Quantum limited measurements of atomic scattering properties

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We propose a method to perform precision measurements of the interaction parameters in systems of $N$ ultracold spin 1/2 atoms. The spectroscopy is realized by first creating a coherent spin superposition of the two relevant internal states of each atom and then letting the atoms evolve under a squeezing Hamiltonian. The non-linear nature of the Hamiltonian decreases the fundamental limit imposed by the Heisenberg uncertainty principle to $N^{-2}$, a factor of $N$ smaller than the fundamental limit achievable with non-interacting atoms. We study the effect of decoherence and show that even with decoherence, entangled states can outperform the signal to noise limit of non-entangled states. We present two possible experimental implementations of the method using Bose-Einstein spinor condensates and fermionic atoms loaded in optical lattices and discuss their advantages and disadvantages.

I. INTRODUCTION

The preparation of many-particle entangled states is becoming a fundamental task in modern quantum physics. Entanglement lies at the heart of quantum communication and quantum information and is also a fundamental resource in precision spectroscopy. With recent advances in the manipulation of trapped ions and neutral atoms, there has been significant progress in the preparation of many-body entanglement states in atomic gases\textsuperscript{1,2,3,4}. Such systems therefore offer rich opportunities to investigate new physics with both practical and fundamental applications.

Here we propose an interferometric method that uses the entanglement that emerges during the many-body dynamics between interacting spinor atoms to perform precision measurements of their interatomic interaction strength, usually parameterized by the $s$-wave scattering length. Our method is a many-body generalization of the technique reported in Ref. \textsuperscript{5}, where information of the scattering properties was obtained by using the entanglement dynamics in pairs of atoms trapped in the ground state of a potential well.

A precise determination of the scattering properties has broad and important applications beyond atomic physics. For example, Ref. \textsuperscript{6} shows that such type of measurements might be useful for fundamental physics, as by monitoring the scattering length on the $10^{-5}$ level one could detect variation in the electron to proton mass ratio on the level of $10^{-11} - 10^{-14}$.

Standard Ramsey spectroscopy starts with a system of $N$ non-interacting spin 1/2 particles initially prepared in the same internal state. Subsequently, a Ramsey pulse prepares it in a coherent superposition of the two internal states and the system is let to evolve freely for a time during which a relative phase accumulates, $\phi$, due to the energy splitting between the two states. The latter is then decoded by using a second Ramsey pulse which maps the phase onto a population difference, thus allowing to perform precision measurements of the atomic transition frequency.

The statistical fluctuations associated with a finite sampling yields a lower limit in the phase accuracy $\delta \phi = \sqrt{1/N}$ called the shot noise limit \textsuperscript{7}. The Heisenberg uncertainty principle, however, allows for phase accuracies consistent with the basic principles of quantum mechanics, as low as $\delta \phi = 1/N$, called the Heisenberg limit. The latter can be achieved by using initially entangled atoms instead of a polarized sample \textsuperscript{8}.

Here we propose a spectroscopy technique which aims to measure the atomic interaction parameters arising form two-body collisions by using similar Ramsey spectroscopic ideas, but applied to interacting atoms instead of non-interacting ones. The many-body interactions during the free evolution build in quantum mechanical correlations between the atoms and generates a large amount of entanglement even from an initially uncorrelated sample \textsuperscript{9}. This scheme provides a resolution which is fundamentally limited to $N^{-2}$, a factor of $N$ smaller than the Heisenberg limit achievable with non-interacting atoms. Moreover, for initially uncorrelated atoms the sensitivity scales as $N^{-3}$, implying a gain in resolution by a factor of $\sqrt{N}$ with respect to the classical shot noise resolution. A more general analysis of beyond-Heisenberg scaling for multi-body collisions or tensor-field interactions has been derived in Ref. \textsuperscript{10}.

Many-particle entangled states, however, are difficult to prepare and maintain since they are extremely fragile: in practice, noise and decoherence rapidly collapse entangled states into classical statistical mixtures. For example in standard Ramsey spectroscopy decoherence destroys the potential gain provided by entanglement, and, when it is taken into account both initially uncorrelated and maximally entangled states provide the same shot noise resolution \textsuperscript{11}. Here we show that decoherence degrades the sensitivity of our method to $N^{-3/2}$ for initially entangled atoms and to $N^{-3/4}$ for initially uncorrelated ones. This scaling is interesting as it demonstrates that the entanglement build up by the many-body interactions helps, even with decoherence, to keep the sensitivity achievable with initially squeezed atoms above the maximally achievable with initially uncorrelated particles.

We also discuss physical implementation of our scheme using a) spinor Bose- Einstein condensates and b) fermionic atoms loaded in optical lattices and compare and contrast the advantage and disadvantages of these two possible set-ups. The paper is organized as follows. In Sec. II we present our interferometry method based on the many-body collective dynamics. In Sec. III we apply it to initially uncorrelated states and derive the type of initially entangled states that lead to
Heisenberg-limited sensitivity. In this section, we also propose a way to generate such states. In Sec. IV we include the effect of decoherence. In Sec. V we discuss the experimental implementation of the interferometry method using spinor condensates and cold fermionic atoms in optical lattices and discuss possible technical limitations. Finally we conclude in Sec. VI.

