Freudenthal theorem and spherical classes in $H_*QS^0$

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Abstract

This note is on spherical classes in $H_*(QS^0; k)$ when $k = \mathbb{Z}, \mathbb{Z}/p$ with a special focus on the case of $p = 2$ related to Curtis conjecture. We apply Freudenthal theorem to prove a vanishing result for the Hurewicz image of elements in $\pi_*^s$ that factor through certain finite spectra. Either in $p$-local or $p$-complete settings, this immediately implies that elements of well known infinite families in $p\pi_*^s$, such as Mahowaldean families, map trivially under the unstable Hurewicz homomorphism $p\pi_*^s \simeq p\pi_*QS^0 \to H_*(QS^0; \mathbb{Z}/p)$. We also observe that the image of the integral unstable Hurewicz homomorphism $\pi_*^s \simeq \pi_*QS^0 \to H_*(QS^0; \mathbb{Z})$ when restricted to the submodule of decomposable elements, is given by $\mathbb{Z}\{h(\eta^2), h(\nu^2), h(\sigma^2)\}$.

We apply this latter to completely determine spherical classes in $H_*(\Omega^nS^n+d; \mathbb{Z}/2)$ for certain values of $n > 0$ and $d > 0$; this verifies a Eccles’ conjecture on spherical classes in $H_*QS^n$, $n > 0$, on finite loop spaces associated to spheres.

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1 Introduction and statement of results

Let $QS^0 = \operatorname{colim} \Omega^iS^i$ be the infinite loop space associated to the sphere spectrum. Curtis conjecture then reads as follows (see [8, Proposition 7.1] and [22] for more discussions).
Conjecture 1.1 (Curtis Conjecture). In positive degrees only the Hopf invariant one and Kervaire invariant one elements survive under the unstable Hurewicz homomorphism $h : 2\pi_* \simeq 2\pi_* QS^0 \to H_* QS^0$.

Here and throughout, we write $p\pi_*^i$ and $p\pi_*$ for the $p$-components of $\pi_*^i$ and $\pi_*$, respectively. We also write $H_*$ for $H_*(-; k)$ where the coefficient ring will be clear from the context with an interest in $k = \mathbb{Z}/p$; although some of our results such as Theorem 1.3 holds when $k$ is an arbitrary $p$-local commutative coefficient ring. We work $p$-locally or $p$-complete which again will be clear from the context.

We wish to examine Conjecture 1.1 in the light of Freudenthal theorem. Suppose $E$ is a connected and connective spectrum, i.e. $\pi_* E \simeq 0$ for $i < 1$. We write $\Omega^\infty E = \text{colim} \Omega^i E_i$ for the infinite loop space associated to $E$, $QX$ for $\Omega^\infty (\Sigma^\infty X)$ when $X$ is a space, and $\epsilon : \Sigma^\infty \Omega^\infty E \to E$ for the evaluation map which is the stable adjoint to the identity $\Omega^\infty E \to \Omega^\infty E$. The map $\epsilon$ induces stable homology suspension $H_* \Omega^\infty E \to H_* E$. The following is a consequence of Freudenthal theorem.

**Proposition 1.2.** Suppose $E$ is $r$-connected with $r > 0$. Then, $\epsilon$ is a weak $(2r + 1)$-equivalence.

**Proof.** First, note that for a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ of maps among pointed spaces, the homotopy fibre of the induced map $\text{Fib}(gf) \to \text{Fib}(g)$, up to weak homotopy equivalence, can be identified with $\text{Fib}(f)$. Here, $\text{Fib}(f)$ denotes the homotopy fibre of $f$. In particular, for $X = Z$, if $gf$ is homotopic to the identity then $\text{Fib}(f)$ is weak homotopy equivalent to $\Omega \text{Fib}(g)$.

Now, let $E$ be an $r$-connected spectrum. Then, $X = \Omega^\infty E$ is an $r$-connected space. Since, $X$ is an infinite loop space then the composition $X \xrightarrow{\iota} QX \xrightarrow{\Omega^\infty \epsilon} X$ equals to identity $\Omega^\infty E$ (see also [3]). It follows that there is a weak homotopy equivalence $\text{Fib}(\Omega^\infty \epsilon) \to \Omega \text{Fib}(\iota)$. On the other hand, by Freudenthal theorem, the map $\iota : X \to QX$ induces an isomorphism $\pi_i X \to \pi_i QX \simeq \pi_i^r X$ for $i \leq 2r$. This means that $\iota$ is a $2r$-equivalence, that is $\pi_i \text{Fib}(\iota) \simeq 0$ for $i \leq 2r$, consequently $\pi_i \text{Fib}(\Omega^\infty \epsilon) \simeq 0$ for $i \leq 2r + 1$. This means that $(\Omega^\infty \epsilon)_* : \pi_* Q \Omega^\infty E \to \pi_* \Omega^\infty E$ is an isomorphism $i \leq 2r + 1$. Equivalently, $\epsilon_* : \pi_* \Sigma^\infty \Omega^\infty E \to \pi_* E$ is an isomorphism for $i \leq 2r + 1$. □

Note that if $E$ is a CW-spectrum, then the weak homotopy equivalence would give a homotopy equivalence. To state our next result, we fix our terminology; for a spectrum $E$ we refer to the Hurewicz homomorphism $h^r : \pi_* E \to H_* E$ as the stable Hurewicz homomorphism whereas, in positive degrees, we refer to $h : \pi_* E \simeq \pi_* \Omega^\infty E \to H_* \Omega^\infty E$ as the unstable Hurewicz homomorphism. We can apply the above equivalence to obtain some information on the image of the unstable Hurewicz homomorphism. We have the following.

**Theorem 1.3.** Suppose $f : S^n \to QS^0$ is given so that for some spectrum $E$ there is a factorisation $S^n \xrightarrow{f_E} \Omega^\infty E \to QS^0$. Then, the following statements hold.

(i) If $E$ is $r$-connected, $n \leq 2r + 1$, and $h^r(f_E) = 0$, then $h(f) = 0$. In particular, if $E$ is a CW-spectrum and $f_E$ maps trivially under $p$-local stable Hurewicz map $h^r(p) : p\pi_* E \to H_* (E; \mathbb{Z}/p)$ only for some prime $p$ then $f$ maps trivially under the $p$-local unstable Hurewicz map $p\pi_* QS^0 \to H_* QS^0$.

A similar statement holds in the $p$-complete setting.

(ii) If the above factorisation is induced by a factorisation of a map of spectra $S^n \xrightarrow{\alpha} E \xrightarrow{c} S^0$ where $E$ is a finite CW-spectrum of dimension $r$, $n \geq 2r + 1$, and $c_* = 0$ then $h(f) = 0$. In particular, if $c_* = 0$ holds $p$-locally only for some prime $p$ then $f$ maps trivially under the $p$-local unstable Hurewicz map $p\pi_* QS^0 \to H_* QS^0$. A similar statement holds in the $p$-complete setting.
As the first application of the $p$-local/$p$-complete version of the above Theorem, we have the following.

