AN ADDEDUM TO THE PAPER "SOME ELEMENTARY ESTIMATES FOR THE NAVIER-STOKES SYSTEM"

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Abstract. In this paper we give a proof of the existence of global regular solutions to the Fourier transformed Navier-Stokes system with small initial data in $\Phi(2)$ via an iteration argument. The proof of the regularity theorem is a minor modification of the proof given in the paper "Some elementary estimates for the Navier-Stokes system", so this paper is intended to be just a complement to the afore mentioned paper.

1. Introduction

A Generalized Navier-Stokes system (with periodic boundary conditions on $[0,1]^3$) is a system of the form

\begin{equation}
\begin{aligned}
\psi^k(\xi,t) &= \psi^k(\xi) \exp \left(-|\xi|^2 t\right) \\
&\quad + \int_0^t \exp \left(-|\xi|^2 (t-s)\right) \sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi) \psi^i(q,s) \psi^j(\xi - q,s) \, ds,
\end{aligned}
\end{equation}

for $\xi \in \mathbb{Z}^3$, and where $M_{ijk}(\xi)$ satisfies the bound

$|M_{ijk}(\xi)| \leq |\xi|$.

To solve this problem it is usual to consider the following iteration scheme

\begin{equation}
\begin{aligned}
\psi_{n+1}^k(\xi,t) &= \psi^k(\xi) \exp \left(-|\xi|^2 t\right) \\
&\quad + \int_0^t \exp \left(-|\xi|^2 (t-s)\right) \sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi) \psi_n^i(q,s) \psi_n^j(\xi - q,s) \, ds.
\end{aligned}
\end{equation}

In what follows we will show the convergence of this method for small initial conditions on $\Phi(2)$ (for the definition of the space $\Phi(2)$ see [4]). More exactly we will show that

\textbf{Theorem 1.} There exists an $\epsilon > 0$ such that if $\|\psi\|_2 < \epsilon$, then (1) has a global regular solution with initial condition $\psi$.

The main purpose on writing this note is for it to serve as a complement to our paper [4], and to show that the free divergence condition, neither the fact of considering Leray-Hopf weak solutions is an issue for the proofs presented in that paper.

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2. Existence

We start with two auxiliary results,

**Lemma 1.** There exists an $\epsilon > 0$ such that if $\|\psi\| < \epsilon$ then the sequence $v^k_n(\xi, t)$ is uniformly bounded on $[0, T]$ for $\xi$ fixed.

**Proof.** To proof this fact it is enough to show that if

$$|v^k_n(\xi, t)| \leq \frac{\epsilon}{|\xi|^2}$$

then

$$\left| \sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi) v^i_n(q, s) v^j_n(\xi - q, s) \right| \leq c \epsilon^2,$$

where $c$ is a universal constant, because then we would have, for any $t \geq 0$ and $\epsilon > 0$ small enough,

$$|v^k_{n+1}(\xi, t)| \leq |\psi^k(\xi)| \exp \left( -|\xi|^2 t \right) + c \int_0^t \exp \left( (t - s) \epsilon^2 ds \right)$$

$$\leq \frac{\epsilon}{|\xi|^2} \exp \left( -|\xi|^2 t \right) + \frac{c \epsilon^2}{|\xi|^2} \left( 1 - \exp \left( -|\xi|^2 t \right) \right)$$

$$\leq \frac{\epsilon}{|\xi|^2} \exp \left( -|\xi|^2 t \right) + \frac{c \epsilon^2}{|\xi|^2} \left( 1 - \exp \left( -|\xi|^2 t \right) \right) = \frac{\epsilon}{|\xi|^2}.$$  

We proceed to show the validity of (2). Write

$$\sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi) v^i_n(q, s) v^j_n(\xi - q, s) = I + II + III,$$

where

$$I = \sum_{1 \leq |q| \leq 2 |\xi|, 1 \leq |\xi - q| \leq \frac{|\xi|}{2}} M_{ijk}(\xi) v^i_n(\xi, t) v^j_n(\xi - q, t),$$

$$II = \sum_{1 \leq |q| \leq 2 |\xi|, |\xi - q| > \frac{|\xi|}{2}} M_{ijk}(\xi) v^i_n(\xi, t) v^j_n(\xi - q, t),$$

