"Vector bundles" over quantum Heisenberg manifolds.

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Abstract: We construct, out of Rieffel projections, projections in certain algebras which are strong-Morita equivalent to the quantum Heisenberg manifolds $D_{c,\bar{\hbar}^{\mu\nu}}$. Then, by means of techniques from the Morita equivalence theory, we get finitely generated and projective modules over the algebras $D_{c,\bar{\hbar}^{\mu\nu}}$. This enables us to show that the group $Z + 2\mu Z + 2\nu Z$ is contained in the range of the trace on $K_0(D_{c,\bar{\hbar}^{\mu\nu}})$.

Preliminaries. Let $G$ be the Heisenberg group,

$$
G = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\},
$$

and, for a positive integer $c$, let $H_c$ be the subgroup of $G$ obtained when $x$, $y$, and $cz$ are integers. The Heisenberg manifold $M_c$ is the quotient $G/H_c$.

Non-zero Poisson brackets on $M_c$ that are invariant under the action $G$ on $M_c$ by left translation can be parametrized by two real numbers $\mu$ and $\nu$, with $\mu^2 + \nu^2 \neq 0$ ([RF3]).

For each positive integer $c$ and real numbers $\mu$ and $\nu$ as above, Rieffel constructed ([RF3]) a deformation quantization $\{D_{c,\bar{\hbar}^{\mu\nu}}\}_{\bar{\hbar} \in \mathbb{R}}$ of $M_c$ in the direction of the Poisson bracket $\Lambda_{\mu\nu}$.

Since $D_{c,\bar{\hbar}^{\mu\nu}}$ is isomorphic to $D_{c,1,\bar{\hbar}^{\mu\nu}}$, and we will not need to keep track of the Planck constant $\bar{\hbar}$, we absorb it from now on into the parameters $\mu$ and $\nu$. Thus we will use $D_{c,\mu\nu}$ to denote $D_{c,1,\bar{\hbar}^{\mu\nu}}$.

As shown in [RF3], the algebra $D_{c,\mu\nu}$ can be described as the generalized fixed-point algebra of the crossed-product $C_0(\mathbb{R} \times T) \rtimes_{\lambda Z} Z$, where $\lambda_{p}(x, y) = (x + 2p\mu, y + 2p\nu)$, for all $p \in \mathbb{Z}$, under the action $\rho$ of $Z$ defined by

$$(\rho_{p}\Phi)(x, y, p) = e(ckp(y - p\nu))\Phi(x + k, y, p),$$

where $k, p \in \mathbb{Z}$, $\Phi \in C_c(\mathbb{R} \times T \times \mathbb{Z})$, and, for any real number $x$, $e(x) = \exp(2\pi ix)$.

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1 The material of this work is part of the author’s Ph.D dissertation, submitted to the University of California at Berkeley in May 1992.
The action ρ defined above corresponds to the action ρ defined in [Rf3, p.539], after taking Fourier transform in the third variable to get the algebra denoted in that paper by \( A_\hbar \), and viewing \( A_\hbar \) as a dense *-subalgebra of \( C_0(R \times T) \times_\lambda Z \) via the embedding \( J \) defined in [Rf3, p.547]. Equivalently, \( D^c_{\mu \nu} \) is the closure in the multiplier algebra of \( C_0(R \times T) \times_\lambda Z \) of the *-subalgebra \( D_0 \) consisting of functions \( \Phi \in C(R \times T \times Z) \) which have compact support on \( Z \) and satisfy

\[
\Phi(x + k, y, p) = e(-ckp(y - p\nu))\Phi(x, y, p),
\]

for all \( k, p \in Z \), and \( (x, y) \in R \times T \) (\( D_0 \) is the image under the embedding \( J \) mentioned above of the subalgebra denoted by \( C^\rho \) in the proof of [Rf3, Thm.5.4]).

