Coherent State Quantization of
SU(N) Non-Abelian Chern-Simons Particles

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Abstract

We present a new method of formulating the classical theory of $SU(N + 1)$ non-Abelian Chern-Simons (NACS) particles for arbitrary $N \geq 1$ using the symplectic reduction of $CP(N)$ manifold from $S^{2N+1}$. Quantizing the theory using BRST formulation and coherent state path integral method, we obtain a quantum mechanical model for $SU(N + 1)$ NACS particles.
The anyons, particles carrying anomalous spin and exhibiting fractional statistics in two spatial dimensions are an interesting theoretical curiosity and may be realized in condensed matter physics [1]. The anyons can be described as particles carrying both charge and magnetic flux and an action for them can be constructed by coupling their charges minimally with the Abelian Chern-Simons gauge field. One can generalize the notion of anyon by introducing a non-Abelian gauge group, i.e., particles which carry non-Abelian charges and interact with each other through the non-Abelian Chern-Simons term [2, 3]. We will call them non-Abelian Chern-Simons (NACS) particles. One can also construct the corresponding classical action for the NACS particles by introducing isospin degrees of freedom and coupling isospin charge with the non-Abelian Chern-Simons gauge fields [4]. Recently, a derivation of quantum mechanical model for SU(2) NACS particles from a classical action has been achieved in ref.[5]. The resulting quantum mechanics showed that the NACS particles also carry anomalous spins and satisfy yet more generalized braid statistics. In this letter, we present a new method of formulating classical theory of SU(N + 1) NACS particles with arbitrary \( N \geq 1 \) and quantize the theory using coherent state path integral [6], and thereby obtain a quantum mechanical model for NACS particles.

A conventional way of formulating a classical action for SU(N + 1) NACS particles is to construct a first order Lagrangean for internal degrees of freedom defined on the SU(N + 1) group manifold and to introduce isospin charges which transform under the adjoint representation of the group [7]. Then, one minimally couples the isospin charges with non-Abelian Chern-Simons gauge field [4]. It turns out that both first and second class constraints [8] arise from this classical Lagrangean for NACS particles [7, 9] and the quantization of the classical action for NACS particles is rather involved.

One way of remedying such complications would be to start with a Lagrangean in which second class constraints do not appear from the beginning if we suitably define the Poisson bracket [10]. For that purpose, it would be better to consider CP(N) manifold [11] instead of SU(N + 1) manifold to describe internal degrees of freedom.
\( CP(N) \) manifold is one of the coadjoint orbits of \( SU(N + 1) \) group and it can be reduced from \( S^{2N+1} \) via the symplectic reduction, i.e., \( CP(N) \simeq S^{2N+1}/U(1) \). In terms of a complex column vector \( Z = (Z_0, Z_1, \cdots, Z_N)^T \) and its complex conjugate \( \bar{Z} = (\bar{Z}_0, \bar{Z}_1, \cdots, \bar{Z}_N) \), \( S^{2N+1} \) is defined by the constraint
\[
\phi = \bar{Z}Z - 1 = 0.
\]

Now, let us consider first order Lagrangian
\[
L_Z = iJ(\bar{Z}\dot{Z} - \dot{\bar{Z}}Z) - h(Z, \bar{Z}) - \lambda\phi
\]
where \( J \) is a constant and \( \lambda \) is a Lagrange multiplier. From this first order Lagrangian, the Poisson bracket can be read as follows:
\[
\{F, G\} = -\frac{i}{2J} \sum_I \left( \frac{\partial F}{\partial Z_I} \frac{\partial G}{\partial \bar{Z}_I} - \frac{\partial F}{\partial \bar{Z}_I} \frac{\partial G}{\partial Z_I} \right)
\]
and the fundamental commutators are given by
\[
\{\bar{Z}_I, Z_K\} = -i \frac{\delta_{IK}}{2J}, \quad \{\bar{Z}_I, \bar{Z}_K\} = \{Z_I, Z_K\} = 0.
\]

