A 5D noncompact Kaluza-Klein cosmology in the presence of Null perfect fluid

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Abstract

For the description of the early inflation, and acceleration expansion of the Universe, compatible with observational data, the 5D noncompact Kaluza–Klein cosmology is investigated. It is proposed that the 5D space is filled with a null perfect fluid, resulting a perfect fluid in 4D universe, plus one along the fifth dimension. By analyzing the reduced field equations for flat FRW model, we show the early inflationary behavior and current acceleration of the universe.

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1. INTRODUCTION

Recent astronomical observations of distant supernovae lightcurves, \cite{1}, \cite{2}, indicate that at early time the universe has gone through an inflationary epoch and currently is undergoing an accelerated phase of its expansion. Within the framework of cosmology, these observations can be explained by dark energy, which occupies almost seventy percent of the content of our universe at present \cite{3}. A simple explanation for dark energy is the cosmological constant, which acts like a perfect fluid with an equation of state, and the energy density is associated with quantum vacuum \cite{4}.

On the other hand, the early time inflation and late time acceleration of the Universe may be explained by the modification of gravitation, such as extra dimensional theories of gravity which recently have received much interest \cite{5}. Of particular interest is the cosmology of 5D pure geometry in noncompact Kaluza-Klein theory, known as space-time-matter (STM) theory, in which one do not need to insert any matter into the 5D manifold by hand provided an appropriate definition is made for the energy-momentum tensor of matter in terms of the extra part of the geometry \cite{6}. As a result, instead, the matter appears in four dimensions induced by the 5D vacuum theory. Physically, the picture behind this interpretation is that curvature in 5D space induces effective properties of matter in 4D space-time. This idea is a consequence of the Campbells theorem which states that any analytic N-dimensional Riemannian manifold can be locally embedded in an (N+1)–dimensional Ricci flat Riemannian manifold \cite{7}, \cite{8}.

One may receive extra motivations to work on this challenging problem by recognizing that the energy density of scalar fields may also contribute to the early inflation and current accelerating expansion of the universe \cite{9}. So instead of pure 5D geometry in noncompact Kaluza-Klein theory, we examine the theory with matter fields. Based on integration of these theories to explain the early inflation and current acceleration, Darabi has studied a 5D universe consists of a 5D geometry subject to a 5D energy momentum tensor in the framework of noncompact Kaluza-Klein theory \cite{10}. In another attempt, in the present study, we investigate the non vacuum 5D space–time subject to a 5D energy momentum tensor which takes the form of a null perfect fluid in the above framework. Physically, a null fluid by itself describes either gravitational radiation, or some kind of nongravitational radiation or a combination of these two. In here, the null fluid induced from higher dimension,
divides itself into a dark energy part and a matter part where for the 5D velocity of the fluid we have $U.U = 0$. From the FRW symmetries of the model, the energy-momentum tensor then is associated with the sum of two perfect fluids, one in 4D space–time and one along the fifth dimension. We then address the early inflation and current acceleration of the universe based on the above prescription.

2. THE MODEL

We will consider a general five-dimensional manifold with coordinates $x^A \,(A = 1, 2, 3, 4, 5)$ and metric tensor $g_{AB}(x^C)$. The 5D interval is then given by

$$dS^2 = g_{AB} dx^A dx^B,$$  \hspace{1cm} (1)

where for simplicity we take

$$g_{5\mu} = 0,$$
$$g_{55} = \epsilon \phi^2(x^\mu),$$
$$g_{\mu\nu} = g_{\mu\nu}(x^\mu).$$  \hspace{1cm} (2)

In here $\epsilon^2 = 1$ and the signature of the scaler part of the metric is left general to allow timelike or spacelike signature for the fifth dimension without loss of generality. The fifth dimensional independency of the scalar field and the 4D metric in our formalism does not contradict the non compactness of the theory. In the compact version of the theory the cylindrical condition imposed to justify the non observability of the 5th dimension. However, in here, the theory is noncompact but for simplicity we assume that the scalar field and the 4D metric is independent of this extra dimension. So it is a matter of how we justify the non observability and independency of the extra dimension.

