On the Completeness of the Black Hole Singularity in 2d Dilaton Theories

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Abstract

The black hole of the widely used ordinary 2d–dilaton model (DBH) deviates from the Schwarzschild black hole (SBH) of General Relativity in one important feature: Whereas non-null extremals or geodesics show the expected incompleteness this turns out \textit{not to be the case for the null extremals}. After a simple analysis in Kruskal coordinates for singularities with power behavior of this – apparently till now overlooked – property we discuss the global structure of a large family of generalized dilaton theories which does not only contain the DBH and SBH but also other proposed dilaton theories as special cases. For large ranges of the parameters such theories are found to be free from this defect and exhibit global SBH behavior.

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1 Introduction

Dilaton models in $1 + 1$ dimensions have been studied extensively in their string theory inspired form [1], as well as in generalized versions [2]. Recently also models with torsion [3] which can be motivated as gauge theories for the zweibein have been seen [4] to be locally equivalent to generalized dilaton theories, although the global properties differ in a characteristic manner.

The prime motivation for investigating such models and especially the ordinary dilaton black hole (DBH), always has been the hope to obtain information concerning problems of the 'genuine' Schwarzschild black hole (SBH) in $d = 4$ General Relativity: the quantum creation of the SBH and its eventual evanescence because of Hawking radiation and the correlated difficulty of information loss by the transformation of pure quantum states into mixed ones, black hole thermodynamics etc. [5].

On the other hand, essential differences between DBH and SBH have been known for a long time. We just quote the Hawking temperature ($T_H$) and specific heat: For the DBH $T_H$ only depends on the cosmological constant instead of a dependence on the mass parameter as in the SBH. The specific heat is positive for DBH and negative for the SBH.

A basis for any application of DBH must be a comparison of its singular behavior with the one of the SBH. However, careful studies of the singularity structure in such theories seem to be scarce. Apart from [3] and our recent work [4] we are not aware of such a comparison. It always seems to have been assumed that the physical features coincide at least qualitatively in all respects. During our recent work [4] we noted that this is not the case: For the ordinary dilaton black hole of [1] null extremals are complete at the singularity. Of course, non–null extremals are incomplete, and so at least that property holds for the DBH, but, from a physical point of view, it seems a strange situation that massive test bodies fall into that singularity at a finite proper time whereas it needs an infinite value of the affine parameter of the null extremal (describing the influx e.g. of massless particles) to arrive. Thus, from the point of view of the genuine SBH serious doubts may be raised against any effort to extract theoretical insight from the usual DBH.

In Section 2 we exhibit the main differences in the presumably most direct manner, namely by treating the DBH as well as the SBH as special cases of a general power behavior of the metric in Kruskal coordinates. We find that the DBH lies at a point just outside the end of the interval of the BH-s which are qualitatively equal to the SBH in the sense that both null and non–null extremals are incomplete.

In order to pave the way for a more realistic modelling of the SBH we then (Section 3) consider a two parameter family of generalized dilaton theories which interpolates between the DBH and other models, several of whom have been already suggested in the literature [1, 2, 3, 4]. The Eddington–Finkelstein (EF) form of the line element, appearing naturally in 2d models when they are expressed as 'Poisson–Sigma models' (PSM) [11] is very helpful in this context. We indeed find large ranges of parameters for which possibly more satisfactory BH models in $d = 2$ may be obtained.
2 Completeness at a Curvature Singularity

Consider a metric expressed in Kruskal coordinates \( u, v \)

\[
(ds)^2 = 2f(uv)dudv \sim 2z^{-a}dudv
\]

where we assume the metric to be dependent only on the product \( uv \) and a simple leading power behavior of \( f(uv) \) near

\[
z = 1 - uv \to 0.
\]

Without loss of generality we consider the space-time where \( f > 0 \) or \( uv < 1 \). The case \( a = 0 \) corresponds to flat Minkowskian space-time and is not considered in the following. This metric covers many interesting cases. For the DBH we have \( f = z^{-1} \) with \( a = 1 \) \[1\]. Neglecting the angular dependence the SBH metric has also this form with \( a = \frac{1}{2} \). This can be extracted easily from the formulas on p. 152 and 153 of ref. \[13\].

From the only nonvanishing components of the affine connection

\[
u^{-1}\Gamma^1_{11} = v^{-1}\Gamma^0_{00} = \frac{d}{d(uv)}(\ln f) = (\ln f)'
\]

the curvature scalar becomes at \( z \to 0 \) with (1)

\[
R = \frac{2}{f} [(\ln f)' + uv(\ln f)''] \xrightarrow{z \to 0} 2az^{a-2}
\]

Thus \( a = 2 \) (de Sitter behavior) represents the border between singular \( (a < 2) \) and vanishing \( (a > 2) \) curvature at this point. Both SBH and DBH belong to the singular range. With (2) the geodesic equations are simply

\[
\ddot{u} + v(\ln f)'u\dot{u}^2 = 0,
\]

\[
\ddot{v} + u(\ln f)'v\dot{v}^2 = 0,
\]

where a dot denotes differentiation with respect to the canonical parameter \( \tau \). The analysis of these equations greatly simplifies by noting that the metric \[1\] admits a Killing vector \( k^\mu, \mu = 0, 1 \), for arbitrary \( f \),

\[
k^\mu = \left( \begin{array}{c} -u \\ v \end{array} \right)
\]

which implies the integral

\[
(\dot{u}v - u\dot{v})f = A_1.
\]

