DYER-LASHOF OPERATIONS IN K-THEORY

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Dyer-Lashof operations were first introduced by Araki and Kudo in [1] in order to calculate $H_*(\Omega^n S^{n+k}; Z_2)$. These operations were later used by Dyer and Lashof to determine $H_*(QY; Z_p)$ as a functor of $H_*(Y; Z_p)$ [5], where $QY = \bigcup_n \Omega^n \Sigma^n Y$. This has had many important applications. Hodgkin and Snaith independently defined a single secondary operation in $K$-homology (for $p$ odd and $p = 2$ respectively) which was analogous to the sequence of Dyer-Lashof operations in ordinary homology [7, 13], and this operation has been used to calculate $K_*(QY; Z_p)$ when $Y$ is a sphere or when $p = 2$ and $Y$ is a real projective space [11, 12]. In this note we describe new primary Dyer-Lashof operations in $K$-theory which completely determine $K_*(QY; Z_p)$ in general.

We shall remove the indeterminacy of the operation by lifting it to higher torsion groups. First we establish notation. $X$ will always denote an $E_\infty$-space [9] and $Y$ will denote an arbitrary space, considered as a subspace of $QY$ via the natural inclusion. We write $K_*(Y; r)$ for $K_0(Y; Z_p)^\oplus K_1(Y; Z_p)$; in particular $K$-theory is $\mathbb{Z}_2$-graded and we write $|x|$ for the mod 2 degree of $x$. There are evident natural maps

$$p_s^* : K_\alpha(Y; r) \rightarrow K_\alpha(Y; r + s)$$

if $s \geq 1$,

$$* : K_\alpha(QY; r) \rightarrow K_\alpha(QY; t)$$

if $1 \leq t \leq r$,

and

$$\beta_r : K_\alpha(Y; r) \rightarrow K_{\alpha-1}(Y; r).$$

THEOREM 1. For each $r \geq 2$ and $\alpha \in \mathbb{Z}_2$ there is an operation

$$Q : K_\alpha(X; r) \rightarrow K_\alpha(X; r - 1)$$

with the following properties, where $x, y \in K_*(X; r)$.

(i) $Q$ is natural with respect to $E_\infty$-maps.

(ii) $Q(x + y) = \begin{cases} Qx + Qy - \pi \left[ \sum_{i=1}^{p-1} \frac{1}{p} \left( 1 - \frac{1}{p} \right) x^i y^{p-i} \right] & \text{if } |x| = |y| = 0, \\
Qx + Qy & \text{if } |x| = |y| = 1. \end{cases}$

(iii) $Q\phi = 0$, where $\phi \in K_0(X; r)$ is the identity element.

$$Q(xy) = \begin{cases} Qx \cdot \pi(y^p) + \pi(x^p) \cdot Qy + p(Qx)(Qy) & \text{if } |x| = |y| = 0, \\
(Qx)(Qy) & \text{if } |x| = |y| = 1. \end{cases}$$
\(\sigma Qx = \begin{cases} Q\sigma x & \text{if } |x| = 0, \\
\pi(\sigma x)^p + pQ\sigma x & \text{if } |x| = 1, \end{cases}\)

where \(\sigma : \tilde{K}(\Omega X; r) \to K_{\alpha+1}(X; r)\) is the homology suspension.

(vi) If \(k\) is prime to \(p\), then \(Q\psi^k = \psi^k Q\), where \(\psi^k\) is the \(k\)th Adams operation.

\[
\beta_{r-1}Qx = \begin{cases} Q\beta_r x + p\pi(x^{p-1}\beta_r x) & \text{if } |x| = 0, \\
\pi(\beta_r x)^p + pQ\beta_r x & \text{if } |x| = 1. \end{cases}
\]

\(Q\pi x = \pi Qx\) if \(r \geq 3\) and

\[
Q\pi x = \begin{cases} x^p & \text{if } |x| = 0, r = 1, \\
p_*Qx - (p^{p-1} - 1)x^p & \text{if } |x| = 0, r \geq 2, \\
0 & \text{if } |x| = 1, r = 1, \\
p_*Qx & \text{if } |x| = 1, r \geq 2. \end{cases}
\]

(ix) Let \(p = 2\). If \(x \in K_1(X; 1)\) then \(Q\beta_2 x = x^2\). If \(x \in K_1(X; 2)\) then
\((\pi x)^2 = (\pi \beta_2 x)^2\); in particular \((\pi x)^2 \in K_0(X; 1)\) is zero if \(x \in K_1(X; r)\) with \(r \geq 3\).

