Birational classification of pointless del Pezzo surfaces of degree 8

Andrey Trepalin

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Abstract
Let \( k \) be a perfect field. Recently Jean-Louis Colliot-Thélène showed that two pointless quadric surfaces over \( k \) are birationally equivalent if and only if they are isomorphic. We show that this result holds for arbitrary del Pezzo surfaces of degree 8 with the Picard number 1, and describe minimal surfaces birationally equivalent to a given pointless del Pezzo surface of degree 8.

Keywords Del Pezzo surfaces · Quadric surfaces · Brauer group · Sarkisov links

Mathematics Subject Classification 14E05 · 14F22 · 14G27 · 14J45

1 Introduction
Let \( k \) be a perfect field, and \( Q \) be a smooth quadric surface in \( \mathbb{P}^3_k \). If \( k \) is algebraically closed then all smooth quadrics are isomorphic, but if \( k \) is not algebraically closed then two quadrics can be non-isomorphic. The biregular classification of quadric surfaces over nonclosed fields is well-known (see Sect. 2 for the details). Moreover, recently in [4], Colliot-Thélène showed that the birational classification is almost the same.

Note that if there is a \( k \)-point on a quadric surface \( Q \) in \( \mathbb{P}^3_k \), then one can consider a projection from this point to \( \mathbb{P}^2_k \). This projection is a birational map \( Q \dasharrow \mathbb{P}^2_k \). Therefore any quadric with a \( k \)-point is \( k \)-rational. In particular, any two such quadrics are birationally equivalent over \( k \). But if there are no \( k \)-points on a quadric then it is
obviously not \( k \)-rational. The birational classification of pointless quadric surfaces is given in the following theorem.

**Theorem 1.1** ([4]) Let \( k \) be an algebraically nonclosed field such that \( \text{char} \, k \neq 2 \), and let \( Q_1 \) and \( Q_2 \) be two pointless quadrics in \( \mathbb{P}^3_k \). Then \( Q_1 \) is birationally equivalent to \( Q_2 \) if and only if \( Q_1 \cong Q_2 \).

The aim of this paper is to obtain an analogue of Theorem 1.1 for the case of arbitrary pointless del Pezzo surfaces of degree 8. Let us recall the definition of a del Pezzo surface.

**Definition 1.2** A del Pezzo surface is a smooth projective surface \( X \) such that the anticanonical class \( -K_X \) is ample. The number \( d = K^2_X \) is called the degree of a del Pezzo surface \( X \).

Any quadric in \( \mathbb{P}^3_k \) is a del Pezzo surface of degree 8. Moreover, over algebraically closed field any del Pezzo surface of degree 8 is either isomorphic to \( \mathbb{P}^1_k \times \mathbb{P}^1_k \), or isomorphic to the blowup of \( \mathbb{P}^2_k \) at a point. Therefore for a perfect field \( k \) and a del Pezzo surface \( X \) of degree 8 there are two possibilities: either \( X(\mathbb{K}) \neq \emptyset \) then \( X \) is isomorphic to a quadric in \( \mathbb{P}^3_k \). In the latter case there is a unique \((-1)\)-curve on \( X \). This curve is defined over \( k \), therefore we can contract this curve and get a morphism \( X \to \mathbb{P}^2_k \). In particular, one has \( X(\mathbb{K}) \neq \emptyset \). Thus for any pointless del Pezzo surface \( X \) of degree 8 the surface \( \overline{X} \) is isomorphic to \( \mathbb{P}^1_k \times \mathbb{P}^1_k \). If \( X \) is a pointless del Pezzo surface of degree 8 then \( \text{Pic}(\overline{X}) = \text{Pic}(\mathbb{P}^1_k \times \mathbb{P}^1_k) \cong \mathbb{Z}^2 \). Therefore the Picard number \( \rho(X) = \text{rk} \, \text{Pic}(X) \) is equal to 1 or 2, since \( \text{Pic}(X) \subset \text{Pic}(\overline{X}) \).

To establish more results about birational classification of pointless del Pezzo surfaces of degree 8 we need the following definition.

**Definition 1.3** A smooth projective surface \( S \) is called minimal if any birational morphism of smooth surfaces \( S \to S' \) is an isomorphism.

The classification of minimal geometrically rational surfaces is well-known.

**Theorem 1.4** (cf. [8, Theorems 1, 4, 5]) Let \( S \) be a minimal geometrically rational surface. Then either \( S \) admits a conic bundle structure over a conic with \( \rho(S) = 2 \), or \( S \) is a del Pezzo surface with \( \rho(S) = 1 \). In the former case \( K^2_S \notin \{3, 5, 6, 7\} \), and for \( K^2_S = 8 \) the surface \( S \) is not isomorphic to the blowup of \( \mathbb{P}^2_k \) at a point.

Note that any pointless del Pezzo surface \( X \) of degree 8 is minimal. If \( \rho(X) = 1 \) then it is obvious, and if \( \rho(X) = 2 \) then \( X \) is isomorphic to \( C_1 \times C_2 \), where \( C_1 \) and \( C_2 \) are smooth conics, and the projections \( X \to C_1 \) and \( X \to C_2 \) define two structures of conic bundles on \( X \).

We want to study which surfaces are birationally equivalent to pointless del Pezzo surfaces of degree 8. Obviously, for any variety one can equivariantly blow up a collection of points defined over \( k \) and get a birationally equivalent variety. Therefore
for a given pointless del Pezzo surface $X$ of degree 8 the natural problem is to find minimal surfaces birationally equivalent to $X$.

For the case $\rho(X) = 2$ the answer seems to be well-known to experts. For example, many results for this case can be found in [10]. For completeness we prove the following theorem.

**Theorem 1.5** Let $k$ be a perfect field, and let $X$ be a pointless del Pezzo surface of degree 8 such that $\rho(X) = 2$. Then $X$ is a product of two smooth conics and there are two following possibilities.

1. The surface $X$ is isomorphic to $C_1 \times C_2$, where $C_1$ and $C_2$ are two non-trivial smooth conics not isomorphic to each other. If the Brauer product $C_3 = C_1 \ast C_2$ (for the definition see Definition 3.1 or [10, 7]) is defined, then any minimal surface birationally equivalent to $X$ is isomorphic to $C_1 \times C_2$, $C_1 \times C_3$ or $C_2 \times C_3$. Otherwise, any minimal surface birationally equivalent to $X$ is isomorphic to $X$.

2. The surface $X$ is isomorphic to $C \times C$ or $C \times \mathbb{P}^1_k$, where $C$ is a non-trivial smooth conic. Then any minimal surface birationally equivalent to $X$ is isomorphic to $C \times C$, $C \times \mathbb{P}^1_k$ or to a $k$-form of Hirzebruch surface $\mathbb{F}_2$ admitting a conic bundle structure over $C$. Such $k$-form is unique up to isomorphism for a given $k$.

For the case $\rho(X) = 1$ we prove the following theorem.

**Theorem 1.6** Let $k$ be a perfect field, and let $X$ be a pointless del Pezzo surface of degree 8 such that $\rho(X) = 1$. Then any minimal surface birationally equivalent to $X$ is isomorphic to $X$.

A geometrically rational surface $X$ admitting a structure of a conic bundle is called a relatively minimal conic bundle if the Picard number of this surface is 2.

Let $X$ be a minimal del Pezzo surface with $\rho(X) = 1$ or a relatively minimal conic bundle. The surface $X$ is called birationally rigid if for any birational map $X \to X'$, where $X'$ is a minimal del Pezzo surface $X'$ with $\rho(X') = 1$ or a relatively minimal conic bundle, one has $X' \cong X$ (for the precise definition of birational rigidity in general case and some properties see e.g. [3]).

In Sect. 4 we show that in general case a pointless del Pezzo surface $X$ of degree 8 with $\rho(X) = 1$ is birationally equivalent to a (non-minimal) del Pezzo surface of degree 6 admitting a structure of relatively minimal conic bundle. For a given surface $X$ these surfaces are parametrised by quadratic extensions $F/k$ such that $X_F(F) \neq \emptyset$. Therefore $X$ is not birationally rigid.

Let us recall that the index $I(V)$ of a variety $V$ is the greatest common divisor of the degrees of closed points on $V$. We give a description of birationally rigid del Pezzo surfaces of degree 8 in the following theorem.

**Theorem 1.7** Let $k$ be a perfect field, and let $X$ be a del Pezzo surface of degree 8. Then the following assertions are equivalent.

(a) One has $I(X) = 4$.

(b) There are no points of degree 2 on $X$.

(c) The Amitsur subgroup $\text{Am}(X) \subset \text{Br}(k)$ (see Definition 2.4 or [12, Definition 2.8]) contains an element that does not correspond to a conic.
(d) The surface $X$ is birationally rigid.

Actually for any pointless del Pezzo surface $X$ of degree 8 we describe all possible minimal del Pezzo surfaces and relatively minimal conic bundles birationally equivalent to $X$. Note that if $X$ is a minimal del Pezzo surface $X$ with $\rho(X) = 1$ of degree 1, 2 or 3, or a pointless del Pezzo surface of degree 4, then $X$ is birationally rigid (see [9, Theorems 4.4 and 4.5]). On the other hand, by [9, Chapter 4] if $X$ is a del Pezzo surface of degree at least 5 with $\rho(X) = 1$ and $X(k) \neq \emptyset$ or $X \to \mathbb{P}^1_k$ is a conic bundle with $\rho(X) = 2$, $X(k) \neq \emptyset$ and $K^2_X \geq 5$ then $X$ is $k$-rational. In particular, all these surfaces are birationally equivalent to each other.

A non-trivial Severi–Brauer surface (i.e. a pointless del Pezzo surface of degree 9) is not birationally equivalent to any conic bundle, and if a minimal del Pezzo surface $X'$ is birationally equivalent to $X$ then either $X' \cong X$, or $X' \cong X^{\text{op}}$, where $X^{\text{op}}$ is the Severi–Brauer surface such that the central simple algebras corresponding to $X$ and $X^{\text{op}}$ are opposite (see [13, Corollary 2.4 and Theorem 2.10] and [17]).

Note that a geometrically rational surface $X$ with $K^2_X = 7$ is never minimal by [8, Theorem 4], and a del Pezzo surface of degree 5 always has a $k$-point (see [16]).

