The large diffusion limit for the heat equation with a dynamical boundary condition

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Abstract

We study the heat equation on a half-space with a linear dynamical boundary condition. Our main aim is to show that, if the diffusion coefficient tends to infinity, then the solutions converge (in a suitable sense) to solutions of the Laplace equation with the same dynamical boundary condition.

Keywords: heat equation, dynamical boundary condition, large diffusion limit

1 Introduction

We consider the problem

\[
\begin{cases}
\varepsilon \partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & x \in \mathbb{R}^N_+, \ t > 0, \\
\partial_t u_\varepsilon + \partial_\nu u_\varepsilon = 0, & x \in \partial \mathbb{R}^N_+, \ t > 0, \\
u_\varepsilon(x,0) = \varphi(x), & x \in \mathbb{R}^N_+, \ 
\end{cases}
\]

(1.1)

where $N \geq 2$, $\mathbb{R}^N_+ := \mathbb{R}^{N-1} \times \mathbb{R}_+$, $\Delta$ is the $N$-dimensional Laplacian (in $x$), $\partial_t := \partial/\partial t$, $\partial_\nu := -\partial/\partial x_N$, $\varepsilon \in (0,1)$ and $\varphi$ and $\varphi_b$ are measurable functions in $\mathbb{R}^N_+$ and $\mathbb{R}^{N-1}$, respectively.
Our main aim is to show that, as \( \varepsilon \to 0 \), it holds that \( u_\varepsilon \to u \) (in a suitable sense), where \( u \) is the solution of

\[
\begin{aligned}
\Delta u &= 0, & x &\in \mathbb{R}_+^N, & t &> 0, \\
\partial_t u + \partial_x u &= 0, & x &\in \partial\mathbb{R}_+^N, & t &> 0, \\
u_\varepsilon(x,0) &= \varphi_0(x'), & x &= (x',0) &\in \partial\mathbb{R}_+^N.
\end{aligned}
\] (1.2)

This convergence does not look unexpected, see [1], but we are not aware of any previous result which would support this natural conjecture. In particular, convergence of this type means that the influence of the initial function \( \varphi \) is lost in the limit, and we shall describe this phenomenon in more detail.

A result in a similar spirit was established in [1] for the eikonal equation with the same dynamical boundary condition as in (1.1). More precisely, the following problem was considered in [1]:

\[
\begin{aligned}
\varepsilon \partial_t u_\varepsilon + |\nabla_x u_\varepsilon| &= 1, & x &\in \Omega, & t &> 0, \\
\partial_t u_\varepsilon + \partial_x u_\varepsilon &= 0, & x &\in \partial\Omega, & t &> 0, \\
u_\varepsilon(x,0) &= \varphi(x), & x &\in \overline{\Omega},
\end{aligned}
\]

where \( \varepsilon \in (0,1) \), \( \Omega \subset \mathbb{R}^N \) is a bounded domain with \( C^1 \)-boundary, and \( \nu \) is the outer normal of \( \partial\Omega \). It was shown in [1] that \( u_\varepsilon \to u \) as \( \varepsilon \to 0 \), where \( u \) is the solution of

\[
\begin{aligned}
|\nabla_x u| &= 1, & x &\in \Omega, & t &> 0, \\
\partial_t u + \partial_x u &= 0, & x &\in \partial\Omega, & t &> 0,
\end{aligned}
\]

with an appropriate initial condition.

In the context of diffusion, the boundary condition from (1.1) describes thermal contact with a perfect conductor or diffusion of solute from a well-stirred fluid or vapour (see e.g. [8]). Various aspects of analysis of parabolic and elliptic equations with dynamical boundary conditions have been treated by many authors, see for example [2]–[7], [9, 11, 12, 21, 22, 25, 27, 29], [33]–[35] for the parabolic case and [10], [13]–[20], [26], [29]–[32], [36, 37] for the elliptic one. Here we demonstrate on the simplest possible linear example how are these two classes of problems linked.

Throughout this paper we often identify \( \mathbb{R}^{N-1} \) with \( \partial\mathbb{R}_+^N \). We introduce some notation. Let \( \Gamma_D = \Gamma_D(x,y,t) \) be the Dirichlet heat kernel on \( \mathbb{R}_+^N \), that is,

\[
\Gamma_D(x,y,t) := (4\pi t)^{-\frac{N}{2}} \left[ \exp \left( -\frac{|x-y|^2}{4t} \right) - \exp \left( -\frac{|x-y_*|^2}{4t} \right) \right]
\] (1.3)

for \( (x,y,t) \in \mathbb{R}_+^N \times \mathbb{R}_+^N \times (0,\infty) \), where \( y_* = (y',-y_N) \) for \( y = (y',y_N) \in \mathbb{R}_+^N \). Define

\[
[S_1(t)\phi](x) := \int_{\mathbb{R}_+^N} \Gamma_D(x,y,t)\phi(y) \, dy, \quad (x,t) \in \mathbb{R}_+^N \times (0,\infty),
\] (1.4)

for any measurable function \( \phi \) in \( \mathbb{R}_+^N \). For \( x = (x',x_N) \in \mathbb{R}_+^N \) and \( t > 0 \), set

\[
P(x',x_N,t) := c_N(x_N+t)^{1-N} \left( 1 + \left| \frac{x'}{x_N+t} \right|^2 \right)^{-\frac{N}{2}},
\]
where $c_N$ is a constant chosen so that

$$
\int_{\mathbb{R}^N} P(x', x_N, t) \, dx' = 1 \quad \text{for all } x_N \geq 0 \text{ and } t > 0.
$$

Then $P = P(x', x_N, t)$ is the fundamental solution of the Laplace equation in $\mathbb{R}^N_+$ with the homogeneous dynamical boundary condition, that is, $P$ satisfies

$$
\begin{cases}
-\Delta P = 0, & x \in \mathbb{R}^N_+, \ t > 0, \\
\partial_t P + \partial_x P = 0, & x \in \partial\mathbb{R}^N_+, \ t > 0, \\
P(x, 0) = \delta(x'), & x = (x', 0) \in \partial\mathbb{R}^N_+,
\end{cases}
$$

where $\delta = \delta(\cdot)$ is the Dirac delta function on $\partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}$. Define

$$[S_2(t)\psi](x) := \int_{\mathbb{R}^N} P(x' - y', x_N, t)\psi(y') \, dy', \quad (x, t) \in \overline{\mathbb{R}^N_+} \times (0, \infty), \tag{1.5}$$

for any measurable function $\psi$ in $\mathbb{R}^{N-1}$.

