Solution of infinite system of ordinary differential equations and fractional hybrid differential equations via measure of noncompactness

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ABSTRACT
In this work, we use the notion of convex power condensing mapping under measure of noncompactness in locally convex spaces and establish some new coupled fixed point results. The results proved herein are the generalization and extension of some widely known results in the existing literature. Furthermore, we apply our results for the existence of solution to infinite system of ordinary differential equations and system of fractional hybrid differential equations.

1. Introduction
The Mathematical model has a vital role in Mathematical analysis and supports plenty of real-world problems. Some of the problems that come up in geology, physical sciences, engineering, mechanics, applied mathematics and economics find its way into mathematical models expressed by differential equations. The following equation can be used to simulate many real-world problems.

\[ f(u) + g(u) = u, \quad u \in A \]  

where \( A \) is a subset of a linear space. The operator form of (1) is

\[ T = f + g, \]  

where \( f \) is a contraction in some sense (expansive or nonexpansive), and \( g \) fulfills some conditions (completely continuous, compactness, convex power condensing, . . . ) and \( T \) itself has neither of these properties. To check the existence of solution, we use fixed point theory, because the problem of existence solution usually turns into the problem of finding a fixed point of a particular mapping. Due to this fact, the results of fixed point theory could be implemented to get results of an operator equation. Thus finding the existence solution of Equation (1) is equivalent to finding the fixed point of the operator equation (2). To find the fixed point of Equation (2), it becomes desirable to develop fixed point theorems for such situations. An early theorem of this type was given by Krasnosel’skii [1], which combined both the Schauder fixed point theorem and the Banach fixed point theorem. In Krasnosel’skii fixed point theorem, the notion of the compactness plays an essential role. To tackle this obstacle, a feasible technique is to use the notion of measure of noncompactness (MNC, for short).

The notion of MNC suggested by Kuratowski [2] is a new area for the researchers. The MNC appears in several contexts and played a key role in various branches of mathematics, especially in nonlinear analysis and determines the existence of solution to non-linear problems. On the other hand, Sun and Zhang [3] launched the notion of convex power condensing mapping under the Kuratowski MNC. This notion was extended by Ezzinbi and Taoudi [4] and then by Shi [5]. Recently, Khchine et al. [6] extended the view of a convex power condensing operator \( T \) in connection with another operator \( S \) of [5] in complete Hausdorff locally convex space. In particular, they relaxed the compactness condition in Krasnosel’skii fixed point theorem by using the notion of MNC.

In fixed point theory, one of the remarkable and pivotal result is a coupled fixed point theorem, which was introduced by Guo and Lakhsmikantham [7]. Bhaskar and Lakhsmikantham [8] were the pioneers who used coupled fixed points results for the existence of unique
solution to a periodic boundary value problem. Many prominent researchers have taken greater interest regarding the application potential of coupled fixed point theory.

Using the notions of MNC and a convex-power condensing mapping, we established coupled fixed point results in complete Hausdorff locally convex space. Further, we apply our results for the existence solution to two classes of infinite system of ordinary differential equations and system of fractional hybrid differential equations. In support we provide an example for the effectiveness of our existence results.

2. Preliminaries

Thoroughly this work, $\mathcal{I} = [0, k]$ with $(k > 0)$ denote the real numbers set by $\mathbb{R}$, the topological dual of a locally convex space $\mathcal{H}$ by $\mathcal{H}^*$, the collection of all bounded subsets of $\mathcal{H}$ by $B(\mathcal{H})$ and the class of semi-norms which produces the topology on $\mathcal{H}$ by $\mathfrak{P} = \{p_\alpha \}_{\alpha \in I}$. Also, $\bar{A}$ stands for the closure of $A$, $\overline{\mathfrak{A}}(A)$ stands for the closure convex hull of $A$. Moreover, let us denote

$$\Phi = \{\beta \mid \beta : [0, \infty) \to [0, \infty) \text{ such that } \beta(u) < u \text{ for } u > 0 \text{ and } \beta(0) = 0\}$$

and

$$\Gamma = \{\gamma \mid \gamma : [0, \infty) \to [0, \infty) \text{ such that } \gamma(u) > u \text{ for } u > 0 \text{ and } \gamma(0) = 0\}.$$ 

Note that the semi-norm $p_\alpha$ for the product space is defined as $p_\alpha(u, v) = p_\alpha(u) + p_\alpha(v)$. Now, we list some basic concepts and essential results.

Definition 2.1 ([9]): Let $(\mathcal{H}, \mathcal{Y})$ be a Hausdorff topological vector space with $0$ as its zero element and $\mathcal{L}$ be a lattice with $\mathcal{Y}$ as its least element. Then a function $\mathcal{Y} : B(\mathcal{H}) \to \mathcal{L}$ is called an MNC on $\mathcal{H}$ if it fulfills the following axioms:

(i) $\mathcal{Y}(\overline{\mathcal{A}}(Q)) = \mathcal{Y}(Q)$ for every $Q \in B(\mathcal{H})$;

(ii) $Q_1 \subset Q_2 \implies \mathcal{Y}(Q_2) \leq \mathcal{Y}(Q_1)$ (Monotonicity);

(iii) $\mathcal{Y}(\{a\} \cup Q) = \mathcal{Y}(Q)$ for any $a \in \mathcal{H}$ and $Q \in B(\mathcal{H})$;

(iv) $\mathcal{Y}(Q) = 0 \iff Q$ is relatively compact in $\mathcal{H}$ (Regularity).

Additionally, if $\mathcal{L}$ is a cone in a linear space over the real field, then the MNC $\mathcal{Y}$ is

(v) homogeneous if $\mathcal{Y}(\lambda Q) = |\lambda| \mathcal{Y}(Q)$ for every $Q \in B(\mathcal{H})$ and $\lambda \in \mathbb{R}$;

(vi) subadditive if $\mathcal{Y}(Q_1 \cup Q_2) \leq \mathcal{Y}(Q_1) + \mathcal{Y}(Q_2)$ for every $Q_1, Q_2 \in B(\mathcal{H})$.

Definition 2.2: Let $M \subseteq \mathcal{H}$ such that $M \neq \emptyset$. Then a mapping $F : M \to \mathcal{H}$ is

(a) $p_\alpha$-contraction if for every $\alpha \in I$, there exist $\kappa_\alpha \in (0, 1)$ such that

$$p_\alpha(Fv_1 - Fv_2) \leq \kappa_\alpha p_\alpha(v_1 - v_2),$$

for all $v_1, v_2 \in M$,

(b) $p_\alpha$-nonexpansive if for every $\alpha \in I$, we have

$$p_\alpha(Fv_1 - Fv_2) \leq p_\alpha(v_1 - v_2),$$

for all $v_1, v_2 \in M$,

(c) $p_\alpha$-expansive if for every $\alpha \in I$, there exists $\kappa_\alpha \in [1, \infty)$ such that

$$p_\alpha(Fv_1 - Fv_2) \geq \kappa_\alpha p_\alpha(v_1 - v_2),$$

for all $v_1, v_2 \in M$.

