Proof of the Gour-Wallach conjecture

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The absolute value of the hyperdeterminant of four qubits is a useful measure of genuine entanglement. We prove a recent conjecture of Gour and Wallach describing the pure maximally entangled four-qubit states with respect to this measure.

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I. INTRODUCTION

Maximal entanglement is an important resource in quantum information science. The Bell state is the maximally entangled two-qubit pure state. It contains one entanglement bit (ebit) measured by the von Neumann entropy. Quantum teleportation requires the cost of one ebit and cannot be faithfully carried out by non-maximally entangled states. To make non-maximally entangled states useful for teleportation, remote state preparation and some other quantum-information tasks, they are converted into Bell states by the local operations and classical communications (LOCC). This process is known as the entanglement purification or distillation. For bipartite pure states of higher dimensions, the maximally entangled state can be used to create any state under LOCC due to the majorization criterion. So maximally entangled states are the universal generators for quantum entanglement.

It is thus expected that the multipartite maximally entangled states will play a similar role in the multipartite systems. The multipartite entanglement is a more universally operational quantum resource than the bipartite entanglement. For this purpose we need to characterize the maximally entangled states in multiqubit systems. Unlike the bipartite case, the multipartite maximally entangled states depend on the choice of the multipartite entanglement measure. It is conjectured that any multiqubit maximally entangled state has maximally mixed state as the one-party reduced density operator when the entanglement is measured by the sum of the negativity over all inequivalent bipartitions. Under a similar condition, the maximal entanglement with respect to the 4-tangle has been characterized. On the other hand, the W state is the maximally entangled three-qubit state with respect to the geometric measure of entanglement, which is a distance-like entanglement measure. In this paper, we explicitly characterize the four-qubit maximally entangled states (see Eq. (7)) under the absolute value of the hyperdeterminant introduced in [14, 15]. In the case of two-qubit and three-qubit pure states, after normalization, the hyperdeterminant becomes the concurrence and 3-tangle, respectively. In the four-qubit case, the hyperdeterminant is an invariant homogeneous polynomial of degree 24, which yields an entanglement measure for 4-qubit genuine entanglement. Our work mainly confirms a conjecture proposed in [16], to which we shall refer as the Gour-Wallach conjecture throughout the paper.

Many multiqubit states, such as the ten-photon Greenberger-Horne-Zeilinger state, four-photon W state, and six-photon Dicke states, have been experimentally realized in recent years. We expect that the maximally entangled 4-qubit state may be also realized by using the present lab techniques.

In Sec. III we provide some background information on the polynomial invariants and in particular on the hyperdeterminant, and state the Gour-Wallach conjecture (Conjecture 1). The proof of the conjecture is given in Sec. IV. In Sec. V we conclude our findings, state an open problem related to the above conjecture, and make a comment on the generators of the algebra of symmetric polynomial invariants.

II. STATEMENT OF THE CONJECTURE

We denote by SL the direct product SL₂×₄ of four copies of SL₂(C). It is well known (see e.g. [19]) that the algebra of polynomial SL-invariants of four qubits is a polynomial algebra, A, in four variables. The four homogeneous generators of A have degrees 2, 4, 4 and 6. We enlarge SL by including the group S₄ of permutations of four qubits, and obtain the semidirect product SL* := SL × S₄. The algebra of polynomial SL*-invariants, B, is also a polynomial algebra in four variables. The four homogeneous generators of B have degrees 2, 6, 8 and 12, see [14, 20]. The vectors ψ in the Hilbert space H of four qubits can be identified with the 2×2×2×2 complex matrices. There is a generalization of determinant to these four-dimensional matrices called hyperdeterminant, see [14] Chapter 14 and [12], which is a homogeneous SL*-invariant of degree 24. We shall denote it by Det(ψ).

