ON DIVERGENCE FORM SPDES WITH GROWING COEFFICIENTS IN $W_2^1$ SPACES WITHOUT WEIGHTS

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Abstract. We consider divergence form uniformly parabolic SPDEs with bounded and measurable leading coefficients and possibly growing lower-order coefficients in the deterministic part of the equations. We look for solutions which are summable to the second power with respect to the usual Lebesgue measure along with their first derivatives with respect to the spatial variable.

1. Introduction

We consider divergence form uniformly parabolic SPDEs with bounded and measurable leading coefficients and possibly growing lower-order coefficients in the deterministic part of the equation. We look for solutions which are summable to the second power with respect to the usual Lebesgue measure along with their first derivatives with respect to the spatial variable. To the best of our knowledge our results are new even for deterministic PDEs when one deletes all stochastic terms in the results below. If there are no stochastic terms and the coefficients are nonrandom and time independent, our results allow one to obtain the corresponding results for elliptic divergence-form equations which also seem to be new. A sample result in this case is the following. Consider the equation

$$D_i (a^{ij}(x) D_j u(x) + b^i(x) u(x)) + b^i(x) D_i u(x) - (c(x) + \lambda) u(x) = D_i f^i(x) + f^0(x)$$

(1.1)

in $\mathbb{R}^d$ which is the Euclidean space of points $x = (x^1, ..., x^d)$. Here and below the summation convention is enforced and

$$D_i = \frac{\partial}{\partial x^i}.$$

Assume that (1.1) is uniformly elliptic, $a^{ij}$ are bounded, and $c \geq 0$. Also assume that $f^j \in L_2 = L_2(\mathbb{R}^d), \ j = 0, ..., d$, and

$$\sup_{|x-y| \leq 1} (|b(x) - b(y)| + |b(x) - b(y)| + |c(x) - c(y)|) < \infty$$

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and that the constant $\lambda > 0$ is large enough. Then equation \((1.1)\) has a unique solution in the class of functions $u \in W^{1}_{2} = W^{1}_{2}(\mathbb{R}^{d})$. Notice that the above condition on $b, b,$ and $c$ allow them to grow linearly as $|x| \to \infty$.

As in [3] one of the main motivations for studying SPDEs with growing first-order coefficients is filtering theory for partially observable diffusion processes.

It is generally believed that introducing weights is the most natural setting for equations with growing coefficients. When the coefficients grow it is quite natural to consider the equations in function spaces with weights that would restrict the set of solutions in such a way that all terms in the equation will be from the same space as the free terms. The present paper seems to be the first one treating the unique solvability of these equations with growing lower-order coefficients in the usual Sobolev spaces $W^{1}_{2}$ without weights and without imposing any special conditions on the relations between the coefficients or on their derivatives.

The theory of SPDEs in Sobolev-Hilbert spaces with weights attracted some attention in the past. We do not use weights and only mention a few papers about stochastic PDEs in $L_{p}$-spaces with weights in which one can find further references: [1] (mild solutions, general $p$), [3], [8], [9] ($p = 2$ in the four last articles).

Many more papers are devoted to the theory of deterministic PDEs with growing coefficients in Sobolev spaces with weights. We cite only a few of them sending the reader to the references therein again because neither do we deal with weights nor use the results of these papers. It is also worth saying that our results do not generalize the results of the above cited papers.

In most of these papers the coefficients are time independent, see [2], [4], [7], [20], [22], part of the result of which are extended in [6] to time-dependent Ornstein-Uhlenbeck operators.

It is worth noting that many issues for deterministic divergence-type equations with time independent growing coefficients in $L_{p}$-spaces with arbitrary $p \in (1, \infty)$ without weights were also treated previously in the literature. This was done mostly by using the semigroup approach which excludes time dependent coefficients and makes it almost impossible to use the results in the more or less general filtering theory. We briefly mention only a few recent papers sending the reader to them for additional information.

In [21] a strongly continuous in $L_{p}$ semigroup is constructed corresponding to elliptic operators with measurable leading coefficients and Lipschitz continuous drift coefficients. In [23] it is assumed that if, for $|x| \to \infty$, the drift coefficients grow, then the zeroth-order coefficient should grow, basically, as the square of the drift. There is also a condition on the divergence of the drift coefficient. In [24] there is no zeroth-order term and the semigroup is constructed under some assumptions one of which translates into the monotonicity of $\pm b(x) - Kx$, for a constant $K$, if the leading term is the Laplacian. In [5] the drift coefficient is assumed to be globally Lipschitz continuous if the zeroth-order coefficient is constant.
Some conclusions in the above cited papers are quite similar to ours but
the corresponding assumptions are not as general in what concerns the reg-
cularity of the coefficients. However, these papers contain a lot of additional
important information not touched upon in the present paper (in particular,
it is shown in \[21\] that the corresponding semigroup is not analytic).

The technique, we apply, originated from \[18\] and \[13\] and uses special
cut-off functions whose support evolves in time in a manner adapted to the
drift. We do not make any regularity assumptions on the coefficients and are
restricted to only treat equations in \(L^2\). Similar, techniques could be used
to consider equations in the spaces \(W^p\) with any \(p \geq 2\). This time one can
use the results of \[11\] and \[14\] where some regularity on the coefficients in \(x\)
variable is needed like, say, the condition that the second order coefficients
be in VMO uniformly with respect to the time variable (see \[14\]). However,
for the sake of brevity and clarity we concentrate only on \(p = 2\). The main
emphasis here is that we allow the first-order coefficients to grow as \(|x| \to \infty\)
and still measure the size of the derivatives with respect to Lebesgue measure
thus avoiding using weights.

It is worth noting that considering divergence form equations in \(L^p\)-spaces
is quite useful in the treatment of filtering problems (see, for instance, \[17\])
especially when the power of summability is taken large and we intend to
treat this issue in a subsequent paper.

The article is organized as follows. In Section 2 we describe the problem,
Section 3 contains the statements of two main results, Theorem 3.1 on an
apriori estimate providing, in particular, uniqueness of solutions and Theo-
rem 3.2 about the existence of solutions. Theorem 3.1 is proved in Section 5
after we prepare the necessary tools in Section 4. Theorem 3.2 is proved in
the last Section 6.

As usual when we speak of “a constant” we always mean “a finite con-
stant”.

2. Setting of the problem

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space with an increasing filtration
\(\{\mathcal{F}_t, t \geq 0\}\) of complete with respect to \((\mathcal{F}, P)\) \(\sigma\)-fields \(\mathcal{F}_t \subset \mathcal{F}\). Denote
by \(\mathcal{P}\) the predictable \(\sigma\)-field in \(\Omega \times (0, \infty)\) associated with \(\{\mathcal{F}_t\}\). Let \(w^k_t\),
\(k = 1, 2, \ldots\), be independent one-dimensional Wiener processes with respect
to \(\{\mathcal{F}_t\}\). Finally, let \(\tau\) be a stopping time with respect to \(\{\mathcal{F}_t\}\).

We consider the second-order operator \(L\)

\[
L_t u_t(x) = D_i \left( a^{ij}_t(x) D_j u_t(x) + b^i_t(x) u_t(x) \right) + b^i_t(x) D_i u_t(x) - c_t(x) u_t(x),
\]

and the first-order operators

\[
\Lambda^k_t u_t(x) = \sigma^{ik}_t(x) D_i u_t(x) + \nu^k_t(x) u_t(x)
\]

acting on functions \(u_t(x)\) defined on \(\Omega \times \mathbb{R}^{d+1}_+\), where \(\mathbb{R}^{d+1}_+ = [0, \infty) \times \mathbb{R}^d\),
and given for \(k = 1, 2, \ldots\) (the summation convention is enforced throughout
the article). We set \(\mathbb{R}_+ = [0, \infty)\).
Our main concern is proving the unique solvability of the equation
\[ du_t = (L_t u_t - \lambda u_t + D_i f^i_t + f^0_t) \, dt + (\Lambda^k_t u_t + g^k_t) \, dw^k_t, \quad t \leq \tau, \]
with an appropriate initial condition at \( t = 0 \), where \( \lambda > 0 \) is a constant. The precise assumptions on the coefficients, free terms, and initial data will be given later. First we introduce appropriate function spaces.

Denote \( C_0^\infty = C_0^\infty(\mathbb{R}^d) \), \( L_2 = L_2(\mathbb{R}^d) \), and let \( \mathcal{W}^1_2 = W^1_2(\mathbb{R}^d) \) be the Sobolev space of functions \( u \) of class \( L_2 \), such that \( Du \in L_2 \), where \( Du \) is the gradient of \( u \). Introduce
\[
\mathcal{L}_2(\tau) = L_2((0, \tau], \mathcal{P}, L_2), \quad \mathcal{W}^1_2(\tau) = L_2((0, \tau], \mathcal{P}, W^1_2),
\]
where \( \mathcal{P} \) is the completion of \( P \) with respect to the product measure. Remember that the elements of \( \mathcal{L}_2(\tau) \) need only belong to \( L_2 \) on a predictable subset of \( (0, \tau] \) of full measure. For the sake of convenience we will always assume that they are defined everywhere on \( (0, \tau] \) at least as generalized functions. Similar situation occurs in the case of \( \mathcal{W}^1_2(\tau) \). We also use the same notation \( \mathcal{L}_2(\tau) \) for \( \ell_2 \)-valued functions like \( g_t = (g^k_t) \). For such a function, naturally,
\[
\|g\|_{\mathcal{L}_2} = \|g\|_{\ell_2} \|_{\mathcal{L}_2} = \left\| \left( \sum_{k=1}^{\infty} (g^k)^2 \right)^{1/2} \right\|_{\mathcal{L}_2} = \left( \sum_{k=1}^{\infty} \int_{\mathbb{R}^d} |g^k|^2 \, dx \right)^{1/2}.
\]

The following definition turns out to be useful if the coefficients of \( L \) and \( \Lambda^k \) are bounded.

**Definition 2.1.** We introduce the space \( \mathcal{W}^1_2(\tau) \), which is the space of functions \( u_t = u_t(\omega, \cdot) \) on \( \{ (\omega, t) : 0 \leq t \leq \tau, t < \infty \} \) with values in the space of generalized functions on \( \mathbb{R}^d \) and having the following properties:

(i) We have \( u_0 \in L_2(\Omega, \mathcal{F}_0, L_2) \);

(ii) We have \( u \in \mathcal{W}^1_2(\tau) \);

(iii) There exist \( f^i \in L_2(\tau) \), \( i = 0, ..., d \), and \( g = (g^1, g^2, ...) \in \mathcal{L}_2(\tau) \) such that for any \( \phi \in C_0^\infty \) with probability 1 for all \( t \in \mathbb{R}_+ \) we have
\[
(u_{t \wedge \tau}, \phi) = (u_0, \phi) + \sum_{k=1}^{\infty} \int_0^{t \wedge \tau} I_{s \leq \tau}(g^k_s, \phi) \, dw^k_s + \int_0^{t \wedge \tau} I_{s \leq \tau}(f^0_s, \phi) - (f^1_s, D_i \phi) \, ds.
\]

In particular, for any \( \phi \in C_0^\infty \), the process \( (u_{t \wedge \tau}, \phi) \) is \( \mathcal{F}_t \)-adapted and (a.s.) continuous. In case that property (iii) holds, we write
\[
du_t = (D_i f^i_t + f^0_t) \, dt + g^k_t \, dw^k_t, \quad t \leq \tau.
\]

It is a standard fact that for \( g \in \mathcal{L}_2(\tau) \) and any \( \phi \in C_0^\infty \) the series in (2.3) converges uniformly on \( \mathbb{R}_+ \) in probability.

