On current contribution to Fronsdal equations

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Abstract

We explore a local form of second-order Vasiliev equations proposed in [arXiv:1706.03718] and obtain an explicit expression for quadratic corrections to bosonic Fronsdal equations, generated by gauge-invariant higher-spin currents. Our analysis is performed for general phase factor, and for the case of parity-invariant theory we find the agreement with expressions for cubic vertices available in the literature. This provides an additional indication that local frame proposed in [arXiv:1706.03718] is the proper one.
1 Introduction

Linear equations that describe a free propagation of massless higher-spin (HS) fields were found a long time ago by Fronsdal and Fang [1, 2]. But to build any consistent nonlinear deformation of them turned out to be extremely nontrivial task. Up to date the only available example of full nonlinear HS gauge theory is provided by Vasiliev equations [3, 4]. They represent an interacting theory of massless fields of all spins over anti-de Sitter (AdS) background. As opposed to e.o.m. of standard field theory, Vasiliev equations are so-called unfolded ones, i.e. they represent first-order differential equations in terms of exterior (0- and 1-) forms. Each field of given spin is described by an infinite number of unfolded fields parametrising all its degrees of freedom. An infinite number of vertices, describing HS interactions, are encoded into the evolution over auxiliary twistor-like variables. In this regard Vasiliev equations can be considered as generating ones (“equations for equations”).

A reconstruction of space-time dynamics from Vasiliev equations is a nontrivial problem, essentially because of the freedom in the choice of resolution operator for twistor-like variables. As usual, the resolution operator is determined up to an arbitrary solution of homogeneous equation, that in terms of physical fields amounts to the freedom in a field redefinition, which can affect the form of e.o.m. For example, in [5] it was found that by nonlocal field redefinitions one can get rid of interactions in 3d HS equations (see also [6, 7] for the proof of pseudolocal-triviality of any 3d HS currents). In [8] it was shown that the simplest choice of the resolution operator lead to nonlocal expressions for 4d cubic HS vertices. All that brings up a question of admissible functional class of field redefinitions [9, 10, 11, 12]. On the other hand, field redefinitions, bringing quadratic equations to the local form, were found in [13] for the sector of 0-forms and in [14] for the sector of 1-forms. These were tested in [15, 16, 17], where it was shown that the resulting local HS equations properly reproduce holographic correlators in accordance with Klebanov–Polyakov HS AdS/CFT conjecture [18]. Later, in [19] it was shown how to construct a proper resolution operator, enforcing the locality at the second order and minimising nonlocality at higher orders. Formally this operator can be considered as the resolution operator of [8], rectified by non-local field redefinitions of [13, 14].

In this note we provide a further analysis of unfolded local quadratic equations of [14] and obtain an explicit form of corrections to bosonic Fronsdal equations that are generated by gauge-invariant HS currents. These should be compared with results of [20] where HS cubic couplings were found in flat space in lightcone formulation, and [21] where they were restored via AdS/CFT from correlators of boundary free scalar theory and later in [22] shown to solve the bulk Noether procedure. Expressions for quadratic corrections we found turn out to be in the full agreement with these results, thus providing one more confirmation that the local frame of [13, 14] is the appropriate one. In addition, we worked out the dependence of vertices on the phase factor entering Vasiliev equations, thus extending previous results to parity-breaking theories. It turns out that there is a specific value of the phase $\varphi = \frac{\pi}{4}$, where leading-derivative vertex maximally breaks parity, which may have interesting implications for dual boundary theory.
2 Higher-Spin Equations

HS equations in four dimensions are \[[4]\]

\[
dW + W \wedge W = -i\theta_{\alpha} \wedge \theta^{\alpha} (1 + \eta B \ast \kappa k) - i\bar{\theta}_{\dot{\alpha}} \wedge \bar{\theta}^{\dot{\alpha}} (1 + \bar{\eta} B \ast \bar{\kappa} \bar{k}),
\]
\[
dB + W \ast B - B \ast W = 0.
\]

Here $d$ is the space-time de Rham differential, $W$ and $B$ are master-fields of the theory (onwards we omit wedge symbol) dependent on space-time coordinates and twistor-like variables $Y^A = (y^\alpha, \bar{y}^{\dot{\alpha}})$, $Z^A = (z^\alpha, \bar{z}^{\dot{\alpha}})$ with two-valued spinor indices $\alpha$ and $\dot{\alpha}$. The $Y$ and $Z$ realise the HS algebra through the noncommutative star product

\[
(f \star g)(Z, Y) = \int d^4U d^4V e^{iU A V^A} f(Z + U, Y + U) g(Z - V, Y + V),
\]

with the integration measure fixed so as $1 \star f = f \star 1 = 1$. Spinor indices are raised and lowered via $sp(2)$-metrics

\[
u^\alpha = \epsilon^{\alpha\beta} v_\beta, \quad v_\alpha = \epsilon_{\beta\alpha} v^\beta, \quad \bar{v}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{v}_{\dot{\beta}}, \quad \bar{v}_{\dot{\alpha}} = \epsilon_{\dot{\beta}\dot{\alpha}} \bar{v}^{\dot{\beta}}.
\]

$sp(4)$-indices are transformed by $\epsilon_{AB}$ built from $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$

\[
V^A = \epsilon^{AB} V_B, \quad V_A = \epsilon_{BA} V^B.
\]

$\kappa$ and $\bar{\kappa}$ in \[(2.1)\] are inner Klein operators, which are specific elements of the star-product algebra

\[
\kappa := \exp (iz^\alpha y^\alpha), \quad \bar{\kappa} := \exp (i\bar{z}^{\dot{\alpha}} \bar{y}^{\dot{\alpha}}),
\]

having the distinguishing properties

\[
\kappa \star \kappa = 1, \quad \kappa \star f (z^\alpha, y^\alpha) = f (-z^\alpha, -y^\alpha) \star \kappa,
\]

\[
f (z, y) \star \kappa = f (-y, -z) e^{iz^\alpha y^\alpha},
\]

and analogously for $\bar{\kappa}$.

Master-field $B$ is a 0-form, while $W$ is a 1-form in a space-time differential $dx^m$ or in an auxiliary differential $\theta^A$ dual to $Z^A$. All differentials anticommute

\[
\{dx^m, dx^n\} = \{dx^m, \theta^A\} = \{\theta^A, \theta^B\} = 0.
\]

Besides the inner Klein operators there is also a pair of exterior Klein operators $K = (k, \bar{k})$ which have similar properties to $(\kappa, \bar{\kappa})$

\[
kk = 1, \quad kf (z^\alpha, y^\alpha) = f (-z^\alpha, -y^\alpha) k,
\]

(analogously for $\bar{k}$), but $k (\bar{k})$ in addition anticommute with $\theta (\bar{\theta})$ differentials that does not permit to realise them as elements of the star-product algebra.

Thus the full arguments of master-fields are

\[
W = W (Z; Y | K|x| \theta^A, dx^m), \quad B = B (Z; Y | K|x).
\]
$K$-dependence of the fields leads to the splitting of the field spectrum into topological and physical sectors. The first one describes finite-dimensional modules and contains $W$ linear in $k$ or $\bar{k}$ and $B$ depending on $k\bar{k}$. We truncate it away. The physical sector describing relativistic fields contains $W$ depending on $k\bar{k}$ and $B$ linear in $k$ or $\bar{k}$. Moreover, in this note we consider a bosonic reduction, which leaves only one field of every integer spin and is reached by setting

$$W (Z; Y|K|x|\theta^A, dx^m) \rightarrow W (Z; Y|x|\theta^A, dx^m) \left(1 + k\bar{k}\right), \quad B (Z; Y|K|x) \rightarrow B (Z; Y|x) \left(k + \bar{k}\right).$$

(2.12)

$\eta$ in (2.1) is a free complex parameter of the theory which can be normalised to be unimodular $\eta\bar{\eta} = 1$, hence representing the phase factor freedom. HS theory is parity-invariant in the two cases of $\eta = 1$ (A-model) and $\eta = i$ (B-model).

