Existence of Global Decaying Solutions to the Cauchy Problem for a Nonlinear Dissipative Wave Equation of Klein-Gordon Type with a Derivative Nonlinearity

By

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Abstract. We prove the existence of global decaying solutions to the Cauchy problem for the wave equation of Klein-Gordon type with a nonlinear dissipation and a derivative nonlinearity. To derive required estimates of solutions we employ a delicate 'loan' method.

Key Words and Phrases. Global solutions, Energy decay, Wave equation, Nonlinear dissipation, Derivative nonlinearity.

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1. Introduction

Let us consider the Cauchy problem of a Klein Gordon type nonlinear wave equation of the form:

\begin{align}
    \frac{\partial^2 u}{\partial t^2} - \Delta u + \rho(u_t) + mu &= f(u, \nabla u, u_t) \quad \text{for } (x, t) \in \mathbb{R}^N \times \mathbb{R}^+ \\
    u(x, 0) &= u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{for } x \in \mathbb{R}^N
\end{align}

where \( m \) is a positive constant, \( \rho(v) \) is a function like \( \rho(v) \approx |v|^r v, \ 0 \leq r \leq \frac{2}{(N-2)^+} \), and \( f(u, \nabla u, u_t) \) is a nonlinear term including derivatives of \( u \). For simplicity of notation we assume \( m = 1 \) throughout the paper. We make the following assumption on the nonlinear term \( f(u, v) \).

Hyp.A. \( f(u, v) \) is a \( C^1 \) class function on \( \mathbb{R} \times \mathbb{R}^{N+1} \) and satisfies:

\begin{align}
    |f(u, v)| &\leq k_0(|u|^\alpha + |u|^\beta |v|), \\
    |f_u(u, v)| &\leq k_0(|u|^\alpha + |u|^\beta |v|)
\end{align}

with \( \alpha > 0, \beta \geq 0 \),

and
\[ |f_t(u,v)| \leq k_0 |u|^\beta + 1 \]

where \( k_0 > 0 \).

A typical example is \( f = \nabla \cdot G(u) \), a nonlinear convection term. Artificial, but, another typical example is

\[ f = |u|^\beta u(d_1|\nabla u|^2/\sqrt{1 + |\nabla u|^2} + d_2 |u_t|^2/\sqrt{1 + |u_t|^2}) \]

For the initial data \((u_0, u_1) \in H_2 \times H_1\) we can show by a standard argument that the problem (1.1)–(1.2) admits a unique local in time solution \( u(t) \in X_2(T) = L^\infty([0,T);H_2) \cap W^{1,\infty}([0,T);H_1) \cap W^{2,\infty}([0,T);L^2) \) for some \( T > 0 \) (cf. Lions and Strauss [2]). Further, if \( \text{supp } u_0 \cup \text{supp } u_1 \subset B(L) \equiv \{ x \in \mathbb{R}^N \mid |x| \leq L \} \) we can show the finite propagation property,

\[ \text{supp } u(t) \subset B(L + t), \quad t > 0. \]

(See John [1].)

In this paper we show the global existence of solutions in \( X_2 \equiv X_2(\infty) \) for a small initial data \((u_0, u_1)\) with \( \text{supp } u_0 \cup \text{supp } u_1 \subset B(L) \equiv \{ x \in \mathbb{R}^N \mid |x| \leq L \}, L > 0. \)

When \( f \) is independent of \( \nabla u, u_t \) and the growth order in \( u \) satisfies \( \alpha \leq 2/(N-2)^+ \) we can apply the potential well method by Sattinger [12] to show the existence and uniqueness of global solution in \( C([0, \infty); H_1) \times C^1([0, \infty); L^2) \) for \((u_0, u_1) \in \mathscr{W}^\alpha\), potential well. The dissipative term \( \rho(u_t) \) plays no role in this case. However, if \( \alpha > 2/(N-2)^+ \) the potential well method can not be applied (as far as the existence is concerned the condition \( \alpha < 4/(N-2)^+ \) is sufficient). For such a case we can use the expected decay property of the energy \( E(t) \) by the effect of the term \( \rho(u_t) \). In fact, letting \( \rho(u_t) = |u_t|^\gamma u_t \), for the problem without \( f(u, \nabla, u_t) \) we know the decay estimate

\[ E(t) \leq \left\{ E(0)^{-r/2} + m_0 \int_0^{(t-1)^+} (1 + t - s)^{-N \gamma/2} ds \right\}^{-2/r} \]

\[ \leq C_0 (1 + t)^{-(2 - N \gamma)/r}, \quad m_0 > 0, \]

provided with \( 0 < r < 2/N \), where we set

\[ E(t) = \frac{1}{2} (\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|^2). \]
Combining this estimate with a rather simple ‘loan’ method as in [6] we can define a ‘modified potential well’ and show the global solutions in $X_2$ if $\|u_0\|_{H_2} + \|u_1\|_{H_1}$ is small (see [10]). But, if $f$ depends on $Vu$ and/or $u_t$ we can not apply the loan method used in [10]. The object of the present paper is to introduce a very delicate ‘loan’ method and show the existence of global decaying solutions in $X_2$ for the problem (1.1)–(1.2), $N = 1, 2, 3$, where $f$ depends on $Vu$ and/or $u_t$. We need to use the relation $H_2 \subset L^\infty$ and it is difficult to treat the higher dimensional case $N \geq 4$.

Quite recently in [9], we have considered the same problem (1.1)–(1.2) (with $m = 0$) in a bounded domain $\Omega$ and shown the existence of global decaying solutions by use of a difference inequality on $E(t)$ and a rather complicated ‘loan’ method. We use again similar ideas in [9], but, the derived difference inequality is much more delicate and we must introduce a more delicate loan method than that in [9].

For the interested reader we give some comments on future research. Let us consider the delicate case where the dissipative term $\rho(x, u_t)$ is space-dependent and localized near infinity. For such a case, when $f(u, Vu, u_t) = 0$, we know a similar decay estimate as (1.3) with $C_0$ replaced by $C_1 = C(\|u_0\|_{H_2} + \|u_1\|_{H_1})$ (cf. [7]). This is known even for the exterior problem with the Dirichlet boundary condition if we assume further that $\rho(x, u_t)$ is effective near a part of the boundary of the obstacle (cf. [8]). Therefore we can expect the existence of global decaying solutions to the initial boundary value problem in the exterior domains with such a nonlinear localized dissipation and a derivative nonlinearity $f(u, Vu, u_t)$. Indeed, we have also treated the equation with a localized dissipation in bounded domains in [9] and it is natural to consider the above problem. It is also an interesting problem whether we can treat the nonlinear wave equation (1.1)–(1.2) without the mass term $mu$. In this case the difficulty is how to control the $L^2$-norm of solutions, and when $\rho(u_t)$ is nonlinear like $\rho(u_t) = |u_t|^2 u_t$, we can expect only very weak decay estimates for $E(t)$, say, logarithmic or very weak algebraic type, (cf. Mochizuki-Motai [3] and Todorova-Yordanov [13]). Since the expected decay of $E(t)$ is so weak it seems impossible to treat such a massless equation with a derivative nonlinearity. In spite of such a disadvantage situation there is still some possibility to derive a global existence result if $\rho(u_t)$ is replaced by $\rho(x, u_t)$, space-dependent dissipation, and $\rho(x, v)$ is linear in $v$ for large $|x|$, because we know a certain algebraic decay estimate of $E(t)$ for such a case of ‘half-linear’ dissipation, (cf. [11]). Another interesting problem is to consider the case where $m = m(x)$ depends on $x$. When $m(x)$ decays at a certain rate as $|x| \to \infty$ it is interesting to find a critical exponent of its decay rate which should assure the global existence of small amplitude solutions to the problem (1.1)–(1.3) as well as the decay estimate of $E(t)$. 

