Solving Three Types of Satellite Gravity Gradient Boundary Value Problems by Least-Squares

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Abstract  The principle and method for solving three types of satellite gravity gradient boundary value problems by least-squares are discussed in detail. Also, kernel function expressions of the least-squares solution of three geodetic boundary value problems with the observations \( \{ \Gamma_{xx}, \Gamma_{yy}, \Gamma_{xy} \} \) and \( \{ \Gamma_{xx} - \Gamma_{yy}, 2\Gamma_{xy} \} \) are presented. From the results of recovering gravity field using simulated gravity gradient tensor data, we can draw a conclusion that satellite gravity gradient integral formulas derived from least-squares are valid and rigorous for recovering the gravity field.

Keywords  least-square; GBVP; kernel function; satellite gravity gradient

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Introduction

The GOCE (Gravity field and steady-state Ocean Circulation Explorer) satellite will be launched by the ESA (European Space Agency) in the later half of 2007 to explore the Earth’s gravity field with high accuracy and resolution\(^1\). Therefore, the research on theories and methods for solving the gravity field using SGG (satellite gravity gradient) observations is very important.

The methods for solving the gravity field using SGG observations is usually divided into two classes: space-wise approach and time-wise approach. Rummel derived the solution of uniquely and overdetermined GBVP (geodetic boundary value problems) of multi-observations with constant radius approximation\(^2\); Rummel and Gelderen, given the relation between disturbing potential and its second derivatives and with the same approximation, proposed the solution of GBVP by least-squares, which is applicable on the condition that the relation between the disturbing potential function and the observations in spectral domain only depends on the degree of their spherical harmonic expansions\(^3,4\). Luo derived the pseudo-solution to satellite gradiometry boundary value problems using the pseudo-solution theory of overdetermined GBVP\(^5\). Li derived rigorous integral formulas and corresponding rigorous kernel functions of solving the gravity disturbance, the disturbing potential, the gravity anomaly and the deflection of the vertical\(^6\).

The principle for solving GBVP by least-squares is studied in the paper. The integral formulas and rigorous kernel functions with respect to three types of gravity gradiometry boundary value problems are given. The simulation and testing results prove the validation and strictness of this method.
1 Mathematical model and principle

1.1 Mathematical model for solving general GBVP by least-squares

An observation equation of a fixed GBVP or free GBVP transformed by Bruns formula can be expressed as:

\[ g = DT \]  

where \( g \) is the boundary function, which is scalar, vector or tensor; \( D \) is the linear differential operator and \( T \) is the disturbing potential function.

From Eq.(1) the boundary function \( g \) can be expressed as a linear function of the disturbing potential \( T \). It is supposed that \( T \in V, \ g(P) \in W \), where \( V, W \) are Hilbert spaces of function on the unit sphere. Using the definition of adjoint operator and the spherical harmonic series of function, we can write:

\[
T(P) = \sum_{lm} \lambda_i c_{lm} y_{lm}(P) \\
g(P) = \sum_{lm} g_{lm}^{(i)} \frac{1}{\lambda_i} D y_{lm}(P)
\]

where \( \lambda_i \) is the singular value of the operator \( D \) and not equal to zero; \( y_{lm}(P) = 1 \sum_{j=1}^{\infty} Y_{lm}(P) \). Based on the orthogonality of spherical harmonic function and Eq.(2) and Eq.(3), the potential coefficients \( c_{lm} \) and the scalar coefficients \( g_{lm} \) of the observation \( g \) are:

\[
c_{lm} = \langle T, y_{lm} \rangle_W \\
g_{lm} = \frac{1}{\lambda_i} \langle g, D y_{lm} \rangle_W
\]

Then, the observation Eq.(1) in spectral domain is:

\[
g_{lm} = \lambda_i c_{lm}, \quad \forall l, m | l \in \mathbb{N}, m \leq l, \lambda_i \neq 0\]  

It is supposed that there are more than one kind of observations. The spectral domain observation equations are:

\[
\begin{bmatrix}
g_{lm}^{i=1} \\
g_{lm}^{i=2} \\
\vdots
\end{bmatrix} = \begin{bmatrix}
\lambda_i^{i=1} \\
\lambda_i^{i=2} \\
\vdots
\end{bmatrix} \begin{bmatrix}
c_{lm}^{i=1} \\
c_{lm}^{i=2} \\
\vdots
\end{bmatrix}
\]

where \( i \) is the number of observation types.