**Definition 2.3:** Let $\mathcal{X}$ be a nonempty set. Then the mapping $G : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ has a coupled fixed point $(v_1, v_2) \in \mathcal{X} \times \mathcal{X}$, if $G(v_1, v_2) = v_1$ and $G(v_2, v_1) = v_2$.

Let $P$ be a non-empty, closed and convex subset of $\mathcal{X}$ with $u_0 \in P$. Let $f : P \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ be two non-linear mappings. Then for any $Q \subseteq P$, set

$$\mathcal{N}(f, g, Q) = \mathcal{N}^{(1,u_0)}(f, g, Q)$$

$$= \{u \in P : u = gu + fv, \text{ for some } v \in Q\}$$

and

$$\mathcal{N}^{(m,u_0)}(f, g, Q) = \mathcal{N}^{(m,u_0)}(f, g, \overline{\mathfrak{A}}(\mathcal{N}^{(m-1,u_0)}(f, g, Q) \cup \{u_0\}))$$

for $m = 2, 3, \ldots$.

**Definition 2.4 ([6]):** Let $P$ be a non-empty, closed and convex subset of a complete Hausdorff locally convex space $\mathcal{X}$ with $u_0 \in P$ and $f, g : \mathcal{X} \to \mathcal{X}$ are two bounded mappings. If $\mathcal{Y}$ is an MNC on $\mathcal{X}$ such that

$$\mathcal{Y}(\mathcal{N}^{(m_0,u_0)}(f, g, Q)) < \mathcal{Y}(Q),$$

where $Q \subseteq P$ is bounded with $\mathcal{Y}(Q) > 0$. Then $f$ is $g$-convex power condensing mapping about $u_0$ and $m_0 \in \mathbb{N}$ under $\mathcal{Y}$. Definitely, $f : P \to P$ is convex power condensing mapping under $\mathcal{Y}$ about $u_0$ and $m_0$ iff $f$ is 0-convex power condensing under $\mathcal{Y}$ about $u_0$ and $m_0$.

Throughout the rest of this work, $\Omega = (\mathcal{H}, \{p_\alpha\}_{\alpha \in I})$ is a Hausdorff locally convex space, $\mathcal{P}$ is a non-empty, convex, complete and bounded subset of $\Omega$ and $\mathcal{Y}$ is an MNC on $\Omega$.

**Theorem 2.5 ([6]):** Let $f : \mathcal{P} \to \Omega$ and $g : \mathcal{H}_a \to \mathcal{H}_a$ be two mappings such that

(C1) $f$ is continuous;

(C2) $g$ is $p_\alpha$-contraction;
there exists a vector \( u_0 \in \mathcal{P} \) and a positive integer \( m_0 \), for which \( f \) is \( g \)-convex power condensing about \( u_0 \) and \( m_0 \) under \( \gamma \);

\((C_4)\) if \( u = gu + fv \) for some \( v \in \mathcal{P} \), then \( u \in \mathcal{P} \).

Then there is at least one fixed point of \( f + g \) in \( \mathcal{P} \).

**Theorem 2.6 ([6]):** Theorem 2.5 is true if we interchange the condition \((C_3)\) by

\((C_3')\) \( p_1 \in f(\mathcal{P}) \) implies \( g(\mathcal{P}) + p_1 \supset \mathcal{P} \), where \( g(\mathcal{P}) + p_1 = \{ p_2 + p_1, p_2 \in g(\mathcal{P}) \} \).

**Definition 2.7 ([10]):** The Riemann–Liouville fractional integral of order \( \sigma > 0 \) of a function \( f \in L^1(\mathbb{R}^+) \) is

\[
I^\sigma f(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t - s)^{\sigma - 1} f(s) \, ds
\]

as long as the right side is pointwise defined on \((0, \infty)\).

**Definition 2.8 ([10]):** Let \( \sigma > 0 \) with \( p - 1 \leq \sigma < p \), \( p \in \mathbb{N} \) and \( f^{(\sigma)}(x) \) exists. The Caputo fractional order derivative of \( f \) is

\[
{}^C D^\sigma f(t) = \frac{1}{\Gamma(p - \sigma)} \int_0^t (t - s)^{p - \sigma - 1} f^{(p)}(s) \, ds
\]

as long as the right side is pointwise defined on \((0, \infty)\), where \( p = [\sigma] + 1 \) and \( [\sigma] \) represent the integral part of \( \sigma \).

3. Fixed point results
In this section, we present coupled fixed point results. To proceed further, let \( \tau, f : \mathcal{P} \rightarrow \Omega \) and \( g : \mathcal{H}_+ \rightarrow \mathcal{H}_+ \) and \( \tilde{\mathcal{H}} = \mathcal{H}_+ \times \mathcal{H}_+ \), \( \mathcal{P} = \mathcal{P} \times \mathcal{P} \). Define \( \tilde{\tau}, \tilde{f} : \tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{H}} \) and \( \tilde{g} : \tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}} \) by

\[
\tilde{\tau}(x_1, x_2) = (\tau x_1, \tau x_2), \quad \tilde{f}(x_1, x_2) = (f x_1, f x_2), \quad \text{and} \quad \tilde{g}(x_1, x_2) = (g x_1, g x_2).
\]

Now, since

\[
(G(x_1, x_2), G(x_2, x_1)) = (f x_1 + g x_2, f x_2 + g x_1)
\]

\[
= (f x_1, f x_2) + (g x_2, g x_1)
\]

\[
= \tilde{f}(x_1, x_2) + \tilde{g}(x_1, x_2).
\]

Thus to prove that \( G(x_1, x_2) \) has a coupled fixed point in \( \tilde{\mathcal{P}} \), it is sufficient to show that \( \tilde{f}(x_1, x_2) + \tilde{g}(x_1, x_2) \) has a fixed point in \( \mathcal{P} \).

**Theorem 3.1:** Let \( f : \mathcal{P} \rightarrow \Omega \) and \( g : \mathcal{H}_+ \rightarrow \mathcal{H}_+ \) be two mappings fulfilling the following conditions:

\((C_1)\) \( f \) is continuous;

\((C_2)\) there exists \( \beta_2 \in \Phi \) and \( \kappa \in [0, 1) \) such that

\[
p_\alpha (gu - gv) \leq \kappa \beta_2 (p_\alpha (u - v)), \quad \text{for all } u, v \in \mathcal{P}.
\]

\((C_3)\) if \( u = gu^* + fv \) for some \( u^*, v \in \mathcal{P} \), then \( u \in \mathcal{P} \).

