THE NUCLEOLUS OF LARGE MAJORITY GAMES

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ABSTRACT. Members of a shareholder meeting or legislative committee have greater or smaller voting power than meets the eye if the nucleolus of the induced majority game differs from the voting weight distribution. We establish a new sufficient condition for the weight and power distributions to be equal; and we characterize the limit behavior of the nucleolus in case all relative weights become small.

Keywords: nucleolus; power measurement; weighted voting games

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1. INTRODUCTION

Among all individually rational and efficient payoff vectors in a game with transferable utility, the nucleolus selects a particularly stable one. It quantifies each coalition’s dissatisfaction with a proposed vector \( x \) as the gap between the coalition’s worth \( v(S) \) and the surplus share \( \sum_{i \in S} x_i \) that is allocated to members of \( S \subseteq N \); then it selects the allocation \( x^* \) which involves lexicographically minimal dissatisfaction. In contrast to other prominent point solutions in cooperative game theory, such as the Shapley value, \( x^* \) is guaranteed to lie in the core of game \((N,v)\) whenever that is non-empty.

Even before Schmeidler’s 1969 article established the definition, existence, uniqueness, and continuity of the nucleolus, Peleg (1968) had applied it to majority games. In these games – also known as weighted voting games (WVG) – the worth of a coalition \( S \) of players is either 1 or 0, i.e., \( S \) is either winning or losing, and there exists a non-negative quota-and-weights representation \([q; w_1, \ldots, w_n]\) such that \( v(S) = 1 \) iff \( \sum_{i \in S} w_i \geq q \).

Peleg highlighted a property of constant-sum WVGs with a homogeneous representation, i.e., one where total weight of any minimal winning coalition equals \( q \): the nucleolus of such WVG coincides with the relative voting weights.\(^1\) Despite

\[^1\]We assume here that any null player, i.e., a voter \( i \) from set of players \( N \) such that \( v(S) = v(S \cup \{i\}) \) for every \( S \subseteq N \setminus \{i\} \), has zero voting weight. A WVG \((N,v)\) is called constant-sum if for any
this early start, the relation between voting weights and the nucleolus of majority games – constant-sum or not, homogeneous or inhomogeneous – has to the best of our knowledge not systematically been studied so far. This paper is a first attempt to fill this gap.

Discrepancies between weights and the nucleolus matter because the nucleolus is an important indicator of influence in collective decision bodies. It emerges as an equilibrium price vector in models that evaluate voters’ attractiveness to competing lobbying groups (Young 1978; Shubik and Young 1978); more recent theoretical work by Montero (2005, 2006) has established it as a focal equilibrium prediction for strategic bargaining games. So large differences between a voter $i$’s weight $w_i$ and nucleolus $x_i^*$ can mean that the real power distribution in a decision body such as a shareholder meeting is hidden from the casual observer. This intransparency can be particularly problematic for political decision bodies, where voting weight arrangements affect the institution’s legitimacy.

This paper investigates absolute and relative differences between the nucleolus and players’ relative voting weights. We determine an upper bound on their $\| \cdot \|_1$-distance which depends only on quota and maximum weight in a given representation in Lemma 1. The lemma allows to conclude that if the relative weight of every individual voter in player set \{1, \ldots, n\} tends to zero, then the ratio $x_i^*/x_j^*$ of two nucleolus components converges to $w_i/w_j$ for all regular voters $i$ and $j$ as $n \to \infty$ (Prop. 1). This complements analogous limit results in the literature on the Shapley value, the Banzhaf value and voter pivotality on intervals (Neyman 1982; Lindner and Machover 2004; Kurz et al. 2013) as well as for stationary equilibrium payoffs in legislative bargaining games à la Baron-Ferejohn (Snyder et al. 2005). We also establish a new sufficient condition for the nucleolus to coincide with relative weights (Prop. 2). It implies that a finite number of replications brings about full coincidence for any given WVG.

