Proof of Cramer’s rule with Dirac delta function

June-Haak Ee, Jungil Lee and Chaehyun Yu

KPOP Collaboration, Department of Physics, Korea University, Seoul 02841, Korea
E-mail: chodigi@gmail.com, jungil@korea.ac.kr and chyu@korea.ac.kr

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Abstract
We present a new proof of Cramer’s rule by interpreting a system of linear equations as a transformation of \( n \)-dimensional Cartesian-coordinate vectors. To find the solution, we carry out the inverse transformation by convolving the original coordinate vector with Dirac delta functions and changing integration variables from the original coordinates to new coordinates. As a byproduct, we derive a generalized version of Cramer’s rule that applies to a partial set of variables, which is new to the best of our knowledge. Our formulation of finding a transformation rule for multi-variable functions shall be particularly useful in changing a partial set of generalized coordinates of a mechanical system.

Keywords: Cramer’s rule, Dirac delta function, generalized coordinates

1. Introduction

Cramer’s rule [1] is a formula for solving a system of linear equations as long as the system has a unique solution. There are various proofs available [2–5] and six different proofs of Cramer’s rule are listed in reference [5].

Dirac delta function \( \delta(x) \) is not a proper function but a distribution defined only through integrals [6–8]:

\[
\int_{-\infty}^{\infty} dx \, \delta(x) f(x) = f(0),
\] (1)

where \( f(x) \) is a smooth function. The Dirac delta function \( \delta(x-a) \) projects out the value of a function \( f(x) \) at a certain point \( x = a \) after integration: \( \int_{-\infty}^{\infty} dx \, f(x) \delta(x-a) = f(a) \). This

* Author to whom any correspondence should be addressed. Director of the Korea Pragmatist Organization for Physics Education (KPOP) Collaboration.

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elementary property of $\delta(x)$ can further be applied to change the variable: $\int_{-\infty}^{\infty} dx f(x) \delta(x - y) = f(y)$. The Dirac delta function is particularly useful in physics. One can make an educated guess that the change of variables can be made for a multi-variable integral by introducing the integration of products of Dirac delta functions whose arguments contain the transformation constraints.

In this paper, we derive Cramer’s rule by making use of the Dirac delta function to change the variables of multi-dimensional integrals. The change of variables corresponds to a coordinate transform $X' = RX$, where $X = (x_1, \ldots, x_n)^T$ and $X' = (x'_1, \ldots, x'_n)^T$ are two $n$-dimensional Cartesian-coordinate column vectors and $R = (R_{ij})$ is the corresponding invertible transformation matrix. By changing the integration variables from $x_i's$ to $x'_i's$ in a trivial identity

$$X = \prod_{k=1}^{n} \left[ \int_{-\infty}^{\infty} dx(x'_k - \sum_{ik=1}^{n} R_{ki} x_k) \right] X,$$

we carry out the inverse transformation $X = R^{-1}X'. To the best of our knowledge, this is a new proof of Cramer’s rule.

Our formulation of finding a transformation rule for multi-variable functions shall be particularly useful in changing a partial set of generalized coordinates of a mechanical system and can further be extended to reorganizing the phase space of a multi-particle interaction in particle phenomenology.

This paper is organized as follows. After providing a description of the notations that are frequently used in the remainder of this paper in section 2, we proceed with the derivation of Cramer’s rule by making use of Dirac delta functions in section 3. Section 4 is devoted to demonstrate how Cramer’s rule for a partial set of variables or coordinates applies to a mechanical system. In section 5, we discuss the implications and possible applications of our results. A rigorous proof of Cramer’s rule for a partial set of variables by employing mathematical induction is presented in the appendix A.

2. Definitions

We define $n$-dimensional column vectors $X$ and $X'$ whose $i$th elements are the Cartesian coordinates $x_i$ and $x'_i$, respectively:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad X' = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}. \quad (2)$$

The two coordinates are related as $x'_i = \sum_{k=1}^{n} R_{ik} x_k$:

$$X' = RX, \quad (3)$$

where the transformation matrix $R$ is an $n \times n$ invertible matrix with known elements. Cramer’s rule states that the $i$th element of $X$ is determined as

$$x_i = \frac{\partial e \cdot t [R(i)(X')]}{\partial e \cdot t [R]}, \quad (4)$$

where $\partial e \cdot t$ represents the determinant, $x_i$ is the $i$th element of $X$, and $R(i)(X')$ is the $n \times n$ matrix identical to $R$ except that the $i$th column is replaced with $X'$. 

