A New Generalized Cassini Determinant

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Abstract

In this paper we extend a notion of Cassini determinant to recently introduced hyperfibonacci sequences. We find $Q$-matrix for the $r$-th generation hyperfibonacci numbers and prove an explicit expression of the Cassini determinant for these sequences.

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1 Introduction

Given the second order recurrence relation, defined by

$$a_{n+2} = \alpha a_{n+1} + \beta a_n,$$

where $\alpha$ and $\beta$ are constants, a sequence $(a_k)_{k \geq 0}$ is called a solution of (1) if its terms satisfy this recurrence. The set of all solutions of (1) forms a linear space, meaning that if $(a_k)_{k \geq 0}$ and $(b_k)_{k \geq 0}$ are two solutions then $(a_k + b_k)_{k \geq 0}$ is also a solution of (1). Furthermore, it holds true that for any constant $c$, $(ca_k)_{k \geq 0}$ is also a solution of (1). Using these basic properties of a linear space one can derive the identity

$$a_m b_{m-1} - a_{m-1} b_m = (-\beta)^{m-1}(a_1 b_0 - a_0 b_1),$$

where $(a_k)_{k \geq 0}$ and $(b_k)_{k \geq 0}$ are two solutions of recurrence (1) [7]. When $\alpha = \beta = 1$ and initial values of the terms are 0 and 1, respectively, the
relation (1) defines the well known Fibonacci sequence \((F_k)_{k\geq 0}\). One can find more on this subject in a classic reference [8]. In case of the Fibonacci sequence relation (2) reduces to
\[
F_{n-1}F_{n+1} - F_n^2 = (-1)^n
\]
and it is called Cassini identity [3, 6, 9]. This relation can also be written in matrix form as
\[
\det \begin{pmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{pmatrix} = (-1)^n.
\]

In this paper we study the hyperfibonacci sequences which are defined by the relation
\[
F^{(r)}_n = \sum_{k=0}^{n} F^{(r-1)}_k, \quad F^{(0)}_n = F_n, \quad F^{(r)}_0 = 0, \quad F^{(r)}_1 = 1,
\]
where \(r \in \mathbb{N}\) and \(F_n\) is the \(n\)-th Fibonacci number. The number \(F^{(r)}_n\) we shall call \(n\)-th hyperfibonacci number of \(r\)-th generation. These sequences are recently introduced by Dill and Mezó [2]. Several interesting theoretical number and combinatorial properties of these sequences are already proven, including those available in [1]. Here we define the matrix
\[
A_{r,n} = \begin{pmatrix}
F^{(r)}_n & F^{(r)}_{n+1} & \cdots & F^{(r)}_{n+r+1} \\
F^{(r)}_{n+1} & F^{(r)}_{n+2} & \cdots & F^{(r)}_{n+r+2} \\
\vdots & \vdots & \ddots & \vdots \\
F^{(r)}_{n+r+1} & F^{(r)}_{n+r+2} & \cdots & F^{(r)}_{n+2r+2}
\end{pmatrix}
\]
and we prove that \(\det(A_{r,n})\) is an extension of (3). Thus, we show that the generalization of the Cassini identity, expressed in a matrix form, holds true for the hyperfibonacci sequences.

2 \(Q\)-matrix of the hyperfibonacci sequences

According to the definition (5) obviously we have
\[
F^{(r)}_{n+1} = F^{(r)}_n + F^{(r-1)}_{n+1}.
\]
In case \(r = 1\) the second term \(F^{(r-1)}_{n+1}\) is determined by the Fibonacci recurrence relation,
\[
F^{(1)}_{n+3} = F^{(1)}_{n+2} + (F^{(1)}_{n+2} - F^{(1)}_{n+1}) + (F^{(1)}_{n+1} - F^{(1)}_n),
\]
thus we have

\[ F_{n+3}^{(1)} = 2F_{n+2}^{(1)} - F_n^{(1)}. \]  

Now, iteratively using (7) we derive the recurrence relation

\[ F_{n+2}^{(1)} = F_{n+1}^{(1)} + F_n^{(1)} + 1, \]  

(8)

\[
F_{n+3}^{(1)} = 2F_{n+2}^{(1)} - F_{n+1}^{(1)} + 2F_n^{(1)} - F_{n-1}^{(1)} - F_{n-2}^{(1)} - F_{n-3}^{(1)} - \cdots - F_n^{(1)} + 1.
\]

When \( r = 2 \) we use the same approach to get recurrence for the second generation of the hyperfibonacci numbers,

\[ F_{n+3}^{(2)} = 2F_{n+2}^{(2)} - F_{n}^{(2)} + n + 2. \]  

(9)

Namely, in this case the second term in (6) is determined by obtained recurrence relation (8). This means that again we can perform the \((n+1)\)-step iterative procedure, this time using

\[ F_{n+3}^{(2)} = 2F_{n+2}^{(2)} - F_n^{(2)} + 1. \]  

(10)

The fact that terms indexed 3 through \( n \) cancel each other and that \((n+1)\) is remains, completes the proof of (9).