II. INTERFEROMETRIC TECHNIQUE

Let us consider a collection of $N$ spin-1/2 interacting particles described by the Hamiltonian

$$\hat{H}_m = \chi \hat{J}_z^2.$$  \hspace{1cm} (1)

We used $\hat{J}_z$ to denote the collective spin operator of the $N$ atoms: $\hat{J}_z = \sum_\alpha \hat{\sigma}_z^\alpha$, where $\alpha = x, y, z$ and $\hat{\sigma}_z^\alpha$ is a Pauli operator acting on the $i^{th}$ atom. We set $\hbar = 1$.

Our interferometric technique follows the same ideas used in standard Ramsey spectroscopy, but replaces the free evolution of the atoms by evolution with $\hat{H}_m$, and the goal is to estimate $\chi$ as accurately as possible. We start with the initial state $|\Phi_0\rangle$ and apply a $\pi/2$ rotation to all particles about the $y$ axis, $|\psi(0)\rangle = e^{i\pi/2\hat{J}_y} |\Phi_0\rangle$. After letting the system evolve for time $t$ under $\hat{H}_m$, $|\psi(t)\rangle = e^{-it\hat{H}_m} |\psi(0)\rangle$, a second $-\pi/2$ pulse is applied, $|\psi_f(t)\rangle = e^{-i\pi/2\hat{J}_z} |\psi(t)\rangle$. Finally the collective spin is measured $\langle \hat{J}_z(t) = \langle \psi_f(t) | \hat{J}_z | \psi_f(t) \rangle$ . If this scheme is repeated during a total time $T$ the achievable sensitivity is given by

$$|\chi\rangle^2 = \frac{t}{T} \frac{\langle \Delta \hat{J}_z(t) \rangle}{(\Delta \hat{J}_z(t))/\chi)^2} = \frac{t}{T} \frac{\langle \psi(t) | \Delta \hat{J}_x | \psi(t) \rangle}{\langle \psi(t) | \hat{J}_z | \psi(t) \rangle / \chi^2},$$  \hspace{1cm} (2)

with variance $\Delta \hat{J}_z = \hat{J}_z^2 - \langle \hat{J}_z \rangle^2$.

In contrast to standard Ramsey spectroscopy, which is fundamentally limited by the Heisenberg exclusion principle to $(tN)^{-1}$ \[811\] per realization, the fundamental limit of this scheme is $2/(tN^2)$. This follows from an application of the time-energy uncertainty principle: $\delta t^2 (\Delta \hat{H}_m) \geq 1/4$ where $\langle \Delta \hat{H}_m \rangle$ is the variance of the Hamiltonian and $\delta t$ is the variance in estimating time from a measurement on the system \[23\]. The time-energy uncertainty can be reexpressed as $\delta \varphi^2 (\Delta \hat{J}_z^2) \geq 1/4$, where $\varphi = \chi t$. We can establish an upper limit $\langle \Delta \hat{J}_z^2 \rangle = \langle \hat{J}_z^2 \rangle - \langle \hat{J}_z \rangle^2 \leq (N^4/16) \[26\] This implies $\delta \varphi \geq 2N^{-1}$ and therefore a fundamental uncertainty in determining $\chi$ after a time $t$ of: $|\chi\rangle \geq 2/(tN^2)$.

III. SPECTROSCOPY

Let us start by outlining the basic properties of $\hat{H}_m$. It commutes with $\hat{J}^2$ and $\hat{J}_z$, so a good basis to describe the dynamics is the one spanned by the collective angular momentum eigenstates. In the $2^N$ dimensional Hilbert space, there are $N + 1$ orthogonal fully symmetric states which we denote by $|J = \frac{N}{2}, M\rangle$ that satisfy: $\hat{J}^2 |\frac{N}{2}, M\rangle = \frac{N}{2}^2 |\frac{N}{2}, M\rangle$, and $\hat{J}_z |\frac{N}{2}, M\rangle = M |\frac{N}{2}, M\rangle$ with $-\frac{N}{2} \leq M \leq \frac{N}{2}$. We refer to the $J = N/2$ states as the $P$ manifold. The quantum numbers $J$ and $M$ are conserved during the dynamics, and therefore, if the initial state $|\Phi_0\rangle$ belongs to $P$, then the subsequent evolution of the system only takes place within $P$.

