REALIZATIONS OF THERMAL SUPERSYMMETRY

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Abstract

We investigate realizations of supersymmetry at finite temperature in terms of thermal superfields, in a thermally constrained superspace: the Grassmann coordinates are promoted to be time-dependent and antiperiodic, with a period given by the inverse temperature. This approach allows to formulate a Kubo-Martin-Schwinger (KMS) condition at the level of thermal superfield propagators. The latter is proven directly in thermal superspace, and is shown to imply the correct (bosonic and fermionic) KMS conditions for the component fields. In thermal superspace, we formulate thermal covariant derivatives and supercharges and derive the thermal super-Poincaré algebra. Finally, we briefly investigate field realizations of this thermal supersymmetry algebra, focussing on the Wess-Zumino model. The thermal superspace formalism is used to characterize the breaking of global supersymmetry at finite temperature.

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1 Introduction

The most popular extension of the Standard Model of strong and electroweak interactions has a broken supersymmetry. The breaking of supersymmetry is in several aspects different from the breaking of, for instance, the $SU(2)_L \times U(1)_Y$ gauge symmetry. Firstly, it is not spontaneous, or it is spontaneous only in a more fundamental theory like supergravity or superstrings acting at much shorter distances. Secondly, in contrast with the common behaviour of spontaneously broken gauge symmetries, supersymmetry is not restored at high temperature. This fact has conceptual and technical implications when studying phase transitions in the minimal supersymmetric standard model or, more generally, cosmology of supersymmetric field theories. One cannot expect that the cooling of the expanding Universe has triggered supersymmetry breaking.

The fate of supersymmetry in thermal environments has been studied with various motivations and purposes in a relatively small number of references [1]–[13], scattered over the last twenty years. The behaviour of supersymmetry at finite temperature is somewhat but not entirely similar to the one of Lorentz symmetry. In a thermal bath, a relativistic field theory can be formulated in a covariant way [14], even if the notion of temperature is certainly not Lorentz invariant since it refers to a particular frame. Poincaré symmetry acts then in a well defined way on the observables of the theory. Supersymmetry adds a new aspect: it exchanges bosons and fermions. At zero temperature, superspace takes care of the difference in statistics by introducing Grassmann anticommuting coordinates, which turn bosonic commutation relations into fermionic anticommutators. At finite temperature, fermion and boson statistics involve in addition the appropriate statistical weight in field theory Green’s functions.

Our first goal in Sections 2 and 3 of the present paper is to develop a superspace covariant formalism for the Green’s functions of a supersymmetric field theory at finite temperature. We formulate the restrictions that thermal effects impose on superspace, in terms of Kubo-Martin-Schwinger (KMS) relations [15] on Green’s functions. In sections 4 and 5, we then use this “thermal superspace” formalism to derive thermal covariant derivatives, thermal supercharges, as well as the thermal covariantizations of the translation and Lorentz generators, in complete analogy with the zero-temperature case. These operators generate the usual super-Poincaré algebra, acting locally on the thermal version of superspace. Notice that the existence of this algebra is not in contradiction with the expected breaking of supersymmetry in finite temperature field theories, which is due to periodicity (for bosons) and antiperiodicity (for fermions) relations in the (complex) time direction. These characterizations are of global character, and the super-Poincaré algebra is not sensitive to the field’s global periodicity properties. But the existence of a super-Poincaré algebra in superspace implies neither the existence of field representations nor the existence of invariant actions.

At a technical level, the present approach rests mainly on the notion of thermal superspace, the properties of which can be motivated through the following, heuristic argument. Consider first supersymmetry at zero temperature. The supersymmetry algebra relates supersymmetries $Q_\alpha, \overline{Q}_\dot{\alpha}$ to translations $P_\mu$ through $\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu$. As a consequence, supersymmetries can be seen as “square roots” of translations, which act...
like the derivative, $P_\mu = -i \partial_\mu$. One could thus try to represent supersymmetries as generalized translations, which would act through derivatives only, in analogy to translations. This is however not possible in usual space-time, but requires an enlarged space including the Grassmann variables that are translated by the supersymmetry generators. Such an enlarged space is provided by superspace, which consists, in addition to the usual space-time coordinates, of Grassmannian objects denoted by $\theta$ and $\bar{\theta}$. A point $X$ in superspace has therefore coordinates $X = (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, and since at zero temperature the parameters of supersymmetry transformations are space-time constant, the zero-temperature superspace coordinates $\theta^\alpha$ and $\bar{\theta}^{\dot{\alpha}}$ are space-time constants as well.

However, as was noted in [2] (see also [3]), if one wants to respect the correct KMS boundary conditions, one cannot make use of constant parameters in supersymmetry transformations rules at finite temperature: the supersymmetry parameters must be time-dependent and antiperiodic in imaginary time on the interval $[0, \beta]$, where $\beta = 1/T$ denotes the inverse temperature. Adapting straightforwardly the zero-temperature argument above, it is natural to require that the variables which are translated by the effect of the thermal supersymmetry transformation bear the same characteristics as the thermal supersymmetry parameters. From this we conclude that the construction of a thermal superspace requires that the Grassmann parameters get promoted to be time-dependent and antiperiodic in imaginary time on the interval $[0, \beta]$. A point in thermal superspace has therefore coordinates

$$\hat{X} = (x^\mu, \hat{\theta}^\alpha(t), \hat{\bar{\theta}}^{\dot{\alpha}}(t)),$$

(1.1)

where a “hat” is used to denote thermal quantities, and $\hat{\theta}^\alpha(t), \hat{\bar{\theta}}^{\dot{\alpha}}(t)$ are subject to the antiperiodicity conditions

$$\hat{\theta}^\alpha(t + i\beta) = -\hat{\theta}^\alpha(t), \quad \hat{\bar{\theta}}^{\dot{\alpha}}(t + i\beta) = -\hat{\bar{\theta}}^{\dot{\alpha}}(t).$$

(1.2)

This argument is at the basis of the development of thermal superspace in the following sections.

The last part of the paper considers field representations of the thermal super-Poincaré algebra, focussing on the simplest case of a chiral supermultiplet (Sections 6 and 7). Thermal fields are characterized by their time/temperature periodicity properties: bosonic fields are periodic, while fermionic fields are antiperiodic. We apply our thermal superspace formalism to the free Wess-Zumino model and briefly study the behaviour of the model under thermal supersymmetry transformations, and the structure of its breaking. Starting from the $T = 0$ Wess-Zumino model, we perform the Matsubara mode expansion on the temperature circle of length $\beta$ and derive the resulting three-dimensional action for thermal modes. We obtain the bosonic and fermionic thermal mass spectra. Finally, we derive the variation of the finite temperature model under thermal supersymmetry, using the supercharges obtained in thermal superspace.

\footnote{For a discussion of the difficulties in realizing the correct boundary conditions for thermal fields with constant supersymmetry parameters, see [3].}
2 Thermal formalism for bosons and fermions

As supersymmetric theories include bosonic and fermionic fields, we start this section by briefly reviewing some fundamentals of thermal field theory for bosons and fermions.

We will mainly consider physical systems with free fields immersed in a thermal bath at non-zero temperature $T$. At the level of field averages, this means that we have to replace the vacuum expectation value $\langle 0 | ... | 0 \rangle$ by the thermal average $\langle ... \rangle_\beta$ (to be defined below). The thermal average involves a weighted summation over all accessible eigenstates $|n\rangle$ of the hamiltonian $H$,

$$H|n\rangle = E_n|n\rangle \quad n = 0, 1, \ldots,$$

the spectrum of which is discrete and infinite. The lowest energy state is $|0\rangle$, with energy $E_0$. We assume that the system does not exchange particles with the heat bath, that is, we set the chemical potential to zero. The relevant statistical partition function is the canonical one, given by the trace of the Boltzmann weight,

$$Z(\beta) = \text{Tr} \, e^{-\beta H},$$

where $\beta = 1/T$ is the inverse temperature (we set the Boltzmann constant $k_B = 1$), and the quantum-mechanical trace is

$$\text{Tr} \, O = \sum_n \langle n|O|n \rangle .$$

The thermal average $\langle ... \rangle_\beta$ of an arbitrary field operator $O$ is then defined as

$$\langle O \rangle_\beta \equiv \frac{1}{Z(\beta)} \, \text{Tr} \left( e^{-\beta H} O \right),$$

with the usual Boltzmann weight factor and the normalization $\langle 1 \rangle_\beta = 1$. We also assume that the lowest energy state has $E_0 = 0$ so that, in the zero-temperature limit,

$$\langle O \rangle_\beta \xrightarrow{\beta \to \infty} \langle 0 | O | 0 \rangle .$$

This assumption is merely a normal ordering prescription.

Through the LSZ reduction formulae, a field theory can be defined by the whole set of its Green’s functions. At finite temperature, the Green’s functions are subject to important periodicity constraints in imaginary time, known as the KMS (Kubo-Martin-Schwinger) conditions [15, 16]. We now review the derivation of these conditions for bosonic and fermionic fields, in view of later on discussing the thermal behaviour of superfields, which contain both field statistics in their components.

Consider firstly a free real scalar field $\phi$ carrying no conserved charges. The hamiltonian $H$ being the time evolution operator, the field operator $\phi$ at $x = (t, x)$ (in the Heisenberg picture and with $\hbar = c = 1$) is

$$\phi(x) = \phi(t, x) = e^{iHt} \phi(0, x) e^{-iHt} ,$$

with the usual Boltzmann weight factor and the normalization $\langle 1 \rangle_\beta = 1$. We also assume that the lowest energy state has $E_0 = 0$ so that, in the zero-temperature limit,

$$\langle O \rangle_\beta \xrightarrow{\beta \to \infty} \langle 0 | O | 0 \rangle .$$

This assumption is merely a normal ordering prescription.
with a time coordinate $x^0 = t$ which is allowed to be complex. We now define the \textit{n-point thermal Green's function} $G_{nC}$ to be the thermal average of the $T_C$-ordered product of the Heisenberg field,

$$G_{nC}(x_1, \ldots, x_n) = \langle T_C \phi(x_1) \ldots \phi(x_n) \rangle_\beta.$$  \hspace{1cm} (2.6)

The “path-ordering” operation\footnote{Path-ordering is the thermal generalization of the usual time-ordering of zero-temperature field theory. It prescribes that the fields be arranged depending on the position of their (complex) time variables along a path $C$ taken in the complex time plane \cite{14}.} denoted by $T_C$ is peculiar to the thermal case. For any scalar field $\phi$, path ordering can be defined through

$$T_C \phi(x_1) \phi(x_2) = \theta_C(t_1 - t_2) \phi(x_1) \phi(x_2) + \theta_C(t_2 - t_1) \phi(x_2) \phi(x_1),$$  \hspace{1cm} (2.7)

where the path Heaviside function $\theta_C$ is defined as $\theta_C(t) \equiv \theta(\tau)$ for a path parametrized by $t = z(\tau), \tau \in \mathbb{R}$. The thermal path-ordered propagator then writes

$$D_C(x_1, x_2) = \theta_C(t_1 - t_2) D_C^<(x_1, x_2) + \theta_C(t_2 - t_1) D_C^<(x_1, x_2),$$  \hspace{1cm} (2.8)

where $D_C^<$, $D_C^>$ denote respectively the thermal bosonic two-point functions

$$D_C^>(x_1, x_2) = \langle \phi(x_1) \phi(x_2) \rangle_\beta,$$ \hspace{1cm} (2.9)

$$D_C^<(x_1, x_2) = \langle \phi(x_2) \phi(x_1) \rangle_\beta.$$ \hspace{1cm} (2.10)

The Boltzmann weight $e^{-\beta H}$ can be interpreted as an evolution operator in imaginary time. Indeed, rewriting the bosonic Heisenberg field \cite{15} for a translation in imaginary time by $t = i\beta$, $\beta \in \mathbb{R}$, one gets

$$e^{-\beta H} \phi(t, x) e^{\beta H} = \phi(t + i\beta, x).$$  \hspace{1cm} (2.11)

