Damage Field Theory and its Application in the Evaluation of Damage Effectiveness of Explode Material

SHA Zhaojun¹², QIAN Linfang¹, SUN Lingling²
¹Nanjing University of Science and Technology, Nanjing 210094, China
²Academy of Artillery and Air Defense, Nanjing 211132, China

Abstract: The evaluation of damage effectiveness of ammunition is an important aspect of research on operation application of weapon systems. In traditional methods, a target is considered as a planar object to calculate its average dimensionality of damage. By applying graph theory, discrete mathematics and terminal effects theory, this paper studies the spatial damage effectiveness of ammunition and puts forward the concept of damage field, damage contour plane, damage contour line, damage gradient, damage arithmetic operators and so forth. Meanwhile, this paper establishes a model for damage field to conduct calculation and comparative analysis of the evaluation of damage effectiveness of ammunition.

1. Definition of Damage Field

When a target is fired at and damaged, there is one specific value of damage probability \( \mu(x, y, z) \) for each point \((x, y, z)\) within the spatial area (represented as \( \Omega \)) around the target. This value is related to location, status, nature and hit area of target, type and impact point of shell, number of fragments, damage power of artillery, as well as surrounding soil property and topographic characteristics and tactical background for the target. According to the field theory, a field can be identified in the space surrounding the target, namely, a damage field, represented as \( \Omega \). Since damage probability is an invariant that does not change over time, damage field is a stable scalar field. In order to discuss the theory of damage field scientifically, rigorously and precisely, mathematical language is used to define and describe a damage field.

According to the descriptive definition of damage field, the damage probability of each point \((x, y, z)\) within the field, represented as \( \mu \), is a function of this point. By setting \( \mu = \mu(x, y, z) \) and establishing a coordinate system \((Oxyz)\) with the origin at the center of target, we can get:

\[
\Omega = \{ (x, y, z) \in \mathbb{R}^3 \mid 0 < \mu(x, y, z) \leq 1 \} \tag{1}
\]

\( \mu(x, y, z) \) is called the damage probability function for target.

Suppose that a numerical function is used to represent a damage field; in order to make it practical in any circumstance, the following assumptions are made:

\[
\mu(x, y, z) \in \mathcal{N}(\Omega) \left( \frac{\partial \mu}{\partial x} \right)^2 + \left( \frac{\partial \mu}{\partial y} \right)^2 + \left( \frac{\partial \mu}{\partial z} \right)^2 \neq 0 \tag{2}
\]

According to the definition of damage field, the existence of certain points \((X)\) within the field makes it possible for \( \mu(X) \) to satisfy the following conditions: \([2]\)

\[
\mu(X) = C \quad X \in \Omega \quad (C \text{ as a constant number})
\]

It can be conceived that, all of these points \((X)\) consist of a curved surface within the space \( \Omega \) and each point on this surface shares the same value of damage probability. The curved surface can be
called the damage contour plane of damage probability $\mu(X)$. Obviously, each point within the space ($\Omega$) lies only on an exclusive damage contour plane; all planes never intersect with each other and form the full spatial completeness of space ($\Omega$).

Besides, a planar target is studied as a particular case within the three-dimensional space, the field of which is called planar damage field and described as follows:

$$X \in \Omega \subset \mathbb{R}^2, \mu = \mu(x, y)$$

(3)

A planar damage field is a two-dimensional curve consisted of all points that satisfy the following conditions, $\mu(X) = C$, $X \in \Omega \subset \mathbb{R}^2$ ($C$ as a constant number).

2. Definition of Damage Gradient and Damage Arithmetic Operators

Since a damage field is defined, some relative notions need to be studied. For a damage field, represented as $\mu = \mu(X)$, surrounding the space of a specific target, the damage probability function $\mu = \mu(X)$ clearly changes in any direction at an arbitrary point $X_0 \in \Omega$, the rate of change in certain direction is known as directional derivative. Assuming the rate of change in the direction of $\chi$, namely, the directional derivative is $\frac{\partial \mu(X)}{\partial \chi}$ at the point $X_0 \in \Omega$, and by setting $\chi = (\cos \alpha, \cos \beta, \cos \gamma)$, we can get:[3]

$$\frac{\partial \mu(X)}{\partial \chi} = J_\mu(X_0) \ast \chi$$

(4)

In which (4):

$$J_\mu(X_0) = \left(\frac{\partial \mu(X)}{\partial x}, \frac{\partial \mu(X)}{\partial y}, \frac{\partial \mu(X)}{\partial z}\right)$$

(5)

In which, $\|J_\mu(X_0)\|$ is the norm of vector $J_\mu(X_0)$, $\cos(J_\mu(X_0), \chi)$ the cosine value of vector $J_\mu(X_0)$ and vector angle $\chi$. There are many types of norm of vector and a variety of methods for its calculation. When the $2^{\text{nd}}$ norm is taken:

$$\|J_\mu(X_0)\| = \left(\frac{\partial \mu(X)}{\partial x}^2 + \frac{\partial \mu(X)}{\partial y}^2 + \frac{\partial \mu(X)}{\partial z}^2\right)^{1/2}$$

(6)

According to this formula, when the direction of $\chi$ is consistent with that of $J_\mu(X_0)$, the directional derivative $\frac{\partial \mu(X)}{\partial \chi}$ has a maximum value $\|J_\mu(X_0)\|$. In this connection, a further notion of damage gradient is to be built.

