Sum Capacity of the Gaussian Interference Channel in the Low Interference Regime

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Abstract—New upper bounds on the sum capacity of the two-user Gaussian interference channel are derived. Using these bounds, it is shown that treating interference as noise achieves the sum capacity if the interference levels are below certain thresholds.

I. INTRODUCTION

Interference is a fundamental issue in the design of communication networks, particularly wireless networks. Unlike thermal noise, interference has a structure since it is generated by other users. Can this structure be exploited to decrease the uncertainty and thus improve the performance of the communication network? If so, what are the optimal signalling strategies? In this paper, we show that exploiting the structure of the interference in a two-user Gaussian interference channel does not improve the overall system throughput in the low interference regime. In other words, one can treat interference as noise and can still achieve the maximum possible throughput, if the interference levels are below certain thresholds.

The capacity region of the two-user Gaussian interference channel is known in the strong interference setting [1], [2], [3], where it is shown that each user can decode the information transmitted to the other user, and in the trivial case when there is no interference. The sum capacity of the interference channel is known for the one-sided interference channel (also called the Z-Channel) [4], [5], [6], where treating interference as noise achieves the sum capacity, and the degraded interference channel [7],[5], where one user treats interference as noise and the other user does interference cancelation.

Establishing the capacity region for a general two-user Gaussian interference channel still remains an open problem. The best known achievable strategy is the Han-Kobayashi scheme [2], where each user splits the information into private and common parts. The common messages are decoded at both the receivers, thereby reducing the level of interference. Although Chong, Motani and Garg have recently derived a simple representation of the Han-Kobayashi achievable region [8], it still remains formidable to compute.

In [6], the capacity region of a general two-user Gaussian interference channel is determined to within one bit by comparing a special case of the Han-Kobayashi scheme to the outer bounds derived in [4] and [6]. The concept of a genie-aided channel is used in deriving the outer bounds, where the receivers are provided with side information by a genie. The side information is chosen in such a way as to facilitate the computation of the capacity region of the genie-aided channel, which is an obvious outer bound to the capacity region of the interference channel.

In this paper, we tighten the outer bound on the sum capacity derived in [6]. In a low interference regime, we establish the existence of a genie, which results in a genie-aided channel whose sum capacity can be computed, and yet does not improve upon the sum capacity of the interference channel. Thus, we establish the sum capacity of the two-user Gaussian interference channel in this low interference regime, where the interference parameters are below certain pre-computable thresholds. In this regime, we further establish that it is optimal for the receivers to employ single user decoders that treat the interference as noise.

II. INTERFERENCE CHANNEL MODEL

The two-user Gaussian interference channel that we study in this paper is in the standard form [9], [2]. Over one symbol period the channel is described by

\[
\begin{align*}
Y_1 &= X_1 + h_{12}X_2 + Z_1 \\
Y_2 &= X_2 + h_{21}X_1 + Z_2
\end{align*}
\]

with inputs \(X_1, X_2\), and corresponding outputs \(Y_1, Y_2\). The receiver noise terms \(Z_1\) and \(Z_2\) are assumed to be independent, zero-mean, unit variance Gaussian random variables, and the interference parameters \(h_{12}\) and \(h_{21}\) are assumed to be real numbers. The transmit power constraints on users 1 and 2 are \(P_1\) and \(P_2\), respectively. The noise terms are assumed to be independent and identically distributed (i.i.d.) in time.

For each user \(i\), let the message index \((m_i)\) be uniformly distributed over \(\{1, 2, \ldots, 2^{nR_i}\}\) and \(C_i(n)\) be a code consisting of an encoding function \(X_i^n : \{1, 2, \ldots, 2^{nR_i}\} \rightarrow \mathbb{R}^n\) satisfying the power constraint

\[\|X_i^n(m_i)\|^2 \leq nP_i, \forall m_i \in \{1, 2, \ldots, 2^{nR_i}\}\]

and a decoding function \(g_i : \mathbb{R}^n \rightarrow \{1, 2, \ldots, 2^{nR_i}\}\). The corresponding probability of decoding error \(\lambda_i(n)\) defined as...
Pr\left[m_i \neq g_i(Y^n)\right]$. A rate pair \((R_1, R_2)\) is said to be achievable if there exists a sequence of codes \(\{C_1(n), C_2(n)\}\) such that the error probabilities \(\lambda_1(n)\) and \(\lambda_2(n)\) go to zero as \(n\) goes to infinity.

