Charged Conformal Killing Spinors

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Abstract

We study the twistor equation on pseudo-Riemannian $Spin^c$-manifolds whose solutions we call charged conformal Killing spinors (CCKS). We derive several integrability conditions for the existence of CCKS and study their relations to spinor bilinears. A construction principle for Lorentzian manifolds admitting CCKS with nontrivial charge starting from CR-geometry is presented. We obtain a partial classification result in the Lorentzian case under the additional assumption that the associated Dirac current is normal conformal and complete. The classification of manifolds admitting CCKS in all dimensions and signatures $\leq 5$ which has recently been initiated in the study of supersymmetric field theories on curved space.

Keywords: Twistor spinors, $Spin^c$-Geometry, Conformal Killing forms

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1. Introduction

The study of pseudo-Riemannian geometries admitting symmetries or conformal symmetries is a classical problem in differential geometry. The spinorial analogue leads to the determination of manifolds on which certain spinor field equations can be solved. The pseudo-Riemannian Berger...
list opens up a way to distinguish the holonomy groups of irreducible geometries admitting parallel spinors, see [1]. Furthermore, Lorentzian manifolds with special holonomy admitting parallel spinors or pseudo-Riemannian geometries with parallel pure spinor fields have been studied intensively in [2, 3, 4, 5]. A list of local normal forms of the metric is known in low dimension, see [6]. Generalizing this, the spinorial analogue of Killing vector fields leads to (geometric) Killing spinors which -at least in the Riemannian and Lorentzian case- have been well-studied in [7, 8, 9, 10] and many construction principles are known. Interest in these objects arose independently from the fact that as shown in [10], on a a compact Riemannian spin manifold the eigenspinors to the minimal possible eigenvalue of the Dirac operator are Killing spinors. Moreover, [11] relates Killing spinors to parallel spinors on the cone. It is natural to consider a generalization of this problem to conformal geometry giving rise to the study of conformal Killing spinors, or twistor spinors. They lie in the kernel of a natural differential operator acting on spinor bundles which can be interpreted as being complementary to the spin Dirac operator. Local geometries admitting twistor spinors have been classified in [7, 12, 13] for the Riemannian and Lorentzian case. However, also the study of the twistor equation in higher signatures is of interest as indicated in [14, 15, 16]. Among other aspects it leads to a spinorial characterization of 5-manifolds admitting generic 2-Distributions and to new construction principles for projective structures. Twistor spinors square to conformal vector fields with the special additional property that they insert trivially into the Weyl-and Cotton tensor, see [7, 12] for which the term normal conformal vector field has become standard in the literature, cf. [17]. A generalization of this property to differential forms has been studied in [18], leading to new classification results for pseudo-Riemannian decomposable conformal holonomy, cf. [19].

The study of these spinor field equations has also been motivated by progress in the understanding of physical theories with supergravity and vice versa. For instance, Riemannian manifolds admitting parallel or Killing spinors allow one to place certain supersymmetric Yang Mills theories on them, see [20, 21]. In physics, the twistor equation first appeared in [22]. Moreover, the generalized Killing spinor equations appearing in the Freund-Rubin product ansatz for 11-dimensional supergravity (cf. [23]) lead to conformal Killing spinor equations on the factors. Recently, it has become a fruitful topic in physics literature to place certain supersymmetric Minkowski-space theories on curved space which may lead to new insights in the computation of observables, see [24, 25, 26, 27, 28]. Requiring that the deformed theory on curved space preserves some supersymmetry again leads to generalized Killing spinor equations. Interestingly, one finds for different theories and signatures, namely Euclidean and Lorentzian 3-and 4 manifolds the same type of spinorial equation, namely a Spin$^c$-analogue of the twistor spinor equation whose solutions have been named charged conformal Killing spinors (CCKS), see for instance [24, 27, 28]. As shown in these references, one can derive this twistor equation also by using the AdS/CFT-correspondence and studying the gravitino-variation near the conformal boundary.

In order to put this into a more mathematical context, consider a space- and time-oriented, connected pseudo-Riemannian Spin$^c$-manifold $(M,g)$ of signature $(p,q)$ with underlying $S^1$-principal bundle $P_1$. One can canonically associate to this setting the complex spinor bundle $S^g$ with its Clifford multiplication, denoted by $\mu : TM \times S^g \rightarrow S^g$. If moreover a connection $A$ on $P_1$ is given, there is a canonically induced covariant derivative $\nabla^A$ on $S^g$. Besides the Dirac operator $D^A$, there is another conformally covariant differential operator acting on spinor fields, obtained by performing the spinor covariant derivative $\nabla^A$ followed by orthogonal projection onto the kernel of Clifford multiplication,

$$P^A : \Gamma(S^g) \xrightarrow{\nabla^A} \Gamma(TM \otimes S^g) \xrightarrow{\mu} \Gamma(TM \otimes S^g) \xrightarrow{\text{proj}} \Gamma(\ker \mu),$$

called the Spin$^c$-twistor operator. Elements of its kernel are precisely CCKSs and they are equivalently characterized as solutions of the conformally covariant Spin$^c$-twistor equation

$$\nabla^A_X \varphi + \frac{1}{n} X \cdot D^A \varphi = 0 \text{ for all } X \in \mathfrak{X}(M).$$ (1)
This article is devoted to the study of the twistor equation on $Spin^c$-manifolds. As we have seen, this is motivated by determining geometries in dimensions 3 and 4 on which supersymmetric field theories can be placed. In these signatures, [1] has been solved locally in [22, 27, 26]. However, we also find purely geometric reasons for the study of [1]. First, it is a natural generalization of $Spin^c$-parallel and Killing spinors which have been investigated in [29]. Their study has led to new spinorial characterizations of Sasakian and pseudo-Kähler structures. Generalizations of the $Spin^c$-Killing spinor equations have been investigated in [31]. Moreover, we have the hope that CCKS might lead to equivalent characterizations of manifolds admitting certain conformal Killing forms. By this, we mean the following. Given a CCKS $\varphi$, one can always form its associated Dirac current $V_\varphi$. In the $Spin^c$-case, i.e. $dA = 0$, $V_\varphi$ is always a normal conformal vector field. However, for Lorentzian 3-manifolds it has been shown in [27] that for every conformal vector field $V$ there is a CCKS $\varphi$ wrt. a generically non-flat connection $A$ such that $V = V_\varphi$. The same holds on Lorentzian 4-manifolds for lightlike conformal vector fields, see [26]. We want to investigate whether this principle carries over also to other signatures. This would lead to spinorial characterizations of manifold admitting certain conformal symmetries. Consequently, the natural generalization of already studied $Spin^c$ spinor field equations together with the question of what the spinorial analogue of conformal, not necessarily normal conformal vector fields might be, leads to the study of the twistor equation on pseudo-Riemannian $Spin^c$-manifolds.

This article starts with the investigation of basic properties of the $Spin^c$-twistor operator. It is straightforward to derive integrability conditions relating the conformal Weyl curvature tensor $W^g$ to the curvature $dA$ of the $S^1$-connection. We then ask for construction principles of Lorentzian manifolds admitting global solutions of the CCKS equation. We are motivated by the following: Every pseudo-Riemannian Ricci-flat Kähler spin manifold admits (at least) 2 parallel spinors, see [1]. Given a Kähler manifold equipped with its canonical $Spin^c$-structure and the $S^1$-connection $A$ canonically induced by the Levi-Civita connection, [29] shows that there is (generically) one $Spin^c$-parallel spinor wrt. $A$ and $dA = 0$ iff the manifold is Ricci flat. It is known that Fefferman spin spaces over strictly pseudoconvex manifolds can be viewed as the Lorentzian and conformal analogue of Calabi-Yau manifolds and that they always admit 2 conformal Killing spinors. This construction is presented in detail in [31] and from a conformal holonomy point of view in [19, 13]. In view of this, it is natural to conjecture that there is a $Spin^c$-analogue. Indeed, we find in Theorem 4.4 that every Fefferman space $(F^{2n+2}, h_\phi)$ over a strictly pseudoconvex manifold $(M^{2n+1}, H, J, \theta)$ admits a canonical $Spin^c$-structure and a natural $S^1$-connection $A$ on the auxiliary bundle induced by the Tanaka Webster connection on $M$ such that there exists a CCKS on $F$. Under additional natural assumptions also the converse direction is true, leading to a characterization of Fefferman space in terms of $Spin^c$-spinor equations, see Theorem 4.5.

Further, we obtain a classification of local Lorentzian geometries admitting CCKS under the additional assumption that the associated conformal vector field is normal conformal in Theorem 5.1. Our study of the $Spin^c$-twistor equation on Lorentzian 5-manifolds leads to a equivalent spinorial characterization of geometries admitting Killing 2-forms of a certain causal type in Theorem 6.5. It is straightforward to obtain similar results in signatures $(0, 5), (2, 2)$ and $(3, 2)$.

This article is organized as follows: In section 2 we introduce the basic ingredients of conformal $Spin^c$-geometry in arbitrary signature and show how CCKS can be described as parallel sections in the double spinor bundle wrt. a suitable connection. Sections 3 investigates the integrability conditions resulting from the CCKS equation, the relations between the Weyl curvature and the curvature of the $S^1$-connection and the properties of the spinor bilinears constructed out of a CCKS. Section 4 is then devoted to CCKS on Fefferman spaces which is precisely the $Spin^c$-analogue of [31]. Based on the results obtained so far, we can then present a partial classification result in section 5. In section 6 we continue the local analysis of the CCKS equation which has been initiated recently in physics literature and end up with a local geometric description of geometries admitting CCKS in signatures $(0, 5), (2, 2)$ and $(3, 2)$. 


2. Spin$^c$-Geometry and the twistor operator

2.1. Spin$^c(p,q)$-groups and spinor representations

For these algebraic preparations we follow [32, 33, 34]. We consider $\mathbb{R}^{p,q}$, that is, $\mathbb{R}^n$, where $n = p + q$, equipped with a scalar product $(\cdot, \cdot)_{p,q}$ of index $p$, given by $(e_1, e_j)_{p,q} = \epsilon_{ij}$, where $(e_1, ..., e_n)$ denotes the standard basis of $\mathbb{R}^n$ and $\epsilon_{ij} = -1$, $\epsilon_{ip} = 1$. Let $c_i := (e_i, e_j)_{p,q} \in (\mathbb{R}^{p,q})^*$. We denote by $Cl_{p,q}$ the Clifford algebra of $(\mathbb{R}^n, (\cdot, \cdot)_{p,q})$ and by $Cl^C_{p,q}$ its complexification. It is the associative real or complex algebra with unit multiplicatively generated by $(e_1, ..., e_n)$ with the relations

$$e_ie_j + e_je_i = -2(e_i, e_j)_{p,q}.$$

It is well-known (cf. [34, 35]) that if $p - q \not\equiv 1 \mod 4$, there is (up to equivalence) exactly one irreducible real representation of $Cl_{p,q}$. If $p - q \equiv 1 \mod 4$, there are precisely two inequivalent real irreducible representations of $Cl_{p,q}$. Furthermore, $Cl^C_{p,q}$ admits up to equivalence exactly one irreducible complex representation in case $n$ is even and two such representations if $n$ is odd. In case that there are two equivalence classes of irreducible real or complex representations, they can be distinguished by the unit volume element as presented in [34]. Let $\omega : e_1, ..., e_n \in Cl_{p,q}$ and $\omega_c := (i)^{\frac{n+1}{2}}\omega \in Cl^C_{p,q}$. If $p - q \equiv 1 \mod 4$, each irreducible real representation of $Cl_{p,q}$ or $Cl^C_{p,q}$ maps $\omega_c$ to $I_d$ or $-I_d$. Both possibilities can occur and the resulting representations are inequivalent. The analogous statements are true in the complex case for $Cl^C_{p,q}$ and $n$ odd (cf. [32]). This opens a way to distinguish a up to equivalence unique real resp. complex irreducible representation for all Clifford algebras $Cl_{p,q}$ and $Cl^C_{p,q}$ by requiring that $\omega$ is mapped to $I_d$ in case $n$ even ($\mathbb{K} = \mathbb{C}$) or $p - q \equiv 1 \mod 4$ ($\mathbb{K} = \mathbb{R}$).

**Remark 2.1.** We later need the following concrete realisation of an irreducible, complex representation of $Cl^C_{p,q}$: Let $E, T, g_1$ and $g_2$ denote the $2 \times 2$ matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad U = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad V = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Furthermore, let $\tau_j = \begin{cases} 1 & e_j = 1, \\ i & e_j = -1. \end{cases}$ Let $n = 2m$. In this case, $Cl^C_{p,q} \cong M_{2m}(\mathbb{C})$ as complex algebras, and an explicit realisation of this isomorphism is given by

$$\Phi_{p,q}(e_{2j-1}) = \tau_{2j-1}E \otimes ... \otimes E \otimes U \otimes T \otimes ... \otimes T,$$

$$\Phi_{p,q}(e_{2j}) = \tau_{2j}E \otimes ... \otimes E \otimes V \otimes T \otimes ... \otimes T.$$

Let $n = 2m + 1$. In this case, there is an isomorphism $\overline{\Phi}_{p,q} : Cl^C_{p,q} \to M_{2m}(\mathbb{C}) \oplus M_{2m}(\mathbb{C})$, given by

$$\overline{\Phi}_{p,q}(e_j) = (\Phi_{p,q-1}(e_j), \Phi_{p,q-1}(e_j)), \quad j = 1, ..., 2m,$$

$$\overline{\Phi}_{p,q}(e_{2m+1}) = \tau_{2m+1}(iT \otimes ... \otimes iT, -iT \otimes ... \otimes T),$$

and $\Phi_{p,q} := pr_1 \circ \overline{\Phi}_{p,q}$ is an irreducible representation mapping $\omega_c$ to $I_d$.