Expressing \cite{2.9} as

$$D_C^>(x_1, x_2) = \frac{1}{Z(\beta)} \text{Tr} \left[ e^{-\beta H} \phi(x_1) \phi(x_2) \right],$$  \hspace{1cm} (2.12)

using the cyclicity of the thermal trace (upon inserting $e^{\beta H} e^{-\beta H} = 1$) and the evolution in imaginary time \cite{2.11}, one deduces the \textit{bosonic KMS condition} \cite{15, 16}. This condition relates $D_C^>$ and $D_C^<$ through a translation in imaginary time,

$$D_C^>(t_1; x_1, t_2; x_2) = D_C^>(t_1 + i\beta; x_1, t_2; x_2).$$ \hspace{1cm} (2.13)

A similar analysis can be performed for fermionic fields. Fermionic path ordering differs from the bosonic case through a sign,

$$T_C \psi_a(x_1) \overline{\psi}_b(x_2) = \theta_C(t_1 - t_2) \psi_a(x_1) \overline{\psi}_b(x_2) - \theta_C(t_2 - t_1) \overline{\psi}_b(x_2) \psi_a(x_1),$$  \hspace{1cm} (2.14)

with $a, b = 1, \ldots, 4$ for Dirac (four-component) spinors. The thermal path-ordered fermion propagator writes similarly to \cite{2.8},

$$S_{C \ ab}(x_1, x_2) = \theta_C(t_1 - t_2) S_{C \ ab}^>(x_1, x_2) + \theta_C(t_2 - t_1) S_{C \ ab}^<(x_1, x_2),$$  \hspace{1cm} (2.15)
where $S_C^\geq, S_C^\leq$ are the thermal fermionic two-point functions conventionally defined as

\begin{align}
S_C^{\geq}(x_1, x_2) &= \langle \psi_a(x_1) \overline{\psi}_b(x_2) \rangle \gamma, \\
S_C^{\leq}(x_1, x_2) &= -\langle \overline{\psi}_b(x_2) \psi_a(x_1) \rangle \gamma.
\end{align}

(2.16)

(2.17)

Following the same procedure as in the bosonic case, one derives the fermionic KMS condition

\begin{align}
S_C^{\geq}(t_1; x_1, t_2; x_2) &= -S_C^{\leq}(t_1 + i\beta; x_1, t_2; x_2),
\end{align}

(2.18)

which differs from the bosonic one, eq. (2.13), by a relative sign.

Superspace is usually formulated using two-component Weyl spinors, $\psi_\alpha$ and $\overline{\psi}^{\dot{\alpha}}$, with respectively left-handed and right-handed chirality. The relation with Dirac spinors is

$$
\psi_\alpha = \begin{pmatrix} \psi_\alpha \\ \overline{\psi}^{\dot{\alpha}} \end{pmatrix}.
$$

It is then useful to translate the KMS condition for Dirac spinors (2.18) into KMS conditions for two-component spinors $\psi_\alpha$, $\overline{\psi}^{\dot{\alpha}}$. Defining the thermal two-point functions $S_C^\geq$, $S_C^\leq$, for two-component spinors as, e.g.,

\begin{align}
S_C^{\geq}(x_1, x_2) &= \langle \psi_\alpha(x_1) \overline{\psi}^{\dot{\alpha}}(x_2) \rangle \gamma, \\
S_C^{\leq}(x_1, x_2) &= -\langle \overline{\psi}^{\dot{\alpha}}(x_2) \psi_\alpha(x_1) \rangle \gamma,
\end{align}

(2.19)

we derive from (2.18) a fermionic KMS condition for two-component Majorana spinors:

\begin{align}
S_C^{\geq}(t_1; x_1, t_2; x_2) &= -S_C^{\leq}(t_1 + i\beta; x_1, t_2; x_2).
\end{align}

(2.20)

This is the only relation we shall need for practical purposes. But similar relations can be derived for $S_C^{\alpha\beta}$, $S_C^{\dot{\alpha}\dot{\beta}}$ and $S_C^{\dot{\alpha}\beta}$.

### 3 Thermal formalism for superfields

#### 3.1 Thermal propagators for superfields

At zero temperature, a superfield formulation of bosons and fermions is by construction supersymmetric. The spinorial generators of supersymmetry transformations – the supercharges – act like generalized derivatives on superfields, which contain in their expansion the bosonic and fermionic fields as components. And the supersymmetry transformation of superfields encodes the transformations of its components.

As discussed previously, the KMS periodicity conditions provide an essential, mandatory characterization of thermal effects at the level of Green’s functions. If superfield propagators can be defined in a thermal environment, they should of course obey some form of KMS condition. And such a superfield KMS condition should be able to reproduce the KMS boundary conditions for the superfields’ bosonic and fermionic components.
Let us start by recalling some simple facts of the zero-temperature case. \( T = 0 \) chiral superfields, noted \( \phi \), and \( T = 0 \) antichiral superfields, denoted by \( \bar{\phi} \), are defined by the conditions
\[
\overline{D}_\alpha \phi = 0, \quad D_\alpha \bar{\phi} = 0, \tag{3.1}
\]
where the \( T = 0 \) covariant derivatives \( D_\alpha, \overline{D}_\alpha \) write
\[
D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i \sigma^\mu_{\alpha \dot{\alpha}} \overline{\theta}^{\dot{\alpha}} \partial_\mu, \quad \overline{D}_\alpha = \frac{\partial}{\partial \theta^\alpha} - i \theta^\alpha \sigma^\mu_{\alpha \dot{\alpha}} \partial_\mu. \tag{3.2}
\]
Clearly, if \( \phi \) is chiral, \( \bar{\phi} = \phi^\dagger \) is antichiral. A point in (zero-temperature) superspace is defined by the space-time coordinates \( x^\mu \) and by anticommuting (Grassmann) spinor coordinates \( \theta^\alpha \) and \( \overline{\theta}^{\dot{\alpha}} \), which are space-time constants. It can be equivalently defined by chiral coordinates \((y^\mu, \theta^\alpha, \overline{\theta}^{\dot{\alpha}})\),
\[
y^\mu = x^\mu - i \theta^\alpha \sigma^\mu_{\alpha \dot{\alpha}} \overline{\theta}^{\dot{\alpha}}, \quad \overline{D}_\alpha y^\mu = 0, \tag{3.3}
\]
or by antichiral coordinates \((\overline{y}^\mu, \theta^\alpha, \overline{\theta}^{\dot{\alpha}})\),
\[
\overline{y}^\mu = x^\mu + i \theta^\alpha \sigma^\mu_{\alpha \dot{\alpha}} \overline{\theta}^{\dot{\alpha}}, \quad D_\alpha y^\mu = 0. \tag{3.4}
\]
In these variables, it follows from their defining equations (3.1) that the expansions of chiral and antichiral superfields are simply
\[
\phi(y, \theta) = z(y) + \sqrt{2} \theta \psi(y) - \theta \theta f(y), \tag{3.5}
\]
\[
\bar{\phi}(\overline{y}, \overline{\theta}) = \overline{z}(\overline{y}) + \sqrt{2} \theta \overline{\psi}(\overline{y}) - \theta \theta \overline{f}(\overline{y}). \tag{3.6}
\]
The components of the superfields \( \phi \) and \( \bar{\phi} \) form a chiral multiplet, which contains two complex scalar fields \( z \) and \( f \) and a Majorana spinor \( \psi \) with Weyl components \( \psi^\alpha \) and \( \overline{\psi}^{\dot{\alpha}} \).

Consider now the superfield propagator \( \langle 0 | T \phi(y_1, \theta_1) \bar{\phi}(\overline{y}_2, \overline{\theta}_2) | 0 \rangle \). Its expansion in powers of the Grassmann coordinates includes the Green’s functions
\[
D(x_1 - x_2) \equiv \langle 0 | T z(x_1) \overline{z}(x_2) | 0 \rangle, \tag{3.8}
\]
\[
\mathcal{F}(x_1 - x_2) \equiv \langle 0 | T f(x_1) \overline{f}(x_2) | 0 \rangle, \tag{3.9}
\]
\[
S^\beta_\alpha (x_1 - x_2) \equiv \langle 0 | T \psi^\alpha(x_1) \overline{\psi}^{\beta}(x_2) | 0 \rangle, \tag{3.10}
\]

\(^3\)The spinorial component of the chiral superfield \( \phi \) is the two-component left-handed Weyl spinor
\[
\psi_L = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}.
\]
For a Majorana spinor, we have the relation (provided by \( \bar{\phi} = \phi^\dagger \))
\[
\psi_R = \begin{pmatrix} 0 \\ \overline{\psi}^{\dot{\alpha}} \end{pmatrix} = (\psi_L)^c = -\gamma^0 C \psi_L^\dagger C \tag{3.7}.
\]

\(^4\)Recall that \( y_1 \) is a function of \( x_1, \theta_1 \) and \( \overline{\theta}_1 \), and similarly \( y_2 \) is a function of \( x_2, \theta_2 \) and \( \overline{\theta}_2 \).
as well as Green’s functions for the first or second derivatives of the component fields:

\[ \langle 0 | T \phi(y_1, \theta_1) \bar{\phi}(y_2, \bar{\theta}_2) | 0 \rangle = D(y_1 - \bar{y}_2) - 2\theta_1^\alpha \bar{\theta}_2^\beta S_\alpha^\beta(y_1 - \bar{y}_2) + \theta_1 \theta_1 \bar{\theta}_2 \bar{\theta}_2 F(y_1 - \bar{y}_2) \]

\[ = D(x_1 - x_2) - 2\theta_1^\alpha \bar{\theta}_2^\beta S_\alpha^\beta(x_1 - x_2) + \theta_1 \theta_1 \bar{\theta}_2 \bar{\theta}_2 F(x_1 - x_2) \]

+ derivative terms.

(3.11)

A similar expansion can be performed for \( \langle 0 | T \phi(y_1, \theta_1) \phi(y_2, \theta_2) | 0 \rangle \) and for the conjugate \( \langle 0 | T \bar{\phi}(\bar{y}_1, \bar{\theta}_1) \bar{\phi}(\bar{y}_2, \bar{\theta}_2) | 0 \rangle \).

We now put this system of propagating component fields in a thermal bath at finite temperature \( T \). The heat bath is expected to affect propagation. The definitions of the thermal propagators are a straightforward “thermalization”, along the lines of Section 2, of eqs. (3.8)-(3.10):

\[ D_C(x_1 - x_2) \equiv \langle T_C z(x_1) \bar{z}(x_2) \rangle_\beta, \]

(3.12)

\[ F_C(x_1 - x_2) \equiv \langle T_C f(x_1) \bar{f}(x_2) \rangle_\beta, \]

(3.13)

\[ S_C^\alpha_\beta(x_1 - x_2) \equiv \langle T_C \psi_\alpha(x_1) \bar{\psi}_\beta(x_2) \rangle_\beta. \]

(3.14)

Simultaneously, one has to require that each of these thermal propagators for the components of chiral and antichiral superfields obey KMS conditions. The relevant condition depends on the statistics of the component fields in the propagator. That is, thermal propagators of scalar components must obey the bosonic KMS condition (2.13),

\[ D_C^\alpha(t_1; x_1, t_2; x_2) = D_C^\alpha(t_1 + i\beta; x_1, t_2; x_2), \]

(3.15)

\[ F_C^\alpha(t_1; x_1, t_2; x_2) = F_C^\alpha(t_1 + i\beta; x_1, t_2; x_2), \]

(3.16)

while the thermal propagator of the fermionic components has to satisfy the fermionic constraint (2.20),

\[ S_C^\alpha_\beta(t_1; x_1, t_2; x_2) = -S_C^\alpha_\beta(t_1 + i\beta; x_1, t_2; x_2). \]

(3.17)

3.2 Super-KMS condition and thermal superspace

In the Introduction, we have motivated the fact that, at finite temperature, the superspace Grassmann variables should be dependent on imaginary time and antiperiodic. Consequently, we promote \( \theta \) and \( \bar{\theta} \) to become time-dependent coordinates, \( \theta \to \hat{\theta} = \hat{\theta}(t) \), \( \bar{\theta} \to \hat{\bar{\theta}} = \hat{\bar{\theta}}(t) \) with the antiperiodicity properties (1.2),

\[ \hat{\theta}(t + i\beta) = -\hat{\theta}(t), \quad \hat{\bar{\theta}}(t + i\beta) = -\hat{\bar{\theta}}(t). \]

(3.18)

These conditions induce a temperature-dependent constraint on the time-dependent superspace Grassmann coordinates \( \hat{\theta}(t) \) and \( \hat{\bar{\theta}}(t) \). Notice that while the introduction of a dependence on time in \( \theta \) is a local statement, which should then be visible in the explicit form of space-time symmetry generators, the above antiperiodicity conditions are global
statements. The latter are not expected to affect symmetry generators, at least in their classical expressions. But they will appear in the quantum theory, in the definition of the space of physical states.