We can get the solution of observation Eq.(7) by least-squares principle:

\[
\hat{c}_{lm} = \frac{\sum l_i A_i^l g_{lm}^{i}}{\sum l_i A_i^l} \tag{8}
\]

where \( l_i \) is the weight of the observation. Based on Eq.(2) and Eq.(8), the expression of the disturbing potential in space domain is:

\[
\hat{T}(P) = \sum_{i} \langle H_i(P, Q), g(Q) \rangle \tag{9}
\]

with

\[
H_i(P, Q) = \sum_{lm} \sum_{j=1}^{\infty} p_i^{j} (\lambda_i^j)^2 D y_{lm}(Q) y_{lm}(P) \tag{10}
\]

1.2 LS solutions of three types of satellite gravity gradient boundary value problems

Three types of the gradiometry observations \( \{\Gamma_{xx}, \Gamma_{xy}, \Gamma_{yy}\} \) and \( \{\Gamma_{xx} - \Gamma_{yy}, 2\Gamma_{xy}\} \) on the spherical surface with height \( h \) can be expressed as an infinite series, and the relation between the coefficients of the series and the disturbing potential spherical harmonic series can be expressed with their corresponding singular values. The spectral observation equations are:

\[
\begin{bmatrix}
d\Gamma_{lm}^x \\
d\Gamma_{lm}^y \\
d\Gamma_{lm}^z
\end{bmatrix} = \begin{bmatrix}
\sigma^{x+s}(l+1)(l+2) \\
\sigma^{y+s}(l-1)(l+2) \sqrt{l(l+1)} \\
\sigma^{z+s}(l+1)(l+2) \sqrt{l(l+1)}
\end{bmatrix} \begin{bmatrix}
c_{lm} \\
c_{lm} \\
c_{lm}
\end{bmatrix}
\]

with \( \sigma = R/(R+h) \) and \( R \) is the mean radius of the Earth.

This is an overdetermined boundary value problem with three types of observations. It is difficult or even impossible to derive the rigorous expressions of the kernel function \( H_i(P, Q)(i=1,2,3) \) in the solution[^4]. This paper only considers the uniquely determined GBVP corresponding to the three observations. We can derive the disturbing potential integral expressions of three observations on the boundary surface by Eq.(9):

\[
\hat{T}_1(P) = \frac{R^2}{4 \pi} \int_{\Sigma} \left[ H(P, Q) \Gamma_{xx}(Q) \right] d\sigma_Q \tag{12}
\]

\[
\hat{T}_2(P) = \frac{R^2}{4 \pi} \int_{\Sigma} \left[ H(P, Q) \cos \alpha \Gamma_{xy}(Q) - \sin \alpha \Gamma_{xy}(Q) \right] d\sigma_Q \tag{13}
\]

\[
\hat{T}_3(P) = \frac{R^2}{4 \pi} \int_{\Sigma} \left[ H(P, Q) \cos \alpha \Gamma_{xx}(Q) - \Gamma_{xy}(Q) - 2 \sin \alpha \Gamma_{xy}(Q) \right] d\sigma_Q \tag{14}
\]
where $R_s$ is the radius of satellite reference orbit surface; $d\sigma_q = \sin \theta_p d\theta d\lambda$; and $\psi_{pq}$ is the spherical surface distance between the computational point $P$ and the moving point $Q$. The variables in the formulas are real dimension observations. Based on the formula from Reference [7], the closed expressions of the kernel functions are:

$$H^1(\psi) = -3 + 6 \sin \frac{\psi}{2} + (3 \cos \psi - 1) \ln \left(1 + \frac{1}{\sin \frac{\psi}{2}}\right)$$