In the relation between the canonical parameter and the line element

\[
\dot{u}\dot{v}f = A_2
\]
the constant $A_2 > 0$, $A_2 = 0$ and $A_2 < 0$ describe timelike, null and spacelike extremals, respectively. The sign of $A_1$ is arbitrary. Expressing $\dot{u}$ and $\dot{v}$ from (7) and (8) in $\dot{z} = -u\dot{v} - v\dot{u}$ yields near the singularity

$$\dot{z} = \pm \sqrt{A_1^2 z^{2a} + 4A_2 z^a (1 - z)}.$$  

(9)

The dependence of $u/v$ on $z$ is determined by

$$\frac{d}{dz} \ln \left| \frac{u}{v} \right| = \pm (1 - z)^{-1} \left( 1 + \frac{4A_2}{A_1^2} (1 - z) z^{-a} \right)^{-\frac{1}{2}}.$$  

(10)

This follows from (7) replacing the $\tau$-dependence by a $z$-dependence according to (9). The simple analysis of equation (9) shows that for $0 < a < 2$ the curvature singularity can be reached at finite affine parameter $\tau$ only by timelike and null extremals, whereas for $a < 0$ this holds for all types of extremals. The difference between the asymptotic behavior of null ($A_2 = 0$) and non-null ($A_2 \neq 0$) extremals is especially transparent from (9). For null extremals which cross the singularity ($A_1 \neq 0$) it takes the form

$$\tau \sim z^{(1-a)}, \quad a \neq 1,$$

(11)

$$\tau \sim \ln z, \quad a = 1.$$  

(12)

So null extremals are incomplete (finite value of the canonical parameter) at the singularity if and only if $a < 1$. At the same time for timelike extremals and $a > 0$ equation (9) yields

$$\tau \sim z^{(1-a/2)}, \quad a \neq 2,$$

(13)

$$\tau \sim \ln z, \quad a = 2.$$  

(14)

We see that timelike extremals are incomplete if $0 < a < 2$. If $a < 0$ then the asymptotic behavior of timelike and spacelike extremals near the singularity coincides with that of null extremals, (11), and they are always incomplete. Thus we have proved that non-null extremals are always incomplete at the curvature singularity, $a < 2$, $a \neq 0$. Null extremals are incomplete only for $a < 1$. So the DBH lies precisely at the border, $a = 1$, where null extremals are still complete at the singularity. This is a qualitative difference with the SBH at which all types of extremals are incomplete.

In order to be able to compare the family of dilaton theories considered in section 3 below, we also give the transformation of (1) into the EF metric

$$(ds)^2 = d\bar{v}(2d\bar{u} + l(\bar{u})d\bar{v})$$  

(15)

which explicitly depends on the norm $l = k^a k_\alpha$ of the Killing vector $\partial/\partial \bar{v}$. Introducing $F'(y) = f(y)$ the necessary diffeomorphism is ($\sigma = \pm - 1$ in order to cover the whole original range of $v$)
\[ u = e^{-\bar{v}}h(\bar{u}) \]  
\[ v = \sigma e^\bar{v} \]  
(16)  
(17)

with \( h(\bar{u}) \) determined from

\[ \bar{u} = F(\sigma h) \]  
(18)

so that in (13)

\[ l(\bar{u}) = -2\sigma h(\bar{u})f(\sigma h(\bar{u})). \]  
(19)

For the power behavior (11) we obtain

\[ l(\bar{u}) \sim |\bar{u}|^{\frac{a}{a-1}} \]  
(20)

to be taken at \( \bar{u} \to 0_- \) for \( a < 1 \) and at \( \bar{u} \to +\infty \) for \( a > 1 \). The DBH case \( a = 1 \) must be treated separately. The result

\[ l_{a=1} \to e^{\bar{u}}, \quad \bar{u} \to \infty, \]  
(21)

as expected, agrees with the familiar DBH behavior [1] [3].

Eq. (13) is particularly convenient to make contact with the PSM formulation which can be obtained for all covariant 2d theories [11]. They may be summarized in a first order action

\[ L = \int X^+ T^- + X^- T^+ + X d\omega - e^- \wedge e^+ V(X) \]  
(22)