Similarly to this definition we understand equation (2.2) in the general case as the requirement that for any \( \phi \in C_0^\infty \) with probability one the
relation
\[(u_{t\wedge\tau}, \phi) = (u_0, \phi) + \sum_{k=1}^{\infty} \int_0^t I_{s\leq\tau} (\sigma^{ik}_s D_i u_s + \nu^k_s u_s + g^k_s, \phi) \, dw^k_s \]
\[+ \int_0^t I_{s\leq\tau} [(b^i_s D_i u_s - (c_s + \lambda) u_s + f^0_i, \phi) - (a^{ij}_s D_j u_s + b^i_s u_s + f^j_i, D_i \phi)] \, ds \quad (2.4)\]
hold for all \(t \in \mathbb{R}_+\).

Observe that at this moment it is not clear that the right-hand side makes sense. Also notice that, if the coefficients of \(L\) and \(\Lambda^k\) are bounded, then any \(u \in \mathcal{W}^1_2(\tau)\) is a solution of (2.2) with appropriate free terms since if (2.3) holds, then (2.2) holds as well with
\[f^i_t - a^{ij}_t D_j u_t - b^i_t u_t, \quad i = 1, \ldots, d, \quad f^0_t + (c_t + \lambda) u_t - b^i_t D_i u_t,\]
\[g^k_t - \sigma^{ik}_t D_i u_t - \nu^k_t u_t\]
in place of \(f^i_t, i = 1, \ldots, d, f^0_t\), and \(g^k_t\), respectively.

3. Main results

For \(\rho > 0\) denote \(B_\rho(x) = \{ y \in \mathbb{R}^d : |x - y| < \rho \}, B_\rho = B_\rho(0)\).

**Assumption 3.1.** (i) The functions \(a^{ij}(x), b^i_t(x), b^j_t(x), c_t(x), \sigma^{ik}(x), \nu^i_t(x)\) are real valued, measurable with respect to \(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+^{d+1}), \mathcal{F}_t\)-adapted for any \(x, c \geq 0\).

(ii) There exist constants \(K, \delta > 0\) such that for all values of arguments and \(\xi \in \mathbb{R}^d\)
\[|a^{ij} - \alpha^{ij}| \xi^i \xi^j \geq \delta |\xi|^2, \quad |a^{ij}| \leq \delta^{-1}, \quad |\nu|_{\ell_2} \leq K,\]
where \(\alpha^{ij} = (1/2)(\sigma^i, \sigma^j)_{\ell_2}\). Also, the constant \(\lambda > 0\).

(iii) For any \(x \in \mathbb{R}^d\) (and \(\omega\)) the function
\[\int_{B_1} (|b_t(x + y)| + |b_t(x + y)| + c_t(x + y)) \, dy\]
is locally square integrable on \(\mathbb{R}_+ = [0, \infty)\).

Notice that the matrix \(a = (a^{ij})\) need not be symmetric. Also notice that in Assumption 3.1 (iii) the ball \(B_1\) can be replaced with any other ball without changing the set of admissible coefficients \(b, b, c\).

We take some \(f^j, g \in \mathbb{L}^2(\tau)\) and before we give the definition of solution of (2.2) we remind the reader that, if \(u \in \mathcal{W}^1_2(\tau)\), then owing to the boundedness of \(\nu\) and \(\sigma\) and the fact that \(Du, u, g \in \mathcal{L}_2(\tau)\), the first series on the right in (2.4) converges uniformly in probability and the series is a continuous local martingale.

**Definition 3.1.** By a solution of (2.2) for \(t \leq \tau\) with initial condition \(u_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{L}_2)\) we mean a function \(u \in \mathcal{W}^1_2(\tau)\) (not \(\mathcal{W}^1_2(\tau)\)) such that

(i) For any \(\phi \in C^\infty_0\) with probability one the integral with respect to \(ds\) in (2.4) is well defined and is finite for all \(t \in \mathbb{R}_+\);
(ii) For any \( \phi \in C_0^\infty \) with probability one equation (2.4) holds for all \( t \in \mathbb{R}_+ \).

For \( d \neq 2 \) define
\[
q = d \lor 2,
\]
and if \( d = 2 \) let \( q \) be a fixed number such that \( q > 2 \). The following assumption contains a parameter \( \gamma \in (0,1] \), whose value will be specified later.

**Assumption 3.2 \((\gamma)\).** There exists a \( \rho_0 \in (0,1] \) such that, for any \( \omega \in \Omega \) and \( b := (b^1, \ldots, b^d) \) and \( b := (b^1, \ldots, b^d) \) and \( (t,x) \in \mathbb{R}^{d+1}_+ \) we have
\[
\begin{align*}
\rho_0^{-d} \int_{B_{\rho_0}} \int_{B_{\rho_0}} |b_t(x + y) - b_t(x + z)|^q \, dy \, dz &\leq \gamma, \\
\rho_0^{-d} \int_{B_{\rho_0}} \int_{B_{\rho_0}} |b_t(x + y) - b_t(x + z)|^q \, dy \, dz &\leq \gamma, \\
\rho_0^{-d} \int_{B_{\rho_0}} \int_{B_{\rho_0}} |c_t(x + y) - c_t(x + z)|^q \, dy \, dz &\leq \gamma.
\end{align*}
\]

Obviously, Assumption 3.2 is satisfied with any \( \gamma \in (0,1] \) if \( b, b, \) and \( c \) are independent of \( x \). Also notice that Assumption 3.2 allows \( b, b, \) and \( c \) growing linearly in \( x \).

**Theorem 3.1.** There exist \( \gamma = \gamma(d, \delta, K) \in (0,1] \),
\[
N = N(d, \delta, K), \quad \lambda_0 = \lambda_0(d, \delta, K, \rho_0) \geq 0
\]
such that, if the above assumptions are satisfied and \( \lambda \geq \lambda_0 \) and \( u \) is a solution of (2.2) with initial condition \( u_0 \) and some \( f^i, g \in L_2(\tau) \), then
\[
\begin{align*}
&\|u\sqrt{\lambda + c}\|^2_{L_2(\tau)} + \|Du\|^2_{L_2(\tau)} \leq N \left( \sum_{i=1}^d \|f^i\|^2_{L_2(\tau)} + \|g\|^2_{L_2(\tau)} + \lambda^{-1}\|f^0\|^2_{L_2(\tau)} + E\|u_0\|^2_{L_2} \right), \\
&\quad + \|f^0\|^2_{L_2(\tau)} + \lambda^{-1}\|f^0\|^2_{L_2(\tau)} + E\|u_0\|^2_{L_2} \right). \tag{3.1}
\end{align*}
\]

This theorem provides an apriori estimate implying uniqueness of solutions \( u \). Observe that the assumption that such a solution exists is quite nontrivial because if \( b_t(x) \equiv x \), it is not true that \( bu \in L_2(\tau) \) for arbitrary \( u \in W_2^1(\tau) \).

To prove the existence we need stronger assumptions because, generally, under the above assumptions the term
\[
D_t(b_t^i u_t) + b_t^i D_t u_t
\]
cannot be written even locally as \( D_t\hat{f}_t^i + \hat{f}_t^0 \) with \( \hat{f}_t^j \in L_2(\tau) \) if we only know that \( u \in W_2^1(\tau) \) even if \( b \) and \( b \) are independent of \( x \). We can only prove our crucial Lemma 6.5 if such a representation is possible.
Assumption 3.3. For any $T, R \in \mathbb{R}_+$, and $\omega \in \Omega$ we have

$$\sup_{t \leq T} \int_{B_R} (|b_t(x)| + |b_t(x)| + c_t(x)) \, dx < \infty.$$ 

Theorem 3.2. Let the above assumptions be satisfied with $\gamma$ taken from Theorem 3.1. Take $\lambda \geq \lambda_0$, where $\lambda_0$ is defined in Theorem 3.1, and take $u_0 \in L_2(\Omega, \mathcal{F}_0, L_2)$. Then there exists a unique solution of (2.2) with initial condition $u_0$.

Remark 3.1. If the stopping time $\tau$ is bounded, then in the above theorem one can take $\lambda = 0$. To show this take a large $\lambda > 0$ and replace the unknown function $u_t$ with $v_t e^{\lambda t}$. This leads to an equation for $v_t$ with the additional term $-\lambda v_t dt$ and the free terms multiplied by $e^{-\lambda t}$. The existence of $v \in W^1_2(\tau)$ will be then equivalent to the existence of $u \in W^1_2(\tau)$ if $\tau$ is bounded.

4. A version of the Itô-Wentzell formula

Let $\mathcal{D}$ be the space of generalized functions on $\mathbb{R}^d$. We remind a definition and a result from [16]. Recall that for any $v \in \mathcal{D}$ and $\phi \in C_0^\infty$ the function $(v, \phi(\cdot - x))$ is infinitely differentiable with respect to $x$, so that the sup in (4.1) below is predictable.

Definition 4.1. Denote by $\mathcal{D}$ the set of all $\mathcal{D}$-valued functions $u$ (written as $u_t(x)$ in a common abuse of notation) on $\Omega \times \mathbb{R}_+$ such that, for any $\phi \in C_0^\infty := C_0^\infty(\mathbb{R}^d)$, the restriction of the function $(u_t, \phi)$ on $\Omega \times (0, \infty)$ is $\mathcal{P}$-measurable and $(u_0, \phi)$ is $\mathcal{F}_0$-measurable. For $p = 1, 2$ denote by $\mathcal{D}^p$ the subset of $\mathcal{D}$ consisting of $u$ such that, for any $\phi \in C_0^\infty$ and $T, R \in \mathbb{R}_+$, we have

$$\int_0^T \sup_{|x| \leq R} |(u_t, \phi(\cdot - x))|^p \, dt < \infty \quad \text{(a.s.)}. \quad (4.1)$$

In the same way, considering $\ell_2$-valued distributions $g$ on $C_0^\infty$, that is linear $\ell_2$-valued functionals such that $(g, \phi)$ is continuous as an $\ell_2$-valued function with respect to the standard convergence of test functions, we define $\mathcal{D}(\ell_2)$ and $\mathcal{D}^2(\ell_2)$ replacing $| \cdot |$ in (4.1) with $p = 2$ by $| \cdot |_{\ell_2}$.

Observe that if $g \in \mathcal{D}^2(\ell_2)$ then for any $\phi \in C_0^\infty$, and $T \in \mathbb{R}_+$

$$\sum_{k=1}^\infty \int_0^T (g^k_t, \phi)^2 \, dt = \int_0^T |(g_t, \phi)|^2_{\ell_2} \, dt < \infty \quad \text{(a.s.)},$$

which, by well known theorems about convergence of series of martingales, implies that the series in (4.3) below converges uniformly on $[0, T]$ in probability for any $T \in \mathbb{R}_+$.

