JOHNSON HOMOMORPHISMS

RICHARD HAIN

Dedicated to the memory of Dennis Johnson

Abstract. Torelli groups are subgroups of mapping class groups that consist of those diffeomorphism classes that act trivially on the homology of the associated closed surface. The Johnson homomorphism, defined by Dennis Johnson, and its generalization, defined by S. Morita, are tools for understanding Torelli groups. This paper surveys work on generalized Johnson homomorphisms and tools for studying them. The goal is to unite several related threads in the literature and to clarify existing results and relationships among them using Hodge theory. We survey the work of Alekseev, Kawazumi, Kuno and Naef on the Goldman–Turaev Lie bialgebra, and the work of various authors on cohomological methods for determining the stable image of generalized Johnson homomorphisms. Various open problems and conjectures are included.

Even though the Johnson homomorphisms were originally defined and studied by topologists, they are important in understanding arithmetic properties of mapping class groups and moduli spaces of curves. We define arithmetic Johnson homomorphisms, which extend the generalized Johnson homomorphisms, and explain how the Turaev cobracket constrains their images.

1. Introduction

The Johnson homomorphism was defined by Dennis Johnson in [46], extending earlier work of Papakyriakopoulos [71] and Sullivan [78]. It is a surjective group homomorphism

\[ \tau : T_{S,\partial S} \to \Lambda^3 H_1(S;\mathbb{Z}), \]

where \( S \) is a surface of genus > 1 with one boundary component and \( T_{S,\partial S} \) is the Torelli subgroup of the mapping class group

\[ \Gamma_{S,\partial S} := \pi_0 \text{Diff}^+(S,\partial S) \]

that consists of isotopy classes of orientation preserving diffeomorphisms of \( S \) that fix \( \partial S \) pointwise. Johnson showed (remarkably) in [48] that \( T_{S,\partial S} \) is finitely generated when \( g \geq 3 \) and in [50] that the kernel of the induced surjection

\[ \tau : H_1(T_{S,\partial S}) \to \Lambda^3 H_1(S;\mathbb{Z}) \]

is a finite group of exponent 2.

Understanding Torelli groups is of fundamental importance in geometric topology (because of their relation to homology spheres), algebraic geometry (as they measure the difference between the moduli space \( \mathcal{M}_g \) of genus \( g \) curves and the
moduli space \( \mathcal{A}_g \) of principally polarized abelian \( g \)-folds and thus encode non-trivial information about the geometry of algebraic curves, and arithmetic geometry (because the stacks \( \mathcal{M}_g \) and \( \mathcal{A}_g \) are defined over \( \mathbb{Z} \) and have everywhere good reduction).

In order to probe deeper into the Torelli groups, Johnson [47] proposed investigating generalizations of the homomorphism (1). One starts with the descending central filtration

\[
T_{S,\partial S} = J^1T_{S,\partial S} \supset J^2T_{S,\partial S} \supset J^3T_{S,\partial S} \supset \cdots
\]

of \( T_{S,\partial S} \), where

\[
J^kT_{S,\partial S} := \ker \{ T_{S,\partial S} \to \text{Aut}(\pi_1(S,x)/L^{k+1}\pi_1(S,x)) \},
\]

\( x \in \partial S \), and \( L^kG \) denotes the \( k \)th term of the lower central series (LCS) of a group \( G \). The higher Johnson homomorphisms are injective maps

\[
\tau_k : \text{Gr}_j^k T_{S,\partial S} \hookrightarrow \text{Hom}_\mathbb{Z}(H_1(S), \text{Gr}_{L^k} \pi_1(S,x)).
\]

Morita [63] determined a constraint on their images. Each \( \tau_k \) is invariant under the natural action of the symplectic group \( \text{Sp}_g(\mathbb{Z}) \) on its source and target. The associated graded

\[
\text{Gr}_j^k T_{S,\partial S} := \bigoplus_{k \geq 0} \text{Gr}_j^k T_{S,\partial S} := \bigoplus_{k \geq 0} J^kT_{S,\partial S}/J^{k+1}T_{S,\partial S}
\]

is a graded Lie algebra over \( \mathbb{Z} \) and the higher Johnson homomorphisms comprise a Lie algebra homomorphism into \( \text{Der} \text{Gr}_j^k \pi_1(S,x) \).

The kernel of the Johnson homomorphism (1) is \( J^2T_{S,\partial S} \) and is called the Johnson subgroup of the Torelli group. Johnson showed [49] that it is generated by Dehn twists on separating simple closed curves for all \( g \geq 2 \). The conventional wisdom, inspired by Mess’s proof [62] that the Torelli group in genus 2 is a countably generated free group, was that \( J^2T_{S,\partial S} \) should have infinite first Betti number. Dimca and Papadima [19], using Hodge theory, defied this conventional wisdom by proving that \( H_1(J^2T_{S,\partial S}; \mathbb{Q}) \) is finite dimensional for all \( g \geq 4 \). More recently, Ershov and He [23] have shown that \( J^2T_{S,\partial S} \) is finitely generated when \( g \geq 12 \) and, more generally, that any subgroup of \( T_{S,\partial S} \) that contains \( L^kT_{S,\partial S} \) has finitely generated abelianization for all \( g \geq 8k - 4 \).

These results illustrate one of the main themes of this paper, discussed in detail in Section 15. Namely that, as the genus increases, each quotient \( T_{S,\partial S}/J^{k+1} \) becomes more regular in the sense that each \( \text{Gr}_j^k T_{S,\partial S} \otimes \mathbb{Q} \) stabilizes in the representation ring of the symplectic group. The larger \( k \), the longer it takes for this quotient to stabilize as a representation.

\[\text{1}\text{These had been defined previously for automorphisms of free groups by Andreadakis [4].}\]

\[\text{2}\text{Here and subsequently } \text{Gr}_j^m \pi_1(S,x) \text{ denotes the } m \text{th graded quotient } L^m\pi_1(S,x)/L^{m+1} \text{ of the LCS.}\]

\[\text{3}\text{Whether or not the first Betti number of the genus 3 Johnson subgroup is finite is still unresolved.}\]

\[\text{4}\text{These results have been improved by Church, Ershov and Putman in [10] where they show that } L^kT_{S,\partial S} \text{ is finitely generated when } g \geq 4 \text{ and } g \geq 2k - 1.\]
1.1. From graded to filtered. Traditionally, the generalized Johnson homomorphism is a homomorphism of graded Lie algebras. However, it is more natural to work with filtered Lie algebras and to pass to the associated graded Lie algebras only when necessary. These filtered Lie algebras are obtained by replacing $\pi_1(S, x)$ by its unipotent completion (also called Malcev completion) and the mapping class group $\Gamma_{S,\partial S}$ by its relative unipotent completion. These are proalgebraic groups over $\mathbb{Q}$. This is the approach taken in [31].

The Lie algebra $p(S, x)$ of the unipotent completion of $\pi_1(S, x)$ is a free pronilpotent Lie algebra. There is a canonical isomorphism

$$[\text{Gr}_\bullet \pi_1(S, x)] \otimes \mathbb{Q} \cong L(H)$$

of the associated graded of the LCS of $\pi_1(S, x)$ (tensored with $\mathbb{Q}$) with the free Lie algebra on $H := H_1(S; \mathbb{Q})$. The Lie algebra of the relative completion of $\Gamma_{S,\partial S}$ is an extension

$$0 \to u_{S,\partial S} \to g_{S,\partial S} \to sp(H) \to 0$$

where $sp(H)$ is the Lie algebra of the symplectic group of $H_1(S)$. The geometric Johnson homomorphism is the homomorphism

$$\tau_{S,\partial S} : g_{S,\partial S} \to \text{Der} p(S, x)$$

induced by the action of $\Gamma_{S,\partial S}$ on $p(S, x)$. The higher Johnson homomorphisms $\tau_k$ are packaged in $\tau_{S,\partial S}$. They can be recovered from the geometric Johnson homomorphism by taking graded quotients:

$$\left( \text{Gr}^k_{\bullet} T_{S,\partial S} \right) \otimes \mathbb{Q} \cong \text{Gr}^k_{\bullet} u_{S,\partial S} \xrightarrow{\text{Der}_k} \text{Gr}^k_{\bullet} p(S, x) = \text{Hom}_\mathbb{Q} \left( H, \text{Gr}^{k+1}_{\bullet} L(H) \right).$$

1.2. The role of Hodge theory. Replacing a filtered object, such as $T_{S,\partial S}$, by its associated graded typically results in a significant loss of information as the associated graded functor is rarely exact. However, many topological invariants of complex algebraic varieties possess a canonical weight filtration that has good exactness properties. Weight filtrations are part of a mixed Hodge structure (MHS) on the invariant. For every MHS $V$, there is a natural isomorphism

$$V \cong \bigoplus_{k \in \mathbb{Z}} \text{Gr}_k^W V$$

that is preserved by morphisms of MHS. A primary theme of this paper is that Hodge theory is a powerful tool for studying Johnson homomorphisms. A brief review of Hodge theory is given in Section 1.

In the current context, invariants such as $p(S, x)$ and $g_{S,\partial S}$ carry a natural MHS (once one has fixed an algebraic structure on $S$). Their weight filtrations satisfy

$$W_{-k} p(S, x) = L^k p(S, x)$$

and

$$W_{-k} g_{S,\partial S} = L^k g_{S,\partial S}$$

The Johnson homomorphism is a morphism of MHS. Moreover, there are natural isomorphisms of each with its associated weight graded Lie algebra so that the

---

5 The basics of relative unipotent completion are recalled in Section 5.
6 The direct sum is replaced by a direct product if $V$ is a pro-MHS, such as $g_{S,\partial S}$ or $p(S, x)$.
7 When $S$ has more than one puncture or boundary component, the weight filtration is no longer the lower central series. The weight filtration is the better filtration as it has good exactness properties, whereas the lower central series does not.
diagram
\[
\begin{align*}
\mathfrak{g}_{S,\partial S} & \xrightarrow{\tau_{S,\partial S}} \text{Der } \mathfrak{p} \\
\prod_{k \geq 0} \text{Gr}^W_{-k} \mathfrak{g}_{S,\partial S} & \xrightarrow{\text{Gr}^W_{-k} \tau_{S,\partial S}} \text{Der } \text{Gr}^W_{-k} \mathfrak{p}(S,x)
\end{align*}
\]
commutes.

1.3. The arithmetic Johnson homomorphism. The category of MHS is equivalent to the category of representations of an affine group over \(\mathbb{Q}\). Every MHS is a representation of this group and morphisms of MHS are equivariant under this group action. In this way, Hodge theory provides hidden symmetries which play a significant role in the story. In Section 9 we use these extra symmetries to construct an enlargement \(\hat{\mathfrak{g}}_{S,\partial S}\) of the image \(\mathfrak{g}_{S,\partial S}\) in \(\text{Der } \mathfrak{p}(S,x)\) and show that the geometric Johnson homomorphism (4) induces a homomorphism
\[
\hat{\tau}_{S,\partial S} : \hat{\mathfrak{g}}_{S,\partial S} \hookrightarrow \text{Der } \mathfrak{p}(S,x)
\]
that we call the arithmetic Johnson homomorphism. The Lie algebra \(\hat{\mathfrak{g}}_{S,\partial S}\) also carries a natural MHS, and thus a weight filtration. The quotient Lie algebra
\[
\text{Gr}^W_{-k} (\hat{\mathfrak{g}}_{S,\partial S}/\mathfrak{g}_{S,\partial S})
\]
is isomorphic to the motivic Lie algebra
\[
\mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \sigma_9, \ldots)^\wedge
\]
associated to mixed Tate motives unramified over \(\mathbb{Z}\). The proof of this result uses the proof of the Oda Conjecture by Takao [79] (built on work of Iha ra, Matsumoto and Nakamura) and Brown’s fundamental result [6].

1.4. Bounding the Johnson image. An important open problem is to determine the “Johnson image”. That is, determine the images of the higher Johnson homomorphisms which, by Hodge theory, is equivalent to the problem of determining the image of the geometric Johnson homomorphism (4). The main results of [31] imply that, when \(g \geq 3\), the Johnson image is generated by the image \(\text{Gr}^W_{-1} \mathfrak{g}_{S,\partial S}\) of the classical Johnson homomorphism. It is equally important to determine the image \(\mathfrak{g}_{S,\partial S}\) of the the arithmetic Johnson homomorphism (5) and explicit generators of its associated weight graded.

The image of the universal and arithmetic Johnson homomorphisms lie in the Lie subalgebra
\[
\text{Der}^\theta \mathfrak{p}(S,x) := \{ D \in \text{Der } \mathfrak{p}(S,x) : D(\log \sigma_o) = 0 \}
\]
of \(\text{Der } \mathfrak{p}(S,x)\), where \(\sigma_o\) is the boundary loop and \(\log \sigma_o \in \mathfrak{p}(S,x)\) is its logarithm. The standard approach to bounding \(\mathfrak{g}_{S,\partial S}\) is to find linear functions on \(\text{Der}^\theta \mathfrak{p}(S,x)\) (or its associated weight graded) that vanish on \(\mathfrak{g}_{S,\partial S}\). The first such maps were Morita’s trace maps that were constructed in [63] and have been generalized by

---

8 The action of this group on a MHS is analogous to the action of the Galois group of a number field \(K\) on the profinite topological invariants of varieties defined over \(K\).

9 This is the problem of determining the image of the \(\sigma_{2m+1}\)'s mod \(\text{Gr}^W_{-k} \mathfrak{g}_{S,\partial S}\) and mod commutators of the \(\sigma_{2m+1}\)'s.

10 One can work with the smaller Lie algebra of special derivations. These annihilate \(\theta\) and are “inner” on logs of boundary loops.
Enomoto and Satoh [21] and Conant [11]. The most conceptually appealing methods for producing such functionals was proposed by Kawazumi and Kuno [56] and uses the Goldman–Turaev Lie bialgebra.

Briefly, the Goldman–Turaev Lie bialgebra associated to a surface \( S \) with a framing \( \xi \) is the free \( \mathbb{Z} \)-module \( \mathbb{Z}\lambda(S) \) generated by the set of conjugacy classes \( \lambda(S) \) of the fundamental group of \( S \). It has a Lie bracket
\[
\{ , \} : \mathbb{Z}\lambda(S) \otimes \mathbb{Z}\lambda(S) \to \mathbb{Z}\lambda(S)
\]
defined by Goldman [27], which does not require a framing, and a cobracket
\[
\delta_\xi : \mathbb{Z}\lambda(S) \to \mathbb{Z}\lambda(S) \otimes \mathbb{Z}\lambda(S)
\]
which does. A preliminary version of the cobracket was defined by Turaev [81]. We use a refined version for framed surfaces introduced by Turaev [82] and Alekseev, Kawazumi, Kuno and Naef [2].

Kawazumi and Kuno introduced the completed Goldman–Turaev Lie bialgebra \( \mathbb{Q}\lambda(S)^\wedge \) of a framed surface. One important property of it is that for surfaces with one boundary component, there is an inclusion
\[
\text{Der}^d p(S, x) \hookrightarrow \mathbb{Q}\lambda(S)^\wedge
\]
of Lie algebras. When \( S \) and its framing are algebraic, the bracket and cobracket are both morphisms of MHS [37, 38]. This allows one to identify each with its associated weight graded in a way that is compatible with the mapping class group actions.

One important property of the completed Goldman–Turaev Lie bialgebra is that the cobracket almost vanishes on the arithmetic Johnson image:
\[
W_{-2gS,\partial S} \subseteq \ker \delta_\xi
\]
and \( \delta_\xi \) induces an inclusion
\[
H_1(\mathbb{L}(\sigma_3, \sigma_5, \ldots)^\wedge) \hookrightarrow \mathbb{Q}\lambda(S)^\wedge / \ker \delta_\xi \to \mathbb{Q}\lambda(S)^\wedge \otimes \mathbb{Q}\lambda(S)^\wedge.
\]
This last property suggests that the cobracket should be a useful tool for identifying the images of the \( \sigma_{2m+1} \)'s in \( \text{Der}^d p(S, x) \) modulo the geometric derivations \( \mathbb{G}_{S,\partial S} \) and commutators of the \( \sigma_{2m+1} \)'s, the best one can hope for without specifying (for example) a pants decomposition of \( S \).

The Goldman–Turaev Lie bialgebra is reviewed in Section 3. Work of Kawazumi and Kuno on how the Turaev cobracket constrains the image of the Johnson homomorphism is discussed and refined in Section 12.

1.5. Cohomology and the Johnson image. Another approach to computing the weight graded quotients of the image of the Johnson homomorphism is via cohomology. As we explain in Section 16 the cohomology of \( u_{S,\partial S} \) plus its weight filtration determine the graded quotients of its lower central series. The problem is to compute the cohomology of \( u_{S,\partial S} \). A more realistic goal is to compute the stable cohomology of \( u_{S,\partial S} \), which will determine the stable graded quotients of \( \mathbb{G}_{S,\partial S} \).

For each \( \text{Sp}(H) \)-module \( V \), there is a homomorphism
\[
[ H^\bullet(u_{S,\partial S}) \otimes V ]^{\text{Sp}(H)} \to H^\bullet(\Gamma_{S,\partial S}, V).
\]

\[11\]This method works for any pronilpotent Lie algebra with a MHS with only negative weights, such as \( p(S, x) \).
The computation \[61\] of the stable cohomology of mapping class groups by Madsen and Weiss and a result \[57\] of Kawazumi and Morita, imply that this map is stably surjective when \(V\) is the trivial representation. Combined with results of Looijenga \[60\] and Garoufalidis and Getzler \[25\], one can show that this map is stably surjective for all \(V\). If one can show that the MHS on the cohomology group \(H^m(u_{S,\partial S})\) is stably pure in the sense that it has weight \(m\) when \(g \gg 0\), then the homomorphism \[59\] will be a stable isomorphism. This will give a computation of the stable value of \(Gr^{W}_{\ast}g_{S,\partial S}\) and an upper bound for the weight graded quotients of the Johnson image \(\overline{G}_{S,\partial S}\). Such a “purity” result for \(H^\ast(u_{S,\partial S})\) is closely related to the question of whether \(H^\ast(u_{S,\partial S})\) is (stably) Koszul dual to the enveloping algebra of \(u_{S,\partial S}\).

1.6. Is the Johnson homomorphism motivic? The geometric and arithmetic Johnson homomorphisms are not just morphisms of MHS, they also have an algebraic de Rham description and are, after tensoring with \(\mathbb{Q}_\ell\), equivariant with respect to the natural Galois actions provided suitable base points are used. This provides strong evidence that they are motivic. One can argue about what this should mean, but Johnson homomorphisms should be closely related to algebraic cycles and \(K\)-theory. There is considerable evidence that this is the case.

Johnson’s original homomorphism \[2\] for an algebraic curve \(C\) is related to the Ceresa cycle in its jacobian \(Jac\ C\) and also the Gross–Schoen cycle in the 3-fold \(C^3\), \[30\]. Kawazumi and Morita \[57\] showed that when \(V\) is the trivial coefficient system \(\mathbb{Q}\), the image of the homomorphism \[59\] in \(H^\ast(M_{g,1}, \mathbb{Q})\) is its (cohomological) tautological subring. More recently, Petersen, Tavakol and Yin \[74\] have defined for each partition \(\mu\) of an integer \(|\mu|\), the twisted Chow group \(\text{CH}^\ast(M_g, V_{\mu})\) of \(M_g\).

There is a cycle class map

\[
\text{CH}^k(M_g, V_{\mu}) \to H^{2k-|\mu|}(M_g; V_{\mu})
\]

where \(V_{\mu}\) denotes the variation of Hodge structure of weight \(-|\mu|\) over \(M_g\) that corresponds to the irreducible \(\text{Sp}(H)\)-module corresponding to \(\mu\). They also define the twisted tautological subgroups \(R^k(M_g, V_{\mu})\) of \(\text{CH}^\ast(M_g, V_{\mu})\).

The constructions of Petersen, Tavakol and Yin and the construction of Kawazumi and Morita imply that the image of

\[
[W_{2k-2|\mu|}H^{2k-|\mu|}(u_g) \otimes V_{\mu}]^{\text{Sp}(H)} \to H^{2k-|\mu|}(M_g; V_{\mu})
\]

equals the image of \(R^k(M_g, V_{\mu}) \otimes \mathbb{Q}\) under the cycle map \(7\). Computations of Petersen–Tavakol–Yin \[74\] and of Morita–Sakasai–Suzuki \[67\] imply that this map is stably an isomorphism when \(2k - |\mu| \leq 6\). This suggests that this map is an isomorphism onto its images in \(H^\ast(M_g; V_{\mu})\) and that there is a deep connection between \(u_g\) and tautological algebraic cycles.

**Problem 1.1.** For each \(g \geq 0\), construct a Voevodsky motive whose Hodge/Betti realization is the coordinate ring \(O(G_g)\) of \(G_g\). Use this to show that the Johnson homomorphism is motivic.

This was established in genus 0 by Deligne and Goncharov in \[18\].

\[12\] With corrections by Petersen \[73\] and alternative proof by Kupers and Randal-Williams \[59\].
1.7. The most optimistic landscape. A summary of the most optimistic statements about the Johnson homomorphisms (for surfaces with one boundary component) that, in the opinion of the author, have a reasonable chance of being true are:

(i) The geometric Johnson homomorphism $g_{S,\partial S} \to \text{Der}(p(S,x))$ is injective, at least stably.

(ii) The kernel of the Turaev cobracket

\[ \delta_\xi : \mathbb{Q}\lambda(S)^\wedge \to \mathbb{Q}\lambda(S)^\wedge \otimes \mathbb{Q}\lambda(S)^\wedge \]

satisfies:

\[ (W_{-2} \ker \delta_\xi) \cap \text{Der}(p(S,\vec{v})) = W_{-2} \mathfrak{g}_{S,\partial S}. \]

(iii) The natural map

\[ [H^\bullet(u_{S,\partial S}) \otimes V_\mu]^{Sp(g)} \to H^\bullet(\Gamma_{S,\partial S}, V_\mu) \]

is stably an isomorphism for each partition $\mu$ and the corresponding $Sp_g$ module $V_\mu$. Equivalently, for each $m \geq 0$, the stable value of $H^m(u_{S,\partial S})$ is a pure Hodge structure of weight $m$.

These statements are discussed in much more detail in the body of the paper.

1.8. What is not included. Due to length constraints, we have not covered every aspect of Johnson homomorphisms. Three notable topics that we have omitted are:

(i) The relationship between derivations of $p(S,x)$ and the homology of outer automorphisms of free groups that is due to Kontsevich \[58\]. For an exposition, see \[12\].

(ii) Drinfeld’s pronilpotent version $GRT$ of the Grothendieck–Teichmüller group, \[20\] and its relation to Johnson homomorphisms.

(iii) Connections to non-commutative Poisson geometry via the Kashiwara–Vergne problem. See \[2\] and the references therein.

Other expositions of the Johnson homomorphism and related topics include Morita’s survey \[65\].

Acknowledgments: I am especially grateful to Florian Naef, who patiently answered my questions about his joint work with Alekseev, Kawazumi and Kuno, and to Dan Petersen for his correspondence regarding the material on the stable cohomology of relative completion in Section \[16\]. I would like to thank Anton Alekseev, Nariya Kawazumi, Yusuke Kuno and Shigeyuki Morita for comments on various drafts of this survey.

2. Topological Setup

Suppose that $\overline{S}$ is a connected, closed, oriented surface of genus $g$ and that $P$ and $Q$ are disjoint finite subsets of $\overline{S}$. Suppose that $\vec{V}$ is a set $\{\vec{v}_q : q \in Q\}$ of non-zero tangent vectors of $\overline{S}$, where $\vec{v}_q \in T_{\vec{v}_q} \overline{S}$. Set

\[ S = \overline{S} - (P \cup Q). \]

We will always assume that $S$ is hyperbolic. That is, that $S$ has negative Euler characteristic:

\[ 2g - 2 + \#P + \#Q > 0. \]
The corresponding mapping class group
\[ \Gamma_{\overline{S}, P+\vec{V}} := \pi_0(\text{Diff}^+(\overline{S}, P \cup \vec{V})) \]
is the group of isotopy classes of orientation preserving diffeomorphisms of \( \overline{S} \) that fix \( P \cup Q \) pointwise and fix each of the tangent vectors \( \vec{v}_q \). Up to inner isomorphism, it depends only on \( g \) and the cardinalities \( n \) of \( P \) and \( r \) of \( \vec{V} \). Often we will denote it by \( \Gamma_{g, n+r} \). We will refer to the \( n + r \) punctured surface with its tangent vectors \((\overline{S}, P, \vec{V})\) as a surface of type \((g, n+r)\).

Note that \( S \) is hyperbolic if and only if \( 2g - 2 + n + r > 0 \). The definition of \( \Gamma_{\overline{S}, P+\vec{V}} \) applies equally to disconnected surfaces, each of whose components is hyperbolic. The product of the mapping class groups of its components is a normal subgroup of finite index.

A complex structure on \((\overline{S}, P, \vec{V})\) is an orientation preserving diffeomorphism \( \phi: (\overline{S}, P, \vec{V}) \to (\overline{X}, Y, \vec{V}') \) where \( X \) is a compact Riemann surface, \( Y \) is a finite subset and \( \vec{V}' \) is a finite set of non-zero tangent vectors. We will refer to \((\overline{X}, Y, \vec{V}')\) as a complex curve of type \((g, n+r)\).

There is a moduli space \( M_{g,n+r} \) of complex curves of type \((g, n+r)\). It will be viewed as an orbifold. As such, it is the classifying space of \( \Gamma_{g,n+r} \). The complex structure \( \phi \) determines a point in its universal covering (Teichmüller space) and a canonical isomorphism \( \phi_*: \Gamma_{\overline{S}, P+\vec{V}} \to \pi_1(M_{g,n+r}, \phi) \).

Note that \( M_{g,n+r} \) is a \((\mathbb{C}^*)^r\) bundle over \( \mu_{g,n+r} \) so that one has a central extension \[ 0 \to \mathbb{Z}^r \to \Gamma_{g,n+r} \to \mu_{g,n+r} \to 1. \]

Remark 2.1. From a topological point of view, each tangent vector \( \vec{v} \in \vec{V} \) corresponds to a boundary component. This can be seen using real oriented blow ups. Rotating the tangent vector \( \vec{v} \) in \( T_{\overline{S}} \) corresponds to a Dehn twist about the corresponding boundary component. See [37, §12.1] for a more detailed discussion.

For each commutative ring \( A \), set
\[ H_A := H_1(\overline{S}; A). \]
The intersection pairing \( \langle , \rangle \) on \( H_A \) is unimodular. Denote the corresponding symplectic group (the automorphisms of \( H_A \) that preserve the intersection pairing) by \( \text{Sp}(H_A) \). The functor \( A \) to \( \text{Sp}(H_A) \) is an affine group that we will denote by \( \text{Sp}(H) \). Likewise, we will regard \( H \) as the unipotent group with \( A \)-rational points \( H_A \).

The action of \( \Gamma_{\overline{S}, P+\vec{V}} \) on \( \overline{S} \) induces an action on \( H_1(\overline{S}) \) that preserves the intersection pairing. This corresponds to a homomorphism \( \rho: \Gamma_{\overline{S}, P+\vec{V}} \to \text{Sp}(H_A) \).

It is well-known to be surjective. The Torelli group of type \((g, n+r)\) is its kernel:
\[ T_{g,n+r} = T_{\overline{S}, P+\vec{V}} := \ker \rho. \]

Sometimes we will be less formal, and refer to \((S, \vec{V})\) as a surface of type \((g, n+r)\).
For each $\vec{v} \in \vec{V}$, one has the fundamental group $\pi_1(S,\vec{v})\] The mapping class group action induces an injective homomorphism $\Gamma_{\vec{V}, p + \vec{v}} \hookrightarrow \text{Aut} \pi_1(S,\vec{v})$.

Its image lies in the subgroup of automorphisms that fix the “boundary loop” $\sigma_o$ that corresponds to rotating $\vec{v}$ once about its anchor point $q$ in the positive direction. (Equivalently, traversing the corresponding boundary loop once.)

In this paper, we are primarily interested in the case where $(g, n + \vec{r}) = (g, \vec{1})$, where $S$ is a surface of genus $g \geq 1$ with one “boundary component”.

3. The Goldman–Turaev Lie bialgebra

Suppose that $\mathbb{k}$ is a commutative ring. Denote the set of free homotopy classes of maps $S^1 \to S$ in the surface $S$ by $\lambda(S)$ and the free $\mathbb{k}$-module it generates by $\mathbb{k}\lambda(S)$. The set $\lambda(S)$ is the set of conjugacy classes of $\pi_1(S, x)$ and $\mathbb{k}\lambda(S)$ can be described algebraically as the “cyclic quotient”

$$\|\mathbb{k}\pi_1(S, x)\| := \mathbb{k}\pi_1(S, x)/\langle uv - vu : u, v \in \mathbb{k}\pi_1(S, x)\rangle$$

of the group algebra. Note that $\langle uv - vu : u, v \in \mathbb{k}\pi_1(S, x)\rangle$ is the subspace spanned by the commutators $uv - vu$. It is not an ideal.

