On the construction of cospectral graphs for the adjacency and the normalized Laplacian matrices

M. Rajesh Kannan and Shivaramakrishna Pragada

ABSTRACT
In [A note about cospectral graphs for the adjacency and normalized Laplacian matrices. Linear Multilinear Algebra. 2010;58(3-4):387–390], Butler constructed a family of bipartite graphs, which are cospectral for both the adjacency and the normalized Laplacian matrices. In this article, we extend this construction for generating larger classes of bipartite graphs, which are cospectral for both the adjacency and the normalized Laplacian matrices. Also, we provide a couple of constructions of non-bipartite graphs, which are cospectral for the adjacency matrices but not necessarily for the normalized Laplacian matrices.

1. Introduction
Spectral graph theory studies some of the properties of a graph by associating various types of matrices with it. The spectrum of the matrices associated with a graph determines some of the structural properties of that graph. However, not all features are revealed by the spectrum of the associated matrices. One such instance is that two non-isomorphic graphs can have the same spectrum. In this paper, we propose some methods which could be used to construct a pair of non-isomorphic graphs sharing the same spectrum.

All graphs considered in this paper are simple and undirected. Let $G = (V, E)$ be a graph with the vertex set $V = \{1, 2, \ldots, n\}$ and the edge set $E$. If two vertices $i$ and $j$ of $G$ are adjacent, we denote it by $i \sim j$. The adjacency matrix $A = [a_{ij}]$ of a graph $G$, on $n$ vertices, is an $n \times n$ matrix defined by

$$a_{ij} = \begin{cases} 1, & \text{if } i \sim j, \\ 0, & \text{otherwise.} \end{cases}$$

The spectrum, denoted by $\text{spec}(G)$, of $G$ is the set of all eigenvalues of $A(G)$, with corresponding multiplicities. It has been a longstanding problem to characterize graphs that are determined by their spectrum [1,2]. For a graph $G$, if $\text{spec}(H) = \text{spec}(G)$ for any other graph $H$ implies $H$ is isomorphic to $G$, then $G$ is determined by its spectrum, or a DS graph.
for short, if it is not the case, then $G$ is not determined by its spectrum or NDS graph for short. In [3], Schwenk proved that almost all trees are cospectral. In [4], Godsil and McKay provided a method for constructing NDS graphs, that is, constructing non-isomorphic graphs with the same spectrum. In [1], van Dam and Haemers conjectured that almost all graphs are DS. For more details, we refer to [1,2,4–7].

For each vertex $i$ of a graph $G$, let $d_i$ denote the degree of the vertex $i$. Let $D$ denote the diagonal matrix whose $(i,i)$-th entry is $d_i$. Then the matrix $L = D - A$ is called the Laplacian matrix of the graph $G$, and the matrix $\mathcal{L} = D^{-1/2}LD^{-1/2}$ is called the normalized Laplacian matrix of a graph $G$ without isolated vertices. For more details, we refer to Chung's book [8]. In [9], Butler and Grout provided various constructions that give families of graphs which are cospectral for the normalized Laplacian matrices, while, in [10], Butler constructed families of bipartite graphs which are cospectral for both the adjacency and the normalized Laplacian matrices.

In this paper, we generalize a construction given by Butler [10], which will provide a construction of large classes of NDS bipartite graphs, which are cospectral for both the adjacency and the normalized Laplacian matrices. Thereupon, we extend this construction in a different way, which leads to a larger class of graphs that are cospectral for the adjacency matrices.

This article is organized as follows: In Section 2, we collect necessary definitions and known results, which will be used later on in the paper. In Section 3, we present the main results.

2. Definitions, notions and known results

In this section, we collect some of the known definitions and results. The direct sum of two matrices $A$ and $B$ is defined to be the block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, where the off-diagonal zero matrices are of an appropriate order. The direct sum of two matrices $A$ and $B$ is denoted by $A \oplus B$.

In [10], Butler established the following theorem.

**Theorem 2.1**: Let $p \geq q$, and let $B$ be a $p \times q$ matrix. If

$$C = \begin{bmatrix} 0 & B & B \\ B^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix}$$

$((2q + p) \times (2q + p) \text{ matrix})$ and

$$E = \begin{bmatrix} 0 & B^T & B^T \\ B & 0 & 0 \\ B & 0 & 0 \end{bmatrix}$$

$((2p + q) \times (2p + q) \text{ matrix})$, then $C \oplus 0_{p-q}$ and $E$ are cospectral.

For a non-negative symmetric matrix $M$ with positive row sums, the normalized Laplacian of the matrix $M$, denoted by $L(M)$, is defined as $L(M) = I - D^{-1/2}MD^{-1/2}$ where $D$ is the diagonal matrix with diagonal entries composed of the row sums of $M$ [10].
In the case of the adjacency matrix corresponding to a graph, zero row sums correspond to the isolated vertices in the graph. So $A$, the adjacency matrix of the graph with zero row sums, could be decomposed as $A = A_1 \oplus 0_k$, where $A_1$ is the adjacency matrix of all components which do not contain isolated vertices, and $k$ is the number of isolated vertices. In this case, the normalized Laplacian is defined as follows: $L(A) = L(A_1 \oplus 0_k) = L(A_1) \oplus 0_k$, where $0_k$ denotes the $k \times k$ zero matrix [8].

**Theorem 2.2 ([10]):** Let $L(M)$ denote the normalized Laplacian of a non-negative symmetric matrix $M$ with positive row sums, then we have the following:

1. A vector $x$ is an eigenvector of $L(M)$ if and only if $y = D^{-1/2}x$ (known as the harmonic eigenvector) satisfies $(D - M)y = \lambda Dy$.
2. If $M$ has 0 as an eigenvalue with multiplicity $q$, then $L(M)$ has 1 as an eigenvalue with multiplicity at least $q$.

**Theorem 2.3 ([10, Lemma 1.1]):** Let $p \geq q$, and let $B$ be a $p \times q$ matrix. If

$$C = \begin{bmatrix} 0 & B & B \\ B^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix}$$

($(2q + p) \times (2q + p)$ matrix) and $E = \begin{bmatrix} 0 & B^T & B^T & 0 \\ B & 0 & 0 & 0 \\ B^T & 0 & 0 & 0 \end{bmatrix}$ ($(2p + q) \times (2p + q)$ matrix), then

$$\operatorname{spec}(L(E)) = \operatorname{spec}(L\left(\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}\right)) \cup \{1, \ldots, 1\}_{p \text{-times}}$$

and

$$\operatorname{spec}(L(C)) = \operatorname{spec}(L\left(\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}\right)) \cup \{1, \ldots, 1\}_{q \text{-times}}.$$
By comparing the second and third components, we get

\[ K(z - y) = -\lambda(z - y). \]

Thus, either \(-\lambda\) is an eigenvalue of the matrix \(K\) or the vectors \(y\) and \(z\) are the same.