Definition 4.2. Let $f, u \in \mathcal{D}$, $g \in \mathcal{D}(\ell_2)$. We say that the equality

$$du_t(x) = f_t(x) \, dt + g^k_t(x) \, dw^k_t, \quad t \leq \tau, \quad (4.2)$$
Corollary 4.2. Under the assumptions of Theorem 4.1 for any \( \phi \in C_0^\infty \), with probability one we have for all \( t \in \mathbb{R}_+ \)

\[
(u_{t\wedge \tau}, \phi) = (u_0, \phi) + \int_0^t I_{s \leq \tau}(f_s, \phi) \, ds + \sum_{k=1}^{\infty} \int_0^t I_{s \leq \tau}(g_{sk}^k, \phi) \, dw_s^k. \tag{4.3}
\]

Let \( x_t \) be an \( \mathbb{R}^d \)-valued stochastic process given by

\[
x_t^i = \int_0^t \hat{b}_s^i \, ds + \sum_{k=1}^{\infty} \int_0^t \hat{\sigma}_{sk}^{ik} \, dw_s^k,
\]

where \( \hat{b}_t = (\hat{b}_t^i) \), \( \hat{\sigma}_t^k = (\hat{\sigma}_t^{ik}) \) are predictable \( \mathbb{R}^d \)-valued processes such that for all \( \omega \) and \( s, T \in \mathbb{R}_+ \) we have \( \text{tr} \, \hat{\alpha}_t < \infty \) and

\[
\int_0^T (|\hat{b}_t| + \text{tr} \, \hat{\alpha}_t) \, dt < \infty,
\]

where \( \hat{\alpha}_t = (\hat{\alpha}_t^{ij}) \) and \( 2\hat{\alpha}_t^{ij} = (\hat{\sigma}_t^{ij}, \hat{\sigma}_t^{ji})_{\mathbb{R}_2} \). Finally, before stating the main result of [10] we remind the reader that for a generalized function \( v \), and any \( \phi \in C_0^\infty \) the function \( (v, \phi(\cdot - x)) \) is infinitely differentiable and for any derivative operator \( D \) of order \( n \) with respect to \( x \) we have

\[
D(v, \phi(\cdot - x)) = (-1)^n (v, (D\phi)(\cdot - x)) =: (Dv, \phi(\cdot - x)) =: ((Dv)(\cdot + x), \phi)
\]

implying, in particular, that \( Du \in \mathcal{D} \) if \( u \in \mathcal{D} \).

Theorem 4.1. Let \( f, u \in \mathcal{D} \), and \( g \in \mathcal{D}(l_2) \). Introduce

\[
v_t(x) = u_t(x + x_t)
\]

and assume that (4.2) holds (in the sense of distributions). Then

\[
dv_t(x) = [f_t(x + x_t) + \hat{L}_t v_t(x) + (D_t g_t(x + x_t), \hat{\sigma}_t^i)_{\mathbb{R}_2}] \, dt
\]

\[
+ [g_t^k(x + x_t) + D_t v_t(x) \hat{\sigma}_t^{ik}] \, dw_t^k, \quad t \leq \tau \tag{4.5}
\]

(in the sense of distributions), where \( \hat{L}_t v_t = \hat{\alpha}_t^{ij} D_i D_j v_t(x) + \hat{b}_t^i D_i v_t(x) \). In particular, the terms on the right in (4.5) belong to the right class of functions.

We remind the reader that the summation convention over the repeated indices \( i, j = 1, \ldots, d \) (and \( k = 1, 2, \ldots \)) is enforced throughout the article. In the main part of this paper we are going to use Theorem 4.1 only for \( \hat{\sigma} \equiv 0 \).

Corollary 4.2. Under the assumptions of Theorem 4.1 for any \( \eta \in C_0^\infty \) we have

\[
d[u_t(x) \eta(x - x_t)] = [g_t^k(x) \eta(x - x_t) - u_t(x) \hat{\sigma}_t^{ik} (D_t \eta)(x - x_t)] \, dw_t^k
\]

\[
+ [f_t(x) \eta(x - x_t) + u_t(x) (\hat{L}_t^* \eta)(x - x_t) - (g_t(x), \hat{\sigma}_t^i (D_t \eta)(x - x_t))_{\mathbb{R}_2}] \, dt, \quad t \leq \tau,
\]

where \( \hat{L}_t^* \) is the formal adjoint to \( \hat{L}_t \).
Indeed, what we claim is that for any \( \phi \in C_0^\infty \) with probability one
\[
((u_t \wedge \tau) \phi)(\cdot + x_t \wedge \tau), \eta) = (u_0 \phi, \eta)
\]
\[
+ \int_0^t I_{s \leq \tau} \left( \left[ g_s \phi + \hat{\sigma}_s^D_i(u_s \phi) \right] (\cdot + x_s), \eta \right) d w_s^k
\]
\[
+ \int_0^t I_{s \leq \tau} \left( \left[ f_s \phi + \hat{L}_s(u_t \phi) + (\hat{\sigma}_s^D_i(u_s \phi))_s \right] (\cdot + x_s), \eta \right) d s
\]
for all \( t \). However, to obtain this result it suffices to write down an obvious equation for \( u_t \phi \), then use Theorem 4.1 and, finally, use Definition 4.2 to interpret the result.

5. Proof of Theorem 3.1

Throughout this section we suppose that the assumptions of Theorem 3.1 are satisfied and start with analyzing the second integral in (2.4). Recall that \( q \) was introduced before Assumption 3.2.

Lemma 5.1. Let \( h \in L_q, v \in L_2, \) and \( u \in W_2^1 \). Then there exist \( V \in W_2^1 \), \( j = 0, 1, ..., d \), such that
\[
hv = D_i V^i + V_0, \quad \sum_{j=0}^d \| V^j \|_{L_2} \leq N \| h \|_{L_q} \| v \|_{L_2},
\]
where \( N \) is independent of \( h \) and \( v \). In particular,
\[
| (hv, u) | \leq N \| h \|_{L_q} \| v \|_{L_2} \| u \|_{W_2^1}. \tag{5.1}
\]
Furthermore, if a number \( \rho > 0 \), then for any ball \( B \) of radius \( \rho \) we have
\[
\| I_B hu \|_{L_2} \leq N \| h \|_{L_q} \| I_B u \|_{L_2} \left( \rho^{1-d/q} \| I_B Du \|_{L_2} + \rho^{-d/q} \| I_B u \|_{L_2} \right), \tag{5.2}
\]
where \( N \) is independent of \( h, u, \rho, \) and \( B \).

Proof. Observe that by Hölder’s inequality for \( r = 2q/(2 + q) \) \( (\in [1, 2]) \) we have
\[
\| hv \|_{L_r} \leq \| h \|_{L_q} \| v \|_{L_2}.
\]
Next we use the classical theory and introduce \( V \in W^2_r \) (note that \( r > 1 \) if \( d \neq 1 \) and \( r = 1 \) if \( d = 1 \)) as a unique solution of
\[
\Delta V - V = hv.
\]
We know that for a constant \( N = N(d, r) \) we have
\[
\| V \|_{W^2_r} \leq N \| hv \|_{L_r}, \quad \| V \|_{W^2_2} \leq N \| V \|_{W^2_2},
\]
where the last inequality follows by embedding theorems \( (2-d/r) \geq 1-d/2 \).
Now to prove the first assertion of the lemma it only remains to combine the above estimates and notice that for \( V^i = D_i V, \ i = 1, ..., d, \ V^0 = -V \) it holds that \( hv = D_i V^i + V^0 \).

To prove the second assertion, first let \( q > 2 \). Observe that by Hölder’s inequality
\[
\| I_B hu \|_{L_2} \leq \| h \|_{L_q} \| I_B u \|_{L_2},
\]
where $s = 2q/(q - 2)$. By embedding theorems (we use the fact that $d/s \geq \frac{d}{2} - 1$)

$$\|I_B u\|_{L^s} \leq N(\rho^{1-d/q}\|I_B Du\|_{L^2} + \rho^{-d/q}\|I_B u\|_{L^2})$$

and the result follows. In the remaining case $q = 2$, which happens only if $d = 1$. In that case the above estimates remain true if we set $s = \infty$. The lemma is proved.

Before we extract some consequences from the lemma we take a nonnegative $\xi \in C^\infty_0(B_{\rho_0})$ with unit integral and define

$$\bar{b}_s(x) = \int_{B_{\rho_0}} \xi(y)b_s(x - y) \, dy, \quad \bar{b}_s(x) = \int_{B_{\rho_0}} \xi(y)b_s(x - y) \, dy,$$

$$\bar{c}_s(x) = \int_{B_{\rho_0}} \xi(y)c_s(x - y) \, dy. \quad (5.3)$$

We may assume that $\|\xi\| \leq N(d)\rho_0^{-d}$.

One obtains the first two assertions of the following corollary from (5.1) and (5.2) by performing estimates like

$$\|I_B u(x)\|_{L^q} = \int_{B_{\rho_0}(x)} |b_t - \tilde{b}_t(x_t)|^q \, dx$$

$$= \int_{B_{\rho_0}(x_1)} \int_{B_{\rho_0}(x_1)} |b_t(x) - b_t(y)|\xi(x_t - y) \, dy \, dy$$

$$\leq N \int_{B_{\rho_0}(x_1)} \rho_0^{-d} \int_{B_{\rho_0}(x_1)} |b_t(x) - b_t(y)| \, dy \, dx$$

$$\leq N \rho_0^{-d} \int_{B_{\rho_0}(x_1)} \int_{B_{\rho_0}(x_1)} |b_t(x) - b_t(y)| \, dy \, dx \leq N\gamma, \quad (5.4)$$

**Corollary 5.2.** Let $u \in W^1_2(\tau)$, let $x_s$ be an $\mathbb{R}^d$-valued predictable process, and let $\eta \in C^\infty_0(B_{\rho_0})$. Set $\eta_s(x) = \eta(x - x_s)$. Then on $(0, \tau]$

(i) For any $v \in W^1_2$

$$|b^s(x_s)|I_{B_{\rho_0}(x_s)} |D_i u_s| \leq N(d)\gamma^{1/q} \|I_{B_{\rho_0}(x_s)} D u_s\|_{L^2} \|v\|_{W^2_2};$$

(ii) We have

$$\|I_{B_{\rho_0}(x_s)} b_s - \tilde{b}_s(x_s)|u_s\|_{L^2} + \|I_{B_{\rho_0}(x_s)} c_s - \bar{c}_s(x_s)|u_s\|_{L^2}$$

$$\leq N(d)\gamma^{1/q} (\rho_0^{1-d/q} \|I_{B_{\rho_0}(x_s)} D u_s\|_{L^2} + \rho_0^{-d/q} \|I_{B_{\rho_0}(x_s)} u_s\|_{L^2});$$

(iii) Almost everywhere on $(0, \tau]$ we have

$$(b^s_i - \tilde{b}^s_i(x_s)) \eta_s D_i u_s = D_i V^i_s + V^0_s, \quad (5.5)$$

$$\sum_{j=0}^d \|V^j_s\|_{L^2} \leq N(d)\gamma^{1/q} \|I_{B_{\rho_0}(x_s)} D u_s\|_{L^2} \sup_{B_{\rho_0}} |\eta|, \quad (5.6)$$

where $V^j_s$, $j = 0, \ldots, d$, are some predictable $L^2$-valued functions on $(0, \tau]$. 


To prove (iii) observe that one can find a predictable set \( A \subset (0, \tau] \) of full measure such that \( I_A D_i u_i, i = 1, \ldots, d \), are well defined as \( L_2 \)-valued predictable functions. Then (5.5) with \( I_A D_i u \) in place of \( D_i u \) and (5.6) follow from (5.4), the first assertion of Lemma 5.1 and the fact that the way \( V^j \) are constructed uses bounded hence continuous operators and translates the measurability of the data to the measurability of the result. Since we are interested in (5.5) and (5.6) holding only almost everywhere on \( (0, \tau] \), there is no actual need for the replacement.

