NON-RIGID REGIONS OF REAL GROTHENDIECK GROUPS
OF GENTLE AND SPECIAL BISERIAL ALGEBRAS

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Abstract. In the representation theory of finite-dimensional algebras $A$ over a field, the classification of 2-term (pre)silting complexes is an important problem. One of the useful tools is the g-vector cones associated to the 2-term presilting complexes in the real Grothendieck group $K_0(\proj A)_R := K_0(\proj A) \otimes_{\mathbb{Z}} \mathbb{R}$. The aim of this paper is to study the complement $\mathcal{NR}$ of the union $\mathcal{Cone}$ of all g-vector cones, which we call the non-rigid region. By the work of Iyama and us, $\mathcal{NR}$ is determined by 2-term presilting complexes and a certain closed subset $R_0 \subset K_0(\proj A)_\mathbb{R}$, which is called the purely non-rigid region. In this paper, we give an explicit description of $R_0$ for complete special biserial algebras in terms of a finite set of maximal nonzero paths in the Gabriel quiver of $A$. We also prove that $\mathcal{NR}$ has some kind of fractal property and that $\mathcal{NR}$ is contained in a union of countably many hyperplanes of codimension one. Thus, any complete special biserial algebra is g-tame, that is, $\mathcal{Cone}$ is dense in $K_0(\proj A)_\mathbb{R}$.

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1. Introduction

In the representation theory of finite-dimensional algebras $A$ over an algebraically closed field $K$, silting theory introduced by Keller-Vossieck [KV] is an important tool to study the derived categories of algebras. They considered complexes called silting complexes satisfying certain conditions in the homotopy category $K^b(\proj A)$ of the bound complexes over the category $\proj A$ of finitely generated projective $A$-modules. Silting complexes contain tilting complexes in $K^b(\proj A)$, which control derived equivalences of finite-dimensional algebras [Ric]. One of the nice properties of silting complexes is that they admit the operation called mutation exchanging any indecomposable direct summand to get another silting complex [AiI].

Since then, many authors found one-to-one correspondences between silting complexes and other notions in the category $\fd A$ of finite-dimensional modules and its bounded derived category $D^b(\fd A)$. For example, Koenig-Yang [KY] constructed bijections called the silting-t-structure correspondences from the set of basic silting complexes in $K^b(\proj A)$ to the sets of bounded t-structures with length heart and simple-minded collections in $D^b(\fd A)$. Moreover, Adachi-Iyama-Reiten [AIR] proved that there exist bijections between the basic 2-term silting complex in $K^b(\proj A)$ and functorially finite torsion classes in $\fd A$, and Brüstle-Yang [BY] obtained

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the 2-term versions of the silting-t-structure correspondences. These bijections have been also developed in terms of wide subcategories \([11, MS]\) and semibricks \([As1]\).

In this paper, we aim to understand 2-term silting complexes in \(K^b(\text{proj } A)\) for complete gentle algebras. Complete gentle algebras are an important class of algebras studied in many works \([AY1, OP, PPP1, PPP2]\) by using marked surfaces as well as \([HKK, LP]\) from the point of view of Fukaya categories. It is also known that 2-term silting complexes for Brauer graph algebras can be investigated in the same way as complete gentle algebras; see the explanation before Corollary \(6.9\). In this paper, we also consider a wider class called complete special biserial algebras, which can be expressed as quotient algebras of complete gentle algebras.

To study 2-term silting complexes in \(K^b(\text{proj } A)\), the Grothendieck group \(K_0(\text{proj } A)\) and the real Grothendieck group \(K_0(\text{proj } A)_{\mathbb{R}} := K_0(\text{proj } A) \otimes_{\mathbb{Z}} \mathbb{R}\) are useful. We write \(2\)-silt \(A\) (resp. \(2\)-psilt \(A\)) for the set of basic 2-term silting (resp. presilting) complexes in \(K^b(\text{proj } A)\). For \(U = \bigoplus_{i=1}^\infty U_i \in 2\)-psilt \(A\) with each \(U_i\) indecomposable, we define the \(g\)-vector cones in \(K_0(\text{proj } A)_{\mathbb{R}}\) by

\[
C(U) := \left\{ \sum_{i=1}^m a_i U_i \mid a_i \in \mathbb{R}_{\geq 0} \right\}, \quad C^+(U) := \left\{ \sum_{i=1}^m a_i U_i \mid a_i \in \mathbb{R}_{> 0} \right\}
\]

as in \([DIJ, BST]\). The elements \([U_1], [U_2], \ldots, [U_m] \in K_0(\text{proj } A)\) are linearly independent by \([A1]\), so \(C(U)\) and \(C^+(U)\) are \(m\)-dimensional. Moreover, if \(U''\) is the maximal common direct summand of \(U, U' \in 2\)-psilt \(A\), then \(C(U'') = C(U) \cap C(U')\) holds \([DIJ]\). Thus, \(g\)-vector cones reflect mutation of 2-term silting complexes.

We also consider the subsets

\[
\text{Cone} := \bigcup_{T \in 2\text{-silt } A} C(T) = \bigcup_{U \in 2\text{-psilt } A} C^+(U), \quad \text{NR} := K_0(\text{proj } A)_{\mathbb{R}} \setminus \text{Cone}.
\]

We call NR the non-rigid region. It is well-known that \(\text{NR} = \emptyset\) holds if and only if \(A\) is \(\tau\)-tilting finite, that is, 2-silt \(A\) is a finite set \([ZZ, Asa2]\).

To study NR, the \(\mathbb{R}\)-bilinear form called the Euler form \(K_0(\text{proj } A)_{\mathbb{R}} \times K_0(\text{fd } A)_{\mathbb{R}} \rightarrow \mathbb{R}\) is helpful. Via the Euler form, we can see \(K_0(\text{proj } A)_{\mathbb{R}}\) as a dual \(\mathbb{R}\)-vector space of \(K_0(\text{fd } A)_{\mathbb{R}}\), and any element \(\theta \in K_0(\text{proj } A)_{\mathbb{R}}\) is identified with \(\theta := \langle \theta, \cdot \rangle : K_0(\text{fd } A)_{\mathbb{R}} \rightarrow \mathbb{R}\). See also the beginning of Subsection \(3.2\).

By using this, King \([Kin]\) introduced stability conditions for modules \(M \in \text{fd } A\) and elements \(\theta \in K_0(\text{proj } A)_{\mathbb{R}}\) as a collection of linear inequalities. In a similar way, Baumann-Kamnitzer-Tingley \([BKT]\) associated two numerical torsion pairs \((\mathcal{T}_\theta, \mathcal{F}_\theta)\) and \((\mathcal{T}_\theta, \mathcal{F}_\theta)\) in \(\text{fd } A\) to each \(\theta \in K_0(\text{proj } A)_{\mathbb{R}}\). In the paper \([Asa2]\), we defined the equivalence relation on \(K_0(\text{proj } A)_{\mathbb{R}}\) called TF equivalence: we say that \(\theta\) and \(\theta'\) are TF equivalent if \(\mathcal{T}_\theta = \mathcal{T}_{\theta'}\) and \(\mathcal{F}_\theta = \mathcal{F}_{\theta'}\).

The numerical torsion pairs are strongly related to silting theory. For any \(U \in 2\)-psilt \(A\), Yurikusa \([Yur]\) and Brüstle-Smith-Treffinger \([BST]\) showed that \(\theta \in C^+(U)\) implies that \(\mathcal{T}_\theta = \mathcal{T}_{[U]}\) and \(\mathcal{F}_\theta = \mathcal{F}_{[U]}\). By modifying their arguments, we showed that \(C^+(U)\) is a TF equivalence class in \([Asa2]\); in other words,

\[
C^+(U) = \{ \theta \in K_0(\text{proj } A)_{\mathbb{R}} \mid \mathcal{T}_\theta = \mathcal{T}_{[U]}, \mathcal{F}_\theta = \mathcal{F}_{[U]} \}.
\]

It is also useful to consider the open neighborhood

\[
N_U = \{ \theta \in K_0(\text{proj } A)_{\mathbb{R}} \mid \mathcal{T}_U \subset \mathcal{T}_\theta, \mathcal{F}_U \subset \mathcal{F}_\theta \}
\]

of \(C^+(U)\) associated to each \(U \in 2\)-psilt \(A\) as in \([Asa2]\). By using this neighborhood, Iyama and we \([As1]\) defined the purely non-rigid region

\[
R_0 := K_0(\text{proj } A)_{\mathbb{R}} \setminus \bigcup_{U \in 2\text{-psilt } A} N_U,
\]
and proved that
\[
\text{NR} = \coprod_{U \in 2\text{-silt} A} \{ \theta_1 + \theta_2 \mid \theta_1 \in C^+(U), \ 0 \neq \theta_2 \in \overline{NU} \cap R_0 \}.
\]

Therefore, it is important to determine \( R_0 \) to understand \( \text{NR} \). In this paper, we describe \( \text{NR} \) in the case that \( A = K\overline{Q}/I \) is a complete special biserial algebra by using the following sets of “maximal” paths and cycles in \( Q \):

- \( \text{MP}(A) \) consists of all paths \( p \) with \( p \neq 0 \) and \( \alpha p = \rho a = 0 \) in \( A \) for any arrow \( \alpha \in Q_1 \);
- \( \overline{\text{MP}}(A) \) is the union of \( \text{MP}(A) \) and the set of paths \( e_i \) of length \( 0 \) for \( i \in \mathbb{Z}_0 \) such that there exists at most one arrow starting at \( i \) and at most one arrow ending at \( i \);
- \( \text{Cyc}(A) \): the set of irreducible cycles in \( Q \) such that \( c^m \neq 0 \) in \( A \) for any \( m \in \mathbb{Z}_{\geq 1} \).

See Definition \( \ref{def:2.2} \) for the precise definition. We write \( M(s) \) for the string module associated to each string \( s \) admitted in \( A \). Moreover, we set \( p_c := \alpha_1 \alpha_2 \cdots \alpha_{i-1} \) if \( c = \alpha_1 \alpha_2 \cdots \alpha_i \in \text{Cyc}(A) \) with \( \alpha_i \in Q_1 \). Then, in the case of complete gentle algebras, we have the following result.

**Theorem 1.1** (Theorem \( \ref{thm:5.1} \)). Let \( A = K\overline{Q}/I \) be a complete gentle algebra. For each \( \theta \in K_0(\text{proj} A)_\mathbb{R} \), the following conditions are equivalent.

(a) The element \( \theta \) belongs to \( R_0 \).
(b) We have \( \theta(M(p_c)) = 0 \) for each \( c \in \text{Cyc}(A) \), and \( M(p) \in W_0 \) for each \( p \in \overline{\text{MP}}(A) \).

In particular, \( R_0 \) is a rational polyhedral cone of \( K_0(\text{proj} A)_\mathbb{R} \), that is, there exist finitely many elements \( \theta_1, \theta_2, \ldots, \theta_m \in K_0(\text{proj} A) \) such that \( R_0 = \sum_{i=1}^m \mathbb{R} \theta_i \).

More generally, we give an explicit description of \( R_0 \) for complete special biserial algebras \( A = K\overline{Q}/I \) in Theorem \( \ref{thm:5.3} \) by taking a smaller ideal \( I \subset \mathcal{I} \) such that \( A := K\overline{Q}/I \) is a complete gentle algebra. In this case, \( R_0 \) is a subset of \( R_0(A) \), but it is not necessarily a rational polyhedral cone or even convex; see Example \( \ref{ex:5.10} \).

As an application, we also prove that any complete special biserial algebra \( A \) is \( g \)-tame, that is, \( \text{Cone} \) is a dense subset of \( K_0(\text{proj} A)_\mathbb{R} \). For each \( U \in \text{2-silt} A \), we write \( \mathcal{W}_U := \mathcal{T}_U \cap \mathcal{F}_U \), which is a wide subcategory of \( \text{fd} A \). Set \( \sim \mathcal{W}_U \) as the set of isoclasses of simple objects of \( \mathcal{W}_U \), and \( h_U \in K_0(\text{fd} A) \) by \( h_U := \sum_{X \in \sim \mathcal{W}_U} [X] \).

**Corollary 1.2** (Corollary \( \ref{cor:6.1} \)). Let \( A \) be a complete special biserial algebra. Then, we have
\[
\text{NR} = K_0(\text{proj} A)_\mathbb{R} \setminus \text{Cone} \subset \bigcup_{U \in \text{2-silt} A} \text{Ker}(?, h_U).
\]

In particular, \( \text{Cone} \) is dense in \( K_0(\text{proj} A)_\mathbb{R} \).

The \( g \)-tameness of complete special biserial algebras has been already proved in \( \cite{AY} \), and it was extended to representation-tame algebras in \( \cite{PY} \). However, we get the stronger result that \( \text{NR} \) is contained in a union of countably many hyperplanes of codimension one, since \( \text{2-silt} A \) is a countable set by \( \cite{DLJ} \).

We also show the following result on mutation of 2-term silting complexes. We remark that the corresponding property does not hold for all connected finite-dimensional algebras \( \cite{Ter} \).

**Corollary 1.3** (following from Corollary \( \ref{cor:5.6} \)). If \( A \) is a connected special biserial algebra, then any 2-term silting complex in \( K^b(\text{proj} A) \) can be obtained by iterated mutations from \( A \) or \( A[1] \).

Our proofs of the two properties above are related to \( \tau \)-tilting reduction by Jasso \( \cite{Jas} \). It is an important tool to study the set \( 2\text{-silt}_U A \) of 2-term silting complexes which have a fixed \( U \in \text{2-silt} A \) as a direct summand. Jasso considered the algebra \( B := \text{End}_A(H^0(T))/[H^0(U)] \), where \( T \in 2\text{-silt} A \) is the Bongartz completion of \( T \), and \( [H^0(U)] \) is the ideal of \( \text{End}_A(H^0(T)) \) consisting of all endomorphisms factoring through some module in \( \text{add} H^0(U) \). Then, Jasso proved that there exist a category equivalence \( \mathcal{W}_U \to \text{fd} B \) of abelian categories and a bijection \( 2\text{-silt}_U A \to 2\text{-silt} B \). In \( \cite{Asa2} \), we proved that there exists a linear projection \( \pi: K_0(\text{proj} A)_\mathbb{R} \to \)
$K_0(\text{proj } B)_\mathbb{R}$ which induces a bijection between the TF equivalence classes in $N_U$ and those in $K_0(\text{proj } B)_\mathbb{R}$; see Proposition 4.4.

To prove the two corollaries above, we also need to understand what the algebra $B$ is. We show that the class of finite-dimensional special biserial algebras is closed under $\tau$-tilting reduction.

**Theorem 1.4 (Theorem 4.12).** Let $A$ be a finite-dimensional special biserial algebra, and $U \in 2\text{-psilt } A$. Then, the algebra $B$ above is isomorphic to a finite-dimensional special biserial algebra.

Therefore, we can say that the non-rigid region $\text{NR}$ has a “fractal” property for a complete special biserial algebra $A$, since $\text{NR}$ is a union of some TF equivalence classes; see Corollary 4.16 for a precise statement.

For example, Iyama and we obtained in [AI, p. 44] that $\text{NR}$ is described by the following picture for the complete gentle algebra of the Markov quiver (Example 5.5 (2)). More precisely, the picture below shows the intersection of $\text{NR}$ and the surface of the standard octahedron

$$\left\{ \theta = \sum_{i=1}^{3} a_i [P_i] \in K_0(\text{proj } A)_\mathbb{R} \mid \sum_{i=1}^{3} |a_i| = 1 \right\},$$

where we omit the two faces $\sum_{i=1}^{3} \mathbb{R}_{>0} [P_i]$ and $\sum_{i=1}^{3} (-\mathbb{R}_{>0}) [P_i]$, since they do not intersect with $\text{NR}$:

Now, we explain the organization of this paper.

In Subsection 2.1, we recall the precise definition of complete special biserial algebras and some basic properties, including the classification of the indecomposable modules [BR, WW] and the homomorphisms between them [CB2, Kra]. Then, we consider complete gentle algebras in Subsection 2.2. We give a combinatorial construction of complete gentle algebras by lines and cycles in Definition 2.10 and Proposition 2.12, which is known to some experts. This construction is compatible with the sets $\text{MP}(A)$ and $\text{Cyc}(A)$, which we use to describe the purely non-rigid region $R_0$ in Theorem 1.1.

Section 3 is devoted to recalling important results on silting theory and canonical decompositions in the case of finite-dimensional algebras and modifying them for our setting of complete special biserial algebras.

In Subsection 3.1, we collect some results on the relationship between 2-term silting complexes in $K^b(\text{proj } A)$ and torsion pairs and bricks in $\text{fd } A$ obtained in $\tau$-tilting theory by [AIR, Asa1]. We also refer to reduction theorems of bricks and silting complexes by [Kim, VG], which enable us to deal with a complete special biserial algebra $A$ almost in the same way as the finite-dimensional algebra $A := A/I_c$, where $I_c$ denotes the ideal of $A$ generated by $\text{Cyc}(A)$.

Then, we recall basic properties of the Grothendieck groups, numerical torsion pairs, and the wall-chamber structures on $K_0(\text{proj } A)_\mathbb{R}$ introduced by [BST, Bri] in Subsection 3.2.

In Subsection 3.3, we refer to the works [DF, Pla] on canonical decompositions of elements in the Grothendieck group $K_0(\text{proj } A)$. Derksen-Fei [DF] introduced a canonical decomposition $\theta = \bigoplus_{i=1}^{m} \theta_i$ for each element $\theta \in K_0(\text{proj } A)_\mathbb{R}$ based on decompositions of 2-term complexes in
Kb(\text{proj } A) into indecomposable direct summands. This is an analogue of canonical decompositions of dimension vectors in quiver representations by Kac \cite{Kac}. Derksen-Fei and Plamondon \cite{Pla} proved that any element \( \theta \in K_0(\text{proj } A) \) has a unique canonical decomposition up to reordering. In this subsection, we also prepare some properties on indecomposable elements in \( K_0(\text{proj } A) \) for complete special biserial algebras, which we need to consider the non-rigid region \( NR \).

We deal with \( \tau \)-tilting reduction in Section 4. We explain basic properties of the neighborhood \( \mathcal{N}_U \) for each \( U \in 2\text{-psilt } A \) and the (purely) non-rigid region \( R_0, NR \) in more detail obtained in the prior researches \cite{Jas, Asa2, AsI}. Then, we prove Theorem 1.4 in Subsection 4.2. We will also show that the class of complete gentle algebras “is closed” under \( \tau \)-tilting reduction in a certain sense; see Corollary 4.17.

In Section 5 we will determine \( R_0 \) and \( \mathcal{N}_U \cap R_0 \) for \( U \in 2\text{-psilt } A \) explicitly for any complete special biserial algebra \( A \). Subsection 5.1 is devoted to stating the descriptions of the purely non-rigid region \( R_0 \) (including Theorem 1.1) and examples, and the proofs are done in Subsections 5.2 and 5.3. The subset \( \mathcal{N}_U \cap R_0 \) for \( U \in 2\text{-psilt } A \) is considered in Subsection 5.4.

We give some applications of our results in Section 6. We first prove Corollary 1.2 on the g-tameness of complete special biserial algebras in Subsection 6.1 by using Theorems 1.1 and 1.3. In Subsection 6.2, we show Corollary 1.3 on mutation of 2-term silting complexes. We will also give other proofs of several known properties by \cite{STV, AAC, Ada} on \( \tau \)-tilting finiteness of special biserial algebras by applying our results in Subsection 6.3. We finally remark that Mousavand \cite{Mou} studied \( \tau \)-tilting finiteness of special biserial algebras in a different way.

1.1. Notations. In this paper, we assume that the base field \( K \) is algebraically closed. A quiver \( Q \) is always a finite quiver, and the composite \( \alpha \beta \) of arrows \( \alpha, \beta \in Q_1 \) is a path from the source of \( \alpha \) to the target of \( \beta \). The symbol \( e_i \) denotes the path of length 0 associated to each vertex \( i \in Q_0 \). We set \( \overline{KQ} \) as the complete path algebra with respect to the ideal \( R := \langle \alpha \mid \alpha \in Q_1 \rangle \) generated by all arrows. Thus, we can see

\[
\overline{KQ} = \bigoplus_{p: \text{all paths in } Q} Kp
\]

as \( K \)-vector spaces. Unless otherwise stated, \( A = \overline{KQ}/I \) is a complete algebra over the field \( K \) with \( I \subset R^2 \) an ideal of \( \overline{KQ} \). For each \( i \in Q_0 \), we write \( P_i := e_i A \) for the indecomposable projective module, and \( S_i \) for its simple top.

We write \( \text{proj } A \) for the category of finitely generated right \( A \)-modules, and \( K^b(\text{proj } A) \) for the homotopy category of bounded complexes on \( \text{proj } A \). Similarly, we define \( \text{fd } A \) as the category of finite-dimensional right \( A \)-modules. Then, \( \text{fd } A \) is an abelian length category; in particular, \( \text{fd } A \) satisfies the Jordan-Hölder property.

All subcategories in this paper are assumed to be full subcategories and closed under isomorphism classes. For any additive category \( \mathcal{C} \) and any object \( M \in \mathcal{C} \), we set \( \text{add } M \) as the subcategory of objects \( C \in \mathcal{C} \) which is isomorphic to a direct summand of \( M^{\oplus s} \) for some \( s \geq 1 \). If \( \mathcal{C} = \text{fd } A \), then \( \text{Fac } M \) (resp. \( \text{Sub } M \)) is defined to be the subcategory of objects \( X \in \mathcal{C} \) which has a surjection from (resp. an injection to) \( M^{\oplus s} \) for some \( s \geq 1 \). Moreover, \( M^\perp \) (resp. \( ^\perp M \)) consists of all \( X \in \text{fd } A \) such that \( \text{Hom}_A(M,X) = 0 \) (resp. \( \text{Hom}_A(X,M) = 0 \)).

2. Basic properties of complete special biserial algebras

2.1. Complete special biserial algebras. We first recall the definition of complete special biserial algebras.

Definition 2.1. We say that \( A = \overline{KQ}/I \) is a complete special biserial algebra if \( Q \) is a finite quiver and \( I \subset \overline{KQ} \) is an ideal and the following conditions are satisfied:

(a) The ideal \( I \) is generated by some finite subset \( X \subset \overline{KQ} \) such that each element of \( X \) is a path of length \( \geq 2 \) in \( Q \) or of the form \( p - q \) with \( p, q \) paths of length \( \geq 2 \) in \( Q \).
(b) For each \(i \in Q_0\), the number of arrows starting at \(i\) in \(Q\) is at most two.
(c) For each \(i \in Q_0\), the number of arrows ending at \(i\) in \(Q\) is at most two.
(d) If \(\alpha \in Q_1\) is an arrow ending at \(i \in Q_0\) and \(\beta \neq \gamma \in Q_1\) are arrows starting at \(i\), then \(\alpha \beta \in I\) or \(\alpha \gamma \in I\).
(e) If \(\alpha \in Q_1\) is an arrow starting at \(i \in Q_0\) and \(\beta \neq \gamma \in Q_1\) are arrows ending at \(i\), then \(\beta \alpha \in I\) or \(\gamma \alpha \in I\).

Moreover, \(A\) is called a \textit{complete string algebra} if we can choose \(X\) so that each element of \(X\) is a path of length \(\geq 2\).

Throughout this paper, we assume that \(A\) is a complete special biserial algebra and \(n := \#Q_0\) unless otherwise stated.

In the definition above, we define \(X_+ \supset X\) by adding two paths \(p,q\) in \(Q\) to \(X\) for each element of the form \(p - q\) in \(X\), and set \(I_+ := \langle X_+ \rangle \supset I\) and \(A_+ := \widehat{KQ}/I_+\). Then, \(A_+\) is a complete string algebra.

In this paper, “maximal nonzero paths” in the quiver \(Q\) play an important role. Thus, we prepare some symbols on paths in \(Q\).

**Definition 2.2.** We define the following symbols.

1. For \(i \in Q_0\), we write \(e_i\) for the path at \(i\) of length 0.
2. We define \(\text{MP}_*(A), \text{MP}^*(A), \text{MP}(A)\) by
   \[
   \begin{align*}
   \text{MP}_*(A) &:= \{p : \text{paths of length } \geq 1 \text{ in } Q \mid p \neq 0 \text{ and } p\alpha = 0 \text{ in } A_+ \text{ for any } \alpha \in Q_1\}, \\
   \text{MP}^*(A) &:= \{p : \text{paths of length } \geq 1 \text{ in } Q \mid p \neq 0 \text{ and } \alpha p = 0 \text{ in } A_+ \text{ for any } \alpha \in Q_1\}, \\
   \text{MP}(A) &:= \text{MP}_*(A) \cap \text{MP}^*(A).
   \end{align*}
   \]
3. We define \(\overline{\text{MP}}_*(A), \overline{\text{MP}}^*(A), \overline{\text{MP}}(A)\) by
   \[
   \begin{align*}
   \overline{\text{MP}}_*(A) &:= \text{MP}_*(A) \cup \{e_i \mid \text{there exists at most one arrow starting at } i\}, \\
   \overline{\text{MP}}^*(A) &:= \text{MP}^*(A) \cup \{e_i \mid \text{there exists at most one arrow ending at } i\}, \\
   \overline{\text{MP}}(A) &:= \text{MP}_*(A) \cap \overline{\text{MP}}^*(A).
   \end{align*}
   \]
4. A cycle \(c\) in \(Q\) is said to be \textit{repeatable} in \(A\) if \(c^m \neq 0\) in \(A\) for all \(m \in \mathbb{Z}_{\geq 1}\). We set \(\text{Cyc}(A)\) as the set of repeatable cycles \(c\) in \(Q\) in \(A\) such that no shorter repeatable cycles \(c'\) in \(Q\) and \(m \in \mathbb{Z}_{\geq 2}\) satisfy \(c = (c')^m\) as paths.

Clearly, the sets defined above do not change when \(A\) is replaced to \(A_+\).

We give an example.

**Example 2.3.** Let \(Q\) be the quiver

![Quiver Diagram]

and \(I \subset \widehat{KQ}\) be the ideal generated by the paths
\[
\begin{align*}
\alpha_3\delta_1, \delta_1\beta_1, \beta_3\delta_2, \alpha_2\delta_2, \beta_1\alpha_3, \delta_2\gamma_1, \gamma_3\delta_3, \beta_2\gamma_2, \gamma_1\beta_3, \delta_4\gamma_3, \delta_1\delta_2\delta_3\delta_4.
\end{align*}
\]

Set \(\alpha_{i+3} := \alpha_i\), and so on. Then, we can check that \(A := \widehat{KQ}/I\) is a complete special biserial algebra. In this case, we get
\[
\begin{align*}
\text{MP}_*(A) &= \{\delta_1\delta_2\delta_3, \delta_2\delta_3\delta_4, \delta_3\delta_4, \delta_1\}\}, \\
\text{MP}^*(A) &= \{\delta_1, \delta_1\delta_2, \delta_1\delta_2\delta_3, \delta_2\delta_3\delta_4\}, \\
\text{MP}(A) &= \{\delta_1\delta_2\delta_3, \delta_2\delta_3\delta_4\}, \\
\overline{\text{MP}}_*(A) &= \text{MP}_*(A) \cup \{e_2, e_3, e_8\}, \\
\overline{\text{MP}}^*(A) &= \text{MP}^*(A) \cup \{e_1, e_2, e_7\}, \\
\overline{\text{MP}}(A) &= \text{MP}(A) \cup \{e_2, e_7\},
\end{align*}
\]
and

\[ \text{Cyc}(A) = \{ \alpha_i \alpha_{i+1} \alpha_{i+2}, \beta_i \beta_{i+1} \beta_{i+2}, \gamma_i \gamma_{i+1} \gamma_{i+2} \mid i \in \{1, 2, 3\} \}. \]

We remark that there may exist cycles which are nonzero in \( A \) but are not repeatable.

**Example 2.4.** Let \( A = K[[x, y]]/(x^2, y^2) \). Then, \( A \) is a complete special biserial algebra, and \( \text{Cyc}(A) = \{ xy, yx \} \). The loops \( x \) and \( y \) are clearly nonzero in \( A \), but neither is a repeatable cycle with respect to \( A \), since \( x^2 = y^2 = 0 \) in \( A \).

It is easy to see that any \( M \in \text{fd} A \) admits a finite-dimensional quotient algebra \( A' \) of \( A \) such that \( \text{Cyc}(M) = \text{Cyc}(A') \), and the indecomposable modules over finite-dimensional special biserial algebras and the homomorphisms between them were completely classified by \([BR, WW, Kra, CB2]\) in a combinatorial way. Though we will recall necessary properties below, we remark that \([GLFS]\) Section 5] is a good summary of modules over special biserial algebras.

We first recall the notion of walks in the quiver \( Q \). We define the double quiver \( Q' \) of \( Q \) so that the vertices set \( Q_0 = Q_0 \), and that the arrows set \( Q_1 \) is

\[ Q_1 \{ \alpha^{-1} : j \to i \mid (\alpha : i \to j) \in Q_1 \}. \]

Then, any path in the double quiver \( Q' \) is called a walk in \( Q \). In other words, a walk can be written as \( \alpha_1^{e_1} \alpha_2^{e_2} \cdots \alpha_l^{e_l} \) with \( l \in \mathbb{Z}_{\geq 0} \), \( \alpha_k \in Q_1 \) and \( e_k \in \{ \pm 1 \} \) such that the target of \( \alpha_k^{e_k} \) is the source of \( \alpha_{k+1}^{e_{k+1}} \) in \( Q' \).

Now, we can define strings and bands in \( A \). First, a walk \( s \) in \( Q \) is called a string admitted in \( A \) if \( s \) does not have a subwalk of the form \( \alpha \alpha^{-1} \), \( \alpha^{-1} \alpha \), \( p \), or \( p^{-1} \), where \( \alpha \in Q_1 \) and \( p \) is a path in \( Q \) such that \( p \in I_+ \). Second, a walk \( b \) in \( Q \) is called a band admitted in \( A \) if \( b^2 \) is a string and \( b \) admits no subwalk \( b' \) such that \( b = (b')^m \) for some \( m \geq 2 \).