**Theorem 1.4.** At the prime $p = 2$, let $f$ be an element of one of the following families (some of which are only known to be potentially existing):

(i) Mahowald’s family, $\eta_i \in 2\pi_2 QS^0$ with $i \geq 3$, detected by $h_1 h_i$ in the ASS;

(ii) Lin’s family in $2\pi^a_{2i+18}$, $i > 10$, detected by $c_1 h_i$ in the Adams spectral sequence;

(iii) Lin’s family in $2\pi^a_{2i+1+37}$, $i > 12$, detected by $h_2^3 h_3 d_1$ in the Adams spectral sequence;

(iv) Bruner’s family, $\tau_i \in 2\pi^a_{2i+1+1}$, $i \geq 5$, detected by $h_2 h_i^2$ in the ASS;

Then, for $h : 2\pi_0^a \simeq 2\pi_0 QS^0 \to H_0 QS^0$ we have $h(f) = 0$.

At an odd prime $p$, the elements of

(vi) Cohen’s family in $p\pi^a_{2(p-1)p^2+2p-5}$ detected by $h_0 b_{j-1}$ map trivially $h : p\pi_0^a \simeq p\pi_0 QS^0 \to H_0 QS^0$.

The families mentioned in the above theorem, are well known families that are known (some only potentially) to exist in $p\pi_0^a$. There exist also hypothetical ways to construct infinite families in $p\pi_0^a$ in the literature which are constructed using certain maps $S^{2i} \to \mathbb{R}P^{2i-3}$ or $S^{2i} \to T(2i-3)$ as well as some elements factoring through $\mathbb{R}P^2$ or $\mathbb{C}P^2$; examples of families constructed by using either one of these constructions can be found in [14] Examples 4.5, 4.9, 5.4, 5.6] at $p = 2$, and [12] Example 3.9] at odd primes. We have the following.

**Theorem 1.5.** ($p = 2$) Let $f$ be an element of one of the families constructed in Kuhn that are detected by

(i) $bh_2 h_i$; (ii) $bh_3 h_{i-1}^2$; (iii) $ah_i^2$

where $a$ and $b$ are certain elements in the ASS as in [14] Theorems 4.4, 5.3] and [14] Theorem 5.5] respectively. Then, $f$ maps trivially into $H_0 QS^0$ under $h$.

(ii) Similar vanishing results hold for the elements that are represented by certain triple Toda brackets [14] Lemma 4.6, Theorem 4.8].

($p > 2$) Let $f$ be an element of the family considered by Hunter and Kuhn [12] Corollary 3.8] detected by $dh_0 h_i$ in the Adams spectral sequences. Then, $h(f) = 0$ for $h : p\pi_0^a \simeq p\pi_0 QS^0 \to H_0 QS^0$.

The proof of this is evident from the construction of the families. So, we omit the proof.

As an example where a factorisation such as required by Theorem [1.3] is not known, we prove a vanishing result for elements of $2\pi_0^a$ which factor through certain Brown-Gitler spectra; this will include a family due to Lin [15].

**Theorem 1.6.** Suppose $r$ is even. Then for any $f : S^{2r} \to \Sigma^r B(r/2) \to S^0$ we have $h(f) = 0$ where $h : 2\pi_{2r} \simeq 2\pi_{2r} QS^0 \to H_{2r} QS^0$ is the unstable Hurewicz homomorphism.

The following this is immediate.

**Corollary 1.7.** The elements of Lin’s family in $2\pi^a_{2i+1+2j+1}$ detected by $h_2(h_{i+1} h_j + h_i h_{j+1})$ in the ASS [13] map trivially under the unstable Hurewicz homomorphism.

**Proof.** For $i, j$ with $4 \leq i < j - 1$ Lin has constructed this family as a composition $S^{2r} \to \Sigma^r B(r/2) \to S^0$ where $r = 2^i + 2^j$. Since $r$ is even, then the result follows from [1.6] $\square$

Next, we consider the image of the unstable Hurewicz homomorphism when restricted to submodule of decomposable elements in $\pi_0^a$.  

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Theorem 1.8. The image of $h : \pi_s \simeq \pi_0 QS^0 \to H_s(QS^0; \mathbb{Z})$ when restricted to the submodule of decomposable elements is given by $\mathbb{Z}\{h(\eta^2), h(\nu^2), h(\sigma^2)\}$.

The integral result, immediately, implies the $p$-local version; the image of $h : p\pi_s \simeq p\pi_0 QS^0 \to H_s(QS^0; \mathbb{Z}/p)$ when restricted to the submodule of decomposable elements is given by the $\mathbb{Z}/p$-vector space $\mathbb{Z}/p\{h(\eta^2), h(\nu^2), h(\sigma^2)\}$. At the prime $p = 2$, the above theorem has a variant on the level of ASS due to Hung and Peterson [11, Proposition 5.4] as follows. If $\varphi_k : \text{Ext}_A^{k,k+i}(\mathbb{Z}/2, \mathbb{Z}/2) \to (\mathbb{Z}/2 \otimes D_k)_k^*$ denotes the Lannes-Zarati homomorphism, then $\varphi := \bigoplus k \varphi_k$ does vanish on decomposable classes when $k > 2$; here the authors take $\varphi$ as an algebraic approximation to the unstable Hurewicz homomorphism which seems to imply our Theorem 1.8 at the prime 2. However, the relation between $\varphi$ and $h$ is not of that linear type. After first preprint versions of this work was posted on arxiv, Nick Kuhn kindly pointed out that, theoretically, it is possible to have the vanishing of the Lannes-Zarati homomorphism on a permanent cycle $c$ converging to an element $f$ with $h(f) \neq 0$ (see also [10]). So, for the purpose of verifying Curtis conjecture, our result seems to provide a stronger evidence, and our proof is more geometric and quick.

As an application of this latter observation, we provide a computation of spherical classes in certain finite loop spaces on spheres. Let’s first start with a conjecture due to Eccles to which our next results are related.

Conjecture 1.9. (Eccles conjecture) Let $X$ be a path connected CW-complex with finitely generated homology. For $n > 0$, suppose $h(f) \neq 0$ where $2\pi_n X \simeq 2\pi_0 QX \to H_* QX$ is the unstable Hurewicz homomorphism. Then, the stable adjoint of $f$ either is detected by homology or is detected by a primary operation in its mapping cone.