In our present case only vanishing torsion

\[ T^\pm = (d \pm \omega) \wedge e^\pm \]  
(23)

as implied by (22) is expressed in terms of light-cone (LC) components for the zweibein one form \( e^a \) and for the spin connection one form \( \omega^a_b = e^a \beta_\omega \) where \( e^{ab} = -e^{ba} \), \( \epsilon^{01} = 1 \) is the totally antisymmetric tensor. The ‘potential’ \( V \) is an arbitrary function of \( X \) and determines the dynamics. It is simply related to the Killing norm \( l \) in the EF gauge because (22) can be solved exactly for any integrable \( V \) (cf. the first ref. in [11]) with the solutions (constant curvature is excluded)

\[ e^+ = X^+ df, \]  
(24)
\[ e^- = \frac{dX}{X^+} + X^- df, \]  
(25)

where \( f \) and \( X \) are arbitrary functions of the coordinates such that \( df \wedge dX \neq 0 \) and \( X^+ \) is an arbitrary nonzero function.
A similar solution for $\omega$ will not be needed in the following. The line element immediately yields the EF form $^{(15)}$ with $\bar u = \frac{X}{2}$ and $\bar v = f$. The Killing norm

$$l = 2 \left[ C - w \right]$$

(26)

follows from a conservation law

$$C = X^+ X^- + \int X V(y) dy = X^+ X^- + w$$

(27)

common to all 2d covariant theories $^{[3]}$ $^{[11]}$ $^{[12]}$ which is related to a global non-linear symmetry $^{[14]}$. The usual dilaton models are produced by the introduction of the dilaton field $\phi$ in $X = 2 \exp(-2\phi)$, together with a Weyl transformation $e^a = \exp(-\phi) \tilde e^a$ of $d\omega$ in $^{(22)}$

$$e^{\mu\nu} \partial_\mu \omega_\nu = -\frac{R \sqrt{-g}}{2},$$

(28)

with the components $\omega_\nu$ expressed by the vanishing of the torsion $^{(23)}$ in terms of the zweibein.

The result

$$\sqrt{-g} R = \sqrt{-\tilde g} \tilde R + 2 \partial_\mu (\sqrt{-\tilde g} \tilde g^{\mu\nu} \partial_\nu \phi)$$

(29)

will be used in the below and in the next section.

From the preceding argument there is now a direct relation between the singularity in terms of Kruskal coordinates $^{(1)}$ through $^{(20)}$ to the singularity in the corresponding action $^{(22)}$ using $^{(26)}$

$$V \to |X|^\frac{1}{1-a}$$

(30)

with the singularity at $X \to \infty$ for $1 < a < 2$ and at $X \to 0$ for $a < 1$. The exponential behavior of $V$ for the DBH ($a = 1$) may be read off from $^{(21)}$.

Let us consider the SBH in a little more detail. The starting point is the Schwarzschild solution in EF coordinates $^{[13]}$

$$ds^2 = 2 dv dr + \left( 1 - \frac{2M}{r} \right) dv^2 - r^2 d\Omega^2,$$

(31)

whose $r - v$ part is of the type $^{[15]}$. Thus the radial variable may be identified with $\bar u$ in $^{[13]}$ and with $X$ in $^{[30]}$. Indeed $a = \frac{1}{2}$ yields the correct singularity behavior and the corresponding PSM action $^{(22)}$ with

$$V = -\frac{M}{X^2}.$$  

(32)

In the second order formulation (after eliminating the fields $X^\pm$, and $X$ and using their equations of motion) the action reads

$$L_{d=2}^{SBH} = \frac{3}{2} \left( \frac{M}{2} \right)^{\frac{3}{2}} R^{\frac{1}{2}} \sqrt{-g}.$$  

(33)
Now the metric remains as the only dynamical variable. This action does not look very attractive because of the fractional power of curvature, but in the Euclidean formulation it is positive definite and thus may be useful for quantization. By construction this model beside the flat Minkowskian solution has the solution of the true SBH. It should be stressed that the model (33) reproduces the \( r, v \)-part of the SBH globally and not only near the singularity.

The above analysis may be extended easily to include also powers of \( \log z \) in (I). For example \( f = z^{-1} \ln z \) improves the DBH to make it null incomplete. In that case \( V \to e^{\sqrt{2X}} \) for \( X \to \infty \), diverges 'slightly' less than for the DBH. For exponential behavior in Kruskal coordinates \( f \to e^{-uv} \) for \( uv \to \infty \), \( R \) diverges exponentially as well. In that case both null and non-null extremals are incomplete. Although satisfactory in this respect the corresponding PSM potential \( V \to \ln(-X) \) at \( X \to 0_- \) does not seem particularly useful for applications in BH models.

The 'pure' PSM model for the SBH with potential (32) is fraught with an important drawback: When matter is added the conserved quantity \( C \) in \((27)\) simply generalizes to a similar conserved one with additional matter contributions (cf. the second ref. \[14\]). Thus even before a BH is formed by the influx of matter an 'eternal' singularity as given e.g. by (32) for the SBH, is present in which the mass \( M \) basically cannot be modified by the additional matter. A general method to produce at the same time a singularity–free ground state with, say, \( C = 0 \) is provided by a Weyl transformation of the original metric. It simply generalizes what is really behind the well-known construction of the DBH theory. Consider the transformation

\[
\tilde{g}_{\mu\nu} = \frac{g_{\mu\nu}}{w(X)}
\] (34)

in \((13)\) with \((26)\) together with a transformation of \( X \)

\[
\frac{d\tilde{X}}{dX} = w(X(\tilde{X})).
\] (35)

