Hensel’s Lemma, Backward Dynamics and $p$-adic Approximations

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Abstract. The problem of backward dynamics over the ring of $p$-adic integers is studied. It is shown that Inverse Limit Theory provides the right framework. Backward iterations of a polynomial with $p$-adic integer coefficients are constructed by solving congruences modulo powers of $p$, which in turn are solved by Hensel’s lifting lemma.

Keywords: $p$-adic integer, Hensel’s lifting, Inverse limits, Symbolic Dynamics.

1 Introduction

Dynamical systems originally arose in the study of systems of differential equations used to model physical phenomena. One simplification in this study is to discretize time, so that the state of the system is observed only at discrete steps of time. This leads to the study of the iterates of a single transformation. One is interested in both quantitative behavior, such as the average time spent in a certain region, and also qualitative behavior, such as whether a state eventually becomes periodic or tends to infinity.

A discrete-time dynamical system consists of a non-empty set $X$ and a map $f : X \to X$. For $n \in \mathbb{N}$, the $n$th iterate of $f$ is the $n$-fold composition $f^n = f \circ f \circ \ldots \circ f$ ($n$ times); $f^0$ is defined to be the identity map. If $f$ is invertible, then $f^{-n} = f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1}$ ($n$ times). Since $f^{n+m} = f^n \circ f^m$, these iterates form a group if $f$ is invertible, and a semigroup otherwise. For a given $\alpha$ in $X$, the forward orbit of $\alpha$ is the set

$$O_{\phi(\alpha)} = \{ \phi^n(\alpha) : n \geq 0 \}$$

If the orbit $O_{\phi(\alpha)}$ is finite then $\alpha$ is said to be a pre-periodic point, otherwise $\alpha$ is said to be a wandering point.
The central problem in dynamics is to classify the points \( \alpha \) in the set \( X \) according to the behavior of their orbits \( \mathcal{O}_{\phi(\alpha)} \). In practice \( X \) usually has additional structure that is preserved by the map \( f \). For example, \((X, f)\) could be a measure space and a measure preserving map; a topological space and a continuous map; a metric space and an isometry; or a smooth manifold and a differentiable map; a finite set (e.g., Finite Field) and a polynomial.

1.1 Symbolic dynamics

Symbolic dynamics arose as an attempt to study such systems by means of discretizing space as well as time. The basic idea is to divide up the set of possible states into a finite number of pieces. Each piece is associated with a “symbol”, and in this way the evolution of the system is described by an infinite sequence of symbols. This leads to a “symbolic” dynamical system that mirrors and helps us to understand the dynamical behavior of the original system. Computer simulations of continuous systems necessarily involve a discretization of space, and results of symbolic dynamics help us understand how well, or how badly, the simulation may mimic the original. Symbolic dynamics by itself has proved a bottomless source of beautiful mathematics and intriguing questions. As polygons and curves are to geometry shift spaces are to symbolic dynamics. The set

\[ \Sigma = \{0, 1, 2, \ldots, m - 1\}^\mathbb{N} \]

is called the sequence space on \( m \) symbols \( 0, 1, \ldots, m - 1 \).

The most important ingredient in the sequence space is the shift map \( \sigma \). The shift map \( \sigma : \Sigma \to \Sigma \) is given by \( \sigma((s_0, s_1, s_2, \ldots)) = (s_1, s_2, s_3, \ldots) \).

The shift map discards the first entry in the sequence and shifts all other entries one place to the left. The distance between two sequences \( s = (s_0, s_1, \ldots) \) and \( t = (t_0, t_1, \ldots) \) is given by \( d(s, t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{m^i} \).

