Research Article

The Construction of Type-2 Fuzzy Reasoning Relations for Type-2 Fuzzy Logic Systems

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Type-2 fuzzy reasoning relations are the type-2 fuzzy relations obtained from a group of type-2 fuzzy reasonings by using extended t-(co)norm, which are essential for implementing type-2 fuzzy logic systems. In this paper an algorithm is provided for constructing type-2 fuzzy reasoning relations of SISO type-2 fuzzy logic systems. First, we give some properties of extended t-(co)norm and simplify the expression of type-2 fuzzy reasoning relations in accordance with different input subdomains under certain conditions. And then different techniques are discussed to solve the simplified expressions on the input subdomains by using the related methods on solving fuzzy relation equations. Besides, it is pointed out that the computation amount level of the proposed algorithm is the same as that of polynomials and the possibility of applying the proposed algorithm in the construction of type-2 fuzzy reasoning relations is illustrated on several examples. Finally, the calculation of an arbitrary extended continuous t-norm can be obtained as the special case of the proposed algorithm.

1. Introduction

Type-2 fuzzy sets first proposed by Zadeh in 1975 [1] are fuzzy sets equipped with ordinary fuzzy subsets of [0, 1] as membership grades, henceforth called fuzzy truth values. Then Mizumoto and Tanaka [2, 3] used Zadeh’s extension principle to extend minimum and maximum both based on minimum for calculating union and intersection on type-2 fuzzy sets, respectively, and showed that the results of the union and intersection keep the convexity and normality. Based on the theory of type-2 fuzzy sets, Karnik et al. [4] proposed a new fuzzy system called type-2 fuzzy system. Up to now, both the theory and application of type-2 fuzzy systems have been widely researched (see, e.g., [5–8]). What is more, type-2 fuzzy neural networks and type-2 fuzzy classification and pattern recognition have been also studied (see, e.g., [9, 10]). However, the computation process of the extended operations on the noninterval type-2 fuzzy sets is more complex than that of ordinary operations on type-1 fuzzy sets, which blocks the wide use of the noninterval type-2 fuzzy logic systems, type-2 fuzzy neural networks, and so on. In recent years, a heated wave of research about the operation on type-2 fuzzy sets has been set off. For example, Karnik and Mendel [11] further generalized these definitions of operations presented by Mizumoto and Tanaka and gave some analytical formulae for extensions of extended maximum and minimum based on minimum or product. Kawaguchi and Miyakoshi [12, 13] showed that extended continuous t-(co)norms based on arbitrary t-norm satisfy the definitions of type-2 t-(co)norms. C. L. Walker and E. A. Walker [14, 15] considered the algebras of fuzzy truth values equipped with extended maximum and minimum based on minimum. Coupland and John [16, 17] presented geometric methods for performing the operations of extended minimum and maximum based on minimum on type-2 fuzzy sets. Starczewski [18] provided analytical expressions for membership functions of five kinds of extended t-norms. Ling and Zhang [19] reconstructed the framework of set-theoretic operations on triangle type-2 fuzzy sets by presenting polygon type-2 fuzzy sets and gave manageable and simplified formulas for operations on triangle type-2 fuzzy sets. Hu and Kwong [20] discussed extended t-norm on a linearly ordered set with a unit interval and a real number set as special cases.

From the above it can be seen that these research works have well contributed to the properties of extended t-(co)norms and gave many useful results for the calculations
of some kinds of extended t-(co)norms. All of these promote the structure of noninterval type-2 fuzzy logic systems since extended t-(co)norms are the important tools in the construction of type-2 fuzzy reasoning relations. Nevertheless, there are still many other extended t-norms whose membership functions lack analytical expressions or feasible algorithms. It hampers the attempt of the construction of type-2 fuzzy reasoning relations by using these extended t-norms. Besides, the work [18] leaves a key problem to us that, except for extended minimum and maximum both based on minimum, no theory guarantees that the results of general extended t-(co)norms on two type-2 fuzzy sets still satisfy the calculation conditions (e.g., convexity and normality). Moreover, there are always more than two fuzzy truth values in the calculation process of the construction of type-2 fuzzy reasoning relations; it may be time-consuming and laborious to proceed the calculation just on two fuzzy truth values each time. It is a natural idea that we can solve the computation in an integral and faster way. This paper is devoted to deal with these problems we have mentioned above. The following rows present our results: we show that the results of extended continuous t-(co)norms based on arbitrary t-norm keep the convexity and normality and simplify the expression of type-2 fuzzy reasoning relations of type-2 fuzzy logic systems with single input and single output (SISO) in accordance with different input subdomains under the condition that all the fuzzy truth values of type-2 fuzzy sets participated in the calculation are required to be convex and normal (Theorem 2). After that, we solve the simplified expressions on three input subdomains (from Theorem 3 to Theorem 9), which demonstrate an algorithm to construct type-2 fuzzy reasoning relations. The complexity of the algorithm is analyzed and it is pointed out that the computation amount level of the proposed algorithm is the same as that of polynomials. And then the possibility of applying the proposed algorithm in the construction of type-2 fuzzy reasoning relations is illustrated on several examples. Besides, the calculation of a class of extended t-norms being broader than those in [18] can be obtained as the special case of the proposed algorithm.

This paper is organized in five sections. The following section contains some preliminary knowledge and the concrete expression of type-2 fuzzy reasoning relations of SISO type-2 fuzzy logic systems. In Section 3 the method for the construction of type-2 fuzzy reasoning relations is investigated under certain conditions on the basis of the properties of extended t-(co)norm and the related methods on solving method of fuzzy relation equations. Section 4 gives several examples by using the presented method. Conclusions are given in Section 5.

2. Preliminaries

A type-2 fuzzy set \( A \) on the domain \( X \) is characterized by a membership function \( \mu_A : X \rightarrow \mathcal{F}([0, 1]) \), \( x \mapsto \mu_A(x) \), where \( \mathcal{F}([0, 1]) = \{ f : [0, 1] \rightarrow [0, 1] \} \), and \( f \) \( \in \mathcal{F}([0, 1]) \) is called a fuzzy truth value. Convenience to the following writing, we denote \( \mu_A(x) \) by \( \mu_{\tilde{A}}(x) \). Moreover, \( f \) is normal if there exists an \( x \in [0, 1] \) such that \( f(x) = 1 \) and convex if, for any \( x_1, x_2 \in [0, 1] \) and each \( \lambda \in [0, 1] \), \( f(\lambda x_1 + (1 - \lambda)x_2) = f(x_1) \wedge f(x_2) \). Let \( \mathcal{F}_{CN}([0, 1]) \) be the set of both convex and normal fuzzy truth values. Assume that \( \tilde{A}, \tilde{B} \in \mathcal{F}(X) \). Let \( \ast \) and \( \ast' \) be t-norm and \( \circ \) t-conorm. Union and intersection on type-2 fuzzy sets are given as follows. For \( \forall x \in X \), \( \forall w \in [0, 1] \),

\[
\tilde{A} \cup \tilde{B} : \mu_{\tilde{A} \cup \tilde{B}}(x) (w) = \sup_{u=v+x} \mu_{\tilde{A}}(x) (u) \wedge \mu_{\tilde{B}}(x) (v),
\]

\[
\tilde{A} \cap \tilde{B} : \mu_{\tilde{A} \cap \tilde{B}}(x) (w) = \sup_{u=v+x} \mu_{\tilde{A}}(x) (u) \circ \mu_{\tilde{B}}(x) (v),
\]

where \( \wedge \) and \( \circ \) are called extended t-conorm and extended t-norm, respectively. Let \( X \times Y \) be a new domain constructed by two domains \( X, Y \). A type-2 fuzzy set \( \tilde{R} \in \mathcal{F}(X \times Y) \) is called a type-2 fuzzy relation between \( X \) and \( Y \), where

\[
\mu_{\tilde{R}} : X \times Y \rightarrow \mathcal{F}([0, 1]), \quad (x, y) \mapsto \mu_{\tilde{R}}(x, y) \subseteq \mathcal{F}_{CN}(X \times Y),
\]

