PREHEATING, PARAMETRIC RESONANCE AND THE EINSTEIN FIELD EQUATIONS

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Abstract. We consider the interaction between perturbations in the inflaton and in the metric during the preheating phase in simple inflationary models. By numerically integrating the Einstein field equations we are able to gauge the impact of non-linear gravitational effects on preheating for the first time. In the $\lambda \phi^4$ model we find a large increase in the amplitude of sub-Hubble metric modes, beyond that due to gravitational collapse alone. There is significant mode-mode coupling and the amplification is eventually terminated by back reaction effects. We suggest that such enhancement of inhomogeneity will change the behaviour of the post-inflationary universe.

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1. Introduction

Inflationary cosmology proposes a period of nearly exponential growth in the primordial universe and in so doing solves a number of the problems that face big bang models. In most implementations of inflation, the post-inflationary universe is a very cold and very empty place, and the first question one has is, where did the all matter seen today came from? The short answer is that it was due to a process called reheating in which the inflaton decayed and released its energy into “normal” matter and radiation. It was realised in 1990, however, that the long answer, the details of reheating, might turn out to be very interesting indeed.

Traschen and Brandenberger[1] showed that if the inflaton field oscillates coherently at the end of inflation, then it may drive a period of explosive particle production through the mechanism of parametric resonance. This process is often termed preheating and its ramifications for cosmology include the areas of non-thermal phase transitions and defect production, baryogenesis, dark matter, gravitational waves and primordial black holes (see [2, 3, 4] and refs. within). In models which allow preheating, reheating is now understood as preheating followed by thermalisation.

In the standard analysis of preheating one assumes a Friedmann-Lemaître-Robertson-Walker (FLRW) universe and then considers, both analytically[5, 6] and numerically[7, 8], the field dynamics in this background. In general one finds that certain modes in the momentum space of the fields are exponentially populated, with the precise structure of these resonance bands being highly model-dependent. The result is that the fields soon deviate from homogeneity and there are significant non-linear effects such as rescattering of the particles, possible symmetry restoration and
back reaction on the inflaton. Although the FLRW scale factor can be calculated self-consistently in these scenarios, it was realised recently that the impact of inhomogeneity and non-linearity on the metric must also be considered.

Our analysis of preheating is new in that we use the full Einstein field equations. This enables us to study non-linear gravitational effects that are not accessible in the perturbative approximation. We find that parametric resonance can induce large metric inhomogeneities and that mode-mode coupling and back reaction is significant. At present, our principal simplifying assumption is that the inhomogeneity is in one spatial direction only. This means we have a 1 + 1-dimensional system of partial differential equations for two metric functions.

The larger goal we are pursuing is to determine the spectrum of amplified fluctuations, in both the fields and the metric, that arise from preheating. At issue is whether predictions of preheating are in accord with cosmological observations. Recently it has been argued that super-Hubble modes can be amplified during preheating. The rationale is that inflation leaves the oscillating inflaton coherent on scales much larger than Hubble radius. If this is the case, one might have to reconsider the standard perturbative account of the fluctuations that lead to large-scale structure formation, especially if mode-mode coupling occurs. On the other hand, it is certainly the case that amplified fluctuations on Hubble and sub-Hubble scales will give rise to gravitational waves, and one would also like to know if primordial black hole formation after inflation is affected.

Here we briefly survey our progress to date. We have considered preheating in two models of chaotic inflation, supposing there are no fields other than the inflaton. We have confirmed an analytical result that the \( m^2 \phi^2 \) model does not undergo parametric resonance. In the \( \lambda \phi^4 \) model where the standard analysis predicts parametric resonance in the field we have found strong amplification of the corresponding metric modes, and coupling of these modes to modes outside the resonance band. The modes which are amplified are sub-Hubble modes. We will report more fully on these results in a forthcoming paper.

2. Metric and initial conditions

For simplicity we have assumed a universe with planar symmetry so the metric functions depend only on time and one spatial co-ordinate. We studied preheating after \( m^2 \phi^2 \) inflation using the metric
\[
ds^2 = dt^2 - A^2(t, z)\, dz^2 - B^2(t, z)\left(dx^2 + dy^2\right) .
\] (1)

In the limit of small spatial inhomogeneity this resembles a flat FLRW metric written in terms of physical time. We could also use this metric for the \( \lambda \phi^4 \) case, but it turns out to be more convenient to introduce a “conformal-like” metric. This is obtained via a co-ordinate transformation: \( \eta = \eta(t, z) \) and \( \zeta = \zeta(t, z) \), which allows (1) to be written as
\[
ds^2 = a^2(\eta, \zeta)\left(d\eta^2 - d\zeta^2\right) - \beta^2(\eta, \zeta)\left(dx^2 + dy^2\right) .
\] (2)

Here, we will concentrate on this second metric as we have discussed elsewhere and the way we choose the initial conditions is the same in both cases.

\[\dagger\] One may think of this as a universe with a flat, 2-dimensional, maximally symmetric subspace; in contrast the FLRW metric has a 3-dimensional, maximally symmetric subspace.
Figure 1. The Fourier transform of $\Phi$, the metric perturbation, for a resonant mode is shown: the left panel gives the perturbative result, while the right panel shows the evolution of the mode derived from the full non-linear analysis.