The Clifford group contains Spin$^*$$(p,q)$, the identity component of the spin group, as well as the unit circle $S^1 \subset \mathbb{C}$ as subgroups. Together they generate the group Spin$^c(p,q)$ and since $S^1 \cap Spin^c(p,q) = \{\pm 1\}$, we have

$$Spin^*(p,q) = Spin^c(p,q) \cdot S^1 = Spin^*(p,q) \times_{\mathbb{Z}_2} S^1.$$

Spin$^c(p,q)$ has various algebraic relations to other groups, see [34]. We let $\lambda : Spin^*(p,q) \to SO^*(p,q)$ denote the two-covering. There are natural maps

$$\lambda^* : Spin^*(p,q) \to SO^*(p,q), [g, z] \mapsto \lambda(g),$$

$$\zeta : Spin^*(p,q) \to SO^*(p,q) \times S^1, [g, z] \mapsto (\lambda(g), z^2),$$

...
where $\zeta$ is a 2-fold covering. The Lie algebras of $Spin^*(p,q)$ and $Spin^c(p,q)$ are given by $\mathfrak{spin}(p,q) = \{ e_i e_j \mid 1 \leq i < j \leq n \}$ and $\mathfrak{spin}^c(p,q) = \mathfrak{spin}(p,q) \oplus i \mathbb{R}$. $\zeta$ turns out to be a Lie algebra isomorphism, given by $\zeta (e_i e_j, it) = (2E_{ij}, 2it)$, where $E_{ij} = -e_i e_j + e_j e_i$ for the standard basis $D_{ij}$ of $\mathfrak{gl}(n, \mathbb{R})$. Finally, for $(p,q) = (2p', 2q')$, the group $Spin^c(p,q)$ is related to the group $U(p', q')$ of pseudo-unitary matrices as follows: Let $\iota : \mathfrak{gl}(m, \mathbb{C}) \to \mathfrak{gl}(2m, \mathbb{R})$ denote the natural inclusion and define $F : U(p', q') \to SO(p,q) \times S^1$ by $f(A) = (iA, \det A)$. Then there is exactly one group homomorphism $l : U(p', q') \to Spin^c(p,q)$ such that

$$\zeta \circ l = F.$$ 

For $n = 2m$ or $n = 2m + 1$, fixing an irreducible complex representation $\rho : Cl_{p,q}^C \to \text{End}(\Delta_{p,q}^C)$ on the space of spinors $\Delta_{p,q}^C = \mathbb{C}^{2^n}$, for instance $\rho = \Phi$ from Remark 2.1, and restricting it to $Spin^c(p,q) \subset Cl_{p,q}^C$ yields the complex spinor representation

$$\rho : Spin^c(p,q) \to \text{End}(\Delta_{p,q}^C),$$

$\rho([g,v]) = \rho(g)(v) = z \cdot \rho(g)(v) = z \cdot g \cdot v.$

In case $n$ odd, the restrictions of the two irreducible Clifford representations to $Spin^c(p,q)$ coincide and yield an irreducible representation whereas in case $n = 2m$ even $\Delta_{p,q}^C$ splits into the sum of two inequivalent $Spin^c(p,q)$ representations $\Delta_{p,q}^{C,\pm}$ according to the $\pm 1$ eigenspaces of $\omega$ (cf. [35 [32]). In our realisation from Remark 2.1, one can find these half spinor modules as follows: Let us denote by $u(\delta) \in \mathbb{C}^2$ the vector $u(\delta) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -\delta \end{array} \right), \delta = \pm 1$, and set $u(\delta_1, \ldots, \delta_m) := u(\delta_1) \otimes \ldots \otimes u(\delta_m)$ for $\delta_j = \pm 1$. Then we have

$$\Delta_{p,q}^{C,\pm} = \text{span} \{ u(\delta_1, \ldots, \delta_m) \mid \prod_{j=1}^m \delta_j = \pm 1 \}.$$ 

Note further that $Cl_{p,q}^C$ acts on $\Delta_{p,q}$ via the representation $\rho$, and as $R^n \subset Cl_{p,q} \subset Cl_{p,q}^C$, this defines the Clifford multiplication $X \cdot \varphi \mapsto X \cdot \varphi := \rho(X)(\varphi)$ of a vector by a spinor. Further, as $Cl(p,q) \cong \Lambda_{p,q}^* \cong \Lambda^*(\mathbb{R}^{p,q})^*$ canonically, forms act on the spinor module in a natural way. We consider the Hermitian inner product $\langle \cdot, \cdot \rangle_{\Delta_{p,q}^C}$ on the spinor module given by

$$\langle u, v \rangle_{\Delta_{p,q}^C} = d \cdot (e_1 \cdots e_p) \cdot u \cdot v \in \mathbb{C},$$

where $d$ is some power of $i$ depending on $p,q$, and the concrete realisation of the representation only. In the realisation from Remark 2.1, we take $d = i^{p(q-1)/2}$. If $p, q > 0$, $\langle \cdot, \cdot \rangle_{\Delta_{p,q}^C}$ has neutral signature and it holds that

$$\langle X \cdot u, v \rangle_{\Delta_{p,q}^C} + (-1)^p \langle u, X \cdot v \rangle_{\Delta_{p,q}^C} = 0$$

for all $u,v \in \Delta_{p,q}^C$ and $X \in \mathbb{R}^n$. In particular, $\langle \cdot, \cdot \rangle_{\Delta_{p,q}^C}$ is invariant under $Spin^c(p,q)$.

To every spinor $\chi \in \Delta_{p,q}^C$, we can associate a possibly trivial-linear subspace $\ker \chi := \{ X \in \Delta_{p,q} \mid X \cdot \chi = 0 \}$. If $\ker \chi$ is of maximal dimension $\min(p,q)$, we call the spinor (partially) pure. Moreover, bilinears can be constructed out of spinors generalizing the well-known Dirac current from the Lorentzian case, which might be trivial in other signatures. Concretely, we associate to spinors $\chi_{1,2} \in \Delta_{p,q}$ a series of forms $\alpha_{k,\chi_1,\chi_2} \in \Lambda_{p,q}^k$, $k \in \mathbb{N}$, given by

$$\langle \alpha_{k,\chi_1,\chi_2}, \alpha \rangle_{p,q} := d_{k,p} \langle \alpha \cdot \chi_1, \chi_2 \rangle_{\Delta_{p,q}} \quad \forall \alpha \in \Lambda_{p,q}^k,$$

(2)

d_{k,p} is a nonzero constant depending on the chosen representation but not depending on $\chi$, ensuring that the so defined form is indeed a real form. We set $\alpha_{k,\chi} := \alpha_{k,\chi,\chi}$. In more invariant notation these forms arise in even dimension as the image of a pair of spinors under the map

$$\Delta \otimes \Delta \to \text{End}(\Delta) \cong Cl^C(p,q) \cong (\Lambda_{p,q}^*)^C \to \Lambda^k(p,q).$$

The following properties of these forms are easily checked:

5
Proposition 2.1. Let $\chi \in \Delta_{p,q}$ and $k \in \mathbb{N}$.

1. $\alpha^k_{\chi} = 0 \Leftrightarrow \chi = 0$
2. $\alpha^k_{\chi} = d_k \phi \sum_{i<s} \epsilon_i \cdots \epsilon_k \omega \epsilon_1 \cdots \epsilon_{i_k} \cdot \chi \Delta_{p,q} \epsilon_{i_k} \wedge \cdots \wedge \epsilon_k$
3. Equivariance: $\alpha^k_{z \cdot \chi} = \lambda(g)(\alpha^k_{\chi})$ for all $k \in \mathbb{N}$, $z \cdot g \in Spin^c(p,q)$ and $\chi \in \Delta_{p,q}$.

2.2. Spin$^c$-structures and spinor bundles

The complex analogue of the well-known notion of pseudo-Riemannian spin structures (see [32]) leads to the study of Spin$^c(p,q)$-structures. Let $(M,g)$ be a space-and-time-oriented, connected pseudo-Riemannian manifold of index $p$ and dimension $n = p+q \geq 3$. By $\mathcal{P}^g$ we denote the $SO^+(p,q)$-principal bundle of all space-and-time-oriented pseudo-orthonormal frames $s = (s_1, \ldots, s_n)$. A Spin$^c$-structure of $(M,g)$ is given by the data $(\mathcal{Q}_c, \mathcal{P}_1, f^c)$, where $\mathcal{P}_1$ is a $S^1$-principal bundle over $M$, $\mathcal{Q}_c$ is a Spin$^c$-principal bundle over $M$ which together with $f^c : \mathcal{Q}_c \rightarrow \mathcal{P}^g \times \mathcal{P}_1$ defines a $\zeta$-reduction of the product $SO^+(p,q) \times S^1$-bundle $\mathcal{P}^g \times \mathcal{P}_1$ to Spin$^c(p,q)$. Existence and uniqueness of Spin$^c$-structures is discussed elsewhere, see [34]. We will from now on assume that $(M,g)$ admits a Spin$^c$-structure (which is locally always guaranteed) and assume that this structure is fixed. Given a Spin$^c$-manifold, the associated bundle $S^9 := \mathcal{Q}_c \times_{\text{Spin}^c(p,q)} \Delta^{C}_{p,q}$ is called the complex spinor bundle. In case even $\epsilon$, it holds that $S^9 = S^9(\pm) \oplus S^9(-\epsilon)$, as $\Delta_{p,q} = \Delta_{p,q} \oplus \Delta_{p,q}$. Sections, i.e. elements of $\Gamma(M,S^9(\pm))$ are called (half-)spinor fields. The algebraic objects introduced in the last section define fibrewise Clifford multiplication $\mu : \Omega^r(M) \otimes S^9 \rightarrow S^9$ and an Hermitian inner product $(\cdot, \cdot)_{S^9}$. Clearly, the properties of $(\cdot, \cdot)_{S^p,q}$ translate into corresponding properties of $(\cdot, \cdot)_{S^9}$. Moreover, pointwise applying the construction of spinor bilinears [32] leads to series of differential forms $\Gamma(M,S^9) \otimes \Gamma(M,S^9) \rightarrow \Omega^2(M)$ associated to a pair of spinor fields. Dualizing this for $k = 1$, leads to the well-known Dirac current $V_{\varphi} \in \mathfrak{X}(M)$.

Let $\omega^g \in \Omega^1(\mathcal{P}^g, \mathfrak{so}(p,q))$ denote the Levi Civita connection $\nabla^g$ on $(M,g)$, considered as a bundle connection. Moreover, fix a connection $A \in \Omega^1(\mathcal{P}_1, i\mathbb{R})$ in the $S^1$-bundle. Together, they form a connection $\omega^g + A$ on $\mathcal{P}^g \times \mathcal{P}_1$, which lifts to $\omega^g + A : \mathcal{Q}_c \rightarrow \mathcal{P}^g \times \mathcal{P}_1$. The covariant derivative $\nabla^A$ on $S^9$ induced by this connection can locally be described as follows: Let $\varphi \in \Gamma(S^9)$ be locally given by $\varphi_{[\cdot]} = [s \times e, v]$, where $s \in \Gamma(U(\mathcal{P}^g), e \in \Gamma(U, \mathcal{P}_1)$ and $s \times e$ is a lifting to $\Gamma(U, \mathcal{Q}_c)$.

$$\nabla^A_X \varphi_{[\cdot]} = \left[s \times e, X(\varphi) + \frac{1}{2} \sum_{1 \leq i \leq n} \epsilon_i e_1 g(\nabla^A_{\epsilon_i} s, s_1) \epsilon_i \cdot e_i \cdot \varphi + \frac{1}{2} A^e \right]$$

The inclusion of a $S^1$-connection $A$ in the construction of this covariant derivative "gauges" the natural $S^1$-action on $S^9$, by which we mean the following: Let $f = e^{-it/2} : M \rightarrow S^1$ be a smooth function. Then we have by [32] that

$$\nabla^A_X (f \cdot \varphi) = \frac{i}{2} dr(X) \cdot f \cdot \varphi + f \cdot \nabla^A_X \varphi = f \cdot \nabla^A_X f \cdot \varphi$$

It is moreover known from [33] that for all $X, Y \in \mathfrak{X}(M)$ and $\varphi, \psi \in \Gamma(S^9)$ we have

$$\nabla^A_X (Y \cdot \varphi) = \nabla^A_X Y \cdot \varphi + Y \cdot \nabla^A_X \varphi,$$

$$X(\varphi, \psi)_{S^9} = (\nabla^A_X \varphi, \psi)_{S^9} + (\varphi, \nabla^A_X \psi)_{S^9}.$$

Let $F_A = dA$ denote the curvature form of $A$, seen as element of $\Omega^2(M, i\mathbb{R})$. Let $R^A$ denote the curvature tensor of $\nabla^A$ and $R^g : \Lambda^2(TM) \rightarrow \Lambda^2(TM)$ the curvature tensor of $(M,g)$. It holds that

$$R^A(X,Y)\varphi = \frac{1}{2} R^g(X,Y) \cdot \varphi + \frac{1}{2} dA \cdot \varphi = \sum_{i} \epsilon_i s_i \cdot R^A(s_i, X) \varphi = \frac{1}{2} Ric(X) \cdot \varphi + (X - dA) \cdot \varphi$$

Remark 2.2. Some examples of manifolds admitting Spin$^c$-structures will become important.

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1In the following, these frames are simply referred to as pseudo-orthonormal.
1. Every spin-manifold is canonically Spin° with trivial auxiliary bundle. Moreover, if one takes for \( A \) the canonically flat connection on \( M \times S^1 \) in this situation, then \( \nabla^A \) corresponds to the connection on \( S^9 \) induced by the Levi Civita connection, see [32].

2. Let \( M \) be a manifold which admits a \( U(p', q') \rightarrow SO^+(p, q) \) reduction \( (\mathcal{P}_U, h : \mathcal{P}_U \rightarrow \mathcal{P}^9) \) of its frame bundle. Then the bundles \( (\mathcal{Q}_c \equiv \mathcal{P}_U \times \text{Spin}^c(p, q), \mathcal{P}_1 := \mathcal{P}_U \times_{\text{det}} S^1) \) together with the map

\[
f^c : \mathcal{Q}_c \rightarrow \mathcal{P}^9 \times \mathcal{P}_1, \quad [q, z] \mapsto ([q, \lambda(g)], [q, z^2])
\]

define a \( \text{Spin}^c(p, q) \) structure on \( M \). In this situation, there are natural reduction maps

\[
\phi_c : \mathcal{P}_U \rightarrow \mathcal{Q}_c, \quad p \mapsto [p, 1]_l \quad \text{(6)}
\]

\[
\phi_1 : \mathcal{P}_U \rightarrow \mathcal{P}_1, \quad p \mapsto [p, 1]_{\text{det}}. \quad \text{(7)}
\]

Moreover, local sections in \( \mathcal{Q}_c \) can be obtained as follows: Let \( s \in \Gamma(U, \mathcal{P}_U) \) be a local section. Then we have that \( \phi_c(s) \in \Gamma(U, \mathcal{Q}_c) \) and

\[
f^c(\phi_c(s)) = s \times e, \quad \text{where} \ e = \phi_1(s). \quad \text{(8)}
\]

2.3. Basic properties of charged conformal Killing spinors

Given a pseudo-Riemannian Spin°-manifold \((M, g)\) together with a connection \( A \) on the underlying \( S^1 \)-bundle there are naturally associated differential operators. The composition of \( \nabla^A \) with Clifford multiplication defines the Dirac operator

\[
D^A : \Gamma(S^9) \overset{\nabla^A}{\rightarrow} \Gamma(T^*M \otimes S^9) \overset{\partial}{\rightarrow} \Gamma(TM \otimes S^9) \overset{\Delta_A}{\rightarrow} \Gamma(S^9),
\]