### A. Notation

The variables \(S_1\) and \(S_2\) denote the side information given to receivers 1 and 2, respectively. The variables \(X_{1G}\) and \(X_{2G}\) denote zero-mean Gaussian random variables with variances \(P_1\) and \(P_2\), respectively. The variables \(Y_{1G}, S_{1G}, Y_{2G}\) and \(S_{2G}\) denote the Gaussian outputs and side information that result when the channel inputs are Gaussian, i.e., when \(X_1 = X_{1G}\) and \(X_2 = X_{2G}\).

### III. Symmetric Interference Channel

The essential ideas and results of this paper are captured in the symmetric interference channel, for which \(P_1 = P_2 = P\) and \(h_{12} = h_{21} = h\). For this channel we shall establish the following result.

**Theorem 1:** For the symmetric interference channel, if the interference parameter \(h\) satisfies the condition

\[
|h + h^2P| \leq .5
\]

then treating interference as noise achieves the sum capacity, which is given by

\[
C_{\text{sum}} = \log \left(1 + \frac{P}{1 + h^2P}\right)
\]

### A. Existing Bounds

A natural way to deal with interference between users is to treat interference as noise if the interference is weak, and to orthogonalize the users if the interference is moderate. Therefore, the sum capacity of the symmetric interference channel is easily seen to be lower bounded as:

\[
C_{\text{sum}} \geq \log \left(1 + \frac{P}{1 + h^2P}\right)
\]

(3)

\[
C_{\text{sum}} \geq \log (1 + 2P)
\]

(4)

The optimality of either of these simple strategies is not clear and has not been established previously. More sophisticated strategies such as splitting power into private and common messages, which require multiuser decoders and knowledge of the interfering users’ codebooks, have been proposed by Han and Kobayashi [2]. A simplified version of the Han-Kobayashi strategy was recently shown to produce an achievable region that is within one bit of the capacity region [6].

Regarding upper bounds on the sum capacity, genie-based arguments have been used in [4], [6] to obtain the following:

\[
C_{\text{sum}} \leq \log \left(1 + h^2P + \frac{P}{1 + h^2P}\right)
\]

(5)

\[
C_{\text{sum}} \leq \frac{1}{2} \log (1 + P) + \frac{1}{2} \log \left(1 + \frac{P}{1 + h^2P}\right)
\]

(6)

The upper bound given in (5), which we refer to as the One-Bit bound, is asymptotically tight in the low interference regime [6]. The upper bound given in (6), the Z-Channel bound, is asymptotically tight in the moderate interference interference regime. (See Fig. 1)

In this paper, the upper bound given in (5) is tightened to establish Theorem 1. Furthermore, the upper bound given in (6) is shown to be a special case of Theorem 3 which extends Theorem 1 to the asymmetric interference channel.

### B. Proof of Theorem 1

To prove Theorem 1 we need to establish an upper bound on \(C_{\text{sum}}\) that matches the lower bound given in (3), when condition (2) is satisfied. As in [4], [6], our upper bound is based on a genie giving side information to the receivers. The genie needs to be chosen wisely in order to produce the tightest possible upper bound. To this end, we introduce the following two qualities of a good genie.

1) **Useful Genie**: Obtaining tight outer bounds on the capacity region of multiuser Gaussian channels is generally hindered by the fact that we cannot assume a simple structure (e.g., Gaussian) for the interference seen from other users. One way around this problem is to let a genie provide side information to the receivers in such a way that outer bounds can be derived for the genie-aided channel. In the context of the two-user interference channel of interest in this paper, we call a genie useful, if the sum capacity of the genie-aided channel can be derived. An example of useful genie is one that provides side information \(S_1 = X_2\) to receiver 1 and side information \(S_2 = X_1\) to receiver 2, because the resulting genie-aided channel has no interference. However, being too generous, such a genie does not result in a tight upper bound. This leads us to the notion of a smart genie.

2) **Smart Genie**: We call a genie smart if it results in a tight upper bound, i.e., it should not give too much information to the receivers. The “smartest” genie, of course, is one that does not interact with the receivers at all; however, it is obviously not useful.

