Hydraulic Potential Energy Model for Hydropower Operation in Mixed Reservoir Systems

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Abstract The forecast-informed hydropower operation for mixed reservoir systems, which consist of parallel and cascade reservoirs, is of considerable importance in practice; however, this operation still lacks an analytical basis in theory. From the perspective of energy, this paper introduces the concept of "hydraulic potential energy" and mathematically derives the energy transformation formula for multi-reservoir hydropower operation. Based on this formula and the rolling single-period forecast, a maximum hydraulic potential energy model (E1 model) is proposed. The presented rigorous proofs demonstrate the deficiencies of the commonly considered objectives or principles of minimizing the outflow-induced energy-cost, while the objective function of the E1 model is demonstrated to be superior. If the constraints of power output and reservoir storage are nonbinding, the derived optimal spatial principle for hydropower operation is (1) to equalize the Relative Marginal Energy (RME) among reservoirs or (2) if this status is not feasible, to release water and generate hydropower first from the reservoirs that have the largest RME values. Considering the uncertainty of future inflow, the E1 model is extended to the two-stage hydraulic potential energy model (E2 model). A case study of a hypothetical mixed three-reservoir system demonstrates the superior performance of the E2 model compared with the conventional K-value principle and the minimum energy-cost model. This paper provides an innovative spatial principle and a practical model for the realization of forecast-informed hydropower operation, which contributes to the optimization of the large-scale energy market.

1. Introduction

With the decelerating construction of large-scale water storage facilities in developing and developed countries (MWR, 2013; WCD, 2000), the integrated operation of multiple reservoirs has been a growing concern for maintaining the operational effectiveness and maximizing the benefits (Labadie, 2004). Second only to the dams for irrigation purposes, hydropower dams are extremely common, which comprise 20% of the large dams worldwide (Herschy, 2012). Furthermore, the reservoir systems can be designed and operated primarily for hydropower generation in some mountainous regions (Zeng et al., 2013). The hydropower operation is a complex nonlinear problem due to the existence of the hydraulic head effects (i.e., the relationship between water level and reservoir storage/release) and the complementary effect (i.e., the release becomes more productive with increasing storage) (Cheng et al., 2008; Zeng et al., 2013). This operation is further complicated by the competition and cooperation among parallel reservoirs and the close hydraulic linkages between upstream-downstream reservoirs (Balibar, 2017; Labadie, 2004; Nathan, 1982).

Given the intricate nature of the multi-reservoir hydropower operation, this issue has been discussed for decades by many researchers, see Yeh (1985), Labadie (2004), or de Queiroz (2016) for reviews. The majority of recent studies concentrated on developing the optimization methods and technical algorithms (Lee & Labadie, 2007; Lima et al., 2013; Liu et al., 2018). Deterministic optimization and stochastic optimization are two basic techniques applied to formulate long-term guidelines or short-term decisions. The deterministic optimization is used to develop operation rules or make decisions in a sense that future inflows are represented by historical long-term observations or synthetically generated series, or the assumed facts known in detail in advance (e.g., Cheng et al., 2008; Keppo, 2007). The stochastic optimization involves the realization of the operation under the probabilistic descriptions of random streamflow processes without the presumption of perfect foreknowledge of future inflows (e.g., Lee & Labadie, 2007; Maria & Mario, 1996). The objective functions of these optimization techniques are typically set as maximizing the (expected) energy...
production or utilities from the production, or minimizing the (expected) cost of satisfying the energy demand, over time-horizon (Liu et al., 2018; Sasireka & Neelakantan, 2017). Until now, extensive optimization algorithms and applications have been carried out to retrieve the (near) optimal solutions, see Ahmad et al. (2014) for a recent review. Despite intensive studies on the application of these optimization techniques, a considerable gap still exists between research and practice, especially in the operations with rolling forecasts (Labadie, 2004). The main reasons for this discrepancy are as follows: (1) Many reservoir system operators are skeptical regarding the retrieved results due to the insufficient background of the physical mechanisms involved in the optimization. (2) To optimize the operation of large hydropower systems, mathematical manipulations must be performed, which often incur a high computational cost (e.g., the curse of dimensionality); however, such manipulations cannot always guarantee the optimality of the solutions (e.g., convergence to the local optimum).

An alternative option to cope with such a complex large hydropower operation problem is to identify a better approach of supplying the energy with forecasting. A few theoretical studies have shown that investigating the properties of hydropower systems is fundamental to reform optimization models, which is helpful to explore the optimal rules and efficient algorithms (e.g., Zhao et al., 2014). Nicolaos and Jakob (1970) and later (Turgeon, 1980) simplified the hydropower system into one equivalent reservoir by aggregating the storage and inflow of the individual reservoirs in units of energy, which effectively surmounted the dimensionality problem in both long-term planning and short-term scheduling processes. Teegavarapu and Simonovic (2000) and Mo et al. (2013) proposed short-term operation models that minimize the system energy loss/consumption by water release. A real-world case study showed that, in general, the solution of maximizing the stored energy was the best short-term strategy (Xu et al., 2015). Accordingly, from the perspective of these energy-based targets, the typical hydropower operation rules such as the K-value principle (Wang et al., 2014) and storage effectiveness index rule (Lund, 1996; Lund & Guzman, 1999) were derived. The objective of maximizing the stored energy in reservoirs has long been regarded as equivalent to minimizing the released energy target (Piekutowski et al., 1993). Unfortunately, the energy principles used in these methods are derived from direct but empirical targets, which necessitates more elaborate theoretical analyses and discussions. Thus, it is imperative to formulate a systematic mathematical description for understanding the principles of mixed hydropower systems, which will facilitate the development of more efficient operation methods. To fill this gap, this paper aims to derive the fundamental spatial principles for hydropower operation in mixed reservoir systems and propose new energy-based operation models that can be the representative of both short-term (i.e., hourly or daily) and medium-term (i.e., weekly or monthly) operations.

This paper is organized as follows: Section 2 introduces the concept of hydraulic potential energy and elaborates the internal relationship among energy forms. Section 3 describes the theoretical maximum hydraulic potential energy model (E1 model) and mathematically derives several important properties; in addition, the optimal spatial principle for mixed reservoir hydropower operation is described, which is followed by the proposal of the two-stage hydraulic potential energy model (E2 model) for practical use. Section 4 evaluates the operational performance of the practical E2 model using a hypothetical three-reservoir case study. Finally, the conclusions are drawn in Section 5.

2. Problem Formulation

For the remainder of this work, the following notations are used:

Notation 1. We denote by \( S = \{1, 2, \ldots, n\} \) the set of reservoir (and power plant) indices within a mixed \( n \)-reservoir system. For convenience, let \( n \) be the index of the most downstream reservoir;

Notation 2. We define a partial order “\( \leq \)" over the set \( S \): we say that \( j \leq i \) if either \( j = i \) or the release from reservoir \( j \) can be recaptured by reservoir \( i \);

Notation 3. Set \( S_{ij} := \{ j \in S | j \leq i \} \) be the set of reservoirs in the upstream of reservoir \( i \) (including reservoir \( i \) itself); \( S_{ji} := \{ j \in S | i \leq j \} \) be the set of reservoirs in the downstream of reservoir \( i \) (including reservoir \( i \)); \( S_{s1} := \{ j \in S | j \leq i \} \) be the set of reservoirs in the downstream of reservoir \( i \) (excluding reservoir \( i \));
Notation 4. We denote by $S(i)$ the set of reservoirs that have hydraulic connection with reservoir $i$, namely $S(i) = S_{\leq i} \cup S_{\geq i}$.

Notation 5. We denote by $S_{up(i)}$ the set of upstream reservoirs immediately adjacent to reservoir $i$. Clearly, $S_{up(i)} \subseteq S_{\geq i}$.

For example, consider the following four-reservoir system (Figure 1). With these notations, we have $S = \{1,2,3,4\}$, $S_{\leq 4} = \{1,2,3,4\}$, $S_{\geq 3} = \{3\}$, $S_{\leq 2} = \{1,2\}$, and $S_{\leq 1} = \{1\}$. We also have $S_{\geq 2} = \{2,4\}$, $S_{\geq 2} = \{4\}$, $S_{\geq 4} = \emptyset$, $S_{\geq 2} = \{1,2,4\}$, and $S_{up(4)} = \{2,3\}$.

### 2.1. Formulation of the Multi-Reservoir Hydropower System Operation

The operation of the multi-reservoir hydropower system must balance the release of multiple periods and multiple reservoirs to maximize the power generation. The release in a single period yields hydropower and affects the carryover storage of the reservoir and those of downstream reservoirs. Furthermore, the release from a single reservoir is not always sufficient for satisfying the system hydropower requirements. Thus, a joint operation is deemed essential.

In this paper, $V_i^t$ denotes the active storage above the reservoir dead storage $V_{i,\text{min}}^t$. The system dynamics can be described by the water conservation equation

$$V_i^t = V_{i-1}^t + I_i^t - R_i^t - W_i^t + \sum_{k \in S_{up(i)}} R_k^t + \sum_{k \in S(i)} W_k^t \quad (i \in S)$$

(1)

where $V_{i-1}^t$ and $V_i^t$ are the beginning and ending active storages of reservoir $i \in S = \{1,2,...,n\}$ in time period $t$ (m$^3$/s/Δt is considered to harmonize the measurement unit, where Δt is the operation time interval); $I_i^t$ is the external (intermediate) inflow that arises from rain and tributary streams that are expected to flow into reservoir $i$ during period $t$ (m$^3$/s), as represented by forecasts; $R_i^t$ and $W_i^t$ are the power release (m$^3$/s) and non-power release (water spill, m$^3$/s), respectively, namely, the sum ($R_i^t + W_i^t$) is the reservoir total outflow; $\Delta_i^t$ is the evaporation, which is a function of the average reservoir surface area and, thus, is expressed as a function of the reservoir average storage; and $S_{up(i)}$ is the set of reservoirs in the upstream region that are immediately adjacent to reservoir $i$ (see Notation 5). This equation shows that the outflows from the upstream reservoirs turn into the inflows into the next downstream reservoirs.

The reservoir hydraulic head varies with the reservoir storage and the tailrace water level (Bayón et al., 2009; Zhao et al., 2014). The hydraulic head of reservoir $i$ at the end of period $t$ and the average hydraulic head along period $t$ are

$$H_i^t(V_i^t; R_i^t; W_i^t) = H_{up}^t(V_i^t) - H_{down}^t(R_i^t + W_i^t)$$

(2)

$$\overline{H}(V_i^t; R_i^t; W_i^t) = \frac{1}{2} \left[ H_{up}^t(V_{i-1}^t) + H_{up}^t(V_i^t) \right] - H_{down}^t(R_i^t + W_i^t)$$

(3)

where $H_{up}^t(V_i^t)$ and $H_{down}^t(R_i^t + W_i^t)$ are the reservoir elevation-storage function and the tailrace elevation-outflow function, respectively. The power output of reservoir $i$ during period $t$ depends bi-linearly on the hydraulic head and power release

$$P_i^t = \eta_i^t R_i^t \overline{H}(V_i^t, R_i^t, W_i^t)$$

(4)

where $\eta_i^t$ is a constant factor that represents the efficiency of power generation.

