Application of Hardy inequalities for some singular parabolic equations

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Abstract

Boundary value problems for non-linear parabolic equations with singular potentials are considered. Existence and non-existence results as an application of different Hardy inequalities are proved. Blow-up conditions are investigated too.

Key words. Singular parabolic equations, Hardy inequalities.

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1 Introduction

We consider boundary value problem for the singular p-heat equation

\[ \begin{cases} u_t - \Delta_p u = \mu W(x)|u|^{p-2}u, & \text{for } x \in \Omega, t > 0, \\ u(x, 0) = u_0(x), & \text{for } x \in \Omega, \\ u(x, t) = 0, & \text{for } x \in \partial \Omega, t > 0. \end{cases} \tag{1.1} \]

Here \( \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) \) is the p-Laplacian, \( \mu = \text{const} \in \mathbb{R}, p > 1, \Omega \) is a bounded domain in \( \mathbb{R}^n, n \geq 2, 0 \in \Omega \) and function \( W(x) \) is singular on \( \partial \Omega \) or/and at 0.

More precisely we consider the following functions \( W(x) \)

\[ i) \quad W(x) = d^{-p}(x), d(x) = \text{dist}(x, \partial \Omega), p > 1, p \neq n; \]
\[ ii) \quad W(x) = d^{-p}(x) + \frac{p}{2(p-1)} d^{-p} \log^{-2} \frac{d(x)}{D}, d(x) = \text{dist}(x, \partial \Omega), p > n, \]
\[ D = \sup_{x \in \Omega} d(x); \]
\[ iii) \quad W(x) = \left( |x| \log \frac{R}{|x|} \right)^{-n} \text{ in } \Omega \subset B_R = \{|x| < R\}, R > e \sup_{x \in \Omega} |x|; \]
\[ iv) \quad W(x) = |x|^{m-n} |\varphi^m(x) - |x|^m|^{-p} \text{ in } \Omega = \{|x| < \varphi(\theta)\}, p > n, m = \frac{p-n}{p-1}, \]
\[ \theta = \frac{x}{|x|} \text{ is the angular variable of } x, \text{ and } \Omega \text{ is star-shaped with respect to an interior ball centered at the origin}, \varphi \in C^{0,1}. \]

The interest to parabolic problems with singular potentials is due to the applications, for example in molecular physics, see Lévy-Leblond [1], quantum cosmology, see Berestycki

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and Esteban [2], quantum mechanics, see Baras and Goldstein [3], electron capture problems, see Giri et al. [4], porous medium of fluids, see Ansini and Giacomelli [5] and the combustion models, see Gelf’and [6]. Also in some reaction-diffusion problems involving the heat equation with critical reaction term, the regularized equation is of the type (1.1).

In the remarkable pioneering paper Baras and Goldstein [3] is proved that problem (1.1) with Hardy potential

\[ W(x) = \frac{1}{|x|^2} \]

and \( p = 2 \) has a global solution for \( \mu \leq \left( \frac{n-2}{2} \right)^2 \).

Moreover if \( \mu > \left( \frac{n-2}{2} \right)^2 \) then the solution of (1.1) blows-up for a finite time.

The results of Baras and Goldstein are extended in different directions. In Vázquez and Zuazua [7] the authors present a complete description of the functional framework in which it is possible to obtain well-posedness for the singular heat equation.

The singular parabolic equation

\[
\begin{cases}
  u_t = \Delta u + \frac{\nabla \rho}{\rho} \cdot \nabla u + \frac{\mu}{|x|^2} u, & \text{in } \mathbb{R}^n \times \mathbb{R}^+, \\
  u(x, 0) = u_0(x) \geq 0, & \text{for } x \in \mathbb{R}^n,
\end{cases}
\]

with Kolmogorov operator is studied in Canale et al. [8], where \( \rho \) is probability density on \( \mathbb{R}^n \).

The quasilinear case

\[
\begin{cases}
  u_t - \text{div} (a(x, t, \nabla u)) = \frac{\mu u}{|x|^2} + f(x, t), & \text{in } \Omega \times (0, T), \\
  u(x, 0) = u_0(x) \geq 0, & \text{for } x \in \Omega, \\
  u(x, t) = 0, & \text{for } (x, t) \in \partial \Omega \times (0, T),
\end{cases}
\]  

(1.3)

is considered in Porzio [9]. In the problem (1.3) function \( W(x) = |x|^{-2} \) is Hardy potential and the asymptotic behavior of the solutions of (1.3) when \( f \neq 0 \) is investigated.