Note that the stable adjoint of $f$ being detected by homology means that $h(f) \in H_* QX$ is stably spherical, i.e. it survives under homology suspension $H_* QX \to H_* X$ induced by the evaluation map $\Sigma^\infty QX \to \Sigma^\infty X$ given by the stable adjoint of the identity $QX \to QX$.

The above conjectures make predictions about the image of spherical classes in the image of unstable Hurewicz homomorphism $h : 2\pi_s X \simeq 2\pi_0 QX \to H_* QX$. By Freudenthal suspension theorem, for any $f \in \pi_n QX$, depending on the connectivity of $X$, we may find some nonnegative integer $i$ so that $f$ does pull back to $\pi_n \Omega^i X$. Note that there exist obvious commutative diagrams

$$
\begin{array}{ccc}
\pi_n \Omega^i X & \xrightarrow{h} & H_n \Omega^i \Sigma^i X \\
\downarrow & & \downarrow \\
\pi_n QX & \xrightarrow{h} & H_n QX.
\end{array}
$$

So it is then natural to look for spherical classes in $H_* \Omega^i \Sigma^i X$. Our next result consider the special case of $X$ being a sphere. For various values of $d$ and $k$, the spaces $\Omega^d S^{d+k}$ determine a lattice with positive integer coordinates. It is easier to state our next result in terms of this lattice. We have the following.
Theorem 1.10. Consider the following lattice where \((a, b)\) corresponds to \(H_\ast \Omega^a S^b\)

The following table completely determines the spherical classes in the above lattice.

| \(b - a\) | \(a\) | \(b\) | spherical classes arise from |
|-----------|------|------|-----------------------------|
| 0         | 2    | 2    | \(\eta, \eta^2\)           |
| 1         | 2    | 3    | \(\eta, b\)                |
| 1         | 1    | 2    | \(\eta, b\)                |
| 1         | 3    | 4    | \(\nu, \eta, b\)          |
| 2         | 1    | 3    | \(b\)                      |
| 3         | \(\leq 4\) | \(a + 3\) | \(\nu, b\)                |
| 3         | 5    | 8    | \(\sigma, \nu, b\)        |
| 7 - \(a\) | \(< 4\) | 7    | \(b\)                      |
| 7         | \(\leq 8\) | \(a + 7\) | \(\sigma, b\)            |
| 15 - \(a\) | \(< 8\) | 15   | \(b\)                      |
| 4 - \(a\) | \(< 3\) | 4    | \(\nu, b\)                |
| 8 - \(a\) | \(< 5\) | 8    | \(\sigma, b\)            |
| other cases |      |      | \(b\)                      |
Here, \( b \) corresponds to the inclusion of the bottom cell in the related loop space, that is \( b : S^k \to \Omega^d S^{d+k} \). The diagonal arrows correspond to (iterated) adjoining down, and horizontal arrows correspond to (iterated) stabilisation map \( E \). Both of this operations, preserve spherical classes.

The stabilisation map \( \Omega^d S^{d+k} \to \Omega^{d+1} S^{d+k+1} \) induces a monomorphism in homology, hence the spherical classes computed in the above theorem survive under this map. However, having determined spherical classes in \( H_* \Omega^d S^{d+k} \) does not completely determine spherical classes in \( H_* \Omega^{d+1} S^{d+k+1} \); for some more discussions see Corollary 1.2.

The techniques that we employ in this paper, mainly geometric. In a sequel, we have used homology of James Hopf maps to completely eliminate spherical classes in \( H_* \Omega^d S^{n+d} \) with \( n > 0 \) and \( d < 4 \) [24].

Completion vs. localisation. Often, we deal with maps \( S^n \to QS^0 \) with \( n > 0 \). So, we may safely replace \( QS^0 \) with its basepoint component \( Q_0 S^0 \). The space \( Q_0 S^0 \) is a path connected infinite loop space with \( \pi_i Q_0 S^0 \simeq \pi_i^* \) being finite Abelian groups. So, by generalities on localisation and completion at a prime \( p \) [5], noting that completion at \( p \) is just \( \mathbb{Z}_p \)-localisation, the effect on either localisation or completion at \( p \) on homology will be the same.

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2 Freudenthal Theorem and the unstable Hurewicz homomorphism

In this section, we prove Theorem 1.3. We fix some notation that is used below. We write \( h^s : \pi_* E \to H_* E \) for the (stable) Hurewicz homomorphism, that is the Hurewicz homomorphism on the spectrum level, and \( h : \pi_* E \simeq \pi_* \Omega^\infty E \to H_* \Omega^\infty E \) for the unstable Hurewicz homomorphism which satisfy \( h^s = \epsilon_* \circ h \).

Proof of Theorem 1.3 (i) If \( h(f) \neq 0 \) then \( (\Sigma^\infty f)_* \neq 0 \), i.e. it maps nontrivially under the Hurewicz map \( h^s : \pi_* \Sigma^\infty QS^0 \to H_* \Sigma^\infty QS^0 \) is nonzero in homology. Consequently \( (\Sigma^\infty f_E)_* \neq 0 \). On the other hand, there is a commutative diagram

\[
\begin{array}{ccc}
\pi_n \Sigma^\infty \Omega^\infty E & \xrightarrow{\epsilon_*} & \pi_n E \\
\downarrow h^s & & \downarrow h^s \\
H_n \Sigma^\infty \Omega^\infty E & \xrightarrow{\epsilon_*} & H_n E
\end{array}
\]

where the rows are isomorphisms. However, this leads to a contradiction as

\[
0 \neq \epsilon_* h^s (\Sigma^\infty f_E) = h^s \epsilon_* (\Sigma^\infty f_E) = 0.
\]
This completes the proof in the integral case. The $p$-local version follows, with analogous techniques, replacing $E$ with its $p$-local version $E_{(p)}$, together with the following facts: (1) the $(2r + 1)$-equivalence $\epsilon : \Sigma^\infty \Omega^\infty E \to E$ of Theorem 1.2 is an integral one, so it holds after $p$-localisation as well; (2) $\Omega^\infty$ and localisation functors commute [4, Theorem 1.1]; (3) infinite loop spaces are nilpotent in the sense of Bousfield, so both of homotopy and homology functors commute with localisation [5, Chapter V, Theorem 3.1]. The proof in the $p$-complete setting is similar.

(ii) Write $\tilde{f} : S^n \to E \to S^0$ for the adjoint of $f$. Apply the $S^1$-duality functor $D$ to obtain

$$D(\tilde{f}) : S^n \to \Sigma^n D(E) \to \Sigma^n D(\tilde{E}) \to S^0.$$ 

The result now follows from part (i) and the fact that the elements of $\pi^s_*$ are self $S^1$-dual. Note that as $E$ and $S^0$ are finite CW-spectra, then $c_* = 0$ if and only if $D(c)_* = 0$, or equivalently $h(D(c)) = 0$, by [13, Lemma A.3].

