Approximate analytical and numerical solutions to the damped pendulum oscillator: Newton–Raphson and moving boundary methods

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1. Introduction

Differential equations and nonlinear physical models occupy central roles for applications in all aspects of life [1–15]. The development of modern life demands the control in nonlinear physical models in order to construct useful applications for humanity [16–18]. Therefore, nonlinear phenomena become important and demand mathematical treatments to obtain the exact solutions, approximate solutions or numerical schemes (if the exact solutions cannot be possibly found) to give a good explanation and description for the physical behaviours and to recognize their properties [1–4, 19–23]. Generally, nonlinear models are expressed in terms of ordinary or partial differential equation [24–27]. Usually, in nonlinear analysis, the exact solutions are little comparison to approximate solutions or numerical methods. The nonlinear vibrations, oscillations, and waves are here important examples for nonlinear physical models [28–30]. The evolution of the oscillator took five centuries from the 16th century the 20th century in order to develop from an idea to an application. It was not really simple, but great efforts were made by some famous scientists like Galileo, Huygens, Hooke, Newton, Leibniz, James Bernoulli, John Bernoulli, Euler, Helmholtz, Strutt, Rayleigh, etc. [31]. The simple pendulum has been investigated experimentally and analytically to be recognized their mechanical characters. For long decades, the simple pendulum has been solved analytically or numerically by numerous methods to find appropriate solutions [1–4, 32–35]. It is interest for researchers as a fertile ground for new ideas and applications. Generally, the damping terms often make differential equations more complicated in solutions [36]. In the early time, it was studied under the influence of weight in the absence of friction and dissipation forces. Consequently, the equation of an undamped motion of simple pendulum without a friction or a dissipation reads

\[
\ddot{\theta} + \kappa^2 \sin \theta = 0,
\]

where \( \dot{\theta} \equiv \theta(t) \) gives the angular displacement of the pendulum with respect to the angle between vertical axis and the pendulum, \( \kappa = \sqrt{g/l} \) represents the angular frequency in unit of rad (1/s), \( g = 9.81 \text{ m/s}^2 \) donates the acceleration of the gravity, and \( l \) gives the length of massless pendulum arm as shown in Figure 1. In this case, the simple pendulum moves with a simple harmonic motion indefinitely without decaying because the only effect on the pendulum motion is the conservative force, so the mechanical energy will remain constant during the movement of the pendulum. But this
the approximate analytical and numerical solutions and the RK4 is estimated.

2. An approximate analytic solution for particular initial conditions

In this section, the simple pendulum oscillator exposes to the linear damping effect which appears in terms of $2y\theta'$ and the nonlinear stiffness term which takes the sine form as $\kappa^2 \sin \theta$, where $t$ is a time dependent and the two parameters $\kappa$ and $\gamma$ are positive. This problem is subjected to the initial conditions; a zero displacement $\theta(0) = 0$ and non-zero velocity $\theta'(0) = \theta_0 \neq 0$. Consequently, the following damped simple pendulum oscillator is expressed according to the previous physical description of the Duffing oscillator as:

$$
\begin{align*}
\theta''(t) + 2\gamma \theta'(t) + \kappa^2 \sin \theta(t) &= 0, \\
\theta(0) &= 0 \& \theta'(0) = \theta_0 \neq 0.
\end{align*}
$$