### 3.1. Construction I

Let \(I\) and 0 denote the identity matrix and the zero matrix of appropriate order, respectively.

**Theorem 3.1:** Let \(B\) be a \(p \times q\) matrix where \(p \geq q\). Let

\[
C = \begin{bmatrix}
0 & B & B & \ldots & B \\
B^T & 0 & 0 & \ldots & 0 \\
B^T & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B^T & 0 & 0 & \ldots & 0
\end{bmatrix}
\]

be an \((nq + p) \times (nq + p)\) matrix, and let

\[
E = \begin{bmatrix}
0 & B^T & B^T & \ldots & B^T \\
B & 0 & 0 & \ldots & 0 \\
B & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B & 0 & 0 & \ldots & 0
\end{bmatrix}
\]

be an \((np + q) \times (np + q)\) matrix. Then the matrices \(E\) and \(C \oplus 0_{(n-1)(p-q)}\) are cospectral.

**Proof:** The matrix \(C\) has 0 as an eigenvalue with multiplicity at least \((n - 1)q\). Let \(\lambda\) be an eigenvalue of \(B^T B\), and \(\begin{bmatrix} x \\ y \end{bmatrix}\) be an eigenvector corresponding to \(\lambda\). Then, \(B^T x = \lambda y\) and \(By = \lambda x\). Now,

\[
\begin{bmatrix}
0 & B & B & \ldots & B \\
B^T & 0 & 0 & \ldots & 0 \\
B^T & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B^T & 0 & 0 & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
\sqrt{n}x \\
y \\
\sqrt{n}B^T x
\end{bmatrix}
= \lambda
\begin{bmatrix}
\sqrt{n}x \\
y \\
\sqrt{n}B^T x
\end{bmatrix}.
\]

Hence, the eigenvalues of \(C\) are \(\lambda \sqrt{n}\) and 0 with multiplicity \((n - 1)q\), where \(\lambda\) is an eigenvalue of the matrix \(B^T B\).
It is easy to see that the matrix $E$ has eigenvalue 0 with multiplicity at least $(n - 1)p$, and the remaining eigenvalues of $E$ are $\lambda \sqrt{n}$, where $\lambda$ is an eigenvalue of the matrix $\begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix}$. For,

$$
\begin{bmatrix}
0 & B^T & B^T & \ldots & B^T \\
B & 0 & 0 & \ldots & 0 \\
B & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B & 0 & 0 & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
\sqrt{ny} \\
x \\
x \\
\vdots \\
x
\end{bmatrix}
= 
\begin{bmatrix}
\sqrt{ny} \\
\sqrt{n}By \\
x \\
\vdots \\
x
\end{bmatrix}
= \lambda \begin{bmatrix}
\sqrt{ny} \\
x \\
x \\
\vdots \\
x
\end{bmatrix}.
$$

So the matrices $C$ and $E$ have the same set of non-zero eigenvalues (including multiplicity). If $p = q$, then $C$ and $E$ are cospectral. If $p \neq q$ and $p > q$, then the matrices $E$ and $C \oplus 0_{(n-1)(p-q)}$ are cospectral.

**Remark 3.1:** In Theorem 3.1, if the entries of the matrix $B$ are either 0 or 1, $p \geq q$ and $B$ is chosen such that the maximum row sum of $B$ is different from the maximum column sum of $B$, then the graphs associated with $E$ and $C \oplus 0_{(n-1)(p-q)}$ as adjacency matrices, respectively, are cospectral. But these graphs are not isomorphic.

**Remark 3.2:** From the graph theoretic perspective, the construction I is an unfolding of a bipartite graph with respect to the vertex bipartition for $n$ times. If $G$ is a bipartite graph with bipartition of the vertex set $V = V_1 \cup V_2$ with $|V_1| = |V_2|$ and edge set $E$, then the graphs generated in Remark 3.1 are the following:

1. $G_1$ is the graph with the vertex set $V_1 \cup V_2 \cup \cdots \cup V_2$, and a vertex in any copy of $V_2$ is adjacent to a vertex of $V_1$ if and only if the corresponding vertices are adjacent in $G$.
2. $G_2$ is the graph with the vertex set $V_2 \cup V_1 \cup \cdots \cup V_1$, and a vertex in any copy of $V_1$ is adjacent to a vertex of $V_2$ if and only if the corresponding vertices are adjacent in $G$.

The construction ensures that the graphs $G_1$ and $G_2$ are cospectral.

Now, let us illustrate the above construction with an example. For $p = 3$, $q = 2$ and $n = 3$, let us generate a pair of non-isomorphic, but cospectral graphs with respect to the adjacency matrices. Let $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. In the graphs illustrated below, vertex labelled $\{1, 2, \ldots, n\}$ corresponds to graph (Figure 1) whose adjacency matrix is $E$, and vertex labelled $\{1', 2', \ldots, n'\}$ corresponds to graph (Figure 2) whose adjacency matrix is $C \oplus 0_2$.

In the next theorem, we establish that the normalized Laplacians of the matrices $E$ and $C$ are cospectral when $p = q$.

**Theorem 3.2:** Let $B$, $C$, and $E$ be three matrices defined as in Theorem 3.1. If the row sums and the column sums of $B$ are positive, then,

$$
\text{spec}(L(C)) = \text{spec} \left( L \left( \begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix} \right) \right) \cup \{ \underbrace{1, \ldots, 1}_{((n-1)q)\text{-times}} \}.
$$
and

$$\text{spec}(L(E)) = \text{spec} \left( L \left( \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \right) \right) \cup \{ 1, \ldots, 1 \} \text{ times}$$

**Proof:** The matrices $E$ and $C$ have $0$ as an eigenvalue with multiplicity at least $(n-1)p$ and $(n-1)q$, respectively. Thus, by Theorem 2.2, the spectrum of both the matrices $L(E)$ and $L(C)$ have $1$ as an eigenvalue with multiplicity $(n-1)p$ and $(n-1)q$, respectively.

Now, the remaining eigenvalues are constructed using the eigenvectors of $L \left( \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \right)$.

