A note on the hyper-sums of powers of integers, hyperharmonic polynomials and \(r\)-Stirling numbers of the first kind

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Abstract
Recently, Kargın et al. (arXiv:2008.00284 [math.NT]) obtained (among many other things) the following formula for the hyper-sums of powers of integers \(S_k^{(m)}(n)\)

\[
S_k^{(m)}(n) = \frac{1}{m!} \sum_{i=0}^{m} (-1)^i \binom{m+n+1}{i+n+1}_{n+1} S_{k+i}(n),
\]

where \(S_k^{(0)}(n) \equiv S_k(n)\) is the ordinary power sum \(1^k + 2^k + \cdots + n^k\). In this note we point out that a formula equivalent to the preceding one was already established in a different form, namely, a form in which \(\binom{m+n+1}{i+n+1}_{n+1}\) is given explicitly as a polynomial in \(n\) of degree \(m - i\). We find out the connection between this polynomial and the so-called \(r\)-Stirling polynomials of the first kind. Furthermore, we determine the hyperharmonic polynomials and their successive derivatives in terms of the \(r\)-Stirling polynomials of the first kind, and show the relationship between the (exponential) complete Bell polynomials and the \(r\)-Stirling numbers of the first kind. Finally, we derive some identities involving the Bernoulli numbers and polynomials, the \(r\)-Stirling numbers of the first kind, the Stirling numbers of both kinds, and the harmonic numbers.

1 Introduction

For integers \(k, n \geq 0\) and \(m \geq 1\), the hyper-sums of powers of integers \(S_k^{(m)}(n)\) are defined recursively by

\[
S_k^{(m)}(n) = \sum_{j=1}^{n} S_k^{(m-1)}(j),
\]

with initial condition \(S_k^{(0)}(n) \equiv S_k(n) = 1^k + 2^k + \cdots + n^k\), and \(S_k^{(m)}(0) = 0\). Recently, Kargın et al., in a notable paper (arXiv:2008.00284 [math.NT]) obtained (among many other things) the following formula for \(S_k^{(m)}(n)\) (see [9, Equation (27)])

\[
S_k^{(m)}(n) = \frac{1}{m!} \sum_{i=0}^{m} (-1)^i \binom{m+n+1}{i+n+1}_{n+1} S_{k+i}(n),
\]  

where \(\binom{m+r}{i+r}\) are the \(r\)-Stirling numbers of the first kind [2]. On the other hand, as shown in [4], it turns out that (see [4, Equation (2)])

\[
S_k^{(m)}(n) = \frac{1}{m!} \sum_{i=0}^{m} (-1)^i q_{m,i}(n) S_{k+i}(n),
\]
where \( q_{m,i}(n) \) is the polynomial in \( n \) of degree \( m - i \) given by (cf. [4, Equation (16)])

\[
q_{m,i}(n) = \sum_{j=0}^{m-i} \binom{i+j}{i} \left[ \binom{m+1}{i+j+1} \right] n^j.
\]

In view of (1) and (2), it follows that

\[
\sum_{i=0}^{m} (-1)^i Q_{m,i}(n) S_{k+i}(n) = 0,
\]

where \( Q_{m,i}(n) = \left[ \binom{m+n+1}{i+n+1} \right]_{n+1} - q_{m,i}(n) \) is independent of \( k \). Moreover, it is to be noticed that (3) holds irrespective of the value of \( k \). Consequently, noting that the power sum polynomials \( S_k(n), S_{k+1}(n), \ldots, S_{m}(n) \) are linearly independent, we are led to conclude that \( Q_{m,i}(n) = 0 \) and then the following (nontrivial) relation must hold.

**Proposition.** For integers \( m, i, n \geq 0 \) and \( m \geq i \), we have

\[
\left[ \binom{m+n+1}{i+n+1} \right]_{n+1} = \sum_{j=0}^{m-i} \binom{i+j}{i} \left[ \binom{m+1}{i+j+1} \right] n^j.
\]

\[\text{(4)}\]

In the next section we give an alternative derivation of this relation, showing that the formulas (1) and (2) are in fact equivalent. In section 3, we show how the polynomial (4) relates to the so-called \( r \)-Stirling polynomials of the first kind. In section 4, we express the hyperharmonic polynomials and their successive derivatives in terms of the \( r \)-Stirling polynomials of the first kind. In section 5, we show the relationship between the (exponential) complete Bell polynomials and the \( r \)-Stirling numbers of the first kind. Finally, in section 6, we derive some identities involving the Bernoulli numbers and polynomials, the \( r \)-Stirling numbers of the first kind, the Stirling numbers of both kinds, and the harmonic numbers.

### 2 Proof of the Proposition

According to [2, Equation (27)] (see also [13, page 224]), \( \left[ \binom{m+n+1}{i+n+1} \right]_{n+1} \) can be expressed as

\[
\left[ \binom{m+n+1}{i+n+1} \right]_{n+1} = \sum_{j=i}^{m} \binom{m}{j} \left[ \binom{j}{i} \right] (n+1)^{j-i},
\]

where \( n^\overline{r} \) denotes the rising factorial \( n(n+1) \cdots (n+k-1) \). Changing the summation variable from \( j \) to \( t \), where \( t = m - j \), results in

\[
\left[ \binom{m+n+1}{i+n+1} \right]_{n+1} = \sum_{t=0}^{m-i} \binom{m}{t} \left[ \binom{m-t}{i} \right] (n+1)^t.
\]

Now, we have that

\[
(n+1)^\overline{r} = \sum_{r=0}^{t} \binom{t}{r} (n+1)^r = \sum_{r=0}^{t} \sum_{s=0}^{r} \binom{r}{s} \binom{t}{r} n^s,
\]

\[\text{(2)}\]
and then
\[
\begin{align*}
\binom{m+n+1}{i+n+1}_{n+1} &= \sum_{t=0}^{m-i} \sum_{r=0}^{t} \sum_{s=0}^{r} \binom{m}{t} \binom{m-t}{i} \binom{r}{s} \binom{t}{r} n^s \\
&= \sum_{t=0}^{m-i} \sum_{s=0}^{t} \sum_{r=s}^{t} \binom{r}{s} \binom{t}{r} \binom{m}{t} \binom{m-t}{i} n^s.
\end{align*}
\]

Using the well-known identity (see, e.g., [8, Equation (6.16)]) \(\sum_{r=s}^{t} \binom{r}{s} = \binom{t+1}{s+1}\), the last equation becomes
\[
\begin{align*}
\binom{m+n+1}{i+n+1}_{n+1} &= \sum_{t=0}^{m-i} \binom{m}{t} \binom{m-t}{i} \binom{t+1}{s+1} n^s \\
&= \sum_{t=0}^{m-i} \sum_{j=0}^{t} \binom{m}{t} \binom{t+1}{j+1} \binom{m-t}{i} n^j,
\end{align*}
\]

where we have renamed the variable \(s\) as \(j\).

