Some chiral rings of N=2 discrete superconformal series induced by SL(2) degenerate conformal field theories

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1 Introduction

Recent discussion of M-theory and string dualities involve N = 2 two dimensional superconformal field theories [1]. Superstrings with N = 2 algebra on the world sheet were shown to describe self-dual Yang-Mills and gravity in a Kähler space-time with signature (2,2) (see e.g. [2] and references therein). Such strings are the exactly solvable four dimensional string theories. Models of string compactification based on N = 2 superconformal models are also known from the work by Gepner [3]. Their key stones are the so-called N = 2 minimal models [4]. The latter are a subclass of N = 2 discrete series [5]. These models are non-unitary and have, in general, an OP algebra of primary fields which is not closed. Nevertheless the presence of singular vectors in the representations of N = 2 algebra for such series provides a strong evidence for exact solvability. One motivation for the present work was to do a step towards an exact solution using the recent progress with SL(2) degenerate conformal field theories [6].

Another motivation was to try to understand the nature of chiral rings [7]. In fact, it is the simplest structure of N = 2 superconformal theories. At first sight, it is rather difficult to extract an origin of chiral rings because the same quantum numbers are shared by conformal dimensions, U(1) charges and weights of quantum group. So I am bound to learn something if I succeed.

The outline of the paper is as follows.

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In sections 2.1 and 2.2, a brief overview of the essential elements of \( SL(2) \) and \( N = 2 \) two-dimensional conformal field theories is given. I will mainly concentrate on a degenerate case when representations contain the so-called singular vectors. In section 2.3 I describe a relation between some of these models, and the use of the fermionic construction of Di Vecchia, Petersen, Yu and Zheng to obtain a proper relation between correlation functions. In particular, I show that the construction works for any complex level \( k \) of \( \hat{sl}_2 \). The main body of this work is presented in sections 3.1 and 3.2. In section 3.1 by computing the OP algebra of primary chiral fields I show that they don’t generate a ring structure. The origin of this disaster is the non-unitarity of the models. In the case at hand the U(1) conservation law doesn’t provide a proper selection rule. It forces one to look for more fine structures. The solution of the problem is given in section 3.2 by introducing Moore-Reshetikhin operators \(^1\). This provides a strong evidence for a quantum group nature of chiral rings. Section 4 will present my conclusions and some open problems. In the appendices I give technical details which are relevant for the explicit construction of the chiral rings.

2 Preliminaries

2.1 \( SL(2) \) degenerate conformal field theories

The theories have \( \hat{sl}_2 \times \hat{sl}_2 \) algebra as the symmetry algebra. The holomorphic form of currents has the following OP algebra \(^2\):

\[
J(x_1, z_1)J(x_2, z_2) = -k \frac{x^2_{12}}{z_{12}^2} - 2 \frac{x_{12}}{z_{12}} J(x_2, z_2) - \frac{x^2_{12}}{z_{12}} \frac{\partial}{\partial x_2} J(x_2, z_2) + O(1) ,
\]

where \( k \) is the level of the algebra, \( z \) - a point on the sphere, \( x \) is an isotopic coordinate of \( sl_2 \), \( z_{ij} = z_i - z_j \), \( x_{ij} = x_i - x_j \). The same OP expansion, of course, is valid for the antiholomorphic form \(^3\).

The standard holomorphic currents are the Taylor series components of \( J(x, z) \)

\[
J(x, z) = J^+(z) - 2xJ^0(z) - x^2J^-(z) .
\]

They form the OP algebra:

\[
J^\alpha(z_1)J^\beta(z_2) = \frac{k q^{\alpha\beta}}{2} - \frac{f^{\alpha\beta}}{z_{12}} J^\gamma(z_2) + O(1) ,
\]

where \( q^{00} = 1, q^{+-} = q^{-+} = 2, f_+^0 = f^-_0 = 1, f_0^{++} = 2; \alpha, \beta = 0, +, - \). It is a little exercise in OP expansions to derive (2.3) from (2.1) and vice versa.

\(^1\)The generators of \( sl_2 \) look like \( S^-_j = \frac{\partial}{\partial z_j} \), \( S^+_j = -x^2 \frac{\partial}{\partial x_j} + 2jx \).

\(^2\)I will not write down antiholomorphic OP expansions when their form follows from holomorphic one.
The stress-energy tensor of the model has two independent components which can be chosen in the Sugawara form

\[ T(z) = \frac{1}{k + 2} q_{\alpha\beta} : J^\alpha(z) J^\beta(z) : \]

\[ \bar{T}(\bar{z}) = \frac{1}{k + 2} q_{\alpha\beta} : \bar{J}^\alpha(\bar{z}) \bar{J}^\beta(\bar{z}) : \]  

(2.4)

It is known that each component provides the Virasoro algebra with the central charge \( c = \frac{3k}{k+2} \).

Define the primary fields as

\[ J(x_1, z_1) \Phi^{j\bar{j}}(x_2, \bar{x}_2, z_2, \bar{z}_2) = -2j \frac{x_{12}}{z_{12}} \Phi^{j\bar{j}}(x_2, \bar{x}_2, z_2, \bar{z}_2) - \frac{x_{12}}{z_{12}} \frac{\partial}{\partial x_2} \Phi^{j\bar{j}}(x_2, \bar{x}_2, z_2, \bar{z}_2) + O(1) \]  

(2.5)

It should be noted that in the general case the primary fields are non-polynomial in \( x, \bar{x} \). Furthermore, \( J(x, z) \), \( \bar{J}(\bar{x}, \bar{z}) \) are not primary.

The complete system of fields involved in the theory includes, besides the primary fields \( \Phi^{j\bar{j}} \), all the fields (descendants) of the form

\[ J_{n_1}^{\alpha_1}(x) \ldots J_{n_N}^{\alpha_N}(x) \bar{J}_{\bar{n}_1}^{\beta_{\bar{1}}}(\bar{x}) \ldots \bar{J}_{\bar{n}_M}^{\beta_{\bar{M}}}(\bar{x}) \Phi^{j\bar{j}}(x, \bar{x}, z, \bar{z}) \]  

(2.6)

where \( J_{n_1}^{\alpha}(x), \bar{J}_{\bar{n}_1}^{\beta}(\bar{x}) \) are the Laurent series components of \( J(x, z) \) and \( \bar{J}(\bar{x}, \bar{z}) \), respectively. From a mathematical point of view the primary fields correspond to the highest weight vectors of \( \hat{sl}_2 \times \hat{sl}_2 \). As to the parameters \( j \)'s, they are the weights of the representations.

I will consider only the diagonal embedding the physical space of states into a tensor product of holomorphic and antiholomorphic sectors. Such models are known as "A" series. Since for these models all primary fields are spinless, i.e. \( j \equiv j(\Delta \equiv \Delta) \), I suppress \( j \)-dependence below.

In [10] Kac and Kazhdan found that the highest weight representation of \( \hat{sl}_2 \) is reducible if the highest weight \( j \) takes the values \( j_{n,m} \) defined by

\[ j_{n,m}^+ = \frac{1}{2} - n(k + 2) + \frac{m - 1}{2} \quad \text{or} \quad j_{n,m}^- = \frac{n}{2} (k + 2) - \frac{m + 1}{2} \]  

(2.7)

with \( k \in \mathbb{C}, \{n, m\} \in \mathbb{N} \). Note that the unitary representations are a subset of the Kac-Kazhdan set namely, they are given by \( j_{1,m}^+ \) with the integer level \( k \).

I will call SL(2) conformal field theories with the primary fields parametrized by the Kac-Kazhdan list as the degenerate SL(2) conformal field theories.

