ON COMONADICITY OF THE
EXTENSION-OF-SCALARS FUNCTORS

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Abstract. A criterion for comonadicity of the extension-of-scalars
functor associated to an extension of (not necessarily commutative)
rings is given. As an application of this criterion, some known re-
results on the comonadicity of such functors are obtained.

1. Introduction

In view of the observation of Caenepeel (see [4]) that noncommu-
tative descent for modules reduces to comonadicity of the correspond-
ing extension-of-scalars functor, it becomes even more sensible to have
manageable tests for comonadicity of the extension-of-scalars functors.
(Although there are several results obtained along these lines (see [2],
[3], [4], [5], [6], [7], [8], [10], [13]) the question of comonadicity of such
functors is not fully answered yet.) The main result of this note gives
such a test.

For the basic definitions of category theory, see [9]

2. Preliminaries

A monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ is called biclosed if for all $X \in \mathrm{Ob}(\mathcal{V}_0)$, the functors

$$- \otimes X, \ X \otimes - : \mathcal{V}_0 \to \mathcal{V}_0$$

have (chosen) right adjoints, denoted $[X, -]$ and $\{X, -\}$, respectively. In other words, a biclosed monoidal category consists of a monoidal
category $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$, equipped with two functors

$$[\cdot, \cdot], \{\cdot, \cdot\} : \mathcal{V}_0^{\text{op}} \times \mathcal{V}_0 \to \mathcal{V}_0,$$

for which there are natural isomorphisms

$$\mathcal{V}_0(X, [Y, Z]) \simeq \mathcal{V}_0(X \otimes Y, Z) \simeq \mathcal{V}_0(Y, \{X, Z\}) \quad (2.1)$$

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comonadicity.
Recall that the adjunctions $X \otimes - \dashv \{X, -\}$ and $- \otimes X \dashv [X, -]$ are internal, in the sense that one has natural isomorphisms
\[(2.2) \quad \{X \otimes Y, Z\} \simeq \{Y, \{X, Z\}\}\]
and
\[(2.3) \quad [X \otimes Y, Z] \simeq [X, [Y, Z]].\]

Let us recall that a morphism in a category $\mathcal{A}$ is a regular monomorphism if it is an equalizer of some pair of morphisms. Recall also that an object $X$ of $\mathcal{A}$ is injective if it is injective with respect to the class of regular monomorphisms of $\mathcal{A}$, that is, if every extension problem
\[
\begin{array}{ccc}
A & \xrightarrow{m} & B \\
\downarrow{f} & \nearrow{\bar{f}} \\
X & & 
\end{array}
\]
with $m$ a regular monomorphism has a solution $\bar{f} : B \to X$ extending $f$ along $m$, i.e., satisfying $fm = f$. (The dual notions are the regular epimorphism and the projective object.)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ be a monoidal category and let $f : X \to Y$ be a morphism in $\mathcal{V}_0$. We say that $f$ is right (resp. left) pure if, for any $Z \in \text{Ob}(\mathcal{V}_0)$, the morphism
\[
f \otimes Z : X \otimes Z \to Y \otimes Z
\]
(resp. $Z \otimes f : Z \otimes X \to Z \otimes Y$)
is a regular monomorphism.

Henceforth, we suppose without explicit mention that $\mathcal{V}$ is a finitely complete and finitely cocomplete monoidal biclosed category whose unit $I$ for the tensor product is projective.

**Theorem 2.1.** Let $Q$ be an object of $\mathcal{V}_0$ for which the functor
\[
\{-, Q\} : \mathcal{V}_0^{op} \to \mathcal{V}_0
\]
is conservative (that is, isomorphism-reflecting) and preserves regular epimorphisms. Then the following properties of a morphism $f : X \to Y$ of $\mathcal{V}_0$ are equivalent:

(i) The morphism $f$ is right pure.

(ii) The morphism $f \otimes \{X, Q\} : X \otimes \{X, Q\} \to Y \otimes \{X, Q\}$ is a regular monomorphism.

