Interactions, Localization, and the Integer Quantum Hall Effect

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Abstract

We report on numerical studies of the influence of Coulomb interactions on localization of electronic wavefunctions in a strong magnetic field. Interactions are treated in the Hartree-Fock approximation. Localization properties are studied both by evaluating participation ratios of Hartree-Fock eigenfunctions and by studying the boundary-condition dependence of Hartree-Fock eigenvalues. We find that localization properties are independent of interactions. Typical energy level spacings near the Fermi level and the sensitivity of those energy levels to boundary condition show similar large enhancements so that the Thouless numbers of the Hartree-Fock eigenvalues are similar to those of non-interacting electrons.

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In studies of the integer quantum Hall effect it is usually assumed without justification that Coulomb interactions between electrons can safely be ignored. At strong magnetic fields, eigenfunctions of the single-particle Hamiltonian are localized by a disorder potential at almost all energies and the localization length diverges as \(|E - E_{cN}|^{-7/3}\) at a discrete set of critical energies \(E_{cN}\) near the center of each disorder-broadened Landau level [1]. The critical energies are similar to the mobility edges of the Anderson metal-insulator transition, except that the localization length is finite both for \(E > E_{cN}\) and for \(E < E_{cN}\). Theory suggests that at \(T = 0\) the Hall conductivity \((\sigma_{xy})\) jumps by \(e^2/h\) each time the Fermi energy \(E_F\) crosses one of the critical energies, that the dissipative conductivity \((\sigma_{xx})\) is zero unless \(E_F = E_{cN}\), and [2–5] that \(\sigma_{xx} = \sigma_{xy} = e^2/2h \equiv \sigma_c\) when \(E_F = E_{cN}\). All these suggestions are supported by experimental [6] studies, except possibly [7] for the predicted universal value of the critical dissipative conductivity \(\sigma_{xx}\). In this Letter, we report on numerical studies of the integer quantum Hall effect which include Coulomb interactions at the lowest consistent level by performing Hartree-Fock calculations in the presence of random disorder potentials. The possible importance of Coulomb interactions is suggested by studies in three-dimensions at zero magnetic field where the metal-insulator transition can be profoundly altered by interactions [8], and by the presence of a Coulomb gap in the integer quantum Hall regime [9,10]. However, general scaling considerations suggest [3,2] that for the quantum Hall effect the critical conductivities are independent of Landau level index \(N\), details of the disorder potential, and of interactions.

The concept of a localization length can be generalized to the case of interacting electrons by defining it in terms of the dependence of the imaginary part of the the one-particle Greens function on spatial coordinates. In the Hartree-Fock approximation, this change is accomplished by replacing the single-particle eigenfunctions in a random potential by the self-consistent Hartree-Fock eigenfunctions. We have studied the localization properties of these eigenfunctions by examining their participation ratios and find no qualitative changes. This result is reasonable since the self-consistent Hartree-Fock Hamiltonian is always equivalent to the single-particle Hamiltonian for a non-interacting system with an
effective random potential which however presumably has smaller long wavelength components because of screening effects. (The exchange potential is non-local but its effect is equivalent [11] to that of a local potential.) Universality of localization properties in the absence of interactions would then seem to imply no change in critical localization properties in the Hartree-Fock approximation.

The influence of interactions on the critical conductivity is difficult to determine, even within the Hartree-Fock approximation. The numerical evaluation of $\sigma_{xx}$ necessitates the inclusion of interaction vertex corrections by summing ladder diagrams [12] in the presence of disorder. We have avoided this considerable obstacle by appealing to the close relationship between $\sigma_{xx}$ and the more easily calculated Thouless numbers which are defined in terms of the dependence of eigenenergies on the boundary conditions of the system. The definition we use for the Thouless number is:

$$g_T(E) \equiv \frac{\langle \delta E \rangle}{\Delta E} \quad (1)$$

where $1/\Delta E = L^2 D(E)$ is the average level spacing calculated from the density of states per unit area $D(E)$, $\langle \delta E \rangle$ is the geometric mean of the eigenvalue differences between periodic and antiperiodic boundary conditions and $L^2$ is the area of the finite system. For non-interacting electrons the Kubo formula for the conductivity can be written as

$$\sigma_{xx} = \pi e^2 L^2 \hbar D^2(E_F) \langle |v_x|^2 \rangle \quad (2)$$

where $\langle |v_x|^2 \rangle$ is an average [13] over matrix elements of the velocity operator between states at energies near $E_F$. It follows from simplifying assumptions about correlations between velocity matrix elements and energy level spacings that