### 3 Perturbation theory

To start a perturbative expansion one has to fix some vacuum solution to (2.1), (2.2). Eq. (2.2) can be solved by setting the vacuum value of $B$ to zero

$$B_0 = 0.$$ 

(3.1)

Then the solution for (2.1) can be chosen as

$$W_0 = \omega_{AdS} + Z_A \theta^A,$$  

(3.2)

with the space-time 1-form of $sp(4)$-connection $\omega_{AdS}$ describing the $AdS_4$ background

$$\omega_{AdS} = -\frac{i}{4} \left(\omega_L^{\alpha\beta} y_\alpha y_\beta + \bar{\omega}_L^\beta \bar{y}_\alpha \bar{y}_\beta + 2\lambda \eta \bar{\eta} y_\alpha \bar{y}_\beta\right),$$

(3.3)

$$d\omega_{AdS} + \omega_{AdS} * \omega_{AdS} = 0,$$  (3.4)

where $\lambda$ is the cosmological parameter (inverse radius of $AdS$).

Performing an expansion of (2.1)-(2.2) around vacuum (3.1)-(3.2) one gets at the linear order

$$\mathcal{D}_{ad}\omega (Y|K|x) = L (C),$$

$$\mathcal{D}_{tw}C (Y|K|x) = 0,$$

(3.5)

(3.6)

where

$$L (C) := \frac{i\lambda}{4} \eta H^{\alpha\beta} \partial_\alpha \bar{\partial}_\beta C (0, \bar{y}|K|x) k + \frac{i\lambda}{4} \bar{\eta} \bar{H}^{\alpha\beta} \partial_\alpha \bar{\partial}_\beta C (y, 0|K|x) \bar{k},$$

(3.7)

$$H^{\alpha\beta} := h^{\alpha\gamma} h^{\beta}_\gamma, \quad \bar{H}^{\alpha\beta} := \bar{h}^{\alpha\gamma} \bar{h}^{\beta}_\gamma$$

(3.8)

$$\partial_\alpha := \frac{\partial}{\partial y_\alpha}, \quad \bar{\partial}_\alpha := \frac{\partial}{\partial \bar{y}_\alpha},$$

(3.9)

$$\mathcal{D}_{ad}f (Y|K|x) := D^L f + \lambda h^{\alpha\beta} (y_\alpha \bar{\partial}_\beta + \partial_\alpha \bar{y}_\beta) f,$$

(3.10)

$$\mathcal{D}_{tw}f (Y|K|x) := D^L f - i\lambda \bar{h}^{\alpha\beta} (\bar{y}_\alpha \partial_\beta - \partial_\alpha \bar{y}_\beta) f,$$

(3.11)

$$D^L f := df + \left(\omega_L^{\alpha\beta} y_\alpha \partial_\beta + \bar{\omega}_L^\beta \bar{y}_\beta \bar{\partial}_\beta\right) f.$$  

(3.12)

\footnote{In \cite{4} it was conjectured that a different situation when $\eta \bar{\eta} = 0$ corresponds to (anti)selfdual HS theory, allowing no nontrivial amplitudes. Here we do not consider this case.}
Eqs. (3.5)-(3.6) represent a so-called unfolded form of Fronsdal equations, describing free propagation of HS fields over $AdS_4$ background. Let us expand HS fields as

$$\omega (Y|K|x) = \sum_{m,n=0}^{\infty} \omega_{m,n} (Y|K|x), \quad C (Y|K|x) = \sum_{m,n=0}^{\infty} C_{m,n} (Y|K|x),$$

(3.13)

where

$$f_{m,n} (Y) := f_{a_1...a_{m},b_1...b_n} y^{a_1}...y^{a_m} \bar{y}^{b_1}...\bar{y}^{b_n}.$$  (3.14)

We also introduce a decomposition into different helicity-sign sectors as

$$\omega = \omega_+ + \omega_- + \omega_0, \quad C = C_+ + C_- + C_0,$$  (3.15)

where

$$f_+ = \sum_{m>n} f_{m,n}, \quad f_- = \sum_{m<n} f_{m,n}, \quad f_0 = \sum_{m=n} f_{m,n}$$  (3.16)

are positive-helicity, negative-helicity and zero-helicity sectors respectively. Then a submodule describing spin-$s$ field consists of $\omega_{m,n}$, $n + m = 2 (s - 1)$ and $C_{m,n}$, $|m - n| = 2s$.

At the second order one should make field redefinitions that brings equations to the local frame, removing infinite higher-derivative tails. Such redefinitions were found in [13, 14]. Applying them one obtains

$$\mathcal{D}_{ad} \omega + [\omega, \omega]_s = L (C) + Q (C, \omega) + \Gamma_{s<s_1+s_2} (J) + \Gamma^{can} (J),$$  (3.17)

$$\mathcal{D}_{tw} C (Y|K|x) + [\omega, C]_s = -\mathcal{H}_q (J) - \mathcal{H}_q (J) + \mathcal{D}_{tw} B^{sum} (J),$$  (3.18)

where

$$J (Y^1, Y^2|K|x) := C (Y^1|K|x) C (Y^2|K|x)$$  (3.19)

is a bilinear HS current. The above-mentioned redefinitions serve to make $J$-dependent terms local. We will analyse the first equation (3.17) that comprise Fronsdal equations with quadratic corrections. These corrections are of the four types: $[\omega, \omega]_s$ term which is completely fixed by HS symmetry algebra; gauge-dependent contribution $Q (C, \omega)$ which is local from the very beginning because $\omega$ is a polynomial in $Y$ of restricted degree for any fixed spin; $\Gamma_{s<s_1+s_2} (J)$ being the current deformation in gauge-dependent sector inside the triangle inequality $s < s_1 + s_2$; $\Gamma^{can} (J)$ which is gauge-invariant current deformation outside the triangle inequality, $s \geq s_1 + s_2$. It is this last contribution that we are interested in.

