SOFT AND HARD WALL IN A STOCHASTIC REACTION DIFFUSION EQUATION

LORENZO BERTINI, STELLA BRASSESCO, AND PAOLO BUTTÀ

ABSTRACT. We consider a stochastically perturbed reaction diffusion equation in a bounded interval, with boundary conditions imposing the two stable phases at the endpoints. We investigate the asymptotic behavior of the front separating the two stable phases, as the intensity of the noise vanishes and the size of the interval diverges. In particular, we prove that, in a suitable scaling limit, the front evolves according to a one-dimensional diffusion process with a non-linear drift accounting for a “soft” repulsion from the boundary. We finally show how a “hard” repulsion can be obtained by an extra diffusive scaling.

1. Introduction

Let $V(m)$ be a smooth, symmetric, double well potential whose minimum is attained at $m = m_\pm$, $V''(m_\pm) > 0$. After the pioneering paper [1], the semi-linear parabolic equation

$$\frac{\partial m}{\partial t} = \frac{1}{2} \Delta m - V'(m)$$

(1.1)

and its stochastic perturbations, have became a basic model in the kinetics of phase separation and interface dynamics for systems with a non conserved order parameter.

Before introducing our results, let us review the main features of (1.1) in the one dimensional case. The corresponding evolution is the $L_2$ gradient flow of the functional

$$\mathcal{F}(m) = \int dx \left[ \frac{1}{2} m'(x)^2 + 2V(m(x)) \right].$$

(1.2)

In the case that (1.1) is considered in the whole line $\mathbb{R}$, there are infinitely many stationary solutions, which are the critical points of $\mathcal{F}$. The most relevant are the constant profiles $m_\pm$, where $\mathcal{F}$ attains its minimum, and $\pm \bar{m}$, where $\bar{m}$ is the solution to

$$\frac{1}{2} \bar{m}''(x) - V'('m) = 0, \quad \lim_{x \to \pm \infty} m(x) = m_\pm, \quad m(0) = 0,$$

(1.3)

together with its translates $\pm \bar{m}_{\zeta}(x) = \pm \bar{m}(x - \zeta)$, $\zeta \in \mathbb{R}$. The profile $\bar{m}_{\zeta}$ is a standing wave of (1.1) that connects the two pure phases $m_\pm$. Note that $\bar{m}_{\zeta}$ minimizes $\mathcal{F}$ under the constraint that $\lim_{x \to \pm \infty} m(x) = m_\pm$. Therefore $\bar{m}_{\zeta}$ is the equilibrium state which has the two pure phases $m_\pm$ coexisting to the right and to the left of $\zeta$. It represents a mesoscopic interface located at $\zeta$. We use the word “mesoscopic” because the interface is diffuse and the transition from one phase to the other, even though exponentially fast, is not sharp. In [11] it is proven that the

2000 Mathematics Subject Classification. 82C26, 60H15, 35K57.
one parameter invariant manifold $\mathcal{M} = \{ m_\zeta : \zeta \in \mathbb{R} \}$ is asymptotically stable for the evolution (1.1).

Referring [15] for a review on stochastic interface models, we outline some results on the stochastic perturbation of (1.1). When a random forcing term of intensity $\sqrt{\varepsilon}$ is added to (1.1) and the initial datum is $m_0$, in [7, 8, 14] it is shown that the solution at times $\varepsilon^{-1} t$ stays close to $m_{\zeta(t)}$ for some $\zeta(t)$ which converges to a Brownian motion as $\varepsilon \to 0$. To explain heuristically this result, let us regard the random forcing term as a source of independent small kicks, which we decompose along the directions parallel and orthogonal to $\mathcal{M}$. The orthogonal component is exponentially damped by the deterministic drift, while the parallel component, associated to the zero eigenvalue of the linearization of (1.1) around $m_\zeta$, is not contrasted and, by independence, sums up to a Brownian motion.

We next discuss the behavior of Allen-Cahn equation on the bounded interval $[-a, b]$. The case of Neumann boundary conditions is considered in [9, 16], where it is shown that there exists a stationary solution $m_{a,b}^*$, close to $m_{(b-a)/2}$ as $a, b$ diverge. The profiles $\pm m_{a,b}^*$ are saddle points of $\mathcal{F}$, each one having a one dimensional unstable manifold connecting it to the stable points $m_{\pm}$. For $a, b$ large, solutions are first attracted by this manifolds and they then move along it toward one of the stable phases, with a velocity exponentially small in the distance from the endpoints. From the analysis in [9, 16], we have that there exists a constant $c_0 > 0$ (depending on the potential $V$) such that, if we take $a = c_0 \log \varepsilon^{-1}$, $b = \varepsilon^{-\beta}$ for some $\beta > 0$, and the initial condition is close to $m_0$, the following holds. As $\varepsilon \to 0$, the solution of (1.1) at times $\varepsilon^{-1} t$, for $t$ small enough, is close to $m_{\zeta(t)}$, where $\zeta(t)$ solves the equation $\dot{\zeta} = -A \varepsilon^{-1} e^{-(\zeta+a)/c_0} = -A e^{-\zeta/c_0}$ for some $A > 0$. When a random forcing term of order $\sqrt{\varepsilon}$ is added to (1.1), by the analysis in [7], it follows that, by taking $a = c\log \varepsilon^{-1}$ with $c \gg c_0$, and looking at the time scale $\varepsilon^{-1}$, the random fluctuations are dominant so that the limiting motion of the interface is still described by a Brownian. On the other hand, for $c < c_0$ the deterministic drift should become dominant, the minority phase shrinking deterministically up to extinction. In the critical case $c = c_0$, at the initial state of the evolution, we should see the effect both of the drift and of the stochastic fluctuations.

In this paper, we consider a stochastic perturbation of (1.1) in a bounded interval with inhomogeneous Dirichlet boundary conditions imposing the two stable phases $m_{\pm}$, and analyze the competition between the stochastic fluctuations and the given boundary conditions on the motion of the interface. Let us first consider the deterministic case, that is, (1.1) in the interval $[-a, b]$ with boundary conditions $m(t, -a) = m_-, m(t, b) = m_+$. The meaning of these conditions is to force the $m_-$ phase, respectively the $m_+$ phase, to the left of $-a$, respectively to the right of $b$. If we think of $m$ as the local magnetization, this choice models the effect of opposite magnetic fields applied at the endpoints. To our knowledge, an analysis along the same lines of [9, 16] has not been carried out in detail. However, in this case, it is straightforward to check that there exists a unique, globally attractive, stationary solution $m_{a,b}^*$ close to $m_{(b-a)/2}$ as $a, b$ diverge. Moreover, as it follows from the analysis of the present paper, there is a slow motion as in the case of Neumann boundary conditions. More precisely, there exists an approximately invariant manifold $\mathcal{M}_{a,b}$ close to $\mathcal{M}$ as $a, b$ diverge. In this limit, the motion near $\mathcal{M}_{a,b}$ can be described in terms of coordinates along and transversal to $\mathcal{M}_{a,b}$. The transversal
component of the flow is exponentially damped uniformly in \( a, b \), while the motion along \( \mathcal{M}_{a,b} \), parametrized by the interface location \( \zeta(t) \), evolves according to
\[
\dot{\zeta} = A \left[ e^{-(\alpha + \beta) / \varepsilon} - e^{-(\beta - \alpha) / \varepsilon} \right]
\]
for \( A \) and \( c_0 \) positive constants. We emphasize that, since the boundary conditions force the presence of an interface, the drift pushes the solution toward \( m^*_{a,b} \), where the two pure phases coexist.

We consider a stochastic perturbation of (1.1), given by a space–time white noise of intensity \( \sqrt{\varepsilon} \). To get a nontrivial scaling limit, and to see the competition between the random fluctuations and the repulsion from one endpoint \((-a)\), we choose \( a = c_0 \log \varepsilon^{-1}, b = \varepsilon^{-\beta} \), for some \( \beta > 0 \), the initial condition close to \( m_0 \), and look at the evolution at times \( \varepsilon^{-1} t \). We prove that, as \( \varepsilon \to 0 \), the solution stays close to \( m_\zeta(t) \), where \( \zeta(t) \) solves the stochastic equation
\[
\dot{\zeta} = A e^{-\zeta / c_0} + \eta,
\]
where \( \eta \) is a white noise. We interpret this result as a “soft wall”, since the repulsion is not sharp. Actually, the solution remains close to \( \mathcal{M}_{a,b} \) also on a slightly longer time scale and performing a further diffusive rescaling of the interface location, we also prove that the soft wall converges to a “hard” one: the interface dynamics behaves as a reflected Brownian motion.

2. Notation and results

Let \( a, b \in \mathbb{R}_+ \), \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \) be a standard filtered probability space, and \( W = \{W(t), t \in \mathbb{R}_+\} \) be the cylindrical Wiener process on \( L_2([-a, b], dx) \). This means that \( W \) is the \( \mathcal{F}_t \)-adapted mean zero Gaussian process such that, for each \( \varphi, \varphi' \in C^\infty([-a, b]) \) and \( t, t' \in \mathbb{R}_+ \),
\[
\mathbb{E} \left( \langle W(t), \varphi \rangle \langle W(t'), \varphi' \rangle \right) = t \wedge t' \langle \varphi, \varphi' \rangle,
\]
where \( \mathbb{E} \) denotes the expectation w.r.t. \( \mathbb{P} \), \( t \wedge t' := \min\{t, t'\} \), and \( \langle \cdot, \cdot \rangle \) is the inner product in \( L_2([-a, b], dx) \).

In this paper we consider the prototypical case of the symmetric double well potential, i.e. we choose
\[
V(m) = \frac{1}{4} (m^2 - 1)^2,
\]
which attains its minimum at \( m = \pm 1 \). Given \( \varepsilon > 0 \), we consider a stochastic perturbation of the one dimensional reaction diffusion equation (1.1) with inhomogeneous Dirichlet boundary conditions at the endpoints. More precisely, we let \( m(t) = m(t, x), (t, x) \in \mathbb{R}_+ \times [-a, b] \) be the solution to
\[
\begin{cases}
\frac{dm(t)}{dt} = \left[ \frac{1}{2} \Delta m(t) - V'(m(t)) \right] dt + \sqrt{\varepsilon} \, dW(t), \\
m(t, -a) = -1, \\
m(t, b) = 1, \\
m(0, x) = m_0(x).
\end{cases}
\]
(2.3)

To give a precise meaning to the above equation for \( m_0 \in C([-a, b]) \) such that \( m_0(-a) = -1 \) and \( m_0(b) = 1 \) let \( \nu(x) = \frac{2a}{a+b} + \frac{a-b}{a+b} \) be the solution of \( \nu''(x) = 0 \), \( x \in (-a, b) \) with the above boundary conditions and denote by \( \mathcal{P}_t \) the heat semigroup on \((-a, b)\) with zero boundary conditions at the endpoints. Then a mild solution of (2.3) is defined as the solution of the integral equation
\[
m(t) = \nu + \mathcal{P}_t m_0 - \nu - \int_0^t ds \mathcal{P}_{t-s} V'(m(s)) + \sqrt{\varepsilon} \int_0^t \mathcal{P}_{t-s} dW(s).
\]
(2.4)
By e.g. [12], there exists a unique \( \mathcal{F}_t \)-adapted process \( m \in C(\mathbb{R}_+; C([-a, b])) \) which solves (2.4).

As explained in the Introduction, let \( m_\zeta(x) \) be the standing wave with “center” \( \zeta \in \mathbb{R} \), i.e. the solution to (1.3). For the specific choice potential (2.2) of the potential we have \( m_\zeta(x) = \text{th}(x - \zeta) \). Note that, if \( a = b = \infty \) and \( \varepsilon = 0 \), then \( \mathcal{M} = \{ m_\zeta, \zeta \in \mathbb{R} \} \) is a one parameter family of stationary solutions of (2.3). Given \( p \in [1, \infty] \) we denote by \( \| \cdot \|_p \) the norm in \( L_p([-a, b], dx) \). We consider \( C(\mathbb{R}_+) \) equipped with the (metrizable) topology of uniform convergence in compacts. Our main results are stated as follows.

**Theorem 2.1.** Given \( \beta > 0 \), set

\[
a := \frac{1}{4} \log \varepsilon^{-1}, \quad b := \varepsilon^{-\beta}, \quad \lambda := \log \varepsilon^{-1},
\]

and denote by \( m^{(\varepsilon)}(t) \) the solution to (2.4) with initial datum \( m_0^{(\varepsilon)} \in C([-a, b]) \), \( m_0^{(\varepsilon)}(-a) = -1, m_0^{(\varepsilon)}(b) = 1 \), such that for each \( \eta > 0 \) we have

\[
\lim_{\varepsilon \to 0} \varepsilon^{-\frac{1}{4} + \eta} \| m_0^{(\varepsilon)} - m_0 \|_{\infty} = 0.
\]

Then:

(i) there exists a \( \mathcal{F}_t \)-adapted real process \( X_\varepsilon \) such that, for each \( \theta, \eta > 0 \),

\[
\lim_{\varepsilon \to 0} \mathbb{P} \left( \sup_{t \in [0, \lambda \varepsilon^{-1} \theta]} \| m^{(\varepsilon)}(t) - m_{X_\varepsilon}(t) \|_{\infty} > \varepsilon^{\frac{1}{4} - \eta} \right) = 0;
\]

(ii) the real process \( Y_\varepsilon(\tau) := X_\varepsilon(\varepsilon^{-1} \tau), \tau \in \mathbb{R}_+ \), converges weakly in \( C(\mathbb{R}_+) \) to the unique strong solution \( Y \) to the stochastic equation

\[
\begin{cases}
    dY(\tau) = 12 \exp\{-4Y(\tau)\} d\tau + dB(\tau), \\
    Y(0) = 0,
\end{cases}
\]

where \( B \) is a Brownian motion with diffusion coefficient \( \frac{3}{4} \); (iii) the real process \( Z_\varepsilon(\theta) := \lambda^{-1/2}X_\varepsilon(\lambda \varepsilon^{-1} \theta), \theta \in \mathbb{R}_+ \), converges weakly in \( C(\mathbb{R}_+) \) to a Brownian motion with diffusion coefficient \( \frac{3}{4} \) reflected at zero.

Item (i) states that, up to times \( \varepsilon^{-1} \log \varepsilon^{-1} \), the solution of (2.3) with initial condition close to the one-dimensional manifold \( \{ m_\zeta; \zeta \in (-a, b) \} \) remains close to that manifold. Items (ii) and (iii) then identify the limiting evolution of the interface \( X_\varepsilon(t) \). On the time scales \( \varepsilon^{-1} \) the interface is at distance \( \frac{1}{4} \log \varepsilon^{-1} + Y_\varepsilon \) from the endpoint \(-a\); moreover \( Y_\varepsilon \) behaves as a Brownian motion with a strong drift toward the right for \( Y_\varepsilon < 0 \) and essentially no drift for \( Y_\varepsilon > 0 \). We interpret this as a “soft wall”. On the longer time scale \( \varepsilon^{-1} \log \varepsilon^{-1} \) the interface is at distance \( \frac{1}{4} \log \varepsilon^{-1} + \sqrt{\log \varepsilon^{-1}} Z_\varepsilon \) from the endpoint \(-a\); on this time scale the repulsion is sharp: \( Z_\varepsilon \) behaves as a Brownian motion reflected at zero. We interpret this as a “hard wall”. We finally remark that the choice of \( \lambda \) in (2.5) has been made for the sake of concreteness: it would have been enough to take \( \lambda \) such that \( \lambda \to \infty \) and \( \sqrt{\lambda} / \log \varepsilon^{-1} \to 0 \) as \( \varepsilon \to 0 \).

We emphasize that this nontrivial behavior is due to the choice \( a = \frac{1}{4} \log \varepsilon^{-1} \) for which there is a competition between the stochastic fluctuations and the drift due to the Dirichlet boundary condition at the endpoint \(-a\). Here the coefficient \( \frac{1}{4} \), as well as the diffusion coefficient \( \frac{3}{4} \) of the Brownian motion, depend on the special choice of the double well potential \( V \) in (2.2). Since \( b = \varepsilon^{-\beta} \gg a \) the right
The solution to (2.3) with \( a \) its limit points. Fix \( \mu \) the compactness of \( z \) convergence. Given \( := X = Q \) defined by linear one, which stays close to the quasi-invariant manifold geometrical point of view, we approximate the flow induced by (2.4) with a piecewise solution to the stochastic differential equation \( Y Q \) to of the convergence to the soft wall in Theorem 2.1 is the weak convergence of \( t \). Hence and uniqueness of the strong solution to (2.9). In this setting, the analogous drift term is not globally Lipschitz, a standard coercivity argument shows the existence and uniqueness of the strong solution to the stochastic differential equation

\[
\begin{align*}
\frac{d\Xi(t)}{dt} &= -24 \log\left(4 \Xi(t)\right) + dB(t), \\
\Xi(0) &= z,
\end{align*}
\] (2.9)

where \( B \) is a Brownian motion with diffusion coefficient \( \frac{1}{4} \). Note that, although the drift term is not globally Lipschitz, a standard coercivity argument shows the existence and uniqueness of the strong solution to (2.9). In this setting, the analogous of the convergence to the soft wall in Theorem 2.1 is the weak convergence of \( Q_{m_0} \) to \( Q_{z_0} \); here \( m_0 \) satisfies \( \|m_0 - m_{z_0}\|_\infty \leq \varepsilon^{\frac{1}{4}-\eta} \) for some \( \eta \) small enough. In [2] we also need such convergence to hold uniformly for \( z_0 \) in compacts; this is the content of the following theorem.

