On the index of unbalanced signed bicyclic graphs

Changxiang He · Yuying Li · Haiying Shan · Wenyan Wang

Received: 8 December 2020 / Revised: 26 March 2021 / Accepted: 27 March 2021 / Published online: 26 April 2021
© SBMAC - Sociedade Brasileira de Matemática Aplicada e Computacional 2021

Abstract
In this paper, we focus on the index (largest eigenvalue) of the adjacency matrix of connected signed graphs. We give some general results on the index when the corresponding signed graph is perturbed. As applications, we determine the first five largest indices among all unbalanced signed bicyclic graphs on $n \geq 36$ vertices together with the corresponding extremal signed graphs whose indices attain these values.

Keywords Eigenvalue · Index · Unbalanced signed graph · Bicyclic graph

Mathematics Subject Classification 05C50 · 05C22

1 Introduction

Given a simple graph $G = (V(G), E(G))$, let $\sigma : E(G) \rightarrow \{+1, -1\}$ be a mapping defined on the set $E(G)$, then we call $\Gamma = (G, \sigma)$ the signed graph with underlying graph $G$ and sign function (or signature) $\sigma$. Obviously, $G$ and $\Gamma$ share the same set of vertices (i.e., $V(\Gamma) = V(G)$) and have equal number of edges (i.e., $|E(\Gamma)| = |E(G)|$). An edge $e$ is positive (negative) if $\sigma(e) = +1$ (resp. $\sigma(e) = -1$).

Actually, each concept defined for the underlying graph can be transferred to signed graph. For example, the degree of a vertex $v$ in $G$ is also its degree in $\Gamma$. Furthermore, if some subgraph of the underlying graph is observed, then the sign function for the signed subgraph is the restriction of the previous one. Thus, if $v \in V(G)$, then $\Gamma - v$ denotes the signed subgraph having $G - v$ as the underlying graph, while its signature is the restriction from $E(G)$ to $E(G - v)$ (note, all edges incident to $v$ are deleted). Let $U \subset V(G)$, then $\Gamma[U]$ or

Communicated by Carlos Hoppen.

Research supported by the Natural Science Foundation of Shanghai (Grant No. 12ZR1420300), National Natural Science Foundation of China (Nos. 11101284, 11201303 and 11301340).

✉ Changxiang He
changxiang-he@163.com

1 College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China
2 Department of Mathematics, Tongji University, Shanghai 200092, China
\(G[U]\) denotes the (signed) induced subgraph arising from \(U\), while \(\Gamma - U = \Gamma[V(G) \setminus U]\).

Let \(C\) be a cycle in \(\Gamma\), the sign of \(C\) is given by \(\sigma(C) = \prod_{e \in C} \sigma(e)\). A cycle whose sign is \(+\) (resp. \(-\)) is called positive (resp. negative). Alternatively, we can say that a cycle is positive if it contains an even number of negative edges. A signed graph is balanced if no negative cycles exist; otherwise it is unbalanced. There has been a variety of applications of balance, see (Roberts 1999).

The adjacency matrix of a signed graph \(\Gamma = (G, \sigma)\) whose vertices are \(v_1, v_2, \ldots, v_n\) is the \(n \times n\) matrix \(A(\Gamma) = (a_{ij})\), where

\[
a_{ij} = \begin{cases} 
\sigma(v_i v_j), & \text{if } v_i v_j \in E(\Gamma), \\
0, & \text{otherwise}.
\end{cases}
\]

Clearly, \(A(\Gamma)\) is real symmetric and so all its eigenvalues are real. The characteristic polynomial \(\det(x I - A(\Gamma))\) of the adjacency matrix \(A(\Gamma)\) of a signed graph \(\Gamma\) is called the characteristic polynomial of \(\Gamma\) and is denoted by \(\phi(\Gamma, x)\). The eigenvalues of \(A(\Gamma)\) are called the eigenvalues of \(\Gamma\). The largest eigenvalue is often called the index, denoted by \(\lambda(\Gamma)\).

Suppose \(\theta : V(G) \rightarrow \{+1, -1\}\) is any sign function. Switching by \(\theta\) means forming a new signed graph \(\Gamma^\theta = (G, \sigma^\theta)\), whose underlying graph is the same as \(G\), but whose sign function is defined on an edge \(uv\) by \(\sigma^\theta(uv) = \theta(u)\sigma(uv)\theta(v)\). Note that switching does not change the signs or balance of the cycles of \(\Gamma\). If we define a (diagonal) signature matrix \(D^\theta\) with \(d_{ij} = \theta(v)\) for each \(v \in V(G)\), then \(A(\Gamma^\theta) = D^\theta A(\Gamma) D^\theta\). Two signed graphs \(\Gamma_1\) and \(\Gamma_2\) are called switching equivalent, denoted by \(\Gamma_1 \sim \Gamma_2\), if there exists a switching function \(\theta\), such that \(\Gamma_2 = \Gamma_1^\theta\), or equivalently \(A(\Gamma_2) = D^\theta A(\Gamma_1) D^\theta\).

**Theorem 1.1** (Zaslavsky 1982) Let \(\Gamma\) be a signed graph. Then, \(\Gamma\) is balanced if and only if \(\Gamma = (G, \sigma) \sim (G, +1)\).

Switching equivalence is a relation of equivalence, and two switching equivalent signed graphs have the same eigenvalues. In fact, the signature on bridges is not relevant, hence the edges which do not lie on some cycles are not relevant for the signature and they will be always considered as positive.

One classical problem of graph spectra is to identify the extremal graphs with respect to the index in some given class of graphs. For signed graphs, since all signatures of a given tree are equivalent, the first nontrivial signature arises for unicyclic graphs, which was considered in Akbari et al. (2019). The authors determined signed graphs achieving the minimal or the maximal index in the class of unbalanced unicyclic graphs of order \(n \geq 3\). In Fan et al. (2013), the authors characterized the unicyclic signed graphs of order \(n\) with nullity \(n - 2\), \(n - 3\), \(n - 4\), \(n - 5\), respectively. For the energy of signed graphs, see Bhat and Pirzada (2017), Bhat et al. (2018), Hafeez et al. (2019), Pirzada and Bhat (2014), Wang and Hou (2018) and Wang and Hou (2019) for details. The index of \(\Gamma\) is not necessarily equal to its spectral radius, i.e., the maximal absolute value of the \(A(\Gamma)\)-eigenvalues. Thus, it is not surprising that, even if there is a certain overlapping between our research and Belardo et al. (2020), the set of the detected extremal graphs is not the same. In Stanić (2018), the authors consider the behaviour of the index of signed graphs under small perturbations like adding a vertex, adding an edge or changing the sign of an edge.

Here, we will consider unbalanced signed bicyclic graphs, and determine the first five largest indices among all unbalanced signed bicyclic graphs with given order \(n \geq 36\) together with the corresponding extremal signed graphs whose indices attain these values.

Here is the remainder of the paper. In Sect. 2, we study the effect of some edges moving on the index of a signed graph. In Sect. 3, we introduce the three classes of signed bicyclic...
graphs. More importantly, we determine the first five graphs in the set of unbalanced signed bicyclic graphs on \( n \geq 36 \), ordered according to their indices in decreasing order.

