Effects of magnetic field induced chiral-spin interactions on quasi-one-dimensional spin systems

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It is known that in certain non-bipartite quasi-one dimensional spin systems in a magnetic field, in addition to the usual Pauli coupling of the spins to the field, new parity breaking three spin interactions, i.e. chiral spin interactions, are induced at higher order due to virtual processes involving the intrinsic electronic nature of the underlying spins. The strength of these interactions depend strongly on the orientation of the field, a feature which can be exploited to detect chiral effects experimentally. In many spin systems, these chiral interactions are generated and should be taken into account before any comparison with experiments can be made. We study the effect of the chiral interactions on certain quasi-one dimensional gapped spin half systems and show that they can potentially alter the physics expected from the Pauli coupling alone. In particular, we demonstrate that these terms alter the universality class of the C-IC transition in spin-tubes. More interestingly, in weakly coupled XX zig-zag ladders, we find that the field induced chiral term can close the singlet gap and drive a second order transition in the non-magnetic singlet sector, which then manifests itself as a two component Luttinger liquid-like behaviour in the spin correlation functions. Finally, we discuss the relevance of our results to experiments.

I. INTRODUCTION

Quasi-one dimensional quantum spin systems have long been known to exhibit a spectrum of behaviours, ranging from the gapless critical behaviour in spin-$\frac{1}{2}$ integer systems, to gapped spin Peierls systems and Haldane phases in integer spin systems. The effect of an external magnetic field is usually described through the Pauli coupling ($\mathbf{h} \cdot \mathbf{S}$) of the spins to the field. These fields can have dramatic effects on the spin systems. For example, in gapped spin systems, the magnetic field induces the commensurate-incommensurate (C-IC) transition, where the IC phase is characterized by gapless spin excitations described by a Luttinger liquid and the corresponding spin correlation functions exhibit power law behaviours whose exponents, however, vary from system to system. However, it is important to remember that in real materials, the spins are electrons whose charge degrees of freedom are frozen by strong electron-electron interactions. In a physical system, the magnetic field couples not only to the spin degrees of freedom, via the Pauli term, but also to the orbital degrees of freedom of the electrons. This results in non-trivial additions to the effective spin hamiltonian. In fact, it is known that in the limit of large Coulomb repulsion, this coupling can in principle induce a three spin chiral interaction between neighbouring spins. Such a term arises in two dimensional systems on the triangular and Kagome lattices. These chiral terms also arise in certain quasi one dimensional systems like zig-zag ladders, frustrated ladders, spin tubes, saw tooth chains, etc, whose geometries permit the formation of a closed plaquette involving three neighbouring spins. The possible generation of such chiral terms in quasi-one dimensional spin systems stemming from the underlying electronic nature of realistic models has been systematically overlooked in the literature. In this paper, we show that these chiral terms are indeed relevant and can change the behaviour of certain spin systems like spin-tubes and zig-zag ladders in magnetic fields. The possible effects engendered by these terms should be studied before sensible comparisons with experiments in a magnetic field can be made. Another important motivation is that the effects of these chiral interactions depend strongly on the direction of the applied field, and could be detected experimentally in measurements of various quantities ranging from specific heat, NMR rates and dynamical structure factor measurements.

One class of systems, where the chiral term will surely be relevant are systems which support chiral excitations in zero field. Examples are the recently studied zig-zag spin $s$ ladders, where certain parity breaking interactions induce spin nematic ground states in zero magnetic field and spin tubes, whose ground states have chiral properties in the absence of a magnetic field. These spin tubes were also shown to exhibit an interesting C-IC transition where the IC phase is described by a two component Luttinger liquid describing gapless spin and chiral excitations. In these systems, the new field induced chiral term can completely alter the physics expected from the Pauli coupling alone. Normally, since we do not know of physical probes which couple directly to the chirality, it is quite hard to perceive experimentally the chiral aspects of the ground state. However, the competition between the chiral term and the
Pauli coupling of the spins to an external magnetic field can be exploited to obtain experimental signals of chirality in certain systems. The main objective of this paper is to study the influence of these chiral terms on the gapless incommensurate phases of certain spin systems. These are especially pertinent given the surfeit of experiments on spin systems in strong magnetic fields.

The paper is organised as follows: we first outline the derivation of the chiral spin term starting from the Hubbard model. We then study the effect of this term on the spin-tube and other quasi-one dimensional systems, in the limit of strong planar interchain couplings. We then discuss the very interesting example of the effects of the chiral interaction on the weakly coupled zig-zag ladder, where the magnetic field induced chiral interaction triggers a transition in the spin singlet sector. Finally, we conclude with a discussion of the experimental consequences of this interaction.

II. CHIRAL INTERACTION

In the presence of an external magnetic field, it is customary to only consider the Pauli interaction. However, this completely ignores the coupling of the field to the orbital part of the motion which can result in non-trivial corrections to the effective spin hamiltonian. We start by considering a system of interacting electrons described by a Hubbard model

\[ H = - \sum_{\langle ij \rangle} t_{ij} (c_i^\dagger c_j + h.c.) + U \sum_i n_{i\uparrow} n_{i\downarrow}, \]