We shall call thermal superspace the space spanned by the variables \([x^\mu, \hat{\theta}(t), \hat{\varphi}(t)]\), with \(\hat{\theta}(t)\) and \(\hat{\varphi}(t)\) obeying the conditions (3.18). Note that in taking superspace Grassmann coordinates \(\hat{\theta}(t), \hat{\varphi}(t)\), we also introduce a formal time-dependence in the second terms of the variables \(y\) and \(\eta\) (3.3)-(3.4). The implications of this fact will be discussed later on. For the moment, we only keep track of the \(t\)-dependence with the notation \(\hat{y}_i(t)\), \(\hat{\eta}_i(t)\):

\[
\hat{y}_i(t) = x^\mu - i\bar{\theta}(t)\sigma^\mu \hat{\varphi}(t), \quad \hat{\eta}_i(t) = x^\mu + i\bar{\theta}(t)\sigma^\mu \hat{\varphi}(t).
\] (3.19)

In the present Section, we show that the use of thermal superspace makes it possible to impose KMS conditions at the level of thermal superfield propagators. These superfield KMS conditions will yield the correct bosonic (3.15)-(3.16) and fermionic (3.17) KMS conditions for the thermal propagators of the superfield components (paragraph 3.4). Furthermore, we will see how the antiperiodicity (3.11) allows for a direct proof of the super-KMS condition in thermal superspace (paragraph 3.3).

To start with, we define chiral and antichiral superfields at finite temperature, denoted by the “hat” notation \(\hat{\phi}\), resp. \(\bar{\phi}\), similarly to (3.5), (3.6), but with the thermal superspace Grassmann coordinates \(\hat{\theta}(t), \hat{\varphi}(t)\) as the expansion parameters. This yields for \(\hat{\phi}\),

\[
\hat{\phi} [\hat{y}_i(t), \hat{\theta}(t)] = z[\hat{y}_i(t)] + \sqrt{2} \bar{\theta}(t) \psi[\hat{y}_i(t)] - \hat{\theta}(t)f[\hat{y}_i(t)],
\] (3.20)

whereas for \(\bar{\phi}\) we write

\[
\bar{\phi} [\hat{y}_i(t), \hat{\theta}(t)] = \bar{z}[\hat{y}_i(t)] + \sqrt{2} \bar{\theta}(t) \bar{\psi}[\hat{y}_i(t)] - \hat{\theta}(t)\bar{f}[\hat{y}_i(t)].
\] (3.21)

Since thermal chiral and antichiral superfields are bosonic objects, and are therefore periodic, these expansions are consistent with the fact that at finite temperature bosonic fields are periodic in imaginary time, while fermionic fields are antiperiodic. Moreover, as in the zero-temperature case, these thermal chiral and antichiral superfields can be seen as solutions of conditions which generalize \(D_\alpha \phi = 0, D_\alpha \bar{\phi} = 0\) [eqs. (3.1)] to the thermal context. We postpone this point to Section 4, in which the thermal covariant derivatives are constructed.

Following the same prescription of making \(\theta\) and \(\varphi\) time-dependent and antiperiodic, we next define the chiral-antichiral superfield propagator at finite temperature to be the thermal generalization of eq. (3.11),

\[
G_C [\hat{y}_1(t_1), \hat{y}_2(t_2); \hat{\theta}_1(t_1), \hat{\theta}_2(t_2)] \equiv \langle T_C \hat{\phi} [\hat{y}_1(t_1), \hat{\theta}_1(t_1)] \bar{\phi} [\hat{y}_2(t_2), \hat{\theta}_2(t_2)] \rangle \phi,
\] (3.22)

and expand it in thermal superspace as

\[
G_C [\hat{y}_1(t_1), \hat{y}_2(t_2); \hat{\theta}_1(t_1), \hat{\theta}_2(t_2)] = D_C [\hat{y}_1(t_1) - \hat{y}_2(t_2)] - 2 \hat{\theta}_1^\alpha(t_1) \hat{\theta}_2^\beta(t_2) S_{C, \alpha \beta} [\hat{y}_1(t_1) - \hat{y}_2(t_2)] + \hat{\theta}_1(t_1) \hat{\theta}_2(t_2) \hat{\theta}_2(t_2) \hat{\theta}_2(t_2) \mathcal{F}_C [\hat{y}_1(t_1) - \hat{y}_2(t_2)].
\]
can be obtained from the following perfield propagators. The correct component’s KMS conditions (3.15)-(3.16) and (3.17) with the time-translated variable \( \hat{\sigma}(t) \). Suppose the theory has a hamiltonian \( H \) with a generator of time translations for the time of the superspace Grassmann variables. The main ingredients of the proof are the cyclicity of the thermal trace, the superfield version of the antiperiodicity conditions (3.18) for the thermal super-space coordinates \( \hat{\sigma}(t) \) and \( \hat{\sigma}(t) \).

3.3 Proof of the super-KMS condition

Clearly, the superfield KMS condition (3.27) is of bosonic type, since chiral and antichiral superfields are bosonic objects. This condition can be proven at the superfield level in a way similar to the case of the scalar field case of Section 4. At the superfield level, the main ingredients of the proof are the cyclicity of the thermal trace, the superfield version of the evolution in imaginary time, and, most essential, the antiperiodicity in imaginary time of the superspace Grassmann variables.

Let us start by formulating the evolution in imaginary time [eq. (2.11)] for a thermal superfield. Suppose the theory has a hamiltonian \( H \) for the fields \( z, \psi \) and \( f \), which is the generator of time translations for the \( x \)-dependence of these fields. Applying this evolution
to the various components in the development of, e.g., a thermal chiral superfield \( \hat{\phi} \) [eq. (3.21)], we get
\[
e^{-\beta H} \hat{\phi}[\hat{y}(t), \hat{\theta}(t)] e^{\beta H} = z[\hat{y}_i^0(t) + i\beta; \hat{y}] + \sqrt{2} \hat{\theta}(t) \psi[\hat{y}_i^0(t) + i\beta; \hat{y}] - \hat{\theta}(t) \hat{\theta}(t) f[\hat{y}_i^0(t) + i\beta; \hat{y}] = \hat{\phi}[\hat{y}_i^0(t) + i\beta; \hat{y}, \hat{\theta}(t)].
\] (3.29)

Similarly, for an antichiral superfield \( \hat{\varphi} \), the imaginary time evolution applied to (3.21) yields
\[
e^{-\beta H} \hat{\varphi}[\hat{y}(t), \hat{\theta}(t)] e^{\beta H} = \hat{\varphi}[\hat{y}_i^0(t) + i\beta; \hat{y}, \hat{\theta}(t)].
\] (3.30)

Because of the antiperiodicity of the Grassmann variables, eq. (3.18), and with eq. (3.28),
\[
\hat{\phi}[\hat{y}_i^0(t) + i\beta; \hat{y}, \hat{\theta}(t)] = \hat{\phi}[\hat{y}_i(t_{i+\beta}), \hat{\theta}(t)],
\]
\[
\hat{\varphi}[\hat{y}_i^0(t) + i\beta; \hat{y}, \hat{\theta}(t)] = \hat{\varphi}[\hat{y}_i(t_{i+\beta}), \hat{\theta}(t)].
\]

As a consequence, the thermal superfield evolution is governed by
\[
e^{-\beta H} \hat{\phi}[\hat{y}(t), \hat{\theta}(t)] e^{\beta H} = z[\hat{y}_i(t_{i+\beta}) + i\beta; \hat{y}] + \sqrt{2} \hat{\theta}(t) \psi[\hat{y}_i(t_{i+\beta}) + i\beta; \hat{y}] - \hat{\theta}(t) \hat{\theta}(t) f[\hat{y}_i(t_{i+\beta})] = \hat{\phi}[\hat{y}_i(t_{i+\beta}), \hat{\theta}(t)]
\] (3.31)
for \( \hat{\phi} \), and by a similar equation for \( \hat{\varphi} \):
\[
e^{-\beta H} \hat{\varphi}[\hat{y}(t), \hat{\theta}(t)] e^{\beta H} = \hat{\varphi}[\hat{y}(t_{i+\beta}), \hat{\theta}(t)].
\] (3.32)

Note that, in the above, the time argument of \( \hat{\theta}(t) \) and \( \hat{\theta}(t) \) has not been shifted. The thermal Grassmann variables – which are coordinates – do not undergo dynamical evolution in imaginary time generated by the hamiltonian, which only acts on fields, i.e.,
\[
e^{-\beta H} \hat{\theta}(t) e^{\beta H} = \hat{\theta}(t), \quad e^{-\beta H} \hat{\theta}(t) e^{\beta H} = \hat{\theta}(t).
\] (3.33)

In order to prove the superfield KMS relation (3.27) in thermal superspace, we start from the thermal superfield two-point function \( G_C^\gamma \) (3.25),
\[
G_C^\gamma [\hat{y}_i(t_1), \hat{y}_j(t_2), \hat{\theta}_i(t_1), \hat{\theta}_j(t_2)] = \frac{1}{Z(\beta)} \text{Tr} \left\{ e^{-\beta H} \hat{\phi}[\hat{y}_i(t_1), \hat{\theta}_i(t_1)] e^{\beta H} \hat{\phi}[\hat{y}_j(t_2), \hat{\theta}_j(t_2)] \right\},
\] (3.34)
and introduce the thermal component expansions for the superfields [eqs. (3.21)-(3.24)]. We then rotate cyclically \( \hat{\phi} \) to the front, and insert the identity \( e^{\beta H} e^{-\beta H} = 1 \) in the right side:
\[
\frac{1}{Z(\beta)} \text{Tr} \left\{ e^{-\beta H} \left( z[\hat{y}_1] + \sqrt{2} \hat{\theta}_1 \psi[\hat{y}_1] - \hat{\theta}_1 \hat{\theta}_1 f[\hat{y}_1] \right) \left( \overline{z}[\hat{y}_2] + \sqrt{2} \overline{\theta}_2 \overline{\psi}[\hat{y}_2] - \overline{\theta}_2 \overline{\theta}_2 \overline{f}[\hat{y}_2] \right) \right\}
\]
\[
= \frac{1}{Z(\beta)} \text{Tr} \left\{ \left( \overline{z}[\hat{y}_2] - \sqrt{2} \overline{\theta}_2 \overline{\psi}[\hat{y}_2] - \overline{\theta}_2 \overline{\theta}_2 \overline{f}[\hat{y}_2] \right) \right. \\
\times e^{-\beta H} \left. \left( z[\hat{y}_1] + \sqrt{2} \hat{\theta}_1 \psi[\hat{y}_1] - \hat{\theta}_1 \hat{\theta}_1 f[\hat{y}_1] \right) e^{\beta H} \right\}.
\]