In the hydropower system, the following constraints must also be considered: the lower and upper limits on reservoir storage ($V_{i,\text{min}}^t$, $V_{i,\text{max}}^t$), power release ($R_{i,\text{min}}^t$, $R_{i,\text{max}}^t$), power output ($P_{i,\text{min}}^t$, $P_{i,\text{max}}^t$), reservoir system firm power output ($P_{\text{min,sys}}$), and nonnegative spill ($W_i^t$). These constraints can be defined as follows

$$0 \leq V_i^t \leq V_{i,\text{max}}^t - V_{i,\text{min}}^t = V_{i,\text{max}}^t$$

(5)

$$R_{i,\text{min}}^t \leq R_i^t \leq R_{i,\text{max}}^t$$

(6)
Therefore, the hydraulic potential energy at the end of period 
energy is mathematically identical to maximizing the gravitational potential energy.

This substitution is mathematically reasonable if the focus is on searching for the extreme values in hydro-

tical centroid of the reservoir waterbody that is above the dead storage; and

The determination of the centroid is complicated, which renders further theoretical analysis dif-

The above equations indicate that the hydropower operation is clearly distinct from the water supply opera-

Nevertheless, the mathematical basis of this representation is not clear.

In this study, we propose a new representation as follows. According to the concept of gravitational potential

Equation 1, and Equations 5–9 are the basic constraint sets of the mixed reservoir hydropower systems, in
which \( V^f_i, R^f_i, \) and \( W^f_i, \ i \in S \) are the decision variables of the current period to be considered. The following
analyses and discussions are based upon these equations. For conciseness, from now on, we write the head
functions \( H^f_i \) and \( H^f \left(V^f_i, R^f_i, W^f_i \right) \) instead of \( H^f \left(V^f_i, R^f_i, W^f_i \right) \).

2.2. Definition of the Hydraulic Potential Energy

The above equations indicate that the hydropower operation is clearly distinct from the water supply opera-
tion, since the former contains the process of energy transfer besides water quantity. Nicolaos and
Jakob (1970) first proposed the idea of “potential energy” of the hydropower system by converting the reser-
voir water stores into their potential energy equivalents and formulating a one-dam model. The follow-up
researchers further clarified the potential energy by estimating the hydropower produced by the complete
depletion of reservoir for given storages (Becker & Yeh, 1974; Secundino & Adriano, 1993; Terry et al., 1986).

In this study, we propose a new representation as follows. According to the concept of gravitational potential
energy in physics, the stored energy depends on the mass and the centroid distance between the upstream
and downstream water bodies. If there is only one reservoir with a storage \( V_i \), the gravitational potential
energy \( U \) is

where \( \rho \) is the density; \( k \) is the storage unit conversion factor; \( g \) is the local gravitational field; \( z_i \) is the geo-
metric centroid of the reservoir waterbody that is above the dead storage; and \( H_{\text{min}} \) is a constant value that
represents the reservoir minimum hydraulic head.

The determination of the centroid is complicated, which renders further theoretical analysis difficult. To
make the mathematical deduction clearer and more concise, this paper transforms the potential energy by
substituting the centroid distance \( (z_i + H_{\text{min}}) \) by the effective hydraulic head \( H_i = h_i + H_{\text{min}} \), where \( h_i \) is the
depth of the reservoir storage), and substituting the constant term \( (\rho kg) \) by the power generation efficiency
(\( \eta \)). Since this simplified term no longer represents the strict potential energy but is a type of power output in
the hydraulic system, we define this term as the “hydraulic potential energy”

This substitution is mathematically reasonable if the focus is on searching for the extreme values in hydropower operation because for any given reservoir, the centroid \( z_i \) is a monotonically increasing function of the
depth \( h_i \), namely, \( h_{11} < h_{12} \iff z_{11} < f(h_{11}) < z_{12} = f(h_{12}) \). The transition function \( f \) depends on the shape of the
reservoir waterbody, which can typically be obtained via regression based on field measurements (a special
case is presented in Appendix I, in which the function \( f \) can be directly derived by approximating the shape of
the waterbody as an inverted pyramidal frustum shape). Therefore, maximizing the hydraulic potential
energy is mathematically identical to maximizing the gravitational potential energy.

Therefore, the hydraulic potential energy at the end of period \( t \) for the complete system is as follows

where \( S_{\text{down}} \) is the set denoting reservoirs (power plants) downstream of reservoir \( i \) (including reservoir \( i \), see
Notation 3). The cumulative effective hydraulic head \( \left( \sum_{j \in S_{\text{down}}} \eta H^f_j \right) \) is adopted because reservoirs in series
allow downstream reservoirs to recapture/reuse the upstream release.
It is natural to generalize the concept of the hydraulic potential energy for various contexts. We define the inflow-hydraulic potential energy as \( \varphi_i^I \), estimating the additional time average of the hydraulic potential energy into the reservoir system by the inflow over time period \( t \).

\[
\varphi_i^I = \sum_{i \in S} \sum_{j \in S_{ij}} \eta_j \varphi_i^I
\]  

(13)

Similarly, we define the power-release-hydraulic potential energy as \( \varphi_i^R \), the spilled-water-hydraulic potential energy as \( \varphi_i^W \), and the evaporation-hydraulic potential energy as \( \varphi_i^E \).

\[
\varphi_i^R = \sum_{i \in S} \eta_i^R \varphi_i^R
\]  

(14)

\[
\varphi_i^W = \sum_{i \in S} \eta_i^W \varphi_i^W
\]  

(15)

\[
\varphi_i^E = \sum_{i \in S} \eta_i^E \varphi_i^E
\]  

(16)

These terms estimate the hydraulic potential energy consumption, waste, and losses from the system by power release, non-power release (water spill), and evaporation, respectively. Note that the reservoir outflow from reservoir \( i \) alone \((R_i + W_i)\) can further be reused by downstream reservoirs and, thus, these energy losses through the release are only related to each reservoir itself. For the sake of brevity, we assume that both \( \eta_i^R \) and \( \varphi_i^E \) are functions of the reservoir initial storages \((\varphi_i^E = \sum_{j \in S} \eta_i^E \sum_{j \in S_{ij}} \eta_j \varphi_i^E)\). The variables of inflow, release and hydraulic head are presented in the average conditions rather than as an integral over time interval because (1) for short-term operation, the variance of inflow/outflow can be overlooked due to the relatively short time step; (2) for medium- or long-term operation, the integration of instantaneous values is overly complicated for further analysis, thereby requiring a simplification. This is a common manipulation when calculating the hydropower production or energy losses (Feltenmark & Lindberg, 1997; Xu et al., 2015).

Subsequently, we define the system hydraulic potential energy increment as \( \varphi_i^V \), which accounts for the impacts of changing storages and hydraulic heads between the beginning and end of period \( t \).

\[
\varphi_i^V = \sum_{i \in S} V_i^{t-1} \sum_{j \in S_{ij}} \eta_j H_i^t - \sum_{i \in S} V_i^t \sum_{j \in S_{ij}} \eta_j H_i^{t-1} - \sum_{i \in S} (V_i^t - V_i^{t-1}) \sum_{j \in S_{ij}} \eta_j \varphi_i^t
\]  

(17)

In particular, if the reservoir head function can be piecewise-linearized, namely, if \( \varphi_i^t = (H_i^t + H_i^{t-1})/2 \), Equation 17 becomes

\[
\varphi_i^V = \sum_{i \in S} V_i^{t-1} \sum_{j \in S_{ij}} \eta_j \left( H_i^t - H_i^{t-1} \right)/2 + \sum_{i \in S} V_i^t \sum_{j \in S_{ij}} \eta_j \left( H_i^t - H_i^{t-1} \right)/2
\]

\[= \sum_{i \in S} \left\{ \left[ (V_i^{t-1} + V_i^t)/2 \right] \sum_{j \in S_{ij}} \eta_j \left( H_i^t - H_i^{t-1} \right) \right\} \]

\[= \sum_{i \in S} V_i^t \sum_{j \in S_{ij}} \eta_j \Delta H_i^t\]

where \( \sum_{j \in S_{ij}} \eta_j \Delta H_i^t \) is the cumulative increment of the downstream hydraulic heads.

Theorem 1. (Energy Transformation Formula) The following identity holds

\[
\varphi_i = \varphi_i^{t+1} + \varphi_i^I - \varphi_i^R - \varphi_i^W - \varphi_i^E + \varphi_i^V
\]  

(18)

where the hydraulic potential energies are defined as in Equations 11–17.
Proof  For any reservoir $i \in S$, the water balance equation is as shown in Equation 1. On both sides of the equation, we multiply the accumulative hydraulic heads ($\sum_{j \in S} \eta \overline{H}_t^j$). Then we obtain $n$ equations, combining which yields

$$
\sum_{i \in S} V_i^t \sum_{j \in S_{2i}} \eta \overline{H}_t^j = \sum_{i \in S} V_{i-1}^t \sum_{j \in S_{2i}} \eta \overline{H}_{t-1}^j + \sum_{i \in S} \sum_{j \in S_{2i}} \eta \overline{H}_t^j
$$

$$
- \sum_{i \in S} R_i^t \sum_{j \in S_{2i}} \eta \overline{H}_t^j - \sum_{i \in S} W_i^t \sum_{j \in S_{2i}} \eta \overline{H}_t^j
$$

$$
+ \sum_{i \in S} \sum_{j \in S_{2i}} R_i^t \eta \overline{H}_t^j + \sum_{i \in S} \sum_{j \in S_{2i}} W_i^t \sum_{j \in S_{2i}} \eta \overline{H}_t^j = 0
$$

A straightforward calculation shows the following identity

$$
\sum_{i \in S} V_i^t \sum_{j \in S_{2i}} \eta \overline{H}_t^j = \sum_{i \in S} V_{i-1}^t \sum_{j \in S_{2i}} \eta \overline{H}_{t-1}^j + \sum_{i \in S} \sum_{j \in S_{2i}} \eta \overline{H}_t^j
$$

$$
- \sum_{i \in S} R_i^t \eta \overline{H}_t^j - \sum_{i \in S} W_i^t \eta \overline{H}_t^j - \sum_{i \in S} \sum_{j \in S_{2i}} \eta \overline{H}_t^j
$$

$$
+ \sum_{i \in S} \sum_{j \in S_{2i}} R_i^t \eta \overline{H}_t^j + \sum_{i \in S} \sum_{j \in S_{2i}} W_i^t \sum_{j \in S_{2i}} \eta \overline{H}_t^j = 0
$$

Hence, we have

$$
\sum_{i \in S} V_i^t \sum_{j \in S_{2i}} \eta \overline{H}_t^j = \sum_{i \in S} V_{i-1}^t \sum_{j \in S_{2i}} \eta \overline{H}_{t-1}^j + \sum_{i \in S} \sum_{j \in S_{2i}} \eta \overline{H}_t^j
$$

$$
- \sum_{i \in S} R_i^t \eta \overline{H}_t^j - \sum_{i \in S} W_i^t \eta \overline{H}_t^j - \sum_{i \in S} \sum_{j \in S_{2i}} \eta \overline{H}_t^j
$$

$$
+ \sum_{i \in S} \sum_{j \in S_{2i}} R_i^t \eta \overline{H}_t^j + \sum_{i \in S} \sum_{j \in S_{2i}} W_i^t \sum_{j \in S_{2i}} \eta \overline{H}_t^j = 0
$$

Adding the term $\sum_{i \in S} V_i^t \sum_{j \in S_{2i}} \eta \overline{H}_t^j + \sum_{i \in S} V_{i-1}^t \sum_{j \in S_{2i}} \eta \overline{H}_{t-1}^j$ to both sides of Equation 19, we get

$$
\sum_{i \in S} V_i^t \sum_{j \in S_{2i}} \eta \overline{H}_t^j = \sum_{i \in S} V_{i-1}^t \sum_{j \in S_{2i}} \eta \overline{H}_{t-1}^j + \sum_{i \in S} \sum_{j \in S_{2i}} \eta \overline{H}_t^j
$$

$$
- \sum_{i \in S} R_i^t \eta \overline{H}_t^j - \sum_{i \in S} W_i^t \eta \overline{H}_t^j - \sum_{i \in S} \sum_{j \in S_{2i}} \eta \overline{H}_t^j
$$

$$
+ \sum_{i \in S} \sum_{j \in S_{2i}} R_i^t \eta \overline{H}_t^j + \sum_{i \in S} \sum_{j \in S_{2i}} W_i^t \sum_{j \in S_{2i}} \eta \overline{H}_t^j
$$

The above identity coincides with Equation 18. Hence, Theorem 6 is proved. □

This formula expresses the transformation relationship of the various hydraulic potential energy types in period $t$ for mixed hydropower systems, thereby referred to as the “energy transformation formula”.

