HOW MANY ROOTS OF A RANDOM POLYNOMIAL
SYSTEM ON A LIE GROUP ARE REAL?

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Abstract. A finite linear combination of matrix elements of a finite di-

dimensional representation $\pi$ of a Lie group $K$ is said to be a $\pi$-polynomial

on $K$. If $K$ is compact then any $\pi$-polynomial uniquely extends to a

holomorphic function on the complexification $K_C$ of $K$. For a system

of $n$ $\pi_i$-polynomials, where $n = \dim(K)$, we consider the proportion of
real roots, that is the ratio of the number of roots in $K$ to the number
of roots in $K_C$. It turns out that for growing representations $\pi_i$ and
random system of $\pi_i$-polynomials, the expected proportion of real roots
converges not to 0, but to a nonzero constant. The limit is calculated in
terms of the volumes of some compact convex sets that determine the
growth of the representations $\pi_i$. For a 1-dimensional torus $K$ the limit
is $1/\sqrt{3}$.

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1. Introduction

1.1. Laurent polynomials. For a random real polynomial of degree $m$
the expected number of its real zeros divided by $m$ asymptotically equals

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zeros, theorem BKK.

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The transition from ordinary polynomials to Laurent ones leads to an opposite result. It turns out that a limit of similar expectation is not $0$, but $\frac{1}{\sqrt{3}}$, see Corollary 1.2 or Example 1.1. For many variables, such an effect is described in [3]. The main results for Laurent polynomials are as follows.

Recall that the Laurent polynomial on a complex torus $(\mathbb{C} \setminus 0)^n$ is a function $P = \sum_{m \in \Lambda \subset \mathbb{Z}^n} a_m z^m$, where $z^m = z_1^{m_1} \cdots z_n^{m_n}$. The finite subset $\Lambda$ of $\mathbb{Z}^n$ is called the support of $f$.

Definition 1.1. A Laurent polynomial is called a real Laurent polynomial if its restriction to the compact torus $T^n = \{ z \in (\mathbb{C} \setminus 0)^n : z = (e^{i\theta_1}, \ldots, e^{i\theta_n}) \}$ is a real-valued function. A zero of a Laurent polynomial in $T^n$ is said to be a real zero of Laurent polynomial.

The next corollary of the Definition 1.1 is straightforward.

Corollary 1.1. (1) The support of a real Laurent polynomial is centrally symmetric.
(2) Laurent polynomial $\sum_m a_m z^m$ is a real Laurent polynomial if and only if $\forall m \in \mathbb{Z}^n : a_m = \overline{a_{-m}}$.
(3) The zero set of a real Laurent polynomial is invariant under the mapping $z \mapsto \overline{z}^{-1}$.

Let $P_m(z) = c + \sum_{1 \leq k \leq m} (\lambda_k z^k + \overline{\lambda_k} z^{-k})$ be a random real Laurent polynomial in one variable of degree $m$. If $a_k = \frac{\lambda_k + \overline{\lambda_k}}{2}$, $b_k = \frac{\lambda_k - \overline{\lambda_k}}{2i}$, then

$$f_m(x) = P_m(e^{ix}) = c + \sum_{1 \leq k \leq m} a_k \cos(kx) + b_k \sin(kx)$$

is a random trigonometric polynomial of degree $m$. A number of real zeros of $P_m$ equals to the number of zeros of $f_m$ in the line segment $[0, 2\pi)$. It is known that under "mild distribution assumptions" on the random coefficients of $f_m$ (see [4]) an expectation of a number of real zeros asymptotically equals to $\frac{2m}{\sqrt{3}}$; see also the bibliography in [4] and Example 1.1 below.

Corollary 1.2. The expectation of a number of real zeros divided by $2m$ asymptotically equals to $\frac{1}{\sqrt{3}}$.

For a system of $n$ real Laurent polynomials the ratio of the number of its common real zeros to the number of all common zeros is said to be the proportion of real roots. Below we use the concept of expected proportion $\text{real}(\Lambda)$ of real roots of a random system of $n$ Laurent polynomials in $n$ variables with support $\Lambda$; see Definition 2.3. The following statement is a multidimensional analogue of Corollary 1.2.

Theorem 1. Let $B_m$ be a ball in $\mathbb{R}^n$ with radius $m$ and centre at the origin. Viewing $\mathbb{Z}^n$ as the integer lattice in $\mathbb{R}^n$, set $\Lambda_m = B_m \cap \mathbb{Z}^n$. Then

$$\lim_{m \to \infty} \text{real}(\Lambda_m) = \left( \frac{\sigma_{n-1}}{\sigma_n} \beta_n \right)^{\frac{n}{2}}$$
where \( \beta_n = \int_{-1}^{1} x^2(1-x^2)^{n-1} dx \), and \( \sigma_k \) is a volume of the \( k \)-dimensional unit ball.

**Remark 1.1.** For \( 1 \leq n \leq 10 \): \( \beta_n = \frac{2}{3} \cdot \frac{\pi}{8}, \frac{16}{81}, \frac{16}{105}, \frac{16}{105}, \frac{16}{105}, \frac{16}{105}, \frac{16}{105}, \frac{16}{105} \).

Recall that the convex hull \( \text{conv}(\Lambda) \) is called the Newton polytope of a Laurent polynomial with support \( \Lambda \). We also define the Newton ellipsoid \( \text{Ell}(\Lambda) \) of the centrally symmetric support \( \Lambda \); see Definition 2.5. The proof of Theorem 1 is based on the formula

\[
\text{real}(\Lambda) = \frac{1}{(2\pi)^n} \frac{\text{vol}(\text{Ell}(\Lambda))}{\text{vol}(\text{conv}(\Lambda))} \tag{1.1}
\]

**Example 1.1.** From Definition 2.5 it follows that, for \( n = 1 \), the Newton ellipsoid \( \text{Ell}(\Lambda) \) is a line segment \([\alpha(\Lambda), \alpha(\Lambda)]\), where \( \alpha(\Lambda) = 2\pi \frac{\sqrt{1}}{\#\Lambda} \sum_{\lambda \in \Lambda} \lambda^2 \); see Example 2.1. If \( \Lambda_m = \{-m, \ldots, -1, 0, 1, \ldots, m\} \) then

\[
\alpha(\Lambda_m) = 2\pi \sqrt{\frac{2(1^2 + \ldots + m^2)}{2m + 1}} = 2\pi \sqrt{\frac{m(m+1)}{3}}
\]

From (1.1) it follows that \( \text{real}(\Lambda_m) = \sqrt{\frac{m+1}{3m}} \), and \( \lim_{m \to \infty} \text{real}(\Lambda_m) = 1/\sqrt{3} \).

For systems of \( n \) random real Laurent polynomials in \((\mathbb{C}\setminus 0)^n\) with supports \( \Lambda_1, \ldots, \Lambda_n \), the proportion of real roots \( \text{real}(\Lambda_1, \ldots, \Lambda_n) \) also defined. In this case, the volumes in numerator and denominator of the ratio (1.1) are replaced by the mixed volumes of the corresponding Newton ellipsoids and Newton polyhedra, namely

\[
\text{real}(\Lambda_1, \ldots, \Lambda_n) = \frac{1}{(2\pi)^n} \frac{\text{vol}(\text{Ell}(\Lambda_1), \ldots, \text{Ell}(\Lambda_n))}{\text{vol}(\text{conv}(\Lambda_1), \ldots, \text{conv}(\Lambda_n))} \tag{1.2}
\]

### 1.2. Polynomials on a compact Lie group.

Let \( \pi \) be a finite dimensional representation of group \( K \). If the representation \( \pi \) is real (resp. complex) then a linear combination of matrix elements with real (resp. complex) coefficients is said to be a \( \pi \)-polynomial. In both cases, we denote the space of \( \pi \)-polynomials by \( \text{Trig}(\pi) \). Further we assume that the group \( K \) is compact, connected, and \( \dim K = n \). Any real representation \( \pi \) uniquely extends to a holomorphic representation \( \pi^c \) of the complexification \( K^c \) of group \( K \), and also any \( \pi \)-polynomial extends to a \( \pi^c \)-polynomial. The extensions of \( \pi \)-polynomials form a real subspace \( \text{Trig}_R(\pi^c) \subseteq \text{Trig}(\pi^c) \) of dimension \( \dim \text{Trig}(\pi^c) = \dim \text{Trig}(\pi) \). The elements of \( \text{Trig}_R(\pi^c) \) are said to be real \( \pi^c \)-polynomials.

