ON VANISHING OF ALL FOURFOLD PRODUCTS OF THE RAY CLASSES IN SYMPLECTIC COBORDISM.

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Abstract. This note provides certain computations with transfer associated with projective bundles of Spin vector bundles. One aspect is to revise the proof of the main result of [2] which says that all fourfold products of the Ray classes are zero in symplectic cobordism.

1. Introduction

The Ray classes \( \phi_i \in MSp_{8i-3} \) are indecomposable torsion elements of order two in symplectic bordism ring. \( \phi_i \) arise from the expansion of Conner-Floyd symplectic Pontryagin class

\[
pf_1((\eta^1 - \mathbb{R}) \otimes_{\mathbb{R}} (\zeta - \mathbb{H})) = s \sum_{i \geq 1} \theta_i pf_1^i(\zeta)
\]

in \( MSp^4(S^4 \wedge BSP(1)) \), where \( s \) is the generator of \( MSp^4(S^1) = \mathbb{Z} \), \( \eta^1 \rightarrow S^1 \) is the non-trivial real line bundle and \( \zeta \rightarrow BSP(1) \) is the canonical \( Sp(1) \) bundle. The notation \( \theta_i = \phi_i \) is used in the literature because \( \theta_{2i+1} = 0 \), for \( i > 1 \) [9].

The classes \( \phi_i \) play an essential role in the torsion of the symplectic cobordism ring [8, 6, 11]. By [6, 7] one has \( \theta_1 \phi_i \phi_j = 0 \) and \( \phi_{2i+3}^i = 0 \). By [11] most ternary products \( \phi_i \phi_j \phi_k \) are nonzero.

In ([2], Prop. 4.1) we proved the following

Theorem 1.1. i) All fourfold products of the Ray classes \( \phi_i \phi_j \phi_k \phi_l \) are zero;

ii) The images of all double products \( \phi_i \phi_j \) in self-conjugate cobordism are zero.

In this note we revise the proof of Theorem 1.1 as follows. In [2] the arguments of Remark 1.11, Lemma 1.12, and the proof of Proposition 1, (1.1) and (1.2), case \( m = 5 \) are inherited from the references and don’t seem to be true. Still, these statements seem to be the consequences of Theorem 3.1 in [4]. However, all these points are used to derive the proof of Proposition 1 of [2], which we cover in Section 3. To do this, we first carry out some calculations with transfer in symplectic cobordism by using only double coset formula of [5]. For the reader’s convenience, in Section 4 we briefly recall the proof of Theorem 1.1 by pointing to the sequence of necessary propositions of [2].

2. Preliminaries

Recall from [11] the groups \( Spin(n) \) and \( Pin(n) \) that operate on \( \mathbb{R}^n \) by vector representation. We will use an octonionic representation of Clifford algebra \( Cl(8,0) \) and refer to [10]. One has the isomorphism of Clifford algebras

\[
Cl^0(q + 1, p) \simeq Cl(p, q) \simeq Cl^0(p, q + 1)
\]

obtained from extending

\[
e_1 e_{k+1} \leftarrow e_k \rightarrow e_k e_{n+1}, \quad (1 \leq k \leq n).
\]

This defines the inclusions of \( Pin(n) = Pin^0(n) + Pin^1(n) \) in \( Spin(n + 1) \), where \( Pin^0(n) = Spin(n) \).
Let \( \{e_0, e_1, \cdots, e_7\} \) be an orthonormal basis of \( V = \mathbb{R}^8 \). Note that we choose indices ranging from 0 to 7. The octonionic algebra \( \mathbb{O} \) is assumed to be given with basis \( \{i_0, i_1, \cdots, i_7\} \) obeying the multiplication table

\[
\begin{align*}
i_0 &= 1, & i_k^2 &= -1, & i_k i_l &= i_m = -i_l i_k, & 1 \leq k \leq 7, & \text{and cyclic for} \\
(k, l, m) &\in P = \{(1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 6, 4), (2, 5, 7), (3, 4, 7), (3, 5, 6)\}.
\end{align*}
\]

One can identify \( V \) with \( \mathbb{O} \) as vector spaces by \( \sum x^k e_k \rightarrow \sum x^k i_k \).

An octonionic representation \( Cl(8, 0) \rightarrow M_{2(\mathbb{O})} \) is given by

\[
\Gamma_k = \gamma_8(e_k) = \begin{pmatrix} 0 & i_k \\ i_k^* & 0 \end{pmatrix}, \quad 0 \leq k \leq 7.
\]

\[
\Rightarrow \gamma_8(x) = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}, \quad x \in V.
\]

The carrier space of the representation is understood to be \( \mathbb{O}^2 \), i.e., the set of columns of two octonions, with \( \gamma_8 \) acting on it by left multiplication.

Restricting the representation \( Cl(8, 0) \) to \( Cl^0(8, 0) = Cl^0(0, 8) \) produces a faithful representation with the generators

\[
\Gamma_0 \Gamma_k = \gamma_8(e_k) = \begin{pmatrix} i_k & 0 \\ 0 & -i_k \end{pmatrix}, \quad 1 \leq k \leq 7.
\]

So \( Cl^0(0, 8) \) is represented by diagonal matrices. This representation decomposes into two irreducible representations given by the two elements on the diagonal. By the isomorphism \( Cl^0(0, 8) = Cl(0, 7) \) these two are also irreducible representations \( Cl(0, 7) \rightarrow \mathbb{O} \).