An interesting problem is to describe all minimal surfaces birationally equivalent to a given del Pezzo surface $X$ with $\rho(X) = 1$, or a conic bundle $X \to C$ with $\rho(X) = 2$. The cases of pointless del Pezzo surfaces of degree 8 and pointless conic bundles with 0 or 2 degenerate fibres are considered in this paper. Therefore the remaining cases are conic bundles with at least four degenerate fibres, pointless del Pezzo surfaces of degree 6 and del Pezzo surfaces of degree 4 with $k$-points. One of the particular questions is whether there exist two birationally equivalent del Pezzo surfaces $X$ and $X'$ of degree 4, such that $X$ and $X'$ are not isomorphic. Note that some useful results about birational classification of minimal conic bundles and minimal del Pezzo surfaces of degree 4 with $k$-points are obtained in [15, Theorem 2.5 and Corollary 3.3].

The plan of this paper is as follows. In Sect. 2 we recall some notions and properties of pointless del Pezzo surfaces of degree 8 and give a definition of Amitsur subgroup $\text{Am}(X) \subset \text{Br}(k)$ (see Definition 2.4), that is a birational invariant. Also we give a description of all geometrically rational surfaces with non-trivial $\text{Am}(X)$, and show how one can restore a del Pezzo surface $X$ of degree 8 with $\rho(X) = 1$ by $\text{Am}(X_L)$, where $L$ is a splitting field of $X$: unique quadratic extension of $k$ such that $\rho(X_L) = 2$.

In Sect. 3 we consider pointless del Pezzo surfaces of degree 8 with the Picard number 2. Any such surface $X$ is isomorphic to a product of two conics. We consider Sarkisov links for these surfaces, describe possibilities for minimal surfaces birationally equivalent to $X$, and show that any other surface is not birationally equivalent to $X$. As a result of this section we prove Theorem 1.5.

In Sect. 4 we consider pointless del Pezzo surfaces of degree 8 with the Picard number 1. We show that for any such surface $X$ any Sarkisov link or sequence of Sarkisov links leads to an isomorphic surface or a certain non-minimal del Pezzo surface of degree 6 obtaining a relatively minimal conic bundle structure. Finally, we prove Theorems 1.6 and 1.7, and give an alternative proof of Theorem 1.1 for the case of a perfect field.
Notation 1.8 Throughout this paper $\mathbb{k}$ is a perfect field, $\overline{\mathbb{k}}$ is its algebraic closure, and the Galois group $\text{Gal}(\overline{\mathbb{k}}/\mathbb{k})$ is denoted by $G_{\mathbb{k}}$. For a surface $X$ we denote $X \otimes \overline{\mathbb{k}}$ by $\overline{X}$. For a surface $X$ we denote the Picard group by $\text{Pic}(X)$. The number $\rho(X) = \text{rk Pic}(X)$ is the Picard number of $X$. If two surfaces $X$ and $Y$ are birationally equivalent then we write $X \approx Y$. If two divisors $A$ and $B$ are linearly equivalent then we write $A \sim B$. The rational ruled (Hirzebruch) surface $\mathbb{P}_{\mathbb{P}^1}^{n}(\mathbb{O} \oplus \mathbb{O}(n))$ is denoted by $F_n$. For a given quadratic extension $\mathbb{L}/\mathbb{k}$ and a conic $C$ over $\mathbb{L}$ we denote by $R_{\mathbb{L}/\mathbb{k}}C$ its Weil restriction of scalars.

2 Preliminaries

In this section we review some results about biregular classification of del Pezzo surfaces of degree 8, and give an alternative description of this classification in terms of the Brauer group.

Throughout this section we assume that $X$ is a del Pezzo surface of degree 8 over a perfect field $\mathbb{k}$, such that $X \approx \mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$. The following lemma gives biregular classification of such surfaces.

Lemma 2.1 (cf. [14, Lemma 3.4 (i and ii)]) Let $\mathbb{k}$ be a perfect field, and let $X$ be a del Pezzo surface of degree 8, such that $X \approx \mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$. The following assertions hold.

(i) The surface $X$ is isomorphic to the product $C_1 \times C_2$ of two conics over $\mathbb{k}$, or to $R_{\mathbb{L}/\mathbb{k}}C$, where $\mathbb{L}$ is a quadratic extension of $\mathbb{k}$ and $C$ is a conic over $\mathbb{L}$. In the former case one has $\rho(X) = 2$, while in the latter $\rho(X) = 1$. Furthermore, in the former case the (non-ordered) pair of conics $\{C_1, C_2\}$ is uniquely determined by $X$; in the latter case the extension $\mathbb{L}/\mathbb{k}$ and the conic $C$ are uniquely determined by $X$ up to conjugation by the Galois group $\text{Gal}(\mathbb{L}/\mathbb{k})$.

(ii) The surface $X$ is isomorphic to a quadric in $\mathbb{P}^3_{\mathbb{k}}$ if and only if $X \approx C \times C$ or $X \approx R_{\mathbb{L}/\mathbb{k}}N_{\mathbb{L}}$, for some conic $N$ over $\mathbb{k}$.

Corollary 2.2 Let $\mathbb{k}$ be a perfect field, and let $X$ be a del Pezzo surface of degree 8 with $\rho(X) = 1$. In this case $X \approx R_{\mathbb{L}/\mathbb{k}}C$. Let $C'$ be a $\text{Gal}(\mathbb{L}/\mathbb{k})$-conjugate conic of $C$. Then $X_{\mathbb{L}} \approx C \times C'$. In particular, $X$ is pointless if and only if $X_{\mathbb{L}}$ is pointless.

For the both cases $\rho(X) = 2$ and $\rho(X) = 1$ we want to give an interpretation of Lemma 2.1 in terms of the Brauer group $\text{Br}(\mathbb{k})$ (for the definition of the Brauer group and properties see, for example [7, Chapter 3]). Let us recall some facts about this group.

Proposition 2.3 ([5, Proposition 5.1]) Let $V$ be a smooth projective geometrically irreducible variety over $\mathbb{k}$. Then there exists an exact sequence

$$0 \to \text{Pic}(V) \to \text{Pic}(\overline{V})^{G_{\mathbb{k}}} \to \text{Br}(\mathbb{k}) \to \text{Br}(\mathbb{k}(V)).$$

Definition 2.4 (see [12, Definition 2.8]) Let $V$ be a smooth projective geometrically irreducible variety over $\mathbb{k}$. The group

$$\text{Am}(V) = \text{Pic}(\overline{V})^{G_{\mathbb{k}}}/\text{Pic}(V) \subset \text{Br}(\mathbb{k})$$
is called Amitsur subgroup of $V$ in $\Br(\mathbb{k})$. 

Note that $\Am(V)$ is a birational invariant since it is the kernel of the map $\Br(\mathbb{k}) \to \Br(\mathbb{k}(V))$ (see [12, Proposition 2.10]).

We want to describe minimal geometrically rational surfaces $S$ with non-trivial $\Am(S)$. These surfaces are described in [5, Proposition 5.3], but we obtain a more detailed description. By Theorem 1.4 the surface $S$ is either a del Pezzo surface with $\rho(S) = 1$, or admits a conic bundle structure over a smooth conic with $\rho(S) = 2$.

Let us start from the case of conic bundle. We say that a conic is non-trivial if it is not isomorphic to $\mathbb{P}^1_k$. Note that for each conic $C$ we can define a class $b(C)$, that is an element in 2-torsion subgroup of the Brauer group $\Br(\mathbb{k})$ (see, for example, [7, Section 3.3]). If two conics are not isomorphic then they have different classes, and $b(C)$ is trivial if and only if $C$ is trivial. Moreover, for a conic $C$ the class $b(C)$ is a generator of $\Am(C)$.

If for a smooth projective geometrically irreducible variety $V$ there exists a map $V \dashrightarrow C$, where $C$ is a smooth conic, then this map induces embeddings $\Pic(C) \hookrightarrow \Pic(V)$ and $\Pic(C)^{G_k} \hookrightarrow \Pic(V)^{G_k}$, since we can consider the preimage of a general geometric point on $C$. Therefore $\Am(C) \subset \Am(V)$. In particular, if $C$ is non-trivial then $\Am(V)$ is non-trivial, since it contains $b(C)$.

Now we can describe conic bundles $S \to C$ with non-trivial $\Am(S)$.

**Proposition 2.5** Let $S$ be a surface admitting a conic bundle structure $S \to C$ over a smooth conic $C$ over a perfect field $\mathbb{k}$ such that $\rho(S) = 2$ and $\Am(S)$ is non-trivial. Then there are five possibilities.

1. The surface $S$ is isomorphic to $C_1 \times C_2$, where $C_1$ and $C_2$ are two non-trivial smooth conics not isomorphic to each other. The group $\Am(S) \cong (\mathbb{Z}/2\mathbb{Z})^2$ is generated by $b(C_1)$ and $b(C_2)$.
2. The surface $S$ is isomorphic to $C \times C$, where $C$ is a non-trivial smooth conic. The group $\Am(S) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by $b(C)$.
3. The surface $S$ is isomorphic to $C \times \mathbb{P}^1_k$, where $C$ is a non-trivial smooth conic. The group $\Am(S) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by $b(C)$.
4. The surface $S$ is a $\mathbb{k}$-form of Hirzebruch surface $\mathbb{F}_{2k}$ admitting a conic bundle structure over a smooth non-trivial conic $C$, where $k$ is a positive integer. The group $\Am(S) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by $b(C)$.
5. The surface $S$ admits a conic bundle structure over a smooth non-trivial conic $C$ with $2k$ degenerate geometric fibres, where $k$ is a positive integer. The group $\Am(S) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by $b(C)$.

**Proof** Note that $\rho(\overline{S})^{G_k} = \rho(S) = 2$. Therefore $\Pic(S)$ is a sublattice of $\Pic(\overline{S})^{G_k}$ of finite index.