and

$$III = \sum_{|q| > 2 |\xi|} M_{ijk}(\xi) v^i_n(\xi, t) v^j_n(\xi - q, t).$$

To estimate $I$ observe that if $|\xi - q| \leq \frac{|\xi|}{2}$, then $|q| \geq \frac{|\xi|}{2}$. Therefore, using that

$$|M_{ijk}(\xi)| \leq c |\xi|$$

and the elementary inequality

$$\sum_{1 \leq |q| < r} \frac{1}{|q|^2} \leq cr$$

(where $c$ is a universal constant) we can bound as follows,

$$|I| \leq c |\xi| \frac{\epsilon^2}{|\xi|^2} \sum_{1 \leq |\xi - q| \leq \frac{|\xi|}{2}} \frac{1}{|\xi - q|^2}$$

$$\leq c \frac{\epsilon^2 |\xi|}{|\xi|^2} = c \epsilon^2.$$
can be estimated in the same way, so we also obtain

\[ |II| \leq c \epsilon^2. \]

To estimate III, first notice that \(|q| > 2 |\xi|\) implies that \(|\xi - q| \geq \frac{1}{2} |q|\). Hence, using the inequality

(4) \[ \sum_{|q| \geq r} \frac{1}{|q|^2} \leq \frac{c}{r}, \]

we can bound as follows,

\[ |III| \leq c |\xi|^2 \sum_{|q| > 2 |\xi|} \frac{1}{|q|^2} |\xi - q|^2 \]
\[ \leq c |\xi|^2 \sum_{|q| > 2 |\xi|} \frac{1}{|q|^2} \]
\[ \leq c |\xi| \epsilon^2 \frac{1}{|\xi|} = c \epsilon^2. \]

This shows the lemma. □

**Lemma 2.** If there is an \( \epsilon > 0 \) such that the sequence \( v^k_n (\xi, t) \) satisfies

\[ \|v^k_n (t)\|_2 < \epsilon \quad \text{for all} \quad t \in [0, T] \]

The sequence \( v^k_n (\xi, t) \) is equicontinuous on \([0, T]\) for \( \xi \) fixed.

**Proof.** Let \( t_1, t_2 \in (\rho, T), \ t_2 > t_1 \). Then we estimate for \( \xi \) fixed

\[ |v^k_{n+1} (\xi, t_2) - v^k_{n+1} (\xi, t_1)| \leq I + II + III \]

where

\[ I = |\psi^k (\xi)| \left| \exp \left( - |\xi|^2 t_2 \right) - \exp \left( - |\xi|^2 t_1 \right) \right|, \]
\[ II = \int_{t_1}^{t_2} \left| \exp \left( - |\xi|^2 (t_2 - s) \right) - \exp \left( - |\xi|^2 (t_1 - s) \right) \right| \]
\[ \sum_{q \in \mathbb{Z}^3} |M_{ijk} (\xi) v^i_n (q, s) v^j_n (\xi - q, s)| \, ds \]

and

\[ III = \int_{t_1}^{t_2} \exp \left( - |\xi|^2 (t - s) \right) \sum_{q \in \mathbb{Z}^3} |M_{ijk} (\xi) v^i_n (q, s) v^j_n (\xi - q, s)| \, ds. \]

Let us bound each of the previous expressions,

\[ I = \left| \exp \left( - |\xi|^2 t_2 \right) \right| \left| 1 - \exp \left( - |\xi|^2 (t_2 - t_1) \right) \right| \leq \frac{\epsilon}{|\xi|^2} |\xi|^2 |t_2 - t_1| = \epsilon |t_2 - t_1|, \]
\[ II \leq \int_0^{t_1} \left| \exp \left(-|\xi|^2(t_2-s)\right) - \exp \left(-|\xi|^2(t_1-s)\right) \right| \epsilon^2 \, ds \]
\[
= \int_{\tau_n}^{t_1} \exp \left(-|\xi|^2(t_1-s)\right) \left| 1 - \exp \left(-|\xi|^2(t_2-t_1)\right) \right| \epsilon^2 \, ds \\
\leq |\xi|^2(t_2-t_1) \frac{1}{|\xi|^2} \left| 1 - \exp \left(-|\xi|^2t_1\right) \right| ,
\]
\[ III \leq \int_{t_1}^{t_2} \epsilon^2 \exp \left(-|\xi|^2(t_2-s)\right) \, ds \]
\[
\leq \frac{\epsilon^2}{|\xi|^2} \left| 1 - \exp \left(-|\xi|^2(t_2-t_1)\right) \right| \leq \frac{1}{|\xi|^2} \epsilon^2 |\xi|^2 |t_2-t_1| ,
\]
and hence
\[
\left| v_{n+1}^k(\xi,t_2) - v_{n+1}^k(\xi,t_1) \right| < C(\epsilon) (t_2-t_1)
\]
for \( n \geq 0 \), and the lemma is proved. \( \square \)

