FREDHOLM OPERATORS ON $C^*$-ALGEBRAS

DRAGOLJUB J. KEČKIĆ AND ZLATKO LAZOVIĆ

Abstract. The aim of this note is to generalize the notion of Fredholm operator to an arbitrary $C^*$-algebra. Namely, we define "finite type" elements in an axiomatic way, and also we define Fredholm type element $a$ as such element of a given $C^*$-algebra for which there are finite type elements $p$ and $q$ such that $(1 - q)a(1 - p)$ is "invertible". We derive index theorem for such operators. In applications we show that classical Fredholm operators on a Hilbert space, Fredholm operators in the sense of Breuer, Atiyah and Singer on a properly infinite von Neumann algebra, and Fredholm operators on Hilbert $C^*$-modules over an unital $C^*$-algebra in the sense of Mishchenko and Fomenko are special cases of our theory.

1. Introduction

Fredholm operators have been investigated for many years. Initially, they were considered as those operators acting on a Banach space with finite dimensional kernel and cokernel. Their index is defined as a difference of dimensions of the kernel and cokernel. The most important properties of Fredholm operators are the following

1. The index theorem. It asserts that given two Fredholm operators $T$ and $S$, the operator $TS$ is also Fredholm and

$$\text{ind}(TS) = \text{ind} T + \text{ind} S$$

2. Theorem of Atkinson. It asserts that the operator $T$ is Fredholm if and only if it is invertible modulo compact operators, or equivalently if and only if its image in the Calkin algebra $B(X)/C(X)$ is invertible, where $C(X)$ denote the algebra of compact operators.

3. Perturbation theorem. If $T$ is a Fredholm operator, then the operator $T + K$ is Fredholm as well, provided that $K$ is compact. In this case $\text{ind}(T + K) = \text{ind} T$.

4. Continuity of index. The Fredholm operators form an open set in the norm topology and the index is constant on each connected component of this set.

The generalization of Fredholm theory to the level of von Neumann algebras was initially done by Breuer [4, 5], but it became famous after Atiyah’s work [2]. In order to generalize earlier result of him and Singer, the well known index theorem, to noncompact manifolds he considered the operators with kernel and cokernel that don’t belong to finite dimensional subspaces, but affiliated to some von Neumann algebra. He defined the dimension of such subspaces as the trace of the corresponding projection in the appropriate von Neumann algebra, see [2]. Soon after that

2010 Mathematics Subject Classification. 47A53, 46L08, 46L80.
Key words and phrases. $C^*$-algebra, Fredholm operators, $K$ group, index.
The authors was supported in part by the Ministry of education and science, Republic of Serbia, Grant #174034.
Mischenko and Fomenko introduced the notion of Fredholm operator in the framework of Hilbert $C^*$ modules, see [9]. It was repeated with a different approach by Mingo, see [8] and also important references therein. Several decades ago, there are, also, some attempts to establish axiomatic Fredholm theory in the framework of von Neumann algebras $G$ (relative to ideal $I$). Fredholm operators are defined, in this case, as invertible elements in the quotient space $G/I$, and the index is defined as an element from $C(\Omega)$ (the set of all continuous function on $\Omega$, where $C(\Omega)$ is isomorphic to the center $Z$ of $G$), see [6], [11] and references therein. There are, also, other attempts in generalization of Fredholm theory, for instance [1].

The aim of this note is to single out the properties of finite rank operators which ensure the development of Fredholm theory. In other words we shall introduce axiomatic foundation of Fredholm theory. We shall deal within a framework of a unital $C^*$-algebra $A$ and its faithful representation $\rho : A \to B(H)$ as the subalgebra $\rho(A)$ of the algebra of all bounded operators on some Hilbert space $H$. Further, we prove that the standard Fredholm operators, Fredholm operators in the sense of Atiyah and Singer on $II_\infty$ factors, and Fredholm operators in the sense of Mishchenko and Fomenko are special cases of our theory.

Throughout this paper we shall assume that a given $C^*$-algebra is always unital, even if this is not explicitly mentioned. By ker (cokernel) we shall denote the kernel (cokernel) of some operator. If $X$ is a closed subspace of some Hilbert space $H$, $P_X$ will denote the orthogonal projection on $X$. Sometimes we shall omit the word orthogonal, i.e. projection will mean orthogonal projection. The letter $I$ will stand for unit element in an abstract algebra, whereas $I$ will stand for identity operator on some Hilbert space. Similarly, we use small letters $a$, $b$, $t$, etc. to denote elements of an abstract algebra, whereas we use capital letters $A$, $B$, $T$, etc. to denote operators on a concrete Hilbert space.

**Definition 1.1.** Let $A$ be an unital $C^*$-algebra, and let $F \subseteq A$ be a subalgebra which satisfies the following conditions:

(i) $F$ is a selfadjoint ideal in $A$, i.e. for all $a \in A$, $b \in F$ there holds $ab, ba \in F$ and $a \in F$ implies $a^* \in F$;

(ii) There is an approximate unit $p_\alpha$ for $F$, consisting of projections;

(iii) If $p, q \in F$ are projections, then there exists $v \in A$, such that $vv^* = q$ and $v^*v \perp p$, i.e. $v^*v + p$ is a projection as well;

Such a family we shall call finite type elements. In further, we shall denote it by $F$.

**Definition 1.2.** Let $A$ be an unital $C^*$-algebra, and let $F \subseteq A$ be an algebra of finite type elements.

In the set $\text{Proj}(F)$ we define the equivalence relation:

$$p \sim q \iff \exists v \in A \ vv^* = p, \ v^*v = q,$$

i.e. Murray - von Neumann equivalence. The set $S(F) = \text{Proj}(F)/\sim$ is a commutative semigroup with respect to addition, and the set $K(F) = G(S(F))$, where $G$ denotes the Grothendic functor, is a commutative group.
2. Results

We divide this section into four subsections. First three of them, *Well known Hilbert space Lemmata, Almost invertibility and Approximate units technique* contain introductory material necessary for the last one, *Index and its properties*, which contains the main results.

**Well known Hilbert space Lemmata.** First, we list, and also prove, three elementary statements concerning operators on a Hilbert space.

The first Lemma establishes that two projections, close enough to each other, are unitarily equivalent, and also that obtained unitary is close to the identity operator.

**Lemma 2.1.** Let $P, Q \in B(H)$, let $H$ be a Hilbert space and let $||P - Q|| < 1$. Then there are partial isometries $V, W$ such that

\begin{equation}
VV^* = P, \quad V^*V = Q,
\end{equation}

\begin{equation}
WW^* = I - P, \quad W^*W = I - Q.
\end{equation}

Moreover, there is a unitary $U$ such that

$$U^*PU = Q, \quad U^*(I - P)U = I - Q.$$  

In addition, $V, W, U \in C^*(I, P, Q)$ - a unital $C^*$-algebra generated by $P$ and $Q$. Finally, if $||P - Q|| < 1/2$, we have the estimate

\begin{equation}
||I - U|| \leq C||P - Q||,
\end{equation}

where $C$ is an absolute constant, i.e. it does not depend neither on $P$ nor on $Q$.

**Proof.** The existence of $V \in B(H)$ with properties (2.1) was proved in [13] §105, page 268. Also, it is given that $V = P(I + P(Q - P)P)^{-1/2}Q \in C^*(I, P, Q)$.

Since $||(I - P) - (I - Q)|| = ||P - Q||$, we can apply previous reasoning to $I - P$ and $I - Q$ to obtain $W$ with required properties. Clearly, $U = V + W$ is a unitary we are looking for.

Finally, to obtain (2.2) we have

$$||Q - V|| = ||Q - P(I + P(Q - P)P)^{-1/2}Q|| \leq ||Q - P(I + P(Q - P)P)^{-1/2}||.$$

Let $\alpha_n$ be the coefficients in Taylor expansion of the function $t \mapsto (1 + t)^{-1/2}$. Then

$$(I + P(Q - P)P)^{-1/2} = I + \sum_{n=1}^{\infty} \alpha_n(P(Q - P)P)^n$$

and hence

$$||Q - V|| \leq ||Q - P - \sum_{n=1}^{\infty} \alpha_n(P(Q - P)P)^n|| \leq ||Q - P|| + \sum_{n=1}^{\infty} |\alpha_n||Q - P||^n =$$

$$||Q - P||(1 + \sum_{n=1}^{\infty} |\alpha_n||Q - P||^{n-1}) \leq C_1||Q - P||,$$

where $C_1 = 1 + \sum_{n=1}^{\infty} |\alpha_n|/2^{n-1} < +\infty$. Thus $||I - U|| = ||I - Q + Q - W - V|| \leq ||Q - V|| + ||I - Q - W|| \leq 2C_1||Q - P||$. (Indeed, since $C_1$ does not depend on $P, Q$, applying the previous reasoning to the projections $I - P, I - Q$ we have

\begin{equation}
||I - Q - W|| \leq C_1||I - Q - (I - P)||.\end{equation}

The second Lemma characterizes "almost orthogonal" projections.
Lemma 2.2. Let $P, Q \in B(H)$ be projections, and let for all $\xi \in P(H)$ and all $\eta \in Q(H)$ holds

$$|\langle \xi, \eta \rangle| \leq c||\xi||||\eta||.$$ 

Then $||PQ|| \leq c$.