Now, we are looking for a solution to Equation (3) in the form:

$$
\theta(t) = A \tan^{-1}(Be^{-\frac{t}{\omega}}sd(\sqrt{\omega}t, m_0 e^{-\rho t})),
$$

where $sd(\sqrt{\omega}t, m_0 \exp(-\rho t))$ is Jacobi elliptic function and $A, B, \lambda, \omega, m_0$, and $\rho$ are real constants, to be determined later. If the first initial condition: $\theta(0) = 0$ is applied to Equation (4), then we can prove that $A \neq 0$, $B \neq 0$, and $sd(0, m_0) = 0$. Also, by applying the second initial condition: $\theta'(0) = \theta_0$ to Equation (4), the constant $A$ is expressed as:

$$
A = \frac{\theta_0}{B \sqrt{\omega}}.
$$

In order to determine the values of the others constants $B, \lambda, \omega, m_0$, and $\rho$ to satisfy Equations (3) and (4) in a reasonable way, let us assume that

$$
R(t) = \theta'' + 2\gamma \theta' + \kappa^2 \sin(\theta(t)),
$$

which is subjected the initial conditions $R(0) = 0$, $R^{(1)}(0) = 0$ where $j = 1 - 4$ and $R^{(4)}$ gives derivatives of $R(t)$ from first to fourth. These conditions are used to determine the constants $B, \lambda, \omega, m_0$, and $\rho$. The first condition $R(0) = 0$, is directly applied to Equations (4) and (6), and consequently, the constant $\lambda$ is determined in term of damping parameter as follows:

$$
\lambda = \gamma.
$$

By applying condition $R^{(1)}(0) = 0$, to Equations (4) and (6) with the help of Equation (7), the value of parameter $\omega$ in terms of $\kappa, \gamma, B$, and $m_0$ is determined as

$$
\omega = \frac{\kappa^2 - \gamma^2}{1 + 2B^2 - 2m_0}.
$$

The condition $R^{(2)}(0) = 0$, is applied to Equations (4) and (6) to find the value of $m_0$ with the help of

Figure 1. Simple Pendulum diagram.
Equation (8) as:

\[ m_0 = \frac{2B^2\gamma}{\rho}. \]  

(9)

Two parameters \( B \) and \( \rho \) can be determined by applying the following two conditions \( R^{13}(0) = 0 \) and \( R^{4}(0) = 0 \) to Equations (4) and (6) with the help of Equation (9). Accordingly, we can get the value of \( \rho \) by solving the following polynomial

\[ (2\gamma - \rho)\left(25\rho^6 - 100\gamma\rho^5 - 10(\kappa^2 - \gamma^2)\rho^4 + 400\gamma^3\rho^3 + \alpha_1\rho^2 + \alpha_2\rho + \alpha_3\right) = 0, \]  

(10)

where \( \alpha_1 = (190k^2\gamma^2 - 575\gamma^4 + 3k^2\theta_0^2 - 15k^4), \) \( \alpha_2 = 12\gamma(3k^4 - 26k^2\gamma^2 + 23\gamma^4 - \kappa^2\theta_0^2), \) and \( \alpha_3 = (12k^2\gamma^2\theta_0^2). \) For sake of simplicity, one of the real root to Equation (10) gives the following value of the parameter \( \rho \) as

\[ \rho = 2\gamma, \]  

(11)

and the value of \( B \) is described by

\[ B = \frac{\dot{\theta}_0k}{2(k^2 - \gamma^2)}. \]  

(12)

Now, all values of the mentioned parameters are expressed in terms of \( \kappa, \gamma \) and \( \dot{\theta}_0 \) as follows

\[ A = \frac{2\sqrt{k^2 - \gamma^2}}{\kappa}, \]

\[ B = \frac{k\dot{\theta}_0}{2(k^2 - \gamma^2)}, \]

\[ \omega = (k^2 - \gamma^2), \]

\[ m_0 = B^2. \]

Thus, an (approximate or exact) analytic solution of Equation (3) is given by:

\[ \theta(t) = 2\tan^{-1}\left[\frac{\kappa\theta_0}{2(k^2 - \gamma^2)}e^{-\gamma t}\right], \]

\[ \times \left[\sqrt{k^2 - \gamma^2}t + \frac{\kappa\theta_0^2}{4(k^2 - \gamma^2)^2}e^{-2\gamma t}\right], \]  

(14)

with \( k^2 \neq \gamma^2. \)