The Schroeder-Lichnerowicz formula (cf. [33]) gives that

\[
D^A,^2 \varphi = \Delta_A \varphi + \frac{R}{4} \varphi + \frac{1}{2} d A \cdot \varphi, \quad \text{(9)}
\]

where \( \Delta_A \varphi = -\sum_i \epsilon_i (\nabla^A_{e_i} \nabla^A_{e_i} \varphi - \text{div}(e_i) \nabla^A_{e_i} \varphi) \) and \( R \) is the scalar curvature of \((M, g)\). A complementary operator is obtained by performing the spinor covariant derivative \( \nabla^A \) followed by orthogonal projection onto the kernel of Clifford multiplication. This gives rise to the Spin° twistor operator \( P^A \)

\[
P^A : \Gamma(S^9) \overset{\nabla^A}{\rightarrow} \Gamma(T^*M \otimes S^9) \overset{\partial}{\rightarrow} \Gamma(TM \otimes S^9) \overset{\text{proj}_{\text{ker}}}{} \rightarrow \Gamma(\text{ker} \mu).
\]

Spinor fields \( \varphi \in \ker P^A \) are called Spin°-twistor spinors. A local calculation shows that they are equivalently characterized as solutions of the twistor equation

\[
\nabla^A X \varphi + \frac{1}{n} X \cdot D^A \varphi = 0 \quad \text{for all} \ X \in \mathfrak{X}(M).
\]

Following the conventions in [26, 27, 28], we shall call Spin°-twistor spinors charged conformal Killing spinors and abbreviate them by CCKS. Let us collect some basic properties:

In analogy to the spin case, CCKS are objects of conformal Spin°-geometry. Let \( f^c_\alpha : \mathcal{Q}^c_\alpha \rightarrow \mathcal{P}^9 \times \mathcal{P}_1 \) be a \( \text{Spin}^c(p, q) \) structure for \((M, g)\) and let \( \tilde{g} = e^{2\alpha} g \) be a conformally equivalent metric. As in the case of Spin structures (cf. [32, 3]), there exists a canonically induced \( \text{Spin}^c \)-structure \( f^c_\alpha : \mathcal{Q}^c_\alpha \rightarrow \mathcal{P}^9 \times \mathcal{P}_1 \) and a \( \text{Spin}^c(p, q) \)-equivariant map \( \phi_\alpha : \mathcal{Q}^c_\alpha \rightarrow \mathcal{Q}^c_\alpha \) such that the diagram

\[
\begin{tikzcd}
\mathcal{Q}^c_\alpha \ar{r}{\tilde{f}_\alpha} \ar{d}{f^c_\alpha} & \mathcal{Q}^c_\alpha \ar{d}{f^c_\alpha} \\
\mathcal{P}^9 \times \mathcal{P}_1 \ar{r}{\phi_\alpha} & \mathcal{P}^9 \times \mathcal{P}_1
\end{tikzcd}
\]
commutes, where \( \phi_\sigma((s_1, \ldots, s_n), e) = ((e^{-\sigma}s_1, \ldots, e^{-\sigma}s_n), e) \). We obtain natural identifications

\[
\begin{align*}
\therefore: S^g & \to S^\bar{g}, \quad \varphi = [\bar{g}, v] \mapsto [\bar{\varphi}_\sigma(\bar{g}), v] = \bar{\varphi}, \\
\therefore: TM & \to TM, \quad X = [q, x] \mapsto [\varphi_\sigma(q), x] = e^{-\sigma}X,
\end{align*}
\]

where the second map is an isometry wrt. \( g \) and \( \bar{g} \). With these identifications, the covariant derivative \( \nabla^A \) on the spinor bundle, the Dirac operator and the twistor operator transform in the following way (the proof is the same as in the spin case, see [3, chapter 1 or [32]):

\[
\begin{align*}
\nabla^A_X \varphi &= e^{-\sigma} \nabla^A_X \varphi - \frac{1}{2} e^{-2\sigma}(X \cdot \text{grad}(e^\sigma) \cdot \varphi + g(X, \text{grad}(e^\sigma)) \cdot \varphi)^- \\
D^{A,B} \varphi &= e^{-\sigma} D^{A,B}(e^{\sigma} \varphi)^- \\
P^{A,B} \varphi &= e^{-\sigma} (P^{A,B}(e^{-\sigma} \varphi))^-
\end{align*}
\]

We see that \( P^{A,B} \) is conformally covariant and \( \varphi \in \ker P^{A,B} \) iff \( e^{\sigma/2} \varphi \in \ker D^{A,B} \). Note that the \( S^1 \)-bundle data, and in particular \( A \) are unaffected by the conformal change. However, \([4]\) directly yields the following additional \( S^1 \)-gauge invariance of the CCKS-equation:

**Proposition 2.2.** Let \( \varphi \in \ker P^{A,B} \) and \( f = e^{\sigma/2} \in C^\infty(M, S^1) \). Then \( f \varphi \in \ker P^{A,B} \) and \( D^{A,B}(f \varphi) = f D^{A,B} \varphi \). Thus, the data needed to define CCKSs are in fact a conformal Spin\(^c\)-structure and a gauge equivalence class of \( S^1 \)-connections in the underlying bundle \( P_1 \).

**Proposition 2.3.** The following hold for \( \varphi \in \Gamma(S^g) \) a CCKS:

\[
\begin{align*}
D^{A,B} \varphi &= \frac{n}{n-1} \left( \frac{R}{1} + 1 \right) + \frac{1}{2} dA \cdot \varphi, \\
\nabla^A_X \varphi &= \frac{n}{2} \left( K^g(X) + \frac{1}{n-2} \left( \frac{1}{n-1} X \cdot dA + X \cdot dA \right) \right), \\
\end{align*}
\]

where \( K^g = \frac{n}{n-1} \left( \frac{R}{1} + 1 \right) \) denotes the Shouten tensor.

**Proof.** All calculations are carried out at a fixed point \( x \in M \). Let \( (s_1, \ldots, s_n) \) be a pseudo-orthonormal frame which is parallel in \( p \) and let \( X \) be a vector field which is parallel in \( x \). We have at \( x \) that

\[
-\Delta_A \varphi + \frac{1}{n} D^{A,2} \varphi = \sum_i \epsilon_i \nabla^A_{s_i} \left( \nabla^A_{s_i} \varphi + \frac{1}{n} s_i \cdot D^A \varphi \right) = 0,
\]

and thus by \([3] \)

\[
\frac{1}{n} D^{A,2} \varphi = \Delta_A \varphi = D^{A,2} \varphi - \frac{1}{2} dA \cdot \varphi,
\]

from which \([10]\) follows. To prove \([11]\), note that the twistor equation yields \( R^A(X, s_i) \varphi = \frac{1}{n} (s_i \nabla^A_X \varphi - X \cdot \nabla_{s_i}^A \varphi) \). Inserting this into \([5]\) implies that

\[
Ric(X) \cdot \varphi = \frac{2}{n} (2 - n) \nabla^A_X D^A \varphi + \frac{2}{n} X \cdot D^{A,2} \varphi + (X \cdot dA) \cdot \varphi
\]

\[
= \frac{2}{n} (2 - n) \nabla^A_X D^A \varphi + \frac{R}{2(n-1)} X \cdot \varphi + \frac{1}{n-1} X \cdot dA \cdot \varphi + (X \cdot dA) \cdot \varphi
\]

Solving for \( \nabla^A_X D^A \varphi \) yields the claim.

**Proposition 2.4** leads to an equivalent characterization of CCKS. To this end, consider the bundle \( E^g := S^g \oplus S^\bar{g} \) together with the covariant derivative

\[
\nabla^{E^g}_X \varphi := \left( \nabla^A_X \psi - \frac{1}{2} (K^g(X) + \frac{1}{n-2} \left( \frac{1}{n-1} X \cdot dA + X \cdot dA \right) \right) \varphi
\]

Consequently, \( \varphi \in \ker P^A \) implies that \( \nabla^{E^g}_X \varphi = 0 \), and on the other hand, if \( \nabla^{E^g}_X \varphi = 0 \), then \( \varphi \in \ker P^A \) and \( \psi = D^A \varphi \). It follows as in the spin case that for a nontrivial CCKS the spinors \( \varphi \) and \( D^A \varphi \) never vanish at the same point and \( \dim \ker P^A \leq 2^{[n/2]} + 1 \).
Remark 2.3. As CCKS are objects of conformal geometry, one might try to construct a first prolongation of a conformal Spin\(^c\)-structure and introduce associated spin tractor bundles and covariant derivatives thereon induced by the normal conformal Cartan connection and the auxiliary connection \(A\) as done in the spin setting in [13]. This is indeed straightforward and possible. However, in contrast to the spin-setting, where twistor spinors are equivalently described as parallel spin tractors, the appearance of the \(dA\)-terms in \([17]\) leads to a tractor equation of the form \(\nabla_X^\psi = E(X) \cdot \psi\) on the first prolongation. It is easy to see that \((\frac{1}{n}X \cdot dA + X \cdot dA) \cdot \psi = 0\) implies that \(dA = 0\). Consequently, in the generic case the spin tractor approach does not lead to a simplification of the CCKS-problem.

3. Integrability conditions and spinor bilinears

We obtain integrability conditions for the existence of CCKS by computing the curvature operator \(R^v^g\) which has to vanish when applied to \((\varphi, D^A\varphi)^T\), where \(\varphi \in \ker P^A\varphi\). Let \(pr_{1,2}\) denote the projections onto the corresponding summands of \(E^g\). We calculate:

\[
pr_1\left(R^v^g(X,Y)\left(\varphi, D^A\varphi\right)\right) = \frac{1}{2} \left(R^g(X,Y) - X \cdot K^g(Y) + Y \cdot K^g(X)\right) \cdot \varphi + \frac{1}{2} dA(X,Y) \cdot \varphi
\]

\[
- \frac{1}{2(n-2)} \left(\frac{1}{n-1}(XY - YX) \cdot dA + (X \cdot (Y \cdot dA) - Y \cdot (X \cdot dA))\right) \cdot \varphi
\]

With the definition of the Weyl tensor \(W^g\) and using the identities

\[
X \cdot \omega = X^\flat \wedge \omega - X \omega,
\]

\[
\omega \cdot X = (-1)^k (X^\flat \wedge \omega + X \omega),
\]

where \(X\) is a vector and \(\omega\) a \(k\)-form, we obtain the integrability condition

\[
0 = \frac{1}{2} W^g(X,Y) \cdot \varphi + \left(\frac{n-3}{2(n-1)} dA(X,Y) - \frac{1}{(n-2)(n-1)} X^\flat \wedge Y^\flat \wedge dA\right) \cdot \varphi
\]

\[
+ \frac{1}{n-2} \left(\frac{1}{n-1} - \frac{1}{2}\right) \left(X^\flat \wedge (Y \cdot dA) - Y^\flat \wedge (X \cdot dA)\right) \cdot \varphi
\]

In particular, \(\ker P^A\) is of maximal possible dimension iff \(W^g = 0\) and \(dA = 0\). The integrability condition resulting from \(pr_2\left(R^v^g(X,Y)\left(\varphi, D^A\varphi\right)\right)\) is with the same formulas and the definition of the Cotton York tensor \(C^g(Y) := (\nabla_X Y)^g - (\nabla_X X)^g\), straightforwardly calculated to be

\[
0 = \frac{1}{2} W^g(X,Y) \cdot D^A\varphi + \frac{n}{2} C(X,Y) \cdot \varphi - \frac{1}{2(n-1)(n-1)} X^\flat \wedge \nabla_X dA - X^\flat \wedge \nabla_Y dA
\]

\[
- \frac{n}{2(n-1)} (g(\nabla_X dA,Y) - g(\nabla_Y dA,X)) \cdot \varphi - \frac{1}{2(n-1)(n-1)} X^\flat \wedge Y^\flat \wedge dA + \frac{n-3}{2(n-1)} dA(X,Y)
\]

\[
+ \frac{1}{n-2} ((X \cdot dA) \wedge Y^\flat - (Y \cdot dA) \wedge X^\flat) \cdot D^A\varphi
\]

Remark 3.1. For Riemannian 4-manifolds these integrability conditions have already appeared in [3d]. Note that taking the Clifford trace of \([13]\) leads only to a trivial result.

We now clarify the relation of CCKS to conformal Killing forms. For this purpose, we introduce the following set of differential forms for a spinor field \(\varphi \in \Gamma(S^g)\) and \(k \in \mathbb{N}\):

\[
g(\alpha^k_\varphi, \alpha) := d_k \cdot (\alpha \cdot \varphi, \varphi)_{S^g}, \quad \alpha \in \Omega^k(M),
\]

\[
g(\alpha_0^{k+1}, \beta) := \frac{2d(-1)^{k-1}}{n} h \left(\beta \cdot D^A\varphi, \varphi\right)_{S^g}, \quad \beta \in \Omega^{k+1}(M)
\]

\[
g(\alpha_\varphi^{k-1}, \gamma) := \frac{2d(-1)^{k-1}}{n} h \left(\gamma \cdot D^A\varphi, \varphi\right)_{S^g}, \quad \gamma \in \Omega^{k-1}(M)
\]
where \( h(z) = \frac{1}{4} \left( Re(z) + (-1)^{k+1} Im(z) \right) \), \( d_k \in U(1) \) are constants, ensuring that these forms are indeed real. A straightforward calculation using only the twistor equation yields that for \( \varphi \in \ker P^A \):

\[
\nabla_X \alpha^k_{\varphi} = \frac{2d_k(-1)^{k-1}}{n} \left( X - \alpha^k_{\varphi} + X^* \wedge \alpha^k_{\varphi}^{-1} \right),
\]

(14)
i.e. \( \alpha^k_{\varphi} \) is a conformal Killing form. Such forms have been studied intensively in [37, 18]. From [14] we deduce that \( \frac{2d_k(k+1)(-1)^{k-1}}{n} \alpha^k_{\varphi} = d\alpha^k_{\varphi} \) and \( \frac{(n-k+1)2d_k(-1)^{k-1}}{n} \alpha^k_{\varphi}^{-1} = d^* \alpha^k_{\varphi} \). Moreover, in case \( k = 1 \) (14) is equivalent to say that \( V_{\varphi} = \left( \alpha^1_{\varphi} \right)^2 \) is a conformal vector field. Note that under a conformal change of the metric with factor \( e^{2\sigma} \), \( \alpha^k_{\varphi} \) transforms with factor \( e^{(k+1)\sigma} \), and thus \( V_{\varphi} \) depends on the conformal class only.

