Einstein Product Metrics in Diverse Dimensions

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Abstract

We use direct products of Einstein Metrics to construct new solutions to Einstein’s Equations with cosmological constant. We illustrate the technique with three families of solutions having the geometries Kerr/de Sitter $\otimes$ de Sitter, Kerr/anti-de Sitter $\otimes$ anti-de Sitter and Kerr $\otimes$ Kerr.

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Shortly after Einstein introduced the cosmological constant to General Relativity, Kasner [1] and Schouten and Struik [2] explored the geometry of Einstein Manifolds: manifolds admitting a metric whose Ricci Tensor is a constant multiple of the metric. These are manifolds of constant curvature and solutions to Einstein’s Equations with zero stress-energy tensor but possibly nonzero cosmological constant. With Hubble’s discovery in 1929 of the expanding universe, Einstein’s motivation for introducing the cosmological constant was lost and the canonical Einstein Manifolds, de Sitter and anti-de Sitter spacetimes, were largely relegated to serve as textbook examples. Then in 1998, measurements of high red-shift Type Ia supernovae [3] indicated that the expansion of the universe was accelerating, possibly due to a positive cosmological constant.

Interest in manifolds with dimension greater than 4 has paralleled that of interest in the cosmological constant. As early as 1914 [4], a fifth dimension was postulated as a mechanism for unifying the electromagnetic and gravitational forces. Kaluza and Klein independently applied this mechanism in the context of General Relativity, but for nearly half a century it was seen as unphysical for reasons which are still obvious. Since then, however, the progress in understanding supergravity and string theories has carried interest in manifolds of diverse dimension throughout the academic community and beyond into the public consciousness.

In this note we construct direct products of Einstein Metrics which are solutions to Einstein’s Equations with cosmological constant in diverse dimensions. For two or more metrics defined on disjoint manifolds, the product manifold carries a metric which is the simple sum

\[ ds^2 = ds_1^2(x) + ds_2^2(y) + \ldots \] (1)

For such metrics, the Christoffel Connection and Riemann and Ricci Tensor components are simple sums of the corresponding components on the submanifolds. In addition, the scalar curvature and the Kretschmann Invariant are simple sums of the corresponding invariants. The utility of these relationships has been known and exploited for over half a century to simplify manual computations of connection and tensor components. Using them we see that for the product metric, Einstein’s Equations almost uncouple into a set of disjoint equations:

\[ R_{i,ab} + (\Lambda - \frac{1}{2} \sum_j R_j) g_{i,ab} = \alpha T_{i,ab} \] (2)
It is clear that if the component metrics are independently solutions to Einstein’s Equations with vanishing scalar curvatures, the product metric is also a solution. This implies, for instance, that any product of Ricci-flat metrics (ie., circles, flat tori, Minkowski and Kerr spacetimes, or their Euclidean counterparts) is a solution.

If the component metrics are Einstein Metrics, we have

\[
R_{i,ab} = \chi_i g_{i,ab} \\
R_i = D_i \chi_i \\
\Lambda_i = \frac{D_i - 2}{2} \chi_i
\]  

(3)

where \( \chi_i \) is a constant and \( D_i \) is the dimension of the \( i \)th metric. Note that for \( D_i < 3 \), \( \Lambda_i = 0 \).

For a product of \( n \) Einstein Metrics \( g_i \), Einstein’s Equations reduce to a set of \( n \) algebraic relations

\[
\chi_i + \Lambda - \frac{1}{2} \sum_j R_j = 0.
\]  

(4)

Obviously the \( \chi_i \) must all be equal for the product metric to be a solution, which fixes \( \Lambda \) and the \( \Lambda_i \) in terms of \( \chi \):

\[
\Lambda = \chi \frac{\sum_j D_j - 2}{2} \\
\Lambda_i = \frac{D_i - 2}{2} \chi
\]  

(5)

It is clear that the cosmological constants must all have the same sign.

The most commonly discussed Einstein Manifolds which are not Ricci-flat are de Sitter, anti-de Sitter and Kerr with nonzero cosmological constant. A Euclidean de Sitter (spherical) metric for \( D_S > 2 \) is

\[
ds_S^2 = \beta^2 \cosh^2 \frac{t}{\beta} d\Omega_{D_S-1}^2 + dt^2
\]  

(6)

where \( d\Omega_i^2 \) is the standard metric on \( S^i \). For either this metric or its Lorentzian counterpart (or for the standard metric on \( S^2 \), \( ds_S^2 = \beta^2 d\Omega_2^2 \)), we find

\[
\chi_S = \frac{D_S - 1}{\beta^2}
\]  

(7)

