DIRECT SUMS OF REPRESENTATIONS AS MODULES OVER THEIR ENDOMORPHISM RINGS

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1. Introduction, terminology, and background

Our investigation into the endo-structure of infinite direct sums $\bigoplus_{i \in I} M_i$ of indecomposable modules $M_i$ – over a ring $R$ with identity – is centered on the following question: If $S = \text{End}_R(\bigoplus_{i \in I} M_i)$, how much pressure, in terms of the $S$-structure of $\bigoplus_{i \in I} M_i$, is required to force the $M_i$ into finitely many isomorphism classes? One of the consequences of our principal result in this direction (Theorem H of Section 4) is as follows. If all of the $M_i$ are endofinite (think, for instance, of finitely generated or generic modules over an Artin algebra) and if $(M_t)_{t \in T}$ is a transversal of the isomorphism types of the $M_i$, then the following conditions (1)-(4) are equivalent: (1) $T$ is finite; (2) $\bigoplus_{i \in I} M_i$ is endo-artinian and $(M_t)_{t \in T}$ is left semi-T-nilpotent; (3) $\bigoplus_{i \in I} M_i$ is endo-noetherian and $(M_t)_{t \in T}$ is right semi-T-nilpotent; (4) $(M_t)_{t \in T}$ is left and right semi-T-nilpotent and, for any cofinite subset $T' \subseteq T$, the endosocle of $\bigoplus_{t \in T'} M_t$ has finite support in $T'$. Here we call a family $(M_i)_{i \in I}$ left semi-T-nilpotent in case, for every sequence $(i_n)$ of distinct indices in $I$, every sequence of non-isomorphisms $f_n \in \text{Hom}_R(M_{i_{n+1}}, M_{i_n})$ and every finitely cogenerated factor module $M_{i_1}/X$ of $M_{i_1}$, the image of $f_1 \cdots f_n$ is contained in $X$ for $n \gg 0$. The familiar dual condition of right semi-T-nilpotence, going back to Harada (see [11]), links our results to well-known theorems addressing the exchange property of direct sums.

The mentioned equivalence applies with particular strength to Artin algebras, one of the reasons for this being the following asset (Proposition L): Given any family $(M_i)$ of finitely generated representations of an Artin algebra, the direct sum $\bigoplus_{i \in I} M_i$ is endo-artinian if and only if it is $\Sigma$-algebraically compact (= $\Sigma$-pure injective).

Our study of $\Sigma$-algebraically compact and, more restrictively, endo-artinian direct sums hinges crucially on an analysis of endosocles. The importance of this invariant in measuring the supply of maps among the $M_i$ emerges clearly in the following consequence of Theorem H: A finite dimensional algebra over an algebraically closed field has finite representation type if and only if the endosocles of all direct sums of indecomposable left representations of constant finite dimension have finite supports (Corollary N).

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Another point of independent interest lies in the general connections between $T$-nilpotence conditions and the endo-structure of direct sums exhibited in Proposition E of Section 3: A family $(M_i)_{i \in I}$ of indecomposable modules is right $T$-nilpotent if and only if $\bigoplus_{i \in I} M_i$ has the descending chain condition for finitely generated endo-submodules; moreover, these conditions are equivalent to the requirement that the direct sum $\bigoplus_{i \in I} M_i$ be semi-artinian over its endomorphism ring, i.e., that all endo-factor modules have essential socles. As a consequence, each $\Sigma$-algebraically compact module $M$ is a direct sum of indecomposables $M_i, i \in I$, with the property that the family $(M_i)_{i \in I}$ is right $T$-nilpotent.

We add a few comments on the background of our project for motivation. It is well known that the structure of non-finitely generated representations of $R$, viewed as modules over their endomorphism rings, is decisive in understanding the behavior of the finitely generated representations. We point to a few specific instances of such connections. As was first observed in [20] and [17], finite representation type of $R$ occurs precisely when all (left) $R$-modules have finite lengths over their endomorphism rings. Moreover, the rings with vanishing left pure global dimension are characterized by certain endo-chain conditions satisfied by their modules. Along a different line, Crawley-Boevey proved that, for tame finite dimensional algebras over an algebraically closed field, the generic modules (that is, the infinite dimensional indecomposable endofinite modules) govern the one-parameter families of finitely generated indecomposable representations in an extremely strong sense [6, 7]. (It appears plausible that the endosocles of direct sums of generic modules should reflect domestic versus non-domestic representation type; see, e.g., Example C(3), due to Ringel [21].) There is a common skein tying the listed scenarios at least loosely together, namely the fact that $\Sigma$-algebraic compactness of an $R$-module is tantamount to the descending chain condition for a selection of its endo-submodules. Other results in a related vein link endo-chain conditions of infinite direct sums of finitely generated representations of Artin algebras to the existence of preprojective or preinjective partitions, the existence of almost split maps, and the structure of direct products (see, e.g., [3, 13, 8, 18, 1] and our concluding remarks). These multiple bridges between endo-chain conditions on one hand, and purely representation-theoretic assets of classes of modules on the other, motivate our present investigation.

Section 2 is devoted to exploring endosocle series of direct sums. In Section 3, we compare $T$-nilpotence conditions for families of modules with endo-chain conditions for their direct sums, and in Section 4, we follow with our main results, applications, and examples.

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Prerequisites

Recall from [23] and [24] that a $p$-functor on $R$-Mod is a subfunctor of the forgetful functor $R$-Mod $\to Z$-Mod which commutes with direct products; here $R$-Mod stands for the category of all left $R$-modules. Special instances of $p$-functors can be described as follows: A pointed matrix over $R$ consists of a row-finite matrix $A = (a_{ij})_{i \in I, j \in J}$ of elements in $R$, paired with a column index $\alpha \in J$. Given a pointed matrix $(A, \alpha)$, we call the following $p$-functor $[A, \alpha]$ on $R$-Mod a matrix-functor: For any $R$-module $M$, the
The subgroup \([\mathcal{A}, \alpha]M\) is defined to be the \(\alpha\)-th projection of the solution set in \(M^J\) of the homogeneous system
\[
\sum_{j \in J} a_{ij} X_j = 0 \quad \text{for all } i \in I.
\]

In other words,
\[
[\mathcal{A}, \alpha]M = \{m \in M \mid \exists \text{ a solution } (m_j) \in M^J \text{ of the above system with } m_\alpha = m\}.
\]

Following [24], we call \([\mathcal{A}, \alpha]M\) a matrix subgroup of \(M\), a finite matrix subgroup in case \(I\) and \(J\) are finite. We refer the reader to [24] or [14] for more information on \(p\)-functors.

One of the most salient reasons for our present interest in \(p\)-functors lies in the following equivalent description of \(\Sigma\)-algebraically compact modules, i.e., of the modules \(M\) with the property that all direct sums of copies of \(M\) are algebraically compact (= pure injective): Namely, \(M\) is \(\Sigma\)-algebraically compact if and only if every descending chain of \(p\)-functorial subgroups of \(M\) becomes stationary, a condition which is in turn equivalent to the DCC for finite matrix subgroups of \(M\) (see [23, 9, 24, 10]); here we call a subgroup \(U\) of the abelian group underlying \(M\) \(p\)-functorial in case there exists a \(p\)-functor \(P : R\text{-Mod} \to \mathbb{Z}\text{-Mod}\) with \(P(M) = U\). Since, clearly, every \(p\)-functorial subgroup of \(M\) is an endo-submodule, this is a well-known instance of a link between the \(R\)- and endo-structures of \(M\). While it guarantees that endo-artinian modules are \(\Sigma\)-algebraically compact, the converse fails in general; see, e.g., [5] or [15, Theorem 5]. By contrast, \(M\) is endo-noetherian if and only if \(M\) has the ascending chain condition for \(p\)-functorial subgroups, since every finitely generated endo-submodule of \(M\) is a matrix subgroup. In case \(M\) is a direct sum of finitely presented modules, the ACC for finite matrix subgroups – not equivalent to the ACC for arbitrary matrix subgroups in general – already suffices to guarantee endo-noetherianness (Observation 8 of [17]). In light of the fact that \(p\)-functors automatically commute with direct sums (as arbitrary subfunctors of the forgetful functor do), one deduces the following observation which will prove useful in connection with dualities.

**Observation.** For any family \((M_i)_{i \in I}\) of finitely presented modules, the following conditions are equivalent:

1. \(\bigoplus_{i \in I} M_i\) is endo-noetherian.
2. \(\bigoplus_{i \in I} M_i\) satisfies the ACC for (finite) matrix subgroups.
3. \(\prod_{i \in I} M_i\) satisfies the ACC for (finite) matrix subgroups.
4. \(\prod_{i \in I} M_i\) is endo-noetherian.

2. **Endosocle series of direct sums of indecomposable modules**

The following elementary observations will be very useful to us in the sequel.

**Lemma A.** Every \(\Sigma\)-algebraically compact \(R\)-module \(M\) is semi-artinian over its endomorphism ring \(S\), meaning that every \(S\)-factor module of \(M\) has essential socle.