## 2 Preliminaries

The purpose of this section is to analyse how the index changes when modifications are made to a signed graph. We start with an important tool which also works in the context of signed graphs. Its general form holds for any principal submatrix of a real symmetric matrix.

**Lemma 2.1** (Cvetkovic et al. 2010) (Cauchy Interlacing Theorem for signed graphs) Let \( \Gamma = (G, \sigma) \) be a signed graph of order \( n \) and \( \Gamma - v \) be the signed graph obtained from \( \Gamma \) by deleting the vertex \( v \). If \( \lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma) \) are the (adjacency) eigenvalues of \( \Gamma \), then

\[
\lambda_1(\Gamma) \geq \lambda_1(\Gamma - v) \geq \lambda_2(\Gamma) \geq \lambda_2(\Gamma - v) \geq \cdots \geq \lambda_{n-1}(\Gamma - v) \geq \lambda_n(\Gamma).
\]

**Lemma 2.2** Let \( \Gamma \) be a signed graph with cut edge \( uv \), and \( x \) be an eigenvector corresponding to the index \( \lambda(\Gamma) \). We have \( \sigma(uv)x_u x_v \geq 0 \).

**Proof** Without loss of generality, we assume that \( x \) is unit and \( \sigma(uv) > 0 \). By contradiction, we suppose that \( x_u x_v < 0 \). Let \( \Gamma_1 \) and \( \Gamma_2 \) be the two connected components of \( \Gamma - uv \), respectively. Set \( x = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \), where \( x_1 \) and \( x_2 \) are the subvectors of \( x \) indexed by the vertices in \( \Gamma_1 \) and \( \Gamma_2 \), respectively. Let \( y = \left( \begin{array}{c} -x_1 \\ x_2 \end{array} \right) \), then

\[
y^T A(\Gamma)y - x^T A(\Gamma)x = -4x_u x_v > 0,
\]

which contradicts the fact that \( x \) maximizes the Rayleigh quotient (Cvetkovic et al. 2010).

\[ \square \]

From the above lemma, it is straightforward to derive the following result.

**Corollary 2.1** Let \( T \) be a vertex induced subtree in a signed graph \( \Gamma \), and \( x \) be an eigenvector corresponding to the index \( \lambda(\Gamma) \). Then, for any edge \( uv \) of \( T \), we have \( \sigma(uv)x_u x_v \geq 0 \).

**Remark 1** If \( T \) is a vertex induced subtree with root \( v \) in a signed graph \( \Gamma \), the above corollary implies that if \( x_v \geq 0 \) we can assume that all edges in \( T \) are positive and all vertices of \( T \) have nonnegative coordinates in \( x \). This is valid, because we can prove it by using suitable switchings to the leaves of the rooted subtree.

We proceed by considering how the index changes when cut edges are moved.

**Lemma 2.3** Let \( u, v \) be two vertices of a signed graph \( \Gamma \), \( vv_1, \ldots, vv_s \) (\( s \geq 1 \)) be cut edges of \( \Gamma \), and \( x \) be an eigenvector corresponding to \( \lambda(\Gamma) \). Let

\[
\Gamma' = \Gamma - vv_1 - \cdots - vv_s + uv_1 + \cdots + uv_s.
\]

The sign of \( uv_i \) in \( \Gamma' \) is \( \sigma(vv_i) \) for \( 1 \leq i \leq s \). If \( |x_u| \geq |x_v| \geq 0 \), we have \( \lambda(\Gamma') \geq \lambda(\Gamma) \).
Proof Without loss of generality, we assume that $x$ is unit. Due to the Rayleigh quotient, we have

$$
\lambda(\Gamma') - \lambda(\Gamma) \geq x^T A(\Gamma') x - x^T A(\Gamma) x = (x_u - x_v) \sum_{i=1}^{s} \sigma(vv_i) x_{v_i}.
$$

Lemma 2.2 tells us that $\sigma(vv_i) x_{v_i} x_v \geq 0$, one can quickly verify that $\lambda(\Gamma') \geq \lambda(\Gamma)$ when $|x_u| \geq |x_v| \geq 0$.

If $vv_1, \ldots, vv_s$ are pendant edges in the above lemma, the eigenvalue equation leads to $\lambda(\Gamma) x_{v_i} = \sigma(vv_i) x_{v_i}$, which implies that $\sigma(vv_i) x_{v_i} x_v > 0$ when $x_v \neq 0$, so we can get a stronger version of the above result.

Lemma 2.4 Let $u$, $v$ be two vertices of a signed graph $\Gamma$, $vv_1, \ldots, vv_s$ $(s \geq 1)$ be pendant edges of $\Gamma$, and $x$ be an eigenvector corresponding to $\lambda(\Gamma)$. Let

$$
\Gamma' = \Gamma - vv_1 - \cdots - vv_s + u v_1 + \cdots + u v_s.
$$

The sign of $vv_i$ in $\Gamma'$ is $\sigma(vv_i)$ for $1 \leq i \leq s$. If $|x_u| \geq |x_v| \geq 0$, we have $\lambda(\Gamma') \geq \lambda(\Gamma)$. Furthermore, if $|x_u| > |x_v| > 0$, then $\lambda(\Gamma') > \lambda(\Gamma)$.

In Lemmas 2.3 and 2.4, the displaced edges are all cut edges. Now the perturbation, $\alpha$-transform, described in the following can be seen in many books and many other papers, which can move noncut edges from one vertex to another.

The following definition can be referred to He and Shan (2010), and we make some changes on this basis.

Definition 2.1 Let $\Gamma$ be a connected signed graph, $uv$ be a nonpendant edge of $\Gamma$ that is not contained in any triangle. Let $N_{\Gamma}(v) \backslash \{u\} = \{v_1, \ldots, v_s\}$ with $s \geq 1$, $N_{\Gamma}(u) \backslash \{v\} = \{u_1, \ldots, u_r\}$ with $r \geq 1$. We define the signed graphs:

$$
\alpha(\Gamma, v \rightarrow u) = \Gamma - uv_1 - vv_2 - \cdots - vv_s + uv_1 + uv_2 + \cdots + uv_s,
$$

$$
\alpha(\Gamma, u \rightarrow v) = \Gamma - uu_1 - uu_2 - \cdots - uu_r + vu_1 + vu_2 + \cdots + vu_r.
$$

The sign of $uv_i$ in $\alpha(\Gamma, v \rightarrow u)$ is $\sigma(vv_i)$ for $1 \leq i \leq s$ and the sign of $uu_i$ in $\alpha(\Gamma, u \rightarrow v)$ is $\sigma(uu_i)$ for $1 \leq i \leq r$.

Remark 2 It is not difficult to verify that $\alpha(\Gamma, v \rightarrow u)$ and $\alpha(\Gamma, u \rightarrow v)$ are isomorphic, and they have the same eigenvalues. Therefore, here, we regard $\alpha(\Gamma, v \rightarrow u)$ and $\alpha(\Gamma, u \rightarrow v)$ as the same signed graph, write as $\alpha(\Gamma, uv)$, and say that $\alpha(\Gamma, uv)$ is an $\alpha$-transform of $\Gamma$ on the edge $uv$. Especially, $\lambda(\alpha(\Gamma, uv)) = \lambda(\alpha(\Gamma, v \rightarrow u)) = \lambda(\alpha(\Gamma, u \rightarrow v))$.