where the first term describes the hopping of electrons with spin \( \sigma \) from sites \( i \) to \( j \) and the second, the on-site Coulomb repulsion between electrons. The Pauli coupling of the electronic spins \( \mathbf{S} \) to the external magnetic field \( \mathbf{h} \) is given by \( \mu_0 \mathbf{h} \cdot \mathbf{S} \), where \( \mu_0 = |e|\hbar/2mc \) is the Bohr magneton and the gyromagnetic ratio \( g \approx 2 \). Hereafter we set \( \gamma_0 = 1 \). The magnetic field also couples to the electronic motion via the hopping matrix elements \( t_{ij} \), which become complex, i.e., it picks up a phase \( t_{ij} \to t_{ij} \exp[ie \int_j^i A \cdot d\mathbf{r}] \), where, \( A \) is the vector potential associated with the external field \( \mathbf{h} \). If the electron hops around a closed loop, we immediately see that the net hopping matrix element is proportional to the magnetic flux enclosed by the loop. Since an external magnetic field explicitly breaks time reversal invariance, it is interesting to study whether time reversal violating terms are generated in the spin hamiltonian in the large \( U \) limit. At second order in perturbation theory, no parity breaking terms are generated and one obtains only the Heisenberg term \( \mathbf{S}_i \cdot \mathbf{S}_j \). Chiral terms, can therefore, be obtained only in higher order perturbation theory. An example of a chiral term which respects all spin symmetries and yet violates time reversal and parity, is the triple product of three spins \( \mathbf{S}_i \cdot \mathbf{S}_j \times \mathbf{S}_k \). Is such a term generated within the Hubbard model with nearest neighbour hopping \( t \) in the presence of an external magnetic field \( \mathbf{h} \)? At half filling, it can be shown that no parity and time-reversal violating terms are induced on bipartite lattices. However, on non-bipartite lattices, such a term is explicitly generated at order \( U^{-2} \) in perturbation.

\[ H_{\text{chiral}} = \frac{24t^3}{U^2} \sin(e\Phi) \sum_{\langle ijk \rangle} \mathbf{S}_i \mathbf{S}_j \times \mathbf{S}_k \]

where the summation is over sets of nearest neighbour sites which form triangles. We have restricted ourselves to nearest neighbour hopping \( t \) in the above. \( \Phi \) is the flux enclosed by the plaquette formed by the sites \( i, j \) and \( k \). In terms of the magnitude of the field \( h \), \( \Phi = Ah \cos(\theta) \) where \( A \) is the area of the plaquette and \( \theta \) is the angle made by the magnetic field with respect to the normal to the plane of the plaquette. Substituting this in \( (2) \), we see that no chiral term is generated for magnetic fields applied parallel to the plane of the plaquette. Additionally, the strength of the chiral term varies from a minimum to a maximum as a function of the direction of the applied field. For real materials, the typical area enclosed by these triangles and hence the enclosed flux being very small, the \( \sin(e\Phi) \) term in \( (2) \) can be expanded to yield

\[ H_{\text{chiral}} = \frac{24t^3}{U^2} eAh \cos \theta \sum_{\langle ijk \rangle} \mathbf{S}_i \mathbf{S}_j \times \mathbf{S}_k \]

Since the chiral term is generated at higher order, its strength is smaller than the Heisenberg exchange as well as the Pauli interaction. Despite this, it was shown \( (3) \) that in certain cases the chiral term may have a larger effect on the ground state than the Pauli coupling of electron spins to the external magnetic field, thereby drastically affecting the ground state properties of the model. One example is the case where the ground state of the system in zero field is doubly degenerate with opposite chiralities. Then, when an external magnetic field is applied, in addition to the
effects of the Pauli term, the chiral degeneracy is broken by (3). The effect could be more pronounced in the case where the ground states are degenerate chiral singlets because here, though the chiral term still breaks the degeneracy, the Pauli interaction term has no effect on the ground state for fields smaller than the singlet-triplet gap.

Since, no chiral interactions are generated for fields applied parallel to the plane of the triangular plaquettes, the behaviour of the system could, in principle, be different depending on whether the field is applied parallel or perpendicular to the plane of the system. This is especially relevant for the critical behaviours in the IC phase, where exponents could show a variation with the field direction. Such a behaviour, provides us with a physical handle for determining whether an isotropic spin system is quasi-one dimensional or not i.e., if the behaviour is different for the two directions of the field, one can presume that this difference stems from chiral interactions which necessarily require the system to be at least quasi-one dimensional!

III. STRONGLY COUPLED SPIN TUBES AND LADDERS

In this section, we discuss the effects of the chiral term on the IC phase of certain strongly coupled spin-gap systems, whose geometry allows for the existence of chiral terms in a magnetic field. In the absence of chiral interactions, the behaviour of these systems in the IC phase, has been well studied and catalogued in the literature. As we will show below, the chiral term does not necessarily alter the nature of the incommensurate phase always. The effects of the chiral interaction are drastic or trivial depending on the nature of the ground and excited states of the system.

A. Spin tubes

The first example we consider is the strongly coupled spin tube (i.e. a three-leg ladder with periodic boundary conditions) studied in Ref. 3. As already mentioned spin tubes, exhibit a new class of C-IC behaviour. Starting from the limit of strong-rung coupling (see Fig 3), the low-energy effective Hamiltonian (LEH) describing the gapless incommensurate phase with finite magnetization was derived in Ref. 4. This low energy Hamiltonian was that of a two-component Luttinger liquid describing gapless spin and chiral excitations in the system. The existence of a two-component Luttinger liquid phase has important physical consequences. Namely, it implies that both chiral and magnetic excitations contribute to a non-zero magnetic susceptibility and a strongly enhanced $T$-linear specific heat. This different behaviour stems basically from the presence of a degenerate ground state with different chiralities, within an isolated triangle. This immediately poses the interesting question, what is the effect of the chiral interaction (3) on the two component Luttinger liquid behaviour? Clearly, we expect the chiral term (3) to induce dramatic changes, because it will break the degeneracy of the ground state. To see this, we derive the low-energy effective Hamiltonian for the spin tube in the presence of the chiral interaction (3). We will use this effective theory to discuss the physics of the IC phase and the resulting spin-spin correlation functions.