The Operator Product of any two operators is given by

\[ \phi^{j_1}(x, \bar{x}, z, \bar{z}) \phi^{j_2}(0, 0, 0, 0) = \sum_{j_3} C_{j_1 j_2}^{j_3} (x, \bar{x}, z, \bar{z}) \phi^{j_3}(0, 0, 0, 0) \]  

(2.8)

It is well-known that all the coefficient functions \( C_{j_1 j_2}^{j_3} (x, \bar{x}, z, \bar{z}) \) in the expansion (2.8) can be expressed via the weights (conformal dimensions) of the primary fields (basic operators) and the
structure constants of Operator Algebra \([11]\). The structure constants are defined as coefficients at the primary fields in the OP expansion

\[
\Phi^{j_1}(x, \bar{x}, z, \bar{z}) \Phi^{j_2}(0, 0, 0, 0) = \sum_{j_3} |x|^{2(j_1 + j_2 - j_3)} |z|^{2(\Delta_{j_1} + \Delta_{j_2} - \Delta_{j_3})} C^{j_1 j_2 j_3}_{j_3} \Phi^{j_3}(0, 0, 0, 0). \tag{2.9}
\]

The normalized two and three point functions of the primary fields can be represented as

\[
\langle \Phi^{j_1}(x_1, \bar{x}_1, z_1, \bar{z}_1) \Phi^{j_2}(x_2, \bar{x}_2, z_2, \bar{z}_2) \rangle = \delta^{j_1 j_2} \frac{|x_{12}|^{4j_1}}{|z_{12}|^{4\Delta_{j_1}}}, \tag{2.10}
\]

\[
\langle \Phi^{j_1}(x_1, \bar{x}_1, z_1, \bar{z}_1) \Phi^{j_2}(x_2, \bar{x}_2, z_2, \bar{z}_2) \Phi^{j_3}(x_3, \bar{x}_3, z_3, \bar{z}_3) \rangle = C^{j_1 j_2 j_3} \prod_{n<m} \frac{|x_{nm}|^{2\gamma_{nm}(j)}}{|z_{nm}|^{2\gamma_{nm}(\Delta)}}, \tag{2.11}
\]

where \(\gamma_{12}(y) = y_1 + y_2 - y_3, \gamma_{13}(y) = y_1 + y_3 - y_2, \gamma_{23}(y) = y_2 + y_3 - y_1\) and \(\Delta_j = \frac{j(j+1)}{k+2}\).

As to the four point function, one can find it in the following form \([3]\)

\[
\langle \Phi^{j_1}(x_1, \bar{x}_1, z_1, \bar{z}_1) \ldots \Phi^{j_4}(x_4, \bar{x}_4, z_4, \bar{z}_4) \rangle = G^{j_1 j_2 j_3 j_4}(x, \bar{x}, z, \bar{z}) \prod_{n<m} \frac{|x_{nm}|^{2\varepsilon_{nm}(j)}}{|z_{nm}|^{2\varepsilon_{nm}(\Delta)}},
\]

with \(\varepsilon_{14}(y) = 2y_1, \varepsilon_{23}(y) = y_1 + y_2 + y_3 - y_4, \varepsilon_{24}(y) = -y_1 + y_2 + y_3 + y_4, \varepsilon_{34}(y) = -y_1 - y_2 + y_3 + y_4\) and

\[
x = \frac{x_{12} x_{34}}{x_{14} x_{32}}, \quad \bar{x} = \frac{\bar{x}_{12} \bar{x}_{34}}{\bar{x}_{14} \bar{x}_{32}}, \quad z = \frac{z_{12} z_{34}}{z_{14} z_{32}}, \quad \bar{z} = \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{14} \bar{z}_{32}}.
\]

The functions \(G^{j_1 j_2 j_3 j_4}(x, \bar{x}, z, \bar{z})\) are given by (see \([3]\) for details)

\[
G^{j_1 j_2 j_3 j_4}(x, \bar{x}, z, \bar{z}) = Z(j_1, j_2, j_3, j_4) |z|^\alpha |1 - z|^\beta \prod_{i=1}^{n_1-1} \int d^2 u_i \int d^2 w_{i'} |u_i - w_{i'}|^{-4} \times \prod_{i=1}^{m_1-1} |u_i|^{4\alpha_1} \prod_{i'=1}^{m_1-1} |w_{i'}|^{4\alpha_2} \times \prod_{i=1}^{m_1-1} |w_{i'}|^{4\alpha_1} \prod_{i'=1}^{m_1-1} |w_{i'}|^{4\alpha_2}.
\]

Here \(a = 4j_1 j_2 \alpha_+^2, b = 4j_1 j_3 \alpha_+^2, \alpha_- = -\sqrt{k+2}, \alpha_+ \alpha_- = -1\), \(\alpha_i\)’s are defined via \(\alpha_i = \frac{1-N_i}{2} \alpha_- + \frac{1-M_i}{2} \alpha_+\). It should be noted that \(N_i\)’s (\(M_i\)’s) are linear combinations of \(n_i\)’s (\(m_i\)’s) and their form depends on the parametrizations (2.7).

In order to take into account a relative normalization between the operators of the Dotsenko-Fateev models and the ones of the SL(2) degenerate conformal field theories one has to introduce the normalization constants \(Z(j_1, j_2, j_3, j_4)\). For their explicit form I refer to the original work \([3]\).
From the set (2.7) it is worth distinguishing the so-called admissible representations [12], which correspond to the rational level \( k \). In the case \( k = -2 + p/q \), with the coprime integers \( p \) and \( q \), it is possible to recover the minimal models (series with \( c < 1 \), [13]) via the Drinfeld-Sokolov reduction. On the other hand \( k = -2 - p/q \) leads to the Liouville series with \( c > 25 \). The second point is finite dimensional representations of the modular group for such representations.

At the rational level \( k = -2 + p/q \) there is a symmetry \( \mathfrak{j}_{n,m}^- = \mathfrak{j}^+_{q-n+1,p-m} \) which allows one to reduce the fields parameterized by \( \mathfrak{j}_{n,m}^- \) to the fields parameterized by \( \mathfrak{j}_{n,m}^+ \). In this case the structure constants of the Operator Product algebra are given by

\[
C^+(n_1,m_1;n_2,m_2;n_3,m_3) = \frac{\Gamma^2[\rho]}{\Gamma^2[1-\rho]} P(\sigma' - \frac{1}{2}, \sigma + \frac{1}{2}) \prod_{\{1,2,3\}} (-)^{n_i-1} \rho^{(1-n_i)} \left( \frac{\Gamma[n_i - m_i \rho]}{\Gamma[1 - n_i + m_i \rho]} \right)^{\frac{1}{2}} P(\sigma' - n_i + \frac{1}{2}, \sigma - m_i + \frac{1}{2}) P(n_i,m_i),
\]

(2.13)

\[
C^-(n_1,m_1;n_2,m_2;n_3,m_3) = \rho^{-\frac{1}{2}} P(\sigma', \sigma + \frac{1}{2}) \prod_{\{1,2,3\}} \rho^{-(n_i-1)(m_i-\frac{1}{2})} P(\sigma' - n_i, \sigma - m_i + \frac{1}{2}).
\]

(2.14)

Here \( \sigma' = \frac{n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2}, \sigma = \frac{m_1}{2} + \frac{m_2}{2} + \frac{m_3}{2}, \rho = \alpha_+^2 \) and \( \rho' = \alpha_-^2 \). The function \( P(n,m) \) is defined by

\[
P(n,m) = \prod_{i=1}^{n-1} \prod_{j=1}^{m-1} \left[ i \rho' - j \right]^{-2} \prod_{i=1}^{n-1} \frac{\Gamma[i \rho']}{\Gamma[1 - i \rho']} \prod_{j=1}^{m-1} \frac{\Gamma[j \rho]}{\Gamma[1 - j \rho]}, \quad P(1,1) = 1.
\]

It should be noted that \( n_3, m_3 \) in (2.14) belong to the field parameterized by \( \mathfrak{j}_{n,m}^- \). Such choice clarifies the quantum group structure \( (U_q osp(2/1), U_q sl(2)) \) of the model [13].