This reproduces the metric \( \tilde{g}_{\mu\nu} \) in EF form

\[
(ds)^2 = 2df \left( d\tilde{X} + \left( \frac{C}{w} - 1 \right) df \right)
\] (36)

with a flat ground–state \( C = 0 \). Integrating out \( X^+ \) and \( X^- \) in \((22)\), and using the identity \((29)\) with \( \phi = \frac{1}{2} \ln w \) one arrives at a generalized dilaton theory

\[
L = \sqrt{-\tilde{g}} \left( \frac{X}{2} R + \frac{Vw}{2} \tilde{g}^{\mu\nu} \partial_\mu \tilde{X} \partial_\nu \tilde{X} - Vw \right)
\] (37)

where \( X \) is to be re-expressed by \( \tilde{X} \) through the integral of \((33)\).

It should be noted that the (minimal) coupling to matter is covariant under this redefinition of fields. Clearly \((37)\) is the most general action in \( d = 2 \) where the flat ground state corresponds to \( C = 0 \). \( V(X) \) may determine an arbitrarily complicated singularity structure. The DBH is the special case \( V = \lambda^2 = \text{const.} \)
Then $\tilde{X}$ is easily seen to be proportional to the dilaton field. The SBH results from the choice $V = X^{-1/2}$. Using (35) and comparing (36) with (31) in that case with the interaction constant in $w$ fixed by $w = \frac{\tilde{X}}{2}$, the conserved quantity $C$ is identified with the mass $M$ of the BH and (37) turns into the action of spherically reduced 4D general relativity \[7\]. Unfortunately such a theory cannot be solved exactly if coupling to matter is introduced. Therefore, in the next section a large class of models is studied which contain the SBH as a special case.

3 Dilaton Models Containing Schwarzschild-like Black Holes

As discussed already in the last section, any PSM model is locally equivalent to a generalized dilaton model \[4\]. In the present section we consider all global solutions of the action

$$L = \int d^2x \sqrt{-g e^{-2\phi}(R + 4a(\nabla\phi)^2 + B e^{2(1-a-b)\phi})}$$

(38)

and compare them with the Schwarzschild black hole. This form of the Lagrangian covers e.g. the CGHS model \[1\] for $a = 1$, $b = 0$, spherically reduced gravity \[4\] $a = \frac{1}{2}$, $b = -\frac{1}{2}$, the Jackiw-Teitelboim model \[15\] $a = 0$, $b = 1$. Lemos and Sa \[6\] give the global solutions for $b = 1 - a$ and all values of $a$. Mignemi \[8\] considers $a = 1$ and all values of $b$. The models of \[9\] correspond to $b = 0$, $a \leq 1$. The different models can be arranged in an $a$ vs. $b$ diagram (Fig.1).

The Lagrangian (38) is obtained from (22) by the ”generalized dilatonization”, as explained in the last section with $\phi$ replaced by $a\phi$ in (24)

$$\epsilon^d_\mu = e^{-a\phi}\tilde{\epsilon}^d_\mu \quad \Leftrightarrow \quad g_{\mu\nu} = e^{-2a\phi}\tilde{g}_{\mu\nu},$$

(39)

where $a$ is an arbitrary constant. We also replace $X$ by the usual dilaton field

$$X = 2e^{-2\phi}$$

(40)

and restrict ourselves to a power behavior in $V$

$$V(X) = B \left(\frac{X}{2}\right)^b,$$

(41)

depending on the parameters $b$ and $B$ \[13\]. Using the general solution (24), (23) and defining coordinates $v = -4f$, $u = \phi$ immediately yields the line element corresponding to (38)

$$(d\tilde{s})^2 = g(\phi) \left(2dv d\phi + \tilde{l}(\phi)dv^2\right),$$

(42)
with
\[ b \neq -1 : \quad \tilde{l}(\phi) = e^{2\phi} \left( C - \frac{2B}{b+1} e^{-2(b+1)\phi} \right), \]  
(43)
\[ b = -1 : \quad \tilde{l}(\phi) = \frac{e^{2\phi}}{8} \left( \tilde{C} + 4B\phi \right), \quad \tilde{C} = C - 2B \ln 2 \]  
(44)