Another generalization of Baras and Goldstein [3] is given in Azorero and Alonso [10] for problem (1.1) with general Hardy potential \( W(x) = |x|^{-p} \), \( 1 < p < n \), \( n \geq 3 \) and \( 0 \in \Omega \). The authors prove existence and nonexistence results when \( \mu \leq \left( \frac{n-p}{p} \right)^p \) or \( \mu > \left( \frac{n-p}{p} \right)^p \) respectively. Here \( \left( \frac{n-p}{p} \right)^p \) is the optimal Hardy constant in the corresponding Hardy inequality.

In Junqiang et al. [11] the more general problem

\[
\begin{cases}
  u_t - \Delta_p u = \frac{\mu u}{|x|^s} |u|^{q-2} u, & \text{in } \Omega \times (0, \infty), \\
  u(x, 0) = u_0(x) \geq 0, & \text{for } x \in \Omega, \\
  u(x, t) = 0, & \text{for } (x, t) \in \partial \Omega \times (0, \infty),
\end{cases}
\]  

(1.4)

with \( 1 < p < n \), \( 0 < s \leq p \leq q \leq \frac{n-s}{n-p} \), \( \Omega \subset \mathbb{R}^n \), \( 0 \in \Omega \) is investigated. By means of Sobolev–Hardy inequality

\[
\left( \int_{\Omega} |\nabla u|^p dx \right)^{q/p} \geq C_{n,p,q,s} \int_{\Omega} \frac{|u|^q}{|x|^s} dx, \quad u \in W_0^{1,p}(\Omega),
\]

existence of weak, very weak and entropy solutions to (1.4) are proved for \( \mu \leq C_{n,p,q,s} \) and finite blow-up for \( \mu > C_{n,p,q,s} \).
In Abdellaoui et al. [12] where singular reaction–diffusion equation for p-Laplace operator with diffusion and reaction term

\[
\begin{cases}
  (u_t^\theta - \Delta_p u = \frac{\mu |u|^{p-1}}{|x|^p} + u^q + f, & \text{in } \Omega \times (0, T), \quad u \geq 0 \\
u(x, 0) = u_0(x) \geq 0, & \text{for } x \in \Omega, \\
u(x, t) = 0, & \text{for } (x, t) \in \partial \Omega \times (0, T),
\end{cases}
\]

is investigated. Here \(\theta = 1\) or \(\theta = p - 1\), \(f(x) \geq 0\), \(1 < p < n\), \(q > 0\). The authors prove existence of a critical exponent \(q^+ (p, \lambda)\) such that for \(q < q^+ (p, \lambda)\) there are global solutions to (1.5) and for \(q > q^+ (p, \lambda)\) there is no solution. In the semilinear case of (1.5) existence of a critical exponent \(q^+ (\lambda)\) of Fujita type, see Fujita [13] is stated and existence of a solution is proved if and only if \(q < q^+ (\lambda)\).

Without a presence of a reaction term, for \(\theta = 1\) and \(\mu = 0\), the one dimensional case of (1.5) without reaction term is studied in Esteban and Vázquez [16].

Finally, let us mention the results in Biccari and Zuazua [17] for the problem

\[
\begin{cases}
  u_t - \Delta u = \frac{\mu}{d^2(x)} u + f, & \text{in } \Omega \times (0, T), \\
u(x, 0) = u_0(x), & \text{for } x \in \Omega, \\
u(x, t) = 0, & \text{for } (x, t) \in \partial \Omega \times (0, T),
\end{cases}
\]

which is singular on \(\partial \Omega\). For \(\mu \leq \frac{1}{4}\) the authors prove existence of a unique weak solution. If \(\mu > \frac{1}{4}\) then there is no control, which means that the blow-up phenomena cannot be prevented. Here \(\frac{1}{4}\) is the optimal constant in the corresponding Hardy inequality.