(see [5].)
2. Preliminaries and statement of result

We use only familiar function spaces and omit the definition of them. But we note that \( \| \cdot \|_p \), \( 1 \leq p \), denotes \( L^p(\mathbb{R}^N) \) norm. We write \( \| \cdot \| \) for \( \| \cdot \|_2 \). For the term \( \rho(v) \) we make the following assumption.

**Hyp.B.** \( \rho(v) \) is differentiable and monotone increasing in \( v \in \mathbb{R} \) and satisfies:

\[
(1) \quad k_1 |v|^{r+2} \leq \rho(v)v \leq k_2(|v|^{r+2} + |v|^2) \quad \text{if } |v| \leq 1
\]

with \( k_1, k_2 > 0 \), \( 0 \leq r < \infty \).

\[
(2) \quad k_1 |v|^{p+2} \leq \rho(v)v \leq k_2|v|^{p+2} \quad \text{if } |v| \geq 1
\]

with some \( k_1, k_2 > 0 \) and \( 0 \leq p \leq 2/(N-2)^+ \).

**Theorem 2.1.** Let \( N = 3 \) and assume Hyp.A and Hyp.B with \( \alpha, \beta \) and \( r \) such that

\[
0 < Nr < 2, \quad \beta + 1 > 4r/(2-Nr) \quad \text{and} \quad 2\alpha - (\alpha-2)^+ > 4r/(2-Nr).
\]

Let \( (u_0, u_1) \in H_2 \times H_1 \) with

\[
\text{supp } u_0 \cup \text{supp } u_1 \subseteq B(L), \quad L > 0,
\]

and take \( \tilde{K}_2 > 0 \) such that

\[
\|u_0\|_{H_2} + \|u_1\|_{H_1} < \tilde{K}_2.
\]

Then there exists \( \delta = \delta(\tilde{K}_2, L) > 0 \) such that if \( E(0) < \delta \), the problem (1.1)–(1.2) admits a unique solution \( u(\cdot) \in X_2 \), satisfying

\[
E(t) \leq C_0(1+t)^{-(2-Nr)/r}
\]

and

\[
\|u_2(t)\| + \|\nabla u(t)\| + \|\Delta u(t)\| \leq C(E(0), \tilde{K}_2) < \infty, \quad 0 \leq t < \infty.
\]

**Corollary 2.1.** When \( N = 1, 2 \), the conclusion of Theorem 2.1 holds under the assumption that \( \beta + 1 > 2r/(2-Nr) \) and \( \alpha > 2r/(2-Nr) \). In particular, if \( N = 1 \), under the assumption that \( (u_0, u_1) \in H_1 \times L^2 \) and \( \|\nabla u_0\| + \|u_1\| \) is small, the problem (1.1)–(1.2) admits a unique solution \( u(\cdot) \in C([0, \infty); H_1 \cap C^1([0, \infty); L^2) \cap W^{1,p+2}_{loc}([0, \infty); L_p^{p+2}) \), satisfying

\[
E(t) \leq C_0(1+t)^{-(2-r)/r}.
\]

**Remark 2.1.** When \( r = 0 \) above results hold with the decay estimate replaced by the usual exponential decay \( E(t) \leq C_0 e^{-\lambda t} \), \( \lambda > 0 \). In this case the compactness of \( \text{supp } u_0 \cup \text{supp } u_1 \) is unnecessary.
Remark 2.2. Let $V$ be a compact set in $\mathbb{R}^N$ and $\Omega$ be the exterior domain $\Omega = \mathbb{R}^N \setminus V$. We can consider the problem (1.1)–(1.2) in $\Omega$ under the boundary condition $u(t)|_{\partial \Omega} = 0$. Then, replacing $H_1(\mathbb{R}^N)$ and $H_2(\mathbb{R}^N)$ by $H_1^0(\Omega)$ and $H_2^0(\Omega) \cap H_1^0(\Omega)$, respectively, the assertion of Theorem holds without any changes.

To derive the decay estimate of $E(t)$ we use the following.

Lemma 2.1 ([4]). Let $\phi(t)$ be a nonnegative function on $[0, T]$, $T > 1$, such that $\phi(t + 1) \leq \phi(t)$ and
$$
\sup_{t \leq s \leq t + 1} \phi(s)^{1+\gamma} \leq C_0 (1 + t)^{\theta} (\phi(t) - \phi(t + 1)), \quad 0 \leq t \leq T - 1
$$
with $C_0 > 0$, $\gamma > 0$ and $0 \leq \theta \leq 1$. Then
$$
\phi(t) \leq \left( \sup_{0 \leq s \leq 1} \phi(s)^{-\gamma} + \frac{\gamma}{C_0} \int_0^{(t-1)^+} (1 + t - s)^{-\theta} ds \right)^{-\frac{1}{\gamma}}, \quad 0 \leq t \leq T.
$$
(When $\gamma = \theta = 0$ we have a usual exponential decay of $\phi(t)$.)

(For a proof see [4].)

3. A loan method

The existence of a local in time solution is standard. Indeed, if $(u_0, u_1) \in H_2 \times H_1$ with $\text{supp } u_0 \cup \text{supp } u_1 \subset B(L)$, then the problem (1.1)–(1.2) admits a unique solution $u(\cdot) \in X_2(T) \equiv L^\infty([0, T); H_2) \cap W^{1, \infty}([0, T); H_1^0) \cap W^{2, \infty}([0, T); L^2)$ for some $T > 0$ and further,
$$
\text{supp } u(t) \subset B(L + t).
$$
If it holds that
$$
\text{ess. sup}_{0 \leq t < T}\left( ||u_t(t)|| + ||u_{tt}(t)||_{H_1} + ||u(t)||_{H_2} \right) < \infty,
$$
the solution exists beyond $t = T$. Thus, to prove the global existence it suffices to derive a priori estimates for the first and second order derivatives of $u$. For this we employ a ‘loan’ method.

