Automated Market-Making for Fiat Currencies

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Abstract

We present an automated market-making (AMM) cross-settlement mechanism for digital assets on interoperable blockchains, focusing on central bank digital currencies (CBDCs) and stable coins. We develop an innovative approach for generating fair exchange rates for on-chain assets consistent with traditional off-chain markets. We illustrate the efficacy of our approach on realized FX rates for G-10 currencies.

1 Introduction

Traditionally, many essays on quantitative finance start with an obligatory reference to the celebrated Black-Scholes paper; see [2]. However, as Bob Dylan put it: “For the times they are a-changin’.” Nowadays, to be au courant with research, one needs to start with referring to the seminal paper by Nakamoto describing the Bitcoin protocol; see [8]. Publication of this short paper by its anonymous author (or authors) started the blockchain revolution; see [7]. However, despite its technical brilliance, the Bitcoin protocol is limited to moving the underlying token, called BTC, from one anonymous address to the next consistently and robustly. Ethereum, Cardano, Polkadot, and Solana expanded Bitcoin horizons by building a Consensus-as-a-Service (CaaS) machine capable of handling the so-called smart contracts.

Over the last two years, Decentralised Finance (DeFi) has seen remarkable growth and quickly established itself as one of the first genuine “killer apps” for smart contracts; see [9]. Using DeFi, agents can create precisely tailored and highly complex economic arrangements and execute them automatically without central intermediaries or other trusted parties. As a result, a DeFi-based financial system will be more robust, inclusive, and equitable than its ossified centralized version. Undeniably, the most valuable tools for DeFi are automated market-making (AMM) protocols. The corresponding protocols power all decentralized exchanges (DEXs), which gradually replace conventional centralized exchanges relying on traditional market-making techniques.

Of course, DeFi crypto platforms are not past teething problems. For instance, recently, Poly Network lost approximately $610 million worth of crypto assets when the hacker targeted a vulnerability in the digital interoperability contract. Eventually, the hacker returned the fund, but not before making her point was made loud and clear.

One of the most exciting instruments using smart contracts are the so-called Central Bank Digital Currencies (CBDCs) and Stable Coins (SCs), which map fiat currencies on blockchains; see [6]. Other exciting instruments are the so-called Non-Fungible Tokens (NFTs). Currently, CBDCs are still being conceptualized and developed by several central banks, which act in isolation. The ability to trade different CBDCs and SCs against each other is critical for their adoption. While CBDCs largely remain on the drawing board, SCs, such as Tether, USDC, Dai, and numerous others, are well-developed and implemented as tokens on their underlying blockchains, for example, ERC-20 tokens in Ethereum. SCs implemented on the same blockchain can be swapped by using a smart contract. CBDCs implemented on different blockchains can be swapped automatically only if the corresponding blockchains are interoperable; see [1] [5] [10].

Below we assume that SCs are implemented on the same blockchain, while CBDCs are implemented on interoperable blockchains, and we develop an arbitrage approach to make the on-chain exchange rate of CBDCs or SCs consistent with the traditional off-chain forex (FX) markets. We argue that both on- and off-chain operations are necessary to make the on-chain exchange rate consistent with off-chain pricing. Finally, we apply our methodology for analyzing hypothetical liquidity pools using actual FX exchange data of G-10 currencies. We make our simulations as realistic as currently possible.
2 Decentralised Exchange (DEX) and Automated Marking Making

Decentralised Exchanges with Automated market-making (AMM) have increased over the past year as a central part of the DeFi eco-system to enable the on-chain exchange of different tokens. Currently, Uniswap is the biggest automated DEX with total locked value (TLV) of liquidity reserves of $7.1B (https://defipulse.com/uniswap as of August 2021).

Conceptually DEX is owned by a pool operator. Liquidity providers post tokens to the liquidity pool. Clients and traders use the pool liquidity to exchange one CBDC for another. The exchange rate depends on the order size based on the Constant Function Market Makers.

The pool operator owns the corresponding smart contract on a blockchain and sets the core parameters of automated pool operations, including the transaction fees. In addition, the pool operator implements the constant function market-making (CFMM), which is implemented using a smart contract, that provides the bid-ask quotes for client orders.

Liquidity providers deposit CBDCs to the pool. In return, liquidity providers get fees generated by the pool trading activity. A liquidity provider may hedge CBDCs exposures in the pool using off-chain instruments, in which case the P&L of the provider is the stream of fees generated by the pool trading activity.

Traders and Clients use the pool for either redeeming tokens from the pool or depositing tokens to the pool in a one-sided transaction. We assume that each client transaction of buying from the pool is charged with the proportional pool fee.

The on-chain operations are implemented using the CFMM that assigns bid and ask prices of two coins in the pool based on the size of the trades. As we show, the marginal exchange rate for a trade is proportional to the consumption of the pool liquidity. Therefore, traders and clients must incentivize pool liquidity providers with fees accrued to the pool and shared between the liquidity providers of the pool.

Pool arbitrageurs enforce the rebalancing of the pool when the CFMM indicative bid-ask quotes are outside some thresholds. Next, we find the optimal thresholds for a set of CFMM rules.