In the present paper we extend the result for (1.1) with \(p = 2\) in Baras and Goldstein [3] and for \(2 < p < n\) in Azorero and Alonso [10] when the Hardy potential \(W(x) = |x|^{-p}\) is singular at the origin, to more general singular parabolic equations with \(W(x)\) given in i) - iv) of (1.2) above. The new Hardy potentials are singular on the boundary of the domain or both at the origin and on the boundary. Let us recall that for the corresponding Hardy inequalities, the Hardy constants are optimal, see (2.1) - (2.4). We prove that for \(\mu\) less than the optimal Hardy constant problem (1.1) has a global weak solution. When \(\mu\) is greater than the optimal Hardy constant than the solution blows-up for a finite time. To our best knowledge these type of singular parabolic equations are not consider in the literature.

In Sect. 2 we recall some Hardy inequalities with weights \(W(x)\) and prove global existence of weak solutions of (1.1) while Sect. 3 deals with finite time blow-up of solutions to (1.1).

## 2 Existence of global solution

In this section we prove existence of a global, generalized solution of (1.1) when \(W(x)\) has one of the forms (1.2)i) - (1.2)iv) and \(\mu < C_p\) or \(\mu < C_{p,n}\) respectively.
For this purpose we recall some well-known Hardy’s inequalities which are important in the proof of the main results.

Inequality with kernel (1.2)i)
\[
\int_{\Omega} |\nabla u|^p dx \geq C_p \int_{\Omega} \left( \frac{|u|^p}{d^p(x)} \right) dx,
\]
for \( u \in W^{1,p}_0(\Omega) \), where \( C_p = \left( \frac{p-1}{p} \right)^p \), \( p > 1, n \neq p, n \geq 2 \), \( d(x) = \text{dist}(x, \partial \Omega) \). Here \( \Omega \) is a bounded \( C^2 \) smooth domain in \( \mathbb{R}^n \) with nonnegative mean curvature \( H(x) \geq 0 \). The constant \( C_p = \left( \frac{p-1}{p} \right)^p \) is optimal. Inequality (2.1) is proved in Lewis et al. [18], see Theorem 1.2.

Inequality with kernel (1.2)ii).
\[
\int_{\Omega} |\nabla u|^p dx \geq C \int_{\Omega} \frac{|u|^p}{d^p(x)} \left( 1 + \frac{1}{2} \left( \frac{p}{p_n - 1} \right) \left( \log^{-2} \frac{d(x)}{D} \right) \right) dx,
\]
for \( u \in W^{1,p}_0(\Omega) \), where \( D \geq \sup_{x \in \Omega} d(x), p > n \geq 2 \) and \( \Omega \subset \mathbb{R}^n \) is bounded domain with \( C^2 \) smooth boundary, satisfying some geometric conditions like the mean convexity of \( \partial \Omega \). The constant \( C_p = \left( \frac{p-1}{p} \right)^p \) is optimal. Inequality (2.2) is proved in Barbatis et al. [19], see Theorem A.

Inequality with kernel (1.2)iii).
\[
\int_{\Omega} |\nabla u|^n dx \geq C_n \int_{\Omega} \frac{|u|^n}{\left( |x| \log \frac{R}{|x|} \right)^n} dx,
\]
for \( u \in W^{1,n}_0(\Omega) \), where \( R \geq \epsilon \sup_{x \in \Omega} |x| \), \( \Omega \subset \mathbb{R}^n, n \geq 2, 0 \in \Omega, \Omega \) is a bounded domain, see Theorem 1.1 in Adimurthi and Esteban [20]. Inequality (2.3) for \( \Omega = B_1 = \{|x| < 1\} \) is proved in Ioku and Ishiwata [21]. The constant \( C_n = \left( \frac{n-1}{n} \right)^n \) is optimal;

Inequality with kernel (1.2)iv).
\[
\int_{\Omega} |\nabla u|^p dx \geq C_{p,n} \int_{\Omega} \frac{|u|^p}{|x|^{n-m} |\varphi^m(x)| - |x|^m} dx,
\]
for \( u \in W^{1,p}_0(\Omega) \), where \( p > n, n \geq 2, \Omega = \{|x| < \varphi(\theta)\} \subset \mathbb{R}^n, \theta = \frac{x}{|x|} \) is angular variable of \( x \) and \( \Omega \) is a star shape domain with respect to a small ball centered at the origin, \( \varphi \in C^{0,1} \). The constant \( C_{p,n} = \left( \frac{p-n}{p} \right)^p \) is optimal, see Fabricant et al. [22] and Kutev and Rangelov [23], Theorem 5.1.