Proof. Let $\xi, \eta \in H$. Then $P\xi \in P(H), Q\eta \in Q(H)$, and

$$|\langle PQ\xi, \eta \rangle| = |\langle P\xi, Q\eta \rangle| \leq c||P\xi||||Q\eta|| \leq c||\xi||||\eta||.$$

Therefore $||PQ|| \leq c$ and also $||PQ|| = ||(QP)^*|| \leq c$. \hfill $\square$

The last Lemma in this subsection deals with the adjoint of an isomorphism between two different subspaces.

Lemma 2.3. Let $H_1, H_2$ be Hilbert spaces, $K \leq H_1, L \leq H_2$. Also, let $T : H_1 \to H_2$ be a bounded mapping that maps $K$ bijectively to $L$ and $T|_{K^\perp} = 0$. Then $T^*$ maps bijectively $L$ to $K$, and $T^*\Big|_{L^\perp} = 0$.

Proof. Since $\ker T^* = (\text{ran } T)^\perp = L^\perp$, it follows $T^*\Big|_{L^\perp} = 0$. In particular, $T^*$ is injective on $L$. Also, $\text{ran } T^* = (\ker T)^\perp = K$, i.e. $\text{ran } T^*$ is dense in $K$.

Since $T$ maps bijectively $K$ to $L$ ($K, L$ closed), by open mapping theorem, $T$ is bounded below on $K$, i.e. there is $c > 0$ such that $||T\xi|| \geq c||\xi||$ for all $\xi \in K$. Let $\eta \in L$. Then $T^*\eta \in K$ and 

$$||T^*\eta|| = \sup_{\xi \in K \atop ||\xi|| = 1} |\langle \xi, T^*\eta \rangle| = \sup_{\xi \in K \atop ||\xi|| = 1} |\langle T\xi, \eta \rangle| = \sup_{\xi \in K \atop ||\xi|| = 1} ||T\xi|| \left|\frac{T\xi}{||T\xi||}, \eta\right| \geq \frac{1}{c} \sup_{\xi \in K \atop ||\xi|| = 1} \left|\frac{T\xi}{||T\xi||}, \eta\right| = c \sup_{\eta' \in L \atop ||\eta'|| = 1} ||\langle \eta', \eta \rangle|| = c||\eta||.
$$

The last equality is due to $T(K) = L$.

Therefore, $T^*$ is topologically injective on $L$. Hence its range is closed and we are done. \hfill $\square$

Almost invertibility. In this subsection, we introduce the notion of almost invertibility, i.e. invertibility up to a pair of projections, following definitions of Fredholm operators in the framework of Hilbert $C^*$-modules [7 Definition 2.7.4], and [15 Chapter 17].

Definition 2.1. Let $a \in A$ and let $p, q \in F$. We say that $a$ is invertible up to pair $(p, q)$ if the element $a' = (1 - q)a(1 - p)$ is invertible, i.e., if there is some $b \in A$ with $b = (1 - p)b(1 - q)$ (and immediately $bq = 0$, $pb = 0$, $b = (1 - p)b = b(1 - q)$) such that 

$$a'b = 1 - q, \quad ba' = 1 - p.$$

We refer to such $b$ as almost inverse of $a$, or $(p, q)$-inverse of $a$.

First, we establish that almost invertible elements make an open set, and that almost inverse of $a$ (in a certain sense) continuously changes with respect to $a$, as well as with respect to $p$ and $q$.

Lemma 2.4. Let $a$ be invertible up to $(p, q)$ and let $b$ be $(p, q)$-inverse of $a$. 
a) The element \( a + c \) is also invertible up to \((p, q)\) for every \( c \in A \), which satisfies \( ||c|| < ||b||^{-1} \). If \( b_1 \) denote \((p, q)\)-inverse of \( a + c \), then
\[
||b_1|| \leq \frac{||b||}{1 - ||b||||c||}.
\] (2.3)

b) If \( ||p - p'||, ||q - q'|| < \min\{1/2, 1/4C||a|| ||b||\} \), where \( C \) is the constant from Lemma 2.5, then \( a \) is also invertible up to \((p', q')\). Moreover, if \( b' \) is \((p', q')\)-inverse of \( a \), then \( ||b'|| \leq 2||b|| \).

Proof. a) Let \( b \) be \((p, q)\)-inverse of \( a \). Then \((1 - q)(a + c)(1 - p) = a' + c' = a'(1 - p) + (1 - q)c = a'(1 - p) + a'bc = a'(1 - p + bc')\), where \( c' = (1 - q)c(1 - p) \). The element \( bc' \) belongs to \((1 - p)A(1 - p)\) and its norm is \( ||bc'|| \leq ||b|| ||c|| \), for \( ||c|| < ||b||^{-1} \). Therefore, \( 1 - p + bc' \) is invertible in the corner algebra \((1 - p)A(1 - p)\).

Denote \( t \) as \( (1 - q)(a + c)(1 - p)tb = a'(1 - p + bc')tb = a'b = 1 - q \) and \( (1 - q)(a + c)(1 - p)tb = tba'(1 - p + bc') = (1 - p)(1 - p + bc') = 1 - p \). Therefore \( b_1 = tb \) is \((p', q')\)-inverse of \( a + c \), for \( ||c|| < ||b||^{-1} \).

Let us prove (2.3). We have
\[
||t|| = \sum_{n=1}^{\infty} (-1)^n (bc')^n \leq 1 + \sum_{n=1}^{\infty} ||b||^n ||c'||^n = \frac{1}{1 - ||b|| ||c'||}.
\]

Since \( ||c'|| \leq ||c|| \) and \( b_1 = tb \), (2.3) follows.

b) By Lemma 2.4 there is a unitary \( u \) such that \( u^*q'u = q \). Then, we have
\[
u^*(1 - q')a(1 - p) = (1 - q)a(1 - p) - u^*(1 - q')(u - 1)(1 - p) = (1 - q)a(1 - p) - c.
\]

Note that \( c \in (1 - q)A(1 - p) \), as well as \( ||c|| < ||u - 1|| ||a|| < C||q - q'|| ||a|| < 1/(4||b||) \). By the previous part a), we have that there is \((p, q)\)-inverse of \( u^*(1 - q')a(1 - p) \), say \( b_1 \). It is easy to check that \( b_1u^* \) is \((p', q')\)-inverse of \( a \), and also
\[
||b_1u^*|| \leq ||b_1|| \leq \frac{||b||}{1 - ||b|| ||c||} \leq \frac{4||b||}{3}.
\]

In a similar way we can substitute \( p \) with \( p' \) to obtain that \( a \) is invertible up to \((p', q')\). Denote its \((p, q)\)-inverse by \( b' \). Also we have the inequality
\[
||b'|| \leq \frac{||b_1||}{1 - ||b_1|| ||c'||} \leq \frac{4||b||/3}{1 - 4||b|| ||c'||/3} \leq 2||b||.
\]

□

The next Lemma will be used in the sequel many times. It allows us to transfer a projection from the right side of an element to its left side, if the considered element is invertible on this projection. In the framework of a representation, it means that we can carry a projection from the domain of an operator to its codomain. The transferred projection is equivalent to the initial one.

Lemma 2.5. Let \( A \) be a unital \( C^* \)-algebra, let \( a \in A \), let \( p, r \in A \) be projections such that \( a \) is left invertible up to \( p \), i.e. there is \( b \in A \) such that \( ba(1 - p) = 1 - p \). Also, let \( r \leq 1 - p \). Then:

(i) \( ar \) has a polar decomposition in \( A \), i.e.
\[
(2.4) \quad ar = v|ar|, \quad v, |ar| \in A, \quad r = vv^*.
\]

Further, \( s = v^*v \in A \) is the minimal projection in \( A \), such that \( sar = ar \). (Obviously, \( s \sim r \), and if \( r \in F \), then \( s \in F \) as well.) Moreover, \( a \) is invertible up to \((1 - r, 1 - s)\).
(ii) If \( A \overset{\varphi}{\to} B(H) \) is any faithful representation of \( A \), then \( L = \rho(\text{ar})(H) \) is a closed subspace, and \( \rho(s) \) is the projection on \( L \). In particular projection on \( L = \rho(\text{ar})(H) \) belongs to \( \rho(A) \).

**Proof.** Let \( A \overset{\varphi}{\to} B(H) \) be some faithful representation of \( C^*-\)algebra \( A \) and let \( \rho(r)(H) = K \). Denote \( A = \rho(a) \), \( B = \rho(b) \), \( P = \rho(p) \), \( R = \rho(r) \). Obviously, \( AR \) has a polar decomposition \( AR = V|AR| \) in \( B(H) \). Since the partial isometry \( v \) is uniquely determined by (2.4), we have only to prove that \( V \in \rho(A) \).

Let \( H_1 = (I - P)(H) \). For \( \xi \in H_1 \) we have
\[
||\xi|| = ||BA(I - P)\xi|| \leq ||B|| ||A(I - P)\xi|| = k^{-1}||A\xi||,
\]
where \( k = ||B||^{-1} \). (We used \( (I - P)\xi = \xi \).) Thus,
\[
||A\xi|| \geq k||\xi||, \quad \xi \in H_1,
\]
i.e. \( A \) is injective and bounded below on \( H_1 \), and consequently on \( K = R(H) \leq H_1 \) (\( r \leq 1 - p \)).