There is a faithful trace ([Rf3]) \( \tau_D \) on \( D^c_{\mu \nu} \) defined for \( \Phi \in D_0 \), by

\[
\tau_D(\Phi) = \int_{T^2} \Phi(x, y, 0) dxdy.
\]

It can be shown ([Ab2]) that the algebra \( D^c_{\mu \nu} \) is strong-Morita equivalent to the generalized fixed-point algebra \( E^c_{\mu \nu} \) of the crossed product \( C_0(R \times T) \times_\sigma Z \) under the action \( \gamma \) of \( Z \), where \( \sigma_k(x, y) = (x - k, y) \) and

\[
(\gamma_p \Phi)(x, y, k) = e(-ckp(y - p\nu))\Phi(x - 2p\mu, y - 2p\nu, k),
\]

for \( k, p \in Z \) and \( \Phi \in C_c(R \times T \times Z) \).

As for the quantum Heisenberg manifolds case, \( E^c_{\mu \nu} \) can also be described (see [Ab2]) as the closure in the multiplier algebra of \( C_0(R \times T) \times_\sigma Z \) of the *-algebra \( E_0 \) consisting of functions \( \Phi \in C(R \times T \times Z) \), with compact support on \( Z \) and such that

\[
\Phi(x - 2p\mu, y - 2p\nu, k) = e(ckp(y - p\nu))\Phi(x, y, k),
\]

for all \( k, p \in Z \), \( (x, y) \in R \times T \). The equivalence \( D^c_{\mu \nu} \)-\( E^c_{\mu \nu} \) bimodule \( X \) constructed in [Ab2] is the completion of \( C_c(R \times T) \) with respect to either one of the norms induced by the \( D^c_{\mu \nu} \) and \( E^c_{\mu \nu} \)-valued inner products given by

\[
< f, g >_D(x, y, p) = \sum_{k \in Z} e(ckp(y - p\nu))f(x + k, y)\overline{g(x - 2p\mu + k, y - 2p\nu)}
\]

and

\[
< f, g >_E(x, y, k) = \sum_{p \in Z} e(-ckp(y - p\nu))\overline{f(x - 2p\mu, y - 2p\nu)}g(x - 2p\mu + k, y - 2p\nu),
\]
respectively, where \( f, g \in C_0(R \times T) \), \( \Phi \in D_0 \), \( \Psi \in E_0 \), \( (x, y) \in R \times T \), and \( k, p \in \mathbb{Z} \).

In what follows we produce finitely generated and projective modules over the algebras \( D_{\mu \nu} \). To do this we apply to the Morita equivalence structure described above the methods for constructing projections provided by the Morita equivalence theory. Finally, we get a partial description of the range of the trace at the level of \( K_0(D_{\mu \nu}) \).

**Remark 1** First notice that both \( D_0 \) and \( E_0 \) have identity elements \( I_D \) and \( I_E \), respectively, defined by

\[
I_D(x, y, p) = \delta_0(p) \quad \text{and} \quad I_E(x, y, k) = \delta_0(k),
\]

for \( (x, y) \in R \times T \) and \( k, p \in \mathbb{Z} \).

Therefore, by [Rf2, Prop. 1.2], if \( P \) is a projection in \( E_0 \), then \( XP \) is a projective finitely generated left module over \( D_{\mu \nu} \), and the corresponding projection in \( M_m(D_{\mu \nu}) \) is given by

\[
Q = \begin{pmatrix}
<y_1, x_1>_D & \ldots & <y_m, x_1>_D \\
\vdots & \ddots & \vdots \\
<y_1, x_m>_D & \ldots & <y_m, x_m>_D
\end{pmatrix}
\]

where, for \( i = 1, \ldots, m \), \( x_i, y_i \in XP \) are such that \( P = \sum_{i=1}^m <x_i, y_i>_E \).

On the other hand ([Rf4, Prop. 2.2]), the trace \( \tau_D \) on \( D_{\mu \nu} \) induces a trace \( \tau_E \) on \( E_{\mu \nu} \) via

\[
\tau_E(<f, g>_E) = \tau_D(<g, f>_D).
\]

A straightforward computation shows that for \( \Psi \in E_0 \) we have

\[
\tau_E(\Psi) = \int_0^{2\mu} \int_0^1 \Psi(x, y, 0) dx dy.
\]

Then, in the notation above we get

\[
\tau_D(Q) = \sum_{i=1}^m \tau_D(<y_i, x_i>_D) = \sum_{i=1}^m \tau_E(<x_i, y_i>_E) = \tau_E(P).
\]

**Theorem 1** The bimodule \( X \) is a finitely generated and projective \( D_{\mu \nu} \)-module of trace \( 2\mu \). If \( \nu \in [0, 1/2] \), and \( \mu > 1 \), then there is a finitely generated projective \( D_{\mu \nu} \)-submodule of \( X \) with trace \( 2\nu \).
Proof:

Let us take \( P = I_E \), in the notation of Remark \( \text{[1]} \). Then \( X = XP \) is finitely generated and projective and its trace is \( \tau E(I_E) = 2\mu \).

