Intuitionistic ($\lambda,\mu$)-fuzzy subalgebras in CI-algebras

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Abstract. The aim of this paper is to introduce the notion of intuitionistic ($\lambda,\mu$)-fuzzy subalgebras in CI-algebra and to investigate some of their properties. Characterizations of an intuitionistic ($\lambda,\mu$)-fuzzy subalgebras are provided. It is shown that the intersection and direct product of two intuitionistic ($\lambda,\mu$)-fuzzy subalgebras of CI-algebra are also intuitionistic ($\lambda,\mu$)-fuzzy subalgebra.

1. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [1], several researches were conducted on the generalizations of the notion of fuzzy sets. The idea of intuitionistic fuzzy sets was first proposed by Atanassov [2,3], as a generalization of the notion of fuzzy sets. Fuzzy sets and intuitionistic fuzzy sets are widely used in various algebraic systems. Only with the membership degrees ranged on the interval [0,1], it is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy sets. Based on these observations, Lee [4] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. He gave two kinds of representations of the notion of bipolar-valued fuzzy sets.

In 1966, Imai and Iseki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. A generalization of a BCK-algebra, Kim and Kim [5] introduced the notion of a BE-algebra, and investigated several properties. In [6,7], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras. In [8], Seok, Young and Kyoung introduced the notion of fuzzy ideals in BE-algebras, and investigate related properties. In [9], B.L.Meng defined the notion of CI-algebra as a generalization of a BE-algebra. In [10], Kim studied this algebra in detail and some fundamental properties of CI-algebras were discussed and studied in many papers [11,12].

In this paper, we introduce the concept of intuitionistic ($\lambda,\mu$)-fuzzy subalgebra in CI-algebras, and investigate related properties. It is shown that the intersection and direct product of two intuitionistic ($\lambda,\mu$)-fuzzy subalgebras of CI-algebra are also intuitionistic ($\lambda,\mu$)-fuzzy subalgebra.

2. Preliminaries

In this section, we review some basic knowledge of CI-algebra (see, [9,10,11]).

An algebra $(X,\ast,1)$ of type $(2,0)$ is called a CI-algebra if it satisfies the following conditions: (1) $x\ast x = 1$, (2) $1 \ast x = x$, (3) $x \ast (y \ast z) = y \ast (x \ast z)$ for all $x,y,z \in X$.

A relation $\leq$ on a CI-algebra $X$ by $x \leq y$ if and only if $x \ast y = 1$. Let $(X,\ast,1)$ be a CI-algebra. Then for all $x,y \in X$, $(x \ast 1) \ast (y \ast 1) = (x \ast y) \ast 1$, $y \ast ((y \ast x) \ast x) = 1$. A non-empty subset $A$ of a CI-algebra $X$ is said to be a subalgebra of $X$ if for any $x,y \in A$, $x \ast y \in A$. 

A fuzzy set in set $X$ is a function $\mu : X \rightarrow [0,1]$ and the complement of $\mu$, denoted by $\overline{\mu}$, is the fuzzy set given by $\overline{\mu}(x) = 1 - \mu(x)$. For $t \in [0,1]$, the set $U(\mu; t) = \{x \in X | \mu(x) \geq t\}$ is called an upper level cut set and $L(\mu; t) = \{x \in X | \mu(x) \leq t\}$ is called a lower level cut set.

**Definition 2.1** A fuzzy set $\mu$ is called a fuzzy subalgebra of CI-algebra $X$ if it satisfies: for all $x, y \in X$, $\mu(x * y) \geq \mu(x) \land \mu(y)$.

**Definition 2.2** A fuzzy set $\mu$ is called a $(\lambda, \mu)$-fuzzy subalgebra of CI-algebra $X$ if it satisfies: for all $x, y \in X$ and $0 \leq \lambda < \mu \leq 1$, $\mu(x * y) \lor \lambda \geq \mu(x) \lor \mu(y) \lor \mu$.