All states in this note will be normalized. Let us intro-
duce the subspace $A \subseteq \mathcal{H}$ with basis

$$
|u_0\rangle = \frac{1}{2} (|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle),
$$

$$
|u_1\rangle = \frac{1}{2} (|0000\rangle - |0011\rangle - |1100\rangle + |1111\rangle),
$$

$$
|u_2\rangle = \frac{1}{2} (|0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle),
$$

$$
|u_3\rangle = \frac{1}{2} (|0101\rangle - |0110\rangle - |1001\rangle + |1010\rangle).
$$

We write an arbitrary vector $z \in A$ as

$$
z = \sum_j z_j |u_j\rangle, \quad z_j \in \mathbb{C}.
$$

The polynomial $\det(\psi)$ is usually defined only up to a nonzero constant factor. We use the same normalization as the one adopted in [16]. It is specified by the restriction $\det|A|$, which is given by the formula

$$
\det(z) = \prod_{0\leq j<k \leq 3} (z_j^2 - z_k^2), \quad z \in A.
$$

(This is not the standard normalization, in which the coefficients of $\det(\psi)$, considered as a polynomial in the the $2 \times 2 \times 2 \times 2$ matrix entries, are relatively prime integers.)

We shall denote the unit sphere of $\mathcal{H}$ by $\Sigma$, and we set $\Omega = \Sigma \cap A$. Gour and Wallach have proposed recently [11] to use the absolute value of the hyperdeterminant as a measure of genuine four-qubit entanglement. The maximally entangled states, with respect to this measure, are of special interest and they have proposed the following conjecture concerning these special states.

**Conjecture 1**

(a) The maximum of $|\det(\psi)|$ over all states $|\psi\rangle \in \mathcal{H}$ is reached at the state

$$
|L\rangle = \frac{1}{\sqrt{3}} (|u_0\rangle + \omega |u_1\rangle + \omega^* |u_2\rangle), \quad \omega = e^{i\pi/3}.
$$

(b) Up to local unitary (LU) transformations, $|L\rangle$ is the unique state with this property.

By using the formula (13) we obtain that $\det(L) = -3^{-9}$. By replacing $\omega$ by $\omega^2$ in the above expression for $|L\rangle$ we obtain a new state, $|L'\rangle \equiv \mathcal{Q}$. It is easy to verify that $|L\rangle$ and $|L'\rangle$ are LU-equivalent. They have been widely studied in [3, 16, 21, 22]. It is known that, up to local unitary transformations, $|L\rangle$ is the only state that maximizes the average Tsallis $\alpha$-entropy of entanglement for all $\alpha > 2$ [3].

Let us recall from [16] the definition and some properties of the generic set $\Omega$. By definition, $\Omega$ is the set of all vectors $\psi \in \mathcal{H}$ such that $\dim(\text{SL} \cdot \psi) = 12$. We have $\Omega = \text{SL} \cdot \Omega_A$ where $\Omega_A := \Omega \cap A$. Moreover $\Omega = \{ \psi \in \mathcal{H} : \det(\psi) \neq 0 \}$, and so $\Omega$ is an open dense subset of $\mathcal{H}$.

### III. PROOF OF THE CONJECTURE

We shall first study the restriction $\det|A|$. Let $z = (z_0, z_1, z_2, z_3)$ be a quadruple of complex variables $z_j = r_j e^{i\theta_j}$, where $r_j \geq 0$ and $\theta_j \in \mathbb{R}$. We denote by $f$ the basic antisymmetric polynomial in these variables, i.e.,

$$
f(z) = \prod_{0 \leq j < k \leq 3} (z_j - z_k).
$$

We shall view $z_j$ as a function of two real variables $r_j$ and $\theta_j$, and so $f$ is a function of eight real variables. We denote by $U_A$ the open region in $A$ defined by the condition $f(z) \neq 0$, i.e., $z_j \neq z_k$ whenever $j \neq k$. Thus if $z \in U_A$ then at most one of the $z_j$ may vanish.