For an integer \( m > 1 \), set \( \mathbb{A}_m = \{1, 2, \ldots, m\} \). Let \( \Sigma_m = \mathbb{A}_m^\mathbb{Z} \) be the set of infinite two sided sequences of symbols in \( \mathbb{A}_m \) and \( \Sigma_m^+ = \mathbb{A}_m^\mathbb{N} \) be the set of infinite one-sided sequences. The pair \( (\Sigma_m, \sigma) \) is called the full two sided shift; \( (\Sigma_m^+, \sigma) \) is called the full one sided shift. The
two-sided shift is invertible. For a one-sided sequence, the leftmost symbol disappears, so the one-sided shift is non-invertible, and every point has $m$ pre-images. Both shifts have $m^n$ periodic points of period $n$. The shift spaces are compact topological spaces in the product topology. This topology has a basis consisting of cylinders $C_{j_1,...,j_k} = \{x = (x_i) : x_{n_i} = j_i, i = 1, 2, ..., k\}$, $n_1 < n_2 < ... < n_k$ are indices in $\mathbb{Z}$ or $\mathbb{N}$, and $j_i \in \mathbb{A}_m$. Since the preimage of a cylinder is a cylinder, $\sigma$ is continuous on $\Sigma^+_m$ and homeomorphism on $\Sigma_m$. The metric $d(x, x') = 2^{-l}$, for $l = \min\{|i| : x_i \neq x'_i\}$ generates the product topology on $\Sigma^+_m$ and $\Sigma_m$. In the product topology periodic points are dense and hence there are dense orbits. A detailed study can be found in [4], [5].

2 Arithmetic Dynamical Systems

Classically, discrete dynamics refers to the study of the iteration of self-maps of the complex plane or real line. Arithmetic Dynamics is the study of number theoretic properties of dynamical systems. Arithmetic dynamics is discrete-time dynamics (function iteration) over arithmetical sets, such as algebraic number rings and fields, finite fields, $p$-adic fields, polynomial rings, algebraic curves, etc. A thorough introduction is given in [1], [3].

In this paper we study the problem of backward dynamics of any polynomial of finite degree over finite rings $\mathbb{Z}/p^n\mathbb{Z}$ using Hensel’s lifting lemma, as usual $p$ denotes a prime. Backward-iteration sequences given by

$$x_n = f(x_{n+1}), \quad n > 0$$

are of a different nature because a point could have infinitely many pre-images as well as none. If the given forward moving map is a quadratic map, the corresponding backward map is a square root map; if the given map is a cubic map the corresponding backward map is the cube root map and so on. Thus essentially we solve $f(x) = 0$ as a polynomial over the defining set. For e.g., the Julia set can be found as the set of limit points of the set of pre-images of (essentially) any given point. Unfortunately, as the number of iterated pre-images grows exponentially, this is not feasible computationally.
when the underlying set is the set of Real or Complex numbers.

In general, maps of higher degree \((\geq 5)\) are not suitable for backward
dynamics over \(\mathbb{R}\) or \(\mathbb{C}\). As there is no explicit formula for solving a
polynomial of degree \(\geq 5\), roots can be found by using standard tech-
niques from Numerical Methods over \(\mathbb{R}\) or \(\mathbb{C}\) and we either arrive at
a null sequence or constant sequence after some backward iteration.
In such cases nothing can be said about the behavior of trajectories.

But when the set is finite (endowed with algebraic structure) and the
map is a polynomial it is possible to retrieve the pre-images (roots),
if they exist, with respect to different prime power moduli and thus
study the structure of pre-images locally at that prime.

We show that this problem can be well understood over the ring of
\(p\)-adic integers \(\mathbb{Z}_p\). This process deals with an important branch of
mathematics called Inverse Limit Theory. Below we discuss some
basics of Inverse Limit Theory and the \(p\)-adic integers.