$$\sigma_{xx} = \frac{e^2 \pi g_T(E)}{\hbar} \quad (3)$$

both at [13] $B = 0$ and at [14,15] strong fields. (In the metallic limit a rigorous relationship between $\sigma_{xx}$ and a suitably defined Thouless number can be established [16] at $B = 0$ for non-interacting electrons.) Our numerical calculation is motivated by the expectation
that within the Hartree-Fock approximation a qualitative relationship between \( g_T(E_F) \) and \( \sigma_{xx} \) will survive interactions. Hartree-Fock eigenvalues change with boundary conditions \[13,14\] directly because of the velocity matrix elements between Hartree-Fock eigenfunctions and indirectly because of the change in the exchange potential produced by the changed eigenfunctions. The former effect is related to the ‘bubble’ diagram for the conductivity just as it is without interactions, while the latter effect is related to vertex corrections in the Hartree-Fock approximation for the conductivity. In the integer quantum Hall regime \( D(E_F) \) is strongly suppressed by interactions \[9,10\] just as it is for disordered electrons at \( B = 0 \).

In view of Eq.(2), this suggests that \( \sigma_c \) could be different in interacting electron systems. We have found that a remarkable cancellation occurs in which the change in \( g_T(E_F) \) due to the decrease in \( D(E_F) \) (increase in \( \Delta E \)) is cancelled by increased sensitivity of the Hartree-Fock eigenvalues to boundary conditions. To the precision of our calculations, \( g_T(E_F) \) is unchanged by interactions.

In the model used for our numerical calculations \( N \) electrons in the lowest Landau level are confined to a square of area \( L^2 = 2\pi\ell^2N \phi = N \phi \Phi_0 / B \). \( N \phi \) is the number of single-particle states in the lowest Landau level, \( \Phi_0 \equiv hc/e \) is the magnetic flux quantum and the magnetic length, \( \ell \) is defined by these relations. Our model disorder consists of a randomly located delta-function ‘impurities’ with a random strength uniformly distributed between \(-\lambda \) and \( \lambda \). For this model \[14\] the energy scale which characterises the Landau level width is \( \Gamma = (\lambda^2N\ell^2L^2)^{1/2} \) where \( N \ell \) is the number of impurities. The relative strength of Coulomb interactions and disorder is specified by the parameter \( \gamma = (e^2/\varepsilon \ell) / \Gamma \) (\( \varepsilon \) is the dielectric constant). The solution of the Hartree-Fock equations is greatly facilitated by the fact that matrix elements of the exchange potential \[11\] can be expressed in terms of the electron density rather than the density matrix; technical details of these ‘self-consistent field’ calculations have been explained elsewhere. \[17,9\]

Localization properties were investigated by evaluating participation ratios for self-consistent Hartree-Fock eigenfunctions, \( \varphi_\alpha \):
\[
P_\alpha \equiv \frac{\int d\vec{r} |\varphi_\alpha(\vec{r})|^2}{L^2 \int d\vec{r} |\varphi_\alpha(\vec{r})|^4}.
\] (4)

Figure 1 displays \(\ln P_\alpha\) as a function of \(\nu\) for \(\gamma = 0\) and \(0.4\) and \(\nu_F = 1/2\). (We reserve the symbol \(\nu_F\) for the filling factor at the Fermi level while \(\nu\) denotes the fraction of the density of states below a particular energy. Most of the results reported here were calculated for the case \(\nu_F = 1/2\) for which the extended state occurs at the Fermi energy.) The participation numbers are shifted slightly upward by interactions near \(\nu = 1/2\) and appear to have a cusp in their dependence on \(\nu\) (Note that no cusp appears in the dependence on energy since the linearly vanishing density of states implies that \(|\nu - 1/2| \propto (E - E_c)^2\) in the presence of interactions.) The fact that participation numbers are not greatly changed when the localization length is smaller than our finite system sizes suggests that localization lengths are quantitatively unchanged by interactions. We infer from this that electron-electron interactions do not affect the critical properties of localization in any significant way and that the exponent remains 7/3, in agreement with recent experiments [6].