We now derive further equations for the Lorentzian case 2 and \( k = 1 \). Note that in this case we may set \( d = 1 \). Let us introduce further forms for \( \varphi \in \Gamma(S^0) \) by setting

\[
g (\alpha^i_{dA}, \alpha) = \frac{1}{(n-2)(n-1)} \cdot Re \langle dA \cdot \varphi, \alpha \cdot \varphi \rangle_{\mathbb{S}^n}, \quad \alpha \in \Omega^1(M),
\]

\[
g (\alpha^0_0, \beta) = \frac{2}{n} \cdot Im \langle \beta \cdot D^A \varphi, \varphi \rangle_{\mathbb{S}^n}, \quad \beta \in \Omega^2(M)
\]

\[
\alpha_0 = \frac{2}{n} \cdot Im \langle D^A \varphi, \varphi \rangle_{\mathbb{S}^n}
\]

The the twistor equation and (11) yield the following system of equations:

\[
\left(
\begin{array}{cccc}
-\nabla_X^0 & -X^\varphi & -X^\varphi & 0 \\
-K(X)^\varphi & \nabla_X^0 & 0 & X^\varphi \\
-K(X)^{\varphi^2} & 0 & \nabla_X^0 & -X^\varphi \\
0 & -K(X)^{\varphi} & K(X)^{\varphi} & \nabla_X^0 \\
\end{array}
\right)
\left(
\begin{array}{c}
\alpha^1 \\
\alpha^0 \\
\alpha^1 \\
\alpha^0 \\
\end{array}
\right)
= \left(
\begin{array}{c}
0 \\
\frac{1}{2} (X - dA) - \alpha^3 - X^\varphi + X^\varphi + d\alpha^3_{dA} \\
X^\varphi - 2 \frac{1}{n-1} (X - dA)^{\varphi} \alpha^0_{dA} \\
\frac{1}{n-1} (X - dA)^{\varphi} \alpha^0_{dA} + \alpha_0 (X - dA)^{\varphi} \\
\end{array}
\right)
\]

(15)

**Remark 3.2.** Elements in the kernel of the operator on the left hand side define normal conformal Killing forms resp. vector fields and have been studied in [37, 18]. For a conformal vector field \( V \), being normal conformal is equivalent to the curvature conditions (see [37, 18])

\[
V \cdot W^g = 0, \quad V \cdot C^g = 0.
\]

Due to the \( dA \)-terms, the associated vector to a CCKS is generically no normal conformal vector field, in contrast to the spin setting. In general, there is no additional equation for \( \alpha^1_{\varphi} \) only except the conformal Killing equation.

We next study the relation of \( V_{\varphi} \) with the two main curvature quantities related to a CCKS, namely \( W^g \) and \( dA \). As before, we will restrict ourselves to the Lorentzian case. First, we show that \( V_{\varphi} \) preserves \( dA \).

**Proposition 3.1.** It holds that

\[
V_{\varphi} = \left( \frac{1}{i} dA \right) = \frac{2d_{(1-n)}}{n} d \left( Im(D^A \varphi, \varphi)_{\mathbb{S}^n} \right).
\]

In particular, we have that

\[
L_{V_{\varphi}} \frac{1}{i} dA = 0.
\]

---

2In fact, all the following equations can be obtained in arbitrary signatures where one has to change some signs and real and imaginary parts.
Proof. Let us write \( \omega = \frac{1}{i} dA \in \Omega^2(M) \). We have for \( Y \in TM \):

\[
(V_{\varphi}^\omega(Y)) = \omega(V_{\varphi}, Y) = -\omega(Y, V_{\varphi}) = -g((Y - \omega)^{\sharp} V_{\varphi})
\]

\[
\overset{11}{=} \frac{1}{i} \frac{2}{n} (1 - n) \langle \nabla Y^A D^A \varphi, \varphi \rangle \sigma_s + \frac{n - 1}{n - 1} \frac{1}{i} \frac{1}{1} (K^g(Y) \cdot \varphi, \varphi) \sigma_s + \frac{n - 1}{1 - n} - \frac{1}{i} \frac{1}{1} ((Y - \omega) \cdot \varphi, \varphi) \sigma_s \in \mathbb{R}
\]

\[
= \frac{2}{n} \left( \frac{n - 1}{n} \right) \text{Im}(\nabla^A Y \varphi, \varphi) = \text{Im}(Y \langle D^A \varphi, \varphi \rangle \sigma_s + \frac{1}{n} \langle Y \cdot D^A \varphi, D^A \varphi \rangle \sigma_s)
\]

\[
= \frac{2}{n} \left( \frac{n - 1}{n} \right) d(\text{Im}(\langle D^A \varphi, \varphi \rangle \sigma_s))(Y).
\]

The second formula follows directly with Cartan's formula \( L = -d \circ d + d \circ \omega \).

\[\hfill \square\]

Remark 3.3. For 4-dimensional Lorentzian manifolds an alternative proof of Proposition 3.7 is given in [24].

Next, we investigate how \( V_{\varphi} \) inserts into the Weyl tensor. We have by definition for \( X, Y, Z \in TM \)

\[
W^g(V_{\varphi}, X, Y, Z) = -\langle \varphi, W^g(X, Y, Z) \cdot \varphi \rangle \sigma_s \overset{12}{=} \langle \varphi, Z \cdot W^g(X, Y) \cdot \varphi \rangle \sigma_s - \langle \varphi, (Z \wedge W^g(X, Y)) \cdot \varphi \rangle \sigma_s \in \mathbb{R}
\]

(16)

In Lorentzian signature, \( \langle \varphi, \omega \cdot \varphi \rangle \sigma_s \in i \mathbb{R} \) for \( \omega \in \Omega^2(M) \). Inserting the integrability condition \[\overset{13}{=}(Z)(X^3 \wedge Y^3 \wedge dA) = \frac{3}{2} \left( Z^2 \wedge (X^3 \wedge (Y - dA) - Y^3 \wedge (X - dA)) \right) \varphi \sigma_s \]

By permuting \( X, Y, Z \), it is pure algebra to conclude that the last expression vanishes for all \( X, Y, Z \in TM \) if and only if \( \langle (X^3 \wedge Y^3 \wedge (Z - dA)) \cdot \varphi, \varphi \rangle \sigma_s = 0 \) for all \( X, Y, Z \in TM \). We can express this as follows:

**Proposition 3.2.** For a Lorentzian CCKS \( \varphi \in \ker P^A \), we have that

\[
V_{\varphi}^\omega W^g = 0 \iff (Z-1)_{\frac{1}{i}} \alpha^A_\varphi = 0 \forall Z \in TM.
\]

In particular, one does not need to compute \( W^g \) to check whether \( V_{\varphi} \) is normal conformal. One obtains another relation between \( dA \) and \( V_{\varphi} \) by requiring the imaginary part of \[\overset{10}{=} \varphi \langle (X^3 \wedge Y^3 \wedge dA) + \frac{3}{2} \langle Z^2 \wedge (X^3 \wedge (Y - dA) - Y^3 \wedge (X - dA)) \rangle \varphi \rangle \sigma_s \]

As a consistency check, note that all integrability conditions including the Weyl curvature become trivial in case \( n = 3 \). Inserting \[\overset{13}{=} \varphi \langle (X^3 \wedge Y^3 \wedge dA) + \frac{3}{2} \langle Z^2 \wedge (X^3 \wedge (Y - dA) - Y^3 \wedge (X - dA)) \rangle \varphi \rangle \sigma_s \]

\[\overset{12}{=} \varphi \langle (X^3 \wedge Y^3 \wedge dA) + \frac{3}{2} \langle Z^2 \wedge (X^3 \wedge (Y - dA) - Y^3 \wedge (X - dA)) \rangle \varphi \rangle \sigma_s \]

\[\overset{10}{=} \varphi \langle (X^3 \wedge Y^3 \wedge dA) + \frac{3}{2} \langle Z^2 \wedge (X^3 \wedge (Y - dA) - Y^3 \wedge (X - dA)) \rangle \varphi \rangle \sigma_s \]

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\[\overset{12}{=} \varphi \langle (X^3 \wedge Y^3 \wedge dA) + \frac{3}{2} \langle Z^2 \wedge (X^3 \wedge (Y - dA) - Y^3 \wedge (X - dA)) \rangle \varphi \rangle \sigma_s \]

We conclude these general observations about CCKS with some remarks regarding the zero set \( Z_{\varphi} \subset M \) of a CCKS \( \varphi \in \ker P^A \). By \overset{11}{=} \varphi \in Z_{\varphi} \) satisfies \( \nabla D^A \varphi(x) = 0 \). This observation allows one to prove literally as in \[\overset{7}{=} \text{and } \overset{20}{=} \] the following:

\[\overset{3}{=} \text{In the following } g((X - dA)^{\sharp}, Y) = i \cdot g((X - dA)^{\sharp}, Y) \in i \mathbb{R} \text{ for } X, Y \in TM.\]
Proposition 3.3. Let $\varphi \in \ker P^A$ be a CCKS on $(M^{p,q}, g)$. If $\gamma : I \to Z_\varphi \subset M$ is a curve which runs in the zero set, then $\gamma$ is isotropic. If $p = 0$, then $Z_\varphi$ consists of a countable union of isolated points. If $p = 1$, then the image of every geodesic $\gamma_v$ starting in $x \in Z_\varphi$ with initial velocity $v$ satisfying that $v \cdot D^\varphi \varphi (x) = 0$ is contained in $Z_\varphi$.

This ends our discussion of general properties of the CCKS-equation and its relations to curvature. We now turn to construction principles, classification results and relations to special geometries in small dimensions.

4. CCKS and CR-Geometry

4.1. The Fefferman metric

The purpose of this section is to give a construction principle of CCKS with nontrivial curvature $dA \in \Omega^2 (M, i\mathbb{R})$ on Lorentzian manifolds $(M^{1,2n+1}, g)$ starting from $2n + 1$-dimensional strictly pseudoconvex structures. This can be viewed as the $\text{Spin}^c$-analogue of [31], and in fact the construction is quite similar. As a motivation, let us recall the following well-known fact:

Consider a pseudo-Riemannian Kähler manifold $(M^{p,q}, g, J)$, where $(p, q) = (2p', 2q')$, $p + q = 2n$, endowed with its canonical $\text{Spin}^c$-structure (cf. Remark [22]), where the $U(p', q')$-reduction $\mathcal{P}_U$ of $\mathcal{P}^g$ is given by considering only pseudo-orthonormal bases of the form $(s_1, J(s_1), ..., s_n, J(s_n))$.

As $J$ is parallel, $\nabla^g$ reduces to a connection $\omega_U^g \in \Omega^1 (\mathcal{P}_U, u(p', q'))$. By Remark [22] $\mathcal{P}_U$ and the $S^1$-bundle $\mathcal{P}_1$ are related by det-reduction,

$$\phi : \mathcal{P}_U \to \mathcal{P}_1 = \mathcal{P}_U \times_{\text{det}} S^1.$$ 

Whence there exists a connection $A \in \Omega^1 (\mathcal{P}_1, i\mathbb{R})$, uniquely determined by

$$(\phi_1 (s))^* A := A^{0,1,s} = \text{tr} \left( \omega_U^g \right)^s$$

for $s \in \Gamma (U, \mathcal{P}_U)$.

One calculates that $dA(X, Y) = i \cdot \text{Ric}^g (X, JY)$.

Proposition 4.1. On every pseudo-Riemannian Kähler manifold $(M^{p,q}, g, J)$ there exists a $\nabla^A$-parallel spinor.

Proof. As known from [41] the complex spinor module $\Delta_2^C$ decomposes into $\Delta_2^C = \theta^0 \otimes \Delta_2^C$, where the $\Delta_2^C$ are eigenspaces of the action of the Kähler form $\Omega = \langle ., J \rangle_{p,q}$ to the eigenvalue $\mu_s = (n - 2k)i$. $\Delta_2^C$ turns out to be one-dimensional, in the notation from Remark [24] it is spanned by $u(-1, ..., -1)$ and acted on trivially by $U(p', q')$, i.e.

$$l(U) \cdot u(-1, ..., -1) = u(-1, ..., -1)$$

for $s \in \Gamma (U, \mathcal{P}_U)$. [17] yields that this is well-defined, i.e. independent of the chosen $s$. Writing $s^* \omega_U^g$ and $(\phi_1 (s))^* A$ in terms of $\nabla^g$ is straightforward and then one directly calculates with [3] that $\nabla^A \varphi = 0$.

The rest of this section is devoted to the conformal analogue of this construction. We closely follow [31] and refer to this article when leaving out steps which are identical in our construction. To start with, let $(M, H, J, \theta)$ be a strictly-pseudoconvex pseudo-hermitian manifold of dimension $2n + 1$. Let $L_\theta$ denote the Levi-form and $T$ the characteristic vector field of the contact form $\theta$, i.e. $\theta (T) \equiv 1$ and $T \cdot d\theta \equiv 0$. It is a standard fact that $g_0 := L_\theta + \theta \circ \theta$ is a Riemannian metric on $M$. Clearly, the $SO^+ (2n+1)$-frame bundle $\mathcal{P}_U$ reduces to the $U(n)$ bundle

$$\mathcal{P}_{U,H} := \{(X_1, JX_1, ..., X_n, JX_n, T) | (X_1, JX_1, ..., X_n, JX_n) \text{ pos. oriented ONB of} (H, L_\theta)\}.$$
where \( U(n) \to SO(2n) \to SO(2n + 1) \). By Remark 2.2 this induces a \( Spin^c(2n + 1) \)-structure \((\mathcal{Q}_F^e = \mathcal{P}_{U,H} \times_1 Spin^c(2n + 1), f^e_M)\) on \((M, g_0)\), where \( Spin^c(2n) \to Spin^c(2n + 1) \), with auxiliary bundle \( \mathcal{P}_{1,M} = \mathcal{P}_{U,H} \times_{\text{det}} S^1 \) and natural reduction maps 

\[ \phi_{c,M} : \mathcal{P}_{U,H} \to \mathcal{Q}_F^e, \phi_{1,M} : \mathcal{P}_{U,H} \to \mathcal{P}_{1,M}. \]

There is a special covariant derivative on a strictly pseudoconvex manifold, the Tanaka Webster connection \( \nabla^W : \Gamma(TM) \to \Gamma(T^*M \oplus TM) \), uniquely determined by requiring it to be metric and the torsion tensor \( \text{Tor}^W \) to satisfy 

\[ \text{Tor}^W(X,Y) = L_\theta(JX,Y) \cdot T, \]
\[ \text{Tor}^W(T,X) = -\frac{1}{2}([T,X] + J[T,JX]) \]

for \( X, Y \in \Gamma(H) \). Let \( \text{Ric}^W \) and \( R^W \) denote the Ricci-and scalar curvature of \( \nabla^W \) (see [31]). As \( \nabla^W g_0 = 0, \nabla^W T = 0 \) and \( \nabla^W J = 0 \), it follows that \( \nabla^W \) descends to a connection \( \omega^W \in \Omega^1(\mathcal{P}_{U,H}, \mathfrak{su}(n)) \). In the standard way, this induces a connection \( A^W \in \Omega^1(\mathcal{P}_{1,M}, i\mathbb{R}) \), uniquely determined by 

\[ (\phi_{1,M}(s))^* A^W = \text{Tr} \left( s^* \omega^W \right) \]

Two connections on an \( S^1 \)-bundle over \( M \) differ by an element of \( \Omega^1(M, i\mathbb{R}) \). Consequently, 

\[ A_\theta := A^W + \frac{i}{2(n+1)} R^W \theta \]

is a connection on \( F := \mathcal{P}_{1,M} \). Setting 

\[ h_\theta := \pi^* L_\theta - i \frac{4}{n+2} \pi^* \theta \circ A_\theta \]

defines a right-invariant Lorentzian metric on \( F \), the Fefferman metric. Its further properties are discussed in [13, 43]. In particular, one finds that the conformal class \([h_\theta]\) does not depend on \( \theta \), which is unique up to multiplication with a positive function, but on the CR-data \((M, H, J, \theta)\) only. In the next section we define a natural \( Spin^c(1,2n+1) \)-structure on the Lorentzian manifold \((F, h_\theta)\) and show that it admits a CCKS for a natural choice of \( A \).