Denote the augmentation ideal of $\mathbb{k}\pi_1(S, x)$ by $I_k$. The decomposition $\mathbb{k}\pi_1(S, x) = \mathbb{k} \oplus I_k$ descends to a canonical decomposition

$$\mathbb{k}\lambda(S) = \mathbb{k} \oplus I_k\lambda(S)$$

where the copy of $\mathbb{k}$ is spanned by the trivial loop.

3.1. The Goldman bracket. Goldman [27] defined a binary operation

$$\{ , \} : \mathbb{Z}\lambda(S) \otimes \mathbb{Z}\lambda(S) \to \mathbb{Z}\lambda(S)$$

giving $\mathbb{Z}\lambda(S)$ the structure of a Lie algebra over $\mathbb{Z}$. Briefly, the bracket of two oriented loops $a$ and $b$ is defined by choosing transverse, immersed representatives $\alpha$ and $\beta$ of the loops, and then defining

$$\{a, b\} = \sum_p \epsilon_p(\alpha, \beta) [\alpha \#_p \beta]$$

where the sum is taken over the points $p$ of intersection of $\alpha$ and $\beta$, $\epsilon_p(\alpha, \beta) \in \{\pm 1\}$ denotes the local intersection number of $\alpha$ and $\beta$ at $p$, and where $[\alpha \#_p \beta]$ denotes the free homotopy class of the oriented loop $\alpha \#_p \beta$ obtained by joining $\alpha$ and $\beta$ at $p$ by a simple surgery.

3.2. The Turaev cobracket. Since $S$ is oriented, framings of $S$ correspond to nowhere vanishing vector fields on $S$. For each choice of a framing $\xi$ (a vector field) of $S$, there is a map

$$\delta_\xi : \mathbb{Z}\lambda(S) \to \mathbb{Z}\lambda(S) \otimes \mathbb{Z}\lambda(S),$$

called the Turaev cobracket, that gives $\mathbb{Z}\lambda(S)$ the structure of a Lie coalgebra. The value of the cobracket on a loop $a \in \lambda(X)$ is obtained by representing it by an

---

14This is Deligne's fundamental group with tangential base point. Using it as a base point is equivalent to replacing the anchor point $q \in Q$ of $\vec{v}$ by a boundary component (via real oriented blow up) and choosing a base point on the boundary component. See [27] §12.1 for details.

15The axioms of a Lie bialgebra $C$ are obtained by reversing the arrows in the definition of a Lie algebra. Specifically, the cobracket is skew symmetric, and the dual Jacobi identity holds.
impressed circle \( \alpha : S^1 \rightarrow X \) with transverse self intersections and trivial winding number relative to \( \xi \). Each double point \( p \) of \( \alpha \) divides it into two loops based at \( p \), which we denote by \( \alpha'_p \) and \( \alpha''_p \). Let \( \epsilon_p = \pm 1 \) be the intersection number of the initial arcs of \( \alpha'_p \) and \( \alpha''_p \). The cobracket of \( \alpha \) is then defined by

\[
\delta \xi(a) = \sum_p \epsilon_p (a'_p \otimes a''_p - a''_p \otimes a'_p),
\]

where \( a'_p \) and \( a''_p \) are the homotopy classes of \( \alpha'_p \) and \( \alpha''_p \), respectively.

The cobracket \( \delta \xi \) and the Goldman bracket endow \( \mathbb{Z}\lambda(S) \) with the structure of a Lie bialgebra \([82, 56]\). This simply means that the cobracket and bracket satisfy

\[
\delta \xi \quad \text{and} \quad \{ \cdot, \cdot \}
\]

vanishes.

The reduced cobracket

\[
\overline{\delta} : \text{IZ}(\mathbb{Z}\lambda(S)) \rightarrow \text{IZ}\lambda(S) \otimes \text{IZ}\lambda(S)
\]

is the map induced by composing the restriction of \( \delta \xi \) to \( \text{IZ}\lambda(S) \) with the square of the projection \( \mathbb{Z}\lambda(S) \rightarrow I\lambda(S) \). It does not depend on the framing \( \xi \). The reduced cobracket was first defined by Turaev \([81]\) on \( \mathbb{Z}\lambda(S)/\mathbb{Z} \cong I\lambda(S) \) and lifted to \( \mathbb{Z}\lambda(S) \) for framed surfaces in \([82, \S18]\) and \([2]\).

Remark 3.1. Framings of \( S \) form a principal homogeneous space under \( H^1(S; \mathbb{Z}) \). The definition of the cobracket \( \delta \xi \) implies that the corresponding cobrackets also form a principal homogeneous space under \( H^1(S; \mathbb{Z}) \). Suppose that \( \xi_0 \) and \( \xi_1 \) are two framings of \( S \). Then \( \xi_1 - \xi_0 = \alpha \in H^1(S; \mathbb{Z}) \). Since the reduced cobracket does not depend on the framing, \( \overline{\delta} \xi_0 = \overline{\delta} \xi_1 \). There is therefore a function \( f_\alpha : \mathbb{Z}\lambda(S) \rightarrow \mathbb{Z}\lambda(S) \) that depends only on \( \alpha \in H^1(S; \mathbb{Z}) \) such that

\[
\delta \xi_1 = \delta \xi_0 + f_\alpha \otimes 1 - 1 \otimes f_\alpha.
\]

Property \([10]\) implies that \( f_\alpha \) is a 1-cocycle on \( \mathbb{Z}\lambda(S) \):

\[
f_\alpha(\{u, v\}) = u \cdot f_\alpha(v) - v \cdot f_\alpha(u)
\]

that determines \( \alpha \).

3.3. The Kawazumi–Kuno Action. Suppose that \( \vec{\gamma}, \vec{\gamma}' \in \vec{V} \). Denote the torsor of homotopy classes of paths in \( S \) from \( \vec{\gamma} \) to \( \vec{\gamma}' \) by \( \pi(S; \vec{\gamma}, \vec{\gamma}') \). There is an “action”

\[
\kappa : \mathbb{Z}\lambda(S) \otimes \mathbb{Z}\pi(S; \vec{\gamma}, \vec{\gamma}') \rightarrow \mathbb{Z}\pi(S; \vec{\gamma}, \vec{\gamma}')
\]

which was defined by Kawazumi and Kuno \([54]\). The definition is similar to the definition of the Goldman bracket given above. It is compatible with path multiplication

\[
\mathbb{Z}\pi(S; \vec{v}_1, \vec{v}_2) \otimes \mathbb{Z}\pi(S; \vec{v}_2, \vec{v}_3) \rightarrow \mathbb{Z}\pi(S; \vec{v}_1, \vec{v}_3) \quad \vec{v}_1, \vec{v}_2, \vec{v}_3 \in \vec{V}
\]

in the sense that

\[
\kappa(\gamma \otimes (bc)) = \kappa(\gamma \otimes b)c + b\kappa(\gamma \otimes c).
\]

16This is standard. The explanation is recalled at the beginning of Section 4.
In particular, when \( \vec{v} = \vec{v}' = \vec{v} \), there is a Lie algebra homomorphism

\[
\kappa_{\vec{v}} : \mathbb{Z}\lambda(S) \to \text{Der}\mathbb{Z}\pi(S, \vec{v})
\]

Its image is contained in the Lie subalgebra

\[
\text{Der}^\theta \mathbb{Z}\pi_1(S, \vec{v}) := \{ D \in \text{Der}^\theta \mathbb{Z}\pi_1(S, \vec{v}) : D(\sigma_o) = 0 \},
\]

where

\[
\text{Der}^\theta \mathbb{Z}\pi_1(S, \vec{v}) := \{ D \in \text{Der}^\theta \mathbb{Z}\pi_1(S, \vec{v}) : D(\sigma_o) = 0 \}
\]

and \( \sigma_o \in \pi_1(S, \vec{v}) \) is the loop that corresponds to rotating \( \vec{v} \) once in the positive direction about its anchor point.

3.4. **Power operations.** For \( n \in \mathbb{Z} \), the power operation \( \psi_n : \mathbb{k}\pi_1(S, \vec{v}) \to \mathbb{k}\pi_1(S, \vec{v}) \) is defined by taking \( \gamma \in \pi_1(S, \vec{v}) \) to \( \gamma^n \). Since \( \Delta(\gamma^n) = \gamma^n \otimes \gamma^n \), it follows that the power operations commute with the coproduct

\[
\Delta \circ \psi_n = (\psi_n \otimes \psi_n) \circ \Delta.
\]

The power map \( \psi_n \) of \( \mathbb{k}\pi_1(S, \vec{v}) \) descends to a power map

\[
\psi_n : \mathbb{k}\lambda(S) \to \mathbb{k}\lambda(S),
\]

It would be useful to know how the power operations \( \psi_n \) interact with the bracket and cobracket. One case where this can be understood is where \( \gamma \) is an an imbedded circle. In this case

\[
\delta_\xi \circ \psi_n(\gamma) = n \cdot \text{rot}_\xi(\gamma)(\psi_n(\gamma) \otimes 1 - 1 \otimes \psi_n(\gamma)).
\]

4. **Mapping class group orbits of framings and the stabilizer of the cobracket**

In this section we assume, for simplicity, that \( (\overline{S}, \vec{V}) \) is a surface of type \((g, \vec{1})\) where \( g > 0 \). Similar results hold in general\(^\text{[15]}\) The cobracket \( \delta_\xi \) depends non-trivially on the framing \( \xi \) and the action of the mapping class group \( \Gamma_{g, \vec{1}} \) on \( \mathbb{Z}\lambda(S) \) does not preserve the cobracket. Here we identify the stabilizer of a framing. It preserves the cobracket.

Since \( \overline{S} \) is oriented, we can (and will) regard its tangent bundle \( T\overline{S} \) as a smooth complex line bundle. If \( \xi_0 \) and \( \xi_1 \) are two framings of \( S \), then \( \xi_1 = f\xi_0 \), where \( f : S \to \mathbb{C}^* \). Their homotopy classes differ by the homotopy class of \( f \), which is an element of \( H^1(\mathbb{S}; \mathbb{Z}) = H^1(\overline{S}; \mathbb{Z}) \).

Since the tangent bundle of \( S \) is trivial, there is a short exact sequence

\[
0 \to \mathbb{Z} \to H_1(T'S; \mathbb{Z}) \to H_1(S; \mathbb{Z}) \to 0,
\]

where \( T'S \) denotes the set of non-zero tangent vectors of \( S \). Each framing \( \xi : S \to T'S \) of \( S \) induces a splitting \( s(\xi) \) of this sequence. The difference between two such splittings is naturally an element of \( H^1(S; \mathbb{Z}) \):

\[
s(\xi_1) - s(\xi_0) \in \text{Hom}(H_1(S; \mathbb{Z}); \mathbb{Z}) \cong H^1(\overline{S}; \mathbb{Z}).
\]

This equals the class of \( f : S \to \mathbb{C}^* \).

The splitting \( s(\xi_0) \) induces an isomorphism

\[
H_1(T'S; \mathbb{Z}) \cong H_1(S; \mathbb{Z}) \oplus \mathbb{Z} = H_\mathbb{Z} \oplus \mathbb{Z}
\]

\(^\text{[15]}\)For the general case, see \([38, \S 11]\).
and an isomorphism of the group of automorphisms of the extension \( \mathbb{I} \) with 
\[
\text{Sp}(H_2) \ltimes H_2.
\]

The action of \( \Gamma_{g,1} \) on \( S \) induces an action of it on \( T'S \) that preserves the sequence \( \mathbb{I} \), and therefore a homomorphism \( \tilde{\rho}_{\xi_0} : \Gamma_{g,1} \rightarrow \text{Sp}(H_2) \ltimes H_2 \). It also induces a left action on framings. Identify \( \text{Sp}(H_2) \) with the subgroup of \( \text{Sp}(H_2) \ltimes H_2 \) consisting of \( (\phi, u) \) with \( u = 0 \).

**Proposition 4.1.** When \( g \geq 2 \), the stabilizer of the homotopy class of \( \xi_0 \) in \( \Gamma_{g,1} \) is the inverse image of \( \text{Sp}(H_2) \) under \( \tilde{\rho}_{\xi_0} \).

Denote the stabilizer of the homotopy class of \( \xi \) by \( \Gamma_{g,1}^\xi \).

**Corollary 4.2.** The action of the mapping class group \( \Gamma_{g,1} \) on \( \mathbb{Z}(S) \) preserves the Goldman bracket and the stabilizer \( \Gamma_{g,1}^\xi \) preserves the cobracket \( \delta_\xi \).

**Proposition 4.3.** The restriction of \( \tilde{\rho}_{\xi_0} \) to the Torelli group is the homomorphism
\[
\begin{array}{c}
T_{g,1} \longrightarrow H_1(T_{g,1}) \xrightarrow{\tau} \Lambda^3 H_2 \xrightarrow{2c} H_2
\end{array}
\]
where \( \tau \) is Johnson’s homomorphism and \( c \) is the \( \text{Sp}(H) \)-invariant contraction
\[
u \wedge v \wedge w \mapsto \langle u, v \rangle w + \langle v, w \rangle u + \langle w, u \rangle v.
\]
The restriction of \( \tilde{\rho}_{\xi_0} : H_1(T_{g,1}) \rightarrow H_2 \) to the image of the “point pushing subgroup” of \( T_{g,1} \) is the multiplication by \( 2g - 2 \) map \( H_2 \rightarrow H_2 \).

**Proposition 4.4.** When \( g \geq 2 \), the map \( \Gamma_{g,1} / \Gamma_{g,0} \rightarrow H_2 \) induced by \( \tilde{\rho}_{\xi_0} \) that takes the coset of \( \phi \) to \( \tilde{\rho}_{\xi_0}(\xi_0) = \phi_* \xi_0 - \xi_0 \) is injective and has image \( (g - 1)H_2 \).

### 4.1. Orbits of framings

Kawazumi [53] determined the mapping class group orbits of framings of a surface with finite topology. We recall the classification in the case of surfaces \( \overline{S} \) of type \( (g, 1) \), where \( g \geq 1 \) and \( \overline{v} \in T_{g,1} \). Set \( S = \overline{S} - \{ q \} \).

For a simple closed curve \( c \) on \( S \), let
\[
f_\xi(c) = 1 + \text{rot}_\xi(c) \mod 2 \in \mathbb{F}_2,
\]
where \( \text{rot}_\xi(c) \) denotes the winding number of \( c \) relative to \( \xi \). The Poincaré–Hopf Theorem implies that the index (local winding number) of a framing of \( S \) at \( q \) is \( 2 - 2g \). This implies that \( f_\xi(c) \) depends only on the homology class of \( c \) in \( H_1(\overline{S}; \mathbb{F}_2) \), so that \( f_\xi \) induces a well define map \( H_1(\overline{S}; \mathbb{F}_2) \rightarrow \mathbb{F}_2 \). It is easily verified to be an \( \mathbb{F}_2 \)-quadratic form. The Arf invariant of \( \xi \) is defined to be the Arf invariant
\[
\text{Arf}(\xi) := \sum_{j=1}^{g} f_\xi(a_j)f_\xi(b_j)
\]
of \( f_\xi \), where \( a_1, \ldots, a_g, b_1, \ldots, b_g \) is a symplectic basis of \( H_{g,2} \). The Arf invariant is constant on mapping class group orbits of framings.

**Theorem 4.5** (Kawazumi [53]). When \( g > 1 \) there there are two \( \Gamma_{g,1} \) orbits of framings of \( S \). These are distinguished by their Arf invariants. If \( g = 1 \), then two framings \( \xi_0 \) and \( \xi_1 \) are in the same \( \Gamma_{1,1} \) orbit if and only if

\[
A(\xi_0) = A(\xi_1),
\]

where
\[
A(\xi) = \gcd \{ \text{rot}_\xi(\gamma) : \gamma \text{ is a non-separating simple closed curve in } S \}.
\]

This is congruent to \( 1 + \text{Arf}(\xi) \mod 2 \).
5. Completions

5.1. Unipotent completion. Suppose that $\pi$ is a discrete group and that $k$ is a commutative ring. The group algebra $k\pi$ is a Hopf algebra with comultiplication $\Delta : k\pi \to k\pi \otimes k\pi$ induced by taking each $\gamma \in \pi$ to $\gamma \otimes \gamma$. As previously, $I_k$ denotes the augmentation ideal of $k\pi$. Its powers define a topology on $k\pi$. The $I$-adic completion of $k\pi$ is

$$k\pi^\wedge := \lim_{\leftarrow n} k\pi / I_k^n.$$ 

It is a complete Hopf algebra with (completed) coproduct

$$\Delta : k\pi^\wedge \to k\pi^\wedge \otimes k\pi^\wedge$$

induced by the coproduct of $k\pi$.

When $\pi$ is finitely generated, the unipotent completion $\pi^\un$ of $\pi$ is the pronipotent $\mathbb{Q}$-group whose group of $k$-rational points (where $k$ is a $\mathbb{Q}$-algebra) is the set of grouplike elements

$$\pi^\un(k) = \{ u \in k\pi^\wedge : \Delta u = u \otimes u \} \subset 1 + I_k^\wedge.$$ 

Its Lie algebra $p$ is the set

$$p := \{ v \in Q\pi^\wedge : \Delta v = v \otimes 1 + 1 \otimes v \} \subset I_Q^\wedge.$$ 

The natural map $\pi \to Q\pi^\wedge$ induces a natural homomorphism $\pi \to \pi^\un(k)$.

The exponential and logarithm maps

$$I_k^\wedge \xrightarrow{\exp} 1 + I_k^\wedge$$

are mutually inverse bijections and restrict to a bijection $p \otimes k \cong \pi^\un(k)$. This bijection is a group isomorphism if we give $p$ the multiplication given by the Baker–Campbell–Hausdorff (BCH) formula. It implies that every element of $\pi^\un(Q)$ has a logarithm that lies in $p$.

The universal enveloping algebra $U p$ is a topological algebra whose $I$-adic completion is naturally isomorphic to $Q\pi^\wedge$ and the coordinate ring of $\pi^\un$ is the continuous dual of $Q\pi$:

$$O(\pi^\un) = \text{Hom}_{\mathbb{Q}}^{\text{cts}}(Q\pi^\wedge, Q) := \lim_{\leftarrow n} \text{Hom}_{\mathbb{Q}}(Q\pi / I^n, \mathbb{Q}).$$

5.2. The completed Goldman–Turaev Lie bialgebra. Now suppose that $(\overline{S}, P)$ is a decorated surface and that $k$ is a $\mathbb{Q}$-algebra. The $I$-adic topology on $k\lambda(S)$ is the quotient topology induced by the quotient map

$$k\pi_1(S, x) \to |k\pi_1(S, x)| = k\lambda(S).$$

It does not depend on the choice of base point $x \in S$. Denote the image of $I_k^\wedge$ by $I_k^\lambda$. The $I$-adic completion of $k\lambda(S)$ is

$$k\lambda(S)^\wedge = \lim_{\leftarrow n} k\lambda(S) / I_k^n \lambda(S).$$

Kawazumi and Kuno showed in [55] that the Goldman bracket is continuous in the $I$-adic topology on $k\lambda(S)^\wedge$ and therefore induces a Lie bracket

$$\{ , \} : k\lambda(S)^\wedge \otimes k\lambda(S)^\wedge \to k\lambda(S)^\wedge.$$
They also proved that when $S$ is framed with framing $\xi$, the cobracket $\delta_\xi$ is continuous so that it induces a cobracket

$$\delta_\xi : k\lambda(S)^\wedge \to k\lambda(S)^\wedge \hat{\otimes} k\lambda(S)^\wedge.$$  

Similarly, for a surface $(S, \vec{v})$, Kawazumi and Kuno show that the action

$$\kappa : k\lambda(S) \otimes k\pi_1(S, \vec{v}) \to k\pi_1(S, \vec{v})$$

is continuous in the $I$-adic topology, and therefore induces a continuous map

$$\hat{\kappa} : k\lambda(S)^\wedge \otimes k\pi_1(S, \vec{v})^\wedge \to k\pi_1(S, \vec{v})^\wedge$$

and a continuous Lie algebra homomorphism

$$\hat{\kappa}_{\vec{v}} : k\lambda(S)^\wedge \to \text{Der}^\theta k\pi_1(S, \vec{v})^\wedge.$$  

5.2.1. Relation to the derivation algebra. Denote the Lie algebra of $\pi_1^\text{un}(S, \vec{v})$ by $p(S, \vec{v})$. The Lie algebra

$$\text{Der}^\theta p(S, \vec{v})$$

of continuous derivations of $p(S, \vec{v})$ that fix $\theta := \log \sigma$, is the recipient of the Johnson homomorphism. It is the Lie subalgebra of $\text{Der}^\theta Q\pi_1(S, \vec{v})^\wedge$ consisting of those derivations of $Q\pi_1(S, \vec{v})^\wedge$ that respect its Hopf algebra structure.

The following is a special case of a result [55, Thm. 6.2.1] of Kawazumi and Kuno.

**Theorem 5.1** (Kawazumi–Kuno). If $(\vec{S}, \vec{v})$ is a surface of type $(g, \vec{a})$, then the Lie algebra homomorphism

$$\hat{\kappa}_{\vec{v}} : Q\lambda(S)^\wedge \to \text{Der}^\theta Q\pi_1(S, \vec{v})^\wedge.$$  

is surjective and has 1-dimensional kernel, which is spanned by the trivial loop.

This result allows the Turaev cobracket to be lifted to $\text{Der}^\theta Q\pi_1(S, \vec{v})^\wedge$. This is key to bounding the size of the Johnson image, which we shall explain in Section 12.

5.2.2. A dual PBW-like decomposition of $Q\lambda(S)^\wedge$. The Poincaré–Birkhoff–Witt Theorem implies that the symmetrization map

$$(15) \prod_{n \geq 0} \text{Sym}^n p(S, \vec{v}) \to Q\pi_1(S, \vec{v})^\wedge$$

defined by taking

$$u_1 u_2 \ldots u_n \in \text{Sym}^n p(S, \vec{v}) \text{ to } \frac{1}{n!} \sum_{\sigma \in \Sigma_n} u_{\sigma(1)} u_{\sigma(2)} \ldots u_{\sigma(n)} \in Q\pi_1(S, \vec{v})^\wedge$$

is a complete coalgebra isomorphism. This induces a PBW-like decomposition of $|Q\lambda(S)^\wedge|$. Denote the image of $\text{Sym}^n p(S, \vec{v})$ in $Q\lambda(S)^\wedge$ by $|\text{Sym}^n p(S, \vec{v})|$.

**Lemma 5.2.** For each $k \geq 1$ and $n \geq 0$, $\psi_k$ acts on $\text{Sym}^n p(S, \vec{v})$ and $|\text{Sym}^n p(S, \vec{v})|$ as multiplication by $k^n$.

**Proof.** Observe that the projection map $Q\pi_1(S, \vec{v})^\wedge \to Q\lambda(S)^\wedge$ commutes with each $\psi_k$. Since $\psi_k$ acts on the group-like elements of $Q\pi_1(S, \vec{v})^\wedge$ by the $k$th-power map, it acts on $p(S, \vec{v})$ as multiplication by $k$. Since $\psi_k$ commutes with the coproduct [12], and since the PBW isomorphism is a coalgebra isomorphism, it follows by
induction on \( n \), that the restriction of \( \psi_k \) to \( \text{Sym}^n p(S, \vec{v}) \) is multiplication by \( k^n \) as the reduced diagonal

\[
\text{Sym}^n p(S, \vec{v}) \to \sum_{a+b=n, a,b>0} \text{Sym}^a p(S, \vec{v}) \otimes \text{Sym}^b p(S, \vec{v})
\]

is injective when \( n > 1 \). The second statement follows as the projection \( \mathbb{Q} \pi_1(S, \vec{v})^\wedge \to \mathbb{Q} \lambda(S)^\wedge \) commutes with \( \psi_k \).

As observed in [37, §8.3], a direct consequence is that \( \mathbb{Q} \lambda(S)^\wedge \) decomposes as the completed product of the \(|\text{Sym}^n p(S, \vec{v})|\).

\textbf{Corollary 5.3.} The PBW isomorphism \([13]\) descends to a direct product decomposition

\[
(16) \quad \mathbb{Q} \lambda(S)^\wedge = |\mathbb{Q} \pi_1(S, \vec{v})^\wedge| \cong \prod_{n \geq 0} |\text{Sym}^n p(S, \vec{v})|.
\]

This decomposition does not depend on \( \vec{v} \).

There is a canonical isomorphism \(|p(S, \vec{v})| \cong H_1(S; \mathbb{Q})\) as \(|u, v| = 0\) for all \( u, v \in p(S, \vec{v}) \). When \((S, \vec{v})\) is of type \((g, \vec{1})\) we can say more. The proof is sketched in subsection 7.4.2.

\textbf{Proposition 5.4.} If \( g \geq 1 \) and \((S, \vec{v})\) is a surface of type \((g, \vec{1})\), then the subspace \(|\text{Sym}^2 p(S, \vec{v})|\) of \( \mathbb{Q} \lambda(S)^\wedge \) is a Lie subalgebra and \( \kappa_\vec{v} \) induces a Lie algebra isomorphism \(|\text{Sym}^2 p(S, \vec{v})| \to \text{Der}^0 p(S, \vec{v})|\).

\textbf{5.3. Relative unipotent completion.} This is a very brief review of relative unipotent completion of discrete groups, especially of mapping class groups. A more detailed exposition can be found in [33], and complete results in [31].

Suppose that:

(i) \( \Gamma \) is a discrete group
(ii) \( k \) is a field of characteristic zero,
(iii) \( R \) is a reductive group over \( k \),
(iv) \( \rho : \Gamma \to R(k) \) is a Zariski dense homomorphism.

The completion of \( \Gamma \) relative to \( \rho \) (or the relative completion of \( \Gamma \)) consists of an affine group \( \overline{G} \) over \( k \) and a homomorphism \( \bar{\rho} : \Gamma \to \overline{G} \). The group \( \overline{G} \) is an extension of \( R \) by a prounipotent group:

\[
1 \to U \to \overline{G} \to R \to 1
\]

where the composition \( \Gamma \to \overline{G} \to R(k) \) is \( \rho \). These have the property that if

\[
1 \to U \to G \to R \to 1
\]

is an extension of affine groups over \( k \), where \( U \) is prounipotent, and if \( \phi : \Gamma \to G(k) \) is a homomorphism through which \( \rho \) factors \( \Gamma \to G(k) \to R(k) \), then there is a unique homomorphism of affine groups \( \overline{G} \to G \) that commutes with projections to \( R \) such that \( \phi \) is the composition \( \Gamma \to \overline{G} \to G \). That is, \( (\overline{G}, \bar{\rho}) \) is an initial object of a category whose objects are pairs \((G, \phi)\) and whose morphisms are appropriately defined.

\textbf{Remark 5.5.} Several comments are in order:

\textit{Equivalently, a proalgebraic group.}
(i) One can also define relative completion using tannakian categories, as explained in [42 §10.1].
(ii) The homomorphism $\hat{\rho} : \Gamma \to G(k)$ is Zariski dense.
(iii) The prounipotent group $U$ is determined by its Lie algebra.
(iv) When $R$ is trivial, relative completion reduces to unipotent completion.

This is the correct way to define unipotent completion for discrete groups $\pi$ with infinite dimensional $H_1(\pi; \mathbb{Q})$. This is relevant as genus 2 Torelli groups are countably generated [62] and have infinite dimensional abelianization.

5.4. Relative completion of mapping class groups. In this section, we take $k = \mathbb{Q}$. The group $\text{Sp}(H)$ is a reductive $\mathbb{Q}$-group. The completion of $\Gamma_{g,n+\mathbb{f}}$ relative to the standard homomorphism $\rho : \Gamma_{g,n+\mathbb{f}} \to \text{Sp}(H_\mathbb{Q})$ will be denoted by $G_{g,n+\mathbb{f}}$.

Denote its prounipotent radical by $U_{g,n+\mathbb{f}}$ and the Lie algebras of $G_{g,n+\mathbb{f}}$ and $U_{g,n+\mathbb{f}}$ by $g_{g,n+\mathbb{f}}$ and $u_{g,n+\mathbb{f}}$. With the help of Hodge theory, we will give presentations of various $g_{g,n+\mathbb{f}}$ in Section 7.2.

Relative completion is just unipotent completion in genus 0 as $H = 0$. The Lie algebras $g_{0,n+\mathbb{f}}$ are well understood. We recall their presentation in Section 8.0.1.