For any string \( s \) in \( Q \), the reversed walk \( s^{-1} \) is also admitted in \( A \), and \( s \) and \( s^{-1} \) are the only strings isomorphic to \( s \). On the other hand, we say that two bands \( b, b' \) in \( A \) are isomorphic as bands in this paper if \( b' \) is a cyclic permutation of \( b \) or \( b^{-1} \).

We associate \( A \)-modules to strings and bands in \( A \) as follows. The construction below is due to \( [BR, WW] \). Let \( s = \alpha_1^{e_1} \alpha_2^{e_2} \cdots \alpha_l^{e_l} \) be a string in \( A \), and \( i_k \) be the source of \( \alpha_k^{e_k} \) for \( k \in \{1, 2, \ldots, l\} \), and \( i_{k+1} \) be the target of \( \alpha_k^{e_k} \). Then, we set a string module \( M(s) \) by

- \( M(s) := \bigoplus_{k=1}^{l+1} V_k \) with \( V_k := K \) as a \( K \)-vector space;
- for each \( k \in \{1, 2, \ldots, l+1\} \), the vector space \( V_k \) is associated to the vertex \( i_k \in Q_0 \);
- for each \( k \in \{1, 2, \ldots, l\} \), the action of the arrow \( \alpha_k \) is defined so that \( \alpha_k \) induces the identity map \( K = V_k \to V_{k+1} = K \) if \( e_k = 1 \), and the identity map \( K = V_{k+1} \to V_k = K \) if \( e_k = -1 \).

Next, let \( b = \alpha_1^{e_1} \alpha_2^{e_2} \cdots \alpha_l^{e_l} \) be a band, and \( i_k \) be the source of \( \alpha_k^{e_k} \) for \( k \in \{1, 2, \ldots, l\} \), and \( m \in \mathbb{Z}_{\geq 1} \), \( \lambda \in K^\times \). We set a band module \( M(b, m, \lambda) \) by

- \( M(b, m, \lambda) := \bigoplus_{k=1}^{l+1} V_k \) with \( V_k := K^m \) as a \( K \)-vector space;
- for each \( k \in \{1, 2, \ldots, l\} \), the vector space \( V_k \) is associated to the vertex \( i_k \in Q_0 \);
- for each \( k \in \{1, 2, \ldots, l-1\} \), the action of the arrow \( \alpha_k \) is defined so that \( \alpha_k \) induces the identity map \( K^m = V_k \to V_{k+1} = K^m \) if \( e_k = 1 \), and the identity map \( K^m = V_{k+1} \to V_k = K^m \) if \( e_k = -1 \);
- the action of the arrow \( \alpha_l \) is defined so that \( \alpha_l \) induces the \( K \)-linear map \( J_{\lambda} : K^m = V_k \to V_{k+1} = K^m \) if \( e_k = 1 \), and the identity map \( J_{\lambda} : K^m = V_{k+1} \to V_k = K^m \) if \( e_k = -1 \), where

\[
J_{\lambda} := \lambda \cdot 1_m + \begin{bmatrix} 0 & 1_{m-1} \\ 0 & 0 \end{bmatrix}
\]

is the Jordan block of size \( m \) and eigenvalue \( \lambda \).

We mainly deal with the case \( m = 1 \) in this paper, so we write \( M(b, \lambda) := M(b, 1, \lambda) \), and call such modules simple band modules.
For example, in the setting of Example 2.3, \( s := \beta_1 \beta_2 \gamma_1^{-1} \delta_3 \delta_4 \) is a string admitted in \( A \), and \( b := \gamma_2^{-1} \gamma_1^{-1} \delta_3 \delta_4 \) is a band admitted in \( A \). The corresponding modules can be depicted as

\[
M(s) = \begin{array}{ccc}
3 & 4 & 5 \\
\downarrow & \downarrow & \downarrow \\
6 & 7 & 8
\end{array}, \quad M(b, \lambda) = \begin{array}{ccc}
5 & 6 & 7 \\
\downarrow & \downarrow & \downarrow \\
8 & \lambda & \lambda
\end{array}.
\]

We have the following classification of indecomposable modules.

**Proposition 2.5.** (BR Section 3) (WW) Proposition 2.3] Let \( M \in \text{fd} A \) be indecomposable. Then, \( M \) is isomorphic to one of the following modules:

(a) the string module \( M(s) \) for some string \( s \) in \( A \);
(b) the band module \( M(b, m, \lambda) \) for some band \( b \) in \( A \), \( m \in \mathbb{Z}_{\geq 1} \) and \( \lambda \in K^\times \);
(c) the indecomposable projective-injective module \( P_i \) for some vertex \( i \in Q_0 \) which admits two paths \( p, q \) from \( i \) such that \( p, q \notin I \) and \( p - q \in I \).

Next, we explain the results on the homomorphisms between indecomposable modules by \( \text{CB}2, \text{Kra} \). Though their results cover all indecomposable modules, we only consider string modules and simple band modules here, which are enough for our purpose. We need the following symbols.

Let \( s = \alpha_1^{\varepsilon_1} \alpha_2^{\varepsilon_2} \cdots \alpha_l^{\varepsilon_l} \) be a string in \( A \). For any \( u, v \in \{1, 2, \ldots, l + 1\} \) with \( u \leq v \), we set

\[
s|\_{u,v} := \alpha_u^{\varepsilon_u} \alpha_{u+1}^{\varepsilon_{u+1}} \cdots \alpha_{v-1}^{\varepsilon_{v-1}}.
\]

This is a string from \( i_u \) to \( i_v \). In particular, \( s|\_{u,u} = \varepsilon_i \).

Let \( b = \alpha_1^{\varepsilon_1} \alpha_2^{\varepsilon_2} \cdots \alpha_l^{\varepsilon_l} \) be a band in \( A \). For any \( u, v \in \{1, 2, \ldots, l\} \), we set

\[
b|\_{u,v} := \begin{cases}
\alpha_u^{\varepsilon_u} \alpha_{u+1}^{\varepsilon_{u+1}} \cdots \alpha_v^{\varepsilon_v} & (u \leq v) \\
\alpha_u^{\varepsilon_u} \alpha_{u+1}^{\varepsilon_{u+1}} \cdots \alpha_{v-1}^{\varepsilon_{v-1}} & (u > v)
\end{cases}
\]

where \( \alpha_{i+1} := \alpha_i \) and \( \varepsilon_{i+1} := \varepsilon_i \). This is a string from \( i_u \) to \( i_v \), and strictly shorter than \( b \). We have \( b|\_{u,u} = \varepsilon_i \) also in this case.

We define the symbols \( \text{fac} M \) and \( \text{sub} M \) as follows to describe the homomorphisms between indecomposable modules.

If \( M = M(s) \) for some string \( s \) in \( A \), we set

\[
\text{fac} M := \{(u, v, s|\_{u,v}) \mid 1 \leq u \leq v \leq l + 1, \varepsilon_{u-1} \in \{-1, 0\}, \varepsilon_v \in \{1, 0\}\},
\]

\[
\text{sub} M := \{(u, v, s|\_{u,v}) \mid 1 \leq u \leq v \leq l + 1, \varepsilon_{u-1} \in \{1, 0\}, \varepsilon_v \in \{-1, 0\}\},
\]

where we set \( \varepsilon_0 = \varepsilon_{l+1} := 0 \).

Similarly, if \( M = M(b, \lambda) \) for some band \( b \) in \( A \), we define

\[
\text{fac} M := \{(u, v, b|\_{u,v}) \mid 1 \leq u, v \leq l, \varepsilon_{u-1} = -1, \varepsilon_v = 1\},
\]

\[
\text{sub} M := \{(u, v, b|\_{u,v}) \mid 1 \leq u, v \leq l, \varepsilon_{u-1} = 1, \varepsilon_v = -1\}.
\]

We finally consider the case \( M \) is indecomposable projective-injective. In this case, we set

\[
\text{fac} M := \text{fac}(M/\text{soc} M), \quad \text{sub} M := \text{sub}(\text{rad} M).
\]

For any \( M, M' \) which are string modules or simple band modules or indecomposable projective-injective modules, we set

\[
H_{M,M'} := \{((u, v, t), (u', v', t')) \in \text{fac} M \times \text{sub} M' \mid t' = t \text{ or } t' = t^{-1}\}.
\]

For each \( h = ((u, v, t), (u', v', t')) \in H_{M,M'} \), we set \( f_h : M \to M(t) \to M' \) if \( t' = t \), and \( f_h : M \to M(t \cong M(t^{-1}) \to M' \) if \( t' = t^{-1} \), where the first map is the canonical surjection and the last map is the canonical injection in either case.
Proposition 2.6. [CB] Section 2, Theorem | [Kra] Theorem] Let $M, M' \in \text{fd} A$ be string modules, simple band modules or indecomposable projective-injective modules.

1. If $M$ and $M'$ are isomorphic and not string modules, then $f_h$ for all $h \in H_{M,M'}$ and the isomorphism $M \cong M'$ give a $K$-basis of $\text{Hom}_A(M,M')$.

2. Otherwise, $f_h$ for all $h \in H_{M,M'}$ give a $K$-basis of $\text{Hom}_A(M,M')$.

The homomorphisms appearing above are called standard homomorphisms.

Also, the Auslander-Reiten translate $\tau M$ of $M \in \text{fd} A$ can be combinatorially described in the case that $A$ is finite-dimensional. To explain this, we prepare some symbols.

Let $s = \alpha_1^{e_1} \alpha_2^{e_2} \cdots \alpha_l^{e_l}$ be a string in $A$.

Define $l_1, l_2 \in \{0, 1, \ldots, l\}$ as the maximal integers such that $e_1 = e_2 = \cdots = e_{l_1} = -1$ and $e_{l_l-2} = \cdots = e_{l_l-1} = e_{l_l} = 1$, respectively, and two paths $p_1, p_2$ in $Q$ by $p_1 := (s|_{1,l_l})^{-1}$ and $p_2 := s|_{l_l+1, l_l+1}$. Then, their lengths are $l_1$ and $l_2$, respectively. We can check that $l_1 + l_2 \neq l - 1$, so $l_1 + l_2 < l$ implies $l_1 + l_2 \leq l - 2$.

Assume $l \geq 1$. If $p_1 \not\in \text{MF}_s(A)$, then we take $\beta_1 \in Q_1$ so that $p_1 \beta_1 \neq 0$, and that $\beta_1 \neq \alpha_1$ if moreover $l_1 = 0$. Such $\beta_1$ is uniquely determined. Otherwise, we do not define $\beta_1$. Similarly, in the case $p_2 \not\in \text{MF}_s(A)$, we define (or do not define) $\beta_2 \in Q_1$ in the same rule.

If $l = 0$ and $s = e_i$, then we choose distinct arrows $\beta_1, \beta_2, \ldots, \beta_m$ so that $\{\beta_i\}_{i=1}^m \in Q_1$ is the set of arrows starting at $i$.

For each $i \in \{1, 2\}$ such that $\beta_i$ is defined, let $j_i$ be the target of $\beta_i$. We can uniquely take a path $q_i \in \text{MF}(A)$ ending at $j_i$ such that the length of $q_i$ is 0 or the last arrow of $q_i$ is not $\beta_i$.

Now, we can explicitly write down the Auslander-Reiten translate of $M(s)$.

Proposition 2.7. [WW] Lemmas 3.1, 3.2] Let $A$ be a finite-dimensional special biserial algebra.

1. Let $s$ be a string in $A$, and $X \subset \{1, 2\}$ be the set of $i \in \{1, 2\}$ such that $\beta_i$ is defined.

   i. If $X = \emptyset$, then $\tau M(s) = \begin{cases} 0 & (l_1 + l_2 = l, M(s) \in \text{proj} A) \\ \text{rad} P & (l_1 + l_2 = l, M(s) = P/\text{soc} P, P \in \text{proj} A \cap \text{inj} A) \\ M(s|_{l_1+2,l_2-l_2}) & (l_1 + l_2 \neq l)\end{cases}$

   ii. If $X = \{2\}$, then $\tau M(s) = \begin{cases} M(q_2^{-1}) & (l_1 = l) \\ M(s|_{l_1+2,l_1+1} \cdot \beta_2 q_2^{-1}) & (l_1 \neq l)\end{cases}$.

   iii. If $X = \{1\}$, then $\tau M(s) = \begin{cases} M(q_1) & (l_2 = l) \\ M(q_1 \beta_1^{-1} \cdot s|_{l_1,l_2}) & (l_2 \neq l)\end{cases}$.

   iv. If $X = \{1, 2\}$, then $\tau M(s) = M(q_1 \beta_1^{-1} \cdot s \cdot \beta_2 q_2^{-1})$.

2. Let $b$ be a band in $A$, and $m \in \mathbb{Z}_{\geq 1}$, $\lambda \in K^\times$. Then, $\tau M(b, m, \lambda) = M(b, m, \lambda)$.

As we have seen above, the original algebra $A$ and the string algebra $A_+$ share almost all properties. One can see that there is little difference between $A$ and $A_+$ also in our results later.

2.2. Complete gentle algebras. To investigate complete special biserial algebras, we will use the nice subclass of complete special biserial algebras called complete gentle algebras.

Definition 2.8. Let $A = \overline{KQ}/I$ be a complete special biserial algebra. Then, we call $A$ a complete gentle algebra if $I$ is generated by some set $X$ of paths in $Q$ of length 2 such that

(a) if $\alpha \in Q_1$ is an arrow ending at $i \in Q_0$ and $\beta \neq \gamma \in Q_1$ are arrows starting at $i$, then $\alpha \beta \not\in X$ or $\alpha \gamma \not\in X$. 


(b) if $\alpha \in Q_1$ is an arrow starting at $i \in Q_0$ and $\beta \neq \gamma \in Q_1$ are arrows ending at $i$, then $\beta \alpha \notin X$ or $\gamma \alpha \notin X$.

The following property is easily deduced.

**Proposition 2.9.** Let $A = \widetilde{KQ}/I$ be a complete special biserial algebra. Then, there exists an admissible ideal $\widetilde{I}$ of $\widetilde{KQ}$ such that $\widetilde{I} \subset I$ and $\tilde{A} := \widetilde{KQ}/\widetilde{I}$ is a complete gentle algebra.

We remark that the choice of the ideal $\widetilde{I}$ is not necessarily unique. This happens, for example, when there exists $i \in Q_0$ such that $\alpha \in Q_1$ is an arrow ending at $i$, $\beta \neq \gamma \in Q_1$ are arrows starting at $i$, and that $\alpha \beta = 0$, $\alpha \gamma = 0$ in $A$.

Complete gentle algebras are constructed also in the following way. This construction is known to experts.

**Definition 2.10.** Suppose that a pair $(\tilde{Q}, \sim)$ satisfies the following conditions:

(a) $\tilde{Q}$ is the disjoint union $\prod_{x \in X} Q^{(x)}$ with $X = X_1 \sqcup X_2$ a finite set and each $Q^{(x)}$ is a quiver

\[
x, 1 \xrightarrow{(x, 2)} \cdots \xrightarrow{(x, n_x)} (x, 1)
\]

with $n_x \in \mathbb{Z}_{\geq 1}$;

(b) $\sim$ is an equivalence relation on the vertices set $\tilde{Q}_0$ such that

- every equivalence class $[(x, i)]$ with respect to $\sim$ has at most two elements; and that
- $[(x, 1)] = \{(x, 1)\}$ if $x \in X_1$ and $n_x = 1$.

Then, we associate a complete gentle algebra $A := \tilde{KQ}/I$ for the pair $(\tilde{Q}, \sim)$ given as follows:

(a) $Q$ is the quiver whose vertices set $Q_0$ and whose arrows set $Q_1$ are

\[
Q_0 := \tilde{Q}_0/\sim, \quad Q_1 := \{\gamma : [(x, i)] \to [(x, i')] \mid (\alpha : (x, i) \to (x, i')) \in \tilde{Q}_1\};
\]

(b) $I$ is the ideal of $\tilde{KQ}$ generated by the paths of the form $\gamma_{\alpha \gamma_{\beta}}$ of length 2 in $Q$ with $\alpha, \beta \in \tilde{Q}_1$ such that the target $(x, i)$ of $\alpha$ and the source $(y, j)$ of $\beta$ in $\tilde{Q}$ satisfy $(x, i) \neq (y, j)$ and $(x, i) \sim (y, j)$ in $\tilde{Q}_0$.

Note that $Q$ has the same number of arrows as $\tilde{Q}$, while it has less vertices than $\tilde{Q}$ unless any equivalence class with respect to $\sim$ has one element. We simply write $\alpha$ for the arrow $\gamma_{\alpha}$ in $Q$ above if there is no confusion.

**Example 2.11.** We define a pair $(\tilde{Q}, \sim)$ as follows:

(a) $\tilde{Q}$ is the quiver

\[
(1, 1) \xrightarrow{a_1} (1, 2) \xrightarrow{a_2} (1, 3) \xrightarrow{a_3} (1, 4) \xrightarrow{a_4} (1, 5) \xrightarrow{a_5} (1, 6)
\]

\[
(2, 1) \xrightarrow{\beta} (2, 2)
\]

\[
(3, 1) \xrightarrow{\gamma_1} (3, 2) \xrightarrow{\gamma_2} (3, 3) \xrightarrow{\gamma_3} (3, 4) \xrightarrow{\gamma_4} (3, 5)
\]

\[
\xrightarrow{\gamma_5}
\]

(b) $\sim$ is the equivalence relation on $\tilde{Q}_0$ such that the nontrivial equivalences are $(1, 2) \sim (1, 5), (1, 3) \sim (3, 1), (1, 6) \sim (2, 1)$ and $(3, 3) \sim (3, 5)$.
Then, this pair gives the gentle algebra \(A := \hat{K}Q/I\), where \(Q\) is the quiver

\[
\begin{array}{c}
1 \\
\alpha_1 \\
2 \\
\alpha_5 \\
3 \xrightarrow{\gamma_1} 7 \xrightarrow{\gamma_2} 8 \xrightarrow{\gamma_3} 9 \\
\alpha_4 \\
\alpha_2 \xrightarrow{\gamma_5} 6 \\
4
\end{array}
\]

and the ideal \(I \subset \hat{K}Q\) is generated by the paths

\(\alpha_1\alpha_5, \alpha_4\alpha_2, \alpha_5\beta, \gamma_1\gamma_5\).

Therefore,

\[
\begin{align*}
\text{MP}(A) &= \{\alpha_1\alpha_5, \alpha_4\alpha_2, \alpha_5\beta, \gamma_1\gamma_5\}, \\
\text{MP}(A) &= \text{MP}(A) \cup \{e_1, e_4, e_5, e_6, e_7, e_9\}, \\
\text{Cyc}(A) &= \{\gamma_i\gamma_{i+1} \cdots \gamma_i \mid i \in \{1, 2, \ldots, 5\}\},
\end{align*}
\]

where \(\gamma_{i+5} := \gamma_i\).

Actually, all complete gentle algebras are obtained in the way above. This fact is well-known among experts, but we include the proof for the convenience of the reader.

**Proposition 2.12.** Any complete gentle algebra is obtained from some pair \((\tilde{Q}, \sim)\) in Definition 2.10.

**Proof.** We may assume that every \(i \in Q_0\) has some arrow starting or ending at \(i\). We construct a pair \((\tilde{Q}, \sim)\).

We consider an equivalence relation on \(\text{Cyc}(A)\) such that \(p, q \in \text{Cyc}(A)\) are equivalent if and only if \(q\) is a cyclic permutation of \(p\). We can take a complete system \(X_2 \subset \text{Cyc}(A)\) of representatives with respect to this equivalence relation. Set \(X_1 := \text{MP}(A)\) and \(X := X_1 \sqcup X_2\).

For each \(p \in X\) of length \(l_p\), we set a quiver \(Q^{(p)}\) by the following rule:

- If \(p \in X_1\), then \(Q^{(p)}\) is
  \[
  (p, 1) \rightarrow (p, 2) \rightarrow \cdots \rightarrow (p, l_p) .
  \]
- If \(p \in X_2\), then \(Q^{(p)}\) is
  \[
  (p, 1) \rightarrow (p, 2) \rightarrow \cdots \rightarrow (p, l_p) .
  \]

Now, we set \(\tilde{Q} := \bigsqcup_{p \in X} Q^{(p)}\). Then, we introduce an equivalence relation \(\sim\) on the vertices set \(\tilde{Q}_0\) so that \((p, u) \sim (q, v)\) if and only if \(i_u = j_v\) in \(Q_0\), where

\[
p = (i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_{l_p} \rightarrow i_1), \quad q = (j_1 \rightarrow j_2 \rightarrow \cdots \rightarrow j_{l_q} \rightarrow j_1),
\]

where the dashed last arrow in \(p\) exist if and only if \(p \in X_2\), and similar for \(q\).

Since \(A\) is a complete gentle algebra, each arrow \(\alpha \in Q_1\) admits exactly one \(p \in \text{MP}(A) \cup \text{Cyc}(A)\) such that \(\alpha\) appears in \(p\). Thus, each vertex \(i \in Q_0\) is involved in at most two \(p \in \text{MP}(A) \cup \text{Cyc}(A)\). This means that each equivalence class with respect to \(\sim\) contains at most two elements.

Now, we define a quiver \(Q\) and an ideal \(I \subset \hat{K}Q\) as in Definition 2.10, then we have \(A \cong \hat{K}Q/I\). \(\square\)
3. Preliminary on silting theory and canonical decompositions

3.1. Torsion pairs and 2-term silting complexes. To study complete special algebras, we would like to use existing results on the real Grothendieck group $K_0(\text{proj} \mathcal{A})$ for finite-dimensional algebras. However, complete special algebras are not necessarily finite-dimensional, so we need some preparation.

Let $\mathcal{T}, \mathcal{F} \subset \mathcal{fd} \mathcal{A}$ be full subcategories. Then, we call the pair $(\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathcal{fd} \mathcal{A}$ if $\mathcal{T} = \perp \mathcal{F}$ and $\mathcal{F} = \mathcal{T}^\perp$. We can check that $(\mathcal{T}, \mathcal{F})$ is a torsion pair if and only if $\text{Hom}_\mathcal{A}(\mathcal{T}, \mathcal{F}) = 0$ and any $M \in \mathcal{fd} \mathcal{A}$ has a short exact sequence $0 \to M' \to M \to M'' \to 0$ such that $M' \in \mathcal{T}$ and $M'' \in \mathcal{F}$. A full subcategory $\mathcal{T} \subset \mathcal{fd} \mathcal{A}$ is called a torsion class if $\mathcal{T}$ is closed under taking extensions and quotients. In this case, $(\mathcal{T}, \mathcal{T}^\perp)$ is a torsion pair, since $\mathcal{fd} \mathcal{A}$ is an abelian length category. Dually, $\mathcal{F} \subset \mathcal{fd} \mathcal{A}$ is called a torsion-free class if $\mathcal{F}$ is closed under taking extensions and submodules. We write $\text{tors} \mathcal{A}$ (resp. $\text{torf} \mathcal{A}$) for the set of torsion classes (resp. torsion-free classes) in $\mathcal{fd} \mathcal{A}$.

A finite-dimensional module $S \in \mathcal{fd} \mathcal{A}$ is called a brick if $\mathcal{End}_\mathcal{A}(S) \cong \mathbb{K}$, and write brick $\mathcal{A}$ for the set of isomorphism classes of bricks in $\mathcal{fd} \mathcal{A}$. Then, the following property originally proved for finite-dimensional algebras hold also in our setting, since $\mathcal{fd} \mathcal{A}$ is an abelian length category.

Lemma 3.1. Let $\mathcal{T} \in \text{tors} \mathcal{A}$ and $\mathcal{F} \in \text{torf} \mathcal{A}$ in $\mathcal{fd} \mathcal{A}$.

1. [DIRRT] Lemma 3.8] If $\mathcal{T} \cap \mathcal{F} \neq \{0\}$, then there exists $S \in \text{brick} \mathcal{A}$ such that $S \in \mathcal{T} \cap \mathcal{F}$.
2. [DIRRT] Lemma 3.9] The torsion class $\mathcal{T}$ is the smallest torsion class containing $\mathcal{T} \cap \text{brick} \mathcal{A}$, and the torsion class $\mathcal{F}$ is the smallest torsion-free class containing $\mathcal{F} \cap \text{brick} \mathcal{A}$.

In the rest of this section, let $J \subset \mathcal{I} := \langle \text{Cyc}(\mathcal{A}) \rangle$, and $\overline{\mathcal{A}} := \mathcal{A}/J$. The next proposition coming from results of Kimura [Kim] (cf. [EJR]) is crucial in our study.

Proposition 3.2. The following properties hold.

1. [Kim] Lemma 5.1] If $S \in \text{brick} \mathcal{A}$, then $SJ = 0$. Thus, $\text{brick} \mathcal{A} = \text{brick} \overline{\mathcal{A}}$.
2. [Kim] Theorem 5.4] There exists a bijection $\text{tors} \mathcal{A} \to \text{tors} \overline{\mathcal{A}}$ given by $\mathcal{T} \mapsto \mathcal{T} \cap \mathcal{fd} \overline{\mathcal{A}}$ preserving inclusions.

Proof. We first consider the element $x := \sum_{c \in \text{Cyc}(\mathcal{A})} c$. We can check that $x$ is in the center $Z(\mathcal{A})$, and the elements $x^k$ for $k \in \{1, 2, \ldots\}$ are linearly independent over $\mathbb{K}$. Thus, $\mathcal{A}$ can be considered as a module-finite $\mathbb{K}[\langle t \rangle]$-algebra by $\mathbb{K}[\langle t \rangle] \ni t \mapsto x \in \mathcal{A}$. Then, we can apply [Kim] Lemma 5.1, Theorem 5.4] to get (1) and (2) for any ideal $J \subset \langle x \rangle$.

We move to the general case $J \subset \mathcal{I}$. Let $c \in \text{Cyc}(\mathcal{A})$ be from $i \in Q_0$ to $i \in Q_0$. If there exists $c' \in \text{Cyc}(\mathcal{A}) \setminus \{c\}$ such that $c'$ is also from $i$ to $i$ (such $c'$ is unique if exists), then $c = -c'$ in $\mathcal{A}/(x)$; otherwise, $c = 0$. Thus, $\mathcal{A}/(x)$ is isomorphic to a finite-dimensional special biserial algebra. If $M$ is an indecomposable module in $\mathcal{fd}(\mathcal{A}/(x))$ which is not in $\mathcal{fd}(\mathcal{A}/\mathcal{I})$, then $M$ is a projective-injective module whose top and socle coincide by Proposition 2.5, so $M$ is not a brick. Therefore, we get $\text{brick}(\mathcal{A}/(x)) = \text{brick}(\mathcal{A}/\mathcal{I})$ and (1). This implies (2) by Lemma 3.1] as in the proof of [Kim] Lemma 5.3, Theorem 5.4].
Next, we recall some notions in silting theory of the homotopy category $K^b(\text{proj } A)$. Since $A/I_i$ is clearly finite-dimensional for all $i$, the homotopy category $K^b(\text{proj } A)$ is Krull-Schmidt by [KMI Corollary 4.6]. Therefore, any $U \in K^b(\text{proj } A)$ has a unique decomposition $U = \bigoplus_{n=1}^{\infty} U_i$ in $K^b(\text{proj } A)$ into indecomposable objects up to reordering. Thus, we write $[U]$ for the number of non-isomorphic indecomposable direct summands of $U \in K^b(\text{proj } A)$, so $n = \#Q_0 = |A|$. In this notation, if $U_i \neq U_j$ for any $i \neq j$, then $U$ is said to be basic. We say that a complex $U \in K^b(\text{proj } A)$ is 2-term if the terms of $U$ except $-1$st and 0th ones vanish.

**Definition 3.3.** Let $U$ be a 2-term complex in $K^b(\text{proj } A)$.

1. The complex $U$ is said to be 2-term presilting if $\text{Hom}_{K^b(\text{proj } A)}(U,U[k]) = 0$ for any $k \in \mathbb{Z}_{>0}$. We write $2\text{-psilt } A$ for the set of isoclasses of basic 2-term presilting complexes in $K^b(\text{proj } A)$.

2. The complex $U$ is said to be 2-term silting if $U$ is 2-term presilting and the smallest thick subcategory generated by $U$ is $K^b(\text{proj } A)$ itself. We write $2\text{-silt } A$ for the set of isoclasses of basic 2-term silting complexes in $K^b(\text{proj } A)$.

Note that we have to check only $\text{Hom}_{K^b(\text{proj } A)}(U,U[1]) = 0$ in (1), since $U$ is assumed to be a 2-term complex.

Thanks to the following properties, we can deal with 2-term silting complexes as in the case of finite-dimensional algebras. In the rest, we set $\overline{U} := U \otimes_A \overline{A}$ for any 2-term complex $U \in K^b(\text{proj } A)$.

**Proposition 3.4.** The following statements hold.

1. [Kim Propositions 4.2, 4.4] For any 2-term complexes $U,V \in K^b(\text{proj } A)$, the condition $\text{Hom}_A(U,V[1]) = 0$ holds if and only if $\text{Hom}_{K^b(\text{proj } A)}(U,V[1]) = 0$.

2. [VG Lemma 2.6] There exist bijections $2\text{-silt } A \rightarrow 2\text{-psilt } \overline{A}$ and $2\text{-psilt } A \rightarrow 2\text{-psilt } \overline{A}$ given by $U \mapsto \overline{U}$. Moreover, $|U| = |\overline{U}|$ holds.

Therefore, we can apply the following fundamental properties of 2-term silting complexes to complete special biserial algebras.

**Proposition 3.5.** For any $U \in 2\text{-psilt } A$, we have the following assertions.

1. [AIR Proposition 2.17] There exists some $T \in 2\text{-silt } A$ which has $U$ as a direct summand.

2. [AIR Proposition 3.3] The condition $U \in 2\text{-silt } A$ holds if and only if $|U| = n$.

3. [AIR Proposition 3.8] If $|U| = n - 1$, then there exist exactly two $T \in 2\text{-silt } A$ which have $U$ as a direct summand.

