ON ERGODIC PROPERTIES OF “ICEBERG” TRANSFORMATIONS. I: APPROXIMATION AND SPECTRAL MULTIPLICITY

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Abstract. We investigate a class of mixing dynamical systems around the concept of iceberg transformation. In brief, an iceberg transformation is defined using symbolic language as follows. We build a sequence of words such that the next word is a concatenation of rotated copies of the previous word. For example, a word CAT can turn into CAT, ATC, TCA, TCA, CAT, ATC, then we repeat the procedure applying it to this new word and so on. Geometrically, given an invertible measure preserving transformation $T$ an iceberg is a union of two icelets for the map $T$, one direct and one reverse with common base set, where icelet is defined in a similar way as Rokhlin tower $B\sqcup TB\sqcup \ldots \sqcup T^{k-1}B$, namely, an icelet is a sequence of disjoint measurable sets $\{B_0, B_1, \ldots, B_{h-1}\}$ such that $B_{j+1} \subseteq TB_j$, in other words, the levels $B_j$ continuously disappear from the base to the top of the icelet. Reverse icelet is defined as icelet for the inverse map $T^{-1}$, and it “grows” towards the past. Iceberg transformation is approximated by a sequence of icebergs, resembling the behaviour of rank one ergodic maps. The definition of iceberg maps essentially involves the notion of interval exchange transformation.

We study combinatorial and ergodic properties for several classes of measure preserving transformations satisfying iceberg approximation including random iceberg maps and explicitly defined iceberg maps, in particular, involving finite fields. We also consider a series of extensions for iceberg approximation property. It is a common phenomenon that iceberg approximation implies local rank property and, hence, finite multiplicity of spectrum. It is show that a class of iceberg transformations almost surely has simple spectrum, $1/4$-local rank property and spectral type $\sigma$ such that $\sigma \ast \sigma \ll \lambda$ where $\lambda$ is the Lebesgue measure on the circle $S^1$.

In memory of V. I. Arnold

1. Iceberg maps at a glance

Let us define rotation operator $\rho_{\alpha}$ on finite words: if $W = W_1W_2$ and the length of the first subword $|W_1| = \alpha$ then we set $\rho_\alpha(W) = W_2W_1$. Observe that in other terms $\rho_{\alpha}$ cuts the word $W$ after $\alpha$ positions and then substitutes $W_2$ and $W_1$. This kind of transform is a discrete variation of the well-known interval exchange map. Starting from a word $W_0$ consider the following rotated words concatenation procedure. A word $W_n$ is repeated $q_n$ times, next, each copy is rotated by given value of positions $\alpha_{n,y}$, and the next word in the sequence is given by the formula

$$W_{n+1} = \rho_{\alpha_{n,0}}(W_n)\rho_{\alpha_{n,1}}(W_n)\ldots\rho_{\alpha_{n,q_n-1}}(W_n).$$

For example, if $W_1$ is the word “CAT”, $q_1 = 6$ and $(\rho_0, \rho_1, \ldots, \rho_5) = (0, 1, 2, 2, 0, 1)$, then

$$\text{CAT} \mapsto \text{CAT}.\text{ATC}.\text{TCA}.\text{TCA}.\text{ATC} = W_2$$

(points “.” are used to distinguish groups of symbols). At the next step we rotate the word $W_2$. The following table shows positions of cutting ($\times$)

$\begin{array}{cc}
\text{CATATCT} & \times \text{CATCACATATC} \\
\text{CATA} & \times \text{TCTCATCACATATC} \\
\text{CATATCTCATACATATC} &
\end{array}$

used to create the word

$$W_3 = \text{CATCACATATC}.\text{CATATCT}.\text{TCTCATCACATATC}.\text{CATATATC}.$$

It is a common phenomenon that this sequence of words generates a dynamical system. Simply speaking the words in the sequence $W_n$ become more and more stationary according to empirical distributions on
words of bounded length, and the dynamical system is associated with the shift map \( T: (x_n) \mapsto (x_{n+1}) \). At the same time there is a simple way to produce a geometrical description of the dynamics for this symbolic system. Indeed, one way of drawing \( \rho_a \) is to fix cut points, like it is shown in the following line

\[
\text{(5) } \quad \text{CAT, C AT, CA T, CA T, CAT, C AT}
\]

which implies mapping

\[
\text{(6) } \quad \text{CAT } \mapsto \text{ CAT, ATC, TCA, TCA, CAT, ATC},
\]

and other way is to think that the word CAT is shifted as a function on the group \( \mathbb{Z}_h \) (in this case shift actually coincides with the rotation operator \( \rho_\alpha \)).

**Figure 1.** Each entrance of the word CAT in \( W_2 \) is rotated by the map \( t \mapsto t + \alpha_{1,y} \) (mod 3).

Suppose now that symbols \( \{C, A, T\} \) correspond to a partition \( \{P_C, P_A, P_T\} \) of the probability space \( (X, \mathcal{A}, \mu) \) (we assume that \( X \) is a Lebesgue space without atoms which is isomorphic by Rokhlin’s theorem to the unit segment \([0, 1]\) with Lebesgue measure), \( X = P_C \sqcup P_A \sqcup P_T \) and \( \mu(P_C) = \mu(P_A) = \mu(P_T) = \frac{1}{3} \).

**Figure 2.** Iceberg associated with the word CAT. Letters should be read in vertical direction. The map \( T \) lifts most part of any elementary set (a square) on the picture to the upper level.

**Definition 1.** Let us draw at fig. [2] all the rotations of the word CAT in a way used to draw a Rokhlin tower, placing same letters to the same level and placing up the letter presumed to be next after the letter of the current level. This picture is called iceberg. The level corresponding to the first letter of the word (letter “C”) is called the base level of the iceberg.

Each column at fig. [2] corresponds to a rotation of the word CAT: \( V_1 \) to ATC, \( V_2 \) to TCA and \( V_3 \) to CAT.

The basic idea of iceberg is to guess that a measure preserving transformation \( T \) maps each elementary set (shown as a square) on the picture to the upper level.

\[
\mu(TV_{3,C} | V_{3,A}) \approx 1 \quad \text{and} \quad \mu(TV_{3,A} | V_{3,T}) \approx 1.
\]
Remark that if we denote levels of the iceberg as $B_{-2}, B_{-1}, B_0, B_1, B_2$ then $P_C = B_0, P_A = B_1 \cup B_{-2}, P_T = B_2 \cap B_{-1}$.

At this point it is completely unknown how $T$ acts on the top elementary set of each column. We specify the dynamics of $T$ with the help of equation (\ref{eq:TCAT}) having the following translation to geometric language. We divide each column $V_k$ into several vertical subcolumns to get one separate subcolumn for one entrance of rotated word “CAT” in (\ref{eq:TCAT}) like it is shown on fig. \ref{fig:CAT}. Then we link the corresponding sets on the boundary marked by indexes 1, \ldots, 6 on fig. \ref{fig:CAT}. After linking we get the next (cyclic) sequence CAT.ATC.TCA.TCA.CAT.ATC and we repeat the procedure. This construction is uniquely determined by rotations $(\alpha_n,0,\ldots,\alpha_{n,q_n-1})$ for each step. Consider the edge of the iceberg, the union $E$ of all bottom sets of columns, and define the Poincaré map $\hat{T}_E: E \to E$, a measure preserving map corresponding to the way of linking subcolumns (see fig. \ref{fig:CAT}). Actually the map $\hat{T}_E$ coincides with the return map $T_E$ induced by $T$ on $E$ up to next step cuttings. Iterating this procedure we get a measure preserving transformation on a Lebesgue space.

Let us consider the space $L^2(X,\mu)$ of measurable functions $f: X \to \mathbb{C}$ with integrable square and define Koopman operator

$$\hat{T}: L^2(X,\mu) \to L^2(X,\mu): f(x) \to f(Tx).$$

The meaning of $\hat{T}$ is translation by 1 step along the trajectory of $T$. It can be easily seen that $\hat{T}$ is a unitary operator in space $L^2(X,\mu)$. By spectral theorem $\hat{T}$ is determined up to unitary equivalence by two invariants: spectral type $\sigma$ (a measure on $S^1$ up to equivalence) and multiplicity function $M_T(z)$ mapping $S^1$ to the set $\mathbb{N} \cup \{\infty\}$. Since $\hat{T}1 = 1$ usually $\hat{T}$ is restricted to the space of functions with zero mean $\{f: \int f \, d\mu = 0\}$. An operator $\hat{T}$ has simple spectrum if there exist a function $e_0$ (cyclic vector) such that the iterations $T^k e_0$ generates the whole $L^2(X,\mu)$, where $k \in \mathbb{Z}$.