The Hamiltonian of the spin tube in the presence of an external magnetic field $h$ and a spin-chiral interaction is given by,

$$H = \sum_{i=1}^{N} \sum_{p=1}^{3} (JS_{i,p} \cdot S_{i+1,p} + J_{\perp} \cdot S_{i,p} \cdot S_{i,p+1} - h \cdot S_{i,p}) + \mu_{c}h \sum_{i=1}^{N} S_{i,1} \cdot (S_{i,2} \times S_{i,3}), \quad (4)$$

where $(i, p)$ denote the site and chain indices respectively and periodic boundary conditions imply that the site $(i, 4)$ is identified with the site $(i, 1)$. The first two terms represent the spin Hamiltonian for the spin tube, the third the Pauli coupling of the spins to the magnetic field and the last term is the chiral term derived earlier. $J$ is the coupling along the chain, $J_{\perp}$ the transverse coupling between the chains and, $\mu_{c} = \frac{eA}{2mc} \cos \theta$ is the strength of the spin-chiral interaction. As in Ref. 3, we consider the limit $J_{\perp} \gg J$. In this limit, the properties of the system can be studied by perturbing in $J$ around the limit of decoupled triangles. For $J = 0$, the system consists of independent triangles. The eight states of a given triangle split into two spin-$\frac{1}{2}$ doublet ground states, with energy $E_{s} = -3J_{\perp}/4$ and an excited spin-$\frac{3}{2}$ quadruplet, with energy $E_{t} = 3J_{\perp}/4$. The spin doublet ground state has chiral eigenvalues $\pm$. For strong magnetic fields applied parallel to the plane of the triangle, no chiral terms are generated. Consequently, the degeneracy in the ground state is only partially lifted and one reverts back to the case studied in Ref. 3. On the other hand, for fields perpendicular to the plane of the triangle, the ensuing Pauli and the spin-chiral terms lift all the degeneracies. For instance, for the spin doublet states, the Pauli and chiral terms contribute $\pm h/2$ and $\pm \mu_{c} \sqrt{3}/4$ respectively, to the energy. Note, however, that the chiral term has no effect on the ferromagnetic spin $\frac{3}{2}$ excited
states. This is expected since the chiral term, which is nothing but a triple product, has a non-zero expectation value only for non-collinear spin configurations. Therefore, for the field applied in the $z$ direction, the resulting energies are

$$
\tilde{E}_i = E_i - h m_{3/2}
$$

$$
\tilde{E}_s = E_s - h m_{1/2} + \frac{\sqrt{3} \mu_c h}{4}
$$

Here $m_s$ just refers to the $S_z$ value of the state considered. There exists a critical field $h_c$, such that for $h < h_c$ the ground state has magnetisation $m = \frac{1}{2}$ and for $h > h_c$ the ground state has $m = \frac{3}{2}$. In the absence of chiral interactions, the critical field $h_c$ required to close the gap is given by $h_c = E_1 - E_s = 3 J_s / 2$. However, in the presence of chiral interactions, the critical field is slightly increased and is given

$$
h_c = \frac{(E_1 - E_s)}{1 - \sqrt{3} \mu_c} = \frac{3 J_s}{2 - \sqrt{3} \mu_c}
$$

(5)

We therefore see that the chiral interaction changes the magnitude of $h_c$ in a subtle way. Note also that the magnitude of the critical field depends on the direction of the applied field. It is minimum for a field applied in the plane of the triangle i.e., $x$ direction where $h_c = 3 J_s / 2$ and is maximum for the field in the $z$ direction. A small non-zero value of the coupling $J$, broadens the abrupt transition from a state with $m = \frac{1}{2}$ to $m = \frac{3}{2}$. This results in two critical fields $H_{c1}$ (where the $m = \frac{1}{2}$ state stops being the ground state) and $H_{c2}$ (where the magnetisation saturates at $m = \frac{3}{2}$) corresponding to a ferromagnetic state) with the magnetisation varying smoothly in between. The difference $H_{c2} - H_{c1}$ is of the order of the coupling $J$. We now write an effective Hamiltonian describing the low energy physics of the spin tube close to the critical magnetic field. Since at $h_c$ the $m_s = \frac{3}{2}$ state becomes degenerate with only one of the chiral states with $m_s = \frac{1}{2}$ we retain only these two states per site:

$$
\alpha_{3/2} = | \uparrow \uparrow \uparrow \rangle
$$

$$
\alpha_{1/2} = \frac{1}{\sqrt{3}} \left[ | \downarrow \uparrow \uparrow \rangle + j | \uparrow \downarrow \uparrow \rangle + j^2 | \uparrow \uparrow \downarrow \rangle \right]
$$

(7)

where $j = \exp(2 \pi i / 3)$. We now introduce a pseudo-spin half operator $T$ which acts on the subspace containing the above two states. By inspection, we can rewrite the original spin operators $S$ in terms of $T$ i.e.,

$$
S_{i,p}^+ = \frac{j^{p-1}}{\sqrt{3}} T_i^+
$$

$$
S_{i,p}^- = \frac{1}{3} (1 + T_i^z)
$$

(9)

Using this, the effective Hamiltonian, to first order in the coupling between the triangles is now given by

$$
H_{eff.} = \frac{J}{2} \sum_{i=1}^{N} (T_i^+ T_{i+1}^{-} + T_i^- T_{i+1}^{+}) + \frac{J}{3} \sum_{i=1}^{N} T_i^z T_{i+1}^z - h_{eff.} \sum_{i=1}^{N} T_i^z
$$

(10)

where $h_{eff.} = h - h_c - \frac{2 J}{3}$ and contains the effects of chirality. The previous derivation is equivalent to that of the low-energy effective Hamiltonian for three coplanar spin-$\frac{1}{2}$ chains considered in Ref. [3]. The model (10) is the simple XXZ chain in a magnetic field. Since the coefficient of the $z$ component of the interaction is smaller than that of the transverse components, we see that the above system corresponds to that of the XXZ chain in its gapless phase. A lot is known about the model and results for the exponents of various correlation functions as a function of the applied field can be obtained from Ref. [3].