Let $U \in 2\text{-psilt } A$. If $A$ is finite-dimensional, then we can consider the Nakayama functor $\nu: K^b(\text{proj } A) \rightarrow K^b(\text{inj } A)$. Then, $H^0(U)$ is a $\tau$-rigid module, and $H^{-1}(\nu U)$ is a $\tau^{-1}$-rigid module, that is, $\text{Hom}_A(H^0(U),\tau H^0(U)) = 0$ and $\text{Hom}_A(\tau^{-1}(H^{-1}(\nu U)), H^{-1}(\nu U)) = 0$ [AIR Lemma 3.4], so we have two functorially finite torsion pairs $(\text{Fac } H^0(U), H^0(U)^\perp)$ in $\text{fd } A$ by [AS Theorem 5.10]. This is extended in our situation as follows.

**Definition 3.6.** Let $U \in 2\text{-psilt } A$, and $J \subset I_e$ satisfy that $\overline{A}$ is finite-dimensional. Then, we set two torsion pairs $(\overline{T}_U, \overline{F}_U)$ and $(\overline{T}_U, \overline{F}_U)$ in $\text{fd } A$ so that

$$
\overline{T}_U \cap \text{fd } \overline{A} = \nu H^{-1}(\nu \overline{U}), \quad \overline{T}_U \cap \text{fd } \overline{A} = \text{Sub } H^{-1}(\nu \overline{U}),
$$
$$
\overline{T}_U \cap \text{fd } \overline{A} = \text{Fac } H^0(\overline{U}), \quad \overline{T}_U \cap \text{fd } \overline{A} = H^0(\overline{U})^\perp,
$$

which are uniquely defined by Proposition 3.2 not depending on the choice of $J$. Moreover, we set $W_U := \overline{T}_U \cap \overline{F}_U$.

These two torsion pairs coincide if and only if $U \in 2\text{-silt } A$ [AIR Proposition 2.16, Theorems 2.12, 3.2]. If $A$ is a finite-dimensional algebra, then the correspondences $T \mapsto \overline{T}_T = \overline{T}_T$ and $T \mapsto \overline{F}_T = \overline{F}_T$ induce bijections $2\text{-silt } A \mapsto f\text{-tors } A$ and $2\text{-silt } A \mapsto f\text{-torf } A$, respectively [AIR].
Theorem 2.7. Here, f-tors $\overline{A}$ (resp. f-torf $\overline{A}$) is the set of functorially finite torsion (resp. torsion-free) classes in $\text{fd} A$. Therefore, for each $U \in 2$-silt $A$, there uniquely exists $T \in 2$-silt $A$ such that $\overline{T}_T = \overline{T}_U$ and dually, there uniquely exists $T' \in 2$-silt $A$ such that $\overline{F}_{T'} = \overline{F}_U$. This property is verified in our setting of complete special biserial algebras by Proposition 3.2.

**Definition 3.7.** Let $U \in 2$-silt $A$. Then, the Bongartz completion of $U$ is defined as the unique $T \in 2$-silt $A$ such that $\overline{T}_T = \overline{T}_U$ obtained from the bijections $2$-silt $A \to 2$-silt $\overline{A} \to \text{f-tors} \overline{A}$.

Similarly, the Bongartz cocompletion of $U$ is set as the unique $T' \in 2$-silt $A$ such that $\overline{F}_{T'} = \overline{F}_U$ given by the bijection $2$-silt $A \to 2$-silt $\overline{A} \to \text{f-torf} \overline{A}$.

Note that neither $(\overline{T}_U, \overline{F}_U)$ nor $(\overline{T}_U, \overline{F}_U)$ depends on the choice of the ideal $J$. We also remark that $(\overline{T}_U, \overline{F}_U) = (\overline{T}_U, \overline{F}_U)$ if and only if $U \in 2$-silt $A$.

To give nice "generators" of the torsion classes and the torsion-free classes, we recall the notion of semibricks. Let $S \subset \text{brick} A$. Then, we say that $S$ is a semibrick if $\text{Hom}_A(S, S') = 0$ for any $S \neq S' \in S$. By Proposition 3.2, we get the following useful property.

**Lemma 3.8.** Let $T \in \text{tors} A$, and $S \subset \text{brick} \overline{A}$ be a semibrick in $\text{fd} \overline{A}$. If $T \cap \text{fd} \overline{A}$ is the smallest torsion class in $\text{fd} \overline{A}$ containing $S$, then $T$ is the smallest torsion class in $\text{fd} A$ containing $S$.

Assume that $\overline{A}$ is finite-dimensional. Since $H^0(\overline{U})$ is a $\tau$-rigid $\overline{A}$-module, by [Asa1, Lemma 2.5 (5)], there uniquely exists a semibrick $S_U \subset \text{brick} \overline{A}$ such that $\overline{T}_U \cap \text{fd} \overline{A} = \text{Fac} H^0(\overline{U})$ is the smallest torsion class in $\text{fd} \overline{A}$ containing $S_U$. Then, $\overline{T}_U$ is the smallest torsion class in $\text{fd} A$ containing $S_U$ by Lemma 3.8. Similarly, we can define a semibrick $S'_U \subset \text{sbrick} \overline{A}$ such that $\overline{F}_U = \text{Sub} H^{-1}(\nu \overline{U})$ is the smallest torsion class in $\text{fd} A$ containing $S'_U$. The explicit descriptions of $S_U$ and $S'_U$ are given as follows.

**Lemma 3.9.** Let $U \in 2$-silt $A$, and $J \subset I_e$ satisfy that $\overline{A}$ is finite-dimensional. Set $B := \text{End}_\overline{A}(H^0(\overline{U}))$ and $B' := \text{End}_\overline{A}(H^{-1}(\nu \overline{U}))$. Decompose $U = \bigoplus_{i=1}^m U_i$ with $U_i$ indecomposable, and define

$$X_i := H^0(\overline{U}_i)/\sum_{f \in \text{rad}_B(H^0(\overline{U}_i))} \text{im} f, \quad X'_i := \bigcap_{f \in \text{rad}_B(H^{-1}(\nu \overline{U}_i), H^{-1}(\nu \overline{U}))} \text{Ker} f$$

for each $i$. Then, the following assertions hold.

1. [Asa1, Theorem 2.3] We have

$$S_U = \text{ind}(H^0(\overline{U})/\text{rad}_B H^0(\overline{U})) = \{X_i \mid i \in \{1, 2, \ldots, m\}\} \setminus \{0\},

S'_U = \text{ind}(\text{soc}_B H^{-1}(\nu \overline{U})) = \{X'_i \mid i \in \{1, 2, \ldots, m\}\} \setminus \{0\}.$$  

Thus, neither $S_U$ nor $S'_U$ depends on the choice of $J$.

2. For any $i, j \in \{1, 2, \ldots, m\}$, we have

$$\text{Hom}_B(H^0(\overline{U}_i), X_j) \cong \begin{cases} K & (i = j, \ X_j \neq 0) \\ 0 & \text{(otherwise)} \end{cases},$$

$$\text{Hom}_B(X'_j, H^{-1}(\nu \overline{U})) \cong \begin{cases} K & (i = j, \ X'_j \neq 0) \\ 0 & \text{(otherwise)} \end{cases}.$$  

3. For any $i$, we have $X_i \neq 0$ or $X'_i = 0$.

**Proof.** (2) Take the Bongartz completion $T \in 2$-silt $A$ of $U$. Then, $\overline{T} \in 2$-silt $\overline{A}$ is the Bongartz completion of $\overline{U}$. Then, we can apply [Asa1, Theorem 3.3, Lemma 3.14] (see also [KY, Lemma 5.3]) to $\overline{T}$, and have

$$\text{Hom}_B(H^0(\overline{U}_i), X_j) \cong \text{Hom}_B(H^0(\text{fd} A)(\overline{U}_i), X_j) \cong \begin{cases} K & (i = j, \ X_j \neq 0) \\ 0 & \text{(otherwise)} \end{cases}.$$  

The other isomorphism is similarly shown by using the Bongartz cocompletion.
(3) Assume that $X_i' = 0$. Then, since we can regard
\[
\text{rad}_A(H^{-1}(\nu\overline{U}_{i})), H^{-1}(\nu\overline{U}_i)) \subset \text{rad}_A(H^{-1}(\nu\overline{U}_i)), H^{-1}(\nu\overline{T}_i)),
\]
we have
\[
f \in \text{rad}_A(H^{-1}(\nu\overline{U}_i)), H^{-1}(\nu\overline{T}_i)) \quad \text{Ker } f = 0.
\]
Obviously, $\overline{U}_i$ is an indecomposable direct summand of $\overline{T}$, so by \cite{Asa1} Lemma 3.13 (1),
\[
H^0(\overline{U}_i) / \sum_{f \in \text{rad}_A(H^0(\overline{T}), H^0(\overline{U}_i))} \text{Im } f \neq 0.
\]
This implies that $X_i \neq 0$. \hfill \square

3.2. Grothendieck groups. We next prepare some notions on Grothendieck groups. We first remark that the definitions we give below are specialized for complete special biserial algebras, and that van Garderen \cite{VG} Section 2.1 summarized important properties of Grothendieck groups for all algebras over complete local Noetherian commutative rings.

In our setting, the complete special biserial algebra $A$ is given by a quiver and relations, so the projective $A$-modules $P_i = e_i A$ for all $i \in Q_0$ are the indecomposable objects of $\text{proj } A$, and the simple tops $S_i$ of $P_i$ are the simple objects of the abelian length category $\text{fd } A$. Thus, $(P_i)_{i \in Q_0}$ is a $\mathbb{Z}$-basis of $K_0(\text{proj } A)$ and $(S_i)_{i \in Q_0}$ is a $\mathbb{Z}$-basis of $K_0(\text{fd } A)$. Then, $\dim_K \text{Hom}_A(P_i, S_j) = \delta_{ij}$.

Therefore, we can define the Euler form \( \langle \cdot , \cdot \rangle : K_0(\text{proj } A) \times K_0(\text{fd } A) \to \mathbb{Z} \) as in the case of finite-dimensional algebras by $(P_i, S_j) := \delta_{ij}$.

The notions above are easily extended to the real Grothendieck groups $K_0(\text{proj } A)_{\mathbb{R}} := K_0(\text{proj } A) \otimes_{\mathbb{Z}} \mathbb{R}$ and $K_0(\text{fd } A)_{\mathbb{R}} := K_0(\text{fd } A) \otimes_{\mathbb{Z}} \mathbb{R}$, and we can regard each $\theta \in K_0(\text{proj } A)_{\mathbb{R}}$ as an $R$-linear form \( \langle \cdot , \cdot \rangle = K_0(\text{fd } A)_{\mathbb{R}} \to \mathbb{R} \).

Now, we can define stability conditions and numerical torsion pairs for complete special algebras $A$.

Definition 3.10. Let $\theta \in K_0(\text{proj } A)_{\mathbb{R}}$.

(1) \cite{Kin} Definition 1.1] Let $M \in \text{fd } A$. Then, $M$ is said to be $\theta$-semistable if $\theta(M) = 0$ and any quotient module $N$ of $M$ satisfies $\theta(N) \geq 0$. We define $\mathcal{W}_\theta \subset \text{fd } A$ as the category of all $\theta$-semistable modules $M \in \text{fd } A$.

(2) \cite{BKT} Subsection 3.1] We define the following torsion pairs $(\mathcal{T}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_0, \mathcal{F}_0)$ of $\text{fd } A$:

\[
\mathcal{T}_\theta := \{ M \in \text{fd } A \mid \text{any quotient module }N \text{ of }M \text{ satisfies }\theta(N) \geq 0 \},
\]

\[
\mathcal{F}_\theta := \{ M \in \text{fd } A \mid \text{any submodule }L \neq 0 \text{ of }M \text{ satisfies }\theta(L) < 0 \},
\]

\[
\mathcal{T}_0 := \{ M \in \text{fd } A \mid \text{any quotient module }N \neq 0 \text{ of }M \text{ satisfies }\theta(N) > 0 \},
\]

\[
\mathcal{F}_0 := \{ M \in \text{fd } A \mid \text{any submodule }L \text{ of }M \text{ satisfies }\theta(L) \leq 0 \}.
\]

By definition, $\mathcal{W}_\theta = \mathcal{T}_\theta \cap \mathcal{F}_\theta$. It is easy to see that $\mathcal{W}_\theta$ is a wide subcategory of $\text{fd } A$, that is, a full subcategory closed under extensions, kernels and cokernels. In particular, $\mathcal{W}_\theta$ is an abelian length category, so any simple object $S$ in $\mathcal{W}_\theta$ is a brick.

Definition 3.11. We define the following notions.

(1) \cite{BST} Definition 3.2] To each $M \in \text{fd } A \setminus \{0\}$, we associate the wall
\[
\Theta_M := \{ \theta \in K_0(\text{proj } A)_{\mathbb{R}} \mid M \in \mathcal{W}_\theta \}.
\]

In the rest, we consider the wall-chamber structure on $K_0(\text{proj } A)_{\mathbb{R}}$ whose walls are $\Theta_M$.

(2) \cite{Asa2} Definition 2.13] Let $\theta, \theta' \in K_0(\text{proj } A)_{\mathbb{R}}$. Then, we say that $\theta$ and $\theta'$ are $\text{TF}$ equivalent if $(\mathcal{T}_\theta, \mathcal{F}_\theta) = (\mathcal{T}_{\theta'}, \mathcal{F}_{\theta'})$ and $(\mathcal{T}_0, \mathcal{F}_0) = (\mathcal{T}_{\theta'}, \mathcal{F}_{\theta'})$.

By Lemma 3.3 and Proposition 3.2, the following properites hold also for complete special biserial algebras.
Proposition 3.12. [Asa2] Proposition 2.8] Let $M \in \text{fd} A \setminus \{0\}$. Then, there exists a brick $S \in \text{br} A$ such that $\Theta_S \supset \Theta_M$.

Proposition 3.13. [Asa2] Theorem 4.7] Let $\theta, \theta' \in K_0(\text{proj} A)_R$ be distinct elements. Then, the following conditions are equivalent.

(a) The elements $\theta$ and $\theta'$ are TF equivalent.

(b) The semistable subcategory $W_{\theta'}$ is constant $\theta$ in the line segment $[\theta, \theta']$.

(c) There exists no finite-dimensional brick $S \in \text{br} A$ such that $[\theta, \theta'] \cap \Theta_S$ is one point.

Recall that $J \subset I_c = (\text{Cyc}(A))$ and $\overline{A} = A/J$. It is clear that the quotient map $A \to \overline{A}$ induces canonical isomorphisms $K_0(\text{proj} A) \cong K_0(\text{proj} \overline{A})$ and $K_0(\text{fd} A) \cong K_0(\text{fd} \overline{A})$. Then, Proposition 3.2 yields that the TF equivalence in $K_0(\text{proj} A)$ is the same as that in $K_0(\text{proj} \overline{A})$.

Lemma 3.14. Let $\theta, \theta' \in K_0(\text{proj} A)_R$. Then, $\theta$ and $\theta'$ are TF equivalent in $K_0(\text{proj} A)_R$ if and only if they are TF equivalent in $K_0(\text{proj} \overline{A})_R$.

For any indecomposable $U \in 2\text{-psilt} A$, the element $[U] \in K_0(\text{proj} A)$ is called the g-vector of $U$. Obviously, the bijection $2\text{-psilt} A \to 2\text{-psilt} \overline{A}$ in Proposition 3.4 preserves the g-vectors of indecomposable 2-term presilting complexes in $K^b(\text{proj} A)$; see also [VG] Lemma 2.7.

Let $U = \bigoplus_{i=1}^m U_i \in 2\text{-psilt} A$ with $U_i$ indecomposable, and consider the g-vector cones

$$C(U) := \left\{ \sum_{i=1}^m a_i [U_i] \mid a_i \in \mathbb{R}_{\geq 0} \right\}, \quad C^+(U) := \left\{ \sum_{i=1}^m a_i [U_i] \mid a_i \in \mathbb{R}_{> 0} \right\}$$

spanned by the g-vectors of the indecomposable direct summands. By Proposition 3.3 and the following property, the cones are $m$-dimensional.

Proposition 3.15. [AI] Theorem 2.27] Let $T = \bigoplus_{i=1}^n T_i \in 2\text{-silt} A$ with $T_i$ indecomposable. Then, the family $\left\{ [T_i] \right\}_{i=1}^n$ is a $\mathbb{Z}$-basis of $K_0(\text{proj} A)$.

Moreover, g-vector cones are compatible with common direct summands of 2-term presilting complexes.

Proposition 3.16. [DL] Corollary 6.7] Let $U, U' \in 2\text{-psilt} A$, and $U''$ be their maximal common direct summands. Then, we have $C(U) \cap C(U') = C(U'')$.

In [Asa2, we proved that $C^+(U)$ is a TF equivalence class based on the results of [Ymr, BST] for finite-dimensional algebras. It can be extended by Propositions 3.2 and 3.3 as follows.

Proposition 3.17. [Asa2] Proposition 3.11] Let $U \in 2\text{-psilt} A$. Then, $C^+(U)$ is a TF equivalence class such that

$$C^+(U) = \{ \theta \in K_0(\text{proj} A)_R \mid (\overline{\mathcal{T}}_\theta, \overline{\mathcal{F}}_\theta) = (\overline{\mathcal{T}}_U, \overline{\mathcal{F}}_U), (\mathcal{T}_\theta, \mathcal{F}_\theta) = (\mathcal{T}_U, \mathcal{F}_U) \}.$$  

In general, we cannot obtain all TF equivalence classes from g-vector cones, because the disjoint union $\bigsqcup_{U \in 2\text{-psilt} A} C^+(U)$ does not necessarily coincide with $K_0(\text{proj} A)_R$. Actually, the following property, which was originally proved for finite-dimensional algebras, holds in our setting thanks to Proposition 3.4. We say that $A$ is $\tau$-tilting finite if $2\text{-silt} A$ is a finite set.

Proposition 3.18. [Asa2] Theorem 4.7] [ZZ] The algebra $A$ is $\tau$-tilting finite if and only if $\bigsqcup_{U \in 2\text{-psilt} A} C^+(U) = K_0(\text{proj} A)_R$ holds.

3.3. Canonical decompositions. As we have seen in the end of the previous subsection, the cones $C^+(U)$ constructed by 2-term presilting complexes are not enough to study the wall-chamber structure of $K_0(\text{proj} A)_R$. One of the aims of this paper is to study the regions of $K_0(\text{proj} A)_R$ where the cones $C^+(U)$ do not exist. For this reason, we first recall the notions of presentation spaces and canonical decompositions established by Derksen-Fei [DF].

Definition 3.19. [DF] Section 1] Let $\theta \in K_0(\text{proj} A)$.

(1) We define $P^{\theta}_0, P^{\theta}_1 \in \text{proj} A$ satisfy $\theta = [P^{\theta}_0] - [P^{\theta}_1]$ and add $P^{\theta}_0 \cap \text{add} P^{\theta}_1 = \{0\}$.
(2) We set \( \text{Hom}(\theta) := \text{Hom}_A(P_1^\theta, P_0^\theta) \), and call it the presentation space of \( \theta \).

(3) For any \( f \in \text{Hom}(\theta) \), we write \( P_f \) for the 2-term complex \( P_1^\theta \to P_0^\theta \) in \( \text{K}^b(\text{proj} A) \).

These notions were originally defined for finite-dimensional algebras. In this case, \( \text{Hom}(\theta) \) is a finite-dimensional \( K \)-vector space, so \( \text{Hom}(\theta) \) can be regarded as an irreducible algebraic variety with Zariski topology. Let \( (C_f) \) be a condition for \( f \in \text{Hom}(\theta) \), then we say that \( (C_f) \) holds for any general element \( f \in \text{Hom}(\theta) \) if the set \( \{ f \in \text{Hom}(\theta) \mid (C_f) \} \) is a dense subset of \( \text{Hom}(\theta) \).

By using presentation spaces, we can consider direct sums in the Grothendieck group. In (2), we naturally identify \( K_0(\text{proj} A) \) and \( K_0(\text{proj} A') \) for any quotient algebra \( A' \) of \( A \) such that \( |A'| = |A| \).

**Definition 3.20.** [DF, Definition 4.3] Let \( \theta, \theta_1, \theta_2, \ldots, \theta_m \in K_0(\text{proj} A) \).

1. Assume that \( A \) is a finite-dimensional algebra.
   (i) We write \( \bigoplus_{i=1}^m \theta_i \) in \( K_0(\text{proj} A) \) if any general \( f \in \text{Hom}(\sum_{i=1}^m \theta_i) \) admits \( f_i \in \text{Hom}(\theta_i) \) such that \( P_f \cong \bigoplus_{i=1}^m P_{f_i} \) in \( \text{K}^b(\text{proj} A) \).
   (ii) We say that \( \theta \) is indecomposable in \( K_0(\text{proj} A) \) if \( P_f \) is indecomposable in \( \text{K}^b(\text{proj} A) \) for any general \( f \in \text{Hom}(\theta) \).

2. Assume that \( A \) is a complete special biserial algebra and that \( A' \) is a finite-dimensional quotient algebra \( A' \) of \( A \) such that \( |A'| = |A| \).
   (i) We write \( \bigoplus_{i=1}^m \theta_i \) in \( K_0(\text{proj} A) \) if \( \bigoplus_{i=1}^m \theta_i \) in \( K_0(\text{proj} A') \) for any \( A' \) above.
   (ii) We say that \( \theta \) is indecomposable in \( K_0(\text{proj} A) \) if \( \theta \) is indecomposable in \( K_0(\text{proj} A') \) for some \( A' \) above.

We recall the important results in the case of finite-dimensional algebras.

**Proposition 3.21.** Let \( A \) be a finite-dimensional algebra.

1. [DF, Theorem 4.4] Let \( \theta, \theta_1, \theta_2, \ldots, \theta_m \in K_0(\text{proj} A) \). The direct sum \( \bigoplus_{i=1}^m \theta_i \) holds if and only if, for any \( i, j \in \{1, 2, \ldots, m\} \) with \( i \neq j \), there exist a pair \( (f, g) \in \text{Hom}(\theta_i) \times \text{Hom}(\theta_j) \) such that
   \[ \text{Hom}_{\text{K}^b(\text{proj} A)}(P_f, P_g[1]) = 0, \]
   \[ \text{Hom}_{\text{K}^b(\text{proj} A)}(P_f, P_g[1]) = 0. \]
   In particular, \( \bigoplus_{i=1}^m \theta_i \) is equivalent to that \( \theta \oplus \theta_j \) holds for any \( i \neq j \).

2. [Pla, Theorem 2.7] [DF] Let \( \theta \in K_0(\text{proj} A) \). Then, there exist indecomposable elements \( \theta_1, \theta_2, \ldots, \theta_m \in K_0(\text{proj} A) \) such that \( \theta = \bigoplus_{i=1}^m \theta_i \) and that \( P_{\theta_i} \) is indecomposable in \( \text{K}^b(\text{proj} A) \) for any \( i \) and general \( f_i \in \text{Hom}(\theta_i) \). Moreover, this decomposition is unique up to reordering.

The unique decomposition \( \theta = \bigoplus_{i=1}^m \theta_i \) above is called the canonical decomposition of \( \theta \) in \( K_0(\text{proj} A) \).

For example, if \( A \) is finite-dimensional and \( U = \bigoplus_{i=1}^m U_i \) is a (not necessarily basic) 2-term presilting complex in \( \text{K}^b(\text{proj} A) \) with \( U_i \) indecomposable, then \( [U] = \bigoplus_{i=1}^m [U_i] \) is a canonical decomposition. If \( U \) is basic, then \( [U_1], [U_2], \ldots, [U_m] \) can be extended to a \( \mathbb{Z} \)-basis of \( K_0(\text{proj} A) \) by Proposition 3.15 so any \( \theta \in C^+(T) \cap K_0(\text{proj} A) \) has a canonical decomposition of the form \( \theta = \bigoplus_{i=1}^m [U_i]^{[l_i]} \) with \( l_i \in \mathbb{Z}_{\geq 1} \). This observation is valid also if \( A \) is a complete special biserial algebra by Proposition 3.3.

To verify the notion of canonical decompositions in the case of complete special biserial algebras, we prepare the following properties.

**Lemma 3.22.** Let \( J \subset I \) be an ideal of \( A \) such that \( \overline{A} = A/J \) is finite-dimensional. Then, the following assertions hold.

1. For any \( \theta, \theta_1, \theta_2, \ldots, \theta_m \in K_0(\text{proj} A) \), the following conditions are equivalent.
   (a) The direct sum \( \bigoplus_{i=1}^m \theta_i \) holds in \( K_0(\text{proj} A) \).
   (b) The direct sum \( \bigoplus_{i=1}^m \theta_i \) holds in \( K_0(\text{proj} \overline{A}) \).
For any \( i, j \in \{1, 2, \ldots, m\} \) with \( i \neq j \), there exist \( f \in \Hom(\theta_i) \) and \( g \in \Hom(\theta_j) \) such that \( \Hom^{\mathbb{K}_{\mathbb{b}}(\proj A)}(P_f, P_g[1]) = 0 \) and \( \Hom^{\mathbb{K}_{\mathbb{b}}(\proj A)}(P_g, P_f[1]) = 0 \).

(2) Let \( \theta \in K_0(\proj A) \). Then, \( \theta \) is indecomposable in \( K_0(\proj A) \) if and only if \( \theta \) is indecomposable in \( K_0(\proj A) \).

Proof. (1) (a) \( \Rightarrow \) (b): It is clear.

(b) \( \Rightarrow \) (c): Let \( i, j \in \{1, 2, \ldots, m\} \) with \( i \neq j \). By assumption, we can take \( f' \in \Hom(\theta_i) \) and \( g' \in \Hom(\theta_j) \) such that \( \Hom^{\mathbb{K}_{\mathbb{b}}(\proj A)}(P_{f'}, P_{g'}[1]) = 0 \) and \( \Hom^{\mathbb{K}_{\mathbb{b}}(\proj A)}(P_{g'}, P_{f'}[1]) = 0 \). It is easy to see that \( \bigcirc A \) induces a surjection \( \Hom(\theta) \to \Hom(\theta) \), so there exist \( f \in \Hom(\theta_i) \) and \( g \in \Hom(\theta_j) \) such that \( \Gamma_f = P_{f'} \) and \( \Gamma_g = P_{g'} \). Then, Proposition 3.21 (1) implies (c).

(c) \( \Rightarrow \) (a): Let \( i, j \in \{1, 2, \ldots, m\} \) with \( i \neq j \) and take \( f, g \) in (c). Let \( A' \) be a finite-dimensional quotient algebra of \( A \), and set \( P_f' := P_f \bigcirc_A A' \) and \( P_g' := P_g[1] \bigcirc_A A' \). Then, by the same argument as \( [\text{AsI}] \) Proposition 3.35, we have

\[
\Hom^{\mathbb{K}_{\mathbb{b}}(\proj A')}(P_f', P_g'[1]) = 0, \quad \Hom^{\mathbb{K}_{\mathbb{b}}(\proj A')}(P_g', P_f'[1]) = 0.
\]

Therefore, Proposition 3.21 (1) implies that \( \bigoplus_{i=1}^n \theta_i \) holds in \( K_0(\proj A') \).

(2) follows from (1) and Proposition 3.21 (2).

By Lemma 3.24 the canonical decomposition of each element \( \theta \) is well-defined also in the case of complete special biserial algebras.

**Proposition 3.23.** Let \( A \) be a complete special biserial algebra. Then, Proposition 3.21 holds.

In our purpose, it is important to distinguish the elements in \( K_0(\proj A) \) corresponding to 2-term presilting complexes from the other elements. Thus, we use the following notions.

**Definition 3.24.** [DF, Definition 4.6] Let \( \theta \in K_0(\proj A) \).

1. We say that \( \theta \) is rigid if there exists some \( f \in \Hom(\theta) \) such that \( P_f \) is 2-term presilting.
2. We set \( \mathrm{IR}(A) \) (resp. \( \mathrm{INR}(A) \)) as the set of all indecomposable rigid (resp. non-rigid) elements in \( K_0(\proj A) \).

[DLL, Theorem 6.5] implies that if (not necessarily basic) 2-term presilting complexes \( U, U' \) in \( K_{\mathbb{b}}(\proj A) \) satisfy \( [U] = [U'] \in K_0(\proj A) \), then \( U \cong U' \in K_{\mathbb{b}}(\proj A) \). In the case that \( A \) is finite-dimensional, they also showed there that \( P_f \cong U \in K_{\mathbb{b}}(\proj A) \) holds for any general \( f \in \Hom([U]) \) by using [Pla, Lemma 2.16]. By this fact and Lemma 3.22 we can check that direct sums of rigid elements in \( K_0(\proj A) \) are nothing but the compatibilities of 2-term presilting complexes.

**Lemma 3.25.** Let \( U_1, U_2 \) be 2-term presilting complexes in \( K_{\mathbb{b}}(\proj A) \). Then, \( [U_1] \oplus [U_2] \) holds in \( K_0(\proj A) \) if and only if \( U_1 \oplus U_2 \) is 2-term presilting.

Recently, a very strong result [PY, Theorem 3.8], which is a restatement of [GLFS, Theorem 3.2], on finite-dimensional representation-tame algebra was found. The points of the proof are results in [CB1] on 1-parameter families of modules over representation-tame algebras. In our setting, this can be written as follows by Proposition 3.24.

**Proposition 3.26.** Any complete special biserial algebra \( A \) is E-tame, that is, any element \( \theta \in K_0(\proj A) \) satisfies \( \theta \oplus \theta \).