We formulate the definition of a solution of (1.1) by the use of the two integral kernels $\Gamma_D$ and $P$. For simplicity, let $\varphi_b = \varphi_b(x')$ and $g = g(x', t)$ be continuous functions in $\mathbb{R}^{N-1}$ and $\mathbb{R}^{N-1} \times (0, \infty)$, respectively, such that $\varphi_b(x')$ and $g(x', t)$ decay rapidly as $|x'| \to \infty$. Then the function

$$w(x, t) = w(x', x_N, t) := [S_2(t)\varphi_b](x) + \int_0^t [S_2(t - s)g(s)](x) \, ds \tag{1.6}$$

can be defined for $x = (x', x_N) \in \mathbb{R}^N_+$ and $t > 0$ and it is a classical solution of the Cauchy problem for the Laplace equation with a nonhomogeneous dynamical boundary condition

$$
\begin{cases}
-\Delta w = 0, & x \in \mathbb{R}^N_+, \ t > 0, \\
\partial_t w + \partial_x w = g, & x \in \partial\mathbb{R}^N_+, \ t > 0, \\
w(x, 0) = \varphi_b(x'), & x = (x', 0) \in \partial\mathbb{R}^N_+.
\end{cases}
$$

It follows from (1.6) that

$$
\partial_t w(x, t) := \int_{\mathbb{R}^N} \partial_t P(x' - y', x_N, t)\varphi_b(y') \, dy' + \int_{\mathbb{R}^N} P(x' - y', x_N, 0)g(y', t) \, dy' + \int_0^t \int_{\mathbb{R}^N} \partial_t P(x' - y', x_N, t - s)g(y', s) \, dy' \, ds \tag{1.8}
$$

for $x = (x', x_N) \in \mathbb{R}^N_+$ and $t \in (0, T)$. Set

$$\Phi(x) := \varphi(x) - [S_2(0)\varphi_b](x). \tag{1.9}$$
Then the function
\[ v_\epsilon(x, t) := \int_0^t [S_1(\epsilon^{-1}(t-s))] \partial_t w(s)](x) \, ds \]
satisfies
\[
\begin{cases}
\epsilon \partial_t v_\epsilon = \Delta v_\epsilon, & x \in \mathbb{R}_+^N, \ t > 0, \\
\partial_t w_\epsilon = \Delta w_\epsilon, & x \in \partial \mathbb{R}_+^N, \ t > 0, \\
v_\epsilon(x, 0) = \Phi(x), & x \in \mathbb{R}_+^N, \\
w_\epsilon(x, 0) = \varphi_b(x'), & x = (x', 0) \in \partial \mathbb{R}_+^N,
\end{cases}
\tag{1.10}
\]
Let \( \partial_{x_N} := \partial / \partial x_N \). If \( g_\epsilon(x', t) := \partial_{x_N} v_\epsilon(x', 0, t) \) for \( x' \in \mathbb{R}^{N-1}, t > 0 \), and \( w_\epsilon \) is defined as in (1.6) with \( g_\epsilon \) instead of \( g \), then it follows from \( (1.7), (1.8) \) and \( (1.10) \) that \( v_\epsilon \) and \( w_\epsilon \) satisfy
\[
\begin{cases}
\epsilon \partial_t v_\epsilon = \Delta v_\epsilon - \epsilon F_1[\varphi_b] - \epsilon F_2[v_\epsilon], & x \in \mathbb{R}_+^N, \ t > 0, \\
\Delta w_\epsilon = 0, & x \in \mathbb{R}_+^N, \ t > 0, \\
v_\epsilon = 0, \ \partial_t w_\epsilon - \partial_{x_N} w_\epsilon = \partial_{x_N} v_\epsilon, & x \in \partial \mathbb{R}_+^N, \ t > 0, \\
v_\epsilon(x, 0) = \Phi(x), & x \in \mathbb{R}_+^N, \\
w_\epsilon(x, 0) = \varphi_b(x'), & x = (x', 0) \in \partial \mathbb{R}_+^N,
\end{cases}
\tag{1.11}
\]
where
\[
F_1[\varphi_b](x, t) := \int_{\mathbb{R}^{N-1}} \partial_t P(x' - y', x_N, t) \varphi_b(y') \, dy',
\tag{1.12}
\]
\[
F_2[v](x, t) := \int_{\mathbb{R}^{N-1}} P(x' - y', x_N, 0) \partial_{x_N} v(y', 0, t) \, dy'
\tag{1.13}
+ \int_0^t \int_{\mathbb{R}^{N-1}} \partial_t P(x' - y', x_N, t-s) \partial_{x_N} v(y', 0, s) \, dy' \, ds.
\]
Furthermore, the function \( u_\epsilon := v_\epsilon + w_\epsilon \) is a classical solution of (1.1). Motivated by this observation, we formulate the definition of a solution of (1.1) via problem (1.11).

**Definition 1.1** Let \( \varphi \) and \( \varphi_b \) be measurable functions in \( \mathbb{R}_+^N \) and \( \mathbb{R}^{N-1} \), respectively. Let \( 0 < T \leq \infty \) and
\[
v_\epsilon, \ \partial_{x_N} v_\epsilon, \ w_\epsilon \in C(\overline{\mathbb{R}_+^N} \times (0, T)).
\]
We call \((v_\epsilon, w_\epsilon)\) a solution of (1.11) in \( \mathbb{R}_+^N \times (0, T) \) if \( v_\epsilon \) and \( w_\epsilon \) satisfy
\[
v_\epsilon(x, t) = [S_1(\epsilon^{-1}t)] \Phi(x) - \int_0^t [S_1(\epsilon^{-1}(t-s))] F_1[\varphi_b](s)(x) \, ds
\tag{1.14}
- \int_0^t [S_1(\epsilon^{-1}(t-s))] F_2[v_\epsilon](s)(x) \, ds,
\]
\[
w_\epsilon(x, t) = [S_2(t)] \varphi_b(x) + \int_0^t [S_2(t-s)] \partial_{x_N} v_\epsilon(s)(x) \, ds,
\tag{1.15}
\]
for \( x \in \overline{\mathbb{R}_+^N} \) and \( t \in (0, T) \). In the case when \( T = \infty \), we call \((v_\epsilon, w_\epsilon)\) a global-in-time solution of (1.11) and \( u_\epsilon \) a global-in-time solution of (1.1).
We are ready to state the main results of this paper. For $1 \leq r \leq \infty$, we write $\| \cdot \|_{L^r} := \| \cdot \|_{L^r(\partial \mathbb{R}^N)}$ and $\| \cdot \|_{L^r} := \| \cdot \|_{L^r(\mathbb{R}^N)}$ for simplicity.