Therefore, \( L = AR(H) = A(K) \) is a closed subspace. (This proves the first claim in (ii).) As \( AR \) maps bijectively \( K \) to \( L \), \( (AR)^* \) maps bijectively \( L \) to \( K - \) Lemma 2.3 and hence, \( RA^*AR = (AR)^*AR \) is an isomorphism of \( K \). Denote its inverse by \( \hat{T} : K \to K \). Let \( T : H \to H \) be the extension of \( \hat{T} \), defined to be 0 on \( K \). Trivially, \( T \geq 0 \), and \( \sqrt{T} \) exists. Also, \( \sqrt{T} \) maps \( K \) to \( K \) and \( \sqrt{T}|_{K^2} = 0 \). We claim that \( V = AR\sqrt{T} \). Indeed, both \( V \) and \( AR\sqrt{T} \) annihilates \( K^2 \), whereas on \( K \), \( \sqrt{T} \) and \( |AR| \) are inverses to each other, since \( |AR| = ((AR)^*AR)^{1/2} \) and \( T \) is inverse for \((AR)^*AR\) Thus, it remains to prove that \( T \in \rho(A) \).

As it is easy to check, \( (AR)^*AR + I - R \) and \( T + I - R \) are inverses to each other. Since \( (AR)^*AR + I - R \in \rho(A) \), its inverse also belongs to \( \rho(A) \). Namely, by Theorem 11.29 from [12] (or Theorem 2.1.11 from [10]) the spectrum of an element is the same with respect to algebra as well as with respect to subalgebra. Apply this conclusion to 0 as the point of spectrum to obtain \( T + I - R \in \rho(A) \) and immediately, \( T \in \rho(A) \). Therefore \( T = \rho(t) \) for some \( t \in A \). Hence \( V = \rho(v) \), where \( v = ar\sqrt{T} \) and \( ar \) has a polar decomposition in \( A \).

To finish the proof, note that \( sar = ar \), since \( s = v^*v \) and \( S = \rho(s) \) is the projection on the subspace \( L = AR(H) \). If for some other projection \( s_1 \), there holds \( s_1ar = ar \) then must be \( S_1(H) \geq L \) and hence \( S_1 \geq S \) implying \( s_1 \geq s \). \( \square \)

The following Lemma (as many others), in fact, deals with 2 \( \times \) 2 matrices. It can be freely reformulated as follows: If \([a_{i,j}]_{i,j=1}^2\) is invertible, \( a_{11} \) is invertible and if this matrix is either triangular \((a_{12} = 0)\) or close to triangular \((a_{12} \) small) then \( a_{22} \) is also invertible.

However, we do not use matrix notation (now and in the sequel) in order to avoid confusions such as with respect to which pair(s) of projections a matrix is formed.

**Lemma 2.6.** Let \( a, p, q \in A \), let \( p, q \) be projections such that \( a \) is invertible up to \((p, q)\) and let \( r_1, r_2, s_1, s_2 \) be projections such that \( 1 - p = r_1 + r_2, 1 - q = s_1 + s_2 \). If \( a \) is invertible up to \((1 - r_1, 1 - s_1)\) and if \(|s_2ar_1| < ||b||^{-1} \) or \(|s_1ar_2| < ||b||^{-1} \), where \( b \) is \((p, q) \) inverse of \( a \), then \( a \) is invertible up to \((1 - r_2, 1 - s_2)\).

**Proof.** 1° case - let \( s_2ar_1 = 0 \). Decompose \( (1 - q)a(1 - p) \) and \( b \) as
\[
(2.5) \quad (1 - q)a(1 - p) = (s_1 + s_2)a(r_1 + r_2) = s_1ar_1 + s_1ar_2 + s_2ar_2.
\]
\[
(2.6) \quad b = (1 - p)b(1 - q) = (r_1 + r_2)b(s_1 + s_2) = r_1bs_1 + r_1bs_2 + r_2bs_1 + r_2bs_2.
\]
Proof. Since
\( \text{Let } s = \text{well defined.} \)
Multiplying the last equality by \( s \) from both, left and right side we obtain:
\[
(2.7) \quad s = s_ar_2b_2s_2.
\]
Multiplying \((2.5)\) and \((2.6)\) in reverse order we get
\[
(2.8) \quad r_1 + r_2 = 1 - p = r_1bs_1ar_1 + r_1bs_1ar_2 + r_1bs_2ar_2 + r_1bs_1ar_1 + r_2bs_1ar_2 + r_2bs_2ar_2.
\]
Multiply \((2.8)\) by \( r_2 \) from the left, and by \( r_1 \) from the right to obtain \( r_2bs_1ar_1 = 0. \)
The element \( a \) has an \((1 - r_1, 1 - s_1)\) inverse, say \( b' \). It holds \( s_1ar_1b' = s_1 \), and we have \( r_2bs_1 = r_2bs_1ar_1b' = 0b' = 0. \)

Finally, multiply \((2.8)\) by \( r_2 \) from both, left and right side to obtain
\[
r_2 = r_2bs_1ar_2 + r_2bs_2ar_2 = 0ar_2 + r_2bs_2ar_2 = r_2bs_2s_2ar_2.
\]
This together with \((2.7)\) means that \( r_2bs_2 \) is \((1 - r_2, 1 - s_2)\)-inverse for \( a \). Therefore, by Lemma \( \ref{lem:2.6} \) applying to \( r_2 \leq r_1 + r_2 \) we found that \( r_2 \sim s_2. \)

2° case - general. Consider \( \tilde{a} = a - s_2ar_1 \). Obviously, \( s_1ar_1 = s_1ar_1, s_2ar_2 = s_2ar_2 \) and \( s_2ar_1 = 0. \) Since \( ||s_2ar_1|| < ||b'||^{-1} \), by Lemma \( \ref{lem:2.4} \), \( \tilde{a} \) is invertible up to \((p, q)\) and we can apply the previous case.

If \( ||s_1ar_2|| < ||b'||^{-1} \), apply the presented proof to \( a^* \).

The next Lemma is a refinement of Lemma \( \ref{lem:2.5} \).

**Lemma 2.7.** Let \( \mathcal{A} \) be a unital C*-algebra, let \( p, q, r \in \mathcal{A} \) be projections such that \( a \) is invertible up to \((p, q)\), and \( r \leq 1 - p \), and let \( s \) be the projection obtained in Lemma \( \ref{lem:2.5} \). If \( qa(1 - p) = 0 \), then \( s \leq 1 - q \), and \( 1 - q - s \sim 1 - p - r. \)

**Proof.** Since \( a \) is invertible up to \((p, q)\), \( a \) is left invertible up to \( p \). So, the projection \( s \) is well defined.

If \( qa(1 - p) = 0 \), then \((1 - q)a(1 - p) = a(1 - p)r = (1 - q)a(1 - p)r = (1 - q)ar \). So, \( s \leq 1 - q \) by minimality of \( s \).

Denote \( r_1 = 1 - p - r \). Apply the previous part of the proof to \( r_1 \leq 1 - p \) to obtain the minimal \( s_1 \leq 1 - q \) such that \( s_1ar_1 = ar_1 \) and \( r_1 \sim s_1. \) Denote \( s_2 = 1 - q - s \). Projections \( s_1 \) and \( s_2 \) might not coincide, but they must be equivalent. More precisely \( s_2 \sim r_1 \sim s_1 \). To show this it is enough to prove that \( s_2 = \) "invertible". However, from \( sar = ar \) we get \( s_2ar = (1 - q)ar - sar = (1 - q)a(1 - p)r - sar = a(1 - p)r - qa(1 - p)r - sar = ar - 0 - sar = 0. \) Thus, it is sufficient to apply Lemma \( \ref{lem:2.6} \). \( \square \)

The last statement in this subsection ensures that almost invertible elements can be triangularized. More precisely, if \( a \) is invertible up to \((p, q)\) then we can substitute \( p \) or \( q \) (one of them - not simultaneously) by an equivalent projection such that almost invertibility is not harmed and such that with respect to new projections the considered element has a triangular form.

**Proposition 2.8.** Let \( \mathcal{A} \) be a unital C*-algebra, let \( p, q \in \mathcal{A} \) and let \( a \) be invertible up to \((p, q)\).

a) Then there is a projection \( q' \in \mathcal{A} \), \( q' \sim q \) such that \( a \) is invertible up to \((p, q')\) and \( q'a(1 - p) = 0; \)

b) Also, there is another projection \( p' \in \mathcal{A} \), \( p' \sim p \) such that \( a \) is invertible up to \((p', q)\) and \((1 - q)a(1 - p) = 0. \)
Proof. a) Let \( b \) be the \( (p, q) \)-inverse of \( a \). Then
\[
(1-q)a(1-p)b = 1-q, \quad b(1-q)a(1-p) = 1-p,
\]
(2.9)
\[
(1-p)b(1-q) = b, \quad pb = bq = 0, \quad b = (1-p)b, \quad b = b(1-q).
\]
Let \( u = 1 + qab \). This element has its inverse, \( u^{-1} = 1 - qab \), which can be easily checked.

Consider the element \( a_1 = u^{-1}a = (1-qab)a = a - qaba \). Using \( b(1-q) = b \) and \( b(1-q)a(1-p) = 1-p \) (see (2.10)), we have
\[
qa_1(1-p) = q(a-qaba)(1-p) = qa(1-p) - qab(1-q)a(1-p) = qa(1-p) - qa(1-p) = 0.
\]
(2.10)
The element \( u^{-1}a \) is invertible together with \( u \) and we can apply Lemma 2.6 to obtain a minimal projection \( q' \in \mathcal{A} \) such that \( q' \sim q \), \( u^{-1}a \) is invertible up to \( (1-q, 1-q') \) and \( q'u^{-1}a = u^{-1}q' \). The last equality implies
\[
qu^{-1} = qu^{-1}q', \quad qu^{-1}(1-q') = 0,
\]
(2.11) which ensures, by Lemma 2.6 that \( u^{-1}a \) is invertible up to \( (1-q', 1-q) \), as well as up to \( (q', q) \).