Now we convert all objects to 0-forms expanding them in terms of vierbeins

$$\omega_{m,n} = h^{\alpha\beta} \omega_{m,n|\alpha\beta}, \quad D^L = h^{\alpha\beta} D_{\alpha\beta}.$$  (3.20)

Then one can rewrite a relevant sector of (3.17) describing current contribution to spin-$s$ field e.o.m. as [14]

$$D_{\alpha\beta} \omega_{s-2,s|\alpha\beta} = -\bar{y}^{\dot{\beta}} \partial_{\alpha} \omega_{s-1,s-1|\alpha|\beta} - y_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}} \omega_{s-2,s-1|\alpha\beta} + \partial_{\alpha} \partial_{\alpha} \mathcal{J}_{s,s},$$  (3.21)

$$D_{\beta\dot{\alpha}} \omega_{s-2,s|\alpha\beta} = -y^{\dot{\beta}} \partial_{\dot{\alpha}} \omega_{s-1,s-1|\beta\dot{\alpha}} - \bar{y}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}} \omega_{s+1,s-3|\beta\dot{\alpha}} + \bar{\partial}_{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}} \mathcal{J}_{s,s},$$  (3.22)

where

$$\mathcal{J}_{s,s} = \frac{i (s-2)!}{8 (2s)!} \sum_{k,m=0}^{s} \frac{(m+k)! (2s-m-k)!}{(s-k)! (s-m)! m!} (y^\alpha \partial^1_\alpha)^m (-y^{\dot{\beta}} \partial^2_{\dot{\beta}})^{s-m} (\bar{y}^{\dot{\gamma}} \partial^1_{\dot{\gamma}})^{s-k} (-\bar{y}^{\dot{\beta}} \partial^2_{\dot{\beta}})^k \left\{ \sum_{n=0}^{s} \frac{n^n}{(s+n-1)!} \left[ (\partial_1^1 \partial_2^{2\gamma})^n + (\partial_1^1 \partial_2^{2\gamma})^n \right] C (Y^1|K|x) C (Y^2|K|x) \right\} \bigg|_{y^1 = y^2 = 0}.$$  (3.23)
4 Currents contribution to Fronsdal equations

Our goal is to develop an explicit expression for quadratic corrections to Fronsdal equations that are generated by (3.21)-(3.22). Double-traceless field of spin-$s$ is described in terms of spinors as $\omega_{\alpha(s-1),\dot{\alpha}(s-1)|\beta}\dot{\beta}$. We make use of the fact that the currents in question are conformal [24], so we can keep track only of totally traceless (in Lorentz tensor language) components of the Fronsdal fields, that in spinor language corresponds to totally symmetric spinor-tensors $\phi_{\alpha(s),\dot{\alpha}(s)}$

$$\phi_{s,s} := \omega_{s-1,s-1|\beta}\dot{\beta}y^{\beta}\dot{y}_{\dot{\beta}}.$$  \hspace{1cm} (4.1)

Next, as we are on-shell we can take our fields to be transverse

$$D_{\alpha\dot{\beta}}\partial^\alpha\bar{\partial}^\dot{\beta}\phi_{s,s} = 0.$$ \hspace{1cm} (4.2)

Finally, an important fact is that although the full nonlinear HS theory does not admit a flat limit, cubic couplings we are studying do admit it (for Fradkin-Vasiliev $2-s-s$ vertex [25, 26] this was shown in [27]; see also [28]). So we can take a flat limit in our equations and consider derivatives to be commuting

$$[D_{\alpha}, D_{\beta\dot{\gamma}}] = 0.$$ \hspace{1cm} (4.3)

In order to do this we rescale HS fields as follows

$$\omega_{m,n} \rightarrow \lambda^{\frac{|m-n|}{2}}\omega_{m,n}, \quad C_{m,n} \rightarrow \lambda^{-\frac{m+n}{2}}C_{m,n}. \hspace{1cm} (4.4)$$

For rescaled fields the flat limit $\lambda \rightarrow 0$ turn covariant derivatives to

$$D_{\alpha\omega} (Y[K]x) \rightarrow D^L\omega + h^{\alpha\dot{\beta}}y_\alpha\bar{\partial}_\dot{\beta}\omega_+ + h^\alpha\partial_\alpha\bar{y}_\dot{\beta}\omega_+,$$ \hspace{1cm} (4.5)

$$D_{\alpha\omega} C (Y[K]x) \rightarrow D^L C + ih^{\alpha\dot{\beta}}\partial_\alpha\bar{\partial}_\dot{\beta}C,$$ \hspace{1cm} (4.6)

where $D^L$ and $h^{\alpha\dot{\beta}}$ are Lorentz-covariant derivative and vierbein of Minkowski space-time. Then one substitutes (4.5)-(4.6) into (3.5)-(3.6) and gets linear equations for HS fields in flat space-time.

$$D^L\omega (Y[K]x) + h^{\alpha\dot{\beta}}y_\alpha\bar{\partial}_\dot{\beta}\omega_+ (Y[K]x) + h^\alpha\partial_\alpha\bar{y}_\dot{\beta}\omega_+ (Y[K]x) = \frac{i}{4}\eta H^{\alpha\dot{\beta}}\partial_\alpha\bar{\partial}_\dot{\beta}C (0, \bar{y}[K]x) k +$$

$$+\frac{i}{4}\eta H^{\alpha\dot{\beta}}\partial_\alpha\partial_\dot{\beta}C (y, 0[K]x) k,$$ \hspace{1cm} (4.7)

$$D^L C (Y[K]x) + ih^{\alpha\dot{\beta}}\partial_\alpha\bar{\partial}_\dot{\beta}C (Y[K]x) = 0.$$ \hspace{1cm} (4.8)

Now the first step is to express $C$ fields in (3.23) via derivatives of Fronsdal fields. To this end one rewrites (4.7) in terms of 0-forms

$$D_{\alpha\dot{\beta}}\omega_{n,m|\beta\dot{\gamma}} = -y^{\dot{\beta}}\bar{\partial}_{\dot{\gamma}}(\omega_{-})_{n-1,m+1|\beta\dot{\gamma}} - \partial^\beta\bar{y}_\dot{\beta}(\omega_+)(n+1,m-1|\beta\dot{\gamma}) + \frac{i}{2}\eta\delta_{n,0}\partial_\alpha\bar{\partial}_\dot{\beta}C_{0,m+2}k; \hspace{1cm} (4.9)$$

$$D_{\alpha\dot{\beta}}\omega_{n,m|\alpha\dot{\gamma}} = -y_\alpha\bar{\partial}_{\dot{\gamma}}(\omega_{-})_{n-1,m+1|\alpha\dot{\gamma}} - \partial^\dot{\gamma}\bar{y}_\dot{\gamma}(\omega_+)(n+1,m-1|\alpha\dot{\gamma}) + \frac{i}{2}\eta\delta_{n,0}\partial_\alpha\partial_\dot{\beta}C_{n+2}k.$$ \hspace{1cm} (4.10)

Contracting (4.9) with $\bar{y}^{\dot{\alpha}}\bar{y}_{\dot{\alpha}}$ and (4.10) with $y^{\alpha}y_\alpha$ yields

$$\bar{y}^{\dot{\alpha}}D_{\alpha\dot{\alpha}}\partial^\alpha\phi_{n,m} = n \cdot m (\phi_-)(n-1,m+1) - \frac{i}{2}\eta\delta_{n,1}m(m+1)C_{0,m+1}k; \hspace{1cm} (4.11)$$

$$y^{\alpha}D_{\alpha\dot{\alpha}}\partial^\alpha\phi_{n,m} = n \cdot m (\phi_+)(n+1,m-1) - \frac{i}{2}\eta\delta_{m,1}n(n+1)C_{n+1}k.$$ \hspace{1cm} (4.12)
From this one finds

\[
C_{2s,0} = \frac{2i\eta}{s \cdot (2s)!} (y^\alpha D_{\alpha\hat{\alpha}} \tilde{\phi})^s \phi_{s,\bar{k}},
\]

(4.13)

\[
C_{0,2s} = \frac{2i\bar{\eta}}{s \cdot (2s)!} (\tilde{y}^\hat{\alpha} D_{\alpha\hat{\alpha}} \phi)^s \phi_{s,k}.
\]

(4.14)