**Theorem 1.1** Let $N \geq 2$, $\varepsilon \in (0, 1)$, $\varphi \in L^\infty(\mathbb{R}^N_+)$ and $\varphi_b \in L^\infty(\mathbb{R}^{N-1})$. Then problem (1.11) possesses a unique global-in-time solution $(v_\varepsilon, w_\varepsilon)$ satisfying

$$\sup_{0 < t < T} \left[ \| v_\varepsilon(t) \|_{L^\infty} + (\varepsilon^{-1} t)^{\frac{1}{2}} \| \partial x_N v_\varepsilon(t) \|_{L^\infty} + \| w_\varepsilon(t) \|_{L^\infty} \right] < \infty$$

(1.14)

for any $T > 0$. Furthermore, $v_\varepsilon$ and $w_\varepsilon$ are bounded and smooth in $\mathbb{R}^N_+ \times I$ for any bounded interval $I \subset (0, \infty)$ and have the following properties for any $\tau > 0$:

(a) There exists $C(\tau) > 0$ such that

$$\sup_{0 < t < \tau} \left[ \| v_\varepsilon(t) \|_{L^\infty} + (\varepsilon^{-1} t)^{\frac{1}{2}} \| \partial x_N v_\varepsilon(t) \|_{L^\infty} + \| w_\varepsilon(t) \|_{L^\infty} \right] \leq C_\tau \left( \| \varphi \|_{L^\infty} + |\varphi_b|_{L^\infty} \right);$$

(b) $\lim_{\varepsilon \to 0} \sup_{0 < t < \tau} t^{\frac{1}{2}} \| v_\varepsilon(t) \|_{L^\infty(\mathbb{R}^{N-1} \times (0,L))} = 0$ for any $L > 0$;

(c) $\lim_{\varepsilon \to 0} \sup_{0 < t < \tau} \| w_\varepsilon(t) - S_2(t) \varphi_b \|_{L^\infty} = 0$.

As a corollary of Theorem 1.1, we see that the solution $u_\varepsilon = v_\varepsilon + w_\varepsilon$ of (1.1) converges to the solution $S_2(t) \varphi_b$ of (1.2).

**Corollary 1.1** Assume the same conditions as in Theorem 1.1. Let $(v_\varepsilon, w_\varepsilon)$ be the solution given in Theorem 1.1. Then $u_\varepsilon = v_\varepsilon + w_\varepsilon$ is a classical solution of (1.1) and it satisfies

$$\lim_{\varepsilon \to 0} \sup_{\tau_1 < t < \tau_2} \| u_\varepsilon(t) - S_2(t) \varphi_b \|_{L^\infty(K)} = 0$$

for any compact set $K$ in $\mathbb{R}^N_+$ and $0 < \tau_1 < \tau_2 < \infty$.

We prepare some useful lemmata in Section 2 and then we give a proof of Theorem 1.1 in Section 3.

## 2 Preliminaries

In this section we prove several lemmata on $S_1(t)\phi$, $F_1[\varphi_b]$ and $F_2[v]$. In what follows, by the letter $C$ we denote generic positive constants (independent of $x$ and $t$) and they may have different values also within the same line.

We first recall the following properties of $S_1(t)\phi$ (see e.g. [24]).
For any $1 \leq q \leq r \leq \infty$,

$$\|S_1(t)\phi\|_{L^r} \leq Ct^{-\frac{N}{q}}\|\phi\|_{L^q}, \quad t > 0,$$

for all $\phi \in L^q(\mathbb{R}^N)$. In particular, if $q = r$, then

$$\sup_{t > 0} \|S_1(t)\phi\|_{L^r} \leq \|\phi\|_{L^r}. \quad (2.1)$$

Let $\phi \in L^q(\mathbb{R}^N)$ with $1 \leq q \leq \infty$. Then, for any $T > 0$, $S_1(t)\phi$ is bounded and smooth in $\mathbb{R}^N_+ \times (T, \infty)$. Furthermore, we have:

**Lemma 2.1** Let $\phi \in L^\infty(\mathbb{R}^N)$. Then

$$\sup_{t > 0} t^{\frac{1}{2}} \|\partial_{x_N}[S_1(t)\phi]\|_{L^\infty} \leq \|\phi\|_{L^\infty}. \quad (2.2)$$

Furthermore,

$$\lim_{\varepsilon \to 0} \sup_{t > 0} t^{\frac{1}{2}} \|S_1(\varepsilon^{-1}t)\phi\|_{L^\infty(\mathbb{R}^{N-1}_+ \times (0, L))} = 0 \quad \text{for any } L > 0. \quad (2.3)$$

**Proof.** It follows from (1.3) that

$$K(x, y, t) := \partial_{x_N} \Gamma_D(x, y, t)$$
$$= \Gamma_{N-1}(x' - y', t) \times$$
$$\times \left(-\frac{x_N - y_N}{2t} \Gamma_1(x_N - y_N, t) + \frac{x_N + y_N}{2t} \Gamma_1(x_N + y_N, t)\right) \quad (2.4)$$

for $(x, y, t) \in \mathbb{R}^N_+ \times \mathbb{R}^N_+ \times (0, \infty)$, where $\Gamma_d (d = 1, 2, \ldots)$ is the Gauss kernel in $\mathbb{R}^d$. Then

$$\int_{\mathbb{R}^N_+} |K(x, y, t)| \, dy$$
$$\leq \int_{0}^{\infty} \left(\frac{|x_N - y_N|}{2t} \Gamma_1(x_N - y_N, t) + \frac{x_N + y_N}{2t} \Gamma_1(x_N + y_N, t)\right) \, dy_N \quad (2.5)$$
$$= (\pi t)^{-\frac{1}{2}} \int_{0}^{\infty} 2\eta e^{-\eta^2} \, d\eta = (\pi t)^{-\frac{1}{2}}$$

for $x \in \mathbb{R}^N_+$ and $t > 0$. By (1.3) and (2.5) we have

$$|\partial_{x_N}[S_1(t)\phi](x)| \leq \int_{\mathbb{R}^N_+} |K(x, y, t)||\phi(y)| \, dy \leq t^{-\frac{1}{2}}\|\phi\|_{L^\infty}$$

for $x \in \mathbb{R}^N_+$ and $t > 0$. This implies (2.2).
On the other hand, for any $L > 0$, it follows from (1.3) that

$$
\int_{\mathbb{R}^N_+} \Gamma_D(x, y, \varepsilon^{-1}t) \, dy \\
= \int_{0}^{\infty} \left( \Gamma_1(x_N - y_N, \varepsilon^{-1}t) - \Gamma_1(x_N + y_N, \varepsilon^{-1}t) \right) \, dy_N \\
= 2(4\pi \varepsilon^{-1}t)^{-\frac{1}{2}} \int_{0}^{\infty} \exp \left( -\frac{\varepsilon^2 y^2}{4t} \right) \, dy \leq 2(4\pi \varepsilon^{-1}t)^{-\frac{1}{2}} L \leq C(\varepsilon^{-1}t)^{-\frac{1}{2}}
$$

