The theory of the strong interactions, Quantum Chromodynamics (QCD), has been addressed by a variety of non-perturbative techniques over the decades since its introduction. We have investigated Hamiltonian formulations with different quantization methods and approximation schemes. In one method, we utilize light-front coordinates to investigate the role of bosonic zero modes in leading to confinement. In another method we are able to obtain spectra for the mesons and baryons using constituent quark masses but no phenomenological confinement. We survey our principal accomplishments to date and indicate our future directions.

1 Introduction and Motivation

For strong interaction physics, one might well ask ‘Why develop Hamiltonian methods for gauge theories - after all, does not the lattice gauge method work well?’

While lattice gauge methods work well for certain observables, we have a wider range of observables in mind. In addition, we are motivated by the desire to work within a Hamiltonian framework which we find more physically intuitive. One final motivation for us is that we believe there are potential advantages of using advanced methods from quantum many-body theory.

On the other hand, we are also quick to acknowledge many challenges which we face. For example, little is known about renormalization and scale
dependence within the Hamiltonian approach to strong interactions. We will lose manifest gauge invariance when we derive a Hamiltonian expressed only in terms of the independent degrees of freedom. Any approximations, such as a truncation, will usually lead to gauge-dependent results. Finally, due to our approximations, we may lose other symmetries respected by the original Lagrangian.

Our purpose here is to outline our recent progress in implementing Hamiltonian approaches and to indicate where we still face major hurdles.

2 Introduction to Many-Body Theory of $H_{\text{eff}}$

Here we discuss certain aspects of recent developments in the theory of effective Hamiltonians for quantum many-body systems which are particularly relevant for our applications to strong interactions. Note that these developments are cast in a form which is independent of the kinematics of the interacting particles.

The basic framework we adopt has been utilized extensively in nuclear physics applications. Our goal is to solve the usual Hamiltonian eigenvalue problem:

$$H |\Psi_\alpha\rangle = E_\alpha |\Psi_\alpha\rangle$$

for the eigenenergies $E_\alpha$ and the eigenstates $|\Psi_\alpha\rangle$ of the many-particle system, where $\alpha$ is some label characterizing the states. But it is impossible to solve this problem in the full Hilbert space $S$ when the number of particles in the system exceeds 3 or 4 because it contains too many degrees of freedom. Consequently, one wishes to truncate the problem to a smaller space $S$ of dimension $d$, in which it becomes tractable to carry out the calculation. Now let $|\Phi_\beta\rangle$ represent the projections of $d$ of the states $|\Psi_\beta\rangle$ into $S$. Thus we define the effective Hamiltonian $H_{\text{eff}}$ in $S$ to satisfy

$$H_{\text{eff}} |\Phi_\beta\rangle = E_\beta |\Phi_\beta\rangle,$$

where the eigenvalues $\{E_\beta\}$ are $d$ of the exact eigenvalues $\{E_\alpha\}$ in Eq. (1). Because the $|\Phi_\beta\rangle$ are projections of the $|\Psi_\alpha\rangle$, they are, in general, not orthogonal.

The question then arises whether an appropriate $H_{\text{eff}}$ exists for any given truncation. One can show this to be true by constructing the biorthogonals to $|\Phi_\beta\rangle$, namely, $|\check{\Phi}_\gamma\rangle$, which satisfy $\langle \check{\Phi}_\gamma | \Phi_\beta\rangle = \delta_{\gamma\beta}$. It then follows that the effective Hamiltonian $H_{\text{eff}}$ always exists and is of the form

$$H_{\text{eff}} = \sum_{\beta \in S} |\Phi_\beta\rangle E_\beta \langle \check{\Phi}_\beta|,$$
which automatically satisfies Eq.(4). As Kirson has emphasized, the question is not whether $H_{\text{eff}}$ exists, but whether it has a simple enough form, so as to be useful.

We use the time-independent-perturbation-theory approach in establishing the connection between $H_{\text{eff}}$ and $H$. The basic idea involves the separation of the Hilbert space into two parts, using the projection operators $P$ and $Q$, where $P$ defines the truncated or ‘model’ space, defined by the eigenstates of an unperturbed Hamiltonian $H_0$, and $Q$ defines the excluded space outside the model space. The projection operators $P$ and $Q$ define non-overlapping spaces, so that $PH_0Q = 0$.

In the full Hilbert space, a typical many-body choice for $H$ is of the form

$$H = \sum_{i=1}^{A} t_i + \sum_{i<j}^{A} v_{ij} = T + V = (T + U) + (V - U) = H_0 + H_I,$$

where $U$ is some single-particle or ‘auxiliary’ potential, $H_0 = T + U$, and $H_I = V - U$ is the residual interaction. Only two-body interactions $v_{ij}$ have been assumed among the $A$-particles, but the method can be generalized to many-body forces.

Using the Feshbach projection method, one can explicitly project $H$ into the $P$ and $Q$ spaces and rewrite the $P$ space equation (omitting the subscript $\alpha$ everywhere) in the form

$$\left[ PHP + PHQ \frac{1}{E - QHQ} QHP \right] P|\Psi\rangle = EP|\Psi\rangle,$$

where $P|\Psi\rangle = |\Phi\rangle$. The term in the square brackets defines the effective Hamiltonian

$$H_{\text{eff}} = PHP + V(E),$$

where

$$V(E) = PHP + PHQ \frac{1}{E - QHQ} QHP$$

is the effective interaction. It should be noted that, in general, $V$ is an $A$-particle operator and the energy $E$ in the denominator corresponds to one of the exact eigenenergies of the $A$-particle system.