3 Hurewicz Image of some infinite families

In this section, we compute image of some infinite families under the unstable Hurewicz homomorphism. Most of these follow immediately from Theorem 1.3. As an example where Theorem 1.3 does not apply, we will compute image of certain family in $\pi^{a}_{2i+1 + 2j + 1}$ due to Lin [15].

3.1 Snaith splitting and related spectra: Proof of Theorem 1.4

Recall that for a path connected space $X$, the space $\Omega^k \Sigma^k X$ is filtered by certain spaces $C_{k,n}(X)$ [18]. By Snaith there is an equivalence of suspension spectra $\Sigma^\infty \Omega^k \Sigma^k X \simeq \bigvee_{n=1}^{+\infty} \Sigma^\infty D_{k,n}X$ (1)

where $D_{k,n}X$ is the cofibre of $C_{k-1,n}(X) \to C_{k,n}(X)$ given by $F(\mathbb{R}^k, n) \ltimes \Sigma_n X^{\wedge n}$ [21]. By construction, $D_{k,n}X$ is a genuine space for any $k$ and $n$ which has its bottom cell in dimension $rn$ if $X$ has its bottom cell in dimension $r$. For a prime $p$ and $t \in \{0, 1\}$, following Hunter and Kuhn [12], at the prime $p$ define

$$T(2n + t) = \Sigma^{2pn + 2t} D(D_{2, pn + t}S^1)$$

where $D(-)$ denotes the $S$-duality functor. For $S^1$ having its bottom cell in dimension 1, the spectrum $T(2n + t)$ has its top cell in dimension $pn + t$. These spectra, at least on homological level, are related to Brown-Gitler spectra. For instance, at the prime $p = 2$, there is a homotopy equivalence $\Sigma^\infty D_{2,n}S^1 \to \Sigma^n B([n/2])$ where $[\cdot]$ is the integer-part function [7, Theorem B]. We are now able to prove Theorem 1.4.

Proof of Theorem 1.4. First, let $p = 2$. Then,

- For $i \geq 3$, $\eta_i$ is constructed as a composition $S^{2^i} \to D_{2, 2^i - 3}S^7 \to S^0$ [17] (working in 2-complete session, we may construct $\eta_i$ as a certain composition $S^{2^i} \to \mathbb{R} P^{2^i - 3} \to S^0$ [13, Remark 2.2 and Theorem 4.1] or as a composition $S^{2^i} \to T(2^{i-3}) \to S^0$ [12, Proposition 2.1(1), Theorem 3.4]);
• Lin’s family of elements detected by \(c_1 h_i\) is constructed as a composition \(S^{2i+18} \to \Sigma^2 P_1^4 \to S^0\) (working in \(2\)-complete session y construct these as a composition \(S^{2i+18} \to \mathbb{R} P^{2i-3} \to S^0\) [14, Section 6]);

• For \(i > 11\), Lin’s family of elements for \(i > 11\) is as a composition

\[
S^{2i+1+37} \xrightarrow{(d_1)} \Sigma^{2i+1+1} \mathbb{C} P^2 \to S^0.
\]

which is detected by \(h_i^2 h_3 d_1\) in the Adams spectral sequence [16] (see also [12, Theorem 7.3])

• working \(2\)-complete \(\tau_{i-1}\) can be constructed as a composition \(S^{2i+1} \to \Sigma T(2^i - 3) \to S^0\) [12, Proposition 2.1(1), Theorem 3.6(1)(b)].

Taking either of the compositions for the afore mentioned families, the dimensional reasons together with Theorem [13] show that these elements map trivially under \(h : 2\pi_*^s \simeq 2\pi_* Q S^0 \to H_* Q S^0\). When \(p\) is odd, appealing to [12, Proposition 2.1(2), Theorem 3.6(2)(b)], the result follows by a similar reasoning.

Let’s note that it is possible to compute \(h(\tau_i)\) using the relation between the Hurewicz homomorphism and homotopy operations arising from \(2\pi_*^s D_2 S^0\) as done in [23, Proposition 0] which does not need \(2\)-completion.

### 3.2 Factorisation through certain \(B(n)\) spectra

This section is devoted to the proof of Theorem [1.6]

It is known that, at the prime \(p = 2\), there is a homotopy equivalence \(\Sigma^\infty D_{2,n} S^1 \to \Sigma^n B([n/2])\) where \([\cdot]\) is the integer-part function [7, Theorem B]. Here, the spaces \(D_{2,n} S^1\) are stable summands of Snaith splitting [11] for \(\Omega^2 S^3\). The composition \(S^{2r} \to \Sigma^r B(r/2) \to S^0\) of maybe written as

\[
S^{2r} \to \Sigma^\infty D_{2,r} S^1 \to S^0
\]

with the map on the left belonging to \(2\pi_*^s D_{2,r} S^1\). By construction, the complex \(D_{2,r} S^1 = F(\mathbb{R}^2, r) \ltimes \Sigma_r (S^1)^{\wedge r}\) is a genuine complex with its bottom cell in dimension \(r\). By Freudenthal theorem, the stable adjoint of the above composition admits a factorisation as

\[
S^{2r} \to \Omega^1 \Sigma D_{2,r} S^1 \to Q D_{2,r} S^1 \to QS^0
\]

where the pull back \(S^{2r} \to \Omega^1 \Sigma D_{2,r} S^1\) is not necessarily unique. The result now follows from the following lemma.

**Lemma 3.1.** For any map \(f : S^{2r} \to \Omega^1 \Sigma D_{2,r} S^1\) we have \(f_* = 0\).

**Proof.** According to James [13] for a path connected space \(X\), there is an isomorphism of algebras \(H_*(\Omega^1 \Sigma X; k) \simeq T_k(\tilde{H}_*(X; k))\) where \(T_k(\cdot)\) is the free tensor algebra over \(k\), and \(k\) is an arbitrary PID. We wish to show that there is no spherical class in \(H_{2r} \Omega^1 \Sigma D_{2,r} S^1 \simeq T_{2r/2}(\tilde{H}_* D_{2,r} S^1)\). The space \(D_{2,r} S^1\) has its bottom cell in dimension \(r\), hence first nontrivial homology living in dimension \(r\). The algebra structure of \(H_* \Omega^1 \Sigma D_{2,r} S^1\) then implies that \(H_{2r} \Omega^1 \Sigma D_{2,r} S^1\) is the vector space generated by typical elements \(\alpha_r \otimes \beta_r\) and \(\gamma_{2r}\) where \(\alpha_r, \beta_r \in H_* D_{2,r} S^1\) and \(\gamma_{2r} \in H_{2r} D_{2,r} S^1\) are arbitrary.
elements. Next, note that the homology \( \tilde{H}_* D_2 r S^1 \) corresponds to the part of homology \( H_* \Omega^2 S^3 \) of height \( r \), and consequently \( H_* \Omega^2 D_2 r S^1 \) will be the free graded associative algebra generated by elements of \( H_* \Omega^2 S^3 \) that are of height \( r \) \cite{20} Theorem 9.4.5. The height \( r \) elements of \( H_* \Omega^2 S^3 \) are precisely those elements that map nontrivially under the projection \((p_r)_* : \tilde{H}_* \Omega^2 S^3 \to \tilde{H}_* D_2 r S^1 \) provided by Snith splitting \( \Sigma^\infty \Omega^2 S^3 \simeq \bigvee_{k=1}^{\infty} \Sigma^\infty D_{2k} S^1 \). We recall the description of \( H_* \Omega^2 S^3 \). At \( p = 2 \), there is an isomorphism of algebras