Note that on $A$ we have $\det(z) = f(z_0^2, \ldots, z_3^2)$, i.e., $\det = (f \circ Q)^2$ where $Q$ is the squaring map $(z_0, \ldots, z_3) \rightarrow (z_0^2, \ldots, z_3^2)$. It follows that $\Omega_A = \{ z \in A : Q(z) \in U_A \}$. Thus, the maximization problem for $|\det(z)|$ on $\Sigma_A$ reduces to the problem of maximizing $|f(z)|^2$ subject to the constraint

$$
r_1 + r_2 + r_3 + r_4 = 1.
$$

Let $\Delta$ denote the closed subset of $A$ defined by this equation. As $\det(z) = 0$ at the points $z \in \Sigma_A \setminus \Omega_A$, the maximum of $|\det|$ on $\Sigma_A$ must be reached at some points $z \in \Omega_A$. In that case $Q(z) \in U_A$ and $|f|^2$ reaches its maximum on $\Delta$ at the point $Q(z)$.

Our goal is to show that at all points $z \in \Delta$ where $|f|^2$ has local maximum on $\Delta$, we have $|f(z)|^2 \leq 3^{-9}$ and at the same time identify the points at which the equality holds. Note that $|f|^2$, Eq. (9) and the region $U_A$ are all invariant under permutation of the variables $z_j$ and the multiplication of all $z_j$ by the same phase factor. Therefore the set of local maxima points that we are looking for is also invariant under these transformations and we shall use this fact to simplify the problem.

We shall first treat the case when all $r_j > 0$. Note that for $z \in U_A$ we have

$$
\frac{1}{f} \frac{\partial f}{\partial r_j} = w_j
$$

and

$$
\frac{1}{f} \frac{\partial f}{\partial \theta_j} = ir_j w_j
$$

where

$$
w_j = e^{i\theta_j} \sum_{k \neq j} \frac{1}{z_j - z_k}.
$$

Note also that

$$
\sum_{j=0}^{3} e^{-i\theta_j} w_j = 0.
$$

Unless stated otherwise, we shall assume that $z \in \Delta$ is a point where $|f|^2$ reaches its maximum on $\Delta$. Then Eqs. (10) and (11) imply that $w_j$ is real and

$$
\frac{1}{f} \frac{\partial f}{\partial r_j} = w_j
$$

and

$$
\frac{1}{f} \frac{\partial f}{\partial \theta_j} = 0
$$

for all $z_0, z_1, z_2, z_3$. This shows that $|f|^2$ is constant on $\Omega_A$ and it follows that $z_1, z_2, z_3$ are all real, and $z_0$ is real and positive. Therefore $z_0 \neq z_j$ for all $j$.

We shall now show that $z_0 > 0$. This is achieved by the formula (9) and the region $U_A$ are all invariant under permutation of the variables $z_j$ and the multiplication of all $z_j$ by the same phase factor. Therefore the set of local maxima points that we are looking for is also invariant under these transformations and we shall use this fact to simplify the problem.

When $z \in U_A$ we have

$$
\frac{1}{f} \frac{\partial f}{\partial r_j} = w_j
$$

and

$$
\frac{1}{f} \frac{\partial f}{\partial \theta_j} = ir_j w_j
$$

where

$$
w_j = e^{i\theta_j} \sum_{k \neq j} \frac{1}{z_j - z_k}.
$$

Note also that

$$
\sum_{j=0}^{3} e^{-i\theta_j} w_j = 0.
$$

Unless stated otherwise, we shall assume that $z \in \Delta$ is a point where $|f|^2$ reaches its maximum on $\Delta$. Then Eqs.
show that the numbers $w_j$ must be real. For $j \neq k$ we can replace $r_j$ and $r_k$ with $r_j + t$ and $r_k - t$, where $t$ is an auxiliary real variable. Note that the constraint equation remains satisfied when $|t|$ is small. Since $z$ is a critical point, Eqs. \textcolor{red}{[10]} imply that
\[ \frac{\partial f}{\partial r_j} - \frac{\partial f}{\partial r_k} = (w_j - w_k)f, \tag{14} \]
and we deduce that numbers $w_j - w_k$ must be purely imaginary. On the other hand the numbers $w_j$ are real, and so we must have $w_j - w_k = 0$. Thus we have shown that $w_0 = w_1 = w_2 = w_3 \in \mathbb{R}$.