Using the Jordan-Wigner transformation we fermionize (10), to obtain

$$
H_{eff.} = \frac{J}{2} \sum_i [c_i^+ c_{i+1} + h.c.] + \frac{J}{3} \sum_i n_i n_{i+1} - \mu \sum_i n_i + \text{const.}
$$

(11)

$\mu = h - h_c - J/3$. The values of the critical fields $H_{c1}$ and $H_{c2}$ for the spin-tube can now be obtained from the effective model (11). When $h = H_{c1}$, the band of spinless fermions is empty. This occurs when the chemical potential $\mu$ equals the band minimum i.e., when $\mu = -J$ which in turn yields $H_{c1} = h_c = 2 J / 3$. Similarly, $H_{c2}$ corresponds to a completely filled band of electrons or equivalently an empty band of holes. The simplest way to describe this
situation is to perform a particle-hole transformation on the Hamiltonian (11): $c_i^+ \rightarrow h_i$. Up to a constant, the new Hamiltonian reads:

$$H_{\text{eff}}^h = -\frac{J}{2} \sum_i [h_i^+ h_{i+1} + \text{h.c.}] + \frac{J}{3} \sum_i n_i^h n_{i+1}^h - \mu h \sum_i n_i^h$$  \hspace{1cm} (12)

where the chemical potential $\mu h = -\mu + 2J/3$. In terms of holes, $H_{\text{eff}}$ corresponds again to the chemical potential where the band stars to fill up, thus we find $H_{\text{eff}} = h + 2J/3$.

The Hamiltonian (11) can be easily bosonized to obtain the various correlation functions and exponents in the incommensurate phase. Using the standard expressions for bosonization

$$T_\pm(x) = -\frac{1}{\pi} \phi + (-1)^{\pi} \frac{c_\phi(x) + 2}{\pi a}$$  \hspace{1cm} (13)

$$T_\pm(x) = \frac{1}{\sqrt{2\pi a}} e^{-i\theta} \left[ (-1)^{\mp} + \cos 2\phi(x) \right]$$  \hspace{1cm} (14)

we obtain the following continuum theory

$$\mathcal{H} = \frac{1}{2\pi} \int dx [uK(\pi \Pi)^2 + \left( \frac{u}{K} \right) \partial_x \phi] + \frac{1}{\pi} \int dx h_{\text{eff}} \partial_x \phi,$$  \hspace{1cm} (15)

where $\Pi$ is the momentum conjugate to the field $\phi$ where $u = \frac{\pi \sqrt{3J^2 - J^2}}{2 \cos^{-1}(\frac{J}{3})}$ is the spin wave velocity, and

$$2K = [1 - \frac{1}{\pi} \cos^{-1}(\frac{J}{J})]^{-1}$$  \hspace{1cm} (16)

This indicates clearly that in the presence of chiral interactions, the spin tube in the IC phase, behaves as a one-component Luttinger liquid, describing spinless excitations alone. In the absence of the chiral term, both spin and chiral excitations were gapless which then led to the two component Luttinger liquid behaviour. Here, we see that the effect of the chiral term is to open a gap for the chiral excitations, which then do not contribute to the low energy behaviour. The magnetic field term, described by the gradient in (15), can be eliminated by a simple shift of the $\phi$ field i.e., $\phi \rightarrow \phi + \pi mx$ where the magnetisation $m \propto h_{\text{eff}} K/u$. For small values of the field $h_{\text{eff}}$, the effective magnetisation $m$ increases linearly with $h$. This field shift results in the appearance of incommensurate modes in the spin-spin correlation functions. Note that $K$ varies from $K = 1$ for the case of the XX- antiferromagnet to $K = \frac{1}{2}$ for the isotropic XXX system. For the present problem, $K = J/3$ resulting in $K = 0.83$ and $u = 1.2J$. It is well known that for this value of $K$ the system is gapless and lies in the XXZ universality class. However, due to presence of marginal operators in the theory, the Luttinger parameter $K$ now varies with the magnetisation. This variation depends also on the anisotropy $J_\perp/J = 1/3$ and can be obtained numerically using the Bethe ansatz equations of Ref. 8.

In the critical incommensurate (IC) region characterized by a finite magnetisation, we can easily calculate the various correlation functions of the effective spin chain and hence that of the original spin tube

$$\chi_p^{zz} = \langle S_p^z(x,t)S_p^z(0,0) \rangle$$  \hspace{1cm} (17)

$$\chi_p^{+-} = \langle S_p^+(x,t)S_p^-(0,0) \rangle$$  \hspace{1cm} (18)

where $p$ is the chain index and $\pm$ refers to the intra and interchain correlation, respectively. From (17), we see that the magnetisation $\tilde{m}$ in the spin tube is related to the magnetisation of the effective system by $\tilde{m} = (1 - m)/3$. The first correlation function is useful for neutron scattering experiments, whereas the correlation function (18) defines the staggered susceptibility that is useful for the calculation of NMR relaxation rates. Using (17) which relate the original spins to the pseudo-spin $\frac{1}{2}$ variables, we obtain

$$\chi_p^{zz} = \frac{1}{9} \left[ T^z(x,t)T^z(0,0) \right]$$  \hspace{1cm} (19)

$$\chi_p^{+-} = \frac{1}{3} \left[ T^+(x,t)T^-(0,0) \right]$$  \hspace{1cm} (20)

which in turn leads to the following results.\[1]
\[ \chi_p^{\pm} \approx \frac{1}{9}(1-m)^2 + \frac{1}{9}\cos \pi x(1-2m)(x^2-u^2t^2)^{-\frac{1}{12}} + \text{const.} \cos(2\pi mx)(x^2-u^2t^2)^{-\frac{(1-K)}{12}}. \]  