A Euclidean anti-de Sitter metric for \( D_A > 2 \) is

\[
ds_A^2 = \alpha^2 (dr^2 + \sinh^2 r d\Omega_{n-2}^2 + \cosh^2 r dt^2)
\]  

(8)
We include the constant factor $\alpha$ which in the product metric becomes a nontrivial parameter. For either this metric or its Lorentzian counterpart,

$$\chi_A = -\frac{D_A - 1}{\alpha^2} \tag{9}$$

Kerr solutions with nonzero cosmological constant exist in any dimension $D_K > 3$. Let

$$\rho^2 = r^2 + a^2 \cos^2 \theta$$
$$\Delta(D_K) = (1 - \frac{2\Lambda_K r^2}{(D_K - 1)(D_K - 2)})(r^2 + a^2) - \frac{\mu}{r^{D_K - 5}}$$
$$\psi(D_K) = 1 + \frac{2a^2 \Lambda_K \cos^2 \theta}{(D_K - 1)(D_K - 2)}$$
$$\Sigma(D_K) = 1 + \frac{2a^2 \Lambda_K}{(D_K - 1)(D_K - 2)} \tag{10}$$

where $\mu$ is proportional to the mass, $a$ is proportional to the angular momentum and $\Lambda_K$ is the cosmological constant. The corresponding Kerr metric is

$$ds^2_K = r^2 \cos^2 \theta d\Omega^2 + \frac{\rho^2}{\Delta(D_K)} dr^2 + \frac{\rho^2}{\psi(D_K)} d\theta^2 + \frac{(\psi(D_K)(r^2 + a^2)^2 - \Delta(D_K)a^2 \sin^2 \theta \sin^2 \theta)}{\Sigma^2(D_K)\rho^2} d\phi^2 + \frac{2a \sin^2 \theta (\psi(D_K)(r^2 + a^2) - \Delta(D_K))}{\Sigma^2(D_K)\rho^2} d\phi dt - \frac{\Delta(D_K) - a^2 \psi(D_K) \sin^2 \theta}{\Sigma^2(D_K)\rho^2} dt^2 \tag{11}$$

where $d\Omega^2$ is the standard metric on $S^{D_K - 4}$. These solutions possess a single rotation axis; more general solutions have been discussed which have the maximal number of rotation parameters. For these metrics or their Euclidean counterparts,

$$\chi_K = \frac{2\Lambda_K}{D_K - 2} \tag{12}$$

Forming the product metric $ds^2 = ds^2_K + ds^2_S$, we set $\chi_K$ equal to $\chi_S$ and find the product is a solution for

$$\Lambda = \frac{(D_K + D_S - 2)\Lambda_K}{D_K - 2}$$
$$\beta = \sqrt{\frac{(D_K - 2)(D_S - 1)}{2\Lambda_K}} \tag{13}$$

Physically, an observer on the Kerr submanifold measures a smaller value for the cosmological constant than an observer living in the entire product spacetime, and the ratio of
the two is purely a function of the dimension of the spherical submanifold. In addition, the 
cosmological constant measured by the Kerr observer is inversely proportional to the square 
of the radius parameter for the sphere. Assuming that $D_K = 4$ and $\Lambda_K$ corresponds to data 
from current observations, we find that $\beta$ is of the order of the radius of the observable 
universe. Since the geodesic equations for a simple product manifold are not coupled, each 
submanifold is a geodesic hypersurface. But without a mechanism to prevent energy and 
momentum transfer between submanifolds, these manifolds are clearly of purely academic 
interest.

This is obviously true as well when more than one submanifold is noncompact. For the 
anti-de Sitter case, $ds^2 = ds^2_K + ds^2_A$ is a solution if

$$\Lambda = \frac{(D_K + D_A - 2)\Lambda_K}{D_K - 2}$$

$$\alpha = \sqrt{-\frac{(D_A - 1)(D_K - 2)}{2\Lambda_K}}$$

(14)

Finally, we form a product of two Kerr metrics with cosmological constant and confirm 
that the solution requires

$$\Lambda = \frac{(D_1 + D_2 - 2)\Lambda_1}{D_1 - 2}$$

$$\Lambda_2 = \frac{(D_2 - 2)\Lambda_1}{D_1 - 2}$$

(15)

We noted previously that the scalar curvature and Kretschmann Invariant for a simple 
product metric are simply the sums of those invariants for the metrics on the submanifolds. 
Therefore if either metric possesses curvature singularities, the product metric possesses 
those same singularities. This means that our example product metrics possess the well-
known ring singularity for $D_K(D_i)$ equal to 4 or 5, and the singularity at $r = 0$ for $D_K(D_i) > 
5$ [9].}
Acknowledgements

We would like to thank Cenalo Vaz and Louis Witten for stimulating discussions on these matters.

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