Then (4.8) gives

\[
C_{2s+d,d} = \frac{2\eta \cdot i^{d+1}}{s \cdot (2s + d)!d!} (y^\beta D_{\beta\hat{\beta}} \tilde{y}^\hat{\beta})^d (y^\alpha D_{\alpha\hat{\alpha}} \tilde{\phi})^s \phi_{s,\bar{k}},
\]

(4.15)

\[
C_{d,2s+d} = \frac{2\bar{\eta} \cdot i^{d+1}}{s \cdot (2s + d)!d!} (\tilde{y}^\hat{\alpha} D_{\alpha\hat{\alpha}} \phi)^d (\tilde{y}^\hat{\beta} D_{\beta\hat{\beta}} \tilde{\phi})^s \phi_{s,k}.
\]

(4.16)

Now one contracts (3.21) with \(y^\alpha y^\alpha\) (or (3.22) with \(\tilde{y}^\hat{\alpha} \tilde{y}^\hat{\alpha}\)) and makes use of (4.12) (or (4.11)) to obtain

\[
\Box \phi_{s,\bar{k}} + \ldots = -s^2 (s - 1) J_{s,s} + \ldots,
\]

(4.17)

where \(\Box = \frac{1}{2} D_{\alpha\hat{\alpha}} D^{\alpha\hat{\alpha}}\), ellipsis on the l.h.s. denotes other terms of Fronsdal kinetic operator besides the box and ellipsis on the r.h.s. denotes other (gauge-noninvariant) sources generated by \(Q(C, \omega)\) and \(\Gamma_{s<s_1+s_2} (J)\) in (3.17).

Now let us consider the current (3.23). We want to extract the term describing \(s - s_1 - s_2\) vertex. A simple counting shows that two kind of terms are presented in (3.23): either two co-directional helicities are coupled (\(C_+ C_+\) or \(C_- C_-\)), then the term has \((s + s_1 + s_2)\) derivatives, or two opposite ones (\(C_+ C_-\) or \(C_- C_+\)), then total number of derivatives is \((s + |s_1 - s_2|)\) (let us remind that we are in \(s \geq s_1 + s_2\) sector). This corresponds to two types of 4d cubic HS vertices found in [29]. Altogether this means there are no higher-derivative improvements to vertices of [29], which could, for instance, affect locality issue in higher orders. (Note that lower-derivative improvements to \((s + s_1 + s_2)\)-term cannot contribute to \((s + |s_1 - s_2|)\)-term because they have different helicity structure.) We will analyse two vertices separately.

### 4.1 Maximal-derivative part

First, let us consider the part of (3.23) with \((s + s_1 + s_2)\) derivatives. This looks as follows

\[
\mathcal{J}_{s-s_1-s_2}^H = i \frac{(s - 2)!}{8(2s)!} \sum_{k,m=0}^{s} \frac{(m + k)! (2s - m - k)!}{(s - k)! (s - m)! m!} (y^\alpha \partial^1_\alpha m (-y^\beta \partial^2_\beta)^s - m (\tilde{y}^\hat{\alpha} \partial^1_{\hat{\alpha}})^s - k (-\tilde{y}^\hat{\beta} \partial^2_{\hat{\beta}})^k
\]

\[
\sum_{n=0}^{s} \frac{i^n}{(s + n - 1)!} \left( (\partial^1_\alpha \partial^2_\gamma)^n + (\tilde{\partial}^1_{\hat{\alpha}} \tilde{\partial}^2_{\hat{\beta}})^n \right) \sum_{d_1,d_2=0}^\infty \left\{ C_{2s_1+d_1,d_1} (Y^1|K|x) C_{2s_2+d_2,d_2} (Y^2|K|x) + C_{2s_2+d_2,d_2} (Y^1|K|x) C_{2s_1+d_1,d_1} (Y^2|K|x) + C_{d_1,2s_1+d_1} (Y^1|K|x) C_{d_2,2s_2+d_2} (Y^2|K|x) + C_{d_2,2s_2+d_2} (Y^1|K|x) C_{d_1,2s_1+d_1} (Y^2|K|x) \right\} \bigg|_{Y^1=Y^2=0}.
\]

(4.18)
That spins $s_1, s_2$ of constituent fields are fixed and all $Y^1$ and $Y^2$ are eventually put to zero reduces the fivefold sum in (4.18) to the single one:

$$\mathcal{J}^H_{s_1-s_2} = \frac{i(s-2)!}{8(2s)!} \sum_{d=0}^{s} \frac{(s+s_1-s_2)! (s-s_1+s_2)!}{(s-d)!d! (-s_1+s_2+d)! (s+s_1-s_2-d)!} \left( y^\alpha \partial_\alpha \right)^{s+s_1-s_2-d} \left( y^\beta \partial_\beta \right)^{s_1+s_2+d} \left( y^\delta \partial_\delta \right)^{s-d} \left( y^\gamma \partial_\gamma \right)^{d} \left( -1 \right)^{s+d} \left( 1 + (-1)^{s+s_1+s_2} \right) \frac{\delta^{s_1+s_2} (-1)^{s+d} (1 + (-1)^{s+s_1+s_2})}{(s+s_1+s_2-1)!} \left( \partial_\gamma ^2 \right)^{s_1+s_2} C_{2s_1+s-d,s-d} \left( Y^1|x \right) C_{2s_2+d,d} \left( Y^2|x \right) + h.c. \right) \bigg|_{Y^1-Y^2=0} \quad (4.19)$$

(here we resolved $K$-dependence using (2.10)), that after evaluating derivatives from the second line yields

$$\mathcal{J}^H_{s_1-s_2} = \frac{(s-2)! (s+s_1-s_2)! (s-s_1+s_2)!}{8(2s)! (s+s_1+s_2-1)!} i^{s_1+s_2+1} \left( 1 + (-1)^{s+s_1+s_2} \right) \cdot \sum_{d=0}^{s} (-1)^{s+d} \left\{ (\partial_\gamma)^{s_1+s_2} C_{2s_1+s-d,s-d} \left( Y^1|x \right) \cdot (\partial_\gamma)^{s_1+s_2} C_{2s_2+d,d} \left( Y^2|x \right) + \right. $$

$$+ \left. (\partial_\gamma)^{s_1+s_2} C_{2s_1+s-d,s-d} \left( Y^1|x \right) \cdot (\partial_\gamma)^{s_1+s_2} C_{2s_2+d,d} \left( Y^2|x \right) \right\} \quad (4.20)$$

Note that due to $\left( 1 + (-1)^{s+s_1+s_2} \right)$ factor, (4.20) vanishes if the total sum of spins is odd. In fact, this is because we have only one field of every spin, similarly to the electrodynamics where one needs two copies of the matter fields to have a nonzero electric current. So if one considers matrix-valued HS fields, the contribution would be nonzero.

Now let us analyse the spinorial expression in (4.20). Our goal is to bring it to the form that can be simply re-expressed in terms of Lorentz tensors.