In the following, we will give the expression of type-2 fuzzy relation from a group of type-2 fuzzy reasoning. This type-2 fuzzy relation is called a type-2 fuzzy reasoning relation. Let \( \{ \tilde{A}_i \}_{i \in \mathbb{N}} \) and \( \{ \tilde{B}_i \}_{i \in \mathbb{N}} \) be, respectively, type-2 fuzzy sets on input domain \( X \) and output domain \( Y \). For a group of type-2 fuzzy reasonings in a SISO type-2 fuzzy logic system

\[
\text{if } x \text{ is } \tilde{A}_i \text{ then } y \text{ is } \tilde{B}_i, \quad i = 1, \ldots, N,
\]

which can be rewritten as \( \{ \tilde{A}_i \rightarrow \tilde{B}_i \}, i = 1, \ldots, N \) and induce the total type-2 fuzzy reasoning relation as follows:

\[
\tilde{R} = \bigcup_{i=1}^{N} \tilde{R}_i = \bigcup_{i=1}^{N} (\tilde{A}_i \rightarrow \tilde{B}_i).
\]

By choosing the suitable \( \bigcup^{(\ast', \ast)} \) and \( \bigcap^{(\ast', \ast)} \) we can obtain that

\[
\mu_{\tilde{R}(x, y)}(w) = \mu_{\bigcup_{i=1}^{N} \mu_{\tilde{R}_i}(x, y)}(w)
\]

\[
= \left( \bigcup_{i=1}^{N} (\mu_{\tilde{A}_i}(x) \bigcap^{(\ast', \ast)} \mu_{\tilde{B}_i}(y)) \right) (w)
\]

\[
= \sup_{V \in \mathcal{F}_{CN}(Y)} \left( \bigcap_{i=1}^{N} (\tilde{V}_i (x)) (u) \ast' \mu_{\tilde{B}_i}(y) (v) \right),
\]

where \( \bigcup' \) and \( \ast' \) indicate the same t-norm. It is clear that the difficulty on the calculation of type-2 fuzzy reasoning relation is to solve the expression (5). For convenience, we first fix \( x \) and \( y \) and denote

\[
\mathbf{u} = (u_1, \ldots, u_N), \quad \mathbf{v} = (v_1, \ldots, v_N),
\]

\[
F(w) = \mu_{\tilde{R}(x, y)}(w),
\]
\[ g_i(u_i) = \mu_{\tilde{A}_i}(u_i), \quad i = 1, \ldots, N, \]
\[ h_i(v_i) = \mu_{\tilde{B}_i}(v_i), \quad i = 1, \ldots, N, \]
\[ f(u, v) = \mathcal{F}^N_{i=1} \left( g(u_i) \ast h(v_i) \right), \]
\[ P_w = \left\{ (u, v) \in [0, 1]^2 \mid \bigwedge_{i=1}^N (u_i \ast v_i) = w \right\}. \]

Then the expression (5) can be rewritten as
\[ F(w) = \sup \left\{ f(u, v) \mid (u, v) \in P_w \right\}. \]

In what follows, we mainly pay attention to working out the expression (7). When \( w \) changes, \( P_w \) and \( F(w) \) change with it. Then in order to solve \( F(w) \), we should reduce the range of \( P_w \) as much as possible and then obtain \( F(w) \) (i.e., the maximum of \( f(u, v) \) in \( P_w \)) according to the characteristic of elements in \( P_w \). Next, we will focus on analyzing the condition \( \bigvee_{i=1}^N (u_i \ast v_i) = w \), which is a fuzzy relation equation if \( u \) is regarded as a coefficient vector and \( v \) is regarded as an unknown vector. It is known that fuzzy relation equation was first presented by Sanchez in 1976 [21]. Following it, a lot of work has focused on the solvability conditions and the solution sets. For example, these works [22–24] have systematically introduced some theories of fuzzy relational equations. Bourke and Fisher [25] gave solution algorithms for fuzzy relational equations with max-product composition. Stambou and Tzafestas [26] discussed the resolution of composite fuzzy relation equations based on Archimedean triangular norms. Wang and Xiong [27] investigated the solution sets of a fuzzy relation equation with sup-conjunctive composition in a complete lattice. Next some conceptions and conclusions on fuzzy relation equations will be given.

Let \( a = (a_1, \ldots, a_N) \), \( b = (b_1, \ldots, b_N) \in [0, 1]^N \). Define the partial order
\[ a \leq b \iff a_i \leq b_i, \quad i = 1, \ldots, N. \] (8)

There exists no partial order relation between \( a \) and \( b \) if and only if \(|a, b| = [b, a] = \emptyset\).

Define
\[ a \lor b \triangleq (a_1 \lor b_1, \ldots, a_N \lor b_N), \]
\[ a \land b \triangleq (a_1 \land b_1, \ldots, a_N \land b_N). \]

The single fuzzy relation equation constituted by composite relation \( \lor \ast \ast \) is as follows:
\[ (a_1 \ast x_1) \lor (a_2 \ast x_2) \lor \cdots \lor (a_N \ast x_N) = b, \] (10)
where \( a = (a_1, \ldots, a_N) \in [0, 1]^N \) is the coefficient vector, \( b \in [0, 1] \) is known, and \( x = (x_1, \ldots, x_N) \in [0, 1]^N \) is unknown. Let \( \mathcal{X}_* \) be the solution set of (10). The greatest and minimal elements in \( \mathcal{X}_* \) are, respectively, called the greatest and minimal solutions of (10). Denote
\[ \mathcal{J}_*(a, b) = \sup \left\{ x \in [0, 1] \mid a \ast x \leq b \right\}, \]
\[ \mathcal{L}_*(a, b) = \inf \left\{ x \in [0, 1] \mid a \ast x \geq b \right\}. \]

Define \( \inf \emptyset = 1 \). Moreover, some necessary interpretations about the two operations are presented in the following.

1. \( \mathcal{J}_*(a, b) \geq b \) since \( a \ast b \in [1 \ast b = b \).
2. If \( a \ast c \leq b \) then \( \mathcal{J}_*(a, b) \geq c \); if \( a \ast c \geq b \), then \( \mathcal{L}_*(a, b) \leq c \).
3. Both \( \mathcal{J}_*(a, b) \) and \( \mathcal{L}_*(a, b) \) are monotone decreasing about the first variable, that is,
\[ a_1 \leq a_2 \implies \mathcal{J}_*(a_1, b) \geq \mathcal{J}_*(a_2, b), \]
\[ \mathcal{L}_*(a_1, b) \geq \mathcal{L}_*(a_2, b), \]

since \( \{ x \in [0, 1] \mid a_1 \ast x \leq b \} \supseteq \{ x \in [0, 1] \mid a_2 \ast x \leq b \} \) and \( \{ x \in [0, 1] \mid a_1 \ast x \geq b \} \subseteq \{ x \in [0, 1] \mid a_2 \ast x \geq b \} \).

For the first variable, that is, there exists \( a_i \geq b \) if and only if \( (10) \) has the greatest solution \( x^1 = (x^1_1, \ldots, x^1_N) = (\mathcal{J}_*(a_1, b), \ldots, \mathcal{J}_*(a_N, b)) \).

2. If \( \mathcal{X}_* \neq \emptyset \), then \( \mathcal{X}_* \) has the minimum solutions where the \( j \)-th minimum solution \( x^j_0 = (x^0_{j1}, \ldots, x^0_{jN}) \) \((1 \leq j \leq |\mathcal{G}_b|)\) is
\[ x^j_0 = \begin{cases} \mathcal{J}_*(a_k, b), & i = k, j, \\ 0, & \text{otherwise}, \end{cases} \quad i = 1, \ldots, N. \] (14)

Furthermore, the solution set of (10) can be written as
\[ \mathcal{X}_* = \bigcup_{j=1}^{G_b} [x^0_{j1}, x^*]. \] (15)

3. The Construction of Type-2 Fuzzy Reasoning Relations

In this section, we will demonstrate the solving process for the expression (7) gradually. First, we will simplify the expression (7) in accordance with three subdomains of \( w \). Importantly, for two of these subdomains we will, respectively, reduce \( P_w \) into its subdomains \( P_{w1} \) and \( P_{w2} \) but keeping the values of \( F(w) \) without change (Theorem 2). Then all the elements in \( P_{w1} \) and \( P_{w2} \) will be found out (Theorem 3). Following it, \( P_{w1} \) and \( P_{w2} \) will be further reduced into smaller subsets \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) still keeping the values of \( F(w) \) without change, respectively (Theorem 4). Finally, some theorems about how to get the exact value of \( F(w) \) will be presented on the basis of the characteristics of the \( f(u, v) \) on \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) (Theorems 7 and 9).
It needs to be stated that the proposed method to solve \( F(w) \) differs from the native algorithm which is just finding the maximal number of \( f(u,v) \) from all the elements in \( P_w \) (or \( P_{w_1} \) and \( P_{w_2} \)). The native algorithm is impractical due to its huge computation. But what form of the elements in \( P_w \) is the key to solving the problem (7). Let \( g \in \mathcal{F}_{CN}([0,1]) \). Denote \([g]_1 = \{u \in [0,1] \mid g(u) = 1\}\).