\footnote{To simplify the notation, we occasionally use \( \hat{y}_i, \hat{\theta}_i \) instead of \( \hat{y}_i(t_i), \hat{\theta}_i(t_i) \) in non-ambiguous situations.}
Here the negative sign in front of one of the fermionic components follows from the anticommutativity of the Grassmann variables. Indeed, while fermion fields do not generate a sign when crossed in a cyclic rotation of the quantum mechanical trace, Grassmann parameters do. We now use time evolution. Upon cyclically rotating a Boltzmann factor to the front, and using the superfield evolution eq. (3.31), our last expression can be rewritten as:

\[
\frac{1}{Z(\beta)} \text{Tr} \left\{ e^{-\beta H} \left( z[\tilde{g}_2] - \sqrt{2} \tilde{\theta}_2 \tilde{\psi} [\tilde{g}_2] - \tilde{\theta}_2 \tilde{\psi} [\tilde{g}_2] \right) \right\} 
\times e^{-\beta H} \left( z[\tilde{y}_1] + \sqrt{2} \tilde{\theta}_1 \tilde{\psi} [\tilde{y}_1] - \tilde{\theta}_1 \tilde{\theta}_1 f[\tilde{y}_1] \right) e^{\beta H} = \frac{1}{Z(\beta)} \text{Tr} \left\{ e^{-\beta H} \left( z[\tilde{g}_2(t_2)] - \sqrt{2} \tilde{\theta}_2(t_2) \tilde{\psi} [\tilde{g}_2(t_2)] - \tilde{\theta}_2(t_2) \tilde{\psi} [\tilde{g}_2(t_2)] \right) \right\} 
\times \left( z[\tilde{y}_1(t_1 + i\beta)] + \sqrt{2} \tilde{\theta}_1(t_1 + i\beta) \tilde{\psi} [\tilde{y}_1(t_1 + i\beta)] - \tilde{\theta}_1(t_1 + i\beta) \tilde{\theta}_1(t_1 + i\beta) f[\tilde{y}_1(t_1 + i\beta)] \right) \right\}. \tag{3.35}
\]

At this point, we make use of the antiperiodicity (3.18) of the Grassmann variables and set \( \tilde{\theta}(t_1) = -\tilde{\theta}(t_1 + i\beta) \). As a result, the right side of (3.35) rewrites as

\[
\frac{1}{Z(\beta)} \text{Tr} \left\{ e^{-\beta H} \left( z[\tilde{g}_2(t_2)] - \sqrt{2} \tilde{\theta}_2(t_2) \tilde{\psi} [\tilde{g}_2(t_2)] - \tilde{\theta}_2(t_2) \tilde{\psi} [\tilde{g}_2(t_2)] \right) \right\} 
\times \left( z[\tilde{y}_1(t_1 + i\beta)] - \sqrt{2} \tilde{\theta}_1(t_1 + i\beta) \tilde{\psi} [\tilde{y}_1(t_1 + i\beta)] - \tilde{\theta}_1(t_1 + i\beta) \tilde{\theta}_1(t_1 + i\beta) f[\tilde{y}_1(t_1 + i\beta)] \right) \right\}. \tag{3.36}
\]

Since fermionic fields do not propagate into bosonic fields, and vice-versa, the two negative signs in front of the fermionic components above can equivalently be replaced by two positive signs. The computation (3.34)-(3.36) therefore yields

\[
G_C^\gamma [\tilde{g}_1(t_1), \tilde{g}_2(t_2), \tilde{\theta}_1(t_1), \tilde{\theta}_2(t_2)] = \frac{1}{Z(\beta)} \text{Tr} \left\{ e^{-\beta H} \left( z[\tilde{g}_2(t_2)] + \sqrt{2} \tilde{\theta}_2(t_2) \tilde{\psi} [\tilde{g}_2(t_2)] - \tilde{\theta}_2(t_2) \tilde{\psi} [\tilde{g}_2(t_2)] \right) \right\} 
\times \left( z[\tilde{y}_1(t_1 + i\beta)] + \sqrt{2} \tilde{\theta}_1(t_1 + i\beta) \tilde{\psi} [\tilde{y}_1(t_1 + i\beta)] - \tilde{\theta}_1(t_1 + i\beta) \tilde{\theta}_1(t_1 + i\beta) f[\tilde{y}_1(t_1 + i\beta)] \right) \right\}. \tag{3.37}
\]

Realizing that the second line is just the thermal superfield \( \tilde{\phi} \) (3.20) with all time arguments shifted by \( i\beta \), we rewrite (3.37) as

\[
G_C^\gamma [\tilde{g}_1(t_1), \tilde{g}_2(t_2), \tilde{\theta}_1(t_1), \tilde{\theta}_2(t_2)] = G_C^\gamma [\tilde{y}_1(t_1 + i\beta), \tilde{g}_2(t_2), \tilde{\theta}_1(t_1 + i\beta), \tilde{\theta}_2(t_2)]. \tag{3.38}
\]

This is the superfield KMS condition (3.27), which is hereby proved.

### 3.4 Component KMS conditions from super-KMS

We verify in this paragraph that the superfield KMS condition (3.27) yields the expected component relations (3.17)-(3.16) and (3.17). As we shall see, the antiperiodicity of the
thermal Grassmann variables is again an essential ingredient. This is done by expanding in eqs. (3.23)–(3.26) the thermal chiral and antichiral superfields \( \hat{\phi} \) and \( \tilde{\phi} \) in components, using eqs. (3.20) and (3.21):

\[
G_C^\wedge [\hat{y}_1(t_1), \hat{y}_2(t_2), \hat{\theta}_1(t_1), \hat{\theta}_2(t_2)] = D_C^\wedge [\hat{y}_1(t_1), \hat{y}_2(t_2)] - 2\hat{\theta}_1^\alpha(t_1)\hat{\theta}_2^\beta(t_2) S_C^\alpha{}\beta [\hat{y}_1(t_1), \hat{y}_2(t_2)]
+ \hat{\theta}_1(t_1)\hat{\theta}_1(t_1)\hat{\theta}_2(t_2)\hat{\theta}_2(t_2) F_C^\wedge [\hat{y}_1(t_1), \hat{y}_2(t_2)],
\]

\[
(3.39)
\]

\[
G_C^\wedge [\hat{y}_1(t_1), \hat{y}_2(t_2), \hat{\theta}_1(t_1), \hat{\theta}_2(t_2)] = D_C^\wedge [\hat{y}_1(t_1), \hat{y}_2(t_2)] - 2\hat{\theta}_1^\alpha(t_1)\hat{\theta}_2^\beta(t_2) S_C^\wedge{}^\alpha \beta [\hat{y}_1(t_1), \hat{y}_2(t_2)]
+ \hat{\theta}_1(t_1)\hat{\theta}_1(t_1)\hat{\theta}_2(t_2)\hat{\theta}_2(t_2) F_C^\wedge [\hat{y}_1(t_1), \hat{y}_2(t_2)].
\]

\[
(3.40)
\]

The superfield KMS condition (3.24) leads then to the following:

(i) For the scalar component,

\[
D_C^\wedge [\hat{y}_1(t_1), \hat{y}_2(t_2)] = D_C^\wedge [\hat{y}_1(t_1+i\beta), \hat{y}_2(t_2)],
\]

which reduces, when returning to variables \( x = (t; x) \), to the bosonic KMS relation (3.13),

\[
D_C(t_1; x_1, t_2; x_2) = D_C(t_1 + i\beta; x_1, t_2; x_2).
\]

(ii) For the fermionic component,

\[
\hat{\theta}_1^\alpha(t_1)\hat{\theta}_2^\beta(t_2) S_C^\alpha \beta [\hat{y}_1(t_1), \hat{y}_2(t_2)] = \hat{\theta}_1^\alpha(t_1 + i\beta)\hat{\theta}_2^\beta(t_2) S_C^\wedge{}^\alpha \beta [\hat{y}_1(t_1+i\beta), \hat{y}_2(t_2)].
\]

\[
(3.42)
\]

With the antiperiodicity condition (3.18), \( \hat{\theta}_1^\alpha(t_1 + i\beta) = -\hat{\theta}_1^\alpha(t_1) \), we obtain

\[
S_C^\alpha \beta [\hat{y}_1(t_1), \hat{y}_2(t_2)] = -S_C^\wedge{}^\alpha \beta [\hat{y}_1(t_1+i\beta), \hat{y}_2(t_2)],
\]

\[
(3.43)
\]

which yields, in the variables \( x = (t; x) \), the fermionic KMS condition (3.17) with the correct relative sign, \( S_C^\wedge{}^\alpha \beta (t_1; x_1, t_2; x_2) = -S_C^\alpha \beta (t_1 + i\beta; x_1, t_2; x_2) \).

(iii) For the auxiliary field, one gets from (3.27):

\[
\hat{\theta}_1(t_1)\hat{\theta}_1(t_1)\hat{\theta}_2(t_2)\hat{\theta}_2(t_2) F_C^\wedge [\hat{y}_1(t_1), \hat{y}_2(t_2)]
= \hat{\theta}_1(t_1 + i\beta)\hat{\theta}_1(t_1 + i\beta)\hat{\theta}_2(t_2)\hat{\theta}_2(t_2) F_C^\wedge [\hat{y}_1(t_1+i\beta), \hat{y}_2(t_2)].
\]

\[
(3.44)
\]

With \( \hat{\theta}_1(t_1 + i\beta) = -\hat{\theta}_1(t_1) \) (3.18) and in the variables \( x = (t; x) \), this is the bosonic KMS condition (3.14), \( F_C(t_1; x_1, t_2; x_2) = F_C(t_1 + i\beta; x_1, t_2; x_2) \).

Finally, we recall that we have considered here only the chiral-antichiral thermal superfield two-point function which, at zero temperature, contains the kinetic propagators for the scalar and spinor fields. Mass contributions to propagators would arise from the chiral-chiral Green’s function \( \langle T_C \hat{\phi} \hat{\phi} \rangle \beta \), or from its conjugate \( \langle T_C \tilde{\phi} \tilde{\phi} \rangle \beta \). To each of these cases, there corresponds a superfield KMS condition. Since their treatment on thermal superspace is entirely similar to the chiral-antichiral case discussed above, we don’t consider them explicitly here.
4 Thermal covariant derivatives and thermal supercharges

Up to this point, our construction of thermal superspace has been somewhat academic. We have simply translated in a superfield formalism results of finite temperature field theory for the fields of the chiral supermultiplet. The price has been to introduce a dependence on time/temperature in Grassmann superspace coordinates, and also in chiral space-time coordinates \( y^\mu \) and \( \overline{y}^\mu \). In contrast to the antiperiodic \( \hat{\theta} \) variables, the latter dependence is periodic in \( t \rightarrow t + i\beta \) since \( y^\mu \) and \( \overline{y}^\mu \) are bosonic quantities. The main interest in superspace lies however in the natural representation it provides for the super-Poincaré algebra in terms of derivatives with respect to superspace coordinates. A function on superspace – a superfield – carries then automatically a representation of supersymmetry. The purpose of this section is to construct the supersymmetry generators and covariant derivatives on thermal superspace. Their existence is not a surprise. The algebra reflects local properties in superspace while the distinction between fermions and bosons at finite temperature is related to a global property: periodicity or antiperiodicity of fields around the temperature circle with radius \( \beta/2\pi \). However, the existence of a supersymmetry algebra on thermal superspace should not be assimilated to a statement that supersymmetry does not break at finite \( T \). That such an algebra exists does not imply that a supersymmetric field theory can be constructed carrying the same symmetry algebra. We shall come back to this point in Section 7.

