THE YONEDA EXT BIFUNCTOR AND ARBITRARY PRODUCTS AND COPRODUCTS IN ABELIAN CATEGORIES

ALEJANDRO ARGUDÍN MONROY

Abstract. There are well known identities that involve the Ext bifunctor, coproducts, and products in Ab4 and Ab4* abelian categories with enough projectives and enough injectives. Namely, for every such category $A$, the isomorphisms

$$\text{Ext}_A^0 \left( \bigoplus_{i \in I} A_i, X \right) \cong \prod_{i \in I} \text{Ext}_A^0 (A_i, X), \quad \text{Ext}_A^0 \left( X, \prod_{i \in I} A_i \right) \cong \prod_{i \in I} \text{Ext}_A^0 (X, A_i)$$

always exist. The goal of this paper is to show similar isomorphisms for the Yoneda Ext in Ab4 and Ab4* abelian categories with not necessarily enough projectives nor injectives. The desired isomorphisms are constructed explicitly by using limits and colimits.

1. Introduction

The study of extensions is a theory that has developed from multiplicative groups [18, 12], with applications ranging from representations of central simple algebras [4, 10] to topology [8].

In this article we will focus on extensions in an abelian category $C$. In this context, an extension of an object $A$ by an object $C$ is a short exact sequence

$$0 \to A \to M \to C \to 0$$

up to equivalence, where two exact sequences are equivalent if there is a morphism from one to another with identity morphisms at the ends. This kind of approach was first made by R. Baer in 1934. On his work [1, 2], Baer defined an addition on the class $\text{Ext}_C^1 (C, A)$ of extensions of an abelian group $A$ by an abelian group $C$. His construction can be easily extended to abelian categories, where it is used to show that the class $\text{Ext}_C^1 (C, A)$ has a natural structure of abelian group. For this reason usually $\text{Ext}_C^1 (C, A)$ is called the group of extensions of $A$ by $C$.

Later on, H. Cartan and S. Eilenberg [7], using methods of homological algebra, showed that the first derived functor of the $\text{Hom}_C (C, -)$ functor, or $\text{Hom}_C (-, A)$ functor, is isomorphic to $\text{Ext}_C^1 (C, -)$, or respectively to $\text{Ext}_C^1 (-, A)$. This result marked the beginning of a series of research works looking for ways of constructing...
the derived functors of the Hom functor without using projective or injective objects, with the spirit that resolutions should be only a calculation tool for derived functors.

One of this attempts, registered in the work of D. Buchsbaum, B. Mitchell, S. Schannel, S. Mac Lane, M.C.R. Butler, and G. Horrocks [13 5 6 14], was based in the ideas of N. Yoneda [20 21], defining what is known today as the theory of $n$-extensions and the functor called as the Yoneda Ext. An $n$-extension of an object $A$ by an object $C$ is an exact sequence of length $n$

$$0 \rightarrow A \rightarrow M_1 \rightarrow \cdots \rightarrow M_n \rightarrow C \rightarrow 0$$

up to equivalence, where the equivalence of exact sequences of length $n > 1$ is defined in a similar way as was defined for length 1. In this theory, the Baer sum can be extended to $n$-extensions, proving that the class $\text{Ext}_n^R (C, A)$ of $n$-extensions of $A$ by $C$ is an abelian group.

Recently, the generalization of homological techniques such as Gorenstein or tilting objects to abstract contexts [16 3], such as abelian categories that do not necessarily have projectives or injectives, claim for the introduction of an Ext functor that can be used without restraints. The only problem is that most of the rich properties of the homological Ext are not known to be valid for the Yoneda Ext. The goal of this work is to make a next step by exploring some properties that the Yoneda Ext shares with the homological Ext.

Namely, we will explore the following property that is well known for module categories:

**Theorem 1.1.** [17 7.21,7.22] Let $R$ be a ring, $M \in \text{Mod}_R$, and $\{N_i\}_{i \in I}$ be a set of $R$-modules. Then, there exist a pair of isomorphisms

$$\text{Ext}_R^n\left(\bigoplus_{i \in I} N_i, M\right) \cong \prod_{i \in I} \text{Ext}_R^n (N_i, M) \quad \text{and} \quad \text{Ext}_R^n\left(M, \prod_{i \in I} N_i\right) \cong \prod_{i \in I} \text{Ext}_R^n (M, N_i).$$

The proof of such theorem can be extended to Ab4 and Ab4* abelian categories with enough projectives and enough injectives. Our goal will be to prove an analogue result for the Yoneda Ext without assuming the existence of enough projectives nor injectives.

Let us now describe the contents of this paper. Section 2 is devoted to review the basic results of the theory of extensions by following the steps of B. Mitchell in [13]. Finally, in section 3 we prove the desired theorem. More precisely, we show that in an Ab4 abelian category we can build the desired bijections explicitly by using limits and colimits.

2. Extensions

In this section we will remember the basic theory of extensions. As was mentioned before, the theory of $n$-extensions was created by Nobuo Yoneda in [20]. In such
paper he worked in a category of modules and most of the results are related with
the homological tools built by projective and injective modules. Since our goal is
to work in an abelian category without depending on the existence of projective
or injective objects, we refer the reader to the work of Barry Mitchell [14] for
an approach in abelian categories without further assumptions. Throughout this
paper, \( \mathcal{C} \) will denote an abelian category.

**Definition 2.1.** [14 VII.1.] Let \( C \in \mathcal{C} \), and \( \alpha : A \to B, \alpha' : A' \to B' \) be morphisms
in \( \mathcal{C} \). We set the following notation:

(a) \( \nabla_C := (1_C 1_C) : C \oplus C \to C \);
(b) \( \Delta_C := (1_C 1_C) : C \to C \oplus C \);
(c) \( \alpha \oplus \alpha' := (\alpha 0 0 \alpha') : A \oplus A' \to B \oplus B' \).

2.1. 1-Extensions. Let us begin by recalling some basic facts and notation about
1-extensions.