Recall that polytopic numbers are generalization of square and triangular numbers. These numbers can be represented by a regular geometrical arrangement of equally spaced points. The \( n \)-th regular \( r \)-topic number \( P_n^{(r)} \) is equal to

\[ P_n^{(r)} = \binom{n + r - 1}{r}. \]  

(11)

When \( r = 3 \), the \( i \)-th step of the iterative procedure described above results with an extra \( i \), which sum to a triangular number \( \binom{n+3}{2} \) after the final \((n+1)\)st iteration. Furthermore, in the next case we add the \( i \)-th triangular number in \( i \)-th step of iteration. According to the properties of polytopic numbers, these numbers sum to the tetrahedral number \( \binom{n+4}{3} \). In general, in the \( i \)-th step of the iteration we add \( i \)-th regular \((r-1)\)-topic number and sum of these numbers after the final step of the procedure is the regular polytopic number \( \binom{n+r}{r-1} \). Now we collect all this reasoning into the following
Lemma 1. The difference between \( n \)-th \( r \)-generation hyperfibonacci number and the sum of its two consecutive predecessors is \( n \)-th regular \((r-1)\)-topic number,

\[
F^{(r)}_{n+2} = F^{(r)}_{n+1} + F^{(r)}_n + \binom{n+r}{r-1}.
\] (12)

We can also write relation (12) as

\[
F^{(r)}_{n+2} = F^{(r)}_{n+1} + F^{(r)}_n + P^{(r-1)}_{n+2}.
\]

Hyperfibonacci sequences can be defined by the vector recurrence relation

\[
\begin{pmatrix}
F^{(r)}_{n+1} \\
F^{(r)}_{n} \\
F^{(r)}_{n+2} \\
\vdots \\
F^{(r)}_{n+r+2}
\end{pmatrix}
= Q_{r+2}
\begin{pmatrix}
F^{(r)}_n \\
F^{(r)}_{n+1} \\
\vdots \\
F^{(r)}_{n+r+1}
\end{pmatrix}
\] (13)

where \( Q_{r+2} \) is a square matrix

\[
Q_{r+2} =
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
q_1 & q_2 & q_3 & \cdots & q_{r+1} & q_{r+2}
\end{pmatrix}
\] (14)

In order to determine elements \( q_1, \ldots, q_{r+2} \) we use the fact that terms from \( -r \) through 0 of the \( r \)-th generation hyperfibonacci numbers takes values 0, \( \ldots \pm 1, 0, 0, \ldots, 0, 1, r+1, \ldots \)

This follows from Lemma II since we have

\[
\binom{(n-2)+r}{r-1} = \frac{n(n+1)(n+2)\cdots(n+r-2)}{(r-1)!}.
\] (15)

These expressions are obviously equal to 0 for \( n = 0, -1, \ldots, -r \).

In particular, when \( n = -r + 2 \) we get

\[
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
q_1 & q_2 & q_3 & \cdots & q_{r+1} & q_{r+2}
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{pmatrix}
\]
meaning that \( q_{r+2} = F_2^{(r)} \). In the same way we obtain relations for all elements of \( Q_{r+2} \),
\[
q_{r+2} = F_2^{(r)} \\
q_{r+1} = F_3^{(r)} - F_2^{(r)} q_{r+2} \\
q_r = F_4^{(r)} - F_3^{(r)} q_{r+2} - F_2^{(r)} q_{r+1} \\
\cdots \\
q_1 = F_{r+3}^{(r)} - F_{r+3}^{(r)} q_{r+2} - \cdots - F_2^{(r)} q_2.
\]
This reasoning gives the next Theorem \( \blacksquare \).

