Dynamic effects of electromagnetic wave on a damped two-level atom: Exact solution via path integral

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Abstract. The spin coherent state path integral describing the dynamics of a two-level system interacting with electromagnetic wave of circular polarization is considered. The propagator is first written in the standard form by replacing the spin by a unit vector aligned along the polar and azimuthal directions. Then it is determined exactly using perturbation methods. Thus, the exact energy spectra with corresponding wave functions are deduced.

1. Introduction
The applications of path-integral formalism have widely increased since a large class of potentials had been resolved [1]. However it is known that the most relativistic interactions are those where the spin, which is a very useful and very important notion in physics, is taken into account. In the framework of non-relativistic theory the phenomena of spin is automatically introduced by the Pauli equation which contains the Schrödinger Hamiltonian and a spin-field interaction. This motivates the research into the solvable Pauli equations which is inevitably useful in applied physics. For instance a well-known example of its direct application is the time-dependent field acting on an atom with two levels whose time-evolution is controlled by the Pauli-type equation. The solution for this equation has made clear the associated transition amplitudes [2]. This and similar [3, 4] types of interaction aside, there are little analytical and exact computations which treat the time-dependent spin-field interaction. Furthermore, if one replaces the time dependence of the exterior field by a space-time dependence or by only space dependence this becomes even more restrictive [5, 6, 7, 8, 9, 10].

Moreover, the problem becomes nearly unsolvable if we try to build these solutions by the path integral formalism because the spin is a discrete quantity. The difficulty here is associated to the fact that the path integral lacks some classical ideas such as trajectories and up to now one does not know how to deal with this kind of technique in this important case. Thus some effort has been made to find a partial solution using the Schwinger’s model of spin and some explicit computations are then carried out [11, 12, 13, 14, 15, 16, 17].

A different model for spin is the use of the spin coherent state path integral which turns out to be helpful in visualizing the quantum dynamics in terms of classical ideas, and to our knowledge few explicit and semiclassical calculations are carried out [18, 19, 20]. If on the other hand the time-dependence of the exterior field is replaced by space-time dependence or
space dependence, to our knowledge, the exact solution in these cases using spin coherent state path integral are very rare [21, 22, 23, 24]. The aim of this paper is to give an attempt for the case below where we present an explicit spin coherent state path integral. For this reason we are devoted to this type of interaction by considering a problem which has recently been treated according to usual quantum mechanics [25]. It acts on an atom which has two levels and which interacts with electromagnetic wave of circular polarization. The same problem has been considered with damping in ref. [26].

The purpose of this article is to deal with the same problem using the formalism of spin coherent path integral. The two-level atom with a mass \( m \) and an angular frequency \( \omega \) has a dipole moment \( D \). The electromagnetic wave has wave vector \( k \) and angular frequency \( \omega_L \), propagating along the \( z \)-axis, and is described by the electric field

\[
E = (A \cos(\omega_L t - kz), -A \sin(\omega_L t - kz), 0),
\]

where \( A \) is the amplitude of \( E \). The dynamics of the atom in interaction with the electromagnetic wave is described by the following Hamiltonian:

\[
H = \frac{p^2}{2m} + \frac{\hbar}{2} \omega \sigma_z - \frac{i\hbar}{2} \left( \begin{array}{cc} \gamma_1 & 0 \\ 0 & \gamma_2 \end{array} \right) + V,
\]

where

- the first term represents the kinetic energy associated with the center–of–mass momentum along the \( z \)-direction,
- the second and the third terms describe the internal movement of the atom with, \( \gamma_1 = \frac{1}{\tau_1} \) and \( \gamma_2 = \frac{1}{\tau_2} \), being the lifetimes of the two atomic levels,
- \( V \) is the interaction energy between the atom and electromagnetic wave represented in the dipole approximation by

\[
V = -DE = \left( \begin{array}{cc} 0 & -\frac{1}{2} \hbar \Omega e^{-i(\omega_L t - kz)} \\ -\frac{1}{2} \hbar \Omega e^{i(\omega_L t - kz)} & 0 \end{array} \right)
\]

\[
\sigma_z = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \sigma_+ = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \quad \sigma_- = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right), \quad I = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)
\]

The Hamiltonian related to our problem has the following form

\[
H = H_0 + H_{int} = \frac{p^2}{2m} + \frac{\hbar}{2} (\omega - \frac{i\hbar}{2}) \sigma_z - \frac{i\hbar}{2} \hbar \sigma_+ + \frac{1}{2} \hbar \Omega e^{-i(\omega_L t - kz)} \sigma_- - \frac{1}{2} \hbar \Omega e^{i(\omega_L t - kz)} \sigma_-, \tag{4}
\]

where

\[
\sigma_z = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \sigma_+ = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \quad \sigma_- = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right), \quad I = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)
\]

are the usual Pauli matrices.