In the absence of damping \((\gamma = 0), \) the solution (14) can be reduced to

\[ \theta(t) = 2\tan^{-1}\left[\frac{\dot{\theta}_0}{2k}\left[\kappa t + \frac{\dot{\theta}_0^2}{4k^2}\right]\right], \]  

(15)

with \( k \neq 0. \)

Equation (15) represents the exact analytic solution to the following undamped Pendulum equation

\[ \begin{cases} \theta'' + \kappa^2\sin(\theta) = 0, \\ \theta(0) = 0 & \theta'(0) = \dot{\theta}_0, \end{cases} \]  

(16)

where \( \kappa > 0 \) and \( \dot{\theta}_0 \neq 0. \)

In this case, the Pendulum can oscillate forever without interruption or decay. We can deduce that our results are consistent with the theory for the undamped Pendulum motion.

Remark 2.1: Recently, Johannesen [34] gives an approximate analytical solution to the problem

\[ \begin{align*}
\psi''(t) + 2\gamma\psi'(t) + (1 + \gamma^2)\sin(\psi(t)) &= 0, \\
\psi(0) = 0 & \psi'(0) = 2\sin\left(\frac{1}{2}\psi_{\text{max}}/2\right),
\end{align*} \]

(17)

in the following form

\[ \psi(t) = 2\arctan\left(\sqrt{m(t)}\sin(\xi(t), m(t))\right), \]  

(18)

with

\[ m(t) = m_0\exp(-2\gamma t), \]

(19)

and

\[ \xi(t) = \left(1 + \frac{1}{4}m(t) + \frac{9}{64}m(t)^2\right)t \\
+ \frac{1}{8\gamma}(m(t) - m_0) + \frac{9}{256\gamma}(m(t)^2 - m_0^2), \]  

(20)

where \( m_0 = \sin^2(\psi_{\text{max}}/2). \)

Now, let us make a comparison between the approximate analytic solutions (14) and (18) as well as the approximate numerical solution using Runge Kutta fourth-order method (RK4) in order to measure the accuracy of the solutions (14) and (18). This comparison is introduced in Figure 2(a). It is observed that the solution (18) is more accurate than the solution (14). However, the solution (14) can be modified by introducing the following new approximation

\[ \begin{cases} \psi(t) = \theta(\xi(t)), \\ \kappa = \sqrt{1 + \gamma^2}. \end{cases} \]

(21)

Note that in this modification/improvement, we replaced \( \theta(t) \) in solution (14) by \( \theta(\xi(t)) \), i.e. \( t \rightarrow \xi(t) \). These results are confirmed in Figure 2(b) which the three approximate analytic solutions (18), (21), and RK4 are compared to each other. Moreover, the distance error for the obtained solution with respect to RK4 solution is estimated. It is found that the new approximate analytic solution (21) is better than both solutions (14) and (18). However, in Johannesen paper [34], the damped motion of simple Pendulum for zero initial angle, i.e. \( \theta(0) = 0 \) is studied, but in the following section, we shall investigate the damped motion of the pendulum for arbitrary initial conditions.
We define an approximate analytical solution to the problem (22) as follows

\[
\theta = 2 \tan^{-1} \left\{ \tan \left( \frac{\theta_0}{2} \right) e^{-\gamma t} \left[ \frac{\gamma}{\sqrt{\kappa^2 - \gamma^2}} \sin \left( \sqrt{\kappa^2 - \gamma^2} t \right) \right] \right. \\
+ \left. \cos \left( \sqrt{\kappa^2 - \gamma^2} t \right) \right\},
\]

where \( \kappa \neq \gamma \).