Let us assume for a moment,
$$
E(t) \leq K_0 E(0),
$$
$$
E(t) \leq \left( K_1^{-r/2} + m_0 \int_0^{(t-1)^+} (1 + t - s)^{-N_r/2} ds \right)^{-2/r}
$$
and
$$
||u_t(t)|| + ||u_{tt}(t)|| \leq K_2, \quad ||u(t)|| \leq K_2
$$
for $0 \leq t \leq \bar{T} (\leq T)$, where $K_0$, $K_1$, $K_2$ and $m_0$ are positive constants specified later. By (3.2) we know $E(t) \leq K_1$ on $[0, \bar{T}]$. We note that these estimates are certainly valid for some $\bar{T} > 0$ if

$$K_0 > 1 \quad \text{and} \quad E(0) < K_1$$

and

$$\|u_{tt}(0)\| + \|\nabla u_t(0)\| < K_2, \quad \|\Delta u(0)\| < K_2$$

where

$$u_{tt}(0) = \Delta u_0 - \rho(u_1) + f(u_0, \nabla u_0, u_1) \in L^2.$$  

(See the next section.)

We begin with taking such $K_0$, $K_1$ and $K_2$.

By use of the above temporarily assumed estimates (loan!) we shall derive the estimates like

$$E(t) < K_0 E(0) \quad \text{(if } E(0) \neq 0),$$

$$E(t) < \left(K_1^{-(r/2)} + m_0 \int_0^{(t-t_1)^+} (1 + t-s)^{-N/2} ds\right)^{-2/r},$$

$$\|u_{tt}(t)\| + \|\nabla u_t(t)\| \leq Q_2(E(0), K_1, K_2)$$

and

$$\|\Delta u(t)\| \leq Q_2(E(0), K_1, K_2)$$

for $0 \leq t \leq \bar{T} (\leq T)$. If we can show that for some $K_1$, $K_2$,

$$Q_2(E(0), K_1, K_2) < K_2,$$

then all of the estimates (3.4)–(3.6)' are strictly better than the assumed estimates (3.1)–(3.3) and we can conclude that the solutions exist globally on $[0, \infty)$ and all of the gained estimates (3.4), (3.5), (3.6) and (3.6)' hold actually on $[0, \infty)$. In fact the required condition (3.7) will be verified if $E(0)$ is sufficiently small.

4. Estimates on $[0, 1]$

Let $K_0 > 1$ and $K_1$, $K_2$ be the constants satisfying (3.1), (3.2) and (3.3) on $[0, \bar{T}]$. In this section we shall derive the estimates for $E(t)$ and the second order derivatives of $u(t)$ on $0 \leq t \leq \min\{\bar{T}, 1\}$. These estimates will show that we can take $T > 1$. The arguments are almost the same as in [9] and we sometimes omit the details. We begin with $E(t)$. 


Proposition 4.1. When \( N = 3 \) we take \( K_0 > 1 \) such that
\[
1 + CK_0(K_1^{2x-(x-2)/4}K_2^{(x-2)/2} + K_1^{(\beta+1)/4}K_2^{(\beta+1)/2}) < K_0,
\]
where \( C \) is a constant independent of \( u(t) \). Then, if \( E(0) \neq 0 \),
\[
E(t) < K_0 E(0) \quad \text{for} \quad 0 \leq t \leq \min\{1, \hat{T}\}.
\]
When \( N = 1, 2 \) we take \( K_0 \), in stead of (4.1), such that
\[
1 + CK_0(K_1^{x} + K_1^{(\beta-1-\delta)/2}K_2^{\delta}) < K_0
\]
where \( \varepsilon = 0 \) if \( N = 1 \) and \( 0 < \varepsilon \ll 1 \) if \( N = 2 \). Then (4.2) holds.

Proof. Assume that
\[
E(t) \leq K_0 E(0) \quad \text{on} \quad [0, \hat{T}]
\]
for some \( \hat{T} \leq \min\{\hat{T}, 1\} \). Such a choice of \( \hat{T} \) is possible since \( K_0 > 1 \). Note that if \( N = 3 \), by use of the Gagliardo-Nirenberg inequality \( \|u\|_\infty \leq C\|u\|_6^{1/2}\|\nabla u\|^{1/2} \leq C\|\nabla u\|^{1/2} \| \Delta u \|^{1/2} \) for \( u \in H_2 \) we see
\[
\|u\|_{2(x+1)}^{x+1} \leq C\|u\|_{\infty}^{x-2}\|\nabla u\|^{3} \leq C\|\nabla u\|^{3+(x-2)/2}\|\Delta u\|^{(x-2)/2} \quad \text{if} \quad x > 2
\]
and also,
\[
\|u\|_{2(x+1)}^{x+1} \leq C\|u\|^{(2-x)/2}\|\nabla u\|^{3x/2} \leq CE(t)^{(x+1)/2} \quad \text{if} \quad 0 < x \leq 2.
\]
Then,
\[
E(t) \leq E(0) + \int_0^t \int_{\mathbb{R}^N} |f(u, \nabla u, u_t)| dx ds
\]
\[
\leq E(0) + C \int_0^t (\|u\|_{2(x+1)}^{x+1}\|u_t\| + \|u\|_\infty^\beta E(s)) ds
\]
\[
\leq (1 + CK_0K_1^{(2x-(x-2)/4)}K_2^{(x-2)/2} + CK_0K_1^{(\beta+1)/4}K_2^{(\beta+1)/2})E(0), \quad 0 \leq t \leq \hat{T}.
\]
This estimate means that under the assumption (4.1), we can take \( \hat{T} = \min\{\hat{T}, 1\} \) and \( E(t) < K_0 E(0) \) for \( 0 \leq t \leq \min\{\hat{T}, 1\} \) if \( E(0) \neq 0 \).

When \( N = 1, 2 \) we modify the above estimation by use of the inequalities
\[
\|u(t)\|_\infty \leq C(\|u(t)\| + \|\nabla u(t)\|)^{-1-\varepsilon}\|\Delta u(t)\|^{-\varepsilon}
\]
with \( \varepsilon = 0 \) if \( N = 1 \) and \( 0 < \varepsilon \ll 1 \), arbitrarily small, if \( N = 2 \), and
\[
\|u(t)\|_{2(x+1)} \leq C(\|u(t)\| + \|\nabla u(t)\|).
\]
For estimation of the second order derivatives we prepare:

**Proposition 4.2.** Let $N = 3$. For $0 \leq t < \bar{T}$ we have, under the assumptions (4.1) and (4.2), that

\[
(4.3) \quad \frac{d}{dt} E_1(t) \leq C(K_2^{(\beta+3)/2} E(t)^{(\beta+1)/4} + CK_0 E(t)^{(2\sigma-(\alpha-2)^+)/4} K_2^{(\alpha-2)^+}/2) \sqrt{E_1(t)}
\]

where we set

\[
E_1(t) = \frac{1}{2} (\|u_{tt}(t)\|^2 + \|\nabla u_{t}(t)\|^2 + |u_t(t)|^2).
\]

When $N = 1, 2$ we have

\[
(4.3)' \quad \frac{d}{dt} E_1(t) \leq C(K_2^{N/2+\varepsilon} E(t)^{(\beta(1-\varepsilon)/2+(4-N)/4)}
\]

\[
+ K_2^{1+(\beta+1)\varepsilon} E(t)^{(\beta+1)(1-\varepsilon)/2} + K_2^{2\varepsilon} E(t)^{(1-\varepsilon)\alpha)/2+1/2}) \sqrt{E_1(t)}
\]

where $\varepsilon = 0$ if $N = 1$ and $0 < \varepsilon \ll 1$ if $N = 2$.

