GLOBAL SOLUTION TO A NON LINEAR WAVE EQUATION OF LIQUID CRYSTAL IN THE CONSTANT ELECTRIC FIELD

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Abstract. We construct a global conservative weak solution to the Cauchy problem for the non-linear variational wave equation

$$v_{tt} - c(v)(c(v)v_x)_x + \frac{1}{2}(v + v^3) = 0$$

where $c(\cdot)$ is any smooth function with uniformly positive bounded value. This wave equation is derived from a wave system modelling nematic liquid crystals in a constant electric field.

1. Introduction

1.1. Physical background.

In the natural world, there are three phases of matter, solid, liquid and gas. Between solid and liquid, there is an intermediate phase called liquid crystal. Some solid materials become liquid crystalline at some temperatures. Liquid crystalline means that they can flow with some liquid property as well as have some optical properties of solids. The liquid crystals molecules have ordered arrangement. We can consider the liquid crystals as fluids made up of long rigid molecules. There are several liquid crystalline phases: solid phase, the molecules have orientation and periodicity; liquid phase, the molecules have no orientation or periodicity; nematic phase, molecules have orientation but no periodicity; and Smectic phase, molecules with orientation with some periodicity. In the nematic phase, the orientation of the molecules can be described by a field of unit vector $n(x,t) \in S^2$, the unit sphere. The nematic crystal molecules are invariant under the inversion $n \rightarrow -n$, thus we call $n$ as a director field.

$n$ is a director field of the unit vector to describe the mean orientation of the long molecules in a nematic liquid crystal. Also the famous Oseen-Frank potential energy density $W$ is associated with the director field $n$ in the system. $W$ is defined by

$$W(n, \nabla n) = \alpha |n \times (\nabla \times n)|^2 + \beta (\nabla \cdot n)^2 + \gamma (n \cdot \nabla \times n)^2.$$  

$\alpha, \beta$ and $\gamma$ are positive elastic constants of the liquid crystal. $\alpha$ represents the splay phenomenon of the nematic liquid crystal, $\beta$ represents the bend phenomenon, and $\gamma$ represents the twist phenomenon. The kinetic energy are usually neglected in studies of nematic liquid crystals. Then, by variational principle, we can obtain an elliptic partial differential equation. In the case when we include the kinetic energy, we can form a non-linear wave equation

$$u_{tt} - c(u)[c(u)u_x]_x = 0,$$
with smooth function $u$ on modelling the nematic liquid crystal in one space dimension without any fields applied.

We study the nematic liquid crystal under the a constant electric field with the electric energy density described by

$$f_{\text{electric}} = -\frac{1}{2}\vec{P} \cdot \vec{E} = -\frac{1}{2}\epsilon_0 \epsilon \perp E^2 - \frac{1}{2}\epsilon_0 \Delta \epsilon (\vec{E} \cdot \hat{n})^2,$$

where $\vec{P}$ is the polarization, $\vec{E}$ is the electric field. We assume that the applied field is neither parallel nor perpendicular to $\hat{n}$, the permittivity is $\epsilon \perp$. And the dielectric constants $\epsilon_0, \epsilon \parallel$, and $\epsilon \perp$ are related to the permittivities by $\epsilon_\parallel = 1 + \chi \parallel$ and $\epsilon_\perp = 1 + \chi \perp$. And thus, $\Delta \epsilon = \chi \parallel - \chi \perp$. [10]

1.2. Known results. For the equation (1.1), Glassey, Hunter, and Zheng [7] shows that the smooth solutions develop singularities in finite time. Bressan and Zheng [5] constructed conservative weak solution, showed that the measure to the energy concentrated time is measure zero, and stated that the solution can be extended beyond the time when the singularity appears see also [8]. Zhang and Zheng [19] also constructed the the global weak solution by using the compactness method.

For the conservative solutions, Bressan and Chen [1] showed that for $C^3$ initial data, the solution is piecewise smooth in $t$-$x$ plane, and [2] constructed a metric that renders the flow uniformly Lipschitz continuous on bounded subsetsof $H^1(\mathbb{R})$. Bressan and Huang [4] also constucted the dissipative solution to the system that the total energy is a monotonely decreasing function with respect to time. Huang and Zheng [9] and Chen and Zheng [6], [11]-[13], [15]-[18] studied the existence and uniqueness of the conservative solutions and the singularity formation of a system of nematic liquid crystals.

1.3. main theorems.

Our main results are stated as follows. For the nematic liquid crystal under electric field, we obtain a Cauchy problem

(1.2) \[ v_t - c(v)(c(v)v_x)_x + \frac{1}{2}(v + v^3) = 0, \]

with the initial data

(1.3) \[ v(0, x) = v_0(x), \quad v_t(0, x) = v_1(x). \]

For the smooth function $c(\cdot)$, we assume that $c : \mathbb{R} \mapsto \mathbb{R}^+$ is a bounded and uniformly positive function.

**Theorem 1.1.** Assume that $c : \mathbb{R} \mapsto [K^{-1}, K]$ is a smooth function for some $K > 1$. $v_0(x)$ and $v_1(x)$ are stated in (1.3). Also assume that the initial data $v_0(x)$ is absolutely continuous, $(v_0(x))_x \in L^2$, and $v_1(x) \in L^2$. Then (1.2)-(1.3) can be considered as a Cauchy problem admitting a weak solution $v(t, x)$ defined for all $(t, x) \in \mathbb{R} \times \mathbb{R}$. Moreover, in the $t$-$x$ plane,
$v(x,t)$ is locally Hölder-$\frac{1}{2}$ continuous. For all $1 \leq p < 2$, the map $t \mapsto v(t,\cdot)$ is continuously differentiable with values in $L^p_{loc}$. The weak solution $v(t,\cdot)$ is Lipschitz continuous with respect to $L^2$ distance. That is for all $t,s \in \mathbb{R}$,

\begin{equation}
\|v(t,\cdot) - v(s,\cdot)\|_{L^2} \leq L|t-s|.
\end{equation}

For all test function $\phi \in C^1_c$, the equation \eqref{1.2} satisfies the following integral:

\begin{equation}
\int\int \phi_t v_t - [c(v)\phi]_x[c(v)v_x] - \frac{\phi}{2}(v + v^3)dxdt = 0.
\end{equation}

We define the energy as:

\begin{equation}
E(t) := \frac{1}{2} \int \left\{ v^2_t(t,x) + c^2(v(t,x))v^2_x(t,x) + \frac{v^2(t,x)}{2} + \frac{v^4(t,x)}{4} \right\} dx.
\end{equation}

The energy is uniformly bounded. And we define $E_0$ as follows:

\begin{equation}
E_0 := \frac{1}{2} \int \left\{ v^2_0(x) + c^2(v_0(x))v^2_0(x) + \frac{v^2_0(x)}{2} + \frac{v^4_0(x)}{4} \right\} dx.
\end{equation}

**Theorem 1.2.** A family of weak solutions to the Cauchy problem \eqref{1.2}-\eqref{1.3} can be obtained with the properties:

\begin{equation}
E(t) \leq E_0.
\end{equation}

Let a sequence of initial condition satisfies:

\begin{align*}
\|(v^n_0(x))_x - (v_0(x))_x\|_{L^2} &\to 0, \\
\|v^n_0(x) - v_1(x)\|_{L^2} &\to 0.
\end{align*}

Also, $u^n \to u$ uniformly on bounded subsets of the $t$-$x$ plane and $v^n_0 \to v_0$ on compact sets as $n \to \infty$.

**Theorem 1.3.** There exist a continuous family of positive Radon measures $\{\mu_t : t \in \mathbb{R}\}$.

This family of positive Radon measure is defined on the real line and it satisfies the following properties:

(i) $\mu_t(\mathbb{R}) = E_0$ for any time $t$.

(ii) With respect to Lebesgue measure, the absolutely continuous part of $\mu_t$ has density $\frac{1}{2}(v^2_t + c^2(v)v^2_x + \frac{1}{2}v^2 + \frac{1}{4}v^4)$.

(iii) the singular part of $\mu_t$ has measure zero on the set where $c'(v) = 0$.

The paper is organized as follows. In section 2 we will derive the energy equation and then introduce a new set of dependent variables. Based on those dependent variables, we can formulate a set of equations in terms of the new variables. This set of equations is equivalent to \eqref{1.2}. In section 3, we use a transformation in a Banach space. In the transformation, we will find the suitable weighted norm. This shows that there is a unique solution to the set of equations in terms of the new variables. In section 4, we will show that the integral \eqref{1.5}
holds and the Hölder-$\frac{1}{2}$ continuous condition holds. In section 5, we show that (1.8) holds and the Lipschitz condition on the map $t \mapsto v(t, \cdot)$. This section will mainly prove theorem 1.2. On section 6, we study the maps of $t \mapsto u_x(t, \cdot)$ and $t \mapsto u_t(t, \cdot)$, and complete the proof of theorem 1.1. Section 7 proves the theorem 1.3.

2. Variable Transformations

2.1. Derivation of (1.2). Equation (1.2) has some physical origins. In the context of nematic liquid crystals, we introduce the famous Oseen-Frank potential energy density $W$ given by

\[ W(n, \nabla n) = \alpha |n \times (\nabla \times n)|^2 + \beta (\nabla \cdot n)^2 + \gamma (n \cdot \nabla \times n)^2. \]

As stated in [10], in an electric field the electric energy of the liquid crystal per unit volume is given by

\[ f_{\text{electric}} = -\frac{1}{2} \varepsilon_0 \chi \cdot \mathbf{E}^2 - \frac{1}{2} \varepsilon_0 \Delta \varepsilon (\mathbf{E} \cdot \mathbf{n})^2. \]

We will discuss that when dielectric anisotropy is negative $\Delta \varepsilon < 0$ which means that the electric energy is low when the applied electric field is normal to the liquid crystal director. We choose $\chi = -\Delta \varepsilon$ so that (2.2) is equivalent to $-\frac{1}{2} |n \cdot E^\perp|^2 + 1$. We can say that the electric energy $= -\frac{1}{2} |n \cdot E^\perp|^2$. By the property of the potential energy, the action can be described as $\frac{1}{2} |n|^2 - W(n, \nabla n) + \frac{1}{2} |n \cdot E^\perp|^2$. We denote that $F = \nabla n$, so that $W(n, \nabla n) = W(n, F)$.

And we define: as for $A, B \in \mathbb{R}^{n \times n}$, $A : B = \sum_{j=1}^{n} \sum_{i=1}^{n} A_{ij} B_{ji}$.