**Theorem 2.2.** Let \( \tau_0 > 0 \). There exists \( \eta_1 > 0 \) such that for any \( \eta \in [0, \eta_1] \) the following holds. For each \( L > 0 \) and each uniformly continuous and bounded function \( F : C([0, \tau_0]; X) \to \mathbb{R} \) we have

\[
\lim_{\varepsilon \to 0} \sup_{z \in [-L, L]} \sup_{m_0 \in N_\varepsilon(z)} |Q_{m_0} \varepsilon(F) - Q_z(F)| = 0,
\] (2.10)

where \( N_\varepsilon(z) := \{ m \in X : \|m - m_z\|_\infty \leq \varepsilon^{\frac{1}{4}-\eta}\} \).

**Outline and basic strategy.** The proof of Theorem 2.2 relies on an iterative scheme, in which we linearize (2.4) around \( m_\zeta \) for a suitable \( \zeta \) recursively defined. From a geometrical point of view, we approximate the flow induced by (2.4) with a piecewise linear one, which stays close to the quasi-invariant manifold \( \mathcal{M}_{a,b} \), and allows to compute the motion along the manifold itself. More precisely, following [3, 7, 8], we split the time axis into intervals of length \( T \), taking \( T \) diverging as \( \varepsilon \to 0 \), yet very small as compared to the macroscopic time \( \varepsilon^{-1} \). For the piecewise linear flow, we compute the displacement of the center, effectively tracking the motion along the quasi-invariant manifold. To this end, sharp estimates on the linear flow are needed. We emphasize that, even if the linearization of (1.1) on the whole line...
around the standing wave $\overline{m}_\zeta$ is very well understood [11], for our purposes the finite size corrections are crucial, the nonlinear drift in (2.8) being indeed due to them. Moreover, to control the difference between the true flow and the piecewise linear one, we need a priori bounds which allow us to neglect the nonlinear terms. Finally, the convergence to the hard wall stated in item (iii) is proven by showing that the interface motion is accurately described by (2.8) also on the time scale $\lambda \varepsilon^{-1}$. The proof then follows by showing that the diffusive scaling of the latter converges to a reflected Brownian motion. The proof of Theorem 2.2 requires only minor modifications and it is sketched in Appendix A.

3. The iterative scheme

The notion of “center” of a function plays an important role in our analysis. Following [7, 8], given a function $f \in C([-a, b])$ we define its center $\zeta$ as a point in $(-a, b)$ such that

$$\langle f - \overline{m}_\zeta, \overline{m}_\zeta \rangle = \int_{-a}^{b} dx [f(x) - \overline{m}_\zeta(x)] \overline{m}_\zeta(x) = 0. \quad (3.1)$$

Referring to [8] for an interpretation of the above definition in terms of the dynamics given by the linearization of (2.8) around $\overline{m}_\zeta$, here we simply note that $\zeta$ minimizes the $L_2$ norm of $f - \overline{m}_z$ as a function of $z$.

Given $\delta, \ell > 0$ we define

$$\Upsilon(\delta, \ell) := \left\{ f \in C([-a, b]) : \|f - \overline{m}_z\|_\infty < \delta \text{ for some } z \in (-a + \ell, b - \ell) \right\}. \quad (3.2)$$

Existence and uniqueness of the center holds for functions in $\Upsilon(\delta, \ell)$ for $\varepsilon, \delta$ small enough and $\ell$ large enough, as precisely stated in the next proposition. Recall that we have chosen $a = \frac{1}{2} \log \varepsilon^{-1}$, $b = \varepsilon^{-\beta}$. The result is analogous to [7, Prop. 3.2] where the whole line is considered, and the proof follows by standard implicit function arguments [7, 8].

**Proposition 3.1.** There are reals $\delta_0, \ell_0 > 0$ such that, for any $\varepsilon$ small enough, if $f \in \Upsilon(\delta_0, \ell_0)$ then $f$ has a unique center $\zeta \in (-a, b)$. Moreover there is a constant $C_0 > 0$ so that if $z \in (-a + \ell_0, b - \ell_0)$ is such that $\|f - \overline{m}_z\|_\infty < \delta_0$ we have

$$|\zeta - z| \leq C_0 \|f - \overline{m}_z\|_\infty$$

and

$$\zeta = z - \frac{3}{4} \langle \overline{m}_z, f - \overline{m}_z \rangle - \frac{9}{16} \langle \overline{m}_z, f - \overline{m}_z \rangle \langle \overline{m}_z, f - \overline{m}_z \rangle + R(z, f),$$

$$|R(z, f)| \leq C_0 \left\{ \|f - \overline{m}_z\|^3_\infty + \left( e^{-2(b - z)} + e^{-2(a + z)} \right) \|f - \overline{m}_z\|_\infty \right\}.$$ 

In the sequel, given $f \in \Upsilon(\delta, \ell)$ with $\delta < \delta_0$ and $\ell > \ell_0$, we denote by $X(f)$ the center of $f$, which is well defined for $\varepsilon$ sufficiently small. From now on we drop however the explicit dependence on $\varepsilon$ from the notation. Let $m(t)$ be the solution to (2.4) with $m_0$ satisfying (2.7) and $\alpha \in (0, 1)$; we define the stopping times

$$S_{\delta, \ell} := \inf \left\{ t \in \mathbb{R}_+ : m(t) \notin \Upsilon(\delta, \ell) \right\}, \quad (3.3)$$

$$S_{\delta, \ell, \alpha} := S_{\delta, \ell} \wedge \inf \left\{ t : |X(m(t))| \geq \alpha \right\}. \quad (3.4)$$

We analyze $m(t)$ as long as it stays in $\Upsilon(\delta, \ell)$ and its center is not too far from the origin, namely we stop the evolution at the time $S_{\delta, \ell, \alpha}$ by considering $m(t \wedge S_{\delta, \ell, \alpha})$. We are going to introduce an iterative procedure in which we linearize the equation...
around $\mathcal{m}_c$ for a suitable $x$ recursively defined. To do so, we need a few definitions.

Given $\zeta \in (-a, b)$, let $\varphi_\zeta \in C^2([-a, b])$ be the solution to

$$\begin{cases}
\frac{1}{2} \varphi_\zeta''(x) - V''(\mathcal{m}_\zeta(x))\varphi_\zeta(x) = 0, \\
\varphi_\zeta(-a) = -1 - \mathcal{m}_\zeta(-a), \\
\varphi_\zeta(b) = 1 - \mathcal{m}_\zeta(b).
\end{cases}$$

(3.5)

An explicit computation yields

$$\varphi_\zeta(x) = \mathcal{m}'_\zeta(x)[c_\zeta q_\zeta(x) + d_\zeta], \quad q_\zeta(x) := \frac{h_\zeta(x) - h_\zeta(-a)}{h_\zeta(b) - h_\zeta(-a)},$$

(3.6)

where

$$h_\zeta(x) := \int_x^x \frac{1}{\mathcal{m}'_\zeta(y)} \frac{1}{2} (x - \zeta) + \frac{3}{8} \mathcal{m}_\zeta(x) - \frac{3}{8} \mathcal{m}_\zeta(x) + \frac{1}{4} \mathcal{m}_\zeta(x)^2, \quad c_\zeta := \frac{1}{1 - \mathcal{m}_\zeta(-a)} + \frac{1}{1 + \mathcal{m}_\zeta(b)}, \quad d_\zeta := -\frac{1}{1 - \mathcal{m}_\zeta(-a)}.$$  

(3.7)

We also introduce the operator $H_\zeta$ on $C_0([-a, b])$, the space of continuous functions vanishing at the endpoints, defined on $C^2([-a, b])$, the space of twice differentiable functions compactly supported in $(-a, b)$, by

$$H_\zeta f(x) := -\frac{1}{2} f''(x) + V''(\mathcal{m}_\zeta(x)) f(x)$$

(3.9)

and denote by $g_t^{(\zeta)} := \exp\{-tH_\zeta\}$ the corresponding semigroup.

Let $t_0 \in \mathbb{R}_+$, and $m(t)$, $t \geq t_0$ be the solution to (2.3) with initial condition $m(t_0) = \mathcal{m}_\zeta + \vartheta$, for some $\zeta \in (-a + t_0, b - t_0)$ and $\vartheta \in C([-a, b])$ such that $\|\vartheta\|_\infty < \delta_0(t_0, \vartheta)$ as in Proposition 3.1. By writing $m(t) = \mathcal{m}_\zeta + v(t)$ and expanding $V'(\mathcal{m}_\zeta + v) = V'('0) + V''('0)v + 3\mathcal{m}_c v^2 + v^3$, it is easy to check from (2.3) that $v(t)$ satisfies the integral equation

$$v(t) = \varphi_\zeta + g_{t-t_0}[\vartheta - \varphi_\zeta + \int_0^t ds g_{t-s}^{(\zeta)}[3\mathcal{m}_\zeta v(s)^2 + v(s)^3] + \sqrt{\varepsilon} \int_0^t g_{t-s}^{(\zeta)} dW(s).$$

(3.10)

Let now $m(t)$, $t \geq 0$, be the solution to (2.3) and consider the partition $\mathbb{R}_+ = \bigcup_{n \geq 0} [T_n, T_{n+1})$, where $T_n = nT$, $n \in \mathbb{N}$ and $T = \varepsilon^{-\gamma}$, $\gamma \in (0, \frac{1}{8})$. We next define, by induction on $n \geq 0$, reals $x_n$ and functions $v_n(t) \equiv \{v_n(t, x), x \in [-a, b]\}, t \in [T_n, T_{n+1})$. They will have the property that for any $t \in [T_n, T_{n+1})$

$$m(t \wedge S_{\delta, t, \alpha}) = \mathcal{m}_{x_n} + v_n(t).$$

(3.11)

Set $x_0 := X(m_0)$, i.e. the center of $m_0$, and let $v_0(t), t \in [0, T]$, be the solution to (3.10) with $t_0 = 0$, $\zeta = x_0$, and $\vartheta = m_0 - \mathcal{m}_{x_0}$, stopped at $S_{\delta, t, \alpha}$. Suppose now, by induction, that we have defined $x_{n-1}$ and $v_{n-1}$. We then define $x_n$ as the center of $m(T_n \wedge S_{\delta, t, \alpha}) = \mathcal{m}_x_{n-1} + v_{n-1}(T_n)$ (which exists by the definition of the stopping time $S_{\delta, t, \alpha}$) and $v_n(t), t \in [T_n, T_{n+1})$, as the solution to (3.10) with $t_0 = T_n$, $\zeta = x_n$, and $\vartheta = m(T_n \wedge S_{\delta, t, \alpha}) - \mathcal{m}_x_{n} \wedge v_{n-1}(T_n),$ stopped at $S_{\delta, t, \alpha}$. We emphasize that in this construction the initial condition $v_n(T_n)$ for the evolution in the interval $[T_n, T_{n+1})$ is related to the final condition $v_{n-1}(T_n)$ of the previous interval by

$$v_n(T_n) = -\mathcal{m}_x + \mathcal{m}_{x_n-1} + v_{n-1}(T_n).$$

(3.12)
We consider the operator $H_\zeta$ defined in [3.9] also as an operator on $L_2([-a,b], dx)$ self-adjoint with domain $W^{2,2}([-a,b], dx) \cap W^{1,2}_0([-a,b], dx)$. The bottom of its spectrum is an isolated eigenvalue $\lambda_{0}^{(\zeta)} > 0$ of multiplicity one. The corresponding eigenfunction, that we denote by $\Psi_0^{(\zeta)}$, is chosen positive. We also introduce the spectral gap of $H_\zeta$ which is defined as $\text{gap}(H_\zeta) := \inf \text{spec}(H_\zeta \upharpoonright (\Psi_0^{(\zeta)})^\perp)$, where $H_\zeta \upharpoonright (\Psi_0^{(\zeta)})^\perp$ denotes the restriction of $H_\zeta$ to the subspace orthogonal to $\Psi_0^{(\zeta)}$.

Recalling $g_t^{(\zeta)} = e^{-tH_\zeta}$, we then define
\begin{align*}
g_t^{(\zeta,\perp)} f & := g_t^{(\zeta)} f - e^{-\lambda_{0}^{(\zeta)} t} (\Psi_0^{(\zeta)}, f) \Psi_0^{(\zeta)}, \quad (3.13) \\
G^{(\zeta,\perp)} & := \int_0^\infty dt g_t^{(\zeta,\perp)}. \quad (3.14)
\end{align*}

Note that $G^{(\zeta,\perp)}$ is well defined as $\lambda_{0}^{(\zeta)} > 0$. We denote by $g_t^{(\zeta,\perp)}(x,y)$, $t > 0$, and $G^{(\zeta,\perp)}(x,y)$, $x,y \in [-a,b]$, the corresponding integral kernels. We shall use the same notation for the semigroups acting on $C([-a,b])$.

Let $\hat{\Pi}_\zeta$ be the same operator as in [3.9], but defined on the whole line $\mathbb{R}$, i.e. as an operator on $C_b(\mathbb{R})$, the space of bounded continuous functions, or on $L_2(\mathbb{R}, dx)$. It is well known that $\hat{\Pi}_\zeta$ has a zero eigenvalue with eigenfunction $\hat{m}_\zeta$ and a strictly positive spectral gap $[11]$. This properties play a crucial role in the analysis of the interface fluctuations for a stochastic reaction diffusion equation on the whole line or, in any case, with the interface sufficiently far from the boundary, see [3,6–8,14]. Analogously, we need sharp bounds on the convergence, in a suitable sense, of $H_\zeta$ to $\hat{\Pi}_\zeta$ as $\varepsilon \to 0$, which are stated below and proved in Section 8. Note that, since $H_\zeta$ and $\hat{\Pi}_\zeta$ are defined in different spaces, these bounds do not follow directly from standard perturbation theory. We introduce
\begin{equation}
\phi^{(\zeta)}(x) := \frac{\hat{m}_\zeta(x)}{\|\hat{m}_\zeta\|_2}, \quad x \in [-a,b]. \quad (3.15)
\end{equation}

**Theorem 3.2.** Set $a$ and $b$ as in the statement of Theorem 2.1. Then, for each $\alpha \in (0,1)$ there exist real $\varepsilon_1, \delta_1, C_1 > 0$ such that, for any $\varepsilon \in (0,\varepsilon_1]$, $|\zeta| < \alpha a$, and $f \in C([-a,b])$
\begin{align*}
\|g_t^{(\zeta)} f\|_\infty & \le C_1 \|f\|_\infty \quad \text{for any } t \ge 0, \quad (3.16) \\
gap(H_\zeta) & \ge \delta_1, \quad (3.17) \\
\|g_t^{(\zeta,\perp)} f\|_\infty & \le C_1 e^{-\delta_1 t} \|f\|_2^{2/3} \|f\|_\infty^{1/3} \quad \text{for any } t \ge 1. \quad (3.18)
\end{align*}

Moreover, for each $\eta > 0$,
\begin{align*}
& \lim_{\varepsilon \to 0} \sup_{|\zeta| < \alpha a} \varepsilon^{-\frac{2}{3}(1-\alpha)+\eta} \|\lambda_{0}^{(\zeta)} - 24 \varepsilon e^{-4\zeta}\| = 0, \quad (3.19) \\
& \lim_{\varepsilon \to 0} \sup_{|\zeta| < \alpha a} \varepsilon^{-\frac{1-\alpha}{2}+\eta} \|\Psi_0^{(\zeta)} - \phi^{(\zeta)}\|_\infty = 0, \quad (3.20) \\
& \lim_{\varepsilon \to 0} \sup_{|\zeta| < \alpha a} \varepsilon^{-\frac{1-\alpha}{2}+\eta} \|\Psi_0^{(\zeta)} - \phi^{(\zeta)}\|_1 = 0, \quad (3.21) \\
& \lim_{\varepsilon \to 0} \sup_{|\zeta| < \alpha a} \varepsilon^{-2(1-\alpha)+\eta} \|\Psi_0^{(\zeta)} - \phi^{(\zeta)}\| = 0, \quad (3.22) \\
& \lim_{\varepsilon \to 0} \sup_{|\zeta| < \alpha a} \varepsilon^{-2(1-\alpha)+\eta} \int_a^b dx \hat{m}_\zeta(x) \bar{m}_\zeta(x) G^{(\zeta,\perp)}(x,x) = 0. \quad (3.23)
\end{align*}
4. A priori bounds

The following lemma captures the correct asymptotic behavior of the first terms on the r.h.s. of \((3.10)\). Recall that \(T = \varepsilon^{-\gamma}, \gamma \in (0, \frac{1}{4})\) and that \(\varphi_\zeta\) is defined in \((3.6)\).