Next, we focus on how the index changes after an $\alpha$-transform is performed.

Lemma 2.5 Let $uv$ be an edge of a signed graph $\Gamma$ and not contained in any triangle, $\Gamma' = \alpha(\Gamma, uv)$ be the signed graph obtained from $\Gamma$ by $\alpha$-transform on the edge $uv$. Let $x$ be an eigenvector corresponding to $\lambda(\Gamma)$. For any edge ab $\in E(\Gamma)$, we write the function $f(a, b) = (x_b - x_a) (\lambda(\Gamma) x_a - \sigma(ab) x_b)$. If $f(v, u) \geq 0$ or $f(u, v) \geq 0$, we have $\lambda(\Gamma') \geq \lambda(\Gamma)$. Furthermore, if $f(v, u) > 0$ or $f(u, v) > 0$, we have $\lambda(\Gamma') > \lambda(\Gamma)$.

Proof Let $N_{\Gamma}(u) \backslash \{v\} = \{u_1, \ldots, u_r\}$ and $N_{\Gamma}(v) \backslash \{u\} = \{v_1, \ldots, v_s\}$. The eigenvalue equation leads to the relations:

$$
\lambda(\Gamma) x_v = \sigma(uv) x_u + \sum_{v_i \in N_{\Gamma}(v) \backslash \{u\}} \sigma(vv_i) x_{v_i},
$$
\[ \lambda(\Gamma)x_u = \sigma(uv)x_v + \sum_{u_i \in N_{\Gamma}(u) \setminus \{v\}} \sigma(uu_i)x_{u_i}. \]

On one hand, \( \lambda(\Gamma') = \lambda(\alpha(\Gamma, v \rightarrow u)) \), then
\[
\begin{align*}
\lambda(\Gamma') - \lambda(\Gamma) &= \lambda(\alpha(\Gamma, v \rightarrow u)) - \lambda(\Gamma) \\
&\geq x^T A(\alpha(\Gamma, v \rightarrow u))x - x^T A(\Gamma)x \\
&= (x_u - x_v) \sum_{v_i \in N_{\Gamma}(v) \setminus \{u\}} \sigma(vv_i)x_{v_i} \\
&= (x_u - x_v)(\lambda(\Gamma)x_v - \sigma(uv)x_u) = f(v, u),
\end{align*}
\]

(1)

On the other hand, \( \lambda(\Gamma') = \lambda(\alpha(\Gamma, u \rightarrow v)) \), then
\[
\begin{align*}
\lambda(\Gamma') - \lambda(\Gamma) &= \lambda(\alpha(\Gamma, u \rightarrow v)) - \lambda(\Gamma) \\
&\geq x^T A(\alpha(\Gamma, u \rightarrow v))x - x^T A(\Gamma)x \\
&= (x_v - x_u) \sum_{u_i \in N_{\Gamma}(u) \setminus \{v\}} \sigma(uu_i)x_{u_i} \\
&= (x_v - x_u)(\lambda(\Gamma)x_u - \sigma(uv)x_v) = f(u, v).
\end{align*}
\]

(2)

Therefore, by (1) and (2), we can know that \( \lambda(\Gamma') \geq \lambda(\Gamma) \) when \( f(v, u) \geq 0 \) or \( f(u, v) \geq 0 \). Furthermore, if \( f(v, u) > 0 \) or \( f(u, v) > 0 \), we have \( \lambda(\Gamma') > \lambda(\Gamma) \).

Remark 3 The conditions in Lemma 2.5 are necessary. For example, the signed graph \( \Gamma \) (as shown in Fig. 1) has index \( \lambda(\Gamma) \approx 2.214 \), the numbers on the vertices of \( \Gamma \) are precisely the coordinates of an unit eigenvector corresponding to \( \lambda(\Gamma) \), the positive edge \( v_2v_3 \) of \( \Gamma \) does not satisfy the conditions in Lemma 2.5. If we let \( \Gamma' = \alpha(\Gamma, v_2v_3) \), then the index \( \lambda(\Gamma') = 2 \) is less than \( \lambda(\Gamma) \).

However, in Lemma 2.5, if \( uv \) is a cut edge, things are easier.

Corollary 2.2 Let \( uv \) be a cut edge of a signed graph \( \Gamma \), and \( \Gamma' = \alpha(\Gamma, uv) \). We have \( \lambda(\Gamma') \geq \lambda(\Gamma) \).

Proof Suppose, without loss of generality, that \( uv \) is positive. Let \( x \) be an unit eigenvector corresponding to \( \lambda(\Gamma) \). By Lemma 2.2, we can know that \( x_u x_v \geq 0 \). Let \( N_{\Gamma}(v) \setminus \{u\} = \{v_1, \ldots, v_s\}, N_{\Gamma}(u) \setminus \{v\} = \{u_1, \ldots, u_r\} \). If \( x_u \geq x_v \geq 0 \), the eigenvalue equation leads to the relation:
\[
\lambda(\Gamma)x_v = x_u + \sum_{v_i \in N_{\Gamma}(u) \setminus \{v\}} \sigma(vv_i)x_{v_i}.
\]

Fig. 1 Example \( \Gamma \) in Remark 3
We claim that \( \sum_{v \in N_G(v) \setminus \{u\}} \sigma(vv_i)x_{v_i} \geq 0 \). Otherwise, we write the component of \( \Gamma - uv \) containing the vertex \( v \) as \( U \). Set \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \), where \( x_1 \) is the subvector of \( x \) indexed by the vertices in \( U - v \). Let \( y = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} \), then \( y^T A(\Gamma) y - x^T A(\Gamma) x = -4x_y \sum_{v \in N_G(v) \setminus \{u\}} \sigma(vv_i)x_{v_i} > 0 \), which contradicts the fact that \( x \) maximizes the Rayleigh quotient.

Since \( \sum_{v \in N_G(v) \setminus \{u\}} \sigma(vv_i)x_{v_i} \geq 0 \), we have \( x_u \leq \lambda(\Gamma)x_v \). By gluing together this inequality with \( x_v \geq x_v \) and Lemma 2.5, we get the assertion.

Similarly, Lemma 2.5 also holds in the case of \( x_u \geq x_u \) and \( x_v < 0 \) (here \( \sum_{v \in N_G(v) \setminus \{u\}} \sigma(vv_i)x_{v_i} \leq 0 \), so \( \lambda(\Gamma') \geq \lambda(\Gamma) \).