REMARKS. (i) There are no Adem relations.

(ii) If \(x \in K_\ast(X; 1)\) has \(\beta x = 0\) then \(x\) lifts to \(y \in K_\ast(X; 2)\). Thus one can define a secondary operation \(Q\) on \(K_\ast\) by \(Qx = Qy\). The element \(y\) is well defined modulo the image of \(p_*\), and thus Theorem 1 (viii) shows that \(Qx\) is well defined modulo \(p\)th powers if \(|x| = 0\) and has no indeterminacy if \(|x| = 1\). This is essentially the operation defined by Hodgkin and Snaith (although their construction is incorrect when \(p\) is odd, as shown in [10]).

The next result shows that, in contrast to ordinary homology, \(K_\ast(QY; 1)\) will in general have nilpotent elements.

**Theorem 2.** \(\pi(\beta_r x)^p = 0\) in \(K_0(X; 1)\) if \(x \in K_1(X; r)\).

If \(x \in K_\ast(Y; r)\), we write \(Q^s x \in K_\ast(QY; r - s)\) for the \(s\)th iterate of \(Q\) when \(s < r\). These elements give a family of indecomposable generators in \(K_\ast(QY; 1)\), but in general there can be other generators as well. For example, if \(x \in K_1(Y; 1)\) with \(\beta x \neq 0\) then \(x(\beta x)^{p-1}\) has zero Bockstein by Theorem 2, hence it lifts to an element \(z \in K_1(QY; 2)\), and it turns out that \(Qz\) is indecomposable (note that we cannot apply the Cartan formula to \(Qz\)). The next theorem allows us to deal systematically with elements like \(z\); in particular it gives the higher Bocksteins of such elements.

**Theorem 3.** For each \(r \geq 1\) there is an operation

\[R : K_1(X; r) \to K_1(X; r + 1)\]

with the following properties, where \(x, y \in K_1(X; r)\).

(i) \(R\) is natural for \(E_\infty\)-maps.

(ii) \(p_*Rx = Rp_*x, \pi Rx = Qp_*x - x(\beta_r x)^{p-1}\), and if \(r \geq 2, R\pi x = Qp_*x - p^{p-1}x(\beta_r x)^{p-1}\).

(iii) \(\beta_{r+1}Rx = Q\beta_{r+2}p^2 x\).

(iv) If \(r \geq 2\), then \(QRx = RQx\).
(v) If $k$ is prime to $p$, then $R\psi^k = \psi^k R$.

(vi) $\sigma R^x = \begin{cases} p_*[(\sigma x)^p] & \text{if } r = 1, \\ p_*[(\sigma x)^p] + p_*^2 Q x & \text{if } r \geq 2. \end{cases}$

$R(x + y) = Rx + Ry - \sum_{i=1}^{p-1} \left[ \frac{1}{p} \binom{p}{i} (p_* x)(\beta_{r+1p_* x})^{i-1}(\beta_{r+1p_* y})^{p-i} \right.$

(vii) $\begin{array}{c} + \left( \frac{p-1}{i} \right) \beta_{r+1p_* (xy)}(\beta_{r+1p_* x})^{i-1}(\beta_{r+1p_* y})^{p-i-1} \end{array}$.

Theorems 1 and 3 imply that $\pi Q^s R^t x$ is decomposable if $x \in K_1(Y;r)$ and $s < r + t - 1$. If $s = r + t - 1$ and $\pi \beta_r x \neq 0$ then this element turns out to be indecomposable.

In order to give a Cartan formula for $R$ and to provide generators for the higher terms of the Bockstein spectral sequence, we next give a $K$-theoretical analogue for the Pontryagin $p$th power introduced in ordinary homology by Madsen [8] and May [4]. Note, however, that by part (viii) of the following theorem this operation does not give rise to new families of indecomposables in $K_*(QY;1)$.

**Theorem 4.** For each $r \geq 1$ there is an operation

$$Q : K_0(X;r) \to K_0(X;r + 1)$$

with the following properties.

(i) $Q$ is natural for $E_\infty$-maps.

(ii) $\pi Q x = x^p$ and $Q p_* x = p^{p-1} p_* Q x$. If $r \geq 2$ then $Q \pi x = x^p$.

(iii) $\pi \beta_{r+1} Q x = x^{p-1} \beta_r x$.

