STABILITY UNDER DEFORMATIONS OF EXTREMAL ALMOST-KÄHLER METRICS IN DIMENSION 4.

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Abstract. Given a path of almost-Kähler metrics compatible with a fixed symplectic form on a compact 4-manifold such that at time zero the almost-Kähler metric is an extremal Kähler one, we prove, for a short time and under a certain hypothesis, the existence of a smooth family of extremal almost-Kähler metrics compatible with the same symplectic form, such that at each time the induced almost-complex structure is diffeomorphic to the one induced by the path.

1. Introduction

An almost-Kähler metric on a 2n-dimensional symplectic manifold (M, ω) is induced by an almost-complex structure J compatible with ω in the sense that the tensor field g(·, ·) = ω(·, J·) is symmetric and positive definite and thus it defines a Riemannian metric on M. The almost-Kähler metric is Kähler if the almost-complex structure J is integrable. Given an almost-Kähler metric, one can define a canonical hermitian connection ∇ (see e.g. [16, 24]). The hermitian scalar curvature s∇ is then obtained by taking a trace and contracting the curvature of ∇ with ω. In the Kähler case, the hermitian scalar curvature coincides with the Riemannian scalar curvature.

A key observation, made by Fujiki [13] in the integrable case and by Donaldson [9] in the general almost-Kähler case, asserts that the natural action of the infinite dimensional group Ham(M, ω) of hamiltonian symplectomorphisms on the space AKω of ω-compatible almost-Kähler metrics is hamiltonian with moment map µ : AKω → (Lie(Ham(M, ω)))∗ given by µJ(f) = ∫Ms∇fωn/n!. The critical points of the norm ∫M (s∇)2ωn/n! are called extremal almost-Kähler metrics. It turns out that the symplectic gradient of s∇ of such metrics is a holomorphic vector field in the sense that its flow preserves the corresponding almost-complex structure. In particular, extremal Kähler metrics in the sense of Calabi [7] and almost-Kähler metrics with constant hermitian scalar curvature are extremal.

The GIT formal picture in [9] suggests the existence and the uniqueness of an extremal almost-Kähler metric, modulo the action of Ham(M, ω), in each ‘stable complexified’ orbit of the action of Ham(M, ω). However, in this formal infinite dimensional setting, a natural complexification of Ham(M, ω) does not exist. When H1(M, ℜ) = 0, an identification of the ‘complexified’ orbit of a Kähler metric (J, g) ∈ AKω is given by considering all Kähler metrics (J, ˜g) in the Kähler class [ω] and applying Moser’s Lemma [9]. In this setting, Fujiki–Schumacher [14] and LeBrun–Simanca [21] showed, in the absence of holomorphic vector fields, that the existence of an extremal Kähler metric is an open condition on the space of such
orbits. Moreover, Apostolov–Calderbank–Gauduchon–Friedman [3] generalized this result by fixing a maximal torus $T$ in the reduced automorphism group of $(M, J)$ and considering $T$-invariant $\omega$-compatible Kähler metrics. In general, for an almost-Kähler metric, a description of these ‘complexified’ orbits is not available, see however [10] for the toric case. Nevertheless, the formal picture suggests that the existence of an extremal Kähler metric should persist for smooth almost-Kähler metrics close to an extremal one.

Thus motivated, we consider in this paper the 4-dimensional case where one can introduce a notion of almost-Kähler potential related to the one defined by Weinkove [27, 28]. In the spirit of [14, 21], we shall apply the Banach Implicit Function Theorem for the hermitian scalar curvature of $T$-invariant $\omega$-compatible almost-Kähler metrics where $T$ is a maximal torus in $\text{Ham}(M, \omega)$. The main technical problem is the regularity of a family of Green operators involved in the definition of the almost-Kähler potential. Using a Kodaira–Spencer result [19, 20], one can resolve this problem if we suppose that the dimension of $g_t$-harmonic $J_t$-anti-invariant 2-forms, denoted by $h^t_{J_t}$ (see [12]), satisfies the condition $h^t_{J_t} = h^t_{\tilde{J}_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$ along the path $(J_t, g_t) \in AK^T_\omega$ in the space of $T$-invariant $\omega$-compatible almost-Kähler metrics. So, our main theorem claims the following

**Theorem 1.1.** Let $(M, \omega)$ be a 4-dimensional compact symplectic manifold and $T$ a maximal torus in $\text{Ham}(M, \omega)$. Let $(J_t, g_t)$ be any smooth family of almost-Kähler metrics in $AK^T_\omega$ such that $(J_0, g_0)$ is an extremal Kähler metric. Suppose that $h^t_{\tilde{J}_t} = h^t_{\tilde{J}_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$. Then, there exists a smooth family $(\tilde{J}_t, \tilde{g}_t)$ of extremal almost-Kähler metrics in $AK^T_\omega$, defined for sufficiently small $t$, with $(\tilde{J}_0, \tilde{g}_0) = (J_0, g_0)$ and such that $\tilde{J}_t$ is equivariantly diffeomorphic to $J_t$.

**Remark 1.2.** (i) The condition that $h^t_{\tilde{J}_t} = h^t_{\tilde{J}_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$ is satisfied in the following cases:

1. When $J_t$ are integrable almost-complex structures for each $t$. Then, $h^t_{\tilde{J}_t} = 2h^{2,0}(M, J_t) = b^+(M) - 1$ by a well-known result of Kodaira [5]. On the other hand, it is unknown whether or not, for an $\omega$-compatible non-integrable almost-complex $J$ on a compact 4-dimensional symplectic manifold $M$ with $b^+(M) \geq 3$, the equality $h^t_{\tilde{J}_t} = b^+(M) - 1$ is possible (see [12]).