Although $V(E)$ is an $A$-particle interaction, the standard assumption is to approximate it in terms of a perturbation-theory expansion using two-body interactions. There are many uncertainties associated with this approach for conventional nuclear physics applications and experience gained in those applications may well assist us in our applications to quantum field theories.
Our approach is to take the $A$-particles as all active (i.e. we do not assume a passive ‘core’), and Eq. (7) may be interpreted as a generalized $A$-particle $G$-matrix equation. For a one-dimensional model space, the exact solutions for the eigenvalues are given by

$$E = E_0 + \mathcal{V}(E), \quad (8)$$

where $E_0$ is the eigenenergy of the unperturbed Hamiltonian $PH_0P$. If $\mathcal{V}(E)$ can be constructed for the $A$-particle system, then Eq. (8) can be solved diagrammatically or iteratively to obtain all the eigenenergies of the $A$-particle system whose eigenstates have non-vanishing projections on the one-dimensional model space. Examples of these procedures with solvable models prove instructive.

It is not generally possible to construct the full $A$-particle $G$ matrix. We approximate it by the two-particle reaction matrix $G_{11}$ plus higher-order terms. The two-particle $G$ matrix is simply the infinite sum of two-particle ladder terms.

The perturbation-theory expansion for $\mathcal{V}(E)$ is now rewritten as a perturbation series in $G(\omega)$ where $\omega$ is an arbitrary energy, the ‘starting’ energy, around which we make an expansion. In our application to no-core model spaces, these corrections at the two-particle level are all of the folded-diagram type. The remaining corrections generate effective many-body interactions which are neglected in the applications we discuss here.

3 Renormalization of $H_{eff}$

In this section, we introduce a way of implementing Wilson renormalization within the context of the theory of effective Hamiltonians. Our renormalization scheme involves manipulations at the level of the generalized $G$–matrix. We show how to calculate the beta function within this context and exhibit our method using simple scale–invariant quantum mechanical systems.

We have argued above that the knowledge of the matrix $G$ allows us to obtain $H_{eff}$. In what follows, we shall therefore restrict our attention only to $G$. For the sake of convenience we choose to work in the momentum representation where the kinetic energy term in the Hamiltonian is diagonal. To introduce the concept of renormalization we shall focus our attention on the one–particle system. The matrix elements of $G$ are here given by

$$G_{kk'} = \langle k|PV P|k' \rangle + \int dp \, dp' \langle k|PVQ|p \rangle \langle p|QH_0Q|p' \rangle \frac{1}{\omega - QH_0Q} \langle p'|QVP|k' \rangle + \cdots \quad (9)$$
where we have set \( U = 0 \) and expanded \( QVQ \) out of the denominator.

Let us suppose that the potential \( V \) depends on a single coupling constant \( \mu_0 \), which we shall call the bare coupling constant. It is clear from Eq. (3) that the matrix element \( G_{kk'} \) will be a function of \( \mu_0 \). The expression in Eq. (3) may, in general, require regularization due to the divergence arising from the integral. The regularization that we choose consists of introducing an ultraviolet cutoff \( \Lambda \). The matrix element in Eq. (3) is now a function of the coupling constant \( \mu_0 \) and the cutoff \( \Lambda \). At the end of the calculation we must remove the cutoff, i.e. we must take \( \Lambda \to \infty \), which, as discussed above, may in general lead to divergence. One way to avoid the divergence is to replace the coupling constant \( \mu_0 \) with a function of \( \Lambda \), which we denote as \( \mu(\Lambda) \), and then require that matrix element in Eq. (3) remain finite and independent of the cutoff as the cutoff is removed. In other words, we demand that
\[
\lim_{\Lambda \to \infty} \frac{d}{d\Lambda} G_{kk'}(\Lambda, \mu(\Lambda)) = 0 \tag{10}
\]
The function \( \mu(\Lambda) \) thus plays the role of the renormalized coupling constant.

The dependence of the coupling constant on the cutoff is usually expressed in terms of the beta function, which is defined by
\[
\beta(\mu) \equiv \Lambda \frac{d\mu}{d\Lambda}. \tag{11}
\]
Within our formalism, Eqs. (10) and (11) can be used to calculate the beta function.

Note that once Eq. (10) is satisfied and \( \mu(\Lambda) \) is determined, then \( H_{\text{eff}} \), based on \( G_{kk'}(\Lambda, \mu(\Lambda)) \), should also be independent of \( \Lambda \) as \( \Lambda \to \infty \). Thus, the complete problem of renormalization is solved.