\[
H_* \Omega^2 S^3 \simeq \mathbb{Z}/2[x_1, x_3, x_7, x_{15}, \ldots]
\]

where \( x_j \in H_j \Omega^2 S^3 \) \cite{20} Proposition 9.4.1. For an arbitrary class \( x_{2^1 - 1}^{j_1} \cdots x_{2^n - 1}^{j_n} \in \tilde{H}_* \Omega^2 S^3 \) with \( j_t > 0 \) we write \( x_{2^1 - 1}^{j_1} \cdots x_{2^n - 1}^{j_n} \) for its image under \((p_r)_*\). The height function \( h : H_* \Omega^2 S^3 \to \mathbb{Z}_+ \) is defined by setting \( h(x_{2^1 - 1}^{j_1} \cdots x_{2^n - 1}^{j_n}) = 2^{i-1} \) and \( h(\eta \xi) = h(\eta) + h(\xi) \) for any \( \eta, \xi \in H_* \Omega^2 S^3 \). The definition of height function implies that for a monomials \( x_{2^1 - 1}^{j_1} \cdots x_{2^n - 1}^{j_n} \) we have

\[
h(x_{2^1 - 1}^{j_1} \cdots x_{2^n - 1}^{j_n}) = \sum_{t=1}^{n} h(x_{2^t - 1}^{j_t}) = \sum_{t=1}^{n} j_t h(x_{2^t - 1}) = \sum_{t=1}^{n} j_t 2^{i-1}.
\]

By inspection, one can see that the only height \( r \) class as above, living in dimension \( r \) is \( x_1^{r} \) giving rise to a nontrivial class \( \widetilde{x}_1^{r} \in \tilde{H}_* D_2 r S^1 \). From this class, we get the nontrivial class \( \widetilde{x}_1^{r} \in H_2 \Omega^2 D_2 r S^1 \). Moreover, for any monomial \( x_{2^1 - 1}^{j_1} \cdots x_{2^n - 1}^{j_n} \) of height \( r \), requiring \( \text{dim}(x_{2^1 - 1}^{j_1} \cdots x_{2^n - 1}^{j_n}) = \sum_{t=1}^{n} j_t (2^t - 1) = 2r \), multiplying the above equality for height by 2, implies that \( \sum_{t=1}^{n} j_t = 0 \), and consequently \( j_t = 0 \) for all \( t = 1, \ldots, n \). That is we don’t have any nontrivial monomial \( x_{2^1 - 1}^{j_1} \cdots x_{2^n - 1}^{j_n} \) of dimension \( 2r \) and height \( r \). This shows that \( H_{2r} \Omega^2 D_2 r \simeq \mathbb{Z}/2(\{\widetilde{x}_1^{r} \} \cup \{ \widetilde{x}_1^{r} \}) \).

Now, if \( f : S^{2r} \to \Omega \Sigma D_2 r S^1 \) is any map with \( f_* \neq 0 \) then \( h(f) = \widetilde{x}_1^{r} \). Consequently, the composition \( S^{2r} \to \Omega \Sigma D_2 r S^1 \to Q D_2 r S^1 \) will behave similarly in homology. By \cite{2} Proposition 5.8] this latter implies that the stable adjoint of \( f, S^{2r} \to \Sigma^\infty D_2 r S^1 \) is detected by \( Sq^{r+1} \) on the bottom dimensional cohomology of its stable mapping cone, i.e. \( Sq^{r+1} \widetilde{x}_1^{r} = g_{2r+1} \) where by abuse of notation \( \widetilde{x}_1^{r} \in H^* C_f \simeq H^* D_2 r S^1 \simeq \mathbb{Z}/2 \) denotes a generator of \( r \)-dimensional cohomology and \( g_{2r+1} \in H^{2r+1} C_f \) denotes a generator coming from the \( 2r + 1 \)-cell attached by \( f \). We show that the equality \( Sq^{r+1} \widetilde{x}_1^{r} = g_{2r+1} \) cannot happen in \( H^* C_f \). Since \( r \) is even then \( Sq^{r+1} = Sq^1 Sq^r \). Moreover, by \cite{3} Theorem 1.3] \( H^* B(r/2) \simeq 0 \) for \( * > r - \alpha(r/2) \geq r - 1 \), where \( \alpha(n) \) is the number of 1’s in binary expansion of \( n \); in particular \( H^* B(r/2) \simeq 0 \). Consequently, \( Sq^r \) acts trivially on the zero dimensional generator of \( B(r/2) \). By stability of Steenrod operations this implies that \( Sq^r \widetilde{x}_1^{r} = 0 \) in \( H^* D_2 r \simeq H^* C_f \) (in dimensions \(* < 2r + 1 \)). This contradicts \( Sq^{r+1} \widetilde{x}_1^{r} = g_{2r+1} \) as \( Sq^{r+1} \widetilde{x}_1^{r} = Sq^1 Sq^r \widetilde{x}_1^{r} = Sq^0 \). Hence, \( f_* = 0 \) and consequently \( h(f) = 0 \).

4 The unstable Hurewicz image of decomposable elements in \( \pi_* S \)

As we said earlier, we are interested in Freudenthal theorem as it is an integral result and comes with no localisation. This section is devoted to the proof of Theorem 1.8.

Proof of Theorem 1.8 For \( f \in \pi_* S \) and \( g \in \pi_* S \), the product \( gf \) in \( \pi_* S \) is determined by the composition of stable maps \( S^{i+j} \xrightarrow{f} S^i \xrightarrow{g} S^j \rightarrow S^0 \). For \( i > j \), we have \( i + j > 2j \) which implies that
\[ i + j \geq 2j + 1. \] The result follows from Theorem \[1.3(ii)\]. The case \( i < j \) follows either by the same reasoning or from commutativity of \( \pi^* \).

