Generalized Ricci solitons on $K$-contact manifolds

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Abstract
The object of the present paper is to study $K$-contact manifold admitting generalized Ricci solitons. We prove that a $K$-contact manifold of dimension $(2n+1)$ satisfying the generalized Ricci soliton equation is an Einstein one. Finally, we obtain several remarks.

Keywords: $K$-contact manifold; Generalized Ricci soliton; Einstein manifold.

AMS Subject Classification (2020): Primary: 53C15; 53C55.

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1. Introduction

Let $M$ be a $(2n+1)$-dimensional differentiable manifold. Suppose that $(\phi, \xi, \eta, g)$ is an almost contact metric structure on $M$. This means that $(\phi, \xi, \eta, g)$ is a quadruple consisting of a $(1,1)$-tensor field $\phi$, an associated vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ on $M$ satisfying the following relations

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.1}$$

where $X, Y$ are smooth vector fields on $M$. In addition, we have

$$\phi \xi = 0, \quad \eta(\phi X) = 0, \quad g(X, \xi) = \eta(X), \quad g(\phi X, Y) = -g(X, \phi Y). \tag{1.2}$$

An almost contact structure is said to be a contact structure if $g(X, \phi Y) = d\eta(X, Y)$. A contact metric structure is said to be normal if the induced almost complex structure $J$ on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by

$$J(X, f \frac{d}{dt}) = (\phi X - f \xi, \eta(X) \frac{d}{dt})$$

is integrable where $X$ is tangent to $M$, $t$ is the coordinate of $\mathbb{R}$ and $f$ is a smooth function on $M^{2n+1} \times \mathbb{R}$. A normal contact metric manifold is called a Sasakian manifold. If $\xi$ is a Killing vector field on a contact metric manifold $(M, g)$, then the manifold is called a $K$-contact metric manifold or simply a $K$-contact manifold ([1], [15]). An almost contact manifold is Sasakian [1], if and only if

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \tag{1.3}$$

where $\nabla$ is the Levi-Civita connection.

A complete regular contact metric manifold $M^{2n+1}$ carries a $K$-contact structure $(\phi, \xi, \eta, g)$, defined in terms of the almost Kähler structure $(J, G)$ of the base manifold $M^{2n+1}$. Here the $K$-contact structure $(\phi, \xi, \eta, g)$ is Sasakian if and only if the base manifold $(M^{2n+1}, J, G)$ is Kählerian. If $(M^{2n+1}, J, G)$ is only almost Kähler, then $(\phi, \xi, \eta, g)$ is only $K$-contact [1]. In a Sasakian manifold, the Ricci operator $Q$ commutes with $\phi$, that is, $Q\phi = \phi Q$. Recently in [11], it has been shown that there exist $K$-contact manifolds with $Q\phi = \phi Q$ which are not Sasakian. It is seen that $K$-contact structure is the intermediate between contact and Sasakian structure. $K$-contact manifolds have
been studied by several authors ([6], [7], [8], [14], [16], [18]) and many others. Given a smooth function $f$ on $M$, the gradient of $f$ is defined by

$$g(\text{grad } f, X) = Xf,$$  

(1.4)

the Hessian of $f$ is defined by

$$(\text{Hess } f)(X, Y) = g(\nabla_X \text{grad } f, Y),$$

(1.5)

for all smooth vector fields $X, Y$. For a smooth vector field $X$, we have ([12], [13])

$$X^b(Y) = g(X, Y).$$

(1.6)

The generalized Ricci soliton equation in a Riemannian manifold $(M, g)$ is defined by [13]

$$\mathcal{L}_X g = -2c_1 X^b \odot X^b + 2c_2 S + 2\lambda g,$$

(1.7)

where $\mathcal{L}_X g$ is the Lie derivative of $g$ along $X$ given by

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y),$$

(1.8)

for all smooth vector fields $X, Y, Z$ and $c_1, c_2, \lambda \in \mathbb{R}$. For different values of $c_1, c_2$ and $\lambda$, equation (1.7) is a generalization of Killing equation ($c_1 = c_2 = \lambda = 0$), equation for homotheties ($c_1 = c_2 = 0$), Ricci soliton ($c_1 = 0, c_2 = -1$), Vacuum near-horizon geometry equation ($c_1 = 1, c_2 = \frac{1}{2}$) etc. For more details we refer to the reader ([3], [4], [5], [9], [13]).