The below Euler-Largrangian equation derived from the least action principle

\[ L(n) = |n|^2 - 2W(n, F) + |n \cdot E^\perp|^2 \]

\[ \hat{L}(n, \lambda) = L(n) - \lambda \cdot (|n|^2 - 1) \]

\[ L(n, \lambda) = \int \int |n|^2 - 2W(n, F) + |n \cdot E^\perp|^2 - \lambda(|n|^2 - 1) dx dt \]

\[ \iff 0 = \delta L(n, \lambda) = \int \int n_t \delta n_t - (\partial_n W \cdot \delta n + \partial_F W : \delta \nabla n) + (n \cdot E^\perp) \delta n \cdot E^\perp - \lambda n \delta n \]

\[- \frac{1}{2} \delta \lambda (|n|^2 - 1) dx dt \]

\[ (2.3) \iff n_{tt} - \partial n W(n, \partial_x n) - \partial x [\partial_{\partial_x} W(n, \partial_x n)] + (n \cdot E^\perp) \cdot E^\perp = \lambda n, \quad n \cdot n = 1. \]

For planar deformations depending on a single space variable $x$, the director field has the form $n = \cos u(x,t) \mathbf{i} + \sin u(x,t) \mathbf{j}$ The dependent variable $u \in \mathbb{R}^1$ measures the angle of the director field to the $x$ direction. In this case, we have the wave speed $c$ given specifically by
\( c^2(u) = \gamma \cos^2 u + \alpha \sin^2 u \) and \( \alpha = \beta, \gamma = 0 \) for \( \delta W(n, \nabla n) \). In one space dimension, (2.3) becomes:

\[
\lambda n_i = \partial_t n_i + \partial_n W(n, \partial_x n) - \partial_x [\partial_{\partial_x n} W(n, \partial_x n)] + (n \cdot E^\perp) \cdot E^\perp,
\]

for \( i=1,2,3 \)

(2.4) \( \lambda n_i = \partial_t n_i + \partial_n W(n, \partial_x n) - \partial_x [\partial_{\partial_x n} W(n, \partial_x n)] + (n \cdot E^\perp) \cdot E^\perp, \)

from (2.6) we can compute that,

(2.6) \( W(n, \partial_x n) = \frac{\alpha}{2} (\partial_x n_1)^2 + \frac{\beta}{2} (\partial_x n_2)^2 + \frac{1}{2} (\gamma - \beta) n_1^2 |\partial_x n|^2. \)

We denote that \( c_1^2 = \alpha + (\gamma - \alpha)n_1^2 \) and \( c_2^2 = \beta + (\gamma - \beta)n_1^2 \). With (2.4), (2.5) and (2.6), we can compute that

(2.7) \( \partial_t n_1 - \partial_x [c_1^2(n_1) \partial_x n_1] + (n \cdot E^\perp) \delta n \cdot E^\perp = \{-|n_i|^2 + (2c_2^2 - \gamma) |n_x|^2 + 2(\alpha - \beta)(\partial_x n_1)^2 - |n \cdot E^\perp|^2\} n_1. \)

In particular, taking \( \alpha = \beta \) in (2.6), we let

\( c^2(u) = c_1^2(n_1) = c_2^2(n_1), \)

and from (2.6) we can compute that,

\( \partial_t n_1 = -\sin(u) u_{tt} - \cos(u) u_t^2, \)

\( -\partial_x (c^2(n_1) \partial_x n_1) = c^2(u) \sin(u) u_{xx} - 2\gamma \cos(u) u_x^2 + 3\gamma \cos^3(u)u_x^2 + 3\alpha \sin^2(u) \cos(u) u_x^2, \)

\( (2c^2(u) - \gamma) |n_x|^2 n_1 = 2\alpha \cos(u) u_x^2 + 2\gamma \cos^3(u)u_x^2 - 2\alpha \cos^3(u)u_x^2 - \gamma \cos(u) u_x^2, \)

\( -|n_t|^2 n_1 = -u_t^2 \cos(u), \)

\( \lambda n_1 = -|n_t|^2 n_1 + (2c^2(u) - \gamma) |n_x|^2 n_1 - (n \cdot E^\perp) n_1, \)

\( \iff \lambda n_1 = \{-|n_t|^2 + (2c_2^2 - \gamma) |n_x|^2 + 2(\alpha - \beta)(\partial_x n_1)^2 - |n \cdot E^\perp|^2\} n_1 = -\cos(u). \)

Thus we plug in the above results into (2.7) and get

\[
0 = -\sin(u) u_{tt} + c^2(u) \sin(u) u_{xx} - \gamma \cos(u) u_x^2 + \gamma \cos^3(u) u_x^2 \\
+ \alpha \cos(u) u_x^2 - \alpha \cos^3(u) u_x^2 + \cos^2(u) + \cos(u) \\
0 = -\sin(u) u_{tt} + c^2(u) \sin(u) u_{xx} - \gamma \sin^2(u) u_x^2 \cos(u) + \alpha \cos(u) u_x^2 \sin(u) + \cos^2(u) + \cos(u) \\
0 = -u_{tt} + c^2(u) u_{xx} - \gamma \cos(u) \sin(u) u_x^2 + \alpha \cos(u) \sin(u) u_x^2 + \frac{\cos^2(u)}{\sin(u)} + \frac{\cos(u)}{\sin(u)} \\
0 = -u_{tt} + c^2(u) u_{xx} + (\gamma \cos(u) \sin(u) + \alpha \cos(u) \sin(u)) u_x^2 + \frac{\cos^2(u)}{\sin(u)} + \frac{\cos(u)}{\sin(u)}.
\]
\[
(2.8) \quad u_{tt} - c(u)(c(u)u_x)_x - \frac{\cos(u)(1 + \cos(u))}{\sin(u)} = 0,
\]

For the asymptotic equation \((2.8)\), we do the Taylor expansion for \(\frac{\cos(u)(1 + \cos(u))}{\sin(u)}\) around point \(u = \pi\) ignoring some high order terms, \((2.8)\) becomes
\[
u_{tt} - c(u)(c(u)\nu_x)_x + \frac{1}{2} (u + \pi)^3 + \frac{1}{2} (u + \pi) = 0.
\]
we let our \(v\) to be \(v = u + \pi\) that give us that
\[
(2.9) \quad \nu_{tt} - c(v)(c(v)\nu_x)_x + \frac{1}{2} (\nu + v^3) = 0.
\]

2.2. Definitions. This subsection will provide definitions used in this paper.

**Definition 2.1.** \(L^p\) space is defined as follows.

Let \(X\) be a measure space. Given a function \(f\), we say \(f \in L^p\) on \(X\) if \(f\) is Lebesgue measurable and if
\[
\int_X |f|^p d\mu < +\infty.
\]
Then \(f\) is \(L^p\)-integrable.

For \(f \in L^p(\mu)\), we define the norm
\[
\|f\|_{L^p} := \left( \int_X |f|^p d\mu \right)^{\frac{1}{p}}.
\]
And call \(\|f\|_{L^p}\) as the \(L^p(\mu)\) norm of \(f\).

**Definition 2.2.** Locally integrable function \(L^p_{loc}\) is defined as follows.

Let \(\Omega\) be an open set in the \(\mathbb{R}^n\) space and \(f : \Omega \to \mathbb{C}\) be a Lebesgue measurable function.

For given \(p\) with \(1 \leq p \leq +\infty\) and this \(p\) satisfies
\[
\int_K |f|^p dx < +\infty,
\]
It belong to the \(L^p(K)\) for all compact subsets \(K\) of \(\Omega\), then \(f\) is called locally \(p\)-integrable.
Then the set of all such function is denoted by \(L^p_{loc}\):
\[
L^p_{loc}(\Omega) = \{ f : \Omega \to \mathbb{C} \text{ measurable} \mid f \in L^p(K), \forall K \subset \Omega, K \text{ is compact} \}.
\]

**Definition 2.3.** Hölder continuous is defined as follows.

We say function \(f(x)\) is Hölder continuous with exponent \(\alpha\) if there exists a constant \(C \geq 0\) and \(\alpha\) such that
\[
|f(x) - f(y)| \leq C|x - y|^{\alpha},
\]
for all \(x\) and \(y\) in the domain of \(f\).
2.3. Derivation of the Energy Equation.

From (2.9) we can compute that

\[
0 = \int v_t v_{tt} - v_t c(v)[c(v)v_x]_x + \frac{v}{2} v_t + \frac{v^3}{2} v_t \, dx \\
= \int \left( \frac{1}{2} v_t^2 \right)_t + \left( \frac{c^2(v) v_x^2}{2} \right)_t + \left( \frac{1}{4} v^2 \right)_t + \left( \frac{1}{8} v^4 \right)_t \, dx,
\]

(2.10)

\[
\int \left( \frac{1}{2} v_t^2 \right)_t + \left( \frac{c^2(v) v_x^2}{2} \right)_t + \left( \frac{1}{4} v^2 \right)_t + \left( \frac{1}{8} v^4 \right)_t \, dx = 0.
\]

And from (2.10), we can get our Energy equation as

(2.11)

\[
E := \frac{1}{2} \left( v_t^2 + c^2(v) v_x^2 + \frac{1}{2} v^2 + \frac{1}{4} v^4 \right).
\]

2.4. Variables transform. In this section we will derive some identities that holds for smooth solutions. We first denote variables:

(2.12)

\[
\begin{align*}
R &:= v_t + c(v)v_x, \\
S &:= v_t - c(v)v_x.
\end{align*}
\]

Thus, we can write \( v_t \) and \( v_x \) as follows

(2.13)

\[
\begin{align*}
v_t &= \frac{R + S}{2}, \\
v_x &= \frac{R - S}{2c}.
\end{align*}
\]

By (1.2), the following identities are valid

(2.14)

\[
\begin{align*}
S_t + cS_x &= \frac{c'}{4c} (S^2 - R^2) - \frac{1}{2} (v + v^3), \\
R_t - cR_x &= \frac{c'}{4c} (R^2 - S^2) - \frac{1}{2} (v + v^3).
\end{align*}
\]

By the following calculation

\[
R_t - cR_x = (u_t + cu_x)_t - c(u_t + cu_x)_x
= -\frac{v}{2} - \frac{v^3}{2} + \frac{c'}{4c} (R^2 - S^2).
\]

And we can compute \( S_t + cS_x \) in the similar way to get (2.14).

We denote energy and momentum as

(2.15)

\[
E := \frac{1}{2} \left( v_t^2 + c^2(v) v_x^2 + \frac{1}{2} v^2 + \frac{1}{4} v^4 \right) = \frac{R^2 + S^2}{4} + \frac{1}{8} (2v^2 + v^4),
\]

(2.16)

\[
M := -u_t u_x = \frac{S^2 - R^2}{4c}.
\]

The analysis of (1.2) has a main difficult that the possible breakdown of the regularity solutions. The quantities \( v_x \) and \( v_t \) can blow up in finite time even with smooth initial
data. Thus we need to introduce a new set of dependent variables to deal with the possible unbounded value $R$ and $S$:

\begin{equation}
(2.17) \quad w := 2 \arctan R, \quad z := 2 \arctan S.
\end{equation}

Thus we have

\begin{equation}
(2.18) \quad R = \tan \left( \frac{w}{2} \right), \quad S = \tan \left( \frac{z}{2} \right).
\end{equation}

By (2.14), we have the equations:

\begin{equation}
(2.19) \quad w_t - cw_x = \frac{2}{1 + R^2} (R_t - cR_x) = \frac{c'}{2} \frac{R^2 - S^2}{1 + R^2} - \frac{1}{1 + R^2} (v + v^3),
\end{equation}

\begin{equation}
(2.20) \quad z_t + cz_x = \frac{2}{1 + S^2} (S_t + cS_x) = \frac{c'}{2} \frac{S^2 - R^2}{1 + S^2} - \frac{1}{1 + S^2} (v + v^3).
\end{equation}

In order to reduce the equation to a semi-linear system, we need to have a further change of variables. The forward characteristics equation and the backward characteristics equation:

\begin{equation}
(2.21) \quad \dot{x^+} = c(v), \quad \dot{x^-} = c(v).
\end{equation}

And we denote the characteristics lines pass through the point $(t, x)$ as

\begin{equation}
(2.22) \quad s \rightarrow x^+(s, t, x), \quad s \rightarrow x^-(s, t, x).
\end{equation}

So we can use a new coordinate system $(X, Y)$ to represent point $(t, x)$ by

\begin{equation}
(2.23) \quad X := \int_0^{x^-(0,t,x)} (1 + R^2(0, x)) dx,
\end{equation}
From (2.26), we can compute that
\[ Y := \int_{x^+(0,t,x)}^{0} (1 + S^2(0,x))dx. \]

And (2.21) and (2.22) implies that
\[ X_t - c(v)X_x = 0, \quad Y_t + c(v)Y_x = 0, \]
\[ (X_x)_t - (c(v)X)_x = 0, \quad (Y_x)_t + (c(v)Y_x)_x = 0. \]

Thus, given any smooth function \( f \), by using (2.23) we have
\[ f_t + c(v)f_x = 2c(v)X_xf_X, \]
\[ f_t - c(v)f_x = 2c(v)Y_xf_Y. \]

