A class of random fields on complete graphs with tractable partition function

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Abstract—The aim of this short note is to draw attention to a method by which the partition function and marginal probabilities for a certain class of random fields on complete graphs can be computed in polynomial time. This class includes Ising models with homogeneous pairwise potentials but arbitrary (inhomogeneous) unary potentials. Similarly, the partition function and marginal probabilities can be computed in polynomial time for random fields on complete bipartite graphs, provided they have homogeneous pairwise potentials. We expect that these tractable classes of large scale random fields can be very useful for the evaluation of approximation algorithms by providing exact error estimates.

Index Terms—Markov random fields

1 INTRODUCTION

The computation of unary and pairwise marginals for probabilistic graphical models is necessary both for inference and learning if these models are used in pattern recognition and computer vision. It is well known that this problem is #P-hard for Markov/Gibbs Random Fields if there are no restrictions for the underlying graph structure of the model [1]. This explains the strong focus on approximation algorithms for the calculation of marginal probabilities.

Ising like models on complete but small graphs are often considered in error estimation experiments, when proposing new or improved methods for approximate calculation of marginal probabilities [2], [3], [4], [5], [6]. We show in this note that the partition function and marginal probabilities for a certain class of random fields on complete graphs can be computed in polynomial time. This class includes Ising models with homogeneous pairwise potentials but arbitrary (inhomogeneous) unary potentials. Similarly, the partition function and marginal probabilities can be computed in polynomial time for random fields on complete bipartite graphs, provided they have homogeneous pairwise potentials.

The main idea is to partition the set of all labellings so that the following holds for each subset: (i) the contribution of the pairwise factors is equal for all labellings in a subset, (ii) the sum of contributions of the unary factors can be computed by dynamic programming over the graph size. This results, e.g., in an algorithm with $O(n^K)$ time complexity for computing the partition sum for $K$-valued random fields on a complete graph with $n$ vertices, provided that the pairwise factors of the GRF are homogeneous.

2 THE MODEL CLASS; COMPUTING THE PARTITION FUNCTION

Let us consider the following class of binary valued random fields on undirected complete graphs

$$p(x) = \frac{1}{Z} \prod_{(ij) \in E} g(x_i, x_j) \prod_{i \in V} q_i(x_i),$$

where $(V, E)$ denote the sets of vertices and edges of a complete graph and $x: V \rightarrow \{0, 1\}$ is a binary valued labelling of the vertices. Notice that we assume that $g: \{0, 1\}^2 \rightarrow \mathbb{R}$ is shared by all pairwise factors. Given the model parameters $g$ and $q$, the task is to compute the partition sum

$$Z = \sum_{x \in \mathcal{X}} \prod_{(ij) \in E} g(x_i, x_j) \prod_{i \in V} q_i(x_i)$$

as well as unary and pairwise marginal probabilities $p(x_i)$ and $p(x_i, x_j)$. We assume without loss of generality that $g$ has the form

$$g(k, k') = \begin{cases} \alpha & \text{if } k \neq k', \\ 1 & \text{otherwise}, \end{cases}$$

and the unary factors have the form

$$q_i(k) = \begin{cases} \beta_i & \text{if } k = 0, \\ 1 & \text{otherwise}. \end{cases}$$

This can be achieved by applying an appropriate re-parametrisation without changing the probability [1].

In order to calculate the partition sum $Z$ for a graph with $n$ vertices, we partition the set $\mathcal{X} = \{0, 1\}^n$ of all labellings into the sets $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_n$, where

$$\mathcal{X}_m = \left\{ x \in \mathcal{X} \mid \sum_{i \in V} x_i = m \right\}$$

denotes the set of all labellings with $m$ vertices labelled by “1”. Accordingly, we denote the partial sums by

$$Z_m = \sum_{x \in \mathcal{X}_m} \prod_{(ij) \in E} g(x_i, x_j) \prod_{i \in V} q_i(x_i).$$
Due to the homogeneity assumption, the pairwise factors in contribute to each summand of $Z_m$ by the same factor $\alpha^{m(n-m)}$. Hence, we can write