2. When $b^+(M) = 1$, $h^t_{\tilde{J}_t} = 0$ for each $t$. This condition is satisfied when $(M, \omega)$ admits a non trivial torus in $\text{Ham}(M, \omega)$ [17].

(ii) Theorem 1.1 holds under the weaker assumption that the torus $T \subset \text{Ham}(M, \omega)$ is maximal in $\text{Ham}(M, \omega) \cap \text{Isom}_0(M, g_0)$, where $\text{Isom}_0(M, g_0)$ denotes the connected component of the isometry group of the initial metric $g_0$. By a known result of Calabi [5], any extremal Kähler metric is invariant under a maximal connected compact subgroup of $\text{Ham}(M, \omega) \cap \text{Aut}(M, J_0)$, where $\text{Aut}(M, J_0)$ is the reduced automorphism group of $(M, J_0)$. Hence, Theorem 1.1 generalizes [14, 21] in the 4-dimensional case.

(iii) It was kindly pointed out to us by T. Drăghici that using a recent result of Donaldson and Remarks (i) and (ii) above, one can further extend Theorem 1.1 in the case when $b^+(M) = 1$ as follows: Let $(M, \omega_0, J_0, g_0)$ be a compact 4-dimensional extremal Kähler manifold with $b^+(M) = 1$ and $T$ be a maximal torus
in \( \text{Ham}(M, \omega) \cap \text{Isom}_0(M, g_0) \). Then, for any smooth family of \( T \)-invariant almost-complex structures \( J(t) \) with \( J(t) = J_0 \), \( J(t) \) is compatible with an extremal almost-Kähler metric \( g_t \) for \( t \in (-\epsilon, \epsilon) \). Indeed, as \( J(t) \) are tamed by \( \omega_0 \) for \( t \in (-\epsilon, \epsilon) \) and \( b^+(M) = 1 \), one can use the openness result of Donaldson [11] (see also [12, Sec. 5]) to show that there exists a smooth family of \( J(t) \)-invariant symplectic forms \( \omega_t \) with \( [\omega_t] = [\omega_0] \). Averaging \( \omega_t \) over the compact group \( T \) and using the equivariant Moser Lemma, we obtain a family \( J_t \) of \( T \)-invariant \( \omega_0 \)-compatible almost-complex structures such that \( J_t \) is \( T \)-equivariantly diffeomorphic to \( J(t) \). We can then apply Theorem [13] to produce compatible extremal metrics

Kim and Sung [18] showed that, in any dimension, if one starts with a Kähler metric of constant scalar curvature with no holomorphic vector fields, one can construct infinite dimensional families of almost-Kähler metrics of constant hermitian scalar curvature which coincide with the initial metric away from an open set. Similar existence result was presented in [22] when the initial Kähler metric is locally toric.

2. Preliminaries

Let \((M, \omega)\) be a compact symplectic manifold of dimension \(2n\). An almost-complex structure \( J \) is compatible with \( \omega \) if the tensor field \( g(\cdot, \cdot) := \omega(\cdot, J \cdot) \) defines a Riemannian metric on \( M \); then, \((J, g)\) is called an \( (\omega \)-compatible\) almost-Kähler metric on \((M, \omega)\). If, additionally, the almost-complex structure \( J \) is integrable, then \((J, g)\) is a Kähler metric on \((M, \omega)\).

The almost-complex structure \( J \) acts on the cotangent bundle \( T^*(M) \) by \( J \alpha(X) = -\alpha(JX) \), where \( \alpha \) is a 1-form and \( X \) a vector field on \( M \). Any section \( \psi \) of the bundle \( \otimes^2 T^*(M) \) admits an orthogonal splitting \( \psi = \psi^{J,+} + \psi^{J,-} \), where \( \psi^{J,+} \) is the \( J \)-invariant part and \( \psi^{J,-} \) is the \( J \)-anti-invariant part, given by

\[
\psi^{J,+}(\cdot, \cdot) = \frac{1}{2} (\psi(\cdot, \cdot) + \psi(J \cdot, J \cdot)) \quad \text{and} \quad \psi^{J,-}(\cdot, \cdot) = \frac{1}{2} (\psi(\cdot, \cdot) - \psi(J \cdot, J \cdot)).
\]

In particular, the bundle of 2-forms decomposes under the action of \( J \)

\[
\Lambda^2(M) = \mathbb{R} \cdot \omega \oplus \Lambda^{J,+}_0(M) \oplus \Lambda^{J,-}(M),
\]

where \( \Lambda^{J,+}_0(M) \) is the subbundle of the primitive \( J \)-invariant 2-forms (i.e. 2-forms pointwise orthogonal to \( \omega \)) and \( \Lambda^{J,-}(M) \) is the subbundle of \( J \)-anti-invariant 2-forms. Hence, the subbundle of primitive 2-forms \( \Lambda^2_0(M) \) admits the splitting

\[
\Lambda^2_0(M) = \Lambda^{J,+}_0(M) \oplus \Lambda^{J,-}(M).
\]

For an \( \omega \)-compatible almost-Kähler metric \((J, g)\), the canonical hermitian connection on the complex tangent bundle \((T(M), J, g)\) is defined by

\[
\nabla_X Y = D_X^g Y - \frac{1}{2} J(D_X^g J) Y,
\]

where \( D^g \) is the Levi-Civita connection with respect to \( g \) and \( X, Y \) are vector fields on \( M \). Denote by \( R^\nabla \) the curvature of \( \nabla \). Then, the hermitian Ricci form \( \rho^\nabla \) is the trace of \( R^\nabla_{X,Y} \) viewed as an anti-hermitian linear operator of \((T(M), J, g)\), i.e.