Then there is a coupled fixed point of \( G(u, v) = fu + gv \) in \( \mathcal{P} \).

**Proof:** One can easily check that \( \tilde{\mathcal{Q}} \) is a Hausdorff locally convex space and \( \tilde{\mathcal{P}} \) is a non-empty, bounded, complete and convex subset of \( \tilde{\mathcal{Q}} \).

We have to show that the mappings \( \tilde{\mathcal{D}} \) and \( \tilde{\mathcal{Q}} \) fulfill all the conditions of Theorem 2.5. First, we check the continuity of \( \tilde{f} \), for this we have

\[
p_\alpha (\tilde{f}(u_1, v_1) - \tilde{f}(u_2, v_2))
\]

\[
= p_\alpha (fu_1 - fu_2, fv_1 - fv_2)
\]

\[
\leq \kappa \beta_2 (p_\alpha (u_1 - u_2, v_1 - v_2))
\]

\[
< \kappa \beta_2 ((p_\alpha (u_1 - u_1) + (p_\alpha (u_2 - v_2)))
\]

\[
= \kappa \beta_2 p_\alpha (u_1 - u_2, v_1 - v_2)
\]

Thus \( \tilde{\mathcal{D}} \) is \( p_\alpha \)-contraction.

Next, we must show that there is a vector \( u_0 \in \tilde{\mathcal{P}} \) and an integer \( m_0 > 0 \) such that the mapping \( f \) is \( \tilde{\mathcal{D}} \)-convex power condensing about \( u_0 \) and \( m_0 \) under \( \gamma \). For this, let \( \tilde{\mathcal{Q}} \subset \tilde{\mathcal{P}} \) such that \( \tilde{\mathcal{Q}} \) is bounded and \( \gamma (\tilde{\mathcal{Q}}) > 0 \) and let \( \tilde{\tau} \) be the map that assigns a unique point in \( \tilde{\mathcal{Q}} \) to each \( (u_1, u_2) \in \tilde{\mathcal{P}} \) such that \( \tilde{\tau}(u_1, u_2) = \tilde{\mathcal{D}}(u_1, u_2) \) + \( \tilde{f}(u_1, u_2) \). Then

\[
(\tau u_1, \tau u_2) \subset \tilde{\mathcal{Q}} \subset \mathcal{P} \quad \text{and} \quad \tilde{\tau}(\tilde{\mathcal{P}}) \subset \tilde{\mathcal{P}} \text{ for each } (u_1, u_2) \in \tilde{\mathcal{P}}, \text{ thus } \tilde{\tau}(\tilde{\mathcal{P}}) \subset \tilde{\mathcal{P}}.
\]

Let us claim that for any positive integer \( m, \)

\[
\overline{\mathcal{D}}\left(N^{(m)}(\tilde{\tau} \tilde{g} \tilde{\tau} (\tilde{\mathcal{Q}}) \cup \{u_0\})\right) = \tilde{\mathcal{Q}}.
\]

To support our claim, we use induction. As \( \overline{\mathcal{D}}( \tilde{\mathcal{Q}} \cup \{u_0\}) \subset \tilde{\mathcal{Q}} \), so that

\[
\tilde{\mathcal{A}}(\overline{\mathcal{D}}( \tilde{\mathcal{Q}} \cup \{u_0\})) \subset \tilde{\mathcal{F}}( \tilde{\mathcal{Q}} \cup \{u_0\})\overline{\mathcal{D}}( \tilde{\mathcal{Q}} \cup \{u_0\}).
\]
That is, \( \mathbb{C}(\tilde{f}, \tilde{g}, \tilde{Q}) \) and hence \( \tilde{Q} \subset \mathbb{C}(\tilde{f}, \tilde{g}, \tilde{Q}) \cup \{u_0\} \). Thus \( \tilde{Q} = \mathbb{C}(\tilde{f}, \tilde{g}, \tilde{Q}) \). This shows that Equation (5) is true for \( m = 1 \). Assume that Equation (5) is true for \( m = k > 1 \), that is, \( \mathbb{C}(N^{(k-1,u_0)}(\tilde{f}, \tilde{g}, \tilde{Q}) \cup \{u_0\}) = \tilde{Q} \). Then

\[
N^{(k+1,u_0)}(\tilde{f}, \tilde{g}, \tilde{Q})
= N^{(1,u_0)}(\tilde{f}, \tilde{g}, \mathbb{C}(N^{(k,u_0)}(\tilde{f}, \tilde{g}, \tilde{Q}) \cup \{u_0\}))
= N^{(1,u_0)}(\tilde{f}, \tilde{g}, \tilde{Q})
= \tilde{Q}.
\]

Thus \( \mathbb{C}(N^{(k+1,u_0)}(\tilde{f}, \tilde{g}, \tilde{Q}) \cup \{u_0\}) = \mathbb{C}(\tilde{Q} \cup \{u_0\}) = \tilde{Q} \).

Hence by induction our claim (5) is true. In particular,

\[
\mathbb{C}(N^{(m,u_0)}(\tilde{f}, \tilde{g}, \tilde{Q}) \cup \{u_0\}) = \tilde{Q}.
\]

(6)

Now, using (6) and the fact that \( N^{(m,u_0)}(\tilde{f}, \tilde{g}, \tilde{Q}) \subset N^{(m_0,u_0)}(\tilde{f}, \tilde{g}, \tilde{Q}) \cup \{u_0\} \), we have

\[
\mathbb{C}(N^{(m,u_0)}(\tilde{f}, \tilde{g}, \tilde{Q})) = \mathbb{C}(N^{(m_0,u_0)}(\tilde{f}, \tilde{g}, \tilde{Q})) < \mathbb{C}(\tilde{Q}) = \tilde{Q}.
\]

Hence \( \tilde{f} \) is \( \tilde{g} \)-convex power condensing about \( u_0 \) and \( m_0 \) under \( \mathbb{C} \).