4.2. \( Spin^c \)-characterization of Fefferman spaces

This subsection is mainly an application of the spinor calculus for \( S^1 \)-bundles with isotropic fibres over strictly pseudoconvex spin manifolds from [31] to our case with slight modifications as we are dealing with \( Spin^c \)-structures. Let \((F, h_\theta)\) denote the Fefferman space of \((M, H, J, \theta)\), where \( F = \mathcal{P}_{1,M} \xrightarrow{\pi} M \) is the \( S^1 \)-bundle. Let \( N \in \mathfrak{X}(M) \) denote the fundamental vector field of \( F \) defined by \( \frac{\alpha^2}{\sqrt{2}} i \in i\mathbb{R} \), i.e. \( N(f) := \frac{\alpha}{2\pi i} \left( f \cdot e^{\frac{\alpha^2}{2}\pi i t} \right) \). For a vector field \( X \in \mathfrak{X}(M) \), let \( X^* \) be its \( \text{A}_\theta \)-horizontal lift. We define the orthogonal timelike and spacelike vectors \( s_1 := \sqrt{2}(N - T^*), s_2 := \sqrt{2}(N + T^*) \) which are of unit length. Let the time orientation of \((F, h_\theta)\) be given by \( s_1 \) and the space orientation by vectors \((s_2, X_1^*, JX_1^*, \ldots, X_n^*, JX_n^*)\), where \((X_1, JX_1, \ldots, X_n, JX_n, T) \in \mathcal{P}_{U,H}\). Obviously, the bundle 

\[ \mathcal{P}_{U,F} := \{(s_1, s_2, X_1^*, JX_1^*, \ldots, X_n^*, JX_n^*) \mid (X_1, JX_1, \ldots, X_n, JX_n, T) \in \mathcal{P}_{U,H}\} \]

is a \( U(n) \to SO^*(1,2n+1) \) reduction of \( \mathcal{P}_{h_\theta}^e \) and \( \mathcal{P}_{U,F} \cong \pi^* \mathcal{P}_{U,H} \). It follows again with Remark 2.2 that there is a canonically induced \( Spin^c(1,2n+1) \)-structure 

\[ (Q^e_F := \mathcal{P}_{U,F} \times_1 Spin^c(1,2n+1), f^e_F, \mathcal{P}_F^e := \mathcal{P}_{U,F} \times_{\text{det}} S^1), \]

\(^5\text{Note that this sign differs from the one in the construction in [31]. We use a different realisation of the canonical line bundle.}\)
where $U(n) \xrightarrow{1} Spin^c(2n) \rightarrow Spin^c(1,2n+1)$, together with reduction maps
\[
\phi_{c,M} : \mathcal{P}_{U,F} \rightarrow Q_F^c, \phi_{1,F} : \mathcal{P}_{U,F} \rightarrow \mathcal{P}_{1,F}.
\]
There are two distinct natural maps between the $S^1$-bundles $\mathcal{P}_{1,F}$ and $F$: Viewing $\mathcal{P}_{1,F}$ as the total space of an $S^1$-bundle over the manifold $F$ gives the projection $\pi_F : \mathcal{P}_{1,F} \rightarrow F$, whereas the isomorphism $\pi^*\mathcal{P}_{U,H} \cong \mathcal{P}_{U,F}$ leads to a natural $S^1$-equivariant bundle map
\[
\bar{\pi}_F : \mathcal{P}_{1,F} \cong \pi^*\mathcal{P}_{U,H} \times_{\det S^1} F \cong \mathcal{P}_{U,H} \times_{\det S^1} F.
\]
The proof of the following statements is a matter of unwinding the definitions:

**Proposition 4.2.** Let $s \in \Gamma(V, \mathcal{P}_{U,H})$ be a local section for some open set $V \subset M$ and define $\bar{s} \in \Gamma(\pi^{-1}(V), \mathcal{P}_{U,F})$ by $\bar{s}(f) := (f, s(f))$. Further, let $\pi_U : \pi^*\mathcal{P}_{U,H} \rightarrow \mathcal{P}_{U,F}$ be the natural projection. Then the following diagram commutes:
\[
\begin{array}{ccc}
F & \xrightarrow{\pi} & \mathcal{P}_{U,F} \\
\downarrow{\bar{s}} & & \downarrow{\phi_{1,F}} \\
M & \xrightarrow{\pi_U} & \mathcal{P}_{1,F} \\
\downarrow{\bar{\pi}_F} & & \downarrow{\pi_F} \\
F = \mathcal{P}_{1,M} & & \mathcal{P}_{1,M}
\end{array}
\]

**Proposition 4.3.** Let $A \in \Omega^1(F,i\mathbb{R})$ be a connection on the $S^1$-bundle $F = \mathcal{P}_{1,M} \xrightarrow{\pi} M$. Then $\bar{\pi}_F^* A \in \Omega^1(\mathcal{P}_{1,F}, i\mathbb{R})$ is a connection on the $S^1$-bundle $\mathcal{P}_{1,F} \xrightarrow{\pi_F} F$. Locally, $A$ and $\bar{\pi}_F^* A$ are related as follows: Let $s \in \Gamma(V, \mathcal{P}_{U,H})$ and let $\bar{s} \in \Gamma(\pi^{-1}(V), \mathcal{P}_{U,F})$ be the induced local section as in Proposition 4.2. It holds that
\[
(\bar{\pi}_F^* A)^{(\phi_{1,M}(s))} = \pi^* \left(A^{\phi_{1,M}(s)}\right) \in \Omega^1(\pi^{-1}(V), i\mathbb{R}).
\]

Let us now turn to spinor fields on $F$. By construction, the $Spin^c(2n+1)$-bundle $Q_F^c \rightarrow M$ reduces to the $Spin^c(2n)$-bundle $Q_F^c := \mathcal{P}_{U,H} \times_{1} Spin^c(2n) \rightarrow M$. We introduce the reduced spinor bundle of $M$,
\[
S_H := S_H^{\text{red}} := Q_F^c \times_{\Phi_{2n}} \Delta^C_{2n} \cong \mathcal{P}_{U,H} \times_{\Phi_{2n}} \Delta^C_{2n}.
\]
This allows us to express the spinor bundle $S_F := S_H^{\text{red}} \rightarrow F$ as
\[
S_F = Q_F^c \times_{\Phi_{2n+1}} \Delta^C_{2n+1} \cong \pi^*\mathcal{P}_{U,H} \times_{\Phi_{2n+1}} \Delta^C_{2n+1} \cong \pi^*S_H \oplus \pi^*S_H.
\]

The second step is purely algebraic and follows from the decomposition of $\Delta^C_{2n+1}$ into the sum $\Delta^C_{2n} \oplus \Delta^C_{2n}$ of $Spin^c(2n) \rightarrow Spin^c(1,2n+1)$-representations as presented in [51], where $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n+2}$ via $x \mapsto (0,0,x)$. This identification allows us to define a global section in $\pi^*S_H \oplus 0 \subset S_F$ in analogy to the Kähler case: $u(-1,\ldots,-1) \in \Delta^C_{2n}$ is the (up to $S^1$-action) unique unit-norm spinor in the Eigenspace of the Kähler form on $\mathbb{R}^{2n}$ to the eigenvalue $-i \cdot n$. Let $s : V \rightarrow \mathcal{P}_{U,H}$ be a local section. We set
\[
\varphi(p) := [\phi_{c,F}(s(p)), u(-1,\ldots,-1)], \quad p \in \pi^{-1}(V).
\]

By (17) this is independent of the choice of $s$. Thus, $\varphi \in \Gamma(F, S_F)$. As last ingredient we introduce the connection
\[
A := \bar{\pi}_F^* A^{W} + A^{W} \in \Omega^1(\mathcal{P}_{1,F}, i\mathbb{R})
\]
on $\mathcal{P}_{1,F} \rightarrow F$\footnote{This is to be read as follows: $\bar{\pi}_F^* A^{W}$ is a connection on $\mathcal{P}_{1,F}$ by Proposition 4.3. Any other connection is obtained by adding an element of $\Omega^1(F,i\mathbb{R})$, which we choose to be the connection $A^{W}$ here, i.e. $A = \bar{\pi}_F^* A^{W} + \pi_F^* A^{W}$.}
Theorem 4.4. The spinor field \( \varphi \in \Gamma(F, S^h_{\theta}) \) is a CCKS wrt. \( A \), i.e. \( \varphi \in \ker P^{A,h_{\theta}} \). The curvature \( dA \in \Omega^2(F, i\mathbb{R}) \) is given by
\[
dA = 2\pi_M^* \text{Ric}^W.
\]
In particular, \( \varphi \) descends to a twistor spinor on a spin manifold iff the Tanaka Webster connection is Ricci flat. The associated vector field \( V_\varphi \) satisfies

1. \( V_\varphi \) is a regular isotropic Killing vector field.
2. \( \nabla^A_\varphi \varphi = \frac{1}{\sqrt{2}} h \varphi \)
3. \( V_\varphi \) is normal, i.e. \( V_\varphi - W_{\theta_\varphi} = 0, V_\varphi - C_{\theta_\varphi} = 0, \) and additionally \( K_{\theta_\varphi}(V_\varphi, V_\varphi) = \text{const.} < 0 \)
4. \( V_\varphi - dA = 0 \)

Proof. Applying the local formula (3) to \( \varphi \) and using Proposition 4.3 we find for a local section \( s = (X_1, \ldots, X_{2n}, T) \in \Gamma(V, \mathcal{F}_{U, T}) \) and a vector \( Y \in \Gamma(\pi^{-1}(V), TF) \) that
\[
\nabla^A_\varphi\varphi|_{\pi^{-1}(V)} = \left( -\frac{n}{4} h^A \varphi + \frac{1}{2} A^W(N) \cdot \varphi, 0 \right),
\]
\[
\nabla^A_\varphi\varphi|_{\pi^{-1}(V)} = \left( \frac{R^W}{4(n + 1)} \varphi - \frac{1}{2} \text{Tr} \omega_s(T) + \frac{1}{2} \left( (A^W)^{\phi_1,\text{\(s\)}}(T) + A^W(T^*) \right) \cdot \varphi, 0 \right),
\]
\[
\nabla^A_\varphi\varphi|_{\pi^{-1}(V)} = \left( -\frac{1}{2} \text{Tr} \omega_s(T) + \frac{1}{2} \left( (A^W)^{\phi_1,\text{\(s\)}}(X) + A^W(X^*) \right) \cdot \varphi, 0 \right) - \frac{1}{4} (X - d\theta)^* \cdot T^* \cdot \varphi.
\]

Here, \( \omega_s := s^* A^W \in \Omega^1(V, \mathfrak{u}(n)) \). By definition, we have that
\[
A^W(N) = i \cdot \frac{n + 2}{2},
\]
\[
(A^W)^{\phi_1,\text{\(s\)}}(T) + A^W(T^*) = \text{Tr} \omega_s(T) - \frac{R^W}{2(n + 1)},
\]
\[
(A^W)^{\phi_1,\text{\(s\)}}(X) + A^W(X^*) = \text{Tr} \omega_s(X).
\]

As for \( X \in \{X_1, \ldots, X_{2n}\} \) the 1-form \( X - d\theta \) acts on the spinor bundle by Clifford multiplication with \( J(X) \), we arrive at
\[
\nabla^A_X \varphi = \frac{1}{2} i \varphi,
\]
\[
\nabla^A_{\varphi} \varphi = 0,
\]
\[
\nabla^A_X \varphi = \left( 0, -\frac{\sqrt{2}}{4} J(X) \cdot \varphi \right).
\]

\footnote{Concretely, in \cite{31}, the induced Webster connection on the line bundle is defined with a different sign which changes the sign of its curvature. Moreover, in \cite{31} the Fefferman spin metric comes with a factor \( \frac{1}{n+2} \) instead of \( \frac{1}{n+2} \).}
As in \cite{31} one concludes that \( h(Y,Y)Y \cdot \nabla Y \phi \) is independent of the vector \( Y \) with length \( \pm 1 \), i.e. \( \phi \in \ker P^A \). Literally as in \cite{31} one calculates that \( V \phi = \sqrt{2}N \) and that \( N \) is Killing. The relation of \( V \phi \) to the curvature of \( h_\theta \) is true for any fundamental vertical vector field on a Fefferman space (see \cite{43}).

Let \( s \in \Gamma(V,\mathcal{P}_U \cdot \mathcal{R}) \). It holds that (cf. \cite{31}) \( dA^W = \text{Tr} \ d\omega_s = \text{Ric}^W \in \Omega^2(M,i\mathbb{R}) \). Considered as a 2-form on \( F \), the curvature \( dA \) is thus using Proposition \ref{prop:4.4} given by

\[
\begin{align*}
   dA &= d\left( \pi_F A^W \right) = \pi^* dA^W = \pi^* d\text{Tr} \omega_s + \pi^* \text{Ric}^W \\
   &= 2\pi^* \text{Ric}^W 
\end{align*}
\]

As \( dA \) is the lift of a 2-form on \( M \), it follows immediately that the fundamental field \( V \phi = \sqrt{2}N \) inserts trivially into \( dA \).

\begin{remark}
Generically, we find only one CCKS on the Fefferman space. One can define another natural global section in \( \mathcal{R} \) in analogy to the spin case in \cite{34}. However, there is in general no \( S^1 \)-connection which turns it into a CCKS. This is in complete analogy to the Kähler case: On a Kähler manifold there is a second natural global section in the spinor bundle constructed out of the eigenspinor to the other extremal eigenvalue of the Kähler form on spinors which in general is no Spin\(^-\)-parallel spinor (cf. \cite{22}).
\end{remark}

As in the Spin-case we can also prove a converse of the last statement:

\begin{theorem}
Let \((B,^2n+1,h)\) be a Lorentzian Spin\(^-\)-manifold. Let \( A \in \Omega^1(\mathcal{P}_1,i\mathbb{R}) \) be a connection on the underlying \( S^1 \)-bundle and let \( \phi \in \Gamma(S^9) \) be a nontrivial CCKS wrt. \( A \) such that

1. The Dirac current \( V := V_{\phi} \) of \( \phi \) is a regular isotropic Killing vector field,
2. \( V \cdot WH = 0 \) and \( V \cdot C^h = 0 \), i.e. \( V \) is a normal conformal vector field,
3. \( V \cdot dA = 0 \),
4. \( \nabla V_{\phi} = ic \phi \), where \( c = \text{const} \in \mathbb{R} \setminus \{0\} \).

