CRITICAL EXponents FOR TOTAL POSITIVITY, INDIVIDUAL KERNEL ENCODERS, AND THE JAIN–KARLIN–SCHOENBERG KERNEL

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Abstract. We prove the converse to a result of Karlin [Trans. Amer. Math. Soc. 1964], and also strengthen his result and two results of Schoenberg [Ann. of Math. 1955]. One of the latter results concerns zeros of Laplace transforms of multiply positive functions. The other results study which powers $\alpha$ of two specific kernels are totally non-negative of order $p \geq 2$ (denoted $\text{TN}_p$); both authors showed this happens for $\alpha \geq p - 2$, and Schoenberg proved that it does not for $\alpha < p - 2$. We show more strongly that for every $p \times p$ submatrix of either kernel, up to a shift, its $\alpha$th power is totally positive of order $p$ ($\text{TP}_p$) for every $\alpha > p - 2$, and is not $\text{TN}_p$ for every $\alpha \in (0, p - 2) \setminus \mathbb{Z}$. In particular, these results reveal a ‘critical exponent’ phenomenon in the theory of total positivity. (The same critical exponent $(p - 2)$ was first discovered by FitzGerald–Horn in [J. Math. Anal. Appl. 1977] for positive semidefiniteness.) We also provide a characterization for Pólya frequency functions of order $p \geq 3$, following Schoenberg’s result for $p = 2$ in [J. d’Analyse Math. 1951]. Our proofs are self-contained, with three exceptions.

We further classify the powers preserving all $\text{TN}_p$ Hankel kernels on intervals, and isolate individual kernels encoding these powers. We then transfer results on preservers by Pólya–Szegő (1925), Loewner/Horn (1969), and Khare–Tao (2017), from positive semidefinite matrices to Hankel $\text{TN}_p$ kernels. An additional application of the proofs is to construct individual matrices that encode the Loewner convex powers. This complements Jain’s results (2020) for Loewner positivity, which we strengthen to total positivity, with self-contained proofs. Remarkably, these (strengthened) results of Jain, those of Schoenberg and Karlin, the latter’s converse, and the aforementioned individual Hankel kernels all arise from a single symmetric rank-two kernel and its powers: max$(1 + xy, 0)$.

Contents

1. Introduction and main results
2. A variant of Descartes’ rule of signs, and homotopy arguments
3. Proof of Theorem A: Critical exponent for PF functions and sequences
4. Hankel $\text{TN}_p$ kernels: preservers, critical exponent, and Theorem B
5. Theorem C: Critical exponent for the Jain–Karlin–Schoenberg kernel
6. Theorem D: Laplace transform of a compactly supported $\text{TN}_p$ function
7. Theorem E: Characterizing $\text{TN}_p$ functions
References
Appendix A. Proofs from previous papers

Notation:
(1) A positive semidefinite matrix is a real symmetric matrix with non-negative eigenvalues. Given $I \subset \mathbb{R}$ and $n \geq 1$, denote the space of such $n \times n$ matrices with entries in $I$ by $\mathbb{P}_n(I)$.
(2) The Loewner ordering on $\mathbb{P}^{n \times n}$ is the partial order where $M \succeq N$ if and only if $M - N \in \mathbb{P}_n$.
(3) Following Schur [32], a function $f : I \to \mathbb{R}$ acts entrywise on $\mathbb{P}_n(I)$ via: $f[A] := (f(a_{jk}))_{j,k=1}^n$.

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(4) We say that a map \( f : I \to \mathbb{R} \) preserves \textbf{Loewner positivity} on \( \mathbb{P}_n(I) \) if \( f[A] \geq 0 \) for all \( A \in \mathbb{P}_n(I) \), i.e., for \( A \succeq 0 \).

(5) We will adopt the convention \( 0^0 := 1 \).

\textbf{Definition.} Let \( X, Y \) be totally ordered sets, and \( p \geq 1 \) an integer.

(1) Define \( X^{p,\uparrow} \) to be the set of all \( p \)-tuples \( x = (x_1, \ldots, x_p) \in X \) with strictly increasing coordinates: \( x_1 < \cdots < x_p \). (In his book \( [21] \), Karlin denotes this open simplex by \( \Delta_p(X) \).)

(2) A kernel \( K : X \times Y \to \mathbb{R} \) is \textbf{totally non-negative of order} \( p \), denoted \( \text{TN}_p \), if for all integers \( 1 \leq r \leq p \) and \( p \)-tuples \( x \in X^{r,\uparrow}, y \in Y^{r,\uparrow} \), the determinant of the matrix

\[
K[x; y] := (K(x_j, y_k))_{j,k=1}^{r}
\]

is non-negative. We say \( K \) is \textbf{totally non-negative (TN)} if \( K \) is \( \text{TN}_p \) for all \( p \geq 1 \).

(3) Analogously, one defines \( \text{TP}_p \) and \( \text{TP} \) kernels. If the domains \( X, Y \) are both finite, then this yields \( \text{TN}_p, \text{TN}, \text{TP}_p, \) or \( \text{TP} \) matrices.

\section{1. Introduction and main results}

In recent joint works \([1, 3]\), we explored the preservers of various classes of positive semidefinite, TN, and \( \text{TP} \) kernels on infinite domains – as well as the preservers of \( \text{TN}_p \) and \( \text{TP}_p \) kernels on finite domains. The present paper studies preservers of \( \text{TN}_p \) kernels, albeit on infinite domains – with emphasis on power functions. In doing so, we end up bringing under this roof, several old and new results on powers preserving Loewner positivity, monotonicity, and convexity as well.

Positive semidefinite matrices, totally positive (TP) matrices, and operations preserving these structures have been widely studied in the literature. More generally, the same question applies to post-composition operators applied to (structured) kernels with the various notions of positivity. An important class of totally non-negative (TN) kernels that has been widely studied in analysis, interpolation theory, differential equations, probability and statistics, combinatorics, and other areas consists of the Pólya frequency functions and sequences \([30]\). More generally, TN and \( \text{TP} \) matrices occur in multiple areas of mathematics, ranging from the aforementioned fields to representation theory, cluster algebras, interacting particle systems, and Gabor analysis. We refer the reader to the survey \([2]\) and references therein – and specifically, to the comprehensive book of Karlin \([23]\) – for more on TN and \( \text{TP} \) matrices and kernels.

\subsection{1.1. The critical exponent \( n - 2 \) in positivity}

A well-studied theme in the matrix positivity literature involves entrywise real powers acting on matrices (say with positive entries), to preserve positive (semi)definiteness or other Loewner properties. This theme owes its origins to Loewner, who was interested in understanding (in connection with the Bieberbach conjecture) which entrywise powers preserve positive semidefiniteness. This was resolved by FitzGerald and Horn:

\textbf{Theorem 1.1} (FitzGerald and Horn, 1977, \([3]\)). Let \( n \geq 2 \) be an integer and \( \alpha \in \mathbb{R} \).

(1) The entrywise map \( x^\alpha \) preserves Loewner positivity on \( \mathbb{P}_n((0, \infty)) \) if and only if \( \alpha \in \mathbb{Z}_{>0} \cup [n - 2, \infty) \).

(2) The entrywise map \( x^\alpha \) preserves Loewner monotonicity on \( \mathbb{P}_n((0, \infty)) \) if and only if \( \alpha \in \mathbb{Z}_{>0} \cup [n - 1, \infty) \). Here, we say a map \( f : I \to \mathbb{R} \) is \textbf{Loewner monotone on} \( \mathbb{P}_n(I) \) if \( f[A] \geq f[B] \) whenever \( A \succeq B \) in \( \mathbb{P}_n(I) \).

This phase transition at \( \alpha = n - 2 \) for positivity preservers (resp. \( \alpha = n - 1 \) for monotonicity preservers) is known as a \textbf{critical exponent} in the matrix analysis literature. See \([21]\) for a survey of the early history of this phenomenon. More recently, a plethora of papers have studied Loewner positive entrywise powers on the domain \( I = (0, \infty) \) or \( \mathbb{R} \), and on test sets of positive matrices constrained by rank and sparsity \([1, 11, 12, 13, 18, 19]\). These have yielded similar critical exponents (including a ‘combinatorial’ one for every graph \([12, 13]\)).
In fact the earliest occurrence of this critical exponent \((n - 2)\) – in the positive semidefiniteness literature – was in Horn’s 1969 article \[16\]. Horn began with an important result of Loewner on continuous maps preserving Loewner positivity on \(\mathbb{P}_n((0, \infty))\) (which remains essentially the only known necessary condition to date, for such maps in fixed dimension) – see Theorem 4.6. From this, Horn deduced the ‘only if’ part of Theorem 1.1(1): for \(\alpha \in (0, n - 2) \setminus \mathbb{Z}\), there exists a matrix \(A_\alpha \in \mathbb{P}_n((0, \infty))\) such that \(A_\alpha^\alpha\) is not positive semidefinite. Horn’s proof was non-constructive; moreover, such a ‘counterexample’ matrix \(A_\alpha\) would \textit{a priori} depend on \(\alpha\), as is also the case in the proof of Theorem 1.1(1),(2). This dependence was recently removed, as we explain presently.

1.2. The critical exponent \(n - 2\) in total positivity. At almost the same time\footnote{In fact, also at the same place (Stanford University); Karlin, Loewner, Pólya, and Szegő had been colleagues, and FitzGerald and Horn were Loewner’s students.} as Horn’s aforementioned article containing Loewner’s result, Karlin had completed his monograph \[23\] on total positivity. One can find in it the same set of powers as above – now acting on a certain Pólya frequency function. In this case, however, Karlin showed (originally in his 1964 paper \[22\]) the ‘reverse’ direction to Horn above:

\textbf{Theorem 1.2} (Karlin, \[22\] – see also \[23, Ch. 4, §4, p. 211\]). Let \(p \geq 2\) be an integer and \(\alpha \geq 0\). Define the Pólya frequency function

\[
\Omega(x) := \begin{cases} 
xe^{-x}, & \text{if } x > 0, \\
0, & \text{otherwise.}
\end{cases}
\]

(1.3)

If \(\alpha \in \mathbb{Z}^{\geq 0} \cup [p - 2, \infty)\), then the function \(\Omega(x)^\alpha\) is \(\text{TN}_p\).

In particular, for every integer \(\alpha > 0\), the function \(\Omega^\alpha\) is a Pólya frequency function – this was originally shown by Schoenberg in 1951 \[30\]. We explain the notation used here and in the sequel:

\textbf{Definition 1.4.} Let \(p \geq 1\) be an integer, and \(\Lambda : \mathbb{R} \to \mathbb{R}\) a Lebesgue measurable function.

1. We say \(\Lambda\) is a \textit{Pólya frequency function} if \(\Lambda\) is Lebesgue integrable on \(\mathbb{R}\), the associated Toeplitz kernel

\[T_\Lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (x, y) \mapsto \Lambda(x - y)\]

is totally non-negative, and \(\Lambda\) does not vanish at least at two points (whence on an interval).

2. We say \(\Lambda\) is \textit{totally non-negative of order} \(p \geq 1\), again denoted \(\text{TN}_p\), if \(T_\Lambda\) is \(\text{TN}_p\). If \(\Lambda\) is \(\text{TN}_p\) for all \(p \geq 1\), then we say \(\Lambda\) is \textit{totally non-negative} (\(\text{TN}\)).

3. Analogously, one defines \(\text{TP}_p\) and \(\text{TP}\) functions.

Karlin’s result is at least the second instance of a critical exponent phenomenon, implicit in the theory of total positivity. Almost a decade earlier, Schoenberg had shown a similar result for powers of a seemingly unrelated kernel, which he termed \textit{Wallis distributions}:

\textbf{Theorem 1.5} (Schoenberg, 1955, \[30\] Theorems 4 and 5]). Define the map

\[W : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} \cos(x), & \text{if } x \in (-\pi/2, \pi/2), \\
0, & \text{otherwise.}
\end{cases}\]

(1.6)

Also suppose \(\alpha \geq 0\) and an integer \(p \geq 2\). Then \(W(x)^\alpha\) is \(\text{TN}_p\) if and only if \(\alpha \geq p - 2\).

(The ‘only if’ part implicitly follows from \[31\] Theorem 4 and was not formulated. We write out how this can be achieved, in Remark 6.2.) Thus, Schoenberg’s result shows a critical exponent phenomenon from total positivity – with the same point \(p - 2\) for a \(\text{TN}_p\) kernel, as for positivity preservers on \(p \times p\) matrices.

In parallel: note that Karlin did not address the non-integer powers below \(p - 2\). We begin by achieving this task, and showing that \(\alpha = p - 2\) is indeed a ‘critical exponent’ for total positivity:
Theorem 1.7. Let $p \geq 2$ be an integer and $\alpha \in (0, p-2) \backslash \mathbb{Z}$. Then $\Omega^\alpha$ is not TN$_p$.

One consequence is that there also exists a sequence of Pólya frequency sequences\footnote{Recall, Pólya frequency sequences are defined to be real sequences $a = (a_n)_{n \in \mathbb{Z}}$ such that the Toeplitz kernel $T_a : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ sending $(m, n) \mapsto a_{m-n}$ is TN.} whose $\alpha$th powers are not TN$_p$ for $\alpha \in (0, p-2) \backslash \mathbb{Z}$. This follows from the continuity of the kernel $\Omega$, via a discretization argument as in our recent joint work\footnote{Briefly, the bilateral Laplace transform of $\Omega^\alpha$ is $\Gamma(\alpha+1)/(s+\alpha)^{\alpha+1}$, and if $\alpha \notin \mathbb{Z}^\alpha$ then its reciprocal is not analytic in $s$ – not in the Laguerre–Pólya class. Thus $\Omega^\alpha$ is not a Pólya frequency function by \cite{Schoenberg}, whence not TN.} by discretization. The assertion can be strengthened to show the existence of TN Toeplitz kernels on more general domains $X \times Y$ than $\mathbb{Z} \times \mathbb{Z}$. These subsets $X, Y$ only need to satisfy: for each $p \geq 1$, there exist equi-spaced arithmetic progressions $x \in X^{p,\uparrow}$ and $y \in Y^{p,\uparrow}$ with $x_2 - x_1 = y_2 - y_1$. A similar argument works for Schoenberg’s powers $W(x)^\alpha$.

Thus, Theorems 1.5 and 1.7 say that for each $\alpha \in (0, p-2) \backslash \mathbb{Z}$, one can find tuples $x, y \in R^{p,\uparrow}$ (or in $Q^{p,\uparrow}$ via discretization), for which the Toeplitz matrices $T_{\Omega^\alpha}[x; y]$ and $T_{W}[x; y]$ each contain a negative minor. Our first main result strengthens this by showing that the above condition is satisfied up to a shift at every pair $x, y$, and simultaneously for all powers $\alpha \in (0, p-2) \backslash \mathbb{Z}$:

Theorem A. Fix an integer $p \geq 2$ and subsets $X, Y \subset \mathbb{R}$ of size at least $p$.

1. There exists $a = a(X, Y) \in \mathbb{R}$ such that the restriction of $T_{\Omega^\alpha}(x, y)^\alpha$ to $X \times Y$ (where $\Omega_a(x) = \Omega(x - a)$), is not TN$_p$ for all $\alpha \in (0, p-2) \backslash \mathbb{Z}$.

2. There exists $m = m(X, Y) \in (0, \infty)$ such that the restriction of $T_{W mn}(x, y)^\alpha$ to $X \times Y$ (where $W_m(x) = W(mx)$), is not TN$_p$ for all $\alpha \in (0, p-2) \backslash \mathbb{Z}$.