**Theorem 2.** Let $T$ be an iceberg transformation given by uniform i.i.d. random rotations $\alpha_{n,k}$, and suppose that $q_n \gg h_n$ grows sufficiently fast. Then the following properties hold almost surely

(i) $T$ has 1/4-local rank (see definition in section 3),
(ii) $\hat{T}$ has simple spectrum,
(iii) $\sigma * \sigma \ll \lambda$, where $\sigma$ is the spectral type of $\hat{T}$ and $\lambda$ is Lebesgue measure on $S^1$,
(iv) For a dense set of functions $f$ with zero mean $\forall \varepsilon > 0$

$$\langle T^t f, f \rangle = O(t^{-1/2+\varepsilon}).$$

In the first part of the paper we discuss combinatorial properties of iceberg transformations and prove statements (i) and (ii) of theorem \ref{thm:2}. We place in the second part the detailed investigation for correlation decay, statements (iii) and (iv). And the third part is devoted to a class of iceberg transformation involving finite field arithmetics.
2. Motivation

2.1. Approximation and spectral invariants. Throughout this paper we consider invertible measure preserving transformations of a Lebesgue space \((X, \mathcal{A}, \mu)\) as well as measure preserving group actions. A measurable map \(T: X \to X\) is called measure preserving if \(\mu(T^{-1}A) = \mu(A)\) for any set \(A \in \mathcal{A}\). If \(T\) is invertible and both \(T\) and \(T^{-1}\) are measure preserving then \(T\) is usually called an automorphism on space \((X, \mathcal{A}, \mu)\).

A serious part of research in spectral theory of dynamical systems is based on the idea of approximation. The method and the first examples \([24, 28, 29, 33]\) originate from paper \([20]\) by A. Katok and A. Stépin where the concept of periodic approximation was introduced. The method of approximation is also actively used in smooth dynamics (D. Anoson and A. Katok \([6]\), E. Sataev \([32]\), A. Kochergin \([23]\)). Further development of this method has led to the notion of rank one approximation (T. Adams and N. Friedman \([5]\), R. Chacon \([9]\), D. Ornstein \([27]\)) and a series of different concepts extending it (see review by S. Ferenczi \([16]\)).

The Koopman operator \(\hat{T}\) associated with an automorphism \(T\) is uniquely determined by the spectral type \(\sigma\) and the multiplicity function \(M_T(z)\) (see \([25]\)). Denote \(m(T)\) the essential maximal value of the multiplicity function \(M_T(z)\). We say that \(T\) has simple spectrum if it is of spectral multiplicity one, \(m(T) = 1\). Let us mention the following open question due to S. Banach.

**Question 3.** Does there exist an automorphism \(T\) having simple spectrum and Lebesgue spectral type \(\sigma\)? In other words, is it possible to find an automorphism \(T\) such that for some \(e_0 \in L^2(X, \mu)\) the sequence

\[
\ldots, \hat{T}^{-1}e_0, e_0, Te_0, T^2e_0, \ldots
\]

satisfies \(T^i e_0 \perp T^j e_0\) for \(i \neq j\), and linear combinations of \(T^j e_0\) are dense in \(L^2(X, \mu)\).

![Figure 4. Cutting-and-stacking construction of a rank one map](image)

A Rokhlin tower of height \(h\) is a sequence of disjoint sets \(\mathcal{T} = \{C, TC, T^2C, \ldots, T^{h-1}C\}\), where \(T^j C\) are called levels of the tower, and \(C\) is called the base set of \(\mathcal{T}\). Denote \(\cup \mathcal{T} = C \cup TC \cup \ldots \cup T^{h-1}C\).

**Definition 4.** A map \(T\) is called rank one transformation if there exists a sequence of Rokhlin towers \(\mathcal{T}_n\) such that \(\mu(\cup \mathcal{T}_n) \to 1\) and the corresponding tower partitions \(\mathcal{T}_n\) approximate the \(\sigma\)-algebra \(\mathcal{A}\), in other words, for any measurable set \(A\) there exist \(\mathcal{T}_n\)-measurable \(^1\) sets \(A_n\) with \(\mu(A_n \triangle A) \to 0\) as \(n \to \infty\).

\(^1\)Given a tower \(\mathcal{T} = \{C, TC, \ldots, T^{h-1}C\}\) we use the same symbol \(\mathcal{T}\) for the partition of the phase space into the levels \(T^j C\) and the remainder set \(X \setminus \cup \mathcal{T}\).

\(^2\)\(A \triangle B\) denotes the symmetric difference \((A \setminus B) \cup (B \setminus A)\).
Theorem 5. Any rank one transformations $T$ has simple spectrum, $\mu(T) = 1$.

In paper [19] I. Kwiatkowski and M. Lemańczyk proved that for any subset $K$ of $\mathbb{N}$ containing $1$ there exists a transformation $T$ having $K$ as a set of essential values of the multiplicity function $M_T(z)$.

2.2. Cutting-and-stacking construction. To build a rank one transformation we start from any tower $\mathcal{T}_n$ and build a sequence of towers $\mathcal{T}_n$. A the $n$-th step the next tower $\mathcal{T}_{n+1}$ is constructed from the previous one as follows. We cut the tower $\mathcal{T}_n$ in vertical direction into $q_n$ equal columns. Then we add $s_{n,y}$ extra levels (spacers) to the top of each column and stack these extended columns $C_{n,y}$ together $\mathcal{T}_{n+1} = C_{n,0}C_{n,1} \ldots C_{n,q_n-1}$ to get next tower $\mathcal{T}_{n+1}$. In terms of dynamics this means that whenever a point comes to the top of a column $C_{n,y}$, next time it goes to the bottom level of the right-hand side column $C_{n,y+1}$. This procedure repeated infinitely many times leads to an ergodic transformation if requirement [25] is satisfied.

2.3. Rank one systems: Mixing and spectral type. D. Ornstein in [27] introduced a class of rank one transformations with random spacers defined as follows: we set $s_{n,y} = \alpha_{n,y+1} - \alpha_{n,y}$ where $\alpha_{n,y} \ll h_n$ are i.i.d. random variables. J. Bourgain has shown that Ornstein transformations are of singular spectral type almost surely [5]. A large class of generalized Ornstein transformations was studied by El H. El Abdalaoui and F. Parreau [11, 12, 3]. I. Klemes has proved singularity of spectral type for a class of staircase constructions [21] defined by the spacer sequence $s_{n,y} = y$. Rank one transformations of Ornstein type as well as staircase constructions are also examples of mixing transformations. We say that $T$ is mixing if $\mu(T^j A \cap B) \rightarrow \mu(A) \mu(B)$ for all measurable sets $A$ and $B$. T. Adams proving Smorodinsky’s conjecture [4] has shown that staircase rank one maps with $q_n = n$ are mixing. This result was extended by D. Creutz and C.E. Silva [11] and A. Danilenko and V. Ryzhikov (e.g. see [12]).

Let us remark that $\hat{T}$ has simple spectrum iff there exists $f \in L^2(X, \mu)$ (cyclic vector) such that

$$L^2(X, \mu) = \overline{\text{Span}\{\hat{T}^k f : k \in \mathbb{Z}\}}.$$  

In this case $\sigma_f \sim \sigma$, where $f$ is a cyclic vector and $\sigma_f$ is defined by the property

$$\int_{S^1} z^k \, d\sigma_f = \left\langle T^k f, f \right\rangle.$$  

The following question concerning the spectral type of rank one transformations is still open.

Question 6. Is the following true: The spectral type of any rank one transformation is singular with respect to the Lebesgue measure $\lambda$ on $S^1$?