First, we use (4.15) to rewrite it as

$$\left\{ (\partial_\gamma)^{s_1+s_2} \left( y^\alpha D_{\alpha\alpha} \gamma^\mu \right)^{s-d} \left( y^\alpha D_{\alpha\alpha} \gamma^\mu \right)^{s_1} \phi_{s_1,s_1} \right\} \left\{ (\partial_\gamma)^{s_1+s_2} \left( y^\beta D_{\beta\beta} \gamma^\mu \right)^{s_2} \phi_{s_2,s_2} \right\} \cdot \left( 2s_1 + s - d \right)! (2s_2 + d)! (s-d)!d! \cdot s_1 \cdot s_2$$

$$\left\{ (\partial_\gamma)^{s_1+s_2} \left( y^\alpha D_{\alpha\alpha} \gamma^\mu \right)^{s-d} \left( y^\alpha D_{\alpha\alpha} \gamma^\mu \right)^{s_1} \phi_{s_1,s_1} \right\} \left\{ (\partial_\gamma)^{s_1+s_2} \left( y^\beta D_{\beta\beta} \gamma^\mu \right)^{s_2} \phi_{s_2,s_2} \right\} = \quad (4.21)$$

Evaluating spinorial derivatives gives

$$\left\{ (\partial_\gamma)^{s_1+s_2} \left( y^\alpha D_{\alpha\alpha} \gamma^\mu \right)^{s-d} \left( y^\alpha D_{\alpha\alpha} \gamma^\mu \right)^{s_1} \phi_{s_1,s_1} \right\} \left\{ (\partial_\gamma)^{s_1+s_2} \left( y^\beta D_{\beta\beta} \gamma^\mu \right)^{s_2} \phi_{s_2,s_2} \right\} = \quad$$

$$\frac{s_1!s_2! (s+2s_1-d)! (2s_2 + d)!}{(s+s_1-s_2-d)! (-s_1+s_2+d)!} \left\{ (\delta_\mu)^{s_1+s_2} \left( y^\mu \right)^{s+s_1-s_2-d} \left( y^\nu \right)^{s-d} \left( D_{\mu\nu} \right)^{s_1} \phi_{s_1,s_1} \right\} \left\{ (\gamma_{\nu})^{s_1+s_2} \gamma^\nu \right\} \left\{ (\delta_\nu)^{s_1+s_2} \left( y^\nu \right)^{s+s_1-s_2+d} \left( y^\nu \right)^{d \left( D_{\nu\nu} \right)^{s_1} \phi_{s_2,s_2} \right\} \right\} \quad (4.22)$$

Due to symmetrisation over $\mu$ (and over $\nu$), $\gamma$ indices after applying $(\delta_\gamma)^{s_1+s_2}$ and $(\epsilon_{\gamma\nu})^{s_1+s_2}$ will hang symmetrically on fields and derivatives. But we are going to arrange gammas in some particular order. To this end we establish some useful relations. The first is (we write down only relevant indices)

$$D_{\alpha\gamma} \gamma^\mu = D_{\beta\gamma} \gamma^\mu + \epsilon_{\alpha\beta} D_{\gamma\gamma} \gamma^\mu \approx D_{\beta\gamma} \gamma^\mu \approx \epsilon_{\alpha\beta} D_{\gamma\gamma} \gamma^\mu \quad (4.23)$$
where the approximate equality symbol means that we lopped off a divergence of the field, as we neglect it in our problem. The second is

$$D_{\alpha\hat{\alpha}}D_{\beta\hat{\beta}}\phi \approx D_{\alpha\hat{\beta}}D_{\beta\hat{\gamma}}\phi,$$

(4.24)

which is modulo boxes (that can be redefined away) and terms with $D_{\alpha\hat{\alpha}}D_{\beta\hat{\beta}}$ $(D_{\beta\hat{\alpha}}D_{\alpha\hat{\beta}})$, which are zeros in flat space. Using these two relations, we obtain the third one

$$D_{\gamma\hat{\gamma}}D_{\alpha\hat{\beta}}\phi_{\beta} \approx D_{\beta\hat{\gamma}}D_{\alpha\hat{\beta}}\phi_{\gamma}. $$

(4.25)

Altogether they imply that we are free to put gammas on any places instead of lower $\mu$ ($\nu$) indices in (4.22) as all combinations are equivalent. So, assuming for definiteness $s_1 \geq s_2$, we rewrite (4.22) as

$$\frac{s_1!s_2!(s + 2s_1 - d)! (2s_2 + d)!}{(s + s_1 - s_2 - d)! (-s_1 + s_2 + d)!} (y^{\mu})^{s} (\bar{y}^{\nu})^{s} \left\{ (D_{\mu\hat{\nu}})^{s-d} (D_{\mu\hat{\beta}})^{s_1-s_2} (D_{\gamma\hat{\beta}})^{s_2} \phi_{\nu(s_1), \hat{\beta}(s_1)} \right\},$$

(4.26)

and, using (4.23), further as

$$\frac{s_1!s_2!(s + 2s_1 - d)! (2s_2 + d)!}{(s + s_1 - s_2 - d)! (-s_1 + s_2 + d)!} (y^{\mu})^{s} (\bar{y}^{\nu})^{s} \left\{ (D_{\mu\hat{\nu}})^{s-d} (D_{\mu\hat{\beta}})^{s_1-s_2} (D_{\delta\hat{\beta}})^{s_2} \phi_{\nu(s_1), \hat{\beta}(s_1)} \right\},$$

(4.27)

Now we want to replace all lower $\hat{\beta}$ in $(D_{\delta\hat{\beta}})^{s_2}$ in the first line with lower $\hat{\alpha}$ so as to make these derivatives to be entirely contracted with the spin-$s_2$ field. We can perform this with the help of $D_{\mu\hat{\beta}}$ in the first line, because

$$D_{\mu\hat{\beta}}D_{\delta\hat{\beta}}\phi_{\delta\hat{\beta}} = D_{\mu\hat{\beta}}D_{\delta\hat{\beta}}\phi_{\delta\hat{\beta}},$$

$$D_{\mu\hat{\beta}}D_{\delta\hat{\beta}}\phi_{\delta\hat{\beta}} = D_{\mu\hat{\beta}}D_{\delta\hat{\beta}}\phi_{\delta\hat{\beta}} + D_{\mu\hat{\beta}}D_{\delta\hat{\beta}}\phi_{\delta\hat{\beta}} + D_{\mu\hat{\beta}}D_{\delta\hat{\beta}}\phi_{\delta\hat{\beta}} \approx$$

$$\approx D_{\mu\hat{\beta}}D_{\delta\hat{\beta}}\phi_{\delta\hat{\beta}} + D_{\mu\hat{\beta}}D_{\delta\hat{\beta}}\phi_{\delta\hat{\beta}} \approx D_{\mu\hat{\beta}}D_{\delta\hat{\beta}}\phi_{\delta\hat{\beta}} + D_{\mu\hat{\beta}}D_{\delta\hat{\beta}}\phi_{\delta\hat{\beta}} \approx$$

(4.28)

where at the penultimate step we used that

$$D_{\alpha\hat{\gamma}}\phi \cdot D_{\beta\hat{\beta}}\phi = D_{\beta\hat{\gamma}}\phi \cdot D_{\alpha\hat{\beta}}\phi + \epsilon_{\alpha\beta}D_{\gamma\hat{\beta}}\phi \cdot D_{\gamma\hat{\beta}}\phi =$$

$$= D_{\beta\hat{\gamma}}\phi \cdot D_{\alpha\hat{\beta}}\phi + \epsilon_{\alpha\beta}(\square \phi \cdot \phi) \approx$$

$$\approx D_{\beta\hat{\gamma}}\phi \cdot D_{\alpha\hat{\beta}}\phi, $$

(4.29)

and at the last step that

$$D_{\alpha\hat{\alpha}}D_{\beta\hat{\beta}}\phi_{\delta\hat{\beta}} \approx D_{\beta\hat{\alpha}}D_{\alpha\hat{\beta}}\phi_{\beta\hat{\beta}} \approx 0.$$