3. Given tuples $x, y \in R^{p,\uparrow}$, there exist $\alpha \in \mathbb{R}$ and $m > 0$ such that the matrices

\[
(\Omega(x_j - y_k - a)^\alpha^p)_{j,k=1} \quad \text{and} \quad (W(mx_j - y_k)^\alpha)^p_{j,k=1}
\]

are TP if $\alpha > p-2$, TN if $\alpha \in \{0, 1, \ldots, p-2\}$, and not TN if $\alpha \in (0, p-2) \backslash \mathbb{Z}$.

Note that the additive/multiplicative shifts $a = a(x, y)$ and $m = m(x, y)$ are independent of $\alpha \in (0, p-2) \backslash \mathbb{Z}$. Hence so are $a(X, Y), m(X, Y)$.

Remark 1.8. The first two assertions in Theorem A(3) strengthen Karlin’s theorem 1.2 and one implication in Schoenberg’s theorem 1.5\footnote{Our proof of Theorem A below is self-contained, shows a stronger result, and avoids these sophisticated tools.} on a suitable part of their domains. The final assertion in Theorem A(3) is the aforementioned strengthening of the ‘converse’ Theorem 1.7 (and of the other implication in Theorem 1.5), and follows from parts (1) and (2) by specializing $X, Y$ to the sets of coordinates of $x, y$ respectively.

Theorems 1.7 and A lead to a Pólya frequency function whose non-integer powers are not TN:

Corollary 1.9. If $\alpha \geq 0$ and the function $\Omega(x)^\alpha$ is TN, then $\alpha$ is an integer.

This was observed e.g. in \cite{Schoenberg}, where the ‘heavy machinery’ of the bilateral Laplace transform was used through deep results of Schoenberg \cite{Schoenberg} \footnote{This was observed e.g. in \cite{Jain}, where the ‘heavy machinery’ of the bilateral Laplace transform was used through deep results of Schoenberg \cite{Schoenberg}.}. Our proof of Theorem A below is self-contained, shows a stronger result, and avoids these sophisticated tools.

Thus, our first contribution shows that critical exponents for total non-negativity – more strongly, ‘total positivity’ phenomena – occur in the study of preservers of Pólya frequency functions, Pólya frequency sequences, and Toeplitz kernels on more general domains, in the above (strengthened) results by Schoenberg and Karlin and their converses – and for all submatrices, up to a shift.

1.3. Single-matrix encoders; Hankel kernels. As seen above, Schoenberg and Karlin studied individual kernels, for which all powers $\geq p-2$ preserve TN$_p$, and no non-integer power $< p-2$ does so. In close analogy with the FitzGerald–Horn theorem 1.1\footnote{In the latter, parallel setting of entrywise powers preserving positivity, such individual matrices were discovered only recently, by Jain \cite{Jain}. Her results are now stated in parallel to Theorem 1.1 and isolate a smallest possible test set for Loewner positive and monotone powers: } the first two assertions in Theorem A(3) strengthen Karlin’s theorem 1.2 and one implication in Schoenberg’s theorem 1.5\footnote{Theorem A(3) is the aforementioned strengthening of the ‘converse’ Theorem 1.7 (and of the other implication in Theorem 1.5), and follows from parts (1) and (2) by specializing $X, Y$ to the sets of coordinates of $x, y$ respectively.} on a suitable part of their domains. The final assertion in Theorem A(3) is the aforementioned strengthening of the ‘converse’ Theorem 1.7 (and of the other implication in Theorem 1.5), and follows from parts (1) and (2) by specializing $X, Y$ to the sets of coordinates of $x, y$ respectively.
Theorem 1.10 (Jain, 2020, [19]). Let \( n \in \mathbb{Z}, n \geq 2 \) and \( \alpha \in \mathbb{R} \). Suppose \( x_1, \ldots, x_n \in \mathbb{R} \) are pairwise distinct, with \( 1 + x_j x_k > 0 \) \( \forall j, k \). Let \( A := (1 + x_j x_k)_{j,k=1}^n \) and \( B := I_{n \times n} \), so \( A \geq B \geq 0 \).

1. The matrix \( A^{\alpha} \) is positive semidefinite if and only if \( \alpha \in \mathbb{Z}^+ \cup [n-2, \infty) \).
2. Suppose all \( x_j \) are non-zero. The matrix \( A^{\alpha} \) is \( B^{\alpha} \), if and only if \( \alpha \in \mathbb{Z}^+ \cup [n-1, \infty) \).

In fact Jain does more in [18, 19]: he computes the inertia of the matrices \( A^{\alpha} \) above, for all real \( \alpha \geq 0 \). Our main theorem \( \mathbf{C} \) below strengthens Theorem 1.10(1), and shows that \( A^{\alpha} \) is not just positive definite for \( \alpha > n-2 \), but totally positive. In particular, as can be shown using Perron’s theorem [26] and the folklore theorem of Kronecker on eigenvalues of compound matrices, \( A^{\alpha} \) has simple, positive eigenvalues for \( \alpha > n-2 \), parallel to Jain.

Before proceeding further, we describe two consequences of the first part of Jain’s theorem 1.10:

1. Set \( x_j := \cot(j \pi/(2n)) \); now \( A^{\alpha} \) is positive semidefinite if and only if so is the matrix

\[
D^{\alpha} A^{\alpha} D^{\alpha} = (DAD)^{\alpha},
\]

where \( D \) is the diagonal matrix with \( (j, j) \) entry \( \sin(j \pi/(2n)) \). But \( DAD \) is the Toeplitz matrix \( (\cos((j-k)\pi/(2n)))_{j,k=1}^n \), so Jain’s result yields a rank-two positive semidefinite Toeplitz matrix which encodes the Loewner positive powers on \( \mathbb{P}_n((0, \infty)) \). Notice this is a restriction of Schoenberg’s kernel \( T_1 \) from Theorem 1.5.

2. Setting \( x_j := u_0^j \) for \( u_0 \in (0, \infty) \setminus \{1\} \), it follows that \( A \) is a rank-two positive semidefinite Hankel matrix, which encodes the Loewner positive and monotone powers on \( \mathbb{P}_n((0, \infty)) \).

This second consequence leads to our next theorem. Recall that Karlin and Schoenberg’s results above, together with Theorem 1.1 studied Toeplit kernels which encoded the (non-integer) powers preserving \( \text{TN}_p \). We next produce a Hankel kernel with this property. Unfortunately, the naive guess of \( K(x, y) = (x+y)^{-\alpha} \) does not work, since this is ‘equivalent’ to \( T_1(x, -y) \), which leads to ‘row-reversal’ and hence a sign of \( (-1)^{p(p-1)/2} \) in \( p \times p \) submatrices drawn from \( K \). (As a specific instance, \( \det T_1[3, 4]; (-2, -1] < 0 \).) However, the ‘rank-two’ kernel \( 1 + u_0^{x+y} \) is TN and exhibits the same critical exponent phenomenon. More strongly, this kernel encodes the powers preserving \( \text{TN}_p \) for all Hankel kernels on \( \mathbb{R} \times \mathbb{R} \) – in other words, the analogues of Theorems 1.1 and 1.10 hold together, for Hankel kernels on \( \mathbb{R} \times \mathbb{R} \). Slightly more strongly, this happens over arbitrary intervals:

**Theorem B.** Let \( p \geq 2 \) be an integer, and fix scalars \( c_0, u_0 > 0, u_0 \neq 1 \) and \( \alpha \geq 0 \). Also fix an interval \( X_0 \subset \mathbb{R} \) with positive measure. The following are equivalent:

1. If \( X \subset \mathbb{R} \) is an interval with positive measure, and \( H : X \times X \to \mathbb{R} \) is a continuous \( \text{TN}_p \) Hankel kernel, then \( H^{\alpha} \) is \( \text{TN}_p \). Here, by a Hankel kernel we mean \( K : X \times X \to \mathbb{R} \) such that there exists a function \( f : X \times X \to \mathbb{R} \) satisfying \( K(x, y) = f(x+y) \) for \( x, y \in X \).
2. Define the Hankel kernel

\[
H_{u_0}^\alpha : X_0 \times X_0 \to \mathbb{R}, \quad (x, y) \mapsto 1 + c_0 u_0^{x+y}.
\]

Then \( H_{u_0}^\alpha \) is \( TP \) on \( X_0 \times X_0 \).
3. \( \alpha \in \mathbb{Z}^+ \cup [p-2, \infty) \).

In particular, every \( \alpha \in \mathbb{Z}^+ \cup [p-2, \infty) \) preserves \( \text{TN} \) Hankel kernels. Moreover, for every \( x, y \in X_0^{p,\dagger} \), the kernel \( H_{u_0}^\alpha \) is \( TP \) if \( \alpha > p-2 \), and not \( TP \) if \( \alpha \in (0, p-2) \setminus \mathbb{Z} \).

This strengthens results in recent work [1, 3], which study powers preserving \( \text{TN}_p \) Hankel kernels. Theorem 1.3 studies power preservers of \( \text{TN}_p \) Hankel kernels, for each \( p \geq 2 \).

### 1.4. The Jain–Karlin–Schoenberg kernel

Our next main result again concerns power-preservers of \( \text{TN}_p \) kernels. We show that remarkably, the multitude of kernels studied above are all related. More precisely, Karlin’s theorem 1.2 and our converse, Schoenberg’s theorem 1.5, the FitzGerald–Horn theorem 1.1, Jain’s theorem 1.10(1), the aforementioned strengthenings of these, the Hankel
kernels $H_n$, and the related critical exponent phenomena all arise from studying a particular symmetric kernel having ‘rank two’ (on part of its domain) – restricted to various sub-domains. In particular, this will explain why the same critical exponent of $p - 2$ (plus, all powers above $p - 2$, and no non-integer power below it) shows up in each of these settings.

We begin by introducing this simple kernel:

**Definition 1.11.** Define the Jain–Karlin–Schoenberg kernel $K_{JKS}$ as follows:

$$K_{JKS} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad (x, y) \mapsto \max(1 + xy, 0).$$

(1.12)

The choice of name is because – as we explain in Remark 5.2 – the restrictions of this kernel to $(-\infty, 0] \times (0, \infty)$, to $(0, \infty) \times (0, \infty)$, and on the full domain $\mathbb{R}^2$, are intimately related to Karlin’s kernel $\Omega$, to Jain’s matrices $(1 + x_jx_k)$, and to Schoenberg’s cosine-kernel $W$, respectively.

Our next result studies the powers of $K_{JKS}$ that are TN$_p$ on the plane or on the $X$ or $Y$ half-planes. Remark 5.2 will then explain how it connects to all of the results stated above.

**Theorem C.** Fix an integer $p \geq 2$, an interval $I \subset \mathbb{R}$, and let a scalar $\alpha \geq 0$.

1. $K^\alpha_{JKS}$ is TN$_p$ on $\mathbb{R} \times \mathbb{R}$ for $\alpha \geq p - 2$.

2. If the power $K^\alpha_{JKS}$ is TN$_p$, then $\alpha \in \mathbb{Z}_{\geq 0} \cup [p - 2, \infty)$. More strongly, given $x, y \in \mathbb{R}^p \uparrow$ such that $1 + x_jy_k > 0 \forall j, k$, the matrix $K_{JKS}^\alpha[x; y]$ is:
   - TP if $\alpha > p - 2$;
   - TN if $\alpha \in \{0, 1, \ldots, p - 2\}$; and
   - not TN if $\alpha \in (0, p - 2) \setminus \mathbb{Z}$.

3. Suppose $I \subset [0, \infty)$ or $I \subset (-\infty, 0]$. The kernel $K^\alpha_{JKS}$ is TN$_p$ on $I \times \mathbb{R}$ (or $\mathbb{R} \times I$) if and only if $\alpha \in \mathbb{Z}_{\geq 0} \cup [p - 2, \infty)$.
   (In particular, $K^\alpha_{JKS}$ is TN on $I \times \mathbb{R}$ or $\mathbb{R} \times I$ for $\alpha \in \mathbb{Z}_{\geq 0}$.)

As an aside, integer powers of the kernel $K_{JKS}$ (more precisely, of $1 + xy$) have featured in the statistics and machine learning literature, as non-homogeneous polynomial kernels of dot-product type. See e.g. [15, 17, 25, 34, 35].

Our final two results deal with (a) more general TN$_p$ functions than powers, and (b) general kernels. A closely related result to Theorem C is a 1955 theorem by Schoenberg [31], which implies that no power $\alpha < p - 2$ of the kernel $W$ is TN$_p$. This is a result on arbitrary compactly supported, multiply positive functions $\Lambda$, and we strengthen it by restricting the domain of $\Lambda$:

**Theorem D.** Suppose $0 < \rho < \rho \leq +\infty$ and $0 < \epsilon \leq \rho - \rho / 2$ are scalars, with $\rho < \infty$. Suppose $p \geq 2$ is an integer, and the integrable function $\Lambda : (-\rho, \rho) \rightarrow \mathbb{R}$ is positive on $(-\rho/2, \rho/2)$, vanishes outside $[-\rho/2, \rho/2]$, and induces the TN$_p$ kernel

$$T_{\Lambda} : [0, \epsilon) \times (-\rho/2, (\rho/2) + \epsilon) \rightarrow \mathbb{R}, \quad (x, y) \mapsto \Lambda(x - y).$$

Then the Fourier–Laplace transform

$$\mathcal{B}\{\Lambda\}(s) := \int_{-\rho/2}^{\rho/2} e^{-sx} \Lambda(x) \, dx, \quad s \in \mathbb{C}$$

has no zeros in the strip $|\Im(s)| < p\pi / \rho$.

Schoenberg proved this result in [31], assuming $\rho = +\infty$ and that $T_{\Lambda} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is TN$_p$. (He also ‘changed variables’ so that $\rho = \pi$.) This means that all minors of order $\leq p$ drawn from $\Lambda$ are required to be non-negative. We arrive at the same conclusions as Schoenberg, using far fewer minors – indeed, the aforementioned domain of $T_{\Lambda}$ means that we only need to work with the restriction of $\Lambda$ to $(-\rho/2 - \epsilon, (\rho/2) + \epsilon)$.

Our final result provides a characterization of TN$_p$ functions (or Pólya frequency functions of order $p$). Recall that such a result was shown for $p = 2$ by Schoenberg in 1951 [30], and Weinberger mentioned in 1983 a variant for $p = 3$ in [36]. To our knowledge, no such characterization is known for $p \geq 4$. This is provided by the next result, by considering only the largest-sized minors:
Theorem E. Let $p \geq 3$ be an integer, and a function $\Lambda : \mathbb{R} \to [0, \infty)$. The following are equivalent.

1. Either $\Lambda(x) = e^{ax+b}$ for $a, b \in \mathbb{R}$, or: (a) $\Lambda$ is Lebesgue measurable; (b) for all scalars $x_0, y_0$, the function $\Lambda(x_0 - y)\Lambda(y - y_0) \to 0$ as $y \to \infty$; and (c) $\det T_\Lambda(x; y) \geq 0$ for all $x, y \in \mathbb{R}^{p\times1}$.

2. The function $\Lambda : \mathbb{R} \to \mathbb{R}$ is $T_{\mathbb{N}_p}$.

The result also holds for $p = 2$, in which case it is a tautology. The proof-technique also yields similar results for ‘Pólya frequency sequences of order $p$’ or more generally, for (not necessarily Toeplitz) $T_{\mathbb{N}_p}$ kernels on $X \times Y$ for general subsets $X, Y \subset \mathbb{R}$ – under similar decay assumptions. See the final section of the paper.