We start from the case, when the conic bundle $S \to C$ has a degenerate geometric fibre. Let us show that $\Pic(\overline{S})^{G_k}$ is generated by $-K_{\overline{S}}$ and the class $F$ of a geometric fibre. Otherwise there exists a class $D$ in $\Pic(\overline{S})^{G_k}$ such that $mD \sim -K_{\overline{S}} + nF$. Let $E$ be an irreducible component of a geometric degenerate fibre. Then $E$ is a $(-1)$-curve, and one has $-K_{\overline{S}} \cdot E = 1$ and $F \cdot E = 0$. Therefore

$$m(D \cdot E) = mD \cdot E = (-K_{\overline{S}} + nF) \cdot E = 1.$$
It is possible if and only if \( m = \pm 1 \), but in this case \( D \) lies in the lattice generated by \(-K_S\) and \( F\).

Note that \( \text{Pic}(S) \) contains \(-K_S\) and \( 2F\), since there is a point of degree 2 on \( C\), and a fibre over this point has class \( 2F\). Therefore \( \text{Am}(S) \) is non-trivial if and only if \( \text{Pic}(S) \) does not contain \( F\), and for this case \( \text{Am}(S) \cong \mathbb{Z}/2\mathbb{Z} \) is generated by \( b(C)\). This is possible if and only if \( C \) is a non-trivial conic. Also note that if \( S \to C \) has degenerate fibres over odd number of geometric points, then \( C \cong \mathbb{P}^1_k\), since there is a point of odd degree on \( C\).

Now assume that the conic bundle \( S \to C \) does not have degenerate geometric fibres. Then either \( S \cong \mathbb{P}^1_k \times \mathbb{P}^1_k \) or \( S \) is a \( k\)-form of a Hirzebruch surface \( \mathbb{F}_m\), where \( m \geq 1\). In the latter case there is a unique section \( H \) of \( S \to C \) such that \( H^2 = -m\).

The group \( \text{Pic}(S)^G_k = \text{Pic}(S) \) is generated by the class of \( H \) and a class \( F \) of a geometric fibre. The group \( \text{Pic}(S) \) contains \( 2F \) and the class of \( H \), since \( H \) is unique and therefore defined over \( k\). Therefore \( \text{Am}(S) \) is non-trivial if and only if \( \text{Pic}(S) \) does not contain \( F\), and for this case one has \( \text{Am}(S) \cong \mathbb{Z}/2\mathbb{Z}\). This is possible if and only if \( C \) is a non-trivial conic. Also note that \(-K_S \cong 2H + (m + 2)F\), therefore \( mF \in \text{Pic}(S)\), and if \( \text{Am}(S) \) is non-trivial then \( m \) is even.

Now assume that \( S \cong C_1 \times C_2 \) and \( S \cong \mathbb{P}^1_k \times \mathbb{P}^1_k\). Note that \( \text{Pic}(\mathbb{P}^1_k \times \mathbb{P}^1_k\) is generated by the classes \( A \) and \( B \) of fibres of the projections on the first and the second factors of \( \mathbb{P}^1_k \times \mathbb{P}^1_k\). If \( \rho(S) = 2 \) then \( \text{Pic}(S) \) contains \( 2A \) and \( 2B\). Therefore if \( \text{Am}(S) \) is non-trivial then \( \text{Pic}(S) \) is \( \langle 2A, 2B \rangle, \langle 2A, A + B \rangle, \langle 2A, B \rangle \) or \( \langle A, 2B \rangle\).

If \( \text{Pic}(S) \) contains \( A \) or \( B \) then \( C_1 \) or \( C_2 \) respectively is isomorphic to \( \mathbb{P}^1_k\), and \( S \cong C \times \mathbb{P}^1_k\), where \( C \) is a non-trivial smooth conic. For this case \( \text{Am}(S) \cong \mathbb{Z}/2\mathbb{Z}\) is generated by \( b(C)\).

Let us show that \( \text{Pic}(S) \) contains \( A + B \) if and only if \( S \) is isomorphic to a quadric in \( \mathbb{P}^3_k\). If \( A + B \) lies in \( \text{Pic}(S)\), then the linear system \( |A + B| \) defines an embedding \( S \hookrightarrow \mathbb{P}^3_k\) and the image is a quadric surface. Conversely, if \( S \) is isomorphic to a quadric in \( \mathbb{P}^3_k\) then any hyperplane section of this quadric has class \( A + B \) in \( \text{Pic}(S)\). Therefore for this case we can apply Lemma 2.1 and get that \( S \) is isomorphic to \( C \times C\), where \( C \) is a smooth conic over \( k\). If \( \text{Am}(S) \) is non-trivial then \( C \) is non-trivial, and for this case \( \text{Am}(S) \cong \mathbb{Z}/2\mathbb{Z}\) is generated by \( b(C)\).

If \( \text{Pic}(S) \) does not contain \( A\), \( B \) and \( A + B\), then \( C_1 \) and \( C_2 \) are non-trivial and not isomorphic to each other. For this case \( \text{Am}(S) \cong (\mathbb{Z}/2\mathbb{Z})^2\) is generated by \( b(C_1)\) and \( b(C_2)\). \(\square\)

The surfaces considered in cases (1)–(4) of Proposition 2.5 widely appear in this paper. In the following example we show how to construct a surface considered in case (5). For this aim we slightly modify the construction of an exceptional conic bundle (see [6, Subsection 5.2]).

**Example 2.6** Let \( C \) be a non-trivial conic, \( p_1, \ldots, p_{2k} \) be a collection of \( 2k \) geometric points defined over \( k\), and \( H \to C \) be a double cover of \( C\) branched into this collection of points. Assume that \( \mathbb{F}\) is a minimal field such that all points \( p_i\) are defined over \( \mathbb{F}\). Consider two conjugate geometric points \( q_1 \) and \( q_2 \) on \( \mathbb{P}^1_k\) defined over a quadratic extension \( \mathbb{L}/k\) and not defined over \( \mathbb{F}\).
Let \( \iota_1 \) be the involution of the double cover \( H \to C \), and \( \iota_2 \) be an involution acting on \( \mathbb{P}^1_k \) such that \( \iota_2(q_i) = q_i \). Consider an involution \( \iota \) on \( H \times \mathbb{P}^1_k \) that acts on \( H \) and \( \mathbb{P}^1_k \) as \( \iota_1 \) and \( \iota_2 \) respectively. The quotient \( Y = (H \times \mathbb{P}^1_k)/\langle \iota \rangle \) admits a structure of a bundle \( Y \to C \cong H/\langle \iota_1 \rangle \) such that its general fibre is a smooth conic. The images of fixed points of \( \iota \) are \( A_1 \)-singularities. Therefore there are \( 4k \) singular points on \( Y \) lying in \( 2k \) fibres of \( Y \to C \) over the points \( p_i \).

Let \( \tilde{Y} \to Y \) be the minimal resolution of singularities, and let \( \tilde{Y} \to S \) be the contraction of the proper transforms of the \( 2k \) fibres of \( Y \to C \) over the points \( p_i \). We obtain a conic bundle \( S \to C \) with \( 2k \) degenerate fibres over the points \( p_i \). The components of these fibres are permuted by the group \( \text{Gal}(\mathbb{L}/\mathbb{k}) \), therefore \( S \) is minimal. The group \( \text{Am}(S) \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \) since it is generated by \( b(C) \).

The group \( \text{Am}(S) \) is a birational invariant, therefore it is obvious that any surface described in case (1) of Proposition 2.5 cannot be birationally equivalent to any of surfaces listed in cases (2)–(5). In Sect. 3 for the surfaces listed in cases (2)–(4) we show that two of them \( S_1 \) and \( S_2 \) are birationally equivalent to each other if and only if \( \text{Am}(S_1) = \text{Am}(S_2) \). Surfaces described in case (5) are not birationally equivalent to surfaces from the other cases.

Note that for a given smooth conic \( C \) and even positive integer \( 2k \) the surface \( S \) from case (4) is unique, since it is isomorphic to \( \mathbb{P}(O_C \oplus O_C(-2k)) \).

Now we describe minimal del Pezzo surfaces \( S \) with \( \rho(S) = 1 \) and non-trivial \( \text{Am}(S) \).

**Proposition 2.7** Let \( S \) be a del Pezzo surface over a perfect field \( \mathbb{k} \) such that \( \rho(S) = 1 \) and \( \text{Am}(S) \) is non-trivial. Then there are two possibilities.

1. The surface \( S \) is a non-trivial Severi–Brauer surface, \( K_S^2 = 9 \), and \( \text{Am}(S) \cong \mathbb{Z}/3\mathbb{Z} \).
2. The surface \( S \) is a pointless del Pezzo surface of degree 8 isomorphic to \( R_{\mathbb{L}/\mathbb{k}} C \), where \( \mathbb{L} \) is a quadratic extension of \( \mathbb{k} \) and \( C \) is a conic over \( \mathbb{L} \) such that for any conic \( N \) over \( \mathbb{k} \) the conics \( C \) and \( N_{\mathbb{L}} \) are not isomorphic. One has \( \text{Am}(S) \cong \mathbb{Z}/2\mathbb{Z} \).

**Proof** Note that \( \rho(\overline{S})G_k = \rho(S) = 1 \). Therefore \( \text{Pic}(\overline{S})G_k \) is generated by a class \( D \) such that for a certain number \( k \) one has \( kD = -K_S \), since \( -K_S \) obviously lies in \( \text{Pic}(S) \subset \text{Pic}(\overline{S})G_k \).

Assume that there is a \((-1)\)-curve \( E \) on \( \overline{S} \), then one has

\[
1 = -K_{\overline{S}} \cdot E = kD \cdot E = k(D \cdot E).
\]

Therefore \( k = 1 \), since \( D \cdot E \) is integer. Thus in this case \( \text{Pic}(S) = \text{Pic}(\overline{S})G_k \), and \( \text{Am}(S) \) is trivial.