**Corollary 5.3.** Let \( u \in W_{2}^1(\tau) \). Then for almost any \((\omega, s)\) the mappings
\[
\phi \rightarrow I_{s \leq \tau}(b^i_s D_i u_s, \phi), \quad I_{s \leq \tau}(b^i_s u_s, D_i \phi), \quad I_{s \leq \tau}(c_s u_s, \phi) \quad (5.7)
\]
are generalized functions on \( \mathbb{R}^d \). Furthermore, for any \( T \in \mathbb{R}_+ \) almost surely
\[
\int_{T}^{s \leq \tau} \left( |(b^i_s D_i u_s, \phi)| + |(b^i_s u_s, D_i \phi)| + |(c_s u_s, \phi)| \right) ds < \infty, \quad (5.8)
\]
so that requirement (i) in Definition 3.1 can be dropped.

Proof. By having in mind partitions of unity we convince ourselves that it suffices to prove that the mappings (5.7) are generalized functions on any ball \( B \) of radius \( r_0 \) and that (5.8) holds if \( \phi \in C_{0}^{\infty}(B) \). Let \( x_0 \) be the center of \( B \) and set \( x_s \equiv x_0 \). Then to prove the first assertion concerning the last two functions in (5.7) it suffices to use the first assertion of Corollary 5.2 along with the observation that, say,
\[
(b^i_s u_s, D_i \phi) = ((b^i_s - b^i_\tau(x_0)) u_s, D_i \phi) + b^i_\tau(x_0)(u_s, D_i \phi).
\]
Similar transformation and Corollary 5.2 (i) prove that the first function in (5.7) is also a generalized function. Assumption 3.1 (iii) and the estimates from Corollary 5.2 also easily imply (5.8) thus finishing the proof of the corollary.

Before we continue with the proof of Theorem 3.1 we notice that, if \( u \in W_{2}^1(\tau) \), then as we know (see, for instance, Theorem 2.1 of [15]), there exists an event \( \Omega' \) of full probability such that \( u_{t \wedge \tau} I_{\Omega'} \) is a continuous \( L_2 \)-valued \( \mathcal{F}_{t} \)-adapted process on \( \mathbb{R}_+ \). Substituting, \( u_{t \wedge \tau} I_{\Omega'} \) in place of \( u \) in our assumptions and assertions does not change them. Furthermore, replacing \( \tau \) with \( \tau \wedge n \) and then sending \( n \) to infinity allows us to assume that \( \tau \) is bounded. Therefore, without losing generality we assume that

(H) If we are considering a \( u \in W_{2}^1(\tau) \), the process \( u_{t \wedge \tau} \) is a continuous \( L_2 \)-valued \( \mathcal{F}_{t} \)-adapted process on \( \mathbb{R}_+ \). The stopping time \( \tau \) is bounded.

Now we are ready to prove Theorem 3.1 in a particular case.

**Lemma 5.4.** Let \( \nu^k \equiv 0 \) and let \( b^i, b^i_0 \) and \( c \) be independent of \( x \). Assume that \( u \) is a solution of (2.2) with some \( f^j, g^k \in L_2(\tau) \) and \( \lambda > 0 \). Then (3.1) holds with \( N = N(d, \delta, K) \).

Proof. We want to use Theorem 1.1 to get rid of the first order terms. Observe that (2.2) reads as
\[
du_t = (\sigma_t^{ik} D_i u_t + g_t^k) dw_t^k
\]
\[ + \left( D_i(a^{ij}_t D_j u_t + [b^i_t + b^i_0]u_t + f^i_t) + f^{00}_t - (c_t + \lambda)u_t \right) dt, \quad t \leq \tau. \quad (5.9) \]

One can find a predictable set \( A \subset (0, \tau] \) of full measure such that \( I_A f^j, j = 0, 1, \ldots, d, \) and \( I_A D_i u, i = 1, \ldots, d, \) are well defined as \( L_2 \)-valued predictable functions satisfying

\[ \int_0^\infty I_A \left( \sum_{j=0}^d \| f^i_j \|_{L_2}^2 + \| D_i u_t \|_{L_2}^2 \right) dt < \infty. \]

Replacing \( f^j \) and \( D_i u \) in \((5.9)\) with \( I_A f^j \) and \( I_A D_i u, \) respectively, will not affect \((5.9)\). Similarly, one can handle the function \( g \) and the terms \( h_t = I_{[0, \tau]}[b^i + b^i_0]u, I_{[0, \tau]} cu \) for which

\[ \int_0^T \| h_t \|_{L_1} dt < \infty \quad (a.s.) \]

for each \( T \in \mathbb{R}^d \) owing to Assumption 3.1 (iii) and the fact that \( u \in W^{1,2}_2(\tau). \)

After these replacements all terms in \((5.9)\) will be of class \( \mathcal{D}_1 \) or \( \mathcal{D}_2(\ell_2) \) as appropriate since \( a \) and \( \sigma \) are bounded. This allows us to apply Theorem 4.1 and for

\[ B^i_t = \int_0^t (b^i_s + b^i_0) ds, \quad \hat{u}_t(x) = u_t(x - B_t) \]

obtain that

\[ d\hat{u}_t = \left( D_i(\hat{a}^{ij}_t D_j \hat{u}_t) - (c_t + \lambda)\hat{u}_t + D_i\hat{f}^i_t + \hat{f}^{00}_t \right) dt \]

\[ + \left( \hat{\sigma}^{ik}_t D_i \hat{u}_t + \hat{g}^k_t \right) dw^k_t, \quad t \leq \tau, \quad (5.10) \]

where

\[ (\hat{a}^{ij}_t, \hat{\sigma}^{ik}_t, \hat{f}^i_t, \hat{g}^k_t)(x) = (a^{ij}_t, \sigma^{ik}_t, f^i_t, g^k_t)(x - B_t). \]

Obviously, \( \hat{u} \) is in \( W^{1,2}_2(\tau) \) and its norm coincides with that of \( u. \) Moreover, having in mind that \( c_1 \) is independent of \( x \) and is locally (square) integrable, one can find stopping times \( \tau_n \uparrow \tau \) such that \( I_{\tau_n \neq \tau} \downarrow 0 \) and

\[ \xi_{\tau_n} \leq n, \quad \xi_t := \int_0^t c_s ds \leq n. \]

Then it follows from from the equation

\[ d(\xi_t \hat{u}_t) = \left( D_i(\xi_t \hat{a}^{ij}_t D_j \hat{u}_t) - \lambda \xi_t \hat{u}_t + D_i \xi_t \hat{f}^i_t + \xi_t \hat{f}^{00}_t \right) dt \]

\[ + \left( \xi_t \hat{\sigma}^{ik}_t D_i \hat{u}_t + \xi_t \hat{g}^k_t \right) dw^k_t, \quad t \leq \tau_n \]

that \( \xi u \in W^{1,2}_2(\tau_n) \) and hence \( \xi_{t \wedge \tau_n} u_{t \wedge \tau_n} \) is a continuous \( L_2 \)-valued function and so are \( u_{t \wedge \tau_n} \) and \( u_{t \wedge \tau}. \)

Furthermore, since \( \tau \) is bounded and \( u_{t \wedge \tau} \) is a continuous \( L_2 \)-valued function and \( c_t \) is independent of \( x \) and is locally square integrable, we have

\[ \int_0^\tau \| c_t \hat{u}_t \|_{L_2}^2 dt = \int_0^\tau c^2_t \| u_t \|_{L_2}^2 dt \leq \sup_{t \leq \tau} \| u_t \|_{L_2}^2 \int_0^\tau c^2_t dt < \infty \quad (5.11) \]
and there is a sequence of, perhaps, different from the above stopping times \( \tau_n \uparrow \tau \) such that for each \( n \)

\[
E \int_0^{\tau_n} \|c_t \hat{u}_t\|_{L_2}^2 \, dt < \infty.
\]  
(5.12)

Then (5.10) implies that \( \hat{u} \in W_2^1(\tau_n) \) for each \( n \). Also observe that if we can prove (3.1) with \( \tau_n \) in place of \( \tau \), then we can let \( n \rightarrow \infty \) and use the monotone convergence theorem to get (3.1) as is. Therefore, in the rest of the proof we assume that (5.12) holds with \( \tau \) in place of \( \tau_n \), that is, \( \hat{u} \in W_2^1(\tau) \).

The next argument is standard (see, for instance, Lemma 3.3 and Corollary 3.2 of [14]). Itô’s formula implies that

\[
E\|u_0\|_{L_2}^2 + E \int_0^{\tau} \int_{\mathbb{R}^d} I_t \, dx dt \geq 0,
\]  
(5.13)

where

\[
I_t := 2 \hat{u}_t(f_t^0 - \lambda \hat{u}_t - c_t \hat{u}_t) - 2(\hat{\sigma}^{ij} D_j \hat{u}_t + \hat{f}_t^1) D_i \hat{u}_t + |\hat{\sigma}^{ij} D_i \hat{u}_t + \hat{g}_t|^2.
\]

We use the inequality

\[
|\hat{\sigma}^{ij} D_i \hat{u}_t + \hat{g}_t|^2 \leq (1 + \varepsilon)|\hat{\sigma}^{ij} D_i \hat{u}_t|^2 + 2\varepsilon^{-1}|\hat{g}_t|^2,
\]

and Assumption 3.1. Then for \( \varepsilon = \varepsilon(\delta) > 0 \) small enough we find

\[
I_t \leq -\delta |D \hat{u}_t|^2 - 2(c_t + \lambda) \hat{u}_t^2 + 2 \hat{u}_t f_t^0 - 2 \hat{f}_t^1 D_i \hat{u}_t + N |\hat{g}_t|^2.
\]

Once again using \( 2 \hat{u}_t f_t^0 \leq \lambda \hat{u}_t^2 + \lambda^{-1} |f_t^0|^2 \) and similarly estimating \( 2 \hat{f}_t^1 D_i \hat{u}_t \) we conclude that

\[
I_t \leq -(\delta/2) |D \hat{u}_t|^2 - (c_t + \lambda) \hat{u}_t^2 + N \sum_{i=1}^d |\hat{f}_t^i|^2 + |\hat{g}_t|^2 + N \lambda^{-1} |f_t^0|^2.
\]

By coming back to (5.13) we obtain

\[
\| \hat{u} \sqrt{c_t + \lambda} \|^2_{L_2(\tau)} + \|D \hat{u}\|^2_{L_2(\tau)} \leq N \sum_{i=1}^d |\hat{f}_t^i|_{L_2(\tau)}^2 + |\hat{g}_t|_{L_2(\tau)}^2 + N \lambda^{-1} |f_t^0|_{L_2(\tau)}^2.
\]

This is equivalent to (3.1) and the lemma is proved.