We now find a projection \( P \) in \( E_0 \) with \( \tau E(P) = 2\nu \), when \( \nu \in [0, 1/2] \) and \( \mu > 1 \), which ends the proof, in view of Remark \( \text{[1]} \).

So let us consider self-adjoint elements \( P \) of the form:

\[
P(x, y, p) = f(x, y)\delta_1(p) + h(x, y)\delta_0(p) + \overline{f}(x - 1, y)\delta_{-1}(p),
\]

where \( h \) and \( f \) are bounded functions on \( R \times T \) and \( h \) is real-valued. Our next step is to get functions \( f \) and \( h \) such that \( P \) is a projection in \( E_{\mu \nu}^c \).

Now,

\[
(P * P)(x, y, p) = \sum_{q \in \mathbb{Z}} P(x, y, q)P(x + q, y, p - q),
\]

and it follows that \( P * P = P \) if and only if, for all \((x, y) \in R \times T\):

1) \( f(x, y)f(x + 1, y) = 0 \)

2) \( f(x, y)[h(x + 1, y) + h(x, y) - 1] = 0 \)

3) \( |f(x, y)|^2 + |f(x - 1, y)|^2 = h(x, y)(1 - h(x, y)) \).

We also want \( P \) to be in \( E_0 \), so we require

\[
P(x, y, p) = e(cp(y + \nu))P(x + 2\mu, y + 2\nu, p), \text{ that is}
\]

4) \( f(x, y) = e(c(y + \nu))f(x + 2\mu, y + 2\nu) \)

and

5) \( h(x, y) = h(x + 2\mu, y + 2\nu) \).

It was shown on \( \text{[11], 1.1} \) that for any \( \zeta \in [0, 1/2] \) there are maps \( F, H \in C(T) \) such that:

1) \( F(t)F(t - \zeta) = 0 \)

2) \( F(t)[1 - H(t) - H(t - \zeta)] = 0 \)

3) \( H(t)[1 - H(t)] = |F(t)|^2 + |F(t + \zeta)|^2 \)

4) \( \int_T H = \zeta \)

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5)’ \( 0 \leq H(t) \leq 1 \) for any \( t \in T \) and \( F \) vanishes on \([1/2, 1]\).

Let us assume that \( \nu \in [0, 1/2] \), \( \mu > 1 \) and let \( F \) and \( H \) be functions satisfying 1)’-5)’ for \( \zeta = \nu / \mu \).

Translation of \( t \) by \( \zeta \) in equations 1)’-5)’ plays the same role as translation of \( x \) by 1 in equations 1)-5), which suggests taking \( \zeta x \) as the variable \( t \).

However, the variable \( y \) will play an important role in getting \( f \) and \( h \) to satisfy 4) and 5), for which we need to take \( t = 1/2 + y - \zeta x \).

So let
\[
    h(x, y) = H(1/2 + y - \zeta x),
\]
so \( h \) is in \( C(R \times T) \), and it is real-valued and bounded.

Also,
\[
    h(x + 2\mu, y + 2\nu) = H(1/2 + y + 2\nu - \zeta x - 2\nu) = H(1/2 + y - \zeta x) = h(x, y),
\]
so \( h \) satisfies 5).

Now, for \((x, y) \in [0, 2\mu] \times [0, 1]\), set
\[
    f(x, y) = \begin{cases} 
        F(1/2 + y - \zeta x) & \text{if } y \leq x/(2\mu) \\
        e(c(y + \nu))F(1/2 + y - \zeta x) & \text{if } y \geq x/(2\mu)
    \end{cases}
\]

To show \( f \) is continuous it suffices to prove that \( F(1/2 + y - \zeta x) = 0 \) when \( y = x/(2\mu) \), and that follows from the fact that \( F \) vanishes on \([1/2, 1]\), and from the conditions on \( \mu \) and \( \nu \).

