Poisson Structures for Aristotelian Model of
Three Body Motion

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Abstract: We present explicitly Poisson structures, for both time-dependent and time-independent Hamiltonians, of a dynamical system with three degrees of freedom introduced and studied by Calogero et al [2005]. For the time-independent case, new constant of motion includes all parameters of the system. This extends the result of Calogero et al [2009] for semi-symmetrical motion. We also discuss the case of three bodies two of which are not interacting with each other but are coupled with the interaction of third one.

1 Introduction

In references [1],[2] Calogero et al. suggested and studied the system of differential equations

\begin{align}
\dot{x} &= i\omega x + \frac{c}{x-y} + \frac{b}{x-z} \\
\dot{y} &= i\omega y + \frac{a}{y-z} + \frac{c}{y-x} \\
\dot{z} &= i\omega z + \frac{b}{z-x} + \frac{a}{z-y}
\end{align}

as a model for motions of three bodies where forces determine velocities (hence the name Aristotelian), rather than accelerations. They were considered to be a prototype for a large class of models exhibiting transitions from simple to complicated motions that can be explained as travel on Riemann surfaces (see also extensive list of references in [1] and the forthcoming article [3]). In this work, following references [4] and [5], we shall present Poisson structures of an equivalent system of differential equations obtained after elimination of the linear one-body forces in Eq. (1).

Differentiation of Eq. (1) implies that they may be interpreted as Newtonian equations for an inverse-cubic force field. Historically, cubic forces were first considered, in the Newtonian context, by Jacobi [6]. Poincaré, in his search of infinitely many periodic solutions of three body problem, replaced the inverse-square force with an inverse-cubic one [7]. The relation of inverse-cubic forces
with integrable three body problem of Newtonian type with inverse-square potential were explained in an appendix of reference [2]. See also [8]-[12] for other treatments of inverse-cubic forces.

The system in Eq. (1) was communicated to us by Yavuz Nutku who also presented its time-dependent conserved Hamiltonian functions [13] (c.f. Eqs.(4)-(9)). A transformation of variables (c.f. Eq.(6)), first appeared in reference [1], takes one of the Hamiltonian functions into time-independent form but the second one still remains time-dependent. Using techniques of reference [4], we shall present, in section (3), a formal Hamiltonian structure with this time-dependent Hamiltonian function.

We shall indicate, in section (4), a potential function by which the dynamical equations can be cast into a gradient flow. A linear change of variables that includes time-independent Hamiltonian function as one of the new coordinates reduces the number of equations into two which are still in gradient form. We shall then be able to integrate for the second time-independent conserved Hamiltonian. For the full-symmetrical and semi-symmetrical cases, characterized by \( a = b = c \) and \( a = b \neq c \), respectively, the bi-Hamiltonian form of equations for Aristotelian model of three body motion will be presented in section (5). It becomes necessary to give a separate treatment of a subcase of the semi-symmetrical motion characterized by \( a = b \neq 0 \) and \( c = 0 \). This corresponds to a model of three body motion in which two of the bodies do not interact with each other, yet their motions are coupled due to their interaction with the third body.

In section (6), we shall construct only the second time-independent Hamiltonian function containing all coupling constant, which is enough to cast the system into bi-Hamiltonian form in three-dimensions. We shall first complete the discussion for non-interacting two body case by considering \( a \neq b \neq 0 \) and \( c = 0 \). We shall then present the general form of Hamiltonian function for generic values of coupling constants. This generalizes a result of reference [2] where a time-independent conserved function was found by a different analysis for the semi-symmetrical case. Present results are contributions with a geometric taste to the ongoing investigations [3] of the authors of references [1] and [2] where they have been presenting full analysis of the dynamics such as methods of integrating Eq.(1), analytic structure of solutions, as well as the geometry of the Riemann surfaces involved.

2 Aristotelian Model of 3-Body Motion

In [1], the motion of three point particles, for which the velocities, rather than accelerations, are determined by forces, was modelled by the equations

\[
\ddot{z}_n = i \omega z_n + \frac{g_{n-2}}{z_n - z_{n-1}} + \frac{g_{n-1}}{z_n - z_{n-2}}
\]

where \( g_n \) are coupling constants, \( z_n \) are (complex) coordinates of particle positions, \( n = 1, 2, 3 \), \( \omega > 0 \), and \( i = \sqrt{-1} \). Overdots denote derivatives with
respect to (real) time variable $t$ and Eq. 2 was referred to as physical model. It was remarked that this systems admits canonical Hamiltonian formalism with the Hamiltonian function

$$H(z, p) = \sum_{n=1}^{3} \left[ -i\omega z_n p_n + g_n \frac{p_{n+1} - p_{n+2}}{z_{n+1} - z_{n+2}} \right]$$  \hspace{1cm} (3)$$

where the momenta $p_n$ look like Lagrange multipliers.

