From semiconductors to superconductors: a simple model for pseudogaps

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Abstract. We consider a two dimensional semiconductor with a local attraction among the carriers. We study the ground state of this system as a function of the semiconductor gap. We find a direct transition from a semiconducting to an insulating phase for no doping at a critical value, the single particle excitations being always gapped. For finite doping we find a smooth crossover. We calculate the critical temperature due to both the particle excitations and the Berezinskii-Kosterlitz-Thouless transition.

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1 Introduction

In the standard BCS theory of superconductors the gap \( \Delta \) in the spectrum of quasiparticles is due to the breakdown of gauge symmetry at the phase transition, in much the same way as the gap in a spin density wave state is due to Bragg scattering off the doubled periodicity. The gap disappears with order and consequently \( \Delta = 1 - \frac{T_c}{T} \) is comparable to the transition temperature \( T_c \). Largely motivated by high temperature superconductivity, interest has grown recently in a “pseudogap” situation, in which a somewhat blurred gap \( \Delta \) persists in the normal state, such that \( \Delta \gg T_c \) [1]. There are many models for such a pseudogap, the simplest one being a transition from the BCS limit towards a “Bose Einstein condensation of preformed pairs”: if the attraction between electrons is strong enough, bound pairs (let us say singlet) form first, with a binding energy \( \epsilon_B \), then below a critical temperature \( T_c \) they Bose condense. It was shown long ago that the evolution between the two limits is smooth, with however a very different physics in the strong binding limit [2]: Then the gap \( \Delta \) is just the binding energy, a “molecular” quantity which has nothing to do with symmetry breaking; in contrast \( T_c \) is controlled by center of mass motion of bound pairs, which are not broken: as a result \( \Delta \gg T_c \), a simple minded interpretation of the pseudogap. A similar evolution occurs for the transition from a paramagnetic metal to an antiferromagnetic insulator in half filled band: for a strong repulsion the Mott gap \( \Delta \) exists irrespective of the magnetic order. The latter is only a secondary feature, while it was a key actor in the formation of \( \Delta \) at weak coupling [3].

In order to build an accurate description of realistic systems one needs an interpolation between these two limits. That means including quantum fluctuations of the order parameter \( \Delta \), (mostly of its phase) which are systematically ignored in a mean field BCS calculation. That is a difficult task and the literature on the subject is large [4]. We will not attempt to review it: as usual in perturbation theories it relies on approximations that are not really controlled. We will just make a few simple qualitative comments. Grossly speaking one expects the crossover between the two limits to occur when the (zero temperature) gap \( \Delta \) is comparable to the Fermi energy \( \epsilon_F \): they are the only energies in the problem. That can be checked explicitly for a two dimensional system, in which phase fluctuations are enhanced, leading to the well known Kosterlitz Thouless transition [5]. The resulting transition temperature \( T_c^{KT} \) is controlled by the phase stiffness of the ground state: it can be compared to the mean field \( T_c^{BCS} \) which accounts only for pair breaking. Such an elementary calculation is sketched in section 4: it shows that the crossover \( T_c^{KT} \approx T_c^{BCS} \) occurs when \( \Delta \approx \epsilon_F/5 \). There a bound pair would sit in the middle of a continuum, which makes no sense. Sure in the very strong coupling limit one returns to preformed bound pairs, but there exists a broad intermediate region near the crossover where phase fluctuations are already dominant without any bound state.

In the above model the same interaction is responsible for the gap \( \Delta \) and for Bose condensation. That may well be the case and the future will tell whether high temperature superconductors pertain to such a mechanism or not. However that is not a necessity: one may also imagine that a primary gap \( \Delta_o \) is due to some other mechanism that has nothing to do with superconductivity. That may be for instance a lattice distortion of some sort or a magnetic instability, as suggested by various authors [6]: the start-
ing point is then a semiconductor rather than a metal. If a BCS attraction between electrons is added to such a system, it may lead to superconductivity, even if there were no free carriers to start from: free pairs appear spontaneously if the gain in binding energy is larger than the cost to produce free electrons and holes across the gap. Such a mechanism is actually familiar: it was introduced long ago by W. Kohn in his theory of excitonic insulators. There the Coulomb interaction is repulsive and the bound pair (the exciton) involves an electron and a hole: excitons appear spontaneously if their binding energy is larger than the gap. What we present here is just the reverse situation in which electrons attract: bound pairs involve two electrons or two holes; at $T = 0$ they Bose condense (bound carriers are then time reversed of each other), hence superconductivity. It is clear that superconductivity will modify the real gap $\Delta_g$ - hence the name “primary gap” for $\Delta_p$.

The possibility of superconductivity in a semiconductor is not new: it appeared since the beginning of the BCS “era”, as emphasized by the review article of M.L. Cohen (for subsequent development, see [1]). However only intraband interactions in a doped material were included, the only new feature being the scarcity of carriers. Here we are concerned with the effect of interband interactions, in which carriers in both bands conspire in establishing superconducting order. That was also considered before [10], but in a somewhat different context of interband pairing, while we consider only the usual intraband pairing of time-reversed states. We show in Appendix B that the latter is energetically preferred. (An obvious question is of course whether they can occur together.)

Many features of such a “superconductivity in a semiconductor” are general, in the sense that they do not depend on the physical mechanism responsible for gap opening. We felt that such an interplay of $\Delta_p$ and superconducting order deserves study, the more as it leads to somewhat unexpected simple conclusions. The present paper offers a very naive discussion of that problem. At that stage it does not matter. Let $\rho(\xi)$ be the density of states (per spin): in a first approximation we take $\rho(\xi)$ constant: everything is symmetric around $\xi = 0$. Since the total number of states must remain the same, opening the gap $\Delta_g$ means that the conduction band shifts from $[0, \omega_m]$ to $[\Delta_o, \Delta_o + \omega_m]$ while the valence band goes to $[-\Delta_o - \omega_m, -\Delta_o]$. The conduction and valence bands are just shifted away from each other. $\omega_m$ is a bandwidth which we assume $\gg \Delta_o$ (the gap is small). The choice of a constant $\rho$ is in accordance with our choice of simplicity, but it is not realistic for two reasons (i) at band edges $\rho$ goes as $\xi^{1/2}$ in three dimensions, (ii) in usual situations states that are repelled off the gap remain in its vicinity, in an energy range which has the same order of magnitude $\Delta_o$ (see Fig. 1); here these states are repelled at infinity. We will show in Appendix A that more realistic choices for $\rho(\xi)$ do not change the results qualitatively: they are not worth the price of complications.