Considering this problem by the path integral approach, our motivation is the following. We show that for interaction with the coupling of spin-field type, the propagator is first written in the standard form \( \sum_{\text{path}} \exp \left( iS(\text{path})/\hbar \right) \), where \( S \) is the action that describes the system, and the discrete variable relative to spin being inserted as the (continuous) path using spin coherent states. With this approach, the formulation that uses the concept of trajectory is more suitable for a discussion of the semiclassical case which is based on the determination of classical
paths. Then we show that for this interaction, the propagator is exactly calculable and the wave functions and the energy spectrum can be extracted.

We note that this problem has been recently studied using fermionic coherent state path integral [27, 28], where the authors use the Schulman-Langhlin procedure to estimate and replace the $(\Delta z_n)^2$ terms present in the action by $i\varepsilon\hbar/m$, which is translated into an effective potential. In contrary, in this article, we avoid this procedure and the effective potential naturally arises in the calculation, which is also another motivation for using the spin coherent state path integral formalism. The difference between the two approaches stems simply from the difference between the two distinct coherent states.

Our paper is organized as follows. In the next section we give some notation and the spin coherent state path integral for spin $\frac{1}{2}$ system for our further computations. In section 3, after setting up a path integral formalism for the propagator, we perform the direct calculations. To integrate over the variable of the exterior motion we introduce a particular rotation in spin-coherent state space which eliminates the phase of the electromagnetic field, then we linearise the kinetic energy term using the phase space. Consequently the effective potential $i\varepsilon\hbar/m$ naturally arises, and then we integrate over $z$. Accordingly, the integration over the spin variables is easy to carry out and the result is given as a perturbation series. These are summed up exactly and the explicit result of the propagator is directly computed and the wave function is then deduced. Finally, in section 4, we present our conclusions.

2. Path-integral Formulation

There are several ways to represent the spin in the path integral formalism [29, 30, 31]. We use the simplest way [32, 19] which consists of:

- replacing $\sigma$ by a unit vector $n$ directed along $(\theta, \varphi)$,
- associating a coherent state $|\Omega\rangle$

$$|\Omega\rangle = |\theta, \varphi\rangle = e^{-i\varphi S_z}e^{-i\theta S_y}|\uparrow\rangle,$$

(6)

obtained from two rotations of the angles $\theta$ and $\phi$ around $z$ and $y$ axes over the state $|\uparrow\rangle$, and whose scalar product and projector are respectively:

$$\langle\Omega|\Omega'\rangle = \cos \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i\frac{\varphi - \varphi'}{2}} + \sin \frac{\theta}{2} \sin \frac{\theta'}{2} e^{-i\frac{\varphi - \varphi'}{2}},$$

(7)

$$\frac{1}{2\pi} \int d\cos(\theta)d\varphi \langle\Omega|\Omega\rangle = 1.$$

(8)

We label by $z$ the real variable that describes the atom position, with the corresponding projector

$$\int |z\rangle\langle z| dz = 1,$$

(9)

and $(\theta, \varphi)$ the polar variables generating the dynamics of the spin.

The transition amplitude from the initial state $|z_i, \theta_i, \varphi_i\rangle$ at $t_i = 0$ to the final state $|z_f, \theta_f, \varphi_f\rangle$ at $t_f = T$ is given by the matrix elements of the time evolution operator

$$K(z_f, \Omega_f, z_i, \Omega_i; T) = \langle z_f, \theta_f, \varphi_f | T_D \exp\left(-\frac{i}{\hbar} \int_0^T H dt\right) | z_i, \theta_i, \varphi_i\rangle,$$