The field equations for 5D non vacuum Einstein equations are,

$$G_{AB} = 8\pi G T_{AB},$$  \hspace{1cm} (3)

where $G_{AB}$ and $T_{AB}$ are the 5D Einstein tensor and energy-momentum tensor respectively and the 5D Einstein tensor is

$$G_{AB} = R_{AB} - \frac{1}{2}g_{AB}R_{(5)}.$$  \hspace{1cm} (4)
By a minimal extension of general relativity, the five dimensional Ricci tensor is defined in terms of the Christoffel symbols exactly as in four dimensions,

\[ R_{AB} = \partial_C \Gamma^C_{AB} - \partial_B \Gamma^C_{AC} + \Gamma^C_{AB} \Gamma^D_{CD} - \Gamma^C_{AD} \Gamma^D_{BC}. \tag{5} \]

By a dimensional reduction and after canceling out derivatives with respect to fifth coordinate \( x^5 \), the 4D part of the 5D Ricci tensor is obtained by putting \( A \rightarrow \mu, B \rightarrow \nu \) in equation (5) and expanding the summed terms on the right hand side by letting \( C \rightarrow \lambda, 4, \) etc. Therefore, we have

\[
\hat{R}_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} + \partial_5 \Gamma^5_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} - \partial_\mu \Gamma^5_{\nu\lambda} + \Gamma^\lambda_{\mu\nu} \Gamma^\eta_{\lambda\mu} \\
+ \Gamma^\lambda_{\mu\nu} \Gamma^5_{\lambda\nu} + \Gamma^5_{\mu\nu} \Gamma^D_{5D} - \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\nu\eta} - \Gamma^5_{\mu\lambda} \Gamma^\lambda_{\nu5} - \Gamma^D_{\mu5} \Gamma^5_{\nuD}, \tag{6}
\]

where " \(^v\)" denotes the 4D part of the 5D quantities. One finds that the 4D Ricci tensor is a part of equation (6) that may be rewritten as

\[
\hat{R}_{\mu\nu} = R_{\mu\nu} + \nabla_\mu \nabla_\nu \phi. \tag{7}
\]

After evaluating the Christoffel symbols in (7), we obtain,

\[
\hat{R}_{\mu\nu} = R_{\mu\nu} - \frac{\nabla_\mu \nabla_\nu \phi}{\phi}. \tag{8}
\]

We also find the 55 component of the Ricci tensor as

\[
R_{55} = \partial_\lambda \Gamma^\lambda_{55} - \partial_5 \Gamma^5_{55} + \Gamma^\lambda_{\mu5} \Gamma^\eta_{\lambda\mu} + \Gamma^5_{\mu\nu} \Gamma^D_{\nuD} - \Gamma^\lambda_{\mu5} \Gamma^\lambda_{\nu5} - \Gamma^D_{\mu5} \Gamma^5_{\nuD}. \tag{9}
\]

Evaluation of the corresponding Christoffel symbols leads to

\[
R_{55} = -\epsilon \phi \Box \phi. \tag{10}
\]

We then find the 5D Ricci scalar \( R_{(5)} \) as

\[
R_{(5)} = g^{AB} R_{AB} = g^{\mu\nu} R_{\mu\nu} + g^{55} R_{55} \\
= g^{\mu\nu} \left( R_{\mu\nu} - \frac{\nabla_\mu \nabla_\nu \phi}{\phi} \right) + \frac{\epsilon}{\phi^2} (-\epsilon \phi \Box \phi) = R - \frac{2}{\phi} \Box \phi, \tag{11}
\]

where the \( \mu5 \) terms vanish and \( R \) is the 4D Ricci scalar.
We are now ready to construct the space-time and 55 components of the Einstein tensor. The space-time components of the Einstein tensor after substituting $R_{\mu\nu}$ and $R^{(5)}$ are

$$\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2} \hat{g}_{\mu\nu} R^{(5)} = G_{\mu\nu} + \frac{1}{\phi} (g_{\mu\nu} \Box \phi - \nabla_{\mu} \nabla_{\nu} \phi).$$  (12)

In the same way, the 55 component after substituting $R_{55}$ and $R^{(5)}$ is

$$G_{55} = R_{55} - \frac{1}{2} R^{(5)} = -\frac{1}{2} \epsilon R \phi^2.$$  (13)

3. THE FRW MODEL

Consider that the 4D space–time metric has the FRW line element,

$$ds^2 = dt^2 - R(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right),$$  (14)

where $k$ takes the values $+1, 0, -1$ according to a close, flat or open universe, respectively.

