Superposition of causal order in quantum walks: non-Markovianity and causal asymmetry

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We set the criteria for a quantum walk to exhibit nontrivial dynamics when placed in an indefinite causal order and study two-period quantum walks when the evolution operator is arranged in a causally ordered sequence and in an indefinite causal order using quantum switch. When either forward or backward causal sequence is implemented, one observes a causal asymmetry in the dynamics, in the sense that the reduced dynamics of the coin state is more non-Markovian for one particular temporal order of operations than that of the other. When the dynamics is defined using evolution operators in a superposition of causal orders, the reduced dynamics of the coin space exhibit higher non-Markovian behavior than either of the definite causal orders. This effect can be interpreted as a Parrondo-like effect in non-Markovianity of the reduced state dynamics of the coin. We further generalize the qualitative description of our results pertaining to the dynamics when the walk has a higher number of periods.

I. Introduction

In the recent years, quantum walks have gained considerable interest as efficient tool to model controlled quantum dynamics [1–7]. Much like a classical random walk, quantum walks also admit discrete-time and continuous-time realizations. The discrete-time quantum walk (DTQW) is defined on a composite Hilbert space consisting of the two, ‘coin’ and ‘position’ Hilbert space where the evolution is defined using a repeated application of quantum coin operation applied on the coin space, followed by the coin dependent position shift operation on the composite space [1]. The continuous-time quantum walk (CTQW), however, is defined solely on the position space, with the evolution operator being dependent on the adjacency matrix of the graph [8]. Both variants of quantum walks have been used to model various algorithms and schemes for quantum simulation [9–18], and for realizing universal quantum computation [19–22]. A ballistic spread in the probability distribution of the walker in position space is also shown by both variants, and each variant spreads quadratically faster than their classical counterparts [23, 24].

The quadratic spread is readily available for use as a resource for quantum algorithms, and demonstrates the viability of quantum walks in implementation of quantum strategies that show speedup over their classical counterparts. In this work, we restrict ourselves to discrete-time quantum walks. A walker executing a one-dimensional DTQW is characterized by a coin operation and a position shift operation in the Hilbert space \( \mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_p \), where \( \mathcal{H}_c \) and \( \mathcal{H}_p \) are the coin and the position Hilbert spaces, respectively. The position space basis is chosen to be columns of the identity matrix, with the basis states given by \( \{ |x\rangle, x \in \mathbb{Z} \} \). The coin space is a finite-dimensional Hilbert space which, in this case, is chosen to be 2-dimensional. The basis of the coin space is chosen to be the set \( \{ |0\rangle, |1\rangle \} \). With the quantum coin operation defined as \( C(\theta) \in SU(2) \), the t-step evolution is defined as,

\[
\psi(t) = \left[ S(C(\theta) \otimes 1_p) \right]^t \psi(0),
\]

where \( C(\theta) = \begin{bmatrix} \cos(\theta) & i \sin(\theta) \\ i \sin(\theta) & \cos(\theta) \end{bmatrix} \), and shift operator \( S = \sum_{x \in \mathbb{Z}} \left[ |0\rangle_c \langle 0| \otimes |x-a\rangle \langle x| + |1\rangle_c \langle 1| \otimes |x+b\rangle \langle x| \right] \).

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Here, $a, b \in \mathbb{Z}$ represent the amount of traversal in the position space experienced by the component of walker’s probability amplitude in the eigenspaces corresponding to the coin space basis vectors $|0\rangle$ and $|1\rangle$, respectively. In this work, for the sake of clarity and convenience, we choose $a = b = 1$. The initial state of the walker, $|\psi(0)\rangle$ is typically chosen to be localized onto a single eigenstate of $\mathcal{H}_p$, and an equal superposition of eigenstates in $\mathcal{H}_c$, i.e.,

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes |x = 0\rangle.$$  

Our comprehension of physical phenomena generally assumes that events happens in fixed causal order. By using the principle of quantum superposition of states, it becomes possible to create a superposition of temporal order between two events, also called indefinite causal order (ICO). The investigation of this phenomenon is a topic of extensive current research by the quantum information community. One of the major possibilities brought out by this phenomena is that of investigation and simulation of quantum phenomena in a spacetime without a definite causal structure, for example in regions where quantum gravity effects are prominent.

