P}_{\mathfrak{b}_+} \)). The Lie groups \( G_{r_{\text{Gauss}}} \) and \( G_{r_{\text{Iwasawa}}} \) are isomorphic to \( B_+ \times N_- \) and \( B_+ \times K \), respectively. The associated factorization problems in the general linear group \( G = GL(n) \) are the Gauss decomposition of matrices,

\[
g = bn^{-1}, b \in B_+, n \in N_-, \tag{5.3}
\]

in the former case, and the Iwasawa decomposition

\[
g = bk^{-1}, b \in B_+, k \in K, \tag{5.4}
\]

in the latter one. (Here \( N_- \subset GL(n) \) denotes the subgroup of lower triangular unipotent matrices, \( B_+ \subset GL(n) \) the subgroup of upper triangular matrices, and \( K = SO(n) \subset GL(n) \) the subgroup of orthogonal matrices.) Note that in the Iwasawa case the product map \( B_+ \times K \to G : (b, k) \mapsto bk^{-1} \) is a bijection onto \( G \) and hence the factorization problem is always solvable. For the Gauss decomposition the image of \( B_+ \times N_- \to G : (b, n) \mapsto bn^{-1} \) is an open dense subset in \( G \), and hence the factorization problem is solvable for almost all (though not for all) initial data.
The dual space $\mathfrak{g}^\ast$ is canonically identified with $\mathfrak{g}$ by means of the invariant inner product
\[ \langle X, Y \rangle = \text{tr } XY; \]
the decompositions (5.1), (5.2) give rise to the biorthogonal decompositions
\[ \mathfrak{g}^\ast = \mathfrak{b}^\perp + \mathfrak{n} = \mathfrak{b}^\perp + \mathfrak{k}^\perp, \]
which provide the models for dual spaces of the subalgebras,
\[ \mathfrak{b}_+^\ast \simeq \mathfrak{n}^\perp = \mathfrak{b}_-, \mathfrak{n}^\ast \simeq \mathfrak{b}_+^\perp = \mathfrak{n}_+ \]
in the case of the Gauss decomposition and
\[ \mathfrak{b}_+^\ast \simeq \mathfrak{t}^\perp = \mathfrak{p}, \mathfrak{t}^\ast \simeq \mathfrak{b}_+^\perp = \mathfrak{n}_+ \]
in the case of the Iwasawa decomposition (here $\mathfrak{p} \subset \text{Mat}(n)$ denotes the subspace of symmetric matrices).

Exercise 5.1. (i) Let us model the dual space $\mathfrak{b}_+^\ast$ on $\mathfrak{b}_-$: for all $q \geq 0$ the subspaces $\mathfrak{b}_+^q = \bigoplus_{p=0}^{q} \mathfrak{d}_p \subset \mathfrak{b}_-$ are invariant with respect to the coadjoint action of $\mathfrak{B}_+$. (ii) Similarly, if $\mathfrak{b}_+^\ast$ is modelled on $\mathfrak{p}$, the subspaces $\mathfrak{p}^q = \bigoplus_{p=0}^{q} (\mathfrak{d}_p + \mathfrak{d}_{-p}) \cap \mathfrak{p}$ are also invariant.

Orbits in the subspace $\mathfrak{b}_0^\ast$ are all trivial (i.e., each point in this subspace is stable with respect to the coadjoint action and hence is a separate orbit): the subspace $\mathfrak{b}_1^\ast$ contains orbits of maximal dimension $2n - 2$ (over $\mathbb{C}$ there is just one such orbit, over the reals there is a finite number of them); the typical example is the orbit $\mathcal{O}_f$ which contains the matrix $f \in \mathfrak{d}_{-1} \subset \mathfrak{b}_-$,
\[ f = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & 1 & 0 \end{pmatrix}; \]

Exercise 5.2. (i) $\mathcal{O}_f$ consists of all matrices of the form
\[ l = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ b_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & p_{n-1} & 0 \\ 0 & \cdots & b_{n-1} & p_n \end{pmatrix}, b_i \neq 0, \sum p_i = 0. \]

---

⁵Mind that the model for the dual of a subalgebra depends not only on the subalgebra itself but also on the choice of its complement in the big Lie algebra (and eventually also on the choice of the inner product whenever it is not unique).

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(Over the reals, in addition, sign $b_i = +1$.) (ii) Lie-Poisson brackets of coordinate functions $p_i, b_j$ on $\mathcal{O}_f$ are given by
\[ \{ p_i, p_j \} = \{ b_i, b_j \} = 0, \{ p_i, b_i \} = -\{ p_{i+1}, b_i \} = b_i.\]

(iii) If we set $b_i = \exp(q_i - q_{i+1})$, the coordinates $q_i$ have canonical Poisson brackets with momenta,
\[ \{ p_i, q_j \} = \delta_{ij}.\]

Thus, as a symplectic manifold, $\mathcal{O}_f$ is isomorphic to the standard phase space $\mathbb{R}^{2n-2}$. In the Iwasawa model, the same orbit is realized by symmetric matrices,
\[
L = \begin{pmatrix}
p_1 & b_1 & \cdots & 0 \\
b_1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & p_{n-1} & b_{n-1} \\
0 & \cdots & b_{n-1} & p_n
\end{pmatrix}.
\]

Recall that our main theorem associates dynamical systems to coadjoint orbits in $\mathfrak{g}^*$; since $\mathfrak{g}_{\text{Gauss}}$ and $\mathfrak{g}_{\text{Iwasawa}}$ are direct sums,
\[ \mathfrak{g}_{\text{Gauss}} \simeq \mathfrak{b}_+ \oplus \mathfrak{n}_-; \quad \mathfrak{g}_{\text{Iwasawa}} \simeq \mathfrak{b}_+ \oplus \mathfrak{k},\]
respectively. Coadjoint orbits of this bigger algebra are Cartesian products of the coadjoint orbits of the factors. In the Iwasawa case the simplest meaningful choice is to take the zero orbit of $K$ in $\mathfrak{k}^* \simeq \mathfrak{n}_+$; with this choice, (5.5) becomes the Lax matrix for the associated dynamical systems. The Hamiltonians are spectral invariants of $L$; taking, for instance,
\[ H = \frac{1}{2} \text{tr} L^2 \]
and expressing it in terms of the canonical variables $p_i, q_i$, we get
\[ H = \frac{1}{2} \sum_i p_i^2 + \sum_i \exp (q_i - q_{i+1}),\]
i.e., the Hamiltonian of the open Toda lattice. In the Gauss case, it is also possible to take the zero orbit of the complementary subalgebra $\mathfrak{n}_-$; with this choice the Lax matrix $l$ will be lower triangular and hence its spectral invariants will depend only on momenta $p_i$ and the corresponding Hamiltonians will be trivial. Luckily, in this case there is another option: we may take any one-point orbit $\{ e \}$ of $N_-$ in $\mathfrak{n}^+_-$ $\simeq \mathfrak{n}_+$; the constant matrix $e$ is simply added to the Lax matrix. This procedure does not add any new degrees of freedom to our system, but it modifies

\[ ^6 \text{It is easy to see that in the real case the signs of all matrix coefficients below the principal diagonal are preserved by the coadjoint action; hence there are exactly } 2^{n-1} \text{ open orbits in our subspace.} \]
the embedding of the “little orbit” $O_f$ into the big algebra and hence the spectral invariants of the Lax matrix. Specifically, set

$$e = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & \ddots & & \\
\vdots & \ddots & \ddots & \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 0
\end{pmatrix}, \quad L_{Gauss} = l + e.$$

**Remark 5.3.** The reader familiar with semisimple Lie algebras will notice that the Jordan matrices $e, f$ are principal nilpotent elements of the general linear algebra, which suggests the way to generalize the above construction to other algebras.

**Exercise 5.4.** (i) $\{e\} \subset \mathfrak{d}_1 \subset \mathfrak{n}_+$ is a one-point coadjoint orbit of $N_-$.  
(ii) In canonical coordinates, the Hamiltonian $h = \frac{1}{2} \text{tr} L_{Gauss}^2$ is given by

$$h = \frac{1}{2} \sum_i p_i^2 \sum_i \exp (q_i - q_{i+1}).$$

Note that the Hamiltonians $H$ and $h$ (which are defined on the same manifold $O_f$) are *different* (although, in this particular case, they happen to be related by a simple canonical change of variables); so are the associated factorization problems which solve the Toda equations.

Other spectral invariants, e.g., $h_p = \text{tr} L_{Gauss}^p, p = 2, 3, \ldots$, form a system of integrals of motion in involution. Obviously, the number of independent integrals does not exceed $n - 1$ (mind that $h_1$ reduces to a constant on $O_f$).

**Exercise 5.5.** (i) Write down explicitly the two factorization problems. (ii) Find the explicit relation between them. (iii) Show that over the reals the group element $\exp t L_{Gauss}(p, q)$ lies in the “big cell” for all $(p, q) \in \mathbb{R}^{2n-2}$ and for all $t \in \mathbb{R}$, and hence the Gauss factorization is always possible. (iv) Check that the integrals $h_2, \ldots, h_n$ remain functionally independent after restriction to $O_f$.

In both cases, the entries of the factors are *rational functions* of $\exp t$ (with coefficients depending on the initial data, i.e., on $p, q$); thus the functional dependence of the solutions on the time variable is fairly simple. It is easy to see that this simple behaviour is characteristic for all Lax equations associated with factorization problems in finite-dimensional Lie groups.

**Remark 5.6.** The subspaces $\mathfrak{d}_p \subset \mathfrak{g}$ satisfy $[\mathfrak{d}_p, \mathfrak{d}_q] \subset \mathfrak{d}_{p+q}$, i.e., they form a *grading* of the matrix algebra. As a consequence, the subspaces $\oplus_{p \geq n} \mathfrak{d}_p, n = 0, 1, 2, \ldots$, are Lie subalgebras and form a decreasing filtration of $\mathfrak{b}_+$; by duality, the subspaces $\mathfrak{b}_n^\perp = \oplus_{p \leq -n} \mathfrak{d}_p \subset \mathfrak{b}_- \simeq \mathfrak{b}_n^\ast, n = 0, -1, -2, \ldots$, form an increasing filtration of $\mathfrak{b}_n^\ast$ by $Ad_{B_+}$-invariant subspaces. The orbit $O_f$ is the “biggest” orbit in the subspace $\mathfrak{v}_1$; its choice is quite natural, since $\dim O_f = 2n - 2$ is twice the number
of independent Hamiltonians, which is precisely the amount needed for complete
integrability. Dynamical systems associated with coadjoint orbits of higher dimen-
sion (which are abundant) have not got enough “obvious” integrals of motion to
assure their Liouville integrability; on the other hand, all these systems are explicit-
itly solvable by means of the factorization problem. This queer situation is due to
the resonance behaviour of these systems: their trajectories span a submanifold in
the phase space with codimension higher than in the generic case.

An important general conclusion to be held from this example is the role of
gradings: it provides a natural decomposition into subalgebras, as well as plenty of
invariant subspaces for the coadjoint action.

6. Loop Algebras and the Riemann Problem

As already mentioned, loop algebras provide a natural environment for the study
of numerous finite-dimensional systems. In this Section we shall briefly outline the
corresponding constructions.

Let \( g \) be a semisimple Lie algebra with invariant inner product \( (g, g) = \text{sl}(n) \) with
the inner product \( (X, Y) = \text{tr} XY \) is a good example; in the sequel we shall mainly
deal with this standard matrix case). Its loop algebra, \( L_g \) is the Lie algebra of
Laurent polynomials with coefficients in \( g \),

\[
L_g = \left[ z, z^{-1} \right] \text{ with pointwise commutator, } [X z^n, Y z^m] = [X, Y] z^{n+m}, \text{ or } [x, y](z) = [x(z), y(z)].
\]

We may regard an element \( x \in \mathfrak{g} \) as a polynomial mapping from the unit circle
\( T \) into \( \mathfrak{g} \). An invariant
inner product on \( \mathfrak{g} \) is given by

\[
\langle x, y \rangle = \int_T \text{tr} x(z) y(z) \frac{dz}{2\pi i z}.
\]  

(6.1)

Let \( g_n = g \cdot z^n \); clearly, \([g_n, g_m] \subset g_{n+m}\), and hence the decomposition
\( L_g = \bigoplus_{n \in \mathbb{Z}} g_n \) defines a grading (the so called standard grading) of the loop
algebra. Set

\[
L_g^+ = \bigoplus_{n \geq 0} g_n = g[z], \quad L_g^- = \bigoplus_{n < 0} g_n = z^{-1} g[z^{-1}] .
\]  

(6.2)

**Proposition 6.1.** (i) \( L_g^+ \) and \( L_g^- \) are graded subalgebras of \( L_g \), and

\[
L_g = L_g^+ \oplus L_g^- .
\]

(ii) The inner product (6.1) sets \( L_g \) into duality with itself; in particular, \( L_g^\pm \) is
set into duality with \( L_g_{\mp} \). (iii) The ring of Casimir functions on \( L_g^* \simeq L_g^* \) is
generated by the functionals

\[
\Phi_{n,m}[L] = \frac{1}{n} \text{Res}_{z=0} \text{tr} L(z)^n z^{-m-1} .
\]
Note that these functionals are smooth in the sense of ordinary calculus of variations and \( \text{grad} \Phi_{n,m} = \frac{1}{m} L^{n-1} \).

Let \( P_{\pm} \) be the projection operators onto \( Lg_{\pm} \) parallel to the complementary subalgebra, \( r = P_{+} - P_{-} \). In analytic terms, \( Lg \) consists of trigonometric polynomials on the circle, and \( r \) is the standard Hilbert transform. The Lie algebra \((Lg)_r\) is isomorphic to the direct sum \( Lg_{+} \oplus Lg_{-} \). For \( X \in Lg \) we set \( X_{\pm} = \pm P_{\pm} X \).

**Proposition 6.2.** The coadjoint action of \((Lg)_r\) on its dual is given by

\[
\text{ad}^* X \cdot L = P_{+} [X_{-}, L] - P_{-} [X_{+}, L];
\]

this action leaves invariant the subspace of polynomial loops \( Lg \subset (Lg)^*_W \).

Notice that both linear operators \( L \mapsto P_{+} [X_{-}, L] \) and \( L \mapsto P_{-} [X_{+}, L] \) are Toeplitz.

Since \( Lg \) is infinite-dimensional, the choice of the associated Lie group becomes non-obvious. A reasonable option is to take the group \( G_W \) of all Wiener maps \( g : \mathbb{T} \to G \). The Lie algebra \( Lg \) may also be replaced by its appropriate completion, e.g. the Wiener algebra \( Lg_W \). Of course, with this choice the full dual space \( Lg^*_W \) becomes a rather complicated object; the point is that the set of polynomial loops \( Lg \subset Lg^*_W \) is invariant with respect to the coadjoint action of \((Lg)_r \) (though of course not with respect to the coadjoint action of \( Lg_W \)). Let \( G^+_W \subset G_W \) be the subgroup of Wiener maps \( g : \mathbb{T} \to G \) which are holomorphic in the unit circle, and \( G^-_W \subset G_W \) the subgroup of maps which are holomorphic outside the unit circle and satisfy the normalization condition \( \lim_{z \to \infty} g(z) = id \). The Lie group which corresponds to \((Lg)_r \) may be identified with \( G^+_W \times G^-_W \); its coadjoint action is given by

\[
(\text{Ad}^* h \cdot L) = P_{+} (h_{-} (z) L (z) h^{-1}_{-} (z)) - P_{-} (h_{+} (z) L (z) h^{-1}_{+} (z));
\]

\[
h = (h_{+}, h_{-}) \in G^+_W \times G^-_W.
\]

**Exercise 6.3.** For \( g = \mathfrak{sl}(n) \) describe all coadjoint orbits of \( G^+_W \) in the subspace of Laurent polynomials of the form

\[
L (z) = l_{-1} z^{-1} + l_0 + l_1 z.
\]

**Remark 6.4.** An analyst may feel disappointed with our choice of the ‘restricted dual’ \( Lg \subset Lg^*_W \) of the Wiener algebra; indeed, the coadjoint orbits which are contained in this space are modelled on (matrix) polynomial functions; even general rational functions are not allowed, to say nothing of more interesting classes of analytic functions. The reason for this deliberate restriction is very simple: we are willing to get dynamical systems which admit a simple parametrization and (possibly) some physical interpretation; practical experience shows that examples which are physically interesting are usually associated with coadjoint orbits of the
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lowest possible dimensions. (By contrast, most of the orbits which lie in the ‘exotic’ part of the full dual space are infinite-dimensional.) This does not mean of course that dynamical systems which are modelled on analytic functions of a more complicated nature are totally uninteresting; but once again, ‘good’ examples are associated not with generic coadjoint orbits in the ‘very big dual’, but rather with well-embedded finite-dimensional ones. (Below we shall see how to construct such examples using a different choice of the basic Lie algebra.) One more reason to single out the finite-dimensional orbits is the possibility to bring into play the highly powerful machinery of Algebraic Geometry: we shall see below that polynomial (or, more generally, rational) Lax matrices give rise to algebraic curves of finite genus and Lax equations are linearized on their Jacobians. Lax matrices associated with infinite-dimensional orbits will lead to curves of infinite genus.