(4.30)

Thus (4.27) turns into

$$\frac{s_1!s_2!(s + 2s_1 - d)! (2s_2 + d)!}{(s + s_1 - s_2 - d)! (-s_1 + s_2 + d)!} (y^{\mu})^{s} (\bar{y}^{\nu})^{s} \left\{ (D_{\mu\hat{\nu}})^{s-d} (D_{\mu\hat{\beta}})^{s_1-s_2} (D_{\delta\hat{\beta}})^{s_2} \phi_{\nu(s_1), \hat{\beta}(s_1)} \right\},$$

(4.31)
Now we want to exchange \( \dot{\beta} \) in \((D_{\mu\beta})^{s_1-s_2}\) in the first line with \( \dot{\mu} \) in \((D_{\gamma\dot{\mu}})^{s_1-s_2}\) in the second line. This can be done by virtue of a relation, similar to (4.28):

\[
D_{\mu\beta} D_{\mu\dot{\beta}} \phi_{\dot{\gamma}\dot{\beta}} \cdot D_{\gamma\dot{\mu}} D_{\mu\dot{\mu}} \phi_{\gamma\dot{\beta}} \cdot D_{\gamma\dot{\alpha}} \phi_{\gamma\dot{\beta}} \cdot D_{\dot{\alpha}} \phi \approx
\approx D_{\mu\beta} D_{\mu\dot{\beta}} \phi_{\dot{\gamma}\dot{\beta}} \cdot D_{\gamma\dot{\mu}} D_{\mu\dot{\mu}} \phi_{\gamma\dot{\beta}} \cdot D_{\gamma\dot{\alpha}} \phi \approx D_{\mu\beta} D_{\mu\dot{\beta}} \phi_{\dot{\gamma}\dot{\beta}} \cdot D_{\gamma\dot{\mu}} D_{\mu\dot{\mu}} \phi_{\gamma\dot{\beta}} \cdot D_{\gamma\dot{\alpha}} \phi.
\] (4.32)

This allows us to perform all necessary exchanges in \((D_{\mu\beta})^{s_1-s_2}\) except for the last one, because to use (4.32) we need at least two \(D_{\mu\beta}\). So we have

\[
\frac{s_1! s_2!(s+2s_1-d)! (2s_2+d)!}{(s+s_1-s_2-d)! (-s_1+s_2+d)!} (y^\mu)^s (\tilde{y}^\dot{\mu})^s \left\{ D_{\mu\beta} (D_{\mu\dot{\mu}})^{s+s_1-s_2-d-1} (D_{\alpha\dot{\alpha}})^{s_2} \phi_{\beta(s_1), \dot{\beta}(s_1)} \right\}
\left\{ D_{\beta\dot{\beta}} (D_{\alpha\dot{\alpha}})^{s_1-1} \phi_{\alpha(s_2), \dot{\alpha}(s_2)} \right\},
\] (4.33)

and the last exchange leads to the expression of the form

\[
\frac{s_1! s_2!(s+2s_1-d)! (2s_2+d)!}{(s+s_1-s_2-d)! (-s_1+s_2+d)!} (y^\mu)^s (\tilde{y}^\dot{\mu})^s (-1)^{s_1}
\left\[ (D_{\mu\dot{\mu}})^{s+s_1-s_2-d} (D_{\alpha\dot{\alpha}})^{s_2} \phi_{\dot{\beta}(s_1), \beta(s_1)} \cdot (D_{\mu\dot{\mu}})^{-s_1+s_2+d} (D_{\beta\dot{\beta}})^{s_1} \phi_{\alpha(s_2), \dot{\alpha}(s_2)} -
- (D_{\mu\dot{\mu}})^{s+s_1-s_2-d} (D_{\alpha\dot{\alpha}})^{s_2} \phi_{\dot{\beta}(s_1), \beta(s_1)} \cdot (D_{\mu\dot{\mu}})^{-s_1+s_2+d} (D_{\beta\dot{\beta}})^{s_1-1} \phi_{\alpha(s_2), \dot{\alpha}(s_2)} \right\].
\] (4.34)

Now, substituting (4.34) for the second line in (4.21), then (4.21) in (4.20), adding conjugate expression and simplifying, one gets

\[
\mathcal{J}^H_{s_1-s_2} = -i \frac{(s-2)!}{4(2s)!} \sum_{d=0}^{s} \binom{s+s_1-s_2}{d} \binom{s-s_1+s_2}{s-d} (s_1-1)! (s_2-1)!
\]

\[
\frac{(\eta^2 + \bar{\eta}^2)(D_{\mu\dot{\mu}})^{s+s_1-s_2-d} (D_{\alpha\dot{\alpha}})^{s_2} \phi_{\dot{\beta}(s_1), \beta(s_1)} \cdot (D_{\mu\dot{\mu}})^{-s_1+s_2+d} (D_{\beta\dot{\beta}})^{s_1} \phi_{\alpha(s_2), \dot{\alpha}(s_2)} +
+ (\eta^2 - \bar{\eta}^2) \left[ D_{\mu\dot{\beta}} (D_{\mu\dot{\mu}})^{s+s_1-s_2-d-1} (D_{\alpha\dot{\alpha}})^{s_2} \phi_{\dot{\beta}(s_1), \beta(s_1)} \cdot D_{\beta\dot{\mu}} (D_{\mu\dot{\mu}})^{-s_1+s_2+d} (D_{\beta\dot{\beta}})^{s_1-1} \phi_{\alpha(s_2), \dot{\alpha}(s_2)} -
- D_{\beta\dot{\mu}} (D_{\mu\dot{\mu}})^{s+s_1-s_2-d-1} (D_{\alpha\dot{\alpha}})^{s_2} \phi_{\dot{\beta}(s_1), \beta(s_1)} \cdot D_{\mu\dot{\beta}} (D_{\mu\dot{\mu}})^{-s_1+s_2+d} (D_{\beta\dot{\beta}})^{s_1-1} \phi_{\alpha(s_2), \dot{\alpha}(s_2)} \right]\].
\] (4.35)

Then, using Vandermonde's identity

\[
\sum_{n=0}^{c} \binom{a}{n} \binom{b}{c-n} = \binom{a+b}{c}
\] (4.36)
and integrating by parts one can perform a summation over \( d \) explicitly, reducing (4.35) to

\[
\mathcal{J}_{s-s_1-s_2}^H = -i^{s+s_1+s_2+1}(s-2)! (s_1-1)! (s_2-1)! \left(1 + (-1)^{s_1} \right) (y^\mu)^s (\dot{y}^\nu)^s \frac{4 \cdot s!}{(s + s_1 + s_2 - 1)!} \left\{ (\eta^2 + \tilde{\eta}^2) (D_{\mu \nu})^s (D_{\alpha \alpha})^{s_2} \phi^\beta(s_1), \tilde{\phi}^{\dot{\beta}}(s_1), (D_{\beta \dot{\beta}})^{s_1} \phi^{\alpha(s_2), \dot{\alpha}(s_2)} + \right.
\]

\[
+ (\eta^2 - \tilde{\eta}^2) \left[ D_{\mu \beta} (D_{\mu \dot{\nu}})^{s-1} (D_{\alpha \alpha})^{s_2} \phi^\beta(s_1), \tilde{\phi}^{\dot{\beta}}(s_1), D_{\dot{\beta} \dot{\nu}} (D_{\beta \beta})^{s_1-1} \phi^{\alpha(s_2), \dot{\alpha}(s_2)} - \right.
\]

\[
- D_{\mu \dot{\nu}} (D_{\mu \dot{\nu}})^{s-1} (D_{\alpha \alpha})^{s_2} \phi^\beta(s_1), \tilde{\phi}^{\dot{\beta}}(s_1), D_{\mu \dot{\nu}} (D_{\beta \beta})^{s_1-1} \phi^{\alpha(s_2), \dot{\alpha}(s_2)} \right\}. \tag{4.37}
\]

Here we reached our goal because this expression can easily be translated into Lorentz tensors as we show below. Now we are going to process another part of the current (3.23) that contains \((s + |s_1 - s_2|)\) derivatives.