**Proof.** Differentiating the equation, we have

\[
u_{ttt} - \Delta u_t + u_t + \rho_v(u_t)u_{tt} = f_uu_t + f_t \cdot (\nabla u_t, u_{tt}).
\]

Multiplying the equation by $u_{tt}$ and integrating, we have

\[
\frac{d}{dt} E_1(t) \leq C \int_{R^n} |u|^2 |u_t| |u_{tt}| dx + C \int_{R^n} |u|^\beta (|\nabla u| + |u_t|)|u_t||u_{tt}| dx
\]

\[
+ C \int_{R^n} |u|^\beta + 1 (|\nabla u_t| + |u_{tt}|)|u_{tt}| dx
\]

\[
\equiv I_1 + I_2 + I_3.
\]

Let $N = 3$. By Gagliardo-Nirenberg inequality,

\[
I_1 = C \|u(t)\|^2_{L^\infty} \sqrt{E(t)} \sqrt{E_1(t)} \leq CK_2^{3/2} E(t)^{(x+2)/4} \sqrt{E_1(t)}.
\]

Similarly,

\[
I_2 \leq C \|u(t)\|^\beta_{L^\infty} (\|\nabla u(t)\|^2_4 + \|u_t\|^2_4) \sqrt{E_1(t)}
\]

\[
\leq CK_2^{(\beta+3)/2} E(t)^{(\beta+1)/4} \sqrt{E_1(t)}
\]

and

\[
I_3 \leq C \|u(t)\|^{\beta+1}_{L^\infty} \sqrt{E_1(t)} \leq CK_2^{(\beta+3)/2} E(t)^{(\beta+1)/4} \sqrt{E_1(t)}.
\]
Thus we obtain (4.3). When $N = 1, 2$ we have only to modify the arguments as in the proof of Proposition 4.1.

Note that by Proposition 4.1, we know $E(t) < K_0 E(0)$ for $0 \leq t \leq \min\{1, \hat{T}\}$. From this and (4.3) or (4.3)' we immediately get the following:

**Proposition 4.3.** If $N = 3$ we have for $0 \leq t \leq \min\{1, \hat{T}\}$,

$$(4.5) \quad \sqrt{E_1(t)} \leq \sqrt{E_1(0)} + C(K_2^{(\beta+3)/2}(K_0 E(0))^{(\beta+1)/4} + K_2^{x/2}(K_0 E(0))^{(x+2)/4})$$

$$\equiv \tilde{Q}_2(K_0 E(0), E_1(0), K_2).$$

When $N = 1, 2$ the above holds with $\tilde{Q}_2$ replaced by

$$\tilde{Q}_2 \equiv \sqrt{E_1(0)} + C(K_2^{2\varepsilon}(E(0))^{(1-\varepsilon)x/2+1/2} + K_2^{N/2+\beta}(K_0 E(0))^{(\beta(1-\varepsilon)/2) + (4-N)/4}$$

$$+ K_2^{1+(\beta+1)\varepsilon}(E(0))^{(\beta+1)(1-\varepsilon)/2})$$

with $\varepsilon = 0$ if $N = 1$ and $0 < \varepsilon \ll 1$ if $N = 2$.

Further, we obtain the following.

**Proposition 4.4.** Assume that the estimates (4.1)–(4.2) holds. If $N = 3$ we have for $0 < t \leq \hat{T}$,

$$(4.6) \quad \|Au(t)\| \leq \sqrt{2E_1(t)} + C\sqrt{E(t)}$$

$$+ C(K_2^{3p/2}E(t)^{(2-p)/4} + K_2^{(\beta+1)/2}E(t)^{(\beta+3)/4})$$

$$+ K_2^{(x-2)^+/2}E(t)^{(2x+1)-(x-2)^+/4}).$$

When $N = 1, 2$ we have

$$(4.6)' \quad \|Au(t)\| \leq \sqrt{2E_1(t)} + C\sqrt{E(t)}$$

$$+ C(K_2^{Np/2}E(t)^{((2-N)p+2)/4}$$

$$+ K_2^{2(\beta+1)\varepsilon}E(t)^{(\beta+1)(1-\varepsilon)+1/2} + CE(t)^{(x+1)/2}).$$

In particular, for $t$, $0 \leq t \leq \min\{\hat{T}, 1\}$, we have

$$(4.7) \quad \|Au(t)\| \leq \sqrt{2\tilde{Q}_2} + C\sqrt{K_0 E(0)}$$

$$+ C(K_2^{3p/2}(K_0 E(0))^{(2-p)/4} + K_2^{(\beta+1)/2}(K_0 E(0))^{(\beta+3)/4})$$

$$+ K_2^{(x-2)^+/2}(K_0 E(0))^{(x+1)/2-(x-2)^+/4})$$

$$\equiv Q_2(K_0 E(0), E_1(0), K_2), \quad N = 3.$$
When $N = 1, 2$ we have instead of (4.7),

$$(4.7') \quad \|\Delta u(t)\| \leq \sqrt{2}Q_2 + C\sqrt{K_0}E(0)$$

$$+ C(K_2^{Np/2}(K_0E(0))^{((2-N)p+2)/4} + K_2^{2(\beta+1)\varepsilon}(K_0E(0))^{(\beta+1)(1-\varepsilon)+1/2}$$

$$+ C(K_0E(0))^{(x+1)/2} \equiv Q_2(K_0E(0), E_1(0), K_2).$$

with $\varepsilon = 0 \ (N = 1)$ or $0 < \varepsilon \ll 1 \ (N = 2)$.

**Proof.** By the equation (4.4) and Hyp.B we have

$$(4.8) \quad \|\Delta u(t)\| \leq \|u_{tt}(t)\| + C\left(\int_{R^N}|u_t|^2 + |u_t|^{2(p+1)}dx\right)^{1/2}$$

$$+ C\left(\int_{R^N}|u_t|^{2(x+1)} + |u_t|^{2(\beta+1)}(|u_t|^2 + |\nabla u|^2)dx\right)^{1/2}.$$

Applying Gagliardo-Nirenberg inequality we easily obtain (4.6) and (4.6)'. (4.7) and (4.7)' follow from (4.5), (4.6) and (4.6)' respectively.

