GENUS ONE ENUMERATIVE INVARIANTS IN $\mathbb{P}^n$
WITH FIXED $j$ INVARIANT

Eleny Ionel

Abstract

We prove recursive formulas for $\tau_d$, the number of degree $d$ elliptic curves with fixed $j$ invariant in $\mathbb{P}^n$. We use analysis to relate the classical invariant $\tau_d$ to the genus one perturbed invariant $RT_1,d$ defined recently by Ruan and Tian (the later invariant can be computed inductively). By considering a sequence of perturbations converging to zero, we then apply Taubes’ Obstruction Bundle method to compute the difference between the two invariants.

0 Introduction.

A classical problem in enumerative algebraic geometry is to compute the number of degree $d$, genus $g$ holomorphic curves in $\mathbb{P}^n$ that pass through a certain number of constraints (points, lines, etc).

Let $\sigma_d$ denote the number of degree $d$ rational curves ($g = 0$) through appropriate constraints. For example $\sigma_1(pt, pt) = 1$ (since 2 points determine a line). The first nontrivial cases were computed around 1875 when Schubert, Halphen, Chasles et al. found $\sigma_2$ for $\mathbb{P}^2$ and $\mathbb{P}^3$. Later, more low degree examples were computed in $\mathbb{P}^2$ and $\mathbb{P}^3$, but the progress was slow. Then in 1993 Kontsevich [KM] predicted, based on ideas of Witten, that the number $\sigma_d$ of degree $d$ rational curves in $\mathbb{P}^2$ through $3d - 1$ points satisfies the following recursive relation:

$$
\sigma_d = \sum_{d_1 + d_2 = d} \left[ \frac{3d - 1}{3d_1 - 1} d_1^2 d_2^2 - \frac{3d - 1}{3d_1 - 2} d_1^3 d_2 \right] \sigma_{d_1} \sigma_{d_2}
$$

where $d_i \neq 0$, and $\sigma_1 = 1$. Ruan-Tian ([RT], 1994) extended these formulas for $\sigma_d$ in any $\mathbb{P}^n$.

When genus $g = 1$, the classical problem splits into two totally different problems: one can count (i) elliptic curves with a fixed complex structure, or (ii) elliptic curves with unspecified complex structure (each satisfying the appropriate number of constraints). This paper gives recursive formulas which completely solve the first of these.

Thus our goal is to compute the number $\tau_d$ of degree $d$ elliptic curves in $\mathbb{P}^n$ with fixed $j$ invariant. Classically, the progress on this problem has been even slower than on the genus one case. Recently, Pandharipande [Pan] found recursive formulas for $\tau_d$ for the 2 dimensional projective space $\mathbb{P}^2$ using the Kontsevich moduli space of stable curves.

We will approach the problem from a different direction, using analysis. Our approach is based on the ideas introduced by Gromov to study symplectic topology. If $(\Sigma, j)$ is a fixed Riemann surface, let

$$
\{ f : \Sigma \to \mathbb{P}^n | \bar{\partial}_j f = 0, [f] = d \cdot l \in H_2(\mathbb{P}^n, \mathbb{Z}) \} / \text{Aut}(\Sigma, j)
$$
be the moduli space of degree \(d\) holomorphic maps \(f: \Sigma \to \mathbb{P}^n\), modulo the automorphisms of \((\Sigma, j)\). Each constraint, such as the requirement that the image of \(f\) passes through a specified point, defines a subset of this moduli space.

Imposing enough constraints gives a 0-dimensional “cutdown moduli” space \(\mathcal{M}_d\). To see whether or not it consists of finitely many points, one looks at its bubble tree compactification \(\overline{\mathcal{M}}_d\). If the constraints cut transversely, then all the boundary strata of \(\overline{\mathcal{M}}_d\) are at least codimension 1, and thus empty. Unfortunately, transversality fails at multiply-covered maps or at constant maps (called ghosts), so \(\overline{\mathcal{M}}_d\) is not a manifold.

This was a real problem until 1994, when Ruan and Tian considered the moduli space \(\mathcal{M}_\nu\) of solutions of the perturbed equation:

\[
\overline{\mathcal{J}}_f = \nu(x, f(x))
\]

and used marked points instead of moding out by \(\text{Aut}(\Sigma, j)\). For a generic perturbation \(\nu\) the moduli space \(\mathcal{M}_\nu\) is smooth and compact, so it consists of finitely many points that, counted with sign, give an invariant \(RT_{d,g}\) (independent of \(\nu\)).

In \(\mathbb{P}^n\), the genus 0 perturbed invariant \(RT_{d,0}\) is equal to the enumerative invariant \(\sigma_d\). The perturbed invariants satisfy a degeneration formula that gives not only recursive formulas for the enumerative invariant \(\sigma_d\) in \(\mathbb{P}^n\), but also expresses the higher genus perturbed invariants in terms of the genus zero invariants \(RT\). For convenience, these formulas are included in the Appendix.

Unfortunately, when \(g = 1\), the perturbed invariant \(RT_{d,1}\) does not equal the enumerative invariant \(\tau_d\). For example, for \(d = 2\) curves in \(\mathbb{P}^2\) the Ruan-Tian invariant is \(RT_{2,1} = 2\) (cf. (A.2)), while \(\tau_2 = 0\) (there are no degree 2 elliptic curves in \(\mathbb{P}^2\)). Thus while the Ruan-Tian invariants are readily computable, they differ from the enumerative invariants \(\tau_d\). One should seek a formula for the difference between the two invariants. For that, we take the obvious approach:

Start with the genus 1 perturbed invariant \(RT_{d,g}\) and consider a sequence of generic perturbations \(\nu \to 0\). A sequence of \((J, \nu)\)-holomorphic maps converges either to a holomorphic torus or to a bubble tree whose base is a constant map (ghost base). Proposition 1.21 shows that the contribution of the \((J, 0)\)-holomorphic tori is a multiple of \(\tau_d\).

We show that the only other contribution comes from bubble trees with ghost base such that the bubble point is equal to the marked point \(x_1 \in T^2\). To compute this contribution, we use the Taubes’ “Obstruction Bundle” method. Proposition 1.7 identifies the moduli space of \((J, \nu)\)-holomorphic maps close to a bubble tree with the zero set of a specific section of the obstruction bundle. Studying the leading order term of this section, we are able to compute the corresponding contribution (Proposition 1.26). Adding both contributions, yields our main analytic result:

**Theorem 0.1** Consider the genus 1 enumerative invariant \(\tau_d(\beta_1, \ldots, \beta_k)\) in \(\mathbb{P}^n\). Let \(\mathcal{U}_d\) be the \(n - 1\) dimensional moduli space of 1-marked rational curves of degree \(d\) in \(\mathbb{P}^n\) passing through \(\beta_1, \ldots, \beta_k\). Let \(L \to \mathcal{U}_d\) be the relative tangent sheaf, and denote by \(\tilde{L} \to \mathcal{U}_d\) its blow up as in Definition 1.17. Then:

\[
n_j \tau_d(\beta_1, \ldots, \beta_k) = RT_{d,1}(\beta_1 | \beta_2, \ldots, \beta_k) - \sum_{i=0}^{n-1} \binom{n+1}{i+2} \text{ev}^*(H^{n-i-1}) c_i^j(\tilde{L}^*)
\]
where $H^i$ is a codimension $i$ hyperplane in $\mathbb{P}^n$, $ev : \mathcal{U}_d \to \mathbb{P}^n$ is the evaluation map corresponding to the special marked point and $n_j = \text{Aut}_x(j)$ is the order of the group of automorphisms of the complex structure $j$ that fix a point.

Theorem 0.1 becomes completely explicit provided we can compute the top power intersections $ev^*(H^{n-i-1})c_1(\tilde{\mathcal{L}}^*)$. We do this in the second part of the paper, in several steps. For simplicity of notation, let

$$x = c_1(L^*) \in H^2(\mathcal{U}_d, \mathbb{Z}), \quad \tilde{x} = c_1(\tilde{L}^*) \in H^2(\tilde{\mathcal{U}}_d, \mathbb{Z})$$  \hspace{1cm} (0.1)

$$y = ev^*(H), \quad y \in H^2(\mathcal{U}_d, \mathbb{Z}) \quad \text{or} \quad y \in H^2(\tilde{\mathcal{U}}_d, \mathbb{Z})$$  \hspace{1cm} (0.2)

depending on the context. In this notation, Theorem 0.1 combined with (A.2) becomes:

$$n_j \tau_d(\cdot) = \sum_{i_1 + i_2 = n} \sigma_d(H^{i_1}, H^{i_2}, \cdot) + \sum_{i=0}^{n-1} \binom{n+1}{i+2} \tilde{x}^i y^{n-1-i} \cdot \tilde{\mathcal{U}}_d$$  \hspace{1cm} (0.3)

Proposition 2.6 explains how to get recursive formulas relating $\tilde{x}^i y^j$ to $x^k y^l$ and Proposition 2.2 gives recursive formulas for $x^i y^j$ in terms of the enumerative invariant $\sigma_d$. Finally, the recursive formulas for $\sigma_d$ are known (see [RT], [KM]), so the right hand side of (0.3) can be recursively computed.

In the end, we give applications of these formulas. We explicitly work out the formulas expressing the number of degree $d$ elliptic curves passing through generic constraints in $\mathbb{P}^2$ and $\mathbb{P}^3$ in terms of the rational enumerative invariant $\sigma_d$. For example:

**Proposition 0.2** For $j \neq 0, 1728$, the number $\tau_d = \tau_d(p^a, l^b)$ of elliptic curves in $\mathbb{P}^3$ with fixed $j$ invariant and passing through $a$ points and $b$ lines (such that $2a + b = 4d - 1$) is given by:

$$\tau_d(\cdot) = \frac{(d-1)(d-2)}{d} \sigma_d(l, \cdot) - \frac{1}{d} \sum_{d_1 + d_2 = d} d_2(2d_1 d_2 - d) \sigma_{d_1}(l, \cdot) \sigma_{d_2}(\cdot)$$  \hspace{1cm} (0.4)

where $\sigma_d(l, \cdot) = \sigma_d(l, p^a, l^b)$ is the number of degree $d$ rational curves in $\mathbb{P}^3$ passing through same conditions as $\tau_d$ plus one more line. The sum above is over all decompositions into a degree $d_1$ and a degree $d_2$ component, $d_i \neq 0$, and all possible ways of distributing the constraints $p^a$, $l^b$ on the two components.

Using a computer program, one then computes specific invariants: for example, the number of degree 10 tori in $\mathbb{P}^3$ with fixed $j$ invariant and passing through 39 lines is:

$$6 \cdot 386805671822029784844530703900638969856$$

when $j \neq 0, 1728$. To get $\tau_d$ for $j = 0$ or $j = 1728$ one simply divides the $\tau_d$ computed for a generic $j$ by 3 or 2 respectively.

**Acknowledgements.** I would like to thank my advisor Prof. Thomas Parker for introducing me to the subject and for the countless hours of discussions.
1 Analysis

1.1 Setup

Let \( \tau_d \) be the genus one degree \( d \) enumerative invariant (with fixed \( j \) invariant) and \( \sigma_d \) be the genus zero degree \( d \) enumerative invariant in \( \mathbb{P}^n \). Using analytic methods, we will compute \( \tau_d \) by relating it to the perturbed invariant \( RT_{d,g} \) introduced by Ruan and Tian [RT]. The later is defined as follows.

Let \((\Sigma, j)\) be a genus \( g \) Riemann surface with a fixed complex structure and \( \nu \) an inhomogeneous term. A \((J, \nu)\)-holomorphic map is a solution \( f : \Sigma \to \mathbb{P}^n \) of the equation

\[
\overline{\partial}_J f(x) = \nu(x, f(x)).
\]

(1.1)

For \( 2g + l \geq 3 \), let \( x_1, \ldots, x_l \) be fixed marked points on \( \Sigma \), and \( \alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_k \) be various codimension submanifolds in \( \mathbb{P}^n \), such that

\[
\text{index } \overline{\partial}_J = (n + 1)d - n(g - 1) = \sum_{i=1}^{l}(n - |\alpha_i|) + \sum_{i=1}^{k}(n - 1 - |\beta_i|)
\]

For a generic \( \nu \), the invariant

\[
RT_{d,g}(\alpha_1, \ldots, \alpha_l | \beta_1, \ldots, \beta_k)
\]

counts the number of \((J, \nu)\)-holomorphic degree \( d \) maps \( f : \Sigma \to \mathbb{P}^n \) that pass through \( \beta_1, \ldots, \beta_k \) with \( f(x_i) \in \alpha_i \) for \( i = 1, \ldots, l \) (for more details see [RT]).

The first part of this paper is devoted to the proof of Theorem 0.1.

Outline of the Proof of Theorem 0.1. The proof is done in several steps. The basic idea is to start with the genus 1 perturbed invariant

\[
RT_{d,1}(\beta_1 | \beta_2, \ldots, \beta_l)
\]

(1.2)

and take a sequence of generic perturbations \( \nu \to 0 \). Denote by \( \mathcal{M}_{d,1,t\nu} \) the moduli space of \((J, t\nu)\)-holomorphic maps satisfying the constraints in (1.2), and let

\[
\mathcal{M}^\nu = \bigcup_{t \geq 0} \mathcal{M}_{d,1,t\nu}.
\]

(1.3)

As \( t \to 0 \), a sequence of \((J, t\nu)\)-holomorphic maps converges to a \((J, 0)\)-holomorphic torus or to a bubble tree ([PW]). Let \( \overline{\mathcal{M}^\nu} \) denote the bubble tree compactification of \( \mathcal{M}^\nu \) (for details on bubble tree compactifications, see [P]).

Proposition 1.21 shows that the number of \((J, t\nu)\)-holomorphic maps converging to a \( J \)-holomorphic torus is equal to

\[
n_j \tau_d(\beta_1, \ldots, \beta_k)
\]

where \( n_j = |Aut_{x_1}(j)| \) is the order of the group of automorphisms of the complex structure \( j \) that fix the point \( x_1 \). Namely,

\[
n_j = \begin{cases} 
2 & \text{if } j \neq 0, 1728 \\
6 & \text{if } j = 0 \\
4 & \text{if } j = 1728
\end{cases}
\]

(1.4)
These multiplicities occur because if $f$ is a $J$-holomorphic map, then so is $f \circ \phi$ for any $\phi \in \text{Aut}_{x_1}(J)$, but they get perturbed to different $(J, t\nu)$-holomorphic maps.

As $t \to 0$, there are also a certain number of solutions converging to bubble trees. Because the moduli space of $(J,0)$-holomorphic tori passing through $\beta_1, \ldots, \beta_k$ is 0 dimensional, the only bubble trees which occur have a multiply-covered or a ghost base (for these transversality fails, so dimensions jump up).

A careful dimension count shows that the multiply-covered base strata are still codimension at least one for genus $g = 1$ maps in $\mathbb{P}^n$. (This is not true for $g \geq 2$.) But at a ghost base bubble tree the dimension jumps up by $n$ so these strata are $n-1$ dimensional. There are actually 2 such pieces, corresponding to bubble tree where (i) the bubble point is at the marked point $x_1$ and (ii) the bubble point is somewhere else. To make this precise, a digression is necessary to set up some notation.

Let

$$\mathcal{M}_d^0 = \{ (f, y_1, \ldots, y_k) \mid f : S^2 \to \mathbb{P}^n \text{ degree } d \text{ holomorphic}, f(y_j) \in \beta_j \}$$

be the moduli space of bubble maps, and $\mathcal{M}_d = \mathcal{M}_d^0 / G$ be the corresponding moduli space of curves, where $G = \text{PSL}(2, \mathbb{C})$. Introduce one special marked point $y \in S^2$ and let

$$\mathcal{U}_d = \{ [f, y, y_1, \ldots, y_k] \mid [f, y_1, \ldots, y_k] \in \mathcal{M}_d \}$$

be the moduli space of 1-marked curves and

$$\text{ev} : \mathcal{U}_d \to \mathbb{P}^n, \quad \text{ev}( [f, y, y_1, \ldots, y_k] ) = f(y).$$

be the corresponding evaluation map. We will use $f(y)$ to record the image of the ghost base.

For generic constraints $\beta_1, \ldots, \beta_k$ the bubble tree compactification of $\mathcal{U}_d$ is a smooth manifold that comes with a natural stratification, depending on the possible splittings into bubble trees and how the degree $d$ and the constraints $\beta_1, \ldots, \beta_k$ distribute on each bubble.

With this, the two "pieces" of the boundary of $\overline{\mathcal{M}}'$ are:

$$\{x_1\} \times \overline{\mathcal{U}}_d \quad \text{and} \quad T^2 \times \text{ev}^*(\beta_1)$$

The first factor records the bubble point, while the image of the ghost base is encoded in the second factor. For generic constraints, each piece, as well as their intersection, is a smooth manifold, again stratified.