If \( x_v \geq x_u \geq 0 \), the eigenvalue equation leads to the relation:
\[
\lambda(\Gamma)x_u = x_v + \sum_{u_i \in N_G(u) \setminus \{v\}} \sigma(uu_i)x_{u_i}.
\]

We claim that \( \sum_{u_i \in N_G(u) \setminus \{v\}} \sigma(uu_i)x_{u_i} \geq 0 \). Otherwise, we write the component of \( \Gamma - uv \) containing the vertex \( u \) as \( U \). Set \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \), where \( x_1 \) is the subvector of \( x \) indexed by the vertices in \( U - u \). Let \( y = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} \), then \( y^T A(\Gamma) y - x^T A(\Gamma) x = -4x_u \sum_{u_i \in N_G(u) \setminus \{v\}} \sigma(uu_i)x_{u_i} > 0 \), which contradicts the fact that \( x \) maximizes the Rayleigh quotient.

Since \( \sum_{u_i \in N_G(u) \setminus \{v\}} \sigma(uu_i)x_{u_i} \geq 0 \), we have \( x_v \leq \lambda(\Gamma)x_u \). By gluing together this inequality with \( x_v \geq x_v \) and Lemma 2.5, we get the assertion.

Similarly, Lemma 2.5 also holds in the case of \( x_u \leq x_u \leq 0 \) (here \( \sum_{u_i \in N_G(u) \setminus \{v\}} \sigma(uu_i)x_{u_i} \leq 0 \), so \( \lambda(\Gamma') \geq \lambda(\Gamma) \).

\[\square\]

### 3 The index of unbalanced signed bicyclic graphs with given order

A graph \( G \) of order \( n \) is called a **bicyclic graph** if \( G \) is connected and the number of edges of \( G \) is \( n + 1 \). A signed graph whose underlying graph is a bicyclic graph, we call it **signed bicyclic graph**.

It is easy to see from the definition that \( G \) is a bicyclic graph if and only if \( G \) can be obtained from a tree \( T \) (with the same order) by adding two new edges to \( T \).

Let \( G \) be a bicyclic graph. The **base of bicyclic graph** \( G \), denoted by \( \hat{G} \), is the (unique) minimal bicyclic subgraph of \( G \). If \( \Gamma = (G, \sigma) \), then we define \( \hat{\Gamma} = (\hat{G}, \sigma|_{\hat{G}}) \) as the **base of signed bicyclic graph** \( \Gamma \). It is easy to see that \( \hat{G} \) is the unique bicyclic subgraph of \( G \) containing no pendant vertices, while \( G \) can be obtained from \( \hat{G} \) by attaching trees to some vertices of \( \hat{G} \).

We now introduce some notation and terminology. In an unsigned context, it has been used, for instance, in He and Shan (2010). It is well-known that there are the following three types of bicyclic graphs containing no pendant vertices:

1. The bicyclic graph \( B(p, q) \) \( (p, q \geq 3) \) obtained from two vertex-disjoint cycles \( C_p \) and \( C_q \) by identifying vertices \( u \) of \( C_p \) and \( v \) of \( C_q \) (see Fig. 2). This type of graph is also known as an **infinity graph** and we write that a graph of this form is an **\(-\infty\)-graph**.
2. The bicyclic graph \( B(p, \ell, q) \) obtained from two vertex-disjoint cycles \( C_p \) and \( C_q \) by joining vertices \( u \) of \( C_p \) and \( v \) of \( C_q \) by a new path \( uu_1u_2\ldots u_{\ell-1}v \) with length \( \ell \) \( (\ell \geq 1) \).
(see Fig. 2). This type of graph is also known as a dumbbell graph; if the cycles are triangles, it also takes the name of hourglass graph.

![Fig. 2 B(p, q) and B(p, l, q)](image)

The bicyclic graph $B(P_k, P_\ell, P_m)$ (1 ≤ $m$ ≤ min\{k, \ell\}) obtained from three pairwise internal disjoint paths from a vertex $x$ to a vertex $y$. These three paths are $xv_1v_2 \ldots v_{k-1}y$ with length $k$, $xu_1u_2 \ldots u_{\ell-1}y$ with length $\ell$ and $xw_1w_2 \ldots w_{m-1}y$ with length $m$ (see Fig. 3). This type of graph is also known as a theta graph and we write that a graph of this form is a $\theta$-graph.

![Fig. 3 B(P_k, P_\ell, P_m)](image)

We are now ready to describe the class $B_n$ of unbalanced signed bicyclic graphs of order $n$.

- $B_n^1 = \{ \Gamma = (G, \sigma) \in B_n \mid \hat{G} = B(p, q), \text{ for some } p, q \geq 3\}$.
- $B_n^2 = \{ \Gamma = (G, \sigma) \in B_n \mid \hat{G} = B(p, \ell, q), \text{ for some } p, q \geq 3 \text{ and } \ell \geq 1\}$.
- $B_n^3 = \{ \Gamma = (G, \sigma) \in B_n \mid \hat{G} = B(P_k, P_\ell, P_m) \text{ for some } 1 \leq m \leq \min\{k, \ell\}\}$.

It is easy to see that

$$B_n = B_n^1 \cup B_n^2 \cup B_n^3.$$  

Next, we deal with the extremal index problem for the class of unbalanced signed bicyclic graphs with order $n$. We will determine the first five graphs in $B_n$, ordered according to their indices in decreasing order.

For unicyclic graphs, there are exactly two switching equivalent classes. If a unicyclic signed graph is balanced, by Theorem 1.1, it is switching equivalent to one with all positive edges. Otherwise, it is switching equivalent to a signed unicyclic graph with exactly one (arbitrary) negative edge on the cycle (Fan et al. 2013). For unbalanced signed bicyclic graphs, we also have similar results.

Even if the following result has been already proved in Belardo et al. (2018), we provide here an alternative proof.

**Lemma 3.1** If $\Gamma \in B_n^1 \cup B_n^2$, then $\Gamma$ is switching equivalent to a signed bicyclic graph with exactly one (arbitrary) negative edge on each unbalanced cycle. If $\Gamma \in B_n^3$, then $\Gamma$ is switching equivalent to a signed bicyclic graph with exactly one (arbitrary) negative edge on the base.
If \( \Gamma \in B^1_n \cup B^2_n \), let \( e_1 \) and \( e_2 \) be two edges of \( \Gamma \) in different cycles, then \( \Gamma - e_1 - e_2 \) is a tree, which is balanced. Therefore, by Theorem 1.1, there exists a sign function \( \theta \) such that \((\Gamma - e_1 - e_2)\theta \) consisting of positive edges. Returning to the graph \( \Gamma \theta \), the edges \( e_1 \) and \( e_2 \) must have a negative sign if only if the cycle to which they respectively belong is negative, since switching does not change the sign of a cycle.

If \( \Gamma \in B^3_n \), let \( e_1, e_2 \) and \( e_3 \) be the three edges of \( \Gamma \) which are incident to a common 3-degree vertex in the base. Similarly, \((\Gamma - e_1 - e_2)\theta \) consisting of positive edges. Returning to the graph \( \Gamma \theta \), if exactly one of \( e_1 \) and \( e_2 \) is negative, the result follows. If both \( e_1 \) and \( e_2 \) are negative, then \( \Gamma \) is switching equivalent to the signed graph which has the same underlying graph as \( \Gamma \), and just has one negative edge \( e_3 \). \( \square \)

The following lemma is a starting point of our discussions.

