Blowing-up points on l.c.K. manifolds.

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Abstract

It is a classical result, due to F. Tricerri, that the blow-up of a manifold of locally conformally Kähler (l.c.K. for short) type at some point is again of l.c.K. type. However, the proof given in [5] is somehow unclear. We give a different argument to prove the result, using "standard tricks" in algebraic geometry.

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1 Introduction

We begin by recalling the basic definitions and facts; details can be found for instance in the book [2].

Definition 1 Let $(X, J)$ be a complex manifold. A hermitian metric $g$ on it is called locally conformally Kähler, l.c. K. for short, if there exists some open cover $U = \{U_\alpha\}_{\alpha \in A}$ of $X$ such that for each $\alpha \in A$ there is some smooth function $f_\alpha$ defined on $U_\alpha$ such that the metric $g_\alpha = e^{-f_\alpha} g$ is Kähler.

A complex manifold $(X, J)$ will be called of l.c.K. type if it admits an l.c.K. metric.

Letting $\omega$ to be the Kähler form associated to $g$ by $\omega(X, Y) = g(X, JY)$, one can immediately show that the above definition is equivalent to the existence of a closed 1–form $\theta$ such that $d\omega = \theta \wedge \omega$. The form $\theta$ is called the Lee form of the metric $g$. It is almost immediate to see that $\theta$ is closed; it is exact iff the metric $g$ is global conformally equivalent to a Kähler metric. Usually, by an l.c.K. manifold one understands a hermitian manifold whose metric is...
not globally conformally Kähler. In particular, the first Betti number of an l.c.K. manifold is always strictly positive; more, for compact Vaisman manifolds (l.c.K. with parallel Lee form) the fundamental group fits into an exact sequence

$$0 \rightarrow G \rightarrow \pi_1(M) \rightarrow \pi_1(X) \rightarrow 0$$

where $\pi_1(X)$ is a fundamental group of a Kähler orbifold, and $G$ a quotient of $\mathbb{Z}^2$ by a subgroup of rank $\geq 1$ (see [4]). Moreover, the l.c.K. class is not stable to small deformations: some Inoue surfaces do not admit l.c.K. structures and they are complex deformations of other Inoue surfaces with l.c.K. metrics (see [5], [1]).

However, l.c.K. manifolds share with the Kähler ones the property of being closed under blowing-up points. To can state the result, let $X$ be a complex manifold and $P \in X$ some fixed point. We denote by $\hat{X}$ the manifold obtained by blowing-up $P$, by $c : \hat{X} \rightarrow X$ the blowing-up map and $E$ the exceptional divisor of $\pi$ (i.e. $E = c^{-1}(\{P\})$). The goal is to prove the following

**Theorem 1** If the complex manifold $X$ carries an l.c.K. metric, then so does its blow-up $\hat{X}$ at any point.

The result was stated in [5], but the proof in this paper has a gap.

For the sake of completeness, we include in the next section some basic facts about blow-up’s of points on complex manifolds. Eventually, in the last section we prove the theorem.

## 2 Basic facts about blow-up’s of points.

This section is entirely standard and is almost an verbatim reproduction of facts from classical texts, as for instance [3].

Let $X$ be a complex, $n$—dimensional manifold. Let $P \in X$ be a point; choose a holomorphic local coordinate system $(x_1, \ldots, x_n)$ defined in some open neighborhood $U$ of $P$ such that $x_1(P) = \cdots = x_n(P) = 0$. Consider the manifold $U \times \mathbb{P}^{n-1}(\mathbb{C})$ and assume $[y_1 : \ldots : y_n]$ is some fixed homogenous coordinate system on $\mathbb{P}^{n-1}(\mathbb{C})$. Let $\hat{U} \subset U \times \mathbb{P}^{n-1}(\mathbb{C})$ be the closed subset defined by the system of equations $x_i y_j = x_j y_i$, $1 \leq i < j \leq n$. One can check that $\hat{U}$ is actually a submanifold of $U \times \mathbb{P}^{n-1}(\mathbb{C})$. Moreover, the restriction of the projection onto the first factor $c : \hat{U} \rightarrow U$ has the following properties: the
fiber of \( c \) above \( P \), \( c^{-1}\{P\} \), is a submanifold \( E \) of \( \hat{U} \) which is biholomorphic to \( \mathbb{P}^{n-1}(\mathbb{C}) \) and the restriction of \( c \) at \( \hat{U} \setminus E \) defines a biholomorphism between \( \hat{U} \setminus E \) and \( U \setminus \{P\} \). Using it, we can glue \( \hat{U} \) to \( X \) along \( U \setminus \{P\} \).