We then add a local attraction $U$ between fermions and we look for a BCS superconducting state in which electrons (or holes) are paired in singlet states with zero total momentum, $(k \uparrow) + (-k \downarrow)$ (one the time reversed of the other). If the bottom of the band is made up of several valleys at band edge, that means pairing of electrons between symmetric valleys. We treat the problem within a mean field approximation: the Hamiltonian contains anomalous terms

$$\Delta_{o} c_{k\uparrow} c_{-k\downarrow}$$

where $\Delta$ is the usual order parameter, not to be confused with gaps, which must be determined self-consistently. Because of electron-hole symmetry the chemical potential $\mu$
of our undoped system remains at midgap and the reduced kinetic energy \( \xi_k = (\varepsilon_k - \mu) \) starts at \( \pm \Delta_o \) instead of 0. The algebra is identical to that of BCS except for that shift in the energy integration. In the absence of a gap, \( \Delta_o = 0 \), the order parameter \( \Delta \) would take the regular metallic value \( \Delta_m \) given by the familiar BCS self-consistency condition:

\[
1 = \frac{\rho U}{2} \int_{-\omega_m}^{\omega_m} \frac{d\xi}{\sqrt{\xi^2 + \Delta_m^2}} = \rho U \log \frac{2\omega_m}{\Delta_m}. \tag{1}
\]

It is convenient to characterize the attraction by \( \Delta_m \) instead of \( U \). Note that for a regular metal the superconducting gap is just \( \Delta_g = \Delta \). In the presence of a semiconductor gap \( \Delta_o \), the self-consistency equation for the new order parameter \( \Delta \) becomes:

\[
1 = \rho U \log \left[ \frac{2\omega_m}{\Delta_o + \sqrt{\Delta_o^2 + \Delta^2}} \right]. \tag{2}
\]

In order for the two equations to have a solution with the same \( U \), we must have \( \Delta_m = \Delta_o + \sqrt{\Delta_o^2 + \Delta^2} \), i.e.

\[
\Delta = \sqrt{\Delta_m (\Delta_m - 2\Delta_o)}. \tag{3}
\]

Superconductivity persists with a reduced order parameter until the gap \( 2\Delta_o \) reaches the metallic BCS gap \( \Delta_m \); then the gain in interaction energy due to superconductivity can no longer overcome the cost of kinetic energy in producing free carriers across the gap. Note the complete analogy with excitonic insulators in which condensation produces electron-hole pairs in the presence of a repulsive \( U \). Such a model provides an elementary example of a direct superconductor-insulator transition, which here is sharp at \( T = 0 \). The quasiparticle gap \( \Delta_g \) is just \( \Delta_o \) in the insulating state \( (\Delta = 0) \). In the superconducting state it is \( \sqrt{\Delta_o^2 + \Delta^2} \), i.e. \( \Delta_g = \Delta_m - \Delta_o \). The situation is illustrated on Fig. 3: note that the gap is finite when superconductivity disappears.

The calculation is easily extended at finite temperatures. The order parameter \( \Delta \) becomes a function of \( T \). We need only introduce the fermionic occupation factors in the gap equation:

\[
\int d\xi \frac{\tanh \beta \sqrt{\xi^2 + \Delta^2}/2}{\xi^2 + \Delta^2} = \int_{-\omega_m}^{\omega_m} \frac{d\xi}{\sqrt{\xi^2 + \Delta_m^2}}. \tag{4}
\]

(we retain the zero temperature \( \Delta_m \) as a measure of the attraction \( U \)). The integration path \( P \) takes into account the absence of states in the middle of the band: \( P = [-(\Delta_o + \omega_m), -\Delta_o] \) and \([+\Delta_o, +\omega_m + \Delta_o)] \). For \( \Delta_o = 0 \), Eq. (3) is the standard BCS gap equation, yielding the usual \( \Delta(T) \) and \( T_c \). At the other end the critical temperature goes to zero at the superconductor-insulator transition \( \Delta^*_o = \Delta_m/2 \) (there \( \Delta = 0 \) at zero temperature: there is no need of a \( T_c \) to make it vanish). The behaviour of \( T_c(\Delta_o) \) is shown on Fig. 2.

The approach to the transition is interesting: we subtract the \( \Delta_o < \Delta^*_o \) Eq. (3) from the same equation at the transition, \( \Delta_o = \Delta^*_o \):

\[
\int_{\Delta^*_o}^{+\infty} \frac{1 - \tanh \beta \xi/2}{\xi} d\xi - \int_{\Delta_o}^{\Delta^*_o} \frac{\tanh \beta \xi/2}{\xi} d\xi = 0. \tag{5}
\]

When \( \Delta_o \rightarrow \Delta^*_o \), \( T_c \) vanishes and we can approximate \( \tanh x \) with \( 1 - 2e^{-2x} \). The first integral reduces then to the Exponential integral function and for large values of
\[ \beta \Delta^*_0 \text{ we obtain:} \]
\[ \frac{2 T_c e^{\Delta^*_0/T_c}}{\Delta^*_0} = \frac{\Delta^*_0 - \Delta_0}{\Delta^*_0} \equiv \tau. \]  
(6)

Defining the function \( w(x) \) as the solution of the transcendental equation \( x = w e^w \), we can write \( T_c \) in terms of \( w(x) \)
\[ T_c = \frac{\Delta^*_0}{w(2/\tau)}. \]  
(7)

The function \( w(x) \) is a generalization of the logarithm and for \( x \gg 1 \) has the following expansion:
\[ w(x) = \ln x - \ln[\ln x - \ln[\ln x - \ln \ldots]]. \]  
(8)

The convergence of that expansion is very slow [shown in (9)], but the first term is enough to provide a rough result: the critical temperature vanishes logarithmically in \( \tau [T_c \sim \Delta^*_0/\ln(2/\tau)] \).

Altogether the presence of a semiconductor gap \( \Delta^*_0 \) acts to decrease \( T_c \), a result which is reasonable. But such a decrease should be compared to that of the zero temperature order parameter \( \Delta^*_0 \) which behaves as \( \sqrt{\tau} \): we see that the ratio \( \Delta(T = 0)/T_c \) becomes extremely small when \( \Delta_0 \rightarrow \Delta^*_0 \) in contrast to the usual BCS result. The physical origin of that behaviour is clear: due to the finite gap \( \Delta^*_0 \) the quasi particle excitations are much less efficient in destroying the superconducting order than they would be in the BCS case, and consequently \( T_c \) increases. Such a behaviour is exemplified in Fig. 3 in which we plot the order parameter \( \Delta(T) \) obtained from solving (6) for various values of \( \Delta^*_0/\Delta^*_m \): the elongated low plateau near the transition is striking.

All that discussion is based on a BCS like mean field approximation, in which condensed pair breaking is the only allowed thermal excitation, ultimately responsible for the destruction of superconductivity at \( T_c \). Such an approximation is by no means obvious and it should be kept in mind. In practice there exist other excitations that might be more efficient in destroying superconductivity. In particular, phase fluctuations are gapless excitations and at least for \( \Delta_0 \) near \( \Delta^*_0 \) they are expected to reduce drastically the critical temperature. We will discuss this point more extensively in section 4 for the two-dimensional system. There we find that phase fluctuations seem to be always responsible for the transition to the normal phase, even for \( \Delta_0 = 0 \), a somewhat unexpected conclusion.

### 3 Doped system

We now extend our problem, allowing for a finite doping. We do it because it leads to new interesting physics: the sharp transition at \( \Delta^*_0 \) becomes a smooth crossover, and superconductivity persists at zero temperature for any value of \( \Delta_0 \). Beyond that crossover, the superconducting behaviour is very unconventional over a broad range of parameters, in contrast to the undoped case in which queer behaviour occurred only at a specific transition point. In that respect the doped system is much more generic.

Starting from the non interacting, metallic system at \( T = 0 \) we suppose that the upper band is populated up to an energy \( \mu_m = \varepsilon \), referred to midgap. When the gap \( \Delta_0 \) is restored, the "semiconductor" is doped by an amount \( \delta N = 2 \rho \varepsilon \) (referred to the filled valence band). In the metallic state \( \Delta_0 = 0 \), chemical potential and superconductivity are unaffected by each other (the BCS picture is symmetric with respect to Fermi level): \( \mu_m \) remains equal to \( \varepsilon \). In the opposite limit of an insulator with no superconductivity, \( \mu = \Delta_0 + \varepsilon \) is pushed into the conduction band, a familiar feature of degenerate semiconductors. In between we expect no sharp superconductor-insulator transition: a weak superconducting order due to the few free carriers in the conduction band should persist at any \( \Delta_0 \). Keeping \( \Delta_m \) and \( \varepsilon \) fixed, we thus expect the zero temperature chemical potential \( \mu \) and order parameter \( \Delta \) to depend on \( \Delta_0 \) as showed respectively on Fig. 3 (a) and (b). The transition around \( \Delta_0 = \Delta_m/2 \) is a crossover, but with a large change of \( \mu \).