Here, we can recover and discuss some different cases. The first one, if \( \theta_0 = 0 \), the solution (14) satisfies the problem (22). The second case when \( \theta_0 = 0 \) and \( \dot{\theta}(0) = \dot{\theta}_0 \), we get

\[
\theta = 2 \tan^{-1} \left( \frac{\theta_0}{2} \right) e^{-\gamma t} \\
+ \cos \left( \sqrt{\kappa^2 - \gamma^2} t \right). \tag{23}
\]

The third case, if we considered \( \theta_0, \dot{\theta}_0 \neq 0 \), in this case, we define an approximate analytical solution to the problem (22) as follows

\[
\theta = 2 \tan^{-1} \left( \frac{\theta_0}{2} \right) e^{-\gamma t} \\
+ \cos \left( \sqrt{\kappa^2 - \gamma^2} t \right). \tag{24}
\]

with

\[
\omega = \sqrt{(p + q\theta_0^2) + q^2}, \\
2\dot{\theta}_0 \kappa^2 \cos (\theta_0) + 2\gamma (\kappa^2 - \gamma^2) \sin (\theta_0) \\
- \dot{\theta}_0 \left( 2\gamma^2 + 6\gamma \dot{\theta}_0 \tan \left( \frac{\theta_0}{2} \right) + \dot{\theta}_0^2 \right) \\
+ \frac{\dot{\theta}_0 \left( 4\gamma \dot{\theta}_0 \tan \left( \frac{\theta_0}{2} \right) - \dot{\theta}_0^2 + 2\kappa^2 \cos (\theta_0) + 2\kappa^2 \right)}{2 (\gamma \sin (\theta_0) + \dot{\theta}_0)}, \\
\dot{y}_0 = \tan \left( \frac{\theta_0}{2} \right). \\
\]

In this case, when \( \kappa = \gamma \), we may use the last expression taking the limit as \( \kappa \to \gamma \). This same expression may also be used in the case when \( \theta_0 = 0 \) or \( \dot{\theta}_0 = 0 \). Note that for \( \theta_0 = \dot{\theta}_0 = 0 \), the trivial solution \( \dot{\theta}(t) \equiv 0 \) is obtained. In the next section, we shall solve the problem (22) numerically using two approaches namely, NRM and moving boundary method (MBM) and after that comparing the obtained approximate analytic and numerical solutions (24).

### 4. Numerical solutions using NRM and MBM

#### 4.1. Newton–Raphson method

First, for applying NRM to the problem (22), let us consider

\[
\psi(t) = \theta(t + C), \tag{25}
\]

where \( \theta(t) \) is defined in Equation (14).

The values of \( C \) and \( \theta_0 \) can be obtained from the following system

\[
\begin{align*}
\psi(0) &= \varphi_0 = \theta(C), \\
\psi'(0) &= \varphi_0' = \theta'(C). \tag{26}
\end{align*}
\]

This system could be solved with the aid of the NRM.

#### 4.2. Moving boundary method

Sometimes, the approximate analytic solution maybe not good for large time intervals. In that case, we may apply the MBM to improve the accuracy of our solutions.
mentioned problem. We proceed by dividing the time interval into smaller intervals of length \( h = T/N \); say \( 0 = t_0 < t_1 < t_2 < \cdots < t_j < \cdots < t_N = T \), where \( t_j = jh \) (j = 0, 1, ..., N).

Let \( \theta_0(t) \) to be the analytical approximation in the interval \( 0 \leq t \leq t_1 \). Also, it is assumed that \( \theta_1(t) \) to be the analytical approximation in the interval \( t_1 \leq t \leq t_2 \), with the following initial conditions

\[
\begin{aligned}
\theta_1(t_1) &= \theta_0(t_1), \\
\theta'_1(t_1) &= \theta'_0(t_1).
\end{aligned}
\]  

(27)

Moreover, suppose we already defined \( \theta_k(t) \) (\( t_k \leq t \leq t_{k+1} \)) for \( k = 0, 1, 2, ..., j - 1 \). Consequently, we can define \( \theta_j(t) \) on \( t_j \leq t \leq t_{j+1} \) as the analytical approximation with initial conditions

\[
\begin{aligned}
\theta_j(t_j) &= \theta_{j-1}(t_j), \\
\theta'_j(t_j) &= \theta'_{j-1}(t_j).
\end{aligned}
\]  

