Strichartz - type Inequalities for Parabolic and Schrödinger Equations in rearrangement invariant Spaces

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Abstract.

In this paper we generalize the classical Strichartz estimation for solutions of initial problem for linear parabolic and Schrödinger PDE on many popular classes pairs of rearrangement invariant(r.i.) spaces and construct some examples in order to show the exactness of our estimations.

Key words: Strichartz inequality, rearrangement invariant (r.i.) spaces and moment rearrangement invariant (m.r.i.) spaces, Orlicz, Lorentz, Marzinkiewitz and Grand Lebesque spaces, Gaussian kernel, fundamental function, upper and low bounds.

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1. Introduction. Notations. Statement of problem.

Problem (P). Let us consider the initial (Cauchy) problem for the non-degenerate linear parabolic equation in the whole $d$ - dimensional space $\mathbb{R}^d$:

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{d} \sum_{j=1}^{d} a_{k,j}(t,x) \frac{\partial^2 u}{\partial x_k \partial x_j} ,$$  \hspace{1cm} (1.1)

where $u = u(t,x)$, $t \in [0, \infty)$, $x = \vec{x} = \{x_1, x_2, \ldots, x_d\}$ be a $d$ - dimensional vector; $x \in \mathbb{R}^d$ with initial condition

$$\lim_{t \to 0^+} u(t,x) = f(x),$$  \hspace{1cm} (1.2)

where the limit is understood in the $L_p$ sense for some $p \in [1, \infty]$.

It is presumed that for the problem (P) (1.1) with (1.2) are satisfied the classical conditions for existence and uniqueness, for instance:
\[ \lambda \sum_{k=1}^{d} \xi_k^2 \leq \sum_{k=1}^{d} \sum_{j=1}^{d} a_{k,j}(t, x) \xi_k \xi_j \leq \Lambda \sum_{k=1}^{d} \xi_k^2 \]  

(1.3)

for some constants \( \lambda, \Lambda : 0 < \lambda \leq \Lambda < \infty \) (the uniform ellipticity and boundeness condition);

\[ \max_{k} \max_{j} |a_{k,j}(t, x) - a_{k,j}(s, y)| \leq M |t - s| + \sum_{l=1}^{d} |x_l - y_l|^\beta \]  

(1.4)

for some finite positive constants \( M, \beta; \beta \in (0, 1] \) (the uniform Hölder condition).

We will denote hereafter as \( C, C_m, C_j(d), C_l(d, a) \) some finite positive non-essential constants.

It is well-known ([16], [12] etc.) for the solution of problem (1.1) - (1.2) under conditions (1.3) - (1.4) the Strichartz-Krylov estimations, which we want to reformulate in the convenient for us form.

We will denote for the solution of (1.1) - (1.2)

\[ u(t, \cdot) = T_t f(\cdot), \]

where \( \{T_t\} \) is a semi-group of linear operators.

Lemma 1.1. For all the values \( r > p, p \geq 1 \) and \( t > 0 \)

\[ |u|_r \leq C(a, d) |f|_p t^{d(\frac{1}{r} - \frac{1}{p})}. \]  

(1.5)

The estimation (1.5) may be obtained as follows. Without loss of generality we can and will assume the function \( x \to f(x) \) to be non-negative and non-trivial: \( |f|_p \in (0, \infty) \).

The solution \( u = u(t, x) \) has a view:

\[ u(t, x) = C_1 t^{-d/2} \int_{R^d} f(y) G(t, x, y) dy, \]  

(1.6)

where the positive function \( G = G(t, x, y) \) is called Heat Potential (HP) and allows the estimation: \( G(t, x, y) \leq G_0(t, x - y) \);

\[ G_0(t, z) = C_2 \exp \left( -C_3 t^{-1} ||z||^2 \right); ||z||^2 \overset{\text{def}}{=} \sum_{k=1}^{d} |z_k|^2. \]

So, we have:
where the convolution \(*\) is understood over the variable \(x\); \(t = \text{const} > 0\).

We obtain using the well-known Hardy-Littlewood-Young inequality:

\[
|u|_r \leq C_4 t^{-d/2} |f|_p |G_0|_q, \quad 1 + 1/r = 1/p + 1/q, \tag{1.7}
\]

\(p, q, r \geq 1, \quad r \geq p \geq 1. \tag{1.8}\)

It is easy to verify by the direct calculation that

\[
|G_0|_q \leq C_5 (d) t^{d/(2q)},
\]

therefore

\[
C_6 |u|_r \leq t^{d(-0.5+0.5/q)} |f|_p = t^{0.5-d(1/r - 1/p)} |f|_p.
\]

**Problem (S).** Let us consider also the initial (Cauchy) problem for the (linear) Schrödinger equation without potential ("free particle") in the whole space \(R^d\):

\[
-i \frac{\partial v}{\partial t} = 0.5 \sum_{k=1}^{d} \frac{\partial^2 v}{\partial x_k^2} = 0.5 \Delta v, \quad (i^2 = -1), \tag{1.9}
\]

\[
\lim_{t \to 0} u(t, x) = f(x) \tag{1.10}
\]

again in the \(L_p\) sense for some \(p \in [1, \infty]\).