A. Initially uncorrelated atoms

First, consider the case when the initial state is fully polarized along $z$: $|\Phi_0\rangle = |\frac{N}{2}, \frac{N}{2}\rangle$. After the first $\pi/2$ pulse it becomes $|\psi(0)\rangle = |\frac{N}{2}, \frac{N}{2}\rangle \approx \sum_{M=-N/2}^{N/2} (\frac{1}{2\pi N})^{1/4} e^{-M^2/2N} |\frac{N}{2}, M\rangle$. The effect of $\hat{H}_m$ on the state is to imprint a phase $\chi t^2$ to each $|\frac{N}{2}, M\rangle$ component. In the limit of many atoms ($N \gg 1$) we may approximate the sums by integrals assuming $M/N$ is a continuous variable and to replace the binomial coefficient by a Gaussian distribution with the same width. In this limit $\langle \hat{J}_z \rangle$ and $\langle \hat{J}_z^2 \rangle$ become

$$\langle \hat{J}_z \rangle = \frac{N}{2} \sum_{k=0,1,2,\ldots} (-1)^k e^{-N/2(\chi t - k\pi)^2},$$  \hspace{1cm} (3)

$$\langle \hat{J}_z^2 \rangle = \frac{N^2 - N}{8} \sum_{k=0,1,2,\ldots} e^{-N(\chi t - k\pi/2)^2} + \frac{N^2 + N}{8},$$  \hspace{1cm} (4)

where we used the notation $\langle \hat{A} \rangle \equiv \langle \psi(t) | \hat{A} | \psi(t) \rangle$. The time evolution of the signal is a series of equally spaced gaussian pulses (see Fig. \[1\]). As the system evolves, at first the different accumulated phases lead to a collapse of $\langle \hat{J}_z \rangle$. For the initial coherent state in consideration, the collapse time, which depends on its variance, is $\chi t_{coll} \approx 3N^{-1/2}$. The evolution continues and at $\chi t = \pi$ all the different components rephase inducing a perfect revival of the initial state, now with opposite polarization: $\langle \hat{J}_z \rangle = -N/2$. Using Eqs. (3) and (4), one can show that the optimal sensitivity is reached at the points with maximal signal intensity, $\chi t = k\pi$. It scales as $N^{-1}$ (see Fig. \[1\]):

$$|\Delta \chi|_{opt} = \frac{1}{N\sqrt{T\chi}}.$$  \hspace{1cm} (5)

B. Initially correlated atoms

Entangled atomic states (e.g. spin squeezed states) potentially allow to significantly improve the sensitivity in precision measurements. For example, consider the initial state

$$|\Phi^0(0)\rangle = \left( |\frac{N}{2}, 0\rangle + a \left| \frac{N}{2}, 2\right\rangle \right) / \sqrt{1 + a^2}.$$  \hspace{1cm} (6)

Here $a$ is a real number of order unity and $N$ is assumed to be even and large. The probability distribution of $|\Phi^0(0)\rangle$ in
Using properties of the Wigner rotation functions, it is possible to show that:

\[
\langle \hat{J}_x \rangle_\psi^N = \frac{N a \sqrt{8}}{1 + a^2} \sum_{k=0}^{N/2} \frac{J_2 [N (\chi t - \pi/2 + k \pi)]}{N (\chi t - \pi/2 + k \pi)} ,
\]

(7)

\[
\langle \hat{J}_z^2 \rangle_\psi^N = N^2 A \sum_{k=0}^{N/2} \left( \frac{J_1 [2 N (\chi t - k \pi/2)]}{(2 N (\chi t - k \pi/2))} \right) + B ,
\]

(8)

with \( A = \frac{2 a^2}{N^6 (1 + a^2)} - \frac{N^2 + 2}{8 N^4 a^2} \) and \( B = \frac{a^2}{(1 + a^2)} + \frac{N (N + 2)}{16} \). As shown in Fig. 2, the system starts at \( t = 0 \) with \( \langle \hat{J}_x \rangle_\psi^N = 0 \), and during the time evolution \( \langle \hat{J}_x \rangle_\psi^N \) grows while oscillating. At \( \chi t \approx \frac{\pi}{2} \pm 2.29 N^{-1} \) the signal reaches a maximum and vanishes again at \( \chi t = \frac{\pi}{2} \). Then it evolves with opposite polarization until it returns to the initial state at \( \chi t = \pi \). Using Eqs. (7) and (8), the optimal sensitivity can be shown to be reached at \( \chi t = k \pi/2 \), with \( k \) an odd integer (see Fig. 3):

\[
|\delta \chi |_{\text{opt}}^a = \sqrt{\frac{32 (1 + a^2)}{T t N^4}} .
\]

(9)