Let us assume an arbitrary temperature \( T \). In order to fix the two parameters \( \mu(T) \) and \( \Delta(T) \) we need two conditions. One is the self consistency equation for the order parameter, which is the same as (9) but for the introduction of the chemical potential in the expression for the quasiparticle energy:
\[ \int_\rho \frac{\tanh \beta E(\xi)/2}{E(\xi)} d\xi = \int_{\omega_m}^{\omega_m} \frac{d\xi}{\sqrt{\xi^2 + \Delta_m^2}}, \]  
(9)

where \( E(\xi) = \sqrt{(\xi - \mu)^2 + \Delta_m^2} \). The other one is the conservation of particle number, stating that \( \delta N \) is unchanged:
\[ 2\rho \int_\rho d\xi \left[ \nu^2(\xi) + f_p(E(\xi)) \frac{\xi - \mu}{E(\xi)} \right] - 2\rho \int_{-\omega_m}^{\omega_m} d\xi = \delta N, \]  
(10)
3.1 Ground state properties

We begin by considering the zero temperature limit. In this case integrations are elementary and we obtain:

\[
\begin{aligned}
\Delta_m^2 &= \left[ \Delta_o - \mu + \sqrt{ (\Delta_o - \mu)^2 + \Delta^2 } \right] \cdot \left[ \Delta_o + \mu + \sqrt{ (\Delta_o + \mu)^2 + \Delta^2 } \right] \\
2\varepsilon &= 2\mu - \sqrt{ (\Delta_o + \mu)^2 + \Delta^2 } + \sqrt{ (\Delta_o - \mu)^2 + \Delta^2 } \\
&= 2\mu - \sqrt{ \Delta^2 + \Delta_o^2 } + \sqrt{ \Delta^2 + \Delta_o^2 } \\
&= 2\mu - 2\Delta_o + 2\mu = 4\Delta_o \\
\end{aligned}
\]

System (12) fixes $\mu$ and $\Delta$ as a function of the gap $\Delta_o$ and doping $\delta N = 2p\varepsilon$. Once they are known, the overall gap for elementary excitations is either $\Delta$ (if $\mu > \Delta_o$) or $\sqrt{ (\Delta_o - \mu)^2 + \Delta^2 }$ (if $\mu < \Delta_o$). Numerical solution of these equations is easy and it is shown in Fig. 4. Simple limits can be assessed analytically as long as $\varepsilon \ll \Delta_m$, shedding some light on the physical meaning of these results.

Consider first the “conducting” side. Here free carriers extend deep into the valence and conduction bands: the chemical potential for small doping will remain close to midgap. We can therefore expand the second equation of (12) into

\[
\varepsilon = \mu \left[ 1 - \frac{\Delta_o}{\sqrt{\Delta_o^2 + \Delta^2}} \right].
\]

Doping is a small perturbation and we can use the undoped result for $\Delta$: we thus find

\[
\mu = \frac{\Delta_m - \Delta_o}{\Delta_m - 2\Delta_o} \varepsilon + O(\varepsilon^2).
\]

The approximation breaks down when $\varepsilon/\Delta^*_o \sim \tau$. The crossover is very close to the superconductor-insulator sharp transition found when $\delta N = 0$. Substituting that first order result for $\mu$ into the first of (12) we find the correction to $\Delta$:

\[
\Delta = 2\Delta^*_o \sqrt{\tau} + \left( \frac{\varepsilon}{\Delta^*_o} \right)^2 \left( \frac{1 + \tau}{8\tau^{3/2}} \right) \Delta^*_o + O(\varepsilon^4).
\]

As long as $\Delta_o < \Delta^*_o$ doping is a minor perturbation.

In the opposite limit, we first note that an insulating state at $T = 0$ is impossible. It would mean a $\Delta = 0$ solution to the gap equation, i.e. $\Delta_o^2 = 4(\Delta^2 - \mu^2)$, which in turn implies $\mu < \Delta_o$. But with no $\Delta$ that means $\delta N = 0$. Thus superconductivity must either extend all the way, or display a first order transition: a sharp second order transition to the insulating state is precluded. We now show that the present model has a smooth crossover.

Deep in the insulating limit, $\Delta_o > \Delta^*_o$, the role of equations is interchanged. The order parameter $\Delta$ will turn out to be small: consequently the gap equation which used to fix $\Delta$ now fixes $\mu$

\[
\Delta_m^2 = 4(\Delta^2 - \mu^2) \Rightarrow \mu = \sqrt{\Delta_m^2 - \Delta^2}. 
\]
Such an expansion is valid as long as \((\Delta_o - \mu) \gg \Delta\), which implies
\[
\frac{\Delta_o^2}{2\Delta_o} \gg \Delta
\]  
(17)

\(\mu\) is still inside the band gap \(\Delta_o\). Granted \(\mu\), the other equation for \(\delta N\) fixes the order parameter \(\Delta\)
\[
\varepsilon = \frac{\delta N}{2\rho} = \frac{2\Delta^2}{\Delta_m^2} \sqrt{\Delta_o^2 - \Delta_o^2} \Rightarrow \Delta = \frac{\Delta_m \sqrt{\varepsilon^2 / 2}}{[\Delta_o^2 - \Delta_o^2]^{1/4}}.
\]  
(18)

As surmised \(\Delta\) is small for small doping \(\varepsilon\), which justifies our calculation a posteriori. The condition \((7)\) now reads
\[
\varepsilon \ll \frac{\Delta_o^2}{2\Delta_o}.
\]  
(19)

Doping should remain small (or the gap \(\Delta_o\) not exceedingly large).

When the gap exceeds that range the expansion in powers of \(\Delta\) is no longer warranted. But as long as doping \(\varepsilon\) is small compared to \(\Delta_o\), we expect the chemical potential to remain close to \(\Delta_o\), either below (inside the gap) or above (inside the continuum). We may then assume safely that \((\Delta_o + \mu) \approx 2\Delta_o \gg \Delta\). The two basic equations \((12)\) simplify into
\[
\begin{align*}
\Delta_o^2 &= \left[ \Delta_o - \mu + \sqrt{(\Delta_o - \mu)^2 + \Delta^2} \right] \cdot \Delta_o \\
2\varepsilon &= \mu - \Delta_o + \sqrt{(\Delta_o - \mu)^2 + \Delta^2}
\end{align*}
\]  
(20)

Multiplying one by the other we obtain the very simple result
\[
\Delta^2 = \frac{2\varepsilon \Delta_o^2}{\Delta_o}
\]

which joins smoothly with our former expansion \((13)\), and which remains valid throughout the crossover \([\Delta_o - \mu] \sim \Delta\). Similarly we may eliminate the square roots from these two equations, which yields the chemical potential \(\mu\)
\[
\mu = \Delta_o + \varepsilon - \frac{\Delta_o^2}{2\Delta_o}.
\]  
(21)

For very large gaps \(\Delta_o \gg \Delta_o^2 / \varepsilon\), the last term is negligible and we recover the usual result for a doped normal semiconductor, \(\mu = \Delta_o + \varepsilon\).

