Network Risk and Forecasting Power in Phase-Flipping Dynamical Networks

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Abstract—In order to model volatile real-world network behavior, we analyze phase-flipping dynamical scale-free network in which nodes and links fail and recover. We investigate how stochasticity in a parameter governing the recovery process affects phase-flipping dynamics, and find the probability that no more than 9% of nodes and links fail. We derive higher moments of the fractions of active nodes and active links, fn(t) and ft(t), and define two estimators to quantify the level of risk in a network. We find hysteresis in the correlations of fn(t) due to failures at the node level, and derive conditional probabilities for phase-flipping in networks. We apply our model to economic and traffic networks.

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Across a broad range of human activities—from medicine, weather, and traffic management to intelligence services and military operations—forecasting theories help us estimate the probability of future outcomes. In general, the greater the uncertainty of outcome, the more crucial that we be able to forecast future behavior.

A recent paper details how the nodes in dynamic regular networks and Erdős Renyi networks (i) inherently fail, (ii) contiguously fail due to the failure of neighboring nodes, and (iii) recover. These networks exhibit phase-flipping between “active” and “inactive” collective network modes. Here we analyze networks with highly heterogeneous degree distributions and we describe scale-free phase-flipping networks in which nodes and links fail and recover. We describe the collective behavior of these networks using two time-dependent network variables: the fraction of active nodes fn(t) and the fraction of active links ft(t). We place an emphasis on forecasting in dynamic networks—we want to calculate how many nodes will fail at any future time t—and quantify how risky networks are.

(i) At each time t any node in the system can independently fail, breaking its links with all other nodes, with a probability p. The internal failure state of node i we denote by spin $|s_i⟩$ (if i is active, $|s_i⟩ = |1⟩$).
(ii) The external failure states we denote by spin $|S_i⟩$, where $|S_i⟩$ is $|1⟩$ if node i has more than $T_h %$ active neighbors, and $|0⟩$ (for a subsequent time $τ' = 1$) with probability $p_2$ if $≤ T_h %$ of i’s neighbors are active. For scale-free networks, a percentage threshold $T_h$ is the more appropriate choice than the constant $T_h$ used in Ref. 1, 23. Node i—described by the two-spin state $|S_i, s_i⟩$—is active only if both spins are up (1), i.e., if $|S_i, s_i⟩ = |1, 1⟩$.
(iii) After a time period $τ$, the nodes recover from internal failure. Usually $τ$ is random, but we also analyze the case when $τ$ is constant 24.

Estimating how far the parameters of a dynamic system are from the area in parameter space characterized by high instability is crucial. For the network described in (i)–(iii) above, we arbitrarily choose parameters $p_1$ (related to p by $p_1 = 1 - \exp(-prT)$ 23) and $T_h$. We then destabilize the network by increasing $p_2$, causing it to transition from phase I with predominantly active nodes to phase II with predominantly inactive nodes. In Fig. 1(a), for varying $p_2$, the first network statistic—the average $f_n(t)$, $⟨f_n⟩$—gradually decreases for $p_2 ∈ (0, p_2')$ and then, at $p_2 ≈ p_2'$, $⟨f_n⟩$ shows a sudden network crash—a first-order phase transition. In Fig. 1(b) for $p_2 ∈ (0, p_2')$ the second network statistic—the standard deviation of $f_n(t)$, $σ_n$—becomes increasingly volatile. During network recovery, in Figs. 1(a) and 1(b) both $⟨f_n⟩$ and $σ_n$ follow a first-order phase transition, but at a value $p_2 = p_2''$, which differs from ($p_2 = p_2'$) obtained during the I–II transition. Because $⟨f_n⟩$ and $σ_n$ are dependent upon the initial node spins in the network, $p_2' ≠ p_2''$ implies the existence of hysteresis 20, 29. To estimate the part of the ($p_1, p_2$) phase space that is unstable, we calculate the discontinuity ($p_2'', p_2''$) values for varying $p_1$ values [see...
Fig. [I(b)]. Figure [I(c)] shows a hysteresis with two discontinuity lines (spinodals) in the \((p_1, p_2)\) space. The closer the \((p_1, p_2)\) of a network is to the left spinodal in Fig. [I(c)], the less stable the network.