Invoking the identity (see [2, Equation (52)] and [13, page 224])
\[
\begin{align*}
\binom{j+i}{i} \binom{m+r+s}{j+i+r+s}_{r+s} &= \sum_{t=j}^{m-i} \binom{m}{t} \binom{t+r}{j+r} \binom{m-t+s}{i+s},
\end{align*}
\]

and specializing to the case in which \(r = 1\) and \(s = 0\), it follows that
\[
\begin{align*}
\binom{i+j}{i} \binom{m+1}{i+j+1} &= \sum_{t=j}^{m-i} \binom{m}{t} \binom{t+1}{j+1} \binom{m-t}{i},
\end{align*}
\]

and then, combining (5) and (6), we get (4).

**Remark 1.** Clearly, the constant term of the polynomial (4) is \(\binom{m+1}{i+1}\). Therefore, setting \(n = 0\) and \(j = 0\) in (5), we deduce the identity (cf. [2, Equation (30)])
\[
\binom{m+1}{i+1} = \sum_{t=0}^{m-i} t! \binom{m-t}{i}.
\]

**Remark 2.** When \(k = 0\), the hyper-sum \(S_k^{(m)}(n)\) is given by \(S_0^{(m)}(n) = \binom{n+m}{m+1}\). Hence, letting \(k = 0\) in (1) yields
\[
\sum_{i=0}^{m} (-1)^i \binom{m+n+1}{i+n+1} S_i(n) = m! \binom{n+m}{m+1}.
\]

In particular, for \(n = 1\) we find that
\[
\sum_{i=0}^{m} (-1)^i \binom{m+2}{i+2} = m!.
\]
3 \( r \)-Stirling polynomials of the first kind

For integers \( 0 \leq i \leq m \), the \( r \)-Stirling polynomials of the first kind \( R_{m,i}(x) \) are defined for arbitrary \( x \) as (see [2, Equation (56)] and [3, Equation (5.3)])

\[ R_{m,i}(x) = \sum_{j=0}^{m-i} \binom{m}{j} \binom{m-j}{i} x^j, \]

which reduces to \( R_{m,i}(r) = [\binom{m+r}{i+r}]_r \) when \( r \) is a nonnegative integer. Following a procedure analogous to that used in the previous section, it can be shown that \( R_{m,i}(x) \) can equally be written in the form (cf. [3, Equation (5.2)])

\[ R_{m,i}(x) = \sum_{j=0}^{m-i} \binom{i+j}{i} \binom{m+1}{i+j+1} x^j. \tag{8} \]

In view of (4), it may be useful to define the following variant of (8)

\[ \mathcal{R}_{m,i}(x) = \sum_{j=0}^{m-i} \binom{i+j}{i} \binom{m+1}{i+j+1} x^j. \]

Observe that, for \( x = -1 \), we have

\[ \mathcal{R}_{m,i}(-1) = \sum_{j=0}^{m-i} (-1)^j \binom{i+j}{i} \binom{m+1}{i+j+1} = \sum_{t=i}^{m} (-1)^{t-i} \binom{t}{i} \binom{m+1}{t+1} = \binom{m}{i}, \]

according to [8, Equation (6.18)]. Moreover, \( \mathcal{R}_{m,i}(0) = [\binom{m+1}{i+1}]_1 \), and, in general, \( \mathcal{R}_{m,i}(r) = [\binom{m+r+1}{i+r+1}]_{r+1} = R_{m,i}(r+1) \) for any integer \( r \geq 0 \). This in turn implies that, for arbitrary \( x \),

\[ \sum_{j=0}^{m-i} \binom{i+j}{i} \binom{m+1}{i+j+1} x^j = \sum_{j=0}^{m-i} \binom{i+j}{i} \binom{m}{i+j} (x+1)^j. \tag{9} \]

Incidentally, when \( i = 0 \), (9) reduces to

\[ \sum_{j=0}^{m} \binom{m+1}{j+1} x^j = \sum_{j=0}^{m} \binom{m}{j} (x+1)^j, \]

with each side of this equation being equal to the product \( (x+1)(x+2) \cdots (x+m) \).

4 Hyperharmonic polynomials and their derivatives

The \( n \)th hyperharmonic number of order \( r \), denoted by \( H_n^{(r)} \), is defined by

\[ H_n^{(r)} = \begin{cases} 
0 & \text{if } n \leq 0 \text{ or } r < 0, \\
\frac{1}{n} & \text{if } n > 0 \text{ and } r = 0, \\
\sum_{i=1}^{n} H_i^{(r-1)} & \text{if } n, r \geq 1.
\end{cases} \]
Note that $H_n^{(1)}$ is the ordinary harmonic number $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. As is well known (see [1, Theorem 2]), $H_n^{(r)}$ can be given in terms of the $r$-Stirling number as follows: $H_n^{(r)} = \frac{1}{n!} [n + r]_r$. Thus, employing (8) allows us to express the $(j + 1)\text{th}$ hyperharmonic number of order $n$, $H_j^{(n)}$, in the form

$$H_j^{(n)} = \frac{R_{j+1,1}(n)}{(j + 1)!} = \frac{1}{(j + 1)!} \sum_{i=0}^{j} (i + 1) \left[ \frac{j + 1}{i + 1} \right] n^i,$$

giving $H_j^{(n)}$ as a polynomial in $n$ of degree $j$. Let us observe that, when $n = 1$, we recover the well-known identity $H_j = \frac{1}{j!} \sum_{i=1}^{j} i [i]$. Of course, $H_j^{(n)}$ can be naturally extended to a polynomial $H_j^{(x)}$ in which $x$ takes any arbitrary value as follows

$$H_j^{(x)} = \frac{R_{j+1,1}(x)}{(j + 1)!} = \frac{1}{(j + 1)!} \sum_{i=0}^{j} (i + 1) \left[ \frac{j + 1}{i + 1} \right] x^i. \tag{10}$$

Likewise, we have

$$H_j^{(x+1)} = \frac{R_{j+1,1}(x)}{(j + 1)!} = \frac{1}{(j + 1)!} \sum_{i=0}^{j} (i + 1) \left[ \frac{j + 2}{i + 2} \right] x^i, \tag{11}$$

giving $H_j^{(x+1)}$ as a polynomial in $x$ of degree $j$.