To see which bubble trees with ghost base appear as a limit of perturbed tori, we use the Taubes’ Obstruction Bundle. This construction must be performed on the link of each strata. We do this first on the top statum of $\{x_1\} \times \overline{\mathcal{U}}_d$, which consists of bubble trees with ghost base and a single bubble.

In Section 1.2 we construct a set of approximate maps by gluing in the bubble. The "gluing data" $[f, y, v]$ at a 1-marked curve $[f, y]$ consists of a nonvanishing vector $v$ tangent to the bubble at the bubble point $y$. Proposition 1.4 shows that the obstruction bundle is then diffeomorphic to $\text{ev}^*(T\mathbb{P}^n)$.

In Section 1.3 we try to correct the approximate maps to make them $(J, t\nu)$-holomorphic by pushing them in a direction normal to the kernel of the linearized equation. Those approximate
maps that can be corrected to solutions of the equation \((1.1)\) are then identified with the zero set of a section \(\psi_t\) of the obstruction bundle. Proposition \(1.7\) shows that actually all the solutions of the equation \((1.1)\) are obtained this way, i.e. the end of the moduli space of \((J,t\nu)\)-holomorphic maps is diffeomorphic to the zero set of the section \(\psi_t\).

One might be tempted now to believe that the difference between the two invariants is simply the euler class of the obstruction bundle. But in fact, even in generic conditions, the section \(\psi_t\) is not a generic section of the obstruction bundle. We will see that the obstruction bundle has a nowhere vanishing section, so it has a trivial euler class, while there are examples in which the difference term is certainly not zero.

Now, to understand the zero set of \(\psi_t\) it is enough to look at the leading order term of its expansion as \(t \to 0\). By Proposition \(1.44\) this has the form \(df_y(v) + t\bar{\nu}\) where \(\bar{\nu}\) is the projection of \(\nu\) on the obstruction bundle.

The construction described above extends naturally to all the other boundary strata. Each bubble \([f_i,y_i]\) comes with “gluing data” \([f_i,y_i,v_i]\), consisting of a vector \(v_i\) tangent to the bubble at the bubble point \(y_i\). But the leading order term of the section \(\psi_t\) depends only on the vectors tangent to the first level of nontrivial bubbles.

More precisely, let \(Z_h \subset \overline{U}_d\) denote the collection of bubble trees for which the image \(u = f(y)\) of the ghost base lies on \(h\) nontrivial bubbles. Geometrically, the image of a bubble tree in \(Z_h\) has \(h\) components \(C_1, \ldots, C_h\) that meet at \(u\). Let \(W|_{Z_h} \to Z_h\) be the bundle whose fiber is \(T_uC_1 \oplus \cdots \oplus T_uC_h\). The leading order term of \(\psi_t\) on \(Z_h\) is a section of \(W\), equal to

\[
a(f,y,v) + t\bar{\nu} \overset{def}{=} df_1(y_1)(v_1) + \ldots + df_h(y_h)(v_h) + t\bar{\nu}\]

where \(((f_i,y_i,v_i))_{i=1}^h\) is the gluing data corresponding to the bubbles \(C_i, i = 1, \ldots, h\).

Unfortunately \(W \to \overline{U}_d\) is not a vector bundle (its rank is not constant). But if we blow up each strata \(Z_h\) starting with the bottom one, then the total space of \(W\) is the same as the total space of \(\overline{L}\), the blow-up of the relative tangent sheaf \(L \to U_d\). The leading order term of \(\psi_t\) descends as a map \(a + t\bar{\nu} : \overline{L} \to ev^*(TP^n)\). Moreover, \(\bar{\nu}\) doesn’t vanish on \(\text{Im}(M) = ev_*(\overline{U}_d)\) so it induces a splitting on the restriction

\[
TP^n/\text{Im}(M) = C(\bar{\nu}) \oplus E.
\]

Finally, we put all these pieces together in Proposition \(1.26\) to prove that the number of \((J,\nu)\)-holomorphic maps converging as \(\nu \to 0\) to the boundary strata \(\{x_1\} \times \overline{U}_d\) is given by the Euler class \(c_{n-1}(ev^*(E) \otimes \overline{L}^*)\).

In Section \(1.8\) we show that the other boundary strata \(T^2 \times ev^*(\beta_1)\) gives trivial contribution, concluding the proof of the Theorem \(1.1\).

### 1.2 The Approximate gluing map

Let \(U_d\) be the moduli space of 1-marked rational curves of degree \(d\) passing through the conditions \(\beta_1, \ldots, \beta_k\). In this section we construct a set of approximate maps starting from \(\{x_1\} \times \overline{U}_d\), the first boundary strata in \(1.8\). We will use a:
Cutoff function. Fix a smooth cutoff function $\beta$ such that $\beta(r) = 0$ for $r \leq 1$ and $\beta(r) = 1$ for $r \geq 2$. Let $\beta_\lambda(r) = \beta(r/\sqrt{\lambda})$. Then $\beta_\lambda$ has the following properties:

$$|\beta_\lambda| \leq 1, \quad |d\beta_\lambda| \leq 2/\sqrt{\lambda} \quad \text{and} \quad d\beta_\lambda \text{ is supported in } \sqrt{\lambda} \leq r \leq 2\sqrt{\lambda}$$

The definition of the approximate gluing map on the top stratum. Let $N$ denote the top stratum of $\{x_1\} \times \overline{U}_d$. First we need to choose a canonical representative of each bubble curve $[f, y] \in N$ (recall that $f(y)$ is the image of the ghost base). Using the $G = \text{PSL}(2, \mathbb{C})$ action, we can assume that $y$ is the North pole and $f$ is centered on the vertical axis, which leaves a $\mathbb{C}^* \cong S^1 \times \mathbb{R}_+$ indeterminacy. To break it off, include as gluing data a unit vector tangent to the domain $S^2$ of the bubble at the bubble point $y$. The frame bundle

$$Fr = \{ [f, y, u] \mid [f, y] \in U_d, \ u \in T_yS^2, \ |u| = 1 \}$$

models the link of $N$. The notation $[f, y, u]$ means the equivalence class under the action of $G$ given by:

$$g \cdot (f, y, u) = (f \circ g^{-1}, \ g(y), \ g(u))$$

where the compact piece $SO(3) \subset G$ acts on the unit frame $u$ by rotations and the noncompact part acts trivially.

Fix a nonzero vector $u_1$ tangent to the torus at $x_1$. This determines an identification $T_{x_1}(T^2) \cong \mathbb{C}$ such that $u_1 = 1$, giving local coordinates on the torus at $x_1 = 0$. Similarly, let $u_0$ be a unit vector tangent to the sphere $S^2$ at the north pole and consider the identification

$$(T_{x_1}T^2, u_1) \cong (T_NS^2, u_0)$$

that induces natural coordinates on the sphere via the stereographical projection (such that $N = 0, \ u_0 = 1$). These choices of local coordinates on the domain of the bubble tree will be used for the rest of the paper. Fix also a metric on $\mathbb{P}^n$ such that we can use normal coordinates up to radius 1.

To glue, one needs to make sure that only a small part of the energy of $f$ is concentrated in a neighbourhood of $y$. The convention in $\text{[PM]}$ is to rescale $f$ until $\varepsilon_0$ of its energy is distributed in $H_y$, the hemisphere centered at $y$.

But since the constructions in the next couple of sections involve quite a few estimates, we prefer to do a different rescaling, that will simplify the analysis. Choose a representative of $[f, y, v]$ such that

$$y = 0, \ u = 1, \ f \text{ centered on the vertical axis}$$

Since on the top strata $[f, y]$ cannot be a ghost, such representative is uniquely determined up to a rescaling factor $r \in \mathbb{R}_+$. We will choose this rescaling factor such that moreover

$$\max\{ |\nabla^2 f(z)|, \ |z| \leq 1 \} \leq 2$$

Note that if the degree of $f$ is not 1, then imposing the extra condition

$$\max\{ |\nabla^2 f(z)|, \ |z| \leq 1 \} = 2$$
determines uniquely the representative. To see this, choose some representative \( \tilde{f} \) as in (1.11) and look for a map \( f(z) = \tilde{f}(r z) \) satisfying also (1.13). The uniqueness comes from the fact that the map \( s(r) = \max \{ |\nabla^2 \tilde{f}(z)|, |z| \leq r \} - 2/r^2 \) is decreasing.

If the degree of \( f \) is 1, (i.e. the image curve is a line), then we could replace (1.13) by say \(|df(0)| = 1 \) and still have (1.12) satisfied.

Finally, the approximate gluing map
\[
\gamma_\varepsilon : Fr \times (0, \varepsilon) \to \text{Maps}(T^2, X)
\]
is constructed as follows: Choose the unique representative of \([f, y, u]\) satisfying (1.11) and (1.13). The approximate map \( f_\lambda \) is obtained by gluing to the constant map \( f(y) \) defined on \( T^2 \) the bubble map \( f \) rescaled by a factor of \( \lambda \) inside a disk \( D(0, \sqrt{\lambda}) \subset T^2 \),
\[
f_\lambda(z) = \beta_\varepsilon(|z|) f \left( \frac{\lambda}{z} \right)
\]
where the multiplication is done in normal coordinates at \( f(0) \). Let \( Gl = Fr \times (0, \varepsilon) \) denote the set of gluing data.

**Weighted Norms.** On the domain of \( f_\lambda \) we will use the rescaled metric \( g_\lambda = \theta_\lambda^{-2} dz d\bar{z} \), where
\[
\theta_\lambda(z) = (1 - \beta_\lambda(z) ) (\lambda + \lambda^{-1} |z|^2) + \beta_\lambda(z)
\]
Define
\[
\|\xi\|_{1,p,\lambda} = \left( \int |\xi|^p \theta_\lambda^{-2} + |\nabla \xi|^p \theta_\lambda^{-2} \right)^{1/p} \quad \text{for } \xi \text{ vector field along } f_\lambda \text{ and}
\]
\[
\|\eta\|_{p,\lambda} = \left( \int |\eta|^p \theta_\lambda^{-2} \right)^{1/p} \quad \text{for } \eta \text{ 1-form along } f_\lambda
\]
The weighted norm of a vector field or 1-form on \( f_\lambda \) equals its usual norm off \( B(0, 2\sqrt{\lambda}) \) and on \( B(0, \sqrt{\lambda}) \) it is equal with the norm of its pulled back on \( S^2 \) via a rescaling of factor \( \lambda \). The usual Sobolev embeddings hold for this weighted norms with constants independent of \( \lambda \).

**Lemma 1.1** There exists \( \varepsilon_0 > 0 \) and constants \( C > 0 \) such that for any \( p \geq 1 \) and \( \lambda \leq \varepsilon_0 \):
\[
|df_\lambda|_{p,\lambda} \leq C \quad \text{and} \quad |\overline{\partial}_J f_\lambda|_{p,\lambda} \leq C \lambda^{1/p}
\]
Moreover on the annulus \( A: \{ \sqrt{\lambda} \leq |z| \leq 2\sqrt{\lambda} \} \) we have the following expansion:
\[
\overline{\partial}_J f_\lambda = \frac{\sqrt{\lambda}}{|z|} d\beta \cdot df(y)(u) + O(\lambda)
\]
The estimates are uniform on \( Gl \to N \).
Proof. Let \( B \) be the disk \( |z| \leq \sqrt{\lambda} \). Note that \( df_\lambda \) vanishes for \( |z| \geq 2\sqrt{\lambda} \) and by the definition of the weighted norm on \( B \),

\[
\|df_\lambda\|_{p,\lambda,B} = \|df\|_{p,B}
\]

But (1.12) implies that

\[
\max\{ |df(z)|, \, |z| \leq 1 \} \leq 2 \quad (1.17)
\]

In the same time, \( \partial J f_\lambda = 0 \) outside \( A \). Hence we need only to consider what happens in \( A \). But on \( A \)

\[
|\partial J f_\lambda| \leq C|df_\lambda| \leq C(\|d\beta_\lambda\|_B |f| + |\beta_\lambda||df|) \frac{\lambda}{|z|^2} \leq C \frac{1}{\sqrt{\lambda}} \sup_B |f| + C \leq C
\]

since \( \sup_B |f(z)| \leq \sqrt{\lambda} \sup_B |df| \leq 2\sqrt{\lambda} \) in normal coordinates on \( \mathbb{P}^n \) at \( f(y) \). This concludes the first part of the proof. For the second part, notice that on \( A \)

\[
\partial J f_\lambda = \partial J f_\lambda + \lambda \beta_\lambda \cdot \partial J f = \frac{1}{\sqrt{\lambda}} d\beta \frac{z}{|z|} \cdot f \left( \frac{\lambda}{z} \right)
\]

since \( f \) is holomorphic. But using (1.12) in normal coordinates on \( \mathbb{P}^n \) at \( f(y) \) and \( y = 0 \), we get

\[
|f(z) - f(0) - df(0)(z)| \leq 2|z|^2
\]

so

\[
f \left( \frac{\lambda}{z} \right) = \frac{\lambda}{z} \cdot df_y(u) + O(\lambda) \quad \text{on} \quad A
\]

Substituting this in the formula for \( \partial J f_\lambda \) we obtain (1.16). \( \square \)

Extending the approximate gluing map. The approximate gluing map extends naturally to the bubble tree compactification \( \overline{U}_d \) of the moduli space of 1-marked curves. For simplicity, let \( \mathcal{N} \) denote some boundary stratum modeled on a bubble tree \( B \) and corresponding to a certain distribution of the degree \( d = d_1 + \ldots + d_m \) on the bubbles. If \([f_i, y_i], i = 1, \ldots, m\) are the bubble curves corresponding to the bubble map \( f : B \to \mathbb{P}^n \), then the gluing data \( Gl \) is a collection of unit vectors tangent to each sphere in the domain at the corresponding bubble point together with gluing parameters:

\[
Gl = \{ ( [f_i, y_i, u_i], \lambda_i )_{i=1}^m \mid u_i \in T_{y_i} S^2, \, |u_i| \neq 0, \lambda_i \leq \varepsilon \} \quad (1.18)
\]

Note that as long as \( f_i \) is not a constant map, then we can choose a unique representative of \([f_i, y_i, u_i] \) as in (1.11), (1.13). Then Lemma 1.1 extends naturally to \( \mathcal{N} \) to give

**Lemma 1.2** With the notations above, let \( f_\lambda \) be an approximate gluing map, and \( A_1, \ldots, A_m \) be the corresponding annuli of radii \( \lambda_i \) in which the cutoff functions are supported. Then for \( \varepsilon \) small enough, there exists a constant \( C \) such that:

\[
\|df_\lambda\|_{p,\lambda} \leq C, \quad \|\partial J f_\lambda\|_{p,\lambda} \leq C \lambda^{1/p}
\]

Moreover, \( \partial J f_\lambda = 0 \) except on the annuli \( A_i \) that correspond to nontrivial bubbles, where

\[
\partial J f_\lambda = -\frac{\sqrt{\lambda_i}}{|z|} d\beta \cdot f_i(y_i)(u_i) + O(\lambda_i) \quad (1.19)
\]

The estimates above are uniform on \( Gl \to \mathcal{N} \).

We will see later that most of the important information is encoded in the first level of nontrivial bubbles.
1.3 The Obstruction Bundle

In order to see which of the approximate maps can be corrected to solutions of the equation \( \overline{\partial}_J f = \nu \) we need first to understand the behaviour of the linearization of this equation over the space of approximate solutions.

Recall that transversality fails at a bubble tree with ghost base, so the linearization at such bubble tree is not onto. The cause of that is the ghost base. Thus we start by analysing the ghost maps:

Consider the moduli space of holomorphic maps \( f : T^2 \to \mathbb{P}^n \) representing \( 0 \in H_2(\mathbb{P}^n) \). Obviously, the only such maps are the constant ones (ghosts). If \( D_u \) is the linearization of the section \( \overline{\partial}_J : Maps(T^2, \mathbb{P}^n) \to \Lambda^{0,1} \) at \( f : T^2 \to \mathbb{P}^n, f(x) = u \) a constant map, then

\[
\text{index } D_u = \dim \text{Ker}D_u - \dim \text{Coker}D_u = c_1(0) + n(1 - 1) = 0
\]

and

\[
\text{Coker}D_u = H^1(T^2, f^*T\mathbb{P}^n) \cong T_u\mathbb{P}^n \quad (\text{canonically})
\]

since \( f^*(T\mathbb{P}^n) \) is a trivial bundle, so the elements \( \omega \in H^1(T^2, f^*T\mathbb{P}^n) \) are constant on the torus, i.e. have the form \( \omega = Xdz \) for some \( X \in T_u\mathbb{P}^n \).