Reference 27 reports that introducing both a dynamic recovery with a (constant) parameter \((\tau \neq 0)\) and a stochastic continuous spreading \((p_2 \neq 1)\) leads to spontaneous collective network phase-flipping phenomena. Figure [I(d)] shows the fraction of active nodes \(f_n(t)\) for constant \(\tau (\Delta \tau = 0)\) that corresponds to the volatile state \(X_A\) shown in Fig. [I(c)]. Figure [I(d)] shows that if \(\tau\) is not constant but a random variable from a homogeneous probability distribution function (pdf), the phase-flipping phenomenon and thus the collective network mode disappears with increasing \(\Delta \tau\) (increasing stochasticity in \(\tau\)). Beginning with the relation \(p_1 = 1 - \exp(-p\tau)\) in Ref. [27] when \(\tau\) is constant, we confirm this result. Suppose a network is initially set at a phase-flipping state \(X_A\) [Fig. [I(c)]. If \(\tau\) follows a homogeneous pdf, \(H(\tau_0 - \Delta \tau, \tau_0 + \Delta \tau)\), we easily derive the average parameter \(p_1^* = p_1(\Delta \tau)\) as

\[
E(p_1^*) = 1 - \exp(-p\tau_0)\sinh(p\Delta \tau)/p\Delta \tau,
\]

and the average deviation of \(p_1^*\) from \(p_1 \equiv p_1(\Delta \tau = 0)\),

\[
E(p_1^*) - p_1 = \exp(-p\tau_0)[1 - \sinh(p\Delta \tau)/p\Delta \tau] < 0.
\]

With increasing \(\Delta \tau\), \(E(p_1^*) - p_1\) decreases and \(E(p_1^*)\) moves from a volatile network regime \((X_A)\) to a more stable network regime. Hence at \(X_A\) the less dispersed \(\tau\) is (and also \(p_1\)), the more pronounced the phase-flipping. Hereafter, we analyze networks with constant \(\tau\).

We next explore the diagnostic and forecasting power of dynamic networks. When internal \((X)\) and external \((Y)\) failures are independent, according to probability theory \(P(X \cup Y) = P(X) + P(Y) - P(X)P(Y)\), from which Ref. [27] calculates the probability \(a = a(p, p_2, T_h) \equiv P(X \cup Y)\) that a randomly chosen node \(i\) is inactive

\[
a = p + p_2(1-p)\sum_k P(k)E(k, m, a),
\]

equal to the fraction of inactive nodes, \(a = 1 - \langle f_n \rangle\). Clearly, the internal and external failures are only approximately independent [27]. Here \(P(k)\) is the degree distribution, \(T_h\), \(p\), and \(p_2\) are described in (i)–(ii) above, \(m \equiv T_hk, E(k, m, a) \equiv \sum_{j=0}^{m}a^j(1-a)^{k-j}\) is the probability that the neighborhood of node \(i\) is critically damaged. For a network with \(N\) nodes, each with probability \((1-a)\) of being active, using a binomial distribution we obtain any moment of \(f_n\) of order \(q\),

\[
\langle f_n^q \rangle \equiv \sum_{j=0}^{N}(\frac{j}{N})^q a^{N-j}(1-a)^j \left(\frac{N}{j}\right),
\]

that is, for large values of \(N\), \(\langle f_n^q \rangle \approx \int dx x^q G(x, \mu = 1-a, \sigma = \sqrt{a(1-a)/N})\)—\(G\) stands for Gaussian. The dependence of \(\langle f_n^q \rangle\) on \(a\) explains why both \(\langle f_n \rangle\) and \(\sigma_n\) in Fig. [I] show discontinuities for the same \(p_2\) values.

We next use the diagnostic power of the (i)-(iii) network to quantify the level of its stability. Using the first two moments of Eq. (3), we define network risk (volatility) as \(\sigma_n \equiv \sqrt{\langle f_n^2 \rangle - \langle f_n \rangle^2}\). Because a network is more stable when \(f_n(t)\) is less volatile \((\sigma_n \to 0)\) and when \(\langle f_n \rangle\) is as close to 1 as possible, we propose another network stability measure, the stability network ratio,

\[
\langle f_n \rangle / \sigma_n,
\]

where the larger the ratio, the more stable the network. Figure [I(a)] shows that for a (i)–(iii) network the ratio exhibits hysteresis behavior, e.g., with increasing instability \((p_2 \to 1)\), \(\langle f_n \rangle / \sigma_n\) decreases. When \(N\) is large, \(\langle f_n \rangle / \sigma_n = \sqrt{(1-a)N/a}\) [see Eq. (3)]. In practice, if two networks have equal \(\langle f_n \rangle\), but different \(\sigma_n\), the one with the larger ratio is more stable. Note that a similar first-to-second moment of a price return is proposed in finance to quantify the performance of a financial asset [30]. Similar signal-to-noise ratio defined as the ratio of mean to standard deviation of a signal is used widely in science and engineering [31].

In addition to estimating network volatility, we also need to forecast, having the initial configuration of active nodes, how many nodes will have failed at any future time \(t\). We allow the (i)–(iii) network in Fig. [I(c)],
initially at stable state $I_0$, to move $\delta$ steps (with $p_1$ changing linearly) to a highly volatile phase-flipping state $X_A$. Figure 2(b) shows a representative $f_n(t)$. We always start from the same initial $I_0$, perform a large number of simulations [see Fig. 2(c)], and obtain the conditional distribution function (cdf), $C(f_n)$, from which we calculate the probability ($f_{0.01q}^1 C(f_n)df_n$) that no more than $q\%$ nodes will be inactive at $t = 2\delta$. In finance, this probability approximates the risk that a substantial fraction of financial system will collapse, the so-called “systemic risk” [22].