3. Hydraulic Potential Energy Models and the Optimal Spatial Principle

3.1. Hydraulic Potential Energy Model

The concept of the hydraulic potential energy offers a unified definition of both energies stored and lost. In this sense, Theorem 6 reflects the energy conversion and conservation law among reservoirs and operation periods. According to the energy transformation formula, by maximizing the hydraulic potential energy at the end of period $t$, it is possible to functionally realize the objective of minimizing the energy losses by the outflow. Intuitively, maximizing the hydraulic potential energy additionally allocates an energy storage; consequently, more potential energy is retained for future operation (a rigorous proof of this statement is provided in Section 3.2).

Therefore, mathematically, the operational objective for mixed hydropower systems that are based on rolling forecasts is to maximize the system carryover hydraulic potential energy, subject to a variety of physical and economic constraints (hereinafter referred to as the “E1 model”).
Lemma 2. For any $i \in S$, let $F_P = \left\{ R_i^t \mid R_{i_{\text{min}}.t} \leq R_{i_{\text{max}}.t} \leq P_{\text{max}.t} \right\}$, $F_R = \left\{ R_i^t \mid R_{i_{\text{min}}.t} \leq R_{i_{\text{max}}.t} \leq R_{i_{\text{id}}.t} \right\}$. $F_P$ is a subset of $F_R$.

Proof. See Appendix III. 

3.2. Properties of the E1 Model

In this section, we focus on the characteristics of the E1 model. More concretely, we prove that

1. The optimal path of the E1 model decision variables $X_t = \{V_t^i, R_t^i, W_t^i\}$, $i \in S$ depends only on the independent variables $V_t^i$, $i \in S$ (see Remark 15);
2. The optimal solution of the E1 model is a reasonable approximation to the optimal strategy of maintaining the overall potential energy (see Remark 18);
3. If head function in Equation 2 can be approximated with concave functions, the optimal solution to the E1 model is unique (see Proposition 19).

Lemma 1. For any $i \in S$, the head function $H_t(V_t^i, R_t^i, W_t^i)$ is an increasing function of the decision variables $V_t^i$, $i \in S$.

Proof. See Appendix II.

Similarly, we can prove that this conclusion also holds for $H_t^{*}$.

Lemma 2. For any $i \in S$, let $F_P = \left\{ R_i^t \mid R_{i_{\text{min}}.t} \leq R_{i_{\text{max}}.t} \leq P_{\text{max}.t} \right\}$, $F_R = \left\{ R_i^t \mid R_{i_{\text{min}}.t} \leq R_{i_{\text{max}}.t} \leq R_{i_{\text{id}}.t} \right\}$. $F_P$ is a subset of $F_R$.

Proof. See Appendix III.

Since $F_P$, $F_R$, the constraints of the power release and power output in Equations 6–7 are equivalent to $F_P$ (Equation 7). Let $X_t^* = \{V_t^i, R_t^i, W_t^i\}$ be an optimal solution to the E1 model. We demonstrate that $X_t^*$ has the following properties.

Proposition 1. For any reservoir $i \in S$, $W_t^i = 0$ if $V_t^i < V_{i_{\text{min}}.t}$ or $P_t^* < P_{\text{max}.t}$.

Proof. Let us prove this proposition by contradiction. We need to consider the following two cases.
1. If there exists $X'_i = \left(V_i^{i*}, R_i^{i*}, W_i^{i*}\right), \ i \in S$ with $W_i^{i*} > 0$ and $V_i^{i*} < V_{\text{max},i}$ for some $i_0 \in S$, we consider the following perturbations

$$W_i^{i*} = W_i^{i*} - \varepsilon \geq 0$$
$$V_i^{i*} = V_i^{i*} + \varepsilon \leq V_{\text{max},i}$$
$$R_i^{i*} = R_i^{i*}$$

where $\varepsilon > 0$ such that $\min\left\{W_i^{i*}, V_{\text{max},i}, V_i^{i*}\right\} \geq \varepsilon$. Clearly the perturbed decision variable $X'_i = \left(V_i^{i*}, R_i^{i*}, W_i^{i*}\right), \ i \in S \left\{i_0\right\}$ still satisfies the constraints of the E1 model.

Since $X'_i$ is an optimal solution, we can assert that

$$\varphi_i(X'_i) \geq \varphi_i(X_{i*})$$

Using Lemma 9, we know that $H_i^{i*} > H_i^{i*}$; thus,

$$\phi_i > \phi_i^{i*} + \varepsilon \sum_{i \in S_{\text{no}}} \eta_i H_i^{i*} > \phi_i^{i*}$$

This leads to a contradiction with the maximality of $\phi_i^{i*}$.

2. If there exists $X'_i = \left(V_i^{i*}, R_i^{i*}, W_i^{i*}\right), \ i \in S$ with $W_i^{i*} > 0$ and $P_i^{i*} < P_{\text{max},i}$ for some $i_0 \in S$, by using Lemma 11, we get

$$R_i^{i*} = P_i^{i*} < \frac{P_{\text{max},i}}{\eta_i T_i^{i*}} \leq R_{\text{max},i}^{i*}$$

Next, we consider the following perturbation

$$W_i^{i*} = W_i^{i*} - \varepsilon \geq 0$$
$$V_i^{i*} = V_i^{i*}$$
$$R_i^{i*} = R_i^{i*} + \varepsilon \leq \frac{P_{\text{max},i}}{\eta_i T_i^{i*}}$$

where $\varepsilon > 0$ such that $\min\left\{W_i^{i*}, P_{\text{max},i}/\eta_i T_i^{i*}, V_i^{i*}\right\} \geq \varepsilon$.

Since the reservoir storage and total outflow remain unchanged, the hydraulic head should still be $H_i^{i*}$. The power output of reservoir $i_0$ is increased by $\eta_i \varepsilon T_i^{i*}$. Consequently, the power output burden of the other reservoirs $\left(\sum_{i \in S \left\{i_0\right\}} P_i\right)$ is reduced by $\eta_i \varepsilon T_i^{i*} > 0$; thus the hydraulic potential energies of these reservoirs can be further enlarged

$$\sum_{i \in S \left\{i_0\right\}} \varphi_i > \sum_{i \in S \left\{i_0\right\}} \varphi_i^{i*}$$

Therefore,

$$\phi_i = \phi_i^{i*} + \sum_{i \in S \left\{i_0\right\}} \varphi_i > \phi_i^{i*}$$

This finding also contradicts the fact that $\phi_i^{i*}$ is the optimal hydraulic potential energy.

This proves Proposition 13. ■
Proposition 13 suggests that the optimization process of the E1 model does not lead to a water spill until both the storage and power output constraints (Equations 5 and 7, respectively) for any \( i \in S \) are binding.

Remark 2. From the continuity constraint in Equation 1, it follows that the decision variables \( (V_i^t, R_i^t, W_i^t, i \in S) \) are not independent and the relationship among them can be expressed as

\[
R_i^t + W_i^t = \sum_{k \in S_i} (V_{k,i+1}^t + \ell_i - V_k^t)
\]

There exist various combinations of \( R_i^t \) and \( W_i^t \) values that can be formed when \( R_i^t + W_i^t \) is specified. Following case (ii) in Proposition 13, for any \( i \in S \), once the variables \( V_k^t, k \in S_i \) have been fixed, the optimal path of the E1 model is to release water for power generation until \( P_i^* = P_{\text{max},i} \), and only the extra water is spilled in this case. \( R_i^t \cdot \text{Path} \) and \( W_i^t \cdot \text{Path} \) denote the corresponding values of this optimal path under the fixed \( V_i^t, i \in S \).

Specifically, for fixed \( V_i^t, i \in S \),

\[
\begin{cases}
R_i^t \cdot \text{Path} = R_i^t + W_i^t; & W_i^t \cdot \text{Path} = 0 \quad \text{if } R_i^t + W_i^t \leq \frac{P_{\text{max},i}}{\eta T_i} \\
R_i^t \cdot \text{Path} = \frac{P_{\text{max},i}}{\eta T_i}; & W_i^t \cdot \text{Path} = R_i^t + W_i^t - R_i^t \cdot \text{Path} \quad \text{otherwise}
\end{cases}
\]

According to this expression, the optimal path of the E1 model decision variables always depends solely upon \( V_i^t, i \in S \). Thus, the decision variable \( X_t = (V_i^t, R_i^t, W_i^t, i \in S) \) can be simply expressed as \( X_t = (V_1^t, V_2^t, \ldots, V_n^t)^T \). The variables \( V_i^t, i \in S \) are now independent.

Proposition 2. If \( X_t^* \) satisfies identities: (i) \( V_i^* < V_i^{\text{max},i} \) for any \( i \in S \), and (ii) \( P_j^* > P_{\text{min},j} \) for all \( j \in S_{\geq i} \), the inequality in Equation 8 becomes an equality

\[
\sum_{i \in S} P_i^* = \sum_{i \in S} \eta R_i^t \cdot \text{Path} = P_{\text{min},i}
\]

Proof. Suppose the contrary proposition holds, that is, there exists \( X_t^* = (V_i^t, R_i^t, W_i^t, i \in S) \) for some \( i_0 \in S \) with

\[
\begin{align*}
V_i^* &< V_i^{\text{max},i} \\
P_i^* &> P_{\text{min},i}, \quad j \in S_{\geq i_0} \\
\sum_{i \in S} \eta R_i^t \cdot \text{Path} &> P_{\text{min},i}
\end{align*}
\]

We have proved that the optimal path of the decision variables to the E1 model relies exclusively on \( V_i^t, i \in S \) (see Remark 15). Subsequently, we consider the following perturbation

\[
V_i^{b_t} = V_i^{b_t} + \varepsilon < V_i^{\text{max},i}
\]

in which the decisions for all other reservoirs remain unchanged \( \left( V_i^t = V_i^{t*}, i \in S \setminus \{i_0\} \right) \). We know from Proposition 13 that \( W_i^{b_t} = 0 \). It follows from Remark 15 that

\[
\frac{P_{\text{min},i}}{\eta T_i} \leq R_i^t = R_i^{t*} - \varepsilon, \quad j \in S_{\geq i_0}
\]

where \( \varepsilon > 0 \) such that \( \min \left\{ V_i^{b_t} - V_i^{t*}, R_i^{t*} - P_{\text{min},i}/\eta T_i \right\} \geq \varepsilon \).
If we recall Equation 4, the power output of reservoir \( i \) is proportional to the product of reservoir release and average hydraulic head \( (P_i^t = \eta^i R_i^t H^t) \). The factors \( a^i = \frac{\partial H^t}{\partial V_i^t} \) and \( b^i = \frac{\partial H^t}{\partial R_i^t} \) denote the slopes of the head function; and the power output \( P_i^t \) depends largely on \( R_i^t \) instead of \( H^t \). It follows from Lemma 9 and condition (ii) in Proposition 13 that

\[
\begin{align*}
\phi_{\text{min}}^i & \leq P_i^t = \eta^i \left( R_i^t - \epsilon \right) \left( H^t + \alpha^i \epsilon + b^i \epsilon \right) < P_i^t^* \\
\phi_{\text{min}}^i & \leq P_i^t = \eta^i \left( R_i^t - \epsilon \right) \left( H^t + b^i \epsilon \right) < P_i^t^*, \quad j \in S_{\text{out}}
\end{align*}
\]

We thus obtain

\[
P_{\text{min}}^t \leq P_i^t < P_i^t^* = \sum_{i \in S} P_i^t^*,
\]

where \( P_i^t = P_i^t + \sum_{j \in S_{\text{out}}} P_i^t + \sum_{j \in S_{\text{in}}} P_i^t \).