2.4. Generalized Riesz products. If a function $f \in L^2(X, \mu)$ is constant on the levels of $n$-th tower $\mathcal{T}_n$ for a rank one transformation then we identify $f$ with a function $f_{(n)} : \mathbb{Z} \rightarrow \mathbb{C}$, $f_{(n)}(j) \equiv f|_{T^j B_{n,0}}$, where $T^j B_{n,0}$ is the level with index $j$ of $n$-th tower. Let us define polynomials

$$P_n(z) = \frac{1}{\sqrt{q_n}} \sum_{y=0}^{q_n-1} z^{\omega_n(y)} \in M_{q_n},$$  

where

$$\omega_n(y) = y h_n + \sum_{j < y} s_{n,j}.$$  

The spectral measure $\sigma_f$ can be represented as an infinite product (up to a constant multiplier) which converges in weak topology [5, 10, 22, 3]:

$$\sigma_f = |f_{(n_0)}|^2 \prod_{n=n_0}^\infty |P_n(z)|^2,$$
2.5. Littlewood polynomials and flatness phenomenon. Let us consider the following classes of polynomials introduced by J. Littlewood [26] (see also [14]):

\[ K_n = \{ P(z) = \frac{\sqrt{n}}{\sqrt{n+1}} \sum_{k=0}^{n} a_k z^k : |a_k| \equiv 1 \} , \] (15)

\[ L_n = \{ P(z) = \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} a_k z^k : a_k \in \{-1, 1\} \} , \] (16)

\[ M_n = \{ P(z) = \frac{1}{\sqrt{n}} (z^{\omega_1} + z^{\omega_2} + \ldots + z^{\omega_n}) : \omega_j \in \mathbb{Z}, \omega_j < \omega_{j+1} \} . \] (17)

Polynomials in the classes \( K_n \) are called polynomials with unimodular coefficients. 

**Question 7** (J. Littlewood, 1966). Is the following true? For any \( \varepsilon > 0 \) there exists a polynomial \( P(z) \in K_n \) such that

\[ \forall z \in S^1 \quad | |P(z)| - 1| < \varepsilon. \] (18)

**Theorem 8** (Kahane, 1980). The answer to question 7 is “yes” with the speed of convergence

\[ \varepsilon_n = O(n^{-1/17} \sqrt{\ln n}). \] (19)

A new progress in explicit constructions of ultra-flat unimodular polynomials on \( S^1 \) is achieved by J. Bourgain and E. Bombieri in [7]. Passing to group \( \mathbb{R} \) let us define class

\[ M_n^\mathbb{R} = \{ P(z) = \frac{1}{\sqrt{n}} (z^{\omega_1} + z^{\omega_2} + \ldots + z^{\omega_n}) : \omega_j \in \mathbb{R}, \omega_j < \omega_{j+1} \} . \] (20)

It is shown in [30] that the answer to the question on \( L^1 \)-flatness is positive in class \( M_n^\mathbb{R} \) if we understand it as flatness on compact sets in \((0, \infty)\). It occurs that flat sums of such kind given by exponential frequency functions

\[ \omega_y = \frac{n}{\varepsilon^2} e^{\varepsilon y/n} \] (21)

satisfy natural conditions needed to be a polynomial in the Riesz product for a rank one flow, which proves the existence of rank one flows with simple Lebesgue spectrum.

**Question 9** (open). Can we see flatness in \( L_n \) or \( M_n^\mathbb{R} \)?

T. Downarowicz and Y. Lacroix [13] has proved that if all continuous binary Morse systems have singular spectra then the merit factors of binary words are bounded (the Turyn’s conjecture holds). In the work [18] M. Guenais has shown that the positive answer to the Littlewood question in \( L_n \) is equivalent to the fact that a class of transformations given by Morse cocycles has a Lebesgue component in spectrum.

2.6. Idea. In this paper we investigate a new class of dynamical systems\(^3\) a hybrid concept extending both Katok-Stépin periodic approximation and rank one property. Actually the idea leading to the new approximation property is purely analytic. It is known that the spectral type of a rank one transformation (a map approximated by a sequence of Rokhlin towers) is given by generalized Riesz product

\[ \sigma \sim |\hat{f}_{(n_0)}| \cdot \prod_{n=1}^{\infty} |P_n(z)|^2 \lambda, \] (22)

\(^3\)To be correct we should say a new approximation property instead of new class of dynamical systems because, in fact, it is unknown how to distinguish it, for example, to prove that there exists a transformation in this new class which is not rank one (cf. questions [37].)
where \(|z| = 1\) and \(\lambda\) is the Lebesgue measure on the unit circle, and the main ingredient in this product is a sequence of trigonometric polynomials \(P_n(z)\) with coefficients 0 and 1 (see \([10, 22, 3]\)). Let us denote this class of polynomials as \(\mathcal{M}\),

\[
\mathcal{M} = \left\{ P(z) = \sum_{y=0}^{q-1} z^{\omega(y)} : \omega(y) \in \mathbb{Z}, \omega(y) < \omega(y + 1) \right\}.
\]

Thus, understanding analytic properties of the class \(\mathcal{M}\) could be a key to spectral properties of rank one transformations. For example, a hypotetic flatness property for a polynomial \(P \in \mathcal{M}\) would help to find a rank one map with absolutely continuous component in the spectrum. A polynomial \(P(z)\) is \(\varepsilon\)-flat if \(\|q^{-1/2}|P| - 1\| < \varepsilon\) in some norm \(\| \cdot \|\). If we deal with a rank one transformation several limitations to the class of polynomials turn out. Points \(\omega(y)\) must be almost equidistant, \(\omega(y + 1) - \omega(y) \sim h\), and constructing a rank one map we cannot involve any polynomial in \(\mathcal{M}\). The main idea of iceberg transformation is to extend the class of dynamical systems in order to make the underlying class of polynomials richer.

2.7. **Rank one maps: Symbolic interpretation.** It was discovered that rank one property can be expressed in purely combinatorial terms: \(T\) is a rank one map if and only if a typical orbit for \(T\) is \(\varepsilon\)-covered by words which are \(\varepsilon\)-close to a single word \(W(\varepsilon)\) for arbitrary \(\varepsilon\). Furthermore, rank one maps can be described using one of the three equivalent definitions: measure-theoretic (definition \([4]\)), symbolic and geometrical definitions see. Since iceberg transformations extend in a sense rank one systems we will find the corresponding three parallel interpretations for this concept.

**Definition 10.** Consider a sequence of words \(W_n\) in alphabet \(A\) such that

\[
W_{n+1} = W_n 1^{s_{n,1}} W_n 1^{s_{n,2}} W_n \cdots 1^{s_{n,q_n}} W_n,
\]

where symbol “1” is used to create spacers between words. Suppose that

\[
\prod_{n=1}^{\infty} \frac{h_{n+1}}{q_n h_n} < \infty.
\]

This sequence of words is the coding of an orbit starting from the base of \(n\)-th tower according to the partition associated with the alphabet \(A\).

3. **Iceberg map: Formal definition**

The purpose of this section is to discuss the formal definition of iceberg transformations. Actually we consider the family of transformations given by the construction discussed in section \([1]\). At the same time an abstract definition of iceberg approximation is introduced (see def. \([15]\) and question \([37]\)ii) is formulated concerning the following: is it possible to construct a refined sequence of icebergs approximating map \(T\) like in the case of rank one transformations?

3.1. **Iceberg.** Consider an invertible measure preserving transformation \(T\) on the standard Lebesgue space \((X, A, \mu)\), and let \(h\) be a positive integer number.

**Definition 11.** An icelet is a sequence of disjoint sets \(\{B_0, B_1, \ldots, B_{h-1}\}\) such that \(B_{j+1} \subseteq TB_j\) and \(B_j \in A\).

**Definition 12.** A generic iceberg is a sequence of disjoint measurable sets

\[
\mathcal{J} = \{B_{-h+1}, \ldots, B_{-1}, B_0, B_1, \ldots, B_{h-1}\}
\]

such that \(B_{j+1} \subseteq TB_j\) for \(j \geq 0\) and \(B_{j-1} \subseteq T^{-1}B_j\) for \(j \leq 0\) (see fig. \([3]\)). Notice that \(\mathcal{J}\) is composed of two icelets with common base \(B_0\), one normal (direct) and one reverse, where reverse icelet is an icelet for \(T^{-1}\). We will use notation \(\cup \mathcal{J} = \bigcup_j B_j\).
Definition 13. Let $\mathcal{I}$ be a generic iceberg. We say that $\mathcal{I}$ is cyclic if for any point $x \in B_0$ the total number of iterations towards future and past until leaving the iceberg is equal to $h$, i.e. $\#\{j \in \mathbb{Z}: T^j x \in B_j\} \equiv h$. Let us define the cyclic iceberg partition
\begin{equation}
\tilde{\mathcal{I}} = \{B_j \cup B_{j-h}: j = 0, 1, \ldots, h-1\},
\end{equation}
which is evidently refined by $\mathcal{I}$ (see fig.6).