(15)

$$H^2(\psi) = \frac{3}{2} \sin \psi \ln \left(1 + \frac{1}{\sin \frac{\psi}{2}}\right) - \frac{3 \sin \psi + 4 \cos \psi}{2(1 + \sin \frac{\psi}{2})}$$

(16)

$$H^3(\psi) = \frac{1}{2}(1 + \cos \psi)$$

(17)

From Eqs.(12)-(17) we can see that the solutions of three types of GBVP, which are similar to the solution of the Stokes boundary value problem, are in the form of integrality and need known observations and their kernel functions on the integral boundary surface for the real computation. If we have continuous observations on a full boundary surface, we could estimate the disturbing potential on the boundary surface or out of the boundary surface.

## 2 Numerical results

### 2.1 Data simulation

The disturbing gravity gradient tensor in the local north-oriented coordinate system (the $X$-axis directed north, $Y$-axis west and $Z$-axis radially outwards), which is the grid point value with a resolution of $1\times1^\circ$ on the spherical surface with height 250 km, is simulated with the spherical harmonic synthesis method[5]. The gravity field model for simulation is EGM96 with the maximum degree 300, which is regarded as the real gravity field. In order to make simulated observation data more actual, the observation noise has been simulated. For simplicity, the different components of simulated gravity gradient tensor is regarded as having the same accuracy and is superimposed the zero-mean white noise with standard deviation $3\times10^{-3}$ E. The accuracy of the diagonal components of gravity gradient tensor satisfying Laplace’s equation is summarized in Table 1.

| Resolution | Max | Mean | STD | RMS |
|------------|-----|------|-----|-----|
| Non-noised | $1\times1^\circ$ | 8.0×10^{-14} | -5.774 9×10^{-11} | 3.146 4×10^{-13} | 4.199 6×10^{-12} | 4.211 3×10^{-12} |
| Noised     | $1\times1^\circ$ | 2.410 4×10^{-2} | -2.010 4×10^{-2} | 2.579 0×10^{-6} | 5.202 6×10^{-3} | 5.194 6×10^{-3} |

In Table 1, the non-noised disturbing gravity gradient tensor simulated with the spherical harmonic synthesis satisfy Laplace’s equation with the accuracy $10^{-12}$ E, which can be ignored with respect to the observation accuracy $10^{-3}$-$10^{-4}$ E of the gradiometer. The simulated noise disturbing gravity gradient tensor satisfies Laplace’s equation with the accuracy $10^{-3}$ E, which is consistent with the order of the simulated noise.

### 2.2 Results and analysis

The $1\times1^\circ$ grid disturbing potential values were estimated from $1\times1^\circ$ three types of observations \{\(\Gamma_{zz}\), \(\Gamma_{xz}, \Gamma_{zy}\)\} and \{\(\Gamma_{xz} - \Gamma_{zy} - 2\Gamma_{xy}\)\} which are components or components combination of the simulated non-noised disturbing gravity gradient tensor. The computed regions are within the longitude 105°-125° and the latitude 20°-40° of East China, the longitude 75°-95° and the latitude 25°-45° of West China. The spherical surface height is 250 km. The results are given respectively with and without interpolation (the cubic spline interpolation is used in the paper) in the spherical surface integration, and the statistical results of the differences in the disturbing potential between the results from simulated data and EGM96 are given in Table 2.

In Table 2, the maximum absolute value of the differences in the disturbing potential between the results solving from $1\times1^\circ$ non-noised observation \{\(\Gamma_{zz}\)\} and EGM96 is 0.09 m²/s². The maximum RMS is 0.053 m²/s², and the corresponding equipotential surface height transformed by Bruns’ formula is 0.6 cm in East China. Although the results in the west are not as good as those in the east, their accuracies are of the same level and the accuracy of the equipotential surface height is better than cm level.
From Table 2, it is clear that the accuracy of the results with interpolation in the spherical surface integration is improved greatly, and better than the one with direct integration with 1-2 order.