Quantum switch is known to be a super-operator capable of creating ICO [25], and has recently been experimentally realized in photonic settings [26–28]. Two non-commuting quantum operations $\Phi_1$ and $\Phi_2$, can act on a quantum system $\rho$, in two possible temporal orders, leading to the states $\Phi_1 [\Phi_2 [\rho]]$ or $\Phi_2 [\Phi_1 [\rho]]$. A quantum switch puts these two time-ordered operations in a quantum superposition as follows [25]:

$$S(\Phi_1, \Phi_2)[\rho \otimes \rho_s] = \sum_{i,j} W_{ij} (\rho \otimes \rho_s) W_{ij}^\dagger,$$

where, 

$$W_{ij} = \sum_{i,j} K_i^{(2)} K_j^{(1)} \otimes |0\rangle_s \langle 0| + K_j^{(1)} K_i^{(2)} \otimes |1\rangle_s \langle 1|,$$

where $K_i^{(l)}$ are Kraus operators of the channel $\Phi_l[\rho] = \sum_i K_i^{(l)} \rho K_i^{(l)}$. The switch state $\rho_s = |0\rangle \langle 0|$ or $\rho_s = |1\rangle \langle 1|$ leads to the implementation of $\Phi_1 [\Phi_2 [\rho]]$ or $\Phi_2 [\Phi_1 [\rho]]$, respectively. When the switch state in a superposition $\rho_s = |\psi_s\rangle \langle \psi_s|$, where $|\psi_s\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$, the action $S(\Phi_1, \Phi_2)$ shown in Eq. (2) effectively creates an indefinite causal order on the system $\rho$ by creating a superposition of the two temporal orders.

In recent years, ICO has been shown to be a resource for quantum computation and communication [29, 30], while in some cases it need not be so [31]. Therefore, it is still an active area of research. On the other hand, quantum non-Markovianity and causality are recent topics that have attracted attention because of their intimate relationship in physical phenomena [32–34]. The relationship between causality [35] and Parrondo-like effect in quantum walks [36–41] within quantum theory has not been addressed so far in the literature. In this manuscript, we take the first steps in this direction by using a quantum switch to control the sequence of quantum walk operators, in the sense that depending on the state of the control qubit (switch), either definite coin sequence is chosen or a superposition of possible permutations of coin sequences is implemented. In this work, we consider two-period DTQWs with single coin parameter and symmetric initial coin state for simplicity. However, our results can be generalized to more involved dynamics, and to that end we make a qualitative generalization of superposition of causal order in higher period walks.

In the next section, we describe the dynamics where two two-period DTQWs are put in a superposition of temporal orders. We use a quantum switch to control the superposition of forward and reverse causal orders. The former case occurs when the state of the switch is set to $\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$, and the latter corresponds to the states $|0\rangle$ and $|1\rangle$ of the switch, respectively.

In the case of quantum walks, the dynamics of the coin state can be interpreted as if the coin state is passing through a quantum channel $\Phi$, and which can also be given a Kraus representation [42]. However, in this work, we make use of a simple method to calculate the non-Markovianity without requiring the Kraus representations. In the situation we consider here, the coin state passes through the channels in definite causal order $\Phi_2 \Phi_1$ or $\Phi_1 \Phi_2$, and indefinite causal order in which case the effective coin channel is given by Eq. (2). In each case, we see varied amounts of non-Markovian behavior in reduced dynamics of the walker in $\mathcal{H}_c$, in the sense that we observe ‘causal asymmetry’ [43] in the dynamics pertaining the amount of non-Markovianity.
We must note that, in any realistic scenario, a walker (quantum system) may be interacting with a ubiquitous environment resulting in the decoherence and dissipation of quantum resources. The dynamical behavior of the walker under the effect of such noise will modeled by completely positive trace preserving (CPTP) maps acting on the walker. In this work, however, we consider noiseless evolution, nevertheless the generalization of our results to noisy dynamics is quite straightforward.