\[
\rho^\nabla(X, Y) = - tr(J \circ R^\nabla_{X,Y}).
\]

Hence, the 2-form \( \rho^\nabla \) is a closed (real) 2-form and it is a \( \text{deRham} \) representative of \( 2\pi c_1(T(M), J) \) in \( H^2(M, \mathbb{R}) \), where \( c_1(T(M), J) \) is the first (real) Chern class. If
the almost-complex structure $J$ is compatible with a symplectic form $\tilde{\omega}$ such that $\tilde{\omega}^n = e^{F} \omega^n$ for some smooth real-valued function $F$ on $M$, then [26, 27]

\begin{equation}
\tilde{\rho}^\nabla = -\frac{1}{2} d J d F + \rho^\nabla,
\end{equation}

where $\tilde{\rho}^\nabla$ is the hermitian Ricci form of the almost-Kähler $(J, \tilde{g})$ (here $\tilde{g}(-,\cdot) = \tilde{\omega}(-,J\cdot)$ is the induced Riemannian metric).

We define the hermitian scalar curvature $s^n$ of an almost-Kähler metric $(J, g)$ as the trace of $\rho^\nabla$ with respect to $\omega$, i.e.

\begin{equation}
\int_M s^n \omega^n = 2n \left( \rho^\nabla \wedge \omega^{n-1} \right).
\end{equation}

The (Riemannian) Hodge operator $*_{g} : \Lambda^p(M) \rightarrow \Lambda^{2n-p}(M)$ is defined to be the unique isomorphism such that $\psi_1 \wedge (*_{g} \psi_2) = g(\psi_1, \psi_2) \frac{\omega^n}{n!}$, for any $p$-forms $\psi_1, \psi_2$. Then, the codifferential $\delta^g$, defined as the formal adjoint of the exterior derivative $d$ with respect to $g$, is related to $d$ by the relation [3, 15]

\begin{equation}
\delta^g = -*_{g} d *_{g}.
\end{equation}

It follows that

\begin{equation}
d = *_{g} \delta^g *_{g}.
\end{equation}

In dimension $2n = 4$, the bundle of 2-forms decomposes as

\begin{equation}
\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M),
\end{equation}

where $\Lambda^\pm(M)$ correspond to the eigenvalue $(\pm 1)$ under the action of the Hodge operator $*_{g}$. This decomposition is related to the splitting (2.1) as follows

\begin{equation}
\Lambda^+(M) = \mathbb{R} \cdot \omega \oplus \Lambda^J_{-} (M) \text{ and } \Lambda^-(M) = \Lambda^J_{0} (M).
\end{equation}

3. Extremal almost-Kähler metrics

Let $(M, \omega)$ be a compact and connected symplectic manifold of dimension $2n$. Any $\omega$-compatible almost-complex structure is identified with the induced Riemannian metric.

Denote by $AK_\omega$ the Fréchet space of $\omega$-compatible almost-complex structures. The space $AK_\omega$ comes naturally equipped with a formal Kähler structure. Let $Ham(M, \omega)$ be the group of hamiltonian symplectomorphisms of $(M^{2n}, \omega)$. The Lie algebra of $Ham(M, \omega)$ is identified with the space of smooth functions on $M$ with zero mean value.

A key observation, made by Fujiki [13] in the integrable case and by Donaldson [9] in the general almost-Kähler case, asserts that the natural action of $Ham(M, \omega)$ on $AK_\omega$ is hamiltonian with momentum given by the hermitian scalar curvature. More precisely, the moment map $\mu : AK_\omega \rightarrow (\text{Lie}(Ham(M, \omega)))^*$ is

\begin{equation}
\mu_J(f) = \int_M s^n f \frac{\omega^n}{n!}
\end{equation}

where $s^n$ is the hermitian scalar curvature of $(J, g)$ and $f$ is a smooth function with zero mean value viewed as an element of $\text{Lie}(Ham(M, \omega))$. The square-norm of the hermitian scalar curvature defines a functional on $AK_\omega$.

\begin{equation}
J \mapsto \int_M (s^n)^2 \frac{\omega^n}{n!}.
\end{equation}
Definition 3.1. The critical points $(J, g)$ of the functional $(3.1)$ are called extremal almost-Kähler metrics.

Proposition 3.2. An almost-Kähler metric $(J, g)$ is a critical point of $(3.1)$ if and only if $\nabla_s^2 s^V$ is a Killing vector field with respect to $g$.

A proof of Proposition 3.2 is given in [4, 15, 22].

3.1. The extremal vector field. We fix a maximal torus $T$ in $\text{Ham}(M, \omega)$ and denote by $t_\omega$ the finite dimensional space of real-valued smooth functions on $M$ which are hamiltonians with zero mean value of elements of $t = \text{Lie}(T)$. Denote by $\Pi^T_\omega$ the $L^2$-orthogonal projection of $T$-invariant smooth functions on $t_\omega$ with respect to the volume form $\omega^n$. Let $\text{AK}^T_\omega$ be the space of $\omega$-compatible $T$-invariant almost-complex structures. Given any $J \in \text{AK}^T_\omega$, we define $z^T_\omega := \Pi^T_\omega s^V$, where $s^V$ is the hermitian scalar curvature of $(J, g)$. Then, we have the following (for more details see [3, 15, 22])

Proposition 3.3. The potential $z^T_\omega$ is independent of $(J, g)$. Furthermore, a $\omega$-compatible $T$-invariant almost-Kähler metric $(J, g)$ is extremal if and only if $\delta^2 s^V = z^T_\omega$, where $\delta^2$ is the integral zero part of the hermitian scalar curvature $s^V$ of $(J, g)$.