**Lemma 3.2** Let \( u_1u_2u_3u_4 \) be an induced path in a signed bicyclic graph \( \Gamma \), and \( d_\Gamma(u_2) = d_\Gamma(u_3) = 2 \). Let \( x \) be an eigenvector corresponding to the index \( \lambda(\Gamma) \) and \( \Gamma' = \alpha(\Gamma, u_2u_3) \). If \( x_{u_2} \geq 0, x_{u_3} \geq 0, \sigma(u_1u_2)x_{u_4} \geq 0 \) and \( \sigma(u_3u_4)x_{u_4} \geq 0 \), then \( \lambda(\Gamma') \geq \lambda(\Gamma) \).

**Proof** From Lemma 2.5, it suffices to consider the case that \( u_2u_3 \) is a positive edge. If \( x_{u_2} \leq x_{u_3} \), the eigenvalue equation for the index \( \lambda(\Gamma) \), when restricted to the vertex \( u_2 \) becomes

\[
\lambda(\Gamma)x_{u_2} = \sigma(u_1u_2)x_{u_4} + \sum_{v_j \notin \mathcal{N}(u_2) \setminus \{u_1, u_3\}} \sigma(u_2v_j)x_{v_j} + x_{u_3}.
\]

The fact that \( \Gamma \) is a signed bicyclic graph and \( d_\Gamma(u_2) = d_\Gamma(u_3) = 2 \) imply that \( u_2v_j \) is a cut edge, and then \( \sigma(u_2v_j)x_{v_j} \geq 0 \) follows from Lemma 2.2. Hence, \( x_{u_3} \leq \lambda(\Gamma)x_{u_2} \). By Lemma 2.5, we can get the desired result.

Similarly, we can prove the assertion when \( x_{u_2} \geq x_{u_3} \), just do the similar proof for equation

\[
\lambda(\Gamma)x_{u_3} = \sigma(u_3u_4)x_{u_4} + \sum_{v_j \in \mathcal{N}(u_3) \setminus \{u_2, u_4\}} \sigma(u_3v_j)x_{v_j} + x_{u_2}.
\]

For convenience, we use \( \Gamma + \tilde{u}v \) and \( \Gamma + uv \) (where \( uv \notin E(\Gamma) \)) to denote the signed graph obtained from \( \Gamma \) by adding a new negative edge \( uv \) and a new positive edge \( uv \), respectively.

**Lemma 3.3** Let \( \Gamma = (G, \sigma) \) be a \( \infty \)-type unbalanced signed bicyclic graph, and \( \tilde{G} \notin B_n(3, 3) \), then there is some \( \infty \)-type unbalanced signed bicyclic graph \( \Gamma' \) such that \( |V(\tilde{G})| < |V(\Gamma')| \) and \( \lambda(\Gamma') > \lambda(\Gamma) \).

**Proof** By Lemma 3.1, we can assume that there is exactly one negative edge in each unbalanced cycle, and all edges in balanced cycle are positive.

Let \( u_1u_2 \ldots u_8 \) be the unbalanced cycle of \( \Gamma \) with larger length, \( u_1u_2 \) be its unique negative edge, \( u_1 \) be the unique vertex belonging to two cycles, and again \( x \) be an unit eigenvector corresponding to \( \lambda(\Gamma) \). Without loss of generality, we assume \( x_{u_1} \geq 0 \).

If \( g_1 = 3 \). Let \( u_1u_2' \ldots u_{g_2}' \) (\( g_2 \geq 4 \)) be another cycle of \( \Gamma \), note that \( u_1u_2u_3 \) is the unbalanced cycle with larger length, and \( \tilde{G} \notin B_n(3, 3) \), we find that \( u_1u_2' \ldots u_{g_2}' \) is balanced. We claim that the subvector \( x_{u_1} \) of \( x \) indexed by vertices in the cycle \( u_1u_2' \ldots u_{g_2}' \) is nonnegative. Otherwise, let \( y \) be the vector obtained from \( x \) by replacing all negative entries in \( x_{u_1} \) with their absolute values, then \( y^TA(\Gamma)y \geq x^TA(\Gamma)x \), with equality if and only if \( y \) is also an eigenvector of \( \lambda(\Gamma) \). Then, we can either get the claim (by choosing \( x \) as \( y \)) or a contradiction with the fact that \( x^TA(\Gamma)x \) maximizes the Rayleigh quotient. Note that \( g_2 \geq 4 \), we can get the desired \( \Gamma' \) by using \( \alpha \)-transform on the edge \( u_2'u_3' \). Therefore, in the next, we assume that \( g_1 \geq 4 \).
If all nonzero elements in \( \{x_{u3}, \ldots, x_{ug_1}\} \) have the same sign, we can get the desired unbalanced signed graph by Lemma 3.2. Now, we consider the case that \( \{x_{u3}, \ldots, x_{ug_1}\} \) have different signs.

If \( x_{u2} \geq 0 \), \( x_{u3} \leq 0 \), then \( \Gamma' = \Gamma - u_2u_3 + \overline{u_1u_3} \) is the desired unbalanced signed graph with unbalanced cycle \( u_1u_3 \ldots u_{g_1} \). If there is some edge \( u_1u_{i+1} \), where \( 3 \leq i \leq g_1 - 1 \), such that \( x_{u_i} \geq 0 \), \( x_{u_{i+1}} \leq 0 \), then \( \Gamma' = \Gamma - u_iu_{i+1} + u_1u_i \) is the desired unbalanced signed graph with unbalanced cycle \( u_1u_2 \ldots u_i \).

To complete the proof, it suffices to consider the case that there is some \( 3 \leq s \leq g_1 \) such that \( x_{u_2} \leq 0, \ldots, x_{u_s} \leq 0 \) and \( x_{u_{s+1}} \geq 0, \ldots, x_{u_{g_1}} \geq 0 \). If \( g_1 \geq 5 \), as the larger of \( s - 1 \) and \( g - (s - 1) \) is at least half of \( g_1 \) (which is equal to or greater than 3), so we can get the desired \( \Gamma' \) by Lemma 3.2. It remains to consider the case that \( g_1 = 4 \) and \( x_{u_2} \leq 0 \), \( x_{u_3} \leq 0 \), \( x_{u_4} \geq 0 \).

Using the switching equivalence, we can get a signed graph with all nonnegative entries corresponding to \( \lambda(\Gamma) \). Using Lemma 3.2 again, we can get the desired signed graph. \( \square \)

**Lemma 3.4** Let \( \Gamma = (G, \sigma) \) be a \( \theta \)-type unbalanced signed bicyclic graph, and \( \hat{G} \notin Br(B_2(P_1, P_2, P_3)) \), then there is some \( \theta \)-type unbalanced signed bicyclic graph \( \Gamma' \), such that \( |V(\Gamma')| < |V(\hat{G})| \) and \( \lambda(\Gamma') \geq \lambda(\Gamma) \).

**Proof** Suppose, without loss of generality, that there is just one negative edge in the base.

Let \( u_1 \) be one of the 3-degree vertices of \( \hat{\Gamma} \), \( u_1u_2 \) be the unique negative edge. Again let \( x \) be an unit eigenvector corresponding to \( \lambda(\Gamma) \) with \( x_{u_1} \geq 0 \).