It follows from Proposition 4.3 and 4.4 that for $0 \leq t \leq \min\{\hat{T}, 1\},$

$$(4.9) \quad \|u_{tt}(t)\| + \|\nabla u_t(t)\| \leq \sqrt{2}Q_2 \leq Q_2(K_0E(0), E_1(0), K_2),$$

$$\|\Delta u(t)\| \leq Q_2(K_0E(0), E_1(0), K_2).$$

Now we make the assumptions:

$$(4.10) \quad K_0E(0) < K_1 \quad \text{and} \quad Q_2(K_0E(0), E_1(0), K_2) < K_2.$$

We fix $K_0 > 1$ and take $K_2$ so large that $E_1(0) < K_2$. (Note that the choice of $K_2$ depends only on $K_2$ appearing in Theorem 2.1.) Then these inequalities are certainly valid if we assume that $E(0)$ is sufficiently small. The earlier required conditions (4.1) and (4.1)' are also satisfied if $K_1$ is sufficiently small, that is, $E(0)$ is sufficiently small. Thus we conclude that if $E(0)$ is sufficiently small, then we can take $\hat{T} > 1$ and the estimates (4.2) and (4.9) are in fact valid for $0 \leq t \leq \hat{T}$ (Note that the estimate (3.2) also holds on $[0, \hat{T}]$ with some $\hat{T} > 1$ if $K_0E(0) < K_1$.)

5. **A difference inequality**

By refining the argument in [5] we derive a difference inequality for $E(t)$ which will be useful to derive the boundedness and decay of $E(t)$ on $[0, \infty)$. We make the assumptions in (4.10) and hence, $u(\cdot) \in X_2(\hat{T}), \hat{T} > \hat{T} > 1.$
Multiplying the equation by $u_t$ and integrating, we have

(5.1) \[ \int_t^{t+1} \int_{\mathbb{R}^N} \rho(u_t) u_t \, dx \, ds = E(t) - E(t + 1) + \int_t^{t+1} \int_{\mathbb{R}^N} F(x, s) u_t \, dx \, ds \]
\[ \equiv D(t)^2, \quad 0 < t \leq T - 1, \]

where $F(x, t) = f(u, \nabla u, u_t)$.

By the assumption on $\rho$, we see

(5.2) \[ \int_t^{t+1} \left( \int_{\Omega_1(t)} |u_t(s)|^{r+2} \, dx + \int_{\Omega_2(t)} |u_t(s)|^{p+2} \, dx \right) \, ds \leq CD(t)^2, \]

where we set

\[ \Omega_1(t) = \{ x \in \mathbb{R}^N \mid |u(x, t)| \leq 1 \} \quad \text{and} \quad \Omega_2(t) = \{ x \in \mathbb{R}^N \mid |u(x, t)| \geq 1 \}. \]

It follows from (5.2) that

(5.3) \[ \int_t^{t+1} \|u_t(s)\|^2 \, ds \leq \left( \int_t^{t+1} \int_{\Omega_1(t)} |u_t(s)|^{r+2} \, dx \, ds \right)^{2/(r+2)} \left( \int_{B(L+t)} \frac{1}{4} \, dx \right)^{r/(r+2)} + \int_t^{t+1} \int_{\Omega_2(t)} |u_t(s)|^{p+2} \, dx \, ds \]
\[ \leq C(C(L)(1 + t)^{N/(r+2)} D(t)^{2/(r+2)} + D(t)^2) \]

and there exist $t_1 \in [t, t + 1/4]$, $t_2 \in [t + 3/4, t + 1]$ such that

\[ \|u_t(t_i)\|^2 \leq C(C(L)(1 + t)^{N/(r+2)} D(t)^{2/(r+2)} + D(t)^2), \quad i = 1, 2. \]

Next, multiplying the equation by $u(t)$ and integrating on $[t_1, t_2] \times \Omega$ we have (note that $\|u(t)\| \leq \sqrt{E(t)}$)

(5.4) \[ \int_{t_1}^{t_2} (\|u(s)\|^2 + \|\nabla u(s)\|^2) \, ds \]
\[ = (u_t(t_1), u(t_1)) - (u_t(t_2), u(t_2)) \]
\[ + \int_{t_1}^{t_2} \|u_t(s)\|^2 \, ds + \int_{t_1}^{t_2} (fu + \rho(x, u_t)u) \, dx \, ds \]
\[ \leq C(C(L)(1 + t)^{N/(2(r+2))} D(t)^{2/(r+2)} + D(t)) \sup_{t \leq s \leq t+1} \sqrt{E(s)} \]
It follows from (5.3) and (5.4) that
\[
\int_{t_1}^{t_2} E(s) ds \leq CA(t)^2
\]
and hence, there exists \( t^* \in [t_1, t_2] \) such that
\[ E(t^*) \leq CA(t)^2. \]
(Note that the constants \( C \) may change from line to line.) Thus we have
\[
\sup_{t \leq s \leq t+1} E(s) \leq E(t^*) + \int_t^{t+1} \int_{\Omega} \rho(u_t(s)) u_t(s) dx ds + \int_t^{t+1} \int_{\Omega} |F(x, s)| |u_t| dx ds.
\]
Applying Young inequality to (5.5) and absorbing \( \sup_{t \leq s \leq t+1} \sqrt{E(s)} \) on the right-hand side into the left-hand side we arrive at the following difference inequality:

**Proposition 5.1.** For \( 0 \leq t < T - 1 \), we have
\[
\sup_{t \leq s \leq t+1} E(s) \leq C(L)(1 + t)^{N/(r+2)} D(t)^{4/(r+2)} + D(t)^2
\]
\[
+ C \left( \int_t^{t+1} \int_{\Omega} |u_t|^2 dx ds \right)^{1/2} \sup_{t \leq s \leq t+1} E(s)
\]
\[
+ \left( \int_t^{t+1} \int_{\Omega} |u_t|^{p+2} dx ds \right)^{(p+2)/(p+1)} \left( \int_t^{t+1} \int_{\Omega} |u_t|^2 dx ds \right)^{1/(p+2)}
\]
\[
+ \left( \int_t^{t+1} \|F(s)\|^2 ds \right)^{1/2} \sup_{t \leq s \leq t+1} \sqrt{E(s)}
\]
\[
\leq C(L) \left( (1 + t)^{N/2(r+2)} D(t)^{2/(r+2)} + D(t)
\right.
\]
\[
\left. + D(t)^{2(p+2)/(p+1)} + \left( \int_t^{t+1} \|F(s)\|^2 ds \right)^{1/2} \right) \sup_{t \leq s \leq t+1} E(s)
\]
\[
\equiv A(t)^2.
\]
where we recall
\[
D(t)^2 = \int_t^{t+1} \int_\Omega \rho(u_t) u_t \, dx \, dt = E(t) - E(t + 1) + \int_t^{t+1} \int_\Omega F_t \, dx \, ds
\]
and
\[
F(x, t) = f(u, \nabla u, u_t).
\]