(iii) The morphism $\{f, Q\} : \{Y, Q\} \to \{X, Q\}$ is a split epimorphism.
Proof. (i) implies (ii) trivially. To see that (ii) implies (iii), let us assume that the morphism

\[ f \otimes \{X, Q\} : X \otimes \{X, Q\} \to Y \otimes \{X, Q\} \]

is a regular monomorphism. Since the functor \(\{-, Q\} : \mathcal{V}_0^{op} \to \mathcal{V}_0\) preserves regular epimorphisms by hypothesis, the morphism

\[ \{\{X, Q\}, \{f, Q\}\} : \{\{X, Q\}, \{Y, Q\}\} \to \{\{X, Q\}, \{X, Q\}\}, \]

which is isomorphic by (2.2) to the morphism

\[ \{f \otimes \{X, Q\}, Q\} : \{Y \otimes \{X, Q\}, Q\} \to \{X \otimes \{X, Q\}, Q\}, \]

is a regular epimorphism in \(\mathcal{V}_0\). Since \(I\) is assumed to be projective in \(\mathcal{V}_0\), the functor

\[ \mathcal{V}_0(I, -) : \mathcal{V}_0 \to \text{Set} \]

takes regular epimorphisms to surjections. It follows that the map

\[ \mathcal{V}_0(\{X, Q\}, \{f, Q\}) \]

of sets, which (using (2.1))) is isomorphic to the map

\[ \mathcal{V}_0(I, \{\{X, Q\}, \{f, Q\}\}). \]

is surjective. But this means that every morphism

\[ \{X, Q\} \to \{X, Q\} \]

factors through \(\{f, Q\}\), that is to say, that \(\{f, Q\}\) is a split epimorphism, as is seen from the special case of the identity morphism \(1_{\{X, Q\}}\).

It remains to show that (iii) implies (i). If the morphism

\[ \{f, Q\} : \{Y, Q\} \to \{X, Q\} \]

is a split epimorphism, then so is

\[ \{Z, \{f, Q\}\} : \{Z, \{Y, Q\}\} \to \{Z, \{X, Q\}\} \]

too, for all \(Z \in \text{Ob}(\mathcal{V}_0)\). Identifying the morphism \(\{Z, \{f, Q\}\}\) (via the isomorphism (2.2)) with \(\{f \otimes Z, Q\}\), we see that the morphism

\[ \{f \otimes Z, Q\} : \{Y \otimes Z, Q\} \to \{X \otimes Z, Q\} \]

is also a split epimorphism. We now observe that, since the functor \(\{-, Q\} : \mathcal{V}_0^{op} \to \mathcal{V}_0\) admits as a right adjoint the functor \([-, Q] : \mathcal{V}_0 \to \mathcal{V}_0^{op}\), as can be seen from the following sequence of natural isomorphisms:

\[ \mathcal{V}_0(X, \{Y, Q\}) \simeq \mathcal{V}_0(Y \otimes X, Q) \simeq \mathcal{V}_0(Y, [X, Q]) \simeq \mathcal{V}_0^{op}([X, Q], Y), \]

to say that \(\{-, Q\}\) is conservative and preserves regular epimorphisms is to say that it preserves and reflects regular epimorphisms. And since any split epimorphism is regular, it follows that the morphism
\[ f \otimes Z : X \otimes Z \to Y \otimes Z \] is a regular monomorphism for all \( Z \in \text{Ob}(\mathcal{V}_0) \). Thus (iii) implies (i). The proof of the theorem is now complete. □

There is of course a dual result:

**Theorem 2.2.** Let \( Q \) be an object of \( \mathcal{V}_0 \) such that the functor
\[ [-, Q] : \mathcal{V}_0^{\text{op}} \to \mathcal{V}_0 \]
is conservative and preserves regular epimorphisms. Then the following properties of a morphism \( f : X \to Y \) of \( \mathcal{V}_0 \) are equivalent:

(i) The morphism \( f \) is left pure.

(ii) The morphism \( [X, Q] \otimes f : [X, Q] \otimes X \to [X, Q] \otimes Y \) is a regular monomorphism.

(iii) The morphism \( [f, Q] : [Y, Q] \to [X, Q] \) is a split epimorphism.