![Fig. 1. Bounds on the sum capacity, \(P = 10\) dB](image-url)
So the essential question is: Is there a genie that is both useful and smart? The question was partly answered in [6], where the genie that results in the upper bound of (5) is useful and asymptotically smart. What we are looking for is a “divine genie” that allows us to prove Theorem 1.

The quest for the divine genie can be simplified by imposing a structure on the side information it provides. Following [6], we set:

\[ S_1 = hX_1 + h\eta W_1 \]
\[ S_2 = hX_2 + h\eta W_2 \] (7)

where \( \eta \) is a positive real number. However, unlike in [6], we allow \( W_1 \) to be correlated to \( Z_1 \) (and \( W_2 \) with \( Z_2 \)), with correlation coefficient \( \rho \).

**Lemma 1 (Useful Genie):** The sum capacity of the genie-aided channel with side information given in (7) is achieved by using Gaussian inputs and by treating interference as noise if the following condition holds.

\[ |h\eta| \leq \sqrt{1 - \rho^2} \] (8)

Hence the sum capacity of the symmetric interference channel described is bounded as

\[ C_{\text{sum}} \leq I(X_{1G}; Y_{1G}, S_{1G}) + I(X_{2G}; Y_{2G}, S_{2G}) \] (9)

**Proof:** Using Fano’s inequality, we have

\[ n(R_1 - \epsilon_n) \leq I(X_1^n; Y_1^n, S_1^n) \]
\[ = I(X_1^n; Y_1^n | S_1^n) + I(X_1^n; Y_1^n | S_2^n) \]
\[ = h(S_1^n) - h(S_1^n | X_1^n) + h(Y_1^n | S_1^n) - h(Y_1^n | S_1^n, X_1^n) \]
\[ \leq h(S_1^n) - \eta h(S_1^n | X_1^n) \]
\[ + h(Y_1^n | S_1^n) - h(Y_1^n | S_1^n, X_1^n) \]
\[ \leq h(S_1^n) - \eta h(S_1^n | X_1^n) + h(Y_1^n | S_1^n) - h(Y_1^n | S_1^n, X_1^n) \]

where step (a) follows from the fact that \( h(S_1^n | X_1^n) = h(h\eta W_1^n) \) is independent of the distribution of \( X_1^n \), and in step (b) we use the facts that 1) the Gaussian distribution maximizes the conditional differential entropy for a given covariance constraint, and 2) the function

\[ h(Y_1^n | S_1^n) = \frac{1}{2} \log \left[ 2\pi e \left( 1 - \rho^2 + h^2 P_2 + \frac{P_1 (\rho_1 - \eta)}{P_1 + \eta^2} \right) \right] \]

is an increasing and concave function in \( P_1 \) and \( P_2 \). Similarly, we have

\[ n(R_2 - \epsilon_n) \leq h(S_2^n) - \eta h(S_2^n | X_2^n) \]
\[ + h(Y_2^n | S_2^n, X_2^n) \]

Thus \( n(R_1 + R_2 - 2\epsilon_n) \) is upper bounded by

\[ h(S_1^n) - h(Y_2^n | S_2^n, X_2^n) - \eta h(S_1^n | X_1^n) + \eta h(Y_1^n | S_1^n) \]
\[ + h(S_2^n) - h(Y_1^n | S_1^n, X_1^n) - \eta h(S_2^n | X_2^n) + h(Y_2^n | S_2^n) \]

Now consider the expression

\[ \tilde{h}(S_1^n) - h(Y_2^n | S_2^n, X_2^n) = h(hX_1^n + h\eta W_1^n) - h(hX_1^n + Z_1^n | W_2^n) \]
\[ = h(hX_1^n + V_1^n) - h(hX_1^n + V_2^n) \]

where \( V_1 \sim \mathcal{N}(0, h^2 \eta^2) \) and \( V_2 \sim \mathcal{N}(0, 1 - \rho^2) \). Let \( V_1 \) and \( V_2 \) be correlated such that \( V_2 = V_1 + V \), for some Gaussian random variable \( V \) independent of \( V_1 \), which is possible if \( 1 - \rho^2 \geq h^2 \eta^2 \), i.e., (8) holds. Thus

\[ h(S_1^n) - h(Y_2^n | S_2^n, X_2^n) = -I(V^n; aX_1^n + V_1^n + V^n) \]
\[ \leq -nI(V; hX_1^n + V_1^n + V) \]
\[ = nh(S_{1G}) - nh(Y_{2G}|S_{2G}, X_{2G}). \]