Next, suppose \( i = j = n \). As elements of \( \pi_*QS^0 \), the product \( gf \) is given by the composition \( S^{2n} \to QS^n \to QS^0 \). By Freudenthal’s theorem, the stable adjoint of \( gf \) factors as

\[
S^{2n} \xrightarrow{f'} \Omega S^{n+1} \longrightarrow QS^n \longrightarrow QS^0
\]

for some \( f' : S^{2n} \to \Omega S^{n+1} \) which is not necessarily unique. Now, \( h(fg) \neq 0 \) implies that \( h(f') \neq 0 \), and consequently \( h(f') = lx^n \) where \( x_n \in H_n(\Omega S^{n+1}; \mathbb{Z}) \) is a generator, and \( l \in \mathbb{Z} - \{0\} \). It is well known (see for example [9, Proposition 6.1.5]) that \( h(f') \neq 0 \) if and only if \( f \) is detected by the unstable Hopf invariant (as defined in many classic books such as [19 Chapter 4]). Similarly, \( g \) is detected by the unstable Hopf invariant. This implies that \( n + 1 \) has to be even, i.e. \( n \) must be odd.

If \( f \) is of even unstable Hopf invariant then \( f' = l[\nu_n, \nu_n] + f'' \) for some \( f'' \in \ker(H^{\text{unstable}}) \) and some even nonzero integer \( l \). Here, \( H^{\text{unstable}} : \pi_2n \Omega S^{n+1} \cong \pi_{2n+1}S^{n+1} \to \mathbb{Z} \) is the unstable Hopf invariant and \( [\nu_n, \nu_n] \) is the Whitehead product. Since the Whitehead product \( [\nu_n, \nu_n] \) belongs to the kernel of suspension map \( E_* : \pi_* \Omega S^{n+1} \to \pi_* \Omega^2 S^{n+2} \), it follows that \( f = E^\infty f' = E^\infty f'' \) where \( E^\infty : \Omega S^{n+1} \to QS^n \) is the stabilisation map. Consequently, this would imply that in homology \( f'' : S^{2n} \to \Omega S^{n+1} \) sends \( x_{2n} \) to \( lx_n^2 \), that is \( f'' \) is of unstable Hopf invariant \( l \) which possible only if \( l = 0 \). But, this is a contradiction. Hence, \( f \) (and similarly \( g \)) cannot be of even unstable Hopf invariant.

Therefore, \( f \) should be of odd unstable Hopf invariant. In this case, reduction mod 2 implies that \( f \) would be an odd multiple of \( \eta, \nu \), or \( \sigma \) which determines \( n \in \{1, 3, 7\} \) by the Hopf invariant one result \[1.1\]. Similarly, we would have \( g \) as an odd multiple of Hopf invariant one elements. On the other hand it is known that \( \eta^2, \nu^2, \) and \( \sigma^2 \) maps nontrivially under \( h : \pi_* \cong \pi_* QS^0 \to H_*(QS^0; \mathbb{Z}) \). This completes the proof.

\[ \square \]

4.1 Spherical classes in some finite loop spaces: Proof of Theorem \[1.10\]

The main goal of this section, as well as the following section, is to provide an application of Theorem \[1.8\] by proving Theorem \[1.10\]. This allows to derive positive evidence for the truth of Curtis and Eccles conjectures when restricted to finite loop spaces. For this reason, during this section, as well as the following section, we will only work at the prime 2 writing \( \pi_* \) and \( H_* \) for the 2-component of \( \pi_* \) and \( H_*(-; \mathbb{Z}/2) \), respectively. We only note that similar techniques might be applied for the odd primary case as well.

We will consider using James fibrations \( S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega^2 S^{n+1} \) referring to \( E \) as the suspension and \( H \) as the second James-Hopf map. We also write \( E^k : X \to QX \) for the inclusion which induces \( \pi_* X \to \pi_* QS X \cong \pi_* X \) sometimes referring to it as the stablisation. We also may write \( E^k \) for the iterated suspension \( X \to \Omega^k X \). We shall use \( H_* \Omega X \cong T(\tilde{H}_*)X \) where \( X \) is any path connected space, \( T(-) \) is the tensor algebra functor and \( \tilde{H}_* \) denotes the reduced homology. We shall write \( \sigma_* : H_* \Omega X \to H_{*+1}X \) for homology suspension induced by the evaluation \( \Sigma \Omega X \to X \) recalling that it kills decomposable elements in the Pontrjagin ring \( H_* \Omega X \). We begin with the following.

**Theorem 4.1.** (i) The only spherical classes in \( H_* \Omega^2 S^3 \) live in dimensions 1 and 2, arising from the identity \( S^0 \to S^1 \) and \( \Sigma \eta : S^1 \to S^3 \). There are no spherical classes in \( H_* \Omega S^3 \) other than the bottom dimensional class given by \( S^2 \to \Omega S^3 \).
(ii) The only spherical classes in $H_*\Omega S^7$ arise in dimensions 3 and 6, and $\Sigma^3\nu : S^{10} \to S^7$. There are no spherical classes in $H_*\Omega S^7$ for $i < 4$ other than $S^{3+i} \to \Omega^{4-i}S^7$ adjoint to the identity $S^7 \to S^7$ that corresponds to the inclusion of the bottom cell.

(iii) The only spherical classes in $H_*\Omega S^8$ arise in dimensions 7 and 14, arising from the identity $S^8 \to S^{15}$ and $\Sigma^7\sigma : S^{22} \to S^{15}$. There are no spherical classes in $H_*\Omega S^8$ for $i < 8$ other than the one arising from the inclusion of the bottom cell corresponding to the identity on $S^{15}$.

**Proof.** Let $k \geq 0$. Consider the stabilisation $S^{i+k} \to QS^{i+k}$ and by abuse of notation write $E_\infty : \Omega^i S^{i+k} \to \Omega^i QS^{i+k} = QS^k$ for its $i$-fold loop. Recall that $E_\infty$ induces a monomorphism in homology. We also write $E : S^k \to \Omega S^{k-1}$ for the suspension map, that induces the suspension homomorphism $\pi_*S^k \to \pi_*\Omega S^{k-1}$ for $k \geq 1$. We proceed as follows.

(i) Obviously, the inclusion of the bottom cell $S^1 \to \Omega^2S^3$ is nontrivial in homology. Let $f : S^n \to \Omega^2S^3$, $n > 1$, be any map with $f_* \neq 0$. First, note that the Hopf fibration $\eta : S^3 \to S^2$ induces a homotopy equivalence $\Omega^2\eta : \Omega^2S^3 \to \Omega_0\Omega S^2$ where $\Omega_0\Omega S^2$ is the base point component of $\Omega^2S^2$. In particular, $\Omega^2\eta$ is an isomorphism in homology. Consequently, the composition $((\Omega^2E_\infty)(\Omega^2\eta)f)_* \neq 0$.