**Definition 1.2.** Let \( \pi_1, \ldots, \pi_n \) be the finite dimensional real representations of \( K \). For a system of \( n \) real \( \pi_i^c \)-polynomials, the ratio of it’s common real zeros number to the number of common zeros in \( K^c \) is said to be *a proportion of real roots*.

Below we use the concepts of 1) the *mean value* \( \mathcal{M}(\pi_1, \ldots, \pi_n) \) of common number of roots (Definition 2.2), and 2) the *expected proportion* \( \text{real}(\pi_1, \ldots, \pi_n) \) of real roots for a random system of real \( \pi_i^c \)-polynomials (Definition 2.3). In what follows we also use the notations \( \mathcal{M}(\pi) = \mathcal{M}(\pi_1, \ldots, \pi_n) \) and \( \text{real}(\pi) = \text{real}(\pi_1, \ldots, \pi_n) \) for \( \pi = \pi_1 = \ldots = \pi_n \).
We consider the geometry related to these concepts (see (1.2) for $K = T^n$), and use it to find the asymptotics of $\text{real}(\pi_1, \ldots, \pi_n)$ for the increasing representations $\pi_i$. In particular we prove the version of Theorem 1 for a simple Lie group; see Theorem 2.

In our proofs we use two results on the number of roots of a system of equations: the theorem on the mean number of common zeros of $n$ smooth functions on an $n$-dimensional differentiable manifold (see [6, 7]), and the Kushnirenko-Bernstein type formula (also called the BKK formula) for complex reductive groups; see [8–11]). Here we need the BKK formula, which gives an answer in the form of the mixed volume of Newton bodies of representations $\pi_1, \ldots, \pi_n$, defined below; see Theorem 8. Its proof is given in Subsection 3.1.

Remark 1.2. A zero set of a real $\pi^C$-polynomial is invariant under the global Cartan involution in $K^C$ with respect to the maximal compact subgroup $K$. For $K = T^n$ it is Corollary 1.1 (3).

1.2.1. Preliminaries from group theory. In what follows (until the end of the paper) we use the standard notation and facts from the group theory:

- $K$, $\mathfrak{k}$ and $\mathfrak{k}^*$ are respectively the connected compact Lie group, the Lie algebra of $K$ and the space of linear functionals on $\mathfrak{k}$;
- $T_k$, $t$ and $t^*$ are respectively the maximal torus in $K$, the Lie algebra of $T_k$ and the space of linear functionals on $t$;
- $Z_k \subset t^*$ is a lattice of differentials of torus characters;
- $W^*$ is a Weyl group in the space $t^*$, $|W|$ is a number of elements in $W^*$;
- $\mathfrak{c}^*$ is a Weyl chamber in $t^*$;
- $R \subset t^*$ is the root system relative to $t$, $R^+$ is the set of positive roots;
- $\rho = \frac{1}{2} \sum_{\beta \in R^+} \beta$ is the half-sum of the positive roots;
- $\tau = (\ast, \ast)$ is an adjoint invariant positive inner product in $\mathfrak{k}$ (and also the dual inner product in $\mathfrak{k}^*$);
- $\nu$ is the corresponding to $\tau$ measure in $K$ (and also in $T^k$); $\text{vol}(T^k)$ and $\text{vol}(K)$ are the corresponding measures of $T^k$ and $K$;
- $d\nu$ is the corresponding to $\tau$ Lebesgue measure in $\mathfrak{k}^*$ (and also the corresponding Lebesgue measure in $t^*$);
- $P(\lambda) = \prod_{\beta \in R^+} (\lambda, \beta)$;
- $\mu_\lambda$ is the irreducible representation of $K$ with the highest weight $\lambda$;
- $K_C$ is a complexification of the group $K$;
- $\mathfrak{k}_C = \mathfrak{k} + i\mathfrak{k}$ is a complexification of the Lie algebra $\mathfrak{k}$.

Proposition 1.1. For any $\lambda \in \mathfrak{c}^*$ there exists the unique $\lambda' \in \mathfrak{c}^*$ such that $\lambda' \in W^*(-\lambda)$.

Note that $\lambda' = \lambda$. If $\lambda \in W^*(-\lambda)$ then $\lambda' = \lambda$. For example, if the group $W^*$ contains the central symmetry mapping, then always $\lambda' = \lambda$.

Definition 1.3. An unordered pair $(\lambda, \lambda')$ is said to be a symmetric pair. The set $\Lambda \subset \mathfrak{c}^*$ is called symmetric if from $\lambda \in \Lambda$ it follows that $\lambda' \in \Lambda$. If $\Lambda$ is symmetric then we denote by $\Lambda'$ the set of symmetric pairs $\{(\lambda, \lambda') : \lambda \in \Lambda\}$.

For example, if the set $B \subset \mathfrak{k}^*$ is invariant under the action $W^*$ and centrally symmetric, then the set $B \cap \mathfrak{c}^*$ is symmetric. For $K = T^n$, the
symmetry condition of the pair $(\lambda, \delta)$ is $\delta = -\lambda$, and the symmetry of the subset in $\mathbb{C}^*$ means that the set is centrally symmetric.

The following statement divides all $\lambda \in \mathbb{C}^* \cap \mathbb{Z}^k$ into three types: real, complex, and quaternionic; see [5], Chapter IX, Appendix II.

**Proposition 1.2.** There exists the one-to-one mapping $(\lambda, \lambda') \mapsto \pi_{\lambda, \lambda'}$ of the set of symmetric pairs $(\lambda, \lambda')$ in $(\mathbb{C}^* \cap \mathbb{Z}^k) \times (\mathbb{C}^* \cap \mathbb{Z}^k)$ onto the set of real irreducible representations $K$ such that one of these is true

(i) real type $\pi_{\lambda, \lambda'}$: $\pi_{\lambda, \lambda'} \otimes \mathbb{R} \mathbb{C} = \mu_{\lambda}$ and $\lambda = \lambda'$

(ii) quaternionic type $\pi_{\lambda, \lambda'}$: $\pi_{\lambda, \lambda'} \otimes \mathbb{R} \mathbb{C} = \mu_{\lambda} \oplus \mu_{\lambda'}$ and $\lambda = \lambda'$

(iii) complex type $\pi_{\lambda, \lambda'}$: $\pi_{\lambda, \lambda'} \otimes \mathbb{R} \mathbb{C} = \mu_{\lambda} \oplus \mu_{\lambda'}$ and $\lambda \neq \lambda'$

1.2.2. Geometry related to the expected proportion of real roots. Let $\pi_1, \ldots, \pi_n$ be finite dimensional real representations of $K$, and let $\mathcal{M}(\pi_1, \ldots, \pi_n)$ be the mean number of common zeros of random systems of $n$ irreducible representations $K$ that one of these is true

In Subsection 3.1, for any finite dimensional holomorphic representation $\mathcal{M}$, we construct the coadjoint invariant compact convex set $\mathcal{N} := \mathcal{M} \otimes \mathbb{C}$ called the Newton ellipsoid of representation $\mathcal{M}$ (see Definition 2.5), and prove that $\mathcal{M}(\pi_1, \ldots, \pi_n)$ equals to the mixed volume of ellipsoids $\text{Ell}(\pi_i)$ multiplied by $n!/(2\pi)^n$; see Theorem 3.

There exists $\mathcal{N}_C(\pi_1, \ldots, \pi_n)$ such that, for almost all systems of $n$ real $\pi_i$-polynomials, the number of their common complex roots equals $\mathcal{N}_C(\pi_1, \ldots, \pi_n)$. In Subsection 3.1 for any finite dimensional holomorphic representation $\mu$ of $K_C$, we construct the coadjoint invariant compact convex set $\mathcal{N} := \text{Ell}(\pi_i) \subset \mathfrak{t}^*$ called the Newton body of representation $\mu$ (see Definition 3.1 in Subsection 3.1), and prove that $\mathcal{N}_C(\pi_1, \ldots, \pi_n)$ equals to the mixed volume of Newton bodies $\mathcal{N}(\pi_1 \times \mathbb{C})$ multiplied by $n!$ and by the constant $c_K$ depending on the group $K$; see Theorem 8. Just like in (1.2), we obtain (see Theorem 9 in Subsection 3.3) that, for a simple group $K$,

$$\text{real}(\pi_1, \ldots, \pi_n) = \frac{P^2(\rho)}{(2\pi)^n} \frac{\text{vol} \left( \text{Ell}(\pi_1), \ldots, \text{Ell}(\pi_n) \right)}{\text{vol} \left( \mathcal{M}(\pi_1 \otimes \mathbb{C}), \ldots, \mathcal{M}(\pi_n \otimes \mathbb{C}) \right)}$$

(1.3)

1.2.3. Asymptotics of an expected proportion of real roots. Let $B$ be a compact convex set in $\mathfrak{t}^*$. We assume that $B$ has central symmetry and invariant under the action of $W^*$. Then the finite set $\Lambda_B = B \cap \mathfrak{t}^* \cap \mathbb{Z}^k$ is symmetric; see Definition 1.3. We consider the representation $\pi(B) = \bigoplus_{(\lambda, \lambda') \in \Lambda_B} \pi_{\lambda, \lambda'}$, and the sequence of representations $\pi_m(B) = \pi(mB)$ for $m \to \infty$.