**Theorem 1.** For the hypefibonacci sequences we have
\[
A_{r,n} = Q^n_{r+2} A_{r,0}.
\] (16)

**Proof.** Relation (13) can be written as \( A_{r,n} = Q_{r+2} A_{r,n-1} \). Now the statement of theorem follows immediately,
\[
A_{r,n} = Q_{r+2} A_{r,n-1} = Q^2_{r+2} A_{r,n-2} = Q^n_{r+2} A_{r,0}.
\]
\( \blacksquare \)

Elements \( q_1, \ldots, q_{r+2} \) can be expressed explicitly. In particular, expressions for \( q_r, q_{r+1}, q_{r+2} \) are
\[
q_{r+2} = 1 + r \\
q_{r+1} = 1 - \left( \frac{r + 1}{2} \right) \\
q_r = \frac{r^3 - 7r}{6}.
\]
As an example we calculate hyperfibonacci numbers \( F_3^{(2)}, F_4^{(2)}, \ldots, F_9^{(2)} \) of the second generation, collected in the matrix \( A_{2,3} \). For the second generation of the hyperfibonacci sequences we have
\[
A_{2,0} = \begin{pmatrix}
0 & 1 & 3 & 7 \\
1 & 3 & 7 & 14 \\
3 & 7 & 14 & 26 \\
7 & 14 & 26 & 46
\end{pmatrix}
\]
\[
Q_4 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & -1 & -2 & 3
\end{pmatrix}
\]
according to \((5)\) and \((14)\). Now we determine the matrix \(A_{2,3}\) by Theorem 1,
\[
A_{2,3} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & -1 & -2 & 3
\end{pmatrix} \begin{pmatrix}
0 & 1 & 3 & 7 \\
1 & 3 & 7 & 14 \\
3 & 7 & 14 & 26 \\
7 & 14 & 26 & 46
\end{pmatrix} = \begin{pmatrix}
7 & 14 & 26 & 46 \\
14 & 26 & 46 & 79 \\
26 & 46 & 79 & 133 \\
46 & 79 & 133 & 221
\end{pmatrix}.
\]

Note that the eigenvalues of \(Q_4\) are \(\phi, 1, 1, \bar{\phi}\), where
\[
\phi = \frac{1 + \sqrt{5}}{2}, \quad \bar{\phi} = \frac{1 - \sqrt{5}}{2}.
\]

A class of matrices \((14)\) has some further interesting properties. Here we point out that the determinant of such a matrix is \(-1\). This is demonstrated in the following

**Lemma 2.** For \(r \in \mathbb{N}\) the determinant of a matrix \(Q_{r+2}\) takes value \(-1\),

\[
\det(Q_{r+2}) = -1.
\]

**Proof.** We prove this statement by means of comparing determinants of matrices \(A_{r,-r}\) and \(A_{r,-r-1}\),
\[
A_{r,-r} = Q_{r+2}A_{r,-r-1}.
\]

For matrix \(A_{r,-r-1}\) we have
\[
\det(A_{r,-r-1}) = \det\begin{pmatrix}
(-1)^r & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 1 & \cdots & F_{r-2}^{(r)} \\
0 & 1 & r+1 & \cdots & F_{r-1}^{(r)}
\end{pmatrix}_{r \times r}
\]
\[
= (-1)^r \det\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & r+1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & \cdots & F_{r-3}^{(r)} & F_{r-2}^{(r)} \\
1 & r+1 & \cdots & F_{r-2}^{(r)} & F_{r-1}^{(r)}
\end{pmatrix}_{(r-1) \times (r-1)}
\]
\[
= (-1)^{r+1}(1-1)^{\lfloor (r-1)/2 \rfloor +1}
\]
\[
= (-1)^{\lfloor r/2 \rfloor}.
\]

On the other hand, \(\det(A_{r,-r}) = (-1)^{\lfloor r/2 \rfloor}\) which proves that
\[
\det(A_{r,-r}) = -\det(A_{r,-r-1}). \quad (17)
\]

Now, the statement of lemma follows immediately by the Binet-Cauchy theorem. \(\square\)
It is worth mentioning that in [5] authors give some properties of the $k$-generalized Fibonacci $Q$-matrix.