The isospin charges are defined as
\[
Q^a = 2J\bar{Z}T^aZ \quad (a = 1, 2, \cdots, N^2 + 2N)
\]
where \( T^a \) are the \((N + 1) \times (N + 1)\) matrices of \( SU(N + 1) \) satisfying \([T^a, T^b] = if^{abc}T^c\) and \( \text{Tr}(T^aT^b) = \frac{1}{2}\delta_{ab} \). Using Eq.(4), we can check that \( Q^a \) satisfy \( SU(N + 1) \) algebra;
\[
\{Q^a, Q^b\} = -f^{abc}Q^c.
\]
Note that the theory described by the Lagrangean Eq.(2) is invariant under \( U(1) \) gauge transformations generated by \( \delta f = \epsilon\{\phi, f\} \) provided \( \{\phi, h\} = 0 \). In our case, \( h(Z, \bar{Z}) \) turns out to be a function of \( Q^a \)'s only and satisfies the condition due to \( \{\phi, Q^a\} = 0 \).

The next step is to introduce standard coordinates on \( CP(N) \) by \( \xi_i = Z_i/Z_0 (Z_0 \neq 0, i = 1, 2, \cdots, N) \) and to make coordinate transformation from \( Z_I \) to \((Z_0, \xi_i)\). To reduce
the internal group manifold to $CP(N)$, we must choose a gauge condition and we find
the following gauge fixing is convenient
\[ \chi = \frac{1}{2}(\bar{Z}_0 - Z_0) = 0. \] (7)
Then, the solution to the constraint Eq. (7) is given by
\[ \bar{Z}_0 = Z_0 = \frac{1}{\sqrt{1 + |\xi|^2}}, \quad |\xi|^2 = \sum_i |\xi_i|^2. \] (8)
By substituting $Z_I = (Z_0, Z_0 \xi_i)$ and Eq. (8) into the Eq. (2), we obtain the following
Lagrangian which describes the particle motion in the internal space $CP(N)$,
\[ L = iJ \frac{\bar{\xi} \dot{\xi} - \dot{\bar{\xi}} \xi}{1 + |\xi|^2} - h'(\xi, \bar{\xi}) \] (9)
where $h'(\xi, \bar{\xi}) = h(Z, \bar{Z}) \big|_{Z_0 = \bar{Z}_0 = 1/\sqrt{1 + |\xi|^2}, Z_i = \xi_i, \bar{Z}_0}.$

Note that the first term in the above Lagrangian is of the form $\int \theta$ where the one
form $\theta$ is given by
\[ \theta = iJ \frac{\bar{\xi} d\xi - d\bar{\xi} \xi}{1 + |\xi|^2}. \] (10)
It yields the well-known symplectic two form $\omega$ on $CP(N)$,
\[ \omega = d\theta = 2i \left[ J \frac{d\xi \wedge d\bar{\xi}}{1 + |\xi|^2} - \frac{(\bar{\xi} d\xi) \wedge (\xi d\bar{\xi})}{(1 + |\xi|^2)^2} \right], \] (11)
and the Fubini-Study metric $ds^2 = \sum_{i,k} g_{ik} d\xi_i d\bar{\xi}_k$,
\[ g_{ik} = \frac{2J}{(1 + |\xi|^2)^2} [(1 + |\xi|^2) \delta_{ik} - \bar{\xi}_k \xi_i], \] (12)
whose inverse $g^{ki}$ satisfying $g_{ik} g^{kj} = \delta_i^j$ is given by
\[ g^{ki} = \frac{1}{2J} (1 + |\xi|^2) (\delta_{ki} + \bar{\xi}_k \xi_i). \] (13)

Using the Fubini-Study metric, we can define Poisson bracket on $CP(N)$ and the
corresponding fundamental commutators as follows
\[ \{F, G\} = -i \sum_{i,k} g^{ki} \left( \frac{\partial F}{\partial \xi_k} \frac{\partial G}{\partial \xi_i} - \frac{\partial F}{\partial \xi_i} \frac{\partial G}{\partial \xi_k} \right), \]
\[ \{\dot{\xi}_i, \xi_j\} = -\frac{i}{2J} (1 + |\xi|^2) (\delta_{ij} + \dot{\xi}_i \xi_j), \]

\[ \{\xi_i, \xi_j\} = \{\tilde{\xi}_i, \tilde{\xi}_j\} = 0. \]