We shall consider the system of equations (2), and not its extension by the addition of momenta, with $z_1 = x$, $z_2 = y$, $z_3 = z$ and $g_1 = a$, $g_2 = b$, $g_3 = c$ resulting in Eq. (1). Nutku [13] indicated two time-dependent conserved Hamiltonian functions

$$H^{(1)}(x, y, z) = e^{-i\omega t}(x + y + z)$$  \hspace{1cm} (4)$$

$$H^{(2)}(x, y, z) = \frac{1}{4}e^{-4i\omega t}(x^2 + y^2 + z^2) - (a + b + c)t$$  \hspace{1cm} (5)$$

for Eq. (1). The choice of variables, first appeared in [1],

$$(u, v, w, \tau) = e^{-i\omega t}(x, y, z, -e^{-i\omega t}2i\omega)$$  \hspace{1cm} (6)$$

factors out the linear forces and transforms the system in Eq. (1) into the one with two-body interactions only

$$\dot{u} = \frac{c}{u-v} + \frac{b}{u-w}$$

$$\dot{v} = \frac{a}{v-w} + \frac{c}{v-u}$$

$$\dot{w} = \frac{b}{w-u} + \frac{a}{w-v}.$$  \hspace{1cm} (7)$$

Here, prime denotes derivative with respect to complexified time $\tau$ and, for this reason Eq. (7) was referred to as auxiliary model [1]. This transformation makes the first Hamiltonian function $H^{(1)}$ independent of time, but nevertheless $H^{(2)}$ is still time-dependent

$$H^{(1)}(u, v, w) = u + v + w$$  \hspace{1cm} (8)$$

$$H^{(2)}(u, v, w) = u^2 + v^2 + w^2 - 2(a + b + c)\tau.$$  \hspace{1cm} (9)$$

These relations were used in [2] to obtain the general solution of the system (7). Level sets of $H^{(1)}$ are planes and of $H^{(2)}$ are spheres expanding with velocity $2(a + b + c)/\sqrt{2(a + b + c)\tau + H^{(2)}_0}$ where $H^{(2)}_0$ is the initial radius of the sphere. $H^{(1)}$ represents the coordinates of center of mass of three bodies. We note that

$$H^{(3)}(u, v, w) = uv + uw + vw + (a + b + c)\tau$$

is also a time-dependent conserved function for Eq. (7), but it is not independent from $H^{(1)}$ and $H^{(2)}$

$$H^{(3)} = \frac{1}{2}((H^{(1)})^2 - H^{(2)}).$$
Moreover, there is another relation between the time-dependent Hamiltonians
\[(u - v)^2 + (v - w)^2 + (w - u)^2 = 2H^{(2)} - 2H^{(3)} + 4(a + b + c)\tau\]
which can be used to eliminate the time variable \(\tau\). This seemed to be useful in obtaining second time-independent Hamiltonian function in somewhat different analysis of reference [2].

Apart from the semi-symmetrical and the full-symmetrical (or integrable) configurations, the two-body case was obtained by the restriction \(a = b = 0\) and \(c \neq 0\) [1] which is not much interesting in the present framework. However, in presenting bi-Hamiltonian structures, we shall treat separately a particular case of semi-symmetrical motion in which the interaction between two of the bodies is neglected. Note also that, by a rescaling of time \(\tau\) one of the coupling constants can be removed from the right hand side of equations (7). In this work, we shall consider the variables \(u = (u, v, w)\) in a real domain. In the sequel \(U = U \cdot \nabla = \dot{u} \cdot \nabla\) will denote the vector field associated with Eq. (7).

3 Time-dependent Poisson structures

We first present the Hamiltonian structure of equations (7) with the time-dependent Hamiltonian function \(H^{(2)}\). This can be achieved by considering Poisson structures in space-time variables \((\tau, u)\). In other words, we add the equation \(\dot{\tau} = 1\) to the system in Eq. (7) (if necessary, after performing a linear change of parametrization by time). The flow of suspended system in four dimensions is then generated by the vector field \(\partial_t + U\) and our task is to find a Hamiltonian representation of this. We shall use contravariant description of Hamiltonian formalism by Poisson structures.

A Poisson structure on a manifold \(N\) is defined by a skew symmetric contravariant bilinear form subjected to the Jacobi identity expressed as the vanishing of Schouten bracket of Poisson tensor with itself [14]-[17]. Following [4], for a Hamiltonian formalism on a time-extended space \(N = I \times M, I \subset \mathbb{R}\) or \(\mathbb{C}\) and \(M \subset \mathbb{R}^3\) for the present context, we take the bi-vector field
\[
\Lambda(\tau, u) = V(\tau, u) \wedge \partial_\tau + \Pi(\tau, u) \tag{10}
\]
where \(V\) and \(\Pi\) are time-dependent vector and bi-vector fields on \(M\), respectively. The Jacobi identity for \(\Lambda\) can be computed in a coordinate independent way using identities of Schouten algebra of multi-vectors.