To proceed further we need a construction. Take \( \bar{b}, \bar{b}, \) and \( \bar{c} \) from (5.3). From Lemma 4.2 of [13] and Assumption 3.2 it follows that, for \( h_t = \bar{b}_t, \bar{b}_t, \bar{c}_t \), it holds that \( |D^\nu h_t| \leq \kappa_n \), where \( \kappa_n = \kappa_n(n, \gamma, d, \rho_0) \geq 1 \) and \( D^\nu h_t \) is any derivative of \( h_t \) of order \( n \geq 1 \) with respect to \( x \). By Corollary 4.3 of [13] we have \( |h_t(x)| \leq K(t)(1 + |x|) \), where for each \( \omega \) the function \( K(t) = K(\omega, t) \) is locally (square) integrable with respect to \( t \) on \( \mathbb{R}_+ \). Owing to these properties the equation

\[
x_t = x_0 - \int_{t_0}^t (\bar{b}_s + \bar{b}_s)(x_s) \, ds, \quad t \geq t_0,
\]  
(5.14)
for any \((ω \text{ and } (t_0, x_0) ∈ \mathbb{R}^{d+1}_+)\) has a unique solution \(x_t = x_{t_0, x_0, t}\). Obviously, the process \(x_{t_0, x_0, t}, t ≥ t_0\), is \(\mathcal{F}_t\)-adapted.

Next, for \(i = 1, 2\) set \(χ^{(i)}(x)\) to be the indicator function of \(B_{ρ_0/i}\) and introduce

\[
χ^{(i)}_{t_0, x_0, t}(x) = χ^{(i)}(x - x_{t_0, x_0, t})I_{t≥t_0}.
\]

Here is a crucial estimate.

**Lemma 5.5.** Assume that \(u\) is a solution of (2.2) with some \(f^i, g ∈ L^2(τ)\). Then for \((t_0, x_0) ∈ \mathbb{R}^{d+1}_+\) and \(λ > 0\) we have

\[
\|χ^{(2)}_{t_0, x_0}u√c + \chi\|_{L^2(τ)}^2 + \|χ^{(2)}_{t_0, x_0}Du\|_{L^2(τ)}^2 ≤ N\left(\sum_{i=1}^d χ^{(1)}_{t_0, x_0}u\|_{L^2(τ)}^2 + χ^{(1)}_{t_0, x_0}Du\|_{L^2(τ)}^2\right)
\]

\[
+ Nλ^{-1}\|χ^{(1)}_{t_0, x_0}f^0\|_{L^2(τ)}^2 + NE\|ut_0I_{B_{ρ_0}(x_0)}I_0 ≤ τ\|_{L^2}^2
\]

\[
+ N\gamma^{2/3}\|χ^{(1)}_{t_0, x_0}Du\|_{L^2(τ)}^2 + N^*λ^{-1}\|χ^{(1)}_{t_0, x_0}Du\|_{L^2(τ)}^2
\]

\[
+ N^*(1 + λ^{-1})\|χ^{(1)}_{t_0, x_0}u\|_{L^2(τ)}^2 + N^*λ^{-1}\sum_{i=1}^d \|χ^{(1)}_{t_0, x_0}f^i\|_{L^2(τ)}^2,
\]

**Proof.** Since we are only concerned with the values of \(u_t\) if \(t_0 ≤ t ≤ τ\), we may start considering (2.2) on \([t_0, τ ∨ t_0]\) and then shifting time allows us to assume that \(t_0 = 0\). Obviously, we may also assume that \(x_0 = 0\). With this stipulations we will drop the subscripts \(t_0, x_0\). Then, we can include the term \(ν^k\) into \(g^k\) and obtain (5.15) by the triangle inequality if we assume that this estimate is true in case \(ν^k ≡ 0\). Thus, without losing generality we assume

\[
t_0 = 0, \quad x_0 = 0, \quad ν^k ≡ 0.
\]

Fix a \(ζ ∈ C^∞_0\) with support in \(B_{ρ_0}\) and such that \(ζ = 1\) on \(B_{ρ_0/2}\) and \(0 ≤ ζ ≤ 1\). Set \(x_t = x_{0,0,t},\)

\[
\hat{b}_t = \hat{b}_t(x_t), \quad \hat{b}_t = \hat{b}_t(x_t), \quad \hat{c}_t = \hat{c}_t(x_t)
\]

\[
η_t(x) = ζ(x - x_t), \quad ν_t(x) = u_t(x)η_t(x).
\]

The most important property of \(η_t\) is that

\[
dη_t = (\hat{b}_t^i + \hat{b}_t^i)D_iη_t dt.
\]

Also observe for the later that we may assume that

\[
χ^{(2)}_t ≤ η_t ≤ χ^{(1)}_t, \quad |Dη_t| ≤ N\rho_0^{-1}\chi^{(1)}_t,
\]

where \(χ^{(i)}_t = χ_{0,0,t}\) and \(N = N(d)\).
By Corollary 4.2 (also see the argument before (5.10)) we obtain that for $t \leq \tau$

$$dv_t = \left[ D_t (\eta_t a^{ij}_t D_j u_t + b^i_t v_t) - (a^{ij}_t D_j u_t + b^i_t u_t) D_t \eta_t \right.$$

$$+ b^i_t \eta_t D_t u_t - (c_t + \lambda) v_t + D_t (f^i_t \eta_t) - f^i_t D_t \eta_t + f^0_t \eta_t$$

$$\left. + (\hat{b}^i_t + \hat{b}^j_t u_t D_i \eta_t) \right] dt + [\sigma^{ik} D_t v_t - \sigma^{ik} u_t D_i \eta_t + g^{ik}_t] dw^k_t.$$

We transform this further by noticing that

$$\eta_t a^{ij}_t D_j u_t = a^{ij}_t D_j v_t - a^{ij}_t u_t D_j \eta_t.$$

To deal with the term $b^i_t \eta_t D_t u_t$ we use Corollary 5.2 and find the corresponding functions $V^i_t$. Then simple arithmetics show that

$$dv_t = (\sigma^{ik} D_t v_t + \hat{g}^k_t) dw^k_t$$

$$+ \left[ D_t (a^{ij}_t D_j v_t + \hat{b}^i_t v_t) - (\hat{c}_t + \lambda) v_t + \hat{b}^i_t D_t v_t + D_t \hat{f}^i_t + \hat{f}^0_t \right] dt,$$

where

$$\hat{f}^0_t = f^0_t \eta_t - f^i_t D_t \eta_t - a^{ij}_t (D_j u_t) D_t \eta_t + (\hat{b}^i_t - \hat{b}^j_t) u_t D_i \eta_t + (\hat{c}_t - c_t) u_t \eta_t + V^0_t,$$

$$\hat{f}^i_t = f^i_t \eta_t - a^{ij}_t u_t D_j \eta_t + (\hat{b}^i_t - \hat{b}^j_t) u_t \eta_t + V^i_t, \quad i = 1, \ldots, d,$$

$$\hat{g}^k_t = -\sigma^{ik} u_t D_i \eta_t + g^{ik}_t \eta_t.$$

It follows by Lemma 5.3 that for $\lambda > 0$

$$\|v \sqrt{\hat{c} + \chi} \|_{L^2_2(\tau)}^2 + \|Dv\|_{L^2_2(\tau)}^2 \leq N \lambda^{-1} \|\hat{f}^0\|_{L^2_2(\tau)}^2$$

$$+ N \left( \sum_{i=1}^d \|\hat{f}^i\|_{L^2_2(\tau)}^2 + \|\hat{g}\|_{L^2_2(\tau)}^2 + E\|v_0\|_{L^2_2}^2 \right). \quad (5.17)$$

Recall that here and below by $N$ we denote generic constants depending only on $\delta, \delta, K$.

Now we start estimating the right-hand side of (5.17). First we deal with $\hat{f}^i_t$ and $\hat{g}^k_t$. Recall (5.16) and observe that obviously, if $\eta_t(x) \neq 0$, then $|x - x_t| \leq \rho_0$. Therefore,

$$\|\hat{g}\|_{L^2_2(\tau)}^2 \leq N^* \|u \chi^{(1)}\|_{L^2_2(\tau)}^2 + N \|g \chi^{(1)}\|_{L^2_2(\tau)}^2 \quad (5.18)$$

(which we remind the reader that by $N^*$ we denote generic constants depending only on $\delta, \delta, K$, and $\rho_0$). By Corollary 5.2

$$\|b^i_t - \hat{b}^i_t\|_{L^2_2} \leq N \gamma^{2/q}(\rho_0^{2(1-d/q)} \|\chi^{(1)}\|_{L^2_2}^2 + \rho_0^{-2d/q} \|\chi^{(1)}\|_{L^2_2}^2). \quad (5.19)$$

Here $\rho_0^{2(1-d/q)} \leq 1$ since $q \geq d$. By adding that

$$\|a^{ij} u D_j \eta\|_{L^2_2(\tau)}^2 \leq N^* \|\chi^{(1)}\|_{L^2_2(\tau)}^2,$$

we derive from (5.6), (5.18), and (5.19) that

$$\sum_{i=1}^d \|\hat{f}^i\|_{L^2_2(\tau)}^2 + \|\hat{g}\|_{L^2_2(\tau)}^2 \leq N \left( \sum_{i=1}^d \|\chi^{(1)} f^i\|_{L^2_2(\tau)}^2 + \|\chi^{(1)} g\|_{L^2_2(\tau)}^2 \right)$$

$$+ N \gamma^{2/q} \|\chi^{(1)}\|_{L^2_2(\tau)}^2 + N^* \|\chi^{(1)}\|_{L^2_2(\tau)}^2. \quad (5.20)$$
While estimating \( \hat{f}^0 \) we use (5.6) again and observe that we can deal with \((b^i_t - b^i_t) u_t D_i \eta_t\) as in (5.19), this time without paying much attention to the dependence of our constants on \( \rho_0 \) and obtain that
\[
\| (b^i - b^i) u_t D_i \eta_t \|_{L^2(\tau)}^2 \leq N^*(\| \chi^1 u \|_{L^2(\tau)}^2 + \| \chi^1 u \|_{L^2(\tau)}^2).
\]
By estimating also roughly the remaining terms in \( \hat{f}^0 \) and combining this with (5.20) and (5.17), we see that the left-hand side of (5.17) is less than the right-hand side of (5.15). However,
\[
|\chi^1_t u_t| \leq |\eta_t D_i \eta_t| \leq |D_i v_t| + |u_t D_i \eta_t| \leq |D_i v_t| + N\rho_0^{-1} |u_t \chi^1_t|
\]
and also
\[
|\chi^1_t u_t|^2 (c_t + \lambda) \leq |\eta_t u_t|^2 (c_t + \lambda) \leq |\chi^1 u|^2 (1 + |c_t - \hat{c}_t|^2).
\]
By combining this with the fact that by Corollary 5.2
\[
\| (\hat{c}^i - c) u_t \|_{L^2(\tau)}^2 \leq N^2 / q \| \chi^1 u \|_{L^2(\tau)}^2 + N^* |\chi^1 u|_{L^2(\tau)}^2
\]
we obtain (5.15). The lemma is proved.

Next, from the result giving “local” in space estimates we derive global in space estimates but for functions having, roughly speaking, small “future” dependence in the time variable.