On the other hand, from [2, Theorem 28], we know that $R_{m,i}(x) = \frac{1}{i!} \frac{d^i}{dx^i} x^m$. Therefore, it follows that

$$H_j^{(x)} = \frac{1}{(j + 1)!} \frac{d}{dx} R_{j+1,i+1}(x) \frac{d^i}{dx^i} x^{j+1} = \frac{1}{(j + 1)!} \frac{d}{dx} \left( x + j \right)^i,$$

in accordance with [5, Corollary 1]. Moreover, the $i$th derivative of $H_j^{(x)}$ with respect to $x$ is given by

$$\frac{d^i}{dx^i} H_j^{(x)} = \frac{1}{(j + 1)!} \frac{d^i}{dx^i} \left( x + j \right)^i = \frac{1}{(j + 1)!} \frac{d^i}{dx^i} \left( R_{j+1,i+1}(x) \right),$$

and then

$$\frac{d^i}{dx^i} H_j^{(x)} = \frac{(i + 1)!}{(j + 1)!} \sum_{t=0}^{j-i} \left( \frac{i + t + 1}{i + 1} \right) \left[ \frac{j + 1}{i + t + 1} \right] x^t. \tag{12}$$

Likewise, we have

$$\frac{d^i}{dx^i} H_j^{(x+1)} = \frac{(i + 1)!}{(j + 1)!} \frac{d^i}{dx^i} \left( R_{j+1,i+1}(x + 1) \right) = \frac{(i + 1)!}{(j + 1)!} \frac{d^i}{dx^i} \left( R_{j+1,i+1}(x) \right),$$

and then

$$\frac{d^i}{dx^i} H_j^{(x+1)} = \frac{(i + 1)!}{(j + 1)!} \sum_{t=0}^{j-i} \left( \frac{i + t + 1}{i + 1} \right) \left[ \frac{j + 2}{i + t + 2} \right] x^t. \tag{13}$$

Note that, as it must be, (12) and (13) reduce respectively to (10) and (11) when $i = 0$. 

5
Furthermore, for nonnegative integer $r$, from (12) and (13) it follows that
\[
\frac{d^i}{dx^i} H^{(x)}_{j+1} \bigg|_{x=r} = \frac{(i+1)!}{(j+1)!} \left[ \frac{j + r + 1}{i + r + 1} \right]_r,
\]
and
\[
\frac{d^i}{dx^i} H^{(x+1)}_{j+1} \bigg|_{x=r} = \frac{(i+1)!}{(j+1)!} \left[ \frac{j + r + 2}{i + r + 2} \right]_{r+1},
\]
In particular, for $r = 0$ we obtain
\[
\frac{d^i}{dx^i} H^{(x)}_{j+1} \bigg|_{x=0} = \frac{(i+1)!}{(j+1)!} \left[ \frac{j + 1}{i + 1} \right],
\]
and
\[
\frac{d^i}{dx^i} H^{(x+1)}_{j+1} \bigg|_{x=0} = \frac{(i+1)!}{(j+1)!} \left[ \frac{j + 2}{i + 2} \right].
\]

**Remark 3.** According to [5, Theorem 2], the hyperharmonic polynomial $H^{(x+1)}_{j+1}$ can be expressed as
\[
H^{(x+1)}_{j+1} = \sum_{t=0}^{j} \frac{1}{j + 1 - t} \binom{x + t}{t}.
\]
Using this representation in combination with (13) gives
\[
\sum_{t=0}^{j} \frac{1}{j + 1 - t} \frac{d^i}{dx^i} \binom{x + t}{t} = \frac{(i+1)!}{(j+1)!} \sum_{t=0}^{j-i} \binom{i + t + 1}{i + 1} \left[ \frac{j + 2}{i + t + 2} \right] x^t.
\]
As a consequence, we have
\[
\sum_{t=0}^{j} \frac{1}{j + 1 - t} \frac{d^i}{dx^i} \binom{x + t}{t} \bigg|_{x=r} = \frac{(i+1)!}{(j+1)!} \left[ \frac{j + r + 2}{i + r + 2} \right]_{r+1},
\]
or, equivalently,
\[
\sum_{t=0}^{j} \frac{1}{t! (j + 1 - t)} \binom{t + r}{i + r} = \frac{i + 1}{(j + 1)!} \left[ \frac{j + r + 1}{i + r + 1} \right]_r, \tag{14}
\]
which holds for any integers $0 \leq i \leq j$ and $r \geq 0$. In particular, substituting $i = r = 1$ in (14) yields the identity
\[
\sum_{t=0}^{j} \frac{H_t}{j + 1 - t} = \frac{2}{(j + 1)!} \left[ \frac{j + 2}{3} \right] = H^{[2]}_{j+1} - H^{[2]}_{j+1},
\]
which can also be found in [10, p. 544], and where the notation $H^{[2]}_j$ means $\sum_{t=1}^{j} 1/t^2$. 

6
5 $r$-Stirling numbers of the first kind and complete Bell polynomials

From the previous section we know that, for $j \geq 1$,
\[
\frac{d^i}{dx^i} H_j^{(x+1)} \bigg|_{x=r} = \frac{(i + 1)!}{j!} \left[ \frac{j + r + 1}{i + r + 2} \right]_{r+1}.
\] (15)

In what follows we are going to show that (15) is consistent with the result found in [9, Equation (19)]. To this end, we first introduce the concepts behind the said equation [9, Equation (19)].