The damage gradient is defined as follows; by assuming a damage field $\mu = \mu(X) \in \Psi(\Omega)$, $\Omega \subset \mathbb{R}^3$, $\forall X_0 \in \Omega$, which is a Jacobi vector field:

$$J_\mu(X_0) = \left(\frac{\partial \mu(X)}{\partial x}, \frac{\partial \mu(X)}{\partial y}, \frac{\partial \mu(X)}{\partial z}\right)$$

(7)

According to the definition of damage gradient, the damage gradient of point $X_0$ is a vector, the direction of which is consistent with the direction of maximum value of $\mu = \mu(X)$, the modulus of which the maximum value of $\|J_\mu(X_0)\|$. As for point $X_0$, the damage probability function $\mu(X)$ increases most rapidly in the direction of point $X_0$’s damage gradient, decreases most rapidly in the opposite direction of it, yet changes most slowly in the direction perpendicular to it. Damage gradient is the normal vector of damage contour plane $\mu(X) = C$ ($X \in \Omega \subset \mathbb{R}^2$, $C$ as a constant number), which passes through point $X_0$, therefore, it is perpendicular to this damage contour plane. In addition to the notion of damage gradient, damage arithmetic operators, represented as $\nabla$:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

(8)
Damage gradient can be represented in the following shorthand formula:

\[ H_{\text{grad}}\mu = \nabla \mu \]  \hspace{1cm} (9)

According to mathematics, there are some aspects of damage gradient \( \nabla \mu \):\(^3\)

a. \( \nabla(\mu) = \nabla \mu (C \text{ as a constant number);} \)
b. \( \nabla(\mu + \mu_z) = \nabla \mu + \nabla \mu_z \);  
c. \( \nabla(\mu \mu_z) = \mu \nabla \mu_z + \mu_z \nabla \mu \);  
d. \( \nabla \left( \frac{\mu_z}{\mu} \right) = \frac{\mu_z \nabla \mu - \mu \nabla \mu_z}{\mu^2} \), \( \mu_z \neq 0 \);  
e. \( \nabla f(\mu) = f^*(\mu) \nabla \mu \), \( f^*(\mu) \nabla \mu \) is a derivative and proved to exist.

3. Cubical Elliptical Damage Rule and the Determination of its Damage Probability Function

When the theory of damage field is applied in the assessment of damage efficiency of ammunition, the trajectory forms by points of equal value of damage probability density function \( G(x, y, z) \) is in a shape of ellipsoidal surface because of the deviation of burst point to target in distance, height and direction, which is known as the cubical elliptical damage rule. Since damage probability density function \( G(x, y, z) \) must satisfy the following conditions:

1. \( 0 \leq \alpha(x, y, z) \leq 1 \);  
2. \( x = y = z = 0, \alpha(x, y, z) = 1 \)  
3. \( x, y, z \to \infty, \alpha(x, y, z) \to 0 \)

By applying the method of undetermined coefficients, the damage probability density function \( G(x, y, z) \) can be transformed to Formula (10), which is:

\[ G(x, y, z) = e^{\left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right\}} \]  \hspace{1cm} (10)

In Formula (10), \( a, b, c \) are undetermined constants, their constant values can be obtained as follows:\(^6\)

\[ V = \int \int \int_{x=0}^{x=\infty} e^{\left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right\}} \, dx \, dy \, dz = a\sqrt{\pi} \cdot b\sqrt{\pi} \cdot c\sqrt{\pi} = \pi^2 abc \]  \hspace{1cm} (11)

\[ \sum x = \int_{x=0}^{x=\infty} e^{\left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right\}} \, dx = a \left( \frac{\pi}{a^2} \right) \cdot a\sqrt{\pi} \]  \hspace{1cm} (12)

When \( y = 0, z = 0 \), \( \sum x \) has a maximum value:

\[ \text{Max} \sum x = a \cdot \sqrt{\pi} \]  \hspace{1cm} (13)

Similarly:

\[ \text{Max} \sum y = b \cdot \sqrt{\pi} \quad , \quad \text{Max} \sum z = c \cdot \sqrt{\pi} \]  \hspace{1cm} (14)

According to Formula (12), (13), (14), the ratio value of damage field \( V \) for distance, direction and height are:\(^2\)

\[ B_{xy} = \frac{a}{b} \quad , \quad B_{xz} = \frac{a}{c} \quad , \quad B_{yz} = \frac{b}{c} \]  \hspace{1cm} (15)