It is easy to see from (2.13) and (2.14) that the OP algebra at the rational level is closed in the grid \( 1 \leq n_i \leq q, \ 1 \leq m_i \leq p - 1 \). The corresponding fusion rules are given by

\[
\begin{cases}
|n_{12}| + 1 \leq n_3 \leq \min(n_1 + n_2 - 1, 2q - n_1 - n_2 + 1), \text{ with } \Delta n_3 = 1, \\
|m_{12}| + 1 \leq m_3 \leq \min(m_1 + m_2 - 1, 2p - m_1 - m_2 - 1), \text{ with } \Delta m_3 = 2.
\end{cases}
\]

(2.15)

In the above \( \Delta \) means a step. These fusion rules were first found in [9, 14] from the differential equations for the conformal blocks.

Let me now define the primary fields of the algebra (2.3) via \( \Phi^j(x, \bar{x}, z, \bar{z}) \) as

\[
\Phi^j_{\mu, \bar{\mu}}(z, \bar{z}) = \frac{1}{\mathcal{N}(j, \mu, \bar{\mu})} \oint_C \oint_{\bar{C}} dx d\bar{x} \ x^{\mu-1-j} \bar{x}^{\bar{\mu}-1-j} \Phi^j(x, \bar{x}, z, \bar{z}),
\]

(2.16)

where \( C, \bar{C} \) are closed contours, \( \mu, \bar{\mu} \) are arbitrary parameters. The normalization factors \( \mathcal{N}(j, \mu, \bar{\mu}) \) are computed in Appendix A. Explicitly

\[
\mathcal{N}(j, \mu, \bar{\mu}) = \Gamma[2j + 1] \{ \Gamma[1 + j + \mu] \Gamma[1 + j - \mu] \Gamma[1 + j + \bar{\mu}] \Gamma[1 + j - \bar{\mu}] \}^{-\frac{1}{2}}.
\]

(2.17)
Using the OP expansion (2.5) as well as (2.2) one arrives at
\[ J^0(z_1)\Phi^j_{\mu,\bar{\mu}}(z_2, \bar{z}_2) = \frac{\mu}{z_{12}^3}\Phi^j_{\mu,\bar{\mu}}(z_2, \bar{z}_2) + O(1) , \]
\[ J^\pm(z_1)\Phi^j_{\mu,\bar{\mu}}(z_2, \bar{z}_2) = \frac{1}{z_{12}^3}\mathcal{M}_\pm\Phi^j_{\mu\pm1,\bar{\mu}}(z_2, \bar{z}_2) + O(1) , \]
with \( \mathcal{M}_\pm = (j + \mu + 1)\mathcal{N}(j, \mu + 1, \bar{\mu})/\mathcal{N}(j, \mu, \bar{\mu}) \).

The highest (lowest) weight vectors of \( \hat{sl}_2 \times \hat{sl}_2 \) algebra can be extracted from (2.16) by setting \( \mu = \bar{\mu} = j (\mu = \bar{\mu} = -j) \). This is an immediate consequence of (2.17) and (2.18).

Before discussing the \( N = 2 \) discrete superconformal field theories, I pause here to emphasize one important point. The primary fields defined in (2.16) depend on contours \( C_i(\bar{C}_i) \) in the isotopic spaces. From this point of view one has the non-local operators. However it hasn’t influence on the main results obtained below.

### 2.2 \( N = 2 \) discrete superconformal field theories

The theory has \( N = 2 \times N = 2 \) algebra as the symmetry algebra. The holomorphic part, \( N = 2 \) superconformal algebra, is generated by four local currents: \( T(z) \), \( G(z) \) and \( J(z) \). The fermionic currents \( G(z) \) have a conformal dimension \((\frac{3}{2}, 0)\), and the bosonic currents \( J(z) \) and \( T(z) \) have \((1, 0)\) and \((2, 0)\), respectively. The current \( T(z) \) is the holomorphic stress-energy tensor.

The algebra is determined by the following Operator Product expansions:

\[ J(z_1)J(z_2) = \frac{c_2/4}{z_{12}^2} + O(1) \quad \cdot \quad T(z_1)J(z_2) = \frac{1}{z_{12}^2}J(z_2) + \frac{1}{z_{12}^2}\partial J(z_2) + O(1) \quad \cdot \quad \]
\[ J(z_1)G^\pm(z_2) = \pm\frac{1/2}{z_{12}}G^\pm(z_2) + O(1) \quad \cdot \quad T(z_1)G^\pm(z_2) = \frac{3/2}{z_{12}^2}G^\pm(z_2) + \frac{1}{z_{12}^2}\partial G^\pm(z_2) + O(1) \quad \cdot \quad \]
\[ T(z_1)T(z_2) = \frac{3c_2/2}{z_{12}^2} + \frac{2}{z_{12}^2}T(z_2) + \frac{1}{z_{12}^2}\partial T(z_2) + O(1) \quad \cdot \quad \]
\[ G^+(z_1)G^+(z_2) = \frac{2c_2}{z_{12}^2} + \frac{4}{z_{12}^2}J(z_2) + \frac{2}{z_{12}^2}(\partial J(z_2) + T(z_2)) + O(1) \quad \cdot \quad \]

\[
\text{(2.19)}
\]

The central charge \( c_2 \) is related to the usual Virasoro\((N = 0)\) central charge \( c \) by \( c_2 = c/3 \). The normalization is fixed so that \( c_2 = 1 \) for the free scalar superfield.

The three sectors of the theory are given by three moddings of the generators, corresponding to three ways of choosing boundary conditions on the cylinder. Because I am interested in chiral rings let me restrict to the Neveu-Schwarz (NS) sector. This sector has integer modes for the bosonic currents, but half-integers for fermionic ones.

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\(^3\) Notice that it is possible to recover ground states of the Ramond sector from the NS sector under the spectral flow mapping \([15]\).
The corresponding primary fields are given by

\[ J(z_1) \phi_{q,q}^{h,h}(z_2, \bar{z}_2) = \frac{q}{z_{12}} \phi_{q,q}^{h,h}(z_2, \bar{z}_2) + O(1) , \]
\[ T(z_1) \phi_{q,q}^{h,h}(z_2, \bar{z}_2) = \frac{h}{z_{12}} \phi_{q,q}^{h,h}(z_2, \bar{z}_2) + \frac{1}{z_{12}} \frac{\partial}{\partial z_2} \phi_{q,q}^{h,h}(z_2, \bar{z}_2) + O(1) , \]  
\[ G^\pm(z_1) \phi_{q,q}^{h,h}(z_2, \bar{z}_2) = \frac{1}{z_{12}} G^\pm \phi_{q,q}^{h,h}(z_2, \bar{z}_2) + O(1) . \]  

Here \( h \) and \( q \) are a conformal dimension and U(1) charge.

The complete system of fields involved in the theory is obtained by acting with the all negative frequency modes of the currents on the primary fields. From mathematical point of view the primary fields correspond to the highest weight vectors of \( \mathcal{N} = 2 \times \mathcal{N} = 2 \) algebra.

As in the previous section I will only consider the diagonal embeddings of the physical space of states into a tensor product of holomorphic and antiholomorphic sectors, the so-called "\( \Lambda \)" series, and due to this reason I will suppress \( \bar{h}, \bar{q} \)-dependence below.

It is known [4, 5] that the highest weight representation is reducible if the conformal dimension takes the values defined by

\[ h^I_{n,m} = \frac{1}{4(k+2)} \left[ (m-(k+2)(n-1))^2 - 1 \right] - (k+2)q^2 , \quad n, m \in \mathbb{N} \]  
\[ h^{II}_p = \frac{1}{4(k+2)} (p^2 - 1) \pm pq , \quad p \in \mathbb{N} \]  

where \( c_2 = 1 - \frac{2}{k+2} , \quad k \in \mathbb{C} \).