Remark 14. If iceberg is cyclic, we can redraw fig.2 moving $h$ points up levels with negative indexes (in fact, all icebergs on the illustrations are cyclic). After the modification a cyclic iceberg looks like a set of columns
\begin{equation}
V_k = \{V_{k,j}: j \in \mathbb{Z}_h\},
\end{equation}
with different cut points at position $k = 0, 1, \ldots, h-1$ as shown on fig.6. The meaning of cutting will be clear in the forthcoming discussion on ergodic map generated by icebergs.

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4 Iceberg $\mathcal{I}_{n+1}$ refines $\mathcal{I}_n$ if $\mathcal{A}(\mathcal{I}_n) \subseteq \mathcal{A}(\mathcal{I}_{n+1})$, where $\mathcal{A}(\mathcal{P})$ is the $\sigma$-algebra generated by partition $\mathcal{P}$.

5 The term icelet is proposed by V. Ryzhikov.
A column before cutting is a sequence of sets $C_{k,j}$ indexed by $j \in \mathbb{Z}$ with $\mu(C_{k,j}) \equiv \mu(C_{k,j+1})$ such that $\mu(TC_{k,j} \mid C_{k,j+1}) \approx 1$, i.e. the set $TC_{k,j}$ is close to $C_{k,j+1}$. When we add cut point at position $k$ this property holds for all $j$ except only one $j_\times = k$, namely, the case when we pass from level $k$ to level $k+1$.

![Figure 8](image)

**Figure 8.** Cyclic rotation of a word indexing levels of the cyclic iceberg leads to another choice of the base set and the letter defined to be the origin.

A principal feature of a cyclic iceberg is the possibility of its cyclic rotation which in fact means that we can take any level as the base set of the iceberg (see fig. 8). Clearly a choice of a base set corresponds to a choice of a point in the homogeneous space $\mathbb{Z}_h$ playing role of origin. Since the acting group $\mathbb{Z}$ is Abelian this homogeneous space is a group. Thus, we have seen that the levels of the iceberg are indexed by elements of the group $\mathbb{Z}_h$. Furthermore, the way we draw an iceberg as a kind of parallelogram is explained by the idea to draw the columns in order making the cut points sequence ascending, and from dynamical point of view an iceberg is just an unordered set of columns as shown on fig. 7, where each column corresponds to a specific choice of the cut point.

What is the meaning of cut point? If we consider a word with glued first and last letters as discrete circle $\mathbb{Z}_h$ then the cut point originates from the boundary of a fundamental domain associated with lattice $h\mathbb{Z}$.

3.2. **Iceberg approximation as dynamical invariant.** In the sequel for simplicity we will use the term iceberg as shorter equivalent of the term cyclic iceberg. Actually the notion of cyclic iceberg is connected with the concept of iceberg approximation explained in the next definition, and defined using rotation operator $\rho_\alpha$.

**Definition 15.** We say that a measure preserving map $T$ admits iceberg approximation if given a finite measurable partition associated with an alphabet $A$ for any $\varepsilon > 0$ there exists a word $W_\varepsilon$ in the alphabet $A$ such that for $(1-\varepsilon)$-fraction of orbits $(x_n)$ the subword of length $N(\varepsilon)$ in $(x_n)$ starting from $x_0$ is $\varepsilon$-covered by rotations $\rho_\alpha(W_\varepsilon)$ of the word $W_\alpha$.

**Theorem 16.** Iceberg approximation property is a dynamical invariant.

A question on finding an invariant property for iceberg transformations is proposed by B. Weiss. To see the difference between rank one and iceberg approximation let us draw compare orbit structure for some rank one and some iceberg transformation, see fig. 9.

**Proof.** If two transformations $T$ and $S$ are equivalent via a measure preserving map $I$ then we can map a finite partition $\mathcal{P}$ for one map to a finite partition $I(\mathcal{P})$ for another, and if the $\mathcal{P}$-coding of $T$-orbits is covered by words $\rho_\alpha(W)$ the same property holds for $I(\mathcal{P})$-coding of $S$-orbits. \qed
Lemma 18. The maps $T$ is a measure preserving invertible transformation of the space $(X, \mu)$.

### 3.3. Construction of iceberg transformation

**Definition 17.** An *iceberg transformation* (without spacers) is approximated by a sequence of icebergs $\mathcal{T}_n$ of height $h_n$ such that any next iceberg in the sequence refines the previous one according to the rotated words concatenation procedure defined in section I. Let us fix heights $h_n$ and rotations $\alpha_{n,y}$, and suppose that

$$h_{n+1} = q_n h_n, \quad h_0 = 1, \quad q_n \in \mathbb{Z}, \quad q_n \geq 2.$$  

Set $X_n = \{0, 1, \ldots, h_n - 1\}$, and consider projections

$$\phi_n : X_{n+1} \to X_n, \quad n = 0, 1, \ldots$$

where

$$\phi_n(y h_n + t) = t - \alpha_{n,y} \pmod{h_n}, \quad 0 \leq t < h_n, \quad y = 0, 1, \ldots, q_n - 1.$$  

Restricted to any interval $[y h_n, (y+1)h_n)$ the map $\phi_n|_{[y h_n, (y+1)h_n)}$ is exactly the inverse map to the rotation operator $\rho_{\alpha_{n,y}}$ in concordance with Figure 9. It is clear that $\mu_{n+1}(\phi_n^{-1} A) = \mu_n(A)$ for any set $A$. Thus, we can endow the inverse limit $X$ of spaces $(X_n, \mu_n)$ with Borel measure $\mu$ such that $\mu(\{x: x_n = t\}) \equiv h_n^{-1}$, where

$$X = \{x = (x_0, x_1, \ldots, x_n, \ldots): \phi_n(x_{n+1}) = x_n\}.$$  

Clearly, the $n$-th coordinate $x_n$ satisfies condition

$$\phi_{n-1}(x_n + 1) = \phi_{n-1}(x_n) + 1$$

with $\mu$-probability $1/h_{n-1}$ (the probability to encounter the cut point). Since $h_n \to \infty$ condition [33] holds for coordinates $x_{n_0}, x_{n_0+1}, \ldots$ starting from some index $n_0$ with $\mu$-probability 1. We call these points $x$ *regular*. Let us define the iceberg transformation map $T$ for a regular point $x$ by formula

$$T : (x_0, \ldots, x_{n_0-1}, x_{n_0}, x_{n_0+1}, \ldots) \mapsto (x_{n_0}^{x_0}, \ldots, x_{n_0-1}^{x_0}, x_{n_0} + 1, x_{n_0+1} + 1, \ldots),$$

where the head of the sequence is recovered in the only way such that $Tx$ becomes a correct sequence with the property $\phi_n((Tx)_{n+1}) = (Tx)_n$, namely we set

$$x_{n_0} + 1 \xrightarrow{\phi_{n_0}} x_{n_0-1}^{x_0} \xrightarrow{\phi_{n_0-2}} x_{n_0-2}^{x_0} \xrightarrow{\phi_{n_0-3}} \ldots \xrightarrow{\phi_{1}} x_{1}^{x_0} \xrightarrow{\phi_{0}} x_{0}^{x_0}.  

**Lemma 18.** The maps $T$ is a measure preserving invertible transformation of the space $(X, \mu)$.  

Figure 9. Comparing rank one and iceberg approximation.
Proof. On the $n$-th level the map $T$ is close to the rotation map $t \mapsto t + 1$ which preserves $\mu_n$, and we can take arbitrary $n$, hence, $T$ is measure preserving.

Invertibility of $T$ is a corollary of an important observation that $T$ is generated by the underlying transformation $t \mapsto t + 1$ of the acting group $\mathbb{Z}$ having $t \mapsto t - 1$ as inverse. Indeed, if look at $n$-th level the transformation $t \mapsto t + 1 \pmod{h_n}$ of the homogeneous space $X_n$ associated with $n$-th iceberg is induced by the shift transformation $t \mapsto t + 1$ of $\mathbb{Z}$. Thus, the inverse map for $T$ is defined in the same way as $T$ but using the map $t \mapsto t - 1$.