### 4.2 Minimal-derivative part

Analysis of the minimal-derivative part of (3.23) which has the form

\[
\mathcal{J}_{s-s_1-s_2}^L = \frac{i (s-2)!}{8 (2s)!} \sum_{m,n=0}^s \frac{(s+k)! (2s-m-k)!}{(s-k)! (s-m)! m!} (y^\alpha \phi_\alpha^1)^m (-y^\beta \phi_\beta^2)^s-m (\dot{y}^\alpha \dot{\phi}_\alpha^1)^s-k (-\dot{y}^\beta \dot{\phi}_\beta^2)^k
\]

\[
\sum_{n=0}^s \frac{i^n}{(s+n-1)!} \left( (\partial_\gamma \partial^2 \gamma)^n + (\tilde{\partial}_\gamma \tilde{\partial}^2 \gamma)^n \right) \sum_{d_1, d_2 = 0}^\infty C_{2s_1 + d_1, d_1} (Y^1 | K | x) C_{d_2, 2s_2 + d_2} (Y^2 | K | x) +
\]

\[
+ C_{2s_2 + d_2, d_2} (Y^1 | K | x) C_{d_1, 2s_1 + d_1} (Y^2 | K | x) + C_{d_1, 2s_1 + d_1} (Y^1 | K | x) C_{2s_2 + d_2, d_2} (Y^2 | K | x) +
\]

\[
+ C_{d_2, 2s_2 + d_2} (Y^1 | K | x) C_{d_1, 2s_1 + d_1} (Y^2 | K | x) \right\} \bigg|_{Y^1 = Y^2 = 0}. \tag{4.38}
\]

practically repeats analysis of the maximal-derivative one.

First, one evaluates derivatives from the first line of (4.38) and simplifies the expression to

\[
\mathcal{J}_{s-s_1-s_2}^L = \frac{(s-2)!}{8 \cdot (2s)!} \frac{(s + s_1 + s_2)! (s - s_1 - s_2)!}{(s + s_1 - s_2 - 1)!} \left(1 + (-1)^{s+s_1+s_2} \right) .
\]

\[
\sum_{d=0}^s (-1)^{s+d+s_1} \left\{ (\partial_\gamma)^{s_1-s_2} C_{2s_1+d,d} (Y | x) \cdot (\tilde{\partial}_\gamma)^{s_1-s_2} C_{s-2s_2-d,s-d} (Y | x) +
\]

\[
+ (\tilde{\partial}_\gamma)^{s_1-s_2} C_{d,2s_1+d} (Y | x) \cdot (\tilde{\partial}_\gamma)^{s_1-s_2} C_{s-d,s-2s_2-d} (Y | x) \right\}. \tag{4.39}
\]

Then, using (4.15), one rewrites the first term in brackets in (4.39) as

\[
(\partial_\gamma)^{s_1-s_2} C_{2s_1+d,d} (Y | x) \cdot (\tilde{\partial}_\gamma)^{s_1-s_2} C_{s-2s_2-d,s-d} (Y | x) = - \frac{4i^s (-1)^{s_2} (s_1-1)! (s_2-1)!}{(s_1 + s_2 + d)! (s - s_1 - s_2 - d)! (s - d)!d!} .
\]

\[
\cdot \left\{ (\delta_\gamma)^{s_1-s_2} (y^\mu)^{s_1+s_2+d} (\dot{y}^\nu)^d (D_{\mu \nu})^{s_1} \phi_{\mu(s_1), \dot{\alpha}(s_1)} \right\} .
\]

\[
\cdot \left\{ (\epsilon_\gamma)^{s_1-s_2} (y^\nu)^{s-s_1-s_2-d} (\dot{y}^\nu)^{s_1-s_2-d} (D_{\nu \nu})^{s_2} (D_{\nu \nu})^{s_2} \phi_{\alpha(s_2), \dot{\nu}(s_2)} \right\} . \tag{4.40}
\]
As in Section 4.1 by means of (4.23)-(4.25) one hangs all gammas in the second line of (4.40) on spin-$s$ field

\[
(\partial_\nu)^{s_1-s_2} C_{2s_1+d,d} (Y|x) \cdot (\partial_\gamma)^{s_1-s_2} C_{s-2s_2-d,d} (Y|x) = \frac{4i^s(-1)^{s_1}(s_1-1)!(s_2-1)!}{(s_1+s_2+d)!(s-s_1-s_2-d)!(s-d)!d!} \cdot (y^\mu)^s \left( (D_{\mu\dot{\mu}})^d (D_{\dot{\mu}\dot{\beta}})^{s_1} \phi_{\mu(s_2)}(2,\beta(s_1)) \right) \left( (D_{\gamma\dot{\mu}})^{s_1-s_2} (D_{\mu\dot{\mu}})^{s-s_1-s_2-d} (D_{\alpha\dot{\mu}})^{s_2} \phi^{\alpha(s_2)}(\mu(s_2)) \right),
\]

and exchanges $(s_1 - s_2)$ pieces of $\beta$ of $D_{\mu\dot{\mu}}$ in the first bracket with $\mu$ of $D_{\gamma\dot{\mu}}$ from the second bracket

\[
\left\{ (D_{\mu\dot{\mu}})^d (D_{\dot{\mu}\dot{\beta}})^{s_1} \phi_{\mu(s_2)}(2,\beta(s_1)) \right\} \left\{ (D_{\gamma\dot{\mu}})^{s_1-s_2} (D_{\mu\dot{\mu}})^{s-s_1-s_2-d} (D_{\alpha\dot{\mu}})^{s_2} \phi^{\alpha(s_2)}(\mu(s_2)) \right\} = \left\{ (D_{\mu\dot{\mu}})^{s_1-s_2+d} (D_{\dot{\mu}\dot{\beta}})^{s_2} \phi_{\mu(s_2)}(2,\beta(s_1)) \right\} \left\{ (D_{\gamma\dot{\mu}})^{s_1-s_2} (D_{\mu\dot{\mu}})^{s-s_1-s_2-d} (D_{\alpha\dot{\mu}})^{s_2} \phi^{\alpha(s_2)}(\mu(s_2)) \right\}.
\]