It is well-known ([33], [34], [5], [16], [6], [7], [17], [29], [30], [31] etc.) for the solution \(v = v(t, x)\) of problem (S) (1.9) - (1.10) under condition \(f(\cdot) \in L_q\) for some \(q \in [1, 2]\) there exists, is unique and satisfies the following assertion.

**Lemma 1.2.** In the case \(p \geq 2\) the following inequality is true:

\[
|v|_p \leq C_7(d) |t|^{d(0.5 - 1/p')} |f|_{p'}. \tag{1.11}
\]

The assertion (1.11) follows from the conservation law:

\[
\forall t > 0 \Rightarrow |v|_2 \leq C_8(d) |f|_2,
\]

explicit formula for \(v(\cdot, \cdot): t > 0 \Rightarrow\)

\[
v(t, x) = C_8(d) t^{-d/2} \int_{R^d} \exp \left(0.5 i \|x - y\|/t\right) f(y) \, dy,
\]

from which it follows the inequality

\[
|v|_\infty \leq C_9(d) t^{-d/2} |f|_1, \quad t > 0,
\]

and from the famous interpolation theorem belonging to Riesz-Thorin.

We will denote also for the solution \(v = v(t, x)\) of the problem (1.9) - (1.10)

\[
v(t, \cdot) = U_t f(\cdot),
\]
where \( \{U_t\} \) is a group of linear operators.

In the physical literature the operators \( \{T_t\} \) and \( \{U_t\} \) are called often the Propagation operators.

Note that the inequalities (1.5) and (1.11) are non-trivial only in the case of sufficiently great values \( t : t \gg 1 \), or more exactly

\[
t \to \infty; \ t > 2.
\]

(1.12)

Further we will assume the condition (1.12) to be satisfied for both considered problems.

Our goal is generalization of the estimations (1.5) and (1.11) on some popular classes of rearrangement invariant (r.i.) spaces, more exactly, so-called moment rearrangement invariant (m.r.i.) spaces.

In detail. Parabolic case. The inequality (1.5) may be rewritten as follows. Let \((X, \| \cdot \|_X)\) be any rearrangement invariant (r.i.) space on the set \( \mathbb{R}^d \); denote by \( \phi(X, \delta) \) its fundamental function

\[
\phi(X, \delta) = \sup_{A, \mu(A) \leq \delta} \| I(A) \|_X, \quad I(A) = I(A, x) = 1, x \in A,
\]

where \( I(A) = I(A, x) = 0 \), \( x \notin A \).

We introduce also for two function r.i. spaces \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) defined over our set \( \mathbb{R}^d \) and for arbitrary finite positive constants \( K_1, K_2 \) and the values \( t > 2 \) the so-called Strichartz Parabolic two-space functional, briefly: SP functional between the spaces \( X \) and \( Y \) as

\[
W_{SP}(X, Y, K_1, K_2; t) \overset{\text{def}}{=} \sup_{f \neq 0} \left[ \frac{\| T_t f \|_Y}{\phi(X, \delta)} : \frac{\| f \|_X}{\phi(Y, \delta)} \right],
\]

\[
W_{SP}(X, Y; t) \overset{\text{def}}{=} W_{SP}(X, Y, 1, 1; t).
\]

Then (1.5) is equivalent to the following inequality:

\[
r > p \geq 1 \Rightarrow \sup_{t > 2} W_{SP}(L_p, L_r; t) < \infty.
\]

(1.13)

Definition 1.

By definition, the pair of r.i. spaces \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) over \( \mathbb{R}^d \) is said to be a (strong) Strichartz Parabolic pair, write: \((X, Y) \in SP\), if the SP functional \( W_{SP}(X, Y; t) \) between \( X \) and \( Y \) is uniform on the variable \( t \), \( t > 2 \) finite:

\[
\sup_{t > 2} W_{SP}(X, Y; t) < \infty
\]

(1.14)

and is called a weak Strichartz Parabolic pair, write \((X, Y) \in wSP\), if for some non-trivial constants \( K_1, K_2 \)
We introduce also for two function r.i. spaces $(X, || \cdot ||_X)$ and $(Y, || \cdot ||_Y)$ defined over our set $R^d$ and for arbitrary finite positive constant $K$ and the values $t > 2$ the so-called **Strichartz Schrödinger two-space functional**, briefly: SR functional between the spaces $X$ and $Y$ as

$$V_{SR}(X, Y, K; t) \overset{\text{def}}{=} \sup_{f \in X, f \neq 0} \frac{t^{-d/2} \cdot ||U_t f||_Y}{||f||_X \cdot \phi(X, K t^{-d})},$$

and define $V_{SR}(X, Y; t) \overset{\text{def}}{=} V_{SR}(X, Y, 1; t)$.

