Quantum fidelity of symmetric multipartite states

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I. INTRODUCTION

In many applications and implementations of quantum information processing, one has to compare two different quantum states. In this context, the quantum fidelity [1, 2] is a very useful tool to measure the “closeness” between two states in the Hilbert space of a quantum system. For two arbitrary states ρ1 and ρ2, it is defined as

\[ F(\rho_1, \rho_2) = \left( \text{Tr} \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}} \right)^2. \]  

(1)

For any pair of pure states |ψ⟩ and |φ⟩, the quantum fidelity reduces to their (squared) overlap, \( F(\psi, \phi) = |\langle \psi | \phi \rangle|^2 \). Although the fidelity does not define a metric on the state space, it is the core ingredient for several of them, like for instance the Bures distance \( d_B(\rho_1, \rho_2) = [2 - 2 \sqrt{F(\rho_1, \rho_2)}]^{1/2} \). The fidelity is also widely used to define various entanglement monotones. The Bures distance \( d_B \) is itself such an example [3]. For multipartite pure states, the geometric measure of entanglement \( E_G(\psi) = 1 - F(\psi, S) \) with

\[ F(\psi, S) = \sup_{|\phi\rangle \text{sep}} F(\psi, \phi) \]  

(2)

is another example that exploits the maximal fidelity between the state |ψ⟩ to characterize the set \( S \) of all fully separable states |φ⟩ = |a⟩ ⊗ |b⟩ ⊗ |c⟩ ⊗ ⋯. The question arises as to how the supremum in Eq. (2) can be computed. In general, it is known to be an NP-hard task [4]. For multipartite states |ψ⟩ that are symmetric with respect to the permutations of the parties it has been shown [5] that this supremum is realized among the symmetric separable states |ψ⟩ = |a⟩ ⊗ |a⟩ ⊗ |a⟩ ⊗ ⋯ only,

\[ F(|\psi⟩, S) = F^S(|\psi⟩, S) = \sup_{\text{symmetric } |\phi\rangle \text{ sep}} F(|\psi⟩, |\phi⟩). \]  

(3)

In fact it can even be proven that for three or more particles the state maximizing the overlap in the definition of \( F(\psi, S), S \) is necessarily symmetric [6]. This nice property considerably simplifies the calculation of the geometric measure of entanglement for symmetric states.

The maximization of the fidelity over sets other than the separable states \( S \) has proven to be very useful for discrimination strategies of inequivalent classes of multipartite entangled states with witnesses [7] or other methods [8]. In this case, the maximization is typically to be performed on sets of states equivalent through either local unitary operations (LU) or stochastic local operations assisted by classical communication (SLOCC). One needs to evaluate the maximal fidelity

\[ F_{\psi, C} = \sup_{|\phi⟩ \in C} F(\psi, \phi) \]  

(4)

with \( C \) any considered LU or SLOCC class of states. Mathematically, these classes are defined as follows: The LU equivalence class of a pure state |χ⟩ is given by all states of the form \( |\phi⟩ = U_1 ⊗ U_2 ⊗ ⋯ ⊗ U_N |\chi⟩ \), where the \( U_k \) are unitary matrices acting on the \( k \)-th party. The SLOCC equivalence class of |χ⟩ is given by normalized states of the form \( |\phi⟩ \sim A_1 ⊗ A_2 ⊗ ⋯ ⊗ A_N |\chi⟩ \), where the \( A_k \) are invertible matrices [9]. The LU and SLOCC equivalence classes of states never coincide, except for the fully separable states that are all both LU and SLOCC equivalent.

In Eq. (4), if \( C \) contains symmetric states, the question naturally arises whether the simplification given by Eq. (3) for the particular case of the SLOCC (and LU) class \( S \) of separable states generalizes similarly. In other words, do we have for any symmetric state |ψ⟩ and any LU or SLOCC classes \( C \) containing symmetric states

\[ F_{|\psi⟩, C} = F^S_{|\psi⟩, C} = \sup_{\text{symmetric } |\phi⟩ \in C} F(|\psi⟩, |\phi⟩)? \]  

(5)

This paper provides answers to this question for multipartite systems. First, in the case of LU classes, the answer is positive and this is formally proven in Sec. II.
Second, for the case of SLOCC classes, the answer is surprisingly negative and spectacular violations of Eq. (5) will be given in Sec. II. In Sec. III we summarize and discuss further open problems.