The specialization of our main theorem to the present situation may be stated as follows:

**Theorem 6.5.** (i) Invariant functionals $\Phi_{n,m}$ give rise to Hamiltonian equations of motion on $Lg \subset (Lg)^*_{r}$ with respect to the Lie-Poisson bracket of $Lg_{r}$; these equations may be written in the Lax form,

$$\frac{dL}{dt} = [L, M_{\pm}], M_{\pm} = \pm \mathcal{P}_{\pm} (\text{grad } \Phi_{n,m}).$$

(ii) The integral curve of (6.3) with origin $L_0$ is given by

$$L(t, z) = g_{\pm}(t, z)^{-1} L_0 g_{\pm}(t, z),$$

where $g_{+}(t), g_{-}(t)$ solve the matrix Riemann problem

$$\exp t \text{grad } \Phi_{n,m} [L_0] (z) = g_{+}(t, z) g_{-}(t, z)^{-1},$$

$$g_{+}(t) \in \mathcal{G}_{W}^{+}, g_{-}(t) \in \mathcal{G}_{W}^{-}.$$ 

Note that $\text{grad } \Phi_{n,m}$ is a Laurent polynomial and hence is regular in the punctured Riemann sphere $\mathbb{CP}_1 \setminus \{0 \cup \{\infty\}\} = \mathbb{C} \setminus \{0\}$. Hence the factorization problem (6.5) has the following geometric meaning. The projective line $\mathbb{CP}_1$ is covered by two domains $U_{+} = \mathbb{CP}_1 \setminus \{\infty\}$ and $U_{-} = \mathbb{CP}_1 \setminus \{0\}$. The function $\exp t \text{dh}(L)$ is regular in $U_{+} \cap U_{-} = \mathbb{C} \setminus \{0\}$ and may be regarded as the transition function of a holomorphic vector bundle over $\mathbb{CP}_1$. Factorization problem (6.5) amounts to an analytic trivialization of this bundle. It is well known (see [12]) that not all vector bundles over $\mathbb{CP}_1$ are analytically trivial: each $n$-dimensional bundle breaks up into a sum of line bundles, and their degrees, $d_{1},...,d_{n} \in \mathbb{Z}$ form a full system of holomorphic invariants of the given bundle. In the language of transition functions this means that $\exp t \text{dh}(L)$ admits a factorization of the form

$$\exp t \text{dh}(L) = g_{+}(z, t) d(z) g_{-}(z, t)^{-1},$$
where \( d(z) = \text{diag}(z^{d_1}, \ldots, z^{d_n}) \). Thus formula (6.4) requires that all partial indices \( d_1, \ldots, d_n \) are zero. One can prove that this is true at least for \( t \) sufficiently small \([13]\). The exceptional values of \( t \in \mathbb{C} \) for which problem (6.5) does not admit a solution form a discrete set in \( \mathbb{C} \); at these points the solution \( L(t) \) has a pole: the trajectory of the Lax equation goes off to infinity.

### 6.1. Riemann Problem and Spectral Curves

Theorem 6.5 provides a link between the Hamiltonian scheme of Section 3 and the algebro-geometric methods of the finite-band integration theory (see \([14]\)). Namely, the formula (6.4) for the trajectories immediately implies that the Lax equations linearize on the Jacobian of the spectral curve associated with the Lax matrix. The proof is so short and simple that I would like to reproduce it here.

Let \( g = \mathfrak{gl}(n, \mathbb{C}) \) and \( L(z) = \sum X_i z^i, X_i \in g \), a matrix-valued Laurent polynomial. Let us consider the algebraic curve \( \Gamma_0 \subset \mathbb{C} \setminus \{0\} \times \mathbb{C} \) defined by the characteristic equation

\[
\det(L(z) - \lambda) = 0; \tag{6.6}
\]

we may regard \( z, \lambda \) as meromorphic functions defined on \( \Gamma_0 \). Assume that the spectrum of \( L(z) \) is simple for generic \( z \) (this key technical assumption is satisfied in most applications). For each nonsingular point \( P \subset \Gamma_0 \) which is not a branching point of \( \lambda \) there is a one-dimensional eigenspace \( E(P) \subset \mathbb{C}^n \) of \( L(z(P)) \) with eigenvalue \( \lambda(P) \). This gives a holomorphic line bundle on \( \Gamma_0 \) defined everywhere except for singular points and branching points. Let \( \Gamma \) be the nonsingular, compact model of \( \Gamma_0 \). One can show that the eigenvector bundle extends to a holomorphic line bundle \( E \rightarrow \Gamma \) on the whole smooth curve \( \Gamma \).

The spectral curve \( \Gamma \) with two distinguished meromorphic functions \( z \) and \( \lambda \) and the line bundle \( E \rightarrow \Gamma \) constitute the set of spectral data for \( L(z) \). The evolution determined by a Lax equation of motion leaves the spectral curve (7.3) invariant, and the dynamics of the line bundle \( E \) is easy to describe. Let \( h \) be the Hamiltonian of our Lax equation; set \( M = dh(L) \); since \([L,M] = 0\) pointwise and the spectrum of \( L \) is simple, the eigenvectors of \( L(z) \) are also the eigenvectors of \( M(z) \),

\[
M(z(P))v = \mu(P)v, \ P \in \Gamma, \ v \in E(P) \subset \mathbb{C}^n,
\]

where \( \mu(P) \) is a meromorphic function on \( \Gamma \). Define the domains \( V_\pm \subset \Gamma \) by \( V_\pm = \left\{ P \in \Gamma; z(P)^{\pm 1} \neq \infty \right\} \). Clearly, \( V_+ \cup V_- = \Gamma \) and \( \mu \) is regular in the intersection \( V_+ \cap V_- = \Gamma_0 \). Recall that a line bundle over a curve is specified by its \textit{transition function} (with values in the multiplicative group \( \mathbb{C}^* \)) with respect to some covering; tensor product of bundles corresponds to the ordinary product.

\[\text{Indeed, the mapping } P \rightarrow E(P) \text{ determines a meromorphic mapping } \Gamma \rightarrow \mathbb{C}P_{n-1} \text{ (this is a corollary of the elementary analytic perturbation theory). By a standard theorem, any such mapping is actually holomorphic, and hence the eigenvector bundle extends to } \Gamma.\]
of transition functions. The equivalence classes of line bundles form an abelian group \( \text{Pic} \Gamma \) with respect to the tensor product. Let \( F_t \) be the line bundle on \( \Gamma \) determined by the transition function \( \exp t \mu (P) \) with respect to the covering \( \{ V_+, V_- \} \). For all \( t \in \mathbb{C} \) the bundles \( F_t \) have degree zero and form a 1-parameter subgroup in the Picard group \( \text{Pic}_0 \Gamma \) of equivalence classes of holomorphic line bundles of degree zero on \( \Gamma \) which, by Abel’s theorem, is canonically isomorphic to \( \text{Jac} \Gamma \), the Jacobian of \( \Gamma \). (See, for instance, [15].)

**Theorem 6.6.** The line bundle \( E \) regarded as a point of \( \text{Pic} \Gamma \) evolves linearly with time, \( E(t) = E \otimes F_{-t} \).

**Proof.** Since the Lax matrix evolves by similarity transformation, its eigenvectors evolve linearly. Let \( g_{\pm} (t) \) be the solution of the Riemann problem (6.5). In view of (6.4), the moving eigenspace \( E_t (P) \) regarded as a subspace of \( \Gamma \times \mathbb{C}^n \), is expressed as

\[
E_t (P) = g_+ (t, z (P))^{-1} E (P)
\]

over \( V_+ \), and

\[
E_t (P) = g_- (t, z (P))^{-1} E (P)
\]

over \( V_- \). In other words, \( g_{\pm} (t, z)^{-1} \) define isomorphisms between \( E (P) \) and \( E_t (P) \) over \( V_\pm \). The transition function in \( V_+ \cap V_- \) which matches these two isomorphisms is \( g_+ (t, z (P))^{-1} g_- (t, z (P)) \big|_{E_t (P)} \). It is easy to check that

\[
g_+ (t, z) g_- (t, z)^{-1} = \exp t dh (L_0)
\]

implies

\[
g_- (t, z) g_+ (t, z)^{-1} = \exp t dh (L (t));
\]

hence,

\[
g_+ (t, z (P))^{-1} g_- (t, z (P)) \big|_{E_t (P)} = e^{-tM (t, z (P))} \big|_{E_t (P)} = e^{-t \mu (P)},
\]

or \( E_t (P) = E (P) \otimes F_{-t} \), as claimed. \( \square \)

The eigenvector \( \psi (t, P) = (\psi_1, ..., \psi_n) \in E (t, P) \subset \mathbb{C}^n \) is called the Baker-Akhiezer function of \( L (z, t) \). From (6.7, 6.8) it follows that the Baker-Akhiezer function \( \psi (t, P) \) in the domains \( V_\pm \subset \Gamma \) may be written in the form

\[
\psi_\pm (t, P) = g_\pm (t, z (P))^{-1} \psi (P),
\]

so that

\[
\psi_+ (t, P) = e^{-t \mu (P)} \psi_- (t, P).
\]

Since \( \partial_t g_{\pm} \cdot g_{\pm}^{-1} = M_{\pm} \) (see the proof of theorem 3.3), we have

\[
\frac{d}{dt} \psi_\pm (t, P) = -M_\pm (t, z (P)) \psi_\pm (t, P).
\]
Using the machinery of Algebraic Geometry, it is possible to construct the Baker-Akhiezer function explicitly, in terms of the Riemann theta functions and Abelian integrals. This, in turn, allows to obtain an explicit solution of the Riemann problem. Let us explain how the matrices $g_{\pm}(t, z)$ may be reconstructed from $\psi_{\pm}(t, P)$. Suppose that $z \in \mathbb{C}$ is not a ramification point of $\Gamma$ (i.e., all eigenvalues of $L(z)$ are distinct); let $P_1, ..., P_n$ be the points of $\Gamma$ which lie over $z$. Let us arrange the column vectors $\psi_+(t, P_1), ..., \psi_+(t, P_n)$ in a $n \times n$ matrix $\hat{\psi}_+(t, z)$. Put

$$g_{\pm}(t, z) = \psi_{\pm}(t, \lambda) \hat{\psi}_{\pm}(0, \lambda)^{-1}.$$  

Note that if we change the ordering of branches $P_1, ..., P_n$, $\hat{\psi}_{\pm}$ is multiplied on the right by a permutation matrix and hence $g_{\pm}$ remains invariant.

**Proposition 6.7.** (i) $g_{\pm}(t, z)$ satisfies the differential equation

$$\frac{dg_{\pm}(t, z)}{dt} = -M_{\pm}(t, z) g_{\pm}(t, z), \quad M_{\pm}(t, z) = (\text{grad } h[L(t, z)])_{\pm}. $$  

(ii) $g_{\pm}$ are entire functions of $z^{\pm 1}$. (iii) $g_{\pm}$ solve the factorization problem

$$g_+(t, z) g_-(t, z)^{-1} = \exp t \text{grad } h[L(0, z)].$$

The key assertion (ii) is an easy consequence of the differential equation (6.11); indeed, by (6.10) $g_{\pm}$ is the fundamental solution of (6.11) normalized by $g_{\pm}(0, z) = Id$; hence it is holomorphic in the domain where $M_{\pm}(t, z)$ is nonsingular, i.e., in $\mathbb{C} P_1 \setminus \{\infty\}$ and $\mathbb{C} P_1 \setminus \{0\}$, respectively.

**7. More Examples**

**7.1. Twisted Loop Algebras**

The decomposition of the loop algebra we used so far is based on its standard grading (i.e., grading by the powers of the loop parameter $z$). This grading is by no means unique, and it is possible to use other gradings to produce new examples of Lax representations. Another possibility (which actually absorbs the former one) is to bring into play twisted loop algebras. Let $\sigma$ be an automorphism of $\mathfrak{g}$ of order $n$. The twisted loop algebra $L(\mathfrak{g}, \sigma)$ is the subalgebra of $L\mathfrak{g}$ defined by

$$L(\mathfrak{g}, \sigma) = \{x \in L\mathfrak{g}; \sigma(x(z)) = x(\epsilon z)\},$$

where $\epsilon = \exp \frac{2\pi i}{n}$ is the root of unity. Equivalently, $L(\mathfrak{g}, \sigma) \subset L\mathfrak{g}$ is the stable subalgebra of the automorphism $\sigma : L\mathfrak{g} \to L\mathfrak{g}$ such that $x^\sigma(z) = \sigma(x(\epsilon z))$ for
all $x \in L\mathfrak{g}$. The isomorphism class of the twisted loop algebra depends on the properties of $\sigma$; one can show that when $\sigma$ is an inner automorphism, the stable subalgebra $L(\mathfrak{g},\sigma)$ is isomorphic to $L\mathfrak{g}$ as an abstract Lie algebra; however, the grading which is induced on $L(\mathfrak{g},\sigma) \subset L\mathfrak{g}$ by the standard grading in $L\mathfrak{g}$ is different. The classical theorem due to V.Kac asserts that all different gradings on $L\mathfrak{g}$ may be obtained in this way. (Among the integrable systems that may be very naturally constructed along these lines one may quote periodic Toda lattices which are associated with the decomposition of loop algebras derived from their principal grading, see [8] for details.)

The most interesting case for applications is when $\sigma$ is an outer automorphism of order 2 (an involution). In this case we may assume that $\mathfrak{g}$ and $L\mathfrak{g}$ are real. Here is the key example: $\mathfrak{g} = \mathfrak{gl}(n), \sigma(X) = -X^t$ ($t$ denotes transposition). The loop algebra $L\mathfrak{g}^\sigma$ consists of Laurent polynomials

$$X(z) = \sum X_k z^k,$$

where $X_{2p} = -X_{2p}^t, X_{2p+1} = X_{2p+1}^t$. Antisymmetric matrices belong to the Lie algebra $\mathfrak{so}(n)$ of the orthogonal group, which describes kinematics of the rigid body; on the other hand, symmetric matrices are reminiscent of the quadratic form associated with kinetic energy (inertia tensor). Thus, the twisted loop algebra seems to be a good candidate to set up the stage for applications to the mechanics of the rigid body. Let us equip $L\mathfrak{g}^\sigma$ with the inner product

$$\langle X, Y \rangle = -\text{Res}_{z=0} \frac{1}{z} \text{tr} X(z) Y(z)$$

which sets $L\mathfrak{g}^\sigma$ into duality with itself (mind the difference with (6.1): the factor $z^{-1}$ makes the coupling respect parity; the minus sign makes the inner product positive on $\mathfrak{so}(n)$). Let $r$ be the classical $r$-matrix associated with the standard decomposition $L\mathfrak{g}^\sigma = L\mathfrak{g}^\sigma_+ + L\mathfrak{g}^\sigma_-$, as in (6.2). Obviously,

$$(L\mathfrak{g}^\sigma_+)^* \simeq (L\mathfrak{g}^\sigma_-)^\perp = \oplus_{k \leq 0} \mathfrak{g} \cdot z^k, \quad (L\mathfrak{g}^\sigma_-)^* \simeq (L\mathfrak{g}^\sigma_+)^\perp = \oplus_{k > 0} \mathfrak{g} \cdot z^k.$$

Coadjoint orbits of $L\mathfrak{g}^\sigma_-$ are direct products of orbits of $L\mathfrak{g}^\sigma_+$ lying in $(L\mathfrak{g}^\sigma_-)^\perp$ and orbits of $L\mathfrak{g}^\sigma_+$ lying in $(L\mathfrak{g}^\sigma_-)^\perp$. Lax matrices describing the motion of the rigid body and related mechanical systems belong to the simplest $ad^*$-invariant subspace $L\mathfrak{g}^\sigma_{-1,1}$ consisting of matrices

$$L(z) = az^{-1} + l + bz, l \in \mathfrak{so}(n), a = a^t, b = b^t.$$

**Exercise 7.1.**

(i) All monomials $bz \in L\mathfrak{g}$ are 1-point orbits of $L\mathfrak{g}^\sigma_+$. (ii) Coadjoint representation of $L\mathfrak{g}^\sigma_+$ in the subspace $\{az^{-1} + l\}$ of $(L\mathfrak{g}^\sigma_-)^*$ factors through its finite-dimensional quotient $L\mathfrak{g}^\sigma_+ / z^2 L\mathfrak{g}^\sigma_+$. (iii) The quotient algebra $L\mathfrak{g}^\sigma_+ / z^2 L\mathfrak{g}^\sigma_+$ is isomorphic to the semidirect product
\( g_0^n = so(n) \ltimes sym(n) \) of the orthogonal algebra \( so(n) \) and the space of symmetric matrices (with zero Lie bracket).

The associated Lie group \( G_0^n \) is the semidirect product of \( SO(n) \) and the additive group of the linear space \( sym(n) \) of symmetric \( n \times n \)-matrices. Its coadjoint orbits are easy to describe; here is a simple example:

**Exercise 7.2.** Fix a unit vector \( e \in \mathbb{R}^n \) and let \( a = e \otimes e \) be the rank one orthogonal projection operator onto \( \mathbb{R} \cdot e \subset \mathbb{R}^n \). Let \( T^*S^{n-1} \) be the cotangent bundle of the sphere \( S^{n-1} = SO(n) \cdot e \subset \mathbb{R}^n \) realized as the subbundle of \( S^{n-1} \times \mathbb{R}^n \),

\[ T^*S^{n-1} = \{(x,p) \in S^{n-1} \times \mathbb{R}^n; \langle p, x \rangle = 0.\} \]

There is natural map \( \pi : T^*S^{n-1} \to \mathcal{O}_g \) onto the coadjoint orbit of \( G_0^n \) passing through the monomial \( z^{-1}a \),

\[ \pi : (x,p) \mapsto z^{-1}x \otimes x + p \wedge x. \]

(\( \pi \) is actually a double covering.)

The Lax matrix associated with this orbit has the form

\[ L(z) = z^{-1}x \otimes x + p \wedge x + zb, \ x \in SO(n), \ p \in \mathbb{R}^n, \ \langle p, x \rangle = 0; \]

the associated phase space describes the point moving on a sphere. The constant matrix \( b \in sym(n) \) does not affect kinematics, but is quite useful to produce interesting Hamiltonians. The simplest Hamiltonian is

\[ H = -\frac{1}{4} \operatorname{Res}_{z=0} \operatorname{tr} \frac{1}{z} L(z)^2 = \frac{1}{2} \langle p, p \rangle - \frac{1}{2} \langle bx, x \rangle; \]

it describes the so called Neumann problem (point moving on a sphere in a quadratic potential).

**Exercise 7.3.** Describe the set of commuting integrals of motion for the Neumann problem which are quadratic in momenta.

We may generalize this example in various ways; a useful remark is that we may avoid a too detailed description of the coadjoint orbits. Instead, we may produce a map \( \pi \) onto such orbit, or a union of orbits, which is compatible with the Poisson structure but need not be a bijection (and so possibly introduces some extra variables). This idea is implemented in the following statement. Set \( K = SO(n), \mathfrak{t} = so(n) \) and let \( T^*K \simeq K \times \mathfrak{t} \) be the cotangent bundle (equipped with its standard Poisson bracket). Fix \( a \in sym(n) \) and consider the mapping

\[ \pi : T^*K \to (g_0^n)^* : (k, \rho) \mapsto k(\rho + a)k^{-1} \]

---

\( ^8 \)We denote by \( p \wedge x \) the \( n \times n \)-matrix with entries \( p_i x_j - p_j x_i \); in a similar way, the entries of \( x \otimes x \) are \( x_i x_j \).
Proposition 7.4. \( \pi \) is a Poisson mapping (i.e., maps canonical Poisson brackets in \( T^*K \) onto Lie-Poisson brackets in \( (\mathfrak{g}_0^\sigma)^* \)); its image is a union of coadjoint orbits of the semidirect product \( G_0^\sigma = K \times \text{sym}(n) \).