Now if \( f : B \to \mathbb{P}^n \) is a bubble tree map whose base is a ghost torus \( u = f(y) \in \mathbb{P}^n \), let \( D_f \) be the linearization at \( f \) of the section \( \overline{\partial}_J : Maps(B, \mathbb{P}^n) \to \Lambda^{0,1} \). Then

\[
\text{index } D_f = \dim \text{Ker}D_f - \dim \text{Coker}D_f = -1
\]

To describe \( \text{Coker}D_f \) we will use the following:

**Definition 1.3** If \( f : B \to \mathbb{P}^n \) is as above, let

\( B_1 \subset B \) consist of the domains of all the ghost bubbles with image \( f(y) \),

\( B_2 = B - B_1 \) and

\( \tilde{B} \subset B \) denote the first level of bubbles that are not in \( B_1 \).

Then \( \text{Coker}D_f \) is \( n \) dimensional, consisting of \( 1 \)-forms \( \omega \) such that

\[
\omega = \begin{cases} 
Xdz & \text{on } B_1 \\
0 & \text{on } B_2
\end{cases}
\]

for some \( X \in T_u\mathbb{P}^n \). In particular, there is a natural isomorphism

\[
\text{Coker}D \cong \text{ev}^*(T\mathbb{P}^n) \quad (1.20)
\]

where \( \text{ev} : \overline{U_d} \to \mathbb{P}^n \) is the evaluation map. Since the moduli space of bubble trees \( \overline{U_d} \) is compact, there exists a constant \( E > 0 \) such that \( D_f D_f^* \) has a zero eigenvalue with multiplicity \( n \), and all the other eigenvalues are greater than \( 2E \) for all \( f \in \overline{U_d} \).

When \( f_\lambda \) is an approximate map, let \( D_\lambda \) be the linearization of

\[
\overline{\partial}_J : Maps(T^2, \mathbb{P}^n) \to \Lambda^{0,1}
\]

at \( f_\lambda \) and \( D_\lambda^* \) its \( L^2 \)-adjoint with respect to the metric \( g_\lambda \) on \( T^2 \). Then \( D_\lambda \) is not uniformly invertible. More precisely,
Lemma 1.4 For $\lambda > 0$ small, the operator $\Delta_\lambda = D_\lambda D_\lambda^*$ has exactly $n$ eigenvalues of order $\sqrt{\lambda}$ and all the others are greater than $E$. Moreover, over the set of gluing data $Gl$, the span of low eigenvalues

$$\Lambda_{\text{low}}^{0,1}(f_\lambda^* TP^n) \hookrightarrow \Lambda_{\text{low}}^{0,1} \downarrow Gl$$

is a $n$-dimensional vector bundle (called the Taubes obstruction bundle), naturally isomorphic to the bundle

$$\text{ev}^*(TP^n) \rightarrow Gl$$

where $\text{ev}: Gl \rightarrow P^n$ is the evaluation map.

Proof. The proof is more or less the same as the one Taubes used for the similar result in the context of Donaldson theory, [T2]. For each gluing data in $Gl$, by cutting and pasting eigenvectors we show that the eigenvalues of $\Delta_\lambda = D_\lambda D_\lambda^*$ are $O(\sqrt{\lambda})$ close to those of $\Delta_u = D_u D_u^*$, where $u$ is the point map in the base of the bubble tree.

Take for example the top stratum of $\mathcal{U}_d$. Choose $\{\omega_i, i = 1, n\}$ a local orthonormal base of $\text{Coker}D \cong \text{ev}^*(TP^n)$ and define

$$\omega_i^j(z) = \beta \left( \frac{z}{2\sqrt{\lambda}} \right) \omega^j(z)$$

(1.21)

A straightforward computation shows that:

$$\|D_\lambda^* \omega_\lambda\|_{2,\lambda} \leq \lambda^{1/4} \|\omega_\lambda\|_{2,\lambda}$$

(1.22)

$$\langle \overline{\omega}_\lambda, \overline{\omega}_\lambda^j \rangle_{2,\lambda} = \delta_{ij} + O(\lambda)$$

(1.23)

The Gramm-Schmidt orthonormalization procedure then provides $n$ eigenvectors $\overline{\omega}_\lambda$ for $\Delta_\lambda$ with eigenvalues $O(\sqrt{\lambda})$ such that

$$\overline{\omega}_\lambda = \omega^i_\lambda + O(\lambda)$$

The construction above extends naturally to the other substrata of $\mathcal{U}_d$. Note that for example when $B_1$ has other components besides $T^2$ then $\overline{\omega}_\lambda$ is equal to $\omega$ not only on the ghost base, but on all $B_1$ and is extended with 0 starting from the first level of nontrivial bubbles.

An adaptation of Taubes argument from [T1] shows that there are at most $n$ low eigenvalues of $\Delta_\lambda$. Therefore there is a well defined splitting

$$\Lambda^{0,1}(f_\lambda^* TP^n) = \Lambda_{\text{low}}^{0,1}(f_\lambda^* TP^n) \oplus \Lambda_{E}^{0,1}(f_\lambda^* TP^n)$$

The definition (1.21) combined with (1.20) provides the isomorphism $\Lambda_{\text{low}}^{0,1} \cong \text{ev}^*(TP^n)$, concluding the proof. $\Box$

The partial right inverse of $D_\lambda$. The restriction of $D_\lambda D_\lambda^*$ to $\Lambda_{E}^{0,1}$ is invertible (since all its eigenvalues are at least $E$). Define $P_\lambda$ to be the composition of the $L^2$-orthogonal projection $\Lambda^{0,1} \rightarrow \Lambda_{E}^{0,1}$ with the operator $D_\lambda^*(D_\lambda D_\lambda^*)^{-1}$ on $\Lambda_{E}^{0,1}$. Then

$$P_\lambda: \Lambda^{0,1}(f_\lambda^* TP^n) \rightarrow \Lambda^{0}(f_\lambda^* TP^n)$$

(1.24)
is the partial right inverse of $D_\lambda$ and satisfies the uniform estimate:

$$\|P_\lambda \eta\|_{1,p,\lambda} \leq E^{-1} \|\eta\|_{p,\lambda}$$

We will denote by $\pi_{f_\lambda} : \Lambda^{0,1}(f_\lambda^*TP^n) \to \Lambda^{0,1}_{\text{low}}(f_\lambda^*TP^n)$ the projection onto the fiber of the obstruction bundle.

### 1.4 The Gluing map

The next step is to correct the approximate gluing map to take values in the moduli space $\mathcal{M}_{t\nu}$ of solutions to the equation

$$\overline{\partial} f(x) = t \cdot \nu(x, f(x))$$

where $\nu$ is generic and fixed and $t$ is a small parameter.

If $f_\lambda$ is an approximate map, use the exponential map to write any nearby map in the form $f = \exp_{f_\lambda}(\xi)$, for some correction $\xi \in \Lambda^0(f_\lambda^*TP^n)$. Let $D_\lambda$ be the linearization of the $\overline{\partial}$-section at $f_\lambda$ so

$$\overline{\partial} f = \overline{\partial} f_\lambda + D_\lambda(\xi) + Q_\lambda(\xi)$$

where $Q_\lambda$ is quadratic in $\xi$. Similarly,

$$\nu(x, f(x)) = \nu(x, f_\lambda(x)) + d\nu(\xi) + \tilde{Q}_\lambda(\xi)$$

so equation (1.26) can be rewritten as:

$$D_\lambda(\xi) + N_\lambda(\xi, t) = t\nu(x, f_\lambda(x)) - \overline{\partial} f_\lambda$$

where $N_\lambda(\xi, t) = Q_\lambda(\xi) - td\nu(\xi) - t\tilde{Q}_\lambda(\xi)$ is quadratic in $(\xi, t)$.

The kernel of $D_\lambda$ models the tangent directions to the space of approximate maps, so it is natural to look for a correction in the normal direction. More precisely, we will consider the solutions of (1.28) of the form

$$f = \exp_{f_\lambda}(P_\lambda \eta)$$

where $\pi_-(\eta) = 0$ (1.29)

Since $D_\lambda(P_\lambda(\eta)) = \eta$ for such $\eta$, then equation (1.28) becomes

$$\eta + N_{\lambda, t}(P_\lambda \eta) = t\nu - \overline{\partial} f_\lambda$$

The existence of a solution of (1.30) is a standard application of the Banach fixed point theorem combined with the estimates in the previous sections.

**Lemma 1.5** There exists a constant $\delta > 0$ (independent of $\lambda, t$) such that for $t$ small enough and for any $\alpha \in \Lambda^{0,1}(f_\lambda^*TP^n)$ so that $\|\alpha\|_{p,\lambda} < \delta/2$ the equation:

$$\eta + N_{\lambda, t}(P_\lambda \eta) = \alpha$$

has a unique small solution $\eta \in \Lambda^{0,1}(f_\lambda^*TP^n)$ with $\|\eta\|_{p,\lambda} < \delta$. Moreover,

$$\|\eta\|_{p,\lambda} < 2\|\alpha\|_{p,\lambda}$$

and if $\alpha$ is $C^\infty$, so in $\eta$. 

12
Proof. Apply the contraction principle to the operator

\[ T_\lambda : \Lambda^{0,1}(f_\lambda^*TP^n) \to \Lambda^{0,1}(f_\lambda^*TP^n) \]

\[ T_\lambda \eta = \alpha - N_{\lambda,t}(P_\lambda \eta) \]
defined on a small ball centered at 0 in the Banach space \( \Lambda^{0,1}(f_\lambda^*TP^n) \) with the weighted Sobolev norm \( L_{\lambda,t}^2 \). To prove that \( T \) is a contraction we note that:

\[ \|T_\lambda \eta_1 - T_\lambda \eta_2\|_{p,\lambda} = \|N_{\lambda,t}(P_\lambda \eta_1) - N_{\lambda,t}(P_\lambda \eta_2)\|_{p,\lambda} \]

and use some estimates of Floer. He proved in [F] that for the quadratic part \( Q \) of (1.27), there exists a constant \( C \) depending only on \( \|df\|_{p,\lambda} \) such that:

\[ \|Q_f(\xi_1) - Q_f(\xi_2)\|_{p,\lambda} \leq C (\|\xi_1\|_{1,p,\lambda} + \|\xi_2\|_{1,p,\lambda}) \|\xi_1 - \xi_2\|_{1,p,\lambda} \]

(1.31)

\[ \|Q_f(\xi)\|_{p,\lambda} \leq C \|\xi\|_{\infty,\lambda} \cdot \|\xi\|_{1,p,\lambda} \]

(1.32)

(Floer’s estimates are for the usual Sobolev norm, but the same proof goes through for the weighted norms.) Since \( \|df_\lambda\|_{p,\lambda} \) is uniformly bounded by Lemma 1.2, the same constant \( C \) works for all \( f_\lambda \in \text{Im}(\gamma_\epsilon) \). Moreover, for \( t \) very small the same estimates hold for the nonlinear part \( N_{\lambda,t} \).

Hence by (1.31):

\[ \|T_\lambda \eta_1 - T_\lambda \eta_2\|_{p,\lambda} \leq C (\|P_\lambda \eta_1\|_{1,p,\lambda} + \|P_\lambda \eta_2\|_{1,p,\lambda}) \|P_\lambda (\eta_1 - \eta_2)\|_{1,p,\lambda} \]

\[ \leq C/E^2 (\|\eta_1\|_{p,\lambda} + \|\eta_2\|_{p,\lambda}) \cdot \|\eta_1 - \eta_2\|_{p,\lambda}. \]

Choosing \( \delta < E^2/(4C) \) this implies

\[ \|T_\lambda \eta_1 - T_\lambda \eta_2\|_{p,\lambda} \leq 1/2 \|\eta_1 - \eta_2\|_{p,\lambda} \]

for any \( \eta_1, \eta_2 \in B(0,\delta) \). Moreover, since \( \|T_\lambda(0)\|_{p,\lambda} \leq \delta/2 \) then \( T_\lambda : B(0,\delta) \to B(0,\delta) \) is a contraction. Therefore \( T_\lambda \) has a unique fixed point \( \eta \) in the ball such that moreover

\[ \|\eta\|_{p,\lambda} \leq \|T_\lambda \eta - T_\lambda(0)\|_{p,\lambda} + \|T_\lambda(0)\|_{p,\lambda} \leq 1/2 \|\eta\|_{p,\lambda} + \|T_\lambda(0)\|_{p,\lambda} \]

so \( \|\eta\|_{p,\lambda} \leq 2 \|T_\lambda(0)\|_{p,\lambda} = 2\|\alpha\|_{p,\lambda} \). Elliptic regularity implies that \( \eta \) is smooth when \( \alpha \) is. \( \square \)

Corollary 1.6 For \( t, \lambda \) small enough, equation (1.30) has a unique small solution \( \|\eta\|_{p,\lambda} \leq \delta \). Moreover,

\[ \|\eta\|_{p,\lambda} \leq C(t|\nu| + \lambda^{\frac{1}{5}}). \]

Proof. Follows immediately from Lemmas 1.2 and 1.3 and the estimate

\[ \|\alpha\|_{p,\lambda} = \|t\nu - \tilde{\partial}_f f_\lambda\|_{p,\lambda} \leq t|\nu| + C\lambda^{\frac{1}{5}}. \]

The gluing map. Let \( Gl \) be the set of gluing data. The gluing map is defined by

\[ \tilde{\gamma}_\epsilon : Gl \to \text{Maps}(T^2, X) \]
\[ \tilde{\gamma}_\varepsilon([f, y, u], \lambda) = \tilde{f}_\lambda = \exp_{f_\lambda}(P_\lambda \eta) \]

where \( \eta = \eta(f, y, u, \lambda) \) is the unique solution to the equation \((1.30)\) given by Corollary \(1.6\).

By construction, \( \tilde{\gamma}_\varepsilon \) is a local diffeomorphism onto its image. Moreover, if \( \pi_{f_\lambda}^{-1}(\eta) = 0 \) then \( \tilde{f}_\lambda \) is actually a solution of \((1.26)\).

**The obstruction to gluing.** The section
\[ \psi_t : GL \to \Lambda_{\text{low}}^{0,1}(f_\lambda^* T \mathbb{P}^n) \]

given by
\[ \psi_t(f, y, u, \lambda) = \pi_{f_\lambda}^{\perp}(\eta) = \pi_{f_\lambda}^{\perp}(t \nu - \overline{\partial}_J f_\lambda) - \pi_{f_\lambda}^{\perp}(N_{\lambda,t}(P_\lambda \eta)) \]

will be called the *obstruction to gluing*. Let \( Z_t = \psi_t^{-1}(0) \) be the zero set of this section. By applying the gluing construction to bubble trees in \( Z_t \) we get a subset of the moduli space \( \mathcal{M}^{t\nu} \).

### 1.5 Completion of the construction

We have seen in the previous section that applying the gluing construction to the bubble trees in the zero set \( Z_t \) we will get elements of the moduli space \( \mathcal{M}_{d,1,t\nu} \). It is not clear yet why all the elements of this moduli space close enough to the boundary stratum \( \mathcal{N} \) can be obtained by the gluing procedure. The purpose of this section is to clarify this issue.

Recall the construction of the gluing map: Starting with a bubble tree we glue in the bubble to obtain an approximate map \( f_\lambda \). Then we correct \( f_\lambda \) by pushing it in a direction normal to the kernel of \( D_\lambda \) in order to get an element of the moduli space \( \mathcal{M}^{t\nu} \). The key fact here is that the kernel of the linearization \( D_\lambda \) models the tangent space to the approximate maps, and therefore, at least in the linear model, it is enough to look for solutions only in a normal direction. For the construction to be complete though, we need to show that the same thing is true for the nonlinear problem.

More precisely, we will show that for \( t \) small, all the elements of the moduli space \( \mathcal{M}_{d,1,t\nu} \) close to the boundary stratum \( \mathcal{N} \) can be reached starting with an approximate map and going out in a normal direction. The proof of the following Theorem is an adaptation of the proof for the same kind of result in the context of Donaldson theory \([DK]\). It is pretty technical and we include it just for continuity of the presentation.

**Theorem 1.7** The end of the moduli space \( \mathcal{M}_{d,1,t\nu} \) close to the boundary strata \( \mathcal{N} \) is diffeomorphic to the zero set of the section \( \psi_t \). More precisely, for \( \delta \) and \( t \) small enough, there exists an isomorphism
\[ \mathcal{M}_{d,1,t\nu} \cap U_\delta \cong \psi_t^{-1}(0) \]

where
\[ U_\delta = \{ f : T^2 \to X \mid \exists f_\lambda \text{ s.t. } f = \exp_{f_\lambda}(\xi), \| \xi \|_{2,\lambda} \leq \delta, \| \overline{\partial}_J f \|_{2,\lambda} \leq \delta^{3/2} \} \]

and \( f_\lambda \in \text{Im} \gamma_\varepsilon \) is some approximate map.