\[ g(\phi) = e^{-2(1-a)\phi} \]  
(45)
where \( C \) is the arbitrary constant defined in (27). Calculations can be performed in this form or by changing to a new variable
\[ a \neq 1 : \quad u = \frac{e^{-2(1-a)\phi}}{2(a-1)} , \]  
(46)
\[ a = 1 : \quad u = \phi, \]  
(47)
to obtain the EF form (13) of the metric with
\[ b \neq -1 : \quad a \neq 1 : \quad l(u) = B_1 |u|^{\frac{a}{a-1}} - B_2 |u|^{1 - \frac{b}{a-1}} , \]  
(48)
\[ a = 1 : \quad l(u) = \frac{1}{8} e^{2u} \left( C - \frac{2B}{b+1} e^{-2(b+1)u} \right) , \]  
(49)
\[ b = -1 : \quad a \neq 1 : \quad l(u) = \frac{1}{8} 2(a-1)u^{\frac{a}{a-1}} \left( C + \frac{2B}{a-1} \ln |2(a-1)u| \right) , \]  
(50)
\[ a = 1 : \quad l(u) = \frac{e^{2u}}{8} \left( \tilde{C} + 4Bu \right) , \]  
(51)
where the constants are given by
\[ B_1 = \frac{C}{8} (2|a-1|)^{\frac{1}{a-1}} \quad B_2 = \frac{B}{4(b+1)} (2|a-1|)^{1 - \frac{b}{a-1}} . \]  
(52)
The range of \( u \) for \( a = 1 \) is \([-\infty, +\infty]\) whereas from (13) one has
\[ 2(a-1)u > 0, \]
and for \( a > 1 \) the range is reduced to \([0, +\infty]\) and for \( a < 1 \) it is \([-\infty, 0]\). Now the scalar curvature in the EF coordinates
\[ R = l_{,uu} \]
can be easily calculated
\[ b \neq -1, \quad a \neq 1 : \quad R = B_1 u^{\frac{2-a}{a-1}} + B_2 u^{\frac{a+b-1}{b-1}} \]  
(53)
\[ b \neq -1, \quad a = 1 : \quad R = \frac{C}{2} e^{2u} - \frac{Bb^2}{b+1} e^{-2bu} \]  
(54)
\[ b = -1, \quad a \neq 1 : \quad R = \frac{a}{2} |2(a-1)u|^{\frac{2-a}{a-1}} \left( C + \frac{2B}{a} + 2B + \frac{2B}{a-1} \ln |2(a-1)u| \right) \]  
(55)
\[ b = -1, \quad a = 1 : \quad R = \frac{1}{2} e^{2u}(\tilde{C} + 4B + 4Bu) . \]  
(56)
with $B'_1$ and $B'_2$ to be determined easily from (52). In the following we will continue our analysis in terms of $\phi$ which can always be transformed to $u$ by means of (46).

We see that depending on the values of $a$ and $b$ the scalar curvature may be singular in the limits $u \to 0, \pm \infty$. In Fig. 2 the dashed regions show singularities of the scalar curvature. In the limit $\phi \to \infty$ the singular regions are different for zero and nonzero value of $C$,

\[
\begin{align*}
\phi & \to +\infty, \quad C \neq 0, \quad |R| \to \infty, \quad (a < 2) \cup (a + b < 1) \cup (a = 2, b = -1), \\
\phi & \to +\infty, \quad C = 0, \quad |R| \to \infty, \quad (a < 1) \cup (a + b < 1) \cup (a = 2, b = -1). 
\end{align*}
\]

(57) \hspace{1cm} (58)

The singularity of $R$ may be positive or negative depending on the values of the constants $a$, $b$, $B$, and $C$ as can be easily seen from (53)–(56). In the limit $\phi \to -\infty$ the scalar curvature is singular when

\[
\phi \to -\infty, \quad \forall C, \quad |R| \to \infty, \quad (a + b > 1) \cup (a > 2) \cup (a = 2, b = -1).
\]

(59)

Now one has to analyse the behavior and completeness of extremals corresponding to the line element (12). For the first term in (18) the behavior at the singularity for $|u| \to 0, \ a < 1$ coincides with (20) if $u = \bar{u}$ and if $a$ denotes the same parameter as in section 2. However for a general discussion of (18) the behavior of both terms and the zeros of $l$ (horizons) are better discussed in terms of the general procedure as outlined in [11] of which a summary also was given in [1]. The equations for extremals for the EF metric read ($l' = dl/du$)

\[
\begin{align*}
\ddot{u} + l' \dot{u} \dot{\dot{v}} + \frac{1}{2}ll' \dot{v}^2 &= 0, \\
\ddot{v} - \frac{1}{2}l' \dot{v}^2 &= 0.
\end{align*}
\]

(60) \hspace{1cm} (61)

Thus the null extremals $v_1 = \text{const}$ are always complete because $\ddot{u} = 0$ follows from (60) and (61). The second null direction

\[
\frac{dv_2}{du} = \frac{2}{l}
\]

(62)

inserted into (60) also yields $\ddot{u} = 0$ or equivalently

\[
d\tau \sim e^{2(a-1)\phi} d\phi.
\]

Thus the completeness of null extremals depends only on $a$:

\[
\begin{align*}
\phi & \to +\infty, \quad a \geq 1, \quad \text{complete,} \\
\phi & \to -\infty, \quad a \leq 1, \quad \text{complete,}
\end{align*}
\]

(63) \hspace{1cm} (64)

as shown in Fig. 3. The physically unreasonable behavior of null extremals in this case, encountered already in section 2, thus has been reconfirmed also here.
The affine parameter of non-null extremals is obtained from an integral similar to (65)
\[ \frac{du}{d\tau} + l \frac{dv}{d\tau} = \sqrt{A} = \text{const.} \] (65)
Identifying the affine parameter $d\tau$ with the $ds$ in (15), these extremals are found to obey
\[ \frac{dv}{du} = -\frac{1}{l} \left[ 1 \mp (1 - 1/A)^{-1/2} \right] \] (66)
by simply solving a quadratic equation. In addition, from (65) and (66) the affine parameter is determined by
\[ \tau(u) = \int u dy \sqrt{A - l(u)}. \] (67)