Definition 3.4. The vector field $Z^T_\omega := \nabla_s z^T_\omega$ is called the extremal vector field relative to $T$.

Proposition 3.5. The vector field $Z^T_\omega$ is invariant under $T$-invariant isotopy of $\omega$.

Remark 3.6. The assumption that $T \subset \text{Ham}(M, \omega)$ is a maximal torus is used only in the second part of Proposition 3.3. Indeed, the arguments in [22] show that $z^T_\omega = \Pi^T_\omega s^V$ is independent of $(J, g)$ for any torus $T \subset \text{Ham}(M, \omega)$ and Proposition 3.5 still holds true for the corresponding vector field $Z^T_\omega = \nabla_s z^T_\omega$.

4. Almost-Kähler potentials in dimension 4

Let $(M, \omega)$ be a compact symplectic manifold of dimension $2n = 4$ and $(J, g)$ a $\omega$-compatible almost-Kähler metric. In order to define the almost-Kähler potentials, we consider the following second order linear differential operator [23] on the smooth sections $\Omega^{J,-}(M)$ of the bundle of $J$-anti-invariant 2-forms.

$$ P : \Omega^{J,-}(M) \longrightarrow \Omega^{J,-}(M), \quad \psi \mapsto (d\delta^g \psi)^{J,-}, $$

where $\delta^g$ is the codifferential with respect to the metric $g$.

Lemma 4.1. $P$ is a self-adjoint strongly elliptic linear operator with kernel the $g$-harmonic $J$-anti-invariant 2-forms.

Proof. The principal symbol of $P$ is given by the linear map $\sigma(P)_{\xi}(\psi) = -\frac{1}{2}\xi|\xi|^2\psi$, $\forall \xi \in T^*_x(M), \psi \in \Omega^{J,-}(M)$. So, $P$ is a self-adjoint elliptic linear operator with respect to the global inner product $\langle \cdot, \cdot \rangle = \int_M g(\cdot, \cdot) \frac{\sqrt{g}}{2}$. Now, let $\psi \in \Omega^{J,-}(M)$ and suppose that $P(\psi) = 0$. Then, $0 = \langle (d\delta^g \psi)^{J,-}, \psi \rangle = \langle d\delta^g \psi, \psi \rangle = \langle \delta^g \psi, \delta^g \psi \rangle$ which means that $\delta^g \psi = 0$. It follows from (2.5) and since $\psi$ is $J$-anti-invariant that $\ast g \psi = \psi$. Using the relation (2.4), we obtain $d\psi = \ast g \delta^g \ast g \psi = \ast g \delta^g \psi = 0$. Hence, $d\psi = \delta^g \psi = 0$ and thus $\psi$ is a $g$-harmonic $J$-anti-invariant 2-form. \qed
Corollary 4.2. For $f \in C^\infty(M,\mathbb{R})$, there exist a unique $\psi_f \in \Omega^{J,-}(M)$ orthogonal to the kernel of $P$ such that $(d\delta^g \psi_f)^{J,-} = (dJdf)^{J,-}$.

Proof. For a smooth real-valued function $f \in C^\infty(M,\mathbb{R})$ and any $\alpha$ in the kernel of $P$, we have $\langle (dJdf)^{J,-}, \alpha \rangle = \langle dJdf, \alpha \rangle = \langle Jdf, \delta^g \alpha \rangle = 0$. By a standard result of elliptic theory \cite{6, 29} and since $P$ is self-adjoint, there exist a smooth section $\psi_f \in \Omega^{J,-}(M)$ such that $P(\psi_f) = (dJdf)^{J,-}$. Moreover, $\psi_f$ is unique if one requires $\psi_f$ be orthogonal to the kernel of $P$.

From Corollary \cite{12} it follows that, for $f \in C^\infty(M,\mathbb{R})$, the symplectic form $\omega_f = \omega + d(Jdf - \delta^g \psi_f)$ is a $J$-invariant closed 2-form. Then, the function $f$ is called an almost-Kähler potential if the induced symmetric tensor $g_f(\cdot, \cdot) := \omega_f(\cdot, J \cdot)$ is a Riemannian metric. This notion of almost-Kähler potential is closely related but different (in general) from the one defined by Weinkove in \cite{28}. More precisely, if the almost-complex structure $J$ is compatible with a symplectic form $\tilde{\omega}$ which is cohomologous to $\omega$ i.e. $\tilde{\omega} - \omega = d\alpha$ (for some 1-form $\alpha$), then the almost-Kähler potential defined by Weinkove is given by the function $\tilde{f}$ which is uniquely determined (up to the addition of constant) by the Hodge decomposition of $\alpha$ with respect to the (self-adjoint elliptic) twisted Laplace operator $\tilde{\Delta} = J\Delta^g J^{-1}$, where $\Delta^g$ is the (Riemannian) Laplace operator with respect to the induced metric $\tilde{g}(\cdot, \cdot) = \tilde{\omega}(\cdot, J \cdot)$. In other words, we have the decomposition $\alpha = \alpha_H + \tilde{\Delta} \tilde{G} \alpha$, where $\tilde{G}$ is the Green operator associated to $\tilde{\Delta}$ and $\alpha_H$ is the harmonic part of $\alpha$ with respect to $\Delta^g$. Thus, $\tilde{f} = -\delta^g J \tilde{G} \alpha$, where $\delta^g$ is the codifferential with respect to the metric $\tilde{g}$.