(28)

This allows us to define an analytical approximate solution \( \theta(t) \) on the whole interval \( 0 \leq t \leq T \) as follows

\[
\theta(t) = \sum_{j=1}^{N} \chi_j(t) \theta_j(t),
\]

(29)

where

\[
\chi_j(t) = \begin{cases} 
1, & \text{for } t_{j-1} \leq t < t_j, \\
0, & \text{otherwise}.
\end{cases}
\]  

(30)

Figure 3 shows the comparison between the numerical approximations (RK4, NRM and MBM) and analytical approximation (24). It is clear from Figure 3(a) that the approximation (24) gives excellent results as compared to the numerical solutions. In Figure 3(b,c), the comparison between the RK4 and MBM solutions is considered. One can see, the near-perfect match between the two numerical solutions, which strengthens the MBM.

5. Conclusion

The damped pendulum differential equation of motion has been solved analytically and numerically. The analytical approximation is introduced in the form of the Jacobean elliptic functions for two cases. In the first case, the problem is solved for certain initial conditions (the initial angle is taken to be zero and non-zero initial speed). In this case, our analytic solution is compared with Johannesen’s solution [34]. It is observed that our solution is less accurate than the Johannesen’s solution [34]. However, we improved our solution and a new solution with accuracy higher than Johannesen’s solution [34] has been obtained. Also, the problem was solved under the same initial conditions numerically using NRM and moving boundary method. The comparison between the analytical and numerical approximations has been carried out. In addition, the distance error for each method with respect to the fourth-order Runge Kutta method has been estimated. It has been noticed that our solution gives good results. With respect to the second case, we solved the pendulum equation of motion with arbitrary initial conditions and got a general solution in the form Jacobean elliptic functions and this solution could be recovered many cases. For example, the general solution could be reduced to the solution of the first case if we use the same initial conditions of the first case. Moreover, the general solution was compared with the numerical solutions. the MBM allows us to obtain an analytical solution as linear combination of characteristics functions of the intervals that correspond to the time interval partition. This method...
is applicable once we obtained an analytical approximated solution for a given arbitrary conditions. It may be employed to solve other nonlinear problems such as the damped and forced cubic Duffing equation as well as the cubic-quintic damped and forced Duffing equation.

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Data availability statement

The data sets generated for this paper are available on request to El-Tantawy.