**Theorem 5.6.** For all $g \geq 2$, $H_1(u_{g,n+\mathbb{f}})$ is finite dimensional, so that $u_{g,n+\mathbb{f}}$ is finitely (topologically) generated. When $g = 1$, $H_1(u_{1,n+\mathbb{f}})$ is infinite dimensional, so that $u_{1,n+\mathbb{f}}$ is not finitely generated.

The genus 1 case is important as it is closely related to classical modular forms. It will be discussed in more detail in Section 8.0.4.

Since the Torelli group $T_{g,n+\mathbb{f}}$ is the kernel of $\rho$, its image in $G_{g,n+\mathbb{f}}(\mathbb{Q})$ lies in $U_{g,n+\mathbb{f}}(\mathbb{Q})$. Since this is prounipotent, the restriction of $\rho$ to the Torelli group factors through relative completion:

$$T_{g,n+\mathbb{f}} \to T_{g,n+\mathbb{f}}^{\text{un}}(\mathbb{Q}) \to U_{g,n+\mathbb{f}}(\mathbb{Q}).$$

Denote the Lie algebra of $T_{g,n+\mathbb{f}}^{\text{un}}$ by $t_{g,n+\mathbb{f}}$.

**Theorem 5.7.** If $g \geq 2$, the homomorphism $T_{g,n+\mathbb{f}}^{\text{un}} \to U_{g,n+\mathbb{f}}$ is surjective. When $g \geq 3$, its kernel is a copy of the additive group $\mathbb{G}_a$ contained in the center of $T_{g,n+\mathbb{f}}^{\text{un}}$, so that there is a central extension

$$0 \to \mathbb{Q} \to t_{g,n+\mathbb{f}} \to u_{g,n+\mathbb{f}} \to 0,$$

of prounipotent Lie algebras. When $g = 2$, the kernel of $T_{g,n+\mathbb{f}}^{\text{un}} \to U_{g,n+\mathbb{f}}$ is infinitely generated and free when $r = n = 0$.

**Remarks on the proof.** I understand this much better than when I first published a proof of the genus $g \geq 3$ case in [29]. In the interests of clarity, and because of its relevance to the study of Johnson homomorphisms, I will give a brief sketch of the argument.

The first step is to note that, when $g \geq 2$, standard cohomology vanishing theorems imply that the completion of $\text{Sp}(H_\mathbb{Q})$ with its inclusion into $\text{Sp}(H_\mathbb{Q})$ is just $\text{Sp}(H)$. (See Example 3.2 in [33].) The second is to use the fact that relative completion is right exact [33 Prop. 3.7] to see the sequence

$$T_{g,n+\mathbb{f}}^{\text{un}} \to G_{g,n+\mathbb{f}} \to \text{Sp}(H) \to 1$$

is exact. This implies that $T_{g,n+\mathbb{f}}^{\text{un}} \to U_{g,n+\mathbb{f}}$ is surjective when $g \geq 2$ as every group is Zariski dense in its unipotent completion.
Now suppose that \( g \geq 3 \). The next ingredient is to use [29, Prop. 4.13] to see that the kernel of this map is central and there is an exact sequence

\[
H_2(\text{Sp}(H_Z; \mathbb{Q})) \rightarrow T_{g,n+r}^\text{un} \rightarrow \mathcal{U}_{g,n+r} \rightarrow 1.
\]

The proof of this requires that \( H_1(T_{g,n+r}; \mathbb{Q}) \) be finite dimensional. This holds when \( g \geq 3 \) by Johnson’s computation of \( H_1(T_{g,1}) \) in [50] and fails in genus 2 in view of Mess’s computation [62]. As explained in [29] the injectivity on the left hand end is equivalent to the non-triviality of the biextension line bundle over \( \mathcal{M}_g \) associated to the Ceresa cycle \( C - C^- \) in the jacobian of a smooth projective curve \( C \) of genus \( g \). This follows from a computation of Morita in [64, 5.8]. Another proof closer to algebraic geometry can be found in [43, Thm. 7].

Mess [62] proved that \( \text{Sp}(H) \) is a countably generated free groups and \( H_1(T_2) \) is the free \( \mathbb{Z} \) module generated by the homology decompositions \( H_Z = A \oplus B \), where the restriction of the intersection pairing to \( A \) and \( B \) are unimodular. This implies that \( H_1(T_{2,n+r}) \) has infinite rank of all \( n \) and \( r \). On the other hand, Watanabe [83], using work of [74], proved that \( u_2 \) is generated by the irreducible \( \text{Sp}(H) \) representation \( \mathcal{V} \) that corresponds to the partition \([2^2] \). This implies that

\[
H_1(u_{2,n+r}) \cong \mathcal{V} \oplus H_{\mathbb{Q}}^{n+r}
\]
as \( \text{Sp}(H) \)-modules. \( \square \)

5.5. **The geometric Johnson homomorphism.** Suppose that \((\mathbf{N}, P, V)\) is a surface of type \((g, n + r)\) and that \( \mathbf{N} \in V \). Suppose that \( \mathbf{N} \in V \). Denote the Lie algebra of the unipotent completion of \( \pi_1(S, \mathbf{N}) \) by \( \mathcal{P}(S, \mathbf{N}) \). Since unipotent completion is a functor, the action of \( \Gamma_{g,n+\mathbf{N}} \) on \( \pi_1(S, \mathbf{N}) \) induces a homomorphism

\[
\phi_\theta : \Gamma_{g,n+\mathbf{N}} \rightarrow \text{Aut}^\theta \mathcal{P}(S, \mathbf{N}).
\]
The universal mapping property of relative completion implies that this factors uniquely through a homomorphism

\[
\phi_\mathbf{N} : \mathcal{G}_{g,n+\mathbf{N}} \rightarrow \text{Aut}^\theta \mathcal{P}(S, \mathbf{N})
\]
of affine \( \mathbb{Q} \)-groups. This induces a Lie algebra homomorphism

\[
d\phi_{\mathbf{N}} : \mathfrak{g}_{g,n+\mathbf{N}} \rightarrow \text{Der}^\theta \mathcal{P}(S, \mathbf{N}).
\]
This is the **geometric Johnson homomorphism**. It factors through \( \mathfrak{g}_{g,n+r} \rightarrow \mathfrak{g}_{g,n+r-1+\mathbf{N}} \) as the action of \( \Gamma_{g,n+r} \) on \( \pi_1(S, \mathbf{N}) \) factors through \( \Gamma_{g,n+r} \rightarrow \Gamma_{g,n+r-1+\mathbf{N}} \).

**Question 5.8.** Suppose that \( g \neq 1 \) and that \((S, \mathbf{N})\) is a surface of type \((g, \mathbf{N})\). Is the geometric Johnson homomorphism \( \mathfrak{g}_{g,\mathbf{N}} \rightarrow \text{Der}^\theta \mathcal{P}(S, \mathbf{N}) \) injective?

The geometric Johnson homomorphism in not injective when \( g = 1 \) as it factors through the quotient map \( \mathfrak{g}_{1,\mathbf{N}}^{\text{MEM}} \rightarrow \mathfrak{g}_{1,\mathbf{N}} \), where \( \mathfrak{g}_{1,\mathbf{N}}^{\text{MEM}} \) is the Lie algebra defined in [42]. In this case, one can ask if \( \mathfrak{g}_{1,\mathbf{N}}^{\text{MEM}} \rightarrow \text{Der}^\theta \mathcal{P}(S, \mathbf{N}) \) is injective.

The geometric Johnson homomorphism factors through the Kawazumi–Kuno action. This is essentially a result of Kawazumi and Kuno [55, Thm. 5.2.1]. The version below is formulated and proved in [57, §13].

\[10\text{This is the highest weight submodule of Sym}^2 \Lambda^2 H.\]
Theorem 5.9. There is a Lie algebra homomorphism
\[ \tilde{\varphi} : \mathfrak{g}_{g,n+r} \to \mathbb{Q}\lambda(S)^\wedge \]
that depends only on \((\mathfrak{g}, P \cup Q)\), such that for each \(v \in \tilde{\mathfrak{V}}\) the geometric Johnson homomorphism (17) factors
\[ \mathfrak{g}_{g,n+r} \xrightarrow{d\varphi_v} \text{Der}^q\mathfrak{p}(S,v) \]
\[ \xrightarrow{\tilde{\varphi}} \mathbb{Q}\lambda(S)^\wedge \xrightarrow{\kappa_v} \text{Der}^q\mathbb{Q}\pi_1(S,v)^\wedge. \]

The main ingredient in the proof is the non-abelian generalization of the classical Picard–Lefschetz formula due to Kawazumi and Kuno [55, Thm. 5.2.1]. Because of its beauty and importance, we state and sketch a proof of it in the next section.

Kawazumi and Kuno [56] observed that this provides an upper bound on the image of the Johnson homomorphism. This will be explained in Section 12.

5.6. The non-abelian Picard–Lefschetz formula. Suppose that \((\mathfrak{g}, P, \mathfrak{V})\) is a surface of type \((g,n+r)\). For each \(\alpha \in \lambda(S)\), \(1 - \alpha\) is in \(I\mathbb{Q}\lambda(S)\), which implies that for all \(n \geq 1,\)
\[ (1 - \alpha)^n := \sum_{j=0}^{n} (-1)^j \binom{n}{j} \psi_j(\alpha) \in I^n\mathbb{Q}\lambda(S). \]
This implies that any power series in \(1 - \alpha\) converges in \(\mathbb{Q}\lambda(S)^\wedge\). In particular, we can define
\[ \log \alpha := -\sum_{n=1}^{\infty} \frac{1}{n} (1 - \alpha)^n \in \mathbb{Q}\lambda(S)^\wedge \]
and its powers, such as
\[ (\log \alpha)^2 := \sum_{n=1}^{\infty} \sum_{j+k=n, j,k>0} \frac{1}{jk} (1 - \alpha)^n \in \mathbb{Q}\lambda(S)^\wedge. \]
Since \(\log \alpha\) lifts naturally to \(\mathbb{Q}\pi_1(S,\tilde{v})^\wedge\), and since this lift lies in \(\mathfrak{p}(S,\tilde{v})\), we see that \((\log \alpha)^n \in [\text{Sym}^n \mathfrak{p}(S,\tilde{v})]\).

When \(\alpha\) is a simple closed curve \((\log \alpha)^2\) has a concrete meaning. Denote by \(t_\alpha \in \Gamma_{g,n+r}\) the (positive) Dehn twist about \(\alpha\). It does not depend on the orientation of \(\alpha\). Since \(t_\alpha\) acts unipotently on \(H_1(S)\), its image in \(\mathcal{G}_{g,n+r}\) lies in a prounipotent subgroup. It therefore has a canonical logarithm \(\log t_\alpha \in \mathfrak{g}_{g,n+r}\).

Theorem 5.10 (Kawazumi–Kuno [55, Thm. 5.2.1]). For each simple closed curve \(\alpha \in \lambda(S)\) and each \(\tilde{v} \in \tilde{\mathfrak{V}}\), we have
\[ \frac{1}{2}\kappa_v((\log \alpha)^2) = d\varphi_v(\log t_\alpha) \in \text{Der}^q\mathbb{Q}\pi_1(S,\tilde{v})^\wedge. \]
For a once punctured surface \(S\), this reduces to the standard Picard–Lefschetz formula \(u \mapsto u + \langle \alpha, u \rangle \alpha\) for the action of \(t_\alpha\) on \(H_1(S)\).

Sketch of Proof. The first step in the proof is to reduce to the case where \(S\) is an annulus \([0,1] \times S^1\), \(\alpha\) is its core \(\{1/2\} \times S^1\), and \(u\) is represented by a path that is a fiber of the projection of the annulus onto \(\alpha\). This follows from the naturality of
the Kawazumi–Kuno action under inclusion of surfaces and the derivation property\(11\). More precisely, we replace\( S \) by a regular neighbourhood\( N \) of\( \alpha \) and\( u \) by the path\( \gamma \) defined by\( \gamma(s) = (s, 1) \).

Denote the Kawazumi–Kuno action on the paths in\( N \) from\( x_0 := (0, 1) \) to\( x_1 := (1, 1) \) by\( \kappa \). The definition of\( \kappa \) implies that
\[
\kappa(\alpha^n)(\gamma) = n t^n_n(\gamma).
\]

Every power series
\[
\Psi(u) = \sum_{n=0}^{\infty} a_n(u - 1)^n \in \mathbb{Q}[[u - 1]]
\]
in\( u - 1 \) can be evaluated on\( \alpha \) to obtain an element\( \Psi(\alpha) \in \mathbb{Q}\lambda(N)\wedge \) and on\( t_\alpha \) to obtain an element
\[
\Psi(t_\alpha) \in \text{End} \mathbb{Q}\pi(N; x_0, x_1)\wedge.
\]
The formula\(18\) implies that
\[
\kappa(\Psi(\alpha))(\gamma) = t_\alpha \Psi'(t_\alpha)(\gamma).
\]
The result follows by taking
\[
\Psi(u) = \frac{(\log u)^2}{2} := \left( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(u - 1)^n}{n} \right)^2
\]
as it has logarithmic derivative\( u\Psi'(u) = \log u \).

5.7. **Centralizers.** For technical reasons, it is important to understand the centralizer of a Dehn twist in the relative completion of a surface. Since relative completion is not, in general, left exact, it is not clear that the centralizer of a Dehn twist in\( \Gamma_S \) is Zariski dense in the stabilizer of the Dehn twist in\( \mathcal{G}_S \).

Suppose that\( A \) is a simple closed curve in\( S \). We allow\( A \) to be a boundary component. Let\( T \) be the surface obtained by cutting\( S \) open along\( A \). Assume that each connected component of of\( S \) with\( A \) removed is hyperbolic. Denote the Dehn twist on\( A \) by\( t_A \). Then it is well known\[24\] Fact 3.8 that the centralizer\( Z(t_A) \) of\( t_A \) in\( \Gamma_{S,\partial S} \) is
\[
Z(t_A) = \{ \phi \in \Gamma_{S,\partial S} : \phi(A) = A \}.
\]
It is an extension by a finite group of the quotient of\( \Gamma_{T,\partial T} \) obtained by identifying the two Dehn twists on the two boundary components of\( T \) that are identified to obtain\( S \). One can ask whether this holds for relative completion:

**Question 5.11.** Is the centralizer\( Z(t_A) \) of\( t_A \) in\( \Gamma_S \) Zariski dense in the centralizer of the image of\( t_A \) in\( \mathcal{G}_S \)? In particular, is the center of\( \mathfrak{g}_{g,n+\bar{f}} \) spanned by the logarithms of the Dehn twists on the boundary components?

6. **Hodge Theory**

Hodge theory provides two important tools for studying the Johnson homomorphism. The first is that, once one has established that a map between topological invariants (such as the Johnson homomorphism, the Goldman bracket, the Turaev cobracket, \ldots) is a morphism of mixed Hodge structure (MHS), one can replace it by the induced map on the associated weight graded objects without loss of (topological) information. The second is that, since the category of (graded polarizable)\( \mathbb{Q} \) mixed Hodge structures is tannakian, every MHS is a module over...
an affine group that we shall denote by $\pi_1(MHS)$, and every morphism of MHS is $\pi_1(MHS)$-equivariant. This provides a large hidden group of symmetries that acts on all invariants once once has chosen a complex structure on $(\overline{\mathcal{M}}, P, \overline{V}, \xi)$.

6.1. Summary of Hodge theory. We will assume the reader is familiar with the basics of mixed Hodge theory. Minimal background suitable for this discussion, as well as further references on Hodge theory, can be found in [37, §10].

We will consider (and need) only $\mathbb{Q}$-MHS. A $\mathbb{Q}$-MHS $V$ consists of a finite dimensional $\mathbb{Q}$ vector space $V_\mathbb{Q}$ endowed with an increasing weight filtration

$$0 = W_n V_\mathbb{Q} \subseteq \cdots \subseteq W_j V_\mathbb{Q} \subseteq W_j V_\mathbb{Q} \subseteq \cdots \subseteq W_N V_\mathbb{Q} = V_\mathbb{Q}$$

and a decreasing Hodge filtration

$$V_c = F^m V_c \supseteq \cdots \supseteq F^p V_c \supseteq F^{p+1} V_c \supseteq \cdots \supseteq F^k V_c = 0$$

of its complexification $V_\mathbb{C}$. These are required to satisfy the condition that, for all $m \in \mathbb{Z}$, the $m$th weight graded quotient $Gr^W_m V$ of $V$, whose underlying $\mathbb{Q}$ vector space is

$$Gr^W_m V := W_m V_\mathbb{Q} / W_{m-1} V_\mathbb{Q}$$

is a Hodge structure of weight $m$. This means simply that

$$Gr^W_m V \mathbb{C} = \bigoplus_{p+q = m} (F^p Gr^W_m V_\mathbb{C}) \cap (F^q Gr^W_m V_\mathbb{C})$$

where $\overline{F^m V_\mathbb{C}}$ is the conjugate of $F^m V_\mathbb{C}$ under the action of complex conjugation on $V_\mathbb{C}$. A MHS $V$ is said to be pure of weight $m \in \mathbb{Z}$ if $Gr^W_r V = 0$ when $r \neq m$. A Hodge structure of weight $m$ is simply a MHS that is pure of weight $m$.

Morphisms of MHS are weight filtration preserving $\mathbb{Q}$-linear maps of the underlying $\mathbb{Q}$-vector spaces which induce Hodge filtration preserving maps with $\mathbb{C}$. The category of MHS is a $\mathbb{Q}$-linear abelian tensor category. This is not obvious, as the category of filtered vector spaces is not an abelian category.

The key property for us is that, for each $m \in \mathbb{Z}$, the functor $Gr^W_m$ from the category of MHS to $Vec_\mathbb{Q}$, the category of $\mathbb{Q}$ vector spaces, is exact. In particular, this implies that if $\phi : V \to V'$ is a morphism of MHS, then there are natural isomorphisms

$$Gr^W_m \ker \phi \cong \ker Gr^W_m \phi \text{ and } Gr^W_m \text{ im } \phi \cong \text{ im } Gr^W_m \phi. \tag{19}$$

Deligne [15, 16] proved that the cohomology of every complex algebraic variety has a natural MHS that is functorial with respect to morphisms of varieties. This was extended to homotopy invariants by Morgan [63] and the author in [28].

Example 6.1. The Hodge structure $\mathbb{Q}(n)$ is a pure Hodge structure of weight $-2n$ and has type $(-n, -n)$. Equivalently, $F^{-n} Q(n)_\mathbb{C} = Q(n)_\mathbb{C}$ and $F^{-n+1} = 0$. One can also define $Q(1) = H_1(\mathbb{C}^*; \mathbb{Q})$ and $Q(n) = Q(1) \otimes^n$ for all $n \geq 0$. When $n < 0$, one defines $Q(n)$ to be the dual of $Q(-n)$. Mixed Hodge structures whose weight graded quotients are direct sums of $Q(n)$’s are called Tate MHSs.

The tensor product of a MHS $V$ with $Q(n)$ is denoted by $V(n)$ and called a Tate twist of $V$.

Example 6.2. If $X$ is a compact Riemann surface of genus $g > 0$, then $H_Q := H_1(X; \mathbb{Q})$ is pure of weight $-1$. The intersection pairing

$$\langle , \rangle : H_1(X) \otimes H_1(X) \to \mathbb{Q}(1)$$
is a morphism of MHS. Poincaré duality is an isomorphism of MHS
\[ H_1(X) \to H^1(X) \otimes \mathbb{Q}(1). \]

6.2. The category MHS. In order to have a good category of mixed Hodge structures, we need to introduce the notion of a polarized Hodge structure. Polarizations generalize the Riemann bilinear relation
\[ i \int_X \omega \wedge \overline{\omega} \geq 0 \] with equality if and only if \( \omega = 0 \)
that is satisfied by holomorphic 1-forms \( \omega \) on a compact Riemann surface \( X \).

In general, a polarized Hodge structure of a Hodge structure \( V \) of weight \( m \) is a \((-1)^m\)-symmetric bilinear form
\[ Q : V \otimes V \to \mathbb{Q} \]
that satisfies the Riemann–Hodge bilinear relations
1. The restriction of \( Q \) to \( V^{p,m-p} \otimes V^{s,m-s} \) vanishes except when \( s = m - p \), where \( V^{p,m-p} := F^p V \cap F^{m-p} V \);
2. The hermitian form
\[ v \otimes \overline{w} \mapsto i^{p-q} Q(v, \overline{w}), \quad v, w \in V^{p,q}, \ p + q = m \]
is positive definite on each \( V^{p,q} \).

Polarizations \( Q \) are non-degenerate.

A polarized Hodge structure is a Hodge structure endowed with a polarization. From our point of view, the significance of polarizations is that if \( A \) is a Hodge substructure of a polarized Hodge structure \( V \), then the restriction of the polarization to \( A \) is non-degenerate (and therefore a polarization) and
\[ V = A \oplus A^\perp \]
as MHS. So polarized Hodge structures are semi-simple objects in the category of mixed Hodge structures.

More generally, a MHS \( V \) is graded polarizable if each of its weight graded quotients \( \Gr^W V \) admits a polarization. All mixed Hodge structures that arise in geometry are graded polarizable, as are all MHS that occur in this paper. Denote the category of graded polarizable \( \mathbb{Q} \)-MHS by \( \text{MHS} \).

6.3. Splittings. The category \( \text{MHS} \) is a \( \mathbb{Q} \)-linear neutral tannakian category with fiber functor \( \text{MHS} \to \text{Vec}_\mathbb{Q} \) that takes an MHS \( V \) to its underlying rational vector space \( V_\mathbb{Q} \). There is therefore an affine \( \mathbb{Q} \)-group
\[ \pi_1(\text{MHS}) := \pi_1(\text{MHS}, \omega) := \text{Aut}^\omega \]
such that (a) every mixed Hodge structure is naturally a \( \pi_1(\text{MHS}) \)-module and (b) the category of finite dimensional representations of \( \pi_1(\text{MHS}) \) is equivalent to \( \text{MHS} \), [15, 17].

For us, the most important aspect of the tannakian picture is that \( \pi_1(\text{MHS}) \) is that one has a diagram
\[ 1 \to \mathcal{U}^{\text{MHS}} \to \pi_1(\text{MHS}) \to \pi_1(\text{MHS}^s) \to 1 \]
with \( \chi \) and \( G_m \).
where $\mathcal{U}^{\text{MHS}}$ is pro-nilpotent and $\text{MHS}^e$ denotes the full sub-category of $\text{MHS}$ whose objects are direct sums of graded polarizable Hodge structures (the semi-simple objects of $\text{MHS}$) and where $\chi$ is a central cocharacter. The copy of $\mathbb{G}_m$ acts on a Hodge structure $V$ of weight $j$ by $\chi(t) : v \mapsto t^j v$. The lifts $\tilde{\chi} : \mathbb{G}_m \to \pi_1(\text{MHS})$ of $\chi$ form a torsor (principal homogeneous space) under $\mathcal{U}^{\text{MHS}}$, which acts by conjugation.

Since every $V$ in $\text{MHS}$ is a $\pi_1(\text{MHS})$-module, each choice of lift $\tilde{\chi}$ makes $V$ into a $\mathbb{G}_m$ module. This determines a natural (not canonical) splitting

$$V_Q \cong \text{Gr}_1^W V_Q := \bigoplus_{m \in \mathbb{Z}} \text{Gr}_m^W V_Q$$

of the weight filtration of each MHS $V$ that is preserved by morphisms of MHS. A more detailed explanation can be found in [37, §10].

Finally, the category of all representations of $\pi_1(\text{MHS})$ is equivalent to the category $\text{ind-MHS}$. Many objects we consider will be pro-objects of $\text{MHS}$ and their continuous duals will be ind-objects.

7. Hodge Theory and Surface Topology

Here we recall basic results about the existence of MHS on invariants of surface groups and their mapping class groups. These MHS require the choice of a complex structure $S, \vec{V}$, and the MHS on each of these invariants depends non-trivially on the complex structure $v$.

The Lie bracket is a morphism of MHS.

The weight filtration satisfies

$$W_{-m} \mathbb{Q} \pi_1(S, \vec{V}) = \mathbb{I}^m \mathbb{Q} \pi_1(S, \vec{V})$$

and $W_{-m} \mathbb{Q} \lambda(S) = \mathbb{I}^m \mathbb{Q} \lambda(S)$.

All of these MHS are functorial with respect to holomorphic maps.

7.1. Surface groups. Each choice of a complex structure $\phi : (\mathbb{S}, P, \vec{V}) \to (\mathbb{X}, Y, \vec{V}')$ on $(\mathbb{S}, P, \vec{V})$ determines a canonical pro-MHS on

$$\mathbb{Q} \pi_1(S, \vec{V})$$

and $\mathbb{P}(S, \vec{V})$.

The pro-MHS on $\mathbb{Q} \pi_1(S, \vec{V})$ descends to a pro-MHS on $\mathbb{Q} \lambda(S)$. These MHS are preserved by the product and coproduct of $\mathbb{Q} \pi_1(S, \vec{V})$ and by the Lie bracket of $\mathbb{P}(S, \vec{V})$.

When $S$ is a surface of type $(g, n + \vec{r})$, where $n + \vec{r} \leq 1$, then $H_1(S)$ is pure of weight $-1$ and the weight filtrations are given by

$$W_{-m} \mathbb{Q} \pi_1(S, \vec{V}) = \mathbb{I}^m \mathbb{Q} \pi_1(S, \vec{V})$$

and $W_{-m} \mathbb{P}(S, \vec{V}) = \mathbb{L}^m \mathbb{P}(S, \vec{V})$, where $\mathbb{L}^m$ denotes the $m$th term of the lower central series. When $S$ is hyperbolic, the MHS on each of these invariants depends non-trivially on the complex structure on $(S, \vec{V})$. In addition

$$W_{-m} \mathbb{Q} \lambda(S) = \mathbb{I}^m \mathbb{Q} \lambda(S).$$

All of these MHS are functorial with respect to holomorphic maps.

7.2. Mapping class groups. Each choice of a complex structure $\phi$ on $(\mathbb{S}, P, \vec{V})$ determines a canonical pro-MHS on the Lie algebra $\mathfrak{g}_{g,n+\vec{r}}$ of the relative completion of the mapping class group $\Gamma_{g,n+\vec{r}} \cong \pi_1(\mathcal{M}_{g,n+\vec{r}}, \phi)$. It depends non-trivially on $\phi$. The weight filtration satisfies

$$W_0 \mathfrak{g}_{g,n+\vec{r}} = \mathfrak{g}_{g,n+\vec{r}}, W_{-1} \mathfrak{g}_{g,n+\vec{r}} = u_{g,n+\vec{r}}, \text{Gr}_0^W \mathfrak{g}_{g,n+\vec{r}} \cong \mathfrak{sp}(H).$$

The Lie bracket is a morphism of MHS.

For each complex structure $\phi$, and for each $\vec{v} \in \vec{V}$, the geometric Johnson homomorphism

$$\mathfrak{g}_{g,n+\vec{r}} \to \text{Der}^\phi \mathbb{P}(S, \vec{V})$$

is a morphism of MHS.
When \( g \geq 3 \), the weight filtration of \( u_{g,n+r} \) is its lower central series
\[
W_{-m}u_{g,n+r} = L^m u_{g,n+r}.
\]
This holds for all non-negative \( n \) and \( r \). This is not true when \( g \leq 2 \).

When \( g \geq 3 \), there is a natural MHS on \( t_{g,n+r} \) such that the surjection \( t_{g,n+r} \to u_{g,n+r} \) is a morphism of MHS with kernel \( Q(1) \). The weight filtration of \( t_{g,n+r} \) is
\[
Gr_{W_{-m}} t_{g,n+r} \cong Gr_{L_{-m}} t_{g,n+r} \cong (Gr_{L_{-m}} T_{g,n+r}) \otimes Q.
\]

7.3. The Goldman–Turaev Lie bialgebra. For each choice of a complex structure \( \phi \) on \((S,P,\vec{V})\), the Goldman bracket
\[
\{ , \} : \mathbb{Q}\lambda(S)^{\wedge} \otimes \mathbb{Q}\lambda(S)^{\wedge} \to \mathbb{Q}\lambda(S)^{\wedge} \otimes \mathbb{Q}(1)
\]
twisted by \( \mathbb{Q}(1) \) is a morphism of MHS, and, for each \( \vec{v} \in \vec{V} \) and the (completed) Kawazumi–Kuno action
\[
(21) \quad \kappa_{\vec{v}} : \mathbb{Q}\lambda(S)^{\wedge} \otimes \mathbb{Q}(-1) \to \text{Der}^0 \mathbb{Q}_1(S,\vec{v})^{\wedge}
\]
is also a morphism of MHS. These assertions are proved in [37].

In order for the cobracket of the framed surface \((S,\xi)\) to be a morphism of MHS, we have to choose a complex structure on \((S,\xi)\). By this we mean a complex structure \( \phi : (S,P,\vec{V}) \to (X,Y,\vec{V}') \)
for which \( \phi_*\xi \) is homotopic to a meromorphic vector field on \( X \) that is holomorphic and nowhere vanishing on \( X := X - Y \).