If there are no \((-1)\)-curves on \( \overline{S} \) then \( \overline{S} \) is either \( \mathbb{P}^2_\mathbb{k} \) or \( \mathbb{P}^1_\mathbb{k} \times \mathbb{P}^1_\mathbb{k} \). In the former case \( S \) is a del Pezzo surface of degree 9, i.e. a Severi–Brauer surface. A trivial Severi–Brauer surface is just \( \mathbb{P}^2_\mathbb{k} \) and \( \text{Am}(\mathbb{P}^2_\mathbb{k}) \) is trivial. For a non-trivial Severi–Brauer surface \( S \) it is well-known that \( \text{Pic}(S) \) is generated by \( -K_S \), and \( \text{Pic}(\overline{S})G_k = \text{Pic}(\mathbb{P}^2_\mathbb{k})G_k = \text{Pic}(\mathbb{P}^2_\mathbb{k}) \) is generated by the class of a line on \( \mathbb{P}^2_\mathbb{k} \). Therefore \( \text{Am}(S) \cong \mathbb{Z}/3\mathbb{Z} \).
If \( \overline{S} \cong \mathbb{P}^1_\mathbb{F}_3 \times \mathbb{P}^1_\mathbb{F}_3 \) then \( K_S^2 = 8 \). As in the case \( \rho(S) = 2 \), the group \( \text{Pic}(\mathbb{P}^1_\mathbb{F}_3 \times \mathbb{P}^1_\mathbb{F}_3) \) is generated by the classes \( A \) and \( B \) of fibres of the projections on the first and the second factors of \( \mathbb{P}^1_\mathbb{F}_3 \times \mathbb{P}^1_\mathbb{F}_3 \), and \( \text{Pic}(S) \) contains \( -K_S \sim 2A + 2B \). If \( \rho(S) = 1 \), then the Galois group \( G_\mathbb{F}_3 \) permutes \( A \) and \( B \), and \( \text{Pic}(\overline{S})^{G_\mathbb{F}_3} \) is generated by the class \( A + B \). Therefore if \( \text{Am}(S) \) is not trivial, then \( \text{Am}(S) \cong \mathbb{Z}/2\mathbb{Z} \) and \( \text{Pic}(S) \) does not contain \( A + B \).

As in the proof of Proposition 2.5 one can easily show that \( \text{Pic}(S) \) contains \( A + B \) if and only if \( S \) is isomorphic to a quadric in \( \mathbb{P}^3_\mathbb{F}_3 \). Therefore for a given del Pezzo surface \( X \) of degree 8 with \( \rho(X) = 1 \) we want to consider more invariants for such surfaces.

### Remark 2.8

It is well-known, that two non-trivial Severi–Brauer surfaces \( S \) and \( S' \) are birationally equivalent if and only if \( S \cong S' \) or \( S' \cong S^{\text{op}} \), where \( S^{\text{op}} \) is the Severi–Brauer surface such that the central simple algebras corresponding to \( S \) and \( S^{\text{op}} \) are opposite (see [1, Corollary 9.5]). Any other minimal surface is not birationally equivalent to \( S \) and \( S' \) by [13, Corollary 2.4 and Theorem 2.10] (see also [17]). Applying Propositions 2.5 and 2.7 we can obtain an alternative proof of this fact.

By Proposition 2.7 a non-trivial Severi–Brauer surface \( S \) has \( \text{Am}(S) \cong \mathbb{Z}/3\mathbb{Z} \) generated by the class \( b(S) \) of \( S \). If a surface \( S' \) is birationally equivalent to \( S \) then \( \text{Am}(S') = \text{Am}(S) \cong \mathbb{Z}/3\mathbb{Z} \). Therefore \( S \) is not birationally equivalent to any relatively minimal conic bundle by Proposition 2.5. If \( S' \) is a minimal del Pezzo surface then \( S' \) is a Severi–Brauer surface by Proposition 2.7, and \( b(S') \) generates \( \text{Am}(S') = \text{Am}(S) \cong \mathbb{Z}/3\mathbb{Z} \). There are two possibilities: either \( b(S') = b(S) \) and \( S' \cong S \), or \( b(S') = -b(S) \) and \( S' \cong S^{\text{op}} \).

Note that non-isomorphic del Pezzo surfaces of degree 8 with \( \rho(X) = 1 \) can have the same group \( \text{Am}(X) \). For example, if \( X \) is a pointless quadric in \( \mathbb{P}^3_\mathbb{F}_3 \) then \( \text{Am}(X) \) is trivial. Therefore we want to consider more invariants for such surfaces.

For a given del Pezzo surface \( X \) of degree 8 with \( \rho(X) = 1 \) there exists a quadratic extension \( \mathbb{L}/\mathbb{F}_3 \), such that \( X_\mathbb{L} \cong C_1 \times C_2 \), where \( C_1 \) and \( C_2 \) are smooth conics over \( \mathbb{L} \). The field \( \mathbb{L} \) is uniquely determined by \( X \), since \( \mathbb{L} = \mathbb{F}_3^K \), where \( K \) is the kernel of the action of \( \text{Gal}(\mathbb{F}_3^*/\mathbb{F}_3) \) on \( \text{Pic}(X) \). For simplicity of notation we will say that \( \mathbb{L} \) is the splitting field of \( X \).

We can consider the group \( \text{Am}(X_\mathbb{L}) \subset \text{Br}(\mathbb{L}) \). This group is obviously a birational invariant. By Lemma 2.1 one has \( X \cong R_{\mathbb{L}/\mathbb{F}_3} \), where \( C \) is a smooth conic over \( \mathbb{L} \). Let \( C' \) be the conjugate conic. Then by Corollary 2.2 one has \( X_\mathbb{L} \cong C \times C' \).

Applying Proposition 2.5 we can see that there are three possibilities for \( \text{Am}(X_\mathbb{L}) \):

- \( \text{Am}(X_\mathbb{L}) \) is trivial if and only if \( C \cong \mathbb{P}^1_\mathbb{F}_3 \);
- \( \text{Am}(X_\mathbb{L}) \cong \mathbb{Z}/2\mathbb{Z} \) if and only if \( C \cong C' \) and \( C \) is non-trivial;
- \( \text{Am}(X_\mathbb{L}) \cong (\mathbb{Z}/2\mathbb{Z})^2 \) if and only if \( C \) and \( C' \) are not isomorphic.

Therefore for a given del Pezzo surface \( X \) of degree 8 with \( \rho(X) = 1 \) we can consider a pair of invariants \((\mathbb{L}, \text{Am}(X_\mathbb{L}))\). We show that this pair is a birational invariant.
Theorem 2.9 Let $X$ and $X'$ be two del Pezzo surfaces of degree 8 with $\rho(X) = \rho(X') = 1$. Then $X$ and $X'$ have the same splitting field $\mathbb{L}$ and $\text{Am}(X_{\mathbb{L}}) = \text{Am}(X'_{\mathbb{L}})$ if and only if $X \cong X'$.

Proof If $X \cong X'$ then they obviously have the same splitting field $\mathbb{L}$ and $\text{Am}(X_{\mathbb{L}}) = \text{Am}(X'_{\mathbb{L}})$.

Now assume that $X$ and $X'$ have the same splitting field $\mathbb{L}$ and $\text{Am}(X_{\mathbb{L}}) = \text{Am}(X'_{\mathbb{L}})$ and show that $X \cong X'$.

Note that $X \cong R_{\mathbb{L}/k}C$, where $b(C) \in \text{Am}(X_{\mathbb{L}})$. For a $\text{Gal}(\mathbb{L}/k)$-conjugate conic $C'$ one has $R_{\mathbb{L}/k}C \cong R_{\mathbb{L}/k}C'$ by Lemma 2.1. Moreover, $X_{\mathbb{L}} \cong C \times C'$ by Corollary 2.2.

If $\text{Am}(X_{\mathbb{L}}) \cong \mathbb{Z}/2\mathbb{Z}$ then $C \cong N_{\mathbb{L}}$, where $N$ is a conic over $k$. Assume that $C$ and $C'$ are not isomorphic. Then $\text{Am}(X_{\mathbb{L}}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. If the third non-trivial element in $\text{Am}(X_{\mathbb{L}})$ corresponds to a conic $C$, then the $\text{Gal}(\mathbb{L}/k)$-conjugate conic is isomorphic to $\widetilde{C}$. Then for $X = R_{\mathbb{L}/k}C$ one has $X_{\mathbb{L}} \cong \widetilde{C} \times \widetilde{C}$, and $\text{Am}(X_{\mathbb{L}}) \cong \mathbb{Z}/2\mathbb{Z}$. Thus if $\text{Am}(X_{\mathbb{L}}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ then we can definitely restore the pair of conjugate conics $C$ and $C'$ such that $X_{\mathbb{L}} \cong C \times C'$ and $X \cong R_{\mathbb{L}/k}C \cong R_{\mathbb{L}/k}C'$.

Therefore in any case $X' \cong R_{\mathbb{L}/k}C \cong X$. □

Question 2.10 Let $C$ be a conic over $\mathbb{L}$, the conic $C'$ be its $\text{Gal}(\mathbb{L}/k)$-conjugate and $X \cong R_{\mathbb{L}/k}C$. Assume that $C$ and $C'$ are not isomorphic. Then $\text{Am}(X_{\mathbb{L}}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. If the third non-trivial element in $\text{Am}(X_{\mathbb{L}})$ corresponds to a conic $C$, how can one describe the surface $R_{\mathbb{L}/k}C$?

In Sect. 4 we show that the pair $(\mathbb{L}, \text{Am}(X_{\mathbb{L}}))$ is also a birational invariant.

Remark 2.11 Actually, the consideration of such pairs was inspired by [14, Lemma 3.4], that originally was obtained to describe the automorphism groups of pointless del Pezzo surfaces of degree 8. For the case $\rho(X) = 1$ we show connection between the groups $\text{Am}(X)$ and $\text{Am}(X_{\mathbb{L}})$ and the structure of the group $\text{Aut}(X)$ in the following table.

| Am $(X)$ | Am $(X_{\mathbb{L}})$ | Quadric | $C$ | Aut $(X)$ |
|----------|----------------------|---------|-----|----------|
| (id)     | (id)                 | Yes     | $C \cong \mathbb{P}^1_{\mathbb{L}}$ | $\text{PGL}_2(\mathbb{L}) \times (\mathbb{Z}/2\mathbb{Z})$ |
| (id)     | $\mathbb{Z}/2\mathbb{Z}$ | Yes     | $C \cong N_{\mathbb{L}}$, where $N$ is a conic over $k$ | $\text{Aut}(C) \times (\mathbb{Z}/2\mathbb{Z})$ |
| $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | No      | $C \cong C'$ is not isomorphic to $N_{\mathbb{L}}$ | $\text{Aut}(X)/\text{Aut}(C) \cong \mathbb{Z}/2\mathbb{Z}$ |
| $\mathbb{Z}/2\mathbb{Z}$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | No      | $C$ is not isomorphic to $C'$ | $\text{Aut}(X) \cong \text{Aut}(C)$ |

In the third column we give an answer to the question whether the surface $X$ is isomorphic to a conic in $\mathbb{P}^3_{k}$. In the fourth row the group $\text{Aut}(X)$ is a non-split
extension of $\text{Aut}(C)$ by $\mathbb{Z}/2\mathbb{Z}$, that means that $\text{Aut}(X)$ contains a normal subgroup $\text{Aut}(C)$ of index 2 but is not isomorphic to any semi-direct product $\text{Aut}(C) \rtimes (\mathbb{Z}/2\mathbb{Z})$.