**Lemma 4.1.** Let \(\alpha \in (0, \gamma)\). Then for each \(\eta > 0\)

\[
\lim_{\varepsilon \to 0} \sup_{|\zeta| < \alpha \varepsilon} \sup_{t \in [0, T]} \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta} \|\varphi_\zeta - g_t^{(\zeta)} \varphi_\zeta\|_\infty = 0. \tag{4.1}
\]

**Proof.** Recalling \((3.6) - (3.9)\) and \((3.15)\) we write

\[
\varphi_\zeta - g_t^{(\zeta)} \varphi_\zeta = c_\zeta [\overline{m}_\zeta' q_\zeta - g_t^{(\zeta)} (\overline{m}_\zeta' q_\zeta)] + d_\zeta \|\overline{m}_\zeta'\|_2 [\phi_\zeta - g_t^{(\zeta)} \phi_\zeta].
\]

Note that \(\|\overline{m}_\zeta'\|_2 \leq \int_\infty^\infty dx \overline{m}_0(x)^2 = \frac{4}{3}\) and max \(|c_\zeta|, |d_\zeta|\) \(\leq 1\) for \(\varepsilon\) sufficiently small, so from \((3.16)\) we get

\[
\|\varphi_\zeta - g_t^{(\zeta)} \varphi_\zeta\|_\infty \leq (1 + C_1) \|\overline{m}_\zeta' q_\zeta\|_\infty + \sqrt{\frac{4}{3}} \|\phi_\zeta - g_t^{(\zeta)} \phi_\zeta\|_\infty.
\]

By using \(g_t^{(\zeta)} \Psi_0^{(\zeta)} = e^{-\lambda_t^{(\zeta)} t} \Psi_0^{(\zeta)}\) and again \((3.16)\),

\[
\|\phi_\zeta - g_t^{(\zeta)} \phi_\zeta\|_\infty \leq (1 + C_1) \|\Psi_0^{(\zeta)} - \phi_\zeta\|_\infty + (1 - e^{-\lambda_0^{(\zeta)} t}) \|\Psi_0^{(\zeta)}\|_\infty.
\]

By \((3.19)\), for each \(\eta > 0\), we have \(1 - e^{-\lambda_0^{(\zeta)} t} \leq e^{1-\alpha - \eta} t\) for any \(t \in [0, T], |\zeta| < \alpha \varepsilon\), and \(\varepsilon\) small enough. Then, using \((3.20)\),

\[
\lim_{\varepsilon \to 0} \sup_{|\zeta| < \alpha \varepsilon} \sup_{t \in [0, T]} \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta} \|\phi_\zeta - g_t^{(\zeta)} \phi_\zeta\|_\infty = 0. \tag{4.2}
\]

We next note that, by \((3.7)\), there exists \(C_2 > 0\) such that

\[
\sup_{|\zeta| < \alpha \varepsilon} \sup_{x \in [-a,b]} \overline{m}_\zeta'(x)^2 |h_\zeta(x)| \leq C_2,
\]

whence there is \(C_3 > 0\) such that, for \(|\zeta| < \alpha \varepsilon\) and \(\varepsilon\) small enough,

\[
|\overline{m}_\zeta'(x) q_\zeta(x)| \leq C_3 \frac{\overline{m}_\zeta'(x)^{-1} - h_\zeta^{-1}(-a)}{h_\zeta(b) - h_\zeta(-a)} \leq C \varepsilon^{-\frac{\alpha}{2}} \exp \{-2 \varepsilon^{-\beta}\}, \tag{4.4}
\]

where we used that for \(\varepsilon\) small enough and \(|\zeta| < \alpha \varepsilon\), \(\overline{m}_\zeta'(x)^{-1}\) achieves its maximum at \(x = b\). The estimate \((4.1)\) follows. \(\square\)

To simplify the notation let us introduce, for \(n \in \mathbb{N}\) and \(t \in [T_n, T_{n+1}]\),

\[
z_n(t) := \int_{T_n}^t g_t^{(x_n)} dW(s), \tag{4.5}
\]

which is the last term that appears in the integral equation for \(v_n\), see \((3.10)\). Given \(\tau \in \mathbb{R}_+\) we let \(n_\tau(\tau) := [\varepsilon^{-1} \tau/T]\) and \(n_{\varepsilon, \delta, t, a}(\tau) := \lceil (\varepsilon^{-1} \tau \wedge S_{\delta, t, a})/T \rceil\). Given \(\eta > 0, \theta \in \mathbb{R}_+\), we define the event

\[
\mathcal{B}_{\varepsilon, \theta, \eta}^{(1)} := \left\{ \sup_{0 \leq n \leq n_{\varepsilon, \delta, t, a}(\tau)} \sup_{t \in [T_n, T_{n+1}]} \|z_n(t)\|_\infty \leq \varepsilon^{-\eta} \sqrt{T} \right\}. \tag{4.6}
\]

Let also

\[
z_n^+(t) := z_n(t) - (\Psi_0^{(x_n)} , z_n(t)) \Psi_0^{(x_n)} = \int_{T_n}^t g_t^{(x_n, -1)} dW(s), \tag{4.7}
\]

\[
v_n^+(t) := v_n(t) - (\Psi_0^{(x_n)} , v_n(t)) \Psi_0^{(x_n)} \tag{4.8}
\]
be the component of \( z_n(t) \), resp. \( v_n(t) \), orthogonal to \( \Psi_0^{(x_n)} \). We define
\[
B^{(2)}_{\varepsilon, \theta, \eta} := \left\{ \sup_{0 \leq n \leq n_\varepsilon(\lambda \theta)} \sup_{t \in [T_n, T_{n+1}]} \| z_n^1(t) \|_\infty \leq \varepsilon^{-\eta} \right\} 
\] (4.9)
and set \( B_{\varepsilon, \theta, \eta} := B^{(1)}_{\varepsilon, \theta, \eta} \cap B^{(2)}_{\varepsilon, \theta, \eta} \). By standard Gaussian estimates, see [3, Appendix B], we have that for each \( \theta, \eta, \eta > 0 \)
\[
P(B_{\varepsilon, \theta, \eta}) \geq 1 - \varepsilon^q 
\] (4.10)
for any \( \varepsilon \) small enough.

**Theorem 4.2.** Let \( \alpha \in (0, \gamma) \); then there exists \( \eta_0 > 0 \) such that, for any \( \theta \in \mathbb{R}_+ \) and \( \eta \in (0, \eta_0) \), on the event \( B_{\varepsilon, \theta, \eta} \) we have
\[
\sup_{0 \leq n \leq n_\varepsilon(\lambda \theta)} \sup_{t \in [T_n, T_{n+1}]} \| v_n(t) \|_\infty \leq \sqrt{\varepsilon T} \varepsilon^{-2\eta}, 
\] (4.11)
\[
\sup_{0 \leq n < n_\varepsilon(\delta, \ell, \alpha)(\lambda \theta)} \left\{ \| v_n^1(T_{n+1}) \|_\infty + \| v_n(T_n) \|_\infty \right\} \leq \varepsilon^{\frac{1}{2}(1-\alpha)-2\eta}, 
\] (4.12)
\[
\sup_{0 \leq n < n_\varepsilon(\delta, \ell, \alpha)(\lambda \theta)} \| v_n(t) - \sqrt{\varepsilon} z_n(t) \|_\infty \leq \varepsilon^{\frac{1}{2}(1-\alpha)-2\eta}, 
\] (4.13)
\[
\sup_{0 \leq n < n_\varepsilon(\delta, \ell, \alpha)(\lambda \theta)} \left| x_{n+1} - \left( x_0 - \frac{3}{4} \sum_{k=0}^n \langle m_{\varepsilon k}, v_k(T_{k+1}) \rangle \right) \right| \leq \varepsilon^{-\frac{n-n_\varepsilon - \eta + 3\eta}{2}} T^{-\frac{1}{2}}, 
\] (4.14)
for any \( \varepsilon \) small enough.

**Proof.** By the recursive definition of \( v_n(t) \), see in particular (3.12) and (5.11), on the event \( B^{(1)}_{\varepsilon, \theta, \eta} \) for \( t \leq \varepsilon^{-1} \lambda \theta \wedge S_{\delta, \ell, \alpha} \) and \( n = [t/T] \) we have (where we understand \( v_{-1}(0) = m_0 - m_{x_0} \))
\[
\| v_n(t) \|_\infty \leq \| \varphi_{x_n} - g_{t-T_n}^{(x_n)} \varphi_{x_n} \|_\infty + C_1 \| m_{x_{n-1}} - m_{x_n} \|_\infty + C_1 \| v_{n-1}(T_n) \|_\infty \\
+ 3C_1 \int_{T_n}^t ds \| v_n(s) \|_\infty^2 [1 + \| v_n(s) \|_\infty] + \sqrt{\varepsilon T} \varepsilon^{-\eta} \\
\leq 2\sqrt{\varepsilon T} \varepsilon^{-\eta} + C_1(C_0 + 1) \| v_{n-1}(T_n) \|_\infty + 3C_1 \int_{T_n}^t ds \| v_n(s) \|_\infty^2 [1 + \| v_n(s) \|_\infty],
\]
where we used Proposition 3.1 and Lemma 4.1; note \( \alpha \in (0, \gamma) \) implies \( \varepsilon^{-\alpha} < T \). On the other hand, for \( t \in (\varepsilon^{-1} \lambda \theta \wedge S_{\delta, \ell, \alpha}, \varepsilon^{-1} \lambda \theta) \) we clearly have \( v_n(t) = v_j(S_{\delta, \ell, \alpha}/T)(S_{\delta, \ell, \alpha}) \). Recalling (2.6), the proof of (4.11) is now completed by a standard bootstrap argument.

By the recursive definition of \( v_n(t) \), Theorem 3.2 and (4.11), for \( n < n_\varepsilon(\delta, \ell, \alpha)(\lambda \theta) \), on the event \( B_{\varepsilon, \theta, \eta} \) we have
\[
\| v_n^1(T_{n+1}) \|_\infty \leq C \| \varphi_{x_n} - g_T^{(x_n)} \varphi_{x_n} \|_\infty + \varepsilon^{\frac{1}{2}-\eta} \\
+ C \varepsilon^{-\delta_{\ell, \alpha} \sup_{v_n(T_n)} \| v_n(T_n) \|_\infty^{1/3}} \| v_n(T_n) \|_2^{1/2} + 4 \varepsilon^{1-4\eta} T^2. 
\]
Using Lemma 4.1 and again (4.11), we can bound the r.h.s. above by \( \varepsilon^{\frac{1}{2}(1-\alpha)-2\eta} \).

Recalling (3.12) we have
\[
v_{n+1}(T_{n+1}) = -m_{x_{n+1}} + m_{x_{n+1}} + \langle \phi^{(x_n)}, v_n(T_{n+1}) \rangle \phi^{(x_n)} + D_n + \varepsilon^{\frac{1}{2}} \phi^{(x_n)}. 
\]
where
\[
D_n := \langle \Psi_0^{(x_n)}, v_n(T_{n+1}) \rangle \Psi_0^{(x_n)} - \langle \phi^{(x_n)}, v_n(T_{n+1}) \rangle \phi^{(x_n)}. 
\]
From Theorem (3.2) and (4.11) it is straightforward to deduce \( \|D_n\|_\infty \leq \frac{1}{6} \bar{\epsilon} (1-\alpha)^{-2\eta} \). To complete the proof of (4.12) it is then enough to show that
\[
\| - \bar{m}_{x_{n+1}} + \bar{m}_{x_n} + \langle \phi(x_n), v_n(T_{n+1}) \rangle \phi(x_n) \|_\infty \leq \frac{1}{6} \bar{\epsilon} (1-\alpha)^{-2\eta},
\]
which follows, by elementary computations, from Proposition 3.1 and (4.11), using that there exists \( C > 0 \) such that, for any \( \bar{\epsilon} > 0 \),
\[
\sup_{|\xi| \leq \alpha} \left[ \int_{-\infty}^{\infty} dx \bar{m}'_\xi(x)^2 - \| \bar{m}'_\xi \|_2^2 \right] = \sup_{|\xi| \leq \alpha} \left[ \frac{4}{3} - \| \bar{m}'_\xi \|_2^2 \right] \leq C\bar{\epsilon}^{1-\alpha}. \tag{4.15}
\]

The bound (4.13) follows, by (3.10), from (4.11), (4.12), and Lemma 4.1.

To prove (4.14), we first note that, by Proposition 3.1, the recursive definition of the center, and (4.11), for \( n < n_{\bar{\epsilon},\ell,\alpha}(\lambda\theta) \), we have
\[
| x_{n+1} - x_n + \frac{3}{4} \langle \bar{m}'_{x_n}, v_n(T_{n+1}) \rangle + \frac{9}{16} \langle \bar{m}'_{x_n}, v_n(T_{n+1}) \rangle \langle \bar{m}'_{x_n}, v_n(T_{n+1}) \rangle \rangle \leq C_0 \left[ \frac{2}{T} e^{-6\bar{\epsilon} T} + 2e^{\frac{2}{T} (1-\alpha)} \bar{\epsilon}^{2-\eta} \sqrt{T} \right].
\]

On the other hand, by writing \( v_n(T_{n+1}) = \langle \psi_0(x_n), v_n(T_{n+1}) \rangle \psi_0(x_n) + v_n(T_{n+1}) \) and using (3.21), the bound \( | \langle \bar{m}''_{x_n}, \phi(x_n) \rangle | \leq \bar{\epsilon}^{1-\alpha} \) together with (4.11) and (4.12), we get
\[
| \langle \bar{m}'_{x_n}, v_n(T_{n+1}) \rangle \langle \bar{m}'_{x_n}, v_n(T_{n+1}) \rangle | \leq \bar{\epsilon}^{1-\alpha} e^{-4\bar{\epsilon} T}.
\]

Putting together the above estimates we get the bound (4.14). \( \square \)

5. Recursive equation for the center and stability

Let \( x_0 \) be the center of the initial condition \( m_0 \) in (2.3), set \( \xi_0 = x_0 \) and
\[
\xi_{n+1} := x_n - \frac{3}{4} \sum_{k=0}^{n-1} \langle \bar{m}'_{x_k}, v_k(T_{k+1}) \rangle,
\]
\[
\sigma_n := -\frac{3}{4} \sqrt{\varepsilon} \langle \bar{m}'_{x_n}, v_n(T_{n+1}) \rangle = -\frac{3}{4} \sqrt{\varepsilon} \int_{T_n}^{T_{n+1}} \langle \bar{m}'_{x_n}, \phi(x_n) \rangle \langle \bar{m}'_{x_n}, g_{n+1}^{(x_n)} - dW(t) \rangle, \tag{5.1}
\]
\[
F_n := \frac{3}{4} \varepsilon \int_{T_n}^{T_{n+1}} dt \langle \bar{m}'_{x_n}, \bar{m}'_{x_n} z_n(t) \rangle \langle \bar{m}'_{x_n}, \bar{m}'_{x_n} z_n(t) \rangle.
\]

Notice that, by the bound (4.11), \( \xi_{n+1} \) is an approximation to the center \( x_{n+1} \) for \( n < [S_{\delta,\ell,\alpha}/T] \). Moreover, conditionally on the centers \( x_0, x_1, \ldots, x_n \), the random variables \( \sigma_n, \ldots, \sigma_n \) are independent Gaussians with mean zero and variance \( \frac{3}{4} \varepsilon T [1 + o(1)] \). The next theorem identifies a recursive equation satisfied by \( \xi_n \).

**Theorem 5.1.** For each \( n < [S_{\delta,\ell,\alpha}/T] \) we have
\[
\xi_{n+1} - \xi_n = \sigma_n + 12 \varepsilon T e^{-4\varepsilon} + F_n + R_n, \tag{5.2}
\]
where the remainder \( R_n \) can be bounded as follows. There exist \( q, \alpha_0, \eta_0 > 0 \) such that for any \( \alpha \in (0, \alpha_0), \eta \in (0, \eta_0) \), and \( \theta \in \mathbb{R}_+ \) on the event \( B_{\varepsilon,\theta,\eta} \) we have
\[
\sup_{0 \leq n \leq [S_{\delta,\ell,\alpha}/T]} | R_n | \leq \varepsilon \lambda^{-1} T \varepsilon^q \tag{5.3}
\]
for any $\varepsilon$ small enough. Moreover, for each $\theta \in \mathbb{R}_+$ there exists $q > 0$ such that

$$\lim_{\varepsilon \to 0} \mathbb{P}\left( \sup_{0 \leq n < n_\varepsilon(\lambda \theta)} \left| \sum_{k=0}^{n} F_k \right| > \varepsilon^q \right) = 0. \quad (5.4)$$

We remark that, while the remainder $R_n$ is deterministically small on the event $B_{\varepsilon, \theta, n}$, the non-linear term $F_n$ becomes negligible in the limit $\varepsilon \to 0$ only in probability. This is due to a cancellation in which we exploit a martingale structure of $F_n$. In other words $F_n$ gives no contribution to the limit equation not because of its magnitude, which would instead give a finite contribution, but because its expected value vanishes in the limit. The same mechanism, which depends on the symmetry of $V$, was already exploited for the stochastic reaction diffusion equation with the interface far from the boundary $[3, 7, 8]$.

Before proving Theorem 5.1 we state a lemma that identifies the leading corrections in Lemma 4.1 for $t = T$, which will be responsible for the non-linear drift in the limiting equation (2.8).

**Lemma 5.2.** Let $\alpha \in (0, \frac{3}{4})$. Then, for each $\eta > 0$,

$$\lim_{\varepsilon \to 0} \sup_{|\zeta| < \alpha \varepsilon} \varepsilon^{-(1-\alpha)+\eta} \left| \langle \tilde{m}', \varphi_\zeta - g_T^{(0)} \varphi_\zeta \rangle - \frac{4}{3} \varepsilon T e^{-4\varepsilon} \right| = 0. \quad (5.5)$$

**Proof.** Recalling (3.15), we write

$$\langle \tilde{m}', \varphi_\zeta - g_T^{(0)} \varphi_\zeta \rangle = \langle \tilde{m}', \varphi_\zeta \rangle - e^{-\lambda_0^{(0)} T} \langle \psi_0^{(0)}, \varphi_\zeta \rangle \langle \tilde{m}', \psi_0^{(0)} \rangle - \langle \tilde{m}', g_T^{(0),+} \varphi_\zeta \rangle - e^{-\lambda_0^{(0)} T} \langle \varphi_\zeta, \psi_0^{(0)}, \varphi_\zeta \rangle - \langle \tilde{m}', g_T^{(0),-} \varphi_\zeta \rangle - \langle \tilde{m}', g_T^{(0),-} \varphi_\zeta \rangle - \langle \tilde{m}', g_T^{(0),+} \varphi_\zeta \rangle.