The resulting manifold will usually be denoted by \( \hat{X} \); the map \( c \) above extends obviously to a map - denoted by the same letter - \( \hat{c} : \hat{X} \to X \).

Notice that on one hand \( \hat{c} \) is a biholomorphic map between \( \hat{X} \setminus E \) and \( X \setminus \{P\} \) and, on the other hand, \( \hat{c} \) "contracts" \( E \), i.e. \( \hat{c}(E) = \{P\} \) (\( E \) is called accordingly the "exceptional divisor" of \( \hat{c} \)).

Let now \( y \in \hat{X} \) be some point. If \( y \notin E \), then the tangent map

\[
c_{*,y} : T_y(\hat{X}) \to T_{c(y)}(X)
\]

is an isomorphism, while if \( y \in E \) then the rank of this map is one and its kernel consists of those vectors that are tangent at \( y \) to \( E \), i.e. \( \text{Ker}(c_{*,y}) = T_y(E) \).

Next, recall that to each closed complex submanifold \( E \) of codimension one of some complex manifold \( X \) one can associate a holomorphic vector bundle, usually denoted \( \mathcal{O}_X(E) \); see e.g. [3], Chapter 1, Section 1. If one chooses a hermitian metric \( h \) in \( \mathcal{O}_X(E) \) there exists and is unique a linear connection \( D \) in the vector bundle which is also compatible with the complex structure (see e.g. the Lemma on page 73, [3]). The curvature \( \Omega_E \) of this connection is a closed \((1,1)\)-form.

We shall next exemplify the computation of the curvature of a metric connection in the special case we are interested in, namely when \( E \) is the exceptional divisor of some blow-up. So let \( X \) be a manifold, \( P \in X \), \( U \) a coordinate neighborhood of \( P \) as in the beginning of the section and \( \hat{X} \) the blow-up of \( X \) at \( P \). For \( \varepsilon \) small enough set

\[
U_{2\varepsilon} \overset{def}{=} Q \in U \mid |x_i(Q)| < 2\varepsilon \text{ for all } i = 1, \ldots, n\}.
\]

Let \( \pi' : U \times \mathbb{P}^{n-1}(\mathbb{C}) \to \mathbb{P}^{n-1}(\mathbb{C}) \) be the projection onto the the second factor; then \( \mathcal{O}_{\hat{U}}(E) = \pi'^*(\mathcal{O}_{\mathbb{P}^{n-1}(\mathbb{C})}(-1)) \). Let \( \omega_{FS} \) be the Kähler form of the Fubini-Study metric on \( \mathbb{P}^{n-1}(\mathbb{C}) \); then \( -\omega_{FS} \) is the curvature of the canonical connection of the natural metric \( h \) in the tautological line bundle \( \mathcal{O}_{\mathbb{P}^{n-1}(\mathbb{C})}(-1) \). Let \( h' \overset{def}{=} \pi'^*(h) \) be the induced metric in \( \mathcal{O}_{\hat{U}}(E) \); then its curvature will be \( \pi'^*(-\omega_{FS}) \). On the other hand, the line bundle \( \mathcal{O}_{\hat{X}}(E) \) is trivial outside \( E \); fix a nowhere vanishing section \( \sigma \) of it and let \( h'' \) be the unique metric making \( \sigma \) into a unitary basis. Let now \( \varrho_1, \varrho_2 \) be a partition of unity such that \( \varrho_1 \equiv 1 \) on \( U_\varepsilon \) and \( \varrho_1 \equiv 0 \) outside \( U_{2\varepsilon} \) and respectively \( \varrho_2 \equiv 0 \)
on \( U_\varepsilon \) and \( \equiv 1 \) outside \( U_{2\varepsilon} \). Let \( h = \varrho_1 h' + \varrho_2 h'' \); it is a hermitian metric on \( O_X(E) \). Its curvature will be zero outside \( U_{2\varepsilon} \) since \( h = h'' \) there. In \( U_\varepsilon \), its curvature will be the pull-back (via \( \pi' \)) of \(-\omega_{FS} \), hence it is semi-negative definite; moreover, its restriction to \( E \) will be negative definite on vectors that are tangent along \( E \), since the restriction of \( \pi' \) to \( E \) is a biholomorphism between \( E \) and \( \mathbb{P}^{n-1}(\mathbb{C}) \).