The pool arbitrageur implements both on-chain and off-chain transactions intending to keep his total balance between assets and liabilities zero. The pool arbitrageur is crucial for "price-discovery" of the on-chain pricing. Pool operators may use designated arbitrageurs incentivized with lower fees.

3 Constant Function Market Makers (CFMMs)

For clarity, we denote the EURUSD FX spot rate by $p_t$ and assume that $p_0 = 1.25$. We assume that the AMM operator creates a pool with with notional $p_0 N_0$ USDC and $N_0$ EUDC. The initial pool balances in USDC, $x_0$, and EUDC $y_0$ are set respectively by:

$$x_0 = p_0 N_0, \quad y_0 = N_0.$$  \hfill (1)

The redemption and deposits of tokens from the AMM pool is determined by the CFMM function, also known as the pool invariant, in the following way:

$$F(x_1, y_1; x_0, y_0, \Theta) = 0,$$  \hfill (2)

where $x_1$ and $y_1$ is the amount of USDC and EUDC tokens after a transaction, respectively; $x_0$ and $y_0$ is the amount of USDC and EUDC tokens right before the transaction. $\Theta$ is the set of constant set of pool parameters, including the proportional fee rate $\epsilon$. After each transaction, pool balances $x_0$ and $y_0$ are updated by $x_1$ and $y_1$, respectively.

3.1 Buying/Redeeming USDC

Buying $\Delta x$ USDC tokens from the pool involves the redemption from the USDC pool by deposing $(1 - \epsilon)\Delta y$ tokens to the EUDC pool. The AMM applies the CFMM (2) to determine the amount of USDC $\Delta x$ redeemed in exchange for $\Delta y$ EUDC deposited as follows:

$$F^{-}(x_0 - \Delta x, y_0 + (1 - \epsilon)\Delta y) = 0.$$  \hfill (3)

Buying USDC from the pool can be done in two ways. First, the trader can redeem known amount of $\Delta x$ USDC from pool by depositing yet unknown amount of $\Delta y$ EUDC, which is found using Eq (3) with given $\Delta x$. 

Electronic copy available at: https://ssrn.com/abstract=3939695
Table 1: Operations for buying USDC from the pool

| Quote   | Order               | USDC balance | EUDC balance | Op for Eq(3) |
|---------|---------------------|--------------|--------------|--------------|
| Ask EUDC| Buy USDC/Sell EUDC  | redeem Δx USDC | deposit Δy EUDC | Δx → Δy       |
| Bid USDC| Buy USDC/Sell EUDC  | redeem Δx USDC | deposit Δy EUDC | Δy → Δx       |

Table 2: Operations for buying EUDC

| Quote   | Order               | USDC balance | EUDC balance | Op for Eq(4) |
|---------|---------------------|--------------|--------------|--------------|
| Ask USDC| Buy EUDC/Sell USDC  | deposit Δx USDC | redeem Δy EUDC | Δy → Δx       |
| Bid EUDC| Buy EUDC/Sell USDC  | deposit Δx USDC | redeem Δy EUDC | Δx → Δy       |

Second, the trader can deposit a known amount of Δy EUDC and redeems Δx amount of USDC, which is set using Eq (3) with given Δy. We summarize these operations in Table 1.

3.2 Buying/Redeeming EUDC

The AMM will apply the CFMM to find the amount of EUDC Δy redeemed in exchange for depositing Δx USDC using CFMM (2) as follows:

\[ F(x_0 + (1 - \epsilon)\Delta x, y_0 - \Delta y) = 0. \] (4)

By analogy, the trader can redeem given amount of Δy EUDC by depositing the amount of Δx USDC found using (4). Alternatively, the trader can deposit given amount of Δx USDC to redeem the amount of Δy EUDC set using Eq. (4). We summarize the operations in Table 2.

3.3 Bid/Ask Matching

In Table (3), we aggregate the operations in Tables 1 and 2 to present the conventional matching of order book.

4 Representative examples

We now present the most important types of CFMMs; among numerous others, see [1, 3].

4.1 Constant sum function

Constant sum rule specifies the following invariant for CFMM (2):

\[ F(x_1, y_1) = x_1 + sy_1 - \Sigma_0 = 0. \] (5)

Here and below s is the fixed level of “equilibrium” conversion rate of y tokens to x tokens, and \( \Sigma_0 = x_0 + sy_0 \).

For redemption of USDC, Eq(3) simplifies to:

\[ \Delta x = s(1 - \epsilon)\Delta y, \] (6)

and for for redemption of EUDC, Eq(3) becomes:

\[ (1 - \epsilon)\Delta x = s\Delta y. \] (7)

In Table 4, we summarise bid/ask matching as defined in Table 3.

In Figure 1, we show the bid ask rates for USDC and EUDC implied by the AMM. It is clear that the constant sum rule does not depend on the order size, so that it is less realistic for AMM.