**Theorem 2.** Let \( w \in [0,1], g_1, \ldots, g_N, h_1, \ldots, h_N \in \mathcal{F}_{CN}([0,1]), \) where \([g]_1 = [m_{g_1}, n_{g_1}], \ldots, [g_N]_1 = [m_{g_N}, n_{g_N}], [h_1]_1 = [m_{h_1}, n_{h_1}], \ldots, [h_N]_1 = [m_{h_N}, n_{h_N}]\).

Denote

\[\alpha = \bigvee_{i=1}^{N} (m_{g_i} * m_{h_i}), \beta = \bigvee_{i=1}^{N} (n_{g_i} * n_{h_i}),\]

\[m_g = (m_{g_1}, \ldots, m_{g_N}), m_h = (m_{h_1}, \ldots, m_{h_N}),\]

\[n_g = (n_{g_1}, \ldots, n_{g_N}), n_h = (n_{h_1}, \ldots, n_{h_N}),\]

\[P_{w_1} = \left\{ (u,v) \in [0,1]^{2N} \mid \bigvee_{i=1}^{N} (u_i * v_i) = w, u \leq m_g, v \leq m_h \right\},\]

\[P_{w_2} = \left\{ (u,v) \in [0,1]^{2N} \mid \bigvee_{i=1}^{N} (u_i * v_i) = w, u \geq n_g, v \geq n_h \right\}.\]

Then the following items hold.

1. If \( w \in [0,a] \), then \( F(w) = \sup \{f(u,v) \mid (u,v) \in P_{w_1}\}.\)
2. If \( w \in [a,\beta] \), then \( F(w) = 1.\)
3. If \( w \in [\beta,1] \), then \( F(w) = \sup \{f(u,v) \mid (u,v) \in P_{w_2}\}.\)

Before the proof of Theorem 2, several conclusions and their proofs will be given in the following and the conclusion (a) is from [18].

(a) Let \( w \in [0,1], f, g \in \mathcal{F}_{CN}([0,1]) \), where \([f]_1 = [m_f, n_f] \) and \([g]_1 = [m_g, n_g] \). Assume that \( \ast \) is continuous. Denote

\[L = \{(u,v) \in [0,1]^2 \mid u \ast v = w\},\]

\[L_1 = \{(u,v) \in [0,1]^2 \mid u \ast v = w, u \leq m_f, v \leq m_g\},\]

\[L_2 = \{(u,v) \in [0,1]^2 \mid u \ast v = w, n_f \leq u, n_g \leq v\}.\]

Then the following items hold.

1. If \( w \in [0,m_f \ast m_g] \), then \((f \cap (\ast, \ast))g(w) = \sup \{f(u) \ast g(v) \mid (u,v) \in L_1\}.\)
2. If \( w \in [m_f \ast m_g, n_f \ast n_g] \), then \((f \cap (\ast, \ast))g(w) = 1.\)
3. If \( w \in [n_f \ast n_g, 1] \), then \((f \cap (\ast, \ast))g(w) = \sup \{f(u) \ast g(v) \mid (u,v) \in L_2\}.\)

(b) Suppose that the conditions is the same as that of (a).

Denote

\[C_1 = \{(u,v) \in [0,1]^2 \mid u \lor v = w, u \leq m_f, v \leq m_g\},\]

\[C_2 = \{(u,v) \in [0,1]^2 \mid u \lor v = w, n_f \leq u, n_g \leq v\}.\]

Then the following items hold.

1. If \( w \in [0,m_f \lor m_g] \), then \((f \cup (\lor, \ast))g(w) = \sup \{f(u) \ast g(v) \mid (u,v) \in C_1\}.\)
2. If \( w \in [m_f \lor m_g, n_f \lor n_g] \), then \((f \cup (\lor, \ast))g(w) = 1.\)
3. If \( w \in [n_f \lor n_g, 1] \), then \((f \cup (\lor, \ast))g(w) = \sup \{f(u) \ast g(v) \mid (u,v) \in C_2\}.\)

Proof. This proof is similar as that of (a) in [18] since \( \lor \) is also monotonous increasing in the first and second variables.

(c) Let \( w_1, w_2, \tau_1, \tau_2 \in [0,1] \), where \( w_1 < w_2 \). Assume that \* is continuous. Denote

\[\mathcal{M}_1 = \{(a,b) \in [0,1]^2 \mid a \ast b = w_1, a \leq \tau_1, b \leq \tau_2\},\]

\[\mathcal{M}_2 = \{(c,d) \in [0,1]^2 \mid c \ast d = w_2, c \leq \tau_1, d \leq \tau_2\},\]

\[\mathcal{M}_3 = \{(c,d) \in [0,1]^2 \mid a \leq \tau_1, a \leq \tau_2, a \leq \tau_2 \leq b\},\]

\[\mathcal{M}_4 = \{(c,d) \in [0,1]^2 \mid c \leq w_2, c \leq \tau_1, d \leq \tau_2 \}.\]

Then for every \((a,b) \in \mathcal{M}_1 \) [resp. \( \mathcal{M}_3 \)], there exists \((c,d) \in \mathcal{M}_2 \) [resp. \( \mathcal{M}_4 \)] such that \(a \leq c \) and \( b \leq d \).

Proof. Let \((a,b) \in \mathcal{M}_1 \) and \((u,v) \in \mathcal{M}_2 \). Since \( w_1 < w_2 \), by the monotonicity of \*, we have \( a \leq u \) or \( b \leq v \). Assume that \( a \leq u \). If \( b \leq v \), then the conclusion is obvious. For the case of \( b > v \), there is

\[a \ast b \leq u \ast v \leq u \ast b,\]

that is,

\[a \ast b \leq w_2 \leq a \ast b.\]
Proof. Let \([f_1 = [m_f, n_f], [g_1 = [m_g, n_g], w_1, w_2, w_3 \in [0, 1]], \) where \(w_1 \leq w_2 \leq w_3\). By the continuity of \(*\) and conclusion (a), we obtain that if \(w_2 \in [m_f \ast m_g, n_f \ast n_g]\), then \((f_\ast g)(w_2) = 1\). For the converse, let \(w \in [0, m_f \ast m_g]\). If there exists \(w_0 \in [0, m_f \ast m_g]\) such that \((f_\ast g)(w_0) = 1\), that is, there exists \((u_0, v_0) \in [0, 1]^2\) such that \(u_0 = w_0 \leq m_f \ast m_g\) and \(f(u_0) \ast g(v_0) = 1\), then there is \(f(u_0) = g(v_0) = 1\). By the monotonicity of \(*\), we have \(u_0 < m_f\) or \(v_0 < m_g\). Without loss of generality, we can assume that \(u_0 < m_f\). Thus \(f(u_0) < (f_\ast g)(1)\), which leads to a contradiction.

Therefore, \((f_\ast g)(w) < 1\) in a similar way, we can prove that if \(w \in (n_f \ast n_g, 1]\), then \((f_\ast g)(w) < 1\). To sum up, we have \([f_\ast g][0, 1] = [m_f \ast m_g, n_f \ast n_g]\).

Next we will give the proof of Theorem 2.