II. CASE OF LOCAL UNITARY TRANSFORMATIONS

When considering LU equivalence classes $\mathbb{C}$ and multiqubit systems in Eq. (5) and since any two LU-equivalent symmetric states can be transformed into each other with the same local unitary acting on each party $[12, 13]$, the question can be rephrased as follows: Do we have, for any $N$-qubit symmetric states $|\psi_S\rangle$ and $|\phi_S\rangle$,

$$\sup_{U_1, \ldots, U_N \in U(2)} |\langle \psi_S | U_1 \otimes \cdots \otimes U_N | \phi_S \rangle|^2 = \sup_{U \in U(2)} |\langle \psi_S | U \otimes \cdots \otimes U_N | \phi_S \rangle|^2,$$

(6)

where $U(2)$ is the group of unitary matrices of dimension $2 \times 2$. Since only the absolute value of the overlap matters, we can choose the phases of the $U_k$ as we like. So, it suffices to take matrices with determinant 1 and consider

$$\sup_{U_1, \ldots, U_N \in U(2)} |\langle \psi_S | U_1 \otimes \cdots \otimes U_N | \phi_S \rangle|^2 = \sup_{U \in SU(2)} |\langle \psi_S | U \otimes \cdots \otimes U_N | \phi_S \rangle|^2,$$

(7)

with $SU(2)$ the group of unitary matrices of determinant 1. The question (6) can thus be rephrased as

$$\sup_{U_1, \ldots, U_N \in SU(2)} |\langle \psi_S | U_1 \otimes \cdots \otimes U_N | \phi_S \rangle|^2 = \sup_{U \in SU(2)} |\langle \psi_S | U \otimes \cdots \otimes U_N | \phi_S \rangle|^2?$$

(8)

To tackle this problem, we note that an arbitrary $SU(2)$ matrix can be written as

$$U = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C} : |\alpha|^2 + |\beta|^2 = 1.$$  

(9)

We then define the function

$$P_{\psi_S, \phi_S} : \mathbb{C}^2 \times \cdots \times \mathbb{C}^2 \to \mathbb{C} : (q_1, \ldots, q_N) \to P_{\psi_S, \phi_S}(q_1, \ldots, q_N) = \langle \psi_S | U_1 \otimes \cdots \otimes U_N | \phi_S \rangle,$$

(10)

with

$$U_i = \begin{pmatrix} \alpha_i & -\beta_i^* \\ \beta_i & \alpha_i^* \end{pmatrix} : (\alpha_i, \beta_i) \equiv q_i,$$

(11)

where $\mathbb{C} \simeq \mathbb{R}^2$ and $\mathbb{C}^2 \simeq \mathbb{R}^4$ are considered as real Hilbert spaces. The function $P_{\psi_S, \phi_S}$ is symmetric with respect to the permutations of the variables, because $|\psi_S\rangle$ and $|\phi_S\rangle$ are symmetric states. Furthermore, it is $\mathbb{R}$-multilinear in the coefficients $q_1, \ldots, q_N$, that is,

$$\forall s, t \in \mathbb{R}, \quad i = 1, \ldots, N :$$

$$P_{\psi_S, \phi_S}(q_1, \ldots, s q_i + t p_i, \ldots, q_N) = s P_{\psi_S, \phi_S}(q_1, \ldots, q_N) + t P_{\psi_S, \phi_S}(q_1, \ldots, p_i, \ldots, q_N).$$

(12)

The multilinearity is only ensured for real $s$ and $t$ and this is the reason why we have to consider the real Hilbert spaces $\mathbb{C}$ and $\mathbb{C}^2$. Under these conditions, Hörmander’s theorem 4 of Ref. [14] and its extension to the case of real Hilbert spaces of any dimension $[14, 15]$ can be applied and we have

$$\sup_{q_1, \ldots, q_N \in \mathbb{C}^2} \frac{|P_{\psi_S, \phi_S}(q_1, \ldots, q_N)|^2}{\|q_1\|^2 \cdots \|q_N\|^2} = \sup_{q \in \mathbb{C}^2} \frac{|P_{\psi_S, \phi_S}(q)|^2}{\|q\|^{2N}},$$

(13)

where $P_{\psi_S, \phi_S}(q) \equiv P_{\psi_S, \phi_S}(q_1, \ldots, q_N)$. Because of the $\mathbb{R}$-multilinearity of $P_{\psi_S, \phi_S}$,

$$\frac{|P_{\psi_S, \phi_S}(q_1, \ldots, q_N)|^2}{\|q_1\|^2 \cdots \|q_N\|^2} = P_{\psi_S, \phi_S}(q_1, \ldots, q_N)^2,$$

(14)

with $q_i = q_i/\|q\|$ and thus

$$\sup_{q_1, \ldots, q_N \in \mathbb{C}^2 : \|q\| = 1} |P_{\psi_S, \phi_S}(q_1, \ldots, q_N)|^2$$

(15)