[In this statement we identified \( \mathfrak{g}_0^\sigma \) with \( \text{Mat}(n) \) as a linear space and also used the inner product \( \langle X, Y \rangle = -\text{tr} XY \) to identify the dual space \( \mathfrak{g}_0^\sigma \) with \( \text{Mat}(n) \).]

The cotangent bundle \( T^*K \) is naturally interpreted as the phase space of a rigid body in \( \mathbb{R}^n \); we get a family of Lax matrices parametrized by points of \( T^*K \):

\[
L(z) = z^{-1}kak^{-1} + kpk^{-1} + zb;
\]

(7.3)

Hamiltonians which may be derived from (7.3) include the so called Manakov case of the motion of a free top in \( \mathbb{R}^n \) (for \( n = 3 \) this is the classical Euler top), or more generally, the Manakov top in a quadratic potential.

Remark 7.5. One may wonder, what is the relation of the low-dimensional Neumann system to the “big” phase space \( T^*K \) (we have \( \dim T^*S^{n-1} = 2n - 2 \) and \( \dim T^*K = n(n-1) \)). The answer is that, for special choices of \( a \in \text{sym}(n) \), the Hamiltonians associated with the Lax matrix (7.3) possess high symmetry (resulting from the redundancy introduced by \( \pi \)). The Neumann system is the result of Hamiltonian reduction of the “big system” with respect to this symmetry group.

One may notice that the use of the twisted loop algebra was indeed crucial: the built-in symmetry of the Lax matrix accounts both for the correct kinematics of the rigid body (antisymmetric matrices) and for the symmetry of the related quadratic forms (notably, of the kinetic energy). Further generalization is straightforward: we must scan the list of semisimple Lie algebras and their involutions and look for nice-looking opportunities. In this way we get the following list:

| Algebra    | Involution | Related systems                      |
|------------|------------|--------------------------------------|
| \( \mathfrak{gl}(n) \) | \( X \mapsto -X^t \) | Manakov top, Neumann system, etc.     |
| \( \mathfrak{so}(n,1) \) | \( X \mapsto -X^t \) | Lagrange top, spherical pendulum, etc. |
| \( \mathfrak{so}(p,q), p > q \geq 2 \) | \( X \mapsto -X^t \) | Kowalevski top and its generalizations |
| \( \mathfrak{so}(n,n) \) | \( X \mapsto -X^t \) | Interacting Manakov tops               |
| \( G_2 \subset \mathfrak{so}(4,3) \) | \( X \mapsto -X^t \) | Exotic integrable top on \( \text{SO}(4) \) |

In all cases, Lax matrices belong to the subspace \( \mathfrak{g}_{0,-1,1}^\sigma \); to get particular examples (for instance, the Kowalevski top) one sometimes has to perform additional Hamiltonian reduction; we refer the reader to [8] for details.

Let us finally discuss the implications of the twisting automorphism for the geometry of the spectral curve and for the linearization theorem.

Proposition 7.6. (i) Let us assume that the Lax matrix \( L(z) \in L\mathfrak{g}^\sigma \) and \( \sigma \) is an inner automorphism of \( \mathfrak{g} \), \( \text{ord} \sigma = m \). In that case the spectral curve \( \Gamma = \{ (z, \lambda) \in \mathbb{C}^2; \det (L(z) - \lambda) = 0 \} \) admits an automorphism \( \hat{\sigma} : (z, \lambda) \mapsto (\epsilon z, \lambda) \)
(here \( \epsilon = \exp 2\pi i/n \) is the root of unity); this automorphism lifts to \( \text{Pic} \Gamma \) and the transition function (6.9) which determines the evolution of the eigenbundle of \( L \) is invariant under \( \hat{\sigma} \). (ii) Suppose that \( \sigma \) is an outer involution, \( \sigma (X) = -X^t \). Then the spectral curve admits an automorphism \( \hat{\sigma} : (z, \lambda) \mapsto (-z, -\lambda) \); the transition function is anti-invariant under \( \hat{\sigma} \), i.e., \( \hat{\sigma} : \exp t\mu(z, \lambda) \mapsto \exp (-t\mu(z, \lambda)) \).

The check of both assertions is obvious: An inner automorphism preserves the eigenvalues of a matrix; by contrast, \( \sigma (X) = -X^t \) changes the sign of the eigenvalues. The logarithm of the transition function \( \mu(z, \lambda) \) is the eigenvalue of the gradient \( \text{grad} H[L](z) \) (more precisely, one of its branches associated with the eigenvector of \( L(z) \) which corresponds to \( \lambda \)). Since \( \text{grad} H[L] \in \mathfrak{g} \sigma \), \( \mu(z, \lambda) \) is invariant when \( \sigma \) is inner and changes sign when it’s derived from transposition.

7.2. Rational Lax Matrices

Let us now describe how to deal with Lax matrices which are rational functions of the spectral parameter. As we mentioned, it is possible to trace down the corresponding coadjoint orbits inside the dual space of an appropriate completion of the standard loop algebra, but it is more practical to choose our basic Lie algebra in a different way.

Let \( D = \{z_1, ..., z_N \} \subset \mathbb{CP}_1 \) be a finite set; we assume that \( \infty \in D \). For \( z_j \in D \) let \( \lambda_j \) be the local parameter on \( \mathbb{CP}_1 \) at \( z = z_j \), i.e., \( \lambda_j = z - z_j \) if \( z_j \neq \infty \) and \( \lambda_\infty = z^{-1} \) for \( z_j = \infty \). We define the local algebra \( \mathfrak{g}_{z_j} \) as the algebra of formal Laurent series in local parameter with coefficients in a little Lie algebra \( \mathfrak{g}, \mathfrak{g}_{z_j} = \mathfrak{g}((\lambda_j)) \). (We may assume that \( \mathfrak{g} \) is the matrix algebra with the standard inner product.) If \( z_j \neq \infty \), let \( \mathfrak{g}_{z_j}^+ \) be the algebra of formal Taylor series in local parameter; for \( z_j = \infty \) we set \( \mathfrak{g}_{z_j}^+ = \lambda_\infty \mathfrak{g}[[\lambda_\infty]] \) (in other words, \( \mathfrak{g}_{z_j}^+ \) consists of formal Taylor series without constant term). Put

\[
\mathfrak{g}_D = \bigoplus_{z_j \in D} \mathfrak{g}_{z_j}, \quad \mathfrak{g}_D^+ = \bigoplus_{z_j \in D} \mathfrak{g}_{z_j}^+
\]

(direct sum of Lie algebras). Let \( \mathfrak{g}(D) \) be the algebra of rational functions on \( \mathbb{CP}_1 \) with coefficients in \( \mathfrak{g} \) which are regular outside \( D \); it is naturally embedded into \( \mathfrak{g}_D \) (the embedding assigns to each \( X \in \mathfrak{g}(D) \) the collection of its Laurent series at each point of \( D \)).

**Proposition 7.7.** (i) \( \mathfrak{g}_D = \mathfrak{g}(D) + \mathfrak{g}_D^+ \) (direct sum of linear spaces).

(ii) The \( \mathbb{C} \)-bilinear inner product on \( \mathfrak{g}_D \)

\[
\langle X, Y \rangle = \sum_{z_j \in D} \text{Res}_{z_j} \text{tr} X_j Y_j d\lambda_j
\]

(7.4)

is invariant and nondegenerate. (iii) \( \mathfrak{g}(D) \) and \( \mathfrak{g}_D^+ \) are isotropic subspaces with
respect to (7.4); moreover, \( g(D) \cong (\mathfrak{g}^+_D)^* \). (iv) Coadjoint orbits of \( \mathfrak{g}^+_D \) in \( g(D) \) are finite-dimensional.

Sketch of a proof. An element \( X = (X_j)_{z_j \in D} \) is a finite collection of Laurent series; stripping each of its positive part we get a set of principal parts at \( z_j \in D \); let \( X^0 \) be the unique rational function with these principal parts; by construction, \( X - X^0 \in \mathfrak{g}^+_D \) (mind the special role of \( \infty \) which fixes the normalization condition!). In brief, we can say that the decomposition \( \mathfrak{g}_D = g(D) + \mathfrak{g}^+_D \) is equivalent to the Mittag-Leffler theorem for rational functions. Isotropy of \( g(D) \) and \( \mathfrak{g}^+_D \) means that the inner product restricted to these subspaces is identically zero; this condition assures that the associated classical \( r \)-matrix is skew symmetric. For \( \mathfrak{g}^+_D \) this assertion is immediate, since (for \( z \neq \infty \)) the product of two Taylor series has zero residue; for \( z = \infty \) the residue disappears because of the normalization condition. The isotropy of \( g(D) \) is a reformulation of the classical theorem: the sum of residues of a rational function is zero.

Since \( \mathfrak{g}^+_D \) is a direct sum of local algebras, its coadjoint orbits are direct products of the coadjoint orbits of each local factor; it is easy to see that the coadjoint orbits of the local algebra \( \mathfrak{g}^+_z \) are modelled on rational functions with a single pole at \( z = z_j \); moreover, the subspace of rational functions with prescribed order of singularity at this point is stable under the coadjoint action of \( \mathfrak{g}^+_z \). Clearly, this subspace is finite-dimensional, which proves (iv).

Exercise 7.8. Describe coadjoint orbits in the subspace of functions admitting only simple poles.

Our main theorem immediately applies in this setting and provides an ample set of integrals of motion in involution for Lax equations with rational Lax matrix.

Remark 7.9. (1) Since we are interested only in Lax operators which are global rational functions on the Riemann sphere, we consider only coadjoint orbits of \( \mathfrak{g}^+_D \subset (\mathfrak{g}_D)_r \); this is legitimate, since \( (\mathfrak{g}_D)_r \) splits into direct sum of two complementary subalgebras,

\[
(\mathfrak{g}_D)_r \cong \mathfrak{g}^+_D \oplus g(D),
\]

and hence its orbits are direct products of orbits lying in \( g(D) \) and in \( \mathfrak{g}^+_D \); in other words, we take orbits which project into zero in \( \mathfrak{g}^+_D \).

(2) The global algebra \( \mathfrak{g}^+_D \) is decomposed into direct sum of local factors, \( \oplus_{z_j \in D} \mathfrak{g}^+_z \); coadjoint orbits of each local algebra are the same that we encountered for the ordinary loop algebra. What makes things different, is the way these orbits are embedded into the bigger algebra; this embedding affects the choice of the invariant Hamiltonians as well as the formulation of the factorization problem.

The use of formal series is well adapted for the study of coadjoint orbits in \( g(D) \); in order to be able to define Lie groups associated with our Lie algebras, we must change the topology by replacing formal series with convergent ones. Let \( G^{\text{sing}}_{z_j} \) be the group of germs of functions with values in \( G \) which are regular in some punctured disc around \( z_j \in \mathbb{C}P_1 \) (with topology of uniform absolute convergence),
its subgroup consisting of functions regular in the entire small disc, and 
$G(D)$ the group consisting of holomorphic mappings $\mathbb{C}P_1 \setminus D \to G$. The
infinitesimal decomposition described in proposition 7.7 (i) corresponds to the
following multiplicative problem:

Given a set of local meromorphic functions $g_1, \ldots, g_N, g_j \in G_W^\ast$, find a global
meromorphic function $g_0$ which is regular in the punctured sphere $\mathbb{C}P_1 \setminus D$ such
that $g_0 g_j^{-1}$ is regular in some small disc around $z_j$.

This is the standard multiplicative Cousin problem; its geometrical meaning is
the same as for the matrix Riemann problem discussed above: it corresponds to
the trivialization of a vector bundle over $\mathbb{C}P_1$ (defined with respect to a different
covering of the sphere).

**Exercise 7.10.** Reformulate the global factorization theorem in this setting.

One is of course tempted to generalize the above construction replacing $\mathbb{C}P_1$
with an arbitrary Riemann surface. There is an obvious obstruction which comes
from the Mittag-Leffler theorem for curves: a global meromorphic function on a
curve $\Gamma$ with prescribed principal parts exists if and only if these principal parts
satisfy a set of linear constraints; roughly, the sum of residues

$$\sum_{z_j \in D} \text{Res}_{z_j} \text{tr} X_j \omega$$

must be zero for all holomorphic differentials $\omega \in H^1(\Gamma) \otimes g$. The trouble is that
the constrained data do not form a Lie subalgebra inside the global algebra $\mathfrak{g}_D$,
and hence one cannot find a complement of $\mathfrak{g}_D^\ast$ which is a Lie subalgebra. When
$\Gamma$ is elliptic, this obstruction may be overcome by imposing additional automorphy
conditions, see e.g. [8].

8. **Zero Curvature Equations**

In applications to integrable PDE’s, Lax matrices are replaced by first order ma-
trix differential operators. The systematic treatment of these applications is based
on the use of double loop algebras, or, more precisely, of their central extensions.
Let us start with discussion of the central extension of the ordinary loop algebra.

8.1. **Central Extensions of Loop Algebras**

Set $\mathfrak{g} = \mathfrak{gl}(n)$ and let $\mathfrak{g} = C^\infty (S^1; \mathfrak{g})$ be the Lie algebra of smooth functions
on the circle with values in $\mathfrak{g}$ and with pointwise commutator. (Mind that in the
present setting we choose topology in our loop algebra in a different way! This is
because we are willing to treat functions of $x \in S^1$ as dynamical variables for our future evolution equations.) We equip $\mathfrak{g}$ with the invariant inner product

$$\langle X, Y \rangle = \int_0^{2\pi} \text{tr} \ XY \, dx;$$

accordingly, we get an embedding $\mathfrak{g} \subset \mathfrak{g}^*$ which defines the smooth dual of $\mathfrak{g}$.

Put

$$\omega(X, Y) = \int_0^{2\pi} \text{tr} \ X \cdot \frac{dY}{dx} \, dx;$$

Exercise 8.1. $\omega$ is a skew symmetric bilinear form which satisfies the cocycle condition

$$\omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0.$$  

Put $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$ and define the Lie bracket in $\hat{\mathfrak{g}}$ by

$$[[X, c], [Y, c']] = ([X, Y], \omega(X, Y)), \quad X, Y \in \mathfrak{g}, \quad c, c' \in \mathbb{C}.$$  

(The Jacobi identity is equivalent to (8.3).) Notice that $c = \{(0, c) ; c \in \mathbb{C}\} \subset \hat{\mathfrak{g}}$ is central in $\hat{\mathfrak{g}}$ and $\mathfrak{g}$ may be identified with the quotient algebra, $\mathfrak{g} = \hat{\mathfrak{g}}/c$. The (smooth) dual of $\mathfrak{g}$ may be identified with $\mathfrak{g} \oplus \mathbb{R}$.

**Proposition 8.2.** The coadjoint representation of $\hat{\mathfrak{g}}$ is given by

$$\text{ad}^* X \cdot (L, e) = ([X, L] + e \partial_x X, 0).$$

(Note that this representation is trivial on the center $c \subset \hat{\mathfrak{g}}$ and therefore it may be regarded as a representation of $\mathfrak{g}$.)

Let $G = C^\infty(S^1; G)$ be the Lie group associated with the Lie algebra $\mathfrak{g}$. The coadjoint representation of $\mathfrak{g}$ in $\hat{\mathfrak{g}}^*$ may easily be integrated to a representation of $G$.

**Proposition 8.3.** The coadjoint representation of $G$ in $\hat{\mathfrak{g}}^*$ is given by

$$\text{Ad}^* g \cdot (L, e) = (gLg^{-1} + e \partial_x g \cdot g^{-1}, e).$$

Proposition 8.3 admits a remarkable geometric interpretation. Consider the auxiliary linear differential equation

$$e \frac{d\psi}{dx} = L\psi$$

(we regard it as a differential equation on the line with periodic potential $L$).
Exercise 8.4. Coadjoint representation (8.5) leaves invariant the hyperplanes $e = \text{const}$ in $\hat{G}^*$; on each hyperplane $e = \text{const} \neq 0$ it is equivalent to the gauge transformation of the potential $L$ in the linear equation (8.6) induced by the natural action of $G = C^\infty(S^1; G)$ on its solutions by left multiplication, $g : \psi \mapsto g \cdot \psi$.

Let $\psi_0 \in C^\infty(\mathbb{R}; G)$ be the fundamental solution of (8.6) normalized by $\psi_0(0) = \text{id}; T_L = \psi_0(2\pi) \in G$ is called the monodromy matrix of $L$.

Theorem 8.5. (Floquet) Two potentials $L, L' \in C^\infty(S^1; g)$ lie on the same coadjoint orbit in $\hat{G}^*$ (with fixed $e \neq 0$) if and only if $T_L$ and $T_{L'}$ are conjugate in $G$.

Sketch of a proof. Without restricting the generality we may assume that $T_L = T_{L'}$. Let $\psi_L, \psi_{L'}$ be the fundamental solutions normalized by $\psi_L(0) = \psi_{L'}(0) = \text{id}$; put $g(x) = \psi_L(x) \psi_{L'}(x)^{-1}$. Clearly, $g$ is $2\pi$-periodic on the line and $g \cdot \psi_{L'} = \psi_L$, hence $\text{Ad}^* g \cdot L' = L$.

Corollary 8.6. All coadjoint orbits lying in the hyperplanes $e = \text{const} \neq 0$ have finite codimension; the ring of Casimir functions is generated by spectral invariants of the monodromy.

The Hamiltonian mechanics in $\hat{G}^*$ may be defined with the help of the elementary calculus of variations. Let $\varphi[L]$ be a smooth functional of the potential $L, X_\varphi = \text{grad}_\varphi [L] \in \mathfrak{g}$ its Frechet derivative defined by

$$
\frac{d}{ds} \varphi [L + s \eta] = \int_0^{2\pi} \text{tr} X_\varphi(x) \eta(x) \, dx.
$$

The Lie-Poisson bracket of two functionals $\varphi_1, \varphi_2$ is given by

$$
\{ \varphi_1, \varphi_2 \} [L] = \int_0^{2\pi} \text{tr} (L(x) [X_{\varphi_1}(x), X_{\varphi_2}(x)] + e X_{\varphi_1} \partial_x X_{\varphi_2}) \, dx.
$$

Proposition 8.7. The Hamiltonian equation of motion on $\hat{G}^*$ with Hamiltonian $\varphi$ is equivalent to the following differential equation for the potential $L$:

$$
\frac{\partial L}{\partial t} = - [X_\varphi, L] - e \frac{\partial X_\varphi}{\partial x}.
$$

Equation (8.7) has a nice geometrical meaning. Let us consider the $\mathfrak{g}$-valued differential form

$$
L dx + X_\varphi dt;
$$

it may be regarded as a connection form of a connection on $\mathbb{R}^2$ (with values in $\mathfrak{g}$); equation (8.7) then means that this connection has zero curvature (hence the term “zero curvature equation”). We would like to use the central extension $\hat{\mathfrak{g}}$ as a building block to construct integrable equation; there are already two reassuring points:
1. The description of coadjoint orbits in $\hat{\mathfrak{g}}^*$ automatically leads to the auxiliary linear problem (8.6).