Let us explore these two regimes in some more detail. When the chemical potential lies inside the gap \(\Delta_o\) the real gap \(\Delta_g\) is just \((\Delta_o - \mu) \gg \Delta\), of order \(\Delta_o^2 / 2\Delta_o\) as \(\mu\) approaches the band edge. The relevant ratio is
\[
\frac{\Delta}{\Delta_g} = \sqrt{\frac{8\varepsilon \Delta_o}{\Delta_o^2}} \ll 1.
\]  
(22)

Note the strong similarity between such a regime and the Bose condensation of preformed pairs \([2]\). In the latter case the chemical potential \(\mu\) is close to the energy of bound pairs, \(\varepsilon_b / 2\) per fermion and therefore it lies inside the gap. Here the reduced kinetic energy \((\xi - \mu)\) likewise never vanishes: \((\Delta_o - \mu)\) plays the role of \(\varepsilon_b / 2\). The resulting behaviour is similar in both cases, with a completely different physics. Here the hierarchy of energy scales is
\[
\varepsilon \ll \Delta \ll \Delta_g
\]

(Indeed one has \(\Delta^2 = 4\varepsilon \Delta_g\)). In the opposite limit the chemical potential enters the conduction band: then the real gap \(\Delta_g\) is just the order parameter \(\Delta\). The relevant ratio is that of \(\Delta\) to the Fermi energy \(\varepsilon\) measured from the bottom of the band
\[
\frac{\Delta}{\varepsilon} = \sqrt{\frac{2\Delta_o^2}{\Delta_o}}.
\]

Except for a numerical factor \(4\) we see that \(\Delta \ll \varepsilon\); we recover an ordinary BCS regime in which superconducting pairing is a small perturbation.

What is unusual is the behaviour of the order parameter \(\Delta \approx \sqrt{\varepsilon}\) in that “insulating regime”, for small doping \(\varepsilon\) and moderate semiconductor gap \(\Delta_o\). Such an unusual dependence \(\Delta \approx \sqrt{\varepsilon}\) is due to the interplay of doping and superconductivity: (i) extrinsic carriers are needed in order to start superconductivity, but (ii) the latter does add further carriers, thereby modifying \(\Delta\). That result for the dependence on doping away from \(\Delta_o^2\) should be compared with the result \(\Delta = 2\Delta_o \sqrt{T}\) for the undoped system close to \(\Delta_o^2\). The two variables \(\tau\) and \(\varepsilon\) act in the same way in driving the system away from the \(T = 0\) critical point (we will find this behaviour again for the critical temperature).

Analytic calculations are possible only in limiting cases. In the crossover region one must resort to numerical solutions, which are quite straightforward. The result is shown in Fig. 1 where system \((12)\) is solved numerically for the values of \(\varepsilon\) indicated. Both \(\Delta\) and \(\mu\) clearly display the transition from a “metallic” to a “semi insulating” regime. In the latter, \(\mu\) crosses the band edge when \(\varepsilon \sim \Delta_o^2 / 2\Delta_o\), separating Bose Einstein like and BCS like behaviours. Note that the chemical potential crosses the band edge \(\Delta_o\) twice when \(\varepsilon\) is smaller than a critical value (see also the discussion after Fig. 1).

### 3.2 Finite temperature behaviour

Having a complete view of the zero temperature doped system, we consider now the effect of a finite temperature. In particular we want to investigate the critical temperature of the system. Setting \(\Delta = 0\) into Eqs. \((1)\) we obtain:
\[
\int_{\Delta_o}^{\infty} d\xi \left[ \frac{\tanh \beta_c (\xi - \mu_c) / 2}{\xi - \mu_c} + \frac{\tanh \beta_c (\xi + \mu_c) / 2}{\xi + \mu_c} \right] = 2 \int_0^{\Delta_o} \frac{d\xi}{\sqrt{\xi^2 + \Delta_m^2}}
\]  
(23)

and
\[
\varepsilon = 2\mu_c + \frac{1}{\beta_c} \log \left[ 1 + e^{\beta_c (\Delta_o - \mu_c)} \right].
\]  
(24)
These equations can be easily solved numerically and the results are shown in Fig. 5 for different values of $\varepsilon$ including the undoped case. As expected from the zero temperature analysis, the critical temperature also exhibits a smooth crossover from a large to a small value, and it never vanishes for finite doping. The region $\Delta_o < \Delta^*_o$ is not much different from a standard BCS superconductor: a comparison of Fig. 5 with Fig. 4 (a) clearly shows that $T_c$ is proportional to $\Delta(T = 0)$ (with the same BCS factor for $\Delta \to 0$).

A new and interesting behaviour is instead found when $\Delta_o > \Delta^*_o$. In this region the critical temperature appears to be much larger than the order parameter $\Delta(T = 0)$ when $\varepsilon/\Delta_m \ll 1$. This point can easily be verified analytically. In this limit we can substitute the ground state chemical potential $\mu_o$ in Eq. (24). Since the critical temperature vanishes for $\varepsilon = 0$ while $\Delta_o - \mu_o > 0$ the exponentials in Eq. (24) are large and we can expand the expressions. At leading order in $\exp((\mu_o - \Delta_o)/2T_c)$ we obtain the following equation for $T_c$:

$$T_c = \varepsilon e^{\frac{\mu_o - \Delta_o}{x_c}}.$$  \hspace{1cm} (25)

Again the solution can be written in terms of $w(x)$:

$$T_c = \frac{\Delta_o - \mu}{w \left[ \frac{\Delta_o - \mu}{\varepsilon} \right]} \sim \frac{\Delta_o - \mu_o}{\ln \left[ \frac{\Delta_o - \mu_o}{\varepsilon} \right]}.$$  \hspace{1cm} (26)

It is then clear that the ratio $T_c/\Delta(T = 0)$ diverges for $\varepsilon$ going to zero. One should note that the behaviour found for both $\Delta(T = 0)$ and $T_c$ as a function of $\tau$ is exactly the same of that found for these two quantities as a function of $\varepsilon$. The origin is the same since the gap at $T_c$ in this case becomes $\Delta_o - \mu$.

We report in Fig. 6 the ratio between the critical temperature $T_c$ and the other two relevant energy scales, the zero temperature order parameter $\Delta(T = 0)$ and excitation gap $\Delta_g(T = 0)$ for different values of $\varepsilon/\Delta_m$ going from 0 to $10^{-1}$. For small $\varepsilon/\Delta_m$ the familiar BCS results of $\varepsilon^2/\pi \approx 0.57$ is recovered, while for $\Delta_o > \Delta_m/2$ a totally different behaviour is found. The critical temperature appears to be in between the other two quantities (for $\varepsilon \ll \Delta_m$)

$$\Delta(T = 0) < T_c < \Delta_g(T = 0).$$  \hspace{1cm} (27)

This can be easily checked analytically, since for $\varepsilon$ going to zero $\Delta_o$ stays finite $[\Delta_g \to \Delta_o - \sqrt{\Delta^2 - \Delta^2_o/4}]$, while $T_c$ vanishes logarithmically and $\Delta(T = 0)$ vanishes like $\sqrt{\varepsilon}$. Thus at mean field level we have found a rather peculiar form of superconductivity, whose behaviour is totally different from the BCS one. We note that the BCS behavior is recovered also for large values of $\Delta_o$ when the chemical potential lies in the conduction band. In Fig. 6 all the lines will join the BCS value 0.57 for fixed $\varepsilon$ and large enough $\Delta_o$. If $\varepsilon/\Delta_m$ is small enough, the parameter $\Delta_o$ drives the system through four different regimes. For very small $\Delta_o$ the system is in a two-band BCS regime. Increasing $\Delta_o$ the chemical potential enters the gap and the smoothed transition from a normal superconductor to a “semi-conductor” superconductor takes place. When $\Delta_o > \Delta^*_o$ and the chemical potential is still in the gap, the carriers in the upper band act as a strong coupling superconductor, while the lower band is nearly decoupled. For even larger $\Delta_o > \Delta^*_o/2$ the chemical potential lies again in the valence band and a BCS behaviour is recovered for the upper band. The schematic phase diagram is plotted in Fig. 6 where the region of positive and negative $\mu - \Delta_o$ are identified on the plane $\Delta_o$, the path described above correspond to the dashed line.