Now,

$$\begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & B^T \\ 0 & B & 0 \end{bmatrix} - \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} D_1 - B \\ -B^T & D_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} D_1x - By \\ D_2y - B^T x \end{bmatrix} = \lambda \begin{bmatrix} D_1x \\ D_2y \end{bmatrix}.$$ 

So the remaining eigenvalues of $L(C)$ are given by

$$\begin{bmatrix} nD_1 & -B & \ldots & -B^T \\ -B^T & D_2 & 0 & \ldots & 0 \\ -B^T & 0 & D_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -B^T & 0 & 0 & \ldots & D_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} nD_1x - nBy \\ D_2y - B^T x \\ D_2y - B^T x \end{bmatrix} = \lambda \begin{bmatrix} D_1x \\ D_2y \end{bmatrix}.$$
Thus

\[
\text{spec}(L(C)) = \text{spec}\left( L\left( \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \right) \right) \cup \{ 1, \ldots, 1 \}_{((n-1)q)-\text{times}}.
\]

The remaining eigenvalues of \( L(E) \) are given by

\[
\begin{bmatrix}
  nD_2 & -B^T & -B^T & \ldots & -B^T \\
  -B & D_1 & 0 & \ldots & 0 \\
  -B & 0 & D_1 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  -B & 0 & 0 & \ldots & D_1 \\
\end{bmatrix}
\begin{bmatrix} y \\ x \\ x \\ \vdots \\ x \end{bmatrix}
= 
\begin{bmatrix}
  nD_2 y - nB^T x \\
  D_1 x - By \\
  D_1 x - By \\
  \vdots \\
  D_1 x - By \\
\end{bmatrix}
= \begin{bmatrix} \lambda \\ D_1 x \\ D_1 x \\ \vdots \\ D_1 x \end{bmatrix}.
\]

Thus,

\[
\text{spec}(L(E)) = \text{spec}\left( L\left( \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \right) \right) \cup \{ 1, \ldots, 1 \}_{((n-1)q)-\text{times}}.
\]

\[\blacksquare\]

**Remark 3.3:** In Theorem 3.1, if the entries of the matrix \( B \) are either 0 or 1, \( p = q \) and the matrix \( B \) is chosen such that the maximum row sum of \( B \) is different from the maximum column sum of \( B \), then the graphs with the adjacency matrices \( E \) and \( C \) are cospectral. Also, by Theorem 3.2, the normalized Laplacian matrices corresponding to these graphs are cospectral. But the graphs are not isomorphic.

Next, we illustrate the above construction with an example. For \( p = q = n = 3 \), let us generate a pair of non-isomorphic, but cospectral graphs with respect to both the adjacency and the normalized Laplacian matrices. Let \( B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \). In the graphs illustrated below, vertex labelled \( \{1, 2, \ldots, n\} \) corresponds to graph (Figure 3) whose adjacency matrix is \( E \), and vertex labelled \( \{1', 2', \ldots, n'\} \) corresponds to graph (Figure 4) whose adjacency matrix is \( C \).

### 3.2. Generalization of construction I

Let \( n \) be a positive integer, and \( \sigma(n) \) denote the number of divisors of \( n \). Let \( B \) be a \( p \times q \) matrix, and \( k \) be a divisor of \( n \). Let us construct the block matrix \( F_k \) of size \( kq + \frac{n}{k}p \) as
follows: First row of the block matrix $F_k$ consists of $k$ number of $B$’s such that the $B$’s are kept in $(1,2)$th, $(1,3)$th, ... , $(1,k+1)$th position of $F_k$, and the remaining $n-k$ blocks of $F_k$ are filled with zero matrices of appropriate order. Symmetrically fill the first column of the block matrix $F_k$. Now, if the $i$th entry of the first column $F_k$ is the zero block, then set the $i$th row of the matrix $F_k$ equals to the first row of $F_k$. Let us fill the remaining entries symmetrically. In this process, the matrix $B$ is used for $n$ times, and hence, by symmetry, the matrix $B^T$ is also used for $n$ times. The matrix $F_k$ constructed as above is the following:

\[
\begin{bmatrix}
0 & B & \ldots & B & 0 & \ldots & 0 \\
B^T & 0 & \ldots & 0 & B^T & \ldots & B^T \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
B^T & 0 & \ldots & 0 & B^T & \ldots & B^T \\
0 & B & \ldots & B & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & B & \ldots & B & 0 & \ldots & 0
\end{bmatrix}
\]

It is easy to see that there are $\sigma(n)$ possibilities for the matrix $F_k$. In the next theorem, we shall show that all these $\sigma(n)$ matrices have all non-zero eigenvalues in common.

**Theorem 3.3:** Let $B$ be a $p \times q$ matrix. Consider the family of symmetric matrices $F_k$ constructed as above. Then the matrices $F_k \oplus 0_{(n-k)p+(1-k)q}$, with $k$ varying over the set of all divisors of $n$, are cospectral.

**Proof:** Let us compute the spectrum of the matrix $F_k$. As we have $k$ columns with the first entry as $B$ are the same, so 0 is an eigenvalue with multiplicity at least $(k-1)q$. Similarly, we have $\frac{n}{k}$ columns with first entry as 0 are the same, so 0 is an eigenvalue with multiplicity at least $(\frac{n}{k} - 1)p$.

The remaining eigenvalues are obtained by using the eigenvectors of the matrix $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$, which is illustrated below.
Let $\lambda$ be an eigenvalue of $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$, and $\begin{bmatrix} x \\ y \end{bmatrix}$ be an eigenvector corresponding to $\lambda$. As, $B^T x = \lambda y$ and $By = \lambda x$, we get

$$\begin{bmatrix}
0 & B & \ldots & B & 0 & \ldots & 0 \\
B^T & 0 & \ldots & 0 & B^T & \ldots & B^T \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
B^T & 0 & \ldots & 0 & B^T & \ldots & B^T \\
0 & B & \ldots & B & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & B & \ldots & B & 0 & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
\sqrt{kx} \\
\sqrt{\frac{n}{k}xy} \\
\sqrt{\frac{n}{k}y} \\
\sqrt{\frac{n}{k}xy} \\
\sqrt{kx} \\
\sqrt{\frac{n}{k}y} \\
\sqrt{\frac{n}{k}xy} \\
\sqrt{\frac{n}{k}y}
\end{bmatrix}
= \begin{bmatrix}
k\sqrt{\frac{n}{k}By} \\
\sqrt{\frac{n}{k}ky} \\
\sqrt{\frac{n}{k}By} \\
\sqrt{\frac{n}{k}ky} \\
k\sqrt{\frac{n}{k}By} \\
\sqrt{\frac{n}{k}ky} \\
k\sqrt{\frac{n}{k}By} \\
\sqrt{\frac{n}{k}ky}
\end{bmatrix}
= \lambda \sqrt{n}
\begin{bmatrix}
\sqrt{kx} \\
\sqrt{\frac{n}{k}xy} \\
\sqrt{\frac{n}{k}y} \\
\sqrt{\frac{n}{k}xy} \\
\sqrt{kx} \\
\sqrt{\frac{n}{k}y} \\
\sqrt{\frac{n}{k}xy} \\
\sqrt{\frac{n}{k}y}
\end{bmatrix}.$$

Thus, the remaining eigenvalues of $F_k$ are $\lambda \sqrt{n}$, where $\lambda$ is an eigenvalue of $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$.