Finally, if \( u = \tilde{g}v + \tilde{f}v \), for some \( v = (v_1, v_2) \in \tilde{P} \), then we have to show that \( u = (u_1, u_2) \in \tilde{P} \). For this, since

\[
(u_1, u_2) = \tilde{g}(u_1, u_2) + \tilde{f}(v_1, v_2) = (gu_2, gu_1) + (fv_1, fv_2)
= (gu_2 + fv_1, gu_1 + fv_2),
\]

which implies that \( u_1 = gu_2 + fv_1 \) and \( u_2 = gu_1 + fv_2 \), by condition \((C_2)\), \( u_1, u_2 \in \mathbb{P} \) and hence \( u \in \tilde{P} \). Thus by Theorem 2.5, there exists at least one fixed point of \( \tilde{f} + \tilde{g} \) in \( \tilde{P} \) and hence there exists at least one coupled fixed point of \( G(u, v) \) in \( \tilde{P} \).

Remark 3.2: The arguments of Theorem 3.1 are the same if we alternatively change the contraction in \((C_2)\) by \( p_u \)-contraction.

From Theorem 3.1, without any hurdle we can derive the following corollary.

Corollary 3.3: Let \( \mathbb{P} \) be a non-empty, convex, bounded and closed subset of a Banach space \( \Omega \). Let \( f : \mathbb{P} \rightarrow \Omega \) and \( g : \Omega \rightarrow \Omega \) be two mappings such that

\[
\begin{align*}
(C_1) \quad & f \text{ is continuous;} \\
(C_2) \quad & \text{there exists } \beta_\mathbb{P} \in \Phi \text{ and } k \in (0, 1) \text{ such that } \\
& \|gu - gv\| \leq k\beta_\mathbb{P}\|u - v\|, \text{ for all } u, v \in \Omega; \\
(C_3) \quad & \text{if } u = gu^* + fv \text{ for some } u^*, v \in \mathbb{P}, \text{ then } u \in \mathbb{P}.
\end{align*}
\]

Then there exists a coupled fixed point of \( G(u, v) = fu + gv \) in \( \tilde{P} \).

Remark 3.4: Corollary 3.3 generalizes Theorem 1 of [11]. In Theorem 1 of [11], \( f : \mathbb{P} \rightarrow \Omega \) is completely continuous, however in Corollary 3.3, \( f : \mathbb{P} \rightarrow \Omega \) is continuous.

If \( \Omega_0 \) is a Banach space equipped with its weak topology, then \( \Omega_0 \) is locally convex induced by the family of seminorms \( p_\mathbb{P}(u) = |f(u)| \) for all \( u \in \Omega^*_0 \). We can deduce corollary from Theorem 3.1 as:

Corollary 3.5: Let \( \mathbb{P} \) be a non-empty, convex, closed and bounded subset of a Banach space \( \Omega_0 \). Let \( f : \mathbb{P} \rightarrow \Omega_0 \) and \( g : \Omega_0 \rightarrow \Omega_0 \) be two mappings such that

\[
\begin{align*}
(C_1) \quad & f \text{ is weakly sequentially continuous;} \\
(C_2) \quad & \text{there exists } \beta_\mathbb{P} \in \Phi \text{ and for each } \phi \in \Omega^*_0, \text{ there exists } 0 < k_\phi < 1 \text{ such that for all } u, v \in \Omega_0, \text{ we have } \\
& \phi(S(u) - S(v)) \leq k_\phi \beta_\mathbb{P} g(S(u) - S(v)); \\
(C_3) \quad & \text{if } u = gu^* + fv \text{ for some } u^*, v \in \mathbb{P}, \text{ then } u \in \mathbb{P}.
\end{align*}
\]

Then there exists a coupled fixed point of \( G(u, v) = fu + gv \) in \( \tilde{P} \).

Theorem 3.6: Theorem 3.1 is true if we interchange the conditions \((C_2)\) and \((C_3)\) by

\[
\begin{align*}
(C_2) \quad & \text{there exists } \beta_\mathbb{P} \in \Phi \text{ and } k_\mathbb{P} \in (0, 1) \text{ such that } \\
& p_u(gu - gv) \geq k_\mathbb{P} \beta_\mathbb{P} g(p_u(u - v)), \text{ for all } u, v \in \mathbb{P}; \\
(C_3) \quad & \text{if } u = gu^* + fv \text{ for some } u^*, v \in \mathbb{P}, \text{ then } u \in \mathbb{P}, \text{ where } g(\mathbb{P}) = \mathbb{P} + p_1 \subset \mathbb{P}, \text{ where } g(\mathbb{P}) = \mathbb{P} + p_1 \subset \mathbb{P}.
\end{align*}
\]

Proof: We need to show that the mappings \( \tilde{f} \) and \( \tilde{g} \) fulfil all the conditions of Theorem 2.6 on \( \tilde{P} \). \( \tilde{f} \) is continuous and \( \tilde{g} \)-convex power condensing under \( \mathbb{C} \) as proved in Theorem 3.1.

Now, we show that \( \tilde{g} \) is \( p_u \)-expansive. To do this, using condition \((C_2)\) we have for every \( u = (u_1, u_2), v = (v_1, v_2) \),

\[
p_u(\tilde{g}u - \tilde{g}v) = p_u(\tilde{g}u_1, \tilde{g}u_2) - (\tilde{g}v_1, \tilde{g}v_2)
= p_u(gu_1 - gv_1, gu_2 - gv_2)
\geq k_\mathbb{P} \gamma_\mathbb{P} (p_u(u_1 - v_1)) + k_\mathbb{P} \gamma_\mathbb{P} (p_u(u_2 - v_2))
> k_\mathbb{P} p_u(u_1 - v_1) + p_u(u_2 - v_2)
= k_\mathbb{P} p_u(u - v).
\]

Thus \( \tilde{g} \) is \( p_u \)-expansive.
Finally, let \( \tilde{w} \in f(\tilde{P}) \). Then we have to show that \( g(\tilde{P}) + \tilde{w} \supset \tilde{P} \). For this, since \((w_1, w_2) = \tilde{w} \in f(\tilde{P})\) implies that \(w_1, w_2 \in f(\tilde{P})\) and thus by condition (C_3), we have

\[
g(\tilde{P}) + w_1 \supset \tilde{P} \quad \text{and} \quad g(\tilde{P}) + w_2 \supset \tilde{P}
\]

Thus by Theorem 2.6, there exists at least one fixed point of \( \tilde{T} + \tilde{g} \) in \( \tilde{P} \) and hence there exists at least one coupled fixed point of \( G(u, v) \) in \( \tilde{P} \).

\[\blacksquare\]

**Remark 3.7:** The arguments of Theorem 3.6 are the same if we change the contraction in \((C_2)\) by \( p_\alpha \)-expansive.