I will call theories with the primary fields parametrized by this set as the discrete \( \mathcal{N} = 2 \) superconformal models. Note that the unitary minimal models are a subset of the discrete ones namely, they are given by \( h^I_{n,m} \) with the integer parameter \( k \).

2.3 \( \mathcal{N}=2 \) via \( \text{SL}(2) \) degenerate conformal field theories

In order to write down correlation functions of the \( \mathcal{N} = 2 \) discrete superconformal field theories it seems very natural to use the fermionic construction proposed by Di Vecchia, Petersen, Yu and Zheng to build the unitary representations of the \( \mathcal{N} = 2 \) superconformal algebra in terms of free fermions and unitary representations of \( \hat{\mathfrak{sl}}_2 \). In fact one can do better: the only difference between the unitary representations of \( \hat{\mathfrak{sl}}_2 \) and degenerate ones is a value of \( k \) (see (2.7)). Therefore one can relate the degenerate representations of \( \hat{\mathfrak{sl}}_2 \) to the discrete representations of \( \mathcal{N} = 2 \).

Let me sketch the main points of this construction. The holomorphic part is described in terms of the free fermions \( \psi^\pm(z) \) and \( \hat{\mathfrak{sl}}_2 \) algebra. The U(1) current and stress-energy tensor of the fermions are given by

\[ j(z) =: \psi^+(z) \psi^-(z) : , \quad T_\psi = \frac{1}{2} : j(z) j(z) : . \]

It is straightforward to see that in the case of a general \( k \) the \( \mathcal{N} = 2 \) currents are also
expressed as

$$J(z) = \frac{1}{2(k + 2)}(2J^0(z) + kj(z)) \quad , \quad G^\pm(z) = \sqrt{\frac{2}{k + 2}}\psi^\pm(z)J^\pm(z) \quad ,$$

$$T(z) = T_{sl^2} + T_\psi - \frac{1}{k + 2} : (J^0(z) - j(z))^2 : \quad ,$$

where $T_{sl^2}(z)$ is the Sugawara stress-energy tensor given by (2.4). The OP expansions of $J^\alpha(z)$ are defined in (2.3).

The primary fields of the $N = 2$ superconformal theories can be written as

$$\Phi^{h}(z, \bar{z}) = \Phi^{j}(z, \bar{z}) \mathbb{1} \quad .$$

Here $\mathbb{1}$ is a trivial field (identity operator) which corresponds to the vacuum of the fermionic system in the NS sector and $\Phi^{j}$'s are the primaries of $\hat{sl}_2 \times \hat{sl}_2$.

The conformal dimensions and $U(1)$ charges are expressed via $j$ and $\mu$ as

$$h = \frac{j(j + 1)}{k + 2} - \frac{\mu^2}{k + 2} \quad , \quad q = \frac{\mu}{k + 2} \quad .$$

To give a relation between correlation functions of the above models, let me now proceed in complete accordance with the derivation of the Knizhnik-Zamolodchikov (KZ) equations [16]. Inserting the constraint

$$(k + 2)L_{-1} = \sum_{n = -\infty}^{+\infty} g_{\alpha\beta} J^\alpha_n J^\beta_{-1-n} : - \sum_{n = -\infty}^{+\infty} J^0_n J^0_{-1-n}$$

into a correlation function, I find

$$(k + 2)\frac{\partial}{\partial z_i} \langle \prod_{i=1}^{N} \Phi^{h_i}(z_i, \bar{z}_i) \rangle = \sum_{i \neq j} g_{\alpha\beta} \frac{\partial}{\partial z_{ij}} \langle \prod_{i=1}^{N} \Phi^{h_i}(z_i, \bar{z}_i) \rangle - 2 \sum_{i \neq j} \frac{\mu_i\mu_j}{z_{ij}} \langle \prod_{i=1}^{N} \Phi^{h_i}(z_i, \bar{z}_i) \rangle \quad .$$

Here $t_i^{\alpha}$'s are generators of $sl(2)$.

The solution of the above equations is given by

$$\langle \prod_{i=1}^{N} \Phi^{h_i}(z_i, \bar{z}_i) \rangle = \prod_{i < j} |z_{ij}|^{-\frac{\mu_i\mu_j}{k + 2}} \langle \prod_{i=1}^{N} \Phi^{j_i}(z_i, \bar{z}_i) \rangle \quad ,$$

where the last factor is a solution of the standard KZ equations for the $SL(2)$ conformal field theory, namely

$$(k + 2)\frac{\partial}{\partial z_i} \langle \prod_{i=1}^{N} \Phi^{j_i}(z_i, \bar{z}_i) \rangle = \sum_{i \neq j} g_{\alpha\beta} \frac{\partial}{\partial z_{ij}} \langle \prod_{i=1}^{N} \Phi^{j_i}(z_i, \bar{z}_i) \rangle \quad .$$

\footnote{Since the field $\mathbb{1}$ has trivial OP expansions I omit modes of $T_\psi(z)$ and $j(z)$.}
So I obtain the relation between the correlation functions. Let me conclude this section by giving a few remarks.

(i) It is clear from (2.7) and (2.25) that one can recover only the first degenerate series $h^I$ of $N = 2$ superconformal algebra via the degenerate representations of $\hat{sl}_2$. However, the primary fields parametrized by the first series $h^I$ form a closed OP algebra, i.e. there is decoupling of the second series $h^{II}$. To see this, it is convenient to use the free field representation. More discussion on this point is given in Appendix B.

(ii) In the case of $N = 2$ unitary minimal models it is possible to derive the relation (2.27) via the Fateev-Zamolodchikov parafermions \cite{17, 18}. However for a non-integer parameter $k$ the algebra of the parafermionic currents is not closed and leads to an ill-defined parafermionic theory. On the other hand, there is a strong indication on a finite number of order parameters in such "parafermionic theory" for a rational $k$ because a proper $SL(2)$ theory has the closed OP algebra of the primary fields in this case.

3 Chiral rings

3.1 Primary chiral fields

Among the primary fields of the Neveu-Schwarz sector of $N = 2$ models it is worth to distinguish the so-called primary chiral fields introduced by Lerche, Vafa and Warner in \cite{7}. Such fields satisfy, in addition to (2.20), the condition

$$G_{-\frac{1}{2}}^+ \Phi^h_q(z, \bar{z}) = 0 \ .$$

(3.1)

The anti-chiral fields are defined by replacing $G_{-\frac{1}{2}}^+ \rightarrow G_{-\frac{1}{2}}^-$. Using (2.19) one can deduce that for such states $h = q$. The equations (2.21-2.22) allow me to find the conformal dimensions in terms of integers as

$$h^I_1 = \frac{1 - n}{2} + \frac{m - 1}{2(k + 2)} \ , \quad h^I_2 = \frac{n - 1}{2} - \frac{m + 1}{2(k + 2)} \ ;$$

$$h^{II}_1 = \frac{p - 1}{4(k + 2)} \ , \quad h^{II}_2 = -\frac{p + 1}{4(k + 2)} \ .$$

(3.2)

(3.3)

On the other hand, the relationship between the $SL(2)$ and $N = 2$ models implies that the primary chiral fields correspond to the highest weight vectors of the $\hat{sl}_2 \times \hat{sl}_2$ algebra. Note that a solution $\mu = -j - 1$ of equations (2.25) with $h = q$ is forbidden because it corresponds to a zero norm state (see (2.17)). As a result, one has the following set of the conformal dimensions provided by $SL(2)$

$$h^+_{n,m} = \frac{1 - n}{2} + \frac{m - 1}{2(k + 2)} \ , \quad h^-_{n,m} = \frac{n}{2} - \frac{m + 1}{2(k + 2)} \ .$$

(3.4)

It is evident that for a general $k$ it is possible to recover dimensions: $h^I_1$, $h^I_2$ with $n > 1$ and $h^{II}_1$ with odd $p$. The other solutions are decoupled. The second series decoupling is discussed\footnote{This is the case for correlation functions too.}
in Appendix B. As to $h^I_2$ with $n = 1$, it is usual zero vectors decoupling in 2d conformal field theories.