The previous Lemmas via the theorem of Arzela-Ascoli, using Cantor’s diagonal procedure, show that there is a well defined \( v \in \Phi(2) \) defined on \([0,T]\) such that,

\[ v^k(\xi,t) = v^k(\xi,t) \exp \left(-|\xi|^2t\right) \]
\[
+ \int_0^t \exp \left(-|\xi|^2(t-s)\right) \sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi) v^i(q,s) v^j(\xi-q,s) \, ds
\]

Let us give a proof of this. To simplify notation, let us assume that the sequence converging uniformly on \([0,T]\) for each \( \xi \) is the sequence \( v_n(\xi,t) \). By what we have shown, there exists a \( D \) not depending on \( t, \xi \) or \( n \) such that
\[
\left| v_n^i(\xi,t) \right| \leq \frac{D}{|\xi|^2}.
\]

Let \( \xi \) be fixed, and let \( \eta > 0 \) arbitrary. the previous estimate allows us to choose a \( Q \) such that
\[
\left| \sum_{|q| \geq Q} M_{ijk}(\xi) v_n^i(\xi,t) v_n^j(\xi,t) \right| \leq \eta.
\]
and also that the same inequality is valid with \( v_n \) replaced by \( v \) (this can be done since the choice of \( Q \) only depends on \( D \)). Hence we have
\[
\left| v_{n+1}^k(\xi,t) - v^k(\xi) \exp \left(-|\xi|^2t\right) \right| \\
- \int_0^t \exp \left(-|\xi|^2(t-s)\right) \sum_{1 \leq |q| < Q} M_{ijk}(\xi) v_n^i(q,s) v_n^j(\xi-q,s) \, ds \right| \\
\leq \eta
Taking $n \to \infty$, we obtain

$$|v^k(\xi, t) - \psi^k(\xi) \exp\left(-|\xi|^2 t\right)|$$

$$- \int_0^t \exp\left(-|\xi|^2 (t - s)\right) \sum_{1 \leq |q| < Q} M_{ijk}(\xi) u^i(q, s) v^j(\xi - q, s) \, ds \leq \eta$$

and from this follows that

$$|v^k(\xi, t) - \psi^k(\xi) \exp\left(-|\xi|^2 t\right)|$$

$$- \int_0^t \exp\left(-|\xi|^2 (t - s)\right) \sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi) u^i(q, s) v^j(\xi - q, s) \, ds \leq 2\eta.$$ 

Since $\eta > 0$ is arbitrary, our claim is proved.

3. Regularity

We shall show now that the solutions produced by the iteration scheme are regular under certain smallness condition. Indeed, we have

**Theorem 2.** Let $v \in L^\infty(0, T; \Phi(2))$ be a solution to (1). There exists an $\epsilon > 0$ such that if there is a $k_0 - 1$ for which $v$ satisfies

$$\sup_{|\xi| \geq k_0} |\xi|^2 |v^k(\xi, t)| < \epsilon \quad \text{for all} \quad t \in (0, T)$$

then $v$ is smooth.

To prove Theorem 2, we will need to estimate the term

$$\sum M_{ijk}(\xi) u^i(q) v^j(\xi - q).$$

This is the content of Lemma 3. But before we state and prove Lemma 3 and in order to express our estimates in a convenient way we will define to sequences of numbers. Namely

$$\left\{ \begin{array}{l} \mu_0 = 1 \\ \mu_1 = 1 \\ \mu_{n+1} = 2\mu_n - 1, \quad n \geq 2 \end{array} \right.$$ 

and

$$k_n = \frac{1}{\epsilon^{2^n}} k_0$$

where $k_0$ is such that

$$\frac{k_1}{k_0} \cdot D < \min \left\{ \epsilon, \frac{1}{2} \right\}$$

and $D = \sup_{(0,T)} \|u(t)\|$. 