Let us, first prove that \( q'a(1-p) = 0 \). Indeed, by (2.10) and (2.11) we have
\[
0 = qa_1(1-p) = qu^{-1}a(1-p) = qu^{-1}q'a(1-p).
\]
(2.12) Let \( t \) is \( (1-q', 1-q) \)-inverse for \( u^{-1} \). We have \( tqu^{-1}q' = q' \). Multiply the equation (2.12) from the left by \( t \) to obtain
\[
0 = tqu^{-1}q'a(1-p) = q'a(1-p).
\]
Finally, let us prove that \( a \) is invertible up to \( (p, q') \). Note that \( (1-q)a_1(1-p) = (1-q)(a-qaba)(1-p) = (1-q)a(1-p) \). Next, by (2.11) we get \( qu^{-1}(1-q') = 0 \) and \( (1-q)u^{-1}(1-q') = u^{-1}(1-q') \) and hence
\[
(1-q')a(1-p) = a(1-p) = uw^{-1}a(1-p) = ua_1(1-p) = u(1-q)a_1(1-p) = u(1-q)a(1-p).
\]
Now, it is easy to check that \( bu^{-1}(1-q') \) is \( (p, q') \)-inverse of \( a \). Indeed
\[
(1-q')a(1-p) \cdot bu^{-1}(1-q') = u(1-q)a(1-p)bu^{-1}(1-q') = u(1-q)u^{-1}(1-q') = uu^{-1}(1-q') = 1 - q'
\]
and
\[
bu^{-1}(1-q') \cdot (1-q')a(1-p) = bu^{-1}u(1-q)a(1-p) = 1 - p.
\]
b) It is enough to apply the previous conclusion to \( a^* \). \( \square \)

**Approximate units technique.** For the development of the abstract Fredholm theory, we required in Definition 1.1 an ideal with an approximate unit consisting of projections. In this subsection we derive some properties of such an ideal.

In the next two Lemmata we obtain that an approximate unit absorbs any other projections and that any finite projection is an element of some approximate unit.

**Lemma 2.9.** Let \( p_\alpha \in \mathcal{F} \) be an approximate unit, and let \( p \in \mathcal{F} \). Then there is some \( \alpha_0 \) and \( \bar{p} \) such that \( p \sim \bar{p} \leq p_\alpha \). Moreover, \( \alpha_0 \) can be chosen such that \( \|p - \bar{p}\| \) is arbitrarily small.

**Proof.** Since \( p_\alpha \) is an approximate unit, we have \( \|p - p_\alpha p\| \to 0 \), as \( \alpha \to \infty \). Therefore, there is \( \alpha_0 \) such that
\[
\|p - p_\alpha p\| \leq \delta < 1, \quad \text{implying} \quad \|p - pp_\alpha p\| \leq \delta.
\]
(2.13)
First, we obtain that $pp_{\alpha}p$ is invertible element of the corner algebra $pA_p$.

Now, pick a faithful representation $\rho$ of $A$ on some Hilbert space $H$. Denote the images of $\rho$ by the corresponding capital letters, $P = \rho(p)$, $P_{\alpha} = \rho(p_{\alpha})$ etc.

Let $K = P(H)$. Since $p_{\alpha}$ is invertible up to $(1 - p, 1 - p)$ we can apply Lemma 2.2(i) to conclude that there is $p' \in A$, $p' \sim p$. By the part (ii) of the same Lemma, we get $p' \leq p_{\alpha}$ ($P'$ is the projection on the range of $P_{\alpha}P$. The last is, obviously, subspace of the range of $P_{\alpha}$).

It remains to prove that $\|p - p'\|$ can be arbitrarily small. To do this, let us prove that $p_{\alpha}$ does not change norm of $\xi \in K$ too much. Indeed, for $\xi \in K$, using (2.13) we have

$$\|p_{\alpha}\xi\| = \|p_{\alpha}p\xi\| = \|p\xi - (p - p_{\alpha}p)\xi\| \leq (1 + \delta)\|\xi\|,$$

and also,

$$\|p_{\alpha}\xi\| = \|p_{\alpha}p\xi\| = \|p\xi - (p - p_{\alpha}p)\xi\| \geq \|\xi\| - \|(p - p_{\alpha}p)\xi\| \geq (1 - \delta)\|\xi\|.$$

Now, we want to prove that $1 - p'$ and $p$ (and similarly $1 - p$ and $p'$) are "almost orthogonal".

Let $\eta = (1 - p)\eta$ and $\zeta = p'\zeta$. Then $\zeta = p_{\alpha}\xi$ for some $\xi = p\xi \in K$, and $\|\zeta\| \geq (1 - \delta)\|\xi\|$ by (2.14). We have

$$\langle \zeta, \eta \rangle = \langle p_{\alpha}p\xi, (1 - p)\eta \rangle = -\langle (p - p_{\alpha}p)\xi, (1 - p)\eta \rangle,$$

and hence

$$|\langle \zeta, \eta \rangle| \leq \|p - p_{\alpha}p\| \|\xi\| \|\eta\| \leq \frac{\delta}{1 - \delta}\|\xi\| \|\eta\|.$$

Therefore, by Lemma 2.2

$$\|p' - (1 - p)\| \leq \delta \frac{1}{1 - \delta}.$$

Now, let $\zeta = (1 - p')\zeta$ and $\eta = p\eta$. Then, $\zeta \perp p_{\alpha}p\eta$ (since $p_{\alpha}p\eta \in p'H$) and therefore

$$|\langle \zeta, \eta \rangle| = |\langle (1 - p')\zeta, p\eta \rangle| = |\langle (1 - p')\zeta, (p - p_{\alpha}p)\eta \rangle| \leq \|p - p_{\alpha}p\| \|\xi\| \|\eta\| \leq \delta \|\zeta\| \|\eta\|,$$

from which we conclude again by Lemma 2.2

$$\|p - p'p\| = \|p - p'p^*\| = \|p - pp'\| \leq \delta.$$

From (2.15) and (2.16) we get

$$\|p - p'\| \leq \|p - pp'\| + \|p' - pp'\| \leq \delta \frac{2 - \delta}{1 - \delta},$$

which can be arbitrarily small. \qed

**Lemma 2.10.** Let $F$ be an algebra of finite type elements. For every projection $p \in F$ there is an approximate unit $p_{\alpha}$ in $F$ such that for all $\alpha$ there holds $p \leq p_{\alpha}$.

**Proof.** Let $p \in F$ and let $p_{\alpha}$ be an approximate unit. By Lemma 2.9 for $\alpha$ large enough, we have $p \sim p' \leq p_{\alpha}$ and $\|p - p'\| < 1$. By Lemma 2.4 there is a unitary $u$ such that $p' = u^*pu$. Then $p_{\alpha}' = u^*p_{\alpha}u$ is an approximate unit that contains $p$. \qed
The following Proposition plays the key role in this paper. It ensures that we can transfer an approximate unit from the right side of an almost invertible element to its left side, retaining some triangular properties. Briefly, if such almost invertible element is upper triangular, with upper left entry invertible, then this entry itself has a triangular form (but lower - not upper) with respect to an approximate unit from the right and its corresponding approximate unit from the left.

**Proposition 2.11.** Let $p, q \in \mathcal{F}$, let $a$ be invertible up to $(p, q)$, and let $qa(1 - p) = 0$. Further, let $p_\alpha \geq p$ be an approximate unit for $\mathcal{F}$. Then there exists an approximate unit $q_\alpha \in \mathcal{F}$, such that $q_\alpha - q \sim p_\alpha - p$, $a$ is invertible up to $(p_\alpha, q_\alpha)$, and

\[(2.17) \quad (1 - q_\alpha)a(p_\alpha - p) = 0,
\]

for all $\alpha$.

**Proof.** a) Since $a' = (1 - q)a(1 - p)$ is invertible, and since $p_\alpha - p \leq 1 - p$, there is (by Lemma 2.6) a minimal projection $q'_\alpha$ such that $q'_\alpha a(p_\alpha - p) = a(p_\alpha - p)$ and $a$ is invertible up to $(1 - (p_\alpha - p), 1 - q'_\alpha)$. By $qa(1 - p) = 0$ and by Lemma 2.7 we have $q'_\alpha \leq 1 - q$. Let $q_\alpha = q + q'_\alpha$. We have

\[(2.18) \quad (q_\alpha - q)a(p_\alpha - p) = a(p_\alpha - p)
\]

and as a consequence (2.17). Indeed, using $qa(1 - p) = 0$, we find $qa = qap$ and hence $qap_\alpha = qap_\alpha = qap$. Thus $(q_\alpha - q)a(p_\alpha - p) = q_\alpha ap_\alpha - q_\alpha ap - qap_\alpha + qap = q_\alpha ap_\alpha - q_\alpha ap = q_\alpha a(p_\alpha - p)$. Thus, (2.18) becomes $q_\alpha a(p_\alpha - p) = a(p_\alpha - p)$ which is equivalent to (2.17).

Since $p_\alpha - p \leq 1 - p$, we have $qa(p_\alpha - p) = 0$, and from (2.17) we have

\[0 = (1 - q_\alpha)a(p_\alpha - p) = (1 - q_\alpha)a(p_\alpha - p) + qa(p_\alpha - p) = (1 - (q_\alpha - q))a(p_\alpha - p).
\]

By Lemma 2.8 $a$ is invertible up to $(p_\alpha, q_\alpha)$. Also, $q'_\alpha \sim p_\alpha - p \in \mathcal{F}$ and therefore $q_\alpha = q'_\alpha + q \in \mathcal{F}$, as well. The first claim is proved.