By means of Hardy inequalities (2.1) - (2.4) we have the following results.

**Theorem 2.1.** Suppose \( \Omega \) is a bounded \( C^2 \) smooth domain in \( \mathbb{R}^n, n \geq 2 \), with nonnegative mean curvature \( H(x) \geq 0 \) and \( p > 1, p \neq n \). Then if \( \mu < \left( \frac{p-1}{p} \right)^p \), \( u_0(x) \in L^2(\Omega) \) problem (1.1) with \( W(x) \) in (1.2)i) has a global solution
\[
u(x,t) \in L^\infty ((0,\tau), L^2(\Omega)) \cup L^p ((0,\tau), W^{1,p}(\Omega)), \quad \text{for all } \tau > 0. \tag{2.5}
\]
Theorem 2.2. Suppose \( \Omega \) is a bounded convex \( C^2 \) smooth domain in \( \mathbb{R}^n, n \geq 2 \). Then if \( \mu < \left( \frac{p-1}{p} \right)^p, p > 1, p \neq n \), \( u_0(x) \in L^2(\Omega) \), problem (1.1) with \( W(x) \) defined in (1.2)ii) has a global solution satisfying (2.5).

Theorem 2.3. Suppose \( \Omega \) is a bounded domain in \( \mathbb{R}^n, n \geq 2 \), \( 0 \in \Omega \). Then if \( \mu < \left( \frac{n-1}{n} \right)^n \), \( u_0(x) \in L^2(\Omega) \), problem (1.1) with \( W(x) \) defined in (1.2)iii) has a global solution satisfying (2.5).

Theorem 2.4. Suppose \( \Omega = \{ |x| < \varphi(x) \} \subset \mathbb{R}^n, n \geq 2 \), is a star-shaped domain with respect to a small ball centered at the origin. Then if \( \mu < \left( \frac{p-n}{p} \right)^p, p > n \), \( u_0(x) \in L^2(\Omega) \), problem (1.1) with \( W(x) \) given by (1.2)iv) has a global solution, satisfying (2.5).

Proof of Theorems 2.1 - 2.4. In order to prove the Theorems 2.1 - 2.4 we consider the truncated function

\[
W_N(x) = \min\{N,W(x)\}, N = 1, 2, \ldots
\]  

(6.2)

where \( W(x) \) is one of the functions (1.2)i) - (1.2)iv). Let \( u_N(x,t) \) be the solution of the truncated problem

\[
\begin{align*}
  u_{N,t} - \Delta_p u_N &= \mu W_N(x)|u_N|^{p-2} u_N, \quad \text{for } x \in \Omega, t > 0, \\
  u_N(x,0) &= u_0(x) \quad \text{for } x \in \Omega, \\
  u_N(x,t) &= 0 \quad \text{for } x \in \partial \Omega, t > 0.
\end{align*}
\]

(7.2)

Multiplying (7.2) with \( u_N \) and integrating by parts we get from the Hardy’s inequalities (2.1) - (2.4) the following estimates for every \( T > 0 \)

\[
\begin{align*}
  \int_{\Omega} |u_N(x,T)|^2 dx + \int_0^T \int_{\Omega} |\nabla u_N(x,t)|^p dxdt &= \int_{\Omega} |u_0(x)|^2 dx + \mu \int_0^T \int_{\Omega} W_N(x)|u_N(x,t)|^p dxdt \\
  &\leq \int_{\Omega} |u_0(x)|^2 dx + \mu C^{-1} \int_0^T \int_{\Omega} |\nabla u_N(x,t)|^p dxdt,
\end{align*}
\]

(8.2)

where

\[
C = \begin{cases} 
  C_p = \left( \frac{p-1}{p} \right) & \text{in Theorems 2.1 and 2.2}, \\
  C_n = \left( \frac{n-1}{n} \right) & \text{in Theorem 2.3}, \\
  C_{n,p} = \left( \frac{p-n}{p} \right) & \text{in Theorem 2.4}.
\end{cases}
\]