**Proof.** The proof consists of 2 steps. First, Lemma \(1.8\) shows that \( U_\delta \) is actually a neighborhood of \( \mathcal{N} \) in the bubble tree convergence topology. Second, recall that in constructing the section \( \psi_t \) we were looking for solutions of the equation \((1.26)\) that have the form
\[ f = \exp_{f_\lambda}(P_\lambda \eta) \]

for some \( \| \eta \|_{2,\lambda} \leq \delta \) \((1.33)\).

To prove the Theorem it is enough to show that for \( t \) small, all the solutions of the equation \((1.26)\) can be written in the form \((1.33)\). This is a consequence of Proposition \(1.9\).
Lemma 1.8 \( U_\delta \cap \overline{\mathcal{M}}^\nu \) is a neighborhood of \( \mathcal{N} \) in the bubble tree convergence topology. More precisely, for any \((J,t_\nu)\)-holomorphic map \( f \) close to the boundary strata \( \mathcal{N} \) there exists an approximate map \( f_\lambda \) such that \( f \) can be written in the form

\[
f = \exp_{f_\lambda}(\xi) \text{ for some } \|\xi\|_{1,2,\lambda} \leq \delta
\]

**Proof.** By contradiction, assume there exists a sequence \( f_n \) of \((J,t_\nu)\)-holomorphic maps for \( t_n \to 0 \) such that \( f_n \) do not have the required property. By the bubble tree convergence Theorem ([PW]) there exists a bubble tree \( f \) such that \( f_n \to f \) uniform on compacts. Moreover, after rescaling the functions \( f_n \) by some \( \lambda_n \), this becomes a \( L^{1,2} \)-convergence (cf. [PW]). But this is equivalent to saying that \( f_n \) is \( L^{1,2,\lambda_n} \) close to \( f \). In particular, for \( \lambda \) small enough, \( f_n \) is \( L^{1,2,\lambda_n} \) close to \( f_\lambda \), which gives a contradiction. \( \square \)

**Proposition 1.9** For small enough \( \delta, t \) any map in \( U_\delta \) can be represented in the form

\[
f = \exp_{f_\lambda}(P_\lambda \eta) \text{ for some } f_\lambda \in \text{Im} \gamma, \|\eta\|_{2,\lambda} < \delta \text{ and } \pi^{f_\lambda}(\eta) = 0
\]

**Proof.** We will use the continuation method. The key fact is that a neighborhood of \( f_\lambda \) in \( \text{Im} \gamma \) is modeled by \( \Lambda_{0,\text{low}}^0 \) and that the image of \( P_\lambda \) spans the normal directions to \( \text{Im} \gamma \).

Let \( f \in U_\delta \). By definition, there is \( f_\lambda \in \text{Im} \gamma \) such that \( f = \exp_{f_\lambda} \xi \), where \( \|\xi\|_{1,2,\lambda} < \delta \). Consider the path \( f_s = \exp_{f_\lambda}(s\xi) \). Let

\[S = \{ s \in [0,1] \mid \exists f_\lambda \text{ and } \|\eta_s\|_{p,\lambda_s} < \delta \text{ such that } f_s = \exp_{f_\lambda}(P_\lambda \eta_s) \}.\] (1.34)

Note that by definition \( f = f_\lambda = \exp_{f_\lambda}(0) \) so \( 0 \in S \). We will show that \( S \) is both open and closed and since it is nonempty, \( 1 \in S \).

**S is closed.** The only open condition in the definition of \( S \) is \( \|\eta_s\|_{p,\lambda_s} < \delta \). But since

\[
\begin{align*}
\overline{\partial}_J f_s &= \overline{\partial}_J f_{\lambda_s} + D_{\lambda_s}(P_\lambda \eta_s) + N_{\lambda}(P_\lambda \eta_s) \quad \text{then} \\
\eta_s &= \overline{\partial}_J f_s - \overline{\partial}_J f_{\lambda_s} - N_{\lambda_s}(P_\lambda \eta_s) \quad \text{so} \\
\|\eta_s\|_{2,\lambda} &\leq \|\overline{\partial}_J f_s\|_{2,\lambda} + \|\overline{\partial}_J f_{\lambda_s}\|_{2,\lambda} + \frac{C}{\mathcal{E}^2} \|\eta_s\|^2_{2,\lambda} \\
&\leq \|\overline{\partial}_J f_s\|_{2,\lambda} + C\sqrt{\lambda} + C \|\eta_s\|^2_{2,\lambda}
\end{align*}
\] (1.35)

We need to estimate \( \|\overline{\partial}_J f_s\|_{2,\lambda} \). But

\[
\begin{align*}
\overline{\partial}_J f_s &= \overline{\partial}_J f_\lambda + sD\lambda(\xi) + N_\lambda(s\xi) \quad \text{and} \\
\overline{\partial}_J f_1 &= \overline{\partial}_J f_\lambda + D\lambda(\xi) + N_\lambda(\xi) \quad \text{so} \\
\overline{\partial}_J f_s &= s\overline{\partial}_J f_1 + (1-s)\overline{\partial}_J f_\lambda + N_\lambda(s\xi) - sN_\lambda(\xi)
\end{align*}
\] (1.36)

The relation (1.36) combined with the estimate (1.32) gives \( \|N_\lambda(\xi)\|_{2,\lambda} \leq C \|\xi\|^2_{1,2,\lambda} \) so

\[
\|\overline{\partial}_J f_s\|_{2,\lambda} \leq \|\overline{\partial}_J f_1\|_{2,\lambda} + \|\overline{\partial}_J f_\lambda\|_{2,\lambda} + 2C \|\xi\|^2_{1,2,\lambda} \leq \sqrt{\lambda} + \delta^{3/2} + C \delta^2
\]

Therefore for \( \lambda \ll \delta \),

\[
\|\overline{\partial}_J f_s\|_{2,\lambda} \leq 2C \delta^{3/2}
\] (1.37)
Using (1.37) in (1.35) we get

\[ \|\eta_s\|_{2,\lambda} \leq 2 C \delta^{3/2} + C \sqrt{\lambda} + C \|\eta_s\|^2_{2,\lambda} \]

For small \( \lambda \leq \delta^3 \), the constraint \( \|\eta_s\|_{2,\lambda} < \delta \) implies \( \|\eta_s\|_{2,\lambda} < \delta/2 \) so it is a closed condition too. \( \square \)

**S** is open. Assume that \( s_0 \in S \), i.e. there exists an approximate map \( f_{s_0} \) such that \( f_{s_0} = \exp_{f_{s_0}}(P_{s_0}(\eta_0)) \). We will show that \( s \in S \) for \( s \) sufficiently close to \( s_0 \). For that we need to find an approximate map \( f_{\lambda_s} \) and an \( \eta_s \in \Lambda_E^0 \) such that:

\[ f_s = \exp_{f_{\lambda_s}}(s\xi) = \exp_{f_{\lambda_s}}(P_{\lambda_s} \eta_s) \]

(1.38)

It is enough to prove that the linearization of the equation (1.38) is onto at \( s_0 \). First we prove that:

**Lemma 1.10** A small neighborhood \( \mathcal{N}_\delta \) of \( f_{\lambda} \) in \( \text{Im} \gamma_\xi \) is modelled by \( \Lambda_{\text{low}}^0 \). More precisely, there is a well defined map \( g : \Lambda_{\text{low}}^0 \rightarrow \Lambda_{\text{low}}^0,1 \) such that any approximate map \( f \) in \( \text{Im} \gamma_\xi \) has the form \( f = \exp_{f_{\lambda}}(\zeta + P_\lambda g(\zeta)) \) for some \( \zeta \in \Lambda_{\text{low}}^0 \), \( \|\zeta\|_{1,2,\lambda} \leq \delta \).

**Proof.** The first statement is an immediate consequence of the way we constructed the approximate maps. For the second part, notice that any \( f \) in \( \text{Im} \gamma_\xi \) close to \( f_{\lambda} \) can be written in the form \( f = \exp_{f_{\lambda}}(\chi) \), with \( \chi \) small. Let \( \chi = \zeta + P_\lambda \eta \) be the orthogonal decomposition of \( \chi \) in \( \Lambda_{\text{low}}^0 \oplus \Lambda_E^0 \), where \( \eta \in \Lambda_E^0 \) (recall that \( P_\lambda : \Lambda_E^0 \rightarrow \Lambda_E^0 \) is an isomorphism). Using the same techniques as in Section 1.4 we can prove that for any \( \zeta \in \Lambda_{\text{low}}^0 \) there exists a unique solution \( \eta = g(\zeta) \) to the equation

\[ \eta + N_\lambda(P_\lambda \eta) = \overline{\partial}_f \]

which concludes the proof of Lemma. \( \square \)

Since the notations are becoming cumbersome, we will illustrate for simplicity the case \( s_0 = 0 \). The general case follows similarly. Using Lemma 1.10 we can regard the equation (1.38) as an equation in \( (\zeta, \eta) \in \Lambda_{\text{low}}^0 \oplus \Lambda_{\text{low}}^0,1 \). More precisely, for a fixed \( s \) small, we need to find \( \zeta \in \Lambda_{\text{low}}^0 \) and \( \eta \in \Lambda_E^0 \) such that the approximate map \( f = \exp_{f_{\lambda}}(\zeta + P_\lambda g(\zeta)) \) solves the equation:

\[ \exp_f(P_f \eta) = \exp_{f_{\lambda}}(s\xi) \]

(1.39)

The linearization of the equation (1.38) at \((0, \eta)\) is \( D : \Lambda_{\text{low}}^0 \oplus \Lambda_{\text{low}}^0,1 \rightarrow \Lambda^0 \),

\[ D_{(0, \eta)}(z, n) = z + P_\lambda \nabla g(z) + P_\lambda n + \Pi(z, \eta) \]

where \( \Pi(z, \eta) \) is the derivative of \( P_\lambda \eta \) with respect to \( f_{\lambda} \).

Our goal is to show that the operator \( D_{(0, \eta)} \) is an isomorphism in some appropriate norms on \( \Lambda_{\text{low}}^0 \oplus \Lambda_{\text{low}}^0,1 \) and \( \Lambda^0 \).

**Definition 1.11** On \( \Lambda_{\text{low}}^0 \oplus \Lambda_{\text{low}}^0,1 \) and \( \Lambda^0 \) define the following norms:

\[
\begin{align*}
\|(z, n)\|_{B_1} &= \|z\|_{1,2,\lambda} + \|n + \nabla g(z)\|_{2,\lambda} \quad \text{for any} \quad (z, n) \in \Lambda_{\text{low}}^0 \oplus \Lambda_{\text{low}}^0,1 \\
\|\xi\|_{B_2} &= \|D_\lambda \xi\|_{2,\lambda} \quad \text{for any} \quad \xi \in \Lambda^0
\end{align*}
\]
Consider the operator $T : \Lambda_{\text{low}}^0 \oplus \Lambda_{E}^0 \rightarrow \Lambda^0$ given by $T(z, n) = z + P_\lambda (n + \nabla g(z))$. Then $T$ is continuous, since

$$
\|T(z, n)\|_{B_2} = \|D_\lambda z + n + \nabla g(z)\|_{2, \lambda} \leq \|D_\lambda z\|_{2, \lambda} + \|n + \nabla g(z)\|_{2, \lambda} \leq C\lambda^{1/4}\|\cdot\|_{1, 2, \lambda} + \|n + \nabla g(z)\|_{2, \lambda} \leq \|T(z, n)\|_{B_1}
$$

for $\lambda$ small enough. Recall that the low eigenvalues of $D_\lambda$ are of order $\lambda^{1/4}$, and thus $\|D_\lambda z\|_{2, \lambda} \leq \lambda^{1/4}\|n\|_{1, 2, \lambda}$ on $\Lambda_{\text{low}}^0$.

**Lemma 1.12** For $\lambda, \delta$ small enough $T$ is invertible, with the operator norm of the inverse uniformly bounded $\|T^{-1}\| \leq C_T$ (independent of $\lambda, \delta$).

**Proof.** Let $\alpha = z + P_\lambda (n + \nabla g(z))$. We need to estimate $\|z\|_{1, 2, \lambda}$ and $\|n + \nabla g(z)\|_{p, \lambda}$ in terms of $\|\alpha\|_{B_2}$. Since $D_\lambda \alpha = D_\lambda z + n + \nabla g(z)$ then

$$
\|n + \nabla g(z)\|_{2, \lambda} \leq \|\alpha\|_{B_2} + \|D_\lambda z\|_{2, \lambda} \leq \|\alpha\|_{B_2} + C\lambda^{1/4}\|\cdot\|_{1, 2, \lambda}
$$

$$
\leq \|\alpha\|_{B_2} + C\lambda^{1/4}\|\alpha - P_\lambda (n + \nabla g(z))\|_{1, 2, \lambda}
$$

$$
\leq \|\alpha\|_{B_2} + C\lambda^{1/4}\|\alpha\|_{B_2} + C\lambda^{1/4}\|n + \nabla g(z)\|_{2, \lambda}
$$

So for $\lambda$ small we get the uniform estimate $\|n + \nabla g(z)\|_{2, \lambda} \leq C_1\|\alpha\|_{B_2}$. This gives

$$
\|z\|_{1, 2, \lambda} = \|\alpha - P_\lambda (n + \nabla g(z))\|_{1, 2, \lambda} \leq \|\alpha\|_{B_2} + C\|n + \nabla g(z)\|_{2, \lambda}
$$

$$
\leq C_2\|\alpha\|_{B_2}
$$

thus

$$
\|(z, n)\|_{B_1} \leq C_T\|T(z, n)\|_{B_2}
$$

So $T$ is an injective linear operator. But by construction inded($T$) = 0 thus $T$ is invertible, with $\|T^{-1}\| \leq C_T$ (independent of $\lambda, \delta$). \(\Box\)

**Lemma 1.13** For $z$ small, $\|\Pi(z, \eta)\|_{B_2} \leq C\|\eta\|_{2, \lambda}(z, 0)\|_{B_1}$.

**Proof.** By differentiating the relation $D_f P_f \eta = \eta$ with respect to $f$ at $f_\lambda$ we get

$$
\partial D_f(P_\lambda \eta)(z) + D_f(\Pi(z, \eta)) = 0 \quad \text{so}
$$

$$
\|D_\lambda(\Pi(z, \eta))\|_{2, \lambda} = \|\partial D_f(P_\lambda \eta)(z)\|_{2, \lambda}.
$$

Using the expansion of

$$
D_f \xi = \frac{1}{2}(\nabla \xi + J(f) \circ \nabla \xi \circ j) + \frac{1}{8}N_f(\partial_j f, \xi)
$$

(cf. [MS]) it is easy to check that

$$
\|\partial D_f(P_\lambda \eta)(z)\|_{2, \lambda} \leq C\|z\|_{\infty, \lambda}\|P_\lambda \eta\|_{1, 2, \lambda}
$$
uniformly in a neighborhood of $f_\lambda$. Therefore

$$
\|\Pi(z, \eta)\|_{B_2} = \|Df(\Pi(z, \eta))\|_{2, \lambda} \leq C\|z\|_{\infty, \lambda}\|P_\lambda \eta\|_{1,2, \lambda}
\leq C\|z\|_{1,2, \lambda}\|\eta\|_{2, \lambda} = C\|z\|_{B_2}\|\eta\|_{2, \lambda}.
$$

If we choose $\delta$ small enough then for $\|\eta\|_{2, \lambda} < \delta$,

$$
\|\Pi(z, \eta)\|_{B_2} \leq \frac{C}{T} \|(z, n)\|_{B_2}
$$

where $C_T$ is the constant in Lemma 1.12 so $D_{(0, \eta)}(z, n) = T(z, n) + \Pi(z, n)$ is still invertible. This concludes the proof of Proposition 1.9.

1.6 The leading order term of the obstruction $\psi_t$ for $t$ small

Next step is to identify the leading order term of the section $\psi_t$ as $t \to 0$. Let $N$ denote some stratum of $U_d$ and $Gl \to N$ denote the gluing data as in (1.18). For the sake of the gluing construction, the gluing data has to be defined on the domain of the bubble tree. But we will see in a moment that the important information is encoded in the image curves. Introduce first some notation: If $u_i \in T_{y_i}S^2$ is a unit frame and $\lambda_i$ is the gluing parameter, let $v_i = \lambda_i \cdot u_i \in T_{y_i}S^2$, $(v_i \neq 0)$ denote the gluing data.

**Definition 1.14** For any $[f, y, v] \in Gl$, such that $f : B \to \mathbb{P}^n$ is an element of $N$, let $([f_i, y_i, v_i])_{i=1}^n$ be the bubble maps together with the gluing data and let $u$ be the image of the ghost base (so $u = f_j(y_j)$ for all $j \in \tilde{B}$). Set

$$
a([f, y, v]) = \sum_{i=1}^n \sum_{j \in \tilde{B}} \langle df_j(y_j)(v_j), X_i \rangle \omega_i
$$

$$
\bar{\nu}(x) = \sum_{i=1}^n \int_{T^2} \langle \nu(z, u), \omega_i(z) \rangle \omega_i
$$

where $\{\omega_i = X_i dz, i = 1, n\}$ is an orthonormal base of $H^1(T^2, u^*TP^n)$, $X_i \in T_u\mathbb{P}^n$ and $\tilde{B}$ is as in Definition 1.3.

