NON-WIEFERICH PRIMES IN NUMBER FIELDS AND

ABC CONJECTURE

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Abstract. Let $K/\mathbb{Q}$ be an algebraic number field of class number
one and $\mathcal{O}_K$ be its ring of integers. We show that there are infin-
itely many non-Wieferich primes with respect to certain units in
$\mathcal{O}_K$ under the assumption of the $abc$ conjecture for number fields.

1. Introduction

An odd rational prime $p$ is called Wieferich prime if

$$2^{p-1} \equiv 1 \pmod{p^2}. \quad (1)$$

A. Wieferich [10] proved that if an odd prime $p$ is non-Wieferich prime,
i.e., $p$ satisfies

$$2^{p-1} \not\equiv 1 \pmod{p^2},$$

then there are no integer solutions to the Fermat equation $x^p + y^p = z^p$,
with $p \nmid xyz$. The known Wieferich primes are 1093 and 3511 and
according to the PrimeGrid project [5], these are the only Wieferich
primes less than $17 \times 10^{15}$. One of the unsolved problems in this area
of research is to determine whether the number of Wieferich or non-
Wieferich primes is finite or infinite. Instead of the base 2 if we take
any base $a$, then $p$ is said to be a Wieferich prime with respect to the
base $a$ if

$$a^{p-1} \equiv 1 \pmod{p^2}, \quad (2)$$

and if the congruence (2) does not hold then we shall say that $p$ is
non-Wieferich prime to the base $a$. Under the famous $abc$ conjecture
(defined below), J. H. Silverman [8] proved that given any integer $a$,
there are infinitely many non-Wieferich primes to the base $a$. He es-

tablished this result by showing that for any fixed $\alpha \in \mathbb{Q}^\times, \alpha \neq \pm 1$, and assuming the truth of $abc$ conjecture,

$$\text{card} \{p \leq x : a^{p-1} \not\equiv 1 \pmod{p^2} \} \gg_{\alpha} \log x \quad \text{as} \quad x \to \infty.$$  

In [2] Hester Graves and M. Ram Murty extended this result to primes
in arithmetical progression by showing that for any $a \geq 2$ and any fixed
$k \geq 2$, there are $\gg \log x / \log \log x$ primes $p \leq x$ such that $a^{p-1} \not\equiv 1 \pmod{p^2}$ and $p \equiv 1 \pmod{k}$, under the assumption of $abc$ conjecture.

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In this paper, we study non-Wieferich primes in algebraic number fields of class number one. More precisely, we prove the following

**Theorem 1.1.** Let $K = \mathbb{Q}(\sqrt{m})$ be a real quadratic field of class number one and assume that the abc conjecture holds true in $K$. Then there are infinitely many non-Wieferich primes in $\mathcal{O}_K$ with respect to the unit $\varepsilon$ satisfying $|\varepsilon| > 1$.

**Theorem 1.2.** Let $K$ be any algebraic number field of class number one and assume that the abc conjecture holds true in $K$. Let $\eta$ be a unit in $\mathcal{O}_K$ satisfying $|\eta| > 1$ and $|\eta^{(j)}| < 1$ for all $j \neq 1$, where $\eta^{(j)}$ is the $j$th conjugate of $\eta$. Then there exist infinitely many non-Wieferich primes in $K$ with respect to the base $\eta$.

The plan of this article is as follows. In section 2, we shall define the abc conjecture for number fields. In section 3, a brief introduction to Wieferich/non-Wieferich primes over number fields will be given and in section 4 and 5, we shall prove Theorem 1.1 and Theorem 1.2, respectively.

## 2. The abc-conjecture

The abc-conjecture propounded by Oesterlé and Masser (1985) states that given any $\delta > 0$ and positive integers $a, b, c$ such that $a + b = c$ with $(a, b) = 1$, we have

$$c \ll_{\delta} (\text{rad}(abc))^{1+\delta},$$

where $\text{rad}(abc) := \prod_{p|abc} p$.

The abc conjecture has several applications, the reader may refer to [9], [3], [6], [7] for details.