Then \((B,h)\) is an \( S^1 \)-bundle over a strictly pseudoconvex manifold \((M^{2n+1},H,J,\theta)\) and \((B,h)\) is locally isometric to the Fefferman space \((F,h_\theta)\) of \((M,H,J,\theta)\).
\end{theorem}

\begin{proof}
The proof runs through the same lines as in the Spin case in \cite{31} and references given there: First, we prove that

\[
K(V,V) = \text{const.} < 0. 
\tag{18}
\]

To this end, we calculate using (11)

\[
V \cdot \nabla V_{\phi} D^A \phi = \frac{\sqrt{2}}{2} V \cdot K^9(V) + c_1 \cdot V \cdot (V \cdot dA) \cdot \phi + c_2 \cdot V \cdot (V \wedge dA) \cdot \phi, 
\]

where the real constants \( c_{1,2} \) are specified by (11). However, as \( V \) is lightlike and \( V \cdot dA = 0 \), the last two summands vanish by (12). Consequently, \( V \cdot \nabla V_{\phi} D^A \phi = \frac{\sqrt{2}}{2} V \cdot K^9(V) = -n \cdot K(V,V) \cdot \phi \).

On the other hand, the twistor equation and our assumptions yield \( V \cdot \nabla V_{\phi} D^A \phi = \frac{\sqrt{2}}{2} \nabla_{\phi}(V \cdot D^A) = -n \cdot \nabla_{\phi} V_{\phi} \cdot \phi \) \( \equiv \) \( -n^2 \cdot \phi \). Consequently, \( K(V,V) = -c^2 \). Regularity of \( V \) implies that there is a natural \( S^1 \)-action on \( B \),

\[
B \times S^1 \ni (p,e^t) \mapsto \gamma^V_t(p) \in B, 
\]

where \( \gamma^V_t(p) \) is the integral curve of \( V \) through \( p \) and \( L \) is the period of the integral curves. Thus, \( M := B/S^1 \) is a \( 2n \times 1 \)-dimensional manifold and \( V \) is the fundamental vector field defined by the element \( \frac{\partial}{\partial t} \in i\mathbb{R} \) in the \( S^1 \)-principal bundle \( (B,\pi,M;S^1) \).

As \( V \) is by assumption normal and satisfies (13), Sparlings characterization of Fefferman spaces applies (see \cite{13}), yielding that there is a strictly pseudoconvex pseudo-hermitian structure \((H,J,\theta)\) on \( M \) such that \( (B,h) \) is locally isometric to the Fefferman space \((F,h_\theta)\) of \((M,H,J,\theta)\). For more details regarding the construction of the local isometries \( \phi_U : B_U \to F_U \cdot \theta \) we refer to \cite{31,43}. \( \square \)

\footnote{From this condition, it follows that \( V \cdot \phi = 0 \).}

\[16\]
5. A partial classification result for the Lorentzian case

We give a complete description of Lorentzian manifolds admitting CCKS under the additional assumption that $V_{\varphi}$ is normal. The proof closely follows the $\text{Spin}^\mathbb{C}$-case from [13]. For a 1-form $\alpha \in \Omega^1(M)$ we define the rank of $\alpha$ to be $\text{rk}(\alpha) := \max \{ n \in \mathbb{N}_0 \mid \alpha \wedge (d\alpha)^n \neq 0 \}$.

**Theorem 5.1.** Let $(M,g)$ be a Lorentzian $\text{Spin}^\mathbb{C}$-manifold admitting a CCKS $\varphi \in \Gamma(S^9)$ wrt. a connection $A \in \Omega^1(P_1,\mathbb{R})$. Assume further that $V := V_{\varphi}$ is a normal conformal vector field. Then locally off a singular set exactly one of the following cases occurs:

1. It holds that $\text{rk}(V^1) = 0$ and $\|V\|^2 = 0$.
   The spinor $\varphi$ is locally conformally equivalent to a $\text{Spin}^\mathbb{C}$-parallel spinor on a Brinkmann space.

2. It holds that $\text{rk}(V^1) = 0$ and $\|V\|^2 < 0$.
   Locally, $[g] = [-dt^2 + h]$, where $h$ is a Riemannian metric admitting a $\text{Spin}^\mathbb{C}$-parallel spinor.
   The latter metrics are completely classified, cf. [24].

3. $n$ is odd and $\text{rk}(V^1) = (n-1)/2$ is maximal.
   $(M,g)$ is locally conformally equivalent to a Lorentzian Einstein Sasaki manifold$^9$. There exist geometric Spin-Killing spinors $\varphi_{1,2}$ on $(M,g)$ which might be different from $\varphi$, but satisfying $V_{\varphi_{1,2}} = V$.

4. $n$ is even and $\text{rk}(V^1) = (n-2)/2$ is maximal.
   In this case, $(M,g)$ is locally conformally equivalent to a Fefferman space.

5. If none of these cases occurs, there exists locally a product metric $g_1 \times g_2 \in [g]$, where $g_1$ is a Lorentzian Einstein Sasaki metric on a space $M_1$ admitting a geometric Killing spinor $\varphi_1$ and $g_2$ is a Riemannian Einstein metric on a space $M_2$ such that $M = M_1 \times M_2$ and $V = V_{\varphi_1}$.

Conversely, given one of the above geometries with a CCKS of the mentioned type, the associated Dirac current $V$ is always normal.

**Proof.** The condition that $V$ is normal is equivalent to say that $\alpha^1_\varphi$ is a normal conformal Killing 1-form (cf. Remark 5.2), which means that the RHS in (13) vanishes. Using tractor calculus for conformal geometries (cf. [14] [15] [16]), we conclude that there exists a 2-form $\alpha \in \Lambda^2_{\mathbb{C}}$ which is fixed by the conformal holonomy representation $\text{Hol}(M,c) \subset SO^+(2n)$. The system of equations (15) allows us to conclude as in (15) that $\alpha = \alpha^\chi_\varphi$ for a spinor $\chi \in \Lambda^2_{\mathbb{C}}$. 2-forms induced by a spinor in signature $(2,n)$ have been classified in [13] and the geometric meaning of a holonomy-reduction imposed by such a fixed $\alpha^\chi_\varphi$ is well-understood. The following possibilities can occur:

$\alpha = l_1 \wedge l_2$ for $l_1,l_2$ mutually orthogonal lightlike vectors. Using nc-Killing form theory as in the proof of Theorem 10 in [13] we conclude that this precisely corresponds to the first case of Theorem 5.1 and that there is locally a metric such that $V$ is parallel. Literally as in in [12] we conclude that also the spinor is $\nabla^A$-parallel in this case.

$\alpha = l \wedge t^i$, where $l$ is a lightlike vector and $t$ a orthogonal timelike vector. Using [18] it follows that there is locally a Ricci-flat metric in the conformal class on which $V$ is parallel. By constantly rescaling the metric, we may assume that $\|V\|^2 = -1$. We have to show that the spinor itself is parallel in this situation. To this end, we calculate:

$$0 = V \omega(V,V) = V(V \cdot \varphi, \varphi) = -\frac{1}{n}((V^2 \cdot D^0 \varphi, \varphi) + (V \cdot \varphi, V \cdot D^0 \varphi))$$

$$= \frac{2}{n} \text{Re}(D^0 \varphi, \varphi)$$

$^9$Note that every simply connected Einstein Sasaki manifold is spin, see [8].
We differentiate this function wrt. an arbitrary vector $X$, use $K^g = 0$ and (11) to obtain

$$0 = \text{Re}(c_1 (X \cdot dA) \cdot \varphi + c_2 (X^3 \wedge dA) \cdot \varphi, \varphi) - \frac{1}{n} \text{Re}(X \cdot D^4 \varphi, D^4 \varphi).$$

The first scalar product vanishes as $\langle (X \cdot dA) \cdot \varphi, \varphi \rangle \in \mathbb{R}$ and $\langle (X^3 \wedge dA) \cdot \varphi, \varphi \rangle = 0$ by Proposition 3.2. Thus, $0 = V_{D \wedge \varphi}$ from which in the Lorentzian case $D^4 \varphi = 0$ follows. It is clear that $\varphi$ descends to a $\text{Spin}^c$–parallel spinor on the Riemannian factor.

$n$ is odd and $\alpha = (\omega_0)_{|V}$, where $V \subset \mathbb{R}^{2,n}$ is a pseudo-Euclidean subspace of signature $(2, n - 1)$ and $\omega_0$ denotes the pseudo-Kähler form on $V$. In this case $\text{Hol}(M, c) \subset SU(1, (n - 1)/2)$. As in [13] we conclude that there is locally a Lorentzian Einstein Sasaki metric $g$ (of negative scalar curvature) in the conformal class. Moreover, $V$ is unit timelike Killing wrt. this metric and belongs to the defining data of the Sasakian structure. It is known from [8] that there are geometric Killing spinors $\varphi_i$ on $(M, g)$ with $V_{\varphi_i} = V$.

$n$ is even and $\alpha = \omega_0$ is the pseudo-Kähler form on $\mathbb{R}^{2,n}$. This corresponds to conformal holonomy in $SU(1, n/2)$ and as known from [13] this is locally equivalent to having a Fefferman space in the conformal class on which a CCKS exists by the preceding section.

$\alpha = (\omega_0)_{|W}$, where $W \subset \mathbb{R}^{2,n}$ is a pseudo-Euclidean subspace of even dimension and signature $(2, k)$, where $4 \leq k < n - 2$ and $\omega_0$ denotes the pseudo-Kähler form on $W$. In this case, the conformal holonomy representation fixes a proper, nondegenerate subspace of dimension $\geq 2$ and is special unitary on the orthogonal complement. As shown in [13] this is exactly the case if locally there is a metric in the conformal class such that $(M, g) = (M_1 \times M_2, g_1 \times g_2)$, where the first factor is Lorentzian Einstein Sasaki. As mentioned before, there exists a geometric $\text{Spin}^c$–Killing spinor inducing $V$ on $M_1$.

Conversely, if one of the geometries from Theorem 5.1 together with a $\text{Spin}^c$–CCKS of mentioned type as in the Theorem is given, it follows that $V_{\varphi}$ is normal conformal: In the first two cases, $\varphi$ is parallel, for which $\text{Ric}(X) \cdot \varphi = 1/2(X \cdot dA) \cdot \varphi$ is known (see [24]). We thus have that $(X - dA)^4 \cdot \alpha^g \in \mathbb{R} \cap \mathbb{R}$. Proposition 3.2 yields that $V_{\varphi} - W^g = 0$. A analogous straightforward but tedious equation yields that $V_{\varphi} - C^g = 0$. In cases 3 and 5 of Theorem 5.1 $V$ is normal as it is induced by a $\text{Spin}^c$–Killing spinor. Case 4 was discussed in the previous section and $V$ is normal by Theorem 4.4.

**Remark 5.1.** We remark that the Spin–Killing spinors $\varphi_i$ in cases 3 and 5 might be different from the spinor $\varphi$ we started with, i.e. it could be the case that on the Lorentzian Einstein Sasaki space, the original spinor $\varphi$ is a CCKS wrt. some nontrivial connection $A$. However, as shown in [24], if $(M, g)$ is an irreducible LES manifold, only $\text{Spin}^c$ structures with $dA = 0$ admit Killing spinors.

**Remark 5.2.** It is easy to think of examples of Lorentzian manifolds admitting a CCKS with non-normal Dirac current: [24] shows that on any Lorentzian 4-manifold admitting a null-conformal vector field $V$, there exists -at least locally- a CCKS $\varphi$ such that $V_{\varphi} = V$. Given a generic null conformal vector field, it will not be normal conformal, and thus the preceding Theorem does not apply. In fact (see [38]), if $V_{\varphi}$ is null and normal conformal on a Lorentzian 4-manifold $(M, g)$, then $(M, g)$ is pointwise conformally flat or of Petrov type N.

**Remark 5.3.** The classification for the Riemannian case seems to differ drastically from the Spin–case. For instance, a CCKS on a Ricci-flat manifold need not be parallel and the CCKS equation does not reduce to the study of parallel or Killing spinors on conformally related metrics as in the spin case. Furthermore, every Riemannian 3-manifold admitting a twistor spinor is conformally flat (see [2]), whereas there are examples of 3-dimensional non-conformally flat $\text{Spin}^c$–manifolds admitting CCKS which can not be rescaled to Killing spinors, see [24].
6. Small dimensions

6.1. A geometric motivation

In physics literature, conformal structures admitting CCKS have been classified for Riemannian and Lorentzian manifolds of dimensions 3 and 4, see [26, 27, 28, 36]. Interestingly, one observes that CCKS yield a spinorial characterization for the existence of certain conformal tensors in these signatures. Let us motivate the classification of low dimensional conformal structures admitting a CCKS from this geometric point of view, taking signature (3, 1) as an example.

Consider the map

$$ l: \Delta_{3,1}^C \setminus \{0\} \cong \mathbb{C}^2 \setminus \{0\} \to L^+ \subset \mathbb{R}^{3,1}, \epsilon \mapsto V_\epsilon, $$

where $L^+$ denotes the forward lightcone. This map is surjective (cf. [38]) and the space $\{\epsilon \in \Delta_{3,1}^C | (\epsilon, \epsilon)_{S^2} = \text{const.}, > 0\}$ is an $S^3$ which is mapped by $l$ to the space of null vectors $z$ with fixed time component $z_0$, i.e. the image is an $S^2$. Thus, $l$ is the Hopf fibration map with fibre $S^1 \cong U(1)$. Similarly one can show that $\Delta_{3,1}^R \setminus \{0\}/S^1 \cong \mathbb{L}^+$. In the spin case, one uses this last observation to prove:

**Theorem 6.1** (see [38], Thm.4.3.8). Let $(M^{3,1}, g)$ be a non-conformally flat Lorentzian manifold admitting a null normal conformal vector field $V$ without zeroes such that its twist $V^1 \wedge dV^3$ vanishes everywhere or nowhere on $M$. Then there exists locally a real twistor spinor $\varphi \in \Gamma(S^2_\mathbb{C})$ such that $V_\varphi = V$.