We now assume that the energy momentum tensor in 5D takes the form of a null anisotropic perfect fluid, where for the velocity of the fluid in 5D we have $U \cdot U = 0$, where $U$ is velocity in terms of 5D proper time. From the symmetries of the metric (14), we also assume that the energy-momentum tensor is associated with the sum of two perfect fluids, one in 4D space–time and one along the fifth dimension. The energy-momentum tensor in 4D space time is

$$T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - pg_{\mu\nu},$$  (15)

where $\rho$ and $p$ are the energy density and pressure in 4D space–time. We also obtain the energy momentum tensor along the fifth dimension after writing the 5th component of the proper velocity in terms of 4D proper time:

$$T_{55} = \frac{\rho + p}{1 - \epsilon \phi^2} u_5 u_5 - \bar{p} g_{55},$$  (16)

where $\bar{\rho}$ and $\bar{p}$ are the energy density and pressure along the fifth dimension. In both (15) and (16) velocities are in terms of 4D proper time.

Using energy- momentum tensors (15), (16), and the 4D and fifth components of Einstein tensor, (12) and (13), the full non-vacuum 4D and the fifth dimensional Einstein equations can be obtained as

$$G_{\mu\nu} = 8\pi G[(\rho + p) u_{\mu} u_{\nu} - pg_{\mu\nu}] + \frac{1}{\phi} [\nabla_{\mu} \nabla_{\nu} \phi - \Box \phi g_{\mu\nu}],$$  (17)

$$R = 16\pi G \left[ \frac{\bar{\rho} + \bar{p}}{1 - \epsilon \phi^2} + \bar{p} \right].$$  (18)
Taking the trace of equation (17) and combining with equation (18), one obtains

$$\Box \phi = \frac{1}{3} \{ 8\pi G (\rho - 3p) + 16\pi G \frac{(\bar{p} + \bar{p})}{1 - \epsilon \phi^2} \} \phi, \quad (19)$$

where '□' is defined as usual in 4D by \( \Box \phi \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu \phi \). For a nonvanishing \( \phi \), from equation (19) one may assume the following replacements

$$\frac{1}{\phi} \Box \phi = \frac{1}{3} \{ 8\pi G (\rho - 3p) + 16\pi G \frac{(\bar{p} + \bar{p})}{1 - \epsilon \phi^2} \}, \quad (20)$$

$$\frac{1}{\phi} \nabla_\mu \nabla_\nu \phi = \frac{1}{3} \{ 8\pi G (\rho - 3p) + 16\pi G \frac{(\bar{p} + \bar{p})}{1 - \epsilon \phi^2} \} u_\mu u_\nu. \quad (21)$$

Substituting equations (20) and (21) into equation (17) leads to

$$G_{\mu\nu} = 8\pi G [(\rho + \bar{p})u_\mu u_\nu - \bar{p} g_{\mu\nu}], \quad (22)$$

where

$$\bar{p} = \frac{1}{3} (\rho + 2(\frac{\bar{p}}{1 - \epsilon \phi^2} + \bar{p})), \quad (23)$$

is the equivalent pressure in 4D. The right hand side of equation (22) describes a perfect fluid with density \( \rho \) and pressure \( \bar{p} \). It is interesting that the contributions of the scalar field at higher dimension cancels out exactly the physics of pressure \( p \) in 4D and instead substitute the pressure \( \bar{p} \) given by equation (23) in terms of the matter density, dark energy density and dark pressure density.