(10)

where $T_D$ is Dyson’s time ordering symbol.
Discretizing the time $\varepsilon = T/(N + l)$, using the Trotter formula and inserting $N$-times the resolution of unity (8) and (9) between each pair of the evolution operator at time $\varepsilon$, we obtain the discretized form of the transition amplitude

$$K(z_f, \Omega_f, z_i, \Omega_i; T) = \lim_{N \to \infty} \frac{(m/2\pi i\varepsilon)^N}{N} \int \prod_{n=1}^{N} dz_n \prod_{n=1}^{N+1} dz_n \times \exp \frac{im}{2\hbar \varepsilon} (z_n - z_{n-1})^2 K^z(\Omega_f, \Omega_i; T),$$

with

$$K^z(\Omega_f, \Omega_i; T) = \lim_{N \to \infty} \int \prod_{n=1}^{N} d\cos(\theta_n) d\varphi_n \times \prod_{n=1}^{N+1} [\Omega_n | \Omega_n - 1] - i\varepsilon (\Omega_n | H_{\text{int}} | \Omega_n - 1)],$$

where

$$z_{N+1} = z_f, \quad \Omega_{N+1} = \Omega_f \quad \text{and} \quad z_0 = z_i, \quad \Omega_0 = \Omega_i.$$  \hfill (13)

It is easy to find that the following matrix elements:

$$|\Omega \rangle \sigma_z |\Omega' \rangle = \cos \frac{\theta}{2} 
\cos \frac{\theta'}{2} e^{\frac{i}{2}(\varphi' - \varphi)} - \sin \frac{\theta}{2} \sin \frac{\theta'}{2} e^{-\frac{i}{2}(\varphi' - \varphi)},$$

$$|\Omega \rangle \sigma_+ |\Omega' \rangle = \cos \frac{\theta}{2} \sin \frac{\theta'}{2} e^{\frac{i}{2}(\varphi' + \varphi)},$$

$$|\Omega \rangle \sigma_- |\Omega' \rangle = \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{-\frac{i}{2}(\varphi + \varphi)},$$

and the propagator related to our problem (11) takes the form of Feynman path integral

$$K = \int D\text{path} \exp(i\text{Action}),$$

which means in our case:

$$K(z_f, \Omega_f, z_i, \Omega_i; T) = \lim_{N \to \infty} \frac{(m/2\pi i\varepsilon)^N}{N} \int \prod_{n=1}^{N} dz_n \prod_{n=1}^{N+1} \exp \frac{im}{2\hbar \varepsilon} (z_n - z_{n-1})^2 \times \prod_{n=1}^{N} \cos(\theta_n) d\varphi_n \exp \left\{ \sum_{n=1}^{N+1} [\log < \Omega_n | \Omega_n - 1] - i\varepsilon (\Omega_n | H_{\text{int}} | \Omega_n - 1)] \right\}. \hfill (18)$$

Having obtained the conventional form it remains to integrate it in order to extract the interesting physical properties. We thus proceed to the calculation of $K(z_f, \Omega_f, z_i, \Omega_i; T)$.

3. The calculation of the propagator

We note that (18) can be rewritten in the form

$$K(z_f, \Omega_f, z_i, \Omega_i; T) = \lim_{N \to \infty} \frac{(m/2\pi i\varepsilon)^N}{N} \int \prod_{n=1}^{N} dz_n \prod_{n=1}^{N+1} \exp \left\{ \sum_{n=1}^{N+1} [\log < \Omega_n | \Omega_n - 1] - i\varepsilon (\Omega_n | H_{\text{int}} | \Omega_n - 1)] \right\}. \hfill (18)$$
Then, the measure 

\[
\times e^{-\frac{T\gamma_n}{4}} \exp \frac{im}{2\hbar} (z_n - z_{n-1})^2 \lim_{N \to \infty} \int \prod_{n=1}^{N} \frac{d\cos(\theta_n) d\varphi_n}{2\pi} 
\]

with 

\[
R(z_n, t_n) = \left( \begin{array}{cc} 1 - i\varepsilon \frac{1}{2} \left( \omega - i\frac{\gamma_n}{2} \right) & i\varepsilon \frac{1}{2} \Omega e^{-i(\omega_L t_n - k z_n)} \\ i\varepsilon \frac{1}{2} \Omega e^{i(\omega_L t_n - k z_n - 1)} & 1 + i\varepsilon \frac{1}{2} \left( \omega - i\frac{\gamma_{n-1}}{2} \right) \end{array} \right). 
\] 

To integrate it is necessary to first eliminate the inconvenient terms \( e^{-i(\omega_L t_n - k z_n)} \) and \( e^{i(\omega_L t_n - k z_n - 1)} \) with the help of the following change of variable:

\[
\varphi_n = \varphi'_n + \omega_L t_n - k z_n.
\]