6. Boundedness and decay of \( E(t) \) on \([0, \bar{T}]\)

From the difference inequality (5.6) we first derive the boundedness of \( E(t) \), \( 0 \leq t \leq \bar{T} \). Assume that \( E(t) \leq E(t + 1) \) for some \( t, 0 \leq t \leq \bar{T} - 1 \). Then, the inequality (5.6) implies

\[
\begin{align*}
\sup_{t \leq s \leq t+1} E(s) \leq C(1 + t)^{N \gamma/(r + 2)} & \left( \int_t^{t+1} \int_{R^N} |F_t|^2 \, dx \, ds \right)^{2/(r+2)} \\
& + C \left( \int_t^{t+1} \int_{R^N} |F_t|^2 \, dx \, ds \right)^{(p+1)/(p+2)} + C \int_t^{t+1} \int_{R^N} |F|^2 \, dx \, ds.
\end{align*}
\]

By the argument in the proof of Proposition 4.1 we know, if \( N = 3 \),

\[
\begin{align*}
\int_t^{t+1} \int_{R^N} |F_t|^2 \, dx \, ds \\
\leq C \left( K_2^{(x-2)^+}/2 \sup_{t \leq s \leq t+1} E(s)^{(2x+4-(x-2)^+)/4} + K_2^{(1+\beta)/2} \sup_{t \leq s \leq t+1} E(s)^{(\beta+5)/4} \right) \\
\end{align*}
\]

and

\[
\begin{align*}
\int_t^{t+1} \int_{R^N} |F|^2 \, dx \, ds \\
\leq C \left( K_2^{(x-2)^+} \sup_{t \leq s \leq t+1} E(s)^{(x+1)-(x-2)^+/2} + K_2^{\beta+1} \sup_{t \leq s \leq t+1} E(s)^{(\beta+3)/2} \right).
\end{align*}
\]

Then, if \( N = 3 \), we have from (6.1),

\[
\begin{align*}
\sup_{t \leq s \leq t+1} E(s) \\
\leq C(1 + t)^{N \gamma/(r+2)} & \left( K_2^{(x-2)^+}/2 \sup_{t \leq s \leq t+1} E(s)^{(2x+4-(x-2)^+)/4} \\
& + K_2^{(\beta+1)/2} \sup_{t \leq s \leq t+1} E(s)^{(\beta+5)/4} \right)^{2/(r+2)}
\end{align*}
\]
\[ + C \left( K_2^{(x-2)/2} \sup_{t \leq s \leq t+1} E(s)^{(2x+4-(x-2))}/4 \right. \\
\left. + K_2^{(\beta+1)/2} \sup_{t \leq s \leq t+1} E(s)^{(\beta+5)/4}\right)^{2(p+1)/(p+2)} \\
+ C \left( K_2^{(x-2)^+} \sup_{t \leq s \leq t+1} E(s)^{x+1-(x-2)^+}/2 \right. \\
\left. + K_2^{\beta+1} \sup_{t \leq s \leq t+1} E(s)^{(\beta+3)/2}\right) \\
\equiv I_1 + I_2 + I_3. \]

By (3.2) we see \( E(t) \leq C(K_1 + m_0^{-2/r})(1 + t)^{-(2-Nr)/r} \). Using this assumption and \( E(t) \leq K_0 E(0), \ 0 \leq t \leq \hat{T}, \) we can treat the first term of the right-hand side as follows.

\[(6.3) \quad I_1 \leq CK_2^{(x-2)^+/(r+2)}(1 + t)^{Nr/(r+2)} \sup_{t \leq s \leq t+1} E(s)^{(2x+4-(x-2)^+)/(2(r+2)-(\mu_1+1))} \]
\[\times \sup_{t \leq s \leq t+1} E(s)^{\mu_1+1} \]
\[+ CK_2^{\beta+1/(r+2)} \sup_{t \leq s \leq t+1} E(s)^{(\beta+3)/(2(r+2) - (\mu_2+1))} \times (1 + t)^{Nr/(r+2) - (2x+4-(x-2)^+)(2-Nr)/(2r(r+2)+{(\mu_1+1)(2-Nr)})/r} \times (K_0 E(0))^\mu \sup_{t \leq s \leq t+1} E(s) \]
\[+ C(K_1, m_0)K_2^{\beta+1/(r+2)}(1 + t)^{Nr/(r+2) - (\beta+5)(2-Nr)/(2r(r+2)+{(\mu_2+1)(2-Nr)})/r} \times (K_0 E(0))^\mu \sup_{t \leq s \leq t+1} E(s) \]
\[\leq C(K_1, m_0)((K_0 E(0))^{\mu_1} + (K_0 E(0))^{\mu_2}) \sup_{t \leq s \leq t+1} E(s) \]

where we have set
\[\mu_1 = (2x - (x - 2)^+)/2(r + 2) - 2r/(r + 2)(2-Nr), \]
\[\mu_2 = (\beta + 1)/2(r + 2) - 2r/(r + 2)(2-Nr) \]
and assumed that $\mu_1, \mu_2 > 0$, that is,

$$2\alpha - (\alpha - 2)^+ > 4r/(2 - Nr) \quad \text{and} \quad \beta + 1 > 4r/(2 - Nr).$$

We easily see,

$$I_2 \leq (K_2^{(\alpha - 2)^+/(p+1)/(p+2)}(K_0E(0))^{(2\alpha - (\alpha - 2)^+)(p+1) - (p+2)/(p+2)}$$

$$+ K_2^{(p+1)/(\beta+1)/(p+2)}(K_0E(0))^{((\beta + 3)p+\beta+1)/2(p+2)} \sup_{t \leq s \leq t+1} E(s)$$

and

$$I_3 \leq (K_2^{(\alpha - 2)^+}(K_0E(0))^{(2\alpha - (\alpha - 2)^+ + 2} + K_2^{\beta+1}(K_0E(0))^{(\beta+1)/2}) \sup_{t \leq s \leq t+1} E(s).$$

Thus we obtain

$$(6.5) \quad \sup_{t \leq s \leq t+1} E(s) \leq Q_0(K_1, K_2, K_0E(0)) \sup_{t \leq s \leq t+1} E(s)$$

where

$$(6.6) \quad Q_0(K_1, K_2, K_0E(0))$$

$$\equiv C(K_1, m_0)(K_2^{(\alpha - 2)^+/(r+2)}(K_0E(0))^{\mu_1} + K_2^{(\beta+1)/(r+2)}(K_0E(0))^{\mu_2})$$

$$+ (K_2^{(\alpha - 2)^+/(p+1)/(p+2)}(K_0E(0))^{(2\alpha - (\alpha - 2)^+)(p+1) - (p+2)/(p+2)}$$

$$+ K_2^{(p+1)/(\beta+1)/(p+2)}(K_0E(0))^{((\beta + 3)p+\beta+1)/2(p+2)}$$

$$+ (K_2^{(\alpha - 2)^+}(K_0E(0))^{(2\alpha - (\alpha - 2)^+ + 2} + K_2^{\beta+1}(K_0E(0))^{(\beta+1)/2}).$$