Let $B$ be a ball with radius 1 and centre at the origin. Then by Definition 3.1 the Newton body $\mathcal{N}(\pi_m \otimes \mathbb{C})$ asymptotically equals the ball with radius $m$ and centre at the origin. If the group $K$ is simple then the Newton ellipsoid $\text{Ell}(\pi_m(B))$ is also a ball; see Corollary 2(5) (2). Therefore, by (1.3), to calculate the limit of $\text{real}(\pi_m(B))$ it suffices to find the asymptotics of the radius of $\text{Ell}(\pi_m(B))$ as $m \to \infty$. The result is analogous to the similar one for a torus; see Theorem 1.
Theorem 2. Assume that the group $K$ is simple. Then

$$\lim_{m \to \infty} \text{real}(\pi_m(B)) = \frac{P^2(\rho)}{(2\pi)^n(n+2)^{n/2}(\alpha, \alpha + 2\rho)^{n/2}}$$

where $\alpha$ be the highest root of $K$ (that is the highest weight of the adjoint representation $\mu_\alpha$).

2. Mean number of real roots

In this Section we consider the random systems of real $\pi$-polynomials, the expectations of their numbers of roots, the asymptotics of the expectation for growing representation $\pi$, and the related geometry of ellipsoids.

2.1. Mean number of roots (definition). Let $\pi$ be a finite dimensional real representation of a connected compact Lie group $K$. We consider the space of $\pi$-polynomials $\text{Trig}(\pi)$ with an inner product taken from the space $L^2_{\mathbb{R}}(\chi)$, where $\chi$ is the normalized invariant measure in $K$.

Definition 2.1. A representation $\pi$ without multiple irreducible components is said to be a flat representation. For any real or complex representation $\pi$, the flat representation $\pi_F$ with the same irreducible components we call the flattening of $\pi$.

Corollary 2.1. For any $\pi$ we have $\text{Trig}(\pi) = \text{Trig}(\pi_F)$.

Proof. Follows from the definitions of $\text{Trig}(\pi)$ and $\text{Trig}(\pi_F)$.

Definition 2.2. Let $\pi_1, \ldots, \pi_n$ be the real representations of $K$, and let $S_i$ be a sphere with radius 1 and centre at the origin in $\text{Trig}(\pi_i)$. For $s_i \in S_i$, we denote by $N(s_1, \ldots, s_n)$ a number of isolated points in the set of common zeros of $\pi_i$-polynomials $s_1, \ldots, s_n$. Let

$$\mathcal{M}(\pi_1, \ldots, \pi_n) = \int_{S_1 \times \ldots \times S_n} N(s_1, \ldots, s_n) \, ds_1 \ldots ds_n, \quad (2.1)$$

where $ds_i$ is the rotation invariant normalized measure on $S_i$. The number $\mathcal{M}(\pi_1, \ldots, \pi_n)$ is called the mean number of common zeros of random $\pi_i$-polynomials. In what follows we also use the notation $\mathcal{M}(\pi) = \mathcal{M}(\pi_1, \ldots, \pi_n)$ for $\pi = \pi_1 = \ldots = \pi_n$.

Corollary 2.2. Let $\pi_1F, \ldots, \pi_nF$ be the flattenings of representations $\pi_1, \ldots, \pi_n$. Then $\mathcal{M}(\pi_1, \ldots, \pi_n) = \mathcal{M}(\pi_1F, \ldots, \pi_nF)$

Proof. Follows from Corollary 2.1.\hfill $\square$

Let $\pi_C$ be an extension of $\pi$ to a holomorphic representation of the complexification $K_C$ of group $K$. For a $\pi$-polynomial $f$ we denote by $f^C$ its holomorphic extension to a $\pi_C$-polynomial on $K_C$. For $\pi_1, \ldots, \pi_n$ and $\pi_i$-polynomials $f_i$ we denote by $N_C(f_1, \ldots, f_n)$ the number of common zeros of $f_1^C, \ldots, f_n^C$. The function $N_C(s_1, \ldots, s_n)$ is constant outside some set of a zero measure in $S_1 \times \ldots \times S_n$; see Proposition 3.1.
Definition 2.3. We define the expected proportion of common real roots of random polynomials \( f_1 \in \text{Trig}(\pi_1), \ldots, f_n \in \text{Trig}(\pi_n) \) as
\[
\text{real}(\pi_1, \ldots, \pi_n) = \int_{S_1 \times \ldots \times S_n} \frac{N(s_1, \ldots, s_n)}{N^c(s_1, \ldots, s_n)} \, ds_1 \ldots ds_n.
\]

2.2. Newton ellipsoids. Using the inner product \((\ast, \ast)\) in \(\text{Trig}(\pi)\) we consider the mapping \(\Theta(\pi): K \to \text{Trig}(\pi)\), such that
\[
\forall f \in \text{Trig}(\pi): \quad (\Theta(\pi)(g), f) = \frac{1}{\sqrt{N}} f(g), \tag{2.2}
\]
where \(N = \dim \text{Trig}(\pi)\).

Lemma 2.1. The set \(\Theta(\pi)(K)\) is contained in the sphere \(S \subset \text{Trig}(\pi)\) with radius 1 and centre at the origin.

Proof. The inner product in \(\text{Trig}(\pi)\) is \(K\)-invariant. Hence the set \(\Theta(\pi)(K)\) is contained in a sphere of some radius \(r\). From (2.2) it follows that for any orthonormal basis \(f_1, \ldots, f_N\) in \(\text{Trig}(\pi)\)
\[
\Theta(\pi)(g) = \frac{1}{\sqrt{N}} (f_1(g)f_1 + \ldots + e_N(g)f_N).
\tag{2.3}
\]
Hence \(r^2 = (\Theta(\pi)(g), \Theta(\pi)(g)) = \frac{1}{N} \sum_i f_i^2(g)\). Integrating over the measure \(d\chi\), we obtain \(r^2 = (f_1, f_1) + \ldots (f_N, f_N))/N = 1\). \(\square\)

Definition 2.4. We define the symmetric bilinear form \(G(\pi)\) in any tangent space \(T_x K\) of \(K\) as a pullback of the scalar product \((\ast, \ast)\) in \(\text{Trig}(\pi)\) by the mapping \(\Theta(\pi)\). Let \(F(\pi)\) be a restriction \(G(\pi)\) onto the Lie algebra \(\mathfrak{t}\) of \(K\), and let \(g(\pi)\) be the corresponding quadratic form in \(\mathfrak{t}\).

Corollary 2.3. For any orthonormal basis \(f_1, \ldots, f_N\) in the space \(\text{Trig}(\pi)\),
\[
F(\pi)(\xi, \eta) = \frac{1}{N} \sum_{1 \leq i \leq N} df_i(\xi) \, df_i(\eta),
\]
where \(df_i\) is a differential of \(f_i\) at the unit of the group \(K\).

The bilinear form \(G(\pi)\) is invariant under the action \((k_1 \times k_2): k \mapsto k_1 kk_2^{-1}\) of the group \(K \times K\) in \(K\). Therefore the non-negative inner product \(F(\pi)\) is invariant under the adjoint action \(K\) in \(\mathfrak{t}\). The form \(g(\pi)\) is non-negative. Hence the function \(h_\pi = \sqrt{g(\pi)}: \mathfrak{t} \to \mathbb{R}\) is positively homogeneous of degree 1 and convex. Recall that a positively homogeneous of degree 1 convex function \(h(x): \mathbb{R}^n \to \mathbb{R}\) is called the support function of the compact convex set \(A \subset \mathbb{R}^n\), if \(h(x) = \max_{a \in A} a(x)\).