By simplifying the damage field \( V \) to a cuboid shape, the length, width and height of which are represented as \( l_x, l_y, l_z \), and bringing \( V = l_x l_y l_z \) into Formula (11), with the aid of Formula (15), it can be concluded that:

\[ l_x = \sqrt{\pi^2 abc \cdot \frac{a}{b} \cdot \frac{a}{c}} = a\sqrt{\pi} \], \( l_y = \sqrt{\pi^2 abc \cdot \frac{b}{a} \cdot \frac{b}{c}} = b\sqrt{\pi} \), \( l_z = \sqrt{\pi^2 abc \cdot \frac{c}{b} \cdot \frac{c}{a}} = c\sqrt{\pi} \]
Therefore, we bring  \( a = \frac{I_x}{\sqrt{\pi}}, b = \frac{I_y}{\sqrt{\pi}}, b = \frac{I_z}{\sqrt{\pi}} \) into Formula (1) to get:

\[
G(x,y,z) = e^{-\left(\frac{x^2 + y^2 + z^2}{\rho^2}ight)}
\]  \hspace{1cm} (16)

Formula (16) is the calculation formula of the damage probability density function for cubical elliptical damage rule; its form is very similar to the probability density function that accords with three-dimensional normal distribution pattern. To facilitate the application of convolution integral for calculation and synthesis of three-dimensional normal probability density function, the above-mentioned formula can be transformed to:

\[
G(x,y,z) = e^{-\left(\frac{x^2 + y^2 + z^2}{\eta_1^2 + \eta_2^2 + \eta_3^2}\right)}
\]  \hspace{1cm} (17)

4. Application of the Model and the Comparison and Analysis of Results

In this section, calculation and analysis of the damage efficiency of 122mm howitzer are carried out by applying both damage field model and traditional computing method. Providing that the distance intermediate error \( E_d \), direction intermediate error \( E_r \) and height intermediate error \( E_z \) of the artillery dispersion error are 30m, 6m and 2m, the distance intermediate error \( E_d \), direction intermediate error \( E_r \) and height intermediate error \( E_z \) of the firing date dispersion error are 15m, 3m and 2m, the height, width and depth of the target are 2m, 6m and 6m, the killing radius of shell \( r \) is 1.5m, a total number of 60 shells are fired and one of the shells hits and damages the target, the damage probabilities are calculated with the two methods respectively. Table 1 shows the results for less number of fired shells, while Table 2 shows the results for more number of fired shells.

| Damage field | Dimensionality of damage | Comparison of results | Percentage of damage probability change |
|--------------|--------------------------|-----------------------|----------------------------------------|
|              |                          |                       | Damage field | Dimensionality of damage |
| 1            | 0.0158275                | 0.0114046             | 38.78%       | /                       |
| 2            | 0.0313611                | 0.0226791             | 38.28%       | 98.14%                  | 98.86% |
| 3            | 0.0466066                | 0.0338251             | 37.79%       | 48.61%                  | 49.15% |
| 4            | 0.0615699                | 0.0448439             | 37.30%       | 32.11%                  | 32.58% |
| 5            | 0.0762567                | 0.0557370             | 36.82%       | 23.85%                  | 19.32% |

A collation map of damage and a change chart are generated from the Comparison of calculation results for less number of fired shells (Table 1).

1) When less number of shells is fired, the results calculated by the two methods differ sharply but the difference gradually reduces as the number of fired shells increases.
(2) When less number of shells is fired, the calculation results of both methods change steadily as the number of fired shells increases, which shows an approximately exponential profile.

Similarly, a collation map of damage and a change chart are generated from the Comparison of calculation results for more number of fired shells (Table 2).

(1) When more number of shells is fired, the results calculated by the two methods still differ sharply but the difference gradually decreases as the number of fired shells increases.

(2) When more number of shells is fired, the calculation results of both methods change steadily as the number of fired shells increases, which shows an approximately exponential profile.

Based on the calculation results and the comparison and analysis of these results, it is concluded that results generated from the two methods differ widely.

(1) Comparison shows that the calculation results from the two methods differ widely, on one hand, the damage field model generates results that are clearly larger comparing with the calculation of average dimensionality of damage, but their difference reduces as the number of fired shells increases. When less number of shells is fired, this trend of change is approximately linear, but loses its pattern to become unpredictable as the number of fired shells increases.

(2) On the other hand, results generated from both methods change steadily in a certain range and shows an approximately exponential profile, yet the change of results can be too volatile to be predicted.

References
[1] G. V. Schipanov. Theory and methods of designing automatic regulators [J]. Automatika in Telemekhanika, 1939.
[2] G. H. Hostetter, J. Meditch. On the generalization of observers to systems with unmeasurable, unknown inputs, [J]. Automatics, 1973.
[3] S. Kwon, W. K. Chung. A Discrete-Time Design and Analysis of Perturbation Observer for Motion Control Applications [J]. IEEE Transactions on Control Systems Technology, 2003.
[4] J. S. Ko, Y. G. Seo, H. S. Kim. Precision Position Control of PMSM using Neural Observer and Parameter Compensator [J]. Journal of Power Electronics, 2003.
[5] L. A. Jones, J. H. Lang. A state observer for the permanent magnet synchronous motor [J]. IEEE Transactions on Industrial Electronics, 1989.
[6] J. Holtz. Sensorless control of induction machines With or without signal injection [J]. IEEE Transactions on Industrial Electronics, 2006.