The last term above is easily bounded by using (3.6) and (3.18). Again by (3.6) and (3.22) it is easy to show, see Lemma 4.1 for analogous computations, that

$$\lim_{\varepsilon \to 0} \sup_{|\zeta| < \alpha \varepsilon} \varepsilon^{-(1-\alpha)+\eta} \langle \psi_0^{(0)}, \varphi_\zeta \rangle = 0.$$

From (3.15) and since $\sup_{|\zeta| < \alpha \varepsilon} \left| d_\zeta + \frac{2}{3} \right| \leq C \varepsilon^{3(1-\alpha)}$, again by (3.6) we have

$$\lim_{\varepsilon \to 0} \sup_{|\zeta| < \alpha \varepsilon} \varepsilon^{-(1-\alpha)+\eta} \left| \langle \tilde{m}', \varphi_\zeta \rangle + \frac{2}{3} \right| = 0.$$

Finally, by (3.19) and (3.22),

$$\lim_{\varepsilon \to 0} \sup_{|\zeta| < \alpha \varepsilon} \varepsilon^{-(1-\alpha)+\eta} \sup_{|\zeta| < \alpha \varepsilon} \left\{ \left| 1 - \langle \psi_0^{(0)}, \varphi_\zeta \rangle \right| + \left| 1 - e^{-\lambda_0^{(0)} T} - 24 \varepsilon T e^{-4\varepsilon} \right| \right\} = 0,$$

which concludes the proof. \qed
Theorem 3.2 for each

for the terms $R_1$, $R_2$, and (4.12), we have the bound (4.4) and Theorem 3.2, for each $\eta > 0$ we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-\frac{1}{4}(1-\alpha)+\eta} \sup_{t \in [0, T]} \sup_{x \leq \alpha \varepsilon} \left\| m'_{\varsigma} - g^{(\xi)}_{\varsigma} m'_{\varsigma} \right\|_1 = 0,$$

so that, by Theorem 4.2 it is enough to prove (4.3) for

$$\tilde{R}_3 := \int_{T_n}^{T_{n+1}} dt \left\langle m'_{x_n}, \bar{m}_{x_n} \left[ v_n(t) - \sqrt{\varepsilon} z_n(t) \right] \left[ v_n(t) + \sqrt{\varepsilon} z_n(t) \right] \right\rangle.$$

We decompose $[T_n, T_{n+1}] = [T_n, T_n + \log^2 T] \cup [T_n + \log^2 T, T_{n+1}]$ and estimate separately the two time integrals. For the first one it is enough to notice that, by (4.11) and (4.13), we have

$$\int_{T_n}^{T_n+\log^2 T} dt \left\langle m'_{x_n}, \bar{m}_{x_n} \left[ v_n(t) - \sqrt{\varepsilon} z_n(t) \right] \left[ v_n(t) + \sqrt{\varepsilon} z_n(t) \right] \right\rangle \leq \varepsilon^{1-\frac{1}{4} \alpha - 4\eta} \sqrt{T} \log^2 T.$$

To bound the second integral we write, from the integral equation for $v_n$, see (3.10) and the iterative definition of $v_n$,

$$v_n(t) - \sqrt{\varepsilon} z_n(t) = \varphi_{x_n} - e^{-\lambda_0^{(x_n)}(t-T_n)} \left\langle \varphi_{x_n}, \varphi_{x_n}^{(x_n)} \right\rangle \varphi_{x_n}^{(x_n)} - g_{t-T_n}^{(x_n-1)} \varphi_{x_n}$$

$$+ e^{-\lambda_0^{(x_n)}(t-T_n)} \left\langle \varphi_{x_n}, \Psi_{0}^{(x_n)} \right\rangle \varphi_{x_n}^{(x_n)} + g_{t-T_n}^{(x_n-1)} v_n(T_n) + D_n(t),$$

where, by Theorem 4.2, sup $\sup_{t \in [T_n, T_{n+1}]} \| D_n(t) \|_\infty \leq 4 T^2 \varepsilon^{1-4\eta}$. By the explicit expression (3.10), the bound (4.14) and Theorem 5.2, for each $\eta > 0$ we have

$$\left\langle m'_{x_n}, \varphi_{x_n} - e^{-\lambda_0^{(x_n)}(t-T_n)} \left\langle \varphi_{x_n}, \varphi_{x_n}^{(x_n)} \right\rangle \varphi_{x_n}^{(x_n)} \right\rangle \leq \varepsilon^{1-\alpha-\eta T}.$$
Putting all the above bounds together and using Theorem 4.2 to bound \(|v_n(t) + \sqrt{\varepsilon}z_n(t)||_{\infty}\), we finally get

\[
\left| \int_{T_n + \log^2 T}^{T_{n+1}} dt \langle \mathcal{m}'_{x_n}, \mathcal{m}_{x_n} \rangle \left[ v_n(t) - \sqrt{\varepsilon}z_n(t) \right] \left[ v_n(t) + \sqrt{\varepsilon}z_n(t) \right] \right| \leq \varepsilon^{\frac{1}{2} - \alpha - 6\eta} T^2,
\]

which concludes the proof of (5.3).

We next prove (5.4). By the Doob decomposition,

\[
\sum_{k=0}^{n-1} F_k = M_n + \sum_{k=0}^{n-1} \gamma_k,
\]

where

\[
\gamma_k := \mathbb{E}(F_k | \mathcal{F}_{T_k})
\]

and \(M_n\) is an \(\mathcal{F}_{T_n}\)-martingale with bracket

\[
\langle M \rangle_n = \sum_{k=0}^{n-1} \left\{ \mathbb{E}(F_k^2 | \mathcal{F}_{T_k}) - \gamma_k^2 \right\}.
\]

Since for \((t, x) \in [T_k, T_{k+1}] \times [-a, b]\)

\[
\mathbb{E}(z_k(t, x)^2 | \mathcal{F}_{T_k}) = \int_{T_k}^{T} ds \, g_{2(t-s)}(x, x),
\]

we have

\[
\gamma_k = -\frac{9}{4} \varepsilon \int_{T_k}^{T_{k+1}} dt \int_{T_k}^{t} ds \int_{-a}^{b} dx \, \mathcal{m}'_{x_k}(x) \mathcal{m}_{x_k}(x) \, g_{2(t-s)}(x, x)
\]

\[
= -\frac{9}{4} \varepsilon \int_{0}^{T} dt \, (T - t) \int_{-a}^{b} dx \, \mathcal{m}'_{x_k}(x) \mathcal{m}_{x_k}(x) \, g_{2t}(x, x) + r_k,
\]

where

\[
r_k := -\frac{9}{4} \varepsilon \int_{0}^{T} dt \, (T - t) \int_{-a}^{b} dx \, \mathcal{m}'_{x_k}(x) \mathcal{m}_{x_k}(x) \exp \left\{ -2\lambda_0(x, t) \right\} \Psi_0(x) \, \|x\|^2.
\]

Since \(\langle \langle \mathcal{m}'_{x_k}, \mathcal{m}_{x_k}(\mathcal{m}'_{x_k}) \rangle \rangle \leq \varepsilon^{\frac{3}{2}(1-\alpha)}\), by (3.19) and (3.21) we have that \(|r_k| \leq \varepsilon^{-\frac{3}{2}(1-\alpha)} T^2\).

Recall that \(G^{(\zeta, \perp)}\) has been defined in (3.4). We claim that

\[
\sup_{|\zeta| < \alpha} \sup_{x \in [-a, b]} \left| \int_{0}^{T} dt \, \frac{T - t}{T} \, g_{2t}^{(\zeta, \perp)}(x, x) - \frac{1}{2} G^{(\zeta, \perp)}(x, x) \right| \leq \frac{C}{T}.
\]

To prove it, we write

\[
g_{2t}^{(\zeta, \perp)}(x, x) = \sum_{i=1}^{\infty} \exp \left\{ -2t \lambda_i^{(\zeta)} \right\} \Psi_i^{(\zeta)}(x)^2,
\]

where \(\lambda_i^{(\zeta)}\), resp. \(\Psi_i^{(\zeta)}\), \(i \geq 0\), are the eigenvalues, resp. the eigenfunctions, of \(H_{\zeta}\).

A straightforward computation yields

\[
\frac{1}{T} \int_{0}^{T} dt \, (T - t) \, g_{2t}^{(\zeta, \perp)}(x, x) = \sum_{i=1}^{\infty} \frac{\Psi_i^{(\zeta)}(x)^2}{2\lambda_i^{(\zeta)}} \left[ 1 - \frac{1 - \exp \left\{ -2\lambda_i^{(\zeta)} T \right\}}{2\lambda_i^{(\zeta)} T} \right].
\]
As $G^{(\zeta,-)}(x, x) = \sum_{n=1}^{\infty} \Psi_1^{(\zeta)}(x)^2 / \lambda_1^{(\zeta)}$, the bound follows from Remark 1 at the end of Section 8. By (3.23) and the previous bounds we finally get that there exists $q > 0$ such that

$$\sum_{k=0}^{n_e(\lambda\theta)} |\gamma_k| \leq n_\varepsilon(\lambda\theta) \sup_{0 \leq n \leq n_\varepsilon(\lambda\theta)} |\gamma_n| \leq \varepsilon^q.$$  

We are left with the bound of the martingale part $M_n$. Given $q > 0$, by Doob’s inequality, recalling (5.5),

$$\mathbb{P}\left( \sup_{0 \leq n \leq n_e(\lambda\theta)} |M_n| \geq \varepsilon^q \right) \leq \varepsilon^{-2q} \mathbb{E}\left( (M)_{n_e(\lambda\theta)} \right) \leq \varepsilon^{-2q} \sum_{k=0}^{n_e(\lambda\theta)} \mathbb{E}\left[ \mathbb{E}(F_k^2 | F_{T_k}) \right] \leq C \varepsilon^{-2q} [n_\varepsilon(\lambda\theta) + 1] \varepsilon T^4,$$

(5.10)

where we used that there exists $C > 0$ such that, for any $\varepsilon > 0$ and $k \leq n_\varepsilon(\lambda\theta)$, we have

$$\sqrt{\mathbb{E}(F_k^2 | F_{T_k})} \leq C \varepsilon \int_{T_k}^{T_{k+1}} dt \int_{-a}^{b} dx \mathcal{M}^2_k(x) \sqrt{\mathbb{E}(z_k(t, x)^4 | F_{T_k})} \leq C \varepsilon T^2,$$

which concludes the proof. □

In the following lemma we prove that $\xi_n$ is bounded with probability close to one. In proving the convergence to the soft wall we need such control for $n \leq (\varepsilon T)^{-1}$, while for the convergence to the hard wall we need that $\xi_n$ grows at most as $\sqrt{\lambda}$ for $n \leq \lambda(\varepsilon T)^{-1}$.

**Lemma 5.3.** For each $\theta \in \mathbb{R}_+$ we have

$$\lim_{L \to \infty} \lim_{\varepsilon \to 0} \mathbb{P}\left( \sup_{0 \leq n \leq n_\varepsilon(\mu\theta)} |\xi_n| \geq L \sqrt{\mu} \right) = 0, \quad \mu = 1, \lambda.$$  

(5.11)

**Proof.** Since for $n \geq [S_{\delta, \varepsilon, \alpha} / T]$, by definition (5.1), $\xi_n = \xi_{[S_{\delta, \varepsilon, \alpha} / T]}$, it is enough to prove the statement for $n < n_{\varepsilon, \delta, \alpha}(\mu\theta)$. Recall (5.2) and let

$$S_n := \sum_{k=0}^{n-1} \sigma_k, \quad A_n := S_n + x_0 + \sum_{k=0}^{n-1} [F_k + R_k].$$

(5.12)

By (2.6) and Proposition 3.1 for each $\eta > 0$ we have that, for any $\varepsilon$ small enough,

$$|x_0| \leq \varepsilon^{\frac{1}{2} - \eta}.$$  

(5.13)

Recalling definition (5.1), it is easy to show that there exists a real $C > 0$ such that, for any $\varepsilon > 0$,

$$\mathbb{E}(\sigma_k | F_{T_k}) = 0, \quad \mathbb{E}(\sigma_k^2 | F_{T_k}) \leq C \varepsilon T.$$  

(5.14)

Given $\theta \in \mathbb{R}_+$, an application of Doob inequality then yields

$$\lim_{L \to \infty} \lim_{\varepsilon \to 0} \mathbb{P}\left( \sup_{0 \leq n \leq n_\varepsilon(\mu\theta)} |S_n| \geq L \sqrt{\mu} \right) = 0.$$  

(5.15)

By Theorem 5.1 (4.10), and (5.15) we have

$$\xi_n = \sum_{k=0}^{n-1} 12 \varepsilon T e^{-4\xi_k} + A_n.$$  

(5.16)
Indeed, recalling (3.3) and (5.1), by (4.11), (4.14), (4.10), and Lemma 5.3 it follows that $\sup_{0 \leq n \leq n_{\epsilon, \delta, \ell, \alpha} (\mu \theta)} |A_n| \leq L_1 \sqrt{\mu}$. We now prove item (i) of Theorem 2.1, item (i). Let us first prove that for each $\nu > 0$ there exists a $\epsilon > 0$ such that

$$
\lim_{\epsilon \to 0} \mathbb{P} \left( \sup_{t \leq \epsilon^{-1} \theta} \left| X_{\epsilon}(t) - \mathbb{E} X_{\epsilon}(t) \right| > \nu \right) = 0.
$$

We now prove item (ii) of Theorem 2.1, item (ii). To prove item (ii) of Theorem 2.1 we shall identify the limiting equation satisfied by $\xi$. To this end we need a few lemmata. Recalling the definition (5.12) of $S_\nu$, we denote by $S_\bar{\nu}(\tau)$ the continuous process defined, as in (6.1), by the linear interpolation of $S_\nu$.

6. Convergence to the soft wall

Recalling that $n_\epsilon (\tau) = [\epsilon^{-1} \tau / T]$, $T = \epsilon^{-\gamma}$, and that $\xi_\nu$ has been defined in (5.11), we define the continuous process $\xi_\nu (\tau)$, $\tau \in \mathbb{R}_+$, as the piecewise linear interpolation of $\xi_\nu$ namely, we set

$$
\xi_\nu(\tau) := \xi_\nu(\tau) + \left[ \tau - \epsilon T \nu(\tau) \right] \left[ \xi_\nu(\tau) + 1 - \xi_\nu(\tau) - 1 \right].
$$

By (5.18), (4.14), and (4.11) we have that for each $\theta \in \mathbb{R}_+$ there exists a $q > 0$ such that

$$
\lim_{\epsilon \to 0} \mathbb{P} \left( \sup_{\tau \in [0, \lambda \theta]} \left| X_{\epsilon}(\epsilon^{-1} \tau) - \xi_\nu(\tau) \right| > \epsilon^q \right) = 0.
$$

To prove item (ii) of Theorem 2.1, item (ii) we shall identify the limiting equation satisfied by $\xi$. To this end we need a few lemmata. Recalling the definition (5.12) of $S_\nu$, we denote by $S_\bar{\nu}(\tau)$ the continuous process defined, as in (6.1), by the linear interpolation of $S_\nu$. 

\[ \lim_{L \to \infty} \lim_{\epsilon \to 0} \mathbb{P} \left( \sup_{0 \leq n \leq n_{\epsilon, \delta, \ell, \alpha} (\mu \theta)} |A_n| > L \sqrt{\mu} \right) = 0. \]
The first lemma relies on standard martingale arguments to show the weak convergence of \( S_n \) to a Brownian motion. For completeness we however present also its proof.

**Lemma 6.1.** As \( \varepsilon \to 0 \), the process \( \{ S_\varepsilon \} \) converges weakly in \( C(\mathbb{R}_+) \) to a Brownian motion with diffusion coefficient \( \frac{3}{4} \).

**Proof.** Recalling (5.14), an application of Doob inequality then yields, for any \( \tau \in \mathbb{R}_+, \eta > 0 \)

\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \mathbb{P} \left( \sup_{\tau_1, \tau_2 \in [0, \tau]} |S_\varepsilon(\tau_2) - S_\varepsilon(\tau_1)| > \eta \right) = 0. \tag{6.3}
\]

Since \( S_\varepsilon(0) = 0 \), by [4, Thm. 8.2], \( \{ S_\varepsilon \} \) is tight.

Let \( S \) be a weak limit of \( S_\varepsilon \), we shall prove that \( S(\tau) \) and \( S(\tau)^2 - \frac{3}{4} \tau \) are martingales. By Levy’s characterization theorem we then get the result. By (5.14) we have that, for each \( \tau \in \mathbb{R}_+ \), \( \mathbb{E}(S_\varepsilon(\tau)^2) \) is bounded uniformly as \( \varepsilon \to 0 \). Let \( 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq \tau_1 < \tau_2 \), \( F \) be a bounded continuous function on \( \mathbb{R}^n \), and consider a subsequence, still denoted by \( \varepsilon \), converging to zero such that \( S_\varepsilon \xrightarrow{\text{w}} S \). We then have, by the boundedness of \( F \) and the uniform integrability of \( S_\varepsilon(\tau) \),

\[
\mathbb{E}\left( (S(\tau_2) - S(\tau_1)) F(S(s_1), \cdots, S(s_n)) \right) = \lim_{\varepsilon \to 0} \mathbb{E}\left( (S_\varepsilon(\tau_2) - S_\varepsilon(\tau_1)) F(S_\varepsilon(s_1), \cdots, S_\varepsilon(s_n)) \right) = 0,
\]

where we used

\[
S_\varepsilon(\tau_2) - S_\varepsilon(\tau_1) = \sum_{k=n_\varepsilon(\tau_1)}^{n_\varepsilon(\tau_2) - 1} \sigma_k + (\tau_2 - \varepsilon T n_\varepsilon(\tau_2)) \sigma_{n_\varepsilon(\tau_2)} - (\tau_1 - \varepsilon T n_\varepsilon(\tau_1)) \sigma_{n_\varepsilon(\tau_1)},
\]

so that \( \mathbb{E}(S_\varepsilon(\tau_2) - S_\varepsilon(\tau_1) | \mathcal{F}_{T_{n_\varepsilon(\tau_1)}}) = 0 \). As \( F, \tau_1 \) and \( \tau_2 \) were arbitrary, we get that \( S(\tau) \) is a martingale.