From (2.23), we have \( X_t + c(v)X_x = 2c(v)X_x \). To get (2.25) we compute directly
\[ f_t + c(v)f_x = f_XX_t + f_YY_t + c(v)f_XX_x + c(v)f_YX_x = (X_t + c(v)X_x)f_X = 2c(v)X_xf_X, \]
\[ f_t - c(v)f_x = f_XX_t + f_YY_t - c(v)f_XX_x - c(v)f_YX_x = (Y_t - c(v)Y_x)f_Y = 2c(v)Y_xf_Y. \]

Introducing new variables
\[ p := \frac{1 + R^2}{X_x}, \quad q := \frac{1 + S^2}{-Y_x}. \]

From (2.26), we can compute that
\[ \frac{1}{X_x} = \frac{p}{1 + R^2} = p \cos^2\left(\frac{w}{2}\right) = \frac{p(1 + \cos w)}{2}, \]
\[ \frac{1}{Y_x} = \frac{q}{1 + S^2} = q \cos^2\left(\frac{z}{2}\right) = \frac{q(1 + \cos z)}{2}. \]

Apply (2.19)-(2.20) to (2.20), we have
\[ w_t - cw_x = 2c \frac{1 + S^2}{q} w_y = \frac{c' R^2 - S^2}{2c} \frac{1}{1 + R^2} + \frac{1}{1 + R^2}(-v - v^3), \]
\[ z_t + cz_x = 2c \frac{1 + R^2}{p} z_y = \frac{c' S^2 - R^2}{2c} \frac{1}{1 + S^2} + \frac{1}{1 + S^2}(-v - v^3). \]

Thus, we can compute \( w_Y \) and \( z_X \) as
\[ w_Y = \frac{c' R^2 - S^2}{4c^2} \frac{q}{1 + R^2} \frac{1}{1 + S^2} - \frac{q}{2c} \frac{1}{1 + S^2} \frac{1}{1 + R^2}(v + v^3), \]
\[ z_X = \frac{c' S^2 - R^2}{4c^2} \frac{p}{1 + S^2} \frac{1}{1 + R^2} - \frac{p}{2c} \frac{1}{1 + S^2} \frac{1}{1 + R^2}(v + v^3). \]

So we have
\[ \begin{align*}
  w_Y &= \frac{c' R^2 - S^2}{4c^2} (\cos z - \cos w) - \frac{q}{2c} (v + v^3)(1 + \cos z)(1 + \cos w), \\
  z_X &= \frac{c' S^2 - R^2}{4c^2} (\cos w - \cos z) - \frac{p}{2c} (v + v^3)(1 + \cos z)(1 + \cos w).
\end{align*} \]
By using (2.24) and (2.27),

\[ p_t - cp_x = \frac{1}{X_x} 2R(R_t - cR_x) - \frac{1}{X_x^2} [(X_x)_t - c(X_x)_x](1 + R^2) \]

\[ = \frac{c'}{2c} \frac{p}{1 + R^2} \left[ S(1 + R^2) - R(1 + S^2) \right] - \frac{p}{1 + R^2} R(v + v^3), \]

\[ q_t + cq_x = \frac{1}{Y_x} 2S(S_t - cS_x) - \frac{1}{Y_x^2} [(-Y_x)_t + c(-Y_x)_x](1 + S^2) \]

\[ = \frac{c'}{2c} \frac{q}{1 + S^2} \left[ R(1 + S^2) - S(1 + R^2) \right] - \frac{q}{1 + S^2} S(v + v^3). \]

By applying (2.25),

\[ p_t - cp_x = -2cY_x p_Y, \]

\[ q_t + cq_x = 2cX_x q_X. \]

And thus,

\[ p_Y = (p_t - cp_x) \frac{1}{-2cY_x} = (p_t - cp_x) \frac{1}{2c} \frac{q}{1 + S^2} \]

\[ = \frac{c'}{8c^2} \left[ \sin z - \sin w \right] pq - \frac{1}{8c} \frac{pq \sin w(v + v^3)(1 + \cos z)}, \]

\[ q_X = (q_t + cq_x) \frac{1}{2cX_x} = (q_t + cq_x) \frac{1}{2c} \frac{p}{1 + R^2} \]

\[ = \frac{c'}{8c^2} \left[ \sin w - \sin z \right] pq - \frac{1}{8c} \frac{pq \sin z(v + v^3)(1 + \cos w)}. \]

So, we have the following identities:

\[ \begin{align*}
    p_Y &= \frac{c'}{8c^2} \left[ \sin z - \sin w \right] pq - \frac{1}{8c} \frac{pq \sin w(v + v^3)(1 + \cos z)}, \\
    q_X &= \frac{c'}{8c^2} \left[ \sin w - \sin z \right] pq - \frac{1}{8c} \frac{pq \sin z(v + v^3)(1 + \cos w)}. 
\end{align*} \tag{2.29} \]

Also, we apply \( v \) into (2.25) and get

\[ \begin{align*}
    v_X &= (v_t + cv_x) \frac{1}{2c} \frac{p}{1 + R^2} = \frac{1}{2c} \left( \tan \frac{w}{2} \cos^2 \frac{w}{2} \right) p = p \frac{1}{4c} \sin w, \\
    v_Y &= (v_t - cv_x) \frac{1}{2c} \frac{q}{1 + S^2} = \frac{1}{2c} \left( \tan \frac{z}{2} \cos^2 \frac{z}{2} \right) q = q \frac{1}{4c} \sin z. 
\end{align*} \tag{2.30} \]

Combining (2.28), (2.29), and (2.30), we obtain a semi-linear hyperbolic system from the non-linear equation (1.2). This system uses X,Y as independent variables with smooth coefficients for the variables v, w, z, p, q

\[ \begin{align*}
    w_Y &= \frac{c'}{8c^2} \left( \cos z - \cos w \right) q - \frac{q}{8c} \left( v + v^3 \right)(1 + \cos z)(1 + \cos w), \\
    z_X &= \frac{c'}{8c^2} \left( \cos w - \cos z \right) p - \frac{p}{8c} \left( v + v^3 \right)(1 + \cos z)(1 + \cos w), \\
    \end{align*} \tag{2.31} \]

\[ \begin{align*}
    p_Y &= \frac{c'}{8c^2} \left[ \sin z - \sin w \right] pq - \frac{1}{8c} \frac{pq \sin w(v + v^3)(1 + \cos z)}, \\
    q_X &= \frac{c'}{8c^2} \left[ \sin w - \sin z \right] pq - \frac{1}{8c} \frac{pq \sin z(v + v^3)(1 + \cos w)}, \\
    \end{align*} \tag{2.32} \]
The system (2.31)-(2.33) should have non-characteristic boundary conditions related to (1.3). From (1.3), \( v_0 \) and \( v_1 \) determine the initial values of \( R \) and \( S \) at time \( t = 0 \). We denote the curve \( \gamma \) as the line in \((X,Y)\) plane at time \( t = 0 \), say \( Y = \varphi(X), \quad X \in \mathbb{R}. \)

And \( Y = \varphi(X) \) if and only if for some \( x \in \mathbb{R}, \)

\[
X = \int_0^x (1 + R^2(0, x))dx, \quad Y = \int_x^0 (1 + S^2(0, x))dx.
\]

By the assumptions of Theorem 1.1, \( v_0 \in H^1, v_1 \in L^2 \). This implies that \( R \in L^2 \) and \( S \in L^2 \). Moreover, we let

\[
E_0 := \frac{1}{4} \int [R^2(0, x) + S^2(0, x)]dx < \infty.
\]

Thus, we have

\[
X(x) := \int_0^x (1 + R^2(0, y))dy, \quad Y(x) := \int_x^0 (1 + S^2(0, y))dy.
\]

are absolutely continuous and well defined functions. Further more, by observing (2.35), \( X \) is increasing and \( Y \) is decreasing. So, we can conclude that the map \( X \mapsto \varphi(X) \) is continuous and decreasing. And from (2.34), we have

\[
|X + \varphi(X)| \leq 4E_0.
\]

We know that \((t, x) \in [0, \infty) \times (-\infty, \infty), \) thus, our new independent variables \((X, Y) \in \Omega^+, \) and the domain is defined as

\[
\Omega^+ := \{(X, Y) : Y \geq \varphi(X)\},
\]

along the curve \( \gamma := \{(X, Y) : Y = \varphi(X)\}. \)

We can have the following boundary data \((\bar{w}, \bar{z}, \bar{p}, \bar{q}, \bar{v}) \in L^\infty, \)

\[
\begin{align*}
(2.36) \quad & \bar{w} = 2 \arctan(R(0, x)), \\
& \bar{z} = 2 \arctan(S(0, x)), \\
(2.37) \quad & \bar{p} \equiv 1, \\
& \bar{q} \equiv 1, \\
(2.38) \quad & \bar{v} = v_0(x).
\end{align*}
\]
3. Construct the integral solution

We will proof the global existence and uniqueness for the semi-linear system \((2.31) - (2.33)\).

**Theorem 3.1.** If the assumptions of Theorem 1.1 holds, then the semi-linear system \((2.31) - (2.33)\) with the boundary conditions \((2.36) - (2.38)\) has a unique solution for all \((X,Y) \in \mathbb{R} \times \mathbb{R}^+\).

We will construct the solution on the region \(\Omega^+\) which is the case that \(Y \geq \varphi(X)\). The proof of the solution on the \(\Omega^-\) which is the case that \(Y \leq \varphi(X)\) can be construct in the similar way. We will show the Lipschitz condition for the system \((2.31) - (2.33)\). To make sure the solution is defined in the region \(\Omega^+\), we need to construct some priori bounds. So that we can show that \(p,q\) are bounded. The Lipschitz condition can be derived as follows.

We first assume that

\[
C_1 := \sup_{v \in \mathbb{R}} \left| \frac{c'(v)}{4c^2(v)} \right| < \infty,
\]

\[
K_1 := \sup_{v \in \mathbb{R}} \int \frac{v^2}{2} + \frac{v^4}{4} \, dx < \infty,
\]

\[
K_0 := \sup_{v \in \mathbb{R}} v + v^3 < \infty.
\]

From \((2.32)\), we have the identity:

\[
q_X + p_Y = \frac{1}{2} \left[ (\frac{v^2}{2} + \frac{v^4}{4})q(1 + \cos z) \right]_X + \frac{1}{2} \left[ (\frac{v^2}{2} + \frac{v^4}{4})p(1 + \cos w) \right]_Y - \frac{1}{2} \left( \frac{v^2}{2} + \frac{v^4}{4} \right) \frac{c'}{8c} (\sin w - \sin z)(\cos z - \cos w)(\frac{1}{c} + 1).
\]

We construct a closed curve \(\Sigma\) for every \((X,Y) \in \Omega^+\) with the vertical line segment connect \((X,Y)\) with \((X,\varphi(X))\), the horizontal line segment connect \((X,Y)\) with \((\varphi^{-1}(Y),Y)\), and a part of the boundary \(\gamma = Y = \varphi(X)\) connecting \((X,\varphi(X))\) with \((\varphi^{-1}(Y),Y)\). The closed curve \(\Sigma = \Gamma_1 + \Gamma_2 + \Gamma_3\).