Organization of the paper. The next section develops a few preliminaries – specifically, novel homotopy arguments that are used in our proofs. The subsequent five sections of the paper prove our main theorems, one per section. The first three of these sections contain three other features: (a) After proving Theorem A, we show akin to Jain’s theorem 1.10 that the same ‘individual’ matrices $(1 + x_j x_k)_{j,k=1}^n$ and $1_{n \times n}$ also encode the powers preserving Loewner convexity. See Section 3.1 for the definition of Loewner convexity as well as the precise result. (b) After proving Theorem B, we present results – now for Hankel $T_{\mathbb{N}_p}$ kernel preservers – parallel to Loewner’s aforementioned necessary condition in [16], to an old observation of Pólya–Szegő [28], and to our recent work with Tao [24] on polynomial preservers of positivity on $p \times p$ matrices. (c) After proving Theorem C, we explain in Remark 5.2 how this result for $K_{J,KS}$ subsumes our results above, as well as those of Karlin, Jain, and some of Schoenberg. The Appendix contains proofs of several results pertaining to power-preservers of $T_{\mathbb{N}_p}$, in order to keep the present paper mostly self-contained.

2. A variant of Descartes’ rule of signs, and homotopy arguments

The proofs of the above results rely on new tools and old. We begin with a variant from [19] of Descartes’ rule of signs, in which exponentials are replaced by powers $(1 + u x_j)^r$. To state this result requires the following notation.

Definition 2.1. Given an integer $n \geq 1$ and a tuple $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define

$$A_x := \begin{cases} -\infty, & \text{if } \max_j x_j \leq 0, \\ -1/\max_j x_j, & \text{otherwise,} \end{cases} \quad B_x := \begin{cases} \infty, & \text{if } \min_j x_j \geq 0, \\ -1/\min_j x_j, & \text{otherwise.} \end{cases}$$

Proposition 2.2 (Jain, [19]). Fix an integer $n \geq 1$ and real tuples $c = (c_1, \ldots, c_n) \neq 0$ and $x = (x_1, \ldots, x_n)$, where the $x_j$ are pairwise distinct. For a real number $r$, define the function

$$\varphi_{x,c,r} : (A_x, B_x) \to \mathbb{R}, \quad u \mapsto \sum_{j=1}^n c_j (1 + u x_j)^r.$$

Then either $\varphi_{x,c,r} \equiv 0$, or it has at most $n - 1$ zeros, counting multiplicities.

In the interest of keeping this paper self-contained, we sketch this proof in a somewhat vestigial Appendix, together with proofs of the results from other works that are used in this paper.

The next step is a (novel) homotopy argument for symmetric matrices; a non-symmetric variant will also be proved and used below.

Proposition 2.3. Fix an integer $n \geq 2$ and real scalars $x_1 < \cdots < x_n$ and $0 < y_1 < \cdots < y_n$, with $1 + x_j x_k > 0 \forall j, k$.

There exists $\delta > 0$ such that for all $0 < \epsilon \leq \delta$, the ‘linear homotopies’ between $x_j$ and $\epsilon y_j$, given by

$$x_j^{(\epsilon)}(t) := x_j + t(\epsilon y_j - x_j), \quad t \in [0, 1]$$

satisfy

$$1 + x_j^{(\epsilon)}(t)x_k^{(\epsilon)}(t) > 0, \quad \forall 1 \leq j, k \leq n, \; t \in [0, 1].$$
Remark 2.4. The above result is (implicitly stated, and explicitly) used in [19] with all $y_j = j$, and without the factor of $\epsilon$. Its use is key if one wishes to avoid using Jain’s prior work [18] in proving Theorem [1.10]. Unfortunately, the factor of $\epsilon$ here is crucial, otherwise the result fails to hold. Here are two explicit examples: in both of them, $n = 2, \epsilon = 1, and (y_1,y_2) = (1,2)$. Suppose first that $(x_1,x_2) = (-199,0)$; then ‘completing the square’ shows that the above assertion fails to hold at ‘most’ times in the homotopy:

$$1 + x_1^{(1)}(t)x_2^{(1)}(t) \leq 0, \quad \forall t \in \left[\frac{398}{800} - \frac{1}{20} \sqrt{\frac{398^2}{40^2} - 1}, \frac{398}{800} + \frac{1}{20} \sqrt{\frac{398^2}{40^2} - 1}\right],$$

and this interval contains $[0.0026, 0.9924]$. As another example, if $(x_1,x_2) = (-8.5,0.1)$, then

$$1 + x_1^{(1)}(t)x_2^{(1)}(t) \leq 0, \quad \forall t \in \left[\frac{8 - \sqrt{61}}{19}, \frac{8 + \sqrt{61}}{19}\right] \supset [0.01, 0.8321].$$

Remark 2.5. Jain has communicated to us [20] a short workaround to the above gap in [19], as follows: if all $x_j \leq 0$ then to prove Theorem [1.10(1)] one can replace all $x_j$ with $-x_j$. If $x_1 < 0 < x_n$ then one lets $0 < y_1 < \cdots < y_n < x_n$, and for these specific $y_j$, the homotopy argument works. However, we then need to show Theorem [1.10(1)] in the special case when all $x_j > 0$ – which is a result in Jain’s prior work; see [18] and the references and results cited therein. These prior results involve strictly sign regular (SSR) matrices and earlier papers. In this paper we avoid SSR matrices, and hence our approach additionally serves to provide a shorter, direct proof of Theorem [1.10].

We now show the above homotopy result.

Proof of Proposition 2.3. We make three clarifying observations to start the proof, with $x_j(t)$ denoting $x_j(t)$ throughout for a fixed $\epsilon > 0$. First, the assumptions imply $x_1(t) < \cdots < x_n(t)$ for all $t \in [0,1]$.

Second, if $x_1 = x_1(0) \geq 0$, then clearly $x_j(t) \geq 0$ for all $t \in [0,1]$ and all $1 \leq j \leq n$, and in this case the result follows at once. We will thus assume in the sequel that $x_1 < 0$.

Third, suppose there exist integers $1 \leq j < k \leq n$ and a time $t \in [0,1]$ such that $1 + x_j(t)x_k(t) \leq 0$, then we have $x_j(t) < 0 < x_k(t)$, and so $x_1(t) < 0 < x_n(t)$. A straightforward computation shows

$$1 + x_1(t)x_n(t) \leq 1 + x_j(t)x_k(t) \leq 0.$$ 

Given these observations, suppose we have initial data $x_j, y_j$, with $x_1 < 0$ from above. It suffices to find $\delta > 0$ such that

$$1 + x_1^{(1)}(t)x_n^{(1)}(t) > 0, \quad \forall \epsilon \in (0,\delta], \quad t \in (0,1).$$

Depending on the sign of $x_n$, we consider two cases:

Case 1: $x_n \geq 0$, in which case $x_n < 1/|x_1|$. We claim that $\delta := 1/(|x_1|y_n)$ works. Indeed, given $0 < \epsilon \leq \delta$, and $t \in (0,1)$, compute:

$$1 + x_1^{(1)}(t)x_n^{(1)}(t) = 1 + (t\epsilon y_1 + (1-t)x_1)(t\epsilon y_n + (1-t)x_n)$$

$$> 1 + (1-t)x_1(t\epsilon y_n + (1-t)x_n)$$

$$> 1 + (1-t)x_1(t\epsilon y_n + (1-t)/|x_1|),$$

with both inequalities strict because $t \in (0,1)$. Now the final expression equals

$$= 1 - (1-t)^2 + t(1-t)\epsilon y_n x_1 \geq t(2 - (1-t)\delta y_n |x_1|) = t > 0.$$  

Case 2: $x_n < 0$. Define the continuous function

$$g(\epsilon) := 1 - \frac{\epsilon^2(x_n y_1 - x_1 y_n)^2}{4(\epsilon y_1 - x_1)(\epsilon y_n - x_n)}, \quad \epsilon \geq 0.$$
Since \( g(0) > 0 \), there exists \( \delta > 0 \) such that \( g \) is positive on \([0, \delta]\). We claim this choice of \( \delta \) works.

Fix \( 0 < \epsilon \leq \delta \), and define

\[
t^{(e)}_j := -x_j/(\epsilon y_j - x_j), \quad \forall j \in [1, n].
\]

It is easy to check that \( x^{(e)}_j(t) \) is positive, zero, or negative when \( t > t^{(e)}_j \), \( t = t^{(e)}_j \), \( t < t^{(e)}_j \) respectively; moreover, since all \( x_j < 0 \), the above observations imply

\[
0 < t^{(e)}_1 < t^{(e)}_2 < \cdots < t^{(e)}_{n-1} < t^{(e)}_n < 1.
\]

In particular, if \( 0 \leq t \leq t^{(e)}_n \) or \( t^{(e)}_1 \leq t \leq 1 \), then \( x^{(e)}_1(t) \) and \( x^{(e)}_n(t) \) both have the same sign, whence \( 1 + x^{(e)}_1(t)x^{(e)}_n(t) \geq 1 \), so is positive. Otherwise \( t^{(e)}_n < t < t^{(e)}_1 \), in which case we first note that

\[
x^{(e)}_j(t) = t\epsilon y_j + (1-t)x_j = (t - t^{(e)}_j)(\epsilon y_j - x_j), \quad \forall j \in [1, n], \; t \in [0, 1].
\]

But now we compute, using the AM–GM inequality and choice of \( \delta \):

\[
1 + x^{(e)}_1(t)x^{(e)}_n(t) = 1 + (t-t^{(e)}_1)(t-t^{(e)}_n)(\epsilon y_1 - x_1)(\epsilon y_n - x_n)
\]

\[
\geq 1 - \frac{1}{4}(t^{(e)}_1 - t^{(e)}_n)^2(\epsilon y_1 - x_1)(\epsilon y_n - x_n) = g(\epsilon) > 0.
\]

Our next result is more widely applicable, at the cost of making the homotopy ‘piecewise linear’:

**Proposition 2.6.** Fix an integer \( n \geq 2 \) and tuples of real scalars

\[
x, y, p, q \in \mathbb{R}^n.
\]

such that \( 1 + x_j y_k > 0 \) \( \forall j, k \) and \( p_1, q_1 > 0 \). Then there exists piecewise linear homotopies

\[
x_j(t), y_j(t) : [0, 1] \to \mathbb{R}, \quad 1 \leq j \leq n
\]

such that \( x(t), y(t) \in \mathbb{R}^n \) for all times \( t \in [0, 1] \), with

\[
x_j(0) = x_j, \quad x_j(1) = p_j, \quad y_j(0) = y_j, \quad x_j(1) = q_j,
\]

and such that \( 1 + x_j(t)y_k(t) > 0 \) for all \( t \in [0, 1] \).

**Proof.** Let \( \delta_1 := \frac{1}{2|y_1|p_1} \) if \( y_1 \neq 0 \), and 1 otherwise. Define

\[
x'_j(t) := x_j + t(\delta_1 p_j - x_j), \quad 1 \leq j \leq n, \; t \in [0, 1].
\]

We claim that \( 1 + x'_j(t)y_k(t) > 0 \) for all \( 1 \leq j, k \leq n \) and \( t \in [0, 1] \). This is true at \( t = 0 \) for all \( j, k \); now suppose it fails for some \( t_0 \in (0, 1] \) and \( j, k \in [1, n] \). If \( y_k \geq 0 \) then \( 0 > x'_j(t_0) \geq x_j \), so

\[
0 \geq 1 + x'_j(t_0)y_k \geq 1 + x_j y_k > 0,
\]

which is impossible. Thus we must have

\[
y_k < 0 < x'_j(0) \leq x'_n(t_0) \leq \max(\delta_1 p_n, x_n).
\]

Using this,

\[
0 \geq 1 + x'_j(t_0)y_k \geq 1 + x'_j(0)y_k \geq 1 + y_1 \max(\delta_1 p_n, x_n) = \min(1 + y_1 x_n, 1 + \delta_1 y_1 p_n) > 0,
\]

which is similarly impossible.

This reasoning shows that one can define a linear homotopy \( x(t), \; t \in [0, 1/3] \) going from \( x \) to \( \delta_1 p \) for some \( \delta_1 > 0 \), such that \( 1 + x_j(t)y_k > 0 \) for all \( t \). Throughout, we define \( y(t) \equiv y \) for \( t \in [0, 1/3] \).

In a similar fashion, we let \( x(t) \equiv \delta_1 p \) for \( t \in [1/3, 2/3] \), and write down a linear homotopy \( y(t) \) from \( y \) to \( \delta_2 q \) for some \( \delta_2 > 0 \), such that \( 1 + x_j(t)y_k(t) > 0 \) for \( t \in [1/3, 2/3] \).

Finally, let \( x(t) \) (respectively \( y(t) \)) for \( t \in [2/3, 1] \) be the linear homotopy from \( \delta_1 p \) to \( p \) (respectively from \( \delta_2 q \) to \( q \)). Since \( p_1, q_1 > 0 \), it is trivially true that \( 1 + x_j(t)y_k(t) > 0 \) for \( t \in [2/3, 1] \). □
3. Proof of Theorem A: Critical exponent for PF functions and sequences

We now show the main results above. The next step is a direct application of Proposition 2.2.

**Proposition 3.1** (Jain, [19]). Suppose \( x_1, \ldots, x_n \in \mathbb{R} \) are pairwise distinct, as are \( y_1, \ldots, y_n \in \mathbb{R} \). If \( 1 + x_j y_k > 0 \) for all \( j, k \), and \( \alpha \in \mathbb{R} \setminus \{ 0, 1, \ldots, n - 2 \} \), then \( S^\alpha \) is non-singular, where \( S := (1 + x_j y_k)^n_{j,k=1} \). If \( \alpha \in \{ 0, 1, \ldots, n - 2 \} \), then \( S^\alpha \) has rank \( \alpha + 1 \).

Once again, the short proof is outlined in the Appendix.

In this section and the next, we will provide applications of Proposition 3.1: to our main theorems, as well as to Jain’s theorem 1.10. All of these applications also rely on the (novel) homotopy argument in Proposition 2.3; this keeps the proofs in this paper self-contained. We begin with Theorem 1.10 as it is used in the subsequent proofs.

**Proof of Theorem 1.10**

(1) If \( \alpha \in \mathbb{Z}^{\geq 0} \cup \{ n - 2, \infty \} \) then \( A^\alpha \in \mathbb{P}_n \) by Theorem 1.1(1) (the proof of which is outlined in the Appendix). We will need a refinement of the converse result, so we sketch this argument, taken from [8]. The result is easily shown for \( \alpha < 0 \), so we suppose \( \alpha \in (0, n - 2) \setminus \mathbb{Z} \) and in particular, \( n \geq 3 \) now. Let \( x = x(\epsilon) := \epsilon (1, 2, \ldots, n)^T \) with \( \epsilon > 0 \), and choose any vector \( v \in \mathbb{R}^n \) that is orthogonal to \( 1, x, x^2, \ldots, x^\alpha \) but not to \( x^{(\alpha + 2)} \). (Here, \( x^m = (1, \ldots, n^m)^T \) for an integer \( m \).) Now using binomial series, one computes:

\[
v^T (1_{n \times n} + xx^T)^\alpha v = \epsilon^{2(\alpha + 2)} \left( \frac{\alpha}{[\alpha] + 2} \right) (v^T x^{(\alpha + 2)})^2 + o(\epsilon^{2(\alpha + 2)}).
\]

Divide by \( \epsilon^{2(\alpha + 2)} \) and let \( \epsilon \to 0^+ \); as the right-hand side has a negative limit, the matrix-power on the left cannot be positive semidefinite.

With this special case at hand, the general case follows, via a more direct argument than in [18, 19]. Given pairwise distinct \( x_j \) such that \( 1 + x_j x_k > 0 \forall j, k \), let \( y_j := \epsilon_j \), where \( \epsilon > 0 \) is small enough to satisfy both the argument in the preceding paragraph, as well as the conclusions of Proposition 2.3. Now let \( x_j(t) := x_j + t(\epsilon_j - x_j) \) and let \( C(t) := (1 + x_j(t)x_k(t))^{\alpha} \). Then the smallest eigenvalue \( \lambda_{\min}(C(1)) = 0 \) from above, and \( C(t) \) is always non-singular by Proposition 3.1. It follows by the continuity of eigenvalues (or a simpler, direct argument) that \( \lambda_{\min}(C(0)) > 0 \), as desired.