Note that $a$ depends only on the gluing data on the first level $\tilde{B}$ of essential bubbles, and $\bar{\nu}$ depends only on the image of the ghost base. Then

**Lemma 1.15** Using the notation above, let $f_{\lambda}$ be an approximate gluing map. Then for $t$ and $|\lambda| = \sqrt{\lambda_1^2 + \ldots + \lambda_i^2}$ small enough,

$$
\pi_{f_{\lambda}}(\nu) = \bar{\nu}(u) + O(|\lambda|)
$$

$$
\pi_{f_{\lambda}}(\partial J f_{\lambda}) = a([f, y, v]) + O(|\lambda|^{3/2}).
$$

and the section $\psi_t$ has the form

$$
\psi_t([f, y, v]) = t\bar{\nu}(u) + a([f, y, v]) + O(|\lambda|^{5/4} + t\sqrt{|\lambda| + t^2}).
$$

The estimates above are uniform on $N$. 

18
Proof. For the first 2 relations, it is enough to check them on components. Assume for simplicity that \( \tilde{B} \) consists of a single bubble \([f, y, v]\). If \( \omega = Xdz \) is an element of the base for \( H^{0,1} \), let \( \overline{\omega}_\lambda \) be the element of the local orthonormal frame for \( \Lambda^{0,1}_{\text{low}}(f^*TP^n) \) provided by Lemma 1.4. Then

\[
|\langle \nu, \omega_\lambda - \overline{\omega}_\lambda \rangle_\lambda| \leq ||\nu||_\infty ||\omega_\lambda - \overline{\omega}_\lambda||_{2,\lambda} \leq C\lambda \quad \text{so}
\]
\[
\langle \nu, \overline{\omega}_\lambda \rangle_\lambda = \langle \nu, \omega_\lambda \rangle_\lambda + O(\lambda)
\]

On the other hand, using the definition of \( \omega_\lambda \) and the fact that \( \langle , \rangle_\lambda \) is the usual inner product on \( T^2 \) off a small ball we get

\[
\langle \nu, \omega_\lambda \rangle_\lambda = \int_{|z| > \sqrt{\lambda}} \langle \nu(z, f(y)), \omega \rangle = \int_{T^2} \langle \nu(z, f(y)), \omega \rangle + O(\lambda) \quad \text{so}
\]
\[
\langle \nu, \overline{\omega}_\lambda \rangle_\lambda = \int_{T^2} \langle \nu(z, f(y)), \omega \rangle + O(\lambda)
\]

which gives (1.42). Similarly,

\[
|\langle \overline{\partial}_f \nu, \omega_\lambda - \overline{\omega}_\lambda \rangle_\lambda| \leq ||\overline{\partial}_f \nu||_{2,\lambda} ||\omega_\lambda - \overline{\omega}_\lambda||_{2,\lambda} \leq C\lambda^{1/2}\lambda \leq C\lambda^{3/2}
\]

and using the estimate (1.14) and the definition of \( \omega_\lambda \) we get

\[
\langle \overline{\partial}_f \nu, \omega_\lambda \rangle_\lambda = \int_{\sqrt{\lambda} \leq |z| \leq 2\sqrt{\lambda}} \frac{\sqrt{\lambda}}{|z|} d\beta \langle df(y)(u), \lambda \rangle + O(\lambda^2)
\]

\[
= \lambda \langle df(y)(u), \lambda \rangle + O(\lambda^2)
\]

Combine the previous 2 relations we get

\[
\langle \overline{\partial}_f \nu, \overline{\omega}_\lambda \rangle_\lambda = \langle df(y)(\lambda u), \lambda \rangle + O(\lambda^{3/2})\langle df(y)(v), \lambda \rangle + O(\lambda^{3/2})
\]

which implies (1.43).

The general case when \( B \) has more bubbles follows in a similar manner using the relation (1.19) and the fact that \( \omega = 0 \) past the first level of nontrivial bubbles.

Finally, the relation (1.44) is a consequence of (1.12) provided we have an estimate of the quadratic part. For that use (1.32) to get

\[
\langle N_\lambda(P_\lambda \eta), \overline{\omega}_\lambda \rangle_\lambda \leq \|N_\lambda(P_\lambda \eta)\|_{4/3,\lambda} \|\overline{\omega}_\lambda\|_{4,\lambda} \leq C\|\eta\|_{2,\lambda} \|\eta\|_{4/3,\lambda} \|\eta\|_4
\]

\[
\leq O(|\lambda|^{1/2} + t) \cdot O(|\lambda|^{3/4} + t^2).
\]

Thus the quadratic part is \( O(|\lambda|^{5/4} + t\sqrt{|\lambda|} + t^2) \). \( \square \)

The definition of \( \tilde{L} \to \tilde{U}_d \). From this point on, since we are going to look at the leading order term, it will become easier if we forget part of the gluing data. We have already observed that
the map \( a \) depends only on the gluing data on the first level \( \tilde{B} \) of essential bubbles. Moreover, if we denote by

\[
w = \sum_{j \in \tilde{B}} df_j(y_j)(v_j) \in T_u P^n
\]  

(1.45)

then the map \( a \) and the linear part \( \tilde{\psi}_t \) of \( \psi_t \) become respectively

\[
a(w) = \sum_{i=1}^{n} \langle w, X_i \rangle \omega_i \quad \text{(1.46)}
\]

\[
\tilde{\psi}_t(w) = t\bar{\nu}(u) + a(w) \quad \text{(1.47)}
\]

Introduce a space \( W \) together with a projection \( \pi : W \to U_d \) such that the fiber of \( \pi \) at a 1-marked curve (possibly with more components) is the span of the tangent planes to all the image bubbles that meet at the marked point. By definition \( w \in W \) so (1.45) defines a projection \( p : \text{Gl} \to W \). Note though that \( \pi : W \to U_d \) is not a vector bundle, and that \( W \) it is equal to the relative tangent bundle \( L \to U_d \) on the top strata of \( \overline{U}_d \).

Here is a more precise description of \( W \). Stratify \( U_d \) by letting \( Z_h \) be the union of all boundary strata such that the image of the marked point is on \( h \) nontrivial bubbles, i.e.

\[
Z_h = \{ f : B \to P^n \mid \tilde{B} \text{ has } h \text{ elements} \} \quad \text{(1.48)}
\]

Each \( Z_h \) is a variety with normal crossings. For transversality arguments we need to use the moduli space \( \hat{Z}_h \) obtained from \( Z_h \) by collapsing all the ghost bubbles up to the first level of essential bubbles. Note that \( \hat{Z}_2 \supset \hat{Z}_3 \supset \ldots \), and the natural projection

\[
q : Z_h \to \hat{Z}_h
\]

has fiber \( U_{0,h} = \mathcal{M}_{0,h+1} \), the moduli space of \( h + 1 \) marked points on the sphere. Moreover,

\[
\dim Z_h = n - h - 1 \quad \text{and} \quad \dim \hat{Z}_h = n - 2h + 1 \quad \text{(1.49)}
\]

In particular, \( Z_h \neq \emptyset \) only for \( h \leq \lceil \frac{n+1}{2} \rceil \).

Let \( L_i \) be the pullback of the relative tangent sheaf to the \( i \)'th factor of \( \hat{Z}_h \). When the constraints \( \beta_1, \ldots, \beta_k \) are in generic position, the fibers of \( L_1, \ldots, L_h \) over a point in \( \hat{Z}_h \) are linearly independent subspaces of \( P^n \). This is because linear dependence imposes \( n + 1 - h \) conditions, and \( \hat{Z}_h \) is only \( n - 2h + 1 \) dimensional. So on \( Z_h \)

\[
W|_{Z_h} = q^*(L_1 \oplus \ldots \oplus L_h) \quad \text{(1.50)}
\]

Remark 1.16 Since not all the gluing parameters can be zero, a dimension count argument similar to the one above shows that \( w \) defined by (1.45) is an element of \( W - \{0\} \), the space nonzero vectors in \( W \), thus \( p : \text{Gl} \to W - \{0\} \).

Note that \( W|_{Z_h} \) is nothing but the normal bundle of \( Z_h \) in \( \overline{U}_d \), for any \( 2 \leq h \leq \lceil \frac{n+1}{2} \rceil \). This observation allows us to get a line bundle out of \( W \) as follows:
Definition 1.17 Let $N = \left\lfloor \frac{n+1}{2} \right\rfloor$. Blow up $\overline{U}_d$ along $Z_N$ (the bottom strata), then blow up the proper transform of $Z_{N-1}$ and so on, all the way up to blowing up the proper transform of $Z_2$ and denote by

$$\rho : \tilde{U}_d \rightarrow U_d$$

the resulting manifold. Similarly, after the first blow up, extend $L$ over the exceptional divisor $E_N$ as the universal line bundle over $P(N_{U_d}Z_N)$, the projectivization of the normal bundle of $Z_N$, and so on. Let $\tilde{L} \rightarrow \tilde{U}_d$ denote the blow up of $L$ constructed above.

By definition, the total space of $\tilde{L} \rightarrow \tilde{U}_d$ is the same as $\rho^*(W)$. From now on, we will make this identification.

Both the map $a$ and the linear part $\tilde{\psi}_t$ of $\psi_t$ pull back to $\tilde{L} - \{0\}$ as

$$a(w) = \sum_{i=1}^{n} \langle w, X_i \rangle X_i$$

(1.51)

$$\tilde{\psi}_t(w) = t\tilde{\nu}(\pi(w)) + a(w)$$

(1.52)

where $\pi : \tilde{L} \rightarrow P^n$ is the composition $\tilde{L} \rightarrow \tilde{U}_d \xrightarrow{\text{ev}} P^n$. For simplicity of notation, we have also denoted by $\text{ev} : U_d \rightarrow P^n$ the composition $U_d \xrightarrow{\text{ev}} P^n$. Note that by definition, $a$ is a linear map but $\tilde{\psi}_t$ is not, and we have the following diagramm:

$$\begin{array}{ccc}
\tilde{L} - \{0\} & \xrightarrow{a, \tilde{\psi}_t} & \text{ev}^*(TP^n) \\
& \xrightarrow{\text{ev}} & \\
\tilde{U}_d & \xrightarrow{\text{ev}} & P^n
\end{array}$$

Proposition 1.18 As $t \rightarrow 0$ the zero set of the section $\psi_t$ is homotopic to the zero set of its leading order term

$$\tilde{\psi}_t : \tilde{L} - \{0\} \rightarrow \text{ev}^*(TP^n)$$

Proof. In generic conditions and for $t$ small enough the zero sets of both sections

$$\psi_t : Gl \rightarrow \text{ev}^*(TP^n)$$

and

$$\rho^*p_*(\psi_t) : \tilde{L} - \{0\} \rightarrow \text{ev}^*(TP^n)$$

consist of points lying on the top stratum of $U_d$ and $\tilde{U}_d$ respectively. But on the top stratum, the projection $pr : Gl \rightarrow \tilde{L} - \{0\}$ is an isomorphism, thus the two zero sets are diffeomorphic for $t$ small. Note that (1.44) gives

$$p_*(\psi_t(w)) = t\tilde{\nu}(u) + a(w) + O(|w|^{5/4} + t\sqrt{|w|} + t^2)$$

Finally, Lemma 1.19 gives that $w = O(t)$ on the zero set of $\psi_t$, so

$$p_*(\psi_t(w)) = t\tilde{\nu}(u) + a(w) + O(t^{5/4})$$

giving the desired homotopy as $t \rightarrow 0$.  

21
Lemma 1.19 The linear map \( a : \tilde{L} - \{0\} \to \text{ev}^*(TP^n) \) defined in \((1.51)\) has no zeros when the constraints \( \beta_1, \ldots, \beta_l \) are in a generic position, thus there exists \( C > 0 \) such that
\[
|a(w)| \geq C|w| \quad (1.53)
\]
Moreover, there exists a uniform constant \( C \) on \( \tilde{L} - \{0\} \) such that the zero set of \( \psi_t \) is contained in \( |w| \leq Ct \).

\textbf{Proof.} First part is a standard transversality argument and dimension count. Note that \( a \) induces a map
\[
a \otimes \text{id} : \tilde{L} \otimes \tilde{L}^* \to \text{ev}^*(TP^n) \otimes \tilde{L}^* \quad \text{i.e.}
\]
\[
a \otimes \text{id} : \tilde{U}_d \times \mathbb{C} \to \text{ev}^*(TP^n) \otimes \tilde{L}^*
\]
Because of the \( \mathbb{C}^* \)-equivariance of \( a \), the zero set of \( a : \tilde{L} - \{0\} \to \text{ev}^*(TP^n) \) is the same as the zero set of the section
\[
\tilde{a} : \tilde{U}_d \to \text{ev}^*(TP^n) \otimes \tilde{L}^*
\]
\[
\tilde{a}(x) = (a \otimes \text{id})(x, 1)
\]
If the constraints \( \beta_1, \ldots, \beta_k \) are in generic position, then \( \tilde{a} \) is transverse to the zero set of \( \text{ev}^*(TP^n) \). But the base \( \tilde{U}_d \) is only \( n - 1 \) dimensional, while the fiber is \( n \) dimensional, so generically \( \tilde{a} \) and thus \( a \) has no zeros.

For the second part, note that on the zero set of \( p_*(\psi_t) \)
\[
0 = p_*(\psi_t) = a(w) + t\tilde{v}(u) + O(|w|^{5/4} + |w|^{1/2} t + t^2) \quad \text{so}
\]
\[
a(w) = -t\tilde{v}(u) - O(|w|^{5/4} + |w|^{1/2} t + t^2)
\]
which combined with \((1.53)\) gives
\[
C|w| \leq |a(w)| \leq t|\tilde{v}(u)| + \tilde{C}(|w|^{5/4} + |w|^{1/2} t + t^2) \quad \text{i.e.}
\]
\[
|w|(C - \tilde{C}|w|^{1/4}) \leq Ct
\]
For \( t \) and \( w \) small, the left hand side is positive, completing the proof. \( \square \)

1.7 The enumerative invariant \( \tau_d \)

Next step is to find the zero set of the leading order term of \( \psi_t \). As a warm-up we will discuss first the limit case \( t = 0 \).

The constructions described in the previous sections apply equally in this case, giving:

\textbf{Proposition 1.20} Let \( \mathcal{N} \) be a ghost base boundary stratum of \( \tilde{U}_d \). Then the moduli space of \( J \)-holomorphic tori close to \( \mathcal{N} \) is isomorphic to the zero set of a section in the obstruction bundle over the space of gluing data
\[
\psi([f_i, y_i, v_i]_{i=1}^m) = a([f_i, y_i, v_i]_{i=1}^m) + O(|\lambda|^{5/4})
\]
where $a$ is defined by (1.40). Moreover, for generic constraints $\beta_1, \ldots, \beta_l$, the number of $J$-holomorphic tori that define the enumerative invariant

$$\tau_d(\beta_1, \ldots, \beta_l)$$

is finite, and the moduli space of these holomorphic tori is at a positive distance from the ghost base boundary strata of the bubble tree compactification.

**Proof.** For the second part, note that $\psi$ and $\lambda^{-1}\psi$ have the same zero set, so as $\lambda \to 0$ the limit of the end of the moduli space of $J$-holomorphic tori is modeled by the zero set of the section $a$. But we have seen that generically $a$ has no zeros, and thus there are no $J$-holomorphic tori in a small neighborhood of that boundary stratum.  

Now we can now evaluate the contribution from the interior:

**Proposition 1.21** For $t$ small, the number of $(J, tv)$-holomorphic maps that satisfy the constraints in the definition of $RT_{d,1}(\beta_1 | \beta_2, \ldots, \beta_l)$ and are close to some $(J,0)$-holomorphic torus is equal to

$$n_j \tau_d(\beta_1, \ldots, \beta_l)$$

where $n_j = |\text{Aut}_{x_1}(j)|$ is the order of the group of automorphisms of the complex structure $j$ that fix the point $x_1$.

**Proof.** Recall that $RT_{d,1}(\beta_1 | \beta_2, \ldots, \beta_l)$ counts the number of solutions of the equation

$$\overline{\partial}_J f(x) = \nu(x, f(x))$$

such that $f(x_1) \in \beta_1$ and $f$ passes through $\beta_2, \ldots, \beta_l$.

A generic path of perturbations converging to $0$ provides a cobordism $M^\nu$ to the solutions of the equation

$$\overline{\partial}_J f(x) = 0$$

such that $f(x_1) \in \beta_1$ and $f$ passes through $\beta_2, \ldots, \beta_l$. A $(J,0)$-holomorphic torus $f : T^2 \to \mathbb{P}^n$ is a smooth point of this cobordism, i.e. all the intersections are transversal and the cokernel $H^{0,1}(T^2, f^*(TP^n))$ vanishes (since $f^*(TP^n)$ is a positive bundle for the standard complex structure).