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We define the $j \times j$ square matrix $R_{[j \times j]}$ whose elements are identical to those of $R$ for $j = 1, \ldots, n$:

$$R_{[j \times j]} = \begin{pmatrix} R_{11} & \cdots & R_{1j} \\ \vdots & \ddots & \vdots \\ R_{j1} & \cdots & R_{jj} \end{pmatrix}.$$  \hspace{1cm} (5)

Thus $R_{[n \times n]} = R$. We define the column vectors $X'_{[j]}$ and $R^{(k)}_{[j]}$ with $j$ rows as

$$X'_{[j]} = \begin{pmatrix} x'_{1} \\ \vdots \\ x'_{j} \end{pmatrix}, \quad R^{(k)}_{[j]} = \begin{pmatrix} R_{1k} \\ \vdots \\ R_{jk} \end{pmatrix}.$$  \hspace{1cm} (6)

We define the partial contribution of $X'_{[j]}$ originated from the higher-dimensional coordinates $x_{j+1}$ through $x_{n}$ as

$$X'_{[j]} = \sum_{k=j+1}^{n} x_{k} \begin{pmatrix} R_{1k} \\ \vdots \\ R_{jk} \end{pmatrix}.$$  \hspace{1cm} (7)

For $j = n$, we set $X'_{[n]} = 0$, the $n$-dimensional null vector.

We denote $\Omega_{j}$ by the determinant of $R_{[j \times j]}$:

$$\Omega_{j} = \varepsilon \epsilon_{[j \times j]} = \sum_{i_{1}=1}^{j} \cdots \sum_{i_{j}=1}^{j} \epsilon_{i_{1}\cdots i_{j}} R_{i_{1}1} R_{i_{2}2} \cdots R_{i_{j}j},$$  \hspace{1cm} (8)

where $\epsilon_{i_{1}\cdots i_{j}}$ is the $j$-dimensional totally antisymmetric Levi-Civita symbol.

We denote $R_{[j \times j]}^{(i)}(V)$ by the $j \times j$ square matrix identical to $R_{[j \times j]}$ defined in equation (5) except that the $i$th column is replaced with a column vector $V$ with $j$ elements:

$$R_{[j \times j]}^{(i)}(V) = \begin{pmatrix} R_{1i}^{(1)} & \cdots & R_{1i}^{(i-1)} & \cdots & R_{1i}^{(i+1)} & \cdots & R_{1i}^{(j)} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ R_{ji}^{(1)} & \cdots & R_{ji}^{(i-1)} & \cdots & R_{ji}^{(i+1)} & \cdots & R_{ji}^{(j)} \end{pmatrix}.$$  \hspace{1cm} (9)

where $R_{[n \times n]}^{(i)}(V) = R^{(i)}(V)$.

3. Proof of Cramer’s rule

In this section, we illustrate the proof of Cramer’s rule with the aid of the Dirac delta function. As a byproduct, we derive a generalized version of Cramer’s rule involving a partial set of variables.

We multiply the unities

$$1 = \int_{-\infty}^{\infty} dx'_{i} \delta \left( x'_{i} - \sum_{k=1}^{n} R_{ik} x_{k} \right)$$  \hspace{1cm} (10)
to $X$ for $i = 1$ through $n$. Then we find that

$$X = \int_{-\infty}^{\infty} dx_1' \cdots \int_{-\infty}^{\infty} dx_n' \prod_{k=1}^{n} \delta \left( x_k' - \sum_{a_k=1}^{n} R_{a_k} x_k \right). \quad (11)$$

The multiple integration over $x_1', \ldots, x_n'$ can be replaced with the integration over the variables $x_1, \ldots, x_n$ as

$$\int_{-\infty}^{\infty} dx_1' \cdots \int_{-\infty}^{\infty} dx_n' = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \mathcal{J}, \quad (12)$$

where $\mathcal{J}$ is the Jacobian:

$$\mathcal{J} = \det \begin{pmatrix} \frac{\partial x_1'}{\partial x_1} & \cdots & \frac{\partial x_1'}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n'}{\partial x_1} & \cdots & \frac{\partial x_n'}{\partial x_n} \end{pmatrix} = |\mathcal{D}et[\mathbb{R}]| = |\mathcal{D}_n|. \quad (13)$$