(2) We show the ‘if’ part of Theorem 1.1(2) from [8] for self-completeness (and also because it is used presently). If \( \alpha \in \mathbb{Z}^{\geq 0} \) and \( C \geq D \geq 0 \) in \( \mathbb{P}_n((0, \infty)) \), then

\[
C^{\alpha} \geq C^{\alpha - 1} \circ D \geq \cdots \geq D^{\alpha},
\]

by the Schur product theorem.\footnote{The Schur product theorem [32] says that if \( A, B \in \mathbb{P}_n(\mathbb{R}) \), then so is their entrywise product \( A \circ B := (a_{jk}b_{jk})^{n}_{j,k=1} \). (For self-completeness: This is easily checked using the spectral eigen-decompositions of \( A, B \).)} If \( \alpha \geq n - 1 \), then by the fundamental theorem of calculus,

\[
C^{\alpha} - D^{\alpha} = \alpha \int_{0}^{1} (C - D) \circ (\lambda C + (1 - \lambda) D)^{\alpha - 1} \ d\lambda.
\]

By Theorem 1.1(1) and the Schur product theorem, the integrand is positive semidefinite, whence we are done.

The ‘only if’ part of Theorem 1.1(2) follows from Theorem 1.10(2), which is immediate from the preceding part: Suppose \( A^{\alpha} \geq B = B^{\alpha} \), with \( A = (1 + x_j x_k)^n_{j,k=1} \) and \( B = 1 \) as given. If \( x' := (x^T, 0)^T \in \mathbb{R}^{n+1} \), then the matrix

\[
A' := 1_{(n+1) \times (n+1)} + x'(x')^T = \begin{pmatrix} A & 1 \\ 1 & 1 \end{pmatrix}
\]
satisfies the hypotheses of part (1). Using Schur complements and part (1), we thus have:

\[ A^{\alpha} \geq 1_{n \times n} \iff \tilde{A}^{\alpha} \in \mathbb{P}_{n+1} \iff \alpha \in \mathbb{Z}^{>0} \cup [n-1, \infty). \]

This concludes a self-contained (modulo the Appendix) proof of Theorem 1.10, avoiding the use of SSR matrices as in \[18\] (which is used in \[19\]). A key corollary, used repeatedly below, now strengthens Theorem 1.10 from positive (semi)definiteness to total positivity, as promised above:

**Corollary 3.2.** Let \( p \geq 2 \) be an integer, and \( x, y \in \mathbb{R}^{p, \uparrow} \) be tuples such that \( 1 + x_j y_k > 0 \) for all \( j, k \). Let the matrix \( C := (1 + x_j y_k)_{j,k=1}^p \).

1. If \( \alpha > p - 2 \) then \( C^{\alpha \circ} \) is TP.
2. If \( \alpha \in \{0, 1, \ldots, p - 2\} \), then \( C^{\alpha \circ} \) has rank \( \alpha + 1 \).
3. If \( \alpha \in (0, p - 2) \setminus \mathbb{Z} \), then \( C^{\alpha \circ} \) is not TN – in fact, it has a principal minor that is negative.

See also Corollary 5.3 below, for a stronger version with more detailed information.

**Proof.** The second part follows from Proposition 3.1. For the third, fix any tuple \( q \in (0, \infty)^{p, \uparrow} \), and use Proposition 2.6 to construct piecewise linear homotopies \( x(t), y(t), t \in [0,1] \) from \( x, y \) to \( q \) respectively, such that \( 1 + x_j(t) y_k(t) > 0 \) for all \( 1 \leq j, k \leq p \) and \( t \in [0,1] \). Let \( C(t) := 1_{p \times p} + x(t) y(t)^T \). Then \( C(1)^{\alpha \circ} = (1_{p \times p} + q q^T)^{\alpha \circ} \) is not positive semidefinite by Theorem 1.10(1), hence has a negative principal minor. Now use Proposition 2.6 to show that the same principal minor of \( C(0)^{\alpha \circ} = C^{\alpha \circ} \) is negative, again by Proposition 3.1.

For the first part, let \( B \) be any square submatrix of \( C \) of order \( p' \in [2, p] \); then one can repeat the preceding argument with \( C = B \) for this part. Thus, let \( q \) and \( C(t)_{p' \times p'} = B(t) \) be as in the previous paragraph. Since now \( \alpha > p' - 2 \), so det \( B(t)^{\alpha \circ} \) does not change sign, by Proposition 3.1. But det \( B(1)^{\alpha \circ} > 0 \) by Theorem 1.10(1). This shows that every minor of the original matrix \( C_{p \times p}^{\alpha \circ} \) is positive, whence \( C^{\alpha \circ} \) is TP. \( \square \)

With this and the preceding ingredients at hand, we show our first main result.

**Proof of Theorem 1.10**

1. We first show the result for \( X = Y = \mathbb{R} \). Notice in this case that the result for any \( a \in \mathbb{R} \) shows the result for any other, so we work with \( a = 0 \). Suppose \( \alpha \in (0, p - 2) \setminus \mathbb{Z} \), and

\[ 0 < v_p < \cdots < v_1 < u_p < \cdots < u_1 \]

are fixed scalars. Set \( x_j := u_j^{-1} \) and \( y_k := -v_k \); thus \( x_j > 0 \) and \( 1 + x_j y_k > 0 \) for all \( j, k \). By (the proof of) Corollary 3.2(3), the matrix \( C := (1 + x_j y_k)^\alpha_{j,k=1}^p \) has a negative minor, hence is not TN. Pre- and post-multiply by diagonal matrices with \( (j, j) \) entry \( u_j^\alpha e^{-\alpha u_j} \) and \( e^{\alpha v_j} \) respectively. This shows, via applying the order-reversing permutation to the rows and to the columns, that given

\[ u' := (u_1', \ldots, u_p'), v' := (v_1', \ldots, v_p') \in \mathbb{R}^{p, \uparrow}, \text{ with } u_p' < u_1', \]

the matrix \( T_{\Omega}[^u';^v'] = T_{\Omega}[^u' - (v_1' - 1)1;^v' - (v_1' - 1)1] \) has a submatrix with negative determinant.

This shows the result when \( X = Y = \mathbb{R} \), e.g. with \( a = 0 \). For arbitrary \( X, Y \subset \mathbb{R} \) of sizes at least \( p \), first choose and fix increasing \( p \)-tuples \( u' \in X^{p, \uparrow}, v' \in Y^{p, \uparrow} \); now choose any \( a < u_1' - v_p' \). By the above proof, the matrix \( T_{\Omega+a[^u';^v']} = T_{\Omega}[^u' - a1;^v'] \) is not TN.\( p \).

This shows the result for all \( X, Y \).

2. Choose \( m > 0 \) and tuples \( x \in X^{p, \uparrow}, y \in Y^{p, \uparrow} \) such that \( |mx_j|, |my_j| < \pi/4 \) for all \( j \). Now,

\[ T_{W_m}[^x;^y]^{\alpha \circ} = (\cos(mx_j - my_k)^\alpha_{j,k=1}^p = D_x(1 + \tan(mx_j) \tan(my_k))^\alpha D_y, \]

where \( D_x \) for a vector \( x \) equals \( \text{diag}(\cos(mx_j)^\alpha) \). Now since \( mx, my \) have increasing coordinates, all in \((-\pi/4, \pi/4)\), Corollary 3.2(3) applies to show that \( T_{W_m}^{\alpha \circ} \) is not TN.\( p \).
(3) The previous two parts in fact show the case of $\alpha \in (0, p - 2) \setminus \mathbb{Z}$. The other two cases follow by using similar arguments, via Corollary 3.2(1),(2). \qed

3.1. Single-matrix encoders of Loewner convexity. As an application of the methods used above, we provide single-matrix encoders of the entrywise powers preserving Loewner convexity. Recall for $I \subset \mathbb{R}$ that a function $f : I \to \mathbb{R}$ preserves Loewner convexity on a set $V \subset \mathbb{P}_n(I)$ if $f[\lambda A + (1 - \lambda)B] \leq \lambda f[A] + (1 - \lambda)f[B]$ whenever $\lambda \in [0, 1]$ and $A \geq B \geq 0$ in $V$.

The powers preserving Loewner convexity were classified by Hiai in 2009:

Theorem 3.3 (Hiai, [14]). Let $n \geq 2$ be an integer and $\alpha \in \mathbb{R}$. The entrywise map $x^\alpha$ preserves Loewner convexity on $\mathbb{P}_n([0, \infty))$ if and only if $\alpha \in \mathbb{Z}_{\geq 0} \cup [n, \infty)$.

In the spirit of Theorem 1.10 we provide single-matrix encoders of these powers:

Theorem 3.4. Let $n \geq 2$ be an integer and $\alpha \in \mathbb{R}$. Suppose $x_1, \ldots, x_n \in \mathbb{R}$ are pairwise distinct, non-zero scalars such that $1 + x_jx_k > 0$ for all $j, k$. Let $A := (1 + x_jx_k)_{j,k=1}^n$ and $B := 1_{n \times n}$. Then $x^\alpha$ preserves Loewner convexity on $A \geq B \geq 0$ if and only if $\alpha \in \mathbb{Z}_{\geq 0} \cup [n, \infty)$.

The proof relies on the following preliminary lemma, which can be shown by an argument of Hiai – see the Appendix.

Lemma 3.5. Let $n \geq 2$ and $A \geq B \geq 0$ in $\mathbb{P}_n(\mathbb{R})$ be such that $A - B = uu^T$ has rank one and no non-zero entries. Choose an open interval $I \subset \mathbb{R}$ containing the entries of $A, B$, and suppose $f : I \to \mathbb{R}$ is differentiable. If the entrywise map $f[-]$ preserves Loewner convexity on the interval $[B, A] := \{\lambda A + (1 - \lambda)B : \lambda \in [0, 1]\}$ then $f'[-]$ preserves Loewner monotonicity on $[B, A]$. The converse holds for arbitrary matrices $0 \leq B \leq A$.

We now prove Theorem 3.4 – in the process also proving Hiai’s result:

Proof of Theorems 3.4 and 3.3. By Lemma 3.5 and Theorem 1.12, $x^\alpha$ preserves Loewner convexity on $\mathbb{P}_n([0, \infty))$ for $\alpha \in \mathbb{Z}_{\geq 0} \cup [n, \infty)$, and obviously so for $\alpha = 0$. The result for $\mathbb{P}_n([0, \infty))$ follows by continuity. Next, if $x^\alpha$ preserves Loewner convexity on $\mathbb{P}_n([0, \infty))$, then it does so on the given matrices $A \geq B \geq 0$. Finally, if the latter condition holds and $\alpha \notin \mathbb{Z}_{\geq 0}$, then Lemma 3.5 applies, so $\alpha \geq n$ via Theorem 1.10(2). \qed

4. Hankel $\mathbb{T}_p$ Kernels: Preservers, Critical Exponent, and Theorem [15]

In this section we first prove Theorem [15] The key tool is a result of Fekete from 1912 [7]:

Lemma 4.1. Suppose $1 \leq p \leq m, n$ are integers, and $A \in \mathbb{R}^{m \times n}$. Then $A$ is $\mathbb{T}_p$ if and only if all contiguous minors of orders $\leq p$ are positive. (Here, ‘contiguous’ means that the rows and columns for the minor are both consecutive.)

The proof is not too long, relying on computational lemmas by Gantmacher and Krein. See [10].

Corollary 4.2. Suppose $1 \leq p \leq n$ are integers and $A \in \mathbb{R}^{n \times n}$ is Hankel. Let $A^{(1)}$ denote the truncation of $A$, i.e. the submatrix with the first row and last column of $A$ removed. Then $A$ is $\mathbb{T}_p$ (respectively $\mathbb{TP}_p$) if and only if every contiguous principal minor of $A$ and of $A^{(1)}$ of size $\leq p$ is non-negative (respectively positive).

This result can be found in [27], Chapter 4 for the TP case, and in [6] for the $\mathbb{TP}_p, \mathbb{TN}, \mathbb{TP}_p$ cases. These sources do not use the word ‘contiguous’ – the advantage of using contiguous (principal) minors is that they are all Hankel. We provide a quick proof of Corollary 4.2 in the Appendix, for self-completeness. For now, we apply this result to prove Theorem [15] and other results. The relevant part of this argument is isolated into the following standalone result.
Proposition 4.3. Suppose $p \geq 2$ is an integer, $X \subset \mathbb{R}$ is an interval with positive measure, and $H : X \times X \to \mathbb{R}$ is a continuous Hankel TN$_p$ kernel. If $f : [0, \infty) \to [0, \infty)$ is continuous at $0^+$, and preserves positive semidefiniteness when acting entrywise on $r \times r$ Hankel matrices for $1 \leq r \leq p$, then $f \circ H : X \times X \to \mathbb{R}$ is continuous, Hankel, and TN$_p$.

Proof. The first step is to show that $f$ is continuous on $(0, \infty)$; this quickly follows e.g. from work of Hiai [14], and is sketched in the Appendix for completeness. (A longer proof is via using a 1929 result of Ostrowski; see e.g. [1].) Thus $f \circ H$ is continuous and Hankel on $X \times X$.

Now let $2 \leq r \leq p$ and choose $\mathbf{x}, \mathbf{y} \in X^r$. We need to show $\det((f \circ H)[\mathbf{x}; \mathbf{y}]) \geq 0$. Let $\mathbf{u} = (u_1, \ldots, u_m)$ denote the ordered tuple whose coordinates are the union of the $x_j, y_k$ (without repetitions). We claim that $(f \circ H)[\mathbf{u}; \mathbf{u}]$ is TN$_p$; this would suffice to complete the proof.

To show the claim, approximate the increasing tuple $\mathbf{u}$ by tuples $\mathbf{u}(k) \in (X \cap \mathbb{Q})^m$ of rational numbers in $X$, with $\mathbf{u}(k) \to \mathbf{u}$ as $k \to \infty$. Choose integers $N_k > 0$ such that $N_k \mathbf{u}(k)$ has integer coordinates. Now if the matrices

$$(f \circ H)[\mathbf{v}; \mathbf{v}], \text{ where } \mathbf{v}_k := (u_1^{(k)}, u_1^{(k)} + \frac{1}{N_k}, \ldots, u_m^{(k)})$$

can be shown to be TN$_p$, then by taking submatrices and the limit as $k \to \infty$, it follows that $(f \circ H)[\mathbf{u}; \mathbf{u}]$ is TN$_p$, as claimed. We use here that $f$ is continuous.

Since each $\mathbf{v}_k$ is an arithmetic progression, it is easy to see that the matrices $A_k := H[\mathbf{v}_k; \mathbf{v}_k]$ are Hankel, and TN$_p$ because $H$ is so. Now observe that all contiguous principal submatrices $C$ of $A_k$ or of $A_k^{(1)}$ of size $2 \leq r \leq p$ are symmetric Hankel positive semidefinite matrices. Thus $f[C]$ is positive semidefinite by assumption, hence has determinant $\geq 0$. It follows by Corollary 4.2 that $(f \circ H)[\mathbf{v}_k; \mathbf{v}_k]$ is TN$_p$ for all $k$, and this completes the proof.

With Proposition 4.3 and the previous results at hand, our next main result follows.