Let \( A \) be a finite-dimensional algebra. For any \( M \in \mathbb{f}d \, A \), \( \theta \in K_0(\proj A) \) and \( f \in \Hom(\theta) \), we can check that

\[
\theta(M) = \dim_K \Hom^{\mathbb{K}_{\mathbb{b}}(\mathbb{f}d \, A)}(P_f, M) - \dim_K \Hom^{\mathbb{K}_{\mathbb{b}}(\mathbb{f}d \, A)}(P_f, M[1]) = \dim_K \Hom^{\mathbb{K}_{\mathbb{b}}(\mathbb{f}d \, A)}(P_f, M) - \dim_K \Hom^{\mathbb{K}_{\mathbb{b}}(\mathbb{f}d \, A)}(M, \nu P_f[-1]) = \dim_K \Hom_A (\mathrm{Coker} \, f, M) - \dim_K \Hom_A (M, \mathrm{Ker} \, \nu f).
\]

Thus, if \( \Hom_A(M, \mathrm{Ker} \, \nu f) = 0 \), then we get \( M \in \Gamma_\theta \). By Proposition 3.26 the converse also holds if \( A \) is a finite-dimensional special biserial algebra.
Proposition 3.27. [FGL Lemma 2.13] (cf. [ASl Theorem 3.20]) Assume that $A$ is a finite-dimensional special biserial algebra. Let $\theta \in K_0(\text{proj} A)$ and $M \in \text{fd} A$.

1. The condition $M \in \bar{T}_\theta$ holds if and only if $\text{Hom}_A(M, \text{Ker} \nu f) = 0$ for some $f \in \text{Hom}(\theta)$. In this case, $\text{Hom}_A(M, \text{Ker} \nu f) = 0$ for any general $f \in \text{Hom}(\theta)$.

2. The condition $M \in \bar{T}_\theta$ holds if and only if $\text{Hom}_A(\text{Coker} f, M) = 0$ for some $f \in \text{Hom}(\theta)$. In this case, $\text{Hom}_A(\text{Coker} f, M) = 0$ for any general $f \in \text{Hom}(\theta)$.

Proof. They directly follow from [FGL Lemma 2.13] together with Proposition 3.26.

Lemma 3.28. [ASl Proposition 3.22] Assume that $A$ is a finite-dimensional special biserial algebra. Let $\theta_1, \theta_2 \in K_0(\text{proj} A)$. Then, $\theta_1 \oplus \theta_2$ holds if and only if $\text{Ker} f \subset \bar{T}_{\theta_2}$ and $\text{Ker} \nu f \subset \bar{T}_{\theta_2}$ for some $f \in \text{Hom}(\theta)$.

The following characterization of direct sums is also useful. Note that it is valid for all complete special biserial algebras by Proposition 3.22.

Proposition 3.29. [ASl Proposition 4.9, Theorem 3.14] Let $\theta_1, \theta_2 \in K_0(\text{proj} A)$. Then, $\theta_1 \oplus \theta_2$ holds if and only if $\bar{T}_{\theta_1} \subset \bar{T}_{\theta_2}$ and $\bar{T}_{\theta_1} \subset \bar{T}_{\theta_2}$. In this case, $\bar{T}_{\theta_1} \subset \bar{T}_{\theta_1+\theta_2} \subset \bar{T}_{\theta_1+\theta_2} \subset \bar{T}_{\theta_1+\theta_2}$, and $\bar{T}_{\theta_1} \subset \bar{T}_{\theta_1+\theta_2} \subset \bar{T}_{\theta_1+\theta_2}$, for any $i \in \{1, 2\}$.

The above proposition implies the following result, which was originally proved by Plamondon [Pla]. We say that $\theta = \sum_{i=1}^n a_i [P_i]$ and $\theta' = \sum_{i=1}^n a_i' [P_i]$ are sign-coherent if $a_i a_i' \geq 0$ holds for all $i \in \{1, 2, \ldots, n\}$.

Proposition 3.30. [Pla Lemma 2.10] Let $\theta_1, \theta_2 \in K_0(\text{proj} A)$. If $\theta_1 \oplus \theta_2$, then $\theta_1$ and $\theta_2$ are sign-coherent.

In the rest of this subsection, we assume that $A$ is a finite-dimensional special biserial algebra, and focus on indecomposable elements in $K_0(\text{proj} A)$. The argument in the proof of Proposition 3.20 in [PY] gives the next explicit results.

Proposition 3.31. The following assertions hold.

1. If $\sigma \in \text{IR}(A)$, then there uniquely exists $U \in 2 \text{-psilt} A$ such that $[U] = \sigma$, and each of $H^0(U)$ and $H^{-1}(\nu U)$ is zero or a string module or a projective-injective module. Moreover, for any general $f \in \text{Hom}(\theta)$, we have $P_f \cong U$ in $K^0(\text{proj} A)$.

2. If $\eta \in \text{INR}(A)$, then there exists a band $b_\eta$ such that, for any general $f \in \text{Hom}(\theta)$, both $\text{Coker} f$ and $\text{Ker} \nu f$ are isomorphic to the band module $M(b_\eta, \lambda)$ with $\lambda \in K^\times$ depending on $f$. Moreover, $b_\eta$ is unique up to isomorphisms of bands, and $M(b_\eta, \lambda)$ is a brick.

Proof. (1) is clear by Lemma 3.25 and the explanation before it.

(2) For each band $b$ in $A$ and $\lambda \in K^\times$, there exists a $K[t]-A$-bimodule $X$ such that $K[t]/(t - \lambda) \otimes_{K[t]} X \cong M(b, \lambda)$ by [WWW Corollary 2.4]. By using this, the proof of [PY] Theorem 3.8 implies the assertion.

We prepare symbols for the modules appearing above.

Definition 3.32. We associate the following modules to indecomposable elements in $K_0(\text{proj} A)$.

1. If $\sigma \in \text{IR}(A)$, then $M_{\sigma} := H^0(U)$ and $M'_\sigma := H^{-1}(\nu U)$.

2. If $\eta \in \text{INR}(A)$, then fix a band $b_\eta$ in Proposition 3.31 (2), and set $M_\eta(\lambda) := M(b_\eta, \lambda)$.

By Lemma 3.28 we obtain the following properties.

Lemma 3.33. Let $\theta \in K_0(\text{proj} A)$.

1. Let $\sigma \in \text{IR}(A)$. Then, $\theta \oplus \sigma$ holds if and only if $M_{\sigma} \in \bar{T}_{\theta}$ and $M'_{\sigma} \in \bar{T}_{\theta}$.

2. Let $\eta \in \text{INR}(A)$. Then, $\theta \oplus \eta$ holds if and only if $M_\eta(\lambda) \in \text{W}_0$ for all $\lambda \in K^\times$.

We also use the next property following from Proposition 3.31 (2) (cf. [ASl Remark 4.12]).
Lemma 3.34. Let $\eta \in \text{INR}(A)$. Then, $M_\eta(\lambda)$ is a simple object in $\mathcal{W}_\eta$.

The following lemma is useful when we want to show an element in $K_0(\text{proj} \ A)$ is in $\text{INR}(A)$.

Lemma 3.35. Let $\eta \in K_0(\text{proj} \ A)$ admit a band $b$ in $A$ such that, $M(b, \lambda)$ is a brick and that there exists $f \in \text{Hom}(\eta)$ such that $\text{Coker} \ f \cong M(b, \lambda) \cong \text{Ker} \ f$ for some $\lambda \in K^\times$. Then, $\eta \in \text{INR}(A)$, and $b$ is isomorphic to $b_\eta$ as bands.

Proof. We can write $b = p_1^{-1}q_1p_2^{-1}q_2 \cdots p_m^{-1}q_m$ with $p_i$ and $q_i$ paths of length $\geq 1$ admitted in $A$. Define $i_k, j_k \in Q_0$ so that $q_i$ is a path from $i_k$ to $j_k$.

For each $\lambda \in K^\times$, a minimal projective presentation of $M(b, \lambda)$ is given by

$$f_\lambda := \begin{pmatrix} (q_1) & 0 & \cdots & 0 & (-p_1) \\ (-p_2) & (q_2) & \cdots & 0 & 0 \\ 0 & (-p_3) & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & (-p_m) & (\lambda q_m) \end{pmatrix} : \bigoplus_{k=1}^m P_{j_k} \to \bigoplus_{k=1}^m P_{i_k}.$$  

Then, $f_\lambda \in \text{Hom}(\eta)$ by assumption.

Let $\mathcal{O}_\lambda := \{g_0 f_\lambda g_1^{-1} | g_i \in \text{Aut}(P_i^\eta)\}$ for each $\lambda \in K^\times$. Then, this set and $X := \bigcup_{\lambda \in K^\times} \mathcal{O}_\lambda = \{g_0 f_\lambda g_1^{-1} | g_i \in \text{Aut}(P_i^\eta), \lambda \in K\}$ are both constructible by Chevalley’s Lemma.

We have

$$\text{Hom}_{K^\nu(\text{proj} \ A)}(P_{f_\lambda}, P_{f_\lambda}[1]) \cong \text{Hom}_{K^\nu(\text{proj} \ A)}(P_{f_\lambda}, \nu P_{f_\lambda}[-1]) \cong \text{Hom}_A(M(b, \lambda), \tau M(b, \lambda)) \cong \text{Hom}_A(M(b, \lambda), M(b, \lambda)),$$

and this is one-dimensional, since $M(b, \lambda)$ is a brick by assumption. Thus, by [Pla] Lemma 2.16, the codimension of $\mathcal{O}_\lambda$ is one in $\text{Hom}(\eta)$ for each $\lambda \in K^\times$.

We can regard the tangent spaces $T_{f_\lambda}(\mathcal{O}_\lambda)$ and $T_{f_\lambda}(X)$ at $f_\lambda$ as $K$-vector subspaces of $\text{Hom}(\eta)$ so that $T_{f_\lambda}(\mathcal{O}_\lambda) \subset T_{f_\lambda}(X) \subset \text{Hom}(\eta)$. Since the codimension of $\mathcal{O}_\lambda$ is one, that of $T_{f_\lambda}(\mathcal{O}_\lambda)$ is also one. Thus, it suffices to show $T_{f_\lambda}(\mathcal{O}_\lambda) \subset T_{f_\lambda}(X)$ to obtain the codimension of $T_{f_\lambda}(X)$ is zero.

Let $\mu \in K^\times \setminus \{\lambda\}$. Then,

$$T_{f_\lambda}(\mathcal{O}_\lambda) = \{f_\lambda h_1 - b_0 f_\lambda | h_i \in \text{End}_A(P_i^\eta)\} \subset \text{Hom}(\eta)$$  

by [Pla] Lemma 2.15. This set does not have $f_\mu - f_\lambda$ as an element; otherwise, the element $f' := [f_\lambda - f_\mu]_0$ satisfies $P_{f'} \cong (P_{f_\lambda})^{\oplus 2}$, but the cokernel of the left-hand side is $M(b, 2, [\lambda_\mu - \lambda_\lambda])$, which is not isomorphic to the cokernel $M(b, \lambda)^{\oplus 2}$ of the right-hand side. On the other hand, the tangent space $T_{f_\lambda}(X)$ of $X$ at $f_\lambda$ has $f_\mu - f_\lambda$ as an element by the construction of $f_\lambda$. Thus, we get $T_{f_\lambda}(\mathcal{O}_\lambda) \subset T_{f_\lambda}(X)$.

Therefore, $T_{f_\lambda}(X)$ is of codimension zero in $\text{Hom}(\eta)$, and so is $X$. Thus, the constructible set $X$ is a dense subset of $\text{Hom}(\eta)$, so $\eta \in \text{INR}(A)$. Now, Lemma 3.31 implies that $b$ must be isomorphic to $b_\eta$. □

Take a $K[t]$-$A$-bimodule $M$ such that $K[t]/(t - \lambda) \otimes_{K[t]} M \cong M(b, \lambda)$ for each band $b$ as in the proof of Proposition 3.31. Then, the construction of $f_\lambda$ above means that we can take a minimal projective resolution

$$K[t] \otimes_K P_0^\eta \xrightarrow{\tilde{f}} K[t] \otimes_K P_0^\eta \to \to 0$$

such that the $(k, l)$-entry $K[t] \otimes_K P_{ji} \xrightarrow{\tilde{f}_{ji}} K[t] \otimes_K P_{ki}$ of the matrix expression of $\tilde{f}$ sends $1 \otimes e_{ji}$ to some element $1 \otimes a_0 + t \otimes a_1$ with $a_0, a_1 \in e_{ji}Ae_{ji}$.  


4. \(\tau\)-Tilting Reduction

In this section, we deal with \(\tau\)-tilting reduction by Jasso [Jas], which is a great method to consider the 2-term (pre)silting complexes which have a fixed direct summand \(U \in 2\text{-silt}A\) by using a certain algebra \(B\) associated to \(U\).

In [Asa2, AsI], we studied the relationship between \(\tau\)-tilting reduction, TF equivalence classes and canonical decompositions, and found that this is useful to describe the union of \(g\)-vector cones. We recall some results in those papers in Subsection 4.1.

Since we use the new algebra \(B\) in \(\tau\)-tilting reduction, it is important to know which kind of algebra \(B\) is. Subsection 4.2 is devoted to showing that \(B\) is a finite-dimensional special biserial algebra if so is \(A\), which is crucial in our study in this paper.

4.1. Neighborhoods associated to 2-term presilting complexes. We start with defining the following subset of \(K_0(\text{proj}A)_{\mathbb{R}}\).

**Definition 4.1.** [Asa2] Subsection 4.1] For any \(U \in 2\text{-silt}A\), we set

\[
N_U = N_{[U]} := \{ \theta \in K_0(\text{proj}A)_{\mathbb{R}} \mid \mathcal{T}_U \subset \mathcal{T}_\theta, \mathcal{F}_U \subset \mathcal{F}_\theta \}.
\]

Recall that we have defined two semibricks \(S_U\) and \(S'_U\) before Lemma 3.9 so that \(\mathcal{T}_U\) is the smallest torsion class containing \(S_U\) and that \(\mathcal{F}_U\) is the smallest torsion class containing \(S'_U\). Therefore,

\[
N_U = \{ \theta \in K_0(\text{proj}A)_{\mathbb{R}} \mid S_U \subset \mathcal{T}_\theta, S'_U \subset \mathcal{F}_\theta \}.
\]

We prepare some basic properties of \(N_U\) from [Asa2, AsI].

**Lemma 4.2.** Let \(U, U' \in 2\text{-silt}A\).

1. [Asa2] Lemma 4.3] The subset \(N_U\) is an open neighborhood of \(C^+(U)\).
2. [AsI] Lemmas 6.4, 6.5] We have

\[
\overline{N_U} = \{ \theta \in K_0(\text{proj}A)_{\mathbb{R}} \mid \mathcal{T}_U \subset \mathcal{T}_\theta, \mathcal{F}_U \subset \mathcal{F}_\theta \},
\]

\[
N_U \cap K_0(\text{proj}A) = \{ \theta \in K_0(\text{proj}A) \mid [U] \text{ is a direct summand of } \theta \},
\]

\[
\overline{N_U} \cap K_0(\text{proj}A) = \{ \theta \in K_0(\text{proj}A) \mid [U] \subset \theta \}.
\]

3. [AsI] Lemma 6.6] The complex \(U \oplus U'\) is 2-term silting if and only if \(N_U \cap N_{U'} \neq \emptyset\).

In this case, \(N_U \cap N_{U'} = N_V\), where \(V\) is the basic 2-term presilting complex given by \(U \oplus U'\).

**Proof.** (1) This is shown in the same way as [Asa2] Lemma 4.3; namely, Proposition 3.17 and \(N_U = \{ \theta \in K_0(\text{proj}A)_{\mathbb{R}} \mid S_U \subset \mathcal{T}_\theta, S'_U \subset \mathcal{F}_\theta \} \) yield the assertion.

(2), (3) The proofs in [AsI] work also in our setting of complete special biserial algebras. We remark that Proposition 3.20 is also needed to deduce the last equality of (2).

Therefore, the neighborhood \(N_U\) is a nice tool when we consider rigid direct summands of a given element \(\theta \in K_0(\text{proj}A)\).

By using Bongartz completions in Definition 3.7, Jasso [Jas] gave a very useful method called \(\tau\)-tilting reduction to investigate the wide subcategory \(\mathcal{W}_U = \mathcal{T}_U \cap \mathcal{F}_U\), which is equal to \(\mathcal{W}_\theta\) for any \(\theta \in C^+(U)\) by Proposition 3.17 and the subset \(2\text{-silt}_UA\) consisting of all \(V \in 2\text{-silt}A\) which have \(U\) as a direct summand. We define \(2\text{-silt}_UA\) in a similar way. The point of \(\tau\)-tilting reduction is the algebra

\[
B := B_U = \text{End}_A(H^0(T))/[H^0(U)],
\]

where \([H^0(U)]\) is the ideal of \(\text{End}_A(H^0(T))\) of all endomorphisms on \(H^0(T)\) factoring through some module in \text{add} \(H^0(U)\).

For any \(M \in \text{fd}A\), recall that we have a short exact sequence \(0 \to t_U M \to M \to t_U M \to 0\) such that \(t_U M \in \mathcal{T}_U\) and \(t_U M \in \mathcal{F}_U\).

Under this preparation, \(\tau\)-tilting reductions are given as follows.
Proposition 4.3. Assume that $A$ is a finite-dimensional algebra. Let $U \in \text{2-psilt} A$.

(1) [Jas, Theorem 3.8] (cf. [DIRRT, Theorem 4.12]) We have an equivalence

$$\Phi := \text{Hom}_A(H^0(T),?) : \mathcal{W}_U \to \text{fd} B$$

of abelian categories.

(2) [Jas, Theorems 3.16, 4.12, Asa2, Proposition 4.2] There exists a bijection

$$\text{red} : 2\text{-psilt}_U A \to 2\text{-psilt} B$$

such that $\Phi(\text{red}(V)) = H^0(\text{red}(V))$ for any $V \in 2\text{-psilt}_U A$.

To describe the relationship between $[V] \in K_0(\text{proj} A)$ and $[\text{red}(V)] \in K_0(\text{proj} B)$ for each $V \in 2\text{-psilt}_U A$, we introduced an $R$-linear projection $\pi : K_0(\text{proj} A)_R \to K_0(\text{proj} B)_R$ in Subsection 4.1. Let $m := |U|$, and write $T = \bigoplus_{i=1}^m T_i$ with $U = \bigoplus_{i=n-m+1}^m T_i$ and $T_i$ indecomposable, and set

$$X_i := H^0(T_i)/ \sum_{f \in \text{rad}_A(H^0(T),H^0(T_i))} \text{Im} f.$$ as in Lemma 3.9 for each $i$. Then, $\{X_1, X_2, \ldots, X_{n-m}\}$ is the set of simple objects in $\mathcal{W}_U$ by [Asa1, Theorem 2.21], so $S_i^B := \Phi(X_i)$ are the simple modules in $\text{fd} B$. We remark that $\{X_1, X_2, \ldots, X_{n-m}\}$ is contained in the semibrick $S_T$. Moreover, for any $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, n-m\}$, we get $\dim_K \text{Hom}_A(T_i, X_j) = \delta_{i,j}$ by Lemma 3.9. Since $X_j \in \mathcal{W}_U$, we also have $\text{Hom}_A(T_i, X_j) = \text{Hom}_A(X_j, \nu T_i[-1]) = 0$. Thus, $\langle T_i, X_j \rangle = \delta_{i,j}$.

Define $P_i^B$ as the projective cover of $S_i^B$. Since $[T_1], [T_2], \ldots, [T_m]$ give a $\mathbb{Z}$-basis of $K_0(\text{proj} A)$ by Proposition 3.13, we can define an $R$-linear surjection $\pi : K_0(\text{proj} A)_R \to K_0(\text{proj} B)_R$ by

$$\pi(\theta) := \sum_{i=1}^m \theta(X_i)[P_i^B].$$

Then, since $\langle T_i, X_j \rangle = \delta_{i,j}$, we have

$$\pi([T_i^B]) = \begin{cases} \delta_{i,n-m} & (i \leq n-m) \\ 0 & (i > n-m) \end{cases}.$$

Under this setting, we have the projection $\pi$ is compatible with the $\tau$-tilting reduction at $U$.

Proposition 4.4. [Asa2, Lemma 4.4, Theorem 4.5] Assume that $A$ is a finite-dimensional algebra. Let $U \in 2\text{-psilt} A$.

(1) We have $\pi(N_U) = K_0(\text{proj} B)_R$.

(2) For any $\theta \in N_U$, we get $\Phi(\mathcal{W}_\theta) = \mathcal{W}_{\pi(\theta)}$.

(3) Let $\theta, \theta' \in N_U$. Then, $\theta$ and $\theta'$ are TF equivalent in $K_0(\text{proj} A)_R$ if and only if $\pi(\theta)$ and $\pi(\theta')$ are TF equivalent in $K_0(\text{proj} B)_R$.

(4) Let $V \in 2\text{-psilt}_U A$. Then, $\pi([V]) = [\text{red}(V)]$ in $K_0(\text{proj} B)_R$, and $\pi^{-1}(N_{\text{red}(V)}(B)) \cap N_U = N_V$ in $N_U$.

Proof. This follows from [Asa2, Lemma 4.4, Theorem 4.5] and Proposition 3.17.

We remark the following property for later use.

Lemma 4.5. Assume that $U, U' \in 2\text{-psilt} A$ satisfy $B_U \cong B_{U'}$ as algebras. Then, there exists an $R$-linear automorphism $\rho : K_0(\text{proj} A)_R \to K_0(\text{proj} A)_R$ which induces a bijection

$$\{\text{TF equivalence classes in } N_U\} \to \{\text{TF equivalence classes in } N_{U'}\}$$

$E \mapsto (\text{the TF equivalence class containing } \rho(E) \cap N_{U'})$.

Proof. Take the Bongartz completions $T, T' \in 2\text{-silt} A$ of $U, U' \in 2\text{-psilt} A$, respectively. Decompose them as $T = \bigoplus_{i=1}^n T_i$ and $T' = \bigoplus_{i=1}^n T'_i$ so that

- $U = \bigoplus_{i=n-m+1}^n T_i$ and $U' = \bigoplus_{i=n-m+1}^n T'_i$;
\textbullet \ \text{red}_U(T_i) \cong \text{red}_{U'}(T'_i) \text{ as projective } B_U \cong B_{U'}-\text{modules for each } i \in \{1, 2, \ldots, n - m\}.

Then, we can define an \( \mathbb{R} \)-linear automorphism \( \rho: K_0(\text{proj } A)_\mathbb{R} \to K_0(\text{proj } A)_\mathbb{R} \) by \( [T_i] \mapsto [T'_i] \) for each \( i \in \{1, 2, \ldots, n\} \). Now, Proposition 4.4 implies the assertion. \( \square \)

For each \( \theta \in K_0(\text{proj } A) \), we can define the maximum rigid direct summand of \( \theta \) by Proposition 3.23. An analogue in \( K_0(\text{proj } A)_{\mathbb{R}} \) was considered in \text{[AS]}.

**Definition 4.6.** [AS] Definition 6.9] For any \( U \in 2-\text{psilt } A \), we set
\[
R_U = R_U(A) := N_U \setminus \bigcup_{V \in (2-\text{psilt } A) \setminus \{U\}} N_V.
\]
We call
\[
R_0 = K_0(\text{proj } A)_{\mathbb{R}} \setminus \bigcup_{V \in (2-\text{psilt } A) \setminus \{0\}} N_V
\]
the \textit{purely non-rigid region} of \( K_0(\text{proj } A)_{\mathbb{R}} \).

By definition, \( K_0(\text{proj } A)_{\mathbb{R}} \) is the disjoint union of \( R_U \). For each \( \theta \in K_0(\text{proj } A) \), the maximum rigid direct summand of \( \theta \) is \( [U] \) if and only if \( \theta \in R_U \) by Lemma 4.2. The purely non-rigid region \( R_0 \) is a closed subset of \( K_0(\text{proj } A)_{\mathbb{R}} \), and \( \theta \in K_0(\text{proj } A) \) has no nonzero rigid direct summand if and only if \( \theta \in R_0 \).

We state the following easy property.

**Lemma 4.7.** Let \( U \in 2-\text{psilt } A \).

1. If \( |U| = n \) (that is, \( U \in 2-\text{silt } A \)), then \( N_U = R_U = C^+(U) \).
2. If \( |U| = n - 1 \) and \( 2-\text{silt } A = \{T, T'\} \), then \( N_U = C^+(T) \cup C^+(T') \cup C^+(U) \) and \( R_U = C^+(U) \).

**Proof.** Both follow from Proposition 4.4. \( \square \)

Since \( N_U, R_U \) are unions of TF equivalence classes, the following property comes from Propositions 3.2 and 3.13. Recall that \( \overline{A} = A/J \) and \( \overline{U} = U \otimes_A \overline{A} \) for any 2-term complex \( U \in K^b(\text{proj } A) \).

**Proposition 4.8.** Let \( A = \overline{KQ}/I \) be a complete special biserial algebra, \( U \in 2-\text{psilt } A \), and \( J \subset I \) be an ideal of \( A \). Then, we have \( N_U(A) = \overline{N_U(\overline{A})} \) and \( R_U(A) = \overline{R_U(\overline{A})} \). In particular, \( R_0(A) = R_0(\overline{A}) \).

We set \( \text{Cone} \) as the union of the cones \( C(T) \):
\[
\text{Cone} := \bigcup_{T \in 2-\text{silt } A} C(T) = \bigsqcup_{U \in 2-\text{psilt } A} C^+(U)
\]
and call its complement \( \text{NR} := K_0(\text{proj } A)_{\mathbb{R}} \setminus \text{Cone} \) the \textit{non-rigid region} in \( K_0(\text{proj } A)_{\mathbb{R}} \). Proposition 3.18 implies that \( \text{NR} = \emptyset \) if and only if \( A \) is \( \tau \)-tilting finite. The non-rigid region \( \text{NR} \) can be described as follows.

**Proposition 4.9.** [AS] Corollary 6.11] The following assertions hold.

1. For each \( U \in 2-\text{psilt } A \), we have
\[
R_U = C^+(U) + (N_U \cap R_0) := \{ \theta_1 + \theta_2 \mid \theta_1 \in C^+(U), \theta_2 \in N_U \cap R_0 \},
\]
and the choice of such \( (\theta_1, \theta_2) \) for each \( \theta \in R_U \) is unique.
2. For any \( U \in 2-\text{psilt } A \), we have
\[
R_U \cap \text{NR} = R_U \setminus C^+(U) = C^+(U) + ((N_U \cap R_0) \setminus \{0\}).
\]
Therefore,
\[
\text{NR} = \bigsqcup_{U \in 2-\text{psilt } A} (C^+(U) + ((N_U \cap R_0) \setminus \{0\})).
The proposition above says that we know $R_U$ without calculating the algebra $B$ or the linear map $\pi : N_U \to K_0(\text{proj } B)_{\mathbb{R}}$ if $R_0$ is explicitly given. Thus, we will determine $R_0$ for complete special biserial algebras in the next section.

We also remark that $R_0 \cap NR = R_0 \setminus \{0\}$, so any nonzero element in the purely non-rigid region $R_0$ belongs to the non-rigid region $NR$, but $0 \notin NR$.

For later use, we state the following property.

**Lemma 4.10.** Let $U \in 2\text{-psilt } A$. The linear projection $\pi : K_0(\text{proj } A)_{\mathbb{R}} \to K_0(\text{proj } B)_{\mathbb{R}}$ restricts to a bijection $\varphi : N_U \cap R_0 \to R_0(B)$.

**Proof.** This follows from Propositions 4.13 and 4.9. □

We end this subsection by recalling the next property, which was obtained in our proof of [Asa2, Theorem 4.7] (Proposition 3.18 in this paper).

**Proposition 4.11.** [Asa2] Theorem 4.7 Let $A$ be a finite-dimensional algebra. Then, the algebra $A$ is $\tau$-tilting finite if and only if $R_0 = \{0\}$.

We remark that it is not known whether $R_0 \cap K_0(\text{proj } A) = \{0\}$ implies that $A$ is $\tau$-tilting finite for general finite-dimensional algebras. In the case of complete special biserial algebras, Schroll-Treffinger-Valdivieso [STV] showed that $R_0 \cap K_0(\text{proj } A) = \{0\}$ is equivalent to that $A$ is $\tau$-tilting finite; see also Corollary 6.8.

### 4.2. \(\tau\)-Tilting reduction of special biserial algebras

In this subsection, we show the following result, that is, the class of finite-dimensional special biserial algebras is closed under $\tau$-tilting reduction.

**Theorem 4.12.** Let $A$ be a finite-dimensional special biserial algebra, and $U \in 2\text{-psilt } A$, and $T \in 2\text{-silt } A$ be its Bongartz completion. Then, $B = B_U = \text{End}_A(H^0(T))/[H^0(U)]$ is isomorphic to a finite-dimensional special biserial algebra.

We first prepare the following property on the structure of the modules $M_i$ in $W_U$.

**Lemma 4.13.** For any $i \in \{1, 2, \ldots, m\}$, there exists a sequence

$$0 = M_{i,k_0} \subset \cdots \subset M_{i,2} \subset M_{i,1} \subset M_{i,0} = M_i,$$

such that, for any $k \in \{0, 1, \ldots, k_0 - 1\}$,

- the module $M_{i,k}$ does not have a band module as a direct summand;
- $M_{i,k}/M_{i,k+1}$ is a nonzero direct sum of simple objects in $W_U$.

**Proof.** By Proposition 1.3(1), the short exact sequences in $W_U$ are sent to those in $\text{fd } B$. Therefore, $\text{Ext}_A^1(M_i, M_i) = 0$, so $M_i$ is not a band module.

For each $k \in \mathbb{Z}_{\geq 1}$, we have

$$\text{rad}^k P_i^B = \sum_{g_1, g_2, \ldots, g_k \in \text{rad}(B, B)} \text{im}(\pi_i^B g_k \cdots g_2 g_1),$$

where $\pi_i^B : B \to P_i^B$ is the canonical projection. Set $\pi_i := \Phi^{-1}(\pi_i^B) : M \to M_i$. Then, Proposition 1.3(1) implies

$$M_{i,k} := \Phi^{-1}(\text{rad}^k P_i^B) = \sum_{f_1, f_2, \ldots, f_k \in \text{rad}(M, M)} \text{im}(\pi_i f_k \cdots f_2 f_1) \subset M_i.$$
By Proposition 2.6, the standard nonisomorphic endomorphisms on $M$ give a basis of $\text{rad}(M, M)$. Thus, $M_{i,k}$ cannot have a band module as a direct summand. Set $k_0$ as the smallest integer satisfying $M_{i,k_0} = 0$. In the sequence

$$0 = M_{i,k_0} \subseteq \cdots \subseteq M_{i,1} \subseteq M_{i,0} = M_i,$$

the subfactor $M_{i,k}/M_{i,k+1}$ is sent to a nonzero semisimple $B$-module $\text{rad}^k P_B^B/\text{rad}^{k+1} P_B^B$ for $k \in \{0, 1, \ldots, k_0 - 1\}$, so $M_{i,k}/M_{i,k+1}$ is a nonzero direct sum of some simple objects in $\mathcal{W}_U$. \qed

Under this property, we define a set $\Gamma$ of standard homomorphisms between indecomposable direct summands of $M$ by $\Gamma = \prod_{i=1}^m \Gamma_i$, where $\Gamma_i$ is given as follows.