Similarly, when $N = 1, 2$ we have

$$(6.6)' \quad \sup_{t \leq s \leq t+1} E(s) \leq C(1 + t)^{Nr/(r+2)}\left( \sup_{t \leq s \leq t+1} E(s)^{(\beta+1)(1-\varepsilon)/2+1} \|\Delta u(s)\|^{(\beta+1)/2} \right)^{2/(r+2)}$$

$$+ \sup_{t \leq s \leq t+1} E(s)^{2/(r+2)}$$

$$+ C\left( \sup_{t \leq s \leq t+1} E(s)^{(\beta+1)(1-\varepsilon)/2+1} \|\Delta u(s)\|^{(\beta+1)/2} \right)^{2(p+1)/(p+2)}$$

$$+ \sup_{t \leq s \leq t+1} E(s)^{2(p+1)/(p+2)}.$$
\[ + C \sup_{t \leq s \leq t+1} E(s)^{(\beta+1)(1-c)/2+1} \|Au(s)\|^{2(\beta+1)c} \]
\[ + \sup_{t \leq s \leq t+1} E(s)^{x+1} \|Au(s)\|^{2(x+1)c} \]
\[ \leq CQ_0(K_0E(0), K_1, K_2) \sup_{t \leq s \leq t+1} E(s), \quad 0 \leq t \leq \tilde{T} - 1. \]

where

\[ Q_0(K_0E(0), K_1, K_2) \]
\[ \equiv C(K_1, m_0)((K_0E(0))^\mu_1 + K_2^{2(\beta+1)c/(r+2)}(K_0E(0))^{\mu_2}) \]
\[ + C((K_0E(0))^{((p+1)(\beta+1)(1-c)+2p)/(p+2)}K_2^{2(\beta+1)c(p+1)/(p+2)}) \]
\[ + C((K_0E(0))^{(\beta+1)(1-c)}K_2^{(\beta+1)c} + (K_0E(0))^{2x}). \]

with \( c = 0 \) if \( N = 1 \) and \( 0 < c < 1 \) if \( N = 2 \), where we have assumed

\[ \mu_1 \equiv (\alpha(2-Nr) - 2r)/(2-Nr)(r+2) > 0, \]

and

\[ \mu_2 \equiv ((\beta+1)(1-c)(2-Nr) - 2r)/(2-Nr)(r+2) > 0, \]

that is,

\[ (6.4)' \quad \beta + 1 > 2r/(2-Nr) \quad \text{and} \quad \alpha > 2r/(2-Nr). \]

We assume (6.4) if \( N = 3 \) and (6.4)' if \( N = 1, 2 \). Then, for a sufficiently small \( E(0) \) we see

\[ (6.7) \quad CQ_0(K_0E(0), K_1, K_2) < 1. \]

Under the assumption (6.7) we have \( \sup_{t \leq s \leq t+1} E(s) = 0 \). That is, if \( E(t) \leq E(t+1) \), then \( E(s) = 0 \), \( t \leq s \leq t+1 \). This implies that

\[ (6.8) \quad E(t+1) \leq E(t) \quad \text{for all} \quad t, 0 \leq t \leq \tilde{T} - 1, \]

and hence from (4.2),

\[ (6.9) \quad E(t) \leq \sup_{0 \leq s \leq 1} E(s) < K_0E(0), \quad 0 \leq t \leq \tilde{T}, \quad \text{if} \quad E(0) \neq 0. \]

Now we can show the decay estimate of \( E(t) \). Returning to the difference inequality (5.6), we have

\[ (6.10) \quad \sup_{t \leq s \leq t+1} E(s) \leq CQ_0(K_0E(0), K_1, K_2) \sup_{t \leq s \leq t+1} E(s) \]
\[ + C\{(1+t)^{Nr/(r+2)}D_0(t)^{4/(r+2)} + D_0(t)^{4(p+1)/(p+2)} + D_0(t)^2\} \]
where
\[ D_0(t)^2 = E(t) - E(t + 1) \geq 0. \]

Here, we make a little stronger assumption than (6.7),

\[(6.7)' \quad CQ_0(K_0E(0), K_1, K_2) \leq \frac{1}{2}. \]

Then, from (6.10),
\[
\sup_{t \leq s \leq t + 1} E(s)^{1+r/2} \leq \hat{C}_0(1 + t)^{Nr/2}(E(t) - E(t + 1))
\]
where \( \hat{C}_0 = C\{(K_0E(0))^r + (K_0E(0))^{(pr + p + r)/(p + 2)} + 1\} \). Since we may assume \( K_0E(0) < 1 \) we can replace \( \hat{C}_0 \) by \( \hat{C} \) which is independent of \( K_0 \) and \( E(0) \). We apply the lemma below to get the decay estimate

\[(6.11) \quad E(t) \leq \left( \sup_{0 \leq s \leq 1} E(s) \right)^{-r/2} + r(2\hat{C})^{-1} \int_0^{(t-1)^+} (t - s)^{-Nr/2} ds \]
\[ 0 \leq t \leq T. \]

We summarize the result in this section.

**Proposition 6.1.** We assume that \( 0 < Nr < 2 \) and

\[ \beta + 1 > 4r/(2 - Nr) \quad \text{and} \quad 2\alpha - (\alpha - 2)^+ > 4r/(2 - Nr) \quad \text{if} \quad N = 3 \]

or

\[ \beta + 1 > 2r/(2 - Nr) \quad \text{and} \quad \alpha > 2r/(2 - Nr) \quad \text{if} \quad N = 1, 2. \]

Then, under the temporary assumptions (‘loan’) (3.1), (3.2) and (3.3) with \( K_0, K_1, K_2 \) satisfying (6.4) (or (6.4)') and (6.7)', we have (6.8) and (6.9). Further, the decay estimate (6.11) holds.

Now we take \( K_1 \) such that

\[(6.12) \quad \left( \sup_{0 \leq s \leq 1} E(s) \right) K_0E(0) < K_1. \]

Then, setting \( m_0 = r(2\hat{C})^{-1} \) we obtain the decay estimate

\[(6.13) \quad E(t) < \left( K_1^{-r/2} + m_0 \int_0^{(t-1)^+} (t - s)^{-Nr/2} ds \right)^{-2/r}, \quad 0 \leq t \leq T. \]

Note that the estimate (6.13) is strictly better than the temporarily made assumption (3.2).