To show the second martingale relationship we first prove the uniform integrability of \( S_\varepsilon(\tau)^2 \). It is enough to show that, for each \( \tau \in \mathbb{R}_+ \),

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sum_{k=0}^{n_\varepsilon(\tau)} \sigma_k^4 \right) < \infty,
\]

which is proven as follows. By (5.12), \( S_n \) is a \( \mathcal{F}_{T_n} \)-martingale with quadratic variation \( [S]_n = \sum_{k=0}^{n-1} \sigma_k^2 \). By the BDG inequality, see e.g. [18, VII, §3], (5.14), and the uniform bound \( \mathbb{E}(\sigma_k^4 | \mathcal{F}_{T_k}) \leq C(\varepsilon T)^2 \) for some \( C > 0 \), which follows by a Gaussian computation, we get the above bound.

By (4.12) we have that, for each \( \tau \in \mathbb{R}_+ \),

\[
\lim_{\varepsilon \to 0} \sup_{0 \leq n \leq n_\varepsilon(\tau)} \frac{1}{\varepsilon T} \left| \mathbb{E}(\sigma_n^2 | \mathcal{F}_{T_n}) - \frac{3}{4} \varepsilon T \right| = 0,
\]

which implies

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sum_{k=n_\varepsilon(\tau_1)}^{n_\varepsilon(\tau_2) - 1} \sigma_k \right)^2 - \frac{3}{4} (\tau_2 - \tau_1) | \mathcal{F}_{T_{n_\varepsilon(\tau_1)}} = 0.
\]

Thanks to the uniform integrability of \( S_\varepsilon(\tau)^2 \), we conclude that \( S(\tau)^2 - \frac{3}{4} \tau \) is a martingale by the same argument used to show that \( S(\tau) \) is a martingale. \( \square \)
Lemma 6.2. For each sequence \( \varepsilon \to 0 \) the process \( \xi_\varepsilon \) is tight in \( C(\mathbb{R}_+) \).

Proof. From (2.6) and Proposition 2.1, \( \xi_\varepsilon(0) \to 0 \), so by [4, Thm. 8.2], it is enough to show that for each \( \tau \in \mathbb{R}_+ \), \( \eta > 0 \) we have
\[
\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \mathbb{P}\left( \sup_{\tau_1, \tau_2 \in [0, \tau]} |\xi_\varepsilon(\tau_2) - \xi_\varepsilon(\tau_1)| > \eta \right) = 0. \tag{6.4}
\]

By (6.1) and (5.11), to prove (6.4) it is enough to show that, for each \( \tau \),
\[
\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \mathbb{P}\left( \sup_{\tau_1, \tau_2 \in [0, \tau]} |\xi_{n_\varepsilon(\tau_1)} - \xi_{n_\varepsilon(\tau_2)}| > \eta, \sup_{0 \leq n \leq n_\varepsilon(\tau)} |\xi_n| \leq L \right) = 0. \tag{6.5}
\]

By Theorem 5.1, (4.10), and (5.18), for \( \tau_1 < \tau_2 \),
\[
\xi_{n_\varepsilon(\tau_2)} - \xi_{n_\varepsilon(\tau_1)} = \sum_{k=n_\varepsilon(\tau_1)+1}^{n_\varepsilon(\tau_2)-1} (12 \varepsilon T e^{-4k\varepsilon} + \sigma_k) + R_\varepsilon(\tau_1, \tau_2),
\]
where for each \( \tau \in \mathbb{R}_+ \) there exists \( q > 0 \) so that
\[
\lim_{\varepsilon \to 0} \mathbb{P}\left( \sup_{\tau_1, \tau_2 \in [0, \tau]} |R_\varepsilon(\tau_1, \tau_2)| > \varepsilon^q \right) = 0.
\]

By (6.3) it is now straightforward to conclude the proof of (6.5). \( \square \)

Lemma 6.3. For each \( \delta > 0, \theta \in \mathbb{R}_+ \)
\[
\lim_{\varepsilon \to 0} \mathbb{P}\left( \sup_{s \in [0, \theta]} |\xi_\varepsilon(s) - S_\varepsilon(s) - \int_0^s du 12 \exp\{-4 \xi_\varepsilon(u)\}| > \delta \sqrt{\mu} \right) = 0, \quad \mu = 1, \lambda.
\]

Remark. In this section the above lemma is used for \( \mu = 1 \); we shall use it with \( \mu = \lambda \) in proving the convergence to the hard wall.

Proof of Lemma 6.3 By Lemma 5.3 it is enough to show that, for each \( L > 0 \),
\[
\lim_{\varepsilon \to 0} \mathbb{P}\left( \sup_{s \in [0, \theta]} |\xi_\varepsilon(s) - S_\varepsilon(s) - \int_0^s du 12 \exp\{-4 \xi_\varepsilon(u)\}| > \delta \sqrt{\mu} \right) = 0, \quad \mu = 1, \lambda. \tag{6.6}
\]

Recalling the definition of \( \xi_n \) in (5.1), the bound (4.11) and Proposition 2.1 yields
\[
|\xi_{n+1} - \xi_n| \leq C e^{\frac{1}{2} - \eta} \sqrt{T} \quad \text{for } n \leq n_\varepsilon(\lambda \theta)
\]
on a set of probability converging to 1 as \( \varepsilon \to 0 \) by (4.10). By definition (6.1), for each \( \theta \in \mathbb{R}_+ \), \( \delta > 0 \), and \( L > 0 \) we have
\[
\lim_{\varepsilon \to 0} \mathbb{P}\left( \sup_{s \in [0, \theta]} \left| \sum_{n=0}^{n_\varepsilon(s)} \varepsilon T e^{-4\xi_n} - \int_0^s du e^{-4\xi_\varepsilon(u)} \right| > \delta \sqrt{\mu} \right) = 0, \quad \mu = 1, \lambda. \tag{6.7}
\]
as it can be easily shown by the change of variable \( u = \varepsilon t \) in the integral and using \( |e^{-4\xi_{n+1}} - e^{-4\xi_n}| \leq 4 e^{4 \max\{|\xi_n|, |\xi_{n+1}|\}} |\xi_{n+1} - \xi_n| \). The proof of (6.6) is now completed by using Theorem 5.1, (4.10), and (5.18).

Proof of Theorem 7.1 item (ii). Thanks to (6.2) it is enough to prove the statement for \( \xi_\varepsilon \) in place of \( Y_\varepsilon \). Let us denote by \( P_\varepsilon \), a probability on \( C(\mathbb{R}_+) \times C(\mathbb{R}_+) \), the
law of the process \((S_\varepsilon, \xi_\varepsilon)\). By Lemmata \([6.1] and [6.2]\) there exists a subsequence \(\varepsilon \to 0\) and a probability \(P\) such that \(P_\varepsilon \Rightarrow P\). By \([18, \text{Thm. III.8.1}]\) there exists a probability space \((\Omega^*, \mathcal{F}^*, \mathbb{P}^*)\) and random elements \((X^*_\varepsilon, Y^*_\varepsilon)\), \((X^*, Y^*)\) with values in \(C(\mathbb{R}_+) \times C(\mathbb{R}_+)\) such that the law of \((X^*_\varepsilon, Y^*_\varepsilon)\), resp. \((X^*, Y^*)\), is \(P_\varepsilon\), resp. \(P\), and \((X^*_\varepsilon, Y^*_\varepsilon)\) converges to \((X^*, Y^*)\) \(\mathbb{P}^*\)-a.s. Moreover, again by Lemma \([5.1]\), \(X^*\) is a Brownian motion with diffusion coefficient \(\frac{1}{2}\). Denoting by \((x(\cdot), y(\cdot))\) the canonical coordinates in \(C(\mathbb{R}_+) \times C(\mathbb{R}_+)\), for each \(\delta > 0\) and \(\tau \in \mathbb{R}_+\), we have

\[
P\left( \sup_{s \in [0, \tau]} \left| y(s) - x(s) - \int_0^s du \, 12 \exp\{-4y(s)\} \right| > \delta \right)
\]

\[
= \mathbb{P}^* \left( \sup_{s \in [0, \tau]} \left| Y^*(s) - X^*(s) - \int_0^s du \, 12 \exp\{-4Y^*(s)\} \right| > \delta \right)
\]

\[
= \lim_{\varepsilon \to 0} \mathbb{P}^* \left( \sup_{s \in [0, \tau]} \left| Y^*_\varepsilon(s) - X^*_\varepsilon(s) - \int_0^s du \, 12 \exp\{-4Y^*_\varepsilon(s)\} \right| > \delta \right)
\]

\[
= \lim_{\varepsilon \to 0} \mathbb{P}_\varepsilon \left( \sup_{s \in [0, \tau]} \left| y(s) - x(s) - \int_0^s du \, 12 \exp\{-4y(s)\} \right| > \delta \right)
\]

\[
= \lim_{\varepsilon \to 0} \mathbb{P}_\varepsilon \left( \sup_{s \in [0, \tau]} \left| \xi_\varepsilon(s) - S_\varepsilon(s) - \int_0^s du \, 12 \exp\{-4\xi_\varepsilon(s)\} \right| > \delta \right) = 0.
\]

where we used, in the second step, the \(\mathbb{P}^*\)-a.s convergence of \((X^*_\varepsilon, Y^*_\varepsilon)\) to \((X^*, Y^*)\) and, in the last step, Lemma \([6.3]\) with \(\mu = 1\). As \(\delta\) and \(\tau\) were arbitrary it follows that any limit point solves \((2.8)\). In fact this also prove existence of a weak solution to \((2.8)\). Since the real function \(y \to 12e^{-4y}\) is locally Lipschitz, by \([17, \text{Thm. 5.2.5}]\) there is path-wise uniqueness of \((2.8)\). By \([17, \text{Cor. 5.3.23}]\) it follows there is a strong solution to \((2.8)\) which is unique in the sense of probability law. We then conclude that \(Y_\gamma\) weakly converges to the unique strong solution of \((2.8)\) \(\square\)

### 7. Convergence to the Hard Wall

To prove item \((iii)\) of Theorem \([2.1]\) we first state and prove an analogous result for the diffusive scaling of the stochastic equation \((2.8)\). To simplify the notation we introduce a probabilistic model not related with the one introduced in Section \([2]\) and denote by \(t\) the macroscopic time variable. Let \(B\) be a Brownian motion on some filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) and \(\gamma\) a positive parameter that will eventually diverge. We suppose given a sequence of \(\mathcal{F}_t\)-adapted continuous processes \(B_\gamma\) such that \(B_\gamma(0) = 0\) and satisfying that for each \(T \geq 0\),

\[
P \left( \lim_{\gamma \to \infty} \sup_{t \in [0, T]} |B_\gamma(t) - B(t)| = 0 \right) = 1. \tag{7.1}
\]

We consider the sequence of processes that solve the equation

\[
Y_\gamma(t) = \gamma \int_0^t ds \, [Y_\gamma(s)]_- + B_\gamma(t), \tag{7.2}
\]

where \([Y]_- = \max\{0, -Y\}\) is the negative part of \(Y\). We shall prove that \(Y_\gamma\) converges to a Brownian motion reflected at the origin. The precise statement is the following.

**Theorem 7.1.** Let

\[
Y(t) := B(t) + \sup_{s \in [0, t]} {-B(s)} \tag{7.3}
\]
Then, for any $T \geq 0$,

$$
P \left( \lim_{\gamma \to \infty} \sup_{t \in [0, T]} |Y_\gamma(t) - Y(t)| = 0 \right) = 1.
$$

Note that, by e.g. [17, Thm. 6.17], $Y$ has the law of a Brownian motion reflected at the origin.

**Proof.** Let

$$
r_\gamma(T) := \sup_{t \in [0, T]} \sup_{\gamma' > \gamma} |B_{\gamma'}(t) - B_{\gamma}(t)|
$$

and note that by (7.4), for each $T \in \mathbb{R}_+$ we have $r_\gamma(T) \to 0$ $\mathcal{P}$-a.s. as $\gamma \to \infty$. We claim that for $\gamma_1 < \gamma_2$, $t \in [0, T]$, we have

$$
Y_{\gamma_2}(t) \geq Y_{\gamma_1}(t) - 2r_{\gamma_2}(T). \quad (7.5)
$$

Indeed, if $Y_{\gamma_2}(t) \geq Y_{\gamma_1}(t)$ there is nothing to prove, otherwise let $\tau = \sup\{s \in [0, t] : Y_{\gamma_2}(s) \geq Y_{\gamma_1}(s)\}$ which exists because $Y_{\gamma_1}(0) = Y_{\gamma_2}(0)$.

By definition, $Y_{\gamma_2}(s) \leq Y_{\gamma_1}(s)$ for $s \in [\tau, t]$; by writing the equation (7.2) in this interval and using the monotonicity of $x \mapsto [x]_-$ the bound (7.5) follows easily.

We next claim that

$$
Y_\gamma(t) \leq B_\gamma(t) + \sup_{s \in [0, t]} \{-B_\gamma(s)\} =: w_\gamma(t). \quad (7.6)
$$

This can be proved as follows. We first note that $w_\gamma \geq 0$. Let $t \geq 0$, if $Y_{\gamma}(t) \leq 0$ there is nothing to prove, otherwise, setting $\tau = \sup\{s \in [0, t] : Y_{\gamma}(s) = 0\}$ we have:

$$
Y_{\gamma}(t) = Y_{\gamma}(t) - Y_{\gamma}(\tau) = B_{\gamma}(t) - B_{\gamma}(\tau) + \int_\tau^t ds \gamma [Y_{\gamma}(s)]_- = B_{\gamma}(t) - B_{\gamma}(\tau) \leq w_{\gamma}(t),
$$

where we used that $[Y_{\gamma}(s)]_- = 0$ for $s \in [\tau, t]$.

Let $Z(t) := \lim_{\gamma \to \infty} Y_{\gamma}(t)$. By (7.4) and (7.6) we have $Z(t) \leq Y(t)$. It is easy to show, by (7.5), that $\mathcal{P}$-a.s. $\lim_{\gamma \to \infty} Y_{\gamma}(t) = Z(t)$. To complete the proof of the theorem we shall prove: $Z$ is a.s. continuous, $Z \geq 0$, there exists a continuous increasing process $\ell$ so that $Z = B + \ell$ and $\int_0^\infty \ell(t) Z(t) = 0$. Then from the Skorohod Lemma, see e.g. [17, Lemma 6.14], it follows $Z = Y$.

For $f \in C(\mathbb{R}_+, \delta > 0$, and $T > 0$, we let $\omega_{\delta,T}(f)$ be the modulus of continuity of the function $f$ on $[0, T]$, i.e.

$$
\omega_{\delta,T}(f) := \sup_{s,t \in [0,T] \atop |t-s| < \delta} |f(t) - f(s)|.
$$

We first show the a priori bound:

$$
\inf_{t \in [0, T]} Y_{\gamma}(t) \geq -2 \omega_{\delta,T}(B_{\gamma}) - 4 e^{-\delta \gamma} \sup_{t \in [0, T]} |B_{\gamma}(t)|. \quad (7.7)
$$

Indeed, pick $\tau \in [0, T]$ such that $\inf_{t \in [0, T]} Y_{\gamma}(t) = Y_{\gamma}(\tau)$. If $Y_{\gamma}(\tau) = 0$ there is nothing to prove, otherwise let $\sigma = \sup\{t \in [0, \tau] : Y_{\gamma}(t) = 0\}$. For $t \in [\sigma, \tau]$ we can
integrate the equation \((7.2)\) getting:

\[
Y_\gamma(t) = B_\gamma(t) - B_\gamma(s) - \int_0^\tau ds \gamma e^{-(\tau-s)\gamma} [B_\gamma(s) - B_\gamma(\sigma)]
\]

\[
= e^{-(\tau-s)\gamma} [B_\gamma(t) - B_\gamma(s)] + \int_0^\tau ds \gamma e^{-(\tau-s)\gamma} [B_\gamma(\sigma) - B_\gamma(s)] + \int_{\sigma}^{\gamma(t)} ds \gamma e^{-(\tau-s)\gamma} [B_\gamma(t) - B_\gamma(s)]
\]

\[
\geq -4e^{-\delta\gamma} \sup \limits_{t \in [0,T]} |B_\gamma(t)| - 2\omega_{\delta,T}(B_\gamma).
\]

We next bound the modulus of continuity of \(Y_\gamma\). We claim that

\[
\omega_{\delta,T}(Y_\gamma) \leq 8 \left[ \omega_{\delta,T}(B_\gamma) + e^{-\delta\gamma} \sup \limits_{t \in [0,T]} |B_\gamma(t)| \right]. \tag{7.8}
\]

Let us fix \(t, s \in [0, T]\) with \(|t-s| < \delta\). We consider first the case in which \(Y_\gamma(u) \leq 0\) for any \(u \in [s, t]\). Solving equation \((7.2)\) in this time interval, we get

\[
Y_\gamma(t) - Y_\gamma(s) = (e^{-(t-s)\gamma} - 1)Y_\gamma(s) + B_\gamma(t) - B_\gamma(s) - \int_s^t du \gamma e^{-(t-u)\gamma} [B_\gamma(u) - B_\gamma(s)],
\]

so that, by \((7.2)\),

\[
|Y_\gamma(t) - Y_\gamma(s)| \leq |Y_\gamma(s)| + 2\omega_{\delta,T}(B_\gamma) \leq 4 \left[ \omega_{\delta,T}(B_\gamma) + e^{-\delta\gamma} \sup \limits_{t \in [0,T]} |B_\gamma(t)| \right]. \tag{7.9}
\]

The case in which \(Y_\gamma(u) \geq 0\) for any \(u \in [s, t]\) we clearly have \(|Y_\gamma(t) - Y_\gamma(s)| \leq \omega_{\delta,T}(B_\gamma)\). The other cases can be reduced to the previous ones. We discuss only the case \(Y_\gamma(s) < 0, Y_\gamma(t) < 0\). Let \(\sigma = \inf \{u > s : Y_\gamma(u) = 0\}\) and \(\tau = \sup \{u < t : Y_\gamma(u) = 0\}\). We then write \(Y_\gamma(t) - Y_\gamma(s) = [Y_\gamma(t) - Y_\gamma(\tau)] + [Y_\gamma(\tau) - Y_\gamma(s)]\) and use the bound \((7.9)\) in the intervals \([s, \sigma]\) and \([\tau, t]\) to get \((7.8)\).