The isospin charge \( Q^a \) of Eq.(13) can be re-expressed in terms of coordinates \( \xi_i \)

\[ Q^a = 2J \sum_{I=0}^{N} \bar{Z}_I T^a_{IK} Z_K \bigg|_{Z_0 = \frac{1}{\sqrt{1 + |\xi|^2}}, \bar{Z}_i = \xi_i}. \]  

(15)

For \( SU(2) \) group with Pauli matrices \( T^a = \sigma^a/2 \), we have

\[ Q^1 = \frac{J(\xi + \bar{\xi})}{1 + |\xi|^2}, \quad Q^2 = \frac{iJ(\bar{\xi} - \xi)}{1 + |\xi|^2}, \quad Q^3 = \frac{J(1 - |\xi|^2)}{1 + |\xi|^2} \]  

(16)

We can easily check that the isospin charges satisfy the same \( SU(N+1) \) algebra Eq.(13) after the reduction.

Now we can write down the Lagrangean for a system of \( N_p \) isospin particles which are interacting with \( SU(N+1) \) Chern-Simons gauge field. Denoting the spatial and internal coordinates of the particles by \( \mathbf{q}_\alpha \) and \( \xi_\alpha^i, \alpha = 1, 2, \cdots, N_p, \quad i = 1, 2, \cdots, N \), we have

\[ L = L_{\text{ptl}} + L_{\text{int}} + L_{\text{C-S}}, \]

\[ L_{\text{ptl}} = \sum_\alpha \left( -\frac{1}{2} m_\alpha \mathbf{q}_\alpha^2 + iJ_\alpha \frac{\bar{\xi}_\alpha \dot{\xi}_\alpha - \dot{\bar{\xi}}_\alpha \xi_\alpha}{1 + |\xi_\alpha|^2} \right), \]

(17)

\[ L_{\text{int}} = \int d^2x \sum_\alpha (A^a_\alpha(t, \mathbf{x}) \cdot \dot{\mathbf{q}}_\alpha + A^a_0(t, \mathbf{x})) Q^a_\alpha \delta(\mathbf{x} - \mathbf{q}_\alpha), \]

\[ L_{\text{C-S}} = \kappa \int d^2x \epsilon^{\mu\nu\lambda} \text{tr} \left( A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right), \]

where \( \kappa = \frac{k}{4\pi} \) with \( k = \text{integer} \), \( A_\mu = A^a_\mu T^a \) and the space-time signature is \((+, -, -)\). The equations of motion obtained from the above Lagrangean contain the Wong’s equation [14].

It is convenient to employ the complex coordinates, \( z = x + iy, \quad \bar{z} = x - iy, \quad z_\alpha = q_\alpha^1 + iq_\alpha^2, \quad \bar{z}_\alpha = q_\alpha^1 - iq_\alpha^2, \quad A^a_1 = \frac{i}{2}(A^1_\alpha - iA_2^\alpha), \quad A^a_2 = \frac{i}{2}(A^1_\alpha + iA_2^\alpha) \) when we proceed to constraint analysis and quantization. Introducing momenta variables, \( p_\alpha^\xi, p_\alpha^\bar{\xi}, \pi^a_\alpha \) and \( p_\alpha^a \)
which are canonical conjugates to \( z_\alpha, \bar{z}_\alpha \) and \( A_0^a \) respectively, we obtain the following first order Lagrangean

\[
L = \sum_\alpha \left( p_\alpha^z \dot{z}_\alpha + p_\alpha^\bar{z} \dot{\bar{z}}_\alpha + iJ_\alpha \frac{\dot{\bar{\xi}}_\alpha \dot{\xi}_\alpha - \dot{\bar{\xi}}_\alpha \dot{\xi}_\alpha}{1 + |\xi_\alpha|^2} \right) \\
+ \int d^2z \left( \pi_\alpha^a \dot{A}_0^a + \frac{\kappa}{2} \left( \dot{A}_z^a A_z^a - \dot{A}_\bar{z}^a A_\bar{z}^a \right) + a_0^a \Phi^a \right) - H, \\
(18)
\]

\[
H = \sum_\alpha \frac{2}{m_\alpha} \left( p_\alpha^z - A_z^a (z_\alpha, \bar{z}_\alpha) Q_\alpha^a \right) \left( p_\alpha^\bar{z} - A_\bar{z}^a (z_\alpha, \bar{z}_\alpha) Q_\alpha^a \right).
\]