**Proposition 1** \(\Lambda\) is a Poisson bi-vector field on \(I \times M\) if and only if
\[
[\Pi, \Pi] = 2V \wedge \frac{\partial \Pi}{\partial \tau}, \quad [\Pi, V] = V \wedge \frac{\partial V}{\partial \tau}. \tag{11}
\]

The Hamiltonian form of the suspended vector field \(\partial_t + U\) on \(I \times M\) will then be
\[
\partial_t + U = \Lambda(dH) = V(H)\partial_t + \Pi(dH) - H_iV \tag{12}
\]
where \( H \) is a time-dependent conserved function of \( U \). Due to non-linearity of Jacobi identity, finding bi-vector \( \Lambda \) from Eqs. (11) and (12) for given Hamiltonian function \( H \), is a difficult task. The next result, which is the linearization of Jacobi identity by Hamiltonian vector fields, may help.

**Proposition 2** For the Hamiltonian vector field in Eq. (12), the Poisson bi-vector (10) satisfies the infinitesimal invariance conditions\

\[
\frac{\partial V}{\partial \tau} + [U, V] = 0, \quad \frac{\partial \Pi}{\partial \tau} + [U, \Pi] = V \wedge \frac{\partial U}{\partial \tau}
\]  

which are equivalent to the Jacobi identity in Eq. (11). Moreover, \( V \) is an infinitesimal automorphism of \( \Lambda \).

This follows from the Jacobi identity (of Schouten algebra) for multi-vectors \((\Lambda, \Lambda, h)\). Conversely, one obtains Eq. (11) by inserting Hamilton’s equations of Eqs. (12) and (13) into Eq. (13). The last conclusion follows from second of Eq. (11).

Thus, the construction of Hamiltonian structure of a system admitting time-dependent Hamiltonian function amounts to solving the linear system consisting of Eqs. (12), (13) and the conservation law for \( H \). As it can be inferred from Eq. (12) the function \( H \) is, in addition, coupled to the vector field \( V \) by the condition \( V(H) = 1 \).

The construction of Hamiltonian structure for the Aristotelian model of dynamical equations relies on the observation that the first of Eq. (13) gives a characterization of the vector field \( V \) as a time-dependent infinitesimal symmetry of \( U \). Denoting

\[
E = u \partial_u + v \partial_v + w \partial_w
\]

the Euler vector field and, noting that \( U \) is homogeneous of degree \(-1\), it is easy to see that the vector field

\[
E - 2\tau U = (u - \frac{2\tau c}{u - v}) \partial_u + (v - \frac{2\tau a}{v - w}) \partial_v + (w - \frac{2\tau b}{w - u}) \partial_w
\]

is a time-dependent infinitesimal symmetry of \( U \), that is,

\[
[\partial_\tau + U, E - 2\tau U] = 0.
\]

This condition is essential to show that the bi-vector field

\[
\Lambda = (E - 2\tau U) \wedge \partial_\tau + E \wedge U
\]

is Poisson \([\Lambda, \Lambda] = 0\). We can then cast the system (7) with time-dependent conserved function in Eq. (9) into an autonomous Hamiltonian system in four-dimensions.

**Proposition 3** The vector field \((1, U)\) is Hamiltonian for the Poisson bi-vector in Eq. (14) and with the Hamiltonian function defined by the time-dependent function in Eq. (9)

\[
\partial_\tau + U = \Lambda(dH), \quad H = \frac{1}{2} \ln H^{(2)}.
\]
This rather formal Hamiltonian structure may be useful in investigation of geometric structure of solution space as well as symmetries and invariants of the flow thereon [4]. For example, it follows that the time-dependent infinitesimal symmetry \( E - 2\tau U \) of \( U \) is a Hamiltonian vector field for \( \Lambda \) with the Hamiltonian function \( \tau \). In other words, the solution space of Eq.(7) may be realized as level sets of Hamiltonian function of an infinitesimal symmetry of motion it describes. The conserved function \( H^{(1)} \) gives the Hamiltonian vector field \( H^{(1)}(\partial_\tau + U) \) whereas \( H^{(3)} \) results in \(-2H^{(3)}U\).

4 Potential Function

The force field of Aristotelian model can be derivable from a logarithmic potential field.

**Proposition 4** The dynamical vector field \( U \) in Eq.(7) generating the Aristotelian motion of three bodies is a gradient field \( U = \nabla F \), with the potential function

\[
F = \ln(v - w)^a (u - w)^b (u - v)^c.
\]

Note that the potential function \( F \) is a solution of the partial differential equation

\[
\partial_u F + \partial_v F + \partial_w F = 0
\]

which is another manifestation of the conservation law for \( H^{(1)} \).