2. Equations of motion associated with $\hat{\mathfrak{g}}$ are zero curvature equations, as desired.

However, there are also two major drawbacks:

1. There is only a finite number of independent Casimirs (one can take e.g. the coefficients of the characteristic polynomial $\det (T_L - \lambda)$).

2. The Casimirs are highly nonlocal functionals of the potential.

By contrast, in order to get integrable PDE’s we need an infinite number of conservation laws; these conservation laws are usually expressed as integrals of local densities which are polynomial in the matrix coefficients of the potential $L$ and its derivatives in $x$. The way to resolve these difficulties is suggested by the auxiliary linear equation (8.6): in order to characterize the potential, we need to know the monodromy for all energies; in other words, we must introduce into (8.6) spectral parameter. Algebraically, this means that we have to modify the choice of our basic Lie algebra.

8.2. Double Loop Algebras

Let us put $\mathfrak{g} = C^\infty (S^1; g)$ as before; let $\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C} ((z))$ be the algebra of formal Laurent series with values in $\mathfrak{g}$. (In other words we can say that $\mathfrak{g}$ is the double loop algebra of $g$; the different roles of the variables $x, z$ are imposed by our choice of its central extension.) We equip $\mathfrak{g}$ with the inner product

$$\langle X, Y \rangle = \text{Res}_{z=0} \int \text{tr} X(x, z) Y(x, z) \, dx = \frac{1}{2\pi i} \int \text{tr} X(x, z) Y(x, z) \, dx \, dz.$$  

(8.8)

The 2-cocycle $\omega$ on $\mathfrak{g}$ is defined by

$$\omega(X, Y) = \left\langle X, \frac{dY}{dx} \right\rangle.$$  

(8.9)

Let $\hat{\mathfrak{g}}$ be the central extension of $\mathfrak{g}$ defined by this cocycle. As before, we may identify the dual of $\hat{\mathfrak{g}}$ with $\mathfrak{g} \oplus \mathbb{C}$. Formula (8.4) for the coadjoint representation remains valid. We conclude that the coadjoint representation for $\hat{\mathfrak{g}}$ coincides with (infinitesimal) gauge transformation associated with the auxiliary linear problem.

---

9In the case of the loop algebra the definition of the 2-cocycle $\omega$ is essentially unique; for the double loop algebra there is a possibility to introduce into (8.9) a weight factor $\phi(z)$ which does not depend on $x$; this weight factor will modify the auxiliary linear problem. This freedom is useful in applications; our choice in (8.9) is the simplest one possible.
where this time $L \in g$, i.e., it is a formal series in $z$ with coefficients in $G$. In other words, $z$ plays the role of spectral parameter in the auxiliary linear problem, as desired. There are some troubles with the definition of the associated Lie group, but let us ignore them for the moment. Notice that if $L$ is a polynomial in $z, z^{-1}$, the monodromy matrix $T_L$ is a well-defined analytic function of $z$ (with values in $G = GL(n)$) which is holomorphic in $\mathbb{C}\setminus \{0\}$.

Our choice of the basic Lie algebra makes it easy to define the other key element of our scheme, the classical $r$-matrix. Set $g_+ = G \otimes \mathbb{C}[z], \, g_- = G \otimes \mathbb{C}[z^{-1}]$. Clearly, $g = g_+ \oplus g_-$ as a linear space. Both subalgebras $g_+$ and $g_-$ are isotropic with respect to the inner product (8.8) which sets them into duality. As before, we put

$$r = P_+ - P_-,$$

where $P_+, P_-$ are the projection operators onto $g_+$ and $g_-$, respectively, and define the $r$-bracket on $g$ by $[X, Y] = \frac{1}{2}([rX, Y] + [X, rY])$. In this way we get the algebra $g_r$, but this is still not quite what we need to apply our main theorem, since our basic algebra is $\hat{g}$, not $g$! As it happens, the theorem survives central extension.

**Lemma 8.8.** Let $\omega$ be a 2-cocycle on $g$ and $r \in \text{End} g$ a linear operator which satisfies the modified Yang-Baxter identity. Set

$$\omega_r (X, Y) = \frac{1}{2} \left( \omega (rX, Y) + \omega (X, rY) \right).$$

Then $\omega_r$ is a 2-cocycle on $g_r$.

**Exercise 8.9.** Prove lemma 8.8 (the proof is abstract and uses only manipulation with the Jacobi and the Yang-Baxter identities).

Let $\hat{g}_r$ be the central extension of $g_r$ associated with $\omega_r$. It is easy to see that $(\hat{g}, \hat{g}_r)$ is a double Lie algebra and we may apply our main idea: *Casimirs of $\hat{g}$ regarded as Hamiltonians with respect to the Lie-Poisson bracket of $\hat{g}_r$ give rise to generalized Lax equations.* Actually, there is one more simplification, which is due to our choice of $\omega$ (see (8.9)):

**Exercise 8.10.** Suppose that $r \in \text{End} g$ is skew symmetric with respect to the inner product on $g$; then $\omega_r = 0$.

The $r$-matrix (8.10) clearly satisfies this condition; hence the algebra $\hat{g}_r = g_r \oplus \mathbb{C}$ splits and the Lie-Poisson brackets for $\hat{g}_r$ and $g_r$ coincide. Explicitly, this means that the Poisson bracket of two smooth functionals $\varphi_1, \varphi_2$ defined on $g_r^* \simeq g$ is given by

$$\{ \varphi_1, \varphi_2 \}_r [L] = \frac{1}{2\pi i} \int \text{tr} \left( [\text{grad} \varphi_1, \text{grad} \varphi_1], L(x, z) \right) dx dz,$$

where $\text{grad} \varphi_i [L] (x, z) \in g, \, i = 1, 2$, is the Frechet derivative.
Remark 8.11. The skew symmetry of $r$ makes the above discussion of the cocycle $\omega$ void; however, as already noticed (see footnote 9), we may modify the cocycle $\omega$ by a weight factor $\phi(z)$, and in that case our Poisson bracket will contain derivatives $\partial_x$ of the gradients.\(^{10}\)

The antisymmetry of $r$ makes the description of coadjoint orbits very simple. In the absence of cocycle we must deal with the orbits of $\mathfrak{g}_r$; note that the $r$-matrix (8.10) is acting only on the variable $z$ in the double loop algebra, and hence the other variable $x$ becomes a parameter. Let us consider the ‘little’ loop algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}(\{z\})$ and the associated algebra $L\mathfrak{g}_r$, which we have already discussed in Section 6. The ‘big’ algebra $\mathfrak{g}_r$ consists of smooth periodic functions on the line with values in $L\mathfrak{g}_r$ and with pointwise commutator.

Proposition 8.12. Fix an orbit $\mathcal{O} \subset (L\mathfrak{g}_r)^*$; then $\mathcal{O} = \text{Map}(S^1, \mathcal{O})$ is an orbit of $\mathfrak{g}_r$.

More generally, we may vary orbits $\mathcal{O}$ lying over different points of $S^1$ (i.e., consider smooth families of orbits in $(L\mathfrak{g}_r)^*$ parametrized by $S^1$).

Example 8.13. Let $\mathfrak{g} = \mathfrak{sl}(2)$; then the matrices $s \in \mathfrak{g}$,

\[
(8.12) \quad s = \begin{pmatrix} s_3 & s_1 + is_2 \\ s_1 - is_2 & -s_3 \end{pmatrix}, \quad s_j \in \mathbb{C}, \quad s_1^2 + s_2^2 + s_3^2 = 1,
\]

form a coadjoint orbit $S_1 \subset \mathfrak{sl}(2)$. Check that $\mathcal{O}_H = \{z^{-1} s, s \in S_1\} \subset L\mathfrak{g} \simeq L\mathfrak{g}_r^*$ is a coadjoint orbit of $L\mathfrak{g}_r \subset L\mathfrak{g}_r$. The corresponding orbit $\mathcal{O}_H \subset \mathfrak{g}$ is parametrized by a triple of $2\pi$-periodic functions $s_j, j = 1,2,3$, satisfying the constraint (8.12). The associated linear differential operator is

\[
(8.13) \quad \frac{d}{dx} - \frac{1}{z}s(x).
\]

One can show that the simplest local Hamiltonian associated with (8.13) is

\[
H = -\frac{1}{2} \text{tr} \int s_x^2 dx;
\]

the corresponding nonlinear equation describes the Heisenberg ferromagnet:

\[
s_t = [s, s_{xx}].
\]

Example 8.14. Set

\[
\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

\(^{10}\)Poisson bracket of functionals which does not contain derivatives $\partial_x$ of the gradients is sometimes called ultralocal; in more complicated cases, Poisson brackets may contain derivatives (non-ultralocal case) or even be non-local, i.e., contain integral operators.
The matrices
\[
U = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} + z\sigma, u, v \in \mathbb{C},
\]
form a coadjoint orbit of the Lie algebra \( Lg_- \subset Lg_r \). The corresponding orbit \( O_S \subset g \) is parametrized by a pair of functions \( u, v \in C^\infty(S^1, \mathbb{C}) \); the associated linear operator is
\[
\frac{d}{dx} - U;
\]
this is (essentially) the Lax operator for the so called split nonlinear Schrödinger equation\(^\text{11}\)
\[
i\partial_t u = -u_{xx} + u^2v, \quad -i\partial_t v = -v_{xx} + v^2u.
\]

In a more general way, for \( g = gl(n) \) we can easily construct coadjoint orbits associated with linear differential operators of the form
\[
\frac{d}{dx} - U(x, z),
\]
where the potential \( U \) is a Laurent polynomial in \( z \).

### 8.3. Monodromy Map

Examples 8.13, 8.14 show that our approach is indeed reasonable: starting with double loop algebras, we arrived at the natural auxiliary linear problems with spectral parameter of the form which is familiar in spectral theory. The next question is to construct an appropriate class of Hamiltonians. Formally, the Hamiltonians are spectral invariants of the auxiliary linear operator, i.e., the spectral invariants of its monodromy matrix. The monodromy matrix \( M_U \) of (8.14) is a holomorphic function in \( \mathbb{C}P^1 \setminus \{\{0\} \cup \{\infty\} \} \); any functional of the monodromy which is invariant under conjugation gives rise to a zero curvature equation on the coadjoint orbits of \( g_r \). The mapping \( M : U \sim M_U \) is the direct spectral transform for (8.14). We may regard \( M \) as a mapping from \( g \) into the group of matrix-valued functions which are regular in the punctured Riemann sphere. When the Poisson structure is ultralocal (i.e., the \( r \)-matrix is skew with respect to the inner product in \( g \)), the spectral transform \( M \) has an important property: the target space carries a natural Poisson bracket and \( M \) is a Poisson mapping (i.e., it preserves Poisson brackets). We shall briefly outline this computation, since it plays an important role in the theory and brings into play an important class of Poisson brackets (the so called Sklyanin brackets). To simplify the notation, let us start with ordinary loop algebras.

\(^{\text{11}}\)For simplicity, in both examples we deal with complex Lie algebras; in order to pass to a real form we must choose an \textit{anti-involution} of our basic loop algebra; this is of course a necessary step in order to make the auxiliary linear operators genuine skew selfadjoint operators.
Let $g = gl(n)$, and $r \in \text{End} g$ a linear operator which satisfies the modified Yang-Baxter identity (4.4) and is skew symmetric with respect to the inner product on $g$. Let $\mathcal{G} = C^\infty(S^1, g)$; we equip $\mathcal{G}$ with the inner product (8.1); the Poisson bracket of functionals on $\mathcal{G}^* \simeq \mathcal{G}$ is given by

\[(8.15) \quad \{\varphi_1, \varphi_2\}_r [L] = \int_0^{2\pi} \text{tr} (\text{grad} \varphi_1 [L] (x), \text{grad} \varphi_2 [L] (x), L (x)) \, dx.\]

For $L \in \mathcal{G}$ let $\psi_L$ be the fundamental solution of (8.6) normalized by $\psi_L(0) = id$ and $M_L \in GL(n)$ the monodromy matrix. Fix a smooth function $\varphi \in C^\infty(GL(n))$ and put $h_\varphi [L] = \varphi (M_L)$. Let $\nabla \varphi, \nabla' \varphi \in g$ be the left and right gradients of $\varphi$ on $G = GL(n)$ defined by

\[\frac{d}{ds}_{s=0} \varphi(e^{sX}x) = \text{tr} (X \cdot \nabla \varphi (x)), \quad \frac{d}{ds}_{s=0} \varphi(xe^{sX}) = \text{tr} (X \cdot \nabla' \varphi (x)), \quad X \in g.\]

**Lemma 8.15.** The Frechet derivative of the functional $h_\varphi$ is given by

\[(8.16) \quad \text{grad} h_\varphi [L] (x) = \psi (x) \nabla' \varphi (M_L) \psi (x)^{-1}.\]

**Corollary 8.16.** The Frechet derivative satisfies the differential equation

\[(8.17) \quad \frac{dX}{dx} = [L, X]\]

and the boundary conditions

\[(8.18) \quad X (0) = \nabla' \varphi (M_L), \quad X (2\pi) = \nabla \varphi (M_L).\]

**Sketch of a proof.** Taking variation of the both sides of (8.6), we get

\[(8.19) \quad \partial_x \delta \psi_L = L \delta \psi_L + (\delta L) \psi_L.\]

Set $\delta \psi_L = \psi_L Y$, where $Y$ is an unknown function, $Y \in C^\infty(\mathbb{R}, g)$, $Y (0) = 0$; (8.19) yields $\partial_x Y = \psi^{-1} \delta L \psi$, whence

\[Y (x) = \int_0^x \psi^{-1}_L (y) \delta L (y) \psi_L (y) \, dy.\]

Since $M_L = \psi_L (2\pi)$, we get

\[M_L^{-1} \delta M_L = \int_0^{2\pi} \psi^{-1}_L \cdot \delta L \cdot \psi_L \, dy.\]

Now,

\[\delta \varphi (M_L) = \text{tr} (\nabla' \varphi (M_L) \cdot M_L^{-1} \delta M_L) = \int_0^{2\pi} \text{tr} \psi_L (y) \nabla' \varphi (M_L) \psi^{-1}_L (y) \cdot \delta L (y) \, dy,\]
which implies (8.16). Taking derivatives of the both sides of (8.16) yields (8.17).

**Proposition 8.17.** The Poisson bracket of two functionals $h_{\varphi_1}, h_{\varphi_2}$ on $G^*$ is given by

$$\{ h_{\varphi_1}, h_{\varphi_2} \} [L] = h_{\{ \varphi_1, \varphi_2 \}} [L],$$

where the Poisson bracket of $\varphi_1, \varphi_2 \in C^\infty(G)$ is defined by

$$\{ \varphi_1, \varphi_2 \}_G = \frac{1}{2} \text{tr} \left( r(\nabla \varphi_1) \nabla \varphi_2 - r(\nabla' \varphi_1) \nabla' \varphi_2 \right).$$

**Corollary 8.18.** Let us equip the group $G = GL(n)$ with the Poisson bracket (8.21). Then the monodromy map $M : G^* \to G : L \mapsto M_L$ preserves Poisson brackets.

**Proof.** Set $X_i = \text{grad} h_{\varphi_i}, i = 1, 2$. We have

$$\begin{align*}
\{ h_{\varphi_1}, h_{\varphi_2} \} [L] &= \int_0^{2\pi} \text{tr} \left( [X_1, X_2] r, L \right) dx \\
&= \frac{1}{2} \int_0^{2\pi} \text{tr} \left( r X_1 X_2 + [X_1, r X_2] \right) L dx \\
&= \frac{1}{2} \int_0^{2\pi} \text{tr} \left( [L, X_2] r X_1 + [L, X_1] r X_2 \right) dx \\
&= \frac{1}{2} \int_0^{2\pi} \frac{d}{dx} \text{tr} \left( r X_1 \cdot X_2 \right) dx,
\end{align*}$$

where we used the definition of the $r$-bracket, the cyclic invariance of trace, the differential equation (8.17) satisfied by $X_i$ and, finally, the skew symmetry of $r$. Evaluating the last integral and taking into account the boundary conditions (8.18) for $X_i$, we get (8.20). \hfill \Box

Formula (8.21) defines a remarkable Poisson bracket (the Sklyanin bracket) on the group manifold itself. The Jacobi identity for this bracket is not obvious (though it follows from our computation). Its properties will be discussed in some detail in Section 9.

**Exercise 8.19.** Show that central functions on $G$ (i.e., functions which satisfy $\varphi(xy) = \varphi(yx)$ for all $x, y \in G$) commute with respect to the Sklyanin bracket.

The above discussion applies to loop algebra $\mathfrak{G} = C^\infty(S^1, \mathfrak{g})$; the generalization to the double loop algebra is straightforward: in our computation, we must replace the finite dimensional algebra $\mathfrak{g}$ with its loop algebra $L\mathfrak{g}$; accordingly, smooth functions $\varphi_1, \varphi_2 \in C^\infty(G)$ are replaced by smooth functionals on the corresponding loop group, their left and right gradients are replaced by the left and right variational derivatives, etc. Spectral invariants of the auxiliary linear problem (8.14) correspond to central functionals on the loop group. An example
of such a functional is given by evaluation functionals $H_{n,w}[U] = \text{tr} M_U^n(w), w \in \mathbb{CP}_1 \setminus \{0 \cup \{\infty}\}$.

Exercise 8.20. Compute the variational derivative of $H_{n,w}$ with respect to $U$.

The fundamental drawback of these functionals is, however, their nonlocality. The description of local functionals is outlined in the next paragraph.