We shall now illustrate the method prescribed above in two simple cases of a Dirac particle in 1 dimension and a Schrodinger particle in 2 dimensions.\[15\][16].
In both these cases the interaction potential will be taken as a delta function in position space:
\[
V(x) = -\mu_0 \delta^{(n)}(x), \tag{12}
\]
where \( n \) is the dimension of configuration space. In the momentum space the interaction potential would simply be a constant, i.e.,
\[
V(k) = -\mu_0. \tag{13}
\]
We will choose \( H_0 \) to be the pure kinetic operator, and our model space to consist of all states with momenta less than \( \lambda \). Thus \( Q \) projects onto the momentum range \([\lambda, \infty)\).
With the choice of the interaction potential described above, the series in Eq. (9) can be summed exactly and is given by
\[ G_{kk'} = -\frac{\mu_0}{1 + \mu_0 I(\omega)} \delta(k - k'), \]  
(14)
where \( I(\omega) \) is given by
\[ I(\omega) = \int_\Lambda^\infty d^np \frac{1}{\omega - E_0(p)}, \]  
(15)
Following the preceding discussion we now introduce an ultraviolet cutoff \( \Lambda \). Replacing \( \mu_0 \) by the renormalized coupling constant \( \mu \) and using Eqs. (10) and (11), we obtain the beta function as
\[ \beta(\mu) = \mu^2 \Lambda \frac{\partial I}{\partial \Lambda}. \]  
(16)
To obtain the explicit expression for the beta function we need to evaluate the integral appearing in Eq. (13). For the 1 dimensional Dirac particle we have \( n = 1, E_0(p) = p + m \) and
\[ I(\omega) = \int_\lambda^\Lambda dp \frac{1}{\omega - (p + m)} = -\ln \left( \frac{\omega - (\Lambda + m)}{\omega - (\lambda + m)} \right). \]  
(17)
The corresponding beta function is given by
\[ \beta(\mu) = -\mu^2. \]  
(18)
For the Schrödinger particle in 2 dimensions we have \( n = 2 \) and \( E_0(p) = p^2/2 \) (we set the mass of the particle to unity.) Proceeding exactly as before, we obtain
\[ I(\omega) = -2\pi \ln \left( \frac{\omega - \Lambda^2}{\omega - \lambda^2} \right) \]  
(19)
and
\[ \beta = -4\pi\mu^2 \]  
(20)
Note that the results in both examples above have the desirable property that the beta function is independent of the model space cutoff, \( \lambda \). The beta functions calculated give rise to asymptotically free theories and generate the generally accepted pattern for the flow of the coupling constant for the two examples described above.
4 Light-Front QCD Application

Rather than take a pure light-front quantization approach, we opt for quantizing on a space-like surface in ‘near light-front coordinates’ in order to maintain contact with many of the known results from the usual equal-time quantization method.

Here we will sketch the dynamics of the gluonic zero modes of the color SU(2) QCD Hamiltonian and show how the strong coupling solutions serve as a basis for the complete problem. Only an external charge density $\rho_m$ is considered here.

4.1 Color SU(2) QCD Hamiltonian

The Lagrangian in the near light front coordinate system reads

$$\mathcal{L} = \frac{1}{2} F_{+/-}^a + \sum_{i=1,2} \left( F_{+/-}^a + \frac{g^2}{2} F_{+/-}^a F_{+/-}^a \right) - \frac{1}{2} F_{12}^2 F_{12}^a - \rho_m A_+^a, \quad (21)$$

where the color index $a$ is summed from 1 to 3, and the transverse coordinates are labeled by $i = 1, 2$. We will also use the matrix notation; for example $A_{-} = A_{-}^a \tau^a / 2$. The field strength tensor contains the commutator of the gauge fields and the coupling constant $g$:

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - ig[A_{\mu}, A_{\nu}]. \quad (22)$$

The $A_+^a$ coordinates have no momenta conjugate to them. As a consequence, the Weyl gauge $A_+^a = 0$ is the most natural starting point for a canonical formulation. The canonical momenta of the dynamical fields $A_{-}, A_i^a$ are given by

$$\Pi_{-}^a = \frac{\partial \mathcal{L}}{\partial F_{+/-}^a} = F_{+/-}^a,$$

$$\Pi_i^a = \frac{\partial \mathcal{L}}{\partial F_{+/-}^a} = F_{-i}^a + g^2 F_{+/-}^a. \quad (23)$$

From this, we get the Weyl gauge Hamiltonian density

$$\mathcal{H}_W = \frac{1}{2} \Pi_{-}^a \Pi_{-}^a + \frac{1}{2} F_{12}^a F_{12}^a + \frac{1}{2g^2} \sum_{i=1,2} \left( \Pi_i^a - F_{-i}^a \right)^2. \quad (24)$$

We choose periodic boundary conditions in $x^- = \frac{x^0 - x^3}{\sqrt{2}}$ and $x_\perp$ on intervals of size $[0, L]$. Using the appropriate periodic delta functions, the quantization
is straightforward. However, the Hamiltonian has to be supplemented by the  
original Euler–Lagrange equation for \( A_+ \) as constraints on the physical states
\[
G^a(x_+, x^-)|\Phi\rangle = \left( D^{ab}_\perp \Pi^b_\perp + D^b_\perp \Pi^b_\perp + g\rho^a_\perp \right) |\Phi\rangle = 0,
\]
with the covariant derivatives
\[
D^{ab}_\perp = \partial_\perp \delta^{ab} + g f^{acb} A^c_-, \\
D^a_\perp = \partial_\perp \delta^a_\perp + g f^{acb} A^c_-, 
\]
where \( f^{acb} \) are the structure constants of \( SU(2) \). These equations are known  
as Gauss' Law constraints. Since the Gauss' Law operator commutes with the  
Hamiltonian
\[
[H_W, G^a(x_+, x^-)] = 0,
\]
time evolution leaves the system in the space of physical states. Furthermore,  
\( H_W \) is invariant under time independent residual gauge transformations whose  
generator is closely connected to Gauss' Law.