Walking by $T$ and $T^{-1}$ it is interesting to observe the reversibility of jumps

$$T: x_n \mapsto x_n^{++}, \quad T^{-1}: \tilde{x}_n \mapsto \tilde{x}_n^{-},$$

where $\tilde{x}_n = x_n^{++}$. Under a jump we understand the case when $x_n^{++} \neq x_n + 1$. In fact, crossing the cut point $(\times)$ after the (first) letter “C” in a join ...ATC$_{\times}$TCA... we jump to the position of letter “T” instead of passing to the next letter “A” in the original order C $\mapsto$ A $\mapsto$ T $\mapsto$ C. □

\[\text{Figure 10. Jumps under the action of Poincaré map}\]

3.4. Interpretation of dynamics. Let us draw on one picture two steps of the iceberg map construction to understand clearly the dynamics of $T$. We are going to start with the picture shown at fig.2 and to gradually add details on this picture. The first step is to interpret the Poincaré map $\tilde{T}_{E_n}$.

\[\text{Figure 11.}\]

Definition 19. Let us call fat columns the vertical columns $V_k$ on figures2,3 and 10. Intersecting fat column $V_k$ with levels we get partition

$$V_k = V_{k,-h+k} \sqcup V_{k,-h+k+1} \sqcup \ldots \sqcup V_{k,k-1}, \quad V_{k,j} = V_k \cap B_j,$$

where $k = 1, 2, \ldots, h$. Observe that the number of all fat columns is $h_n$. Further, let us use the term thin column for a vertical subcolumn of a fact column corresponding to one rotated copy of the word $W_n$ inside the word $W_{n+1}$ (see fig.12). In other wordth, a thin column corresponds to an interval $[yh_n, (y+1)h_{n+1})$ in $X_{n+1}$. There are totally $q_n$ thin columns.
Figure 12. Iceberg transformation dynamics.

Figure 13. Grayed columns contain jumps. If a point $x \in B_j$ and if $x$ is located in the white area (body) then $T x \in B_{j+1}$.

Clearly the meaning of Poincaré map is to express the connection of two thin columns corresponding to a pair of adjacent rotated copies $\rho_{n,y}(W_n)\rho_{n,y+1}(W_n)$ which are subwords of $W_{n+1}$. For example, if this pair is CAT.ATC the Poincaré map sends the top set in a thin column included in fat column $V_3$ ("CAT") to the bottom set of some thin column in fat column $V_1$ ("ATC"). From the point of view of observer watching the coordinate $x_n$ in the homogeneous space $\mathbb{Z}_{h_n}$ the following occurs: with probability $1 - \varepsilon$ point $x_n$ moves one step forward $x_n \mapsto x_n + 1$, and with small probability $\varepsilon$ it jumps to any other point in $\mathbb{Z}_{h_n}$. At this level we cannot say precisely how this jumpings are distributed (this is another definition of iceberg, cyclicity means that forward and backward tracks before jumping have common length $h_n$). In terms of iceberg we can explain this behavior as follows. A point $x$ under $T$ moves up with probability $(1 - \varepsilon_2)$, and reaching the top level in a fat column it make a random jump to some bottom set in another fat column (see fig. 13). In other words we are going to show geometrically what happens when we rotate and concatenate copies of word $W_2 = \text{CAT}.\text{ATC.}\text{TCA.}\text{TCA.}\text{CAT.}\text{ATC}$. 
Suppose that \( q_2 = 3 \) and \( W_3 \) is the result of concatenation of the following three rotated copies
\[
\text{(38)} \quad \text{CATATCT} \times \text{CATCACATATC} \\
\text{CATATCT} \times \text{CATACATATC} \times \text{TCTCATACATATC} \\
\text{CATATCTCATACATATC}
\]
namely,
\[
\text{(39)} \quad \text{W}_3 = \text{CATACATATC} \times \text{TCTCATACATATC} \times \text{CATACATATC} \times \text{TCTCATACATATC}
\]
At figure 12 both first and second Poincaré maps are shown, and we can continue this procedure. Let us mark the columns containing jumps as grayed (see fig. 13).

3.5. Approximation and rank.

**Lemma 20.** Let \( T \) be an iceberg transformation. A function \( f \in L^2(X, \mu) \) can be approximated by a sequence of \( \mathcal{F}_n \)-measurable functions. In other words, \( T \) admits iceberg approximation. In particular, function \( f \) can be approximated by a sequence of functions \( f_n \), where \( f_n \) is constant on levels of icebergs \( \mathcal{I}_n \).

**Remark 21.** Recall that \( \mathcal{I}_n \)-measurable functions are constant on levels \( \{B_{-h+1}, \ldots, B_0, \ldots, B_{-1}\} \) of iceberg \( \mathcal{I}_n \), and \( \mathcal{I}_n \)-measurable functions are constant on levels \( B_j \) and in addition it take same value both on \( B_{j-h} \) and \( B_j \). Simply speaking, different values of \( f \) correspond to different letters on fig. 2.

**Lemma 22.** Suppose that \( q_n \to \infty \). There exist sub-icebers \( \mathcal{I}_n^* \) for an iceberg transformation \( T \) such that \( T \) lifts any point \( x \in \mathcal{I}_n^* \) exactly to the upper level, and \( \mu(\bigcup \mathcal{I}_n^* | \bigcup \mathcal{I}_n) \to 1 \) as \( n \to \infty \). In other words, there exist sets \( V_{k,j}^* \subseteq V_{k,j} \) such that
\[
TV_{k,j}^* = V_{k+1, j+1}^*, \quad j = -h + k, \ldots, k - 1,
\]
and \( \mu(\bigcup \mathcal{I}_n^* | \bigcup \mathcal{I}_n) \to 1 \), where
\[
\bigcup \mathcal{I}_n^* = \bigcup_{k=1}^{\infty} \bigcup_{j=-h+k}^{k-1} V_{n,k,j}^*.
\]

It can be easily seen from the condition that each set \( \mathcal{I}_n^* \) (we call it body) can be choosen as the maximal measurable set with the property point make no jumps until reaching the top set in a column.

**Proof.** Consider the thin vertical subcolumns corresponding to the rotated copies of \( W_n \) in the word \( W_{n+r} \). There are totally \( Q_{n,n+r} = q_n q_{n+1} \cdots q_{n+r-1} \) such columns. Let us watch the number \( Q_{n,n+r}^* \) of thin subcolumns at level \( n+r \) which is not touched by cut points. For example, \( Q_{n,n+1} = Q_{n,n+1}^* = q_n \) and
\[
Q_{n,n+2} = q_n q_{n+1}, \quad Q_{n,n+2}^* \geq Q_{n,n+2} - q_{n+1},
\]
since passing from level \( n+1 \) to level \( n+2 \) the word \( W_{n+2} \) is a sequence of rotated copies of \( W_{n+1} \), and potentially we can see \( q_{n+1} \) cut point touching small words originated from \( W_n \). More precisely, passing from level \( n+r \) to \( n+r+1 \) we can get one additional cutting inside one thin subcolumn at level \( n+r \), i.e.
\[
Q_{n,n+r+1}^* \geq Q_{n,n+r}^* q_{n+r} - q_{n+r} = (Q_{n,n+r}^* - 1) q_{n+r}.
\]
Let us denote as \( A_{n,n+r} \) the union of all thin subcolumns build at level \( n+r \),
\[
\mu(A_{n,n+r+1} | \bigcup \mathcal{I}_n) = \frac{Q_{n,n+r+1}^*}{Q_{n,n+r+1}} = \frac{Q_{n,n+r+1}^*/Q_{n,n+r}^* Q_{n,n+r}^*}{q_{n+r}} \geq \left( 1 - \frac{1}{Q_{n,n+r}^*} \right) \mu(A_{n,n+r} | \bigcup \mathcal{I}_n).
\]
Since \( q_n \to \infty \) we can assume that \( q_{n+r} \geq 3 \), hence, \( Q_{n,n+r} \to \infty \) and \( Q_{n,n+r} - 1 \geq \frac{1}{2} Q_{n,n+r} \). Therefore
\[
Q_{n,n+r+1}^* \geq Q_{n,n+r}^* \cdot \frac{q_{n+r}}{2} \geq Q_{n,n+r}^* \cdot \frac{3}{2} \geq \left( \frac{3}{2} \right)^r,
\]
and lemma follows from estimate (44). □

Remark 23. Let us note that existence of body $\mathcal{J}_n^*$ such that $\mu(\cup \mathcal{J}_n^* | \cup \mathcal{J}_n) \to 1$ is stronger than correctness of the definition of $T$. The method used to prove lemma 22 is purely combinatorial and requires $q_n \to 0$. At the same time, applying ergodic theorem (we only need condition $\sum_n h_n^{-1} < \infty$) it is not hard to see that this requirement can be omitted, but we not put this proof here.