All that discussion is based on a mean field, BCS like formulation. We should now ask whether such an approximation is warranted. In the next section we inquire about the effect of phase fluctuations in two dimensions, which are known to provide an alternate mechanism for the crit-
BCS and phase fluctuations \[ J^{MF} \] vanishes at the mean field transition temperature \( T^{MF}_c \), hence two possibilities: (i) \( J^{MF}(0) \gg T_c \): phase fluctuations are essentially negligible except very close to \( T_c \); then \( T^{KT}_c(T) \) becomes comparable to \( T \) and BCS turns into a Kosterlitz-Thouless regime; the ultimate transition is definitely in the Kosterlitz-Thouless universality class, but the relevant range is narrow and the real \( T_c \) is very close to \( T^{MF} \). Basically BCS is correct. (ii) \( J^{MF}(0) \ll T_c \): then the Kosterlitz-Thouless transition occurs well before pair breaking fluctuations become relevant; but for renormalization corrections \( T_c \) is \( T^{KT}_c(0) \).

What we need is an estimate of the stiffness \( J(0) \). That is easily achieved considering a superfluid flow in which pairs are condensed with center of mass momentum \( Q/2 \) instead of \( 0 \). The corresponding phase of the order parameter is \( \phi = Q \cdot r \) and its gradient is \( Q \). In order to calculate the energy of that flow we need a specific model of the band structure in momentum space: the knowledge of the density of states is not enough. Let us first consider the standard model of a 2d free electron gas, with no gap at all. The Fermi surface is a sphere with radius \( k_F \), the density of electrons with either spin is \( N = k_F^2/2\pi \). For a parabolic dispersion law, with mass \( m \), the center of mass and relative motion of a pair are completely decoupled: the cost in kinetic energy of the superfluid flow is \( \hbar^2 Q^2/8m \). The phase stiffness is consequently

\[ J = \frac{Nk^2}{4m} = \frac{\varepsilon_F}{4\pi} \]

The corresponding transition temperature is \( T^{KT}_c = \varepsilon_F/8 \), comparable to the Fermi energy, but noticeably smaller due to the accumulation of numerical factors. That must be compared to the BCS \( T_c \approx 0.57 \Delta \); as mentioned in the introduction the crossover occurs when \( \Delta \) is about a fifth of \( \varepsilon_F \), making the idea of a preformed bound state meaningless. Usual low temperature superconductors are way below that crossover and phase fluctuations are accordingly negligible.

We now turn to our gapped semiconductor: we need to specify the detailed form of the two bands. The simplest guess assumes valence and conduction bands that have a well defined extremum in the Brillouin zone, whether at the same place (direct gap), or at different places (indirect gap). Near the band edge the dispersion is parabolic, characterized by two (different) masses (the values of these masses will not enter in the final expression of the critical temperature). In a 2d system the density of states at band edge is finite, and the assumption of a constant \( \rho \) is not crucial: up to that point, our model raises no difficulty. It does however when we try to calculate the phase stiffness \( J \). Before attempting explicit calculations, let us explain the point qualitatively. In order to set up a superfluid flow in a free electron gas, we shifted the Fermi surface in momentum space, thereby obtaining a superfluid density \( n_s \approx N \) and a phase stiffness \( \sim \varepsilon_F \). If we do the same in a filled band we get nothing! A filled band is a single state with no degree of freedom, and shifting the distribution \( n_k = 1 \) produces the same state. Hence the

4 Phase stiffness and Kosterlitz–Thouless fluctuations

We limit our discussion to a two dimensional system for which they are known to be crucial, leading to the so called Kosterlitz-Thouless transition \[ J \]. Although there is no strict long range order in 2d, the low temperature decay of the order parameter is algebraic up to a sharp critical temperature \( T^{KT}_c \) at which the decay becomes exponential: such a transition from a "quasi superfluid" phase to a normal phase is due to the unbinding of vortex pairs. In the most naive approximation one may assume that the modulus of the order parameter is constant, the only relevant variable being its phase \( \phi(x) \). The corresponding fluctuations are controlled by the phase stiffness \( J \) defined by the following term in the Hamiltonian

\[ \Delta H = \frac{J}{2} \int d^2x |\nabla \phi(x)|^2 \]  

(28)

\( J \) measures the resistance of the system to phase gradients. From \( J \) one infers in 2d the well known exact critical temperature \[ T^{KT}_c \]

\[ T^{KT}_c = \frac{\pi}{2} J. \]  

(29)

The stiffness that enters \[ (29) \] is actually temperature dependent, \( J(T) \). That dependence occurs both at the mean field level, through pair breaking thermal fluctuations, and via thermal fluctuations of the phase itself, which can be described within a renormalization scheme. We disregard the latter and we discuss briefly the mean field \( J^{MF}(T) \) which provides a neat picture of the competition between

\[ \mu > \Delta_0 \]

Fig. 7. Phase diagram for the chemical potential. The shaded part is the region with \( \mu < \Delta_0 \). The thick line represents the insulating region of the phase diagram, for any other value of \( \varepsilon \) and \( \Delta_0 \) the system is superconducting. The dashed line represents the path in the phase diagram described in the text.
obvious conclusion: a filled band has no stiffness. $J$ can only result either from doping or from the order parameter $\Delta$ that creates free carriers even if there were none to start from. As a result the phase fluctuation $T^{K_T}$ is severely reduced: it becomes of order $\Delta$, comparable to the BCS pair breaking $T_c$ in every case. The difference with free carriers is dramatic, due to the narrow range of free carriers in momentum space.

One more general comment is in order: is it possible to go smoothly from the semiconductor to the metal? Put another way, does a reduction of the gap $\Delta_n$ to 0 drive us back to a regular Fermi liquid metal? As long as we were dealing with the density of states $\rho$ only, the answer was "yes" in 2d: where free carriers sat in the Brillouin zone was of no concern. When calculating the phase stiffness $J$ the answer is "no". The standard situation is an indirect gap which closes when the minimum of the conduction band is lower than the top of the valence band. Thereafter free carriers exist even in the normal state, but they remain localized near the corresponding band extrema. The insulator with no Fermi surface will not turn suddenly into a metal with a large Fermi surface (a sphere for free carriers). Instead a small Fermi surface will appear near the band extrema, responsible for the very small superfluid density (and phase stiffness). Such a behaviour should be contrasted with the 1d case, in which a smooth transition to a commensurate spin density wave at zone edge: the resulting antiferromagnetic order produces Bragg scattering and opens a gap $\Delta_n$. Near the gap the band dispersion is hyperbolic, returning to the normal metal behaviour when the distance $(\varepsilon - \varepsilon_0)$ to the band bottom $\varepsilon_0$ exceds $\Delta_n$: in such a case the transition metal $\to$ semiconductor is smooth. But then the physics is completely different, as usual in 1d geometries. We see no realistic 2d system that could provide a smooth transition and a large superfluid density (that would mean a gap spread over the whole Fermi surface and translating together with it).