In addition, the higher water storage in reservoir \( i \) yields a larger hydraulic potential energy \( (\phi_i^0(V_i^t) > \phi_i^0(V_i^t^*)) \). The system hydraulic potential energy thus becomes

\[
\phi^0 > \phi^*_i \\
\phi^0 = \phi^0_i + \sum_{i \in S} V_i^t \sum_{j \in S} \eta^i_j H^t_j
\]

This finding contradicts the fact that \( \phi^*_i \) is the optimal hydraulic potential energy. Therefore our assumption does not hold, and Equation 8 must thus be an equality. Proposition 16 is thus proved.

It follows from Theorem 6 that the objective function of the E1 model is

\[
\max_{X_t} \varphi_t = \max_{X_t} \{ \varphi_{t1} + \varphi_{t2} + \varphi_{t3} - \varphi_{t4} - \varphi_{t5} + \varphi_{t6} \}
\]

where \( X_t = (V_t^1, V_t^2, \ldots, V_t^n)^T \) accounts for the decision variable. The optimal values of the other two variable sets \( R_t, W_t, i \in S \) under a fixed \( V_t^i, i \in S \) can be determined easily by using Equations 21 and 22. Note that both \( \varphi_{t1} \) and \( \varphi_{t1} \) are known quantities at the beginning of decision because they are functions of the reservoir initial storages. The E1 model objective can be rephrased as

\[
\max_{X_t} \varphi_t = \max_{X_t} \{ -\varphi_{t6} + \varphi_{t5} + \varphi_{t4} \}
\]

Remark 3. Combining Proposition 13 and Proposition 16 yields the following: if \( W_i^t > 0 \) for any \( i \in S \) or \( \sum_{i \in S} P_i^t > P_{\text{min}}^t \), the reservoir must already be full or bound by release/power-output constraints. In other words, if the reservoir systems are strictly regulated within the feasible regulatory regions, the spilled-water-hydraulic potential energy satisfies \( \varphi_i^{W^*} = 0 \) and power-release-hydraulic potential energy satisfies \( \varphi_i^{R^*} = P_{\text{min}}^t \). Thus, the energy losses that are associated with the reservoir outflow of the optimal solution to the E1 model are minimal \( (\varphi_i^{R^*} + \varphi_i^{W^*} = P_{\text{min}}^t) \). Furthermore, the active constraints (Equations 5 and 7) ensure that the hydraulic potential energy target does not diverge from the target of minimizing the energy-cost due to the outflow. In addition to this energy-cost target, the proposed E1 model additionally maximizes the sum of the inflow-hydraulic potential energy and the system energy increment \( (\varphi_t^* + \varphi_t^*), \) in Equation 23. If Remark 8 holds, we confidently conclude that the optimal solution to the proposed multi-reservoir hydropower operation model (the E1 model) is a reasonable approximation to the optimal strategy of maintaining the overall potential energy.

To further analyze the E1 model, some convex properties are expected. However, this model is not guaranteed to be a convex problem. Similar to the objective formula of the E1 model, the non-convexity of hydropower production has long been regarded as a key obstacle in hydropower problem (Feltenmark & Lindberg, 1997; Goor et al., 2011). The approximation methods, such as the piecewise linear
approximation (Borghetti et al., 2008), convex function approximation (Zhao et al., 2014), are typically applied to bypass this difficulty and enhance the computational efficiency.

The convex function approximation can take a variety of forms, for example, it can be assumed that the head varies in a piece-wise linear manner with storage or outflow (Lima et al., 2013) or the upstream or downstream level is constant (El-Hawary & Christensen, 1979). As a mathematical operation of concave functions, the sum of a concave function and a linear function is concave. It is easy to prove that, under both the conditions, the reservoir head functions are also concave functions of the decision variable \( X_t = (V^1_t, V^2_t, \cdots, V^n_t)^T \). Following these methods, we can denote the approximation to the head function defined in Equation 2 by concave functions. Subsequently, the following property can be obtained.

Proposition 3. If \( H^i(V^1_t, R^i_t, W^i_t), i \in S \) can be approximated with concave functions, the E1 model is a convex optimization problem.

Recall that a convex problem is of the form

\[
\max_{X_t} \quad \varphi_i(X_t) \text{s.t.} \quad X_t \in C
\]

where \( C \) is a convex set, and \( \varphi_i \) is a concave function over \( C \).

Proof. We first check the concavity of the constraints. From Equation 21, for any \( i \in S \),

\[
\overline{H}^i(V^k_t, R^i_t, W^i_t) = \overline{H}^i(V^k_t, k \in S_{S_i}) = \frac{1}{2} \left( H^i_{\text{up}}(V^i_{t-1}) + H^i_{\text{down}}(V^i_t) \right) - H^i_{\text{down}}(V^k_t, k \in S_{S_i})
\]

Given the approximated concavity of \( H^i(V_t, R^i_t, W^i_t) \), the reservoir average head function \( \overline{H}^i(V^k_t, k \in S_{S_i}) \), along with \( H^i(V^k_t, k \in S_{S_i}) \), is also concave functions of the decision variable \( X_t = (V^1_t, V^2_t, \cdots, V^n_t)^T \), as determined by obtaining the second-order partial derivative of the variable.

Given the negative linear correlation between \( R^i_t \) and \( V^i_t \) illustrated in Equation 21, \( V^i_t H^i_t \) is a concave function while \( R^i_t H^i_t \) is a convex function. Hence, \( P^i_t = \eta^i_t R^i_t \overline{H}^i_t \) should be a convex function of \( X^i_t \). Constraint (7) and their combination constraint (8) are also nonlinear convex constraints. It suffices to show that the intersection of a collection of sets consisting of convex nonlinear constraints, linear equality and inequality constraints is a convex set.

Now, we consider the objective. Since \( H^i_t \) is a concave function of \( X^i_t \), the non-negative combination \( \sum_{j \in S_{S_i}} \eta^j H^j_t \) is also concave. Thus the objective function \( \left( \sum_{j \in S_{S_i}} V^j_t \sum_{j \in S_{S_i}} \eta^j H^j_t \right) \) of the E1 model is a concave function of \( X^i_t \). To conclude, the E1 model is a convex optimization problem of maximizing a concave function over a convex set.

As an important aspect of a convex problem, any local maximum of the E1 model is a global maximum. Since the objective function is a strictly concave function, this model has a unique optimal solution \( X^*_t \) on compact sets.

### 3.3. Optimal Spatial Principle for Multi-Reservoir Hydropower Operation

The theoretical reservoir operation rule is useful for practical operations and for understanding the operation of multiple reservoir systems. This section derives the allocation rule for hydropower system operation.

Theorem 2. (Relative Marginal Energy principle) If both the power output and the storage capacity constraints are nonbinding, the necessary condition for hydropower operation is to equalize the Relative Marginal Energy (RME) over the reservoirs, where
(Marginal Energy Index Principle) In particular, if the tail-water levels vary linearly with the outflow, namely, if \( H_{down,i} = a^i Q_{out,i} + b_i \), the RME principle is to equalize the Marginal Energy Index \( MEI(V^i_t) \) values among reservoirs, where

\[
MEI(V^i_t) := \sum_{j \in S_{ij}} H^i_j - \frac{\partial H^{up,j}_t}{\partial V^i_j} \sum_{k \in S_{jk}} M^i_k + \sum_{m \in S_{im}} (V^m_i - 2M^m_i) \sum_{n \in S_{im}(m)} a^n \tag{25}
\]

Proof: Recall that \( X_t = (V^i_t, R^i_t, W^i_t, \; i \in S) \) denotes the decision variable to the E1 model. The E1 model can be rewritten as

\[
\begin{align*}
\max_{X_t} & \quad \varphi_t(X_t) \\
\text{s.t.} & \quad h^i_1(X_t) = R^i_t + W^i_t - \sum_{k \in S_{ij}} (V^k_{t-1} + \beta^k_i - \beta^k_t - V^i_t) = 0 \\
& \quad g^i_0(X_t) = P_{min,t} - \sum_{i \in S} \eta^i R^i_t T^i_t \leq 0 \\
& \quad g^i_1(X_t) = V^i_t - V^i_{max,t} \leq 0 \\
& \quad g^i_2(X_t) = -V^i_t \leq 0 \\
& \quad g^i_3(X_t) = \eta^i R^i_t T^i_t - P^i_{max,t} \leq 0 \\
& \quad g^i_4(X_t) = P^i_{min,t} - \eta^i R^i_t T^i_t \leq 0 \\
& \quad g^i_5(X_t) = -W^i_t \leq 0
\end{align*}
\]

where \( \varphi_t(X_t) = \sum_{i \in S} V^i_t \sum_{j \in S_{ij}} \eta^i H^i_j \) (see Equation 20).

We convert this primal problem into the Lagrangian dual problem

\[
\begin{align*}
\max_{X_t} & \quad L(X_t) = \varphi_t(X_t) - \sum_{i \in S} \alpha^i h^i_1(X_t) - \sum_{i \in S} \beta^i_0 g^i_0(X_t) \\
\text{s.t.} & \quad \beta^i_1, \beta^i_0 \geq 0, \; i \in S, j \in \{1, 2, \ldots S\}
\end{align*}
\]

Provided that the functions of the objective and constraints are continuously differentiable, the Karush-Kuhn-Tucker (KKT) conditions can be used to obtain the necessary condition for the optimal decision; namely, there exist \( \alpha^i \) and \( \beta^i_1, \beta^i_0 \geq 0 \; i \in S, j \in \{1, 2, \ldots S\} \), such that

\[
\begin{align*}
\forall \varphi_t(X^*_t) - \sum_{i \in S} \alpha^i \nabla h^i_1(X^*_t) - \sum_{i \in S} \beta^i_0 \nabla g^i_0(X^*_t) - \beta^i_0 g^i_0(X^*_t) &= 0 \\
h^i_1(X^*_t) &= 0 \\
\beta^i_0 g^i_0(X^*_t) &= 0, \; j = 1, 2, \ldots S \\
\beta^i_0 g^i_0(X^*_t) &= 0 \\
g^i_1(X^*_t) &\leq 0, \; j = 1, 2, \ldots S \\
g^i_5(X^*_t) &\leq 0
\end{align*}
\]

where \( \alpha^i, \beta^i_1, \beta^i_0 \) are parameters defined with the KKT conditions.
Because the KKT conditions are difficult to be physically interpreted owing to the complex form, we assume that, at the optimum state, all the storage and power release/output constraints are not binding \((g_i^*(X_i) < 0, g_j^*(X_j) < 0, g_k^*(X_k) < 0, \text{for any } i \in S)\). Consequently, we obtain \(\beta_j^* = 0\) for any \(i \in S, j \in \{1, 2, \ldots, S\}\).