Proof of Theorem 1.3. The first step is to verify that $H_{u_0}$ is TN; this is easy because $H_{u_0}$ has ‘rank two’, being the moment sequence/kernel of the two-point measure $\delta_1 + c_0 \delta_{u_0}$, so all $r \times r$ minors vanish for $r \geq 3$. We next prove a chain of cyclic implications. Clearly $(1) \implies (2)$, and $(3) \implies (1)$ by Proposition 4.3 and Theorem 1.1(1). Finally, suppose $\alpha \notin \{0, 1, \ldots, p - 2\}$. Choose tuples $x, y \in X^p$ and apply Corollary 3.2 with $x_j, y_j$ replaced by $\sqrt{c_0} u_0^{x_j}, \sqrt{c_0} u_0^{y_j}$ respectively; we also reverse the rows and columns if $u_0 \in (0, 1)$. This yields a TN matrix $H_{u_0}[x; y]$, whose $\alpha$th entrywise power is not TN$_p$ if $\alpha \in (0, p - 2) \setminus \mathbb{Z}$, and is TP if $\alpha > p - 2$. This shows both $(2) \implies (3)$ as well as the remaining assertions.

For the curious reader, Theorem 1.3 leads to a question about Toeplitz analogues that may be of theoretical interest. One can ask if this ‘clean’ phenomenon holds for the parallel class of Toeplitz kernels – namely, if for all integers $p \geq 2$, the TN$_p$-preserving powers $x^\alpha$ are precisely $\alpha \in \mathbb{Z}^+ \cup \{p - 2, \infty\}$. This is easily verified to hold for $p = 2$; see e.g. [30] (or Lemma 6.1 below), where TN$_2$ functions are characterized as exponentials of concave functions. However, such a clean result fails to hold in general. Specifically, considering the question from the ‘dual’ viewpoint of the powers $\alpha$: while $x^\alpha$ for $\alpha = 0, 1$ obviously preserves TN$_p$ for all $p$, this fails to hold for every other integer $\alpha \geq 2$. Namely, one can find a TN$_p$ kernel (for some $p \geq 0$), whose $\alpha$th power is not TN$_p$. This can be refined further, to work with a single kernel – which is moreover TN – that provides a counterexample for all integer powers:

Lemma 4.4. There exists a Pólya frequency function $M : \mathbb{R} \to \mathbb{R}$, such that for every integer power $\alpha \geq 2$, there exists an integer $p(\alpha) \geq 1$ satisfying: $M^\alpha$ is not TN$_{p(\alpha)}$.

Proof. Let $M(x) := 2e^{-|x|} - e^{-2|x|}$ for $x \in \mathbb{R}$. It was shown in [3] that $M^\alpha$ is not TN for any $\alpha \geq 2$, while $M$ is. (See the Appendix for details.) This proves the result.
In light of Lemma 4.4 one can ask more refined questions, e.g. if all non-integer powers $\alpha > p - 2$ preserve $\mathbb{T}^p$ Toeplitz functions/kernels, with $p \geq 4$. A challenge in tackling such questions comes from the lack of a well-developed theory for Pólya frequency functions of finite order, i.e., integrable $\mathbb{T}^p$ functions. For instance, to our knowledge there is no known characterization to date of Pólya frequency functions of order $p = 4, 5, \ldots$. (While Theorem 4.3 provides one such characterization, it does not turn out to be effective enough to help here.)

**Remark 4.5.** In light of Lemma 4.4 and the above results, one can also ask about the classification of powers – or more generally, arbitrary functions – that preserve the class of $\mathbb{T}^p$ kernels, whether Hankel or Toeplitz, upon composing. These characterizations were recently achieved in joint work [3]: for continuous Hankel kernels, the preservers are precisely the convergent power series with non-negative Maclaurin coefficients (see also Lemma 4.8), while for Toeplitz kernels, the preservers are precisely constants $c$ or homotheties $cx$ or Heaviside functions $c1_{x > 0}$, with $c \geq 0$.

4.1. **Connection to fixed-dimension results on positivity preservers.** Given an integer $p \geq 1$ and a subset $I \subset \mathbb{R}$, let $\mathbb{P}_p(I)$ denote the set of real symmetric $p \times p$ matrices, which are positive semidefinite and have all entries in $I$. The critical exponent phenomena studied above suggest that $\mathbb{T}^p$-preservers are closely related to entrywise functions preserving positive definiteness on $\mathbb{P}_p((0, \infty))$ – especially for Hankel kernels, in light of Proposition 4.3. Although our focus in this paper is on powers, we briefly digress to point out a few such connections. The first is Loewner’s necessary condition for preserving positivity on such matrices:

**Theorem 4.6** (Loewner / Horn, 1969, [16]). Suppose $I = (0, \infty)$, $f : I \to \mathbb{R}$ is continuous, and $p \geq 3$ is an integer such that $f[-]$ applied entrywise to matrices in $\mathbb{P}_p(I)$ preserves positivity. Then $f \in C^{p-3}(I)$, $f^{(p-3)}$ is convex on $I$, and $f, f', \ldots, f^{(p-3)} \geq 0$ on $I$. If in particular $f \in C^{p-1}(I)$, then $f^{(p-2)}, f^{(p-1)} \geq 0$ on $I$ as well.

We claim that the same conclusions hold if $f$ preserves the $\mathbb{T}^p$ Hankel kernels – in fact on a far smaller test set, and without the continuity assumption from [16]:

**Theorem 4.7.** Suppose $I = (0, \infty)$, $f : I \to \mathbb{R}$, and $X_0 \subset \mathbb{R}$ is any interval with positive measure. Suppose $p \geq 3$ is an integer such that the post-composition transform $f \circ -$ preserves $\mathbb{T}^p$ on Hankel $\mathbb{T}^p$ kernels corresponding to non-negative measures supported on at most two points. Then the conclusions of Theorem 4.6 hold.

That this result is sharp – in the number of non-negative derivatives $f, \ldots, f^{(p-1)}$ on $I$ – follows from Theorem 4.3 by considering a suitable power function $f$.

**Proof.** We appeal to results in [1], which assert that if $f[-]$ preserves positivity on the matrices

$$(a_0 + c_0 u_0^{j+k})^{p-1}_{j,k=0}, \quad a_0, c_0 \geq 0, \quad a_0 + c_0 > 0,$$

$$\begin{pmatrix} a & b \\ b & b \end{pmatrix}, \begin{pmatrix} c^2 & cd \\ cd & d^2 \end{pmatrix}, \quad a, b, c, d > 0, \quad a > b > 0, \quad c \geq d > 0$$

for some fixed $u_0 \in (0, 1)$, then $f$ satisfies the conclusions of Theorem 4.6. It thus suffices to embed these test matrices in $\mathbb{T}^p$ Hankel kernels. We do so on $\mathbb{R} \times \mathbb{R}$; the restriction to $X_0 \times X_0$ follows by a linear change of variables that contains an appropriate compact sub-interval of $\mathbb{R}$. The first class of test matrices above embeds in the Hankel kernels

$$H_{a_0, c_0}(x, y) := a_0 + c_0 u_0^{x+y}, \quad x, y \in \mathbb{R},$$

for $a_0, c_0 \geq 0$, while the ‘rank-one’ matrices above embed in the kernel $H_{c,0}$ if $c = d$, and in $H_{0,c^2}$ with $u_0 = d/c$, if $c > d > 0$. Recently in [3], the final class of matrices $\begin{pmatrix} a & b \\ b & b \end{pmatrix}$ above was shown to
embed in the following ‘rank-two’ TN Hankel kernel, which completes the proof:
\[
\frac{(2a - b)^2}{4a - 3b} \left( \frac{b}{2a - b} \right)^{x+y} + \frac{b(a - b)}{4a - 3b} 2^{x+y}, \quad x, y \in \mathbb{R}.
\]

The next connection is to an even older observation of Pólya and Szegő [28] from 1925:

**Lemma 4.8.** Suppose \( f_0 \) is the restriction to \([0, \infty)\) of an entire function with non-negative Maclaurin coefficients. Then \( f_0 \circ - \) preserves the class of continuous TN\( _p \) Hankel kernels on \( X \times X \), for all integers \( p \geq 1 \) and intervals \( X \subset \mathbb{R} \).

**Proof.** By the Schur product theorem, \( x^k \) entrywise preserves positivity on \( \mathbb{P}_p([0, \infty)) \) for all integers \( k \geq 0 \); here we set \( 0^0 = 1 \). Since \( \mathbb{P}_p([0, \infty)) \) is a closed convex cone, it follows that all functions \( f_0 \) as in the lemma share the same property. We are now done by Proposition 4.3 \( \square \)

Our third connection is to entrywise polynomials that preserve TN\( _p \). By the preceding lemma, all power series with non-negative coefficients preserve TN\( _p \) on continuous Hankel TN\( _p \) kernels. It is natural to ask if a wider class of polynomials shares this property.\(^5\) We conclude this section by providing a positive answer, essentially coming from recent joint work with Tao [24]:

**Theorem 4.9.** Let \( p > 0 \) and \( 0 \leq n_0 < \cdots < n_{p-1} < M < n_p < \cdots < n_{2p-1} \) be integers, and let \( c_{n_0}, \ldots, c_{n_{2p-1}} > 0 \) be reals. There exists a negative number \( c_M \) such that the polynomial
\[
x \mapsto c_{n_0}x^{n_0} + c_{n_1}x^{n_1} + \cdots + c_{n_{p-1}}x^{n_{p-1}} + c_Mx^M + c_{n_p}x^{n_p} + \cdots + c_{n_{2p-1}}x^{n_{2p-1}},
\]

preserves the continuous Hankel TN\( _p \) kernels on \( X \times X \), for intervals \( X \subset \mathbb{R} \) with positive measure.

Via Proposition 4.3, Theorem 4.9 follows from [24], because such a polynomial was shown in loc. cit. to preserve Loewner positivity on \( \mathbb{P}_p([0, \infty)) \). Theorem 4.9 also admits extensions to power series and more general preservers; we refer the interested reader to [24] for further details.

5. **Theorem C** Critical exponent for the Jain–Karlin–Schoenberg kernel

We next show Theorem C on the total non-negativity of the powers of the kernel \( K_{JKS} \), and explain how it connects to the (total) positivity results stated before in the opening section.

**Proof of Theorem C.** The second part follows from Corollary 3.2. For the first, begin with the basic trigonometric fact: If \(-\pi/2 < \varphi < \theta < \pi/2\), then \( \tan(\theta)\tan(\varphi) > -1 \) if and only if \( \theta - \varphi < \pi/2 \).

Now let \( x, y \in \mathbb{R}^{p\uparrow} \) and let \( u_j := \tan^{-1}(x_j), \quad v_j := \tan^{-1}(y_j) \). Then \( u, v \in (-\pi/2, \pi/2)^{p\uparrow} \), so:
\[
K_{JKS}(x_j, y_k) = (1 + \tan(u_j)\tan(v_k))I_{\tan(u_j)\tan(v_k) > -1}
\]
\[
= (1 + \tan(u_j)\tan(v_k))I_{|u_j - v_k| < \pi/2}
\]
\[
= \sec(u_j)\sec(v_k) \left[ \cos(u_j - v_k)I_{|u_j - v_k| < \pi/2} \right]
\]
\[
= \sec(u_j)\sec(v_k)T_W(u_j, v_k).
\]

It follows that
\[
K_{JKS}[x; y]^\alpha = D_u^\alpha T_W[u; v]^\alpha D_v^\alpha, \quad \forall \alpha \geq 0
\]
where \( D_u \) for a vector \( u \in (-\pi/2, \pi/2)^{p\uparrow} \) is the diagonal matrix with \((j, j)\) entry \( \sec(u_j) \). Theorem 4.5 now implies that this matrix is TN if \( \alpha \geq p - 2 \), proving the first part.

Finally, we show the third part. Since the kernel \( K_{JKS} \) is invariant under the automorphism group generated by the involutions \( x \leftrightarrow y \) and \((x, y) \leftrightarrow (-x, -y)\), it suffices to show that the restriction to \([0, \infty) \times \mathbb{R} \) of \( K_{JKS}^\alpha \) is TN\( _p \) if and only if \( \alpha \in \mathbb{Z}_{\geq 0} \cup [p - 2, \infty) \). This already holds for

\(^5\)In the original setting of entrywise polynomials and power series preserving positivity on \( \mathbb{P}_p([0, \infty)) \), no examples were known for \( p \geq 3 \), until recent joint work [24].
The final sub-case is when \( \alpha \in \mathbb{Z}^{\geq 0} \). Let \( x \in [0, \infty)^{p,\uparrow} \) and \( y \in \mathbb{R}^{p,\uparrow} \); we need to show that
\[
det C^\alpha \geq 0, \quad \text{where} \quad C := (\max(1 + x_j y_k, 0))^p_{j,k=1}.
\]
By the continuity of the function \( K_{JKS} \), we may assume \( x_1 > 0 \). Now,
\[
C^\alpha = \text{diag}(x_1^\alpha)_j \max(0, x_1^{-1} - (-y_k)^\alpha)_k = \text{diag}(x_1^\alpha e^{\alpha x_1^{-1}})_j (\Omega(x_1^{-1} - (-y_k))^\alpha \text{diag}(e^{\alpha y_k})
\]
Reversing the rows and the columns, we are done by Theorem 1.2.
\[\square\]

**Remark 5.2.** We now explain how Theorem C implies many of the results in Section 1

1. Given scalars \( 0 < x_1 < \ldots < x_p \) and \( y_1 < \ldots < y_p \), the Karlin-kernel \( \Omega \) is a specialization of the Jain–Karlin–Schoenberg kernel, up to multiplying by diagonal matrices and reversing rows and columns:
\[
(T_\Omega[x, y]^\alpha)^T = D^\alpha K_{JKS}[y', x]^\alpha D_1^\alpha,
\]
where \( y' = (-y_1, \ldots, -y_p) \), \( x' = (1/x_1, \ldots, 1/x_p) \), and \( D_1, D \) are diagonal matrices
\[
D_1 = \text{diag}(x_1 e^{-x_1}, \ldots, x_p e^{-x_1}), \quad D = \text{diag}(e^{y_1}, \ldots, e^{y_p}),
\]

2. Similarly, the proof of Theorem C(1) shows how, via the transformation arctan, the Jain–Karlin–Schoenberg kernel is intimately related to the Schoenberg-kernel \( T_W \). These observations show how Theorem C about (the powers of) the Jain–Karlin–Schoenberg kernel is related to Theorem A and to Theorems 1.15 and 1.2 of Schoenberg and Karlin, respectively.

3. Given an integer \( n \geq 2 \), the kernel \( K_{JKS} \) clearly specializes on the set of bi-tuples
\[
\{(x, y) \in (\mathbb{R}^{n,\uparrow})^2 : 1 + x_j y_k > 0 \ \forall j, k = 1, \ldots, n\}
\]
to Jain’s theorem 1.10(1) – in fact, to the stronger TN assertion in Theorem C(2).

4. Restricting the kernel \( K_{JKS} \) to \( (0, \infty)^2 \) via the transform \( u^\alpha \), we see that Theorem C implies the equivalence (2) \iff (3) in Theorem B.

5. Our methods have provided an alternate proof above to Karlin’s theorem 1.2. Indeed, as discussed during the proof of Theorem C(3), the result is shown in the Appendix for integer powers, and for non-integer powers \( \alpha > p - 2 \) it is a special case of Schoenberg’s theorem – restricting the domain from \( (-\pi/2, \pi/2)^2 \) to \( \mathbb{R}^2 \) via arctan, then to \( \mathbb{R} \times [0, \infty) \). Here we use the identifications of \( K_{JKS} \) with Schoenberg and Karlin’s kernels.