**Definition 2.2.** [14 VII.1.] Let \( \alpha : N \to N', \beta : M \to M', \) and \( \gamma : K \to K' \) be
morphisms in \( \mathcal{C} \), and \( \eta : 0 \to N \xrightarrow{f} M \xrightarrow{g} K \to 0, \eta' : 0 \to N' \xrightarrow{f'} M' \xrightarrow{g'} K' \to 0 \) be
short exact sequences in \( \mathcal{C} \).

(a) We say that \( (\alpha, \beta, \gamma) : \eta \to \eta' \) is a morphism of short exact sequences if
\( \beta f = f' \alpha \) and \( \gamma g = g' \beta \).

(b) We denote by \( \eta \oplus \eta' \) to the short exact sequence
\( 0 \to N \oplus N' \xrightarrow{f \oplus f'} M \oplus M' \xrightarrow{g \oplus g'} K \oplus K' \to 0 \).

**Definition 2.3.** [14 VII.1.] For \( A, C \in \mathcal{C} \), let \( \text{Ext}^1_C(C, A) \) denote the class of short
exact sequences of the form \( 0 \to N \to M \to K \to 0 \).

**Remark 2.4.** Let \( A, C \in \mathcal{C} \) and \( \eta, \eta' \in \text{Ext}^1_C(C, A) \). Consider the relation \( \equiv \eta \equiv \eta' \) given
by the existence of a short exact sequence morphism \( (1_A, \beta, 1_C) : \eta \to \eta' \). By the
snake lemma, we know that \( \beta \) is an isomorphism, and hence \( \equiv \) is an equivalence
relation on \( \text{Ext}^1_C(K, N) \).

**Definition 2.5.** [14 VII.1.] Consider \( A, C \in \mathcal{C} \).

(a) Let \( \text{Ext}^1_C(C, A) := \text{Ext}^1_C(C, A) / \equiv \);
(b) Each object of \( \text{Ext}^1_C(C, A) \) is refered as an extension from \( A \) to \( C \);
(c) Every extension from \( A \) to \( C \) will be denoted with a capital letter \( E \), or by
\( \overline{\eta} \), in case \( \eta \) is a representative of the class \( E \).
(d) Given \( \overline{\eta} \in \text{Ext}^1_C(C, A) \) and \( \overline{\eta'} \in \text{Ext}^1_C(C', A') \), we will call extension morphism
from \( \overline{\eta} \) to \( \overline{\eta'} \), to every short exact sequence morphism \( \eta \to \eta' \).
(e) If \( (\alpha, \beta, \gamma) : E \to E' \) and \( (\alpha', \beta', \gamma') : E' \to E'' \) are extension morphisms,
we define the composition morphism as
\( (\alpha', \beta', \gamma')(\alpha, \beta, \gamma) := (\alpha' \alpha, \beta' \beta, \gamma' \gamma) \).

**Remark 2.6.** An essential comment made by B. Mitchell in [14 VII] is that the
class \( \text{Ext}^1_C(C, A) \) may not be a set. Considering this fact, we should be cautious
when we talk about correspondences between extensions classes. Nevertheless, by simplicity we will say that a correspondence

$$\Phi : \text{Ext}_C^1(C', A') \to \text{Ext}_C^1(C, A)$$

is a function, if it associates to each $$\eta \in \text{Ext}_C^1(C', A')$$ a single element $$\Phi(\eta)$$ in $$\text{Ext}_C^1(C, A)$$.

Remember the following result.

**Proposition 2.7.** [13, III.1.2.] Let $$\alpha : X \to K$$ be a morphism and $$0 \to N \overset{f}{\to} M \overset{g}{\to} K \to 0$$ be an exact sequence in $$C$$. If $$(E, \alpha', g')$$ is the pull-back diagram of the morphisms $$g$$ and $$\alpha$$, then there is an exact short sequence $$\eta$$ and a morphism $$(1, \alpha', \alpha) : \eta \to \eta$$.

Of course the construction described above defines a correspondence between the extension classes.

**Proposition 2.8.** [14, VII.1.2.] Let $$\eta \in \mathcal{E}_C(C, A)$$ and $$\gamma \in \text{Hom}_C(C', C)$$. Then, the correspondence $$\Phi_{\gamma} : \text{Ext}_C^1(C, A) \to \text{Ext}_C^1(C', A), \eta \mapsto \eta\gamma$$, is a function.

By duality, given a morphism $$\alpha : N \to X$$ and an exact sequence

$$\eta : 0 \to N \overset{f}{\to} M \overset{g}{\to} K \to 0,$$

the pushout of the morphisms $$f$$ and $$\alpha$$, gives us an exact sequence $$\eta^\alpha$$ together with a morphism $$(\alpha, \alpha', 1) : \eta \to \eta^\alpha$$. Moreover, we also have that the correspondence $$\Phi^\alpha : \text{Ext}_C^1(K, N) \to \text{Ext}_C^1(K, X), \eta \mapsto \eta^\alpha$$, is a function.

**Definition 2.9.** [14, VII.1.] For $$E \in \text{Ext}_C^1(C, A)$$, $$\alpha : A \to A'$$ and $$\gamma : C' \to C$$ in $$C$$, we set $$E\gamma := \Phi_\gamma(E)$$, and $$\alpha E := \Phi^\alpha(E)$$.

As we have described, there exists a natural action of the morphisms on the extension classes. These actions are associative and respect identities.

**Lemma 2.10.** [14 VII.1.3.] Let $$E \in \text{Ext}_C^1(C, A)$$, $$\alpha : A \to A'$$, $$\alpha' : A' \to A''$$, $$\gamma : C' \to C$$, and $$\gamma' : C'' \to C'$$ be morphisms in $$C$$. Then,

(a) $$1_A E = E$$ and $$E 1_C = E$$;
(b) $$(\alpha' \alpha) E = \alpha' (\alpha E)$$ and $$E (\gamma' \gamma) = (E\gamma) \gamma'$$;
(c) $$(\alpha E) \gamma = \alpha (E\gamma)$$.