Let us prove that $q_\alpha$ is a left approximate unit for $\mathcal{F}$. Let $a' = (1 - q)a(1 - p)$, and let $f \in \mathcal{F}$. Since $qa(1 - p) = 0$, there holds $a' = a - (1 - q)ap - qap$. Therefore $(1 - q_\alpha)a' = (1 - q_\alpha)a - (1 - q_\alpha)ap$, since $q_\alpha \geq q$. Now, by (2.17), we have $(1 - q_\alpha)f = (1 - q_\alpha)a(1 - p)f = (1 - q_\alpha)a(1 - p_\alpha)f \to 0$ in norm topology for any $f \in \mathcal{F}$, since $(1 - q_\alpha)$ is norm bounded and $p_\alpha$ is an approximate unit. Thus $q_\alpha$ is a left approximate unit for $a'\mathcal{F}$. However, any $f \in \mathcal{F}$ can be expressed as $f = (1 - q)f + qf = a'bf + qf$. Using $q \leq q_\alpha$ we get $(1 - q_\alpha)q = 0$ and therefore

\[(1 - q_\alpha)f = (1 - q_\alpha)a'bf \to 0, \quad \text{in norm.}
\]

To finish the proof, note that the mapping $p_\alpha \mapsto q_\alpha$ preserves order. Indeed, if $p_\beta \geq p_\alpha$, we have

\[(q_\alpha - q)a(p_\alpha - p) = a(p_\alpha - p), \quad (q_\beta - q)a(p_\beta - p) = a(p_\beta - p),
\]

where $q_\alpha - q, q_\beta - q$ have minimal property. Multiply the second equality by $(p_\alpha - p)$ and using $(p_\beta - p)(p_\alpha - p) = p_\alpha - p$ we find $(q_\beta - q)a(p_\alpha - p) = a(p_\alpha - p)$, which implies $q_\beta \geq q_\alpha$ by minimal property.

Since $\mathcal{F}$ is a $*$-ideal, we have that $q_\alpha$ is a right approximate unit, as well. □
**Index and its properties.** First, we establish that the difference \([p] - [q]\) will not change as long as a fixed \(a\) is invertible up to \((p, q)\). We need such a result to define the index, exactly to be the mentioned difference in the sequel Definition.

**Proposition 2.12.** Let \(a \in A\) be invertible up to \((p, q)\), and also invertible up to \((p', q')\), and let \(p, q, p', q' \in \mathcal{F}\). Then in \(K(\mathcal{F})\) we have

\[ [p] - [q] = [p'] - [q']. \]

**Proof.** Due to the Proposition [2.8], we may assume that \(qa(1 - p) = q'a(1 - p') = 0\). Since the same proposition follows that assumption will not change classes \([q]\) and \([q']\).

By Lemma [2.10] there is an approximate unit \(p_\alpha\) of \(\mathcal{F}\) containing \(p\). For all \(\alpha\) we have

\[ p_\alpha = p + r_\alpha. \]

By Lemma [2.9] choose \(\alpha\) large enough, such that \(p'' \sim p', \ \|p'' - p'\| < \delta\) and

\[ p_\alpha = p'' + r'_\alpha, \]

where \(\delta < 1/2, \delta < 1/(4C||a|| ||b'||)\), \(C\) is the absolute constant from (2.2) and \(b'\) is \((p', q')\)-inverse of \(a\).

The previous two displayed formulae yields

(2.19) \[ [p] + [r_\alpha] = [p'] + [r'_\alpha] \]

By Proposition [2.11] there is another approximate unit \(q_\alpha \geq q\) such that \(q_\alpha - q \sim p_\alpha - p\) and such that (2.17) holds. Let \(s_\alpha = q_\alpha - q\). Then

(2.20) \[ q_\alpha = q + s_\alpha, \quad r_\alpha \sim s_\alpha. \]

Enlarging \(\alpha\), if necessary, there is \(q'' \leq q_\alpha\) such that \(q'' \sim q'\) and \(||q'' - q'|| < \delta\) and consequently

\[ q_\alpha = q'' + s'_\alpha. \]

Thus we have, also

(2.21) \[ [q] + [s_\alpha] = [q'] + [s'_\alpha]. \]

By Lemma [2.4]b), using \(||p'' - p'||, ||q'' - q'|| < 1/(4C||a|| ||b'||)\), we conclude that \(a\) is invertible up to \((p'', q'')\). Also, if \(b''\) is \((p'', q'')\)-inverse of \(a\), then \(||b''|| \leq 2||b'||\).

Next, we want to estimate \(||(1 - q_\alpha)a(p_\alpha - p'')||\). Using (2.17) we have

\[
(1 - q_\alpha)a(p_\alpha - p'') = (1 - q_\alpha)a(p_\alpha - p + p - p' + p' - p'') \\
= (1 - q_\alpha)a(p_\alpha - p) + (1 - q_\alpha)a(p - p') + (1 - q_\alpha)a(p' - p'') \\
= (1 - q_\alpha)a(p - p') + (1 - q_\alpha)a(p' - p'').
\]

Enlarging \(\alpha\), once again, we can assume that \(||(1 - q_\alpha)a(p - p')|| < ||a|| \delta\). Thus, we have \(||(1 - q_\alpha)a(p_\alpha - p'')|| < 2||a|| \delta < 1/(2C||b''||) \leq 1/(C||b''||)\). Since \(C > 1\), we can apply Lemma [2.6] to obtain \(q_\alpha - q'' \sim p_\alpha - p'\), i.e.

(2.22) \[ r'_\alpha \sim s'_\alpha. \]

Subtracting (2.19) and (2.21) we obtain

\[ [p] + [r_\alpha] - [q] - [s_\alpha] = [p'] + [r'_\alpha] - [q'] - [s'_\alpha] \]

which finishes the proof, because \([r_\alpha] - [s_\alpha] = 0\) by (2.20), and \([r'_\alpha] - [s'_\alpha] = 0\) by (2.22). \(\square\)
Remark 2.1. Note that in all previous proofs, we obtain Murray von Neumann equivalency, whereas, at this moment, we include the cancellation law in $K$ group, which produces that it can be $[p] = [q]$ though $p$ and $q$ might not be Murray - von Neumann equivalent. Namely, the Grothendick functor expand initial equivalence relation, if the underlying semigroup does not satisfy the cancellation law. This fact is known in $K$ theory as "stable equivalency".

Definition 2.2. Let $F$ be finite type elements. We say that $a ∈ A$ is of Fredholm type (or abstract Fredholm element) if there are $p, q ∈ F$ such that $a$ is invertible up to $(p, q)$. The index of the element $a$ (or abstract index) is the element of the group $K(F)$ defined by

$$\text{ind}(a) = ([p], [q]) ∈ K(F),$$

or less formally

$$\text{ind}(a) = [p] − [q].$$

Note that the index is well defined due to the Proposition 2.12.

Now, we proceed deriving the properties of the abstract index, defined in the previous Definition.

Proposition 2.13. The set of Fredholm type elements is open in $A$ and the index is a locally constant function.

Proof. It follows immediately from Lemma 2.3. □

Proposition 2.14. a) Let $a ∈ A$ be of Fredholm type, and let $f ∈ F$. Then $a + f$ is also of Fredholm type, and $\text{ind}(a + f) = \text{ind} a$.

b) If $f ∈ F$, then $1 + f$ is of Fredholm type, and $\text{ind}(1 + f) = 0$. Moreover, there is $p ∈ F$ such that $1 + f$ is invertible up to $(p, p)$.

Proof. a) Let $p, q ∈ F$ be projections such that $a$ is invertible up to $(p, q)$. By Proposition 2.11 there are approximate units $p_α ≥ p$, $q_α ≥ q$ such that $a$ is invertible up to $(p_α, q_α)$. Choose $α$ large enough such that $||f|| ≤ ||b_α||^{-1}$, where $b_α$ is $(p_α, q_α)$-inverse of $a$. Then,

$$(1 − q_α)(a + f)(1 − p_α) = (1 − q_α)a(1 − p_α) + (1 − q_α)f(1 − p_α)$$

and $||(1 − q_α)f(1 − p_α)|| ≤ ||f|| ≤ ||b_α||^{-1}$. By Lemma 2.4, $a + f$ is also invertible up to $(p_α, q_α)$ and hence $\text{ind}(a + f) = [p_α] − [q_α] = \text{ind} a$.

b) In this special case, $α = 1$ and we can choose $q_α = p_α$. □

Proposition 2.15. a) If $a$ is of Fredholm type, then $a$ is invertible modulo $F$;

b) Conversely, if $a$ is invertible modulo $F$, then $a$ is of Fredholm type.

Proof. a) Assume that $a$ is invertible up to $(p, q)$. Let $b$ be $(p, q)$-inverse of $a$. Then $ab = (1 − q)a + qab = (1 − q)a(1 − p)b + qab = 1 − q + qab ∈ 1 + F$, and also $ba = b(1 − p) + ba = b(1 − q)a(1 − p) + bab = 1 − p + bab ∈ 1 + F$.

b) Let $ab_1 = 1 + f_1$, $b_2a = 1 + f_2$, $f_1, f_2 ∈ F$. By the previous Proposition, there is $p ∈ F$ such that $ab_1$ is invertible up to $(p, p)$. It follows that $a^* = a^*(1 − p)b_1^*$, where $c ∈ (1 − p)A(1 − p)$ is $(p, p)$-inverse of $ab_1$. Therefore, by Lemma 2.3, there is a projection $1 − r ∈ A$ such that $(1 − r)a^*(1 − p) = a^*(1 − p)$, or equivalently,

$$(2.23) \quad (1 − p)a = (1 − p)a(1 − r).$$

Furthermore, $a$ is invertible up to $(p, r)$. It remains to prove $r ∈ F$. 