**Proof.** Suppose that \(M\) is a \(\Sigma\)-algebraically compact \(R\)-module. As we pointed out above, \(M\) then satisfies the descending chain condition for matrix subgroups which, in particular,
yields the DCC for finitely generated $S$-submodules of $M$. But this latter condition clearly entails our claim: Indeed, given $S$-submodules $U \subseteq V$ of $M$, there exists a minimal $S$-submodule $W$ of $V$ not contained in $U$, and any such choice of $W$ gives rise to a simple submodule $(W + U)/U \subseteq V/U$. □

The converse of Observation A fails in general: Let $R$ be any left perfect ring which fails to be left $\Sigma$-algebraically compact (for examples see [25]); then all left $R^{\text{op}}$-modules have essential socles, and therefore the regular left $R$-module is semi-artinian over its endomorphism ring.

Recall that the ascending socle series of an $S$-module $M$ is defined as follows: $\text{soc}_0 M = 0$, $\text{soc}_1 M = \text{soc}_S M$, and, if $\alpha = \beta + 1$ is a successor ordinal, $\text{soc}_\alpha M$ is the preimage of $\text{soc}_S(M/\text{soc}_\beta M)$ under the canonical epimorphism $M \to M/\text{soc}_\beta M$; for a limit ordinal $\alpha$, finally, $\text{soc}_\alpha M$ is defined to be the union $\bigcup_{\beta < \alpha} \text{soc}_\beta M$. Accordingly, the socle length of $M$ is the least ordinal $\mu$ with the property that $\text{soc}_{\mu+1} M = \text{soc}_\mu M$.

Given any left $R$-module $M$ with endomorphism ring $S$, we will refer to the $S$-socle of $M$ as the endosocle, denoted by endosoc $M$, and to the $S$-socle series as the endosocle series; a generic term of this series will be labeled endosoc $\alpha M$. The endosocle length of $M$, finally, is the length of its $S$-socle series. Whenever we speak of endosoc $M$ as having a property (X), we are referring to the $S$-structure, not the $R$-structure.

Since indecomposable algebraically compact modules have local endomorphism rings ([15]), Lemma A yields

**Corollary A'**. Suppose that $M$ is a $\Sigma$-algebraically compact $R$-module with endosocle length $\mu$. Then endosoc $\mu M = M$.

Hence: If the module $M$ is moreover indecomposable with endomorphism ring $S$, it contains an exhaustive, well-ordered chain of $S$-submodules with consecutive factors of the form $(S/J(S))^{(A)}$ (that is, a chain $(M_\alpha)_{\alpha<\tau}$ of $S$-submodules, where $\tau$ is an ordinal, with $M_\alpha \subseteq M_\beta$ for $\alpha < \beta < \tau$ such that $\bigcup_{\alpha<\tau} M_\alpha = M$ and all consecutive factors $M_{\alpha+1}/M_\alpha$ are direct sums of copies of $S/J(S)$). □

In contrast to the preceding observation, direct sums of $\Sigma$-algebraically compact modules may have trivial endosocles – think of $M = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/(p^n)$, for instance, where $p$ is prime. This can be seen as a pronounced failure of the direct sum of $\Sigma$-algebraically compact modules to inherit crucial properties implied by $\Sigma$-algebraic compactness.

Recall that any $\Sigma$-algebraically compact module $M$ is a direct sum of submodules with local endomorphism rings. Our main interest being in families $(M_i)$ of $\Sigma$-algebraically compact or, more restrictively, endo-artinian modules, most of our discussion will therefore focus on the situation where all of the $M_i$ have local endomorphism rings. In this setting, we record how the endosocle of $\bigoplus_{i \in I} M_i$ relates to the endosocles of the individual $M_i$. In case all of the $M_i$ are pairwise isomorphic, the connection is straightforward, otherwise it reflects the behavior of maps among the summands $M_i$ as follows.

**Lemma B.** Let $(M_i)_{i \in I}$ be a family of modules with local endomorphism rings, and $S = \text{End}_R(\bigoplus_{i \in I} M_i)$. 


First suppose that \( M_i = N \) for all \( i \in I \). Then:

- endosoc \((N^{(I)}) = \text{endosoc}(N)^{(I)} \), and the endosocle of \( N^{(I)} \) consists of a single homogeneous component.
- The endosocle of \( N^{(I)} \) is finitely generated as an endo-submodule if and only if the same is true for the endosocle of \( N \).

(2) In general, endosoc \( (\bigoplus_{i \in I} M_i) = \bigoplus_{i \in I} B_i \), where each \( B_i \) is the \( \text{End}_R(M_i) \)-submodule of \( M_i \) consisting precisely of those elements of \( M_i \) which are annihilated by all non-isomorphisms in \( \bigcup_{j \in I} \text{Hom}_R(M_i, M_j) \).

In particular: If \( M_i \not\cong M_j \) for \( i \neq j \), then each \( B_i \) is a semisimple \( \text{End}_R(M_i) \)-submodule of \( M_i \), as well as a semisimple \( S \)-submodule of the direct sum \( \bigoplus_{i \in I} M_i \).

(3) Whenever \( (M_n)_{n \in \mathbb{N}} \) is a family of pairwise nonisomorphic modules with local endomorphism rings which permits a sequence of monomorphisms \( M_1 \hookrightarrow M_2 \hookrightarrow M_3 \hookrightarrow \cdots \), the direct sum \( \bigoplus_{n \in \mathbb{N}} M_n \) has trivial endosocle.

**Proof.** Set \( M = \bigoplus_{i \in I} M_i \), and observe that each \( S \)-submodule \( U \) of \( M \) is of the form \( U = \bigoplus_{i \in I} (U \cap M_i) \), where the intersection \( U \cap M_i \) is an \( \text{End}_R(M_i) \)-submodule of \( M_i \). We deduce that each simple \( S \)-submodule of \( M \) can be written in the form \( Sx_i \) for a nonzero element \( x_i \) of some \( M_i \).

Part (1) is immediate from these remarks.

(2) Set \( B_i = (\text{endosoc } M) \cap M_i \). To see that \( B_i \) is contained in the annihilator of the set of non-isomorphisms in \( \bigcup_{j \in I} \text{Hom}_R(M_i, M_j) \), observe that, due to the locality of the rings \( \text{End}_R(M_i) \), each of the indicated non-isomorphisms belongs to the Jacobson radical of \( S \).

For the other inclusion, fix \( i \in I \), set \( I(i) = \{ j \in I \mid M_j \cong M_i \} \), and \( S(i) = \text{End}_R(\bigoplus_{j \in I(i)} M_j) \). If \( x \in M_i \) is annihilated by all non-isomorphisms in the above union, then \( x \) clearly belongs to the \( \text{End}_R(M_i) \)-socle of \( M_i \), and consequently \( S(i)x \) is contained in the \( S(i) \)-socle of \( \bigoplus_{j \in I(i)} M_j \) by part (1). Since \( Sx = S(i)x \), this shows that \( Sx \subseteq \text{endosoc } M \), which yields \( x \in B_i \) as required.

The final assertion under (2) is now obvious, as is (3). \( \square \)

**Examples C.**

(1) Let \( R = \mathbb{Z} \) and \( p \) a prime. The terms of the endosocle series of \( M = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/(p^n) \oplus \mathbb{Z}(p^\infty) \) are as follows: \( \text{endosoc}_\alpha M \) equals the copy of \( \mathbb{Z}/(p^\alpha) \) inside \( \mathbb{Z}(p^\infty) \) for \( \alpha < \omega \), and \( \text{endosoc}_\alpha M = \mathbb{Z}(p^\infty) \) for \( \alpha \geq \omega \).

(2) Let \( \Lambda \) be the Kronecker algebra. If \( (M_n)_{n \in \mathbb{N}} \) is the family of all preprojective modules in \( R \)-ind, then \( \text{endosoc}(\bigoplus_{n \in \mathbb{N}} M_n) = 0 \); in fact, \( \text{endosoc}(\bigoplus_{n \geq m} M_n) = 0 \) for all \( m \). On the other hand, if \( M' = \bigoplus_{n \in \mathbb{N}} M_n' \) is the direct sum of all preinjective modules, with length \( (M_n') = 2n - 1 \), the endosocle of \( M' \) equals \( M_1' \oplus \text{soc}_\Lambda M_2' \), and, for \( m \geq 2 \), we have \( \text{endosoc}(\bigoplus_{n \geq m} M_n') = M_m' \).