Let $Y_n(x_1, x_2, \ldots, x_n)$ be the $n$th (exponential) complete Bell polynomial defined by $Y_0 = 1$ and ([6, page 134])
\[
\exp \left( \sum_{j=1}^{\infty} x_j \frac{t^j}{j!} \right) = 1 + \sum_{p=1}^{\infty} Y_p(x_1, x_2, \ldots, x_p) \frac{t^p}{p!}.
\]

Now, for integers $r \geq 0$ and $j, k \geq 1$, let us define $H(j, k; r)$ as
\[
H(j, k; r) = \sum_{i=1}^{j} \frac{1}{(i + r)^k}.
\]

Note that $H(j, 1; 0) = H_j$. Moreover, following Spieß (see [14, Equation (6)]), we introduce the numbers $P(i, j + r, r)$ as
\[
P(i, j + r, r) = P_i(H(j, 1; r), H(j, 2; r), \ldots, H(j, i; r)),
\]
where the polynomial $P_i(x_1, x_2, \ldots, x_i)$ is defined by
\[
P_i(x_1, x_2, \ldots, x_i) = (-1)^i Y_i(-0!x_1, -1!x_2, \ldots, -(i - 1)!x_i),
\]
or, equivalently,
\[
P_i(x_1, x_2, \ldots, x_i) = Y_i(0!x_1, -1!x_2, \ldots, -(i - 1)!x_i).
\]

The first five polynomials $P_i(x_1, x_2, \ldots, x_i)$ are given by
\[
P_1(x_1) = x_1,
\]
\[
P_2(x_1, x_2) = x_1^2 - x_2,
\]
\[
P_3(x_1, x_2, x_3) = x_1^3 - 3x_1x_2 + 2x_3,
\]
\[
P_4(x_1, x_2, x_3, x_4) = x_1^4 - 6x_1^2x_2 + 8x_1x_3 + 3x_2^2 - 6x_4,
\]
\[
P_5(x_1, x_2, x_3, x_4, x_5) = x_1^5 - 10x_1^3x_2 + 20x_1^2x_3 + 15x_1x_2^2 - 30x_1x_4 - 20x_2x_3 + 24x_5.
\]

With these ingredients at hand, we are ready to state the result established in [9, Equation (19)], namely
\[
\frac{d^i}{dx^i} H_j^{(x+1)} \bigg|_{x=r} = \binom{j + r}{i + r} P(i + 1, j + r, r). \quad (16)
\]

Next we show that the right-hand sides of (15) and (16) are the same. This is a direct consequence of the following theorem set forth by Kölbl in [12, Theorem].
Theorem (Kölbig, 1994). Let \( \alpha \in \mathbb{R} \) with \( \alpha \neq -1, -2, \ldots, -j \), and

\[
H(j, k; \alpha) = \sum_{t=1}^{j} \frac{1}{(t + \alpha)^k},
\]

for integers \( j, k \geq 1 \). Then we have

\[
P_q(H(j, 1; \alpha), H(j, 2; \alpha), \ldots, H(j, q; \alpha)) = \begin{cases} 
\frac{q!}{(1 + \alpha)^j} S(j, q; \alpha), & q \leq j, \\
0, & q > j,
\end{cases}
\]

where

\[
S(j, q; \alpha) = \sum_{t=q}^{j} \binom{t}{q} \left[ \begin{array}{c} j \\ t \end{array} \right] (1 + \alpha)^{t-q}.
\]

Hence, taking \( \alpha = r \) in Kölbig’s theorem (with integer \( r \geq 0 \)), it follows that

\[
\binom{j+r}{r} P(i+1, j+r, r) = \binom{j+r}{r} P_{i+1}(H(j, 1; r), H(j, 2; r), \ldots, H(j, i+1; r))
\]

\[
= \frac{(i+1)!}{(1+r)^j} \binom{j+r}{r} S(j, i+1; r),
\]

where we assume that \( j \geq 1 \) and \( j \geq i+1 \), with \( i = 0, 1, 2, \ldots \). Furthermore, it turns out that

\[
\binom{j+r}{r} = \frac{1}{j!} (1 + r)^j,
\]

and then

\[
\binom{j+r}{r} P(i+1, j+r, r) = \frac{(i+1)!}{j!} S(j, i+1; r),
\]

where

\[
S(j, i+1; r) = \sum_{t=i+1}^{j} \binom{t}{i+1} \left[ \begin{array}{c} j \\ t \end{array} \right] (1 + r)^{t-i-1}.
\]

On the other hand, we have

\[
\left[ \begin{array}{c} j+r+1 \\ i+r+2 \end{array} \right]_{r+1} = R_{j,i+1}(r+1) = \sum_{s=0}^{j-i-1} \binom{i+s+1}{i+1} \binom{j}{i+s+1} (r+1)^s
\]

\[
= \sum_{t=i+1}^{j} \binom{t}{i+1} \left[ \begin{array}{c} j \\ t \end{array} \right] (r+1)^{t-i-1}.
\]

Therefore, we conclude that \( S(j, i+1; r) = \left[ \begin{array}{c} j+r+1 \\ i+r+2 \end{array} \right]_{r+1} \), and we are done.

Remark 4. It should be remarked that the identity (16) was already established in [16, Equation (4.2)] in the equivalent form

\[
\frac{d^i}{dx^i} \left. x^{-r} \right|_{x=r+1} = j! \binom{j+r}{r} P(i, j+r, r).
\]
Furthermore, in [17, p. 270] we find that
\[ \frac{d^n}{dx^n} \left( \frac{x + n}{m} \right) \bigg|_{x=0} = \left( \frac{n}{m} \right) P(i, n, n - m), \quad n \geq m > 0, \]
which, as can be easily verified, is also equivalent to (16).

**Remark 5.** For any integer \( r \geq 1 \), when \( i = 0 \) and \( r \to r - 1 \), (16) reduces to
\[ H_j^{(x+1)} \bigg|_{x=r-1} = H_j^{(r)} = \binom{j + r - 1}{r - 1} H(j, 1; r - 1) = \binom{j + r - 1}{r - 1} \sum_{t=1}^{j} \frac{1}{t + r - 1} = \binom{j + r - 1}{r - 1} (H_{j+r-1} - H_{r-1}), \]
thus retrieving the well-known formula connecting hyperharmonic numbers with ordinary harmonic numbers put forward by Conway and Guy [7, p. 258].

Equating the right-hand sides of (15) and (16), and letting \( i \to i - 1 \), we obtain the identity
\[ \binom{j + r - 1}{r - 1} P(i, j + r, r) = \frac{j!}{j!} \left[ \binom{j + r + 1}{i + r + 1} \right]_{r+1}, \quad (17) \]
connecting the complete Bell polynomials and the \( r \)-Stirling numbers of the first kind (cf. [16, Equations (4.2) and (4.36)]). When \( r = 0 \), we recover the well-known result (see [6, Equation (7b), p. 217])
\[ \binom{j + 1}{i + 1} = \frac{j!}{i!} P(i, j, 0) = \frac{j!}{i!} P_i(H_j^{[1]}, H_j^{[2]}, \ldots, H_j^{[i]}), \]
where \( H_j^{[i]} \equiv H(j, i; 0) = \sum_{t=1}^{j} 1/t^i \), and \( H_j^{[1]} = H_j \).