### 2.1 Inverse Limits

Let \(X_0, X_1, X_2, \ldots\) be a countable collection of spaces, and suppose
that, for each \(n > 0\), there is a continuous mapping \(f_n : X_n \to X_{n-1}\).
The sequence of spaces and mappings \(\{X_n, f_n\}\) is called an inverse
limit sequence and may be represented as

\[
\ldots \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \ldots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0
\]

Clearly, if \(n > m\), there is a continuous mapping \(f_{n,m} : X_n \to X_m\)
given by the composition \(f_{n,m} = f_{m+1} \cdot f_{m+2} \cdots f_{n-1} \cdot f_n\).
Consider a sequence \((x_0, x_1, \ldots, x_n, \ldots)\) such that each \(x_n\) is a point of
the space \(X_n\) and such that \(x_n = f_{n+1}(x_{n+1})\), \(n \geq 0\). Such a sequence
can be identified in the product space \(\prod_{n=0}^\infty X_n\) by considering a
function \(\phi\) from the nonnegative integers into \(\prod_{n=0}^\infty X_n\), given by
\(\phi(n) = x_n\). Thus the set of all sequences is a subset of \(\prod_{n=0}^\infty X_n\) and
has a topology as a subspace. This topological space is the inverse
limit space of of the sequence \(\{X_n, f_n\}\) denoted by \(X = \lim_{\leftarrow} (X_n, f_n)\).

**Theorem 1.** Suppose that each space \(X_n\) in the inverse limit se-
quence \(\{X_n, f_n\}\) is a compact Hausdorff space. Then \(X\) is not empty
\([7]\).
Theorem 2. A space $X$ is a compact Hausdorff space with $\dim(X) \leq 0$ if and only if $X$ is an inverse limit of finite discrete spaces [7].

A finite discrete space is totally disconnected, compact and Hausdorff and all those properties carry over to inverse limits too. A detailed study of inverse limit spaces can be found in [6].

2.2 $p$-adic Integers

An important class of such inverse limits is given by rings of $p$-adic integers $\mathbb{Z}_p$. For every $n \geq 1$, let $A_n = \mathbb{Z}/p^n\mathbb{Z}$. An element of $A_n$ defines in an obvious way an element of $A_{n-1}$ and the homomorphism

$$\phi_n : A_n \to A_{n-1}$$

is surjective and the kernel is $p^{n-1}A_n$. The sequence

$$... \to A_n \to A_{n-1} \to A_2 \to A_1$$

forms a "projective system" indexed by the integers $\geq 1$. Inverse limit of this inverse system is $\mathbb{Z}_p = \lim_{\leftarrow} (A_n, \phi_n)$. Refer [10] for details.

Definition 1. The ring of $p$-adic integers, $\mathbb{Z}_p$, is the projective limit (inverse limit) of the system $(A_n, \phi_n)$.

An element of $\mathbb{Z}_p = \lim_{\leftarrow} (A_n, \phi_n)$ is a sequence $x = (..., x_n, ..., x_1)$ with $x_n \in A_n$ and $\phi_n(x_n) = x_{n-1}$ if $n \geq 2$. Addition and multiplication in $\mathbb{Z}_p$ are defined co-ordinate wise. In other words, $\mathbb{Z}_p$ is a subring of the product $\prod_{n \geq 1} A_n$. If $A_n$ is endowed with discrete topology and $\prod_{n \geq 1} A_n$ the product topology, the ring $\mathbb{Z}_p$ inherits a topology which turns it into a compact space.

Let $p$ be a prime. For $n \in \mathbb{Z}$, let $\nu_p(n)$ denote the exponent of highest power of $p$ that divides $n$, and $\nu_p(0) = \infty$. More formally, $\nu_p(n)$ is the unique natural number such that $n = p^{\nu_p(n)}u$ with $p \nmid u$. This definition can be extended to $\mathbb{Q}$ by letting $\nu_p(\frac{a}{b}) = \nu_p(a) - \nu_p(b)$. The $p$-adic absolute value is defined by $|x|_p = p^{-\nu_p(x)}$ for any $x \in \mathbb{Q}$. Then $d_p(x, y) = |x - y|_p$ defines a metric on $\mathbb{Q}$. The metric space $(\mathbb{Q}, d_p)$ is not complete, and its completion is the $p$-adic number field $\mathbb{Q}_p$. This absolute value is non-Archimedean as, in the place of
triangle inequality, the stronger relation \(|x + y|_p \leq \operatorname{Max}\{|x|_p, |y|_p\} \)
also known as ultrametric inequality holds. The non-Archimedean property is equivalent to the assertion that, \(\sup\{|n| : n \in \mathbb{Z}\} = 1\)
As a consequence (in stark contrast to the Euclidean norm), the \(p\)-adic norm does not permit accumulation of errors in the following sense: if each of \(k\) elements \(\{x_1, x_2, ..., x_k\}\) have \(p\)-adic norm at most \(\epsilon\), then \(|x_1 + x_2 + ... + x_k|_p \leq \epsilon\) as well. This property justifies extensive
use of modular arithmetic (\(p\)-adic estimation) in \(p\)-adic calculations.