II. Two-period DTQWs under a superposition of causal orders

A two-period DTQW is a variant of DTQW where the walker at each step alternates between two different coins. The \( N \)-step walk operation is then characterized by the following unitary:

\[
U = (SC_2)^{N \mod 2} (SC_1.SC_2)^{\lfloor N/2 \rfloor},
\]

where \( S \) is the shift operator as shown in Eq. (1), and \( C_1, C_2 \) are single-parameter coin operators, where the subscript is used to differentiate the parameters, i.e. \( C_i \equiv C(\theta_i) \otimes I \). Here, \( \lfloor N \rfloor \) represents the greatest integer less than or equal to \( N \). As is apparent from the form of Eq. (3), the 2-period DTQW in reverse temporal order is equivalent to exchanging the two coin operations. In fact, for two-period walks, there can be only two sequences (or, permutations) of coin operations which do not commute with each other, allowing for a superposition of causal order in quantum walk unitaries. This can be seen from a more generalized point of view, as a DTQW is a unitary operator that can be expressed as an ordered sequence of one or more one-step DTQWs, also known as ‘DTQW steps’. Quite generally, for a quantum walk to exhibit nontrivial dynamics under superposition of causal orders, it must satisfy two criteria:

(i) It must have at least 2 steps, and

(ii) At least one of the DTQW steps must not commute with the other(s).

These criteria follow naturally from the conditions of a sequence of unitary operations commuting with elements of the set of its permutations (or a subset thereof). In order to construct quantum walk steps that do not commute, we use the following lemma (see Appendix for proof).

**Lemma 1** Two DTQW steps having identical shift operations of the form shown in Eq. (1) and different single-parameter coins characterized by the parameters \( \theta_i \), where \( i = 1, 2 \) shall only commute if \( \theta_1 \) and \( \theta_2 \) differ by an integral multiple of \( \pi \).

These conditions serve to ensure that at least some permutations of the sequence do not commute with each other, and thus exhibit the possibility of being put under superposition of causal orders. Thus the only DTQWs capable of exhibiting indefinite causal order are either DTQWs with disorders or multi-period DTQWs.

In this section, we consider a two-period DTQW in superposition of temporal order with its own temporally reversed form, a construction known as ‘causal activation’. As long as the parameters \( \theta_1, \theta_2 \) are not separated by an integral multiple of \( \pi \) (cf. Lemma 1), the forward and reverse temporal forms of the DTQW will not commute and exhibit nontrivial behavior under causal activation. We compare the dynamical behavior of the DTQWs under both forward and reverse temporal orders, and an equal superposition of these temporal orders implemented using a quantum switch.

The two temporally ordered variants of the 2-period DTQW with the same parameters will thus be of the form,

\[
U_1 = (SC_2)^{N \mod 2} (SC_1.SC_2)^{\lfloor N/2 \rfloor} \\
U_2 = (SC_1)^{N \mod 2} (SC_2.SC_1)^{\lfloor N/2 \rfloor}
\]

This formulation admits two different variations in the implementation of indefiniteness in causal order, namely, the entire two-period walk being causally activated, or each step of the walker individually being put in a superposition of causal order. We consider each case below.
A. Superposition of causal order at each step

In this case, we consider that each step of the walk has been switched. In case of the 2-period walk, this may be seen as each step is either $SC_1$ or $SC_2$. Now, in order to effect indefiniteness in the causal order of quantum walk steps, care needs to be taken in the definition of the construction which is being put under the superposition of causal order. Interestingly, for a two-period walk, when the switch is used at each step, it will not create indefiniteness in the causal order as the effective operator applied to the walker simply reduces to a superposition of two different walk steps, which in turn cannot be decomposed into a sequence of walk steps that are temporally reversed with respect to each other. This can be seen easily, since upon measurement of the switch qubit, the effective operation on the walker will be

$$U_{\text{eff}} = \cos(\theta_s)SC_1 + \sin(\theta_s)SC_2,$$

where the subscript $s$ denotes the parameter of the switch state, here assumed to be $|\psi_s\rangle = \cos(\theta_s)|0\rangle + \sin(\theta_s)|1\rangle$. In order to use quantum switch each ‘step’, one must consider superposition of the walk two steps at a time. Thus, the effective operator applied on the walk will become,

$$U_{\text{SwStep}} = (\cos(\theta_s)SC_2 + \sin(\theta_s)SC_1)^N \mod 2 \cdot (\cos(\theta_s)SC_1SC_2 + \sin(\theta_s)SC_2SC_1)^{\lfloor N/2 \rfloor} \tag{5}$$