Definition 2.5. The compact convex set \(\text{Ell}(\pi) \subset \mathfrak{t}^*\) defined by the support function \(h_\pi = \sqrt{g(\pi)}\) is called the Newton ellipsoid of the representation \(\pi\). The set \(\text{Ell}(\pi)\) is an ellipsoid in the subspace of \(\mathfrak{t}^*\), orthogonal to the kernel of the quadratic form \(g(\pi)\).

Example 2.1. Consider in detail the real representation \(\nu\) of the circle \(T^1 = \{ e^{2\pi i \theta} | 0 \leq \theta \leq 1 \}\) with the spectrum \(\{-m, \ldots, 0, \ldots, m\}\); see Example 1.1. For \(j = 1, \ldots, m\), the basis 1, \(\sqrt{2} \cos(j \theta), \sqrt{2} \sin(j \theta)\) of the space \(\text{Trig}(\nu)\)
is orthonormal. Since $\frac{d}{d\theta} \cos \theta = -2\pi i \sin \theta$, $\frac{d}{d\theta} \sin \theta = 2\pi i \cos \theta$, then from Corollary 2.3 and Definition 2.5 it follows that

$$h_\nu(\xi) = 2\pi|\xi|/(2m + 1)\sqrt{2(1^+ \ldots + m^2)} = 2\pi|\xi|\sqrt{m(m + 1)/3}.$$  

The Newton ellipsoid $\text{Ell}(\nu)$ is the interval with endpoints $\pm h_\nu(1)$. Since $h_\nu(1) = 2\pi\sqrt{m(m + 1)/3}$, then from Theorem 3 (see below in Subsection 2.3) it follows that $\mathcal{M}(\nu) = 2\sqrt{m(m + 1)/3}$.

**Corollary 2.4.** Ellipsoid $\text{Ell}(\pi)$ is invariant under the coadjoint action of $K$.

Follows from adjoint invariance of the function $h_\pi$.

**Corollary 2.5.** (1) The ellipsoid $\text{Ell}(\pi)$ is a ball of some coadjoint invariant metric in the subspace $\ker^\perp \subset \mathfrak{k}^*$, where $\ker^\perp$ is an orthogonal complement to $\ker(d\pi)$.

(2) If the group $K$ is simple, then the ellipsoid $\text{Ell}(\pi)$ is a ball in $\mathfrak{k}^*$ for any coadjoint invariant metric.

**Proof.** The statement (1) follows from Corollary 2.4. For a simple group, any two coadjoint invariant metrics differ by a constant factor; hence the statement (2) follows.

**Corollary 2.6.** If $\pi_F$ is a flattening of representation $\pi$ then $\text{Ell}(\pi_F) = \text{Ell}(\pi)$.

**Proof.** Follows from Corollary 2.1.

### 2.3. Mixed volumes of Newton ellipsoids.

Let $\Omega$ be an invariant normalized volume form on $K$, associated with the Haar measure $\chi$, and $\omega$ the associated volume form on $\mathfrak{k}$. We denote by $\gamma$ the dual to $\omega$ volume form on $\mathfrak{k}^*$. Namely, if the form $\omega$ is defined as a covector $\xi^* = \xi^1_1 \wedge \ldots \wedge \xi^*_n \in \bigwedge^n \mathfrak{k}^*$, then $\gamma(\xi^*) = 1$. In the following theorem we measure the mixed volume $\text{vol}$ in $\mathfrak{k}^*$ using the volume form $\gamma$.

**Theorem 3.** For any finite dimensional real representations $\pi_1, \ldots, \pi_n$

$$\mathcal{M}(\pi_1, \ldots, \pi_n) = \frac{n!}{(2\pi)^n} \text{vol}(\text{Ell}(\pi_1), \ldots, \text{Ell}(\pi_n)).$$

**Proof.** Recall that a Banach body $B$ in the differentiable manifold $X$ is a family of centrally symmetric compact convex sets $B(x)$ in the fibers $T_x^*X$ of the cotangent bundle to $X$; see [11, 7]. The volume of the Banach body $B$ is defined as the volume of $\bigcup_{x \in X} B(x) \subset T^*_xX$ with respect to the symplectic structure on the cotangent bundle $T^*X$. Let $\dim X = n$. Then we use a volume form $\omega^n/n!$, where $\omega$ is the standard symplectic form in $T^*_xX$. Using Minkowski sums and homotheties in the spaces $T^*_xX$, we define the linear combination $\lambda_1 B_1 + \ldots + \lambda_k B_k$ of Banach bodies as

$$(\lambda_1 B_1 + \ldots + \lambda_k B_k)(x) = \lambda_1 B_1(x) + \ldots + \lambda_k B_k(x).$$

The volume of $\lambda_1 B_1 + \ldots + \lambda_k B_k$ is a homogeneous polynomial of degree $n$ in $\lambda_1, \ldots, \lambda_k$. If $k = n$ then we denote by $\mathcal{V}(B_1, \ldots, B_n)$ the coefficient of the volume polynomial at the monomial $\lambda_1 \cdot \ldots \cdot \lambda_n$ divided by $n!$. We call $\mathcal{V}(B_1, \ldots, B_n)$ the mixed symplectic volume of Banach bodies $B_1, \ldots, B_n$. 


Let $V \subset C^\infty(X)$ be a finite-dimensional vector space with the scalar product $(\cdot, \cdot)$, and let
\[
\forall x \in X \exists f \in V: f(x) \neq 0. \tag{2.4}
\]
Consider the mappings $\theta: X \to V$ and $\Theta: X \to V$ such that
\[
\forall(f \in V, x \in X): (\theta(x), f) = f(x) \text{ and } \Theta(x) = \theta/\sqrt{(\theta(x), \theta(x))}.
\]
We define the Banach body $\mathcal{B}_V$ as
\[
\mathcal{B}_V(x) = d^*_{xV}(x) \Theta(\mathcal{B}_V(x)),
\]
where $\mathcal{B}_\subset V$ is the ball with radius 1 and centre at the origin, and $d^*_{xV}: V \to T^*_x V$ is a linear operator adjoint to the differential $d\Theta$ at the point $x \in X$. In this way, for any $x \in X$, the compact set $\mathcal{B}_V(x)$ is an ellipsoid.

For any finite-dimensional Euclidean spaces $V_1, \ldots, V_n \subset C^\infty(X)$, the mean number of common zeros $M(V_1, \ldots, V_n)$ of random functions $f_1 \in V_1, \ldots, f_n \in V_n$ is defined, and under conditions (2.4) for each $V_i$, proved (see [6, Theorem 1]) that
\[
M(V_1, \ldots, V_n) = \frac{n!}{(2\pi)^n} \mathcal{V}(\mathcal{B}_{V_1}, \ldots, \mathcal{B}_{V_n}). \tag{2.5}
\]

Let $X = K$ and $V_i = \text{Trig}(\tau_i)$. The Newton ellipsoid $\text{Ell}(\tau)$ is located in the cotangent space $T^*_e K$ at the unit element $e$ of group $K$. Let us denote by $\mathcal{B}(\tau_i)$ the Banach body in $K$, consisting of the left shifts of $\text{Ell}(\tau_i)$. From the definition of Newton ellipsoid and from Lemma 2.1 it follows that $\mathcal{B}(\tau_i) = \mathcal{B}_{V_i}$. Applying (2.5) we get
\[
M(\tau_1, \ldots, \tau_n) = \frac{n!}{(2\pi)^n} \mathcal{V}(\mathcal{B}(\tau_1), \ldots, \mathcal{B}(\tau_n)).
\]
From the left invariance of $\mathcal{B}(\tau_i)$ and from $\int_K d\chi = 1$ it follows that
\[
M(\tau_1, \ldots, \tau_n) = \frac{n!}{(2\pi)^n} \mathcal{V}(\mathcal{B}(\tau_1), \ldots, \mathcal{B}(\tau_n)) = \frac{n!}{(2\pi)^n} \mathcal{V} \left( \frac{\mathcal{B}(\tau_1)}{\mathcal{B}(\tau_1)}, \ldots, \frac{\mathcal{B}(\tau_n)}{\mathcal{B}(\tau_n)} \right).
\]
The proof is completed.