**Definition 2.** A \(p\)-adic integer is a formal series \(\sum_{i \geq 0} a_i p^i\) with integral coefficients \(a_i\) satisfying \(0 \leq a_i \leq p - 1\).
The subset of \(\mathbb{Q}_p\) defined by \(\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}\) is called the integer ring and is an integral domain. The set of invertible elements in \(\mathbb{Z}_p\), called its group of units is \(\mathbb{Z}_p^* = \{x \in \mathbb{Z}_p : |x|_p = 1\}\). The ring \(\mathbb{Z}_p\) contains a unique maximal ideal \(p\mathbb{Z}_p = \{x \in \mathbb{Z}_p : |x|_p < 1\}\). The quotient of \(\mathbb{Z}_p\) by this maximal ideal is a field, which can be identified with the finite field of \(p\) elements \(\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}\) in the obvious way. \(\mathbb{F}_p\) is called the residue class field of \(\mathbb{Q}_p\). The field \(\mathbb{Q}_p\) is unorderable, in essence due to the modular arithmetic of \(\mathbb{F}_p\). It has characteristic 0 since it contains \(\mathbb{Q}\) as a subfield. Indeed, \(\mathbb{Q}\) is a dense proper subset of \(\mathbb{Q}_p\) and embeds into \(\mathbb{Q}_p\) as the set of elements whose \(p\)adics are eventually periodic. Topologically \(\mathbb{Q}_p\) is a Cantor set: totally disconnected but not discrete. In algebraic sense too \(p\)adics are full of holes: There is no finite extension of \(\mathbb{Q}_p\), which is algebraically closed. Ostrowski’s theorem states that any non-trivial absolute value on the rational numbers \(\mathbb{Q}\) is equivalent to either the usual real absolute value or a \(p\)-adic absolute value. Refer [11], [8] for details.

### 3 \(p\)-adic Dynamics

It is natural to have accumulated truncation errors or round-off errors in a dynamical system even with a small perturbation. These are unavoidable since even with some repetitive operations, the number of digits of the result can increase so much that the result cannot be held fully in the registers available in the computer. Such errors accumulate one after another from iteration to iteration generating new
errors. These difficulties motivated to look for an alternate number system which possesses the best features as well as the advantages of both the $p$-ary and residue number system. Such a number system is the $p$-adic number system, discovered by Kurt Hensel in 1897 in the course of his work on finding new completions of the rational numbers. Hensel’s original description of the $p$-adic numbers involved an analogy between the ring of integers and the ring of polynomials over the complex numbers, the crux of which was the development of a representation of rational numbers analogous to that of Laurent expansions of rational functions namely, the $p$-adic expansion. This idea was motivated by the existence of real expansions of rational numbers with respect to a $p$-scale:

$$x = \sum_{n=-\infty}^{k} \alpha_n p^n, \quad \alpha_n = 0, \ldots, p - 1.$$ 

Such manipulations with rational numbers and series generated the idea that there exists some algebraic structure similar to the system of real numbers $\mathbb{R}$. Thus each $\mathbb{Q}_p$ has the structure of a number field. In fact, the fields of $p$-adic numbers, $\mathbb{Q}_p$, were the first examples of infinite fields that differs from $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and corresponding fields of rational functions. The following definition is useful in identifying a $p$-adic integer.