Since $h^\pm$ are parameterized by two integers $(n, m)$ it is useful to denote the primary chiral fields $\Phi^I_{h^\pm}(z, \bar{z})$ as $\Phi^I_{n,m}(z, \bar{z})$.

The correlation functions of the primary chiral fields parameterized by (3.4) are computable by the relation (2.27) in terms of the correlation functions of the highest weight vectors. For instance, a small calculation shows that the four point function of $\Phi^I_{n,m}$ is given by

$$\langle \Phi^I_{n_1,m_1}(z_1, \bar{z}_1) \Phi^I_{n_2,m_2}(z_2, \bar{z}_2) \Phi^I_{n_3,m_3}(z_3, \bar{z}_3) \Phi^I_{n_4,m_4}(z_4, \bar{z}_4) \rangle = Z(h_1, h_2, h_3, h_4) \prod_{i=1}^{3} |z_i|^2 h_i \times$$

$$\times \prod_{i=1}^{m_1-1} \int d^2 w_i \prod_{i=1}^{n_1-1} |u_i|^{4|\alpha_1\alpha_-|} |1 - u_i|^{4|\alpha_2\alpha_-|} |x - u_i|^{4|\alpha_2\alpha_+|} |z - u_i|^{4|\alpha_3\alpha_-|} \prod_{i<j}^{n_1-1} |u_{ij}|^{4\alpha_-} \times$$

$$\times \prod_{i=1}^{m_1-1} \prod_{i' = 1}^{n_1-1} |w_i|^{4|\alpha_1\alpha_+|} |1 - w_i|^{4|\alpha_2\alpha_+|} |x - w_i|^{4|\alpha_2\alpha_-|} |z - w_i|^{4|\alpha_3\alpha_+|} \prod_{i<j}^{m_1-1} |w_{ij}|^{4\alpha_+} \prod_{i=1}^{n_1-1} \prod_{i' = 1}^{n_1-1} |u_i - w_{ij}|^{-4} .$$

(3.5)

Here $\alpha_i = \frac{1-N_1}{2} \alpha_- + \frac{1-M_1}{2} \alpha_+$ with

$$N_1 = \frac{n_1}{2} + \frac{n_2}{2} - \frac{n_3}{2} - \frac{n_4}{2} ; \quad M_1 = \frac{m_1}{2} + \frac{m_2}{2} - \frac{m_3}{2} - \frac{m_4}{2} ;$$

$$N_2 = \frac{n_1}{2} - \frac{n_2}{2} + \frac{n_3}{2} - \frac{n_4}{2} ; \quad M_2 = \frac{m_1}{2} + \frac{m_2}{2} - \frac{m_3}{2} + \frac{m_4}{2} ;$$

$$N_3 = \frac{n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2} + \frac{n_4}{2} - 1 ; \quad M_3 = \frac{m_1}{2} + \frac{m_2}{2} + \frac{m_3}{2} + \frac{m_4}{2} .$$

A conjugate field $\Phi^\dagger_{n,m}$ is defined as $\Phi^\dagger_{n,m}(z, \bar{z}) = \Phi^I_{-h}(z, \bar{z})$. Note that the U(1) conservation law provides $h_4 = h_1 + h_2 + h_3$.

One can try to analyze singularities of (3.5) in order to learn the OP algebra of the primary chiral fields. However, due to the contours $C_i(C_i)$, it is a difficult task. On the other hand, it is enough to set $n_1 = m_1 = 1$ into a 4-point function

$$\langle \Phi^I_{n_1,m_1}(z_1, \bar{z}_1) \Phi^I_{n_2,m_2}(z_2, \bar{z}_2) \Phi^I_{n_3,m_3}(z_3, \bar{z}_3) \Phi^I_{n_4,m_4}(z_4, \bar{z}_4) \rangle$$

to find the structure constants of OP algebra via the corresponding three point correlation functions. The three point function of interest is given by

$$\langle \Phi^I_{n_1,m_1}(z_1, \bar{z}_1) \Phi^I_{n_2,m_2}(z_2, \bar{z}_2) \Phi^I_{n_3,m_3}(z_3, \bar{z}_3) \Phi^I_{n_4,m_4}(z_4, \bar{z}_4) \rangle = C(n_1, m_1; n_2, m_2; n_3, m_3) \prod_{i<j} |z_{ij}|^{-2n_{ij}(h)} , \quad (3.6)$$

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where \( q_3 = q_1 + q_2 \) and the structure constants \( C \) are written as

\[
C(n_1, m_1; n_2, m_2; n_3, m_3) = C^+(n_1, m_1; n_2, m_2; n_3, m_3) \prod_{i=1}^{2} \frac{\Gamma[1 + \rho'(h_1 + h_2 + h_3 - 2h_i)]}{\prod_{i=1}^{3} \Gamma[1 + 2\rho' h_i]} \times \frac{\Gamma[1 + \rho'(h_1 + h_2 + h_3)]}{\prod_{i=1}^{3} \Gamma[1 + \rho'(h_3 - h_1 - h_2)]},
\]

with the coefficients \( C^+ \) defined in (2.13). The \( \Gamma \)-functions come from the normalization factor \( \mathcal{N}(j_3, -j_1 - j_2) \) as well as multiple integral over \( x_i \). The latter is computed in Appendix C. Note that after setting \( n_1 = m_1 = 1 \), the integrals over \( u_i, w_i \) are eliminated and the integral over \( x_1 \) is decoupled, so the only multiple integral of interest is an integral over \( x_i, i = \{2, 3, 4\} \).

It is easy to see from (3.6) that the chiral primary fields don’t form a closed OP algebra at the rational level \( k = -2 + p/q \), with the coprime integers \( p \) and \( q \). The only exception is the unitary series which correspond to \( q = 1 \).

It should be noted that (3.6) represents the three point functions when all conformal dimensions are parameterized by \( h^+ \). There are, of course, three point functions with \( h^- \). This is similar for the case of the \( SL(2) \) degenerate conformal fields theories (see ref.\[6\]). The fusion rules then become

\[
n_3 = n_1 + n_2 - 1 \quad , \quad m_3 = m_1 + m_2 - 1
\]  

and

\[
\begin{align*}
|n_{12}| + 1 \leq n_3 \leq \min (n_1 + n_2 - 2, 2q - n_1 - n_2 + 1), \ & \text{with} \ \Delta n_3 = 1 , \\
|m_{12}| + 1 \leq m_3 \leq \min (m_1 + m_2 - 1, 2p - m_1 - m_2 - 1), \ & \text{with} \ \Delta m_3 = 2 .
\end{align*}
\]

In above, only the first selection rule corresponds to the primary chiral field. As to the others, they correspond to the primary fields which are no longer chiral. It is due to the \( U(1) \) conservation law \( q_3 = q_1 + q_2 \).

### 3.2 Chiral rings

The results of section 3.1 are forced me to look for new objects which have a ring structure. In attempting to do this it is advantageous to use operators introduced by Moore and Reshetikhin \[8\].

According to \[8\] define holomorphic vertex operators, \( ^\alpha \Phi_h^q(z) \), associated to a triple \( (h, q, \alpha) \), where \( h \) and \( q \) are the conformal dimension and \( U(1) \) charge, respectively. As to \( \alpha \), it means a pair of states in the highest weight representations of the quantum groups \( (U_qosp(2/1), U_qsl(2)) \). In fact, I need a structure which manages the fusion of \( (n, m) \), i.e. \( (U_qosp(2/1), U_qsl(2)) \) (see (2.15) and ref.\[13\]).