Proof. It is known that \(F(w) = \langle \bigvee_{i=1}^{N} (f_i \ast g_i) \rangle(w)\). From conclusion (d) it can be obtained that \([f_\ast g]_1 = [f_\ast g]_2 = \cdots = [f_\ast g]_N \in \mathcal{F}_{CN}(0, 1)\) and \([f_i \ast g_i]_1 = [f_i \ast g_i]_2 = \cdots = [f_i \ast g_i]_N = \{m_f, m_g, n_f, n_g\}, i = 1, \ldots, N\).

By conclusion (c), there exists \((u, v) \in \mathcal{N}_1\) such that \(u \leq m_f \ast m_g\), and \(v \leq n_f \ast n_g\). Because \(f(u) \leq f(u_0)\) and \(g(v) \leq g(v_0)\) by the convexity of \(f\) and \(g\), we have

\[
(f(u) \ast g(v))_1 \leq (f(u_0) \ast g(v_0))_1 \leq (f_\ast g)(w)_1 \leq 1.
\]

From (24) and (25), we get

\[
(f_\ast g)(w)_1 = (f_\ast g)(w)_2 \leq 1
\]

which implies that (22) holds. If \(w_2 \in (n_f \ast n_g, 1]\), then \(n_f \ast n_g \leq w_2 \leq w_3\). In a similar way, we can prove that \((f_\ast g)(w_2) \leq (f_\ast g)(w_3)\). Thus (22) holds. To sum up, there is \(f_\ast g \in \mathcal{F}_{CN}(0, 1)\). It is easy to prove that the conclusion (c) is valid if \(*\) is replaced with \(\ast\) since \(\ast\) is also monotone increasing in the first and second variables. Therefore, in a similar way, we can give the proof of \(f_\ast g \in \mathcal{F}_{CN}(0, 1)\).

□

From Theorem 2, it can be seen that when \(w \in [\alpha, \beta]\), we can omit the calculation process of \(F(w)\) since \(F(w) = 1\), and for other situations \(F(w)\) can be obtained from \(P_{u_1}\) or \(P_{v_1}\) independently. From now on, we will focus on analyzing the cases of \(w \in [0, \alpha]\) and \(w \in [\beta, 1]\) and assume that
\( g_i(t), h_i(t) \in F_{\mathbb{C}^N}(0, 1) \), where \( [g_i]_1 = [m_{g_i}, n_{g_i}], [h_i]_1 = [m_{h_i}, n_{h_i}], i = 1, \ldots, N \). Denote
\[
P_{o_1} = \{ u \in [0, 1]^N | u \leq m_g \},
\]
\[
P'_{o_1} = \{ v \in [0, 1]^N | v \leq m_h \},
\]
\[
P_{o_2} = \{ u \in [0, 1]^N | u \geq n_g \},
\]
\[
P'_{o_2} = \{ v \in [0, 1]^N | v \geq n_h \}.
\]
(30)
The idea about how to find the elements in \( P_{o_1} \) and \( P_{o_2} \) is to solve the fuzzy relation equation
\[
(u_1 \ast x_1) \lor (u_2 \ast x_2) \lor \cdots \lor (u_N \ast x_N) = w,
\]
by taking \( u \in P_{o_1} \) and then obtain the solution \( x \) in \( P'_{o_2} \). Thus \( (u, x) \in P_{o_1} \times P_{o_2} \). In this way, all the elements in \( P_{o_1} \times P_{o_2} \) can be found. Denote
\[
G_w = \{ i \in [1, \ldots, N] | u_i \geq w \}.
\]
(32)
Now we will provide all the elements of \( P_{o_1} \) and \( P_{o_2} \).

**Theorem 3.** Assume that \( * \) is continuous. For every \( u \in P_{o_1} \) or \( P_{o_2} \) denote the greatest solution of (31) in \( [0, 1]^N \) as \( x_u^* \) and minimal solution of (31) in \( [0, 1]^N \) as \( x_u^0, \ldots, x_u^{N_{G_w}} \) (if any).

The following items hold.

1. Suppose that \( w \in [0, \alpha] \). Then for every \( u \in P_{o_1} \) the solution set of (31) in \( P_{o_1} \) is \( \cup_{j=1}^{N_{w}} [x_{u_j}, x_u^* \wedge m_h] \) denoted by \( Y^1_u \) and
\[ P_{o_1} = \{ (u, v) \in [0, 1]^{2N} | v \in Y^1_u, u \leq m_g \}. \]
(33)
2. Suppose that \( w \in (\beta, 1] \). Then for every \( u \in P_{o_2} \) the solution set of (31) in \( P_{o_2} \) is \( \cup_{j=1}^{N_{w}} [x_{u_j}^0, v_h, x_u^*] \) denoted by \( Y^2_u \) and
\[ P_{o_2} = \{ (u, v) \in [0, 1]^{2N} | v \in Y^2_u, n_g \leq u \}. \]
(34)

**Proof.** (1) From Lemma 1 it is obvious that \( Y^1_u \) is the solution set of (31) in \( P_{o_1} \). For every \( (u, v) \in \{ (u, v) \in [0, 1]^{2N} | v \in Y^1_u, u \leq m_g \} \), we have \( u \in P_{o_1} \) and \( v \in Y^1_u \leq P'_{o_1} \). Then from Lemma 1 it can be inferred that \( \bigvee_{j=1}^{N_{w}} (u_j \ast v_j) = w \). Thus \( (u, v) \in P_{o_1} \); that is, \( \{ (u, v) \in [0, 1]^{2N} | v \in Y^1_u, u \leq m_g \} \subset P_{o_1} \). For the converse case, let \( (u, v) \in P_{o_1} \). Then \( \bigvee_{j=1}^{N_{w}} (u_j \ast v_j) = w \). Obviously \( v \leq m_g \), and \( v \) is a solution of (31) with the coefficient vector \( u \). Denote the solution set of (31) in \( [0, 1]^N \) as \( \{ x_{u_j}^0, x_u^* \} \). Clearly there exists \( j \in [1, \ldots, N_{G_u}] \) such that \( v \in \{ x_{u_j}^0, x_u^* \} \). Clearly, \( \bigvee_{j=1}^{N_{w}} (x_{u_j}^0 \wedge m_h) \leq Y^1_u \), that is, \( (u, v) \in \{ (u, v) \in [0, 1]^{2N} | v \in Y^1_u, u \leq m_g \} \); that is, \( (u, v) \in \{ x_{u_j}^0 \wedge m_h \} \). To sum up, the conclusion (1) holds. In a similar way, we can prove the case (2). □

**Corollary 4.** Assume that \( * \) is continuous. Let \( u \in P_{o_1} \) or \( P_{o_2} \). Denote the greatest solution of (31) in \( [0, 1]^N \) as \( x_u^* \) and minimal solution of (31) in \( [0, 1]^N \) as \( x_u^0, \ldots, x_u^{N_{G_u}} \) (if any).

The following hold.

1. Let \( u \in P_{o_1} \). Equation (31) has a solution in \( P'_{o_1} \) if and only if there exists \( j \in G_u \) such that \( x_{u_j}^* \leq m_h \).
2. Let \( u \in P_{o_2} \). Equation (31) has a solution in \( P'_{o_2} \) if and only if \( n_h \leq x_u^* \).

**Proof.** (1) Equation (31) has a solution in \( P'_{o_1} \), if and only if \( \bigvee u_j \neq 0 \), and if and only if there exists \( j \in G_u \) such that \( x_{u_j}^* \leq m_h \).

(2) Equation (31) has a solution in \( P'_{o_2} \), if and only if \( \bigvee u_j \neq 0 \), and if and only if \( n_h \leq x_u^* \).

Next, on the basis of Theorem 3 we will further find subsets of \( P_{o_1} \) and \( P_{o_2} \) but keeping the values of \( F(w) \) without change.