In fact, it is possible to refine the above results even more. Given integers \( 1 \leq p \leq n \), matrices \( C = (1 + x_j y_k)^n_{j,k=1} \) with positive entries, and powers \( \alpha \geq 0 \), one can show that all \( p \times p \) minors of \( C^\alpha \) have the same sign – which depends only on \( n, p, \alpha \) but not on \( x_j, y_k \). This follows from above for \( \alpha \in \mathbb{Z}^{\geq 0} \cup [p-2, \infty) \). If \( \alpha \in (0, p-2) \setminus \mathbb{Z} \), this follows by using SSR (strictly sign regular) matrices and kernels, found in Karlin’s book [23] and Jain’s works [18, 19]. In fact, the following holds, e.g. by Propositions 2.6 and 3.1 and [18, 4.1.4.4, Theorem 2.4]:

**Corollary 5.3.** Given a scalar \( \alpha \geq 0 \), an integer \( n \geq 2 \), and tuples \( x, y \in \mathbb{R}^{n,\uparrow} \) such that \( 1 + x_j y_k > 0 \) for all \( j, k \), the power-matrix \( C^\alpha \) studied above is sign regular, with signature given as follows:
\[
\text{signature}((1 + x_j y_k)^\alpha)_j,k=1 = \begin{cases} \mathbb{C} \{((-1)^{\frac{1}{2}} e_{\rho,\alpha})_{\rho=1}^{\alpha} \}, & \text{if } \alpha \not\in \{0, 1, \ldots, n-2\}, \\ \mathbb{C} \{((-1)^{\frac{1}{2}} e_{\rho,\alpha})_{\rho=1}^{\alpha+1} \}, & \text{otherwise}, \end{cases}
\]

This part of Karlin’s result, for integer powers \( \alpha \geq 0 \), was already shown by Schoenberg in 1951 [20]. For the interested reader, his direct proof is included in the Appendix.
That is, the sign of any $p \times p$ minor of $((1 + x_jy_k)^{\alpha})_{j,k=1}^n$ depends only on $n, p, \alpha$; here, $\varepsilon_{p,\alpha}$ equals

$$
\varepsilon_{p,\alpha} = \begin{cases} 
(-1)^{[p/2]}, & \text{if } \alpha > p - 2, \\
(-1)^{p-s+1}, & \text{if } 2s < \alpha < 2s + 1 \leq p - 2, \ s \in \mathbb{Z}^\geq 0, \\
(-1)^{s+1}, & \text{if } 2s + 1 < \alpha < 2s + 2 \leq p - 2, \ s \in \mathbb{Z}^\geq 0, \\
0, & \text{if } \alpha = 0, 1, \ldots, p - 2.
\end{cases}
$$

To conclude this section, note that Theorem [D] completely classifies the powers of $K_{JKS}$ preserving $\text{TN}_p$ on $\mathbb{R} \times [0, \infty)$. The same question on the full domain $\mathbb{R}^2$ of $K_{JKS}$ remains, but only for integers $\alpha \in \{0, 1, \ldots, p - 2\}$. This is equivalent to the following

**Question 5.4.** For an integer $\alpha \geq 1$, can the kernel $K_{JKS}^\alpha$ be shown to be not $\text{TN}_{\alpha+3}$ on $\mathbb{R} \times \mathbb{R}$? More strongly, can it be shown to be not ‘positive semidefinite’, i.e. using $x = y \in \mathbb{R}^{\alpha+3}$?

A complete resolution of Question [5.4] would complete the classification of powers of the Jain–Karlin–Schoenberg kernel $K_{JKS}$ that are totally non-negative of each order $p \geq 2$. Note that the question has a ‘positive’ answer for $\alpha = 1$, so that $K_{JKS}$ is not $\text{TN}_4$. Indeed,

$$
x = y = \frac{1}{\sqrt{2}}(-2, -1, 1, 2) \in \mathbb{R}^4 \implies \det K_{JKS}[x; y] = \det \begin{pmatrix} 3 & 2 & 0 & 0 \\ 2 & 3/2 & 1/2 & 0 \\ 0 & 1/2 & 3/2 & 2 \\ 0 & 0 & 2 & 3 \end{pmatrix} = -2.
$$

6. Theorem [D] Laplace Transform of a Compactly Supported $\text{TN}_p$ Function

We now show Theorem [D]. The first step toward proving the result is to characterize $\text{TN}_2$ functions $\Lambda$ on a sub-interval $I \subset \mathbb{R}$, instead of on all of $\mathbb{R}$ as is prevalent in the literature. We provide a proof of this result along the lines of [31], but with a few modifications for more general $I$:

**Lemma 6.1.** Suppose $J \subset \mathbb{R}$ is an interval strictly containing the origin, and $\Lambda : J \rightarrow J \rightarrow \mathbb{R}$ is Lebesgue measurable. The following are equivalent:

1. The nonzero-locus of $\Lambda$ is an interval $I \subset J - J$, on which $\Lambda > 0$ and $\log \Lambda$ is concave.
2. The Toeplitz kernel $T_{\Lambda} : J \times J \rightarrow \mathbb{R}$ is $\text{TN}_2$.

Thus $\Lambda$ is continuous on the interior of $I$, whence discontinuous on $J - J$ at most at two points.

In particular, this applies to $I = (-\rho/2, \rho/2) \subset J = (-\rho, \rho)$, as in Theorem [D].

**Proof.** The result is straightforward if $\Lambda$ does not vanish at most at one point, so we suppose henceforth that $\Lambda \neq 0$ at least at two points.

1. $\implies$ (2): Given scalars $\alpha < \beta$ and $\gamma < \delta$ in $J$, note that $\alpha - \gamma, \beta - \delta \in (\alpha - \delta, \beta - \gamma)$. If $\alpha - \gamma$ or $\beta - \delta$ lie outside $I$, the matrix $M := \begin{pmatrix} \Lambda(\alpha - \gamma) & \Lambda(\alpha - \delta) \\ \Lambda(\beta - \gamma) & \Lambda(\beta - \delta) \end{pmatrix}$ has a zero row or zero column. Else $\alpha - \gamma, \beta - \delta \in I$; if now one of $\alpha - \delta, \beta - \gamma$ is not in $I$ then $M$ is triangular, whence again $\det(M) > 0$. Else $M$ has all positive entries; now the concavity of $\log \Lambda$ implies $\det(M) > 0$.

2. $\implies$ (1): Since $\Lambda$ is $\text{TN}_2$, we have $\Lambda \geq 0$ on $J - J$. Fix $\delta > 0$ such that $J$ contains either $[0, \delta]$ or $(-\delta, 0]$. Suppose $\Lambda(x_0) > 0$. We claim that if $x_1 > x_0$ in $J - J$ and $\Lambda(x_1) = 0$, then $\Lambda$ vanishes on $(J - J) \cap [x_1, \infty)$; and similarly for $x_1 < x_0$ in $J - J$. It suffices to show that $\Lambda(y) = 0$ for $y \in (J - J) \cap (x_1, x_1 + \delta)$. If $J \supset (-\delta, 0]$, this is because

$$
0 \leq \det T_{\Lambda}([x_0, x_1); (x_1 - y, 0)] = \det \begin{pmatrix} \Lambda(x_0 - x_1 + y) & \Lambda(x_0) \\ \Lambda(y) & \Lambda(x_1) \end{pmatrix} = -\Lambda(x_0)\Lambda(y);
$$

here, $x_0 - x_1 + y \in (x_0, y) \subset J - J$. Similarly, if $J \supset [0, \delta)$, then we instead use

$$
0 \leq \det T_{\Lambda}((x_0 - x_1 + y); (0, y - x_1)] = \det \begin{pmatrix} \Lambda(x_0 - x_1 + y) & \Lambda(x_0) \\ \Lambda(y) & \Lambda(x_1) \end{pmatrix} = -\Lambda(x_0)\Lambda(y).
$$
This produces the interval $I$; now given points $y - \epsilon < y < y + \epsilon$ of $I$, we show that $\Lambda(y) \geq \sqrt{\Lambda(y + \epsilon)\Lambda(y - \epsilon)}$ using discrete-time, finite state-space Markov chains. Let $n_0 := 2[\epsilon/\delta]$, so that $\epsilon/n_0 \in (0, \delta)$. Let $z_k := \Lambda(y + k\epsilon/n_0)$ for $-n_0 \leq k \leq n_0$; then $z_k > 0$. Now if $J \supset (-\delta, 0]$, then

$$0 \leq \det T_\Lambda[(y - (k + 1)\epsilon/n_0, y - k\epsilon/n_0); (-\epsilon/n_0, 0)] = z_k^2 - z_{k-1}z_{k+1}, \quad \forall -n_0 < k < n_0.$$  

If instead $J \supset [0, \delta)$ then we use $0 \leq \det T_\Lambda[(y - k\epsilon/n_0, y - (k - 1)\epsilon/n_0); (0, \epsilon/n_0)]$ for the same values of $k$, to obtain the same conclusions. From each case, it follows inductively that

$$z_0 \geq (z_1z_{-1})^{1/2} \geq (z_2z_0z_{-2})^{1/4} \geq \cdots \geq \prod_{j=0}^{n_0} (z_j^{n_0})^{-2n_0} \geq \cdots$$

At each step, no power of $z_{\pm n_0}$ is changed, while the remaining powers $z_j^{\gamma}$ are lower-bounded by $(z_{j-1}z_{j+1})^{\gamma/2}$. The exponents of the $z_j$ give probability distributions on $S := \{z_{-n_0}, \ldots, 0, 1, \ldots, n_0\}$ corresponding to the symmetric gambler’s ruin, i.e. a simple random walk on the state space $S$ with absorbing barriers $z_{\pm n_0}$. The transition probabilities here for all other states $z_j$ are 1/2 for $z_j \mapsto z_{j+1}$. Since at each stage we moreover have equal powers of $z_{\pm n_0}$, it follows by Markov chain theory (or one can show via a direct argument that $z_0 \geq \sqrt[n_0]{z_{-n_0}z_{n_0}}$). Hence $-\log \Lambda$ is midpoint-convex and measurable on $I$. It follows by Sierpiński’s well-known result [33] that $-\log \Lambda$ is continuous on the interior of $I$, whence convex, and so $\Lambda$ is also continuous on the interior of $I$.

Finally, to show $-\log \Lambda$ is convex on $I$, it suffices to show for $a, b \in I$ and $\lambda \in (0, 1)$ that $\log \Lambda(\lambda a + (1 - \lambda)b) \geq \lambda \log(a) + (1 - \lambda)\log(b)$. But this can be shown by approximating $\lambda$ by dyadic rationals $\lambda_n \in (0, 1)$ for all $n \geq 1$. For each of these, the above mid-convexity implies:

$$\log \Lambda(\lambda_na + (1 - \lambda_n)b) \geq \lambda_n \log(a) + (1 - \lambda_n)\log(b), \quad \forall n \geq 1.$$  

Letting $n \to \infty$, since $\Lambda$ is continuous on the interior of $I$, it follows that $-\log \Lambda$ is convex on $I$. This completes the proof of (2) $\implies$ (1).

\[\text{Proof of Theorem 1}\] We modify as follows, the arguments of Schoenberg’s proof of Theorem 4 in [31]. Let $0 < \epsilon < \rho/2 < \bar{\rho} - \rho/2$, and work with integers $m > (p - 1)\rho/\epsilon$. Then the following increasing, equi-spaced arithmetic progressions fall in the specified domains:

$$x := (0, \frac{\rho}{m + 1}, \frac{2\rho}{m + 1}, \ldots, \frac{(p - 1)\rho}{m + 1}) \in [0, \epsilon)^{p+\uparrow}$$

$$y := (-m\rho, \frac{-m(2m - 2)\rho}{2m + 2}, \ldots, \frac{(m + 2p - 2)\rho}{2m + 2}) \in (-\rho/2, (\rho/2) + \epsilon)^{m+p+\uparrow}.$$  

Hence the matrix $T_\Lambda[x; y]$ is TN; reversing the order of the rows and columns, the matrix

$$A_m := \begin{pmatrix}
a_0 & a_1 & \cdots & \cdots & a_{m-1} & a_m & 0 & 0 & \cdots & 0 \\
0 & a_0 & \cdots & \cdots & a_{m-1} & a_m & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & a_{m-p+1} & a_{m-p+2} & a_{m-p+3} & \cdots & a_m & \end{pmatrix}_{p \times (m+p)}$$

is TN, where we define $a_\nu := \Lambda\left(\frac{(2\nu - m)\rho}{2m+2}\right) > 0$ for $\nu = 0, 1, \ldots, m$.

Once this matrix is constructed, repeat the proof of [31, Theorem 1]. This shows the polynomial

$$p_m(z) := \frac{\rho}{m + 1} \sum_{\nu=0}^{m} \Lambda((2\nu - m)\rho/(2m+2))z^\nu$$

\[\text{Indeed, if } c_t \text{ denotes the sum of the exponents for } z_{-(n_0-1)}, \ldots, z_t, z_t, \ldots, z_{n_0-1} \text{ at 'time' } t, \text{ then one shows via the AM–GM inequality that } c_{t+2(n_0-1)} \leq c_t(1 + 2^{-1-n_0}). \text{ Now let } t = m(2n_0 - 1), \text{ with } n_0 \to \infty.$$

\[\text{This proof can be found in Karlin’s book – see [29, Chap. 8, Theorem 3.1] – and uses the variation-diminishing property of the TN matrix } A_m, \text{ as shown by Schoenberg [29].}\]
has no roots in the sector $|\arg(z)| < p\pi/(m + p - 1)$.

Now given $s \in \mathbb{C}$ and $m \geq 1$, let $z = e^{-sp/(m+1)}$, and consider the holomorphic function

$$F_m(s) := \frac{p}{m+1} \sum_{\nu=0}^{m} e^{-s(2\nu-m)p/(2m+2)} \Lambda((2\nu - m)p/(2m + 2)) = e^{smp/(2m+2)}\rho_m(z), \quad s \in \mathbb{C}. $$

From above, $F_m(s)$ has no zeros in the strip

$$|\Im(s)| < \frac{p\pi(m+1)}{p(m + p - 1)} = \frac{p\pi}{p} \left(1 - \frac{p-2}{m + p - 1}\right)$$

for all $m$ sufficiently large. If $p = 2$ then this concludes the proof; else fixing $\delta \in (0, p\pi/p)$, $F_m$ has no zeros $s$ satisfying: $|\Im(s)| < (p\pi/p) - \delta$. By Lemma 6.1 the holomorphic Riemann sums $F_m(s)$ converge to $B\{\Lambda\}(s)$ uniformly on each bounded domain, so by Hurwitz’s theorem, $B\{\Lambda\} \neq 0$ also has no root $s$ with $|\Im(s)| < (p\pi/p) - \delta$. As this holds for all $\delta \in (0, p\pi/p)$, the proof is complete. □

**Remark 6.2.** As noted following Theorem E, the hypotheses therein require using that the restriction of $\Lambda$ to the interval $I(\epsilon) := (-\epsilon, \epsilon) \subset (0, \infty)$ is $\mathcal{N}_p$. If this can be strengthened to using only $I(0) = (-\rho/2, \rho/2)$, then this would answer Question 5.4 in the affirmative, by specializing to $\Lambda = W$, $\rho = \pi$, and translating from $T_W$ to $K_{\mathcal{K}S}$ via $\operatorname{arctan}$ as above. Indeed, the above strengthening would imply that the following function has no roots $s$ with $|\Im(s)| < p$:

$$B\{W^\alpha\}(s) = \int_{-\pi/2}^{\pi/2} e^{-sx} \cos(x)^\alpha \, dx.$$ 

Since $\alpha \in [0, \infty)$ here, the right-hand side can be computed using a well-known, classical formula of Cauchy [31 pp. 40], or directly as in [31 §10], to yield:

$$B\{W^\alpha\}(s) = \frac{\pi \Gamma(\alpha + 1)}{2^n \Gamma\left(\frac{1}{2}(\alpha + 2 + si)\right) \Gamma\left(\frac{1}{2}(\alpha + 2 - si)\right)},$$

and this has roots at $s = \pm(\alpha + 2)i$. It follows that $\alpha + 2 = |\alpha + 2| \geq p$. This also explains how Schoenberg’s work implies that $T_{W^\alpha}$ is not $\mathcal{N}_p$ for $\alpha \in (0, p-2)$.