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$S \subseteq N$ either $v(S) = 1$ or $v(N \setminus S) = 1$. $S \subseteq N$ is a minimal winning coalition (MWC) if $v(S) = 1$ and $v(T) = 0$ for any $T \subset S$.

2 Corresponding experimental lab evidence is mixed. See Montero et al. (2008).

3 See Le Breton et al. (2012) for nucleolus-based power analysis of the European Union’s Council; an early-day weight arrangement meant that Luxembourg had a relative voting weight of 1/17 but zero voting power. - In general, the power-to-weight ratio can differ arbitrarily from 1. For instance, the nucleolus of the WVG with representation $[0.5; (1 - \varepsilon)/2, (1 - \varepsilon)/2, \varepsilon]$ is $x^* = (1/3, 1/3, 1/3)$ for any $\varepsilon \in (0; 0.5)$.
2. Nucleolus

Consider a WVG \((N,v)\) with representation \([q; w_1, \ldots, w_n]\). Using notation \(x(S) = \sum_{i \in S} x_i\), a vector \(x \in \mathbb{R}^n\) with \(x_i \geq v\{i\}\) and \(x(N) = v(N)\) is called an imputation. For any coalition \(S \subseteq N\) and imputation \(x\), call \(e(S, x) = v(S) - x(S)\) the excess of \(S\) at \(x\). It can be interpreted as quantifying the coalition’s dissatisfaction and potential opposition to an agreement on allocation \(x\). For any fixed \(x\) let \(S_1, \ldots, S_{2^n}\) be an ordering of all coalitions such that the excesses at \(x\) are weakly decreasing, and denote these ordered excesses by \(E(x) = (e(S_k, x))_{k=1,\ldots,2^n}\). Imputation \(x\) is lexicographically less than imputation \(y\) if \(E_k(x) < E_k(y)\) for the smallest component \(k\) with \(E_k(x) \neq E_k(y)\). The nucleolus of \((N,v)\) is then uniquely defined as the lexicographically minimal imputation\(^4\).

As an example, consider \((N,v)\) with representation \([q; w] = [8; 6, 4, 3, 2]\). The nucleolus can be computed as \(x^* = (2/5, 1/5, 1/5, 1/5)\) by solving a sequence of linear programs – or by appealing to the sufficient condition of Peleg (1968) after noting that the game is constant-sum and permits a homogeneous representation \([q'; w'] = [3; 2, 1, 1, 1]\). Denoting the normalization of weight vector \(w\) by \(\bar{w}\), i.e., \(\bar{w} = w / \sum w_i\), the respective total differences between relative weights and the nucleolus are \(\|\bar{w} - x^*\|_1 = 2/15\) for the first and \(\|\bar{w}' - x^*\|_1 = 0\) for the second representation (with \(\|x\|_1 = \sum |x_i|\)).

3. Results

Saying that representation \([q; w]\) is normalized if \(w = \bar{w}\), we have\(^5\)

**Lemma 1.** Consider a normalized representation \([q; w]\) with \(0 < q < 1\) and \(w_1 \geq \cdots \geq w_n \geq 0\) and let \(x^*\) be the nucleolus of this WVG. Then

\[
\|x^* - w\|_1 \leq \frac{2w_1}{\min\{q, 1 - q\}}.
\]

\(^4\)Schmeidler’s (1969) original definition did not restrict the considered vectors to be imputations but is usually specialized this way. The set of imputations that minimize just the largest excess, \(E_1(x)\), is called the nucleus of \((N,v)\) by Montero (2006). Our results are stated for the nucleolus but extend to every element of the nucleus (with slight adaptation of the proof of Prop. 2).

