On $C_J$ and $C_T$ in the Gross–Neveu and $O(N)$ models

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Abstract

We apply large $N$ diagrammatic techniques for theories with double-trace interactions to the leading corrections to $C_J$, the coefficient of a conserved current two-point function, and $C_T$, the coefficient of the stress–energy tensor two-point function. We study in detail two famous conformal field theories in continuous dimensions, the scalar $O(N)$ model and the Gross–Neveu (GN) model. For the $O(N)$ model, where the answers for the leading large $N$ corrections to $C_J$ and $C_T$ were derived long ago using analytic bootstrap, we show that the diagrammatic approach reproduces them correctly. We also carry out a new perturbative test of these results using the $O(N)$ symmetric cubic scalar theory in $6 - \epsilon$ dimensions. We go on to apply the diagrammatic method to the GN model, finding explicit formulae for the leading corrections to $C_J$ and $C_T$ as a function of dimension. We check these large $N$ results using regular perturbation theory for the GN model in $2 + \epsilon$ dimensions and the Gross–Neveu–Yukawa model in $4 - \epsilon$ dimensions. For small values of $N$, we use Padé approximants based on the $4 - \epsilon$ and $2 + \epsilon$ expansions to estimate the values of $C_J$ and $C_T$ in $d = 3$. For the $O(N)$ model our estimates are close to those found using the conformal bootstrap. For the GN model, our estimates suggest that, even when $N$ is small, $C_T$ differs by no more than 2% from that in the theory of free fermions. We find that the inequality $C_T^{UV} > C_T^{JR}$ applies both to the GN and the scalar $O(N)$ models in $d = 3$.

Keywords: conformal field theory, renormalization group, large $N$ expansion

(Some figures may appear in colour only in the online journal)
1. Introduction and summary

The essential data characterizing a \(d\)-dimensional conformal field theory (CFT) includes the scaling dimensions of conformal primary operators and their operator product coefficients [1, 2]. In general, the normalizations of operators may be chosen arbitrarily; therefore, the normalizations of their two-point functions are not physical observables. Exceptions to this are provided by the conserved currents: their insertions into correlations functions of other operators are determined by the Ward identities which fix the normalizations of the currents. Therefore, the coefficients of the two-point functions of conserved currents are physically meaningful. The most commonly encountered ones are \(C_J\), which refers to the conserved spin-1 currents \(J^a_\mu, a = 1, \ldots, \text{dim}(G)\), associated with a global symmetry of the theory with group \(G\), and \(C_T\), which refers to the stress–energy tensor \(T_{\mu\nu}\) [3]:

\[
\langle J^a_\mu(x_1)J^b_\nu(x_2) \rangle = C_J \frac{I_{\mu\nu}(x_{12})}{(x_{12}^2)^{d-1}},
\]

\[
\langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2) \rangle = C_T \frac{I_{\mu\nu,\rho\sigma}(x_{12})}{(x_{12}^2)^d},
\]

where

\[
I_{\mu\nu}(x) \equiv \delta_{\mu\nu} - 2 \frac{g_{\mu\nu} x^\rho x^\sigma}{x^2},
\]

\[
I_{\mu\nu,\rho\sigma}(x) \equiv \frac{1}{2} (I_{\mu\lambda}(x)I_{\rho\sigma}(x) + I_{\mu\rho}(x)I_{\nu\sigma}(x)) - \frac{1}{d} \delta_{\mu\nu} \delta_{\rho\sigma}.
\]

These quantities have various applications: \(C_J\) determines the universal charge or spin conductivity \([4, 5]\); \(C_T\) appears in many contexts, including some properties of the Rényi and entanglement entropies \([6, 7]\). For example, \(C_T\) determines the leading response of the entanglement entropy across a sphere to small variations in its shape \([7]\); in particular, in \(d = 3\) it determines its limiting behavior for entangling contours with cusps \([8]\). \(C_T\) is also one of the natural measures of the number of degrees of freedom, and in two dimensions it satisfies the famous Zamolodchikov theorem \([9]\). In higher dimensions there are counter-examples to the monotonicity of \(C_T\) \([10–12]\), but it is still interesting to study its behavior under RG flow.

A number of results about \(C_J\) and \(C_T\) are available for CFTs in \(d > 2\) \([4, 5, 11, 13, 14]\). Of special interest to us is the work by Petkou \([14]\), who used large \(N\) methods and operator product expansions to determine the leading \(1/N\) corrections to \(C_J\) and \(C_T\) for the critical scalar \(O(N)\) model with quartic interaction \((\phi \phi)^2\). Defining

\[
C_J = C_{J0} \left(1 + \frac{C_{J1}}{N} + \frac{C_{J2}}{N^2} + O(1/N^3)\right),
\]

\[
C_T = C_{T0} \left(1 + \frac{C_{T1}}{N} + \frac{C_{T2}}{N^2} + O(1/N^3)\right),
\]

Petkou found \([14]\)

\[
C_{J1}^{O(N)} = -\frac{8(d-1)}{d(d-2)} C_{J1}^{O(N)},
\]

\[
C_{T1}^{O(N)} = -2 \left(\frac{2C_{O(N)}(d)}{d+2} + \frac{d^2 + 6d - 8}{d(d^2 - 4)}\right) C_{T1}^{O(N)}.
\]
Here

\[ \gamma_1^{O(N)} = \frac{2 \Gamma(d - 2) \sin \left( \frac{\pi d}{2} \right)}{\pi \Gamma \left( \frac{d}{2} - 2 \right) \Gamma \left( \frac{d}{2} + 1 \right)} \]  

(1.7)

is the 1/N correction to the dimension of the fundamental field \( \phi' \), and

\[ C_{O(N)}(d) = \psi \left( 3 - \frac{d}{2} \right) + \psi(d - 1) - \psi(1) - \psi \left( \frac{d}{2} \right) \]  

(1.8)

where \( \psi(x) = \Gamma'(x)/\Gamma(x) \) is the digamma function. In \( d = 3, \) these results yield

\[ C_{T}^{O(N)} \big|_{d=3} = C_{T}^{O(N)} \big|_{d-3} = \frac{116}{9 \pi^2 N} + \mathcal{O}(1/N^2). \]  

(1.9)

The critical \( O(N) \) model with the quartic interaction \( (\phi' \phi')^2 \) is weakly coupled in \( 4 - \varepsilon \) dimensions [15], and the results (1.5) and (1.6) agree with the \( \varepsilon \) expansions found from conventional perturbation theory [11, 16]. In recent works [12, 17, 18] it was shown that, for sufficiently large \( N \), the \( O(N) \) model has another weakly coupled description in \( 6 - \varepsilon \) dimensions. It involves an additional scalar field \( \sigma \) with the action

\[ \int d^dx \left( \frac{1}{2} (\partial \phi')^2 + \frac{1}{2} (\partial \sigma)^2 + \frac{1}{2} g_{1} \phi' \phi' + \frac{1}{6} g_{2} \sigma^3 \right). \]  

(1.10)

In section 3 we will use this cubic \( O(N) \) symmetric theory to develop the \( 6 - \varepsilon \) expansion of \( C_T \) and \( C_F \), providing additional checks of the large \( N \) results (1.5) and (1.6). In particular, for \( d = 6 \) the large \( N \) result (1.6) yields [12]

\[ C_{T}^{O(N)} \big|_{d=6} = 1, \]  

(1.11)

which precisely reproduces the contribution of a 6d canonical scalar field. More generally, in even dimensions \( d \), generalizing the arguments leading to (1.10), we expect to find a (non-unitary) free theory of \( N \) canonical scalars \( \phi' \) and a \( \Delta = 2 \) scalar with local kinetic term \( \sim \sigma (\partial^2 \phi')^2 \sigma \). For instance, for \( d = 8 \) this was recently discussed in [19]. Here

\[ C_{T}^{O(N)} \big|_{d=8} = -4. \]  

(1.12)

This implies that the ratio of the \( C_T \) of a free 4-derivative scalar to that of a canonical scalar is \(-4\). The value of \( C_{T}^{O(N)} \) for general even \( d \) is given in [20] and in equation (3.54).

In section 4 we will derive formulae for \( C_T \) and \( C_F \) in the \( d \)-dimensional Gross–Neveu (GN) model [21], which has the action

\[ S_{GN} = -\int d^d x \left( \bar{\psi_i} \gamma^\mu \partial_\mu \psi^i + \frac{g}{2} (\bar{\psi}_i \psi^i)^2 \right). \]  

(1.13)

We will take \( \psi^i \) with \( i = 1, 2, \ldots, \tilde{N} \) to be a collection of \( \tilde{N} \) Dirac fermions, and we will denote \( \tilde{N} = N - 1 \). Where \( \text{Tr} \) is the trace of the identity operator on the vector space on which the Dirac matrices act. Since this factor can be absorbed into the expansion parameter \( N \), one may keep it arbitrary in intermediate steps of the calculation, and set it to the desired value at the end. For instance, for the case of \( \tilde{N} \) 2-component Dirac fermions in \( d = 3 \), one should take \( \text{Tr} = 2 \), i.e. \( N = 2\tilde{N} \). In \( 2 \leq d \leq 4 \), it is natural to take \( \psi^i \) to be 4-component fermions, i.e. \( N = \tilde{N} \text{Tr} = 4\tilde{N} \). This allows us to smoothly connect to the GNY model in
\( d = 4 - \epsilon \) described below. The 4-component fermion notation also appears naturally in \( d = 3 \) in the condensed matter applications of models involving fermions, see for instance [22–26].

The perturbing operator \( O(x) = \frac{1}{2}(\bar{\psi}i\psi)^2 \) in (1.13) has dimension \( \Delta = 2(d - 1) \) in the free theory. In \( d = 2 \) the GN model is asymptotically free, while for \( d > 2 \) it is free in the IR and has an interacting UV fixed point (it is unitary for \( 2 < d < 4 \)). For this interacting CFT we will find, after lengthy calculations \(^3\)

\[
C_{j1}^{\text{GN}} = \frac{8(d-1)}{d(d-2)} \gamma_1^{\text{GN}},
\]

\[
C_{T1}^{\text{GN}} = -4\gamma_1^{\text{GN}} \left( \frac{C_{\text{GN}}(d)}{d+2} + \frac{(d-2)}{d(d+2)(d-1)} \right),
\]

where

\[
\gamma_1^{\text{GN}} = \frac{\Gamma(d-1)(d-2)^2}{4\Gamma(2-d)^2 \Gamma(d+1) \Gamma(d+1/2)^2}.
\]

is the \( 1/N \) correction to the dimension of the fundamental fermion field \( \psi \), and

\[
C_{\text{GN}}(d) = \psi \left( 2 - \frac{d}{2} \right) + \psi(d-1) - \psi(1) - \psi \left( \frac{d}{2} \right).
\]

In \( d = 3 \), we find

\[
C_{j1}^{\text{GN}|_{d=3}} = C_{j0}^{\text{GN}} \left( 1 - \frac{64}{9\pi^2N} + \mathcal{O}(1/N^2) \right),
\]

\[
C_{T1}^{\text{GN}|_{d=3}} = C_{T0}^{\text{GN}} \left( 1 + \frac{8}{9\pi^2N} + \mathcal{O}(1/N^2) \right).
\]

We will derive these results using a large \( N \) diagrammatic approach similar to that used in [4, 5, 33–36] (for a review, see [37]). We will also use the diagrammatic method to rederive the formulae (1.5) and (1.6) for the scalar \( O(N) \) model, finding complete agreement with the bootstrap method of [14]; these calculations are presented in section 3.3. The diagrammatic approach has also been used to calculate \( C_{j1} \) and \( C_{T1} \) in 3-dimensional QED [4, 5]. The paper [38], which is a follow-up to the present one, uses the diagrammatic approach to calculate the \( C_{j1} \) and \( C_{T1} \) in \( d \)-dimensional conformal QED and compare the results with the \( \epsilon \) expansions. An important feature of the diagrammatic approach, which we will uncover, is the necessity of a divergent multiplicative ‘renormalization’ \( Z_T \) for the stress–energy tensor (for the conserved current such a renormalization is not needed). Despite this renormalization, the anomalous dimension of the stress-tensor is, of course, exactly zero.

The interacting GN CFT has different perturbative \( \epsilon \) expansions near 2 and 4 dimensions. In \( 2 + \epsilon \) dimensions, where the theory has a weakly coupled UV fixed point, it involves the original GN formulation (1.13) with the quartic interaction. There is an alternate, Gross–Neveu–Yukawa (GNY) formulation of the theory [39, 40] which contains an additional real scalar field \( \sigma \) with a Yukawa coupling to the \( \bar{N} \) Dirac fermions:

\(^3\) Besides their intrinsic interest, formulae (1.5), (1.6), (1.14) and (1.15) may have applications to the higher-spin AdS/CFT dualities which relate the \( d \)-dimensional \( O(N) \) [27] or GN models [28, 29] to Vasiliev theories [30, 31] in AdS\(_{d+1}\) (for a review, see [32]).
This theory, which may be regarded as the UV completion of the GN model, has a weakly coupled IR fixed point in $d = 4 - \epsilon$. Using these tools, we develop the $2 + \epsilon$ and $4 - \epsilon$ expansions of $C_T$ and $C_J$ for the GN. In the large $N$ limit these expansions agree with (1.14) and (1.15), providing their important perturbative checks. In particular, we see that for $d = 4$, the large $N$ result (1.15) yields

$$C_{T1}^{GN}|_{d=4} = \frac{2}{3},$$

(1.21)

which precisely reproduces the contribution of a 4d free scalar field. More generally, in even dimensions $d$, generalizing the arguments leading to (1.20), we expect to find a (non-unitary) free theory of $N$ Dirac fermions and a free scalar with $\Delta = 1$ and local kinetic term $\sim (\partial^2 \tilde{h})^{-1} \sigma$. For instance, in $d = 6$ we find

$$C_{T1}^{GN}|_{d=6} = -2,$$

(1.22)

which implies that $C_T = -6/S_6^2$ for the 4-derivative scalar field in $d = 6$ (in units where $C_T = 6/(5S_6^2)$ for the ordinary 2-derivative scalar). The ratio of the $C_T$ of a free $(d-2)$-derivative scalar to that of a canonical scalar is given in all even dimensions in equation (4.28). Interestingly, it is always an integer.

Using the $2 + \epsilon$ and $4 - \epsilon$ expansions, in section 4.5 we carry out two-sided Padé extrapolations and find estimates for $C_T$ and $C_J$ in $d = 3$ for small values of $N$. The values of $C_T$ we find are typically just $1\%–2\%$ above those for the theory of free fermions. Our estimates suggest that, as the $d = 3$ theory flows from the interacting GN fixed point to the free fermion theory, $C_T$ decreases for all $N$. There is a supersymmetric counter-example to the $d = 3$ ‘$C_T$-theorem’ [10], but we find that the inequality $C_{T}^{2N} > C_{T}^{IR}$ applies both to the GN and the scalar $O(N)$ models in $d = 3$. However, as we discuss in section 4.2, for the GN model with large $N$ it is violated for $2 < d \lesssim 2.3$.

2. Change of $C_J$ and $C_T$ under double-trace perturbations

In this section we work out the general structure of the change in the $C_J$ and $C_T$ coefficients under RG flows in large $N$ theories, which are induced by double-trace operators $O^2$. Both the critical scalar and the GN model are of this type, and in later sections we will carry out specific calculations for these models.

Before proceeding, let us introduce some useful notation that we will use in the rest of the paper. To deal efficiently with the tensor structures in stress–energy tensor and current correlators, it is convenient to introduce an auxiliary null vector $z^\mu$, satisfying

$$z^2 = z^\mu z^\nu \delta_{\mu\nu} = 0.$$  

(2.1)

We work in flat $d$-dimensional Euclidean space, so such a null vector is complex, but we will never need to specify an explicit form of $z^\mu$. It is convenient to define the stress–energy tensor and current projected onto the auxiliary null vector

$^4$ Recall that in dimension $d$ a free scalar has $C_T^d = \frac{d}{d-10}$ and a free fermion $C_T^{2d} = \text{Tr} \frac{d}{d-10}$ [3]. In $d = 4$, we then have $C_T^d / (\sqrt{2} C_T^{2d}) = \frac{2}{5\pi}$. 

5
From (1.2), we see that the two-point functions of $T$ and $J$ take the simple form

$$
\langle T(x)T(0) \rangle = \frac{4C_T}{(x^2)^{d}} x^4,
$$

$$
\langle J^a(x)J^b(0) \rangle = \delta^{ab} - 2C_J \frac{x^2}{(x^2)^{2-d}}.
$$

(2.3)

where we have introduced the notation $x_c \equiv \zeta^\mu x^\mu$. Using the Fourier transform

$$
\int \frac{d^dp}{(2\pi)^d} e^{-ipx} \frac{1}{(p^2)^{\frac{d}{2}}} = \frac{\Gamma\left(\frac{d}{2} - \alpha\right)}{4^\alpha \pi^\frac{d}{2}} \frac{1}{(x^2)^{\frac{d}{2} - \alpha}},
$$

(2.4)

$$
\int d^d\chi e^{-ip\chi} (\chi^a)^{d} = \frac{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2} - \alpha\right)}{4^\alpha \pi^\frac{d}{2}} \frac{1}{(p^2)^{\frac{d}{2} - \alpha}},
$$

(2.5)

we find in momentum space

$$
\langle T_{\mu\nu}(p)T_{\lambda\rho}(-p) \rangle = C_T \frac{\pi^2 \Gamma\left(1 - \frac{d}{2}\right)}{2^{d-2} \Gamma(d+2)} (p^2)^{\frac{d}{2}} \delta_{\mu\nu,\lambda\rho}(p),
$$

$$
\langle J^a_{\mu}(p)J^b_{\nu}(-p) \rangle = -C_J \frac{\pi^2 \Gamma\left(2 - \frac{d}{2}\right)}{2^{d-1} \Gamma(d)} (p^2)^{\frac{d}{2} - 1} \Pi_{\mu\nu}(p) \delta^{ab},
$$

(2.6)

where $\Pi_{\mu\nu}(p) = \delta_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2}$ and

$$
\delta_{\mu\nu,\lambda\rho}(p) \equiv \frac{1}{2} \Pi_{\mu\nu}(p) \Pi_{\lambda\rho}(p) - \frac{d-1}{4} (\Pi_{\mu\lambda}(p) \Pi_{\nu\rho}(p) + \Pi_{\mu\rho}(p) \Pi_{\nu\lambda}(p)).
$$

(2.7)

Therefore

$$
\langle T(p)T(-p) \rangle = C_T \frac{\pi^2 \Gamma\left(2 - \frac{d}{2}\right)}{2^{d-2} \Gamma(d+2)} \frac{p^4}{(p^2)^{2 - \frac{d}{2}}},
$$

$$
\langle J^a(p)J^b(-p) \rangle = C_J \frac{\pi^2 \Gamma\left(2 - \frac{d}{2}\right)}{2^{d-1} \Gamma(d)} \frac{p^2}{(p^2)^{2 - \frac{d}{2}} \delta^{ab}},
$$

(2.8)

where $p^2 \equiv \zeta^\mu p^\mu$.

Let us consider a general CFT$_0$ in $d$ Euclidean dimensions, and assume that it admits a large $N$ expansion with the usual properties. Given a single trace operator $O(x)$ of dimension $D$, we can consider the double-trace deformation

$$
S_h = S_{\text{CFT}_0} + \lambda \int d^d\chi O(x)^2.
$$

(2.9)

When $\Delta_D < d/2$, the deformation is relevant and there is a RG flow from CFT$_0$ to a new CFT where $\Delta_{D}^{\text{IR}} = d - \Delta_D + O(1/N)$ [41, 42]. When $\Delta_D > d/2$, the deformation is irrelevant, but one may show that there is a large $N$ UV fixed point, where $\Delta_{D}^{\text{UV}} = d - \Delta_D + O(1/N)$, and the RG flow leads to CFT$_0$ in the IR. A well-known example of the IR fixed point is the scalar $O(N)$ model, i.e. the theory of $N$ massless scalar fields $\phi^i$ perturbed by the $(\phi^i\phi^i)^2$ operator; we will discuss the calculation of $C_T$ in this theory in section 3. A well-known example of the UV fixed point is the GN model (1.13); it will be
discussed in section 4. To be definite when writing powers of $N$, we will assume below that
the unperturbed CFT$_0$ is vector-like, i.e. $C_O \sim N$ and $\langle TT\rangle_0 \sim N$.