\[ \chi_p^{\pm} \approx \frac{1}{3} \left[ (-)^{n/2}(x^2-u^2t^2)^{-\frac{1}{12}} + \text{const.} \cos(2\pi mx)(x^2-u^2t^2)^{-\frac{(1-K)}{12}} \right]. \]

where, \( u \) is the spin mode velocity. The correlation functions so found show a power-law decay in space and time. As anticipated, we see that two incommensurate modes appear at the wave vectors \( Q = 2\pi(1-m) \), for \( \chi^{zz} \) and \( Q = 2\pi m \) for \( \chi^{\pm} \), resulting in a different behaviour in the plane perpendicular to the field and along the field. The exponents of these power-laws now vary with the magnetisation \( m \). These can be obtained by numerically solving the Bethe ansatz equations of Ref. [3]. As the magnetisation \( m \to 0.5 \) or equivalently when the average number of fermions goes to 1, the fermion band is completely filled and we expect to recover the physics of free fermions whose \( \chi^{zz} \approx 1 \).

The behaviour found for the spin-spin correlation functions in the present case, is wholly different from that found in Ref. [2].

\[ \chi_p^{\pm} \approx \left( \frac{1}{x^2-(u_a t)^2} \right)^{K_a + \frac{1}{12}} \cos(Q_1(x,t)) + \cos(Q_2(x,t)) \left( \frac{1}{x^2-(u_a t)^2} \right)^{\frac{3K_a}{2a}} + \cos(Q_3(x,t)) \left( \frac{1}{x^2-(u_b t)^2} \right)^{\frac{3K_b}{2b}}. \]

Here, \( u_a, u_b \) are the velocities of the spin and chiral excitations and \( K_a, K_b \) are the Luttinger parameters associated with these excitations. \( Q_1, Q_2, Q_3, Q_4 \) are functions which fix the incommensuration. One of the primary differences between [23] and [22] is that the former shows an explicit dependence on two different velocities \( u_a \) and \( u_b \) related to the magnetic and chirality modes and exhibits incommensuration at many more wavevectors. These velocities were different and in addition to the sum of the power law correlations exhibited by the two components, there is an interference between the two which contributes a third term with a different exponent in the correlation function for the spins. The two components will be distinguished when an extra field which couples to the spin current in the transverse direction is applied.

To summarize, when the field is applied parallel to the plane of the triangles, no chiral interaction is induced and the system exhibits the two component Luttinger behaviour of [23]. However, when the field is applied along the length of the tube i.e., perpendicular to the plane of the triangles, the chiral interaction plays a dominant role and we recover the single component behaviour indicated in [22]. The exponents of the various power laws in the correlation functions will also be different in the two cases. Moreover, the value of the critical fields \( H_{c1} \) and \( H_{c2} \) are also different for the different field directions. These predictions are strong signatures of the chiral term and they could be verified experimentally. Moreover, in the past there have been a lot of discussions of how to experimentally establish the existence of chirality in systems. In spin tubes and other systems, the change in critical behaviour can be perceived as an indirect test of such chiral exchanges in realistic systems.

1. Frustrated spin ladders

We have also analysed the effect of these chiral interactions on certain strongly coupled frustrated ladders considered by Mila[1] and shown in Fig 4. In all of these cases, we find that the chiral term has no effect whatsoever on the C-IC physics seen in these systems. This is due to the fact that in each of the examples considered, the unperturbed ground state (equivalent of the spin \( \frac{1}{2} \) states on the triangles in the spin-tube) has no interesting chiral properties i.e., they have no low lying chiral modes. As shown above, we introduce pseudo-spin operators to derive the relevant effective hamiltonians close to \( h_c \). In all these models, it turns out that the chiral term is effectively zero when written in terms of these operators. Consequently, the physics of these strongly coupled frustrated ladders is completely unaffected by the chiral term.

IV. WEAKLY COUPLED ZIG-ZAG LADDERS

In the previous section, we considered the effect of the chiral interaction on the spin-tube as an example of a strongly coupled quasi-one dimensional systems. Here, the possible non-trivial effects generated by the chiral term could be
deduced by merely studying the the chiral properties of the relevant “unperturbed” ground state in the absence of the external field. Such an approach fails for weakly coupled spin systems like the zig-zag chains and frustrated ladders. Here, to begin with, one lacks a schematic picture of the ground state(s) which is highly correlated. Nevertheless, in such cases, one can use bosonization methods to study the effects of the chiral terms on the long wavelength physics of these systems. Here, we consider the example of the zig-zag ladder, where the chiral terms are relatively simpler than those generated in frustrated ladders. We will restrict ourselves to the limit of very strong anisotropy that corresponds to two coupled XX chains. (We assume that the anisotropies are generated by spin-orbit couplings, which do not affect the form of the chiral interaction.) The XX zig-zag ladder was studied in Ref. 3 using a mean-field treatment of the bosonized hamiltonian. Here, we analyze the effect of the chiral term on the mean-field solution via a similar treatment.

The zig-zag ladder (Fig. 3) has spins \( S \) on chain 1 and spins \( T \) on chain 2. The bosonized hamiltonian for the zig zag ladder has been derived by Nersesyan et al. The bosonic fields \( \phi_1 \) and \( \phi_2 \) and their duals \( \theta_1 \) and \( \theta_2 \) are introduced to describe the spins in the two chains, \( S \) and \( T \). In terms of the the symmetric (antisymmetric) combinations, \( \phi_{s,a} = (\phi_1 \pm \phi_2)/\sqrt{2} \) and \( \theta_{s,a} = (\theta_1 \pm \theta_2)/\sqrt{2} \) which describe the triplet (singlet) sector, the hamiltonian describing the long wavelength physics of the anisotropic XX zig-zag ladder is given by

\[
H = H_0 + H_{int} + H_{mag}
\]

where,

\[
H_0 = \sum_{\alpha = s,a} \frac{v}{2} \left[ (\partial_x \theta_\alpha)^2 + (\partial_x \phi_\alpha)^2 \right]
\]

\[
H_{int} = \gamma \partial_x \theta_s \sin \sqrt{2} \theta_a
\]

and the Pauli coupling of the spins to an external field \( h \) (which affects only the triplet sector) is given by

\[
H_{mag} = h \frac{\sqrt{2}}{\pi} \partial_x \phi_s
\]