The \( N = 2 \) primary fields are given by

\[
\Phi_h^q(z, \bar{z}) = \sum_\alpha ^\alpha \Phi_h^q(z) ^\alpha \Phi_h^q(\bar{z}) . \quad (3.7)
\]

\(^6\)A similar construction was also considered by Cremmer, Gervais and Roussel \[19\].
New features induced by the quantum groups are the corresponding Wigner symbols in correlation functions and the Clebsch-Gordan coefficients in the OP expansions. This implies, in particular, that 3-point functions of operators \( \Phi_{q_1}^h(z, \bar{z}) = \Phi_{q_2}^h(z) \Phi_{q_3}^h(\bar{z}) \) are given by

\[
\langle \alpha_1 \Phi_{n_1,m_1}^h(z_1, \bar{z}_1) \alpha_2 \Phi_{n_2,m_2}^h(z_2, \bar{z}_2) \alpha_3 \Phi_{n_3,m_3}^h(z_3, \bar{z}_3) \rangle = J(\alpha_1, \alpha_2, \alpha_3) C(h_1, q_1; h_2, q_2; h_3, q_3) \prod_{i<j} |z_{ij}|^{-2\gamma_{ij}(h)},
\]

(3.8)

where \( J(\alpha_1, \alpha_2, \alpha_3) \) are the squares of the Wigner symbols, \( C(h_1, q_1; h_2, q_2; h_3, q_3) \) are structure constants of the OP algebra of the primary fields.

If one denotes the vertices corresponding to the primary chiral fields by \( \Phi_{n,m} \), then the 3-point functions of interest are

\[
\langle \alpha_1 \Phi_{n_1,m_1}^h(z_1, \bar{z}_1) \alpha_2 \Phi_{n_2,m_2}^h(z_2, \bar{z}_2) \alpha_3 \Phi_{n_3,m_3}^h(z_3, \bar{z}_3) \rangle = J(\alpha_1, \alpha_2, \alpha_3) C(n_1, m_1; n_2, m_2; n_3, m_3) \prod_{i<j} |z_{ij}|^{-2\gamma_{ij}(h)},
\]

(3.9)

with

\[
C(n_1, m_1; n_2, m_2; n_3, m_3) = C^\pm(n_1, m_1; n_2, m_2; n_3, m_3) \prod_{i=1}^3 \Gamma \left[ 1 + \rho'(h_1 + h_2 + h_3 - 2h_i) \right] / \prod_{i=1}^3 \Gamma \left[ 1 + 2\rho'h_i \right],
\]

and

\[
The coefficients \( C^\pm \) depend on the parameterization of \( h_3 \) namely, the sign plus means \( h_3 = h_{n_3,m_3}^+ \), the sign minus - \( h_3 = h_{n_3,m_3}^- \) (see (2.13) and (2.14) for details).

If the states \( \alpha_i, i = \{1, 2, 3\} \), are the highest weight vectors then it is easy to find all non-zero correlation functions. In the case of the rational level \( k = -2 + p/q \) the field \( \alpha_3 \Phi_{q_3}^h \) is uniquely determined by

\[
\begin{align*}
n_3 &= n_1 + n_2 - 1, \\
m_3 &= m_1 + m_2 - 1, \\
1 &\leq n_3 \leq q, \\
1 &\leq m_3 \leq p - 1, \\
h_3 &= h_1 + h_2, \\
q_3 &= q_1 + q_2, \\
\alpha_3 &\text{ - a pair of the lowest weight vectors.}
\end{align*}
\]

The above result implies that the operators \( \Phi_{n,m} \) generate the ring

\[
\alpha_1 \Phi_{n_1,m_1} \times \alpha_2 \Phi_{n_2,m_2} = \begin{cases} \alpha_3 \Phi_{n_1+n_2-1,m_1+m_2-1}, & n_1 + n_2 - 1 \leq q, \quad m_1 + m_2 \leq p, \\
0, & n_1 + n_2 - 1 > q, \quad m_1 + m_2 > p. \end{cases}
\]

(3.10)

At this point a few comments are in order:

(i) Because the highest weights \((j, j')\) of \((U_{osp}(2/1), U_{sl}(2))\) are expressed in terms of \((n, m)\) as \((j = \frac{n-1}{2}, j' = \frac{m-1}{2})\), the Wigner symbol (Clebsch-Gordan coefficient) provides \( j_3 = j_1 + j_2, j_3' = j_1' + j_2' \) or \( n_3 = n_1 + n_2 - 1, m_3 = m_1 + m_2 - 1 \) [20].

(ii) One can use the relation between the chiral primary fields and the highest weight vectors of \( sl_2 \) in order to see that in a general case the chiral primary fields don’t form the closed OP algebra because the corresponding highest weight vectors don’t do this [21]. However if one doesn’t use screening operators, that means that only the highest weight vectors of the quantum group are allowed [22], the fusion of \((n, m)\) is precisely \( n_1 \times n_2 \rightarrow n_1 + n_2 - 1, m_1 \times m_2 \rightarrow m_1 + m_2 - 1 \).

(iii) The operators \( \Phi_{n,m} \) which define the ring obey the OP expansions (2.20) as well as (3.1), i.e. they are primary and chiral.
4 Conclusions and remarks

First, let me say a few words about the results.

In this work I have found the relation between the \( SL(2) \) degenerate conformal field theories on one side and some \( N = 2 \) discrete superconformal series on the other side. This generalized fermionic construction allows me to investigate the properties of the primary chiral fields in the \( N = 2 \) models. As a result, the OP algebra of such fields was computed. It turned out that the primary chiral fields don’t generate the ring. The origin of the disaster is the non-unitarity of the models. Next the Moore-Reshetikhin operators were introduced to solve the problem. This solution gives a strong evidence that a quantum group underlies the ring. It is disguised in the unitary case in virtue of the U(1) conservation law, but it is becomes clear in the non-unitary case. The experience with the fermionic construction also shows that one has to take into account more exotic modules over \( \hat{sl}_2 \) to recover the all highest weight modules over \( N = 2 \) (see point (iii) below for details).

Let me conclude by mentioning some open problems:

(i) It is clear that techniques developed in sections 2.1 and 2.2 allow one to consider any four point function

\[
\langle \prod_{i=1}^{4} \Phi_{q_i}(z_i, \bar{z}_i) \rangle = \prod_{i<j} |z_{ij}|^{-2 + \mu_i + \mu_j} \prod_{i=1}^{4} N^{-1}(j_i, \mu_i) \int_{C_i} \frac{dx_i}{x_i^{j_i+1} - \mu_i} \int_{\bar{C}_i} \frac{d\bar{x}_i}{\bar{x}_i^{j_i+1} - \mu_i} \langle \prod_{i=1}^{4} \Phi^h(x_i, \bar{x}_i, z_i, \bar{z}_i) \rangle ,
\]

with \( h_i, q_i \) defined in (2.25).

The correct contours \( C_i(\bar{C}_i) \), for a particular conformal block, should be chosen by the correct singularities at \( z_{ij} \to 0 \), which should fit to an OP algebra obtained by setting \( n_1 = m_1 = 1 \). An exact prescription, for picking up the correct contours is lacking at this time.

In fact, the problem is closely connected with generalized Dotsenko-Fateev integrals. In the simplest case such integrals look like

\[
I(a_1, a_2; b_1, b_2; c) = \prod_{i=1}^{2} \int_{C_i} dx_i x_i^{a_i} (1 - x_i)^{b_i} x_i^{c} ,
\]

with some real parameters \( a_i, b_i, c \).

Note that the integral (4.2) reduces to the Dotsenko-Fateev one under \( a_1 = a_2 \) and \( b_1 = b_2 \).