When we use $f_n$ we are assuming that every node is equally important. This is frequently not hold for real-world networks [2, 3, 33], e.g., when large banks become dysfunctional they affect the overall financial network much more than dysfunctioning small banks. In the (i)–(iii) network the importance of each node is governed by network topology—the time-dependent node degree, $k(t)$. A randomly chosen link is active if both its nodes are active and so the probability that the link is active is $(1 - a)^2$. The average number of active links is

$$\langle L \rangle = (1 - a)^2 L_T,$$

where $L_T \equiv 1/2\Sigma_{i=1}^N k_i$ denotes the total number of links when all links are active. Similar to Eq. (3) for a network with $L_T$ links, each with probability $u = (1 - a)^2$ of being active, a $q$–order moment of $f(t) = L(t)/L_T$—the fraction of active links—is

$$\langle f(t)^q \rangle = \Sigma_{j=0}^{L_T} \frac{j}{L_T}^q u^j (1 - u)^{L_T - j} \left(\frac{L_T}{j}\right).$$

Figure 2(b) shows a representative $f(t)$, and Fig. 2(c) shows $C(f(t))$ (broader than $C(f_s)$) from which we can calculate the probability ($f_{0.01q}^1 C(f_s)df_s$) that no more than $q\%$ of links will be inactive at $t$. Figure 2(d) shows the pdf of the relative change in $L(t)$ and its exponential fit—which is potentially important information for network management. Note that $L(t) = L_T[1 - f(t)]$ denotes the loss of a network's links. Using Eq. (5) we obtain $(L) \equiv L_T - \langle L \rangle = a(2 - a)L_T$.

The (i)–(iii) network model offers one more potentially important forecasting property. Suppose a network set in a state $X_B$ (see Fig. 1(c)) within the hysteresis regime is predominantly inactive. Reference 22 defines a local time-dependent parameter $p_{2,\lambda}(t) = \frac{1}{N}\Sigma_{i=1}^N p_{2,\lambda}(t + 1 - i)$ as the average fraction of externally failed nodes over the most recent interval of length $\lambda$. When $p_{2,\lambda}(t)$ crosses the “left” spinodal, the network shifts from the inactive phase II to the active phase I. Similarly, $p_{1,\lambda}(t) = \frac{1}{N}\Sigma_{i=1}^N p_{1,\lambda}(t + 1 - i)$. In Ref. [25] the pdf of $p_{2,\lambda}(t)$ ($p_{1,\lambda}(t)$) determines the average lifetime of the system in I and II. Here we find that $p_{2}(t)$ follows a binomial distribution that can be approximated for large samples $n$ with the normal distribution $N(\mu = p_{2}, \sigma^2 = p_{2}(1 - p_{2})/n) = P(2p(t)) \sim \exp[-n(p_{2}(t) - p_{2})^2/(2p_{2}(1 - p_{2}))^2]$, where $n = NE(\alpha_2, p_{2}, k, m)$ [see Eq. (2)]. From $p_{2,\lambda}(t) = \frac{1}{N}\Sigma_{i=1}^{\lambda} p_{2,\lambda+1-i}$ we easily derive $p_{2,\lambda}(t) = p_{2}(t)/\lambda + p_{2,\lambda}(t - 1) - p_{2}(t - \lambda)/\lambda$.

Thus, having information about the previous $p_{2,\lambda}$, $p_{2,\lambda}(t - 1)$, we can forecast the current value, where the
closer $p_{2,\lambda}$ is to a spinodal, the larger the probability that the phase will flip. We quantify this probability using the conditional distribution function (cdf) $C(p_{2,\lambda}(t)) \sim \exp\left[-\frac{N\lambda^2 E(a(p_1p_2),k,m)(p_{2,\lambda}(t)-p_2(t-\lambda))}{2p_2(1-p_2)}\right]$.

This probability can be used to estimate, given the most recent local state $p_{2,\lambda}(t-1)$ and $p_2(t-\lambda)$, the probability $P(x \leq p_{2,\lambda}|p_{2,\lambda}(t-1),p_2(t-\lambda))$ that the network will move from being predominantly inactive, II, to predominantly active, I—here, as in [25], $p_{2s}$ is a spinodal value where the network phase-flips from II to I [Fig. 1(c)]. Similarly, if $p_{1s}$ defines a spinodal value at which the network phase-flips from phase I to II [Fig. 1(c)], from the cdf $C(p_{1,\lambda}(t)) \sim \exp\left[-\frac{N\lambda^2(p_{1,\lambda}(t)-p_{1,\lambda}(t-1)+p_1(t-\lambda)-p_{1,\lambda}(t-\lambda))}{2p_1(1-p_1)}\right]$ we can estimate the probability $P(x \geq p_{1s}|p_{1,\lambda}(t-1),p_1(t-\lambda))$ that the network will fall into the mainly inactive phase (e.g., as in an economic recession) within the next period.