(iv) Let $p$ be odd. Then

$$R(xy) = \begin{cases} (Rx)(Qy) & \text{if } |x| = 1, |y| = 0 \text{ and } r = 1, \\ (Rx)(Qy) + p_*^2 [(Qx)(Qy)] & \text{if } |x| = 1, |y| = 0 \text{ and } r \geq 2. \end{cases}$$

$$Q(xy) = (Qx)(Qy) \quad \text{if } |x| = |y| = 0.$$ (v) If $k$ is prime to $p$, $\psi^k Q = Q \psi^k$.

(vi) $Q(x + y) = Q x + Q y + \sum_{i=1}^{p-1} \left[ \frac{1}{p} \binom{p}{i} p_* (x^i y^{p-i}) \right]$.

(vii) $\sigma Q x = \begin{cases} 0 & \text{if } p \text{ is odd}, \\ 2^{r-1} 2_*[(\sigma x)(\beta_r \sigma x)] & \text{if } p = 2. \end{cases}$

(viii) $Q Q x = \begin{cases} 0 & \text{if } r = 1, \\ \sum_{i=1}^{p} \binom{p}{i} p^{i-2} x^{p^2 - ip} p_* [(Qx)^i] & \text{if } r \geq 2. \end{cases}$
REMARK. The formulas in part (iv) have analogues when $p = 2$, but some of the coefficients in this case have not yet been determined.

Using the operations $Q$ and $R$ we can completely describe $K_\ast(QY; 1)$. We shall assume that $Y$ is a finite complex, although this condition can be avoided. First recall the construction $CY$ from [9]. By [4, Theorem 1.5.10] we have $K_\ast(QY; 1) = (\pi_0 Y)^{-1} K_\ast(CY; 1)$, and so it suffices to give $K_\ast(CY; 1)$.

Next recall the reduced $K$-theory Bockstein spectral sequence $E^2 Y$ from [2]. If $Y$ is a finite complex we have $E^2 Y = E^\infty Y$ for some $n$, and we can choose a subset $A_\infty \subseteq \tilde{K}_\ast(Y; Z)$ projecting to a basis for $E^\infty Y$. Proceeding inductively, we can choose subsets $A_r \subseteq \tilde{K}_\ast(Y; r)$ such that

$$A_\infty \cup A_{n-1} \cup \beta_{n-1}(A_{n-1}) \cup \cdots \cup A_r \cup \beta_r(A_r)$$

projects to a basis of $E^r Y$ for $1 \leq r \leq n - 1$. We write $A_{r0}$ and $A_{r1}$ for the zero- and one-dimensional subsets of $A_r$. Let $BY$ be the quotient of the free strictly commutative algebra generated by the four sets

$$\{ \pi Q^s x | x \in A_r, 0 \leq s < r \}, \quad \{ \pi \beta_{r-s} Q^s x | x \in A_r, 0 \leq s < r < \infty \},$$

$$\{ Q^{r+s} R^{s+1} x | x \in A_{r1}, r < \infty, 0 \leq s \}, \quad \{ \pi \beta_{r+s} R^s x | x \in A_{r1}, r < \infty, 0 \leq s \}$$

by the ideal generated by the set

$$\{ (\pi \beta_{r+s} R^s x)^{p^{r+s}} | x \in A_{r1}, r < \infty, 0 \leq s \}.$$

The Dyer-Lashof operations $Q$ and $R$ give an additive homomorphism $\lambda: BY \to K_\ast(CY; 1)$, which is a ring homomorphism if $p$ is odd but not if $p = 2$. Our main theorem is

**THEOREM 5.** $\lambda$ is an isomorphism.

REMARKS. (i) Theorems 1, 3, and 5 also give the ring structure of $K_\ast(CY; 1)$ when $p = 2$. First recall that mod 2 $K$-theory is noncommutative [2], in fact the commutator of $x$ and $y$ is $(\beta x)(\beta y)$.

Now

$$\beta(Q^{r+s} R^{s+1} x) = (\beta_{r+s+1} R^{s+1} x)^{2^{r+s}}$$

if $x \in A_{r1}$ with $r < \infty$ and $s \geq -1$, and all other generators (except $Q^{r-1} x$ for $x \in A_{r0}$, $r < \infty$, whose Bockstein is the generator $\beta Q^{r-1} x$) have zero Bockstein and hence lie in the center. Further, all odd-dimensional generators have square zero except in the following cases:

$$\left( \pi Q^{r-2} x \right)^2 = (\beta_r x)^{2^{r-1}} \quad \text{if } x \in A_{r1}, 2 \leq r < \infty;$$

$$\left( Q^{r+s} R^{s+1} x \right)^2 = (\pi \beta_{r+s+2} R^{s+2} x)^{2^{r+s}} \quad \text{if } x \in A_{r1}, r < \infty, s \geq -1.$$ 

These facts, together with Theorem 5, determine the ring structure.