In passing, we note that we do not introduce conjugate momenta \( f \) or the variable which already appear as phase space variables. Here \( \Phi^a \)'s are the Gauss constraints given by

\[
\Phi^a (z) = \kappa F^a_{z\bar{z}} + \sum_\alpha Q_\alpha^a \delta (z - z_\alpha) = 0. \\
(19)
\]

The above first order Lagrangean Eq.(18) leads us to define the Poisson bracket as in ref.[3] along with Eq.(14). As expected, we find that \( \Phi^a \)'s satisfy the \( SU(N+1) \) algebra and no further constraints arise

\[
\{ \Phi^a (z), \Phi^b (z') \} = -f^{abc} \Phi^c \delta (z - z'), \quad \{ H, \Phi^a (z) \} = 0. \\
(20)
\]

If the Gauss’ constraint is solved explicitly, dynamics of the NACS particles can be described solely by the quantum mechanical Hamiltonian \( H \) in Eq.(18). Thus it is desirable to choose a gauge condition where the Gauss’ constraint can be solved explicitly. Taking advantage of the nature of 2+1 dimensions, one may realize that the axial gauge works. Choosing an axial gauge condition, say, \( A_1^a = 0 \), we get a solution for the Gauss’ constraint

\[
A_1^a (x) = 0, \quad A_2^a (x) = \frac{1}{\kappa} \sum_\alpha Q_\alpha^a \theta (x - x_\alpha) \delta (y - y_\alpha) + f (y) \\
(21)
\]

where \( f (y) \) is an arbitrary function of \( y \). Unfortunately, it is rather awkward to describe the dynamics of the NACS particles with this axial gauge solution because of the strings attached to the particles.
A better gauge condition can be found if we adopt the coherent state quantization \[13, 16\] for the gauge fields. Adopting the coherent state quantization effectively enlarges the gauge orbit space in that \(A^a_z\) and \(\bar{A}^a_{\bar{z}}\) are treated as independent variables. This enables us to choose \(A^a_z = 0\) as a gauge condition \[5\] in the framework of the coherent state quantization. Since the gauge fields have only holomorphic parts in this gauge, it may be called holomorphic gauge. In this gauge the solution for the Gauss' constraint is obtained \[5, 17\] as

\[
A^a_z(z, \bar{z}) = 0, \quad \bar{A}^a_{\bar{z}}(z, \bar{z}) = \frac{i}{2\pi K} \sum_{\alpha} Q^a_{\alpha} \frac{1}{z - z_{\alpha}}. \tag{22}
\]

The BRST invariant physical transition amplitude is represented by a path integral

\[
Z_\Psi = \int Dp^z Dq^\bar{z} Dp^\bar{z} Dq^z D\mu(\xi, \bar{\xi}) DA_z DA_{\bar{z}} D\pi DA_0 Db Dc D\bar{b} D\bar{c}
\exp \left\{ -\kappa i \int d^2 z \left( A_z^i A^i_{\bar{z}} + A_{\bar{z}}^i A^i_z \right) \right\} \exp \left\{ i \int_{t_i}^{t_f} dt \left( K + \{ \Psi, \Omega \} + \int d^2 z \left( \bar{c}^a \bar{b}^a + i\bar{c}^a c^a \right) \right) \right\}
= \int d^2 z \left( \pi^a A^a_0 + \frac{\kappa}{2} \left( \dot{A}^a_z A_{\bar{z}}^a - \dot{A}^a_{\bar{z}} A^a_z \right) \right) - H. \tag{23}
\]

where \(K = \sum_{\alpha} \left( p^z_\alpha \dot{z}_{\alpha} + p^{\bar{z}}_\alpha \dot{\bar{z}}_{\alpha} + iJ_a \frac{\dot{c}^a_{\alpha} \dot{z}_{\alpha} \bar{c}^a_{\alpha} \bar{z}_{\alpha}}{1 + |\xi_{\alpha}|^2} \right) + \int d^2 z \left( \pi^a A^a_0 + \frac{\kappa}{2} \left( \dot{A}^a_z A_{\bar{z}}^a - \dot{A}^a_{\bar{z}} A^a_z \right) \right) - H. \]

Also \(\Omega\) is the nilpotent BRST charge defined by

\[
\Omega = \int d^2 z (c^a \Phi^a - i\bar{b}^a \pi^a - \frac{1}{2} f^{abc} c^a c^b \bar{b}^c) \tag{24}
\]

and \(\Psi\) is the Grassmann odd gauge function whose explicit expression depends on the gauge condition chosen. It should be understood that it is the BRST invariance which guarantees the equivalence of the path integral in the holomorphic gauge to those in more conventional gauges such as Coulomb and axial gauges: \(Z_\Psi\) is independent of \(\Psi\) \[18\].