When \( \xi \) is algebraic, the twisted cobracket
\[
\delta_{\xi} : \mathbb{Q}\lambda(S)^{\wedge} \otimes \mathbb{Q}(-1) \to \mathbb{Q}\lambda(S)^{\wedge} \otimes \mathbb{Q}\lambda(S)^{\wedge}
\]
is a morphism of MHS. This is proved in [38].

Provided that \( n + r > 0 \), there is always a complex structure on \((S,P,\vec{V})\) that admits an algebraic framing. For example, one has the framing \( y\partial/\partial x \) of the genus \( g \) affine hyperelliptic curve
\[
y^2 = \prod_{j=0}^{2g} (x - a_j).
\]

More generally, when \( g \neq 1 \), every topological framing \( \xi \) is homotopic to a quasi-algebraic framing. See [38, §9].

Finally, observe that since \( p(S,\vec{v}) \) is a sub-MHS of \( \mathbb{Q}_1(S,\vec{v})^{\wedge} \), each factor \( \text{Sym}^n p(S,\vec{v}) \) in the PBW decomposition \([16]\) of \( \mathbb{Q}\lambda(S)^{\wedge} \) is a sub-MHS. This implies that \([16]\) is also a decomposition of pro-MHS.

---

20Actually, in order for \( \delta_{\xi} \) to be a morphisms of MHS, one need only require that \( \phi_*\xi \) be homotopic to a meromorphic section of \( TX \otimes L \), the holomorphic tangent bundle of \( X \) twisted by a torsion line bundle \( L \) over \( X \). Such framings are called quasi-algebraic in [38].
7.4. **Splittings.** As explained in Section 6.3 each choice of a lift \( \tilde{\chi} : G_m \to \pi_1(\text{MHS}) \) of the central cocharacter \( G_m \to \pi_1(\text{MHS}^{ss}) \) determines, for every \( V \) in \( \text{MHS} \), an isomorphism

\[
V_Q \xrightarrow{\sim} \text{Gr}_W V_Q = \bigoplus_{m \in \mathbb{Z}} \text{Gr}_m W V_Q
\]

of the rational vector space underlying \( V \) with its associated graded. These isomorphisms commute with maps induced by morphisms of \( \text{MHS} \) and are compatible with tensor products and duals. This extends verbatim to ind-objects of \( \text{MHS} \). For pro-objects of \( \text{MHS} \), each lift \( \tilde{\chi} \) determines isomorphisms

\[
V_Q \xrightarrow{\sim} (\text{Gr}_W V_Q)^\wedge := \prod_{m \in \mathbb{Z}} \text{Gr}_m W V_Q
\]

that is natural for morphisms of pro-MHS. In all cases, the effect of a twist by \( \mathbb{Q}(r) \) is to shift this grading by \( 2r \):

\[
\text{Gr}_m W V(r) = \text{Gr}_{m+2r} W V.
\]

In particular, for each complex structure \( \phi \) on \( (S, P, \vec{V}) \), the choice of a lift \( \tilde{\chi} \) of the central cocharacter \( \chi \), determines natural Lie algebra isomorphisms

\[
\mathbb{Q}\lambda(S)^\wedge \cong (\text{Gr}_W \mathbb{Q}\lambda(S))^\wedge := \prod_{m \geq 0} \text{Gr}_{-m} \mathbb{Q}\lambda(S)^\wedge
\]

\[
p(S, \vec{v}) \cong (\text{Gr}_W p(S, \vec{v}))^\wedge := \prod_{m \geq 0} \text{Gr}_{-m} p(S, \vec{v})
\]

\[
\mathfrak{g}_{g,n+r} \cong (\text{Gr}_W \mathfrak{g}_{g,n+r})^\wedge := \prod_{m \geq 0} \text{Gr}_{-m} \mathfrak{g}_{g,n+r}
\]

\[
\text{Der}^\theta p(S, \vec{v}) \cong \prod_{m \geq 0} \text{Gr}_{-m} \text{Der}^\theta p(S, \vec{v}) =: (\text{Der}^\theta \text{Gr}_W p(S, \vec{v}))^\wedge
\]

(22) \[
\text{Der}^\theta \mathbb{Q}\pi_1(S, \vec{v})^\wedge \cong \prod_{m \geq 0} \text{Gr}_{-m} \text{Der}^\theta \mathbb{Q}\pi_1(S, \vec{v})^\wedge =: (\text{Der}^\theta \text{Gr}_W \mathbb{Q}\pi_1(S, \vec{v}))^\wedge
\]

Here

\[
\text{Der}^\theta \text{Gr}_{-m} p(S, \vec{v}) = \{ D \in \text{Der} \text{Gr}_W p(S, \vec{v}) : D(\theta) = 0 \},
\]

where \( \theta = \sum [a_j, b_j] \) and \( \{ a_1, \ldots, a_g, b_1, \ldots, b_g \} \) is a symplectic basis of \( \text{Gr}_{-1} p(S, \vec{v}) = H \). Similarly for \( \text{Der}^\theta \mathbb{Q}\pi_1(S, \vec{v})^\wedge \).

When \( S \) is of type \((g, \bar{1})\), there is a **canonical** graded Lie algebra isomorphism

\[
\text{Gr}_W p(S, \vec{v}) \cong L(H)
\]

of the associated graded of \( p(S, \vec{v}) \) with the free Lie algebra generated by \( H \)\(^{[2]}\)

In all but the first isomorphism above, the bracket of the associated weight graded Lie algebra preserves the weights. In the first case the bracket increases weights by 2 because of the Tate twist:

\[
\{ , \} : \text{Gr}_{-a}^{W_a} \mathbb{Q}\lambda(S)^\wedge \otimes \text{Gr}_{-b}^{W_b} \mathbb{Q}\lambda(S)^\wedge \to \text{Gr}_{-a-b}^{W_{a-b}} \mathbb{Q}\lambda(S)^\wedge.
\]

\(^{[2]}\) Elements of \( L(H) \) can be represented by linear combinations of rooted, planar, trivalent trees (modulo the IHX relation) whose leaves are labelled by elements of \( H \). The weight is minus the number of leaves. In the \((g, \bar{1})\) case, elements of \( \text{Der}^\theta L(H) \) can be represented as linear combinations of planar trivalent graphs (modulo IHX) whose leaves are labelled by elements of \( H \). Derivations of weight \(-m\) have \( m \) internal vertices and \( m + 2 \) leaves. A precise description of this correspondence can be found in \([20]\).
When we also have an algebraic framing $\xi$ of $S$, the Goldman–Turaev Lie bialgebra is isomorphic to its associated weight graded Lie bialgebra, where the bracket and cobracket

$$\delta_\xi : \text{Gr}^W_{-m} \mathbb{Q}\lambda(S) \to \bigoplus_{a+b=m-2} \text{Gr}^W_a \mathbb{Q}\lambda(S) \otimes \text{Gr}^W_{-b} \mathbb{Q}\lambda(S)$$

both increase weights by 2.

Since the geometric Johnson homomorphism (17) is a morphism of MHS, we have for each $\vec{v} \in \vec{V}$ a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{g}_{g,n} & \xrightarrow{d\varphi_{\vec{v}}} & \text{Der}^\theta p(S,\vec{v}) \\
\downarrow \cong & & \downarrow \cong \\
(\text{Gr}^W_{g,n} \mathfrak{g}_{g,n}) & \xrightarrow{\text{Gr}^W_{d\varphi_{\vec{v}}}} & (\text{Der}^\theta \text{Gr}^W_{g,n} p(S,\vec{v}))
\end{array}
\]

We will call the homomorphism

\[\text{Gr}^W_{d\varphi_{\vec{v}}} : \text{Gr}^W_{g,\vec{v}} \to \text{Der}^\theta \text{Gr}^W_{g,\vec{v}} p(S,\vec{v})\]

in the $(g,\vec{1})$ case the graded Johnson homomorphism.

**Remark 7.1.** The exactness property (19) of morphisms of MHS implies that

$$\text{Gr}^W_{d\varphi_{\vec{v}}} \text{im} d\varphi_{\vec{v}} = \text{im} \text{Gr}^W_{d\varphi_{\vec{v}}}.$$

Morita’s higher Johnson homomorphism is the inclusion

$$\text{im} \text{Gr}^W_{d\varphi_{\vec{v}}} \hookrightarrow \text{Der}^\theta \text{Gr}^W_{g,\vec{v}} p(S,\vec{v}).$$

Exactness also implies that $t_{g,n+\vec{r}} \to \text{Der}^\theta p(S,\vec{v})$ is not injective as $\ker \text{Gr}^W_{d\varphi_{\vec{v}}}$ is non trivial. See [31, §14] for a complete discussion.

The final observation is that the diagram in Theorem 5.9 is isomorphic to the diagram

\[
\begin{array}{ccc}
\text{Gr}^W_{g,n+\vec{r}} & \xrightarrow{\text{Gr}^W_{d\varphi_{\vec{v}}}} & \text{Der}^\theta \text{Gr}^W_{g,n} p(S,\vec{v}) \\
\downarrow \text{Gr}^W_{d\varphi_{\vec{v}}} & & \downarrow \\
\text{Gr}^W_{g,n+\vec{r}} & \xrightarrow{\text{Gr}^W_{d\varphi_{\vec{v}}}} & \text{Der}^\theta \text{Gr}^W_{g,n} p(S,\vec{v})
\end{array}
\]

of associated weight graded Lie algebras.

The following is a weaker version of Question 5.8

**Question 7.2.** Is the Johnson homomorphism stably injective? That is, is the map

$$\text{Gr}^W_{g,\vec{v}} d\varphi_{\vec{v}} : \text{Gr}^W_{g,\vec{v}} \to \text{Der}^\theta \text{Gr}^W_{g,\vec{v}} p(S,\vec{v})$$

when $g \gg -m$?

If it is stably injective, Proposition 15.1 will imply that it is injective when $g \geq -m/3$. When $-m \leq 6$ and $g \geq -3m$, the computations in [67] imply that the Johnson homomorphism is injective.
7.4.1. The Lie bialgebra $\text{Gr}^W \mathbb{Q}\lambda(S)$. Fix a complex structure on $(\mathcal{S}, \mathcal{P}, \mathcal{V})$ and a lift $\tilde{\chi}$ of $\chi$. These determine an isomorphism

$$\mathbb{Q}\lambda(S)^\wedge \cong \prod_{n \geq 0} \text{Gr}^W_n \mathbb{Q}\lambda(S).$$

Under this isomorphism, the Goldman bracket is graded and increases weights by 2. For each choice of $\mathcal{V} \in \mathcal{V}$, the map

$$\text{Gr}^W \kappa_{\mathcal{V}} : \text{Gr}^W \mathbb{Q}\lambda(S) \to \text{Der}^\theta \text{Gr}^W_{n+2} \mathbb{Q}\pi_1(S, \mathcal{V})$$

induced by (21) increases weights by 2 and is a Lie algebra homomorphism.

For each algebraic framing $\xi$ of $\mathcal{S}$, the cobracket $\delta_{\xi}$ is also graded and increases weights by 2, making $\text{Gr}^W \mathbb{Q}\lambda(S)$ into a graded Lie bialgebra.

The bracket and cobracket on $\text{Gr}^W \mathbb{Q}\lambda(S)$ have a combinatorial description, which can be found in [2, §4.2]. In the $(g, \hat{1})$ it is Schedler’s Lie bialgebra [76, §2]. To explain it, we need some notation. For the rest of this subsection, $\mathcal{S}$ is a surface of type $(g, \hat{1})$.

Suppose that $A$ is an associative algebra. Denote the image of $a \in A$ in its cyclic quotient

$$|A| := A/\langle uv - vu : u, v \in A \rangle$$

by $|a|$. When

$$A = T(H) = \mathbb{Q}\langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle \cong \text{Gr}^W \mathbb{Q}\pi_1(S, \mathcal{V})$$

the image of $x_1 x_2 \ldots x_m \in H^\otimes m$ is the cyclic word

$$|x_1 x_2 \ldots x_m| \in |\text{Gr}^W \mathbb{Q}\pi_1(S, \mathcal{V})| = \text{Gr}^W |\mathbb{Q}\lambda(S)^\wedge|.$$

With this notation, the graded Goldman bracket is

$$\{,\} : |x_1 x_2 \ldots x_n| \otimes |y_1 y_2 \ldots y_m| \mapsto \sum_{j=1}^n \sum_{k=1}^m \langle x_j, y_k \rangle |x_{j+1} \ldots x_{n+1} x_1 \ldots x_{j-1} y_{k+1} \ldots y_{m+1} y_1 \ldots y_{k-1}||$$

where all $x_j, y_k \in H$. The graded cobracket $\delta_{\xi}$ is given by

$$\delta_{\xi} : |x_1 \ldots x_n| \mapsto \sum_{j>k} \langle x_j, x_k \rangle \left( |x_{j+1} \ldots x_{k-1}| \otimes |x_{j+1} \ldots x_n x_1 \ldots x_{j-1}| - |x_{k+1} \ldots x_n x_1 \ldots x_{j-1}| \otimes |x_{j+1} \ldots x_{k-1}| \right)$$

where each $x_j \in H$.

The action $\kappa_{\mathcal{V}} : |T(H)| \to \text{Der}^\theta T(H)$ also has a simple combinatorial description:

$$\kappa_{\mathcal{V}} : |x_1 \ldots x_n| \otimes y_1 \ldots y_m \mapsto \sum_{j=1}^n \sum_{k=1}^m \langle x_j, y_k \rangle |y_1 \ldots y_{k-1} x_{j+1} \ldots x_n x_1 \ldots x_{j-1} y_{k+1} \ldots y_m|$$

\text{\footnotesize{\textsuperscript{22}One may be concerned that this formula does not appear to depend on the framing. But, in order to get a splitting of the weight filtration via Hodge theory, we need a complex structure on the surface. But a once punctured compact Riemann surface $X = \mathcal{S} - \{p\}$ has at most one algebraic framing as two algebraic framings of $X$ differ by a rational function whose divisor is supported at $p$, and is therefore constant.}}}
7.4.2. Sketch of Proof of Proposition 7.4. To prove the proposition, it suffices to prove the graded version. We use the well-known description\(^{26}\) of elements of \( L(H) \) as rooted planar trees whose vertices are labelled by element of \( H \), and elements of \( \text{Gr}^W_{-n} \text{Der}^\theta L(H) \) is spanned by planar trivalent graphs with \( n \) vertices whose leaves are labelled by elements of \( H \) modulo the IHX relation. Cutting the graph that represents a derivation divides it into 2 rooted trees that are well-defined mod IHX, and thus gives an element of \( \text{Sym}^2 L(H) \). See figure 1. This element of

\[
|\text{Sym}^2 L(H)| \quad \text{is well-defined as, for } U, V, W \in L(H) \quad \text{and so, rooted trees} \quad \frac{[U, V]W}{[V][W]} = |UVW| - |VUW| = |U[V, W]|
\]

The formula\(^ {26}\) implies that the \( \kappa \) action of \( |\text{Sym}^2 L(H)| \) on \( T(H) \) is easily seen to coincide with the standard action of decorated planar trivalent graphs on \( L(H) \).

8. Presentations of the Lie Algebras \( \mathfrak{g}_{g,n} + \vec{r} \)

The restrictions\(^ {20}\) on the weight filtration of \( \mathfrak{g}_{g,n} + \vec{r} \) imply that there is a graded Lie algebra isomorphism\(^ {23}\)

\[
\text{Gr}^W_\bullet \mathfrak{g}_{g,n} + \vec{r} \cong \text{sp}(H) \ltimes \text{Gr}^W_\bullet u_{g,n} + \vec{r}.
\]

The choice of a complex structure \( \phi \) and a lift \( \tilde{\chi} \) determines a Lie algebra isomorphism

\[
\mathfrak{g}_{g,n} + \vec{r} \cong \text{sp}(H) \ltimes \prod_{m>0} \text{Gr}^W_{-m} u_{g,n} + \vec{r}.
\]

So, to give a presentation of \( \mathfrak{g}_{g,n} + \vec{r} \), it suffices to give a presentation of the graded Lie algebra \( \text{Gr}^W_\bullet u_{g,n} + \vec{r} \) in the category of \( \text{Sp}(H) \)-modules.

In this section, we recall the presentations of \( \text{Gr}^W_\bullet u_{g,n} + \vec{r} \) when \( n + \vec{r} \leq 1 \) for all \( g \neq 2 \). When \( g = 2 \), the only case where we do not have complete presentations, we give a partial presentation.

8.0.1. Genus 0. When \( g = 0 \), the group \( \mathfrak{g}_{0,n} + \vec{r} \) is unipotent completion of \( \Gamma_{0,n} + \vec{r} \). When \( r = 0 \), its associated weight graded Lie algebra \( \text{Gr}^W_\bullet \mathfrak{g}_{0,n+1} \) is the quotient of the free Lie algebra

\[
\mathbb{L}(e_{j,k} : \{j, k\} \text{ is a 2-element subset of } \{0, \ldots, n\})
\]

\(^{23}\)This isomorphism integrates to an isomorphism \( \mathfrak{g}_{g,n} + \vec{r} \cong \text{Sp}(H) \ltimes U_{g,n} + \vec{r} \) as one gets from Levi’s Theorem.
by the ideal of relations generated by

\[ \sum_{j=0}^{n} e_{j,k} = 0, \text{ for each } k \in \{0, \ldots, n\}, \]

\[ [e_{j,k}, e_{s,t}] = 0 \text{ when } j, k, s, t \text{ distinct}, \]

\[ [e_{j,\ell} + e_{\ell,k}, e_{j,k}] = 0 \text{ when } j, k, \ell \text{ distinct.} \]

Here we are using the convention that each \( e_{j,j} = 0 \).

Each generator \( e_{j,k} \) has weight \(-2\). The symmetric group \( \Sigma_{n+1} \) acts on \( \mathfrak{g}_{0,n+1} \) by permuting the indices. The first homology \( H_1(\mathfrak{g}_{0,n+1}) \) is the irreducible \( \Sigma_{n+1} \)-module corresponding to the partition \([n-1, 2]\). It has dimension

\[ \dim H_1(\mathfrak{g}_{0,n+1}) = \binom{n}{2} - 1. \]

The Lie algebra of \( \mathfrak{g}_{0,n+1} \) is

\[ \mathfrak{g}_{0,n+1} = \mathfrak{g}_{0,n+1} \oplus \mathbb{Q}z_0 \]

where \( z_0 \) is central and spans a copy of \( \mathbb{Q}(1) \). There is a natural isomorphism of the weight graded of the Lie algebra of \( \pi_1^{alg}(\mathbb{P}^1 - \{p_0, \ldots, p_n\}, \bar{\nu}) \) with

\[ L(H_1(\mathbb{P}^1 - \{p_0, \ldots, p_n\})) \cong L(e_0, \ldots, e_n)/(e_0 + \cdots + e_n) \]

where \( e_j \) is the homology class of a small loop encircling \( x_j \) and \( \bar{\nu} \in T_{p_0} \mathbb{P}^1 \). The homomorphism

\[ G_{\bar{\nu}}^W \mathfrak{g}_{0,n+1} \to \text{Der} L(e_0, \ldots, e_n)/(e_0 + \cdots + e_n) \]

induces by the action of \( \Gamma_{0,n+1} \) on \( \pi_1^{alg}(\mathbb{P}^1 - \{p_0, \ldots, p_n\}, \bar{\nu}) \), where \( \bar{\nu} \in T_{p_0} \mathbb{P}^1 \), is given by

\[ e_{j,k}: e_t \mapsto (\delta_{j,t} - \delta_{k,t})[e_j, e_k] \text{ and } z_0: e_t \mapsto [e_0, e_t]. \]

8.0.2. Representation theory preliminaries. There are several \( \text{Sp}(H) \)-modules that play a prominent role when \( g \) is positive, particularly when \( g \geq 3 \). First set

\[ \theta = \sum_{j=1}^{g} a_j \wedge b_j \in \Lambda^2 H \]

where \( a_1, \ldots, a_g, b_1, \ldots, b_g \) is a symplectic basis of \( H \). It spans a copy of the trivial representation in \( \Lambda^2 H \).

When \( g \geq 2 \), the \( \text{Sp}(H) \) map \( \theta \wedge \cdot : H \to \Lambda^3 H \) is an \( \text{Sp}(H) \)-invariant injection. When \( g = 2 \), it is an isomorphism, but when \( g \geq 3 \), it has an irreducible \( \text{Sp}(H) \)-invariant complement that we denote by \( \Lambda^3 H \). It is the kernel of the contraction

\[ \Lambda^3 H \to H, \quad u \wedge v \wedge w \mapsto (u, v)w + (v, w)u + (w, u)v. \]

Remark 8.1. Johnson’s computation of the abelianization of the Torelli groups implies that for all \( g \geq 3 \) and all \( n, r \), there are \( \text{Sp}(H^r) \)-invariant isomorphisms

\[ H_1(T_{g,n+r}; \mathbb{Q}) \cong H_1(u_{g,n+r}) \cong \Lambda^3 H_{\mathbb{Q}} \oplus H_{\mathbb{Q}}^{g,n}. \]

This and Hodge theory imply that \( H_1(u_{g,n+r}) \) is pure of weight \(-1\) for all \( g \geq 3 \).
For all \( g \geq 2 \), there is an irreducible \( \text{Sp}(H) \)-module corresponding to the partition \([2, 2]\). We denote it by \( V_{\mathbb{H}} \). It is the highest weight submodule of \( \text{Sym}^2 \Lambda^2 H \). When \( g \geq 3 \), there is a unique copy of \( V_{\mathbb{H}} \) in \( \Lambda^2 \Lambda_0^3 H \) and in \( \Lambda^2 \Lambda^3 H \). When \( g = 3 \),

\[
V_{\mathbb{H}} = \mathbb{Q} \oplus \Lambda^2 \Lambda_0^3 H.
\]

For all \( g \geq 2 \), we have

\[
\text{Gr}_{-m}^W \text{Der}^\theta \mathcal{L}(\Lambda^3 H) \cong \begin{cases} 
\text{sp}(H) & m = 0, \\
\Lambda^3 H & m = 1, \\
V_{\mathbb{H}} \oplus \Lambda^2 H & m = 2.
\end{cases}
\]

When \( g \geq 3 \), the homomorphism

\[
\text{Gr}_W^* d\varphi : \text{Gr}_W^* u_{g,\overline{1}} \to \text{Der}^\theta \text{Gr}_* \mathcal{L}(S, \overline{v})
\]

is an isomorphism in weights \( m = 0, -1, -2 \). See [31, §10]. It also holds in genus 2 as we explain below.

8.0.3. Genus \( g \geq 3 \). In this case, all generators of \( u_{g,n+\overline{1}} \) have weight \(-1\) and \( \text{Gr}_W^* u_{g,n+\overline{1}} \) is quadratically presented for all \( g \geq 4 \). In genus 3, there are quadratic and cubic relations.

**Theorem 8.2** (Hain [31, 35]). For all \( g > 3 \) and all \( n, r \geq 0 \), the graded Lie algebra \( \text{Gr}_W^* u_{g,n+\overline{1}} \) is quadratically presented. In particular,

\[
\text{Gr}_W^* u_{g,\overline{1}} \cong \mathbb{L}(\Lambda^3 H)/(R_2)
\]

where \( R_2 \) is the kernel of the Lie bracket \( \Lambda^2 \text{Gr}_{-1}^W u_{g,\overline{1}} \to \text{Gr}_{-2}^W \text{Der}^\theta \mathcal{L}(H) \). In genus 3, the graded Lie algebra \( u_{3,n+\overline{1}} \) has non-trivial quadratic and cubic relations. These are determined by the condition that \( \text{Gr}_W^* u_{3,\overline{1}} \to \text{Der}^\theta \text{Gr}_* \mathcal{L}(H) \) is an isomorphism in weights \(-1, -2\), and injective when \( m = -3 \).

Explicit relations can be found in [31, 35] and, from a different point of view, in [34, §9].

**Corollary 8.3.** When \( g \geq 3 \), the image of the graded Johnson homomorphism \([24]\) is generated by

\[
\text{Gr}_{-1}^W u_{g,\overline{1}} \cong \Lambda^3 H.
\]

The action of \( \text{Gr}_W^* u_{g,\overline{1}} \) on \( \text{Gr}_W^* \mathcal{L}(S, \overline{v}) \cong \mathbb{L}(H) \) is determined by the action of \( \text{Gr}_{-1}^W u_{g,\overline{1}} \cong \Lambda^3 H \) on \( \mathbb{L}(H) \). This is given by

\[
\{ u_1 \wedge u_2 \wedge u_3 \mapsto - \langle v, \{ u_1, v \} [u_2, u_3] + \langle u_2, v \} [u_3, u_1] + \langle u_3, v \} [u_1, u_2] \}
\in \text{Hom}(H, \Lambda^2 H) \subseteq \text{Der} \mathbb{L}(H).
\]

In the graphical version [26], this derivation corresponds to the planar trivalent graph with one vertex whose 3 leaves are labelled by \( u_1, u_2, u_3 \).
8.0.4. Genus 1. While the genus 0 story is combinatorial (a feature of varieties that are closely related to hyperplane complements) and the \( g \geq 3 \) story, which is both geometric (via its relation the the Ceresa cycle and the Johnson isomorphisms), the genus 1 story is rich, with a distinctly arithmetic flavour because of its connection to classical modular forms. Here we will give a brief introduction. Full details can be found in [36, 42].

One has the central extension
\[
0 \to \mathbb{Q}(1) \to g_{1,\overline{1}} \to g_{1,1} \to 0.
\]

The first difference between genus 1 and all other genera is that neither \( u_{1,1} \) nor \( u_{1,\overline{1}} \) is finitely generated. The Lie algebra \( \Gr W_{*}u_{1,\overline{1}} \) is generated by the graded \( \text{SL}(H) \)-module \( V \) with
\[
\Gr_{-1} V = \bigoplus_{n>0} H_{\text{cusp}}^1(\text{SL}_2(\mathbb{Z}); \text{Sym}^{2n} H)^{\vee} \otimes \text{Sym}^{2n} H \text{ and } \Gr_{-2n} V = \text{Sym}^{2n-2} H.
\]

Here \(( )^{\vee}\) denotes dual. The only relations in \( \Gr W_{*}u_{1,\overline{1}} \) is that the generator \( e_2 \) of \( \Gr W_{-2}L(V) \) is central, so that \( \Gr W_{*}u_{1,\overline{1}} \) is a free Lie algebra. There is a natural torus in \( \text{SL}_2 \) (explained in [36, §10]) and the natural choice of a highest weight vector \( e_{2n} \) in \( \text{Sym}^{2n-2} H \). It is dual to the normalized Eisenstein series of weight \(-2n\).

Let \((S,\vec{v})\) be a surface of type \((1,\overline{1})\). Identify \( \Gr W_{*}p(S,\vec{v}) \) with \( L(H) \) via the canonical isomorphism. The representation
\[
\Gr_{*} W_{1} \to \text{Der}^\theta L(H)
\]
is far from injective. Its kernel is not even finitely generated as it contains the infinite dimensional vector space \( \Gr_{-1} V \). However, the images of the \( e_{2n} \) are non-zero. In fact, the monodromy homomorphism factors canonically
\[
g_{1,\overline{1}} \to \Gr_{1,1}^{\text{MEM}} \to \text{Der}^\theta p(S,\vec{v})
\]
through a Lie algebra \( \Gr_{1,1}^{\text{MEM}} \) that has a MHS and is generated topologically by
\[
\bigoplus_{n \geq 0} \text{Sym}^{2n} H.
\]

We abuse notation and denote the highest weight vector of \( \text{Sym}^{2n-2} H \) by \( e_{2n} \). It has weight \(-2n\). The fact that \( \Gr_{1,1}^{\text{MEM}} \) has a mixed Hodge structure forces a countable set of relations to hold between the \( e_{2n} \) in \( \Gr_{1,1}^{\text{MEM}} \). So that \( \Gr_{1,1}^{\text{MEM}} \) is far from being free; each normalized Hecke eigenform \( f \) of \( \text{SL}_2(\mathbb{Z}) \) determines a countable set of independent relations, one of each “degree” \( \geq 2 \). These are the “Pollack relations”.

Fix a symplectic basis of \( a, b \) of \( H \). For each \( n \geq 0 \), there is a unique derivation\(^{24}\)
\[
e_{2n} \in \Gr_{-2n} \text{Der}^\theta L(H) \text{ of } L(a, b) = L(H) \text{ satisfying }
\]
\[
e_{2n}(\theta) = 0 \text{ and } e_{2n}(b) = \text{ad}^{2n}_b(a).
\]

The graded monodromy representation \( \Gr p_{1,\overline{1}}^{\text{MEM}} \to \text{Der}^\theta \Gr_{*} W_{*}p(S,\vec{v}) \) takes \( e_{2n} \) to \( 2e_{2n}/(2n-2)! \).