The spin-wave velocity \( v \propto J_a \) and \( \gamma \) is the strength of the zig-zag coupling. Note that the interaction term in \( H_{int} \) is marginal. Due to the anisotropy, we only consider the case where the field is applied along the \( z \)-axis perpendicular to the plane of the zig-zag ladder. A magnetic field applied in the plane of the zig zag chain is very complicated, since even the simple Pauli coupling generates a gap in the triplet sector. In the absence of any field, the ground state of this model was shown to be a spin nematic with gapless triplet excitations and gapped singlet excitations. This state was also characterized by a power law decay of the staggered transverse spin correlations at an incommensurate wave vector \( Q = \pi - \delta \), where \( \delta \) depends on \( v \) and \( \gamma \). A Pauli coupling to the magnetic field in the \( z \) direction has no effect on the staggered part of the correlation and induces only a trivial incommensuration in the uniform part of the spin correlation function. To the hamiltonian \( H_{24} \) we now add the chiral interaction term \( H_{27} \) involving the spins on every triangular plaquette. It takes the form

\[
H_{chiral} = \mu_c h (S_i - T_{i+1}) \cdot T_i \times S_{i+1} \tag{28}
\]

for every pair of spins \( S_i, T_i \). In terms of the continuum fields,

\[
H_{chiral} = -\mu_c h \int dx \left[ \frac{1}{2\pi^2 a^2} \sin \sqrt{2} \theta_a(x)(1 + 2\sqrt{2}a\partial_x \phi_a) + \frac{1}{(2\pi)^2} (\partial_x \phi_a \partial_x \theta_s - \partial_x \phi_s \partial_x \theta_a) \right], \tag{29}
\]

We see that the first term in \( 29 \) is a relevant operator of dimension one that can have the effect of eliminating the gap in the singlet sector. The other terms are all marginal. The first two terms act exclusively on the singlet sector, whereas the last two terms couple the singlet and triplet sectors. Moreover, these terms will compete with \( H_{int} \) in \( 25 \). On physical grounds we speculate that the coupling of the chiral interaction to the singlet sector arises from the hidden chirality in the ground state. The presence of so many competing operators renders the problem highly complex. Here, since we are interested only in the qualitative effects of the chiral term, we restrict ourselves to a simple self-consistent mean-field analysis which leads us to the following Hamiltonian

\[
H_{MF} = H_s + H_a \tag{31}
\]
with
\[ H_s = \frac{v}{2} \left[ (\partial_x \theta_s)^2 + (\partial_x \phi_s)^2 \right] + (h + \beta L_3) \partial_x \phi_s + (\gamma K_1 - \beta K_2) \partial_x \theta_s, \tag{32} \]
\[ H_a = \frac{v}{2} \left[ (\partial_x \theta_a)^2 + (\partial_x \phi_a)^2 \right] + (\alpha_1 + \alpha_2 K_2 + \gamma L_2) \sin \sqrt{2} \theta_a + (\alpha_2 K_1 - \beta L_2) \partial_x \phi_a + \beta L_4 \partial_x \theta_a, \tag{33} \]
where the mean-field parameters
\[ K_1 = \langle \sin \sqrt{2} \theta_a \rangle, \quad K_2 = \langle \partial_x \phi_a \rangle, \tag{34} \]
\[ L_1 = K_2 = \langle \partial_x \phi_a \rangle, \quad L_2 = \langle \partial_x \theta_s \rangle, \tag{35} \]
\[ L_3 = \langle \partial_x \theta_s \rangle, \quad L_4 = \langle \partial_x \phi_a \rangle, \tag{36} \]
and \( \alpha_1 = -\mu_c h/2\pi^2, \alpha_2 = -\sqrt{2} \mu_c h/\pi^2, \beta = \mu_c/2\pi^2. \) The triplet channel \((s)\) can be solved easily by eliminating the terms linear in \(\theta_s\) and \(\phi_s\) through the field shifts: \(\theta_s \rightarrow \theta_s + (\gamma K_1 - \beta K_2)x/2,\) and \(\phi_s \rightarrow \phi_s + (h + \beta L_3)x/2.\) Analogously in the \(a\) channel, the gradient of \(\phi_a\) can be eliminated by a simple shift \(\phi_a \rightarrow \phi_a + (\alpha_2 K_1 - \beta L_2)x/2.\) Self-consistency of the solution then leads to the following relations
\[ K_2 = \frac{2\alpha_2 + \beta \gamma}{\beta^2 - 4} K_1, \quad L_2 = \frac{1}{2} (\beta K_2 - \gamma K_1), \tag{37} \]
\[ L_3 = -\frac{1}{2} (h + \beta L_3). \tag{38} \]

Here we have assumed that the ground state of the system is found in the sector with nonzero spin current \(\partial_x \theta_a.\)

From the very structure of the mean field Hamiltonian (33), we see that though triplet excitations remain gapless, the situation is not so clear for singlet excitations. The Hamiltonian in the singlet sector takes the form
\[ H_a = \frac{v}{2} \left[ (\partial_x \theta_a)^2 + (\partial_x \phi_a)^2 \right] + \mu \sin \sqrt{2} \theta_a + \beta L_4 \partial_x \theta_a \tag{39} \]
where the coefficient of the sine term which generates a gap is given by
\[ \mu = \alpha_1 + \frac{2(\alpha_2^2 + \alpha_2 \beta \gamma + \gamma^2)}{\beta^2 - 4} K_1, \tag{40} \]