Next, we recall the definition of the Baer sum.

**Definition 2.11.** [14 VII.1.] For $$E, E' \in \text{Ext}_C^1(C, A)$$, the sum extension of $$E$$ and $$E'$$ is $$E + E' := \nabla_A (E \oplus E') \Delta_C$$.

This sum operation is well behaved with the actions before described and gives a structure of abelian group to the extension classes.
Theorem 2.12. VII.1.4.-1.5. For any $A, C \in \mathcal{C}$, we have that the pair $(\text{Ext}^1_C(C,A), +)$ is an abelian group, where the identity element is the extension $E_0$ given by the exact sequences that split. Furthermore, let $E \in \text{Ext}^1_C(C,A)$, $E' \in \text{Ext}^1_C(C',A')$, $\alpha \in \text{Hom}_\mathcal{C}(A,X)$, $\alpha' \in \text{Hom}_\mathcal{C}(A',X')$, $\gamma \in \text{Hom}_\mathcal{C}(Y,C)$ and $\gamma' \in \text{Hom}_\mathcal{C}(Y',C')$. Then, the following equalities hold true:

(a) $(\alpha + \alpha') (E \oplus E') = \alpha E \oplus \alpha' E'$;
(b) $(\alpha + \alpha') E = \alpha E + \alpha' E$;
(c) $\alpha (E + E') = \alpha E + \alpha E'$;
(d) $(E \oplus E') (\gamma + \gamma') = E\gamma \oplus E'\gamma'$;
(e) $E (\gamma + \gamma') = E\gamma + E\gamma'$;
(f) $(E + E') \gamma = E\gamma + E'\gamma$;
(g) $0E = E0 = E0$ for every $E \in \text{Ext}^1_C(C,A)$.

2.2. $n$-Extensions. We are ready for recalling the definition of $n$-extensions. It is a well known fact that short exact sequences can be stuck together in order to construct a long exact sequence. Following this thought, the spirit of $n$-extensions is to define a well behaved 1-extensions composition that constructs long extensions.

Definition 2.13. VII.3.] We will make use of the following considerations.

(a) For an exact sequence $\eta : 0 \rightarrow A \rightarrow B_{n-1} \rightarrow \cdots \rightarrow B_0 \rightarrow C \rightarrow 0$ in $\mathcal{C}$ we say that $\eta$ is an exact sequence of length $n$, and $A$ and $C$ are the left and right ends of $\eta$, respectively.
(b) Let $\mathcal{E}^n_\mathcal{C}(L,N)$ denote the class of exact sequences of length $n$ with $L$ and $N$ as right and left ends.
(c) Consider the following exact sequences in $\mathcal{C}$

$$
\eta : 0 \rightarrow N \xrightarrow{\mu} B_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} B_0 \xrightarrow{\pi} K \rightarrow 0,$n
\eta' : 0 \rightarrow N' \xrightarrow{\mu'} B'_{n-1} \xrightarrow{f'_{n-1}} \cdots \xrightarrow{f'_1} B'_0 \xrightarrow{\pi'} K' \rightarrow 0.
$$

A morphism $\eta \rightarrow \eta'$ is a collection of $n + 2$ morphisms $(\alpha, \beta_{n-1}, \cdots, \beta_0, \gamma)$ in $\mathcal{C}$, where $\alpha : N \rightarrow N'$, $\gamma : K \rightarrow K'$, and $\beta_i : B_i \rightarrow B'_i, \forall i \in [0, n-1]$ are such that

$$
\beta_{n-1} \mu = \mu' \alpha, \quad \gamma \pi = \pi' \beta_0 \quad \text{and} \quad \beta_{i-1} f_i = f'_i \beta_i, \forall i \in [0, n-1].
$$

Equivalently, we can say that a morphism of exact sequences of length $n$ is a commutative diagram

$$
\begin{array}{cccccccccc}
0 & \rightarrow & N & \rightarrow & B_{n-1} & \rightarrow & \cdots & \rightarrow & B_0 & \rightarrow & K & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & N' & \rightarrow & B'_{n-1} & \rightarrow & \cdots & \rightarrow & B'_0 & \rightarrow & K' & \rightarrow & 0
\end{array}
$$

In the following lines, we define an equivalence relation for studying the classes of exact sequences of length $n$. As we did for the case with $n = 1$, we start by saying that two exact sequences $\eta, \eta' \in \mathcal{E}^n_\mathcal{C}(C,A)$ are related, and denoted also by $\eta \equiv \eta'$,
if there is a morphism \((1_A, \beta_{n-1}, \ldots, \beta_0, 1_C) : \eta \to \eta'\). In this case, we say also that this morphism has fixed ends. Observe that, in contrast with the case \(n = 1\), this relation needs not to be symmetric. Thus, for achieving our goal, we most consider the equivalence relation \(\equiv\) induced by \(\preceq\). Namely, we write \(\eta \equiv \eta'\) if there are exact sequences \(\eta_1, \ldots, \eta_k\) such that

\[\eta = \eta_1, \quad \eta_i \preceq \eta_{i+1} \text{ or } \eta_{i+1} \preceq \eta_i, \quad \text{and} \quad \eta' = \eta_k.\]

**Definition 2.14.** \([11, IV.9.]\) For \(n \geq 1\) and \(A, C \in C\), we consider the class \(\text{Ext}^n_C(C, A) := E^n_C(C, A) / \equiv\), whose elements will be called extensions of length \(n\) with \(C\) and \(A\) as right and left ends. Let \(\overline{\eta}\) denote the equivalence class of \(\eta \in E^n_C(C, A)\). An extension morphism from \(\eta\) to \(\eta'\) is just a morphism from \(\eta\) to \(\eta'\).

**Remark 2.15.** The definition of the equivalence relation above might seem naive. But actually the relation is built with the purpose of making the composition of extensions associate properly when there is a morphism acting in the involved extensions \([14, VII.3.],[13, III.5.]\). In the following lines, we will discuss briefly such matter.