for $x \in \mathbb{R}^{N-1} \times (0, L)$, $t > 0$ and $\varepsilon > 0$. For any $\phi \in L^\infty(\mathbb{R}^N_+)$, by (1.4) and (2.6) we have

$$
|S_1(\varepsilon^{-1}t)\phi(x)| \leq \int_{\mathbb{R}^N_+} \Gamma_D(x, y, \varepsilon^{-1}t) |\phi(y)| \, dy \leq C(\varepsilon^{-1}t)^{-\frac{1}{2}} \|\phi\|_{L^\infty}
$$

for $x \in \mathbb{R}^{N-1} \times (0, L)$, $t > 0$ and $\varepsilon > 0$. This implies (2.3), and the proof of Lemma 2.1 is complete. \(\blacksquare\)

Next we recall some properties of $S_2(t)\psi$.

(P1) Let $\psi \in L^r(\mathbb{R}^{N-1})$ for some $r \in [1, \infty]$ and $t$, $t' > 0$. Then

$$
[S_2(t)\psi](x', x_N) = [S_2(t + x_N)\psi](x', 0), \\
[S_2(t + t')\psi](x) = [S_2(t)(S_2(t')\psi)](x),
$$

for $x = (x', x_N) \in \overline{\mathbb{R}_+^{N}}$. Furthermore,

$$
\lim_{t \to 0} |S_2(t)\psi - \psi|_r = 0 \quad \text{if} \ 1 \leq r < \infty;
$$

(P2) For any $1 \leq q \leq r \leq \infty$,

$$
|S_2(t)\psi|_{L^r} \leq C t^{-(N-1)(\frac{1}{q} - \frac{1}{r})} |\psi|_{L^q}, \quad t > 0,
$$

for all $\psi \in L^q(\mathbb{R}^{N-1})$. In particular, if $q = r$, then

$$
\sup_{t > 0} |S_2(t)\psi|_{L^r} \leq |\psi|_{L^r}. \tag{2.7}
$$

Properties (P1) and (P2) easily follow from [15] (see e.g. [15]) and imply that

$$
\sup_{t > 0} \|S_2(t)\psi\|_{L^\infty} \leq |\psi|_{L^\infty} \tag{2.8}
$$

for all $\psi \in L^\infty(\mathbb{R}^{N-1})$. Furthermore, by a similar argument as in the proof of property $(G_2)$ we have:

(P3) Let $\psi \in L^q(\mathbb{R}^{N-1})$ with $1 \leq q \leq \infty$. Then, for any $T > 0$, $S_2(t)\psi$ is bounded and smooth in $\overline{\mathbb{R}_+^{N}} \times (T, \infty)$.  

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Lemma 2.2 Let $\psi \in L^\infty(\mathbb{R}^{N-1})$. Set

$$D_\varepsilon[\psi](x,t) := \int_0^t [S_1(\varepsilon^{-1}(t-s))F_1[\psi](s)](x) \, ds$$

(2.9)

for $x \in \mathbb{R}^N_+$, $t > 0$ and $\varepsilon > 0$. Then $D_\varepsilon[\psi]$ and $\partial_x D_\varepsilon[\psi]$ are bounded and smooth in $\mathbb{R}^N_+ \times (T, \infty)$ for any $T > 0$. Furthermore, there exists $C > 0$ such that

$$\|D_\varepsilon[\psi](t)\|_{L^\infty} \leq Ct^{\frac{1}{2}}(\varepsilon^{\frac{1}{2}} + t^{\frac{3}{4}})|\psi|_{L^\infty}$$

(2.10)

for $t > 0$ and $\varepsilon > 0$. Moreover,

$$\lim_{\varepsilon \to 0, t \in (0, T_1)} \|D_\varepsilon[\psi](t)\|_{L^\infty(\mathbb{R}^{N-1} \times (0, L))} = 0$$

(2.11)

for $T_1 > 0$ and $L > 0$.

Proof. We prove (2.10) first. Since

$$\partial_t P(x', x_N, t) = \frac{1}{x_N + t} \frac{|x'|^2 - (N - 1)(x_N + t)^2}{|x'|^2 + (x_N + t)^2} P(x', x_N, t),$$

it follows that

$$|\partial_t P(x', x_N, t)| \leq C(x_N + t)^{-1} P(x', x_N, t).$$

(2.12)

By (1.12), (2.7) and (2.12) we have

$$\|F_1[\psi](\cdot, y_N, s)\|_{L^\infty(\mathbb{R}^{N-1})} \leq C(y_N + s)^{-1}\|S_2(s + y_N)\psi\|_{L^\infty(\mathbb{R}^{N-1})}$$

$$\leq C(y_N + s)^{-1}|\psi|_{L^\infty}$$

(2.13)

for $y_N \in [0, \infty)$ and $s > 0$. Since

$$(y_N + s)^{-1} \leq \begin{cases} y_N^{-\frac{3}{4}}s^{-\frac{1}{4}} & \text{for } 0 \leq y_N \leq 1, \\ 1 & \text{for } y_N > 1, \end{cases}$$

(2.14)

by (1.3), (2.9) and (2.13) we see that

$$|D_\varepsilon[\psi](x,t)| \leq \int_0^t \int_{\mathbb{R}^N_+} \Gamma_D(x, y, \tau_\varepsilon)|F_1[\psi](y, s)| \, dy \, ds$$

$$\leq C \int_0^t \int_0^\infty \Gamma_1(x_N - y_N, \tau_\varepsilon)|F_1[\psi](\cdot, y_N, s)|_{L^\infty(\mathbb{R}^{N-1})} \, dy_N \, ds$$

$$\leq C|\psi|_{L^\infty} \int_0^t \int_0^{\infty} \tau_\varepsilon^{-\frac{1}{2}} \exp\left(-\frac{(x_N - y_N)^2}{4\tau_\varepsilon}\right) (y_N + s)^{-1} \, dy_N \, ds$$

$$\leq C|\psi|_{L^\infty} \left\{ \int_0^t \tau_\varepsilon^{-\frac{1}{2}} y_N^{-\frac{1}{4}} y_N^{\frac{1}{4}} \, dy_N \, ds + \int_0^t ds \right\}$$

$$\leq C|\psi|_{L^\infty} \left\{ \frac{1}{4} \int_0^t (t - s)^{-\frac{3}{4}} s^{\frac{1}{4}} \, ds + t \right\}$$

$$\leq C|\psi|_{L^\infty}(\varepsilon^{\frac{1}{2}} t^{\frac{3}{4}} + t) = C|\psi|_{L^\infty}t^{\frac{1}{4}}(\varepsilon^{\frac{1}{2}} + t^{\frac{3}{4}})$$

(2.15)
for $x \in \mathbb{R}^N_+$, $t > 0$ and $\varepsilon > 0$, where $\tau_\varepsilon := \varepsilon^{-1}(t-s)$. Here $\Gamma_1$ is the one-dimensional Gauss kernel. This implies (2.10).