**Theorem 5.** Assume that \( w \) is continuous. The following holds.

1. Suppose that \( w \in [0, \alpha] \) and for every \( u \in P_{o_1} \) the greatest solution of (31) in \( [0, 1]^N \) is \( x_u^* \) (if any). Denote
\[ U_1 = \{ u \in P_{o_1} | \exists v \in P'_{o_1}, s.t. (u, v) \in P_{o_1} \}, \]
\[ X_1 = \{ (u, v) \in [0, 1]^{2N} | v = x_u^* \wedge m_h, u \in U_1 \}. \]
(35)
Then
\[ F(w) = \sup \{ f(u, v) | (u, v) \in X_1 \}. \]
(36)

2. Suppose that \( w \in (\beta, 1] \) and for every \( u \in P_{o_2} \) minimal solutions of (31) in \( [0, 1]^N \) are \( x_u^0, j = 1, \ldots, |G_u| \), (if any). Denote
\[ U_2 = \{ u \in P_{o_2} | \exists v \in P'_{o_2}, s.t. (u, v) \in P_{o_2} \}, \]
\[ X_2 = \{ (u, v) \in [0, 1]^{2N} | v \in \{ x_u^0 \vee m_h, j = 1, \ldots, |G_u| \}, u \in U_2 \}. \]
(37)
Then
\[ F(w) = \sup \{ f(u, v) | (u, v) \in X_2 \}. \]
(38)

**Proof.** (1) Clearly, for every \( u \in U_1 \), (31) has a solution in \( P'_{o_1} \). From Theorem 3, there is \( (u, x_u^* \wedge m_h) \in P_{o_1} \); that is, \( X_1 \subseteq P_{o_1} \). Then
\[ \sup \{ f(u, v) | (u, v) \in X_1 \} \leq \sup \{ f(u, v) | (u, v) \in P_{o_1} \} = F(w). \]
(39)
For the converse case, let \( (u, v) \in P_{o_1} \). Then \( u \in U_1 \) and \( v \in Y_u \). We have \( v \leq x_u^* \wedge m_h \). Denote \( v_u = x_u^* \wedge m_h \). From the convexity of \( g_i \) and \( h_i \) and the monotonicity of \( t \)-norm, there is
\[ f(u, v) = T_N \bigg( g_i(u_j) \ast h_i(v_j) \bigg) \]
\[ \leq T_N \bigg( g_i(u_j) \ast h_i(v_j) \bigg) = f(u, v_u). \]
(40)
That is, for every \((u,v) \in P_u\) there exists \((u,v_u) \in X_1\), such that \(f(u,v) \leq f(u,v_u)\). Thus
\[
 F(w) = \sup \{ f(u,v) \mid (u,v) \in P_{w1} \} \\
 \leq \sup \{ f(u,v) \mid (u,v) \in X_1 \}.
\]
(41)
Combined with (39) and (41), it can be shown that the conclusion (1) holds.

(2) Clearly, for every \(u \in U_2\), \((31)\) has a solution in \(P_{o2}\).

From Theorem 3, there is \((u, x_u) \vee n_u \in P_{o2}\), where \(x_u^0 \in \{x_{u,j}^0 \mid j = 1, \ldots, |G_u|\}\); that is, \(X_1 \subseteq P_{o2}\). Thus
\[
 \sup \{ f(u,v) \mid (u,v) \in X_1 \} \\
 \leq \sup \{ f(u,v) \mid (u,v) \in P_{o2} \} = F(w).
\]
(42)
For the converse case, let \((u, v) \in P_{o2}\). Then \(u \in U_2\) and \(v \in V_u^2\). There exists \(x_u \in \{x_{u,j}^0 \mid j = 1, \ldots, |G_u|\}\), such that \(x_u \vee n_u \leq v\). Denote \(v_u = x_u^0 \vee n_u^0\). From the convexity of \(g_j\) and \(h_1\) and the monotonicity of \(t\)-norm, there is
\[
f(u,v) = \sup_{i=1}^{N} \left\{ g_i(u_i) \star h_1(v_i) \right\} \\
\leq \sup_{i=1}^{N} \left\{ g_i(u_i) \star h_1(v_i) \right\} = f(u,v_u).
\]
(43)
That is, for every \((u,v) \in P_{o2}\), there exists \((u,v_u) \in X_2\), such that \(f(u,v) \leq f(u,v_u)\). Thus
\[
 F(w) = \sup \{ f(u,v) \mid (u,v) \in P_{o2} \} \\
 \leq \sup \{ f(u,v) \mid (u,v) \in X_2 \}.
\]
(44)
Combined with (42) and (44), conclusion (2) holds.

If \(w \in [0, \alpha)\), from Theorem 5 it can be seen that all of the elements in \(X_1\) can be obtained when all of the elements in \(U_j\) and the greatest solutions of the corresponding equation \((31)\) in \([0,1]^N\) are obtained. The following lemma describes the characteristics of the elements of \(X_1\). Denote
\[
 J = \left\{ i \in \{1, \ldots, N\} \mid m_{g_j} \star m_{h_1} \geq w \right\},
\]
\[
 \tilde{u}_j = \inf \left\{ x \in [0, m_{g_j}] \mid \mathcal{L}_j(x, w) \leq m_{h_j} \right\}, \quad j \in J.
\]
Lemma 6. Let \(w \in [0, \alpha)\) and \(u \in P_{oj}\). Assume that \(*\) is continuous. Then \((31)\) has a solution in \(P_{o1}\) if and only if there exists \(j \in J\) such that
\[
 (0, \ldots, 0, \tilde{u}_j \vee w, 0, \ldots, 0) \leq u \leq m_u.
\]
(46)
Proof. For the first, we will prove that \(w \leq \tilde{u}_j \vee w \leq m_{g_j}, \quad j \in J\).

It can be seen that \(\mathcal{L}_j(m_{g_j}, w) \leq m_{h_j}\) since \(m_{g_j} \star m_{h_j} \geq w\) for every \(j \in J\). Therefore, \(m_{g_j} \in \{x \in [0, m_{g_j}] \mid \mathcal{L}_j(x, w) \leq m_{h_j}\}\); that is, \(\tilde{u}_j = \inf \left\{ x \in [0, m_{g_j}] \mid \mathcal{L}_j(x, w) \leq m_{h_j} \right\}\). Obviously \(w \leq \tilde{u}_j \vee w \leq m_{g_j}\).

Let \(u \) satisfy (46). Since \(w \leq \tilde{u}_j \vee w \leq u_j\), it can be inferred that \((31)\) is solvable in \([0,1]^N\) and \(x_u^0\) is a minimal solution in \([0,1]^N\) from Lemma 1, where \(x_u^0 = (0, \ldots, 0, \mathcal{L}_j(u_j, w), 0, \ldots, 0)\). Then we have \(\mathcal{L}_j(u_j, w) \leq \mathcal{L}_j(u_j, w) \leq m_{h_j}\) since \(u_j \leq u_j \vee w \leq u_j \leq m_{g_j}\). That is, \(x_u^0 \leq m_u\), which verifies that \((31)\) has a solution in \(P_{o1}\) by Corollary 4.

For the converse case, let \(v \in P_{o1}\) be a solution of \((31)\). Then \((u_1 + v_1) \vee \cdots \cdots \vee (u_N + v_N) = w\). There exists \(j \in \{1, \ldots, N\}\) such that \(u_j + v_j = w\). Therefore \(m_{g_j} \geq u_j \geq w, m_{h_j} \geq v_j \geq w\) and \(m_{g_j} \star m_{h_j} \geq u_j \star v_j = w\). Thus \(j \in J\). It can be seen that equation \(u_j \star x = w\) is solvable and its minimal solution is \(\mathcal{L}_j(u_j, w)\) by Lemma 1. Because \(v_j\) is also a solution, we have \(\mathcal{L}_j(u_j, w) \leq v_j \leq m_{h_j}\). So \(u_j \in \{x \in [0, m_{g_j}] \mid \mathcal{L}_j(x, w) \leq m_{h_j}\}\). Therefore, \(u_j \leq u_j \leq m_{g_j}\). To sum up, \(u_j \vee w \leq u_j \leq m_{g_j}\); that is, \(u\) satisfies (46).