In order to get rid of the first term, one may simply restrict $N$ to even integral values. Assuming that $\theta_2 - \theta_1 \neq n\pi$, this may be expanded as,

$$U_{\text{SwStep}} = \sum_{j=0}^{N/2} \cos\frac{N}{2}j(\theta_s) \sin(j(\theta_s)) \left[ \sum_{\alpha_j} U_{j,\alpha_j}(\theta_1, \theta_2) \right], \tag{6}$$

where $\alpha_j$ is a $j$-dimensional column vector such that no two of its components are equal, and for a general $l^{th}$ component $\alpha_{j_l}$, $\alpha_{j_l} \in (0, \lfloor \frac{N}{2} \rfloor)$, and $\alpha_{j_l} \in \mathbb{Z}$. Without loss of generality, we also have, $l < m \implies \alpha_{j_l} < \alpha_{j_m}$. $U_{j,\alpha_j}(\theta_1, \theta_2)$ represents a possible $\lfloor \frac{N}{2} \rfloor$-step quantum walk with the unitary $\tilde{U}_1 = SC_1SC_2$, such that the operator at $j$ of the $\lfloor \frac{N}{2} \rfloor$ steps, specified the components of $\alpha_j$, are replaced by the unitary $\tilde{U}_2 = SC_2SC_1$. It may be seen by observation that $U_{j,\alpha_j}(\theta_1, \theta_2) = U_{\tilde{j},\alpha_j}(\theta_2, \theta_1)$, where $\tilde{\alpha}_j$ is a ‘dual’ of $\alpha_j$, and consists of all points in the interval $(0, \lfloor \frac{N}{2} \rfloor)$ that are not contained in $\alpha_j$. This may be interpreted as the fact that a quantum walk of the type $U_{1/\tau}^{\lfloor \frac{N}{2} \rfloor}$, where the operations at $j$ of the steps are replaced by $\tilde{U}_2$, may be equivalently seen as a quantum walk of type $U_{2/\tau}^{\lfloor \frac{N}{2} \rfloor}$, where the other $\lfloor \frac{N}{2} \rfloor - j$ steps are replaced with $\tilde{U}_1$. In the case where $\theta_2 - \theta_1 = n\pi$, the walk steps commute (cf. Lemma 1), and the walk operator may be simplified by using a binomial expansion of Eq. (5), and we obtain,

$$U_{\text{SwStep}} = \sum_{j=0}^{\lfloor N/2 \rfloor} \left[ \left( \frac{N}{2} \right)_j \right] \cos\frac{N}{2}j(\theta_s) \sin^j(\theta_s) \cdot \left( \tilde{U}_1 \right)^{(\frac{N}{2})-j} \left( \tilde{U}_2 \right)^j \tag{7}$$

where $\left( \frac{N}{2} \right)_j$ is the binomial coefficient, and $\tilde{U}_1, \tilde{U}_2$ are as defined above. A plot of numerical results obtained from the simulation of this scenario is detailed in Figs. (1)-(2).

In case the measurement and subsequent tracing out is done after all the steps have been executed, the two cases are equivalent. This process may be considered as an alternative numerical recipe to simulate generation of indefinite causal order by using a switch operation over the entire dynamics of the walk.

B. Causal activation of a two-period quantum walk

The dynamical behavior of the switch qubit is defined to be,

$$U_{sw} = |0\rangle_s \langle 0| \otimes U_1 + |1\rangle_s \langle 1| \otimes U_2, \tag{8}$$
Figure 1: Plots showing the variation in standard deviation of a two-period quantum walk where the switch is used at each step, for $\theta_1 < \theta_2$. It can be seen that there is a temporary advantage for a short time, and then the walker begins to localize. For this plot, the values used were $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{\pi}{4}$.

Figure 2: Plots showing the variation in standard deviation of a two-period quantum walk where the switch is used at each step, for $\theta_1 > \theta_2$. Exactly as in the case of $\theta_1 < \theta_2$ as shown in Fig. 1, the walker begins to localize after a short period of advantage. The values used for this figure is $\theta_1 = \frac{\pi}{4}$, $\theta_2 = \frac{\pi}{6}$.