We are now ready to state and prove,

**Lemma 3.** Assume that for all $\xi$ such that $|\xi| \geq k_0$

$$|v^k(\xi, s)| \leq \frac{\epsilon}{|\xi|^2}.$$
and if $|\xi| \geq k_m$

$$|v^k(\xi, s)| \leq \frac{\epsilon_{\mu_m}}{|\xi|^3}$$

Then for $|\xi| \geq k_{m+1}$ it holds that,

$$\left| \sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi) v^i(q, s) v^j(\xi - q, s) \right| \leq \epsilon_{\mu_m+1}.$$

Proof. First recall that $|M_{ijk}(\xi)| \leq c|\xi|$.

(7)\[ I_k \leq |\xi| \sum_{1 \leq |q| < k_{m-1}} |v^i(q, s) v^j(\xi - q, s)| + |\xi| \sum_{k_{m-1} \leq |q| < k_m} |v^i(q, s) v^j(\xi - q, s)| + |\xi| \sum_{|q| \geq k_m} |v^i(q, s) v^j(\xi - q, s)|\]

We estimate the first sum. Observe that $k_{m-1} \leq \frac{|\xi|}{4}$, so if $|q| < k_{m-1}$, we must have $|\xi - q| \geq \frac{|\xi|}{2}$. Hence, using the elementary inequality (3), we can bound

$$\sum_{1 \leq |q| < k_{m-1}} |v^i(q, s) v^j(\xi - q, s)| \leq \frac{4 \epsilon_{\mu_m}}{|\xi|^3} \sum_{1 \leq |q| < k_{m-1}} \frac{D}{|q|^2} \leq 4 \epsilon_{\mu_m} \frac{k_{m-1}}{k_m} \leq 4 \epsilon c^{2 \mu_m}.$$

To estimate the second sum, notice that if $|\xi| \geq k_{m+1}$ and $|q| \leq k_m$, then $|\xi - q| \geq \frac{|\xi|}{2} \geq k_m$. All this said, using inequality (3) again we obtain,

$$\sum_{k_{m-1} \leq |q| < k_m} |v^i(q, s) v^j(\xi - q, s)| \leq \frac{4 \epsilon_{\mu_m}}{|\xi|^3} \sum_{k_{m-1} \leq |q| < k_m} \frac{\epsilon}{|q|^2} \leq \frac{4 \epsilon_{\mu_m}}{|\xi|^3} \epsilon k_m.$$

Observe now that $\frac{k_m}{k_{m+1}} \leq c^{2m} \leq \epsilon_{\mu_m}$. This yields the bound,

$$\sum_{1 \leq |q| < k_m} |v^i(q, s) v^j(\xi - q, s)| \leq \frac{4 \epsilon_{\mu_m}}{|\xi|} \frac{k_m}{k_{m+1}} \leq \frac{4 \epsilon_{\mu_m}}{|\xi|} \epsilon^2 k_m.$$

To estimate the second sum on the righthandside of (7) we split it into three sums, namely

(8)\[ \sum_{|q| \geq k_m} |v^i(q, s) v^j(\xi - q, s)| = \sum_{k_m \leq |q| < \frac{|\xi|}{4}} |v^i(q, s) v^j(\xi - q, s)| + \sum_{\frac{|\xi|}{4} \leq |q| < 2|\xi|} |v^i(q, s) v^j(\xi - q, s)| + \sum_{|q| \geq 2|\xi|} |v^i(q, s) v^j(\xi - q, s)| \]

Estimating the three sums on the right hand side separately. Observe that if $|q| \leq \frac{|\xi|}{4}$ then we must have $|\xi - q| \geq \frac{|\xi|}{2} > k_m$. Therefore, using inequality (3), we get
implies that

Proof of Theorem 2.

3.1. □ and the Lemma is proved.

and if we assume $0 < \varepsilon < T$, then

To estimate the second sum we split it into two sums,

Estimating the first sum on the right-hand side of the previous equality,

The estimation of the second sum proceeds in exactly the same way as the estimation of the first sum on the right-hand side of (11), and hence we obtain

Now we estimate the third sum in the right-hand side of (13). Using that $|q| \geq 2|\xi|$ implies that $|\xi - q| \geq \frac{1}{2} |q|$, and inequality (11) we can bound,

Putting all the previous estimations together, we arrive at

and if we assume $0 < \varepsilon < \frac{1}{28}$, the previous inequality reads as

and the Lemma is proved.