Equation (15) then yields

$$\sup_{q_1, \ldots, q_N \in \mathbb{C}^2 : \|q\| = 1} |P_{\psi_S, \phi_S}(q_1, \ldots, q_N)|^2 = \sup_{q \in \mathbb{C}^2 : \|q\| = 1} |P_{\psi_S, \phi_S}(q)|^2,$$

(16)

that is to say,

$$\sup_{U_1, \ldots, U_N \in SU(2)} |\langle \psi_S | U_1 \otimes \cdots \otimes U_N | \phi_S \rangle|^2 = \sup_{U \in SU(2)} |\langle \psi_S | U \otimes \cdots \otimes U_N | \phi_S \rangle|^2.$$

(17)

So, the answer to question posed in Eq. (8) and in Eq. (6) is clearly positive. This finishes the proof.

III. CASE OF SLOCC TRANSFORMATIONS

Now we turn to the case when SLOCC classes are considered in Eq. (5). Since any two SLOCC-equivalent symmetric states can be transformed into each other with the same invertible local operation (ILO) acting on each party $[13, 16]$, the question addressed here for multiqubit systems is the following: Do we have, for any $N$-qubit symmetric states $|\psi_S\rangle$ and $|\phi_S\rangle$,}

$$\sup_{A_1, \ldots, A_N \in GL(2)} \frac{|\langle \psi_S | A_1 \otimes \cdots \otimes A_N | \phi_S \rangle|^2}{\|A_1 \otimes \cdots \otimes A_N \| \|\phi_S\|^2}$$

(18)

where $GL(2)$ is the group of invertible matrices of dimension $2 \times 2$. 

Here, the expression to be maximized contains a normalization constant that also depends on the state $|\psi_S\rangle$. Contrary to the LU case, the left-hand side term of Eq. (18) cannot be cast in a multilinear form divided by a product of norms like the left-hand side term of Eq. (13). Hörmander’s theorem therefore cannot be exploited to tentatively prove Eq. (18). Actually, the counterexamples to this equation identified hereafter prove that any attempt in this direction would be doomed to failure.

As the first observation, for three-qubit systems, extensive numerical simulations showed no violation of Eq. (5). Similarly, numerical simulations for $N$ up to 8 gave indications that Eq. (5) seems also to hold for the classes of states SLOCC-equivalent to the 1-excitation Dicke states $|D_N^{(1)}\rangle$ [17], hereafter denoted by the classes $W_N^{(1)}$. These classes gather both symmetric and nonsymmetric states.

When restricted to the symmetric subspace, they merely identify to the $D_{N-1,1}$ families of symmetric states of Ref. [18]. This brings us to conjecture that the equality

$$F_{|\psi_S\rangle,W_N^{(1)}} = F^S_{|\psi_S\rangle,W_N^{(1)}}$$

actually holds for any $N$ and any symmetric state $|\psi_S\rangle$.

### A. First counterexample

The generalization of Eq. (18) to arbitrary SLOCC classes containing symmetric states, however, is not correct. Spectacular violations are obtained when considering the classes of states SLOCC-equivalent to the $k$-excitation Dicke states $|D_N^{(k)}\rangle$ [17], hereafter denoted by the classes $W_N^{(k)}$, for $N \geq 4$ and $k = 2, \ldots, \lfloor N/2 \rfloor$. All these classes gather both symmetric and nonsymmetric states. When restricted to the symmetric subspace, they identify to the $D_{N-k,k}$ families of symmetric states of Ref. [18]. For the values of $N$ and $k$ considered, there are symmetric states $|\psi_S\rangle$ for which

$$F_{|\psi_S\rangle,W_N^{(k)}} > F^S_{|\psi_S\rangle,W_N^{(k)}}.$$

To prove this result, we first consider the state $|\psi_S\rangle = |D_N^{(1)}\rangle \in W_N^{(1)}$. For all aforementioned $N$ and $k$, $W_N^{(1)} \neq W_N^{(k)}$ [17] and one gets (see Appendix A for a detailed calculation)

$$F^S_{|D_N^{(1)}\rangle,W_N^{(k)}} = NC^k_N \sup_{x,x',y} f(x,x',y)$$

with

$$f(x,x',y) = (1-x^2)^{N-k-1}(1-x'^2)^{k-1} \left( \frac{N-k}{N} \right)^2 x^2(1-x^2) + \left( \frac{k}{N} \right)^2 x'^2(1-x'^2) + 2 \frac{k}{N} x x' \sqrt{1-x^2} \sqrt{1-x'^2} \sum_{j=0}^{N-k \lfloor j \rfloor \geq 0} C^j_N C^j_{N-k} \left[ x^2 x'^2 + (1-x^2)(1-x'^2) + 2 y x x' \sqrt{1-x^2}(1-x'^2) \right].$$