From \((3.3)\) we can compute \(\iint q_X + p_Y \, dA = \int -p \, dX + \int q \, dY\) and denote that

\[
\int \int q_X + p_Y \, dA = \int -p \, dX + \int q \, dY,
\]

\[
Q_X := \frac{1}{2} \left[ (\frac{v^2}{2} + \frac{v^4}{4})q(1 + \cos z) \right]_X,
\]

\[
P_Y := \frac{1}{2} \left[ (\frac{v^2}{2} + \frac{v^4}{4})p(1 + \cos w) \right]_Y,
\]

\[
\xi := \left( \frac{v^2}{2} + \frac{v^4}{4} \right) \frac{c'}{8c} (\sin w - \sin z)(\cos z - \cos w)(\frac{1}{c} + 1).
\]
So we have

\[ \int \int_{\Sigma} p_y + q_X dX dY = \int_{\Sigma} -pdX + \int_{\Sigma} qdY = \int_{\Sigma} Q_X + P_Y - \frac{1}{2} \xi dX dY, \]

\[ \iff \int_{\Sigma} -pdX + \int_{\Sigma} qdY = \int_{\Sigma} -PdX + \int_{\Sigma} QdY - \frac{1}{2} \int \int_{\Sigma} \xi dX dY. \]

\[ \iff \frac{1}{2} \int \int_{\Sigma} \xi dX dY = \int_{\Sigma} p - PdX + \int_{\Sigma} Q - qdY, \]

\[ = \int_{\Sigma} p - \frac{1}{2} \left( \frac{v^2}{2} + \frac{v^4}{4} \right) p(1 + \cos w) dX + \int_{\Sigma} -q + \frac{1}{2} \left( \frac{v^2}{2} + \frac{v^4}{4} \right) q(1 + \cos z) dX. \]

Since \( \Sigma = \Gamma_1 + \Gamma_2 + \Gamma_3 \), so we compute the integral of \( \Gamma_1 \) directly and in the way \( \Gamma_1 = - (\Gamma_2 + \Gamma_3) \).

\[ \begin{align*}
\int_{\Gamma_1} 1 - \frac{1}{2} \left( \frac{v^2}{2} + \frac{v^4}{4} \right)(1 + \cos w) dX + \int_{\Gamma_1} -1 + \frac{1}{2} \left( \frac{v^2}{2} + \frac{v^4}{4} \right)(1 + \cos z) dY,
\end{align*} \]

and we have

\[ dX = \frac{2}{1 + \cos w} dx, \quad dY = \frac{2}{1 + \cos z} dx. \]
Thus \((1 + \cos w)dx = 2dx\), and \((1 + \cos w)dy = 2dy\).

So (3.8) becomes
\[
\int_{\Gamma_1} 1 - \frac{1}{2}(\frac{v^2}{2} + \frac{v^4}{4})(1 + \cos w)dx + \int_{\Gamma_1} -1 + \frac{1}{2}(\frac{v^2}{2} + \frac{v^4}{4})(1 + \cos z)dy \\
\leq 2(|X| + |Y| + 4E_0) + K_1.
\]

And also,
\[
\int_{\Gamma_2} 1 - \frac{1}{2}(\frac{v^2}{2} + \frac{v^4}{4})(1 + \cos w)dx + \int_{\Gamma_2} -1 + \frac{1}{2}(\frac{v^2}{2} + \frac{v^4}{4})(1 + \cos z)dy \\
\leq 0 - Y + \varphi(X) + \frac{K_1}{2},
\]
\[
\int_{\Gamma_3} 1 - \frac{1}{2}(\frac{v^2}{2} + \frac{v^4}{4})(1 + \cos w)dx + \int_{\Gamma_3} -1 + \frac{1}{2}(\frac{v^2}{2} + \frac{v^4}{4})(1 + \cos z)dy \\
\leq \varphi^{-1}(Y) - X - 0 + \frac{K_1}{2}.
\]

As a result, we can have the following identity
\[
(3.10) \quad \int_{\varphi^{-1}(Y)}^{X} p(X', Y)dx' + \int_{\varphi(X)}^{Y} q(X, Y')dy' \leq 2(|X| + |Y| + 4E_0) + K_1
\]

By observing the boundary conditions (2.36) - (2.38), \(p, q > 0\). And by (2.32),
\[
p_y = \frac{1}{8c} pq \left\{ \frac{c'}{c} [\sin z - \sin w] - \sin w(v + v^3)(1 + \cos z) \right\},
\]
\[
p(X, Y) = \exp \left\{ \int_{\varphi(X)}^{Y} \frac{1}{8c} \frac{c'}{c} [\sin z - \sin w] - \sin w(v + v^3)(1 + \cos z)q(X, Y')dy' \right\}
\]
\[
\leq \exp \left\{ C_1 \int_{\varphi(X)}^{Y} q(X, Y')dy' \right\}
\]
\[
\leq \exp \left\{ 2C_1(|X| + |Y| + 4E_0) + C_1 K_1 \right\}.
\]

Similarly, we have
\[
q(X, Y) \leq \exp \left\{ 2C_1(|X| + |Y| + 4E_0) + C_1 K_1 \right\}.
\]

Now, we will show that on any bounded sets in \(X-Y\) plane, we can construct the solution for the system of the equations (2.31) - (2.33) with boundary condition (2.36) - (2.38) by the fixed point of a constructive map. For any \(r > 0\), we can construct a bounded domain
\[
\Omega_r := \{ (X, Y) : Y \leq \varphi(X), X \leq r, Y \leq r \}.
\]

And also introduce the function space :
\[
(3.11) \quad \Lambda_r := \{ f : \Omega_r \to \mathbb{R} : \| f \|_r := ess \sup_{(X,Y) \in \Omega_r} e^{-K(X+Y)}|f(X,Y)| < \infty \}.
\]
Where $K$ is a suitably big constant and it will be determined later. And for $(w, z, p, q, v) \in \Lambda_r$, we can construct a map $\tau(w, z, p, q, v) = (\tilde{w}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{v})$. And this map is define as follows.

\begin{equation}
\begin{aligned}
(3.12) \\
\left\{ \\
\tilde{w}(X, Y) = \tilde{w}(X, \varphi(X)) + \int_{\varphi(X)}^Y \frac{\varepsilon}{c^2} (\cos z - \cos w) q - \frac{q}{4c} (v + v^3)(1 + \cos z)(1 + \cos w) dY, \\
\tilde{z}(X, Y) = \tilde{z}(\varphi^{-1}(Y), Y) + \int_{\varphi^{-1}(Y)}^X \frac{\varepsilon}{c^2} (\cos w - \cos z) p - \frac{p}{4c} (v + v^3)(1 + \cos z)(1 + \cos w) dX,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
(3.13) \\
\left\{ \\
\tilde{p}(X, Y) = 1 + \int_{\varphi(X)}^Y \frac{1}{4c} pq \left\{ \frac{\varepsilon}{c} [\sin z - \sin w] - \sin w(v + v^3)(1 + \cos z) \right\} dY, \\
\tilde{q}(X, Y) = 1 + \int_{\varphi^{-1}(Y)}^X \frac{1}{4c} pq \left\{ \frac{\varepsilon}{c} [\sin w - \sin z] - \sin z(v + v^3)(1 + \cos w) \right\} dX,
\end{aligned}
\end{equation}

\begin{equation}
(3.14) \quad \tilde{v}(X, Y) = \tilde{v}(X, \varphi(X)) \int_{\varphi(X)}^Y \frac{1}{4c} \sin z q dY.
\end{equation}

We want to prove the uniform Lipschitz condition. First, we define that

$$ \Phi_r := \Lambda_r \times \Lambda_r \times \Lambda_r \times \Lambda_r \times \Lambda_r. $$

For some properly chosen distance $D : \Phi_r \times \Phi_r \to \mathbb{R}$. We want to show that

$$ D((\tilde{w}_1, \tilde{z}_1, \tilde{p}_1, \tilde{q}_1, \tilde{v}_1), (\tilde{w}_2, \tilde{z}_2, \tilde{p}_2, \tilde{q}_2, \tilde{v}_2)) < L \times D((w_1, z_1, p_1, q_1, v_1), (w_2, z_2, p_2, q_2, v_2)), $$

for some Lipschitz constant $L \leq 1$. In fact, define the distance as:

$$ D((\tilde{w}_1, \tilde{z}_1, \tilde{p}_1, \tilde{q}_1, \tilde{v}_1), (\tilde{w}_2, \tilde{z}_2, \tilde{p}_2, \tilde{q}_2, \tilde{v}_2)) := \max\{\|\tilde{w}_1 - \tilde{w}_2\|_{\ast}, \|\tilde{z}_1 - \tilde{z}_2\|_{\ast}, \|\tilde{p}_1 - \tilde{p}_2\|_{\ast}, \|\tilde{q}_1 - \tilde{q}_2\|_{\ast}, \|\tilde{v}_1 - \tilde{v}_2\|_{\ast}\}, $$

and the norm $\|\cdot\|_{\ast}$ is defined in \((3.11)\).

A straightforward computation shows that $L = \frac{C(E_0, K)}{K}$, where $C(E_0, K)$ is a constant that depends on $E_0$ and $K$. By choosing $K$ sufficiently large, we can guarantee $L < 1$.

Hence, the uniform Lipschitz condition is proved. By the fixed point theorem, the solution in the $X$-$Y$ plane exists and is unique. \hfill \Box

If the initial data in \((1.2)\) are smooth, then the solutions of \((2.31) - (2.33)\) with boundary condition \((2.36) - (2.38)\) are smooth functions with variables $(X,Y)$.

Also, if there is a sequence of smooth functions $(v_{0}^{m}(x), v_{1}^{m}(x))_{m \geq 1}$ with the following conditions:

$$ v_{0}^{m}(x) \to v_{0}(x), \ v_{1}^{m}(x) \to v_{1}(x), \ (v_{0}^{m}(x))_{x} \to (v_{0}(x))_{x}, $$

uniformly on a compact subset of $\mathbb{R}$. Then

$$ (p^{m}, q^{m}, w^{m}, z^{m}, v^{m}) \to (p, q, w, z, v), $$

uniformly on some bounded subsets of $X$-$Y$ plane.
4. Weak solutions

In this section, we want to construct a map \( v(X,Y) \rightarrow v(t,x) \). Which is to write \( (X,Y) \) in terms of \( (t,x) \) so that we can obtain a solution to the Cauchy problem (1.2), (1.3). It will provides a proof of Theorem 1.1.

The map \( (X,Y) \mapsto (t,x) \) can be obtain in the following way. We plug in \( f = x \) and \( f = t \) into the equation (2.25), and get

\[
\begin{align*}
    c &= 2cX_x x_X, \\
    -c &= -2cY_x x_Y, \\
    1 &= 2cX_x t_X, \\
    1 &= -2cY_x t_Y.
\end{align*}
\]

And by applying (2.27) we have

\[
\begin{align*}
    X_x &= \frac{2}{(1+\cos w)p}, \\
    Y_x &= \frac{-2}{(1+\cos z)q}, \\
    X_t &= \frac{2c}{(1+\cos w)p}, \\
    Y_t &= \frac{-2c}{(1+\cos z)q}.
\end{align*}
\]

We assume that the partial derivatives above valid for points that \( w, z \neq -\pi \). Thus, we have

\[
\begin{align*}
    x_X &= \frac{1}{2X_x} = \frac{(1+\cos w)p}{4}, \\
    x_Y &= \frac{1}{2Y_x} = \frac{-(1+\cos z)q}{4}.
\end{align*}
\]

\[
\begin{align*}
    t_X &= \frac{1}{2X_x} = \frac{(1+\cos w)p}{4c}, \\
    t_Y &= \frac{1}{-2Y_x} = \frac{(1+\cos z)q}{4c}.
\end{align*}
\]
An easy computation shows that \( x_{XY} = x_{YX} \) and \( t_{XY} = t_{YX} \)

\[
x_{XY} = \frac{(1 + \cos w) p v}{4} - \frac{p \sin w p}{4} w y
\]

\[
= \frac{(1 + \cos w) c'}{8 c^2} \left[ \sin z - \sin w \right] p q - \frac{(1 + \cos w) 1}{8 c} p \sin w (v + v^3)(1 + \cos z)
\]

\[
- \frac{p \sin w q}{4} \cos z c' - \frac{q \sin w}{4} (v + v^3)(1 + \cos z)(1 + \cos w)
\]

\[
= \frac{c' p q}{32 c} \left[ \sin z - \sin w + \sin(z - w) \right],
\]

\[
x_{YX} = \frac{(1 + \cos z) q v}{4} - \frac{q \sin w z y}{4}
\]

\[
= \frac{(1 + \cos w) c'}{8 c^2} \left[ \sin w - \sin z \right] p q - \frac{(1 + \cos w) 1}{8 c} p \sin z (v + v^3)(1 + \cos w)
\]

\[
- \frac{q \sin z c'}{4} \cos w c' + \frac{q \sin z p}{4} (v + v^3)(1 + \cos z)(1 + \cos w)
\]

\[
= \frac{c' p q}{32 c} \left[ \sin z - \sin w + \sin(z - w) \right],
\]

\[
\Longleftrightarrow x_{XY} = x_{YX}.
\]

And similarly, we can compute that \( t_{XY} = t_{YX} \). So, the two equation in (4.3) are equivalent: \( x_{XY} = x_{YX} \). And the two equation in (4.4) are equivalent since \( t_{XY} = t_{YX} \). We can recover the solution in terms of \((t, x)\) with function \( x = x(X,Y) \) by integrating one of the equation in (4.3). Also, we can write \( t = t(X,Y) \) by integrating one of the equation in (4.4).