\(^{24}\)These derivations were first considered by Tsunogai [80].
Remark 8.4. Pollack [73] found all quadratic relations between the $\epsilon_{2n}$ in $\text{Der}^g \mathbb{L}(H)$ and all higher degree relations modulo the third term of the “elliptic depth filtration” of $\text{Der}^g \mathbb{L}(H)$ in [73]. All relations were proved to lift to $\text{Der}^g \mathbb{L}(H)$ and to be motivic in [122 §25]. It is not known whether these generate all relations between the $\epsilon_{2n}$ in either $u_{1,\Gamma}^{\text{MEM}}$ or their images in $\text{Der}^g \mathbb{L}(H)$.

8.0.5. Genus 2. This is the only genus in which we do not have a complete presentation of $\text{Gr}_*^W u_{g,n+r}$, although Watanabe has made significant progress. He has computed $H_1(u_2)$ and proved that $u_2$ is finitely presented. This is surprising as $T_2$ is countably generated free group.

Theorem 8.5 (Watanabe [83]). There is an isomorphism of $\text{Sp}(H)$-modules
$$\text{Gr}_*^W u_2 \cong V_{\mathbb{H}}$$
where $V_{\mathbb{H}}$ is in weight $-2$ and $\text{Gr}_*^W u_2$ has a minimal presentation
$$\text{Gr}_*^W u_2 = L(V_{\mathbb{H}})/(R_4, R_6, R_8, R_{10}, R_{14})$$
where $R_m$ is in weight $-m$.

These relations have yet to be determined. A conjectural description is given below.

Exactness properties of relative completion imply that
$$0 \to H^{n+r} \to H_1(u_{2,n+r}) \to H_1(u_2) \to 0$$
is exact. This implies that
$$0 \to H^{n+r} \to \text{Gr}_*^W H_1(u_{2,n+r}) \to \text{Gr}_*^W H_1(u_2) \to 0$$
is exact and that $\text{Gr}_*^W u_{2,n+r}$ is finitely presented for all $n$ and $r$.

Corollary 8.6. In genus 2, the image of the graded Johnson homomorphism [24] is generated by
$$\text{Gr}_{-1}^W u_{2,1} \oplus \text{Gr}_{-2}^W u_2 \cong H \oplus V_{\mathbb{H}}.$$
Once one has a presentation of $\Gr^W u_2$, it is straightforward to get a presentation for all $u_{2,n+r}$. In the $r = 0$ case, this is because the kernel of $u_{2,n+r} \to u_2$ is the Lie algebra $p_{2,n}$ of the configuration space of $n$ points on a genus 2 surface. This has a natural MHS which is generated in weight $-1$ and has a well-known presentation due to Bezrukavnikov \footnote{See \cite{hain00} \S 2,
\S 12} So, to lift the presentation of $\Gr^W u_2$ to a presentation of $\Gr^W u_{2,n}$, one only has to determine the action

$$\Gr^W_{-2} u_2 \otimes \Gr^W_{-1} p_{2,n} \to \Gr^W_{-3} p_{2,n}$$

which one can compute using the graded Johnson homomorphism.

9. The Arithmetic Johnson Homomorphism

The geometric Johnson homomorphism \footnote{See \cite{hain00} \S 2, \S 12} extends to a larger Lie algebra that can be constructed using the action of the arithmetic fundamental group of the moduli stack $\mathcal{M}_{g,n+r}/\mathbb{Q}$ on $\pi_1^{\text{ar}}(S, \vec{V})$. One way to do this is to use the weighted completion of arithmetic mapping class groups, \footnote{See \cite{hain00} \S 12}. Here we construct it using Hodge theory.

Throughout this section, $\hat{(S, P, \vec{V})}$ is a surface of type $(g, n+r)$. Fix $\vec{v} \in \vec{V}$. Denote the image of the geometric Johnson homomorphism

$$d\varphi_\vec{v} : \mathfrak{g}_{g,n+r} \to \text{Der}^\theta p(S, \vec{v})$$

by $\mathfrak{g}_{g,n+r}$.

The MHS on $p(S, \vec{v})$ associated to a complex structure $\phi$ on $\hat{(S, P, \vec{V})}$ is determined by a homomorphism

$$\pi_1(\text{MHS}) \to \text{Aut}^\theta p(S, \vec{v}).$$

The image of this homomorphism is (by definition) the Mumford–Tate group $\text{MT}_\phi$ of the MHS on $p(S, \vec{v})$ associated to $\phi$. Denote its Lie algebra by $m^\phi$. Since the geometric Johnson homomorphism is a morphism of MHS, its image $\mathfrak{g}_{g,n+r}$ is a sub-MHS of $\text{Der}^\theta p(S, \vec{v})$. It is therefore normalized by $m^\phi$.

Define the Lie algebra $\mathfrak{g}_{g,n+r}^\phi$ to be the Lie subalgebra of $\text{Der}^\theta p(S, \vec{v})$ generated by $\mathfrak{g}_{g,n+r}$ and $m^\phi$. Since $m^\phi$ normalizes $\mathfrak{g}_{g,n+r}$, $\mathfrak{g}_{g,n+r}^\phi$ is an ideal of $\mathfrak{g}_{g,n+r}$ and $\mathfrak{g}_{g,n+r}^\phi / \mathfrak{g}_{g,n+r}$ is a quotient of $m^\phi$.

Denote the category of mixed Tate motives, unramified over $\mathbb{Z}$, by $\text{MTM}$. This category is constructed in \cite{hain00}. It is a tannakian category and therefore the category of representations of a group, $\pi_1(\text{MTM}, \omega^B)$, where $\omega^B$ is the Betti fiber functor \footnote{Note that the size of $m^\phi$ depends on $\phi$.}

Its Lie algebra $\text{mtm}$ is an extension

$$0 \to \ell \to \text{mtm} \to \mathbb{Q} \to 0$$

where $\mathbb{Q}$ is the Lie algebra of $\mathbb{G}_m$ and $\ell \cong L(\sigma_3, \sigma_5, \sigma_7, \sigma_9, \ldots)^\wedge$, with $\sigma_{2n+1} \in \text{Gr}^W_{4n-2} \ell$.

Remark 9.1. Note that this isomorphism is not canonical as the splitting of the weight filtration depends on the choice of a lift $\hat{\chi}$, and also because, apart from rescaling the $\sigma_{2n+1}$’s, we can replace $\sigma_{11}$ by $\sigma_{11} + [\sigma_3, [\sigma_3, \sigma_3]]$, etc.
Theorem 9.2 (Brown [6]). The Mumford–Tate group of $\pi_1^m(\mathbb{P}^1 - \{0, 1, \infty\}, \bar{\nu})$ is $\pi_1(\text{MTM})$.

Takao’s affirmation [79] of the Oda Conjecture [70], Brown’s result above, and the proof of [38, Prop. 13.1] show that, although $m^\phi$ can be large, the quotient $m^\phi / (\mathfrak{T} \cap m^\phi)$ is constant and isomorphic to $\text{mim}$.

Theorem 9.3 (Oda Conjecture). The Lie subalgebra $\hat{\mathfrak{g}}_{g,n+\vec{r}}^\phi$ of $\text{Der}^\phi p(S, \bar{\nu})$ does not depend on the complex structure $\phi$ on $(S, P, \vec{V})$. It is an extension

$$0 \to \mathfrak{g}_{g,n+\vec{r}} \to \hat{\mathfrak{g}}_{g,n+\vec{r}} \to \text{mim} \to 0$$

where $\text{mim}$ denotes the Lie algebra of the category $\text{MTM}$ of mixed Tate motives unramified over $\mathbb{Z}$.

The following result is proved using the argument in the proof of [37, Prop. 13.3], the main ingredient of which is the injectivity result [55, Thm. 5.2.1] of Kawazumi and Kuno.

Proposition 9.4. The inclusion of $\hat{\mathfrak{g}}_{g,n+\vec{r}}$ into $\text{Der}^\phi p(S, \bar{\nu})$ lifts to a Lie algebra homomorphism

$$\hat{\varphi} : \hat{\mathfrak{g}}_{g,n+\vec{r}} \to \mathbb{Q}\lambda(S)^\wedge$$

such that the diagram

$$\begin{array}{ccc} \hat{\mathfrak{g}}_{g,n+\vec{r}} & \longrightarrow & \text{Der}^\phi p(S, \bar{\nu}) \\ \downarrow \hat{\varphi} & & \downarrow \delta^\phi \\ \mathbb{Q}\lambda(S)^\wedge & \longrightarrow & \mathbb{Q}\pi_1(S, \bar{\nu})^\wedge \end{array}$$

commutes.

10. Hodge Theory and Decomposition of Surfaces

It is often useful to decompose a surface into subsurfaces, such as when one takes a pants decomposition. In this section, we explain how to make such constructions compatible with Hodge theory. The basic idea to consider the decomposed surface as a model of a first order smoothing of the nodal surface obtained by contracting each circle in the decomposition to a point. The mixed Hodge structure on an invariant of the decomposed surface is a “limit MHS”.

If $T$ is a subsurface of a closed surface $S$, then the induced map $\mathbb{Z}\lambda(T) \hookrightarrow \mathbb{Z}\lambda(S)$ preserves the Goldman bracket. If $\xi$ is a framing of $S$ (and thus of $T$), then this inclusion also preserves the cobracket $\delta_\xi$ and induces a Lie bialgebra homomorphism $\mathbb{Q}\lambda(T)^\wedge \to \mathbb{Q}\lambda(S)^\wedge$. More generally, we can consider how the Goldman–Turaev Lie bialgebra behaves when an oriented surface is decomposed into a union of closed subsurfaces by cutting along a finite set of disjoint simple closed curves.

Since surfaces with boundary are not algebraic curves, the map $T \to S$ induced by a decomposition of $S$ into subsurfaces does not appear to be compatible with algebraic geometry and Hodge theory. In this section, we explain briefly how decompositions $T \to S$ do induce morphisms of MHS. The essential point is to regard

---

28I suspect that, now that Brown’s Theorem is known, a shorter proof of the pronipotent version of Oda’s Conjecture can be given.

29So we will henceforth drop the $\phi$ from the notation.
the decomposed surface as a first order smoothing of a nodal surface. The MHSs that arise are “limit MHS”. Material in this section is explained in detail in [39].

10.1. Smoothings of nodal surfaces. Suppose that $S$ is an oriented surface. Note that we are not assuming that $S$ be connected. A non-zero tangent vector $\mathbf{v}$ of $S$ at a point $p$ determines a marked point $[\mathbf{v}]$ on the boundary circle of the real oriented blowup $\text{Bl}_p S$ of the surface at $p$. Given two non-zero tangent vectors $\mathbf{v} \in T_p S$ and $\mathbf{w} \in T_q S$ anchored at two distinct points $p, q \in S$, we can “smooth” the nodal surface

$$R_0 = S/(p \sim q)$$

by gluing the the two boundary circles of the real oriented blowup $\hat{S} := \text{Bl}_{p,q} S$ of $S$ that lie over $p$ and $q$ so that $[\mathbf{v}] \in \partial \hat{S}$ is identified with $[\mathbf{w}] \in \partial \hat{S}$. Denote the resulting surface by $R_{\mathbf{v} \otimes \mathbf{w}}$. We call it the (topological) smoothing associated to $\mathbf{v} \otimes \mathbf{w}$ of the nodal surface $R_0$. The image of the boundary circles of $\hat{S}$ in $R_{\mathbf{v} \otimes \mathbf{w}}$ is the vanishing cycle. The (positive) Dehn twist on the vanishing cycle is called the monodromy operator. The quotient of $R_{\mathbf{v} \otimes \mathbf{w}}$ obtained by collapsing the vanishing cycle to a point is canonically homeomorphic to $R_0$.

This construction generalizes to nodal surfaces $R_0$ that are obtained by identifying disjoint pairs of distinct points $p_j$ and $q_j$ ($j = 1, \ldots, m$) of $S$. A smoothing is determined by the set $\{\mathbf{v}_j \otimes \mathbf{w}_j : j = 1, \ldots, m\}$, where $\mathbf{v}_j \in T_{p_j} S$ and $\mathbf{w}_j \in T_{q_j} S$ are non-zero. There is one vanishing cycle for each node. The monodromy operator is the product of the Dehn twists about the vanishing cycles. We will denote the smoothed surface by $R_{\mathbf{v}}$, where $\mathbf{v} = \sum_{j=1}^m \mathbf{v}_j \otimes \mathbf{w}_j$.\footnote{The family over $\prod_{j=1}^m (T_{p_j} S \otimes T_{q_j} S - \{0\})$ whose fiber over $\mathbf{v}$ is $R_{\mathbf{v}}$ is topologically locally trivial. The monodromy about the $j$th coordinate hyperplane $\mathbf{v}_j \otimes \mathbf{w}_j = 0$ is the Dehn twist about the $j$th vanishing cycle.}

A complex structure on $R_0$ is, by definition, the structure of a nodal algebraic curve. Denote this algebraic curve by $X_0$. Each topological smoothing $R_{\mathbf{v}}$ of $R_0$ corresponds to a first order smoothing $X_{\mathbf{v}}$ of $X_0$ as an algebraic curve.\footnote{One should think of $R_{\mathbf{v}}$ as the topological space underlying $X_{\mathbf{v}}$.} For each complex structure on $R_0$, each of the topological invariants of surfaces that we are considering: $\mathbb{Q}\pi_1(R_{\mathbf{v}}, x)^\wedge$, $\mathbb{Q}\lambda(R_{\mathbf{v}})^\wedge$, $\mathfrak{g}_{R_{\mathbf{v}}, \partial R_{\mathbf{v}}}, \ldots$, has a natural limit mixed Hodge structure which depends non-trivially on the collection $\{\mathbf{v}_j \otimes \mathbf{w}_j : j = 1, \ldots, m\}$.\footnote{These limit MHS form a nilpotent orbit of MHS over $\prod_{j=1}^m (T_{p_j} S \otimes T_{q_j} S - \{0\})$.}

![Figure 2. Smoothing a node](image)
Such a limit MHS has two weight filtrations: $W_\bullet$, which we have already encountered, and the relative weight filtration $M_\bullet$, which is constructed from the weight filtration $W_\bullet$ and the action of the monodromy operator $M_\bullet$.

A morphism $R'_\lambda \to R_{\lambda}$ between topological smoothings of nodal surfaces is a map between the underlying topological spaces that is a generic inclusion. More precisely,

(i) $R'_0 = R_0 - \Sigma$, where $\Sigma$ is finite and may contain nodes of $R_0$,

(ii) the smoothings of $R'_0$ and $R_0$ agree at each node of $R'_0$. That is, if $\vec{v}'_j \otimes \vec{w}'_j$ is the vector corresponding to the $j$th node of $R'_0$ and $\vec{v}_j \otimes \vec{w}_j$ is the vector corresponding to its image in $R_0$, then $\vec{v}'_j \otimes \vec{w}'_j = \vec{v}_j \otimes \vec{w}_j$.

A morphism between $X'_\lambda \to X_\lambda$ between smoothings of nodal algebraic curves is a morphism $X'_0 \to X_0$ of the algebraic curves together with a morphism $R'_\lambda \to R_{\lambda}$ of the underlying topological spaces. Morphisms of smoothed nodal curves induce map morphisms of (limit) MHS on the standard invariants we are considering.

**Theorem 10.1.** The homomorphism $\mathbb{Q}\lambda(X'_\lambda)^{\wedge} \to \mathbb{Q}\lambda(X_{\lambda})^{\wedge}$ induced by a morphism $X'_\lambda \to X_{\lambda}$ of smoothed nodal curves induces a morphism of MHS. Moreover the functors $\operatorname{Gr}^{M}_{\bullet}$ and $\operatorname{Gr}^{W}_{\bullet}$ are both exact.

10.2. **Indexed pants decompositions.** An indexed pants decomposition of a surface is a pants decomposition in which there are only two ways to identify two boundary circles. In the next section, we will see that they correspond to very special smoothings of maximally degenerate nodal curves.

The starting point is to note that the Riemann sphere has 6 canonical tangent vectors, namely $\partial / \partial z \in T_0 \mathbb{P}^1$ and its images under the canonical action of the symmetric group $\Sigma_3$ on $(\mathbb{P}^1, \{0, 1, \infty\})$. An indexed pair of pants is the real oriented blow up

$$\hat{\mathbb{P}} := \text{Bl}_{0,1,\infty} \mathbb{P}^1$$

of $\mathbb{P}^1$ at $\{0, 1, \infty\}$ together with the 6 boundary points (2 on each boundary circle) that correspond to the 6 canonical tangent vectors.

![Figure 3. The canonical tangent vectors of $\mathbb{P}^1(\mathbb{R})$ and the blowup $\hat{\mathbb{P}}$](image)

An indexed pants decomposition of a surface $S$ is a decomposition of $S$ into indexed pairs of pants in which the indexing of adjacent pants match on each common boundary component. The associated nodal curve $R_0$ is obtained from

---

33 The weight filtration of the limit MHS is $M_\bullet$, not $W_\bullet$. However, each $W_r$ is a sub MHS of the limit MHS. See [33] for an exposition of relative weight filtrations for topologists.

34 Indexed pants decompositions correspond to isotopy classes of quilt decompositions of a pants decomposition as defined by Nakamura and Schneps in [69].
S by collapsing the boundaries of all pairs of pants. Collapsed boundary circles should be regarded as marked points on $R_0$.

10.3. Ihara curves. A maximally degenerate algebraic curve of type $(g, n)$ is a nodal algebraic curve $X_0$, each of whose components is a copy of $\mathbb{P}^1$ and where the number of nodes plus the number of marked points on each component of $X_0$ is 3. Tangent vectors can be added at the marked points to obtain a maximally degenerate curve of type $(g, m + r)$, where $n = m + r$.

An Ihara curve of type $(g, n)$ is a first order smoothing $\mathcal{R}_\varphi$ of a maximally degenerate algebraic curve $X_0$ of type $(g, n)$ where

$$\varphi = \sum \varphi_p \otimes \varphi_p' \in \bigoplus_{\text{nodes } p \text{ of } R_0} T_p U' \otimes T_p U''.$$  

Here $U'$ and $U''$ are the two analytic branches of $X_0$ at the node $p$, $\varphi'$ is a canonical tangent vector on the copy of $\mathbb{P}^1$ that contains $U'$, and $\varphi''$ is a canonical tangent vector on the copy of $\mathbb{P}^1$ that contains $U''$. An Ihara curve of type $(g, m + r)$ is an Ihara curve of type $(g, m + r)$, with $r$ additional canonical tangent vectors added at $r$ distinct points of $X_0$. Ihara curves correspond to indexed pants decompositions of a surface of type $(g, n)$.

Ihara and Nakamura \[44\] constructed a canonical formal smoothing $X/\mathbb{Z}[[q_p : p \text{ a node of } X_0]]$ of each maximally degenerate nodal curve $X_0$. These generalize Tate’s elliptic curve in genus 1. The Ihara curve $X_\varphi$ is the fiber of $X$ over a tangent vector $\varphi = \sum_p \partial/\partial q_p$. For us, the importance of Ihara curves is that their invariants are mixed Tate motives, unramified over $\mathbb{Z}$.

**Theorem 10.2.** If $(X, \varphi)$ is an Ihara curve, then each of the invariants $p(X, \varphi)$, $\mathbb{Q}\lambda(X)$, $\mathfrak{g}$, $\hat{\mathfrak{g}}$, ... is a pro-object of MTM and therefore has a canonical $\mathfrak{k}$ action. The action of $\mathfrak{k}$ commutes with the Dehn twist on each vanishing cycle.

**Corollary 10.3.** An indexed pants decomposition $X = \bigcup_{T \in \mathcal{R}} T$ of an Ihara curve induces a Lie bialgebra homomorphism

$$\bigoplus_{T \in \mathcal{R}} \mathbb{Q}\lambda(T)^\wedge \to \mathbb{Q}\lambda(X)^\wedge$$

whose kernel is spanned by the powers $\{\psi_n \log \nu : n \geq 1\}$ of the logarithms of the vanishing cycles $\nu$. It is a morphism of mixed Tate motives (and thus also, MHS). The homomorphism $\mathfrak{k} \to \mathbb{Q}\lambda(X)^\wedge$ is “decomposable” in the sense that the diagram

$$\begin{array}{ccc}
\mathfrak{k} \\
\downarrow \varphi_T \\
\bigoplus_{T \in \mathcal{R}} \mathbb{Q}\lambda(T)^\wedge \\
\downarrow \varphi_X \\
\mathbb{Q}\lambda(X)^\wedge
\end{array}$$

commutes.

11. Special Derivations and Edge Homomorphisms

A major point of the works \[1, 2\] of Alekseev–Kawazumi–Kuno–Naef is that much of the algebra of $\text{Gr}_W^* \mathbb{Q}\lambda(S)$ can be expressed in terms of several formal, non-commutative analogues of the classical divergence. This allows them to prove...
several key formulas for the Turaev cobracket on $\Gr_{\bullet}^W \mathbb{Q} \lambda(S)$. This section is a review of one aspect of their work useful for understanding the image of $\kappa_2 : \hat{\frak{g}}_{g,n+1} \to \text{Der}^\theta \mathfrak{p}(S)$. This section is quite long and technical. It is needed in for the proof of Theorem 12.6 and can be skipped if one is prepared to believe that result.

### 11.1. Special derivations

Suppose that $(S, \vec{v})$ is a surface of type $(g, n + \hat{1})$. Index the punctures by the integers $j$ with $0 \leq j \leq n$. Choose the indexing so that the distinguished tangent vector $\vec{v}$ is anchored at the 0th puncture. Let $\mu_j$ ($0 \leq j \leq n$) be a circle that bounds a small disk centered at the $j$th puncture. For each $j \geq 1$, choose a path $\gamma_j$ in $S$ from $\vec{v}$ to a point on $\gamma_j$. Set $s_0 = \mu_0$, where we consider $\mu_0$ to be a loop based at $\vec{v}$, and

$$s_j := \gamma_j \mu_j \gamma_j^{-1} \in \pi_1(S, \vec{v}) \text{ when } j = 1, \ldots, n.$$  

Denote $\pi^m_i(S, \vec{v})$ by $\mathcal{P}$ and its Lie algebra by $\mathfrak{p}$. Identify $s_j$ with its image in $\mathcal{P}(\mathbb{Q})$. Define the group of \textit{special automorphisms} of $\mathcal{P}$ by

$$\text{SAut} \mathcal{P} = \{ \phi \in \text{Aut} \mathcal{P} : \phi(s_0) = s_0 \text{ and } \phi(s_j) \sim s_j, \ j = 1, \ldots, n\},$$

where $\sim$ denotes “conjugate to”. This is an affine $\mathbb{Q}$-group which does not depend on the choice of the $\gamma_j$. The Lie algebra of $\text{SAut} \mathcal{P}$ is the Lie algebra $\text{SDer} \mathfrak{p} := \{ D \in \text{Der} \mathfrak{p} : D(\log s_0) = 0, \ D(\log s_j) = [u_j, \log s_j] \text{ where } u_j \in \mathfrak{p}, \ j > 0 \}$ of \textit{special derivations} of $\mathfrak{p}$. This is a Lie subalgebra of $\text{Der}^\theta \mathfrak{p}$.

**Proposition 11.1.** For each complex structure (possibly a first order smoothing) on $(S, \vec{v})$, there are canonical MHSs on $\text{SAut} \mathcal{P}$ and $\text{SDer} \mathcal{P}$.

**Proof.** We prove the assertion for $\text{SDer} \mathfrak{p}$, which we will need, and sketch the proof for $\text{SAut} \mathcal{P}$, which we will not. The canonical homomorphisms

$$\mathfrak{p} \to \mathbb{Q} \pi_1(S, \vec{v})^\wedge \to \mathbb{Q} \lambda(S)^\wedge$$

are morphisms of MHS. Each derivation of $\mathfrak{p}$ induces a derivation of its enveloping algebra $\mathbb{Q} \pi_1(S, \vec{v})^\wedge$, and thus an endomorphism of $\mathbb{Q} \lambda(S)^\wedge$. The corresponding linear map

$$\text{Der} \mathfrak{p} \to \text{End} \mathbb{Q} \lambda(S)^\wedge$$

is a morphism of MHS. Standard Hodge theory implies that each log $|\mu_j|$ spans a copy of $\mathbb{Q}(1)$ in $\mathbb{Q} \lambda(S)^\wedge$. Consequently, the derivations of $\mathfrak{p}$ that annihilate each log $|\mu_j|$ form a sub MHS of $\text{Der} \mathfrak{p}$. Since $|\log s_j| = |\log |\mu_j||$, $\text{SDer} \mathfrak{p}$ is the intersection of this Lie algebra with $\text{Der}^\theta \mathfrak{p}$. It follows that $\text{SDer} \mathfrak{p}$ is a sub MHS of $\text{Der} \mathfrak{p}$.

The corresponding statement for $\text{SAut} \mathcal{P}$ is proved similarly using the action of $\pi_1(\text{MHS})$ on $\mathcal{P}$ and the fact that it acts on each log $s_j$ via the canonical character $\pi_1(\text{MHS}) \to \text{Aut} \mathbb{Q}(1) \cong \mathbb{G}_m$. Just use the fact that the canonical maps $\mathcal{P} \to \mathbb{Q} \pi_1(S, \vec{v})^\wedge \to \mathbb{Q} \lambda(S)^\wedge$ are $\pi_1(\text{MHS})$-equivariant.

For each $j = 0, \ldots, n$, set $z_j = \log s_j \in \Gr_{-2}^W \mathfrak{p}$. Define $\text{SDer} \Gr_{\bullet}^W \mathfrak{p} = \{ D \in \text{Der} \Gr_{\bullet}^W \mathfrak{p} : \text{ for all } j, \ D(z_j) = [u_j, z_j] \text{ for some } u_j \in \Gr_{\bullet}^W \mathfrak{p} \}$. As a consequence of the exactness properties of $\Gr_{\bullet}^W$ we have:

**Proposition 11.2.** For each complex structure (possibly a first order smoothing) on $(S, \vec{v})$ and each lift of the canonical central cocharacter $\chi : \mathbb{G}_m \to \pi_1(\text{MHS})$, there is a natural Lie algebra isomorphism

$$\Gr_{\bullet}^W \text{SDer} \mathfrak{p} \cong \text{SDer} \Gr_{\bullet}^W \mathfrak{p}.$$
The image of the natural homomorphism \( \varphi_{\gamma} : \hat{G}_{g,n+1} \to \text{Aut} \mathcal{P} \) lies in \( \text{SAut} \mathcal{P} \). This implies that the image of \( \hat{G}_{g,n+1} \) lies in \( \text{SDer} \mathfrak{p} \). For each complex structure on \((S,\vec{v})\) (possibly a smoothing), the homomorphisms \( \varphi_{\gamma} : \hat{G}_{g,n+1} \to \text{SAut} \mathcal{P} \) and \( d\varphi_{\gamma} : \hat{G}_{g,n+1} \to \text{SDer} \mathfrak{p} \) are morphisms of MHS. We can therefore replace \( \text{Der}^{g}_{p} \) by \( \text{SDer} \mathfrak{p} \) in the diagram \( \boxtimes \).

Similarly, one defines \( \text{SDer} \mathbb{Q} \pi_{1}(S,\vec{v})^{\wedge} \) and \( \text{SDer} \mathbb{Q} \pi_{1}(S,v) \). Both have a natural MHS for each complex structure on \((S,\vec{v})\). The image of \( \kappa_{\gamma} \) is easily seen to lie in \( \text{SDer} \mathbb{Q} \pi_{1}(S,\vec{v})^{\wedge} \), so we will regard it as a homomorphism

\[
\kappa_{\gamma} : \mathbb{Q}\lambda(S)^{\wedge} \to \text{SDer} \mathbb{Q}\pi_{1}(S,\vec{v})^{\wedge}.
\]

It is a morphism of MHS for each choice of complex structure on \((S,\vec{v})\). One can thus replace it by its associated weight graded.

### 11.2. The edge homomorphism

As above, \((S,\vec{v},\xi)\) is a framed surface of type \((g,n+1)\). The **edge map** \( \text{edg}_{\xi} \) is defined to be the composite

\[
\mathbb{Q}\lambda(S)^{\wedge} \xrightarrow{\delta_{\xi}} \mathbb{Q}\lambda(S)^{\wedge} \otimes \mathbb{Q}\lambda(S)^{\wedge} \xrightarrow{\text{id} \otimes \epsilon} \mathbb{Q}\lambda(S)^{\wedge},
\]

where \( \epsilon : \mathbb{Q}\lambda(S)^{\wedge} \to \mathbb{Q} \) is the map induced by the augmentation. It is a morphism of MHS for each choice of complex structure on \( S \). Its dependence on the framing is easily determined.