Substituting all this into (4.39), adding conjugate term and allowing for

\[
D_{\mu\dot{\mu}}^\mu \phi^\alpha \phi_{\mu(\beta)} = D_{\mu\dot{\mu}}^\alpha \phi^\alpha \phi_{\mu(\beta)} - D_{\mu\dot{\mu}}^\mu \phi^\beta \phi_{\mu(\beta)} - D_{\mu\dot{\mu}}^\mu \phi^\alpha \phi_{\mu(\beta)} - D_{\mu\dot{\mu}}\phi^\alpha \phi_{\mu(\beta)}
\]

leads to the following expression for the minimal-derivative part of the current

\[
\mathcal{J}^L_{s-s_1-s_2} = -i\left( s - 2 \right)! \frac{i^{s+s_1+s_2} \left( -1 \right)^{s+s_1+s_2}}{(2s)!} \frac{(s-1)!}{(s+s_1-s_2-1)!} \cdot \sum_{d=0}^{s} (-1)^d \left( \begin{array}{cc} s + s_1 + s_2 \\ s - d \\ d \\ \end{array} \right) \left( y^\mu \right)^s \left( \bar{y}^{\dot{\mu}} \right)^s \left( D_{\mu\dot{\mu}} \right)^d \phi^{\alpha(s_1),\dot{\alpha}(s_1)}
\]

\[
\cdot \sum_{n=0}^{s_2} (-1)^n \left( \begin{array}{cc} s_2 \\ n \\ \end{array} \right) \left( D_{\mu\dot{\mu}} \right)^{s-d-n} \left( D_{\alpha\dot{\alpha}} \right)^{s_1-s_2+n} \phi^{\alpha(s_2-n)\mu(n),\dot{\alpha}(s_2-n)\dot{\mu}(n)}
\]

As in the Section 4.1 using Vandermonde's identity (4.36) and integrating by parts one can evaluate the sum over $d$, obtaining

\[
\mathcal{J}^L_{s-s_1-s_2} = -\frac{(s-2)!}{s!} \frac{(s_1-1)!}{(s_1+s_2-1)!} \frac{(s_2-1)!}{(s_2+s_1-s_2-1)!} \left( 1 + (-1)^{s+s_1+s_2} \right) \left( y^\mu \right)^s \left( \bar{y}^{\dot{\mu}} \right)^s \phi^{\alpha(s_1),\dot{\alpha}(s_1)}
\]

\[
\cdot \sum_{n=0}^{s_2} (-1)^n \left( \begin{array}{cc} s_2 \\ n \\ \end{array} \right) \left( D_{\mu\dot{\mu}} \right)^{s-n} \left( D_{\alpha\dot{\alpha}} \right)^{s_1-s_2+n} \phi^{\alpha(s_2-n)\mu(n),\dot{\alpha}(s_2-n)\dot{\mu}(n)}
\]

This completes the analysis of minimal-derivative part of the current.

### 4.3 Fronsdal equations with HS current corrections

Now we are ready to make a final step and write down a current contribution to quadratic HS equations in Lorentz tensor language. From (4.17) we have

\[
\Box \phi_{\mu(s),\dot{\mu}(s)} \left( y^\mu \right)^s \left( \bar{y}^{\dot{\mu}} \right)^s + \ldots = -s^2 \sum_{s_1+s_2 \leq s} (\mathcal{J}^H_{s-s_1-s_2} + \mathcal{J}^L_{s-s_1-s_2}) + \ldots
\]

\[(4.46)\]
where $\mathcal{J}_{s-s_1-s_2}^H$ and $\mathcal{J}_{s-s_1-s_2}^L$ are given in (4.37) and (4.45) respectively. After removing twistor variables $y$ and $\bar{y}$, tensor indices are restored via $\sigma$-matrices, that gives by virtue of

$$\text{Tr} \{ \sigma_a \bar{\sigma}_b \} = 2\eta_{ab}, \quad (\sigma^a \bar{\sigma}^b \sigma^c - \sigma^c \bar{\sigma}^b \sigma^a) = 2i\epsilon^{abcd} \sigma_d$$

the following result

$$\Box \phi_{a(s)} + \ldots = \sum_{s_1+s_2 \leq s} \frac{(s_1-1)!}{(s_1+s_2+1)!} \left[ 1 + (-1)^{s_1+s_2+1} \right] \left( \begin{array}{c} 2s_1 \cr s_1+s_2 \end{array} \right) \left( s_2 \right) \left( \begin{array}{c} s_2 \cr n \end{array} \right) \mathcal{D}^{s_1 \rightarrow s_2} \mathcal{D}^{s_2 \rightarrow s_1} \phi_{a(s)} + \ldots$$

To simplify the form of this equation one can rescale fields as

$$\phi_{a(n)} \rightarrow \frac{2^{-\frac{n}{2}}}{(n-1)!^{(n+1)}} \phi_{a(n)}, \quad \text{(4.49)}$$

then (4.48) turns into

$$\Box \phi_{a(s)} + \ldots = \sum_{s_1+s_2 \leq s} \left( 1 + (-1)^{s_1+s_2+1} \right) \frac{2^{s_1+s_2}}{2} \left( \begin{array}{c} 2s_1 \cr s_1+s_2 \end{array} \right) \left( \begin{array}{c} s_2 \cr n \end{array} \right) \mathcal{D}^{s_1 \rightarrow s_2} \mathcal{D}^{s_2 \rightarrow s_1} \phi_{a(s)} + \ldots$$

where

$$k_n := 2^{-s_2} \left( \begin{array}{c} s_2 \cr n \end{array} \right), \quad \sum_{n=0}^{s_2} k_n = 1, \quad \text{(4.51)}$$

and we introduced a 'phase angle' $\varphi$

$$\eta = \exp (i\varphi). \quad \text{(4.52)}$$

Let us discuss (4.50), which is the main result of the paper, in some more details. First of all, let us remind that ellipsis on the l.h.s. denotes the rest of kinetic Fronsdal operator, while ellipsis on the r.h.s. denotes contributions in $s < s_1 + s_2$ domain and the contributions of HS currents outside the transverse-traceless (TT) sector. The non-TT part is completely fixed by the TT one, which we have found (the procedure of completion of TT part to the full Lagrangian AdS HS cubic vertex were demonstrated in [21-30]).
Next, we see that minimal-derivative part of the current (the last term in brackets) is $\varphi$-independent, while maximal-derivative part consists of two different $\varphi$-dependent terms. Term proportional to $\sin (2\varphi)$ contains Levi-Civita symbol and thus is parity-violating, so it expectedly vanishes in parity-invariant $A$- and $B$-models ($\varphi = 0$ and $\varphi = \frac{\pi}{2}$). For parity-invariant models vertices in (4.50) coincide up to a normalisation with the expressions available in the literature [20, 21], confirming the correctness of local frame of Vasiliev equations found in [14]. Another peculiar situation is $\varphi = \frac{\pi}{4}$ model. In this case the first term with $\cos (2\varphi)$ is absent, so the maximal-derivative part of the vertex is in whole proportional to Levi-Civita symbol, being somewhat of 'maximally parity-breaking'. It would be interesting to see the implication of that for dual theory, which is conjectured to be 3d Chern-Simons theory coupled to scalar fields [31, 32].

5 Conclusion

In the note we obtained quadratic corrections to bosonic Fronsdal equations generated by gauge-invariant HS currents, starting with the local second-order Vasiliev equations of [13, 14]. The result agrees with previously known expressions [20, 21] for HS cubic vertices in case of parity-invariant models. This gives an additional confirmation that the local frame of HS equations, proposed in [13, 14], is the appropriate one. For the case of $\varphi = \frac{\pi}{4}$ model we found that maximal-derivative part of the vertex is proportional to Levi-Civita symbol, being maximally parity-breaking, that may have interesting consequences for dual boundary Chern-Simons theory. It would be interesting also to study the theories with fermions as well as to find the contribution of gauge-dependent sector, that would allow one to write down the full quadratic HS equations.

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