Hence, the collection of matrices $F_k$ have all non-zero eigenvalues in common and the collection of matrices generated by $F_k \oplus 0_{(n-\frac{n}{k})p+(1-k)q}$ with $k$ varying over the divisors of $n$ are cospectral.

**Remark 3.4:** In Theorem 3.3, if the entries of the matrix $B$ are either 0 or 1, $p \geq q$ and if $B$ is such that the maximum row sum of $B$ is different from the maximum column sum of $B$, then the family of graphs associated with $F_k \oplus 0_{(n-\frac{n}{k})p+(1-k)q}$ as adjacency matrices, respectively, are cospectral. But these graphs are not isomorphic. In the construction of matrix $F_k$ even if we permute 0’s and B’s, we get the same graph with permuted labels. ■

Next, we illustrate the above construction with an example. For $p = 3$, $q = 2$ and $n = 4$. As the number of divisors of 4 is 3, let us generate 3 graphs which are cospectral with respect to the adjacency matrices, but non-isomorphic.

Let $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$. In the graphs illustrated below, vertex labelled $\{1, 2, \ldots, n\}$ corresponds to the graph (Figure 5) of the adjacency matrix $F_1$ with $k = 1$, vertex labelled $\{1', 2', \ldots, n'\}$ corresponds to the graph (Figure 6) of the adjacency matrix $F_4 \oplus 0_3$ with $k = 4$, and vertex labelled $\{1'', 2'', \ldots, n''\}$ corresponds to the graph (Figure 7) of the adjacency matrix $F_2 \oplus 0_4$ with $k = 2$. 
3.3. Construction II

**Theorem 3.4**: Let $B$ be a $p \times q$ matrix, let $K$ be a $q \times q$ symmetric matrix and $K'$ be a $p \times p$ symmetric matrix. Let

\[
A = \begin{bmatrix}
K' & B & B \\
B^T & 0 & K \\
B^T & K & 0
\end{bmatrix}
\]

be a $(p + 2q) \times (p + 2q)$ matrix and

\[
C = \begin{bmatrix}
K & B^T & B^T \\
B & 0 & K' \\
B & K' & 0
\end{bmatrix}
\]

be a $(2p + q) \times (2p + q)$ matrix. Then $A \oplus \begin{bmatrix} 0 & K' \\ K & 0 \end{bmatrix} \oplus K$ and $C \oplus \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} \oplus K'$ are cospectral matrices.

**Proof**: If $\lambda$ is an eigenvalue of the matrix $K$ with an eigenvector $y$, then

\[
\begin{bmatrix}
K' & B & B \\
B^T & 0 & K \\
B^T & K & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
y \\
-y
\end{bmatrix}
= \begin{bmatrix}
0 \\
-Ky \\
Ky
\end{bmatrix}
= -\lambda \begin{bmatrix}
0 \\
y \\
-y
\end{bmatrix}.
\]
Thus, $-\lambda$ is an eigenvalue of the matrix $A$, whenever $\lambda$ is an eigenvalue of $K$.

If $\mu$ is an eigenvalue of the matrix $K'$ with an eigenvector $z$, then

\[
\begin{bmatrix}
K & B^T & B^T \\
B & 0 & K' \\
B & K' & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
z \\
-z
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
-K'z \\
K'z
\end{bmatrix}
= -\mu
\begin{bmatrix}
0 \\
z \\
-z
\end{bmatrix}.
\]

Thus, $-\mu$ is an eigenvalue of the matrix $C$, whenever $\mu$ is an eigenvalue of $K'$. The remaining eigenvalues of $A$ are given as follows:

\[
\begin{bmatrix}
K' & B & B \\
B^T & 0 & K \\
B^T & K & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
y
\end{bmatrix}
= 
\begin{bmatrix}
K'x + 2By \\
B^Tx + Ky \\
B^Tx + Ky
\end{bmatrix}
= \lambda
\begin{bmatrix}
x \\
y \\
y
\end{bmatrix}. \tag{1}
\]

The remaining eigenvalues of $C$ are the same as the remaining eigenvalues of $A$. For, from Equation (1), we have $K'x + 2By = \lambda x$ and $B^Tx + Ky = \lambda y$, and hence

\[
\begin{bmatrix}
K & B^T & B^T \\
B & 0 & K' \\
B & K' & 0
\end{bmatrix}
\begin{bmatrix}
y \\
\frac{y}{2} \\
\frac{y}{2}
\end{bmatrix}
= 
\begin{bmatrix}
Ky + B^Tx \\
By + K'\frac{y}{2} \\
By + K'\frac{y}{2}
\end{bmatrix}
= \lambda
\begin{bmatrix}
y \\
\frac{y}{2} \\
\frac{y}{2}
\end{bmatrix}.
\]

Thus, the remaining eigenvalues of both matrices $A$ and $C$ are the same.

It is easy to see that, the matrices $A \oplus \begin{bmatrix} 0 & K' \\ K & 0 \end{bmatrix}$ and $C \oplus \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$ are cospectral.

\[\blacksquare\]

**Remark 3.5:** In Theorem 3.4, if the entries of the matrix $B$ are either 0 or 1, $p \geq q$, if $B$ is such that the maximum row sum of $B$ is different from the maximum column sum of $B$ and if both $K$ and $K'$ are adjacency matrices of some graphs, then the graphs associated with $A \oplus \begin{bmatrix} 0 & K' \\ K & 0 \end{bmatrix}$ and $C \oplus \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$ as adjacency matrices, respectively, are cospectral. But these graphs are not isomorphic.

Next, we illustrate the above construction with an example. For $p = 3$ and $q = 2$, let us generate a pair of non-isomorphic but cospectral graphs with respect to adjacency matrix. Let $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$, $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $K' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. The corresponding normalized Laplacian matrices are clearly not cospectral as the graphs have a different number of connected components [8, Lemma 1.7].