Deriving expressions for the supercharges and the covariant derivatives on thermal superspace can be done simply by performing the change of variables from usual, zero temperature, superspace to thermal superspace, \( i.e., : \)

\[
(x^\mu, \theta, \overline{\theta}) \rightarrow \left( x'^\mu = x^\mu, \theta' = \hat{\theta}(t), \overline{\theta}' = \hat{\overline{\theta}}(t) \right),
\]

with \( t = x^0 \). Under this change of variables, the partial derivatives with respect to \( x, \theta \) and \( \overline{\theta} \) transform trivially,

\[
\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x'} = \frac{\partial}{\partial x}, (4.1)
\]

\[
\frac{\partial}{\partial \theta^\alpha} \rightarrow \frac{\partial}{\partial \theta'^\alpha} = \frac{\partial}{\partial \theta^\alpha}, (4.2)
\]

\[
\frac{\partial}{\partial \overline{\theta}^\dot{\alpha}} \rightarrow \frac{\partial}{\partial \overline{\theta}'^\dot{\alpha}} = \frac{\partial}{\partial \overline{\theta}^\dot{\alpha}}, (4.3)
\]

while the time derivative has to take the time-dependence of the thermal Grassmann variables into account:

\[
\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t'} + \frac{\partial \theta'^\alpha}{\partial t} \frac{\partial}{\partial \theta^\alpha} + \frac{\partial \overline{\theta}'^{\dot{\alpha}}}{\partial t} \frac{\partial}{\partial \overline{\theta}^\dot{\alpha}}, \quad \left( \frac{\partial t'}{\partial t} = 1 \right). (4.4)
\]

Consequently, we define the partial time derivative at finite temperature as

\[
\frac{\hat{\partial}}{\partial t} \equiv \frac{\partial}{\partial t} - \Delta. (4.5)
\]
We call this object the thermal time derivative; $\Delta$ accounts for the thermal corrections,

$$\Delta = \frac{\partial \hat{\theta}^\alpha}{\partial t} \frac{\partial}{\partial \hat{\theta}^\alpha} + \frac{\partial \hat{\theta}^{\dot{\alpha}}}{\partial t} \frac{\partial}{\partial \hat{\theta}^{\dot{\alpha}}}. \quad (4.6)$$

Accordingly, we also define a thermal space-time derivative as

$$\hat{\partial}_\mu = \left( \frac{\partial}{\partial t} - \Delta ; \frac{\partial}{\partial x} \right). \quad (4.7)$$

Let us now construct the thermal covariant derivatives. We proceed by replacing, in the expressions of the (zero-temperature) covariant derivatives (3.2), the zero-temperature Grassmann variables and derivative operators by their thermal counterparts. This means that (i) we replace the zero-temperature, constant Grassmann parameters $\theta, \bar{\theta}$ by the thermal, time-dependent and antiperiodic parameters $\hat{\theta}, \hat{\bar{\theta}}$, and that (ii) the derivative operators $\partial/\partial \theta$, $\partial/\partial \bar{\theta}$ and $\partial/\partial \hat{\theta}$, $\partial/\partial \hat{\bar{\theta}}$ are replaced by $\hat{\partial}^\mu$, $\partial/\partial \hat{\theta}$ and $\partial/\partial \hat{\bar{\theta}}$. This yields thermal covariant derivatives $\hat{D}_\alpha$ and $\hat{D}^{\dot{\alpha}}$ in the form:

$$\hat{D}_\alpha = \frac{\partial}{\partial \hat{\theta}^\alpha} - i \sigma^\mu_{\alpha\dot{\alpha}} \hat{\partial}_\mu, \quad \hat{D}^{\dot{\alpha}} = \frac{\partial}{\partial \hat{\bar{\theta}}^{\dot{\alpha}}} - i \hat{\theta}^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \hat{\partial}_\mu, \quad (4.8)$$

which write explicitly, using eqs. (4.7) and (4.6), as

$$\hat{D}_\alpha = \frac{\partial}{\partial \hat{\theta}^\alpha} - i \sigma^\mu_{\alpha\dot{\alpha}} \hat{\partial}_\mu + i \sigma^\alpha_{\alpha\dot{\gamma}} \left( \frac{\partial \hat{\theta}^\gamma}{\partial t} \frac{\partial}{\partial \hat{\theta}^\gamma} + \frac{\partial \hat{\bar{\theta}}^{\dot{\gamma}}}{\partial t} \frac{\partial}{\partial \hat{\bar{\theta}}^{\dot{\gamma}}} \right), \quad (4.9)$$

$$\hat{D}^{\dot{\alpha}} = \frac{\partial}{\partial \hat{\bar{\theta}}^{\dot{\alpha}}} - i \hat{\theta}^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \hat{\partial}_\mu + i \sigma^\alpha_{\alpha\dot{\gamma}} \left( \frac{\partial \hat{\theta}^\gamma}{\partial t} \frac{\partial}{\partial \hat{\theta}^\gamma} + \frac{\partial \hat{\bar{\theta}}^{\dot{\gamma}}}{\partial t} \frac{\partial}{\partial \hat{\bar{\theta}}^{\dot{\gamma}}} \right). \quad (4.10)$$

In order to validate these expressions, we observe that they play, in thermal superspace, the same role as the usual covariant derivatives of supersymmetry in $T = 0$ superspace.

Firstly, the thermal covariant derivatives obey the same anticommutation relations as at $T = 0$. This can readily be checked by direct computation of the anticommutators. One obtains, in perfect analogy to the $T = 0$ case,

$$\{\hat{D}_\alpha, \hat{D}^{\dot{\alpha}}\} = -2i \sigma^\mu_{\alpha\dot{\alpha}} \hat{\partial}_\mu, \quad \{\hat{D}_\alpha, \hat{D}_\beta\} = \{\hat{D}^{\dot{\alpha}}, \hat{D}^{\dot{\beta}}\} = 0. \quad (4.11)$$

This is actually obvious upon noticing that the thermal space-time derivative $\hat{\partial}_\mu$ gives zero when acting on the $t$-dependent variables $\hat{\theta}, \hat{\bar{\theta}}$, since

$$\hat{\partial}_0 \hat{\theta}^\alpha = \frac{\partial \hat{\theta}^\alpha}{\partial t} - \frac{\partial \hat{\theta}^\gamma}{\partial t} \delta_\alpha^\gamma = 0, \quad \hat{\partial}_0 \hat{\bar{\theta}}^{\dot{\alpha}} = \frac{\partial \hat{\bar{\theta}}^{\dot{\alpha}}}{\partial t} - \frac{\partial \hat{\bar{\theta}}^{\dot{\gamma}}}{\partial t} \delta^{\dot{\alpha}}_{\dot{\gamma}} = 0, \quad (4.12)$$

and plays therefore the same role for the thermal Grassmann variables as that of the usual space-time derivative for the $t$-independent, non thermal $\theta$ and $\bar{\theta}$. In this sense, the
thermal time (and consequently the thermal space-time) derivative is a covariantization, with respect to thermal superspace, of the zero-temperature time (space-time) derivative.

Secondly, the thermal covariant derivatives $\hat{D}_\alpha$ and $\hat{D}_\dot{\alpha}$ provide a definition of the thermal chiral and antichiral superfields $\hat{\phi}$ and $\hat{\bar{\phi}}$, eqs. (3.20) and (3.21), as the solution to the thermal generalization of the conditions (3.1):

$$\hat{D}_\alpha \hat{\phi} = 0, \quad \hat{D}_{\dot{\alpha}} \hat{\phi} = 0.$$  
(4.13)

We could have actually derived the thermal covariant derivatives directly from the requirements

$$\hat{D}_\alpha \hat{y}^\mu = 0, \quad \hat{D}_{\dot{\alpha}} \hat{y}^\mu = 0,$$

which are equivalent to the chirality conditions (4.13).

Quite naturally, our next aim is to derive the thermal supersymmetry charges. The zero-temperature supercharges are, in our conventions:

$$Q_\alpha = -i \frac{\partial}{\partial \theta^\alpha} + \sigma^\mu_{\alpha a} \hat{\bar{\theta}}^a \partial_\mu, \quad \bar{Q}_{\dot{\alpha}} = i \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - \theta^\alpha \sigma^\mu_{\alpha a} \partial_\mu.$$  
(4.14)

The corresponding thermal objects are constructed using the same procedure as the one used above for the thermal covariant derivatives. This yields the following expressions for the thermal supercharges:

$$\hat{Q}_\alpha = -i \frac{\partial}{\partial \hat{\theta}^\alpha} + \sigma^\mu_{\alpha a} \hat{\bar{\theta}}^a \partial_\mu, \quad \hat{Q}_{\dot{\alpha}} = i \frac{\partial}{\partial \hat{\bar{\theta}}^{\dot{\alpha}}} - \hat{\theta}^{\dot{\alpha}} \sigma^\mu_{\dot{\alpha} a} \partial_\mu.$$  
(4.15)

or, in a compact form,

$$\hat{Q}_\alpha = -i \frac{\partial}{\partial \hat{\theta}^\alpha} + \sigma^\mu_{\alpha a} \hat{\bar{\theta}}^a \partial_\mu, \quad \hat{Q}_{\dot{\alpha}} = i \frac{\partial}{\partial \hat{\bar{\theta}}^{\dot{\alpha}}} - \hat{\theta}^{\dot{\alpha}} \sigma^\mu_{\dot{\alpha} a} \partial_\mu.$$  
(4.16)

It is again straightforward to check that thermal supercharges obey the same anticommutation relations with thermal covariant derivatives as at $T = 0$:

$$\{\hat{Q}_\alpha, \hat{D}_{\dot{\beta}}\} = \{\hat{Q}_{\dot{\alpha}}, \hat{D}_\beta\} = \{\hat{Q}_\alpha, \hat{D}_\dot{\beta}\} = \{\hat{Q}_{\dot{\alpha}}, \hat{D}_\beta\} = 0.$$  
(4.18)

Furthermore we have

$$\{\hat{Q}_\alpha, \hat{Q}_{\dot{\alpha}}\} = 2i \sigma^\mu_{\alpha a} \partial_\mu, \quad \{\hat{Q}_\alpha, \hat{Q}_\beta\} = \{\hat{Q}_{\dot{\alpha}}, \hat{Q}_{\dot{\beta}}\} = 0.$$  
(4.19)
5 The thermal super-Poincaré algebra

In order to compute the full thermal super-Poincaré algebra, we need, in addition to the thermal supercharges constructed in the previous section, expressions for the thermal translations and thermal Lorentz generators. Let us start by writing the expressions in the zero-temperature case, i.e., for the generators of the $T=0$ supersymmetry algebra acting on $T=0$ superfields. The $T=0$ translation generators act on a chiral superfield $\phi$ simply as:

$$[P^\mu, \phi(x, \theta, \bar{\theta})] = -i \partial^\mu \phi(x, \theta, \bar{\theta}),$$

(5.1)

while the Lorentz generators entail a $\theta, \bar{\theta}$-dependent part:

$$[M^{\mu\nu}, \phi(x, \theta, \bar{\theta})] = \left[ i (x^\mu \partial^\nu - x^\nu \partial^\mu) + i 2 (\sigma^{\mu\nu})^\alpha_\beta \theta_\beta \partial_\alpha + i 2 (\sigma^{\nu\mu})^{\dot{\alpha}}_{\dot{\beta}} \bar{\theta}_{\dot{\beta}} \partial_{\dot{\alpha}} \right] \phi(x, \theta, \bar{\theta}).$$

(5.2)

Notice that the above equations display Poincaré transformations of the (Lorentz) scalar superfield at a fixed superspace coordinate point $(x, \theta, \bar{\theta})$. In other words, they display $\phi'(x, \theta, \bar{\theta}) - \phi(x, \theta, \bar{\theta})$ for a Poincaré transformation $(x, \theta, \bar{\theta}) \rightarrow (x', \theta', \bar{\theta})$, $\phi(x, \theta, \bar{\theta}) \rightarrow \phi'(x', \theta', \bar{\theta})$.