If \( x_{u_2} \geq 0 \), similar to the proof of the case \( g_1 = 3 \) in Lemma 3.3, \( x \) is nonnegative, we can get the desired \( \Gamma' \) by using \( \alpha \)-transform.

Consequently, if \( x_{u_2} < 0 \). Let \( u_1u_2^\prime \ldots u_{p}u_2 \) be the longest path from \( u_1 \) to \( u_2 \). If there is some edge \( u_iu_{i+1} \) such that \( x_{u_i} \leq 0 \), \( x_{u_{i+1}} \geq 0 \), then \( \Gamma' = \Gamma - u_iu_{i+1} + u_2u_i^\prime \) is the desired signed graph. If there is some edge \( u_iu_{i+1} \) such that \( x_{u_i} \geq 0 \), \( x_{u_{i+1}} \leq 0 \), then \( \Gamma' = \Gamma - u_iu_{i+1} + \overline{u_2u_i^\prime} \) is the desired signed graph. If all nonzero entries in \( \{x_{u_2^\prime}, \ldots, x_{u_p}\} \) have the same sign, as before, we can set \( \Gamma' = \alpha(\Gamma, u_2u_3) \).

By Lemmas 3.3 and 3.4, we can narrow the range of the extremal signed graphs. We recall from Belardo et al. (2010) the following Schwenk’s formula.

**Lemma 3.5** Let \( v \) be a fixed vertex of a signed graph \( \Gamma \). Then

\[
\Phi(\Gamma, x) = x\Phi(\Gamma - v, x) - \sum_{u \in E(\Gamma)} \Phi(\Gamma - u - v, x) - 2 \sum_{C \in C_v} \sigma(C) \Phi(\Gamma - C, x),
\]

where \( C_v \) is the set of signed cycles passing through \( v \), and \( \Gamma - C \) is the signed graph obtained from \( \Gamma \) by deleting \( C \).

![Fig. 4 Five signed graphs with maximum index in \( B_n \)](image-url)
Lemma 3.6 Let \( \Gamma_i \in \mathcal{B}_n \) (where \( i = 1, 2, \ldots, 5 \)) be the unbalanced signed graphs as shown in Fig. 4, then \( \lambda(\Gamma_i) \) is the largest root of the equation \( f_i(x) = 0 \), where
\[
\begin{align*}
  f_1(x) &= x^4 - nx^2 + n - 5, \\
  f_2(x) &= x^4 - (n + 1)x^2 + 2n - 4, \\
  f_3(x) &= x^4 - (n + 1)x^2 + 4x + 2n - 8, \\
  f_4(x) &= x^3 + x^2 - (n - 1)x - n + 5, \\
  f_5(x) &= x^3 - x^2 - (n - 2)x + n - 4.
\end{align*}
\]

Furthermore, we have \( \lambda(\Gamma_1) > \lambda(\Gamma_2) > \lambda(\Gamma_3) > \lambda(\Gamma_4) > \lambda(\Gamma_5) \) when \( n \geq 36 \).

**Proof** By Lemma 3.5, we can get the characteristic polynomials of \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5 \) by direct calculation
\[
\Phi(\Gamma_1, x) = x^{n-6}(x^2 - 1)(x^4 - nx^2 + n - 5),
\]
\[ \Phi(\Gamma_2, x) = x^{n-4}(x^4 - (n + 1)x^2 + 2n - 4), \]
\[ \Phi(\Gamma_3, x) = x^{n-4}(x^4 - (n + 1)x^2 + 4x + 2n - 8), \]
\[ \Phi(\Gamma_4, x) = x^{n-6}(x + 1)(x - 1)^2[x^3 + x^2 - (n - 1)x + n + 5], \]
\[ \Phi(\Gamma_5, x) = x^{n-5}(x + 2)(x - 1)[x^3 - x^2 - (n - 2)x + n - 4]. \]

By comparing the indices of these graphs and applying equations above, we have
\[ \Phi(\Gamma_2, x) - \Phi(\Gamma_1, x) = x^{n-6}(x^2 + n - 5) > 0, \]
\[ \Phi(\Gamma_3, x) - \Phi(\Gamma_2, x) = 4x^{n-4}(x - 1), \]
\[ \Phi(\Gamma_4, x) - \Phi(\Gamma_3, x) = x^{n-6}(3x^2 - 4x - n + 5). \]

Lemma 2.1 implies that \( \lambda(\Gamma_i) > \sqrt{n-2} > 1 \) for \( i = 2, 3 \). It is not difficult to see that
\[ \Phi(\Gamma_3, x) > \Phi(\Gamma_2, x) \] when \( x \geq \lambda(\Gamma_2) \) and \( \Phi(\Gamma_4, x) > \Phi(\Gamma_3, x) \) when \( x \geq \lambda(\Gamma_3) \). These are exactly what we need here, \( \lambda(\Gamma_1) > \lambda(\Gamma_2) > \lambda(\Gamma_3) > \lambda(\Gamma_4) \).

To compare \( \lambda(\Gamma_4) \) and \( \lambda(\Gamma_5) \), we let
\[ f_4(x) = x^3 + x^2 - (n - 1)x - n + 5, \]
\[ f_5(x) = x^3 - x^2 - (n - 2)x + n - 4. \]

Then \( g(x) = f_4(x) - f_5(x) = 2x^2 - x - 2n + 9 \) has the largest root \( \frac{1 + \sqrt{16n - 71}}{4} \). One can check directly \( f_5(-\infty) < 0, f_5(0) = n - 4 > 0, f_5(1) = -2 < 0 \) and \( f_5(\frac{1 + \sqrt{16n - 71}}{4}) > 0 \) when \( n \geq 36 \). Hence, the largest root of \( f_5(x) = 0 \) is less than \( \frac{1 + \sqrt{16n - 71}}{4} \), which implies that \( f_4(x) < 0 \) when \( x \) is the largest root of \( f_5(x) = 0 \). Therefore, we have \( \lambda(\Gamma_4) > \lambda(\Gamma_5) \). This completes the proof.

**Lemma 3.7** If \( \Gamma \in \mathcal{B}_n \setminus \{\Gamma_1, \Gamma_4\} \) is an \( \infty \)-type unbalanced signed graph, then \( \lambda(\Gamma) < \lambda(\Gamma_5) \).