7. **Theorem [1]** Characterizing $\mathcal{N}_p$ functions

Finally, we come to Theorem E and a few related variants, which characterize not only $\mathcal{N}_p$ functions $\Lambda : \mathbb{R} \rightarrow \mathbb{R}$, but also $\mathcal{N}_p$ kernels $K : X \times Y \rightarrow (0, \infty)$ for general $X,Y \subset \mathbb{R}$. A characterization of $\mathcal{N}_p$ functions is known for $p = 2$ by Schoenberg [30]; for $p = 3$ an analogous result can be found in Weinberger’s work [33], but it seems to have a small gap, owing to the following lemma.

**Lemma 7.1.** For all $d \in [0,1]$, the following ‘Heaviside’ function is $\mathcal{N}$, whence $\mathcal{N}_3$:

$$H_d(x) = \begin{cases} 0, & x < 0, \\ d, & x = 0, \\ 1, & x > 0. \end{cases} \quad (7.2)$$

In particular, the function $\lambda_d(x) := e^{-x}H_d(x)$ is a Pólya frequency function.

Weinberger’s result [33, Theorem 1] asserts in particular that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is $\mathcal{N}_3$, then either $f(x) = H_1(ax+b)e^{cx+c'}$ for suitable scalars $a,b,c,c' \in \mathbb{R}$, or the nonzero-locus of $f$ is an open interval. However, $H_d, \lambda_d$ are nonzero on $[0, \infty)$ and are $\mathcal{N}$ for $d \in (0,1)$ as well.

Lemma 7.1 was stated and used in [3] without a proof; moreover, we were unable to find the functions $H_d, \lambda_d$ in the text of Karlin [23]. Even Schoenberg, in [30 Corollary 2], mentions that the only discontinuous Pólya frequency function is “essentially equivalent to” $\lambda(x) = e^{-x}1_{x \geq 0}$. In
particular, Schoenberg does not mention $H_d, \lambda_d$ either. Thus, in the interest of future clarity, we record a proof of Lemma 7.1.

Proof. Let $p \geq 1$ and $x, y \in \mathbb{R}^p$; define $M := T_{H_d}[x; y]$. We prove that $\det M \geq 0$ by induction on $p$. The base case $p = 1$ is clear; for the induction step, assume $p \geq 2$ and consider various sub-cases:

1. If $x_1 < y_2$, then all entries in the first row vanish, except at most the first entry. Hence,
   \[ \det T_{H_d}[x; y] = H_d(x_1 - y_1) \det T_{H_d}[x'; y'], \quad \text{where } x' = (x_2, \ldots, x_p), \ y' = (y_2, \ldots, y_p). \]

   Now the induction hypothesis implies $\det T_{H_d}[x; y] \geq 0$.

2. Otherwise, suppose henceforth that $y_1 < y_2 \leq x_1$. First suppose $y_2 = x_1$; subtracting the second row of the matrix $M$ from the first yields a matrix with first column $(1 - d, 0, \ldots, 0)^T$.

   Now expand along the first column and use the induction hypothesis.

3. Finally, if $y_1 < y_2 < x_1$, then the two first columns of $T_{H_d}[x; y]$ are identical, so $\det M = 0$.

Finally, given any $T_N^p$ function $f(x)$ for $p \geq 1$, and scalars $a, b \in \mathbb{R}$, the function $e^{ax + b}f(x)$ is also $T_N^p$, since for all $1 \leq r \leq p$ and $x, y \in \mathbb{R}^{r \times 1}$, the matrix
   \[ T_{e^{ax + b}}[x; y] = D \cdot T_f[x; y] \cdot D', \]

where $D, D'$ are diagonal $r \times r$ matrices with $(j, j)$ entries $e^{ax_j + b}$ and $e^{-ay_j}$ respectively. In particular, the matrix on the left again has non-negative determinant. Hence $\lambda_d$ is also $T_N$.

To our knowledge, there are no other characterization results for $T_N^p$ functions in the literature, prior to Theorem 7.1. This result will follow from a more general formulation:

Proposition 7.3. Let $t_*, \rho \in \mathbb{R}$ and fix a subset $Y \subset \mathbb{R}$ that is not bounded above. Suppose $X \subset \mathbb{R}$ contains $t_* + y$ for all $\rho < y \in Y$. Let $\Lambda : X - Y \to [0, \infty)$ be such that $\Lambda(t_*) > 0$ and
   \[ \lim_{y \in Y, \rho < y \to +\infty} \Lambda(x_0 - y)\Lambda(t_* + y - y_0) \to 0, \quad \forall x_0 \in X, \ y_0 \in Y. \]

If $\det T_{\Lambda}[x; y] \geq 0$ for all $x \in X^{p, \uparrow}, y \in Y^{p, \uparrow}$, then the kernel $T_{\Lambda}$ is $T_N^p$.

Proposition 7.3 extends a recent result of Förster–Kieburg–Kösters [9] in two ways: first, it works over a large class of domains $X, Y \subset \mathbb{R}$, whereas the result in [9] requires $X = Y = \mathbb{R}$. Second, even assuming $X = Y = \mathbb{R}$, the result in [9] requires $\Lambda$ to be integrable; however, Proposition 7.3 works for all $T_N^p$ functions, such as (via Remark 7.1)
   \[ \Lambda(x) = \begin{cases} ce^{\beta(x - x_0)}, & \text{if } x \leq x_0, \\ ce^{\alpha(x - x_0)}, & \text{if } x > x_0, \end{cases} \quad \text{where } -\infty \leq \alpha < \beta \leq +\infty, \ c > 0. \] (7.4)

If now $\alpha, \beta \geq 0$, then $\Lambda$ is not integrable, but the hypotheses in Proposition 7.3 are satisfied.

Proof of Proposition 7.3. We show by downward induction on $1 \leq r \leq p$ that all $r \times r$ minors of $T_{\Lambda}$ on $X \times Y$ are non-negative. The $r = p$ case is obvious, and it suffices to deduce from it the $r = p - 1$ case. Thus, fix $x' \in X^{p-1, \uparrow}$ and $y' \in Y^{p-1, \uparrow}$. We are to show that
   \[ \psi(x_p, y_p) := \det T_{\Lambda}[(x', y'); (y', y_p)] \geq 0 \quad \forall x_p > x_{p-1}, y_p > y_{p-1} \implies \det T_{\Lambda}[x'; y'] \geq 0. \]

We now refine the argument in [9]. Begin by defining the $(p - 1) \times (p - 1)$ matrix $A := T\Lambda[x'; y']$, and let $A^{j,k}$ denote the submatrix obtained by removing the $j$th row and $k$th column of $A$. (Since $p \geq 3$, these matrices are at least $1 \times 1$.) Now the following scalar does not depend on $x_p, y_p$:
   \[ L := \max_{1 \leq j, k \leq p-1} |\det A^{j,k}| \geq 0. \] (7.5)

Next, define $t_m \in Y$ for all $m \geq 1$ such that $t_m > \max\{x_{p-1} - t_*, y_{p-1}, \rho\}$ and
   \[ \Lambda(x_j - t_m)\Lambda(t_s + t_m - y_k) < 1/m, \quad \forall 0 < j, k < p. \]
With these choices made, we turn to the proof. Begin by expanding $\psi(x_p, y_p)$ along the final row, and excluding the cofactor for $(p, p)$, expand all other cofactors along the final column, to get:

$$
\psi(x_p, y_p) = \Lambda(x_p - y_p) \det(A) + \sum_{j,k=1}^{p-1} (-1)^{j+k-1} \Lambda(x_j - y_p) \Lambda(x_p - y_k) \det A^{(j,k)}.
$$

Define $y_p^{(m)} := t_m$ and $x_p^{(m)} := t_* + t_m$, with $t_*, t_m$ as above. Then

$$
x_p^{(m)} \in X, \quad x_p^{(m)} > x_{p-1}, \quad y_p^{(m)} \in Y, \quad y_p^{(m)} > y_{p-1}.
$$

Moreover, since $\psi(x_p^{(m)}, y_p^{(m)}) \geq 0$, we compute for $m \geq 1$:

$$
\Lambda(t_*) \det(A) \geq \psi(x_p^{(m)}, y_p^{(m)}) - L \sum_{j,k=1}^{p-1} \Lambda(x_j - y_p^{(m)}) \Lambda(x_p^{(m)} - y_k) \geq -L \frac{(p-1)^2}{m}.
$$

Now taking $m \to \infty$ concludes the proof, since $\Lambda(t_*) > 0$ by assumption. \hfill \Box

**Remark 7.6.** Proposition 7.3 specializes to $X = Y = G$, an arbitrary additive subgroup of $(\mathbb{R}, +)$. E.g. for $G = \mathbb{Z}$, we obtain a result – whence a characterization, akin to Theorem E and results below – for Pólya frequency sequences of order $p'$ that vanish at $\pm \infty$. Here, $t_*$ would be an integer.

With Proposition 7.3 at hand, the final outstanding proof follows.

**Proof of Theorem 2.** If $\Lambda \equiv 0$ then the result is immediate. If $\Lambda(x) = e^{ax+b}$ then the result is again easy, since by the argument to show Lemma 7.1 it suffices to show the case of $a = b = 0$, which is obvious. Now suppose $\Lambda$ is not of the form $ce^{ax}$ for $a \in \mathbb{R}$ and $c \geq 0$. Then (2) follows by Proposition 7.3 with arbitrary $\rho \in \mathbb{R}$.

Conversely, suppose $\Lambda$ is not of the form $ce^{ax}$ for $a \in \mathbb{R}$ and $c \geq 0$. Since it is TN$_p$, clearly (1)(a),(c) follow. In particular, since $\Lambda$ is also TN$_2$, $g(x) := \log(\Lambda(x))$ is concave on $\mathbb{R}$ (in the generalized sense, i.e., it is allowed to take the value $-\infty$), by Lemma 6.1. Now let $I$ be the nonzero-locus of $\Lambda$. If $I$ is not all of $\mathbb{R}$, then (1)(b) is immediate. If instead $\Lambda(x) > 0$ for all $x \in \mathbb{R}$, then since $\Lambda$ is not an exponential, $g(x)$ is not linear from above. Hence a short argument of Schoenberg [30] shows that there exist $\beta, \gamma \in \mathbb{R}$ and $\delta > 0$ such that

$$
e^{-\gamma x} \Lambda(x) \leq e^{\beta - \delta |x|}, \quad \text{as } x \to \pm \infty.
$$

From this, the decay property (1)(b) immediately follows. \hfill \Box

We conclude by extending the above result to arbitrary positive-valued kernels on $X \times Y$:

**Proposition 7.7.** Let $X,Y \subset \mathbb{R}$ be non-empty, and $K : X \times Y \to (0,\infty)$ a kernel satisfying any of the following decay conditions:

- $\sup Y \not\in \mathbb{R}$, $\lim_{y \to (\sup Y)^-} K(x_0, y) = 0$, $\forall x_0 \in X$,
- $\inf Y \not\in \mathbb{R}$, $\lim_{y \to (\inf Y)^+} K(x_0, y) = 0$, $\forall x_0 \in X$,
- $\sup X \not\in \mathbb{R}$, $\lim_{x \to (\sup X)^-} K(x, y_0) = 0$, $\forall y_0 \in Y$,
- $\inf X \not\in \mathbb{R}$, $\lim_{x \to (\inf X)^+} K(x, y_0) = 0$, $\forall y_0 \in Y$.

Given an integer $p \geq 2$, the kernel $K$ is TN$_p$ on $X \times Y$, if and only if every $p \times p$ minor of $K$ is non-negative.

---

9Since $g$ is concave, $g'$ exists and is non-increasing on a co-countable subset of $\mathbb{R}$. Since $g'$ is not constant, there exist scalars $x_- < x_+$ and $c_\pm$ such that $g'(x_-) > g'(x_+)$ and $\log \Lambda(x) \leq g'(x_\pm) x + c_\pm$. Choose $\gamma, \delta \in \mathbb{R}$ such that $g'(x_+) < \gamma - \delta < \gamma + \delta < g'(x_-)$. Then $\log \Lambda(x) - \gamma x$ is bounded above by $(g'(x_\pm) - \gamma)x + c_\pm$, for $\pm x > 0$. 
For instance, this can be specialized to kernels over \( X = Y = G \), an additive subgroup of \((\mathbb{R}, +)\).

**Proof.** One implication is immediate. Conversely, as in the preceding proofs it suffices to show that \( K[x'; y'] \geq 0 \) for all tuples \( x' \in X^{p-1}, y' \in Y^{p-1} \). We show this under the fourth decay condition; the other cases are similar to this proof and the proofs above. Fix increasing tuples \( x := (x_2, \ldots, x_p) \in X^{p-1}, y := (y_2, \ldots, y_p) \in Y^{p-1} \) as well as \( y_1 \in (-\infty, y_2) \cap Y \). Let \( A = K[x'; y'] \) and define \( L \geq 0 \) as in (7.5) above. Also choose for each \( m \geq 1 \) an element \( x_1^{(m)} \in X \), such that \( x_1^{(m)} < x_2 \) and \( K(x_1^{(m)}, y_k) < 1/m \) for \( 2 \leq k \leq p \). Now compute as in the proof of Proposition 7.3, this time expanding the determinant along the first row and column:

\[
K(x_1^{(m)}, y_1) \det(A) \geq \det K[(x_1^{(m)}, x'); (y_1, y')] - L \sum_{j,k=2}^{p} K(x_j, y_1)K(x_1^{(m)}, y_k)
\]

\[
\geq \det K[(x_1^{(m)}, x'); (y_1, y')] - \frac{L(p-1)}{m} \sum_{j=2}^{p} K(x_j, y_1).
\]

As \( \det K[(x_1^{(m)}, x'); (y_1, y')] \geq 0 \) and \( K(x_1^{(m)}, y_1) > 0 \), the result follows by letting \( m \to \infty \). \(\square\)

**Remark 7.8.** We have tried to keep the proofs of the results in our main theme self-contained (modulo the Appendix) – specifically, for the results related to powers preserving TN\(_p\). The only three such proofs that use prior results are those of Theorems [B] Theorem C(1); and Theorem D which use Lemma 4.1 (Fekete); Theorem 1.5 (Schoenberg); and Schoenberg’s [31, Theorem 1] plus Sierpiński’s result [33], respectively.

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In the interest of keeping this paper as self-contained as possible, this Appendix contains short proofs (from the original papers) of the results which are stated above and are used in proving our main theorems. The reader is welcome to skip these proofs (certainly in a first reading).