**Definition 2.**

**By definition,** the pair of r.i. spaces $(X, || \cdot ||_X)$ and $(Y, || \cdot ||_Y)$ over $R^d$ is said to be a (strong) Strichartz Schrödinger pair, write: $(X, Y) \in SR$, if the $SR$ functional $V_{SR}(X, Y; t)$ between $X$ and $Y$ is uniform on the variable $t$, $t > 2$ finite:

$$\sup_{t > 2} V_{SR}(X, Y; t) < \infty \quad (1.16)$$

and is called a **weak Strichartz Schrödinger pair**, write $(X, Y) \in wSR$, if for some positive non-trivial constants $K$

$$\sup_{t > 2} V_{SR}(X, Y; t) < \infty. \quad (1.17)$$

Roughly speaking, we will prove that the most of popular pairs of r.i. spaces are strong, or at least weak Strichartz pairs, Parabolic or Schrödinger.

The paper is organized as follows. In the next section we recall and describe a new class of r.i. spaces, namely, so-called moment rearrangement invariant spaces, briefly, m.r.i. spaces. In the section 3 we formulate and prove the main result of paper for m.r.i. spaces.

In the section 4 we offer some examples of our results. In the section 5 we consider some low bounds for introduced functionals in order to show the precision of obtained estimations.

In the last section 6 we describe some generalizations of results of the section 3.

### 2. Auxiliary facts. Moment rearrangement invariant spaces.

The complete investigation of the theory of r.i. spaces see, e.g., in [3], chapters 1,2; [19], chapter 1.

We recall here only that the Banach function space $X$ equipped with the norm $|| \cdot ||_X$ over the set, e.g., $R^d$ is called rearrangement invariant (r.i.) space, if the norm in this space dependent only on the distribution function of $f$:
\[ \| f \|_X = R(Q_f(\cdot)) , \]

where \( Q_f(\cdot) \) is the distribution function for the (measurable) function \( f \):

\[ Q_f(s) = m\{ x, x \in R^d, \ |f(x)| > s \}; \ s \in (0, \infty) \]

and \( R(\cdot) \) is some functional.

For instance, many popular functional spaces: \( L_p \) spaces, Orlicz, Lorentz, Marzinkiewitz spaces are r.i. spaces.

Let \( (X, \| \cdot \|_X) \) be the r.i. space, where \( X \) is linear subset on the space of all measurable function \( R^d \to R \) with norm \( \| \cdot \|_X \).

**Definition 3.**

We will say that the space \( X \) with the norm \( \| \cdot \|_X \) is **moment rearrangement invariant space**, briefly: m.r.i. space, or \( X = (X, \| \cdot \|_X) \in m.r.i. \), if there exist a real constants \( a, b; 1 \leq a < b \leq \infty \), and some *rearrangement invariant norm* \( < \cdot > \) defined on the space of a real functions defined on the interval \((a, b)\), non necessary to be finite on all the functions, such that

\[ \forall f \in X \Rightarrow \| f \|_X =< h(\cdot) >, \ h(p) = |f|^p. \quad (2.1) \]

We will say that the space \( X \) with the norm \( \| \cdot \|_X \) is **weak moment rearrangement space**, briefly, w.m.r.i. space, or \( X = (X, \| \cdot \|_X) \in w.m.r.i. \), if there exist a constants \( a, b; 1 \leq a < b \leq \infty \), and some functional \( F \), defined on the space of a real functions defined on the interval \((a, b)\), non necessary to be finite on all the functions, such that

\[ \forall f \in X \Rightarrow \| f \|_X = F( h(\cdot) ), \ h(p) = |f|^p. \quad (2.2) \]

Roughly speaking, the functional space \( X \) is called m.r.i. space or w.m.r.i. space, if the norm in this space dependent only on the some family of \( L_p \) norms of considering function.

We will write for considered w.m.r.i. and m.r.i. spaces \( (X, \| \cdot \|_X) \)

\[ (a, b) \overset{def}{=} msupp(X) , \quad (2.3) \]

(moment support; not necessary to be uniquely defined) and define for other such a space \( Y = (Y, \| \cdot \|_Y) \) with \( (c, d) = msupp(Y) \)

\[ msupp(Y) \gg msupp(X) , \]

or equally, \( msupp(X) \ll msupp(Y) \), iff \( \max(a, b) \leq \min(c, d) \).