**Corollary 2.7.** Let $\tau$ be an adjoint invariant positive inner product in $\mathfrak{k}$, and let $d\nu$ be the corresponding Lebesgue measure in $\mathfrak{k}^*$ (see Subsection 1.2.1). Then
\[
M(\tau_1, \ldots, \tau_n) = \mathcal{V}(\mathcal{B}(\tau_1), \ldots, \mathcal{B}(\tau_n)) = \frac{n!}{(2\pi)^n} \mathcal{V}(\mathcal{B}(\tau_1), \ldots, \mathcal{B}(\tau_n)) \int_K d\chi.
\]
**Proof.** For the volume form $\mathcal{V}$ in $\mathfrak{k}^*$ corresponding to the normalized Haar measure $\chi$ in $K$ we have $\mathcal{V}(\mathcal{B}(\tau_1), \ldots, \mathcal{B}(\tau_n)) = \mathcal{V}(\mathcal{B}(\tau_1), \ldots, \mathcal{B}(\tau_n))$. Hence the statement follows from Theorem 3. □

The following two statements are not used further and are given for completeness.

**Corollary 2.8.** Let $U$ be an open domain in $K$. Then the mean number of contained in $U$ common zeros of $n$ random $\pi_i$-polynomials equals
\[
\frac{n!}{(2\pi)^n} \mathcal{V}(\mathcal{B}(\tau_1), \ldots, \mathcal{B}(\tau_n)) \int_U d\chi.
\]
**Proof.** Follows from [6, Theorem 1]. □
Corollary 2.9. For any representations $\pi_1, \ldots, \pi_n$

1. $M^2(\pi_1, \ldots, \pi_n) \geq M(\pi_1, \ldots, \pi_{n-1}, \pi_n) \cdot M(\pi_1, \ldots, \pi_n)$,
2. $M^n(\pi_1, \ldots, \pi_n) \geq M(\pi_1) \cdot \cdots \cdot M(\pi_n)$
3. For a simple group $K$, the above inequalities turn into the equalities.

Proof. (1) and (2) follow from the Alexandrov-Fenchel inequalities for mixed volumes of ellipsoids $\text{Ell}(\pi_i)$; see [12]. The mixed volume of balls is equal to the product of their radii multiplied by $\sigma_n$. Hence (3) follows from Corollary 2.5 (2). \hfill \Box

Remark 2.1. The inequalities from Corollary 2.9 are the analogs of the Hodge inequalities for the intersection indices of projective algebraic varieties; see, for example, [13].

2.4. Use of complexification. To consider the mean number of roots in more detail we use the complexifications of real representations of $K$. For a complex representation $\mu$ of $K$, we consider the complex vector space $\text{Trig}(\mu)$ of $\mu$-polynomials on $K$. This is a vector space of linear combinations of matrix elements $\mu$ equipped with a Hermitian inner product $\langle \ast, \ast \rangle$ taken from the space $L^2(\mathcal{D}(d\chi))$.

Lemma 2.2. Let $\pi$ be a real representation of $K$, $\mu = \pi \otimes_{\mathbb{R}} \mathbb{C}$. Then:

1. The space $\text{Trig}(\pi)$ is a real subspace in the complex vector space $\text{Trig}(\mu)$;
2. The restriction of the Hermitian inner product $\langle \ast, \ast \rangle$ to $\text{Trig}(\pi)$ is equal to the inner product $\langle \ast, \ast \rangle$ taken from $L^2(\mathcal{D}(d\chi))$;
3. The orthonormal basis in $\text{Trig}(\pi)$ is also orthonormal in $\text{Trig}(\mu)$;
4. Denote by $R(f)$ the real part of the function $f: K \to \mathbb{C}$, and let $\text{ReTrig}(\mu) = \{R(f): f \in \text{Trig}(\mu)\}$. Then $\text{Trig}(\pi) = \text{ReTrig}(\mu)$.

Proof. Follows directly from the definitions of $\pi \otimes_{\mathbb{R}} \mathbb{C}$ and $\langle \ast, \ast \rangle$.

Corollary 2.10. For the real irreducible representation $\pi_{\lambda,\lambda'}$ (see Proposition 1.2), $\text{Trig}(\pi_{\lambda,\lambda'}) = \text{ReTrig}(\mu_{\lambda})$.

Proof. If the representations $\mu, \nu$ are dual to each other then $\text{ReTrig}(\mu) = \text{ReTrig}(\nu)$. Hence (see Subsection 1.2) $\text{ReTrig}(\mu_{\lambda}) = \text{ReTrig}(\mu_{\lambda'})$. Now the statement follows from Lemma 2.2 (4) and Proposition 1.2. \hfill \Box

For a complex representation $\mu$, as for real representations in (2.2), we consider the mapping $\Theta(\mu): K \to \text{Trig}(\mu)$ defined by

$$\forall f \in \text{Trig}(\mu): \langle f, \Theta(\mu)(g) \rangle = \frac{1}{\sqrt{N}} f(g), \quad (2.6)$$

where $N = \dim_{\mathbb{C}} \text{Trig}(\mu)$. As in (2.3), for any orthonormal basis $f_1, \ldots, f_N$ in $\text{Trig}(\mu)$, we have

$$\Theta(\mu)(g) = \frac{1}{\sqrt{N}} (f_1(g)f_1 + \cdots + f_N(g)f_N). \quad (2.7)$$

As for the real representation $\pi$ (see Definition 2.4), we define the bilinear form $F(\mu)$ in algebra $\mathfrak{t}$ as

$$F(\mu)(\xi, \eta) = \langle d\Theta(\mu)(\xi), d\Theta(\mu)(\eta) \rangle = \frac{1}{N} \sum_i df_i(\xi) df_i(\eta). \quad (2.8)$$
From Lemma 2.2 (3) it follows that
\[ F(\pi \otimes_{\mathbb{R}} \mathbb{C}) = F(\pi). \]  
(2.9)
As in the real case,
\[ F(\mu_F) = F(\mu), \]  
(2.10)
where \( \mu_F \) is the flattening of representation \( \mu \); see Definition 2.1.

Let \( \Lambda \) be a symmetric finite set in \( \mathbb{Z}^k \cap \mathbb{C}^* \), and
\[
\pi = \bigoplus_{(\lambda, \lambda') \in \Lambda^2} m_{\lambda, \lambda'} \pi_{\lambda, \lambda'}, \quad \mu = \bigoplus_{\lambda \in \Lambda} m_{\lambda} \mu_{\lambda}
\]
(see Subsection 1.2.1). Next the set \( \Lambda \) is said to be the spectrum of \( \pi \).

From Proposition 1.2 it follows that \( \pi \otimes_{\mathbb{R}} \mathbb{C} = \mu + \sum_{\lambda \in \mathbb{Q}(\Lambda)} m_{\lambda} \mu_{\lambda} \), where \( \mathbb{Q}(\Lambda) \) is the set of quaternionic weights in \( \Lambda \). Hence \( (\pi \otimes_{\mathbb{R}} \mathbb{C})_F = \mu_F \). From (2.10) and (2.9) it follows that
\[ F(\pi) = F(\mu) \]  
(2.11)
Below, in the proofs we assume by default (based on the equality (2.10) and on Corollary 2.6) that all representations are flat.

Lemma 2.3. For the representation \( \mu_{\lambda} \) with highest weight \( \lambda \)
\[ F(\mu_{\lambda})(\xi, \eta) = -\frac{1}{\dim \mu_{\lambda}} \text{Tr}(d\mu_{\lambda}(\xi) \cdot d\mu_{\lambda}(\eta)) \]

Proof. Let \( \{t_{i,j}^{\lambda}\} \) be a unitary matrix defining the representation \( \mu_{\lambda} \). From the orthogonality relations for matrix elements \( t_{i,j}^{\lambda} \), it follows that the functions \( \sqrt{\dim(\mu_{\lambda})} t_{i,j}^{\lambda} \) form an orthonormal basis in \( \text{Trig}(\mu_{\lambda}) \). Then by (2.8)
\[ F(\mu_{\lambda})(\xi, \eta) = \frac{1}{\dim(\mu_{\lambda})} \sum_{i,j} dt_{i,j}^{\lambda}(\xi) dt_{i,j}^{\lambda}(\eta) = -\frac{1}{\dim \mu_{\lambda}} \text{Tr}(d\mu_{\lambda}(\xi) \cdot d\mu_{\lambda}(\eta)) \]

Corollary 2.11. Let \( \alpha \) be the highest root of the simple group \( K \). Then
\[ \forall \xi, \eta \in \mathfrak{k}: \quad F(\pi_{\alpha,\alpha'})(\xi, \eta) = F(\mu_{\alpha})(\xi, \eta) = -n^{-1} \mathcal{K}(\xi, \eta), \]
where \( \mathcal{K} \) is the Killing form in \( \mathfrak{k} \) (recall that \( \mathcal{K}(\xi, \eta) = \text{Tr}(d\mu_{\alpha}(\xi) \cdot d\mu_{\alpha}(\eta)) \)), where \( \mu_{\alpha} \) is the adjoint representation of \( \mathfrak{k} \).