In order to obtain a Hamiltonian formulated in terms of unconstrained  
variables, we eliminate the \( A_- \). Classically this would correspond to the light  
front gauge \( A_- = 0 \). However, this choice is not compatible with gauge  
invariance and periodic boundary conditions. Only the (classical) Coulomb light  
front gauge (\( \partial_- A_- = 0 \)) is legitimate. The reason is that \( A_- \) carries information on gauge invariant quantities, such as the eigenvalues of the spatial Polyakov (Wilson) loop
\[
P(x_+) = P \exp \left[ ig \int d^2 x^- A_-(x_+, x^-) \right], 
\]
which can be written in terms of a diagonal matrix \( a_-(x_+) \)
\[
P(x_+) = V \exp \left[ ig a_-(x_+) \right] V^\dagger. 
\]
Thus, we obviously need to keep these ‘zero modes’ \( a_-(x_+) \) as dynamical variables. In order to eliminate the conjugate momentum, \( \Pi_- \) of \( A_- \), by  
means of Gauss' Law, one needs to ‘invert’ the covariant derivative \( D_- \). After  
an unitary transformation \( D_- \) simplifies significantly (compare to Eq. (26))
\[
D_\perp \to d_\perp = \partial_\perp - ig [a_-, \quad .
\]
Now Gauss' Law can be readily resolved: in the physical space one can make  
the replacement
\[
\Pi_-(x_+, x_-) \to p_-(x_+) - (d^-_\perp) G_\perp(x_+, y^-).
\]
The zero mode operator \( p_-(x_\perp) \) is the conjugate momentum to \( a_-(x_\perp) \). It has eigenvalue zero with respect to \( d_- \), i.e., \( d_- p_- = 0 \), and is therefore not constrained.

The appearance of the zero modes also implies residual Gauss’ Law constraints. In the space of transformed physical states \( |\chi\rangle \), they read

\[
\int dx^- G^3_\perp |\chi\rangle = \int dx^- \left( D^3_\perp \Pi^3_\perp + g \rho^3_\perp \right) |\chi\rangle = 0.
\]

These two-dimensional constraints can be handled in full analogy to QED, since they correspond to the diagonal part of color space. This further gauge fixing in the \( SU(2) \) 3-direction is done via another gauge fixing transformation, which leads to the Coulomb gauge representation in the transverse plane for the neutral fields. In other words, we eliminate the color neutral, \( x^- \)-independent, two-dimensional longitudinal gauge fields

\[
a^\ell_\perp(x_\perp) = \frac{1}{L} \int dy^- dy_\perp d(x_\perp - y_\perp) \nabla_\perp \left( \nabla_\perp \cdot A^a_\perp(y_\perp, y^-) \right) \frac{\tau^3}{2}.
\]

Here we use the periodic Greens function of the two dimensional Laplace operator

\[
d(z_\perp) = \frac{1}{L^2} \sum_{\vec{n} \neq 0} \frac{1}{p_n^2} e^{ip_n z_\perp}, \quad p_n = \frac{2\pi}{L} \vec{n},
\]

where \( \vec{n} = (n_1, n_2) \) and \( n_1, n_2 \) are integers. The conjugate momenta of these fields, \( p^\ell_\perp(x_\perp) \), are defined analogously. Resolution of the residual Gauss’ Law allows one to replace them, in the sector of the transformed physical space \( |\Phi'\rangle \), by the neutral chromo-electric field

\[
e_\perp(x_\perp) = g \nabla_\perp \int dy^- dy_\perp d(x_\perp - y_\perp) \{ f^{ab}_3 A^a_\perp(y_\perp, y^-) \Pi^b_\perp(y_\perp, y^-) \\
+ \rho^3_\perp(y_\perp, y^-) \} \frac{\tau^3}{2}. \tag{35}
\]

It is convenient to introduce the unconstrained gauge fields and their conjugate momenta:

\[
A'_\perp(x_\perp, x^-) = A_\perp(x_\perp, x^-) - a^\ell_\perp(x_\perp),
\]

\[
\Pi'_\perp(x_\perp, x^-) = \Pi_\perp(x_\perp, x^-) - \frac{1}{L} p^\ell_\perp(x_\perp). \tag{36}
\]

These relations turn out to be important for neutral gluon exchange; recall that the subtracted fields are diagonal in color space. Note that the physical
degrees of freedom $A'_\perp$ and $\Pi'_\perp$ still contain $(x_\perp, x^-)$–independent, color neutral, modes. Therefore, there is a remnant of the local Gauss’ Law constraints – the global condition

$$Q^3 |\Phi'\rangle = \int dy^- dy_\perp \left\{ f^{3ab} A'^a_\perp (y_\perp, y^-) \Pi'^b_\perp (y_\perp, y^-) + \rho^3_m (y_\perp, y^-) \right\} |\Phi'\rangle = 0.$$  \hspace{1cm} (37)

Its physical meaning is that the neutral component of the total color charge, including external matter as well as gluonic contributions, must vanish in the sector of physical states.

The final Hamiltonian density in the physical sector explicitly reads

$$\mathcal{H} = \text{tr} \left[ \partial_1 A'_2 - \partial_2 A'_1 - ig[A'_1, A'_2]\right]^2 + \frac{1}{\eta^2} \text{tr} \left[ \Pi'_\perp - (\partial_- A'_\perp - ig[a_-, A'_\perp])\right]^2$$

$$+ \frac{1}{\eta^2} \text{tr} \left[ \frac{1}{L} e_\perp - \nabla_\perp a_- \right]^2 + \frac{1}{2L^2} p^2_\perp (x_\perp)$$

$$+ \frac{1}{L^2} \int_0^L dz^- \int_0^L dy^- \sum_{p,q,n} G'_{\perp pq}(x_\perp, z^-) G'_{\perp qp}(x_\perp, y^-) \left[ \frac{2\pi n}{L} + g(a_- (x_\perp) - a_-(x_\perp)) \right] e^{i2\pi n(z^- - y^-)/L},$$

where $p$ and $q$ are matrix labels for rows and columns, $a_- = (a_-)_{qq}$ and the prime indicates that the summation is restricted to $n \neq 0$ if $p = q$. The operator $G'_\perp(x_\perp, x^-)$ is defined as

$$G'_\perp(x_\perp, x^-) = \nabla_\perp \Pi'_\perp + g f^{abc} \frac{x^a}{2p} A'^b_\perp \left( \Pi'_\perp - \frac{1}{L} e_\perp \right) + g \rho_m.$$  \hspace{1cm} (39)

### 4.2 Zero Mode Dynamics

The principal advantage of an exact light front formulation is the apparent triviality of the ground state which simplifies calculations of the hadron spectrum. The light front vacuum, however, is not guaranteed to be trivial in the zero mode sector.