We prove (2.11). Let $L > 0$. Similarly to (2.15), we obtain

$$|D_\varepsilon[\psi](x, t)| \leq \int_0^t \int_{\mathbb{R}^N} \Gamma_D(x, y, \tau_\varepsilon)|F_1[\psi](y, s)| \, dy \, ds$$

$$\leq C \int_0^t \int_0^\infty \left( \Gamma_1(x_N - y_N, \tau_\varepsilon) - \Gamma_1(x_N + y_N, \varepsilon^{-1}(t-s)) \right)$$

$$\times ||F_1[\psi](\cdot, y_N, s)||_{L_\infty(\mathbb{R}^{N-1})} \, dy_N \, ds$$

(2.16)

for $x \in \mathbb{R}^N_+, t > 0$ and $\varepsilon > 0$. This together with (2.6), (2.13) and (2.14) implies that

$$|D_\varepsilon[\psi](x, t)|$$

$$\leq C|\psi|_{L_\infty} \left\{ \int_0^t \tau_\varepsilon^{-\frac{1}{2}} s^{-\frac{1}{4}} \int_0^1 y_N^{-\frac{1}{4}} \, dy_N \, ds ight\}$$

$$+ \int_0^t \tau_\varepsilon^{-\frac{1}{2}} \int_1^\infty \left[ \exp \left( -\frac{(x_N - y_N)^2}{4\tau_\varepsilon} \right) - \exp \left( -\frac{(x_N + y_N)^2}{4\tau_\varepsilon} \right) \right] \, dy_N \, ds$$

$$\leq C|\psi|_{L_\infty} + \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{4}} \, ds$$

$$\leq C|\psi|_{L_\infty}(\varepsilon^{-1} + (\varepsilon t)^{-\frac{1}{2}})$$

for $x \in \mathbb{R}^{N-1} \times (0, L)$, $t > 0$ and $\varepsilon > 0$. Thus (2.11) holds.

On the other hand, it follows from the semigroup property of $S_1(t)$ that

$$D_\varepsilon[\psi](x, t) = \int_0^t \left[ S_1(\varepsilon^{-1}(t-s))F_1[\psi](s) \right](x) \, ds$$

$$= S_1(\varepsilon^{-1}(t-T/2))D_\varepsilon[\psi](x, T/2) + \int_{T/2}^t \left[ S_1(\varepsilon^{-1}(t-s))F_1[\psi](s) \right](x) \, ds$$

for $x \in \mathbb{R}^N_+$ and $0 < T < t < \infty$. Then, by (2.9) and (G_2) we see that

$$S_1(\varepsilon^{-1}(t-T/2))D_\varepsilon[\psi](x, T/2)$$

is bounded and smooth in $\mathbb{R}^N_+ \times (T, \infty)$. Furthermore, by (2.13) we apply the same argument as in [23, Section 3, Chapter 1] to see that

$$\int_{T/2}^t \left[ S_1(\varepsilon^{-1}(t-s))F_1[\psi](s) \right](x) \, ds$$
is also bounded and smooth in $\mathbb{R}_+^N \times (T, \infty)$. (See also [14, Proposition 5.2] and [28, Lemma 2.1].) Therefore we deduce that $D_\varepsilon[\psi]$ and $\partial_{x_N} D_\varepsilon[\psi]$ are bounded and smooth in $\mathbb{R}_+^N \times (T, \infty)$. Thus Lemma 2.2 follows. □

**Lemma 2.3** Let $0 \leq \alpha < 1$. Then there exists $C > 0$ such that
\[
\int_0^\infty \frac{|x \pm y|}{t} \Gamma_1(x \pm y, t)y^{-\alpha} dy \leq Ct^{-\frac{\alpha+1}{2}}
\]  
for $x \geq 0$ and $t > 0$. Here $\Gamma_1$ is the one-dimensional Gauss kernel.

**Proof.** Let $x \geq 0$ and $t > 0$. It follows that
\[
\int_0^\infty \frac{|x - y|}{t} \Gamma_1(x - y, t)y^{-\alpha} dy = (4\pi t)^{\frac{1}{2}} \left( \int_0^{x/2} + \int_{x/2}^\infty \right) \frac{|x - y|}{t} \exp \left( -\frac{|x - y|^2}{4t} \right) y^{-\alpha} dy.
\]
Since $y^{-1} \leq |x - y|^{-1}$ for $0 \leq x \leq 2y$, we have
\[
\int_0^\infty \frac{|x - y|}{t} \Gamma_1(x - y, t)y^{-\alpha} dy 
\leq Ct^{-\frac{1}{2}} \int_0^{x/2} \frac{x}{t} \exp \left( -\frac{x^2}{16t} \right) y^{-\alpha} dy + Ct^{-\frac{1}{2}} \int_{x/2}^\infty \frac{1}{t^{1/2}} \exp \left( -\frac{|x - y|^2}{4t} \right) dy
\]
\[
\leq Ct^{-\frac{3}{2}} x^{2-\alpha} \exp \left( -\frac{x^2}{16t} \right) + Ct^{-1+\frac{1}{2}} \leq Ct^{-\frac{\alpha+1}{2}}.
\]
Since $y^{-1} \leq 2(x + y)^{-1}$ for $0 \leq x \leq y$, similarly to (2.18), we obtain
\[
\int_0^\infty \frac{x + y}{t} \Gamma_1(x + y, t)y^{-\alpha} dy \leq Ct^{-\frac{\alpha+1}{2}}.
\]
Thus (2.17) holds and Lemma 2.3 follows. □

**Lemma 2.4** Let $\psi \in L^\infty(\mathbb{R}^{N-1})$. Then there exists $C > 0$ such that
\[
\|\partial_{x_N} D_\varepsilon[\psi](t)\|_{L^\infty} \leq C\varepsilon^{\frac{2}{\alpha}} t^{-\frac{\alpha}{4}} \|\psi\|_{L^\infty}
\]  
for $t > 0$ and $\varepsilon > 0$.