| Strategies | Max     | Min     | Mean    | STD    | RMS    |
|------------|---------|---------|---------|--------|--------|
| Direct integration | 3.566   | -1.682  | 0.341   | 0.824  | 0.891  |
| Integration | 0.709   | -0.655  | 0.045   | 0.186  | 0.191  |
| after interpolation | 0.430   | -0.479  | -0.030  | 0.145  | 0.144  |
| East       |         |         |         |        |        |
| Direct integration | 5.706   | -5.391  | 0.232   | 2.505  | 2.513  |
| Integration | 2.418   | -3.278  | -0.003  | 1.001  | 1.000  |
| after interpolation | 0.043   | -0.065  | -0.010  | 0.017  | 0.020  |
| West       |         |         |         |        |        |
| Direct integration | 2.155   | -0.215  | 0.079   | 0.174  | 0.192  |
| Integration | 0.049   | -0.165  | 0.056   | 0.045  | 0.071  |
| after interpolation | 0.137   | -0.171  | -0.000  | 0.050  | 0.050  |

Theoretically, if the non-noised SGG observations cover the spherical surface fully and continuously, the estimated results should be consistent with the one from the model. But in reality, we cannot obtain globally continuous observations. So, computing the spherical surface integration using those discrete observations with one resolution will cause integral discrete errors in the results. We can densify the observations using interpolation method to reduce the integral discrete errors. In the paper, the cubic spline interpolation is adopted to interpolate observations in the 1° × 1° area with 1, 3, 5, and 9 interpolated points. The 1° × 1° grid point disturbing potential values are estimated in the west. The RMS of the differences between these results with different number of interpolated points and EGM96 is summarized in Table 3.

| Number of interpolation | 0 | 1 | 3 | 5 | 9 |
|-------------------------|---|---|---|---|---|
| \( \{ \Gamma_x \} \)    | 2.513 | 0.404 | 0.071 | 0.061 | 0.063 |
| \( \{ \Gamma_x, \Gamma_y \} \) | 1.000 | 0.231 | 0.050 | 0.023 | 0.016 |
| \( \{ \Gamma_x - \Gamma_y, 2 \Gamma_y \} \) | 0.305 | 0.090 | 0.041 | 0.034 | 0.032 |

In Table 3, although the interpolation can clearly improve the accuracy of the solutions, the accuracy is not continually improved with the increasing number of interpolated points. The solutions with 3, 5, and 9 interpolated points have nearly the same accuracy, which illustrates that the interpolation will cause error in the computation, and the more interpolation points possibly cause more interpolation error.

We also estimate the 1° × 1° grid disturbing potential values in the longitude 75°-95° and the latitude 25°-45° of West China from the 1° × 1° grid point noised disturbing SGG observations. The statistical results compared to the results from EGM96 are given in Table 4 and illustrated in Fig.1.
From Table 4 and Fig.1, the differences between the results solved from noised observations superimposed the zero-mean white noise with standard deviation $3 \times 10^{-3}$ m and the one from EGM96 are mostly in the range of $-0.2$-$0.2$ m/s$^2$. The maximum RMS is $0.192$ m/s$^2$, and the corresponding equipotential surface height is about $2.1$ cm, which reaches to centimeter level although the accuracy is lower than the one solved from the non-noised observations.

## 3 Conclusions

The theory and method of solving GBVP corresponding to the three types of the observations $\{\Gamma_x\}, \{\Gamma_{xx}, \Gamma_{yy}\}$ and $\{\Gamma_{xx} - \Gamma_{yy}, 2\Gamma_{xy}\}$ by least-squares is discussed in the paper. To validate the correctness and possibility of the method, we simulate grid point disturbing gravity gradient tensor on the spherical surface of the satellite height, and add the zero-mean white noise to observations. The testing results show that the satellite gravity gradient integral formula derived from least-squares is valid and rigorous for recovering the gravity field. At the same time, the proper interpolation method should be applied to improve the space resolution of the observation data for reducing the integral discrete error. It should be mentioned that the simulation of satellite orbit and altitude was not considered in the paper, and to actually realize the object of centimeter level geoid, this method should be tested further using the in situ satellite gravity gradient observations.

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