where the subscript $s$ is used to designate the projection operator applied on the switch qubit, and the operators $U_1$ and $U_2$ are defined for the walker as in Eq. (4). We assume that the switch was prepared in the state

$$|\psi_s\rangle = \cos(\theta_s) |0\rangle + \sin(\theta_s) |1\rangle.$$  

Upon measuring the switch state in the computational basis $\{|0\rangle, |1\rangle\}$ and tracing out of switch qubit, we see that the effective operation on the walker space becomes,

$$U_s = \cos(\theta_s)U_1 + \sin(\theta_s)U_2$$  

It is to be noted that $U_s$ is not a unitary operator but a CP map. Its effect may be interpreted as a form
Figure 3: Illustration of the probability distribution of a quantum walk in position space after 100 steps of a discrete time quantum walk in forward temporal order (labeled $|\psi_s\rangle = |0\rangle$), reversed temporal order (labeled $|\psi_s\rangle = |1\rangle$), and equal superposition of temporal order (labeled $|\psi_s\rangle = |+\rangle$). The walker is chosen to be initially localized in the state $|\psi_0\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |x = 0\rangle)$, and the coin parameters are chosen to be $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{6}$. It can be clearly seen that the dynamics of the three cases are different with respect to each other.

of post-selection (in which case, it must be normalized), or as a CP map acting on reduced dynamics of the walker which can be made unitary by adding the dimension of an auxiliary qubit to the walker Hilbert space.

III. Multi-period discrete-time quantum walks under indefinite causal order

Reversal of temporal order of a multi-period DTQW creates another DTQW, however, since the values of the coin parameters $\theta_i$ remain constant, the spread of each variant remains the same, defined in [44]. It can be seen from the following figure that the three dynamics are unique. The action of the quantum switch on the two-period DTQW can be represented by the operator $U_s$ shown in Eq. 9. Here the parameter $\theta_s$ determines the contribution of each temporal order in the superposition. Setting $\theta_s = 0$ results in the usual causal order, while $\theta_s = \frac{\pi}{2}$ implements the temporally reversed form of this DTQW.

In case of a $k$-period DTQW ($k > 2$), we define partial causal activation as the coherent superposition of a variant of this DTQW with its own temporally reversed form. Similarly, one may define full causal activation as the coherent superposition of all possible (up to $k!$) variants of a $k$-period DTQW such that each variant has the same number of steps and shares the same unordered set of coin parameters. In this work, we have considered the partial causal activation of $k > 2$-period DTQWs.
However, the effective unitary operator applied to the walker in case of a fully causally activated $k$-period walk may be given by

$$U_{\text{eff}} = \frac{1}{\sqrt{k!}} \sum_{\text{all sequences}} \left( \prod_{m=1}^{k} \tilde{U}_m \right)^{N/k},$$

(10)

where $\tilde{U}_m$ represents the quantum walk step $S(C(\theta) \otimes I)$, such that $\theta \in \{\theta_1, \theta_2, ..., \theta_k\}$ and the sum is over all unique $k$-step sequences of such steps. It is possible to modify this form for the case where the number of steps is not a multiple of $k$.

For a 2-period walk, a Parrondo-like effect is seen in the temporal variation of standard deviation in case of causal activation. This is illustrated in Fig. (4). The spread of a walker executing DTQW is measured with the standard deviation of the walker’s position at each time step. In case of a DTQW with definite causal order, the spread increases linearly with time, as expected from known results, however, the DTQW under causal activation shows a higher spread after some time has elapsed. This is a Parrondo-like effect, as by choosing to execute a DTQW under causal activation, one is able to generate an asymptotic advantage in the ballistic nature of the walker’s spread in position space. This has significant implications in the use of DTQW-based search algorithms.

A similar effect is also seen in the 3-period case, however, the degree of advantage offered by the partially causally activated DTQW is dependent on the order of walk operations. In the most general case, when all parameters are chosen to be different, the (asymptotic) advantage observed in spread is highest when the operators are applied in ascending order of coin parameters, and similarly, it is found to be lowest when the operators are applied in descending order of coin parameters. In other words, assuming that $\tilde{U}_i := S(C(\theta_i) \otimes I)$, and $\theta_1 < \theta_2 < \theta_3$, then the DTQW with highest and lowest advantages under partial causal activation will be given by,

$$U_{\text{max}} = \left[ \tilde{U}_1 \tilde{U}_2 \tilde{U}_3 \right]^{\frac{N}{3}} \prod_{j=1}^{N \mod 3} \tilde{U}_j,$$

$$U_{\text{min}} = \left[ \tilde{U}_2 \tilde{U}_2 \tilde{U}_1 \right]^{\frac{N}{3}} \prod_{j=1}^{N \mod 3} \tilde{U}_{(N \mod 3) - j},$$

(11)

These two cases are illustrated in Fig. 5. This effect is only seen in case of periods greater than 3, because in case of the two-period DTQW, the case of increasing and decreasing orders of coin parameters are temporal reverses of each other, and under causal activation, offer the same advantage.