Next, we will prove that the function \( v \) is a weak solution to (1.2). By (2.25), we want to show that

\[
0 = \int \int \phi_t v_t - [c(v) \phi]_x [c(v) v_x] - \frac{\phi}{2} (v + v^3) dx dt.
\]

In fact, it is equivalent to prove:

\[
0 = \int \int \left( (v_t + cv_x) [\phi_t - (c(v) \phi)_x] + (v_t - cv_x) [\phi_t + (c(v) \phi)_x] - \phi(v + v^3) \right) dx dt
\]

\[
= \int \int -\frac{\sin w}{2} p \phi - \frac{\sin z}{2} q \phi + \frac{c' p q}{8 c} \left[ \sin w \frac{1 + \cos z}{2} - \sin z \frac{1 + \cos w}{2} \right] \phi \tan \frac{z}{2} - \tan \frac{w}{2} dX dY
\]

\[
- \int \int \phi(v + v^3) dx dt
\]

\[
= I + II,
\]

where in the last step, we have used (4.2),

\[
d x dt = \left| \begin{array}{cc} \frac{dx}{dX} & \frac{dx}{dt} \\ \frac{dx}{dY} & \frac{dx}{dt} \end{array} \right| dX dY = \frac{pq}{2c(1 + R^2)(1 + S^2)} dX dY.
\]
And used the following identities derived from (2.33),

\[
\begin{align*}
\frac{1}{1+R^2} &= \frac{1+\cos w}{2}, \\
\frac{1}{1+S^2} &= \frac{1+\cos z}{2},
\end{align*}
\]

(4.5)

\[
\begin{align*}
\frac{R}{1+R^2} &= \frac{\sin w}{2}, \\
\frac{S}{1+S^2} &= \frac{\sin z}{2}.
\end{align*}
\]

(4.6)

We denote I and II as follows

\[
\begin{align*}
I &= \int \int (-\frac{\sin w}{2} p) y \phi - (-\frac{\sin z}{2} q)x \phi + \frac{c'pq}{8c} [\sin w \frac{1+\cos z}{2} - \sin z \frac{1+\cos w}{2}] \phi (\tan z - \tan \frac{w}{2}) dX dY, \\
II &= \int \int \phi (v + v^3) dx dt.
\end{align*}
\]

(4.7)

We compute I with (2.31) - (2.33),

\[
\begin{align*}
I &= \int \int (-\frac{\cos w}{2} w y p + \frac{\sin w}{2} p y) \phi - (-\frac{\cos z}{2} z x q + \frac{\sin z}{2} q x) \phi + \frac{c'pq}{8c^2} [\cos(w + z) - 1] \phi dX dY \\
&= \int \int \frac{pq}{16c} \phi (v + v^3) (\cos w + \cos z + 2 + 2 \cos z \cos w \\
&\quad + \cos^2 w \cos z + \cos w \cos^2 z + \sin^2 w \cos z + \sin^2 z \cos w) dX dY \\
&= \int \int \frac{pq}{8c} \phi (v + v^3) (1 + \cos z + \cos w + \cos z \cos w) dX dY.
\end{align*}
\]

A computation on II shows that

\[
\begin{align*}
II &= \int \int \phi (v + v^3) \frac{pq}{2c(1 + s^2)(1 + R^2)} dX dY \\
&= \int \int \frac{pq}{8c} \phi (v + v^3) (1 + \cos z + \cos w + \cos z \cos w) dX dY.
\end{align*}
\]

Clearly, I = II. Thus the integral (1.5) is

\[
\begin{align*}
0 &= \int \int (-\frac{\sin w}{2} p) y \phi - (-\frac{\sin z}{2} q)x \phi + \frac{c'pq}{8c} [\sin w \frac{1+\cos z}{2} - \sin z \frac{1+\cos w}{2}] \phi (\tan z - \tan \frac{w}{2}) dX dY \\
&\quad - \int \int \phi (v + v^3) dx dt \\
0 &= I - II,
\end{align*}
\]

holds, where I is defined in (4.7), and II is defined in (4.8).

Next, we will define \( v \) as a function in terms of the original variables (\( t, x \)). We will invert the map \( (X, Y) \mapsto (t, x) \) and then we will have \( v(t, x) = v(X(t, x), Y(t, x)) \). This map may not be one to one since there might be more than one point on the X-Y plane maps to the same point in the t-x plane. But this does not cause any difficulty. Given arbitrary \( (t^*, x^*) \) in
the $tx$ plane, we can choose arbitrary point $(X^*, Y^*)$ in $X$-$Y$ plane such that $t^* = t(X^*, Y^*)$ and $x^* = x(X^*, Y^*)$. We also define that $v(t^*, x^*) = v(X^*, Y^*)$. And we also assume that there are two different points $(t(X_1, Y_1), x(X_1, Y_1)) = (t(X_2, Y_2), x(X_2, Y_2)) = (t^*, x^*)$. We will consider two cases: case 1 $X_1 \leq X_2, Y_1 \leq Y_2$, and case 2 $X_1 \leq X_2, Y_1 \geq Y_2$.

Case 1: We consider the set
\[
\Gamma_{x^*} := \{(X, Y) : x(X, Y) \leq x^*\},
\]
and we denote $\partial \Gamma_{x^*}$ as the boundary of $\Gamma_{x^*}$. By (4.3), we can observe that $x$ is increasing with $X$ increasing and $x$ is decreasing with $Y$ increasing. Thus, this boundary can be written as a Lipschitz continuous function denoted as $X - Y = \phi(X - Y)$.

We construct the Lipschitz continuous curve $\gamma$ with the following properties:

- a horizontal line segment connecting $(X_1, Y_1)$ with a point $P = (X_P, Y_P) \in \partial \Gamma_{x^*}$ and $Y_P = Y_1$.
- a vertical line segment connecting $(X_2, Y_2)$ with a point $Q = (X_Q, Y_Q) \in \partial \Gamma_{x^*}$ and $X_Q = X_2$ a part of $\partial \Gamma_{x^*}$.

Thus, we can obtain a Lipschitz continuous parametrization of the curve $\gamma : [\xi_1, \xi_2] \mapsto \mathbb{R} \times \mathbb{R}$ where the parameter $\xi = X + Y$. By observing, the map $(X, Y) \mapsto (t, x)$ is constant along the curve $\gamma$. And (4.3) - (4.4) implies that

\[
(1 + \cos w)X_\xi = (1 + \cos z)Y_\xi = 0, \tag{4.9}
\]
\[
\iff \sin wX_\xi = \sin zY_\xi = 0. \tag{4.10}
\]
Thus, by (4.10)

\[ v(X_2, Y_2) - v(X_1, Y_1) = \int_{\gamma} (v_X dX + v_Y dY) \]

\[ = \int_{\xi_2}^{\xi_1} \left( \frac{p \sin w}{4c} X_\xi - \frac{q \sin z}{4c} Y_\xi \right) d\xi = 0. \]

So we have proved case 1.

Case 2: \( X_1 \leq X_2, Y_1 \geq Y_2 \). We consider the set:

\[ \Gamma_{t^*} := \{(X, Y) : t(X, Y) \leq t^*\}. \]

And we do the same process as we did in case 1. Construct \( \gamma \) connecting \((X_1, X_2)\) and \((X_2, y_2)\) as Figure 3 case 2 indicates.

Next, we will prove the function \( v(t, x) = v(X(t, x), Y(t, x)) \) is Hölder-\( \frac{1}{2} \) continuous on the bounded sets.

To prove this, we need to consider characteristic curve such that \( t \mapsto X^+(t) \) with \( \bar{x}^+ = c(v) \).

For some fixed \( \bar{Y} \) this can be parametrized by the function \( X \mapsto (t(X, \bar{Y}), x(X, \bar{Y})) \). By (2.23), (2.25), (2.27) and (2.33), we have

\[ \int_0^\tau \left[ v_t + c(v) v_x \right]^2 dt = \int_{X_0}^{X_\tau} \left( 2cX_x^2 v_x \right)^2 \frac{1}{2X_t} dX \]

\[ = \int_{X_0}^{X_\tau} \frac{p}{2c} \sin^2 \left( \frac{w}{2} \right) dX \leq \int_{X_0}^{X_\tau} \frac{p}{2c} dX \leq C_\tau. \]

Thus, we obtain that

\[ (4.11) \quad \int_0^\tau \left[ v_t + c(v) v_x \right]^2 dt \leq C_\tau. \]

Similarly, we can integrate along backward characteristics curves \( t \mapsto x^-(t) \) and find out that

\[ (4.12) \quad \int_0^\tau \left[ v_t - c(v) v_x \right]^2 dt \leq C_\tau. \]

Thus, since the speed of the characteristic curve is \( +c(v) \) or \( -c(v) \) and \( c(v) \) is uniformly positive bounded. With the bounds (4.11) and (4.12), the function \( v(t, x) \) is Hölder-\( \frac{1}{2} \) continuous. \qed
5. Conserved Quantities

Recalling (2.15) and (2.16), an straightforward computation shows that

\[
E_t = \left( \frac{1}{2} v_t^2 + \frac{1}{2} c^2 v_x^2 + \frac{v^4}{4} + \frac{v^4}{8} \right)_t
= v_{tt} v_t + cc' v_t v_x^2 + c^2 v_x v_{xt} + \frac{1}{2} (v + v^3) v_t,
\]

\[(c^2 M)_x = (-c^2 v_t v_x)_x = -2 cc' v_t v_x + c^2 v_{tx} v_x - c^2 v_t v_{xx},\]

\[E_t + (c^2 M)_x = v_t (v_t - cc' v_x - c^2 v_{xx} + \frac{1}{2} (v + v^3)) = 0,\]

and

\[M_t = -v_{tt} v_x - v_t v_{xt},\]

\[E_x = v_t v_{tx} + cc' v_x v_x^2 + c^2 v_x v_{xx} + \frac{1}{2} (v + v^3) v_x,\]

\[M_t + E_x = -v_x \left( v_t - cc' v_x - c^2 v_{xx} - \frac{1}{2} (v + v^3) \right) = 0.\]

Thus, we have

\[(5.1) \quad \begin{cases} E_t + (c^2 M)_x = 0, \\ M_t + E_x = 0. \end{cases}\]

We also have

\[(5.2) \quad dx = \frac{(1 + \cos w)}{4} dX - \frac{(1 + \cos z)}{4} dY,\]

\[(5.3) \quad dt = \frac{(1 + \cos w)}{4c} dX + \frac{(1 + \cos z)}{4c} dY,\]

which is closed. We want to show that \(Edx - (c^2 M)dt, \ Mdx - Edt\) are closed. Recalling (2.31) - (2.33), we will write them in terms of \(X, Y\), and show that they are closed.