Considering \((1 - p)a\) instead of \(a\), we find that
\[
b_2(1 - p)a = b_2a - b_2pa = 1 + f_2 - b_2pa = 1 + f_2' \in 1 + \mathcal{F}.
\]
Again, by the previous Proposition, there is a \(q \in \mathcal{F}\) such that \(b_2(1 - p)a\) is invertible up to \((q, q)\). By Proposition 2.23 there is a projection \(q' \in \mathcal{F}\) such that \(b_2(1 - p)a\) is invertible up to \((q', q)\) and such that \((1 - q)b_2(1 - p)aq' = 0\). The last is equivalent to
\[
(1 - q)b_2(1 - p)a(1 - q') = (1 - q)b_2(1 - p)a.
\]
If \(t\) is \((q', q)\)-inverse of \(b_2(1 - p)a\), then
\[
(1 - q)r = t(1 - q)b_2(1 - p)a(1 - q')r = t(1 - q)b_2(1 - p)ar = 0,
\]
since from (2.23) we easily find \((1 - p)ar = 0\). Thus, \(r = qr \in \mathcal{F}\), since \(\mathcal{F}\) is an ideal.

\textbf{Theorem 2.16} (index theorem). Let \(\mathcal{A}\) be a \(C^*\)-algebra, and let \(\mathcal{F} \subseteq \mathcal{A}\) be an algebra of finite type elements. If \(t_1\) and \(t_2\) are Fredholm type elements then \(t_1t_2\) is of Fredholm type as well. Moreover there holds
\[
\text{ind}(t_1t_2) = \text{ind} t_1 + \text{ind} t_2.
\]
In other words, If we denote the set of all Fredholm type elements by \(\text{Fred}(\mathcal{F})\), then \(\text{Fred}(\mathcal{F})\) is a semigroup (with unit) with respect to multiplication, and the mapping \(\text{ind}\) is a homomorphism from \((\text{Fred}(\mathcal{F}), \cdot)\) to \((\mathcal{K}(\mathcal{F}), +)\).

\textbf{Proof.} Let \(p_1, q_1, p_2, q_2 \in \mathcal{F}\) be projections such that \(t_1\) is invertible up to \((p_1, q_1)\) and \(t_2\) is invertible up to \((p_2, q_2)\). By Proposition 2.11 there are approximate units \(p_\alpha \geq p_2, q_\alpha \geq q_2\) such that \(t_2\) is also invertible up to \((p_\alpha, q_\alpha)\) and
\[
\text{ind} t_2 = \lfloor p_2 \rfloor - \lfloor q_2 \rfloor = \lfloor p_\alpha \rfloor - \lfloor q_\alpha \rfloor.
\]
By Lemma 2.23 there is an \(\alpha\) and \(p' \sim p_1\) such that \(p' \leq q_\alpha\) and \(\|p_1 - p'\| < 1/2\).
By Lemma 2.24 b), \(t_1\) is invertible up to \((p', q_1)\), and therefore \(\text{ind} t_1 = \lfloor p' \rfloor - \lfloor q_1 \rfloor\).
Next, by Proposition 2.8 there is a projection \(q' \sim q_1\) such that \(t_1\) is invertible up to \((p', q')\) and
\[
q't_1(1 - p') = 0.
\]
It follows
\[
\text{ind} t_1 = \lfloor p' \rfloor - \lfloor q' \rfloor.
\]
Let \(r = q_\alpha - p'\). We have \(r \leq 1 - p'\), and by Lemma 2.25 there is a minimal projection \(s\) such that
\[
st_1r = t_1r.
\]
Also \(r \sim s\) and \(t_1\) is invertible up to \((1 - r, 1 - s)\). By (2.26) and Lemma 2.27 we conclude \(s \leq 1 - q'\). In particular, \(1 - q' - s\) is a projection and \(1 - q' - s \leq 1 - s\).
So, from (2.24) we conclude that \((1 - s)t_1r = 0\) and hence \((1 - q' - s)t_1r = (1 - q' - s)(1 - s)t_1r = 0\). Therefore, by Lemma 2.4 we obtain that \(t_1\) is invertible up to \((q_\alpha, q' + s)\).
Now, it is easy to derive that \(t_1(1 - q_\alpha)t_2\) is invertible up to \((p_\alpha, q' + s)\). Indeed, if \(v_1\) is \((q_\alpha, q' + s)\)-inverse of \(t_1\), and \(v_2\) is \((p_\alpha, q_\alpha)\)-inverse of \(t_2\), a straightforward
calculation yields that \( v_2v_1 \) is \((p_\alpha, q' + s)\)-inverse of \( t_1(1 - q_\alpha)t_2 \). Thus \( t_1(1 - q_\alpha)t_2 \) is of Fredholm type and

\[
\text{ind}(t_1(1 - q_\alpha)t_2) = [p_\alpha] - [q'] - [s] = [p_\alpha] - [q_\alpha] + [q'] - [r] = [p_\alpha] - [q_\alpha] + [q'] - [q'] = \text{ind} t_1 + \text{ind} t_2.
\]

Since \( t_1t_2 \) and \( t_1(1 - q_\alpha)t_2 \) differ by \( t_1q_\alpha t_2 \in F \), by Proposition 2.14 a) \( t_1t_2 \) is also of Fredholm type and its index is the same as that of \( t_1(1 - q_\alpha)t_2 \). The proof is complete.

\[\square\]

3. Applications

In this section we apply results obtained in the previous section in three different ways. Namely, we shall prove corollaries concerning the ordinary Fredholm operators, the Fredholm operators in the sense of Atiyah and Singer over the \( II_\infty \) factors and Fredholm operators on Hilbert \( C^* \)-modules over a \( C^* \)-algebra. All of these corollaries have been known for a long time. Nevertheless, we shall derive them in order to show that results obtained in the previous section are a generalization of all of them.

Classic Fredholm operators on a Hilbert space.

**Corollary 3.1.** Let \( \mathcal{A} \) be the full algebra of all bounded operators on some infinite dimensional Hilbert space \( H \), and let \( F \) be the ideal of all compact operators. Then the couple \( (\mathcal{A}, F) \) satisfy the conditions (i) – (iii) of Definition 1.1. Hence, ordinary Fredholm operators are the special case of Fredholm type elements defined in this note.

**Proof.** (i) The ideal of all compact operators is a selfadjoint ideal in \( B(H) \).

(ii) As it is well known, the set of all finite rank projections is an approximate unit for compact operators. Even more, if the Hilbert space \( H \) is separable, there is a countable approximate unit for \( F \). In more details if \( P_n \) is the projection on the space generated by \( \{e_1, \ldots, e_n\} \), where \( \{e_j\} \) is an countable complete orthonormal system, then \( \{P_n \mid n \in \mathbb{N}\} \) is a countable approximate unit;

(iii) Any compact projection \( P \in F \) is a finite rank projection. Then \( H \ominus PH \cong H \). Let \( V : H \to H \ominus PH \) be an isomorphism. Then \( VQV^* \) is a projection equivalent to \( Q \) such that \( VQV^*(H) \perp P(H) \). Therefore, \( VQV^* + P \) is projection. \[\square\]

Fredholm operators on a von Neumann algebra. Our second application is devoted to Fredholm operators in the sense of Breuer [4, 5] on a properly infinite von Neumann algebra. First, we give the Breuer definition

**Definition 3.1.** Let \( \mathcal{A} \) be a von Neumann algebra, let \( \text{Proj}(\mathcal{A}) \) be the set of all projections belonging to \( \mathcal{A} \), and let \( \text{Proj}_0(\mathcal{A}) \) be the set of all finite projections in \( \mathcal{A} \) (i.e. those projections that are not Murray von Neumann equivalent to any its proper subprojection).

The operator \( T \in \mathcal{A} \) is said to be \( \mathcal{A} \)-Fredholm if (i) \( P_{\ker T} \in \text{Proj}_0(\mathcal{A}) \), where \( P_{\ker T} \) is the projection to the subspace \( \ker T \) and (ii) There is a projection \( E \in \text{Proj}_0(\mathcal{A}) \) such that \( \text{ran}(I - E) \subseteq \text{ran}T \). The second condition ensures that \( P_{\ker T} \) also belongs to \( \text{Proj}_0(\mathcal{A}) \).

The index of an \( \mathcal{A} \)-Fredholm operator \( T \) is defined as

\[
\text{ind} T = \dim(\ker T) - \dim(\ker T^*) \in I(\mathcal{A}).
\]
Here, $I(A)$ is the so called index group of a von Neumann algebra $A$ defined as the Grothendieck group of the commutative monoid of all representations of the commutant $\mathcal{A}'$ generated by representations of the form $\mathcal{A}' \ni S \mapsto ES = \pi_E(S)$ for some $E \in \text{Proj}_0(A)$ [4, Section 2]. For a subspace $L$, its dimension $\dim L$ is defined as the class $[\pi_{P_L}] \in I(A)$ of the representation $\pi_{P_L}$, where $P_L$ is the projection to $L$.

Before proving that the Breuer’s $A$-Fredholm operators is a special case of abstract Fredholm operators, we list some well known facts concerning von Neumann algebras.