An object \( Q \) of a monoidal biclosed category
\[ \mathcal{V} = (\mathcal{V}_0, \otimes, I, [\cdot, \cdot], \{-, -\}) \]
is said to be cyclic if the functors \( \{-, Q\} \) and \( [-, Q] \) are naturally isomorphic. If \( Q \) is such an object, we shall denote by \([[\cdot, Q]]\) the functor \( [-, Q] \simeq \{-, Q\} \).

Combining Theorems 2.1 and 2.2, we get:

**Theorem 2.3.** Let \( Q \) be a cyclic object of \( \mathcal{V}_0 \) for which the functor
\[ [[\cdot, Q]] : \mathcal{V}_0^{\text{op}} \to \mathcal{V}_0 \]
is conservative and preserves regular epimorphisms (equivalently, preserves and reflects regular epimorphisms). Then the following properties of a morphism \( f : X \to Y \) of \( \mathcal{V}_0 \) are equivalent:

(i) The morphism \( f \) is left pure.

(ii) The morphism \( f \) is right pure.

(iii) The morphism \([[X, Q]] \otimes f : [[X, Q]] \otimes X \to [[X, Q]] \otimes Y \) is a regular monomorphism.

(iv) The morphism \( f \otimes [[X, Q]] : X \otimes [[X, Q]] \to Y \otimes [[X, Q]] \) is a regular monomorphism.

(v) The morphism \([[f, Q]] : [[Y, Q]] \to [[X, Q]] \) is a split epimorphism.
3. A Criterion for Comonadicity of Extension-of-Scalars Functors

In this section we present our main result.

Let us fix a commutative ring $K$ with unit ($K = \mathbb{Z}$, the ring of integers, inclusive). All rings under consideration are associative unital $K$-algebras. A right or left module means a unital module. All bimodules are assumed to be $K$-symmetric. The $K$-categories of left and right modules over a ring $A$ are denoted by $_A\text{Mod}$ and $\text{Mod}_A$, respectively; while the category of $(A, B)$-bimodules is $\text{Mod}_{A,B}$. We will use the notation $\mathcal{B}M_A$ to indicate that $M$ is a left $\mathcal{B}$, right $A$-module.

It is a well-known fact that, for a fixed ring $A$, the category $\text{Mod}_A$ is a monoidal category with tensor product of two $(A, A)$-bimodules being their usual tensor product over $A$ and the unit for this tensor product being the $(A, A)$-bimodule $A$. Moreover, this monoidal category is biclosed: If $M$ and $N$ are two $(A, A)$-bimodules, then $[M, N] = \text{Mod}_A(M, N)$ and $\{M, N\} = \text{Mod}(M, N)$.

For any $(A, A)$-bimodule $M$, the character $(A, A)$-bimodule of $M$ is defined to be $M^+ = \text{Ab}(M, \mathbb{Q}/\mathbb{Z})$ (where $\text{Ab}$ is the category of abelian groups and $\mathbb{Q}/\mathbb{Z}$ is the rational circle abelian group). This is an $(A, A)$-bimodule via the actions $(af\alpha')(m) = f(\alpha'ma)$.

**Lemma 3.1.** The character bimodule $A^+$ of the $(A, A)$-bimodule $A$ is a cyclic object of the monoidal biclosed category $\text{Mod}_A$ of $(A, A)$-bimodules.

**Proof.** The following string of natural isomorphisms

$$\{\cdot, A^+\} = \{\cdot, \text{Ab}(A, \mathbb{Q}/\mathbb{Z})\} = \text{Mod}_A(\cdot, \text{Ab}(A, \mathbb{Q}/\mathbb{Z})) \simeq \text{Ab}(A \otimes_A \cdot, \mathbb{Q}/\mathbb{Z}) \simeq \text{Ab}(\cdot, \mathbb{Q}/\mathbb{Z}) \simeq \text{Ab}(\cdot \otimes_A A, \mathbb{Q}/\mathbb{Z}) \simeq \text{Mod}_A(\cdot, \text{Ab}(A, \mathbb{Q}/\mathbb{Z})) = [-, A^+]$$

shows that the functors

$$\{\cdot, A^+\}, [-, A^+] : (\text{Mod}_A)^{op} \to \text{Mod}_A$$

are naturally equivalent. \qed

Since the functor $[[-, A^+]]$ is naturally equivalent to $\text{Ab}(\cdot, \mathbb{Q}/\mathbb{Z})$ and since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in $\text{Ab}$, we have that