In the following example, the graph (Figure 8) whose vertices are labelled by the index set $\{1, 2, \ldots, n\}$ is the graph whose adjacency matrix is given by $C \oplus \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$ and the graph (Figure 9) whose vertices are labelled by the index set $\{1', 2', \ldots, n'\}$ is the graph whose adjacency matrix is given by $A \oplus \begin{bmatrix} 0 & K' \\ K' & 0 \end{bmatrix}$.
3.4. Construction III

Let $B$ be a $p \times q$ matrix with $p \geq q$. Let

$$C = \begin{bmatrix}
0 & B & B & \ldots & B \\
B^T & 0 & I & \ldots & I \\
B^T & I & 0 & \ldots & I \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B^T & I & I & \ldots & 0
\end{bmatrix}$$

be a $(nq + p) \times (nq + p)$ matrix, and

$$E = \begin{bmatrix}
0 & B^T & B^T & \ldots & B^T \\
B & 0 & I & \ldots & I \\
B & I & 0 & \ldots & I \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B & I & I & \ldots & 0
\end{bmatrix}$$

be a $(np + q) \times (np + q)$ matrix.

**Lemma 3.1:** The matrix $C$, defined in Equation (2), has $-1$ as an eigenvalue with multiplicity at least $(n - 1)q$, and the matrix $E$, defined in Equation (3), has $-1$ as an eigenvalue with multiplicity at least $(n - 1)p$. 
Proof: For any non-zero vector \( y \) of size \( q \), it is easy to see that
\[
\begin{bmatrix}
0 & B & B & \ldots & B \\
B^T & 0 & I & \ldots & I \\
B^T & I & 0 & \ldots & I \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B^T & I & I & \ldots & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
y \\
0 \\
\vdots \\
-y \\
\end{bmatrix} = \begin{bmatrix}
0 \\
y \\
0 \\
\vdots \\
-y \\
\end{bmatrix}.
\]
By placing the vector \( y \) from the 2nd position to the \( n \)th position, we get \( (n - 1) \) linearly independent eigenvectors corresponding to the eigenvalue \(-1\). As each \( y \) is a \( q \times 1 \) vector, there are \( (n - 1)q \) linearly independent eigenvectors associated with the eigenvalue \(-1\). Similarly, \( E \) has \( (n - 1)p \) linearly independent eigenvectors associated with the eigenvalue \(-1\).

The following lemmas are about the remaining eigenvalues of the matrices \( C \) and \( E \).

Lemma 3.2: If \( \lambda \) is an eigenvalue of the matrix \( C \), then \( (n - \lambda - 1) \) is an eigenvalue of the matrix \( E \).

Proof: Let \( \lambda \) be an eigenvalue of \( C \), \( \lambda \neq -1 \). Then,
\[
\begin{bmatrix}
0 & B & B & \ldots & B \\
B^T & 0 & I & \ldots & I \\
B^T & I & 0 & \ldots & I \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B^T & I & I & \ldots & 0 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
0 \\
\vdots \\
y \\
\end{bmatrix} = \begin{bmatrix}
x \\
y \\
x \\
\vdots \\
y \\
\end{bmatrix}.
\] (4)
As, from Equation (4), we have \( B^T x + (n - 1) y = \lambda y \) and \( n y = \lambda x \) and hence
\[
\begin{bmatrix}
0 & B^T & B^T & \ldots & B^T \\
B & 0 & I & \ldots & I \\
B & I & 0 & \ldots & I \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B & I & I & \ldots & 0 \\
\end{bmatrix}
\begin{bmatrix}
y \\
-x/n \lambda \\
-x/n \lambda \\
\vdots \\
-x/n \lambda \\
\end{bmatrix} = (n - 1 - \lambda) \begin{bmatrix}
y \\
-x/n \lambda \\
-x/n \lambda \\
\vdots \\
-x/n \lambda \\
\end{bmatrix}.
\]

Lemma 3.3: If \( \lambda \) is an eigenvalue of the matrix \( C \) other than 0, then \( (n - \lambda - 1) \) is an eigenvalue of the matrix \( C \).

Proof: Let \( \lambda \) be an eigenvalue of \( C \), \( \lambda \neq 0 \) and \( \lambda \neq -1 \). Then,
\[
\begin{bmatrix}
0 & B & B & \ldots & B \\
B^T & 0 & I & \ldots & I \\
B^T & I & 0 & \ldots & I \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B^T & I & I & \ldots & 0 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
0 \\
\vdots \\
y \\
\end{bmatrix} = \begin{bmatrix}
x \\
y \\
x \\
\vdots \\
y \\
\end{bmatrix}.
\] (5)
For, from Equation (5), we have $B^T x + (n - 1) y = \lambda y$ and $n B y = \lambda x$ and hence

$$
\begin{bmatrix}
0 & B & B & \ldots & B \\
B^T & 0 & I & \ldots & I \\
B^T & I & 0 & \ldots & I \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B^T & I & I & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
x \\
(n-1-\lambda) y \\
(n-1-\lambda) y \\
\vdots \\
(n-1-\lambda) y \\
(n-1-\lambda) y
\end{bmatrix}
= (n - 1 - \lambda)
\begin{bmatrix}
x \\
(n-1-\lambda) y \\
(n-1-\lambda) y \\
\vdots \\
(n-1-\lambda) y \\
(n-1-\lambda) y
\end{bmatrix}
.$$  

Lemma 3.4: If $\lambda$ is an eigenvalue of $E$ other than $n - 1$, then $n - \lambda - 1$ is an eigenvalue of $E$.

Proof: The proof is similar to that of Lemma 3.3. $\blacksquare$

Lemma 3.5: Let the matrix $B^T$ have full row rank. Then, the matrix $C$ has 0 as an eigenvalue with multiplicity at least $p - q$, and the matrix $E$ has $n - 1$ as an eigenvalue with multiplicity at least $p - q$.