The $1/N$ expansion in the perturbed CFT may be developed with the aid of a Hubbard–Stratonovich auxiliary field. We may rewrite the perturbed action as

$$S_\lambda = S_{\text{CFT}_0} + \int d^4 \chi \sigma O - \frac{1}{4\lambda} \int d^4 \chi \sigma^2.$$ (2.10)

The equation of motion of $\sigma$ imposes $\sigma = 2\lambda O$ and leads to the original action. However, by performing the path integral in the CFT$_0$, one may derive an effective action for $\sigma$. At large $N$, we have

$$\left\langle e^{-\int d^4 \chi \sigma O}\right\rangle_0 \approx e^{\int d^4 \chi d^4 y \sigma(x)\sigma(y)\langle O(x)O(y)\rangle_0 + O(\sigma^3)},$$ (2.11)

so the quadratic term in the $\sigma$ effective action is

$$S[\sigma] = -\frac{1}{2} \int d^4 x d^4 y \sigma(x)\sigma(y)\langle O(x)O(y)\rangle_0 - \frac{1}{4\lambda} \int d^4 \chi \sigma^2,$$ (2.12)

$$= -\frac{1}{2} \int \frac{d^dp}{(2\pi)^d} \sigma(p)\sigma(-p) \left(C_O \frac{(4\pi)^{d/2} \Gamma(d/2 - \Delta_O)}{4 \Delta_O \Gamma(\Delta_O)} \langle O^2\rangle_{\Delta_O - d/2} + \frac{1}{2\lambda}\right),$$ (2.13)

where we have used

$$\langle O(x)O(y)\rangle_0 = \frac{C_O}{|x - y|^{2\Delta_O}} = C_O \frac{(4\pi)^{d/2} \Gamma(d/2 - \Delta_O)}{4 \Delta_O \Gamma(\Delta_O)} \int \frac{d^dp}{(2\pi)^d} e^{ip(x-y)} \langle p^2\rangle_{\Delta_O - d/2}.$$ (2.14)

When $\Delta_O < d/2$, we see that the second term in (2.13) can be dropped in the IR limit (and when $\Delta_O > d/2$, it can be dropped in the UV limit), and so at the perturbed fixed point we get the two-point function of $\sigma$, at leading order in $1/N$, to be

$$G_\sigma(p) = \langle \sigma(p)\sigma(-p) \rangle = -\frac{4\Delta_O \Gamma(\Delta_O)}{C_O (4\pi)^{d/2} \Gamma(d/2 - \Delta_O)} \langle p^2\rangle_{\Delta_O - d/2} \equiv \tilde{C}_\sigma \langle p^2\rangle_{\Delta_O - d/2}$$ (2.15)

or, in coordinate space

$$G_\sigma(x, y) = \frac{(d/2 - \Delta_O) \sin((d/2 - \Delta_O)\pi) \Gamma(d - \Delta_O) \Gamma(\Delta_O)}{\pi^{d+1} C_O |x - y|^{2(d - \Delta_O)}} \equiv \frac{C_\sigma}{|x - y|^{2(d - \Delta_O)}}.$$ (2.16)

This shows that the scalar operator $\sigma \sim O$ now has dimension $d - \Delta_O + \mathcal{O}(1/N)$. At the perturbed fixed point, we may hence omit the last term in (2.10) and work with the action

$$S_{\text{crit}} = S_{\text{CFT}_0} + \int d^4 \chi \sigma O.$$ (2.17)

A $1/N$ diagrammatic expansion can be obtained using this action and the effective $\sigma$ propagator (2.16) (with the prescription that the planar bubble diagrams contributing to $\langle \sigma^2 \rangle$ should not be included as they are already taken into account by the effective propagator).
The two-point function of the stress–energy tensor may be then computed as

\[
\langle T(x)T(0) \rangle_{\text{crit}} = \int D\sigma \left\langle T(x)T(0)e^{-f\sigma} \right\rangle_0 \\
= \langle T(x)T(0) \rangle_0 + \frac{1}{2} \int dz_1 dz_2 G_{\sigma}(z_1, z_2) \langle T(x)T(0)O(z_1)O(z_2) \rangle_0 \\
+ \frac{1}{2} \int dz_1 dz_2 dz_3 dz_4 \langle T(x)T(0)O(z_1)G_{\sigma}(z_3, z_4)G_{\sigma}(z_2, z_4)O(z_2)O(z_3) \rangle_0 + \mathcal{O}(1/N),
\]

(2.18)

where to obtain the ‘Aslamazov–Larkin term’ [43] in the last line we have used the large \( N \) approximation to rewrite the 6-point function as a product of 3-point functions. Note that since \( C_0 \sim N \), both of the contributions above are of order \( N^0 \). By conformal invariance, we may write

\[
\frac{1}{2} \int dz_1 dz_2 G_{\sigma}(z_1, z_2) \langle T(x)T(0)O(z_1)O(z_2) \rangle_0 = I_{\langle TTDO \rangle} \left( \frac{x_\sigma}{\langle x^2 \rangle^{d+2}} \right),
\]

\[
\frac{1}{2} \int dz_1 \cdots dz_4 G_{\sigma}(z_1, z_3)G_{\sigma}(z_2, z_4) \langle T(x)O(z_1)O(z_2) \rangle_0 \times \langle T(0)O(z_3)O(z_4) \rangle_0 = I_{\langle TOO \rangle^2} \left( \frac{x_\sigma}{\langle x^2 \rangle^{d+2}} \right)
\]

and so

\[
\langle T(x)T(0) \rangle_{\text{crit}} = (4C_T + I_{\langle TTDO \rangle} + I_{\langle TOO \rangle^2} + \mathcal{O}(1/N)) \left( \frac{x_\sigma}{\langle x^2 \rangle^{d+2}} \right).
\]

(2.20)

Thus, we see that the change in \( C_T \) to leading order in \( 1/N \) receives contributions from both integrated 4-point and 3-point functions in the unperturbed CFT. While \( \langle TOO \rangle \) has a universal form that only depends on \( \Delta_\sigma \) due to the conformal Ward identity, the 4-point function \( \langle TOO \rangle \) does not have a universal form. Therefore, unlike the sphere free energy [42, 44], we do not expect a simple universal formula for the change in \( C_T \) that only depends on the dimension of the perturbing operator.

So far we have ignored the issues of regularization, but in fact the result (2.20) by itself is not well-defined, since the contributions \( I_{\langle TTDO \rangle} \) and \( I_{\langle TOO \rangle^2} \) are divergent and require regularization. The usual dimensional continuation does not work in this case, because the vertex in (2.17) is critical for all \( d \) within the \( 1/N \) expansion. One may use a simple momentum cutoff, however this makes the integrals hard to compute in general \( d \). A regulator that is often employed, and which we will use in the paper, is to formally shift the dimension of \( \sigma \) by a small parameter \( \Delta \) that is taken to zero at the end of the calculation [33–35, 45]. Explicitly, we take the propagator in the regularized theory to be

\[
G_{\sigma}(p) = \hat{\sigma}(p^2)^{d/2-\Delta_\sigma-\Delta}, \quad \Delta \to 0.
\]

(2.21)

This makes the vertex dimensionful, \( S_{\text{vertex}} = \mu^2 \int \sigma O \), where we introduced an arbitrary renormalization scale \( \mu \) to compensate dimensions. Then, the integrals (2.19) in the regularized theory take the form

\( 5 \) The regulator parameter \( \Delta \), which is sent to zero at the end, should not be confused with the scaling dimension.
\[ I_{(TTOO)} = (x^2 \mu^2)^\Delta \left( \frac{1}{\Delta} I_{(TTOO)}^{(1)} + I_{(TTOO)}^{(0)} + \mathcal{O}(\Delta) \right), \]
\[ I_{(TDOO)} = (x^2 \mu^2)^{2\Delta} \left( \frac{1}{\Delta} I_{(TDOO)}^{(1)} + I_{(TDOO)}^{(0)} + \mathcal{O}(\Delta) \right). \quad (2.22) \]

Importantly, we see that the two contributions carry a different power of the renormalization scale, since they involve two and four vertices respectively. Then, we find
\[ I_{(TTOO)} + I_{(TDOO)} = \frac{1}{\Delta} (I_{(TTOO)}^{(1)} + I_{(TDOO)}^{(1)}) \]
\[ + \log(\mu^2 x^2) (I_{(TTOO)}^{(1)} + 2 I_{(TDOO)}^{(1)}) + I_{(TTOO)}^{(0)} + I_{(TDOO)}^{(0)} + \mathcal{O}(\Delta). \quad (2.23) \]

Absence of an anomalous dimension for \( T \) requires \( I_{(TTOO)}^{(1)} + 2 I_{(TDOO)}^{(1)} = 0 \), so that the logarithmic term vanishes. We will see in the explicit examples below that this is indeed the case, as expected. However, we see that the \( 1/\Delta \) pole cannot cancel by itself, since it involves a different combination of the coefficients (unless both contributions are finite by themselves, but in all examples we studied, this does not appear to be the case). A resolution of this issue is to allow for a divergent ‘\( Z \)-factor’ renormalization of the stress tensor so that the poles are cancelled
\[ T_{\text{ren}}(x) = Z_T \ T(x), \quad Z_T = 1 + \frac{1}{N} \left( \frac{Z_{T1}}{\Delta} + Z_{T1}' + \mathcal{O}(\Delta) \right) + \mathcal{O}(1/N^2). \quad (2.24) \]

The pole coefficient \( Z_{T1} \) is fixed by cancellation of the \( 1/\Delta \) divergence in (2.23). In addition, we will find that a non-trivial finite shift \( Z_{T1}' \) is required in order for the conformal Ward identity to hold. This peculiar stress tensor ‘renormalization’ is presumably due to the unusual features of the regularized \( 1/N \) perturbation theory, at least within the regularization scheme we employ. Putting everything together, one arrives at the following final answer for the shift in \( C_T \) to leading order at large \( N \) (recall that \( C_{TO} \sim N \)):
\[ C_T = C_{T0} + \frac{1}{4} \left( I_{(TTOO)}^{(0)} + I_{(TDOO)}^{(0)} + \frac{8}{N} C_{T0} Z_{T1}' \right) + \mathcal{O}(1/N). \quad (2.25) \]

As we will see below, the shift proportional to \( Z_{T1}' \) is essential for reproducing the result of [46] for the scalar \( O(N) \) model, and also for matching the \( 4 - \epsilon \) and \( 2 + \epsilon \) expansions for the GN model.

One may study in a similar way the current two point function \( \langle JJ \rangle \). Assuming for simplicity that the perturbing operator is neutral under the symmetry generated by \( J \), following analogous steps as above, one ends up with
\[ \langle J^a(x) J^b(0) \rangle_{\text{crit}} = \int D\sigma \langle J^a(x) J^b(0) \rangle e^{-\sigma \mathcal{O}} |_0 
\[ = \langle J^a(x) J^b(0) \rangle |_0 + \frac{\mu^2 \Delta}{2} \int d^2z_1 d^2z_2 G_{\sigma}(z_1, z_2) \]
\[ \times \langle J^a(x) J^b(0) O(z_1) O(z_2) \rangle |_0 + \mathcal{O}(1/N). \quad (2.26) \]

This yields
\[ \langle J^a(x) J^b(0) \rangle_{\text{crit}} = \delta^{ab} \left(-2 C_{J0} + (x^2 \mu^2)^\Delta \left( \frac{1}{\Delta} I_{(TDOO)}^{(1)} + I_{(TDOO)}^{(0)} + \mathcal{O}(\Delta) \right)\right)(x_2)^2 \quad (2.27) \]

In this case, since the only contribution is given by the integrated 4-point function, the absence of the anomalous dimension of \( J \) requires that \( I_{(TDOO)}^{(1)} = 0 \). Therefore, no ‘\( Z \)-factor’ is needed, at least to this order in the \( 1/N \) expansion (examining the Ward identities for \( J \), we
will find that a finite shift analogous to the one in (2.24) is not needed either\(^6\). Then, the final result is

\[
C_f = C_{f0} - \frac{1}{2} I_{f(00)}^{(1)} + O(1/N).
\]  

(2.28)

3. Scalar \(O(N)\) model

3.1. Scalar with cubic interaction in \(6 - \epsilon\) dimensions

In this section, we will consider a theory of \(N\) scalar fields \(\phi^i\) transforming under an internal \(O(N)\) symmetry group and a scalar \(\sigma\) in \(6 - \epsilon\) dimensions described by the action (1.10). Dimensional analysis implies that the interactions are relevant for \(d < 6\), so we expect that there should exist a non-trivial infrared fixed point. We are interested in the case where \(d = 6 - \epsilon\). For small \(\epsilon\) and sufficiently large \(N\), this fixed point indeed exists, and the coupling constants at that fixed point have been computed to \(\epsilon^3\) order by [12, 17, 18]. The answer they obtained at leading \(\epsilon\)-order was:

\[
g_{1*} = \sqrt{\frac{6\epsilon (4\pi)^3}{(N - 44)\xi(N)^2 + 1}} \xi(N), \quad g_{2*} = \sqrt{\frac{6\epsilon (4\pi)^3}{(N - 44)\xi(N)^2 + 1}} (1 + 6\xi(N)),
\]

(3.1)

where \(\xi(N)\) is the solution to the cubic equation

\[
840\xi^3 - (N - 464)\xi^2 + 84\xi + 5 = 0,
\]

(3.2)

which asymptotically tends to \(\xi = N/(840) + \cdots\) at large\(^7\) \(N\). Such a solution exists for \(N > 1038\) [12].

The solution for the fixed point couplings (3.1) is valid for finite \(N\), but its explicit form is somewhat cumbersome. Expanding in powers of \(1/N\), one gets:

\[
g_{1*} = \sqrt{\frac{6\epsilon (4\pi)^3}{N}} \left(1 + \frac{22}{N} + \frac{726}{N^2} + \cdots\right),
\]

(3.3)

\[
g_{2*} = \sqrt{\frac{6\epsilon (4\pi)^3}{N}} \left(1 + \frac{162}{N} + \frac{68766}{N^2} + \cdots\right).
\]

(3.4)

Our goal is to compute the two-point function of the stress-energy tensor and of a conserved spin-1 current at order \(\epsilon\), and in particular compare with the large \(N\) results (1.5) and (1.6) obtained in [46].

The spin-1 current corresponding to the global \(O(N)\) symmetry of the model is given by

\[
J^a_\mu(x) = \phi^i t^a_{ij} \partial_\mu \phi^j.
\]

(3.5)

Here, the matrices \(t^a\) are the generators of the internal \(O(N)\) symmetry group. Since the two point function of this current is proportional to \(\delta_{ab}\), we may as well pick a convenient generator. We will choose:

\(^6\) One may study a different model where double-trace perturbations include the product \(OO^*\) of an operator that is charged under the symmetry associated to \(J\) and its conjugate. In this case, an Aslamazov–Larkin contribution will be present, and one will need a ‘\(Z_J\) factor’ analogous to the \(Z_T\) discussed above.

\(^7\) The other roots correspond to fixed points with unstable directions (in the RG sense) that are not related to the \(O(N)\) theory with \(\xi(\phi,\bar{\phi})^2\) interaction.
To the first non-trivial order in the $\epsilon$-expansion, we find

$$J(p) = J_{(+)}(p) + J_{(-)}(p) + \mathcal{O}(\epsilon^2),$$

where the necessary diagrams are shown in figure 1. The solid lines here denote the propagators, the dotted line the $\sigma$ propagators, and the arrows here simply denote the flow of momentum. The explicit integrands for $D_0, D_1, D_2$ and the result of the integrations are given in appendix E. After Fourier transforming to position space and dividing by the free field contribution, we obtain the result

$$\langle J(p) J(-p) \rangle = D_0 + D_1 + D_2 + \mathcal{O}(\epsilon^2),$$

where in the second step we have substituted the large $N$ expansion (3.3) of the critical coupling. One may check that this precisely agrees with the $6 - \epsilon$ expansion (3.46) of the large $N$ result (1.5) obtained in [46].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram1.pdf}
\caption{Diagrams for $C_J$ up to order $\epsilon$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram2.pdf}
\caption{Diagrams for $C_T$ up to order $\epsilon$.}
\end{figure}

It is important to divide by $D_0$ and take the $\epsilon \to 0$ after performing the Fourier transform. This is because the leading order behavior of the $\Gamma$ functions arising from the Fourier transform (which are regularized by expanding in $d = 6 - \epsilon$) are proportional to $\epsilon/2$ for the second-order diagrams $D_1$ and $D_2$, but to $\epsilon$ for the one-loop diagram $D_0$. Effectively, this results in an ‘enhancement’ of $D_1$ and $D_2$ by a factor of 2 relative to $D_0$. 

\[ J(x) = z^\mu J_\mu(x) = z^\mu(\phi^i \partial_\mu \phi^i - \phi^2 \partial_\mu \phi^4). \]
Let us now move to the calculation of CT. The stress–energy tensor may be split into its φ and σ contributions, $T = z^{\mu \nu} T_{\mu \nu} = T_\phi + T_\sigma$, where

$$T_\phi = z^{\mu \nu} \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{d} \frac{2}{d-1} \partial_\mu \partial_\nu (\phi \phi) \right),$$

$$T_\sigma = z^{\mu \nu} \left( \partial_\mu \sigma \partial_\nu \sigma - \frac{1}{d} \frac{2}{d-1} \partial_\mu \partial_\nu (\sigma^2) \right).$$

Here we have dropped terms proportional to $d_{\mu \nu}$ (including terms involving the interactions), since we work with the projected stress tensor along the null vector $z^{\mu}$.

We may write $\langle T(p) T(-p) \rangle = \langle T_\phi(p) T_\phi(-p) \rangle + 2 \langle T_\phi(p) T_\sigma(-p) \rangle + 2 \langle T_\sigma(p) T_\sigma(-p) \rangle$, and the diagrams contributing to each term are shown in figure 2. The explicit integrands and results are given in appendix E. Putting everything together, the final result is:

$$\frac{C_T^{O(N)}}{C_{T,\text{free}}} = 1 + \frac{1}{N} + \left( \frac{1}{N} \frac{D_1 + D_2}{ND_0} \right) \left( 3N g_{1*}^2 + 3N g_{2*}^2 \right)$$

$$= 1 + \frac{1}{N} + \left( \frac{7}{4608 \pi^3} + \mathcal{O}(\epsilon) \right) \left( 3N g_{1*}^2 + 3N g_{2*}^2 \right)$$

$$= 1 + \frac{1}{N} + \epsilon \left( \frac{7}{4N} - \frac{98}{N^2} - \frac{10192}{N^3} + \mathcal{O} \left( \frac{1}{N^4} \right) \right) + \mathcal{O}(\epsilon^2).$$

Again, we find that this agrees with the $6 - \epsilon$ expansion (3.53) of Petkou’s result (1.6).

### 3.2. 1/N expansion

The $1/N$ expansion of the O(N) model can be developed using the Hubbard–Stratonovich transformation, as reviewed in section 2. After introducing the Hubbard–Stratonovich auxiliary field and dropping the term quadratic in σ in the IR limit, we effectively have the following action, expressed in terms of bare fields:

$$S_{\text{crit-scal}} = \frac{1}{2} \int d^d x \left( (\partial \phi_0)^2 + \frac{1}{\sqrt{N}} \sigma_0 \phi_0 \phi_0 \right).$$

The propagator of the $\phi_0^i$ field reads

$$\langle \phi_0^i(p) \phi_0^j(-p) \rangle_0 = \delta^{ij}/p^2.$$  

After integrating over the fundamental fields $\phi_0^i$, the auxiliary field $\sigma_0$ develops a non-local kinetic term with an effective propagator

$$\langle \sigma_0(p) \sigma_0(-p) \rangle_0 = \tilde{C}_{\sigma_0}/(p^2)^{\Delta-2+\Delta},$$
where

\[ \mathcal{C}_{\sigma 0} \equiv 2^{d+1}(4\pi)^{\varepsilon/2}\Gamma\left(\frac{d-1}{2}\right)\sin\left(\frac{\pi d}{2}\right). \]  

(3.14)

and we have already introduced a regulator \( \Delta \) \([33–35, 45]\), as described in section 2. This regulator essentially works analogously to \( \theta \) in dimensional regularization, but there are some subtleties, which we will discuss in this section.