Observe how in general, for \(\eta \in E^1_C(C, A)\), \(\eta' \in E^1_D(C', A')\) and \(\beta : C \to C'\) in \(C\), it is false that \((\eta \beta) \eta' = \eta (\beta \eta')\). The only affirmation that can be made is that there is an extension morphism \((\eta \beta) \eta' \to \eta (\beta \eta')\). To show such morphism, we remember that \(\beta\) induces morphisms \(\eta \beta \to \eta\) and \(\eta' \to \beta \eta'\). Hence, we can build the morphisms

\[(\eta \beta) \eta' \to \eta \eta' \quad \text{and} \quad \eta \eta' \to \eta (\beta \eta'),\]

whose composition gives the wanted morphism. Therefore, even if \((\eta \beta) \eta' \neq \eta (\beta \eta')\) we can conclude that \((\eta \beta) \eta' = \eta (\beta \eta')\).

**Definition 2.16.** \([14, VII.3.]\) Consider the following exact sequences of length \(n\) and \(m\), respectively

\[
\eta : 0 \to N \xrightarrow{\mu} B_n \to \cdots \to B_1 \xrightarrow{\pi} K \to 0,
\]

\[
\eta' : 0 \to K \xrightarrow{\nu} B'_n \to \cdots \to B'_1 \xrightarrow{\pi'} L \to 0.
\]

The composition sequence \(\eta \eta'\), of \(\eta\) with \(\eta'\), is the exact sequence

\[0 \to N \xrightarrow{\mu} B_n \to \cdots \to B_1 \xrightarrow{\nu \pi} B'_m \to \cdots \to B'_1 \xrightarrow{\pi'} L \to 0.
\]

**Remark 2.17.** Note that each exact sequence in \(C\)

\[
\kappa : 0 \to A \to B_n \to \cdots \to B_1 \to C \to 0
\]

can be written as a composition of \(n\) short exact sequences \(\kappa = \eta_n \cdots \eta_1\), where

\[
\eta_i := 0 \to K_{i+1} \to B_i \to K_i \to 0,
\]

with \(K_{n+1} := A, K_1 := C\) and \(K_i = \text{Im}(B_i \to B_{i-1}) \forall i \in [1, n-1]\). We will refer to such factorization of \(\kappa\) as its natural decomposition.

Of course, the composition of exact sequences induces a composition of extensions.
Definition 2.18. [14 VII.3.1.] Let $m, n > 0$, and $A, C, D \in \mathcal{C}$. Then, the correspondence $\Phi : \text{Ext}^n_{\mathcal{C}} (C, A) \times \text{Ext}^m_{\mathcal{C}} (D, C) \to \text{Ext}^{n+m}_{\mathcal{C}} (D, A)$, $(\eta, \eta') \mapsto \overline{\eta \eta'}$, is a function.

We can now define without ambiguity the composition of extensions.

Definition 2.19. Let $E \in \text{Ext}^n_{\mathcal{C}} (C, A)$ and $E' \in \text{Ext}^m_{\mathcal{C}} (D, C)$. For $E = \eta$ and $E' = \eta'$, we define the composition extension $EE'$ of $E$ with $E'$, as $EE' := \eta \eta'$. If $\eta = \eta_n \cdots \eta_1$ is the natural decomposition of $\eta$, the induced extension factorization $E = \overline{\eta_m} \cdots \overline{\eta_1}$ is known as a natural decomposition of $E$.

In the same way, an $n$-extension can be factored into simpler extensions; a morphism of $n$-extensions can be factored into a composition of $n$ simpler morphisms. The next lemma shows the basic fact in this matter.

Lemma 2.20. [14 VII.1.1.] Let $(\alpha, \beta, \gamma) : \eta' \to \eta$ be a morphism of short exact sequences, with
\[
\begin{array}{c}
\eta : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \text{ and} \\
\eta' : 0 \to A' \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0.
\end{array}
\]
Then, $\eta \gamma = \alpha \eta'$ and $(\alpha, \beta, \gamma)$ factors through $\eta \gamma$ as
\[
(\alpha, \beta, \gamma) = (1, \beta', \gamma)(\alpha, \beta'', 1).
\]

In general, we can make the following affirmation.

Corollary 2.21. Let $\eta, \eta' \in \mathcal{E}^n_{\mathcal{C}} (C, A)$ be exact sequences with natural decompositions $\eta = \eta_n \cdots \eta_1$ and $\eta' = \eta'_n \cdots \eta'_1$. Then, the following statements hold true.

(a) There is an exact sequence morphism $(\alpha_{n-1}, \cdots, \beta_0, \gamma) : \eta \to \eta'$ if, and only if, there is a collection of extension morphisms
\[
(\alpha_i, \beta_{i-1}, \alpha_{i-1}) : \overline{\eta_i} \to \eta_i' \forall i \in [1, n]
\]
where $\alpha_n = \alpha$ and $\alpha_0 = \gamma$.

(b) If there is an exact sequence morphism $(\alpha_{n-1}, \cdots, \beta_0, \gamma) : \eta \to \eta'$, then there is a collection of morphisms $\alpha_{n-1}, \cdots, \alpha_1$ in $\mathcal{C}$ satisfying the following equalities:

\[
\begin{align*}
& (b1) \overline{\eta_n} \cdots \overline{\eta_i} \gamma = \alpha \overline{\eta_i} \cdots \overline{\eta_1}, \text{ and} \\
& (b2) \overline{\eta_i} \cdots \overline{\eta_1} \gamma = \alpha_i \eta_i \cdots \eta_1 \text{ and } \overline{\eta_n} \cdots \overline{\eta_{i+1}} \alpha_i = \alpha \eta_n \cdots \eta_{i+1} \forall i \in [1, n-1].
\end{align*}
\]

Proof. It follows from (2.20) \qed

By Lemma 2.18 the following actions are well defined.