(9.2)

Since \( \mu C^{-1} < 1 \) we get from (8.2) the energy estimate

\[
\int_{\Omega} |u_N(x,T)|^2 dx + (1 - \mu C^{-1}) \int_0^T \int_{\Omega} |\nabla u_N(x,t)|^p dxdt \leq \int_{\Omega} |u_0(x)|^2 dx.
\]

From the comparison principle \( u_N(x,t) \) is a nondecreasing sequence of functions because \( W_N(x,t) \geq W_M(x,t) \) for every \( x \in \Omega, t > 0 \) and \( N \geq M \). We can pass to the limit \( N \to \infty \) by using Theorem 4.1 in Boccardo and Murat [24]. Thus the global solution \( u(x,t) \) of (1.1) is defined as a limit of the solution \( u_N(x,t) \) of the truncated problem (7.2) and \( u(x,t) \) has the regularity properties given in (2.5). \( \square \)
3 Finite time blow-up

In this section we prove finite time blow-up of the solutions to the problem (1.1) with $u_0(x) > 0$ for $x \in \Omega$, i.e.,

$$
\begin{align*}
\left\{ \begin{array}{l}
u_t - \Delta_p u &= \mu W(x)|u|^{p-2}u, \quad \text{for } x \in \Omega, t > 0, \\
u(x, 0) &= u_0(x) > 0, \quad \text{for } x \in \bar{\Omega}, \\
u(x, t) &= 0, \quad \text{for } x \in \partial \Omega, t > 0,
\end{array} \right.
\end{align*}
$$

(3.1)

for $W(x)$ defined in (1.2)i) - (1.2)iv).

If $\lambda_{1N}, \phi_{1N}(x)$ are the first eigenvalue and the first eigenfunction of the problem

$$
\begin{align*}
\left\{ \begin{array}{l}
-\Delta_p \phi_{1N} &= \lambda_{1N} W_N(x)|\phi_{1N}|^{p-2} \phi_{1N}, \quad x \in \Omega, \\
\phi_{1N} &= 0, \quad x \in \partial \Omega,
\end{array} \right.
\end{align*}
$$

where $W_N(x)$ is defined in (2.6), then the following result holds.

**Lemma 3.1.** Suppose $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 2$ and $\lambda > C$, where $C$ is given in (2.9). Then we have

$$
\lambda_{1N} \geq C, \quad N = 1, 2, \ldots
$$

(3.2)

$$
\lim_{N \to \infty} \lambda_{1N} = C
$$

(3.3)

**Proof.** Inequality (3.2) follows from the Rayleigh quotient and Hardy inequalities (2.1) - (2.4). Indeed

$$
\lambda_{1N} = \inf_{v \in W_0^{1,p}(\Omega)} \frac{\int_{\Omega} |\nabla v|^p dx}{\int_{\Omega} W_N(x)|v|^p dx} \geq \inf_{v \in W_0^{1,p}(\Omega)} \frac{\int_{\Omega} |\nabla v|^p dx}{\int_{\Omega} W(x)|v|^p dx} \geq C
$$

In order to proof (3.3) we suppose by contradiction that

$$
\lim_{N \to \infty} \lambda_{1N} = C + 2\delta, \quad \text{for some} \ \delta > 0.
$$

(3.4)

From the optimality of the constant $C$ in Hardy inequalities (2.1) - (2.4) there exists $v_N \in W_0^{1,p}(\Omega)$ such that

$$
\frac{\int_{\Omega} |\nabla v_N|^p dx}{\int_{\Omega} W_N(x)|v_N|^p dx} < C + \delta
$$

Thus $\lambda_{1N} \leq C + \delta$ for every $N = 1, 2, \ldots$ which contradicts (3.4). \hfill \Box

We will construct a positive sub-solution to (3.1) by the method of separation of variables, i.e., a sub-solution $v(x, t) = X(x)T(t)$ where $X(x)$ and $T(t)$ satisfy the problems.