**Lemma 3.2.** The functor $[[-, A^+]] : (\text{Mod}_A)^{op} \to \text{Mod}_A$ is exact and conservative.

Before we prove our main result we recall a result from [10]:

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Theorem 3.3. Let \( i : A \to B \) be a homomorphism of rings. If the induced morphism \( i^+: B^+ \to A^+ \) is a split epimorphism of \((A,A)\)-bimodules, then the functors
\[
- \otimes_A B : \text{Mod}_A \to \text{Mod}_B
\]
and
\[
B \otimes_A - : \text{AMod} \to B \text{Mod}
\]
are both comonadic.

Recall (for example from [12]) that a morphism \( f : M \to N \) of right \( A \)-modules is called pure if \( f \otimes_A 1_L : M \otimes_A L \to N \otimes_A L \) is injective for every left \( A \)-module \( L \). Pure morphisms in the category of left \( A \)-modules are defined analogously.

The main result of this note is contained in the following:

Theorem 3.4. Let \( i : A \to B \) be a homomorphism of rings. If \( A \) is a separable \( K \)-algebra, then the following are equivalent:

(i) \( i \) is a pure morphism of left \( A \)-modules.

(ii) \( i \) is a pure morphism of right \( A \)-modules.

(iii) \( i^+: B^+ \to A^+ \) is a split epimorphism of \((A,A)\)-bimodules.

(iv) The functor \(- \otimes_A B : \text{Mod}_A \to \text{Mod}_B\) is comonadic.

(v) The functor \( B \otimes_A - : \text{AMod} \to B \text{Mod}\) is comonadic.

Proof. We remark first that, by left-right symmetry, it suffices to prove the equivalence of (i), (iii) and (v).

Using that the forgetful functor \( \text{AMod}_A \to \text{Mod}_A \) preserves and reflects monomorphisms and tensor products, it is easy to see that if \( i \) is a pure morphism of left \( A \)-modules, then it is left pure in \( \text{AMod}_A \). And since assuming \( A \) be \( K \)-separable is, just by definition, the same as assuming \( A \) be projective in \( \text{AMod}_A \), it follows from Theorem 2.3 that the morphism \([i, A^+] \simeq i^+\) is a split epimorphism of \((A,A)\)-bimodules, provided that \( i \) is a pure morphism of left \( A \)-modules. Thus (i) implies (iii).

(iii) implies (v) by Theorem 3.3.

It is well known that the functor
\[
- \otimes_A B : \text{Mod}_A \to \text{Mod}_B
\]
admits as a right adjoint the functor
\[
\text{Mod}_B(B, -) : \text{Mod}_B \to \text{Mod}_A
\]
and that the unit \( \eta \) of this adjunction has components
\[
\eta_X : X \otimes_A i : X \simeq X \otimes_A A \to X \otimes_A B, \ X \in \text{Mod}_A.
\]
Thus $i$ is a pure morphism of left $A$-modules precisely when $\eta$ is componentwise a monomorphism. According to Theorem 9 of Section 2.3 of [1], this is in particular the case when the functor $- \otimes_A B$ is comonadic. So (v) implies (i). This completes the proof of the theorem. \(\square\)

4. Applications

In this section we state some consequences of our main theorem. To state the first one, we need a definition. Let $A$, $B$ be rings. Recall [5] that an $(A, B)$-bimodule $M$ is said to be totally faithful as a left $A$-module if the morphism

$$X \to \text{Mod}_B(M, X \otimes_A M), \ m \to x \otimes_A m,$$

is injective for every $X \in \text{Mod}_A$, or equivalently, if the unit of the adjunction

$$- \otimes_A M \dashv \text{Mod}_B(M, -) : \text{Mod}_B \to \text{Mod}_A$$

is pointwise a monomorphism.

**Theorem 4.1.** Let $A$ and $B$ be rings, $M$ an $(A, B)$-bimodule with $M_B$ finitely generated and projective, $\mathcal{E}_M = \text{Mod}_B(M, M)$ the right endomorphism ring of $M_B$ and

$$i_M : A \to \mathcal{E}_M, \ a \to [m \to am]$$

the corresponding ring homomorphism. If $A$ is $K$-separable, then the following are equivalent:

(i) The bimodule $A M_B$ is totally faithful as a left $A$-module.