An Intuitionistic fuzzy set (briefly, IFS) $A$ in a nonempty set $X$ is an object having the form $A = \{(x, \alpha_A(x), \beta_A(x)) | x \in X\}$, and $0 \leq \alpha_A(x), \beta_A(x) \leq 1$, $0 \leq \alpha_A(x) + \beta_A(x) \leq 1$ for all $x \in X$.

**Definition 2.3** An IFS $A = (\alpha_A, \beta_A)$ of $X$ is called the intuitionistic fuzzy subalgebras of $X$ if it satisfies: for all $x, y, z \in X$,

(1) $\alpha_A(x * y) \geq \alpha_A(x) \land \alpha_A(y)$,  
(2) $\beta_A(x * y) \leq \beta_A(x) \lor \beta_A(y)$.  

3. Intuitionistic $(\lambda, \mu)$-fuzzy subalgebras of CI-algebras

In what follows, let $X$ denote a CI-algebra and $0 \leq \lambda < \mu \leq 1$ unless otherwise specified.

**Definition 3.1** An IFS $A = (\alpha_A, \beta_A)$ of $X$ is called the intuitionistic $(\lambda, \mu)$-fuzzy subalgebras of $X$ if it satisfies: for all $x, y \in X$,

(1) $\alpha_A(x * y) \lor \lambda \geq \alpha_A(x) \lor \alpha_A(y) \lor \mu$,  
(2) $\beta_A(x * y) \land \mu \leq \beta_A(x) \land \beta_A(y) \lor \lambda$.

**Example 3.1** Let $X = \{1, a, b, c\}$ in which $*$ is defined by

| *    | 1 | a | b | c |
|------|---|---|---|---|
| 1    | 1 | a | b | c |
| a    | 1 | 1 | b | b |
| b    | 1 | a | 1 | a |
| c    | 1 | 1 | 1 | 1 |

Then $X$ is a CI-algebra. Let $\alpha_A(1) = \alpha_A(b) = \alpha_A(c) = 1$, $\alpha_A(a) = 0$, $\beta_A(1) = \beta_A(b) = \beta_A(c) = 0$, $\beta_A(a) = 1$, then $A = (\alpha_A, \beta_A)$ is an intuitionistic $(\lambda, \mu)$-fuzzy subalgebras of $X$.

If $\alpha_B(1) = \alpha_B(b) = 0.4$, $\alpha_B(a) = \alpha_B(c) = 0.6$, and $\beta_B(1) = \beta_B(b) = 0.6$, $\beta_B(a) = \beta_B(c) = 0.4$, then $B = (\alpha_B, \beta_B)$ is not an intuitionistic $(\lambda, \mu)$-fuzzy subalgebras of $X$.

**Theorem 3.1** Any intuitionistic fuzzy subalgebras is an intuitionistic $(\lambda, \mu)$-fuzzy subalgebras of $X$.

**Theorem 3.2** An IFS $A = (\alpha_A, \beta_A)$ is an intuitionistic $(\lambda, \mu)$-fuzzy subalgebras of $X$ if and only if the fuzzy sets $\alpha_A$ and $\overline{\beta_A}$ are $(\lambda, \mu)$-fuzzy subalgebra of $X$.

**Proof** Let IFS $A = (\alpha_A, \beta_A)$ is an intuitionistic $(\lambda, \mu)$-fuzzy subalgebras of $X$, clearly $\alpha_A$ is a $(\lambda, \mu)$-fuzzy subalgebra of $X$. For all $x, y \in X$ and $0 \leq \lambda < \mu \leq 1$,

$\overline{\beta_A}(x * y) \lor \lambda = (1 - \beta_A(x * y)) \lor \lambda = 1 - \beta_A(x * y) \lor (1 - \lambda) \leq 1 - \beta_A(x) \lor (1 - \lambda) \lor (1 - \mu)$

$= (1 - \beta_A(x) \lor \beta_A(y)) \lor \mu = (1 - \beta_A(x) \lor (1 - \beta_A(y))) \lor \mu = \overline{\beta_A}(x) \lor \overline{\beta_A}(y) \lor \mu.$

By definition 2.2, it holds that $\overline{\beta_A}$ is a $(\lambda, \mu)$-fuzzy subalgebra of $X$.