Now we will solve the formula (5) with the situation of \(w \in (0, \alpha)\). For every \(u \in [0,1]^N\) denote the greatest solution of \((31)\) in \([0,1]^N\) by \(x_u^*\) (if any), where \(x_u^* = (\mathcal{F}_j(u_1j, w), \ldots, \mathcal{F}_j(x_u, w))\). From Theorem 5 and Lemma 6 it can be inferred that
\[
 X_1 = \{ (u,v) \in [0,1]^2 \mid \forall j \in J \}
\]
(47)
Denote
\[
 \mathcal{F}_1 = \{ g_1(u_1) \star \cdots \star g_N(u_N) \star h_1(v_1) \}
\]
(48)
\[
 \star \cdots \star h_N(v_N) \mid (u,v) \in X_1 \}
\]
Obviously \(X_1\) can be viewed as a union of \(|J|\) subsets, where the \(j\)th subset is as follows:
\[
 \mathcal{G}_j = \{ g_1(u_1) \star \cdots \star g_N(u_N) \star h_1(u_1, w) \}
\]
\[
 \star \cdots \star h_N(u_N, w) \mid \{0,0, \tilde{u}_j \vee w, \ldots, 0\} \leq u \leq m_u, \quad j \in J.
\]
(49)
That is, \(\mathcal{G}_j = \bigcup_{j \in J} \mathcal{G}_j\). Notice that, for any \(j_1, j_2 \in J\) and \(j_1 \neq j_2\), it may appear that \(\mathcal{G}_j \cap \mathcal{G}_j \neq \emptyset\). However, it will not affect our final results. The following theorem provides a method to obtain \(F(w)\) when \(w \in [0, \alpha]\).

Theorem 7. Let \(w \in (0, \alpha)\). Assume that \(*\) is continuous. Denote
\[
 F_j^1 = \sup \{ g_i(u_i) \star h_1(u_i, w) \mid u_i \in [0, m_{g_i}] \},
\]
\[
 i \in \{1, \ldots, N\},
\]
If \( w \in \beta, 1 \), then by Theorem 5 all elements in \( \mathbb{X}_2 \) can be obtained when all of the elements in \( \mathcal{U}_2 \) and minimal solutions of the corresponding equation (31) in \([0, 1]^N\) are obtained. The following lemma describes the characteristics of the elements in \( \mathcal{U}_2 \). Denote
\[
\tilde{u}_i = \sup \{ x \in [n_{i_0}, 1] \mid \mathcal{I}_+ (x, w) \geq n_{i_0} \},
\]
\( i = 1, \ldots, N. \)

**Lemma 8.** Let \( w \in [\beta, 1] \) and \( u \in P_{o2} \). Assume that \( * \) is continuous. Then (31) has a solution in \( P_{o2} \) and if and only if there exists \( i \in \{1, \ldots, N\} \) such that
\[
(n_{1_0}, \ldots, n_{i_0}, n_{i_0} \vee w, n_{i_0}, \ldots, n_{i_0}) \subseteq u \subseteq (\tilde{u}_{i_0}, \tilde{u}_{i_0}, \tilde{u}_{i_0}, \ldots, \tilde{u}_{i_0}).
\]

**Proof.** For the first, we will give the proof of \( u \leq n_{i_0} \vee w \leq u_i \) for every \( i \in \{1, \ldots, N\} \). Note that \( n_{i_0} \leq u \leq n_{i_0} \) since \( \forall_{i=1}^{N}(n_{i_0} \vee n_{i_0}) \leq w \). Then \( \mathcal{I}_+ (n_{i_0}, w) \geq n_{i_0} \). Therefore, \( \mathcal{I}_+ (n_{i_0} \vee w, w) \geq n_{i_0} \) since \( \mathcal{I}_+ (w, w) = 1 \geq n_{i_0} \). We obtain that \( n_{i_0} \vee w \in [x \in [n_{i_0}, 1] \mid \mathcal{I}_+ (x, w) \geq n_{i_0}] \), which indicates that \( u_i \) always exists and \( u_i \geq n_{i_0} \vee w \). Obviously \( w \leq n_{i_0} \vee w \leq u_i \).

Let \( u \) satisfy (55). Because \( w \leq n_{i_0} \vee w \leq u_i \) from Lemma 1 it can be inferred that (31) has a solution in \([0, 1]^N\) and the greatest solution is
\[
x_u^* = (\mathcal{I}_+ (u_1, w), \ldots, \mathcal{I}_+ (u_i, w), \ldots, \mathcal{I}_+ (u_N, w)).
\]
Furthermore, for every \( i \in \{1, \ldots, N\} \) we have \( \mathcal{I}_+ (u_i, w) \geq \mathcal{I}_+ (u_i, w) \geq n_{i_0} \) since \( u \subseteq (\tilde{u}_{i_0}, \tilde{u}_{i_0}, \ldots, \tilde{u}_{i_0}) \). Then \( x_u^* \geq n_{i_0} \), which indicates that (31) has a solution in \( P_{o2} \) by Corollary 4.

For the conversion, assume that (31) has a solution in \( P_{o2} \). There exists \( x_0 \) such that \( x_0 \geq w \) for every \( x_0 \geq n_{i_0} \vee w \). Then \( u_i \geq n_{i_0} \vee w \) since \( u_i \geq n_{i_0} \vee w \). That is to say, \( u \geq (n_{i_0}, \ldots, n_{i_0}, n_{i_0} \vee w, n_{i_0}, \ldots, n_{i_0}) \). On the other hand, there is \( x_u^* \geq n_{i_0} \) by Corollary 4. Thus for every \( j \in \{1, \ldots, N\} \), we have \( u_j \in [x \in [n_{i_0}, 1] \mid \mathcal{I}_+ (x, w) \geq n_{i_0}] \), which indicates that \( u_i \leq \tilde{u}_i \).

Therefore, \( u \subseteq (\tilde{u}_{i_0}, \tilde{u}_{i_0}, \ldots, \tilde{u}_{i_0}) \).

Next we will solve the formula (5) with the situation of \( w \in (\beta, 1) \). For every \( u \in [0, 1]^N \), denote minimal solutions of (31) in \([0, 1]^N\) by \( x_u^0, j \in \{1, \ldots, [G_u]\} \) (any). From Theorem 5 and Lemma 8, it can be seen that
\[
\mathbb{X}_2 = \left\{ (u, v) \in [0, 1]^N \mid v \in [x_u^0] \vee n_{i_0}, j = 1, \ldots, [G_u]\right\} \cdot
\]

For every \( u \) satisfying (55) there must exist \( x_u^0 \in [x_u^0], j = 1, \ldots, [G_u]\) such that it has the following form:
\[
x_u^0 = (0, \ldots, 0, \mathcal{I}_+ (u_i, w), 0, 0, \ldots, 0).
\]
Then
\[ v = x^0 \cup n_h = (n_{h_1}, \ldots, n_{h_{N-1}}, L^+ (u_i, w) \cup n_{h_i}, \ldots, n_{h_{N}}) . \tag{59} \]

Denote
\[ \mathcal{F}_2 = \{ g_1 (u_i) \ast \cdots \ast g_N (u_N) \ast' h_1 (v_i) \ast' \cdots \ast' h_N (v_N) \mid (u, v) \in \mathcal{F}_2 \} , \tag{60} \]
\[ \delta_i = h_1 (n_{h_i}) \ast \cdots \ast h_{i-1} (n_{h_{i-1}}) \ast' h_i (n_{h_i}) \ast' \cdots \ast' h_N (n_{h_N}) , \tag{61} \]