**Remark 6.** The generating function of the numbers \( \binom{j + r}{r} P(i, j + r, r) \) is given by (see, e.g., [16, Equation (1.6)])
\[ \sum_{j=1}^{\infty} \binom{j + r}{r} P(i, j + r, r) t^j = \frac{\ln(1 - t)^i}{(1 - t)^{r+1}}. \]
Therefore, using (17) and letting \( r \to r - 1 \), we readily obtain the exponential generating function of the \( r \)-Stirling numbers of the first kind, namely
\[ \sum_{j=1}^{\infty} \left[ \binom{j + r}{i + r} \right]_{r+1} t^j = \frac{1}{i!} \left[ \frac{\ln(1 - t)^i}{(1 - t)^r} \right]. \]
In particular, for \( r = 0 \) and \( i \geq 1 \), we have
\[ (- \ln(1 - t))^i = \sum_{j=1}^{\infty} \frac{j!}{j!} \left[ \frac{j}{i} \right] t^j = \sum_{j=1}^{\infty} \frac{i}{j} P(i - 1, j - 1, 0) t^j, \]
in agreement with [14, Theorem 9]. Note that setting \( i = 1 \) in last equation yields the Maclaurin series of the natural logarithm
\[ \ln(1 - t) = - \sum_{j=1}^{\infty} \frac{t^j}{j} = -t - \frac{t^2}{2} - \frac{t^3}{3} - \ldots. \]
Remark 7. According to [14, Theorem 16], it turns out that, for \( m, r \geq 0 \),

\[
\sum_{k=0}^{m} \frac{P(r,k,0)}{k+1} = \frac{P(r+1,m+1,0)}{r+1}.
\]

Recalling that \( P(r,k,0) = \frac{r^{k}}{k! r^{k+1}} \), it is easily seen that the above identity is equivalent to (cf. [8, Equation (6.21)])

\[
\sum_{k=0}^{m} \frac{1}{k!} \binom{k}{r} [m+1] = m! \sum_{k=0}^{m} \frac{1}{k!} \binom{k}{r}.
\]

Remark 8. By virtue of (17), we can reformulate [9, Theorem 6] as follows:

\[
\sum_{k=l}^{n} \left[ \frac{n+r}{k+r} \right] _{r+1} \binom{k}{l} B_{k-l}(q) = \frac{l+1}{n+1} \left[ \frac{n+q+r}{l+q+r} \right] _{q+r-1},
\]  

(18)

which may also be written as

\[
\sum_{k=l}^{n} \left[ \frac{n+r+1}{k+r+1} \right] _{r+1} \binom{k}{l} B_{k-l}(q) = \frac{l+1}{n+1} \left[ \frac{n+q+r+1}{l+q+r+1} \right] _{q+r},
\]  

(19)

or else,

\[
\sum_{k=l}^{n} \left[ \frac{n+r}{k+r} \right] _{r} \binom{k}{l} B_{k-l}(q+1) = \frac{l+1}{n+1} \left[ \frac{n+q+r+1}{l+q+r+1} \right] _{q+r},
\]  

(20)

for nonnegative integers \( l, q, r \), and \( n \geq l \). As an example, putting \( l = 2 \), \( q = 0 \), and \( r = 1 \) in (20), we get

\[
\sum_{k=2}^{n} (-1)^{k} \left[ \frac{n+1}{k+1} \right] k(k-1)B_{k-2} = \frac{6}{n+1} \left[ \frac{n+2}{4} \right] _{3}
\]

\[
= n! \left( H_{n+1}^{3} - 3H_{n+1}^{[2]} + 2H_{n+1}^{[3]} \right),
\]

in accordance with the particular identity found in [9, page 8]. By the way, from (19) and (20), we find that

\[
\sum_{k=l}^{n} \left[ \frac{n+r+1}{k+r+1} \right] _{r+1} \binom{k}{l} B_{k-l}(x) = \sum_{k=l}^{n} \left[ \frac{n+r}{k+r} \right] _{r} \binom{k}{l} B_{k-l}(x+1),
\]

which holds for arbitrary \( x \). In particular, for \( r = 0 \), and renaming the indices \( k-l \rightarrow j \), \( l \rightarrow i \), and \( n \rightarrow m \), the last identity becomes

\[
\sum_{j=0}^{m-i} \binom{i+j}{i} \left[ \frac{m+1}{i+j+1} \right] B_{j}(x) = \sum_{j=0}^{m-i} \binom{i+j}{i} \left[ \frac{m}{i+j} \right] B_{j}(x+1),
\]

which may be compared with (9).

Moreover, (18) can be generalized as follows:

\[
\sum_{k=l}^{n} \left[ \frac{n+r}{k+r} \right] _{r} \binom{k}{l} B_{k-l}(x) = \frac{l+1}{n+1} \sum_{k=l}^{n} \left[ \frac{k+1}{l+1} \right] \left[ \frac{n+1}{k+1} \right] (x+r-1)_{k-l},
\]  

(21)
which holds for arbitrary $x$. For $l = 0$, (21) can be compactly written as
\[
\sum_{k=0}^{n} \binom{n + r}{k + r} B_k(x) = n! H_{n+1}^{(x+r-1)}.
\] (22)

Conversely, (21) can be obtained by performing the $l$th derivative with respect to $x$ of both sides of (22). For $x = -r$, (22) reads
\[
\sum_{k=0}^{n} \binom{n + r}{k + r} B_k(-r) = n! H_{n+1}^{(-1)}.
\]

Now, according to [5, Equation (42)], we have
\[
H_{n+1}^{(-1)} = \begin{cases} 
\frac{1}{n(n+1)}, & n \geq 1, \\
1, & n = 0,
\end{cases}
\]
and then we get the identity
\[
\sum_{k=0}^{n} \binom{n + r}{k + r} B_k(-r) = -\frac{n!}{n(n+1)},
\]
which holds for any integers $r \geq 0$ and $n \geq 1$. In particular, for $r = 0$ we obtain
\[
\sum_{k=0}^{n} \binom{n}{k} B_k = -\frac{(n-1)!}{n+1}, \quad n \geq 1.
\]

Remark 9. Using (17) into (7) leads to the relation
\[
\sum_{i=0}^{m} \frac{(-1)^i}{i!} P(i, m+n, n) S_i(n) = \frac{n}{m+1},
\]
and, in particular,
\[
\sum_{i=0}^{m} \frac{(-1)^i}{i!} P(i, m+1, 1) = \frac{1}{m+1}.
\]

6 Further connections with $r$-Stirling numbers of the first kind

In this section we will provide further identities involving the $r$-Stirling numbers of the first kind, the Bernoulli numbers and polynomials, the ordinary Stirling numbers of the first and second kinds, and the harmonic numbers.