Equation (21) cannot be reduced analytically. It can, however, be straightforwardly evaluated numerically. We illustrate it in Fig. 1 for $N = 4, \ldots, 100$ and $k = 2, \ldots, \lfloor N/2 \rfloor$. The figure clearly shows that $F^S_{|D_N^{(1)}\rangle,W_N^{(k)}}$ remains significantly below one with an asymptotic behavior as $N$ tends to infinity for fixed $k$. For fixed $N$, the considered fidelities decrease with increasing $k$. The largest ones are obtained for $k \geq 2$ with an asymptotic value for large $N$ of $\sim 0.63$.

By contrast, surprisingly we have

$$F_{|D_N^{(1)}\rangle,W_N^{(k)}} = 1, \quad \forall N \geq 4, k = 2, \ldots, \lfloor N/2 \rfloor,$$

which means that the Dicke state $|D_N^{(1)}\rangle$ can be approached as closely as desired by non-symmetric SLOCC inequivalent $W_N^{(k)}$ states. We thus clearly have

$$F_{|D_N^{(1)}\rangle,W_N^{(k)}} > F^S_{|D_N^{(1)}\rangle,W_N^{(k)}},$$

$$\forall N \geq 4, k = 2, \ldots, \lfloor N/2 \rfloor.$$
The state $|\psi^{(k)}_N(\epsilon)\rangle$ is also non-symmetric since a non-symmetric ILO acting on $|D_N^{(k)}\rangle$ always yields a non-symmetric state for $k \geq 2$. A detailed calculation then gives, for any $\epsilon \neq 0$ [19], (see Appendix [23])

$$|\psi^{(k)}_N(\epsilon)\rangle = \mathcal{N}_\epsilon(|D_{N-1}^{(1)}\rangle + |\psi_\epsilon\rangle),$$

where $\mathcal{N}_\epsilon = 1/\sqrt{1 + \|\psi_\epsilon\|^2}$ and

$$|\psi_\epsilon\rangle = \sum_{j=1}^{N-k} (-\epsilon)^j a_j (b_j |D_{N-1}^{(j-1)}\rangle \otimes |0\rangle + c_j |D_{N-1}^{(j-1)}\rangle \otimes |1\rangle),$$

with

$$a_j = \frac{C_{N-j-1}^{k-1}}{C_{N-2}^k} \sqrt{C_{N-1}^{j+1} / N},$$

$$b_j = (N-j-k) \sqrt{\frac{j+1}{N-j-1}},$$

$$c_j = \frac{(j-k) - 2N(j-1)}{N-1}.$$  

We have $\lim_{\epsilon \to 0} |\psi_\epsilon\rangle = 0$ and thus

$$\lim_{\epsilon \to 0} |\psi^{(k)}_N(\epsilon)\rangle = |D_N^{(1)}\rangle.$$

This implies that

$$\lim_{\epsilon \to 0} |\langle D_N^{(1)} | \psi^{(k)}_N(\epsilon) \rangle|^2 = 1^-,$$

and this proves Eq. (24).

These results show that for any $N \geq 4$ and $k = 2, \ldots, N-2$, a non-symmetric ILO of the form $A^\otimes N-1 \otimes B$ can transform the Dicke state $|D_N^{(k)}\rangle$ into a non-symmetric state located as closely as desired to the Dicke state $|D_N^{(1)}\rangle$, even though $|D_N^{(k)}\rangle$ and $|D_N^{(1)}\rangle$ are SLOCC inequivalent. This result cannot be achieved when only symmetric SLOCC operations are considered.

As an aside and out of curiosity, it is interesting to study the inverse operation $g_{N,k}(\epsilon)^{-1}$ acting on the $|D_N^{(1)}\rangle$ state. While one obviously has

$$\frac{g_{N,k}(\epsilon)^{-1} |\psi^{(k)}_N\rangle}{\|g_{N,k}(\epsilon)^{-1} |\psi^{(k)}_N\rangle\|} = |D_N^{(k)}\rangle,$$

the state

$$|\psi^{(1)}_{N,k}(\epsilon)\rangle = \frac{g_{N,k}(\epsilon)^{-1} |D_N^{(1)}\rangle}{\|g_{N,k}(\epsilon)^{-1} |D_N^{(1)}\rangle\|}$$

reads, up to a normalization constant,

$$|\psi^{(1)}_{N,k}(\epsilon)\rangle =$$

$$\left((N-2)k|1,\ldots,1\rangle + (N-3)(N-k)|1,\ldots,1,0\rangle - |D_{N-1}^{(N-2)}\rangle \otimes \frac{(N-2)(N-k)}{\sqrt{N-1}} |0\rangle + \sqrt{N-1}k|1\rangle \right),$$

This state is totally independent of $\epsilon$ and differs significantly from $|D_N^{(k)}\rangle$. It has only components onto states with at least $N-2$ excitations. Therefore, except for $N = 4\alpha dk = 2$, $|\psi^{(1)}_{N,k}(\epsilon)\rangle$ has no overlap with $|D_N^{(k)}\rangle$.