## 3 Cassini identity in a matrix form

**Lemma 3.**

\[
\det \begin{pmatrix}
F_n - 1 & F_{n+1} - 1 & F_{n+2} - 1 \\
F_{n+1} - 1 & F_{n+2} - 1 & F_{n+3} - 1 \\
F_{n+2} - 1 & F_{n+3} - 1 & F_{n+4} - 1
\end{pmatrix} = (-1)^n, \quad n \geq 0. \quad (18)
\]

**Proof.** Using the definition of Fibonacci numbers and elementary transformations on rows and columns of determinants we get

\[
\det \begin{pmatrix}
F_n - 1 & F_{n+1} - 1 & F_{n+2} - 1 \\
F_{n+1} - 1 & F_{n+2} - 1 & F_{n+3} - 1 \\
F_{n+2} - 1 & F_{n+3} - 1 & F_{n+4} - 1
\end{pmatrix}
= \det \begin{pmatrix}
F_n - 1 & F_{n+1} - 1 & F_{n+2} - 1 \\
F_{n+1} - 1 & F_{n+2} - 1 & F_{n+3} - 1 \\
F_n + F_{n+1} - 1 & F_{n+1} + F_{n+2} - 1 & F_{n+2} + F_{n+3} - 1
\end{pmatrix}
= \det \begin{pmatrix}
F_n - 1 & F_{n+1} - 1 & F_n + F_{n+1} - 1 \\
F_{n+1} - 1 & F_{n+2} - 1 & F_{n+1} + F_{n+2} - 1 \\
1 & 1 & 1
\end{pmatrix}
= \det \begin{pmatrix}
F_n - 1 & F_{n+1} - 1 & 1 \\
F_{n+1} - 1 & F_{n+2} - 1 & 1 \\
1 & 1 & -1
\end{pmatrix}
= -(F_n F_{n+2} - F_{n+1}^2) = (-1)^n \quad \Box
\]

**Lemma 4.** For the first generation of hyperfibonacci sequences $(F_n^{(1)})_{n \geq 0}$

\[
\det \begin{pmatrix}
F_n^{(1)} & F_{n+1}^{(1)} & F_{n+2}^{(1)} \\
F_{n+1}^{(1)} & F_{n+2}^{(1)} & F_{n+3}^{(1)} \\
F_{n+2}^{(1)} & F_{n+3}^{(1)} & F_{n+4}^{(1)}
\end{pmatrix} = (-1)^n.
\]

**Proof.** By using relation

\[
F_n^{(1)} = F_{n+2} - 1 \quad (19)
\]

(that immediately follows from the elementary Fibonacci identity $\sum_{k=0}^n F_k = F_{n+2} - 1$, $n \geq 0$) and Lemma 3 we have

\[
\det \begin{pmatrix}
F_n^{(1)} & F_{n+1}^{(1)} & F_{n+2}^{(1)} \\
F_{n+1}^{(1)} & F_{n+2}^{(1)} & F_{n+3}^{(1)} \\
F_{n+2}^{(1)} & F_{n+3}^{(1)} & F_{n+4}^{(1)}
\end{pmatrix}
= \det \begin{pmatrix}
F_{n+2} - 1 & F_{n+3} - 1 & F_{n+4} - 1 \\
F_{n+3} - 1 & F_{n+4} - 1 & F_{n+5} - 1 \\
F_{n+4} - 1 & F_{n+5} - 1 & F_{n+6} - 1
\end{pmatrix} = (-1)^n \quad \Box
\]
Theorem 2. For the sequence \((F^{(r)}_k)_{k \geq 0}, \ r \in \mathbb{N}\) and \(n \in \mathbb{Z}\) a determinant of a matrix \(A_{r,n}\) takes values \pm 1,
\[
\det(A_{r,n}) = (-1)^{n+\left\lfloor \frac{r+3}{2} \right\rfloor}. \quad (20)
\]

Proof. Using elementary transformations on matrix and Lemma 2 we get
\[
\det(A_{r,0}) = \det \begin{pmatrix}
F_0^{(r)} & F_1^{(r)} & \cdots & F_{r-2}^{(r)} & F_{r-1}^{(r)} \\
F_1^{(r)} & F_2^{(r)} & \cdots & F_{r-1}^{(r)} & F_{r}^{(r)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
F_{r-2}^{(r)} & F_{r-1}^{(r)} & \cdots & F_{2r-4}^{(r)} & F_{2r-3}^{(r)} \\
F_{r-1}^{(r)} & F_{r}^{(r)} & \cdots & F_{2r-3}^{(r)} & F_{2r-2}^{(r)}
\end{pmatrix}
= - \det \begin{pmatrix}
0 & F_0^{(r)} & \cdots & F_{r-3}^{(r)} & F_{r-2}^{(r)} \\
F_0^{(r)} & F_1^{(r)} & \cdots & F_{r-2}^{(r)} & F_{r-1}^{(r)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
F_{r-3}^{(r)} & F_{r-2}^{(r)} & \cdots & F_{2r-5}^{(r)} & F_{2r-4}^{(r)} \\
F_{r-2}^{(r)} & F_{r-1}^{(r)} & \cdots & F_{2r-4}^{(r)} & F_{2r-3}^{(r)}
\end{pmatrix}
= (-1)^r \det \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & F_2^{(r)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & \cdots & F_{r-2}^{(r)} & F_{r-1}^{(r)} \\
1 & F_2^{(r)} & \cdots & F_{r-1}^{(r)} & F_{r}^{(r)}
\end{pmatrix}
= (-1)^r(-1)^{\left\lfloor \frac{r+2}{2} \right\rfloor} \det \begin{pmatrix}
1 & F_2^{(r)} & \cdots & F_{r-1}^{(r)} & F_{r}^{(r)} \\
0 & 1 & \cdots & F_{r-2}^{(r)} & F_{r-1}^{(r)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & F_2^{(r)} \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}
= (-1)^{\left\lfloor \frac{r+3}{2} \right\rfloor}.
\]