3 Sarkisov links for the case $\rho(X) = 2$

In this section we consider Sarkisov links for pointless del Pezzo surfaces of degree 8 with the Picard number 2, and prove Theorem 1.5.

Let $X$ be a pointless del Pezzo surface of degree 8 with the Picard number 2. By Lemma 2.1 in this case $X$ is birationally equivalent to $C$ with the Picard number 2, and prove Theorem 1.5.

In this section we consider Sarkisov links for pointless del Pezzo surfaces of degree 8 by extension of $\text{Aut}(C)$ by $\mathbb{Z}/2\mathbb{Z}$, that means that $\text{Aut}(X)$ contains a normal subgroup $\text{Aut}(C)$ of index 2 but is not isomorphic to any semi-direct product $\text{Aut}(C) \rtimes (\mathbb{Z}/2\mathbb{Z})$.

3 Sarkisov links for the case $\rho(X) = 2$

In this section we consider Sarkisov links for pointless del Pezzo surfaces of degree 8 with the Picard number 2, and prove Theorem 1.5.

Let $X$ be a pointless del Pezzo surface of degree 8 with the Picard number 2. By Lemma 2.1 in this case $X$ is isomorphic to a product of two smooth conics, such that at least one of these conics is non-trivial. Therefore by Proposition 2.5 either $\text{Am}(X) \cong (\mathbb{Z}/2\mathbb{Z})^2$, or $\text{Am}(X) \cong \mathbb{Z}/2\mathbb{Z}$.

We start from the case $\text{Am}(X) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Assume that there exists a minimal surface $X'$, and a birational map $X \dasharrow X'$. Then $\text{Am}(X') = \text{Am}(X) \cong (\mathbb{Z}/2\mathbb{Z})^2$. By Propositions 2.5 and 2.7 it is possible only if $X' \cong C_1' \times C_2'$, where $C_1'$ and $C_2'$ are two smooth non-trivial non-isomorphic conics. Moreover, [10, Theorem 2] implies that if for a minimal surface $X' \cong C_1' \times C_2'$ one has $\text{Am}(X') = \text{Am}(X) \cong (\mathbb{Z}/2\mathbb{Z})^2$, then $X'$ is birationally equivalent to $X$.

Actually we want to remind some results and constructions from [10] to describe birational maps from $X$ more explicitly.

Definition 3.1 Let $C_1$ and $C_2$ be two smooth conics, and $b(C_1)$ and $b(C_2)$ be the classes of these conics in $\text{Br}(k)$. If there exists a conic $C_3$ such that $b(C_3) = b(C_1) + b(C_2)$, then $C_3$ is called the Brauer product of $C_1$ and $C_2$. We denote the Brauer product of $C_1$ and $C_2$ by $C_1 \ast C_2$.

We want to show that $C_1 \ast C_2$ is defined if and only if there exists a quadratic extension $F/k$ such that $C_1 \otimes F \cong C_2 \otimes F \cong \mathbb{P}^1_F$. In the other words, there are points of degree 2 on $C_1$ and $C_2$ such that the corresponding geometric points are defined over $F$. Note that this condition holds if $C_1 \cong C_2$, or if $C_1$ or $C_2$ is trivial.

Lemma 3.2 Let $C_1$ and $C_2$ be two smooth conics. Suppose that there exist points of degree 2 on $C_1$ and $C_2$ such that the corresponding geometric points are defined over a quadratic extension $F/k$. Then there exists a conic $C_3 = C_1 \ast C_2$, and $X \cong C_1 \times C_2$ is birationally equivalent to $C_1 \ast C_3$ and $C_2 \ast C_3$.

Proof Let $\pi_1: X \rightarrow C_1$ and $\pi_2: X \rightarrow C_2$ be the projections. Let $A$ and $B$ be the classes in $\text{Pic}(X)$ of geometric fibres of $\pi_1$ and $\pi_2$ respectively. Let $A_1$ and $A_2$ be conjugate fibres of $\pi_1$ defined over $F$, and $B_1$ and $B_2$ be conjugate fibres of $\pi_2$ defined over $F$. Then the pair of points $A_1 \cap B_1$ and $A_2 \cap B_2$ are defined over $F$. We can equivariantly blow up this pair of points, contract the proper transforms of $A_1$ and $A_2$, and get a surface $X'$.

Note that the birational map $X \dasharrow X'$ respects the projection $\pi_1: X \rightarrow C_1$. Therefore $X'$ admits a structure of a conic bundle over $C_1$. Moreover, $K_{X'}^2 = K_X^2 = 8$, therefore $X'$ is a $k$-form of a Hirzebruch surface $\mathbb{F}_n$. Also note that the proper transforms of $B_1$ and $B_2$ are sections of the conic bundle $X' \rightarrow C_1$ with selfintersection number 0. For $n > 0$ there are no sections of $\mathbb{F}_n \rightarrow C_1$ with selfintersection number 0. Thus $X' \cong \mathbb{F}_0$ that is a product of two conics.
One of these conics is \( C_1 \), since \( X' \) admits a structure of a conic bundle over \( C_1 \). Denote the other conic by \( C_3 \). Note that preimage of a geometric point on \( C_3 \) under the sequence of maps \( X \rightarrow X' \rightarrow C_3 \) has class \( A + B \) in \( \text{Pic}(X) \). Therefore \( b(C_3) \in \text{Br}(k) \) is \( b(C_1) + b(C_2) \), since the preimages of geometric points on \( C_1 \) and \( C_2 \) under the projections \( \pi_1 \) and \( \pi_2 \) have classes \( A \) and \( B \) in \( \text{Pic}(X) \) respectively. Thus \( C_3 = C_1 \times C_2 \), and \( X \cong C_1 \times C_2 \) is birationally equivalent to \( X' \cong C_1 \times C_3 \). In the same way we can show that \( X \) is birationally equivalent to \( C_2 \times C_3 \).

The following lemma directly follows from the proof of [10, Lemma 8].

**Lemma 3.3** Let \( C_1 \) and \( C_2 \) be two smooth conics such that for any pair of points of degree 2 on \( C_1 \) and \( C_2 \) the corresponding geometric points are not defined over any quadratic extension \( \mathbb{F}/k \). Then a conic with class \( b(C_1) + b(C_2) \) does not exist.

One can find an example of such pair of conics in [2, Example 5.4].

Now consider the case \( \text{Am}(X) \cong \mathbb{Z}/2\mathbb{Z} \). By Proposition 2.5 the surface \( X \) is isomorphic either to \( C \times \mathbb{C} \), or to \( C \times \mathbb{P}^1_k \) where \( C \) is a smooth non-trivial conic. Moreover, by Lemma 3.2 for a given smooth non-trivial conic \( C \) the surfaces \( C \times C \) and \( C \times \mathbb{P}^1_k \) are birationally equivalent. In the following lemma we show that these surfaces are birationally equivalent to certain \( k \)-forms of Hirzebruch surfaces.

**Lemma 3.4** Let \( C \) be a smooth non-trivial conic. Then the surfaces \( C \times C \), \( C \times \mathbb{P}^1_k \), and \( k \)-forms of Hirzebruch surfaces \( \mathbb{F}_{2k} \) admitting a structure of conic bundle over \( C \) are birationally equivalent.

**Proof** The surfaces \( C \times C \) and \( C \times \mathbb{P}^1_k \) are birationally equivalent by Lemma 3.2, since \( C \times C \cong \mathbb{P}^1_k \).

Now consider \( X \cong C \times \mathbb{P}^1_k \). Let \( \pi_1 : X \rightarrow C \) and \( \pi_2 : X \rightarrow \mathbb{P}^1_k \) be the projections. Then for a \( k \)-point \( p \in \mathbb{P}^1_k \), the preimage \( B = \pi_2^{-1}(p) \) is a section of \( \pi_2 \). Let \( A_1 \) and \( A_2 \) be conjugate fibres of \( \pi_1 \). Then the points \( A_1 \cap B \) and \( A_2 \cap B \) are conjugate. We can equivariantly blow up this pair of points, contract the proper transforms of \( A_1 \) and \( A_2 \), and get a surface \( X' \).

The birational map \( X \rightarrow X' \) respects the projection \( \pi_1 : X \rightarrow C \). Therefore \( X' \) admits a structure of a conic bundle over \( C \). Moreover, \( K_{X'}^2 = K_X^2 = 8 \), therefore \( X' \) is a \( k \)-form of a Hirzebruch surface \( \mathbb{F}_n \). Note that the proper transform of \( B \) is a section of the conic bundle \( X' \rightarrow C \) with selfintersection number \(-2 \). Therefore \( n = 2 \). Thus \( C \times \mathbb{P}^1_k \) is birationally equivalent to a \( k \)-form of a Hirzebruch surface \( \mathbb{F}_2 \) admitting a structure of a conic bundle over \( C \).

Now consider a \( k \)-form \( S \) of a Hirzebruch surface \( \mathbb{F}_{2k} \) admitting a structure of a conic bundle \( \pi : S \rightarrow C \). There is a unique section \( H \) of \( \pi \) such that \( H^2 = -2k \). Let \( A_1 \) and \( A_2 \) be conjugate fibres of \( \pi \). Then the points \( A_1 \cap H \) and \( A_2 \cap H \) are conjugate. We can equivariantly blow up this pair of points, contract the proper transforms of \( A_1 \) and \( A_2 \), and get a surface \( S' \).