References

[1] Wazwaz AM. Partial differential equations and solitary waves theory: higher education. Beijing, Berlin: Springer; 2009.
[2] Wazwaz AM. Partial differential equations and solitary waves theorem. Berlin: Springer and HEP; 2009.
[3] El-Tantawy SA. Ion-acoustic waves in ultracold neutral plasmas: modulational instability and dissipative rogue waves. Phys Lett A. 2017;381:787–791.
[4] Wazwaz AM, El-Tantawy SA. Solving the (3+1)-dimensional KP–Boussinesq and BKP–Boussinesq equations by the simplified Hirota’s method. Nonlinear Dyn. 2017;88:3017–3021.
[5] Lü X, Ma WX, Yu J, et al. Solitary waves with the modulational fluid description: a generalized derivative nonlinear schrodinger equation. Commun Nonlinear Sci Numer Simulat. 2016;31:30.
[6] Wang G. A novel (3+1)-dimensional sine-Gorden and sinh-Gorden equation: derivation, symmetries and conservation laws. Appl Math Lett. 2021;114:956.
[7] Wang G. A nev (3+1)-dimensional Schrödinger equation: derivation, soliton solutions and conservation laws. Nonlinear Dyn. 2021;104:1595.
[8] Wang G. Symmetry analysis, analytical solutions and conservation laws of a generalized derivative nonlinear Kuramoto equation and its fractional version. Fractals. 2021;29: Article ID 2150101.
[9] Wang G, Yang K, Gu H, et al. A (2+1)-dimensional sine-Gordon and sinh-Gordon equations with symmetries and kink wave solutions. Nucl Phys B. 2020;953:Article ID 114956.
[10] Wang G, Kara AH. A (2+1)-dimensional KdV equation and mKdV equation: symmetries, group invariant solutions and conservation laws. Phys Lett A. 2019;383:72.
[11] Wang G, Liu Y, Wu Y, et al. Symmetry analysis for a seventh-order generalized KdV equation and its fractional version in fluid mechanics. Fractals. 2020;28:Article ID 2050044.
[12] Wang G, Kara AH. Conservation laws, multipliers, adjoint equations and Lagrangians for Jaulent-Miodek and some families of systems of KdV type equations. Nonlinear Dyn. 2015;81:753.
[13] Rozina C, Hira A, Ali S, et al. Low-frequency plasma waves in a radiative dusty magnetoplasma. Phys Scr. 2020;95:Article ID 045608.
[14] Drumheller DS. Introduction to wave propagation in nonlinear fluids and solids. Cambridge University Press; 1998.
[15] Krack M, Gross J. Harmonic balance for nonlinear vibration problems. Springer; 2019.
[16] Caviglia G, Morro A. Inhomogeneous waves in solids and fluids. World Scientific; 1992.
[17] Gladwell GML. Inverse problems in vibration. Kluwer Academic Publishers; 2005.
[18] Samelson RM, Wiggins S. Lagrangian transport in geophysical jets and waves the dynamical systems approach. Springer; 2006.
[19] Sachdev PL. Self-similarity and beyond exact solutions of nonlinear problems. Chapman & Hall/CRC; 2000.
[20] Grossinho MR, Ramos M, Rebelo C, et al. Progress in nonlinear differential equations and their applications. Birkhäuser; 2001.
[21] Cheban DN. Asymptotically almost periodic solutions of differential equations. Hindawi Publishing Corporation; 2009.
[22] Hermann M, Saravi M. Nonlinear ordinary differential equations: analytical approximation and numerical methods. Springer; 2016.
[23] Butcher JC. Numerical methods for ordinary differential equations. 3rd ed. Wiley; 2016.
[24] Ray SS. Nonlinear differential equations: novel methods for finding solutions. Springer; 2020.
[25] Debnath L. Nonlinear partial differential equations in physics: novel methods and thependulumequation. ApplMathSci. 2014;8:8781.
[26] Berti M. Nonlinear oscillations of hamiltonian PDEs. Birkhäuser; 2007.
[27] Yoshida N. Oscillation theory of partial differential equations. World Scientific; 2008.
[28] Kovacic I, Brennan MJ. The duffing equation: nonlinear oscillators and their behaviour. John Wiley & Sons, Ltd.; 2011.
[29] Jazar RN. Advanced vibrations – a modern approach. Springer; 2013.
[30] Johannesssen K. An approximate solution to the equation of motion for large-angle oscillations of the simple pendulum with initial velocity. Eur J Phys. 2010;31:511.
[31] Johannesssen K. An analytical solution to the equation of motion for the damped nonlinear pendulum. Eur J Phys. 2014;35:Article ID 035014.
[32] Salas AH, Castillo JE. Exact solution to duffing equation and the pendulum equation. Appl Math Sci. 2014;8:8781.
[33] Veselić K. Damped oscillations of linear systems: A mathematical introduction. Springer; 2011.