**Proof** By Lemma 2.3, 3.3 and Corollary 2.2, it is not difficult to see that, we only need to prove that if \( \Gamma \in \{\Gamma_6, \Gamma_7, \Gamma_8, \Gamma_1^r, \Gamma_4^r\} \), where \( 1 \leq i \leq 6 \) and \( 1 \leq j \leq 4 \) (as shown in Fig. 5), then \( \lambda(\Gamma) < \lambda(\Gamma_5) \). By direct computations, we can prove that
\[ \lambda(\Gamma_5) > \lambda(\Gamma_6) = \max\{\lambda(\Gamma_6), \lambda(\Gamma_7), \lambda(\Gamma_8)\}, \]
\[ \lambda(\Gamma_5) > \lambda(\Gamma_1^r) = \max\{\lambda(\Gamma_1^r), \ldots, \lambda(\Gamma_6^r)\}, \]

and
\[ \lambda(\Gamma_5) > \lambda(\Gamma_4^r) > \lambda(\Gamma_4^r), \lambda(\Gamma_5) > \lambda(\Gamma_4^r) > \lambda(\Gamma_4^r). \]

Hence, we can get the desired result (Fig. 6). \( \square \)

**Fig. 6** Signed graphs considered in proof of Lemma 3.8

\[ \begin{align*}
\Gamma_{11} & \quad \Gamma_{12} & \quad \Gamma_{13}
\end{align*} \]

**Lemma 3.8** If \( \Gamma \) is a dumbbell-type unbalanced signed graph, then \( \lambda(\Gamma) < \lambda(\Gamma_5) \).
Proof By Lemma 2.3, 3.3 and Corollary 2.2, we know that for any $\Gamma \in B_n(p, \ell, q)$, the index of $\lambda(\Gamma) \leq \max\{\lambda(\Gamma_{11}), \lambda(\Gamma_{12}), \lambda(\Gamma_{13})\} < \lambda(\Gamma_5)$.

Lemma 3.9 If $\Gamma \in B_n\setminus\{\Gamma_2, \Gamma_3, \Gamma_5\}$ is a $\theta$-type unbalanced signed graph, then $\lambda(\Gamma) < \lambda(\Gamma_5)$.

Proof By Lemmas 2.3, 3.4 and Corollary 2.2, it is not difficult to see that, we only need to consider the case that $\Gamma \in \{\Gamma_9, \Gamma_{10}, \Gamma_2^i, \Gamma_3^j\}$, where $1 \leq i \leq 7$ and $1 \leq j \leq 5$ (as shown in Fig. 5). By direct computations, we can prove that

- $\lambda(\Gamma_5) > \lambda(\Gamma_9)$, $\lambda(\Gamma_5) > \lambda(\Gamma_{10})$,
- $\lambda(\Gamma_5) > \lambda(\Gamma_2^i) = \max\{\lambda(\Gamma_3^1), \ldots, \lambda(\Gamma_3^5)\}$,
- $\lambda(\Gamma_5) > \lambda(\Gamma_3^1) = \max\{\lambda(\Gamma_3^1), \lambda(\Gamma_3^2), \lambda(\Gamma_3^3), \lambda(\Gamma_3^4)\}$, $\lambda(\Gamma_5) > \lambda(\Gamma_3^2) > \lambda(\Gamma_3^4)$.

Hence, we can get the desired result.

Combining Lemmas 3.6, 3.7, 3.8 and 3.9, we can get the following result immediately. In this theorem, we determine the extremal signed graphs corresponding to the first five largest indices of $B_n$ on $n \geq 36$.

Theorem 3.1 Let $\Gamma_i \in B_n$ (where $i = 1, 2, \ldots, 5$) be the unbalanced signed graphs as shown in Fig. 4, then

1. The index $\lambda(\Gamma_i)$ is the largest root of the equation $f_i(x) = 0$, where

   $f_1(x) = x^4 - nx^2 + n - 5,$
   $f_2(x) = x^4 - (n + 1)x^2 + 2n - 4,$
   $f_3(x) = x^4 - (n + 1)x^2 + 4x + 2n - 8,$
   $f_4(x) = x^3 + x^2 - (n - 1)x - n + 5,$
   $f_5(x) = x^3 - x^2 - (n - 2)x + n - 4.$

2. For $n \geq 36$, we have $\lambda(\Gamma_1) > \lambda(\Gamma_2) > \lambda(\Gamma_3) > \lambda(\Gamma_4) > \lambda(\Gamma_5)$.

3. If $\Gamma \in B_n\setminus\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$, we have $\lambda(\Gamma) < \lambda(\Gamma_5)$.

Remark 4 In the proof of Theorem 3.1, we need $n \geq 36$, because Lemma 3.6 does not work when $n \leq 35$. Which are the extremal signed graphs when the order is between 4 and 35? This is worth thinking about. For $n \in \{4, 5\}$, it is easy. However, for $6 \leq n \leq 35$, maybe $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$ (as shown in Fig. 4) are also the extremal signed graphs, but verifying it need a lot of computational work.

4 Conclusion

In this paper, we proved the conditions increasing the index of unbalanced signed graphs under some edge perturbations. As applications, we determined the first five largest indices among all unbalanced signed bicyclic graphs on $n \geq 36$ vertices together with the corresponding extremal signed graphs. To find these extremal signed graphs, we proved Lemma 3.3 and Lemma 3.4 to narrow the range of the extremal signed graphs. After computation, we determined the structures of extremal signed graphs and their characteristic polynomials in Theorem 3.1.
Appendix

See Table 1.