Note that the structure of \(H_a\) is similar to that of systems with gapped triplet excitations in an external magnetic field (the equivalent of the “external” magnetic field in this case being the mean field parameter \(\beta L_4.\)) The bare gap in this case is \(\Delta \propto \mu^{2/3}.\)

From the structure of the mean field equations, we see that since \(\mu\) depends on the external magnetic field, the gap is a varying function of \(h.\) The field \(L_4\) tends to reduce this gap. Moreover, it is known that these systems exhibit the C-IC transition when the strength of the field attains a critical value which equals the magnitude of the gap. For fields greater than the critical field \(h_c\) (not estimated here), the system has gapless excitations at incommensurate values of the wave vector. We, therefore, expect within mean field two kinds of solutions: in the first one the gap induced by the sine term survives and, consequently, there is no net “magnetisation” i.e., \(\langle \partial_x \theta_a \rangle = 0.\) This implies that \(K_1 \neq 0\) and \(L_3 = 0.\) Using (38), we see that \(L_4 = -\mu_c/2\) in the gapped phase. The second solution describes the “incommensurate” phase with a finite “magnetisation” \(m_a\) i.e., \(L_3 = \langle \partial_x \theta_a \rangle = m_a.\) In this phase, the effective field in the singlet sector is given by \(L_4 = -\mu_c/(h + \beta m_a).\) One or the other solution will be energetically favoured as the field is tuned. In the gapless IC phase, the gradient in \(\theta_a\) can be absorbed into the quadratic part by the shift \(\theta_a \rightarrow \theta_a + \beta L_4 x/2;\) the singlet sector is now effectively a Luttinger liquid described by
\[ H_a = \frac{v_a(h)}{2} \left[ K_a(h) (\partial_x \phi_a)^2 + \frac{1}{K_a(h)} (\partial_x \theta_a)^2 \right] \tag{41} \]
where the new velocity and the Luttinger liquid parameters \(v_a, K_a\) both vary with the field. The results obtained earlier in the context of the normal C-IC transition (44) can be used to obtain \(K_a(h).\) In fact, \(K_a(h = h_c) = 2.\) In conjunction with the gapless spin excitations described by \(H_a,\) this results in a two component Luttinger liquid like behaviour! The possibility of such a transition, however, depends crucially on the mean field solutions. The calculation of the regime of validity of the two solutions requires the knowledge of the exact expectation value of the mass term \(\langle \sin \sqrt{2} \theta_a \rangle\) as a function of the prefactor of the “magnetic field” like term \(\partial_x \theta_a.\) This expectation value is
rather difficult to estimate and has only been calculated either for zero magnetic field or in the limit of fields larger than the gap opened by the sine-Gordon term i.e., in the gapless phase where $\Delta/\beta L^4 \ll 1$. Here, we assume that the chiral term has indeed driven the singlet sector across the transition into the gapless “incommensurate” phase and solve for the mean field equations. In this limit,

$$\langle \sin \sqrt{2}\theta_a \rangle = c\beta L_4 \mu$$

(42)

where $c$ is some constant and $\mu$ given by (40) is a function of the other mean field parameters. Using (42) and (36) we can now solve explicitly for $K_1$ to obtain

$$K_1 = \frac{\alpha_1}{f^{-1}(h) - \frac{2}{\beta^2-4}(\alpha_2^2 + \alpha_2 \beta \gamma + \gamma^2)}$$

(43)

$$\mu = \alpha_1 + \frac{2\alpha_1(\alpha_2^2 + \alpha_2 \beta \gamma + \gamma^2)}{f^{-1}(h)(\beta^2 - 4) - 2(\alpha_2^2 + \alpha_2 \beta \gamma + \gamma^2)},$$

where $f(h) = (ch)^{-1}$. We find that the mean field solutions are self-consistent with the limit $\Delta/\beta L^4 \ll 1$ used to obtain them. This limit translates to the condition $h \gg \gamma/\mu$, and allows for a finite $h_c$.

In conclusion, we have shown that, while the Pauli interaction with the field leaves the physics of the XX ladder completely unchanged, at the mean field level the chiral term radically alters the physics of the zig-zag chain. It generates a transition to a gapless regime in the singlet sector for large enough values of the magnetic field. The gapless singlet excitations manifest themselves in specific heat measurements and more importantly, in the spin correlation functions. A new peak appears in the specific heat as a function of the magnetic field. Such a transition in the singlet space will result in a change of the exponents of power laws in the transverse spin correlation functions, as we will see below.

A. The correlation functions and relevance to experimental quantities

We conclude the analysis of the effect of the chiral interactions on the zig-zag chain with the calculation of the spin-spin correlation functions. They could be useful for the neutron scattering experiments and the NMR relaxation rates measurements. From the previous mean-field solution we get the following shifts of the fields

$$\theta_s \rightarrow \theta_s + \frac{p K_1}{2v} x$$

$$\theta_a \rightarrow \theta_a + \frac{p'}{2v} x$$

$$\phi_a \rightarrow \phi_a + \frac{\delta K_1}{2v} x$$

$$\phi_s \rightarrow \phi_s + \frac{\delta'}{2v} x$$

where

$$p = \frac{\gamma(\beta^2 - 4) - \beta(\alpha_2 + \beta/2\gamma)}{\beta^2 - 4}$$

$$\delta = \alpha_2 + \frac{\beta v}{2}$$

for all values of the field $h$ and

$$p' = 0; \quad \delta' = h \quad \text{for} \quad h < h_c$$

$$p' = -\frac{1}{2}(h + \beta m_a); \quad \delta' = h + \beta m_a \quad \text{for} \quad h > h_c$$