Definition 2.22. [14 VII.3.] Consider $\eta, \eta' \in \mathcal{E}^n_{\mathcal{C}} (C, A)$, $E := \eta \in \text{Ext}^n_{\mathcal{C}} (C, A)$, $E' := \eta' \in \text{Ext}^n_{\mathcal{C}} (C, A)$, and let $\eta = \eta_n \cdots \eta_1$ and $\eta' = \eta'_n \cdots \eta'_1$ be the natural decompositions of $\eta$ and $\eta'$.
(a) Given $\alpha \in \text{Hom}_C(A, A')$, we define $\alpha E := \alpha \eta_{m} \cdots \eta_{1}$.
(b) Given $\gamma \in \text{Hom}_C(C', C)$, we define $E\gamma := \eta_{m} \cdots \eta_{1}$.
(c) We define the sum of extensions of length $n$ in the following way

\[ E + E' := \nabla_A (E \oplus E') \Delta_C. \]

Most of the properties, proved earlier for extensions of length 1, can be naturally extended, as can be seen in the following lines.

**Corollary 2.23.** [14, VII.3.2-3.3.] Let $n > 0$.

(a) If $E \in \text{Ext}_C^n(C, A)$, $E' \in \text{Ext}_C^m(D, C')$, $\beta \in \text{Hom}_C(C', C)$, $\beta' \in \text{Hom}_C(C'', C')$, $\alpha \in \text{Hom}_C(A, A')$, and $\alpha' \in \text{Hom}_C(A', A'')$, then the following equalities hold true:

1. $(E \beta) E' = E (\beta E')$;
2. $1_A E = E = E1_C$;
3. $E (\beta \beta') = (E \beta) \beta'$;
4. $(\alpha \alpha') E = \alpha' (\alpha E)$.

(b) If $E \in \text{Ext}_C^n(C, A)$, $E' \in \text{Ext}_C^m(C', A')$, $F \in \text{Ext}_C^m(D, C)$, $F' \in \text{Ext}_C^m(D', C')$, $\alpha \in \text{Hom}_C(A, X)$, $\alpha' \in \text{Hom}_C(A', X')$, $\gamma \in \text{Hom}_C(Y, C)$, and $\gamma' \in \text{Hom}_C(Y', C')$, then the following equalities hold true:

1. $(\alpha \oplus \alpha') (E \oplus E') = \alpha E \oplus \alpha' E'$ and $(E \oplus E') (\gamma \oplus \gamma') = \gamma E \oplus \gamma' E'$;
2. $(E \oplus E') (F \oplus F') = EF \oplus EF'$;
3. $(E + E') F = EF + E'F$ and $E (F + F') = EF + EF'$;
4. $(\alpha \oplus \alpha') E = \alpha E + \alpha' E$ and $E (\gamma + \gamma') = E \gamma + E \gamma'$; and
5. $(E + E') = \alpha E + \alpha E'$ and $(E + E') \gamma = E \gamma + E \gamma'$.

(c) The pair $(\text{Ext}_C^n(C, A), +)$ is an abelian group, where the identity element is the extension $E_0$ given by the exact sequence, in case $n \geq 2$,

\[ 0 \to A \xrightarrow{1} A \xrightarrow{0} \cdots \xrightarrow{0} C \xrightarrow{1} C \to 0. \]

We conclude this section with the following theorem that focuses on characterizing the trivial extensions.

**Theorem 2.24.** [14, VII.4.2.] Let $n > 1$ and $\eta \in \mathcal{E}_C^n(C, A)$ with a natural decomposition $\eta = \eta_{m} \cdots \eta_{1}$. Then, the following statements hold true:

(a) $\eta = 0$;
(b) there is an extension $\kappa \in \mathcal{E}_C^n(C, A)$ and a pair of morphisms with fixed ends

\[ 0 \leftarrow \kappa \to \eta. \]

(c) there is an extension $\kappa' \in \mathcal{E}_C^n(C, A)$ and a pair of morphisms with fixed ends

\[ 0 \to \kappa' \leftarrow \eta. \]

3. **Additional structure**

In this section we will approach our problem dealing with arbitrary products and coproducts. Of course, an abelian category does not necessarily have arbitrary
products and coproducts. Hence, we will review briefly the theory of abelian categories with additional structure introduced by A. Grothendieck in \cite{9}. For further reading we suggest \cite{15} 2.8.

### 3.1. Limits and colimits.

**Definition 3.1.** \cite{15} 1.4.] Let $\mathcal{C}$ and $I$ be categories, where $I$ is small (that is the class of objects of $I$ is a set). Let $F : I \to \mathcal{C}$ be a functor and $X \in \mathcal{C}$. A family of morphisms $\{\alpha_i : F(i) \to X\}_{i \in I}$ in $\mathcal{C}$ is co-compatible with $F$, if $\alpha_i = \alpha_j F(\lambda)$ for every $\lambda : i \to j$ in $I$.

The colimit (or inductive limit) of $F$ is an object $\text{colim} \ F$ with a co-compatible family of morphisms in $\mathcal{C}$ $\{\mu_i : F(i) \to \text{colim} \ F\}_{i \in I}$, such that for every co-compatible family of morphisms $\{\gamma_i : F(i) \to X\}_{i \in I}$, there is a unique morphism $\gamma : \text{colim} \ F \to X$ such that $\gamma_i = \gamma \mu_i$ for every $i \in I$.

Let $I$ be a small category and $\lambda : i \to j$ be a morphism in $I$. The following notation will be useful $s(\lambda) := i$ and $t(\lambda) := j$.

**Proposition 3.2.** \cite{19} IV.8.4] Let $\mathcal{C}$ be a preadditive category with coproducts and cokernels, $I$ be a small category, $F : I \to \mathcal{C}$ be a functor, and $u_k : F(k) \to \bigoplus_{i \in I} F(i)$, $\forall k \in I$, $v_\lambda : F(s(\lambda)) \to \bigoplus_{\gamma \in H} F(s(\gamma))$, $\forall \lambda \in H := \text{Hom}_I$ be the respective canonical inclusions into the coproducts. Then,

$$\text{colim} \ F = \text{Coker} \left( \bigoplus_{\gamma \in H} F(s(\gamma)) \xrightarrow{\varphi} \bigoplus_{i \in I} F(i) \right),$$

where $\varphi$ is the morphism induced by the universal property of coproducts applied to the family of morphisms $\{\varphi_\lambda := u_{s(\lambda)} - u_{t(\lambda)} F(\lambda)\}_{\lambda \in H}$.