For $N = 4\alpha dk = 2$, the fidelity with the state $|D_N^{(k)}\rangle$ amounts only to $1/14$.

### B. More general counterexamples

Equation (23) can even be generalized. Any state $|\psi_{N,1}\rangle$ of the $\mathcal{W}_N^{(1)}$ SLOCC class satisfies

$$F_{|\psi_{N,1}\rangle,\mathcal{W}_N^{(1)}} = 1, \quad \forall N \geq 4, k = 2, \ldots, [N/2].$$  

![FIG. 1. (Color online) Quantum fidelities $F_{\mathcal{D}_N^{(k)},\mathcal{W}_N^{(k)}}$ (green and blue dotted points) and $F_{\mathcal{D}_N^{(k)},\mathcal{W}_N^{(1)}}$ (red diamond points) as a function of $N \geq 4$ for each $k = 2, \ldots, [N/2]$. For a given $k$ value, the first plotted quantum fidelity $F^S$ is for $N = 2k$.](image-url)
Indeed, for such states an ILO $h$ can be found such that
\[ |\psi_{N,1}⟩ = hD_N(1). \] (38)

We then define for any $\epsilon \neq 0$ the $W_N^{(k)}$ states
\[ |\psi'_{N}(\epsilon)⟩ = \frac{h|\psi_{N}(\epsilon)⟩}{\|h|\psi_{N}(\epsilon)⟩\|}, \] (39)
with $|\psi_{N}(\epsilon)⟩$ as defined by Eq. (29). We get trivially, up to a normalization constant,
\[ |\psi'_{N}(\epsilon)⟩ = |\psi_{N,1}⟩ + h|ψ_{r}\rangle, \] (40)
and thus
\[ \lim_{\epsilon \to 0} |\psi'_{N}(\epsilon)⟩ = |\psi_{N,1}⟩. \] (41)

It follows that
\[ \lim_{\epsilon \to 0} (|\psi_{N,1}⟩|\psi'_{N}(\epsilon)⟩|^2 = 1^-, \] (42)
and this implies Eq. (37).

This result shows that all states of the $W_N^{(1)}$ SLOCC class can be approached as closely as desired by non-symmetric states of any of the $W_N^{(k)}$ SLOCC classes, for $N \geq 4$ and $k = 2, \ldots, [N/2]$. Topologically, this means that the $W_N^{(1)}$ SLOCC class of states lies at the boundary of the non-symmetric side of any of the $W_N^{(k)}$ classes. This result sheds an additional light on the general topology of the SLOCC classes of multiqubit systems, whose restriction on the only symmetric subspace was established in Ref. [20].

The converse of the previous result is by far not true: The states of the $W_N^{(k)}$ SLOCC classes cannot be approached as closely as desired by $W_N^{(1)}$ states, even if they are non-symmetric. As an example, a detailed calculation yields (see Appendix A)
\[ F_{|D_N^{(s)},W_N^{(1)}⟩} = C_N \hat{k}^{-1} \left[ 1 - \hat{k}_r \right]^{N-k-1} \left[ \hat{k} + (1 - 2\hat{k})\hat{k}_r \right] \] (43)
with $\hat{k} = k/N$ and $\hat{k}_r = \sqrt{\hat{k}(1 - \hat{k})/(N - 1)}$ and extensive numerical simulations [see our conjecture in Eq. (19)] showed that this should also correspond to $F_{|D_N^{(s)},W_N^{(1)}⟩}$. We exemplify some values of $F_{|D_N^{(s)},W_N^{(1)}⟩}$ in Table I. It is noteworthy to mention that these fidelities decrease with increasing $N$ and $k$. For fixed $k$, we have
\[ \lim_{N \to \infty} F_{|D_N^{(s)},W_N^{(1)}⟩} = \frac{1}{k!^2} \sqrt{\pi}^{-k}(k - \sqrt{k})^k \frac{2\sqrt{k} - 1}{\sqrt{k} - 1}. \] (44)

Explicitly, for $k = 2, \ldots, 8$, this limit reads 0.422, 0.322, 0.271, 0.238, 0.215, 0.197, and 0.183, respectively. All these results clearly show that while the $|D_N^{(1)}⟩$ state can be approached as closely as desired by non-symmetric states that are SLOCC equivalent to any of the $|D_N^{(k)}⟩$ states ($k = 2, \ldots, [N/2]$), none of these latter can be closely approached by any state SLOCC equivalent to the Dicke state $|D_N^{(1)}⟩$.