As can be seen from the dispersion relation for massless particles on the light front, $p_+ = \frac{p^2}{2p}$, soft momentum modes become high energy states. In this way, high energy physics becomes tied to long range physics, contrary to the equal time formulation. This physics appears in deep inelastic scattering at small scaling variable and is related to the long distance features of the proton. We will focus on the zero mode sector in order to try to acquire some insight into its dynamics.

From the comparison of abelian and non-abelian theories, striking differences show up in the zero mode sector. Recently, in the equal time formalism,
the zero mode sector in QCD has been claimed to be relevant for the confinement phenomenon. On the level of approximations and restrictions followed below, the formal differences between light front and equal time approach are rather small and, consequently, results and methods are similar.

In this work we do not restrict ourselves to the strong coupling approximation. We will, however, start with the strongly coupled theory to define our set of basis functions. We will restrict ourselves to pure gluonic SU(2) Yang-Mills theory. It already has the typical non-abelian features such as the Coulomb term which explicitly contains the zero modes in the denominator and the non-standard kinetic energy for the zero modes.

The zero mode degrees of freedom couple to the three-dimensional gluon fields via the second term in\(\mathcal{H}\) shifting the longitudinal momenta of the transverse gluon fields (Eq. (39)). We neglect these couplings and consider the pure zero mode Hamiltonian

\[
\mathcal{H} = \int d^2x \left[ \frac{1}{2L} p_3^\dagger(x_\perp)p_3(x_\perp) + \frac{L}{2}\eta^2(\nabla_\perp a_3(x_\perp))^2 \right].
\]

(40)

We recognize ‘electric’ and ‘magnetic’ contributions in \(\mathcal{H}\), the zero mode Hamiltonian – the first and second term, respectively. The light front variables mix the ordinary spatial and time variables so the labeling above is to be understood in analogy with the equal time Hamiltonian. Even at this level of approximation, this zero mode Hamiltonian differs from the corresponding one in QED. The reason is the hermiticity defect of the canonical momentum: \(p_3^\dagger \neq p_3\).

We now omit the color index and introduce dimensionless variables

\[
\varphi(x_\perp) = \frac{gL}{2} \eta(x_\perp).
\]

(41)

In the Schrödinger representation we then obtain

\[
\mathcal{H} = \int d^2 x_\perp \left[ -\frac{g^2 L}{8J(\varphi(x_\perp))} \frac{\delta}{\delta \varphi(x_\perp)}J(\varphi(x_\perp))\frac{\delta}{\delta \varphi(x_\perp)} + \frac{\delta}{\delta \varphi(x_\perp)}J(\varphi(x_\perp))\frac{\delta}{\delta \varphi(x_\perp)} + \frac{2}{\eta^2 g^2 L}(\nabla_\perp \varphi(x_\perp))^2 \right],
\]

(42)

where \(J(\varphi)\) is the Jacobian and equals the Haar measure of \(SU(2)\)

\[
J(\varphi(x_\perp)) = \sin^2(\varphi(x_\perp)).
\]

(43)

The Jacobian is connected to the hermiticity defect of \(p_3\); which stems from the gauge fixing procedure taking into account the curvilinear coordinates. The measure also appears in the integration volume element for calculating matrix elements. As in earlier approaches, \(\varphi\) will be treated as a compact variable, \(0 \leq \varphi < \pi\).
At this stage it is necessary to appeal to the physics of the infinite momentum frame to factorize the reduced true energy $h_{\text{red}}$ and the Lorentz boost factor $\frac{1}{\sqrt{1 - \eta^2}}$, since essentially $h$ is a light front energy, and it is well known how it behaves under a Lorentz transformation. Thus we rewrite $h_{\text{red}} = 2\eta h$. Since the integral over transverse coordinates can contain arbitrarily small wavelengths, we regularize $h_{\text{red}}$ by introducing a lattice to evaluate the transverse integral. The lattice vector $\vec{b}$ numbers the lattice sites, and $\vec{e}_1$ and $\vec{e}_2$ are the two unit vectors on the two–dimensional lattice. In order to have standard commutation relations on the lattice the derivative on the lattice becomes $\frac{\delta}{\delta \phi_{\vec{b}}} = \frac{\delta}{\delta \phi_{(x,1)}} a^2$. We further explicitly pull out the dependence on the lattice cutoff by defining a new reduced Hamiltonian $\hat{h}_{\text{red}} = a h_{\text{red}} = 2\eta a h$ and substituting $\eta = \frac{1}{\sqrt{2}} a M$, where $M$ is a typical hadronic mass. This yields

$$\hat{h}_{\text{red}} = \sum_{\vec{b}} \hat{h}^e_{\vec{b}} + \sum_{\vec{b}} \hat{h}^m_{\vec{b}},$$

with the electric contribution

$$\hat{h}^e_{\vec{b}} = -g_{\text{eff}}^2 \frac{1}{\delta} \frac{\delta}{\delta \phi_{\vec{b}}} J \frac{\delta}{\delta \phi_{\vec{b}}} J,$$

and the magnetic term

$$\hat{h}^m_{\vec{b}} = \frac{1}{g_{\text{eff}}} \sum_{\vec{e}} (\phi_{\vec{b}} - \phi_{\vec{b}+\vec{e}})^2.$$