(a) If $M_i$ is a string module $M(s)$, then from Lemma 4.13, the string $s$ is of the form

$$s_{-\nu \alpha_{-\nu}^{-1} \cdots s_{-2 \alpha_{-2}^{-1} s_{-1}^{-1} \cdots s_0 \cdot \alpha_1 \alpha_2 s_2 \cdots \alpha_l s_l}$$

such that $l, \nu \in \mathbb{Z}_{\geq 0}$, $\alpha_1 \in Q_1$ and that $M(s_0) = X_i$.

If $l \geq 1$, then take $j \in \{1, 2, \ldots, m\}$ such that $M(s_1) \cong X_j$. It is easy to check that there uniquely exists a standard homomorphism $\gamma_i : M_j \to M_i$ satisfying $\text{Im} \gamma_i = M(s_1 \alpha_2 s_2 \cdots \alpha_l s_l)$.

If $l' \geq 1$, then we can similarly take $j' \in \{1, 2, \ldots, m\}$ so that $M(s_{-1}) \cong X_{j'}$, and define $\gamma'_i : M_{j'} \to M_i$ such that $\text{Im} \gamma'_i = M(s_{-1} \alpha_2 s_2 \cdots s_{-2} \alpha_{-2}^{-1} s_{-1})$. Thus, we set

$$\Gamma_i := \{ \{ \gamma_i, \gamma'_i \} \mid (l \geq 1, \ l' \geq 1) \}$$. 

(b) Otherwise, $M_i$ is a projective-injective module. In this case, take strings $s, t$ such that $M/M_{i,k_0-1} \cong M(s)$ and $M_i/k_0-1 \cong M(t)$. Note that $M(t)$ is a simple object in $\mathcal{W}_U$. If $M(s) = 0$, then $\Gamma_i := \emptyset$. Otherwise, $M(s)$ is a string module. Write

$$s = s_{-\nu \alpha_{-\nu}^{-1} \cdots s_{-2 \alpha_{-2}^{-1} s_{-1}^{-1} \cdots s_0 \cdot \alpha_1 \alpha_2 s_2 \cdots \alpha_l s_l}$$

as above.

If $l \geq 1$, then we take $j$ so that $M(s_1) \cong X_j$ and define $\gamma_i : M_j \to M_i$ so that $\text{Im} \gamma_i = M(s_1 \alpha_2 s_2 \cdots \alpha_l s_l)$ for some arrow $\alpha_{l+1} \in Q_1$, where we replace $t$ by $t^{-1}$ if necessary. If $l' \geq 1$, then we similarly take $j'$ so that $M(s_{-1}) \cong X_{j'}$ and define $\gamma'_i : M_{j'} \to M_i$ so that $\text{Im} \gamma'_i = M(t^{-1} \alpha_{-t}^{-1} s_{-t} \alpha_{-t}^{-1} \cdots s_{-2} \alpha_{-2}^{-1} s_{-1})$ for some arrow $\alpha_{-t-1} \in Q_1$. If $l = l' = 0$, then we take $j$ such that $M(t) \cong X_j$ in $\mathcal{W}_U$ and the unique standard homomorphism $\gamma'' : M_j \to M_i$ such that $\text{Im} \gamma'' = M(t)$. Thus, we set

$$\Gamma_i := \{ \{ \gamma_i, \gamma_i' \} \mid (l \geq 1, \ l' \geq 1) \}$$. 

Dually, by using the indecomposable injective $B$-modules $I_i^B := \nu_B P_i^B$, we define a set $\Delta$ of standard homomorphisms between indecomposable direct summands $M_i' := \Phi^{-1}(I_i^B)$ of $M' := \Phi^{-1}(DB)$.

We define $\tilde{\Gamma}$ (resp. $\tilde{\Delta}$) as the set of standard homomorphisms between indecomposable direct summands of $M$ (resp. $DM$). Clearly, $\tilde{\Gamma} \supset \Gamma$ and $\tilde{\Delta} \supset \Delta$.

Lemma 4.14. There exists a bijection $\tilde{\Gamma} \to \tilde{\Delta}$ which restricts to a bijection $\Gamma \to \Delta$. 

Proof. Let \((\gamma: M_j \to M_i) \in \tilde{\Gamma}\). By construction, the standard homomorphism \(\gamma\) uniquely induces a standard homomorphism \((\delta: M'_j \to M'_i) \in \tilde{\Delta}\) which satisfies the commutative diagram

\[
\begin{array}{ccc}
M_j & \xrightarrow{\gamma} & M_i \\
\downarrow & & \downarrow 1 \\
X_j & \xrightarrow{\text{P.O.}} & X_i
\end{array}
\quad
\begin{array}{ccc}
M'_j & \xrightarrow{\delta} & M'_i \\
\downarrow & & \downarrow \text{P.O.} \\
M'_j & \xrightarrow{1} & M'_i
\end{array}
\]

of standard homomorphisms (P.O. means push outs). Thus, we obtain a map \(\tilde{\Gamma} \to \tilde{\Delta}\). By the same commutative diagram and pullbacks, we can define a map \(\tilde{\Delta} \to \tilde{\Gamma}\).

We can check these maps are mutually inverse, and restrict to bijections between \(\Gamma\) and \(\Delta\). \(\square\)

We remark that the bijection \(\tilde{\Gamma} \to \tilde{\Delta}\) can be extended to an equivalence \(\text{add } M \to \text{add } M'\).

Proof of Theorem 4.12. Since each element in \(\tilde{\Gamma}\) is a composite of some elements in \(\Gamma\), the ideal \(\text{rad } B\) is generated by \(\Phi(\Gamma)\); here, we regard each \(\gamma \in \Gamma\) as an endomorphism on \(M\) in an obvious way. Thus, the quiver \(Q_B\) defined by \((Q_B)_0 := \{1, 2, \ldots, n - m\}\) and \((Q_B)_1 := \{\beta_i: 0 \to j \mid (\gamma: M_j \to M_i) \in \Gamma\}\) is the Gabriel quiver of \(B\), and we have the surjective homomorphism \(\hat{K}Q_B \to B\) of algebras given by \(\beta_i \mapsto \gamma\) for each \(\gamma \in \Gamma\). We check the conditions (a)–(e) in Definition 2.1.

(a) For each \(i \in (Q_B)_0\), there are at most two elements in \(\Gamma\) whose codomains are \(M_i\), which corresponds to the arrows starting at \(i\). Thus, (a) is satisfied.

(c) For any \((\gamma: M_j \to M_i) \in \Gamma\) such that \(#\Gamma_j = 2\), the construction implies that \(\gamma\gamma' = 0\) for at least one \(\gamma' \in \Gamma_j\). This means (c).

(b) and (d) are similarly shown to (a) and (c) by using \(\Delta\) and Lemma 4.14.

Set \(J_B \subseteq \hat{K}Q_B\) as the ideal generated by \(\{\beta_i, \beta_{i'}, \gamma, \gamma' \in \Gamma, \gamma\gamma' = 0\}\), whose elements are paths in \(Q_B\) of length 2.

(e) Let \(i \in (Q_B)_0\). We write \(E_i\) for the set of elements in \(\tilde{\Gamma}\) such that their codomains are \(M_i\) and their images are simple objects in \(\mathcal{W}_U\).

We first assume that \(M_i\) is isomorphic to a string module. Then, for each element \(\gamma = \gamma_1 \gamma_2 \cdots \gamma_l \in E_i\) with each \(\gamma_k \in \Gamma\), we define a path \(p_\gamma\) in \(Q_B\) by \(p_\gamma := \beta_{\gamma_1} \beta_{\gamma_2} \cdots \beta_{\gamma_l} \beta_{\gamma_{l+1}}\) if there exists \(\gamma_{l+1} \in \Gamma\) such that \(\gamma\gamma_{l+1} = 0\) (such \(\gamma_{l+1}\) is unique if exists by (c)); and \(p_\gamma = 0\) otherwise.

Otherwise, \(M_i\) is projective-injective. In this case, \(#E_i = 1\), so take the unique element \(\gamma \in E_i\). We can write \(\gamma = \gamma_1 \gamma_2 \cdots \gamma_l = \gamma'_1 \gamma'_2 \cdots \gamma'_{l'}\) with \(\gamma_k, \gamma'_k \in \Gamma, \gamma_1 \neq \gamma'_1\). Set \(p_\gamma\) in \(\hat{K}Q_B\) by \(p_\gamma := \beta_{\gamma_1} \beta_{\gamma_2} \cdots \beta_{\gamma_l} - \beta_{\gamma'_1} \beta_{\gamma'_2} \cdots \beta_{\gamma'_{l'}}\).

We can check that the kernel of the surjective homomorphism \(\hat{K}Q_B \to B\) of algebras is the ideal generated by \(J_B\) and \(p_\gamma\) for \(\tilde{\Gamma}_i\)’s. Thus, (e) is satisfied. \(\square\)

We give a way to calculate the algebra \(B\) explicitly in the following example.

Example 4.15. Define \(A'\) as \(A\) in the complete special biserial algebra appearing in Example 2.3 and we here set \(A := A'/I_A\), which is a finite-dimensional special biserial algebra.

(1) It is straightforward to check that \(U = U_1 \oplus U_2 \oplus U_3 \oplus U_4 \in 2\text{-psilt } A\), where

\[
U_1 := (P_2 \xrightarrow{(\alpha_1)} P_1), \quad U_2 := (P_4 \xrightarrow{(\beta_1)} P_3), \quad U_3 := (P_6 \xrightarrow{(\gamma_1)} P_5), \quad U_4 := (P_8 \xrightarrow{(\delta_1)} P_7).
\]
The simple objects of $\mathcal{W}_U = \mathcal{W}_{|U|}$ are

$$X_1 := \begin{array}{c} 1 \\ 2 \end{array}, \quad X_2 := \begin{array}{c} 3 \\ 4 \end{array}, \quad X_3 := \begin{array}{c} 5 \\ 6 \end{array}, \quad X_4 := \begin{array}{c} 7 \\ 8 \end{array}.$$  

These are sent to $S_i^B \in \text{fd} B$ by $\Phi$. The module $M_i = \Phi^{-1}(S_i^B)$ is the unique module $M_i$ in $\mathcal{W}_U$ which has $X_i = \Phi^{-1}(S_i^B)$ as a quotient module and $\text{Ext}_A^1(M_i, \mathcal{W}_U) = 0$. By this characterization, we get

$$M_1 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}, \quad M_2 = \begin{array}{c} 4 \\ 6 \\ 5 \\ 7 \end{array}, \quad M_3 = \begin{array}{c} 8 \\ 7 \\ 6 \\ 5 \end{array}, \quad M_4 = \begin{array}{c} 8 \\ 8 \end{array}.$$  

Thus, $B = \text{End}_A(M)$ is isomorphic to $KQ_B/I_B$, where

$$Q_B = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4,$$  

$I_B = (\alpha\beta, \beta\delta)$.

As an application of the proof of Theorem 4.12, we can show some kind of “fractal” property of the wall-chamber structure on $K_0(\text{proj} A)_R$.

**Corollary 4.16.** Let $A$ be a finite-dimensional special biserial algebra which is not $\tau$-tilting finite, and $m \in \{1, 2, \ldots, n\}$.

1. There exists an infinite subset $\mathcal{U}$ of $2$-$\text{psilt} A$ such that $|U| = |U'| = m$ and $B_U \cong B_{U'}$ as algebras for any $U, U' \in \mathcal{U}$.
2. Let $\mathcal{U}$ as in (1). Then, for any $U, U' \in \mathcal{U}$, there exists an $\mathbb{R}$-linear automorphism $\rho_{U:U'}: K_0(\text{proj} A)_R \to K_0(\text{proj} A)_R$ which induces a bijection

$$\{\text{TF equivalence classes in } N_U\} \to \{\text{TF equivalence classes in } N_{U'}\}$$  

$E \mapsto (\text{the TF equivalence class containing } \rho(E) \cap N_{U'})$.

**Proof.** (1) Take $l \in \mathbb{Z}_{\geq 2}$ so that the admissible ideal $I$ in $A = \overline{KQ}/I$ satisfies $R^l \subset I$, where $R$ is the arrow ideal of $\overline{KQ}$. Then, for any $U \in 2$-$\text{psilt} A$, the proof of Theorem 4.12 implies that $B_U$ is isomorphic to a finite-dimensional special biserial algebra $\overline{KQ}_B/I_B$ with $I_B$ an ideal satisfying $(R')^l \subset I_B$ for the arrow ideal $R'$ of $\overline{KQ}_B$. Thus, the set $\{B_U \mid U \in 2$-$\text{psilt} A\}$ contains only finitely many isoclasses of finite-dimensional special biserial algebras.

On the other hand, there exist infinitely many $U \in 2$-$\text{psilt} A$ with $|U| = m$, since $A$ is not $\tau$-tilting finite.

Thus, we have the assertion.

(2) follows from Lemma 4.15. \qed

We end this section with showing that complete gentle algebras are closed under $\tau$-tilting reduction in the following sense. This will be crucial in Section 6.

**Corollary 4.17.** Let $A$ be a complete gentle algebra, and $U \in 2$-$\text{psilt} A$. For each $k \in \mathbb{Z}_{\geq 1}$, we set $A_k := A/I_k^B$ and $B_k := \text{End}_A(H^0(T_k))/[H^0(U_k)]$, where $T_k \in 2$-$\text{psilt} A_k$ is the Bongartz completion of $U_k := U \otimes_A A_k$. Then, there exists a complete gentle algebra $B$ such that we have epimorphisms $B \to B_k \to B/I_k^B$ of algebras for sufficiently large $k$.

**Proof.** We have seen that $B_k$ is a string algebra for any $k$ in Theorem 4.12. Moreover, the simple objects of $\mathcal{W}_{U_k} \subset \text{fd} A_k$ do not depend on $k$ by Proposition 3.2.

If $k$ is sufficiently large, then the quiver $Q_{B_k}$ of $B_k$ is constant, so we call it $Q'$. We define an admissible ideal $I'$ of the complete algebra $\overline{KQ}'$ so that $I'$ is generated by the paths $p$ of length $\geq 2$ in $Q'$ which is not admitted in $B_k$ for any $k$. Then, since $A$ is a complete gentle algebra, the generators of $I'$ are paths of exactly length 2, so $B := \overline{KQ}'/I'$ is a complete gentle algebra.
It remains to show the existence of $k_0$ such that any path in $Q'$ admitted in $B/I_c^B$ is admitted also in $B_{k_0}$. We can take $l_0 \in \mathbb{Z}_{\geq 1}$ such that any path admitted in $B/I_c^B$ is of length $\leq l_0$, and $k \in \mathbb{Z}_{\geq 1}$ such that any path of length $2$ admitted in $B$ is admitted in $B_k$.

Let $p' = \beta_1 \beta_2 \cdots \beta_l$ be a path admitted in $B/I_c^B$ with $\beta_i \in Q_i'$ and $l \leq l_0$. Then, $\beta_i \beta_{i+1}$ is admitted in $B_k$ for each $i \in \{1, 2, \ldots, l-1\}$, and we can consider the modules $M(\beta_i \beta_{i+1}) \in \text{fd } B_k$ and $\Phi_k^{-1}(M(\beta_i \beta_{i+1})) \in \mathcal{W}_{U_k} \subset \text{fd } A_k$, where $\Phi_k : \mathcal{W}_{U_k} \to \text{fd } B_k$ is the equivalence. In the proof of Theorem 4.12, each arrow in $S_{\text{OT}}$, $A_{\text{SAI}}$ and $B$ admitted in $Q'$ corresponds to some standard homomorphism in $\text{fd } A_k$ with $A_k$ a string algebra. Therefore, there exist strings $s_0, s_1, \ldots, s_l$ admitted in $A_k$ and arrows $\alpha_1, \alpha_2, \ldots, \alpha_l \in Q_1$ such that, for each $i \in \{1, 2, \ldots, l-1\}$, we have $\Phi_k^{-1}(M(\beta_i \beta_{i+1})) = M(s_{i-1} \alpha_i s_i \alpha_{i+1} s_{i+1})$ in $\text{fd } A_k$. By definition of $I_c$, we can check that $s_0 \alpha_1 s_1 \cdots \alpha_l s_l$ is an admitted string in $A_{k_{l_0}}$. This means that $\beta_1 \beta_2 \cdots \beta_l$ is a path admitted in $B_{k_{l_0}}$. By setting $k_0 := k_{l_0}$, we have proved that any path in $Q'$ admitted in $B/I_c^B$ is admitted also in $B_{k_0}$ as desired. □

5. Descriptions of the non-rigid regions

Our aim in this section is to describe the non-rigid region $\text{NR}$ for complete special biserial algebras $A$. As we have seen in Proposition 4.3, the purely non-rigid region $R_0$ plays an important role, so we explicitly determine it for all complete special biserial algebras.

In Subsection 5.1, we give our main results and some examples on $R_0$. Roughly speaking, our description of $R_0$ claims that $R_0$ is expressed by finitely many stability conditions for string modules associated to certain paths admitted in $A$. Subsection 5.2 is devoted to the key arguments, and we complete the proof of our main results in Subsection 5.3. The argument in these subsections is a modification of that for tame hereditary algebras in [AS1] Subsection 7.1. We also describe $\mathcal{N}_U \cap R_0$ for $U \in 2\text{-psilt } A$ in terms of $\tau$-rigid and $\tau^{-1}$-rigid modules in Subsection 5.4.

5.1. Results on the purely non-rigid regions. We first give the purely non-rigid region $R_0$ for complete gentle algebras.

For any $c \in \text{Cyc}(A)$, there uniquely exist a path $p$ in $Q$ and $\beta \in Q_1$ such that $c = p\beta$. We write $p_c$ for this $p$.

Theorem 5.1. Let $A$ be a complete gentle algebra. For each $\theta \in K_0(\text{proj } A)_{\mathbb{R}}$, the following conditions are equivalent.

(a) The element $\theta$ belongs to $R_0$.
(b) For any $c \in \text{Cyc}(A)$, there exists some cyclic permutation $d \in \text{Cyc}(A)$ of $c$ such that $\theta(M(p_d)) \in \mathcal{W}_\theta$, and for any $p \in \text{MP}(A)$, we have $M(p) \in \mathcal{W}_\theta$.
(b') For any $c \in \text{Cyc}(A)$, we have $\theta(M(p_c)) = 0$, and for any $p \in \text{MP}(A)$, we get $M(p) \in \mathcal{W}_\theta$. In particular, $R_0$ is a rational polyhedral cone of $K_0(\text{proj } A)_{\mathbb{R}}$, that is, there exist finitely many elements $\theta_1, \theta_2, \ldots, \theta_m \in K_0(\text{proj } A)$ such that $R_0 = \sum_{i=1}^m \mathbb{R}\theta_i$.

Note that, if $c' \in \text{Cyc}(A)$ is a cyclic permutation of $c \in \text{Cyc}(A)$, then $\theta(M(p_{c'})) = 0$ is equivalent to $\theta(M(p_c)) = 0$, since these two modules have the same dimension vector.

Set $h := \sum_{i \in Q_0} [S_i] \in K_0(\text{fd } A)$, then we obtain the following nice property from the theorem.

Corollary 5.2. Let $A$ be a complete gentle algebra. Then, $R_0 \subset \text{Ker}(?, h)$.

Proof. We define an equivalence relation $\sim$ on $\text{Cyc}(A)$ so that $c \sim c'$ if and only if $c'$ is a cyclic permutation of $c$, and take a complete system $X \subset \text{Cyc}(A)$ of representatives with respect to $\sim$. By Proposition 2.12, we can check that

$$2h = \sum_{d \in X} [M(p_d)] + \sum_{p \in \text{MP}(A)} a_p[M(p)],$$

where $a_p$ is given as follows:

- $a_p = 0$ if $p = e_i$ and $i \in Q_0$ has exactly one arrow $\alpha$ ending at $i$ and exactly one arrow $\beta$ starting at $i$ such that $\alpha \beta = 0$ in $A$;
Calculating these, we have

Similarly, $\eta_i \in \text{INR}(A)$, by Lemma 4.2, we have

We can check that $A := \widehat{KQ}/I$ is a complete gentle algebra. In this case,

We give some examples.

Example 5.3. Let $Q$ be the quiver

(\text{which is the same as Example 2.3}) and $I \subset KQ$ be the ideal generated by the paths

We can check that $A := \widehat{KQ}/I$ is a complete gentle algebra. In this case,

where $\alpha_i^{i+3} = \alpha_i$, and so on. Theorem 5.1 tells us that $R_0$ consists of all $\theta = \sum_{i=1}^8 a_i [P_i]$ satisfying

Calculating these, we have

The element $\eta_1$ is in $\text{INR}(A)$; indeed if $\eta_1 = \theta \oplus \theta'$, then since $\eta_1$ has no rigid direct summand by Lemma 4.2, we have $\theta, \theta' \in K_0(\text{proj} \ A) \cap R_0 = \{x\eta_1 + y\eta_2 \mid x, y \in \mathbb{R}_{\geq 0}\}$, so $\theta$ or $\theta'$ must be 0.

Similarly, $\eta_2 \in \text{INR}(A)$. The bands $b_{\eta_1}$ and $b_{\eta_2}$ are

respectively. On the other hand, $\eta_1$ and $\eta_2$ are not sign coherent, so $\eta_1 \oplus \eta_2$ does not hold in $K_0(\text{proj} \ A)$. Actually, $\eta_3 := \eta_1 + \eta_2 = [P_1] - [P_3] + [P_6] - [P_8]$ also belongs to $\text{INR}(A)$, and the band $b_{\eta_3}$ is
We can check $\eta_1 + \eta_3 = \eta_1 \oplus \eta_3$ and $\eta_3 + \eta_2 = \eta_3 \oplus \eta_2$ by Lemma 3.33. Since
\[
R_0 = \{x\eta_1 + z\eta_3 \mid x, z \in \mathbb{R}_{\geq 0}\} \cup \{z\eta_3 + y\eta_2 \mid z, y \in \mathbb{R}_{\geq 0}\},
\]
we get that $\text{INR}(A) = \{\eta_1, \eta_2, \eta_3\}$.

We obtain the following observation if we apply our result to hereditary algebras of type $\tilde{A}_{n-1}$.

**Example 5.4.** Let $Q$ be quiver of type $\tilde{A}_{n-1}$.

If $Q$ is acyclic, then $A = \overline{KQ}$ is a finite-dimensional gentle algebra. Set $(Q_0)_+, (Q_0)_-$ as the set of sources and the set of sinks, respectively. Then, Theorem 5.1 gives
\[
R_0 = \mathbb{R}_{\geq 0} \left( \sum_{i \in (Q_0)_+} [P_i] - \sum_{i \in (Q_0)_-} [P_i] \right).
\]

We remark that the modules $M(p)$ for $p \in \overline{\text{MP}}(A)$ are nothing but the string modules in the mouths of the regular tubes.

For example, let $Q$ be the quiver

```
\begin{array}{c}
4 \leftarrow 3 \\
\downarrow \quad \downarrow \\
\alpha_4 \quad \beta_3
\end{array}
```

Then, $\overline{\text{MP}}(A)$ consists of the paths
\[
e_1, \alpha_2 \alpha_3 \alpha_4, \alpha_6 \alpha_7 \alpha_8; \quad \beta_1 \beta_9, e_8, e_7, e_5, e_4, e_3.
\]

Thus, $R_0 = \mathbb{R}_{\geq 0}([P_2] + [P_6] - [P_5] - [P_9])$ by Theorem 5.1. The band associated to the unique element $[P_2] + [P_6] - [P_5] - [P_9]$ in $\text{INR}(A)$ is the quiver $Q$ itself:

```
\begin{array}{c}
4 \leftarrow 3 \\
\downarrow \quad \downarrow \\
\alpha_4 \quad \beta_3
\end{array}
```

On the other hand, if $Q$ is cyclic, then $A = \overline{KQ}$ is a complete gentle algebra. In this case, any $e_i$ belongs to $\overline{\text{MP}}(A)$, so Theorem 5.1 implies $R_0 = \{0\}$. Thus, any element in $K_0(\text{proj} A)$ is rigid.

This can be explained also in the following way. Clearly, the algebra $A/I_c$ is a Nakayama algebra, so $A/I_c$ is representation-finite and $\tau$-tilting finite. Therefore, $R_0 = R_0(A/I_c) = \{0\}$ by Proposition 4.8.

The following complete algebras from [AsI] Subsections 7.2, 7.3 suggest that $R_0$ itself is not enough to determine $\text{INR}(A)$.

**Example 5.5.** We consider the following two gentle algebras.
(1) Let $Q$ be the quiver

![Quiver](image)

and $I \subset \hat{KQ}$ be the ideal generated by the paths

$$\alpha_1 \beta_2, \alpha_2 \beta_3, \alpha_3 \beta_1, \beta_1 \alpha_3, \beta_2 \alpha_1, \beta_3 \alpha_2$$

Then, $A := \hat{KQ}/I$ is a complete gentle algebra. In this case,

$$\overline{\text{MP}}(A) = \emptyset,$$

$$\text{Cyc}(A) = \{ \alpha_i \alpha_{i+1} \alpha_{i+2}, \beta_i \beta_{i+1} \beta_{i+2} \}.$$ 

Therefore, $R_0 = \text{Ker}(?; h)$. By considering the quotient algebra $A/I_c$, we have

$$\text{INR}(A) = \{ \theta_{i,j} \mid i, j \in \{1, 2, 3\}, i \neq j \}$$

from [AsI, Theorem 7.6].

(2) Let $Q$ be the quiver

![Quiver](image)

and $I \subset \hat{KQ}$ be the ideal generated by the paths

$$\alpha_1 \beta_2, \alpha_2 \beta_3, \alpha_3 \beta_1, \beta_1 \alpha_3, \beta_2 \alpha_1, \beta_3 \alpha_1$$

Then, $A := \hat{KQ}/I$ is a complete gentle algebra. In this case,

$$\overline{\text{MP}}(A) = \emptyset,$$

$$\text{Cyc}(A) = \{ \alpha_i \alpha_{i+1} \alpha_{i+2}, \beta_i \beta_{i+1} \beta_{i+2} \}.$$ 

We have $R_0 = \text{Ker}(?; h)$. This is the same as in (1). However, $\text{INR}(A)$ has infinitely many elements by [AsI, Theorem 7.9]:

$$\text{INR}(A) = \{ a_1[P_1] + a_2[P_2] + a_3[P_3] \mid a_1, a_2, a_3 \in \mathbb{Z}, a_1 + a_2 + a_3 = 0, \gcd(a_1, a_2, a_3) = 1 \}.$$ 

Next, we consider $R_0$ for any complete special biserial algebra $A$. We fix an ideal $\tilde{I} \subset I$ such that $\tilde{A}$ is a complete gentle algebra by Proposition [2.9].

For each $\theta \in K_0(\text{proj } A)_R$, we set $\tilde{W}_{\theta}$ ($\subset \tilde{W}_{\theta} \subset \text{fd } \tilde{A}$) as the subcategory of all $M \in \text{fd } \tilde{A}$ such that $M$ admit filtrations $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ with $M_i/M_{i-1} \cong \tilde{W}_{\theta} \cap \text{fd } A$.

For our purpose, we do not need to fully understand the category $\tilde{W}_{\theta}$; we are interested only in whether the string module $M(p)$ associated to a path $p$ belongs to $\tilde{W}_{\theta}$ or not. In this context, the following properties are useful.

**Lemma 5.6.** Let $p$ be a path in $Q$ with $p \neq 0$ in $A$. Then, $M(p) \in \tilde{W}_{\theta}$ if and only if there exist paths $q_1, q_2, \ldots, q_k$ in $Q$ and arrows $\alpha_1, \alpha_2, \ldots, \alpha_{k-1} \in Q_1$ such that
\begin{itemize}
  \item $p = q_1 \alpha_1 q_2 \alpha_2 \cdots q_{i-1} \alpha_{i-1} q_i$ as paths;
  \item $M(q_i)$ is a simple object in $\mathcal{W}_\theta \cap \text{fd} A$;
\end{itemize}

In particular, \{\theta \in K_0(\text{proj} A)_R \mid M(p) \in \mathcal{W}_\theta\} is a union of finitely many rational polyhedral cones.

Proof. This directly follows from the definition. \hfill \Box

\begin{lemma}
Let \( \tilde{p} \) be a path which is a string in \( \tilde{A} \). If \( M(\tilde{p}) \in \mathcal{W}_\theta \) and any \( p \in \overline{\text{MP}}_*(A) \) satisfies \( M(p) \not\in T_\theta \), then \( M(\tilde{p}) \in \mathcal{W}_\theta \).
\end{lemma}

Proof. If there exists \( p \in \overline{\text{MP}}_*(A) \) such that \( M(\tilde{p}) \) is a quotient module of \( M(p) \), then it is done.

Otherwise, we can take subpaths \( p_1, \tilde{s}_1 \) of length \( \geq 1 \) such that \( p_1 \in \text{MP}_*(A) \) and \( \tilde{p} = p_1 \tilde{s}_1 \). By assumption, there exist some subpaths \( q_1, r_1 \) of \( p_1 \) such that \( M(q_1) \in \mathcal{F}_\theta \cap \text{fd} A \) and that \( p_1 = q_1 r_1 \).