To examine the behavior of the localization length in more detail we have investigated the system-size dependence of our results. The left panel of Fig. 2 shows the dependence of \(\ln P_\alpha\) on \(\ln N_{\varphi}\) in the absence of electron-electron interactions for \(\nu = 0.5\), \(\nu = 0.066\) and \(\nu = 0.0022\). The right panel of Fig. 2 shows similar results in the presence of electron-electron interactions. Results are calculated at the filling factors \(0.5, 0.081\) and \(0.056\) for \(\nu_F = 1/2\) and at \(0.538, 0.147\) and \(0.019\) for \(\nu_F = 1/5\). For localized states we expect that \(P \propto (\xi/L)^2 \propto \xi^2/N_{\varphi}\). The results displayed in the left and right panels show that states in the Landau level tails are well localized whether the Fermi energy is at \(E_c\) or located in the Landau level tail. The results shown at \(\nu = 0.5\) and \(0.538\) are in a regime where the localization length exceeds the system size. In this regime the participation ratio for non-interacting electrons is known [18] to have a powerlaw dependence on system size, \(P \propto N_{\varphi}^{-\lambda}\) where the exponent \(\lambda\) is related to the multifractal character [19] of the extended states responsible for the anomalous diffusion [20] which occurs when \(E_F = E_c\). The results in Figure 2 show that the self-consistent Hartree-Fock eigenfunctions in the critical regime have
the same fractal properties as the critical wavefunctions for non-interacting electrons. $\lambda$ is
given by $D[2] = 2(1 - \lambda)$ where $D[2]$ is a measure of the structure of the multifractal. From
Fig. 2 we estimate that $D[2] \sim 1.6$ in agreement with earlier work [18,19].

In Figure 3 we show, $\Gamma/\Delta E$, $\delta E/\Gamma$ and $g_T$ as a function of $\nu$ for $\nu_F = 1/2$, $N_\varphi = 72$
and $N_D = 1412$ for both interacting and non-interacting electrons. Our results for non-
interacting electrons are in agreement with earlier [14,15] work. As expected, interactions
lead to an increase in $\Delta E$ near $E_F$ corresponding to the density-of-states suppression. How-
ever, $\delta E$ is also increased and the net result is that the Thouless number is unchanged to
within the accuracy of our calculations. In agreement with Hanna et al. we find that the
peak Thouless numbers appear to already approach their thermodynamic limit for $N_\varphi \sim 50$.
Our numerical results suggest that the peak Thouless numbers are independent of interac-
tions. The Thouless numbers away from $E_F$ are larger for finite-size systems in the presence
of the interactions, presumably because the screening changes the nature of the disorder po-
tential. For bigger system sizes we expect [3] that $\Delta E \propto 1/L$ at $E_F$ which implies that $\delta E$
is also $\propto 1/L$. If Eq.( 3) were exact, the critical conductivity would be proportional to the
peak Thouless number. Our Thouless numbers, both with and without interactions, would
correspond to peak conductivities $\approx 0.2(e^2/h)$. This value is smaller than both the expected
critical conductivity $e^2/2h$ and the self-consistent Born approximation peak conductivity,
$e^2/h\pi$. To understand these quantitative relationships the connection between $g_T$ and $\sigma_{xx}$
must be established under conditions more general that those considered in [16].

In conclusion, we have shown that for the metal-insulator transition of the integer quan-
tum Hall effect Thouless numbers defined in terms of the sensitivity of Hartree-Fock eigenval-
ues to changes in boundary conditions, have a critical value which appears to be independent
of the strength of electron-electron interactions despite large changes in the density of states
at the Fermi energy. Our work adds further motivation to efforts to clarify the connection
between Thouless numbers and conductivities, in the presence of interactions and at a finite
magnetic field.

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REFERENCES

[1] H. Aoki and T. Ando Phys. Rev. Lett. 54, 831 (1985); G. V. Mil’nikov and I. M. Sokolov, Pis’ma Zh. Eksp. Teor. Fiz 48, 494 (1988) [JETP Lett. 48, 536 (1988)]; J.T. Chalker and P.D. Coddington, J. Phys. C21, 2665 (1988); B. Huckestein and B. Kramer, Phys.Rev. Lett. 64, 1437 (1990); Y. Huo and R. N. Bhatt, Phys. Rev. Lett. 68, 1375 (1992); D.Z. Liu and S. Das Sarma, Phys. Rev. B 49, 2677(1994). For a recent review see M. Janssen, O. Viehweger, U. Fastenrath, and J. Hajdu, Introduction to the Theory of the Integer Quantum Hall Effect, (VCH, Weinheim, 1994).

[2] D. H. Lee, Z. Wang, and S. Kivelson, Phys. Rev. Lett. 70, 4130 (1993).

[3] M. P. A. Fisher, G. Grinstein, and S. M. Girvin, Phys. Rev. Lett. 64, 587 (1990).

[4] Y. Huo, R. E. Hetzel, and R. N. Bhatt, Phys. Rev. Lett. 70, 481 (1993).