We now turn to the calculation of $J$, putting all the pairs in a state of total momentum $Q$ and looking at the variation of the total energy. The BCS calculation for the energy is unmodified but for the following transformation

$$\xi_k \to \Xi_k(Q) = (\xi_{k+Q/2} + \xi_{-k+Q/2})/2.$$  

where $\xi_k$ is the dispersion relation of the band. This transformation changes only the kinetic energy: in our case we have for the two bands:

$$K(Q) = 2 \sum_{k \in c} u_k^2 \Xi_k(Q) - \mu + 2 \sum_{k \in c} v_k^2 \Xi_k(Q) - \mu.$$  

Since the shift of $Q$ of a filled band does not change the kinetic energy of the band we can write:

$$K(Q) = 2 \sum_{k \in c} (\xi_k - \mu) - 2 \sum_{k \in c} u_k^2 \Xi_k(Q) - \mu + 2 \sum_{k \in c} v_k^2 \Xi_k(Q) - \mu.$$  

We use at this point the hypothesis that in the region where $u_k$ and $v_k$ are significantly different from zero the two bands are parabolic. We can thus expand $\Xi$ for small $Q$:

$$\Xi_k^c(Q) \simeq \xi_k^c - \frac{Q^2}{8m^c}$$  

and

$$\Xi_k^v(Q) \simeq \xi_k^v + \frac{Q^2}{8m^v}$$  

where $m_{c/v}$ are the effective masses of the two bands. The expression for $K$ becomes:

$$K(Q) = K(0) + \frac{Q^2}{4m^c} \sum_{k \in c} u_k^2 + \frac{Q^2}{4m^v} \sum_{k \in c} v_k^2.$$  

By introducing the 2-dimensional density of states per spin: $\rho = m/2\pi$ we obtain:

$$K(Q) - K(0) = \frac{Q^2}{8\pi} \left[ \int_{-\Delta_n}^{\Delta_n} u(\xi)^2 \xi d\xi + \int_{\Delta_n}^{+\infty} v(\xi)^2 \xi d\xi \right],$$  

and the masses simplify from the expression. We thus obtain for $J$:

$$J = \int_{\Delta_n}^{+\infty} d\xi \left[ 2 \frac{\xi + \mu}{(\xi + \mu)^2 + \Delta^2} - \frac{\xi - \mu}{(\xi - \mu)^2 + \Delta^2} \right],$$  

and since the integration is elementary we can write a closed expression for $T^{K_T}$

$$T^{K_T} = \sqrt{\frac{(\Delta_n - \mu)^2 + \Delta^2}{16}} + \sqrt{\frac{\mu^2}{\Delta_n} + \Delta^2} - 2\Delta_n.$$  

We comment this result in the two cases of $\varepsilon = 0$ and of finite doping.

In the undoped case the expression simplifies even more. In the superconducting region, $\Delta_n < \Delta_0$, we find

$$T^{K_T} = (\Delta_m - 2\Delta_n)/8.$$  

$T^{K_T}$ is thus comparable to the order parameter $\Delta$, in contrast to a BCS superconductor, a spectacular consequence of the severe reduction in phase stiffness. The behavior near the transition point is easily understood. There the gap is large as compared to $\Delta$ and the density of free carriers $v^2$ lies in the tail of the BCS distribution: it is of the order $\rho\Delta^2/\Delta_m$. The stiffness is accordingly of the same order, leading to (88). Our conclusion is that the transition to the normal state is always due to unbinding of vortices, in sharp contrast to the weak coupling BCS limit.

We consider now the case of large $\Delta_n$. We can use the approximation used in (21) and keeping the lowest order in $\Delta/\Delta_n$ in Eq. (17) we find:

$$T^{K_T} = \frac{\varepsilon}{8} \left[ 1 + \frac{\Delta_n^2}{2\Delta^2} \right].$$  

As expected $T^{K_T}$ is proportional to the stiffness of a single band with Fermi energy = $\varepsilon$, this result is true for any sign of $\mu - \Delta_n$ provided $\mu \sim \Delta_n \gg \Delta$. The factor of proportionality is greater than one indicating that the lower band is contributing to the stiffness. This result implies that even
Conclusions

Superconductivity is usually concerned with a regular metal in which free carriers exist at the Fermi level. We show here that superconductivity can also occur in semiconductors as soon as carriers in either band feel an attraction. The semiconductor is characterized by a gap \( \Delta_o \), the origin of which is not specified. The attraction is characterized by the superconducting gap \( \Delta_m \) that would exist in a regular metal. The relevant parameters are the ratio \( \Delta_o/\Delta_m \) and the doping of the semiconductor.

Superconductivity may happen even in an undoped semiconductor in which no free carriers exist in the normal state at \( T = 0 \), as soon as the gain in superconducting energy \( \sim \Delta_m \) exceeds the cost in producing carriers \( \sim \Delta_o \). Such a situation is completely equivalent to excitonic insulators, in which electrons repel instead of attracting: then electrons and holes attract and they form bound excitons. If the binding energy is larger than the gap these excitons appear spontaneously. In our model we find a sharp transition at a specific value \( \Delta_o^* \sim \Delta_m \) (the precise value depends on details of the density of states). The ground state is superconducting if \( \Delta_o < \Delta_o^* \), insulating if \( \Delta_o > \Delta_o^* \). Near the transition it departs from the familiar BCS regime: the order parameter \( \Delta \), the real quasiparticle gap \( \Delta_q \) and the critical temperature \( T_c \) are such that

\[
\Delta \ll T_c \ll \Delta_q
\]

That may be viewed as a “pseudogap” behaviour, monitored by the competition between semiconducting and superconducting features.

In the undoped case only the vicinity of the transition point \( \Delta_o^* \) is unusual. This is no longer true when the semiconductor is doped: then the sharp transition becomes a crossover and superconductivity extends all the way. Nothing very spectacular happens when \( \Delta_o < \Delta_o^* \): the effect of a small doping is minor. In contrast the “semimetallic region” \( \Delta_o > \Delta_o^* \) is extremely unusual: that is the main result of our paper. Past the crossover the chemical potential first remains inside the band gap. We then find a superconducting state very reminiscent of the Bose Einstein condensation of preformed pairs. The order parameter \( \Delta \) is proportional to the square root of doping, and the characteristic energies display the same pseudogap ordering as in [2]. That behaviour prevails until the gap reaches \( \Delta_o^{*2}/\varepsilon \), where \( \varepsilon \) is a typical Fermi energy in the conduction band. Then the chemical potential enters the conduction band and, after a transient region, the system returns to a single band BCS superconductor that involves only extrinsic carriers. The range of anomalous behaviour is very broad, and thus generic.

The main body of the paper is concerned with the simplest conceivable model, with a constant density of states \( \rho \) (a sensible choice for a 2d system), and a mean field treatment “à la BCS”, both at zero and at finite temperatures. The only new feature is the presence of the semiconductor gap \( \Delta_s \). The problem is then studied both analytically and numerically. Both simplifications can be questioned: we now address the issues in succession.

A constant density of states is not realistic for two reasons: (i) it ships the gap states to infinity, while standard mechanisms for a small gap leave the states close to where they originated from, in an energy range \( \sim \Delta_o \)\( \Delta_o/\Delta_m \). Near the transition the quasiparticle gap \( \Delta_q \) is very broad, and thus generic.

(ii) in 3d the density of states \( \rho(\xi) \) starts as \( \sqrt{\xi} \) near band edge. Improvements are explored in Appendix A, where other densities of states are tried, still within a mean field picture. Numerical results show that the qualitative evolution remains the same. In the undoped case the sharp superconductor-insulator transition is just shifted, the square root behaviour of \( \Delta \) being unaffected. In the doped case the calculation can be carried out explicitly in the anomalous region, if

\[
\Delta \ll \Delta_o - \mu \ll \Delta_o
\]

Free carriers at \( T = 0 \) are then mostly in the conduction band and their density \( v_F^2 \) is small, of order \( \Delta^2 \). Our main result, \( \Delta \) proportional to the square root of doping, is maintained (with modified coefficients). We conclude that the physics described in this paper is not too sensitive to details of \( \rho(\xi) \). We did not consider the case of a Fermi surface close to a Van Hove singularity, as found in a nested 2d system. That could definitely be done.