\[(5.4) \quad Edx - (c^2 M)dt = \left[ \tan^2 \left( \frac{w}{2} \right) + \tan^2 \left( \frac{z}{2} \right) \right] dx + c^2 \tan^2 \left( \frac{w}{2} \right) - \tan^2 \left( \frac{z}{2} \right) dt\]

\[(5.5) \quad = \left[ \frac{(1 - \cos w)}{8} + \frac{(1 + \cos w)}{32} (2v^2 + v^4) \right] p dX - \left[ \frac{(1 - \cos z)}{8} + \frac{1 + \cos z}{32} (2v^2 + v^4) \right] q dY,\]

\[Mdx + Edt = \left[ \tan^2 \left( \frac{w}{2} \right) - \tan^2 \left( \frac{z}{2} \right) \right] dx + \left[ \frac{1}{8} (2 \tan^2 \left( \frac{w}{2} \right) + 2 \tan^2 \left( \frac{z}{2} \right)) + \frac{1}{4} (v^2 + \frac{1}{2} v^4) \right] dt\]

\[= \left\{ \frac{(1 - \cos w)}{8c} + \frac{(1 + \cos w)}{32c} (2v^2 + v^4) \right\} p dX + \left\{ \frac{1 - \cos z}{8c} + \frac{1 + \cos z}{32c} (2v^2 + v^4) \right\} q dY.\]
And we can compute that

\[
\begin{align*}
&\left\{ \left[ \frac{(1 - \cos w)}{8} + \frac{(1 + \cos w)}{32}(2v^2 + v^4) \right] p \right\} \bigg|_x = -\frac{\sin w(2v^2 + v^4)p}{32} \cdot \frac{c'}{8c^2}(\cos z - \cos w)q - \frac{(1 + \cos w)(2v + v^3)}{32} \cdot \frac{1}{4c} \sin zq \\
&+ \frac{1 + \cos w}{32} \cdot \frac{c'}{8c^2}[\sin z - \sin w]pq \\
&= -\left\{ \left[ \frac{(1 - \cos z)}{8} + \frac{1 + \cos z}{32}(2v^2 + v^4) \right] q \right\} \bigg|_x,
\end{align*}
\]

and

\[
\begin{align*}
&\left\{ \left[ \frac{(1 - \cos w)}{8c} + \frac{(1 + \cos w)}{32c}(2v^2 + v^4) \right] p \right\} \bigg|_y = \frac{\sin wp}{64c^3} \cdot \frac{c'}{64c^2}(\cos z - \cos w)q - \frac{\sin wpq}{64c^2}(v + v^3)(1 + \cos z)(1 + \cos w) \\
&+ \frac{(1 - \cos w)c'}{64c^3}[\sin z - \sin w]pq - \frac{(1 - \cos w)pq}{64c^2} \cdot \frac{\sin w(v + v^3)}{4c} \cdot (1 + \cos z) \\
&- \frac{\sin w}{32c} \cdot \frac{p(2v^2 + v^4)c'}{8c^2}(\cos z - \cos w)q + \frac{(1 + \cos w)}{32c} \cdot \frac{1}{4c} \sin zq \\
&\cdot \frac{c'}{8c^2}[\sin z - \sin w]pq + \frac{1 + \cos w}{32c} \cdot \frac{p(4v + 4v^3)}{4c} \cdot \frac{1}{4c} \sin zq \\
&= \left\{ \left[ \frac{(1 - \cos z)}{8c} + \frac{1 + \cos z}{32c}(2v^2 + v^4) \right] q \right\} \bigg|_x.
\end{align*}
\]

Thus \(\{Edx - (c^2M)dt\}, \{Mdx - Edt\}\) are closed.
To prove the inequality (1.8), we fixed some \( \tau > 0 \), and the case \( \tau < 0 \) is identical. We assume that for an arbitrary large \( r > 0 \). We define the set

\[
\Gamma := \{(X, Y) : 0 \leq t(X, Y) \leq \tau, X \leq r, Y \leq r\}.
\]

We form the map \((X, Y) \mapsto (t, x)\) in the following pattern:

\[
A \mapsto (\tau, a), \quad B \mapsto (\tau, b), \quad C \mapsto (0, c), \quad D \mapsto (0, d),
\]

such that \( a < b \) and \( c > d \). Then, we can integrate the (5.5) along \( \partial \Gamma \), the boundary of \( \Gamma \). Then we have

\[
\int_{AB} \left\{ \frac{1 - \cos w}{8} p + \frac{1 + \cos w}{32} p (2v^2 + v^4) \right\} dX - \left\{ \frac{1 - \cos z}{8} q + \frac{1 + \cos z}{32} q (2v^2 + v^4) \right\} dY
\]

\[
= \int_{DC} \left\{ \frac{1 - \cos w}{8} p + \frac{1 + \cos w}{32} p (2v^2 + v^4) \right\} dX - \left\{ \frac{1 - \cos z}{8} q + \frac{1 + \cos z}{32} q (2v^2 + v^4) \right\} dY
\]

\[
- \int_{DA} \left\{ \frac{1 - \cos w}{8} p + \frac{1 + \cos w}{32} p (2v^2 + v^4) \right\} dX
\]

\[
- \int_{CB} \left\{ \frac{1 - \cos z}{8} q + \frac{1 + \cos z}{32} q (2v^2 + v^4) \right\} dY
\]

\[
\leq \int_{DC} \left\{ \frac{1 - \cos w}{8} p + \frac{1 + \cos w}{32} p (2v^2 + v^4) \right\} dX - \left\{ \frac{1 - \cos z}{8} q + \frac{1 + \cos z}{32} q (2v^2 + v^4) \right\} dY
\]

\[
\leq \int_{d}^{c} \frac{1}{2} \left[ v_{1}^{2}(0, x) + c^{2}(v(0, x))v_{2}^{2}(0, x) + \frac{1}{2}(v(0, x) + v^{3}(0, x)) \right] dx.
\]

We also have

\[
\int_{a}^{b} \frac{1}{2} \left[ v_{1}^{2}(0, x) + c^{2}(v(0, x))v_{2}^{2}(0, x) + \frac{1}{2}(v(0, x) + v^{3}(0, x)) \right] dx
\]

\[
= \int_{AB \cap \{ \cos w \neq -1 \}} \left\{ \frac{1 - \cos w}{8} p + \frac{1 + \cos w}{32} p (2v^2 + v^4) \right\} dX
\]

\[
- \left\{ \frac{1 - \cos z}{8} q + \frac{1 + \cos z}{32} q (2v^2 + v^4) \right\} dY
\]

\[
\leq E_{0}.
\]

Let \( r \to \infty \), we have \( a \to -\infty \) and \( b \to +\infty \). We conclude that \( E(t) \leq E_{0} \). Thus, the inequity (1.8) is proved.

Now, we will prove the Lipschitz condition on the map \( t \mapsto v(t, \cdot) \) in the \( L^{2} \) distance. First, for any fixed time \( \tau \), we define \( \mu_{\tau} := \mu_{\tau}^{-} + \mu_{\tau}^{+} \) and \( \mu_{\tau} \) is the positive measure on the real lines. We define \( \mu_{\tau}^{-}, \mu_{\tau}^{+} \) as follows.

we define \( \Gamma_{\tau} := \{(X, Y) : t(X, Y) \leq \tau\} \) and let \( \gamma_{\tau} \) be the boundary of \( \Gamma_{\tau} \).

For any open interval \([a, b]\), we define \( A = (X_{A}, Y_{A}), B = (X_{B}, Y_{B}) \) be points on the \( \gamma_{\tau} \) such that

\( x(A) = a, \) and \( X_{P} - Y_{P} \leq X_{A} - Y_{A} \) for all points \( P \in \gamma_{\tau} \) and \( x(P) \leq a, \)
\[ x(B) = b, \text{ and } X_B - Y_B \leq X_P - Y_P \text{ for all points } P \in \gamma_t \text{ and } x(P) \geq b. \]

Then we have

\[ \mu_t := \mu_t^-(a,b] + \mu_t^+(a,b], \tag{5.7} \]

and we define that in general case

\[ \mu_t^-(a,b] := \int_{AB} \left\{ \frac{(1 - \cos w)p}{8} + \frac{(1 + \cos w)p}{32}(2v^2 + v^4) \right\} dX, \tag{5.8} \]
\[ \mu_t^+(a,b] := \int_{AB} \left\{ \frac{(1 - \cos z)q}{8} + \frac{(1 + \cos z)q}{32}(2v^2 + v^4) \right\} dY. \tag{5.9} \]

And in the smooth case:

\[ \mu_t^-(a,b] := \frac{1}{4} \int_a^b R^2(\tau, x)dx, \tag{5.10} \]
\[ \mu_t^+(a,b] := \frac{1}{4} \int_a^b S^2(\tau, x)dx. \tag{5.11} \]
Clearly, $\mu^+$ and $\mu^-$ are bounded positive measure. And for all $\tau$, we have $\mu_\tau(\mathbb{R}) = E_0$ by (5.5). By (5.7)-(5.9) and (2.16) we can compute that

$$\int_a^b c^2 u^2_x dx = \int_a^b \frac{c^2(R - S)^2}{4c^2} dx = \int_a^b \frac{R^2 - 2RS + S^2}{4} dx \leq \int_a^b \frac{R^2 + S^2}{2} dx = 2\mu([a, b]) .$$

Thus, for arbitrary $a, b$ with $a < b$ we have

$$(5.12) \quad |v(\tau, b) - v(\tau, a)|^2 \leq |b - a| \int_a^b v_x(\tau, y) dy \leq |b - a|2K^2\mu_\tau([a, b]).$$

For given $y \in R$ and $h > 0$, our goal is to estimate the $|v(\tau + h, y) - v(\tau, y)|$. We first denote that $\Gamma_{\tau+h}$ as the set $\Gamma_{\tau+h} := \{(X, Y) : t(X, Y) \leq \tau + h\}$ and we denote that $\gamma_{\tau+h}$ to be the boundary of the set $\Gamma_{\tau+h}$.

Let $P = (P_X, P_Y)$ be points on $\gamma_{\tau+h}$ (as the figure 5(a) shows) such that $x(P) = y$, and $X_P - Y_P \leq X_P - Y_P$ for all $\tilde{P} \in \gamma_\tau$, $x(\tilde{P}) \leq x(P)$.

Let $Q = (Q_X, Q_Y)$ be points on $\gamma_{\tau+h}$ such that $x(Q) = y$ and $X_Q - Y_Q \leq X_Q - Y_Q$ for all $\tilde{Q} \in \gamma_{\tau+h}$, $x(\tilde{Q}) \leq x(Q)$.

So we have $X_P \leq X_Q$ and $Y_P \leq Y_Q$. Let $P^+ = (X_Q, Y^+) \in \gamma_\tau$ and $P^- = (X^-, Y_Q) \in \gamma_\tau$.

As is shown in the figure 5, since the point $(\tau, x(P^+))$ lies on some characteristic curve with the speed $c(v) \leq K$ and go through the point $(\tau + h, y)$, so $x(P^+) \in ]y, y + Kh[$.