$$Z = \sum_{m=0}^{n} Z_m = \sum_{m=0}^{n} \alpha^{m(n-m)} H_V(m),$$

where

$$H_V(m) = \sum_{x \in X_m} \prod_{i \in V} q_i(x_i) = \sum_{x \in X_m} \prod_{i \in V} \beta^{1-x_i}$$

denotes the sum of all unary contributions to the partial sum $Z_m$. These quantities can be computed recursively over the size of the graph. Let us denote by $H_U(m)$, $m = 0, 1, \ldots, |U|$ the corresponding quantities for the complete sub-graph induced by the vertex set $U \subset V$. If $i \in U$ is a vertex of this graph, then

$$H_U(m) = \beta_i H_{U \setminus \{i\}}(m) + H_{U \setminus \{i\}}(m-1).$$

This equation follows from the simple observation that any labelling of $U$ with $m$ vertices labelled by “1” either has the vertex $i$ labelled by “1” and $m-1$ of the remaining vertices labelled by “1” or has the vertex $i$ labelled by “0” and consequently $m$ of the remaining vertices labelled by “1”. Hence, the quantities $H_V(m)$ can be computed by dynamic programming over the size of the graph, what eventually leads to an algorithm for computing $Z$ with $O(n^2)$ time complexity.

It is similarly easy to compute marginal probabilities because the mapping defined by is invertible. In order to compute e.g. the unary marginal probabilities for the vertex $i \in V$

$$p(x_i = 1) \propto \sum_{m=1}^{n} H_{V \setminus \{i\}}(m-1) \alpha^{m(n-m)}$$

$$p(x_i = 0) \propto \beta_i \sum_{m=0}^{n-1} H_{V \setminus \{i\}}(m) \alpha^{m(n-m)},$$

we need the quantities $H_{V \setminus \{i\}}(m)$, which can be computed from those for the whole graph with $O(n)$ time complexity by

$$H_{V \setminus \{i\}}(m) = \frac{1}{\beta_i} [H_V(m) - H_{V \setminus \{i\}}(m-1)].$$

This results in an $O(n^2)$ algorithm for calculating the unary marginals for all vertices of the graph.

The proposed approach can be generalised to $K$-valued random fields on complete graphs with homogeneous pairwise interactions. The probability and partition function are given by as before, but $x$ denotes now a realisation of a field of $K$-valued random variables. Notice that the pairwise factors $g(\cdot, \cdot)$ are assumed to be symmetric up to a re-parametrisation. We partition the set of all labellings $\mathcal{X}$ into sets $\mathcal{X}_m$, where $m = (m_1, \ldots, m_K)$ is a vector, the components of which denote the number of variables taking the respective label value, i.e.,

$$\mathcal{X}_m = \{ x \in \mathcal{X} \mid \sum_{i \in V} \delta_{x_i,i} = m_1, \ldots, \sum_{i \in V} \delta_{x_i,K} = m_K \},$$

where $\delta_{ij}$ denotes the Kronecker delta. The corresponding contributions $H_V(m)$ of all unary terms to the partial sums can be again computed recursively by

$$H_U(m) = \sum_{k=1}^{K} q_k(m) H_{U \setminus \{i\}}(m - e_k),$$

where $e_k$ denotes the standard basis vector for the component $k$. Since there are $O(nK^{-1})$ quantities $H_V(m)$ and each of them must be recomputed $n$ times, the overall time complexity for computing them is $O(nK^2)$. Finally, the partition sum is obtained using the pairwise factors $g(\cdot, \cdot)$

$$Z = \sum_{m} H_V(m) \prod_{k \leq k'} g(k, k')^{n(m_k, m_{k'})},$$

where

$$n(m_k, m_{k'}) = \begin{cases} m_k m_{k'} & \text{if } k \neq k', \\ m_k (m_k - 1)/2 & \text{otherwise} \end{cases}$$

is the number of edges connecting vertices labelled by $k$ and $k'$ for labellings in $\mathcal{X}_m$.

In order to compute all unary marginal probabilities

$$p(x_i = k) \propto q_k(m) \times \sum_{m} H_{V \setminus \{i\}}(m) \prod_{k' \leq k''} g(k', k'')^{n(m_k, m_{k'})},$$

it is necessary to compute the quantities $H_{V \setminus \{i\}}(m)$ for all $i \in V$ and all $m$. This can be done efficiently because the mapping $H_{V \setminus \{i\}}(\cdot) \mapsto H_V(\cdot)$ is invertible as before in the case of binary valued random fields. Hence, each of the $O(nK)$ quantities $H_{V \setminus \{i\}}(m)$ can be computed in constant time from the previously computed quantities $H_V(m)$.