Now we turn to estimation of spectral multiplicity of $T$.

**Definition 24.** Given a Rokhlin tower $\mathcal{F}$ let us use the notation $\mathcal{F}^\varepsilon$ for the measurable partition into levels of the tower $\mathcal{F}$ and distinct points outside $\cup \mathcal{F}$. Notice that $\mathcal{F} \preceq \mathcal{F}^\varepsilon$.

**Definition 25.** Local rank $\beta(T)$ of a measure preserving transformation $T$ is the maximal value $\beta$ with the following property: there exists a sequence of towers $\mathcal{F}_n$ such that $\mu(\cup \mathcal{F}_n) \to \beta$ as $n \to \infty$ and for any measurable set $A$ there exists $\mathcal{F}_n^\varepsilon$-measurable sets $A_n$ with $\mu(A_n \Delta A) \to 0$ as $n \to \infty$.

The following theorem is an immediate geometrical corollary of lemma 22.

**Theorem 26.** If $T$ is an iceberg transformation with $q_n \to \infty$ then $T$ is of $1/4$-local rank.

**Remark 27.** V. Ryzhikov observed [31] that $\beta(T \times T) = 1/4$ for a typical $T$.

**Remark 28.** In the proof we essentially use the fact that $T$ is approximated by cyclic icebergs.

**Proof.** The idea of the proof is based on the following observation: since $\mathcal{J}_n^*$ approximates $\mathcal{J}_n$ then we can use $\mathcal{J}_n^*$ in investigation of dynamics of $T$. The $1/4$-local rank property can be proved using the following illustration. Consider a pair $(x, y)$, where $0 \leq x, y \leq 1$, and a subset of the unit square (see fig. 14)

\[ G = \left\{ (x, y) \in [0, 1] \times [0, 1]: \ (1 - x)y \geq \frac{1}{2} \text{ or } x(1 - y) \geq \frac{1}{2} \right\}. \]

Now we play the following geometric game. Starting from $(0, 0)$ we have to reach $(1, 1)$ moving continuously along horizontal or vertical lines (right or up). Evidently, one can find a turn point $p$ on the trajectory such that $p \in G$. Moreover, we can also choose such a point $p$ with the following additional property: after the next step it will fall into the upper grayed area. This point $p$ describes a subtower $\mathcal{T}_n$ of $\mathcal{J}_n^*$ such that $\mu(\cup \mathcal{T}_n) \geq \frac{1}{4} \mu(\cup \mathcal{J}_n^*)$. The second requirement to the point $p$ provides a subtower of height $\tilde{h}_n \geq h_n$. Finally, recall that $\mu(\cup \mathcal{J}_n^*) \to 1$. 

![Figure 14. Proving 1/4-local rank property. One vertex on the path visits the white area inside the unit square. This vertex corresponds to a sub tower of the iceberg shown as (green) rectangular area greater than 1/4. On the left picture an iceberg is shown corresponding to the path in the unit square.](image-url)
So, a sequence of Rokhlin subtowers $\Sigma_n$ is found such that levels of $\Sigma_n$ are included into levels of $\mathcal{I}_n$. We know that for any measurable set $A$ there exist sets $A_n$ which is a union of levels of $\mathcal{I}_n$ such that $\mu(A_n \Delta A) \to 0$ as $n \to \infty$. Hence, the intersection of the set $A_n$ with the tower $\cup \Sigma_n$ is a union of levels of this tower. □

Remark 29. This corollary remains true for $T$ having property of iceberg approximation. Actually the requirement $\sum_n h_n^{-1} < \infty$ is needed to achieve certainly the property $\beta(T) \geq 1/4$.

Theorem 30. An automorphism $T$ with the property of iceberg approximation has $1/4$-local rank.

Remark 31. The proof of the above corollary is not so evident. So, to give a simple explanation of the effect let us consider an iceberg with large number of fat columns having measure close to $1/h_n$. In this case the iceberg looks like parallelogram (see fig. 15) which is composed of two triangles with vertices $(0,0)$, $(0,h)$, $(-h,0)$, $(-h,-h)$. This illustration will be often used in the sequel.

Definition 32. Let us call the sequence of icebergs $\mathcal{I}_n$ uniform if the vector $(\mu(V_n,1), \ldots, \mu(V_n,h_n))$ is $l_1$-close to the uniform distribution $(h_n^{-1}, \ldots, h_n^{-1})$.

Example 33. We say that an iceberg transformation $T$ has Morse property if $h_n^{-1} | \alpha_{n,y}$ for any $y = 0, \ldots, q_n - 1$. Remark that Morse property implies $\cup \mathcal{I}_n^* = \cup \mathcal{I}_n$. To give an example of Morse iceberg system, suppose that for any $n$ the word $W_n$ is split into a pair of equal blocks: $W_n = B_n A_n$, $|A_n| = |B_n| = h_n^{-1}$, then by definition we set

(47) $W_{n+1} = A_n B_n B_n A_n$.

Clearly, starting with $W_0 = 01$ we get the classical Morse sequence. Further, if $W_n = A_n B_n C_n$, we can consider Morse iceberg map $T_{(3)}$ of order 3:

(48) $W_{n+1} = A_n B_n C_n . B_n C_n A_n . C_n A_n B_n$

and so on. Observe that theorem 26 can be applied to iceberg transformations with Morse property and we see that

(49) $\beta(T_{(r)}) = \left(\frac{r + 1}{2r}\right)^2$ if $r = 2k + 1$

and

(50) $\beta(T_{(r)}) = \frac{r(r + 2)}{4r^2}$ if $r = 2k$.

Definition 34. We say that a measure preserving map $T$ has iceberg rank $r$ if there exists a sequence of partitions $\mathcal{P}_n = \mathcal{I}_n^{(1)} \vee \cdots \vee \mathcal{I}_n^{(r)}$ such that any measurable set $A$ is approximated by $\mathcal{P}_n$-measurable sets $A_n$. Let us denote $r_1(T)$ the minimal $r$ with this property.

Definition 35. Let us define $\beta_1(T)$ to be the maximal value $\beta$ such that a sequence of icebergs $\mathcal{I}_n$ approaches $\sigma$-algebra and $\mu(\cup \mathcal{I}_n) \to \beta$ as $n \to \infty$. 
3.6. Iceberg transformations with spacers. Let us improve the definition adding spacers between rotated copies of word $W_n$:

$$W_{n+1} = \rho_{\alpha_n,0}(W_n)1^{s_n,0}\rho_{\alpha_n,1}(W_n)1^{s_n,1}\cdots\rho_{\alpha_n,q_n}(W_n)1^{s_n,q_n-1}. \quad (51)$$

If we require condition (25) this procedure generates a measure preserving transformation on a Lebesgue space.

Remark 36. There is one special case of iceberg map with spacers: we do not put spacers between adjacent words $\rho_{\alpha_n,0}(W_n)1^{s_n,0}\rho_{\alpha_n,1}(W_n)1^{s_n,1}$ but we add only one group of spacers $1^{s_n,q_n-1}$ to the tail of $W_{n+1}$. This method helps to provide $h_n$ with special arithmetic properties.

Question 37. Is the following true or not?

(i) There exists a map with iceberg approximation but not rank one?

(ii) Any map with iceberg approximation is isomorphic to a map built using iceberg construction with spacers? In other word, is it always possible to refine the sequence of icebergs for a map with iceberg approximation?

(iii) The classes of iceberg maps with and without spacers are not identical?

(iv) There exists a map $T$ with iceberg approximation such that $T^{jk} \rightarrow (\text{Id} + \hat{T})/2$?

(v) What are $r_f(T)$ and $\beta_f(T)$ for a typical $T$?

(vi) What is the value of $\beta_f(T \times T)$ for a typical $T$?

(vii) For any map $T$ with iceberg approximation $T \times T$ is of local rank one?

(ix) The entropy $h(T) > 0$ if $\beta_f(T) > 0$?

(x) Given a transformation $T$ with iceberg approximation is it true that symbolic complexity of $T$ is always sub-exponential (see [15])?