To state the analogue of abc-conjecture for number fields, we need some preparations, which we do below. The interested reader may refer to [9], [3] for more details.

Let $K$ be an algebraic number field and let $V_K$ denote the set of primes on $K$, that is, any $v$ in $V_K$ is an equivalence class of norm on $K$ (finite or infinite). Let $||x||_v := N_{K/\mathbb{Q}}(p)^{v_p(x)}$, if $v$ is a prime defined by the prime ideal $p$ of the ring of integers $\mathcal{O}_K$ in $K$ and $v_p$ is the corresponding valuation, where $N_{K/\mathbb{Q}}$ is the absolute value norm. Let $||x||_v := |g(x)|^e$ for all non-conjugate embeddings $g : K \rightarrow \mathbb{C}$ with $e = 1$ if $g$ is real and $e = 2$ if $g$ is complex. Define the height of any triple $a, b, c \in K^\times$ as

$$H_K(a, b, c) := \prod_{v \in V_K} \max(||a||_v, ||b||_v, ||c||_v),$$
and the radical of \((a, b, c)\) by

\[
\text{rad}_K(a, b, c) := \prod_{p \in I_K(a, b, c)} N_{K/Q}(p)^{v_p(p)},
\]

where \(p\) is a rational prime with \(p\mathbb{Z} = p \cap \mathbb{Z}\) and \(I_K(a, b, c)\) is the set of all primes \(p\) of \(O_K\) for which \(||a||_v, ||b||_v, ||c||_v\) are not equal.

The abc conjecture for algebraic number fields is stated as follows: For any \(\delta > 0\), we have

\[
H_K(a, b, c) \ll_{\delta, K} \left(\text{rad}_K(a, b, c)\right)^{1+\delta},
\]

for all \(a, b, c, \in K^\times\) satisfying \(a+b+c = 0\), the implied constant depends on \(K\) and \(\delta\).

### 3. Wieferich/non-Wieferich primes in number fields

Let \(K\) be an algebraic number field and \(O_K\) be its ring of integers. A prime \(\pi \in O_K\) is called Wieferich prime with respect to the base \(\varepsilon \in O_K^\times\) if

\[
\varepsilon N(\pi)^{-1} \equiv 1 \pmod{\pi^2},
\]

where \(N(.)\) is the absolute value norm. If the congruence (4) does not hold for a prime \(\pi \in O_K\), then it is called non-Wieferich prime to the base \(\varepsilon\).

**Notation:** In what follows, \(\varepsilon\) will denote a unit in \(O_K\) and we shall write \(\varepsilon^n - 1 = u_nv_n\), where \(u_n\) is the square free part and \(v_n\) is the squarefull part, i.e., if \(\pi | v_n\) then \(\pi^2 | v_n\). We shall denote absolute value norm on \(K\) by \(N\).

### 4. Proof of theorem (1.1)

Let \(K = \mathbb{Q}(\sqrt{m}), m > 0\) be a real quadratic field and \(O_K\) be its ring of integers. Let \(\varepsilon \in O_K^\times\) be a unit with \(|\varepsilon| > 1\). The results of Silverman [8], Ram Murty and Hester [2] elucidated in the introduction use a key lemma of Silverman (Lemma 3, [8]). We derive an analogue of Silverman’s lemma for number fields which will play a fundamental role in the proof of the main theorems.

**Lemma 4.1.** Let \(K = \mathbb{Q}(\sqrt{m})\) be a real quadratic field of class number one. Let \(\varepsilon \in O_K^\times\) be a unit. If \(\varepsilon^n - 1 = u_nv_n\), then every prime divisor \(\pi\) of \(u_n\) is a non-Wieferich prime with respect to the base \(\varepsilon\).

**Proof.** The assumption that \(K\) has class number one allows us to write the element \(\varepsilon^n - 1 \in O_K\) as a product of primes uniquely. Accordingly, we shall write

\[
\varepsilon^n - 1 = u_nv_n
\]

for \(n \in \mathbb{N}\). Then

\[
\varepsilon^n = 1 + \pi w,
\]

(5)
with \(\pi|u_n\) and \(\pi\) and \(w\) are coprime. As \(\pi\) is a prime, we have \(N(\pi) = p\) or \(p^2\), \(p\) is a rational prime.