Thus, in signature (3, 1) real twistor spinors locally characterize the existence of normal conformal null vector fields with a certain twist condition. In view of this, we ask whether the existence of a generic null conformal vector field on $(M^{3,1}, g)$ which is not necessarily normal conformal can be characterized in terms of spinor fields. As passing from a null vector field $V$ to a complex half spinor field $\varphi \in \Gamma(S^2_\mathbb{C})$ via the map $l$ comes with a $U(1)$-ambiguity at each point, i.e. $V = V_\varphi$ iff $V = V_{f\varphi}$ for every $f : M \to S^1$ it seems natural to include a gauge field which precisely gauges this symmetry, which by (4) leads to Spin$^c$-geometry. Indeed, one can now prove the following:

**Proposition 6.2** ([26]). Let $V$ be a null conformal vector field without zeroes on a Lorentzian manifold $(M^{3,1}, g)$. Then there exists locally a connection $A$ and a CCKS $\varphi \in \Gamma(S^2_\mathbb{C})$ wrt. $A$ such that $V = V_\varphi$.

With the same methods, one proves that on a 3-dimensional Lorentzian manifold the existence of a CCKS without zeroes is locally equivalent to the existence of a conformal vector field (cf. [27]). Also in Riemannian signature (4, 0) and (3, 0) the existence of a CCKS yields an equivalent spinorial characterization of natural geometric structures, see [28]. The signatures (2, 1) and (3, 1) in mind, we hope that also in higher (Lorentzian) signatures CCKS might locally characterize the existence of certain conformal, but not necessarily normal conformal tensors. This is indeed the case as we shall see in the next sections.

**Remark 6.1.** In the following, all of our considerations will be local on some open, simply connected set $U \subset M$, i.e. we can always assume that there is a uniquely determined Spin$^c$-structure, the $S^1$-bundle is trivial and $A$ corresponds to a 1-form $A \in \Omega^1(U, \mathbb{R})$.

6.2. 5-dimensional Lorentzian manifolds with a CCKS

The spinor representation in signature (1, 4) is quaternionic, i.e. $\Delta_{1,4}^C \cong \mathbb{H}^2$. However, we prefer to work with complex quantities. We choose a Clifford representation on $\mathbb{C}^4$:

$$
\begin{align*}
e_1 &= \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix},& e_2 &= \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix},& e_3 &= \begin{pmatrix} i & -1 \\ -i & -1 \end{pmatrix},& e_4 &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},& e_0 &= \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.
\end{align*}
$$

(19)
The \( \text{Spin}^c(1,4) \)-invariant scalar product is given by \( \langle v, w \rangle_{\Delta_{1,4}} = (e_0 \cdot v, w)_{S^4} \). According to [3], the nonzero orbits of the action of \( \text{Spin}^c(1,4) \cong \text{Sp}(1,1) \) on \( \Delta_{1,4} \) are given by the level sets of \( v \mapsto \langle v, v \rangle_{\Delta_{1,4}} \in \mathbb{R} \). Consider the spinors \( u_1 = (1 \ 0 \ 0 \ 0)^T, u_0 = (1 \ 1 \ 0 \ 0)^T \in \Delta_{1,4} \).

Straightforward calculation shows:

\[
\begin{align*}
(u_1, u_1)_{\Delta_{1,4}} &= 1, V_{u_1} = e_0, \alpha^2_{u_1} = e_1^+ \wedge e_2^+ + e_3^+ \wedge e_4^+, \alpha^2_{u_1} \cdot u_1 = 2i \cdot u_1,
(u_0, u_0)_{\Delta_{1,4}} &= 0, V_{u_0} = -2(e_0 + e_2), \alpha^2_{u_0} = 2(e_1^+ \wedge (e_0^+ + e_1^+), \alpha^2_{u_0} \cdot u_0 = 0. \tag{20}
\end{align*}
\]

Here, \( \langle \alpha^2_{u_0}, \alpha \rangle_{\Delta_{1,4}} = \frac{1}{2} (\alpha \cdot u, u)_{\Delta_{1,4}} \in \mathbb{R} \) for \( \alpha \in \Lambda^2_{1,4} \).

Let \( (M^{1,4}, g) \) be a Lorentzian \( \text{Spin}^c \)-manifold admitting a CCKS \( \varphi \) wrt. a \( S^1 \)-connection \( A \). Locally, around a given point, one has by omitting singular points either that \( \langle \varphi, \varphi \rangle \neq 0 \) or \( \langle \varphi, \varphi \rangle = 0 \).

In the first case let us assume that \( \langle \varphi, \varphi \rangle > 0 \). The analysis for CCKS of negative length is completely analogous. Thus, locally there are only two cases to consider:

In the first case, we may after rescaling the metric assume that \( \varphi \in \Gamma(S^4) \) is a CCKS with \( \langle \varphi, \varphi \rangle = 1 \). Differentiating the length function and inserting the twistor equation yields that

\[
\text{Re}(X \cdot \varphi, \eta) \equiv 0,
\]

for functions \( b, g, h : U \to \mathbb{R} \). With this preparation, the conformal Killing equation for \( \alpha^2_{\varphi} \) (cf. [15]) is calculated to be

\[
\nabla_{\varphi} \alpha^2_{\varphi} = \text{const.} \cdot \text{Im} \left( \langle \varphi, D^b \varphi \rangle_{S^4} \cdot X^b \wedge V_{\varphi} \right)
\]

In particular, \( \nabla_{\varphi} \alpha^2_{\varphi} = 0 \). We now differentiate \( \alpha^2_{\varphi} \cdot \varphi = 2i \cdot \varphi \) wrt. \( V_{\varphi} \) to obtain \( \alpha^2_{\varphi} \cdot \nabla_{\varphi} V_{\varphi} = 2i \nabla_{\varphi}^V \varphi \). We multiply this equation by \( V_{\varphi} \). By \( \varphi \cdot \nabla_{\varphi} V_{\varphi} = \eta \) by the twistor equation, leading to

\[
\alpha^2_{\varphi} \cdot \eta = 2i \left[ \varphi, (ib \ 0 \ 0 \ g + ih)^T \right] = 2i \eta \left[ \varphi, (ib \ 0 \ 0 \ g + ih)^T \right].
\]

Consequently, \( D^4 \varphi = -5ib \cdot \varphi \), and thus \( \nabla_{\varphi} A_{\varphi} = ib \cdot X \cdot \varphi \). However, it is proved in [16] that this forces \( b \) to be constant, i.e. \( \varphi \) is a \( \text{Spin}^c \)-Killing spinor or a \( \text{Spin}^c \)-parallel spinor. In the second case, \( V_{\varphi} \) is parallel and the metric splits into a product \( (\mathbb{R}, -dt^2) \times (N, h) \) where the Riemannian 4-manifold \( (N, h) \) admits a parallel \( \text{Spin}^c \)-spinor. As moreover \( \alpha^2_{\varphi} \) descends to a parallel 2-form on \( (N, h) \) of Kähler type, we conclude that \( (N, h) \) is Kähler. Conversely, every Kähler \( \text{Spin}^c \)-manifold endowed with its canonical \( \text{Spin}^c \)-structure admits parallel spinors. In the first case, \( \text{Re}(\varphi, D^4 \varphi) = 0, \) thus \( V_{\varphi} \) is a timelike Killing vector field of unit length satisfying \( V_{\varphi} \cdot \varphi = \varphi \). By a constant rescaling of the metric we may moreover assume that the Killing constant is given by \( \pm \frac{1}{\sqrt{2}} \).

Then it is known from [8], Thm 46 that \( V_{\varphi} \) defines a (not necessarily Einstein) Lorentzian Sasaki structure. Conversely, by [29] every Lorentzian Sasaki structure endowed with its canonical \( \text{Spin}^c \)-structure admits imaginary \( \text{Spin}^c \)-Killing spinors.

Let us turn to the second case, i.e. the CCKS satisfies \( \langle \varphi, \varphi \rangle = 0 \). We first remark that in the \( \text{Spin} \)-case, i.e. \( A \equiv 0 \), this always implies that the spinor is locally conformally equivalent to a parallel spinor off a singular set (see [32], Lema 4.4.6). As we shall see, in the \( \text{Spin}^c \)-case something
more interesting happens: By passing to a dense subset we may assume that \( \varphi \) and \( V \varphi \) have no zeroes. We locally rescale the metric such that \( V \varphi \) becomes Killing\(^{10}\) which is by (14) equivalent to

\[
\operatorname{Re}(\varphi, D^A \varphi) = 0
\]

in this metric \( g \). In the chosen metric we also have (see (20) that \( \alpha^2 = r^4 \wedge l^3 \), where \( r \) is a spacelike vector field of constant length orthogonal to \( V \varphi \). Proceeding exactly as in the first case, i.e. locally evaluating the conditions \( \operatorname{Re}(X \cdot \varphi, D^A \varphi) = 0 \) and (22) and inserting this into the conformal Killing form equation (14) leads to

\[
\alpha^2 = \text{const.}\cdot d^* \alpha^2 = \text{const.}\cdot \operatorname{Im}(\langle \varphi, D^A \varphi \rangle)_{S^2} \cdot V^3,
\]

for some real constant. Conversely, a local computation shows that given a conformal Killing form \( \alpha = r^4 \wedge l^3 \) such that \( \alpha^2 = f \cdot l^3 \) and \( r \) is spacelike, orthogonal to \( l \) and of constant length, then \( l \) has to be a Killing vector field. We summarize:

**Proposition 6.3.** Given a CCKS \( \varphi \in \ker P^A \) without zeroes such that \( \langle \varphi, \varphi \rangle = 0 \), the conformal Killing form \( \alpha^2 \) satisfies \( \alpha^2 = r^4 \wedge l^3 \) for a spacelike vector field \( r \). There is a local metric \( g \in c \) such that \( \alpha^2 = \text{const.}\cdot \operatorname{Im}(\langle \varphi, D^A \varphi \rangle)_{S^2} \cdot V^3 \). In this scale, \( V \varphi \) is Killing.

We will now prove that the converse is also true, i.e. given a zero-free conformal Killing 2-form \( \alpha = r^4 \wedge l^3 \in \Omega^{4}(M) \) where \( r \) is a spacelike vector of unit length, \( l \) is a lightlike vector field on \((M, g)\) such that \( d^* \alpha = f \cdot l^3 \), for some function \( f \), then there exists a 1-form \( A \in \Omega^1(U, \mathbb{R}) \) and a CCKS \( \varphi \in \Gamma(U, S^2) \wedge l^3 \) such that \( \alpha^2 = \alpha \) and \( f = \text{const.}\cdot \operatorname{Im}(\langle \varphi, D^A \varphi \rangle)_{S^2} \). We proceed as follows: There exists a local orthonormal frame \( s = (s_0, ..., s_4) \) such that locally \( \alpha = [s, \alpha^2] \). Defining \( \varphi = [s, u_0] \), where \( s \) is the local lift of \( s \) to the spin structure shows that \( \alpha^2 = \alpha \). It is a purely algebraic observation that \( \varphi \) is the up to local \( U(1) \)-action unique spinor field with this property, i.e. the surjective map

\[
\Delta^C_{\lambda, \alpha} : \{ \epsilon | \langle \epsilon, \epsilon \rangle_{\Delta} = 0 \} \to \alpha^2 \in \{ \alpha | \alpha \equiv r^4 \wedge l^3, \| r \|_{\alpha}^2 = 1, \| l \|_{\alpha}^2 = 0, \langle r, l \rangle_{\alpha} = 0 \} \subset \Lambda^2_{\alpha}
\]

is an \( S^1 \)-fibration. Locally, the mentioned properties of the conformal Killing form \( \alpha \) give a linear system of equations for the local connection coefficients \( \omega_{ij} \). By the local formula (22), the property of \( \varphi \) being a CCKS becomes a linear system of equations for the \( \omega_{ij} \) and the \( A_i = A(s_i) \in C^\infty(U, \mathbb{R}) \). A tedious but straightforward computation shows that there is a unique choice of \( A \) such that these equations are indeed satisfied. In our chosen gauge one has that

\[
A_1 = -2i\omega_{34}(s_1), A_2 = -2i\omega_{34}(s_2), A_3 = -2i(\omega_{34}(s_3) + \omega_{14}(s_3)), A_4 = -2i(\omega_{34}(s_4) + \omega_{14}(s_4)), A_0 = -2i\omega_{34}(s_0)
\]

Details of this calculation can be found in the appendix. We summarize our observations:

**Theorem 6.4.** Let \( \varphi \in \Gamma(M, S^2) \) be a CCKS wrt. a connection \( A \) on a Lorentzian 5-manifold \((M, g)\). Locally and off a singular set the metric can be rescaled such that exactly one of the following cases occurs:

1. The spinor is of nonzero length and a parallel Spin\(^c\) spinor on a metric product \( \mathbb{R} \times N \), where \( N \) is a Riemannian 4-Kähler manifold with parallel spinor.
2. \( \varphi \) is an imaginary Spin\(^c\)-Killing spinor of nonzero length, its vector field \( V \varphi \) is Killing and defines a Sasakian structure.
3. \( |\varphi|^2 \equiv 0 \). The conformal Killing form \( \alpha^2 = \alpha \) can be written as \( \alpha \equiv r^4 \wedge l^3 \). There is a scale in which \( d^* \alpha = f \cdot l^3 \), for some function \( f \).

Conversely, for all the geometries listed in 1.-3. there exists (in case 3. only locally) a Spin\(^c\) structure, a \( S^1 \)-connection \( A \) and a CCKS \( \varphi \in \ker P^A \).

\(^{10}\)Choose local coordinates such that \( V = \partial_1 \). If \( g \) is any metric in the conformal class, we have that \( LV g = \lambda g \) for a function \( \lambda \). \( V \) being Killing wrt. \( e^{2\sigma} g \) is equivalent to \( \partial_1 \sigma = -\frac{1}{2} \) which can be solved locally for \( \sigma \).
It is easy to verify that the correspondence in the third part of this Theorem descends to parallel objects, i.e. on a Lorentzian Spin-c-manifold \((M^{1,4}, g)\) there exist a Spin-c-parallel spinor of zero length if and only if there is a parallel 2-form of type \(\alpha = l^b \wedge r^b\). This can be understood well from a holonomy-point of view: The Spin\(^*\)(1,4)-stabilizer of an isotropic spinor in signature (1,4) is by \([5]\) isomorphic to \(\mathbb{R}^3\), its stabilizer under the Spin\(^*\)(1,4) action is thus given by \(SO(2) \times \mathbb{R}^3 \subset SO^*(1,4)\) which is precisely the stabilizer of a 2-form \(\alpha\) as above. Moreover, \([26]\) leads to the spinorial description of geometries admitting certain Killing forms:

**Theorem 6.5.** On every Lorentzian 5-manifold admitting a Killing 2-form of type \(r^b \wedge l^b\) and a Killing vector field \(r\) of unit length and an orthogonal lightlike vector field \(l\), there exists (locally) a CCKS with \(\langle \varphi, D^A \varphi \rangle_{S^5} = 0\) and vice versa.

### 6.3. Other signatures

We investigate the CCKS-equation on manifolds of signature \((0,5), (2,2)\) and \((3,2)\). Together with the last section and the results from \([26, 27, 28]\) this yields a complete local description of geometries admitting CCKS in all signatures for dimension \(\leq 5\). Many steps will be analogous to those carried out in the previous section for the Lorentzian case and we will therefore be brief.