Now, since \( f(x, 1) = f(x, 0) \), \( f \) is continuous on \([0, 2\mu] \times T\). We want to extend \( f \) to \( R \times T \) by letting
\[
    f(x + 2\mu, y) = e(-c(y - \nu))f(x, y - 2\nu),
\]
so as to have \( f \) satisfy 4). We only need to show that
\[
    f(2\mu, y) = e(-c(y - \nu))f(0, y - 2\nu) \text{ for any } y \in T.
\]

For an arbitrary \( y \in R \), let \( k, l \in Z \) be such that \( y + k, y - 2\nu + l \in [0, 1] \).

Then,
\[
    f(2\mu, y) = F(1/2 + y + k - 2\nu) = F(1/2 + y - 2\nu), \text{ and }
\]
\[
    f(0, y - 2\nu) = f(0, y - 2\nu + l) = e(c(y - \nu + l))F(1/2 + y - 2\nu) =
\]
\[
    = e(c(y - \nu))f(2\mu, y),
\]

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as wanted, and f, extended to $R \times T$ as above, satisfies 4). It remains to show that f and g satisfy 1), 2) and 3):

1) $|f(x, y)f(x + 1, y)| = |F(1/2 + y - \zeta x)F(1/2 + y - \zeta x - \zeta) = 0$, by 1)'.

2) $|f(x, y)[h(x + 1, y) + h(x, y) - 1]| = |F(1/2 + y - \zeta x)[H(1/2 + y - \zeta x - \zeta) + H(1/2 + y - \zeta x) - 1]| = 0$, by 2)'.

3) $|f(x, y)|^2 + |f(x - 1, y)|^2 = |F(1/2 + y - \zeta x)|^2 + |F(1/2 + y - \zeta x + \zeta)|^2 = H(1/2 + y - \zeta x)[1 - H(1/2 + y - \zeta x)] = h(x, y)(1 - h(x, y))$, by 3)'.

Therefore P is a projection on $E_0$. Besides,

$$\tau_E(P) = \int_0^{2\mu} \int_T h(x, y)dydx = \int_0^{2\mu} (\int_T H(1/2 + y - \zeta x)dy)dx = \int_0^{2\mu} \zeta = 2\mu \zeta = 2\nu$$

by 5)'.

Q.E.D.

The following propositions enable us to extend the previous results by dropping the restrictions on $\mu$ and $\nu$.

Notation: In Propositions 1 and 2, $\Pi$ denotes the faithful representation of $D_{\mu, \nu}^c$ on $L^2(R \times T \times Z)$ obtained by restriction of the left regular representation of the multiplier algebra of $C_0(R \times T) \times \lambda Z$ on $L^2(R \times T \times Z)$, i.e.

$$(\Pi_\phi \xi)(x, y, p) = \sum_{q \in Z} \Phi(x + 2p\mu, y + 2p\nu, q)\xi(x, y, p - q),$$

for $\Phi \in D_0$, $\xi \in L^2(R \times T \times Z)$, and $(x, y, p) \in R \times T \times Z$.

Notice that $\Pi$ is faithful because $Z$ is amenable ([Pd, 7.7.5 and 7.7.7]).

Proposition 1 There is a trace-preserving isomorphism between $D_{\mu, \nu}^c$ and $D_{\mu+k, \nu+l}^c$ for all $k, l \in Z$.

Proof:

It is clear that $\Phi \mapsto \Phi$ is an isomorphism between $D_{\mu, \nu}^c$ and $D_{\mu, \nu+l}^c$, so let us assume $l = 0, k = 1$.