(3) The following is due to Ringel [21]: If \( \Lambda \) is a string algebra (i.e., a finite dimensional monomial relation algebra which is special biserial), but fails to be of domestic representation type, then there exists a family \( (M_n)_{n \in \mathbb{N}} \) of pairwise non-isomorphic generic \( \Lambda \)-modules allowing for consecutive embeddings \( M_1 \hookrightarrow M_2 \hookrightarrow \cdots \); by Lemma B(3), we infer that the endosocle of the direct sum of the \( M_n \)'s is zero.
We continue to assume that \( M = \bigoplus_{i \in I} M_i \), where all \( M_i \) have local endomorphism rings. In exploring the endo-structure of \( M \), a different sequence of iterated endosocles – not forming an ascending chain – is frequently more helpful than the series we just discussed. This is due to the fact that the traditional endosocle series may be infinite and still get stalled within a finite subsum of \( M_i \)'s (see Example C(1)). By contrast, any infinite sequence of nonzero terms of the following relative endosocle series involves infinitely many summands \( M_i \). This often makes it a more effective tool in studying homomorphisms among non-isomorphic \( M_i \)'s.

As usual, we start with endosoc\(_1'\) \( M = 0 \); moreover, we set \( I_0 = \emptyset \). Next, we set endosoc\(_1'\) \( M = \text{endosoc} \ M \), and denote by \( I_1 \) the support of endosoc\(_1'\) \( M \) in \( I \). Assume that the terms endosoc\(_\beta'\) \( M \) have already been defined for all ordinal numbers \( \beta < \alpha \). In case \( \alpha \) is a limit ordinal, we set endosoc\(_\alpha'\) \( M = 0 \). If, on the other hand, \( \alpha \) is a successor ordinal, say \( \alpha = \beta + 1 \), we let \( I_\beta \) be the support of \( \sum_{\gamma \leq \beta} \text{endosoc} \_\gamma M \), and define endosoc\(_\alpha'\) \( M \) to be the endosocle of the trimmed direct sum \( \bigoplus_{I_\beta} M_i \). In particular, we see that \( I_\alpha \) is defined for each ordinal \( \alpha \) and equals the support of \( \sum_{\beta \leq \alpha} \text{endosoc} \_\beta M = \bigoplus_{\beta \leq \alpha} \text{endosoc} \_\beta M \). Finally, we refer to the least ordinal number \( \mu \) such that endosoc\(_{\mu + 1} M = 0 \) (or, equivalently, the least ordinal number \( \mu \) with \( I_{\mu + 1} = I_\mu \)) as the relative endosocle length of \( M \).

Clearly, the isomorphism types of the \( R \)-submodules endosoc\(_\alpha'\) \( M \) are isomorphism invariants of \( M \), and the sum \( \sum_{\alpha \leq \mu} \text{endosoc} \_\alpha M \) is direct by construction. The second of the following observations will be used repeatedly in the next section. Both are straightforward from the definitions.

**Lemma D.** Let \( M = \bigoplus_{i \in I} M_i \), where each \( M_i \) has local endomorphism ring.

1. If each nontrivial subsum of \( M_i \)'s has nontrivial endosocle, then \( I_\mu = I \), where \( \mu \) is the relative endosocle length of \( M \).
2. Let \( \alpha \) be a successor ordinal, say \( \alpha = \beta + 1 \), such that \( \bigoplus_{i \in I \backslash I_\beta} M_i \) has essential endosocle. If \( U \) is an endo-submodule of \( M \) which is not contained in \( \bigoplus_{i \in I_\alpha} M_i \), then \( (J(S)U) \cap \text{endosoc} \_\alpha M \neq 0 \). \( \square \)

Observe that all hypotheses of Lemma D are satisfied if \( M \) is \( \Sigma \)-algebraically compact.

3. T-nilpotence versus endo-chain conditions

Harada [11] called a family \((M_i)_{i \in I}\) of indecomposable \( R \)-modules *semi-T-nilpotent* in case, for any sequence \((i_n)_{n \in \mathbb{N}}\) of distinct indices in \( I \), any family of non-isomorphisms \( f_n \in \text{Hom}_R(M_{i_n}, M_{i_{n+1}}) \), and any finitely generated \( R \)-submodule \( X \) of \( M_{i_1} \), there exists \( n_0 \in \mathbb{N} \) with

\[
    f_{n_0}f_{n_0-1} \cdots f_1(X) = 0.
\]

Obviously, this is a condition which can be tested ‘pointwise’, i.e., the family \((M_i)\) satisfies it precisely when, for each family of indices and non-isomorphisms \( f_n \) as above and any element \( x \in M_{i_1} \), there exists a number \( n_0 \) with \( f_{n_0}f_{n_0-1} \cdots f_1(x) = 0 \). To make room for a dual concept, we will refer to the described condition as *right semi-T-nilpotence*. 
The following twin concept of left semi-T-nilpotence of a family \((M_i)_{i \in I}\) of indecomposables requires that, for any sequence \((i_n)_{n \in \mathbb{N}}\) of distinct indices in \(I\), any family of non-isomorphisms \(f_n \in \text{Hom}_R(M_{i_{n+1}}, M_{i_n})\), and any finitely cogenerated factor module \(M_i/X\) of the \(R\)-module \(M_i\), there exists a natural number \(n_0\) such that

\[\text{Im}(f_1f_2 \cdots f_{n_0}) \subseteq X.\]

Left semi-T-nilpotence clearly implies the following elementwise condition: For each family of indices and homomorphisms \(f_n\) as specified above, and for each nonzero \(y \in M_i\), there exists \(n_0 \in \mathbb{N}\) such that \(y \notin \text{Im}(f_1f_2 \cdots f_{n_0})\). While the converse fails in general, it does hold if all of the \(M_i\)'s are finitely cogenerated – so, in particular, if they have finite length – in which case either condition is equivalent to the requirement that the family of homomorphisms among the \(M_i\)'s be artinian in the sense of Auslander (see below).

The stronger conditions of right/left T-nilpotence, finally, call for the same conclusions as the corresponding semi-T-nilpotence conditions, but on waiving the premise that the families of indices considered be free of repetitions.

We point out that, for the special case of finitely generated indecomposable modules \(M_i\) over an Artin algebra, Auslander introduced alternate terms for these nilpotence conditions (see [2]). He labeled a family of morphisms among such modules noetherian (resp., artinian) in case, for every countable sequence of non-isomorphisms \((f_n)\) selected within the family – consecutively composable in the appropriate sense – there exists an index \(n_0\) with \(f_{n_0} \cdots f_1 = 0\) (resp., \(f_1 \cdots f_{n_0} = 0\)). So, in this situation, the family \((M_i)\) is right semi-T-nilpotent if and only if it is right T-nilpotent, if and only if the family of homomorphisms among the \(M_i\) is noetherian. (We alert the reader to the fact that this condition automatically comes with endo-artinianness of \(\bigoplus_{i \in I} M_i\) – see Proposition E below – but may fail for endo-noetherian direct sums \(\bigoplus_{i \in I} M_i\).)

The second of the following known characterizations of right semi-T-nilpotent families of modules will be used freely in the sequel, while the third will enter into an application of our main theorem. The implications \(‘(3) \Rightarrow (2) \iff (1)’\) are due to Harada ([11]), and the remaining one was filled in by W. Zimmermann and the first-named author [16]. Recall that a module \(M\) is said to have the exchange property in case, for every equality of the form \(M' \oplus A = \bigoplus_{i \in L} A_i\) with \(M' \cong M\), there exist submodules \(B_i \subseteq A_i\) such that \(M' \oplus A = M' \oplus \bigoplus_{i \in L} B_i\).

**Known facts.** Suppose that all \(M_i\) have local endomorphism rings. Then the following conditions are equivalent:

1. The family \((M_i)_{i \in I}\) is right semi-T-nilpotent.
2. The Jacobson radical of \(\text{End}_R(\bigoplus_{i \in I} M_i)\) coincides with the set of those endomorphisms \(f\) which have the property that all compositions \(\text{pr}_j \cdot f \cdot \text{in}_i\) are non-isomorphisms (here \(\text{pr}_j\) and \(\text{in}_i\) denote the canonical projections and injections, respectively).
3. The direct sum \(\bigoplus_{i \in I} M_i\) has the exchange property.

We start by relating right semi-T-nilpotence to the endo-semi-artinian condition we encountered in Lemma A. These connections will not only feed into the proof of our main result, but are of interest as supplements. Part (3) of the following proposition is a variant
Moreover, we note that the implications (a) \implies (c) \implies (b) hold without the blanket hypothesis under (2).

**Proposition E.** Suppose that \((M_i)_{i \in I}\) is a family of indecomposable left \(R\)-modules, and set \(M = \bigoplus_{i \in I} M_i\).

(1) If \((M_i)\) is a right \(T\)-nilpotent family, then all of the \(M_i\) have local endomorphism rings.

(2) If all \(M_i\) have local endomorphism rings, the following conditions are equivalent:

(a) The family \((M_i)\) is right \(T\)-nilpotent.
(b) The direct sum \(M\) is semi-artinian over its endomorphism ring.
(c) \(M\) satisfies the DCC for cyclic endo-submodules (or, equivalently, the DCC for finitely generated endo-submodules).

As a consequence, every \(\Sigma\)-algebraically compact \(R\)-module \(M\) is a direct sum ranging over a right \(T\)-nilpotent family of indecomposable modules.

