Fermions
in Geodesic Witten Diagrams

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Outline

1. Introduction

2. Detail of our result
Conformal field theory in theoretical physics
e.g. String theory, Critical phenomena, RG flow, …

Recent progress of CFT
(They are not directly related to our paper.)
Numerical conformal bootstrap
[Rattazzi, Rychkov, Tonni, Vichi, 2008], …
[El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi, 2012], …

Out of time order correlation function in CFT
[Fitzpatrick, Kaplan, Walters, 2014], [Roberts, Stanford, 2014], …
Conformal partial wave $W_{\Delta,l}(x_i)$

Basis of CFT’s correlation function

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4) \rangle = \sum_{\mathcal{O}} C_{12}\mathcal{O}C_{34}\mathcal{O}W_{\Delta,l}(x_i)$$

It depends on the theory. It doesn’t depend on the theory.

Conformal partial wave is important for the recent progress of CFT.
Q What is the gravity dual of conformal partial wave?

A 4-point geodesic Witten diagram (GWD)

Diagram for amplitude in AdS spacetime which is integrated over geodesics

[Q. Hijano, P. Kraus, E. Perlmutter, R. Sniverly, 2015]
Our purpose

Generalization to fermion fields

Our motivation

- Conformal partial wave for CFT with fermion

- Towards super conformal partial wave

[Refs: E. Hijano, P. Kraus, E. Perlmutter, R. Sniverly, 2015, ...]

our target

Spinor propagator
Our result
Geodesic Witten diagrams
with spinor fields
in odd dimensional AdS

- Embedding formalism
  for spinor propagators
  in odd dimensional AdS

- Checking that 4-point GWD
  with fermion exchange
  satisfies the conformal Casimir
  equation for conformal partial wave

- GWD expansion
  of fermion exchange 4-point Witten diagram
Outline

1. Introduction

2. Detail of our result
Embedding formalism

[Dirac, 1936], …, [Costa, Penedones, Poland, Rychkov, 2011], …

In the embedding formalism, d+1-dim bulk and d-dim boundary coordinates can be described by d+2-dim Minkowski spacetime.

Bulk coordinate

\[ X^A = \frac{1}{z} (1, z^2 + x^2, x^a) \]

Boundary coordinate

\[ P^A = (1, x^2, x^a) \]

In the embedding formalism, conformal symmetry and AdS’s isometry can be represented by simple Lorentz symmetry.
AdS spinor propagators
in the embedding formalism

We consider odd dim AdS and even dim CFT.

Spinor bulk-boundary
propagator

\[ G_{b\bar{\partial}}^{\Delta,\frac{1}{2}}(X, \bar{S}_b; P, \Sigma_{\partial}) = C_{\Delta,\frac{1}{2}} \frac{\langle \bar{S}_b \Pi_- S_{\partial} \rangle}{(-2X \cdot P)^{\Delta + \frac{1}{2}}} \]

Spinor bulk-bulk propagator

\[ G_{bb}^{\Delta,\frac{1}{2}}(X, \bar{S}_b; Y, T_b) = \langle \bar{S}_b \Pi_+ T_b \rangle \left( \frac{d}{du} G_{bb}^{\Delta,0}(u) \right) + \langle \bar{S}_b \Pi_- T_b \rangle \left( \frac{d}{du} G_{bb}^{\Delta,0}(u) \right) \]

They are consistent with the expressions without the embedding formalism in [M. Henningson, K. Sfetsos, 1998], [T. Kawano, K. Okuyama, 1999].

We construct (geodesic) Witten diagram by using these propagators.
Conformal Casimir equation and equation of AdS propagator

Conformal partial wave $W(\Delta, \Delta_i; P_i)$ satisfies the conformal Casimir equation.

$$-\frac{1}{2} (L_1 + L_2)^2 W_4(\Delta, \Delta_i; P_i) = C_{\Delta, s} W_4(\Delta, \Delta_i; P_i)$$

In the view point of GWD, the conformal Casimir equation is related to the equation of the bulk-bulk propagator $(X \neq Y)$.

$$[\left(\Gamma^A \nabla_A\right)^2 - m^2] \Psi(X) = \left(-\frac{1}{2} L^{AB} L_{AB} - C_{\Delta, \frac{1}{2}}\right) \Psi(X) = 0$$

$$m = \Delta - \frac{d}{2}$$
Amplitude of 4-point GWD with fermion exchange

$$\mathcal{W}_4(\Delta, \Delta_i) = \int_{-\infty}^{\infty} d\lambda F_\Delta[P_1, P_2, Y(\lambda), \bar{S}_{1\partial}, T_b]$$

$$\times (\partial_{T_b} (1 + Y(\lambda)) \partial_{\bar{T}_b}) G_{b\partial}^{\Delta_4, \frac{1}{2}}(Y(\lambda), \bar{T}_b; P_4, S_{4\partial}) G_{b\partial}^{\Delta_3, 0}(Y(\lambda); P_3)$$

$$F_\Delta[P_1, P_2, Y, \bar{S}_{1\partial}, T_b]$$

$$:= \int_{-\infty}^{\infty} d\lambda \bar{G}_{b\partial}^{\Delta_1, \frac{1}{2}}(X(\lambda), S_b; P_1, \bar{S}_{1\partial}) G_{b\partial}^{\Delta_2, 0}(X(\lambda); P_2) (\partial_S (1 + X(\lambda)) \partial_{\bar{S}}) G_{bb}^{\Delta, \frac{1}{2}}(X(\lambda), \bar{S}_b; Y, T_b)$$