8.4. Formal Diagonalization and Local Conservation Laws

In contrast to evaluation functionals, local conservation laws are related to the asymptotic expansions of the monodromy matrix at its essential singularities, i.e., for $z = 0, \infty$. This implies some additional difficulties:

1. These functionals are not defined everywhere on the double loop algebra.
2. The associated formal series are in most cases divergent.

Let us assume that the potential $U(x,z)$ in the auxiliary linear problem (8.14) is a Laurent polynomial,

$$U = \sum_{-N}^{M} U_k z^k, U_k \in C^\infty(S^1, \mathfrak{gl}(n));$$

let $J_0 = U_{-N}, J_\infty = U_M$ be its lowest and highest coefficients.

Definition 8.21. $U$ is called regular if

(i) the matrices $J_0(x), J_\infty(x)$ are semisimple,

(ii) the centralizers of $J_0(x), J_\infty(x)$ in $\mathfrak{g} = \mathfrak{gl}(n)$ are conjugate for all $x \in S^1$.

We have seen that the set of Laurent polynomials of fixed degree is a Poisson subspace for the r-bracket. It is easy to check that the regularity condition holds for entire coadjoint orbits in this subspace, and hence it is a characteristic of our phase space. Our next theorem allows to construct for regular Lax operators two series of local Hamiltonians which are associated with the poles of $U$ on $\mathbb{CP}_1$. For concreteness, we shall describe the construction of the series associated with the pole at infinity. Performing a suitable gauge transformation we may assume that the leading coefficient at infinity $J_\infty(x)$ satisfies a stronger condition:

(iii') the centralizer of $J_\infty$ in $\mathfrak{g} = \mathfrak{gl}(n)$ is a fixed subalgebra $\mathfrak{g}^{J_\infty} \subset \mathfrak{g}$ which does not depend on $x$.

(Note that since, by construction, local Hamiltonians are gauge invariant, this stronger condition does not restrict generality.) Let $\mathfrak{g}_{J_\infty} \subset \mathfrak{g}^{J_\infty}$ be the commutant of $\mathfrak{g}^{J_\infty}$,

$$\mathfrak{g}_{J_\infty} = \{ X \in \mathfrak{g}; [X,Y] = 0 \text{ for all } Y \in \mathfrak{g}^{J_\infty} \}.$$
**Theorem 8.22. (On normal form at infinity)** There exists a formal gauge transformation

\[ \Phi^\infty = \text{Id} + \sum_{m=1}^\infty \Phi_m z^{-m}, \Phi_m \in C^\infty(S^1, \text{Mat}(n)), \]

which transforms the differential operator \( \partial_x - U \) into normal form,

\[ (\Phi^\infty)^{-1} \circ \left( \frac{d}{dx} - U \right) \circ \Phi^\infty = \frac{d}{dx} - D^\infty, \]  

where

\[ D^\infty = \sum_{m=-M}^\infty D^\infty_m z^{-m}, \quad D^\infty_m \in C^\infty(S^1, \mathfrak{g}^{J\infty}), \quad D^\infty_{-M} = J_{\infty}; \]

matrix coefficients of \( \Phi^\infty_m, D^\infty_m \) are expressed as polynomials of the coefficients of \( U \) and its derivatives in \( x \).

**Sketch of a proof.** The intertwining relation (8.22) is equivalent to the differential equation

\[ \left( \frac{d}{dx} - U \right) \Phi^\infty = -\Phi^\infty D^\infty; \]  

which may be solved recurrently in powers of the local parameter \( z^{-1} \). The first nontrivial coefficients \( \Phi_1, D^\infty_{-M+1} \) satisfy

\[ J_{\infty} \Phi_1^\infty - \Phi_1^\infty J_{\infty} = D^\infty_{-M+1} - U_{M-1}. \]  

This equation for \( \Phi_1 \) admits a solution if and only if the r.h.s. is in the image of \( \text{ad} J_{\infty} \in \text{End} \mathfrak{g} \). Assumption (i) above implies that

\[ \mathfrak{g} = \text{Im} \text{ad} J_{\infty} \mathfrak{g} \mathfrak{g} + \ker \text{ad} J_{\infty}, \]

moreover, by (ii') \( \text{Im} \text{ad} J_{\infty} \) and \( \ker \text{ad} J_{\infty} = \mathfrak{g}^{J\infty} \) do not depend on \( x \). Hence, \( D^\infty_{-M+1} \in \mathfrak{g}^{J\infty} \) is uniquely determined from the solvability condition of (8.24) and

\[ \Phi_1^\infty = \left( \text{ad} J_{\infty} \right)^{-1} \left( D^\infty_{-M+1} - U_{M-1} \right). \]

If the coefficients \( \Phi^\infty_1, ..., \Phi^\infty_m, D^\infty_{-M+1}, ..., D^\infty_{-M+m} \) are already determined, we get for \( \Phi^\infty_{m+1} \) the relation of the form

\[ \text{ad} J_{\infty} \cdot \Phi^\infty_{m+1} = -F_m \left( U, \Phi^\infty_1, ..., \Phi^\infty_m, D^\infty_{-M+1}, ..., D^\infty_{-M+m} \right), \]  

where \( F_m \) depends on \( U \) and on the already determined coefficients and their derivatives. By the same argument, (8.25) allows to determine \( D^\infty_{-M+m+1}, \Phi^\infty_{m+1} \).

---

12In various applications \( J_0, J_\infty \) are regular matrices with distinct eigenvalues. In that case \( \mathfrak{g}^{J_0} = \mathfrak{g}_{J_0} \) and \( \mathfrak{g}^{J\infty} = \mathfrak{g}_{J_{\infty}} \) are abelian subalgebras; hence theorem 8.22 means that the potential \( U \) may be transformed to diagonal form by a formal gauge transformation. When the eigenvalues of \( J_0, J_{\infty} \) have multiplicities, the potential may be transformed only to block diagonal form.
Remark 8.23. The coefficients $\Phi^\infty_m, D^\infty_m$ are determined from (8.25) not completely canonically, since we must fix in some way the operator $(ad J^\infty)^{-1}$. One can show that this freedom corresponds to the possibility to perform gauge transformations

\begin{equation}
\frac{d}{dx} - D^\infty \rightsquigarrow \exp(-\phi) \circ \left( \frac{d}{dx} - D^\infty \right) \circ \exp \phi, \tag{8.26}
\end{equation}

$$\phi = Id + \sum_{m=1}^{\infty} \phi_m z^{-m}, \phi_m \in g^J.$$

The formal series $\Phi^\infty$ is sometimes called the formal Baker function at infinity of the operator $L = \partial_x - U$. For $\alpha \in g^J \otimes \mathbb{C}[z, z^{-1}]$ and put

\begin{equation}
H^\infty_\alpha[U] = \text{Res}_{z=0} \int_0^{2\pi} \text{tr} \alpha(z) D^\infty(x, z) \, dx. \tag{8.27}
\end{equation}

Theorem 8.24.

(i) Functionals $H^\infty_\alpha$ do not depend on the freedom in the definition of the normal form.

(ii) All functionals $H^\infty_\alpha$ are in involution with respect to the Poisson bracket (8.15) on $g^*_r$.

(iii) Hamiltonian equation of motion defined by $H^\infty_\alpha$ on $g^*_r$ have the form of zero curvature equations.

Lemma 8.25. Gauge transformations (8.26) leave the density $\text{tr} \alpha(z) D^\infty(x, z)$ invariant up to a total derivative.

Sketch of a proof. Gauge transformations (8.26) map $D^\infty$ into $e^{-\phi} D^\infty e^\phi - e^{-\phi} \partial_x e^\phi$. By a standard formula,

$$\partial_x (\exp \phi) = \frac{e^{-ad\phi} - Id}{-ad\phi} \cdot \partial_x \phi = \left( Id - \frac{1}{2} ad\phi + \frac{1}{3!} (ad\phi)^2 + ... \right) \cdot \partial_x \phi.$$

Hence

$$\text{tr} \alpha \exp(-\phi) \partial_x (\exp \phi) = \text{tr} \alpha \cdot \left( \frac{e^{-ad\phi} - Id}{-ad\phi} \cdot \partial_x \phi \right) =$$

$$\text{tr} \left( \frac{e^{-ad\phi} - Id}{-ad\phi} \cdot \alpha \right) \cdot \partial_x \phi = \text{tr} \alpha \cdot \partial_x \phi = \partial_x (\text{tr} \alpha \phi).$$

where we also used the invariance of trace, the condition $\alpha \in g^J$, which assures that it commutes with $\phi$ and, finally, the condition $\partial_x \alpha = 0$. 

Lemma 8.26. The Frechet derivative of $H_\alpha^\infty$ is given by

\begin{equation}
\text{grad } H_\alpha^\infty = \Phi^\infty \alpha (\Phi^\infty)^{-1},
\end{equation}

where $\Phi^\infty$ is the formal Baker function.

Sketch of a proof. Taking variations of both sides of (8.23), we get

\[ \delta D^\infty = (\Phi^\infty)^{-1} \delta U \Phi^\infty + \left[ D^\infty, (\Phi^\infty)^{-1} \delta \Phi^\infty \right] - \partial_x \left( (\Phi^\infty)^{-1} \delta \Phi^\infty \right). \]

Hence

\[ \delta H_\alpha^\infty = \text{Res}_{z=0} \int_0^{2\pi} \left\{ \text{tr } \Phi^\infty \alpha (\Phi^\infty)^{-1} \delta U + \text{tr} \left( \partial_x \alpha - [D^\infty, \alpha] \right) (\Phi^\infty)^{-1} \delta \Phi^\infty \right\} dx, \]

where we used the invariance of trace and integrated by parts; the contribution of the second term vanishes, since $\partial_x \alpha = [D^\infty, \alpha] = 0$. \(\square\)

Corollary 8.27. The Frechet derivative $X = \text{grad } H_\alpha^\infty$ satisfies the differential equation

\begin{equation}
\partial_x X = [U, X].
\end{equation}

Indeed, (8.28) and (8.23) imply

\[ \partial_x X = (U \Phi^\infty - \Phi^\infty D^\infty) \alpha (\Phi^\infty)^{-1} - \Phi^\infty \alpha (\Phi^\infty)^{-1} (U \Phi^\infty - \Phi^\infty D^\infty) (\Phi^\infty)^{-1} \]

\[ = [U, X] - \Phi^\infty [D^\infty, \alpha] (\Phi^\infty)^{-1} = [U, X]. \]

Note that geometrically (8.29) is equivalent to

\begin{equation}
\text{ad}^*_\mathfrak{g} \text{grad } H^\infty_\alpha \cdot U = 0,
\end{equation}

where $\text{ad}^*_\mathfrak{g}$ is the coadjoint representation of the Lie algebra $\hat{\mathfrak{g}}$ (the central extension of $\mathfrak{g}$); this is precisely the property which characterizes the Casimirs of a Lie algebra (cf. proposition 2.6). In the present setting $H^\infty_\alpha$ is not a true Casimir function: it is defined only for regular elements $U \in \mathfrak{g}$ with fixed highest coefficient. However, a short inspection of the proof of theorem 3.1 shows that it uses only (8.30); the last assertion of theorem 8.24 now follows.

In a similar way, we may define the second series of local Hamiltonians which is associated with the pole at $z = 0$; one can show that the Hamiltonians from these two families mutually commute (this does not follow immediately from the arguments above, but may be proved in a similar way). In a more general way, if the potential $U$ is a rational function on $\mathbb{C}P_1$ (cf. Section 7.2), we may associate a series of local Hamiltonians to each of its poles; the corresponding Frechet derivatives are formal Laurent series in local parameter at the pole.
8.5. Higher order differential operators

In applications, it is quite common to deal with Lax representations which contain higher order differential operators; the most famous example is the KdV equation associated with the Schroedinger operator on the line

\[ D_2 = -\frac{d^2}{dx^2} + u(x). \]

In order to put these operators into our framework, one needs extra work. We shall outline the procedure without going into details. First of all, an \( n \)-th order differential equation

\[ D_n = \frac{d^n}{dx^n} \psi + u_{n-2} \frac{d^{n-2}}{dx^{n-2}} \psi + \ldots + u_0 \psi + z \psi = 0 \quad (8.31) \]

may be written as a first order matrix equation,

\[ \frac{d}{dx} \varphi + L \varphi = 0, \quad (8.32) \]

where

\[ L = \begin{pmatrix}
0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 1 \\
u_0 + z & \ldots & u_{n-2} & 0
\end{pmatrix}, \quad \varphi = \begin{pmatrix}
\psi \\
\psi' \\
\vdots \\
\psi^{(n-1)}
\end{pmatrix}. \quad (8.33) \]

However, the companion matrix in (8.33) contains “too much zeros” and cannot be directly associated with a coadjoint orbit of the loop algebra. Let us observe first of all that choosing the column vector \( \varphi \) in (8.33) in this particular form is not quite canonical: we may add to \( \psi^{(k)}, k = 1, 2, \ldots, n-1 \), an arbitrary linear combination (possibly, with variable coefficients) of \( \psi, \psi', \ldots, \psi^{(k-1)} \); this freedom amounts to a gauge transformation \( \varphi (x) \mapsto n(x) \cdot \varphi (x) \), where \( n \) is a lower triangular (unipotent) matrix. The potential \( L \) in (8.33) becomes an arbitrary matrix of the form

\[ L = \begin{pmatrix}
* & 1 & 0 & \ldots & 0 \\
* & * & 1 & \ddots & \vdots \\
\vdots & \ddots & * & \ddots & 0 \\
& \ddots & \ddots & 1 \\
* & \ldots & * & * & *
\end{pmatrix} \quad (8.34) \]

The companion matrix in (8.33) is the result of gauge fixing; indeed, we have

**Proposition 8.28.** For each potential \( L \) of the form (8.34) there exists a unique lower triangular gauge transformation which transforms it into the canonical form (8.33).
Conclusion: The space of $n$-th order differential operators is the quotient space of the set of all potentials $L$ of the form (8.34) modulo the gauge action of the lower triangular group. This quotient space is modelled on the set of companion matrices. With a little skill in symplectic geometry one may describe this quotient space in terms of Hamiltonian reduction (the key point is to observe that potentials of the form (8.34) form a level surface for the moment map associated with our gauge action).

There is one more difficulty: the term of highest degree in $z$ in the potential $L(z)$ is a nilpotent matrix, and so the expansion procedure which yields local integrals of motion does not work. To tackle with this trouble let us recall that the loop parameter $z$ is in fact associated with a special grading (the standard grading) of the loop algebra; in this grading, the constant matrix in (8.32) is

$$
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & 1 \\
z & \cdots & 0 & 0
\end{pmatrix}
$$

(8.35)

If we use the so called principal grading of the loop algebra instead, (8.35) is replaced, after rescaling, with

$$
\zeta \cdot 
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & 1 \\
1 & \cdots & 0 & 0
\end{pmatrix}
$$

which is already a semisimple matrix with different eigenvalues; now local conservation laws may constructed by expansion in $\zeta^{-1}$ in the usual way.

### 8.6. Dressing and Solutions of Zero Curvature Equations

Formula (8.28) for the Frechet derivative of a local Hamiltonian makes it clear that the direct use of the global factorization theorem (theorem 3.3) to solve zero curvature equations is impossible. Indeed, $\text{grad} H_{\alpha}^{\infty} [U]$ is a formal series in local parameter $z^{-1}$; it is therefore impossible even to define the 1-parameter subgroup $\exp t \text{grad} H_{\alpha}^{\infty} [U]$; this reflects real analytic difficulties which exist in the study of the initial value problem with arbitrary initial data for integrable PDE’s. Our way around this difficulty is to introduce the following definition.
Definition 8.29. A differential operator $\partial_x - U$ is called **strongly regular at zero** (at infinity) if $U$ satisfies the conditions imposed in definition 8.21 and, moreover, its formal Baker function at zero (at infinity) is convergent.

One may of course strongly doubt the merits of this definition. The point is, however, that strongly regular operators form a homogeneous space: there is a natural action of the loop group (called dressing transformations) on the set of potential which preserves strong regularity. Moreover, commuting Hamiltonian flows generated by local Hamiltonians are naturally included into the dressing group as a maximal commutative subgroup. Examples of strongly regular operators include solitonic and finite-band solutions; on the other hand, it is very difficult to give their complete characteristics in local terms (i.e., to tell which initial data in the space $C^\infty(S^1, G)$ correspond to strongly regular operators). The group action in question was discovered by Zakharov and Shabat [16] (though at the time they did not notice the composition law), and later by Sato and his school [17]. A subtle question in the theory of dressing transformations is the treatment of boundary conditions. In our discussion above we used periodic boundary conditions; the main motivation was the Floquet theorem which gives an accurate description of coadjoint orbits and Casimir functions for the central extension of the loop algebra. Dressing transformations do not preserve periodicity.13 One more interesting point is the relation between dressing and the Poisson brackets: since the dressing action is defined in terms of the Riemann problem and, on the other hand, the Poisson structure is derived from the classical r-matrix related to the same Riemann problem, one anticipates some relation between the two matters. However, the simplest guess appears to be wrong: dressing transformations do not preserve Poisson brackets on the phase space of integrable PDE’s. The exact relation is more subtle: the loop group itself carries a natural Poisson bracket, again derived from the classical r-matrix, and dressing is an example of a Poisson group action [18]. The same Poisson structure on the loop group plays an important role in the study of difference Lax equations, and also yields a semiclassical approximation in the theory of Quantum groups.

In this paragraph we shall discuss the simplest facts about dressing transformations. We start with the motivation of the main definitions.

Let $G$ be the double loop group consisting of maps

$$g : \mathbb{R} \times \mathbb{C} \mathbb{P}^1 \setminus \{0 \cup \{\infty\}\} \rightarrow GL(n)$$

which are holomorphic with respect to the 2nd argument. The Lie algebra $\mathfrak{g}$ of $G$ consists of maps

$$U : \mathbb{R} \times \mathbb{C} \mathbb{P}^1 \setminus \{0 \cup \{\infty\}\} \rightarrow gl(n)$$

which are holomorphic with respect to the 2nd argument. Informally, we may call

13There is a different version of dressing which uses the Riemann problem in the half-plane; under some additional restrictions, this version is adapted to the study of rapidly decreasing solutions.
elements of \( G \) wave functions. Define the mapping

\[
p : G \to g : \psi \mapsto U_\psi = \partial_x \psi \cdot \psi^{-1};
\]

\( G \) acts on itself by left multiplications; we have

\[
g \cdot U_\psi \overset{\text{def}}{=} U_{g \psi} = g U \psi g^{-1} + \partial_x \psi \cdot g^{-1},
\]

in other words, left multiplication on \( G \) induces gauge transformations on the set of “potentials” \( U \). Conversely, if \( U \in g \), let \( \psi_U \) be the fundamental solution of the differential equation on the line

\[
\frac{d\psi}{dx} = U(x, z) \psi,
\]

normalized by \( \psi_U(0) = \text{Id} \). The mapping

\[
\psi : g \to G : U \mapsto \psi_U
\]

is a right inverse of \( p \). Let \( G \subset G \) be the subgroup consisting of maps which do not depend on \( x \in \mathbb{R} \).