IV. CONCLUSION

In this paper we have analyzed the maximization of the quantum fidelity between symmetric multiqubit states and sets of LU- or SLOCC-equivalent states that contain symmetric states. We have shown that the open question in Eq. (5) admits a positive answer when LU classes of qubit states are considered, while the answer is negative when turning to SLOCC classes of states.

In the case of LU sets, the positive answer simplifies considerably the calculation of the desired maximal overlap. For SLOCC classes of states, we have shown significant violations of Eq. (5), whereas for some states $|\psi_{S}⟩$ and classes $C$ the fidelity $F_{|\psi_{S},C⟩}$ takes the maximal possible value 1, while the symmetric restriction $F_{|\psi_{S},C⟩}^{S}$ has only significantly much lower values. This is in particular the case when considering any states $|\psi_{S}⟩$ of the $W_N^{(1)}$ SLOCC class in combination with any of the $W_N^{(k)}$ SLOCC classes, for $N \geq 4$ and $k = 2, \ldots, [N/2]$. Finally, extensive numerical simulations have also lead us to conjecture that Eq. (5) seems to be correct when the considered SLOCC class $C$ identifies to $W_N^{(1)}$, whatever the state $|\psi_{S}⟩$.

There are several directions in which our work can be generalized. First, concerning LU equivalence classes, it would be highly desirable to prove our result also for higher-dimensional systems and not only for qubits. In our proof, we made use of the simple parametrization of SU(2) matrices, which is not so simple in higher dimensional systems. Second, for SLOCC equivalence classes it would be very useful to find out under which additional conditions the optimization over symmetric states is enough. Based on numerical evidence we identified some examples where this seems to be the case, but so far no clear understanding has been reached. From a more general perspective, our work presents examples where symmetries can help to solve optimization problems related to the numerical range [21,22]. Understanding further the role of symmetry in such problems is clearly a challenging task, nevertheless it will have a significant impact on various problems in quantum information theory.

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For \( j = 0, \ldots, k' \),

\[
c_j(x, x') = C_N^{k'} C_{N-k'}^{k-j} x^{k-j} \sqrt{1 - x^2}^{N-k'+j} \times x'^j \sqrt{1 - x'^2}^{k-j}.
\]  

(A2)

where \( C_m^n \equiv \binom{m}{n} \) is the binomial coefficient between \( m \) and \( n \), with the usual convention \( C_m^n = 0 \) if \( n < 0 \) or \( n > m \). \( \delta \) denotes the Kronecker delta, \( T_j(y) \) is the \( j \)-th degree Chebyshev polynomial of the first kind, and, for \( j = 0, \ldots, k' \),

\[
F_{[D_N^{(k')}, W_N^{(k')}]}^S = \frac{C_N^k}{C_N^N} \sup_{x, x' \in [0,1]} \sum_{j=0}^{k'} \left[ (2 - \delta_j, 0) \sum_{j'=j}^{k'} c_{j'}(x, x') c_{j'-j}(x, x') T_j(y) \right],
\]  

(A1)

In particular, for \( k' = 0 \), one gets the well known result

\[
F_{[D_N^{(k')}, W_N^{(k')}]}^S = C_N^k \sup_{x \in [0,1]} x^{2k}(1 - x^2)^{N-k} = C_N^k(1 - \tilde{k})^{N-k},
\]  

(A3)

with \( \tilde{k} = k/N \) the fractional excitation of the Dicke state \( |D_N^{(k')}\rangle \). For \( k' = 1 \), one gets

\[
F_{[D_N^{(k')}, W_N^{(k')}]}^S = \frac{C_N^k}{C_N^N} \sup_{x, x' \in [0,1]} c_0(x, x')^2 + c_1(x, x')^2 + c_0(x, x') c_1(x, x') y
\]

\[
= C_N^k (1 - \tilde{k})^{N-k-1} \left[ \tilde{k} + (1 - 2\tilde{k}) \tilde{k}_r \right],
\]  

(A4)

with

\[
\tilde{k}_r = \tilde{k} - \sqrt{\frac{1 - \tilde{k}}{N-1}}.
\]

(A5)

For \( k' > 1 \), Eq. (A1) cannot be reduced analytically and it must be evaluated numerically. We noticed that for all tested cases the supremum was systematically obtained for \( y = -1 \).