Suppose that \( \xi_{1} \) and \( \xi_{0} \) are two framings of \( S \) and that \( \xi_{1} - \xi_{0} = \phi \in H^{1}(S;\mathbb{Z}) \). Set \( \text{edg}_{\phi}(\gamma) := \text{edg}_{\xi_{1}}(\gamma) - \text{edg}_{\xi_{0}}(\gamma) \).

**Lemma 11.3.** If \( \xi_{1} - \xi_{0} = \phi \in H^{1}(S) \), then on \( \mathbb{Q}\lambda(S)^{\wedge} \), we have

\[
\text{edg}_{\phi} \circ \psi_{n} = n \psi_{n} \circ \text{edg}_{\phi}.
\]

Consequently, \( \text{edg}_{\phi} \) maps the subspace \(|\text{Sym}^{k} \mathfrak{p}|\) of \( \mathbb{Q}\lambda(S)^{\wedge} \) into \(|\text{Sym}^{k-1} \mathfrak{p}|\).

**Proof.** For all immersed loops \( \gamma \) in \( S \) we have

\[
(29) \quad \text{edg}_{\phi}(\gamma) = \text{edg}_{\xi_{1}}(\gamma) - \text{edg}_{\xi_{0}}(\gamma) = (\text{rot}_{\xi_{1}}(\gamma) - \text{rot}_{\xi_{0}}(\gamma)) \gamma = \phi(\gamma) \gamma.
\]

This implies that

\[
\text{edg}_{\phi}(\psi_{n}(\gamma)) = \phi(\psi_{n}(\gamma)) \psi_{n}(\gamma) = n \phi(\gamma) \psi_{n}(\gamma) = n \psi_{n}(\text{edg}_{\phi}(\gamma))
\]

for all \( \gamma \in \lambda(S) \). Since \( \text{edg}_{\phi} \) is continuous, this formula also holds in \( \mathbb{Q}\lambda(S)^{\wedge} \). The second assertion follows from this and Corollary \( \boxtimes \). \( \square \)

**Corollary 11.4.** If \( u, v \in \mathfrak{p} \), then

\[
\text{edg}_{\phi}(uv) = (\phi(u)|v| + \phi(v)|u|)/2 \in H_{1}(S).
\]

**Proof.** First note that there is a canonical isomorphism \(|\text{Sym}^{2} \mathfrak{p}| \cong H_{1}(S) \), so Lemma \( \boxtimes \) implies that \( \text{edg}_{\phi} : |\text{Sym}^{2} \mathfrak{p}| \to H_{1}(S) \). It also implies that

\[
\text{edg}_{\phi}(uv) = \text{edg}_{\phi}(1 + |u + v| + |uv + u^{2}/2 + v^{2}/2| + \cdots)
\]

\[
= \phi(u + v)(1 + |u + v| + |uv + u^{2}/2 + v^{2}/2| + \cdots)
\]

which implies that

\[
\text{edg}_{\phi}(uv + u^{2}/2 + v^{2}/2) = \phi(u + v)|u + v|.
\]

The lemma also implies that \( \text{edg}_{\phi}|u^{2}/2 + v^{2}/2| = \phi(u)|u| + \phi(v)|v| \). Together these imply that \( \text{edg}_{\phi}(uv) = \phi(u)|v| + \phi(v)|u| \). \( \square \)
11.3. **Genus 0 setup.** Suppose that $n \geq 2$ and $S = \mathbb{P}^1 - \{p_0, \ldots, p_n\}$. We use the notation of \(^{[26]}\), where $p_0 = \infty^{[23]}$. In particular, $H_1(S)$ is spanned by $e_0, \ldots, e_n$, with the single relation $e_0 + \cdots + e_n = 0$. There is a vector space inclusion

$$\text{SDer } Gr^W_1 \cong \text{SDer } \left( L(e_0, \ldots, e_n) / (e_0 + \cdots + e_n) \right) \hookrightarrow L(H_1(S))^{n+1}.$$  

It takes the special derivation $D$ of $Gr^W_1$ of weight $-2m$ to the vector

$$u = (u_1, \ldots, u_n) \in \left( Gr^W_{2m+2} p \right)^n$$

where $D(e_j) = [u_j, e_j]$ \(^{[26]}\). The image consists of those $u$ satisfying

$$(30) \quad [u_1, e_1] + \cdots + [u_n, e_n] = -D(e_0) = 0.$$  

Denote this derivation by $D_u$.

Similarly, the Lie algebra of special derivations of

$$Gr^W_1 \mathbb{Q} \pi_1(S, \vec{v}) \cong \mathbb{Q}(e_0, \ldots, e_n) / (e_0 + \cdots + e_n)$$

is the Lie algebra of derivations that satisfy

$$D_u : e_j \mapsto [u_j, e_j] := u_j e_j - e_j u_j, \quad j = 1, \ldots, n$$

where each $u_j \in \mathbb{Q}(e_0, \ldots, e_n) / (e_0 + \cdots + e_n)$, $u_0 = 0$ and \(^{[26]}\) holds.

Alekseev, Kazawumi, Kuno and Naef \(^{[1]}\) show that, in genus 0, the graded Goldman–Turaev Lie bialgebra is essentially isomorphic to $\text{SDer } Gr^W_1 \mathbb{Q} \pi_1(S, \vec{v})$.

**Proposition 11.5** (\(^{[1]}\) Lem. 8.3). If $(S, \vec{v})$ is a surface of type $(0, n + \vec{1})$, then the Lie algebra homomorphism

$$\kappa_\vec{v} : Gr^W_1 \mathbb{Q} \lambda(S) \to \text{SDer } Gr^W_1 \mathbb{Q} \pi_1(S, \vec{v})$$

is surjective with 1-dimensional kernel spanned by the trivial loop. The isomorphism $\kappa_\vec{v} : Gr^W_1 \mathbb{I} \mathbb{Q} \lambda(S) \to \text{SDer } Gr^W_1 \mathbb{Q} \pi_1(S, \vec{v})$ restricts to an isomorphism

$$\text{SDer } Gr^W_1 p \cong \bigoplus_{j=1}^n [e_j L(e_1, \ldots, e_n)] \subset Gr^W_1 | \text{Sym}^2 p].$$

Under this isomorphism, $D_u \in \text{SDer } Gr^W_1 p$ corresponds to $\sum_{j=1}^n |e_j u_j| \in |\text{Sym}^2 Gr^W_1 p|$. Consequently, for each framing $\xi$ of $S$, the edge maps descends to a map

$$\text{edg}_\xi : \text{SDer } Gr^W_1 \mathbb{Q} \pi_1(S, \vec{v}) \to Gr^W_1 \mathbb{Q} \lambda(S).$$

**Remark 11.6.** Since $\kappa_\vec{v}$ is a morphism of MHS, this result implies that for all complex structures on $(S, \vec{v})$, and all (necessarily algebraic) framings $\xi$, the cobracket descends to a MHS morphism $\delta_\xi : \text{SDer } \mathbb{Q} \pi_1(S, \vec{v}) \to \mathbb{Q} \lambda(S)$. It also implies that there is a canonical isomorphism

$$\kappa_0 : \bigoplus_{j=1}^n |\log \mu_j| p_j \xrightarrow{\cong} \text{SDer } p.$$  

where $p_j$ denotes the Lie algebra of $\pi_1^{\text{an}}(S, \vec{v}_j)$, where each $\vec{v}_j \in T_{p_j} \mathbb{P}^1$ is non-zero. Note that $p_0 = p$.

\(^{35}\) So a complex structure on $S$ corresponds to a choice of the points $p_1, \ldots, p_n \in \mathbb{C}$.

\(^{36}\) When $m \neq 1$, $\mathbf{u}$ is uniquely determined by $D$. When $m = -1$, $\mathbf{u}$ is uniquely determined if we insist that each $u_j$ be orthogonal to $e_j$ with respect to the natural $\Sigma_{n+1}$-invariant inner product on $H_1(S)$ defined by $e_j : e_k = -1$ when $j \neq k$ and $n$ when $j = k$.

\(^{37}\) Note that $[D_u, D_v] = D_{\omega}$, where $\omega_j = D_u(v_j) - D_v(u_j) - [u_j, v_j], \ j = 1, \ldots, n.$
Corollary 11.7. Fix a complex structure on $(S, \vec{v})$. If $\xi_0$ and $\xi_1$ are two (necessarily algebraic) framings of $S$ that differ by $\phi \in H^1(S)$, then $\text{edg}_{\phi}: \text{SDer} p \to H_1(S)$ is a morphism of MHS. The induced map on associated weight gradeds is

$$\text{edg}_{\phi}: D_u \mapsto \sum_{j=1}^n \phi(e_j)|u_j|.$$ 

**Proof.** The first assertion follows from Proposition 11.5 and Lemma 11.3. That $\text{edg}_{\phi}$ is a morphism of MHS follows from the fact that $\delta_\xi$ is a morphisms of MHS for all algebraic framings (Section 7.1) and, in genus 0, every framing is homotopic to an algebraic framing.

The condition (30) implies that $\sum_{j=1}^n (e_j u_j - u_j e_j) = 0 \in p \otimes \mathbb{Q}$. This implies that

$$\sum_{j=1}^n \phi(e_j)|u_j| = \sum_{j=1}^n \phi(u_j)e_j.$$ 

Since $\phi(\mu_j) = \phi(e_j)$, Proposition 11.5 and Corollary 11.4 imply that

$$\text{edg}_{\phi}: D_u \mapsto \sum_{j=1}^n (\phi(u_j)|e_j| + \phi(e_j)|u_j|)/2 = \sum_{j=1}^n \phi(e_j)|u_j|.$$ 

\[ \square \]

11.4. The divergence cocycle in genus 0. For each surface $(S, \vec{v})$ of type $(0, n + 1)$, Alekseev, Kawazumi, Kuno and Naef construct [1, §3.2] a Lie algebra 1-cocycle $\text{div}_0: \text{SDer} \ Gr^W \ p \to \text{Gr}^W \ Q\lambda(S)$.

Write $u_j = \sum_{k=1}^n e_k u^{(k)}_{j} \in Q(\vec{e}_1, \ldots, \vec{e}_n)$. Then

$$\text{div}_0 D_u := \sum_{j=1}^n |e_j u^{(j)}_j|.$$ 

For each framing $\xi$ of $S$, define $r_{0, \xi}: \text{SDer} \ Gr^W \ p \to \text{Gr}^W \ Q\lambda(S)$ by

$$r_{0, \xi}: D_u \mapsto \sum_{j=1}^n \text{rot}_\xi(\mu_j)|u_j| \in |p| \cong H_1(S).$$ 

It is a homomorphism $\text{SDer} \ Gr^W \ p \to H_1(S)$. The subscript 0 in $\text{div}_0$ and $r_{0, \xi}$ indicates the distinguished boundary component of $S$. Note that $p = p_0$.

**Theorem 11.8** (Alekseev–Kawazumi–Kuno–Naef). For each framed surface $(S, \vec{v}, \xi)$ of type $(0, n + 1)$, the restriction of $\text{edg}_\xi$ to $\text{SDer} \ Gr^W_\ast \ p$ is a 1-cocycle equal to $\text{div}_0 + r_{0, \xi}$. More precisely, the diagram

$$\begin{array}{ccc}
\text{Gr}^W_\ast \ Q\lambda(S) & \xrightarrow{\text{edg}_\xi} & \text{Gr}^W_\ast \ Q\lambda(S) \\
\downarrow \kappa_\vec{v} & & \downarrow \kappa_\vec{v}^{-1} \\
\text{SDer} \ Gr^W_\ast \ Q\pi_1(S, \vec{v}) & \xrightarrow{\text{div}_0 + r_{0, \xi}} & \text{SDer} \ Gr^W_\ast \ p
\end{array}$$

commutes. Consequently, the restriction of $\text{edg}_\xi: \text{SDer} p \to Q\lambda(S)^\wedge$ to $W_{-3} \text{SDer} p$ does not depend on the choice of the framing $\xi$. 
Remarks on the Proof. First observe that the cocycle $c$ of [1] Lem. 3.2 is $r_{0,\xi_0}$, where $\xi_0 := \partial/\partial z$ is the “blackboard framing” of $S$. Note that the cobracket $\delta^+$ in [1] denotes $\delta_0$.

The result holds for the blackboard framing. This is [1] Prop. 6.7], which follows from [1] Prop. 3.5] and the discussion at the beginning of [1, §5.3]. To deduce the result for all framings $\xi$, note that Corollary [1,1.7] implies that

$$r_{0,\xi}(D_u) - r_{0,\xi_0}(D_u) = \sum_{j=1}^n \phi(\mu_j)|u_j| = \text{edg}_\xi(D_\mu) - \text{edg}_{\xi_0}(D_\mu),$$

where $\xi - \xi_0 = \phi$. Finally, the last statement follows as $r_{0,\xi}$ vanishes on $W_{-3}$ as $W_{-3}H_1(S) = 0$. \hfill \Box

Remark 11.9. At least when $S$ has genus 0, [1] Prop. 3.5] implies that the restriction

$$\delta_\xi : \text{SDer} p \to \mathbb{Q}\lambda(S)^\wedge \otimes \mathbb{Q}\lambda(S)^\wedge$$

of the cobracket to $\text{SDer} p$ is determined by the edge homomorphism $\text{edg}_\xi$.

Problem 11.10. Give a clear conceptual account of the results of this section for algebraic curves of type $(g, n + 1)$ with a quasi-algebraic framing for all $g \geq 0$.

12. Constraints on the Arithmetic Johnson Image

In this section $(S, \bar{\nu})$ is a hyperbolic surface of type $(g, n + 1)$ with framing $\xi$. Set $\pi = \pi_1(S, \bar{\nu})$, $p = p(S, \bar{\nu})$, $\mathfrak{g} = \mathfrak{g}_{g,n+1}$ and $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_{g,n+1}$, etc. We explain how the Turaev cobracket gives an upper bound on the image of the arithmetic Johnson image $\hat{\mathfrak{g}}$ in $\text{Der}^\theta p$. The solution of the Oda Conjecture, Theorem 9.3] implies that this also gives a bound on the size of the geometric Johnson image $\mathfrak{g}$, if not its precise location in $\text{Der}^\theta p$.

Fix a complex structure $\phi$ on $(S, \bar{\nu}, \xi)$ and a lift $\tilde{\chi} : G_m \to \pi_1(\text{MHS})$. These fix natural isomorphisms between a MHS and its associated weight graded quotients. In particular, they fix natural isomorphisms

$$p \cong (\text{Gr}_W^* p)^\wedge, \quad \hat{\mathfrak{g}} \cong (\text{Gr}_W^* \hat{\mathfrak{g}})^\wedge, \quad \mathbb{Q}\lambda(S)^\wedge \cong (\text{Gr}_W^* \mathbb{Q}\lambda(S))^\wedge, \ldots$$

compatible with their respective algebraic structures (Lie algebra, Lie bialgebra, . . . ). This means that all results in this section apply equally to the proalgebraic objects (such as $\hat{\mathfrak{g}}$, $\text{Der}^\theta p$ and their associated weight graded objects).

Denote the normalizer of the geometric Johnson image $\mathfrak{g}$ in $\text{Der}^\theta \mathbb{Q}\pi^\wedge$ by $n/\mathfrak{g}$. Observe that $\hat{\mathfrak{g}}$ is contained in $n$ as $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$. Since the adjoint action of $\mathfrak{g}$ on $n/\mathfrak{g}$ is trivial, the weight graded Lie algebra

$$\text{Gr}_W^* n / \text{Gr}_W^* \mathfrak{g}$$

is a trivial $\mathfrak{sp}(H)$ module. The action $\mathfrak{m}^\phi \to \text{Der}^\theta \mathbb{Q}\pi^\wedge$ of the Mumford–Tate Lie algebra induces a homomorphism

$$(32) \quad \mathfrak{e} \to n/\mathfrak{g}$$

which is injective by the solution of the Oda Conjecture. This implies that $n/\mathfrak{g}$ is a pro-object of $\text{MTM}(\mathbb{Z})$. Since $\mathfrak{g}$ acts trivially on $n/\mathfrak{g}$, the Theorem of the Fixed Part implies that the MHS on $n/\mathfrak{g}$ does not depend on the complex structure $\phi$.

\[38\text{Note that this is the normalizer of } \mathfrak{g}. \text{ This is strictly smaller than the normalizer of } \mathfrak{g}.\]
Denote the stabilizer of \( \xi \) in \( \hat{g} \) by \( \hat{g}^{\xi} \). Note that \( \hat{g}^{\xi} \) stabilizes \( \delta_\xi \) in the sense that
\[
\delta_\xi(\{u, v\}) = u \cdot \delta_\xi(v)
\]
for all \( u \in \hat{g}^{\xi} \) and all \( v \in Q\lambda(S)^\wedge \).

**Lemma 12.1.** Suppose that \( L \) is a Lie bialgebra with cobracket \( \delta \). If \( h \) is a Lie subalgebra of \( L \) that preserves the bracket in the sense that
\[
\delta([h, u]) = h \cdot \delta u
\]
for all \( u \in L \), \( h \in h \), then \( [h, h] \subseteq \ker \delta_\xi \).

**Proof.** Since \( \delta \) is \( h \)-invariant, \( \delta_\xi([h', h'']) = h' \cdot \delta h'' = -h'' \cdot \delta h' \) for all \( h', h'' \in h \). On the other hand, since \( L \) is a Lie bialgebra, \( \delta([h', h'']) = h' \cdot \delta h'' - h'' \cdot \delta h' \). Combining these two statements, we see that \( \delta([h', h'']) = 0 \) all \( h', h'' \in h \).

Denote the Lie algebra of the pronilpotent radical of \( m_\phi \) by \( u_{MT}^\phi \). Since \( \delta \) acts by a Tate twist, it is not invariant under all of \( \pi_1(M(T)) \). It is, however, invariant under its pronilpotent radical, and therefore under \( u_{MT}^\phi \).

**Corollary 12.2.** If \( g \geq 1 \), then \( [u_{MT}^\phi, u_{MT}^\phi] + \hat{g}^{\xi} \subseteq \ker \delta_\xi \). Consequently, the homomorphism (32) induces a homomorphism
\[
H_1(k) \to Q\lambda(S)^\wedge / \ker \delta_\xi
\]
which is a morphism of MHS for all complex structures on \( (S, \vec{v}) \).

**Proof.** The cobracket is a morphism of MHS for each complex structure \( \phi \). This implies that it is \( u_{MT}^\phi \) invariant. Proposition 9.4 implies that there is an inclusion \( u_{MT}^\phi \to Q\lambda(S)^\wedge \) and that it is a morphism of MHS. The first assertion follows as Lemma 12.1 implies that \( [u_{MT}^\phi, u_{MT}^\phi] \subseteq \ker \delta_\xi \). It also implies that the inclusion of \( u_{MT}^\phi \) into \( Q\lambda(S)^\wedge \) induces a homomorphism
\[
H_1(u_{MT}^\phi) \to Q\lambda(S)^\wedge / \ker \delta_\xi
\]
There is a canonical canonical projection \( u_{MT}^\phi \to \mathfrak{k} \) for all complex structures \( \phi \). Corollary 12.2 implies that (34) factors through \( H_1(u_{MT}^\phi) \to H_1(\mathfrak{k}) \).

Since \( \mathfrak{k} = W_{-6} \), Theorem 11.8 implies that \( \text{edg}^S_\xi \) is just the divergence and so does not depend on the choice of a framing.

**Corollary 12.3.** The composition of the canonical homomorphism \( \varphi \) of Proposition 9.4 with the edge homomorphism induces a homomorphism
\[
\text{edg}^S_\xi : H_1(\mathfrak{k}) \to Q\lambda(S)^\wedge
\]
that is a morphism of MHS for all complex structures on \( S \). It is invariant under the mapping class group of \( S \) and does not depend on the complex structure on \( S \) or on the framing \( \xi \).
12.1. Decomposability. Our goal in this section is to prove that \( \text{edg}^S \) is injective for all hyperbolic surfaces \( S \). The first step is to prove that \( \text{edg}^S \) behaves well when \( S \) is decomposed.

**Lemma 12.4.** If \( S = S' \cup S'' \) is a decomposition of \( S \) into two closed subsurfaces, then

\[
\text{edg}^S = \text{edg}^{S'} + \text{edg}^{S''} \in \mathbb{Q}\lambda(S')^\wedge + \mathbb{Q}\lambda(S'')^\wedge \subset \mathbb{Q}\lambda(S)^\wedge.
\]

Furthermore, if \( S \) has genus 0, then the image of (35) is contained in the center of \( \mathbb{Q}\lambda(S)^\wedge \).

**Proof.** Since \( \text{edg}^S \) does not depend on the complex structure on \( S \), we can take \( S \) to be an Ihara curve. We can further assume that \( S = S' \cup S'' \) is a decomposition of \( S \) into two Ihara curves. The first statement is then a direct consequence of Corollary 10.3.

To prove the second, observe that since the cobracket is \( \mathfrak{t} \)-linear, we have

\[
\sigma \cdot \delta_\xi(u) = \delta_\xi(\{\sigma, u\}) = \sigma \cdot \delta_\xi(u) - u \cdot \delta_\xi(\sigma),
\]

for all \( u \in \mathbb{Q}\lambda(S)^\wedge \) and \( \sigma \in \mathfrak{t} \). This implies that \( u \cdot \delta_\xi(\sigma) = 0 \) for all \( u \) and \( \sigma \). Since \( \delta_\xi \) increases weights by 2, and since \( S \) has genus 0, we have

\[
(id \otimes \varepsilon)(u \cdot \delta_\xi(\sigma)) = u \cdot (id \otimes \varepsilon)(\delta_\xi(\sigma)).
\]

Combining these, we see that \( u \cdot \text{edg}_\xi(\sigma) = 0 \) for all \( u \in \mathbb{Q}\lambda(S)^\wedge \). \( \square \)

Recall that \( \mu_j, j \in \{0, \ldots, n\} \), is a small positive loop about the \( j \)th puncture. For each \( k > 0 \)

\[
(36) \sum_{j=0}^{n} (\log \mu_j)^k \in \mathbb{Q}\lambda(S)^\wedge.
\]

Note that this is a Hodge class of weight \(-2k\) for all complex structures on \( S \).

The weight graded version of the following result and key. It is proved in [13] and reproved in [2, Thm.5.4].

**Proposition 12.5.** These elements span the center of \( \mathbb{Q}\lambda(S)^\wedge \).

**Proof.** The statement for the completed version follows from Hodge theory as the exactness of \( \text{Gr}_{W}^\bullet \mathbb{Q}\lambda(S)^\wedge \) implies that

\[
Z \text{Gr}_{W}^\bullet \mathbb{Q}\lambda(S)^\wedge \cong \text{Gr}_{W}^\bullet \mathbb{Q}\lambda(S)^\wedge.
\]

where \( Z \) denotes center. \( \square \)

12.2. Injectivity of \( \text{edg}^S \). The abelianization of \( \mathfrak{t} \) is the semi-simple pro-Hodge structure

\[
H_1(\mathfrak{t}) = \prod_{m \geq 1} \mathbb{Q}[\sigma_{2m+1}] = \prod_{m \geq 1} \mathbb{Q}(2m + 1).
\]

The following theorem is essentially due to Alekseev, Kawazumi, Kuno and Naef [2]. It was explained to me by Florian Naef. A precursor is [3, Prop. 4.10] of Alekseev and Torossian.

**Theorem 12.6.** If \( S \) is a hyperbolic surface, then the image of the generator \([\sigma_{2m+1}]\) of \( H_1(\mathfrak{t}) \) (suitably normalized) under the homomorphism (32) satisfies

\[
\text{edg}^S(\sigma_{2m+1}) = \sum_{j=0}^{n} (\log \mu_j)^{2m+1} \in \mathbb{Q}\lambda(S)^\wedge.
\]
In particular, \( \text{edg}_S \) is injective for all hyperbolic surfaces \( S \), so that \((33)\) is also injective.

The proof is somewhat technical and is relegated to Appendix A.

12.3. Concluding remarks. For surfaces of type \((g, \vec{1})\) with \( g > 0 \), Enomoto and Satoh [21] constructed \( \text{Sp}(H) \)-invariant trace maps
\[
\text{Tr}_{\mathcal{E}_S} : \text{Gr}_{W}^{-m} \text{Der}_{\theta}^\theta p \to |H^{\otimes m}| \quad m \geq 1
\]
that generalize Morita’s trace maps and vanish on \( \text{Gr}_{\mathcal{W}}^{-m} \). In [2, §8.2], it is shown that \( \text{Tr}_{\mathcal{E}_S} \) is the weight graded of the edge map defined in Section 11. The trace map in genus 0 is the divergence \( \text{div}_0 \). The “reconstruction formula” mentioned in Remark 11.9 then implies the kernels of \( \text{Tr}_{\mathcal{E}_S} \) and \( \delta_\xi \) on \( \text{Der}_\theta \mathcal{L}(H) \) are identical.

In another paper [22] with Kuno, they consider the kernel of the reduced cobracket on \( \text{Der}_\theta \mathcal{L}(H) \). They show that
\[
\text{Gr}_{\mathcal{W}}^{-m} \subseteq \ker \text{Tr}_{\mathcal{E}_S} \subseteq \ker \text{Gr}_{\mathcal{W}}^{-m} \delta
\]
when \( 2 \leq m \leq 2g - 2 \). They also show [22 Thm. 2] that \( (\text{Gr}_{\mathcal{W}}^{-m} \delta) / \ker \text{Tr}_{\mathcal{E}_S} \) contains a non-trivial \( \text{Sp}(H) \)-module in weight \(-8\) and when \( m > 5 \) is congruent to 1 mod 5. Since \( \text{Gr}_{\mathcal{W}}^{-m} \) is a trivial \( \text{Sp}(H) \)-module, this implies that \( \ker \delta \) does not normalize \( \mathfrak{g} \).

In \((g, \vec{1})\) case, we have the following diagram in \( \mathbb{Q}(\lambda(S))^\wedge \):

\[
\begin{array}{c}
\ker \delta_\xi \\
n \cap \text{Der}_\theta^\theta p \\
n \cap \ker \delta_\xi \cap \text{Der}_\theta^\theta p \\
\mathfrak{g}_\xi
\end{array}
\]

This raises the following question, which is related to Drinfeld’s GRT, [20].

**Question 12.7.** Is \( \mathfrak{k} = (\ker \delta_\xi \cap \text{Der}_\theta^\theta p) / \mathfrak{g}_\xi \)? This would follow if one knew that \( \mathfrak{g}_\xi = n \cap \ker \delta_\xi \cap \text{Der}_\theta^\theta p \) and that \( \ker \delta_\xi \cap \text{Der}_\theta^\theta p \) normalizes \( \mathfrak{g}_\xi \).

13. Remarks on a Conjecture of Morita

Suppose that \((S, \vec{\nu})\) is a surface of type \((g, \vec{1})\) with an indexed pants decomposition. As explained in Section 10, such a decomposition corresponds to an Ihara curve and thus determines an action of the motivic Lie algebra \( \mathfrak{mtm} \) on \( p(S, \vec{\nu}) \). Morita [66] has proposed a location of the image of the generator \( \sigma_{2n+1} \) of \( \mathfrak{k} \) in \( \text{Gr}_{W}^{-4n-2} \text{Der}_\theta^\theta p(S, \vec{\nu}) \). Here we discuss Morita’s proposal. As in the previous section, we set \( p = p(S, \vec{\nu}), \bar{\mathfrak{g}} = \bar{\mathfrak{g}}_{g, \vec{1}} \) and \( \mathfrak{g} = \mathfrak{g}_{g, \vec{1}} \).

Since these generators \( \sigma_{2n+1} \) of \( \mathfrak{k} \) are not canonical (see Remark 9.1), the best we can hope for is that we can determine the image of each \( \sigma_{2n+1} \) mod \( [t, \bar{t}] \). Since the image of \( t \to \text{Der}_\theta^\theta p \) depends on the pants decomposition, the best we can hope for if we ignore the pants decomposition is to determine the image of \( \sigma_{2n+1} \) mod \( [t, \bar{t}] + \bar{\mathfrak{g}} \). This is because the isomorphism \( \mathfrak{g} \cong \mathfrak{mtm} \times \bar{\mathfrak{g}} \) depends non-trivially on 39So these inclusions hold stably.
the puncts decomposition and because the image of $\widetilde{\gamma}$ in Det$^\theta p$ does not depend on the complex structure by Theorem \[\ref{thm:independence} \] Finally, since the image of $\mathfrak{t}$ is contained in $W_{-6}$ Det$^\theta p$, and since $W_{-2}\mathfrak{f} = W_{-2}\mathfrak{g}$ by Corollary \[\ref{cor:abelian} \], the image of $\sigma_{2n+1}$ is well defined mod ker $\delta$. Morita’s starting point is the following observation of Nakamura (unpublished).