(ii) The bimodule $B M^*_A$ is totally faithful as a right $A$-module.

(Here we denote by $M^*$ the dual $\text{Mod}_B(M, B)$ of $M_B$ which is a $(B, A)$-bimodule in a canonical way.)

(iii) The morphism $(i_M)^+ : (\mathcal{E}_M)^+ \to A^+$ is a split epimorphism of $(A, A)$-bimodules.

(iv) The functor $- \otimes_A M : \text{Mod}_A \to \text{Mod}_B$ is comonadic.

(v) The functor $M^* \otimes_A - : \text{AMod} \to \text{BMod}$ is comonadic.

**Proof.** Immediate from Theorem 3.4 using that:

- $A M_B$ (resp. $B M_A^*$) is totally faithful as a left (resp. a right) $A$-module if and only if $i_M : A \to \mathcal{E}_M$ is a pure morphism of left (resp. right) $A$-modules (see Lemma 2.2 in [5], or Proposition 7.3 in [10]);
- the functor $- \otimes_A M : \text{Mod}_A \to \text{Mod}_B$ (resp. $M^* \otimes_A - : \text{AMod} \to \text{BMod}$) is comonadic if and only if the functor $- \otimes_A \mathcal{E}_M : \text{Mod}_A \to \text{Mod}_{\mathcal{E}_M}$ (resp. $\mathcal{E}_M \otimes_A - : \text{AMod} \to \mathcal{E}_M \text{Mod}$) is so (see Theorem 7.5 in [10]).
As a special case of Theorem 4.1 one can take $A = K$. Then, since obviously $K$ is $K$-separable, we recover a result by Caenepeel, De Groot and Vercruysse [5].

**Theorem 4.2** (Caenepeel, De Groot and Vercruysse [5]). Let $A$ be a ring and let $M$ be a $(K, A)$-bimodule with $M_A$ finitely generated and projective. Then the following are equivalent:

(i) The morphism $i_M : K \to \mathcal{E}_M = \text{Mod}_A(M, M)$ is a pure morphism of left $K$-modules.

(ii) The morphism $i_M : K \to \mathcal{E}_M = \text{Mod}_A(M, M)$ is a pure morphism of right $K$-modules.

(iii) The bimodule $A M_B$ is totally faithful as a left $A$-module.

(iv) The bimodule $B M_A^*$ is totally faithful as a right $A$-module.

(v) The functor $- \otimes_A M : \text{Mod}_A \to \text{Mod}_B$ is comonadic.

(vi) The functor $M^* \otimes_A - : \text{AMod} \to \text{BMod}$ is comonadic.

For the special case in which $M = A$, we recapture easily the following result of Joyal and Tierney (unpublished, but see [11]). Recall (for example from [8]) that a homomorphism $i : K \to A$ of commutative rings is said to be effective for descent if the extension-of-scalars functor

$$ A \otimes_K - : \text{Mod}_K \to \text{Mod}_A $$

is comonadic.

**Theorem 4.3** (Joyal and Tierney). A homomorphism $i : K \to A$ of commutative rings is effective for descent if and only if it is a pure morphism of (say left) $K$-modules.

We end this note with an interesting consequence of Theorem 3.4. Let us write $\mathbb{M}_n(K)$ for the ring of $n \times n$ matrices over $K$.

**Theorem 4.4.** The following are equivalent for a homomorphism $i : \mathbb{M}_n(K) \to A$ of rings:

(i) $i$ is a pure morphism of left $A$-modules;

(ii) $i$ is a pure morphism of right $A$-modules;

(iii) the functor $- \otimes_{\mathbb{M}_n(K)} A : \text{Mod}_{\mathbb{M}_n(K)} \to \text{Mod}_A$ is comonadic;

(iv) the functor $A \otimes_{\mathbb{M}_n(K)} - : \mathbb{M}_n(K)\text{Mod} \to \text{AMod}$ is comonadic.

*Proof.* Immediate from Theorem 3.4, since for any $n \in \mathbb{N}$, the ring $\mathbb{M}_n(K)$ is $K$-separable. \qed
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