Table 3: Bid/Ask quoting as function of order sizes using CFMM

| Quote | USDC                  | EUDC                  |
|-------|-----------------------|-----------------------|
| Bid   | Buy USDC/Sell EUDC: \( F^{(-, -)}: \Delta y \to \Delta x \) | Buy EUDC/Sell USDC: \( F^{(+, -)}: \Delta x \to \Delta y \) |
| Ask   | Buy EUDC/Sell USDC: \( F^{(+, -)}: \Delta y \to \Delta x \) | Buy USDC/Sell EUDC: \( F^{(-, +)}: \Delta x \to \Delta y \) |
Deposit USDC: \( F^{(+,-)} / \Delta y = \frac{s}{s(1-\epsilon)} \)

Deposit EUDC: \( F^{(-,+)} / \Delta x = \frac{1}{s(1-\epsilon)} \)

Redeem: \( F^{(-,-)} / \Delta y = \frac{1}{s(1-\epsilon)} \)

Redeem EUDC: \( F^{(+,-)} / \Delta x = \frac{s}{s(1-\epsilon)} \)

Table 4: Bid/Ask order quoting for constant product function

Figure 1: Left: the AMM rate to buy/sell EUDC as function of \( \Delta x \) USDC, \( \Delta x = [1, 2, \ldots, 100] \), with x-axis is pool utilization rate \( \Delta x / x_0 \). Right: the AMM rate to buy/sell USDC as function of \( \Delta y \) EUDC, \( \Delta y = [1, 2, \ldots, 100] \), with y-axis is pool utilization rate \( \Delta y / y_0 \). Initial parameters \( s = p_0 = 1.25 \) (EUR/USD FX spot) and \( 1/p_0 = 0.80 \) (USD/EUR FX spot), \( N_0 = 10000 \), \( \epsilon = 1bp \).

4.2 Constant product function

Constant product function specifies the following invariant for CFMM \( (2) \):

\[
F(x_1, y_1) \equiv sx_1y_1 - \Pi_0 = 0.
\]

The AMM will apply the CFMM to find the amount of USDC \( \Delta x \) redeemed by the trader in exchange for depositing \( \Delta y \) EUDC as follows:

\[
F^{(-,+)}(x_0 - \Delta x, y_0 + (1-\epsilon)\Delta y) = \Pi_0 / s.
\]

Accordingly, for the operation of redeeming a given amount of USDC \( \Delta x \) the amount of EUDC received \( \Delta y \) is found by:

\[
\Delta y = \frac{1}{s(x_0 - \Delta x)} \left( \frac{\Pi_0}{s(x_0 - \Delta x)} - y_0 \right).
\]

For the operation of depositing a given amount of EUDC \( \Delta y \), the amount of USDC \( \Delta x \) is set by:

\[
\Delta x = x_0 - \frac{\Pi_0}{s(y_0 + (1-\epsilon)\Delta y)}.
\]

Similarly, we apply the invariant for redeeming a given amount of EUDC \( \Delta y \). In Table 5, we provide the summary for order matching.

In Figure 2, we show the bid-ask rates for USDC and EUDC implied by the constant product AMM. The constant product rule makes large orders prohibitively expensive.

Table 5: Bid/Ask order matching for constant product function
Figure 2: Left: the AMM rate to buy/sell EUDC implied by the constant product AMM as function of pool utilization rate $\Delta x/x_0$. Right: the AMM rate to buy/sell USDC as function of pool utilization $\Delta y/x_0$. Initial parameters: $p_0 = 1.25$, $s = p_0$, $\Pi_0 = 10000$, $\epsilon = 1bp$.

| Quote | USDC | EUDC |
|-------|------|------|
| Bid   | Redeem: $\Delta \tilde{x} = x_0 - x_1$, $\Delta \tilde{y} = \frac{(y_1 - y_0)}{(1 - \epsilon)}$ | Redeem: $\Delta \tilde{x} = \frac{(x_1 - x_0)}{(1 - \epsilon)}$, $\Delta \tilde{y} = y_0 - \tilde{y}_1$ |
| Ask   | Deposit: $\Delta \tilde{x} = \frac{(x_1 - x_0)}{(1 - \epsilon)}$, $\Delta y = y_0 - \tilde{y}_1$ | Deposit: $\Delta \tilde{x} = x_0 - \tilde{x}_1$, $\Delta y = \frac{(y_1 - y_0)}{(1 - \epsilon)}$ |

Table 6: Bid/Ask order quoting for mixed rule AMM

4.3 Mixed rule function

Mixed rule function specifies the following invariant for CFMM [2]:

$$F(x_1, y_1) \equiv \left( \frac{\Pi_0}{sx_1y_1} - 1 \right) - \alpha \left( \frac{x_1 + sy_1}{\Sigma_0} - 1 \right) = 0,$$

(12)

where the pool composition is defined by sum and product:

$$x_0 + sy_0 = \Sigma_0, \quad sx_0y_0 = \Pi_0.$$

(13)

It is clear that the mixed rule is symmetric under the change $(x, sy) \rightarrow (sx, y)$.