\(^5\)All proofs are provided in the Mathematical Appendix.
If we consider a sequence \( \{(\{1, \ldots, n\}, v^{(n)})\}_{n \in \mathbb{N}} \) of \( n \)-player WVGs with representations \([q^{(n)}; w^{(n)}]\) such that the normalized quota \( \bar{q}^{(n)} \) is bounded away from 0 and 1 (or, more generally, 0 and 1 are no cluster points of \( \{\bar{q}^{(n)}\}_{n \in \mathbb{N}} \)), and each player \( i \)'s normalized weight \( \bar{w}_i^{(n)} \) vanishes as \( n \to \infty \) then Lemma 1 implies

\[
\lim_{n \to \infty} \|x^{(n)} - \bar{w}^{(n)}\|_1 \to 0.
\]

Convergence to zero of the total difference between nucleolus components \( x_i^{(n)} \) and relative voting weights \( \bar{w}_i^{(n)} \) does not yet guarantee that the nucleolus is asymptotically proportional to the weight vector, i.e., that each ratio \( x_i^{(n)}/x_j^{(n)} \) converges to \( w_i/w_j \). This can be seen, e.g., by considering

\[
[q^{(n)}; w^{(n)}] = \left[\frac{2n - 1}{2}; 1, 2, \ldots, \frac{2}{n-1}\right].
\]

The nucleolus either equals \((0, \frac{1}{n-1}, \ldots, \frac{1}{n-1})\) or \((\frac{1}{n}, \ldots, \frac{1}{n})\) depending on whether \( n \) is even or odd; ratio \( x_1^{(n)}/x_2^{(n)} \) alternates between 0 and 1.

But such pathologies are ruled out for players \( i \) and \( j \) whose weights are “nonsingular” in the weight sequence \( \{w^{(n)}\}_{n \in \mathbb{N}} \). Specifically, denote the total number of players \( i \in \{1, \ldots, n\} \) with an identical weight of \( w_i^{(n)} = \omega \) by \( \tilde{m}_\omega(n) = \omega_n(n)/n \). We say that a player \( j \) with weight \( w_j \) is regular if \( \tilde{m}_\omega(n) \cdot \bar{w}_j^{(n)} \) is bounded away from 0 by some constant \( \varepsilon > 0 \). Lemma 1 then implies:

**Proposition 1.** Consider a sequence \( \{[q^{(n)}; (w_1, \ldots, w_n)]\}_{n \in \mathbb{N}} \) with corresponding normalized quotas that exclude 0 and 1 as cluster points and with normalized weights satisfying \( \bar{w}_k^{(n)} \downarrow 0 \) for every \( k \in \mathbb{N} \) as \( n \to \infty \). Then the nucleolus \( x^{(n)} \) of the WVG represented by \([q^{(n)}; (w_1, \ldots, w_n)]\) satisfies

\[
\lim_{n \to \infty} \frac{x_i^{(n)}}{x_j^{(n)}} = \frac{w_i}{w_j}
\]

for any regular players \( i \) and \( j \).

\[\text{We assume } w_i^{(n)} = w_j \text{ in our exposition. Adaptations to cases where } q^{(n)} \text{ and } w_j^{(n)} \text{ vary in } n \text{ are straightforward. The essential regularity requirement is that a voter type’s aggregate relative weight does not vanish.}\]
For a considerable class of games, asymptotic equality of nucleolus and normalized weights can be strengthened to actual equality. Namely, for a fixed $n$-player WVG with representation $[q; w_1, \ldots, w_n]$ let $m_\omega$ denote the number of players that have weight $\omega$; so

$$m^\circ = \min_{i \in \{1, \ldots, n\}} m_{w_i} \geq 1$$

is the number of occurrences of the rarest weight in vector $w = (w_1, \ldots, w_n)$.