By taking the limit as \(\gamma \to \infty\) in \((7.8)\) we get that the limiting process \(Z\) is continuous. Let \(\overline{Y}_\gamma(t) := \inf \{\gamma \geq \gamma \in \mathbb{R} : Y_\gamma(t) \geq 0\}\), so that \(\overline{Y}_\gamma(t) \uparrow Z(t)\). By the continuity of \(Z\), the previous convergence is in fact uniform for \(t\) on compacts. By using \((7.5)\) we get that \(Y_\gamma(t) - \overline{Y}_\gamma(t)\) converges, \(P\)-a.s., to zero uniformly for \(t \in [0, T]\). Hence \(Y_\gamma\) converges to \(Z\) uniformly on compacts.

To show that \(Z \geq 0\) we note that

\[
\int_0^t ds [Y_\gamma(s)]_- = \frac{1}{\gamma} [Y_\gamma(t) - B_\gamma(t)],
\]

which, by taking first the limit \(\gamma \to \infty\) and then \(t \to \infty\), implies \(\int_0^\infty ds [Z(s)]_- = 0\), whence \(Z \geq 0\) by the continuity of \(Z\).

Let us introduce the increasing process

\[
\ell_\gamma(t) := \int_0^t ds \gamma [Y_\gamma(s)]_- = Y_\gamma(t) - B_\gamma(t).
\]

By the convergence of \(Y_\gamma\) to the continuous process \(Z\),

\[
\ell(t) := \lim_{\gamma \to \infty} \ell_\gamma(t) = Z(t) - B(t)
\]
Lemma 7.3. \( \theta \) the macroscopic time variable and recall (6.1), let \( \zeta \). Thanks to (6.2) it is enough to prove the statement.

Proof of Theorem 2.1, item (iii). □

which concludes the proof.

Given \( \gamma > 0 \), let \( X_\gamma \) be the solution of the equation

\[ X_\gamma(t) = \gamma \int_0^t ds \exp\{-4\gamma X_\gamma(s)\} + B_\gamma(t). \]  

(7.10)

Note that if \( Y(\tau) \) solves (2.8) then \( X_\lambda(t) := \lambda^{-1/2} Y(\lambda t) \) solves (7.10) in law with \( \gamma = \sqrt{\lambda} \) and \( B_\gamma \) a Brownian motion for each \( \gamma \).

Corollary 7.2. As \( \gamma \to \infty \) the process \( X_\gamma \) converges \( \mathcal{P} \) almost surely to the continuous process \( Y \) defined by (2.3).

Proof. For given \( \delta > 0 \), set \( c_{\delta,\gamma} := 12\gamma e^{-4\gamma \delta} \) and define the continuous process \( Z_{\delta,\gamma} \) as

\[ Z_{\delta,\gamma}(t) := \delta + B_\gamma(t) + c_{\delta,\gamma} t + \sup_{s \in [0,t]} \left[-B_\gamma(s) - c_{\delta,\gamma} s\right]. \]  

(7.11)

Note that \( Z_{\delta,\gamma}(0) = \delta \).\( Recall that \( Y_\gamma \) is the solution of (7.12). By arguing as in the proof of Theorem 7.1 the following comparison holds. For each \( \delta > 0 \) and \( \gamma > 1 \), we have, \( \mathcal{P} \) almost surely,

\[ Y_\gamma \leq X_\gamma \leq Z_{\delta,\gamma}, \]  

(7.12)

from which, by using Theorem 7.3, the statement follows by taking first the limit as \( \gamma \to \infty \) and then as \( \delta \to 0 \). □

We are now ready to conclude the proof of our main result. We next denote by \( \theta \) the macroscopic time variable and recall \( \lambda = \log \varepsilon^{-1} \). Recalling \( \xi_\varepsilon \) is defined in (6.1), let \( \zeta_\varepsilon \) be the continuous process defined as

\[ \zeta_\varepsilon(\theta) := \lambda^{-1/2} \xi_\varepsilon(\lambda \theta). \]

Lemma 7.3. Let \( B_\varepsilon(\theta) := \zeta_\varepsilon(\theta) - \sqrt{\lambda} \int_0^\theta ds 12 \exp\{-4\sqrt{\lambda} \zeta_\varepsilon(s)\}. \)  

(7.13)

The process \( B_\varepsilon \) weakly converges in \( C(\mathbb{R}_+) \) to a Brownian motion with diffusion coefficient \( \frac{4}{\lambda} \).

Proof. Recalling \( S_\varepsilon \) is the linear interpolation of the sequence \( S_0 \) defined in (6.12), let \( \overline{S}_\varepsilon(\theta) := \lambda^{-1/2} S_\varepsilon(\lambda \theta) \). By arguing exactly as in Lemma 6.3 one shows that the process \( \overline{S}_\varepsilon \) weakly converge in \( C(\mathbb{R}_+) \) to a Brownian motion with diffusion \( \frac{4}{\lambda} \). Moreover, by Lemma 6.3 with \( \mu = \lambda \), for each \( \delta > 0 \), \( \theta \in \mathbb{R}_+ \), we have

\[ \lim_{\varepsilon \to 0} \mathbb{P} \left( \sup_{s \in [0,\theta]} \left| \zeta_\varepsilon(s) - \overline{S}_\varepsilon(s) - \sqrt{\lambda} \int_0^\theta du 12 e^{-4\sqrt{\lambda} \zeta_\varepsilon(u)} \right| > \delta \right) = 0, \]

which concludes the proof. □

Proof of Theorem 7.1 item (iii). Thanks to (6.2) it is enough to prove the statement for \( \zeta_\varepsilon \) in place of \( Z_\varepsilon \). By Lemma 7.3 and [18, Thm. III.8.1] there exists a probability space \( (\Omega^*, \mathcal{F}^*, \mathbb{P}^*) \) and random elements \( B^*, B_\varepsilon^* \), with values in \( C(\mathbb{R}_+) \) such that
\( B^* \) is a Brownian motion with diffusion coefficient \( \frac{3}{4} \), the law of \( B^* \) equals the one of \( B \), defined in (7.13), and \( B^* \) converges, \( \mathbb{P}^* \) almost surely, to \( B^* \). We now define \( \zeta^*_\varepsilon \) as the solution of the equation

\[
\zeta^*_\varepsilon(\theta) = B^*_\varepsilon(\theta) - \sqrt{\lambda} \int_0^\theta ds 12 \exp\{-4\sqrt{\lambda} \zeta^*_\varepsilon(s)\}.
\]

By uniqueness of its solution, the law of \( \zeta^*_\varepsilon \) equals the one of \( \zeta_\varepsilon \). By Corollary 7.2 \( \zeta^*_\varepsilon(\theta) \) converges, \( \mathbb{P}^* \) almost surely, to \( B^*_\varepsilon(\theta) + \sup_{s \leq \theta} \left\{ -B^*_\varepsilon(s) \right\} \), whose law is that of a Brownian motion with diffusion coefficient \( \frac{3}{4} \) reflected at the origin. \( \square \)

8. Spectral Analysis

In this section we prove Theorem 3.2. To keep the notation simple we shall define the operator

\[
H = -\frac{1}{2} \Delta + V''(m), \quad m(x) := \text{th}(x),
\]

acting on \( L^2([-a,b]) \) with Dirichlet boundary conditions. We denote by \( \lambda_0 < \lambda_1 < \ldots < \lambda_i < \ldots \), resp. \( \Psi_i \) (recall \( \Psi_0 \) is chosen positive), \( i \geq 0 \), the eigenvalues, resp. the eigenfunctions, of \( H \) and by \( g_t := \exp\{-tH\} \) the corresponding semigroup. The operators \( g_t^\perp \) and \( G_t^\perp \) are defined as in (3.13) and (3.14).

By standard techniques it is not difficult to compute the Green operator \( G = H^{-1} \) for the quartic double well potential \( V \) in (2.2) obtaining that its integral kernel is given by:

\[
G(x,y) = \frac{2m'(x)m'(y)}{h(b) + h(a)} \begin{cases} [h(x) + h(a)] [h(b) - h(y)] & \text{if } -a \leq x \leq y \leq b \\ [h(y) + h(a)] [h(b) - h(x)] & \text{if } -a \leq y < x \leq b \end{cases}
\]

where, recalling (8.3),

\[
h(x) := h_0(x) = \frac{3}{8} x + \frac{3}{8} \frac{m(x)}{m'(x)} + \frac{1}{4} \frac{m(x)}{m'(x)^2}.
\]

Notation warning. In the sequel we will denote by \( C \) a generic positive constant, independent of \( a,b \), whose numerical value may change from line to line and from one side to the other in an inequality.

We first obtain some rougher estimates by following the approach in [10, Lemma 2.1] where analogous bounds are proven in the case of Neumann boundary conditions.

**Lemma 8.1.** There exists \( K > 0 \) and \( a_* > 0 \) such that, for any \( b \geq a \geq a_* \),

\[
0 \leq \lambda_0 \leq Ke^{-4a},
\]

\[
\langle \Psi_0, m' \rangle \geq \frac{1}{K},
\]

\[
\lambda_1 - \lambda_0 \geq \frac{1}{K},
\]

\[
\| \Psi_0 \|_\infty + \| \Psi_0' \|_\infty \leq K.
\]

**Sketch of the proof.**

**Step 1.** An elementary computation shows that, for each \( f \in C_0^\infty([-a,b]) \),

\[
\langle f, Hf \rangle = \frac{1}{2} \int_{-a}^b dx \ m'(x)^2 \left[ \frac{d}{dx} \frac{f(x)}{m'(x)} \right]^2 \geq 0,
\]
which in particular implies \( \lambda_0 \geq 0 \). On the other hand, by using \( G\mathbf{m}' \) as test function in the variational characterization of the smallest eigenvalue,

\[
\lambda_0 \leq \frac{\langle G\mathbf{m}', H G\mathbf{m}' \rangle}{\| G\mathbf{m}' \|_2^2} = \frac{\langle \mathbf{m}', G\mathbf{m}' \rangle}{\| G\mathbf{m}' \|_2^2} \tag{8.8}
\]

From (8.2) we now get

\[
G\mathbf{m}'(x) = 2h(a) \| \mathbf{m}' \|_2^2 \mathbf{m}'(x) + 2\mathbf{m}'(x)B(x) + A(x),
\]

where

\[
A(x) = \frac{-2\mathbf{m}'(x) [h(x) + h(a)]}{h(b) + h(a)} \int_{-a}^b dy \, \mathbf{m}'(y)^2 [h(y) + h(a)],
\]

\[
B(x) = \int_{-a}^x dy \, \mathbf{m}'(y)^2 h(y) + h(x) \int_x^b dy \, \mathbf{m}'(y)^2.
\]

Then:

\[
\langle \mathbf{m}', G\mathbf{m}' \rangle = 2h(a) \| \mathbf{m}' \|_2^2 + 2\langle \mathbf{m}', \mathbf{m}'B \rangle + \langle \mathbf{m}', A \rangle,
\]

\[
\| G\mathbf{m}' \|_2^2 = 4h^2(a) \| \mathbf{m}' \|_2^2 + 8h(a) \| \mathbf{m}' \|_2 \langle \mathbf{m}', \mathbf{m}'B \rangle + 4\| \mathbf{m}'B \|_2^2
\]

\[+ 4h(a) \| \mathbf{m}' \|_2 \langle \mathbf{m}', A \rangle + 4\| \mathbf{m}'B, A \| + \| A \|_2. \tag{8.13}
\]

From (8.10) and (8.20) we get

\[
\| A \|_\infty \leq C b \frac{h^2(a)}{h(b)^{1/2}}, \quad \| A \|_2 \leq C b \frac{h^2(a)}{h(b)^{1/2}},
\]

and, from (8.11) and (8.23),

\[
|\mathbf{m}'(x)B(x)| \leq |\mathbf{m}'(x)(x + a)| + Ce^{-2x},
\]

so that, after integrating,

\[
\langle \mathbf{m}', \mathbf{m}'B \rangle \leq C a, \quad \| \mathbf{m}'B \|_2^2 \leq C e^{4a}. \tag{8.16}
\]

Substituting (8.12) and (8.13) in the last quotient in (8.8), after estimating the terms with the aid of (8.11), (8.10) and (8.23), the bound (8.3) follows.

Step 2. Let \( \psi \) be an eigenfunction associated to an eigenvalue \( \lambda \leq 1/2 \) and choose a real \( \ell_0 \) such that \( \inf_{|x| \geq \ell_0} V''(\mathbf{m}(x)) \geq 3/2 \). By a comparison principle, we get:

\[
|\psi(x)| \leq |\psi(\ell)| \frac{\text{sh}(\sqrt{2}(b - x))}{\text{sh}(\sqrt{2}(b - \ell))} \quad \forall \ell \leq x \leq b,
\]

\[
|\psi(x)| \leq |\psi(-\ell)| \frac{\text{sh}(\sqrt{2}(x + a))}{\text{sh}(\sqrt{2}(a - \ell))} \quad \forall -a \leq x \leq -\ell \leq -\ell_0.
\]

Since \( \psi \) is normalized there exist reals \( \ell_\pm, \ell_+ \in [\ell_0, \ell_0 + 1] \) and \( \ell_- \in [-\ell_0 - 1, -\ell_0] \) such that \( |\psi(\ell_{\pm})| \leq 1 \). Hence, for any \( b \geq a > \ell_0 + 1 \),

\[
|\psi(x)| \leq C \exp\{-\sqrt{2}|x|\} \quad \forall |x| \geq \ell_0 + 1.
\]

Step 3. By (8.18) there exist reals \( \ell^*, a^* > 0 \) such that \( \int_{-\ell^*}^{\ell^*} dx \Psi_0(x)^2 \geq 1/2 \) for any \( a > a^* \). Since \( \lambda_0 \) is uniformly bounded by (8.4), by the Harnack inequality applied to the equation \( [H - \lambda_0]\Psi_0 = 0 \) in the interval \( [-\ell^* - 1, \ell^* + 1] \) we get that, for any \( b \geq a \geq a^* \), we have

\[
\inf_{|x| \leq \ell^*} \Psi_0(x) \geq C \sup_{|x| \leq \ell^*} \Psi_0(x) \geq C \left[ \frac{1}{2\ell^*} \int_{-\ell^*}^{\ell^*} dx \Psi_0(x)^2 \right]^{1/2} \geq \frac{C}{2\sqrt{\ell^*}}.
\]
The above bound and \( m'(x) = \text{ch}(x)^{-2} \) yields \( \text{[3.5]} \).

**Step 4.** We can assume \( \lambda_1 \leq 1/2 \). As well known, the corresponding eigenfunction \( \Psi_1 \) has a unique zero \( x_0 \) in the open interval \((-a, b)\); moreover, by \( \text{[3.17]} \), \( x_0 < \ell_0 \).

Integration by parts and \( H \text{[3.16]} - \text{[3.23]} \) for the operator \( H \) have

\[
\lambda_1 \geq \lambda_1 \int_a^b dx \Psi_1(x) \, m'(x) = \frac{1}{2} \left| \left( \Psi'_1 m' \right)(x_0) - \left( \Psi'_1 m' \right)(b) \right| \geq \frac{1}{2} \left| \left( \Psi'_1 m' \right)(x_0) \right|
\]

since \( \text{sgn}\left( \left( \Psi_1 \right)'(x_0) \right) = -1 \).

By the same argument as in Step 3, we have that either \( \int_{x_0}^b dx \Psi_1(x)^2 \geq 1/4 \) or \( \int_{x_0}^b dx \Psi_1(x)^2 \geq 1/4 \). By using the Hopf maximum principle we then deduce a lower bound on \( |\Psi_1(x_0)| \) which is uniform in \( b \geq a \geq a^\ast \). The estimate \( \text{[3.6]} \) follows.

**Step 5.** A uniform bound for \( \|\Psi_0\|_\infty \) follows from \( \text{[8.18]} \) and a comparison argument in the interval \([-\ell_0 - 1, \ell_0 + 1]\). Finally, since \( H\Psi_0 = \lambda_0 \Psi_0 \) and \( |V''(m)| \leq 2 \), we have \( |\Psi_0'(x)| \leq C \Psi_0(x) \). The bound \( \text{[8.7]} \) follows.

**Proof of Theorem 3.2.** We observe that, given \( \alpha \in (0, 1) \) it is equivalent to prove \( \text{[3.10]} \) - \( \text{[3.23]} \) for the operator \( H \) with

\[
a = \frac{1}{4} \log \varepsilon^{-1} + \zeta, \quad b = \varepsilon^{-\beta} - \zeta, \quad |\zeta| \leq \frac{1}{4} \alpha \log \varepsilon^{-1}, \quad (8.19)
\]

and that Lemma \( \text{[8.1]} \) clearly holds for these values of \( a \) and \( b \).