Thus \( \mathcal{F}_2 \) can be viewed as a union of \( N \) subsets, where the \( i \)th subset is as follows:
\[ \mathcal{F}_i = \{ g_1 (u_i) \ast \cdots \ast g_i (u_i) \ast' \cdots \ast' g_N (u_N) \ast' h_i (L^+ (u_i, w) \cup n_{h_i}) \ast' \cdots \ast' h_N (v_N) \mid (u, v) \in \mathcal{F}_2 \} , \]

that is, \( \mathcal{F}_2 = \bigcup_{i=1}^{N} \mathcal{F}_i \). In fact, it is natural that \( \mathcal{F}_2 \supseteq \bigcup_{i=1}^{N} \mathcal{F}_i \) since \( \mathcal{F}_2 \supseteq \mathcal{F}_i \), \( i = 1, \ldots, N \). For the converse, take a \( g_1 (u_i) \ast \cdots \ast g_N (u_N) \ast' h_1 (v_1) \ast' \cdots \ast' h_N (v_N) \in \mathcal{F}_2 \). From the form of the elements in \( \mathcal{F}_2 \), there exists \( h_i \in \{ 1, \ldots, N \} \) such that
\[ v = (n_{h_1}, \ldots, n_{h_{i-1}}, L^+ (u_{i-1}, w) \cup n_{h_{i-1}}, n_{h_{i+1}}, \ldots, n_{h_{N}}) , \tag{63} \]

which indicates that \( u_{i-1} \geq w \) by Lemma 1. Clearly \( u_{i-1} \geq n_{g_{i-1}} \). Therefore,
\[ (n_{g_1}, \ldots, n_{g_{i-1}}, n_{g_{i}}, n_{g_{i+1}}, \ldots, n_{g_{N}}) \leq u \leq (u_{i-1}, \ldots, u_{i-1}, u_{i-2}, u_{i-3}, \ldots, u_N) . \tag{64} \]

From the above, we have \( g_1 (u_1) \ast \cdots \ast g_N (u_N) \ast' h_1 (v_1) \ast' \cdots \ast' h_N (v_N) \in \mathcal{F}_{i-1} \) that is, \( g_1 (u_1) \ast \cdots \ast g_N (u_N) \ast' h_1 (v_1) \ast' \cdots \ast' h_N (v_N) \in \bigcup_{i=1}^{N} \mathcal{F}_i \). Thus \( \mathcal{F}_2 \subseteq \bigcup_{i=1}^{N} \mathcal{F}_i \). To sum up, we obtain that \( \mathcal{F}_2 = \bigcup_{i=1}^{N} \mathcal{F}_i \).

Notice that, for any \( i_1, i_2 \in \{ 1, 2, \ldots, N \} \) and \( i_1 \neq i_2 \), it may appear that \( \mathcal{F}_{i_1} \cap \mathcal{F}_{i_2} \neq \emptyset \). However, it will not affect our final results. The following theorem provides a method to obtain \( F(w) \) when \( w \in (\beta, 1] \).

**Theorem 9.** Let \( w \in (\beta, 1] \). Assume that \( \ast \) is continuous. Then the following items hold:

1. \( \sup \mathcal{F}_i = \sup \{ g_i (u_i) \ast' h_1 (L^+ (u_i, w) \cup n_{h_i}) \mid u_i \in [n_{g_i} \cup w, u_i] \} , i = 1, \ldots, N \),
2. \( F(w) = \sup \{ \sup \mathcal{F}_i , i = 1, \ldots, N \} \).

**Proof.** (1) Similar to the proof of Theorem 7, it can be obtained that there exists \( u_i \in [n_{g_i} \cup w, u_i] \) such that
\[ g_i (u_i) \ast' h_1 (L^+ (u_i, w) \cup n_{h_i}) = \sup \{ g_i (u_i) \ast' h_1 (L^+ (u_i, w) \cup n_{h_i}) \mid u_i \in [n_{g_i} \cup w, u_i] \} , \tag{65} \]

Without loss of the generality, we prove the situation of \( \mathcal{F}_1 \). From Lemma 8, it is easy to see that
\[ (u_1, n_{g_1}, \ldots, n_{g_N}, L^+ (u_1, w) \cup n_{h_1}, n_{h_2}, \ldots, n_{h_N}) \in \mathcal{F}_2 . \tag{66} \]

Denote
\[ \sigma_1 = g_2 (n_{g_2}) \ast' \cdots \ast' g_N (n_{g_N}) , \tag{67} \]
\[ g_2 (u_2) \ast' \cdots \ast' g_N (u_N) \leq g_2 (n_{g_2}) \ast' \cdots \ast' g_N (n_{g_N}) . \tag{68} \]

Moreover, from the assumptions, there is
\[ g_1 (u_1) \ast' h_1 (L^+ (u_1, w)) \leq g_1 (u_1) \ast' h_1 (L^+ (u_1, w) \cup n_{h_1}) . \tag{69} \]

Thus
\[ g_1 (u_1) \ast' g_2 (u_2) \ast' \cdots \ast' g_N (u_N) \ast' h_1 (L^+ (u_1, w) \cup n_{h_1}) \ast' \delta_1 \leq g_1 (u_1) \ast' g_2 (n_{g_2}) \ast' \cdots \ast' g_N (n_{g_N}) \ast' h_1 (L^+ (u_1, w) \cup n_{h_1}) \ast' \delta_1 , \tag{70} \]

It can be obtained that
\[ \sup \mathcal{F}_1 = \sup \{ g_1 (u_1) \ast' g_2 (n_{g_2}) \ast' \cdots \ast' g_N (n_{g_N}) \ast' h_1 (L^+ (u_1, w) \cup n_{h_1}) \ast' \delta_1 \} = \sup \{ g_1 (u_1) \ast' h_1 (L^+ (u_1, w)) \ast' \delta_1 \ast' \sigma_1 \} = g_1 (u_1) \ast' h_1 (L^+ (u_1, w)) . \tag{71} \]

(2) It is clear that \( \sup \mathcal{F}_i = \sup \{ \sup \mathcal{F}_1 , i = 1, \ldots, N \} \) since \( \mathcal{F}_2 = \bigcup_{i=1}^{N} \mathcal{F}_1 \) and \( F(w) = \sup \mathcal{F}_2 \).

Up to now, we can get all the values of expression (5). That is to say, for every fixed \( x \in X \) and \( y \in Y \), we can calculate the function \( \mu_{R(x,y)} (w) \) when \( \ast \) is continuous and \( \mu_{R(x)} \) and \( \mu_{R(y)} \) are both convex and normal. On the basis of Theorems 2, 7, and 9, we will give the implementation procedures in the following.
Algorithm 10. Consider the following (Figure 1):

Step 1. Calculate \( \alpha = \bigvee_{i=1}^{N} (m_{g_i} \cdot m_{h_i}) \), \( \beta = \bigvee_{i=1}^{N} (n_{g_i} \cdot n_{h_i}) \). For every \( w \in [0,1] \) employ Step 2–Step 4.

Step 2. When \( w \in [0,\alpha) \), step size \( \Delta \) and calculate the variables

\[
J = \{ i \in \{1, \ldots, N\} \mid m_{g_i} \cdot m_{h_i} \geq w \},
\]
\[
\hat{u}_j = \inf \left\{ x \in [0, m_{g_j}] \mid \mathcal{L}_x(x, w) \leq m_{h_j} \right\}, \quad j \in J.
\]

Find the greatest value of \( g_j(u_i) \ast h_j(J_j(u_i, w) \land m_{h_i}) \) in \([0, m_{g_i}]\), denoted by \( F^1_i \), \( i \in \{1, \ldots, N\} \); find the greatest value of \( g_j(u_i) \ast h_j(J_j(u_i, w) \land m_{h_i}) \) in \([\hat{u}_j \lor w, m_{g_i}]\), denoted by \( F^2_i \), \( i \in \{1, \ldots, N\} \). Let

\[
\lambda_j = F^1_j \ast F^2_{j+1} \ast \ldots \ast F^1_N, \quad j \in J.
\]

Then \( F(w) = \sup \{ \lambda_j \}, j \in J \).

Step 3. When \( w \in [\alpha, \beta] \), \( F(w) = 1 \).

Step 4. When \( w \in (\beta, 1] \), step size \( \Delta \) and calculate the variables

\[
\hat{u}_i = \sup \left\{ x \in [n_{g_i}, 1] \mid \mathcal{I}_x(x, w) \geq n_{h_i} \right\}, \quad i = 1, \ldots, N.
\]

Figure 1: The structure of Algorithm 10.

For every \( i \in \{1, \ldots, N\} \), find the greatest value of \( g_i(u_i) \ast h_i([\mathcal{I}_x(u_i, w) \lor n_{h_i}], \hat{u}_i] \), denoted by \( F^2_i \), \( i \in \{1, \ldots, N\} \). Then \( F(w) = \sup \{ F^2_i, i = 1, \ldots, N \} \).