The dual notion of colimit is the limit.

**Definition 3.3.** \cite{15} 1.4.] Let $\mathcal{C}$ and $I$ be categories, with $I$ small. Let $F : I \to \mathcal{C}$ be a functor and $X \in \mathcal{C}$. A family of morphisms $\{\alpha_i : X \to F(i)\}_{i \in I}$ in $\mathcal{C}$ is compatible with $F$, if $\alpha_i = \alpha_j F(\lambda)$ for every $\lambda : i \to j$ in $I$.

The limit (or projective limit) of $F$ is an object $\lim \ F \in \mathcal{C}$ together with a compatible family of morphisms $\{\mu_i : \lim \ F \to F(i)\}_{i \in I}$ such that for any compatible family of morphisms $\{\gamma_i : X \to F(i)\}_{i \in I}$ there is a unique morphism $\gamma : X \to \lim \ F$ such that $\gamma_i = \mu_i \gamma$ for every $i \in I$. 

\[\xymatrix{ F(i) \ar@{>->}[r]_(0.3){\mu_i} & F(j) \\
\gamma_i \ar@{>->}[r] & \gamma_j \ar[u]_F} \]
Proposition 3.4. [19, IV.8.2] Let $\mathcal{C}$ be a preadditive category with products and kernels, $I$ be a small category, $F : I \rightarrow \mathcal{C}$ be a functor, and

$u_k : \prod_{i \in I} F(i) \rightarrow F(k) \quad \forall k \in I, \quad v_\lambda : \prod_{\gamma \in H} F(t(\gamma)) \rightarrow F(t(\lambda)) \quad \forall \lambda \in H := \text{Hom}_I$

be the respective canonical projections out of the products. Then,

$$\lim F = \text{Ker} \left( \prod_{i \in I} F(i) \xrightarrow{\phi} \prod_{\gamma \in H} F(t(\gamma)) \right),$$

where $\phi$ is the morphism induced by the universal property of products applied to the family of morphisms

$$\{ \varphi_\lambda := F(\lambda)u_{s(\lambda)} - u_{t(\lambda)} \}_{\lambda \in H}.$$

Definition 3.5. Let $I$ be a small category and $\mathcal{C}$ be an abelian category. It is said that a family of objects and morphisms

$$(M_i, f_\alpha)_{i \in I, \alpha \in \text{Hom}_I}$$

is a direct system if there is a functor $F : I \rightarrow \mathcal{C}$ such that $F(i) = M_i \forall i \in I$ and $F(\alpha) = f_\alpha \forall \alpha \in \text{Hom}_I$.

3.2. Ab3 and Ab4 Categories.

Definition 3.6. [15, 2.8.] An Ab3 category is an abelian category satisfying the following condition:

$$(\text{Ab3})$$

For every set of objects $\{A_i\}_{i \in I}$ in $\mathcal{C}$ the coproduct $\bigoplus_{i \in I} A_i$ exists.

We will refer to the dual condition as Ab3*.

We remember the following well known fact.

Proposition 3.7. [15, 2.8.] Let $\mathcal{C}$ be an Ab3 category and

$$\left\{ X_i' \xrightarrow{f_i} X_i \xrightarrow{g_i} X_i'' \xrightarrow{0} \right\}_{i \in I}$$

be a set of exact sequences in $\mathcal{C}$. Then,

$$\bigoplus_{i \in I} X_i' \xrightarrow{\bigoplus f_i} \bigoplus_{i \in I} X_i \xrightarrow{\bigoplus g_i} \bigoplus_{i \in I} X_i'' \rightarrow 0$$

is an exact sequence in $\mathcal{C}$.

In general, it is not possible to prove that $\bigoplus f_i$ is a monomorphism if each $f_i$ is a monomorphism. For this reason, the following Grothendieck’s condition arised.

Definition 3.8. [15, 2.8.3.] An Ab4 category is an Ab3 category $\mathcal{C}$ satisfying the following condition:
(Ab4): for every set of monomorphisms \( \{ f_i : X_i \to Y_i \}_{i \in I} \) in \( C \), the morphism \( \bigoplus_{i \in I} f_i \) is a monomorphism.

Remark 3.9. Let \( C \) be an Ab4 category. Then, for every sets of objects \( \{ A_i \}_{i \in I} \) and \( \{ B_i \}_{i \in I} \) in \( C \), the correspondence

\[
C : \prod_{i \in I} \text{Ext}^n_{C}(A_i, B_i) \to \text{Ext}^n_{C}\left( \bigoplus_{i \in I} A_i, \bigoplus_{i \in I} B_i \right), (\eta_i) \mapsto \bigoplus_{i \in I} \eta_i,
\]

is a well defined morphism of abelian groups.

3.3. Ext groups and arbitrary products and coproducts. We are finally ready to proceed in our goal’s direction.

Lemma 3.10. Let \( C \) be an Ab4 category, and

\[
\left\{ \eta_i : 0 \to B \xrightarrow{\beta} A_i \xrightarrow{\mu_i} C_i \to 0 \right\}_{i \in I}
\]

be a set of short exact sequences in \( C \). Then, there is a short exact sequence

\[
\eta : 0 \to B \xrightarrow{\beta} \text{colim}(f_i) \xrightarrow{\Phi} \bigoplus_{i \in I} C_i \to 0
\]

such that \( \eta_{\mu_i} = \eta_i \forall i \in I \), where \( \{ \mu_i : C_i \to \bigoplus_{i \in I} C_i \}_{i \in I} \) is the family of canonical inclusions.