Lemma 2.4. Let \( p(\lambda) = \dim(\mu_{\lambda}) \). Then
\[ F(\pi) = \frac{\sum_{\lambda \in \Lambda} p^2(\lambda) F(\mu_{\lambda})}{\sum_{\lambda \in \Lambda} p^2(\lambda)} \]

Proof. The collection \( \{\sqrt{p(\lambda)} t_{i,j}^{\lambda}\}_{\lambda \in \Lambda} \) is an orthonormal basis in \( \text{Trig}(\mu) \). Since \( \dim \text{Trig}(\mu_{\lambda}) = p^2(\lambda) \) and \( \dim \text{Trig}(\mu) = \sum_{\lambda \in \Lambda} p^2(\lambda) \), then from (2.3) it follows that \( F(\mu) = (\sum_{\lambda \in \Lambda} p^2(\lambda) F(\mu_{\lambda})) / \sum_{\lambda \in \Lambda} p^2(\lambda) \). Now the statement is a consequence of (2.11).

Corollary 2.12. For any \( \xi, \eta \in \mathfrak{k} \)
\[ F(\pi)(\xi, \eta) = -\frac{\sum_{\lambda \in \Lambda} p(\lambda) \text{Tr}(d\mu_{\lambda}(\xi) \cdot d\mu_{\lambda}(\eta))}{\sum_{\lambda \in \Lambda} p^2(\lambda)} \]

Proof. Follows from Lemmas 2.4, 2.3 and from equality (2.11).
2.5. Simple group. Here we assume the group \( K \) to be simple. For a simple group the Newton ellipsoid \( \text{Ell}(\pi) \) is a ball centered at the origin, see Corollary 2.5 (2). Here we compute the radius of \( \text{Ell}(\pi) \).

We will use the notations from Subsection 1.2.1. Moreover we use the Killing inner product \( \langle \xi, \eta \rangle = -\text{Tr} (d\mu_\alpha(\xi) \cdot d\mu_\alpha(\eta)) \) (or the Killing metric) in \( \mathfrak{k} \), and we keep the term Killing metric for the dual inner product (or resp. for the dual metric) in \( \mathfrak{k}^* \). The radius of the ball \( B \subset \mathfrak{k}^* \) with respect to the Killing metric we call the Killing radius of \( B \).

Theorem 4. Let \( r_K(\Lambda) \) be a Killing radius of \( \text{Ell}(\pi) \). Then

\[
r_K^2(\Lambda) = \frac{\sum_{\lambda \in \Lambda} p^2(\lambda) (\lambda, \lambda + 2\rho)}{n(\alpha, \alpha + 2\rho) \sum_{\lambda \in \Lambda} p^2(\lambda)}
\]

where \( \alpha \) is the highest root of \( K \) (that is a highest weight of the adjoint representation \( \mu_\alpha \)).

For a simple group, any two coadjoint invariant metrics differ by a constant factor. Hence, for any complex representation \( \mu \), there exists a constant \( l(\mu) > 0 \), such that for any \( \xi, \eta \in \mathfrak{k} \)

\[
\frac{\text{Tr} (d\mu(\xi) \cdot d\mu(\eta))}{\text{dim}(\mu)} = l(\mu) \frac{\text{Tr} (d\mu_\alpha(\xi) \cdot d\mu_\alpha(\eta))}{n}
\]

(2.12)

It is known (see for example [16, (8)]) that

\[
l(\mu_\lambda) = \frac{\langle \lambda, \lambda + 2\rho \rangle}{\langle \alpha, \alpha + 2\rho \rangle},
\]

(2.13)

Remark 2.2. The equality (2.13) is equivalent to the expression \( \langle \lambda, \lambda + 2\rho \rangle \) for the eigenvalue of the Casimir operator in the space \( \text{Trig}(\mu_\lambda) \); see [5].

Lemma 2.5. Let \( \pi \) be a real representation \( K \) with the spectrum \( \Lambda \), and let

\[
\nu_K = \frac{1}{n(\alpha, \alpha + 2\rho)} \frac{\sum_{\lambda \in \Lambda} p^2(\lambda) (\lambda, \lambda + 2\rho)}{\sum_{\lambda \in \Lambda} p^2(\lambda)}
\]

Then \( \nu(\pi)(\xi, \xi) = \nu_K |\xi|^2 \).

Proof. Follows from Corollary 2.12 for \( \xi = \eta = \zeta \) and from equalities (2.12), (2.13).

Proof of Theorem 4. Recall that \( h_\pi = \sqrt{\nu(\pi)} \) is a support function of the Newton ellipsoid \( \text{Ell}(\pi) \), see Definition 2.5. Since \( r_K = h_\pi(\zeta) \), where \( |\zeta| = 1 \), then Lemma 2.5 states that \( r_K^2 = \nu_K \), and the required statement follows.

2.6. The asymptotics of mean number of roots. In this Subsection the group \( K \) is also assumed to be simple. Accordingly, the metric \( \nu \) in \( \mathfrak{k} \) is assumed to be Killing metric. The dual metric in \( \mathfrak{k}^* \) also denoted as \( \nu \). Recall that any of the corresponding Lebesgue measures in the spaces \( \mathfrak{k}, \mathfrak{k}^* \) and in \( \mathfrak{t}^* \) are denoted as \( d\nu \); see Subsection 1.2.1. The corresponding to \( \nu \) left invariant metric on \( K \) is denoted by \( \tau \).

Let \( \Delta \) be a compact convex set in \( \mathfrak{t}^* \). We assume that \( \Delta \) is centrally symmetric and invariant under the action of the Weyl group \( W^\ast \). (The examples of such type convex sets are the ball with centre at the origin, and
the weighted polyhedron of complexification of a finite dimensional representation \( K \), see Definition 3.1). Then the finite set \( \Lambda = \Delta \cap \mathfrak{c}^* \cap \mathbb{Z}^k \) is symmetric; see Definition 1.3. We consider the representation

\[
\pi(\Delta) = \bigoplus_{(\lambda, \lambda') \in \Lambda'} \pi_{\lambda, \lambda'},
\]

where \( \Lambda' \) is a set of unordered symmetric pairs \((\lambda, \lambda')\) with \( \lambda \in \Lambda \); see Definition 1.3. Then, for \( m \to \infty \), we calculate the asymptotics of the mean number of roots \( \mathfrak{M}(\pi_m(\Delta)) \), where

\[
\pi_m(\Delta) = \pi(m\Delta).
\]

In the following theorem we use the compact convex set \( \mathfrak{N}(\Delta) \subset \mathfrak{t}^* \) defined as the union of all coadjoint orbits \( K \) intersecting \( \Delta \). In particular, if \( \Delta \) is a ball in \( \mathfrak{t}^* \) then \( \mathfrak{N}(\Delta) \) is a ball with the same radius in \( \mathfrak{t}^* \).

The proof of the theorem is based on the estimate for the asymptotics of a Killing radius of the ball \( \text{Ell}(\pi_m(\Delta)) \).

**Theorem 5.** The sequence of compact convex sets \( \frac{1}{m} \text{Ell}(\pi_m(\Delta)) \), with respect to Hausdorff topology, converges to the ball with the Killing radius \( r_K(\Delta) \) and centre at the origin, where

\[
r^2_K(\Delta) = \frac{\int_{\mathfrak{N}(\Delta)} (\xi, \xi) \, d\nu(\xi)}{n(\alpha, \alpha + 2\rho) \, \text{vol}(\mathfrak{N}(\Delta))}
\]

(recall that the Newton ellipsoids \( \text{Ell}(\pi_m(\Delta)) \) themselves are also balls; see Corollary 2.13 (2)).