Then, the measure 

\[
\prod_{n=1}^{N} \frac{d\cos(\theta_n) d\varphi_n}{2\pi} = \prod_{n=1}^{N} \frac{d\cos(\theta_n) d\varphi'_n}{2\pi}, 
\]

remains unchanged. The expression (19) becomes 

\[
K \left( z_f, \Omega_f, z_i, \Omega_i; T \right) = \lim_{N \to \infty} \frac{(m/2\pi i\hbar)^{\frac{N}{2}}}{N!} \int \prod_{n=1}^{N} d z_n \prod_{n=1}^{N+1} d z_{n-1} 
\]

\[
\times e^{-\frac{T\gamma_n}{4}} \exp \frac{im}{2\hbar} (z_n - z_{n-1})^2 \lim_{N \to \infty} \int \prod_{n=1}^{N} \frac{d\cos(\theta_n) d\varphi'_n}{2\pi} 
\]

\[
\times \prod_{n=1}^{N+1} \left( \cos \frac{\theta_n}{2} e^{\frac{i}{2} \varphi_n} \sin \frac{\theta_n}{2} e^{-\frac{i}{2} \varphi_n} \right) R_1(z_n, t_n) \left( \begin{array}{cc} \cos \frac{\theta_{n-1}}{2} e^{-\frac{i}{2} \varphi'_{n-1}} & i\varepsilon \frac{1}{2} \Omega \\ i\varepsilon \frac{1}{2} \Omega & 1 + i\varepsilon \frac{1}{2} \left( \Delta \omega - i\frac{\gamma_{n-1}}{2} \right) e^{i\frac{k}{2} \Delta z_n} \end{array} \right), 
\]

where 

\[
R_1(z_n, t_n) = \left( \begin{array}{cc} 1 - i\varepsilon \frac{1}{2} \left( \Delta \omega - i\frac{\gamma_n}{2} \right) e^{-i\frac{k}{2} \Delta z_n} & i\varepsilon \frac{1}{2} \Omega \\ i\varepsilon \frac{1}{2} \Omega & 1 + i\varepsilon \frac{1}{2} \left( \Delta \omega - i\frac{\gamma_{n-1}}{2} \right) e^{i\frac{k}{2} \Delta z_n} \end{array} \right) 
\] 

and \( \Delta \omega = \omega - \omega_L \).

Then we use the following identity:

\[
\int_{-\infty}^{+\infty} \frac{dp_n}{2\pi \hbar} \exp \left[ -i\varepsilon \frac{p_n^2}{2m \hbar^2} + \frac{i}{\hbar} p_n \left( \Delta z_n + \frac{\varepsilon \hbar}{2m k} \right) \right] = \sqrt{\frac{m}{2\pi i \varepsilon \hbar}} \exp \left[ im \left( \Delta z_n + \frac{\varepsilon \hbar}{2m k} \right)^2 \right]. 
\]
Hence the propagator (25) takes the following form:

\[
K\left(z_f, \Omega_f, z_i, \Omega_i; T\right) = \lim_{N \to \infty} \int_{-\infty}^{+\infty} \prod_{n=1}^{N} dz_n \prod_{n=1}^{N} \int_{-\infty}^{+\infty} \frac{dp_n}{2\pi \hbar} \times \exp \left[ \sum_{n=1}^{N+1} \left[ -\frac{i \varepsilon_n}{2m} p_n^2 + \frac{i}{\hbar} p_n \Delta z_n - \frac{i \varepsilon_n \hbar^2}{8m} \right] \right] \lim_{N \to \infty} \int \prod_{n=1}^{N} \frac{d\cos(\theta_n) d\varphi_n^\prime}{2\pi} \prod_{n=1}^{N+1} \left( \cos \frac{\theta_n^\prime}{2} e^{+\frac{i}{2} \varphi_n^\prime} \sin \frac{\theta_n^\prime}{2} e^{-\frac{i}{2} \varphi_n^\prime} \right) R_2(z_n, t_n) \left( \cos \frac{\theta_n^\prime-1}{2} e^{-\frac{i}{2} \varphi_n^\prime-1} \sin \frac{\theta_n^\prime-1}{2} e^{+\frac{i}{2} \varphi_n^\prime-1} \right),
\]

where

\[
R_2(p_n, t_n) = \left( 1 - \varepsilon \left( \frac{1}{2} \left( \Delta \omega - \frac{ib \gamma}{2} \right) + \frac{p_n k}{2m} \right) \right) \left( 1 + \varepsilon \left( \frac{1}{2} \left( \Delta \omega - \frac{ib \gamma}{2} \right) + \frac{p_n k}{2m} \right) \right).
\]