**Lemma 8.30.** (i) \( G \subset G \) is the stationary subgroup of the zero potential on the line. (ii) \( G^U = \psi_U g \psi_U^{-1} \subset G \) is the stationary subgroup of \( U \in g \).

Let \( G_+ \subset G \) be the subgroup consisting of functions which are regular in \( \mathbb{C}P_1 \setminus \{\infty\} \) with respect to the second argument, and \( G_- \subset G \) the subgroup of functions which are regular in \( \mathbb{C}P_1 \setminus \{0\} \) and satisfy the normalization condition \( g_-(\infty) = \text{Id} \). The factorization problem in \( G \) consists in representing \( g \in G \) as \( g = g_+ g_- \), \( g_\pm \in G_\pm \); the first argument of \( g(x, z) \) is regarded as a parameter.

**Theorem 8.31.** Formula

\[
(8.38) \quad dr_{(x,y)} \psi = \left( \psi xy^{-1} \psi^{-1} \right)_+^{-1} \psi x = \left( \psi xy^{-1} \psi^{-1} \right)_-^{-1} \psi y
\]

defines a right group action \( dr : (G \times G) \times G \to G \).

The definition looks rather exotic; in particular, the composition law for dressing transformations is not at all obvious. Note that the equality in (8.38) is closely related to the fact that \( \psi xy^{-1} \psi^{-1} \in G^U \) and hence the two factors \( \left( \psi xy^{-1} \psi^{-1} \right)_\pm \) define the same gauge transformation of the potential \( U_\psi \). To check the composition law we shall give a geometric interpretation of (8.38). To avoid lengthy notation we shall use a model example.

Let \( K \) be a group admitting a factorization into product of its subgroups \( K_\pm \); set \( \mathcal{D}(K) = K \times K \). Let \( K^\delta \subset K \times K \) be the diagonal subgroup, \( K^\delta = \{(x, x) : x \in K\} \), and \( K_r = K_+ \times K_- \).
Lemma 8.32. $D(K) = K_r \cdot K^\delta$; in other words, the factorization problem in
$D(K)$,
\[(x, y) = (\eta_+, \eta_-) \cdot (\xi, \xi), \eta_\pm \in K_\pm, \xi \in K,\]
is uniquely solvable.\footnote{14}

Indeed, we have
\[
\eta_\pm = (xy^{-1})_\pm, \xi = (xy^{-1})_+ x = (xy^{-1})_- y.
\]

Corollary 8.33. The quotient space $K_r \backslash D(K)$ of left coset classes mod $K_r$ is
modelled on the diagonal subgroup $K^\delta$; the projection $\pi : D(K) \to K^\delta$ is given by
\[
\pi(x, y) = (xy^{-1})_+ x = (xy^{-1})_- y.
\]

The group $D(K)$ acts on itself by right translations. Consider the commutative
\[
\begin{array}{cccc}
D(K) \times D(K) & \xrightarrow{m} & D(K) \\
\downarrow \pi \times id & & \downarrow \pi \\
K_r \backslash D(K) \times D(K) & \xrightarrow{dr} & K_r \backslash D(K) \\
\downarrow & & \downarrow \\
K \times D(K) & \xrightarrow{dr} & K
\end{array}
\]

Proposition 8.34. The right action $K \times D(K) \xrightarrow{dr} K$ induced by the identifica-
tion of $K^\delta \subset D(K)$ with the coset space $K_r \backslash D(K)$ is given by
\[(8.40) \quad dr(x, y) : k \mapsto (kxy^{-1}k^{-1})_+ kx = (kxy^{-1}k^{-1})_- ky.
\]

A comparison of $(8.38)$ and $(8.40)$ explains the mystery around the definition.
In our model setting we assumed for simplicity that the factorization problem is
globally solvable. In general this is of course not true; however, under reasonable
conditions it is solvable on an open dense subset of the big group, and hence the
diagonal subgroup may be identified with a “big cell” in the quotient space. Thus
the situation is not much different from the treatment of e.g. fractional linear
transformations on the line.

Returning back to $(8.38)$, note that the diagonal subgroup $G^\delta \subset D(G)$ acts by
$dr(g, g) : \psi \mapsto \psi g$; this action amounts to a simple change of the normalization

\footnote{14}$D(K)$ is called the double of $K$; we have already used a similar construction for Lie algebras
in Section 4. One may notice that the construction below uses only factorization in $D(K)$ and
the subgroups $K_\pm$ need not be complementary in $K$. In Section 9 we shall once again encounter
this construction in the study of Poisson Lie groups.
of the wave function and does not affect the potential \( U = \partial_x \psi \psi^{-1} \). On the other hand, the subgroup \( \mathbb{G}_r = \mathbb{G}_+ \times \mathbb{G}_- \) preserves the normalization condition \( \psi(0) = \text{Id} \). Hence we may define an action \( \mathbb{G}_r \times \mathfrak{g} \to \mathfrak{g} \) on the space of potentials with the help of commutative diagram

\[
\begin{array}{ccc}
\mathbb{G}_r \times \mathbb{G} & \xrightarrow{\text{dr}} & \mathbb{G} \\
\downarrow \text{id} \times \psi & & \downarrow \psi \\
\mathbb{G}_r \times \mathfrak{g} & \xrightarrow{\text{dr}} & \mathfrak{g}
\end{array}
\]

(8.41)

Let \( \mathfrak{g}_{M,N} \subset \mathfrak{g} \) be the subspace of Laurent polynomials with the degrees of pole at zero (at infinity) not exceeding \( M \) (resp., \( N \))

**Proposition 8.35.** *Dressing action on \( \mathfrak{g} \) preserves \( \mathfrak{g}_{M,N} \).*

*Sketch of a proof.* Compare two equivalent formulae for dressing which follow from (8.38); the first one shows that dressing does not increase the degree of pole at 0, the second one, that it does not affect infinity.

This argument explains the key idea of the dressing method: indeed, the most striking property of dressing is the fact that it preserves the structure of poles of the Lax operator. A slight refinement of the same argument shows that dressing preserves symplectic leaves of the r-bracket in \( \mathfrak{g}_{M,N} \subset \mathfrak{g} \cong \mathfrak{g}_r^* \) (here \( r \) is the standard r-matrix associated with the factorization problem in \( \mathbb{G} \)).

**Proposition 8.36.** *Dressing preserves strong regularity.*

*Sketch of a proof.* Formal Baker functions at zero and at infinity of the dressed operator are given by

\[
\Phi_{0}^{g} = \left( \psi g_{+} g_{-}^{-1} \psi^{-1} \right)^{-1}_{+} \Phi_{0}, \\
\Phi_{\infty}^{g} = \left( \psi g_{+} g_{-}^{-1} \psi^{-1} \right)_{-}^{-1} \Phi_{\infty}.
\]

Clearly the gauge factors \( \left( \psi g_{+} g_{-}^{-1} \psi^{-1} \right)^{-1} \) expand into convergent series in local parameter around zero and infinity, respectively; hence the same is true for the dressed wave functions.

In applications, dressing is usually applied to *trivial*, or free, Lax matrices. Let us assume that the leading coefficient at infinity is a diagonal matrix with distinct eigenvalues. By definition, free Lax operator has the form

\[
L_{\text{free}} = \frac{d}{dx} - D(z),
\]

where \( D(z) \) is a constant diagonal matrix (with coefficients which are polynomial in \( z \)). Our next assertion shows that the factorization theorem survives for regular potentials; moreover, the dynamical flows associated with all Lax equations
(derived from a given Lax operator) correspond to the action of an abelian subgroup of the “big” dressing group (essentially, the group of diagonal loops which are regular at infinity).

**Proposition 8.37.** Assume that \( L \) is obtained from \( L_{\text{free}} \) by dressing, \( L = L_{\text{free}}^g \).

The integral curve of the Hamiltonian equation of motion with the Hamiltonian \( H_\alpha \) defined (8.27) which starts at \( L \) is given by

\[
L(t) = g_\pm(t)^{-1} \circ L \circ g_\pm(t),
\]

where \( g_\pm(t) \) are regarded as multiplication operators on the line and \( g_+(t,x), g_-(t,x) \) are the solutions of the factorization problem

\[
g_+(t,x) g_-(t,x)^{-1} = \psi_{\text{free}}(x) \exp t \alpha (z^{-1}) \cdot g \cdot \psi_{\text{free}}(x)^{-1},
\]

\[
\psi_{\text{free}}(x) = \exp x D(z).
\]

9. Difference Equations and Poisson Lie Groups

9.1. Motivation: Zero curvature equations on the lattice

We have already mentioned that integrable systems which are associated with difference operators require a special treatment; in this case the underlying Poisson structures are nonlinear, and hence the geometric setting we considered so far, based on the use of the Lie-Poisson brackets, must be generalized. Nonlinear equations associated with a finite difference operator may be regarded as lattice analogues of zero curvature equations. They are usually written in the form

\[
\frac{dL_m}{dt} = L_m M_{m+1} - M_m L_m, \ m \in \mathbb{Z}. \tag{9.1}
\]

Equation (9.1) is the compatibility condition for the linear system

\[
\frac{d\psi_{m+1}}{dt} = L_m \psi_m, \qquad \frac{d\psi_m}{dt} = M_m \psi_m, \ m \in \mathbb{Z}. \tag{9.2}
\]

This system of equations is covariant under the gauge transformations of the form

\[
\psi_m \mapsto g_m \psi_m, \quad L_m \mapsto g_{m+1} L_m g_m^{-1}, \quad M_m \mapsto g_m M_m g_m^{-1} + \partial_t g_m \cdot g_m^{-1}. \tag{9.3}
\]

\[\text{This problem is nontrivial, since in order to get a nonzero Hamiltonian } H_\alpha, \text{ the constant diagonal matrix } \alpha (z^{-1}) \text{ must have pole at zero.}\]
The use of difference operators associated with a 1-dimensional lattice is particularly well-suited for the study of “multi-particle” problems. Let us assume that “potentials” $L_m$ are periodic, $L_{m+N} = L_m$; the period $N$ may be interpreted as the number of copies of an elementary system. In this way we get families of Hamiltonians which remain integrable for all $N$. The phase spaces for such systems are direct products of “one-particle” phase spaces.

It is natural to suppose that the dynamics associated with difference Lax equations develops on submanifolds of a matrix Lie group $G$ (or of a loop group, if there is a spectral parameter), rather than on Lie algebras or their duals. As before, we are looking for a geometric theory which should simultaneously account for the Poisson structure of the phase space, the origin of conservation laws, and the reduction of dynamics to factorization problems. An extension of the geometric scheme described in Section 3 to lattice systems is based on the theory of Poisson Lie groups introduced by Drinfeld [20] following the pioneering work of Sklyanin [19]. Unlike the Lie-Poisson brackets discussed before, this new class of Poisson brackets was virtually unknown in geometry.

Very briefly, the motivation for the formal definitions we are going to discuss is as follows. As in the continuous case, the natural Hamiltonians associated with the zero curvature equations should be gauge invariant. Let us assume that $L_{m+N} = L_m$. Consider the monodromy mapping which assigns to the set of local Lax matrices their ordered product,

$$ T : G^N \to G : (L_0, ..., L_{N-1}) \mapsto T_L = \prod_k L_k. $$

A version of the Floquet theorem asserts that two difference operators with periodic coefficients are gauge equivalent if and only if their monodromy matrices are conjugate. Thus one expects the integrals of motion of equation (9.1) to be of the form

$$ h_k = \text{tr} T^k_L. $$

This will hold if the monodromy itself satisfies a Lax equation,

$$ \frac{dT_L}{dt} = [T_L, A_L]. \tag{9.4} $$

In more formal terms, let $F_t : G^N \to G^N$ be the dynamical flow associated with (9.1) and $\hat{F}_t : G \to G$ the corresponding flow associated with (9.4); then the

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16Typical dynamical systems of this kind are the classical analogues of lattice models in quantum statistics, although some of the systems which we mentioned earlier (e.g., Toda lattices, certain tops, etc.) also admit difference Lax representations.

17This latter paper was in turn motivated by the Quantum Inverse Scattering Method developed by Faddeev, Takhtajan and Sklyanin (cf. [22]) and by the work of Baxter on Quantum Statistical Mechanics (cf. [21]).
The following diagram should be commutative:

\[
\begin{array}{ccc}
G^N & \xrightarrow{F_t} & G^N \\
\downarrow{T_L} & & \downarrow{T_L} \\
G & \xleftarrow{\bar{F}_t} & G
\end{array}
\]  

(9.5)

We want to equip our phase spaces with Poisson structures which are compatible with all mappings in this diagram. Moreover, we would like to keep to our geometric picture suggested by theorem 3.1; this means that we must find two Poisson brackets in each space, so that

(i) **Spectral invariants of the monodromy are Casimir functions for the first structure.**

(ii) **They are in involution with respect to the second one and generate difference Lax equations (respectively, ordinary Lax equations for the monodromy).**

(iii) **The flows \(F_t, \bar{F}_t\) preserve intersections of symplectic leaves for the two brackets.**

(iv) **Vertical arrows in the diagram (9.5) are Poisson mappings with respect to both structures.**

(v) **Finally, the equations of motion (both upstairs and downstairs) should reduce to a factorization problem.**

It is remarkable that all these conditions may be satisfied with the help of a single ingredient, the classical r-matrix, the same one which is responsible for the factorization problem. As compared to the previous case (that of Lie algebras), we need only one extra property (which actually is satisfied in most of the examples we considered beforehand): the Lie algebra \(\mathfrak{g}\) of our Lie group \(G\) must be equipped with an invariant inner product and the r-matrix \(r \in \text{End}\mathfrak{g}\) must be a skew-symmetric operator. The construction which provides Poisson brackets satisfying all these conditions is rather nontrivial (in fact, an important message is that this is possible at all!); it may be naturally divided into two separate problems:

1. **Given an r-matrix, find the brackets on \(G = G^N\) and on \(G\) which have spectral invariants of the monodromy as their Casimir functions.**

2. **Find a Poisson bracket on \(G = G^N\) which yields zero curvature equations (9.1) as the equations of motion and assures that the monodromy map is compatible with the Poisson brackets.**

The key point in both questions is that the r-matrix is fixed *in advance* and we must arrange the brackets with its help (otherwise, there are too many options and the problem is not well posed!).
The second question is better known than the first one; in fact, it is this question that has led to the theory of Poisson groups. The key step is the following simplifying assumption:

- **Dynamical variables associated with different factors in** \( G = G \times ... \times G \) **commute with each other.**

By induction, the monodromy \( T : G^N \rightarrow G \) is a Poisson mapping if the product map \( m : G \times G \rightarrow G \) preserves Poisson brackets.

**Definition 9.1.** Poisson bracket on a Lie group \( G \) satisfying the property above is called **multiplicative**; a Lie group equipped with multiplicative bracket is called a Poisson Lie group.

Let us explain this condition in a more explicit way. Let \( \phi, \psi \in C^\infty(G) \); put \( \Phi(x,y) = \phi(xy), \Psi(x,y) = \psi(xy), \Phi, \Psi \in C^\infty(G \times G) \). In order to compute the Poisson bracket \( \{ \Phi, \Psi \} \) we regard them as functions of two variables, that is, we compute derivatives of \( \Phi, \Psi \) with respect to \( x \) for fixed \( y \) and with respect to \( y \) for fixed \( x \) and take the sum of these two terms; on the other hand we may compute the bracket \( \{ \phi, \psi \} \) for functions of one variable \( z \in G \) and then insert \( z = xy \). Multiplicativity means that the two results coincide.

**9.2. Key example: Sklyanin bracket**

Let us assume that the Lie algebra \( g \) of \( G \) carries an invariant inner product and \( r \in \text{End} \ g \) is skew and satisfies the modified Yang-Baxter identity. For \( \phi \in C^\infty(G) \) let \( \nabla \phi, \nabla' \phi \in g \) be its **left and right gradients** defined by

\[
\langle \nabla \phi(x), X \rangle = \left( \frac{d}{ds} \right)_{s=0} \phi(e^{sX} \cdot x),
\]

\[
\langle \nabla' \phi(x), X \rangle = \left( \frac{d}{ds} \right)_{s=0} \phi(x \cdot e^{sX}), X \in g.
\]

**Proposition 9.2.** The bracket on \( G \)

\[
\{ \phi, \psi \} = \frac{1}{2} \left( \langle r(\nabla \phi), \nabla \psi \rangle - \langle r(\nabla' \phi), \nabla' \psi \rangle \right)
\]

(9.6)

is multiplicative and satisfies the Jacobi identity.\(^{18}\)

---

\(^{18}\)Formula (9.6) seems to be not the simplest bracket which can be arranged using an antisymmetric operator: why not take \( \{ \phi, \psi \}^g(x) = \langle r(\nabla \phi), \nabla \psi \rangle \), or \( \{ \phi, \psi \}^d(x) = \langle r(\nabla' \phi), \nabla' \psi \rangle \)? The reason is this: when \( r \) satisfies the modified Yang-Baxter identity, neither of these brackets satisfies Jacobi. However, the obstructions cancel each other when we take the difference, or the sum of the two (and precisely in these two cases)! We shall return to this question in Section 9.4 below.
Remark 9.3. Formula (9.6) coincides with (8.21), which we deduced from the study of the monodromy map in the continuous case.