**Proposition 2.** Consider a WVG representation $[q; w]$ with integer weights $w_1 \geq \ldots \geq w_n \geq 0$ and normalization $[\bar{q}; \bar{w}]$. Denoting the number of numerically distinct values in $w$ by $1 \leq t \leq n$, the nucleolus $x^*$ of this WVG satisfies

$$x^* = \bar{w} \text{ if } \min\{\bar{q}, 1 - \bar{q}\} \cdot m^\circ > 2t w_1^2.$$  

The proposition refers to integer weights. Even though it is not difficult to obtain an integer representation from any given $[q; w]$ with non-integer values, this is an important restriction. In particular, it is not possible to rescale a given weight vector $w$ so as to make the maximal weight $w_1$ arbitrarily small.

The right-hand side of the inequality in condition (6) is smaller, the smaller the number of different weights in the representation, and the smaller the involved integers (particularly $w_1$). Similarly, the left-hand side is larger, the greater the number of occurrences of the rarest weight. It follows that condition (6) is most easily met when null players (where $x^*_i = 0$ is known) are removed from the WVG in question and a minimal integer representation is considered. This is automatically also a homogeneous representation if any exists.

Our sufficient condition for $x^* = \bar{w}$ is, however, independent of the known homogeneity-based one. The normalization of weights in $[3; 2, 1, 1, 1]$ must, according to Peleg (1968), coincide with the WVG’s nucleolus because the game is constant-sum; but our condition (6) is violated. In contrast, the representation $[q; w] = [1500; 4, \ldots, 4, 3, \ldots, 3, 2, \ldots, 2]$ of a 900-player WVG where each of the non-null players has a positive nucleolus value in this case – in contrast to WVGs in general. So we implicitly establish a sufficient condition for $w_1 > 0 \Rightarrow x^*_i > 0$.

Note also that inequality (6) must be violated if two symmetric players of $(N, v)$ have different weights because $x^* = \bar{w}$ would then contradict the symmetry property of the nucleolus. So as a subtle implication of the integer requirement, weight changes which would destroy a given symmetric or ‘type-preserving’ representation and satisfy (6) are impossible.

Uniqueness and other properties of minimal integer representations of WVG are investigated by Kurz (2012).
Proof of Lemma 1.\[108x440\]

Majority games, voting weight and power are the same.

In contrast, $x^* = w^*$. Neither Peleg’s condition can be brought to bear\[108x386\] nor are standard solution techniques helpful for so large a game.

The left-hand side in condition (6) equals at most half the number of occurrences of the rarest weight, $m^*$, and the right-hand side is bounded below by 2. This, first, implies that the condition cannot be met by WVGs where only one instance of some weight type is involved. This limits Prop. 2’s applicability for small-scale games such as $[3; 2, 1, 1, 1]$. But, second, it means that if we consider $\rho$-replicas of any given $n$-player WVG with integer representation $[q; w]$, i.e., a WVG with quota $\rho q$ and $\rho$ instances of any of the $n$ voters in $[q; w]$, then one can compute an explicit number $\tilde{\rho}$ from (6) such that the nucleolus of the resulting $\rho n$-player WVG must coincide with the corresponding normalized weight vector for every $\rho \geq \tilde{\rho}$.

This observation echoes the coincidence result obtained by Snyder et al. (2005) for WVG replicas under Baron-Ferejohn bargaining\[12\] at least in sufficiently large majority games, voting weight and power are the same.

Mathematical Appendix

Proof of Lemma 1.\[108x319\] Define $w(S) = \sum_{i \in S} w_i$ and $x^*(S) = \sum_{i \in S} x^*_i$. Let $S^+ = \{i \in N \mid x^*_i > w_i\}$ and $S^- = \{i \in N \mid x^*_i \leq w_i\}$. We have $w(S^-) > 0$ because weights cannot exceed nucleolus values for all $i \in N$ given $w(N) = x^*(N) = 1$.

If $w(S^-) = 1$ then $\|x^* - w\|_1 = 0$.