Also, $x(P^-) \in ]y - Kh, y[$, since point $(\tau, y)$ lies on some characteristic curve with the speed $-c(v) \geq -K$ and go through the point $(\tau + h, y)$.
Thus, and by (2.33), we can compute that

$$|v(Q) - v(P^+)| \leq \int_{Y^+}^Y |v_Y(X_Q, Y)| dY$$

$$= \int_{Y^+}^Y |\sin z q| dY$$

$$= \int_{Y^+}^Y \left( \frac{1 + \cos z}{4c} q \right)^{\frac{1}{2}} \left( \frac{1 - \cos z}{4c} q \right)^{\frac{1}{2}} dY$$

$$\leq \left( \int_{Y^+}^Y \frac{1 + \cos z}{4c} q dY \right)^{\frac{1}{2}} \left( \int_{Y^+}^Y \frac{1 - \cos z}{4c} q dY \right)^{\frac{1}{2}}$$

$$\leq \left( \int_{Y^+}^Y \frac{1 + \cos z}{4c} q dY + \frac{1 + \cos w}{4c} p dX \right)^{\frac{1}{2}} \left( \int_{Y^+}^Y \frac{1 - \cos z}{4c} q dY + \frac{1 - \cos w}{4c} p dY \right)^{\frac{1}{2}}$$

$$\leq \left( \int_{Y^+}^Y \frac{1 + \cos z}{4c} q dY + \frac{1 + \cos w}{4c} p dY \right)^{\frac{1}{2}}$$

Thus we have

$$\left| v(Q) - v(P^+) \right| \leq h^{\frac{1}{2}} \left( \int_{P^-}^P \frac{1 - \cos z}{4c} q dY + \frac{1 - \cos w}{4c} p dY \right)^{\frac{1}{2}}.$$  \hspace{1cm} (5.13)

So, by (5.12) and (5.13) we can compute that

$$|v(\tau + h, x) - v(\tau, x)|^2 = |v(\tau + h, x) - v(t(P^+), x(P^+)) + v(t(P^+), x(P^+)) - v(\tau, x)|^2$$

$$\leq 2 \{v(\tau + h, x) - v(t(P^+), x(P^+))\}^2 + 2 \{v(t(P^+), x(P^+)) - v(\tau, x)\}^2$$

$$\leq 2 \{v(Q) - v(P^+)\}^2 + 2 \{v(P^+) - v(P)\}^2$$

$$\leq 2 \left[ h^{\frac{1}{2}} \left( \int_{P^-}^P \frac{1 - \cos z}{4c} q dY + \frac{1 - \cos w}{4c} p dY \right)^{\frac{1}{2}} \right]^2$$

$$+ 2 \left[ 2 \mathcal{K}^2 (\mathcal{K} h) \mu_r([x, x + h]) \right]$$

$$\leq 4 h \mu_r([x - \mathcal{K} h, x + \mathcal{K} h]) + 4 \mathcal{K}^3 h \mu_r([x, x + h])$$

$$\leq 4 h \mu_r([x - \mathcal{K} h, x + \mathcal{K} h])(1 + \mathcal{K}^3).$$
Thus, for all $h > 0$,

$$
\|v(\tau + h, \cdot) - v(\tau, \cdot)\|_{L^2} = \left\{ \int |v(\tau + h, x) - v(\tau, x)|^2 \, dx \right\}^{\frac{1}{2}} \\
\leq \left\{ \int 4(1 + \mathcal{K}^3)h\mu_\tau(|x - \mathcal{K}h, x + \mathcal{K}h|) \right\}^{\frac{1}{2}} \\
\leq \left\{ 4(\mathcal{K}^3 + 1)h^2\mu_\tau(\mathbb{R}) \right\}^{\frac{1}{2}} \\
\leq h[4(\mathcal{K}^3 + 1)E_0]^{\frac{1}{2}} \\
\leq |\tau + h - \tau| L,
$$

(5.14) \hspace{1cm} \|v(\tau + h, \cdot) - v(\tau, \cdot)\|_{L^2} \leq h[4(\mathcal{K}^3 + 1)E_0]^{\frac{1}{2}}.

Where $L = [4(\mathcal{K}^3 + 1)E_0]^{\frac{1}{2}}$ is the Lipschitz constant. So, this proves the uniform Lipschitz continuous of the maps $t \mapsto v(t, \cdot)$.

\[\square\]

6. Regularity of trajectories

In this section, we want to show that continuity of functions $t \mapsto v_l(t, \cdot)$ and the $t \mapsto v_x(t, \cdot)$ as functions with function value in $L^2$. This will finish the proof of theorem 1.1.

We consider the that the initial data $(v_0)_x$ and $v_1$ are smooth functions with compact support. In this situation, the solution $v(X, Y)$ is smooth on the $X$-$Y$ plane.

Fix some time $\tau$ and denote that $\Gamma_\tau := \{(X, Y) : t(X, Y) \leq \tau\}$ and $\gamma_\tau$ is the boundary of set $\Gamma_\tau$. Then we claim that

$$
\frac{d}{dt} v(t, \cdot)|_{t=\tau} = v_\tau(\tau, \cdot).
$$

(6.1) \hspace{1cm} \frac{d}{dt} v(t, \cdot)|_{t=\tau} = v_\tau(\tau, \cdot).

And by \[2.24\], \[2.27\], and \[2.33\], we have

$$
v_l(\tau, x) := v_XX_t + v_YY_t
$$

(6.2) \hspace{1cm} v_l(\tau, x) := v_XX_t + v_YY_t

$$
= \frac{\sin w}{4c} - \frac{2c}{4c}\frac{p}{p(1 + \cos w)} + \frac{\sin z}{4c}\frac{q}{q(1 + \cos z)}
$$

(6.3) \hspace{1cm} v_l(\tau, x) = \frac{\sin w}{2(1 + \cos w)} + \frac{\sin z}{2(1 + \cos z)}.

(6.4) \hspace{1cm} v_l(\tau, x) = \frac{\sin w}{2(1 + \cos w)} + \frac{\sin z}{2(1 + \cos z)}.

(6.2) - (6.4) define the value of $v_l(\tau, \cdot)$ at almost all the point of $x \in \mathbb{R}$. And by the inequity \[1.8\] and $c(v) \geq \mathcal{K}^{-1}$, we have

$$
\int_{\mathbb{R}} |v_l(\tau, x)|^2 \, dx \leq \mathcal{K}^2 \mathcal{E}(\tau) \leq \mathcal{K}^2 E_0.
$$

(6.5) \hspace{1cm} \int_{\mathbb{R}} |v_l(\tau, x)|^2 \, dx \leq \mathcal{K}^2 \mathcal{E}(\tau) \leq \mathcal{K}^2 E_0.

Next, to prove (6.1), given $\epsilon > 0$, there exists finitely many disjoint intervals $[a_i, b_i]$ subsets of $\mathbb{R}$ with $i = 1, 2, \ldots N$. We call the $A_i, B_i \in \gamma_\tau$ with $x(A_i) = a_i$ , $x(B_i) = b_i$. Then at every
point \( P \) in the arcs \( A_iB_i \) while \( 1 + \cos(w(P)) > \epsilon \) and \( 1 + \cos(z(P)) > \epsilon \). We have
\[
\min\{1 + \cos(w(P)), 1 + \cos(z(P))\} \leq 2\epsilon.
\]

We call that \( J := \bigcup_{1 \leq i \leq N} [a_i, b_i] \) as the points \( P \) along the curve \( \gamma_\tau \) that does not contain in any of the arcs \( A_iB_i \). And denote that \( J' := \mathbb{R} \setminus J \). Since \( v(t, x) \) is smooth in a neighbourhood of the set \( \{\tau\} \times J' \) and by the differentiability of \( v \) and apply the Minkowski’s inequality, we have
\[
\lim_{h \to 0} \frac{1}{h} \left\{ \int_\mathbb{R} |v(\tau + h, x) - v(\tau, x) - hv_t(\tau, x)|^p dx \right\}^{\frac{1}{p}}
\leq \lim_{h \to 0} \frac{1}{h} \left\{ \int_\gamma |v(\tau + h, x) - v(\tau, x)|^p dx \right\}^{\frac{1}{p}} + \left\{ \int_\gamma |v_t(\tau, x)|^p dx \right\}^{\frac{1}{p}}.
\]

Now, we estimate the measure of the bad set \( J \). Since \( (1 + \cos w) < 2\epsilon(1 - \cos w) \) and \( (1 + \cos z) < 2\epsilon(1 - \cos z) \), \( \text{meas}(J) = \int_\gamma dx = \Sigma_i \int_{A_iB_i} \frac{(1 + \cos w)p}{4} dX - \frac{(1 + \cos z)q}{4} dY \)
\leq 2\epsilon \Sigma_i \int_{A_iB_i} \frac{(1 - \cos w)p}{4} dX - \frac{(1 - \cos z)q}{4} dY
\leq 2\epsilon \int_{\gamma_\tau} \frac{(1 - \cos w)p}{4} dX - \frac{(1 - \cos z)q}{4} dY
\leq 2\epsilon E_0.

Using Hölder’s inequality with exponents \( \frac{2}{p} \) and \( q \) and choose \( q = \frac{2}{2 - p} \) so that \( \frac{p}{2} + \frac{1}{q} = 1 \). By (5.14), we obtain
\[
\int_\gamma |v(\tau + h, x) - v(\tau, x)|^p dx \leq \text{meas}(J)^{\frac{1}{2}} \left\{ \int_\gamma |v(\tau, x) - v(\tau, x)|^q dx \right\}^{\frac{p}{q}}
\leq [2\epsilon E_0]^{\frac{1}{2}} + \left\{ h[4(K^3 + 1)E_0]^{\frac{1}{2}} \right\}^{\frac{p}{q}}.
\]

Thus we have
\[
\lim_{h \to 0} \sup \frac{1}{h} \left\{ \int_\gamma |v(\tau + h, x) - v(\tau, x)|^p dx \right\}^{\frac{1}{p}}
\leq [2\epsilon E_0]^{\frac{1}{2}} + h[4(K^3 + 1)E_0]^{\frac{1}{2}}.
\]

Similarly, and by (6.5) we estimate that
\[
\int_\gamma |v_t(\tau, x)|^p dx \leq [\text{meas}(J)]^{\frac{1}{2}} \left\{ \int_\gamma |v_t(\tau, x)|^q dx \right\}^{\frac{p}{q}},
\]
\[
\Leftrightarrow \left\{ \int_\gamma |v_t(\tau, x)|^p dx \right\}^{\frac{1}{p}} \leq [2\epsilon E_0]^{\frac{1}{2}} [K^2 E_0]^{\frac{1}{2}}.
\]
Since $\epsilon > 0$ is arbitrary, so we can conclude that
\[
\lim_{h \to 0} \frac{1}{h} \left\{ \int_{\mathbb{R}} |v(\tau + h, x) - v(\tau, x) - hv(\tau, x)|^p dx \right\}^{\frac{1}{p}} = 0.
\]

Next, we will prove the continuity of the map $t \mapsto v_t$. First, we fix $\epsilon > 0$ and consider disjoint intervals $[a_i, b_i]$ subsets of $\mathbb{R}$ with $i = 1, 2, \ldots, N$. We call the $A_i, B_i \in \gamma_{\tau}$ with $x(A_i) = a_i$, $x(B_i) = b_i$. Since $v$ is a smooth function on the neighbourhood of $\{\tau\} \times J'$. By Hölder's inequality and Minkowski's inequality, we can estimate that
\[
\lim_{\sup h \to 0} \int |v_t(\tau + h, x) - v_t(\tau, x)|^p dx \leq \lim_{\sup h \to 0} \int |v_t(\tau + h, x) - v_t(\tau, x)|^2 dx \leq \lim_{\sup h \to 0} \left[ \frac{2\epsilon E_0}{h} \right]^\frac{1}{2} \left\{ \|v_t(\tau + h, \cdot)\|_{L^2}^2 + \|v_t(\tau, \cdot)\|_{L^2}^2 \right\}^\frac{q}{2} \leq \left[ 2\epsilon E_0 \right]^\frac{1}{4} \left[ 4E_0 \right]^p.
\]
Since the $\epsilon > 0$ is arbitrary, so the continuity is proved.

For general initial data $(v_0)_x, v_1 \in L^2$, we can consider a sequence of initial data $v_0^n \to v_0$, $(v_0^n)_x \to (v_0)_x$, and $v_1^n \to v_1$ in $L^2$ and for all $n \in \mathbb{N}$, $(v_0^n)_x, v_0^n, v_1^n \in C_c^\infty$.

The continuity of the map $t \mapsto v_x(t, \cdot)$ with values in $L^p$ and $1 \leq p < 2$ can be proved in the same way as above.

7. Energy conservation

In this section, we will provide proof of Theorem 1.3.

First, we defined the wave interaction potential
\[
(7.1) \quad \Lambda(t) := (\mu^-_t \otimes \mu^+_t) \{(x, y) : x > y\},
\]
with the $\mu^-_t$ and $\mu^+_t$ defined in (5.17) and (5.18). And since $\mu^-_t$ and $\mu^+_t$ are absolutely continuous in Lebesgue measure, so (5.19) and (5.20) holds and this implies that
\[
(7.2) \quad (7.1) \iff \Lambda(t) = \frac{1}{4} \iint_{x>y} R^2(t, x) S^2(t, x) dx dy.
\]

Lemma 7.1. There exist a Lipschitz constant $L_0$ such that
\[
\Lambda(t) - \Lambda(s) \leq L_0(t - s).
\]

So the map $t \mapsto \Lambda(t)$ has bounded variation.
The Lemma will be proved later.