(3) The following condition is sufficient for right semi-\(T\)-nilpotence of the family \((M_i)\): Each \(M_i\) has local endomorphism ring, and every descending chain of matrix functors on \(R\text{-Mod}\) becomes uniformly stationary on almost all of the \(M_i\)’s.

**Proof.** Set \(S = \text{End}_R(M)\).

(1) Suppose that \(M_i\) belongs to a right \(T\)-nilpotent family, and let \(f \in \text{End}_R(M_i)\) be a non-isomorphism. Then \(f\) fails to be a monomorphism, for, given any element \(x \in M\), there exists \(m \in \mathbb{N}\) with \(f^m(x) = 0\). To see that, for any \(g \in \text{End}_R(M_i)\), the map \(\text{id}_{M_i} - gf\) is an isomorphism, it thus suffices to check injectivity. To that end, we observe that \(x = gf(x)\) implies \(x = (gf)^n(x)\) for all \(n \in \mathbb{N}\), and hence \(x = 0\) by our \(T\)-nilpotence hypothesis.

(2) Start by recalling that the descending chain condition for cyclic submodules is equivalent to that for finitely generated submodules, due to Bjork [4]. The canonical embeddings and projections coming with the direct sum \(\bigoplus_{i \in I} M_i\) will be denoted by \(\text{in}_i\) and \(\text{pr}_i\), respectively.

To back up ‘(a) \implies (c)’, suppose (a) holds.

First we show that, for any element \(m = (m_i)_{i \in I} \in M\), the cyclic \(S\)-module \(Sm/J(S)m\) is semisimple. Indeed, right semi-\(T\)-nilpotence of \((M_i)\) guarantees that the Jacobson radical \(J(S)\) of \(S\) contains all endomorphisms \(f\) with the property that the compositions \(\text{pr}_i f \text{in}_j\) are non-isomorphisms for arbitrary \(i, j \in I\). We will make the assumption that \(M_i \not= M_j\) whenever \(i \neq j\); this does not affect the generality of our argument, for every endo-submodule \(U\) of a direct sum of powers \(\bigoplus_{i \in I} M_i^{(L_i)}\) is of the form \(\bigoplus_{i \in I} U_i^{(L_i)}\) for a suitable endo-submodule \(U_i\) of \(M_i\). Consequently,

\[ Sm/J(S)m = \bigoplus_{\text{finite}} \text{End}_R(M_i)m_i/J(\text{End}_R(M_i))m_i, \]

where each of the summands on the right is an \(S\)-module, the \(S\)-structure of which coincides with the pertinent \(\text{End}_R(M_i)/J(\text{End}_R(M_i))\)-structure.

To the contrary of our claim assume that \(M\) contains a strictly descending chain \(Sx_1 \supsetneq Sx_2 \supsetneq Sx_3 \supsetneq \cdots\) of cyclic \(S\)-submodules. Now the induced sequence in the semisimple
factor module $Sx_1/J(S)x_1$ does become stationary, yielding an integer $m$ such that $Sx_m \subseteq Sx_{m+1} + J(S)x_1$. By possibly dropping some terms from our descending chain, we may assume that $Sx_2 \subseteq Sx_3 + J(S)x_1$. We apply the same argument to $Sx_2/J(S)x_2$ and iterate, whence an obvious induction permits us to thin out our original sequence of cyclics so as to assure $Sx_n \subseteq Sx_{n+1} + J(S)x_{n-1}$ for all $n \geq 2$. For each $n \geq 2$ we thus obtain an equality $x_n = f_{n-1}(x_{n+1}) + g_{n-1}(x_{n-1})$ with $f_{n-1} \in S$ and $g_{n-1} \in J(S)$. By substituting $x_2 = f_3(x_3) + g_1(x_1)$ into the equality $x_3 = f_4(x_4) + g_2(x_2)$, we deduce that $(1 - g_2f_3)(x_3) = f_4(x_4) + g_2g_1(x_1)$, and multiplicity by the inverse of $1 - g_2f_3$ yields an equality of the form $x_3 = f'_4(x_4) + h_1(x_1)$ with $h_1 \in J(S)$. In a next step, we insert this expression for $x_3$ into $x_4 = f_5(x_5) + g_3(x_3)$, thus obtaining an element $h_2 \in J(S)$ with the property that $x_4 = f'_5(x_5) + h_2h_1(x_1)$ for a suitable $f'_5 \in S$. Again we iterate, and arrive at a sequence of elements $h_1, h_2, h_3, \ldots \in J(S)$ satisfying $x_n = f'_n(x_{n+1}) + h_{n-2} \cdot \cdots \cdot h_2h_1(x_1)$ for suitable choices of $f'_n \in S$.

T-nilpotence of the family $(M_i)_{i \in I}$, combined with Koenig’s graph theorem, will now provide us with a natural number $N$ such that $h_N = \cdots = h_2h_1(x_1) = 0$. For, if we had $h_n \cdots h_2h_1(x_1) \neq 0$ for all $n$, the following graph $G$ would have paths of arbitrary length: the roots of $G$ are the nonzero components $x_i \in M_i$ of the element $x \in \text{finite in number, say } x_{i, i(1)}, \ldots, x_{i, i(m)}$; and the edge paths of length $l$ in $G$ correspond to the nonzero evaluations

$$(\text{pr}_{j(l)} h_l \text{ in } j(l-1)) (\text{pr}_{j(l-1)} h_{l-1} \text{ in } j(l-2)) \cdots (\text{pr}_{j(1)} h_1 \text{ in } j(0))(x_{1,j(0)}),$$

for suitable indices $j(s) \in I$ such that $j(0) \in \{i(1), \ldots, i(m)\}$. Koenig’s graph theorem would then imply the existence of an infinite path. But this is incompatible with our T-nilpotence condition, since each of the maps $\text{pr}_{j(s)} h_s \text{ in } j(s-1) : M_{j(s-1)} \rightarrow M_{j(s)}$ is a non-isomorphism by our choice of the $h_n$.

Thus, we obtain a number $N$ as described and infer $x_{N+2} \in Sx_{N+3}$, contrary to our assumption that the chain of $Sx_n$ be strictly descending.

To justify the implication ‘(c) $\implies$ (b)’, carry over the argument proving Lemma A. For ‘(b) $\implies$ (a)’, see [22, Chapter VIII].

(3) Suppose that $(i_n)_{n \in \mathbb{N}}$ is a sequence of distinct elements of $I$, and $f_{i_n} : M_{i_n} \rightarrow M_{i_{n+1}}$ a non-isomorphism for $n \in \mathbb{N}$. Given $x \in M_{i_1}$, consider the following chain of principal $S$-submodules of $M$:

$$Sf_{i_1}x \supseteq Sf_{i_2}f_{i_1}x \supseteq Sf_{i_3}f_{i_2}f_{i_1}x \supseteq \cdots.$$ 

All terms of this chain are matrix subgroups of the $R$-module $M$ (see Section 1), say $Sf_{i_n} \cdots f_{i_1}x = P_n(M)$, where $P_n$ is a matrix functor. Since the class of matrix functors on $R$-Mod is closed under intersections, we can normalize to the situation where $P_1 \supseteq P_2 \supseteq P_3 \supseteq \cdots$. Assuming that this chain becomes uniformly stationary on almost all of the $M_i$’s, we obtain a natural number $N$, together with a finite subset $I_0 \subseteq I$, such that $P_n(\bigoplus_{i \in I_0} M_i) = P_N(\bigoplus_{i \in I_0} M_i)$ for all $n \geq N$. Since the $i_n$ are pairwise different, we can adjust $N$ upwards, if necessary, so as to guarantee that, moreover, $f_{i_n} \cdots f_{i_1}x \in$
\[ \bigoplus_{i \in I \setminus I_0} M_i \] for all \( n \geq N \). Denoting the \( R \)-endomorphism ring of \( \bigoplus_{i \in I \setminus I_0} M_i \) by \( S' \), we infer that the chain

\[ S'f_{i_n} \cdots f_{i_{N+1}} (f_{i_N} f_{i_{N-1}} \cdots f_{i_1} x), \quad n \geq N \]

is stationary. We will deduce that \( f_{i_N} \cdots f_{i_1} x = 0 \). Indeed, our setup yields a map \( g \in S' \) with \( gf_{i_{N+1}} f_{i_N} \cdots f_{i_1} x = f_{i_N} \cdots f_{i_1} x \). Clearly, we may assume that \( g \) is a map from \( M_{i_{N+2}} \) to \( M_{i_{N+1}} \). If \( f_{i_N} \cdots f_{i_1} x \) were nonzero, Nakayama’s Lemma would exclude \( g f_{i_{N+1}} \) from the Jacobson radical of \( \text{End}_R(M_{i_N}) \), and locality of this endomorphism ring would guarantee \( g f_{i_{N+1}} \) to be a unit of \( \text{End}_R(M_{i_N}) \). But this is impossible, since \( M_{i_{N+2}} \) is indecomposable and \( f_{i_{N+1}} \) a non-isomorphism. Hence \( f_{i_N} \cdots f_{i_1} x = 0 \) as claimed, and the argument is complete. \( \square \)

Part (2) of the preceding proposition can actually be de-specified from the situation of direct sums regarded as modules over their endomorphism rings: If \( S \) is any ring and \( M \) a left \( S \)-module, then the DCC for finitely generated submodules forces \( M \) to be semiartinian; the latter condition in turn implies that, given any \( m \in M \) and any sequence \((s_1, s_2, \ldots)\) in \( J(S) \), we have \( s_n \cdots s_1 m = 0 \) for \( n \gg 0 \). Moreover, these three conditions are equivalent in case \( Sm/J(S)m \) is semisimple for every element \( m \in M \).