Let $J : D_{\mu+1, \nu}^c \longrightarrow D_{\mu, \nu}^c$ be defined at the level of functions in $D_0$ by:

$$(J\Phi)(x, y, p) = e(c(4p^3\nu/3 - p^2 y))\Phi(x, y, p).$$
It is easily checked that $J\Phi \in D_{\mu\nu}^c$ for all $\Phi \in D_{\mu+1,\nu}^c$. Besides, the unitary operator $U : L^2(R \times T \times Z) \rightarrow L^2(R \times T \times Z)$ given by

$$U\xi(x, y, p) = e(c(-4p^3\nu/3 - p^2y))\xi(x, y, p)$$

intertwines $\Pi_\Phi$ and $\Pi_{J\Phi}$:

$$(\Pi_{J\Phi}U\xi)(x, y, p) = \sum_{q \in \mathbb{Z}} \Phi(x + 2p(\mu + 1), y + 2p\nu, q)U\xi(x, y, p - q) =$$

$$= \sum_{q \in \mathbb{Z}} e(-2pcq(y + (2p - q)\nu)e(c((-4\nu/3)(p - q)^3 - (p - q)^2y)).$$

$$\cdot \Phi(x + 2p\mu, y + 2p\nu, q)\xi(x, y, p - q) =$$

$$= e(c(-4\nu p^3/3-p^2y)) \sum_{q \in \mathbb{Z}} e(c(4q^3\nu/3-q^2(y+2p\nu))\Phi(x+2p\mu, y+2p\nu, q)\xi(x, y, p-q) =$$

$$= (U\Pi_{J\Phi}\xi)(x, y, p).$$

Also,

$$\tau(J\Phi) = \int_0^1 \int_T J\Phi(x, y, 0) = \int_0^1 \Phi(x, y, 0) = \tau(\Phi).$$

Q.E.D.

**Proposition 2** There is a trace-preserving isomorphism between $D_{\mu\nu}^c$ and $D_{-\mu,-\nu}^c$.

**Proof:**

Let $J : D_{\mu\nu}^c \rightarrow D_{-\mu,-\nu}^c$ be defined, at the level of functions, by:

$$(J\Phi)(x, y, p) = \Phi(-x, -y, p).$$

It is easily checked that $J\Phi \in D_{-\mu,-\nu}^c$. Besides, the unitary operator $U : L^2(R \times T \times Z) \rightarrow L^2(R \times T \times Z)$ defined by

$$(U\xi)(x, y, p) = \xi(-x, -y, p)$$

intertwines $\Pi_\Phi$ and $\Pi_{J\Phi}$:

$$[\Pi_{J\Phi}(U\xi)](x, y, p) = \sum_{q \in \mathbb{Z}} (J\Phi)(x - 2p\mu, y - 2p\nu, q)\xi(-x, -y, p - q) =$$
\[
\sum_{q \in \mathbb{Z}} \Phi(-x + 2p\mu, -y + 2p\nu, q)\xi(-x, -y, p - q) = \\
= (\Pi_{\Phi}\xi)(-x, -y, p) = (U\Pi_{\Phi}\xi)(x, y, p).
\]

Finally, \( J \) preserves the trace:

\[
\tau(J\Phi) = \int_{T^2} \Phi(-x, -y, 0) = \tau(\Phi).
\]

Q.E.D.

**Theorem 2** The range of the trace on \( K_0(D_{\mu\nu}^c) \) contains the set \( \mathbb{Z} + 2\mu\mathbb{Z} + 2\nu\mathbb{Z} \).

**Proof:**

We obviously have \( \mathbb{Z} \subseteq \tau(K_0(D_{\mu\nu}^c)) \), since \( D_{\mu
u}^c \) has an identity element. Besides, it follows from Theorem 1 that \( 2\mu\mathbb{Z} \subseteq \tau(K_0(D_{\mu\nu}^c)) \). So it only remains to show that \( 2\nu\mathbb{Z} \subseteq \tau(K_0(D_{\mu\nu}^c)) \).

Let \( k \in \mathbb{Z} \) be such that \( \nu' = \pm\nu + k \) and \( \nu' \in [0, 1/2] \). Then one can find \( l \in \mathbb{Z} \) and \( \mu' = \pm\mu + l \) such that \( \mu' \geq 1 \). Thus, by Propositions 1 and 2 we have that \( \tau(K_0(D_{\mu'\nu'}^c)) = \tau(K_0(D_{\mu\nu}^c)) \).

Now, by Theorem 1 there is a projection in \( M_m(D_{\mu'\nu'}^c) \) for some positive integer \( m \) with trace \( 2\nu' = \pm 2\nu + 2k \), which ends the proof.

Q.E.D.

**Remark.** It can be shown ([Ab1]) that the inclusion in the previous theorem is actually an identity.

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