By integrating over the \( N \) variables \( z_n \), we clearly get Dirac functions \( \delta (\dot{p}) \) which reflect the conservation of the atom impulsion during the movement, i.e:

\[
p_1 = p_2 = \cdots = p_{N+1} = p.
\]

Hence the propagator (25) takes the following form:

\[
K\left(z_f, \Omega_f, z_i, \Omega_i; T\right) = \left[ i \hbar p(z_f - z_i) - \frac{i T p^2}{2m \hbar} - i T \hbar k^2 \right] \prod_{n=1}^{N+1} \left( \cos \frac{\theta_n^\prime}{2} e^{+\frac{i}{2} \varphi_n^\prime} \sin \frac{\theta_n^\prime}{2} e^{-\frac{i}{2} \varphi_n^\prime} \right) R_2(p, t_n) \left( \cos \frac{\theta_n^\prime-1}{2} e^{-\frac{i}{2} \varphi_n^\prime-1} \sin \frac{\theta_n^\prime-1}{2} e^{+\frac{i}{2} \varphi_n^\prime-1} \right),
\]

where

\[
R_2(p, t_n) = \left( 1 - \varepsilon \left( \frac{1}{2} \left( \Delta \omega - \frac{ib \gamma}{2} \right) + \frac{p k}{2m} \right) \right) \left( 1 + \varepsilon \left( \frac{1}{2} \left( \Delta \omega - \frac{ib \gamma}{2} \right) + \frac{p k}{2m} \right) \right).
\]

let us integrate on the angular variables \( \Theta_n \)et \( \varphi_n^\prime \)

the propagator takes the following form:

\[
K(f, i; T) = e^{-\frac{T \gamma}{4}} \left[ i \hbar p(z_f - z_i) - \frac{i T \gamma p^2}{2m \hbar} - i T \hbar k^2 \right] \prod_{n=1}^{N+1} \left( \cos \frac{\theta_n^\prime}{2} e^{+\frac{i}{2} \varphi_n^\prime} \sin \frac{\theta_n^\prime}{2} e^{-\frac{i}{2} \varphi_n^\prime} \right) R(p, T) \left( \cos \frac{\theta_n^\prime}{2} e^{-\frac{i}{2} \varphi_n^\prime} \sin \frac{\theta_n^\prime}{2} e^{+\frac{i}{2} \varphi_n^\prime} \right),
\]

with

\[
R(p, T) = \lim_{N \to \infty} (-1)^N \prod_{n=1}^{N+1} R_2(p_n, t_n).
\]
The arrow under the product symbol indicates the time ordering operation. We concentrate on the evaluation of the matrix elements $R_{ij}(p, T)$. To this aim the matrix $R(p, t_n)$ is written as a sum of a diagonal matrix and an off-diagonal matrix. To first order in $\varepsilon$ we have

$$R(p, t_n) = e^{-i\varepsilon \omega(j) \sigma_z} + i \varepsilon K(n),$$

where the off-diagonal matrix is given by

$$K(n) = \begin{pmatrix} 0 & u(n) \\ u(n) & 0 \end{pmatrix},$$

(33)

with

$$\omega(n) = \frac{1}{2} \left( \Delta \omega - \frac{i \hbar \gamma_+}{2} \right) + \frac{kp}{2m}, \text{ and } u(n) = \frac{\Omega}{2},$$