I leave the analysis of these problems for future study.

(ii) The second problem is interesting too. It concerns the conjecture that \( N = 2 \) superconformal field theories in two dimensions are critical points of super-renormalizable Landau-Ginzburg (LG) models. This conjecture followed a discussion of usual minimal models (N=0) by Zamolodchikov [25] and in the context of the \( N = 2 \) minimal models was further developed by many authors (see e.g. [26] and refs therein). In the case of the \( N = 2 \) discrete series it seems natural to follow the same procedure. Introducing two chiral fields \( X, Y \) which correspond to the fundamental fields \( \Phi_{1,2} \) and \( \Phi_{2,1} \), one can write down an equation for the superpotential \( W(X,Y) \)

\[
W(X,Y) = \frac{1}{k + 2} X \frac{\partial}{\partial X} W(X,Y) - Y \frac{\partial}{\partial Y} W(X,Y) , \quad k + 2 = \frac{p}{q} .
\]
However this equation has an infinite set of solutions. The solution consistent with the $N = 2$ minimal models ($q = 1$) can be written in the form

$$W(X,Y) = X^p Y^{q-1} + \tilde{W}(X,Y), \quad \tilde{W}(X,Y)|_{q=1} = 0 \quad \text{(or } \tilde{W}(X,Y)|_{q=1} = f(Y)) \quad (4.4)$$

The main problem here is to find $\tilde{W}(X,Y)$. To do this one can try to use the $\varepsilon$-expansion as it was done by Howe and West in the case of the minimal models [27]. On the other hand, it would be interesting to apply the Witten elliptic genus calculations [28] to the problem at hand. A natural question which also arises: which algebraic varieties do give $W(X,Y)$? They are well-known for the minimal models (see e.g. [29]).

(iii) One has seen in section 3.1 that it is not enough to use only the highest(lowest) weight representations of $\hat{sl}_2$ as well as intermediate ones to describe all highest weight representations of $N = 2$ for the discrete series. The first series $h^I$ is recovered by considering modules over $\hat{sl}_2$ with $\mu = -j - 1$. Such modules contain two parts: non-normalizable states and normalizable ones. In the context of free field representations of $\hat{sl}_2$ similar modules were discussed in [30]. They reveal an interesting submodule structure which is a mixture of the Verma and Wakimoto structures. However a conformal field theory with primary fields correspond to these modules is lacking at the moment.

It should be noted that a similar problem is considered from the mathematical point of view in [31] where an equivalence between some categories of modules over $\hat{sl}_2$ and topological $N = 2$ algebra is proven. The latter is closely connected with the standard $N = 2$ algebra and its chiral rings (see e.g. [32] for details).

(iv) An immediate consequence of section 3.1 is that a quantum group structure underlying the $N = 2$ discrete series is larger then $(U_qosp(2/1), U_qsl(2))$. It is due to contributions from $h^I_j, \ n = 1$ and $h^{II}$ sectors. The problem is to find it exactly.

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Appendix A

The purpose of this appendix is to compute the normalization factors relevant for the primary fields (2.16).

Let me normalize the two-point function as

$$\langle \Phi^j_{\mu,\bar{\mu}}(z_1, \bar{z}_1) \Phi^j_{-\mu,-\bar{\mu}}(z_2, \bar{z}_2) \rangle = \frac{(-)^{\mu-\bar{\mu}}}{|z_{12}|^{2\Delta_j}} \quad (A.1)$$

Using the corresponding normalization the $\Phi^j(x, \bar{x}, \bar{z})$ fields (see (2.10)) one easily gets

$$(-)^{\mu-\bar{\mu}} N(j, \mu, \bar{\mu}) N(j, -\mu, -\bar{\mu}) = \oint_{C_1} \oint_{C_2} dx_1 dx_2 x_1^{\mu-1-j} x_2^{-\mu-1-j} x_{12}^{2j} \times (c.c) \quad (A.2)$$

In above (c.c) means integrals with $x_i \to \bar{x}_i$ and $\mu \to \bar{\mu}$.  

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The substitutions $x_2 = tx_1$ and $\bar{x}_2 = \bar{t}\bar{x}_1$ lead to

$$(-)^{\bar{\mu}} N(j, \mu, \bar{\mu}) N(j, -\mu, -\bar{\mu}) = \oint_{C_t} dt \ t^{\mu - 1 - j} (1 - t)^{2j} \times (c.c) \quad .$$  \hspace{1cm} (A.3)

Choosing the contours $C_t$ and $\bar{C}_t$ as shown in Fig.1 and using the definition of the B-function, one finds

$$N(j, \mu, \bar{\mu}) N(j, -\mu, -\bar{\mu}) = \Gamma[2j + 1] \{ \Gamma[1 + j + \mu] \Gamma[1 + j - \mu] \Gamma[1 + j + \bar{\mu}] \Gamma[1 + j - \bar{\mu}] \}^{-1} \quad .$$  \hspace{1cm} (A.4)

Finally, the normalization factors are given by

$$N(j, \mu, \bar{\mu}) = N(j, -\mu, -\bar{\mu}) = \Gamma[2j + 1] \{ \Gamma[1 + j + \mu] \Gamma[1 + j - \mu] \Gamma[1 + j + \bar{\mu}] \Gamma[1 + j - \bar{\mu}] \}^{-\frac{1}{2}} \quad .$$  \hspace{1cm} (A.5)

For "A" series (A.5) reduces to

$$N(j, \mu) = \frac{\Gamma[2j + 1]}{\Gamma[1 + j + \mu] \Gamma[1 + j - \mu]} \quad .$$  \hspace{1cm} (A.6)

Appendix B

It turns out that it is easy to show that the primary fields of the $N = 2$ models parameterized by the first series $h^I$ form a closed OP algebra, i.e. there is decoupling of the second series $h^{II}$. To do this, I will use the free field representation of $N = 2$ algebra constructed by Yu and Zheng \cite{23}.

In the NS sector the holomorphic part is described by two free scalar chiral superfields $\phi^\pm$ coupled to a background charge. In terms of the component fields, complex scalars and fermions, one has

$$\phi^+(z) = \varphi^+(z) + \sqrt{2} \theta^+ \psi^+(z) + \theta^- \phi^+(z) \quad , \quad \phi^-(z) = \varphi^-(z) + \sqrt{2} \theta^+ \psi^-(z) - \theta^- \phi^-(z) \quad ,$$

But use the following normalization: $\oint_{C_a} \frac{dz}{z} = 1$. 

Fig.1 Contours used in the definition of the normalization factors $N(j, \mu, \bar{\mu})$. 

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where $z = (z, \theta^+, \theta^-)$ is a point on the supersphere.
The chirality means $D^\pm \phi^\pm(z) = D^\pm \phi^\mp(z) = 0$ with the superderivatives $D^\pm = \frac{\partial}{\partial z^\pm} + \theta^\pm \frac{\partial}{\partial \theta^\pm}$.
The two point functions of the component fields are normalized as
\[
\langle \phi^+(z_1)\phi^-(z_2) \rangle = -\log z_{12} \quad \langle \psi^+(z_1)\psi^-(z_2) \rangle = \frac{1}{z_{12}}.
\]

The holomorphic supercurrent is given by
\[
\mathcal{J}(z) = \frac{1}{2} D^+ \phi^+ D^- \phi^-(z) + i \alpha_0 \left( \partial \phi^+ - \partial \phi^- \right) .
\] (B.1)

It is an exercise on OP expansions to check that $\mathcal{J}$ has the following OP algebra
\[
\mathcal{J}(z_1)\mathcal{J}(z_2) = \frac{c_2/4}{z_{12}^2} + \left( \frac{\theta_{12}^- \theta_{12}^+}{z_{12}^2} + \frac{1}{2} \frac{\theta_{12}^+}{z_{12}} D^+ - \frac{1}{2} \frac{\theta_{12}^-}{z_{12}} D^- + \frac{\theta_{12}^+ \theta_{12}^-}{z_{12}} \frac{\partial}{\partial z_{12}} \right) \mathcal{J}(z_2) + \ldots ,
\] (B.2)

with $z_{12} = z_1 - \theta_1^- \theta_2^+ - \theta_1^+ \theta_2^-$ and $c_2 = 1 - 2\alpha_0^2$.