4. Main results and examples

Theorem H below will zero in on our primary concern: Namely to significantly weaken conditions on the endo-structure of \( \bigoplus_{i \in I} M_i \) which are known to guarantee that the \( M_i \) fall into finitely many isomorphism classes. An instance of such a condition is endofiniteness of \( \bigoplus_{i \in I} M_i \); see [7]. Our result also provides background for the following well-known characterization of the Artin algebras \( \Lambda \) having finite representation type, in terms of morphisms among the indecomposable finitely generated left \( \Lambda \)-modules, due to Auslander ([2]): Namely, \( \Lambda \) has finite representation type precisely when every countable family of non-isomorphisms is both artinian and noetherian in the sense given at the beginning of Section 2. Related results for families of finitely generated indecomposable modules were recently obtained by Dung [8, Theorem 3.3 and Corollary 3.12]. We, too, are particularly interested in the situation where all of the \( M_i \)'s have finite length over \( R \) – next to the case where the \( M_i \)'s are generic – and will address it in subsequent corollaries and examples.

**Lemma F.** Let \((M_i)_{i \in I}\) be a right semi-\( T \)-nilpotent family of pairwise non-isomorphic modules with local endomorphism rings. If, for each cofinite subset \( I' \) of \( I \), the direct sum \( \bigoplus_{i \in I'} M_i \) is finitely generated over its endomorphism ring, then \( I \) is finite.

**Proof.** Suppose that the endo-condition of our claim is satisfied, and set \( M = \bigoplus_{i \in I} M_i \). For any subset \( I' \subseteq I \), we denote by \( \text{pr}_{I'} \), the canonical projection \( M \to \bigoplus_{i \in I'} M_i \) along \( \bigoplus_{i \in I \setminus I'} M_i \), preserving the convention \( \text{pr}_i = \text{pr}_{\{i\}} \) for \( i \in I \). Moreover, we set \( S = \text{End}_R(\bigoplus_{i \in I} M_i) \). Keeping in mind that every \( S \)-submodule \( U \) of \( M \) is of the form \( \bigoplus_{i \in I} (M_i \cap U) \), where each \( M_i \cap U \) is an \( \text{End}_R(M_i) \) submodule of \( M_i \), we find that, given any \( S \)-generating set \( G \) of such a module \( U \), the collection \( \{ \text{pr}_i(x) \mid i \in I, x \in G \} \) of projections again belongs to \( U \) (and thus generates \( U \)).
By hypothesis, we can therefore pick a finite family of elements \((x_{l_1})_{l_1 \in L_1}\) in \(\bigcup_{i \in I} M_i\) with the property that \(M = \sum_{i \in I} S x_{l_i}\); by \(K_1 \subseteq I\) we denote the (finite) union of the supports of the \(x_{l_i}\). Setting \(S_1 = \text{End}_R(\bigoplus_{i \in I \setminus K_1} M_i)\) and applying our hypothesis to the pared-down direct sum, we see that each of the modules \(S_1 \text{pr}_{I \setminus K_1}(S x_{l_i})\) can be written as a finite sum, \(S_1 \text{pr}_{I \setminus K_1}(S x_{l_i}) = \sum_{l_2 \in L_2} S_1 f_{l_2l_1}(x_{l_i})\), where \(L_2\) is a finite index set disjoint from \(L_1\) and the \(f_{l_2l_1}\) are homomorphisms from suitable summands \(M_i\) of \(\bigoplus_{k \in K_1} M_k\) to summands \(M_j\) of \(\bigoplus_{k \in I \setminus K_1} M_k\); here the sources and targets of the \(f_{l_2l_1}\) may appear repeatedly. By allowing zero maps among the \(f_{l_1l_2}\), we can make the same index set \(L_2\) work for all \(l_1 \in L_1\) simultaneously. We thus obtain

\[
M = \bigoplus_{k \in K_1} M_k + \sum_{l_1 \in L_1} \text{pr}_{I \setminus K_1}(S x_{l_i})
= \bigoplus_{k \in K_1} M_k + \sum_{l_1 \in L_1, l_2 \in L_2} S_1 f_{l_2l_1}(x_{l_i}).
\]

Next we let \(K_2 \subseteq I\) be a finite set containing \(K_1\) and the supports of the elements \(f_{l_2l_1}(x_{l_i})\) for all choices of \(l_1, l_2\). Now we set \(S_2 = \text{End}_R(\bigoplus_{i \in I \setminus K_2} M_i)\) and, using the same guidelines as for the choice of \(L_1\) and the \(f_{l_2l_1}\)’s, we choose a finite index set \(L_3\), disjoint from \(L_1 \cup L_2\), and homomorphisms \(f_{l_3l_2}\), each from a suitable summand \(M_i\) of \(\bigoplus_{k \in K_2} M_k\) to a summand \(M_j\) of \(\bigoplus_{k \in I \setminus K_2} M_k\), such that

\[
S_2 \text{pr}_{I \setminus K_2}(S_1 f_{l_2l_1}(x_{l_i})) = \sum_{l_3 \in L_3} S_2 f_{l_3l_2} f_{l_2l_1}(x_{l_i}),
\]

for all \(l_1 \in L_1\) and \(l_2 \in L_2\). This yields the equality

\[
M = \bigoplus_{k \in K_2} M_k + \sum_{l_1 \in L_1, l_2, l_3 \in L_2} S_2 f_{l_3l_2} f_{l_2l_1}(x_{l_i}).
\]

We repeat the above procedure to inductively obtain pairwise disjoint finite index sets \(L_1, L_2, L_3, \ldots\), finite subsets \(K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots\) of \(I\), and homomorphisms \(f_{l_n+1l_n} -\) for \(l_n \in L_n\) and \(l_{n+1} \in L_{n+1}\) – from summands \(M_i\) of \(\bigoplus_{k \in K_n} M_k\) to summands \(M_j\) of \(\bigoplus_{i \in I \setminus K_n} M_i\), respectively, such that for each \(n \in \mathbb{N}\),

\[
M = \bigoplus_{k \in K_n} M_k + \sum_{l_m \in L_m, 1 \leq m \leq n} S_n f_{l_{n+1}l_n} \cdots f_{l_2l_1}(x_{l_i}),
\]

where \(S_n = \text{End}_R(\bigoplus_{i \in I \setminus K_n} M_i)\).

Let \(G\) be the graph having as vertices the indices in \(\bigcup_{i \in I} K_i\) and as edge paths those concatenations \(f_{l_{n+1}l_n} \cdots f_{l_2l_1}\) which have the property that \(f_{l_{n+1}l_n} \cdots f_{l_2l_1}(x_{l_i}) \neq 0\). Since all of the maps \(f_{l_i+1l_i}\) are non-isomorphisms by construction, our T-nilpotence hypothesis excludes the occurrence of an infinite path. Therefore, König’s Graph Theorem guarantees the existence of an upper bound on the lengths of paths in \(G\) – N say – which shows that \(M = \bigoplus_{k \in K_N} M_k\). But this says that \(I = K_N\) is finite as required.

We now come to our key lemma. In spite of the dual formats of Lemmas F and G, the latter cannot be obtained by dualizing the preceding argument.
Lemma G. Let \((M_i)_{i \in I}\) be a left semi-T-nilpotent family of modules with local endomorphism rings. If, for each cofinite subset \(I'\) of \(I\), the direct sum \(\bigoplus_{i \in I'} M_i\) is finitely cogenerated over its endomorphism ring, then \(I\) is finite.

Proof. Again suppose that the endo-condition of our claim is satisfied, and set \(M = \bigoplus_{i \in I} M_i\), \(S = \text{End}_R(M)\). Assume that \(I\) is infinite. In order to deduce that the family \((M_i)_{i \in I}\) then fails to be left semi-T-nilpotent, we consider the relative endosocle series \((\text{endosoc}'_\alpha M)\_\alpha\), as introduced at the end of Section 2. As before, we denote the support of \(\bigoplus_{\beta \leq \alpha} \text{endosoc}'_\beta M\) by \(I_\alpha\). By hypothesis, endosoc \(M\) is finitely generated, whence the support \(I_1\) is finite by Lemma B(2). An obvious induction, based on the hypothesis that the endosocle of any cofinite subsum \(\bigoplus_{i \in I'} M_i\) has finite support in \(I'\), thus yields finiteness of \(I_k\) for all \(k \in \mathbb{N}\).