In order to impose the holomorphic gauge condition, we take

\[
\Psi = \int d^2 z \left( \frac{i}{\beta} c^a A^a_z + \bar{b}^a A^a_0 \right) \tag{25}
\]

where \(\beta\) is a parameter to be taken zero at the end. Integrating out the BRST ghosts and the gauge fields as in ref. \[5\], we end up with a quantum mechanical description of
the NACS particles:

\[
Z = \int Dp^z Dq^z Dp^\bar{z} Dq^\bar{z} D\mu(\bar{\xi}, \xi)
\times \exp \left\{ i \int_{t_i}^{t_f} dt \left( \sum_{\alpha} \left( p^z_{\alpha} \dot{z}_{\alpha} + p^\bar{z}_{\alpha} \dot{\bar{z}}_{\alpha} + iJ_{\alpha} \frac{\bar{\xi}_{\alpha} \dot{\xi}_{\alpha} - \dot{\bar{\xi}}_{\alpha} \xi_{\alpha}}{1 + |\xi_{\alpha}|^2} \right) - H \right) \right\}
\]

(26)

\[
H = \sum_{\alpha} \frac{2}{m_{\alpha}} p^z_{\alpha} \left( p^z_{\alpha} - \frac{i}{2\pi \kappa} \sum_{\beta} Q_{\beta} \frac{1}{z_{\alpha} - z_{\beta}} \right)
\]

where \(D\mu(\bar{\xi}, \xi)\) is the Liouville measure on \(CP(N)\) given by

\[
D\mu(\bar{\xi}, \xi) = \prod_{\alpha} D\mu(\bar{\xi}_{\alpha}, \xi_{\alpha}) = \prod_{\alpha} \frac{c \, d|\xi_{\alpha}|}{(1 + |\xi_{\alpha}|^2)^{N+1}}
\]

(27)

where \(c\) is a normalization constant \([19]\).

Now, we only need to show that the above path integral is equal to the following transition amplitude governed by the Hamiltonian operator \(\hat{H}\)

\[
Z = \langle \xi_F, \zeta_F | \exp \{-i\hat{H}(t_F - t_I)\} | \zeta_I, \xi_I \rangle
\]

(28)

where \(\zeta\) and \(\xi\) denote collectively \((z_\alpha, \bar{z}_\alpha)\) and \((\xi_\alpha, \bar{\xi}_\alpha)\), and \(|\zeta, \xi| = \Pi \alpha \, |z_\alpha > \otimes |\bar{z}_\alpha > \otimes |\xi_\alpha > \otimes |\bar{\xi}_\alpha >\). The Hamiltonian operator \(\hat{H}\) is the operator version of the classical Hamiltonian Eq.(26). We achieve this by using coherent states path integral on \(CP(N)\) \([6]\).

We first construct coherent states on \(CP(N)\). Let us consider \(|0\rangle\), the highest weight state annihilated by all positive roots of \(SU(N + 1)\) algebra in Cartan basis. Then for \(CP(N)\) with given \(P_\alpha \equiv 2J_\alpha \ (P_\alpha \in \mathbb{Z}^+)\) we have an irreducible representation \((P_\alpha, 0, \cdots, 0)\) of \(SU(N + 1)\) group according to Borel-Weil-Bott theorem \([12]\) and there are precisely \(N\) negative roots \(E_\gamma, \gamma = 1, 2, \cdots, N\) such that \(E_\gamma \, |0 \rangle \neq 0 \rangle\) for the irreducible representations \((P_\alpha, 0, \cdots, 0)\). Let us label \(\{E_i\} = \{E_1\}\). Representing a point on \(CP(N)\) by \(\xi^\alpha \equiv (\xi_1^\alpha, \xi_2^\alpha, \cdots, \xi_N^\alpha)\), we construct a coherent state on \(CP(N)\) by

\[
| P_\alpha, \xi^\alpha > = \frac{1}{(1 + |\xi^\alpha|^2)^{J_\alpha}} \exp(\sum_i E_i |E_i\rangle) \, |0\rangle
\]