We shall consider a linear transformation of Cartesian coordinates \((u, v, w)\) by which the conserved Hamiltonian function \( H^{(1)} \) is eliminated and the potential function \( F \) becomes a function of two variables. The reduced system in two variables, which is still a gradient flow, admits a non-canonical symplectic structure similar to the one for the Lotka-Volterra system in [18].

**Proposition 5** In the orthonormal coordinates

\[
\zeta = \frac{1}{\sqrt{3}} (u + v + w), \quad \eta = \frac{1}{\sqrt{2}} (u - v), \quad \xi = \frac{1}{\sqrt{6}} (u + v - 2w)
\]

the potential function \( F \) becomes a function of \((\eta, \xi)\) only

\[
F(\eta, \xi) = \ln \eta^c (\sqrt{3}\xi + \eta)^b (\sqrt{3}\xi - \eta)^a.
\]

In the new coordinates the right handed orthonormal basis vectors are

\[
e_1 = \nabla \zeta, \quad e_2 = \nabla \eta, \quad e_3 = \nabla \xi.
\]

From the inversion of Eq.(15) we observe that the combinations

\[
u - v = \sqrt{2}\eta, \quad v - w = \frac{1}{\sqrt{2}} (\sqrt{3}\xi - \eta), \quad w - u = -\frac{1}{\sqrt{2}} (\sqrt{3}\xi + \eta)
\]

are independent of \( \zeta = H^{(1)}/\sqrt{3} \).
Proposition 6 On level sets of Hamiltonian function $H^{(1)}$ with coordinates $(\eta, \xi)$, the dynamical system in Eq.(7) becomes

\begin{align*}
\dot{\eta} &= \frac{c}{\eta} + \frac{b}{\sqrt{3}\xi + \eta} - \frac{a}{\sqrt{3}\xi - \eta}, \\
\dot{\xi} &= \sqrt{3}\left(\frac{b}{\sqrt{3}\xi + \eta} + \frac{a}{\sqrt{3}\xi - \eta}\right)
\end{align*}

(17)

which is also a gradient system with the potential in Eq.(16).

These are the reduced equations of motion after elimination of the motion of center of mass. General theory of reduction implies that the reduced space must be symplectic \[17\]. In fact, as any orientable two dimensional space is symplectic, we can take an area form $\phi(\eta, \xi) d\xi \wedge d\eta$ as the symplectic two-form for Eq.(17). In the context of local structure of Poisson manifolds, as described in reference \[15\], this reduction gives the symplectic foliation of the three dimensional space of variables $(u, v, w)$. Liouville theorem for the adapted symplectic form and Eq.(17) implies the invariance of the volume elements in the sense that the volume density $\phi(\eta, \xi)$ satisfies

\begin{align*}
\dot{\eta} \frac{\partial \phi}{\partial \eta} + \dot{\xi} \frac{\partial \phi}{\partial \xi} + \phi \nabla^2 F(\eta, \xi) &= 0
\end{align*}

where $\nabla$ denotes $(\partial_{\eta}, \partial_{\xi})$. The characteristics of this equation is the second time-independent conserved Hamiltonian we are looking for. It is also the Hamiltonian function for Eq.(17) with respect to the symplectic structure introduced above.

5 Bi-Hamiltonian structures in $\mathbb{R}^3$

The construction of bi-Hamiltonian structure in three dimensions requires two (time-independent) Hamiltonian functions $H_1, H_2$ and a conformal factor $\phi$ (this will be seen to be the same function involving symplectic structure mentioned above) such that Hamilton’s equations take the form

$$\dot{u} = \phi \nabla H_1 \times \nabla H_2.$$ 

The Poisson tensors can be identified with vectors $\phi \nabla H_1$ and $-\phi \nabla H_2$ using the isomorphism between three vectors in $\mathbb{R}^3$ and $3 \times 3$ skew-symmetric matrices. The corresponding Hamiltonian functions are $H_2$ and $H_1$, respectively. Poisson tensors constructed this way can always be made into a compatible pair. A priori unspecified function $\phi$ is related to an invariance property of the Jacobi identity in three dimensions. Namely, any multiple of a Poisson tensor with an arbitrary function is also a Poisson tensor. Thus, to cast a system into bi-Hamiltonian form, we need to find fundamental conserved quantities and then determine the multiplicative function $\phi$. See references \[5\],\[19\]-\[27\] for more details and various explicit examples.