**Proof.** By (2.4), (2.9) and (2.13) we see that
\[
|\partial_{x_N} D_\varepsilon[\psi](x, t)| 
\leq \int_0^t \int_{\mathbb{R}_+^N} |K(x, y, \tau\varepsilon)||F_1[\psi](y, s)| \, dy \, ds
\]
\[
\leq C \int_0^t \int_0^\infty \tilde{K}(x_N, y_N, \tau\varepsilon)||F_1[\psi](\cdot, y_N, s)||_{L^\infty(\mathbb{R}^{N-1})} \, dy_N \, ds
\]
\[
\leq C|\psi|_{L^\infty} \int_0^t s^{-\frac{1}{2}} \int_0^\infty \tilde{K}(x_N, y_N, \tau\varepsilon) y_N^{-\frac{1}{2}} \, dy_N \, ds
\]
for \( x \in \mathbb{R}_+^N, t > 0 \) and \( \varepsilon > 0 \), where \( \tau_\varepsilon := \varepsilon^{-1}(t - s) \) and
\[
\tilde{K}(x_N, y_N, t) = \frac{|x_N - y_N|}{t} \Gamma_1(x_N - y_N, t) + \frac{x_N + y_N}{t} \Gamma_1(x_N + y_N, t)
\] (2.21)
for \( x_N \geq 0, y_N > 0 \) and \( t > 0 \). By (2.17) with \( \alpha = 1/2 \) and (2.20) we deduce that
\[
|\partial_{x_N} D_\varepsilon[v](x, t)| \leq C|v|_{L^\infty} \int_0^t s^{-\frac{1}{2}} \tau_\varepsilon^{-\frac{3}{4}} ds
\]
\[
= C|v|_{L^\infty} \int_0^t s^{-\frac{1}{2}} (\varepsilon^{-1}(t - s))^{-\frac{3}{4}} ds \leq C\varepsilon^\frac{3}{4} t^{-\frac{1}{4}} |v|_{L^\infty}
\]
for \( x \in \mathbb{R}_+^N, t > 0 \) and \( \varepsilon > 0 \). Thus (2.19) follows. \( \square \)

3 Proof of Theorem 1.1

We introduce some notation. Let \( T > 0 \) and \( \varepsilon \in (0, 1) \). Set
\[
X_T := \left\{ v, \partial_{x_N} v \in C(\mathbb{R}_+^N \times (0, T)) : \|v\|_{X_T} < \infty \right\}, \quad \|v\|_{X_T} := \sup_{0 < t < T} E_\varepsilon[v](t),
\]
where
\[
E_\varepsilon[v](t) := \|v(t)\|_{L^\infty} + (\varepsilon^{-1}t)^\frac{1}{2} \|\partial_{x_N} v(t)\|_{L^\infty}.
\]
Then \( X_T \) is a Banach space equipped with the norm \( \| \cdot \|_{X_T} \). For the proof of Theorem 1.1 we apply the Banach contraction mapping principle in \( X_T \) to find a fixed point of
\[
Q_\varepsilon[v](t) := S_1(\varepsilon^{-1}t)\Phi - D_\varepsilon[b](t) - \int_0^t S_1(\varepsilon^{-1}(t - s)) F_2[v](s) ds,
\] (3.1)
where \( \Phi, F_2[v] \) and \( D_\varepsilon[b] \) are as in (1.9), (1.13) and (2.9), respectively.

Lemma 3.1 There exists \( C > 0 \) such that
\[
F_2[v](x, t) \leq C\varepsilon^\frac{3}{4} \left( t^{\frac{1}{2}} + h(x_N, t) \right) \|v\|_{X_T}
\] (3.2)
for \( x \in \mathbb{R}_+^N, 0 < t < T, \varepsilon \in (0, 1) \) and \( v \in X_T \). Here
\[
h(x_N, t) := x_N^{-\frac{3}{4}} t^{\frac{1}{4}} \quad \text{if} \quad 0 < x_N \leq 1 \quad \text{and} \quad h(x_N, t) := x_N^{-\frac{1}{2}} \quad \text{if} \quad x_N > 1.
\]

Proof. Let \( T > 0, \varepsilon \in (0, 1) \) and \( v \in X_T \). It follows from (1.13) that
\[
F_2[v](x, t) = F_2'[v](x, t) + F_2''[v](x, t)
\] (3.3)
for \( x \in \mathbb{R}_+^N \) and \( t > 0 \), where
\[
F_2'[v](x, t) := \int_{\mathbb{R}_+^{N-1}} P(x' - y', x_N, 0) \partial_{x_N} v(y', 0, t) dy',
\]
\[
F_2''[v](x, t) := \int_0^t \int_{\mathbb{R}_+^{N-1}} \partial_t P(x' - y', x_N, t - s) \partial_{x_N} v(y', 0, s) dy' ds.
\]
Then there exists $T_*$ for $x \in \mathbb{R}_+^N$ and $t > 0$. On the other hand, it follows from (3.2) that

$$\|\partial_t P(x', x_N, t)\| \leq \begin{cases} CP(x', x_N, t) x_N^{-\frac{3}{2}} t^{-\frac{1}{2}} & \text{if } x_N \leq 1, \\ CP(x', x_N, t) x_N^{-\frac{1}{2}} t^{-\frac{1}{2}} & \text{if } x_N > 1, \end{cases}$$

for $x \in \mathbb{R}_+^N$ and $t > 0$. Therefore, by (3.3), (3.4), (3.5) and (3.6) we obtain

$$\|F''_2[v](x, t)\| \leq C \int_0^t (t - s)^{-\frac{1}{2}} \int_{\mathbb{R}_+^N} P(x' - y', x_N, t - s) |\partial_{x_N} v(y', 0, s)| dy' ds,$$

for $x' \in \mathbb{R}^{N-1}$, $0 < x_N \leq 1$ and $0 < t < T$. Similarly, we deduce that

$$\|F''_2[v](x, t)\| = C \int_0^t (t - s)^{-\frac{1}{2}} \int_{\mathbb{R}_+^N} P(x' - y', x_N, t - s) |\partial_{x_N} v(y', 0, s)| dy' ds,$$

for $x' \in \mathbb{R}^{N-1}$, $x_N > 1$ and $0 < t < T$. Therefore, by (3.3), (3.4), (3.5) and (3.6) we obtain (3.2). Thus Lemma 3.1 follows. □

**Lemma 3.2** For any $v \in X_T$ and $\varepsilon \in (0, 1)$, set

$$\tilde{D}_\varepsilon[v](t) := \int_0^t S_1(\varepsilon^{-1}(t - s)) F_2[v](s) ds.$$

Then there exists $T_* = T_*(N) > 0$ such that

$$\|\tilde{D}_\varepsilon[v]\|_{X_{T_*}} \leq \frac{1}{4} \|v\|_{X_{T_*}}$$

for $v \in X_{T_*}$ and $\varepsilon \in (0, 1)$. Furthermore, $\tilde{D}_\varepsilon[v]$ and $\partial_{x_N} \tilde{D}_\varepsilon[v]$ are bounded and smooth in $\mathbb{R}_+^N \times (\tau, T_*)$ for any $0 < \tau < T_*$. 