To prove Theorem 1.3, we need to consider three sets

\[
\Omega_1 := \{(X,Y) : w(X,Y) = -\pi, z(X,Y) \neq -\pi, c'(v(X,Y)) \neq 0\},
\]
\[
\Omega_2 := \{(X,Y) : w(X,Y) \neq \pi, z(X,Y) = -\pi, c'(v(X,Y)) \neq 0\},
\]
\[
\Omega_3 := \{(X,Y) : w(X,Y) = -\pi, z(X,Y) = -\pi, c'(v(X,Y)) \neq 0\}.
\]

From (2.31) and since \(w_Y \neq 0\) on \(\Omega_1\) and \(z_X \neq 0\) on \(\Omega_2\). So that \(\text{meas}(\Omega_1) = 0\) and \(\text{meas}(\Omega_2) = 0\).

We define \(\Omega^*_3\) be the set of Lebesgue points of \(\Omega_3\). And we want to show that

\[
\text{meas}\{(t(X,Y) : (X,Y) \in \Omega^*_3)\} = 0
\]

First, we fix point \(P^* \in \Omega^*_3\) and \(P^* := (X^*,Y^*)\). And we claim that for \(h,k > 0\)

\[
\lim_{h,k \to 0+} \frac{\Lambda(\tau - h) - \Lambda(\tau + k)}{h + k} = +\infty.
\]

For arbitrary \(\epsilon > 0\), \(\epsilon\) arbitrary small, we can find \(\delta > 0\) such that for any square \(Q\) with length \(l \leq \delta\) center at \(P^*\), there exist a vertical segment \(\sigma\) satisfying \(\text{meas}(\Omega_3 \cup \sigma) \geq (1-\epsilon)l\), and a horizontal segment \(\sigma'\) satisfying \(\text{meas}(\Omega_3 \cup \sigma') \geq (1-\epsilon)l\).

We define that

\[
\begin{align*}
t^+ &:= \max\{t(X,Y) : (X,Y) \in \sigma \cup \sigma'\}, \\
t^- &:= \min\{t(X,Y) : (X,Y) \in \sigma \cup \sigma'\}.
\end{align*}
\]

By (4.4), for some constant \(c_0 > 0\)

\[
t^+ - t^- \leq \int_\sigma \frac{(1 + \cos w)p}{4c} dX + \int_{\sigma'} \frac{(1 + \cos z)q}{4c} dY \leq c_0(\epsilon l)^2.
\]

(7.9) is Lipschitz continuous and vanished outside of a set of measure \(\epsilon l\). Also, for some constant \(c_1, c_2 > 0\),

\[
\Lambda(t^-) - \Lambda(t^+) \geq c_1(1-\epsilon)^2 l^2 - c_2(t^+ + t^-).
\]

Since the choose of \(\epsilon > 0\) is arbitrary, so this implies (7.4). And by the Lemma 1, the map \(t \mapsto \Lambda\) has bounded variation, so (7.4) implies (7.3).

Thus, the singular part of the \(\mu_t\) is not trivial only if the set \(\Omega_4 := \{P \in \gamma_t : w(P) = -\pi, z(P) = -\pi\}\) has positive one-dimensional measure. By the above analysis, this is restricted to a set where \(c' \neq 0\) and only happen for a set of time with measure zero.

**Proof of Lemma 1.**

We first claim that

\[
\begin{align*}
(R)_t^2 - (cR)^2 x &= \frac{c^2}{2c}(R^2 S - S^2 R) - R(v + v^3), \\
(S)_t^2 + (cS)^2 x &= -\frac{c^2}{2c}(R^2 S - S^2 R) - S(v + v^3),
\end{align*}
\]
and (7.9) is obtained by

From (2.14),

\[ R_t - cr_x = \frac{c}{4c}(R^2 - S^2) - \frac{1}{2}(v + v^3), \]

\[ R_t R - cR_x = \frac{c}{4c}R(R^2 - S^2) - \frac{R}{2}(v + v^3), \]

\[ \frac{1}{2}(R^2)_t - \frac{1}{2}(cR^2)_x + \frac{1}{2}c'v_x R^2 = \frac{c}{4c}R^3 - \frac{c}{4c}RS^2 - \frac{R}{2}(v + v^3), \]

\[ (R^2)_t - (cR^2)_x = R^2(-c'v_x + \frac{c}{2c}R) - \frac{c}{2c}RS^2 - R(v + v^3), \]

\[ = \frac{c}{2c}(R^2S - RS^2) - R(v + v^3). \]

Similarly, we obtain \((S)_t^2 + (cS^2)_x = -\frac{c}{xz}(R^2S - S^2R) - S(v + v^3).\)

We first provide an argument valid for \(v = v(t, x)\) is smooth. (7.9) implies that

\[
\frac{d}{dt}(4\Lambda(t)) = \frac{d}{dt} \int \int_{x>y} R^2(t, x)S^2(t, y)\,dx\,dy
+ 2S(t, y)[cS_x(t, y) + \frac{c'}{4c}(S^2(t, x) - R^2(t, x)) - \frac{1}{2}(v(t, y) + v^3(t, y))]R^2(t, x)dxdy
= \int \int 2R(t, x)S^2(t, y)cR_x(t, x) + 2R(t, x)S^2(t, y)\frac{c'}{4c}(R^2(t, x) - S^2(t, x))
- 2R(t, x)S^2(t, y)\frac{1}{2}(v(t, x) + v^3(t, x)) + 2S(t, y)R^2(t, x)cS_x(t, y)
+ 2S(t, y)R^2(t, x)\frac{c'}{4c}(S^2(t, x) - R^2(t, x)) - 2S(t, y)R^2(t, x)\frac{1}{2}(v(t, y) + v^3(t, y))dxdy
\le \int \int c(S^2R^2)_x + \frac{c'}{2c}(R^2 - S^2)(RS^2 - S^2R)
- R(t, x)S^2(t, y)[v(t, x) - v^2(t, x)] - R^2(t, x)S(t, y)[v(t, y) - v^2(t, y)]dxdy
\le -2 \int cR^2S^2\,dx + \int (R^2 + S^2)\,dx \cdot \int \frac{c'}{2c}|R^2S - S^2R|\,dx
- \int \int R(t, x)S^2(t, y)[v(t, x) - v^2(t, x)] - R^2(t, x)S(t, y)[v(t, y) - v^2(t, y)]dxdy.
And
\[
\left| \int \int R(t, x) S^2(t, y)[v(t, x) - v^2(t, x)]dxdy \right|
\leq \int S^2(t, y)dy \int \left| R(t, x)(v(t, x) + v^3(t, x)) \right| dx
\leq E_0 \| R(t, x) \|_{L^2} \| v(t, x) + v^3(t, x) \|_{L^2}
\leq E_0 E_0^{\frac{1}{2}} \| v(t, x) \|_{L^2} \| 1 + v(t, x) \|_{L^\infty}
\leq E_0^2 \| 1 + v \|_{L^\infty}.
\]

Thus we obtain that
\[
(7.10) \quad \frac{d}{dt}(4\Lambda(t)) \leq -2K^{-1} \int R^2 S^2 dx + 4E_0 \left\| \frac{c'}{2c} \right\|_{L^\infty} \int |R^2 S - S^2 R| dx + 2E_0^2 \| 1 + v \|_{L^\infty}.
\]

Where $K^{-1}$ is the lower bound for the speed $c(v)$. And for each $\epsilon > 0$, we have $|R| \leq \epsilon^{-\frac{1}{2}} + \epsilon^{\frac{1}{2}} R^2$. And pick any $\epsilon > 0$ such that $K^{-1} > 4E_0 \left\| \frac{c'}{2c} \right\|_{L^\infty} \cdot 2\sqrt{\epsilon}$.

Thus we obtain
\[
(7.11) \quad \frac{d}{dt}(4\Lambda(t)) \leq -K^{-1} \int R^2 S^2 dx + \frac{16E_0^2}{\sqrt{\epsilon}} \left\| \frac{c'}{2c} \right\|_{L^\infty} + 2E_0^2 \| 1 + v \|_{L^\infty}.
\]

The smooth case is proved. To proof Lemma 1 in general cases, for every $\epsilon > 0$, there exist a constant $K_\epsilon$ satisfying that for all $w, z$,
\[
| \sin z(1 - \cos w) - \sin w(1 - \cos z) | \leq K_\epsilon \left[ \tan^2 \left( \frac{w}{2} \right) + \tan^2 \left( \frac{z}{2} \right) \right] (1 + \cos w)(1 + \cos z) + \epsilon(1 - \cos w)(1 - \cos z),
\]

for fixed $0 \leq s < t$, consider the sets $\Gamma_s$ and $\Gamma_t$ as we defined in (5.6) and define $\Gamma_{st} := \Gamma_t \setminus \Gamma_s$. And recall that
\[
dxdt = \frac{pq}{8c}(1 + \cos w)(1 + \cos z)dXdY.
\]

We can write that
\[
(7.14) \quad \int_{s}^{t} \int_{-\infty}^{+\infty} \frac{1}{4}(R^2 - S^2) dxdt = (t - s) E_0
\]
\[
(7.15) \quad = \int_{\Gamma_{st}} \frac{pq}{32c}(1 + \cos w)(1 + \cos z)[\tan^2 \left( \frac{z}{2} \right) + \tan^2 \left( \frac{w}{2} \right)]dXdY.
\]
(7.14) holds only on the case that \( v(t,x) \) is smooth while (7.15) holds for all cases. Combine (5.5), (5.17), and (5.18) and apply (7.12)-(7.15), we obtain that

\[
\Gamma(t) - \Gamma(s) \leq \int_{\Gamma_{st}} \frac{(1 - \cos w)(1 - \cos z) pq}{64} dX dY \\
+ E_0 \int_{\Gamma_{st}} \frac{c'pq}{64c^2} \sin z(1 - \cos w) - \sin w(1 - \cos z) dX dY \\
+ E_0 \int_{\Gamma_{st}} \left\{ - \frac{pq}{32c^2} (v + v^3)(1 + \cos z) \sin w - \frac{\sin w(2v^2 + v^4)p}{8c} \frac{c'}{8c} (\cos z - \cos w)q \\
+ \frac{(1 + csow)(4v + 4v^3)p}{32} \frac{1}{4c} \sin zq + \frac{(1 + \cos w)}{32} \frac{c'}{8c^2} (2v^2 + v^4)[\sin z - \sin w] pq \right\} dX dY \\
\leq \frac{1}{64} \int_{\Gamma_{st}} (1 - \cos w)(1 - \cos z) pq dX dY \\
+ E_0 \int_{\Gamma_{st}} \frac{c'}{64c^2} pq \left[ K_\epsilon \left( \tan^2 \left( \frac{w}{2} \right) + \tan^2 \left( \frac{z}{2} \right) \right) (1 + \cos w)(1 + \cos z) \\
+ \epsilon (1 - \cos w)(1 - \cos z) \right] dX dY + E_0 \int_{\Gamma_{st}} \left\{ - \frac{pq}{32c^2} (v + v^3)(1 + \cos z) \sin w \\
- \frac{\sin w(2v^2 + v^4)p}{32} \frac{c'}{8c} (\cos z - \cos w)q + \frac{(1 + csow)(4v + 4v^3)p}{32} \frac{1}{4c} \sin zq \\
+ \frac{(1 + \cos cos w)}{32} \frac{c'}{8c^2} (2v^2 + v^4)[\sin z - \sin w]pq \right\} dX dY \\
\leq K(t - s),
\]

for a suitable constant \( K \). And this proved the Lemma 1. \( \square \)

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