At finite temperature, the translation and Lorentz generators above are to be modified – similarly to the thermal covariant derivatives and the thermal supercharges – by replacing $\partial_\mu$ with $\hat{\partial}_\mu$, and $\theta, \bar{\theta}$ with $\hat{\theta}(t), \bar{\bar{\theta}}(t)$. Therefore we define the action of thermal translation and thermal Lorentz generators on a thermal scalar superfield through

$$[\hat{P}^\mu, \hat{\phi}(x, \hat{\theta}, \bar{\bar{\theta}})] = -i \hat{\partial}^\mu \hat{\phi}(x, \hat{\theta}, \bar{\bar{\theta}}),$$

(5.3)

and

$$[\hat{M}^{\mu\nu}, \hat{\phi}(x, \hat{\theta}, \bar{\bar{\theta}})] = \left[ i (x^\mu \hat{\partial}^\nu - x^\nu \hat{\partial}^\mu) + i 2 (\sigma^{\mu\nu})^\alpha_\beta \hat{\theta}_\beta \partial_\alpha + i 2 (\sigma^{\nu\mu})^{\dot{\alpha}}_{\dot{\beta}} \hat{\bar{\theta}}_{\dot{\beta}} \partial_{\dot{\alpha}} \right] \hat{\phi}(x, \hat{\theta}, \bar{\bar{\theta}}).$$

(5.4)

As only the derivative in the time direction is modified at finite temperature, we now distinguish between the generators which are genuinely thermal [that is, which involve the operator $\Delta$ introduced in eq. (4.6)] and those generators of which the only thermal character comes from the superspace coordinates being time-dependent. We drop the “hat” for the latter operators, and hence decompose the thermal translations $\hat{P}^\mu$ into $\hat{P}^0$ and $\hat{P}^i$, while the thermal Lorentz generators $\hat{M}^{\mu\nu}$ are separated into thermal Lorentz boosts $\hat{M}^{0i}$ and rotations $M^{ij}$. A straightforward computation of the commutation rules yields the thermal Poincaré algebra – the bosonic sector of the thermal super-Poincaré
algebra:

\[[\hat{M}^{0i}, \hat{P}^{0}] = -i P^i, \tag{5.5}\]

\[[\hat{M}^{0i}, P^j] = i \eta^{ij} \hat{P}^0, \tag{5.6}\]

\[[M^{ij}, \hat{P}^{0}] = 0, \tag{5.7}\]

\[[M^{ij}, P^k] = -i (\eta^{ik} P^j - \eta^{jk} P^i), \tag{5.8}\]

\[[M^{ij}, M^{kl}] = -i (\eta^{ik} M^{jl} + \eta^{jl} M^{ik} - \eta^{il} M^{jk} - \eta^{jk} M^{il}), \tag{5.9}\]

\[[\hat{M}^{0i}, M^{0j}] = -i (\eta^{ik} \hat{M}^{0j} - \eta^{ij} \hat{M}^{0k}), \tag{5.10}\]

\[[\hat{M}^{0i}, \hat{M}^{0j}] = -i M^{ij}, \tag{5.11}\]

\[[\hat{P}^{0}, \hat{P}^{0}] = [\hat{P}^{0}, P^i] = [P^i, P^j] = 0. \tag{5.12}\]

The fermionic sector is given by

\[\{\hat{Q}_\alpha, \hat{Q}_{\dot{\beta}}\} = -2 \left( \sigma^0_{\alpha\dot{\beta}} \hat{P}^0 - \sigma^i_{\alpha\dot{\beta}} P^i \right), \tag{5.13}\]

\[[\hat{P}^{0}, \hat{Q}_\alpha] = [\hat{P}^{0i}, \hat{Q}_{\dot{\alpha}}] = [P^i, \hat{Q}_\alpha] = [P^i, \hat{Q}_{\dot{\alpha}}] = 0, \tag{5.14}\]

\[[\hat{M}^{0i}, \hat{Q}_\alpha] = -\frac{i}{2} (\sigma^{0i})_{\alpha \beta} \hat{Q}_{\beta}, \tag{5.15}\]

\[[\hat{M}^{0i}, \hat{Q}_{\dot{\alpha}}] = -\frac{i}{2} (\dot{\sigma}^{0i})_{\dot{\alpha} \dot{\beta}} \hat{Q}^{\dot{\beta}}, \tag{5.16}\]

\[[M^{ij}, \hat{Q}_\alpha] = -\frac{i}{2} (\sigma^{ij})_{\alpha \beta} \hat{Q}_{\beta}, \tag{5.17}\]

\[[M^{ij}, \hat{Q}_{\dot{\alpha}}] = -\frac{i}{2} (\dot{\sigma}^{ij})_{\dot{\alpha} \dot{\beta}} \hat{Q}^{\dot{\beta}}. \tag{5.18}\]

The thermal super-Poincaré algebra (5.5)–(5.18) has hence the same structure as at \(T = 0\), and contains the same number of supercharges. This is a natural result of our construction which introduces a local dependence on time/temperature on superspace coordinates. The algebra is maintained by the appropriate covariantizations.

An interpretation of the role of the thermal time translation operator \(\hat{\bar{P}}^0 = -i \bar{\partial}^0\) – the thermal covariantization of \(P^0\) – can be given as follows. Clearly, in the thermal Poincaré algebra (5.5)–(5.12), the thermal Lorentz boosts \(\hat{M}^{0i}\) mix space- and time-translations. Even if a covariant formulation can be given \[\[\tag{14}\]\], immersing the field theory in a heat bath via periodicity or antiperiodicity conditions on (imaginary) time removes invariance under the Lorentz boosts \[\[\tag{17, 18}\]\]. Without these boosts, the Poincaré algebra reduces to spatial rotations \(M^{ij}\) and translations \(\hat{\bar{P}}^\mu\), which are true symmetries of finite temperature field theory. The thermal supersymmetry generators \(\hat{\bar{Q}}_\alpha\) and \(\hat{Q}_{\dot{\alpha}}\) then add to these space-time symmetries to form a three-dimensional supersymmetry algebra in which the thermal time translation operator \(\hat{\bar{P}}^0\) is a central charge: it commutes with all operators \(P^i, M^{ij}, \hat{Q}_\alpha\) and \(\hat{Q}_{\dot{\alpha}}\).
6 Thermal supersymmetry transformations of bosons and fermions

We wish here to compute the transformations of thermal superfield components under the thermal supersymmetry transformations generated by the thermal supercharges \( \mathcal{Q}_\alpha \) and \( \mathcal{Q}_{\dot{\alpha}} \), eqs. (4.13)-(4.16). This means translating into component language the thermal superfield transformation \( \delta \) given by

\[
\delta \phi(x, \theta, \bar{\theta}) = i \left( \bar{\epsilon}^\alpha \mathcal{Q}_\alpha + \bar{\epsilon}_{\dot{\alpha}} \mathcal{Q}_{\dot{\alpha}} \right) \phi(x, \theta, \bar{\theta}). 
\]  
(6.1)

Recall that at zero temperature, this is most easily done for a chiral superfield \( \phi \) in chiral variables \( y, \theta \). Since

\[
Q_\alpha y^\mu = 0, \quad Q_\alpha \sigma^\mu = 2(\sigma^\mu \theta)_{\alpha}, \quad \mathcal{Q}_{\dot{\alpha}} y^\mu = -2(\theta \sigma^\mu)_{\dot{\alpha}} 
\]  
(6.2)

one easily deduces that the \( T = 0 \) supercharges acting on \( \phi(y, \theta) \) are

\[
Q_\alpha \phi(y, \theta) = -i \frac{\partial}{\partial \theta} \phi(y, \theta), \quad \mathcal{Q}_{\dot{\alpha}} \phi(y, \theta) = -2(\theta \sigma^\mu)_{\dot{\alpha}} \frac{\partial}{\partial y^\mu} \phi(y, \theta). 
\]  
(6.3)

It is then immediate to compute the supersymmetry transformations of the components of \( \phi \) in chiral variables by expanding \( \delta \phi(y, \theta) = i(\epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}} \mathcal{Q}_{\dot{\alpha}}) \phi(y, \theta) \).

The aim here is to investigate the thermal situation. By construction, thermal supercharges in the thermal chiral variables \( \bar{y}, \tilde{\theta}, \bar{\theta} \) are analogous to the \( T = 0 \) expressions above. We also have

\[
\mathcal{Q}_\alpha \tilde{\theta} = -i \delta^\beta, \quad \mathcal{Q}_{\dot{\alpha}} \tilde{\theta} = i \delta^\beta, \quad \mathcal{Q}_\alpha \bar{\theta} = 0, \quad \mathcal{Q}_{\dot{\alpha}} \bar{\theta} = 0, 
\]  
(6.4)

and

\[
\partial_\nu \tilde{y}^\mu = \delta^\nu, \quad \partial_\nu \mathcal{Q}_{\dot{\alpha}} \tilde{y}^\mu = \delta^\nu. 
\]  
(6.5)

As a consequence, one sees that

\[
\mathcal{Q}_\alpha \tilde{y}^\mu = 0, \quad \mathcal{Q}_{\dot{\alpha}} \tilde{y}^\mu = 2(\sigma^\mu \tilde{\theta})_{\dot{\alpha}}, \quad \mathcal{Q}_\alpha \bar{y}^\mu = -2(\theta \sigma^\mu)_{\dot{\alpha}}. 
\]  
(6.6)

Therefore, the supercharges \( \mathcal{Q}_\alpha \) and \( \mathcal{Q}_{\dot{\alpha}} \), when acting on a general thermal superfield denoted \( \tilde{F} \) expressed in the chiral variables, yield

\[
\mathcal{Q}_\alpha \tilde{F}(\tilde{y}, \tilde{\theta}, \bar{\theta}) = -i \partial_\nu \tilde{F}(\tilde{y}, \tilde{\theta}, \bar{\theta}), 
\]  
(6.7)

\[
\mathcal{Q}_{\dot{\alpha}} \tilde{F}(\tilde{y}, \tilde{\theta}, \bar{\theta}) = -i \left( \partial_\nu - 2i(\bar{\theta} \sigma^\mu)_{\dot{\alpha}} \partial_\nu \tilde{y}^\mu \right) \tilde{F}(\tilde{y}, \tilde{\theta}, \bar{\theta}). 
\]  
(6.8)

\(^a\)A function of the thermal superspace coordinates \( \tilde{y}, \tilde{\theta} \) and \( \bar{\theta} \).
On a thermal chiral superfield \( \hat{\phi}(\hat{y}, \hat{\theta}) \), these thermal supercharges obviously reduce to

\[
\hat{Q}_\alpha \hat{\phi}(\hat{y}, \hat{\theta}) = -i \frac{\partial}{\partial \hat{\theta}_\alpha} \hat{\phi}(\hat{y}, \hat{\theta}), \quad \hat{\mathbf{Q}}_{\hat{\alpha}} \hat{\phi}(\hat{y}, \hat{\theta}) = -2(\hat{\theta} \sigma^\mu)_{\hat{\alpha}} \frac{\partial}{\partial \hat{y}^\mu} \hat{\phi}(\hat{y}, \hat{\theta}),
\]

which is nothing but the thermal version of eqs. (5.3). Analogous expressions can be constructed for the thermal supercharges acting on antichiral thermal superfields as functions of variables \( \hat{\theta} \) and \( \hat{\theta} \).

Inserting the supercharges (6.9) into the thermal supersymmetry transformation (5.1) leads to

\[
\hat{\delta} \hat{\phi}(\hat{y}, \hat{\theta}) = \left( \hat{\epsilon}^\alpha \frac{\partial}{\partial \hat{\theta}_\alpha} - 2i(\hat{\theta} \sigma^\mu \hat{\epsilon}) \frac{\partial}{\partial \hat{y}^\mu} \right) \hat{\phi}(\hat{y}, \hat{\theta}).
\]

Defining then \( \frac{\partial}{\partial \hat{y}^\mu} \varphi(\hat{y}) \equiv \partial_\mu \varphi \), for \( \varphi = z \) or \( \psi \), we get:

\[
\hat{\delta} \hat{\phi}(\hat{y}, \hat{\theta}) = \hat{\epsilon}^\alpha \left[ \sqrt{2} \psi_\alpha(\hat{y}) - 2\theta_\alpha f(\hat{y}) \right] - 2i(\hat{\theta} \sigma^\mu \hat{\epsilon}) \left[ \partial_\mu z(\hat{y}) + \sqrt{2} \hat{\theta} \sigma^\mu \psi_\alpha(\hat{y}) \right].
\]

Comparison with the component expansion of \( \hat{\delta} \hat{\phi}(\hat{y}, \hat{\theta}) \) immediately leads to:

\[
\begin{align*}
\hat{\delta} z &= \sqrt{2} \hat{\epsilon}^\alpha \psi_\alpha, \\
\hat{\delta} \psi_\alpha &= -\sqrt{2} \hat{\epsilon}^\alpha f - i \sqrt{2} (\sigma^\mu \hat{\epsilon})_\alpha (\partial_\mu z), \\
\hat{\delta} f &= -i \sqrt{2} (\sigma^\mu \hat{\epsilon})^\alpha (\partial_\mu \psi_\alpha).
\end{align*}
\]

In these transformations of the chiral multiplet, the unique difference with the case of zero temperature is the appearance of the thermal spinorial parameter \( \hat{\epsilon} \) in place of the constant spinorial parameter \( \epsilon \) of \( T = 0 \) supersymmetry.