It is obvious that arbitrary m.r.i. space is r.i. space.

There are many r.i. spaces satisfied the condition (2.2): exponential Orlicz’s spaces, some Martzinkiewitz spaces, interpolation spaces (see [1], [15], [8], [32], [27] etc.)
In the article [21] are introduced the so-called $G(p, \alpha)$ spaces consisted on all measurable function $f : T \to R$ with finite norm

$$\|f\|_{p, \alpha} = \left[ \int_{1}^{\infty} \left( \frac{|f(x)|}{x^{\alpha}} \right)^{p} \nu(dx) \right]^{1/p},$$

where $\nu$ is some Borelian measure.

Astashkin in [2] proved that the space $G(p, \alpha)$ in the case $T = [0, 1]$ and $\nu = m$, $m$ is usually Lebesgue measure, coincides with the Lorentz $\Lambda_{p}(\log_{1-p\alpha}(2/s))$ space. Therefore, both this spaces are m.r.i. spaces.

Another examples. Recently (see [8], [10], [11], [13], [14], [18], [23], [24], [25], [26], [27], [28] etc.) appear the so-called Grand Lebesque Spaces $GLS = G(\psi) = G(\psi; a, b)$ spaces consisting on all the measurable functions $f : R^{d} \to R$ with finite norms

$$\|f\|_{G(\psi)} \overset{\text{def}}{=} \sup_{p \in (a, b)} [\|f\|_{p}/\psi(p)]. \quad (2.4)$$

Here $\psi(\cdot)$ is some positive continuous on the open interval $(a, b)$ function such that

$$\inf_{p \in (a, b)} \psi(p) > 0. \quad (2.5)$$

It is evident that $G(\psi; a, b)$ is m.r.i. space and msupp$(G(\psi(a, b))) = (a, b)$.

We will write in this case $\psi \in \Psi(a, b)$.

This spaces are used, for example, in the theory of probability ([18], [20], [23], [24], [25], [26], [27], [28] etc.), theory of PDE ([11], [14]), functional analysis ([1], [2], [8], [10], [15], theory of Fourier series ([27]), theory of martingales ([23]), [28]) etc.

We can consider the classical Lebesgue spaces $L_{s}$, $s \geq 1$ as an extremal case of $G(\psi)$ spaces, namely, define a function

$$\psi_{s}(p) = 1$, $p = s$, $\psi_{s}(p) = +\infty$, $p \neq s.$$

If we define formally $\infty/\infty = \infty$, then

$$|f|_{s} = \|f\|_{G(\psi_{s})}.$$ 

See in detail [24], chapters 1,2.

Let us consider as an example now the (generalized) Zygmund’s spaces $L_{p} Log^{r}L$, which may be defined as an Orlicz’s spaces over some subset of the space $R^{d}$ with non-empty interior and with $N-$ Orlicz function of a view

$$\Phi(u) = |u|^{p} \log^{r}(C + |u|)$, $p \geq 1$, $r \neq 0.$$

**Lemma 2.1.**

1. All the spaces $L_{p} Log^{r}L$ over real line with measure $m$ with condition $r \neq 0$ are not m.r.i. spaces.
2. If \( r \) is positive and integer, then the spaces \( L_p \log^r L \) are w.m.r.i. space.

**Proof. 1.** It is sufficient to consider the case \( d = 1 \) with the classical Lebesgue measure \( m \) and the case \( p > 1 \).

There exists a function \( f_0 = f_0(x) \) belonging to the space \( L_p \log^r L \), for example, for which

\[
\int_T |f_0|^p \log^r(C + |f_0|) \, dx < \infty,
\]

but such that for all sufficiently small values \( \epsilon > 0 \)

\[
\int_T |f_0|^{p+\epsilon} \, dx = \infty
\]

in the case \( p > 1 \) and

\[
\int_T |f_0|^{p+\epsilon} \, dx = \infty
\]

in the case \( p = 1 \).
Therefore, the interval \((a, b)\) in the definition of m.r.i. spaces does not exists.
The affirmation 2 it follows from the formula

\[
|f|^p \log |f|^k = d^k |f|^p / dp^k, \quad k = 1, 2, \ldots.
\]

**Lemma 2.2** There exists an r.i. space without the w.m.r.i. property.

**Proof.** On the interval \( T = [0, 1] \) with usual Lebesgue measure \( m \) there exists a function \( f \) with standard normal (Gaussian) distribution. This implies, for example, that

\[
\int_T \exp(pf(x)) \, dx = \exp \left( 0.5 p^2 \right), \quad p \in \mathbb{R}.
\]

There exist a functions \( g : \mathbb{R} \to \mathbb{R} \) such that the function \( h(x) = g(f(x)) \) which distribution can not be uniquely defined by means of all positive moments, for instance, \( h(x) = g(f(x)) = [f(x)]^3 \) or \( g(x) = \exp(f(x)) \).