But the invariant $\tau_d(\beta_1, \ldots, \beta_l)$ counts the number of such solutions mod the automorphism group of $j$. Imposing the condition $f(x_1) \in \beta_1$ reduces the stabilizer to just $\text{Aut}_{x_1}(j)$.  

**Remark 1.22** Note that the pertubed invariant counts the number of $(J, \nu)$-holomorphic maps with sign. This sign is determined by the spectral flow of the linearization $D_f$ to $\overline{\partial}_J$. In the limit, when $\nu = 0$, we have $D_f = \overline{\partial}_J$ thus all $(J,0)$-tori have a positive sign. This agrees with the way they were counted classically to obtain $\tau_d$.

**Lemma 1.23** For generic $\nu$ the section $\nu : \text{ev}_*(\overline{U}_d) \to TP^n$ defined by (1.44) has no zeros.
Proof. For generic \( \nu \), the section \( \bar{\nu} \) is transverse to the zero section. But the fiber of \( TP^n \) is \( n \) dimensional, and the base \( \operatorname{Im}(M) = ev_*(\bar{U}_d) \) is only \( n - 1 \) dimensional, so \( \bar{\nu} \) has no zeros generically. \( \square \)

Remark 1.24 The zeros \( u \in P^n \) of \( \bar{\nu} \) give the location of the point maps \( u \) that can be perturbed away to get genus one \((J, \nu)\)-holomorphic maps representing \( 0 \in H_2(P^n) \). Since \( \text{index}=0 \) then generically \( \bar{\nu} \) has finitely many zeros. But \( \operatorname{Im}(M) \) is a codimension 1 subvariety in \( P^n \) that doesn’t depend on \( \nu \). Then we can choose \( \nu \) generic so that its zeros do not lie in \( \operatorname{Im}(M) \), and thus \( \bar{\nu}(f(y)) \neq 0 \) for any \([f, y] \in U_d\).

Moreover, Lemma 1.18 showed that as \( t \to 0 \) the zero set \( Z_t \) of \( \psi_t \) is homotopic to the zero set \( Z_0 \) of the map

\[
\psi_0 : \tilde{L} - \{0\} \to \operatorname{ev}^*(TP^n) \\
\psi_0(w) = \bar{\nu}(\pi(w)) + a(w)
\]

where \( a, \bar{\nu} \) are defined in \( (1.51), (1.41) \) and \( \pi : \tilde{L} \to P^n \) is the composition \( \tilde{L} \to \bar{U}_d \xrightarrow{ev} P^n \). We have made a change of variables \( w \to w/t \).

Next we identify the zero set \( Z_0 \). Since \( \bar{\nu}(u) \neq 0 \) on \( \operatorname{Im}(M) \) then it induces a splitting of the obstruction bundle:

\[
TP^n|_{\operatorname{Im}(M)} = C < \bar{\nu} > \oplus E
\]

where \( E \) is an \( n - 1 \) dimensional bundle, so

\[
\operatorname{ev}^*(TP^n) = C < \bar{\nu} > \oplus \operatorname{ev}^* E
\]

Lemma 1.25 The number of zeros (counted with multiplicity) of \( \psi_0 \) is equal to

\[
c_{n-1}(\operatorname{ev}^*(E) \otimes \tilde{L}^*)
\]

Proof. Using \( (1.55) \) map \( \psi_0 : \tilde{L} - \{0\} \to \operatorname{ev}^*(TP^n) \) splits as

\[
\psi_1(w) = \bar{\nu}(\pi(w)) + a_1(w) \quad (1.56) \\
\psi_2(w) = a_2(w) \quad (1.57)
\]

where \( a_i \) denote the projections of \( a(w) \). The map \( a_2 : \tilde{L} - \{0\} \to \operatorname{ev}^*(E) \) is \( C^* \)-equivariant, so tensored with the identity on \( \tilde{L}^* \) induces a \( C^* \)-equivariant map

\[
\bar{a}_2 : \bar{U}_d \times C^* \to \operatorname{ev}^*(E) \otimes \tilde{L}^*
\]

that has the same zero set as \( a_2 \). Let

\[
\bar{a}_2 : \bar{U}_d \to \operatorname{ev}^*(E) \otimes \tilde{L}^* \quad \text{given by} \quad \bar{a}_2(x) = \bar{a}_2(x, 1)
\]

Then the zero set of \( a_2 \) is equal to \( Z(\bar{a}_2) \times C^* \). To find the zero set of \( \psi_0 \), for any \((x, v) \in Z(\bar{a}_2) \times C^* \) solve the equation

\[
0 = \psi_1(x, v) = \bar{\nu}(x) + a_1(x, v) = \bar{\nu}(x) + v \cdot a_1(x, 1)
\]

24
Note that $a_1 \neq 0$ on $Z(a_2)$ since $a$ has no zeros, so for any $x \in Z(\tilde{a}_2)$ there exists a unique $v \in \mathbb{C}^*$ such that

$$-\bar{\nu}(x) = v \cdot a_1(x, 1)$$

This implies that there exists an isomorphism between the zero set of $\psi_0$ and the zero set of $\tilde{a}_2$.

To complete the proof, note that for generic $\nu$ the section $\tilde{a}_2$ is transversal to the zero section of $ev^*(E) \otimes \tilde{L}^*$, so its zero set is given by the Euler class of $ev^*(E) \otimes \tilde{L}^*$.  \[\square\]

Finally, we can compute the boundary contribution:

**Proposition 1.26** For $t$ small, the number of $(J, tv)$-holomorphic maps that satisfy the constraints in the definition of $RT_{d, 1}(\beta_1 | \beta_2, \ldots, \beta_l)$ and are close to the boundary strata $\{x_1\} \times \overline{U}_d$ is equal to

$$\sum_{i=0}^{n-1} \binom{n+1}{i+2} ev^*(H^{n-1-i}) \cdot c_i(\tilde{L}^*)$$

where $\tilde{L}$ is the blow up of the relative tangent sheaf $L$ as in Definition 1.17.

**Proof.** As we have seen previously, the moduli space of $(J, tv)$-holomorphic maps that satisfy the constraints in the definition of $RT_{d, 1}(\beta_1 | \beta_2, \ldots, \beta_l)$ and are close to the boundary strata $\{x_1\} \times \overline{U}_d$ is diffeomorphic to the zero set of the section $\psi_0$. Using Lemma 1.25, the later is equal to

$$c_{n-1}(ev^*(E) \otimes \tilde{L}^*) = \sum_{i=0}^{n-1} ev^*(c_{n-i-1}(E)) \cdot c_i(\tilde{L}^*)$$

But by definition $c_i(E) = c_i(TP^n) = \binom{n+1}{i} H^i$, completing the proof. \[\square\]

### 1.8 The other contribution

In the previous sections we have described in great length the gluing construction corresponding to the strata $\{x_1\} \times \overline{U}_d$, that consists of a ghost base and a bubble at the marked point $x_1$. Finally, it is the time to sketch the gluing construction corresponding to other boundary stratum $T^2 \times ev^*(\beta_1)$, and to explain why it does not give any contribution.

**Proposition 1.27** For $t$ small, the number of $(J, tv)$-holomorphic maps that satisfy the constraints in the definition of $RT_{d, 1}(\beta_1 | \beta_2, \ldots, \beta_l)$ and are close to the boundary strata $T^2 \times ev^*(\beta_1)$ is equal to 0.

**Proof.** Construct first the space of approximate maps. The only difference from the gluing construction described in Section 1.2 is that we need to allow the bubble point $x \in T^2$ to vary. Since the tangent bundle of the torus is trivial, choose an isomorphism

$$TT^2 \cong T^2 \times \mathbb{C}$$

which gives an identification $T_x T^2 \cong \mathbb{C}$ for all $x \in T^2$ (providing local coordinates on $T^2$). The set of gluing data will then be modeled on:

$$T^2 \times Fr \times (0, \varepsilon)$$

25
where
\[ Fr = \{ [f, y, u] \mid [f, y] \in \ev^*(\beta_1), \ u \in T_y S^2 \mid u = 1 \} \]
is the restriction of the frame bundle over \( U_d \) defined by (1.4).

To glue, use the unit frame \( u \in T_y S^2 \) to identify \( T_x T^2 \cong T_y S^2 \) which will induce natural coordinates on the sphere via the stereographic projection.

Then all the constructions described in Sections 1.2-1.7 extend to this case. Since the holomorphic 1-form \( \omega \in H^{0,1}(T^2, \mathbb{C}) \) is constant along the torus, then the isomorphism between the obstruction bundle and \( \ev^* (T P^n) \) is independent of the bubble point, so
\[
\begin{array}{c}
H^{0,1} \cong \rho^* \ev^*(TP^n) \\
\downarrow \\
T^2 \times \ev^*(\beta_1) \quad \rightarrow \quad \ev^*(\beta_1)
\end{array}
\]
Moreover, the linear part of the section \( \psi_t \) that models the end of the moduli space is also independent of the bubble point. But a dimension count shows that the zero set of a \( T^2 \)-equivariant section in the obstruction bundle must be empty generically. \( \square \)

2 Computations

In this second part of the paper we explain how one can compute the top power intersections \( c_1^i(L^*)\ev^*(H^{n-1-i}) \) involved in Theorem 0.1. The program is simple: first we find recursive formulas for the top intersections \( c_1^i(L^*)\ev^*(H^{n-1-i}) \) (Proposition 2.2), where \( L \) is the relative tangent sheaf of \( U_d \), an object well known to the algebraic geometers. Next we can exploit the fact that \( \tilde{L} \) is a blow up of \( L \) to compute its corresponding top intersections.

Unfortunately, the notation becomes quickly pretty complicated if we insist on keeping track of all the information, so we chose to indicate at each step only the new changes, leaving out the data that stays the same.

**Notations.** If \( \beta_0, \ldots, \beta_k \) are various codimension constraints let
\[
U_d(\beta_0 ; \beta_1, \ldots, \beta_k) = \ev^*(\beta_0) \left[ U_d(\beta_1, \ldots, \beta_k) \right]
\]
denote the moduli space of 1-marked curves in \( \mathbb{P}^n \) passing through \( \beta_0, \ldots, \beta_k \), such that the special marked point is on \( \beta_0 \) and let
\[
\mathcal{M}_d(\beta_0, \beta_1, \ldots, \beta_k)
\]
denote the corresponding moduli space of curves (in which we forget the special marked point).

In particular, let \( U_d = U_d(\beta_1, \ldots, \beta_k) \) be the moduli space of 1-marked curves that appears in Theorem 0.1. If \( i, j \geq 0 \) are such that \( i + j = \dim U_d \) then let
\[
\phi_d(i, j | \beta_1, \ldots, \beta_k) = c_1^i(L^*) \ev^*(H^j) \left[ U_d \right] \quad (2.1)
\]
denote the top intersection. Moreover, if \( \tilde{U}_d \) is the blow-up \( U_d \) as in Definition 1.17 let
\[
x = c_1(L^*) \in H^2(U_d, \mathbb{Z}), \quad \bar{x} = c_1(\tilde{L}^*) \in H^2(\tilde{U}_d, \mathbb{Z}), \quad y = \ev^*(H) \quad (2.2)
\]
where \( y \in H^2(\mathcal{U}_d, \mathbb{Z}) \) or \( y \in H^2(\bar{\mathcal{U}}_d, \mathbb{Z}) \) depending on the context. Note that
\[
\phi_d(i, j \mid \cdot) = x^i y^j \lfloor \mathcal{U}_d \rfloor = x^i \lfloor \mathcal{U}_d(H^j; \cdot) \rfloor
\] (2.3)

Using the notation above and the degeneration formula (A.2), Theorem 0.1 becomes
\[
n_j \tau_d(\cdot) = \sum_{i_1 + i_2 = n} \sigma_d(H^{i_1}, H^{i_2}, \cdot) - \sum_{i=0}^{n-1} \left( \frac{n+1}{i+2} \right) \bar{x}^i y^n - i \lfloor \bar{\mathcal{U}}_d \rfloor
\] (2.4)

**Remark 2.1** To compute a particular value for \( \tau_d \) in \( \mathbb{P}^n \) one should use a computer program based on the following four steps:

1. Find \( \sigma_d \) using the recursive formula (A.3)
2. Find \( \phi_d(i, j \mid \cdot) = c_1(L^*)\text{ev}^*(H^j)[\mathcal{U}_d] \) using the recursive formulas of Proposition 2.2
3. Find recursive formulas for \( \bar{x}^i \cdot y^j = c_1(\bar{L}^*)\text{ev}^*(H^j)[\bar{\mathcal{U}}_d] \) as outlined in Proposition 2.6
4. Finally, use (2.4) to get \( \tau_d \).

**2.1 Recursive formulas for \( c_1(L^*)\text{ev}^*(H^j) \)**

Let \( \mathcal{U}_d \) be some \( r \)-dimensional moduli space of 1-marked curves of degree \( d \) through some constraints \( \beta_1, \ldots, \beta_k \) (not necessarily the same as in Theorem 0.1) and let \( L \to \mathcal{U}_d \) be its relative tangent sheaf. In this section we give recursive formulas for top intersections
\[
\phi_d(i, j \mid \cdot) = c_1(L^*)\text{ev}^*(H^j)[\mathcal{U}_d]
\]
where \( i + j = r \) and the constraints \( \beta_1, \ldots, \beta_k \) are dropped from the notation.

**Proposition 2.2** For every \( r \)-dimensional moduli space \( \mathcal{U}_d \) of any degree \( d \geq 1 \), there are the following recursive relations for the top intersections:
\[
\phi_d(0, j \mid \cdot) = \sigma_d(H^j, \cdot)
\] (2.5)
\[
\phi_d(i + 1, j \mid \cdot) = -\frac{2}{d^2} \phi_d(i, j + 1 \mid \cdot) + \frac{1}{d^2} \phi_d(i, j \mid H^2, \cdot) + \sum_{d_1 + d_2 = d \atop i_1 + i_2 = n} \frac{d_1^2}{d^2} \phi_d(i, j \mid H^{i_1}, \cdot) \cdot \sigma_d(H^{i_2}, \cdot)
\] (2.6)

for any \( i \geq 0 \), where the sums above are over all possible distributions of the constraints \( \beta_1, \ldots, \beta_k \) on the two factors and \( d_1, d_2 \neq 0 \). When \( i = 0 \), the last term in (2.6) is missing.

**Proof.** The first relation follows by definition, and provides the initial step of the recursion. The second one requires more work. In what follows, we will identify a cohomology class like \( c_1(L) \) with a divisor representing it. Then:
Lemma 2.3 On $\mathcal{U}_d$, we have the following relation:

$$c_1(L^*) = \frac{1}{d^2} \mathcal{H} - \frac{2}{d} \text{ev}^*(H) + \frac{1}{d^2} \sum_{d_1+d_2=d} d_2^2 \mathcal{M}_{d_1,d_2}$$  \hspace{1cm} \text{(2.7)}$$

where $\mathcal{H}$ denotes the extra condition that the curve passes through $H^2$, and $\mathcal{M}_{d_1,d_2}$ denotes the boundary stratum corresponding to the splittings in a degree $d_1$ 1-marked curve and a degree $d_2$ curve, for $d_i \neq 0$ (for all possible distributions of the constraints $\beta_1, \ldots, \beta_k$ on the two components).

Proof. Fix 2 hyperplanes in generic position in $\mathbb{P}^n$. Each curve in $\mathcal{U}_d$ intersects a hyperplane in $d$ points. Then the moduli space $Y = \text{ev}_{k+1}^*(H) \cap \text{ev}_{k+2}^*(H)$ of 1-marked curves passing through $\beta_1, \ldots, \beta_k, H, H$ is a $d^2$ fold cover of $\mathcal{U}_d$:

$$\pi : Y \to \mathcal{U}_d, \quad [f, y_1, \ldots, y_k, a, b ; y] \to [f, y_1, \ldots, y_k ; y]$$

Define the section

$$s([f, y_1, \ldots, y_k, a, b ; y]) = \frac{(a - b)dy}{(y - a)(y - b)}$$

Then $s$ is a section in the relative cotangent sheaf $L^*$, and it extends to the compactification $\overline{\mathcal{U}_d}$. As $a$ and $b$ are getting closer together, the section $s$ converges to 0. Thus its zero set is the sum of the divisors $\{a = b\}$ and $\mathcal{M}(y ; a, b)$, where $\mathcal{M}(y ; a, b)$ is the sum of all boundary strata corresponding to splittings into a degree $d_1$ 1-marked bubble and a degree $d_2$ bubble containing $a, b$ for $d = d_1 + d_2$. Note that $d_i \neq 0$. The infinity divisor is $\{y = a\} + \{y = b\}$. Thus

$$\pi^*(c_1(L^*)) = \{a = b\} + \mathcal{M}(y ; a, b) - \{y = a\} - \{y = b\}$$

Note that

$$d^2 c_1(L^*) = \pi_* \pi^*(c_1(L^*))$$

When projecting down to $\overline{\mathcal{U}_d}$, the divisor $\{a = b\}$ becomes $\mathcal{H}$, and the divisors $\{y = a\}, \{y = b\}$ become each $d \cdot \text{ev}^*(H)$. The rest amounts to summing over all codimension 1 boundary strata. The boundary strata $\mathcal{M}_{d_1,d_2}$ appears with coefficient $d_2^2$ in $\pi_*(\mathcal{M}(y, a ; b))$. Combining all the pieces together completes the proof of Lemma. \qed

Remark 2.4 We could have chosen any 2 marked points out of the already existent ones, and then express $c_1(L)$ in terms of them. But then this expression would not look independent of choice. Nevertheless, with some work, one can actually see that all these divisors are homotopic. We have chosen to introduce 2 new marked points to avoid this issue.