On the other hand, note that we have a commutative diagram

$$
\begin{array}{ccc}
\Omega^2S^3 & \xrightarrow{\Omega^2E_\infty} & QS^1 \\
\| & & \| \\
\Omega^2S^2 & \xrightarrow{\Omega^2\eta} & Q_0S^0
\end{array}
$$

where the right vertical map is obtained from $\pi : QS^3 \to QS^2$ and we write $Q_0S^0$ for the base point component of $QS^0$. This implies that $(\Omega^2\eta)_*(\Omega^2E_\infty)f_* \neq 0$. This implies that for $E_\infty f$ and $\eta$ as elements of $\pi_*S^k$, we have $h(\eta(E_\infty f)) \neq 0$ where $h$ is the unstable Hurewicz homomorphism. By Theorem 1.8 if $n > 0$ then $E_\infty f$ and $\eta$ are in the same dimension. That is $E_\infty f \in \pi_1$, consequently $n = 2$. Therefore, $f \in \pi_2\Omega^2S^3 \simeq \pi_4S^3 \simeq \mathbb{Z}/2$, which implies that $f = E\eta$ viewing $\eta : S^3 \to S^2$.

Moreover, note that by Freudenthal’s theorem $f$ pulls back to $\pi_3S^2$ allowing to consider $f$ as $f : S^2 \to \Omega S^2$. The 2-dimensional class in $H_2\Omega S^2$ is given by $x_1^2$ which maps to $x_1 \in H_1\Omega S^1$ a generator.

Moreover, suppose $f$ is any map $S^{n+1} \to \Omega S^3$ which is nontrivial in homology. By adjoining down, we have a map $f' : S^n \to \Omega^2S^3$ which is nontrivial in homology. We have $f_*\sigma_1x = \sigma_1f'_*x$ for any homotopy class $x$. However, by the above computation, if $f'_*x \neq 0$ then $f'_*x = x_1^2$ which is a decomposable class that is killed under homology suspension. This contradicts the claim that $f_* \neq 0$ for $f : S^{n+1} \to \Omega S^3$.

(ii) The argument here is almost the same. We consider the Hopf fibration

$$
\Omega S^7 \to \Omega S^4 \to S^3 \to S^7 \to S^4
$$

noting that $H(\nu) = 1$ where $H : \Omega S^4 \to \Omega S^7$ is the second James-Hopf map. This provides us with a decomposition $\Omega S^4 \to S^3 \times \Omega S^7$. A choice of decomposition may be given by $\Omega S^7 \cong S^3 \times \Omega S^7 \xrightarrow{E+\Omega\nu} S^4$ and $H : \Omega S^4 \to \Omega S^7$ in the other direction where $+$ is the loop sum in $\Omega S^4$. The decomposition of 3-fold loop spaces, in particular implies that there is monomorphism in homology given by $\Omega^3\nu : \Omega^3S^7 \to \Omega^3S^4$.

Note that $H_*\Omega S^7$ is nontrivial for $n < 3$. Similar to the previous part, the inclusion of the bottom cell $S^3 \to \Omega S^7$, adjoint to the identity $S^7 \to S^7$ is nontrivial in homology. Suppose $f : S^n \to \Omega^3S^7$ with $n > 3$ and $f_* \neq 0$ is given. Then, similar as above, we have $((\Omega^3E_\infty)(\Omega^3\nu)f)_* \neq 0$. Consider
the commutative diagram

\[
\begin{array}{ccc}
S^7 & \xrightarrow{E^\infty} & QS^7 \\
\downarrow \nu & & \downarrow \nu \\
S^4 & \xrightarrow{E^\infty} & QS^4
\end{array}
\]

and then loop it 4-times. Moreover, using the splitting of \( \Omega S^4 \) as above, we may rewrite the quadruple loop of the left vertical arrow and obtain a commutative diagram of 4-fold loop maps as

\[
\begin{array}{ccc}
\Omega^4 S^7 & \xrightarrow{\Omega^4 E^\infty} & QS^3 \\
\Omega^4 S^7 & \xrightarrow{\Omega^4 E^\infty} & QS^3 \\
\Omega^4 S^7 & \xrightarrow{\Omega^4 E^\infty} & QS^3 \\
\Omega^4 S^7 & \xrightarrow{\Omega^4 E^\infty} & QS^3
\end{array}
\]

noting that the composition of vertical arrows on the left is just \( \Omega^4 \nu \). By commutativity of the above diagram, we see that \( h(\nu E^\infty f) \neq 0 \). Consequently, by Theorem 1.8 as elements of \( \pi_*^s \), \( E^\infty f \) is in the same dimension as \( \nu \) implying that \( n = 6 \). Therefore, \( f \in 2\pi_6 \Omega^4 S^7 \simeq 2\pi_{10} S^7 \simeq \mathbb{Z}/8 \) and \( f = E^3 \nu \) where \( E^3 \) is the iterated suspension (strictly speaking, \( f \) will be three fold suspension of an odd multiple of \( \nu \) in \( 2\pi_3^s \), but since we are working at the prime 2, we take it as \( \nu \)). By Freudenthal suspension theorem it pulls back to \( \pi_7 S^4 \simeq \pi_6 \Omega S^4 \). Hence, \( f \) does factorise as \( f : S^6 \to \Omega S^4 \to \Omega^1 S^7 \). The map \( S^6 \to \Omega^4 S^4 \) has square of a three dimensional class as its Hurewicz image. Hence, as previous part, eliminating classes in \( H_* \Omega^i S^7 \) with \( i < 4 \) from being spherical.

(iii) Similar to the previous part, one has to use a decomposition \( \Omega S^8 \simeq S^7 \times \Omega S^{15} \). We leave the rest to the reader.

The above has an application to the finite case of Curtis conjecture.

**Corollary 4.2.** Suppose \( f \in H_* \Omega^8 S^8 \) is a spherical class. Then, \( f \) either corresponds to a spherical class in \( \Omega^8 S^{15} \) or is suspension of a spherical class in \( \Omega^7 S^7 \) under stablisation map \( \Omega^7 S^7 \to \Omega^8 S^8 \).

In particular, there are spherical classes arising from Hopf invariant one elements \( \eta, \nu, \sigma \) and Kervaire invariant one elements \( \eta^2, \nu^2, \sigma^2 \) in \( H_* \Omega^8 S^8 \).

Note that spherical classes in \( \Omega^8 S^{15} \) arise from 1 and \( \sigma \) which map to \( \sigma \) and \( \sigma^2 \) under \( \Omega \sigma : \Omega S^{15} \to \Omega S^8 \). By previous theorem spherical classes in \( \Omega^8 S^{15} \) are known. Hence, the computations in the above theorem will be complete once we know spherical classes in \( \Omega^7 S^7 \).