If $x_1$ is given, Eq. (12) is solved for $\tilde{y}_1^{(\alpha)}(x_1)$:

$$\tilde{y}_1^{(\alpha)}(x) = \frac{1}{2\tilde{\alpha}} \left( -((1 - \alpha) + \tilde{\alpha}x) + \sqrt{D} \right), \quad D = ((1 - \alpha) + \tilde{\alpha}x)^2 + \frac{4\tilde{\alpha}\Pi}{x},$$

(14)

where $\tilde{\alpha} \equiv \alpha/\Sigma$. If $y_1$ is given, Eq. (12) is solved for $\tilde{x}_1^{(\alpha)}(y_1)$:

$$\tilde{x}_1^{(\alpha)}(y) = \frac{1}{2\tilde{\alpha}} \left( -((1 - \alpha) + \tilde{\alpha}sy) + \sqrt{D} \right), \quad D = ((1 - \alpha) + \tilde{\alpha}sy)^2 + \frac{4\tilde{\alpha}\Pi}{sy}.$$

(15)

To redeem given $\Delta x$ USDC from the pool by depositing $\Delta \tilde{y}$ EUDC, we solve Eq (14) for $\tilde{y}_1$ using $x_1 = x_0 - \Delta x$ and set:

$$\Delta x = x_0 - x_1, \quad \Delta \tilde{y} = \frac{(\tilde{y}_1 - y_0)}{(1 - \epsilon)}.$$

(16)

By analogy we consider the other 3 transactions and summarise the outputs in Table 6.

In Figure 3, we show the bid/ask rates as functions of order size relative to the pool liquidity using different values of the parameter $\alpha$. Here $\alpha = 0$ and $\alpha = \infty$ correspond to the sum and product rules, respectively. Using $\alpha$, we can control the marginal exchange rate as a function of order size.
Figure 3: Left: the AMM rate to buy/sell EUDC implied by the mixed rule AMM as function of pool utilization rate $\Delta x/x_0$. Right: the AMM rate to buy/sell USDC as function of pool utilization $\Delta y/x_0$. Initial parameters: $p_0 = 1.25$, $s = p_0$, $\Pi_0 = 10000$, $\epsilon = 1bp$.

| Operation            | Type    | USD cash       | EUR cash   | USDC | EUDC |
|----------------------|---------|----------------|------------|------|------|
| 1. Borrow USD / Buy EUR | OFF     | $-P_{E/U}\Delta y$ | $\Delta y$ | 0   | 0   |
| 2. Convert EUR to EUDC | OFF$\rightarrow$ON | $-P_{E/U}\Delta y$ | 0   | 0   | $\Delta y$ |
| 3. Redeem USDC/deposit EUDC | SC     | $-P_{E/U}\Delta y$ | 0   | $\Delta x$ | 0   |
| 4. Convert USDC to USD | ON$\rightarrow$OFF | $\Delta x - P_{E/U}\Delta y$ | 0   | 0   | 0   |

Table 7: Operations for arbitrage high USDC/EUDC rate implied by the pool. $P_{E/U}$ is the spot FX EUR/USD rate. Type OFF is off-chain, ON is on-chain, and SC is smart contract.

5 AMM arbitrage

The arbitrage of the pool exchange rate includes both off-chain transactions of buying/selling FX spots and on-chain redemption/deposit of DC.

5.1 Operations

We start with USDC/EUDC arbitrage which can be executed if the marginal rate of USDC/EUDC is above the FX rate. The arbitrageur will borrow USD cash to buy EUR cash off-chain, convert it to EUDC, exchange EUDC to USDC, and finally convert USDC back to USD cash to cover the initial USD margin to buy EUR cash. The operations are summarised in Table 7.

Conversely, EUDC/USDC can be executed if the marginal rate of EUDC/USDC is above the FX rate. The arbitrageur will borrow EUR cash to buy USD cash off-chain, convert it to USDC, exchange USDC to EUDC, and finally convert EUDC back to EUR cash to cover the initial EUR margin to buy USD cash. We can summarise the operations similarly to Table 7.

Pool arbitrageurs are necessary to adjust the pool compositions so that the AMM rate is fair and attractive for clients. AMM operators may incentivize arbitrageurs to enforce the pool’s price discovery through a sequence of on- and off-chain transactions.

5.2 Formulation

5.2.1 USDC arbitrage

For USDC arbitrage, we use the CFMM [3] to deposit given amount $\Delta y$ EUDC redeem $\Delta x$ USDC and deposit EUDC. We use operations in Table 7 to derive the arbitrage P&L as function of $\Delta y$:

$$\Omega(\Delta y; p_1, x_0, y_0) = \Delta x - p_1\Delta y,$$

s.t. $F(x_0 - \Delta x, y_0 + (1 - \epsilon')\Delta y) = 0,$

(17)
where $p_1$ is EUR/USD spot rate. Here we assume that the $e'$ is the fee paid by arbitrageur. The optimal arbitrage trade size is:

$$\Delta y^* = \arg\max_{\Delta y} \Omega(\Delta y; p_1, x_0, y_0),$$

(18)

and the rebalancing condition is $\Delta y^* > 0$.