Remark 2.5 Note that the base locus of the line bundle $L$ is exactly the union of the divisors $Z_b$. Doing the intersection theory in the blow up along the base locus is the same as considering the excess intersection (see [Ful]).

Relation (2.7) provides the basic relation for proving (2.6):

$$c_1^{i+1}(L^*) = -\frac{2}{d} c_1^i(L^*) \cdot \text{ev}^*(H) + \frac{1}{d^2} c_1^i(L^*) \cdot \mathcal{H} + \sum_{d_1+d_2=d} d_2^2 c_1^i(L^*) \cdot \mathcal{M}_{d_1,d_2}$$

28
so taking a cup product with \( \text{ev}^* (H^j) \) we get:

\[
\phi_d(i + 1, j \mid \cdot) = \frac{2}{d} \phi_d(i, j + 1 \mid \cdot) + \frac{1}{d^2} \phi_d(i, j \mid H^2, \cdot) \tag{2.8}
\]

\[
+ \sum_{d_1 + d_2 = d} \frac{d_1^2}{d^2} c_1^d (L^*) \text{ev}^* (H^j) \cdot \mathcal{M}_{d_1, d_2}
\]

Next, we need to understand the restriction of \( L^* \) to the boundary stratum \( \mathcal{M}_{d_1, d_2} \). Let \( p : \mathcal{M}_{d_1, d_2} \to \mathcal{U}_{d_1} \) be the projection on the first component (the one that contains the special marked point \( y \)). If \( A, B \) are the 2 special points of \( \mathcal{M}_{d_1, d_2} \) (where the 2 components meet), let \( \text{ev}_{A} \times \text{ev}_{B} : \mathcal{U}_{d_1} \times \mathcal{M}_{d_2} \to \mathbb{P}^n \times \mathbb{P}^n \) be the corresponding evaluation map. Then by definition

\[
\mathcal{M}_{d_1, d_2} = (\text{ev}_{A} \times \text{ev}_{B})^* ([\Delta]) \tag{2.9}
\]

where \( \Delta \) is the diagonal of \( \mathbb{P}^n \times \mathbb{P}^n \). Moreover, it is known that as divisors,

\[
c_1(L^*)/\mathcal{M}_{d_1, d_2} = p^* c_1(L_A^*) + \{ y = A \} \tag{2.10}
\]

where \( L_A = L|_{\mathcal{U}_{d_1}} \) is the relative tangent sheaf of \( \mathcal{U}_{d_1} \). Next step is to find

\[
c_1^i (L^*) / \mathcal{M}_{d_1, d_2} = \sum_{l=0}^{i} \binom{i}{l} p^* c_1^{i-l} (L_A^*) \cdot \{ y = A \}^l \tag{2.11}
\]

For the self intersection of the divisor \( \{ y = A \} \) note that its normal bundle \( N \) inside \( \mathcal{M}_{d_1, d_2} \) is nothing but \( p^* (L_A)/\{ y = A \} \), so for \( l > 0 \),

\[
\{ y = A \}^l = c_1(N)^{l-1} = (-1)^{l-1} p^* c_1^{l-1} (L_A^*) \cdot \{ y = A \}
\]

Substituting in (2.11) and after some algebraic manipulations we get:

\[
c_1^i (L^*) \cdot [\mathcal{M}_{d_1, d_2}] = p^* c_1^i (L_A^*) + p^* c_1^{i-1} (L_A^*) \cdot \{ y = A \} \tag{2.12}
\]

We will do the intersection theory inside \( \mathcal{U}_{d_1} \times \mathcal{M}_{d_2} \). The relation (2.9) combined with \( [\Delta] = \sum_{i_1 + i_2 = n} H^{i_1} \times H^{i_2} \) gives

\[
\text{ev}^* (H^j) \cdot [\mathcal{M}_{d_1, d_2}] = \sum_{i_1 + i_2 = n} \mathcal{U}_{d_1} (H^j; H^{i_1}, \cdot) \times \mathcal{M}_{d_2} (H^{i_2}, \cdot)
\]

\[
\text{ev}^* (H^j) \cdot \{ y = A \} = \sum_{i_1 + i_2 = n+j} \mathcal{U}_{d_1} (H^{i_1}; \cdot) \times \mathcal{M}_{d_2} (H^{i_2}, \cdot)
\]

29
where we sum over all possible distributions of the constraints on the two components. The relations above imply

$$\text{ev}^*(H^j) \cdot p^*c_1(L_A^*) \cdot [\mathcal{M}_{d_1,d_2}] $$

$$= \sum_{i_1 + i_2 = n} (c_1(L_A) \cdot \mathcal{U}_{d_1}(H^j; H^{i_1}, \cdot ) ) \times \mathcal{M}_{d_2}(H^{i_2}, \cdot )$$

$$= \sum_{i_1 + i_2 = n} \phi_{d_1}(i,j \mid H^{i_1}, \cdot ) \cdot \sigma_{d_2}(H^{i_2}, \cdot )$$

(2.13)

$$\text{ev}^*(H^j) \cdot p^*c_1(L_A^*) \cdot [(y = y_A)] $$

$$= \sum_{i_1 + i_2 = n+j} (c_1(L_A) \cdot \mathcal{U}_{d_1}(H^{i_1}, \cdot ) ) \times \mathcal{M}_{d_2}(H^{i_2}, \cdot )$$

$$= \sum_{i_1 + i_2 = n+j} \phi_{d_1}(i-1,i_1 \mid \cdot ) \cdot \sigma_{d_2}(H^{i_2}, \cdot )$$

(2.14)

Substituting these relations in (2.8) using (2.12) we get (2.6), which concludes the proof of Proposition 2.2.

\[ \square \]

2.2 Recursive formulas for $c_1^i(\tilde{L}^*) \cdot \text{ev}^*(H^j)$

Next step is to express the top intersections involving the first Chern class of $\tilde{L}$, the blow up of $L$, in terms of the top intersections involving the first Chern class of $L$. The program for such kind of computations is very nicely outlined in [Ful], which we will follow closely. Although recursive formulas can be found for any $n$, the more strata we need to blow up, the longer and more complicated looking these formulas become.

For simplicity of the presentation, in this section we will give only the general principles of the algorithm, without working out completely the recursive formulas. In the next section we will use this algorithm to obtain recursive formulas for small values of $n$ (i.e. $n \leq 4$).

Let $\mathcal{U}_d = \mathcal{U}_d( ; \beta_1, \ldots, \beta_k)$ the some $r$-dimensional moduli space of 1-marked curves. Recall the construction of $\mathcal{U}_d$: starting with $\mathcal{U}_d$, we first blow up along $\mathcal{Z}_N$, then we blow up the proper transform of $\mathcal{Z}_{N-1}$ and so on, up to blowing up the proper transform of $\mathcal{Z}_2$. Since $\tilde{L}$ extends as the blow up of $L$ then

$$c_1(\tilde{L}) = c_1(L) + \sum_{h=2}^{N} E_h$$

(2.15)

where $E_h$ is the exceptional divisor corresponding to the proper transform of $\mathcal{Z}_h$.

**Proposition 2.6** Using the notations above, the top intersections

$$c_1^i(\tilde{L}^*) \cdot \text{ev}^*(H^j)$$

(2.16)

on $\tilde{U}_d$ can be recursively expressed in terms of the top intersections

$$\phi_{d}(k,l \mid \cdot ) = c_1^k(L^*) \cdot \text{ev}^*(H^l)$$

on possibly lower dimensional moduli spaces $\mathcal{U}_{d'}$.  

30
Proof. The idea is of course to do inductively one blow up at a time. Although the fact there exist such recursive formulas is not that hard to see, writing them down becomes pretty complicated very quickly. So we explain why such formulas exist, leaving their derivation for later. By definition,

$$\tilde{x} = c_1(\tilde{L}^*) = x - \sum_{h=2}^{N} E_h$$

Let

$$x(h) = x - \sum_{l=h}^{N} E_l$$

so $\tilde{x}(N+1) = x$ and $\tilde{x}(2) = \tilde{x}$. Using $x(h) = x(h+1) - E_h$, and expanding,

$$x(h)^i \cdot y^j = x(h+1)^i \cdot y^j + \sum_{l=1}^{i} \left( \begin{array}{c} i \\ l \end{array} \right) x(h+1)^{i-l} y^j (-1)^l E_h^l$$

$$= x(h+1)^i y^j - \sum_{l=h}^{i} \left( \begin{array}{c} i \\ l \end{array} \right) x(h+1)^{i-l} y^j s_{l-h} (N_{\tilde{U}_d} \tilde{Z}_h) [\tilde{Z}_h]$$

(2.18)

The last equality is a consequence of the following

Fact 2.7 (cf. [Ful]) Assume $X \subset Y$ regular imbedding of codimension $a$, where $\text{dim } Y = r$. Let $\pi : \tilde{Y} \rightarrow Y$ be the blow up of $Y$ along $X$ and $E = \mathbb{P}(N_Y X)$ be the exceptional curve. For any $\alpha \in H^{r-1}(Y)$, the top intersection

$$(\pi^* (\alpha) \cup E)^l \cap [\tilde{Y}] = (-1)^{l-1} (\alpha \cup s_{l-a} (N_Y X)) \cap [Y]$$

(2.19)

as integers, where $l \geq 1$ and $s(N)$ is the seegre class of the normal bundle $N$.

Next we need to understand how $x(h+1)$ or equivalently $E_m$ restricts to $\tilde{Z}_h$, and also we need to find $s(N_{\tilde{U}_d} \tilde{Z}_d)$. First, we find:

The normal bundle of $Z_h$ in $U_d$. Recall that $Z_h$ consists of bubble trees with $h$ essential components meeting at the image of the ghost base. So in particular, $Z_h$ has components indexed by the different distributions of the degree on the $h$ bubbles:

$$Z_{d_1, \ldots, d_h} = \text{ev}_0^*([\Delta]) \subset \overline{\mathcal{M}}_{0,h+1} \times U_{d_1} \times \ldots \times U_{d_h}$$

(2.20)

where $d_i \neq 0$ for $i = 1, \ldots, h$, $\Delta$ is the small diagonal in $(\mathbb{P}^n)^h$ and

$$\text{ev}_0 : U_{d_1} \times \ldots \times U_{d_h} \rightarrow (\mathbb{P}^n)^h$$

(2.21)

$$\text{ev}_0([f_1, y_1], \ldots, [f_h, y_h]) = (f_1(y_1), \ldots, f_h(y_h))$$

(2.22)

is the evaluation map. Then

$$Z_h = \frac{1}{h!} \bigcup_{d_1 + \ldots + d_h = d} Z_{d_1, \ldots, d_h}$$

(2.23)

31
where the factor of $h!$ comes from the action of the symmetric group that permutes the order of the $h$ bubbles (yielding the same bubble tree). Let

$$\mathcal{M}_{0,h+1} \times \mathcal{U}_d \times \cdots \times \mathcal{U}_d \xrightarrow{\pi_i} \mathcal{U}_d$$

be the projection and $L_i$ be the relative tangent sheaf of the $i$'th factor. It is easy to check that

**Lemma 2.8** Using the notations above, the normal bundle $N_{\mathcal{Z}}$ of $\mathcal{Z}$ in $\mathcal{U}_d$ is isomorphic to

$$p_1^*L_1 \oplus \cdots \oplus p_h^*L_h$$

on each component (2.24)

so

$$s(N_{\mathcal{Z}}) = \frac{1}{(1-x_1) \cdots (1-x_h)}$$

(2.25)

where $x_i = p_i^*c_1(L_i)$.

**Remark 2.9** One word of caution: so far we have defined $x_i$ on each component $\mathcal{Z}_{d_1}, \ldots, \mathcal{Z}_{d_h}$ of $\mathcal{Z}$, but these definitions do not match on the intersection of two components. Nevertheless, after we blow up $\mathcal{U}_d$ as in Definition 1.17, all the components of $\mathcal{Z}$ become disjoint, so doing the intersection theory in the blow up allows up to treat each component separately as if they were disjoint.

Next, use the following

**Fact 2.10** (cf. [Ful]) Assume $X, Y \subset Z$ are regular imbeddings. Let $\tilde{Z} = Bl_X Z$ be the blow up of $Z$ along $X$ and $\tilde{Y} = Bl_{X \cap Y} Y$ be the proper transform of $Y$. Denote by $E = \mathbb{P}(N_Z X)$ the exceptional curve in $\tilde{Z}$, and let $F = \mathbb{P}(N_Y (X \cap Y)$ be the exceptional curve in $\tilde{Y}$. Then:

$$E \cap \tilde{Y} = F$$

(2.26)

$$N_{\tilde{Z}} \tilde{Y} \cong \pi^*(N_{Z} Y) \otimes O(-F)$$

so

(2.27)

$$sp(N_{\tilde{Z}} \tilde{Y}) = \sum_{i=0}^{p-a} \binom{a+p}{a+i} s_i(N_{Z} Y) F^{p-i}$$

(2.28)

where $a = \text{rank } N_{Z} Y$.

So

$$E_l \cap \tilde{Z}_h = E_{h,l}$$

(2.29)

is the exceptional curve in the blow up of $\mathcal{Z}_h$ along the proper transform of $\mathcal{Z}_h \cap \mathcal{Z}_l$ and

$$N_{\mathcal{U}_d} \tilde{Z}_h \cong \pi^*(N_{\mathcal{U}_d} \mathcal{Z}) \otimes O \left( - \sum_{l=h+1}^{N} E_{h,l} \right)$$

(2.30)

Note that $\mathcal{Z}_h$ has less dimensions than $\mathcal{U}_d$ and it is stratified by subvarieties $\mathcal{Z}_h \cap \mathcal{Z}_l$ for $l \geq h + 1$ the same way $\mathcal{U}_d$ is stratified by the sets $\mathcal{Z}_l$. Thus we can repeat the same construction.
Inductively, the intersection theory takes place inside a strata of form $Z_{h_1} \cap \ldots \cap Z_{h_t}$. Now, it is not that easy to list and parametrize all the possible bubble tree configurations in this intersection. But a closer look reveals that although the combinatorics involved is complicated, all these intersections have components of the form

$$ev_0^*([\Delta]) \subset Z \times U_{d_1} \times \ldots \times U_{d_m}$$

(2.31)

where $Z$ is some substrata of $\overline{M}_{0,m+1}$, and $\Delta$ is the small diagonal in $(\mathbb{P}^n)^m$. Such component comes in with a coefficient of one over the order of the subgroup of permutations that preserve the same bubble tree configuration.

The normal bundle of (2.31) has the same form as in Lemma 2.8, and using the arguments outlined above we are inductively decreasing either the number of exceptional curves or the dimension of the moduli space we do the intersection theory over. In either case, the process terminates in finite time, reducing the top intersections $\tilde{x}_i \tilde{y}_j$ on $\mathcal{U}_d$ we started with to sums of top intersections of the form

$$x_1^{i_1} \ldots x_m^{i_m} y_1^{j_1} \ldots y_m^{j_m} \text{ on } ev_0(\Delta) \subset Z \times U_{d_1} \times \ldots \times U_{d_m}$$

(2.32)

Finally, let

$$\pi: Z \times U_{d_1} \times \ldots \times U_{d_m} \to U_{d_1} \times \ldots \times U_{d_m}$$

be the projection. Since all the classes in (2.32) are pull-backs by $\pi$ then the top intersection (2.32) vanishes unless $Z \subset M_{0,m+1}$ is 0 dimensional. When $Z$ is 0-dimensional then using the decomposition of the diagonal

$$[\Delta] = \sum_{i_1+\ldots+i_m=n} H^{i_1} \times \ldots \times H^{i_m}$$

and letting $y_j = ev^*(H)$ on the $j$'th factor we get

$$ev_0^*([\Delta]) = \sum_{j_1+\ldots+j_m=n} y_1^{j_1} \ldots y_m^{j_m} [U_{d_1} \times \ldots \times U_{d_m}] \quad \text{and}$$

(2.33)

$$y^j ev_0^*([\Delta]) = \sum_{j_1+\ldots+j_m=n+j} y_1^{j_1} \ldots y_m^{j_m} [U_{d_1} \times \ldots \times U_{d_m}]$$

(2.34)

so (2.32) is equal to

$$\sum_{j_1+\ldots+j_m=n+j} (x_1^{i_1} y_1^{j_1} [U_{d_1}]) \times \ldots \times (x_m^{i_m} y_m^{j_m} [U_{d_m}])$$

(2.35)

$$= \sum_{i_1+\ldots+i_m=n+j} \phi_d(i_1,j_1) \cdot \ldots \cdot \phi_d(i_m,j_m)$$

(2.36)

giving Proposition 2.6.