**Lemma 5.6.** Assume that \( u \) is a solution of (2.2) with some \( f^i, g \in L^2(\tau) \) and assume that \( u_0 = 0 \) if \( t_0 + \kappa_1^{-1} \leq t \leq \tau \) with \( \kappa_1 = \kappa_1(\gamma, d, \rho_0) \geq 1 \) introduced before (5.13) and some (nonrandom) \( t_0 \geq 0 \) (nothing is required for those \( \omega \) for which \( \tau < t_0 + \kappa_1^{-1} \)). Then for \( \lambda > 0 \) and \( I_{t_0} := I_{[t_0, \infty)} \)
\[
\| I_{t_0} u \chi \|_{L^2(\tau)}^2 + \| I_{t_0} D u \|_{L^2(\tau)}^2 \leq N \left( \sum_{i=1}^d \| I_{t_0} f^i \|_{L^2(\tau)}^2 + \| I_{t_0} g \|_{L^2(\tau)}^2 \right) + N\lambda^{-1} \| I_{t_0} f^0 \|_{L^2(\tau)}^2 + NE \| u_0 I_{t_0 \leq \tau} \|_{L^2}^2
\]
\[
+ N^2 / q \| I_{t_0} D u \|_{L^2(\tau)}^2 + N^* |\chi^1 u|_{L^2(\tau)}^2 + N^*(1 + \lambda)^{-1} \| I_{t_0} u \|_{L^2(\tau)}^2 + N^* \lambda^{-1} \sum_{i=1}^d \| I_{t_0} f^i \|_{L^2(\tau)}^2, \quad (5.21)
\]
where and below in the proof by \( N \) we denote generic constants depending only on \( d, \delta, \) and \( K \) and by \( N^* \) constants depending only on the same objects and \( \rho_0 \).

Proof. Take \( x_0 \in \mathbb{R}^d \) and use the notation introduced before Lemma 5.5. One knows that for each \( t \geq t_0 \), the mapping \( x_0 \to x_{t_0, x_0, t} \) is a diffeomorphism with Jacobian determinant given by
\[
\left| \frac{\partial x_{t_0, x_0, t}}{\partial x_0} \right| = \exp \left( - \int_{t_0}^t \sum_{i=1}^d D_i [b^i_s + \bar{b}^i_s] (x_{t_0, x_0, s}) \, ds \right).
\]
By the way the constant $\kappa_1$ is introduced, we have

$$e^{-N\kappa_1 (t-t_0)} \leq \left| \frac{\partial x_{t_0,x_0,t}}{\partial x_0} \right| \leq e^N\kappa_1 (t-t_0),$$

where $N$ depends only on $d$. Therefore, for any nonnegative Lebesgue measurable function $w(x)$ it holds that

$$e^{-N\kappa_1 (t-t_0)} \int_{\mathbb{R}^d} w(y) \, dy \leq \int_{\mathbb{R}^d} w(x_{t_0,x_0,t}) \, dx_0 \leq e^N\kappa_1 (t-t_0) \int_{\mathbb{R}^d} w(y) \, dy.$$ 

In particular, since

$$\left| \chi^{(i)}_{t_0,x_0,t}(x) \right|^2 dx_0 = \int_{\mathbb{R}^d} \left| \chi^{(i)}(x-x_{t_0,x_0,t}) \right|^2 dx_0,$$

we have

$$e^{-N\kappa_1 (t-t_0)} = N_1^* e^{-N\kappa_1 (t-t_0)} \int_{\mathbb{R}^d} \left| \chi^{(i)}(x-y) \right|^2 dy \leq N_1^* \int_{\mathbb{R}^d} \left| \chi^{(i)}_{t_0,x_0,t}(x) \right|^2 dx_0 \leq N_1^* e^{N\kappa_1 (t-t_0)} \int_{\mathbb{R}^d} \left| \chi^{(i)}(x-y) \right|^2 dy = e^{N\kappa_1 (t-t_0)},$$

where $N_1^* = |B_1|^{-d}/q$ and $|B_1|$ is the volume of $B_1$. It follows that

$$(N_1^*)^{-1} e^{N\kappa_1 (t-t_0)} \leq \int_{\mathbb{R}^d} \left| \chi^{(1)}_{t_0,x_0,t}(x) \right|^2 dx_0.$$

Furthermore, since $u_0 = 0$ if $\tau \geq t \geq t_0 + \kappa_1^{-1}$ and $\chi^{(i)}_{t_0,x_0,t} = 0$ if $t < t_0$, in evaluating the norms in (5.15) we need not integrate with respect to $t$ such that $\kappa_1 (t-t_0) \geq 1$, so that for all $t$ really involved we have

$$\int_{\mathbb{R}^d} \left| \chi^{(1)}_{t_0,x_0,t}(x) \right|^2 dx_0 \leq (N_1^*)^{-1} e^N, \quad (N_2^*)^{-1} e^{-N} \leq \int_{\mathbb{R}^d} \left| \chi^{(2)}_{t_0,x_0,t}(x) \right|^2 dx_0.$$

After this observation it only remains to integrate (5.15) through with respect to $x_0$ and use the fact that $N_1^* = 2^{-d} N_2^*$. The lemma is proved.

**Proof of Theorem 3.1** First we show how to choose $\gamma = \gamma(d, \delta, K) > 0$. Call $N_0$ the constant factor of $\gamma^{2/q} \|I_0 Du\|^2_{L_2(\tau)}$ in (5.21). We know that $N_0 = N_0(d, \delta, K)$ and we choose $\gamma \in (0, 1]$ so that $N_0 \gamma^{2/q} \leq 1/2$. Then under the conditions of Lemma 5.6 for $\lambda \geq 1$ we have

$$\|I_0 u \sqrt{c + \lambda} \|_{L_2(\tau)}^2 + \|I_0 Du \|^2_{L_2(\tau)} \leq N \left( \sum_{i=1}^d \|I_0 f^i \|^2_{L_2(\tau)} + \|I_0 g \|^2_{L_2(\tau)} \right)$$

$$+ N \lambda^{-1} \|I_0 f^0 \|^2_{L_2(\tau)} + NE \|u_0, I_0 \leq \tau \|^2_{L_2} + N \lambda^{-1} \|I_0 Du \|^2_{L_2(\tau)}$$

$$+ N^* \|I_0 u \|^2_{L_2(\tau)} + N^* \lambda^{-1} \sum_{i=1}^d \|I_0 f^i \|^2_{L_2(\tau)}. \quad (5.22)$$
After $\gamma$ has been fixed we have $\kappa_1 = \kappa_1(d, \delta, K, \rho_0)$ and we take a $\zeta \in C_0^\infty(\mathbb{R})$ with support in $(0, \kappa_1^{-1})$ such that
\[
\int_{-\infty}^\infty \zeta^2(t) \, dt = 1.
\tag{5.23}
\]
For $s \in \mathbb{R}$ define $\zeta_t^s = \zeta(t - s)$, $u_t^s(x) = u_t(x)\zeta_t^s$. Obviously $u_t^s = 0$ if $s_+ + \kappa_1^{-1} \leq t \leq \tau$. Therefore, we can apply (5.22) to $u_t^s$ with $t_0 = s_+$ observing that
\[
du_t^s = (\sigma_{t_i}^k D_i u_t^s + \nu_t u_t^s + \zeta_t^s g^k) \, dw_t^k
\]
+ $$(D_i(a_i^j D_j u_t^s + b_i^j u_t^s) + b_i^k u_t^s - (c_t + \lambda) u_t^s + D_i(\zeta_t^s f_t^i) + (\zeta_t^s f_t^0) + (\zeta_t^s)’ u_t) \, dt.$$ Then from (5.22) we obtain
\[
|I_{s+} \zeta^s u(\sqrt{c + \lambda}|^2_{L^2(\tau)} + \|I_{s+} \zeta^s Du||^2_{L^2(\tau)})
\leq N\left(\sum_{i=1}^d \|I_{s+} \zeta^s f^i||^2_{L^2(\tau)} + \|I_{s+} \zeta^s g||^2_{L^2(\tau)}\right)
+ N\lambda^{-1}\left(\|I_{s+} \zeta^s f^0||^2_{L^2(\tau)} + \|I_{s+} (\zeta^s)' u||^2_{L^2(\tau)}\right) + NE\|u_{s+} \zeta^s \|_{L^2(\tau)}^2
\]
+ $N^*\lambda^{-1}\|I_{s+} \zeta^s Du||^2_{L^2(\tau)} + N^*\|I_{s+} \zeta^s u||^2_{L^2(\tau)} + N^*\lambda^{-1}\|I_{s+} \zeta^s f^i||^2_{L^2(\tau)}.$
\tag{5.24}
Here $I_{s+}$ can be dropped since $I_{s+} I_{(0, \tau)} = I_s I_{(0, \tau)}$ and $I_s \zeta^s = \zeta^s$. After dropping $I_{s+}$ we integrate through (5.24) with respect to $s \in \mathbb{R}$, use (5.23), and observe that, since $\kappa_1$ depends only on $d, \delta, K, \rho_0$, we have
\[
\int_{-\infty}^\infty |\zeta^s(s)|^2 \, ds = N^*.
\]
We also use the fact that $\zeta^s_{s+} \neq 0$ only if $s_+ = 0$ and $-\kappa_1^{-1} \leq s \leq 0$ whereas
\[
\int_{-\kappa_1^{-1}}^0 (\zeta^s_{s+})^2 \, ds = 1.
\]
Then we conclude
\[
\lambda\|u||^2_{L^2(\tau)} + \|u\sqrt{c}||^2_{L^2(\tau)} + \|Du||^2_{L^2(\tau)}
\leq N_1\left(\sum_{i=1}^d \|f^i||^2_{L^2(\tau)} + \|g||^2_{L^2(\tau)} + E\|u_0||^2_{L^2}\right)
+ N_1\lambda^{-1}\|f^0||^2_{L^2(\tau)} + \|u||^2_{L^2(\tau)}\right) + N_1^*\lambda^{-1}\|Du||^2_{L^2(\tau)}
\]
+ $N_1^*\|u||^2_{L^2(\tau)} + N_1^*\lambda^{-1}\sum_{i=1}^d \|f^i||^2_{L^2(\tau)}.$
Without losing generality we assume that $N_1 \geq 1$ and we show how to choose $\lambda_0 = \lambda_0(d, \delta, K, \rho_0)$. We take it so that $\lambda_0 \geq 4N_1^*$, $\lambda_0^2 \geq 4N_1$. Then we obviously come to (5.1) with $N = 4N_1$. The theorem is proved.
6. Proof of Theorem 3.2

We may assume in this section that \( F_t = F_{t+} \) for all \( t \in \mathbb{R}_+ \). This does not restrict generality because replacing \( F_t \) with \( F_{t+} \) makes our assumptions weaker and does not affect our assertions because the solutions are continuous in time. Furthermore, having in mind setting all data equal to zero for \( t > \tau \), we see that without loss of generality we may assume that \( \tau = \infty \). Set

\[
\mathbb{L}_2 = \mathbb{L}_2(\infty), \quad \mathcal{W}^1_2 = \mathcal{W}^1_2(\infty), \quad \mathcal{W}_2 = \mathcal{W}^1_2(\infty).
\]

We need a few auxiliary results.

**Lemma 6.1.** For any \( T, R \in \mathbb{R}_+ \), and \( \omega \in \Omega \) we have

\[
\sup_{t \leq T} \int_{B_R} (|b_t(x)|^q + |b_t(x)|^q + c_t^q(x)) \, dx < \infty. \tag{6.1}
\]

Proof. Obviously it suffices to prove (6.1) with \( B_{\rho_0}(x_0) \) in place of \( B_R \) for any \( x_0 \). In that case, for instance,

\[
\int_{B_{\rho_0}(x_0)} |b_t(x)|^q \, dx \leq 2^q \int_{B_{\rho_0}(x_0)} |b_t(x) - \bar{b}_t(x_0)|^q \, dx + N |\bar{b}_t(x_0)|^q
\]

and we conclude estimating the left-hand side as in (5.4) also relying on Assumption 3.3. Similarly, \( b_t \) and \( c_t \) are treated. The lemma is proved.