(ii) The effect of $(Q f)_*: K_\ast(QY; 1) \to K_\ast(QZ; 1)$ for any $f: Y \to Z$ can be ascertained from Theorems 1, 3, and 5 if $f_*: K_\ast(Y; r) \to K_\ast(Z; r)$ is known for all $r \geq 1$ (although the formulas can become complicated unless $f_*$ takes the chosen sets $A_r$ for $Y$ into the corresponding sets for $Z$). In particular if $f: S^2 \to S^2$ is the degree $p$ map then Theorem 1 (ii) implies that $(Q f)_*$ is nonzero on $K_\ast(QS^2; 1)$. Thus $K_\ast(QY; 1)$ is not a functor of $K_\ast(Y; 1)$, a fact first noticed by Hodgkin [7].
(iii) Theorem 5 specializes to give an independent proof of the computations of Hodgkin [6] and Miller and Snaith [11, 12]. The operation $R$ did not arise in those computations since in the cases considered $A_{r+1}$ was empty for all $r < \infty$.

Finally, we describe the Bockstein spectral sequence for $CY$.

**Theorem 6.** For $1 \leq m < \infty$, $E_\infty^m(CY)^+$ is additively isomorphic to the quotient of the free strictly commutative algebra generated by the six sets

\[
\{ \pi Q^s x | x \in A_r, m \leq r-s, s \geq 0 \},
\{ \pi \beta_{r-s} Q^s x | x \in A_{r+1}, m \leq r-s < \infty, s \geq 0 \},
\{ \pi Q^{m-r+s} Q^s x | x \in A_{r+1}, 1 \leq r-s < m, s \geq 0 \},
\{ \pi \beta_m Q^{m-r+s} Q^s x | x \in A_{r+1}, 1 \leq r-s < m, s \geq 0 \},
\{ \pi Q^{t-m} R^{t-r} x | x \in A_{r+1}, t \geq \max(m, r+1), r < \infty \},
\{ \pi \beta_{t-r} R^{t-r} | x \in A_{r+1}, t \geq \max(m, r), r < \infty \}
\]

by the ideal generated by the set

\[
\{ (\pi \beta_{t-r} R^{t-r} x)^{p+1-m} | x \in A_{r+1}, t \geq \max(m, r), r < \infty \}.
\]

If $p$ is odd or $m \geq 3$ the isomorphism is multiplicative. The differential in $E_\infty^m(CY)^+$ is determined by the formula

\[
\pi \beta_m Q^{t-m} R^{t-r} x = (\pi \beta_{t-r} R^{t-r} x)^{p-1-m}
\]
for $x \in A_{r+1}, t \geq \max(m, r), r < \infty$.

The construction of the operations is as follows. Let $M_r$ be the Moore spectrum $S^{-1} \cup_{p^r} e^0$ and let $K$ be the integral $K$-theory spectrum. By definition, any $x \in K_\alpha(X; r)$ is represented by a stable map

\[
x : S^\alpha \to K \wedge \Sigma M_r \wedge X.
\]
Since the dual of $\Sigma M_r$ is $M_r$, such a map induces

\[
x' : \Sigma^\alpha M_r \to K \wedge X.
\]
Applying the stable extended power functor $D_p$ and using the fact that $K \wedge X$ is an $H_\infty$ ring spectrum [3] one obtains a composite

\[
x'' : D_p \Sigma^\alpha M_r \to D_p(K \wedge X) \to K \wedge X.
\]
Finally, if $e \in K_\alpha(D_p \Sigma^\alpha M_r; s)$ for some $s$ one has the composite

\[
\Sigma^\alpha M_s \to K \wedge D_p \Sigma^\alpha M_r \to K \wedge X \to K \wedge X,
\]
where $\mu$ is the $K$-theory product. This composite represents an element of $K_\alpha(X; s)$ depending only on $e$ and $x$. The operations $Qx$, $Qx$ and $Rx$ are obtained in this way for various choices of $e$, and the proofs of Theorems 1, 3, and 4 reduce in each case to the analysis of $e$. The construction has the further advantage that the proof of Theorem 5 is reduced, after some diagram chasing, to the universal case $Y = \Sigma^\alpha M_r$. Details will appear in [3].
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