Then the integral (11) reduces into

$$X = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n |\mathcal{D}_n| \prod_{k=1}^{n} \delta (x_k' - \sum_{a_k=1}^{n} R_{a_k} x_k). \quad (14)$$

The integration over $x_1$ in equation (14) can be carried out as

$$\int_{-\infty}^{\infty} dx_1 |\mathcal{D}_n| X_1 \delta (x_1' - \sum_{k=1}^{n} R_{1k} x_k)$$

$$= \int_{-\infty}^{\infty} dx_1 |\mathcal{D}_n| X_1 \delta \left[ \mathcal{D}et[\mathbb{R}_{1 \times 1}] (X_1'[1]_1 - X_1'[1]) - \mathcal{D}_1 x_1 \right]$$

$$= \int_{-\infty}^{\infty} dx_1 \frac{|\mathcal{D}_n|}{|\mathcal{D}_1|} X_1 \delta \left[ x_1 - \frac{\mathcal{D}et[\mathbb{R}_{1 \times 1}] (X_1'[1]_1 - X_1'[1])}{\mathcal{D}_1} \right]$$

$$= \frac{|\mathcal{D}_n|}{|\mathcal{D}_1|} X_1^{(1)}, \quad (15)$$

where $\mathcal{D}_1 = \mathcal{D}et[\mathbb{R}_{1 \times 1}] = R_{11}$ and

$$X_1^{(1)} \equiv \begin{pmatrix} x_1(x_1', x_2, \ldots, x_n) \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \quad (16)$$

The notation $X_1^{(1)}$ indicates that $x_1$ in $X$ is replaced with the expression in terms of $x_1'$ and $x_k$ for $k = 2, \ldots, n$ as
We shall find that, after carrying out the integrations over \( x_1 \) through \( x_j \), these coordinates are replaced with the corresponding linear combinations of \( x'_1 \) through \( x'_j \) and \( x_{j+1} \) through \( x_n \).

Thus we define \( X^{(j)} \) by the generalized version of \( X^{(1)} \) for \( j = 1, \ldots, n \) as

\[
X^{(j)} = \begin{pmatrix}
  x_1(x'_1, \ldots, x'_j, x_{j+1}, \ldots, x_n) \\
  \vdots \\
  x_j(x'_1, \ldots, x'_j, x_{j+1}, \ldots, x_n) \\
  x_{j+1} \\
  \vdots \\
  x_n
\end{pmatrix}.
\] (18)

In a similar manner, the integration over \( x_2 \) can be carried out as

\[
\int_{-\infty}^{\infty} dx_2 \left| \frac{\partial n}{\partial x_2} \right| \delta(x'_2 - \sum_{k=1}^{n} R_{2k}x_k)
= \int_{-\infty}^{\infty} dx_2 \left| \frac{\partial n}{\partial x_2} \right| \delta \left[ \frac{1}{\partial x_2} \left( \partial e t \left[ R_{2 \times 2}^{(2)}(x'_2 - X'_{[2]}) - X'_{[2]} \right] \right) - x_2 \right]
= \left| \frac{\partial n}{\partial x_2} \right| X^{(2)}.
\] (19)

Because the derivation of equation (19) is rather tricky, we explain how we have obtained the results in detail:

\[
x'_2 - \sum_{k=1}^{n} R_{2k}x_k
= \frac{1}{\partial x_1} \left( R_{11}x'_2 - R_{11}R_{21}x_1 - R_{11} \sum_{k=2}^{n} R_{1k}x_k \right)
= \frac{1}{\partial x_1} \left( (R_{11}x'_2 - R_{11}x'_1) - \sum_{k=3}^{n} (R_{11}R_{2k} - R_{21}R_{1k}) x_k - (R_{11}R_{22} - R_{21}R_{12}) x_2 \right)
= \frac{1}{\partial x_1} \left[ \partial e t \left( R_{11} x'_2 - R_{11} x'_1 \right) - \sum_{k=3}^{n} \partial e t \left( R_{11} R_{2k} x_k - R_{21} R_{1k} x_k \right) - \partial e t \left( R_{11} R_{22} x_2 \right) \right]
= \frac{1}{\partial x_1} \left[ \partial e t \left[ R_{2 \times 2}^{(2)}(X'_{[2]} - X'_{[2]}) - X'_{[2]} \right] - \partial e t \left[ R_{2 \times 2}^{(2)}(X'_{[2]} - X'_{[2]}) - X'_{[2]} \right] \right]
= \frac{1}{\partial x_1} \left[ \partial e t \left[ R_{2 \times 2}^{(2)}(X'_{[2]} - X'_{[2]}) - X'_{[2]} \right] - \partial e t \left[ R_{2 \times 2}^{(2)}(X'_{[2]} - X'_{[2]}) - X'_{[2]} \right] \right].
\] (20)
On the right side of the first line of equation (20), we have imposed from equation (8) and pulled out the \( k = 1 \) term from the summation over \( k \). According to equations (16) and (17), we replace \( x_1 \) with a linear combination of \( x'_1 \) and \( x_2, \ldots, x_n \). Collecting the primed-coordinate, \( x_2 \), and unprimed-coordinate contributions of \( k \geq 3 \) separately, we obtain the second line of equation (20). Then the remaining derivation from the third to the last line of equation (20) is straightforward.

After evaluating the \( x_2 \) integral, we end up with \( \mathbb{X}^{(2)} \) defined in equation (18) up to an overall factor of determinant ratio \( |\mathcal{D}_n|/|\mathcal{D}_2| \). After the double integrations over \( x_1 \) in equation (15) and over \( x_2 \) in equation (19), \( x_1 \) and \( x_2 \) are expressed in terms of \( x'_1, x'_2, \) and \( x_k \)'s for \( k = 3, \ldots, n \)

\[
x_i = \frac{1}{\mathcal{D}_2} \mathcal{D} \text{e} \text{t} \left[ R_{[i \times 2]}^{(i)} \left( \mathbb{X}_{[2]}' - \mathbb{X}_{\perp[2]}' \right) \right], \quad i = 1, 2,
\]

(21)

where the expression for \( x_1 \) is obtained by inserting \( x_2 \) into equation (17).

In appendix A, we show that the integration over \( x_j \) can be carried out by employing mathematical induction as

\[
\int_{-\infty}^{\infty} dx_j \frac{\mathbb{R}_{[n]} |\mathcal{D}_{j-1}|}{|\mathcal{D}_j|} \mathbb{X}^{(j-1)} \delta (x_j' - \sum_{k=1}^{n} R_{jk} x_k) \\
= \int_{-\infty}^{\infty} dx_j \frac{\mathbb{R}_{[n]} |\mathcal{D}_{j-1}|}{|\mathcal{D}_j|} \mathbb{X}^{(j-1)} \delta \left[ \frac{1}{\mathcal{D}_{j-1}} \left( \mathcal{D} \text{e} \text{t} \left[ R_{[i \times j]}^{(i)} \left( \mathbb{X}_{[j]}' - \mathbb{X}_{\perp[j]}' \right) \right] \right) \right] \\
= \int_{-\infty}^{\infty} dx_j \frac{\mathbb{R}_{[n]} |\mathcal{D}_{j-1}|}{|\mathcal{D}_j|} \mathbb{X}^{(j-1)} \delta \left[ x_j - \frac{\mathcal{D} \text{e} \text{t} \left[ R_{[i \times j]}^{(i)} \left( \mathbb{X}_{[j]}' - \mathbb{X}_{\perp[j]}' \right) \right] }{\mathcal{D}_j} \right] \\
= \frac{|\mathbb{R}_{[n]}|}{|\mathcal{D}_j|} \mathbb{X}^{(i)},
\]

(22)

where \( \mathbb{R}_{[i \times j]}^{(i)} (\mathcal{V}) \) and \( \mathbb{X}_{\perp[j]}' \) are defined in equations (9) and (7), respectively. We note that \( \mathbb{X}_{\perp[j]}' \) depends only on \( x_{j+1} \) through \( x_n \).