In order to cancel the divergences as \( \Delta \to 0 \) we have to renormalize the bare fields \( \phi_0 \) and \( \sigma_0 \):

\[ \phi = Z_{\phi}^{1/2} \phi_0, \quad \sigma = Z_{\sigma}^{1/2} \sigma_0, \]  

(3.15)

where \( Z_{\phi} \) and \( Z_{\sigma} \) have only poles in \( \Delta \) (using a ‘minimal subtraction’ scheme), and read

\[ Z_{\phi} = 1 + \frac{1}{N} \frac{Z_{\phi 1}}{\Delta} + \mathcal{O}(1/N^2), \quad Z_{\sigma} = 1 + \frac{1}{N} \frac{Z_{\sigma 1}}{\Delta} + \mathcal{O}(1/N^2). \]  

(3.16)

The full propagators of the renormalized fields in momentum space read

\[ \langle \phi'(p)\phi'(-p) \rangle = \frac{\delta ij}{(p^2)^{\Delta_{ij}}}, \quad \langle \sigma(p)\sigma(-p) \rangle = \frac{\mathcal{C}_\sigma}{(p^2)^{\Delta_\sigma}}, \]  

(3.17)

where we introduced anomalous dimensions \( \Delta_{ij} \) and \( \Delta_\sigma \) and two point constants \( \mathcal{C}_\phi \) and \( \mathcal{C}_\sigma \) in the momentum space. All of them can be represented as series in \( 1/N \):

\[ \Delta_{ij} = \frac{d}{2} - 1 + \gamma^{O(N)}_{ij}, \quad \Delta_\sigma = 2 - \gamma^{O(N)}_{\sigma} - \kappa^{O(N)}, \]  

(3.18)

where \( \gamma^{O(N)} = \gamma_1^{O(N)}/N + \gamma_2^{O(N)}/N^2 + \mathcal{O}(1/N^3), \quad \kappa^{O(N)} = \kappa_1^{O(N)}/N + \kappa_2^{O(N)}/N^2 + \mathcal{O}(1/N^3) \) and

\[ \mathcal{C}_{\phi} = 1 + \frac{\tilde{C}_{\phi 1}}{N} + \frac{\tilde{C}_{\phi 2}}{N^2} + \mathcal{O}(1/N^3), \quad \mathcal{C}_{\sigma} = \tilde{C}_{\sigma 0} + \frac{\tilde{C}_{\sigma 1}}{N} + \frac{\tilde{C}_{\sigma 2}}{N^2} + \mathcal{O}(1/N^3). \]  

(3.19)

Recalling that we may drop all terms proportional to \( \delta_{\mu\nu} \) since \( z^\mu \) is null, the stress–energy tensor and the \( O(N) \) current are:

\[ T(x) = z^{ij} \partial_i \phi_0^j \partial_j \phi_0^i - \frac{1}{4} \frac{d - 2}{d - 1} \partial_i \partial_j (\phi_0^i \phi_0^j), \]

\[ J^\alpha(x) = z^\alpha \phi_0^i (r^\gamma \partial_j \partial_j \phi_0^i). \]  

(3.20)

Figure 4. One loop correction to the \( \langle \phi'(p)\phi'(\bar{p}) \rangle \) propagator.
In momentum space:

\[
T(p) = \frac{1}{2} \int \frac{d^d p_1}{(2\pi)^d} \left( 2p_{1z}(p_{1z} + p_z) + c p_z^2 \right) \phi_0^j(p + p_1) \phi_0^i(-p_1),
\]

\[
J^a(p) = \frac{1}{2} \int \frac{d^d p_1}{(2\pi)^d} i(2p_{1z} + p_z) \phi_0^0(-p_1)(t^a)^i_j \phi_0^j(p + p_1),
\]

(3.21)

where \( c \equiv \frac{d-2}{2(d-1)} \) and the diagrams for (3.21) are shown in figure 3.

For the Ward identity calculation performed below, we will first need to find \( \tilde{C}_{\phi^1} \), \( \gamma_1 \) and \( Z_{\phi^1} \). To compute them we have to consider the one loop diagram for the renormalization of the \( \langle \phi\phi \rangle \)-propagator, see figure 4.

Computing this diagram, we find the result (1.7), and

\[
\tilde{C}_{\phi^1} = -\frac{1}{2} \left( 3d^2 - 12d + 8 \right) \frac{\sin \left( \frac{\pi d}{2} \right) \Gamma(d - 2)}{\pi \Gamma \left( \frac{d}{2} + 1 \right)^2}.
\]

(3.22)

As discussed in section 2, in order to cancel \( 1/\Delta \) poles in correlation functions involving \( T \) and \( J \), one may introduce \( 'Z_T' \) and \( 'Z_J' \) factors as

\[
T_{\mu\nu}^{\text{ren}} = Z_T T_{\mu\nu}, \quad J_{\mu}^{\text{ren}, a} = Z_J J_{\mu}^a,
\]

(3.23)

which admit the following decomposition:

\[
Z_T = 1 + \frac{1}{N} \left( \frac{Z_{T1} + Z_{T1}'}{\Delta} \right) + O(1/N^2), \quad Z_J = 1 + \frac{1}{N} \left( \frac{Z_{J1} + Z_{J1}'}{\Delta} \right) + O(1/N^2).
\]

(3.24)

The explicit form of these factors can be obtained from Ward identities. Let us consider \( Z_T \) first. For this, we can examine the three point function \( \langle T_{\mu\nu}^{\text{ren}} \phi^i \phi^j \rangle \). Its structure is fixed by conformal symmetry and current conservation to be [3]

\[
\langle T_{\mu\nu}^{\text{ren}}(x_1)\phi^i(x_2)\phi^j(x_3) \rangle = \frac{-C_{T\phi\phi}}{(x_1^2 x_3^2)^{-1}(x_2 x_3^2)^{-\frac{d}{2}+1}} \left( (X_{23})_\mu (X_{23})_\nu - \frac{1}{d} \delta_{\mu\nu} (X_{23})^2 \right) \delta^{ij},
\]

(3.25)

where

\[
(X_{23})_\mu = \frac{(x_{12})_\mu}{x_{12}^2} - \frac{(x_{13})_\mu}{x_{13}^2}.
\]

(3.26)

The structure constant \( C_{T\phi\phi} \) is not arbitrary and is related to \( C_{\phi} \) by the Ward identity. To show this, we note that for the infinitesimal scaling transformation \( \varepsilon_\mu = \varepsilon x_\mu \):

\[
\langle \delta_\varepsilon \phi^i(x_2)\phi^j(x_3) \rangle = -\varepsilon \int d^d \Omega r^{d-2} \rho_\varepsilon (T_{\mu\nu}^{\text{ren}}(x_1)\phi^i(x_2)\phi^j(x_3)),
\]

(3.27)

where \( r = |x_1 - x_2| \) and \( \delta_\varepsilon \phi^i(x) = \varepsilon (\Delta_\phi + x_\mu \partial_\mu) \phi^i(x) \). Performing the integral in the limit \( r \to 0 \) we find
Figure 5. Diagrams contributing to \((T(0)\phi_0(p)\phi_0(-p))\) up to order \(1/N\).

\[
C_{T\phi\phi} = \frac{1}{S_d} \frac{d\Delta}{d - 1} C_{\phi},
\]

where \(S_d = 2\pi^{d/2}/\Gamma(d/2)\) and \(C_{\phi}\) is the two-point function constant in coordinate space; it is related to \(\tilde{C}_{\phi}\) in momentum space \((3.19)\) through the Fourier transform\(^9\). Taking the Fourier transform of \((3.25)\) and using \((3.28)\) we find\(^{10}\)

\[
\langle T^{\text{ren}}(0)\phi(p)\phi(-p) \rangle = (d - 2\Delta_{\phi})\tilde{C}_{\phi} \frac{p_j^2}{(p^2)^{\Delta_{\phi}+1}},
\]

where we took the stress–energy tensor at zero momentum for simplicity. Now, to fix \(Z_T\) we compute \((3.29)\) using a direct Feynman diagram calculation:

\[
\langle T^{\text{ren}}(0)\phi(p)\phi(-p) \rangle = Z_T Z_{\phi}(T(0)\phi_0(p)\phi_0(-p)).
\]

To \(1/N\) order we have four diagrams

\[
\langle T(0)\phi_0(p)\phi_0(-p) \rangle = D_0 + D_1 + D_2 + D_3 + \mathcal{O}(1/N^2),
\]

which are shown in figure 5 and given explicitly in appendix E.

Computing these diagrams and using \((3.29)\) and \((3.30)\), we find

\[
Z_{T1} = \frac{2\gamma_{1}^{(O(N))}}{d + 2}, \quad Z'_{T1} = \frac{8\gamma_{1}^{(O(N))}}{(d + 2)(d - 4)},
\]

where \(\gamma_{1}^{(O(N))}\) is given in \((1.7)\). These renormalization constants will be of great importance for the \(C_T\) calculation.

To find \(Z_i\), we again consider the three-point function \(\langle J_{\mu}^a \phi^i \phi^j \rangle\), which is fixed by conformal invariance and current conservation \[3]\]

\[
\langle J_{\mu}^a(x) \phi^i(x_2) \phi^j(x_3) \rangle = \frac{C_{\phi\phi}}{(x_{12} \cdot x_{13})^{\Delta_{\phi} + 1/2} (x_{23} \cdot x_{23})^{\Delta_{\phi} + 1/2}} (X_{23})_a (t_a)^i, \tag{3.33}
\]

and again the structure constant \(C_{\phi\phi}\) is exactly related to \(C_{\phi}\) by the Ward identity. To show this, we perform an infinitesimal \(O(N)\) rotation of fields \(\delta \phi^i = \varepsilon (t^a)^i \delta^i\), and we get

\[
\langle \delta_i \phi^i(x_2) \phi^j(x_3) \rangle = \varepsilon \int d^dQ \lambda^{d-2} r_p \left(J_{\mu}^{\text{ren},a}(x_1) \phi^i(x_2) \phi^j(x_3) \right),
\]

where \(r = |x_1 - x_2|\). Using \((3.33)\) and performing the integral in the limit \(r \rightarrow 0\) we find

\[
C_{\phi\phi} = \frac{1}{S_d} C_{\phi},
\]

\(^9\) Notice that it is important that we define \(C_{\phi}\) in \((3.14)\) in momentum space. Thus, \(C_{\phi}\) in the coordinate space will depend on \(\Delta\). This dependence will affect the loop calculations in coordinate space.

\(^{10}\) Here we fix some field, say \(\phi = \phi^i\), and do not write the \(O(N)\)-index explicitly.
Taking the Fourier transform of (3.33) and using (3.35), we get
\[ \langle \text{ren.}^a(0) \phi^i(p) \phi^j(-p) \rangle = i(d - 2\Delta)\mathcal{C}_0 \frac{P_1}{(p^2)^{\frac{d}{2} - \Delta} + 1} (t^a)^{ij}, \]
(3.36)
where again we took the current at zero momentum to simplify the calculation. Now to fix \( Z_J \), we can compute (3.36) by a direct perturbative calculation
\[ \langle \text{ren.}^a(0) \phi^i(p) \phi^j(-p) \rangle = Z_J Z_0 \langle \text{ren.}^a(0) \phi^i(p) \phi^j(-p) \rangle, \]
(3.37)
and to \( 1/N \) order we have three diagrams
\[ \langle \text{ren.}^a(0) \phi^i(p) \phi^j(-p) \rangle = D_0 + D_1 + D_2 + \mathcal{O}(1/N^2), \]
(3.38)
which are shown in figure 6. Computing these diagrams and using (3.36) and (3.37), we find
\[ Z_J = 1 + \mathcal{O}(1/N^2). \]
(3.39)
Therefore \( Z_J \) is trivial to order \( 1/N \) and will not affect the \( C_{J1} \) calculation.

### 3.3. Calculation of \( C_{J1}^{(1/N)} \) and \( C_{T1}^{(1/N)} \)

There are three diagrams contributing to the \( 1/N \) correction to \( C_J \), depicted in figure 7. The current two-point function up to order \( 1/N \) is then
\[ \langle J^a(p) J^b(-p) \rangle = D_0 + D_1 + D_2 + \mathcal{O}(1/N^2). \]
(3.40)
The sum of \( D_1 \) and \( D_2 \) corresponds to the contribution denoted \( I_{(100)} \) in section 2. The explicit integrands and results for each diagram are given in appendix E. To compute these diagrams, we use standard techniques to perform tensor reductions and partial fraction decompositions of the integrand, which are discussed in appendix A. This results in a sum of simpler scalar integrals which involve either the product of two elementary one-loop integrals of the form
\[ \int \frac{dp_1}{(2\pi)^d} \frac{1}{p_1^{2\alpha} (p + p_1)^{2\beta}} = \frac{\Gamma\left(\frac{d}{2} - \alpha\right)\Gamma\left(\frac{d}{2} - \beta\right)\Gamma\left(\alpha + \beta - d\right)}{(4\pi)^{d/2}\Gamma(\alpha)\Gamma(\beta)\Gamma(d - \alpha - \beta)} (p^2)^{d/2 - \alpha - \beta} \]
\[ \equiv I(\alpha, \beta)(p^2)^{d/2 - \alpha - \beta}, \]
(3.41)
or the two-loop ‘kite’ diagram with the topology of $D_2$ and general power of the middle line

$$K(a) = \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} \frac{1}{p_1^3 (p + p_1)^2 p_2^2 (p + p_2)^2 (p_1 - p_2)^2}.$$  \hspace{1cm} (3.42)

The result for this integral as a function of $d$ and $a$ can be obtained, for instance, by using the Gegenbauer polynomial technique \cite{47, 48}. Putting all contributions together, the final result is

$$\langle J^a(p) J^b(-p) \rangle = \frac{\pi^2 \Gamma \left( 2 - \frac{d}{2} \right)^2 C_{J0}^{O(N)} \left( 1 - \frac{1}{N} \frac{8(d - 1)}{d(d - 2)} \gamma_1^{O(N)} + O(1/N^2) \right) \left( \frac{p^2}{(p^2)^{2 - \frac{d}{2}}} \right)^2}{2^d - 3 \Gamma(d)}.$$  \hspace{1cm} (3.43)

where $\gamma_1^{O(N)}$ is given in (1.7) and

$$C_{J0}^{O(N)} = -\frac{\text{tr} \left( t^a t^b \right)}{(d - 2) S_f^2}.$$  \hspace{1cm} (3.44)

Using that in this case $Z_f = 1 + O(1/N^2)$, we find

$$C_{J1}^{O(N)} = -\frac{8(d - 1)}{d(d - 2)} \gamma_1^{O(N)}.$$  \hspace{1cm} (3.45)
This agrees with the result of [46], who derived it using the conformal bootstrap technique. We can verify that $C_{J_1}$ is negative throughout the range $2 < d < 6$, as shown in figure 8. The value in $d = 3$ is given in equation (1.9), and from (3.45) one can also get

$$C_{J_1}^{(N)}|_{d=2+\epsilon} = -2 + \epsilon + \frac{\epsilon^2}{2}, \quad C_{J_1}^{(N)}|_{d=4-\epsilon} = -\frac{3\epsilon^2}{4} - \frac{\epsilon^3}{8}. \quad (3.46)$$

We note that the $d = 6 - \epsilon$ expansion precisely agrees with the result (3.8) that we derived above from the cubic model.

Let us now turn to the calculation of $C_T$. There are four diagrams contributing to $\langle TT \rangle$ to order $N^0$

$$\langle T(p)T(-p) \rangle = D_0 + D_1 + D_2 + D_3 + O(1/N), \quad (3.47)$$

including the three-loop diagram of Aslamazov–Larkin type [43], which was not present in the calculation of $C_T$, as shown in figure 9. After tensor reductions, one obtains a large sum of scalar integrals that, in addition to (3.41) and (3.42), involve three-loop ladder scalar integrals with various powers of the propagator lines. The evaluation of this type of integrals is discussed in detail in appendix B, and the results for the individual diagrams are listed in appendix E. After a very laborious computation, we obtain

$$\langle T(p)T(-p) \rangle = \frac{\pi^2 \Gamma\left(2 - \frac{d}{2}\right)}{2^{d-2}\Gamma(d+2)} C_{T0}^{(N)} \left(1 - \frac{1}{N} \left(\frac{4\gamma_{11}^{(N)}}{\Delta (d+2)} + \gamma_{11}^{(N)} \left(\frac{4C_{O(N)}(d)}{d+2}\right)\right)ight) + \mathcal{O}(1/N^2) \left(\frac{p_s^4}{(p^2)^{d-2-\epsilon/2}}\right). \quad (3.48)$$

where $C_{O(N)}(d) = \psi\left(3 - \frac{d}{2}\right) + \psi(d - 1) - \psi(1) - \psi\left(\frac{d}{2}\right)$ and $\gamma_{11}^{(N)}$ is given in (1.7), and

$$C_{T0}^{(N)} = \frac{Nd}{(d - 1)S_d^2}. \quad (3.49)$$

As we have already discussed, the $1/\Delta$-pole is present, but there is no $\log(p^2/\mu^2)$ term, as expected since the stress–energy tensor is exactly conserved and cannot develop an anomalous dimension. In order to get an expression free of the $1/\Delta$ poles, we have to use ‘renormalized’ stress–energy tensor $T^{\text{ren}}_{\mu\nu} = Z_T T_{\mu\nu}$, where $Z_T$ was derived above and given in (3.32). Therefore, we obtain

$$\langle T^{\text{ren}}(p)T^{\text{ren}}(-p) \rangle = Z_T^2 \langle T(p)T(-p) \rangle = \frac{\pi^2 \Gamma\left(2 - \frac{d}{2}\right)}{2^{d-2}\Gamma(d+2)} C_{T0}^{(N)} \left(1 - \frac{1}{N} \left(\frac{4C_{O(N)}(d)}{d+2}\right)\right) + \mathcal{O}(1/N^2) \left(\frac{p_s^4}{(p^2)^{d-2-\epsilon/2}}\right). \quad (3.50)$$

Note that, as desired, the $1/\Delta$ pole was cancelled. This is a non-trivial consistency check of our procedure, since the $Z_T$ factor was obtained above from an independent Ward identity calculation. From (3.50), we thus find
\[ C_{T_1}^{(N)} = -2 \gamma_1^{(N)} \left( \frac{2C_{O(N)}^{(d)}(d)}{d + 2} + \frac{d^2 + 6d - 8}{(d - 2)d(d + 2)} \right), \]  

which exactly agrees with the result of \cite{46}. We note that we may also write this result in a simpler form as

\[ C_{T_1}^{(N)} = -2 \gamma_1^{(N)} \left( \frac{2\Psi_{O(N)}(d)}{d + 2} + \frac{d + 4}{d(d + 2)} \right), \]  

where \( \Psi_{O(N)}(d) \equiv \psi \left( 3 - \frac{d}{2} \right) + \psi(d - 1) - \psi(1) - \psi \left( \frac{d}{2} - 1 \right). \)

A plot of \( C_{T_1}^{(N)} \) in \( 2 < d < 6 \) is given in figure 10. The value in \( d = 3 \) was already given in (1.9). From (3.51), one can also get

\[ C_{T_1}^{(N)} \mid_{d=2+\varepsilon} = -1 + \frac{3\varepsilon^2}{4}, \quad C_{T_1}^{(N)} \mid_{d=4-\varepsilon} = -\frac{5\varepsilon^2}{12} - \frac{7\varepsilon^3}{36}, \quad C_{T_1}^{(N)} \mid_{d=6-\varepsilon} = 1 - \frac{7\varepsilon}{4} + \frac{23\varepsilon^2}{288}. \]  

We note that the result for \( C_{T_1}^{(O(N))} \) expanded in \( d = 6 - \varepsilon \) precisely agrees with the the calculation in the cubic model, see (3.10). This constitutes a new perturbative check of the formula (3.51) for \( C_{T_1}^{(O(N))} \). Note that the leading term in \( d = 6 - \varepsilon \) is just the contribution of the free scalar field \( \sigma \) in the cubic model. As discussed in the Introduction, for all even \( d \), the critical \( O(N) \) model is expected to reduce to a free theory of \( N \) ordinary conformal scalars, plus a \( \Delta = 2 \) scalar with kinetic term \( \sim \sigma (\partial^2 - \gamma^2) \sigma \), see equation (3.13). From (3.51) it follows that

\[ C_{T_1}^{(O(N))} \mid_{\text{even } d} = \left( -1 \right)^{d+1} \frac{(d-4)(d-2)!}{\left( \frac{d}{2} + 1 \right)! \left( \frac{d}{2} - 1 \right)!} = \left( -1 \right)^{d+1} \left( \frac{d-4}{d-5} - \frac{d-4}{d-3} \right). \]  

Interestingly, this is an integer for all even dimensions \cite{20}\footnote{In fact, we note that (3.54) appears to be equal (for \( d > 4 \)) to \( (-1)^{d/2+1} \) times the dimension of the irreducible representation of \( S^d(d-4) \) labelled by the Young tableaux \{11, ..., \}, \( d/2 \leq 3 \).}. The formula (3.54) is the ratio of the \( C_T \) of a free \( (d-4) \)-derivative scalar to that of a canonical scalar. This means that
It would be interesting to check this result via an explicit calculation using the action for a higher derivative scalar.