### 5.2.2 EUDC arbitrage

For EUDC arbitrage, we use the CFMM to deposit given amount $\Delta x$ USDC to redeem $\Delta y$ USDC. We derive the arbitrage P&L as function of $\Delta x$ as follows:

$$\Omega(\Delta x; p_1, x_0, y_0) = p_1 \Delta y - \Delta x,$$

s.t. $F(x_0 + (1 - e')\Delta x, y_0 - \Delta y) = 0,$

(19)

where $p_1$ is EUR/USD spot rate. Here $e'$ is the fees to add EUUSD to the AMM pool with the total fees incurred by the arbitrage for all 4 operations in Table 7.

The optimal arbitrage trade size is:

$$\Delta x^* = \arg\max_{\Delta x} \Omega(\Delta x; p_1, x_0, y_0).$$

(20)

### 5.3 Arbitrage with Constant product rule

For USDC arbitrage, we use Eq(17) with Eq (11) for $\Delta y$:

$$\Omega(\Delta y) = \left( x_0 - \frac{\Pi_0}{s(y_0 + (1 - e')\Delta y)} \right) - p_1 \Delta y,$$

(21)

so that

$$\Delta y^* = \left( \sqrt{\frac{\Pi_0}{s(1 - e')p_1}} - \frac{y_0}{(1 - e')} \right).$$

(22)

Similarly for EUDC Arbitrage, we obtain the optimal trade size:

$$\Delta x^* = \left( \sqrt{\frac{p_1 \Pi_0}{s(1 - e')}} - \frac{x_0}{(1 - e'p_1)} \right).$$

(23)

Rebalancing condition $\Delta y^* > 0$ and $\Delta x^* > 0$ imply the following bands:

$$\frac{1}{(1 - e')} \frac{x_0}{y_0} < p_1 < (1 - e') \frac{x_0}{y_0}.$$  

(24)

### 5.4 Arbitrage with Mixed Rule

For USDC arbitrage, we use Eq(17) with $\Delta x$:

$$\Omega(\Delta y) = \left( x_0 - x_1^{(\alpha)}(y_0 + (1 - e')\Delta y) \right) - p_1 \Delta y,$$

(25)

where $x_1^{(\alpha)}(y)$ solves Eq (15).

The optimality condition is

$$\Omega'(\Delta y) = -(1 - e')\partial_y x_1^{(\alpha)}(y) - p_1,$$

(26)

where $y = y_0 + (1 - e')\Delta y$.

Accordingly, we solve for $y^*$ using Newton-Raphson:

$$y_{n+1}^* = y_n^* - \frac{\partial_y x_1^{(\alpha)}(y_n^*) + p_1/(1 - e')}{\partial_y^2 x_1^{(\alpha)}(y_n^*)},$$

(27)
Figure 4: Optimal P&L and rebalancing as function of \( \alpha \) using fee 1bp

and take

\[
\Delta y^* = \frac{(y_n^* - y_0^*)}{1 - \epsilon}.
\] (28)

Similarly, we derive the optimal trade size for EUxDC arbitrage.

In Figure 4, we show the optimal arbitrage P&L and rebalancing as function of \( \alpha \). The constant product rule is the most expensive for arbitrages, so the optimal trade must be small. High alpha values produce smaller marginal costs of using AMM so that the arbitrages can trade in larger sizes with higher profit potential.

6 Pool simulations using FX Data

We apply the actual FX data, assuming it is representative of DCs, for simulation of the dynamics of the pool, pool spreads, and P&L of liquidity providers.

6.1 FX Data and Normalization

We use FX data downloaded from the Dukascopy Bank SA platform. The FX data represent one-minute open-high-low-close quotes and traded volumes for both bid and ask trades. These bid/ask data are well suited for our purposes because we need to apply both the price and the volume data for realistic simulations of the FX pools. We chose one-minute bars because they would be representative of an average block mining time.

We consider each 24-hour trading session a complete life-cycle of the FX pool, independent of previous sessions. At the start of each trading session, the initial pool balances are reset, and, at the end of the trading session, selected variables of the pool activity are recorded. We use the last three years of the FX data as the most representative. As a result, we obtain 780 (= 3 \times 260) independent realizations of daily pool variables for each FX pool. Each 24-hour session has a total of 1440 (= 24 \times 60) data points.

We use 10 FX pairs for G-10 currencies and Chinese yuan including the following FX pairs: 'EURUSD', 'GBPUSD', 'USDJPY', 'USDCHF', 'AUDUSD', 'NZDUSD', 'USDCAD', 'USDNOK', 'USDSEK', 'USDCNH'.