Case (1): Suppose \(N(\pi) = p\).

From equation (5), we get
\[\varepsilon^{n(p-1)} \equiv 1 + (p-1)\pi w \not\equiv 1 \pmod{\pi^2}.
\]

Case (2): Suppose \(N(\pi) = p^2\).

Again from equation (5), we obtain
\[\varepsilon^{n(p^2-1)} = \varepsilon^{n(N(\pi)-1)} = (1 + \pi w)^{p^2-1} \equiv 1 + \pi w(p^2-1) \not\equiv 1 \pmod{\pi^2}.
\]

Thus in either case,
\[\varepsilon^{(N(\pi)-1)} \not\equiv 1 \pmod{\pi^2},
\]

and hence \(\pi\) is a non-Wieferich prime to the base \(\varepsilon\).

The above lemma shows that whenever a prime \(\pi\) divides \(u_n\) for some positive integer \(n\), then \(\pi\) is a non-Wieferich prime with respect to the base \(\varepsilon\). Thus, if we can show that the set \(\{N(u_n) : n \in \mathbb{N}\}\) is unbounded, then this will imply that the set \(\{\pi : \pi|u_n, n \in \mathbb{N}\}\) is an infinite set. Consequently, this establishes the fact that there are infinitely many non-Wieferich primes in every real quadratic field of class number one with respect to the unit \(\varepsilon\), with \(|\varepsilon| > 1\). Therefore, we need only to show the following

**Lemma 4.2.** Let \(\mathbb{Q}(\sqrt{m})\) be a real quadratic field of class number one. Let \(\varepsilon \in \mathcal{O}_K^\times\) be a unit with \(|\varepsilon| > 1\). Then under abc-conjecture for number fields, the set \(\{N(u_n) : n \in \mathbb{N}\}\) is unbounded.

**Proof.** Invoking the abc-conjecture (3) to the equation
\[\varepsilon^n = 1 + u_nv_n\]  \hspace{1cm} (6)

yields
\[|\varepsilon^n| \ll \left( \prod_{p|u_nv_n} N(p)^{v_p(p)} \right)^{1+\delta} = \left( \prod_{p|u_n} N(p)^{v_p(p)} \right)^{1+\delta} \prod_{p|v_n} N(p)^{v_p(p)}\]  \hspace{1cm} (7)

for some \(\delta > 0\). Here the implied constant depends on \(K\) and \(\delta\).

As \(v_p(p) \leq 2\) for any prime ideal \(p\) lying above the rational prime \(p\), we have
\[\prod_{p|u_n} N(p)^{v_p(p)} \leq N(u_n)^2.\]  \hspace{1cm} (8)

For a prime ideal \(p|v_n\), let \(e_p\) be the largest exponent of \(p\) dividing \(v_n\), i.e., \(p^{e_p}|v_n\). As \(v_n\) is the square-full part of \(\varepsilon^n - 1\), we have \(e_p \geq 2\). Hence,