These states lead to \( 1/N^2 \) sensitivity. The best accuracy is obtained for \( a \to 0 \), in which case \( |\delta \chi |_{\text{opt}}^a = \sqrt{\frac{32 T}{T t N^4}} \). Note, however, that estimating \( \chi \) by measuring \( \hat{J}_z \) might be experimentally impractical due to the fact that not only the signal vanishes at \( \chi t = k \pi/2 \) (regardless of the value of \( a \)) but also \( |\delta \chi | \) is very sharply peaked at \( \chi t = k \pi/2 \) and hence very sensitive to small variations of \( \chi t \). To overcome this limitation we propose, following the ideas exposed in Ref. [12], to measure \( \hat{J}_z \) instead of \( \hat{J}_x \). In contrast to \( \hat{J}_z \), the average square signal does not vanishes for finite \( a \) (it goes like \( \frac{4 a^2}{1 + a^2} \)) and, as shown in Fig. 3, measuring \( \chi \) by means of \( \hat{J}_z^2 \) provides the same \( N^{-2} \) accuracy with the advantage of a broader profile around the optimal value which becomes slightly shifted from \( \chi t = k \pi/2 \). We calculated \( |\delta \chi | \) numerically according to Eq. (2) but replacing \( \hat{J}_x \) by \( \hat{J}_z \). This idea, however, has the drawback that \( |\delta \chi |_{\hat{J}_z \to \hat{J}_z^2} \) diverges exactly at \( \chi t = k \pi/2 \). Nevertheless as shown in Fig. 3 this is a extremely narrow divergence and almost a one point behavior.

Let us now discuss how to generate states of the form given by Eq. (6). These states are highly squeezed, and a robust method to generate them can have also applications in Heisenberg limited phase detection [12]. The idea is to adiabatically convert \( |M, \frac{3}{2} \rangle \) into \( |M, \frac{1}{2} \rangle \) by gradually increasing \( \lambda_\zeta \) from 0 to 1 in the following time dependent Hamiltonian [13]:

\[
H_{\text{adi}} (\lambda_\zeta) = -\zeta \hat{J}_z^2 + (1 - \lambda_\zeta) \omega_\zeta \hat{J}_x + \lambda_\zeta \chi \hat{J}_z .
\]

(10)

The first term proportional to \( \zeta \) is required to keep the system within \( \mathcal{P} \). As shown in Fig. 4, there is always a finite gap between the (non-degenerate) ground state [27] and the lowest excited state, so the adiabatic passage can be satisfied. The experimental implementation Eq. (10) will be discussed in Sec. VB.

Once \( |M, \frac{3}{2} \rangle \) is obtained, the \( |M, \frac{1}{2}, \pm 2 \rangle \) components can be generated by evolving \( |M, \frac{1}{2}, 0 \rangle \) with the Hamiltonian \( H_{2ct} = \).
\(i\xi (J_x^2 - J_z^2)\). As proposed in Ref. [13], \(H_{2xt}\) can be implemented by using two laser beams with frequency difference twice the hyperfine level splitting which coherently flips pairs of aligned spins [28].

**IV. EFFECTS OF DECOHERENCE**

In realistic experiments decoherence effects are inevitably present. The main type of decoherence is dephasing due to processes that cause random changes in the relative phase of the quantum states, while preserving the total atomic population in the atomic levels. Among the mechanisms that result in dephasing effects one can consider collisions, stray fields, and laser instabilities. We model the phase decoherence by adding a term to the Hamiltonian given by

\[
\hat{H}_{\text{env}} = \frac{1}{2} \sum_i \hat{h}_i(t) \sigma_i^z + \text{h.c.}
\]

where \(\hat{h}_i(t)\) are independent random gaussian variables with zero mean and white noise correlation functions

\[
\langle \hat{h}_i(t) \hat{h}_j(\tau) \rangle = 2\Gamma \delta(t - \tau)
\]

that represent couplings to the environment. Here the bar denotes averaging over the different outcomes of the random variables. This model is equivalent to considering the time evolution of the reduced density operator for a single ion \(\hat{\rho}\) as given by the master equation

\[
\dot{\hat{\rho}} = i[\hat{\rho}, \hat{H}_m] - \Gamma/2(\sigma_z \hat{\rho} \sigma_z - \hat{\rho}).
\]

To include the effect of decoherence it is simpler to go to the uncoupled basis which diagonalizes both \(\hat{H}_{\text{env}}\) and \(\hat{H}_m\). Each state in this basis can be label as \(|h_j(k)\rangle = |s_{1}^{k}, s_{2}^{k}, \ldots, s_{N}^{k}\rangle\), where \(s_{i}^{k} = \pm 1\) and \(k = 1, \ldots, 2^N\). Using the uncorrelated nature of the \(h_i\) variables one can show by straightforward calculations that

\[
\langle \hat{J}_x \rangle_{\psi, \Gamma} = e^{-\Gamma t} \langle \hat{J}_x \rangle_{\psi, 0},
\]

\[
\langle \hat{J}_x^2 \rangle_{\psi, \Gamma} = e^{-2\Gamma t} \langle \hat{J}_x^2 \rangle_{\psi, 0} + \frac{N}{4} (1 - e^{-2\Gamma t}),
\]

\[
\langle \hat{J}_z \rangle_{\psi, \Gamma} = e^{-2\Gamma t} \langle \hat{J}_z \rangle_{\psi, 0} + g(t) \left( \langle \hat{J}_x^2 \rangle_{\psi, 0} - \frac{N}{4} \right) + f(t),
\]

with

\[
g(t) = \frac{3N^2 - 2N}{16} (e^{-2\Gamma t} - e^{-4\Gamma t})
\]

and

\[
f(t) = \frac{3N^2 - 2N}{16} (1 - e^{-4\Gamma t}).
\]

Note that these equations are valid for an arbitrary initial state. The noise induces an exponential decay of the average signal and its higher moments. However, unlike the average signal the higher moments are also affected by the presence of diffusion terms.