Conversely, assume that $\alpha_A$ and $\overline{\beta_A}$ are $(\lambda, \mu)$-fuzzy subalgebras of $X$, then the (1) of Definition 3.1 is true. For all $x, y \in X$, we have
\[
\beta_A(x*y) \land \mu = (1 - \overline{\beta}_A(x*y)) \land \mu = 1 - \overline{\beta}_A(x*y) \lor (1 - \mu) \leq 1 - \overline{\beta}_A(x) \lor \overline{\beta}_A(y) \lor (1 - \lambda)
\]

\[
= (1 - \overline{\beta}_A(x)) \lor (1 - \overline{\beta}_A(y)) \lor \lambda = \beta_A(x) \land \beta_A(y) \land \lambda.
\]

It follows from Definition 3.1 that \(A = (\alpha_A, \beta_A)\) is an intuitionistic \((\lambda, \mu)\)-fuzzy subalgebras of \(X\).

**Theorem 3.3** An IFS \(A = (\alpha_A, \beta_A)\) is an intuitionistic \((\lambda, \mu)\)-fuzzy subalgebras of \(X\) if and only if intuitionistic fuzzy sets \((\alpha_A, \overline{\alpha}_A)\) and \((\overline{\beta}_A, \beta_A)\) are intuitionistic \((\lambda, \mu)\)-fuzzy subalgebras of \(X\).

**Theorem 3.4** An IFS \(A = (\alpha_A, \beta_A)\) is an intuitionistic \((\lambda, \mu)\)-fuzzy subalgebras of \(X\) if and only if for all \(s,t \in [0,1]\), the nonempty sets \(U(\alpha_A; t)\) and \(L(\beta_A; s)\) are subalgebras of \(X\).

**Proof** Let \(A = (\alpha_A, \beta_A)\) is an intuitionistic \((\lambda, \mu)\)-fuzzy subalgebras of \(X\). For \(a \in [\lambda, \mu]\), if \(x, y \in U(\alpha_A; a)\), then \(\alpha_A(x) \geq a, \alpha_A(y) \geq a\), so \(\alpha_A(x*y) \lor \lambda \geq \alpha_A(x) \land \alpha_A(y) \lor \mu \geq a\). That is \(\alpha_A(x*y) \lor \lambda \geq a\), thus \(x*y \in U(\alpha_A; a)\). Therefore \(U(\alpha_A; t)\) is subalgebra of \(X\).

For all \(b \in [\lambda, \mu]\), if \(x, y \in L(\beta_A; b)\), then \(\beta_A(x) \leq b\) and \(\beta_A(y) \leq b\), which implies that \(\beta_A(x*y) \land \mu \leq \beta_A(x) \lor \beta_A(y) \lor \lambda \leq b\). Hence \(\beta_A(x*y) \leq b\) and \(x*y \in L(\beta_A; b)\). Therefore \(L(\beta_A; s)\) is a subalgebra of \(X\).

Conversely, assume that for each \(s,t \in [\lambda, \mu]\), the nonempty sets \(U(\alpha_A; t)\) and \(L(\beta_A; s)\) are subalgebras of \(X\). If there exist \(a, b \in X\) such that \(\alpha_A(a*b) \lor \lambda < \alpha_A(a) \land \alpha_A(b) \lor \mu\), then takin \(t_0 = (\alpha_A(a*b) \lor \lambda + \alpha_A(a) \land \alpha_A(b) \lor \mu)/2\), we have \(a, b \in U(\alpha_A; t_0)\) and \(a*b \in U(\alpha_A; t_0)\). But \(U(\alpha_A; t_0)\) is a subalgebra of \(X\). This is a contradiction. Hence for all \(x, y \in X\),

\[
\alpha_A(x*y) \lor \lambda < \alpha_A(x) \land \alpha_A(y) \lor \mu.
\]

In the same way, we have for all \(x, y \in X\), \(\beta_A(x*y) \land \mu \leq \beta_A(x) \lor \beta_A(y) \lor \lambda\). Therefore IFS \(A = (\alpha_A, \beta_A)\) is an intuitionistic \((\lambda, \mu)\)-fuzzy subalgebras of \(X\).