Note that $(dJdf)^{J,-} = D^g_{Jdf} \omega$ (see e.g. \cite{15}), where $\sharp_g$ stands for the isomorphism between $T^*(M)$ and $T(M)$ induced by $g^{-1}$. Hence, in the Kähler case, $(dJdf)^{J,-} = 0$ which implies that $\psi_f = 0$ and thus this almost-Kähler potential coincides with the usual Kähler one.

5. Main Theorem

Let $(M, \omega)$ be a compact and connected symplectic manifold of dimension $2n = 4$ and $J_t \in AK_{\omega}$ be a smooth path of $\omega$-compatible almost-complex structures. We define the following family of differential operators associated to $J_t$

$$P_t : \Omega^2_0(M) \rightarrow \Omega^2_0(M)$$

$$\psi \mapsto \frac{1}{2} \Delta^g \psi - \frac{1}{2} g_t(\Delta^g \psi, \omega) \omega,$$

where $\Omega^2_0(M)$ is the space of smooth sections of the bundle $\Lambda^2_0(M)$ of primitive 2-forms (pointwise orthogonal to $\omega$) and $\Delta^g$ is the (Riemannian) Laplacian with respect to the metric $g_t(\cdot, \cdot) = \omega(\cdot, J_t \cdot)$ (here we use the convention $g_t(\omega, \omega) = 2$).

One can easily check that $P_t$ preserves the decomposition

$$\Omega^2_0(M) = \Omega^2_{0,+}(M) \oplus \Omega^2_{0,-}(M).$$

Furthermore,

$$P_t|_{\Omega^2_{0,-}(M)}(\psi) = (d\delta^g \psi)^{J_t,-}$$

and $P_t|_{\Omega^2_{0,+}(M)}(\psi) = \frac{1}{2} \Delta^g \psi$.

It follows that the kernel of $P_t$ consists of primitive harmonic 2-forms which splits as anti-selfdual and $J_t$-anti-invariant ones so we have

$$\dim \ker(P_t) = b^{-}(M) + h_{J_t}^-,$$
where $h_{t}^{-}$ is introduced by Drăghici–Li–Zhang in [12].

Moreover, $P_{t} - \frac{1}{2} \Delta^{g_{t}}$ is a linear differential operator of order 1. Indeed, a direct computation shows that

$$
\left( P_{t} - \frac{1}{2} \Delta^{g_{t}} \right) (\psi) = \frac{1}{2} \left[ \frac{1}{2} \delta_{g_{t}} (D^{g_{t}} \omega(\psi)) - \frac{1}{2} g_{t} (D^{g_{t}} \psi, D^{g_{t}} \omega) + \frac{s_{g_{t}}}{6} g_{t}(\omega, \psi) - W^{g_{t}}(\omega, \psi) \right] \omega,
$$

where $W^{g_{t}}$ stands for the Weyl tensor (see e.g. [6]), $D^{g_{t}}$ (resp. $\delta^{g_{t}}$) for the Levi-Civita connection (resp. the codifferential) with respect to the metric $g_{t}$ and $s_{g_{t}}$ for the Riemannian scalar curvature defined as the trace of the (Riemannian) tensor.

The operator $P_{t}$ is a self-adjoint strongly elliptic linear operator of order 2. We obtain then a family of Green operators $G_{t}$ associated to $P_{t}$. If $h_{t}^{-} = h_{t}^{-} = b^{+}(M) - 1$ for $t \in (-\epsilon, \epsilon)$, then $G_{t}$ is $C^{\infty}$ differentiable in $t \in (-\epsilon, \epsilon)$ [19, 20], meaning that $G_{t}(\psi_{t})$ is a smooth family of sections of $\Lambda_{0}^{2}(M)$ for any smooth sections $\psi_{t}$.

To show Theorem [13], we consider the extension of $G_{t}$ to the Sobolev spaces $W^{k,p}(M, \Lambda_{0}^{2}(M))$ involving derivatives up to $k$.

**Lemma 5.1.** Let $G_{t}: \Omega_{0}^{2}(M) \rightarrow \Omega_{0}^{2}(M)$ the family of the above Green operators associated to $P_{t}$ and suppose that $h_{t}^{-} = h_{t}^{-} = b^{+}(M) - 1$ for $t \in (-\epsilon, \epsilon)$. Then, the extension of $G_{t}$ to Sobolev spaces, still denoted by $G_{t}$, defines a $C^{1}$ map $G: (-\epsilon, \epsilon) \times W^{k,p}(M, \Lambda_{0}^{2}(M)) \rightarrow W^{k+2,p}(M, \Lambda_{0}^{2}(M))$.