Proof: Since the matrix $B^T$ has full row rank, so the dimension of the null space of $B^T$ is $p - q$. Now, for a vector $x$ in the null space of $B^T$

$$
\begin{bmatrix}
0 & B & B & \ldots & B \\
B^T & 0 & I & \ldots & I \\
B^T & I & 0 & \ldots & I \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B^T & I & I & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
x \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
= 0.
$$

Thus $C$ has 0 as an eigenvalue with multiplicity at least $(p - q)$, and hence, by Lemma 3.2, $E$ has $(n - 1)$ as an eigenvalue with multiplicity at least $(p - q)$. $\blacksquare$

Now, let us compute the remaining eigenvalues of the matrix $C$. Consider

$$
\begin{bmatrix}
0 & B & B & \ldots & B \\
B^T & 0 & I & \ldots & I \\
B^T & I & 0 & \ldots & I \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B^T & I & I & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
y \\
\vdots \\
y \\
y
\end{bmatrix}
= \lambda
\begin{bmatrix}
x \\
y \\
y \\
\vdots \\
y \\
y
\end{bmatrix}.
$$

Then $B^T x + (n - 1) y = \lambda y$ and $n B y = \lambda x$. It is easy to see that the eigenvalue equations of the $2 \times 2$ block matrix

$$
\begin{bmatrix}
0 & n B \\
B^T (n-1) I
\end{bmatrix}
$$

are $B^T x + (n - 1) y = \lambda y$ and $n B y = \lambda x$.

By applying the Schur complement formula with respect to (1, 1)th block, we get

$$
det \left( \begin{bmatrix} \lambda I & -n B \\ -B^T & \lambda I - (n - 1) I \end{bmatrix} \right) = \lambda^{(p-q)} \det(\lambda (\lambda - n + 1) I - n B^T B).
$$

This giving the remaining eigenvalues of the matrix $C$. (As we assumed that $\lambda$ is not zero, and since $B$ has full row rank, $B^T B$ is a full rank matrix thus it does not have zero eigenvalues, justifying the usage of Schur complement formula.)
Now for the matrix \( E \) similar analysis can be done, and it is easy to show that the remaining eigenvalues of the matrix \( E \) corresponds to the block matrix

\[
\begin{bmatrix}
0 & nB^T \\
B & (n-1)I
\end{bmatrix}
\]

By the Schur complement formula with respect to \((2,2)\)th block, we get

\[
det \left( \begin{bmatrix}
\lambda I & -nB^T \\
-B & \lambda I - (n-1)I
\end{bmatrix} \right) = (n - 1 - \lambda)^{(p-q)} \det(\lambda(\lambda - n + 1)I - nB^TB).
\]

This gives the remaining eigenvalues of the matrix \( E \).

**Theorem 3.5:** The matrices \( E \oplus 0_{(p-q)} \) and \( C \oplus (J-I)_n \oplus \cdots \oplus (J-I)_n \) are cospectral.

**Proof:** Follows from the previous lemmas.

### 3.5. Generalization of construction III

Next, we provide an alternate proof of Theorem 3.5 without the full row rank assumption. The advantage of the previous construction is an insight into the eigenvalues and the structure of the eigenvectors, which is not transparent in this construction. Let \( \mathbb{C}(\lambda) \) denote the field of rational functions.

By Lemma 3.1, the matrix \( C \) has \(-1\) as an eigenvalue with multiplicity at least \((n-1)q\), and the matrix \( E \) has \(-1\) as an eigenvalue with multiplicity at least \((n-1)p\). The remaining eigenvalues of the matrix \( C \) are given by

\[
\begin{bmatrix}
0 & B & B & \ldots & B \\
B^T & 0 & I & \ldots & I \\
B^T & I & 0 & \ldots & I \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B^T & I & I & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
y \\
\vdots \\
y
\end{bmatrix}
= \begin{bmatrix}
nBy \\
B^Tx + (n-1)y \\
B^Tx + (n-1)y \\
\vdots \\
B^Tx + (n-1)y
\end{bmatrix}
= \lambda \begin{bmatrix}
x \\
y \\
y \\
\vdots \\
y
\end{bmatrix}.
\]

The eigenvalue equations of the above matrix lead to that of the \(2 \times 2\) block matrix

\[
\begin{bmatrix}
0 & nB \\
B^T & (n-1)I
\end{bmatrix}.
\]

By the Schur complement formula (in \( \mathbb{C}(\lambda) \)) with respect to the \((1,1)\)th block, we obtain

\[
det \left( \begin{bmatrix}
\lambda I & -nB^T \\
-B & \lambda I - (n-1)I
\end{bmatrix} \right)
= \lambda^p \det((\lambda - n + 1)I - nB^T(\lambda I)^{-1}B) = \lambda^{p-q} \det(\lambda(\lambda - n + 1)I - nB^TB).
\]

As \( p \geq q \), we get a polynomial at the end of the above computation, which is the indeed characteristic polynomial of the given matrix. This giving the remaining eigenvalues of the matrix \( C \).

Now, by the same argument, the remaining eigenvalues of \( E \) correspond to the \(2 \times 2\) block matrix

\[
\begin{bmatrix}
0 & nB^T \\
B & (n-1)I
\end{bmatrix}.
\]

By the Schur complement formula (in \( \mathbb{C}(\lambda) \)) with respect to
(2, 2)th block, we get
\[
\det \left( \begin{bmatrix} \lambda I & -nB^T \\ -B & \lambda I - (n-1)I \end{bmatrix} \right) = (n - \lambda - 1)^p \det(\lambda I - nB^T(\lambda I - (n-1)I)^{-1}B) \\
= (n - \lambda - 1)^{(p-q)} \det(\lambda (\lambda - n + 1)I - nB^TB).
\]

As \( p \geq q \), we get a polynomial at the end of the above computation, which is the characteristic polynomial of a given matrix. This gives the remaining eigenvalues of the matrix \( E \).

It is easy to see that the matrices \( E \oplus 0_{(p-q)} \) and \( C \oplus (J - I)_n \oplus \cdots \oplus (J - I)_n \) are cospectral.

**Remark 3.6:** In Theorem 3.5, if the entries of the matrix \( B \) are either 0 or 1, \( p \geq q \) and the matrix \( B \) is chosen such that the maximum row sum of \( B \) is different from the maximum column sum of \( B \), then the graphs associated with \( E \oplus 0_{(p-q)} \) and \( C \oplus (J - I)_n \oplus \cdots \oplus (J - I)_n \) as adjacency matrices, respectively, are cospectral. But these graphs are not isomorphic.

Next, we illustrate the above construction with two examples.

(1) For \( p = 3 \), \( q = 2 \) and \( n = 2 \), let us generate a pair of non-isomorphic but cospectral graphs with respect to the adjacency matrix, but not with respect to the normalized Laplacian matrix. Let \( B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \). In the following example, the graph (Figure 11) whose vertices are labelled by the index set \( \{1', 2', \ldots, n'\} \) is the graph whose adjacency matrix is given by \( E \oplus 0_1 \), and the graph (Figure 10) whose vertices are labelled by the index set \( \{1, 2, \ldots, n\} \) is the graph whose adjacency matrix is given by \( C \oplus (J - I)_2 \). But the spectrum of matrix \( L(E \oplus 0_1) \) is \( \{0, 0, 0.6667, 0.6667, 1, 1.3333, 1.3333, 1.3333, 1.6667\} \) (decimal approximation of the eigenvalues), and the spectrum of matrix \( L(C \oplus (J - I)_2) \) is \( \{0, 0, 0.75, 1, 1, 1.25, 1.25, 1.75, 2\} \).