So let $0 < w(S^-) < 1$ and define $0 \leq \delta \leq 1$ by $x^*(S^-) = (1 - \delta)w(S^-)$. We have $x^*(S^+) = w(S^+) + \delta w(S^-)$ and the respective substitutions in decomposition

\[108x352\]

\[108x386\]

\[108x409\]

\[108x431\]

\[108x456\]

\[108x473\]

\[108x490\]

\[108x507\]

\[108x523\]

\[108x540\]

\[108x557\]

\[108x574\]

\[108x590\]

\[108x607\]

\[108x624\]

\[108x641\]

\[108x657\]

\[108x674\]

\[108x698\]

\[108x702\]

\[108x724\]

\[108x740\]

\[108x756\]

\[108x772\]
\[ \|x^* - w\|_1 = \sum_{i \in S^+} (x^*_i - w_i) + \sum_{j \in S^-} (w_j - x^*_j) \text{ yield} \]

(7) \[ \|x^* - w\|_1 = 2\delta w(S^-). \]

Let \( T \) be a MWC which is generated by starting with \( S = \emptyset \) and successively adding a remaining player \( i \) with minimal \( x^*_i / \bar{w}_i \) until \( w(T) \geq q \).

In case \( w(S^-) \geq q \) we then have \( x^*(T) / w(T) \leq x^*(S^-) / w(S^-) = 1 - \delta \).

Multiplying by \( w(T) \), using \( q \leq w(T) \leq q + w_1 \) and finally \( \delta \leq 1 \) yields

(8) \[ x^*(T) \leq (1 - \delta) w(T) \leq (1 - \delta)(q + w_1) \leq q(1 - \delta) + w_1. \]

This and \( q \leq w(T) \leq x^*(T) \) yield \( \delta \leq w_1 / q. \) Applying this and \( w(S^-) < 1 \) in equation (7) gives \( \|x^* - w\|_1 \leq \frac{2w_1}{q}. \)

In case \( w(S^-) < q \), note that moving from \( S^- \) to \( T \) involves the weight addition \( w(T) - w(S^-) \) which comes with a nucleolus per weight unit of at most \( x^*(S^+) / w(S^+) \).

So

\[
x^*(T) = x^*(S^-) + x^*(T \setminus S^-)
\]

\[
\leq (1 - \delta) w(S^-) + \frac{x^*(S^+)}{w(S^+)} \cdot (w(T) - w(S^-))
\]

(9) \[ \leq (1 - \delta) w(S^-) + \frac{x^*(S^+)}{w(S^+)} \cdot (q - w(S^-) + w_1) \]

where the last inequality uses \( w(T) \leq q + w_1 \). Rearranging with \( x^*(S^+) = w(S^+) + \delta w(S^-) \) and \( w(S^-) = 1 - w(S^+) \) yields

(10) \[ x^*(T) \leq q + \frac{w(S^+) + \delta w(S^-)}{w(S^+)} \cdot w_1 - \frac{(1 - q)\delta w(S^-)}{w(S^+)} \]

Since \( \delta \leq 1 \) the right hand side of (10) is at most \( q + (w_1 - (1 - q)\delta w(S^-)) / w(S^+) \). So \( q \leq x^*(T) \) implies \( (1 - q)\delta w(S^-) \leq w_1 \). Hence \( \|x - w\|_1 \leq \frac{2w_1}{1 - q}. \)

\[ \square \]

**Proof of Proposition 1.** If \( x^*_i / \bar{w}_i \geq 1 + \delta \) or \( x^*_i / \bar{w}_i \leq 1 - \delta \) then \( \|x^*_i - \bar{w}_i\|_1 \geq \delta \cdot \bar{m}_i(n) \cdot \bar{w}_i(n) \geq \delta \varepsilon \) for some \( \varepsilon > 0 \) if \( i \) is regular. But \( \lim_{n \to \infty} \|x^*_i - \bar{w}_i\|_1 = 0 \) (Lemma 1). So \( \lim_{n \to \infty} x^*_i / \bar{w}_i = 1 \) and hence