1. \(N(p)^{2v_p(p)} \leq N(p)^{2+e_p}\) for all prime ideals \(p\) with \(v_p(p) = 2\).
2. \(N(p)^{2v_p(p)} \leq N(p)^{e_p}\) for all prime ideals \(p\) with \(v_p(p) = 1\).
Thus
\[
\prod_{p|\nu_n} N(p)^{2v_p(p)} \leq \prod_{p|\nu_n} N(p)^{2v_p(p)+e_p(p)} \prod_{p|\nu_n} \prod_{v_p(p)=1} N(p)^{e_p(p)}
\]
\[
\leq \prod_{p|\nu_n} N(p)^{2} \prod_{v_p(p)=2} N(p)^{e_p(p)} \prod_{v_p(p)=1} N(p)^{e_p(p)}
\]
\[
\leq \prod_{p} N(p)^{2} \prod_{v_p(p)=2} N(p)^{e_p(p)} \prod_{v_p(p)=1} N(p)^{e_p(p)},
\]
where ' indicates that the product is over all primes \( p \) in \( \mathcal{O}_K \) such that \( v_p(p) = 2 \). As it is well known that there are only finitely many ramified primes in a number field, it follows that the product is bounded by a constant \( A \) (say). Thus, we have
\[
\prod_{p|\nu_n} N(p)^{v_p(p)} \leq \sqrt{AN(v_n)}.
\]
Combining equations (7), (8) and (9), we get
\[
|\varepsilon^n| \ll \left( N(u_n)^2 \sqrt{N(v_n)} \right)^{1+\delta}.
\]
Now, as \( |\varepsilon| > 1 \),
\[
N(u_n)N(v_n) = N(\varepsilon^n - 1) \leq 2|\varepsilon^n - 1| < 2|\varepsilon|^n,
\]
i.e.,
\[
N(v_n) < 2|\varepsilon|^n / N(u_n).
\]
Substituting the above expression in (10), we obtain
\[
|\varepsilon^n| \ll \left( N(u_n)^2 \frac{|\varepsilon|^{n/2}}{\sqrt{N(u_n)}} \right)^{(1+\delta)}.
\]
Thus,
\[
(N(u_n))^{\frac{3(1+\delta)}{2}} \gg |\varepsilon|^{\frac{n(1-\delta)}{2}}.
\]
Thus, for a fixed \( \delta \), \( N(u_n) \to \infty \) as \( n \to \infty \). This proves the lemma and hence completes the proof of the theorem. \( \square \)

5. Non-Wieferich primes in algebraic number fields

In this section, we generalize the arguments of previous section to arbitrary number fields. From now onwards, \( K \) will always denote an algebraic number field of degree \( [K: \mathbb{Q}] = l \) over \( \mathbb{Q} \) of class number one. Let \( r_1 \) and \( r_2 \) be the number of real and non-conjugate complex embeddings of \( K \) into \( \mathbb{C} \) respectively, so that \( l = r_1 + 2r_2 \). We begin with an analogue of Lemma (4.1).
Lemma 5.1. Let \( \varepsilon \) be a unit in \( \mathcal{O}_K \). If \( \varepsilon^n - 1 = u_nv_n \), then every prime divisor \( \pi \) of \( u_n \) is a non-Wieferich prime with respect to the base \( \varepsilon \).

Proof. Let \( N(\pi) = p^k \), where \( p \) is a rational prime and \( k \) is a positive integer. Then
\[
\varepsilon^{n(N(\pi)-1)} = \varepsilon^{n(p^k-1)} = (1 + w\pi)^{(p^k-1)} \equiv 1 + (p^k-1)w\pi \not\equiv 1 \pmod{\pi^2}.
\]
This implies \( \varepsilon^{N(\pi)-1} \not\equiv 1 \pmod{\pi^2} \).

Thus, the lemma shows that \( \pi \) is a non-Wieferich prime to the base \( \varepsilon \) whenever the hypothesis of the lemma is met. Now, under the \( abc \) conjecture for number fields, we show below the existence of infinitely many non-Wieferich primes.

Lemma 5.2. The set \( \{N(u_n) : n \in \mathbb{N}\} \) is unbounded, where \( u_n \)'s are as defined in Lemma (5.1).

Proof. By the hypothesis of the lemma, we have \( \varepsilon^n = 1 + u_nv_n \), where \( \varepsilon^n, 1, u_n, v_n \in K^\times \). Applying the \( abc \) conjecture for number fields to the above equation, we obtain
\[
\prod_{v \in V_K} \max(|u_nv_n|_v, |1|_v, |\varepsilon^n|_v) \ll \left( \prod_{p|u_nv_n} N(p)^{v_p(p)+1+\delta} \right),
\]  \hspace{1cm} (11)
for some \( \delta > 0 \).