Our hypothesis moreover guarantees that each of the relative endosocles \(\text{endosoc}'_{k+1} M\) is essential in the corresponding direct sum \(\bigoplus_{i \in I / I_k} M_i\) viewed as a module over its endomorphism ring \(S_k\). Since, under the canonical embedding of \(S_k\) into \(S\), we have \(J(S_k) \subseteq J(S)\), we deduce from Lemma D(2) that, for any nonzero \(S_k\)-submodule \(U\) of \(\bigoplus_{i \in I / I_k} M_i\), the intersection \((J(S)U) \cap \text{endosoc}'_k M\) is nonzero. For each \(k \in \mathbb{N}\), we now consider a descending chain \(A_{k1} \supseteq A_{k2} \supseteq A_{k3} \supseteq \ldots\) of nonzero \(S_k\)-submodules of \(\text{endosoc}'_k M\), where the \(A_{ki}\) are inductively defined as follows: \(A_{k1} = (J(S) \text{endosoc}'_{k+1} M) \cap \text{endosoc}'_k M\) for \(k \geq 1\), and \(A_{k,i+1} = (J(S)A_{k+1,i}) \cap \text{endosoc}'_k M\).

Being contained in the finitely generated semisimple \(S_{k-1}\)-module \(\text{endosoc}'_k M\), the chain \((A_{ki})_{i \geq 1}\) becomes stationary. Furthermore, by Lemma D(2), it consists of nonzero terms. Therefore it converges to a nonzero \(S_{k-1}\)-submodule \(C_k \subseteq \text{endosoc}'_k M\). From the definition of the \(A_{ki}\), we moreover derive \(C_k = (J(S)C_{k+1}) \cap \text{endosoc}'_k M\) for all \(k \in \mathbb{N}\), which yields a sequence of inclusions

\[C_1 \subseteq J(S)C_2, \quad C_2 \subseteq J(S)C_3, \ldots.\]

Since \(C_1\) is an \(S\)-module, we can pick \(i_1 \in I_1\) so that \(M_{i_1} \cap C_1\) contains a nonzero element \(x\). The above string of inclusions then yields disjoint finite sets \(L_n\) for \(n \in \mathbb{N}\), together with families of homomorphisms \((f_{i_n})_{l_n \in L_n}\) in \(J(S)\), each of the form \(f_{i_n} = \text{pr}_{v(l_n)} \cdot f_{i_n} \cdot \text{im}_{u(l_n)} \in \text{Hom}_R(M_{u(l_n)}, M_{v(l_n)})\) for suitable indices \(u(l_n) \in I_{n+1} \setminus I_n, v(l_n) \in I_n,\) and \(v(l_1) = i_1\) for all \(l_1 \in L_1\), such that

\[x \in \text{Im} \left( \sum_{l_1 \in L_1, \ldots, l_n \in L_n} f_{i_1} f_{i_2} \cdots f_{i_n} \right)\]

for each \(n\). Here all of the \(f_{i_n}\)'s are non-isomorphisms from \(M_{u(l_n)}\) to \(M_{v(l_n)}\), respectively, since they belong to \(J(S)\) (as usual, we identify homomorphisms \(M_i \rightarrow M_j\) with elements of \(S\)). Note that the sets \(\{u(l_n) \mid l_n \in L_n\}\) for \(n \in \mathbb{N}\) are pairwise disjoint by construction; on the other hand, we permit repetitions \(u(l_k) = u(l'_k)\) and \(v(l_k) = v(l'_k)\) for \(l_k \neq l'_k\).

Next, we let \(Y \subseteq M_{i_1}\) be a maximal \(R\)-submodule of \(M_{i_1}\) with \(x \notin Y\), and consider the tree having as root the index \(i_1\) and as branches all those concatenations \(f_{i_1} f_{i_2} \cdots f_{i_n}\) the image of which is not contained in \(Y\). By construction, there is no upper bound on the lengths of the edge paths in this graph, and therefore König’s graph theorem guarantees
the existence of a path of infinite length. But, since $M_{i_1}/Y$ is a finitely cogenerated factor module of $M_{i_1}$, this means that the family $(M_i)$ fails to be left semi-$T$-nilpotent. Thus, our hypothesis ensures finiteness of $I$. □

This smooths the road to

**Theorem H.** Let $(M_i)_{i \in I}$ be a family of indecomposable $R$-modules and $(M_t)_{t \in T}$ a transversal of the isomorphism classes of the $M_i$.

(I) If all of the $M_i$ have local endomorphism rings, the following statements are equivalent:

1. $\bigoplus_{i \in I} M_i$ is endo-noetherian, and $(M_t)_{t \in T}$ is right semi-$T$-nilpotent.
2. $T$ is finite, and each $M_t$ is endo-noetherian.

(II) The following statements are equivalent:

1. $\bigoplus_{i \in I} M_i$ is endo-artinian, and $(M_t)_{t \in T}$ is left semi-$T$-nilpotent.
2. $T$ is finite, and each $M_t$ is endo-artinian.

(III) Suppose that $(M_t)_{t \in T}$ is a left semi-$T$-nilpotent and right $T$-nilpotent family satisfying the following additional finiteness condition: (●) For any cofinite subset $T' \subseteq T$, the endosocle of $\bigoplus_{t \in T'} M_t$ is finitely generated.

Then $T$ is finite.

Proof. Start by observing that, in each of the first two parts, the implication ‘(2) $\implies$ (1)’ is trivial. Indeed, it suffices to note that both endo-noetherianness and endo-artinianness are inherited by arbitrary powers and finite direct sums. Moreover, note that, in proving the remaining implications, it is innocuous to assume that $I = T$, since the conditions involved are stable under passage from $\bigoplus_{i \in I} M_i$ to $\bigoplus_{t \in T} M_t$. The implications ‘(1) $\implies$ (2)’ under (I) and (II) now follow from Lemma F and Lemma G, respectively; note that (II.1) guarantees all of the $M_i$ to have local endomorphism rings, while this is a blanket hypothesis for (I).

In order to prove (III), adopt all of the listed hypotheses. By part (1) of Proposition E, the $M_i$ again have local endomorphism rings. Moreover, for each cofinite subset $T' \subseteq T$, the direct sum $\bigoplus_{t \in T'} M_t$ is finitely cogenerated over its endomorphism ring: Indeed, in view of Proposition E(2), right $T$-nilpotence of $(M_t)_{t \in T'}$ ensures that this direct sum has essential endosocle and, by condition (●), this endosocle is finitely generated. Thus, once more, Lemma G yields finiteness of $T$. □

The first part of the following corollary results from a combination of Theorem H(I,II) with Proposition E; the second part is an immediate consequence of that proposition and Theorem H(III).

**Corollary I.** Let $(M_i)_{i \in I}$ be a family of indecomposable $R$-modules and $(M_t)_{t \in T}$ a transversal of the isomorphism classes of the $M_i$.

(I) Then the following statements are equivalent:

1. $\bigoplus_{i \in I} M_i$ is endo-noetherian, and $(M_t)_{t \in T}$ is right $T$-nilpotent.
2. $T$ is finite, and each $M_t$ is endofinite.

(II) Suppose that $\bigoplus_{i \in I} M_i$ is $\Sigma$-algebraically compact, $(M_t)_{t \in T}$ left semi-$T$-nilpotent,
and

\[(\bullet) \text{ for each cofinite subset } T' \subseteq T, \text{ the endosocle of } \bigoplus_{t \in T'} M_t \text{ is finitely generated.}\]

Then \( T \) is finite. \( \square \)

Remarks. None of the semi-T-nilpotence conditions in the various parts of Theorem H and Corollary I are redundant, not even when all of the \( M_i \) are finitely generated over a finite dimensional algebra – see Examples O below.

As for condition \((\bullet)\) in Theorem H(III) and Corollary I(II): Dropping it means jeopardizing the conclusion in either case, as is illustrated by Example J(a) below. In case the individual modules \( M_i \) have endosocles of finite length, condition \((\bullet)\) can clearly be weakened to the requirement that, for each cofinite subset \( T' \subseteq T \), the endosocle of \( \bigoplus_{t \in T'} M_t \) has finite support. We do not know whether, in the above statements, condition \((\bullet)\) can always be relaxed in this fashion, but point out that finitely generated endosocles are not automatic in \( \Sigma \)-algebraically compact modules (see Example J(b)).

Examples J. (a) The direct sum \( M = \bigoplus_p \mathbb{Z}(p\infty) \) is a \( \Sigma \)-algebraically compact \( \mathbb{Z} \)-module having as endomorphism ring the direct product of the rings of \( p \)-adic integers, where \( p \) runs through all primes. Moreover, the family \( (\mathbb{Z}(p\infty))_{p \text{ prime}} \) is clearly left semi-T-nilpotent and right T-nilpotent. However, the endosocle of \( M \) coincides with the \( \mathbb{Z} \)-socle, and thus has infinite support. This shows that the endosocle condition in Theorem H(III) and Corollary I(II) is crucial.