[5] B.M. Gammel and W. Brenig, Phys. Rev. Lett., 73, 3286 (1994).

[6] H. P. Wei, D. C. Tsui, M.A. Paalanen. and A.M.M. Pruisken, Phys. Rev. Lett., 61, 1294 (1988); S. Koch, K. von Klitzing and K. Ploog, Phys. Rev. Lett., 67, 883 (1991).

[7] P.T. Coleridge, Phys. Rev. Lett. 72, 3917 (1994); D.-H. Lee, S.A. Kivelson, Z. Wang, and S.-C. Zhang, Phys. Rev. Lett. 72, 3918 (1994); P.L. McEuen, et al., Phys. Rev. Lett. 64, 2062 (1990); H.P. Wei, S.Y. Lin, D.C. Tsui, and A.M.M. Pruisken, Phys. Rev. B 45, 3926 (1992).

[8] For a recent review see D. Belitz and T. R. Kirkpatrick, Rev. of Mod. Phys. 66, 261 (1994).

[9] S.-R. Eric Yang and A. H. MacDonald, Phys. Rev. Lett. 70, 4110 (1993). The density of states appears to vanish linearly at the Fermi level in the Hartree-Fock approximation.

[10] For experimental support see R. C. Ashoori, J. A. Lebens, N. P. Bigelow, and R. H. Silsbee, Phys. Rev. Lett. 64, 681 (1990).
[11] A.H. MacDonald and S.M. Girvin, Phys. Rev. B 38, 6295 (1988).

[12] P. Nozieres, Theory of interacting Fermi systems (W.A. Benjamin, New York), 1964.

[13] J. T. Edwards and D. J. Thouless, J. Phys. C 5, 807 (1972).

[14] Tsuneya Ando, J. Phys. Soc. Jpn. 52, 1740 (1983); ibid 53, 3101 (1984).

[15] C.B. Hanna, D.P. Arovas, K. Mullen, and S.M. Girvin, submitted to Phys. Rev. B (1994).

[16] E. Akkermans and G. Montambaux, Phys. Rev. Lett. 68, 642 (1992).

[17] A.H. MacDonald and G.C. Aers, Phys. Rev. B 34, 2906 (1986).

[18] H. Aoki, Phys. Rev. B 33, 7310 (1986).

[19] B. Huckestein, and L. Sweitzer, Phys. Rev. Lett. 72, 713 (1994); W. Pook and M. Janssen, Z. Phys. B82, 295 (1991).

[20] J.T. Chalker and G.J. Daniell, Phys. Rev. B 61, 593 (1988); J.T. Chalker, J. Phys. C 21, L119 (1988).

[21] See M. Janssen, preprint (1994) and work cited therein.
FIGURES

FIG. 1. Participation ratios as a function of $\nu$ for $\gamma = 0$ (crosses) and $\gamma = 0.4$ (squares). The Hartree-Fock results were obtained for the case of $\nu_F = 1/2$ and $N_\phi = 200$ by averaging over $N_D = 16$ disorder realizations.

FIG. 2. Left panel: $\ln P_\alpha$ vs. $\ln N_\phi$ for $E = 0, -0.35, -0.6\Gamma$ in the absence of Coulomb interactions ($\gamma = 0$). The filling factors for these energies are 0.5, 0.066 and 0.0022 respectively. The slope of the lines is $\approx -1.0$ for $E \neq 0$ and $\approx 0.2$ for $E = 0$. Right panel: Same as in the left panel but for $\gamma = 0.4$. Results were calculated at the filling factors 0.5, 0.081 and 0.056 for $\nu_F = 1/2$ and at 0.538, 0.147 and 0.019 for $\nu_F = 1/5$. The Hartree-Fock eigenenergies at these filling factors are $-0.225$, $-0.6$ and $-0.8\Gamma$ for $\nu_F = 1/2$ and $-0.05$, $-0.40$ and $-0.65\Gamma$ for $\nu_F = 1/5$. The square, circle, and triangle symbols show results for $\nu_F = 1/2$ while the pluse, cross, and diamond symbols show results for $\nu_F = 1/5$.

FIG. 3. $L^2D_{\Delta E}(E)/N_\phi$, $\langle \delta E \rangle /\Gamma$ and $0.5g_T\pi^2$ as a function of $\nu$ for $\gamma = 0$ (left panels) and $\gamma = 0.4$ (right panels). $L^2D_{\Delta E}(E)$ is the average number of states in the interval $\Delta E$ around $E$ per disorder realization. Here $N_D = 1412$ and $\nu_F = 1/2$. 