Since we have the time-independent conserved Hamiltonian $H^{(1)}$ given by Eq.(8), our first task is to find a second one. We want to search for the second
time-independent conserved quantity for the system in Eq.(7) or, equivalently, in Eq.(17) using the reduced gradient flow of the latter. Note that, if there exist a time-independent conserved Hamiltonian function \( H(\eta, \xi) \), then we can write

\[
\nabla F = \phi \nabla \zeta \times \nabla H = e_1 \times \phi \nabla H
\]

for some function \( \phi \). It then follows that the non-zero part of Eq.(17) admit (noncanonical) symplectic formulation in the \((\eta, \xi)\)-variables

\[
\partial_{\eta} F = -\phi \partial_{\xi} H, \quad \partial_{\xi} F = \phi \partial_{\eta} H
\]

with the symplectic two-form \( \phi(\eta, \xi)d\xi \wedge d\eta \) which is always closed and non-degenerate in two dimensions. Thus, the linear change of coordinates in Eq.(15) enables us to realize the local structure of Poisson manifold that we are going to construct. More precisely, the symplectic foliation of the space of variables \((\zeta, \eta, \xi)\) consists of coordinate planes \( \zeta = \text{constant} \), or equivalently, the level sets of the conserved Hamiltonian function \( H^{(1)} \).

The Hamiltonian function \( H(\eta, \xi) \) will be a function of characteristic curves defined by the Hamiltonian system in Eq.(18) and it may depend arbitrarily on the variable \( \zeta \). To find the characteristic curves of Eq.(18), or what we shall call the fundamental conserved quantity, we eliminate the time derivatives in Eq.(17) and obtain the ordinary differential equation

\[
\frac{d\xi}{d\eta} = \frac{-\sqrt{3}(a-b)\eta^2 + 3(a+b)\eta\xi}{(a+b+c)\eta^2 + \sqrt{3}(a-b)\eta\xi - 3c\xi^2}
\]

which is homogeneous of degree two. So, for \( \theta = \xi/\eta \), its solution defines the characteristics

\[
\text{constant} = \ln \eta + \int \frac{(a + b + c) + \sqrt{3}(a-b)\theta - 3c\theta^2}{\sqrt{3}(a-b) + (4a+4b+c)\theta + \sqrt{3}(a-b)\theta^2 - 3c\theta^3} d\theta
\]

or, equivalently, the fundamental conserved quantity for the flow. That means, the time-independent Hamiltonian function we are seeking is a function of \( \zeta \) and the function defined by Eq.(20). Starting from the simplest, we shall first present explicitly bi-Hamiltonian structures of symmetrical cases and then proceed, in the next section, to analyse the generic solution for the fundamental conserved quantity in Eq.(20). We shall see that, in semi-symmetrical motion, the subcase \( c = 0 \) requires a separate treatment and we shall do this in the last subsection.

5.1 Full-symmetrical case

We set \( a = b = c \) for which Eq.(19) becomes independent of coupling constants

\[
\frac{d\xi}{d\eta} = -\frac{2\eta\xi}{\eta^2 - \xi^2}
\]

and the solution gives

\[
h_f(\xi, \eta) = \xi^3 - 3\xi\eta^2
\]
which can easily be verified to be conserved under the flow of Eq. \((17)\). Simple manipulations show that

\[
\phi_f(\xi, \eta) = \frac{1}{\sqrt{3\eta(3\xi^2 - \eta^2)}}
\]

is the conformal factor entering the definition of symplectic structure. As a function of variables \((u, v, w)\) in Eq. \((7)\), we find

**Proposition 7** For the Aristotelian model with equal coupling constants, the dynamical equations

\[
\begin{align*}
\dot{u} &= \frac{1}{u-v} + \frac{1}{u-w} \\
\dot{v} &= \frac{1}{v-w} + \frac{1}{v-u} \\
\dot{w} &= \frac{1}{w-u} + \frac{1}{w-v}
\end{align*}
\]

admit bi-Hamiltonian structure

\[
\dot{u} = P_{f1}(u)\nabla H_f(u) = P_{f2}(u)\nabla H^{(1)}(u)
\]

with the following pairs of Poisson matrices and Hamiltonian functions

\[
P_{f1}(u) = \frac{1}{\sqrt{6(u-v)(v-w)(w-u)}} \begin{pmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{pmatrix},
\]

\[
H_f(u) = \frac{1}{6\sqrt{6}}(u+v-2w)[(u+v-2w)^2 - 9(u-v)^2],
\]

\[
P_{f2}(u) = \frac{1}{6}\left[ \frac{(u-v)}{(v-w)(w-u)} + \frac{2}{u-v} \right] \begin{pmatrix}
0 & -2 & -1 \\
2 & 0 & 1 \\
1 & -1 & 0
\end{pmatrix} + \frac{1}{2}\left[ \frac{1}{v-w} - \frac{1}{w-u} \right] \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
-1 & -1 & 0
\end{pmatrix},
\]

\[
H^{(1)}(u) = u + v + w.
\]

The conformal factor for these Poisson structures is the function \(\phi_f(u)\) given as the coefficient of the constant matrix of \(P_{f1}(u)\). This function constitutes an invariant volume density for the solution space of variables \(u\) in the sense that the three-form \(\phi_f(u)du \wedge dv \wedge dw\) has vanishing Lie derivative with respect to the vector field defined by the right hand side of Eq. \((23)\).