Let us consider a two such a functions \( f_1 \) and \( f_2 \) with different distributions, but with at the same moments, for example:

\[
\int_T |f_1|^p \, dx = \int_T |f_2|^p \, dx = \int_T [\exp(f)]^p \, dx = \exp(p^2 / 2), \quad p \in \mathbb{R}.
\]

We choose the (quasi) - concave positive strictly increasing continuous function \( \theta(\cdot), \ \theta(0+) = 0 \), for which

\[
\int_0^\infty \theta(m\{x : |f_1(x)| > \lambda\}) \, d\lambda = \infty,
\]

but

\[
\int_0^\infty \theta(m\{x : |f_2(x)| > \lambda\}) \, d\lambda < \infty.
\]

The Lorentz r.i. space \( \Lambda(T, \theta) \) over \( T = [0, 1] \) with the function \( \theta(\cdot) \) and the classical norm (see [3], chapter 2, section 2)
\[ \|f\|_{L(T, \theta)} = \int_0^\infty \theta(m\{x : |f(x)| > \lambda\}) \, d\lambda \]

is not w.m.r.i. space.

3. Main result. Strichartz inequalities for the pairs of m.r.i. spaces.

**Theorem 3.1.** Let \((X, \| \cdot \|_X)\) be any m.r.i. space over the space \(\mathbb{R}^d\) with moment support \(\text{msupp}(X) = (a_1, b_1)\) relatively the auxiliary norm \(< \cdot >\), and let \((Y, \| \cdot \|_Y)\) be another m.r.i. space over at the same set \(\mathbb{R}^d\) relatively the second auxiliary norm \(<< \cdot >>\) and with \(\text{msupp}(Y) = (a_2, b_2)\), where \((a_1, b_1) << (a_2, b_2)\).

Then the pair of m.r.i. spaces \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) is the (strong) Parabolic Strichartz pair:

\[
\sup_{t > 2} \mathcal{W}_{SP}(X, Y; t) = C_{SP}(X, Y) < \infty. \quad (3.1)
\]

**Note** that the restriction \((a_1, b_1) << (a_2, b_2)\) is not loss of generality.

**Theorem 3.2.** Let \((X, \| \cdot \|_X)\) be any m.r.i. space over the space \(\mathbb{R}^d\) with moment support \(\text{msupp}(X) = (a_1, b_1)\) relatively the auxiliary norm \(< \cdot >\), and let \((Y, \| \cdot \|_Y)\) be another m.r.i. space over at the same set \(\mathbb{R}^d\) relatively the second auxiliary norm \(<< \cdot >>\) and with \(\text{msupp}(Y) = (a_2, b_2)\), where \((a_1, b_1) << (a_2, b_2)\), \(a_2 > 2\).

Then the pair of m.r.i. spaces \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) is the (strong) Schrödinger Strichartz pair:

\[
\sup_{t > 2} \mathcal{V}_{SR}(X, Y; t) = C_{SR}(X, Y) < \infty. \quad (3.2)
\]

**Proofs.** **Theorem 3.1.** It follows from the inequalities (1.5) for the values \(p \in (a_1, b_1)\) and \(r \in (a_2, b_2)\) correspondingly:

\[
|T_t f|_r \cdot t^{d/(2r)} \leq C(a, d) \|f\|_p \cdot t^{d/(2r)}. \quad (3.3)
\]

Tacking into account the monotonicity of the norm \(< \cdot >\) and equality

\[ \phi(X, t^{d/2}) = z(\cdot), \]

where for all admissible values \(r\)

\[ z(r) = t^{d/(2r)}, \]

we get from (3.3) tacking the norm \(< \cdot >\):

\[
|T_t f|_r \cdot \phi(X, t^{d/2}) \leq C(a, d) \|f\|_X \cdot t^{d/(2r)}. \quad (3.4)
\]

Tacking analogously from the bide - side of inequality (3.4) the norm \(<< \cdot >>\), we conclude

\[
\|T_t f\|_Y \cdot \phi(X, t^{d/2}) \leq C(a, d) \|f\|_X \cdot \phi(Y, t^{d/2}),
\]
which completes the proof of Theorem 3.1.

**Proof of Theorem 3.2** is analogously. We use the assertion (1.11) of the Lemma 1.2:

$$|U_t f| \leq C(d) t^{d/2} |f|.$$

Tacking the norm $< \cdot, \cdot >$, we obtain:

$$|U_t f| \cdot \phi(X, t^{d/2}) \leq C(d) t^{d/2} \cdot ||f||X.$$

Tacking the norm $<< \cdot, \cdot >>$, we obtain:

$$||U_t f||Y \cdot \phi(X, t^{d/2}) \leq C(d) t^{d/2} \cdot ||f||X.$$

This completes the proof of Theorem 3.2.