Assume that $w_0 = 0$. Then $w_j = 0$ for all $j$, i.e., we have
\[ \sum_{j \neq k} \frac{1}{z_j - z_k} = 0, \quad j = 0, 1, 2, 3. \tag{15} \]
By simplifying the first three of these equations we obtain the system
\[ \begin{align*}
3z_0^2 - 2z_0(z_1 + z_2 + z_3) + (z_1z_2 + z_1z_3 + z_2z_3) &= 0, \\
3z_1^2 - 2z_1(z_0 + z_2 + z_3) + (z_0z_2 + z_0z_3 + z_2z_3) &= 0, \\
3z_2^2 - 2z_2(z_0 + z_1 + z_3) + (z_0z_1 + z_0z_3 + z_1z_3) &= 0.
\end{align*} \]
as the $z_j$ are pairwise distinct, these three equations lead to a contradiction. We conclude that $w_0 \neq 0$.

Consequently, Eq. \textcolor{red}{[13]} implies that
\[ e^{i\theta_0} + e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} = 0. \tag{16} \]
A simple geometric argument shows that we may assume that $r_0 = \max_j r_j$ and that
\[ \theta_0 = \theta, \quad \theta_1 = \pi - \theta, \quad \theta_2 = \pi + \theta, \quad \theta_3 = -\theta \tag{17} \]
for some $\theta \in [0, \pi/4]$. By plugging in these expressions into Eq. \textcolor{red}{[12]}, we obtain the formulae
\[ \begin{align*}
w_0 &= \frac{1}{r_0 + r_2} + \frac{1}{r_0 + r_1u^{-1}} + \frac{1}{r_0 - r_3u^{-1}}, \\
w_1 &= \frac{1}{r_1 + r_3} + \frac{1}{r_1 + r_0u} + \frac{1}{r_1 - r_2u}, \\
w_2 &= \frac{1}{r_0 + r_2} + \frac{1}{r_2 + r_1u^{-1}} + \frac{1}{r_2 + r_3u^{-1}}, \\
w_3 &= \frac{1}{r_1 + r_3} + \frac{1}{r_3 - r_0u} + \frac{1}{r_3 + r_2u},
\end{align*} \tag{18-21} \]
where $u = e^{2i\theta}$. Since the $w_j$ are real, we have
\[ \begin{align*}
(r_1|r_0 - r_3u^{-2} - r_3|r_0 + r_1u^{2})\sin 2\theta &= 0, \\
(r_0|r_1 - r_2u^{-2} - r_2|r_1 + r_0u^{2})\sin 2\theta &= 0, \\
(r_1|r_2 + r_3u^{-2} - r_3|r_2 + r_1u^{2})\sin 2\theta &= 0, \\
(r_0|r_3 + r_2u^{-2} - r_2|r_3 - r_0u^{2})\sin 2\theta &= 0.
\end{align*} \]
If $\theta > 0$ then the above four equations imply that
\[ 4r_0r_1r_2r_3 \cos 2\theta = r_2(r_1 - r_3)(r_0^2 - r_1r_3) = r_3(r_0 - r_2)(r_1^2 - r_0r_2) = r_0(r_1 - r_3)(r_1r_3 - r_2^2) = r_1(r_0 - r_2)(r_0r_2 - r_3^2). \tag{22} \]
Assume first that $\theta = 0$ and thus $u = 1$. From $w_0 = w_1$ and $w_0 = w_2$ we obtain that
\[ \begin{align*}
(r_0 - r_1 + r_2 - r_3)(r_0r_3 + r_1r_2) - (r_0r_2 + r_1r_3) + 2(r_0r_1 + r_2r_3) &= 0, \\
(r_0 + r_1 - r_2 - r_3)(r_0r_3 + r_1r_2) + 2(r_0r_2 + r_1r_3) - (r_0r_1 + r_2r_3) &= 0.
\end{align*} \]
Since $z_0 \neq z_3$ we have $r_0 \neq r_3$ and so either $r_0 + r_1 - r_2 - r_3 = (r_0r_3 + r_1r_2) + 2(r_0r_2 + r_1r_3) - (r_0r_1 + r_2r_3) = 0$ or $r_0 - r_1 + r_2 - r_3 = (r_0r_3 + r_1r_2) - (r_0r_2 + r_1r_3) + 2(r_0r_1 + r_2r_3) = 0$. Since $r_0 + r_1 + r_2 + r_3 = 1$, in both cases we obtain that $|f(z)|^2 = 2(2^2 - 2^2 - 6 - 6) < 3^9$.