**Lemma 6.2.** For any \( R \in \mathbb{R}_+ \) there exists a sequence of stopping times \( \tau_n \uparrow \infty \) such that for any \( n = 1, 2, \ldots \) and \( \omega \) for almost any \( t \leq \tau_n \) we have

\[
\int_{B_R} (|b_t| + |c_t|) \, dx \leq n. \tag{6.2}
\]

Proof. For each \( t, R > 0 \), and \( \omega \) define

\[
\beta_{t,R} = \int_{B_R} (|b_t| + |c_t|) \, dx,
\]

\[
\psi_{t,R} = \lim_{s_2 - s_1 \to 0} \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \beta_{s,R} \, ds.
\]

As is easy to see, \( \psi_{t,R} \) is an increasing, left-continuous, and \( \mathcal{F}_t \)-adapted process. It follows that

\[
\tau_n := \inf \{ t \geq 0 : \psi_{t,R} > n \}
\]

are stopping times with respect to \( \mathcal{F}_{t+} (= \mathcal{F}_t) \) and \( \psi_{t,R} \leq n \) for \( t < \tau_n \). Furthermore, by Lemma 6.1 we have \( \tau_n \uparrow \infty \) as \( n \to \infty \). By Lebesgue differentiation theorem we conclude that (for any \( \omega \)) for almost all \( t \leq \tau_n \) we have (6.2). This proves the lemma.

By combining this lemma with Lemma 5.1 we obtain the following.
Corollary 6.3. If \( \psi \in C_0^\infty \) has support in \( B_R \), then for \( \tau_n \) from Lemma 6.2 for each \( n = 1, 2, \ldots \), for almost all \( t \leq \tau_n \), for any \( u \in W^1_2 \) and \( v \in W^1_2 \) we have

\[
|\langle b^i_t D_i (v \psi), u \rangle| \leq N \|v\|_{W^1_2} \|u\|_{W^1_2}, \quad |\langle b^i_t D_i u, v \psi \rangle| \leq N \|v\|_{W^1_2} \|u\|_{W^1_2}, \\
|\langle c_t v \psi, u \rangle| \leq N \|v\|_{L^2_2} \|u\|_{W^1_2},
\]

(6.3)

where the constant \( N = N(n, d) \).

Since bounded linear operators are continuous we obtain the following.

Corollary 6.4. If \( \phi \in C_0^\infty \) has support in \( B_R \), then for \( \tau_n \) from Lemma 6.2 and each \( n \) the operators

\[
\begin{align*}
&u \to \langle b^i_t D_i u, \phi \rangle, \quad u \to \langle b^i_t u, D_i \phi \rangle, \quad u \to \langle c u, \phi \rangle, \\
&u \to \int_0^t \langle b^i_t D_i u, \phi \rangle \, dt, \quad u \to \int_0^t \langle b^i_t u, D_i \phi \rangle \, dt, \quad u \to \int_0^t \langle c u, \phi \rangle \, dt
\end{align*}
\]

are continuous as operators from \( \mathbb{W}^1_2 \) to \( L^2_2([0, n \wedge \tau_n]) = L^2_2((0, n \wedge \tau_n], \mathbb{R}) \).

In the proof of Theorem 3.2 we are going to use sequences which converge weakly in \( \mathbb{W}^1_2 \). Therefore, the following result is relevant.

Lemma 6.5. Assume that for some \( f^j \in \mathbb{L}_2 \), \( j = 0, \ldots, d \), \( g = (g^k) \in \mathbb{L}_2 \), \( u \in \mathbb{W}^1_2 \), and any \( \phi \in C_0^\infty \) equation (2.4) with \( u_0 \in L_2(\Omega, \mathcal{F}_0, L_2) \) holds for almost all \((\omega, t)\). Then there exists a function \( \tilde{u} \in \mathbb{W}^1_2 \) solving equation (2.2) (for all \( t \)) with initial data \( u_0 \) in the sense of Definition 3.1.

Proof. We split the proof into two steps.

Step 1. Modifying \( u_t \psi \). We recall some facts from the theory of Itô stochastic integrals in a separable Hilbert space, say \( H \) and some other results, which can be found, for instance, in [19] and [12]. Integrating \( H \)-valued processes with respect to a one-dimensional Wiener process presents no difficulties and leads to strongly continuous \( H \)-valued locally square-integrable martingales with natural isometry. If \( g = (g^k) \in \mathbb{L}_2 \), then by Doob’s inequality

\[
E \sup_t \left\| \sum_{k=n}^m \int_0^t g^k_s \, dw^k_s \right\|_{L_2}^2 \leq 4E \int_0^\infty \sum_{k=n}^m \|g^k_s\|_{L_2}^2 \, ds \to 0
\]
as \( m \geq n \to \infty \). Therefore,

\[
m_t = \sum_{k=1}^\infty \int_0^t g^k_s \, dw^k_s
\]
is well defined as a continuous \( L_2 \)-valued square-integrable martingale. Furthermore, for any \( \phi \in C_0^\infty \) with probability one we have

\[
(m_t, \phi) = \sum_{k=1}^\infty \int_0^t \langle g^k_s, \phi \rangle \, dw^k_s
\]
for all \( t \) and the series on the right converges uniformly in probability on \( \mathbb{R}_+ \). If \( g \in L_2(\tau_n), n = 1, 2, \ldots \), and stopping times \( \tau_n \uparrow \infty \), then

\[
m_t = \sum_{k=1}^{\infty} \int_0^t g^k_s \, dw^k_s
\]

is well defined as a locally square-integrable \( L_2 \)-valued continuous martingale. Again for any \( \phi \in C_0^\infty \) with probability one we have

\[
(m_t, \phi) = \sum_{k=1}^{\infty} \int_0^t (g^k_s, \phi) \, dw^k_s
\]

(6.4)

for all \( t \) and the series on the right converges uniformly in probability on every finite interval of time.

We fix a \( \psi \in C_0^\infty \) and apply the above to

\[
h_t^{\psi} := \sum_{k=1}^{\infty} \int_0^t \psi(s^k D_i u_s + \nu^k_s u_s + g^k_s) \, dw^k_s.
\]

Observe that, by assumption, for any \( v \in C_0^\infty \) for almost all \((\omega, t)\)

\[
(u_t \psi, v) = (u_0 \psi, v) + \int_0^t \langle F_s, v \rangle \, ds + \langle h_t^{\psi}, v \rangle,
\]

(6.5)

where

\[
\langle F_t, v \rangle = (b^i_t D_i u_t - (c_t + \lambda) u_t + f^0_t, \psi) - (a^{ij}_t D_j u_t + b^i_t u_t + f^i_t, D_i(\psi)).
\]

We also define \( V = W_2^1 \), and notice that if \( \|v\|_V \leq 1 \), then by Corollary 6.3 for any \( T \in \mathbb{R}_+ \) for almost any \((\omega, t) \in \Omega \times [0, T] \) we have

\[
|\langle F_t, v \rangle| \leq N \left( \sum_{j=0}^{d} \|f^j_t\|_{L_2} + \|u_t\|_{W_2^1} \right),
\]

where \( N \) is independent of \( v, t \) (but may depend on \( \omega \) and \( T \)). It follows that, for \( V^* \) defined as the dual of \( V \), the \( V^* \)-norm of \( F_t \) is in \( L_2([0, T]) \) (a.s.) for every \( T \in \mathbb{R}_+ \). It also follows that (6.5) holds for almost all \((\omega, t) \) for each \( v \in V \) rather than only for \( v \in C_0^\infty \).

By Theorem 3.1 of [19] there exists a set \( \Omega_\psi \) of full probability and an \( L_2 \)-valued function \( \tilde{u}_t^{\psi} \) on \( \Omega \times \mathbb{R}_+ \) such that \( \tilde{u}_t^{\psi} \) is \( F_t \)-measurable, \( \tilde{u}_t^{\psi} \) is \( L_2 \)-continuous in \( t \) for every \( \omega \) and \( \tilde{u}_t^{\psi} = u_t \psi \) for almost all \((\omega, t) \). Furthermore, for \( \omega \in \Omega_\psi, t \geq 0, \) and \( \phi \in C_0^\infty \) we have

\[
(\tilde{u}_t^{\psi}, \phi) = (h_t^{\psi}, \phi) + \int_0^t (b^i_t D_i u_s - (c_s + \lambda) u_s + f^0_s, \phi \psi) \, ds
\]

\[
- \int_0^t (a^{ij}_s D_j u_s + b^i_s u_s + f^i_s, D_i(\phi \psi)) \, ds.
\]

(6.6)
Step 2. Constructing $\bar{u}_t$. Let $\psi \in C_0^\infty$ be such that $\psi = 1$ on $B_1$ and set $\psi_n(x) = \psi(x/n)$, $n = 1, 2, \ldots$. Define $\bar{u}_t^n = \bar{u}_t^{\psi_n}$ and notice that by the above for $m \geq n$ and almost all $(\omega, t)$

$$
\bar{u}_t^n I_{B_n} = u_t \psi_m I_{B_n} = u_t I_{B_n} = \bar{u}_t^n I_{B_n}
$$

as $L_2$-elements. Since the extreme terms are $L_2$-continuous functions of $t$, there exist sets $\Omega_{nm}$, $m \geq n$, of full probability such that for $\omega \in \Omega_{nm}$ we have $\bar{u}_t^n I_{B_n} = \bar{u}_t^n I_{B_n}$ as $L_2$-elements for all $t$.

Then for $t \geq 0$ and $\omega \in \Omega' := \bigcap_{m \geq n} \Omega_{nm}$ the formula

$$
\bar{u}_t = I_{\Omega'} \sum_{n=0}^{\infty} \bar{u}_t^{n+1} I_{B_n+1 \setminus B_n}
$$

defines a distribution such that $\bar{u}_t I_{B_n} = \bar{u}_t^n I_{B_n}$ as $L_2$-elements for any $\omega \in \Omega'$, $t \geq 0$, and $n$. It follows that $\bar{u}_t = u_t$ as distributions for almost any $(\omega, t)$, hence, $\bar{u} \in \mathcal{W}^1_2$ and there exists an event $\Omega'' \subset \Omega'$ of full probability such that for any $\omega \in \Omega''$ and almost any $t \geq 0$ we have $\bar{u}_t = u_t$. Now (6.6) implies that if $\phi \in C_0^\infty$ is such that $\phi(x) = 0$ for $|x| \geq n$, then for $\omega \in \Omega'' \cap \Omega_{nm}$ and all $t \geq 0$ we have

$$
(\bar{u}_t, \phi) = (\bar{u}_t^n, \phi) = (b_t^{\psi_n}, \phi) + \int_0^t (b_s^i D_i \bar{u}_s - (c_s + \lambda)\bar{u}_s + f_s^0, \phi) \, ds
$$

$$
- \int_0^t (a_s^{ij} D_j \bar{u}_s + b_s^i \bar{u}_s + f_s^i, D_i \phi) \, ds.
$$

(6.7)

By recalling what was said about (6.4) and using Corollary 6.3, we see that indeed the requirements of Definition 3.1 are satisfied with $\bar{u}$ and $\infty$ in place of $u$ and $\tau$, respectively. The lemma is proved.