**Proof of \( \text{[3.10]} \).** By the Feynman-Kac formula, see e.g. [13, Theorem 2.3], we have that, for any \( f \in C([-a, b]) \), \( t > 0 \), and \( x \in (-a, b) \),

\[
(g_t f)(x) = \mathbb{E}\left( f(B_t^{(x)}) \mathbb{I}_{\tau_x > t} \exp\left\{ \int_0^t ds V''(m(B_s^{(x)})) \right\} \right), \quad (8.20)
\]

where \( \{B_t^{(x)}, t \geq 0\} \) is a Brownian motion starting at \( x \) and \( \tau_x := \inf\{t \geq 0 : B_t^{(x)} \notin (-a, b)\} \). The above representation permits to compare \( g_t \) with the semigroup \( \exp\{-t \mathcal{P}_0\} \), defined on the whole line \( \mathbb{R} \). For the latter the analogous estimate has been proved in [5, Prop. A.8], whence

\[
|g_t f(x)| \leq (g_t |f|)(x) \leq \left( \exp\{-t \mathcal{P}_0\} |f| \right)(x) \leq C \|f\|_\infty.
\]

**Proof of \( \text{[3.17]} \).** It is a restatement of \( \text{[8.6]} \).

**Proof of \( \text{[3.18]} \).** We will use an interpolation inequality, see [11, Lemma 5.1], that holds for each \( F \in C^1([-a, b]) \) such that \( F(a) = F(b) = 0 \),

\[
\|F\|_\infty^3 \leq \frac{3}{2} \|\nabla F\|_\infty \|F\|_2^2. \quad (8.21)
\]

Recalling \( p_t^0 \) denotes the heat semigroup with zero boundary conditions at the endpoints of \([-a, b] \), we have:

\[
\nabla g_t f = \nabla p_t^0 f - \int_0^t ds \nabla p_{t-s}^0 V''(m) g_s f.
\]

Since \( \|\nabla p_t^0 f\|_\infty \leq C t^{-1/2} \|f\|_\infty \) by \( \text{[3.16]} \) and the above identity we conclude that \( \|\nabla g_t f\|_\infty \leq C \sqrt{t} \|f\|_\infty \) for any \( t \geq 1 \). By choosing \( F = g_t^1 f \) in \( \text{[8.21]} \), the estimate \( \text{[3.18]} \) follows from \( \text{[3.13]} \), \( \text{[8.7]} \) and \( \text{[3.17]} \). \( \square \)
To prove the estimates (8.19)–(8.23), we will use the Kellogg method, see e.g. [19], to obtain successive approximations of the eigenvalues and eigenvectors by iterations of the Green operator $G$ applied to the function $\phi(x) := \phi^{(0)}(x) = \|m'\|^2_2 m'(x)$, $x \in [-a, b]$. Let $f_0 := \phi$, $f_1 := Gf_0$, $f_2 := Gf_1$ and $e_1 := f_1/\|f_1\|_2$, $e_2 := f_2/\|f_2\|_2$ be their $L_2$-normalizations. Also, let

$$
\mu := \frac{\|f_1\|_2^2}{\|f_2\|_2^2}, \quad R^2 := \sup_{x \in [-a, b]} \int_a^b dy G(x, y)^2, \quad c := (\Psi_0, \phi). \quad (8.22)
$$

Then, by [19, §28.1], we have the estimates

$$
0 \leq \mu - \lambda_0 \leq \frac{\lambda_0}{2} \frac{\lambda_1}{\lambda_1} \frac{(1 - c^2)^3}{c^2},
$$

$$
\|\Psi_0 - e_1\|_2 \leq \frac{\lambda_0}{\lambda_1} \frac{\sqrt{1 - c^2}}{c}, \quad (8.23)
$$

$$
\|\Psi_0 - e_2\|_\infty \leq R \lambda_1 \frac{\lambda_0}{\lambda_1} \frac{\sqrt{1 - c^2}}{c}.
$$

To use the above estimates, we will need expressions for $e_i$, $i = 1, 2$, and $\mu$. They are given in terms of the following formulae. From (8.22) and (8.23), we have:

$$
G^2 \overline{m}'(x) = P(x) + U(x),
$$

where

$$
P(x) = 4h^2(a) \|\overline{m}'\|_2^2 \overline{m}'(x) + 4h(a) \|\overline{m}'\|_2 \overline{m}'(x) B(x) + 2G(\overline{m}'B)(x),
$$

$$
U(x) = 2h(a) \|\overline{m}'\|_2^2 A(x) + GA(x).
$$

Also,

$$
\|G^2 \overline{m}'\|_2^2 = 16h^4(a) \|\overline{m}'\|_2^{10} + 32h^3(a) \|\overline{m}'\|_2^6 \overline{m}'B + 16h^2(a) \|\overline{m}'\|_2^4 \|\overline{m}'B\|_2^2 + 16h^2(a) \|\overline{m}'\|_2 \langle G\overline{m}', \overline{m}'B \rangle + 16h(a) \|\overline{m}'\|_2 \langle \overline{m}'B, G(\overline{m}') \rangle + 4\|G(\overline{m}'B)\|_2^2 + \|U\|_2^2 + 2(P, U).
$$

We finally remark that, by (8.28), $c = \|\overline{m}'\|_2^{-1}\langle \Psi_0, \overline{m}' \rangle$ is uniformly bounded from below by some positive constant.

**Proof of (8.19).** By (8.4) and (8.19) we have that, for each $\eta > 0$,

$$
\lim_{\varepsilon \to 0} \sup_{|\xi| < \frac{\varepsilon}{4\alpha \log \varepsilon^{-1}}} \varepsilon^{-(1-\alpha)+\eta} \lambda_0 = 0. \quad (8.27)
$$

From (8.23), (8.17), and (8.27), to prove (8.19) it is enough to show that

$$
\lim_{\varepsilon \to 0} \sup_{|\xi| < \frac{\varepsilon}{4\alpha \log \varepsilon^{-1}}} \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta} \mu + 4\varepsilon - 24 \varepsilon e^{-4\varepsilon K} = 0. \quad (8.28)
$$

From (8.13), the estimates (8.14), (8.16), and (8.19), it follows that

$$
\|G\overline{m}'\|_2 = 2h(a) \|\overline{m}'\|_2^2 (1 + \Delta_1), \quad \lim_{\varepsilon \to 0} \sup_{|\xi| < \frac{\varepsilon}{4\alpha \log \varepsilon^{-1}}} \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta} \Delta_1 = 0. \quad (8.29)
$$

Analogously, from (8.23) and the estimate $\|G(\overline{m}'B)\|_2^2 \leq C a^4 h^2(a)$ (that follows from (8.2) and (8.15)), together with (8.14) and (8.16),

$$
\|G^2 \overline{m}'\|_2 = 4h^2(a) \|\overline{m}'\|_2^5 (1 + \Delta_2), \quad \lim_{\varepsilon \to 0} \sup_{|\xi| < \frac{\varepsilon}{4\alpha \log \varepsilon^{-1}}} \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta} \Delta_2 = 0. \quad (8.30)
$$
Substitution of the previous expressions in the definition of $\mu$ yields
\[
\mu = \frac{1}{2h(a)} \frac{1}{\|m'\|^2_2} \frac{1}{1 + \Delta_1} \frac{1}{1 + \Delta_2}, \tag{8.31}
\]
from which (8.28) follows since, by (8.19), $e^{-4c} = e^{-4a}$ and, by (8.3),
\[
\lim_{\varepsilon \to 0} \sup_{|\xi| < \frac{3}{4} \alpha \log \varepsilon^{-1}} \varepsilon^{-\frac{3}{2} (1-\alpha) + \eta} \frac{1}{2h(a) \|m'\|^2_2} - 24 e^{-4a} = 0. \tag{8.32}
\]

Proof of (3.20). By (8.22) and (8.2) we have $R^2 = \sup_{x \in [-a,b]} G(x, x) \leq C h(a)$. From (8.23), (8.27), and (8.17), to prove (3.20) it is then enough to show that, for each $\eta > 0$,
\[
\lim_{\varepsilon \to 0} \sup_{|\xi| < \frac{3}{4} \alpha \log \varepsilon^{-1}} \varepsilon^{-\frac{3}{2} (1-\alpha) + \eta} \|e_2 - \phi\|_\infty = 0. \tag{8.33}
\]
From the definition of $e_2$, (8.24), (8.25), and (8.30), we have:
\[
e_2(x) - \phi(x) = \frac{G^2(m'(x))}{\|G^2m'\|_2} - \phi(x) \]
\[
= \left( \frac{m'(x)B(x)}{h(a) \|m'\|^2_2} + \frac{G(m'(B)(x)}{2h^2(a) \|m'\|^2_2} + \frac{U(x)}{4h^2(a) \|m'\|^2_2} \right) \frac{1}{1 + \Delta_2} - \frac{\Delta_2 \phi(x)}{1 + \Delta_2}. \tag{8.34}
\]
Now, by (8.14), (8.15), (8.25), and using the definition (8.2), it is easy to show that:
\[
\lim_{\varepsilon \to 0} \sup_{|\xi| < \frac{3}{4} \alpha \log \varepsilon^{-1}} \varepsilon^{-\frac{3}{2} (1-\alpha) + \eta} \|m' B\|_\infty \|h(a)\] = 0,
\[
\lim_{\varepsilon \to 0} \sup_{|\xi| < \frac{3}{4} \alpha \log \varepsilon^{-1}} \varepsilon^{-\frac{3}{2} (1-\alpha) + \eta} \|U\|_\infty + \|G(m' B)\|_\infty \|h^2(a)\] = 0, \tag{8.35}
\]
from which (8.33) follows.

Proof of (3.21). For any $\ell < a$ we have
\[
\|\Psi_0 - \phi\|_1 \leq 2\ell \|\Psi_0 - \phi\|_\infty + \int_{[-a,b] \setminus [-\ell, \ell]} dx \left( \Psi_0(x) + \phi(x) \right).
\]
Then, by (3.20), (8.18), and recalling $\phi(x) \leq e^{-2|x|}$, we get (3.21) by choosing e.g. $\ell = \log^4 \varepsilon^{-1}$.

Proof of (3.22). To prove (3.22) recall that, from (8.23), (8.27), and (8.17), it is sufficient to show that, for each $\eta > 0$,
\[
\lim_{\varepsilon \to 0} \sup_{|\xi| < \frac{3}{4} \alpha \log \varepsilon^{-1}} \varepsilon^{-(1-\alpha) + \eta} \langle |e_2 - \phi|, \phi \rangle = 0.
\]
Substituting in $\langle |e_2 - \phi|, \phi \rangle$ the expression (8.34), since $\|\phi\|_1 \leq C$, the limit above follows from (8.35) and the first estimate in (8.16).

Proof of (3.23). From the definition (8.2), (8.7), and recalling that $\lambda_0$ is the first eigenvalue of $H$, we have:
\[
G^\perp(x, x) = 2m'(x)^2 \left[ h(x) + h(a) \right] \left[ 1 - \frac{h(x) + h(a)}{h(b) + h(a)} \right] \frac{\Psi_0^2(x)}{\lambda_0}. \tag{8.36}
\]
Taking the square in (8.9) and substituting into (8.39), from (8.14), (8.15), and (8.16),

We next notice that, by (3.22) and the definition (8.22) of \( c \), for each \( \eta > 0 \),

\[
\lim_{\varepsilon \to 0} \sup_{|\zeta| < \frac{c}{\alpha} \log \varepsilon^{-1}} \varepsilon^{-\frac{\varepsilon}{2} (1-\alpha) + \eta} \sqrt{1-c^2} = 0,
\]

so that, by (8.22), (8.27), and (8.37),

\[
\lim_{\varepsilon \to 0} \sup_{|\zeta| < \frac{c}{\alpha} \log \varepsilon^{-1}} \varepsilon^{-\frac{\varepsilon}{2} (1-\alpha) + \eta} \int_{-a}^{b} dx \, \tilde{m}'(x) |\tilde{m}(x)| \left| \frac{\Psi_{\alpha}(x)}{\lambda_{0}} - \frac{c_{1}(x)}{\mu} \right| = 0. \tag{8.38}
\]

On the other hand, from (8.29), (8.30), and (8.31),

\[
e_{1}(x) = \frac{1 + \Delta_{3}}{2h(\alpha) \| \tilde{m} \|_{2}^{2}} \| G(m')(x) \|^{2}, \quad \lim_{\varepsilon \to 0} \sup_{|\zeta| < \frac{c}{\alpha} \log \varepsilon^{-1}} \varepsilon^{-\frac{\varepsilon}{2} (1-\alpha) + \eta} \Delta_{3} = 0. \tag{8.39}
\]

Taking the square in (8.39) and substituting into (8.39), from (8.14), (8.15), (8.16) and (8.31), it follows that

\[
\frac{e_{1}(x)\varepsilon}{\mu} = 2h(\alpha)(1 + \Delta_{3}) \tilde{m}'(x)^{2} + \frac{4B(x) \tilde{m}'(x)^{2}}{\| \tilde{m} \|_{2}^{2}} + W(x),
\]

with

\[
\lim_{\varepsilon \to 0} \sup_{|\zeta| < \frac{c}{\alpha} \log \varepsilon^{-1}} \varepsilon^{-\frac{\varepsilon}{2} (1-\alpha) + \eta} \| (m')^{2} m W \|_{1} = 0.
\]

By (8.36), (8.37), (8.38), and the above limit, we are reduced to prove that, for each \( \eta > 0 \),

\[
\lim_{\varepsilon \to 0} \sup_{|\zeta| < \frac{c}{\alpha} \log \varepsilon^{-1}} \varepsilon^{-\frac{\varepsilon}{2} (1-\alpha) + \eta} Q(\zeta, \varepsilon) = 0, \tag{8.40}
\]

where

\[
Q(\zeta, \varepsilon) := \left| \int_{-a}^{b} dx \tilde{m}'(x)^{3} \tilde{m}(x) \left( 2h(x) - \frac{4B(x)}{\| \tilde{m} \|_{2}^{2}} - 2\Delta_{3} h(\alpha) \right) \right|. \tag{8.41}
\]

Now, since \( (\tilde{m}')^{3} \tilde{m} \) is an odd function, we have

\[
\left| \int_{-a}^{b} dx \tilde{m}'(x)^{3} \tilde{m}(x) \right| = \left| \int_{a}^{b} dx \tilde{m}'(x)^{3} \tilde{m}(x) \right| \leq C e^{4a}. \tag{8.42}
\]

From (8.11),

\[
h(x) - \frac{2B(x)}{\| \tilde{m} \|_{2}^{2}} = h(x) \left[ 1 - 2 \int_{x}^{b} dy \frac{\tilde{m}'(y)^{2}}{\| \tilde{m} \|_{2}^{2}} \right] + 2 \int_{a}^{x} dy \frac{\tilde{m}'(y)^{2}}{\| \tilde{m} \|_{2}^{2}} h(y).
\]

Since \( \tilde{m}(x) = \tilde{m}(x) \),

\[
2 \int_{x}^{b} dy \frac{\tilde{m}'(y)^{2}}{\| \tilde{m} \|_{2}^{2}} = \frac{3}{2} \int_{x}^{\infty} dy \tilde{m}'(y)^{2} + D(x) = 1 + \frac{\tilde{m}'^{3}(x) - 3\tilde{m}(x)}{2} + D(x),
\]
with $|D(x)| \leq Ce^{-4a}$. Then, recalling $h(x)\bar{m}'(x)^2 \leq C$,

$$
\left| \int_{-a}^{b} dx \, \bar{m}'(x)^3 \bar{m}(x) h(x) \left[ 1 - 2 \int_{x}^{b} dy \, \frac{\bar{m}'(y)^2}{\|\bar{m}'\|_2^2} \right] \right|
\leq C e^{-4a} + \left| \int_{-a}^{b} dx \, \bar{m}'(x)^3 \bar{m}(x) h(x) \frac{\bar{m}^3(x) - 3\bar{m}(x)}{2} \right|
\leq C e^{-2a},
$$

(8.43)

where we have used that the integrand in the last integral is an odd function, see (8.3). Finally, observing that $-6(\bar{m}')^3\bar{m} = [(1 - \bar{m}^2)^3]'$, integration by parts in the remaining integral yields

$$
\left| \int_{-a}^{b} dx \, \bar{m}'(x)^3 \bar{m}(x) \int_{-a}^{x} dy \, \bar{m}'(y)^2 h(y) \right| \leq \left| \frac{(1 - \bar{m}^2(b))^3}{6} \int_{-a}^{b} dy \, \bar{m}'(y)^2 h(y) \right|
+ \left| \int_{-a}^{b} dx \, \frac{(1 - \bar{m}^2(x))^3}{6} \bar{m}'(x)^2 h(x) \right| \leq C e^{-6a}.
$$

(8.44)

To estimate the last integral we have used again $h(x)\bar{m}'(x)^2 \leq C$ and that the integrand is an odd function. From (8.42), (8.43), and (8.44) the limit (8.40) follows.

**Remark 1.** Proceeding as in the proof of (3.23), from (3.19), (3.20), and (8.23) it can be shown that $\sup_{x \in [-a,b]} G^+(x,x) < \infty$.

**Remark 2.** From the previous computations, it follows that $G^+(x,y)$ converges pointwise, as $\varepsilon \to 0$, to the kernel of the generalized Green function $\mathcal{G}$ which inverts $\bar{H}_0$ on the subspace orthogonal to $\bar{m}'$. This kernel is

$$
\mathcal{G}(x,y) := \begin{cases} 
\frac{3}{4} \bar{m}'(x) \bar{m}'(y) \left[ u(x) + u(-y) + \frac{5}{12} \right] & \text{if } x \leq y \\
\frac{3}{4} \bar{m}'(x) \bar{m}'(y) \left[ u(-x) + u(y) + \frac{5}{12} \right] & \text{if } x > y 
\end{cases}
$$

(8.45)

where

$$
u(x) := \frac{1}{24} e^{4x} + \frac{1}{3} e^{2x} + \frac{1}{2} x - \frac{3}{8}.
$$

(8.46)

This expression has been obtained in [6, Prop. 3.3] where however the constant $\frac{5}{12}$ should read $\frac{5}{12}$.

**Appendix A. Fluctuations of a localized interface**

In this section we sketch the proof of Theorem 2.2 which describes the asymptotic behavior of the interface when $a = b = \frac{1}{4} \log \varepsilon^{-1}$, by pointing out the relevant differences w.r.t. the case $a = b = \frac{1}{4} \log \varepsilon^{-1}$, $b \gg a$. We then explain how to get the uniformity w.r.t. the initial condition.