**Remark 4.3.** Let's note that the existence of Hopf fibrations, as well as James-Hopf maps together with Freudenthal suspension theorem all work well integrally. Also, the computations of Herewicz map on decomposable elements maybe carried out integrally as done in Theorem 1.8. We then conclude that the above results may be stated integrally, possibly with some modifications due to omitting the odd torsion.

**Corollary 4.4.** (i) The only spherical class in \( H_* \Omega S^2 \) corresponds to \( \eta \in \pi_1^s \simeq \pi_2 QS^1 \). In fact there is such a class given by \( S^2 \to \Omega S^2 \) provided by the adjoint of \( \eta : S^0 \to S^2 \).

(ii) The only spherical class in \( H_* \Omega^1 S^{3+i} \), \( 1 \leq i \leq 3 \), corresponds to \( \nu \in \pi_3^s \simeq \pi_0 QS^3 \). In fact there is such a class give by \( S^6 \to \Omega S^4 \xrightarrow{E^i} \Omega^i S^{4+i} \). Moreover, there is no spherical class in \( H_* \Omega^i S^{4+i} \).
for \( j < i \).

(iii) The only spherical class in \( H_* \Omega^i S^{7+i} \), \( 1 \leq i \leq 7 \), corresponds to \( \sigma \in \pi^2_{i+1} \simeq \pi_{i+4} QS^7 \). In fact there is such a class given by \( S^{14} \to \Omega S^8 \xrightarrow{E_1^{-1}} \Omega^i S^{8+i} \). Moreover, there is no spherical class in \( H_* \Omega^i S^{8+i} \) for \( j < i \).

Proof. All of these follow from the above theorem together with the fact that the suspension map \( \Omega S^{k+1} \to \Omega^i S^{k+i} \) induces a monomorphism in homology, hence preserving spherical classes. The nonexistence part, follows similar to the previous theorem by adjoining down. We do (ii) for illustration. Consider the iterated suspension map \( \Omega^i S^{3+i} \to \Omega^4 S^7 \). Hence, for \( f : S^n \to \Omega^i S^{3+i} \) with \( f_* \neq 0 \) the composition \( S^n \to \Omega^i S^{4+i} \to \Omega^4 S^7 \) is nontrivial in homology. Therefore, \( n = 6 \) and \( f : S^6 \to \Omega^4 S^{7+i} \) which maps to \( \nu \) by the above theorem. In fact, for \( i > 1 \), \( f \in \pi_{6+i} S^{3+i} \simeq \pi^2_8 \simeq \mathbb{Z}/8 \). Hence, \( f = E^i \nu \). For \( i = 1 \), \( f : S^6 \to \Omega S^4 \) is nontrivial, if and only if \( f_* = g_2^2 \), hence the adjoint of \( f \) is detected by the unstable Hopf invariant, which working at the prime 2 means it is detected by \( S^q_4 \) in its mapping cone. This means \( f \) is an odd multiple of \( S^7 \to S^4 \). \( \square \)

Finally, we conclude by computing spherical classes in \( H_* \Omega^2 S^2 \).

**Theorem 4.5.** The only spherical classes in \( H_* \Omega^2 S^2 \) arise from the Hopf invariant one and Kervaire invariant one elements \( \eta \) and \( \eta^2 \), respectively.

Proof. We apply the homotopy equivalence \( \Omega^2 \eta \). By this equivalence, any spherical class in \( H_* \Omega^2 S^2 \) is image of a spherical class in \( H_* \Omega^2 S^3 \). By the above computations, spherical classes in \( H_* \Omega^2 S^3 \) arise from \( 1, \eta \in \pi_* \Omega^2 S^3 \) which map to \( \eta \) and \( \eta^2 \) in \( \pi_* \Omega^2 S^2 \) under \( (\Omega^2 \eta)_* : \pi_* \Omega^2 S^3 \to \pi_* \Omega^2 S^2 \). This completes the proof.

\( \square \)

### 4.2 Further computations

The aim of this section is to compute spherical classes in more loop spaces associated to spheres. For instance, the decomposition \( \Omega S^4 \simeq S^3 \times \Omega S^7 \) provides a decomposition of 2-loop spaces \( \Omega^3 S^4 \simeq \Omega^2 S^3 \times \Omega^3 S^7 \). The spherical classes in the second factor arise from inclusion of the bottom cell. Noting that the decomposition of \( \Omega S^7 \) off \( \Omega S^4 \) is provided by \( \Omega \nu : \Omega S^7 \to \Omega S^4 \), the bottom cell of \( \Omega S^7 \) gives rise to a spherical class in \( \Omega^3 S^4 \) detected by \( \nu \). The other spherical classes arise from the first factor, that is the image of \( E : \Omega^2 S^3 \to \Omega^3 S^4 \). Moreover, note that by adjoining down, we may use a spherical classes in \( \Omega^i S^4 \), with \( i < 3 \), to get a spherical classes in homology of \( \Omega^3 S^4 \). Since Hurewicz image of \( \eta : S^2 \to \Omega S^2 \) is a square, and since \( \Omega E : \Omega S^2 \to \Omega^3 S^4 \) is loop map, then the Hurewicz image of \( \eta : S^2 \to \Omega^3 S^4 \) remains a square which will die under homology suspension \( H_* \Omega^2 S^4 \to H_{*+1} \Omega^2 S^4 \). We then have proved the following.

**Lemma 4.6.** The only spherical classes in \( H_* \Omega^3 S^4 \) arise from \( \eta, \nu \) and the inclusion of the bottom cell \( S^1 \to \Omega^3 S^4 \). Moreover, the only spherical classes in \( H_* \Omega^i S^4 \) with \( 1 \leq i < 3 \) arise from the inclusion of the bottom cell and \( \nu \).

A similar reasoning as above, using the decomposition \( \Omega S^8 \simeq S^7 \times \Omega S^{15} \), proves the following.

**Lemma 4.7.** The only spherical classes in \( H_* \Omega^5 S^8 \) arise from \( \sigma, \nu \) and the inclusion of the bottom cell \( S^3 \to \Omega^5 S^8 \). Moreover, the only spherical classes in \( H_* \Omega^i S^4 \) with \( 1 \leq i < 5 \) arise from the inclusion of the bottom cell and \( \sigma \).
Next, note that as the suspension $\Omega^3S^4 \to \Omega^7S^8$ induces a monomorphism, hence we get spherical classes given by the inclusion of the bottom cell, $\eta$ and $\nu$. We also obtain, a spherical class given $\sigma$ by adjoining down the spherical class given by $\sigma$ in $\Omega^5S^8$.

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