**Lemma 13.1.** For all $g \geq 2$ and $n \geq 0$, there is unique copy of $\text{Sym}^{2n+1} H$ in $Gr_{-2n-1}^W$ Det$^\theta p$. The restriction of Morita’s trace map to it is an isomorphism.

**Sketch of Proof.** Morita’s trace map implies that there is at least one copy of $\text{Sym}^{2n+1} H$ in $Gr_{-2n-1}^W$ Det$^\theta \mathbb{L}(H)$. We need to prove uniqueness. Recall that elements of Det$^\theta \mathbb{L}(H)$ of weight $-2n-1$ are represented by planar trivalent graphs with $2n+1$ vertices modulo the IHX relation. The unique copy of $\text{Sym}^{2n+1} H$ is the image of the map that takes $x_0 x_1 \ldots x_{2n} \in \text{Sym}^{2n+1} H$ to

$$
\sum_{j=1}^g \sum_{\sigma \in \Sigma_{2n+1}} a_j \overbrace{\ldots}^{x_{a(j)} x_{b(j)} x_{c(j)}} b_j \in Gr_{-2n-1}^W \text{Det}^\theta \mathbb{L}(H).
$$

The copy of $H$ in $Gr_{-1}^W$ p lies in the image of $Gr_{-1}^W \mathfrak{g}_{g,\mathfrak{f}}$ in Det$^\theta p$ and is thus “geometric”. Morita \[\ref{morita} \] has shown that $\text{Sym}^{2n+1} H$ is not geometric for all $n \geq 1$.

Fix an Sp$(H)$ equivariant inclusion $\mu_{2n+1} : \text{Sym}^{2n+1} H \to \text{Det}^\theta Gr^W_p$. Since $\text{Sym}^{2n+1} H$ is an irreducible Sp$(H)$-module, there is a unique copy of the trivial representation in $\Lambda^2 \text{Sym}^{2n+1} H$. Denote the image of a generator of this under the bracket

$$
\left[ \right] : \left[ \Lambda^2 \text{Sym}^{2n+1} H \right]^{\text{Sp}(H)} \to Gr_{-4n-2}^W \text{Det}^\theta p
$$

by $\mu_{2n+1}^2$. The following is a refinement of Morita’s proposal \[\ref{morita} \] for the image of the $\sigma_{2n+1}$.

**Question 13.2.** Is it true that for each $n \geq 1$, after rescaling $\mu_{2n+1}$ if necessary, $\sigma_{2n+1} \equiv \mu_{2n+1}^2 \mod \ker \delta$.

An affirmative answer to Questions \[\ref{q:geometric} \] and \[\ref{q:abelian} \] would imply that in $Gr_{-4n-2}^W$ Det$^\theta p$, we have

$$
\sigma_{2n+1} \equiv \mu_{2n+1}^2 \mod [\mathfrak{t}, \mathfrak{f}] + \mathfrak{g}.
$$

for each indexed pants decomposition of $S$. A negative answer to Question \[\ref{q:abelian} \] would imply that this is false.

**14. The Cohomology of $u_g$**

In the remaining sections, we explain the cohomological approach to computing the weight graded quotients of $u_{g,n+r}$, at least stably. The basic tool, explained in the appendix, is that $Gr^W_{-1} H^*(u_{g,n+r})$ determines $Gr_{-1}^W u_{g,n+r}$ by Möbius inversion. The converse is true in a range of weights $w \geq -m$ when $H^*(u_{g,n+r})$ satisfies a purity condition in degrees $\leq m$.

In this section, we recall a few facts about the cohomology of the $u_{g,n+r}$. The first is that the choice of a complex structure on a reference surface of type $(g, n+r)$ determines a natural isomorphism

$$
Gr_{-1}^W H^*(u_{g,n+r}) \cong H^*(u_{g,n+r})
$$

for each indexed pants decomposition of $S$. A negative answer to Question \[\ref{q:abelian} \] would imply that this is false.
and that the exactness properties of the weight filtration implies that there is a natural isomorphism
\[ \text{Gr}^W H^\bullet(u_{g,n+r}) \cong H^\bullet(\text{Gr}_W^\bullet u_{g,n+r}). \]
In addition, we shall need the following result, which is proved in [31, 35].

**Lemma 14.1.** For all \( g, W_j H^j(u_{g,n+r}) \) vanishes. The lowest weight subring
\[ \bigoplus_{j \geq 0} W_j H^j(u_{g,n+r}) \]
of \( H^\bullet(u_{g,n+r}) \) is quadratically presented for all \( g \geq 3 \). In particular, the lowest weight subring of \( u_{g,*} \), where \( * \in \{0, 1, \bar{1}\} \) have quadratic presentations
\[
\bigoplus_{j \geq 0} W_j H^j(u_g) = \Lambda^\bullet(\Lambda^3_0 H)/(V_{\mathbb{M}}),
\]
\[
\bigoplus_{j \geq 0} W_j H^j(u_{g,1}) = \Lambda^\bullet(\Lambda^3_0 H)/(V_{\mathbb{M}} + \Lambda^2 H_0 + \mathbb{Q})
\]
\[
\bigoplus_{j \geq 0} W_j H^j(u_{g,\bar{1}}) = \Lambda^\bullet(\Lambda^3_0 H)/(V_{\mathbb{M}} + \Lambda^2 H),
\]
where the generators have Hodge weight 1, and the generators of the ideal of relations have weight 2.

Here \( \Lambda^3_0 H \) denotes the \( \text{Sp}(H) \)-invariant complement of the trivial representation in \( \Lambda^2 H \). Note that in \( \Lambda^2 \Lambda^3_0 H, \Lambda^3_0 H \) has multiplicity 3 and the trivial representation \( \mathbb{Q} \) has multiplicity 2 when \( g \geq 6 \). Precise locations of the relations above can be deduced from the quadratic relations in \( u_{g,*} \), determined in [31, §11]. (See also [34, §9].)

Theorem 5.7 implies that \( t_g \) is a non-trivial central extension of \( u_g \). Since the Gysin sequence of this extension is an exact sequence of MHS, the lowest weight subring of \( H^\bullet(t_g) \) is the lowest weight subring of \( u_g \) mod the ideal generated by the class \( \kappa_1 \in H^2(u_g) \) of the central extension.

**Corollary 14.2.** For all \( g, W_{j-1} H^j(t_{g,n+r}) \) vanishes. The lowest weight subring
\[ \bigoplus_{j \geq 0} W_j H^j(t_{g,n+r}) \]
of \( H^\bullet(t_{g,n+r}) \) is quadratically presented for all \( g \geq 3 \). In particular, the lowest weight subring of \( t_g \) has quadratic presentation
\[ \Lambda^\bullet(\Lambda^3_0 H)/(V_{\mathbb{M}} + \mathbb{Q}\kappa_1) \]
where the generators \( \Lambda^3_0 H \) have Hodge weight 1, and the relations have weight 2. The class \( \kappa_1 \) spans the unique copy of the trivial representation in \( \Lambda^3 \Lambda^3_0 H \).

**Remark 14.3.** This result implies that \( W_j H^j(u_g) \) is non-zero for all \( j \leq \binom{g}{3} \) when \( g \geq 3 \). This is because the generating set \( V := \Lambda^3_0 H(-1) \) is a Hodge structure of weight \(-1\) and level \( 3 \). The set of relations \( V_{\mathbb{M}} \) is a Hodge structure of weight \(-2\) and level \( 4 \), as it is contained in \( H^\otimes 4(-1) \). This implies that \( \Lambda^j V \) is a Hodge structure of level \( 3j \) when
\[ 0 \leq j \leq \dim \Lambda^j V^{2,-1} = \binom{g}{3} \]
and that the ideal of relations has level \( \leq 3j - 2 \) in degrees \( j \geq 2 \). Similarly, the lowest weight subring of the Lie algebra \( t_g \) of the Torelli group is non-zero in all degrees \( \leq \binom{g}{3} \). Since the homological dimension of \( T_g \) is \( 3g - 5 \) when \( g \geq 2 \) (by [8]), the canonical homomorphism

\[
H^j(t_g) \to H^j(T_g; \mathbb{Q})
\]

is not an isomorphism when \( 3g - 5 < j < \binom{g}{3} \).

14.1. **Injectivity of the Johnson homomorphism.** Injectivity of the Johnson homomorphism \( g_0, \vec{1} \to \text{Der}_p(S, \vec{v}) \) can be expressed cohomologically. The following result is a Lie algebra analogue of Stallings' Theorem [77]. It follows directly from the self-contained discussion in [42, §18].

**Proposition 14.4.** Suppose that \( g \geq 3 \). The truncated Johnson homomorphism \( g_0, \vec{1}/W_{m-1} \to g_0, \vec{1}/W_{m-1} \) is an isomorphism if and only if

\[
W_mH^2(\bar{\mathfrak{u}}_{g, \vec{1}}) \to W_mH^2(\mathfrak{u}_{g, \vec{1}})
\]

is injective (and therefore an isomorphism).

15. **Representation Stability**

As previously remarked, the behaviour each weight graded quotient of each of the invariants we are considering, viewed as an \( \text{Sp}(H) \) module, becomes more regular as the genus increases. In particular, the highest weights that occur in such representations do not depend on the genus once it is large enough. A precise formulation of representation stability in the representation ring \( \mathcal{R} = \mathcal{R}(\text{Sp}(H)) \) of \( \text{Sp}(H) \) is given in [31, §6] 40.

Suppose that \((S, \mathcal{P}, \vec{V})\) is a surface of type \((g, n+\vec{r} + \vec{1})\). By successively attaching surfaces of type \((1, \vec{2})\), we obtain a sequence

\[
S_g \subset S_{g+1} \subset S_{g+2} \subset \cdots
\]

where \( S_k \) is of type \((k, n+\vec{r} + \vec{1})\). This induces a sequence of inclusions

\[
\Gamma S_0, \partial S_0 \hookrightarrow \Gamma S_1, \partial S_1 \hookrightarrow \Gamma S_2, \partial S_2 \hookrightarrow \cdots
\]

which induces sequences such as

\[
\mathfrak{g}_{S_0, \partial S_0} \to \mathfrak{g}_{S_1, \partial S_1} \to \mathfrak{g}_{S_2, \partial S_2} \to \cdots, \quad \bar{\mathfrak{u}}_{S_0, \partial S_0} \to \bar{\mathfrak{u}}_{S_1, \partial S_1} \to \bar{\mathfrak{u}}_{S_2, \partial S_2} \to \cdots
\]

and

\[
H^\bullet(\mathfrak{g}_{S_0, \partial S_0}) \to H^\bullet(\mathfrak{g}_{S_1, \partial S_1}) \to H^\bullet(\mathfrak{g}_{S_2, \partial S_2}) \to \cdots
\]

\[
H^\bullet(\bar{\mathfrak{u}}_{S_0, \partial S_0}) \to H^\bullet(\bar{\mathfrak{u}}_{S_1, \partial S_1}) \to H^\bullet(\bar{\mathfrak{u}}_{S_2, \partial S_2}) \to \cdots
\]

Using limit MHS, we can arrange for all of these stabilization maps to be morphisms of MHS [13]. Consequently, the stabilizations of each of these invariants admits an ind- (or pro-) MHS. This implies that the weight graded quotients of each of these invariants stabilizes as a Hodge structure. Each of these invariants also stabilizes in the representation ring of \( \text{Sp}(H) \). The following is an incomplete list of invariants

---

40 The representation ring \( \mathcal{R}(G) \) of an affine group \( G \) is defined to be the Grothendieck group of the category \( \text{Rep}(G) \) of its finite dimensional representations. When \( G \) is reductive, it is the free \( \mathbb{Z} \)-module generated by the isomorphism classes of irreducible \( G \)-modules.

41 Technical point: Since the vanishing cycles are homologically trivial, the relative weight filtration \( M_\bullet \) and the weight filtration \( W_\bullet \) coincide.
associated to completed mapping class groups that stabilize in the representation ring of $\text{Sp}(H)$.

**Proposition 15.1.** Suppose that $(S, P, \bar{V})$ is a surface of type $(g, n + r + 1)$ with $n, r \geq 0$. For $V$ being one of $p(S)$, $\text{Der}^0 p(S, \bar{V}), g_{S,\bar{S}}, \mathbb{P}_S, H^\bullet(u_{S,\bar{S}})$, $H^\bullet(\mathbb{P}_S)$, the weight graded quotient $\text{Gr}_V^m V$ stabilizes in the representation ring of $\text{Sp}(H)$ as $g \to \infty$ and $n$ and $r$ remain constant. The stable value of $\text{Gr}_V^m V$ occurs when

$$g \geq \begin{cases} |m| & V = p(S, \bar{V}), \\ |m| + 2 & V = g_{S,\bar{S}}, \text{Der}^0 p(S, \bar{V}), \\ |m|/3 & V = g_{S,\bar{S}}, H^j(u_{S,\bar{S}}), H^j(\mathbb{P}_S). \end{cases}$$

These stability ranges are not optimal. For example, for $m$ for which $\text{Gr}_V^m g_{S,\bar{S}} = \text{Gr}_V^m \mathbb{P}_S$ (e.g., $-6 \leq m \leq -1$ when $g \gg 0$), stabilization of $\text{Gr}_V^m g_{S,\bar{S}}$ occurs when $g \geq -m + 2$.

**Sketch of Proof.** We use the notation above and we apply the conventions (e.g., choice of torus, positive roots) of [31], where $a_1, \ldots, a_k, b_1, \ldots, b_k$ is a symplectic basis of $\text{Gr}_V^m H_1(S_k)$. Suppose that $V_k$ is an $\text{Sp}_{12}$-module $(k = h, h + 1)$ and that $V_h \to V_{h+1}$ that is equivariant with respect to the inclusion $\text{Sp}_h \to \text{Sp}_{h+1}$. Then $V_h \to V_{h+1}$ is an isomorphism in the stable representation ring $\text{R}(\text{Sp})$ if it induces a bijection on highest weight vectors. In particular, $V_h$ and $V_{h+1}$ have the same highest weight decompositions.

Denote the Schur functor associated to a partition $\mu$ by $S_{\mu}$. The key point is to observe that if $V_h \to V_{h+1}$ is an isomorphism in the stable representation ring $\text{R}(\text{Sp})$ and if $h$ is sufficiently large relative to $\mu$, then by [51], $S_{\mu} V_h \to S_{\mu} V_{h+1}$ is also an isomorphism in $\text{R}(\text{Sp})$.

Suppose that

$$V_g \to V_{g+1} \to V_{g+2} \to \cdots$$

is a sequence of finite dimensional graded $\text{Sp}$-modules that stabilizes in $\text{R}(\text{Sp})$. Assume that $\text{Gr}_V^m V_g = 0$ when $m \geq 0$. This induces a sequence

$$\mathbb{L}(V_g) \to \mathbb{L}(V_{g+1}) \to \mathbb{L}(V_{g+2}) \to \cdots$$

of graded Lie algebras in $\text{R}(\text{Sp})$. Representation stability implies that each weight graded quotient of the graded tensor algebra $T(V_g)$ stabilizes in $\text{R}(\text{Sp})$. The PBW theorem and the stability of Schur functors then implies that each weight graded quotient of $\mathbb{L}(V_k)$ stabilizes in $\text{R}(\text{Sp})$.

We first apply this when $V_k = \text{Gr}_V^m H_1(S_k)$. Since $\text{Gr}_V^m p(S_k, \bar{V})$ is canonically a quotient of $\text{Gr}_V^m \mathbb{L}(\text{Gr}_V^m H_1(S_k))$, $\text{Gr}_V^m p(S_k, \bar{V}) \to \text{Gr}_V^m p(S_{k+1}, \bar{V})$ is surjective on highest weight vectors when $k$ is sufficiently large. Denote the kernel of the canonical quotient map

$$\mathbb{L}(\text{Gr}_V^m H_1(S_k)) \to p(S_k, \bar{V})$$

by $r_k$. When $k \geq 2$ this ideal is generated by elements of weight $-2$ by [5] and [31, §12]. The surjectivity of the bracket map $\text{Gr}_V^m r_k \otimes \text{Gr}_V^m H_1(S_k) \to \text{Gr}_V^m r_k$ implies that $\text{Gr}_V^m r_k \to \text{Gr}_V^m r_{k+1}$ is surjective on highest weight vectors when $k$$^{42}$Kabanov [51] proves that if $V_\nu$ is the irreducible representation of $\text{Sp}_h$ corresponding to the partition $\nu$, then $S_{\mu} V_\nu$ stabilizes in $\text{R}(\text{Sp})$ when $h \geq |\mu||\nu|$.
is sufficiently large. Since (as Sp\_g+k-modules) v_k \oplus Gr\_W^r p(S_k, v) = L(Gr\_W^r H_1(S_k)), we conclude that p(S_k, v) stabilizes in R(Sp).

Similarly taking V_k = H_1(uS_k, \partial S_k) and using the fact that each Gr\_W^* uS_k, \partial S_k is quadratically presented when k \geq 4, we see that each weight graded quotient of uS_k, \partial S_k stabilizes in R(Sp).

The results on cohomology of uS, \partial S follow similarly. Stability of Gr\_W^* H^* (uS_k, \partial S_k) follows as each weight graded quotient of the complex C^\_W (Gr\_W^* uS_k, \partial S_k) (vector spaces and maps) stabilizes in R(Sp) by the stability of Schur functors [51]. □

The stability results also hold in the case where there are no boundary components — that is, when r = 0.

**Corollary 15.2.** For all n \geq 0, the weight graded quotients of g\_g,n and H\_n (g\_g,n) stabilize in the representation ring of Sp(H).

**Proof.** The weight graded quotients of g\_g,n stabilize in the representation ring of Sp(H) when n > 0 as the Gysin sequence

\[
0 \to \mathbb{Q}(1) \to g\_g,n-1+I \to g\_g,n \to 0.
\]

is an exact sequence of MHS. Exactness of Gr\_W then implies that the weight graded quotients of g\_g,n-1+I and g\_g,n are isomorphic except in weight -2, where they differ by a copy of the trivial representation. Similarly, representation stability of the weight graded quotients is easily deduced from the exactness of the sequence

\[
0 \to p_g \to g\_g,1 \to g\_g \to 0
\]

where p_g is the Lie algebra of the unipotent completion of the fundamental group of a smooth compact surface of genus g. Its weight graded quotients stabilize in R(Sp(H)). □

For example the first 3 graded quotients are

\[
Gr\_W^m u_g \cong \begin{cases} 
\Lambda^3_0 H & m = 1, g \geq 3, \\
V_2 & m = 2, g \geq 3, \\
V_{[312]} & m = 3, g \geq 3.
\end{cases}
\]

(See [51] Prop. 9.6.) The next 3 stable values of Gr\_W^m u_g can be found in [67]. This is the current state of the art. However, if the stable cohomology of u_g satisfies a purity condition, one can compute Gr\_W^m u_g as will be explained in the next section.

**16. Stable Cohomology**

The cohomology of a (negatively weighted) pronilpotent Lie algebra u in the category of MHS determines the associated weight graded Lie algebra Gr\_W^* u. In particular, the stable cohomology of u\_g,n+\_r determines the stable value of Gr\_W^* u\_g,n+\_r, and the stable cohomology of u\_g,n+\_r determines the stable value of Gr\_W^* u\_g,n+\_r.

Here we discuss evidence that the stable cohomology of u\_g,n+\_r equals the stable cohomology groups of \Gamma\_g,n+\_r with symplectic coefficients. If this is the case, it provides another avenue for determining the image of the Johnson homomorphism.

---

43 The computations in [67] compute Gr\_W^m \pi, but imply that W_6 H^2(\pi) = W_6 H^2(u). So \pi/W_{-7} \cong u/W_{-7} by Proposition [14.3].
16.1. **Comparison with** $H^\bullet(\Gamma_{g,n+r}; V_\mu)$. The cohomology of an affine (i.e., pro-algebraic) group over a field $k$ with coefficients in a $G$-module $V$ is defined by

$$H^j(G; V) := \text{Ext}^j(k, V),$$

where the Ext is taken in the category $\text{Rep}(G)$ of $G$-modules \[15\]. A homomorphism $\Gamma \to G(k)$ from a discrete group into the $k$-rational points of $G$ induces a homomorphism $H^\bullet(G; V) \to H^\bullet(\Gamma; V)$ for each $G$-module $V$. In particular, for each $\text{Sp}(H)$-module $V$, we have the homomorphism

$$H^\bullet(\mathcal{G}_{g,n+r}; V) \to H^\bullet(\Gamma_{g,n+r}; V)$$

induced by $\Gamma_{g,n+r} \to \mathcal{G}_{g,n+r}(\mathbb{Q})$. General properties of relative completion imply that this map is an isomorphism in degrees $\leq 1$ and is injective in degree $2$ \[44\]. Standard facts about the cohomology of algebraic groups \[45\] imply that

$$H^j(\mathcal{G}_{g,n+r}; V) \cong \left[ H^j(u_{g,n+r}) \otimes V \right]^{\text{Sp}(H)}.$$

These homomorphisms are compatible with Hodge theory in a sense to be explained below.

The map $\left(37\right)$ is an isomorphism in genus 0 (in all degrees) as each $\mathcal{M}_{0,n}$ is a hyperplane complement of fiber type, and therefore a rational $K(\pi, 1)$. One can check that it is an isomorphism in genus 1 in all degrees as well. A proof can be deduced from results in Sections 9 and 10 of \[42\]. However, it fails to be an isomorphism for all $g \geq 3$. This follows from the discussion in Remark 14.3: since $\Gamma_g$ has virtual cohomological dimension $4g - 5$, $\left(37\right)$ cannot be an isomorphism in degrees $4g - 4 \leq j \leq \left(\frac{g}{2}\right)$ for at least one coefficient module $V$ in each degree.

16.2. **Towards the stable cohomology of** $\mathcal{G}_{g,n+r}$. Isomorphism classes of irreducible $\text{Sp}_g = \text{Sp}(H)$ modules are in bijective correspondence with partitions

$$\mu : \mu_1 + \mu_2 + \cdots + \mu_g, \quad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_g \geq 0$$

of positive integers into $\leq g$ pieces. Set $|\mu| := \mu_1 + \cdots + \mu_r$. For each partition $\mu$ fix an irreducible $\text{Sp}(H)$-module $V_\mu$ in the corresponding isomorphism class. Each Hodge structure on $H$ determines a Hodge structure on $V_\mu$ of weight $-|\mu|$.

**Conjecture** 16.1 (cf. \[10\] 39). The homomorphism $\left(37\right)$ with $V = V_\mu$ is stably an isomorphism for all $\mu$.

The local system $V_\mu$ over $\mathcal{M}_{g,n+r}$ corresponding to $V_\mu$ has a unique structure of a polarized variation of Hodge structure of weight $-|\mu|$ over $\mathcal{M}_{g,n+r}$. The natural isomorphism

$$H^j(\Gamma_{g,n+r}; V_\mu) \cong H^j(\mathcal{M}_{g,n+r}; V_\mu)$$

induces a MHS on $H^j(\Gamma_{g,n+r}; V_\mu)$ with weights $\geq j - |\mu|$. Each choice of a complex structure $\phi$ on $(\overline{\mathcal{S}}, P, \mathcal{V})$, the MHS on $u_{g,n+r}$ determines MHSs on $H^\bullet(u_{g,n+r})$ and $V_\mu$ via linear algebra. With these MHSs, the natural map

$$H^j(\mathcal{G}_{g,n+r}; V_\mu) \cong \left[ H^j(u_{g,n+r}) \otimes V_\mu \right]^{\text{Sp}(H)} \to H^j(\Gamma_{g,n+r}; V_\mu)$$

is a morphism of MHS. For a proof, see \[31\] 37.

If $\left(37\right)$ is stably an isomorphism it will force $H^j(u_{g,n+r})$ to be pure of weight $j$. This leads us to:

---

\[44\] One source is \[31\] 37, but simpler proofs exist.

\[45\] This is very well known. A proof can be found in \[42\].
Conjecture 16.2 (Purity). The weight $m$ stable cohomology of $u_{g,n+r}$ is of degree $m$. That is,
\[ \text{Gr}_m^W H^*(u_{g,n+r}) = \text{Gr}_m^W H^m(u_{g,n+r}) \text{ for all } g \geq m/3. \]

The relevance of this conjecture is that purity of $H^*(u_{g,n+r})$ in degrees $\leq m$ allows the computation of $\text{Gr}_j^W u_{g,n+r}$ in the representation ring of $\text{Sp}(H)$ for $j \leq m$ by the formula in the Appendix. Purity is known when $m = 1$ (Johnson [50], [51]) and $m = 2$ (Garoufalidis, Petersen [52], Hain [31]), and $2 < m \leq 6$ (Morita–Sakasai–Suzuki [67]). The general case would follow if one can show that the enveloping algebra of $u_{g,n+r}$ is stably Koszul dual to its cohomology. (Cf. [40, Q. 9.14].)

Further evidence for these conjectures is provided by the following result, which implies that the lowest weight cohomology of $\Gamma_{g,*}$ is isomorphic to the stable cohomology of $\Gamma_{g,*}$ for $* \in \{0, 1, \tilde{1}\}$.

Theorem 16.3 (Garoufalidis–Getzler, Petersen, Kupers–Randal-Williams). For all partitions $\mu$ and $* \in \{0, 1, \tilde{1}\}$, the homomorphism
\[
\nu_* : \bigoplus_{m \geq 0} W_{m-|\mu|} H^m(G_{g,*}, V_\mu) \cong \left[ \bigoplus_{m \geq 0} \Lambda^* W_m H^m(u_{g,*}) \otimes V_\mu \right]^{\text{Sp}(H)} \to H^*(\Gamma_{g,*}; V_\mu)
\]
is an isomorphism when $g \gg 0$.

The history of this theorem is tangled and still evolving. Garoufalidis and Getzler [50] claim it is true when $* = 0, 1$. However, as pointed out to me by Dan Petersen, their proof is incomplete. Petersen [73] has given a corrected proof in this case. The $* = \tilde{1}$ case can be deduced from results of Kupers and Randal-Williams in [59]. The goal of the following proposition is to clarify the relationship between the 3 cases of the theorem. In particular, prove that the $* = 0$ and $* = 1$ cases are equivalent.

Proposition 16.4. The following hold:

(i) $\nu_0$ is a stable isomorphism if and only if $\nu_1$ is a stable isomorphism.
(ii) If $\nu_1$ a stable isomorphism, then $\nu_\tilde{1}$ is a stable isomorphism.

Sketch of Proof. We will focus on the first assertion as there are several proofs of the $* = 1, \tilde{1}$ cases. First note that if $S$ is a compact surface of genus $g$, then the natural map $H^*(\text{Sp}(S)) \to H^*(S; \mathbb{Q})$ is an isomorphism. Since $H^2(G_{g,1}) \to H^2(\Gamma_{g,1}; \mathbb{Q})$ is an isomorphism for all $g \geq 3$, there is a class
\[
\psi \in H^2(G_{g,1}) \cong H^2(u_{g,1})^{\text{Sp}(H)}
\]
that corresponds to the first Chern class of the relative tangent bundle of $M_{g,1}$ to $M_g$. Its restriction to $H^2(p)$ is non-zero. Choose a closed, $\text{Sp}(H)$-invariant representative of it in $\Lambda^2 \text{Gr}_1^W u_{g,1}$. We’ll also denote it by $\psi$. The homomorphism $\Gamma_{g,1} \to G_{g,1}(\mathbb{Q})$ induces a map of extensions
\[
\begin{array}{cccccc}
1 & \longrightarrow & \pi_1(S, x_0) & \longrightarrow & \Gamma_{g,1} & \longrightarrow & \Gamma_g & \longrightarrow & 1 \\
1 & \longrightarrow & \mathcal{P} & \longrightarrow & G_{g,1} & \longrightarrow & G_g & \longrightarrow & 1
\end{array}
\]

[46] This is long well-known. A proof can be found in [37, §5.1].
and thus of the corresponding spectral sequences that compute cohomology with coefficients in \( V_{\mu} \). Since \( \mathcal{M}_{g,1} \to \mathcal{M}_g \) has smooth projective fibers, the spectral sequence of the top extension collapses at \( E_2 \) by Deligne’s theorem [14]. Since \( H^\bullet(p) \to H^\bullet(S; \mathbb{Q}) \) is an isomorphism, Deligne’s argument also implies that the spectral sequence of the lower extension collapses at \( E_2 \). This is because cupping with \( \psi \in E_r^{\bullet, \bullet} \) commutes with \( d_r \) and induces an isomorphism \( E_r^{\bullet, 0} \to E_r^{\bullet, 2} \). It is now an exercise to prove the first claim.