Lemma 3.2. The set $\text{Proj}(A) = \{ p \in A \mid p \text{ is a projection} \}$ is a complete lattice. In particular, if $p, q \in \text{Proj}(A)$, then $p \lor q$ (the least upper bound of the set $\{ p, q \}$) belongs to $\text{Proj}(A)$.

Proof. The proof can be found in [14, Proposition V.1.1] or [3, I.9.2.1(ii)]. □

The operator $T \in \mathcal{A}$ is called finite if the projection to the closure of its range $P_{\text{ran}T} \in \text{Proj}_0(\mathcal{A})$. The set of all finite operators is denoted by $m_0$, and its norm closure is denoted by $m$.

Lemma 3.3. The set $m$ is a selfadjoint, two sided ideal of $\mathcal{A}$. Also, $m$ is generated (as a closed selfadjoint two sided ideal) by $\text{Proj}_0(\mathcal{A})$.

Proof. This was proved in the Section 3 of [4]. □

Lemma 3.4. If $p, q \in \text{Proj}_0(A)$ then $p \lor q \in \text{Proj}_0(A)$ as well.

Proof. The proof can be found in [14, Proposition V.1.6] or [3, III.1.1.3]. □

Lemma 3.5. Let $T \in \mathcal{A}$ and let $T = U|T|$ be its polar decomposition. Then both $U, |T| \in \mathcal{A}$.

Proof. The proof can be found in [14, Proposition II.3.14] or [3, I.9.2.1(iii)]. □

Lemma 3.6. Let $0 \leq T \in \mathcal{A}$. Then all spectral projections of $T$ also belongs to $\mathcal{A}$.

Proof. The proof can be found in [3, I.9.2.1(iv)]. □

Lemma 3.7. Let $\mathcal{A}$ be a von Neumann algebra which is not of finite type. Then the operator $T \in \mathcal{A}$ is $A$-Fredholm if and only if $T$ is invertible modulo $m$.

Proof. This is [5, Theorem 1]. □

Lemma 3.8. Let $\mathcal{A}$ be a von Neumann algebra which is not of finite type, and let $T$ be a $A$-Fredholm operator. Then $|T|$ is $A$-Fredholm as well.

Proof. Let $T = V|T|$ be the polar decomposition of $T$. By Lemma 3.5 $|T| \in \mathcal{A}$, and by Lemma 3.7 there is $S \in \mathcal{A}$ such that $ST, TS \in 1 + m$, i.e. $S$ is $m$-inverse of $T$. Then, as it easy to see, $SV$ is a left $m$-inverse of $|T|$, whereas $V^*S^*$ is a right inverse of $|T|$. Therefore, $|T|$ is invertible modulo $m$, and again by Lemma 3.7 $|T|$ is $A$-Fredholm. □

Lemma 3.9. Let $T \geq 0$ be an $A$-Fredholm operator, and let $E_T$ be its spectral measure. Then, there is $\delta > 0$ such that $E_T([0, \delta)) \in \text{Proj}_0(A)$. 

Proof. By Definition 3.1 there is a closed subspace $L \subseteq \text{ran } T$, which complement is finite (i.e. $I - P_L \in \text{Proj}_0(A)$). Consider the restriction $T_1 = T|_{\text{ker } T^\perp}$, and define $L_1 = T_1^{-1}(L)$. Since both $L_1$ and $L$ are closed, by open mapping theorem $T_1$ realizes a topological isomorphism between $L_1$ and $L$. Hence, there is $\delta > 0$ such that for all $\xi \in L_1$ there holds $\|T\xi\| > \delta\|\xi\|$. However, for $\xi \in \text{ran } E_T([0, \delta))$ we have $\|T\xi\| \leq \delta\|\xi\|$, i.e. $L_1 \cap \text{ran } E_T([0, \delta)) = \{0\}$. Thus, it suffices to prove that $L_1^\perp$ is finite.

Consider $L_2 = (\ker T)^\perp \oplus L_1$. Obviously, $T(L_2) \cap L = \{0\}$, implying that $(I - P_L)T$ maps $L_2$ injectively to a subspace of $L^\perp$. By Lemma 3.3 applied to $(I - P_L)TP_{L_2}$ there is a partial isometry $W \in A$ that connects $L_2$ and the closure $L_3$ of $(I - P_L)T(L_2)$ which is a subspace of $L^\perp$. Since $L^\perp$ is finite, both $L_3 \subseteq L^\perp$ and $L_2 \sim L_3$ are also finite. Since by Definition 3.1 $\ker T$ is finite as well, it follows that $L_1^\perp$ is finite. The proof is complete, because $E_T([0, \delta)) \leq P_{L_1^\perp}$.

Corollary 3.10. Let $A$ be a properly infinite von Neumann algebra acting on a Hilbert space $H$, and let $m$ be defined as above. Then the couple $(A, m)$ satisfies the conditions (i)–(iii) of Definition 1.1.

Moreover, abstract Fredholm elements are generalized Fredholm operators in the sense of Breuer.

Proof. By Lemma 3.3 $m$ is an $*$-ideal which proves (i) in Definition 1.1.

Let us prove that the set $\text{Proj}_0(A)$ is an approximate unit. First, this is a directed set. Indeed, let $p, q \in \text{Proj}_0(A)$. Since projection in a von Neumann algebra makes a complete lattice (Lemma 3.2), the projection $p \lor q \in A$. By Lemma 3.3 $p \lor q \in \text{Proj}_0(A)$ as well. Let $T \in m$, and let $T = U[T]$ be its polar decomposition. By Lemma 3.3 $U, T \in A$, as well as any spectral projection of $|T|$ also belongs to $A$. Pick $\varepsilon > 0$, and denote $P_\varepsilon = E_{|T|}(\varepsilon, +\infty)$, where $E_{|T|}$ stands for the spectral measure of $|T|$. Then $P_\varepsilon \leq P_{\text{ran } T}$ implying $P_\varepsilon \in \text{Proj}_0(A)$. Also, $\| |T|(I - P_\varepsilon)\| \leq \varepsilon$, since the function $\lambda \mapsto \lambda(1 - \chi_{[\varepsilon, +\infty)})$ is bounded by $\varepsilon$. For all $P \geq P_\varepsilon$ we have $I - P \leq I - P_\varepsilon$ and hence, $\|U[T](I - P)\| = \|U[T](I - P_\varepsilon)(I - P)\| \leq \varepsilon$. We proved (ii) in Definition 1.1.

Since $A$ is properly infinite, by [3] Lemma 8, for $P, Q \in \text{Proj}_0(A)$, there holds $I - P \sim I$, i.e. there is a a partial isometry $V$ such that $V^*V = I$, and $VV^* = I - P$. Then $VQ$ is a partial isometry, required in the property (iii) od Definition 1.1.

Using Proposition 2.15 and Lemma 3.7 we see that abstract Fredholm operators with respect to $(A, m)$ coincide with $A$-Fredholm operators in the sense of Breuer. Let us prove that their indices are equal.

Let $T$ be Fredholm operator. By Lemma 3.3 there is a $\delta > 0$ such that $P = E_{|T|}([0, \delta]) \in \text{Proj}_0(A)$. The operator $|T|$ is bounded below on the space $E_{|T|}([\delta, \infty))H = (I - P)H$, hence $T = V|T|$ is also bounded below on $(I - P)H$. Let $I - Q$ be the projection on the closure of $T(I - P)H$. Then, obviously, $T$ is invertible up to $(P, Q)$.

We have $P = P_{\ker T} \oplus E_{|T|}([0, \delta]) = P_{\ker T} \oplus P_1$, and $N_1 = P_1H$ is invariant for $|T|$. Since $V$ acts as a partial isometry on $N, T$ maps $N_1$ some subspace $N_2 \sim N_1$. Thus, $P(H) = \ker T \oplus P_{N_1}$ and $Q = \ker T^* \oplus P_{N_2}$, and $P_{N_1} \sim P_{N_2}$ and the result follows.

Remark 3.1. The condition (iii) in Definition 1.1 is not substantial. If it is suppressed, we have not a semigroup, but a conditional addition. However, we can also
form a corresponding $K$-group, starting with a free group with $\text{Proj}_0(A)$ as generators and then using appropriate identifications. Obtained group will be isomorphic to $I(A)$, see [4, Section 2].

**Fredholm operators on the standard Hilbert module.** The last corollary is devoted to Fredholm theory on the standard Hilbert $C^*$-module $l^2(\mathcal{B})$ over a unital $C^*$-algebra $\mathcal{B}$. We list some definitions and known facts. For references and further details the reader is referred to [7].

Let $\mathcal{B}$ be a unital $C^*$-algebra. We define the right Hilbert module $l^2(\mathcal{B})$ as 

$$l^2(\mathcal{B}) = \{(a_n)_{n=1}^{\infty} | a_n \in \mathcal{B}, \sum_n \langle a_n, a_n \rangle \text{ converges in norm}\}$$

equipped with the right action of $\mathcal{B}$ $(a_n)b = (a_nb)$ and with the inner product $\langle \cdot, \cdot \rangle : l^2(\mathcal{B}) \times l^2(\mathcal{B}) \to \mathcal{B}$ given by

$$\langle (a_n), (b_n) \rangle = \sum_{n=1}^{\infty} a_n^*b_n.$$ 

Let $B(l^2(\mathcal{B}))$ denote the set of all bounded operators on $l^2(\mathcal{B})$, and let $B^a(l^2(\mathcal{B}))$ denote the algebra of all operators from $B(l^2(\mathcal{B}))$ that have the adjoint.