**Example 3.2** Let \(X = \{1, a, b, c\}\) in which \(*\) is defined by

|      | 1   | a   | b   | c   |
|------|-----|-----|-----|-----|
| 1    | 1   | a   | b   | c   |
| a    | 1   | 1   | b   | c   |
| b    | 1   | a   | 1   | c   |
| c    | c   | c   | c   | 1   |

Then \((X, *, 1)\) is a CI-algebra. Let \(\alpha_A(1) = \alpha_A(c) = 0.7, \alpha_A(a) = \alpha_A(b) = 0.2,\) and \(\beta_A(1) = 0.2, \beta_A(c) = 0.2, \beta_A(a) = \beta_A(b) = 0.7\), then \(A\) is an intuitionistic \((\lambda, \mu)\)-fuzzy subalgebras of \(X\).

Clearly, \(U(\alpha_A; 0.2) = L(\beta_A; 0.7) = \{1, a, b, c\}\) and \(U(\alpha_A; 0.7) = L(\beta_A; 0.2) = \{1, c\}\) are subalgebra of \(X\).

Suppose that \(A = (\alpha_A, \beta_A)\) and \(B = (\alpha_B, \beta_B)\) be two intuitionistic fuzzy sets of \(X\), the intersection \(A \cap B = (\alpha_{A \cap B}, \beta_{A \cap B})\) of \(A\) and \(B\) is defined by

\[
\alpha_{A \cap B}(x) = \alpha_A(x) \land \alpha_B(x), \quad \beta_{A \cap B}(x) = \beta_A(x) \lor \beta_B(x).
\]

**Theorem 3.5** Let \(A = (\alpha_A, \beta_A)\) and \(B = (\alpha_B, \beta_B)\) be two intuitionistic \((\lambda, \mu)\)-fuzzy subalgebras of \(X\). Then intersection \(A \cap B\) is also intuitionistic \((\lambda, \mu)\)-fuzzy subalgebra of \(X\).

**Proof** Assume \(A = (\alpha_A, \beta_A)\) and \(B = (\alpha_B, \beta_B)\) be two intuitionistic fuzzy subalgebras of \(X\), then, for all \(x, y, z \in X\),
\[ \alpha_a(x \ast y) \vee \lambda \geq \alpha_a(x) \wedge \alpha_a(y) \wedge \mu, \quad \alpha_b(x \ast y) \vee \lambda \geq \alpha_b(x) \wedge \alpha_b(y) \wedge \mu \]
\[ \beta_a(x \ast y) \wedge \mu \leq \beta_a(x) \vee \beta_a(y) \vee \lambda, \quad \beta_b(x \ast y) \wedge \mu \leq \beta_b(x) \vee \beta_b(y) \vee \lambda. \]
Therefore
\[ \alpha_{a \ast b}(x \ast y) \vee \lambda = \alpha_a(x \ast y) \wedge \alpha_b(x \ast y) \vee \lambda = (\alpha_a(x \ast y) \vee \lambda) \wedge (\alpha_b(x \ast y) \vee \lambda) \]
\[ \geq (\alpha_a(x) \wedge \alpha_a(y) \wedge \mu) \wedge (\alpha_b(x) \wedge \alpha_b(y) \wedge \mu) \]
\[ = (\alpha_a(x) \wedge \alpha_b(x)) \wedge (\alpha_a(y) \wedge \alpha_b(y)) \wedge \mu = \alpha_{a \ast b}(x) \wedge \alpha_{a \ast b}(y) \wedge \mu. \]
\[ \beta_{a \ast b}(x \ast y) \wedge \mu = \beta_a(x \ast y) \vee \beta_b(x \ast y) \wedge \mu = (\beta_a(x \ast y) \vee \lambda) \wedge (\beta_b(x \ast y) \wedge \mu) \]
\[ \leq (\beta_a(x) \vee \beta_a(y) \vee \lambda) \wedge (\beta_b(x) \vee \beta_b(y) \vee \lambda) \]
\[ = (\beta_a(x) \vee \beta_b(x)) \vee (\beta_a(y) \vee \beta_b(y)) \vee \lambda = \beta_{a \ast b}(x) \vee \beta_{a \ast b}(y) \vee \lambda. \]
Hence \( A \cap B \) is also intuitionistic \((\lambda, \mu)\)-fuzzy subalgebra of \( X \).