Finally we examine the emerging hysteresis in correlations of $f_n(t)$ due to network dysfunctionality at the node level. For each time series $f_n(t)$ ($\tau = \text{const}$) in Figs. 4(a) and 4(b) we apply detrended fluctuation analysis (DFA) $F^2(t) \propto t^{2\alpha}$. Figure 3(a) shows that $f_n(t)$ exhibits finite-range correlations of the random-walk type ($\alpha \approx 1.5$) with a clear first-order phase transition, in which a sudden change in the correlation exponent $\alpha$ occurs when $p_2$ approaches the value at which we expect network collapse (see Fig. 1). An approximate explanation of the correlations in $f_n(t)$ is that correlations in $f_n(t)$ and its hysteresis behavior are due to correlations in the fractions of externally failed nodes $p_2(t)$ and internally failed nodes $p_1(t)$. Figure 3(a) confirms this assumption by showing only correlations in $p_2(t)$. The existence of hysteresis [27] in Fig. 3(b) indicates that the correlations in collective modes are not the same when the network approaches network collapse and when the network recovers—if, e.g., our network models the global economy, then when the economy moves from “bad” to “good” years, “good” years are never as good as the previous “good” years.

To demonstrate the utility of the (i)–(iii) network model when analyzing real-world networks, we first analyze a small economic network of 19 developed countries [35], and use an output measurement of trading dynamics, per capita gross domestic product—$\text{gdp}$. For each country and for each year $t$ between 1870 and 2012 [36], a country (node) is active if the $\text{gdp}$ growth is non-negative (if it has been a “good” year). Figure 4(a) shows the fraction of active countries $f_n(t)$ that are becoming increasingly interdependent due to globalization [the non-stationary analyzed in Fig. 2(b)]. When we disregard this non-stationarity we find model parameters ($p = 0.082 \pm 0.02$, $p_2 = 0.77 \pm 0.03$, $\tau = 1.33 \pm 0.5$, and $T_h = 56 \pm 3\%$) for which the $\langle f_n \rangle$ of our model and the second network moment $\sigma_n$ best fit the empirical moments. Figure 4(b) shows the $f_n(t)$ of the model. From $p_1 = 1 - \exp(-p\tau)$ [25], $p_1 = 0.103$ suggests that any randomly chosen developed country will experience recession (failure) approximately every ten years, since $p_1$ represents the average fraction of internally failed nodes [25]. The parameter $p_2 = 0.77$ means that there is an $\approx 77\%$ probability that a country will undergo recession if its trading partners have recently experienced recession. Figure 4(c) shows the hysteresis [27] in $(p_1, p_2)$ space for the (i)–(iii) network model with $\tau = 2$ and $T_h = 50\%$. We find that developed countries with $p_1 = 0.106 \pm 0.01$, $p_2 = 0.82 \pm 0.02$ lie close to a critical hysteresis line—an indication that the world economy is highly unstable. We next analyze $f_n(t)$ for 23 Latin American countries, 25 EU countries, and 25 Asian countries for each year since 1980. We calculate the network stability ratio $\frac{\langle f_n \rangle}{\sigma_n}$ of Eq. (4) for each group and obtain the values 3.25, 4.15, and 6.95, implying that Asian countries are best performers.

We next analyze the airport traffic network [37] in the Northeastern United States (The Library of Congress definition) comprising 66 airports (nodes), and we consider only those flights (links) within the Northeast. For each day during the period 6/1/2012 – 5/31/2013 we calculate the fraction of failed airports $1 - f_n(t)$. We arbitrarily define failed airports as those in which more than $T_h = 40\%$ flights have been canceled for the day. The air traffic network in Fig. 4(d) shows a much higher level of nodes’s stability than is typical of economic networks—rarely does $1 - f_n(t)$ drop to $40\%$. Since air traffic
network is known to be scale-free \cite{22}, we apply the (i)–(iii) network model to fit the empirical $\langle f_n \rangle$ and $\sigma_n$ data to the model's parameters—\( p = 0.011 \pm 0.003 \) and \( p_2 = 0.92 \pm 0.03 \). Note that in air traffic network although it is common for links to fail (for flights to be canceled), airports still function properly. This implies that the (i)–(iii) dynamic network model could be extended to introduce item (iv), the probability, \( p_3 \), that each link can fail.

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