**Remark 6.1.** Note that \( Q_0(K_0E(0), K_1, K_2) \) appearing in (6.4) does not include \( K_2 \) if \( N = 1 \). Also the condition (4.1)' is independent of \( K_2 \). So,
without any information on $K_2$ these conitions are satisfied if $K_1$ is small, that is, $E(0)$ is small. Thus we have proved the latter assertion of Corollary 2.1.

7. Completion of the proof of Theorem

Finally, we estimate the second order derivatives of $u(t)$ on $[0, \tilde{T}]$ which will in fact yield the boundedness of the second order derivatives on $[0, \infty)$ and complete the proof of Theorem 2.1.

We return to the differentiated equation,

$$u_{ttt} - \Delta u_t + \rho'(u_t)u_{tt} = f_u u_t + \nabla \cdot (\nabla u_t, u_{tt}).$$

We already know the estimates (4.3) and (4.3)'.

We take the exponent $n$ such that $n > r/(2 - Nr)$. Then from the estimate (6.11) for $E(t)$, we have

$$(7.1) \quad \int_0^{\tilde{T}} E(t)^v dt$$

$$\leq \int_0^\infty \left( \left( \sup_{0 \leq t \leq 1} E(s) \right)^{-\nu/2} + r/(2C_0) \int_0^{(t-1)} (t + 1 - s)^{-\nu/2} ds \right)^{-2\nu/r} dt$$

$$\leq 2(K_0E(0))^v + \int_2^\infty \left( (K_0E(0))^{-\nu/2} + 2/(2 - Nr)(t + 1)^{(2-Nr)/2} - 2^{(2-Nr)/2} \right)^{-2\nu/r} dt$$

$$\leq (K_0E(0))^v + \int_2^\infty \left( (K_0E(0))^{-\nu/2} + 2m_1/2 - Nr(t - 1)^{(2-Nr)/2} \right)^{-2\nu/r} dt$$

$$\leq 2(K_0E(0))^v + \int_1^\infty \left\{ \left( 2m_1/2 - Nr \right)^{-2\nu/r} \right\} dt$$

$$\leq 2(K_0E(0))^v + \int_1^\infty \left\{ \left( 2m_1/2 - Nr \right)^{-2\nu/r} \right\} dt$$

$$= 2K_0E(0)^v + m_2(K_0E(0))^{-\nu/r(2-Nr)},$$
where \( m_1 = 3^{(2-Nr)/2} - 2^{(2-Nr)/2} \) and \( m_2 \) is a certain positive constant depending on \( m_1 \) and \( r \). It follows from (4.3) and (7.1) that if \( N = 3 \),

\[
(7.2) \quad \sqrt{E_1(t)} \leq \sqrt{E_1(0)} + C \{ K_2^{\beta/2} ((K_0 E(0))^{(x+2)/4} + (K_0 E(0))^{(x+2)/(4-r(2-Nr))}) \\
+ K_2^{(\beta+3)/2} ((K_0 E(0))^{(\beta+1)/4} \\
+ (K_0 E(0))^{(\beta+1)/(4-r(2-Nr))}) \}
\]

\[
\equiv \tilde{Q}_2(K_0 E(0), K_2)
\]

provided that \( \beta + 1 > 4r/(2-Nr) \) and \( x + 2 > 4r/(2-Nr) \), which are valid under (6.4).

When \( N = 1,2 \) we have instead of (7.2),

\[
(7.2)' \quad \sqrt{E_1(t)} \leq \sqrt{E_1(0)} + C \{ K_2^{N/2+\beta} ((K_0 E(0))^{\beta(1-\varepsilon)/2+(4-N)/4} \\
+ (K_0 E(0))^{\beta(1-\varepsilon)/2+(4-N)/4-r(2-Nr)}) \\
+ K_2^{1+(\beta+1)/2} ((K_0 E(0))^{(\beta+1)(1-\varepsilon)/2} + (K_0 E(0))^{(\beta+1)(1-\varepsilon)/2-r(2-Nr)}) \\
+ K_2^{\infty}((K_0 E(0))^{1-\varepsilon-x)/2+1/2} + (K_0 E(0))^{(1-\varepsilon-x)/2+1/2-r(2-Nr)}) \}
\]

\[
\equiv \tilde{Q}_2(K_0 E(0), K_2)
\]

provided that \( \beta + 1 > 2r/(2-Nr) \) and \( x + 1 > 2r/(2-Nr) \), which are valid under (6.4)'.

Thus we obtain

\[
(7.3) \quad \lVert u_t(t) \rVert + \lVert \nabla u(t) \rVert \leq 2 \tilde{Q}_2(K_0 E(0), K_2)
\]

Further, we see that the estimates (4.7) and (4.7)' holds for \( 0 \leq t \leq \tilde{T} \). That is,

\[
(7.4) \quad \lVert \Delta u(t) \rVert \leq \sqrt{2} \tilde{Q}_2 + C \sqrt{K_0 E(0)} \\
+ C(K_2^{3p/2}(K_0 E(0))^{(2-p)/4} + K_2^{(\beta+1)/2}(K_0 E(0))^{(\beta+3)/4} \\
+ K_2^{(\infty-\varepsilon)/2}(K_0 E(0))^{(x+1)/2-(x-2)/4})
\]

\[
\equiv Q_2(K_0 E(0), K_2), \quad 0 \leq t \leq \tilde{T}.
\]

When \( N = 1,2 \) we have

\[
(7.4)' \quad \lVert \Delta u(t) \rVert \leq \sqrt{2} \tilde{Q}_2 + C \sqrt{K_0 E(0)} \\
+ C(K_2^{N/2}(K_0 E(0))^{(2-Np)/4} + K_2^{2(\beta+1)x}(K_0 E(0))^{(\beta+1)(1-\varepsilon)+1/2} \\
+ C(K_0 E(0))^{(x+1)/2} \equiv Q_2(K_0 E(0), K_2).
\]
Here, we make the additional assumption

\[(7.5) \quad 2\tilde{Q}_2(K_0 E(0), K_2) < K_2 \quad \text{and} \quad Q_2(K_0 E(0), K_2) < K_2.\]

Note that (7.5) is true if \(E(0)\) is sufficiently small. In conclusion we take \(K_0, K_1\) and \(K_2\) such that

\[K_0 > 1, \quad K_0 E(0) < K_1 \quad \text{and} \quad \|u(0)\| + \|\nabla u_1\| < K_2, \quad \|Au_0\| < K_2\]

and assume that \(E(0)\) is so small that (4.1) (or (4.1)'), (6.7)' and (7.5) may hold. Then the solution exists on \([0, \infty)\) and all of the estimates derived for \(E(t), \|u(t)\| + \|\nabla u(t)\|\) and \(\|Au(t)\|\) are valid for \(t, 0 \leq t < \infty\).

The proof of Theorem 2.1 is now complete.

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