Firstly, from [5, Equation (69)], it follows that, for $k \geq 0$, the Bernoulli polynomials $B_k(x)$ can be expressed in terms of $H_{j+1}^{(x)}$ as
\[
B_k(x) = (-1)^k \sum_{j=0}^{k} (-1)^j j! \binom{k + 1}{j + 1} H_{j+1}^{(x)},
\] (23)
where \( \{k\} \) are the Stirling numbers of the second kind. Hence, combining (10) and (23) gives

\[
B_k(x) = (-1)^k \sum_{i=0}^{k} \sum_{j=i}^{k} (-1)^j \frac{i+1}{j+1} \{k+1\}_{j+1} \left[ \frac{j+1}{i+1} \right] x^i,
\]

and, for \( x = 0 \),

\[
B_k = (-1)^k \sum_{j=0}^{k} (-1)^j \frac{j!}{j+1} \{k+1\}_{j+1}.
\]

Furthermore, taking successive derivatives of (23) we find that

\[
\frac{d^n}{dx^n} B_k(x) = i! \binom{k}{i} B_{k-i}(x)
\]

\[
= (-1)^k (i+1)! \sum_{j=i}^{k} \frac{(-1)^j}{j+1} \{k+1\}_{j+1} R_{j+1,i+1}(x),
\]

where we have used that \( R_{j,i}(x) = 0 \) for \( i > j \). When \( r \) is a nonnegative integer, we obtain

\[
\left. \frac{d^n}{dx^n} B_k(x) \right|_{x=r} = (-1)^k (i+1)! \sum_{j=i}^{k} \frac{(-1)^j}{j+1} \{k+1\}_{j+1} \left[ \frac{j+r+1}{i+r+1} \right]_r
\]

and, in particular,

\[
B_k(r) = (-1)^k \sum_{j=0}^{k} \frac{(-1)^j}{j+1} \{k+1\}_{j+1} \left[ \frac{j+r+1}{r+1} \right]_r.
\]

On the other hand, Spieß [14] and Wang [16] derived a number of identities involving the numbers \( P(i, j+r, r) \). Next, making use of (17), we rewrite some of these identities in terms of the \( r \)-Stirling numbers \( \{j+r+1\}_{i+r+1}_{r+1} \).

- From [14, Theorem 10] and (17), we get

\[
\sum_{k=0}^{m} \frac{(-1)^k}{k!} \left( \frac{r+1}{m-k} \right) \left[ \frac{k+r+1}{i+r+1} \right]_r = \frac{(-1)^m}{m!} \left[ \frac{m}{i} \right].
\]

When \( r = 0 \), this identity yields the well-known recurrence relation \( \left[ \frac{m+1}{i+1} \right] = \left[ \frac{m}{i} \right] + m \left[ \frac{m}{i+1} \right] \).

- From [14, Theorem 13] and (17), we get

\[
\sum_{k=r}^{m+1-s} \frac{r!s!}{k!(m+1-k)!} \left[ \frac{k}{m+1-k} \right] \left[ \frac{m+1-k}{s} \right] = \frac{(r+s)!}{(m+1)!} \left[ \frac{m+1}{r+s} \right].
\]

In particular, for \( s = 1 \) we obtain

\[
\sum_{k=r}^{m} \frac{1}{k!(m+1-k)} \left[ \frac{k}{m+1-k} \right] \left[ \frac{m+1-k}{1} \right] = \frac{(r+1)!}{(m+1)!} \left[ \frac{m+1}{r+1} \right],
\]

which is just the identity in (14) evaluated for \( r = 0 \).
From [14, Theorem 15] and (17), and after a few simple manipulation, we get
\[
q^r \sum_{k=1}^{m} \frac{1}{(k+q)!} \left[ \begin{array}{l} k \\ r \end{array} \right] = \frac{1}{q!} - \frac{1}{(m+q)!} \sum_{j=1}^{r} q^{j-1} \left[ \begin{array}{l} m+1 \\ j \end{array} \right],
\]
which is valid for any integers \( q \geq 1 \) and \( 1 \leq r \leq m \). Setting \( r = 1, 2, \) and \( 3 \) in (24) gives
\[
\sum_{k=1}^{m} \frac{1}{k(k+1)\cdots(k+q)} = \frac{1}{q \cdot q!} - \frac{1}{q(m+1)\cdots(m+q)},
\]
\[
\sum_{k=1}^{m} \frac{H_{k-1}}{k(k+1)\cdots(k+q)} = \frac{1}{q^2 \cdot q!} - \frac{1}{q^2(m+1)\cdots(m+q)},
\]
and
\[
\sum_{k=1}^{m} \frac{H_{k-1}^2 - H_{k-1}^{[2]}}{k(k+1)\cdots(k+q)} = \frac{2 + 2qH_m + q^2(H_m^2 - H_m^{[2]})}{q^3(m+1)\cdots(m+q)},
\]
respectively. The above three identities are to be compared with the corresponding Examples 1, 2, and 3 previously obtained in [14, p. 849]. Moreover, it is worth pointing out that, for the case in which \( r = m \), (24) yields the horizontal generating function for the Stirling numbers \( \left[ m+1 \atop j+1 \right] \), namely
\[
\sum_{j=0}^{m} \left[ m+1 \atop j+1 \right] q^j = (q+1)(q+2)\cdots(q+m),
\]
which holds for arbitrary \( q \).

- From [16, Equation (3.4)] and (17), we get
\[
\sum_{k=0}^{m} \frac{(-1)^k}{k!} \left( \begin{array}{l} m \\ k \end{array} \right) \left( \begin{array}{l} k+r \\ r \end{array} \right)^{-1} \left[ \begin{array}{l} k+r+1 \\ i+r+1 \end{array} \right]_{r+1} = \frac{(-1)^i}{m!} \left( \begin{array}{l} m+r \\ r \end{array} \right)^{-1} \left[ \begin{array}{l} m \\ i \end{array} \right].
\]
Letting \( r = i = 1 \) in the above relation leads to
\[
\sum_{k=0}^{m} (-1)^k \left( \begin{array}{l} m \\ k \end{array} \right) H_{k+1} = -\frac{1}{m(m+1)},
\]
which may be compared with the more commonly known identity (see, e.g., [17, Equation (3.2)]) \( \sum_{k=0}^{m} (-1)^k \left( \begin{array}{l} m \\ k \end{array} \right) H_k = -\frac{1}{m} \), where \( m \geq 1 \).