**Definition 3.** Let $p$ be some prime number. A sequence of integers

$$\{x_n\} = \{x_0, x_1, \ldots, x_n, \ldots\}$$

satisfying

$$x_n \equiv x_{n-1} \pmod{p^n}$$

for all $n \geq 1$, is called a $p$-adic integer. Two sequences and $\{x_n\}$ and $\{x'_n\}$ determine the same $p$-adic integer if and only if $x_n \equiv x'_n \pmod{p^n}$ for all $n \geq 0$.

This definition can be easily identified with the definition 2.

The connection between congruences and equations is based on the simple remark that, if the equation

$$F(x_1, x_2, \ldots, x_n) = 0$$
where $F$ is a polynomial with integral coefficients, has a solution in integers, then the congruence

$$F(x_1, x_2, ..., x_n) \equiv 0 \pmod{m}$$

is solvable for any value of the modulus $m$. On the other hand the situation is more complicated for congruences. For any modulus $m > 1$, there are polynomial congruences having no solutions. For eg., the congruence $x^p - x + 1 \equiv 0 \pmod{m}$ has no solution if $p$ is any prime factor of $m$ by Fermat’s theorem. Where as a congruence can have more solutions than its degree, for eg., $x^2 - 7x + 2 \equiv 0 \pmod{10}$ has four solutions $x = 3, 4, 8, 9$. But if the modulus is a prime, a congruence cannot have more solutions than its degree.

The following results are basic, see [9].

**Theorem 3.** If the degree $n$ of $f(x) \equiv 0 \pmod{p}$ is greater than or equal to $p$ (mod $p$) or there is a polynomial $g(x)$ having integral coefficients, with leading coefficient 1, such that, $g(x) \equiv 0 \pmod{p}$ is of degree less than $p$ and the solutions of $g(x) \equiv 0 \pmod{p}$ are precisely those of $f(x) \equiv 0 \pmod{p}$.

**Theorem 4.** The congruence $f(x) \equiv 0 \pmod{p}$ of degree $n$ has atmost $n$ solutions.

**Corollary 1.** If $b_n x^n + b_{n-1} x^{n-1} + ... + b_0 \equiv 0 \pmod{p}$ has more than $n$ solutions, then all the coefficients $b_j$ are divisible by $p$.

**Theorem 5.** The congruence $f(x) \equiv 0 \pmod{p}$ of degree $n$, with leading coefficient $a_n = 1$, has $n$ solutions if and only if $f(x)$ is a factor of $x^p - x$ modulo $p$, that is, if and only if $x^p - x = f(x)q(x) + ps(x)$, where $q(x)$ and $s(x)$ have integral coefficients, $q(x)$ has degree $p - n$ and leading coefficient 1, and where $s(x)$ is a polynomial of degree less than $n$ or $s(x)$ is zero.

The proofs of the above theorems are simple consequences of Fermat’s little theorem and its application which can be found in many books on number theory.
3.1 Hensel’s Lifting Lemma

As both \( \mathbb{Q} \) and \( \mathbb{R} \) are not algebraically closed, they do not always contain all roots of polynomials with integer coefficients. Though \( \mathbb{C} \) is algebraically closed, as the degree of the polynomial increases, finding roots of the polynomial is computationally not feasible, as it requires high level of precision for machine computation. To this end, we now turn our attention to solving polynomial congruences modulo prime powers.

We note that for any polynomial \( f(x) \in \mathbb{Z}[x] \) and any integer \( r \), there is a polynomial \( g_r(x) \in \mathbb{Z}[x] \) with \( f(x+r) = f(r) + x f'(r) + x^2 g_r(x) \). This can be seen either through the Taylor expansion for \( f(x+r) \) or through the binomial theorem in the form

\[
(x + r)^d = r^d + dr^{d-1}x + x^2 \sum_{j=2}^{d} \binom{d}{j} r^{d-j} x^{j-2}
\]

The above standard results mentioned in the form of Theorems can be used to find the solutions to \( f(x) \equiv 0 \pmod{p} \). The question is how we might be able to lift a solution to one modulo \( p^k \) for various exponents \( k \). With regard to this, Hensel’s Lemma is a powerful tool which relates the roots of a given polynomial to its solution modulo a prime. The lemma and its proof both rely on iterative procedures that return an agreeable solution if supplied with a well-behaved seed.