Next assume that $\theta = \pi/4$ and so $\cos 2\theta = 0$. If $r_1 > r_3$ then $r_0r_2 = r_1^2 - r_3^2$ and $r_0 > r_1$. A computation shows that $w_3 - w_0 = 3(r_0 - r_1)^2/2(r_0^2 + r_1^2) > 0$. Thus we have a contradiction. We conclude that $r_0 = r_2$. The equality $(r_1 - r_3)(r_0r_3 - r_1r_2) = 0$ implies that $r_1 = r_3$. If also $r_0 = r_1$ then $r_2 = 1/4$ and we have $f(z) = -1/256$. Otherwise $r_0 > r_1$ and $w_0 = w_3 = (r_0 - r_1)^2 - 4r_0r_1 + r_3^2) = 0$ implies that $r_0 = (3 + \sqrt{3})/12$ and $r_2 = (3 - \sqrt{3})/12$. Then a computation shows that $|f(z)|^2 = 6^6 - 6^9 < 3^9$.

Finally assume that $0 < \theta < \pi/4$. Then Eqs. \textcolor{red}{[22]} imply that $r_0r_2 = r_1r_3$, $r_0 \geq r_1 \geq r_3 \geq 2$, and $\cos 2\theta = (r_0 - r_2)(r_1 - r_3)/4r_0r_2$. By using the equation \textcolor{red}{[11]} and $r_0r_2 = r_1r_3$, we obtain that
\[ r_2 = \frac{r_1(1 - r_0 - r_1)}{r_0 + r_1}, \quad r_3 = \frac{r_0(1 - r_0 - r_1)}{r_0 + r_1}. \tag{23} \]
A tedious computation now shows that $w_0 \neq w_1$ and so we have a contradiction.

Next we consider the case when some $z_j = 0$. As $z \in U_{\delta}$, at most one of the $z_j$ may vanish. Without any loss of generality we may assume that $z_3 = 0$, and so $z_0z_1z_2 \neq 0$. We proceed as in the previous case but we have to make some essential changes. The equations \textcolor{red}{[10]}, \textcolor{red}{[11]} and \textcolor{red}{[12]} are now valid only for $j = 0, 1, 2$. As $r_3 = 0$, $\theta_3$ can be chosen arbitrarily and so $w_3$ is not defined. Consequently, Eq. \textcolor{red}{[13]} is not meaningful but we have the following substitute
\[ \sum_{j=0}^{2} \left( w_j - \frac{1}{r_j} \right) e^{-i\theta_j} = 0. \tag{24} \]
The proof of the assertion $w_0 = w_1 = w_2 \in \mathbb{R}$ remains valid. In the proof of the claim that $w_0 \neq 0$ we used Eq. \textcolor{red}{[11]} only for $j = 0, 1, 2$, and so this proof remains valid.

Note that the maximum of $|f(z_0, z_1, z_2, 0)|$ when $z_0, z_1, z_2 \in \mathbb{R}$ and $r_0 + r_1 + r_2 = 1$ is equal to the maximum of $x_0 x_3 x_2 (x_0 - x_1)(x_0 + x_2)(x_1 + x_2)$ where $x_0 \geq x_1 \geq 0$, $x_2 \geq 0$ and $x_0 + x_1 + x_2 = 1$. By using the method of Lagrange multipliers, it is easy to verify that the latter maximum is equal to $2^{-8}$. (The maximum occurs at the point $x_0 = 1/2, x_1 = (2 - \sqrt{2})/4, and x_2 = \sqrt{2}/4$.) Hence, we can dismiss the cases where all $z_j$ are real.
We may assume that $r_0 \geq r_1 \geq r_2$, $\theta_0 = 0$ and at least one of the phase factors $s_1 = e^{i\theta_1}$ and $s_2 = e^{i\theta_2}$ is not real. Eq. (24) then implies that neither $s_1$ nor $s_2$ is real.