This is of course not a coincidence. To explain why the Poisson bracket for the monodromy on the circle should be multiplicative let us consider the auxiliary problems (8.6) with potentials $L$ consisting of two separate patches, so that $\text{supp } L$ is the union of two disjoint intervals, $\text{supp } L = I' \cup I''$. Let us denote by $\mathfrak{G}_{I'}, \mathfrak{G}_{I''} \subset \mathfrak{G}$ the set of all potentials supported on $I', I''$, respectively; then $\mathfrak{G}_{I'} \cap \mathfrak{G}_{I''} \subset \mathfrak{G}$ are Poisson submanifolds, and moreover, $\mathfrak{G}_{I',I''} = \mathfrak{G}_{I'} \times \mathfrak{G}_{I''}$, again as Poisson manifolds (which means that $L' \in \mathfrak{G}_{I'}, L' \in \mathfrak{G}_{I''}$ may be treated as independent variables with respect to our Poisson structure).\(^{19}\) Clearly, for $L = L' + L''$, $L' \in \mathfrak{G}_{I'}, L'' \in \mathfrak{G}_{I''}$ we have $M_L = M_{L'} M_{L''}$ and the Poisson bracket for the monodromy matrix may be computed in two different ways: either by computing the Poisson brackets for the monodromy matrices $M_{L'}, M_{L''}$ regarded as independent variables, or alternatively by computing directly the monodromy $M_L$ for the potential $L = L' + L''$ decomposed into two separate patches. The two results should of course coincide, and this means precisely that the Poisson bracket for the monodromies should be multiplicative.

Note that $\nabla \varphi(x) = Ad x \cdot \nabla' \varphi(x)$, or, in the matrix case, $\nabla \varphi(x) = x \cdot \nabla' \varphi(x) \cdot x^{-1}$, so we may rewrite (9.6) using only left gradients:

$$\{ \varphi, \psi \}(x) = \text{tr} (\eta_r(x) \cdot (\nabla \varphi \wedge \nabla \psi)),$$

where we set

$$\eta_r(x) = \frac{1}{2} (r - Ad x^{-1} \circ r \circ Ad x)$$

and identify $\nabla \varphi \wedge \nabla \psi \in \mathfrak{g} \wedge \mathfrak{g}$ with an antisymmetric linear operator on $\mathfrak{g}$, using the inner product. The function $\eta_r : \mathfrak{g} \rightarrow \text{End } \mathfrak{g}$ satisfies the following remarkable functional equation:\(^{20}\)

$$\eta_r(xy) = \eta_r(x) + Ad x^{-1} \circ \eta_r(y) \circ Ad x,$$

One may check that this functional equation is exactly equivalent to the multiplicativity of (9.6).

Assume that $G = GL(n)$ is a matrix group. Let us consider “tautological” functions $\phi_{ij}$ on $G$ which assign to a matrix $L \in G$ its matrix coefficients, $\phi_{ij}(L) = L_{ij}$; clearly, the ring of polynomials $\mathbb{C} [\phi_{ij}]$ is dense in $C^\infty(G)$, and the Poisson bracket on $G$ is completely specified by its values on the “generators” $\phi_{ij}$. Let us identify $r \in \text{End } \mathfrak{gl}(n)$ with an element of $\mathfrak{gl}(n) \otimes \mathfrak{gl}(n) \simeq \text{Mat}(n^2)$.

Proposition 9.4. The Poisson bracket (9.6) of matrix coefficients is given by

$$\{ \phi_{ij}, \phi_{km} \}(L) = [r, L \otimes L]_{ikjm}.$$

\(^{19}\)At this point we use the crucial property of (8.11): the Poisson operator is a multiplication operator.

\(^{20}\)This condition is usually expressed by saying that $\eta_r$ is a 1-cocycle (in fact, a coboundary) on $G$. In these lectures we shall not use this language.
The commutator in the r.h.s is computed in $Mat(n^2)$. Usually people do not distinguish $\phi_{ij}$ and its values and write this formula (with suppressed matrix indices!) as

$$\{ L \otimes L \} = [r, L \otimes L].$$

(9.9)

Formula (9.9) has served as the original definition of the *Sklyanin bracket*. Note that the r.h.s. in (9.9) is a *quadratic* expression in matrix coefficients (this is to be compared with the Lie-Poisson bracket which is *linear*).

Let us note some important properties of the Sklyanin bracket.

(i) **The bracket is identically zero at the unit element of the group.** (Indeed at $x = e$ right and left gradients coincide).

(ii) **Linearizing the bracket at the origin of group, we get the structure of a Lie algebra in the cotangent space $T^*_e G = g^*$**: if $\xi, \zeta \in g^*, X \in g$, choose $\varphi, \psi \in C^\infty(G)$ in such a way that $\nabla \varphi(e) = \xi, \nabla \psi(e) = \zeta$ and set

$$\langle [\xi, \zeta], X \rangle = \left( \frac{d}{ds} \right)_{s=0} \{ \varphi, \psi \} \left( e^{sX} \right) = \left( \frac{d}{ds} \right)_{s=0} \text{tr} \left( \eta_r(e^{sX}) \cdot (\xi \wedge \zeta) \right)$$

(the second formula checks that the bracket $[\xi, \zeta]_*$ does not really depend on the choice of $\varphi, \psi$ and so is well defined).

**Proposition 9.5.** The bracket $[\xi, \zeta]_*$ coincides with the r-bracket (up to the identification of $g$ and $g^*$ induced by the inner product$^{21}$):

$$[\xi, \zeta]_* = \frac{1}{2} \left( [r \xi, \zeta] + [\xi, r \zeta] \right).$$

(9.10)

Up to dualization, (9.10) coincides with (3.1) which was our starting point in Section 3. In the present setting we get some extra properties which follow from multiplicativity of the bracket. Set

$$\delta_r(X) = \left( \frac{d}{ds} \right)_{s=0} \eta_r(e^{sX}).$$

Explicitly, we get

$$\delta_r(X) = adX \circ r - r \circ adX.$$

**Proposition 9.6.** (i) **We have**

$$\text{tr} \left( \delta_r(X) \circ (\xi \wedge \zeta) \right) = \langle [\xi, \zeta], X \rangle.$$  

(9.11)

$^{21}$Formula (9.10) explains the choice of normalization in (9.6): we wanted to get the same thing as in (3.1).
(ii) The mapping $\delta_r : \mathfrak{g} \to \text{End} \mathfrak{g}$ satisfies the functional equation

\begin{equation}
\delta_r ([X,Y]) = [\text{ad}X, \delta_r (Y)] - [\text{ad}Y, \delta_r (X)].
\end{equation}

Equation (9.11) shows that $\delta_r$ is the dual of the commutator map $\mathfrak{g}^* \land \mathfrak{g}^* \to \mathfrak{g}^*$.22

**Definition 9.7.** A pair $(\mathfrak{g}, \mathfrak{g}^*)$ is called a *Lie bialgebra* if (i) $\mathfrak{g}$ and $\mathfrak{g}^*$ are set in duality as linear spaces, (ii) both $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie algebras, (iii) the dual of the commutator map $[,] : \mathfrak{g} \land \mathfrak{g} \to \mathfrak{g}$ satisfies the functional equation (9.11).

**Remark 9.8.** One can show that (iii) implies that in, the dual way, the mapping $\delta_* : \mathfrak{g}^* \to \mathfrak{g}^* \land \mathfrak{g}^*$ which dualizes the commutator $[,] : \mathfrak{g} \land \mathfrak{g} \to \mathfrak{g}$ is a 1-cocycle on $\mathfrak{g}^*$, and so this definition is symmetric.

It is instructive to compare the definitions of Lie bialgebras and of the double Lie algebras introduced in Section 3. These definitions are *different* and use different notions of the classical $r$-matrix. In the case of double Lie algebras there are two Lie brackets on the *same underlying linear space*; the classical $r$-matrix is a linear operator $r \in \text{End} \mathfrak{g}$; in the case of Lie bialgebras there are two Lie brackets which are defined on *dual spaces* $\mathfrak{g}$ and $\mathfrak{g}^*$. The motivation for these definitions are very much different as well: as we saw, double Lie algebras provide a natural setting for the Involutivity theorem (theorem 3.1); Lie bialgebras naturally arise in the study of multiplicative Poisson brackets on Lie groups. Proposition 9.5 specifies the setting in which these two notions merge together: we must assume that $\mathfrak{g}$ carries an *invariant inner product* which allows to identify $\mathfrak{g}$ and $\mathfrak{g}^*$ and that $r \in \text{End} \mathfrak{g}$ is *skew*. One more natural condition is the modified *Yang-Baxter equation* (which assures that there is an underlying factorization problem). When all three conditions are satisfied, we say that $(\mathfrak{g}, \mathfrak{g}^*)$ is a *factorizable Lie bialgebra*. Factorizable Lie bialgebras and the associated Poisson Lie groups provide a natural environment for all applications to lattice integrable systems.

Before turning to lattice zero curvature equations let us discuss ordinary Lax equations on Lie groups. Here is a version of the factorization theorem (theorem 3.3) which applies in this setting. Let $G$ be a matrix Lie group; we assume that the Poisson bracket on $G$ is given by (9.6) and that its tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is factorizable. Let $I (G) \subset C^\infty (G)$ be the algebra of central functions on $G$ ($\varphi \in C^\infty (G)$ is central if $\varphi (xy) = \varphi (yx)$ for all $x, y \in G$).

**Theorem 9.9.** (i) All central functions are in involution with respect to the Sklyanin bracket (9.6). (ii) Hamiltonian equation on $G$ with Hamiltonian $h \in I (G)$

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22In formal terms, equation (9.12) means that $\delta_r$ is a 1-cocycle on $\mathfrak{g}$ (with values in $\text{End} \mathfrak{g}$).
may be written in Lax form\textsuperscript{23}

\begin{equation}
\frac{dL}{dt} = LM_{\pm} - M_{\pm}L,
\end{equation}

where \( M_{\pm} = r_{\pm} (\nabla h (L)) \).

(iii) The integral curve \( L(t) \) of (9.13) with \( L(0) = L_0 \) is given by

\begin{equation}
L(t) = g_\pm (t)^{-1} L_0 g_\pm (t),
\end{equation}

where \( g_+ (t), g_- (t) \) are the solutions of the factorization problem in \( G \)

\begin{equation}
g_+(t) g_-(t)^{-1} = \exp t \nabla h (L_0)
\end{equation}

associated with \( r \).

As before, the direct proof of theorem 9.9 is easy: to check that (9.14) is an integral curve of (9.13) just compute the derivative of the r.h.s in (9.14). As in Section 3.3, there exists also a geometric proof which explains the background machinery. Below, we shall briefly outline the corresponding construction.

\section*{9.3. Duality for Poisson Lie groups}

As already noted, Lie bialgebras possess a remarkable symmetry: if \((\mathfrak{g}, \mathfrak{g}^*)\) is a Lie bialgebra, the same is true for \((\mathfrak{g}^*, \mathfrak{g})\). Hence the dual group \( G^* \) (which corresponds to \( G^* \)) also carries a multiplicative Poisson bracket. In the case of factorizable Poisson groups this dual bracket may be pushed forward to \( G \) by means of the factorization map. Thus we get two brackets on \( G \) which fit into our geometric treatment of Lax equations. The best way to understand this duality is to notice that both \( G \) and \( G^* \) are Poisson subgroups of a bigger Poisson group, the double of \( G \).

Let \((\mathfrak{g}, \mathfrak{g}^*)\) be a Lie bialgebra; the linear space \( \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^* \) carries a natural inner product

\begin{equation}
\langle \langle (X,F), (X',F') \rangle \rangle = \langle F, X' \rangle + \langle F', X \rangle.
\end{equation}

The following key theorem was discovered by Drinfeld.

\textbf{Theorem 9.10.} There exists a unique structure of the Lie algebra on \( \mathfrak{d} \) such that:

\begin{itemize}
  \item[(i)] \( \mathfrak{g}, \mathfrak{g}^* \subset \mathfrak{d} \) are Lie subalgebras.
  \item[(ii)] The inner product (9.16) is invariant.
\end{itemize}

\textbf{Corollary 9.11.} Let \( P_\mathfrak{g}, P_\mathfrak{g}^* \) be the projection operators onto \( \mathfrak{g}, \mathfrak{g}^* \) in the decomposition \( \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^* \). Set \( r_\mathfrak{g} = P_\mathfrak{g} - P_\mathfrak{g}^* \); then \( r_\mathfrak{g} \) defines on \( \mathfrak{d} \) the structure of a factorizable Lie bialgebra.

\textsuperscript{23}In this formula the velocity vector \( dL/dt \) belongs to the tangent space \( TLG \); in order to be more accurate, we may rewrite (9.13) as an equality in the Lie algebra:

\[ L^{-1} \frac{dL}{dt} = M_{\pm} - \text{Ad} L^{-1} \cdot M_{\pm} \]
The pair \((\mathfrak{d}, \mathfrak{d}^*)\) is called the Drinfeld double of \((\mathfrak{g}, \mathfrak{g}^*)\). When the initial Lie bialgebra \((\mathfrak{g}, \mathfrak{g}^*)\) is itself factorizable, the description of the double is very simple. Consider the Lie algebra \(\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}\) (direct sum of two copies of \(\mathfrak{g}\)) and equip it with the inner product
\[
\langle\langle (X, Y), (X', Y') \rangle\rangle = \langle X, X' \rangle - \langle Y, Y' \rangle,
\]
where \(\langle \cdot, \cdot \rangle\) is the invariant inner product on \(\mathfrak{g}\).

**Proposition 9.12.** The double of a factorizable Lie algebra is canonically isomorphic to \(\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}\).

**Sketch of a proof.** We have already seen in Section 4 that there are two natural homomorphisms \(r_{\pm} : \mathfrak{g}^* \to \mathfrak{g}\) given by (4.5); their combination yields an embedding \(\mathfrak{g}^* \subset \mathfrak{g} \oplus \mathfrak{g}\). Let \(\mathfrak{g}^\delta \subset \mathfrak{g} \oplus \mathfrak{g}\) be the diagonal subalgebra, \(\mathfrak{g}^\delta = \{(X, X); X \in \mathfrak{g}\}\). As discussed in Section 4, \(\mathfrak{d} = \mathfrak{g}^\delta + \mathfrak{g}^*\); it is easy to check that the skew symmetry of \(r\) and the choice of the inner product in \(\mathfrak{d}\) imply that \(\mathfrak{g}^*\) and \(\mathfrak{g}^\delta\) are isotropic with respect to the inner product (9.17); this is equivalent to the skew symmetry of \(r_\delta = P_\mathfrak{g} - P_{\mathfrak{g}^*}\). In matrix notation, \(r_\delta \in \text{End} (\mathfrak{g} \oplus \mathfrak{g})\) is given by a 2 \(\times\) 2 block matrix:
\[
r_\delta = \begin{pmatrix} r & 2r_+ \\ 2r_- & -r \end{pmatrix}.
\]

The Lie group which corresponds to \(\mathfrak{d}\) is \(D(G) = G \times G\). Let \(G^\delta, G^*\) be its subgroups which correspond to \(\mathfrak{g}^\delta, \mathfrak{g}^*\). Clearly, \(G^\delta \subset D(G)\) is the diagonal subgroup. As in (8.39), we may associate with the \(r\)-matrix \(r_\delta\) a factorization problem in \(D(G)\). Let us assume for simplicity that it is globally solvable, i.e. \(D(G) \simeq G \cdot G^*\).

**Proposition 9.13.** Let us equip \(D(G)\) with the Sklyanin bracket associated with \(r_\delta\). Then \(G^\delta, G^* \subset D(G)\) are Poisson subgroups (i.e. they are Poisson submanifolds and the induced Poisson structure is multiplicative).\(^{24}\)

The bracket induced on \(G^\delta\) coincides with the original Sklyanin bracket associated with \(r\); the Poisson bracket on the dual group \(G^*\) is described in the following way. Consider the mapping \(m : D(G) \to G : (x, y) \mapsto xy^{-1}\); its restriction to \(G^* \subset D(G)\) is a diffeomorphism.

**Proposition 9.14.** The Poisson bracket on \(G\) induced by \(m : G^* \to G\) is given by
\[
\{\varphi, \psi\}_* = 1/2 (r \nabla \varphi, \nabla \psi) + 1/2 (r' \nabla' \varphi, \nabla' \psi)
\]
\[
- \langle r_+, \nabla \varphi, \nabla' \psi \rangle - \langle r_-, \nabla' \varphi, \nabla \psi \rangle,
\]
where \(\nabla \varphi, \nabla \psi\) and \(\nabla' \varphi, \nabla' \psi\) are left and right gradients of \(\varphi, \psi\).

Formula (9.19) looks rather complicated; however, the bracket \(\{\cdot, \cdot\}_*\) is very remarkable.

\(^{24}\)More precisely, the bracket induced on \(G^* \subset D(G)\) has opposite sign, due to the minus sign in \(r_\delta = P_\mathfrak{g} - P_{\mathfrak{g}^*}\).
Proposition 9.15. (i) Symplectic leaves of $\{ \cdot, \cdot \}$ coincide with conjugacy classes in $G$. (ii) Casimir functions of $\{ \cdot, \cdot \}$ are precisely the central functions on $G$. (iii) The bracket (9.19) vanishes at the unit element $e \in G$; the induced Lie bracket on the tangent space coincides with the original Lie bracket on $g$.

Thus the bracket (9.19) provides the missing ingredient of our geometric picture: we have got now two Poisson structures on the same underlying manifold $G$ and Lax equations preserve intersections of two systems of symplectic leaves.

The symplectic leaves of the Sklyanin bracket also admit a description in terms of the factorization problem. Let us identify $G$ with the quotient space $\mathcal{D}(G)/G^*$ using the factorization $\mathcal{D}(G) = G^* \cdot G$. Let us denote by $\pi$ the canonical projection $\mathcal{D}(G) \to \mathcal{D}(G)/G^*$; define the action $G^* \times G \to G$ using the commutative diagram

(Here $m$ is the group multiplication in $\mathcal{D}(G)$ restricted to the subgroup $G^* \subset \mathcal{D}(G)$.) By analogy with the definition of dressing transformations, this action is called dressing action.

Proposition 9.16. Symplectic leaves of the Sklyanin bracket in $G$ coincide with the orbits of $G^*$ in $G$ with respect to the dressing action.

More explicitly, the dressing prescription is as follows: given $x \in G$, $(h_+, h_-) \in G^*$ solve the factorization problem in $\mathcal{D}(G)$:

$$(h_+x, h_-x) = (x'g_+, x'g_-), x' \in G, (g_+, g_-) \in G^*;$$

then $\text{Dress}(h_+, h_-) \cdot x = x'$. This immediately yields the following formula in terms of the factorization problem in $G$:

$$\text{Dress}(h_+, h_-) \cdot x = h_+x \left( x^{-1}h_+^{-1}h_-x \right)_+ = h_-x \left( x^{-1}h_+^{-1}h_-x \right)_-, (9.20)$$

where $x^{-1}h_+^{-1}h_-x = (x^{-1}h_+^{-1}h_-x)_+ \cdot (x^{-1}h_+^{-1}h_-x)_-^{-1}$ is the factorization in $G$ associated with the original $r$-matrix.