After the multiple integrations over \( x_1 \) in equation (15) through over \( x_j \) in equation (22), \( x_i \) for \( i = 1 \) through \( j \) are expressed in terms of \( x'_1, \ldots, x'_j \) and \( x_k \)'s for \( k = j + 1, \ldots, n \) as

\[
x_i = \frac{1}{\mathcal{D}_j} \mathcal{D} \text{e} \text{t} \left[ R_{[i \times j]}^{(i)} \left( \mathbb{X}_{[j]}' - \mathbb{X}_{\perp[j]}' \right) \right], \quad i = 1, 2, \ldots, j.
\]

(23)

Equation (23) implies that a similar form to Cramer’s rule holds in the change of a partial set of variables. It turns out that Cramer’s rule is actually a special case for \( j = n \) of the general formula (23), which we call Cramer’s rule for a partial set of variables or coordinates. To our best knowledge, the generalized formula (23) is new. In the following section, we demonstrate how the rule (23) applies with a simple mechanical system.

We repeat the same procedure until we reach \( j = n \) at which \( \mathbb{X}_{\perp[n]} \) vanishes:
As a result, for \( i = 1, \ldots, n \),

\[
\begin{align*}
\mathbf{x}_i &= \frac{\mathbf{D} \mathbf{e} \mathbf{t} [\mathbb{R}^{(i)}(\mathbf{X}')]}{\mathbf{D} \mathbf{e} \mathbf{t} [\mathbb{R}]} , \\
&= \mathbf{X}^{(n)} , \tag{25}
\end{align*}
\]

where \( \mathbb{R}^{(i)}(\mathbf{X}') = \mathbb{R}^{(i)}[\mathbf{X}_{[i]}(\mathbf{X}')] \). This completes the proof of Cramer’s rule (4).

### 4. Application

As an example of the physical application of Cramer’s rule for a partial set of variables in equation (23), we consider the accelerating motion of a series of \( n \) blocks connected with ropes of negligible mass as shown in figure 1. We assume that every rope is tight and do not consider the restoring force of the rope. Newton’s second law states that

\[
\begin{pmatrix}
m_1 \\
m_2 \\
m_3 \\
\vdots \\
m_{n-2} \\
m_{n-1} \\
m_n
\end{pmatrix}
\begin{pmatrix}
a \\
\vdots \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
1 & -1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & -1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & -1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
T_1 \\
T_2 \\
T_3 \\
\vdots \\
T_{n-2} \\
T_{n-1} \\
T_n
\end{pmatrix} , \tag{26}
\]

where \( a \) is the acceleration of the system, \( m_i \) is the mass of the \( i \)th block, and \( T_j \) is the tension of the rope between \( m_j \) and \( m_{j-1} \), \( \mathbf{X}', \mathbf{R}, \) and \( \mathbf{X} \) in equation (3) can be read off from equation (26) to find that

\[
x_i = T_i , \quad \text{and} \quad x'_i = ma , \quad \text{for} \quad i = 1, \ldots, n . \quad \tag{27}
\]

Let us choose the partial set of variables as \( T_1, T_2, \) and \( T_3 \) with \( j = 3 \). By making use of Cramer’s rule for a partial set of variables in equation (23), we can express \( T_1, T_2, \) and \( T_3 \) in terms of \( m_1a, m_2a, \) and \( m_3a \) and \( T_4, \ldots, T_n \) as follows:

\[
\begin{align*}
T_1 &= \frac{1}{\mathbf{D} \mathbf{e} \mathbf{t} [\mathbb{R}^{(3)}]} \mathbf{D} \mathbf{e} \mathbf{t} [\mathbb{R}^{(1)}(\mathbf{X}'_{[i]} - \mathbf{X}'_{[i]})] , \\
T_2 &= \frac{1}{\mathbf{D} \mathbf{e} \mathbf{t} [\mathbb{R}^{(3)}]} \mathbf{D} \mathbf{e} \mathbf{t} [\mathbb{R}^{(2)}(\mathbf{X}'_{[i]} - \mathbf{X}'_{[i]})] , \\
T_3 &= \frac{1}{\mathbf{D} \mathbf{e} \mathbf{t} [\mathbb{R}^{(3)}]} \mathbf{D} \mathbf{e} \mathbf{t} [\mathbb{R}^{(3)}(\mathbf{X}'_{[i]} - \mathbf{X}'_{[i]})] , \tag{28}
\end{align*}
\]

where
Figure 1. A series of \( n \) blocks connected with ropes of negligible mass moving with the acceleration \( a \). \( T_i \) is the tension of the rope connecting the blocks of mass \( m_i \) and \( m_{i-1} \).

\[
\mathbf{T} = \begin{pmatrix}
m_1 & T_n & m_{n-1} & T_{n-1} & \ldots & T_3 & m_2 & T_2 & m_1 & T_1 & a
\end{pmatrix}
\]