IV. Non-Markovianity, causal asymmetry, and Parrondo-like effect due to indefinite causal order

Quantum non-Markovianity is one of the topics that has attracted much attention of quantum information community. Various methods have been proposed \[33, 45, 46\] to characterize and quantify quantum non-Markovianity, yet its exact and complete treatment still remains an open problem. In this work we make use of the definition of non-Markovianity based on trace distance \[47\].

Here we give a brief account of how the reduced dynamics of the coin state can be computed in a simple way. In order to calculate the trace distance based measure of non-Markovianity, we consider two orthogonal initial coin states which are chosen to be $|+\rangle_c$ and $|\rangle_c$. The initial density matrices of the (reduced) coin space are thus given by $\rho_+(0) = |+\rangle \langle +|$ and $\rho_-(0) = |\rangle \langle |$. The density matrices at any point $t$ are easily calculated by executing the walk for $t$ steps and tracing out the position space from the resulting density matrix of the walker, i.e.

$$\rho_c(t) = \text{Tr}_p \left[ O (\rho_c(0) \otimes \rho_p(0)) O^\dagger \right],$$

(12)

where $\rho_p(0) = |x = 0\rangle \langle x = 0|$, and $O$ is an operator that represents the dynamics of quantum walks caused due to particular definite causal order and indefinite causal order of coin operations. In the present work, as noted in the previous sections, we consider two cases: one, where each step of the periodic walks are put in superposition of causal order, and two, where the entire walk sequences are put in superposition of
Figure 4: An illustration of the spread of the two-period walk with and without causal activation. (a) shows a plot of the spread as it varies with number of steps of the walk, and (b) shows the difference of the spread between the case when the walk is causally activated and under a definite causal order. It may be observed that one asymptotically gets a significant advantage in spread in the case of indefinite causal order. The variation in the spread may be attributed to the fact that we are using periodic quantum walks. In both cases, the parameters are chosen to be $\theta_1 = \frac{\pi}{4}$, $\theta_2 = \frac{\pi}{6}$.

causal order. In this section, we consider only two-period and three-period walks. The trace distance can be calculated as

$$D(t) = \frac{1}{2} \| \rho_+(t) - \rho_-(t) \|_1, \quad (13)$$

and the measure due to Breuer-Laine-Piilo (BLP) [47] is simply an integral over the positive slope of the $D(t)$, which is given by

$$\mathcal{N} = \max_{\rho_1, \rho_2} \int_{\rho_0}^{\rho_+} \frac{dD(t)}{dt} dt. \quad (14)$$

It is known that BLP measure requires optimization over the initial pair of states so that the measure is maximized over those pair of states. It was shown [47, 48] that the $|+\rangle$ and $|-\rangle$ happen to be the pair of states that maximize the BLP measure, hence the choice of our coin states.

Now, we show that the reduced dynamics of the coin is more non-Markovian according BLP measure when the walk is causally activated. When the quantum walks obey a specific causal order with switch state being in either $|0\rangle$ or $|1\rangle$, then we observe causal asymmetry [43] in the dynamics of reduced coin state. That is, a specific temporal order of coin arrangement causes dynamics to be more non-Markovian than the other, in our case the reverse of the former. However, we see that when the dynamics are used in a superposition of temporal order, we observe a Parrondo-like effect for non-Markovian dynamics. This behavior continues as the period of the walk increases, and the measure of non-Markovianity is found to saturate to a value as number of periods is increased. This is illustrated in Fig. (7).