Appendix. Elliptic functions in small points

Here, the simple brief is presented about elliptic functions. In the beginning, three elliptic integrals are introduced and hence elliptic functions. The incomplete elliptic integral of the
first kind is defined as:

\[ z = F(\phi, \kappa) = \int_{0}^{\varphi} \frac{d\theta}{\sqrt{1 - \kappa^2 \sin^2 \theta}} \]  
(A1)

where \( \kappa \) is the elliptic modulus such that \( 0 < \kappa^2 < 1 \) and \(-\infty < \phi < \infty \) if \( 0 < \kappa^2 < 1 \) and \(-\pi/2 < \phi < \pi/2 \) if \( \kappa = 1 \). \( F(\phi, \kappa) \) is an odd and increasing function of \( \phi \). Other forms of the incomplete elliptic integral of the first kind are defined as:

\[ z = \phi \int_{0}^{\phi} \frac{dx}{\sqrt{(1 - \kappa^2 x^2)(1 - x^2)}}. \]  
(A2)

where \( x = \sin \theta \) or

\[ z = F(\phi, \kappa) = \int_{0}^{\tan \theta} \frac{dt}{\sqrt{(1 + \kappa^2 t^2)(1 + t^2)}}. \]  
(A3)

where \( t = \tan \theta \) and \( \kappa^{-2} = 1 - \kappa^2 \) is the complementary elliptic modulus. The complete elliptic integral of the first kind is defined as:

\[ \kappa(\kappa) = F(\frac{\pi}{2}, \kappa) = \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - \kappa^2 \sin^2 \theta}} \]

\[ = \frac{\pi}{2} _{2}F_{1}\left(\frac{1}{2}, \frac{1}{2}; 1; \kappa^2\right). \]  
(A4)

where \( _{2}F_{1}(a; b; c; x) \) is the hypergeometric function. The incomplete elliptic integral of the second kind is defined as:

\[ E(\phi, \kappa) = \int_{0}^{\phi} \frac{1}{\sqrt{1 - \kappa^2 \sin^2 \theta}} d\theta. \]  
(A5)

where \( \kappa \) is the elliptic modulus such that \( 0 < \kappa^2 < 1 \) and if \( x = \sin \theta \), it is rewritten as:

\[ E(\phi, \kappa) = \int_{0}^{\sin \phi} \frac{1}{\sqrt{1 - \kappa^2 x^2}} dx. \]  
(A6)

The complete elliptic integral of the second kind is defined as:

\[ E(\kappa) = E\left(\frac{\pi}{2}, \kappa\right) = \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - \kappa^2 \sin^2 \theta}} \]

\[ = \frac{\pi}{2} _{2}F_{1}\left(\frac{1}{2}, \frac{1}{2}; 1; \kappa^2\right). \]  
(A7)

The incomplete elliptic integral of the third kind is defined as:

\[ \Pi(n; \phi, \kappa) = \int_{0}^{\phi} \frac{d\theta}{(1 - n \sin^2 \theta) \sqrt{1 - \kappa^2 \sin^2 \theta}}. \]  
(A8)

where \( \kappa \) is the elliptic modulus such that \( 0 < \kappa^2 < 1 \) and \( n \) is the elliptic characteristic. Other form is described by:

\[ \Pi(n; \phi, \kappa) = \int_{0}^{\sin \phi} \frac{dx}{(1 - nx^2) \sqrt{(1 - \kappa^2 x^2)(1 - x^2)}}. \]  
(A9)

The complete elliptic integral of the third kind is defined as:

\[ \Pi(n, \kappa) = \Pi\left(\frac{\pi}{2}, \kappa\right) = \int_{0}^{\pi/2} \frac{d\theta}{(1 - n \sin^2 \theta) \sqrt{1 - \kappa^2 \sin^2 \theta}} \]

\[ = \frac{\pi}{2} _{2}F_{1}\left(1; \frac{1}{2}; 1; 1; m, n\right). \]  
(A10)

The Jacobi amplitude function \( \phi = am(z, \kappa) = F^{-1}(\phi, \kappa) \) is Jacobi amplitude and \( F^{-1} \) is the inverse function of the first incomplete elliptic integral.