For each trading session, we normalize the 1-minute close prices as follows:

\[
P^{mid}(t) = \frac{1}{2} (P^{bid}(t) + P^{ask}(t)), \quad p^i(t) = \frac{P^i(t)}{P^{mid}(0)}, \quad i = mid, bid, ask,
\] (29)

where capital \( P \) stands for natural prices and small \( p \) stands for normalized prices.
The natural bid and ask volumes represent the price-weighted volumes of sell and buy trades, respectively. We normalize the volume data for each session to add up to one as follows:

\[ \hat{V} = \left( \sum_t V^{\text{bid}}(t) + \sum_t V^{\text{ask}}(t) \right), \quad v^{\text{bid}}(t) = \frac{1}{V} V^{\text{bid}}(t), \quad v^{\text{ask}}(t) = \frac{1}{V} V^{\text{ask}}(t). \]  

(30)

where capital \( V \) stands for natural volumes and small \( v \) stands for normalized volumes. It follows that normalized bid and ask volumes represent the unit-based buy and sell trades. Thus, the total amount of daily volume sums up to 100%.

### 6.2 Pool Specification

For each FX pair, we set \( y \) to represent the pool balances for the foreign DC and \( x \) represent the pool balances for the domestic DC. The price \( p^{\text{mid}}(t) \) is set to be the market rate of exchange of one unit of the foreign currency to \( p^{\text{mid}}(t) \) units of the domestic currency. For example, for the ‘EURUSD’ FX pair, EUR (EUDC) is the foreign currency with the pool balances equal \( y \), and USD (USDC) is the domestic currency with the pool balance equal \( x \).

The trade volumes to redeem (buy) the foreign DC from the pool balances \( y \) are set to \( \Delta y^{\text{bid}}(t) = v^{\text{bid}}(t) \) with the matched volumes to deposit (sell) the domestic DC to pool balances \( x \) are set to \( \Delta x^{\text{ask}}(t) \), which is the output from the CFMM. Conversely, the trade volumes to redeem (buy) the foreign DC from the pool balances \( x \) is set to \( \Delta x^{\text{bid}}(t) = v^{\text{ask}}(t) \), with the matched volumes to deposit (sell) the foreign DC to the pool are set to \( \Delta y^{\text{ask}}(t) \), which is the output from the CFMM.

At each trading session, the pool balances are initialized as follows:

\[ x_0 = 1, \quad y_0 = 1, \quad s = 1, \quad \Pi_0 = sx_0y_0 = 1, \quad \Sigma_0 = x_0 + sy_0 = 2. \]  

(31)

Given that the initial FX rate for \( y \) prices in \( x \) is normalized to unity, the starting value of the pool (also known as total value locked or TVL) is 1. Because the normalized volumes add up to 1, we make an explicit assumption that the daily turnover of the pool is 100% (without accounting for intraday FX rate fluctuations) for each of the DC pools.

The actual pool capacity and pool turnover are expected to be varying on daily basis, so we keep the pool turnover as an endogenous variable set to unity. We assume that single liquidity providers the pool liquidity, so that the pool accrued fees and the P&L correspond to the liquidity provider with 100% ownership of the pool.

### 6.3 Intraday Simulation of the Pool

For each trading session, the time stamps \( t_n \) represent one-minute intervals. The 1-minute realizations of the intraday dynamics of the pool are simulated as follows.

1. **Initialization.** The pool is initialized at the beginning of each trading session using Eq (31).

2. **Order arrivals.** We assume that all bid and ask orders are aggregated into the respective single bid and ask order using normalized bid and ask trade volumes at \( n \)-th period. We assume one lag delay, so bid and ask orders are processed first using the pool data at \((n-1)\)-th period.

   The bid volumes for foreign DC \( \Delta y_n^{\text{bid}} = v_n^{\text{bid}} \) correspond to redeeming/buying foreign DC from balances \( y \) with matched deposit size \( \Delta x_n \) to domestic DC balance \( x \) processed using CFMM specified in Eq. (3):

   \[ F^{(+,-)}(x_{n-1} + (1-\epsilon)\Delta y_n^{\text{ask}}, y_{n-1} - \Delta y_n^{\text{bid}}) = 0, \]  

(32)

where \( \Delta y_n^{\text{bid}} = v_n^{\text{bid}} \).

The bid volumes for domestic DC \( \Delta x_n^{\text{bid}} = v_n^{\text{ask}} \) correspond to buying/redeeming domestic DC for the pool balances \( x \) with matched deposit size \( \Delta y_n^{\text{ask}} \) to foreign DC balance \( y \) processed using CFMM specified in Eq. (4):

\[ F^{(-,+)}(x_{n-1} - \Delta x_n^{\text{bid}}, y_{n-1} + (1-\epsilon)\Delta y_n^{\text{ask}}) = 0, \]  

(33)
where $\Delta x_n^{bid} = \Delta y_n^{ask}$. The pool composition at the end of period $n$ is then updated as follows:

$$
x_n = x_{n-1} - \Delta x_n^{bid} + \Delta x_n^{ask},
$$
$$
y_n = y_{n-1} - \Delta y_n^{bid} + \Delta y_n^{ask}.
$$

(34)

The marginal rate for the AMM bid/ask order fills is given by:

$$
p_{n}^{AMM,bid} = \frac{\Delta x_n^{bid}}{\Delta y_n^{ask}}, \quad p_{n}^{AMM,ask} = \frac{\Delta x_n^{ask}}{\Delta y_n^{bid}},
$$

(35)