The birational map \( S \rightarrow S' \) respects the projection \( \pi : S \rightarrow C \). Therefore \( S' \) admits a structure of a conic bundle over \( C \). Moreover, \( K_{S'}^2 = K_S^2 = 8 \), therefore \( S' \) is a \( k \)-form of a Hirzebruch surface \( \mathbb{F}_n \). Note that the proper transform of \( H \) is a section of the conic bundle \( S' \rightarrow C \) with selfintersection number \(-2k - 2 \). Therefore
$n = 2k + 2$. Thus $S$ is birationally equivalent to a $\mathbb{k}$-form of a Hirzebruch surface $\mathbb{F}_{2k+2}$ admitting a structure of a conic bundle over $C$.

Thus we see that all surfaces listed in Lemma 3.4 are birationally equivalent. $\square$ $\square$

Now we prove Theorem 1.5.

**Proof of Theorem 1.5** Let $X$ be a pointless del Pezzo surface of degree 8 with $\rho(X) = 2$. Then $\overline{X} \cong \mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$, and $X$ is a product of two conics. If $\text{Am}(X)$ is trivial then $X \cong \mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$ and $X(\mathbb{k}) \neq \emptyset$. Therefore $\text{Am}(X)$ is non-trivial, and by Proposition 2.5 either $\text{Am}(X) \cong (\mathbb{Z}/2\mathbb{Z})^2$, or $\text{Am}(X) \cong \mathbb{Z}/2\mathbb{Z}$.

If $\text{Am}(X) \cong (\mathbb{Z}/2\mathbb{Z})^2$ then $X$ is a product of two non-trivial smooth conics $C_1$ and $C_2$, not isomorphic to each other. If the Brauer product $C_3 = C_1 \ast C_2$ is defined then $X \cong C_1 \times C_3 \cong C_2 \times C_3$ by Lemma 3.2.

If a minimal surface $X'$ is birationally equivalent to $X$ then

$$\text{Am}(X') = \text{Am}(X) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$  

By Propositions 2.5 and 2.7 this is possible only if $X'$ is a product of two smooth conics $C'_1$ and $C'_2$. Moreover, $b(C'_1)$ and $b(C'_2)$ are distinct non-trivial elements of $\text{Am}(X)$. Therefore if $C_3$ is defined then $X'$ is isomorphic to $C_1 \times C_2$, $C_1 \times C_2$ or $C_2 \times C_3$, and otherwise $X' \cong C_1 \times C_2 \cong X$. We have proved Theorem 1.5 (1).

If $\text{Am}(X) \cong \mathbb{Z}/2\mathbb{Z}$ then either $X \cong C \times C$, or $X \cong C \times \mathbb{P}^1_{\mathbb{k}}$, where $C$ is a non-trivial smooth conic. By Lemma 3.4 the surface $X$ is birationally equivalent to $C \times C$, $C \times \mathbb{P}^1_{\mathbb{k}}$, and $\mathbb{k}$-forms of Hirzebruch surfaces $\mathbb{F}_{2k}$ admitting a structure of a conic bundle over $C$.

Assume that a minimal surface $X'$ is birationally equivalent to $X$ and not listed before. Then the map $X \rightarrow X'$ can be decomposed into a sequence of Sarkisov links:

$$X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow X_n = X'.$$

In particular there exists a Sarkisov link $f: S \rightarrow S'$ such that $S$ is listed in Lemma 3.4, and $S'$ is not listed. For geometrically rational surfaces Sarkisov links are described in [9, Theorem 2.6]. All surfaces listed in Lemma 3.4 admit a structure of a conic bundle over $C$. Therefore $f$ has type II, III or IV. Links of types II and IV preserve $K^2_S$ and $\text{Am}(S)$. Therefore in this case $S'$ is one of the surfaces listed in Proposition 2.5 (2)–(4), and also in Lemma 3.4. For $S$ with $\rho(S) = 2$ and $K^2_S = 8$ the only possible link of type III is a contraction of $(-1)$-section on a $\mathbb{k}$-form of a Hirzebruch surface $\mathbb{F}_1$. But such surface is not listed in Lemma 3.4, therefore this link is impossible, and $X'$ must be isomorphic to a surface listed in Lemma 3.4. We have proved Theorem 1.5 (2). $\square$

4 Sarkisov links for the case $\rho(X) = 1$  

In this section we consider Sarkisov links for pointless del Pezzo surfaces of degree 8 with the Picard number 1, and prove Theorem 1.6.
Let $X$ be a pointless del Pezzo surface of degree 8 with the Picard number 1. By Lemma 2.1 in this case $X$ is isomorphic to $R_{L/k}C$, where $L$ is a quadratic extension of $k$ and $C$ is a conic over $L$. The field $L$ is uniquely determined by $X$. The pair $(L, \text{Am}(X_L))$ is a biregular invariant of $X$ by Theorem 2.9. We want to find possibilities for minimal surfaces $X'$ birationally equivalent to $X$.

Note that by Corollary 2.2 one has $X_L \cong C \times C'$, where $C'$ is a conic conjugate to $C$ under the action of $\text{Gal}(L/k)$. Note that $X$ is pointless if and only if $X_L$ is pointless too. In particular, if $X$ is pointless then $\text{Am}(X_L)$ is non-trivial.

**Lemma 4.1** Let $X$ be a pointless del Pezzo surface of degree 8 with $\rho(X) = 1$. Then any point on $X$ has even degree.

**Proof** Assume that there exists a point of odd degree on $X$. Then there exists a point of odd degree on $X_L$, and images of this point under the projection on $C$ and $C'$ are points of odd degree. There are no points of odd degree on any non-trivial conic. Therefore $C \cong C' \cong \mathbb{P}^1_k$, and thus $X(k) \neq \emptyset$.

We obtain a contradiction. Therefore any point on $X$ has even degree. \hfill \Box

Now we want to consider Sarkisov links for the surface $X$. These links are described in [9, Theorem 2.6 (i and ii)]. In our case such a link must have type I or II, since $X$ is a del Pezzo surface with $\rho(X) = 1$. Any link of type I is a blowup $X_1 \to X$ of a point of degree $d$, where $X_1$ has a structure of a conic bundle over a smooth conic and $\rho(X_1) = 2$. Any link of type II is a composition $\sigma_1 \circ \sigma^{-1}$, where $\sigma : Z \to X$ is a blowup of a point of degree $d$, and $\sigma_1 : Z \to X_1$ is a contraction of the set of disjoint conjugate $(-1)$-curves, that differs from the exceptional divisor of $\sigma$. In our case $d = 2$, $d = 4$ or $d = 6$ by Lemma 4.1. For $d = 2$ the corresponding link has type I, and for $d = 4$ and $d = 6$ the corresponding link has type II. For each of these three links we want to describe $X_1$.

**Lemma 4.2** Let $X$ be a del Pezzo surface of degree 8 with $\rho(X) = 1$. Let $X \dashrightarrow X_1$ be a Sarkisov link corresponding to the blowup of a point of degree 6. Then $X_1 \cong X$.

**Proof** This case is considered in [9, Theorem 2.6 (ii), $K_X^2 = 8$, $d = 6$], and there is explicitly written that $X_1 \cong X$. \hfill \Box

**Lemma 4.3** Let $X$ be a del Pezzo surface of degree 8 with $\rho(X) = 1$. Let $X \dashrightarrow X_1$ be a Sarkisov link corresponding to the blowup of a point of degree 4. Then $X_1 \cong X$.

**Proof** This case is considered in [9, Theorem 2.6 (ii), $K_X^2 = 8$, $d = 4$], and for this case $K_{X_1}^2 = 8$ and $\rho(X_1) = 1$. Therefore by Lemma 2.1 one has $R_{L_1/k}C_1$, where $L_1$ is the splitting field of $X_1$ and $C_1$ is a smooth conic over $L_1$. Let us show that $L_1 = L$. It is sufficient to prove that $\rho((X_1)_{L_1}) = 2$.

Let $\pi_1 : X_L \to C$ and $\pi_2 : X_L \to C'$ be the projections. Let $A$ and $B$ be the classes in $\text{Pic}(X)$ of geometric fibres of $\pi_1$ and $\pi_2$ respectively. The map $X \dashrightarrow X_1$ is a composition $\sigma_1 \circ \sigma^{-1}$, where $\sigma : Z \to X$ is a blowup of a point of degree 4 and $\sigma_1 : Z \to X_1$ is a contraction of the set of disjoint conjugate $(-1)$-curves. Let $E_1$, $E_2$, $E_3$ and $E_4$ be the classes in $\text{Pic}(Z)$ of geometrically irreducible components of the exceptional divisor of $\sigma$.
The surface $Z$ is a del Pezzo surface of degree 4, and thus there are 16 ($-1$)-curves on $Z$, having classes $E_i$, $A - E_i$, $B - E_i$ and $A + B + E_i - \sum_{j=1}^{4} E_j$ in $\text{Pic}(\overline{Z})$. The four ($-1$)-curves with classes $A + B + E_i - \sum_{j=1}^{4} E_j$ are conjugate and disjoint, therefore $\sigma_1 : Z \to X_1$ is the contraction of these curves.

Let $k$ be the number of $\text{Gal}(\overline{k}/\mathbb{L})$-orbits on the set $E_i$ (actually $k = 1$ or $k = 2$). Then the number of $\text{Gal}(\overline{k}/\mathbb{L})$-orbits on the set $A + B + E_i - \sum_{j=1}^{4} E_j$ is also $k$, since $A + B - \sum_{j=1}^{4} E_j$ is $\text{Gal}(\overline{k}/\mathbb{L})$-invariant. Therefore

$$\rho((X_1)_L) = \rho(Z_L) - k = \rho(X_L) = 2.$$ 

The del Pezzo surfaces $X$ and $X_1$ of degree 8 with $\rho(X) = \rho(X_1) = 1$ have the same splitting field and $\text{Am}(X_L) = \text{Am}((X_1)_L)$, since $X \approx X_1$. Therefore $X \cong X_1$ by Theorem 2.9. \hfill \Box

**Remark 4.4** Note that in Lemmas 4.2 and 4.3 the surface $X$ is not necessary pointless.

The link of type I corresponding to the blowup $Z \to X$ of two conjugate geometric points is considered in [9, Theorem 2.6 (i), $K^2_X = 8$]. In this case $Z$ admits a structure of a conic bundle $\pi : Z \to C$ with two degenerate geometric fibres, where $C$ is a conic such that $b(C)$ is the generator of $\text{Am}(X)$ (in particular, $C \cong \mathbb{P}^1_k$ if $\text{Am}(X)$ is trivial).