Exercise 9.17. Check this formula using the definition of dressing action.

Dressing action may be regarded as a nonlinear analog of the coadjoint representation, as it is clear from the following simple assertion.

Proposition 9.18. Dressing action leaves the unit $e \in G$ invariant; the linearization of the dressing action in the tangent space $T_eG \simeq g$ coincides with the coadjoint representation of $G^*$ in $g^{**} \simeq g$. 

In applications it is natural to assume that $G$ is an algebraic loop group consisting of matrices whose coefficients are rational functions of $z$. Orbits of the dressing action of $G^*$ in this loop group are finite-dimensional, and we get a description of phase spaces for Lax equations which is largely parallel to the case of coadjoint orbits of $G_r$ discussed in Section 6.

One more application of the dual Poisson structure described by (9.19) is the accurate description of the Poisson properties of the dressing transformations from Section 8.6. We keep to the notation introduced in lemma 8.30. Let $G$ be the loop group; its Lie algebra $Lg = g[z, z^{-1}]$ is equipped with the inner product

$$\langle X, Y \rangle = \text{Res}_{z=0} \text{tr} \left( X(z) Y(z) \right)$$

and with the standard $r$-matrix $r = P_+ - P_-$ associated with the Riemann problem. The loop group $G$ equipped with the corresponding Sklyanin bracket becomes a Lie-Poisson group. Let $G_r \simeq G^* \times G_-$ be the dual group equipped with its natural Poisson structure (9.19).

**Theorem 9.19.** The dressing action described by the commutative diagram (8.41) is a Poisson group action.

(We shall not reproduce the proof here; see [18], [23].)

### 9.4. Symplectic double and the free dynamics

One more important ingredient of the geometric picture outlined in Section 3.3 is the “big phase space” with “free” dynamical flow. Its counterpart in the present setting is provided by the so called symplectic double of $G$. Let again $D(G) = G \times G$ be the double of $G$; the Sklyanin bracket on $D(G)$ is given by

$$\{ \varphi, \psi \} = \langle (r \nabla \varphi, \nabla \psi) \rangle - \langle (r' \nabla' \varphi, \nabla' \psi) \rangle \overset{\text{def}}{=} \{ \varphi, \psi \}^g - \{ \varphi, \psi \}^d \langle (\nabla \varphi_1, \nabla \varphi_2), \nabla \varphi_3 \rangle$$

As noticed in footnote 18, the terms with left and right gradients separately do not satisfy the Jacobi identity; the obstructions cancel when the two are combined together. Explicitly,

$$\{ \{ \varphi_1, \varphi_2 \}^g, \varphi_3 \}^g + \text{c.p.} = \langle [\nabla \varphi_1, \nabla \varphi_2], \nabla \varphi_3 \rangle,$$

(9.22)

$$\{ \{ \varphi_1, \varphi_2 \}^d, \varphi_3 \}^d + \text{c.p.} = -\langle [\nabla' \varphi_1, \nabla' \varphi_2], \nabla' \varphi_3 \rangle.$$

The two terms cancel, since $\nabla \varphi(x) = x \cdot \nabla' \varphi(x) \cdot x^{-1}$ and the inner product is $Ad$-invariant. It is important to notice that the crucial minus sign in (9.22) is due not to the minus in (9.21), but rather to the fact that the action of a group by right translations is its anti-representation. (There are no terms of “mixed chirality” in the obstruction, since left and right translations commute with each other.) Thus we get the following assertion:
Proposition 9.20. Let \( r, r' \in \text{End } \mathfrak{d} \) be two arbitrary classical \( r \)-matrices satisfying the modified Yang-Baxter equation; the bracket

\[
\{ \varphi, \psi \}_{r,r'} = \langle \langle r \nabla \varphi, \nabla \psi \rangle \rangle + \langle \langle r' \nabla' \varphi, \nabla' \psi \rangle \rangle
\]  

(9.23)

satisfies the Jacobi identity.

Specifically, let us take \( r = r' = r_0 \); the resulting Poisson structure is non-degenerate (at least if we assume – as we always do in this Section – that the factorization problem in \( D \) is globally solvable), and hence defines a symplectic structure on \( D(G) \).

Definition 9.21. The manifold \( D(G) \) with the Poisson bracket

\[
\{ \varphi, \psi \}_+ = \langle \langle r_0 \nabla \varphi, \nabla \psi \rangle \rangle + \langle \langle r_0' \nabla' \varphi, \nabla' \psi \rangle \rangle
\]

(9.24)
is called the symplectic double of \( G \). We shall denote it \( D(G)_+ \) in order to distinguish it from the Drinfeld double equipped with the Sklyanin bracket.

The bracket (9.24) is not multiplicative, so from the point of view of the theory of Poisson groups \( D(G)_+ \) is not a group! Instead, we have the following property which shows that \( D(G)_+ \) is a principal homogeneous space for \( D(G) \):

Proposition 9.22. Left and right multiplication in \( D(G) \) induce Poisson mappings \( D(G) \times D(G)_+ \rightarrow D(G)_+ \), \( D(G)_+ \times D(G) \rightarrow D(G)_+ \).

Proposition 9.22 paves the way to use the reduction technique in our present setting. Let us recall the point of view on reduction adopted in Section 3.3: if \( M \) is symplectic and \( K \times M \rightarrow M \) is a group action, the reduction is the natural projection \( \pi : M \rightarrow M/K \) onto the space of \( K \)-orbits in \( M \). The key property which we need to get a Poisson bracket on \( M/K \) is this: Poisson bracket of two \( G \)-invariant functions on \( M \) is again \( G \)-invariant. Let us discuss briefly how can one control this property. For \( \varphi \in \mathcal{C}^\infty(M) \) let us denote by \( \xi_\varphi \in \text{Vect } M \) the linear functional defined by

\[
\langle \xi_\varphi(x), X \rangle = \frac{d}{dt}_{t=0} \varphi(\exp tX \cdot x).
\]

When vector fields \( \hat{X} \in \text{Vect } M \) are Hamiltonian we have simply

\[
\hat{X} \{ \varphi, \psi \} = \{ \hat{X} \varphi, \psi \} + \{ \varphi, \hat{X} \psi \} = 0.
\]

In the case of Poisson group actions vector fields \( \hat{X} \) are no longer Hamiltonian; however, the rate of nonconservation of Poisson brackets by these vector fields may be characterized very sharply. For \( \varphi \in \mathcal{C}^\infty(M) \), \( x \in M \), let us denote by \( \xi_\varphi(x) \in \mathfrak{k}^* \) the linear functional defined by

\[
\langle \xi_\varphi(x), X \rangle = \frac{d}{dt}_{t=0} \varphi(\exp tX \cdot x).
\]
Proposition 9.23. Let us assume that $K$ is a Poisson Lie group with Lie bialgebra $(\mathfrak{t}, \mathfrak{t}^*)$. The mapping $K \times M \longrightarrow M$ is a Poisson mapping if and only if
\[
\hat{X}\{\varphi, \psi\} - \{\hat{X}\varphi, \psi\} - \{\varphi, \hat{X}\psi\} = \langle [\xi_\varphi, \xi_\psi], X \rangle.
\]

When $\hat{X}\varphi = \hat{X}\psi = 0$ for all $x \in \mathfrak{t}$, $\xi_\varphi = \xi_\psi \equiv 0$, and hence $\hat{X}\{\varphi, \psi\} = 0$, which assures the possibility of reduction. In a more general way, let us say that
\[
\hat{X}\{\varphi, \psi\} = 0
\]
when $\hat{X}\varphi = \hat{X}\psi = 0$ for all $x \in \mathfrak{h}$ implies that $\xi_\varphi, \xi_\psi \in \mathfrak{h}^\perp$. When $\mathfrak{h}^\perp \subset \mathfrak{t}^*$ is a Lie subalgebra, $\langle [\xi_\varphi, \xi_\psi], X \rangle = 0$ for all $x \in \mathfrak{h}$ and hence $\hat{X}\{\varphi, \psi\} = 0$.

As a first example of reduction, let us derive from (9.24) the dual Poisson bracket (9.25) on $G$.

Proposition 9.24. $H \subset K$ is admissible $\iff \mathfrak{h}^\perp \subset \mathfrak{t}^*$ is a Lie subalgebra.

Sketch of a proof. $\hat{X}\varphi = \hat{X}\psi = 0$ for all $x \in \mathfrak{h}$ implies that $\xi_\varphi, \xi_\psi \in \mathfrak{h}^\perp$. When $\mathfrak{h}^\perp \subset \mathfrak{t}^*$ is a Lie subalgebra, $\langle [\xi_\varphi, \xi_\psi], X \rangle = 0$ for all $x \in \mathfrak{h}$ and hence $\hat{X}\{\varphi, \psi\} = 0$.

Lemma 9.26. (On free dynamics) The Hamiltonian flow on $D(G)_+$ defined by $h_\varphi$ is given by
\[
F_t : (x, y) \longrightarrow (x e^{tx}, ye^{tx}), \quad X = \nabla \varphi (x^{-1}y).
\]

\footnote{Gradients in the l.h.s are computed with respect to two variables $x, y$; in the r.h.s they are computed with respect to a single variable.}
Theorem 9.27. (i) The Hamiltonian $h_\varphi$ is invariant with respect to the group $G^*$ which is acting on $D(G)_+$ via

$$(h_+, h_-) : (x, y) \mapsto (h_+xh_-^{-1}, h_yh_-^{-1})$$

(ii) $G^*$ is an admissible subgroup in $D(G) \times D(G)$ (which acts on $D(G)_+$ by left and right translations). (iii) The mapping

$$\pi : D(G)_+ \rightarrow G : (x, y) \mapsto yx^{-1}$$

is constant on $G^*$-orbits in $D(G)_+$ and allows to identify the quotient space $D(G)_+/G^*$ with the subgroup $G = \{(x, e) : x \in G\} \subset D(G)_+$.

(iv) The quotient Poisson structure on $D(G)_+/G^*$ coincides with the Sklyanin bracket.

(v) The quotient flow $\bar{F}_t$ on $G$ is given by $\bar{F}_t : x \mapsto g_\pm(t)^{-1}xg_\pm(t)$, where $g_+(t), g_-(t)$ solve the factorization problem $\exp t\nabla \varphi(x) = g_+(t)g_-(t)^{-1}$, and satisfies the Lax equation (9.13).

9.5. Lattice zero curvature equations and the twisted double

So far our geometric construction is restricted to ordinary Lax equations on a single copy of $G$. In order to put lattice zero curvature equations into our framework we need one more effort. First of all, let us state the factorization theorem which applies in this case (cf. Section 9.1). Let again

$$T : G^N \rightarrow G : (L_0, ..., L_N) \mapsto T_L = \prod_k L_k$$

be the monodromy map; choose $\varphi \in I(G)$ and set $H_\varphi = \varphi \circ T$. We define the “wave function” $\psi_m$ associated with the auxiliary linear problem (9.2) by

$$\psi_m = \prod_{0 \leq k \leq m-1} L_k$$

The Poisson structure on $G^N$ is defined as the direct product of Sklyanin brackets on each factor. As usual, we assume that our basic r-matrix is skew and satisfies the modified Yang-Baxter equation, so that $(g, g^*)$ is a factorizable Lie bialgebra.

Theorem 9.28. (i) The Hamiltonian equation of motion on $G^N$ with Hamiltonian $H_\varphi$ may be written as

$$\frac{dL_m}{dt} = L_m M_{m+1}^\pm - M_m^\pm L_m,$$

\text{(9.26)}

\text{As usual, for } g \in G \text{ we denote by } g_+, g_- \text{ the solutions of the factorization problem } g = g_+g_-^{-1}.$
where
\[(9.27)\]
\[M_m^\pm = r_\pm \left( \psi_m \nabla \varphi (T_L) \psi_m^{-1} \right).\]

(ii) Its integral curve with origin \(L^0 = (L^0_0, ..., L^0_{N-1})\) is given by
\[(9.28)\]
\[L_m(t) = g_m^\pm(t)^{-1} L^0_m g_m^\pm(t),\]
where \(g_m^+(t), g_m^-(t)\) are the solutions of the factorization problem
\[(9.29)\]
\[g_m^+(t), g_m^-(t)^{-1} = \psi_0^m \exp t \nabla \varphi (T_L^m) \cdot (\psi_0^m)^{-1}.\]

As usual, a direct proof of theorem 9.28 is easy. One point worth to be mentioned
is the computation of the gradients of the Hamiltonian: for that end we may use the
“variation of constants” in the auxiliary linear problem (9.2), similarly to the case
of differential operators on the circle discussed in Section 8.3. A geometric deriva-
tion is not so straightforward. Let us introduce the following notation in order to
simplify the bulky formulae. We set \(G = G^N, \mathfrak{g} = \bigoplus^N \mathfrak{g}, L = (L_0, ..., L_{N-1}) \in \mathfrak{g} \). Let \(\tau\) be the automorphism of \(G\) induced by cyclic permutation of indices,
\[(L_0, ..., L_{N-1})^\tau = (L_1, L_2, ..., L_{N-1}, L_0);\]
we denote the corresponding automor-
phism of \(\mathfrak{g}\) by the same letter.\(^{27}\) Equations (9.26), (9.28), (9.29) may be rewritten
as
\[(9.30)\]
\[\frac{dL}{dt} = LM^\tau_\pm - M_\pm L,\]
\[L(t) = g_\pm(t)^{-1} L^0_m g_\pm(t)^\tau,\]
where
\[g_+(t) g_-(t)^{-1} = \psi_0^m \exp t \nabla \varphi (T_L^m) \cdot (\psi_0^m)^{-1}.\]

\(^{27}\)The twisting automorphism \(\tau\) plays the role which is similar to that of the derivation \(\partial_t\) for
ordinary zero curvature equations; we can say that its use allows to reproduce for lattice systems
the effects of central extension of loop algebras discussed in Section 8.1.
in block notation we have

\[ r_D = \begin{pmatrix} r & 2r_+ \\ 2r_- & -r \end{pmatrix} \]

(cf. (9.18)). Let us define the automorphism \( \tau \in \text{Aut}(G \times G) \) by \((x, y)^\tau = (x, y^\tau)\); the corresponding automorphism of \( \mathcal{D} = \mathfrak{g} \oplus \mathfrak{g} \) is again denoted by the same letter. Put \( r_D^\tau = \tau \circ r_D \circ \tau^{-1} \); in block notation we have

\[ r_D^\tau = \begin{pmatrix} r & 2r_+ \circ \tau^{-1} \\ 2r_- \circ \tau^{-1} & -r \end{pmatrix}. \]

Let us define the twisted Poisson structure on \( D(G) \) by

\[ \{\varphi, \psi\}_\tau = \left\langle \langle \nabla \varphi, \nabla \psi \rangle \right\rangle + \left\langle \langle r_D^\tau \nabla' \varphi, \nabla' \psi \rangle \right\rangle. \]

The Jacobi identity for (9.31) follows from proposition 9.20. Our next assertion is the twisted version of proposition 9.25.

**Proposition 9.29.** The action \( G \times D(G)_\tau \to D(G)_\tau \) : \( h : (x, y) \mapsto (hx, hy) \) is admissible; the projection map \( p : D(G)_\tau \to G : (x, y) \mapsto x^{-1}y \) is constant on its orbits. The quotient Poisson bracket on \( D(G)_+ / G \cong G \) is given by

\[ \{\varphi, \psi\} = \left\langle \left\langle r_D^\tau \left( \begin{array}{c} \nabla \varphi \\ -\nabla' \varphi \end{array} \right), \left( \begin{array}{c} \nabla \psi \\ -\nabla' \psi \end{array} \right) \right\rangle \right\rangle. \]

The proof is parallel to that of proposition 9.25. The properties of the quotient bracket are also quite remarkable. (Note that unlike the Sklyanin bracket on \( G = G \times \ldots \times G \) the bracket (9.32) is non-local, due to presence of the twist \( \tau \).)

Let us consider the monodromy map \( T : G \to G \). Assume that \( G \) carries the Poisson bracket (9.32) and \( G \) is equipped with the bracket (9.19).

**Proposition 9.30.** (i) The monodromy \( T : G \to G \) is a Poisson mapping. (ii) The spectral invariants of the monodromy are the Casimirs of the quotient bracket. (iii) Define the gauge action \( G \times G \to G : x : g \mapsto xg(x^\tau)^{-1} \); when \( N \) is odd, its orbits coincide with symplectic leaves of the quotient bracket.

In brief, we see that the bracket (9.32) provides us with the missing ingredient for our geometric picture: it is the sought for second bracket which we need for a geometric treatment of lattice zero curvature equations.

**Lemma 9.31.** (on free dynamics) Let \( \varphi \in I(G) \); set \( h_\varphi = \varphi \circ T \in C^\infty(G) \), \( H_\varphi = h_\varphi \circ p \in C^\infty(D(G)) \). The integral curves of the Hamiltonian \( H_\varphi \) in \( D(G)_\tau \) are given by

\[ (x(t), y(t)) = (x_0 e^{tX}, y_0 e^{tX}), X = \nabla \varphi(\tau^{-1}(x_0)). \]
Theorem 9.32. Set $G^* = (G^*)^N$ and consider the action $G^* \times D(G)_{\tau} \to D(G)_{\tau}$ given by

$$(h_+, h_-) : (x, y) \mapsto \left( h_+ x h_-^{-1}, h_+ y (h_-)^{-1} \right).$$

This action is admissible; the quotient space $D(G)_{\tau}/G^*$ may be identified with $G$ by means of the map

$$\pi : D(G) \to G : (x, y) \mapsto y_+^{-1} x y_-^{-1}$$

which is constant on the orbits of $G^*$, and the quotient Poisson structure coincides with the Sklyanin bracket. The flow (9.33) admits reduction with respect to this action; the quotient flow is coincides with (9.30).

Theorem 9.32 fills the last gap in our geometric picture and shows that the qualitative behaviour of difference zero curvature equations is the same as in the case of linear phase spaces.

Acknowledgements

It is my personal pleasure to thank the Organizing Committee and in particular Professor A. Ferreira dos Santos and Dr. N. Manojlovic for their kind invitation and for the inspiring atmosphere they have created during the School.

The present work was partially supported by the INTAS Open 00-00055 grant.

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2000 Mathematics Subject Classification. Primary 81R12; Secondary 17Bxx, 37J35, 37K10

Received