We claim that $r_0 = r_1 = r_2 = 1/3$. Since $w_0 = w_1 = w_2$ the resultants of $w_0 - w_1$ and $w_0 - w_2$ with respect to the variable $s_1$ and $s_2$ (separately) must vanish. We obtain the equations

\[
\begin{align*}
r_0^2(5r_0 - r_1 - r_2)(r_0 + r_1 - r_2) & - r_0r_2(5r_0^2 - 5r_1^2 + 5r_2^2 + 2r_0r_1 + 2r_1r_2 - 14r_0r_2)s_2 \\
+ r_0^3(r_0 - r_1 - r_2)(r_0 + r_1 - 5r_2) & = 0, \quad (25) \\
r_0^2(5r_0 - r_1 - r_2)(r_0 - r_1 + r_2)s_2 & - r_0r_1(5r_0^2 + 5r_1^2 - 3r_2^2 + 2r_0r_2 + 2r_1r_2 - 14r_0r_1)s_1 \\
+ r_0^3(r_0 - r_1 - r_2)(r_0 + r_2 - 5r_1) & = 0, \quad (26)
\end{align*}
\]

respectively. As $s_1, s_2 \notin R$ and $|s_1| = |s_2| = 1$, the leading and constant terms must be equal in each of these two equations. Thus we obtain the following two equations

\[
\begin{align*}
(r_0^3 + 4r_0r_1r_2 + r_2^3 - (r_0 + r_2)(5r_0r_2 + r_2^2)) & (r_0 - r_2) = 0, \quad (27) \\
(r_0^3 + 4r_0r_1r_2 + r_2^3 - (r_0 + r_1)(5r_0r_1 + r_2^2)) & (r_0 - r_1) = 0. \quad (28)
\end{align*}
\]

If $r_0 = r_1$ then Eq. (27) implies that $r_0 = r_2$ and so our claim holds. Assume now that $r_0 > r_1$. After dropping the factors $r_0 - r_2$ and $r_0 - r_1$ from the left hand sides of the above equations, the resultant with respect to $r_0$ of the two remaining polynomials is equal to $288r_1r_2(r_1 - r_2)^3(r_1 + r_2)^4$. Thus, we obtain that $r_1 = r_2$, and so $r_0 = 1 - 2r_1$ and $r_1 < 1/3$. Eq. (27) now gives that $r_1 = (9 - \sqrt{33})/24$. However, then the roots of Eq. (26) are real and we have a contradiction. Thus, our claim is proved.

After setting $r_0 = r_1 = r_2 = 1/3$ in Eqs. (25) and (26), we deduce that $s_1, s_2 \notin 1$ are cube roots of 1. As $z_1 \neq z_2$ we have $s_1 \neq s_2$.

To summarize, we have proved the following lemma.

**Lemma 2** The inequality $|f(z)|^2 \leq 3^{-9}$ holds for all $z \in A$ such that $\sum_j |z_j| = 1$. The equality holds if and only if exactly one $z_k = 0$ while the other three $z_j$ form vertices of an equilateral triangle inscribed in the circle of radius 1/3 and centered at the origin 0.

We now shift our focus from Det$_A$ to Det itself. The following lemma plays a crucial role.

**Lemma 3** Let $z \in A$ and let $O$ denote the $\text{SL}$-orbit through $z$. Then $||z|| \leq \|v\|$ for all points $v \in O$ and the equality holds if and only if $v \in \text{SU} \cdot z$, where $\text{SU} := \text{SU}(2)^4$ (a maximal compact subgroup of $\text{SL}$).