**Corollary 2.13.** Let \( \Delta \) be a ball in \( \mathfrak{t}^* \) with radius \( r \). Then

\[
r^2_K(\Delta) = \frac{r^2}{n(\alpha, \alpha + 2\rho)}
\]

**Proof.** Since \( \mathfrak{N}(\Delta) \) equals to the ball \( B_r \) in \( \mathfrak{t}^* \) with radius \( r \) and centre at the origin, then

\[
r^2_K(\Delta) = \frac{1}{n(\alpha, \alpha + 2\rho)} \int_{B_r} |x|^2 \, dx = \frac{r^2}{n(\alpha, \alpha + 2\rho)}
\]

**Proof of Theorem 5.** By the Weyl dimension formula

\[
\dim(\mu_\lambda) = \frac{P(\lambda + \rho)}{P(\rho)},
\]

where \( \mu_\lambda \) is an irreducible representation with the highest weight \( \lambda \), and \( P(\lambda) = \prod_{\beta \in R^+} (\lambda, \beta) \); see [5, Theorem 5 in Ch. IX, §7, n. 3]). Then applying Lemma 2.5 we obtain

\[
\frac{1}{m^2} F(\pi_m(\Delta))(\zeta, \zeta) = \frac{|\zeta|^2}{n(\alpha, \alpha + 2\rho)} \sum_{\lambda \in \lambda m} (\lambda, \lambda + 2\rho/m) P^2(\lambda + \rho/m) / \sum_{\lambda \in \lambda m} P^2(\lambda + \rho/m)
\]

Hence for a large \( m \) we have

\[
\frac{1}{m^2} F(\pi_m(\Delta))(\zeta, \zeta) \approx \frac{|\zeta|^2}{n(\alpha, \alpha + 2\rho)} \sum_{\lambda \in \lambda m} P^2(\lambda) (\lambda, \lambda) / \sum_{\lambda \in \lambda m} P^2(\lambda)
\]
Denote by $s$ the volume of a fundamental cube of the integer lattice $\mathbb{Z}^k \subset t^*$ measured with respect to the Killing metric. Then the sums
\[
\sum_{\lambda \in \Delta_m} P^2(\lambda) \cdot \frac{s}{m^k} \quad \text{and} \quad \sum_{\lambda \in \Delta_m} P^2(\lambda, \lambda) \cdot \frac{s}{m^k}
\]
are the Riemann sums for the integrals respectively $\int_{\Delta} P^2(\lambda) d\nu$ and $\int_{\Delta} P^2(\lambda, \lambda) d\nu$ with the partition given by $\frac{1}{m}\mathbb{Z}^k$. Hence
\[
\lim_{m \to \infty} \frac{1}{m^2} \int_{\Delta} F(\pi_m(\Delta))(\zeta, \zeta) = \frac{|\zeta|^2}{n(\alpha, \alpha + 2)} \frac{\int_{\Delta} P^2(\lambda, \lambda) d\nu}{\int_{\Delta} P^2(\lambda) d\nu}
\]
Now we recall the Weyl’s integration formula for the coadjoint invariant function $f: t^* \to \mathbb{R}$ \cite{[3]} Proposition 3 in Ch. IX, §3, n. 3
\[
\frac{\text{vol}(K)}{|W| \text{vol}(T^k)} \int_{t^*} P^2(\lambda) f \, d\nu = \int_{t^*} f \, d\nu.
\]  
(2.14)
Let $f$ be the characteristic function of the Newton body $\mathfrak{N}(\Delta)$. Applying \cite{[3]} to the functions $f$ and $(\xi, \xi)f$, we obtain that
\[
\lim_{m \to \infty} \frac{1}{m^2} \int_{\Delta} F(\pi_m(\Delta))(\zeta, \zeta) = \frac{|\zeta|^2}{n(\alpha, \alpha + 2) \text{vol}(\mathfrak{N}(\Delta))} \int_{\mathfrak{N}(\Delta)} \xi, \xi \, d\nu
\]
Hence, since $h_{\pi_m(\Delta)} = \sqrt{F(\pi_m(\Delta))}$, then
\[
\lim_{m \to \infty} \frac{1}{m} h_{\pi_m(\Delta)}(\zeta) = \sqrt{\frac{1}{n(\alpha, \alpha + 2) \text{vol}(\mathfrak{N}(\Delta))} \int_{\mathfrak{N}(\Delta)} \xi, \xi \, d\nu}
\]
and for $|\zeta| = 1$ we get the required equality.
\[\square\]
**Theorem 6.** Let $\Delta \subset t^*$ be the Killing ball with radius $r$ and centre at the origin. Then for increasing $m$ we have
\[
\mathfrak{N}(\pi_m(\Delta)) \asymp \frac{\sigma_n m^n}{(2\pi)^n (n + 2)^{n/2}(\alpha, \alpha + 2)^{n/2}} r^n,
\]
de where $\text{vol}_K(K)$ is a Killing volume of the group $K$.

**Proof.** From Theorem \cite{[3]} and Corollary \cite{[2.13]} it follows that the sequence $\frac{1}{m} \text{Ell}(\pi_m(\Delta))$ converges to the ball with Killing radius
\[
(n + 2)^{-1/2}(\alpha, \alpha + 2)^{-1/2} r.
\]
Now it remains to apply Theorem \cite{[3]} and Corollary \cite{[2.7]} \[\square\]
**Corollary 2.14.** Let $g$ be an invariant inner product $(\ast, \ast)_\tau$ in $\mathfrak{k}$ such that $\text{vol}_K(K) = 1$, and let $\Delta \subset t^*$ be the corresponding ball with radius $r$ and centre at the origin. Then
\[
\mathfrak{N}(\pi_m(\Delta)) \asymp \frac{n!}{(2\pi)^n (n^2 + 2)^{n/2}(\alpha, \alpha + 2)^{n/2}} r^n
\]

**Proof.** Let $(\ast, \ast)_K = c_r(\ast, \ast)_\tau$, where $(\ast, \ast)_K$ is the Killing inner product in $K$. Then $r_K(\Delta) = c^{-1} r$. Since $\text{vol}_K(K) = c_r^2 \text{vol}_r(K)$, then applying Theorem \cite{[3]} we obtain the required equality. \[\square\]
3. Proportion of real roots

In this Section we 1) formulate and prove the BKK reductive theorem in the form convenient for what follows, 2) use it to calculate the expected proportion of real roots of \( \pi \)-polynomial systems, 3) use the calculation to prove Theorem 2.

3.1. Theorem BKK for reductive groups. We denote by \( K_C \) a complexification of \( K \). That is a complex connected \( n \)-dimensional reductive Lie group, such that \( K \) is a maximal compact subgroup in \( K_C \). We consider the finite-dimensional holomorphic representations \( \mu_1, \ldots, \mu_n \) of \( K_C \), and the complex vector spaces of \( \mu_i \)-polynomials \( \text{Trig}(\mu_i) \) (recall that a \( \mu_i \)-polynomial is a linear combination of matrix elements of \( \mu_i \)). To any system of \( n \) non-zero \( \mu_i \)-polynomials \( f_i \in \text{Trig}(\Pi_i) \) we associate the point \( \iota(f_1, \ldots, f_n) = (Cf_1) \times \ldots \times (Cf_n) \in \mathbb{P}_1 \times \ldots \times \mathbb{P}_n \), where \( \mathbb{P}_i \) is a complex projective space of one-dimensional subspaces in \( \text{Trig}(\mu_i) \). Next, we will use the statement that is standard in algebraic geometry.

Proposition 3.1. There exist the number \( N(\mu_1, \ldots, \mu_n) \) and the algebraic hypersurface \( H \) in \( \mathbb{P}_1 \times \ldots \times \mathbb{P}_n \) such that the following is true. For any \( n \) \( \mu_i \)-polynomials \( f_i \in \text{Trig}(\Pi_i) \) with \( \iota(f_1, \ldots, f_n) \notin H \), the number of their common zeros equals \( N(\mu_1, \ldots, \mu_n) \).

Now we give the geometric formula for \( N(\mu_1, \ldots, \mu_n) \) (Theorem 8). Let \( \mu \) be a finite dimensional complex representation of \( K \), and let \( \mu = \bigoplus_{\lambda \in \Lambda \subseteq \mathbb{Z}^k \cap \mathbb{C}^*} \sum_{0 < m_\lambda \in \mathbb{Z}} m_\lambda \mu_\lambda \) be the decomposition of \( \mu \) into the sum of irreducible representations \( \mu_\lambda \) with the highest weight \( \lambda \) and the multiplicity \( m_\lambda \).