Since the effective coupling constant,

$$g_{\text{eff}}^2 = \frac{g^2 LM}{4\sqrt{2}},$$

contains the large factor $LM$, the product of lattice size in the longitudinal direction and the hadron mass, a strong coupling approach seems to be a good starting point. Note that we avoid introducing ‘radial wave functions’ or effective potentials as others have done. For each lattice site $\vec{b}$, $\hat{h}^e_{\vec{b}}$ (the kinetic energy) has the Gegenbauer polynomials $C_{n_\vec{b}}(\phi_{\vec{b}})$ for eigenfunctions:

$$\hat{h}^e_{\vec{b}} C_{n_\vec{b}}(\phi_{\vec{b}}) = g_{\text{eff}}^2 n_\vec{b} (n_\vec{b} + 2) C_{n_\vec{b}}(\phi_{\vec{b}}),$$

with

$$C_{n_\vec{b}}(\phi_{\vec{b}}) = \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin((n_\vec{b} + 1)\phi_{\vec{b}})}{\sin \phi_{\vec{b}}} \right\}.$$
\[ \int_0^\pi J(\varphi) C_n(\varphi) C_m(\varphi) d\varphi = \delta_{nm}. \quad (50) \]

The strong coupling wave functions of the full transverse lattice are product states characterized by a set of quantum numbers \( \{n\} = \{n_{\vec{b}}\} \),

\[ \Psi_{\{n\}}(\varphi) = \prod_{\vec{b}} C_{n_{\vec{b}}}(\varphi_{\vec{b}}). \quad (51) \]

These functions form a complete and orthonormal basis for the zero mode sector. They satisfy the energy eigenvalue equation

\[ \sum_{\vec{b}} \hat{h}_{\vec{b}} \Psi_{\{n\}}(\varphi) = g_{\text{eff}}^2 \sum_{\vec{b}} n_{\vec{b}}(n_{\vec{b}}+2) \Psi_{\{n\}}(\varphi). \quad (52) \]

The ground state in this limit corresponds to all \( n_{\vec{b}} = 0 \) – a constant wave function

\[ \Psi_{\{0\}}(\varphi) = \prod_{\vec{b}} \sqrt{\frac{2}{\pi}}, \quad (53) \]

and the ground state energy is zero

\[ E_0 = 0. \quad (54) \]

The first excited energy level is \( N_1^2 \)-fold degenerate - an excitation at a single lattice point

\[ \Psi_{\{1\}}(\varphi) = \sqrt{\frac{2}{\pi}} \frac{\sin (2\varphi_{\vec{b}})}{\sin \varphi_{\vec{b}}} \prod_{\vec{b} \neq \vec{b}_0} \sqrt{\frac{2}{\pi}}. \quad (55) \]

In strong coupling this level is separated by a large amount from the ground state energy

\[ E_1 = 3g_{\text{eff}}^2. \quad (56) \]

So far our results are equivalent to those reported by others to within re–definitions of wave functions and integration measures. Weak coupling variational solutions for the full \( SU(2) \) lattice Hamiltonian have also been given. Furthermore, studies in \( (1+1) \)-dimensional Yang-Mills theory give formal extensions to construct wave functions for \( SU(N) \) gauge theories.

The magnetic term of the Hamiltonian couples nearest neighbor lattice points. In the strong coupling limit its contribution may be obtained perturbatively (as it has the coefficient \( 1/g_{\text{eff}}^2 \)) by evaluating it with the basis function
of the ground state. The result of this is
\[ \langle \Psi_0 | \sum_{\vec{b}} \hat{h}^{\vec{b}}_0 | \Psi_0 \rangle = \frac{1}{g_{\text{eff}}} \left( \frac{\pi^2}{6} - 1 \right) \cdot (2N^2_\perp). \]  
(57)

Since this energy is proportional to \( N^2_\perp = (L/a)^2 \) and \( g_{\text{eff}}^2 \) grows linearly with \( L \), this part of the zero mode dynamics represents a negligible surface effect for the three-dimensional system in the strong coupling approximation.

Next, we discuss the weak coupling limit \( g^2 \to 0 \). In this case we can simplify the kinetic term of the Hamiltonian by defining new variables \( \alpha_{\vec{b}} \):
\[ \alpha_{\vec{b}} = \frac{\varphi_{\vec{b}}}{\kappa g}, \]  
(58)
with \( 8\kappa^2 = \sqrt{2LM} \). After expanding in \( g \), the reduced Hamiltonian becomes
\[ \hat{h}_{\text{red}} = \sum_{\vec{b}} \left\{ - \left( \frac{\partial^2}{\partial \alpha_{\vec{b}}^2} + \frac{2}{\alpha_{\vec{b}}^2} \frac{\partial}{\partial \alpha_{\vec{b}}} \right) + \sum_{\vec{c}} (\alpha_{\vec{c}} - \alpha_{\vec{b}+\vec{c}})^2 \right\}. \]  
(59)

Going over to Fourier momentum representation,
\[ \alpha_{\vec{b}} = \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{b}}, \]  
(60)
with \( k_i = 2\pi n_i/N_\perp a \) and \( n_i = 0, \pm 1, \pm 2, \ldots \) we have
\[ \hat{h}_{\text{red}} = \sum_{\vec{k}} \left\{ - \left( \frac{\partial^2}{\partial R_{\vec{k}}^2} + \frac{2}{R_{\vec{k}}^2} \frac{\partial}{\partial R_{\vec{k}}} \right) + 4\sum_{\vec{c}} \sin^2 \frac{\vec{k} \cdot \vec{c}}{2} R_{\vec{k}} R_{-\vec{k}} \right\}. \]  
(61)