**Definition 4.** If \( f(a) \equiv 0 \pmod{p} \), then the root \( a \) is called nonsingular if \( f'(a) \not\equiv 0 \pmod{p} \); otherwise it is singular.

Two versions of Hensel’s Lemma are stated below.

**Theorem 6.** Hensel’s Lemma over the ring of integers.
Suppose that \( f(x) \) is a polynomial with integral coefficients. If \( f(a) \equiv 0 \pmod{p} \) and \( f'(a) \not\equiv 0 \pmod{p} \), then there is a unique \( t \pmod{p} \) such that \( f(a + tp^j) \equiv 0 \pmod{p^{j+1}} \).

**Theorem 7.** Hensel’s Lemma over the ring of \( p \)-adic integers.
Let \( f \in \mathbb{Z}_p[x] \) be monic. If \( a_0 \in \mathbb{Z} \) is a simple root of \( f(x) \equiv 0 \pmod{p} \), then \( \exists y \in \mathbb{Z}_p \) such that \( y \equiv a_0 \pmod{p} \) and \( f(y) = 0 \).
Proof. Suppose that $\exists a_n$ such that $f(a_n) \equiv 0 \pmod{p^n}$. We must show that $a_n$ can be lifted uniquely to $a_{n+1} \pmod{p^{n+1}}$ such that $a_{n+1} \equiv a_n \pmod{p^n}$ and $f(a_{n+1}) \equiv 0 \pmod{p^{n+1}}$, then $y$ is the limit of this sequence of $(\pmod{p^k})$ solutions.
Since $f$ is a polynomial we can write it in the form $f(x) = \sum c_i x^i$. Also consider $tp^n + a_n$ as a possible lift of $a_n$. Then
\[
f(a_n + tp^n) = \sum_i c_i (tp^n + a_n)^i \equiv f(a_n) + p^n tf'(a_n) \pmod{p^{n+1}}.
\] (2) The equivalence above is a result of Taylor series expansion. Now, solve for $t$ in
\[
p^n tf'(a_n) + f(a_n) \equiv 0 \pmod{p^{n+1}}.
\]

Thus
\[
tf'(a_n) \equiv -\left(\frac{f(a_n)}{p^n}\right) \pmod{p}.
\]

Since $f(a_n) \equiv 0 \pmod{p^n}$, (since $a_n \equiv a_0 \pmod{p}$) and $f'(a_n) \not\equiv 0 \pmod{p}$ (simple root), then $t$ has a unique solution $\pmod{p}$. Thus $a_{n+1} = a_n + tp^n$ is a unique lift of $a_n \pmod{p}$. Thus we have constructed an infinite sequence of $a_i$ such that $f(a_i) \equiv 0 \pmod{p^i}$, such that, $f'(a_i) \not\equiv 0 \pmod{p^i}$ and $a_{i+1} \equiv a_i \pmod{p}$. This sequence is Cauchy, and therefore converges to a unique limit $y \in \mathbb{Z}_p$.

Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, proof of Theorem follows directly from Theorem.

It can be verified that, proof of Hensel’s Lemma is entirely analogous to Newton’s method for locating the root of a differentiable function. Let us recall Newton’s method from calculus as a method of finding roots to a polynomial by choosing a seed and then making better and better approximations based on the polynomial’s derivative at that point. In the case of Newton’s method, the condition on the seed is that the derivative at that point be non-zero, otherwise it supplies no useful information for improving at each iteration. Hensel’s Lemma is similar, it takes a polynomial with coefficients in $\mathbb{Z}_p$ and instead of requiring a guess at a possible root, it requires a $p$-adic integer that
is a root mod $p$, i.e. some $\alpha$ such that the polynomial $f(x)$ evaluated at $\alpha$ is

$$f(\alpha) \equiv 0 \pmod {p\mathbb{Z}_p}$$

This method will then return roots mod $p, p^2, p^3, \ldots$ until the desired root of the equation is found.