Definition 3.1. The compact convex set

\[
\Delta(\mu) = \text{conv}(\bigcup_{\lambda \in \Lambda} W^*(\lambda))
\]

is called the weighted polyhedron of the representation \( \mu \). We denote by \( \mathfrak{N}(\mu) \subset t^* \) the union of coadjoint orbits \( K \) intersecting the weighted polyhedron \( \Delta(\mu) \) (we identify \( t^* \) with the fixed points subspace of the coadjoint action \( T^k \) in \( t^* \)). In what follows \( \mathfrak{N}(\mu) \) is called the Newton body of the representation \( \mu \).

Corollary 3.1. (1) Let \( \mu_T \) be a restriction of representation \( \mu \) onto the maximal subtorus \( T^k \) of \( K \), and let \( \Lambda_T \subset t^* \) be a set of weights of representation \( \mu_T \) of the torus \( T^k \). Then \( \Delta(\mu) = \text{conv}(\Lambda_T) \).

(2) The set \( \mathfrak{N}(\mu) \) is convex.

(3) \( \Delta(\mu) = p(\mathfrak{N}(\mu)) \), where \( p \) is the standard projection mapping \( t^* \rightarrow t^* \).

Proof. The statement (1) follows from the highest weight theory. It is known that for any \( \zeta \in t^* \) the projection \( p \) of the coadjoint orbit \( \text{Ad}(K)(\zeta) \) to \( t^* \) is equal to a convex hull of the orbit \( W^*\zeta \) of the Weyl group \( W^* \); see [14]. This implies the statements (2) and (3).
Let $F$ be a homogeneous polynomial in $t^*$ of degree $p$. For any convex polytope $\Delta \subset t^*$ we set

$$I(\Delta; F) = \int_\Delta F \, d\mu,$$

where the measure $\mu$ in $K$ is chosen so that the Lebesgue measure $d\mu$ of a fundamental cube of the character lattice in $t^*$ equals 1. It is known (for example [10]) that, for a fixed polynomial $F$, the function $I(\Delta; F)$ is a homogeneous polynomial of degree $k + p$ in the space of virtual convex polyhedra in $t^*$. Denote by $J(\Delta_1, \ldots, \Delta_{k+p}; F)$ its polarization, that is a symmetric multi-linear function on the space of virtual convex polyhedra in $t^*$ such that $J(\Delta, \ldots, \Delta; F) = I(\Delta; F)$.

First we present the reductive BKK theorem in standard formulation; see [10]. According to the Weyl dimension formula the dimension of representation $\mu_\lambda$ with the highest weight $\lambda$ is equal to $F_W(\lambda)$, where $F_W$ is the polynomial on $t^*$ of degree $(2n - k)/2$. We denote the homogeneous component of highest degree of $F_W$ by $\phi$.

**Theorem 7.** For any finite dimensional holomorphic representations $\mu_1, \ldots, \mu_n$ of $K_C$ we have

$$N(\mu_1, \ldots, \mu_n) = \frac{n!}{|W|} J(\Delta(\mu_1), \ldots, \Delta(\mu_n); \phi^2),$$

where $\Delta(\mu_i)$ is the weighted polyhedron of $\mu_i$ (see Definition 3.7).

Let $\mu$ be the normalized Haar measure in $K$. We use it for the measurement of the volume $\text{vol}(K)$ of the group $K$ and the volume of the Newton body $\mathfrak{N}(\mu)$ of representation $\mu$; see Subsection 2.3.

**Theorem 8.** Let the group $K_C$ be simple. Then for any $n$ holomorphic representations $\mu_1, \ldots, \mu_n$ of $K_C$ we have

$$N(\mu_1, \ldots, \mu_n) = \frac{n!}{P^2(\rho)} \text{vol}(\mathfrak{N}(\mu_1), \ldots, \mathfrak{N}(\mu_n))$$

Proof. For a simple group, $F_W(\lambda) = P(\lambda + \rho)/P(\rho)$. Hence $\phi^2(\lambda) = P^2(\lambda)/P^2(\rho)$. Since $\left( \int_\Delta F d\nu \right) = \text{vol}(T^k) I(\Delta; F)$ then by Theorem 7 we have

$$N(\mu) = \frac{n!}{|W| \text{vol}(T^k) P^2(\rho)} \int_{\Delta(\mu)} P^2(\lambda) \, d\nu$$

Next we use the notion from Subsection 1.2.1. Applying the Weyl’s integration formula (see (2.14) in Subsect. 2.6) to the characteristic function $f$ of the set $\mathfrak{N}(\mu) \subset t^*$, we obtain that

$$N(\mu) = \frac{n!}{P^2(\rho) \text{vol}(K)} \text{vol}(\mathfrak{N}(\mu)) = \frac{n!}{P^2(\rho) \text{vol}(\mathfrak{N}(\mu))}$$

Now using the standard polarization formula for the homogeneous polynomial $I(\Delta; \phi^2)$ (see for example [13]) and also for the volume polynomial in the space of virtual convex compact sets in $t^*$, we obtain the required statement. □
3.2. Expected proportion of real roots. Further, we assume that the metric \( \tau \) is chosen so that \( \text{vol}(K) = 1 \).

**Theorem 9.** Let \( \pi_1, \ldots, \pi_n \) be the real representations of the simple Lie group \( K \). Denote by \( \mu_1, \ldots, \mu_n \) their extensions to holomorphic representations of \( K_\mathbb{C} \). Then

\[
\text{real}(\pi_1, \ldots, \pi_n) = \frac{P^2(\rho)}{(2\pi)^n} \frac{\text{vol}(\text{Ell}(\pi_1), \ldots, \text{Ell}(\pi_n))}{\text{vol}(\mathfrak{N}(\mu_1), \ldots, \mathfrak{N}(\mu_n))},
\]

Proof. The vector space \( \text{Trig}(\pi_i) \) is a real subspace of the complex vector space \( \text{Trig}(\Pi_i) \), which is dense with respect to Zariski topology. Let us consider the projection mapping \( \bar{\kappa} : S^1 \times \ldots \times S^n \rightarrow P^1 \times \ldots \times P^n \) (see Definition 2.2). The image of \( \bar{\kappa} \) is dense in \( P^1 \times \ldots \times P^n \). Therefore, \( \bar{\kappa}^{-1}H \) is contained in some closed real hypersurface in \( S^1 \times \ldots \times S^n \). Therefore the function \( \mathfrak{N}_\mathbb{C}(s_1, \ldots, s_n) \) in \( S^1 \times \ldots \times S^n \) (see Definition 2.3) coincides with \( \mathfrak{N}(\mu_1, \ldots, \mu_n) \) almost everywhere. Hence

\[
\text{real}(\pi_1, \ldots, \pi_n) = \frac{\mathfrak{M}(\pi_1, \ldots, \pi_n)}{\mathfrak{N}(\mu_1, \ldots, \mu_n)},
\]

and the required statement follows from Theorems 3, 8. \( \square \)

3.3. The asymptotics of expected proportion. First we recall the Theorem 2. Let \( B \) be a ball with radius 1 and centre at the origin, \( \Lambda_m = mB \cap \mathbb{Z}^k \cap \mathbb{C}^* \), and

\[
\pi_m(B) = \bigoplus_{(\lambda, \lambda') \in \Lambda'_m} \pi_{\lambda, \lambda'}.
\]

Theorem 2 states that, for a simple group \( K \),

\[
\lim_{m \to \infty} \text{real}(\pi_m(B)) = \frac{P^2(\rho)}{(2\pi)^n (n+2)^{n/2} (\alpha, \alpha+2\rho)^{n/2}} \quad (3.1)
\]

Proof of Theorem 2. By definition

\[
\text{real}(\pi_m(B)) = \frac{\mathfrak{M}(\pi_m(B))}{\mathfrak{N}(\pi_m(B))}.
\]

From Corollary 2,14 it follows that

\[
\mathfrak{M}(\pi_m(B)) \asymp \frac{n!}{(2\pi)^n (n+2)^{n/2} (\alpha, \alpha+2\rho)^{n/2}} \sigma_n m^n
\]

Since \( \mathfrak{N}(\pi_m(B)) \asymp mB \) then, by Theorem 8

\[
\mathfrak{N}(\pi_m(B)) \asymp \frac{n!}{P^2(\rho)} \sigma_n m^n.
\]

Hence the required equality follows. \( \square \)

**Remark 3.1.** Using the results [17, 19] it is possible to present the answer (3.1) in a more topological form. But the topological answer seems to be more complicated than the given one.
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