Proof. Consider the set \( \{ f_i : B \to A_i \}_{i \in I} \) as a direct system. Observe that the set of morphisms of exact sequences

\[
\{(1_B, f_i, 0) : \beta \to \eta_i \}_{i \in I}
\]

with \( \beta := 0 \to B \xrightarrow{\beta} B \xrightarrow{0} 0 \to 0 \),

is a direct system of exact sequences. We will consider the colimit of such system and prove that, as result, we get a short exact sequence. To this end, we observe that \( (B, 1_B : B \to B) \in I \) is the colimit of the system \( \{ 1_B : B \to B \}_{i \in I} \) and that \( (\bigoplus_{i \in I} C_i, \mu_i : C_i \to \bigoplus_{i \in I} C_i) \) is the colimit of the system \( \{ 0 : 0 \to C_i \}_{i \in I} \).

Hence, by 3.2 we build the diagram beside, where the columns are the morphism mentioned in 3.2 the upper and central rows are coproducts of the sequences \( \beta \) and \( \eta_i \) respectively, and the bottom row is the result of the colimits. Thus, by the snake lemma we get the exact sequence

\[
\eta : 0 \to B \to \text{colim}(f_i) \to \bigoplus_{i \in I} C_i \to 0.
\]
Furthermore, the families of morphisms associated to such colimits give us the exact sequence morphisms
\[(1, \mu'_i, \mu_i) : \eta_i \to \eta \forall i \in I,\]
which proves the statement. □

**Proposition 3.11.** Let \(\mathcal{C}\) be an Ab4 category and \(\{A_i\}_{i \in I}\) a set of objects in \(\mathcal{C}\). Consider the coproduct with the canonical inclusions \((\mu_i : A_i \to \bigoplus_{i \in I} A_i)_{i \in I}\). Then, the correspondence \(\Psi : \text{Ext}_{\mathcal{C}}^1(\bigoplus A_i, B) \to \prod_{i \in I} \text{Ext}_{\mathcal{C}}^1(A_i, B), E \mapsto (E\mu_i)_{i \in I}\), is an isomorphism for every \(B \in \mathcal{C}\).

**Proof.** We will proceed by proving the following steps:

(a) The correspondence \(\Psi\) is an abelian group morphism; 
(b) \(\Psi\) is injective; 
(c) given \((\eta_i) \in \prod_{i \in I} \text{Ext}_{\mathcal{C}}^1(A_i, B)\) with \(\eta_i : 0 \to B \overset{f_i}{\to} C_i \overset{g_i}{\to} A_i \to 0\), there is \(E \in \text{Ext}_{\mathcal{C}}^1(\bigoplus A_i, B)\) such that \(\Psi(E) = (\eta_i)\).

Clearly proving such statements are enough to conclude the desired lemma.

(a) It follows by 2.21.
(b) Suppose that \(E\) is an extension with representative
\[\eta : 0 \to B \overset{f}{\to} C \overset{g}{\to} \bigoplus_{i \in I} A_i \to 0\]
such that \(E\mu_i = 0\) for every \(i \in I\). Suppose that \((1, p_i, \mu_i) : E\mu_i \to E\) is the morphism induced by \(\mu_i\), and that each extension \(E\mu_i\) has as representative the exact sequence
\[\eta_i : 0 \to B \overset{f_i}{\to} C_i \overset{g_i}{\to} A_i \to 0.\]
By definition, there is a morphism \(h_i : A_i \to C_i\) such that \(g_i h_i = 1_{A_i}\).

\[
\begin{array}{c}
0 \to B \overset{f}{\to} C \overset{g}{\to} \bigoplus_{i \in I} A_i \to 0 \\
0 \to B \overset{f}{\to} C \overset{g}{\to} \bigoplus_{i \in I} A_i \to 0
\end{array}
\]
Thus, by the coproduct universal property, there is a unique morphism \(h : \bigoplus_{i \in I} A_i \to C\) such that \(h\mu_i = p_i h_i \forall i \in \{1, 2\}\). Therefore, by the following equalities
\[gh\mu_i = gp_i h_i = \mu_i g_i h_i = \mu_i \forall i \in I\]
we have that \(gh = 1_{\bigoplus_{i \in I} A_i}\) by the coproduct universal property; and thus, \(E = 0\).
(c) It follows by 3.10.

THEOREM 3.12. Let \(\mathcal{C}\) be an Ab4 category, \(n \geq 1\), and \(\{A_i\}_{i \in I}\) be a set of objects in \(\mathcal{C}\). Consider the coproduct with the canonical inclusions \((\mu_i : A_i \to \bigoplus_{i \in I} A_i)_{i \in I}\). Then, the correspondence \(\Psi_n : \text{Ext}_{\mathcal{C}}^n(\bigoplus A_i, B) \to \prod_{i \in I} \text{Ext}_{\mathcal{C}}^n(A_i, B), E \mapsto (E\mu_i)_{i \in I}\), is an isomorphism of abelian groups for every \(B \in \mathcal{C}\).
Proof. We will proceed by proving the following statements:

(a) the correspondence $\Psi_n$ is a morphism of abelian groups;
(b) $\Psi_n$ is injective;
(c) given $\eta \in \prod_{i \in I} \text{Ext}_n^C (A_i, B)$, there is $E \in \text{Ext}_1^C (\bigoplus A_i, B)$ such that $\Psi_n (E) = (\eta_i)$.

It is worth to mention that the result was already proved in 3.11 for $n = 1$. Furthermore, in the proof of 3.11(c) it was shown explicitly the inverse function of $\Psi_1$. We will denote such correspondence as $\Psi_1^{-1}$.

(a) It follows by 2.21.
(b) Suppose that $\eta$ is an extension with a natural decomposition $\eta = \eta_n \cdots \eta_1$ such that $\eta_i = 0 \forall i \in I$. By 2.24 this means that for every $i \in I$ there is a pair of exact sequences morphisms with fixed ends $\eta_i \leftarrow \kappa_i \rightarrow 0$. Suppose that each exact sequence $\kappa_i$ has as natural decomposition the expression $\kappa_i = \kappa(i)_n \cdots \kappa(i)_1$.