### 3.2 Backward Iterations, Inverse Limits and $p$-adic Approximations

Now we find the backward iteration of any given polynomial $f(x)$ of arbitrary degree say $n$, at any point, say $x_n$ of the forward iterating orbit, i.e., we would like to solve

$$f(x) \equiv x_n \pmod {p^k} \quad (4)$$

for some $k \in \mathbb{N}$. For this, we first solve the congruence $f(x) \equiv x_n \pmod {p}$, such that the nonzero coefficients of $f(x)$ are relatively prime to $p$ and $x_n$ is chosen as above, so that the corresponding backward iterating orbit of $x_n$ are found modulo $p^j, j \rightarrow \infty$.

If the degree $n$ of $f(x)$ is greater than $p$, then by Theorem 3, $f(x)$ is divided by $(x^p - x) \pmod p$ and solutions of the resulting polynomial $g(x)$ are the same as those of $f(x) \equiv x_n \pmod p$. Since the modulus is prime, the congruence cannot have more solutions than its degree. Let $a_1, a_2, \ldots, a_l$ be the roots obtained, by solving the congruence $f(x) \equiv x_n \pmod p$ where $(l \leq \deg(g(x)$ by Theorem 4). Now each $a_i$ is lifted modulo $p^j, j = 2, 3, \ldots$ whenever $f'(a_i) \not\equiv 0 \pmod p$, $(i = 1, 2, \ldots, l)$ i.e., whenever the $a_i'$s are nonsingular. We work up to a fixed precision say $j = k$. Once the roots are lifted modulo $p^k$, $x_n$ is replaced by the lifted root and the congruence equation (4) is solved now with respect to the new root. The process of replacing the old root by the new lifted root is repeated for some finite number of steps. Thus with respect to each nonsingular $a_i$, we obtain the corresponding backward sequence, generated from the single seed $x_n$. Thus if there are $n$ nonsingular roots, after the $m$th step of replacing the old root by the new lifted root, there are atmost $n^m$ backward iterating points generated from the single seed $x_n$. Hence there exists a tree like structure, the roots may be called as leaves and the branches are formed at each new lifted root. A simple code for this program can
be written on python, which computes the sequences effectively. Sequences thus obtained belong to the inverse limit space by definition. It can be verified that the sequence space formed by the above backward iterations is totally disconnected and discrete. Also, if the original spaces are discrete, then the inverse limit space $\lim\{X, f\}$ is totally disconnected. This is one way of realizing the $p$-adic numbers and the Cantor set.

To study the long time behaviour of a dynamical system it is necessary to introduce a suitable metric. A natural choice would be the one given in section 1, i.e., the distance between two sequences $s = (s_0, s_1, ...)$ and $t = (t_0, t_1, ...)$. is given by

$$d(s, t) = \sum_{i=0}^{\infty} \frac{s_i - t_i}{p^i}.$$ 

Thus by the introduction of a metric, sequence space of backward iteration becomes a compact metric space. The sequence space thus obtained can be identified with the ring of $p$-adic integers in view of the following theorems:

**Theorem 8.** Any two totally disconnected perfect compact metric spaces are homeomorphic.

**Theorem 9.** Let $M$ be a compact, totally disconnected metric space. Then $M$ is homeomorphic to the inverse limit space of an inverse limit sequence of finite discrete spaces [6].

### 4 Conclusion

The problem of backward iteration is studied by solving congruences, which in turn are solved by Hensel’s lifting. Sequences generated by such solutions form, naturally, elements of an Inverse Limit Space. These spaces have also been characterised.

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