(29)
Having defined the coherent states on $CP(N)$, we find that they have the following two important properties which are necessary ingredients to take the next step forward. One is the resolution of unity,

$$
\int D\mu(\bar{\xi}^\alpha, \xi^\alpha) \mid P_\alpha, \xi^\alpha \rangle \langle P_\alpha, \xi^\alpha \mid = I,
$$

and the other is reproducing kernel,

$$
\langle P_\alpha, \xi^\alpha | P_\beta, \xi^\beta \rangle = \frac{(1 + \bar{\xi}^\alpha \xi^\alpha)^{2J_\alpha}}{(1 + |\xi^\alpha|^2)^{J_\alpha}(1 + |\xi^\alpha|^2)^{J_\alpha}} \delta_{\alpha\beta}.
$$

Now we are in a position to show the equivalence between Eqs. (26) and (28). Since the part of the spatial degrees of freedom can be treated by canonical way, we only need to prove the equivalence in the sector of the internal degrees of freedom. Consider

$$
Z = \langle P, \xi_F \mid \exp\{-i\hat{H}(t_F - t_I)\} \mid P, \xi_I \rangle
$$

where $|P, \xi \rangle = \prod_\alpha |P_\alpha, \xi^\alpha \rangle$. Let us divide the time interval by $M + 1$ steps and $\epsilon = \frac{t_F - t_I}{M + 1}$. Also $|P, \xi_I \rangle \equiv |P, \xi(0) \rangle$ and $|P, \xi_F \rangle \equiv |P, \xi(M + 1) \rangle$. Inserting the identity Eq. (30) repeatedly, we have

$$
Z = \lim_{M \to \infty} \int \cdots \int \prod_{\alpha} \prod_{n=1}^{M} D\mu(\bar{\xi}^\alpha(n), \xi^\alpha(n)) \prod_{n=1}^{M+1} < P, \xi(n) | P, \xi(n-1) > \\
	imes \left(1 - i\epsilon \frac{< P, \xi(n) | \hat{H} | P, \xi(n-1) >}{< P, \xi(n) | P, \xi(n-1) >} \right).
$$

Using the Eq. (31) with $\xi^\alpha(n-1) = \xi^\alpha(n) - d\xi^\alpha(n)$, we find

$$
\prod_{n=1}^{M+1} < P, \xi(n) | P, \xi(n-1) > \\
= \prod_\alpha \prod_{n=1}^{M+1} \left(1 - J_\alpha \frac{\bar{\xi}^\alpha(n) \cdot d\xi^\alpha(n) - d\bar{\xi}^\alpha(n) \cdot \xi^\alpha(n) + O((d\xi^\alpha(n))^2)}{1 + |\xi^\alpha(n)|^2} \right)
$$

$$
\approx \exp \left\{ \sum_\alpha \sum_{n=1}^{M+1} \epsilon \left[ J_\alpha \frac{(d\bar{\xi}^\alpha(n) \cdot \xi^\alpha(n) - \bar{\xi}^\alpha(n) \cdot d\xi^\alpha(n))}{\epsilon} \right] \right\}
$$

Hence in the limit $M \to \infty$,

$$
Z = \int D\mu(\bar{\xi}, \xi) \exp \left\{ i \sum_\alpha \int_{t_i}^{t_f} dt \left[ J_\alpha \frac{(\bar{\xi}^\alpha \dot{\xi}^\alpha - \bar{\xi}^\alpha \dot{\xi}^\alpha)}{1 + |\xi^\alpha|^2} - \mathcal{H}(Q^\alpha) \right] \right\}
$$
where $\mathcal{H}(Q^a) =$ $\langle P, \xi \mid \hat{H}(Q^a) \mid P, \xi \rangle$. Using the classical limit of operators in coherent state [13], we can easily show that $\langle P, \xi \mid \hat{Q}^a \mid P, \xi \rangle =$ $Q^a(\xi, \xi)$ of Eq.(13). Then it follows that $\mathcal{H}$ is simply equal to classical Hamiltonian $H$ of Eq.(28). Thus, we proved the equivalence between the coherent state path integral representation Eq.(26) and the operator representation Eq.(28).