3.1. Proof of Theorem 2. Given $0 < \rho < T$, we will first show that for a constant $K(\rho)$, there exists a constant $D$ such that if $|\xi| \geq K(\rho)$ then

Define

\[ \tau_m = \rho - \frac{\rho}{2^m}. \]
We will show by induction that
\[(P) \quad v^k(\xi,t) \leq \frac{\epsilon^{\mu_n}}{|\xi|^2} \quad \text{if} \quad t > \tau_n \quad \text{and} \quad |\xi| \geq k_n.\]

For \(n = 0\), our choice of \(k_0\) guarantees that \((P)\) holds. Assume that \((P)\) holds for \(n = m\). First observe that \(v^k\) satisfies
\[
v^k(\xi,t) = v^k(\xi,\tau_m) \exp \left( -|\xi|^2 (t-\tau_m) \right)
+ \int_{\tau_m}^{t} \exp \left( -|\xi|^2 (t-s) \right) \sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi) v^i(q,s) v^j(\xi - q, s) \, ds.
\]

Using this identity, we bound as follows,
\[
v^k(\xi,t) \leq v^k(\xi,\tau_m) \exp \left( -|\xi|^2 (t-\tau_m) \right)
+ \int_{\tau_m}^{t} \exp \left( -|\xi|^2 (t-s) \right) \epsilon^{2\mu_m} \, ds
\]
\[
\leq \frac{\epsilon^{\mu_m}}{|\xi|^2} \exp \left( -k_{m+1} (\tau_{m+1} - \tau_m) \right)
+ \frac{\epsilon^{2\mu_m}}{|\xi|^2} \left( \exp \left( -|\xi|^2 \tau_m \right) - \exp \left( -|\xi|^2 t \right) \right)
\]
\[
\leq \frac{\epsilon^{\mu_m}}{|\xi|^2} + \frac{\epsilon^{2\mu_m}}{|\xi|^2}.
\]

From this last bound it follows that if \(t \geq \rho > \rho - \frac{\rho}{2^{n+1}}\), then if \(k_m \leq |\xi| < k_{m+1}\) it holds that
\[
\left| v^k(\xi,t) \right| \leq \frac{\epsilon^{\mu_m}}{|\xi|^2}.
\]

Since \(\mu_m \geq 2^{n-1}\) and \(k_m = \frac{k_n}{2^{n+1}}\), it is easy to check that \(\epsilon^{\mu_m} \leq \frac{k_n^{\frac{1}{2}}}{|\xi|^2}\). Hence for all \(t \geq \rho\) the following estimate holds,
\[
\left| v^k(\xi,t) \right| \leq \frac{D}{|\xi|^{2+\eta}}.
\]

The following Lemma will then finish the proof of Theorem 2.

**Lemma 4.** Let \(v\) be a solution to \((FNS)\) such that for all \(t \in (0,T)\) satisfies
\[
|v^k(\xi,t)| \leq \frac{D}{|\xi|^{2+\eta}}
\]
with \(D\) and \(\eta > 0\) independent of \(t\). Then \(v\) is smooth.

**Proof.** Let \(\rho > 0\). Under the hypothesis of the Lemma, we will show that there exists a constant \(K := K(\rho)\) such that if \(t > T\) and \(|\xi| > K\), then for a constant \(E\) independent of time,
\[
|v^k(\xi,t)| \leq \frac{E}{|\xi|^{2+\min\left(\frac{1}{2},\eta\right)}}.
\]

Since \(\rho > 0\) is arbitrary, a finite number of applications of the previous claim shows that for any \(\rho > 0\), the Fourier transform of \(v\) decays faster than any polynomial, and this shows the lemma.
First, we will estimate the term

\[ S = \sum_{q \in \mathbb{Z}^3} M_{ijk}(\xi) v^i(\xi, s) v^j(\xi, s) \]

under the hypothesis of the lemma. In order to do this we write,

\[ S = I_a + I_b + II_a + II_b + III_a + III_b + IV_a + IV_b \]

where

\[ I_a = \sum_{1 \leq |q| \leq \sqrt{\xi}} M_{ijk}(\xi) v^i(\xi, s) v^j(\xi, s), \]

\[ II_a = \sum_{\sqrt{\xi} < |q| \leq \frac{\sqrt{\xi}}{2}} M_{ijk}(\xi) v^i(\xi, s) v^j(\xi, s), \]

\[ III_a = \sum_{|q| \geq \frac{\xi}{2}, 1 \leq |\xi - q| < 2|\xi|} M_{ijk}(\xi) v^i(\xi, s) v^j(\xi, s), \]

and

\[ IV_a = \sum_{|q| \geq \frac{\sqrt{\xi}}{2}, |\xi - q| \geq 2|\xi|} M_{ijk}(\xi) v^i(\xi, s) v^j(\xi, s) \]

The corresponding \( I_b, II_b, III_b \) and \( IV_b \) are the same as their \( a \) counterparts, except that the role of \( q \) and \( \xi - q \) is interchanged. Noticed that by the triangular inequality not both \( q \) and \( \xi - q \) can be less than \( \frac{|\xi|}{2} \), and hence all possible cases are covered.