Substituting the values in equation (29) into equation (28), we obtain

\[
\mathbf{X} = \begin{pmatrix}
x'_1 \\
x'_2 \\
x'_3 \\
\vdots \\
x'_{n-2} \\
x'_{n-1} \\
x'_n
\end{pmatrix} - \sum_{k=4}^{n} x_k \begin{pmatrix}
R_{kk} \\
R_{kk}
\end{pmatrix} \begin{pmatrix}
x'_1 \\
x'_2 \\
x'_3 \\
\vdots \\
x'_{n-2} \\
x'_{n-1} \\
x'_n
\end{pmatrix} - x_4 \begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
m_1 a \\
m_2 a \\
m_3 a + T_4
\end{pmatrix},
\]

and

\[
\mathbf{X}' = \mathbf{X} - \mathbf{X}_0
\]

Substituting the values in equation (29) into equation (28), we obtain

\[
T_1 = (m_1 + m_2 + m_3)a + T_4,
\]

\[
T_2 = (m_2 + m_3)a + T_4,
\]

\[
T_3 = m_3a + T_4.
\]

Therefore, using equation (23), we could express the selected set of variables \( T_1, T_2, T_3 \) in terms of the transformed variables, \( m_1a, m_2a, m_3a \), and could keep the remaining set of variables \( T_k \) for \( k \geq 4 \) intact. This partial transformation enabled by Cramer’s rule for a partial set of variables in equation (23) could also be used in more general cases involving a large number of variables some of which do not have to be specified in terms of the transformed variables.

Recursive applications of the Cramer’s rule for a partial set of variables in equation (23) through \( j = n \) lead to

\[
\begin{pmatrix}
T_1 \\
T_2 \\
T_3 \\
\vdots \\
T_{n-2} \\
T_{n-1} \\
T_n
\end{pmatrix} = a \begin{pmatrix}
1 & 1 & \ldots & 1 & 1 & 1 \\
0 & 1 & \ldots & 1 & 1 & 1 \\
0 & 0 & 1 & \ldots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 1 & 1 \\
0 & 0 & \ldots & 0 & 1 & 1 \\
0 & 0 & \ldots & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
m_1 \\
m_2 \\
m_3 \\
\vdots \\
m_{n-2} \\
m_{n-1} \\
m_n
\end{pmatrix}.
\]
As a result, $T_i$’s are completely solved as

$$T_i = \left( \sum_{k=i}^{n} m_j \right) a.$$  
(32)

While the solution (32) provides complete information of every tension, the solution (30) for a partial set of variables in which one is interested contains every specific information that one wants to know. Thus the solution (30) should be particularly economical as the number of variables becomes very large.

5. Discussion

We have derived Cramer’s rule for a solution of the system of linear equations, $X' = RX$, where $X$ and $X'$ are interpreted as $n$-dimensional Cartesian-coordinate column vectors in equation (2) and $R$ is the corresponding transformation matrix. The introduction of a trivial identity (11) involving Dirac delta functions has leaded to a way to reach the systematic transformation of the original coordinates $x_i$’s into new coordinates $x_i'$’s in a straightforward manner resulting in the completion of the proof of Cramer’s rule (4). To our best knowledge, this proof is new and, furthermore, could be useful to the pedagogical training of recursive application of integration of Dirac delta functions.

In an intermediate step, we have derived equation (23), where $x_i$ ($i = 1, \ldots, j$) is expressed in terms of $x_i'$ and $x_k$ ($k = j + 1, \ldots, n$) with the transformation matrix $R$. In order to find the solution, we have to know information for the elements of the submatrix $R_{ixj}$ as well as those of another $j \times (n - j)$ matrix, $R_{kxj}$. We call equation (23) as Cramer’s rule for a partial set of variables or coordinates. This rule can be immediately applied to change a partial set of generalized coordinates of a mechanical system. The original Cramer’s rule in equation (4) is actually a special case for $j = n$ of the general formula (23). This rule can further be extended to any linear problems, such as Kirchhoff’s circuit laws, and any oscillating systems whose displacements are small.