Let us start with the Riemannian 5-case. A Clifford representation of \(Cl_{0,5}\) on \(\Delta_{0,5} = \mathbb{C}^4\) is given by \([10]\) where one has to replace the \(c_0\)-matrix by \(-i \cdot c_0\) (see \([7]\)). The \(\text{Spin}^*(0,5) \cong \text{Sp}(2)\)-invariant scalar product on \(\Delta_{0,5}^\pm\) is just the usual hermitian product on \(\mathbb{C}^4\) and the nonzero orbits of the \(\text{Spin}^*(0,5)\) action on spinors are given by its level sets. Let us consider the spinor \(u = (1 \ 0 \ 0 \ 0)\). We have that \(V_u = c_0, \alpha_u^2\) is the Kähler form on span \(\{e_1, \ldots, e_4\}\) and \(\alpha_u^2 \cdot u = 2i \cdot u\). Now exactly the same considerations as carried out for spinors of nonzero length in the Lorentzian case in the Lorentzian case reveal the following:

**Theorem 6.6.** Let \(\varphi \in \Gamma(S^3)\) be a CCKS of constant length on a 5-dimensional Riemannian Spin-c-manifold \((M, g)\). Locally, exactly one of the following cases occurs:

1. There is a metric split of \((M, g)\) into a line and a 4-dimensional Kähler manifold on which \(\varphi\) is parallel.
2. After a rescaling of the metric, \(\varphi\) is a Spin-c-Killing spinor to Killing number \(\pm \frac{1}{2}\). \(V_{\varphi}\) is a unit-norm Killing vector field which defines a Sasakian structure.

Conversely, these geometries, equipped with their canonical Spin-c-structures, admit Spin-c-parallel/Killing spinors.

Consequently, CCKS in signature \((0,5)\) locally equivalently characterize the existence of Sasakian structures or splits into a line and a Kähler 4-manifold in the conformal class.

Let us finally study some signatures of higher index: \(Cl_{2,2} \cong \text{gl}(4, \mathbb{R})\), and thus the complex representation of \(Cl^c_{2,2}\) on \(\Delta^c_{2,2} = \mathbb{C}^4\) arises as a complexification of the real representation

\[
e_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}
\]

of \(Cl_{2,2}\) on \(\Delta^\mathbb{R}_{2,2} = \mathbb{R}^4\). In this realisation, \(\text{Spin}^*(2,2) \cong \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})\) and the indefinite scalar product on \(\Delta^\mathbb{C}_{2,2}\) given by \(\langle e_1 \cdot e_2 \cdot v, w \rangle_{\mathbb{C}}\) satisfies \(\langle v, v \rangle_{\Delta} \in i \mathbb{R}\). The nonzero orbits of the \(\text{Spin}^*(2,2)\)-action on \(\Delta^\mathbb{C}_{2,2}\) are given by the level sets of \(\langle \cdot, \cdot \rangle_{\Delta}\) where half spinors of zero length are precisely the real half spinors \(\Delta^\mathbb{R}_{2,2}\) multiplied by elements of \(S^1 \subset \mathbb{C}\). These algebraic observations lead to the following local analysis:

Let \((M^{2,2}, g)\) be a Spin-c(2,2)-manifold admitting a nontrivial CCKS halfspinor \(\varphi \in \Gamma(S^3_{C,2})\) wrt. the \(S^1\)-connection \(A\). As we are only interested in local considerations, we may (after passing to open neighborhoods of a given point and omitting a singular set) assume that \(\|\varphi\|_A = 0\) or \(\|\varphi\|_A \neq 0\).
defines a pseudo-Sasakian structure. A manifold, or a real or imaginary Killing spinor and it follows exactly as in the Lorentzian 

\[ \langle \nabla s_i, \varphi \rangle = \epsilon_j s_j \cdot \nabla A \varphi \in \Gamma(S_{C_2,4}^q \otimes iS_{C_2,4}^q) \forall 1 \leq i, j \leq 4 \]  

(25)

Using the local formula 3 and splitting (25) into real and imaginary part, we arrive at \( \epsilon_i A(s_i) \cdot s_i = \epsilon_j A(s_j) \cdot s_j \) which have been shown if \( A \equiv 0 \). Consequently, we are dealing with real \( Spin^c(2, 2) \) twistor half spinors which have been shown to be locally conformally equivalent to parallel spinors, see [16].

If, on the other hand, the spinor norm is nonvanishing, we may rescale the metric such that \( \| \varphi \|^2 = \pm 1 \). Differentiating yields that \( \text{Im} \, (X \cdot \varphi, D^A \varphi)_{S_4} \equiv 0 \). It is purely algebraic to check that this is possible only if \( D^A \varphi = 0 \). Moreover, \( \alpha^C \) is a constant multiple of the pseudo-Kähler form, i.e. \( \varphi \) is a \( Spin^c \)-parallel half spinor on a Kähler manifold of signature \( (2, 2) \). We summarize:

**Theorem 6.7.** Let \( \varphi \in \Gamma(S_{C_2,4}^q) \) be a CCKS on a \( Spin^c \)-manifold \( (M^{2,2}, g) \). Locally, one of the following cases occurs:

1. \( \| \varphi \|^2 = 0 \). This implies \( A \equiv 0 \). The spinor can be locally rescaled to a parallel spinor with normal form of the metric given in [6,3] .

2. There is a scale such that \( \| \varphi \|^2 = \text{const} \). In this case, \( \varphi \) is a parallel \( Spin^c \)-CCKS on a pseudo-Kähler manifold.

In particular, CCKS half spinors of nonzero length equivalently characterize the existence of pseudo-Kähler metrics in the conformal class.

Finally, we present a classification of local geometries admitting CCKS in signature \( (3,2) \). A real representation of \( Cl_{3,2} \) on \( \Delta_{3,2}^C = \mathbb{R}^4 \) is given by

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

The complex representation on \( \Delta_{3,2}^C \cong \mathbb{C}^4 \) arises by complexification and in this realisation \( Spin^c(3, 2) \cong Sp(2, \mathbb{R}) \). The scalar product \( \langle \cdot, \cdot \rangle_{\Delta_{3,2}^C} \) is given by \( \langle v, w \rangle_{\Delta_{3,2}^C} = v^T J w \), where \( J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \).

Note that \( \langle v, w \rangle_{\Delta_{3,2}^C} \in i\mathbb{R} \). Orbit representatives for the action of \( Spin^c(3, 2) \) on \( \Delta_{3,2}^C \) are \( u := (1 \ 0 \ 0 \ 0), u_0 := (i \ 1 \ 0 \ 0) \) and \( u_b := \sqrt{2} \begin{pmatrix} 1 & 0 & i & b \end{pmatrix} \), where \( b \in \mathbb{R} \setminus \{0\} \). One calculates that

\[
\begin{align*}
\langle u, u \rangle_{\Delta_{3,2}^C} &= 0, \quad V_a = 0, \quad \alpha^2 = (e^1_b - e^1_0) \wedge (e^1_b - e^1_0), \quad \alpha^2_a \cdot u = 0, \\
\langle u_0, u_0 \rangle_{\Delta_{3,2}^C} &= \langle u_b, u_b \rangle_{\Delta_{3,2}^C} = 0, \quad V_{u_0} = 2(e^1_0 - e^1_b), \quad \alpha^2_{u_0} = 2e^1_0 \wedge (e^1_b - e^1_0), \quad \alpha^2_{u_0} \cdot u_0 = 0, \\
\langle u_1, u_1 \rangle_{\Delta_{3,2}^C} &= -i, \quad V_{u_1} = -e^1_2, \quad \alpha^2_{u_1} = (e^1_2 - e^1_3) \wedge (e^1_2 + e^1_3), \quad \alpha^2_{u_1} \cdot u_1 = -2i \cdot u_1.
\end{align*}
\]

(26)

Let \( (M^{3,2}, g) \) be a \( Spin^c \)-manifold with CCKS \( \varphi \in \ker P^A \). In our local analysis, we have two cases to consider: In the first case, we find a metric \( g \) in the conformal class such that \( \| \varphi \|^2 = \pm i \). Using \( P^A \) it follows exactly as in the Lorentzian \((1, 4)\)-case that after constantly rescaling the metric, \( \varphi \) is either parallel, in which case by \( (26) \) the metric splits into a timelike line and a pseudo-Kähler manifold, or a real or imaginary Killing spinor and \( V_\varphi \) which is a timelike unit Killing vector field, defines a pseudo-Sasakian structure.
In the second case, we have that $\|\varphi\|^2 \equiv 0$. If $\varphi$ is of orbit type $u \in \Delta^5_2$ on an open set, it follows exactly as in the signature $(2, 2)$ case that $A \not\equiv 0$, i.e., $\varphi$ is an ordi- nary Spin–twistor. The local analysis for this case has been carried out in [13, 10]. Thus, we are left with the case that $\varphi$ is locally of orbit type $u_{00}$. However, the analysis of this case is completely analogous to the case of Lorentzian Spin$^c$ CCKS of nonzero length and one gets a one-to-one correspondence to certain conformal Killing forms. Carrying out these steps is straightforward and we arrive at

**Theorem 6.8.** Let $(M^{3, 2}, g)$ be a Spin$^c$–manifold of signature $(3, 2)$ and let $\varphi \in \Gamma(S^9)$ be a CCKS wrt. a non-trivial $S^1$–connection $A$ satisfying $\|\varphi\|^2 \equiv 0$. Then there is a scale in which the conformal Killing form $\alpha_2^\varphi$ writes as $\alpha_2^\varphi = r^i \wedge V^i_\varphi$, where $r$ is a spacelike vector field of constant length, $V_\varphi$ is orthogonal to $r$ and lightlike Killing and moreover $d^* \alpha_2^\varphi = \text{const} \cdot \text{Im}(D^A\varphi, \varphi)_{S^9} \cdot V^i_\varphi$. Conversely, if $\alpha = r^i \wedge l^i$ is a conformal Killing form such that $r$ is of constant positive length, $l$ is lightlike and orthogonal to $r$ and $d^* \alpha = f \cdot l^i$ for some function $f$, then there exists a non-trivial $S^1$–connection $A$ and a up to $S^1$–action unique CCKS $\varphi$ wrt. A such that $\alpha_2^\varphi = \alpha$ and $f = \text{const} \cdot \text{Im}(D^A\varphi, \varphi)_{S^9}$.

7. Appendix

We fill in the details of the argument of section [62]. In the notation of this section, we show that if the locally given 2–form $\alpha = \alpha_2^\varphi = [s, \alpha_2^\varphi_s] = s^i_\varphi \wedge (s^3_\varphi + s^3_\varphi^s)$ where $s = (s_0, \ldots, s_4)$ is a local orthonormal frame, is a conformal Killing 2-form such that $\alpha_s = \tilde{f} \cdot V^i_\varphi = \tilde{f} \cdot (s^i_\varphi + s^i_\varphi^s) = \tilde{f} \cdot (s^i_\varphi + s^i_\varphi^s)$ for some function $\tilde{f}$, then there is a uniquely determined 1-form $A \in \Omega^1(U, \mathbb{R})$ such that the spinor $\varphi = [\bar{s}, u_0]$ is a CCKS wrt. $A$. To this end, note that by the equivalent characterization of conformal Killing forms in [71], the requirement on $\alpha$ is equivalent to

$$X - \nabla_X^g \alpha = f \cdot (X - (Y^i \wedge V^i_\varphi) + Y - (X^i \wedge V^i_\varphi)) \quad \forall X, Y \in TM,$$

(27)

where $f = \text{const} \cdot \tilde{f}$. We let $X, Y$ run over the local ONB $(s_0, s_1, s_2, s_3, s_4)$ and use the formula

$$\nabla_X^g (s^i_\varphi \wedge s^j_\varphi) = \sum_j \epsilon_{i j k} \omega_{ij}(X)s^k_\varphi + \sum_j \epsilon_{k j i} \omega_{kj}(X)s^k_\varphi \wedge s^j_\varphi.$$

to obtain that (27) is equivalent to the following system of linear equations in the functions $\omega_{ij}^k := \epsilon_{ij k} g(\nabla_s s_i, s_j)$:

- $\omega_{0 20}^1 = f_0 \omega_{1 23} + \omega_{1 30}^1 = 0, \omega_{1 24}^1 + \omega_{1 40}^1 = 0,$
- $\omega_{2 02} = 0, \omega_{2 12} - \omega_{1 10} = 0, \omega_{2 24} + \omega_{2 40} = 0, \omega_{2 30} = \omega_{1 13},$
- $\omega_{3 10} = -\omega_{3 20}^1, \omega_{3 23} + \omega_{3 30} = 0, \omega_{3 24} + \omega_{3 40} = 0,$
- $\omega_{1 4}^1 = 0, \omega_{2 4}^1 = 0, \omega_{2 2} - \omega_{2 0} = f,$
- $\omega_{2 3}^1 + \omega_{3 0} = -\omega_{2 1}^1, \omega_{1 1}^1 = -f, \omega_{2 4}^1 = 0, \omega_{3 4}^1 = 0, \omega_{3 2}^1 = -f,$
- $\omega_{3 2} = 0, \omega_{3 1} = 0, \omega_{3 0} = 0, \omega_{3 0}^1 = 0,$
- $\omega_{4 3} = -\omega_{4 3}^1,$
- $\omega_{4 0} = 0, \omega_{4 0} = 0, \omega_{4 1} = 0, \omega_{4 2} = 0, \omega_{4 3} = 0, \omega_{4 4} = 0,$
- $\omega_{4 4} + \omega_{4 0} = 0, \omega_{4 1} = 0, \omega_{4 2} = 0, \omega_{4 3} = 0, \omega_{4 4} = 0,$
- $\omega_{4 2} = 0, \omega_{4 3} = 0, \omega_{4 4} = 0.$

11By [23] we then necessarily have that $\tilde{f}$ is a constant multiple of $\text{Im} \langle D^A\varphi, \varphi \rangle_{S^9}$.
Note that these equations already show that $V_\varphi$ is a Killing vector field, which in this frame is equivalent to the equations
\[ \epsilon_j (\omega^i_{2j} - \omega^i_{0j}) + \epsilon_i (\omega^i_{2i} - \omega^i_{0i}) = 0. \]

On the other hand, by the local formula (3), the twistor equation for $\varphi$ is equivalent to the equations
\[ \epsilon_i e_i \cdot \sum_{k<l} \omega^i_{kl} e_k \cdot e_l \cdot u_0 + A_i \cdot u_0 = \epsilon_j e_j \cdot \sum_{k<l} \omega^j_{kl} e_k \cdot e_l \cdot u_0 + A_j \cdot u_0, \tag{28} \]
for $0 \leq i < j \leq 4$ and $A_i := A(s_i) : U \to \mathbb{R}$. Inserting the above $\alpha$–equations, it is pure linear algebra to check that (28) holds if and only if we set the local functions $A_i$ as given in (24).

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