According to Theorem 1 we obtain
\[
\det(A_{r,n}) = \det(Q_{r+2}^n \det(A_{r,0}) = (-1)^n \det(A_{r,0})
= (-1)^n(-1)^{\left\lfloor \frac{r+3}{2} \right\rfloor} = (-1)^n+\left\lfloor \frac{r+3}{2} \right\rfloor,
\]
which completes the statement of the theorem. \(\square\)

Let \(M = M^{(m,n,r)}\) be a matrix with \(M_{i,j} = F^{(r)}_{n+i+j-2}, 1 \leq i, j \leq m\). Theorem 2 can be restated as
\[
\det(M^{(n,r,r+2)}) = (-1)^n+\left\lfloor \frac{r+4}{2} \right\rfloor.
\]
At the end let us show that for $m > r + 2$ the following equality holds:

$$\det (M^{(m,n,r)}) = 0. \quad (21)$$

The proof of (21) consists of performing elementary transformations on $M^{(m,n,r)}$ leading to a matrix having one column consisting of zeroes. Take a look at the $i - th$ row of $M^{(m,n,r)}$:

$$\begin{bmatrix}
F_{n+i-1}^{(r)} & F_{n+i}^{(r)} & F_{n+i+1}^{(r)} & \cdots & F_{n+i+j-2}^{(r)} & \cdots & F_{n+i+m-3}^{(r)} & F_{n+i+m-2}^{(r)}
\end{bmatrix}.$$ 

Using (6) and subtracting $j - th$ element from $(j + 1) - st$ for $j = m - 1, m - 2, \ldots, 2, 1$ (thus simulating subtracting a column $j$ from column $j + 1$ in a matrix $M^{(m,n,r)}$) we get

$$\begin{bmatrix}
F_{n+i-1}^{(r)} & F_{n+i}^{(r-1)} & F_{n+i+1}^{(r-1)} & \cdots & F_{n+i+2}^{(r-1)} & \cdots & F_{n+i+m-3}^{(r-1)} & F_{n+i+m-2}^{(r-1)}
\end{bmatrix}.$$ 

We can repeat the process for $j = m - 1, m - 2, \ldots, 3, 2$ and get

$$\begin{bmatrix}
F_{n+i-1}^{(r)} & F_{n+i}^{(r-1)} & F_{n+i+1}^{(r-2)} & \cdots & F_{n+i+m-3}^{(r-2)} & F_{n+i+m-2}^{(r-2)}
\end{bmatrix}.$$ 

After repeating the process $r - th$ time (for $j = m - 1, m - 2, \ldots, r$), we get

$$\begin{bmatrix}
F_{n+i-1}^{(r)} & F_{n+i}^{(r-1)} & F_{n+i+1}^{(r-2)} & \cdots & F_{n+i+m-2}^{(1)} & F_{n+i+m-1} & \cdots & F_{n+i+m-2}
\end{bmatrix}.$$ 

Since $m > r + 2$ we have $n + i + r - 1 \leq n + i + m - 4$ so the above row contains

$$\begin{bmatrix}
\cdots F_{n+i+r-1} & F_{n+i+r} & F_{n+i+r+1} & \cdots
\end{bmatrix}$$ 

at positions $r - 1$, $r$ and $r + 1$. Subtracting first two elements from the third, we get

$$\begin{bmatrix}
\cdots F_{n+i+r-1} & F_{n+i+r} & 0 & \cdots
\end{bmatrix}.$$ 

That way we arrive at a matrix with column consisting of zeroes whose determinant is therefore zero.

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