Note now as a particular case the case when $X = G(\psi)$, $\psi \in \Psi(a_1, b_1)$; $Y = g(\nu)$, $\nu \in \Psi(a_2, b_2)$, $b_1 < a_2$:

\[
sup_{f \in G(\psi), f \neq 0} \sup_{t > 2} \left\{ \frac{||T_t f||G(\nu)|}{\phi(G(\nu), t^{d/2})} : \frac{||f||G(\psi)|}{\phi(G(\psi), t^{d/2})} \right\} = C_1(\psi, \nu) < \infty \quad (3.5)
\]

and

\[
sup_{f \in G(\psi), f \neq 0} \sup_{t > 2} \frac{t^{-d/2}||U_t f||G(\nu)}}{||f||G(\psi) \cdot \phi(G(\psi), t^{-d})} = C_2(\psi, \nu) < \infty. \quad (3.6)
\]

**4. Examples.**

We consider now a very important for applications examples of $G(\psi)$ spaces. Let $a = const \geq 1, b = const \in (a, \infty)$; $\alpha, \beta = const$. Assume also that at $b < \infty \min(\alpha, \beta) \geq 0$ and denote by $h$ the (unique) root of equation

$$(h - a)^\alpha = (b - h)^\beta, \ a < h < b; \ \zeta(p) = \zeta(a, b, \alpha, \beta; p) =
\]

$$(p - a)^\alpha, \ p \in (a, h); \ \zeta(a, b, \alpha, \beta; p) = (b - p)^\beta, \ p \in [h, b);$$

and in the case $b = \infty$ assume that $\alpha \geq 0, \beta < 0$; denote by $h$ the (unique) root of equation $(h - a)^\alpha = h^\beta, h > a$; define in this case

$$\zeta(p) = \zeta(a, b, \alpha, \beta; p) = (p - a)^\alpha, \ p \in (a, h); \ p \geq h \Rightarrow \zeta(p) = p^\beta.$$

Here and further $p \in (a, b) \Rightarrow \psi(p) \asymp \nu(p)$ denotes that

$$0 < \inf_{p \in (a, b)} \frac{\psi(p)}{\nu(p)} \leq \sup_{p \in (a, b)} \frac{\psi(p)}{\nu(p)} < \infty.$$

The space $G = G_{H^d}(a, b; \alpha, \beta) = G(a, b; \alpha, \beta)$ consists by definition on all the measurable functions $f : T \to R$ with finite norm:

$$||f||G(a, b; \alpha, \beta) = \sup_{p \in (a, b)} [||f||_p \cdot \zeta(p)]. \quad (4.1)$$
On the other words, $G(a, b; \alpha, \beta)$ is the $G(\psi; a, b)$ space with $\psi(p) = 1/\zeta(p)$.

These spaces was introduced in [18], [24], [27]; and in the two last articles was also calculated its fundamental functions.

We rewrite here only the asymptotical expression for $\phi(G(a, b; \alpha, \beta) \delta)$ for two cases: $\delta \to 0^+$ and $\delta \to \infty$.

1. As $\delta \to 0^+$:
\[
\phi(G(a, b; \alpha, \beta) \delta) \sim (\beta b^2/e)^{\alpha} \cdot \delta^{1/\beta} |\log \delta|^{-\beta},
\]
(4.2)

$1 \leq a < b < \infty, \alpha, \beta \geq 0$;

\[
\phi(G(c, \infty; \alpha, - \beta) \delta) \sim (\beta)^{|\beta|} |\log \delta|^{-|\beta|},
\]
(4.3)

$1 \leq c, \alpha \geq 0, \beta > 0$;

2. As $\delta \to \infty$:
\[
\phi(G(a, \infty; \alpha, - \beta) \delta) \sim (a^2 \alpha/e)^{\alpha} \delta^{1/\alpha} (|\log \delta|)^{-\alpha},
\]
(4.4)

$1 \leq a < b \leq \infty$.

We choose in this pilcrow

\[X = G(a_1, b_1; \alpha_1, \beta_1), \quad Y = G(a_2, b_2; \alpha_2, \beta_2),\]

where $1 \leq a_1 < b_1 < a_2 < b_2 \leq \infty$.