3.4. Padé approximations

For any quantity \( f(d) \) known in the \( \epsilon = 4 - d \) and \( \epsilon = d - 2 \) expansions up to a given order, we can construct a Padé approximant

\[
P_{[m,n]}(d) = \frac{A_0 + A_1 d + A_2 d^2 + \cdots + A_m d^m}{1 + B_1 d + B_2 d^2 + \cdots + B_n d^n},
\]

where the coefficients \( A_i, B_i \) are fixed by requiring that the expansion of \( (3.56) \) agrees with the known terms in \( f(4 - \epsilon) \) and \( f(2 + \epsilon) \) obtained by perturbation theory. For the \( O(N) \) model the \( 4 - \epsilon \) expansion can be developed for any integer \( N \) using the weakly coupled Wilson–Fisher IR fixed point [15]. The \( 2 + \epsilon \) expansion can be developed using standard

---

**Figure 11.** Plot of \( N(C_J^{(N)})/C_{J,\text{free}}^{(N)} - 1 \) for \( \text{Padé}_{[2,2]} \).

**Figure 12.** Plot of \( C_J^{(N)}/C_{J,\text{free}}^{(N)} \) in \( d = 3 \).
perturbation theory only for \( N > 2 \), because this is when the \( O(N) \) nonlinear \( \sigma \) model has a weakly coupled UV fixed point [37, 49, 50].

For \( C_J^{O(N)}/C_J^{\text{free}} \), the \( \epsilon \) expansions read (the \( \epsilon/N \) correction in \( d = 2 + \epsilon \) was guessed on the basis of the large \( N \) results and plausible assumptions, and the \( d = 4 - \epsilon \) expansion can be found in [16, 46]):

\[
C_J^{O(N)}/C_J^{\text{free}}(d) = \begin{cases} 
\frac{N-2}{N} + \frac{\epsilon}{N} + \mathcal{O}(\epsilon^2) & \text{in } d = 2 + \epsilon, \\
1 - \frac{3(N+1)x^2}{4(N+N+8)^2} + \mathcal{O}(\epsilon^3) & \text{in } d = 4 - \epsilon.
\end{cases}
\]  
(3.57)

In this case we find that only the approximant Padé\(_{2,2}\) is well-behaved, being free of poles and in good agreement at large \( N \) with the result (3.45) in \( 2 < d < 4 \). We plot Padé\(_{2,2}\) for different values of \( N \) in figure 11, and list a few of its numerical values in \( d = 3 \) in table 1.

We observe that the results we find are close to the \( C_J \) values obtained using the conformal bootstrap [51]. This is shown in figure 12. The quoted bootstrap value \( C_J^{(3)}/C_J^{\text{free}} = 0.9065(27) \) should be compared with our Padé\(_{2,2}\) result 0.9096, and the bootstrap value \( C_J^{(20)}/C_J^{\text{free}} = 0.9674(8) \) with our Padé\(_{2,2}\) result 0.9686.

For the \( C_T^{O(N)}/C_T^{\text{free}} \) we use the following \( \epsilon \)-expansions:

\[
C_T^{O(N)}/C_T^{\text{free}}(d) = \begin{cases} 
1 - \frac{1}{N} + \frac{3(N-1)x^2}{4N(N-2)} + \mathcal{O}(\epsilon^3) & \text{in } d = 2 + \epsilon, \\
1 - \frac{5(N+2)x^2}{12(N+N+8)^2} + \mathcal{O}(\epsilon^3) & \text{in } d = 4 - \epsilon.
\end{cases}
\]  
(3.58)

The leading correction in \( d = 4 - \epsilon \) can be found in [11, 14, 16]. To determine the \( 2 + \epsilon \) expansion we used the fact that there is a \( R_{abcd} \) correction to the central charge in the \( d = 2 \) sigma model with general target space curvature [52, 53]. After specializing to the case of \( N - 1 \) dimensional sphere, we find that this term \( \sim (N - 1)(N - 2)g^2 \). The \( O(N) \) sigma model has a UV fixed point in \( d = 2 + \epsilon \) for \( N > 2 \) [37, 49, 50]. Setting the sigma model coupling \( g \) to its fixed point value \( \sim \frac{\epsilon}{N-2} \), and using the large \( N \) result to normalize the correction, we find the result above.
The best approximant we find is Padé\textsubscript{3,2}; it does not have poles and approaches the large \( N \) result (3.51) quite well. We plot Padé\textsubscript{3,2} for different \( N \) in figure 13. Also, we give the values of \( C_{T,\text{free}}^{(N)} / C_{T,\text{free}}^{(3)} \) for different \( N \) in table 2.

The results we find are close to the \( C_T \) values obtained using the conformal bootstrap \cite{54}. The quoted bootstrap values (see table 3 in \cite{54}) are in good agreement with our Padé\textsubscript{3,2}. This is shown in figure 14, where we also include the result of an ‘improved’ Padé\textsubscript{3,2} approximant obtained by imposing exact agreement with the large \( N \) result (3.51) in \( 2 < d < 4 \). Explicitly, this may be defined as

\[
\text{Improved-Padé}(d, N) = \text{Padé}(d, N) + \frac{1}{N} \left( C_T - \lim_{N \to \infty} (N(\text{Padé}(d, N) - 1)) \right), \tag{3.59}
\]

which by construction exactly approaches the large \( N \) result when \( N \) goes to infinity. From figure 14, we see that it fits the bootstrap data even better than the regular Padé.

\begin{table}[h]
\centering
\caption{List of Padé\textsubscript{1,2} extrapolations for \( C_{T,\text{free}}^{(N)} / C_{T,\text{free}}^{(3)} \) for \( d = 3 \). The second line corresponds to the large \( N \) result (3.45) in \( d = 3 \).}
\begin{tabular}{cccccccc}
\hline
\( N \) & 3 & 4 & 5 & 8 & 12 & 20 & 50 \\
\hline
Padé\textsubscript{1,2} & 0.9096 & 0.9167 & 0.9234 & 0.9395 & 0.9535 & 0.9686 & 0.9860 \\
1 - \frac{64}{97N} & 0.7598 & 0.8199 & 0.8559 & 0.9099 & 0.9400 & 0.9640 & 0.9856 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{List of Padé\textsubscript{3,2} extrapolations for \( C_{T,\text{free}}^{(N)} / C_{T,\text{free}}^{(3)} \) in \( d = 3 \). The second line is the large \( N \) result (3.51) in \( d = 3 \).}
\begin{tabular}{cccccccc}
\hline
\( N \) & 3 & 4 & 5 & 8 & 12 & 20 & 50 \\
\hline
Padé\textsubscript{3,2} & 0.9477 & 0.9501 & 0.9543 & 0.9647 & 0.9732 & 0.9819 & 0.9919 \\
1 - \frac{40}{97N} & 0.8499 & 0.8874 & 0.9099 & 0.9437 & 0.9625 & 0.9775 & 0.9910 \\
\hline
\end{tabular}
\end{table}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure14.png}
\caption{Plot of \( C_{T,\text{free}}^{(N)} / C_{T,\text{free}}^{(3)} \) in \( d = 3 \).}
\end{figure}
4. GN model

4.1. $1/N$ expansion

The Hubbard–Stratonovich analysis reviewed in section 2 can be also applied to the GN model. Introducing the auxiliary field $\sigma$, and dropping the quadratic term in the critical limit, we have the action

$$
S_{\text{crit term}} = \int d^d x \left( -\bar{\psi}_0^{(i)} \not\partial \psi_0^{(i)} + \frac{1}{\sqrt{N}} \sigma_0 \bar{\psi}_0^{(i)} \psi_0^{(i)} \right),
$$

where $i = 1, \ldots, \tilde{N}$ and $N = \tilde{N} \text{Tr} I$. The propagator of the $\psi_0^{(i)}$ field reads

$$
\langle \psi_0^{(i)}(p) \bar{\psi}_0^{(i)}(-p) \rangle_0 = \delta_j^i \frac{ip^2}{p^2}.
$$

The $\sigma$ effective propagator obtained after integrating over the fundamental fields $\psi_0^{(i)}$ reads

$$
\langle \sigma_0(p) \sigma_0(-p) \rangle_0 = \tilde{C} \sigma_0/(p^2)^{2-1+\Delta},
$$

where

$$
\tilde{C} \sigma_0 \equiv -2^{d+1}(4\pi)^{d/2} \Gamma \left( \frac{d-1}{2} \right) \sin \left( \frac{\pi d}{2} \right)
$$

and we have introduced the regulator $\Delta$. Note that the power of $p^2$ in the propagator is $d/2 - 1 + \Delta$ instead of $d/2 - 2 + \Delta$ found in the scalar case. In order to cancel the divergences as $\Delta \to 0$ we have to renormalize the bare fields $\psi_0$ and $\sigma_0$:

$$
\psi = Z_{\psi}^{1/2} \psi_0, \quad \sigma = Z_{\sigma}^{1/2} \sigma_0,
$$

where

$$
Z_{\psi} = 1 + \frac{1}{N} Z_{\psi 1} \frac{1}{\Delta} \mathcal{O}(1/N^2), \quad Z_{\sigma} = 1 + \frac{1}{N} Z_{\sigma 1} \frac{1}{\Delta} \mathcal{O}(1/N^2).
$$

The full propagators of the renormalized fields read

$$
\langle \psi^{(i)}(p) \bar{\psi}^{(i)}(-p) \rangle = \delta_j^i \frac{ip^2}{(p^2)^{2-\Delta_{\psi}+1/2}}, \quad \langle \sigma(p) \sigma(-p) \rangle = \frac{\tilde{C}_\sigma}{(p^2)^{2-\Delta_{\sigma}}},
$$

where we introduced anomalous dimensions $\Delta_{\psi}$ and $\Delta_{\sigma}$ and two-point function normalizations $\tilde{C}_\psi$ and $\tilde{C}_\sigma$ in momentum space. Each of them may be represented as

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig15}
\caption{Momentum space Feynman rules for $T(p)$ and $J^a(p)$.}
\end{figure}
series in $1/N$:
\[
\Delta_{\psi} = \frac{d}{2} - \frac{1}{2} + \gamma_{\psi}^{GN}, \quad \Delta_{\sigma} = 1 - \gamma_{\sigma}^{GN} - \kappa_{\sigma}^{GN},
\]
where $\gamma_{\psi}^{GN} = \gamma_{1}^{GN}/N + \frac{\gamma_{2}^{GN}}{N^2} + O(1/N^3)$, $\kappa_{\psi}^{GN} = \kappa_{1}^{GN}/N + \frac{\kappa_{2}^{GN}}{N^2} + O(1/N^3)$ and
\[
\Delta_{\psi} = 1 + \frac{\tilde{\mathcal{C}}_{\psi 1}}{N} + \frac{\tilde{\mathcal{C}}_{\psi 2}}{N^2} + O(1/N^3), \quad \Delta_{\sigma} = \tilde{\mathcal{C}}_{\sigma 0} + \frac{\tilde{\mathcal{C}}_{\sigma 1}}{N} + \frac{\tilde{\mathcal{C}}_{\sigma 2}}{N^2} + O(1/N^3).
\]

The stress–energy tensor and the current are
\[
T = -\frac{1}{2}(\bar{\psi}_0 \gamma_{\mu} \partial_{\nu} \psi_0^i - \partial_{\mu} \bar{\psi}_0 \gamma_{\nu} \psi_0^j)z^\mu z^\nu,
\]
\[
J^a = -z^\mu \bar{\psi}_0^i (r)^{\mu} j_{\mu} \psi_0^j
\]
and in momentum space
\[
T(p) = -\frac{1}{2} \int \frac{d^dp_1}{(2\pi)^d} \bar{\psi}_{0}(p) \gamma_{\mu} (2p_{1\mu} + p_1) \psi_0^j (p + p_1),
\]
\[
J^a(p) = -\int \frac{d^dp_1}{(2\pi)^d} \bar{\psi}_{0}(p) (r)^{\mu} j_{\mu} \psi_0^j (p + p_1).
\]

The diagrammatic representation is shown in figure 15.

As in the scalar case, we define
\[
T_{\mu \nu}^{\text{ren}} = Z_T T_{\mu \nu}, \quad J_{\mu}^{\text{ren},a} = Z_{J_a} J_{\mu}^{a},
\]
where
\[ Z_T = 1 + \frac{1}{N} \left( \frac{Z_{T1}}{\Delta} + Z_{T1}' \right) + \mathcal{O}(1/N^2), \quad Z_T = 1 + \frac{1}{N} \left( \frac{Z_{T1}}{\Delta} + Z_{T1}' \right) + \mathcal{O}(1/N^2). \] (4.13)

By a direct calculation presented in appendices C and D, we show that Ward identities fix
\[ Z_{T1} = 2 \gamma_{11}^{GN} d + 2, \quad Z_{T1}' = \frac{8 \gamma_{11}^{GN}}{(d + 2)(d - 2)}, \] (4.14)

where \( \gamma_{11}^{GN} \) is defined in (4.8) and reads
\[ \gamma_{11}^{GN} = \frac{\Gamma (d - 1) \left( \frac{d}{2} - 1 \right)^2}{\Gamma \left( 2 - \frac{d}{2} \right) \Gamma \left( \frac{d}{2} + 1 \right) \Gamma \left( \frac{d}{2} \right)^2}. \] (4.15)

For the spin 1 current, we find \( Z_T = 1 + \mathcal{O}(1/N^2) \), which means that it does not affect the \( C_{J1} \) calculation.

4.2. Calculation of \( C_{J1}^{GN} \) and \( C_{T1}^{GN} \)

There are again three diagrams contributing to \( C_{J1}/C_{J0} \) up to order \( 1/N \), given in figure 16. They are identical to the ones for the critical scalar, except the solid lines are fermionic instead of scalar.

To compute the diagrams we use the same methods as for the case of the \( O(N) \) model (see appendices A and B). We find that the \( 1/\Delta \) divergence is canceled in the combination \( D_1 + D_2 \), yielding the result (see appendix E for the integrands and results for each diagram):
\[ \langle J^a(p)J^b(-p) \rangle = D_0 + D_1 + D_2 + \mathcal{O}(1/N^2) \]
\[ = \frac{\pi^2 \Gamma \left( 2 - \frac{d}{2} \right)}{2^{d-1} \Gamma (d)} C_{J0}^{GN} \left[ 1 - \frac{1}{N} \frac{8(d - 1) \gamma_{11}^{GN}}{d(d - 2)} + \mathcal{O}(1/N^2) \right] \frac{p^2}{(p^2)^2 - \frac{4}{\Delta^2}}, \] (4.16)

where \( \gamma_{11}^{GN} \) is given in (4.15) and
\[ C_{J0}^{GN} = -\text{tr} \langle e^{i \phi} \rangle \text{Tr} \frac{1}{S_d}. \] (4.17)

Therefore, we find the final result
\[ C_{J1}^{GN} = 8(d - 1) \gamma_{11}^{GN}, \] (4.18)
We see in figure 17 that $C_{J}^{\text{GN}}$ for the critical fermion is always negative in the range $2 < d < 4$, thus a ‘CJ-theorem’ inequality $C_{J}^{\text{UV}} > C_{J}^{\text{IR}}$ does not hold for the flow from the UV fixed point to the free fermions in the IR.

In $d = 3$, we obtain the value reported in equation (1.18). In $d = 2 + \epsilon$ and $d = 4 - \epsilon$ dimensions, we find

$$C_{J,1}^{\text{GN}}|_{d=2+\epsilon} = - \epsilon + \frac{\epsilon^3}{4} + \mathcal{O}(\epsilon^4),$$

$$C_{J,1}^{\text{GN}}|_{d=4-\epsilon} = - \frac{3\epsilon}{2} + \frac{\epsilon^2}{2} + \frac{15\epsilon^3}{32} + \mathcal{O}(\epsilon^4).$$

(4.19)

We will show that these values are in precise agreement with our $C_{J}$ calculations for the GN and GNY models performed in sections 4.3 and 4.4 below.

The diagrams contributing to the stress tensor two-point function

$$\langle T(p)T(-p) \rangle = D_0 + D_1 + D_2 + D_3 + \mathcal{O}(1/N^2),$$

(4.20)

are shown in figure 18 (see appendix E for the results). After a very laborious computation, the details of which are discussed in the appendices, we obtain the final result

$$\langle T(p)T(-p) \rangle = \frac{\pi^2}{2d-1}(d+2)C_{T0}^{\text{GN}}\left(1 - \frac{1}{\Delta} + \frac{4}{d(d+2)} + \mathcal{O}(1/N^2)\right) + \frac{4\left(5d^2 - 8d + 4\right)}{(d-2)(d-1)d(d+2)} + \mathcal{O}(1/N^2),$$

(4.21)

where $C_{\text{GN}}(d) \equiv \psi\left(2 - \frac{d}{2}\right) + \psi(d-1) - \psi(1) - \psi\left(\frac{d}{2}\right)\gamma_{1}^{\text{GN}}$ is given in (4.15) and

$$C_{T0}^{\text{GN}} = \frac{Nd}{2\Delta^2}. $$

(4.22)

As we already discussed, we see that $1/\Delta$-pole is present, but the $\log(p^2/\mu^2)$ term cancels out; this means that, as expected, the stress tensor does not have an anomalous dimension, because it is exactly conserved. In order to get a finite expression we have to use the renormalized stress-energy tensor $T_{\mu\nu}^{\text{ren}} = Z_T T_{\mu\nu}$, where $Z_T$ is given in (4.13) and (4.14).
Therefore, we obtain

\[
\langle T^{\text{ren}}(p)T^{\text{ren}}(-p) \rangle = Z_t^2 \langle T(p)T(-p) \rangle
\]

\[
= \frac{\pi^21(2 - \frac{d}{2})}{2^{d-2}(d+2)} C_{T0}^{\text{GN}} \left(1 - \frac{\gamma_1^{\text{GN}}}{N} \left(\frac{4C_{\text{GN}}(d)}{d+2}\right)
\right)
\]

\[
+ \frac{4(d-2)}{(d-1)d(d+2)} + \mathcal{O}(1/N^2)\left(\frac{p^2}{p^2-\frac{d}{2}}\right). \tag{4.23}
\]

As in the scalar case discussed earlier, it is a non-trivial test of our procedure that the $Z_T$ factor fixed by Ward identities has precisely the correct pole to cancel the $1/\Delta$ divergence in $\langle TT \rangle$. From (4.23), we then find one of our main results

\[
C_{T1}^{\text{GN}} = -4\gamma_1^{\text{GN}} \left(\frac{C_{\text{GN}}(d)}{d+2} + \frac{d-2}{(d-1)d(d+2)}\right). \tag{4.24}
\]

In $d = 3$, we get the result quoted in equation (1.19). It is interesting that $C_{T1}^{\text{GN}} > 0$ in $d = 3$. This means that the 'CT-theorem' inequality $C_{T1}^{\text{GN}} > C_{T1}^{\text{IR}}$ applies to the large $N$ GN model in $d = 3$. However, as plot in figure 19 shows, this inequality is violated for $2 < d < 2.3$.