The Einstein field equations, $G^{\mu\nu} = -\kappa T^{\mu\nu}$, give equations of motion, i.e. equations involving second derivatives in time, for $\alpha$ and $\beta$. In addition one obtains two constraint equations which contain only first derivatives in time. The equation of motion for $\phi$ follows from $T^{\mu\nu}_{\cdot\cdot} = 0$. (We refer the reader to [16] for the explicit form of these rather lengthy equations.) We numerically integrate the equations of motion using the same techniques of [3] and, in addition, simultaneously compute the co-ordinate transformation which takes us back to metric (1). This allows us to determine $A$ and $B$ as well.

Initial conditions must be chosen to satisfy the constraint equations. This is, in general, quite difficult to do, but we focus on a relatively simple scenario where a single mode only is initially excited. Then it is possible to let $\phi(0, \zeta) = \phi_0$ and $\alpha(0, \zeta) = \beta(0, \zeta) = 1$. Inhomogeneity is put into the system via the inflaton’s momentum

$$\phi_{,\eta}(0, \zeta) = \dot{\phi}_0 + \epsilon \sin \left(\frac{2\pi k\zeta}{Z}\right),$$

where $k$ is the number of times the fluctuation will fit into our simulation “box” of length $Z$. The size of the perturbation is governed by $\epsilon$ and is not required to be small. We start our simulations at the end of inflation and this dictates the values of $\phi_0$ and $\dot{\phi}_0$. Our choice of rescaling makes the initial density perturbation of order $\epsilon^2$.

The constraint equations then determine that

$$\alpha_{,\eta}(0, \zeta) = \frac{\kappa}{2C} \left(\frac{1}{2} \phi_{,\eta}^2 + V(\phi_0)\right) - \frac{C'}{2},$$

$$\beta_{,\eta}(0, \zeta) = \sqrt{\frac{\kappa}{3} \left(\frac{1}{2} \phi_{,\eta}^2 + V(\phi_0)\right)} \equiv C,$$

where $\langle \cdots \rangle$ denotes a spatial average. In the limit of no perturbation these initial conditions ensure the metric reverts to a FLRW type.
Figure 2. The Fourier transform of $\Phi$ for a resonant mode (larger amplitude) and its second harmonic (smaller amplitude), which is outside the resonance band. It grows because, in the non-linear analysis, it is coupled to the resonant mode.

3. Results and discussion

In the standard analysis, where one ignores metric perturbations, there is no parametric resonance in the $m^2 \phi^2$ model. We re-examined this model using our new approach and were able to show that this null result continues to hold. This was in agreement with a perturbative analysis[14] which served as a check on our method. In addition, we began to explore the non-linear regime by making the initial perturbations artificially large. Now mode-mode coupling becomes noticeable and we conjectured that the effect would be to broaden the instability bands in a resonant system. We have been able to confirm this conjecture in our latest work[16].

The $\lambda \phi^4$ model is the simplest model which exhibits parametric resonance and is therefore an important test case for our approach. Tantalisingly, the modes in resonance are initially just inside the Hubble radius. The Hubble radius grows subsequently, so we only observe sub-Hubble mode amplification in this model.

Fig. 1 compares the amplification of a resonant mode in the perturbative and in the non-linear analysis. To facilitate the comparison we plot the gauge invariant metric perturbation $\Phi[18]$, which we extract from our numerical data. The first thing to notice is that the perturbative analysis of a resonant system always fails; the resonant growth takes the system away from the perturbative regime. The full analysis reveals what really happens: resonance terminates due to back reaction effects. An example of the coupling between the modes that the perturbative analysis misses can be seen in Fig. 2.

Perhaps the most dramatic ramification of resonance can be see in the evolution of the metric component $A$. When we simulated the effect of an initial perturbation with wavelength not in the resonance band, we found only a small increase in the inhomogeneity due to collapse of the over-dense regions. In the left panel of Fig. 3 one sees the result: $A \sim \eta$, which is what one expects for a radiation dominated FLRW universe. The situation was quite different, however, when we considered a mode in the resonance band. Here the growth in inhomogeneity is initially driven by

† The universe is initially radiation dominated in the $\lambda \phi^4$ model.
parametric resonance and leads to a large density contrast, $\delta \rho/\rho \sim 1$. After parametric resonance terminates, gravitational collapse continues to enhance the inhomogeneity. It is because the system is initially driven that we think the formation of primordial black holes might need to be reconsidered in preheating scenarios. In the usual understanding, one requires $\delta \rho/\rho \geq 1$ as the mode comes inside the Hubble volume, and black hole formation proceeds by gravitational collapse. In the $\lambda \phi^4$ model, even though the relevant metric modes are always sub-Hubble, they are being amplified resonantly, not gravitationally.

We are now extending the initial results mentioned above\cite{16}. First we will incorporate general initial conditions and then study more realistic models which include other fields beside the inflaton. Once we have determined the spectrum of amplified fluctuations we will be able to consider whether or not cosmological observations can be used to rule out certain preheating models.

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