(2) For \( p = 3 \), \( q = 3 \) and \( n = 2 \), let us generate a pair of non-isomorphic but cospectral graphs with respect to the adjacency matrix but not the normalized Laplacian matrix. Let \( B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \). In the graphs illustrated below, vertices labelled by the \( \{1', 2', \ldots, n'\} \) corresponds to graph (Figure 12) of adjacency matrix \( E \) and vertices labelled by the \( \{1, 2, \ldots, n\} \) corresponds to graph (Figure 13) of adjacency matrix \( C \). The spectrum of the matrix \( L(E) \) is \( \{0, 0.2034, 0.6738, 1, 1.25, 1.3333, 1.3478, 1.5, 1.6917\} \), and the spectrum of the matrix \( L(C) \) is \( \{0, 0.2324, 0.6667, 1, 1.3333, 1.3333, 1.3333, 1.4343, 1.6667\} \).
3.6. Construction IV

Let $B$ be a $p \times q$ matrix, $K$ be a $q \times q$ symmetric matrix, and $K'$ be a $p \times p$ symmetric matrix. Let

$$A = \begin{bmatrix} K' & B & B \\ B^T & 0 & K \\ B^T & K & 0 \end{bmatrix}$$
be a \((p + 2q) \times (p + 2q)\) matrix, and let
\[
C = \begin{bmatrix}
K' & B & B \\
B^T & K & 0 \\
B^T & 0 & K
\end{bmatrix}
\]
be a \((p + 2q) \times (p + 2q)\) matrix.

**Theorem 3.6:** The matrices \(A \oplus \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}\) and \(C \oplus \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}\) are cospectral.

**Proof:** If \(\lambda\) is an eigenvalue of the matrix \(K\) with an eigenvector \(y\), then
\[
\begin{bmatrix}
K' & B & B \\
B^T & 0 & K \\
B^T & 0 & K
\end{bmatrix} \begin{bmatrix} 0 \\ y \\ -y \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda y \\ Ky \end{bmatrix} = -\lambda \begin{bmatrix} 0 \\ y \\ -y \end{bmatrix}.
\]
Thus, \(-\lambda\) is an eigenvalue of the matrix \(A\), whenever \(\lambda\) is an eigenvalue of \(K\).

If \(\mu\) is an eigenvalue of the matrix \(K\) with an eigenvector \(z\), then
\[
\begin{bmatrix}
K' & B & B \\
B^T & 0 & K \\
B^T & 0 & K
\end{bmatrix} \begin{bmatrix} 0 \\ z \\ -z \end{bmatrix} = \begin{bmatrix} 0 \\ Kz \\ -Kz \end{bmatrix} = \mu \begin{bmatrix} 0 \\ z \\ -z \end{bmatrix}.
\]
Thus, \(\mu\) is an eigenvalue of the matrix \(C\), whenever \(\mu\) is an eigenvalue of \(K\). The remaining eigenvalues of the matrix \(A\) are given as follows:
\[
\begin{bmatrix}
K' & B & B \\
B^T & 0 & K \\
B^T & 0 & K
\end{bmatrix} \begin{bmatrix} x \\ y \\ y \end{bmatrix} = \begin{bmatrix} K'x + 2By \\ B^T x + Ky \\ B^T x + Ky \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ y \end{bmatrix}.
\]
(6)

The remaining eigenvalues of the matrix \(C\) are the same as the remaining eigenvalues of the matrix \(A\). For, from Equation (6), we have \(K'x + 2By = \lambda x\) and \(B^T x + Ky = \lambda y\), and hence
\[
\begin{bmatrix}
K' & B & B \\
B^T & 0 & K \\
B^T & 0 & K
\end{bmatrix} \begin{bmatrix} x \\ y \\ y \end{bmatrix} = \begin{bmatrix} K'x + 2By \\ B^T x + Ky \\ B^T x + Ky \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ y \end{bmatrix}.
\]
Thus, the remaining eigenvalues of both matrices \(A\) and \(C\) are the same.

It is easy to see that the matrices \(A \oplus \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}\) and \(C \oplus \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}\) are cospectral. ■

### 3.7. Generalization of construction IV

Let \(B\) be a \(p \times q\) matrix, \(K\) be a \(q \times q\) symmetric matrix, and \(K'\) be a \(p \times p\) symmetric matrix. Let
\[
A = \begin{bmatrix}
K' & B & B \\
B^T & 0 & K \\
B^T & K & 0
\end{bmatrix}
\]
be a \((p+2q) \times (p+2q)\) matrix. Let \(E\) and \(F\) be \(q \times q\) symmetric matrices with the property \(E + F = K\). Define

\[
H = \begin{bmatrix}
K' & B & B \\
B^T & E & F \\
B^T & F & E
\end{bmatrix}
\]

be a \((p+2q) \times (p+2q)\) matrix.

**Theorem 3.7:** The matrices \(A \oplus \begin{bmatrix} E & F \\ F & E \end{bmatrix}\) and \(H \oplus \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}\) are cospectral.

**Proof:** If \(\lambda\) is an eigenvalue of the matrix \(K\) with an eigenvector \(y\), then

\[
\begin{bmatrix}
K' & B & B \\
B^T & 0 & K \\
B^T & K & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
y \\
-\lambda
\end{bmatrix}
= \begin{bmatrix}
0 \\
-Ky \\
Ky
\end{bmatrix} = -\lambda
\begin{bmatrix}
0 \\
y \\
-\lambda
\end{bmatrix}.
\]

Thus, \(-\lambda\) is an eigenvalue of the matrix \(A\), whenever \(\lambda\) is an eigenvalue of \(K\).