In conclusion, the dynamics of the NACS particles are governed by the Hamiltonian $\hat{H}$ of Eq.(28)

$$\hat{H} = -\sum_{\alpha} \frac{1}{m_{\alpha}} (\nabla_{z_{\alpha}} \nabla_{z_{\alpha}} + \nabla_{\bar{z}_{\alpha}} \nabla_{\bar{z}_{\alpha}})$$

$$\nabla_{z_{\alpha}} = \frac{\partial}{\partial z_{\alpha}} + \frac{1}{2\pi \kappa} \left( \sum_{\beta \neq \alpha} \hat{Q}^a_{\alpha} \hat{Q}^b_{\beta} \frac{1}{z_{\alpha} - z_{\beta}} + \hat{Q}^2_{\alpha} a_z(z_{\alpha}) \right)$$

$$\nabla_{\bar{z}_{\alpha}} = \frac{\partial}{\partial \bar{z}_{\alpha}}$$

where $a_z(z_{\alpha}) = \lim_{z \to z_{\alpha}} 1/(z - z_{\alpha})$ and the isospin operators $\hat{Q}^a$'s satisfy the $SU(N + 1)$ algebra, $[\hat{Q}^a_{\alpha}, \hat{Q}^b_{\beta}] = if^{abc}_d \hat{Q}^c_d \delta_{\alpha\beta}$ upon quantizing the classical algebra Eq.(3). This Hamiltonian $\hat{H}$ has been suggested also in ref.[20]. The second term and the third term in $\nabla_{z_{\alpha}}$ are responsible for the non-Abelian statistics and the anomalous spins of the NACS particles respectively [3]. The connection $\nabla_{z_{\alpha}}$ without the third term is the Knizhnik-Zamolodchikov (KZ) connection [21] which has been extensively studied in association with the braid groups [22] and the rational conformal field theories [23]. For the detailed discussion on the physical properties of the NACS particles, we suggest to consult refs.[5, 24]. The non-Abelian Chern-Simons quantum mechanics described by the Hamiltonian Eq.(36) will be useful in studying the fractional quantum Hall effect [25], the non-Abelian Aharanov-Bohm effect [20, 24, 26], etc.

We conclude this letter with a couple of remarks. Firstly, in performing the path integral, we did not consider the quantum fluctuation of the gauge fields around the classical solution Eq.(22). If we take into account it either by the background method [3] or by the perturbative method [27], we will find a shift in the coefficient $\kappa$ of the KZ connection Eq.(36) by $\kappa \to \kappa + \frac{c_4}{4\pi} = \kappa + \frac{N}{4\pi}$. Secondly, we may treat the constraint...
Eq. (1) which reduces $S^{2N+1}$ to $CP(N)$ on an equal footing with the Gauss’ constraint Eq. (19), and consider the unreduced Lagrangean

$$L' = L'_{ptl} + L_{int} + L_{C-S}$$

where

$$L'_{ptl} = \sum_{\alpha} \left( -\frac{1}{2} m_\alpha \dot{q}_\alpha^2 + iJ_\alpha (Z^\alpha \dot{Z}^\alpha - \dot{Z}^\alpha Z^\alpha) - \lambda^\alpha \phi^\alpha \right)$$

(37)

The Lagrangean $L'_{ptl}$ has an advantage that the $SU(N+1)$ gauge symmetry is linearly realized,

$$Z \rightarrow gZ, \quad Q \rightarrow gQg^{-1}$$

$$A \rightarrow gAg^{-1} + igdg^{-1}, \quad g \in SU(N+1).$$

(38)

It also has $U(1)^N_p$ gauge symmetry given by $\delta_{\epsilon, f} = \epsilon_\alpha \{ \phi^\alpha, f \}$. However, the BRST quantization based on the Lagrangean Eq. (37) requires to introduce extra ghosts corresponding to $U(1)^N_p$ symmetry and it makes the quantization procedure cumbersome.

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