The nature of \( \hat{\epsilon} \) is however deeply related to finite temperature. Reversing the argument given in the Introduction, it is clear from (5.9) that the supercharge \( \hat{Q}_\alpha \) acting on \( \hat{\phi} \) translates \( \hat{\theta} \) by an amount \( \hat{\epsilon} \). Since however the Grassmann coordinate \( \hat{\theta} \) is time-dependent and antiperiodic, one must then assume that \( \hat{\epsilon} \) itself is a time-dependent, antiperiodic spinor:

\[
\hat{\epsilon}(t + i\beta) = -\hat{\epsilon}(t).
\]

This is confirmed by the action on \( \hat{\phi} \) of the conjugate supercharge \( \hat{Q}_{\hat{\alpha}} \) [eq. (5.9)], which translates \( \hat{y}^\mu \) by an amount \( -2i(\hat{\theta} \sigma^\mu \hat{\epsilon}) \). In order that the time-dependence of \( \hat{y} \) remains periodic, \( \hat{\epsilon} \) must be antiperiodic, as in eq. (6.15).

The time-dependence of \( \hat{\epsilon} \) is, in this thermal superspace formalism, the manifestation of the breaking of global supersymmetry at finite temperature. Computing the commutator \( [\hat{\delta}_1, \hat{\delta}_2] \) of two thermal supersymmetry transformations \( \hat{\delta}_1, \hat{\delta}_2 \) will not close the algebra we have derived simply because

\[
\{ \hat{Q}_\alpha, \hat{\epsilon}(t) \} \neq 0 \neq \{ \hat{Q}_{\hat{\alpha}}, \hat{\epsilon}(t) \},
\]

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in contrast with the zero-temperature case in which $\epsilon$ is a constant spinor. Notice also that the supersymmetry transformation of the highest component $f$ of the superfield $\hat{\phi}$ is not a space-time derivative since $\partial_\mu \epsilon \neq 0$. The method usually applied to construct invariant lagrangians by tensor calculus will no longer hold at finite temperature. And actions invariant under supersymmetry at zero temperature will exhibit a breaking pattern which can easily be studied using thermal superspace.

7 Thermal supersymmetry and the Wess-Zumino model

In looking for realizations of thermal supersymmetry, we shall be applying the transformations generated by the thermal charges to systems of thermal fields. And thermal fields are known to be characterized by global periodicity conditions which distinguish bosons from fermions. Indeed, a thermal bosonic field (say $z$) is periodic in imaginary time, while a thermal fermionic field (denoted $\psi$) is antiperiodic:

$$z(t + i\beta, \mathbf{x}) = z(t, \mathbf{x}), \quad \psi(t + i\beta, \mathbf{x}) = -\psi(t, \mathbf{x}). \quad (7.1)$$

These thermal characterizations are actually equivalent to the bosonic, resp. fermionic KMS conditions \((2.13), (2.18)\), which are hence of global nature as well. Therefore, we expect to see signs of supersymmetry breaking when realizing the thermal supersymmetry algebra, which is a local structure, on periodic (bosonic), and antiperiodic (fermionic) fields. A common way of introducing the fields’ global periodicity properties is to develop them thermally à la Matsubara. In the Matsubara expansion, bosons are expanded in thermal modes as

$$z(t, \mathbf{x}) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{n=\infty} z_n(\mathbf{x}) e^{i\omega_n^B t}, \quad (7.2)$$

where

$$\omega_n^B = \frac{2n\pi}{\beta} \quad (7.3)$$

are the bosonic Matsubara frequencies, while fermions are developed as

$$\psi(t, \mathbf{x}) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{n=\infty} \psi_n(\mathbf{x}) e^{i\omega_n^F t}, \quad (7.4)$$

with the fermionic Matsubara frequencies

$$\omega_n^F = \frac{(2n+1)\pi}{\beta}. \quad (7.5)$$

Clearly, these developments contain the information on the periodicity in time.

The Matsubara expansion, after rotation to euclidean time, is a realization of the imaginary time formalism for finite temperature field theory. It is an expansion on $S^1 \times \mathbb{R}^3$, the circle $S^1$ having length $\beta = 1/T$. In a supergravity theory, it could be regarded as a particular Scherk-Schwarz compactification \([19]\) scheme of the time direction.
Since we have considered only non-interacting scalar and fermionic matter fields described by chiral and antichiral superfields, the natural zero-temperature limiting lagrangian density – to be studied now at finite temperature – is that of the free (off-shell) Wess-Zumino model

\[ S_{d=4} = \int d^4x \left( \mathcal{L}_{\text{kin}}^{d=4} + \mathcal{L}_{\text{mass}}^{d=4} \right), \]  

with kinetic and mass lagrangians given by,

\[ \mathcal{L}_{\text{kin}}^{d=4} = \frac{1}{2} (\partial_\mu A)^2 + \frac{1}{2} (\partial_\mu B)^2 + i \bar{\psi} \gamma^\mu \partial_\mu \psi + \frac{1}{2} (F^2 + G^2), \]  

\[ \mathcal{L}_{\text{mass}}^{d=4} = -M_4 \left( \frac{1}{2} \bar{\psi} \psi + AF + BG \right). \]

\( M_4 \) is the mass, \( \psi \) a four-component Majorana fermion, \( A, B, F \) and \( G \) are real scalars. The auxiliary fields \( F \) and \( G \) obey the equations of motion

\[ F = M_4 A, \quad G = M_4 B. \]  

One can equivalently use complex scalar fields \( z \) and \( f \) with

\[ z(x) = \frac{1}{\sqrt{2}} [A(x) + iB(x)], \quad f(x) = \frac{1}{\sqrt{2}} [F(x) + iG(x)]. \]  

The supersymmetry transformations at \( T = 0 \) are given by:

\[ \delta A = \bar{\tau} \psi, \quad \delta B = i \bar{\tau} \gamma_5 \psi, \]

\[ \delta F = i \bar{\tau} \gamma^\mu \partial_\mu \psi, \quad \delta G = -\bar{\tau} \gamma_5 \gamma^\mu \partial_\mu \psi, \]

\[ \delta \bar{\psi} = -[i \gamma^\mu \partial_\mu (A + iB\gamma_5) + F + iG\gamma_5] \epsilon, \]

\[ \delta \bar{\psi} = -\bar{\tau} [i \gamma^\mu \partial_\mu (-A + iB\gamma_5) + F + iG\gamma_5]. \]  

Recall that under these \( T = 0 \) transformations, the kinetic and mass contributions to the action \( S_{d=4} \) are separately invariant, i.e., \( \delta \int d^4x \mathcal{L}_{\text{kin}}^{d=4} = \delta \int d^4x \mathcal{L}_{\text{mass}}^{d=4} = 0 \). Concretely, omitting in each case a space-time derivative which integrates to zero,

\[ \delta \int d^4x \mathcal{L}_{\text{kin}}^{d=4} = \int d^4x \bar{\psi} \gamma^\nu \gamma^\mu [\partial_\mu (A + iB\gamma_5)] \partial_\nu \epsilon, \]

\[ \delta \int d^4x \mathcal{L}_{\text{mass}}^{d=4} = -iM_4 \int d^4x \bar{\psi} \gamma^\mu (A + iB\gamma_5) \partial_\mu \epsilon, \]  

which of course vanishes at zero temperature where \( \epsilon \) is a constant spinor.

We now proceed with the thermal mode expansion of the action (7.6), into which we insert the Matsubara developments. For bosons, the mode expansion follows from (7.2) and (7.10):

\[ \Gamma(t, x) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{n=\infty} \Gamma_n(x) e^{i\omega_n B t}, \quad \Gamma_{-n} = \Gamma_n^*, \quad \Gamma = A, B, F, G. \]

More care is needed for the thermal modes of fermionic fields. Since the mode expansion in the time direction effectively reduces the space-time dimension to three, the \( d = 4 \)
Majorana spinor $\psi$ must be rewritten in terms of a single $d = 3$ (euclidean) spinor $\lambda$. In our conventions, the expression is:

$$\psi = \left( \frac{\lambda}{i\sigma^2 \lambda} \right), \quad \psi^\dagger = (\lambda^\dagger - i\lambda^T \sigma^2), \quad \langle \psi \rangle = \psi^\dagger \sigma^0. \quad (7.14)$$

For the two-component fermions $\lambda, \lambda^T$, we hence set

$$\lambda(t, x) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{n=\infty} \lambda_n(x) e^{i\omega_n^T t}, \quad \lambda^T(t, x) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{n=\infty} \lambda^T_n(x) e^{i\omega_n^T t}, \quad (7.15)$$

while the conjugates $\lambda^*, \lambda^\dagger$ are developed similarly, with opposite frequencies. Inserting these expansions into the $T = 0$ supersymmetric action (7.3), one gets straightforwardly the euclidean action at temperature $T = 1/\beta$:

$$S_\beta = \frac{1}{\beta} \int_0^\beta dt \int d^3 x \sum_{m, n=-\infty}^{+\infty} \left\{ \frac{1}{2} \partial^i A_n \partial_i A_n + \frac{1}{2} \partial^i B_m \partial_i B_m + \frac{1}{2} (\omega_n^B)^2 (A_n A_n + B_m B_m) - \frac{1}{2} (F_m F_n + G_m G_n + M_4(A_m F_n + B_m G_n)) e^{i(\omega_m^B + \omega_n^B)t} + \frac{1}{2} \left[ \lambda^T_m \sigma^i \partial_i - \omega_n^F \lambda_n \right] e^{i(\omega_n^F - \omega_m^T)t} + M_4 \lambda^T_m i\sigma^2 \lambda^*_n e^{-i(\omega_n^F + \omega_m^T)t} \right\} \text{h.c.}. \quad (7.16)$$

Integrating on $t$, we get a three-dimensional theory of the $x$-dependent field modes $A_n, B_n, F_n, G_n, \lambda_n, \lambda^T_n, \lambda^*_n$ and $\lambda^\dagger_n$ given by the Matsubara action:

$$S^{d=3} = \int d^3 x \sum_{n=-\infty}^{+\infty} \left\{ \frac{1}{2} \partial^i A_n \partial_i A_n + \frac{1}{2} \partial^i B_n \partial_i B_n + \frac{1}{2} (\omega_n^B)^2 (A_n A_n + B_n B_n) - \frac{1}{2} (F_n F_n + G_n G_n) + \frac{1}{2} \left[ \lambda^T_n (\sigma^i \partial_i - \omega_n^F) \lambda_n \right] + \frac{1}{2} M_4 \left[ \lambda^T_{n-1} i\sigma^2 \lambda^*_n \right] + \text{h.c.} \right\}. \quad (7.17)$$

The equations of motion are given here by the thermal modes of the $T = 0$ equations (7.3),

$$F_n = M_4 A_n, \quad G_n = M_4 B_n. \quad (7.18)$$

The fields $A, B, F, G$ being real, we have for their thermal modes the relations

$$A_{-n} = A_n^*, \quad B_{-n} = B_n^*, \quad F_{-n} = F_n^*, \quad G_{-n} = G_n^*. \quad (7.19)$$

Upon using the equations of motion (7.18) and replacing (7.19), we get for the thermal expansion of the $d = 4, T = 0$ supersymmetric action (7.6) the euclidean expression

$$S^{d=3} = \int d^3 x \sum_{n=-\infty}^{+\infty} \left\{ \frac{1}{2} \partial^i A_n^* \partial_i A_n + \frac{1}{2} \partial^i B_n^* \partial_i B_n + \frac{1}{2} (M_{3,n}^B)^2 (A_n^* A_n + B_n^* B_n) + \frac{1}{2} \left[ \lambda^T_n (\sigma^i \partial_i - \omega_n^F) \lambda_n + M_4 \lambda^T_{n-1} i\sigma^2 \lambda^*_n \right] + \text{h.c.} \right\}, \quad (7.20)$$

The finite temperature expansion uses periodic imaginary time and the relevant quantity to analyze the theory is the euclidean action.
where \((M_{3,n}^B)^2\) stands for the (squared) thermal mass of the \(n\)-th \(d = 3\) bosonic mode,
\[
(M_{3,n}^B)^2 = M_4^2 + (\omega_n^B)^2, \quad (\omega_n^B)^2 = \frac{4\pi^2 n^2}{\beta^2}, \quad n = -\infty, \ldots, +\infty. \tag{7.21}
\]

For fermions, since \(\omega_{n-1}^F = -\omega_n^F\), \(\lambda_n\) and \(\lambda_{n-1}^\dagger\) are associated with the same time-dependent phase. The mass matrix in (7.20) writes then:
\[
\mathcal{L}_{d=3}^{\text{fermions}} = \frac{1}{2} \sum_n \left( \lambda_n^\dagger \lambda_{n-1} \right) \begin{pmatrix} \omega_n^F & M_4 \\ M_4 & -\omega_n^F \end{pmatrix} \begin{pmatrix} \lambda_n \\ \lambda_{n-1}^\dagger \end{pmatrix} + \text{h.c.} \tag{7.22}
\]

This mass matrix has two opposite eigenvalues \(\pm M_{3,n}^F\), verifying the relation for the (squared) thermal mass of the \(n\)-th \(d = 3\) fermionic mode
\[
(M_{3,n}^F)^2 = M_4^2 + (\omega_n^F)^2, \quad (\omega_n^F)^2 = \frac{\pi^2 (2n + 1)^2}{\beta^2}, \quad n = -\infty, \ldots, +\infty, \tag{7.23}
\]
as expected. The eigenstates are linear combinations of \(\lambda_n\) and \(\lambda_{n-1}^\dagger\). From eqs. (7.21), (7.22), it is clear that thermal effects lift the mass degeneracy characteristic of \(T = 0\) supersymmetry.