Departures from mean field behaviour is a more delicate problem. Thermal fluctuations of the phase of \( \Delta \) are controlled by the phase stiffness \( J \). In ordinary metals \( J \) is comparable to the Fermi energy; phase fluctuations are small and a mean field picture is correct except very close to \( T_{MF} \) (where fluctuations turn to a Kosterlitz-Thouless transition in 2d). In our semiconductor we show that \( J \) is only due to free carriers (it would be \( \equiv 0 \) for a filled band). \( J \) is comparable to the Fermi energy and thus small: it follows that the phase fluctuation mechanism for \( T_c \) always prevails (at least in 2d and for \( \Delta_o < \Delta_o^{*2}/\varepsilon \)) that is the second unexpected result of our analysis. Simple considerations are enough in order to provide orders of magnitude, but a real quantitative theory is still lacking.

Granted these difficulties, we only tackled the problem, our goal being to emphasize simple physical ideas. Many important questions remain open. For instance one should worry about the origin of the semiconducting gap \( \Delta_o \). If it arises from another instability (lattice distortion, magnetism, etc...), does superconductivity react on that primary instability? If it does we face a coupled problem, in which \( \Delta \) depends on \( \Delta_s \) and in reverse \( \Delta_s \) depends on \( \Delta \). A self consistent calculation is required, and it may lead to unexpected fixed points [12]. Another issue is the relevance of interband pairing, as envisaged in [10]. Even if it is not the primary effect, it may affect the quantitative results. (It seems that such am interplay of intra and interband pairing is crucial in understanding the smooth
transition of the stiffness when the gap $\Delta_o$ closes to zero in a 1D band.) These remarks open exciting possibilities, which we did not approach. As it stands our paper is only an exploration.

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**A Non constant density of states**

In this appendix we consider a more realistic form for the density of states, in which the states removed from the energy region $[-\Delta_o, \Delta_o]$ remain in the vicinity of the edge. We explore two simple cases: a discontinuous density of states (typical of 2d system) and a density of states that vanishes at $\Delta_o$ with a square root of $\xi - \Delta_o$ (typical of a 3d case). These two possibilities are shown in Fig. 1. We solve the equations numerically showing that the qualitative behaviour found for constant density of states is not changed.

Eqs. (10) and (11) can be easily generalized when the density of states is not constant:

\[
\begin{align*}
\int_{P'} \rho(\xi) \left[ v^2(\xi) + f \frac{|E(\xi)|}{E(\xi)} \xi \right] d\xi &= \int_{-\epsilon}^{\epsilon} \rho_m(\xi) d\xi, \\
\int_{P'} \frac{\tanh \beta E(\xi)/2}{E(\xi)} d\xi &= \int_{-\omega_m}^{+\omega_m} \frac{\rho_m(\xi)}{\sqrt{\Delta_m^2 + \xi^2}} d\xi,
\end{align*}
\]

(40)

In this equations $\rho_m(\xi)$ and $\rho(\xi)$ are the densities of states of the metallic and insulating phase respectively. The integration path is now $P' = [-\omega_m, -\Delta_o]$ and $[\Delta_o, +\omega_m]$, since we conserve the number of states by deforming $\rho$ and not by shifting the whole band. The function $\rho(\xi)(\geq 0)$ is thus constrained by the conservation of the number of states when passing from the metallic to the insulating phase:

\[
\int_{P'} \rho(\xi) d\xi = \int_{-\omega_m}^{+\omega_m} \rho_m(\xi) d\xi.
\]

(41)

It is clear that in order to fulfill Eq. (41) $\rho(\xi)$ must depend on $\Delta_o$ too.

Equations (40) are quite general. We will consider two specific simple cases. We assume that $\rho_m$ is constant, and that $\rho(\xi)$ is defined in the two cases by the following expressions:

\[
\begin{align*}
\rho_1(\xi)/\rho_m &= 1 + e^{-x}, \\
\rho_2(\xi)/\rho_m &= \frac{2 + \pi}{\sqrt{\pi}} \frac{x^2}{1 + x^2}.
\end{align*}
\]

(42)

where $x = (|\xi| - \Delta_o)/\Delta_o$ and $\rho = 0$ for $|\xi| < \Delta_o$ [cf. Fig. 1]. The dotted line correspond to $\rho_1$ and the dashed line to $\rho_2$. The numerical results for $\Delta$ and $\mu$ in the zero temperature case are reported in Fig. 8 for the discontinuous density of states ($\rho_1$), and in Fig. 3 for the second one. There is no qualitative difference with Fig. 1. Different shapes of the density of states correspond to different values of the ratio $\Delta^*/\Delta_m$, always keeping it of order $1$.

Most of the results can be understood analytically. Let us first consider how the order parameter vanishes at the transition for no doping. In this case $\Delta$ is small and we can expand $E$ in the second of (40). After the equation for the critical $\Delta_o$ is subtracted we obtain:

\[
0 = \int_{\Delta_o}^{+\infty} \frac{\rho \Delta^*_o(\xi) - \rho \Delta_o(\xi)}{\xi} d\xi
\]

(43)

Although the energy range over which $\rho(\xi)$ varies is of order $\Delta_o$, we have tried a more general form in which that energy scale is different from $\Delta_o$. The maximal variation of $\Delta^*_o$ obtained is an increase of a factor $e \approx 2.71$ when all the removed states are accumulated at the edge.
For $\Delta_o$ large enough $\mu$ sits again in the valence band, and the limit $\Delta_o \gg \mu - \Delta \gg \Delta$ is reached. In this case the BCS theory applies in the upper band and one recovers the familiar exponential behavior of $\Delta$ on the inverse of the density of states at the Fermi energy.

Concerning the critical temperature we have verified that using the density of states $\rho_1$ one recovers the same behaviour found for a constant density of states: near the transition point and for small doping it vanishes like $1/w(1/\tau)$ or $1/w(1/\varepsilon)$. In the BCS region ($\Delta_o \gg \Delta_m^{\ast}/\varepsilon$) the usual BCS $T_c$ is recovered.

We can readily calculate the phase stiffness if we assume that the density of states depends only on $k^2$. In this case we can generalize the results for the variation of the kinetic energy:

$$\Xi_k(Q) = \xi_k + \xi_k^{\ast} \frac{Q^2}{4} + \xi_k^{\ast} \left(\frac{kQ}{2}\right)^2 + O(Q^4),$$

(45)

where $\xi_k^\prime \equiv d\xi(k^2)/dk^2$. Using this expression for the kinetic energy and the invariance under rotation we obtain for each band:

$$\frac{dE(Q)}{dQ^2} = 2 \sum_{k \in \text{band}} \nu_k^2 \left[ \frac{\xi_k^\prime}{4} + \frac{\xi_k^{\ast} (k^2)}{2} \right].$$

(46)

The derivatives of the dispersion relation can be related to the density of states: $\xi_k^\prime = 1/4\pi \rho$ and $\xi_k^{\ast} = -\rho^\prime/16\pi^2 \rho^3$. Putting everything together we obtain for our problem the same phase stiffness $J$ as in (38), but with an additional factor $A(\xi)$ in the integrand of (36) provided $\rho(\xi) = \rho(-\xi)$:

$$A(\xi) = 1 - \frac{\rho^\prime(\xi)}{\rho(\xi)^2} \int_{-\xi}^{\xi} \rho(x) dx.$$

(47)

When the density of states is constant $A(\xi) = 1$. One can show that the integral of $[A(\xi) - 1]$ is equal to $\Delta_o$: it thus gives the contribution of the additional states to the phase stiffness. The various contributions to the phase stiffness are just rearranged and we find that the shape of $\rho(\xi)$ does not affect qualitatively the dependence of the phase stiffness on the important parameters $\Delta_o$ and $\varepsilon$. Moreover the quantitative change is not enough to let $T_c$ become larger than the $T_c$ given by particle excitations. Thus the phase fluctuations should always be responsible for the transition to the normal phase. (Except for very large $\Delta_o$ at fixed $\varepsilon$ where the BCS limit is recovered.)