**Parabolic example.**

We obtain using the theorems 3.1 for the values $t > 2$:

\[
\sup_{f : f \neq 0, f \in X} \left[ |\|T_t f||_{G(Y)} / |\|f||_{G(X)} | \right] \leq C_1(d; a_1, b_1, a_2, b_2; \alpha_1, \alpha_2, \beta_1, \beta_2) t^{-\frac{n}{a_1} - \frac{n}{a_2}} (\log t)^{\alpha_2 - \alpha_1}.
\]
(4.5)

**Schrödinger example.**

We consider again the case when

\[X = G(a_1, b_1; \alpha_1, \beta_1), \quad Y = G(a_2, b_2; \alpha_2, \beta_2),\]

where $1 \leq a_1 < b_1 < a_2 < b_2 \leq \infty$, but assume in addition $b_1 \leq 2, a_2 \geq 2$ (the cases $b_1 > 2$ or $a_2 < 2$ are trivial).

We obtain using the theorems 3.2 for at the same values $t > 2$:

\[
\sup_{f : f \neq 0, f \in X} \left[ |\|U_t f||_{G(Y)} / |\|f||_{G(X)} | \right] \leq C_2(d; a_1, b_1, a_2, b_2; \alpha_1, \alpha_2, \beta_1, \beta_2) t^{\frac{n}{\alpha_1} - \frac{n}{\alpha_2}} (\log t)^{-\beta_1}.
\]
(4.6)
5. Low bounds.

In this subsections we will construct some examples in order to illustrate the
exactness of result of section 3, for example, the exactness of inequalities (4.5) and
(4.6).

**Theorem 5.1.** Let \( X = L_1(R^d) \) and \( Y = G(\nu) \), \( \nu \) is arbitrary function from
the space \( \Psi(a,b) : \nu(\cdot) \in \Psi(a,b), a > 1, a < b \leq \infty \) be two examples of r.i. spaces. We assert that

\[
\lim_{t \to \infty} \sup_{f: f \neq 0, f \in X} W_{SP}(X, G(\nu); t) = C_{SP}^{(1)} > 0. \tag{5.1}
\]

**Theorem 5.2.** Let \( X = L_1(R^d) \) and \( Y = L_\infty(R^d) \) be two examples of \( G(\Psi) \) spaces. We assert that

\[
\lim_{t \to \infty} \sup_{f: f \neq 0, f \in G(\psi)} V_{SR}(X, Y; t) = C_{SR}^{(2)} > 0. \tag{5.2}
\]

**Proof of theorem 5.1.**

1. It is sufficient to consider here in the problem (P) only the case if equation
(1.9) has a view

\[
\frac{\partial u}{\partial t} = 0.5 \sum_{k=1}^{d} \frac{\partial^2 u}{\partial x_k^2}. \tag{5.1}
\]

2. Let us consider the following function (multidimensional normal density) for
the values \( x = \vec{x} \in R^d, \sigma = \text{const} > 0 \):

\[
g(x) = g_\sigma(x) = (2\pi \sigma)^{-d/2} \exp \left( -\|x\|^2/(2\sigma^2) \right). \tag{5.2}
\]

We get after direct calculation for \( q \in [1, \infty] \):

\[
|g|_q \asymp C(d) |\sigma|^{-d(\frac{1}{2} - \frac{1}{q})}. \tag{5.3}
\]

3. Let \( \nu \in G(\Psi), \Psi \in (a,b) \) and \( f(x) = g(x) \). We have:

\[
||f||X = |g|_1 = 1; \quad \phi(X; \delta) = \delta, \delta > 0;
\]

therefore

\[
\phi \left( X; t^{d/2} \right) = t^{d/2}.
\]

4. As long as the solution of the equation (5.1) has a view:

\[
u = u(t, x) = g_{1+t}(x),
\]

we have for sufficiently greatest values \( t, t > 2 \) and \( r \in (a,b) \):

\[
|u|_r \asymp t^{\frac{d}{2}(\frac{1}{r} - 1)};
\]
\[ \|u\| G(\nu) \leq \sup_{r \in (a,b)} \left[ \frac{t^{d/2} (\frac{1}{r} - 1)}{\nu(r)} \right] / \nu(r) = t^{-d/2} \phi \left( G(\nu); t^{-d/2} \right) . \]

5. Substituting into the expression for \( W_{SP}(X, G(\nu); t) \), we conclude that for \( t \geq 2 \) and \( f = g \)

\[
W_{SP}(X, G(\nu); t) \geq C \left[ \frac{\|T_t g\|_Y}{\phi(Y, t^{d/2})} : \frac{\|g\|_X}{\phi(X, t^{d/2})} \right] \geq C \left[ t^{-d/2} \phi \left( G(\nu); t^{-d/2} \right) \right] / \phi(Y, t^{d/2}) = C \left[ \frac{1}{t^{d/2}} \right] = C > 0.
\]

**Proof of theorem 5.2.** We choose as a function \( f(x) = u(0, x) \) again the function \( f(x) = g(x) \) and obtain:

\[ u = u(t, x) = g_{1+it}(x). \]

Note that the formula (5.3) remains true for the complex values \( \sigma, |\sigma| \geq 2 \).