In Section 3 we have shown that component transformations under thermal supersymmetry have the same form as at \(T = 0\), but with the space-time constant supersymmetry parameter \(\epsilon\) replaced by the thermal, time-dependent and antiperiodic quantity \(\hat{\epsilon}\). This allows us to identify immediately the thermal version of the transformations (7.11):
\[
\hat{\delta}A = \hat{\epsilon}\psi, \quad \hat{\delta}B = i\hat{\epsilon}\gamma_5\psi, \\
\hat{\delta}F = i\hat{\epsilon}\gamma^\mu(\partial_\mu\psi), \quad \hat{\delta}G = -\hat{\epsilon}\gamma_5\gamma^\mu(\partial_\mu\psi), \\
\hat{\delta}\psi = -[i\gamma^\mu(\partial_\mu(A + iB\gamma_5)) + F + iG\gamma_5]\hat{\epsilon}, \\
\hat{\delta}\bar{\psi} = -\hat{\epsilon}[i\gamma^\mu(\partial_\mu(-A + iB\gamma_5)) + F + iG\gamma_5]. \tag{7.24}
\]

These expressions can be easily translated into transformations of the Matsubara modes. The thermal supersymmetry parameter is to be expressed in terms of a three-dimensional two-component spinor \(\hat{e}(t)\) as
\[
\hat{\epsilon}(t) = \begin{pmatrix} \hat{\epsilon}(t) \\ i\sigma^2\hat{\epsilon}^\ast(t) \end{pmatrix}, \quad \hat{\epsilon}^\dagger(t) = \begin{pmatrix} \hat{\epsilon}^\dagger(t) & -i\hat{\epsilon}^T(t)\sigma^2 \end{pmatrix}, \tag{7.25}
\]
and \(\hat{\epsilon}, \hat{\epsilon}^T, \hat{\epsilon}^\ast\) and \(\hat{\epsilon}^\dagger\) are to be expanded thermally. Unlike in eqs. (7.13), the two-component fermions here only depend on time. They therefore develop into frequency sums with constant thermal modes \(e_n, e_n^T, e_n^\ast\) and \(e_n^\dagger\), as
\[
\hat{\epsilon}(t) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{n=\infty} e_n e^{i\omega_n^t}, \quad \hat{\epsilon}^T(t) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{n=\infty} e_n^T e^{i\omega_n^t}, \tag{7.26}
\]
and similarly for \(e_n^\ast\) and \(e_n^\dagger\), with opposite frequencies. Inserting the mode expansions in eqs. (7.24), the transformations of bosonic Matsubara modes write
\[
\delta A_k = \frac{1}{\sqrt{\beta}} \sum_m \left( ie_{k-m-1}^\dagger \sigma^2 \lambda_m^\ast - ie_{k-m-1}^T \sigma^2 \lambda_m \right), \tag{7.27}
\]
\[23\]
\[
\delta B_k = \frac{1}{\sqrt{\beta}} \sum_m \left( e^\dagger_{k-m-1} \sigma^2 \lambda^*_m + e^T_{k-m-1} \sigma^2 \lambda_m \right), \tag{7.28}
\]
\[
\delta F_k = \frac{1}{\sqrt{\beta}} \sum_m \left( -ie^\dagger_{m-k} [(\sigma^I \partial_I - i \omega^F_m) \lambda_m] - ie^T_{m+k} [(\sigma^I \partial_I + i \omega^F_m) \lambda^*_m] \right), \tag{7.29}
\]
\[
\delta G_k = \frac{1}{\sqrt{\beta}} \sum_m \left( -e^\dagger_{m-k} [(\sigma^I \partial_I - i \omega^F_m) \lambda_m] + e^T_{m+k} [(\sigma^I \partial_I + i \omega^F_m) \lambda^*_m] \right), \tag{7.30}
\]
while for fermionic Matsubara modes, upon introducing for the scalars \( \zeta_m = \sqrt{2} z_m \), \( \varphi_m = \sqrt{2} f_m \), with
\[
\zeta_m = A_m + i B_m, \quad \bar{\zeta}_m = A_m - i B_m, \quad \varphi_m = F_m + i G_m, \quad \bar{\varphi}_m = F_m - i G_m, \tag{7.31}
\]
we get
\[
\delta \lambda_k = \frac{1}{\sqrt{\beta}} \sum_m \left( [(\sigma^I \partial_I + i \omega^B_m) \bar{\zeta}_m] \sigma^2 e^\dagger_{m-k-1} - e^\dagger_{k-m} \varphi_m \right), \tag{7.32}
\]
\[
\delta \lambda^*_k = \frac{1}{\sqrt{\beta}} \sum_m \left( \sigma^2 [ (\sigma^I \partial_I - i \omega^B_m) \zeta_m] e^\dagger_{m-k-1} - e^*_{k+m} \bar{\varphi}_m \right), \tag{7.33}
\]
\[
\delta \lambda^T_k = \frac{1}{\sqrt{\beta}} \sum_m \left( e^\dagger_{m-k-1} [(\sigma^I \partial_I - i \omega^B_m) \bar{\zeta}_m] \sigma^2 - e^T_{k-m} \varphi_m \right), \tag{7.34}
\]
\[
\delta \lambda^\dagger_k = \frac{1}{\sqrt{\beta}} \sum_m \left( e^\dagger_{k-m-1} \sigma^2 [(\sigma^I \partial_I + i \omega^B_m) \zeta_m] - e^\dagger_{k+m} \bar{\varphi}_m \right). \tag{7.35}
\]

As the parameter \( \hat{\epsilon} \) has been expanded in non-trivial Matsubara modes, these transformations mix in general boson and fermion modes with different levels \( n \), in contrast to the simple dimensional reduction case where \( \epsilon \) would be a zero-mode constant spinor.

In analogy with the zero-temperature transformation (7.12), the thermal action will have the following non trivial variation under thermal supersymmetry (7.24):
\[
\hat{\delta} \int d^4 x \mathcal{L}^{d=4}_{\text{kin}} = \int d^4 x \psi^\dagger \gamma^\mu \partial_\mu (A_i + i B \gamma_5) \partial_t \hat{\epsilon},
\hat{\delta} \int d^4 x \mathcal{L}^{d=4}_{\text{mass}} = -i M_4 \int d^4 x \psi^\dagger (A_i + i B \gamma_5) \partial_\theta \hat{\epsilon}, \tag{7.36}
\]
where \( \partial_\theta \hat{\epsilon} \) does not vanish. In these expressions, it is understood that a rotation to imaginary (euclidean) time is performed, and that time is integrated over the interval \([0, \beta] \) only. Again, inserting the Matsubara mode expansions leads to
\[
\hat{\delta} \int d^4 x \mathcal{L}^{d=4}_{\text{kin}} = \frac{i}{\sqrt{\beta}} \int d^3 x \sum_{m,n} \omega^F_n \lambda^\dagger_{m-n-1} (\omega^B_m + i \vec{\sigma} \cdot \vec{\nabla}) (A_m - i B_m) \sigma^2 \epsilon^*_n + \text{h.c.},
\hat{\delta} \int d^4 x \mathcal{L}^{d=4}_{\text{mass}} = \frac{M_4}{\sqrt{\beta}} \int d^3 x \sum_{m,n} \omega^F_n (A_m - i B_m) \lambda^\dagger_{n+m} \epsilon_n + \text{h.c.} \tag{7.37}
\]
Clearly, neither the kinetic action nor the mass action are invariant under the thermal supersymmetry transformations. However, the variations \( \delta S^{d=4}_{\text{kin}} \) and \( \delta S^{d=4}_{\text{mass}} \) vanish separately in the \( T \to 0 \) limit, as expected. The variation of the total action is a combination of two terms proportional\(^8 \) to \( \omega^F_n \sim T \).

\(^8\)The prefactor \( \beta^{-1/2} \) is a normalization of the mode expansion which disappears in the \( T \to 0 \) limit.
8 Conclusions

Our conclusions are twofold. Firstly, superspace can be modified to satisfy the constraints imposed by thermal effects and statistics. In particular, superspace Green’s functions verifying the KMS conditions for bosons and fermions can be written. The modified thermal superspace admits a super-Poincaré algebra. It should be noted that this result is closely similar to the Lorentz covariant formulation of finite temperature field theory [14]. The algebras of space-time or superspace transformations are essentially local, they generate infinitesimal transformations and are therefore not affected by global conditions like periodicity or antiperiodicity along the time/temperature circle.

Secondly, in contrast to the case of Lorentz symmetry, the different statistics strongly affect realizations of supersymmetry on multiplets of fields at finite temperature. Thermal supersymmetry of superfields is broken because of the temperature-dependent constraints we impose on the Grassmann coordinates of thermal superspace. These constraints imply a covariantization of the $T = 0$ superspace operators with respect to temperature, which in turn indicates that a simple superfield will not be sufficient to represent the thermal version of the super-Poincaré algebra.

The discussion of the KMS conditions provides thermal superspace with a more formal background. The requirement of antiperiodicity of the Grassmann coordinates is essential both in proving a KMS condition at the level of superfields, and in showing that the latter implies the correct bosonic and fermionic boundary conditions for the superfield components[^9]. Notice that Green’s functions involving general superfields can be treated following the method used here for chiral superfields only.

Signs of thermal supersymmetry breaking are seen only after the boundary conditions that characterize thermal fields have been implemented. These boundary conditions can be formulated in various equivalent ways, either in terms of KMS conditions, or as periodicity and antiperiodicity requirements on the fields, or equivalently upon expanding these fields thermally. Irrespective of the form in which they are implemented, the boundary conditions carry information on the behaviour of thermal fields at distant regions of space-time, and are in this sense of global nature. They induce a strong differentiation between bosons and fermions, as bosonic fields are periodic in imaginary time, while fermionic ones are antiperiodic[^10] and generate obstructions – in terms of lifting of the $T = 0$ mass degeneracy and of non-invariance of the Matsubara action – when trying to realize the thermal super-Poincaré algebra on systems of thermal fields. But thermal superspace can be used to analyze these obstructions.

[^9]: A formulation of the KMS conditions at the superfield level has been attempted in [4]. In that work, the superspace Grassmann coordinates are taken constant and therefore the naive superfield KMS conditions are seen not to hold. They are then reformulated in a “superthermal” approach inspired from [4].

[^10]: Previous studies of thermal supersymmetry breaking, considered either as explicit or as spontaneous, e.g., [1, 2, 3, 6, 9, 13], or of thermal Lorentz breaking [17, 18], have been conducted at the level of thermal fields/states with this global periodicity/antiperiodicity distinction.
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