$$
\begin{align*}
\left\{ \begin{array}{l}
-\Delta_p X(x) - \mu W_N(x)X^{p-1}(x) &= -X(x), \quad x \in \Omega, \\
X(x) &= 0, \quad x \in \partial \Omega,
\end{array} \right.
\end{align*}
$$

(3.5)

$$
\begin{align*}
\left\{ \begin{array}{l}
T'(t) &= T^{p-1}(t), \quad t > 0, \\
T(0) &= T_0.
\end{array} \right.
\end{align*}
$$

(3.6)

Since $T(t) = T_0 \left[1 - (p - 2)T_0^{p-2}t\right]^{-\frac{1}{p-2}}$ for $p > 2$, then $T(t)$ blows-up for $t = \left[(p - 2)T_0^{p-2}\right]^{-1}$. 6
For $a^{p-2} = \mu$, after the change of the functions $aX = Y$, problem (3.5) becomes
\[
\begin{align*}
-\Delta_p Y(x) &= \mu (W_N(x) Y^{p-1}(x) - Y(x)), \quad x \in \Omega, \\
Y(x) &= 0, \quad x \in \partial \Omega.
\end{align*}
\]
(3.7)

We need the following lemma (see Lemma 5.1 in Azorero and Alonso [10])

**Lemma 3.2.** (1) There exists a constant $A > 0$ such that if (3.7) has a positive solution $Y(\Omega)$ then $\|Y\|_{L^\infty(\Omega)} > A$;

(2) If $\lambda_{1N}$ is the first eigenvalue of $-\Delta_p$ with weight $W_N(x)$ then $\lambda_{1N}$ is the unique bifurcation point from infinity for the problem (3.7)

As it is mention in Azorero and Alonso [10] the unbounded bifurcation branch of positive solutions given in Lemma 3.2(2) cannot cross the level $\|Y\|_{L^\infty(\Omega)} = A$ from Lemma 3.2(1), neither, obviously, the hyperplane $\mu = 0$ for $\mu > \lambda_{1N}$. Hence, problem (3.7) has at least one positive solution for $\mu > \lambda_{1N}$, i.e., from (3.3) for $N > N_0$, $N_0$ sufficiently large.

**Theorem 3.1** (Comparison principle). Suppose $u(x,t)$, defined in (2.5) is solution of (2.7) and $v(x,t)$ is a positive sub-solution to (2.7), i.e.,
\[
\begin{align*}
v_t - \Delta_p v &\leq \lambda W_N(x) v^{p-1}, \quad \text{for } x \in \Omega, t \in [0,\tau), \\
v(x,0) &\leq u_0(x), \quad u_0(x) > 0 \quad \text{for } x \in \Omega, \\
v(x,t) &\leq 0 \quad \text{for } x \in \partial \Omega, t \in [0,\tau),
\end{align*}
\]
(3.8)

for a fixed $\tau$. Then
\[
v(x,t) \leq u(x,t), \quad \text{for } x \in \Omega, t \in [0,\tau).
\]
(3.9)

**Proof.** Suppose by contradiction that (3.9) fails, i.e., the set $Q = \Omega_+(s) \times [0,\tau)$, $\Omega_+(s) = \{x \in \Omega, s \in [0,\tau); v(x,s) > u(x,s)\}$ is not empty. Multiplying (3.8) and (2.7) with $(v-u)_+$

where
\[
(v-u)_+ = \begin{cases}
  v-u, & \text{for } v > u, \\
  0, & \text{for } v \leq u
\end{cases}, \quad (v-u)_+ \in W^{1,p}_0(\Omega \times [0,\tau),
\]

and integrating in $\Omega \times [0,t], t < \tau$ we get for their difference
\[
0 \geq \int_0^t \int_{\Omega_+(s)} (v-u)_t(v-u)_+ \, dx dt
\]
\[
- \int_0^t \int_{\Omega_+(s)} \text{div} \left[ |\nabla v|^{p-2} \nabla v - |\nabla u|^{p-2} \nabla u \right] (v-u)_+ \, dx dt
\]
\[
= \frac{1}{2} \int_\Omega (v(x,t) - u(x,t))^2 dx - \frac{1}{2} \int_{\Omega_+(0)} (v(x,0) - u(x,0))^2 dx
\]
\[
+ \int_0^t \int_{\Omega_+(s)} \left[ |\nabla v|^{p-2} v_x(v_x - u_x) - |\nabla u|^{p-2} u_x(v_x - u_x) \right] \, dx dt
\]
(3.10)