where step (a) uses the worst case noise result for the additive noise channel [10]: Gaussian i.i.d. noise with the maximum allowable variance maximizes the mutual information when the input distribution is i.i.d. Gaussian. Therefore \( n(R_1 + R_2 - 2\epsilon_n) \) is upper bounded by

\[ nh(S_{1G}) - nh(Y_{2G}|S_{2G}, X_{2G}) + nh(S_{2G}) - nh(Y_{1G}|S_{1G}, X_{1G}) \]
\[ - nh(S_{1G}|X_{1G}) + nh(Y_{1G}|S_{1G}) - nh(S_{2G}|X_{2G}) + nh(Y_{2G}|S_{2G}) \]
\[ = nI(X_{1G}; Y_{1G}, S_{1G}) + nI(X_{2G}; Y_{2G}, S_{2G}) \]

and the lemma follows by letting \( n \rightarrow \infty \) with \( \epsilon_n \rightarrow 0 \). ■

**Remark 1:** If the genie does not satisfy (8), it might still be useful. Lemma 1 only claims the ‘if’ part, and not the ‘only if’ part.

**Lemma 2 (Smart Genie):** If Gaussian inputs are used, the interference is treated as noise, and the following condition holds

\[ \eta \rho = 1 + h^2 P \] (10)

then the genie does not increase the achievable sum rate, i.e.,

\[ I(X_{1G}; Y_{1G}, S_{1G}) = I(X_{1G}; Y_{1G}) \]
\[ I(X_{2G}; Y_{2G}, S_{2G}) = I(X_{2G}; Y_{2G}) \] (11)

**Proof:** Since

\[ I(X_{1G}; Y_{1G}, S_{1G}) = I(X_{1G}; Y_{1G}) + I(X_{1G}; S_{1G}|Y_{1G}) \]

we need to determine when \( I(X_{1G}; S_{1G}|Y_{1G}) = 0 \). Now,

\[ I(X_{1G}; S_{1G}|Y_{1G}) = I(X_{1G}; X_{1G} + \eta W_1 | X_{1G} + hX_{2G} + Z_{1G}) \]

Hence \( I(X_{1G}; S_{1G}|Y_{1G}) = 0 \), if \( \eta W_1 \) is a degraded version of \( hX_{2G} + Z_{1G} \), i.e., if

\[ \mathbb{E}[\eta W_1 (hX_{2G} + Z_1)] = \mathbb{E}[hX_{2G} + Z_1] \]

which happens when \( \eta \rho = 1 + h^2 P \). ■

The genie is smart and useful if it meets the conditions of both Lemma 1 and Lemma 2 i.e., when there exists a \( \rho \in [0, 1] \) such that

\[ |h + h^3 P| \leq |\rho| \sqrt{1 - \rho^2} \]
which is possible if
\[ |h + h^3 P| \leq 0.5 \]
This completes the proof of Theorem 1.

C. Geometric Interpretation

We now provide a geometric interpretation of the construction of the genie that was used in proving Theorem 1. We begin with an evaluation of the mutual information terms on the RHS of (9). The term \( I(X_{1G}; Y_{1G}, S_{1G}) \) can be expressed as
\[
I(X_{1G}; Y_{1G}, S_{1G}) = I(X_{1G}; X_{1G} + hX_{2G} + Z_1, hX_{1G} + h\eta W_1) \\
= I(X_{1G}; X_{1G} + hX_{2G} + Z_1, X_{1G} + h\eta W_1)
\]
which is the mutual information between a Gaussian random variable and two observations of this random variable in correlated Gaussian noise. The following lemma leads to a geometric interpretation of this mutual information.

**Lemma 3:** Let \( E_i = X_G + N_i, i = 1 \ldots m, \) be noisy observations of a zero-mean Gaussian random variable \( X_G \) with variance \( P \), where the variables \( N_i \) are arbitrary correlated zero mean Gaussian random variables. Then
\[
I(X_G, E) = \frac{1}{2} \log \left( 1 + \frac{P}{\sigma^2} \right)
\]
where \( E = [E_1 \ldots E_m]^T \) and
\[
\sigma^2 = \frac{1}{k} \sum_{i=1}^{m} \sum_{b_i=1}^{\infty} E \left( (b_i^T E - X_G)^2 \right)
\]
The proof of the lemma is relegated to the Appendix. A geometric interpretation of the lemma is provided in Fig. 2. Specializing Lemma 3 to the case \( m = 2 \) we get the following result for the mutual information term on the RHS of (9).