**Proof.** Denote by $\Pi_{t}$ the $L^{2}$-orthogonal projection to the kernel of $P_{t}$ with respect to $\langle \cdot, \cdot \rangle_{L^{2}_{g_{t}}}$ $= \int_{M} g_{t}(\cdot, \cdot) \frac{\omega}{g_{t}}$. We claim that $G_{t} \circ \Pi_{0}$ and $\Pi_{0} \circ G : (-\epsilon, \epsilon) \times W^{k,p}(M, \Lambda_{0}^{2}(M)) \rightarrow W^{k+2,p}(M, \Lambda_{0}^{2}(M))$ are $C^{1}$ maps. Indeed, let $\{ \psi_{t} \}$ be an orthonormal basis of the kernel of $P_{0}$ with respect to $\langle \cdot, \cdot \rangle_{L^{2}_{g_{0}}}$. Note that $\psi_{0}$ are smooth since $P_{0}$ is elliptic. Then, we have

$$(G_{t} \circ \Pi_{0})(\psi) = \sum_{i} \langle \psi, \psi_{0}^{i} \rangle_{L^{2}_{g_{0}}} G_{t}(\psi_{0}^{i}),$$

$$(\Pi_{0} \circ G_{t})(\psi) = \sum_{i} \langle G_{t}(\psi), (\psi_{0}^{i})^{J_{0},+} + (\psi_{0}^{i})^{J_{0},-} \rangle_{L^{2}_{g_{0}}} \psi_{0}^{i}$$

$$= \sum_{i} \left( \int_{M} -G_{t}(\psi) \wedge (\psi_{0}^{i})^{J_{0},+} + G_{t}(\psi) \wedge (\psi_{0}^{i})^{J_{0},-} \right) \psi_{0}^{i}$$

$$= \sum_{i} \left[ \left( \int_{M} -G_{t}(\psi) \wedge (\psi_{0}^{i})^{J_{0},+} \right)^{J_{0},+} + G_{t}(\psi) \wedge (\psi_{0}^{i})^{J_{0},-} \right] \psi_{0}^{i}$$

$$= \sum_{i} \left[ \langle \psi, G_{t} \left( \left( (\psi_{0}^{i})^{J_{0},+} \right)^{J_{0},+} \right) \rangle_{L^{2}_{g_{t}}} - \langle \psi, G_{t} \left( \left( (\psi_{0}^{i})^{J_{0},+} \right)^{J_{0},-} \right) \rangle_{L^{2}_{g_{t}}} \right] \psi_{0}^{i}$$

$$- \langle \psi, G_{t} \left( \left( (\psi_{0}^{i})^{J_{0},+} \right)^{J_{0},-} \right) \rangle_{L^{2}_{g_{t}}} + \langle \psi, G_{t} \left( \left( (\psi_{0}^{i})^{J_{0},-} \right)^{J_{0},+} \right) \rangle_{L^{2}_{g_{t}}} \right] \psi_{0}^{i}$$

(in the latter equality, we used the fact that $G_{t}$ is self-adjoint with respect to $L^{2}_{g_{t}}$).

The claim follows from the result of Kodaira–Spencer [19, 20].
Denote by $W^{k,p}(M, \Lambda^2(M))^\perp$ the space of 2-forms in $W^{k,p}(M, \Lambda^2(M))$ which are orthogonal to the kernel of $P_0$ with respect to $L^2_{g_0}$ and consider the map

$$\Phi: (-\epsilon, \epsilon) \times W^{k+2,p}(M, \Lambda^2(M))^\perp \rightarrow (-\epsilon, \epsilon) \times W^{k,p}(M, \Lambda^2(M))^\perp \quad \text{such that} \quad (t, \psi) \mapsto (t, (I\!d - \Pi_0)P_t(\psi)).$$

Clearly, the map $\Phi$ is of class $C^1$ and its differential at $(0, \psi)$ is an isomorphism so by the inverse function theorem for Banach spaces there exist a neighborhood $V$ of $(0, \psi)$ such that $\Phi|_V$ admits an inverse of class $C^1$. By the The Kodaira–Spencer result [19] [20], the map $\Pi : (-\epsilon, \epsilon) \times W^{k,p}(M, \Lambda^2(M)) \rightarrow W^{k,p}(M, \Lambda^2(M))$ is $C^1$ and thus the map $P_t(\Pi_0 - \Pi_0)G_t(Id - \Pi_0) \equiv (I\!d - \Pi_0)(I\!d - \Pi_0) - P_t(\Pi_0G_t)(I\!d - \Pi_0)$ is clearly $C^1$ since it is a composition of such operators. Then, the map

$$\Phi|^{-1}_V(t, (I\!d - \Pi_0)G_t(Id - \Pi_0)) = (t, (I\!d - \Pi_0)G_t(Id - \Pi_0))$$

is $C^1$ and hence $G_t$ is $C^1$. 

\[\square\]

**Proof of Theorem** [17] Let $(M, \omega)$ be a 4-dimensional compact and connected symplectic manifold and $T$ a maximal torus in $Ham(M, \omega)$. Let $(J_t, g_t)$ a smooth family of $\omega$-compatible almost-Kähler metrics in $AK^C_2(M)$ such that $(J_0, g_0)$ is an extremal Kähler metric.

Following [21], we consider the almost-Kähler deformations

$$\omega_{t,f} = \omega + d(J_t df - \delta^{\omega} \psi^f_t),$$

where $f$ belongs to the Fréchet space $C_\infty^\infty(M, \mathbb{R})$ of $T$-invariant smooth functions (with zero integral), which are $L^2$-orthogonal, with respect to $\frac{\omega^2}{2}$, to $\omega$, and where the 2-form $\psi^f_t$ is given by Corollary [4,2]

Let $\mathcal{U}$ be an open set in $\mathbb{R} \times C_\infty^\infty(M, \mathbb{R})$ containing $(0, 0)$ such that the symmetric tensor $g_t(\cdot, \cdot) := \omega_{t,f}(\cdot, J_t \cdot)$ is a Riemannian metric.