**Proof.** We may assume that $z \neq 0$. The function $\text{Det} : O \to R$ whose value at any $v \in O$ is equal to the norm $||v||$ is a smooth function. Let $\mathfrak{sl} = \mathfrak{sl}(2)^4$ denote the Lie algebra of the group $\text{SL}$. By using Maple we have verified that the vector $z$ is orthogonal to the tangent space $\mathfrak{sl}$ of $O$ at the point $z$. This means that $z$ is a critical point of the function $\text{Det}$. By the Kempf-Ness theorem (see [23, Theorem 6.18]  or [9, Appendix A]) the orbit $O$ is closed, the function $\text{Det}$ has minimum at the point $z$, and all critical points of $\text{Det}$ correspond to a minimum and constitute a single $\text{SU}$-orbit. This completes the proof.

Now we can prove the conjecture. Let $\psi \in \Sigma$ be an arbitrary state. If $\psi \notin \Omega$ then $\text{Det}(\psi) = 0$, so we may assume that $\psi \in \Omega$. Any such $\psi$ can be written as $\psi = g \cdot z$ for some $g \in \text{SL}$ and some $z \in \Omega$. Since Det is a homogeneous polynomial of degree 24 which is $\text{SL}$-invariant, by using Lemma 2 and Lemma 3 we have

\[
\begin{align*}
|\text{Det}(\psi)| & = |\text{Det}(g \cdot z)| \\
& = |\text{Det}(z)| \\
& = ||z||^{24} |\text{Det}(z/||z||)| \\
& \leq 3^{-9} ||z||^{24} \\
& \leq 3^{-9} ||g \cdot z||^{24} \\
& = 3^{-9}.
\end{align*}
\]

Hence, part (a) of the conjecture is proved.

In order to prove part (b), assume that $|\text{Det}(\psi)| = 3^{-9}$. Then the inequality (29) implies that $||z|| = 1$ and by Lemma 3 we have $\psi \in \text{SU} \cdot z$. (See also [6, Proposition 17.1].) Hence, part (b) of the conjecture follows from the following lemma.

**Lemma 4** All states $|z\rangle = \sum_{j=0}^3 z_j |u_j\rangle \in A$ such that $|\text{Det}(z)| = 3^{-9}$ are $\text{SU}$-equivalent to each other.

**Proof.** First we claim that all permutations of the $|u_i\rangle$, $i = 0, 1, 2, 3$, can be performed by $\text{LU}$-transformations. Consider the $\text{LU}$-operators:

\[
\begin{align*}
U_0 & = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
U_1 & = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
U_2 & = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
U_3 & = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{align*}
\]

It is easy to verify that $U_i$ interchanges $|u_i\rangle$ and $|u_{i+1}\rangle$ and fixes the other two $|u_j\rangle$. Thus our claim is proved. Hence, by Lemma 2 we may assume that $z_3 = 0$ and $(z_0^2, z_1^2, z_2^2) = \frac{1}{4}(1, \omega^2, \omega^4)$ where $\omega = e^{2\pi i/3}$. If $z_0 = -1/\sqrt{3}$ we can multiply $|z\rangle$ with the phase factor $-1$. Thus, we can assume that $z_0 = +1/\sqrt{3}$. There are now only four cases to consider. If $z_1z_2 = -1/3$ (there are two such cases) then we can multiply $|z\rangle$ with a suitable phase factor and permute the first three $|u_i\rangle$ to obtain the state $|L\rangle$. For instance, if $z_1 = \omega/\sqrt{3}$ and $z_2 = \omega^2/\sqrt{3}$ then we would multiply $|z\rangle$ with $\omega^{-1}$. Thus, we may assume that $z_1z_2 = 1/3$, i.e., $|z\rangle = |L\rangle$ or $|L'\rangle$. As mentioned in the Introduction, $|L\rangle$ and $|L'\rangle$ are $\text{SU}$-equivalent. This completes the proof. □
IV. CONCLUSION AND DISCUSSION

In this paper we have proved the Gour-Wallah conjecture. Thus, the state \( \rho \) maximizes the absolute value of the hyperdeterminant and, up to local unitary transformations, it is the unique state with this property. In this sense it is the maximally entangled state of four qubits.