Bulk-boundary spinor propagator $G_{b\partial}^{\Delta_1, \frac{1}{2}}$

Bulk-bulk spinor propagator $G_{bb}^{\Delta, \frac{1}{2}}$

Bulk-boundary scalar propagator $G_{b\partial}^{\Delta_2, 0}$

Integration over the geodesics

$F_\Delta$ is invariant under the rotation.

$$(L_1 + L_2) F_\Delta = F_\Delta \bar{L}_Y$$
4-point GWD satisfies the conformal Casimir equation.

Amplitude of 4-point GWD with fermion exchange

\[ W_4(\Delta, \Delta_i) = \int_{-\infty}^{\infty} d\lambda F_\Delta[P_1, P_2, Y(\lambda), \bar{S}_{1\partial}, T_b](\bar{\partial}_{T_b} (1 + Y(\lambda)) \bar{\partial}_{\bar{T}_b}) G_{b\bar{b}}^{\Delta_4, \frac{i}{2}}(Y(\lambda), \bar{T}_b; P_4, S_{4\partial}) G_{b\bar{b}}^{\Delta_3, 0}(Y(\lambda); P_3) \]

\[ F_\Delta[P_1, P_2, Y, \bar{S}_{1\partial}, T_b] \]

\[ := \int_{-\infty}^{\infty} d\lambda G_{b\bar{b}}^{\Delta_1, \frac{i}{2}}(X(\lambda), S_b; P_1, \bar{S}_{1\partial}) G_{b\bar{b}}^{\Delta_2, 0}(X(\lambda); P_2)(\bar{\partial}_{S_b} (1 + X(\lambda)) \bar{\partial}_{\bar{S}_b}) G_{b\bar{b}}^{\Delta_4, \frac{i}{2}}(X(\lambda), \bar{S}_b; Y, T_b) \]

Because of the rotation invariance of \( F_\Delta \) and the equation of the bulk-bulk propagator, the 4-point GWD satisfies the conformal Casimir equation.

\[ -\frac{1}{2}(L_1 + L_2)^2 W_4(\Delta, \Delta_i) = C_\Delta, \frac{1}{2} W_4(\Delta, \Delta_i) \]
Ratio between 3-point GWD and Witten diagram

Geodesic Witten diagram $\mathcal{W}_3$ (Integration over the geodesic)

Witten diagram $A_3$ (Integration over the whole AdS)

The ratio is useful for GWD expansion of the Witten diagram.

$$\frac{A_3}{\mathcal{W}_3(\gamma_{31})} = \pi^{d/2} \Gamma \left( \frac{1}{2} (-d + \Delta_1 + \Delta_2 + \Delta_3 + 1) \right) \frac{\Gamma \left( \frac{1}{2} (\Delta_3 + \Delta_1 - \Delta_2) \right)}{\Gamma(\Delta_3) \Gamma(\Delta_1 + 1/2)}$$
4-point Witten diagram is expressed as the product of two 3-point diagrams.

\[
\Delta_1 \bigtriangleup \Delta_0 \bigtriangleup \Delta_2 \bigtriangleup \Delta_3
\]

\[
\Delta_1 \bigtriangleup \Delta_0 \bigtriangleup \Delta_2 \bigtriangleup \Delta_3 = \frac{i}{2\pi} \int \frac{d\nu}{\nu + i(\Delta_0 - d/2)} \int_{\partial \text{AdS}_{d+1}} [dP]
\]

\[
= \frac{i}{2\pi} \int \frac{d\nu (A_3/W_3)(A'_3/W'_3)}{\nu + i(\Delta_0 - d/2)} \int_{\partial \text{AdS}_{d+1}} [dP]
\]
GWD expansion
of fermion exchange 4-point Witten diagram

$\Delta_1 \quad \Delta_4$
$\Delta_0 \quad \Delta_3$

$= \frac{i}{2\pi} \int d\nu \frac{(A_3/W_3)(A'_3/W'_3)}{\nu + i(\Delta_0 - d/2)}$

Poles of $\frac{(A_3/W_3)(A'_3/W'_3)}{\nu + i(\Delta_0 - d/2)}$ determine conformal dimensions of intermediate states in GWD (conformal partial wave) expansion.

$d/2 + i\nu = \Delta_0$, (Single trace operator)

$d/2 + i\nu = \Delta_1 + \Delta_2 + 2m$, $d/2 + i\nu = \Delta_3 + \Delta_4 + 2m$, $(m = 0, 1, \ldots)$

(Double trace operators)
Summary

- We study geodesic Witten diagram with spinor fields in odd dimensional AdS.
- We show that 4-point geodesic Witten diagram with fermion exchange satisfies the conformal Casimir equation for conformal partial wave.
- We compute the ratio between 3-point GWD and Witten diagram and consider GWD expansion of the 4-point fermion exchange Witten diagram.