Table 1  Characteristic polynomials of signed graphs in Sect. 3

| Signed graph | Characteristic polynomial |
|--------------|--------------------------|
| $\Gamma_1$   | $\Phi(\Gamma_1, x) = x^{n-6}(x^2 - 1)(x^4 - nx^2 + n - 5)$ |
| $\Gamma_2$   | $\Phi(\Gamma_2, x) = x^{n-4}[x^4 - (n + 1)x^2 + 2n - 4]$ |
| $\Gamma_3$   | $\Phi(\Gamma_3, x) = x^{n-4}[x^4 - (n + 1)x^2 + 4x + 2n - 8]$ |
| $\Gamma_4$   | $\Phi(\Gamma_4, x) = x^{n-6}(x + 1)(x - 1)^2[x^3 + x^2 - (n - 1)x - n + 5]$ |
| $\Gamma_5$   | $\Phi(\Gamma_5, x) = x^{n-3}(x + 2)(x - 1)[x^3 - x^2 - (n - 2)x + n - 4]$ |
| $\Gamma_6$   | $\Phi(\Gamma_6, x) = x^{n-6}(x - 1)[x^5 + x^4 - nx^3 - nx^2 + (3n - 15)x + n - 5]$ |
| $\Gamma_7$   | $\Phi(\Gamma_7, x) = x^{n-6}(x + 1)[x^5 - x^4 - nx^3 + nx^2 + (3n - 15)x + n - 5]$ |
| $\Gamma_8$   | $\Phi(\Gamma_8, x) = x^{n-6}(x - 1)[x^5 + x^4 - nx^3 - (n - 4)x^2 + (3n - 11)x + n - 5]$ |
| $\Gamma_9$   | $\Phi(\Gamma_9, x) = x^{n-5}(x + 1)[x^4 + x^3 - nx^2 - (n - 4)x + 2n + 8]$ |
| $\Gamma_{10}$| $\Phi(\Gamma_{10}, x) = x^{n-5}(x - 2)(x + 1)[x^3 + x^2 - (n - 2)x + n - 4]$ |
| $\Gamma_{11}$| $\Phi(\Gamma_{11}, x) = x^{n-7}(x - 1)^2(x + 1)[x^4 + x^3 - (n - 1)x^2 - (n - 1)x + 2n - 12]$ |
| $\Gamma_{12}$| $\Phi(\Gamma_{12}, x) = x^{n-7}(x - 1)(x + 1)[x^4 - x^3 - (n - 1)x^2 + (n - 1)x + 2n - 12]$ |
| $\Gamma_{13}$| $\Phi(\Gamma_{13}, x) = x^{n-7}(x - 1)^2[x^5 + 2x^4 - (n - 2)x^3 -(2n - 6)x^2 + (n - 3)x + 2n - 12]$ |
| $\Gamma_{14}$| $\Phi(\Gamma_{14}, x) = x^{n-8}(x - 1)[x^5 + x^4 - nx^3 - nx^2 + (2n - 9)x + 2n + 11]$ |
| $\Gamma_{15}$| $\Phi(\Gamma_{15}, x) = x^{n-6}(x - 1)[x^5 + x^4 - nx^3 - nx^2 + (4n - 23)x + 2n + 11]$ |
| $\Gamma_{16}$| $\Phi(\Gamma_{16}, x) = x^{n-6}(x - 1)[x^5 - x^4 - nx^3 + nx^2 + (2n - 9)x - 2n + 11]$ |
| $\Gamma_{17}$| $\Phi(\Gamma_{17}, x) = x^{n-6}(x - 1)[x^5 - x^4 - nx^3 + nx^2 + (4n - 23)x - 2n + 11]$ |
| $\Gamma_{18}$| $\Phi(\Gamma_{18}, x) = x^{n-6}(x + 1)(x - 1)[x^4 - nx^2 + 5n - 29]$ |
| $\Gamma_{19}$| $\Phi(\Gamma_{19}, x) = x^{n-6}[x^6 - (n + 1)x^4 + (3n - 7)x^2 - 2n + 8]$ |
| $\Gamma_{20}$| $\Phi(\Gamma_{20}, x) = x^{n-6}[x^6 - (n + 1)x^4 + (3n - 8)x^2 + 2x + n + 5]$ |
| $\Gamma_{21}$| $\Phi(\Gamma_{21}, x) = x^{n-6}[x^6 - (n + 1)x^4 + (4n - 14)x^2 + (2n - 10)x - n + 5]$ |
| $\Gamma_{22}$| $\Phi(\Gamma_{22}, x) = x^{n-6}[x^6 - (n + 1)x^4 + (2n - 9)x^2 + (2n - 10)x - n + 5]$ |
| $\Gamma_{23}$| $\Phi(\Gamma_{23}, x) = x^{n-6}[x^6 - (n + 1)x^4 + (4n - 14)x^2 + (2n - 10)x - n + 5]$ |
| $\Gamma_{24}$| $\Phi(\Gamma_{24}, x) = x^{n-4}[x^4 - (n + 1)x^2 + 3n - 9]$ |
| $\Gamma_{25}$| $\Phi(\Gamma_{25}, x) = x^{n-4}[x^4 - (n - 1)x^2 + 2n + 20]$ |
| $\Gamma_{26}$| $\Phi(\Gamma_{26}, x) = x^{n-4}[x^6 - (n + 1)x^4 + (5n - 19)x^2 - 4n + 20]$ |
| $\Gamma_{27}$| $\Phi(\Gamma_{27}, x) = x^{n-6}[x^6 - (n + 1)x^4 + 4x^3 + (3n - 11)x^2 - 4x - 2n + 12]$ |
| $\Gamma_{28}$| $\Phi(\Gamma_{28}, x) = x^{n-6}[x^6 - (n + 1)x^4 + 4x^3 + (3n - 12)x^2 - 2x - n + 6]$ |
| $\Gamma_{29}$| $\Phi(\Gamma_{29}, x) = x^{n-6}[x^6 - (n + 1)x^4 + 4x^3 + (4n - 18)x^2 - (2n - 10)x - n + 5]$ |
| $\Gamma_{30}$| $\Phi(\Gamma_{30}, x) = x^{n-4}[x^4 - (n + 1)x^2 + 3n - 13]$ |
| $\Gamma_{31}$| $\Phi(\Gamma_{31}, x) = x^{n-5}[x^5 - (n + 1)x^3 + 4x^2 + (5n - 23)x - 4n + 20]$ |
| $\Gamma_{32}$| $\Phi(\Gamma_{32}, x) = x^{n-6}(x - 1)[x^5 + x^4 - nx^3 - (n - 4)x^2 + (2n - 5)x + 2n - 11]$ |
| $\Gamma_{33}$| $\Phi(\Gamma_{33}, x) = x^{n-6}(x - 1)[x^5 + x^4 - nx^3 - (n - 4)x^2 + (4n - 19)x + 2n - 11]$ |
| $\Gamma_{34}$| $\Phi(\Gamma_{34}, x) = x^{n-8}(x - 1)^2[x^4 - (n - 1)x^2 + 4x + n + 7]$ |
| $\Gamma_{35}$| $\Phi(\Gamma_{35}, x) = x^{n-7}(x - 1)^2(x + 1)[x^4 + x^3 - (n - 1)x^2 - (n - 5)x + 4n - 24]$ |
References

Akbari S, Belardo F, Heydari F, Maghasedi M, Souri M (2019) On the largest eigenvalue of signed unicyclic graphs. Linear Algebra Appl 581:145–162
Belardo F, Li Marzi EM, Simic SK (2010) Combinatorial approach for computing the characteristic polynomial of a matrix. Linear Algebra Appl 433:1513–1523
Belardo F, Brunetti M, Ciampella A (2018) Signed bicyclic graphs minimizing the least Laplacian eigenvalue. Linear Algebra Appl 557:201–233
Belardo F, Brunetti M, Ciampella A (2020) Unbalanced unicyclic and bicyclic graphs with extremal spectral radius. Czech Math J. https://doi.org/10.21136/CMJ.2020.0403-19 (in press)
Bhat MA, Pirzada S (2017) Unicyclic signed graphs with minimal energy. Discret Appl Math 226:32–39
Bhat MA, Samee U, Pirzada S (2018) Bicyclic signed graphs with minimal and second minimal energy. Linear Algebra Appl 551:18–35
Cvetkovic DM, Rowlinson P, Simic SK (2010) An introduction to the theory of graph spectra. Cambridge University Press, Cambridge
Fan Y, Wang Y, Wang Y (2013) A note on the nullity of unicyclic signed graphs. Linear Algebra Appl 438:1193–1200
Hafeeza S, Farooq R, Khan M (2019) Bicyclic signed digraphs with maximal energy. Appl Math Comput 347:702–711
He CX, Shan HY (2010) On the Laplacian coefficients of bicyclic graphs. Discret Math 310:3404–3412
Pirzada S, Bhat Mushtaq A (2014) Energy of signed digraphs. Discret Appl Math 169:195–205
Roberts FS (1999) On balanced signed graphs and consistent marked graphs. Electron Notes Discret Math 2:94–105
Stanic Z (2018) Pertubations in a signed graph and its index. Discuss Math Graph Theory 38:841–852
Wang D, Hou Y (2018) Unicyclic signed graphs with maximal energy. arXiv:1809.06206
Wang D, Hou Y (2019) Bicyclic signed graphs with at most one odd cycle and maximal energy. Discret Appl Math 260:244–255
Zaslavsky T (1982) Signed graphs. Discret Appl Math 4:47–74

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.