The classes \( \psi \) are also the first Chern classes of the central extensions

\[ 0 \to \mathbb{Z} \to \Gamma_{g,1} \to \Gamma_{g,1} \to 1 \quad \text{and} \quad 0 \to \mathcal{G}_a \to \mathcal{G}_{g,1} \to \mathcal{G}_{g,1} \to 1. \]

Their Gysin sequences are long exact sequences of MHS. The exactness properties

the weight filtration implies that the rows of the diagram

\[
\begin{array}{cccccc}
W_{m-2}H^{j-2}(\mathcal{G}_{g,1}, V_{\mu}) & \xrightarrow{\psi} & W_m H^j(\mathcal{G}_{g,1}, V_{\mu}) & \xrightarrow{\nu_1^{j-2}} & W_m H^j(\mathcal{G}_{g,1}, V_{\mu}) & \xrightarrow{\nu_1^j} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
W_{m-2}H^{j-2}(\Gamma_{g,1}, V_{\mu}) & \xrightarrow{\psi} & W_m H^j(\Gamma_{g,1}, V_{\mu}) & \xrightarrow{\nu_1^{j-2}} & W_m H^j(\Gamma_{g,1}, V_{\mu}) & \xrightarrow{\nu_1^j} & 0
\end{array}
\]

are exact, where \( m = j - |\mu| \). So if \( \nu_1 \) is stably an isomorphism, then so is \( \nu_1^j \).

Since the sequence

\[ 0 \to \pi_1^m(S, x_0) \to \mathcal{G}_{g,n+1} \to \mathcal{G}_{g,n} \to 1 \]

is exact, where \( S \) is a surface of type \((g, n)\), and since \( H^\bullet(\pi_1^m(S, x_0)) \to H^\bullet(S; \mathbb{Q}) \) is an isomorphism, Theorem 10.3 implies, by induction on \( n \), that

**Corollary 16.5.** Conjecture 16.2 is equivalent to Conjecture 16.1.

**Appendix A. Proof of Theorem 12.6**

Lemma 12.3 implies that we need only prove the result when \( S \) is \( \mathbb{P}^1 - \{0, 1, \infty\} \). Denote \( \partial / \partial z \in T_1 \mathbb{P}^1 \) by \( \bar{v}_1 \). For \( a \in \{0, 1, \infty\} \), let \( \bar{v}_a \in T_a \mathbb{P}^1 \) be an image of \( \bar{v}_1 \) under the action of the symmetric group action on \( \mathbb{P}^1 - \{0, 1, \infty\} \). (These are well defined up to a sign.) Set \( p_a = p(\mathbb{P}^1 - \{0, 1, \infty\}, \bar{v}_a) \).

By Corollary 12.3 edg\(^x\) does not depend on the framing, so we will ignore it.

**A.1. The symmetric depth filtration of** \( \mathbb{Q}\lambda(\mathbb{P}^1 - \{0, 1, \infty\})^\wedge \). The inclusion \( \mathbb{P}^1 - \{0, 1, \infty\} \hookrightarrow G_m \) induces a Lie algebra homomorphism

\[ \phi_1 : p_1 \to p(G_m, 1) \]

in MHS. The depth filtration \( D_\phi^\bullet \) of \( p_1 \) is defined by \( D_0^\phi p_1 = p_1 \) and \( D_k^\phi p_1 = L_k \ker \phi_1 \) when \( k \geq 1 \), the \( k \)th term of the lower central series of \( \ker \phi_1 \). The depth filtration of \( p_1 \) induces one on its enveloping algebra \( \mathbb{Q}\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \bar{v}_1)^\wedge \) and one on \( \mathbb{Q}\lambda(\mathbb{P}^1 - \{0, 1, \infty\})^\wedge \) via the quotient map. Denote all of these depth filtrations by \( D_\phi^\bullet \).

The depth filtration \( D_3^\bullet \) of \( \mathbb{Q}\lambda(\mathbb{P}^1 - \{0, 1, \infty\})^\wedge \) depends on the choice of the “boundary component” \( 1 \in \{0, 1, \infty\} \) where the tangent vector \( \bar{v}_1 \) is anchored. Similarly, we have the depth filtrations \( D_0^\bullet \) and \( D_\infty^\bullet \) of \( \mathbb{Q}\lambda(\mathbb{P}^1 - \{0, 1, \infty\})^\wedge \). These are filtrations in MHS and are permuted by the \( \Sigma_3 \) action on \( \mathbb{P}^1 - \{0, 1, \infty\} \).
Define the symmetric depth filtration $\mathcal{D}^\bullet$ of $\mathbb{Q}\lambda(\mathbb{P}^1 - \{0, 1, \infty\})^\wedge$ to be the intersection of these 3 filtrations:

$$
\mathcal{D}^k \mathbb{Q}\lambda(\mathbb{P}^1 - \{0, 1, \infty\})^\wedge := (D^k_0 \cap D^k_1 \cap D^k_\infty) \mathbb{Q}\lambda(\mathbb{P}^1 - \{0, 1, \infty\})^\wedge.
$$

It is a filtration by MHSs and is $\Sigma_3$-invariant. It is therefore determined by the induced filtration on $Gr^W_\bullet \mathbb{Q}\lambda(S)^\wedge$. We now describe this explicitly.

For each $a \in \{0, 1, \infty\}$, there is a canonical isomorphism

$$
Gr^W_\bullet p_a \cong \mathbb{L}(e_0, e_1, e_\infty)/(e_0 + e_1 + e_\infty).
$$

The associated weight graded of $p_1 \to p(G_m, 1)$ is the homomorphism

$$
\mathbb{L}(e_0, e_1, e_\infty)/(e_0 + e_1 + e_\infty) \to \mathbb{L}(e_0, e_\infty)/(e_0 + e_\infty) \cong \mathbb{L}(e_0)
$$

that sends $e_1$ to 0. The filtration $D^k_1$ of $Gr^W_1 p_1$ is the filtration

$$
D^k_1 Gr^W_1 p_1 = \{\text{Lie words in $e_0, e_1$ of degree $\geq k$ in $e_0$}\}
\quad = \{\text{Lie words in $e_1, e_\infty$ of degree $\geq k$ in $e_\infty$}\}.
$$

Observe that

$$
Gr^1_{D^1_1} p_1 = \bigoplus_{n=0}^\infty \mathbb{Q} \text{ad}^n_{e_1} e_0 = \bigoplus_{n=0}^\infty \mathbb{Q} \text{ad}^n_{e_1} e_\infty.
$$

Formulas for $D^k_1$ and $D^k_\infty$ are obtained by permuting the indices 0, 1, $\infty$. As a consequence

$$
\mathcal{D}^k Gr^W_{-2a} \mathbb{Q}\lambda(\mathbb{P}^1 - \{0, 1, \infty\})^\wedge
\quad = \{f \in | \text{Sym}^d H_1(\mathbb{P}^1 - \{0, 1, \infty\})| : \deg_{e_a} f \geq k, \ a = 0, 1, \infty\},
$$

where we regard $e_0, e_1, e_\infty$ as elements of $H_1(\mathbb{P}^1 - \{0, 1, \infty\})$, which is canonically isomorphic to $Gr^W_{-2} p_a$ for $a \in \{0, 1, \infty\}$.

A.2. The depth filtrations of $\text{SDer} p_1$ and $\mathfrak{t}$. In order to compute $\text{edg}^S(\sigma_{2m+1})$ mod $\mathcal{D}^2 \mathbb{Q}\lambda(S)^\wedge$ we need to relate $\mathcal{D}^\bullet$ to the standard depth filtration $D^\bullet$ of $\text{SDer} p_a$, which is defined by

$$
D^k \text{SDer} p_a := \{\sigma \in \text{SDer} p_a : \sigma(D^j_a p_a) \subseteq D^{j+k}_a p_a \text{ all } j \geq 0\}.
$$

It satisfies $[D^j, D^k] \subseteq D^{j+k}$. The depth filtration $D^\bullet$ of $\mathfrak{t}$ is defined to be its restriction to $\mathfrak{t}$ under the inclusion $\mathfrak{t} \hookrightarrow \text{SDer} p_a$. It does not depend on $a$ and is a central filtration, so that $D^k \mathfrak{t} \supseteq L^k \mathfrak{t}$.\(^{47}\)

Since the depth filtration of $\text{SDer} p_a$ is a filtration by MHS, it is determined by the depth filtration on $\text{SDer} Gr^W_\bullet p_a$. Observe that $D_a \in D^k \text{SDer} Gr^W_\bullet p_1$ if and only if $u_0$ and $u_\infty$ are both in $D^k Gr^W_1 p_1$. Combined with Proposition 11.3 this implies that

$$
D^k \text{SDer} p_a = \text{SDer} p_a \cap \mathcal{D}^k \mathbb{Q}\lambda(\mathbb{P}^1 - \{0, 1, \infty\})^\wedge
$$

which implies that $D^k \mathfrak{t} = \mathfrak{t} \cap \mathcal{D}^k \mathbb{Q}\lambda(\mathbb{P}^1 - \{0, 1, \infty\})^\wedge$.

---

\(^{47}\)These are not, in general, equal. An important open problem is to understand the generators and relations of the Lie algebra $Gr^W_1 \mathfrak{t}$. The relations in depth two are known to correspond to classical cusp forms of level 1. See \[7\] for precise statements and conjectures.
A.3. The computation of $\text{edg}^S(\sigma_{2m+1}) \mod \mathcal{D}^2$. The key point for us is that $\mathfrak{t} = D^1 \mathfrak{t}$ and $[\mathfrak{t}, \mathfrak{t}] = D^2 \mathfrak{t}$. The second statement is well known and follows from the action of $\mathfrak{t}$ on the “polylog quotient” $p_1/D^2p_1$ of $p_1$. It implies that $\text{edg}^S$ vanishes on $D^2 \mathfrak{t}$.

The action of $\sigma_{2m+1}$ on the polylog quotient $p_1/D^2_1 \mathfrak{t}$ implies that the homomorphism $H_1(\mathfrak{t}) \to \text{Gr}^1_D \text{SDer} \text{Gr}_W^1 p_1$ takes $[\sigma_{2m+1}]$ (suitably scaled) to the derivation $e_0 \mapsto \text{ad}_{e_0}^{2m+1} e_1 = [e_0, \text{ad}_{e_0}^{2m} e_1]$ and $e_\infty \mapsto \text{ad}_{e_\infty}^{2m+1} e_1 = [e_\infty, \text{ad}_{e_\infty}^{2m} e_1]$. This implies that if $\sigma_{2m+1} \mapsto D_u$, where $u = (u_0, u_\infty)$, then, mod $D^2 \mathfrak{t}$,

$$u_0 \equiv \text{ad}_{e_0}^{m} e_1 = -\text{ad}_{e_0}^{m} e_\infty \quad \text{and} \quad u_\infty \equiv \text{ad}_{e_\infty}^{m} e_1 = -\text{ad}_{e_\infty}^{m} e_0.$$ Thus, mod $D^2 \mathbb{Q} \langle e_0, e_\infty \rangle$, we have

$$u_0^{(2)} \equiv - \sum_{k=1}^{2m} (-1)^k \binom{2m}{k} e_\infty^k e_0^{2m-k} \quad \text{and} \quad u_\infty^{(2)} \equiv - \sum_{k=1}^{2m} (-1)^k \binom{2m}{k} e_\infty^k e_0^{2m-k}.$$ Since $e_0 + e_1 + e_\infty = 0$,

$$\text{edg}^S D_u = \text{div}_1 D_u \equiv |e_0^{2m} e_\infty| + |e_\infty^{2m} e_0| \equiv - \frac{1}{2m+1} (|e_0^{2m+1}| + |e_\infty^{2m+1}| + |e_\infty^{2m+1}|)$$

mod $\mathcal{D}^2 \text{Gr}_W^1 \mathbb{Q}\lambda(S)^\wedge$. Since $\text{edg}^S(\sigma_{2m+1})$ is central by Lemma 12.4, Proposition 12.5 implies that, after rescaling,

$$\text{edg}^S(\sigma_{2m+1}) = (\log \mu_0)^{2m+1} + (\log \mu_1)^{2m+1} + (\log \mu_\infty)^{2m+1}.$$ 

**Appendix B. Möbius inversion**

Suppose that $k$ is a field of characteristic zero, $G$ is a reductive $k$-group, such as $\text{Sp}(H)$ and that

$$\mathfrak{h} = \bigoplus_{n \geq 1} \mathfrak{h}_n$$
is a graded Lie algebra in the category of $G$-modules. The Chevalley–Eilenberg chain complex

$$C_\bullet(\mathfrak{h}) := \Lambda^\bullet \mathfrak{h}$$
has a natural grading induced by the grading of $\mathfrak{h}$ that is preserved by the differential. This implies that $H_\bullet(\mathfrak{h})$ is graded and that

$$\chi(\text{Gr}_n H_\bullet(\mathfrak{h})) = \chi(\text{Gr}_n \Lambda^\bullet)$$
in the representation ring $R(G)$ of $G$, where $\chi$ denotes Euler characteristic. For example, it implies that

$$\text{Gr}_3 H_\bullet(\mathfrak{h}) = \chi(\Lambda^3 \mathfrak{h}_1 \to \mathfrak{h}_1 \otimes \mathfrak{h}_2 \to \mathfrak{h}_3),$$
where the first map in the complex is the “Jacobi identity”

$$x \wedge y \wedge z \to x \otimes [y, z] + y \otimes [z, x] + z \otimes [x, y] \in \mathfrak{h}_1 \otimes \mathfrak{h}_2$$
and the second map is the bracket. So, if one knows $\text{Gr}_3 H_\bullet(\mathfrak{h})$ and $\mathfrak{h}_1$ and $\mathfrak{h}_2$, then one can compute $\mathfrak{h}_3$ in the representation ring of $G$. This is the method used in [31] [8] to compute the graded quotients of the lower central series of $p(S, x)$.

---

48 Use the formula $\text{ad}_x^k y = \sum_{j=0}^k (-1)^j \binom{k}{j} x^{k-j} y x^j$. 
One can invert this formula to obtain a formula for $h_\bullet$ in terms of $\Gr_\bullet H_\bullet(h)$, a fact I learned from Looijenga. To this end, set
\[ \Phi(x) = \sum_{n \geq 0} \chi(\Gr_n H_\bullet(h)) x^n \in \mathbb{R}(G)[[x]]. \]

**Proposition B.1.** Define $\Psi_n(h) \in \mathbb{R}(G)$ to be the coefficients of the power series
\[ \sum_{n \geq 0} \Psi_n(h) x^n = -\frac{x \Phi'(x)}{\Phi(x)} \in \mathbb{R}(G)[[x]]. \]
Then in $\mathbb{R}(G)$, we have
\[ h_n = \frac{1}{n} \sum_{d|n} \mu(d) \psi^d \Psi_{n/d}(h) \]
where $\psi^d$ denotes the $d$th Adam’s operation and where $\mu$ is the Möbius function.

When every $G$-module is isomorphic to its dual (as it is when $G = \text{Sp}(H)$), homology can be replaced by cohomology in this formula.

If the cohomology of $\mathfrak{h}$ is pure in degrees $\leq m$ in the sense that
\[ H_j(h) = \Gr_j H_j(h), \quad \text{for all } j \leq m, \]
then
\[ \Phi(x) \equiv \sum_{n \geq 0} (-1)^n H_n(h) x^n \in \mathbb{R}(G)[[x]] \text{ mod } (x^{m+1}). \]
This simplifies $\Psi(x) \text{ mod } (x^{m+1})$ and implies that $H^j(h)$ is determined by $H_k(h)$ ($k \leq j$) for all $j \leq m$.

**REFERENCES**

[1] A. Alekseev, N. Kawazumi, Y. Kuno, F. Naef: *The Goldman-Turaev Lie bialgebra in genus zero and the Kashiwara-Vergne problem*, Adv. Math. 326 (2018), 1–53. [arXiv:1703.05813]

[2] A. Alekseev, N. Kawazumi, Y. Kuno, F. Naef: *The Goldman-Turaev Lie bialgebra and the Kashiwara-Vergne problem in higher genera*, [arXiv:1804.09566]

[3] A. Alekseev, C. Torossian: *The Kashiwara–Vergne conjecture and Drinfeld’s associators*, Ann. of Math. (2) 175 (2012), 415–463.

[4] S. Andreadakis: *On the automorphisms of free groups and free nilpotent groups*, Proc. London Math. Soc. 15 (1965), 239–268.

[5] R. Bezrukavnikov: *Koszul DG-algebras arising from configuration spaces*, Geom. Funct. Anal. 4 (1994), 119–135.

[6] F. Brown: *Mixed Tate motives over $\mathbb{Z}$*, Ann. of Math. (2) 175 (2012), 949–976.

[7] F. Brown: *Depth-graded motivic multiple zeta values*, [arXiv:1301.3053]

[8] M. Bestvina, K.-U. Bux, D. Margalit: *The dimension of the Torelli group*, J. Amer. Math. Soc. 23 (2010), 61–105.

[9] M. Chas: *Combinatorial Lie bialgebras of curves on surfaces*, Topology 43 (2004), 543–568.

[10] T. Church, M. Ershov, A. Putman: *On finite generation of the Johnson filtrations*, [arXiv:1711.04779]

[11] J. Conant: *The Johnson cokernel and the Enomoto-Satoh invariant*, Algebr. Geom. Topol. 15 (2015), 801–821.

[12] J. Conant, K. Vogtmann: *On a theorem of Kontsevich*, Algebr. Geom. Topol. 3 (2003), 1167–1224.

[13] W. Crawley-Boevey, P. Etingof, V. Ginzburg: *Noncommutative geometry and quiver algebras*, Adv. Math. 209 (2007), 274–336.

[14] P. Deligne: *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. No. 35 (1968), 259–278.

[15] P. Deligne: *Théorie de Hodge, II*, Inst. Hautes Études Sci. Publ. Math. No. 40 (1971), 5–57.
56 RICHARD HAIN

[16] P. Deligne: *Théorie de Hodge, III*, Inst. Hautes Études Sci. Publ. Math. No. 44 (1974), 5–77.

[17] P. Deligne: *Catégories tannakiennes*, The Grothendieck Festschrift, Vol. II, 111–195, Progr. Math., 87, Birkhäuser Boston, 1990.

[18] P. Deligne, A. Goncharov: *Groupes fondamentaux motiviques de Tate mixte*, Ann. Sci. École Norm. Sup. (4) 38 (2005), 1–56.

[19] A. Dimca, S. Papadima: *Arithmetic group symmetry and finiteness properties of Torelli groups*, Ann. of Math. (2) 177 (2013), 395–423.

[20] V. Drinfeld: *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with* Gal(\(\overline{\mathbb{Q}}/\mathbb{Q}\))*, (Russian) Algebra i Analiz 2 (1990), 149–181; translation in Leningrad Math. J. 2 (1991), 829–860.

[21] N. Enomoto, T. Satoh: *New series in the Johnson cokernels of the mapping class groups of surfaces*, Algebr. Geom. Topol. 14 (2014), 627–669.

[22] N. Enomoto, Y. Kuno, T. Satoh: *A comparison of classes in the Johnson cokernels of the mapping class groups of surfaces*, [arXiv:1805.02563]

[23] M. Ershov, S. He: *On finiteness properties of the Johnson filtrations* Duke Math. J. 167 (2018), 1713–1759.

[24] B. Farb, D. Margalit: *A primer on mapping class groups*, Princeton Mathematical Series 49, Princeton University Press, 2012.

[25] S. Garoufalidis, E. Getzler: *Graph complexes and the symplectic character of the Torelli group*, [arXiv:1712.03606]

[26] S. Garoufalidis, J. Levine: *Tree-level invariants of three-manifolds, Massey products and the Johnson homomorphism*, Graphs and patterns in mathematics and theoretical physics, 173–203, Proc. Sympos. Pure Math., 73, Amer. Math. Soc., 2005.

[27] W. Goldman: *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, Invent. Math. 85 (1986), 263–302.

[28] R. Hain: *The de Rham homotopy theory of complex algebraic varieties, I*, K-Theory 1 (1987), 271–324.

[29] R. Hain: *Completions of mapping class groups and the cycle C\(\rightarrow \mathbb{C}\)−*, in *Mapping Class Groups and Moduli Spaces of Riemann Surfaces*, C.-F. Bödigheimer and R. Hain, editors, Contemp. Math. 150 (1993), 75–105.

[30] R. Hain: *Torelli groups and geometry of moduli spaces of curves*, Current topics in complex algebraic geometry (Berkeley, CA, 1992/93), 97–143, Math. Sci. Res. Inst. Publ., 28, Cambridge Univ. Press, 1995.

[31] R. Hain: *Infinitesimal presentations of the Torelli groups*, J. Amer. Math. Soc. 10 (1997), 597–651.

[32] R. Hain: *Deligne-Beilinson cohomology of affine groups*, Hodge theory and L\(^2\)-analysis, 377–418, Adv. Lect. Math. (ALM), 39, 2017.

[33] R. Hain: *Relative weight filtrations on completions of mapping class groups*, in “Groups of diffeomorphisms”, 309–368, Adv. Stud. Pure Math., 52, Math. Soc. Japan, Tokyo, 2008. [arXiv:0802.0814]

[34] R. Hain: *Rational points of universal curves*, J. Amer. Math. Soc. 24 (2011), 709–769.

[35] R. Hain: *Genus 3 mapping class groups are not Kähler*, J. Topol. 8 (2015), 213–246.

[36] R. Hain: *The Hodge Rham theory of modular groups*, Recent advances in Hodge theory, 422–514, London Math. Soc. Lecture Note Ser., 427, Cambridge Univ. Press, 2016.

[37] R. Hain: *Hodge theory of the Goldman bracket*, [arXiv:1710.06053]

[38] R. Hain: *Hodge Theory of the Turaev Cobracket and the Kashiwara–Vergne Problem*, [arxiv:1807.09209]

[39] R. Hain: *Unipotent Path Torsors of Ihara Curves*, in preparation.

[40] R. Hain, E. Looijenga: *Mapping class groups and moduli spaces of curves*, Algebraic geometrySanta Cruz 1995, 97–142, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., 1997

[41] R. Hain, M. Matsumoto: *Galois actions on fundamental groups of curves and the cycle C− C−*, J. Inst. Math. Jussieu 4 (2005), 363–403.

[42] R. Hain, M. Matsumoto: *Universal mixed elliptic motives*, J. Inst. Math. Jussieu (2018), to appear. [arXiv:1512.03975]

[43] R. Hain, D. Reed: *Geometric proofs of some results of Morita*, J. Algebraic Geom. 10 (2001), 199–217.
[44] Y. Ihara, H. Nakamura: On deformation of maximally degenerate stable marked curves and Oda’s problem, J. Reine Angew. Math. 487 (1997), 125–151.
[45] J. C. Jantzen, Representations of Algebraic Groups, Pure and Applied Mathematics Vol. 131, Academic Press, 1987.
[46] D. Johnson: An abelian quotient of the mapping class group $I_g$, Math. Ann. 249 (1980), 225–242.
[47] D. Johnson: A survey of the Torelli group, Low-dimensional topology (San Francisco, Calif., 1981), 165–179, Contemp. Math., 20, Amer. Math. Soc., 1983.
[48] D. Johnson: The structure of the Torelli group, I: A finite set of generators for $I$, Ann. of Math. (2) 118 (1983), 423–442.
[49] D. Johnson: The structure of the Torelli group. II, A characterization of the group generated by twists on bounding curves, Topology 24 (1985), 113–26.
[50] D. Johnson: The structure of the Torelli group. III: The abelianization of $I$, Topology 24 (1985), 127–144.
[51] A. Kabanov: Stability of Schur functors, J. Algebra 195 (1997), 233–240.
[52] A. Kabanov: The second cohomology with symplectic coefficients of the moduli space of smooth projective curves, Compositio Math. 110 (1998), 163–186.
[53] N. Kawazumi: The mapping class group orbits in the framings of compact surfaces, Q. J. Math. 69 (2018), 1287–1302, [arXiv:1703.02258]
[54] N. Kawazumi, Y. Kuno: The logarithms of Dehn twists, Quantum Topol. 5 (2014), 347–423. [arXiv:1008.5017]
[55] N. Kawazumi, Y. Kuno: Groupoid-theoretical methods in the mapping class groups of surfaces, [arXiv:1109.6479]
[56] N. Kawazumi, Y. Kuno: Intersections of curves on surfaces and their applications to mapping class groups, Ann. Inst. Fourier (Grenoble) 65 (2015), 2711–2762. [arXiv:1112.3841]
[57] N. Kawazumi, S. Morita: The primary approximation to the cohomology of the moduli space of curves and cocycles for the stable characteristic classes, Math. Res. Lett. 3 (1996), 629–641.
[58] M. Kontsevich: Formal (non)commutative symplectic geometry, The Gelfand Mathematical Seminars, 1990-1992, 173–187, Birkhäuser Boston, 1993.
[59] A. Kupers, O. Randal-Williams: On the cohomology of Torelli groups, [arXiv:1901.01862]
[60] E. Looijenga: Stable cohomology of the mapping class group with symplectic coefficients and of the universal Abel–Jacobi map, J. Algebraic Geom. 5 (1996), 135–150.
[61] I. Madsen, M. Weiss The stable moduli space of Riemann surfaces: Mumford’s conjecture, Ann. of Math. (2) 165 (2007), 843–941.
[62] G. Mess: The Torelli groups for genus 2 and 3 surfaces, Topology 31 (1992), 775–790.
[63] S. Morita: Abelian quotients of subgroups of the mapping class group of surfaces, Duke Math. J. 70 (1993), 699–726.
[64] S. Morita: A linear representation of the mapping class group of orientable surfaces and characteristic classes of surface bundles, in the Proceedings of the 37th Taniguchi Symposium on Topology and Teichmüller Spaces, July 1995, S. Kojima et al editors, World Scientific (1996), 159–186.
[65] S. Morita: Structure of the mapping class groups of surfaces: a survey and a prospect, Proceedings of the Kirbyfest (Berkeley, CA, 1998), 349–406, Geom. Topol. Monogr., 2, 1999.
[66] S. Morita: Characteristic classes of moduli spaces—Riemann surface, graph, homology cobordism, (translation of a paper in Sugaku 69-2 (2017), 113–136, Mathematical Society of Japan). [to be updated]
[67] S. Morita, T. Sakasai, M. Suzuki: Structure of symplectic invariant Lie subalgebras of symplectic derivation Lie algebras, Adv. Math. 282 (2015), 291–334.
[68] J. Morgan: The algebraic topology of smooth algebraic varieties, Inst. Hautes Études Sci. Publ. Math. No. 48 (1978), 137–204; Correction: Inst. Hautes Études Sci. Publ. Math. No. 64 (1986), 185.
[69] H. Nakamura, L. Schneps: On a subgroup of the Grothendieck–Teichmüller group acting on the tower of profinite Teichmüller modular groups, Invent. Math. 141 (2000), 503–560.
T. Oda: The universal monodromy representations on the pro-nilpotent fundamental groups of algebraic curves, Mathematische Arbeitstagung (Neue Serie) June 1993, Max-Planck-Institute preprint MPI/93-57.

C. Papakyriakopoulos: Planar regular coverings of orientable closed surfaces, Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox), 261–292. Ann. of Math. Studies, No. 84, Princeton Univ. Press, 1975.

D. Petersen: Cohomology of local systems on the moduli of principally polarized abelian surfaces, Pacific J. Math. 275 (2015), 39–61. [arXiv:1310.2508]

D. Petersen: Personal communication, January, 2018 and August, 2019.

D. Petersen, M. Tavakol, Q. Yin: Tautological classes with twisted coefficients, [arXiv:1705.08875]

A. Pollack: Relations between derivations arising from modular forms, undergraduate thesis, Duke University, 2009.

T. Schedler: A Hopf algebra quantizing a necklace Lie algebra canonically associated to a quiver, Int. Math. Res. Not. 2005, 725–760.

J. Stallings: Homology and central series of groups, J. Algebra 2 (1965), 170–181.

D. Sullivan: On the intersection ring of compact three manifolds, Topology 14 (1975), 275–277.

N. Takao: Braid monodromies on proper curves and pro-$\ell$ Galois representations, J. Inst. Math. Jussieu 11 (2012), 161–181.

H. Tsunogai: On some derivations of Lie algebras related to Galois representations, Publ. Res. Inst. Math. Sci. 31 (1995), 113–134.

V. Turaev: Intersections of loops in two-dimensional manifolds, Mat. Sb. 106 (148) (1978), 566–588.

V. Turaev: Skein quantization of Poisson algebras of loops on surfaces, Ann. Sci. École Norm. Sup. (4) 24 (1991), 635–704.

T. Watanabe: On the completion of the mapping class group of genus two, J. Algebra 501 (2018), 303–327. [arXiv:1609.05552]

Department of Mathematics, Duke University, Durham, NC 27708-0320
E-mail address: hain@math.duke.edu

Available at https://www.mpim-bonn.mpg.de/preblob/4902
Available at: http://dukespace.lib.duke.edu/dspace/handle/10161/1281