(11) \[ 1 = \lim_{n \to \infty} \frac{x^*_i}{\bar{w}_i} = \lim_{n \to \infty} \frac{\bar{w}_j}{x^*_j} = \frac{\bar{w}_j}{x^*_j} \text{ if } i \text{ and } j \text{ are regular}. \]

\[ \square \]
Proof of Proposition 2. It suffices to prove the result in case $w_n > 0$ because $w_i = 0$ directly implies $x_i^* = 0$. We may also assume $0 < \bar{q} < 1$. For each $k \in \{1, \ldots, t\}$ let $\omega_k$ denote the normalized weight of a voter $i$ with type $k$ (i.e., $\bar{w}_i = \omega_k$) and, with slight abuse of notation, let $x_k^*$ be this voter/type’s nucleolus. Define $r_k = x_k^*/\omega_k$ and w.l.o.g. assume $r_1 \geq \ldots \geq r_t$. Let $a$ denote the largest index such that $r_1 = r_a$ and $b$ be the smallest such that $r_b = r_t$. The claim is true if $a \geq b$. So we suppose $a < b$ and establish a contradiction by showing that we can construct an imputation $x^{**}$ with maximum excess $E_1(x^{**})$ smaller than $E_1(x^*)$.

Writing $\varepsilon = \frac{1}{2} \cdot \min\{\bar{q}, 1 - \bar{q}\}$ and $n_k = m_{\omega_k}$, the premise and $t, w_1 \geq 1$ imply

\begin{equation}
12 \quad w_1 \leq tw_1^2 < \varepsilon m^o \leq \varepsilon n_k
\end{equation}

for each $k \in \{1, \ldots, t\}$. Considering $\omega_k$-weighted sums of (12) we obtain

(I) $\sum_{k \leq a} w_1 \omega_k < \varepsilon \sum_{k \leq a} n_k \omega_k$ and

(II) $\sum_{k \geq b} w_1 \omega_k < \varepsilon \sum_{k \geq b} n_k \omega_k$.

Moreover, we have

(III) $\sum_{k < b} w_1 \omega_k < \varepsilon \sum_{k \geq b} n_k \omega_k$ and

(IV) $\bar{w}_1 + \sum_{k > a} w_1 \omega_k < \varepsilon \sum_{k \leq a} n_k \omega_k$.

Inequality (III) follows from

\begin{equation}
13 \quad \sum_{k < b} w_1 \omega_k < tw_1 \bar{w}_1 = \frac{tw_1^2}{w(N)} < \frac{\varepsilon m^o}{w(N)} \leq \varepsilon \sum_{k \geq b} n_k \omega_k
\end{equation}

using $1/w(N) \leq \omega_k \leq \bar{w}_1$ and (12). Similarly, (IV) follows from

\begin{equation}
14 \quad \bar{w}_1 + \sum_{k > a} w_1 \omega_k \leq \bar{w}_1 + (t - 1)w_1 \bar{w}_1 \leq tw_1^2 < \frac{\varepsilon m^o}{w(N)} \leq \varepsilon \sum_{k \leq a} n_k \omega_k.
\end{equation}

Let $n_k^T$ denote the number of $k$-type voters in a coalition $T \subseteq N$ and define

\begin{equation}
15 \quad D(T) = \frac{\sum_{k \leq a} n_k^T \omega_k}{\sum_{k \leq a} n_k \omega_k} \quad \text{and} \quad I(T) = \frac{\sum_{k \geq b} n_k^T \omega_k}{\sum_{k \geq b} n_k \omega_k}.
\end{equation}