**Theorem 3.6** Let \( f \) be an endomorphism of \( X \). If \( A = (\alpha_a, \beta_a) \) is an intuitionistic \((\lambda, \mu)\)-fuzzy subalgebra of \( X \). Then \((\alpha'_a, \beta'_a)\) is also intuitionistic \((\lambda, \mu)\)-fuzzy subalgebra of \( X \). Where \( \alpha'_a(x) = \alpha_a(f(x)) \) and \( \beta'_a(x) = \beta_a(f(x)) \).

**Proof** Assume \( f \) be an endomorphism of \( X \), then for \( x, y \in X \), \( f(x \ast y) = f(x) \ast f(y) \). If \( A = (\alpha_a, \beta_a) \) is an intuitionistic \((\lambda, \mu)\)-fuzzy subalgebra of \( X \), then
\[ \alpha'_a(x \ast y) \vee \lambda = \alpha_a(f(x \ast y)) \vee \lambda = \alpha_a(f(x) \ast f(y)) \vee \lambda \]
\[ \geq \alpha_a(f(x)) \wedge \alpha_a(f(y)) \wedge \mu = \alpha'_a(x) \wedge \alpha'_a(y) \wedge \mu, \]
and
\[ \beta'_a(x \ast y) \wedge \mu = \beta_a(f(x \ast y)) \wedge \mu = \beta_a(f(x) \ast f(y)) \wedge \mu \]
\[ \leq \beta_a(f(x)) \vee \beta_a(f(y)) \vee \lambda = \beta'_a(x) \vee \beta'_a(y) \vee \lambda. \]
It follows from Definition3.1 that \((\alpha'_a, \beta'_a)\) is an intuitionistic \((\lambda, \mu)\)-fuzzy subalgebra of \( X \).

**Theorem 3.7** Let \( f : X \to Y \) be an epimorphism of CI-algebras. If \( A = (\alpha_a, \beta_a) \) is an intuitionistic fuzzy set in \( Y \). Then \( B = (\alpha'_a, \beta'_a) \) is an intuitionistic \((\lambda, \mu)\)-fuzzy subalgebra of \( X \) if and only if \( A = (\alpha_a, \beta_a) \) is an intuitionistic fuzzy \((\lambda, \mu)\)-fuzzy subalgebra of \( Y \).