We compute the bid/ask spread as follows:

$$
s_{n}^{AMM,bid} = p_{n}^{mid} - p_{n}^{AMM,bid}, \quad s_{n}^{AMM,ask} = p_{n}^{AMM,ask} - p_{n}^{mid}.
$$

(36)

3. Arbitrage Operation

The arbitrage is implemented by checking the optimal arbitrage trade sizes in Eq (18) and Eq (20) for the optimal arbitrage trade sizes:

$$
\Delta y_n^* = \text{argmax}_{y} \Omega(\Delta y; p_{n-1}^{mid}, x_n, y_n).
$$

$$
\Delta x_n^* = \text{argmax}_{x} \Omega(\Delta x; p_{n-1}^{mid}, x_n, y_n),
$$

(37)

using updated pool compositions in Eq (34). It is clear that either $\Delta y_n^* > 0$ or $\Delta x_n^* > 0$, but not both, so that the arbitrageur must only make a one-sided rebalancing and the matching sizes $\Delta x_n^*$ and $\Delta y_n^*$ are computed using the CFMM, respectively.

Further we assume that the arbitrage immediately implements the arbitrage adjustments (37) with the fill price for the FX spot transaction given by $p_{n}^{mid}$. Thus, the arbitrage profit as specified by Eq. (17) and (19) for the $n$-th period is computed by:

$$
\Omega(\Delta y_n^*; p_{n-1}^{mid}, x_n, y_n) = \Delta x_n^* - p_{n-1}^{mid} \Delta y_n^*,
$$

$$
\Omega(\Delta x_n^*; p_{n-1}^{mid}, x_n, y_n) = p_{n-1}^{mid} \Delta y_n^* - \Delta x_n^*.
$$

(38)

4. Update of the pool composition

At the end of each $n$-th period, the pool composition is updated by accounting for volume-driven rebalancing in Eq. (34) and the arbitrage-driven rebalancing in Eq (37):

$$
x_n = x_{n-1} + (-\Delta x_n^{bid} + \Delta x_n^{ask} + (\Delta x_n^* - \Delta x_n^*) ,
$$

$$
y_n = y_{n-1} + (-\Delta y_n^{bid} + \Delta y_n^{ask} + (\Delta y_n^* - \Delta y_n^*) .
$$

(39)

5. Record of pool variables at the end of the trading session

At the end of each trading session with $n = N$, we compute the following key variables.

- Volume-weighted average AMM bid-ask spread in basis points is computed by:

$$
s_N = \frac{10000}{2} \left( \sum_{n'=1}^{N} v_{n'}^{bid} s_{n'}^{AMM,bid} + \sum_{n'=1}^{N} v_{n'}^{ask} s_{n'}^{AMM,ask} \right),
$$

(40)

where $s_{n}^{AMM,bid}$ and $s_{n}^{AMM,ask}$ are defined in Eq. (36) and 10000 is the basis point scaler.

- Arbitrage profits are computed using Eq. (38):

$$
p^{arb}_N = \sum_{n'=1}^{N} \Omega(\Delta y_{n'}^*; p_{n'}^{mid}, x_{n'}, y_{n'}) + \sum_{n'=1}^{N} \Omega(\Delta x_{n'}^*; p_{n'}^{mid}, x_{n'}, y_{n'}). \quad (41)
$$
• Total pool fees are computed using pool trades set in Eq. (32) and (33):

\[ f_N = \epsilon \left( \sum_{n'=1}^{N} \Delta p^{ask}_{n'} + \sum_{n'=1}^{N} p^{mid}_{n'} \Delta p^{ask}_{n'} \right). \]  

(42)

• Total P&L for the pool liquidity provider (with 100% pool ownership) including the impermanent loss p&L and total fees is computed by:

\[ p&l_{N}^{lp} = (x_N + p^{mid}_N y_N) - (x_0 + p^{mid}_0 y_0) + f_N. \]  

(43)

• Total P&L for the pool liquidity provider with hedging including the impermanent loss p&L with hedging and total fees is computed by:

\[ p&l_{N}^{lp-hedged} = (x_N + p^{mid}_N y_N) - (x_0 + p^{mid}_0 y_0) + f_N. \]  

(44)

6.4 Illustration of Intraday dynamics

In Figure (5), we illustrate the simulated variables from the intraday simulation of the pool for the 'EURUSD' FX pair observed on 3rd June 2021 using mixed product CFMM with \( \alpha = 5 \) and fee \( \epsilon = 1bp \). We show the following variables.

• 1a). The FX spot is mid-price \( p^{mid}_n \) computed using Eq (29) from the market data. AMM Bid and AMM Ask are given by \( p^{AMM, bid}_n \) and \( p^{AMM, ask}_n \), respectively, computed using Eq. (35).

• 2a). Pool balances for USDC, \( x_n \), and and EUDC, \( y_n \), at the end of \( n \)-th period computed using Eq. (39).

• 3a). AMM Bid, \( s^{AMM, bid}_n \), and AMM Ask, \( s^{AMM, ask}_n \), spreads are computed using Eq. (36).