For $A > 0$, resp. $A < 0$ the parameter $s$ is a timelike, resp. spacelike quantity. As for the null extremals our results are given in terms of $\phi$. Non null extremals turn out to be complete for
\begin{align*}
\phi &\to -\infty, \quad (a \leq 1) \cap (a + b \leq 1), \\
\phi &\to +\infty, \quad C \neq 0, (a \geq 2) \cap (a + b \geq 1), \\
\phi &\to +\infty, \quad C = 0, (a \geq 1) \cap (a + b \geq 1).
\end{align*}
(68)-(70)
as depicted in Fig.3b,c. One observes that only for $\phi \to +\infty$ the region of incompleteness coincides with the one for $|R| \to \infty$ except for the point $a = 2, b = -1$ where non null extremals are complete at the curvature singularity. As depicted in Fig.2 we find for some distinct values of $a$ and $b$ de Sitter space or flat space-time solutions:
\[ \text{deSitter : } a = 2, b = 0, 1 \quad a = 0, b = 1 \] (71)
\[ \text{flat : } \quad a = 0, b = 0 \] (72)
Furthermore we get the special regions:
\begin{align*}
R = C' : & \quad a = 2 \quad \cap \quad B = 0 \\
R = 0 : & \quad B = 0 \quad \cap \quad C = 0 \\
R = -2 \frac{Bb^2}{b+1} : & \quad (C = 0) \cap (a + b = 1) \cap (B \neq 0) \cap (b \neq 0)
\end{align*}
(73)-(75)
Now elegant methods [11] exist to find the global structure of 2D models. Equation (62) determines the shape for a certain patch, e.g. the typical one for a black hole drawn in Fig.4a. The fully extended global solution across the dotted lines is obtained by gluing together patches of this type identifying lines with $u = \text{const.}$
and exchanging the two null directions \[1\] by an (always existing) diffeomorphism in the overlapping square or triangle. This simply amounts to using reflected or rotated building blocks of the same structure, so as to arrive e.g. for the BH at its characteristic Penrose diagram Fig.4b. For definiteness we consider the case \(B \geq 0\). The case \(B < 0\) is obtained by rotating the diagrams for \(B > 0\) by 90 degrees, because formally both cases are related to each other by exchanging the coordinates of space and time. Up to a rotation there are six types of Penrose diagrams \(D_1, \ldots, D_6\), as shown in Fig.5. The boundaries of the diagrams correspond to infinite values of the dilaton field \(\phi = \pm \infty\) as indicated. There the scalar curvature may be singular or nonsingular. For the moment we concentrate on the shape of the diagrams which is defined entirely by the function \(\tilde{l}(43)\) or (44). Therefore it does not depend on the value of \(a\). We summarize the types of Penrose diagrams corresponding to different values of the constants in the following table:

\[
\begin{align*}
B > 0 & \\
D1 : & C = 0, b > 0; \\
D2 : & C < 0, -1 < b \leq 0; \\
D3 : & C < 0, b > 0; \\
D4 : & C > 0, b > 0; \\
D5 : & C < 0, b \leq -1; \\
D6 : & C = 0, b = 0.
\end{align*}
\] (76)

For \(B = 0\), \(C \neq 0\) a global solution is of the type \(D2\), whereas for \(B = C = 0\) one has a flat solution. Using (53)–(59), or equivalently Fig.2 it is straightforward to verify where boundaries are singular, asymptotically flat or correspond to constant curvature. Completeness at the boundaries follows immediately from (53)–(58).

In order to admit a Schwarzschild like global solution, the Penrose diagram must be of the type \(D5\) with incomplete spacelike singular boundary \(\phi \to +\infty\) and complete asymptotically flat boundary \(\phi \to -\infty\). The above analysis proves that this happens if and only if the parameters satisfy the following conditions

\[
\begin{align*}
B > 0, & \\
& a < 1, \\
& C < 0, \\
& b \leq -1, \\
& C = 0, \\
& b = -1, \\
& C > 0, \\
& -1 \leq b \leq 0.
\end{align*}
\] (77)

In Fig.6 we have tried to summarize all possible cases with one horizon. The range of (77) corresponds to the lower left corner \(a < 1, b \leq 0\).

Known soluble models with couplings to scalar matter \[1, 2, 3, 4\] always start from an 'undilatonized' (PSM–type) action with \(V = \lambda^2\), i.e. from a flat theory in which scalar fields are introduced. The subsequent dilatonization moves the singularity to the factor of \(C\), as explained in the previous section and \(B_1 = C = 0\) should be a Minkowskian space-time.