4. Simplicity of spectrum

We know by theorem 26 that iceberg approximation is stronger than $1/4$-local rank approximation, therefore, spectral multiplicity of an iceberg transformation $m(T) \leq 4$ (see corollary 40). At the same time, surprisingly, iceberg approximation implies simplicity of spectrum for a wide class of transformations. We are going to prove this for a class of maps with randomized jumps and for a class of maps with jumps given by a pseudo-random substitution on the set of columns.

Lemma 38. Let $U$ be a unitary operator in a separable Hilbert space $H$, let $\sigma$ be the maximal spectral type measure, and let $M(z)$ be the multiplicity function of the operator $U$. If $M(z) \geq m$ on a set of positive $\sigma$-measure than one can find $m$ orthogonal elements of unit length $f_1, \ldots, f_m$ such that for any cyclic subspace $Z \subset H$ (with respect to $U$) and for any $m$ elements $g_1, \ldots, g_m \in Z$, of equal length $\|g_i\| \equiv a$ the following is true

$$\sum_{i=1}^{m} \|f_i - g_i\|^2 \geq m(1 + a^2 - 2a/\sqrt{m}). \quad (52)$$

Corollary 39. If $T$ is a $\beta$-local rank transformation then $m(T) \leq 1/\beta$. In particular, any rank one transformation have simple spectrum.

Corollary 40. Since any transformation $T$ with iceberg approximation has $1/4$-local rank property, $m(T) \leq 4$. Further, for any map $T$ with $\beta$-icerberg approximation property, $m(T) \leq [4/\beta]$.

The discussion throughout this section concerns simplicity of spectrum for iceberg transformation.

4.1. Preliminary calculations. Suppose that the maximal spectral multiplicity $m(T) \geq 2$. So, applying lemma 38 we see that two functions $f_1$ and $f_2$ should exist such that any cyclic subspace $Z$ contains elements $g_1, g_2$ with the property $\|g_1\| = \|g_2\| = a$ and satisfying

$$\|f_1 - g_1\|^2 + \|f_2 - g_2\|^2 \geq 2(1 + a^2 - 2a/\sqrt{2}). \quad (53)$$
We will show that for a class of transformations there exist a cyclic subspace approximating both $f_1$ and $f_2$, and the contradiction will follow. Functions $f_i$ can be approximated by $\mathcal{F}_n$-measurable functions, and without loss of generality we can assume that $f_1$ are $f_2$ are $\mathcal{F}_n$-measurable. Since the arguments concerning approximation precision are the same both for $f_1$ and $f_2$, we will work with one function $f_1$. Recall that $f_1$ is $\mathcal{F}_n$-measurable for any $n \geq n_0$. Let us denote $f_\alpha$ the lifting of the function $f_1$ to the iceberg $\mathcal{F}_n$. Let $b_n = 1_{B_{n,0}}$ be the indicator of the base set $B_{n,0}$ of the iceberg $\mathcal{F}_n$. We have

$$f = \sum_{j=0}^{h_n-1} f_\alpha(n) S^j b_n = \sum_{j=-(h_n-1)/2}^{(h_n-1)/2} f_\alpha(n) S^j b_n, \quad n \geq n_0,$$

where by definition $SB_j = B_{j+1}$ is the operator in $L^2(X_n)$ corresponding to the rotation $t \mapsto t + 1$.

**Figure 16. Proving simplicity of spectrum**

*Idea.* The following way of proving simplicity of spectrum one can call “3/4-strategy”. We approximate function $f$ by the iterations $T^j b_n$ of the indication of the base level $B_{n,0}$, where $j = -(h_n - 1)/2, \ldots, (h_n - 1)/2$, on the subset of the phase space $X$ of measure $\approx 3/4$ (see fig. 16). Assume for simplicity that $h_n = 2\mathbb{Z} + 1$. Remark that the estimates below used by this approach is not a priori necessary for simplicity.

Without loss of generality we can assume that $\int f \, d\mu = 0$. Consider sets $G_n$ and $E_n$ of measure $\approx 3/4$ and $\approx 1/4$ respectively, where $G_n$ corresponds to the central area on figure 16 and $E_n$ is the union of two triangular areas remote from the base level,

$$\cup \mathcal{F}_n = G_n \cup E_n, \quad G_n = \bigcup_{j=-(h_n-1)/2}^{(h_n-1)/2} B_{n,j}, \quad E_n = \bigcup_{j=-(h_n-1)/2}^{-(h_n-1)/2-1} B_{n,j} \cup \bigcup_{j=(h_n-1)/2+1}^{h_n-1} B_{n,j}.$$

The function $f$ is approximated by the function

$$g = \sum_{j=-(h_n-1)/2}^{(h_n-1)/2} f_\alpha(n) T^j b_n$$

and $g$ can be represented in the following way:

$$g = f - u + v, \quad f - u = f|_{G_n}, \quad u = f|_{E_n},$$

and $v$ is uniquely defined from the equation. The meaning of the function $v$ can be explained as follows. Walking from the base of the iceberg under action of $T$ a point $x$ is moving in the vertical direction, and the value $f(T^j x)$ is recovered by the index of level, namely, $f(T^j x) = f|_{B_j}$. Though, when $x$ approaches to the top set of a fat column it makes jump to the bottom set of some column and continue vertical motion in that column. So, the function $v$ collects the parts of the iterates $T^j b_n$.
leaving the set $G_n$. More precisely, we can write

$$T^j b_n = 1_{B_{n,j}} + \xi_j, \quad v = \sum_{j=-(h_n-1)/2}^{(h_n-1)/2} f(n) \langle j \rangle \xi_j,$$

and

$$u = \sum_{j=-(h_n-1)/2}^{-(h_n-1)/2+1} f(n) \langle j \rangle 1_{B_j} + \sum_{j=-(h_n-1)/2+1}^{h_n-1} f(n) 1_{B_j}.$$

If possible we omit index $n$ for simplicity.

**Lemma 41.** Suppose that $\langle u, v \rangle \to 0$ and $\langle f, v \rangle \to 0$ as $n \to \infty$ then asymptotically $\|f-g\|^2 \to 1/2$ and $\|g\| = (1 + o(1))\|f\|$. In particular, $T$ has simple spectrum.

**Proof.** Assume that $f$ has zero mean and $\|f\| = 1$. Let us notice that $v(x) = u(\Phi x)$ for some measure preserving invertible map $\Phi$, hence, $\|v\|^2 = \|u\|^2$. Further, we have

$$a^2 = \|g\|^2 = \|f - u + v\|^2 = \|f\|^2 + \|u\|^2 + \|v\|^2 - 2 \text{Re} \langle f, u \rangle + 2 \text{Re} \langle f, v \rangle - 2 \text{Re} \langle u, v \rangle \approx \|f\|^2 + 2\|u\|^2 - 2 \text{Re} \langle f, u \rangle = \|f\|^2 = 1,$$

and

$$\|u\|^2 \approx \frac{1}{4}\|f\|^2, \quad \langle f, u \rangle = \|u\|^2.$$

Now let us estimate $\|f-g\|:$

$$\|f-g\|^2 = \| - u + v\|^2 \approx \|u\|^2 + \|v\|^2 \approx \frac{1}{2}\|f\|^2 = \frac{1}{2}.$$

To establish the second statement of the lemma let us divide over $m$ left and right side of inequality 52 in lemma 48 and take into account that we use the same estimates both for $f_1$ and $f_2$. Thus we have to analyze the following inequality:

$$\|f-g\|^2 \geq 1 + a^2 - a\sqrt{2}.$$

Using results of the above calculations we have

$$\|f-g\|^2 \geq 1 + a^2 - a\sqrt{2},$$

$$\frac{1}{2} \geq 2 - \sqrt{2},$$

$$\sqrt{2} \geq \frac{3}{2},$$

and we come to contradiction. \qed

**Definition 42.** If an automorphism $T$ is approximated by a sequence of icebergs $\mathcal{F}_n$ and $\mu(\cup \mathcal{J}_n) \to \beta$ as $n \to \infty$ then we say that $T$ has $\beta$-local iceberg rank.

**Remark 43.** The method of lemma 11 works for $T$ if

$$\beta_1(T) > 8/9.$$

The reason is a small gap in equation 66.