In $d = 2 + \epsilon$ and $d = 4 - \epsilon$, we find

\[
C_{T1}^{\text{GN}} |_{d=2+\epsilon} = -\frac{\epsilon^3}{8} + \mathcal{O}(\epsilon^4), \quad C_{T1}^{\text{GN}} |_{d=4-\epsilon} = \frac{2}{3} - \frac{11\epsilon}{18} - \frac{17\epsilon^2}{54} + \mathcal{O}(\epsilon^3). \tag{4.25}
\]

As we show below, these precisely agree with the results obtained using the $\epsilon$ expansion in the GN and GNY models, respectively.

It is also interesting to look at general even dimensions $d$. In this case, the GN model is expected to be equivalent to a theory of $\tilde{N}$ free fermions plus a higher derivative scalar with local kinetic term $\sim \sigma (\partial^2)^{\frac{d}{2}-1} \sigma$ (see the form of the induced propagator (4.3)). The contribution to $C_T$ of such a free scalar can be obtained from (4.24), which has a finite non-zero limit for all even $d > 2$

\[
C_{T1}^{\text{GN}} |_{\text{even } d} = \frac{(-1)^{\frac{d}{2}} (d-2)(d-2)!}{(\frac{d}{2}+1)(\frac{d}{2}-1)!}. \tag{4.26}
\]

From this, after multiplying by the overall free fermion factor (4.22), one may read off the $C_T$ coefficient of the $(d-2)$-derivative scalar for all even $d$:

\[
C_T^{d-2-\text{deriv. scalar}} |_{\text{even } d} = \frac{(-1)^{\frac{d}{2}} d(d-2)(d-2)!}{2(\frac{d}{2}+1)(\frac{d}{2}-1)!} S_d^2. \tag{4.27}
\]

Its ratio to $C_T$ of a canonical scalar is

\[
\frac{(-1)^{\frac{d}{2}} (d-1)(d-2)(d-2)!}{2(\frac{d}{2}+1)(\frac{d}{2}-1)!} = (-1)^{\frac{d}{2}} \left(\frac{d-1}{d-2}\right). \tag{4.28}
\]
Interestingly, this is an integer; in $d = 6, 8, 10, ...$ we find $-5, 21, -84, ...$. It would be interesting to check the formula (4.28) by a direct calculation using the stress–energy tensor of the free $(d - 2)$-derivative scalar.

### 4.3. GNY model and 4–$\epsilon$ expansions of $C_J$ and $C_T$

In this section we consider the GNY model [39, 40]. It is a theory of $N$ Dirac fermions $\psi^{ij}$ transforming under an internal $U(N)$ symmetry group and a scalar field $\sigma$ in $d = 4 - \epsilon$ dimensions described by the action (1.20). As above, we define $N = \tilde{N} \text{Tr} 1$, where 1 is the identity matrix for the Dirac representation. The model has a weakly coupled fixed point in $d = 4 - \epsilon$, with the coupling constants given by, to leading order in $\epsilon$ [37]

$$g_1 = \frac{16\pi^2 \epsilon}{\sqrt{N + 6}}, \quad (4.29)$$

$$g_2 = 16\pi^2 \epsilon \frac{24N}{(N + 6)((N - 6) + \sqrt{N^2 + 132N + 36})}. \quad (4.30)$$

As before, we will compute $C_J$ and $C_T$ up to two-loop level. We have not found such a calculation in the literature, so our results appear to be new.

For simplicity, we will consider the two-point function of the $U(1)$ current

$$J = z^\mu \bar{\psi} \gamma^\mu \psi^i,$$ \quad (4.31)

which, in the notation used above in equation (4.10), just corresponds to a particular choice of generator of $U(N)$ (the one proportional to the identity). The diagrams contributing to

$$\langle J(p)J(-p) \rangle = D_0 + D_1 + D_2 + \mathcal{O}(1/N^2), \quad (4.32)$$

are shown in figure 20 (see appendix E for the integrands and results). The arrows are fermionic arrows, and we have defined our momenta in such a way that the flow of momentum coincides with the fermionic arrows. As before, the dashed line denotes the $\sigma$ field.

After evaluating the integrals, Fourier transforming to position space, substituting the fixed-point values (4.29) and (4.30) of the coupling constants, and extracting the $C_J$ coefficient from each term according to (2.3), we obtain:

$$C_J^{\text{GNY}} = \frac{1}{S_3} \left( N - \frac{3N\epsilon}{2(N + 6)} + \mathcal{O}((\epsilon^2) \right). \quad (4.33)$$

These correspond to $\pm$ the dimensions of the rank-$(d/2 - 2)$ totally antisymmetric representations of $SO(d - 1)$.
where \( S_d = 2\pi^{d/2}/\Gamma(d/2) \) is the volume of the \((d - 1)\)-dimensional sphere (evaluated here in \( d = 4 - \epsilon \)). Normalizing by the free field contribution, we find

\[
\frac{C_{\text{GNY}}}{C_{\text{free}}} = 1 - \frac{3\epsilon}{2(N + 6)} + \mathcal{O}(\epsilon^2),
\]

which precisely agrees, to leading order at large \( N \), with the result (4.18) expanded in \( d = 4 - \epsilon \), see equation (4.19).

To study \( C_J \) we write \( T = T_\psi + T_\sigma \), where

\[
T_\psi = -\frac{1}{2} (\bar{\psi} \gamma_\mu \partial_\mu \psi + \bar{\psi} \gamma_\mu \gamma_5 \psi) z^{\mu} x^\nu,
\]

and \( T_\sigma \) is given in (3.9). We have \( \langle TT \rangle = \langle T_\psi T_\psi \rangle + 2 \langle T_\psi T_\sigma \rangle + \langle T_\sigma T_\sigma \rangle \).

At leading order, \( \langle T_\psi T_\psi \rangle = 0 \), while \( \langle T_\psi T_\sigma \rangle \) and \( \langle T_\sigma T_\sigma \rangle \) are given by the free field one-loop integrals. At the next to leading order we have four diagrams, which we call \( D_1, D_2, D_3, \) and \( D_4 \); they are shown in figure 21 (see appendix E for the explicit results).

After evaluating the integrals, Fourier transforming to position space, and plugging in the expression (4.29) for the coupling constant \( g_1 \) at the fixed point, we get

\[
C_{\text{GNY}}^{\langle T \rangle} = \frac{d}{S_d} \left( \frac{N}{2} + \frac{1}{d - 1} - \frac{5N\epsilon}{12(N + 6)} \right)
\]

To compare to the large \( N \) calculation in the previous section, we should normalize this result by the contribution of \( \tilde{N} \) free Dirac fermions. Using (4.22), we find

\[
\frac{C_T^{\langle T \rangle}}{NC_{\text{GNY}}^{\langle T \rangle}} = 1 + 2 \frac{\tilde{N}}{3N} - \frac{11N - 24}{18N(N + 6)} \epsilon + \mathcal{O}(\epsilon^2). \tag{4.37}
\]

Comparing with (4.25), we again find precise agreement with our large \( N \) result (4.24) expanded in \( d = 4 - \epsilon \).

### 4.4. 2 + \epsilon expansion of \( C_J \) and \( C_T \)

In this section, we will consider the GN model (1.13) in \( d = 2 + \epsilon \). The beta function and the critical value of \( g \) at the UV fixed point are [37]
\[
\beta = \epsilon g - (N - 2) \frac{g^2}{2\pi} + (N - 2) \frac{g^3}{4\pi^2} + (N - 2)(N - 7) \frac{g^4}{32\pi^3} + \mathcal{O}(g^5),
\]
\[
g_\psi = \frac{2\pi}{N - 2} \epsilon + \frac{2\pi}{(N - 2)^2} \epsilon^2 + \frac{(N + 1)\pi}{2(N - 2)^3} \epsilon^3 + \mathcal{O}(\epsilon^4),
\]  
(4.38)

where \( N = \tilde{N} \text{Tr} 1 \). From the beta function we can also deduce the relation between the bare and renormalized couplings (here \( \mu \) denotes the renormalization scale):
\[
g_0 = \mu^{-\epsilon} \left( g + \frac{N - 2}{2\pi} \frac{g^2}{\epsilon} - \frac{N - 2}{8\pi^2} \frac{g^3}{\epsilon} + \frac{(N - 2)^2}{4\pi^3} \frac{g^4}{\epsilon^2} + \mathcal{O}(g^5) \right). \]
(4.39)

The UV fixed point of this model is related to the IR fixed point of the GNY model. One can check this by comparing the anomalous dimensions of the \( \psi \) and \( \sigma \) fields as in [37]. In this section we will derive \( C_J \) and \( C_T \) for the critical fermionic theory at to next-to-leading order.

To extract \( C_J \), we may calculate the two-point function of the \( U(1) \) current (4.31). The leading order contribution to \( C_J \) is the same diagram \( D_0 \) as in the GNY model, and the contribution of order \( g \) is depicted in figure 22. The diagrams contributing to \( g^2 \) order are shown in figure 23\(^{13}\). There are three different topologies, and multiple ways of directing the fermion lines within each. As before, the arrows are fermionic arrows, and we have defined momenta in such a way that the flow of momentum coincides with the fermionic arrows.

\(^{13}\) We did not draw some of the diagrams with the \( D_4 \) topology because they cancel each other after using the formula \( \text{Tr} (A \cdot B \cdot \mathcal{P}) = \text{Tr} (A \cdot B \cdot \mathcal{P}) \), but diagrams with such a topology do appear in the \( (TT) \) computation. Also, the second diagram for \( D_4 \) in the figure has a partner with different orientation of the fermion line, but one can show that these diagrams are equal, therefore we have a factor of 2 for the integral of this diagram in formula (E.16).
Notice that each insertion of \( J \) carries a \( \gamma_c \), and we have omitted the diagrams that are zero due to having an odd number of \( \gamma' \)'s in the trace.

The explicit results for the diagrams \( D_0, \ldots, D_4 \) are collected in appendix E. After plugging in the critical coupling from (4.38) and normalizing by the free field contribution, we find

\[
\frac{C_J^{\text{GN}}}{C_{J,\text{free}}} = \frac{D_0 + D_1 + D_2 + 2D_3 + D_4}{D_0} = 1 - \frac{\epsilon}{N-2} - \frac{\epsilon^2}{2(N-2)^2} + \mathcal{O}(\epsilon^3).
\]

This agrees with our large-\( N \)-formula (4.18) for \( C_J^{\text{GN}} \) of the critical fermionic theory, expanded in \( d = 2 + \epsilon \) to \( \mathcal{O}(\epsilon^2) \).

The calculation of \( C_T \) proceeds similarly to the computation for \( C_J \) in the previous section. All the diagrams have identical topologies, with the difference that instead of \( J \) we insert the stress–energy tensor (4.35). The two-loop diagram \( D_1 \) with the same topology as the one in figure 22 actually vanishes; see equation (E.19). Computing the three-loop diagrams in figure 24 (see appendix E) and normalizing by the free field contribution, we find the following contribution to \( \frac{C_T^{\text{GN}}}{C_{T,\text{free}}} \):

\[
\frac{D_0 + D_2 + 2D_3 + D_4}{D_0} = 1 + g^2 \left( \frac{3(N-1)}{8(2\pi)^2} \epsilon + \mathcal{O}(\epsilon^2) \right).
\]

Note that this \( \mathcal{O}(g^2) \) term vanishes in \( d = 2 \). Therefore, for \( g = g_4 \) the leading correction is of order \( \epsilon^2 \); this is consistent with the vanishing of the \( \mathcal{O}(\epsilon^2) \) term in our large-\( N \) result (4.25) for \( C_{T_1} \).

In order to determine the coefficient of the \( \mathcal{O}(\epsilon^3) \) correction to \( \frac{C_T^{\text{GN}}}{C_{T,\text{free}}} \) at the critical point, we also need the \( g^3 \) term, which comes from four-loop Feynman diagrams. We will not perform this calculation directly, but rather use a shortcut involving the conformal perturbation theory in \( d = 2 \). The GN-model involves the free Dirac fermions perturbed by a marginal operator \( O = \frac{1}{2}(\bar{\psi} \gamma^i \psi)^2 \) with the scaling dimension \( \Delta_O = 2 + \mathcal{O}(g) \)

\[
S = S_{\text{free fermion}} + g \int d^2 x \, O(x).
\]
The Zamolodchikov c-function is defined as follows [9, 55]:

\[ c(g) = C(g) + 4\beta(g)H(g) - 6\beta^2(g)G(g), \]

where

\[ C(g) = 2\omega^2 \langle T_{ww}(x) T_{ww}(0) \rangle_{x^2 = x_0^2}, \]
\[ H(g) = \frac{\omega^2 x^2}{2} \langle T_{ww}(x) O(0) \rangle_{x^2 = x_0^2}, \]
\[ G(g) = x^4 \langle O(x) O(0) \rangle_{x^2 = x_0^2}. \]

(4.44)

Here \( \omega = x^1 + ix^2 \), \( T_{ww} = T_{11} - T_{22} - 2iT_{12} \), and

\[ \beta(g) = -(N - 2) \frac{g^3}{2\pi} + O(g^3). \]

(4.45)

We notice that

\[ C_T \propto C(g). \]

(4.46)

Therefore, to find the \( g^3 \) term in \( C_T \) we have to find the central charge \( c(g) \) to order \( g^3 \) and the function \( H(g) \) to order \( g \). The term \( \beta^2 G \) obviously does not contribute to this order. Thus, we have

\[ C_T(g) \propto c(g) - 4\beta(g)H(g) + O(g^4). \]

(4.47)

Let us find \( H(g) \) to order \( g \). Using (4.42) we get

\[ H(g) = g\omega^2 x^2 \int d^2y \langle T_{ww}(x) O(0) O(y) \rangle_{x^2 = x_0^2} + O(g^2). \]

(4.48)

To compute this integral it is convenient to use dimensional regularization. We have

\[ \langle T_{\mu\nu}(x) O(0) O(y) \rangle = \frac{-C_{T\mu0})}{(x^2 - y^2)\delta^{d-4}} \left( X_\mu X_\nu - \frac{1}{d}\delta_{\mu\nu} x^2 \right), \]

(4.49)

where \( X_\mu \equiv x_\mu / x^2 - (x - y)_\mu / (x - y)^2 \). Therefore, we find

\[ \int d^2y \langle T_{\mu\nu}(x) O(0) O(y) \rangle = -\frac{2C_{T\mu0}}{d(d - 2\Delta_0)} \frac{\pi^d}{(d/2 + 1)} \frac{1}{(x^2)^{\Delta_0}} \left( \delta_{\mu\nu} - \frac{x_\mu x_\nu}{x^2} \right). \]

(4.50)

In \( d = 2 \) we obtain

\[ H(g) = g\omega^2 x^2 C_{T\mu0} \frac{\pi (2 - \Delta_0)}{1 - \Delta_0} \frac{w^2}{(x^2)^{\Delta_0 + 1}} + O(g^2) \]
\[ = gC_{T\mu0} \frac{\pi (2 - \Delta_0)}{1 - \Delta_0} \frac{1}{(x^2)^{\Delta_0 - 2}} + O(g^2). \]

(4.51)

Since the operator \( O \) is marginal, \( \Delta_0 = 2 + O(g) \), we have \( H(g) = O(g^2) \). This implies

\[ C_T(g) \propto c(g) + O(g^4). \]

(4.52)

So we can write

\[ C_T^{d=2}/C_T^{\text{free}} = 1 + (c(g) - c_{\text{free}})/c_{\text{free}} = 1 + \delta \tilde{F}/\tilde{F}_{\text{free}}, \]

(4.53)

where \( \tilde{F} = -\sin(\pi d/2)F \), and \( F \) is the free energy on the \( d \)-dimensional sphere [56]. For a CFT in \( d = 2 \) we have \( \tilde{F} = \pi c/6 \); therefore, \( c(g) = 6\tilde{F}(g)/\pi \). For the free fermion free-energy in \( d = 2 \) we have [56]
corresponding to the standard value for the free fermion central charge, \( c_{\text{free}} = N/2 \). For the change of the free-energy we find \([57, 58]\)

\[
\delta \mathcal{F} = \pi^3 \int_0^g G(g) \beta(g) dg, \tag{4.55}
\]

where from \((4.44)\) we have \( G(g) = N(N-1)/(2(2\pi)^4) + \mathcal{O}(g) \) and \( \beta(g) = -(N-2)g^2/(2\pi) + \mathcal{O}(g^3) \). Therefore, we obtain for \((4.53)\)

\[
C_T^{d=2} / C_T^{\text{free}} = 1 - \frac{\pi(N-1)(N-2)}{(2\pi)^4} g^3. \tag{4.56}
\]

Thus, in \( d = 2 \) the leading correction is of order \( g^3 \). In \( d = 2 + \epsilon \) this term, evaluated at the fixed point \( g^* = 2\pi\epsilon/(N-2) \), gives a correction of order \( \epsilon^3 \). Adding this correction to the one coming from the order \( g^2 \epsilon \) term \((4.41)\) we finally find

\[
C_T^{GN} / C_T^{\text{free, GN}} = 1 - \frac{(N-1)}{8(N-2)^2} \epsilon^3 + \mathcal{O}(\epsilon^4), \tag{4.57}
\]

In the large \( N \) limit this agrees with \((4.25)\), providing a check of our large \( N \) calculation. The negative sign of the correction in \((4.57)\) means that in \( 2 + \epsilon \) dimensions the \( C_T \) theorem is violated for the GN model with all \( N > 2 \).

### 4.5. Padé approximations

We have the following \( \epsilon \) expansions for \( C_j^{GN} / C_{j,\text{free}}^{GN} \).
In this case we find that only the approximant Padé [2,2] has no poles; it approaches the target result well. We plot Padé [2,2] for different $N$ in figure 25. We also give the $d = 3$ values of $C_T^{\text{GN}} / C_{T,\text{free}}^{\text{GN}}$ for different $N$ in table 3. We consider integer values of $\tilde{N}$, so that $N = N_{\text{tr1}} = 4\tilde{N}$ is a multiple of 4.

We have the following $\epsilon$-expansions for $C_T^{\text{GN}} / C_{T,\text{free}}^{\text{GN}} (d)$

$$C_T^{\text{GN}} / C_{T,\text{free}}^{\text{GN}} (d) = \begin{cases} 0.58 & \frac{1}{2} - \frac{1}{2} \left( \frac{N-1}{N-2} \epsilon^3 + \mathcal{O}(\epsilon^4) \right) \quad \text{in} \quad d = 2 + \epsilon, \\ 0.59 & 1 + \frac{2}{3N} - \frac{11N - 24}{18N (N + 6)} \epsilon + \mathcal{O}(\epsilon^2) \quad \text{in} \quad d = 4 - \epsilon. \end{cases} \quad (4.58)$$

In this case we find that only the approximant Padé [2,2] has no poles; it approaches the target result well. We plot Padé [2,2] for different $N$ in figure 25. We also give the $d = 3$ values of $C_T^{\text{GN}} / C_{T,\text{free}}^{\text{GN}}$ for different $N$ in table 3. We consider integer values of $\tilde{N}$, so that $N = N_{\text{tr1}} = 4\tilde{N}$ is a multiple of 4.