We have:

\[ |g|_1 = 1, \ |g|_\infty \leq C, \]

\[ |U_t g|_1 \leq C, \ |U_t g|_\infty \leq t^{-d/2}, \]

and at \( r \in (1, \infty) \)

\[ |U_t g|_r \geq t^{-d\left(\frac{1}{2} - \frac{1}{r} \right)}. \]  \hspace{1cm} (5.4)

Therefore at \( t > 2 \)

\[ V(L_1, L_\infty; t) \geq C \frac{t^{-d/2} \cdot t^{-d/2}}{t^{-d}} = C > 0, \]

QED.

6. Concluding remarks.

A. Mix estimations.

Let for \( T > 0 \) \( S = S_T = (0, T) \) and \( \theta = \theta(\cdot) = \theta(t) \), \( t \in S_T \in \Psi(A, B) \), \( 1 \leq A < B \leq \infty \). We denote the norm on the space consisting on all the measurable functions \( h : S_T \to R G(\theta) \) as \( ||| h ||| G(\theta) \) and introduce the so-called mix norm as

\[ ||| T_t f ||| G(Y) \times G(\theta_T) \overset{def}{=} ||| u(\cdot) ||| G(Y) ||| G(\theta_T). \]
Theorem 6.1. Let $X, Y$ be m.r.i. spaces such that $\text{msupp}(X) \ll \text{msupp}(Y)$. It follows from the theorem 3.1 that

$$||| u(\cdot, \cdot)||| G(Y) \times G(\theta_T) \leq C ||| f ||| X, \frac{\phi(Y, t^{d/2})}{\phi(X, t^{d/2})} ||| G(\theta_T), \quad (6.1)$$

if of course the last norm $||| \cdot |||$ is finite.

B. Generalizations.

Let us consider some generalization of Schrödinger equation of a view:

$$\partial_t u + (-\Delta)^{\alpha/2} u = 0, \quad 0 < \alpha \leq 2. \quad (6.3)$$

$$u(0, x) = f(x),$$

the so-called dispersive equation, non-local diffusion equation or model of Keller - Segel, see [4], [9].

We denote the (unique) solution of (6.3) as $u = S_\alpha(t) f$.

In the article [4] is proved the estimation for $u(t, x)$ of a view:

$$|S_\alpha(t) f|_r \leq Ct^\frac{\alpha}{2} (\frac{1}{r} - \frac{1}{p}) |f|_p, \quad (6.4)$$

$1 \leq p \leq r \leq \infty$, and

$$|\Delta S_\alpha(t) f|_r \leq Ct^\frac{\alpha}{2} (\frac{1}{r} - \frac{1}{p}) - \frac{1}{\alpha} |f|_p. \quad (6.5)$$

We conclude repeating the proof of theorem 3.1 and using the inequality 6.4:

Theorem 6.2 We have under the condition of theorem 3.1

$$\sup_{f \in X, f \neq 0} \sup_{t > 2} \left( \left[ \frac{||S_\alpha(t) f||_Y}{\phi(Y, t^{d/\alpha})} \right] : \left[ \frac{||f||_X}{\phi(X, t^{d/\alpha})} \right] \right) = C_1(\alpha, X, Y) < \infty. \quad (6.6)$$

The inverse assertion to the theorem 6.2 is also true in the following sense:

Theorem 6.3. Let $\alpha = 2$, $X = L^1(R^d)$ and $Y = G(\nu)$, $\nu$ is arbitrary function from the space $\Psi(a, b) : \nu(\cdot) \in \Psi(a, b), a > 1, a < b \leq \infty$ be two examples of r.i. spaces. We assert that

$$\lim_{t \to \infty} \sup_{f \neq 0, f \in X} \left( \left[ \frac{||S_2(t) f||_Y}{\phi(Y, t^{d/2})} \right] : \left[ \frac{||f||_X}{\phi(X, t^{d/2})} \right] \right) = C_{(2), SR}(X, Y) > 0. \quad (6.7)$$

The proof used the inequality (5.4) and is completely alike to the proof of the theorem 5.2. For instance, we can choose instead the function $f$ the function $f(x) = g(x)$ etc.

C. Derivatives.
**Theorem 6.4** We have under the condition of theorem 3.1 using the estimation (6.5)

\[
\sup_{f \in X, f \neq 0} \sup_{t > 2} t^{1/\alpha} \left[ \frac{\| \Delta S_\alpha(t) f \|_Y}{\phi(Y, t^{d/\alpha})} : \frac{\| f \|_X}{\phi(X, t^{d/\alpha})} \right] = C_3(\alpha, X, Y) < \infty. \quad (6.8)
\]

This estimations (6.8) is exact as in the theorem 6.3.

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