In the following proposition we collect some facts about such conic bundles following [8, 9].

**Proposition 4.5** Let $\pi : S \to C$ be a conic bundle over a smooth conic $C$ with two degenerate geometric fibres such that $\rho(S) = 2$. Then $K^2_S = 6$, and $S$ has the following properties.

(i) The surface $S$ is a del Pezzo surface of degree 6 (see [8, Theorem 5]).

(ii) There are six ($-1$)-curves on $S$: four components of the singular fibres, and two disjoint conjugate sections $E_1$ and $E_2$ of $\pi$. These curves form a hexagon.

(iii) The surface $S$ is not minimal (see [8, Theorem 4]).

(iv) There exists a unique link $S \to Y$ of type III that is the contraction of the pair $E_1$ and $E_2$. The surface $Y$ is a del Pezzo surface of degree 8 with $\rho(Y) = 1$ (see [9, Theorem 2.6 (iii)]).

(v) Any other link $S \to S_1$ has type II, and preserves a structure of a conic bundle over $C$ (see [9, Theorem 2.6]). Such link is a composition $\sigma_1 \circ \sigma^{-1}$, where $\sigma$ is a blowup of a point of degree $d$ such that the corresponding geometric points do not lie on the degenerate fibres, and $\sigma_1$ is the contraction of the proper transforms of $d$ fibres containing these points. In particular, the conic bundles $S \to C$ and $S_1 \to C$ have degenerate fibres over the same points on $C$, $\rho(S_1) = 2$ and $K^2_{S_1} = 6$. Therefore all described above properties hold for $S_1$.

Now we can consider the link of type I.

**Lemma 4.6** Let $X$ be a pointless del Pezzo surface of degree 8 with $\rho(X) = 1$. Consider a sequence of Sarkisov links

$$X \leftarrow Z = Z_0 \dashrightarrow Z_1 \dashrightarrow \cdots \dashrightarrow Z_{n-1} \dashrightarrow Z_n = Z' \to X'.$$
where the first link has type I, the last link has type III, and all other links have type II. Then $X' \cong X$.

**Proof** The surface $Z$ admits a structure of a conic bundle over a smooth conic $\pi : Z \rightarrow C$ with two degenerate fibres. Denote the components of these fibres by $A_1$, $A_2$, $B_1$ and $B_2$. These components are $(-1)$-curves on $Z$. By Proposition 4.5(ii) there are two disjoint conjugate sections $E_1$ and $E_2$ of $\pi$ that are $(-1)$-curves. Without loss of generality we can assume that

$$A_1 \cdot E_1 = E_1 \cdot B_1 = B_1 \cdot A_2 = A_2 \cdot E_2 = E_2 \cdot B_2 = B_2 \cdot A_1 = 1.$$ 

Let $\Gamma$ be the kernel of the action of $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ on the set of $(-1)$-curves on $Z$, and $\mathbb{F} = \overline{\mathbb{F}}/\Gamma$. Then $\rho(Z_{\mathbb{F}}) = \rho(\overline{Z}) = 4$, and $\rho(X_{\mathbb{F}}) = 2$. Therefore $\mathbb{L}$ is a subfield of $\mathbb{F}$. Note that the pairs $A_1$ and $A_2$, $B_1$ and $B_2$, and $E_1$ and $E_2$ are defined over $\mathbb{L}$, but no one of these curves is defined over $\mathbb{L}$, since $X_{\mathbb{L}}$ is pointless. Therefore the group $\text{Gal}(\mathbb{F}/\mathbb{L})$ has order 2 and pairwisely permutes $E_1$ and $E_2$, $A_1$ and $A_2$, and $B_1$ and $B_2$.

By Proposition 4.5(v) all surfaces $Z_i$ are del Pezzo surfaces of degree 6 admitting a structure of a conic bundle $Z_i \rightarrow C$ with two degenerate fibres. Let $f : Z \rightarrow Z'$ be the composition of the Sarkisov links $Z_i \rightarrow Z_{i+1}$. Then $A'_1 = f(A_1)$, $A'_2 = f(A_2)$, $B'_1 = f(B_1)$ and $B'_2 = f(B_2)$ are components of the degenerate fibres of $Z' \rightarrow C$. By Proposition 4.5(ii) there are two disjoint conjugate sections $E'_1$ and $E'_2$ of $\pi' : Z' \rightarrow C$ that are $(-1)$-curves. We can assume that $A'_1 \cdot E'_1 = 1$, and want to show that $E'_1 \cdot B'_1 = 1$.

Let $R = f^{-1}_w(E'_1)$ be the proper transform of $E'_1$ on $Z$. If we blow up a point $p$ on $\overline{Z}$ and contract a proper transform of the fibre containing $p$, then for the proper transform $R$ of $R$ one has $R^2 - R^2 = \pm 1$ ($-1$ if $p \in R$, and 1 otherwise). By Proposition 4.5(v) any link $Z_i \rightarrow Z_{i+1}$ is a composition $\sigma_1 \circ \sigma^{-1}$, where $\sigma$ is a blowup of a point of degree $d$ such that the corresponding geometric points do not lie on degenerate fibres, and $\sigma_1$ is the contraction of the proper transforms of $d$ fibres containing these points. Moreover, $d$ is even, since otherwise there is a point of odd degree on $X$ that is impossible by Lemma 4.1. Thus $R^2 - (E'_1)^2$ is even, and $R^2$ is odd.

Note that the group $\text{Pic}(\overline{Z}) \cong \mathbb{Z}^4$ is generated by $A_1$, $E_1$, $B_1$ and the class of a fibre $F \sim A_1 + B_2 \sim A_2 + B_1$. Therefore $R \sim aA_1 + eE_1 + bB_1 + fF$. The curve $R$ is a section of $\pi : Z \rightarrow C$, thus $e = 1$, since $F^2 = F \cdot A_1 = F \cdot B_1 = 0$ and $F \cdot E_1 = 1$. One has $A_1 \cdot R = A'_1 \cdot E'_1 = 1$, thus $a = 0$, since $A_1 \cdot B_1 = A_1 \cdot F = 0$, $A_1 \cdot E_1 = 1$ and $A'_2 = -1$. Therefore $R \sim E_1 + bB_1 + fF$, and

$$R^2 = E_1^2 + b^2B_1^2 + f^2F^2 + 2bE_1 \cdot B_1 + 2fE_1 \cdot F + 2bfB_1 \cdot F = -1 - b^2 + 2b + 2f.$$ 

The number $b$ is even, since the intersection number $R^2$ is odd. One has

$$B_1 \cdot R = B_1 \cdot (E_1 + bB_1 + fF) = 1 - b.$$ 

Therefore $b = 0$, since $b$ is even and $B_1 \cdot R \geq 0$. Thus $E'_1 \cdot B'_1 = R \cdot B_1 = 1$. 

\[ Springer\]
Now we see that for the set of $(-1)$-curves on $Z'$ we have
\[ A'_1 \cdot E'_1 = E'_1 \cdot B'_1 = B'_1 \cdot A'_2 = A'_2 \cdot E'_2 = E'_2 \cdot B'_2 = B'_2 \cdot A'_1 = 1. \]
The group $\text{Gal}(\mathbb{F}/\mathbb{L})$ has order 2 and pairwisely permutes $A'_1$ and $A'_2$, $B'_1$ and $B'_2$, and also $E'_1$ and $E'_2$. Therefore $\rho(Z'_L) = 3$. By Proposition 4.5(iv) the map $Z' \to X'$ is the contraction of the disjoint conjugate curves $E'_1$ and $E'_2$. Each of those curves is not defined over $\mathbb{L}$, therefore $\rho(X'_L) = 2$. It means that $\mathbb{L}$ is the splitting field of $X'$. One has $\text{Am}(X_L) = \text{Am}(X'_L)$, since $X \approx X'$. Therefore $X \cong X'$ by Theorem 2.9. □

**Remark 4.7** Note that Lemma 4.6 does not work if $X(\mathbb{k}) \neq \emptyset$. Consider a sequence of Sarkisov links
\[ X \leftarrow Z \rightarrow Z' \rightarrow X', \]
where the first link has type I and the last link has type III. The second link corresponds to the blowup of a $\mathbb{k}$-point $p$ on a smooth fibre of $Z \to C$ and the contraction of the proper transform of the fibre passing through $p$, and has type II. Let us show that $X$ and $X'$ are not isomorphic to each other. We use the notation established in the proof of Lemma 4.6 for the $(-1)$-curves on $Z$ and $Z'$. We want to show that $E'_1 \cdot B'_1 = 0$.

Let $R$ be the proper transform of $E'_1$ on $Z$. Then $R^2$ is even, and $R \sim E_1 + bB_1 + fF$. Thus
\[
R^2 = E_1^2 + b^2B_1^2 + f^2F^2 + 2bE_1 \cdot B_1 + 2fE_1 \cdot F + 2bfB_1 \cdot F = -1 - b^2 + 2b + 2f.
\]
The number $b$ is odd, since the intersection number $R^2$ is even. One has
\[ B_1 \cdot R = B_1 \cdot (E_1 + bB_1 + fF) = 1 - b. \]
Therefore $b = 1$, since $b$ is odd and $B_1 \cdot R \geq 0$. Thus $E'_1 \cdot B'_1 = R \cdot B_1 = 0$.

Now we see that for the set of $(-1)$-curves on $Z'$ we have
\[ A'_1 \cdot E'_1 = E'_1 \cdot A'_2 = A'_2 \cdot B'_1 = B'_1 \cdot E'_2 = E'_2 \cdot B'_2 = B'_2 \cdot A'_1 = 1. \]
The group $\text{Gal}(\mathbb{F}/\mathbb{L})$ has order 2 and pairwisely permutes $A'_1$ and $A'_2$, $B'_1$ and $B'_2$, and the $(-1)$-curves $E'_1$ and $E'_2$ are defined over $\mathbb{L}$. Therefore $\rho(Z'_L) = 3$. By Proposition 4.5(iv) the map $Z' \to X'$ is the contraction of the disjoint conjugate curves $E'_1$ and $E'_2$. Each of those curves is defined over $\mathbb{L}$, therefore $\rho(X'_L) = 1$, that means that $\mathbb{L}$ is not the splitting field of $X'$. Hence $X$ and $X'$ are not isomorphic to each other by Theorem 2.9.