By Proposition 13 and Proposition 16,

\[
W_i^* = 0, \\
R_i^* + W_i^* = \sum_{k \in S} \left( V_{i-1}^k + I_i^k - \phi_i^k - V_i^k \right) \sum_{i \in S} \eta_i^* R_i^* = P_{\min, i}
\]

(28)

It can thus be determined that the E1 model decision variables \(X_i = (V_i^1, V_i^2, \ldots, V_i^n)\) (all constraints of the E1 model become inequalities), and these variables \(V_i^1, i \in S\) are independent (see Remark 15). Substituting Equation 27 with Equation 28, the KKT conditions can be simplified as

\[
\begin{aligned}
\nabla \varphi(X_i^*) - \beta_0 \nabla g_0(X_i^*) &= 0, \\
g_0(X_i^*) &= P_{\min, i} - \sum_{i \in S} \sum_{k \in S} \left( V_{i-1}^k + I_i^k - \phi_i^k - V_i^k \right) = 0 \beta_0 > 0
\end{aligned}
\]

(29)

where

\[
\nabla \varphi(X_i) = \begin{pmatrix} \partial \varphi_1 / \partial V_1^i \\ \partial \varphi_2 / \partial V_2^i \\ \vdots \\ \partial \varphi_n / \partial V_n^i \\ \end{pmatrix}^T
\]

(30)

\[
\nabla g_0(X_i) = \begin{pmatrix} \partial g_0 / \partial V_1^i \\ \partial g_0 / \partial V_2^i \\ \vdots \\ \partial g_0 / \partial V_n^i \\ \end{pmatrix}^T
\]

(31)

The optimal spatial principle for the mixed reservoir hydropower operation is to equalize the following term for each reservoir, i.e.,

\[
\beta_0 = \frac{\partial \varphi_1}{\partial \phi_1} = \frac{\partial \varphi_2}{\partial \phi_2} = \cdots = \frac{\partial \varphi_n}{\partial \phi_n}
\]

(32)

Substituting \(g_0\) and \(\phi_i^k\) for the above equation, we get Equation 24.

In particular, if the reservoir tail-water levels have a linear relationship with the outflow, the above optimal principle becomes solvable. Specifically, the explicit expressions of \(\varphi(X_i), g_0(X_i)\) and \(\beta_0\) are given in Appendix IV.

Equivalently, we find

\[
\frac{1}{\beta_0} = \frac{1}{2} + \frac{MEI(V_i^1)}{2}
\]

This formula implies that, without the binding constraints of the power output and storage capacity, the optimal spatial principle is equivalent to the equalization of \(MEI(V_i^1)\) values. Recalling Proposition 19, we can conclude that this MEI principle is globally necessary and sufficient on the condition that the tail-water rises linearly with the outflow.

This proves Theorem 21. ■

It should be emphasized that only the MEI principle depends on the tail-water assumption; the RME principle requires no implicit assumption. Considering the duality principle, irrespective of the convexity of the primal maximization problem (the E1 model), the solution to the dual problem (Equation 26) always provides an upper bound to the solution of the primal problem. In other words, even if the tail-water assumption does not hold or the E1 model is a non-convex optimization problem, the KKT conditions in Equation 27 and
For parallel reservoirs, an illustrative example is provided. Consider a hypothetical two-cascade reservoir system in which all reservoirs have the same characteristics and hydrological conditions. For instance, $a^t$ is assumed to be zero. At the beginning of the time period, Theorem 21 (the MEI principle) yields

$$\text{MEI} (V^t_i) = \frac{(n - i + 1)H_i^t + \frac{\partial H_i^t}{\partial V_i^t} (V_i^t - e_i^t)}{(n - i + 1)H_i^t + \frac{\partial H_i^t}{\partial V_i^t} I_i^t}$$

where $\eta^t$ (respectively, $H_i^t, V_i^t, I_i^t, e_i^t$) are the same for all $i \in S$. A straightforward calculation obtains the following expression

$$\text{MEI} (V^t_i) - \text{MEI} (V^{t+1}_i) = \frac{\frac{\partial H_i^t}{\partial V_i^t} (n + 1)[H_i^t + (V_i^t - e_i^t)H_i^t]}{(n - i + 1)H_i^t + \frac{\partial H_i^t}{\partial V_i^t} I_i^t} > 0$$

This relation $\text{MEI} (V^t_i) > \text{MEI} (V^{t+1}_i)$ implies that under identical conditions, a marginal storage change of in reservoir $i$ produces a larger relative marginal energy supply compared with that of the subsequent downstream reservoir $i+1$. Thus, we conclude that the E1 model has a tendency to release water from the upstream reservoirs first; however, the restoration occurs from the downstream reservoirs first according to the MEI ($V^t_i$) value.

Theorem 21, along with Remark 23, indicates that the maximum hydraulic potential energy model tends to release more water from the reservoirs with a relatively higher hydropower productivity, and a lower system potential energy loss (a larger RME or MEI) is incurred. In particular, the observation of Remark 24 is in agreement with the operation rule identified in the previous studies focusing on reservoirs in series (e.g., Lund & Guzman, 1999; Xu et al., 2015).

For parallel reservoirs, an illustrative example is provided. Consider a hypothetical two-parallel-reservoir system, as described in Table 1 and Figure 2(a)-2(b). Let the system firm power output be 50 MW. The average values of the 24-month inflow are 25 m$^3$/s and 32 m$^3$/s in the two cases. Let the reservoir elevation-storage relationships be polynomial functions and the tailrace elevation-outflow relationships be
linear functions. The decision is made to satisfy the stipulated energy demand based on the rolling single-period forecast (see Remark 8).

To evaluate the effectiveness of the MEI principle, two objectives, namely, (1) maximizing the system hydraulic potential energy (the E1 model, which is defined in Equation 20), and (2) minimizing the difference between the MEI values for each reservoir (the MEI principle in Theorem 21), are compared. The traversing method is applied to calculate the optimal power generation combination.

The results demonstrate that the optimal solutions under the two objectives are exactly the same. The optimal operation processes and the corresponding marginal energy index values (Equation 25) for each reservoir are presented in Figure 2(c)–(d). Within periods 3 to 5, the formula $\text{MEI}(V_1^t) > \text{MEI}(V_2^t)$ always holds, and the optimal operations correspond to releasing all water from Reservoir 1; however, the opposite trend is

![Figure 1. Illustration of the notations.](image)

| Table 1 | Illustrative example of a two-parallel-reservoir system |
|---------|--------------------------------------------------------|
| Parameter | Reservoir characteristic |
| Storage capacity $V_{\text{max},t}$ | 400 | 400 |
| Power output boundary $P_{\text{max},t}$ and $P'_{\text{max},t}$ | 10; 0 | 10; 0 |
| Operation initial storage | 150 | 350 |
| Water level function $H_{\text{up}}$, $H_{\text{down}}$ | $H_1^t$ in Figure 2(a) | $H_2^t$ in Figure 2(a) |
| 24-month inflow series | $I_{\text{up}}^t$ in Figure 2(b) | $I_{\text{down}}^t$ in Figure 2(b) |
observed after period 6. In periods 2 and 6, the optimal operation is to collaborate for power generation for the two reservoirs and to equalize the MEI values

\[ \text{MEI} V_1^t / C_{16} / C_{17} = \text{MEI} V_2^t / C_{16} / C_{17} \]

These results are consistent with the implications of Remark 23 and Theorem 21. After period 20, the storage of Reservoir 1 reaches its upper boundary. The optimal operations correspond to the release of the entire inflow from Reservoir 1, while the remaining power demand is fulfilled by Reservoir 2.

### 3.4. Comparison With Typical Hydropower Operation Methods

The hydraulic potential energy model and the optimal spatial principle are presented from the mathematical perspective. This section compares the proposed model and principle with certain typical energy-based hydropower operation models and principles that have long been utilized in similar studies. These principles include the minimization of the energy-cost with power release and spill (e.g., Mo et al., 2013; Teegavarapu & Simonovic, 2000), the K-value principle (e.g., Wang et al., 2014), and the Storage Effectiveness Index (SEI) principle (Lund, 1996).

The minimum energy-cost model involves minimizing the cost of energy release and the surrogate cost that is associated with water spill while satisfying energy demand. The objective function based on the rolling forecasts can be expressed as follows (Teegavarapu & Simonovic, 2000; Xu et al., 2015)

\[
\min \sum_{i \in S} \eta_i R_i^t T_i^t + C_i W_i^t
\]

The minimum energy-cost model involves minimizing the cost of energy release and the surrogate cost that is associated with water spill while satisfying energy demand. The objective function based on the rolling forecasts can be expressed as follows (Teegavarapu & Simonovic, 2000; Xu et al., 2015)

\[
\min \sum_{i \in S} \eta_i R_i^t T_i^t + C_i W_i^t
\]

where \( C_i \) is a constant that represents the surrogate cost of for each unit of spill from reservoir \( i \). This model can be rewritten informally as \( \varphi_i^R + \varphi_i^W \).
As stated in Remark 18, the objective function for the E1 model, which differs from the traditional objective function, can eventually realize the minimization of the energy losses by the outflow. That is, the (unique) optimal solution \( X_i^* = \left( V_{1,i}^*, R_{1,i}^*, W_{1,i}^* \right) \) to the E1 model is also an optimal solution to \( \min \varphi_f^i + \varphi_w^i \). In contrast, the optimal solutions to \( \min \varphi_f^i + \varphi_w^i \) for \( n > 1 \) are not unique. For instance, consider \( n = 2 \), the optimal solutions to the E1 model satisfy:

\[
\begin{align*}
V_{1,i}^* &= V_{1,i-1} + I_{1,i} - \bar{c}_1 - R_{1,i}^* - W_{1,i}^* \\
V_{2,i}^* &= V_{2,i-1} + I_{2,i} - \bar{c}_2 - R_{2,i}^* - W_{2,i}^* + \sum_{k \in S_{u,i}} R_{k,i}^* + \sum_{k \in S_{s,i}} W_{k,i}^* \\
\eta^1 R_{1,i}^* H_{1,i}^* + \eta^2 R_{2,i}^* H_{2,i}^* &= P_{\min,i} \\
W_{1,i}^* &= 0 \\
W_{2,i}^* &= 0
\end{align*}
\]

In this case, we have only five equalities but six decision variables \( \left( V_{1,i}, V_{2,i}, R_{1,i}, R_{2,i}, W_{1,i}, W_{2,i} \right) \). Considering the degrees of freedom, the optimal solutions are not unique. This finding suggests that such conventional objectives that focus only on the outflow-energy-cost are not optimal; however, this aspect represents a necessary-but-not-sufficient condition for the system energy maintenance. In contrast, the E1 model possesses significant advantages over the conventional models, in terms of at least additionally maximizing the storage (or energy) allocation among individual reservoirs in the system.