Basic Jacobi elliptic functions \( sn(z, \kappa), cn(z, \kappa), \) and \( dn(z, \kappa) \) are introduced as follows:

\[ sn(z, \kappa) = \sin \phi, \quad cn(z, \kappa) = \cos \phi, \]

\[ dn(z, \kappa) = \sqrt{1 - \kappa^2 \sin^2 \phi} = \sqrt{1 - \kappa^2 sn^2(z, \kappa)}. \]  
(A11)

where \( sn \) is an odd function of \( z \) and \( cn \) and \( dn \) are even functions of \( z \). If \( \kappa = 0 \), then \( z = \phi \), we have \( sn(z, 0) = \sin z, \quad cn(z, 0) = \cos z, \quad dn(z, 0) = 1 \), and \( \kappa(0) = \frac{\pi}{2} \). If \( \kappa = 1 \), and \(-\pi/2 < \phi < \pi/2 \), we have \( z = \int_{0}^{\phi} \sec \theta d\theta = \ln(\sec \phi + \tan \phi) \), \( sn(z, 1) = \tanh z, \quad cn(z, 1) = \sech z, \quad dn(z, k) = \sech z \) and \( \kappa(1) = \infty \).

Nine other Jacobi elliptic functions \( cd, cs, ds, dc, sc, sd, nc, \) and \( nd \) are introduced by:

\[ cd(z, \kappa) = \frac{CN(z, \kappa)}{DN(z, \kappa)}, \quad dc(z, \kappa) = \frac{1}{cd(z, \kappa)} = \frac{dn(z, \kappa)}{cn(z, \kappa)}, \]

\[ cs(z, \kappa) = \frac{CN(z, \kappa)}{SN(z, \kappa)}, \quad sc(z, \kappa) = \frac{1}{cs(z, \kappa)} = \frac{sn(z, \kappa)}{cn(z, \kappa)}, \]

\[ ds(z, \kappa) = \frac{SN(z, \kappa)}{DN(z, \kappa)}, \quad sd(z, \kappa) = \frac{1}{ds(z, \kappa)} = \frac{sn(z, \kappa)}{dn(z, \kappa)}, \]

\[ nc(z, \kappa) = \frac{1}{nc(z, \kappa)} = \frac{SN(z, \kappa)}{cn(z, \kappa)}, \]

\[ nd(z, \kappa) = \frac{1}{nd(z, \kappa)} = \frac{DN(z, \kappa)}{dn(z, \kappa)}. \]  
(A12)

Representations of derivatives of Jacobi elliptic functions with respect to \( z \) are given by:

\[ am'(z, \kappa) = dn(z, \kappa), \quad sn'(z, \kappa) = cn(z, \kappa) \cdot dn(z, \kappa), \]

\[ ns'(z, \kappa) = -cs(z, \kappa) \cdot ds(z, \kappa), \quad cn'(z, \kappa) = -sn(z, \kappa) \cdot dn(z, \kappa), \]

\[ nc'(z, \kappa) = dc(z, \kappa) \cdot sc(z, \kappa), \quad sn'(z, \kappa) = -cs(z, \kappa) \cdot sn(z, \kappa), \]

\[ nd'(z, \kappa) = k^2 cd(z, \kappa) \cdot sd(z, \kappa), \]  
(A13)

\[ cd'(z, \kappa) = (k^2 - 1) \cdot nd(z, \kappa) \cdot sd(z, \kappa), \]

\[ dc'(z, \kappa) = -(k^2 - 1) \cdot nc(z, \kappa) \cdot sc(z, \kappa), \]

\[ cs'(z, \kappa) = -ds(z, \kappa) \cdot sn(z, \kappa), \]

\[ sc'(z, \kappa) = dc(z, \kappa) \cdot nc(z, \kappa), \]

\[ ds'(z, \kappa) = -cs(z, \kappa) \cdot sn(z, \kappa), \]

\[ sd'(z, \kappa) = cd(z, \kappa) \cdot nd(z, \kappa). \]