**Proof.** Suppose that $\beta_1(T) \geq \beta$ and consider a sequence of icebergs with approximation property and $\mu(\cup \mathcal{J}_n) \to \beta$. Let us define $g$ in the same way as in the proof of lemma 11 and assume that $\|f\| = 1$. Asymptotically we have

$$\|f-g\|^2 \leq \|u\|^2 + \|v\|^2 + (1-\beta)\|f\|^2 \approx \left(\frac{1}{2}\beta + 1 - \beta\right)\|f\|^2,$$
since we have no information about $f$ outside $\cup \mathcal{J}_n$ except $\|f|_{\cup \mathcal{J}_n}\|^2 \to \beta \|f\|^2$ (in force of ergodic theorem). Further, $a^2 = \|g\|^2 \approx \beta \|f\|^2$, and the following two values must be compared

$$1 - \frac{1}{2} \beta \quad \text{and} \quad 1 + a^2 - a\sqrt{2} \approx 1 + \beta - \sqrt{2\beta}.$$  

Solving equation

$$1 - \frac{1}{2} \beta = 1 + \beta - \sqrt{2\beta}$$
we find the critical value of $\beta = 8/9$.

**Question 44.** Is it possible to find $T$ with non-simple spectrum and $\beta_T(T) \leq 8/9$?

\[\text{Figure 17. Iterates of the base level of an iceberg.}\]

### 4.2. Simplicity of spectrum for randomized iceberg transformations.

**Lemma 45.** Suppose that

$$\mu(TV_{n,k,k-1} | V_{n,s,s-h}) - \frac{1}{h_n} = o\left(\frac{1}{h_n}\right).$$

Then $\langle u, v \rangle \to 0$ and $\langle f, v \rangle \to 0$ as $n \to \infty$.

This lemma immediately implies that $T$ has simple spectrum (see lemma [II]).

**Proof.** It is not hard to see that without loss of generality we can assume the following condition: $\mu(TV_{n,k,k-1} | V_{n,s,s-h}) - \frac{1}{h_n} \equiv 0$. Since $q_n \to \infty$ we can restrict $f$ to the body $\mathcal{J}_n^*$, where $\mu(\cup \mathcal{J}_n^* | \cup \mathcal{J}_n) \to 1$. Suppose that $f$ is $\tilde{\mathcal{J}}_{n-1}$-measurable. Then for any top set in a fat column $V_{n,k,k-1}$ we see that $T^j V_{n,k,k-1}$ intersects the sequence of sets $V_{n,s,s-h+j}$, and

$$\mu(T^j V_{n,k,k-1} \cap V_{n,s,s-h+j}) = \frac{1}{h_n^2}.$$  

Hence, the function $v$ takes value $f_{(n)}(j)$ on each set $V_{n,s,s-h+j}$. At the same time

$$f|_{V_{n,s,s-h+j}} \equiv f_{(n)}(s - h + j),$$

and

$$\langle v|_{\cup V_{n,k,k-1}}, f \rangle = \sum_{s=1}^{h_n} f_{(n)}(j) f_{(n)}(s - h + j) \cdot \frac{1}{h_n^2} = O\left(\frac{h_n}{h_n^2}\right) = 0.$$
Here we use the property $\sum_{\kappa=0}^{h_n-1} f_{(n-1)}(\kappa) = 0$ which is invariant under rotation map $t \mapsto t + 1$, hence, the sequence
\begin{equation}
(f_{(n)}(s - h + j))_{s=1}^{h_n-1} = (f_{(n)}(1 - h + j), \ldots, f_{(n)}(j))
\end{equation}
is split into blocks of length $h_{n-1}$ such that the sum of the elements in each group equals zero. \hfill \square

**Theorem 46.** Iceberg transformations given by uniformly distributed on $\mathbb{Z}_{h_n}$ i.i.d. random rotations have simple spectrum almost surely, if $q_n/h_n \to \infty$.

**Proof.** To prove the theorem it is enough to apply lemma 45 and to observe that estimate (71) follows from independence of $\alpha_{n,y}$ and $\alpha_{n,y+1}$.

\hfill \square

5. CORRELATION DECAY

In this section we discuss estimation of correlation decay for iceberg transformations with uniform i.i.d. rotations $\rho_{\alpha_{n,y}}$ (statements (i) and (ii) of theorem 2). A detailed proof is included in the second part of the paper.

**Remark 47.** Notice that Ornstein rank one transformations is a case of iceberg transformations with random rotations $\alpha_{n,y}$ distributed on an interval $[0, \alpha_n^{\text{max}}]$ and satisfying restriction $\alpha_n^{\text{max}} \ll h_n$.

Let us recall that correlations associated with a measure preserving transformation $T$ is given by the formula
\begin{equation}
R(t) = \langle T^t f, g \rangle = \int_X f(x) g(x) d\mu,
\end{equation}
where $f, g \in L^2(X, \mu)$ are fixed. One can observe that $R(t)$ is approximated by the following functions
\begin{equation}
R_n^c(t) = \frac{1}{h_n} \sum_{j \in \mathbb{Z}_{h_n}} f_{(n)}(j) \bar{g}_{(n)}(j - t)
\end{equation}
if the sequence of icebergs $\mathcal{I}_n$ are uniform (measure of any fat column is close to $h_n^{-1}$). Let us estimate $R_{n+1}^c(t)$ for $t = sh_n$, $1 \leq s \leq q_n - 1$. This assumption does not influence to the merits of case but seriously simplifies calculations. The common case is studied in the detailed proof. We have
\begin{equation}
R_{n+1}^c(sh_n) = \frac{1}{h_n + 1} \sum_{j \in \mathbb{Z}_{h_n+1}} f_{(n+1)}(j) \bar{g}_{(n+1)}(j - sh_n) = \frac{1}{q_n} \sum_{y=0}^{q_n-1} R_n^c(\alpha_{n,y} - \alpha_{n,y-s}).
\end{equation}
The sequence $(\alpha_{n,y})_{y=0}^{q_n-1}$ is i.i.d. and we can expect that $\alpha_{n,y} - \alpha_{n,y-s}$ is a mixing process with fast rate of mixing. Well, in fact this sequence is almost non-correlated and it turns out that this property is enough to see the desired effect. Since $R_{n+1}^c(sh_n)$ is similar to a sum of i.i.d. random variables, we get
\begin{equation}
|R_{n+1}^c(sh_n)| \sim \frac{1}{\sqrt{q_n}} \sqrt{DR_n^c(t)}.
\end{equation}
In other terms,
\begin{equation}
DR_{n+1}^c(sh_n) \sim \frac{1}{h_n + 1} (h_n DR_n^c(t) + (q_n - 1) h_n \frac{1}{q_n} DR_n^c(t)) \sim 2DR_n^c(t),
\end{equation}
and iterating this estimate and using equality $h_{n+1} = q_nh_n$ we have
\begin{equation}
DR_n^c(t) = O\left(2^n \frac{1}{h_n}\right), \quad n \to \infty.
\end{equation}
Thus, for $t \sim h_n$ we have $R(t) = O(2^n h_n^{-1/2}) = O(t^{-1/2+\varepsilon})$ for any $\varepsilon > 0$ (statement (iii) in theorem 2).
Remark 48. One can observe that the proof is based on “simple effects”. We have used only momnt estimations or, in other words, we deal with random sums in a Hilbert space, and central limit theorem is not used throughout the proof. At the same time there are no indications to new effects if we apply CPT. Probably, this estimate cannot be improved.

Further, notice that the estimate \( R(t) \sim O(t^{-a+\varepsilon}) \) with \( a < 1/2 \) would imply \( R \in L^2(\mathbb{Z}) \) and Lebesgue spectrum of \( T \), and 1/2 is the optimal value of \( a \) in the estimate. Nevertheless, observe that \( \sigma \) can be absolutely continuous even in the case when estimate \( R(t) \sim O(h_n^{-1/2+\varepsilon}) \) is the best possible. Moreover, one can expect that for a typical \( \mathfrak{S}_n \)-measurable function \( f \) the density of a hypothetical Lebesgue spectral measure \( \sigma_f \) has singularities. Such effect is observed for densities of spectral measures for rank one flows studied in [30].

Remark 49. Let \( \sigma \) be a spectral measure corresponding to an \( \mathfrak{S}_n \)-measurable function \( f \). Then \( \sigma \ast \sigma \ll \lambda \) since it has \( L^2 \) Fourier coefficients, \( R^2 \in L^2(\mathbb{Z}) \). Thus, \( \sigma = \sigma_{ac} + \sigma_s \), where \( \sigma_{ac} \ll \lambda \), \( \sigma_s \perp \lambda \), and \( \text{supp } \sigma_s \) is not a semi-Kronecker (see [17]).

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