In terms of the operation principle, the K-value principle involves minimizing the impact of the hydropower revenue while avoiding the release of all upstream available inflows. The K-value estimates the percentage reduction in hydropower head, and it is defined for reservoirs in parallel and in series in Equations 34 and 35, respectively. The operation of hydropower reservoirs follows the rule that the reservoirs with a small K-value supply water first and store last (Wang et al., 2014).

\[
K_i^i = \frac{\text{EV}(C_{f,i}^i)}{F_{i-1}^i H_{i-1}^i}
\]

\[
K_i^i = \sum_{k \in S_{u,i}} \frac{\text{EV}(C_{f,k}^i)}{F_{i-1}^k} + \sum_{k \in S_{s,i}} \frac{V_{k,i-1}}{F_{i-1}^k H_{i-1}^k}
\]

where \( C_{f,i}^i \) is the cumulative intermediate inflow to reservoir \( i \) from the current period \( t \) to the end of a drawdown cycle (i.e., between October \( 1^\text{st} \) of one year and March \( 31^\text{st} \) of the next year) or a refill cycle (i.e., between April \( 1^\text{st} \) and September \( 30^\text{th} \)); \( \text{EV}(C_{f,i}^i) \) is the expected value of the cumulative inflow to reservoir \( i \), represented by the means of historical statistics; \( F_{i-1}^i \) and \( H_{i-1}^i \) are the surface area and hydraulic head of reservoir \( i \) at the beginning of period \( t \).

The details of the SEI principle can be found in Lund (1996). The K-value and SEI principles share a similar operating mechanism. If the system natural inflows are not sufficient for satisfying the system power requirement in the drawdown season, these principles are used to estimate the ratio of energy loss that is caused by the extra release requirement and the power shortfall of each reservoir. The reservoirs with the lowest ratios must be drawn down first. To illustrate the three methods, a system that consists of two parallel reservoirs is considered (Figure 3). One is a large reservoir (R1) with a mild elevation-storage relationship, and the other is a smaller reservoir (R1) with a substantially steeper elevation-storage relationship. The firm energy requirement for the current operation step is \( P_{\min,r} \).

Figure 3 (a) illustrates the operation strategies of the K-value and SEI principles. In these strategies, all natural inflow is first released \( (R_i^2 \text{ and } R_i^2) \) and the shortfall of the firm hydropower production due to the insufficient inflow is subsequently estimated as

\[
\Delta P = P_{\min,i} - \sum_{i=1}^2 P_{ij,i}
\]

where \( P_{ij,i} \) is the power output produced by releasing the natural inflow. Since the additional release from R1 results in a substantially smaller head reduction compared with that of R2, eliminating the same power
shortfall also leads to a smaller energy loss. Therefore, an additional release is required from R1 (R₂). The final operation decision is to release R₁ + R₃ from R1 and release R₂ from R2.

In contrast to the above method, the proposed E1 model directly determines the optimal release decisions (R₁ and R₂) by maximizing the system hydraulic potential energy while meeting the power output target (see Figure 3(b)). Theoretically, the operation priority can also be judged by the value of MEI (V_i) at the beginning of and during the operation. Considering the large size difference between the two reservoirs, the strategy of satisfying the system power requirement entirely by R1 while maintaining R2 at a higher level substantially outperforms the simultaneous release of both reservoirs (the K-value and SEI operation strategies).

The model of minimizing the outflow-energy-cost, along with the K-value and SEI principles, implicitly assumes that the initial storage distribution among reservoirs is already at the optimal level. Consequently, releasing all the inflow and minimizing the energy loss by an extra drawdown lead to the maximization of the system stored energy. However, this assumption is not necessarily correct, and the opposite phenomenon may occur in some cases. The additional shortcomings of these commonly used methods are as follows: (1) The energy cost, along with the stored energy, lacks a formal definition and has not been extensively discussed. (2) The release quantity allocation among reservoirs is not ascribed if the K-value or SEI are close. (3) Release priorities that are based on only the initial condition are insufficient, since the changes in the water level within the operation period might change the priority. The proposed E1 model possesses the necessary characteristics for overcoming these limitations.

### 3.5. Two-Stage Hydraulic Potential Energy Model

The E1 model is constructed based only on the current-period forecast and not on the distant future inflow (see Remark 8), which involves a high uncertainty. Although the optimal solution to the E1 model can lead to a higher system hydraulic potential energy (see Remark 18), it does not consider the future risks of reservoir overdraft or overflow. For instance, if the inflows of the next period are low, the upstream reservoirs may not have sufficient water to satisfy the minimum power output (overdraft), which would decrease the reliability of power-generation in the long run; alternatively, if the following inflows are high, the downstream reservoirs may need to spill water due to the proximity to their storage capacity (overflow). This merits special attention for arid regions, in which the stored water resource within the system is reserved not merely for hydropower but also for the provision of the water supply for coping with potential future droughts. Therefore, a reasonable operation strategy need to consider both the current-period energy and the associated risks for the subsequent periods, namely, the failure to satisfy the minimum output and the risk of water spillage.
To reduce these risks, the upstream reservoirs should not release excessively (the likelihood of overdraft increases due to the deficit $V^i_t$) and the downstream reservoirs should not restore excessively (the likelihood of overflow increases due to the surplus $V^i_t$). In other words, the hydraulic potential energy must be maximized not over the whole effective reservoir storage $\{V^i_{\min}, V^i_{\max}\}$ but in a sub-region of it. Thus, the reservoir storage is divided into three regions: the minimum-output guaranteed region $\{V^i_{\min,t}, U^i_{\min,t}\}$, the feasible regulatory region $\{U^i_{\min,t}, U^i_{\max,t}\}$, and the spill-control region $\{U^i_{\max,t}, V^i_{\max,t}\}$. A schematic representation of the relationship among the three regions is presented in Figure 4.

The upper boundary of the minimum-output guaranteed region $U^i_{\min,t}$ can be determined via the empirical envelope curve method. The envelope curve method involves operating reservoirs from the dead water level at the end of the drawdown season and proceeding backward from the upstream to downstream reservoirs, releasing water according to the corresponding minimum power-output of each time period, until the end of the refill season; subsequently, the typical reservoir storage paths under typical inflow combinations are obtained; finally, $U^i_{\min,t}$ is regarded as the upper envelope of the storage paths of each reservoir.

For the spill-control region, an excessive emphasis on lowering the reservoir level substantially influences the reservoir energy storage. Furthermore, it is difficult to define a clear spill-control region for each reservoir since the amount of water that should be spilled is interlinked among reservoirs during operation. Therefore, an implicit spill-control region can be determined by minimizing the cumulative loss of the spilled-water-hydraulic potential energy in the current and next periods

$$\min \left\{ \sum_{i \in S} W_i^t H_i^t + \sum_{i \in S} W_i^{t+1} H_i^{t+1} \right\}$$

(36)

Therefore, it is possible to incorporate the operation regions into the E1 model (Equation 20); via this approach, we can obtain a modified multi-reservoir hydropower operation model (referred to as the “E2 model”). Let $S_t := \{ i \in S | U^i_{\min,t} = V^i_{\min} \leq V^i_{t-1} \leq V^i_{\max,t}, \text{ for any } t \}$ and $S_{t1} := \{ i \in S | 0 \leq V^i_{t-1} < U^i_{\min,t} - V^i_{\max,t}, \text{ for any } t \}$. Based on two period rolling forecasts, the E2 model involves maximizing the carryover system hydraulic potential energy and minimizing the expected next-period energy losses from the overflow or overdraft, subject to a variety of constraints in the upcoming two periods, as follows

$$\max_{S_t} \left\{ \sum_{i \in S_t} \sum_{j \in S_{t1}} \eta_i' H_i^t - \sum_{i \in S_t} W_i^{t+1} H_i^{max,t+1} - \sum_{i \in S_{t1}} \left( U^i_{\min,t} - V^i_{\min} - V^i_t \right) H_i^{t+1} M_i' \right\}$$

(37)

Subject to

$$V^i_t = V^i_{t-1} + I^i_t + \sum_{k \in S_{t0}} R^k_i - \sum_{k \in S_{t0}} W^k_i - R^i_t - e^i_t$$

$$V^i_{t+1} = V^i_t + I^i_{t+1} + \sum_{k \in S_{t1}} R^k_{t+1} + \sum_{k \in S_{t1}} W^k_{t+1} - R^i_{t+1} - e^i_{t+1}$$

(38)

$$P^i_{\min,t} \leq \eta_i^t E_i^t \leq P^i_{\max,t}$$

(39)

$$P^i_{\min,t+1} \leq \eta_i^{t+1} E_i^{t+1} \leq P^i_{\max,t+1}$$

Subject to

$$P^i_{\min,t} \leq \eta_i^t E_i^t \leq P^i_{\max,t}$$

(39)

$$P^i_{\min,t+1} \leq \eta_i^{t+1} E_i^{t+1} \leq P^i_{\max,t+1}$$

(39)
The E2 model explores the maximum mathematic expectation of the energy storage by considering the problem with a unique optimal solution. The E2 model also represents a convex optimization economic benefit of the hydraulic potential energy in period $t$. This case is based on real-world data that were collected from two reservoirs in Southwest China and one reservoir in Southeast China. Reservoirs with distinctive storage capacities and hydrological conditions are selected from power systems to better evaluate the adaptability of the E2 model. The results and discussions are subsequently presented.

### 4. Case Study and Discussion

In this section, the proposed forecast-informed E2 model is applied to a hypothetical three-reservoir hydropower system. This case is based on real-world data that were collected from two reservoirs in Southwest China and one reservoir in Southeast China. Reservoirs with distinctive storage capacities and hydrological conditions are selected from power systems to better evaluate the adaptability of the E2 model. The results and discussions are subsequently presented.

#### 4.1. Experimental Setting

The structure of the three-reservoir system under consideration is illustrated in Figure 5(a), and the reservoir elevation-storage curves and elevation-outflow curves are presented in Figure 5(b) – (d). The basic characteristic data regarding the hydropower system are given in Table 2.

The E2 model is applied to optimize the hydropower operation based on inflow forecasts. The historical inflow records for the three-reservoir system are utilized as the input for rolling forecasts. Due to the limited availability of the inflow data, the monthly step inflow data from the years 1991 to 2010 are employed for medium-term operation test, while the daily step data from the years 2006 to 2009 are adopted for short-term operation test. Upon calculation, the rolling inflow forecast for the current operation period (per month or day) is assumed to be perfect as prescribed in Section 3, whereas that for the next period contains no information other than the monthly mean values of historical statistics. The numerical solution of the E2 model is calculated using the Particle Swarm Optimization algorithm (Kennedy & Eberhart, 1995). The operation decision is made at the beginning of each operation period but based on the inflow forecasts of the upcoming two periods. Apart from the E2 model, the same system with forecasting is solved again using the conventional K-value principle (Equations 34 and 35) and the minimum energy-cost model (Equation 33).

For comparison, the ideal solution is further calculated using dynamic programming (DP) with the objective of maximizing the long-term hydropower generation, as expressed below.
The total number of operation periods is $T$; and the reservoir storage increments are set at $5.27 \times 10^6$ and $1.73 \times 10^6$ m$^3$ for the medium- and short-term operation tests, respectively. The Open Multi-Processing (OpenMP) parallel programming interface (Hermanns, 2002) is used to increase the optimization efficiency. The ideal solution that is obtained via DP is the deterministic optimal trajectories under the perfect prediction of long-term inflow sequences; consequently, this solution is regarded as the upper bound for all the operation results.

\[
\max \sum_{t=1}^{T} \sum_{i \in S} \eta_i R_i^t H_i^t \Delta t
\]  

(42)

Figure 5. A schematic representation of (a) the hypothetical three-reservoir hydropower system, and (b-d) the corresponding elevation-storage curves and tailrace elevation-outflow curves.

| Table 2 | The basic data and model variable bounds of the hypothetical hydropower system |
|---------|-------------------------------|-------------------------------|-------------------------------|
| Parameter | Reservoir 1 | Reservoir 2 | Reservoir 3 |
| Normal pool level (m) | 940.00 | 704.00 | 108.00 |
| Dead water level (m) | 870.00 | 682.35 | 86.00 |
| Annual mean inflow (m$^3$/s) | 147.56 | 128.47 | 368.44 |
| Minimum tailrace water level (m) | 700.00 | 614.00 | 22.50 |
| Initial water level (m) | 920.00 | 704.00 | 98.00 |
| Reservoir storage limits (10$^8$ m$^3$) | 11.60–35.00 | 1.53–3.43 | 75.74–178.40 |
| Power release limits (m$^3$/s) | 88.62–521.12 | 82.35–661.56 | 195.94–1367.39 |
| Power output limits (MW) | 180.8–660.0 | 63.0–300.0 | 142.4–662.5 |
| System firm power output (MW) | 620.0 | | |
4.2. Medium-Term Operation Results

Figure 6 presents the medium-term reservoir water level regulation processes of the three energy-based operation methods (E2 model, K-value principle, and minimum energy-cost model), and the ideal solution obtained via DP. The corresponding system output processes are presented in Figure 7. All three reservoirs exhibit inter-annual variability: The reservoir storages are relatively low in the low-flow seasons and high in the high-flow seasons. Comparing the regulation processes of the three reservoirs, Reservoir 2 is nearly full or empty on most occasions while Reservoirs 1 and 3 seldom reach their storage boundaries.