If $X = \text{grad } f$, then the generalized Ricci soliton equation is given by

$$\text{Hess } f = -c_1 df \odot df + c_2 S + \lambda g.$$  

(1.9)

### 2. Preliminaries

In an $(2n + 1)$-dimensional $K$-contact manifold, the following relations hold ([1], [17])

$$\nabla_X \xi = -\phi X,  

(2.1)$$

$$g(R(\xi, X)Y, \xi) = \eta(R(\xi, X)Y) = g(X, Y) - \eta(X)\eta(Y),  

(2.2)$$

$$R(\xi, X)\xi = -X + \eta(X)\xi,  

(2.3)$$

$$S(X, \xi) = 2n\eta(X),  

(2.4)$$

$$\nabla_X \phi Y = R(\xi, X)Y,  

(2.5)$$

for any vector fields $X, Y \in \chi(M)$. A $K$-contact manifold $M$ of dimension $\geq 3$ is said to be Einstein if its Ricci tensor $S$ is of the form $S = ag$, where $a$ is a constant.

In this case we have

$$S(X, Y) = ag(X, Y).$$

(2.6)

Substituting $X = Y = \xi$ in (2.6) and then using (2.4) and (1.2), we get

$$a = 2n.  

(2.7)$$

Thus using (2.7) we obtain from (2.6)

$$S(X, Y) = 2ng(X, Y).$$

(2.8)

Again from (2.8) we infer that

$$QX = 2nX.$$

(2.9)
3. Generalized Ricci soliton

In this section we characterize $K$-contact manifolds admitting generalized Ricci soliton. First we prove the following Lemma:

**Lemma 3.1.** Let $(M, \phi, \xi, \eta, g)$ be a $K$-contact manifold. Then

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y, \xi) = g(X, Y) + g(\nabla_\xi \nabla_\xi X, Y) + Yg(\nabla_\xi X, \xi),$$

for all smooth vector fields $X, Y$ with $Y$ orthogonal to $\xi$.

*Proof.* We have

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y, \xi) = \xi((\mathcal{L}_X g)(Y, \xi)) - (\mathcal{L}_X g)(\mathcal{L}_\xi Y, \xi) - (\mathcal{L}_\xi g)(\mathcal{L}_Y X, \xi)$$

Using (1.8) in (3.2) yields

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y, \xi) = \xi(g(\nabla_Y X, \xi) - g(\nabla_\xi \nabla_\xi X, Y) - Yg(\nabla_\xi X, \xi)).$$

Now by the definition of Riemannian curvature tensor, from (3.2) it follows that

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y, \xi) = g(\nabla_\xi \nabla_\xi X, Y) + Yg(\nabla_\xi X, \xi).$$

Using (2.2) in (3.3) and with $Y$ orthogonal to $\xi$, we infer that

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y, \xi) = g(X, Y) + g(\nabla_\xi \nabla_\xi X, Y) + Yg(\nabla_\xi X, \xi).$$

**Lemma 3.2.** [12] Let $(M, g)$ be a Riemannian manifold and let $f$ be a smooth function. Then

$$(\mathcal{L}_\xi(df \odot df))(Y, \xi) = Y(\xi(f))\xi(f) + Yf\xi(\xi(f)),$$

for every vector field $Y$.

**Lemma 3.3.** Let $(M, \phi, \xi, \eta, g)$ be a $K$-contact manifold which satisfies the generalized Ricci soliton equation. Then

$$\nabla_\xi \text{grad } f = (\lambda + 2c_2n)\xi - c_1\xi(\xi(f)) \text{grad } f.$$

*Proof.* Using (2.4) we have

$$\lambda \eta(Y) + c_2 S(\xi, Y) = (\lambda + 2c_2n)\eta(Y).$$

(3.4)

Making use of (1.9) and (3.4) implies

$$(\text{Hess } f)(\xi, Y) = -c_1\xi(f)g(\text{grad } f, Y) + (\lambda + 2c_2n)\eta(Y).$$

(3.5)

Hence the Lemma follows from (3.5) and the definition of the Hessian (1.9).