We want to stress again that in the presence of matter it is this factor (the 'mass' parameter) which is changed by the influx of matter, starting say with a
ground state $C = 0$. The only possibility to start with a Minkowskian space-time before matter flows in ($C = 0 \rightarrow C \neq 0$) is the case when the $B_2$-term of (48) loses its $u$ dependence completely, which implies that $b = a - 1$. From Fig.6 we see immediately that the CGHS model as well as spherically reduced 4D gravity lie on that line. If one is interested to reproduce a SBH like model this means that one has to start out with an action of the restricted form

$$L = \int d^2x \sqrt{-g} e^{-2\phi}(R + 4a(\nabla \phi)^2 + B_2 e^{4(1-a)\phi}), \quad a < 1$$

(78)

Similarly we see that for $b = 0$ one inevitably obtains Rindler space-time [9] except for the CGHS model ($a = 1$).

The soluble models with matter couplings (corresponding to $b = 0$ [1, 9]) are compatible with $b = a - 1$ only at $a = 1$, i.e. for the DBH. Although the curvature vanishes for the asymptotic Rindler like space-times in these models this could be interpreted as observing a black hole formation and its Hawking radiation from an accelerating frame — which does not seem very satisfactory.

It may be added that also the problem of a mass independent Hawking temperature $\kappa$ appears for $b = 0$ models. We take its geometric definition [13] from the norm of the Killing vector $k^2 = l$ at the horizon, determined by $l(u_h) = 0$,

$$\partial_\mu k^2|_{u_h} = 2\kappa k^2|_{u_h}$$

i.e.

$$2\kappa = l'(u_h)$$

(79)

(80)

Using (48) one easily finds the Hawking temperature

$$b \neq -1 : \quad 2\kappa = \pm \frac{b + 1}{1 - a} B_1^{\frac{1}{1+b}} |B_2|^{\frac{1}{1+b}}, \quad (81)$$

$$b = -1 : \quad 2\kappa = \frac{B_2}{2} e^{-C}$$

(82)

where minus and plus signs in eq. (81) correspond to the cases $B_1 > 0$, $B_2 > 0$ and $B_1 < 0$, $B_2 < 0$ respectively. In the case $B_1 B_2 < 0$ the black hole solution is absent as follows from (77). We see that the models with $b = 0$ share the unpleasant feature with the DBH that $\kappa$ is then independent of the conserved quantity $C$ interpreted as the mass since $B_1$ disappears. Another interesting consequence of eqs. (81), (82) is the restriction on the coupling constants following from the positiveness of the temperature: only the first two cases in (77) survive.

4 Summary and Outlook

Because systematic investigations of the global properties of BH models are still relatively rare, an important defect of the ordinary dilaton black hole relative to the
genuine Schwarzschild one seems to have been overlooked until now: Its singularity is only incomplete with respect to non-null extremals. The special role of the DBH has been put into perspective by first embedding it into a family of singularities, to be analyzed according to a power behavior in Kruskal coordinates near the singularity. We also found a large class of models which comprises well known theories and which possesses BHs for a certain range of the parameters. Demanding furthermore that for $C = 0$ (no matter) we have Minkowski space reduces the allowed region to a straigth line $b = a - 1$ (cf. Fig.6). Not surprisingly spherically reduced 4D gravity corresponds to one point on that line ($a = \frac{1}{2}$), as well as the DBH ($a = 1$). For exactly solvable models with matter, within this class of theories the asymptotic background inevitably is of Rindler type and thus may lead to problems of interpretation. In our opinion a soluble model (after interaction with scalar matter is added) with qualitative features coinciding completely with the SBH still waits to be discovered.

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References

[1] E. Witten, Phys. Rev. D 44 (1991) 314; C. G. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, Phys. Rev. D, 45 (1992) 1005; V. P. Frolov, Phys. Rev. D 46 (1992) 5383; J. G. Russo and A. A. Tseytlin, Nucl. Phys. B382 (1992) 259; J. Russo, L. Susskind, and L. Thorlacius, Phys. Lett. B292 (1992) 13; T. Banks, A. Dabholkar, M. Douglas, and M. O’Laughlin, Phys. Rev. D 45 (1992) 3607; S. P. deAlwis, Phys. Lett. B289 (1992) 278.

[2] R.B. Mann, A. Shiekh and L. Tarasov, Nucl. Phys. B341 (1990) 134; D. Banks and M. O’Loughlin, Nucl. Phys. B362 (1991) 649; H.J. Schmidt, J. Math. Phys. 32, (1991) 1562; S.D. Odintsov and I.J. Shapiro, Phys. Lett. B263 (1991) 183 and Mod. Phys. Lett. A7 (1992) 437; I.G. Russo and A.A. Tseytlin, Nucl. Phys. B382 (1992) 259; Volovich, Mod. Phys. Lett. A (1992) 1827; R.P. Mann, Phys. Rev. D47 (1993) 4438; D. Louis–Martinez, J. Gegenberg and G. Kunstatter, Phys. Lett. B321 (1994) 193; D. Louis–Martinez and G. Kunstatter, Phys. Rev. D49 (1994) 5227; J.S. Lemos and P.M. Sa, Phys. Rev. D49 (1994) 2897; S. Mignemi, ”Exact solutions, symmetries and quantization of 2-dim higher derivative gravity with dynamical torsion”, preprint IRS-9502, gr-qc/9508003.
[3] M.O. Katanaev and I. V. Volovich; Phys. Lett., 175B, (1986) 413; Ann. Phys., 197 (1990) 1; W. Kummer and D.J. Schwarz, Phys. Rev., D45, (1992) 3628; H. Grosse, W. Kummer, P. Prešnajder, and D.J. Schwarz, J. Math. Phys., 33 (1992) 3892; T. Strobl, Int. J. Mod. Phys., A8 (1993) 1383; S.N. Solodukhin, Phys. Lett., B319 (1993) 87; E.W. Mielke, F. Gronwald, Yu. N. Obukhov, R. Tresguerres, and F.W. Hehl, Phys. Rev. D, bf 48 (1993) 3648; F. Haider and W. Kummer; Int. J. Mod. Phys. 9 (1994) 207; M. O. Katanaev, Nucl. Phys. B, 416 (1994) 563; N. Ikeda, Ann. Phys., 235 (1994) 435.