The results that are obtained from the E2 model (see the blue lines in Figure 6 and Figure 7) exhibit an inclination to release more water from Reservoirs 1 and 3 while maintaining Reservoir 2 at a high level. By using a piecewise linear function to approximate the reservoir tailrace elevation-outflow relationship, the MEI values of the E2 model solutions are obtained and plotted, as shown in Figure 8. The processes of $MEI(V_{i}^{1})$ and $MEI(V_{i}^{3})$ are similar and intersect each other several times, thereby suggesting the occurrence of an alternating release priority between Reservoirs 1 and 3 over time. In most periods, the MEI values cannot be equal because the constraints of reservoir power output or storage are binding. A gap always exists.

Figure 6. Operation process of (a) reservoir 1, (b) reservoir 2, and (c) reservoir 3, represented by reservoir water level from 1991 to 2010. The solutions derived from DP algorithm (ideal solution), E2 model, K-value principle (K-value), and minimum energy-cost model (min E-cost) are represented by the black dashed line, blue full line, green full line, and orange full line, respectively.
between $MEI(V_1^{*})$ and the other two MEI values. This phenomenon occurs because Reservoir 2 has a substantially smaller storage and its storage capacity constraint is always binding. According to Figure 7, the E2 model tends to satisfy the system firm output (620 MW) as much as possible, and it maintains reservoirs at high water levels once the output target has been satisfied. Only when the expected next-period inflow is large or the reservoir is expected to overflow, does the system release extra water compared to the output demand, thereby producing extra power. These two traits support the theoretical properties proved in Section 3.2.

The K-value principle (green lines in Figure 7) tends to release extra water, which often exceeds the demand. The main similarity between the K-value principle and the E2 model is that they both maintain a high level for Reservoir 2. This occurs because Reservoir 2 is located downstream of Reservoir 1 and it has a substantially steeper elevation-storage relationship (see Figure 5 (c)) and a smaller storage capacity, thereby demonstrating a higher power efficiency. However, a drawback of the K-value principle is that the reservoirs with small K-values are made to consistently release water, which leads to more rapid and frequent water level decreases in the large reservoirs. Ultimately, the system sometimes fails to satisfy the power requirement at later periods.

As analyzed in Section 3.4, the optimal solutions to the minimum energy-cost model are not unique. The optimal operation processes differ substantially among optimization runs, with the long-term mean system power output ranging from 679.5 MW to 697.6 MW. One of the superior solutions is adopted and represented by the orange lines. The operation process of Reservoir 3 is similar to that resulting from the K-value principle, whereas the overuse of Reservoir 2 is clearly distinctive from the behavior of the E2 model and the K-value principle. The model of minimizing the energy-cost tends to release more water in the downstream
To evaluate the efficiency and effectiveness of the four solutions that are described above, Table 3 compares several key indicators that reflect the operational reliability and economic benefits. The results demonstrate that the ideal solution obtained using DP, the E2 model solution, and the minimum energy-cost solution can guarantee the firm output; however, the K-value result exhibits failure in some dry years. The E2 model can realize satisfactory hydropower productivity and overall revenue. Compared with that of the minimum energy-cost solution, the power generation of the E2 solution is increased by 162.06 GWh, while the spill rate is decreased by 1.75%. The power generation of the K-value principle lies between that of the E2 model and that of the minimum energy-cost model. This is because the K-value principle distinguishes drawdown periods from refill periods. The corresponding performance is similar to that of the minimum energy-cost model in drawdown periods; however, the performance approaches that of the E2 model in refill periods. In the results regarding the power contribution, only subtle differences are observed among the three reservoirs. On average, Reservoir 1 exhibits the largest proportion, followed by Reservoir 3 and Reservoir 2. Interestingly, compared with the regulation processes that are presented in Figure 6, the reservoirs that exhibit larger fluctuations generate less hydropower electricity. This aspect is attributable to the complementary effect between storage and release (Li et al., 2009). A one-unit release from a hydropower reservoir becomes less productive with the decrease of the reservoir water level. This suggests that the preservation of high water levels in certain reservoirs is necessary.

The marked differences between the ideal solution (DP framework) and the three forecast-informed solutions lie in the availability of long-term inflow forecasts. The former solution is based on the perfect prediction of long-term inflows, and, thus can best utilize all future possible spilled water, thereby yielding the best solution that corresponds to the maximum total revenue. In contrast, since the perfect long-term prediction is not available in practice, water spillage is mostly inevitable in practical hydropower operation. Maintaining high levels to promote the unit release hydropower productivity is a compromise strategy for short-term and medium-term hydropower operations. If the operators want to increase the water utilization, thereby leading to higher power generation and less water spillage, extension of the forecast lead-time and improvement of the inflow forecast accuracy might be helpful.

4.3. Short-Term Operation Results

The daily inflow data of each reservoir were collected from 2006 to 2009 for short-term operation test. In this period, approximately 67.2% of the annual total precipitation occurs in the refill season (May 1st to October 31st), and the daily mean inflows of the three reservoirs are 123.67, 101.06, and 393.14 m³/s.

Table 4 compares the performances regarding hydropower output and reliability (the probability that output is not lower than the minimum demand) of DP, the E2 model and the minimum energy-cost model. The K-value principle is not compared since it is primarily applied to medium-term operation or coupled with operation charts. On average, the system daily hydropower output from 2006 to 2009 always exceeds the minimum output (620 MW). In contrast to the substantial differences in the inflows between the refill and drawdown seasons, the system power generation in the refill season accounts for only 51.84% (Min E-cost model result) to 53.38% (DP result). The E2 model outperforms the minimum energy-cost model overall. The system power generation of the E2 model is 6.8% higher than that of the other model, and the

---

Table 3

| Operation result                           | Ideal solution | E2 model | K-value principle | Min E-cost |
|-------------------------------------------|----------------|----------|-------------------|------------|
| Minimum system output (MW)                | 620.0          | 620.0    | 184.0             | 620.0      |
| Annual system power generation (MW·year)  | 758.7          | 711.8    | 701.1             | 693.3      |
| System water spill rate (%)               | 1.02           | 2.99     | 4.79              | 4.74       |
| Output contribution of Reservoir 1 (%)    | 39.03          | 40.23    | 38.14             | 40.82      |
| Output contribution of Reservoir 2 (%)    | 26.80          | 26.72    | 28.23             | 24.68      |
| Output contribution of Reservoir 3 (%)    | 34.17          | 33.05    | 33.63             | 34.50      |
corresponding reservoirs provide a more reliable hydroelectricity supply. The frequency of failures of the minimum energy-cost model is increased from less than 0.5% to 12.6% for R1 and to 4.2% for R2.

4.4. Limitations and Discussion

This study is a single-objective analysis of multi-reservoir hydropower operation. Although many mountainous reservoirs (e.g., Jiang et al., 2018) are covered by this case, decisions pertaining to water resources management often inevitably involve tradeoffs among competing goals or objectives. In addition to hydropower, most reservoirs implement a multi-objective operation, which typically contributes to services such as water supply, irrigation, flood control, navigation, fisheries, and ecological maintenance. For these cases, the proposed model must be further investigated so that it can be merged into multi-objective optimization models to represent the hydropower aspects.

In a multi-objective operation, objective functions subject to a set of constraints may be weighed and later solved via multi-objective optimization techniques. These techniques can be classified into two main categories: (1) aggregation of the objectives under predefined objective priorities/weights, and (2) Pareto domination approaches if no preference information is available/considered (Burke & Silva, 2006; Olukanni et al., 2018). The simple expression of the hydraulic potential energy proposed in this paper renders the hydropower objective easily accessible in combination with the other operation objectives. Meanwhile, the observed characteristics and laws in hydropower operation can substantially facilitate the solution of multi-objective operation problem.

The second limitation of this study is that only two period inflow forecasts are employed, and the current forecast is assumed to be perfect. This assumption seems to limit the application of the proposed models (the E1 and E2 models) since inherent forecast uncertainty is inevitable. Typically, a longer forecast horizon corresponds to more uncertain inflow information; thus, the forecast uncertainty may become the dominating factor that guides the reservoir operation decision (Zhao et al., 2012). Nevertheless, this issue can be addressed by improving the nowcast and short-term forecast information, or by maximizing the expected objective value. A more extensive analysis on this is left for future work.

5. Conclusions

This paper presents a neat mathematical structure and forecast-informed operation model for mixed reservoir hydropower systems by introducing the concept of the hydraulic potential energy. The maximum hydraulic potential energy model (E1 model) is proposed. The major findings are summarized as follows: First, if the power output and storage constraints are nonbinding, the derived optimal spatial principle is (1) to equalize the Relative Marginal Energy (or Marginal Energy Index if the tail-water levels vary linearly with the outflow) among reservoirs or (2) if this is not feasible, to release water from the reservoirs that have the largest RME (or MEI) values first and to store water in the reverse order. Second, the objectives that focus on only the outflow-energy-cost and principles that are based on this concept are not optimal but represent the necessary-but-not-sufficient condition for system energy maintenance. Third, the E1 model can realize the objective of minimizing the energy-cost and the additional objectives that correspond to the maximization of the storage effect on the inflow-hydraulic potential energy and the system energy increment; the latter quantity has rarely been clarified or studied. Fourth, by considering the long-term water and energy supplies of the system, the two-stage hydraulic potential energy model (E2 model) can enhance the hydropower supply reliability and water reserves.
Appendix I: Monotonicity of the centroid of a reservoir waterbody

Suppose that (i) the reservoir waterbody is an inverted pyramidal frustum and (ii) the water density above the reservoir dead storage is evenly distributed.