**Theorem 3.1.** Suppose that $(M, \phi, \xi, \eta, g)$ is a $K$-contact manifold of dimension $(2n + 1)$ which satisfies the generalized gradient Ricci soliton equation with $c_1(\lambda + 2c_2n) \neq -1$. Then $f$ is a constant function. Furthermore, if $c_2 \neq 0$, then the manifold is an Einstein one.
Proof. Suppose that \( Y \) is orthogonal to \( \xi \). Then from Lemma 3.1 with \( X = \text{grad} \, f \), we have

\[
2(\mathcal{L}_\xi(\text{Hess} \, f))(Y, \xi) = Y(f) + g(\nabla_\xi \nabla_\xi \text{grad} \, f, Y) + Yg(\nabla_\xi \text{grad} \, f, \xi). \tag{3.6}
\]

Using Lemma 3.3 in (3.6) yields

\[
2(\mathcal{L}_\xi(\text{Hess} \, f))(Y, \xi) = Y(f) + (\lambda + 2c_2n)g(\nabla_\xi \xi, Y)
- c_1g(\nabla_\xi (\xi(f) \text{grad} \, f), Y) + (\lambda + 2c_2n)Y - c_1Y(\xi(f)^2)
= Y(f) - c_1g(\nabla_\xi (\xi(f) \text{grad} \, f), Y) + (\lambda + 2c_2n)Y - c_1Y(\xi(f)^2). \tag{3.7}
\]

Again using Lemma 3.3 with \( Y \) orthogonal to \( \xi \), from (3.7) it follows that

\[
2(\mathcal{L}_\xi(\text{Hess} \, f))(Y, \xi) = Y(f) - c_1\xi(\xi(f))Y(f) + c_1^2\xi(f)^2Y(f)
- 2c_1\xi(f)Y(\xi(f)). \tag{3.8}
\]

Since \( \xi \) is a Killing vector field, so \( \mathcal{L}_\xi g = 0 \), it implies \( \mathcal{L}_\xi S = 0 \). Using the above fact and taking the Lie derivative to the generalized Ricci soliton equation (1.9) yields

\[
2(\mathcal{L}_\xi(\text{Hess} \, f))(Y, \xi) = -2c_1(\mathcal{L}_\xi(\text{df} \otimes \text{df}))(Y, \xi). \tag{3.9}
\]

Using (3.8), (3.9) and Lemma 3.2 we infer that

\[
Y(f)[1 + c_1\xi(\xi(f)) + c_1\xi(f^2)] = 0. \tag{3.10}
\]

According to Lemma 3.3 we have

\[
c_1\xi(\xi(f)) = c_1\xi g(\xi, \text{grad} \, f)
= c_1g(\xi, \nabla_\xi \text{grad} \, f)
= c_1(\lambda + 2c_2n) - c_1^2\xi(f)^2. \tag{3.11}
\]

Making use of (3.10) and (3.11), we obtain

\[
Y(f)[1 + c_1(\lambda + 2c_2n)] = 0,
\]

which implies

\[
Yf = 0,
\]

provided \( 1 + c_1(\lambda + 2c_2n) \neq 0 \). Therefore, \( \text{grad} \, f \) is parallel to \( \xi \). Hence \( \text{grad} \, f = 0 \) as \( d = \text{ker} \, \eta \) is nowhere integrable, that is, \( f \) is a constant function. Thus the manifold is an Einstein one follows from (1.9).

Remark 3.1. We know that [10] every Sasakian manifold is \( K \)-contact, but the converse is not true in general. However, a 3-dimensional \( K \)-contact manifold is Sasakian. Thus our main Theorem 3.1 is the generalization of Theorem 1.1 of [12].

Remark 3.2. Since a compact \( K \)-contact Einstein manifold is Sasakian [2], therefore a compact \( K \)-contact manifold admitting generalized Ricci solitons is Sasakian.

Acknowledgements

The authors are grateful to the referees for their valuable comments and suggestions.

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