• 4a). Bid, \( v^{bid}_n \), and ask, \( v^{ask}_n \), volumes are computed using the given market data with Eq. (30).

• 1b). The intraday dynamics of the impermanent loss and hedged impermanent loss of the liquidity provider of the pool corresponds to Eq. (43) and (44), respectively, without accounting for total pool fees \( f_n \).

• 2b). The intraday dynamics of the total pool fees, \( f_n \), are computed using Eq (42).

• 3b). The intraday dynamics of the arbitrage P&L, \( p&l_{N}^{arb} \), is computed using Eq (41).

• 4b). The product of the pool balances is computed using \( \Pi_n = x_n y_n \) and the sum is computed using \( \Sigma_n = (x_n + y_n)/2 \).

7 Analysis

For each trading session we compute the four key realizations:

1. AMM Spread in bp is computed using Eq. (40).

2. Arbitrage P&L Annual % is computed using Eq. (41) with simple annualization of the daily p&l equal to 260\( p&l_{N}^{arb} \).

3. Pool P&L Annual % is computed using Eq. (43) with simple annualization of the daily p&l equal to 260\( p&l_{N}^{lp} \).

4. Pool Hedged P&L Annual % is computed using Eq. (44) with simple annualization of the daily p&l equal to 260\( p&l_{N}^{lp-hedged} \).

The simulation of the pool provides path realizations of independent key variables with the sample size of 780, which we apply for the sensitivity analysis. In particular, we analyze the sensitivity to the pool specification including parameter \( \alpha \) of the CFMM, pool fees \( \epsilon \), and pool liquidity.
Figure 5: Simulated intraday dynamics of the pool using 'EURUSD' data on 3rd June 2021.
7.1 USDC/EUDC pool

7.1.1 Impact of the CFMM parameter $\alpha$

In Figure (6), we show the realizations of the 4 key variables for 780 daily realizations of the USDC/EUDC pool as functions of the CFMM parameter $\alpha$. We show the sample statistics of the median and median absolute deviation (mad). For $\alpha = 0$, the AMM realized spreads are the highest, while the median of the arbitrage P&L, the pool P&L, and the pool hedged P&L are the highest, and so is the deviation. As expected, the volatility of the pool P&L weakly depends on $\alpha$, as the price dynamics mostly influence the P&L. The volatility of hedged P&L is reduced for different levels of $\alpha$.

In Figure (7), we show the results of the same simulation presented as the boxplot of realizations for crucial variables. From the simulation of the AMM spread, we can select an optimal alpha that minimizes the AMM spreads as costs to customers and, simultaneously, maximizes the expected P&L for pool liquidity providers. We find an acceptable value of $\alpha$ equal to 5 as an optimal trade-off that produces median AMM spread of 3bp, which breaks down to 1bp of direct fees and 2bp of the pool liquidity costs.

7.1.2 Impact of end-of-day return and realized volume variance

In Figure (8), we show the sensitivity of daily pool P&L and hedged P&L to the daily return of EURUSD mid-price and the intraday variance of bid-ask volume. It is evident that the impermanent loss and the daily pool P&L has a significant sensitivity with the beta to the realized return, with the estimate of the beta slightly above one as expected. On the other hand, the delta-hedged pool has an insignificant beta to the realized returns. What is also evident is that the daily P&L, in particular, for the hedged pool is significantly influenced by the intraday variance of bid and ask flows.

7.2 G-10 pools

In Figure (9), we show PDFs and boxplots of the core pool variables for the G-10 FX universe. We use $\alpha = 5.0$ for the mixed product AMM and a fee of $\epsilon = 1$bp. We see that PDFs and box plots of realized variables are close to each other. As a result, our approach is general and can be applied to different DC pools.
Figure 7: Boxplot of core pool variables for simulated USDC/EUDC pool as functions of CFMM $\alpha$, fee $\epsilon = 1$bp. Median is the path median and mad is the median absolute deviation.

Figure 8: The boxplot of core variables of the simulated USDC/EUDC pool as functions of pool liquidity multiple using $\alpha = 5$ and fee $\epsilon = 1$bp. Median value is shown inside the bars.

Electronic copy available at: https://ssrn.com/abstract=3939695
Figure 9: Simulated outputs of core pool variables for G-10 universe using $\alpha = 5.0$ and fee $\epsilon = 1$bp. Top panel: PDFs, bottom panel: boxplots with the median value shown.
8 Conclusions

Foreign exchange trading has been practiced since antiquity and continues to be extremely important in our
days. Canonical gospels of the New Testament eloquently describe how Jesus expelled the money changers from
the Temple. Modern-day descendants of the money changers are known as forex market makers. This paper
described an original, novel, and powerful approach to cleansing the Temple of high finance. Specifically, we
showed that using CaaS providers, such as Ethereum, Cardano, and Solana protocols, one can exchange fiat
currencies via AMMs, thus eliminating the need for intermediaries.

Space does not allow us to consider exchanges of fiat currencies into baskets of other fiat currencies, which
are helpful for trading blocks pursuing multinational endeavors such as digital trade coins. However, we shall
discuss such exchanges in a forthcoming paper.

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