Let $V$ be the reservoir storage, $h$ be the corresponding water depth above the dead storage, $A_b$ be the bottom area (the reservoir surface area of dead storage) and $A_t$ be the top area (the reservoir surface area of storage $V$). The following relations exist between the reservoir storage and these factors

\[ h = H_{up}(V) - H_{up}(V = 0) \leq h_{\text{max}} = H_{up}(V_{\text{max}}) - H_{up}(V = 0) \]
\[ V = \frac{h}{3}(A_b + \sqrt{A_bA_t} + A_t) \leq V_{\text{max}} \]

The geometric centroid (the center of mass of the reservoir waterbody) lies above the bottom base at a height (Harris & Stöcker, 1998)

\[ z = \frac{h(3A_b + 2\sqrt{A_bA_t} + A_t)}{4(A_b + \sqrt{A_bA_t} + A_t)} \leq h \]

Consider a marginal increase in the water depth $\Delta h$. The centroid $z'$ of this individual storage $\Delta V$ lies at

\[ z' = h + \Delta z = h + \frac{\Delta h(3A_t + 2\sqrt{A_tA_t'} + A_t')}{4(A_t + \sqrt{A_tA_t'} + A_t')} \geq z + \Delta z \]

The centroid of the compound shape thus becomes

\[ z_{\text{compound}} = \frac{zV + z'\Delta V}{V + \Delta V} \geq z + \frac{\Delta z \Delta V}{V + \Delta V} > z \]

Evidently, the centroid of the reservoir waterbody always monotonically increases with the water depth. The domain of the centroid definition is.

\[ \lim_{h \to 0, z_{\text{max}}} z = \lim_{h 
\to h_{\text{max}}} \frac{h_{\text{max}}(3A_b + 2\sqrt{A_bA_{\text{max}}} + A_{\text{max}})}{4(A_b + \sqrt{A_bA_{\text{max}}} + A_{\text{max}})} = \frac{h(3A_b + 2\sqrt{A_bA_t} + A_t)}{4(A_b + \sqrt{A_bA_t} + A_t)} = \lim_{t \to 0} \frac{h(3A_b + 2\sqrt{A_bA_t} + A_t)}{4(A_b + \sqrt{A_bA_t} + A_t)} \]

L'Hospital’s rule provides a technique to evaluate the above limit, as follows
Thus, for any given reservoir, on the interval \([0, h_{\text{max}}]\), the waterbody centroid \(z\), as a function of the water depth \(h\), is monotonically increasing on the domain \([0, z_{\text{max}}]\).

**Appendix II: Proof of Lemma 9**

Set \(Q^i_{\text{out},t} = R^i_t + W^i_t\). Let \(S_{\leq i}\) be the set of upstream reservoirs of reservoir \(i\) (including reservoir \(i\), see Notation 3). Iterating Equation 1 yields, for any \(i \in S\),

\[
Q^i_{\text{out},t} = R^i_t + W^i_t = \sum_{k \in S_{\leq i}} (V^k_{t-1} + I^k_t - e^k_t - V^k_t)
\]

Thus,

\[
T^i (V^i_t, R^i_t, W^i_t) = \frac{1}{2} \left[ H_{\text{up}}^i (V^i_{t-1}) + H_{\text{up}}^i (V^i_t) \right] - H_{\text{down}}^i (Q^i_{\text{out},t})
\]

\[
= \frac{1}{2} \left[ H_{\text{up}}^i (V^i_{t-1}) + H_{\text{up}}^i (V^i_t) \right] - H_{\text{down}}^i \left( \sum_{k \in S_{\leq i}} (V^k_{t-1} + I^k_t - e^k_t - V^k_t) \right)
\]

According to this equation, \(T^i (V^i_t, R^i_t, W^i_t)\) is a function of the decision variables \(V^i_t, k \in S_{\leq i}\). Since \(V^k_{t-1}, I^k_t,\) and \(e^k_t\) are known quantities, for brevity, we denote

\[
T^i (V^k_t, k \in S_{\leq i}) = \frac{1}{2} \left[ H_{\text{up}}^i (V^k_{t-1}) + H_{\text{up}}^i (V^k_t) \right] - H_{\text{down}}^i (V^k_t, k \in S_{\leq i})
\]

Variables \(V^k_t, k \in S_{\leq i}\) are independent in the function \(T^i (V^k_t, k \in S_{\leq i})\).

In accordance with the empirical studies on hydropower (Bayón et al., 2009; Zhao et al., 2014), the reservoir elevation-storage curve and tailrace elevation-outflow curve exhibit the following properties

\[
\frac{\partial H_{\text{up}}^i (V^i_t)}{\partial V^i_t} > 0, \quad \frac{\partial^2 H_{\text{up}}^i (V^i_t)}{\partial V^i_t^2} < 0
\]

\[
\frac{\partial H_{\text{down}}^i (Q^i_{\text{out},t})}{\partial Q^i_{\text{out},t}} > 0, \quad \frac{\partial^2 H_{\text{down}}^i (Q^i_{\text{out},t})}{\partial Q^i_{\text{out},t}^2} \leq 0
\]

The first derivative of \(T^i (V^k_t, k \in S_{\leq i})\) can be calculated

\[
\frac{\partial T^i (V^k_t, k \in S_{\leq i})}{\partial V^i_t} = \frac{1}{2} \frac{\partial H_{\text{up}}^i (V^i_t)}{\partial V^i_t} - \sum_{k \in S_{\leq i}} (V^k_{t-1} + I^k_t - e^k_t - V^k_t) \frac{\partial H_{\text{down}}^i (Q^i_{\text{out},t})}{\partial Q^i_{\text{out},t}}
\]

\[
= \frac{1}{2} \frac{\partial H_{\text{up}}^i (V^i_t)}{\partial V^i_t} + \frac{\partial H_{\text{down}}^i (Q^i_{\text{out},t})}{\partial Q^i_{\text{out},t}} > 0
\]

For any \(V^{j_0}_{t_0}, k_0 \in S_{\leq i}, V^{j_0}_{t_0}, j_0 \in S \{k | k \in S_{\leq i}\}\)
The inherent relationships between them are empirically determined based on the boundaries of the power release.

A straightforward calculation yields the following expression:

\[
\frac{\partial H_i(V_t^k, k \in S_{i\ell})}{\partial V_t^k} = \frac{1}{2} \frac{\partial H_{up}(V_t^k)}{\partial V_t^k} - \sum_{k \in S_{i\ell}} \left( V_{t-1}^k + t_i^k - V_t^k \right) \frac{\partial H_{down}(Q_{out,t})}{\partial Q_{out,t}} + \frac{3}{2} H_{up}(V_t^k, k \in S_{i\ell})
\]

Hence, \( H_i(V_t^i, R_t^i, W_t^i) \) is an increasing function of \( V_t^i, i \in S \).

Appendix III: Proof of Lemma 11

According to Lemma 9, \( H_i(V_t^i, R_t^i, W_t^i) \) increases with \( V_t^i \). Likewise, \( H_i(V_t^i, R_t^i, W_t^i) \) decreases with \( R_t^i + W_t^i \).

Since \( W_t^i \geq 0 \) and water spill does occur if \( V_t^i \) is low. Consequently, for any \( i \in S \),

\[
H_i\left(V_t^i = 0, R_t^i_{\text{max},t}, 0\right) \leq H_i\left(V_t^i = V_t^i_{\text{max},t}, R_t^i_{\text{min},t}, 0\right)
\]

Empirically, the upper and lower bounds of the power release are determined based on the boundaries of the power output via trial and error. The inherent relationships between them are

\[
R_t^i_{\text{min},t} = \frac{P_t^i_{\text{min},t}}{\eta H_t(V_t^i_{\text{max},t}, R_t^i_{\text{min},t}, 0)}
\]

\[
R_t^i_{\text{max},t} = \frac{P_t^i_{\text{max},t}}{\eta H_t(0, R_t^i_{\text{max},t}, 0)}
\]

A straightforward calculation yields the following expression:

\[
P_t^i_{\text{min},t} \leq \frac{P_t^i_{\text{min},t}}{\eta H_t}
\]

\[
P_t^i_{\text{max},t} \geq \frac{P_t^i_{\text{max},t}}{\eta H_t}
\]

Thus, \( F_P = \left\{ R_t^i | P_t^i_{\text{min},t} \leq R_t^i \leq P_t^i_{\text{max},t} \right\} \subseteq F_R = \left\{ R_t^i | P_t^i_{\text{min},t} \leq R_t^i \leq P_t^i_{\text{max},t} \right\} \), and \( F_P \) is a subset of \( F_R \).

Appendix IV: Calculation of parameter \( \beta_0 \)

If the reservoir tail-water levels are linearly related to the outflow, it follows from the proof of Lemma 9 that

\[
\frac{\partial H_t}{\partial V_t^i} = \frac{1}{2} \frac{\partial H_{up}(V_t^i)}{\partial V_t^i} + a_i, \quad \frac{\partial H_t}{\partial V_t^j} = \frac{\partial H_{up}(V_t^i)}{\partial V_t^j} + a_j, \quad i \in S
\]

\[
\frac{\partial H_t}{\partial V_t^k} = \frac{\partial H_{up}(V_t^k)}{\partial V_t^k}, \quad k \in S_{<i}
\]

\[
\frac{\partial H_t}{\partial V_t^l} = 0, \quad l \in S_{k \geq i}
\]

Let \( S_{\Omega} \) be the set of reservoirs that have a hydraulic connection with reservoir \( i \) (see Notation 4). We have the partial derivative of \( \varphi(X_t) \) at \( X_t = (V_t^1, V_t^2, \ldots, V_t^n)^T \) with respect to the variable \( V_t^i \).
\[ \frac{\partial \phi}{\partial V_t^i} = \frac{\partial}{\partial V_t^i} \left( \sum_{j \in S_{tn}} \eta_j^T H_i^j + \eta_j^T \frac{\partial H_{up}}{\partial V_t^j} \sum_{k \in S_{tn}} V_t^k + \sum_{m \in S_{tn}} V_t^m \sum_{n \in S_{nmax(u,n)}} \eta_n^a a^n \right) \]

For brevity, we denote

\[ M_{t-1}^i = V_{t-1}^i + I_{t-1}^i - e_t^i \]

We obtain the partial derivative of \( g_0(X_t) \) with respect to \( V_t^i \)

\[ \frac{\partial g_0}{\partial V_t^i} = \frac{\partial}{\partial V_t^i} \left[ \sum_{i \in S} \eta_i^T T_i - \sum_{j \in S_{tn}} (M_j^i - V_t^j) \right] \]

\[ = \frac{\partial}{\partial V_t^i} \left[ \sum_{i \in S} (V_t^i - M_j^i) \right] - \sum_{j \in S_{tn}} \left( M_j^i - V_t^j \right) \]

After substituting Equation 32 into the above two formulas, a straightforward calculation yields

\[ 1 - \frac{1}{\rho_0} = 1 + \frac{1}{2} \left( \sum_{j \in S_{tn}} \eta_j^T H_{t-1} + \eta_j^T \frac{\partial H_{up}}{\partial V_t^j} \sum_{k \in S_{tn}} V_t^k + \sum_{m \in S_{tn}} V_t^m \sum_{n \in S_{nmax(u,n)}} \eta_n^a a^n \right) \]

\[ \cdot \left( \sum_{j \in S_{tn}} \eta_j^T T_j - \sum_{j \in S_{tn}} (M_j^i - V_t^j) \right) \]

\[ \cdot \left( \sum_{m \in S_{tn}} V_t^m \sum_{n \in S_{nmax(u,n)}} \eta_n^a a^n \right) \]

\[ \cdot \left( \sum_{m \in S_{tn}} V_t^m \sum_{n \in S_{nmax(u,n)}} \eta_n^a a^n \right) \]

Acknowledgments

The authors would like to thank Dr. Zhengfeng Wang for checking all the mathematical proofs. We also thank the editors and three anonymous reviewers for their insightful and constructive comments, which helped to substantially improve the manuscript. This study was jointly supported by the National Key Research and Development Program of China (2017YFC0404300, 2016YFC0401302, 2016YFC0402003) and the National Natural Science Foundation of China (91747208, 51861125102). The data necessary to reproduce the work are available in the public repository of figshare (https://doi.org/10.6084/m9.figshare.11230444.v2).

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