### 5.2 Semi-symmetrical case

Setting \(a = b \neq c\), we have

\[
\frac{d\xi}{d\eta} = -\frac{6a\eta\xi}{(2a+c)\eta^2 - 3c\xi^2}
\]

\[(24)\]
and its integration gives

\[ \eta \left(\frac{\xi}{\eta}\right)^{\mu} \left(\frac{\xi}{\eta}\right)^2 \frac{1}{4\mu - 1} (1-\mu/2) = \text{constant} \quad (25) \]

where, following reference [2], we introduce the constant \( \mu = (2a + c)/(8a + c) \).

Various powers of the function in Eq. (25) can be adapted as the fundamental conserved quantity. In any case, we have to exclude the values \( \mu = 1/4 \) (or \( c = 0 \)) and \( \mu = 1/2 \) (or \( a = 0 \)) in the following discussions. For \( a = 0 \), the evolution in the variable \( w \) disappears (c.f. Eq. (26)) and we are left with a two dimensional system. For \( c = 0 \), we have the special case of three body motion in which two of the bodies do not interact with each other. This necessarily requires separate treatment which we will take up in the next subsection. We refer to extensive discussion in reference [2] where it was shown that values of the constant \( \mu \), in particular, the real rational values, play important role in determining dynamical evolution of the model.

For the purposes of having a generalization of the full-symmetrical case and the same invariant volume density, we choose the fundamental conserved quantity for semi-symmetrical case to be

\[ h_s(\xi, \eta) = \xi^{2\mu/(1-\mu)} \left(\frac{\xi}{\eta}\right)^2 \frac{1}{4\mu - 1} (1-\mu/2), \quad \mu \neq 1, 1/4 \]

so that, when \( a = c \), or equivalently, \( \mu = 1/3 \) we obtain the function \( h_f(\xi, \eta) \) in Eq. (22).

**Proposition 8** For the Aristotelian model with two coupling constants \( a \neq 0 \) and \( c \neq 0 \), the dynamical equations

\[ \begin{align*}
\dot{u} &= \frac{c}{u-v} + \frac{a}{u-w} \\
\dot{v} &= \frac{a}{v-w} + \frac{c}{v-u} \\
\dot{w} &= \frac{a}{w-u} + \frac{a}{w-v}
\end{align*} \quad (26) \]

admit Hamiltonian structure with the Poisson tensor

\[ P_{11}(u) = -\frac{3c}{2} \frac{(u+v-2w)^{(1-3\mu)/(1-\mu)}}{(u-v)(v-w)(w-u)} \begin{pmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{pmatrix}, \]

and with the Hamiltonian function

\[ H_s(u) = (u + v - 2w)^{2\mu/(1-\mu)} [(u + v - 2w)^2 - \frac{3}{4\mu - 1} (u - v)^2]. \]

The second Hamiltonian structure is defined by the Hamiltonian function \( H^{(1)} = \)
\[ u + v + w \text{ and the Poisson tensor} \]

\[
P_{s2}(u) = -\frac{c}{12} \frac{1}{u-v} \left( \frac{w-w}{w-u} + \frac{w-u}{v-w} - 2 \right) \left( \begin{array}{ccc} 0 & 2 & 1 \\ -2 & 0 & -1 \\ -1 & 1 & 0 \end{array} \right) \\
+ \frac{c}{12} \frac{3 \mu}{4 \mu - 1} \frac{u-v}{(v-w)(w-u)} \left( \begin{array}{ccc} 0 & 2 & 1 \\ -2 & 0 & -1 \\ -1 & 1 & 0 \end{array} \right) \\
+ \frac{c}{4 \mu - 1} \left[ \frac{1}{w-u} - \frac{1}{v-w} \right] \left( \begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{array} \right).
\]

Note that, in the limit \( \mu = 1/3 \) (or \( a = c \)) both \( \phi_s(u) \) (defined as the multiplicative factor in \( P_1(u) \)) and \( H_s(u) \) reduce to some constant multiples of \( \phi_f(u) \) and \( H_f(u) \), respectively.