(34)

let us pass to the calculation of produce matrix according to [11]

$$\prod_{n=1}^{N} (e^{-i\varepsilon \omega(n) \sigma_z} + i \varepsilon K(n)) =$$

$$= e^{-i \sum_{j=1}^{N} \varepsilon \omega(j) \sigma_z} + \sum_{l=1}^{N} (i\varepsilon)e^{-i \sum_{l+1}^{N} \varepsilon \omega(k) \sigma_z} K(l)e^{-i \sum_{l+1}^{N} \varepsilon \omega(k) \sigma_z} + ...$$

$$+ \sum_{l_1=1}^{N} \sum_{l_2=1}^{l_1-1} (i\varepsilon)^2 e^{-i \sum_{l_1+1}^{N} \varepsilon \omega(k) \sigma_z} K(l_1)e^{-i \sum_{l_1+1}^{N} \varepsilon \omega(k) \sigma_z} K(l_2)e^{-i \sum_{l_1+1}^{N} \varepsilon \omega(k) \sigma_z} + ...$$

$$+ \sum_{l_1=1}^{N} \sum_{l_2=1}^{l_1-1} \sum_{l_3=1}^{N-2l_1+1} ... \sum_{l_{N-1}=1}^{N-1} \sum_{l_{N}=1}^{1} (i\varepsilon)^N e^{-i \sum_{l_{N}+1}^{N} \varepsilon \omega(k) \sigma_z} K(l_{N-1})e^{-i \sum_{l_{N}+1}^{N} \varepsilon \omega(k) \sigma_z} K(l_N)e^{-i \sum_{l_{N}+1}^{N} \varepsilon \omega(k) \sigma_z}.$$

(35)

Taking the limit $N \rightarrow +\infty$ of (36) leads to

$$R(p, T) = e^{-i \int_{0}^{T} ds_dw(p, s) \sigma_z} + i \int_{0}^{T} ds_d e^{-i \int_{0}^{T} ds_dw(p, s) \sigma_z} K(s_1)e^{-i \int_{0}^{T} ds_dw(p, s) \sigma_z} e^{-i \int_{0}^{T} ds_dw(p, s) \sigma_z} K(s_2)$$

$$+ (i)^2 \int_{0}^{T} ds_1 \int_{0}^{s_1} ds_2 e^{-i \int_{s_1}^{T} ds_dw(p, s) \sigma_z} K(s_1)e^{-i \int_{s_1}^{T} ds_dw(p, s) \sigma_z} K(s_2)$$

$$+ ... (i)^N \int_{0}^{T} ds_1 \int_{0}^{s_1} ds_2 \int_{0}^{s_2} ds_3 ... \int_{0}^{s_{N-1}} ds_N e^{-i \int_{s_N}^{T} ds_dw(p, s) \sigma_z} K(s_1)$$

$$\times e^{-i \int_{s_N}^{T} ds_dw(p, s) \sigma_z} K(s_2)...K(s_{N-1})e^{-i \int_{s_{N-1}}^{T} ds_dw(p, s) \sigma_z} K(s_N)e^{-i \int_{s_N}^{T} ds_dw(p, s) \sigma_z} + ...$$

(36)

a simple calculation shows us that the terms odd respectively even are the element antidiagonaux respectively diagonal of $R$.

$$R_{11}(p, T) = e^{-i \int_{0}^{T} ds_dw(p, s)} + \sum_{n=1}^{N} 2^n \int_{0}^{T} ds_1 \int_{0}^{s_1} ds_2 \int_{0}^{s_2} ds_3 ... \int_{0}^{s_{2n-1}} ds_{2n} \times$$

$$e^{-i \int_{0}^{T} ds_dw(p, s) u(p(s_1), s_1) e^{+i \int_{0}^{T} ds_dw(p, s) u(p(s_2), s_2)} \times ... e^{+i \int_{0}^{T} ds_dw(p, s) u(p(s_{2n}), s_{2n})} e^{-i \int_{0}^{T} ds_dw(p, s)}$$

(37)
and

\[ R_{12}(p, T) = i \int_0^T ds_1 e^{-i p \int_0^{s_1} ds_2} u(p(s_1), s_1) R_{22}(p(s_1), s_1), \quad (38) \]

\[ R_{22}(p, T) = R^*_1(p, T), \quad (39) \]

\[ R_{21}(p, T) = -R^*_2(p, T). \]

For instance

\[ R_{11}(p, T) = \left[ 1 + \sum_{n=1}^{\infty} \left( \frac{i \Omega}{2} \right)^{2n} \int_0^T e^{i \Delta s_1} ds_1 \int_0^{s_1} e^{-i \Delta s_2} ds_2 \right] e^{-i T \Delta}, \quad (40) \]

with

\[ \Delta = 2 \omega(n, s), \quad (41) \]

\[ \omega(n, s) = \frac{\Delta \omega}{2} - \frac{i \gamma}{4} + \frac{kp}{2m}. \quad (42) \]