We have the following $\epsilon$-expansions for $C_T^{\text{GN}} / C_{T,\text{free}}^{\text{GN}} (d)$

$$C_T^{\text{GN}} / C_{T,\text{free}}^{\text{GN}} (d) = \begin{cases} 0.58 & \frac{1}{2} - \frac{1}{2} \left( \frac{N-1}{N-2} \epsilon^3 + \mathcal{O}(\epsilon^4) \right) \quad \text{in} \quad d = 2 + \epsilon, \\ 0.59 & 1 + \frac{2}{3N} - \frac{11N - 24}{18N (N + 6)} \epsilon + \mathcal{O}(\epsilon^2) \quad \text{in} \quad d = 4 - \epsilon. \end{cases} \quad (4.59)$$

In this case we find that all two-sided Padé approximants have poles. One reason for this behavior is the non-monotonicity of the function we are trying to approximate. To make our approximation better, we apply instead the Padé procedure to the following combination

$$f (d) \equiv \left( \frac{N}{2} (C_T^{\text{GN}} / C_{T,\text{free}}^{\text{GN}} (d) - 1) - \frac{1}{d - 1} \right) = \left( \frac{N}{2} + \frac{1}{d - 1} \right) \quad (4.60)$$

This combination is natural from the point of view of the GNY model. It corresponds to writing $C_T^{\text{GNV}} = C_T^{\text{GNV}} (1 + f (d))$, where $C_T^{\text{GNV}} = \left( \frac{N}{2} + \frac{d}{d - 1} \right) / S_d^2$ is the contribution of the $\tilde{N}$ free fermions and the single free scalar. We find that $f (d)$ is now a monotonic function at
large $N$, and has the $\epsilon$ expansions

$$ f(d) = \begin{cases} 
\frac{2}{N+2} - \frac{2N\epsilon}{(N+2)r^2} - \frac{2N^2\epsilon^2}{(N+2)^3} + \mathcal{O}(\epsilon^3) & \text{in } d = 2 + \epsilon, \\
\frac{15N^3 - 60N^4 + 58N^5 + 4N^5 + 8N\epsilon}{8(N-2)^2(N+2)^4} + \mathcal{O}(\epsilon^2) & \text{in } d = 4 - \epsilon,
\end{cases} \quad (4.61) $$

Applying Padé approximation to this function we find that Padé$[1,4]$ and Padé$[4,1]$ do not have poles for $N \geq 4$ and are in a good agreement with the large $N$ result. Now we may return to the function $\frac{C_{T}^{\text{GN}}/C_{T,\text{tree}}^{\text{GN}}(d) - 1}{N}$. We plot Padé$[1] = (\text{Padé}[1,4] + \text{Padé}[4,1])/2$ for different $N$ in figure 26. We also give the $d = 3$ values of $C_{T}^{\text{GN}}/C_{T,\text{tree}}^{\text{GN}}$ for various choices of $N$ in table 4; the approach to the large $N$ approximation (1.19) appears to be very fast. These values differ from one by only around a percent even for small $N$.

Let us note that $C_{T}^{\text{GN}} > C_{T,\text{tree}}^{\text{GN}}$ for all $N$ in $d = 3$. Thus, the $C_T$ decreases for the RG flow caused by adding the scalar mass term $m^2\sigma^2/2$ in the GNY theory. We may also consider the case $m^2 < 0$ which causes $\sigma$ to acquire a vacuum expectation value. In this vacuum the fluctuations of the $\sigma$ field are massive, and the fermions acquire a mass, so that the time reversal symmetry is broken. The continuous symmetry acting on the fermions remains unbroken; therefore, there are no Nambu–Goldstone bosons. Since the IR theory has only massive excitations, $C_{T}^{\text{IR}} = 0$, and the inequality $C_{T}^{\text{UV}} > C_{T}^{\text{IR}}$ is obviously satisfied for the flow caused by $m^2 < 0$.

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Note added in proof. After the first version of this paper appeared, we were informed by H. Osborn and A. Stergiou that, via a direct calculation [59], they obtained values of $C_T$ for the higher-derivative scalar fields that agree with (3.55) and (4.27). The latter agreement provides additional evidence in favor of our results for the GN model.

Appendix A. Tensor reduction

In this appendix we describe the standard tensor reduction for Feynman integrals in general dimension (see for example [60]). We use this type of reduction because it does not change
the dimension of the integrals, but unfortunately it sometimes adds new denominators to the integrals\textsuperscript{14}. On the other hand, there is another type of tensor reduction called Davydychev recursion relations \cite{Davydychev1995, Davydychev1996}. This method does not add new denominators to Feynman integrals, but it changes their dimension. This type of reduction was applied in the papers \cite{Diab2014, Diab2016} for a very similar computations in $d = 3$.

Let us first briefly review the main logic. Suppose we are trying to evaluate an $m$-loop Feynman integral with loop momenta $p_i$, where $i = 1, \ldots, m$, a single external momentum $p$, and $n$ uncontracted Euclidean indices:

\[
I^{\mu_1 \cdots \mu_k}(p) = \int_{p_{i_1}, \ldots, p_n} \frac{p^{\mu_1}_i \cdots p^{\mu_k}_i \mathrm{Numer}}{\mathrm{Denom}},
\]  

(A.1)

where $\int_p \equiv \int \frac{d^dp}{(2\pi)^d}$ and the (Numer) denotes some function of $(p_i \cdot p), (p_i \cdot p_j)$ and $p^2$. We would like to convert this into a sum of scalar integrals only. First, we define the components of the loop momenta transverse to the external momentum as:

\[
p^{\perp}_i \equiv p^\mu_i - \frac{p \cdot p^\mu_i}{p^2} p^\mu_i.
\]  

(A.2)

Using this formula in (A.1), we get that the original integral $I^{\mu_1 \cdots \mu_k}(p)$ is equal to a sum of integrals of the following form:

\[
I^{\perp \mu_1 \cdots \perp \mu_k}(p) = \int_{p_{i_1}, \ldots, p_n} \frac{p^{\perp}_{\mu_1} \cdots p^{\perp}_{\mu_k} \mathrm{Numer}}{\mathrm{Denom}}.
\]  

(A.3)

Now we notice that the tensor $I^{\perp \mu_1 \cdots \perp \mu_k}$ is transverse with respect to all its indices:

\[
p_{\mu_i} I^{\perp \mu_1 \cdots \perp \mu_k}(p) = 0, \quad \text{for all } i = 1, \ldots, k.
\]  

(A.4)

At the same time $I^{\mu_1 \cdots \mu_k}$ can be expressed only from the external momentum $p^\mu$ and the Kronecker delta $\delta^\mu_\mu$. Notice that if $k$ is odd, then the integral is zero, because for instance there must be a term $p^{\mu_1} \delta^{\mu_2 \mu_3} \cdots \delta^{\mu_{k-1} \mu_k}$, and $I^{\perp \mu_1 \cdots \perp \mu_k}$ cannot be made transverse to $p^{\mu_k}$. Therefore, we can focus only on even $k$.

In this paper we are dealing with the cases of $k = 2$ and $k = 4$. Let us start with the case of $k = 2$, so we have

\[
I^{\perp \mu_1 \perp \mu_2}(p) = \int_{p_{i_1}, \ldots, p_n} \frac{p^{\perp}_{\mu_1} p^{\perp}_{\mu_2} \mathrm{Numer}}{\mathrm{Denom}} = (\delta^{\mu_2 \mu_1} - p^{\mu_1} p^{\mu_2}/p^2)I(p),
\]  

(A.5)

where $I(p)$ is some scalar function and $j_1, j_2$ can be $1, \ldots, m$. Now if we contract (A.5) with $\delta^{\mu_1 \mu_2}$ we can easily find

\[
I(p) = \frac{1}{d - 1} \int_{p_{i_1}, \ldots, p_n} \frac{(p^{\perp}_{j_1} \cdot p^{\perp}_{j_2}) \mathrm{Numer}}{\mathrm{Denom}}.
\]  

(A.6)

Further reduction to usual scalar integrals can be made by using:

\[
p^{\perp}_{i} \cdot p^{\perp}_{j} = p^{\perp}_{i} \cdot p^{\perp}_{j} = \frac{1}{p^2} (p^{\perp}_{i} \cdot p^{\perp}_{j}), \quad (p^{\perp}_{i} \cdot p) = \frac{1}{2} ((p + p^{\perp}_{i})^2 - p^2 - p^2).
\]  

(A.7)

Now consider the case of $k = 4$. We have

\textsuperscript{14} This is why in our Aslamazov–Larkin (ladder) type diagrams we have $a_9$ index (see (B.1)). In order to bring this index to zero we apply a complicated recursion relation, discussed in appendix B.
\[ L^i (a_1, a_2, a_3, a_4 | a_5, a_6) = \]

\[ \begin{array}{c}
\text{Numer} \\
\text{Denom} \\
1234
\end{array} \]

Figure B1. Example of a general ladder diagram \((p_1 - p_2 \text{ and } a_9 \text{ are not included})\).

\[ I^j_{\mu_1 \mu_2 \mu_3 \mu_4} (p) = \int_{p_1, \ldots, p_n} \frac{p_{\mu_1} p_{\mu_2} p_{\mu_3} p_{\mu_4}}{(p_1^2)} (\text{Denom}) \]

\[ = (\delta_{\mu_1 \mu_2} p_{\mu_1} p_{\mu_2} + \delta_{\mu_3 \mu_4} p_{\mu_3} p_{\mu_4} - p_1^2 \delta_{\mu_1 \mu_2} \delta_{\mu_3 \mu_4} - p_{\mu_1} p_{\mu_2} p_{\mu_3} p_{\mu_4}) I_1 (p) \]

\[ + (\delta_{\mu_1 \mu_3} p_{\mu_1} p_{\mu_3} + \delta_{\mu_2 \mu_4} p_{\mu_2} p_{\mu_4} - p_1^2 \delta_{\mu_1 \mu_3} \delta_{\mu_2 \mu_4} - p_{\mu_1} p_{\mu_3} p_{\mu_2} p_{\mu_4}) I_2 (p) \]

\[ + (\delta_{\mu_1 \mu_4} p_{\mu_1} p_{\mu_4} + \delta_{\mu_2 \mu_3} p_{\mu_2} p_{\mu_3} - p_1^2 \delta_{\mu_1 \mu_4} \delta_{\mu_2 \mu_3} - p_{\mu_1} p_{\mu_4} p_{\mu_2} p_{\mu_3}) I_3 (p), \]  

(A.8)

where \( j_1, \ldots, j_4 \) can be 1, \ldots, \( m \) and \( I_1, I_2, I_3 \) are some scalar functions. The particular combination of tensor structures in front of them are fixed by the fact that they should vanish when contracted with \( p_{\mu_1} \) (or \( p_{\mu_2} \)) and \( p_{\mu_3} \) (or \( p_{\mu_4} \)). These are the only three structures with four Euclidean indices, constructed from \( p_{\mu} \) and \( \delta_{\mu \nu} \), and transverse with respect to all indices, so this decomposition is general.

Now, if we contract (A.8) with \( \delta_{\mu_1 \mu_2} \delta_{\mu_3 \mu_4}, \delta_{\mu_1 \mu_3} \delta_{\mu_2 \mu_4}, \) and \( \delta_{\mu_1 \mu_4} \delta_{\mu_2 \mu_3} \), we get three equations, which have the solution

\[ I_1 = \frac{1}{(d-1)(d-2)p_1^2} \int_{p_1, \ldots, p_n} ((p_{\mu_1} p_{\mu_2}) (p_{\mu_3} p_{\mu_4}) + (p_{\mu_1} p_{\mu_3}) (p_{\mu_2} p_{\mu_4}) - d (p_{\mu_1} p_{\mu_2}) (p_{\mu_3} p_{\mu_4}))(\text{Numer}) \]

(A.9)

and \( I_2 \) and \( I_3 \) can be obtained from \( I_1 \) by replacements \( j_2 \leftrightarrow j_4 \) and \( j_2 \leftrightarrow j_4 \) correspondingly. Further reduction can be made by using formulas (A.7) and finally, everything reduces to scalar integrals.

**Appendix B. Recursion relations**

The most difficult part of the calculation is the three-loop ladder (Aslamazov–Larkin) diagram with some non-trivial numerator. After the tensor reduction we are required to compute
integrals of the form:

\[
\int \frac{1}{p_1^{2a_1}(p + p_1)^{2a_2}(p_1 - p_2)^{2a_3}(p + p_2)^{2a_4}(p_2 - p_1)^{2a_5}}. 
\]

(B.1)

where \( \int \equiv \int \frac{dp}{(2\pi)^d} \) and \( p \) is the external momentum. The indices \( a_1 \) to \( a_5 \) correspond to lines shown in figure B1. Note that \( a_9 \), which corresponds to the momentum combination \( p_1 - p_2 \), does not appear in the figure. It is generated by tensor reductions and it can only be a negative integer in our calculation. It is not feasible to evaluate such a large number of diagrams individually. Therefore, we seek to reduce these into a small number of 'master integrals' through integration by parts relations. However, programs such as FIRE [63] does not work well when multiple non-integer indices are included. We therefore need to implement our own reduction relations.

We would first like to use some recursion relation to reduce to \( a_9 = 0 \).

The non-trivial general relation to reduce \( a_9 \) is:

\[
L\left(\begin{array}{c|c|c|c|c|c}
  a_1 & a_2 & a_3 & a_4 & a_5 & a_9 \\
  a_6 & a_7 & a_8 & a_9 & a_9 & a_9
\end{array}\right) = \int_{p_1,p_2,p_3} \frac{1}{p_1^{2a_1}(p + p_1)^{2a_2}(p_1 - p_2)^{2a_3}(p + p_2)^{2a_4}(p_2 - p_1)^{2a_5}}. 
\]

where \( a_{\text{nonl}} \equiv a_n + a_m + a_l + ... \). The relation (B.2) is expected to hold for arbitrary indices. This relation can be used to reduce all integrals to have \( a_9 = 0 \). We will denote the ladder diagrams with \( a_9 = 0 \) as \( L\left(\begin{array}{c|c|c|c|c|c}
  a_1 & a_2 & a_3 & a_4 & a_5 & a_9 \\
  a_6 & a_7 & a_8 & a_9 & a_9 & a_9
\end{array}\right) \equiv L\left(\begin{array}{c|c|c|c|c|c}
  a_1 & a_2 & a_3 & a_4 & a_5 & a_9 \\
  a_6 & a_7 & a_8 & a_9 & a_9 & a_9
\end{array}\right) \). After the reduction of \( a_9 \) the majority of the integrals can be reduced to two-loop integrals of the form

\[
K(a_1, a_2, a_3, a_4, a_5) = \int_{p_1,p_2,p_3} \frac{1}{p_1^{2a_1}(p + p_1)^{2a_2}(p_1 - p_2)^{2a_3}(p + p_2)^{2a_4}(p_2 - p_1)^{2a_5}}. 
\]

(B.3)

This integral is shown in figure B2.

There is an extensive literature about different methods for the computation of this type of integrals [35, 36, 47, 48, 64–67]. The other diagrams can be reduced to the diagram of type...
\[ L \left( \begin{array}{c|c|c} \alpha \\ \hline \beta \\ \hline \end{array} \right) , \text{ where} \]
\[ \alpha = \frac{d}{2} - n + \Delta, \quad \beta = \frac{d}{2} - m + \Delta \] (B.4)
and \( n \) and \( m \) are some integers. The diagram with \( \alpha = \beta = \frac{d}{2} - 2 + \Delta \) was originally computed in [35] and the result reads
\[
L \left( \begin{array}{c|c|c} \alpha \\ \hline \beta \\ \hline \end{array} \right) = \left( p^2 \right)^{d/2 - 2 + \Delta} \left( \frac{\Gamma(\frac{d}{2})}{\Gamma(\alpha)} \right)^2 \frac{\pi^{\frac{d}{2}} 2^{2(d+2-\Delta)}}{a(2\Delta - \frac{d}{2} + 2)} \]
\[
\times \left( a(\frac{d}{2} - 1) \right) \left( \frac{1}{\Delta} + 4B(2) - B(d - 3) - 3B \left( \frac{d}{2} - 1 \right) \right). \] (B.5)
where
\[ a(\alpha) = \Gamma \left( \frac{d}{2} - \alpha \right) / \Gamma(\alpha), \quad B(x) = \psi(x) + \psi \left( \frac{d}{2} - x \right). \] (B.6)
We consider this integral as the master integral. All other diagrams of this type can be related to this master integral using a non-trivial recursion relation\(^{15}\):
\[
L \left( \begin{array}{c|c|c} \alpha \\ \hline \beta \\ \hline \end{array} \right) = \left( \frac{d - 2 - \alpha - \beta}{d - 3 - \alpha} \right) \left( \frac{3d/2 - 4 - \alpha - \beta}{d/2 - 1 - \alpha} \right) L \left( \begin{array}{c|c|c} \alpha - 1 \\ \hline \beta \\ \hline \end{array} \right) \]
\[ - \left( \frac{d - 2 - \alpha - \beta}{d - 3 - \alpha} \right) L \left( \begin{array}{c|c|c} \alpha - 1 \\ \hline \beta \\ \hline \end{array} \right) L \left( \begin{array}{c|c|c} \alpha - 1 \\ \hline \beta \\ \hline \end{array} \right) \]
\[ + \left( \frac{2d - 5 - 2\alpha - \beta}{d - 3 - \alpha} \right) L \left( \begin{array}{c|c|c} \alpha - 1 \\ \hline \beta \\ \hline \end{array} \right) L \left( \begin{array}{c|c|c} \alpha - 1 \\ \hline \beta \\ \hline \end{array} \right) \]
\[ - \left( \frac{d - 3}{d/2 - 1 - \alpha} \right) \left( \frac{1}{a(\alpha)} \right) L \left( \begin{array}{c|c|c} \alpha - 1 \\ \hline \beta \\ \hline \end{array} \right). \] (B.7)
where \( \alpha, \beta \) can be arbitrary non-integer and the integrals of the type \( L \left( \begin{array}{c|c|c} \alpha \\ \hline \beta \\ \hline \end{array} \right) \) and etc can be reduced to the \( K(\alpha, \ldots, \alpha) \) integrals.

**Appendix C. Z\(_T\) factor calculation for the critical fermion**

In this appendix we present different methods for the computation of the \( Z\(_T\) \) factor for the stress–energy tensor in the GN model. For what follows, it is important for us to know \( \hat{C}_{\psi 1}, \hat{C}_{\psi 1}, C_{\psi 1}^{\gamma_{GN}}, \gamma_{GN}, \psi_{\psi 1}, Z_{\psi 1}, Z_{\psi 1} \) defined in (4.6), (4.8) and (4.9). To compute \( \hat{C}_{\psi 1}, \gamma_{GN}, \psi_{\psi 1}, Z_{\psi 1} \)

\(^{15}\) Notice that for some \( \alpha \) and \( \beta \) in order to correctly apply this recursion relation one has to take into account \( O(\Delta) \) terms in the integrals.
we have to consider the one loop diagram for the renormalization of the \(\langle \psi \bar{\psi} \rangle\) propagator. The diagram is depicted in figure C1 and reads

\[
\int \frac{d^d p_1}{(2\pi)^d} \frac{i(\mathbf{p} + p_1)}{(p_1^2 - \Delta)(p + p_1)^2}.
\]

Using the integral (3.41) we find

\[
\gamma_{1}^\text{GN} = Z_{\sigma 1} = \frac{\Gamma(d - 1) \left(\frac{d}{2} - 1\right)^2}{\Gamma(d - 2) \Gamma\left(\frac{d}{2} + 1\right) \Gamma\left(\frac{d}{2}\right)}.
\]

and

\[
\tilde{C}_{\sigma 1} = \frac{2^d - 1 \sin\left(\frac{\pi d}{2}\right) \Gamma\left(\frac{d + 1}{2}\right)}{\pi^{d/2} \Gamma\left(\frac{d}{2}\right)}.
\]

To find \(\tilde{C}_{\sigma}\) and \(\Delta_{\sigma}\) and \(Z_{\sigma}\) to the \(1/N\) order we have to compute the diagrams for the \(\langle \sigma_0(p)\sigma_0(-p) \rangle\) propagator represented in figure C2.