If \(\mu\) is an eigenvalue of the matrix \(E-F\) with an eigenvector \(z\), then

\[
\begin{bmatrix}
K' & B & B \\
B^T & E & F \\
B^T & F & E
\end{bmatrix}
\begin{bmatrix}
0 \\
z \\
-\mu
\end{bmatrix}
= \begin{bmatrix}
0 \\
(E-F)z \\
-(E-F)z
\end{bmatrix} = \mu
\begin{bmatrix}
0 \\
z \\
-\mu
\end{bmatrix}.
\]

Thus, \(\mu\) is an eigenvalue of the matrix \(H\), whenever \(\mu\) is an eigenvalue of \(E-F\). The remaining eigenvalues of the matrix \(A\) are given as follows:

\[
\begin{bmatrix}
K' & B & B \\
B^T & 0 & K \\
B^T & K & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
\lambda
\end{bmatrix}
= \begin{bmatrix}
K'x + 2By \\
B^Tx + Ky \\
\lambda
\end{bmatrix} = \lambda
\begin{bmatrix}
x \\
y \\
\lambda
\end{bmatrix}. \quad (7)
\]

The remaining eigenvalues of the matrix \(H\) are the same as the remaining eigenvalues of the matrix \(A\). For, from Equation (7), we have \(K'x + 2By = \lambda x\) and \(B^Tx + Ky = \lambda y\), and hence

\[
\begin{bmatrix}
K' & B & B \\
B^T & E & F \\
B^T & F & E
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
\lambda
\end{bmatrix}
= \begin{bmatrix}
K'x + 2By \\
B^Tx + (E+F)y \\
\lambda
\end{bmatrix} = \lambda
\begin{bmatrix}
x \\
y \\
\lambda
\end{bmatrix}.
\]

Thus, the remaining eigenvalues of both matrices \(A\) and \(H\) are the same.

It is easy to see that, the matrices \(A \oplus \begin{bmatrix} E & F \\ F & E \end{bmatrix}\) and \(H \oplus \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}\) are cospectral. \(\blacksquare\)

**Remark 3.7:** In Theorem 3.6, if the entries of the matrix \(B\) are either 0 or 1, if \(B\) is such that the maximum row sum of \(B\) is different from the maximum column sum of \(B\) and if both \(K\) and \(K'\) are adjacency matrices of some simple undirected graphs, then the graphs associated with \(A \oplus \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}\) and \(C \oplus \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}\) as adjacency matrices, respectively, are cospectral. But these graphs are not isomorphic.

In Theorem 3.7, if the entries of the matrix \(B\) are either 0 or 1, if \(B\) is such that maximum row sum of \(B\) is different from maximum column sum of \(B\) and if both \(K\) and \(K'\) are adjacency matrices of some simple undirected graphs, and if the entries of the matrices \(E\)
and $F$ are either 0 or 1, along with $E$ having zero diagonal entries, then the graphs associated with $A \oplus \begin{bmatrix} E & F \\ F & E \end{bmatrix}$ and $H \oplus \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$ as adjacency matrices, respectively, are cospectral. But these graphs are not isomorphic.

Next, we illustrate the above construction with an example. For $p = 2$, $q = 3$, let us generate a pair of non-isomorphic but cospectral graphs with respect to adjacency matrix. Let $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $K' = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $K = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. The corresponding normalized Laplacian matrices are not cospectral as the graphs have a different number of connected components. In the graphs illustrated below, vertices labelled by the $\{1, 2, \ldots, n\}$ corresponds to graph (Figure 14) whose adjacency matrix is $C \oplus \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$, and vertices labelled by the $\{1', 2', \ldots, n'\}$ corresponds to graph (Figure 15) whose adjacency matrix is $A \oplus \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$.

4. Conclusions

We extended the construction of cospectral graphs given by Butler. Construction I generates larger classes of bipartite graphs, which are cospectral for both the adjacency and the normalized Laplacian matrices. On the other hand, the other constructions generate non-bipartite graphs which are cospectral for the adjacency matrices but not necessarily for the normalized Laplacian matrices. One of the main questions we considered is the following: given a $p \times q$ matrix $B$, how to create several block matrices which are cospectral? (With entries are the matrix $B$, the matrix $B^T$, the identity matrix, and the zero matrix.) Constructions I and III generate a large class of cospectral matrices. Using this, we construct families
of cospectral graphs. In constructions II and IV, the role of the identity matrix is replaced by the adjacency matrices of some graphs of appropriate order. Enumerating the number of cospectral graphs generated by the proposed constructions is an interesting problem to consider.

Note
1. We would like to thank one of the reviewers for informing us about the updated version of Chung's book available at: http://www.math.ucsd.edu/~fan/research/revised.html.

Acknowledgments
The authors would like to thank the handling editor and the anonymous reviewers for their careful reading of the manuscript. Their constructive criticism has greatly improved this manuscript. M. Rajesh Kannan would like to thank the Department of Science and Technology, India, for financial support through the projects MATRICS (MTR/2018/000986) and Early Career Research Award (ECR/2017/000643).

Disclosure statement
No potential conflict of interest was reported by the author(s).

Funding
M. Rajesh Kannan would like to thank the Science and Engineering Research Board (Department of Science and Technology, India), for financial support through the projects MATRICS (MTR/2018/000986) and Early Career Research Award (ECR/2017/000643).

ORCID
M. Rajesh Kannan http://orcid.org/0000-0001-8038-1795

References
[1] van Dam ER, Haemers WH. Which graphs are determined by their spectrum? Linear Algebra Appl. 2003;373:241–272. Special issue on the Combinatorial Matrix Theory Conference (Pohang, 2002).
[2] van Dam ER, Haemers WH. Developments on spectral characterizations of graphs. Discrete Math. 2009;309(3):576–586.
[3] Schwenk AJ. Almost all trees are cospectral. In New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, MI, 1971); 1973. p. 275–307.
[4] Godsil CD, McKay BD. Constructing cospectral graphs. Aequationes Math. 1982;25(2–3):257–268.
[5] Brouwer AE, Haemers WH. Spectra of graphs. New York (NY): Springer; 2012. (Universitext).
[6] Cvetković DM, Doob M, Sachs H. Spectra of graphs. New York (NY): Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers]; 1980. (Pure and applied mathematics; vol. 87). Theory and application.
[7] Haemers WH, Spence E. Enumeration of cospectral graphs. European J Combin. 2004;25(2):199–211.
[8] Fan RK Chung. Spectral graph theory. Providence (RI): American Mathematical Society; 1997. (CBMS regional conference series in mathematics; vol. 92). Published for the Conference Board of the Mathematical Sciences, Washington, DC.

[9] Butler S, Grout J. A construction of cospectral graphs for the normalized Laplacian. Electron J Combin. 2011;18(1):Paper 231, 20pp.

[10] Butler S. A note about cospectral graphs for the adjacency and normalized Laplacian matrices. Linear Multilinear Algebra. 2010;58(3–4):387–390.