Note that for the absolute value \( |\cdot| \) in \( V_K \), we have
\[
|\varepsilon^n| \leq \prod_{v \in V_K} \max(|u_nv_n|_v, |1|_v, |\varepsilon^n|_v).
\]  \hspace{1cm} (12)

As \( v_p(p) \leq l \) for any prime ideal \( p \) lying above the rational prime \( p \), we have
\[
\prod_{p|u_n} N(p)^{v_p(p)} \leq N(u_n)^l.
\]  \hspace{1cm} (13)

As before, we denote by \( e_p \) the largest exponent of \( p \) which divides \( v_n \), i.e., \( p^{e_p} || v_n \). Clearly \( e_p \geq 2 \). Then
\[
\prod_{p|v_n} N(p)^{2v_p(p)} \leq \prod_{p|v_n} N(p)^{2l-e_p(p)} \prod_{p|v_n} N(p)^{e_p(p)}
\]
\[
\leq \prod_{p|v_n} N(p)^{2l} \prod_{p|v_n} N(p)^{e_p(p)} \prod_{p|v_n} N(p)^{e_p(p)}
\]
\[
\leq \prod_{p} N(p)^{2l} \prod_{p} N(p)^{e_p(p)} \prod_{p} N(p)^{e_p(p)},
\]  \hspace{1cm} where ' indicates that the product is over all primes \( p \) in \( \mathcal{O}_K \) such that \( v_p(p) \geq 2 \). As there are only finitely many ramified primes in a number field, it is bounded by a constant \( B \) (say). Thus, we have
\[ \prod_{p \mid v_n} N(p)^{v_p(p)} \leq \sqrt{BN(v_n)}. \tag{14} \]

Therefore, the equations (11) - (14) yield
\[ |\varepsilon^n| \ll \left( N(u_n)^l \sqrt{N(v_n)} \right)^{1+\delta}. \tag{15} \]

Note that in the case of real quadratic fields, the unit \( \varepsilon \) satisfies \(|\varepsilon| > 1\) and this information was crucial in proving Theorem 1. However, in the case of general number fields, the following result (see Lemma 8.1.5, [1]) comes to our rescue. We state this result as Lemma 5.3.

**Lemma 5.3.** Let \( E = \{ k \in \mathbb{Z} : 1 \leq k \leq r_1 + r_2 \} \). Let \( E = A \cup B \) be a proper partition of \( E \). There exists a unit \( \eta \in \mathcal{O}_K \) with \(|\eta| < 1\), for \( k \in A \) and \(|\eta^{(k)}| > 1\), for \( k \in B \).

Taking \( A = \{ k : 1 < k \leq r_1 + r_2 \} \) and \( B = \{1\} \), Lemma 5.3 produces a unit \( \eta \in \mathcal{O}_K^* \) such that \(|\eta| > 1\) and \(|\eta^{(k)}| < 1\), where \( \eta^{(k)} \) denotes the \( k \)th conjugate of \( \eta \), \( k \neq 1 \). Since, every unit satisfies (15), replacing \( \varepsilon \) with \( \eta \) in (15), we obtain
\[ |\eta^n| \ll \left( N(u_n)^l \sqrt{N(v_n)} \right)^{1+\delta}, \tag{16} \]

where, by abuse of notation, we shall denote \( \eta^n - 1 = u_nv_n \), with \( u_n \) and \( v_n \) denoting the same quantities as defined earlier.

Now,
\[ N(u_n)N(v_n) = N(\eta^n - 1) = (\eta^n - 1)(\eta^{(2)n} - 1)(\eta^{(3)n} - 1) \cdots (\eta^{(l)n} - 1). \]

By Lemma 5.3, \(|\eta^{(j)n} - 1| < 2 \) for all \( j, 2 \leq j \leq l \).

Thus,
\[ N(u_n)N(v_n) < C|\eta^n| \quad \text{or} \quad N(v_n) < C|\eta^n|/N(u_n). \]

Now, (16) can be written as
\[ (N(u_n))^{\frac{(2l-1)(1+\delta)}{2}} \gg |\eta|^{n \frac{1-\delta}{2}}. \tag{17} \]

For a fixed \( \delta \), the right hand side of (17) tends to \( \infty \) as \( n \to \infty \). Therefore the set \( \{ N(u_n) : n \in \mathbb{N} \} \) is unbounded. This shows that there are infinitely many non-Wieferich primes in \( K \) with respect to the base \( \eta \). \( \square \)

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