To prove Eq. (A1), we first observe that in the computational basis the Dicke state \( |D_N^{(k')}\rangle \) merely reads

\[
\frac{1}{\sqrt{C_N^k}} \sum_{\epsilon_0, \ldots, \epsilon_k} |0, \ldots, 0, 1, \ldots, 1\rangle,
\]

where the sum runs over all multimqubit states with any \( k \) qubits in the |1\rangle state and the remaining \( N - k \) qubits in the |0\rangle state. We then notice that any symmetric state of the \( W_N^{(k')} \) SLOCC class, i.e., states of the \( D_{N-k', k'} \) family, can be written in the form

\[
|\psi\rangle = \frac{1}{N} \mathcal{N} \sum_{\epsilon, \ldots, \epsilon'} \epsilon |\epsilon \rangle, \ldots, \epsilon' \rangle,
\]

where \( \mathcal{N} \) is a normalization constant and the sum runs over all multimqubit states with any \( k' \) qubits in |\epsilon\rangle state and the remaining \( N - k' \) qubits in a distinct |\epsilon'\rangle \neq |\epsilon\rangle state. Writing the single qubit states |\epsilon\rangle and |\epsilon'\rangle in

\[
F_{[D_N^{(k')}, W_N^{(k')}]}^S = \sup_{x, x' \in [0,1]} N^2 C_N^k \sum_{j=0}^{k'} c_j(x, x') e^{i j \Delta \phi} \left| \sum_{\Delta \phi \in [0, 2\pi]} \right| 2,
\]  

(A6)

with \( c_j(x, x') \) as given by Eq. (A2). Finally, using the identity \( \cos(j \Delta \phi) = T_j(\cos \Delta \phi) \) and setting \( y = \cos \Delta \phi \) yields Eq. (A1).
Appendix B

In this appendix we prove Eq. (29). We first observe that $A(\epsilon)|0\rangle = |0\rangle - \epsilon|1\rangle$, $A(\epsilon)|1\rangle = |0\rangle$, and

$$B_{N,k}(\epsilon)|0\rangle = |0\rangle + \epsilon|1\rangle,$$

$$B_{N,k}(\epsilon)|1\rangle = \left(1 - \frac{N}{k}\right)(|0\rangle + \frac{N-2}{N-1}|1\rangle). \quad (B1)$$

We then define the unnormalized Dicke states $|u_N^{(k)}\rangle = \sqrt{C_N^k} |D_N^{(k)}\rangle$, which satisfy

$$|u_N^{(k)}\rangle = \left(|u_{N-1}^{(k-1)}\rangle \otimes |1\rangle + |u_{N-1}^{(k)}\rangle \otimes |0\rangle\right). \quad (B2)$$

It follows that

$$g_{N,k}(\epsilon)|u_N^{(k)}\rangle = \left[A(\epsilon)^{\otimes N-1} |u_{N-1}^{(k-1)}\rangle \otimes B_{N,k}(\epsilon)|1\rangle + A(\epsilon)^{\otimes N-1} |u_{N-1}^{(k)}\rangle \otimes B_{N,k}(\epsilon)|0\rangle\right]. \quad (B3)$$

Inserting Eq. (B1) in Eq. (B3) and observing that for any $N$ and $k$ the symmetric state $A(\epsilon)^{\otimes N}|u_N^{(k)}\rangle$ reads in the Dicke state basis

$$A(\epsilon)^{\otimes N}|u_N^{(k)}\rangle = C_N^k \sum_{j=0}^{N-k} (-\epsilon)^j \frac{C_{N-k}^j}{\sqrt{C_N^j}} |D_N^{(j)}\rangle \quad (B4)$$

yields straightforwardly

$$g_{N,k}(\epsilon)|D_N^{(k)}\rangle = \epsilon^{\frac{\alpha_{N-1,k}}{\alpha_{N,k}}} \times \sum_{j=0}^{N-k} (-\epsilon)^j a_j \left(b_j |D_{N-1}^{(j+1)}\rangle \otimes |0\rangle + c_j |D_{N-1}^{(j)}\rangle \otimes |1\rangle\right) \quad (B5)$$

with $a_j$, $b_j$, and $c_j$ as given by Eq. (51) and $\alpha_{N,k} = C_N^k/N$. In the sum over $j$ in Eq. (B5), the first term $j = 0$ merely yields the states $(|D_{N-1}^{(0)}\rangle \otimes |1\rangle + \sqrt{N-1} |D_{N-1}^{(1)}\rangle \otimes |0\rangle)/\sqrt{N}$, which is nothing but the Dicke state $|D_N^{(1)}\rangle$ [see Eq. (B2)]. The rest of the sum from $j = 1$ to $j = N-k$ is by definition the state $|\psi_k\rangle$ [Eq. (31)]. We thus get