Let $\kappa(i)_n = \kappa(i)_1' : 0 \rightarrow B \xrightarrow{f_i} Y_i \xrightarrow{g_i} X_i \rightarrow 0$.

By 3.11(c), we know $\Psi_1^{-1}(\kappa(i)_1') \in \text{Ext}_1^C (\bigoplus_{i \in I} X_i, B)$ is an extension such that $\Psi_1^{-1}(\kappa(i)_1') = \kappa(i)_1' \forall i \in I$, where each $\kappa(i)_1' : X_i \rightarrow \bigoplus_{i \in I} X_i$ is one of the canonic inclusions of the coproduct. Let

$$\kappa := \Psi_1^{-1}(\kappa(i)_1') \left( \bigoplus_{i \in I} \kappa(i)_1 \right) \cdots \left( \bigoplus_{i \in I} \kappa(i)_1 \right).$$

We will show that there is a pair of exact sequence morphisms with fixed ends $\eta \leftarrow \kappa \rightarrow 0$, which will prove (b) by 2.24. Indeed, by the fact that for every $i \in I$ there is a morphism with fixed ends $\eta_i \leftarrow \kappa_i$, it follows that there is a morphism with fixed right end

$$\eta_n \cdots \eta_1 \mu_i \leftarrow \kappa(i)_n \cdots \kappa(i)_1,$$

inducing by the coproduct universal property a morphism with fixed right end

$$\eta_n \cdots \eta_1 \leftarrow \left( \bigoplus_{i \in I} \kappa(i)_n \right) \cdots \left( \bigoplus_{i \in I} \kappa(i)_1 \right).$$

Furthermore, by the proof of 3.11 we know that $\Psi_1^{-1}(\kappa(i))$ has as representative the exact sequence

$$0 \rightarrow B \xrightarrow{f_i} \text{colim}(f_i) \xrightarrow{g_i} \bigoplus_{i \in I} X_i \rightarrow 0.$$

Hence, using the colimit universal property, is easy to see there is a morphism with fixed left end

$$\eta_n \leftarrow \Psi_1^{-1}(\kappa(i)).$$
Therefore, with the last morphisms we can build a morphism with fixed ends
\[ \eta \leftarrow \kappa. \]
For showing the existence of a morphism with fixed ends \( \kappa \rightarrow 0 \), it is enough to show that \( f \) is a splitting monomorphism, which follows straightforward from the colimit universal property together with the fact that every \( f_i \) is a splitting monomorphism. The later explanation is illustrated in the diagram below.

(c) Let \( (\eta_i) \in \prod_{i \in I} \text{Ext}^0_i(\mathcal{A}_i, \mathcal{B}) \). We observe the following for each \( i \in I \).
Suppose \( \eta_i = \kappa_i^n \cdots \kappa_i^1 \) is a natural decomposition, where
\[
\kappa_i^k : 0 \rightarrow B_i^{k+1} \rightarrow C_i^k \rightarrow B_i^k \rightarrow 0 \quad \forall k \in \{n, \cdots, 1\}.
\]
Consider the coproduct canonic inclusions \( u_i^k : B_i^k \rightarrow \bigoplus_{i \in I} B_i^k \). Observe that \( u_i^1 = \mu_i \forall i \in I \). By 2.20 we can see that
\[
\bigoplus_{i \in I} \kappa_i^k u_i^k = u_i^{k+1} \kappa_i^k \forall k \in \{1, \cdots, n + 1\}.
\]
Hence, by 3.11(c), the extension
\[
\eta := \Psi^{-1}_1(\kappa_i^n) \bigoplus_{i \in I} \kappa_i^{n-1} \cdots \bigoplus_{i \in I} \kappa_i^1,
\]
satisfies by recursion the following equalities
\[
\eta \mu_i = \eta u_i^1
\]
\[
= \Psi^{-1}_1(\kappa_i^n) \bigoplus_{i \in I} \kappa_i^{n-1} \cdots \bigoplus_{i \in I} \kappa_i^1 u_i^1
\]
\[
= \Psi^{-1}_1(\kappa_i^n) \bigoplus_{i \in I} \kappa_i^{n-1} \cdots \bigoplus_{i \in I} \kappa_i^2 u_2 \kappa_i^1
\]
\[
= \Psi^{-1}_1(\kappa_i^n) \bigoplus_{i \in I} \kappa_i^{n-1} \cdots \bigoplus_{i \in I} \kappa_i^3 u_3 \kappa_i^2 \kappa_i^1
\]
\[
\vdots
\]
\[
= \Psi^{-1}_1(\kappa_i^n) \bigoplus_{i \in I} \kappa_i^{n-1} u_{n-1} \kappa_i^{n-2} \cdots \kappa_i^1
\]
\[
= \Psi^{-1}_1(\kappa_i^n) u_n \kappa_i^{n-1} \cdots \kappa_i^1
\]
\[
= \kappa_i^n \kappa_i^{n-1} \cdots \kappa_i^1
\]
\[
= \eta_i.
\]
\[\square\]
Diagram illustrating the proof of 3.12(b).
By duality we have the following result.

**Theorem 3.13.** Let $\mathcal{C}$ be an Ab4* category, $n \geq 1$, and $\{A_i\}_{i \in I}$ be a set of objects in $\mathcal{C}$. Consider the product with the canonical projections $(\pi_i : \prod_{i \in I} A_i \rightarrow A_i)_{i \in I}$. Then, the correspondence $\Phi_n : \text{Ext}^n_{\mathcal{C}}(B, \prod_{i \in I} A_i) \rightarrow \prod_{i \in I} \text{Ext}^n_{\mathcal{C}}(B, A_i)$, defined by $E \mapsto (\pi_i E)_{i \in I}$, is an isomorphism of abelian groups for every $B \in \mathcal{C}$.

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**Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Ciudad Universitaria, C.P.04510 Mexico City, Mexico.**

**E-mail address:** argudin@ciencias.unam.mx