- From [16, Equation (3.22)] and (17), we get
\[
\left[ \begin{array}{l} m+r+1 \\ i+r+1 \end{array} \right]_{r+1} = m! \sum_{k=0}^{m} \frac{1}{k!} \left( \begin{array}{l} m-k+r \\ r \end{array} \right) \left[ \begin{array}{l} k \\ i \end{array} \right] = m! \sum_{k=0}^{m} (-1)^{k-i} \frac{1}{k!} \left( \begin{array}{l} m+r \\ k \end{array} \right) \left[ \begin{array}{l} k \\ i \end{array} \right].
\]
In particular, for \( r = 0 \) we have (cf. [16, Equation (3.25)])
\[
\left[ \begin{array}{l} m+1 \\ i+1 \end{array} \right] = m! \sum_{k=0}^{m} \frac{1}{k!} \left[ \begin{array}{l} k \\ i \end{array} \right] = m! \sum_{k=0}^{m} (-1)^{k-i} \frac{1}{k!} \left( \begin{array}{l} m \\ k \end{array} \right) \left[ \begin{array}{l} k \\ i \end{array} \right].
\]
• From [16, Equations (3.32) and (3.33)] and (17), we get

\[ \sum_{k=0}^{m} \frac{1}{k!} \binom{r + m - k - 1}{m - k} \binom{k + s + 1}{i + s + 1}_{s+1} = \frac{1}{m!} \binom{m + r + s + 1}{i + r + s + 1}_{r+s+1}, \tag{25} \]

and

\[ \binom{m + r + 1}{i + r + 1}_{r+1} = m! \sum_{k=0}^{m} \frac{1}{k!} \binom{r + m - k - 1}{m - k} \binom{k + 1}{i + 1}, \tag{26} \]

respectively. Notice that, from (25) and (26), one quickly obtains

\[ \sum_{k=0}^{m} \frac{1}{k!} \binom{r + m - k - 1}{m - k} \binom{k + s + 1}{i + s + 1}_{s+1} = \sum_{k=0}^{m} \frac{1}{k!} \binom{r + s + m - k - 1}{m - k} \binom{k + 1}{i + 1}. \]

Furthermore, regarding (26), for \( r = 1 \) it reads

\[ \binom{m + 2}{i + 2}_{2} = m! \sum_{k=0}^{m} \frac{1}{k!} \binom{k + 1}{i + 1}, \]

or, equivalently,

\[ P_{i}(H(m, 1; 1), H(m, 2; 1), \ldots, H(m, i; 1)) = \frac{i!}{m + 1} \sum_{k=0}^{m} \frac{1}{k!} \binom{k + 1}{i + 1}. \]

In particular, when \( i = 1 \) we recover the well-known identity

\[ \sum_{k=0}^{m} H_{k} = (m + 1)H_{m} - m. \]

• From [16, Equations (4.1) and (4.3)] and (17), we get

\[ \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{k + r + 1}{i + r + 1}_{r+1} = \binom{m}{i} (r + 1)^{m-i}, \tag{27} \]

and

\[ \sum_{k=i}^{m} (-r - 1)^{k-i} \binom{k}{i} \binom{m + r + 1}{k + r + 1}_{r+1} = \binom{m}{i}, \tag{28} \]

respectively. When \( r = 0 \), (27) and (28) reduce to

\[ \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{k + 1}{i + 1} = \binom{m}{i}, \]

and

\[ \sum_{k=i}^{m} (-1)^{k-i} \binom{k}{i} \binom{m + 1}{k + 1} = \binom{m}{i}, \]
respectively. Furthermore, when \( r = i = 1 \), from (27) we obtain
\[
\sum_{k=0}^{m} (-1)^{m-k}(k+1)! \binom{m}{k} (H_{k+1} - 1) = m2^{m-1}.
\]

On the other hand, putting \( i = 1 \) in [16, Equation (4.5)] yields
\[
\sum_{k=0}^{m} (-1)^{m-k}(k+1)! \binom{m}{k} = 2^m.
\]

Therefore, combining the last two identities, we get
\[
\sum_{k=0}^{m} (-1)^{m-k}(k+1)! \binom{m}{k} H_{k+1} = (m + 2) 2^{m-1},
\]
which may be compared with [16, Equation (4.10)].

- From [16, Equation (4.40)] and (17), we get
\[
\sum_{k=0}^{m} (-1)^{k-i}\binom{k}{i} \left[ \begin{array}{c} m + r + 1 \\ k + r + 1 \end{array} \right]_{r+1} = m! \left( \begin{array}{c} r + m - i \\ m - i \end{array} \right),
\]
which, for \( r = 0 \), reduces to
\[
\sum_{k=0}^{m} (-1)^{k-i}\binom{k}{i} \left[ \begin{array}{c} m + 1 \\ k + 1 \end{array} \right] = m!.
\]

- For \( k \geq 0 \) and \( i \geq 1 \), the higher-order Bernoulli numbers \( B_k^{(i)} \) are defined by the generating function
\[
\left( \frac{t}{e^t - 1} \right)^i = \sum_{k=0}^{\infty} B_k^{(i)} \frac{t^k}{k!},
\]
where \( B_k^{(1)} = B_k \) are the ordinary Bernoulli numbers. From [16, Equations (5.2)] and (17), we get
\[
\sum_{k=0}^{m} (-1)^k \left[ \begin{array}{c} m + r + 1 \\ k + r + 1 \end{array} \right] B_k^{(i)} = \left( \begin{array}{c} m + i \\ i \end{array} \right)^{-1} \left[ \begin{array}{c} m + i + r + 1 \\ i + r + 1 \end{array} \right]_{r+1}.
\]
In particular, for \( i = 1 \) we obtain
\[
\sum_{k=0}^{m} (-1)^k \left[ \begin{array}{c} m + r + 1 \\ k + r + 1 \end{array} \right] B_k = \frac{1}{m+1} \left[ \begin{array}{c} m + r + 2 \\ r + 2 \end{array} \right]_{r+1},
\]
which, for \( r = 0 \), reduces to the well-known identity (see, e.g., [16, Equation (5.6)])
\[
\sum_{k=0}^{m} (-1)^k \left[ \begin{array}{c} m + 1 \\ k + 1 \end{array} \right] B_k = m!H_{m+1}.
\]
Moreover, employing the following representation for $B_k^{(i)}$ obtained by Kim et al. [11]