The eigensolutions \( \psi_{\vec{K}} \) of \( \hat{h}_{\text{red}} \) in the weak coupling approximation are decoupled harmonic oscillators for each \( \vec{k} \), with frequencies
\[ \omega_{\vec{k}}^2 = 4 \sum_{\vec{c}} \sin^2 \frac{\vec{k} \cdot \vec{c}}{2}. \]  
(62)

Because of the ‘radial Laplacian’ it looks as if the eigenfunctions would have to vanish at the origin to be normalizable. However, as in the Schrödinger equation in three dimensions, the Jacobian \( J \) allows a constant wave function at the origin. Consequently, the eigenvalue of \( \psi_{\vec{K}} \) is given by the sum over the modes:
\[ \Omega_\vec{K} = \sum_{\vec{k}} \sqrt{4 \sum_{\vec{c}} \sin^2 \left( \frac{\vec{k} \cdot \vec{c}}{2} \right)}, \]  
(63)
which gives in the $N_{1} \rightarrow \infty$ limit spin waves with $\omega_{\vec{k}} = \sqrt{k_{1}^{2} + k_{2}^{2}}$.

In the weak coupling limit the zero mode Hamiltonian supports solutions similar to QED. The strong coupling limit, however, yields different results: ‘gluonic’ excitations are suppressed because of large energy gaps. This is due to the Jacobian, which can be traced back to non-abelian self interactions in the original Lagrangian.

4.3 Effective Hamiltonian for Two-Site Truncation

We will now invoke an effective interaction approach embedded in a cluster expansion, starting with the simplest, two-site cluster, in which either site (or both) can be excited to high energy states. We will obtain the solution for the low-lying spectra of the system approximated as a low density of excitable two-site clusters over the entire range of coupling. This method can be envisaged as the starting point of more ambitious renormalization group techniques.

We work in the representation of the strong coupling solution of $\hat{h}_{e}$ and divide the two-site subspace into a $P$ and $Q$ space, such that $P + Q = 1$ with

$$P = \{ |0,0\rangle \},$$

$$Q = \{ |n,m\rangle; \; n, m \neq 0,0 \} ,$$

(64)

where $n, m$ represent the indices of the Gegenbauer polynomials. Then the two-site energy $E_{2}$ is given by the non-perturbative solution of the Hamiltonian in Eq. (44), truncated to two lattice sites and labelled $h_{2}$. Explicitly, this Hamiltonian is:

$$\hat{h}_{2} = \hat{h}^{e} + \hat{h}^{m},$$

(65)

with

$$\hat{h}^{e} = -g_{\text{eff}}^{2} \left\{ \frac{1}{J} \frac{\delta}{\delta \varphi_{1}} J \frac{\delta}{\delta \varphi_{1}} + \frac{1}{J} \frac{\delta}{\delta \varphi_{2}} J \frac{\delta}{\delta \varphi_{2}} \right\} ,$$

(66)

and

$$\hat{h}^{m} = \frac{1}{g_{\text{eff}}^{2}} (\varphi_{1} - \varphi_{2})^{2} ,$$

(67)

where the subscripts label the sites.

Within the effective Hamiltonian method, the two-site energy is given by

$$E_{2} = P \hat{h}_{2} P + P \hat{h}_{2} Q \frac{1}{E_{2} - Q \hat{h}_{2} Q} Q \hat{h}_{2} P .$$

(68)

The self-consistent solutions of this equation provide the low-lying spectra in this method. The strong coupling basis states are eigenstates of $\hat{h}^{e}$:

$$\hat{h}^{e} |n, m\rangle = g_{\text{eff}}^{2} \{ n(n+2) + m(m+2) \} |n, m\rangle .$$

(69)
Thus, the non–trivial matrix elements are those of $\hat{h}^m$, and are of the form
$\langle n,m| (\varphi_1 - \varphi_2)^2 |n',m'\rangle$, where the subscripts label the sites. The matrix
element reduces to sums and products of one–site matrix elements, which are
given as:

$$\langle n|\varphi|n'\rangle = \begin{cases} \frac{\pi}{2} & \text{for } n = n' \\ \frac{\pi}{(n+n'+2)} - \frac{1}{(n-n')^2} & \text{for } n + n' = \text{odd}, n \neq n' \\ 0 & \text{for } n + n' = \text{even}, n \neq n' \end{cases} \quad \text{for } n = n' \quad \text{(70)}$$

$$\langle n|\varphi^2|n'\rangle = \begin{cases} \frac{\pi^3}{6} - \frac{\pi}{2(n+1)^2} & \text{for } n = n' \\ \pi(-1)^{n+n'} \left[\frac{1}{(n-n')^2} - \frac{1}{(n+n+2)^2}\right] & \text{for } n \neq n' \end{cases} \quad \text{(71)}$$

In the strong coupling limit we obtain the result of perturbation theory in $1/g_{\text{eff}}$

$$E_2 = \langle 0,0|\hat{h}^m|0,0\rangle = \frac{1}{g_{\text{eff}}} \left\{ \langle 0|\varphi^2|0\rangle - 2 \left( \langle 0|\varphi|0\rangle \right)^2 \right\} = \frac{1}{g_{\text{eff}}} \left( \frac{\pi^2}{6} - 1 \right). \quad \text{(71)}$$

In the weak coupling limit we can solve the Schrödinger equation for the
two–site version of the earlier spin wave Hamiltonian, Eq. (59). We call the
respective variables $\alpha_{\vec{b}_1} = x$ and $\alpha_{\vec{b}_2} = y$, then solve:

$$\hat{h}_{\text{red}} = - \left( \frac{\partial^2}{\partial x^2} + \frac{2}{x} \frac{\partial}{\partial x} \right) - \left( \frac{\partial^2}{\partial y^2} + \frac{2}{y} \frac{\partial}{\partial y} \right) + (x-y)^2. \quad \text{(72)}$$

Since this Hamiltonian is invariant under $x \leftrightarrow y$, the eigenfunctions $\Psi_2(x,y)$
can be chosen to be symmetric under the interchange of $x$ and $y$ ($\Psi_{2s}(x,y)$), or
antisymmetric ($\Psi_{2a}(x,y)$). The first symmetric excited state becomes degen-
erate with the ground state of the original problem in the weak coupling limit,
and the first antisymmetric state has a greater energy than the first symmetric
state.