### 3 Proof of the theorem.

**Proof.** First, let us fix the terminology. We will say that a \((1,1)\)-form \( \omega \) on a complex manifold \((M, J_M)\) is positive (semi-)definite if for any point \( m \in M \) and any non-zero tangent vector \( v \in T_mM \) one has \( \omega(v, J_M v) > 0 \) (respectively \( \geq 0 \)), in other words if it is the Kähler form of some hermitian metric on \( M \).

Let now \( \omega \) be the Kähler form of an l.c.K. metric on \( X \). We see \( c^*(\omega) \) is a \((1,1)\)-form on \( \hat{X} \) which is positive definite on \( X \setminus E \) and satisfies \( dc^*(\omega) = c^*(\theta) \wedge c^*(\omega) \), where \( \theta \) is the Lee form of the given l.c.K. metric on \( X \). As \( E \) is simply connected we see (e.g. by using Lemma 4.4 in [3]) there exists an open neighborhood \( U \) of \( E \) and a smooth function \( f : \hat{X} \to \mathbb{R} \) such that \( \omega \overset{def}{=} c^* c^*(\omega) \) satisfies \( d\omega = \theta' \wedge \omega' \) and such that \( \theta'_{|U} \equiv 0 \).

On the other hand, we can find a hermitian metric in the holomorphic line bundle \( O_{\hat{X}}(E) \) on \( \hat{X} \) associated to \( E \) such that the curvature \( \Omega_E \) of its canonical connection is negative definite along \( E \) (i.e. \( \Omega_E(v, J_{\hat{X}} v) < 0 \) for every non-vanishing vector \( v \in T_P(E) \) and for every \( P \in E \), is negatively semidefinite at points of \( E \) (i.e. \( \Omega_E(v, J_{\hat{X}} v) \leq 0 \) for any \( P \in E \) and any \( v \in T_P(\hat{X}) \)) and is zero outside \( U \) (cf. e.g. [3], pp 185-187). Notice that \( \Omega_E \) is closed.

We infer that for some positive integer \( N \) the \((1,1)\)-form \( h \overset{def}{=} N \omega' - \Omega_E \) is positive definite.

Indeed, this is obvious outside \( U \) as \( \Omega_E \) vanishes here and \( N \omega' \) is positive definite for any \( N > 0 \).

Along \( E \), as both \( \omega' \) and \( -\Omega_E \) are positive semidefinite, we have only to check the definiteness of \( h \). Let \( y \in E \) be some point and \( v \in T_y(\hat{X}) \). Assume \( h(v, J_{\hat{X}} v) = 0 \); since both \( \omega' \) and \( -\Omega_E \) are positive semidefinite, we get \( \omega'(v, J_{\hat{X}} v) = \Omega_E(v, J_{\hat{X}} v) = 0 \). But \( \omega'(v, J_{\hat{X}} v) = 0 \) implies \( c^*(\omega)(v, J_{\hat{X}} v) = 0 \); so \( \omega(c_{*,y}(v), J_{\hat{X}} c_{*,y}(v)) = 0 \) hence \( v \in Ker(c_{*,y}) \). As \( Ker(c_{*,y}) = T_y(E) \) we
get that $v \in T_y(E)$; but as $-\Omega_E(v, J \vec{x} v) = 0$ we see that $v = 0$

To check the assertion on $U$, it suffices to notice that for each point $x$ in $U$ there exists some $n_x$ such that $N \omega' - \Omega_E$ is positive definite at $x$ for all $N \geq n_x$, hence also positive definite on a neighborhood of $x$; since $U$ is relatively compact, we can cover it by finitely many such neighborhoods, and take the maximum of the corresponding $n'_x$s.

Last, let us see that $N \omega' - \Omega_E$ is l.c.k. One has 
\[ d(N \omega' - \Omega_E) = N d\omega' = \theta' \wedge N \omega'. \]

But $\theta' \wedge \Omega_E = 0$ since their supports are disjoint, so we have 
\[ d(N \omega' - \Omega_E) = \theta' \wedge N \omega' - \theta' \wedge \Omega_E = \theta' \wedge (N \omega' - \Omega_E). \]

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