**Lemma 6.6.** Let $\phi \in C_0^\infty$ be supported in $B_R$ and take $\tau_n$ from Lemma 6.3. Let $u^n, u \in \mathcal{W}^1_2$, $n = 1, 2, \ldots$, be such that $u^n \to u$ weakly in $\mathcal{W}^1_2$. For $n = 1, 2, \ldots$ define $\chi_n(t) = (-n) \lor t \land n$, $b_{nt}^i = \chi_n(b_t^i)$, $b_{nt} = \chi_n(b_t)$ and set $c_{ns} = n \land c_s$. Then for any $m = 1, 2, \ldots$

$$
\int_0^t \left( \int_0^t \left( \int_0^t \left( (b_{ns}^i D_i u^n_s, \phi) - (b_{ns}^i u^n_s, D_i \phi) - (c_{ns} u^n_s, \phi) \right) \right) ds \right) dt
$$

$$
\to 0
$$

weakly in the space $L_2([0, m \land \tau_m])$ as $n \to \infty$.

Proof. By Corollary 6.3 and by the fact that (strongly) continuous operators are weakly continuous we obtain that

$$
\int_0^t \left( \int_0^t \left( \int_0^t \left( (b_{nt}^i D_i u^n_s, \phi) - (b_{nt}^i u^n_s, D_i \phi) - (c_{nt} u^n_s, \phi) \right) \right) ds \right) dt
$$

$$
\to 0
$$

weakly in the space $L_2([0, m \land \tau_m])$ as $n \to \infty$.
as \( n \to \infty \) weakly in the space \( \mathcal{L}_2((0, m \land \tau_m]) \) for any \( m \). Therefore, it suffices to show that

\[
\int_0^t \left[ (D_t \nu_s^n, (b_s^n - b_s^{n_s}) \phi) - (\nu_s^n, (b_s^n - b_s^{n_s}) D_t \phi + (c_s^n - c_s^{n_s}) \phi) \right] ds \to 0
\]

weakly in \( \mathcal{L}_2((0, m \land \tau_m]) \) for any \( m \). In other words, it suffices to show that for any \( \xi \in \mathcal{L}_2((0, m \land \tau_m]) \)

\[
E \int_0^{m \land \tau_m} \xi_t \left( \int_0^t \left[ (D_t \nu_s^n, (b_s^n - b_s^{n_s}) \phi) - (\nu_s^n, (b_s^n - b_s^{n_s}) D_t \phi + (c_s^n - c_s^{n_s}) \phi) \right] ds \right) dt \to 0.
\]

This relation is rewritten as

\[
E \int_0^{m \land \tau_m} \left[ (D_t \nu_s^n, \eta_s (b_s^n - b_s^{n_s}) \phi) - (\eta_t u_s^n, (b_s^n - b_s^{n_s}) D_t \phi + (c_s^n - c_s^{n_s}) \phi) \right] ds \to 0,
\]

where the process

\[ \eta_s := \int_s^{m \land \tau_m} \xi_t dt \]

is of class \( \mathcal{L}_2((0, m \land \tau_m]) \) since \( m \land \tau_m \) is bounded (\( \leq m \)).

However, by the choice of \( \tau_m \) and the dominated convergence theorem,

\[ \eta_s(b_s^n - b_s^{n_s}) D_t \phi \to 0, \quad \eta_s(b_s^n - b_s^{n_s}) \phi \to 0, \quad \eta_s(c_s^n - c_s^{n_s}) \phi \to 0 \]

as \( n \to \infty \) strongly in \( L_2((0, m \land \tau_m]) \) (use the fact that \( q \geq 2 \)) and by assumption \( u^n \to u \) and \( Du^n \to Du \) weakly in \( L_2((0, \tau_m)] \). This implies (6.9) for any \( m \) and the lemma is proved.

**Proof of Theorem 3.2** Define \( b_{nt}, b_{nt}, \) and \( c_{nt} \) as in Lemma 6.6 and consider equation (2.2) with \( b_{nt}, b_{nt}, \) and \( c_{nt} \) in place of \( b_t, b_t, \) and \( c_t \), respectively, and with \( \tau = n \). By a classical result there exists a unique \( u^n \in W^2_2(n) \) satisfying the modified equation with initial condition \( u_0 \). Obviously, \( b_{nt}, b_{nt}, \) and \( c_{nt} \) satisfy Assumption 3.2 with the same \( \gamma \) as \( b_t, b_t, \) and \( c_t \) do.

By Theorem 3.1 for \( \lambda \geq \lambda_0(d, \delta, K, \rho_0) \) we have

\[ \| u^n \|_{L_2(n)} + \| Du^n \|_{L_2(n)} \leq N, \]

where \( N \) is independent of \( n \). Hence the sequence of functions \( u^n \) converges weakly in the Hilbert space \( W^2_2 \) and consequently has a weak limit point \( u \in W^2_2 \). For simplicity of presentation we assume that the whole sequence \( u^n I_{t \leq n} \) converges weakly to \( u \). Take a \( \phi \in C_0^\infty \). Then by Lemma 6.6 for appropriate \( \tau_m \) we have that (6.8) holds weakly in \( L_2((0, m \land \tau_m]) \) for any \( m \). Since

\[ u = u_t \to \sum_{k=1}^\infty \int_0^t (\Lambda_s^k u_s, \phi) dw_s^k \]
is a continuous operator from $\mathbb{W}_2^1$ to $L_2((0,m])$, it is weakly continuous, so that
\[
\sum_{k=1}^{\infty} \int_0^t \left( \Lambda_k^k u^n_s, \phi \right) dw_k^k \to \sum_{k=1}^{\infty} \int_0^t \left( \Lambda_k^k u_s, \phi \right) dw_k^k
\]
weakly in $L_2((0,m])$ for any $m$. Obviously, the same is true for $(u^n_t, \phi) \to (u_t, \phi)$ and the remaining terms entering the equation for $u^n_s$. Hence by passing to the weak limit in the equation for $u^n_t$ we see that $u$ satisfies the assumptions of Lemma 6.5 applying which finishes the proof of the theorem.

References

[1] S. Assing and R. Manthey, Invariant measures for stochastic heat equations with unbounded coefficients, Stochastic Process. Appl., Vol. 103 (2003), No. 2, 237-256.
[2] P. Cannarsa and V. Vespri, Generation of analytic semigroups by elliptic operators with unbounded coefficients, SIAM J. Math. Anal., Vol. 18 (1987), No. 3, 857-872.
[3] P. Cannarsa and V. Vespri, Existence and uniqueness results for a nonlinear stochastic partial differential equation, in Stochastic Partial Differential Equations and Applications Proceedings, G. Da Prato and L. Tubaro (eds.), Lecture Notes in Math., Vol. 1236, pp. 1-24, Springer Verlag, 1987.
[4] A. Chojnowska-Michalik and B. Goldys, Generalized symmetric Ornstein-Uhlenbeck semigroups in $L_p$: Littlewood-Paley-Stein inequalities and domains of generators, J. Funct. Anal., Vol. 182 (2001), 243-279.
[5] G. Cupini and S. Fornaro, Maximal regularity in $L_p(R^N)$ for a class of elliptic operators with unbounded coefficients, Differential Integral Equations, Vol. 17 (2004), No. 3-4, 259-296.
[6] M. Geissert and A. Lunardi, Invariant measures and maximal L^2 regularity for nonautonomous Ornstein-Uhlenbeck equations, J. Lond. Math. Soc. (2), Vol. 77 (2008), No. 3, 719-740.
[7] B. Farkas and A. Lunardi, Maximal regularity for Kolmogorov operators in $L^2$ spaces with respect to invariant measures, J. Math. Pures Appl., Vol. 86 (2006), 310-321.
[8] I. Gyöngy, Stochastic partial differential equations on Manifolds, I, Potential Analysis, Vol. 2 (1993), 101-113.
[9] I. Gyöngy, Stochastic partial differential equations manifolds II. Nonlinear filtering, Potential Analysis, Vol. 6 (1997), 39-56.
[10] I. Gyöngy and N.V. Krylov, On stochastic partial differential equations with unbounded coefficients, Potential Analysis, Vol. 1 (1992), No. 3, 233-256.
[11] Kyeong-Hun Kim, On $L_p$-theory of stochastic partial differential equations of divergence form in $C^1$ domains, Probab. Theory Related Fields, Vol. 130 (2004), No. 4, 473-492.
[12] N.V. Krylov, An analytic approach to SPDEs, pp. 185-242 in Stochastic Partial Differential Equations: Six Perspectives, Mathematical Surveys and Monographs, Vol. 64, AMS, Providence, RI, 1999.
[13] N.V. Krylov, On linear elliptic and parabolic equations with growing drift in Sobolev spaces without weights, Problemy Matematicheskogo Analiza, Vol. 40 (2009), 77-90, in Russian; English version in Journal of Mathematical Sciences, Vol. 159 (2009), No. 1, 75-90, Springer.
[14] N.V. Krylov, On divergence form SPDEs with VMO coefficients, SIAM J. Math. Anal. Vol. 40 (2009), No. 6, 2262-2285.
[15] N.V. Krylov, Itô’s formula for the $L_p$-norm of stochastic $W^1_p$-valued processes, to appear in Probab. Theory Related Fields, http://arxiv.org/abs/0806.1557
[16] N.V. Krylov, *On the Itô-Wentzell formula for distribution-valued processes and related topics*, submitted to Probab. Theory Related Fields, http://arxiv.org/abs/0904.2752

[17] N.V. Krylov, *Filtering equations for partially observable diffusion processes with Lipschitz continuous coefficients*, to appear in “The Oxford Handbook of Nonlinear Filtering”, Oxford University Press, http://arxiv.org/abs/0908.1935

[18] N.V. Krylov and E. Priola, *Elliptic and parabolic second-order PDEs with growing coefficients*, to appear in Comm. in PDEs, http://arXiv.org/abs/0806.3100

[19] N.V. Krylov and B.L. Rozovskii, *Stochastic evolution equations*, pp. 71-146 in “Itogy nauki i tekhniki”, Vol. 14, VINITI, Moscow, 1979, in Russian; English translation: J. Soviet Math., Vol. 16 (1981), No. 4, 1233-1277.

[20] A. Lunardi, *Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in $\mathbb{R}^n$*, Ann. Sc. Norm. Super Pisa, Ser. IV., Vol. 24 (1997), 13164.

[21] A. Lunardi and V. Vespri, *Generation of strongly continuous semigroups by elliptic operators with unbounded coefficients in $L^p(\mathbb{R}^n)$*, Rend. Istit. Mat. Univ. Trieste 28 (1996), suppl., 251-279 (1997).

[22] G. Metafune, J. Prüss, A. Rhandi, and R. Schnaubelt, *The domain of the Ornstein-Uhlenbeck operator on an $L^p$-space with invariant measure*, Ann. Sc. Norm. Super. Pisa, Cl. Sci., (5) 1 (2002), 471-485.

[23] G. Metafune, J. Prüss, A. Rhandi, and R. Schnaubelt, *$L^p$-regularity for elliptic operators with unbounded coefficients*, Adv. Differential Equations, Vol. 10 (2005), No. 10, 1131-1164.

[24] J. Prüss, A. Rhandi, and R. Schnaubelt, *The domain of elliptic operators on $L^p(\mathbb{R}^d)$ with unbounded drift coefficients*, Houston J. Math., Vol. 32 (2006), No. 2, 563-576.

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