**Sketch of the proof of Theorem 2.2** Fix $\tau_0 > 0$. Throughout this section we denote by $m(t;m_0)$, $t \in [0,\varepsilon^{-1}\tau_0]$, the solution to (2.3) with $a = b = \frac{1}{4} \log \varepsilon^{-1}$, to emphasize its dependence on the initial condition $m_0 \in \mathcal{X}_\varepsilon$. Accordingly, we let $X(m_0)$, resp. $X(t;m_0)$, be the center of $m_0$, resp. $m(t \wedge S_{k,t,\alpha};m_0)$, see (3.4). Recalling the set $\mathcal{N}_{\varepsilon}^{\tau_0}(z)$ is defined in the statement of the theorem, for each $L > 0$ we define $\mathcal{N}_{\varepsilon}^{\tau_0,L} := \bigcup_{z \in [-L,L]} \mathcal{N}_{\varepsilon}^{\tau_0}(z)$. The iterative scheme of Section 3 is repeated with no changes in the present setting.
Step 1. Spectral analysis. We claim that Theorem 3.2 holds with the only change that the asymptotic (8.19) for the smallest eigenvalue has to be replaced by
\[
\lim_{\varepsilon \to 0} \sup_{|\zeta| < \alpha} \varepsilon^{-\frac{3}{2}(1-\alpha) + \eta} |\lambda_0^{(\zeta)} - 48 \varepsilon \chi(4\zeta)| = 0. \tag{A.1}
\]
As in Section 8, we fix the center at the origin and study the operator (8.1) in the interval \(-\ell - \zeta, \ell - \zeta\). The asymptotic of the eigenvalue \(\lambda^{(\zeta)}_0\) can be obtained as in (8.21). The asymptotic of \(\mu\), as defined (8.22), is obtained as follows. Instead of (8.30) we here decompose
\[
Gm'(x) = 2 \|m'\|_2^2 \frac{h(\ell + \zeta) h(\ell - \zeta)}{h(\ell + \zeta) + h(\ell - \zeta)} m'(x)
\]
and get
\[
\|Gm'\|_2 = 2 \frac{h(\ell + \zeta) h(\ell - \zeta)}{h(\ell + \zeta) + h(\ell - \zeta)} \|m'\|_3^2 (1 + \bar{\Delta}_1),
\]
\[
\|G^2m'\|_2 = 4 \left[ \frac{h(\ell + \zeta) h(\ell - \zeta)}{h(\ell + \zeta) + h(\ell - \zeta)} \right]^2 \|m'\|_5^2 (1 + \bar{\Delta}_2),
\]
where \(\bar{\Delta}_1\) and \(\bar{\Delta}_2\) satisfy the estimates stated in (8.29) and (8.30) for \(\Delta_1\) and \(\Delta_2\).

The bound (A.1) now follows by direct computations, see (8.31) and (8.32).

Step 2. A priori bounds and recursion equation for the center. The a priori bounds of Section 4 depend only on \(\beta \geq a\) and therefore hold also in the present setting. Moreover, there exists \(\eta_1 > 0\) such that the following holds. For each \(L > 0\) and \(\eta \in (0, \eta_1]\) there exists \(\eta_0 > 0\) such that the bounds stated in Theorem 4.2 hold for \(\eta \in (0, \eta_0)\) uniformly w.r.t. \(m_0\) in the set \(N_{\eta', L}, \eta' \in [0, \eta_1]\).

The key estimate (5.5) in Lemma 5.2 for the identification of the nonlinear drift is here replaced by
\[
\lim_{\varepsilon \to 0} \sup_{|\zeta| < \alpha} \varepsilon^{-\frac{3}{2}(1-\alpha) + \eta} \left| \langle m'_\zeta, \varphi_\zeta - g_{\zeta T}^{(\zeta)} \varphi_\zeta \rangle + \frac{4}{3} 24 \varepsilon T \text{sh}(4\zeta) \right| = 0, \tag{A.2}
\]
which is proven as follows. Recalling (3.6), we have
\[
\sup_{|\zeta| < \alpha} \left| d \zeta + \frac{1}{2} \right| \leq C \varepsilon \frac{3}{2}(1-\alpha)
\]
\[
\sup_{|\zeta| < \alpha} \left| \zeta - 1 \right| \leq C \varepsilon \frac{3}{2}(1-\alpha)
\]
\[
\left| \frac{h(\ell + \zeta) - h(\ell - \zeta)}{h(\ell + \zeta) + h(\ell - \zeta)} - \frac{1}{2} + \frac{1}{2} \text{th}(4\zeta) \right| \leq C \varepsilon \frac{3}{2}(1-\alpha),
\]
whence
\[
\lim_{\varepsilon \to 0} \sup_{|\zeta| < \alpha} \varepsilon^{-\frac{3}{2}(1-\alpha) + \eta} \left| \langle m'_\zeta, \varphi_\zeta \rangle + \frac{2}{3} \text{th}(4\zeta) \right| = 0.
\]
In view of this bound and \((A.1)\), we can repeat the computations in Lemma 5.2 and get \((A.2)\).

Let \(\xi_n\) and \(\sigma_n\) be defined as in \((5.1)\). We emphasize that \(\xi_0 = x_0 = X(m_0)\) so that the whole sequence \(\xi_n\) depends on the initial condition \(m_0\). By using \((A.2)\) and following the same steps as in Theorem 5.1 it is easy to prove its analogue in the present setting with a uniform control on \(m_0 \in \mathcal{N}^{\varepsilon,L}_{\eta} \), \(\eta \in [0, \eta_1]\). Set \(b(x) := -24 \text{sh}(4x)\), then

\[
\xi_{n+1} - \xi_n = \sigma_n + \varepsilon T b(\xi_n) + \Theta_n,
\]

where, for each \(L \in \mathbb{R}_+\), there exists \(q > 0\) such that

\[
\lim_{\varepsilon \to 0} \sup_{z_0 \in \mathcal{N}_{\eta}^{\varepsilon,L}} \mathbb{P}\left( \sup_{0 \leq n < \varepsilon^{-1}\tau_0/T} \left| \sum_{k=0}^{n} \Theta_k \right| > \varepsilon^q \right) = 0.
\]

Moreover, by the same argument as in Lemma 5.3 the above statement implies that, for each \(L \in \mathbb{R}_+\), we have

\[
\lim_{K \to \infty} \lim_{\varepsilon \to 0} \sup_{z_0 \in \mathcal{N}_{\eta}^{\varepsilon,L}} \mathbb{P}\left( \sup_{0 \leq n \leq \varepsilon^{-1}\tau_0} |\xi_n| > K \right) = 0,
\]

which yields, see the end of Section 5

\[
\lim_{\varepsilon \to 0} \sup_{z_0 \in \mathcal{N}_{\eta}^{\varepsilon,L}} \mathbb{P}\left( \sup_{t \in [0, \varepsilon^{-1}\tau_0]} \|m(t; m_0) - \overline{m}_X(t; m_0)\|_{\infty} > \varepsilon^q \right) = 0.
\]

**Step 3. A coupling argument.** By \((A.7)\), the uniform convergence \((A.10)\) follows once we show there exists \(\eta_1 > 0\) such that for each \(\eta \in [0, \eta_1]\), \(L > 0\), and each uniformly continuous and bounded function \(F : C([0, \tau_0]; \mathcal{X}) \to \mathbb{R}\), we have

\[
\lim_{\varepsilon \to 0} \sup_{z_0 \in [-L, L]} \sup_{m_0 \in \mathcal{N}_{\eta}^{\varepsilon,L}} \|\mathbb{E} F(\mathcal{M}_{Y(\cdot; m_0)}) - \mathbb{E} F(\mathcal{M}_{\Xi_{\eta}(\cdot)})\| = 0
\]

where \(Y(t; m_0) := X(\varepsilon^{-1}t; m_0)\), \(t \in [0, \tau_0]\), and \(E\) denotes the expectation w.r.t. the Brownian motion \(B\) in \((2.9)\). Let \(\xi_\varepsilon(\cdot; m_0)\) be as defined in \((6.1)\). The estimate \((6.2)\) holds uniformly, namely

\[
\lim_{\varepsilon \to 0} \sup_{m_0 \in \mathcal{N}_{\eta}^{\varepsilon,L}} \mathbb{P}\left( \sup_{\tau \in [0, \tau_0]} |X_{\varepsilon}(\varepsilon^{-1}\tau; m_0) - \xi_\varepsilon(\tau; m_0)| > \varepsilon^q \right) = 0.
\]

Let \(\zeta_\varepsilon := \Xi_{2\varepsilon}(\varepsilon T_n)\) and denote by \(\xi_\varepsilon(\cdot; z_0)\) its piecewise linear interpolation as in \((6.1)\). By \((A.9)\) and the continuity of \(\Xi_{2\varepsilon}\), \((A.8)\) is proven once we show

\[
\lim_{\varepsilon \to 0} \sup_{z_0 \in [-L, L]} \sup_{m_0 \in \mathcal{N}_{\eta}^{\varepsilon,L}} \|\mathbb{E} F(\mathcal{M}_{\xi_\varepsilon(\cdot; m_0)}) - \mathbb{E} F(\mathcal{M}_{\zeta_\varepsilon(\cdot; z_0)})\| = 0
\]

Given the random variables \(\sigma_0, \ldots, \sigma_n\), we define the sequence \(\beta_n\) by the recursive relation \(\beta_{n+1} = \beta_n + \varepsilon T b(\beta_n) + \sigma_n\), with \(\beta_0 = \xi_0 = X(m_0)\). The recursive relation \((A.3)\), the bounds \((A.4)\) and \((A.5)\) imply, by a standard Gronwall argument,

\[
\lim_{\varepsilon \to 0} \sup_{m_0 \in \mathcal{N}_{\eta}^{\varepsilon,L}} \|\mathbb{E} F(\mathcal{M}_{\beta_\varepsilon(\cdot; m_0)}) - \mathbb{E} F(\mathcal{M}_{\beta(\cdot; m_0)})\| = 0
\]

where \(\beta_\varepsilon(\cdot; m_0)\) is the piecewise linear interpolation of the sequence \(\beta_n\).
Recall that \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) is the filtered probability space where the cylindrical Wiener process lives. We denote by \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}})\) the filtered probability space where the Brownian motion \(B\) appearing in (2.9) lives. We then set \(\hat{\Omega} := \Omega \times \hat{\Omega}, \hat{\mathcal{F}} := \mathcal{F} \times \hat{\mathcal{F}}, \hat{\mathcal{F}}_t := \mathcal{F}_t \times \hat{\mathcal{F}}_t, \hat{\mathbb{P}} := \mathbb{P} \times \hat{\mathbb{P}}\). On this probability space we define the sequence \(\tilde{\beta}_n\) as

\[
\begin{cases}
\tilde{\beta}_{n+1} = \tilde{\beta}_n + \varepsilon T b(\tilde{\beta}_n) + \sqrt{\frac{3n}{\varepsilon T}} [B(\varepsilon T_{n+1}) - B(\varepsilon T_n)], \\
\tilde{\beta}_0 = \beta_0 = \xi_0 = X(m_0),
\end{cases}
\]

where

\[
s_n := s_n(x_n) := \frac{4}{3} \mathbb{E}[\sigma_n^2|x_n] = \frac{3}{4} \mathbb{E} \int_0^T \langle m'_n, g^{(x_n)}_n \rangle d\tau,
\]

Since, conditionally on the centers \(x_0, \ldots, x_n\), the random variables \(\sigma_0, \ldots, \sigma_n\) are independent Gaussians with variance \(\frac{3}{4} s_0, \ldots, \frac{3}{4} s_n\), the sequence \(\beta_n\) and \(\tilde{\beta}_n\) have the same law. By (A.11), to prove (A.10) it is enough to show that

\[
\lim_{\varepsilon \to 0} \sup_{z_0 \in [-L, L]} \sup_{m_0 \in \mathcal{N}_n^L(z_0)} \mathbb{E} \left| F(\hat{\mathcal{M}}_{\tilde{\beta}_n}(z_0)) - F(\hat{\mathcal{M}}_{\beta_n}(z_0)) \right| = 0,
\]

where \(\tilde{\beta}_n(z_0)\) is the piecewise linear interpolation of the sequence \(\tilde{\beta}_n\). Set \(\varrho_n := \tilde{\beta}_n - \xi_n\); it satisfies the recursive equation

\[
\varrho_{n+1} = \varrho_n + \varepsilon T \left[ b(\tilde{\beta}_n) - b(\xi_n) \right] + R^{(1)}_n + R^{(2)}_n,
\]

where

\[
R^{(1)}_n = \varepsilon T b(\xi_n) - \int_{\varepsilon T_n}^{\varepsilon T_{n+1}} d\tau b(\Xi^{z_0}(\tau)),
\]

\[
R^{(2)}_n = \left( \sqrt{\frac{3n}{\varepsilon T}} - 1 \right) [B(\varepsilon T_{n+1}) - B(\varepsilon T_n)].
\]

Finally, since \(\varrho_0 = X(m_0) - z_0\), for each \(L > 0\) we have

\[
\lim_{\varepsilon \to 0} \sup_{z_0 \in [-L, L]} \sup_{m_0 \in \mathcal{N}_n^L(z_0)} |\varrho_0| = 0.
\]

By simple estimates on \(R^{(i)}_n, i = 1, 2\) and Doob’s inequality, a Gronwall argument shows that, for each \(\delta > 0\),

\[
\lim_{\varepsilon \to 0} \sup_{z_0 \in [-L, L]} \sup_{m_0 \in \mathcal{N}_n^L(z_0)} \hat{\mathbb{P}} \left( \sup_{k \leq n, \tau_k} |\varrho_k| > \delta \right) = 0,
\]

which yields (A.13).

\[\Box\]

**Acknowledgments**

It is a great pleasure to thank E. Presutti for suggesting us the problem discussed in this paper and for his collaboration at the initial stage of the work. We are in debt to L. Zambotti for explaining us Theorem 7.1. L.B. and P.B. acknowledge the partial support of COFIN-MIUR. S.B. acknowledges the hospitality at the Mathematics Department of the University of Rome ‘La Sapienza’.

32 L. BERTINI, S. BRASSESCO, AND P. BUTTA
References

[1] S. Allen, J. Cahn: A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. Acta Metall. 27, 1084–1095 (1979).
[2] L. Bertini, S. Brassesco, P. Buttà: Dobrushin states in the $\phi^4$ model. Preprint 2006.
[3] L. Bertini, S. Brassesco, P. Buttà, E. Presutti: Front fluctuations in one dimensional stochastic phase field equations. Ann. Henri Poincaré 3, 29–86 (2002).
[4] P. Billingsley: Convergence of Probability Measures. New York: Wiley 1968.
[5] S. Brassesco: Stability of the instanton under small random perturbations. Stoch. Proc. Appl. 54, 309–330 (1994).
[6] S. Brassesco, P. Buttà: Interface fluctuations for the D=1 Stochastic Ginzburg–Landau equation with non–symmetric reaction term. J. Statist. Phys. 93, 1111–1142 (1998).
[7] S. Brassesco, P. Buttà, A. De Masi, E. Presutti: Interface fluctuations and couplings in the $d = 1$ Ginzburg–Landau equation with noise. J. Theoret. Probab. 11, 25–80 (1998).
[8] S. Brassesco, A. De Masi, E. Presutti: Brownian fluctuations of the interface in the $d = 1$ Ginzburg–Landau equation with noise. Annal. Inst. H. Poincaré 31, 81–118 (1995).
[9] J. Carr, B. Pego: Metastable patterns in solutions of $u_t = \varepsilon^2 u_{xx} + u(1 - u^2)$. Commun. Pure Applied Math. 42, 523–576 (1989).
[10] X. Chen: Spectrum for the Allen-Chan, Cahn-Hilliard, and phase-field equations for generic interfaces. Commun. Partial Diff. Eqs. H. 19, 1371–1395 (1994).
[11] P.C. Fife, J.B. McLeod: The approach of solutions of nonlinear diffusion equations to travelling front solutions. Arch. Ration. Mech. Anal. 65, 335–361 (1977).
[12] W.G. Faris, G. Jona-Lasinio: Large fluctuations for a nonlinear heat equation with noise. J. Phys. A 15, 3025–3055 (1982).
[13] M. Freidlin: Functional integration and partial differential equations. Princeton: Princeton University Press 1985.
[14] T. Funaki: The scaling limit for a stochastic PDE and the separation of phases. Prob. Theory Relat. Fields 102, 221–288 (1995).
[15] T. Funaki: Stochastic interface models. Lectures on probability theory and statistics. Lecture Notes in Math. 1869, 103–274. Berlin: Springer 2005.
[16] G. Fusco, J. Hale: Slow-motion manifolds, dormant instability and singular perturbations. J. Dynamics Differential equations 1, 75–94 (1989).
[17] I. Karatzas, S.E. Shreve: Brownian motion and stochastic calculus. Second edition. New York: Springer 1991.
[18] A.N. Shiryaev: Probability. Second edition. New York: Springer 1996.
[19] V.S. Vladimirov: Equations of mathematical physics. Moscow: Mir publishers 1984.

Lorenzo Bertini, Dipartimento di Matematica, Università di Roma ‘La Sapienza’, P.le Aldo Moro 2, 00185 Roma, Italy
E-mail address: bertini@mat.uniroma1.it

Stella Brassesco, Departamento de Matemáticas, Instituto Venezolano de Investigaciones Científicas, Apartado Postal 21827, Caracas 1020–A, Venezuela
E-mail address: sbasses@ivic.ve

Paolo Buttà, Dipartimento di Matematica, Università di Roma ‘La Sapienza’, P.le Aldo Moro 2, 00185 Roma, Italy
E-mail address: buttà@mat.uniroma1.it