By possibly replacing $\mathcal{U}$ with a smaller open set, we may assume as in [21] that the kernel of the operator $(I\!d - \Pi^T_\omega) \circ (I\!d - \Pi^T_{\omega_{t,f}})$ is equal to the kernel of $(I\!d - \Pi^T_{\omega_{t,f}})$. Indeed, let $\{X_1, \ldots, X_n\}$ be a basis of $\mathcal{t} = \text{Lie}(T)$. Then, the corresponding Hamiltonians with zero mean value $\{\xi^1_\omega, \ldots, \xi^n_\omega\}$ resp. $\{\xi^1_{\omega_{t,f}}, \ldots, \xi^n_{\omega_{t,f}}\}$, with respect to $\omega$ resp. $\omega_{t,f}$, form a basis of $\mathcal{t}_\omega$ resp. $\mathcal{t}_{\omega_{t,f}}$. Let $\{\hat{\xi}^1_{\omega_{t,f}}, \ldots, \hat{\xi}^n_{\omega_{t,f}}\}$ resp. $\{\hat{\xi}^1_{\omega_{t,f}}, \ldots, \hat{\xi}^n_{\omega_{t,f}}\}$ the corresponding orthonormal basis obtained by the Gram–Schmidt procedure. Since $\text{det} \left[ \begin{array}{ll} \hat{\xi}^1_{\omega_{t,f}} & \hat{\xi}^1_{\omega_{t,f}} \\ \vdots & \vdots \\ \hat{\xi}^n_{\omega_{t,f}} & \hat{\xi}^n_{\omega_{t,f}} \end{array} \right]$ defines a continuous function on $\mathcal{U}$, then we may suppose that $\text{det} \left[ \begin{array}{ll} \hat{\xi}^1_{\omega_{t,f}} & \hat{\xi}^1_{\omega_{t,f}} \\ \vdots & \vdots \\ \hat{\xi}^n_{\omega_{t,f}} & \hat{\xi}^n_{\omega_{t,f}} \end{array} \right] \neq 0$ on an eventually smaller open set than $\mathcal{U}$ (here $\langle \cdot, \cdot \rangle$ denotes the $L^2$ product with respect to the volume form $\frac{\omega^2}{2}$). So, if $u \in \ker \left((I\!d - \Pi^T_\omega) \circ (I\!d - \Pi^T_{\omega_{t,f}})\right)$ then $v \in \mathcal{t}_\omega \cap \mathcal{t}_{\omega_{t,f}}$ such that $v = (I\!d - \Pi^T_{\omega_{t,f}})u$. But the hypothesis $\text{det} \left[ \begin{array}{ll} \hat{\xi}^1_{\omega_{t,f}} & \hat{\xi}^1_{\omega_{t,f}} \\ \vdots & \vdots \\ \hat{\xi}^n_{\omega_{t,f}} & \hat{\xi}^n_{\omega_{t,f}} \end{array} \right] \neq 0$ implies that $v \equiv 0$ and then $\ker \left((I\!d - \Pi^T_\omega) \circ (I\!d - \Pi^T_{\omega_{t,f}})\right) = \ker (I\!d - \Pi^T_{\omega_{t,f}})$.

We then consider the map:
\[ \Psi: \quad U \rightarrow \mathbb{R} \times \tilde{C}_T^\infty(M, \mathbb{R}) \]
\[ (t, f) \mapsto \left( t, (Id - \Pi_{t,f}^T) \circ (Id - \Pi_{t,f}^T)(s^{\nabla_{t,f}}) \right), \]
where \( s^{\nabla_{t,f}} \) is the zero integral part of the hermitian scalar curvature \( s^{\nabla_{t,f}} \) of \((J_t, g_{t,f})\).

It follows from Proposition 3.3 that \( \Psi(t, f) = (t, 0) \) if and only if \((J_t, g_{t,f})\) is an extremal almost-Kähler metric. In particular, \( \Psi(0, 0) = (0, 0) \).

Let \( \alpha_{t,f} = J_t df - \delta^0 \psi^f \) where \( \delta^0 \) is the Green operator associated to the elliptic operator \( P_t : \Omega^{J_t,-}(M) \rightarrow \Omega^{J_t,-}(M) \). In order to extend the map \( \Psi \) to Sobolev spaces, we give an explicit expression of \((Id - \Pi_{t,f}^T)(s^{\nabla_{t,f}})\). A direct computation using (2.2) shows that
\[ s^{\nabla_{t,f}} = \Delta^{g_{t,f}} F_{t,f} + g_{t,f}(\nabla_{t,f}, \omega_{t,f}), \]
where \( F_{t,f} = \log \left( \frac{1}{2} \left( 1 + \frac{g_t(\partial \omega_{t,f}, \omega) + g_t(\partial \omega_{t,f}, \omega)}{2} \right) \right) \) satisfying the relation \( \omega_{t,f}^2 = e^F_{t,f} \omega^2 \). Then
\[ (Id - \Pi_{t,f}^T)(s^{\nabla_{t,f}}) = \Delta^{g_{t,f}} F_{t,f} + g_{t,f}(\nabla_{t,f}, \omega_{t,f}) - \sum_j \left\langle s^{\nabla_{t,f}, \xi_j}, \xi_j \right\rangle \xi_j. \]

Let \( \tilde{W}^{p,k}_T \) be the completion of \( \tilde{C}_T^\infty(M, \mathbb{R}) \) with respect to the Sobolev norm \( \| \cdot \|_{p,k} \) involving derivatives up to order \( k \). We choose \( p, k \) such that \( pk > 2n \) and the corresponding Sobolev space \( \tilde{W}^{p,k}_T \subset C_T^2(M, \mathbb{R}) \) so that all coefficients are \( C_T^2(M, \mathbb{R}) \).