**Lemma 4:**
\[
I(X_{1G}; Y_{1G}, S_{1G}) = \frac{1}{2} \log \left( 1 + \frac{P}{\sigma^2} \right)
\]

where \( \sigma \) is the distance from origin to the line joining the points \( Q_{Y_1} \) and \( Q_{S_1} \), corresponding to \( Y_{1G} \) and \( S_{1G} \). In polar coordinates (see Fig. 3),
\[
Q_{Y_1} = (\sqrt{1 + h^2 P}, 0) \\
Q_{S_1} = (\eta, \theta)
\]
where \( \cos \theta \) is the correlation coefficient between \( hX_{2G} + Z_1 \) and \( \eta W_1 \), i.e.,
\[
\cos \theta = \frac{\mathbb{E}[W_1(hX_{2G} + Z_1)]}{\sqrt{\mathbb{E}[(hX_{2G} + Z_1)(hX_{2G} + Z_1)]}} = \frac{\rho}{\sqrt{1 + h^2 P}}
\]

**Remark 2:** \( (\eta, \theta) \) is an alternate description of the genie that is equivalent to the description \( (\eta, \rho) \).

The conditions for the genie to be useful (8) and smart (10) can be transformed into the following conditions (12) and (13), respectively.

- **Useful Genie:** The genie is useful, if the \( (\eta, \theta) \) lies inside the dashed curve in Fig. 3. This region is specified by
\[
h^2 \eta^2 + (1 + h^2 P) \cos^2 \theta \leq 1 \quad (12)
\]

- **Smart Genie:** From Lemma 4 the genie is smart if \( (\eta, \theta) \) lies on the line parallel to \( y \)-axis passing through the point \( (\sqrt{1 + h^2 P}, 0) \), i.e., if
\[
\eta \cos \theta = \sqrt{1 + h^2 P} \quad (13)
\]

There exists a useful and smart genie if the region specified by (12) intersects with that specified by (13), which is true if (2) holds.
The distance where \( \theta \) is given by (7) obtain an upper bound on the sum capacity. The following theorem uses such a genie to Fig. 4. Geometric derivation of the upper bound on the sum capacity when \( \phi \) does not hold.

D. Upper bound when \( \phi \) does not hold

The importance of the geometric intuition will be more evident when the condition \( \phi \) is not met, i.e., when the solid line and the dashed curve do not intersect in Fig. 5. In this case, it is of interest to pick the best genie within the class specified in (7). The following theorem uses such a genie to obtain an upper bound on the sum capacity.

**Theorem 2:** If \( h + h^3 P > 0.5 \)

\[
C_{\text{sum}} \leq \log \left[ 1 + \frac{P}{1 + h^2 P} \left( 1 + \frac{1}{\mu^2} \right) \right] \tag{14}
\]

where \( \mu \) is the slope of the tangent from \( (\sqrt{1 + h^2 P}, 0) \) to the curve. \( \text{(12)} \)

**Proof:** As illustrated in Fig. 4 we choose the genie corresponding to the point where the tangent touches the curve. Let \( y = \mu x + c \) be equation of the tangent. Since the line passes through \( (\sqrt{1 + h^2 P}, 0) \), we have \( c^2 = \mu^2 (1 + h^2 P) \). The distance \( \sigma \) from origin to the tangent satisfies

\[
\sigma^2 = \frac{c^2}{\mu^2 + 1} = \left( 1 + h^2 P \right) \frac{\mu^2}{\mu^2 + 1}
\]

Thus, by Lemma 4 the result follows.