$$g_{N,k}(\epsilon)|D_N^{(k)}\rangle = \epsilon^{\frac{\alpha_{N-1,k}}{\alpha_{N,k}}} \left(|D_{N}^{(1)}\rangle + |\psi_k\rangle\right), \quad (B6)$$

from which Eq. (29) immediately follows for any $\epsilon \neq 0$. For $\epsilon = 0$, $g_{N,k}(\epsilon)|D_N^{(k)}\rangle = 0$ and the normalized state $|\psi_N^{(k)}(\epsilon)\rangle$ is not defined.

[1] A. Uhlmann, Rep. Math. Phys. 9, 273 (1976).
[2] R. Jozsa, J. Mod. Optics 41, 2315 (1994).
[3] D. J. C. Bures, Trans. Am. Math. Soc. 135, 199 (1969).
[4] M. Hübner, Phys. Lett. A 163, 239 (1992); 179, 226.
[5] H.-J. Sommers and K. Życzkowski, J. Phys. A: Math. Gen. 36, 10083 (2003).
[6] T.-C. Wei and P. M. Goldbart, Phys. Rev. A 68, 042307 (2003).
[7] Y. Huang, New. J. Phys. 16, 033027 (2014).
[8] R. Hübenner, M. Kleinmann, T.-C. Wei, C. González-Guillén, and O. Gühne, Phys. Rev. A 80, 032324 (2009).
[9] A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera Phys. Rev. Lett. 87, 040401 (2001).
[10] S. Niekamp, M. Kleinmann, and O. Gühne, Phys. Rev. A 82, 022322 (2010).
[11] W. Dür, G. Vidal, and J. I. Cirac Phys. Rev. A 62, 062314 (2000).
[12] C. D. Cenci, D. W. Lyons, S. N. Walck, arXiv:1011.5220 [Theory of Quantum Computation, Communication and Cryptography, edited by D. Bacon, M. Martin-Delgado, and M. Roetteler, Lecture Notes in Computer Science, Vol. 6745 (Springer, Berlin, 2014), p. 198.
[13] P. Migdal, J. Rodriguez-Laguna, and M. Lewenstein, Phys. Rev. A 88, 012335 (2013).
[14] L. Hörmander, Math. Scand. 2, 55 (1954).
[15] O. D. Kellogg, Math. Z. 27, 55 (1928).
[16] P. Mathonet, S. Krins, M. Godefroid, L. Lamata, E. Solano, and T. Bastin, Phys. Rev. A 81, 052315 (2010).

[17] We recall that the $k$-excitation Dicke states ($k = 0, \ldots, N$) are defined as $|D_N^{(k)}\rangle = 1/\sqrt{C_N^k} \sum_{\pi} |\pi, 0, \ldots, 0, 1, \ldots, 1\rangle$, where $C_N^k$ denotes the binomial coefficient $\binom{N}{k}$, the multiqubit states in the sum contain $k$ qubits in state $|1\rangle$, and $\pi$ denotes all permutations of the qubits leading to different terms in the sum. All $|D_N^{(k)}\rangle$ Dicke states ($k = 0, \ldots, N$) are symmetric and they form an orthonormal basis in the symmetric subspace of the multiqubit system. For $k = 0, \ldots, \lfloor N/2 \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function, the Dicke states $|D_N^{(k)}\rangle$ are SLOCC inequivalent between each others [18]. In contrast, the $|D_N^{(N-k)}\rangle$ and $|D_N^{(k)}\rangle$ states are LU equivalent.

[18] T. Bastin, S. Krins, P. Mathonet, M. Godefroid, L. Lamata, and E. Solano, Phys. Rev. Lett. 103, 070503 (2009).

[19] Although the $N$-qubit local operation $g_{N,k}(\epsilon)$ remains defined for $\epsilon = 0$, though not invertible in this case, the normalized state $|\psi_N^{(k)}(\epsilon)\rangle$ is in contrast not defined in this case since $g_{N,k}(\epsilon = 0)|D_N^{(k)}\rangle = 0$ (see Appendix B).

[20] T. Bastin, P. Mathonet, and E. Solano, Phys. Rev. A 91, 022310 (2015).

[21] P. Gawron, Z. Puchała, J.A. Miszczak, L. Skowronek, and K. Życzkowski, J. Math. Phys. 51, 102204 (2010).
[22] Z. Puchała, P. Gawron, J.A. Miszczak, L. Skowronek, M.-D. Choi, K. Życzkowski, Linear Algebra Appl. 434, 327 (2011).