Let $C_0(l^2(\mathcal{B}))$ denote the algebra generated by all operators of the form 

$$\theta_{x,y}(z) = x \langle y, z \rangle.$$ 

Its closure we will denote by $C(l^2(\mathcal{B}))$. Although such operators might not map bounded into relatively compact sets it is common to call them compact operators.

Next, we quote definitions and statements given by Mingo [8] and Mischenko and Fomenko [9].

**Definition 3.2.** $T \in B^a(l^2(\mathcal{B}))$ is Fredholm in the sense of Mingo, if it is invertible modulo compact operators. [8, §1.1. Definition].

**Proposition 3.11.** $T$ is Fredholm in the sense of Mingo if and only if there is a compact perturbation $S$ of $T$ (i.e. $T - S$ is compact) such that $\ker S$ and $\ker S^*$ are finitely generated projective. The difference

$$\ker S - \ker S^* \in K_0(\mathcal{B})$$

does not depend on the choice of $S$ [8 §1.4.].

**Definition 3.3.** The index of a Fredholm operator $T$ is defined to be the difference [8, §2] where $S$ is any compact perturbation of $T$ [8 §1.4.].

**Definition 3.4.** $T$ is Fredholm in the sense of Mischenko and Fomenko (MF sense in further) if (i) $T$ is adjointable; (ii) there are decompositions of the domain $l^2(\mathcal{B}) = \mathcal{M}_1 \oplus \mathcal{N}_1$ and of the codomain $l^2(\mathcal{B}) = \mathcal{M}_2 \oplus \mathcal{N}_2$, where $\mathcal{N}_1$, $\mathcal{N}_2$ are finitely generated projective submodules and $T$ with respect to such decompositions has a matrix form $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, with $T_1$ is an isomorphism. [7 Definition 2.7.4].

The index of $T$ is $\text{ind} \ T = [\mathcal{N}_1] - [\mathcal{N}_2]$ [7 Definition 2.7.8].

**Proposition 3.12.** If $T$ is invertible modulo compacts then $T$ is Fredholm in MF sense. [7 Theorem 2.7.14]
Note that the other implication trivially holds. Namely, if $T$ is Fredholm in MF sense, then \( \begin{pmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \) is the inverse of $T$ modulo compacts.

We also need the following Lemma.

**Lemma 3.13.** Let $l^2(B) = M \oplus N$, where $N$ is finitely generated and projective and $M$ is complemented. Next, let $P$ be the orthogonal projection to $N$. Then $I - P$ is bounded below on $M$.

**Proof.** Suppose this is not true, i.e. that there is a sequence $x_n \in M$, $||x_n|| = 1$, $(I - P)x_n \to 0$. Decompose $(I - P)x_n$ as

\[
(I - P)x_n = y_n' + y_n'', \quad y_n' \in M, y_n'' \in N.
\]

Then, $y_n', y_n'' \to 0$, since $M$ and $N$ are complemented. However, from (3.3) we obtain $M \ni x_n - y_n = Px_n + y_n'' \in N$. Thus both sides are equal to 0, since the sum $M \oplus N$ is direct. Hence $x_n = y_n' \to 0$ which gives a contradiction with $||x_n|| = 1$. $\square$

**Corollary 3.14.** Let $A = l^2(B)$ be the standard Hilbert module over some unital $C^*$-algebra $B$ and let $F = C(l^2(B))$, the set of all compact operators on $l^2(B)$. Then the couple $(A, F)$ satisfies conditions in Definition 1.1.

Moreover, the set of abstract Fredholm operators, Fredholm operators in the sense of Mingo, and in MF sense coincide. Finally all three indices are equal.

**Proof.** (i) It is well known that $F$ is a closed ideal.

(ii) Let $l^2(B)$ possesses the standard basis $(e_i), i \in \mathbb{N}$, where $e_i = (0, ..., 0, 1, 0, ...)$ with the unit being the $i$-th entry.

Let $P_n \in F$ be sequence of projections on $B^n$, $P_n(x_1, x_2, \ldots) = (x_1, \ldots, x_n, 0, \ldots)$ and let $K \in C(l^2(B))$. By [14, Proposition 2.2.1] we have

\[
||K - KP_n|| \to 0, \quad n \to \infty.
\]

Therefore, $P_n$ is an approximate unit.

(iii) Let $P, Q \in F$ be projections. By [15, Theorem 15.4.2], $P(A)$ and $Q(A)$ are isomorphic (as modules) to some direct summand in $B^n$ and $B^m$, respectively, for some $m, n \in \mathbb{N}$. Hence, there is partial isometry

\[
V : l^2(B) \to l^2(B), \quad V(x_1, x_2, ...) = e_{m+1}x_1 + e_{m+2}x_2 + ... + e_{m+n}x_n.
\]

Then $P_1 = VPV^*$ is orthogonal to $Q$. Consider the operator $U = VP$. We have

\[
UU^* = VPV^* = P_1 \quad \text{and} \quad U^*U = PV^*VP = P,
\]

since $V^*V = P_{e_1, \ldots, e_n} \geq P$. Also $(UU^* + Q)(UU^* + Q) = VPV^*VPV^* + Q^2 = VPV^* + Q$ implying that $UU^* + Q$ is a projection. Thus, property (iii) is proved.

Proposition 2.12, Proposition 3.12 together with a comment after it and Definition 4.2 proves that all three kind of Fredholm operators (abstract, in the sense of Mingo and in MF sense) coincide. Let us prove that their indices are equal.

Let $T$ be a Fredholm operator, and let $T$ be invertible up to $(P, Q)$. Then $T' = (I - Q)T(I - P)$ is a compact perturbation of $T$ (its difference is $QT(I - P) + (I - Q)TP + QTP$), and $\ker T' = P(l^2(B))$, $\ker T^{\ast} = Q(l^2(B))$. Hence, abstract index is equal to Mingo’s index.

Let $T$ be a Fredholm operator, and let $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ with respect to decompositions $l^2(B) = M_1 \oplus N_1 = M_2 \oplus N_2$ of the domain and the codomain, respectively.
By [7, Theorem 2.7.6], $M_1$, $N_1$, $M_2$, $N_2$ can be chosen such that all of them are complemented, $N_1$ and $N_2$ are finitely generated projective and also $M_1 = N_1^\perp$.

(In truth, the last equality is not emphasized in the statement, but it follows from the proof.) In the same proof we can find that $T|_{M_1}$ is bounded below. Denote by $P$ and $Q$ the orthogonal projections to $N_1$ and $N_2$. By Lemma 3.13 $I - Q$ is bounded below on the space $M_2 = T(M_1) = \text{ran} T(I - P)$. This ensures that $(I - Q)T(I - P)$ is an isomorphism from $M_1 = N_1^\perp$ onto $N_2^\perp$. Thus the abstract index is $[P] - [Q] = [N_1] - [N_2]$, i.e. it is equal to MF index. □

REFERENCES

[1] T. Alvarez and M. Onieva. Generalized Fredholm operators. *Arch. Math.*, 44:270–277, 1985.
[2] M. F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. *Asterisque*, 32–33:43–72, 1976.
[3] B. Blackadar. *Operator Algebras - Theory of $C^*$-algebras and von Neumann algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer, Berlin, Heidelberg, 2006.
[4] Manfred Breuer. Fredholm theories in von neumann algebras I. *Math. Annalen*, 178(3):243–254, 1968.
[5] Manfred Breuer. Fredholm theories in von neumann algebras II. *Math. Annalen*, 180(4):313–325, 1969.
[6] L. A. Coburn, R. G. Douglas, D. G. Schaeffer, and I. M. Singer. $C^*$-algebras of operators on a half-space, ii Index theory. *Arch. Publications mathematiques de l'I.H.É.S.*, 40:68–79, 1971.
[7] V. M. Manuilov and E. V. Troitsky. *Hilbert $C^*$-modules*. Translations of mathematical monographs Vol. 226. AMS, Providence, Rhode Island, 2005.
[8] James A. Mingo. $K$-theory and multipliers of stable $C^*$-algebras. *Trans. Amer. Math. Soc.*, 299(1):397–411, 1987.
[9] A. S. Mishchenko and A. T. Fomenko. The index of elliptic operators over $C^*$-algebras. *Math. USSR Izv.*, 15(1):87–112, 1980.
[10] G. J. Murphy. *$C^*$-algebras and operator theory*. Academic press, London, 1990.
[11] Catherine L. Olsen. Index theory in von Neumann algebras. *Mem. Amer. Math. Soc.*, 47(294):1–71, 1984.
[12] W. Rudin. *Functional Analysis*. McGraw-Hill Book Company, 1973.
[13] B. Sz.-Nagy and F. Riesz. *Functional analysis*. Frederick Ungar Publishing Co., New York, 1955.
[14] M. Takesaki. *Theory of Operator Algebras I*, volume 124 of *Encyclopaedia of Mathematical Sciences*. Springer, Berlin, Heidelberg, etc., 2001.
[15] N. E. Wegge-Olsen. *$K$-theory and $C^*$-algebras - A friendly approach*. Oxford University Press, Oxford, New York, Tokyo, 1993.

University of Belgrade, Faculty of Mathematics, Studentski trg 16-18, 11000 Beograd, Serbia

E-mail address: keckic@matf.bg.ac.rs

University of Belgrade, Faculty of Mathematics, Studentski trg 16-18, 11000 Beograd, Serbia

E-mail address: zlatkol@matf.bg.ac.rs