### 4. Existence results

#### 4.1. Infinite systems of differential equations

Let \( E = C(I, X) \) be the space of continuous functions from \( I \) to \( X \) and \( \tilde{P} = (p_\alpha) \) each \( p_\alpha \) is semi-norms defined by \( p_\alpha = \max_{t \in I} \alpha(a(t)) \), for each \( a \in E \). Note that, \( E \) equipped with the topology induced by the class \( \tilde{P} \) is a complete Hausdorff locally convex space. In this section, we are interested with the existence solution to the following two classes of infinite system of ordinary differential equations:

\[
\frac{d}{dt} \begin{bmatrix} a_i (t) \\ b_i (t) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^i k_{ij} a_j (t) \\ \sum_{j=1}^i k_{ij} b_j (t) \end{bmatrix} = \begin{bmatrix} \psi_i (t, b_i (t)) \\ \psi_i (t, a_i (t)) \end{bmatrix}, \quad i = 1, 2, 3, \ldots, \quad a_0 (t) = b_0 (t) = 0, \quad i = 1, 2, 3, \ldots,
\]

where \( t \in I, k_{ij} \geq 0 \) and \( \psi_i (i = 1, 2, \ldots) \) are continuous mappings on \( I \times \mathbb{R}^m \times \mathbb{R}^m = \prod_{\alpha \in \mathbb{N}} X_\alpha \) is the countable Cartesian product of \( X_\alpha = \mathbb{R} \) and take real values. Note that the derivative in (7) measures the speed of change in time for every parameter/coordinate.

**Lemma 4.1:** Integral representation of the class of two infinite systems (7) is

\[
a(t) = \phi (a(t)) + \int_0^t \psi(s, b(s)) \, ds, \\
b(t) = \phi (b(t)) + \int_0^t \psi(s, a(s)) \, ds,
\]

where \( a = (a_i) \in \mathbb{R}^m, b(t) = (b_i) \in \mathbb{R}^m, \psi = (\psi_i) \) and \( \phi = (\phi_i) \) with \( \phi_i (a) = \sum_{j=1}^i k_{ij} a_j (t) \) \( i = 1, 2, \ldots \) for each \( a(t) = (a_i (t)) \in \mathbb{R}^m \).

**Theorem 4.2:** The class of two infinite systems (7) has at least one solution in \( C(I, \mathbb{R}^m) \) if the following conditions hold:

\[
(A_1) \quad \text{for } k_{ij} \geq 0 \text{ and } \Lambda \geq 2, \text{ we have}
\]

\[
\sum_{j=1}^i k_{ij} \leq \frac{p_\alpha \alpha (a(t) - a(s))}{{\Lambda}^2 \max_{t \in I} p_\alpha (a(t) - b(t))},
\]

\[
(A_2) \quad \text{there exists a continuous function } \xi \in C(I, \mathbb{R}) \text{ such that}
\]

\[
\psi (t, a(t)) \leq \xi (t), \quad a \in \mathbb{R}^m, \quad t \in I.
\]

**Proof:** Define \( M \subset E \) by

\[
M = \bigcap_{a \in E} \{ a \in M_a : p_\alpha \alpha (a(t) - a(s)) \leq \nabla, t, s \in I \},
\]

where \( M_a = \{ a \in E : p_\alpha (a) \leq \Xi \} \) with \( \Xi = \frac{1}{{\Lambda}^2} \max_{t \in I} p_\alpha (a(t) - a(t)) + 2k \rho_\alpha (\xi) \). Clearly \( M \) is a convex, closed, bounded and complete.

Now, since \( a(t) \) is a solution of (7) if and only if \( a(t) \) satisfies (8). Thus to show the existence solution of (7), it is enough to show the existence solution of (8). For this, define \( S : E \to E \) and \( T : M \to E \) by

\[
Sa(t) = \phi (a(t)); \\
Ta(t) = \int_0^t \psi(s, a(s)) \, ds.
\]

The system (8) is turned into the system

\[
a(t) = Sa(t) + Tb(t); \\
b(t) = Sb(t) + Ta(t), \quad t \in I.
\]  

(9)

We have to show that the system (9) fulfils all the conditions of Theorem 3.1. First we show the continuity of \( T \). For this, let us take the sequence \( \{a_n\} \) in \( M \) such that \( a_n \to a \) in \( E \) as \( n \to \infty \). For \( a \in I \) and \( t \in I \), one can write

\[
p_\alpha (Ta_n (t) - Ta(t)) \\
= p_\alpha \left( \int_0^t (\psi(s, a_n(s)) - \psi(s, a(s))) \, ds \right) \\
\leq \int_0^t p_\alpha (\psi(s, a_n(s)) - \psi(s, a(s))) \, ds \\
= \int_0^t p_\alpha (\psi(s, a_n(s)) - \psi(s, a(s))) \, ds.
\]

But \( \psi \) is continuous, so \( p_\alpha (\psi(s, a_n(s)) - \psi(s, a(s))) \to 0 \) as \( n \to \infty \) and by Lebesgue Dominated convergent theorem \( p_\alpha (Ta_n (t) - Ta(t)) \to 0 \) as \( n \to \infty \). Thus \( Ta_n \to Ta \) as \( n \to \infty \) and hence \( T \) is continuous.
Next, we show condition (C₂) of Theorem 3.1. For this, let a, b ∈ E and t ∈ Ω, we have

\[ p_α(Sa - Tb) = p_α(φ(a(t)) - φ(b(t))) \]
\[ = p_α \left( \sum_{j=1}^{j} k_j a_j(t) - \sum_{j=1}^{j} k_j b_j(t) \right) \]
\[ ≤ \sum_{j=1}^{j} k_j p_α(a_j(t) - b_j(t)) \]
\[ ≤ \sum_{j=1}^{j} k_j \max_{t \in Ω} p_α(a_j(t) - b_j(t)) \]
\[ = \max_{t \in Ω} p_α(a(t) - b(t)) \]
\[ ≤ \frac{1}{λ^2} p_α(a(t) - b(t)) \]
\[ = k_α \beta_α \Phi(λ, a - b), \]

where \( β_α(λ) = x/λ \in Φ \) is a control function and \( k_α = 1/λ. \)

Finally, we prove condition (C₂) of Theorem 3.1. For this, let \( a^*, b \in M \) such that \( a = Sa^* + Tb \), then for \( α \in l, t \in Ω \) and using (A₁), we have

\[ p_α(a(t)) = p_α(Sa^*(t) + Tb(t)) \]
\[ ≤ p_α(φ(a^*(t))) + p_α \left( \int_{0}^{t} φ(r, b(r)) \, dr \right) \]
\[ ≤ p_α \left( \sum_{j=1}^{j} k_j a_j(t) \right) + \int_{0}^{t} p_α(φ(r, b(r))) \, dr \]
\[ ≤ \sum_{j=1}^{j} k_j p_α(a_j(t)) + \int_{0}^{t} p_α(φ(r, b(r))) \, dr \]
\[ ≤ \sum_{j=1}^{j} k_j p_α(a_j(t)) + t p_α(φ(λ, x, y)) \]
\[ ≤ \sum_{j=1}^{j} k_j p_α(a_j(t)) + k p_α(φ(λ, x, y)) \]