4. Summation of the perturbation series

Let us put

\[ F(0, T) = \sum_{n=1}^{\infty} \left( \frac{i \Omega}{2} \right)^{2n} \int_0^T e^{i \Delta s_1} ds_1 \int_0^{s_1} e^{-i \Delta s_2} ds_2 \int_0^{s_2} e^{-i \Delta s_3} ds_3 \]

The evaluation of this multiple integral which has the form of a repeated convolution product, proceeds by taking the Laplace transform of all functions with respect to the time differences in their arguments and using the convolution theorem of Laplace transform [11]. The result is

\[ \tilde{F}(0, q) = \int_0^{+\infty} dT e^{-qT} F(0, T) dT = \frac{1}{q} \sum_{n=1}^{\infty} \left[ \frac{-\Omega^2/4}{q(q-i \Delta)} \right]^n \]

the sum over \( N \) is carried out with the result

\[ \tilde{F}(0, q) = \frac{q-i \Delta}{q(q-i \Delta) + \Omega^2/4} - \frac{1}{q} \]

this result is valid for \( |\frac{(i \Omega)^2}{q(q-i \Delta)}| < 1 \). We notice that it is possible to get a contour of the inverse Laplace integration where the condition above is verified. Taking the inverse Laplace transform we obtain for the element \( R_{11}(p, T) \) the expression

\[ R_{11}(p, T) = \cos(\Omega'T) - \frac{i \Delta}{2 \Omega'} \sin(\Omega'T), \quad \text{with} \quad \Omega' = \sqrt{\frac{\Delta^2}{4} + \frac{\Omega^2}{4}}. \]

Following the same method of calculations, the remaining elements are calculated and listed below

\[ R_{12}(p, T) = \frac{i \Omega}{2 \Omega'} \sin(\Omega'T), \quad (43) \]

\[ R_{21}(p, T) = \frac{i \Omega}{2 \Omega'} \sin(\Omega'T), \quad (44) \]

\[ R_{22}(p, T) = \cos(\Omega'T) + \frac{i \Delta}{2 \Omega'} \sin(\Omega'T). \quad (45) \]
With the help of the completeness relations, this amplitude becomes proper states of the spin. We take an example of the matrix element 5. The wave functions propagator is the following:

\[ S_{\omega} \]

where the elements of matrix concerning to our problem is the following

\[ R(p, T) = \begin{pmatrix} \cos(\Omega'T) - \frac{i \Delta}{2 \Omega} \sin(\Omega'T) & -\frac{\Delta}{2 \Omega} \sin(\Omega'T) \\ -\frac{\Delta}{2 \Omega} \sin(\Omega'T) & \cos(\Omega'T) + \frac{i \Delta}{2 \Omega} \sin(\Omega'T) \end{pmatrix} . \]

Now we come back to the old angular variables \((\theta, \varphi)\). So, the exact expression of the propagator concerning to our problem is the following

\[ K(f, i; T) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi \hbar} \exp \left[ -i T p^2 \frac{2m}{2m} - i \hbar k^2 T \frac{m}{8m} + i \hbar p(z_f - z_i) \right] \times \left( \cos \frac{\theta_f}{2} e^{i \frac{\varphi_f}{2}} \sin \frac{\theta_f}{2} e^{-i \frac{\varphi_f}{2}} \right) R(p, T) \left( \cos \frac{\theta_i}{2} e^{-i \frac{\varphi_i}{2}} \sin \frac{\theta_i}{2} e^{i \frac{\varphi_i}{2}} \right) , \]

with

\[ R(p, T) = \begin{pmatrix} \cos(\Omega'T) - \frac{i \Delta}{2 \Omega} \sin(\Omega'T) & -\frac{\Delta}{2 \Omega} \sin(\Omega'T) \\ -\frac{\Delta}{2 \Omega} \sin(\Omega'T) & \cos(\Omega'T) + \frac{i \Delta}{2 \Omega} \sin(\Omega'T) \end{pmatrix} . \]