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Proof. Let $T > 0$. By (2.9) and (3.7) we see that $\tilde{D}_\varepsilon$ is defined analogously as $D_\varepsilon$ with $F_1$ replaced by $F_2$. Then it follows from (2.15) and (3.2) that

\[
|\tilde{D}_\varepsilon[v](x,t)| \\
\leq C \int_0^t \int_0^\infty \Gamma_1(x_N - y_N, \tau_\varepsilon) \|F_2[v](\cdot, y_N, s)\|_{L^\infty(\mathbb{R}^{N-1})} dy_N ds \\
\leq C \varepsilon^{\frac{1}{2}} \|v\|_{X_T} \int_0^t \int_0^\infty \tau_\varepsilon^{-\frac{1}{2}} \exp \left( -\frac{(x_N - y_N)^2}{4\tau_\varepsilon} \right) \left( s^{-\frac{1}{2}} + h(y_N, s) \right) dy_N ds \\
\leq C \varepsilon^{\frac{1}{2}} \|v\|_{X_T} \left\{ t^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} s^{\frac{1}{2}} ds + t \right\} \leq C \varepsilon^{\frac{1}{2}} \|v\|_{X_T} \left( t^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} t^{\frac{1}{2}} + t \right)
\]

for $x \in \mathbb{R}^N_+$ and $0 < t < T$, where $\tau_\varepsilon = \varepsilon^{-1}(t-s)$. Then, taking a sufficiently small $T > 0$ if necessary, we obtain

\[
\sup_{0 < t < T} \|\tilde{D}_\varepsilon[v](t)\|_{L^\infty} \leq \frac{1}{8} \|v\|_{X_T}.
\] (3.9)

On the other hand, similarly to (2.20), by (3.2) we see that

\[
\left| \partial_{x_N} \tilde{D}_\varepsilon[v](x,t) \right| \\
\leq C \int_0^t \int_0^\infty \tilde{K}(x_N, y_N, \tau_\varepsilon) \|F_2[v](\cdot, y_N, s)\|_{L^\infty(\mathbb{R}^{N-1})} dy_N ds \\
\leq C \varepsilon^{\frac{1}{2}} \|v\|_{X_T} \int_0^t \int_0^\infty \tilde{K}(x_N, y_N, \tau_\varepsilon) \left( s^{-\frac{1}{2}} + h(y_N, s) \right) dy_N ds \\
\leq C \varepsilon^{\frac{1}{2}} \|v\|_{X_T} \int_0^t \int_0^\infty \tilde{K}(x_N, y_N, \tau_\varepsilon) \left( s^{-\frac{1}{2}} + y_N^{-\frac{1}{4}} s^{\frac{1}{4}} + y_N^{\frac{1}{2}} \right) dy_N ds
\]

for $x \in \mathbb{R}^N_+$ and $0 < t < T$, where $\tilde{K}$ is the function given by (2.21). By (2.14) we have

\[
\left| \partial_{x_N} \tilde{D}_\varepsilon[v](x,t) \right| \leq C \varepsilon^{\frac{1}{2}} \|v\|_{X_T} \left( \int_0^t s^{-\frac{1}{2}} \tau_\varepsilon^{-\frac{1}{2}} ds + \int_0^t s^{\frac{1}{2}} \tau_\varepsilon^{\frac{1}{4}} ds + \int_0^t \tau_\varepsilon^{-\frac{3}{4}} ds \right) \\
= C \varepsilon^{\frac{1}{2}} \|v\|_{X_T} \left( \int_0^t s^{-\frac{1}{2}} (\varepsilon^{-1}(t-s))^{-\frac{1}{2}} ds \\
+ \int_0^t s^{\frac{1}{2}} (\varepsilon^{-1}(t-s))^{-\frac{1}{2}} ds + \int_0^t (\varepsilon^{-1}(t-s))^{-\frac{1}{2}} ds \right) \\
\leq C \varepsilon^{\frac{1}{2}} \|v\|_{X_T} \left( \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} t^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} t^{\frac{1}{2}} \right) \\
\leq C (\varepsilon^{-1} t)^{-\frac{1}{2}} \|v\|_{X_T} \left( (\varepsilon t)^{\frac{1}{2}} + (\varepsilon t)^{\frac{1}{2}} + (\varepsilon t)^{\frac{1}{2}} \right)
\]

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for \( x \in \mathbb{R}_+^N \) and \( 0 < t < T \). Taking a sufficiently small \( T > 0 \) if necessary, we see that

\[
\sup_{0 < t < T} (\varepsilon^{-1}t)^{\frac{1}{2}} \| \partial_{x_N} \tilde{D}_\varepsilon[v](t) \|_{L^\infty} \leq \frac{1}{8} \| v \|_{X_T}.
\]  
(3.10)

Therefore, by (3.5) and (3.10) we have (3.8). Furthermore, by (3.2) we apply a similar argument as in the proof of Lemma 2.2 and deduce that \( \tilde{D}_\varepsilon[v] \) and \( \partial_{x_N} \tilde{D}_\varepsilon[v] \) are bounded and smooth in \( \mathbb{R}_+^N \times (\tau, T) \) for any \( 0 < \tau < T \). Thus Lemma 3.2 follows. \( \square \)

Now we are ready to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1**

Let

\[
m := 16 \max \{ \| \varphi \|_{L^\infty}, \| \varphi_b \|_{L^\infty} \}.
\]  
(3.11)