**Proof** For any \( y_1, y_2 \in Y \), there exists \( x_1, x_2 \in X \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \). Then
\[ \alpha_a(y_1 \ast y_2) \vee \lambda = \alpha_a(f(x_1) \ast f(x_2)) \vee \lambda = \alpha_a(f(x_1 \ast x_2)) \vee \lambda \]
\[ = \alpha'_a(x_1 \ast x_2) \vee \lambda \geq \alpha'_a(x_1) \wedge \alpha'_a(x_2) \wedge \mu \]
\[ = \alpha_a(f(x_1)) \wedge \alpha_a(f(x_2)) \wedge \mu = \alpha_a(y_1) \wedge \alpha_a(y_2) \wedge \mu. \]
\[ \beta_a(y_1 \ast y_2) \wedge \mu = \beta_a(f(x_1) \ast f(x_2)) \wedge \mu = \beta_a(f(x_1 \ast x_2)) \wedge \mu \]
\[ = \beta'_a(x_1 \ast x_2) \wedge \mu \leq \beta'_a(x_1) \vee \beta'_a(x_2) \vee \lambda \]
\[ = \beta_a(f(x_1)) \vee \beta_a(f(x_2)) \vee \lambda = \beta_a(y_1) \vee \beta_a(y_2) \vee \lambda. \]
Therefore \( A = (\alpha_a, \beta_a) \) is an intuitionistic fuzzy \((\lambda, \mu)\)-fuzzy subalgebra of \( Y \).
Conversely, for any \( x_1, x_2 \in X \), there exists \( y_1, y_2 \in Y \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \). Then
\[ \alpha'_a(x_1 \ast x_2) \vee \lambda = \alpha_a(f(x_1 \ast x_2)) \vee \lambda = \alpha_a(f(x_1) \ast f(x_2)) \vee \lambda \]
\[ = \alpha_a(y_1 \ast y_2) \vee \lambda \geq \alpha_a(y_1) \wedge \alpha_a(y_2) \wedge \mu \]
\[= \alpha_A(f(x_1)) \wedge \alpha_A(f(x_2)) \wedge \mu = \alpha_A(f(x_1)) \wedge \alpha_A(f(x_2)) \wedge \mu.\]

\[\beta_A'(x_1 \ast x_2) \wedge \mu = \beta_A'(f(x_1 \ast x_2)) \wedge \mu = \beta_A'(f(x_1) \ast f(x_2)) \wedge \mu\]

\[= \beta_A(y_1 \ast y_2) \wedge \mu \leq \beta_A(y_1) \vee \beta_A(y_2) \vee \lambda\]

\[= \beta_A(f(x_1)) \vee \beta_A(f(x_2)) \vee \lambda = \beta_A'(x_1) \vee \beta_A'(x_2) \vee \lambda.\]

It follows from Definition 3.1 that \(B = (\alpha_A', \beta_A')\) is an intuitionistic \((\lambda, \mu)\)-fuzzy subalgebra of \(X\).

**Theorem 3.8** Let \(f : X \to Y\) be an epimorphism of CI-algebras. If \(A = (\alpha_A, \beta_A)\) is an intuitionistic fuzzy set in \(X\). If \(A = (\alpha_A, \beta_A)\) is an intuitionistic fuzzy \((\lambda, \mu)\)-fuzzy subalgebra of \(X\). Then \(f(A) = (\alpha_{f(A)}, \beta_{f(A)})\) is an intuitionistic \((\lambda, \mu)\)-fuzzy subalgebra of \(Y\). Where

\[\alpha_{f(A)}(y) = \sup \{\alpha_A(x) \mid f(x) = y\}\]

and \(\beta_{f(A)}(y) = \inf \{\beta_A(x) \mid f(x) = y\}\).

**Proof** Suppose that \(A = (\alpha_A, \beta_A)\) is an intuitionistic fuzzy \((\lambda, \mu)\)-fuzzy subalgebra of \(X\) and \(f : X \to Y\) be an epimorphism of CI-algebras. Then for any \(y_1, y_2 \in Y\), there exists \(x_1, x_2 \in X\) such that \(f(x_1) = y_1\) and \(f(x_2) = y_2\). Then

\[\alpha_{f(A)}(y_1 \ast y_2) \vee \lambda = \sup \{\alpha_A(x_1 \ast x_2) \mid f(x_1 \ast x_2) = y_1 \ast y_2\} \vee \lambda\]

\[= \sup \{\alpha_A(x_1 \ast x_2) \wedge \lambda \mid f(x_1) \ast f(x_2) = y_1 \ast y_2\}\]

\[= \sup \{\alpha_A(x_1 \ast x_2) \wedge \lambda \mid f(x_1) = y_1, f(x_2) = y_2\}\]

\[\geq \sup \{\alpha_A(x_1) \wedge \alpha_A(x_2) \wedge \mu \mid f(x_1) = y_1, f(x_2) = y_2\}\]

\[= \sup \{\alpha_A(x_1) \wedge \alpha_A(x_2) \mid f(x_1) = y_1, f(x_2) = y_2\} \wedge \mu\]