The expressions for the diagrams are\(^\text{16}\)

\[
D_0 = \frac{\tilde{C}_{\sigma 0}}{(p^2)^{\frac{d}{2} - 1}},
\]

\[
D_1 = 2 \left(\frac{\tilde{C}_{\sigma 0}}{(p^2)^{\frac{d}{2} - 1}}\right)^2 \mu^{2\Delta} \int \frac{d^d p_1}{(2\pi)^d} \frac{(-1)^{\text{Tr}((\mathbf{p} + p_1)\mathbf{p}_1\mathbf{p}_2\mathbf{p}_1)\tilde{C}_{\sigma 0}}}{(p + p_1)^2(p_1^2 - \Delta)(p_1 - p_2)^2(p_2^2 - \Delta),}
\]

\[
D_2 = \left(\frac{\tilde{C}_{\sigma 0}}{(p^2)^{\frac{d}{2} - 1}}\right)^2 \mu^{2\Delta} \int \frac{d^d p_1}{(2\pi)^d} \frac{(-1)^{\text{Tr}((\mathbf{p} + p_1)\mathbf{p}_1\mathbf{p}_2\mathbf{p}_1)\tilde{C}_{\sigma 0}}}{(p + p_1)^2(p + p_2)^2(p_1^2 - \Delta)(p_1 - p_2)^2(p_2^2 - \Delta),}
\]

and

\[
\langle \sigma(p)\sigma(-p) \rangle = Z_{\sigma} \langle \sigma_0(p)\sigma_0(-p) \rangle = Z_{\sigma} (D_0 + D_1 + D_2 + \mathcal{O}(1/N^2)).
\]

Computing these diagrams one finds

\[
Z_{\sigma 1} = \frac{4^d \sin(\pi d/2) \Gamma\left(\frac{d}{2} + 1\right)}{\pi^{d/2} \Gamma\left(\frac{d}{2} + 1\right)}, \quad \Delta_{\sigma} = 1 + \frac{4^d \sin(\pi d/2) \Gamma\left(\frac{d}{2} + 1\right)}{\pi^{d/2} \Gamma\left(\frac{d}{2} + 1\right)} \frac{1}{N} + \mathcal{O}(1/N^2)
\]

\(^{16}\) Note that it is very important that we do not shift the power in the \(\langle \sigma_0\rangle\)-external lines by \(\Delta\). One can explain this by noticing that the \(\Delta\) shift in a \(\langle \sigma_0\rangle\)-propagator under a Feynman integral is analogous to changing the dimension of the integral \(d \rightarrow d^' = d - 2\Delta\), while keeping intact the power of \(\langle \sigma_0\rangle\)-propagator [68, 69].
\[ g_1 = \frac{1}{N} \gamma_1 \left( \mathcal{C}_{GN}(d) + \frac{4(d-1)}{d(d-2)} \right) + \mathcal{O}(1/N^2) \]  
(C.7)

where \( \Delta_\sigma = 1 + \gamma_\sigma \) and \( \gamma_\sigma = \gamma_\sigma_1/N + \mathcal{O}(1/N^2) \) and \( \gamma_\epsilon = -\gamma^{GN} - \kappa^{GN} \).

We recall that the ‘bare’ stress–energy tensor \( T_{\mu\nu} \) is related to ‘renormalized’ one \( T_{\mu\nu}^{\text{ren}} \) as

\[ T_{\mu\nu}^{\text{ren}}(x) = Z_T T_{\mu\nu}(x), \]  
(C.8)

where \( Z_T = 1 + \left( Z_T / \Delta \right) + Z_T^p / N + \mathcal{O}(1/N^2) \). Let us first use the three-point function \( \langle T_{\mu\nu}^{\text{ren}}(x_1)\sigma(x_2)\sigma(x_3) \rangle \) to determine \( Z_T \) at \( 1/N \) order. Using conformal invariance and stress–energy tensor conservation one has the general expression for the three-point function

\[ \langle T_{\mu\nu}^{\text{ren}}(x_1)\sigma(x_2)\sigma(x_3) \rangle = \frac{-C_{T\sigma\sigma}}{(x_{12}^2 x_{13}^2) \sqrt{1 - (x_{23}^2)^{\Delta_\sigma - \frac{4}{d} + 1}}} \left( (X_{23})_{\mu}(X_{23})_{\nu} - \frac{1}{d} \delta_{\mu\nu} (X_{23})^2 \right), \]  
(C.9)

where

\[ (X_{23})_{\nu} = \frac{(x_{2})_{\nu}}{x_{12}^2} - \frac{(x_{3})_{\nu}}{x_{13}^2} \]  
(C.10)

and the Ward identity can be used to relate \( C_{T\sigma\sigma} \) with \( C_{\sigma} \)

\[ C_{T\sigma\sigma} = \frac{1}{S_\sigma} \frac{d \Delta_\sigma}{d} C_\sigma, \]  
(C.11)

where \( \Delta_\sigma \) is the anomalous dimension of the field \( \sigma \) and \( C_\sigma \) is the two-point constant of \( \langle \sigma \sigma \rangle \)-propagator in the coordinate space. Taking the Fourier transform of (C.9) and setting the momentum of the stress–energy tensor to zero one finds, in terms of \( T = z^{\mu\nu} \varepsilon^{\nu\sigma} T_{\mu\nu}^{\text{ren}} \)

\[ \langle T_{\mu\nu}^{\text{ren}}(0)\sigma(p)\sigma(-p) \rangle = (d - 2\Delta_\sigma) \tilde{C}_\sigma \frac{p^2}{(p^2)^{\Delta_\sigma - \frac{4}{d} + 1}}, \]  
(C.12)

where \( \tilde{C}_\sigma \) is the normalization of the two-point function \( \langle \sigma \sigma \rangle \) in momentum space. Now we can compute the three-point function \( \langle T_{\mu\nu}^{\text{ren}}(0)\sigma(p)\sigma(-p) \rangle \) directly using Feynman diagrams.
We write

$$\langle T^{\text{ren}}(0)\sigma(p)\sigma(-p) \rangle = Z_T Z_o \langle T(0)\sigma_0(p)\sigma_0(-p) \rangle$$

(C.13)

and the diagrams contributing to $$\langle T(0)\sigma_0(p)\sigma_0(-p) \rangle$$ up to order $$1/N$$ are shown in figure C3. Note that for some topologies we did not draw explicitly diagrams with the opposite fermion loop direction, but they have to be included.

Computing these diagrams and equating the expression (C.12) with the diagrammatic result for the expression (C.13) we find

$$Z_{T1} = 2^{\gamma^\text{GN}_1} \frac{d + 2}{d + 2}, \quad Z'_{T1} = \frac{8\gamma^\text{GN}_1}{(d + 2)(d - 2)},$$

(C.14)

where $$\gamma^\text{GN}_1$$ is given in (C.2).

Alternatively, we can consider the three-point function $$\langle T^{\text{ren}}\bar{\psi}(p)\psi(-p) \rangle$$. Unfortunately, as far as we know, the general form of it in the coordinate space in general $$d$$ is not known. But from general analysis and from our diagrammatic results we argue that in momentum space and setting $$T$$ at zero momentum, it has the form:

$$\langle T^{\text{ren}}(0)\psi(p)\bar{\psi}(-p) \rangle = i\tilde{C}_0 \left( \frac{\gamma_p p}{(p^2)^{\frac{d}{2} - \Delta + \frac{1}{2}}} - (d - 2\Delta + 1) \frac{p^2}{(p^2)^{\frac{d}{2} - \Delta + \frac{1}{2}}} \right).$$

(C.15)

On the other hand we can compute $$\langle T^{\text{ren}}(0)\psi(p)\bar{\psi}(-p) \rangle$$ directly by Feynman diagrams

$$\langle T^{\text{ren}}(0)\psi(p)\bar{\psi}(-p) \rangle = Z_T Z_o \langle T(0)\psi_0(p)\bar{\psi}_0(-p) \rangle,$$

(C.16)

where the diagrams contributing to $$\langle T(0)\psi_0(p)\bar{\psi}_0(-p) \rangle = D_0 + D_1 + D_2 + D_3 + O(1/N^2)$$ are given in figure C4 and read

$$D_0 = \frac{ip}{p^2} \frac{i\bar{\psi}}{2(p_{\xi}^2)} \gamma_\xi \gamma \frac{ip^2}{(p^2)^2} \left( \frac{\gamma_p p}{p^2} - \frac{2p^2}{(p^2)^2} \right),$$

$$D_1 = \frac{i\mu^\Delta}{N} \int \frac{dp_1}{(2\pi)^d} \frac{p \gamma_1 p \gamma_2 \gamma_3 \gamma_0 \bar{C}_{\xi 0}}{(p^2)^2(p_1^2 - p_1 p_2^2(1 - \Delta))},$$

$$D_2 = \frac{i\mu^\Delta}{N} \int \frac{dp_1}{(2\pi)^d} \frac{p \gamma_1 p \gamma_1 \gamma_0 \bar{C}_{\xi 0}}{(p^2)^2(p_1^2 - p_1 p_2^2(1 - \Delta))},$$

$$D_3 = \frac{i\mu^\Delta}{N} \int \frac{dp_1 dp_2}{(2\pi)^d} \frac{p (p - p_2) \gamma_0 \bar{C}_{\xi 0} \text{Tr} (p_1 \gamma_p p_1 (p - p_2))}{(p^2)^2(p - p_2)^2(p_1^2 - p_1 p_2^2(1 - \Delta))},$$

(C.17)

Computing these diagrams and using (C.15) and (C.16), we find the same result (C.14) obtained above.

17 Here we fix some field, say $$\psi = \psi^i$$ and do not write the flavor index explicitly.
Appendix D. \(Z_J\) factor calculation for the critical fermion

We can consider the three-point function \(\langle J^\mu(p_1)\bar{\psi}(p_2)\bar{\psi}(p_3)\rangle\), which is fixed by conformal invariance and current conservation \([70, 71]\)

\[
\langle J^\mu (x_1)\psi^i (x_2)\bar{\psi}_j (x_3) \rangle = - \left( c_{12}\frac{\langle x_1 x_2 x_3 \rangle}{(x_2 x_3)^2} + c_{23}\frac{\langle x_2 x_3 x_1 \rangle}{(x_2 x_3)^2} \right) \langle \bar{\psi}_j \psi_i \rangle. \tag{D.1}
\]

The Ward identity gives a relation between the structure constants \(C_{J\bar{\psi}}^{(1)}\) and \(C_{J\bar{\psi}}^{(2)}\)

\[
\langle \delta \psi^i (x_2)\bar{\psi}_j (x_3) \rangle = - \varepsilon \int d^d\Omega \delta^{d-2}\rho_{\mu} \langle J^\mu (x_1)\psi^i (x_2)\bar{\psi}_j (x_3) \rangle,
\]

where \(\rho_{\mu}\) is the stress-energy tensor. Performing the integral in the limit \(r \rightarrow 0\) we find

\[
C_{J\bar{\psi}}^{(1)} + C_{J\bar{\psi}}^{(2)} = \frac{C_{\psi}}{S_d} \tag{D.3}
\]

Taking the Fourier transform of (D.1) and using (D.3) we get for \(J^\mu^{ren} \) at zero momentum

\[
\langle J^\mu^{ren} (0)\psi^i (p)\bar{\psi}_j (p) \rangle = C_{i} \left( d - 2\Delta_\psi + 1 - \frac{\mu p^2}{(p^2)^2} - \frac{\gamma_{\psi}}{p^2} \right) \langle \bar{\psi}_j \psi_i \rangle. \tag{D.4}
\]

Now to fix \(Z_J\) we compute \(\langle J^\mu \bar{\psi} \rangle\) using Feynman diagrams

\[
\langle J^\mu^{ren} (0)\psi^i (p)\bar{\psi}_j (p) \rangle = Z_J \langle J^\mu (0)\psi^i (p)\bar{\psi}_j (p) \rangle, \tag{D.5}
\]

and to 1/N order we have three diagrams contributing to (D.5), which are shown in figure D1.

The diagrams read

\[
D_0 = \frac{i\mu}{p^2}(-\gamma_\psi)\frac{2p\mu}{(p^2)^2} - \frac{\gamma_{\psi}}{p^2},
\]

\[
D_1 = \frac{2(\mu^{2\Delta})}{N} \int \frac{dp_1}{2\pi^d} \frac{p^2 - (\gamma_{\psi})p_1}{p^2} \frac{p}{p^2} \mathcal{C}_{\psi} \langle \bar{\psi}_j \psi_i \rangle, \tag{D.6}
\]

\[
D_2 = \frac{2(\mu^{2\Delta})}{N} \int \frac{dp_1}{(2\pi)^d} \frac{p^2 - (\gamma_{\psi})p_1}{p^2} \frac{p}{p^2} \mathcal{C}_{\psi} \langle \bar{\psi}_j \psi_i \rangle
\]

and

\[
\langle J^\mu(0)\psi^i_0(p)\bar{\psi}_j_0(-p) \rangle = D_0 + D_1 + D_2 + \mathcal{O}(1/N^2). \tag{D.7}
\]

Computing the diagrams and using (D.4) and (D.5) we find

\[
Z_J = 1 + \mathcal{O}(1/N^2). \tag{D.8}
\]
Appendix E. Integrals and results

E.1. Integrals for $C_J$ for the $O(N)$ scalar theory in $6 - \epsilon$ (figure 1)

Explicitly, the diagrams are:

\[
D_0 = \int \frac{d^d p_1}{(2\pi)^d} \frac{(p_1 + 2p)^2}{p_1^2(p + p_1)^2} = \frac{\pi^\frac{d}{2} \Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{d}{2} - 1\right)}{2^{d-1} (d-1) \Gamma(d-2)} \frac{p^2}{(p^2)^{\frac{d-4}{2}}}.
\]

\[
D_1 = 2g_1^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^d} \frac{(p_1 + 2p_2)^2}{(p_1 + p_2)^2} \frac{(p_1 + p_2)^2}{p_1^2} = \frac{2g_1^2}{d-6d + 2d^2} \frac{p_1^2}{p_2^2}.
\]

\[
D_2 = g_1^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^d} \frac{(p_1 + 2p_2)^2}{(p_1 + p_2)^2} \frac{(p_1 + p_2)^2}{p_1^2} \frac{(p_1 - p_2)^2}{(p_2 - p_1)^2} = \frac{g_1}{(d-4)^2(d-1)} \frac{p_1^2}{p_2^2}.
\]

We perform tensor reduction to get rid of the $z$ indices, converting each integral into a sum of many scalar integrals with integer indices. Using FIRE [63], which implements integration by parts relations, we can reduce these into a small number of 'master integrals’. In the two loop case, the master integrals $I_1$ and $I_2$ can be easily evaluated:

\[
I_1 = I_1(1, 1) l\left(1, 2 - \frac{d}{2}\right) \frac{1}{(p^2)^{d-4}}, \quad I_2 = I_1(1, 1) l\left(1, 1\right) \frac{1}{(p^2)^{d-4}}.
\]

where $l(\alpha, \beta)$ is the integral defined in (3.41).

E.2. Integrals for $C_T$ for the $O(N)$ scalar theory in $d = 6 - \epsilon$ (figure 2)

Explicitly, the diagrams are:

\[
D_0 = \int \frac{d^d p_1}{(2\pi)^d} \frac{(2p_1 + p_1 + p_2 + c^2)^2}{(p_1 + p_2)^2(p_1 + p)^2} = \frac{(d-2)\pi^\frac{d}{2} \Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{d}{2} - 1\right)}{2^{d+2}(d-1)^2(d-2) \Gamma(d-2)} \frac{p^4}{(p^2)^{d-5}}.
\]

\[
D_1 = 2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^d} \frac{(2p_1 + p_1 + p_2 + c^2)^2}{(p_1 + p_2)^2(p_1 + p)^2(p_1 + p_2)^2} \frac{(p_1 + p_2)^2}{(p_1 + p)^2} = \frac{-768 + 864d + 232d^2 + 36d^3 + 6d^4 + d^5}{d^3} \frac{p^4}{p^2}.
\]

\[
D_2 = g_2^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^d} \frac{(2p_1 + p_1 + p_2 + c^2)^2}{(p_1 + p_2)^2(p_1 + p)^2(p_1 + p_2)^2} = \frac{-768 + 608d + 536d^2 + 700d^3 + 238d^4 + 29d^5 + d^6}{d^3} l\left(1, 1\right) \frac{1}{(p^2)^{d-4}}.
\]

(E.1)
E.3. Integral for the anomalous dimension $\gamma$ of $\phi$-field (figure 4)

The diagram reads

$$D_1 = \frac{i \mu^{2\Delta}}{N} \int \frac{d^d p_1}{(2\pi)^d} \frac{\tilde{C}_{c=0} 2p_1^2}{(p_1^2)^{\frac{d}{2} - 2 + \Delta} (p + p_1)^2}$$  \hspace{1cm} (E.4)$$

and can be easily computed using the integral (3.41).

E.4. Integrals for $Z_T$-factor for the critical scalar (figure 5)

$$D_0 = \frac{2p_2^2}{(p^2)^2},$$

$$D_1 = \frac{2\mu^{2\Delta}}{N} \int \frac{d^d p_1}{(2\pi)^d} \frac{\tilde{C}_{c=0} 2p_1^2}{(p_1^2)^2 (p + p_1)^2 (p_1^2)^{\frac{d}{2} - 2 + \Delta}},$$

$$D_2 = \frac{\mu^{2\Delta}}{N} \int \frac{d^d p_1}{(2\pi)^d} \frac{\tilde{C}_{c=0} 2p_1^2}{(p_1^2)^2 (p_1^2)^2 (2\xi - 2 + \Delta)},$$

$$D_3 = \frac{\mu^{4\Delta}}{N} \int \frac{d^d p_1 d^d p_2}{(2\pi)^d} \frac{\tilde{C}_{c=0} 2p_2^2}{(p_2^2)^2 (p - p_2)^2 (p_2^2)^{2(\xi - 2 + \Delta)} (p_2 - p_1)^2 (p_1^2)^2}.$$  \hspace{1cm} (E.5)$$

These diagrams can be easily calculated with the use of elementary integral (3.41).

E.5. Integrals for $Z_J$-factor for the critical scalar (figure 6)

$$D_0 = \frac{i2p_2}{(p^2)^2} (\tau^a)^{ij},$$

$$D_1 = \frac{2\mu^{2\Delta}}{N} \int \frac{d^d p_1}{(2\pi)^d} \frac{\tilde{C}_{c=0} 2p_2^2}{(p_1^2)^{2(\xi - 2 + \Delta)} (p_1^2)^2} (\tau^a)^{ij},$$

$$D_2 = \frac{\mu^{2\Delta}}{N} \int \frac{d^d p_1}{(2\pi)^d} \frac{\tilde{C}_{c=0} 2p_1^2}{(p_1^2)^2 (p_1^2)^2} (\tau^a)^{ij}.$$  \hspace{1cm} (E.6)$$

These diagrams can be easily calculated with the use of elementary integral (3.41).
E.6. Integrals for $C_J$ for the critical scalar (figure 7)

Explicitly, the diagrams are:

$$D_0 = \frac{1}{2} \tr (r^{\eta,b}(i)) \int \frac{d^dp_1 (2p_{1z} + p_z)^2}{(2\pi)^d p_1^2 (p_1 + p)^2} = \frac{1}{2} \tr (r^{\eta,b}) \frac{4^{1-d/2} \pi^{d-4}}{\sin(\pi d/2) \Gamma(\frac{d+1}{2})} \frac{p_z^2}{(p^2)^{d-2}}.$$

$$D_1 = 2 \cdot \frac{1}{2} \tr (r^{\eta,b}(i)) \mu_{\Delta} \int \frac{d^dp_1 d^dp_2}{(2\pi)^{2d}} \frac{\tilde{C}_\text{c.o.}(2p_{1z} + p_z)^2}{(p_1 + p)^2 (p_1^2)^2 (p_1 - p_2)^2 (p_2^2)^{d-2+\Delta}}$$

$$= \frac{1}{N} \gamma_{\eta}^{(0)} D_0 \left( -2 \frac{1}{\Delta} - \log(p^2/\mu^2) \right)$$

$$- \left( 2 \mathcal{C}_\text{o.c.}(d) + \frac{2(10d^3 - 47d^2 + 56d - 16)}{(d - 4)(d - 2)(d - 1)d} \right).$$

$$D_2 = \frac{1}{2} \tr (r^{\eta,b}(i)) \mu_{\Delta} \int \frac{d^dp_1 d^dp_2}{(2\pi)^{2d}} \frac{\tilde{C}_\text{c.o.}(2p_{1z} + p_z)(2p_{2z} + p_z)}{(p_1 + p)^2 (p_1^2)^2 (p_2^2)^2 (p_1 - p_2)^2 (p_2^2)^{d-2+\Delta}}$$

$$= \frac{1}{N} \gamma_{\eta}^{(0)} D_0 \left( -2 \frac{1}{\Delta} - \log(p^2/\mu^2) \right) + \left( 2 \mathcal{C}_\text{o.c.}(d) + \frac{2(2d - 5)(3d - 4)}{(d - 4)(d - 2)(d - 1)} \right).$$

(E.7)

E.7. Integrals for $C_t$ for the critical scalar (figure 9)

Explicitly, the diagrams are:

$$D_0 = \frac{N}{2} \int \frac{d^dp_1 (2p_{1z} + p_z) + c p_z^2}{(2\pi)^d} p_1^2 (p_1 + p)^2$$

$$= N \frac{(d - 1)^2}{2(4\pi)^d (d - 1)^2 (d + 1) \Gamma(d/2)} \frac{p_z^4}{(p^2)^{d-2}}.$$

$$D_1 = \mu_{\Delta} \int \frac{d^dp_1 d^dp_2}{(2\pi)^{2d}} \frac{\tilde{C}_\text{c.o.}(2p_{1z} + p_z) + c p_z^2}{(p_1 + p)^2 (p_1^2)^2 (p_1 - p_2)^2 (p_2^2)^{d-2+\Delta}}$$

$$= \gamma_{\eta}^{(0)} D_0 \left( -2 \frac{1}{\Delta} - \log(p^2/\mu^2) \right)$$

$$- 2 \left( \mathcal{C}_\text{o.c.}(d) + \frac{11d^4 - 45d^3 + 26d^2 + 36d - 16}{(d - 4)(d - 2)(d - 1)d} \right).$$

$$D_2 = \frac{\mu_{\Delta}}{2} \int \frac{d^dp_1 d^dp_2}{(2\pi)^{2d}} \frac{\tilde{C}_\text{c.o.}(2p_{1z} + p_z) + c p_z^2}{(p_1 + p)^2 (p_1^2)^2 (p_2^2)^2 (p_1 - p_2)^2 (p_2^2)^{d-2+\Delta}}$$

$$= \gamma_{\eta}^{(0)} D_0 \left( -2 \frac{1}{\Delta} - \log(p^2/\mu^2) \right) \left( \frac{d - 2}{d + 2} \right)$$

$$+ 2 \left( \frac{d - 2}{d} \right) \frac{\mathcal{C}_\text{o.c.}(d) + \frac{3(3d^3 - 11d^2 + 4d + 8)}{(d - 4)(d - 1)(d + 1)(d + 2)}}{(d - 4)(d - 2)(d - 1)d}.$$ (E.8)
and

\[ D_3 = \frac{\mu^{4\Delta}}{2} \int \frac{d^dp_1 d^dp_2 d^dp_3}{(2\pi)^{3d}} \]
\[ \times \frac{\tilde{c}_1^2(2p_1 \cdot (p_1 + p_2) + cp_2^2)(2p_2 \cdot (p_2 + p_3) + cp_3^2)}{p_1^2(p + p_1)^2(p_2 - p_3)^2(2\pi)^{2+2\Delta}(p_3 + p)^{2(\frac{d}{2} - 2 + \Delta)}p_2^2(p + p_2)^2(p_2 - p_3)^2} \]
\[ = \gamma_1^{(N)} D_0 \left( \frac{1}{\Delta} - 2 \log(p^2/\mu^2) \right) \left( \frac{4}{d + 2} \right) \]
\[ + \left( \frac{4}{d + 2} C_{GN}(d) + \frac{2(d^4 + 18d^3 - 93d^2 + 66d + 56)}{(d-4)(d-2)(d-1)(d+1)(d+2)} \right) \]

where \( c \equiv \frac{d-2}{2(d-1)} \).