Where the elements of matrix \( S(p, T) \) is given by:

\[ S_{11}(p, T) = \left[ \cos \Omega'T - \frac{i \Delta}{2 \Omega} \sin \Omega'T \right] e^{-\frac{\gamma_{z-}}{2}} \exp i \frac{1}{2} (kz_f - \omega_L T - k z_i) , \]

\[ S_{22}(p, T) = \left[ \cos \Omega'T + \frac{i \Delta}{2 \Omega} \sin \Omega'T \right] e^{-\frac{\gamma_{z+}}{2}} \exp -i \frac{1}{2} (kz_f - \omega_L T - k z_i) , \]

\[ S_{12}(p, T) = -\frac{i \Omega}{2 \Omega} \sin \Omega'T e^{-\frac{\gamma_{z+}}{2}} \exp i \frac{1}{2} (kz_f - \omega_L T + k z_i) , \]

\[ S_{21}(p, T) = -\frac{i \Omega}{2 \Omega} \sin \Omega'T e^{-\frac{\gamma_{z-}}{2}} \exp -i \frac{1}{2} (kz_f - \omega_L T + k z_i) . \]

The angles \( \theta, \varphi \) are allowed to vary only in the limited domains \([0, 2\pi]\) and \([0, 4\pi]\). Our propagator is the following:

\[ K(f, i; T) = \sum_{n=\infty}^{+\infty} K(z_f, \theta_f + 2n \pi, z_i, \varphi_f + 4n \pi, \theta_i, \varphi_i; T) \]

\[ = K(z_f, \Omega_f, z_i, \Omega_i; T) \]

It is an expression for the propagator in the spin coherent state representation.

5. The wave functions

Let us now eliminate the coherent states by computing the transition amplitude between the proper states of the spin. We take an example of the matrix element

\[ K_{\uparrow\uparrow}(z_f, z_i; T) = \langle \uparrow | K(z_f, z_i; T) | \uparrow \rangle . \]

With the help of the completeness relations, this amplitude becomes

\[ K_{\uparrow\uparrow}(z_f, z_i; T) = \frac{1}{2\pi} \int d \cos(\theta_f) d \varphi_f \frac{1}{2\pi} \int d \cos(\theta_i) d \varphi_i \]
\[ \times \langle \uparrow | e^{-\frac{i}{2}\varphi_f} \rangle \]

If we fix the initial state of the atom as \( |m_i\rangle = |\uparrow\rangle \), and the final state as \( |m_f\rangle = |\uparrow\rangle \), we obtain

\[ \langle \uparrow | e^{-\frac{i}{2}\varphi_f} \rangle = \cos \frac{\theta_f}{2} e^{-\frac{i}{2}\varphi_f}, \quad \text{and} \quad \langle \Omega_i | \uparrow \rangle = \cos \frac{\theta_i}{2} e^{\frac{i}{2}\varphi_i}. \]  

Substituting (56) in (55) and integrating over polar coordinates, (55) takes the following form

\[ K_{\uparrow\uparrow}(z_f, z_i; T) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} \exp \left[ \frac{i}{\hbar} p(z_f - z_i) - \frac{i}{2m\hbar} \frac{T^2}{T^2 - \Delta\epsilon_p} \right] S_{11}. \]  

In the same manner, we proceed for the remaining elements. The result takes the following matrix form

\[ K(z_f, z_i; T) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} \exp \left[ \frac{i}{\hbar} p(z_f - z_i) - \frac{i}{2m\hbar} \frac{T^2}{T^2 - \Delta\epsilon_p} \right] S(T), \]  

which coincide with those obtained in [28] using fermionic coherent states path integral.

From this propagator, the atom state at time \( T \) is deduced from the initial state via the evolution equation

\[ \Psi(z, T) = \left( \begin{array}{c} \Psi_1(z, T) \\ \Psi_2(z, T) \end{array} \right) = \int_{-\infty}^{+\infty} K(z, y; T) \psi(y, 0) dy. \]  

The same physical features can thus be reobtained as has been done in [25] and in the case with out damping done in [26].

6. Conclusion

We have given an exact treatment using the path-integral formalism to the problem of the two-level atom in interaction with an electromagnetic wave. Thanks to the two variables \((\theta, \varphi)\) replacing the spin, the propagator has been written, first in the conventional form \( \int D(\text{path}) \exp \frac{i}{\hbar} S(\text{path}) \), then determined with exactitude. Thus the wave function has been deduced via the evolution equation. Our results through the path-integral approach are in accordance with those obtained in [25].

Finally let us note that the expression (58) has the form \( \sum_{\text{path}} \exp (iS(\text{path}) / \hbar) \) where the sum runs over all classical paths (parameterized here by the momentum). This form, remarkable for this problem, shows that a traditional semi-treatment can lead to an exact result.

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