(b) Set \( R = K[X_n \mid n \in \mathbb{N}]/(X_n \mid n \in \mathbb{N})^k \), where \( K \) is a field and \( k \geq 2 \). Then the regular \( R \)-module is \( \Sigma \)-algebraically compact by [15]. On the other hand, its endosocle, \( \text{endosoc } R = \text{soc}_R R = J^{k-1} \), fails to be finitely generated.

Prime targets for applications of the following consequence of Theorem H are families of generic or finitely generated modules over Artin algebras. In light of the subsequent proposition addressing families of finitely generated modules over Artin algebras, the picture can be sharpened in that situation.

Corollary K. For any family \( (M_i)_{i \in I} \) of indecomposable endofinite modules the following statements are equivalent:

1. The number of isomorphism types of \( M_i \) is finite.
2. \( \bigoplus_{i \in I} M_i \) is endo-noetherian, and the family \( (M_i) \) is right semi-T-nilpotent.
3. \( \bigoplus_{i \in I} M_i \) is endo-artinian, and the family \( (M_i) \) is left semi-T-nilpotent.
4. If \((M_i)_{i \in T}\) is a transversal of the isomorphism types of the \( M_i \), then \( (M_i) \) is left and right semi-T-nilpotent and, for any cofinite subset \( T' \subseteq T \), the endosocle of \( \bigoplus_{t \in T'} M_t \) has finite support in \( T \).

Proof. In view of Theorem H, only the implication ‘(4) \( \Rightarrow \) (1)’ requires some justification: Since the \( M_i \) are endofinite, each singleton family \( \{M_i\} \) is right T-nilpotent by Proposition E(2), whence right T-nilpotence of the family \( (M_i) \) follows from right semi-T-nilpotence. \( \square \)

The next result exhibits a simplification of the picture for Artin algebras. Namely, a direct sum \( \bigoplus_{i \in I} M_i \) of finitely generated modules \( M_i \) is \( \Sigma \)-algebraically compact precisely
when it is endo-artinian. In other words, for Artin algebras, Proposition L provides an ‘endo-artinian counterpart’ to the final observation of Section 1: This observation remains true on replacing ‘endo-noetherian’ by ‘endo-artinian’, and substituting the ACC for (finite) matrix subgroups by the corresponding DCC.

Throughout our excursion to the narrowed setting, \( \Lambda \) will be an Artin algebra with center \( C \), and \( D : \Lambda \text{-mod} \to \text{mod-} \Lambda \) will denote the standard duality from the category of finitely generated left to the category of finitely generated right \( \Lambda \)-modules, i.e., \( D = \text{Hom}_C(-, E) \), where \( E \) is the \( C \)-injective envelope of \( C/J(C) \). Note that \( D \) induces a Morita duality on \( C \)-mod as well; this obvious fact is useful, because endo-submodules of \( \Lambda \)-modules are of course \( C \)-submodules.

**Proposition L.** If \( \Lambda \) is an Artin algebra and \( (M_i)_{i \in I} \) a family of finitely generated left \( \Lambda \)-modules, the following conditions are equivalent:

1. \( \bigoplus_{i \in I} M_i \) is \( \Sigma \)-algebraically compact.
2. \( \bigoplus_{i \in I} M_i \) is endo-artinian.
3. \( \bigoplus_{i \in I} D(M_i) \) is endo-noetherian.

**Proof.** To verify ‘(1) \( \implies \) (3)’, we adopt (1). This means that \( M = \bigoplus_{i \in I} M_i \) satisfies the ACC for finite matrix subgroups, whence, by Proposition 3 of [17], the dual \( D(M) = \prod_{i \in I} D(M_i) \) has the ACC for finite matrix subgroups. In view of the final observation of Section 1, this entails (3).

To prove ‘(3) \( \implies \) (2)’, we assume that \( \bigoplus_{i \in I} D(M_i) \) is endo-noetherian. We begin by noting that a \( C \)-submodule \( U \) of \( \bigoplus_{i \in I} M_i \) is an endo-submodule of \( \bigoplus_{i \in I} M_i \) if and only if \( U = \bigoplus_{i \in I} U_i \), where the \( U_i \) are submodules of the \( M_i \) with the property that, for all \( i, j \in I \) and all \( f \in \text{Hom}_C(M_i, M_j) \), the image \( f(U_i) \) is contained in \( U_j \). Now suppose that \( U \) is an endo-submodule of \( \bigoplus_{i \in I} M_i \), and, for \( i \in I \), let \( u_i : U_i \hookrightarrow M_i \) be the canonical embedding. Moreover, let \( V_i \subseteq D(M_i) \) denote the kernel of the dual map \( D(u_i) \); i.e., \( V_i = \{ \rho \in \text{Hom}_C(M_i, E) \mid \rho|_{U_i} = 0 \} \). We claim that \( V = \bigoplus_{i \in I} V_i \) is an endo-submodule of \( \bigoplus_{i \in I} D(M_i) \). Clearly, \( V \) is a \( C \)-submodule, whence it suffices to check stability under homomorphisms \( g \in \text{Hom}_C(D(M_i), D(M_j)) \). Suppose \( g = D(f) \) with \( f \in \text{Hom}_C(M_j, M_i) \). Dualizing the fact that \( f \cdot u_j = u_i \cdot (f|_{U_j}) \), we obtain, for each \( \rho \in V_i \), the equality \( g(\rho)|_{U_j} = (\rho \cdot f)|_{U_j} = (\rho|_{U_i}) \cdot (f|_{U_j}) = 0 \); in other words, \( g(\rho) \) belongs to \( V_j \) as required.

Let \( U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots \) be a descending chain of endo-submodules of \( \bigoplus_{i \in I} M_i \), and label the embedding \( U_{k+1} \hookrightarrow U_k \) by \( \phi_k \). Extending the above notation, we moreover write the embedding \( U_{k_i} = M_i \cap U_k \hookrightarrow M_i \) as \( u_{ki} \), and the kernel of the \( C \)-module epimorphism \( D(u_{ki}) \) as \( V_{ki} \). The preceding paragraph shows that each of the direct sums \( V_k = \bigoplus_{i \in I} V_{ki} \) is in fact an endo-submodule of \( \bigoplus_{i \in I} D(M_i) \). Clearly, the \( V_k \) form an ascending chain which, by our hypothesis, becomes stationary. This means that the ascending chains \( V_{1i} \subset V_{2i} \subset \cdots \) become uniformly stationary on \( I \), and, in view of the exact sequences

\[
0 \to V_{ki} \to D(M_i) \to D(U_{ki}) \to 0,
\]

we see that it is again uniformly on \( I \) that the maps \( D(\phi_k|_{U_{ki}}) - \) and hence also the \( \phi_k|_{U_{ki}} - \) become isomorphisms. This proves our initial descending chain \( (U_k)_{k \in \mathbb{N}} \) to become stationary.
In view of the characterization of $\Sigma$-algebraic compactness quoted in Section 1, the implication ‘(2) $\Rightarrow$ (1)’ is clear. □

The next corollary results from a combination of points K and L with the Harada-Sai Lemma [12].

**Corollary M.** Suppose $\Lambda$ is an Artin algebra and $(M_i)_{i \in I}$ a family of left $\Lambda$-modules having uniformly bounded (finite) composition lengths. Moreover, let $(M'_t)_{t \in T}$ be a transversal of the isomorphism types of the indecomposable direct summands of the $M_i$. Then the following conditions are equivalent:

1. $T$ is finite.
2. $\bigoplus_{i \in I} M_i$ is endo-noetherian.
3. $\bigoplus_{i \in I} M_i$ is endo-artinian.
4. $\bigoplus_{i \in I} M_i$ is $\Sigma$-algebraically compact.
5. For any cofinite subset $T' \subseteq T$, the endosocle of $\bigoplus_{t \in T'} M'_t$ has finite support.

**Proof.** It suffices to observe that, due to the Harada-Sai Lemma, our blanket hypothesis forces the family $(M'_t)$ to be both left and right T-nilpotent. □

The final corollary in this series highlights the role played by the endosocles of direct sums of indecomposable representations. It is an immediate consequence of Corollary M and the (confirmed) second Brauer-Thrall conjecture.

**Corollary N.** For any finite dimensional algebra $\Lambda$ over an algebraically closed field, the following conditions are equivalent:

1. $\Lambda$ has finite representation type.
2. For every family $(M_i)_{i \in I}$ of pairwise non-isomorphic indecomposable modules of fixed finite dimension, the endosocle of $\bigoplus_{i \in I} M_i$ has finite support. □

The following examples attest to the fact that none of the T-nilpotence conditions in Theorem H and Corollaries I, K can be dropped without penalty.