3 Applications to $\mathbb{P}^n$, $n \leq 4$

Finally, we apply the inductive algorithm described in the previous section to obtain recursive formulas for the elliptic enumerative invariant $\tau_d$ in $\mathbb{P}^n$, for $n \leq 4$. In this case the story is quite simple, since $N = \left\lfloor \frac{n+1}{2} \right\rfloor \leq 2$, so we need to blow up at most one strata, $Z_2$. 

33
Explicit formulas for $c_1(\tilde{L}^*) \cdot \text{ev}^* (H^j)$. Note that for $n = 2$ there is nothing to blow up, so

$$
\tilde{L} = L \quad \text{for} \quad n = 2
$$

(3.37)

A little more work gives:

**Lemma 3.1** Using the notations in Theorem 0.1, when $n = 3, 4$, we have the following relations:

$$
c_1(\tilde{L}) \cdot \text{ev}^* (H^{n-2}) = \phi_d(1, n - 2 | \cdot )
$$

(3.38)

$$
c_1^2(\tilde{L}) \cdot \text{ev}^* (H^{n-3}) = \phi_d(2, n - 3 | \cdot ) - \sum_{d_1 + d_2 = d \atop i_1 + i_2 = 2n - 3} \sigma_{d_1}(H^{i_1}) \cdot \sigma_{d_2}(H^{i_2})
$$

(3.39)

$$
c_1^3(\tilde{L}) = \phi_d(3, 0 | \cdot ) - \sum_{d_1 + d_2 = d \atop i_1 + i_2 = 2n - 3} \sigma_{d_1}(1, i_1 | \cdot ) \cdot \sigma_{d_2}(H^{i_2}) \quad \text{for} \quad n = 4
$$

(3.40)

where the sums above are over all possible distributions of the constraints on the two components and $d_i \neq 0$.

**Proof.** When $n = 3, 4$ we need to blow up only $Z_2$. Use (2.18) to get:

$$
\overline{x}^i y^{n-1-i} [U_d] = x^i y^{n-1-i} [U_d] - \sum_{l=2}^i \binom{i}{l} x^{i-l} s_{l-2}(N_{U_d} Z_2) y^{n-1-i} [Z_2]
$$

(3.41)

Note that for $i = 1$ the sum in (3.41) is indexed by the empty set, thus giving (3.38). When $i = 2$ the sum reduces to:

$$
x^0 \cdot s_0(N_{U_d} Z_2) \cdot y^{n-3} [Z_2] = y^{n-3} [Z_2] = \frac{1}{2} \sum_{d_1 + d_2 = d \atop i_1 + i_2 = 2n - 3} \sigma_{d_1}(H^{i_1}) \cdot \sigma_{d_2}(H^{i_2})
$$

by (2.36), giving (3.39).

When $n = 4$ and $i = 3$ then the sum in (3.41) becomes

$$
(3x \cdot s_0(N_{U_d} Z_2) + s_1(N_{U_d} Z_2))[Z_2]
$$

(3.42)

But note that $x|[Z_2] = 0$ and Lemma 2.8 gives

$$
s(N_{U_d} Z_2) = \frac{1}{(1 - x_1)(1 - x_2)} \quad \text{thus} \quad s_1(N_{U_d} Z_2) = x_1 + x_2
$$

So (3.42) becomes

$$
(x_1 + x_2) [Z_2] = \frac{1}{2} \sum_{d_1 + d_2 = d \atop j_1 + j_2 = n} (x_1 + x_2) y_1^{j_1} y_2^{j_2} [U_d_1 \times U_{d_2}]
$$

$$
= \sum_{d_1 + d_2 = d \atop j_1 + j_2 = n} (x_1 y_1^{j_1} [U_{d_1}]) \cdot (y_2^{j_2} [U_{d_2}]) = \sum_{d_1 + d_2 = d \atop j_1 + j_2 = n} \phi_{d_1}(1, j_1 | \cdot ) \cdot \sigma_{d_2}(H^{j_2}, \cdot )
$$

using again (2.36). □

Now we can prove for example that:
Proposition 3.2 The number $\tau_d(p^{3d-1})$ of degree $d$ elliptic curves in $\mathbb{P}^2$ with fixed $j$ invariant and passing through $3d-1$ points is

$$\tau_d(p^{3d-1}) = \frac{2}{n_j} \binom{d-1}{2} \sigma_d(p^{3d-1})$$  \hspace{1cm} (3.43)

where $\sigma_d$ is the number of rational curves through $3d$ points, and $n_j$ is the order of the group of automorphisms of the complex structure $j$ fixing a point.

Proof. For $n = 2$, relation (1.3) combined with (3.37) gives:

$$n_j \tau_d(p^{3d-1}) = \sigma_d(l, l, p^{3d-1}) - 3ev^*(H) - c_1(L^*)$$  \hspace{1cm} (3.44)

where $L \rightarrow U_d$ is the relative tangent sheaf over the moduli space of 1-marked rational curves of degree $d$ passing through $3d - 1$. The moduli space $M_d$ of unmarked curves is $n - 2 = 0$ dimensional, consisting of $\sigma_d(p^{3d-1})$ curves. Using (2.2) (or easier by inspection)

$$c_1(L^*) = -\frac{2}{d} \sigma_d(l, p^{3d-1}) = -2 \sigma_d(p^{3d-1})$$

$$ev^*(H) = \sigma_d(l, p^{3d-1}) = d \sigma_d(p^{3d-1}) \quad \text{and} \quad \sigma_d(l, l, p^{3d-1}) = d^2 \sigma_d(p^{3d-1})$$

So plugging them back in (3.44) we obtain

$$\tau_d(p^{3d-1}) = \frac{1}{n_j}(d^2 - 3d + 2) \sigma_d$$

which gives (3.2). \hspace{1cm} \Box

In particular,

$$\tau_d(p^{3d-1}) = \begin{cases} \binom{d-1}{2} \sigma_d & \text{if } j \neq 0, 1728 \\ \frac{1}{2} \binom{d-1}{2} \sigma_d & \text{if } j = 0 \\ \frac{1}{3} \binom{d-1}{2} \sigma_d & \text{if } j = 1728 \end{cases}$$  \hspace{1cm} (3.45)

This formula was recently obtained by Panharipande [Pan] using different methods.

Next we can prove that:

Proposition 3.3 The number $\tau_d = \tau_d(p^a, l^b)$ of elliptic curves in $\mathbb{P}^3$ with fixed $j$ invariant and passing through $a$ points and $b$ lines (such that $2a + b = 4d - 1$) is given by:

$$\tau_d = \frac{2(d - 1)(d - 2)}{dn_j} \sigma_d(l) - \frac{2}{dn_j} \sum_{d_1 + d_2 = d} d_2(2d_1d_2 - d) \sigma_{d_1}(l) \sigma_{d_2}$$  \hspace{1cm} (3.46)

where $\sigma_d(l) = \sigma_d(p^a, l^b, l)$ is the number of degree $d$ rational curves in $\mathbb{P}^3$ passing through same conditions as $\tau_d$ plus one more line. By the term $\sigma_{d_1}(l)\sigma_{d_2}$ we understand the sum over all decompositions into a degree $d_1$ and a degree $d_2$ bubble such that the constraints are distributed in all possible ways on the bubbles, and $d_i \neq 0$.  

35
Proof. When \( n = 3 \), Theorem 1.1 gives:

\[
n_j \tau_d(p^a, l^b) = \sum_{i_1 + i_2 = 3} \sigma_d(H^{i_1}, H^{i_2}, p^a, l^b) - 6ev^*(H^2) - 4ev^*(H)c_1(\tilde{L}^*) - c_1^2(\tilde{L}^*)
\] (3.47)

The moduli space \( \mathcal{M}_d \) of degree \( d \) unmarked curves passing through \( a \) points and \( b \) lines is of degree \( n - 2 = 1 \) dimensional, with a finite number of bubble trees in the boundary. Then Proposition 3.3 is a consequence of (3.47) and the following

Lemma 3.4 In \( \mathbb{P}^3 \),

\[
\sum_{i_1 + i_2 = 3} \sigma_d(H^{i_1}, H^{i_2}, p^a, l^b) = 2d \cdot \sigma_d(p^a, l^{b+1})
\]

\[
ev^*(H^2) = \sigma_d(p^a, l^{b+1})
\]

\[
ev^*(H) \cdot c_1(\tilde{L}^*) = ev^*(H) \cdot c_1(L^*) = -\frac{1}{d} \sigma_d(l) + \frac{1}{d} \sum_{d_1 + d_2 = d} d_1 d_2 \sigma_{d_1}(l) \sigma_{d_2}
\]

\[
c_1(L^*)^2 = -\sum_{d_1 + d_2 = d} d_2 \sigma_{d_1}(l) \sigma_{d_2} \quad \text{and} \quad c_1(\tilde{L}^*)^2 = -2 \sum_{d_1 + d_2 = d} d_2 \sigma_{d_1}(l) \sigma_{d_2}
\]

Proof. The relations above follow either by definition, or by applying several times (2.7) combined with (3.38) or (3.39) (and of course some simple algebraic manipulations).

Remark 3.5 If we distribute the constraints in Proposition 3.3 in all possible ways, formula (3.46) becomes:

\[
\tau_d(p^a, l^b) = \frac{2(d-1)(d-2)}{n_d} \sigma_d(p^a, l^{b+1}) + \frac{2}{n_d} \sum_{d_1 + d_2 = d} \sum_{a_1 = 0}^{a} \sum_{b_1 = 0}^{b} \left( \begin{array}{c} a \\ 2 \end{array} \right) d_2 (2d_1 d_2 - d) \sigma_{d_1}(p^{a_1}, l^{b_1+1}) \cdot \sigma_{d_2}(p^{a_2}, l^{b_2})
\]

where in the sum above \( d_1 + d_2 = d \), \( a_1 + a_2 = a \) and \( b_1 + b_2 = b \).

Example 1. Using a computer program based on (3.48) and the recursive formulas (A.3) for \( \sigma_d \), one recovers for example that in \( \mathbb{P}^3 \) all the degree 2 elliptic invariants are 0 (fact known for a very long time) but also one gets new examples, like:

| \( j \neq 0, 1728 \) | \( j = 0 \) | \( j = 1728 \) |
|-----------------|-----------------|-----------------|
| \( \tau_3(l^{11}) \) | \( \tau_3(p, l^{17}) \) | \( \tau_6(p^{11}, l) \) |
| 6 \cdot 25920 | 6 \cdot 15856790593536 | 6 \cdot 13260 |
| 2 \cdot 25920 | 2 \cdot 15856790593536 | 2 \cdot 13260 |
| 3 \cdot 25920 | 3 \cdot 15856790593536 | 3 \cdot 13260 |

Example 2. Similarly, when \( n = 4 \), one can use a computer program based on the four steps described in the Section 2 to get for example:

| \( j \neq 0, 1728 \) | \( j = 0 \) | \( j = 1728 \) |
|-----------------|-----------------|-----------------|
| \( \tau_3(5H^2, 3l, p) \) | \( \tau_3(12H^2, l) \) | \( \tau_3(14H^2) \) |
| 6 \cdot 42 | 6 \cdot 202680 | 6 \cdot 1305640 |
| 2 \cdot 42 | 2 \cdot 202680 | 2 \cdot 1305640 |
| 3 \cdot 42 | 3 \cdot 202680 | 3 \cdot 1305640 |
Remark 3.6 Unfortunately, the number of steps involved in computing the elliptic invariant $\tau_d$ in $\mathbb{P}^n$ increases extremely fast with $n$. For example, one can write down the recursive formulas for $n = 5, 6$ that do not look that complicated (we need to blow up only two strata, $Z_3$ and $Z_2$). But the amount of time necessary to run the corresponding program is too long to produce interesting examples.

4 Appendix

The genus zero perturbed invariant and the genus zero enumerative invariant are equal in $\mathbb{P}^n$ (cf. [RT]), i.e.

$$\sigma_d(H^{j_1}, H^{j_2}, \ldots, H^{j_k}) = RT_{d,0}(H^{j_1}, H^{j_2}, H^{j_3}|H^{j_4}, \ldots, H^{j_k})$$

(A.1)

Consequences of Ruan-Tian degeneration formula are:

$$RT_{d,1}(\beta_1 | \beta_2, \ldots, \beta_l) = \sum_{i_1+i_2=n} \sigma_d(H^{i_1}, H^{i_2}, \beta_1, \ldots, \beta_l)$$

(A.2)

and that $\sigma_d$ in $\mathbb{P}^n$ satisfies the following recursive formula: for $j_1 \geq j_2 \geq \ldots \geq j_k \geq 2$,

$$\sigma_d(H^{j_1}, H^{j_2}, H^{j_3}) = d\sigma_d(H^{j_1+j_3-1}, H^{j_2}) - d\sigma_d(H^{j_1+j_2}, H^{j_3-1}) - \sigma_d(H^{j_1}, H^{j_2+1}, H^{j_3-1})$$

$$\sum_{d_1=1}^{d-1} \sum_{i=0}^{n} \sigma_{d_1}(H^{j_1}, H^{j_2}, H^i)\sigma_{d_2}(H^{j_3-1}, H^{n-i}) \sum_{d_1=1}^{d-1} \sum_{i=0}^{n} \sigma_{d_1}(H^{j_1}, H^{j_3-1}, H^i)\sigma_{d_2}(H^{j_2}, H^{n-i})$$

(A.3)

where $\sigma_d(H^{j_1}, H^{j_2}, H^{j_3}) = \sigma_d(H^{j_1}, H^{j_2}, H^{j_3}, H^{j_4}, \ldots, H^{j_k})$ and the conditions $H^{j_4}, \ldots, H^{j_k}$ are distributed in the right hand side in all possible ways. Note that $\sigma_1(pt, pt) = 1$ gives the initial step of the recursion.

References

[Al] Aluffi, How many smooth plane cubics with given $j$-invariant are tangent to 8 lines in general position?, Contemp. Math 123 (1991), 15-29.

[AM] P.S. Aspinwall, D.R. Morrison, Topological field theory and rational curves, Comm. Math. Phys. 151 (1993), 245-262.

[D] S. K. Donaldson, The orientation of Yang-Mills moduli spaces and 4-manifold topology, JDG 26 (1987), 397-428.

[DK] S. K. Donaldson, P. Kronheimer, ”The geometry of Four Manifolds”, Oxford Math. Monographs, Oxford Univ. Press, Oxford, (1990).

[F] A. Floer, The Unregularized gradient flow of the symplectic action, Comm. Pure. Appl. Math. XLI (1988), 775-813.

[Ful] W. Fulton, ”Intersection Theory”, Springer-Verlag, Berlin, (1984).
[GH] P. Griffiths, J. Harris, "Principles of Algebraic Geometry", Wiley, New York (1978).

[KM] M. Kontsevich, Y. Manin, Gromov-Witten classes, quantum cohomology and enumerative geometry, Comm.Math.Phys. 164 (1994), 525-562.

[MS] D. McDuff, D. Salamon, "J-holomorphic Curves and Quantum Cohomology", Univ. Lect. Series, AMS, Providence, Rhode Island, 1994.

[Pan] R. Panharipande, A note on elliptic plane curves with fixed j invariant, alg-geom e-print, May 1995.

[PW] T. H. Parker, J. Wolfson, Holomorphic maps and bubble trees, J. Geom. Analysis, 3 (1993), 63-97.

[P] T. H. Parker, Compactified moduli spaces of pseudo-holomorphic curves, preprint.

[RT] Y. Ruan, G. Tian, A mathematical theory of quantum cohomology, JDG 42 (1995), 259-367.

[T1] C. H. Taubes, Self-dual connections on non-self dual 4-manifolds, JDG 17 (1982), 139-170.

[T2] C. H. Taubes, Self-dual connections on 4-manifolds with indefinite intersection form, JDG 19 (1984), 517-560.

Address: Eleny Ionel, MSRI, 1000 Centennial Drive, Berkeley, CA 94720-5070
e-mail: ionel@msri.org