Let \( T_* > 0 \) be as in Lemma 3.2 and \( v \in X_{T_*} \) with \( \| v \|_{X_{T_*}} \leq m \). Then, by property \((G_2)\), Lemmata 2.2 and 3.2 we see that \( Q_\varepsilon[v] \in X_{T_*} \). Since it follows from (1.9) and (2.8) that

\[
\| \Phi \|_{L^\infty} \leq \| \varphi \|_{L^\infty} + \| \varphi_b \|_{L^\infty},
\]

by (2.1), (2.2) and (3.11) we have

\[
\| S_1(\varepsilon^{-1}t)\Phi \|_{L^\infty} + (\varepsilon^{-1}t)^{\frac{1}{2}} \| \partial_{x_N}[S_1(\varepsilon^{-1}t)\Phi] \|_{L^\infty} \leq 2\| \Phi \|_{L^\infty} \leq \frac{m}{4}
\]  
(3.12)

for \( 0 < t < T_* \). Furthermore, by (2.10) and (2.19), taking a sufficiently small \( T_* > 0 \) if necessary, we see that

\[
\| D_\varepsilon[\varphi_b](t) \|_{L^\infty} + (\varepsilon^{-1}t)^{\frac{1}{2}} \| \partial_{x_N} D_\varepsilon[\varphi_b](t) \|_{L^\infty} \leq CT_*^{\frac{3}{2}}(1 + T_*^2)|\varphi_b|_{\infty} \leq \frac{m}{4}
\]  
(3.13)

for \( 0 < t < T_* \). Lemma 3.2 together with (3.11), (3.12) and (3.13) implies that

\[
\| Q_\varepsilon[v] \|_{X_{T_*}} \leq \| S_1(\varepsilon^{-1}t)\dot{\varphi} \|_{X_{T_*}} + \| D_\varepsilon[\varphi_b] \|_{X_{T_*}} + \| \tilde{D}_\varepsilon[v] \|_{X_{T_*}} \leq \frac{3m}{4} < m.
\]

Similarly, we obtain

\[
\| Q_\varepsilon[v_i] - Q_\varepsilon[v_2] \|_{X_{T_*}} = \| \tilde{D}_\varepsilon[v_1] - \tilde{D}_\varepsilon[v_2] \|_{X_{T_*}} \leq \frac{1}{4} \| v_1 - v_2 \|_{X_{T_*}}
\]

for \( v_i \in X_{T_*} \) with \( \| v_i \|_{X_{T_*}} \leq m \) \((i = 1, 2)\). Then, the contraction mapping theorem ensures that there exists a unique \( v_\varepsilon \in X_{T_*} \) with \( \| v_\varepsilon \|_{X_{T_*}} \leq m \) and

\[
v_\varepsilon = Q_\varepsilon[v_\varepsilon] = S_1(\varepsilon^{-1}t)\dot{\varphi} - D_\varepsilon[\varphi_b](t) - \tilde{D}_\varepsilon[v_\varepsilon](t) \quad \text{in} \quad X_{T_*}.
\]  
(3.14)

In particular, we see that

\[
\| v_\varepsilon \|_{X_{T_*}} \leq C(\| \varphi \|_{L^\infty} + |\varphi_b|_{L^\infty}).
\]  
(3.15)

Furthermore, by \((G_2)\) and Lemmata 2.2 and 3.2, we see that \( v_\varepsilon \) is bounded and smooth in \( \mathbb{R}_+^N \times (T_1, T_*) \) for any \( 0 < T_1 < T_* \). As before, set

\[
w_\varepsilon(x, t) = [S_2(t)\dot{\varphi}_b](x) + \int_0^t [S_2(t-s)\partial_{x_N}v_\varepsilon(s)](x) \, ds
\]
for $x \in \mathbb{R}_+^N$ and $t \in (0, T_*)$. By (2.8) and (3.11) we obtain

$$
\|w_\varepsilon(t)\|_{L^\infty} \leq \|S_2(t)\varphi_b\|_{L^\infty} + \int_0^t \|S_2(t-s) \partial_{x_N} v_\varepsilon(s)\|_{L^\infty} \, ds
$$

$$
\leq |\varphi_b|_{L^\infty} + \int_0^t |\partial_{x_N} v_\varepsilon(s)|_{L^\infty} \, ds
$$

$$
\leq \frac{m}{16} + \varepsilon^{\frac{1}{2}} m \int_0^t s^{-\frac{1}{2}} \, ds
$$

$$
\leq C(1 + T_*^\frac{1}{2}) \leq C(1 + T_*^\frac{1}{2})(\|\varphi\|_{L^\infty} + |\varphi_b|_{L^\infty}) < \infty
$$

(3.16)

for all $0 < t < T_*$. Furthermore, by (P_3) we apply a similar argument as in Lemma 2.2 and see that $w_\varepsilon$ is bounded and smooth in $\mathbb{R}_+^N \times (T_1, T_*)$ for any $0 < T_1 < T_*$. Therefore we deduce that $(v_\varepsilon, w_\varepsilon)$ is a solution of (1.11) in $\mathbb{R}_+^N \times (0, T_*)$. Since $T_*$ is independent of $m$, due to the semigroup properties of $S_1(t)$ and $S_2(t)$, we see that $(v_\varepsilon, w_\varepsilon)$ is a global-in-time solution of (1.11) and it satisfies assertion (a) for any $\tau > 0$.

Let $(\tilde{v}_\varepsilon, \tilde{w}_\varepsilon)$ be a global-in-time solution of (1.11) satisfying (1.14). Since

$$
v_\varepsilon - \tilde{v}_\varepsilon = Q_\varepsilon[v_\varepsilon] - Q_\varepsilon[\tilde{v}_\varepsilon] = \tilde{D}_\varepsilon[v_\varepsilon - \tilde{v}_\varepsilon] \quad \text{in} \quad X_T,$$

by (3.8) we have

$$
\|v_\varepsilon - \tilde{v}_\varepsilon\|_{X_T} \leq \frac{1}{4} \|v_\varepsilon - \tilde{v}_\varepsilon\|_{X_T}.
$$

This implies that $v_\varepsilon = \tilde{v}_\varepsilon$ in $X_T$. Repeating this argument, we see that $v_\varepsilon = \tilde{v}_\varepsilon$ in $X_T$ for any $T > 0$. Therefore we deduce that $(v_\varepsilon, w_\varepsilon)$ is a unique global-in-time solution of (1.11) satisfying (1.14).

It remains to prove assertions (b) and (c). Let $T' > 0$ and $L > 0$. By (1.14) and (2.8) we have

$$
\|w_\varepsilon(t) - S_2(t)\varphi_b\|_{L^\infty} \leq \int_0^t \|S_2(t-s) \partial_{x_N} v_\varepsilon(s)\|_{L^\infty} \, ds
$$

$$
\leq \int_0^t |\partial_{x_N} v_\varepsilon(s)|_{L^\infty} \, ds \leq C\|v_\varepsilon\|_{X_{T'}} \varepsilon^{\frac{1}{2}} \int_0^t s^{-\frac{1}{2}} \, ds \leq C\|v_\varepsilon\|_{X_{T'}} \varepsilon^{\frac{1}{2}} T'^{\frac{1}{2}}
$$

for all $t \in (0, T')$. This implies assertion (c). On the other hand, since $\tilde{D}_\varepsilon[v_\varepsilon]$ is given with
Proof of Corollary 1.1. Corollary 1.1 immediately follows from Theorem 1.1 and Definition 1.1. □

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