$D(T)$ is the share of the total weight of the $a$ “most over-represented” types (all having maximal nucleolus-to-relative weight ratio $x_1^*/\omega_1$) which they contribute in coalition $T$. Similarly, $I(T)$ is the respective share for the $t - b + 1$ “most under-represented” types.
Given a suitably large coalition $S \subseteq N$, replacing $w_h$ members of type $k$–all with absolute weight $w_k$ – by $w_k$ players of type $h$ yields a coalition $S'$ with $w(S') = w(S)$. But if $r_k > r_h$, such replacement yields $x^*(S') < x^*(S)$. Thus, for a MWC $T$ with maximum excess at $x^*$, i.e., with excess $1 - x^*(T) \geq v(S) - x^*(S)$ for all $S \subseteq N$, there are no $k, h$ with $r_k > r_h$ such that (i) $w_1$ or more type $k$-players belong to $T$ and (ii) $w_1$ or more type $h$-players do not belong to $T$. This consideration restricts the numbers of members $n_k^T$ of players of type $k$ in any MWC $T$ with maximum excess. There are three cases, for each of which we show $I(T) - D(T) > 0$:

**Case 1:** $n_k^T < w_1$ for all types $1 \leq k < b$.

Then the relative weight $\sum_{k \leq a} n_k^T \omega_k$ in $T$ of the most over-represented types is less than $\sum_{k \leq a} w_1 \omega_k$. So inequality (I) implies $D(T) < \epsilon$. Since $T$ is a winning coalition, the weight $\sum_{k \geq b} n_k^T \omega_k$ in $T$ of the most under-represented types is greater than $\bar{q} - \sum_{k < b} w_1 \omega_k$. Due to (III) and $\sum_{k \geq b} n_k \omega_k \leq 1$ we have $I(T) > \bar{q} - \epsilon$. So $I(T) - D(T) > \bar{q} - 2\epsilon \geq 0$.

**Case 2:** $n_k^T \geq w_1$ for some $1 \leq k \leq a$ but $n_h - n_h^T < w_1$ for all $a < h \leq t$.

Using that $T$ is a MWC, the relative weight $\sum_{k \leq a} n_k^T \omega_k$ in $T$ of the most over-represented types is less than $\bar{q} + \bar{w}_1 - \sum_{k > a} (n_k - w_1) \omega_k$ in this case. So inequality (IV) and $\sum_{k \leq a} n_k \omega_k \leq 1$ imply $D(T) < \bar{q} + \epsilon$. Moreover, the weight $\sum_{k \geq b} n_k^T \omega_k$ in $T$ of the most under-represented types is greater than $\sum_{k \geq b} (n_k - w_1) \omega_k$ and inequality (II) implies $I(T) > 1 - \epsilon$. So $I(T) - D(T) > 1 - \bar{q} - 2\epsilon \geq 0$.

**Case 3:** $n_l^T < w_1$ for all $1 \leq l \leq a$ and $n_k^T \geq w_1$ for some $a < k < b$ but $n_h - n_h^T < w_1$ for all $b < h \leq t$.

The relative weight $\sum_{l \leq a} n_l^T \omega_l$ in $T$ of the most over-represented types is then less than $\sum_{l \leq a} w_1 \omega_l$. So inequality (I) implies $D(T) < \epsilon$. Similarly, the total weight of the players of types $b \leq h \leq t$ is greater than $\sum_{h \geq b} (n_h - w_1) \omega_h$. Inequality (II) then implies $I(T) > 1 - \epsilon$ and we have $I(T) - D(T) > 1 - 2\epsilon > \bar{q} - 2\epsilon \geq 0$.

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If Case 1 does not apply, there is a smallest index $1 \leq k < b$ with $n_k^T \geq w_1$. Assume $k \leq a$ first. Because $r_k > r_h$ for all $a < h < t$, the number $n_h - n_h^T$ of $h$-types outside coalition $T$ is less than $w_1$: otherwise the indicated replacement would yield a MWC $T'$ with $x^*(T') < x^*(T)$, contradicting the maximum-excess property of $T$. This is the description of Case 2. The remaining Case 3 involves $a < k < b$ where $r_k > r_h$ for all $b \leq h \leq t$. Then, analogously, $n_h - n_h^T < w_1$ must hold.
Recall that $x^*_k \geq w_k$ for all $1 \leq k \leq a$. So for sufficiently small $\sigma > 0$