Since \( |q| < \sqrt{\xi} < \frac{|\xi|}{2} \), and hence \( |\xi - q| \geq \frac{|\xi|}{2} \). Hence we have,

\[ |I_a| \leq |\xi| \sum_{1 \leq |q| \leq \sqrt{\xi}} \frac{D}{|q|^{2+\eta}} \frac{D}{|\xi - q|^{2+\eta}} \]

\[ \leq |\xi| \frac{2^{2+\eta} D^2}{|\xi|^{2+\eta}} \sum_{1 \leq |q| \leq \sqrt{\xi}} \frac{D}{|q|^2} \]

and by inequality \( [3] \)

\[ \leq |\xi| \frac{2^{2+\eta} D^2}{|\xi|^{2+\eta}} \sqrt{|\xi|} = \frac{2^{2+\eta} D^2}{|\xi|^{\eta+\eta}}. \]

Estimating \( II_a \) and \( III_a \) is pretty straightforward, via the inequality

\[ \sum_{1 \leq |q| < r} 1 \leq cr^3. \]

Indeed,

\[ |II_a| \leq |\xi| \frac{2^{2+\eta} D}{|\xi|^{1+\eta}} \cdot \frac{D}{\sqrt{|\xi|}} \frac{D}{|\xi|^{2+\eta}} \left( \sum_{|q| \leq \frac{|\xi|}{2}} 1 \right) \]

\[ \leq \frac{2^{2+\eta} D}{|\xi|^{1+\eta}} \cdot \frac{D}{|\xi|^{1+\eta}} |\xi|^3 = \frac{2^{2+\eta} D^2}{|\xi|^{2+\eta}}. \]
\[|III_a| \leq |\xi| \sum_{|q| \geq \frac{|\xi|}{2}, |\xi-q| < 2|\xi|} \frac{D}{|q|^{2+\eta}} \frac{D}{|\xi-q|^{2+\eta}} \]
\[\leq |\xi| \frac{2^{2+\eta} D^2}{|\xi|^{1+2\eta}} \left( \sum_{1 \leq |\xi-q| < 2|\xi|} 1 \right) \]
\[\leq \frac{2^{2+\eta}}{|\xi|^{2\eta}} \sum_{|q| \geq \frac{|\xi|}{2}, |\xi-q| < 2|\xi|} \frac{D}{|q|^{2+\eta}} \frac{D}{|\xi-q|^{2+\eta}} \]
\[\leq \frac{2^{2+\eta}}{|\xi|^{2\eta}} \left( \frac{3}{2} \right)^{2+\eta} \sum_{|q| \geq \frac{|\xi|}{2}} \frac{D^2}{|q|^4} \]
\[\leq |\xi| \frac{1}{|\xi|^{2\eta}} \frac{D^2}{|\xi|^{2\eta}} = \frac{D^2}{|\xi|^{2\eta}}. \]

Finally, using that \(|\xi-q| \geq 2|\xi|\) and \(|q| \geq \frac{|\xi|}{2}\) imply that \(|q| \geq \frac{2}{9} |\xi-q|\) and inequality (1) we can bound \(IV_a\) as follows,
\[|IV_a| \leq |\xi| \sum_{|q| \geq \frac{|\xi|}{2}, |\xi-q| \geq 2|\xi|} \frac{D}{|q|^{2+\eta}} \frac{D}{|\xi-q|^{2+\eta}} \]
\[\leq \frac{2^{2+\eta}}{|\xi|^{2\eta}} \left( \frac{3}{2} \right)^{2+\eta} \sum_{|q| \geq \frac{|\xi|}{2}} \frac{D^2}{|q|^4} \]
\[\leq |\xi| \frac{1}{|\xi|^{2\eta}} \frac{D^2}{|\xi|^{2\eta}} = \frac{D^2}{|\xi|^{2\eta}}. \]

The proof is now complete. \(\Box\)

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