As usual one factorizes the wave function $\Psi_2(x,y) = \frac{1}{xy} \Phi_2(x,y)$, resulting
in the Schrödinger equation

$$\hat{h}_{\text{red}} \Phi_2(x,y) = \left\{ - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + (x-y)^2 \right\} \Phi_2(x,y). \quad \text{(73)}$$

The center–of–mass motion is then separated: $\Phi_2(x,y) = e^{iPR} \chi_2(r)$, with
$R = (x+y)/2$ and $r = x - y$. The Hamiltonian corresponding to the relative
motion \( (r) \) is a simple radial harmonic oscillator:

\[
\hat{h}_\text{red} (r) = \frac{-2}{\partial r^2} + r^2.
\]  

(74)

The lowest states of the symmetric and antisymmetric ‘towers’ are solutions to this Hamiltonian. The energies of these states can be read directly from Eq. (74): \( E_{2s} = \sqrt{2} \), and \( E_{2a} = 3\sqrt{2} \), respectively, giving a energy gap between the states of \( 2\sqrt{2} \).

Thus, the results for the energy gaps of the low–lying states in the weak coupling limit are

\[
\begin{align*}
E_{2s} - E_{\text{ground}} &= 0, \\
E_{2a} - E_{\text{ground}} &= 2\sqrt{2}.
\end{align*}
\]

(75)

We now proceed to calculate the low–lying spectra via the effective Hamiltonian method. In the numerical calculations, we truncate the \( Q \) space at a certain two–site energy, calculate \( E_2 \), then increase the energy until we reach convergence for each choice of coupling constant \( g_{\text{eff}} \). The typical number of two–site states required for the \( Q \) space at convergence was 300.

The numerical solution of Eq. (68) for \( E_2 \) has now been published. In the strong coupling limit (large \( g_{\text{eff}}^2 \)) the large gaps in energy are evident, and the numbers agree with the unperturbed energy of the states, given in Eq. (69). In this same limit, the slope of the ground state energy as a function of the inverse square coupling agrees with the analytic calculation of Eq. (71). In the weak coupling limit, the results for the gap energies were Richardson extrapolated for the \( 1/g_{\text{eff}}^2 \rightarrow \infty \) limit. This extrapolation matched the analytical results of Eq. (73) to five significant figures. Thus, we have obtained two–site solutions in the strong coupling basis for the entire range of coupling which agree with analytic results in the both the weak and strong coupling limits.

It is remarkable that we succeeded with a one–dimensional strong coupling basis throughout the range of coupling strengths. Results for the spectra of \( N \)–sites are straightforwardly obtained and should be valid as long as the number of excited two–site clusters is small compared with \( N/2 \). This is the ‘low density’ approximation.

5 Variational Tamm-Dancoff Application

Over the past several years, we have also investigated the use of relativistic wave equation formulations of quantum field theory in the equal time quantization scheme. Here, we would like to briefly summarize our approach as viewed from the effective Hamiltonian framework.
We begin by selecting the Coulomb gauge Hamiltonian of QCD and we choose to work in a finite momentum space domain. We eliminate all interactions except the fermion-gluon vertex and the triple gluon coupling terms. We fix the scale by giving the quarks a constituent quark mass.

For our unperturbed Hamiltonian we select the kinetic energy operators for the quarks and the gluons. In our earlier work, we chose a cubic B-spline basis expansion for our quark and gluon amplitudes. More recently, we select a harmonic oscillator basis with a variable oscillator parameter, resulting in a non-orthogonal basis.

For our P-space, we select a small finite set of quark-antiquark states for mesons and 3-quark states for baryons.

Our procedure is to optimize our P-space using a variational approach and to determine an effective interaction for that basis. The effective interaction emerges as a ladder series as given by the G-matrix defined earlier. However, there are a number of simplifying assumptions needed to evaluate this effective interaction.

Finally, we solve the resulting Eq. (8) which is the Bethe-Salpeter equation with the kernel defined by the effective interaction we have derived. For fixed constituent quark masses and a fixed strong coupling constant, the solutions of this equation yield spectra in reasonable accord with data.

The most remarkable feature of this approach is found in the shape of the effective interaction which emerges. At short distances it closely approximates the one gluon exchange effective potential and, at large distances, it closely approximates linear confinement.

The linear confining feature is traced to the non-linear role of the triple-gluon coupling term and its effect in our variational treatment. The combination of the choice of Coulomb gauge and the variational approach appear to be ultimately responsible for this result. We believe that this feature will persist as we make our approach more general.

6 Conclusions

The well established framework of effective Hamiltonians for quantum many-body theory with strong interactions appears adaptable to non-perturbative applications in quantum field theory. Our initial results on simple renormalization problems are encouraging and the initial applications to (simplified) QCD problems also inspire us to proceed. However, we realize that considerable effort is required for major additional advances to be achieved.
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