Since \( M(q_1) \) is a quotient module of \( M(p) \in \mathcal{W}_\theta \), we have \( M(q_1) \in \mathcal{W}_\theta \cap \text{fd} A \). We define \( \alpha_1 \in Q_1 \) and a path \( \tilde{p}_1 \) in \( Q \) so that \( r_1 \tilde{s}_1 = \alpha_1 \tilde{p}_1 \). Then, we get \( M(\tilde{p}_1) = M(\tilde{p})/M(p_1) \in \mathcal{W}_\theta \). Applying the argument above to \( \tilde{p}_1 \), we have some paths \( q_2, r_2, \tilde{s}_2 \) such that \( M(q_2) \in \mathcal{W}_\theta \cap \text{fd} A \) and \( \tilde{p}_1 = q_2 r_2 \tilde{s}_2 \). We set \( \alpha_2 \in Q_1 \) and a path \( \tilde{p}_2 \) in \( Q \) so that \( r_2 \tilde{s}_2 = \alpha_2 \tilde{p}_2 \).

Repeating this procedure finitely many times, we finally get paths \( q_1, q_2, \ldots, q_k \) in \( Q \) and arrows \( \alpha_1, \alpha_2, \ldots, \alpha_{k-1} \in Q_1 \) satisfying the conditions of Lemma \ref{lemma:5.6}. Thus, \( M(\tilde{p}) \in \mathcal{W}_\theta \). \hfill \Box

We aim to prove the following result. For any \( \tilde{c} \in \text{Cyc}(\tilde{A}) \), there uniquely exist a path \( \tilde{p} \) in \( Q \) and \( \beta \in Q_1 \) such that \( \tilde{c} = \tilde{p} \beta \). We write \( \tilde{c}^\beta \) for this \( \tilde{p} \) as in Theorem \ref{thm:5.1}.

\begin{theorem}
Let \( A = K\mathcal{Q}/I \) be a complete special biserial algebra, and take an ideal \( \tilde{I} \subset I \) such that \( \tilde{A} = K\tilde{\mathcal{Q}}/\tilde{I} \) is a complete gentle algebra. Then, for each \( \theta \in K_0(\text{proj} A)_R \), the following conditions are equivalent.

(a) The element \( \theta \) belongs to \( R_0 \).
(b) The element \( \theta \) belongs to \( R_0(\tilde{A}) \), and if \( p \in \overline{\text{MP}}_*(A) \), then \( M(p) \not\in T_\theta \).
(b') The element \( \theta \) belongs to \( R_0(\tilde{A}) \), and if \( p \in \overline{\text{MP}}_*(A) \), then \( M(p) \not= T_\theta \).
(c) For any \( \tilde{c} \in \text{Cyc}(\tilde{A}) \), there exists a cyclic permutation \( \tilde{d} \) of \( \tilde{c} \) such that \( M(\tilde{d}) \in \mathcal{W}_\theta \), and for any \( \tilde{p} \in \overline{\text{MP}}(\tilde{A}) \), we have \( M(\tilde{p}) \in \mathcal{W}_\theta \).

In particular, \( R_0 \) is a union of finitely many rational polyhedral cones in \( K_0(\text{proj} A)_R \). Moreover, \( R_0 \subset \text{Ker}(?, h) \).

We remark that \( R_0 \) is not necessarily a rational polyhedral cone; see Example \ref{example:5.10}. Note also that this theorem implies that projective-injective modules do not matter on \( R_0 \); more precisely, if two different paths \( p \not= q \) in \( Q \) satisfy \( p - q \in I \) and \( p, q \not\in I \), then \( R_0 = R_0(A/(p - q)) \).

We write the following property for later use.

\begin{corollary}
Let \( A = K\mathcal{Q}/I \) be a complete special biserial algebra. Then, there exist a set \( Z \) of paths admitted in \( A \) and \( k_p \in \{0, 1, 2\} \) for all \( p \in Z \) satisfying the following conditions:

\begin{itemize}
  \item \( M(p) \) is a simple object in \( \mathcal{W}_\theta \) for each \( p \in Z \);
  \item \( \sum_{p \in Z} k_p |M(p)| = 2h \in K_0(\text{fd} A) \);
  \item \( k_p = 1 \) if \( p \) is of length \( \geq 1 \).
\end{itemize}

Proof. Take the complete gentle algebra \( \tilde{A} \) as in Theorem \ref{thm:5.8}. By Theorem \ref{thm:5.8}, we can take a complete system \( X \subset \text{Cyc}(\tilde{A}) \) of representatives with respect to cyclic permutations so that \( M(\tilde{d}) \in \mathcal{W}_\theta \) for any \( \tilde{d} \in X \).

For each \( \tilde{d} \in X \), we take the paths \( q_i \) for \( i \in \{1, 2, \ldots, l\} \) admitted in \( A \) applying Lemma \ref{lemma:5.6} to \( M(\tilde{d}), \) and rewrite these paths as \( q(\tilde{d}, i) \) for \( i \in \{1, 2, \ldots, l(\tilde{d})\} \). In a similar way, we apply Lemma \ref{lemma:5.6} to \( M(\tilde{p}) \) for each \( \tilde{p} \in \overline{\text{MP}}(\tilde{A}) \), and define \( q(\tilde{p}, i) \) for \( i \in \{1, 2, \ldots, l(\tilde{p})\} \).
We set $a_{\tilde{d}}$ as in the proof of Corollary 5.2, then
\[
2h = \sum_{\tilde{d} \in X} [M(\tilde{\sigma})] + \sum_{\tilde{p} \in \overline{MP}(\tilde{A})} a_{\tilde{p}}[M(\tilde{p})].
\]

For each $\tilde{d} \in X$, we define $a_{\tilde{d}} = 1$. Then, we get
\[
2h = \sum_{\tilde{p} \in \overline{MP}(\tilde{A}) \cup X} \sum_{i=1}^{l(\tilde{p})} a_{\tilde{p}}[M(\tilde{q}(\tilde{p}, i))].
\]

We set $Y$ and $Z$ as the set of all pairs $(\tilde{p}, i)$ and paths $\tilde{q}(\tilde{p}, i)$ appearing above for $\tilde{p} \in \overline{MP}(\tilde{A}) \cup X$, respectively. Namely,
\[
Y := \{ (\tilde{p}, i) \mid \tilde{p} \in \overline{MP}(\tilde{A}) \cup X, i \in \{1, 2, \ldots, l(\tilde{p})\}, \}
\]
\[
Z := \{ \tilde{q}(\tilde{p}, i) \mid (\tilde{p}, i) \in Y \}.
\]

Then, for any $p \in Z$, the path $p$ is admitted in $A$, and $M(p)$ is a simple object in $\mathcal{W}_0$ for any $p \in Z$. Note that the map $Y \ni (\tilde{p}, i) \mapsto q(\tilde{p}, i) \in Z$ is not necessarily injective. Thus, for each $p \in Z$, we define $Y_p := \{ (\tilde{p}, i) \in Y \mid p = q(\tilde{p}, i) \}$ and $k_p := \sum_{(\tilde{p}, i) \in Y_p} a_{\tilde{p}}$. Then, $\sum_{p \in Z} k_p[M(p)] = 2h \in K_0(\text{fd} A)$.

To finish the proof, let $p \in Z$. By definition, we have
- if $\#Y_p \geq 2$, then $p$ is of length $0$, $\#Y_p = 2$, and $a_{\tilde{p}} = 1$ for any $(\tilde{p}, i) \in Y_p$;
- if there exists $(\tilde{p}, i) \in Y_p$ such that $\tilde{p}$ is of length $0$, then $\#Y_p = 1$; and
- if $p$ is of length $\geq 1$, then $\#Y_p = 1$, and $a_{\tilde{p}} = 1$ for the unique element $(\tilde{p}, i) \in Y_p$.

These allow us to check that $k_p = \{0, 1, 2\}$ for any $p \in Z$, and that $k_p = 1$ if $p$ is of length $\geq 1$. \qed

We give some examples.

**Example 5.10.** We consider the algebra $A$ in Example 5.3. Define $\tilde{A}$ as the complete gentle algebra in Example 5.3. Then, $A$ is the quotient algebra of $\tilde{A}$ by the additional relation $\delta_1\delta_2\delta_3\delta_4 = 0$.

We have seen
\[
R_0(\tilde{A}) = \{ x\eta_1 + y\eta_2 \mid x, y \in \mathbb{R}_{\geq 0} \},
\]
\[
\eta_1 = [P_1] - [P_3] - [P_5] + [P_6], \quad \eta_2 = [P_3] - [P_5],
\]
and Theorem 5.8 tells us that $R_0(A) = \{ \theta \in R_0(\tilde{A}) \mid M(\delta_1\delta_2\delta_3\delta_4) \in \overline{\mathcal{W}}_0 \}$. The condition $M(\delta_1\delta_2\delta_3\delta_4) \in \mathcal{W}_0$ implies that $M(e_4), M(\delta_4), M(\delta_3\delta_4)$ or $M(\delta_2\delta_3\delta_4)$ is $\theta$-semistable. If $\theta \in R_0(\tilde{A})$ satisfies this condition, then $\theta \in \mathbb{R}_{\geq 0}\eta_1$ or $\theta \in \mathbb{R}_{\geq 0}\eta_2$. Conversely, if $\theta \in \mathbb{R}_{\geq 0}\eta_1 \cup \mathbb{R}_{\geq 0}\eta_2$, then we can check $M(\delta_1\delta_2\delta_3\delta_4) \in \overline{\mathcal{W}}_0$. Therefore, $R_0 = \mathbb{R}_{\geq 0}\eta_1 \cup \mathbb{R}_{\geq 0}\eta_2$. In particular, $R_0$ itself is not a rational polyhedral cone.

The band $\tilde{A}$-modules corresponding to $\eta_1$ and $\eta_2$ are $A$-modules, so $\eta_1$ and $\eta_2$ are in $\text{INR}(A)$. On the other hand, $\sigma = \eta_1 + \eta_2 \notin R_0$, so $\sigma \notin \text{INR}(A)$. Actually, $\sigma \in \text{IR}(A)$, and the corresponding $\tau$-rigid $(A/I_c)$-module and $\tau^{-1}$-rigid $(A/I_c)$-module are...
\textbf{Example 5.11.} We consider the quiver $Q$ in Example 5.5 (1), and set the admissible ideal $\hat{I}$ of $\hat{K}Q$ as $I$ in Example 5.3 (1). We also define $I := \hat{I} + (\alpha_1\alpha + 1) \supseteq \hat{I}$. For the complete gentle algebra $A := \hat{K}Q/\hat{I}$, we have $\text{MP}(A) = 0$ and $\text{Cyc}(\hat{A}) = \{\alpha_1\alpha_1 + 2, \beta_1\beta_1 + 1, \beta_1 + 2 | i \in \{1, 2, 3\}\}$.

Thus, by Theorem 5.8, $\theta \in R_0$ holds if and only if $M(\beta_i\beta_{i+1}) \in \mathcal{W}_9 \subset \text{fd} \hat{A}$ for some $i$ and $M(\beta_i\beta_{i+1}) \in \mathcal{W}_9 \subset \text{fd} \hat{A}$ for some $i$.

Let $\theta = a_1[P_1] + a_2[P_2] + a_3[P_3] \in K_0(\text{proj} A)$. The second condition is equivalent to $M(\beta_i\beta_{i+1}) \in \mathcal{W}_9$, since $M(\beta_i\beta_{i+1}) \in \text{fd} A$. This holds if and only if $a_1 + a_2 + a_3 = 0$. On the other hand, the first condition is equivalent to that $M(\alpha_1) \in \mathcal{W}_9$ and $S_{i+2} \in \mathcal{W}_9$ for some $i$. This precisely means that $a_i = 0, a_i \geq 0$ and $a_i + 1 = 0$. Therefore, $R_0 = \mathbb{R}_{\geq 0}([P_1] - [P_2]) + \mathbb{R}_{\geq 0}([P_2] - [P_3]) + \mathbb{R}_{\geq 0}([P_3] - [P_1])$.

5.2. Key arguments. In this subsection, we show the following property, which is crucial to prove our theorems.

\textbf{Proposition 5.12.} Let $\theta \in R_0$. Then, for any $p \in \text{MP}_*(A)$, we have $M(p) \notin \mathcal{T}_9$.

First, we define a technical symbol.

\textbf{Definition 5.13.} For any $p \in \text{MP}_*(A)$ and $\theta \in K_0(\text{proj} A)$, we set a subpath $s_{p, \theta}$ of $p$ so that $p = s_{p, \theta} t$ for some path $t$ and that $M(s_{p, \theta}) \cong M(p)/M$, where $M$ is the maximum submodule of $\text{rad} M(p)$ belonging to $\mathcal{T}_9$.

Then, we can prove the next property by using the modules in Definition 5.12.

\textbf{Lemma 5.14.} Assume that $A$ is a finite-dimensional special biserial algebra. Let $p \in \text{MP}_*(A)$. Then, for any $\eta \in \text{INR}(A)$ such that $M(p) \in \mathcal{T}_9$, we have $M(s_{p, \eta}) \in \mathcal{W}_9$.

\textbf{Proof.} Clearly, $M(s_{p, \eta}) \in \mathcal{T}_9$. By Proposition 5.2, it suffices to show $\text{Hom}_A(M(\eta), M(s_{p, \eta})) = 0$ for some $\lambda \in K^\times$. Take $\lambda \in K^\times$. Assume that there exists nonzero $\varphi \in \text{Hom}_A(M(\eta), M(s_{p, \eta}))$.

By definition, any proper module $L \subset M(s_{p, \eta})$ belongs to $\mathcal{T}_9$. Since $L$ is a string module, by using Proposition 5.2, we get $\text{Hom}_A(M(\eta), L) = 0$ for any $\lambda \in K^\times$. This implies that any $\varphi \in \text{Hom}_A(M(\eta), M(s_{p, \eta}))$ is zero or surjective.

If a surjection $\varphi \in \text{Hom}_A(M(\eta), M(s_{p, \eta}))$ exists, then there exist two arrows $\alpha, \alpha' \in Q_1$ and two paths $r, r'$ in $Q$ such that $M((\alpha r')^{-1} s_{p, \eta})$ is isomorphic to a submodule of $M(\eta)$. This and $p \in \text{MP}_*(A)$ imply $s_{p, \eta} \neq p$, and we have $M(r) \in \mathcal{T}_9$ by definition.

Thus, there exists a nonzero homomorphism $M(\eta) \to M(r)$. On the other hand, $M(r)$ is a proper submodule of $M(\eta)$. Thus, we have a nonzero non-isomorphism $M(\eta) \to M(\eta)$. It contradicts Lemma 5.3.

Now, we have obtained $\text{Hom}_A(M(\eta), M(s_{p, \eta})) = 0$. Therefore, $M(s_{p, \eta}) \in \mathcal{W}_9$. \hfill \Box

We also need an analogous result for indecomposable rigid elements. The next notions are useful.

\textbf{Definition 5.15.} Assume that $A$ is a finite-dimensional special biserial algebra. Let $\sigma \in \text{IR}(A)$. We define a module $L_\sigma$ as follows:

- if $\sigma(h) = 1$, then $L_\sigma := M'_\sigma$;
- if $\sigma(h) = 0$, then take the unique $p \in \text{MP}_*(A)$ which admits the canonical surjection $\pi: M_\sigma \to M(p)$, and set $L_\sigma := \text{Ker} \pi$;
- if $\sigma(h) = -1$, then $L_\sigma := M_\sigma$.

We can check that $L_\sigma$ is zero or indecomposable. Moreover, $L_\sigma$ is a submodule of $M_\sigma$. Then, we can show the following property.

\textbf{Lemma 5.16.} Assume that $A$ is a finite-dimensional special biserial algebra. Let $p \in \text{MP}_*(A)$. Then, there exists $E \in \mathbb{Z}_{\geq 1}$ satisfying the following condition: for any $\theta \in \text{IR}(A)$ such that $M(p) \in \mathcal{T}_\sigma$, we have $0 \leq \sigma(M(s_{p, \eta})) \leq E$. 
Proof. Clearly, \( M(s_{p,\sigma}) \in \mathcal{T}_\sigma \). We have \( \sigma(M(s_{p,\sigma})) = \dim_K \text{Hom}_A(M_\sigma, M(s_{p,\sigma})) \) by Proposition 3.27.

To evaluate this, we show that \( \varphi(\mathcal{L}_\sigma) = 0 \) for any \( \varphi: M_\sigma \to M(s_{p,\sigma}) \). If \( \varphi(\mathcal{L}_\sigma) \neq 0 \) for some \( \varphi: M_\sigma \to M(s_{p,\sigma}) \), then we can show that \( \text{Hom}_A(M(s_{p,\sigma}), M'_\sigma) \neq 0 \) as in Lemma 5.14 and this implies \( M(s_{p,\sigma}) \notin \mathcal{T}_\sigma \) by Proposition 3.27 again. This contradicts the assumption, so \( \varphi(\mathcal{L}_\sigma) = 0 \). Thus,

\[
\dim_K \text{Hom}_A(M_\sigma, M(s_{p,\sigma})) = \dim_K \text{Hom}_A(M_\sigma / L_\sigma, M(s_{p,\sigma})).
\]

Since \( M_\sigma / L_\sigma \) is an indecomposable projective module or of the form \( \bigoplus_{j=1}^{1+\sigma(h)} M(p_j) \) with each \( p_j \in \mathcal{M}_p(A) \), there exists \( E \in \mathbb{Z}_{\geq 1} \) such that \( 0 \leq \sigma(M(s_{p,\sigma})) \leq E \), where \( E \) does not depend on \( \sigma \).

If \( \theta' \) is a direct summand of \( \theta \) in \( K_0(\text{proj} A) \), the paths \( s_{p,\theta} \) and \( s_{p,\theta'} \) do not coincide in general. However, this is not a problem by the following property.

Lemma 5.17. Assume that \( A \) is a finite-dimensional special biserial algebra. Let \( p \in \mathcal{M}_p(A) \) with \( M(p) \in \mathcal{T}_\theta \), and \( \theta = \bigoplus_{i=1}^m \theta_i \) in \( K_0(\text{proj} A) \). Then, for any \( i \), we have \( \theta_i(M(s_{p,\theta})) = \theta_i(M(s_{p,\theta_i})) \).

Proof. Let \( i \in \{1, 2, \ldots, m\} \). We consider the canonical surjections \( \pi: M(p) \to M(s_{p,\theta}) \) and \( \pi_i: M(p) \to M(s_{p,\theta_i}) \). By Proposition 3.29, \( s_{p,\theta} \) is shorter than or equal to \( s_{p,\theta_i} \), so we have a surjection \( M(s_{p,\theta_i}) \to M(s_{p,\theta}) \). Define \( X_i \) as its kernel, then \( X_i \) is a proper submodule of \( M(s_{p,\theta_i}) \), so \( X_i \in \mathcal{T}_{\theta_i} \). On the other hand, \( X_i \cong \ker \pi / \ker \pi_i \) and \( \ker \pi \in \mathcal{T}_\theta \) hold, so \( X_i \in \mathcal{T}_{\theta_i} \).

By Proposition 3.29, we get \( X_i \in \mathcal{T}_{\theta_i} \). Therefore, \( X_i \in \mathcal{W}_{\theta_i} \), and \( \theta_i(M(s_{p,\theta})) = \theta_i(M(s_{p,\theta_i})) - \theta_i(X_i) = \theta_i(M(s_{p,\theta_i})) \).

For \( \theta = \sum_{i \in Q_0} a_i [P_i] \in K_0(\text{proj} A)_{\mathbb{R}}, \) we set the 1-norm of \( \theta \) by

\[
||\theta||_1 := \sum_{i \in Q_0} |a_i|
\]
as usual. By using this norm, we have the following crucial lemma.

Lemma 5.18. Let \( p \in \mathcal{M}_p(A) \) and \( \varepsilon \in \mathbb{R}_{>0} \), set

\[
Y := \{ \theta \in K_0(\text{proj} A)_{\mathbb{R}} \mid \theta(X) > \varepsilon ||\theta||_1 \text{ for any nonzero quotient } X \text{ of } M(p) \}.
\]

Then, there exists a finite subset \( F \subset \mathcal{IR}(A) \) such that

\[
Y \subset \bigcup_{\sigma \in F} N_{\sigma}.
\]

Proof. We may assume that \( A \) is a finite-dimensional special biserial algebra by Proposition 4.8. We set \( F := \mathcal{IR}(A) \cap Y \). Then, \( \sigma \in F \) implies \( ||\sigma||_1 < E/\varepsilon \) for \( E \) in Lemma 5.10 so \( F \) is a finite set.

We show that this finite set \( F \) satisfies the assertion. Since each \( N_{\sigma} \) and \( Y \) are both defined by linear inequalities whose coefficients are integers, it suffices to prove

\[
Y \cap K_0(\text{proj} A) \subset \bigcup_{\sigma \in F} N_{\sigma}.
\]

Let \( \theta \in Y \cap K_0(\text{proj} A) \), and \( \theta = \bigoplus_{i=1}^m \theta_i \) be the canonical decomposition. By Proposition 3.30, \( \theta_1, \theta_2, \ldots, \theta_m \) are sign-coherent. This and \( \theta \in Y \) imply that there must exist \( i \in \{1, 2, \ldots, m\} \) such that \( \theta_i(M(s_{p,\theta})) > \varepsilon ||\theta_i||_1 \). By Lemma 5.14, \( \theta_i(M(s_{p,\theta})) > \varepsilon ||\theta_i||_1 > 0 \). Since \( M(p) \in \mathcal{T}_{\theta} \) by \( \theta \in Y \), we have \( M(p) \in \mathcal{T}_{\theta_i} \) by Proposition 3.29. Thus, Lemma 5.14 yields \( \theta_i \notin \mathcal{INR}(A) \); hence, \( \theta_i \in \mathcal{IR}(A) \). We show \( \theta_i \in F \). Let \( X \) be a nonzero quotient module of \( M(p) \). By the definition of \( M(s_{p,\theta_i}) \), we have \( \theta_i(X) \geq \theta_i(M(s_{p,\theta_i})) > \varepsilon ||\theta_i||_1 \), so \( \theta_i \in F \). By Lemma 4.2, we have \( \theta \in N_{\theta_i} \).

Therefore, \( Y \cap K_0(\text{proj} A) \subset \bigcup_{\sigma \in F} N_{\sigma} \), and we have the assertion.

Now, we are ready to prove Proposition 5.12.
Proof of Proposition 5.12. We assume \( \theta \in R_0 \) and \( M(p) \in T_\theta \), and deduce a contradiction. Since \( M(p) \in T_\theta \), there exists \( \varepsilon \in \mathbb{R}_{>0} \) such that \( \theta(M(p)) > \varepsilon \| \theta \|_1 \). Then, Lemma 5.18 implies \( \theta \in N_\sigma \) for some \( \sigma \in IR(A) \). This contradicts \( \theta \in R_0 \). \( \square \)

5.3. Proof of Theorems 5.1 and 5.8. In this section, we prove Theorems 5.1 and 5.8. We first introduce the following notion.

Definition 5.19. Let \( A = \overline{KQ}/I \) be a complete special biserial algebra. Then, \( A \) is called a truncated gentle algebra if there exist a complete gentle algebra \( \tilde{A} \) and \( m \geq 1 \) such that \( A \cong \tilde{A}/(I_c^m) \) and \( ml_c \geq 3 \) for all \( c \in \text{Cyc}(\tilde{A}) \), where \( l_c \) is the length of \( c \).

By Proposition 5.18, it suffices to determine \( R_0 \) for truncated gentle algebras. The assumption \( ml_c \geq 3 \) is needed to make the following property, which gentle algebras also satisfy, hold true for truncated gentle algebras.

Lemma 5.20. Let \( A \) be a truncated gentle algebra.

1. If \( \alpha \in Q_1 \) is an arrow ending at \( i \in Q_0 \) and \( \beta \neq \gamma \in Q_1 \) are arrows starting at \( i \), then \( \alpha \beta \notin I \) or \( \alpha \gamma \notin I \).

2. If \( \alpha \in Q_1 \) is an arrow starting at \( i \in Q_0 \) and \( \beta \neq \gamma \in Q_1 \) are arrows ending at \( i \), then \( \beta \alpha \notin I \) or \( \gamma \alpha \notin I \).

In the truncation process of gentle algebras, the sets \( \text{MP} \) and \( \text{Cyc} \) change as follows.

Lemma 5.21. Let \( \tilde{A} \) be a complete gentle algebra and \( m \geq 1 \). If \( A := \tilde{A}/(I_c^m) \) is a truncated gentle algebra, then \( A \) satisfies

\[
\begin{align*}
\text{MP}_*(A) &= \text{MP}_*(\tilde{A}) \cup C, & \text{MP}^*(A) &= \text{MP}^*(\tilde{A}) \cup C, & \text{MP}(A) &= \text{MP}(\tilde{A}) \cup C, \\
\overline{\text{MP}}_*(A) &= \overline{\text{MP}}_*(\tilde{A}) \cup C, & \overline{\text{MP}}^*(A) &= \overline{\text{MP}}^*(\tilde{A}) \cup C, & \overline{\text{MP}}(A) &= \overline{\text{MP}}(\tilde{A}) \cup C,
\end{align*}
\]

where \( C := \{e^{m-1}p_c \mid c \in \text{Cyc}(\tilde{A}) \} \).

The following properties directly follow from Lemmas 2.13 and 5.21. Note that the paths in (1)(ii) below for truncated gentle algebras \( A \) are precisely the elements in \( C \) of Lemma 5.21.

Proposition 5.22. Let \( A \) be a truncated gentle algebra. In (i) and (ii), \( \alpha \) and \( \beta \) denote the first and the last arrows of \( p \).

1. The set \( \overline{\text{MP}}_*(A) \) consists of the following paths in \( Q \).
   (i) The paths \( p \) of length \( \geq 1 \) satisfying \( p \neq 0 \) and \( \beta \gamma = 0 \) in \( A \) for any arrow \( \gamma \in Q_1 \).
   (ii) The paths \( p \) of length \( \geq 2 \) satisfying \( p \neq 0 \) in \( A \) and that there exists an arrow \( \gamma \in Q_1 \) such that
      * \( c_p := \gamma p \) and \( c^p := \gamma p \) are cycles in \( Q \).
      * \( \beta \gamma \neq 0, \gamma \alpha \neq 0, \gamma p = 0 \) and \( \gamma p = 0 \) in \( A \).
   (iii) The paths \( c_i \) for \( i \in Q_0 \) satisfying that there exists at most one arrow starting at \( i \).

2. The set \( \overline{\text{MP}}(A) \) consists of the following paths in \( Q \).
   (i) The paths \( p \) of length \( \geq 1 \) satisfying \( p \neq 0 \) and \( \gamma \alpha = 0 \) in \( A \) for any arrow \( \gamma \in Q_1 \).
   (ii) The same paths as (1)(ii).
   (iii) The paths \( c_i \) for \( i \in Q_0 \) satisfying that there exists at most one arrow ending at \( i \).

The next property is easily deduced, but will be important later.

Lemma 5.23. Let \( A \) be a truncated gentle algebra, \( p, q, q' \) be paths in \( Q \) such that \( p = q'q \) is of length \( \geq 1 \), and \( \alpha, \beta \in Q_1 \) be the first and the last arrows of \( p \).

1. Let \( p \in \text{MP}_*(A) \). If \( \beta \gamma = 0 \) in \( A \) for any \( \gamma \in Q_1 \), then \( q \in \overline{\text{MP}}_*(A) \). Otherwise, \( p \in \text{MP}(A) \) and is of type (ii) in Proposition 5.22.
(2) Let \( p \in MP^* (A) \). If \( \gamma \alpha = 0 \) in \( A \) for any \( \gamma \in Q_1 \), then \( q' \in MP^* (A) \). Otherwise, \( p \in MP(A) \) and is of type (ii) in Proposition \ref{prop5.22}.

**Proof.** This follows from Proposition \ref{prop5.22}. \(\square\)

We have the following result.

**Theorem 5.24.** Let \( A \) be a truncated gentle algebra. For each \( \theta \in K_0 (\text{proj} A)_R \), the following conditions are equivalent.

(a) The element \( \theta \) belongs to \( R_0 \).

(b) Any \( p \in MP_+(A) \) satisfies \( M(p) \notin T_\theta \), and any \( p \in MP^* (A) \) satisfies \( M(p) \notin F_\theta \).

(c) For any \( p \in MP(A) \), if \( p \) is of type (ii), then there exists a path \( q \in MP(A) \) such that \( c_q \) is a cyclic permutation of \( c_p \) and that \( M(q) \in W_\theta \); otherwise, \( M(p) \in W_\theta \).

\((c')\) For any \( p \in MP(A) \), if \( p \) is of type (ii), then \( \theta (M(p)) = 0 \); otherwise, \( M(p) \in W_\theta \).

In particular, \( R_0 \) is a rational polyhedral cone of \( K_0 (\text{proj} A)_R \).

**Proof.** (a) \(\Rightarrow\) (b): This is Proposition \ref{prop5.12} and its dual.

(b) \(\Rightarrow\) (c): Consider the case that \( p \) is of type (ii) first. Take \( \alpha_1, \alpha_2, \ldots, \alpha_l \in Q_1 \) such that the cycle \( c_p \) is \( \alpha_1 \alpha_2 \cdots \alpha_l \). Thus, \( p = \alpha_1 \alpha_2 \cdots \alpha_{l-1} \). Let \( i_k \in Q_0 \) denote the source of \( \alpha_k \) for each \( k \in \{1, 2, \ldots, l\} \), and for \( u, v \in \{1, 2, \ldots, l\} \), we set

\[
[[u, v]] := \begin{cases} 
\{u, u + 1, \ldots, v\} & (u \leq v) \\
\{u, u + 1, \ldots, l\} \cup \{1, 2, \ldots, v\} & (u > v)
\end{cases}
\]

and

\[
f(u, v) := \sum_{k \in [[u, v]]} \theta(S_{i_k})
\]

If \( \theta (M(p)) \geq 0 \), then we have a pair \( (u, v) \) such that \( [[u, v]] \) has the least elements among the pairs giving the maximum value of \( f \), and a path \( q \) such that \( c_q = (\alpha_{u+1} \cdots \alpha_l) \cdot (\alpha_1 \alpha_2 \cdots \alpha_{u-1}) \). Then, we can check that \( M(q) \in T_\theta \) by \( \theta (M(p)) \geq 0 \). Moreover, if \( \theta (M(p)) > 0 \), then \( M(q) \in T_\theta \), but it contradicts (b).