Consequently, \( \tilde{p}_α(a) ≤ Ξ. \) Moreover, for every \( 0 ≤ t_1 ≤ t_2 ≤ k_α \), we can write

\[ p_α(a(t_1) - a(t_2)) \]
\[ = p_α(Sa^*(t_1) + Tb(t_1) - Sa^*(t_2) - Tb(t_2)) \]
\[ ≤ p_α(Sa^*(t_1) - Sa^*(t_2)) + p_α(Tb(t_1) - Tb(t_2)) \]
\[ = p_α(φ(a^*(t_1)) - φ(a^*(t_2))) \]
\[ + p_α \left( \int_{0}^{t_1} φ(r, b(r)) \, dr - \int_{0}^{t_2} φ(r, b(r)) \, dr \right) \]
\[ = p_α(α^*(t_1)) - φ(a^*(t_2)) \]

which implies that

\[ p_α(a(t_1) - a(t_2)) ≤ Ξ. \]

Thus \( a \in M. \) Therefore, by Theorem (3.1), there exists at least one coupled fixed point of \( G(x, y) = Tx + Sy \) in \( M. \) Consequently, the class of two infinite systems (7) has at least one solution \( a(t) = (a_i(t)) \in C(Ω, R^m), \) where \( i = 1, 2, 3, \ldots \)

4.2. Fractional hybrid differential equations

Let \( X = C(Ω, R) \) endowed with the supremum norm. Then \( X \) is a Banach space with respect to supremum norm and pointwise operations. Throughout this section, \( C(Ω, R) \) and \( C(Ω, R × R) \) represent the class of continuous functions \( f : Ω × R → R \) and the class of functions \( g : Ω × R × R → R, \) respectively, such that for each \( a, b \in R \)

(i) the map \( t → g(γ, a, b) \) is measurable,
(ii) the maps \( a → g(γ, a, b) \) and \( b → g(γ, x_1, b) \) are continuous.

Bashiri et al. [11] studied the existence solution of fractional order hybrid differential equations:

\[ D^p_a[α(γ) - f(γ, a(γ))] = g(γ, b(γ), a(γ)), \quad \text{a.e. } γ \in Ω, \]
\[ D^p_b[γ(γ) - f(γ, b(γ))] = g(γ, a(γ), a(γ)), \quad \text{a.e. } γ \in Ω, \]
\[ a(0) = 0, b(0) = 0, \]
where $\sigma > 0$, $p \in (0, 1)$, and the functions $f : I \times \mathbb{R} \to \mathbb{R}$, $f(0,0) = 0$ and $g : I \times \mathbb{R} \to \mathbb{R}$ satisfy certain conditions. $D^p$ is the Riemann–Liouville fractional order derivative.

We will discuss the existence solution of fractional order hybrid differential equations:

$$
D^p [a(\gamma) - f(\gamma, a(\gamma))] = g(\gamma, b(\gamma), \varpi b(\gamma)), \quad \text{a.e. } \gamma \in I,
$$

$$
D^p [b(\gamma) - f(\gamma, b(\gamma))] = g(\gamma, a(\gamma), b(\gamma)), \quad \text{a.e. } \gamma \in I,
$$

$$
a(0) = \zeta(a(\eta)), \quad b(0) = \zeta(b(\eta)),
$$

where $\sigma > 0$, $p \in (0, 1)$, $\eta \in I$, $\zeta : \mathbb{R} \to I$ is continuous function and the functions $f : I \times \mathbb{R} \to \mathbb{R}$, $f(0, a(0)) = 0$, $f(0, b(0)) = 0$ and $g : I \times \mathbb{R} \to \mathbb{R}$ satisfy specific conditions. $D^p$ is the Caputo fractional order derivative.

To proceed further, assume that the mappings $f : I \times \mathbb{R} \to I$ and $g : I \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

(A1) the function $a \to a - f(\kappa, a)$ is increasing in $\mathbb{R}$ for all $\kappa \in I$;

(A2) for every $\kappa, a, \varpi a \in I \times \mathbb{R} \times \mathbb{R}$, there exist positive constants $l_2$ and $\Delta_g$ such that

$$
|g(\kappa, a, \varpi a)| \leq l_2|a| + \Delta_g;
$$

(A3) there exists $\lambda \geq 1$ and $\Lambda \geq 2$ such that

$$
|f(\kappa, a(\kappa)) - f(\kappa, b(\kappa))| \leq \frac{1}{\Lambda^2(\lambda + |a(\kappa) - b(\kappa)|)}, \quad \text{for all } \kappa \in \mathbb{R}
$$

and for all $\kappa \in I$;

(A4) $\zeta : \mathbb{R} \to I$ is continuous function such that

$$
|\zeta(a(\kappa)) - \zeta(b(\kappa))| \leq \frac{1}{\Lambda^2} |a(\kappa) - b(\kappa)|, \quad \text{for all } a, b \in \mathbb{R},
$$

where $\Lambda \geq 2$;

(A5) there exists a continuous function $h \in C(I, \mathbb{R})$ such that

$$
g(\kappa, a(\kappa), b(\kappa)) \leq h(\kappa), \quad a, b \in \mathbb{R}, \kappa \in I.
$$

Using Lemma 4.3, we can write system (11) as

$$
a(\kappa) = f(\kappa, a(\kappa)) + \zeta(a(\eta)) + \frac{1}{\Gamma(p)} \int_0^\kappa (\kappa - \gamma)^{p-1}
$$

$$
\times g(\gamma, b(\gamma), \varpi b(\gamma)) \, d\gamma, \quad \kappa \in I.
$$

Now, we present the existence result.

Theorem 4.4: The system (11) has a solution defined on $I$ under the hypothesis (A2) – (A5).