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Appendix A. Proof of equation (22) by mathematical induction

In this appendix A, we present a proof of equations (22) and (23) by mathematical induction. We have proved the statement for $j = 1$ in equations (15) and (17). We assume that equations (22) and (23) are satisfied for a given $j = p - 1$. In order to prove that the counterparts of equations (15) and (17) for $j = p$ are true, it is crucial to verify that the following
relation, which involves the argument of the delta function in the left side of equation (22), is true:

$$\mathcal{Q}_{p-1} \left[ x'_p - \sum_{k=1}^{n} R_{pk} x_k \right] = \mathcal{Q} \epsilon \mathcal{t} \left[ \mathbb{R}_{[p \times p]} \left( \mathcal{X}'_{[p]} - \mathcal{X}_{[p]} \right) \right] - \mathcal{Q}_p x_p.$$  \hspace{1cm} (A1)

The left side of equation (A1) can be rewritten as

$$\mathcal{Q}_{p-1} \left[ x'_p - \sum_{k=1}^{n} R_{pk} x_k \right] = \mathcal{Q}_{p-1} \left[ x'_p - \sum_{k=1}^{p-1} R_{pk} x_k - R_{pp} x_p - \sum_{k=p+1}^{n} R_{pk} x_k \right]$$

$$= \mathcal{Q}_{p-1} \left[ x'_p - \sum_{k=1}^{p-1} R_{pk} x_k - R_{pp} x_p - \sum_{k=p+1}^{n} R_{pk} x_k \right]$$

$$= \mathcal{Q}_{p-1} \left[ x'_p - \sum_{k=1}^{p-1} R_{pk} \left( \frac{1}{\mathcal{Q}_{p-1}} \mathcal{Q} \epsilon \mathcal{t} \left[ \mathbb{R}_{(p-1) \times (p-1)} \left( \mathcal{X}'_{[p-1]} - \mathcal{X}_{[p-1]} \right) \right] \right) - R_{pp} x_p - \sum_{k=p+1}^{n} R_{pk} x_k \right]$$

$$= A(x'_1, \ldots, x'_p) + B(x_p) + C(x_{p+1}, \ldots, x_n),$$  \hspace{1cm} (A2)

where $x_k$ for $k = 1, \ldots, p - 1$ in the second line is replaced with the relation (23). In the last line, we collect the primed-coordinate, $x_p$, and unprimed-coordinate contributions of $k \geq p + 1$ separately, which are defined by functions, $A(x'_1, \ldots, x'_p)$, $B(x_p)$, and $C(x_{p+1}, \ldots, x_n)$, respectively.

If we list up the matrix representation for $\mathbb{R}_{(p-1) \times (p-1)} \left( \mathcal{X}'_{[p]} - \mathcal{X}_{[p]} \right)$, then we find that

$$\mathbb{R}_{(p-1) \times (p-1)} \left( \mathcal{X}'_{[p]} - \mathcal{X}_{[p]} \right)$$

$$= \begin{bmatrix}
\begin{array}{cccc}
R_{11} & \cdots & R_{1, k-1} & x'_1 - \sum_{\ell=1}^{n} R_{1 \ell} x_\ell \\
\vdots & \ddots & \vdots & \vdots \\
R_{p-1, 1} & \cdots & R_{p-1, k-1} & x'_{p-1} - \sum_{\ell=p}^{n} R_{p-1, \ell} x_\ell \\
\end{array}
\end{bmatrix}.$$  \hspace{1cm} (A3)

The primed-coordinate contribution $A(x'_1, \ldots, x'_p)$ can be collected from equations (A2) and (A3) as

$$A(x'_1, \ldots, x'_p)$$

$$= \mathcal{Q}_{p-1} x'_p - \sum_{k=1}^{p-1} R_{pk} \mathcal{Q} \epsilon \mathcal{t} \begin{bmatrix}
\begin{array}{cccc}
R_{11} & \cdots & R_{1, k-1} & x'_1 \\
\vdots & \ddots & \vdots & \vdots \\
R_{p-1, 1} & \cdots & R_{p-1, k-1} & x'_{p-1} \\
\end{array}
\end{bmatrix}$$

$$= \mathcal{Q} \epsilon \mathcal{t} \left[ \mathbb{R}_{[p \times p]} \left( \mathcal{X}'_{[p]} \right) \right],$$  \hspace{1cm} (A4)
where we have made use of the following two properties of the determinant: (i) $\det(\mathbf{A} + \mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B})$ for any numbers $a_1$ and $a_2$. (ii) An exchange of two distinct columns of a matrix flips the sign of the determinant.