Similarly, we can deduce a contradiction from \( \theta (M(p)) < 0 \).

Therefore, \( \theta (M(p)) = 0 \) and the path \( q \) above satisfies \( M(q) \in W_\eta \). Clearly, \( c_q \) is a cyclic permutation of \( c_p \).

If \( p \) is of type (i) or (iii), (b) and Lemma \ref{lemma5.23} immediately give the assertion.

(c) \(\Rightarrow\) (c'): It is obvious.

(c') \(\Rightarrow\) (a): We suppose that \( \theta \notin R_0 \), and show that (c') is not satisfied. Then, we can take some indecomposable rigid element \( \sigma \in \text{IR}(A) \) such that \( \theta \in N_\sigma \). If \( \sigma(h) \geq 0 \), then we can find \( q \in MP_+(A) \) such that \( M(q) \) is a quotient module of \( \sigma \). Since \( \theta \in N_\sigma \), we have \( M(q) \in T_\theta \). If \( q \) is of type (ii), then \( q \in MP(A) \) and \( M(q) > 0 \), which denies (c'). Otherwise, there exists some path \( r \) such that \( p := rq \in MP(A) \) is of type (i) or (iii). Then, \( M(q) \in T_\theta \) implies \( M(p) \notin W_\theta \), which contradicts (c'). We can check that (c') is not satisfied in the case \( \sigma(h) < 0 \), either.

Thus, (c) implies that \( \theta \in R_0 \). \(\square\)

Now, Theorem \ref{thm5.1} follows almost immediately.

**Proof of Theorem 5.1** Proposition \ref{prop4.8} allows us to consider a truncated gentle algebra \( A/I_c^m \) instead of the complete gentle algebra \( A \). We apply Theorem \ref{thm5.24} to the truncated gentle algebra \( A/I_c^m \). Then, by Lemma \ref{lemma5.24} and Proposition \ref{prop5.22}, the conditions (c) and (c') in Theorem \ref{thm5.24} are equivalent to (b) and (b') in Theorem \ref{thm5.1} respectively. \(\square\)

We can also prove Theorem \ref{thm5.8}.
Proposition 4.8. By Proposition 4.8, we may assume that \( A \) is a finite-dimensional special biserial algebra.

(b) \( \Rightarrow \) (c): It follows from Theorem 5.1 and Lemma 5.7.

(c) \( \Rightarrow \) (b): First, \( \theta \in R_0(\tilde{A}) \) follows from Theorem 5.1 and (c).

Let \( p \in \widetilde{MP}_s(A) \). If \( p \) is of length 0, then we can take \( \tilde{p} \in \widetilde{MP}(\tilde{A}) \) ending with \( p \). Then, \( M(\tilde{p}) \in \tilde{W}_\theta \) by (c), and since \( M(p) = \text{soc} M(\tilde{p}) \), we get \( M(p) \notin T_\theta \).

In the rest, we assume that \( p \in MP_s(A) \) of length \( \geq 1 \). We first show that some path \( \tilde{p} \) admitted in \( \tilde{A} \) satisfies that \( M(\tilde{p}) \in \tilde{W}_\theta \) and that \( p \) is a subpath of \( \tilde{p} \).

If there exists some \( \tilde{p} \in \widetilde{MP}(\tilde{A}) \) ending with \( p \), then \( M(\tilde{p}) \in \tilde{W}_\theta \) by (c).

Otherwise, there exists some \( \tilde{c} \in \text{Cyc}(\tilde{A}) \) such that \( (\tilde{c})^m \) ends with \( p \) for some \( m \geq 1 \). Then, take a cyclic permutation \( \tilde{d} \) of \( \tilde{c} \) such that that \( M(\tilde{d}^m) \in \tilde{W}_\theta \) by (c). We can check that \( p \) is a subpath of \( \tilde{p} := (d)^m \tilde{q} \), where \( \tilde{d} = \tilde{q} \beta \) for some \( \beta \in Q_1 \). Moreover, \( M(\tilde{p}) \in \tilde{W}_\theta \).

In any case, we have found \( \tilde{p} \) admitted in \( \tilde{A} \) such that \( M(\tilde{p}) \in \tilde{W}_\theta \) and that \( p \) is a subpath of \( \tilde{p} \). Write \( \tilde{p} = \tilde{a}p \).

By applying Lemma 5.6 to \( \tilde{p} \), we can take paths \( q_1, q_2, \ldots, q_k \) that are strings in \( A \) and arrows \( \alpha_1, \alpha_2, \ldots, \alpha_{k-1} \in Q_1 \) such that \( M(q_j) \in \tilde{W}_\cap \text{Id}_{\tilde{A}} \) and \( \tilde{p} = q_1 \alpha_1 q_2 \alpha_2 \cdots q_k \). Set \( j \in \{1, 2, \ldots, k\} \) as the minimum \( j \) such that \( q_1 \alpha_1 q_2 \alpha_2 \cdots q_j \) is strictly longer than or equal to \( \tilde{s} \). For this \( j \), we can check that there exists a nonzero quotient module of \( M(p) \) that is a submodule of \( M(q_j) \) by using \( p \in MP_s(A) \). Since \( M(q_j) \in \tilde{W}_\theta \), we get \( M(p) \notin T_\theta \).

Dually, we can prove \( (b') \Leftrightarrow \text{(c)} \). It remains to show \( (a) \Leftrightarrow ((b) \text{ and } (b')) \).

(a) \( \Rightarrow ((b) \text{ and } (b')) \): It is clear that \( R_0 \subset R_0(\tilde{A}) \). The other part follows from Proposition 5.12 and its dual.

((b) and (b')) \( \Rightarrow (a) \): Assume that \( \theta \notin R_0 \), and show that (b) or (b') is not satisfied. By definition, there exists some indecomposable rigid \( \sigma \in IR(A) \) such that \( \theta \in N_\sigma \).

If \( \sigma(h) \geq 0 \), then we can take \( p \in MP_s(A) \) such that \( M(p) \) is a quotient module of \( M_\sigma \). Since \( \theta \in N_\sigma \), we get \( M(p) \in T_\theta \), which contradicts (b). Similarly, in the case \( \sigma(h) < 0 \), we find \( p \in MP_s(A) \) such that \( M(p) \in \tilde{F}_\theta \), which means that (b') is not satisfied.

Thus, \( \theta \in R_0 \).

The last statement follows from Corollary 5.2. \( \square \)

5.4. Results on the non-rigid regions. In this subsection, we determine \( R_U \) for each \( U \in 2\text{-psilt} A \) in terms of \( R_0 \). We recall \( R_U = C^+(U) + (N_U \cap R_0) \) from Proposition 4.9, so it suffices to describe \( N_U \cap R_0 \) explicitly.

In this subsection, we use the \( \tau \)-rigid module \( H^0(U) \) and the \( \tau^{-1} \)-rigid module \( H^{-1}(\nu U) \) corresponding to \( U \), so unless otherwise stated, we assume that \( A \) is a finite-dimensional special biserial algebra. However, we can apply our results to all complete special biserial algebras by Proposition 4.8.

For a technical reason, we first define a certain quotient algebra of \( A \).

Definition 5.25. For each \( \theta \in R_0 \), we define a quotient algebra \( A_\theta := A/I_\theta \), where the ideal \( I_\theta \) of \( A \) is generated by the following paths \( s_p, t_p \) and the paths \( p, q \) such that \( p = q \neq 0 \) in \( A \):

- For each \( p \in MP_s(A) \) such that \( p \notin \tilde{T}_\theta \), take subpaths \( q, r \) of \( p \) and \( \alpha \in Q_1 \) so that \( p = qar \), \( M(r) \in T_\theta \) and that \( M(q) \in \tilde{T}_\theta \). We set \( s_p := qa \).
- For each \( p \in MP_s(A) \) such that \( p \notin \tilde{T}_\theta \), take subpaths \( q, r \) of \( p \) and \( \alpha \in Q_1 \) so that \( p = qar \), \( M(r) \in T_\theta \), and that \( M(q) \in \tilde{T}_\theta \). We set \( t_p := ar \).

Clearly, \( A_\theta \) is also a finite-dimensional string algebra. Note that \( A_\theta = A \) if \( A \) is a truncated gentle algebra by Theorem 5.24, since any \( s_p \) or \( t_p \) is not defined. The next properties follow from the definition of \( A_\theta \).

Lemma 5.26. Let \( \theta \in R_0 \). If \( p \) is a path admitted in \( A \) and \( M(p) \in \tilde{W}_\theta \), then \( p \) is admitted also in \( A_\theta \).
Lemma 5.27. Let \( \theta \in R_0 \). Then, the following statements hold.

1. For any \( p \in \mathcal{MP}_\sigma(A_\theta) \), we have \( M(p) \in \mathcal{T}_\theta \).
2. For any \( p \in \mathcal{MP}^\sigma(A_\theta) \), we have \( M(p) \in \mathcal{T}_\theta \).

We give an example.

Example 5.28. We use the complete special biserial algebra \( A \) in Example 6.10. The elements of \( \mathcal{MP}_\sigma(A/I_c) \) are

\[
\alpha_i \beta_{i+1}, \gamma_{i+1} (i \in \{1, 2, 3\}), \delta_1 \delta_2 \delta_3, \delta_2 \delta_3 \delta_4, \delta_3 \delta_4, \delta_4, c_8,
\]

and \( \mathcal{MP}^\sigma(A/I_c) \) are

\[
\alpha_i \beta_{i+1}, \gamma_{i+1} (i \in \{1, 2, 3\}), \epsilon_1, \delta_1, \delta_2 \delta_3, \delta_2 \delta_3 \delta_4.
\]

We have obtained \( R_0 = \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\geq 0} \), where \( \eta_1 := [P_1] - [P_3] - [P_5] + [P_6] \) and \( \eta_2 := [P_3] - [P_5] \).

By direct calculation,

\[
(A/I_c)_{\eta_1} = (A/I_c)/\langle \alpha_3, \beta_2, \gamma_1, \delta_3 \delta_4 \rangle, \quad (A/I_c)_{\eta_2} = (A/I_c)/\langle \gamma_3, \delta_1 \delta_2 \rangle.
\]

By using these, we get useful information on \( \mathcal{N}_\sigma \cap R_0 \).

Lemma 5.29. Let \( \sigma \in \mathcal{IR}(A) \setminus \{ \pm [P] \} \) and \( \theta \in \mathcal{N}_\sigma \cap R_0 \). Take a string \( s \) admitted in \( A \) such that \( M(s) \cong M_\sigma \), and write \( s = p_1^{-1}p_2p_3^{-1}p_4 \cdots p_{2k-1}^{-1}p_{2k} \) with \( p_i \) paths in \( Q \) admitted in \( A \) and \( p_i \) is of length \( \geq 1 \) if \( i \neq 1, 2k \).

1. For any \( i \in \{2, 3, \ldots, 2k-1\} \), the path \( p_i \) is admitted in \( A_\theta \).
2. For any \( i \in \{1, 2k\} \), the path \( p_i \) belongs to \( \mathcal{MP}_\sigma(A) \) or \( p_i \) is admitted in \( A_\theta \).

Proof. (1) Let \( i \in \{2, 3, \ldots, 2k-1\} \), and assume that \( p_i \) is not admitted in \( A_\theta \). We take the longest path \( q \) admitted in \( A_\theta \) such that \( p_i = q \alpha \) for some arrow \( \alpha \in Q_1 \) and some path \( r \). By the definition of \( A_\theta \), we get \( M(r) \in \mathcal{T}_\theta \). We can see that \( M(r) \) is a submodule of \( \tau M(s) = M'_\sigma \)
by Proposition 2.7. Then, \( M(r) \in \mathcal{T}_\theta \) implies \( M'_\sigma \notin \mathcal{T}_\theta \), but it contradicts \( \theta \in \mathcal{N}_\sigma \) by Lemma 4.2.

Therefore, \( p_i = q \), and \( p_2, p_3, \ldots, p_{2k-1} \) are admitted in \( A_\theta \).

(2) By symmetry, we only prove the case \( i = 1 \).

Assume that \( p_1 \notin \mathcal{MP}_\sigma(A) \). Then, we can take \( \alpha \in Q_1 \) and \( p_0 \in \mathcal{MP}^\sigma(A) \) such that \( p_0 \alpha^{-1} p_1^{-1} \) is admitted in \( A \). By Proposition 2.7, \( M(p_0) \) is a submodule of \( \tau M_\sigma = M'_\sigma \), so \( \theta \in \mathcal{N}_{\sigma} \) implies \( M(p_0) \in \mathcal{T}_\theta \) by Lemma 4.2. On the other hand, \( \theta \in R_0 \) and Theorem 5.8 imply \( M(p_0) \notin \mathcal{T}_\theta \). Thus, we can find the shortest path \( r_0 \) such that \( p_0 \) ends with \( r_0 \) and that \( M(r_0) \in \mathcal{W}_\theta \). Since \( M(p_0) \) is a quotient module of \( M_\sigma \), we get \( M(p_1) \in \mathcal{T}_\sigma \).

If \( p_1 \) is not admitted in \( A_\theta \), then the same argument for \( p_i \) with \( i \in \{2, 3, \ldots, 2k-1\} \) gives a decomposition \( p_1 = q_1 \alpha r_1 \) such that \( M(r_1) \in \mathcal{T}_\theta \). Then, by the minimality of \( r_0 \), we have \( M(r_0 \alpha^{-1} r_1^{-1}) \in \mathcal{T}_\theta \). However, \( M(r_0 \alpha^{-1} r_1^{-1}) \) is a submodule of \( \tau M(s) = M'_\sigma \) by Proposition 2.7.

Thus, \( M(r_0 \alpha^{-1} r_1^{-1}) \in \mathcal{T}_\theta \) implies \( \theta \notin \mathcal{N}_\sigma \), which contradicts our assumption.

Therefore, \( p_1 \in \mathcal{MP}_\sigma(A) \) or \( p_i \) is admitted in \( A_\theta \).

We can obtain \( \theta \)-semistable modules from 2-term presilting objects \( U \in 2-psilt A \) such that \( \theta \in \mathcal{N}_U \cap R_0 \) as follows.

Proposition 5.30. Let \( U \in 2-psilt A \) and \( \theta \in \mathcal{N}_U \cap R_0 \). Set \( U_\theta := U \otimes_A A_\theta \).

1. We have \( U_\theta \in 2-psilt A_\theta \) and \( C^+(U_\theta) = C^+(U) \).
2. The modules \( H^0(U_\theta) \) and \( H^1(\nu_\theta U_\theta) \) belong to \( \mathcal{W}_\theta \), where \( \nu_\theta \) is the Nakayama functor \( K^b(\text{proj } A_\theta) \to K^b(\text{inj } A_\theta) \).
3. The semibricks \( S_{U_\theta} \) and \( S_{U_\theta}^{m} \) are contained in \( \mathcal{W}_\theta \).
4. Let \( U = \bigoplus_{i=1}^m U_i \) with \( U_i \) indecomposable and \( \sigma_i := [U_i] \). Then, \( L_{\sigma_i} \in \mathcal{W}_\theta \cap \text{fd } A_\theta \) holds for all \( i \).
Proof. We may assume that $U$ is indecomposable in the proof of (1) and (2). Set $\sigma := [U]$, then $M_{\sigma} = H^0(U)$ and $M'_{\sigma} = H^{-1}(\nu U)$.

(1) The first statement is clear.

For the second statement, it suffices to show that $U_\emptyset$ is still indecomposable. In the case that $\sigma = \pm [P_i]$ for some $i \in Q_0$, this is clear. Otherwise, $M_{\sigma}$ is a string module $M(s)$ in $\text{fd } A$. By Lemma 5.29, $M(s) \otimes_A A_\emptyset$ is indecomposable, and it is $H^0(U_\emptyset)$. Since $U_\emptyset$ is not of the form $P_i[1]$, we get that $U_\emptyset$ is indecomposable as desired.

(2) We set $(M_{\sigma})_\emptyset := M_{\sigma} \otimes_A A_\emptyset = H^0(U_\emptyset)$ and $(M'_{\sigma})_\emptyset := \text{Hom}_A(A_\emptyset,M'_{\sigma}) = H^{-1}(\nu U_\emptyset)$.

By Lemma 5.29 we have an injection $\psi : L_\sigma \to (M_{\sigma})_\emptyset$ and a surjection $\psi' : (M'_{\sigma})_\emptyset \to L_\sigma$. Since $\theta \in \overline{U}$, we get $(M_{\sigma})_\emptyset \in \mathcal{T}_\emptyset$ and $(M'_{\sigma})_\emptyset \in \mathcal{T}_\emptyset$, so $L_\sigma \in \mathcal{W}_\emptyset$. If $\text{Coker } \psi, \text{Ker } \psi' \in \mathcal{W}_\emptyset$, then we obtain the assertion. We only prove $\text{Coker } \psi \in \mathcal{W}_\emptyset$, since the other one can be shown similarly.

First, we assume that $\sigma = [P_i]$ for some $i \in Q_0$, then $L_\sigma = 0$ and $\text{Coker } \psi_1 \cong (P_i)_\emptyset$. There exists a short exact sequence

$$0 \to (P_i)_\emptyset \to M(p_1) \oplus M(p_2) \to S_i \to 0,$$

with $p_1,p_2 \in \overline{MP}_i(A_\emptyset)$. Since $M(p_1),M(p_2) \in \mathcal{T}_\emptyset$ by Lemma 5.27, we have $(P_i)_\emptyset \in \mathcal{T}_\emptyset$. On the other hand, $(P_i)_\emptyset = (M_{\sigma})_\emptyset \in \mathcal{T}_\emptyset$. Thus, $\text{Coker } \psi \cong (P_i)_\emptyset \in \mathcal{W}_\emptyset$.

Otherwise, we can see that $\text{Coker } \psi$ is of the form $\bigoplus_{j=1}^{1+\sigma(b)} M(p_j)$ (including the case $\sigma = -[P_i]$), where each $p_j$ is in $\overline{MP}_i(A_i)$. Then, $\text{Coker } \psi \in \mathcal{T}_\emptyset$ by Lemma 5.27. Since $\text{Coker } \psi$ is a quotient module of $(M_{\sigma})_\emptyset \in \mathcal{T}_\emptyset$, we get $\text{Coker } \psi \in \mathcal{T}_\emptyset$. Thus, $\text{Coker } \psi \in \mathcal{W}_\emptyset$.

By the argument above, we have $(M_{\sigma})_\emptyset, (M'_{\sigma})_\emptyset \in \mathcal{W}_\emptyset$ as desired. (3) follows from the definition of the two semibricks and (2). (4) has been shown in the proof of (2).

We can finally describe $\overline{U} \cap R_0$ explicitly by using stability conditions.

Theorem 5.31. Let $U = \bigoplus_{i=1}^m U_i \in 2\text{-psilt } A$ with $U_i$ indecomposable and $\sigma_i := [U_i]$, and consider the injection $\psi_i : L_{\sigma_i} \to M_{\sigma_i}$ and the surjection $\psi'_i : M'_{\sigma_i} \to L_{\sigma_i}$ for each $i$. Then, for any $\theta \in R_0$, the following conditions are equivalent.

(a) The element $\theta$ belongs to $\overline{U} \cap R_0$.

(b) For any $i$, the conditions $L_{\sigma_i} \in \mathcal{W}_\emptyset$, $\text{Coker } \psi_i \in \mathcal{T}_\emptyset$ and $\text{Ker } \psi'_i \in \mathcal{T}_\emptyset$ hold.

(c) Both $H^0(U) \in \mathcal{T}_\emptyset$ and $H^{-1}(\nu U) \in \mathcal{T}_\emptyset$ hold.

(d) Both $S_U \in \mathcal{T}_\emptyset$ and $S'_U \in \mathcal{T}_\emptyset$.

Moreover, if $A$ is a truncated gentle algebra, then the conditions above are also equivalent to the following one.

(b') For any $i$, the conditions $L_{\sigma_i}, \text{Coker } \psi_i, \text{Ker } \psi'_i \in \mathcal{W}_\emptyset$ hold.

(c') The modules $H^0(U)$ and $H^{-1}(\nu U)$ belong to $\mathcal{W}_\emptyset$.

(d') The semibricks $S_U$ and $S'_U$ are contained in $\mathcal{W}_\emptyset$.

Proof. (a) $\Rightarrow$ (b): In Proposition 5.30, $L_{\sigma_1} \in \mathcal{W}_\emptyset$ is shown. The remaining conditions are clear, since $\text{Coker } \psi_i$ is a quotient module of $M_{\sigma_i} \in \mathcal{T}_\emptyset$ and $\text{Ker } \psi'_i$ is a submodule of $M'_{\sigma_i} \in \mathcal{T}_\emptyset$.

(b) $\Rightarrow$ (a) and (a) $\Leftrightarrow$ (c) and (a) $\Leftrightarrow$ (d) follow from Lemma 4.12.

If $A$ is a truncated gentle algebra, then (a) implies (b'), (c') and (d') by Proposition 5.30 since $A_\emptyset = A$ for any $\theta \in R_0$.

(b') $\Rightarrow$ (b), (c') $\Rightarrow$ (c) and (d') $\Rightarrow$ (d) are obvious.

We give an example that (b') is not equivalent to the other conditions.

Example 5.32. We continue Example 5.28. Recall that $\eta_1 = [P_1] - [P_4] - [P_3] + [P_6]$ and $\eta_2 = [P_3] - [P_8]$.
Let \( \sigma := [P] - [P_1] \in K_0(\text{proj } A) \), which also belongs to \( \text{IR}(A) \). Take \( U \in 2\text{-psilt } A \) satisfying \( [U] = \sigma \), and set \( \mathcal{U} := U \otimes_A (A/I_c) \). Then,

\[
M_\sigma = H^0(\mathcal{U}) = 2 < 3 \rightarrow 5 \rightarrow 7, \quad M'_\sigma = H^{-1}(\nu \mathcal{U}) = 6 \rightarrow 3 \rightarrow 4 \rightarrow 2.
\]

In this case, we have \( N_U \cap R_0 = \mathbb{R}_{\geq 0} \eta_1 \cup \mathbb{R}_{\geq 0} \eta_2 \).

Moreover,

\[
L_\sigma = S_2, \quad \text{Coker } \psi = 1 \rightarrow 3 \rightarrow 5 \rightarrow 7, \quad \text{Ker } \psi' = 6 \rightarrow 3 \rightarrow 4.
\]

We can see that all elements in \( \mathbb{R}_{\geq 0} \eta_1 \cup \mathbb{R}_{\geq 0} \eta_2 \) satisfy \( L_\sigma \in \mathcal{W}_0 \), \( \text{Coker } \psi \in \mathcal{W}_0 \) and \( \text{Ker } \psi' \in \mathcal{F}_0 \). However, we also have \( \text{Coker } \psi \notin \mathcal{W}_n \). Thus, \( (b') \) is not equivalent to \( (a) \) and \( (b) \) in Theorem 5.31.

Proposition 5.30 is still valid; we can check that both because

\[
(M_\sigma)_{\eta_2} = 2 < 1 \rightarrow 3 \rightarrow 7, \quad (M'_\sigma)_{\eta_2} = 6 \rightarrow 3 \rightarrow 4 \rightarrow 2
\]

both belong to \( \mathcal{W}_n \).

In Corollary 6.6, we will show that any \( T \in 2\text{-silt } A \) can be obtained by iterated mutations from \( A \) or \( A[1] \) in \( 2\text{-silt } A \) if \( A \) is a complete special biserial algebra. Therefore, we can obtain each \( U \in 2\text{-psilt } A \) by mutations. By applying Theorems 5.8 and 5.31 for each \( U \in 2\text{-psilt } A \), we can “eventually” determine the non-rigid region \( \text{NR} \subset K_0(\text{proj } A)_R \) by Proposition 4.3. The following property for truncated gentle algebras is important to prove Corollary 6.6.

**Corollary 5.33.** Let \( A \) be a truncated gentle algebra, and \( U \in 2\text{-psilt } A \). We define \( B := \text{End}_A(H^0(T))/[H^0(U)] \), where \( T \in 2\text{-silt } A \) is the Bongartz completion of \( U \). Then, the inverse \( \varphi^{-1}: R_0(B) \rightarrow N_U \cap R_0 \) of the bijection \( \varphi: N_U \cap R_0 \rightarrow R_0(B) \) in Lemma 4.10 is a restriction of an \( \mathbb{R} \)-linear map \( K_0(\text{proj } B)_R \rightarrow K_0(\text{proj } A)_R \).

**Proof.** Decompose \( U = \bigoplus_{i=1}^m U_i \) and \( T = U \oplus \left( \bigoplus_{j=1}^{n-m} T_j \right) \) so that \( U_i \) and \( T_j \) are indecomposable. We set \( \sigma_i := [U_i] \) for \( i \in \{1, 2, \ldots, m\} \) and \( \sigma'_j := [T_j] \) for \( j \in \{1, 2, \ldots, n-m\} \).

For each \( i \in \{1, 2, \ldots, m\} \), we define the modules \( X_i \) and \( X'_i \) as in Lemma 3.9 for \( U \in 2\text{-psilt } A \); namely,

\[
X_i := H^0(\mathcal{U})/ \sum_{f \in \text{rad}_A(H^0(\mathcal{U}), H^0(\mathcal{U}))} \text{Im } f, \quad X'_i := \bigcap_{f \in \text{rad}_A(H^{-1}(\nu \mathcal{U}), H^{-1}(\nu \mathcal{U}))} \text{Ker } f.
\]

Then, at least \( X_i \) or \( X'_i \) is nonzero for each \( i \in \{1, 2, \ldots, m\} \). Thus, we set

\[
x_i := \begin{cases} [X_i] & (X_i \neq 0) \\ [-X'_i] & (X_i = 0) \end{cases} \in K_0(\text{fd } A) \setminus \{0\}.
\]

Then, for any \( i, k \in \{1, 2, \ldots, m\} \), Lemma 3.9 tells us \( \sigma_k(x_i) = \delta_k,i \). We also have \( \theta(x_i) = 0 \) for all \( \theta \in N_U \cap R_0 \) by Theorem 5.31.

Recall that \( \varphi \) is a restriction of the linear projection \( K_0(\text{proj } A)_R \rightarrow K_0(\text{proj } B)_R \) such that \( \sigma_i \mapsto 0 \) for each \( i \in \{1, 2, \ldots, m\} \) and \( \sigma'_j \mapsto [P^{B}]_j \) for each \( j \in \{1, 2, \ldots, n-m\} \), where \( P^{B}_j \) is the
indecomposable projective $B$-module corresponding to $T_j$. Thus, if $\theta^B = \sum_{j=1}^{n-m} b_j P_j^B \in R_0(B)$, then
\[
\theta := \varphi^{-1}(\theta^B) = \sum_{j=1}^{n-m} b_j \sigma_j' - \sum_{i=1}^{m} c_{\theta,i} \sigma_i \in \overline{N_U \cap R_0}
\]
for some $c_{\theta,i} \in \mathbb{R}$. By $\theta(x_i) = 0$ and $\sigma_k(x_i) = \delta_{k,i}$ for $i, k \in \{1, 2, \ldots, m\}$, we have $c_{\theta,i} = \sum_{j=1}^{n-m} b_j \sigma_j'(x_i)$. Therefore,
\[
\varphi^{-1} \left( \sum_{j=1}^{n-m} b_j P_j^B \right) = \sum_{j=1}^{n-m} \sum_{i=1}^{m} b_j (\sigma_j' - \sigma_j'(x_i) \cdot \sigma_i).
\]
in particular, it is a restriction of an $\mathbb{R}$-linear map $K_0(\text{proj } B)_\mathbb{R} \to K_0(\text{proj } A)_\mathbb{R}$.

6. Applications

In this section, we show some applications of our results in the previous section. For any $U \in 2$-psilt $A$ and any finite-dimensional algebra $B$, we set
\[
h_U := \sum_{X \in \text{sim } \mathcal{W}_U} |X|, \quad h^B := \sum_{S^B \in \text{sim } (\text{fd } B)} [S^B].
\]

6.1. $g$-tameness. For any complete special biserial algebra, we have shown that $R_0 \subset \text{Ker}(?, h)$ in Theorem 5.8. Moreover, the class of finite-dimensional special biserial algebras are closed under $\tau$-tilting reduction by Theorem 4.12. From these results, we can show that the non-rigid region is contained in a union of countably many hyperplanes of codimension one.

Corollary 6.1. Let $A$ be a complete special biserial algebra. Then, we have
\[
\text{NR} = K_0(\text{proj } A)_\mathbb{R} \setminus \text{Cone} \subset \bigcup_{U \in 2\text{-psilt } A, |U| \leq n - 2} \text{Ker}(?, h_U).
\]
In particular, Cone is dense in $K_0(\text{proj } A)_\mathbb{R}$.

Proof. We may assume that $A$ is finite-dimensional by Proposition 3.2. If $U \in 2$-psilt $A$ with $|U| \geq n - 1$, then $R_U = C^+(U) \subset \text{Cone}$ by Lemma 4.7. Thus, it suffices to show that $R_U \subset \text{Ker}(?, h_U)$ for all $U \in 2$-psilt $A$ with $|U| \leq n - 2$ by Proposition 4.9.

Fix $U \in 2$-psilt $A$ with $|U| \leq n - 2$, take its Bongartz completion $T \in 2$-silt $A$, and set $B$ as the algebra $\text{End}_A(H^0(T))/[H^0(U)]$. The linear map $\pi: K_0(\text{proj } A)_\mathbb{R} \to K_0(\text{proj } B)_\mathbb{R}$ in Proposition 3.3 satisfies $R_U = \pi^{-1}(R_0(B)) \cap N_U$. By Theorem 4.12, $B$ is also a special biserial algebra, so Theorem 5.8 gives $R_0(B) \subset \text{Ker}(?, h^B)$. Proposition 3.3 implies that $\Phi(\text{sim } \mathcal{W}_U) = \text{sim } (\text{fd } B)$, so we get
\[
R_U \subset \pi^{-1}(R_0(B)) \subset \pi^{-1}(\text{Ker}(?, h^B)) \subset \text{Ker}(?, h_U),
\]
where we use Proposition 3.3 (1) for the last inclusion.