$$B_k^{(i)} = \sum_{j=0}^{k} (-1)^j {i+j\choose i}^{-1} \{k\} \binom{m+r}{i} \binom{k+r}{j},$$

and setting $r \to r - 1$ in (29), we get

$$\sum_{k=0}^{m} \sum_{j=0}^{k} (-1)^{k+j} {i+j\choose i}^{-1} \{k\} \binom{m+r}{i} \binom{k+r}{j} = \binom{m+i}{i}^{-1} \binom{m+i+r}{i+r}.$$  \hspace{1cm} (30)

In particular, substituting $i = 2$ and $r = 0$ in (30) yields

$$\sum_{k=0}^{m} \sum_{j=0}^{k} (-1)^{k+j} \frac{j!}{j+2} \{k\} \binom{m}{k} H_{j+1} = \frac{m!}{m+2} \binom{m-2}{i}. \hspace{1cm} (31)$$

On the other hand, employing the representation for $B_k^{(i)}$ given in [15, Equation (15)]

$$B_k^{(i)} = \sum_{j=0}^{k} (-1)^j \{k\} \binom{m+r}{i} \binom{k+r}{j},$$

and making $r \to r - 1$ in (29), we get

$$\sum_{k=0}^{m} \sum_{j=0}^{k} (-1)^{k+j} \frac{1}{i+j} \{k\} \binom{m+r}{i} \binom{k+r}{j} = \binom{m+i}{i}^{-1} \binom{m+i+r}{i+r}.$$ \hspace{1cm} (32)

In particular, when $i = 3$ and $r = 0$, (31) implies the relation

$$\sum_{k=0}^{m} \sum_{j=0}^{k} (-1)^{k+j} \frac{1}{i+j} \{k\} \binom{m+r}{i} \binom{k+r}{j} = \frac{6 \cdot m!}{m+3} (H_{m+2} - H_{m+2}^2).$$

- From [16, Equations (5.3)] and (17), we get

$$\sum_{k=0}^{m} \binom{m+r+1}{k+r+1} B_k(i+1) = m! \binom{m+i+r+1}{i+r} (H_{m+i+r+1} - H_{i+r}). \hspace{1cm} (33)$$

Taking $r \to r - 1$ and $i \to i - 1$ in (32) gives

$$\sum_{k=0}^{m} \binom{m+r}{k+r} B_k(i) = m! \binom{m+i+r-1}{i+r-2} (H_{m+i+r-1} - H_{i+r-2}) = m!H_{m+1}^{(i+r-1)},$$

which is just the identity (22) with $x$ replaced by $i$. 

16
• In [16, p. 1508], we find the identity

$$
\sum_{k=0}^{m} P(k, m + r + i, r + i) \frac{B_k^{(i)}}{k!} = \binom{r + i}{m + i}^{-1} P(i, m + r + i, r).
$$

Using (17), we can write this identity in the equivalent form

$$
\sum_{k=0}^{m} \binom{m + r + i + 1}{k + r + i + 1} \frac{B_k^{(i)}}{r + i + 1} = \binom{m + i}{i + r + 1}^{-1} \binom{m + i + r + 1}{i + r + 1}.
$$

In particular, setting $r = 0$ and $i = 1$ in (33) gives

$$
\sum_{k=0}^{m} P_k(H(m, 1; 1), H(m, 2; 1), \ldots, H(m, k; 1)) \frac{B_k}{m + 1} = \frac{H_{m+1}}{m + 1}.
$$

Incidentally, from (29) and (33) it follows that

$$
\sum_{k=0}^{m} \binom{m + r + i + 1}{k + r + i + 1} \frac{B_k^{(i)}}{r + i + 1} = \sum_{k=0}^{m} (-1)^k \binom{m + r + 1}{k + r + 1} B_k^{(i)}.
$$

• Combining the recurrence appearing in [16, p. 1505]

$$
\binom{j + r}{r} P(i, j + r, r) = \binom{j + r - 1}{r - 1} P(i, j + r - 1, r - 1) + \binom{j + r - 1}{r} P(i, j + r - 1, r)
$$

of the numbers $\binom{j + r}{r} P(i, j + r, r)$, and (17), we readily obtain the corresponding recurrence for the $r$-Stirling numbers of the first kind

$$
\binom{j + r + 1}{i + r + 1} = \binom{j + r}{i + r} + j \binom{j + r}{i + r + 1}.
$$

As a final point to this section, from [18, Equation (4)] and (17), we can get the identity

$$
\sum_{j=1}^{m} \frac{1}{m + 1 - j} \frac{1}{(j - 1)!} \binom{j + r}{i + r} = \frac{i}{m!} \binom{m + r + 1}{i + r + 1},
$$

which holds for any integers $1 \leq i \leq m$ and $r \geq 0$. Note the close resemblance of this identity to (14). In particular, for $r = 0$ we find that

$$
\sum_{j=i}^{m} \frac{1}{m + 1 - j} \frac{1}{(j - 1)!} \binom{j}{i} = \frac{i}{m!} \binom{m + 1}{i + 1}.
$$

7 Conclusion

In conclusion, in this note we have shown that the above formulas (1) and (2) for the hyper-sums of powers of integers are in fact equivalent, bringing to light an explicit formula for $\binom{m+n+1}{i+n+1} \frac{n+1}{n+1}$ as a polynomial in $n$. Moreover, relying on the $r$-Stirling polynomials of the first kind $R_{j+1,i+1}(x)$
and \( R_{j+1,i+1}(x) \), we have expressed the ith derivative of the hyperharmonic polynomials \( H_{j+1}^{(x)} \) and \( H_{j+1}^{(x+1)} \) as a polynomial in \( x \), complementing the formula given in [9, Equation (19)]. Furthermore, we have shown the relationship between the (exponential) complete Bell polynomials and the \( r \)-Stirling numbers of the first kind, and we have derived some identities involving the Bernoulli numbers and polynomials, the \( r \)-Stirling numbers of the first kind, the Stirling numbers of both kinds, and the harmonic numbers.

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