**E.8. Integrals for \( C_J \) for the critical fermion (figure 16)**

The integrals are

\[ D_0 = ( -1 ) i^2 \text{tr} (t^\mu t^\nu) \int \frac{d^d p_1}{(2\pi)^d} \frac{\text{Tr}(\gamma_1 \gamma_4 p + p_1) \gamma_2}{p^2(p + p_1)^2} \]
\[ = \frac{\text{tr}(t^\mu t^\nu) \text{tr}_1 \Gamma^{1 - \frac{2}{d}} \Gamma \left( \frac{d}{2} \right)}{4^\frac{d}{2}(d - 1) \Gamma(d - 2) \sin(\pi d/2) (p^2)^{1 - \frac{d}{2}}} \]
\[ D_1 = \frac{2}{N} \gamma_1^{GN} D_0 \left( \frac{1}{\Delta} - 2 \log(p^2/\mu^2) \right) \left( \frac{4d^2 - 10d + 4}{(d-2)(d-1)d} \right) C_{GN}(d) \]
\[ D_2 = \frac{2}{N} \gamma_1^{GN} D_0 \left( \frac{1}{\Delta} - 2 \log(p^2/\mu^2) \right) \left( \frac{4d^2 - 10d + 4}{(d-2)(d-1)d} \right) C_{GN}(d) \]

\[ \text{(E.9)} \]

**E.9. Integrals for \( C_T \) for the critical fermion (figure 18)**

The integrals are
The three-loop Aslamazov–Larkin contribution is\(^{18}\)

\[
D_0 = \frac{(-1)^i N}{2^{2d}} \int \frac{d^d p_1}{(2\pi)^d} \left( \frac{2(p_{1z} + p_z)^2}{p_1^2(p + p_1)} \right) \text{Tr} \left( \gamma_\mu (p + p_1) \gamma_\nu \right) \\
= -N_\mu \frac{1}{2} \Gamma \left( \frac{d}{2} \right) p_z^4 \\
= \frac{4^{d+1}(d-1)(d+1)\Gamma(d-2)\sin(\pi d/2)}{(p^2)^{2-\Delta}} \\
D_i = \frac{(-1)^i N \mu^{2\Delta}}{2N} \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} \left( \frac{2(p_{1z} + p_z)^2}{p_1^2(p + p_1)} \right) \text{Tr} \left( \gamma_\mu (p + p_1) \gamma_\nu \right) \gamma_\mu \gamma_\nu \left( \gamma_\nu \right) \left( \gamma_\nu \right) \\
= \gamma_i^\text{GN} D_0 \left( -2 \left( \frac{1}{\Delta} - \log(p^2/\mu^2) \right) + 2 \left( C_\text{GN}(d) + \frac{2(3d^3 - 4d^2 - 2d)}{(d-2)(d+1)} \right) \right) (E.10) \\
D_2 = \frac{(-1)^i N \mu^{2\Delta}}{2^{2d}N} \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} \left( \frac{2(p_{1z} + p_z)^2}{p_1^2(p + p_1)} \right) \text{Tr} \left( \gamma_\mu (p + p_1) \right) \gamma_\mu \left( \gamma_\mu \right) \gamma_\nu \left( \gamma_\nu \right) \gamma_\mu \left( \gamma_\mu \right) \\
= \gamma_i^\text{GN} D_0 \left( \frac{1}{\Delta} - \log(p^2/\mu^2) \right) \left( \frac{d-2}{d+2} \right) + 2 \left( \frac{d-2}{d+2} C_\text{GN}(d) + \frac{2(2d^2 - 2d - 1)}{(d-1)(d+1)(d+2)} \right) \\
D_3 = \frac{(-1)^i N \mu^{4\Delta}}{2^{2d}N} \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^{4d}} \left( \frac{2(p_{1z} + p_z)^2}{p_1^2(p + p_1)} \right) \text{Tr} \left( \gamma_\mu (p + p_1) \right) \gamma_\mu \left( \gamma_\mu \right) \gamma_\mu \left( \gamma_\mu \right) \\
= \gamma_i^\text{GN} D_0 \left( \frac{1}{\Delta} - 2 \log(p^2/\mu^2) \right) \left( \frac{2}{d+2} \right) + 2 \left( \frac{2}{d+2} C_\text{GN}(d) + \frac{2d(d+5)}{(d-1)(d+1)(d+2)} \right) (E.11) \\
\]

\(^{18}\) We used the fact that in \(D_3\) the two diagrams with different orientation of the fermion loop are equal due to the identity \(\text{Tr} \left( A \gamma_\mu \gamma_\nu B \right) = \text{Tr} \left( A \gamma_\nu \gamma_\mu B \right) = \text{Tr} \left( A \gamma_\mu \gamma_\nu B \right) = \text{Tr} \left( A \gamma_\nu \gamma_\mu B \right)\).
E.10. Integrals for $C_J$ for the GNY model in $d = 4 - \epsilon$ (figure 20)

\[ D_0 = - \tilde{N} \int \frac{d^k p}{(2\pi)^d} \frac{\text{Tr}(p_i p_i (p_J + p_J) \gamma_i)}{p^2 (p + p_J)^2} \]
\[ = \tilde{N} \frac{(d - 2) \pi^{d-1} \Gamma \left( \frac{d}{2} - 1 \right) \Gamma \left( \frac{d}{2} - 1 \right)}{2^{d-1} \Gamma(d - 2)} \frac{p^2}{(p + p_J)^2}, \]
\[ D_1 = 2\tilde{N}g^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^d} \frac{\text{Tr}(\gamma_i (p_J + p_J) \gamma_i (p_1 + p_2) \gamma_j)}{(p + p_1)^2 (p + p_2)^2 (p_1 - p_2)^2} \]
\[ = - N \frac{\tilde{N} g^2}{(d - 3) (d - 4) p^2} \frac{p_1^4}{p^2}, \]
\[ D_2 = \tilde{N}g^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^d} \frac{\text{Tr}(\gamma_i (p_J + p_J) \gamma_i (p_1 + p_2) \gamma_j)}{(p + p_1)^2 (p + p_2)^2 (p_1 - p_2)^2} \]
\[ = N \frac{\tilde{N} g^2 (d - 3) (4 - 3d + 3p^2 L_i) p_1^4}{6(d - 1)^2 (3d - 4)(3d - 2)}, \]
\[ \quad \times \frac{p^2}{p_1^4}. \]
\[ D_3 = 2\tilde{N}g^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^d} \frac{1}{2} \frac{(2p_1 + p_2)^2 \text{Tr}(\gamma_i (p_J + p_J) \gamma_j)}{(p + p_1)^2 (p + p_2)^2 (p_1 - p_2)^2} \]
\[ = N \frac{(d - 4) (d - 2) (d - 4) (d - 2) (d - 4) (d - 2) (d - 4) p_1^4}{12(d - 1)^4 (3d - 4)(3d - 2) (d - 4)(3d - 2)}, \]
\[ D_4 = - \tilde{N} g^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^d} \frac{(2p_1 + p_2 + c p_2^2) \text{Tr}(\gamma_j (p_J + p_J) \gamma_j)}{(p + p_1)^2 (p + p_2)^2 (p_1 - p_2)^2} \]
\[ = - N \frac{(d - 2)^2 p^2 (d + 2)(d + 4) L_i}{24(d - 4)(d - 1)^2 (3d - 4)(3d - 2) p_1^2}. \]

E.11. Integrals for $C_T$ for the GNY model in $d = 4 - \epsilon$ (figure 21)

These integrals are equal to:

\[ D_1 = 2\tilde{N}g^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^d} \frac{1}{2} \frac{(2p_1 + p_2)^2 \text{Tr}(\gamma_i (p_J + p_J) \gamma_j)}{(p + p_1)^2 (p + p_2)^2 (p_1 - p_2)^2} \]
\[ = - N \frac{(d - 3) (d - 4) (d - 2) (d - 2) p_1^4}{12(d - 1)^4 (3d - 4)(3d - 2)}, \]
\[ D_2 = \tilde{N}g^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^d} \frac{1}{2} \frac{(2p_1 + p_2)^2 \text{Tr}(\gamma_i (p_J + p_J) \gamma_j)}{(p + p_1)^2 (p + p_2)^2 (p_1 - p_2)^2} \]
\[ = N \frac{(d - 3) (d - 4) (d - 2) (d - 2) p_1^4}{12(d - 1)^4 (3d - 4)(3d - 2)}, \]
\[ \times \frac{p^2}{p_1^4}. \]
\[ D_3 = 2\tilde{N}g^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^d} \frac{1}{2} \frac{(2p_1 + p_2 + c p_2^2) \text{Tr}(\gamma_i (p_J + p_J) \gamma_j)}{(p + p_1)^2 (p + p_2)^2 (p_1 - p_2)^2} \]
\[ = - N \frac{(d - 3) (d - 4) (d - 2) (d - 2) p_1^4}{24(d - 4)(d - 1)^2 (3d - 4)(3d - 2)}, \]
\[ D_4 = - \tilde{N} g^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^d} \frac{(2p_1 + p_2 + c p_2^2) \text{Tr}(\gamma_j (p_J + p_J) \gamma_j)}{(p + p_1)^2 (p + p_2)^2 (p_1 - p_2)^2} \]
\[ = - N \frac{(d - 3) (d - 4) (d - 2) (d + 4) L_i}{24(d - 4)(d - 1)^2 (3d - 4)(3d - 2) p_1^2}. \]
As before, we have a factor of 2 in the diagram $D_1$ to account for the fact that the loops may renormalize either the top or bottom line.

### E.12. Integrals for $C_J$ for the GN model in $d = 2 + \epsilon$ (figures 22 and 23)

We have:

\[
D_0 = 2N \int \frac{d^d k}{(2\pi)^d} \text{Tr} \left( \gamma \gamma (\mathbf{p}_1 + \mathbf{p}_2) \gamma \gamma \right) = -N \frac{\pi^{1-d} \csc(\pi d/2) \Gamma(d/2)}{2^d(d-1) \Gamma(d-2) (p^2)^{d-2}},
\]

\[
D_1 = gN \int \frac{d^d p_1 d^d p_2}{(2\pi)^d} \frac{\text{Tr} \left( \gamma (\mathbf{p}_1 + \mathbf{p}_2) \gamma \gamma \gamma (\mathbf{p}_2 + \mathbf{p}_1) \gamma \gamma \gamma \right)}{(p + p_1)^2 (p + p_2)^2 p_1^2 p_2^2} = -gN \frac{\pi^{d-2} \csc(\pi d/2) \Gamma(d/2)}{4(1-d) \Gamma(d-2) (p^2)^{3-d}}.
\]

\[
D_2 = g^2 N \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^d} \frac{\text{Tr} \left( \gamma (\mathbf{p}_1 + \mathbf{p}_2) \gamma \gamma \gamma (\mathbf{p}_1 + \mathbf{p}_3) \gamma \gamma \right) \text{Tr} \left( \gamma (\mathbf{p}_2 + \mathbf{p}_1) \gamma \gamma \right)}{(p_1 + p_2)^2 (p_1 + p_3)^2 (p_2 + p_3)^2 (p_1 + p_2 + p_3)^2} = g^2 N (d-2)^{d/2} \left( \frac{1}{M_2} \right).
\]

\[
D_3 = g^2 N \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^d} \frac{\text{Tr} \left( \gamma (\mathbf{p}_1 + \mathbf{p}_2) \gamma \gamma \gamma (\mathbf{p}_1 + \mathbf{p}_3) \gamma \gamma \right) \text{Tr} \left( \gamma (\mathbf{p}_2 + \mathbf{p}_1) \gamma \gamma \right) \text{Tr} \left( \gamma (\mathbf{p}_3 + \mathbf{p}_2) \gamma \gamma \right)}{(p_1 + p_2)^2 (p_1 + p_3)^2 (p_2 + p_3)^2 (p_1 + p_2 + p_3)^2} = g^2 N (d-2)^{d/2} \left( \frac{1}{M_1} \right).
\]

and

\[
D_4 = g^2 N \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^d} \frac{\text{Tr} \left( \gamma (\mathbf{p}_1 + \mathbf{p}_2) \gamma \gamma \gamma (\mathbf{p}_1 + \mathbf{p}_3) \gamma \gamma \right) \text{Tr} \left( \gamma (\mathbf{p}_2 + \mathbf{p}_1) \gamma \gamma \right) \text{Tr} \left( \gamma (\mathbf{p}_3 + \mathbf{p}_2) \gamma \gamma \right)}{(p_1 + p_2)^2 (p_1 + p_3)^2 (p_2 + p_3)^2 (p_1 + p_2 + p_3)^2} = g^2 N (d-2)^{d/2} \left( \frac{1}{M_0} \right).
\]

where:

\[
f_1(d) = 86112 - 260472d + 307525d^2 - 176601d^3 + 49203d^4 - 5319d^5 \]

\[
f_2(d) = 3(-7776 + 24912d - 30833d^2 + 18395d^3 - 5283d^4 + 585d^5).
\]

After evaluating the traces and performing tensor reduction, each integral becomes a sum of many scalar integrals of the ladder-type with integer indices. Using FIRE [63] to apply integration by parts relations, we can convert all of them into a sum of three master integrals, $M_1$, $M_2$, and $M_3$ as shown in figure E1.
The first two master integrals are primitive and can be readily evaluated with the use of the integral (3.41). The integral $K(1, 1, 1, 1, 2 - d/2)$ (defined in appendix B (B.3)) in the third master integral can be evaluated using the Gegenbauer polynomial technique [47, 48]. Its expansion in $d = 2 + \epsilon$ is

$$
K(1, 1, 1, 1, 2 - d/2) = \frac{7}{6\pi^2\epsilon^2} + \frac{14(\gamma - \log(4\pi)) - 25}{12\pi^2\epsilon} + \frac{84(\gamma - \log(4\pi))^2 - 300(\gamma - \log(4\pi)) - 7\pi^2 - 228}{144\pi^2} + \mathcal{O}(\epsilon). \quad (E.18)
$$

E.13. Integrals for $C_T$ for the GN model in $d = 2 + \epsilon$ (figure 24)

The integrals are

$$
D_0 = \tilde{N} \int \frac{d^d p_1}{(2\pi)^{d/2}} \frac{1}{4(2p_1 + p_2)^2} \text{Tr} \left( \gamma_z(p + p_1)\gamma_z p_1 \right) = -\frac{\pi^{d/2} \csc \left( \frac{d}{2} \right)}{4^{d+1} (d^2 - 1) \Gamma(d - 2) (p^2)^{d/2} - \frac{1}{2}},
$$

$$
D_1 = -g\tilde{N} \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} \frac{1}{4(2p_1 + p_2)(2p_2 + p_1)} \times \frac{\text{Tr} \left( \gamma_z(p + p_1)(p + p_2)\gamma_z p_2 p_1 \right)}{p_1^2(p_1 + p)^2p_2^2(p_2 + p)^2} = 0. \quad (E.19)
$$

Using the Gegenbauer polynomial technique for the integral $K(1, 1, 1, 1, 2 - d/2)$ we obtain an analytic expression for any $d$. This expression includes a hypergeometric function. To expand the hypergeometric function in $d = 2 + \epsilon$ we used the program HypExp [72].
And

\[ D_2 = g^2 N \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^d} \left(\frac{1}{4} (2p_{1z} + p_z)(2p_{2z} + p_z) \right. \]
\[ \times \left. \frac{\text{Tr} \left( \gamma_5 (\not{p} + \not{p}_2)(\not{p} + \not{p}_3)(\not{p} + \not{p}_2) \gamma_5 \not{p}_2 \not{p}_3 \not{p}_1 \right)}{p_z^2 (p_z + p_1)^2 (p_z + p_2)^2 (p_z - p_3)^2} \right) = 0, \]

\[ D_3 = g^2 N^2 \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^d} \left(\frac{1}{4} (2p_{1z} + p_z)^2 \right. \]
\[ \times \left. \frac{\text{Tr} \left( \gamma_5 (\not{p} + \not{p}_2) \gamma_5 \not{p}_1 \gamma_5 (\not{p}_3 - \not{p}_2) \gamma_5 \not{p}_2 \gamma_5 (\not{p}_3 - \not{p}_1) \gamma_5 \not{p}_3 \gamma_5 \not{p}_1 \right)}{(p_z + p_1)^2 (p_z - p_3)^2 (p_z - p_2)^2} \right) \]

\[ = g^2 N (N - 1)(d - 2)^2(2d - 2 + d^2) M_0 p_z^4 \quad (E.20) \]

and

\[ D_4 = \frac{g^2 N^2}{4} \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^d} \left(\frac{1}{4} (2p_{1z} + p_z)(2p_{2z} + p_z) \right. \]
\[ \times \left. \frac{\text{Tr} \left( \gamma_5 (\not{p} + \not{p}_2)(\not{p} + \not{p}_3)(\not{p} + \not{p}_2) \gamma_5 \not{p}_2 \gamma_5 (\not{p}_3 - \not{p}_2) \gamma_5 \not{p}_3 \gamma_5 \not{p}_1 \right)}{(p_z + p_1)^2 (p_z - p_3)^2 (p_z - p_2)^2} \right) \]
\[ = \frac{g^2 N (N - 1)(d - 2)^2(2d - 2 + d^2) M_0 p_z^4}{288(3d - 4)^2(2d - 3)(2d - 1)(2d - 1)(2d - 3)(2d - 8)(3d - 2)} M_0 p_z^4. \quad (E.21) \]

where

\[ f_3(d) = -28800 + 164832d - 368444d^2 + 406366d^3 - 234072d^4 \]
\[ + 67473d^5 - 7884d^6 + 81d^7. \quad (E.22) \]

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