However, it should be noted that the interpretation of Parrondo effect might be context dependent. It is known that the Markovian decoherence effect is caused due to increasing entanglement between system and the environment. On the contrary, when the reduced dynamics of a system is non-Markovian, the entanglement between the system and the environment decreases. In our case, we see that this is consistent with the fact that the entanglement between the coin and position space of the walk dynamics reduces when the reduced coin dynamics is non-Markovian, i.e., there is an information back-flow from position space to the coin space. Therefore, we note that there need not be any relationship between indefinite causal order and Parrondo effect when it comes to entanglement between position and coin space. On the other hand,
Figure 5: A figure illustrating the two extreme cases of 3-period quantum walk under causal activation. It is seen clearly that when coin parameters are chosen to be in ascending order (shown in (a) and (b)), the advantage over the case with definite causal order is clearer and more readily seen as compared to the case where the coin parameters are chosen to be in the descending order (shown in (c) and (d)). It is to be noted that the causal activation of a quantum walk shows an advantage even in the worst choice of coin parameters, i.e. when the parameters are in descending order. The walk is executed for $N = 100$ steps, with the coin parameters set to $\pi/6$, $\pi/4$, and $5\pi/12$.

Parrondo-like effect is generally seen when one looks at the non-Markovianity of the reduced coin state. We expand on this issue in the next section and give an instance of the context-dependence of Parrondo-like effect in quantum walks.

V. Entanglement Measures under indefinite causal order

It is clear by observation of the shift operation that it cannot be expressed in a separable form over the bi-partition consisting of $\mathcal{H}_c$ and $\mathcal{H}_p$ of the walker’s Hilbert space $\mathcal{H}$. Thus, it causes an entanglement of the walker’s dynamics in $\mathcal{H}_c$ and $\mathcal{H}_p$, a phenomenon that is also responsible for the non-Markovian behavior of reduced dynamics in the coin subspace. In this section, we investigate the behavior of concurrence and entanglement entropy between the spaces $\mathcal{H}_c$ and $\mathcal{H}_p$ for the walker executing a 2-period DTQW under a
Figure 6: BLP measure for the cases with definite causal order and the for the case with indefinite causal order of two-period quantum walks. Here, we note that when causal indefiniteness is introduced, the non-Markovianity surpasses that when only definite causal order is introduced. Interestingly, we also observe a type of causal asymmetry in the dynamics when definite causal ordering is preferred in the sense that the non-Markovianity need not amount to be same for both temporal orders.

Figure 7: Figure illustrating the normalized BLP measure for a 50-step walk with different periods, under two definite causal orders, and an equal temporal superposition. (a) considers a walk where $\theta_1 = \frac{\pi}{6}$, and $\theta_i = \frac{\pi}{4}$ for $i > 1$, and (b) shows the results for $\theta_1 = \frac{5\pi}{12}$, $\theta_i = \frac{\pi}{4}$ $\forall 50 > i > 1$. In both cases, the BLP measure saturates to a high value, however, the case of indefinite causal order is significantly more non-Markovian than that with definite causal orders.

standard causal order, its temporally reversed version, and the equal superposition of these two orders. We also show results of simulation for the 3-period DTQW under the extreme cases for most and least advantage in spread, as considered in Sec. III. In case of the 3-period DTQW, a reverse Parrondo-like effect is seen. Both the entanglement entropy and concurrence show a higher decrease when the parameters of the walk are chosen to be in decreasing order, and the case of increasing order of parameters exhibits the lowest decrease of entanglement measures. This is illustrated in Figs. (9) and (10).
Figure 8: Plots showing the concurrence and entanglement entropy correlations for a walker executing a 2-period DTQW for 100 steps, with coin parameters set to $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{3}$. It is seen that for both the entanglement measures, the value in case of indefinite causal order remains bounded between the values for each of the definite causal orders.

Figure 9: Plots showing the entanglement entropy correlations for a walker executing a 3-period DTQW for 100 steps, with coin parameters set to $\theta_1 = \frac{\pi}{3}$, $\theta_2 = \frac{\pi}{4}$, or $\frac{5\pi}{12}$. (a) and (b) show the two extremes of ascending and descending order of parameters, where the walk is executed as in Eq. (11).