**Proof of Theorem 1.1(1).** We show the ‘if’ part; the converse was shown in the proof of Theorem 1.10(1). If \( \alpha \in \mathbb{Z}_{\geq 0} \) then \( x^\alpha \) preserves Loewner positivity by the Schur product theorem \[32\]. If \( \alpha \geq n - 2 \), we show the result by induction on \( n \geq 2 \), with the \( n = 2 \) case obvious. Suppose \( n \geq 3 \) and \( A \in \mathbb{P}_n((0, \infty)) \). Let \( \zeta \) denote the last column of \( A \), and \( B := a_{nn}^{-1} \zeta \zeta^T \). Then \( B \succeq 0 \); moreover, \( A - B \) has last row and column zero, and is itself positive semidefinite via Schur complements. Now
FitzGerald–Horn employ a useful ‘integration trick’: by the Fundamental Theorem of Calculus,

\[ A^\alpha = B^\alpha + \alpha \int_0^1 (A - B) \circ (\lambda A + (1 - \lambda)B)^{(\alpha - 1)} \, d\lambda. \]

But \( A - B \) has last row/column zero, and the leading principal \((n - 1) \times (n - 1)\) submatrix of the integrand is in \( \mathbb{P}_{n-1}(\mathbb{R}) \) by the induction hypothesis. We are done by induction. \( \Box \)

**Proof of Theorem 7.4** for integer powers. For integers \( \alpha \geq 0 \), the proof that \( x^\alpha e^{-\alpha x} 1_{x \geq 0} \) is a Pólya frequency function is in steps. We first show that the kernel \( K(x, y) := 1_{x \geq y} \) is TN on \( \mathbb{R} \times \mathbb{R} \). This is a direct calculation; e.g., Karlin [23, pp. 16] checks for the ‘transpose’ kernel \( K(x, y) := 1_{x \leq y} \):

\[ \det K[x; y] = 1(x_1 \leq y_1 < x_2 < \cdots < x_p \leq y_p), \]

for all \( p \geq 1 \) and tuples \( x, y \in \mathbb{R}^p \). Now pre- and post-multiplying with diagonal matrices with \((k, k)\) entries \( e^{-x_k} \) and \( e^{y_k} \) respectively, shows that the kernel \( \Omega_0(x) := e^{-x} 1_{x \geq 0} \) is a Pólya frequency function. Next, the ‘Basic Composition Formula’ of Pólya–Szegö (see e.g. [23] pp. 17] shows that the class of Pólya frequency functions is closed under convolution. But for any integer \( \alpha \geq 1 \), the \( \alpha\)-fold convolution of \( \Omega_0(x) \) with itself, yields precisely \( x^{\alpha - 1} e^{-x} 1_{x \geq 0} \). Finally, multiplying with a suitable exponential function shows \( \Omega^\alpha \) is still integrable, so also a Pólya frequency function. \( \Box \)

**Remark A.1.** Let \( \Lambda(x) \) be as in \( \text{(7.4)} \). First if \( |\alpha| \) or \( |\beta| \) is infinite, then \( \Lambda \) is essentially \( \lambda_0 \) or \( \lambda_1 \) (up to a linear change of variables), and hence is TN. Similarly if \( \alpha = \beta \) then \( \Lambda \) is an exponential – up to rescaling – so any submatrix drawn from it is a rank-one matrix. Hence \( \Lambda \) is TN. Finally, suppose \( \alpha < \beta \in \mathbb{R} \). As explained in Lemma \( \text{(7.1)} \) the function \( \lambda_1(x) = e^{-x} 1_{x \geq 0} \) is TN, whence so is \( \lambda_1(-x) \). As in the preceding proof, the Basic Composition Formula implies that \( \lambda_1(x) * \lambda_1(-x) = e^{-|x|}/2 \) is also TN. By a linear change of variables, the function \( e^{(\alpha - \beta)|x|/2} \) is TN. Multiplying by \( e^{(\alpha + \beta)|x/2} \), the function in \( \text{(7.4)} \) is also TN.

**Proof of Proposition 2.2.** In this proof-sketch, we also address a small gap in \( \text{(19)} \). The first step is to observe that \( 1 + u x_j > 0 \) for all \( j \) if and only if \( u \in (A_X, B_X) \). Also note that

\[ A_{-X} = -B_X, \quad \text{and} \quad A_X < 0 < B_X, \quad \forall x \in \mathbb{R}^n. \quad \text{(A.2)} \]

We now sketch the proof in \( \text{(19)} \). If \( r = 0 \) then the result is immediate, so we suppose henceforth that \( r \neq 0 \). Denote by \( s \leq n - 1 \) the number of sign changes in \( c \) after removing the zero coordinates. We then claim that the number of zeros is at most \( s \); the proof is by induction on \( n \geq 1 \) and then on \( s \geq 0 \). The base cases of \( n = 1 \), and \( s = 0 \) for any \( n \geq 1 \), are easy to show. For the induction step, we may suppose all \( c_j \) are non-zero, and the \( x_j \) are in increasing order.

The first case is that whenever there is a sign change in \( c \), i.e. \( c_{k-1} c_k < 0 \), we always have \( x_k \leq 0 \). (This is a small clarification that was not addressed in \( \text{(19)} \); on a related note, \( \text{(A.2)} \) does not appear there.) In this case we simply replace \( x \) by \( -x \) and \( c \) by \( c' := (c_n, \ldots, c_1) \). So the assertion for \( \varphi_{-x, c', r} : (-B_X, -A_X) \to \mathbb{R} \) (via \( \text{(A.2)} \)) would show the result for \( \varphi_{x, c, r} \).

Thus there exists \( k \) with \( c_{k-1} c_k < 0 < x_k \). In turn, there exists \( u > 0 \) with \( 1 - v x_k < 0 < 1 - v x_{k-1} \), so that the sequence \( c_j(1 - v x_j), j = 1, \ldots, n \) has one less sign change than \( c \). Now define

\[ \psi(u) := \sum_{j=1}^n c_j (1 - v x_j)(1 + u x_j)^{r-1}, \quad h(u) := (u + v)^{r} \varphi_{x, c, r}(u), \quad u \in (A_X, B_X), \]

so the induction hypothesis applies to \( \psi \). But a straightforward computation yields

\[ \psi(u) = -\frac{(u + v)^{r+1}}{r} h'(u), \quad \text{and} \quad u + v > 0, \quad \forall u \in (A_X, B_X), \]

so by the induction hypothesis, \( h' \) has at most \( s - 1 \) zeros. We are done by Rolle’s theorem. \( \Box \)
Proof-sketch of Proposition 3.1. Suppose $\alpha \in \mathbb{R} \setminus \{0, 1, \ldots, n - 2\}$, and $S^{\alpha} c^T = 0$ for a tuple $c = (c_1, \ldots, c_n) \neq 0$. Rewriting this in the language of Proposition 2.2 yields:

$$\varphi_{x, c, \alpha}(y_k) = \sum_{j=1}^{n} c_j (1 + y_k x_j)^\alpha = 0, \quad \forall 1 \leq k \leq n.$$ 

By assumption, $y_k \in (A_x, B_x)$ for all $k$ (see the line preceding (A.2)), so Proposition 2.2 implies $\varphi_{x, c, \alpha} \equiv 0$ on $(A_x, B_x)$. By (A.2), $\varphi_{x, c, \alpha}(0) = 0$, $\forall 0 \leq k \leq n - 1$. This system can be written as

$$W^{(n-1)}_x D c^T = 0, \quad \text{where } W^{(r)}_x := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^r & x_2^r & \cdots & x_n^r \end{pmatrix}, \quad r \in \mathbb{Z}^>0$$

and $D$ is the diagonal matrix with diagonal entries $1, \alpha, \alpha - 1, \ldots, \alpha - 1 \cdots (\alpha - n + 2)$. By assumption on $\alpha$, the matrix $D$ is non-singular, as is the (usual) Vandermonde matrix $W^{(n-1)}_x$. Hence $c = 0$, and so $S^{\alpha} c$ is non-singular.

Finally, if $\alpha \in \{0, \ldots, n - 2\}$, then $S^{\alpha} = (W^{(\alpha)}_x)^T D W^{(\alpha)}_x$, where $W^{(\alpha)}_x$ was defined above, and $D$ is a diagonal $(\alpha + 1) \times (\alpha + 1)$ matrix with $(k, k)$ entry $\binom{n}{k}$. Since these matrices are each of maximal possible rank, the result follows. \(\square\)

Proof of Lemma 3.5. First suppose $0 \leq B \leq A$ are as claimed. For $\lambda \in (0, 1)$, the Loewner convexity condition can be reformulated in two ways:

$$\frac{f[B + \lambda (A - B)] - f[B]}{\lambda} \leq f[A] - f[B],$$

$$\frac{f[A + (1 - \lambda)(B - A)] - f[A]}{1 - \lambda} \leq f[B] - f[A].$$

Now let $\lambda \to 0^+$ and $\lambda \to 1^-$, respectively. We obtain:

$$(A - B) \circ f'[B] \leq f[A] - f[B], \quad (B - A) \circ f'[A] \leq f[B] - f[A].$$

Summing these inequalities gives $(A - B) \circ (f'[A] - f'[B]) \geq 0$. Since $A - B$ has only non-zero entries, it has a positive semidefinite ‘Schur-inverse’. Take the Schur product with this matrix to obtain $f'[A] \geq f'[B]$, as claimed. Adapting the same argument shows that $f'[A\lambda] \geq f'[A\mu] \forall 0 \leq \mu \leq \lambda \leq 1$, where $A\lambda := \lambda A + (1 - \lambda)B$.

Conversely, suppose $0 \leq B \leq A$ in $\mathbb{P}_n((0, \infty))$ are arbitrary, and $f'$ preserves Loewner monotonicity on $[B, A]$. In the spirit of previous proofs for powers preserving Loewner positivity and monotonicity (see above), another ‘integration trick’ yields:

$$f[(A + B)/2] - f[B] = \frac{1}{2} \int_0^1 (A - B) \circ f' \left[ \lambda \frac{A + B}{2} + (1 - \lambda)B \right] d\lambda,$$

$$f[A] + f[B] \over 2 - f[B] = \frac{f[A] - f[B]}{2} = \frac{1}{2} \int_0^1 (A - B) \circ f' \left[ \lambda A + (1 - \lambda)B \right] d\lambda.$$  \hspace{1cm} (A.3)

Using the Schur product theorem and the hypotheses on $f'$,

$$(A - B) \circ f' \left[ \lambda \frac{A + B}{2} + (1 - \lambda)B \right] \leq (A - B) \circ f'[\lambda A + (1 - \lambda)B].$$

Together with (A.3), this yields $f[(A + B)/2] \leq \frac{1}{2}(f[A] + f[B])$. Now an easy induction argument, first on $m \geq 1$ and then on $k \in [1, 2^m]$, yields

$$f \left[ \frac{k}{2^m} A + \left( 1 - \frac{k}{2^m} \right) B \right] \leq \frac{k}{2^m} f[A] + \left( 1 - \frac{k}{2^m} \right) f[B], \quad \forall m \geq 1, \ 1 \leq k \leq 2^m.$$
Finally, given \( \lambda \in (0,1) \) we approximate \( \lambda \) by a sequence of dyadic rationals of the form \( k/2^n \).

Now the preceding inequality and the continuity of \( f \) allows us to deduce that \( f \) preserves Loewner convexity on \( \{ A, B \} \). The same arguments can be adapted, as in the preceding half of this proof, to show that \( f \) preserves Loewner convexity on \( \{ A_\lambda, A_\mu \} \) for \( 0 \leq \mu \leq \lambda \leq 1 \).

\( \square \)

**Proof of Corollary 4.2.** For the ‘if’ part, note that every contiguous minor of a Hankel matrix \( A \) is a contiguous principal minor of either \( A \) or \( A^{(1)} \). This shows the result for TP(\( p \)) by Fekete’s lemma 4.1. For TN(\( p \)), first let \( B \) be a matrix drawn from the Gaussian kernel, say \( B = (e^{-(x_j - y_k)^2})_{j,k=1}^n \), with \( x, y \in \mathbb{R}^{n \times 1} \). Then \( B = D_x V D_y \), where \( D_x \) for a vector \( x \) is the diagonal matrix with \( (k, k) \) entry \( e^{-x_k^2} \), and \( V \) is the generalized Vandermonde matrix with \( (j, k) \) entry \( e^{2x_j y_k} = (e^{2x_j})^{y_k} \), whence non-singular. As every submatrix of \( B \) is of this form, it follows that \( B \) is TP.

Now given \( A_{n \times n} \) Hankel as specified, we have that all contiguous minors of \( A \) of order \( \leq p \) are non-negative. Since the corresponding submatrices are symmetric (and Hankel), it follows that they are all positive semidefinite. Let \( B := (e^{-(j-k)^2})_{j,k=1}^n \); then \( B \) is TP from above. It follows for \( \epsilon > 0 \) that every contiguous submatrix of \( A + \epsilon B \) of order \( \leq p \) is positive definite. By Fekete’s result, \( A + \epsilon B \) is TP. Letting \( \epsilon \to 0^+ \), \( A \) is TN(\( p \)). The ‘only if’ part follows by definition. \( \square \)

**Proof of continuity in Proposition 4.3.** We claim that \( f \equiv 0 \) or \( f > 0 \) on \( (0, \infty) \). Indeed, suppose \( f(x_0) = 0 \) for some \( x_0 > 0 \). Choose \( 0 < x < x_0 < y \), apply \( f \) entrywise to the Hankel TN matrices

\[
\begin{pmatrix}
x_0 & x \\
x & x_0
\end{pmatrix}, \quad \begin{pmatrix}
x_0 & y \\
y & x_0
\end{pmatrix}
\]

and take determinants. It follows that \( f(x) = f(y) = 0 \), as desired. Using the first of the above test matrices also shows that \( f \) is non-decreasing on \( (0, \infty) \).

Now suppose \( f > 0 \) on \( (0, \infty) \), and fix \( t > 0 \). We present Hiai’s argument from [14] to show \( f \) is continuous at \( t \). For \( \epsilon \in (0, t/5) \), we have \( 0 < t + \epsilon \leq \sqrt{(t + 4\epsilon)(t - \epsilon)} \). It follows that

\[
f(t + \epsilon) \leq f \left( \sqrt{(t + 4\epsilon)(t - \epsilon)} \right) \leq \sqrt{f(t + 4\epsilon)f(t - \epsilon)},
\]

where the second inequality follows by taking the determinant, after applying \( f \) entrywise to the matrix

\[
\begin{pmatrix}
\frac{t + 4\epsilon}{\sqrt{(t + 4\epsilon)(t - \epsilon)}} & \sqrt{(t + 4\epsilon)(t - \epsilon)} \\
\sqrt{(t + 4\epsilon)(t - \epsilon)} & \frac{t - \epsilon}{\sqrt{(t + 4\epsilon)(t - \epsilon)}}
\end{pmatrix}.
\]

Now take \( \epsilon \to 0^- \); then continuity follows, since \( f \) is positive and non-decreasing on \( (0, \infty) \):

\[
0 < f(t) \leq f(t^-) \leq f(t) \leq f(t), \quad \forall t > 0.
\]

\( \square \)

**Proof of Lemma 4.4.** Let the function \( M(x) = 2e^{-|x|} - e^{-2|x|} \) for \( x \in \mathbb{R} \). For all integers \( n \geq 1 \),

\[
B\{M^n\}(s) = 2 \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \frac{2^{n-k}(n+k)}{s^2 - (n+k)^2} = \frac{p_n(s)}{q_n(s)},
\]

say, is the bilateral Laplace transform of \( M(x)^n \). Here the polynomial \( q_n(s) = \prod_{k=0}^{n} (s^2 - (n+k)^2) \) has all simple roots, and degree \( 2n + 2 \). It is easy to check that \( \text{deg}(p_n) \leq 2n \).

Now for \( n = 1 \) this yields \( 12/(s^2-1)(s^2-4) \), whose reciprocal is a polynomial, so classical results of Schoenberg [30] imply that \( M(x) \) is a Pólya frequency function. Also note that \( \text{deg}(p_n) \leq 2n \), and one checks by direct evaluation that \( p_n(\pm(n+k)) \) is non-zero for \( 0 \leq k \leq n \), so \( p_n \) does not vanish at any root \( \pm(n+k) \) of \( q_n \). Finally, \( p_n(n)/p_n(2n) \) is also checked to be \( > 1 \). Hence the rational function \( q_n/p_n \) is not a polynomial for \( n > 1 \) – in fact, not in the Laguerre–Pólya class. The aforementioned results of Schoenberg now imply that \( M(x)^n \) is not a Pólya frequency function. As \( M(x)^n \) is integrable and non-vanishing at two points, it follows that \( M(x)^n \) is not TN for \( n > 1 \).

\( \square \)