\[= (\sup \{\alpha_A(x_1) \mid f(x_1) = y_1\}) \wedge (\sup \{\alpha_A(x_2) \mid f(x_2) = y_2\}) \wedge \mu\]

\[= \alpha_{f(A)}(y_1) \wedge \alpha_{f(A)}(y_2) \wedge \mu.\]

\[\beta_{f(A)}(y_1 \ast y_2) \wedge \mu = \inf \{\beta_A(x_1 \ast x_2) \mid f(x_1 \ast x_2) = y_1 \ast y_2\} \wedge \mu\]

\[= \inf \{\beta_A(x_1 \ast x_2) \wedge \mu \mid f(x_1) \ast f(x_2) = y_1 \ast y_2\}\]

\[= \inf \{\beta_A(x_1 \ast x_2) \wedge \mu \mid f(x_1) = y_1, f(x_2) = y_2\}\]

\[\leq \inf \{\beta_A(x_1) \vee \beta_A(x_2) \mid f(x_1) = y_1, f(x_2) = y_2\} \vee \lambda\]

\[= \inf \{\beta_A(x_1) \vee \beta_A(x_2) \mid f(x_1) = y_1, f(x_2) = y_2\} \vee \lambda\]

\[= (\inf \{\beta_A(x_1) \mid f(x_1) = y_1\}) \vee (\inf \{\beta_A(x_2) \mid f(x_2) = y_2\}) \vee \lambda\]

\[= \beta_{f(A)}(y_1) \vee \beta_{f(A)}(y_2) \vee \lambda.\]

Then \(f(A) = (\alpha_{f(A)}, \beta_{f(A)})\) is an intuitionistic \((\lambda, \mu)\)-fuzzy subalgebra of \(Y\).

In the following, the Cartesian product of intuitionistic \((\lambda, \mu)\)-fuzzy subalgebras are defined and some results are discussed.

**Theorem 3.9** Let \(X\) be CI-algebra and Cartesian product of \(X\) is \(X \times X = \{(x, y) \mid x \in X, y \in X\}\). Then \((X \times X, \circ, (1, 1))\) is also a CI-algebra under the binary operation \(\circ\) defined by

\[(a, b) \circ (c, d) = (a \ast c, b \ast d)\]

for all \((a, b), (c, d) \in X \times X\).

**Definition 3.2** Let \(A = (\alpha_A, \beta_A)\) and \(B = (\alpha_B, \beta_B)\) are intuitionistic fuzzy set of \(X\). The Cartesian product \(A \times B = (\alpha_{A \times B}, \beta_{A \times B})\) of \(A\) and \(B\) is defined by

\[\alpha_{A \times B}(x, y) = \alpha_A(x) \wedge \alpha_B(y),\]

\[\beta_{A \times B}(x, y) = \beta_A(x) \vee \beta_B(y),\]

for all \((x, y) \in X \times X\).
Theorem 3.10 Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are intuitionistic $(\lambda, \mu)$-fuzzy subalgebras of $X$. Then the Cartesian product $A \times B$ of $A$ and $B$ is an intuitionistic $(\lambda, \mu)$-fuzzy subalgebra of CI-algebra $(X \times X, o, (1,1))$.

Let $(X, \ast, 1)$ and $(Y, \ast, 1)$ be two CI-algebras and $A \times B = (\alpha_{A \times B}, \beta_{A \times B})$ be an intuitionistic fuzzy set of $X \times Y$. Define $\tilde{A} = (\alpha_A, \beta_A)$ and $\tilde{B} = (\alpha_B, \beta_B)$ by  
\[
\alpha_A(x) = \sup_{z \in Y} \alpha_{A \times B}(x, z), \quad \beta_A(x) = \inf_{z \in Y} \beta_{A \times B}(x, z), \quad \alpha_B(y) = \sup_{z \in X} \alpha_{A \times B}(z, y), \quad \beta_B(y) = \inf_{z \in X} \beta_{A \times B}(z, y).
\]

Theorem 3.11 Let $A \times B = (\alpha_{A \times B}, \beta_{A \times B})$ be an intuitionistic fuzzy set of CI-algebra $(X \times Y, o, (1,1))$. Then $\tilde{A} = (\alpha_A, \beta_A)$ is an intuitionistic $(\lambda, \mu)$-fuzzy subalgebra of $X$.