B Interband pairing

In this paper we have considered pairing of time reversed state within the same band. If we indicate with $a_{\sigma \sigma}$ and $b_{\sigma \tau}$ the destruction operators for the lower and upper band respectively, the possible pairings considered in the

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig9.png}
\caption{Numerical results for the density of states $\rho = \rho_2$ (square root). Conventions are the same of Fig. 5.}
\end{figure}

\begin{equation}
- \frac{1}{\Delta_o} \int_{-\Delta_o}^{\Delta_o} \rho(\xi) d\xi + \Delta^2 \int_{\Delta_c}^{+\infty} \frac{\rho_\ast}{2\xi^3}. \tag{43}
\end{equation}

The last term for $\tau \rightarrow 0$ ($\Delta_o \rightarrow \Delta_o^\ast$) yields simply $\Delta^2(\tau)$ multiplied by a constant. The second term vanishes like $\tau^{1+\alpha}$ if $\rho(\xi) \sim (\xi - \Delta_o)^{\alpha}$. This term is always subleading compared to the first term that vanishes linearly in $\tau$. We thus conclude that the dependence $\Delta \sim \tau^{1/2}$ is quite general. This is not surprising for a mean field theory.

For finite doping and $T = 0$ it is interesting to study the dependence of $\Delta$ on $\varepsilon$. In particular we can readily obtain this dependence analytically when $\Delta_o \gg \Delta_m$. We distinguish the two cases $\mu < \Delta_o$ and $\mu > \Delta_o$. When $\Delta_o \gg \Delta_o - \mu \gg \Delta$ the lower band is nearly decoupled and the conservation of particles gives:

$$\delta N = 2 \varepsilon \rho_m \approx \int_{-\Delta_o}^{\Delta_o} \rho(\xi) \frac{\Delta^2}{2(\xi - \mu)^2} \approx \frac{\Delta^2}{2} \frac{\rho(\Delta_o - \mu)}{\Delta_o - \mu}. \tag{44}$$

We thus recover $\Delta \sim \varepsilon^{1/2}$ for a general density of states, again a mean field exponent. Note that the coefficient involves the density of states on an energy scale $\sim (\Delta_o - \mu)$ (measured from the bottom of the conduction band). This fact does not invalidate the conclusion since the chemical potential is practically independent of $\varepsilon$, determined as usual by the gap equation.

For $\Delta_o$ large enough $\mu$ sits again in the valence band, and the limit $\Delta_o \gg \mu - \Delta \gg \Delta$ is reached. In this case the BCS theory applies in the upper band and one recovers the familiar exponential behavior of $\Delta$ on the inverse of the density of states at the Fermi energy.
main body of the paper are: \( \langle a_{k^+} a_{-k^+} \rangle \) and \( \langle b_{k^+} b_{-k^+} \rangle \). The two bands are then coupled by the interband interaction, which is assumed identical to the intraband interaction.

Another model of pairing in a two band semiconductor was proposed some years ago by Kohmoto and Takada \cite{10} (see also \cite{14}). The main difference with our proposal is that in that case the paired states are not time-reversed one of the other, specifically electrons of different bands are paired: \( \langle a_{k^+} b_{-k^+} \rangle \). (The spin was not included in the original formulation and it is introduced here for comparison with our work). The mean field calculations predicts at \( T = 0 \) a first order transitions from an insulator to a superconductor. We discuss briefly in this Appendix this model \cite{10} and we compare it to ours in one simple case.

The main difference arise from the fact that the kinetic energies of the paired states are different: one expects that the energy cost should accordingly be larger. Although it is possible to have interactions that favor only one of the two possible states, when the strength of the interaction is the same in the two channels the intraband pairing should be favoured due to the lower (kinetic) energy cost. We demonstrate this fact in a simple case.

We consider a local interaction of the form

\[
-U \int dx \rho(x) \rho(x),
\]

where \( \rho(x) = \sum_\sigma \psi^\dagger(\sigma,x) \psi (\sigma,x) \) and \( \psi_\sigma(x) = a_\sigma(x) + b_\sigma(x) \). The antiparallel spin component of this interaction can be reduced to two terms:

\[
V_1 = -2U \int dx \left( a_{-}^\dagger a_{+}^\dagger + b_{-}^\dagger b_{+}^\dagger \right) \left( a_{+} a_{+} + b_{+} b_{+} \right),
\]

\[
V_2 = -2U \int dx \left( a_{+}^\dagger b_{-}^\dagger - a_{-}^\dagger b_{+}^\dagger \right) \left( b_{+} a_{+} - b_{+} a_{+} \right).
\]

\( V_1 \) would induce the pairing suggested in this paper, while \( V_2 \) is needed in order to obtain the state proposed by Kohmoto and Takada. We consider the simplest case of constant density of states \( \rho \), symmetric bands, and no doping (\( \mu = 0 \)). We also keep the energy cut-off \( \Delta_o + \omega_c = \omega_m \) finite. We can then evaluate the Hamiltonian on the respective trial wave functions. For the intraband pairing proposed in this paper the ground state wavefunction is:

\[
|\psi_1\rangle = \prod_k \left( u_k + v_k \left( b_{+}^\dagger - b_{-}^\dagger \right) \right) \left( v_k + u_k \left( a_{+}^\dagger - a_{-}^\dagger \right) \right) |\text{vac}\rangle,
\]

while for the state proposed in Ref. \cite{10} the wavefunction takes the form:

\[
|\psi_2\rangle = \prod_k \left( \frac{1 + a_{+}^\dagger b_{+}^\dagger}{\sqrt{2}} \right) \left( \frac{1 - a_{+}^\dagger b_{+}^\dagger}{\sqrt{2}} \right) |\text{vac}\rangle.
\]

Here the singlet state has been chosen to take maximal advantage of the interaction. The energy difference with respect to the normal state (\( \delta E \)) is in the two cases:

\[
\delta E_1 = \rho \left[ 2\Delta_o \omega_c + \omega_c^2 - (\Delta_o + \omega_c) \sqrt{ (\Delta_o + \omega_c)^2 + \Delta^2 } \right],
\]

\[
\delta E_2 = \rho \omega_c \left[ 2\Delta_o + \omega_c - 2U \omega_c \rho \right],
\]

where the value of \( \Delta \) in \cite{22} is given by the usual gap equation: \( 1 = 2\rho U \int \delta E / \Delta_o \). The minimum value of the interaction (\( U_{min} \)) for which \( \delta E \) changes sign determines the transition from the insulating to the superconducting phase. For the two states we have respectively \( \rho U_{min} = 1/2 \ln(1 + \omega_c / \Delta_o) \) and \( \rho U_{min} = 1/2 + \Delta_o / \omega_c \).

Thus \( U_{min} < U_{min} \) always. Moreover explicit evaluation of the energy shows that the energy gain of the intraband pairing state is always larger than the interband one. In Fig. \cite{10} we plotted \( \delta E \) in the two cases when \( \omega_c / \Delta_o = 1 \). The transition is second order in case (1), first order in case (2). We report also the average of the order parameter over the states that participate to the pairing:

\[
\Delta_{av} = \frac{\sum_k u_k v_k}{\sum_k 1},
\]

which clearly displays the nature of the transition.

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