[4] M.O. Katanaev, W. Kummer and H. Liebl, Geometric interpretation and classification of global solutions in generalized dilaton gravity, TU Vienna prep. TUW–95–09 (gr-qc/9511009), (to be published in Phys. Rev. D)

[5] A selection of recent reviews is e.g. S.P. de Alvis and D.A. McIntire, Lessons of quantum 2d dilaton gravity, prep. COLO–HEP–241, hep–th/941003; L. Thorlacius, Black hole evolution, prep. NSF–ITP–94–109, [hep–th/9411020]; T. Banks, Lectures on black holes and information loss, prep. RU–94–91, [hep–th/9412131]

[6] J.S. Lemos and P.M. Sa, Phys. Rev. D49 (1994) 2897

[7] P. Thomi, B. Isaak and P. Hajicek, Phys. Rev. D30 (1984), 1168; P. Hajicek, Phys. Rev. D30 (1984), 1178 S.R. Lau ”On the canonical reduction of spherically symmetric gravity”, Techn. Univ. Wien prep. TUW 95-21 (gr–qc/9508028)

[8] S. Mignemi, Phys. Rev. D50, R4733 (1994)

[9] A. Fabbri and J.G. Russo, ‘Soluble models in 2d dilaton gravity’, prep. CERN TH/95-267, [hep-th/9510109].

[10] A very comprehensive global analysis of generally covariant 2d models, without concentrating especially on dilaton theories, has only appeared recently [11], whereas models with torsion had been discussed in detail already before [12].

[11] T. Strobl, Thesis, Tech. Univ. Vienna 1994; T. Klösch and T. Strobl, ”Classical and Quantum Gravity in 1+1 Dimensions: Part I: A unifying approach”, Techn. Univ. Wien prep. TUW-95-16 (gr–qc/9508020), to be published in Classical and Quantum Gravity; ”Classical and Quantum Gravity in 1+1 Dimensions: Part II: All universal coverings”, Techn. Univ. Wien prep. TUW-95-23, (gr–qc/9511081); An early version of this method can be found in N. Walker, J. Math. Phys. 11 (1970) 2280.

[12] M.O. Katanaev, J. Math. Phys. 34 (1993) 700.

[13] Cf. e.g. R.M. Wald, ”General Relativity”, University of Chicago Press, 1984
[14] W. Kummer and P. Widerin, Mod. Phys. Lett. A9 (1994) 1407; W. Kummer and P. Widerin, Phys. Rev. D (1995), to be published. The latter reference also contains a short summary of the PSM in its introductory section.

[15] C. Teitelboim, Phys. Lett. **126B** (1983) 41; R. Jackiw, 1984 Quantum Theory of Gravity, ed S. Christensen (Bristol: Hilger) p 403

[16] For standard dilaton gravity the role of the restriction on X by the redefinitions (39) has been analyzed recently also by M. Cadoni and S. Mignemi, “On the conformal equivalence between 2d black holes and Rindler spacetime”, Univ. Cagliari prep. INFNCA-TH9516, May 1995, (gr-qc/9505032). The possibility to generalize the conformal transformation of the metric has been introduced also in ref. [9].
Fig. 1 Different models in the a-b parameter plane

Fig. 2 Unshaded regions correspond to curvature singularities as $|\phi| \to \infty$. The boundaries belong to the shaded region. Full dots show de Sitter space-times whereas empty dots indicate flat space-times.
a.) Null completeness
b.) Non-null completeness at $\phi = \infty$

c.) Non-null completeness at $\phi = \infty $, $C \neq 0$
d.) Non-null completeness at $\phi = \infty $, $C = 0$

Fig. 3 Completeness of extremals

a.)

Fig. 4 a: Basic patch for a black hole; b: Generic black hole diagram
Fig. 5  Penrose diagrams for a generic black hole (38). The boundaries correspond to infinite values of the dilaton field $\phi \to \pm \infty$ as indicated. Dashed lines inside a diagram denote horizons.

Fig. 6  Shape of the Penrose diagrams with horizon in dependence of the values of the parameters $a$ and $b$. The thick line $b = a - 1$ indicates the region of (78).