Remark 11. The above algorithm can be applied in calculating extended continuous t-norm based on arbitrary t-norm on two type-2 fuzzy sets once setting \( N = 1 \) and extended maximum based on arbitrary t-norm on \( N \) type-2 fuzzy sets once setting \( h_i(u) = 1, i = 1, \ldots, N \). Hence the type-2 fuzzy reasoning relations of type-2 fuzzy logic systems with multiple input and single output can be calculated.

Remark 12. It can be seen from the operation steps above that the presented method to calculate the formula (5) is much simpler than the native algorithm (i.e., finding the maximum of \( f(u, v) \) from all of the combination \((u, v)\) in \( P_{g_1} \) (or \( P_{g_2} \) and \( P_{w_2} \)) which is a huge operation process undoubtedly. Take \( w \) from \([0,1]\) with step size \( \Delta_0 \). Then the amount of computation is no more than

\[
4N + \frac{\alpha}{\Delta_0} \left( \sum_{i=1}^{N} 4m_{g_i} + N^2 \right) + \frac{1 - \beta}{\Delta_0} \left( \sum_{i=1}^{N} \frac{(1 - \hat{u}_i) + 3( \hat{u}_i - n_{g_i} \lor w)}{\Delta} + N \right),
\]

(75)
where $\mathcal{F}_*(a, b), \mathcal{L}_*(a, b)$, $\star$ or $\star'$ is considered one computation and the step size $\Delta$ is small enough. According to above analysis, we can draw the following conclusions: the computation amount level of the proposed algorithm is the same as that of polynomials.

4. Examples

In this section some concrete examples for the construction of type-2 fuzzy reasoning relations of SISO type-2 fuzzy logic systems on the proposed algorithm will be given. All of them are realized by using MATLAB2010 (b).

Example 1. Let input domain $X = \{x\}$ and output domain $Y = \{y\}$. Then each type-2 fuzzy reasoning relation $\tilde{R}_i$ ($i \in \{1, \ldots, N\}$) and the total type-2 fuzzy reasoning relation $\tilde{R}$ are only defined on $X \times Y = \{(x, y)\}$. In the group of type-2 fuzzy reasonings (8) we choose $\tilde{A}_i, \tilde{B}_i, i = 1, \ldots, 7$ as follows:

\[
\begin{align*}
\mu_{\tilde{A}_i}(x) &= \exp \left( -\frac{(x - 0.3)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{A}_i}(x) &= \exp \left( -\frac{(x - 0.34)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{A}_i}(x) &= \exp \left( -\frac{(x - 0.36)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{A}_i}(x) &= \exp \left( -\frac{(x - 0.4)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{A}_i}(x) &= \exp \left( -\frac{(x - 0.5)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{A}_i}(x) &= \exp \left( -\frac{(x - 0.55)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{B}_i}(x) &= \exp \left( -\frac{(x - 0.6)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{B}_i}(x) &= \exp \left( -\frac{(x - 0.5)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{B}_i}(x) &= \exp \left( -\frac{(x - 0.55)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{B}_i}(x) &= \exp \left( -\frac{(x - 0.6)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{B}_i}(x) &= \exp \left( -\frac{(x - 0.65)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{B}_i}(x) &= \exp \left( -\frac{(x - 0.68)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{B}_i}(x) &= \exp \left( -\frac{(x - 0.7)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{B}_i}(x) &= \exp \left( -\frac{(x - 0.75)^2}{2 \times 0.2^2} \right).
\end{align*}
\]  

(76)

Choose $\mathcal{U}^{(\lor,\star')} = \mathcal{U}^{(\lor,\oplus)}$ and $\mathcal{R}^{(\lor,\star')} = \mathcal{R}^{(\oplus,\oplus)}$, where $\alpha \oplus b = 0 \lor (a + b - 1)$, $\bar{A}_i$ and $\bar{B}_i$ as stated in Example 1, $i = 1, \ldots, 7$. Then expression (5) is reduced as

\[
\begin{align*}
\mu_{\tilde{R}_i}(x, y)(w) &= \sup_{\bigvee_{i=1}^{7} (0 \lor (x_i + y_i - 1)) = w} \left( \bigoplus_{i=1}^{7} \left( \mu_{\tilde{A}_i}(x_i) \circ \mu_{\tilde{B}_i}(y_i) \right) \right),
\end{align*}
\]  

(77)

where $\bigoplus$ and $\circ$ indicate the same t-norm. Here we shall calculate (77) by using our method. Clearly $\mathcal{F}_*(a, b) = \mathcal{L}_*(a, b) = 1 \land (b - a + 1)$ and $\alpha = \beta = 0.35$. The function graph of $\mu_{\tilde{R}_i}(x, y)(w)$ in (77) is shown in Figure 2.

Example 2. Choose $\mathcal{U}^{(\lor,\star')} = \mathcal{U}^{(\lor,\otimes)}$ and $\mathcal{R}^{(\lor,\star')} = \mathcal{R}^{(\otimes,\otimes)}$, where

\[
a \otimes b = \begin{cases} a \land b, & a \lor b = 1, \\ 0, & a \lor b < 1. \end{cases}
\]  

(78)

Let $\bar{A}_i$ and $\bar{B}_i$ be the same as stated in Example 1, $i = 1, \ldots, 7$. Then expression (5) is reduced as

\[
\begin{align*}
\mu_{\tilde{R}_i}(x, y)(w) &= \sup_{\bigvee_{i=1}^{7} (0 \lor (x_i + y_i - 1)) = w} \left( \bigodot_{i=1}^{7} \left( \mu_{\tilde{A}_i}(x_i) \otimes \mu_{\tilde{B}_i}(y_i) \right) \right),
\end{align*}
\]  

(79)

where $\bigodot$ and $\otimes$ indicate the same t-norm. Here we will calculate (79) by using our method. Clearly $\alpha = \beta = 0.35$. The function graph of $\mu_{\tilde{R}_i}(x, y)(w)$ in (79) is shown in Figure 3.
Figure 3: The function graph of $\mu_{R(x,y)}(w)$ in (79).

Figure 4: The function graph of $\mu_{R(x,y)}(w)$ in (80).

Example 3. Choose $\square(\lor,\star)$ = $\square(\lor,\odot)$ and $\sqcap(\star,\star)$ = $\sqcap(\cdot,\odot)$, where $a \cdot b = ab$. Let $\tilde{A}_1$ and $\tilde{B}_1$ be the same as stated in Example 1, $i = 1, \ldots, 7$. Then expression (5) is reduced as

$$\mu_{R(x,y)}(w) = \sup_{\chi_{x,y}(u,v)} \left( \bigwedge_{i=1}^{7} \left( \mu_{\tilde{A}_i(x)}(u_i) \odot \mu_{\tilde{B}_i(y)}(v_i) \right) \right).$$  (80)

Here we will calculate (80) by using our method. Clearly $\mathcal{F}(a,b) = \mathcal{L}(a,b) = 1 \land (b/a)$ and $\alpha = \beta = 0.45$. The function graph of $\mu_{R(x,y)}(w)$ in (80) is shown in Figure 4.

5. Conclusions

In this paper, an algorithm for constructing type-2 fuzzy reasoning relations of SISO type-2 fuzzy logic systems has been given under certain conditions. The results may serve to establish many new type-2 fuzzy logic systems by using different extended t-(co)norms. An important conclusion has been given that the results of extended continuous t-(co)norms based on arbitrary t-norms keep the convexity and normality, which guarantees the operation conditions of extended t-(co)norms for the next turn. It can be seen that the proposed algorithm deals with the antecedents and consequents of the group of type-2 fuzzy reasoning in an integral way and the computation amount level of the proposed algorithm is the same as that of polynomials, which indicates that the proposed algorithm may be well applied in type-2 fuzzy logic systems. Besides, it can be seen that the calculations of an extended continuous t-norm based on arbitrary t-norms can be obtained as the special case of the proposed algorithm, which is a new idea to calculate the membership functions of a class of extended t-norm. However, all the fuzzy truth values of type-2 fuzzy sets that participated in the calculation are required to be convex and normal. Obviously, by using our proposed algorithm more applications about noninterval type-2 fuzzy logic system and type-2 fuzzy neural network could be attempted.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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