If we assume an initially coherent state with \(\langle \hat{J}_x \rangle_{\psi, 0}\) and \(\langle \hat{J}_x^2 \rangle_{\psi, 0}\) given by Eqs. (3) and (4), one can calculate from Eqs. (11) and (12) the sensitivity in the presence of decoherence. In contrast to the ideal dynamics where the longer the interrogation time \(t\) the better is the phase accuracy, when decoherence is accounted for, one has to optimize with respect to both \(t\) and \(\chi t\), in order to obtain the best sensitivity. Provided that

\[
T > 1.6/(2\Gamma),
\]

standard minimization yields:

\[
\delta \chi_{\Gamma}^{\text{opt}} = \frac{\sqrt{2\sqrt{2\Gamma/\tau_{\text{opt}}}}}{T N^{3/2}} = \sqrt{\frac{3.5\Gamma}{T N^{3/2}}},
\]

at \(\chi t \approx k\pi \pm 1/2\sqrt{T/\Gamma}\), where \(\tau_{\text{opt}} \approx 2\Gamma/\tau_{\text{opt}} \approx 1.6\).

Note that even with initially uncorrelated atoms decoherence reduces the sensitivity by a factor of \(N^{1/4}\). This has to be contrasted with standard Ramsey spectroscopy done with non-interacting atoms, where the phase resolution with initially uncorrelated atoms has the same classical shot noise resolution so is not much affected by decoherence. The reason of the different behavior is the fact that \(H_m\) builds up particle correlations during the evolution and transforms uncorrelated states into highly squeezed ones which are vulnerable to decoherence. More specifically, as shown in Ref. [14], \(\hat{H}_m\) ideally transforms initially coherent states into maximally correlated \(N\)-particle GHZ states [16] after a period of evolution of \(\chi t = \pi/2\). These states are very fragile and decohere \(N\) times faster than uncorrelated particles.

Let us now consider the case when the initial state is \(|\Phi^+(0)\rangle\). Using Eqs. (7) and (8) in Eqs. (11) and (12), one gets an optimal sensitivity, provided \(T > \tau_{\text{opt}}/(2\Gamma)\), given by:

\[
\delta \chi_{\text{opt}}^{\alpha \Gamma} = \frac{\sqrt{4\Gamma(1 + a^2) e^{-\tau_{\text{opt}}}}}{T a^2 N^{3/2}},
\]

at \((\chi t) = k\pi/2\), and \(\tau_{\text{opt}} = \sqrt{\frac{2a^2}{1 + a^2} N}\). Eq. (14) is very interesting as it demonstrates that by using many-body interactions, entangled states can outperform the signal to noise limit of classically unentangled states even in the presence of decoherence. The sensitivity of the squeezed states is a factor \(N^{1/4}\) larger than the maximal achievable with uncorrelated atoms, see Eq. (13). In Fig. 3 we plot the optimal sensitivity in the presence of decoherence both by measuring \(\hat{J}_x\) and \(\hat{J}_z\).

For practical reasons, as the ones discussed above, measuring \(\hat{J}_z\) instead of \(\hat{J}_x\) might be experimentally convenient also with decoherence. Moreover, note that while states with \(a = 0\) provide the best accuracy in the ideal dynamics, states with \(a \neq 0\) are optimal with decoherence.
V. EXPERIMENTAL IMPLEMENTATIONS

A. Spinor Condensates

\( \chi \hat{J}_z^2 \) describes the spin dynamics of a \( F = 1 \) Bose Einstein condensate, if one associates the internal \( M_F = \pm 1 \) hyperfine states of the atoms as the two states of an effective spin-1/2 particle (the \( M_F = 0 \) state can be decoupled by introducing a detuning much larger than the chemical potential) \[17\]. In this system the coupling strength \( \chi \) is proportional to \( 4\pi/(mV)(a_{↓↑} + a_{↑↓} - a_{↓↓}) \) with \( a_{σσ'} \) the s wave scattering length between particles of type \( σ \) and \( σ' \), \( m \) the atom mass and \( V \) the volume.

Due to the dependence of \( \chi \) on the elastic scattering length difference, \( δa \equiv a_{↓↑} + a_{↑↓} - a_{↓↓} \), using an inter-state Feshbach resonance \[17\], such as the one predicted between the \( F = 2 \) and \( F = 1 \) states in \( ^{87}\text{Rb} \) atoms, one could widely tune \( \chi \) by varying an external magnetic field. Close to the Feshbach resonance there is a singularity in the scattering length, \( a_{↓↑}(B) = a_{bg}^{a↑}(1 - \Delta B/(B - B_o)) \) with \( \Delta B \) being the resonance width, \( B_o \) the zero energy resonance and \( a_{bg}^{a↑} \) the background scattering length. This singularity is due to the near degeneracy of the collision energy of the atoms with the binding energy of an extremely loose diatomic molecular state.