The first step in our proof of this conjecture was to maximize \( |f(z_0, z_1, z_2, z_3)|^2 \) subject to the constraint \( \sum |z_j| = 1 \), where \( f(z_0, z_1, z_2, z_3) \) is the Van der Monde determinant on the complex variables \( z_0, z_1, z_2, z_3 \). As an interesting generalization, we would like to propose the following open problem.

Find the maximum, \( \mu_n \), of the absolute value of the Van der Monde determinant

\[
V_n(z_0, \ldots, z_{n-1}) = \prod_{0 \leq j < k \leq n-1} (z_k - z_j)
\]

on the complex variables \( z_0, \ldots, z_{n-1} \) subject to the constraint

\[
\sum_{j=0}^{n-1} |z_j| = 1.
\]

The value of \( |V_n(z_0, z_1, \ldots, z_{n-1})| \) at the point where \( z_{n-1} = 0 \) and \( z_0, \ldots, z_{n-2} \) are vertices of a regular \( (n-1) \)-gon with center at the origin and radius \( 1/(n-1) \), e.g.,

\[
z_j = \frac{1}{n-1} e^{2\pi i j/(n-1)}, \quad j = 0, 1, \ldots, n-2,
\]

is equal to \( \lambda_n := (n-1)^{(n-1)/2} / 2 \). For \( n = 2, 3, 4 \) we have \( \mu_n = \lambda_n \). The proof for \( n = 2 \) is trivial. For \( n = 4 \) see Lemma 2 and its proof. The case \( n = 3 \) is much easier and can be proved by the same method. If \( n = 2 \) the maximum is also attained at the points \( (z_0, z_1) \) with \( 0 < |z_0| \leq 1 \) and \( z_1 = z_0 - z_0/|z_0| \).

However, for \( n = 7 \) we have \( \mu_7 > \lambda_7 \).

There are two sets of generators of symmetric polynomial invariants of four qubits which have been proposed recently [16, 20]. Recall the algebra of symmetric invariants \( B \) mentioned in the introduction. It is generated by four algebraically independent homogeneous polynomials of degrees 2, 6, 8, 12. We shall express the generators \( F_1, F_3, F_4, F_6 \) of \( B \) constructed in [16] as polynomials in the generators \( H, \Gamma, \Sigma, \Pi \) constructed in [20]. By using the restrictions of the generators to the subspace \( A \), one can easily verify that

\[
F_1 = 2H,
\]

\[
F_3 = 4(3H^3 - 4\Gamma),
\]

\[
F_4 = \frac{4}{3} (33H^4 - 104H\Gamma + 40\Sigma),
\]

\[
F_6 = \frac{4}{3} (513H^6 - 3012H^3\Gamma + 2180H^2\Sigma + 488\Gamma^2 + 480\Pi).
\]

We point out that in the formulae given in [20] Table 6 the invariants \( H, \Gamma, \Sigma, \Pi \) are evaluated at the generic point of \( A \), namely \( a|u_0 \rangle + d|u_1 \rangle + b|u_2 \rangle + c|u_3 \rangle \). Thus, one should set \( a = z_0, b = z_2, c = z_3, d = z_1 \) to get agreement with Eq. (2).

For the hyperdeterminant \( \text{Det} \), normalized as in [10], we have the equality

\[
\text{Det} = \frac{64}{27} (4H^3\Gamma^3 - 4H^6\Pi + 3H^4\Sigma^2 - 6H^5\Gamma\Sigma + 48H^3\Pi \Gamma - 48H^2\Sigma\Pi - 96H\Gamma\Sigma^2 - 96\Gamma^2\Sigma + 32\Sigma^3 - 64\Pi^2 + 60H^2\Gamma^2 - 36\Gamma^4).
\]

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