5.3 Non-interacting two body case

We consider \( a = b \neq 0 \) and \( c = 0 \), that is, \( \mu = 1/4 \). This case corresponds to a situation where two of the three bodies do not interact with each other but with the third one only. Or, to a situation where the distance \( |u-v| \) is so large that the interaction between bodies at \( u \) and \( v \) can be neglected. Setting the only constant \( a = 1 \), we have

**Proposition 9** Aristotelian equations of motion

\[
\dot{u} = \frac{1}{u-w}, \quad \dot{v} = \frac{1}{v-w}, \quad \dot{w} = \frac{1}{w-u} + \frac{1}{w-v}
\]
for two bodies at positions \( u \) and \( v \) interacting with a third one at \( w \) are bi-Hamiltonian with

\[
P_{n1}(u) = \frac{2}{(u-v)^2(v-w)(w-u)} \left( \begin{array}{ccc} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{array} \right),
\]

\[
H_n(u) = \frac{1}{4} (u + v - 2w)(u-v)^3
\]

\[
P_{n2}(u) = \frac{1}{6 (v-w)(w-u)} \left( \begin{array}{ccc} 0 & 2 & 1 \\ -2 & 0 & -1 \\ -1 & 1 & 0 \end{array} \right) \\
+ \frac{1}{2} \left[ \frac{1}{w-u} - \frac{1}{v-w} \right] \left( \begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{array} \right),
\]

\[
H^{(1)}(u) = u + v + w.
\]

There remains the case of non-interacting two bodies each of which interact with the third one with different coupling constants \( a, b, a \neq b \). We will give the form of Hamiltonian function of this general non-interacting two body case in the next section.

11
6 General Form of Hamiltonian Function

We have seen that casting the dynamical system modelling Aristotelian motion of three bodies into bi-Hamiltonian form requires first the integration of Eq. (20) to find the time-independent Hamiltonian function and then, the invariant volume density. In this section, we shall discuss the general form of characteristics and their domain of validity. Our discussion will, by no means, be exhaustive.

We first give the form of Hamiltonian function for motions of three bodies two of which are not interacting with each other. The remaining third body interacts with them with different coupling constants \(a\) and \(b\). Integration in Eq. (20) gives

**Proposition 10** Let two non-interacting bodies at positions \(u\) and \(v\) interact with the third one at \(w\). Then, the Aristotelian equations of motion

\[
\dot{u} = \frac{b}{u - w}, \quad \dot{v} = \frac{a}{v - w}, \quad \dot{w} = \frac{b}{w - u} + \frac{a}{w - v}
\]

admit the conserved Hamiltonians \(H^{(1)}\) and

\[
H(u) = 2\sqrt{4k^2 - 1}\ln\frac{u - v}{\sqrt{2}} + \frac{(\sqrt{4k^2 - 1} - k)}{\sqrt{3(u - v)}} - \frac{\sqrt{4k^2 - 1} + 2k}{\sqrt{3(u - v)}} \quad \text{(27)}
\]

where we introduce the constant \(k = (a + b)/(\sqrt{3(a - b)})\).

Returning to the general case, we first cast Eq. (20) into the form

\[
h(u) = \ln \eta + \int \frac{(\theta - \theta_+)(\theta - \theta_-)}{(\theta - \theta_1)(\theta - \theta_2)(\theta - \theta_3)} d\theta \quad \text{(28)}
\]

and, after some manipulations, obtain

**Proposition 11** Let \(\theta_i\), \(i = 1, 2, 3\) be the roots of quadratic term in numerator and, let \(\theta_i\), \(i = 1, 2, 3\) be the roots of cubic polynomial in denominator of the integral in Eq. (20). Then, the second fundamental (time-independent) conserved quantity for Aristotelian model of three body motion is

\[
h(u) = (\theta_1 - \theta_2)(\theta_2 - \theta_3)(\theta_3 - \theta_1)\ln\frac{u - v}{\sqrt{2}} + (\theta_1 - \theta_+)(\theta_2 - \theta_3)(\theta_1 - \theta_-)\ln\frac{u + v - 2w}{\sqrt{3(u - v)}} - \frac{\theta_1}{\theta_1} + (\theta_2 - \theta_+)(\theta_2 - \theta_-)(\theta_3 - \theta_1)\ln\frac{u + v - 2w}{\sqrt{3(u - v)}} - \frac{\theta_2}{\theta_2} + (\theta_1 - \theta_2)(\theta_3 - \theta_+)(\theta_4 - \theta_-)\ln\frac{u + v - 2w}{\sqrt{3(u - v)}} - \frac{\theta_3}{\theta_3}. \quad \text{(29)}
\]
To this end, we want to discuss examples of restrictions on the domain of definition of the fundamental Hamiltonian function $h$. The integral in Eq.(23) is singular for a real root $\theta_{\text{real}}$ of the cubic polynomial in denominator. That is, for the case $\theta = \xi/\eta = \theta_{\text{real}}$. In the real variables $(u, v, w)$ this implies

$$u(1 - \sqrt{3}\theta_{\text{real}}) + v(1 + \sqrt{3}\theta_{\text{real}}) - 2w = 0.$$ 