IV. ASYMMETRIC INTERFERENCE CHANNEL

**Theorem 3:** For the asymmetric interference channel with interference parameters \( h_{12} \) and \( h_{21} \), suppose there exist \( \rho_1 \in [0, 1] \) and \( \rho_2 \in [0, 1] \) such that

\[
|h_{12}(1 + h_{21}^2 P_1)| \leq \rho_2 \sqrt{1 - \rho_1^2} \tag{15}
\]

\[
|h_{21}(1 + h_{12}^2 P_2)| \leq \rho_1 \sqrt{1 - \rho_2^2}
\]

Then treating interference as noise achieves sum capacity, which is given by

\[
C_{\text{sum}} = \frac{1}{2} \log \left( 1 + \frac{P_1}{1 + h_{12}^2 P_2} \right) + \frac{1}{2} \log \left( 1 + \frac{P_2}{1 + h_{21}^2 P_1} \right)
\]

**Proof:** The proof is similar to that for the symmetric interference channel. We set the genie-aided side information as:

\[
S_1 = h_{21}(X_1 + \eta_1 W_1) \\
S_2 = h_{12}(X_2 + \eta_2 W_2)
\]

Let \( \rho_1 \) be the correlation between \( Z_1 \) and \( W_1 \) (and \( \rho_2 \) the correlation between \( Z_2 \) and \( W_2 \)). Using the same arguments as in Lemma 1 the genie is useful if

\[
|h_{21} \eta_1| \leq \sqrt{1 - \rho_2^2} \\
|h_{12} \eta_2| \leq \sqrt{1 - \rho_1^2}
\]

Also, as in Lemma 2 the genie is smart if

\[
\eta_1 \rho_1 = 1 + h_{12}^2 P_2 \\
\eta_2 \rho_2 = 1 + h_{21}^2 P_1
\]

**Remark 3:** The condition \( \text{(15)} \) is equivalent to

\[
|h_{12}(1 + h_{21}^2 P_1)| + |h_{21}(1 + h_{12}^2 P_2)| \leq 1 \tag{16}
\]

**Proof:** Set \( \rho_1 = \cos \phi_1 \) and \( \rho_2 = \cos \phi_2 \). Then

\[
\rho_2 \sqrt{1 - \rho_1^2} + \rho_1 \sqrt{1 - \rho_2^2} = \sin(\phi_1 + \phi_2) \leq 1
\]

Thus (15) implies (16). On the other hand, if (16) is satisfied, we can find \( \phi \) such that

\[
|h_{12}(1 + h_{21}^2 P_1)| \leq \cos^2 \phi \leq 1 - |h_{21}(1 + h_{12}^2 P_2)|
\]

i.e.,

\[
|h_{12}(1 + h_{21}^2 P_1)| \leq \cos^2 \phi \\
|h_{21}(1 + h_{12}^2 P_1)| \leq \sin^2 \phi
\]

Setting \( \rho_1 = \sin \phi \) and \( \rho_2 = \cos \phi \), we have (15).

**Remark 4:** The sum capacity of the one-sided interference channel \([4], [5], [6]\) and hence the Z-channel outer bound on the sum capacity of the symmetric interference channel \([6]\) are immediate corollaries of Theorem 3.

V. CONCLUSION

We used a genie-aided channel to derive new upper bounds on the sum capacity of the two-user Gaussian interference channel. We introduced the notions of useful genie and smart genie. A genie is useful if the sum capacity of the genie-aided channel can easily be derived, and smart if the sum capacity of the genie-aided channel is the same as that of the interference channel. We showed that when the interference levels are below certain thresholds, we can construct a genie that is both useful and smart. Thus we established the sum capacity of the interference channel in the low interference regime, and furthermore showed that it is optimal for the receivers to treat the interference as noise in this regime. We were recently informed by G. Kramer that Theorem 3 has been independently established in \([11], [12]\).

The notion of a useful and smart genie is generalizible to interference channels with more than two users. We are currently working on establishing sum capacity results for such interference channels.
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APPENDIX

Proof of Lemma 3: From Data processing inequality, it follows that

\[ I(X_G; E) \geq I(X_G; b^T E), \quad \forall b \]

i.e., that

\[ I(X_G; E) \geq \sup_{b \in \mathbb{R}^n} I(X_G; b^T E) \]

Since \( X_G \) and \( N \) are Gaussian, the minimum mean squared-error (MMSE) estimator of the random variable \( X_G \) based on \( E \) is a linear function of \( E \) and is also a sufficient statistic. Hence \( I(X_G, E) = I(X_G; b^T E) \) for some \( b \). Therefore,

\[ I(X_G, E) = \sup_{b \in \mathbb{R}^n} I(X_G; b^T E) \]

and the lemma follows.

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