One can use our spectroscopy technique to measure \( δa \) in such system. However, in order for the method to work, it is important to reduce two and three body particle losses across the resonance, as they can considerably affect the sensitivity. We estimate for example the effect of two-body losses by solving the following master equation:

\[
\dot{\hat{ρ}} = i[\hat{ρ}, \hat{H}_m] - \gamma/2(\hat{C}^\dagger \hat{C} \hat{ρ} + \hat{ρ} \hat{C}^\dagger \hat{C} - 2\hat{C}^\dagger \hat{C} \hat{ρ}),
\]

with \( \hat{C} = a_{↑↓} a_{↑↓}^\dagger \) the operator that destroys a pair atoms in different spin states and which account for inelastic spin relaxation processes across the resonance. Here the constant \( \gamma \) is related to the two body loss rate density, \( \gamma \approx 2K_2/\chi \), which satisfies the relation

\[
N = \frac{K_2}{\chi} N^2.
\]

\( K_2 \) depends on the imaginary part of \( a_{↓↑} \) as \( 16\pi \text{Im}(a_{↓↑})/m \) and \( \text{Im}(a_{↓↑}) \) varies with the applied magnetic field, the difference in magnetic moment between the Feshbach resonance state and a pair of atoms in the entrance channel, \( \mu_{\text{res}} \) and the inverse molecular lifetime, \( \gamma_{\text{res}} = 1/τ_{\text{res}} \), as \[18\]:

\[
\text{Im}(a_{↓↑}) = a_{bg}^{a↓} \Delta B \frac{\gamma_{\text{res}}/\Gamma_{\text{res}}}{(B - B_o)^2 + (\gamma_{\text{res}}/\Gamma_{\text{res}})^2}.
\]

In Fig.5 we summarize our numerical solutions by plotting the scaling exponent \( b \) of the optimal sensitivity with the initial number of atoms, \( \text{log}[\delta \chi/\chi] = -b \text{log}[N(t = 0)] + \text{const} \), versus \( \gamma/\chi \) for both initially uncorrelated atoms and initially squeezed atoms. The plot demonstrates the drastic reduction of the sensitivity with \( \gamma \). For both initially uncorrelated atoms and initially squeezed atoms one requires to limit \( \gamma/\chi < 0.03 \) to have at least shot noise resolution, i.e \( b = 1/2 \).

Experimentally both two and three body particle losses contribute. As three body losses scale as the four power of the scattering length \[19\] they decrease faster than two-body losses away from the resonance. As such, the sensitivity of the method can be optimized by, on one hand, tuning the magnetic field far enough from the resonance so that mainly two-body losses are relevant and, on the other, by reducing \( K_2 \) by using a very narrow Feshbach resonance, and by considering species of atoms that belong to the lowest magnetic manifold so they possess long relaxation times.

B. Fermionic atoms in optical lattices

Recently the dynamics of bosonic atoms with two relevant internal states loaded in a deep optical lattice has been used to perform precision measurements of atomic scattering properties \[5\]. Explicitly, the experiment detected the modification in the Ramsey fringes caused by frequency shifts induced by interatomic interactions in wells occupied with two atoms, and used it to measure the elastic scattering length difference. As the various wells in the optical lattice behave as independent wells due to the suppressed tunneling between them, the experimental phase sensitivity was limited to the shot noise resolution associated to statistical fluctuations in a finite number of wells. In practice however, the sensitivity was much lower due to the two-body losses close to the Feshbach resonance in the double occupied wells.

We now discuss how to generalize this ideas by using the
collective spin dynamics (instead of the single well one) of fermionic atoms in an optical lattice loaded with one atom per site. Specifically we demonstrate how to engineer an effective $J_z^2$ Hamiltonian in such systems. The latter can be used to perform measurements of the scattering properties via the entanglement dynamics but without the limitation of particle losses at the resonance.

The effective dynamics of a system of spinor ultra-cold atoms confined in a unit filled lattice, deep in the Mott Insulator regime [20] can be described in terms of effective spin atoms confined in a unit filled lattice, deep in the Mott Entanglement dynamics but without the limitation of particle to perform measurements of the scattering properties via the operators, $\hat{J}_+ = \hat{\sigma}_+ \hat{a}^\dagger_{\sigma,j}$, $\hat{J}_- = \hat{\sigma}_- \hat{a}_{\sigma,j}$, where $\hat{a}_{\sigma,j}$ are the annihilation operator of a particle of type $\sigma = \uparrow, \downarrow$ at site $j$, and $\hat{n}_{\sigma,j} = \hat{a}_{\sigma,j}^\dagger \hat{a}_{\sigma,j}$ are number operators. In such spin variables the effective Hamiltonian maps to a spin XXZ Hamiltonian [21]:

$$\hat{H}_{\text{lat}} = \hat{H}_H + \hat{H}_I,$$

$$\hat{H}_H = \sum_{(i,j)} \tilde{\lambda} \hat{\sigma}_i^z \hat{\sigma}_j^z,$$

$$\hat{H}_I = \sum_{(i,j)} \tilde{\chi} \hat{\sigma}_i^+ \hat{\sigma}_j^-,$$

where $\langle i,j \rangle$ means that the sum is over nearest neighbors, $t_{\sigma}$ are spin-dependent tunneling energies and $U_{\sigma\sigma'}$ are on-site interaction energies proportional to the s-wave scattering lengths between the various components. For fermions $U_{\uparrow\downarrow}, U_{\downarrow\uparrow} \gg U_{\uparrow\uparrow}$ due to the Pauli exclusion principle. The Heisenberg, $\hat{H}_H$, and Ising $\hat{H}_I$, coupling constants are given by

$$\tilde{\lambda} = \pm \frac{t_\uparrow t_\downarrow}{U_{\uparrow\downarrow}}, \quad \tilde{\chi} = \frac{(t_\uparrow + t_\downarrow)^2}{2U_{\uparrow\downarrow}} - \frac{t_\uparrow^2}{U_{\uparrow\uparrow}} - \frac{t_\downarrow^2}{U_{\downarrow\downarrow}},$$

where the upper and lower signs are for fermionic and bosonic atoms respectively.

Using spin dependent optical potentials one can tune the various coupling constants and in particular one can engineer the condition $\tilde{\lambda} \gg \tilde{\chi}$. In this limit, we may treat the effect of the Ising term, $\hat{H}_I$, by means of perturbation theory. Assuming that at $t = 0$ the initial state is prepared within the $J = N/2$ manifold, $\mathcal{P}$, a perturbative analysis predicts that for times $t$ such that $\tilde{\chi} t < \tilde{\lambda}/\tilde{\chi}$, $\hat{H}_H$ confines the dynamics to $\mathcal{P}$ and transitions outside it can be neglected. As a consequence, only the projection of $\hat{H}_I$ on it, which corresponds to

$$\text{Tr}_\mathcal{P} \hat{H}_I = \chi \hat{J}_z^2 + \text{const} \quad \chi \equiv \frac{2\tilde{\chi}}{N - 1}$$

is effective and $\hat{H}_I$ acts as a long range Hamiltonian [22]. In Eq. (22) $z$ is the number of nearest neighbors. With this effective Hamiltonian the spectroscopic method described above can be applied in lattice systems to measure interactions.

Moreover, due to the fact that the large Heisenberg term restricts the dynamics to the $\mathcal{P}$ manifold, $\hat{H}_H$ acts as an effective $J_z^2$. Therefore, by adding a $\Omega_0 \hat{J}_z$ term via a Raman transition with effective two-photon Rabi frequency $\Omega_0$ one can also implement in such lattice systems Eq. (10). This Hamiltonian allows, as described above, for the squeezed state preparation.

The realization of $J_z^2$ in unit filled optical lattices has the strong advantage that two and three body losses are suppressed and thus they are not longer a limitation in these systems. It is important to note, however, that only fermionic atoms can be used if $\chi$ is going to be tuned by means of a Feshbach resonance. The reason is that for bosons, as $a_{\uparrow\downarrow}$ grows, $\chi$ also grows and at some point the condition $\tilde{\lambda} \gg \tilde{\chi}$ is violated. On the other hand, for fermions, as $a_{\uparrow\downarrow}$ grows, the ratio $\tilde{\lambda}/\tilde{\chi}$ remains constant.

The drawback of the lattice implementation is the factor of $N$ in the denominator of the effective $\chi$, because the slower dynamics limits the sensitivity of our method. Typical experiments that use lattices deep enough for the effective spin model to be valid, work in a parameter regime where $t_\uparrow t_\downarrow / U_{\uparrow\downarrow} \sim 10^{-1}$ kHz - 1 kHz. Using spin dependent optical lattices [23] one can control the tunneling rates and set $(t_\uparrow - t_\downarrow)^2 \sim 0.1 t_\uparrow t_\downarrow$. Additionally close to the resonance $U_{\uparrow\downarrow}$ can be enhanced up to about 10 times its off resonance value [19]. Therefore the revival time in these set-ups varies with the magnetic field between $N \times (10 \text{ ms} - 10^5 \text{ ms})$. Thus, in order to keep the measurement time in a reasonable experimental time scale, one should limit the experiment to 1D lattice systems which have of the order of 20 atoms per tube [24]. The scalability problem certainly limits the achievable sensitivity. Nevertheless, even with these reduced number of atoms, if initially squeezed atoms are used, the phase accuracy $\delta \chi / \chi \propto \delta a_{\uparrow\downarrow} / a_{\uparrow\downarrow} \sim 10^{-2}$ is within the 1% sensitivity required to test the proton-electron mass ratio variation on the level of $10^{-11}$, in a narrow Feshbach resonance ($\sim 1 \text{ mG}$)[6].