Proof (1) For all $x_1, x_2 \in X$, since $A \times B = (\alpha_{A \times B}, \beta_{A \times B})$ be an intuitionistic fuzzy set of CI-algebra $(X \times X, o, (1,1))$, then  
\[
\begin{align*}
\alpha_A(x_1 \ast x_2) \vee \lambda &= \sup_{z_1, z_2 \in Y} \alpha_{A \times B}(x_1, z_2, z) \vee \lambda = \sup_{z_1, z_2 \in Y} \alpha_{A \times B}(x_1 \ast x_2, z_1) \vee \mu \\
&= \sup_{z_1, z_2 \in Y} \alpha_{A \times B}(x_1, z_2, z) \vee \lambda \geq \sup_{z_1, z_2 \in Y} \alpha_{A \times B}(x_1, z_1) \wedge \alpha_{A \times B}(x_2, z_2) \wedge \mu \\
&= (\sup_{z_1, z_2 \in Y} \alpha_{A \times B}(x_1, z_1)) \wedge (\sup_{z_1, z_2 \in Y} \alpha_{A \times B}(x_2, z_2)) \wedge \mu = \alpha_A(x_1) \wedge \alpha_A(x_2) \wedge \mu.
\end{align*}
\]

Therefore $\tilde{A} = (\alpha_A, \beta_A)$ is an intuitionistic $(\lambda, \mu)$-fuzzy subalgebra of $X$.

(2) For all $y_1, y_2 \in Y$, since $A \times B = (\alpha_{A \times B}, \beta_{A \times B})$ be an intuitionistic fuzzy set of CI-algebra $(X \times Y, o, (1,1))$, then  
\[
\begin{align*}
\alpha_B(y_1 \ast y_2) \vee \lambda &= \sup_{z \in X} \alpha_{A \times B}(z, y_1 \ast y_2) \vee \lambda = \sup_{z \in X} \alpha_{A \times B}(z \ast y_2, y_1) \vee \lambda \\
&= \sup_{z \in X} \alpha_{A \times B}(z \ast y_2, y_1) \vee \lambda \\
&\leq \sup_{z \in X} \alpha_{A \times B}(z, y_1) \wedge \alpha_{A \times B}(z, y_2) \wedge \mu \\
&= (\sup_{z \in X} \alpha_{A \times B}(z, y_1)) \wedge (\sup_{z \in X} \alpha_{A \times B}(z, y_2)) \wedge \mu = \alpha_B(y_1) \wedge \alpha_B(y_2) \wedge \mu.
\end{align*}
\]

Therefore $\tilde{B} = (\alpha_B, \beta_B)$ is an intuitionistic $(\lambda, \mu)$-fuzzy subalgebra of $Y$. 

Hence $\tilde{B} = (\alpha_B, \beta_B)$ is an intuitionistic $(\lambda, \mu)$-fuzzy subalgebra of $Y$. 

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Theorem 3.12 Let $A \times B = (\alpha_{A \times B}, \beta_{A \times B})$ be an intuitionistic fuzzy set of CI-algebra $(X \times Y, \circ, (1,1))$. Then

1. $\alpha_A$ and $\beta_A = 1 - \beta_A$ are $(\lambda, \mu)$-fuzzy subalgebra of $X$.
2. $\alpha_B$ and $\beta_B = 1 - \beta_B$ are $(\lambda, \mu)$-fuzzy subalgebra of $Y$.

4. Conclusions
In this paper, the notion of intuitionistic $(\lambda, \mu)$-fuzzy subalgebras in CI-algebra are introduced. Some of their properties are investigated. Some properties of intuitionistic $(\lambda, \mu)$-fuzzy subalgebras in CI-algebra are obtained. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems.

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