The field equations (22) for the metric (14) lead to two independent equations,

$$3 \frac{\ddot{a}^2 + k}{a^2} = 8\pi G \rho, \quad (24)$$

$$2a\ddot{a} + \dot{a}^2 + k \frac{a}{a^2} = -8\pi G \bar{p}. \quad (25)$$

Differentiating equation (24) and combining with acceleration equation (25) leads to the conservation equation,

$$\frac{d}{dt} (\rho a^3) + \bar{p} \frac{d}{dt} (a^3) = 0. \quad (26)$$

Using equation (24), the acceleration equation (25) can be rewritten as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3\bar{p}) = -\frac{8\pi G}{3} (\rho + \bar{p} + 2\bar{p}). \quad (27)$$

From equation (18) we obtain

$$-6(\kappa + \ddot{a}^2 + \ddot{a} a) \frac{a}{a^2} = 16\pi G (\dot{\bar{p}} + 2\bar{p}). \quad (28)$$
Using power law behaviors for scale factor, dark pressure and dark density,

\[ a(t) = a_0 t^\alpha, \quad \bar{p}(t) = \bar{p}_0 t^\beta, \quad \bar{\rho}(t) = \bar{\rho}_0 t^\eta \]

in equation (28), one can easily find that \( \beta = \eta = -2 \).

We also find that for a time dependent scalar field, the equation (19) becomes

\[ \ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} = \frac{1}{3} \left\{ 8\pi G (\rho - 3p) + 16\pi G (\bar{p} + 2\bar{\rho}) \right\} \phi. \]  

(29)

Using power law behavior for scalar field and matter density,

\[ \phi(t) = \phi_0 t^\gamma, \quad \rho(t) = \rho_0 t^\delta (\rho_0 > 0) \]

with the equation of state for matter pressure, \( p = \omega \rho \), and dark pressure, \( \bar{p} = \Omega \bar{\rho} \), for a flat universe \((k = 0)\), we rewrite the acceleration equation, (27), scalar field equation, (29), and conservation equation, (26), respectively in the following form

\[ \alpha(\alpha - 1) + \frac{8\pi G}{3} (\rho_0 + \bar{\rho}_0 (1 + 2\Omega)) = 0, \]  

(30)

\[ \gamma(\gamma - 1) + 3\alpha \gamma - \frac{8\pi G}{3} (\rho_0 (1 - 3\omega) + 2\bar{\rho}_0 (1 + 2\Omega)) = 0, \]  

(31)

\[ 2\rho_0 (2\alpha - 1) + 2\bar{\rho}_0 \alpha (1 + 2\Omega) = 0, \]  

(32)

where due to consistency in the conservation equation we have \( \delta = -2 \).

If we also define a new equation of state, \( \bar{p} = \Gamma \rho \), then, from equation (23), we obtain

\[ \Gamma = \frac{2 - 3\alpha}{3\alpha}. \]  

(33)

One can simply check that in radiation dominant era where \( \alpha = 1/2 \), we have \( \Gamma = 1/3 \), and in matter dominant era where \( \alpha = 2/3 \), we have \( \Gamma = 0 \), as expected. One also find that in early inflationary era where \( \alpha >> 1 \), we have \( \Gamma \approx -1 \) and in late time acceleration that \( \alpha > 1 \) we have \( \Gamma \approx -1/3 \). Therefore, the new equivalent equation of state, \( \bar{p} = \Gamma \rho \) instead of \( p = \omega \rho \) or \( \bar{p} = \Omega \bar{\rho} \), satisfies the physical conditions on different epochs.

From acceleration equation (27), for an accelerating universe, and with the help of equation (32) we require that \( \Omega < -1/2 \) and \( \alpha > 1 \) which accounts for a negative dark energy and accelerating universe.

In radiation dominant era where \( \alpha = 1/2 \), from equations (31) and (32) one finds that

\[ -1/2 < \gamma < 0. \]  

(34)
Also, in matter dominant era where $\alpha = 2/3$, we find that

$$-1 < \gamma < 0.$$  \hspace{1cm} (35)

As it can be seen, the larger values of $\alpha$ lead to more negative value of $\gamma$ which means that as the universe evolves in time from radiation to matter state the scalar field is more suppressed. One also find in acceleration equation (27) that for larger values of $|\Omega|$, the value of $\alpha$ is larger and larger which means that the more negative pressure we have, the more accelerated universe expected.