**Examples O.** Let $\Lambda = K\Gamma$ be the Kronecker algebra over a field $K$, where $\Gamma$ is the quiver

![Quiver Diagram](image)

(1) Moreover, let $(M_n)_{n \in \mathbb{N}}$ be the family of all preinjective left $\Lambda$-modules, i.e., $M_1 = \Lambda e_1/J(\Lambda)e_1$, and for $n \geq 2$, $M_n$ is the string module with graph

![String Module Diagram](image)
having \( n \) summands \( S_i \) in its top. Then the direct sum \( \bigoplus_{n \in \mathbb{N}} M_n \) is \( \Sigma \)-algebraically compact by [19, Theorem 4.6], and Proposition L guarantees that \( \bigoplus_{n \in \mathbb{N}} M_n \) is even endo-artinian.

(2) Let \( \Lambda \) still be the Kronecker algebra, but this time consider the family \( (N_n)_{n \in \mathbb{N}} \) of all preprojective left \( \Lambda \)-modules, i.e., \( N_n = D(M'_n) \), where \( M'_n \) is the preinjective right \( \Lambda \)-module with top dimension \( n \). Since \( \bigoplus_{n \in \mathbb{N}} M'_n \) is endo-artinian (see (1)), the direct sum \( \bigoplus_{n \in \mathbb{N}} N_n \) of the duals of the \( M'_n \) is endo-noetherian, again by Proposition L.

At this point, we return to an arbitrary base ring. In light of our remarks concerning the exchange property of direct sums (in Section 3), we finally obtain the following consequence of Theorem H.

**Corollary P.** An endo-noetherian direct sum \( \bigoplus_{i \in I} M_i \) of indecomposable modules \( M_i \) (over any ring \( R \)) has the exchange property if and only if each \( M_i \) has local endomorphism ring and the \( M_i \) fall into finitely many isomorphism classes. \( \square \)

Viewing the Kronecker examples in light of this fact, we see that the direct sum of the preprojective modules over the Kronecker algebra fails to have the exchange property. On the other hand, the direct sum ranging over the preinjective modules does enjoy this property, as do all algebraically compact modules (see [16]).

Variants of the ACC and DCC for endo-submodules have made numerous appearances in the representation theory of Artin algebras. Our concluding remarks, relating ascending endo-chain conditions of direct sums to preprojective partitions and left almost split morphisms, are merely the ‘tip of the iceberg’. (Naturally, certain relaxed descending endo-chain conditions are linked up with preinjective partitions and the existence of right almost split maps in a dual fashion.)

**Remarks.**

Due to [3] and [13], respectively, we have the following connections between weakened endo-noetherian conditions for direct sums \( \bigoplus_{i \in I} M_i \) of finitely generated indecomposable modules on one hand and the existence of preprojective partitions and left almost split morphisms, are merely the ‘tip of the iceberg’. (Naturally, certain relaxed descending endo-chain conditions are linked up with preinjective partitions and the existence of right almost split maps in a dual fashion.)

Recall from [3] that a **preprojective partition** of \( (M_i)_{i \in I} \) is a partition

\[
\{M_t \mid t \in T\} = \bigsqcup_{n \leq \omega} \mathcal{C}_n
\]

such that, for all \( n < \omega \), the set \( \mathcal{C}_n \) is a minimal finite generating set for \( \bigsqcup_{n \leq m \leq \omega} \mathcal{C}_m \). A **strong preprojective partition** extends this principle to arbitrary ordinals, i.e., it is a partition \( \{M_t \mid t \in T\} = \bigsqcup_{\alpha \leq \gamma} \mathcal{C}_\alpha \), where \( \gamma \) is an ordinal number and each \( \mathcal{C}_\alpha \) is a minimal finite generating set for \( \bigsqcup_{\alpha \leq \beta \leq \gamma} \mathcal{C}_\beta \). If the \( M_i \) have perfect endomorphism rings, such a (strong) preprojective partition is unique in case of existence. The existence of a preprojective partition of \( (M_i) \), on the other hand, is equivalent to the following endo-condition for \( \bigoplus_{i \in I} M_i \): For each cofinite subset \( T_1 \subseteq T \), there exists a cofinite subset
$T_2 \subseteq T_1$ such that
\[ \bigoplus_{t \in T_1} M_t \subseteq \left( \text{End}_R\left( \bigoplus_{t \in T} M_t \right) \right) \cdot \left( \bigoplus_{t \in T_1 \setminus T_2} M_t \right). \]

Moreover, the condition that $\bigoplus_{i \in I} M_i$ be endo-noetherian implies the existence of strong preprojective partitions for arbitrary subfamilies of $(M_i)_{i \in I}$.

The connection between endofiniteness conditions and strong preprojective partitions was recently refined by Dung [8, 3.9 and 3.11] for the case where the $M_i$ have perfect endomorphism rings. Namely, he established the following bridges among the conditions labeled (a), (b), (c) below:

(a) $\text{Hom}_R(M_k, \bigoplus_{i \in I} M_i)$ is noetherian as a left module over $\text{End}_R(\bigoplus_{i \in I} M_i)$ for all $k \in I$.

(b) For every subfamily $C$ of $(M_i)$, the full subcategory $\text{add} C$ of the category of all finitely generated modules has left almost split morphisms.

(c) Every subfamily of $(M_i)$ has a strong preprojective partition.

Under the given hypotheses for the $M_i$, condition (a) implies (b). If, moreover, the direct sum $\bigoplus_{i \in I} M_i$ is finitely generated over its endomorphism ring, then (b) implies (c).

References

1. L. Angeleri Hügel, Direct summands of products, Preprint.
2. M. Auslander, Representation theory of Artin Algebras II, Commun. in Algebra 1 (1974), 293-310.
3. M. Auslander and S. O. Smalø, Preprojective modules over Artin algebras, J. Algebra 66 (1980), 61-122.
4. J.-E. Bjørk, Rings satisfying a minimum condition on principal ideals, J. reine angew. Math. 236 (1969), 112-119.
5. J. H. Cozzens, Homological properties of the ring of differential polynomials, Bull. Amer. Math. Soc. 76 (1970), 75-79.
6. W. W. Crawley-Boevey, Tame algebras and generic modules, Proc. London Math. Soc. 63 (1991), 241-264.
7. , Modules of finite length over their endomorphism ring, in Representations of Algebras and Related Topics (S. Brenner and H. Tachikawa, Eds., eds.), London Math. Soc. Lec. Note Series 168, Cambridge Univ. Press, Cambridge, 1992, pp. 127-184.
8. N. V. Dung, Strong preinjective partitions and almost split morphisms, J. Pure Appl. Algebra 158 (2001), 131-150.
9. L. Gruson and C. U. Jensen, Deux applications de la notion de L-dimension, C. R. Acad. Sci Paris, Sér. A 282 (1976), 23-24.
10. , Dimensions cohomologiques reliées aux foncteurs $\lim^{(i)}$, in Sém. d’Algèbre P. Dubreil et M.-P. Malliavin, Lecture Notes in Math. 867, Springer-Verlag, Berlin, 1981, pp. 234-249.
11. M. Harada, Factor categories with applications to direct decomposition of modules, Lecture Notes in Pure and Applied Math. 88, Dekker, New York, 1983.
12. M. Harada and Y. Sai, On categories of indecomposable modules. I, Osaka J. Math. 7 (1970), 323-344.
13. B. Huisgen-Zimmermann, Strong preinjective partitions and representation type of artinian rings, Proc. Amer. Math. Soc. 109 (1990), 309-322.
14. , Purity, algebraic compactness, direct sum decompositions, and representation type, in Proc. Euroconf. on Infinite Length Modules (Bielefeld 1998) (H. Krause, H. Lenzing and C. M. Ringel, eds.).
15. B. Huisgen-Zimmermann and W. Zimmermann, Algebraically compact rings and modules, Math. Zeitschr. 161 (1978), 81-93.
16. , Classes of modules with the exchange property, J. Algebra 88 (1984), 416-434.
17. ______, *On the sparsity of representations of rings of pure global dimension zero*, Trans. Amer. Math. Soc. 320 (1990), 695-711.

18. H. Krause and M. Saorín, *On minimal approximations of modules*, in Trends in the Representation Theory of Finite Dimensional Algebras (E.L. Green and B. Huisgen-Zimmermann, Eds.), Contemp. Math. 229 (1998), 227-236.

19. H. Lenzing, *Homological transfer from finitely presented to infinite modules*, in Abelian group theory, Proc. Honolulu 1982/83, Lecture Notes in Math. 1006, Springer-Verlag, Berlin, 1983, pp. 734-761.

20. M. Prest, *Duality and pure semisimple rings*, J. London Math. Soc. 38 (1988), 403-409.

21. C. M. Ringel, *On generic modules for string algebras*, Preprint.

22. B. T. Stenström, *Rings of Quotients*, Grundlehren der math. Wissenschaften 217, Springer-Verlag, Berlin, 1975.

23. W. Zimmermann, *Rein-injektive direkte Summen von Moduln*, Habilitationsschrift, Universität München, 1975.

24. ______, *Rein-injektive direkte Summen von Moduln*, Commuc. in Algebra 5 (1977), 1083-1117.

25. ______, *($\Sigma$-) algebraic compactness of rings*, J. Pure Appl. Algebra 23 (1982), 319-328.

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