\begin{equation}
x^*_k(\sigma) = \begin{cases} 
  x^*_k - \sigma \omega_k & \text{if } 1 \leq k \leq a, \\
  x^*_k & \text{if } a < k < b, \text{ and} \\
  x^*_k + \delta \sigma \omega_k & \text{if } b \leq k \leq t
\end{cases}
\end{equation}

with $\delta = \sum_{k \leq a} n_k \omega_k / \sum_{l \geq b} n_l \omega_l > 0$ is an imputation. $x^*_k(\sigma)$’s continuity implies existence of $\sigma > 0$ so that no $S$ with $e(S, x^*) < E_1(x^*)$ has maximum excess at $x^*(\sigma)$. We fix such a value of $\sigma$ and write $x^* = x^*(\sigma)$.

It then suffices to consider coalitions $T'$ with maximum excess at $x^*$ in order to show the contradiction $E_1(x^*) < E_1(x^*)$. Such $T'$ has to be winning, and for any MWC $T \subseteq T'$ it must be true that $e(x^*, T') = e(x^*, T) = E_1(x^*)$. Since $T$ and $T'$ both are winning we have $e(x^*, T') \leq e(x^*, T)$ and

\begin{equation}
E_1(x^*) = \max \{ e(x^*, T) : T \text{ is MWC and } e(x^*, T) = E_1(x^*) \}.
\end{equation}

Moreover, for every $T$ on the right-hand side of equation (17)

\begin{equation}
e(x^*, T) - E_1(x^*) = e(x^*, T) - e(x^*, T) = -\sigma \cdot \left( I(T) - D(T) \right) \cdot \sum_{k \leq a} n_k \omega_k < 0
\end{equation}

implies $e(x^*, T) < E_1(x^*)$, so that $E_1(x^*) < E_1(x^*)$.

\begin{thebibliography}{9}

Kurz, S. (2012). On minimum sum representations for weighted voting games. *Annals of Operations Research* 196(1), 361–369.

Kurz, S., N. Maaser, and S. Napel (2013, March). On the democratic weights of nations. Working Paper 66, Public Choice Research Centre, University of Turku.

Le Breton, M., M. Montero, and V. Zaporozhets (2012). Voting power in the EU Council of Ministers and fair decision making in distributive politics. *Mathematical Social Sciences* 63(2), 159–173.

Lindner, I. and M. Machover (2004). L. S. Penrose’s limit theorem: Proof of some special cases. *Mathematical Social Sciences* 47(1), 37–49.

Montero, M. (2005). On the nucleolus as a power index. *Homo Oeconomicus* 22(4), 551–567.

Montero, M. (2006). Noncooperative foundations of the nucleolus in majority games. *Games and Economic Behavior* 54(2), 380–397.

\end{thebibliography}
References

Montero, M., M. Sefton, and P. Zhang (2008). Enlargement and the balance of power: An experimental study. *Social Choice and Welfare* 30(1), 69–87.

Neyman, A. (1982). Renewal theory for sampling without replacement. *Annals of Probability* 10(2), 464–481.

Peleg, B. (1968). On weights of constant-sum majority games. *SIAM Journal on Applied Mathematics* 16(3), 527–532.

Schmeidler, D. (1969). The nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics* 17(6), 1163–1170.

Shubik, M. and H. P. Young (1978). The nucleolus as a noncooperative game solution. In P. C. Ordeshook (Ed.), *Game Theory and Political Science*, pp. 511–527. New York, NY: New York University Press.

Snyder, James M., J., M. T. Ting, and S. Ansolabehere (2005). Legislative bargaining under weighted voting. *American Economic Review* 95(4), 981–1004.

Young, H. P. (1978). The allocation of funds in lobbying and campaigning. *Behavioral Science* 23(1), 21–31.

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