VI. Conclusions

It is known that the reduced dynamics of the coin state is non-Markovian [42]. In this work, we have shown that indefinite causal order in action of walk operations on the state of a walker executing a two-period DTQW results in more non-Markovianity in the dynamics of the reduced state of the coin. We have also shown that definite causal order pertains to varied amounts of non-Markovianity in these dynamics. That is, the two-period DTQW with particular temporal order of coin arrangement exhibits causal asymmetry. We point out that the observed causal asymmetry is an inherent feature of periodic quantum walks, regardless of the number of periods (equal to or greater than two). Moreover, when the two-period DTQW is causally activated (that is when the quantum switch is in the state $|+\rangle$), non-Markovianity of the reduced coin state
Figure 10: Plots showing the entanglement entropy correlations for a walker executing a 3-period DTQW for 100 steps, with coin parameters set to $\theta_1 = \frac{\pi}{3}$, $\theta_2 = \frac{\pi}{4}$, or $\frac{5\pi}{12}$. (a) and (b) show the two extremes of ascending and descending order of parameters, where the walk is executed as in Eq. (11).

exceeds that of walks with definite causal order, which is interpreted as a Parrondo-like effect. We also note that in the case where the switch is used at each step, the effect of switch reduces to the superposition of the sequence of unitary operators, which in turn cannot be decomposed into sequences of quantum walk which are temporal reverses of each other. Effectively, this cannot be interpreted as a superposition of temporal order, it is merely a coherent superposition of two distinct steps of a quantum walk. In addition, we show that our results also hold for a higher number of periods. However, an increase in the number of periods corresponds to an increase in the number of permutations of the periodic sequences of DTQW steps, thus making the dynamics much more involved. To this end, we also show that as the number of periods of a quantum walk increase, the non-Markovianity of the reduced coin-state dynamics saturates to a constant value.

A number of possible prospects can be stated. Understanding non-Markovianity in the light of temporal correlations has gained renewed interest in the topic [32, 33, 49]. It would be an interesting future exercise to quantify the temporal quantum correlations [50] in the quantum walk dynamics due to indefinite causal order, and also quantify non-Markovianity of reduced dynamics of the coin using measures as proposed in [33, 49]. Since quantum walks are known to be important tools to model dynamics in quantum networks, it would be interesting to explore the role played by non-Markovianity, indefinite causal order and temporal correlations in a quantum walk in quantum networks.

Acknowledgments

We acknowledge support from the Interdisciplinary Cyber Physical Systems (ICPS) Programme of the Department of Science and Technology, Government of India. Grant No. DST/ICPS/QuST/Theme-1/2019. PC and US contributed equally to this work.
Appendix: Proof of Lemma 1

Proof: An elementary proof may be constructed by considering a matrix representation of the coin Hilbert space. The representation of the shift operation will then be,

\[
S = \sum_{x \in \mathbb{Z}} \begin{bmatrix} |0\rangle_c \langle 0| & |x-1\rangle \langle x| & |1\rangle_c \langle 1| & |x+1\rangle \langle x| \end{bmatrix}
\]

\[
= \begin{bmatrix} \sum_{x \in \mathbb{Z}} |x-1\rangle \langle x| & 0 \\ 0 & \sum_{x \in \mathbb{Z}} |x+1\rangle \langle x| \end{bmatrix}
\]

(15)

where the second equation is due to the fact that \( S \) is unitary. For this proof, we consider a generalized \( SU(2) \) operator for the coin, given as,

\[
C(\theta, \xi, \zeta) = \begin{bmatrix} e^{i\xi \cos(\theta)} & e^{i\xi \sin(\theta)} \\ -e^{-i\xi \sin(\theta)} & e^{-i\xi \cos(\theta)} \end{bmatrix}.
\]

(16)

Now, the commutator of the quantum walks will look like,

\[
[SC_1, SC_2] = \left[ \begin{array}{cc} -2i \sin(\xi_1 - \xi_1) \sin(\theta_1) \sin(\theta_2) & (1 - T_+^2) \left( e^{i(\xi_2 + \xi_1) \cos(\theta_1) \sin(\theta_2) - e^{-i(\xi_2 - \xi_1) \cos(\theta_1) \cos(\theta_2)} \right) \\ -i(1 - T_+^2) \sin(\theta_1 - \theta_2) & 0 \end{array} \right]
\]

(17)

where the second equation is due to the fact that \( \xi_1 = \xi_2 = 0 \) and \( \zeta_1 = \zeta_2 = \frac{\pi}{2} \) to get the single-parameter form of the coin as in Eq. 1. Thus, we obtain that,

\[
[SC_1, SC_2] = 0 \\
\implies \sin(\theta_2 - \theta_1) = 0 \\
\implies \theta_2 = \theta_1 + n\pi, \quad n \in \mathbb{Z},
\]

(18)

which completes the proof. \(\square\)

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