Proof: Let $X = C(I, \mathbb{R})$. Define $M \subset X$ by

$$
M = \{u \in X : \|u\| \leq S\},
$$

where $S \geq 1 + \Delta_0 + (k_0^p/\Gamma(p + 1))\|h\|_1$ with $\Delta_0 = \max_{\kappa \in I}|f(\kappa, 0)| + \max_{\kappa \in \mathbb{R}}|\zeta(\kappa)|$. Then, clearly $M$ is nonempty convex, bounded and closed subset of $X$. Now, since $u(\kappa)$ is a solution of the system (11) if and only if $u(\kappa)$ satisfies the system (12). Thus finding the existence solution of system (11) is equivalent to finding the existence solution of (12). For this, define the operators $S : X \to X$ and $T : M \to X$ by

$$
Su(\kappa) = f(\kappa, u(\kappa)) + \zeta(u(\eta)),
$$

$$
Tu(\kappa) = \frac{1}{\Gamma(p)} \int_0^\kappa (\kappa - \gamma)^{p-1} g(\gamma, u(\gamma), \varpi u(\gamma)) \, d\gamma, \quad \kappa \in I,
$$

so, the system of integral equations (12) is transformed into the system of the following operator equations:

$$
u(\kappa) = Su(\kappa) + Tv(\kappa),
$$

$$
v(\kappa) = Sv(\kappa) + Tu(\kappa), \quad \kappa \in I.
$$

We have to show that the system (13) satisfies all the conditions of Corollary 3.3. First we check the continuity of $T$. For this, let $\{u_n\}$ be a sequence in $M$ such that $u_n \to u$ as $n \to \infty$. Now, consider

$$
|Tu_n(\kappa) - Tu(\kappa)|
$$

$$
= \frac{1}{\Gamma(p)} \left| \int_0^\kappa (\kappa - \gamma)^{p-1} g(\gamma, u_n(\gamma), \varpi u_n(\gamma)) \, d\gamma - \int_0^\kappa (\kappa - \gamma)^{p-1} g(\gamma, u(\gamma), \varpi u(\gamma)) \, d\gamma \right|
$$

$$
\leq \frac{1}{\Gamma(p)} \int_0^\kappa (\kappa - \gamma)^{p-1} \left| g(\gamma, u_n(\gamma), \varpi u_n(\gamma)) - g(\gamma, u(\gamma), \varpi u(\gamma)) \right| \, d\gamma.
$$

It follows from continuity of $g$ that

$$
g(\gamma, u_n(\gamma), \varpi u_n(\gamma)) \to g(\gamma, u(\gamma), \varpi u(\gamma)) \text{ as } n \to \infty.
$$
Using (A2), we have
\[
\begin{align*}
|g(\gamma, u_0(\gamma), f'(u_0(\gamma))) - g(\gamma, u(\gamma), f'(u(\gamma)))| \\
\leq |g(\gamma, u_0(\gamma), f'(u_0(\gamma))) + g(\gamma, u(\gamma), f'(u(\gamma)))| \\
\leq I_g \|u_0\| + \Delta g + I_g \|u\| + \Delta g \\
\leq I_g \Xi + I_g \Xi + 2\Delta g \\
\leq 2[I_g \Xi + \Delta g],
\end{align*}
\]
which implies that
\[
\begin{align*}
(k - \gamma)^{p-1} |g(\gamma, u_0(\gamma), f'(u_0(\gamma))) \\
- g(\gamma, u(\gamma), f'(u(\gamma)))| \leq (k - \gamma)^{p-1} 2[I_g \Xi + \Delta g].
\end{align*}
\]
(15)

That is the left side is integrable. With the help of Lebesgue Dominated convergence theorem, we get
\[
\frac{1}{\Gamma(p)} \int_0^\kappa (k - \gamma)^{p-1} |g(\gamma, u_0(\gamma), f'(u_0(\gamma))) \\
- g(\gamma, u(\gamma), f'(u(\gamma)))| \, d\gamma \to 0 \text{ as } n \to \infty
\]
or
\[
\|T u_n(\kappa) - T u(\kappa)\| \to 0 \text{ as } n \to \infty.
\]
Therefore, \( T u_n \to T u \) as \( n \to \infty \), which implies that \( T \) is continuous.

Next, we show condition (C2) of Theorem 3.3. For this, let \( a, b \in X \), we have
\[
|S(a) - S(b)| = |f(k, a(\kappa)) - \xi(\eta) - f(k, b(\kappa)) + \xi(\eta) |
\]
\[
\leq |f(k, a(\kappa)) - f(k, b(\kappa))| + |\xi(\eta) - \xi(\eta)|
\]
\[
\leq |f(k, a(\kappa)) - f(k, b(\kappa))| + \frac{|a(\kappa) - b(\kappa)|}{\Lambda^2} + \frac{|a(\eta) - b(\eta)|}{\Lambda^2}
\]
\[
\leq \frac{|a - b|}{\Lambda^2(\lambda + |a - b|)} + \frac{|a - b|}{\Lambda^2} + \frac{|a - b|}{\Lambda^2(\lambda + |a - b|)}
\]
\[
= \frac{|a - b|}{\Lambda^2(\lambda + |a - b|)}. 
\]

Define a control function \( \varphi \) by \( \varphi(r) = r(1 + \lambda + \gamma)/\Lambda(\lambda + \gamma) \) and take \( k = 1/\Lambda \), we can write
\[
|S(a) - S(b)| \leq k \varphi(|a - b|).
\]

Finally, we have to prove condition (C3) of Corollary 3.3, let \( u^*, v \in M \) such that \( u = Su^* + T v \), by assumptions \((A_3)\) and \((A_4)\), we have
\[
|u(\kappa)| = |Su^*(\kappa) + T v(\kappa)|
\]
\[
\leq |f(k, u^*(\kappa))| + \left| \frac{1}{\Gamma(p)} \int_0^\kappa (k - \gamma)^{p-1} \\
\times g(\gamma, v(\gamma), f'(v(\gamma))) \, d\gamma \right|
\]
\[
\leq |f(k, u^*(\kappa)) - f(k, 0)| + \frac{\max_{\eta \in \Xi} |\xi(u^*(\eta))|}{\Lambda^2(\lambda + |u^*(\kappa)|)} \\
+ \frac{1}{\Gamma(p)} \int_0^\kappa (k - \gamma)^{p-1} \times |g(\gamma, v(\gamma), f'(v(\gamma)))| \, d\gamma
\]
\[
\leq \frac{|u^*(\kappa)|}{\Lambda^2(\lambda + |u^*(\kappa)|)} + \Delta_0 \\
+ \frac{1}{\Gamma(p)} \int_0^\kappa (k - \gamma)^{p-1} \times |h(\gamma)| \\ \\
\times \delta_0 + \frac{\lambda^p}{\Gamma(p + 1)} \|h\|_{L^1},
\]
which implies that
\[
\|u(\kappa)\| \leq 1 + \Delta_0 + \frac{\lambda^p}{\Gamma(p + 1)} \|h\|_{L^1} \leq \Xi.
\]

That is \( u \in M \). Thus condition (C3) of Corollary 3.3 holds. Therefore by Corollary 3.3, the operator \( G(u, v) = Tu + Sv \) has a coupled fixed point in \( M \). Accordingly, the system (11) has solution in \( T \).}

To illustrate the existence result Theorem 4.4, we present an example.