Altogether this leads to the following theorem.

**Theorem 1.** Suppose a $K$-valued random field defined on a complete graph with $n$ vertices has homogeneous pairwise factors and arbitrary (inhomogeneous) unary factors. Then its partition sum as well as all its unary marginals can be computed with $O(n^2)$ time complexity.

**Remark 1.** Note that this theorem is not in contradiction with the dichotomy found by Bulatov and Grohe because here we restrict the graph structure.

### 3 Models on Complete Bipartite Graphs

The applicability of the proposed approach can be extended even further, e.g., for the calculation of the partition function and marginal probabilities for random fields defined on complete bipartite graphs. Here, again, it is required that the pairwise interactions are homogeneous. We will consider binary valued random fields for the sake of simplicity.
Let \( G(A \cup B, E) \) be a complete bipartite graph such that \( n_1 = |A|, n_2 = |B| \). Consider the class of binary valued random fields \( x : A \cup B \rightarrow \{0, 1\} \)

\[
p(x) = \frac{1}{Z} \prod_{i \in A} g(x_i, x_j) \prod_{i \in A \cup B} q_i(x_i).
\]

As discussed in the previous section, we may assume that the factors \( g, q_i \) have the form \( \textcircled{3} \) and \( \textcircled{4} \) respectively.

In order to compute \( Z \), we partition the set of all labellings \( X \) into the sets

\[
X_m = \{ x \in X \mid \sum_{i \in A} x_i = m_1, \sum_{j \in B} x_j = m_2 \},
\]

where \( m = (m_1, m_2) \) counts the number of vertices labelled by \( "1" \) in the first and second part, respectively. It is clear that the pairwise factors contribute to all summands of \( Z_m \) by the same factor \( \alpha^\kappa \), where \( \kappa = m_1(n_2 - m_2) + m_2(n_1 - m_1) \). Hence, as before, the problem reduces to the computation of the contributions

\[
H_{AB}(m) = \sum_{x \in X_m} \prod_{i \in A \cup B} \beta_i^{1-x_i},
\]

of the unary factors. They can be computed recursively over the graph size. Similar to \( \textcircled{7} \) we have here

\[
H_{UV}(m) = \beta_i \cdot H_{U \setminus V}(m) + H_{U \setminus V}(m - e_1),
\]

\[
H_{UV}(m) = \beta_j \cdot H_{U \setminus V}(m) + H_{U \setminus V}(m - e_2).
\]

This results in an \( \mathcal{O}(n_1^K n_2^K) \) algorithm for computing the partition sum \( Z \). Both mappings \( \textcircled{21} \), \( \textcircled{22} \) are invertible, which yields an algorithm for calculating all unary marginal probabilities with the same time complexity. Generalising this to \( K \)-valued labellings as in the previous section yields the following result.

**Theorem 2.** Suppose a \( K \)-valued random field defined on a complete bipartite graph with \( n_1 + n_2 \) vertices has homogeneous pairwise factors and arbitrary (inhomogeneous) unary factors. Then its partition sum as well as all its unary marginals can be computed with \( \mathcal{O}(n_1^K n_2^K) \) time complexity.

**4 Conclusion**

We have shown that the partition sum and marginal probabilities can be efficiently computed for random fields on complete graphs if they have homogeneous pairwise factors. Similarly, the partition sum and marginal probabilities can be efficiently computed for random fields on complete bipartite graphs, provided they have homogeneous pairwise factors. We do not expect these two model classes to be directly relevant for computer vision applications. We expect, however, that they can be very useful to evaluate approximation algorithms for computing marginal probabilities. To the best of our knowledge, they are the only known classes of random fields on graphs with large tree-width and with arbitrary unary factors \( \textcircled{7} \) for which the marginal probabilities can be computed in polynomial time, thus providing exact error estimates for approximation algorithms.

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1. The method proposed in [7] requires outer-planar graphs, i.e. tree-width two, if applied for the case of arbitrary unary factors.