From equation (32), we define the ratio of dark energy density to matter density as

$$r = \frac{\bar{\rho}_0}{\rho_0} = \frac{1 - 2\alpha}{\alpha(1 + 2\Omega)}. \hspace{1cm} (36)$$

From equation (36) in radiation dominated era where $\alpha = 1/2$, we obtain $r = 0$ or the energy density $\bar{\rho} = 0$ which means that the time evolution for the scale factor is not affected by the dark energy. In addition, in the matter dominated era, where $\alpha = 2/3$, in order to have a positive dark energy density, we find that $\Omega < -1/2$. On the other hand, in the early universe inflationary era for $\alpha >> 22$, we have $\Omega \simeq -1/2 - 1/(4r)$, and for an accelerating universe in the late time where $\bar{\rho} > \rho$ and $\alpha > 1$, we have $\Omega < -1/2 - 1/(2\alpha)$.

4. CONCLUSION

In this paper, we present a 5D universe subject to a 5D energy-momentum tensor in the framework of noncompact Kaluza-Klein theory. For simplicity we assume that the metric of the space does not depend explicitly on the extra coordinate and also $g_{\mu\nu} = 0$. We also represent the energy momentum tensor in 5D space–time as a null perfect fluid. The geometry of the 4D space–time is taken to be FRW subject to the conventional perfect fluid with density $\rho$ and pressure $p$ while the extra dimensional part endowed by a scalar field is subject to another perfect fluid with dark density density $\bar{\rho}$ and dark energy pressure $\bar{p}$. By writing down the reduced 4D and extra-dimensional components of 5D Einstein equations we find that the 4D universe corresponds to the vacuum states of the scalar field. It turned out that the contribution of the non–vacuum states of the scalar field to the 4D cosmology cancels out exactly the physics of pressure $p$ and instead require a new equivalent pressure $\tilde{p}$ in 4D that leads to non-zero late acceleration of the universe and also satisfies some of
the observational constraints including equation of state constants in matter and radiation dominated era and early inflation.

[1] Riess, A. G. et al., Astron. J. 116, 1009 (1998); Perlmutter, S. et al., Astrophys. J. 517, 565 (1999).
[2] Astier, P. et al., Astron. Astrophys. 447, 31 (2006).
[3] Tomi Koivisto, Hannu Kurki-Suonio, Class. Quantum Grav. 23, 2355 (2006)
[4] Eric V. Linder, Phys. Rev. D 70, 061302 (2004)
[5] P. S. Wesson, Astrophys. J. 436, 547 (1994); S. Capozziello, Int. J. Mod. Phys. D 11, 483 (2002); S. Nojiri and S. D. Odintsov, J. Phys. Conf. Ser. 66, 012005 (2007); M. E. Soussa and R. P. Woodard, Gen. Rel. Grav. 36, 855 (2004); V. Faraoni, Phys. Rev. D 75, 067302 (2007); G. Allemandi et. al., Gen. Rel. Grav. 37, 1891(2005);
[6] P.S.Wesson,Gen.Relativ.Gravi. 16,193 (1984); Space-Time-Matter:Modern Kaluza-Klein Theory,(World Scientific.Singeapor1999); P. S. Wesson, Space-Time-Matter (Singapore: World Scientific) 1999; M. Bellini, Nucl. Phys. B660, 389 (2003); P. S. Wesson, Space-Time-Matter (Singapore: World Scientific) (1999); J. M. Overduin and P. S. Wesson, Phys. Rept. 283, 303(1997).
[7] H. Y. Liu and P. S. Wesson, Astrophys. J. 562 (2001)
[8] S. S. Seahra and P. S. Wesson, J. Math. Phys 44, 5664 (2003)
[9] Takeshi Chiba and Masahide Yamaguchi, Phys. Rev. D 61, 027304 (1999)
[10] F Darabi, [arXiv:0901.0835] F Darabi, Gen. Rel. Grav, DOI 10.1007/s10714-008-0685-6, [arXiv:0708.3835].