**Example 4.5:** Consider the fractional order hybrid differential equations:
\[
D^{1/2} \left[ a(\kappa) - \frac{e^{-t}|a(\kappa)|}{7 + |a(\kappa)|} \right] = \frac{t^3}{3} - \left( \sin |b(\kappa)| + \sin |b^{3/2}(\kappa)| \right),
\]
\[
D^{1/2} \left[ b(\kappa) - \frac{e^{-t}|b(\kappa)|}{7 + |b(\kappa)|} \right] = \left( \frac{t^3}{3} - \sin |a(\kappa)| + \sin |b^{3/2}(\kappa)| \right),
\]
\[
a(0) = \xi(\eta), \quad b(0) = \xi(\eta),
\]
where \( \kappa, \eta \in [0, \pi] \), \( f : [0, \pi] \times \mathbb{R} \to \mathbb{R} \), \( f(0, a(0)) = 0 \) and \( g : [0, \pi] \times \mathbb{R} \to \mathbb{R} \). \( D^{1/2} \) is the Caputo fractional order derivative.

Here
\[
f(k, a(\kappa)) = \frac{e^{-t}|a(\kappa)|}{7 + |a(\kappa)|},
\]
\[
g(k, a(\kappa), f'(a(\kappa))) = \left( \frac{t^3}{3} - \sin |a(\kappa)| + \sin |b^{3/2}(\kappa)| \right),
\]
\[
\xi(\eta) = \frac{a}{5}.
\]
So
\[
\Delta_0 = \max_{\kappa \in \Xi} |f(k, 0)| + \max_{v \in \Xi} |\xi(u^*(\eta))| = \pi,
\]
and take $h(\kappa) = t^3/3$, then
\[ \|h(\kappa)\| = \sup_{0 \leq t \leq \pi/3} \frac{t^3}{3} = \frac{\pi^2}{12}. \]

Thus we can write
\[ 1 + \Delta_0 + \frac{\lambda^3}{2} \|h\|_{L^1} \]
\[ = 1 + \pi + \frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{2} + 1\right)} \pi^{\frac{3}{2}} \leq 21, \quad \text{i.e., } \Delta \geq 21. \]

Now, to verify the conditions of Theorem 4.4, we have
\[ |g(\kappa, a(\kappa), t^{3/2}a(\kappa))| \]
\[ = \left| \frac{t^3}{3} - \left( \sin |a(\kappa)| + \sin |t^{3/2}a(\kappa)| \right) \right| \]
\[ \leq \left| \frac{t^3}{3} + \sin |a(\kappa)| + \sin |t^{3/2}a(\kappa)| \right| \]
\[ \leq \frac{\pi^3}{3} + |a(\kappa)| + |t^{3/2}a(\kappa)| \]
\[ \leq \frac{\pi^3}{3} + |a(\kappa)| + \left| \int_0^\kappa (\kappa - \gamma)^{3/2-1}a(\gamma) \, d\gamma \right| \]
\[ \leq \frac{\pi^3}{3} + |a(\kappa)| + \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^\kappa (\kappa - \gamma) a(\gamma) \, d\gamma \]
\[ \leq \frac{\pi^3}{3} + |a(\kappa)| + \frac{1}{\Gamma\left(\frac{3}{2}\right)} \|a\| \int_0^\kappa (\kappa - \gamma)^{1/2} \, d\gamma \]
\[ \leq \frac{\pi^3}{3} + |a(\kappa)| + \frac{2}{\sqrt{\pi}} \|a\| \left( \frac{1 + \frac{\Lambda^2}{3}}{3} \right) \]
\[ \leq l_0 \|a\| + \frac{\Lambda^2}{3}, \]
where $l_0 = 1 + \Lambda^2/3$ and $\Delta_0 = \pi^3/3$.

Next, for $\lambda = 1$ and $\Lambda = 2$, we have
\[ |f(\kappa, a(\kappa)) - f(\kappa, b(\kappa))| \]
\[ = \left| \frac{e^{-|a(\kappa)|}}{7(1 + |a(\kappa)|)} - \frac{e^{-|b(\kappa)|}}{7(1 + |b(\kappa)|)} \right| \]
\[ \leq \left| \frac{|a(\kappa) - b(\kappa)| + |b(\kappa)|}{7(1 + |a(\kappa)|) + |b(\kappa)|} \right| \]
\[ \leq \left| \frac{|a(\kappa) - b(\kappa)| + |b(\kappa)|}{7(1 + |a(\kappa)|) + |b(\kappa)|} \right| \]
\[ \leq \left| \frac{|a(\kappa) - b(\kappa)|}{7(1 + |a(\kappa)|) + |b(\kappa)|} \right| \]
\[ \leq \left| \frac{|a(\kappa) - b(\kappa)|}{7(1 + |a(\kappa)|) + |b(\kappa)|} \right| \]
\[ \leq \left| \frac{|a(\kappa) - b(\kappa)|}{7(1 + |a(\kappa)|) + |b(\kappa)|} \right| \]
\[ \leq \left| \frac{|a(\kappa) - b(\kappa)|}{7(1 + |a(\kappa)|) + |b(\kappa)|} \right| \]
\[ \leq \frac{|a(\kappa) - b(\kappa)|}{7(1 + |a(\kappa)|) + |b(\kappa)|} \]
\[ \leq \frac{|a(\kappa) - b(\kappa)|}{\lambda^2(1 + |a(\kappa)| - |b(\kappa)|)} \]
and
\[ |\zeta(a(\kappa)) - \zeta(b(\kappa))| = \frac{|a(\kappa) - b(\kappa)|}{5} \leq \frac{|a(\kappa) - b(\kappa)|}{\Lambda^2}. \]

Finally,
\[ g(\kappa, a(\kappa), t^{3/2}a(\kappa)) = \frac{t^3}{3} - \left( \sin |a(\kappa)| + \sin |t^{3/2}a(\kappa)| \right) \]
\[ \leq \frac{t^3}{3} = h(\kappa), \]
which implies that there exists $h \in C([0, \pi], \mathbb{R})$ such that
\[ g(\kappa, a(\kappa), b(\kappa)) \leq h(\kappa), a, b \in \mathbb{R}, \quad \kappa \in [0, \pi]. \]

Thus it follows that all the assumptions of Theorem 4.4 are satisfied. Therefore, we conclude that the problem (16) has a solution.

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