Note that $h^B$ and $h_U$ are nonzero, since $|B| = n - |U| \geq 2$. Since 2-psilt $A$ is a countable set, $\text{NR} = K_0(\text{proj } A)_\mathbb{R} \setminus \text{Cone}$ is contained in a union of countably many hyperplanes of codimension one. Therefore, Cone is dense in $K_0(\text{proj } A)_\mathbb{R}$.

We remark that a finite-dimensional algebra $A$ is said to be $g$-tame if Cone is a dense subset of $K_0(\text{proj } A)_\mathbb{R}$. There are other proofs of the $g$-tameness of special biserial algebras. For example, Aoki-Yurikusa [AY] obtained the $g$-tameness by using Dehn twists in the marked surfaces associated to complete gentle algebras. Also, this was one of the motivations of the paper [PY], which showed that any representation-tame (or finite) algebra is $g$-tame in the same context as Proposition 3.26.

We also have the following axiomatical description of the subset Cone.

Proposition 6.2. Assume that $A$ is a complete special biserial algebra. Let $\theta \in K_0(\text{proj } A)_\mathbb{R}$. Then, the following conditions are equivalent.
(a) The element $\theta$ belongs to $\text{Cone}$.  
(b) The family $\{[\gamma_i]\}_{\gamma_i \in \text{sim} \mathcal{W}_\theta}$ of elements in $K_0(\text{fd} A)$ can be extended to a $\mathbb{Z}$-basis of $K_0(\text{fd} A)$.

**Proof.** We may assume that $A$ is finite-dimensional by Proposition 6.2.

(a) $\Rightarrow$ (b): This follows from the argument after Proposition 4.3, more precisely, in the notation there, $T_1, T_2, \ldots, T_n \in K_0(\text{proj} A)$ for the Bongartz completion $T \in 2\text{-silt} A$ give a $\mathbb{Z}$-basis of $K_0(\text{proj} A)$, and $\text{sim} \mathcal{W}_\theta = \{X_1, X_2, \ldots, X_{n-m}\}$ satisfy $(T_i, X_j) = \delta_{i,j}$ for any $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, n-m\}$, so (b) holds.

(b) $\Rightarrow$ (a): Let $\theta$ satisfy the condition (b). Take the unique $U \in 2\text{-psilt} A$ such that $\theta \in R_U$. We use the notation of Propositions 4.4 and 4.5. By Theorem 3.2 $B$ is isomorphic to a finite-dimensional special biserial algebra.

We apply Corollary 5.9 to $B$, and take $Z$ and $k_p$ for all $p \in Z$ there. Then,

$$\sum_{p \in Z} k_p[M(p)^B] = 2h^B \in K_0(\text{fd} B).$$

Since $\Phi^{-1}(\mathcal{W}_{\pi(\theta)}) = \mathcal{W}_\theta$ by Proposition 4.4, $Y_p := \Phi^{-1}(M(p)^B)$ is a simple object of $\mathcal{W}_\theta$ for each $p \in Z$, and we get

$$\sum_{p \in Z} k_p[Y_p] = 2h_U \in K_0(\text{fd} A).$$

Now, by the assumption (b), we get that $\{[Y_p]\}_{p \in Z}$ can be extended to some $\mathbb{Z}$-basis of $K_0(\text{fd} A)$. Thus, the equation above implies that $k_p \neq 1$ for every $p \in Z$. By the definition of $k_p$, every $p \in Z$ must be of length 0. Thus, the equation (6.1) implies $\{M(p)^B \mid p \in Z\} = \text{sim}(\text{fd} B)$. Since $\text{sim}(\text{fd} B) = \{M(p)^B \mid p \in Z\} \subset \mathcal{W}_{\pi(\theta)}$, we get $\pi(\theta) = 0$, which yields $\theta \in C^+(U)$ as desired.

We remark that (a) $\Rightarrow$ (b) holds for all finite-dimensional algebras. On the other hand, (b) $\Rightarrow$ (a) does not hold in general; for example, if $A$ is the $3$-Kronecker algebra $K(\begin{array}{ccc} 1 & 1 & 1 \\ & 2 & 0 \end{array})$ and $\theta = [P_1] - r[P_2]$ with $r \in \mathbb{R} \setminus \mathbb{Q}$ and $(3 - \sqrt{5})/2 < r < (3 + \sqrt{5})/2$, then [Asa2, Section 5] implies that $\mathcal{W}_\theta = \{0\}$ (so, (b) is satisfied) and that $\theta \notin \text{Cone}$.

6.2. Connected components of the exchange quiver. In this subsection, we consider mutations and the exchange quiver of 2-term silting complexes in $K^b(\text{proj} A)$ for complete special biserial algebras. We first recall necessary properties.

Let $T \neq T' \in 2\text{-silt} A$ be two non-isomorphic basic 2-term silting complexes. Then, we say that $T'$ is a mutation of $T$ if there exists $U \in 2\text{-psilt} A$ such that $|U| = n - 1$ and that $U$ is a common direct summand of $T$ and $T'$. In this case, we can uniquely take an indecomposable direct summand $V$ of $T$ which is not a direct summand of $T'$. Then, $T'$ is called a mutation of $T$ at $V$. For any indecomposable direct summand of $T$, there uniquely exists a mutation $T'$ of $T$ at $V$ by Proposition 3.5.

Recall that we associated the torsion class $\overline{T}_T = T_T$ for $T \in 2\text{-silt} A$ in Definition 3.6. If $T'$ is a mutation of $T$, then $\overline{T}_T \subset \overline{T}_{T'}$ or $\overline{T}_{T'} \supset \overline{T}_T$ holds, since we can extend the property [AIR, Definition-Proposition 2.28, Theorem 3.2] for finite-dimensional algebras to complete special biserial algebras by Proposition 3.2. If the first condition holds, then we say that $T'$ is a left mutation of $T$; otherwise a right mutation of $T$.

Under this preparation, we can define the exchange quiver of $2\text{-silt} A$.

**Definition 6.3.** We define the exchange quiver of $2\text{-silt} A$ as the quiver such that its vertices set is $2\text{-silt} A$ and that there exists an arrow $T \to T'$ if and only if $T'$ is a left mutation of $T$.

By Proposition 3.10, it is easy to see that the number of connected components of the exchange quiver coincides that of

$$\text{Cone}_{\geq n-1} := \bigcap_{U \in 2\text{-psilt} A, \ |U| \geq n-1} C^+(U) = \bigcup_{U \in 2\text{-psilt} A, \ |U|=n-1} N_U.$$
The main result of this section is the following one. We set $H^+ := \{ \theta \in K_0(\proj A)_R \mid \theta(h) > 0 \}$ and $H^- := \{ \theta \in K_0(\proj A)_R \mid \theta(h) < 0 \}$.

**Theorem 6.4.** Let $A$ be a connected complete gentle algebra. Then, the following assertions hold.

1. Let $T \in 2$-silt $A$. If $C^+(T) \cap H^+ \neq \emptyset$, then $T$ belongs to the same connected component as $A$ in the exchange quiver of $2$-silt $A$, and if $C^+(T) \cap H^- \neq \emptyset$, then $T$ belongs to the same connected component as $A[1]$. 

2. The number of connected components of the exchange quiver of $2$-silt $A$ is 

$$
\begin{cases} 
1 & (R_0 \neq \ker(?, h^B)) \\
2 & (R_0 = \ker(?, h^B)).
\end{cases}
$$

To prove this theorem, we will use $\tau$-tilting reduction, so we need the following property. Here, if a complete gentle algebra $B$ is isomorphic to a direct product $\prod_{i=1}^{n} B_i$ with $B_i$ a connected algebra, we call each $B_i$ a block of $B$.

**Proposition 6.5.** Let $A$ be a connected complete gentle algebra, and $U \in 2$-psilt $A \setminus \{0\}$. Consider the complete gentle algebra $B$ given in Corollary 4.17. Then, for any block $B'$ of $B$, we have $R_0(B') \neq \ker(?, h^B')$.

**Proof.** In the notation of Corollary 4.17 this corollary implies that we can take $k \in \mathbb{Z}_{\geq 1}$ such that there exist epimorphisms $B \to B_k \to B/I_k B$ of algebras. By Propositions 5.3 and 5.8 we may reset $A := A_k$, $U := U \otimes A_k$ and $B := B_k$ (we do not use the original $A, U, B$ in the rest of the proof).

Take the Bongartz completion $T$ of $U$.

1. We first consider the case that $B$ is connected. We assume that $R_0(B) = \ker(?, h^B)$ and deduce a contradiction.

The assumption $R_0(B) = \ker(?, h^B)$ implies that $R_0(B)$ is a $(|B| - 1)$-dimensional $\mathbb{R}$-vector subspace of $K_0(\proj B)_R$. Since the bijection $\varphi^{-1}: R_0(B) \to N_U \cap R_0$ is a restriction of an $\mathbb{R}$-linear map by Corollary 5.33 we get $N_U \cap R_0$ is a $(|B| - 1)$-dimensional $\mathbb{R}$-vector subspace of $K_0(\proj B)_R$.

Thus, for any $\theta \in N_U \cap R_0$, we have $\theta, -\theta \in N_U$, so $H^0(U) \in \mathcal{T}_\theta \cap \mathcal{T}_{-\theta}$ and $H^{-1}(\nu U) \in \mathcal{F}_\theta \cap \mathcal{F}_{-\theta}$ by the definition of $N_U$. Therefore, $H^0(U), H^{-1}(\nu U) \in \mathcal{W}_\theta \cap \mathcal{W}_{-\theta}$. Set

$$
J := \left\{ i \in Q_0 \mid \text{there exists } \sum_{j \in Q_0} a_i |P_i| \in N_U \cap R_0 \text{ such that } a_i \neq 0 \right\}.
$$

By [Asa2] Lemma 2.5, for any $i \in J$, then $S_i$ is not a composition factor of $H^0(U) \text{ or } H^{-1}(\nu U)$. Thus, the simple modules $S_i$ for all $i \in J$ are in $W_U$, and $\{ S_i \mid i \in J \} \subset \sim W_U$. Since $\overline{N_U} \cap R_0$ is a $(|B| - 1)$-dimensional $\mathbb{R}$-vector subspace of $\ker(?, h)$ by Theorem 5.8, we get $J > |B| - 1 = \# \text{sim}(\text{fd}B) - 1 = \# \text{sim} W_U - 1$. Therefore, $\{ S_i \mid i \in J \} \subset \sim W_U$ and $|B| = \# J$, so $\text{fd} B \cong W_U \cong \text{fd} A'$, where $A' := A/(1 - \sum_{i \in J} e_i)$. Thus, the basic algebra $B$ is isomorphic to $A'$, and $R_0(A') = \ker(?, \sum_{i \in J} S_i)$ follows from $R_0(B) = \ker(?, h^B)$.

Let $Q'$ be the full subquiver of $Q$ whose vertices set is $J$. Then, by Theorem 5.1, $R_0(A') = \ker(?, \sum_{i \in J} S_i)$ implies that every $j \in Q_0 \setminus J$ has two arrows starting at $j$ and two arrows ending at $j$ in $Q'$. Since $\# J = |B| = |A| - |U| < |A|$ and $A = KQ/I$ is a connected special biserial algebra, we have $J = \emptyset$. This clearly yields that $B = 0$, but it contradicts that $B$ is connected.

Therefore, $R_0(B) \neq \ker(?, h^B)$.

2. We proceed to the general case. Decompose $B = B' \times B''$ as algebras.

Since $\text{red}(T) = B$ in Proposition 4.3 there uniquely exists $V \in 2$-psilt $U A$ such that $\text{red}(V) = B''$. Then, the Bongartz completion of $V$ is also $T$, and we have $\text{End}_A(H^0(T))/[H^0(V)] \cong B'$. By (i), $R_0(B') \neq \ker(?, h)$. \qed
Now, we are able to prove Theorem 6.4.

Proof of Theorem 6.4. We prove both (1) and (2) at once by induction on \( n = |A| \). If \( n = 0 \), then the assertions are clear. Assume that \( n \geq 1 \) in the rest.

(1) We only consider the first case \( C^+(T) \cap H^+ \neq \emptyset \), since similar arguments work in the other case. Let \( X \) be the connected component of \( \text{Cone}_{\geq n-1} \) containing \( C^+(T) \). It suffices to show that \( C^+(A) \subset X \).

Consider the continuous map \( f : K_0(\text{proj} A)_{\mathbb{R}} \setminus \{0\} \to [-1, 1] \) defined by

\[
f(\theta) := \theta(h)/\|\theta\|_1.
\]

Set \( s := \sup f(X) \) and \( l := \max \{m \in \mathbb{Z}_{\geq 1} \mid s \leq 1/m \} \). Then, we can take \( \theta \in X \) such that \( f(x) > 1/(l + 1) \). By definition, \( \theta \) belongs to \( C^+(U) \) for some \( U \in 2\text{-psilt} A \) with \( |U| \geq n - 1 \). Decompose \( U = \bigoplus_{i=1}^m U_i \) into the indecomposable direct summands. By Lemma 3.26 and Proposition 3.30, \([U_1], [U_2], \ldots, [U_m]\) are sign-coherent. Thus, \( f(\theta) > 1/(l + 1) \) implies that there must exist \( i \in \{1, 2, \ldots, m\} \) with \( f(U_i) > 1/(l + 1) \). Set \( V := U_i \). Since \([V](h) \in \{-1, 0, 1\} \), we get \( f([V]) \in \pm(1/\mathbb{Z}_{\geq 1}) \cup \{0\} \), so \( f([V]) > 1/(l + 1) \) yields \( f([V]) \geq 1/l \geq s \).

We show that \( \text{Cone}_{\geq n-1} \cap N_V \) is connected. This is the union of \( C^+(W) \) for all \( W \in 2\text{-psilt} A \) such that \( |W| \geq n - 1 \). Apply Corollary 4.17 to \( V \), and take the complete gentle algebra \( B \) there. By Proposition 5.3, \( R_2(B') \neq \text{Ker}(\langle ?, h B \rangle) \subset K_0(\text{proj} B')_{\mathbb{R}} \) for all blocks \( B' \), so the exchange quiver of \( 2\text{-silt} B \) is connected by the induction hypothesis for (2). Thus, the exchange quiver of \( 2\text{-silt} A \) is also connected, and \( \text{Cone}_{\geq n-1} \cap N_V \) is connected.

Since \( \text{Cone}_{\geq n-1} \cap N_V \) is connected and \( C^+(U) \subset X \cap (\text{Cone}_{\geq n-1} \cap N_V) \), we have \( \text{Cone}_{\geq n-1} \cap N_V \) is contained in the connected component \( X \). We can take \( \varepsilon > 0 \) such that \([V] + \varepsilon[A] \in N_V \), and by the definition of Bongartz completions, \([V] + \varepsilon[A] \in C^+(T_V) \) for the Bongartz completion \( T_V \) of \( V \). Thus, \([V] + \varepsilon[A] \subset X \). Since \([V] \) and \([V] + \varepsilon[A] \in N_V \) are sign-coherent, we get \( f([V]) \leq f([V] + \varepsilon[A]) \leq \sup f(X) = s \leq f([V]) \). Therefore, we have \( 0 < f([V]) = f([V] + \varepsilon[A]) \), which implies that \( V = P_i \) for some \( i \in Q_0 \). The Bongartz completion \( T_V \) is obviously \( A \), so \( C^+(A) \subset \text{Cone}_{\geq n-1} \cap N_V \subset X \).

Thus, if \( T \in 2\text{-silt} A \) satisfies \( C^+(T) \cap H^+ \neq \emptyset \) and \( X \) is a connected component of \( \text{Cone}_{\geq n-1} \), then \( C^+(T) \) and \( C^+(A) \) are contained in \( X \), so \( T \) and \( A \) belong to the same connected component of the exchange quiver.

(2) By (1), both \( \text{Cone}_{\geq n-1} \cap H^+ \) and \( \text{Cone}_{\geq n-1} \cap H^- \) are connected.

If \( R_0 = \text{Ker}(\langle ?, h \rangle) \), then \( \text{Cone}_{\geq n-1} \subset H^+ \cap H^- \), so we get the assertion.

Otherwise, \( R_0 \neq \text{Ker}(\langle ?, h \rangle) \). Then there exists \( U \in 2\text{-psilt} A \setminus \{0\} \) such that \( C^+(U) \cap H \neq \emptyset \). Then, \( C^+(T) \subset \text{Cone}_{\geq n-1} \cap N_U \cap H^+ \) and \( C^+(T') \subset \text{Cone}_{\geq n-1} \cap N_U \cap H^- \) hold for the Bongartz completion \( T \) and the Bongartz co-completion \( T' \) of \( U \). Thus, it suffices to show that \( T \) and \( T' \) belong to the same connected component of the exchange quiver of \( 2\text{-silt} A \). By Corollary 4.17, there exists a complete gentle algebra \( B \) such that \( |B| = |A| = n \) and the exchange quivers of \( 2\text{-silt} A \) and \( 2\text{-silt} B \) are isomorphic. Then, Proposition 6.3 and the induction hypothesis for (2) imply that the exchange quiver of \( 2\text{-silt} A \) is connected. Therefore, \( 2\text{-silt} A \) is connected.

Now, the induction process is complete. \( \square \)

We have the following result for complete special biserial algebras.

Corollary 6.6. Let \( A \) be a connected complete special biserial algebra. Then, the following assertions hold.

(1) If \( C^+(T) \cap H^+ \neq \emptyset \), then \( T \) belongs to the same connected component as \( A \), and if \( C^+(T) \cap H^- \neq \emptyset \), then \( T \) belongs to the same connected component as \( A[1] \) in the exchange quiver.
(2) The number of connected components of the exchange quiver of 2-tilt $A$ is

$$
\begin{align*}
1 & \quad (\text{Cone}_{\geq -1} \cap \text{Ker}(?, h) \neq \emptyset) \\
2 & \quad (\text{Cone}_{\geq -1} \cap \text{Ker}(?, h) = \emptyset). \\
\end{align*}
$$

Proof. (1) Let $T \in 2$-tilt $A$. We only consider the first case $C^+(T) \cap H^+ \neq \emptyset$.

We take a complete gentle algebra $\tilde{A}$ such that $A$ is a quotient algebra of $\tilde{A}$ and that $|A| = |\tilde{A}|$. Since $\text{Cone}_{\geq -1}(\tilde{A})$ is dense in $\text{Cone}(\tilde{A})$, Corollary 6.4 tells us that $\text{Cone}_{\geq -1}(A)$ is dense in $K_0(\text{proj} \ A) = K_0(\text{proj} \ \tilde{A})$. In particular, $C^+(T) \cap H^+ \cap \text{Cone}_{\geq -1}(A) \neq \emptyset$. Take $\theta \in C^+(T) \cap H^+ \cap \text{Cone}_{\geq -1}(A)$. By Theorem 6.3 (1), $\theta$ belongs to the connected component of $\text{Cone}_{\geq -1}(A)$ containing $C^+(A)$, so it is in the connected component of $\text{Cone}_{\geq -1}$ containing $C^+(A)$. Since $\theta \in C^+(T)$, this means that $T$ and $A$ are in the same connected component of the exchange quiver.

(2) This is obvious from (1).

We remark that the following example shows that we cannot replace the condition $\text{Cone}_{\geq -1} \cap \text{Ker}(?, h) \neq \emptyset$ to $R_0 \subset \text{Ker}(?, h)$ in Corollary 6.6.

Example 6.7. We use the setting of Example 5.11. Recall that we obtained

$$
R_0 = \mathbb{R}_{\geq 0}([P_1] - [P_2]) \cup \mathbb{R}_{\geq 0}([P_2] - [P_3]) \cup \mathbb{R}_{\geq 0}([P_3] - [P_1]).
$$

Thus, $R_0 \neq \text{Ker}(?, h)$. Moreover, $\eta_{1,2} := [P_1] - [P_2]$, $\eta_{2,3} := [P_2] - [P_3]$, $\eta_{3,1} := [P_3] - [P_1]$ belong to $\text{InR}(A)$ by Lemma 1.2.

On the other hand, we show $\text{Cone}_{\geq 2} \cap \text{Ker}(?, h) = \emptyset$. It is easy to see that $\sigma_{1,3} := [P_1] - [P_3]$, $\sigma_{2,3} := [P_3] - [P_2]$, $\sigma_{2,1} := [P_2] - [P_1]$ are in $\text{IR}(A)$. From direct calculation, we have $\eta_{1,2}, \eta_{2,3} \in \mathbb{N}_{\sigma_{1,3}}$. Thus, by Proposition 4.9,

$$(\mathbb{R}_{\geq 0} \sigma_{1,3} \oplus \mathbb{R}_{\geq 0} \eta_{1,2}) \cap \text{Cone}_{\geq 2} = \emptyset.$$

By applying this argument also to $\sigma_{3,2}$ and $\sigma_{2,1}$, we can conclude that $\text{Cone}_{\geq 2} \cap \text{Ker}(?, h) = \emptyset$.

Therefore, there exist two connected components of the exchange quiver of 2-tilt $A$ by Corollary 6.6.

6.3. $\tau$-tilting finiteness. In this section, we give other proofs to some known results on $\tau$-tilting finiteness of special biserial algebras.

First, as an application of Theorem 5.8, we have the following result, where the equivalence of (a) and (b) has already been proved in [STV] Theorem 5.1. More explicitly, our results give a proof of (b) $\Rightarrow$ (a) different from theirs.

Corollary 6.8. For any complete special biserial algebra $A$, the following conditions are equivalent.

(a) The algebra $A$ is $\tau$-tilting finite.
(b) There exists no band $A$-module which is a brick.
(c) Any indecomposable $\theta \in K_0(\text{proj} A)$ is rigid.
(d) The subset $R_0 \cap K_0(\text{proj} A)$ is $\{0\}$.
(e) The subset $R_0$ is $\{0\}$.

Proof. (a) $\Rightarrow$ (b) follows from [STV] Proposition 3.1.
(b) $\Rightarrow$ (e) follows from Proposition 3.3.4.
(c) $\Rightarrow$ (d) is immediate by Lemma 4.12.
(d) $\Rightarrow$ (e) follows from Theorem 5.8.
(e) $\Rightarrow$ (a) is Proposition 4.11.

Our main result also gives another proof for the characterization of $\tau$-tilting Brauer graph algebras by [AAC]. We briefly recall the definition of Brauer graphs and Brauer graph algebras.

Let $G = (V, E)$ be a finite unoriented graph with $V$ the vertices set and $E$ the edges set. We call $G$ a Brauer graph if the following information is additionally given:
Proof. (1) We can take a complete gentle algebra \( \tilde{A} := \overline{KQ}/I_1 \) above. For any \( v \in V \) and \( i \in \{1, 2, \ldots, l_v \} \), we set \( c(v, i) := \alpha_{v,i} \alpha_{v,i+1} \cdots \alpha_{v,i+l_v-1} \). Then, \( \text{Cyc}(\tilde{A}) = \{c(v, i) \mid v \in V, i \in \{1, 2, \ldots, l_v \}\} \), and \( \overline{M}(\tilde{A}) = \emptyset \).

Now, Theorem 5.1 implies that
\[
R_0 = R_0(\tilde{A}) = \bigcap_{v \in V} \text{Ker}(?, x_v).
\]

(2) By (1) and Corollary 6.8, \( A \) is \( \tau \)-tilting finite if and only if the \( \mathbb{R} \)-vector subspace \( X := \sum_{v \in V} \mathbb{R} x_v \) is \( K_0(\text{fd} A)_{\mathbb{R}} \).

We have \( \dim_{\mathbb{R}} X \leq \#V \) and \( \dim_{\mathbb{R}} K_0(\text{fd} A)_{\mathbb{R}} = \#E \). Thus, if \( A \) is \( \tau \)-tilting finite, then \( \#V \geq \#E \), so \( G \) has at most one cycle.

Corollary 6.9. Let \( A \) be the Brauer graph algebra of a connected Brauer graph \( G = (V, E) \). Then, \( A \) is \( \tau \)-tilting finite if and only if \( G \) contains no even cycle and at most one odd cycle.

\[
\begin{align*}
&\text{Corollary 6.9.} \\
&\text{Let } A \text{ be the Brauer graph algebra of a connected Brauer graph } G = (V, E). \\
&(1) \text{ Set } x_v := [S_{e_{v,1}}] + [S_{e_{v,2}}] + \cdots + [S_{e_{v,l_v}}] \in K_0(\text{fd} A) \text{ for each } v \in V. \text{ Then, we have } \\
&R_0 = \bigcap_{v \in V} \text{Ker}(?, x_v). \\
&\text{In particular, } R_0 \text{ is an } \mathbb{R}\text{-vector subspace of } K_0(\text{proj} A)_{\mathbb{R}}. \\
&(2) \text{ [AAC]} \text{ Theorem 6.7] Then, } A \text{ is } \tau \text{-tilting finite if and only if } G \text{ contains no even cycle and at most one odd cycle.}
\end{align*}
\]
If \( v_0 \) is a vertex in \( G \) and there exists only one edge \( e_0 \) involving \( v_0 \), then we have a new Brauer graph \( G' \) by deleting \( v_0 \) and \( e_0 \). Consider its Brauer graph algebra \( A' \), then we can check that \( X = K_0(\text{fd} A)^R \) holds if \( \sum_{v \in V \setminus \{v_0\}} Rx_v = K_0(\text{fd} A')^R \), where \( x_v' := x_v - [S_v] \) if \( e_0 \) involves \( v \), and \( x_v' := x_v \) otherwise.

If \( G \) contains no cycle, then by repeating this process, we get that \( X = K_0(\text{fd} A)^R \) holds if \( \sum_{e \in \mathcal{E}} Rx_e = 0 \), which is obviously true, so we obtain \( X = K_0(\text{fd} A)^R \). Thus, \( A \) is \( \tau \)-tilting finite.

Otherwise, \( G \) contains exactly one cycle. By induction, we may assume that \( G \) itself is a cycle to determine whether \( A \) is \( \tau \)-tilting finite or not. If \( G \) consists of \( l \) distinct vertices \( v_1, v_2, \ldots, v_l \) and edges \( e_1, e_2, \ldots, e_l \) with \( e_i \) connects \( v_i \) and \( v_{i+1} \) \((v_{l+1} := v_1)\), then we can observe that \( X \) is generated by the elements

\[
[S_{e_1}] + [S_{e_{l+1}}] \in K_0(\text{fd} A)
\]

for all \( i \in \{1, 2, \ldots, l\} \) \((e_{l+1} := e_1)\). We can check that \( X = K_0(\text{fd} A)^R \) holds if and only if \( l \) is odd. Therefore, if \( G \) contains no even cycle and a unique odd cycle, then \( A \) is \( \tau \)-tilting finite; on the other hand, if \( G \) has an even cycle and no odd cycle, then \( A \) is not \( \tau \)-tilting finite.

Now, the proof is complete. \( \Box \)

Finally, we apply our results to special biserial radical square zero algebras. Assume that a finite quiver \( Q \) satisfies (b) and (c) in Definition 2.11 that is, the number of arrows starting (resp. ending) at each vertex \( i \in Q_0 \) is at most two, and set \( I \) as the ideal generated by all the paths of length 2. Then, \( A := K Q / I \) is a finite-dimensional special biserial algebra such that \( \text{rad}^2 A = 0 \). Conversely, any special biserial radical square zero algebra is obtained in this way.

As in [Ada, Aok], we can define the separated quiver \( Q^s \) of \( Q \); namely, the vertices set is \( Q^s_0 := \{i^+, i^- \mid i \in Q_0\} \) and the arrows set is \( (Q^s)_1 := \{i^+ \rightarrow j^- \mid (i \rightarrow j) \in Q_1\} \). We say that a subquiver \( Q' \) of \( Q^s \) is a single subquiver if no \( i \in Q_0 \) satisfies \( i^+, i^- \in Q^s_0 \). Then, we can recover [Ada, Theorem 3.1] in the case that \( A \) is a special biserial radical square zero algebra.

**Proposition 6.10.** Let \( A = K Q / I \) be a special biserial algebra with \( \text{rad}^2 A = 0 \). Then, \( A \) is \( \tau \)-tilting finite if and only if any single subquiver of \( Q^s \) is a disjoint union of quivers of type \( \Lambda \).

**Proof.** We first prove the “only if” part. We assume that there exists a single subquiver of \( Q^s \) that is not a disjoint union of \( \Lambda \). It suffices to show that \( A \) is not \( \tau \)-tilting finite. Since \( Q \) satisfies (b) and (c) in Definition 2.11 \( Q^s \) has a single subquiver \( Q' \) of type \( \Lambda \), and we can consider \( Q' \) as a band \( b \). Since \( Q' \) is a single subquiver, the band module \( M(b, \lambda) \) is a brick. Thus, \( A \) is not \( \tau \)-tilting finite by Corollary 6.8.

We next prove the “if” part. Suppose that \( A \) is not \( \tau \)-tilting finite, then by Corollary 6.8 we can take some \( \eta \in \text{INR}(A) \). Take a band \( b_\eta \) in Proposition 3.31. Write \( \eta = \sum_{i=1}^n a_i [P_i] \). By using \( M(b_\eta, \lambda) \cdot \text{rad}^2 A = 0 \), each simple module \( S \) appears in the composition factors of \( M(b_\eta, \lambda) \) at most once, because the band \( b_\eta \) does not admit a shorter string \( s \) such that \( b_\eta = s^m \) with \( m \geq 2 \). The arrows and the inverse arrows appearing in the band \( b_\eta \) define a subquiver \( Q' \) of \( Q^s \). In particular, \( a_i \in \{-1, 0, 1\} \) holds for each \( i \).

The vertices set of \( Q' \) is \( \{i_+ \mid i \in Q_0, a_i = 1\} \) or \( \{i_- \mid i \in Q_0, a_i = -1\} \), so \( Q' \) is a single subquiver of \( Q^s \) of type \( \Lambda \), so it is not a disjoint union of quivers of type \( \Lambda \). \( \Box \)

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