The primary fields are given by
\[
V_{\alpha^+,\alpha^-}(z) = e^{i(\alpha^+ \phi^-(z) + \alpha^- \phi^+(z))}.
\] (B.3)

They have the following OP expansions with the supercurrent
\[
\mathcal{J}(z_1)V_{\alpha^+,\alpha^-}(z_2) = \left( h \frac{\theta_{12}^- \theta_{12}^+}{z_{12}^2} + \frac{1}{2} \frac{\theta_{12}^+}{z_{12}} D^+ - \frac{1}{2} \frac{\theta_{12}^-}{z_{12}} D^- + \frac{\theta_{12}^+ \theta_{12}^-}{z_{12}} \frac{\partial}{\partial z_{12}} + \frac{q}{z_{12}} \right) V_{\alpha^+,\alpha^-}(z_2) + \ldots ,
\] (B.4)

where the conformal dimension $h = \alpha^+ \alpha^- - \frac{\alpha_0^2}{2}(\alpha^+ + \alpha^-)$ and U(1) charge $q = \frac{\alpha_0}{2}(\alpha^+ - \alpha^-)$.

The screening operators are expressed as
\[
S = \oint dz d\theta^+ d\theta^- V_{\alpha_+,\alpha_+}(z) \quad F^+ = \oint dz d\theta^+ V_{\alpha_-,\alpha_0}(z) \quad F^- = \oint dz d\theta^- V_{0,\alpha_-}(z) ,
\] (B.5)

with $\alpha_- = -\sqrt{k+2}$, $\alpha_+ \alpha_- = -1$.

In the free field representation the first series $h^I$ is described by \[23\]
\[
\alpha^+ + \alpha^- + n\alpha_- + m\alpha_+ = 0 \quad q = \frac{\alpha_0}{2}(\alpha^+ - \alpha^-) .
\] (B.6)

As to the second, it corresponds to
\[
\alpha^+ = -\frac{1}{2} m\alpha_+ \quad q = \frac{\alpha_0}{2}(\alpha^+ - \alpha^-) .
\] (B.7)

Now let me look at the three point function which contains two primaries from the first series $h^I$ and one from the second series $h^{II}$. The free field representation results in
\[
\langle \Phi_{q_1}^I(z_1)\Phi_{q_2}^I(z_2)\Phi_{q_3}^{II}(z_3) \rangle = \langle V_{\alpha_1^+ \alpha_1^-}(z_1)\tilde{V}_{\alpha_2^+ \alpha_2^-}(z_2)\tilde{V}_{\alpha_3^+ \alpha_3^-}(z_3) \rangle \prod_{i=1}^{s} S_i \prod_{j=1}^{f^+} F_j^+ \prod_{l=1}^{f^-} F_l^- ,
\] (B.8)
with a conjugate vertex operator $\tilde{V}_{\alpha^+,\alpha^-}(z) = V_{\alpha_0-\alpha_0,\alpha_0-\alpha^-}(z)$ \cite{24, 23} and $\{s, f^\pm\} \in \mathbb{N}$.

The balance of charges (zero modes) leads to

$$\begin{align*}
\begin{cases}
\alpha_1^+ + \alpha_1^- - \alpha_2^- + \alpha_3^+ + \alpha_5^- + 2s\alpha_+ + (f^+ + f^-)\alpha_- = 0, \\
-\alpha_1^+ + \alpha_1^- + \alpha_2^- - \alpha_3^+ + \alpha_5^- + (f^+ - f^-)\alpha_- = 0.
\end{cases}
\end{align*}$$ (B.9)

Taking the first equation, combined with (B.6) and (B.7), I find

$$q_3 = \frac{1}{2}(n_1 - n_2 - f^+ - f^-) + \frac{1}{2}(-m_1 + m_2 - m_3 + 2s)(k + 2)^{-1},$$ (B.10)

which implies that the conformal block isn’t zero if the charge $q_3$ is quantized like the weights \((2.7)\) up to a factor $k + 2$. Since in this case the series $h^{II}$ is equivalent to the $h^I$ one namely, $h^{II}_p = h^{I}_{\alpha, p+\beta}$ with $q = \frac{1-a}{2} + \frac{\beta}{2}(k + 2)^{-1}$; $\{\alpha, \beta\} \in \mathbb{N}$, it means decoupling of the second series.

In the above, I have considered the 3-point conformal block. However, the generalization to a n-point one is straightforward.

**Appendix C**

In this Appendix I will compute a multiple contour integral used in sections 3.1, 3.2 in order to build the ring structure.

Let me consider an integral

$$I(a, b, c) = \prod_{i=1}^{2} \oint_{C_i} \frac{dx_i}{x_i} \oint_{C_3} \frac{dx_3}{x_3^{1+a+b+c}} x_3^{a} x_2^{b} x_1^{c} \times (c.c),$$ (C.1)

with some real parameters $a$, $b$, $c$.

The substitutions $x_1 = t_1x_3$, $x_2 = t_2x_3$ ($\bar{x}_1 = \bar{t}_1\bar{x}_3$, $\bar{x}_2 = \bar{t}_2\bar{x}_3$) lead to

$$I(a, b, c) = \prod_{i=1}^{2} \oint_{C_i} \frac{dt_i}{t_i} (1 - t_1)^a(1 - t_2)^b \times (c.c).$$ (C.2)

Note that the holomorphic integral depends on contours $C_i$. In a general case $a \neq b$ there is no symmetry $C_1 \rightarrow C_2$, $C_2 \rightarrow C_1$, i.e. the integral is not the Dotsenko-Fateev type. At the case at hand one has $C_1 \rightarrow C_2$, $C_2 \rightarrow C_1$, $a \rightarrow b$, $b \rightarrow a$. Due to this reason there are two possibilities for contours $C_i$ namely:\[8]

\[8\] As in the case of Appendix A contours $C_i$ are taken in such way to cancel relative phases of integrals over $x_i$ and $\bar{x}_i$. 

\[17\]
As a result, one has
\[ I^{(A)}(a,b,c) = \frac{\Gamma^2[1 + a]}{\Gamma^2[1 - c] \Gamma^2[1 + a + c]} , \quad I^{(B)}(a,b,c) = \frac{\Gamma^2[1 + b]}{\Gamma^2[1 - c] \Gamma^2[1 + b + c]} \quad \text{(C.3)} \]

Since all three point functions considered in sections 3.1 and 3.2 are symmetric under \((n_1,m_1) \rightarrow (n_2,m_2), (n_2,m_2) \rightarrow (n_1,m_1)\) one is free to symmetrize a factor which comes from \(\mathcal{N}(j_3,-j_1-j_2)\) as well as \(I^{(i)}\). I use the following ansatz
\[ \mathcal{N}^{-1}(j_3,-j_1-j_2)I^{(i)} \rightarrow \frac{\Gamma[1+j_1+j_2+j_3]}{\Gamma[1-j_1-j_2+j_3]} \prod_{i=1}^{2} \Gamma[1+j_1+j_2+j_3-2j_i] \prod_{i=1}^{3} \Gamma[1+2j_i] . \quad \text{(C.4)} \]

It should be stressed that the first factor in the above is universal under any symmetrization prescription due to its explicit symmetry. On the other hand, it is the most important one since this is an origin for a truncation of fusion rules. From this point of view the results (fusion rules) are independent on the contours \(C_i(\bar{C}_i)\).

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