VI. CONCLUSIONS

In summary, here we proposed an interferometric method that relays on the quantum dynamics of interacting spins to perform precision measurements of the scattering properties with a resolution fundamentally limited to $N^{-2}$. We discussed the class of squeezed states that are required to achieve Heisenberg sensitivity and proposed a method for its generation. We studied the effect of decoherence and showed that it affects the achievable sensitivity even with initially uncorrelated atoms as they become vulnerable due the many-particle entanglement builded during the many-body dynamics. We also showed that the dynamically induced quantum correlations help to keep the sensitivity of initially squeezed states above the uncorrelated atom threshold even with decoherence. This has to be contrasted with standard Ramsey spectroscopy with non-interacting atoms where all the potential gain in sensitivity due to initial quantum correlation is lost in the presence of decoherence.

Finally we discussed possible physical implementation of the proposed spectroscopy using spinor condensates and fermions in unit filled optical lattices. We showed that particle losses close to the Feshbach resonance limits the sensitiv-
ity in the BEC implementation, and thus the use of very narrow Feshbach resonances and atomic species with a large spin relaxation time are required to optimize the sensitivity of the method. Our implementation of the scheme in optical lattice set-ups is insensitive to atomic losses but on the other is limited by the slow dynamics. Regardless of such problems, the proposed spectroscopy is a new scheme that takes advantages of many-particle entanglement to perform high precision measurements of the interaction parameters in spin mixtures. We are optimistic that emerging technology in optical and magnetic Feshbach resonances might overcome the current experimental limitations in such systems or that new developments in the field might provide alternative set-ups for the experimental application of the presented method and make of it an useful spectroscopic technique.

VII. ACKNOWLEDGEMENTS

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[1] B. Julsgaard, A. Kozhekin and E.S. Polzik, Nature 413, 400 (2001)
[2] D. Leibfried et al., Science 304, 1476 (2004).
[3] D. Leibfried et al., Nature 438, 639-642 (2005).
[4] O. Mandel et al., Nature 425, 937-940 (2003).
[5] A. Widera et al., Phys. Rev. lett 95, 190405 (2005).
[6] C. Chin and V. V. Flambaum Phys. Rev. Lett. 96, 230801 (2006).
[7] W. H. Itano et al., Phys. Rev. A 47, 3554 (1993)
[8] J.J. Bollinger, W.M. Itano and D.J. Wineland Phys. Rev. A 54, R4649 (1996)
[9] A. Sørensen, L.M. Duan, J.I. Cirac and P. Zoller Nature 409, 63 (2001)
[10] S. Boixo, S.T. Flammia, C.M. Caves, and JM Geremia, Phys. Rev. Lett. 98, 090401 (2007)
[11] C. D. Huelga et al. Phys. Rev. Lett. 79, 3865 (1997)
[12] P. Bouyer and M.A. Kasevich Phys. Rev. A 56, R1083 (1997)
[13] A. Andre and M.D. Lukin Phys. Rev. A 65, 053819 (2002)
[14] Mølmer K. and Sørensen A, Phys. Rev. Lett. 82, 1835 (1999)
[15] R.G. Unanyan and M. Fleischhauer Phys. Rev. Lett. 90, 133601 (2003)
[16] D.M. Greenberger, M.A. Horne, A. Shimony and A. Zeilinger Am. J. Phys. 58, 1131 (1990).
[17] S. L. Cornish et al. Phys. Rev. Lett. 85, 1795 (2000).
[18] T. Köhler, E. Tiesinga and P.S. Julienne Phys. Rev. Lett. 94, 020402 (2005)
[19] J.L. Roberts, N.R. Claussen, S.L. Cornish, and C.E. Wieman Phys. Rev. Lett. 85, 728 (2000)
[20] M.P.A. Fisher, P.B. Weichman, G. Grinstein and D.S. Fisher Phys. Rev. B 40, 546-570 (1989).
[21] L.M. Duan, E. Demler, and M.D. Lukin Phys. Rev. Lett. 88, 243602 (2002)
[22] A. M. Rey, L. Jiang, M. Fleischhauer, E. Demler and M.D. Lukin preprint [cond-mat/0703108]
[23] W. V. Liu, F. Wilczek and P. Zoller Phys. Rev. A 70 033603 (2004).
[24] B. Paredes et al., Nature 429, 277 (2004).
[25] Measurements of an operator $\hat{A}$ can be used to determine time with an uncertainty $\sqrt{\langle \Delta \hat{A} \rangle / d\langle \hat{A} \rangle / dt}$
[26] The inequality follows because the maximum eigenvalue of $\hat{J}_z$ is $N^4/16$
[27] The non-degeneracy requires the particle number to be even. In the case of odd particle numbers, the degeneracy of the ground state can be lifted by introducing $\lambda_{x} \hat{J}_z$.
[28] For large $a$ comparable to unity, excitations to states with $M = \pm 4, \pm 6, \cdots$, should also be considered.