As a by-product of Lemma 4.6 we obtain the following proposition.

**Proposition 4.8** Let $Z$ be a pointless del Pezzo surface of degree 6 admitting a structure of a relatively minimal conic bundle $Z \to C$, and $Z \dashrightarrow Z_1$ be a Sarkisov link of type II. Then $Z \cong Z_1$. 
We decompose the proof into several lemmas that are interesting themselves.

**Lemma 4.9** Let $C$ be a conic over a perfect field $\mathbb{L}$. Then the group $\text{Aut}(C)$ transitively acts on the set of points of degree 2 such that the corresponding geometric points are defined over a quadratic extension $\mathbb{L}(\zeta)$.

**Proof** Assume that $\zeta$ satisfies the equation $\zeta^2 + a\zeta + b = 0$, and $\bar{\zeta}$ be the other root of this equation.

Let $(p_1, p_2)$ and $(q_1, q_2)$ be two pairs of conjugate points on $C$ such that the points $p_1, p_2, q_1$ and $q_2$ are defined over $\mathbb{L}(\zeta)$. Consider an embedding $C \hookrightarrow \mathbb{P}_L^2$. The two lines passing through $(p_1, p_2)$ and $(q_1, q_2)$ are defined over $\mathbb{L}$. Therefore we can choose coordinates in $\mathbb{P}_L^2$ such that these lines are given by $x = 0$ and $y = 0$, and the homogeneous coordinates of $p_1, p_2, q_1$ and $q_2$ are

$$(0 : \zeta : 1), \quad (0 : \bar{\zeta} : 1), \quad (\zeta : 0 : 1), \quad (\bar{\zeta} : 0 : 1)$$

respectively. Then $C$ is given by the equation

$$x^2 + y^2 + axz + ayz + b\bar{z}^2 = \theta xy$$

for some $\theta \in \mathbb{L}$, and the automorphism $x \leftrightarrow y$ acts on $C$ and permutes the pairs $(p_1, p_2)$ and $(q_1, q_2)$. \hfill \Box

**Lemma 4.10** Let $S$ be a pointless del Pezzo surface of degree 8 with $\rho(S) = 1$ over a perfect field $\mathbb{K}$. Then the group $\text{Aut}(S)$ transitively acts on the set of points of degree 2 such that the corresponding geometric points are defined over a quadratic extension $\mathbb{K}(\zeta)$.

**Proof** By Lemma 2.1 one has $S \cong R_{L/\mathbb{K}}C$, where $\mathbb{L}$ is the splitting field of $S$ and $C$ is a conic over $\mathbb{L}$. Let $(p_1, p_2)$ and $(q_1, q_2)$ be two pairs of conjugate points on $C$ such that the points $p_1, p_2, q_1$ and $q_2$ are defined over $\mathbb{K}(\zeta)$. Note that $\zeta \notin \mathbb{L}$ since $S_L$ is pointless by Corollary 2.2.

By Corollary 2.2 one has $S_L \cong C \times C'$, where $C'$ is the Gal($\mathbb{L}/\mathbb{K}$)-conjugate conic of $C$. Let $\pi_1 : S_L \to C$ be the projection. One has $\text{Aut}^0(S) \cong \text{Aut}(C)$ by Remark 2.11, and the group $\text{Aut}^0(S)$ faithfully acts on $C$, since $S \cong R_{L/\mathbb{K}}C$. Consider an element $g \in \text{Aut}^0(S)$ such that the induced action of $g$ on $C$ maps the pair $(\pi_1(p_1), \pi_1(p_2))$ to the pair $(\pi_1(q_1), \pi_1(q_2))$ (such $g$ exists by Lemma 4.9).

Note that the Gal($\mathbb{L}(\zeta)/\mathbb{K}$)-orbit of the fibre $F = \pi_1^{-1}(\pi_1(q_1))$ consists of the two fibres of $S_L \to C$ passing through $q_1$ and $q_2$, and the two fibres of $S_L \to C'$ passing through $q_1$ and $q_2$. These fibres have exactly four common geometric points: the conjugate points $q_1$ and $q_2$, and the two other conjugate points $r_1$ and $r_2$. Note that $r_1$ and $r_2$ are not defined over $\mathbb{K}(\zeta)$, since otherwise $F$ is defined over $\mathbb{K}(\zeta)$ and $S$ splits over $\mathbb{K}(\zeta) \neq \mathbb{L}$. Therefore the pair $(g(p_1), g(p_2))$ coincides with the pair $(q_1, q_2)$, and we are done. \hfill \Box

Now we can prove Proposition 4.8.
Proof of Proposition 4.8 Consider the contractions of the negative sections $Z \rightarrow X$ and $Z_1 \rightarrow X_1$ (see Proposition 4.5(iv)). The del Pezzo surfaces $X$ and $X_1$ are isomorphic by Lemma 4.6. By Lemma 4.10 there exists an automorphism of $X$ that maps the points of the blowup of $Z \rightarrow X$ to the points of the blowup $Z_1 \rightarrow X_1$. Therefore $Z \cong Z_1$. \hfill $\square$

Now we prove Theorems 1.6 and 1.7, and give an alternative proof of Theorem 1.1 for the case of a perfect field.

Proof of Theorem 1.6 Let $X$ be a pointless del Pezzo surface of degree $8$ with $\rho(X) = 1$. Assume that a minimal surface $X''$ is birationally equivalent to $X$. Then $X \dasharrow X''$ can be decomposed into a sequence of Sarkisov links:

$$X = X_0 \dasharrow X_1 \dasharrow \cdots \dasharrow X_{n-1} \dasharrow X_n = X''.$$ 

By Lemma 4.1 any Sarkisov link corresponds to the blowup of $X$ at a point of degree $6$, $4$ or $2$. For the cases of degree $6$ and $4$ this link immediately gives a surface $X_1 \cong X$ by Lemmas 4.2 and 4.3 respectively. For the case of degree $2$ the sequence of links passes through some non-minimal del Pezzo surfaces of degree $6$ (see Proposition 4.5 (iii)) and terminates at a del Pezzo surface $X' \cong X$ by Lemma 4.6. Therefore any minimal del Pezzo surface, obtained in a sequence of Sarkisov links starting from $X$, is isomorphic to $X$. In particular, $X'' \cong X$. \hfill $\square$

Proof of Theorem 1.7 Note that by [12, Theorem 5.1] there is an embedding $X \hookrightarrow P$, where $P$ is a Severi–Brauer threefold such that the class $b(P)$ lies in $\text{Am}(X) \subset \text{Br}(k)$.

One can easily find a point of degree $4$ on $X$, therefore $I(X)$ divides $4$. Moreover, $I(P)$ divides $I(X)$ since any point on $X$ is also a point on $P$. There is a point of degree $I(P)$ on $P$ by [11, Theorem 53].

Assume that $I(P) \leq 2$, then there is a point of degree $2$ on $P$. The line $\overline{L}$ passing through the pair of the corresponding points in $\mathbb{P}^3_k = \overline{P}$ is invariant under the action of the Galois group $\text{Gal}(\overline{k}/k)$. Therefore there is a smooth curve $L \subset P$ of degree $1$. One has $L \cdot X = 2$. Thus there is a point of degree $2$ on $X$, and $I(X) = 2$ or $I(X) = 1$.

Note that the classes $b(L)$ and $b(P)$ coincide in $\text{Br}(k)$ by [11, Definition-Lemma 31]. Moreover these classes coincide with the image of the class $-\frac{1}{2}K_X$ in $\text{Am}(X)$. Therefore if $\rho(X) = 1$ then any element of $\text{Am}(X)$ corresponds to a conic since $\text{Am}(X)$ is generated by the image of the class $-\frac{1}{2}K_X$. Moreover, in this case $X$ is not birationally rigid since one can blow up a point of degree $2$ and get a del Pezzo surface of degree $6$ with a structure of a relatively minimal conic bundle. If $\rho(X) = 2$ then any element of $\text{Am}(X)$ corresponds to a conic, and $X$ is not birationally rigid by Lemma 3.2.

Now assume that $I(P) = 4$, then $P$ cannot contain a conic by Theorem [11, Theorem 53], and the element $b(P) \in \text{Am}(X)$ does not correspond to a conic. Moreover, one has $I(X) = 4$, and there are no points of degree $2$ on $X$ since $I(P)$ divides $I(X)$.

If $\rho(X) = 2$ then $X \cong C_1 \times C_2$ and $\text{Am}(X) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Any minimal surface $X'$ birationally equivalent to $X$ must be a product $C_1' \times C_2'$ of two non-isomorphic smooth non-trivial conics by Propositions 2.5 and 2.7. Therefore $b(C_1')$ and $b(C_2')$ are...
not trivial and are not equal to $b(P)$. Thus the sets $\{b(C'_1), b(C'_2)\}$ and $\{b(C_1), b(C_2)\}$ coincide, and $X' \cong X$. Hence $X$ is birationally rigid.

If $\rho(X) = 1$ then the only possible Sarkisov link is described in Lemma 4.3, and therefore $X$ is birationally rigid. □

Proof of Theorem 1.1 for the case of a perfect field Consider two pointless birationally equivalent quadric surfaces $Q_1$ and $Q_2$. These surfaces are minimal. If $\rho(Q_1) = 1$ then by Theorem 1.5 one has $Q_1 \cong Q_2$.

If $\rho(Q_1) = 2$ then by Lemma 2.1 one has $Q_1 \cong C_1 \times C_1$. By Theorem 1.5 any minimal surface birationally equivalent to $Q_1$ has Picard number 2, and therefore $Q_2 \cong C_2 \times C_2$ by Lemma 2.1. Thus by Theorem 1.5 (2) one has $Q_1 \cong Q_2$. □

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