Simple computation give us
\[
|\nabla v|^{p-2} v_x(v_x - u_x) - |\nabla u|^{p-2} u_x(v_x - u_x)
\]
\[
= |\nabla v|^{p} - |\nabla v|^{p-2} v_x u_x + |\nabla u|^{p} - |\nabla u|^{p-2} v_x u_x
\]
\[
= \frac{1}{2} |\nabla v|^{p} + \frac{1}{2} |\nabla u|^{p-2} |\nabla (v-u)|^2 + \frac{1}{2} |\nabla u|^{p}
\]
\[
+ \frac{1}{2} |\nabla u|^{p-2} |\nabla (v-u)|^2 - \frac{1}{2} |\nabla v|^{p-2} |\nabla u|^{2} - \frac{1}{2} |\nabla u|^{p-2} |\nabla v|^{2}
\]
\[
= \frac{1}{2} \left[ |\nabla v|^{p-2} + |\nabla u|^{p-2} \right] |\nabla (u - v)|^2
\]
\[
+ \frac{1}{2} \left[ |\nabla v|^{p-2} - |\nabla u|^{p-2} \right] \left[ |\nabla v|^2 - |\nabla u|^2 \right] \geq 0.
\]
(3.11)
Since
\[
\int_{\Omega^+(0)} [v(x,0) - u(x,0)]^2 \, dx = 0,
\]
from (3.10) and (3.11) we get for every \( t \in [0, \tau) \)
\[
\int_{\Omega^+(t)} [v(x,t) - u(x,t)]^2 \, dx = 0.
\]
Thus \( v(x,t) = u(x,t) \) for every \( x \in \Omega^+(t) \) and every \( t \in [0, \tau) \), i.e., for \( (x,t) \in Q \) and \( Q = \emptyset \) which contradicts our assumption. \( \square \)

The following theorem is the main result in this section.

**Theorem 3.2.** Suppose \( p > 2 \), \( u_0 \in L^\infty(\Omega) \), \( u_0(x) > 0 \) in \( \overline{\Omega} \) and \( \mu > C \), where \( C \) is given in (2.9). Then the solution of (2.7) blows-up for a finite time.

**Proof.** Let \( X(x) \) be the positive solution of (3.5) according to Lemma 3.2 and \( T(t) \) be the solution of (3.6) with \( T_0 = \varepsilon \), \( \varepsilon \) small enough such that
\[
\varepsilon X(x) \leq u_0(x), \quad \text{for } x \in \Omega \tag{3.12}
\]
Thus \( v(x,t) = X(x)T(t) \) is a positive solution of the problem
\[
\begin{cases}
  v_t - \Delta_p v = \mu W_N v^{p-1}, & \text{for } x \in \Omega, t \in (0, T_{\text{max}}), \\
  v(x,0) = \varepsilon X(x) \geq 0 & \text{for } x \in \Omega, \\
  v(x,t) = 0 & \text{for } x \in \partial \Omega, t \in (0, T_{\text{max}}).
\end{cases}
\tag{3.13}
\]
Here \( T_{\text{max}} \) is defined as
\[
T_{\text{max}} = \left[ (p-2)\varepsilon^{p-2} \right]^{-1},
\]
and \( [0, T_{\text{max}}) \) is the maximal existence time interval for the solution of (3.13).

From (3.12) it follows that \( v(x,t) \) is a positive sub-solution to (2.7).

According the comparison principle, Theorem 3.1, we get
\[
u N(x, t) \geq X(x)T(t) = \varepsilon X(x) \left[ 1 - (p-2)\varepsilon^{p-2} t \right]^{-1/(p-2)} \to \infty \quad \text{for } t \to \left[ (p-2)\varepsilon^{p-2} \right]^{-1}. \tag{3.14}
\]
Since the solutions of (1.1) are defined as the limit of the solutions \( u_N(x,t) \) of the truncated problem (2.7), then from (3.14) it follows that \( u(x,t) \) blows-up for a finite time. \( \square \)

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