Since \( \tilde{W}^{p,k}_T \) form an algebra relative to the standard multiplication of functions \( \mathbf{1} \), we deduce from the expression (5.2) that the extension of \( \Psi \) to the Sobolev completion of \( \tilde{C}_T^\infty(M, \mathbb{R}) \) is a map \( \Psi^{(p,k)} : \tilde{U} \subset \mathbb{R} \times \tilde{W}^{p,k}_T \rightarrow \mathbb{R} \times \tilde{W}^{p,k}_T \).

Clearly \( \Psi^{(p,k)} \) is a \( C^1 \) map (in a small enough open around \( (0, 0) \)). Indeed, it is obtained by a composition of \( C^1 \) maps by Lemma 5.1 and 5.2.

As in [21] and using Proposition 3.3, the differential of \( \Psi^{(p,k)} \) at \((0,0)\) is given by
\[ \left( \mathbf{T}_{(0,0)} \Psi^{(p,k)} \right) (t, f) = (t, \delta^0 \delta^0 h - \frac{2}{\delta^0} \delta^0 (D^0 df)^{J_0,-}), \]
where \( \delta^0 h = \frac{d}{dt}|_{t=0} g_t \).

The operator \( L := \delta^0 |_{(0,0)} \) given by \( L(f) = -2\delta^0 \delta^0 (D^0 df)^{J_0,-} \) is called the \textit{Lichnerowicz operator}. It is a 4-th order self-adjoint \( T \)-invariant elliptic linear operator leaving invariant \( \mathfrak{t}_\omega \) since \( L(f) = 0 \) for any \( f \in \mathfrak{t}_\omega \). By a known result of the elliptic theory [3, 29], we obtain the \( L^2 \)-orthogonal splitting \( \tilde{C}_T^\infty(M, \mathbb{R}) = \ker(L) \oplus \text{Im}(L) \). Following the argument in [3, Lemma 4], any \( f \in \ker(L) \) gives rise to a Killing vector field in the centralizer of \( \mathfrak{t} = \text{Lie}(T) \). By the maximality of the torus \( T, \) \( f \in \mathfrak{t}_\omega \). It follows that \( L \) is an isomorphism of \( \tilde{C}_T^\infty(M, \mathbb{R}) \) and also from \( \tilde{W}^{p,k+4}_T \) to \( \tilde{W}^{p,k}_T \). Thus, \( \mathbf{T}_{(0,0)} \Psi^{(p,k)} \) is an isomorphism from \( \mathbb{R} \oplus \tilde{W}^{p,k+4}_T \) to \( \mathbb{R} \oplus \tilde{W}^{p,k}_T \).

It follows from the inverse function theorem for Banach manifolds that \( \Psi^{(p,k)} \) determines an isomorphism from an open neighbourhood of \((0,0)\) to an open neighbourhood of \((0,0)\). In particular, there exists \( \mu > 0 \) such that for \( |t| < \mu, \) \( \Psi^{(p,k)} \left( \Psi^{(p,k)} \right)^{-1}(t, 0) = (t, 0) \). By Sobolev embedding, we can choose a \( k \) large enough, such that \( \tilde{W}^{p,k+4}_T \subset \tilde{C}_T^\infty(M, \mathbb{R}) \). Thus, for \( |t| < \mu, \) \( (J_t, g_{\Psi^{(p,k)}(t,0)}) \)
is an extremal almost-Kähler metric of regularity at least \( C^4 \) (so we ensure, in this case, that \( \text{grad}\, s^{\nabla_{t, f}} \) is of regularity \( C^3 \)).

By Proposition 3.5, the extremal vector field \( Z^T_{\omega_{t, f}} = \nabla_{t, f} Z^T_{\omega} \) is smooth for any almost-Kähler metric \((J_t, g_{t, f})\). In particular, for an extremal almost-Kähler metric \((J_t, g_{t, f})\) of regularity \( C^4 \), the dual \( ds^{\nabla_{t, f}} \) of \( Z^T_{\omega_{t, f}} \) with respect to \( \omega_{t, f} \) is of regularity \( C^4 \). From (5.1), it follows that the hermitian scalar curvature \( s^{\nabla_{t, f}} \) of \((J_t, g_{t, f})\) is of regularity \( C^5 \).

From (5.1), it follows that the hermitian scalar curvature is given by the pair of equations
\[
s^{\nabla_{t, f}} - g_{t, f}(\nabla_{t, f}, \omega_{t, f}) = \Delta g_{t, f}(u),
\]
\[
e^u = \frac{\omega_{t, f}^2}{\omega^2}.
\]

From (5.3), using the ellipticity \( \Delta \) of the (Riemannian) Laplacian \( \Delta g_{t, f} \) and since the l.h.s of (5.3) is of Hölder class \( C^{5, \beta} \) for any \( \beta \in (0, 1) \), it follows that \( u \) is of class \( C^{5, \beta} \). Following [11, 28], the linearisation of the equation (5.3) \( (\omega + d\alpha) \wedge d\alpha = 0 \) together with the constraints \( \delta_{t, f} \alpha = 0 \) and \( (d\alpha)^{t, f} = 0 \) form a linear elliptic system in \( \alpha \). Elliptic theory [2, 6] ensures that the almost-Kähler metric \( g_{t, f} \) is of class \( C^{5, \beta} \) as the volume form and we can prove that any extremal almost-Kähler metric of regularity \( C^4 \) is smooth by a bootstrapping argument (in the Kähler case see [21]).

We obtain then a smooth family of \( T \)-invariant extremal almost-Kähler structures \( (J_t, \omega_t = \omega + d\alpha_t) \) defined for \( |t| < \mu \). The main theorem follows from the Moser Lemma [25].

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