The cases described by the conditions $u = v$ (or $\eta = 0$), $v = w$ and $w = u$ are particularly included in this equation. Moreover, for non-zero values of $\theta_{\text{real}}$, by adding and subtracting $u$ and $v$ to the above equation we can obtain the singular cases

$$v - w = 0, \quad 1 - \sqrt{3}\theta_{\text{real}} = 0$$
$$w - u = 0, \quad 1 + \sqrt{3}\theta_{\text{real}} = 0$$

which let the variables $u$ and $v$, respectively, be free but bring restrictions on the coupling constants $a, b, c$. Finally, the present form of roots of cubic equation imposes some conditions on the coupling constants related to the discriminant of cubic polynomial.

**Proposition 12** For the values of the vector field $u$ not perpendicular to the constant vector

$$(1 - \sqrt{3}\theta_{\text{real}}, 1 + \sqrt{3}\theta_{\text{real}}, -2),$$

the dynamical system in Eq.(7) admits bi-Hamiltonian structure with Hamiltonian functions in Eqs.(8) and (29).

In the particular value $\theta_{\text{real}} = 0$ of the real root, that may occur for example in semi- and full-symmetrical configurations, the condition of above proposition simply prevents the adapted coordinate $\xi$ to become zero. For generic values of $\theta_{\text{real}}$ it just requires us to be away from the line $\xi = \theta_{\text{real}}\eta$ on level sets of $H^{(1)}$.

The discussion on domain of validity of bi-Hamiltonian structure will become more conclusive if we adapt the parameters

$$\vartheta = \theta - \frac{p}{3}, \quad p = \frac{a - b}{\sqrt{3}c}, \quad q = \frac{a + b}{3c}$$

for the integration of Eq.(20). In these parameters, the semi-symmetrical case is characterized by

$$p = 0, \quad q = \frac{2a}{3c} = \frac{1}{3} \frac{1 - \mu}{4\mu - 1}$$

and for the full-symmetrical case we have $p = 0, q = 2/3$. Note that, the non-interactive two body case ($c = 0$) is necessarily excluded from the present discussion. The integral for the fundamental conserved quantity takes the form

$$h(\xi, \eta) = \ln \eta + \int \frac{-2p^2 + 9q + 3)/9}{-2p(p^2 + 18q + 15)/27 - ((p^2 + 12q + 1)/3)\vartheta^2 + \vartheta^3} d\vartheta \quad (30)$$
which can be put into the form of Eq. (28) with the roots
\[ \vartheta_1 = -\frac{1}{3\lambda^{1/3}}(1 + p^2 + 12q + \lambda^{2/3}) \]
\[ \vartheta_{\pm} = \frac{1}{6\lambda^{1/3}}(\varsigma_{\pm}(1 + p^2 + 12q) + \varsigma_{\mp}\lambda^{2/3}) \]
of the cubic denominator. Here, \( \varsigma_{\pm} = 1 \pm i\sqrt{3} \) and
\[ \lambda = -p(p^2 + 18q + 15) + \sqrt{27p^4 + 6p^2(6(13 - 3q)q + 37)} - \left(1 + 12q\right)^3. \]

In the semi-symmetrical case, we have \( \lambda = (-1 - 12q)^{3/2}, q = 2a/3c \) and so \( \vartheta_1 = 0, \vartheta_{\pm} = \sqrt{1 + 12q}/\sqrt{3} \). In the full-symmetrical case, \( q = 2/3, \lambda = 27i \) and \( \vartheta_1 = 0, \vartheta_{\pm} = \sqrt{3} \).

As a final example of this analysis, we want to present the connection between values of the discriminant and the constants appearing in the integral of Eq. (20). Recall the discriminant
\[ \triangle = 4(-27p^4 + 6p^2(-37 + 6q(-13 + 3q)) + (1 + 12q^3)/27 \]
of cubic polynomial in Eq. (30) which is the negative of the square-rooted term in \( \lambda \). Regarding the right hand side as a quadratic polynomial in \( p^2 \), we find its discriminant to be
\[ -16777216(1 + 3q)^2(7 + 3q)^6(1 + 12q^3)/177147. \]
\( \triangle \) will be a perfect square at the values of \( q = -1/3, -3/7, -1/12 \) and these correspond to the following sets of values of the coupling constants
\[ a + b + c = 0, \quad 7(a + b) + 9c = 0, \quad 4(a + b) + c = 0 \]
two of which are seen in Eq. (20). First two restrictions imply, for the semi-symmetrical motion, \( \mu = 0 \) and \( \mu = 1/29 \), respectively, the last one, however, is not possible.

We expect the results of the present work to be useful in an exhaustive analysis of various cases of Aristotelian model of three body motion as well as in understanding the physics behind this model.

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