In a similar manner, $B(x_p)$ and $C(x_{p+1}, \ldots, x_n)$ can be collected from equations (A2) and (A3) and simplified as

$$B(x_p) = \sum_{k=1}^{p-1} R_{pk} \Delta t \begin{pmatrix} R_{11} & \cdots & R_{1k-1} & R_{1p-1} \\ R_{21} & \cdots & R_{2k-1} & R_{2p-1} \\ \vdots & \ddots & \vdots & \vdots \\ R_{p-1,1} & \cdots & R_{p-1,p-1} & R_{pp} \end{pmatrix} = -x_p \Delta t \begin{pmatrix} R_{11} & \cdots & R_{1p-1} & R_{1p} \\ R_{21} & \cdots & R_{2p-1} & R_{2p} \\ \vdots & \ddots & \vdots & \vdots \\ R_{p-1,1} & \cdots & R_{p-1,p-1} & R_{pp} \end{pmatrix}$$

$$= -x_p \Delta t \begin{pmatrix} R_{11} & \cdots & R_{1p-1} & R_{1p} \\ R_{21} & \cdots & R_{2p-1} & R_{2p} \\ \vdots & \ddots & \vdots & \vdots \\ R_{p-1,1} & \cdots & R_{p-1,p-1} & R_{pp} \end{pmatrix}$$

$$= -\Delta t \begin{pmatrix} R_{11} & \cdots & R_{1p-1} & R_{1p} \\ R_{21} & \cdots & R_{2p-1} & R_{2p} \\ \vdots & \ddots & \vdots & \vdots \\ R_{p-1,1} & \cdots & R_{p-1,p-1} & R_{pp} \end{pmatrix} \begin{pmatrix} x_{p+1} \\ \vdots \\ x_n \end{pmatrix}$$

$$= -\Delta t \begin{pmatrix} R_{11} & \cdots & R_{1p-1} & R_{1p} \\ R_{21} & \cdots & R_{2p-1} & R_{2p} \\ \vdots & \ddots & \vdots & \vdots \\ R_{p-1,1} & \cdots & R_{p-1,p-1} & R_{pp} \end{pmatrix} \begin{pmatrix} \sum_{t=p+1}^{n} R_{pt} x_t \\ \vdots \\ \sum_{t=p+1}^{n} R_{pt} x_t \end{pmatrix}$$

$$= -\Delta t \begin{pmatrix} R_{11} & \cdots & R_{1p-1} & R_{1p} \\ R_{21} & \cdots & R_{2p-1} & R_{2p} \\ \vdots & \ddots & \vdots & \vdots \\ R_{p-1,1} & \cdots & R_{p-1,p-1} & R_{pp} \end{pmatrix} \begin{pmatrix} \sum_{t=p+1}^{n} R_{pt} x_t \\ \vdots \\ \sum_{t=p+1}^{n} R_{pt} x_t \end{pmatrix}$$

$$= -\Delta t \begin{pmatrix} R_{11} & \cdots & R_{1p-1} & R_{1p} \\ R_{21} & \cdots & R_{2p-1} & R_{2p} \\ \vdots & \ddots & \vdots & \vdots \\ R_{p-1,1} & \cdots & R_{p-1,p-1} & R_{pp} \end{pmatrix} \begin{pmatrix} \sum_{t=p+1}^{n} R_{pt} x_t \\ \vdots \\ \sum_{t=p+1}^{n} R_{pt} x_t \end{pmatrix}$$

$$= -\Delta t \begin{pmatrix} R_{11} & \cdots & R_{1p-1} & R_{1p} \\ R_{21} & \cdots & R_{2p-1} & R_{2p} \\ \vdots & \ddots & \vdots & \vdots \\ R_{p-1,1} & \cdots & R_{p-1,p-1} & R_{pp} \end{pmatrix} \begin{pmatrix} \sum_{t=p+1}^{n} R_{pt} x_t \\ \vdots \\ \sum_{t=p+1}^{n} R_{pt} x_t \end{pmatrix}$$

Substituting equations (A4)–(A6) into (A2), we obtain (A1). As a result, the relations (22) and (23) hold for any $j$ for $j = 1$ through $n$ by mathematical induction.

**ORCID iDs**

Jungil Lee https://orcid.org/0000-0003-3824-1769

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