In the C phase ($h < h_c$), with gapless triplet excitations and gapped singlet excitations, $\langle \theta_s \rangle = 0$ and $\theta_a$ has a non-zero expectation value determined by the position of the minima of the Sine-Gordon potential i.e., $\langle \theta_a \rangle = \sqrt{\pi/8}\text{sign}(\mu)$. This allows us to express the dual fields in the chains as

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\[ \theta_{1,2} = \frac{1}{\sqrt{2}} \phi^0(x) + \frac{pK_1}{2\sqrt{2}v}x \pm \frac{\sqrt{\pi}}{4}, \] 

Using (44), the staggered part of the transverse spin correlation functions are now given by

\[ \chi_{\pm}(x, t) \equiv \langle S_{1,2}^+ S_1^- \rangle \sim (-1)^{x/a} e^{-iBx(x^2 - v^2t^2)^{-\frac{1}{2}}} \] 

where \( B = \left( \frac{pK_1}{2\sqrt{2}v} \right) \). The transverse correlations still fall off as a power-law with exponent 1/4, but they are incommensurate with a field dependent characteristic vector:

\[ Q(h) = \pi - \left( \frac{pK_1}{2\sqrt{2}v} \right). \]

Comparing (45,44) with the results of Ref. 3, which corresponds to setting \( \mu_c = 0 \) in the above equations, we find that the exponents are the same in both cases and the only effect of the chiral term is to induce a magnetic field dependent incommensuration. Moreover, the Pauli term does not induce any additional incommensuration in the staggered transverse spin correlation functions. Therefore, a varying \( Q \) should serve as a good experimental probe of the chiral interactions.

In the gapless incommensurate phase of the singlet \((h > h_c)\), the singlet excitations are described by a Luttinger liquid [11]. In this case, both the gapless triplet and singlet excitations (which are probably chiral) contribute to \( \chi_{\pm} \). Here, \( \theta_\alpha \) also has a zero expectation value and the expression (44) is invalid in this phase. Instead,

\[ \theta_{1,2} = \frac{1}{\sqrt{2}} (\theta_a(x) \pm \theta_a(x)) + \frac{pK_1 \pm p'}{2\sqrt{2}v}x. \]

which leads to the following asymptotic behaviour of the transverse spin-spin correlation:

\[ \chi_{\pm}(x, t) \equiv \langle S_{1,2}^+ S_1^- \rangle \sim (-1)^{x/a} e^{-iB_{\pm}x(x^2 - v^2t^2)^{-\frac{1}{2}}} (x^2 - v^2(hc)t^2)^{-\frac{1}{2}} \]

where \( B_{\pm} = \left( \pm \frac{pK_1 \pm p'}{2\sqrt{2}v} \right) \). Note that \( v \) is the velocity of the gapless triplet excitations and \( v_\alpha(h) \) is the velocity of the gapless singlet excitations. This behaviour is indeed reminiscent of a two component Luttinger liquid. The equal-time transverse spin correlations on the two chains fall off as a power-law with exponent \( K(h) = (1 + K_\alpha(h))/4 \), but they are incommensurate with characteristic vectors:

\[ Q_{\pm} = \pi - \left( \pm \frac{pK_1 \pm p'}{2\sqrt{2}v} \right). \]

At the critical field \( h = h_c \), since \( K_\alpha(h = h_c) = 2 \), the spin correlation function

\[ \chi_{\pm}(x, t) \sim e^{iQ_{\pm}x(x^2 - v^2t^2)^{-\frac{1}{2}}} (x^2 - v^2(h_c)t^2)^{-\frac{1}{2}} \]

From (49) and (45), we see that the transition will manifest itself in the equal time spin correlation function as an abrupt change in the exponent i.e., it jumps from \( \frac{1}{4} \) to \( \frac{1}{2} \) and then varies slowly with increasing field. Such a dramatic change in the exponent could be easily seen in NMR and susceptibility measurements.

For strong zig-zag interactions \( \gamma \), we anticipate that the chiral term is irrelevant and that the singlet sector remains gapped. It remains to be seen whether the transition in the singlet space survives when fluctuations are taken into account. Also, these are complicated sine-Gordon theories and one should worry about solitons and kinks. An interesting question is whether such a physics survives in the isotropic case or when there is a finite \( J_z \) coupling.

**V. CONCLUSIONS**

Chiral interactions induced by the magnetic field appear in a variety of non-bipartite quasi one-dimensional spin systems. We have shown that these terms could dramatically alter the physics of systems such as strongly coupled spin tubes and weakly coupled zig-zag ladders. The effect of these chiral interactions depend strongly on the direction of the applied field and could be measured experimentally. In the case of spin-tubes in a magnetic field along the tube, we showed that these chiral terms change the critical behaviour in the incommensurate phase, from that of a two component Luttinger liquid (in the absence of chiral interactions) to the usual single component Luttinger liquid.
However, for fields perpendicular to the axis of the tube, no chiral terms are generated and one recovers the usual two component Luttinger liquid. On the other hand, these terms have no effect on certain strongly coupled zig-zag and frustrated ladders. For the weakly coupled zig-zag ladder, we showed that the chiral term does something entirely novel i.e., an applied magnetic field can drive a C-IC like transition in the non-magnetic singlet sector! This results in gapless singlet and triplet excitations described by a two component Luttinger liquid. It is very remarkable that the chiral term destroys the two component Luttinger liquid in spin tubes, but induces a novel two component Luttinger liquid behaviour in the weak zig zag ladder. These effects should be visible in measurements of various experimental quantities ranging from simple specific heat, NMR rates and dynamical structure factor measurements.

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FIG. 1. Cylindrical three-leg ladder (spin-tube).
FIG. 2. Examples of frustrated ladders in the strong-rung coupling limit.
FIG. 3. Zig-zag ladder in the weak coupling limit. The arrows indicate the closed three-spins path.