Let

\[
\gamma_7 : Cl(0, 7) \rightarrow \mathbb{O}
\]

be the irreducible representation given by the generators

\[
\gamma_7(e_k) = i_k, \quad 1 \leq k \leq 7,
\]

\[
\Leftrightarrow \gamma_7(x) = Imx, \quad x \in Im\mathbb{O},
\]

which act by successive left multiplication on the carrier space \( \mathbb{O} \).

Orthogonal transformations are generated by unit vectors \( u \in Im\mathbb{O} \):

\[
x' = \phi_{(\gamma_7(u))}(x) = u x u^{-1} = -u x u, \quad x \in \mathbb{O}.
\]

By the isomorphism \( Cl^0(0, 7) \cong Cl(0, 6) \), we obtain a faithful and irreducible representation

\[
\gamma_6 : Cl(0, 6) \rightarrow \mathbb{O}:
\]

\[
\gamma_6(e_k) = i_k i_7, \quad 1 \leq k \leq 6,
\]

\[
\gamma_6(u) = u i_7, \quad u \in \mathbb{R}^6.
\]

Orthogonal transformations are generated by

\[
x' = i_k (i_7 x i_7) i_k = i_k ((i_7 x i_7) i_k), \quad x \in \mathbb{O}.
\]

Using (2.1) for \( n = 6, 5, 4, 3 \) one can see how \( Spin(n) \) operates in \( R^n \) identified with the imaginary subspace of \( \mathbb{O} \) with vanishing \( n + 1, \cdots, 7 \)-components.
3. Spin bundles

The following bundles induced by the inclusion of groups
\begin{align*}
(3.1) & \quad i_m : BSpin(m) \to BSpin(m + 1), \\
(3.2) & \quad j_m : BPin(m) \to BSpin(m + 1)
\end{align*}
can be considered as the sphere bundle and the projective bundle of the universal $Spin(m)$ bundle
\begin{equation}
(3.3) \quad \xi^m \to BSpin(m)
\end{equation}
respectively.

Denote the sphere bundle and the projective bundle of a vector bundle $\xi$ by $S(\xi)$ and $P(\xi)$ respectively. In particular we have
\[
S(\xi^m) = BSpin(m), \quad P(\xi^m) = BPin(m)
\]
and the pullback bundles induced by inclusion $Spin(m) \hookrightarrow Spin(m + 1)$,
\begin{align*}
(3.4) & \quad S(\xi^m + 1) \to BSpin(m), \\
(3.5) & \quad P(\xi^m + 1) \to BSpin(m).
\end{align*}

**Lemma 3.1.** Let $\xi^7 \to BSpin(7)$ be the universal $Spin(7)$ bundle as above and let $\xi = 1 + \xi^7$. Let $\pi : P(\xi) \to BSpin(7)$ be the projective bundle of $\xi$:
\[
P(\xi) = ESpin(7) \times_{Spin(7)} RP^7
\]
and let $T_\pi(x)$ be the tangent bundle along the fibers of $\pi$. Then
\[
T_\pi(x) = \pi^* (\xi^7).
\]

**Proof.** Clearly $\phi_7(w)$ induces the action of $Spin(7)$ on $\mathbb{O}$, also on the real projective space
\[
RP^7 = \{ \{\pm x\} | x \in \mathbb{O}, |x| = 1 \}
\]
and on the tangent bundle of $RP^7$:
\[
T_\pi(x) = RP^7 \times R^7 = \{ (x, v(x)) | v(x) = t_1 i_1 x + \cdots + t_7 i_7 x, t_1, \cdots, t_7 \in \mathbb{R} \}.
\]

$Spin(7)$ acts trivially on the line in $\mathbb{R}^8 = \mathbb{O}$ directed by $i_0$. The action on pure octonions defines the universal $Spin(7)$ bundle $\xi^7$.

This defines
\[
T_\pi(x) = ESpin(7) \times_{Spin(7)} \tau_F
\]
and the bundle map
\[
ESpin(7) \times_{Spin(7)} \tau_F \to ESpin(7) \times_{Spin(7)} R^7 = \xi^7,
\]
which classifies $\pi^* (\xi^7)$. \qed

It is well known that $RP^7$ is parallelizable, i.e., admits 7 linearly independent tangent vector fields $\langle \{ \pm p, \pm p i_1 \}, \cdots, \{ \pm p, \pm p i_7 \} \rangle$, where $i_k$ are the octonionic units.

**Lemma 3.2.** There are $7 - k$ number $Spin(k)$-equivariant linearly independent tangent vector fields on $RP^7$, namely $\langle \{ \pm p, \pm p i_{k+1} \}, \cdots, \{ \pm p, \pm p i_7 \} \rangle$, where $k = 2, \cdots, 6$.

**Proof.** Let $k = 6$ and let us check that the vector field $\langle \{ \pm p, \pm p i_7 \} \rangle$ on $RP^7$ is invariant under action of $Spin(6) \subset SO_6^0$. Using Moufang identities
\begin{align*}
(3.6) & \quad (xyz)z = x(y(xz)); \\
(3.7) & \quad z(xy) = ((zx)y)x; \\
(3.8) & \quad x(yz)x = (xy)(zx)
\end{align*}
one has for \( \{ \pm p \} = \pm \{ t_0 i_0 + t_1 i_1 + \cdots t_\tau i_\tau \} \)
\[
\phi_{\gamma_0(e_k)}(p) = i_k((i_\tau p i_\tau) i_k) = i_k(i_\tau(t_0 i_0 + t_1 i_1 + \cdots t_\tau i_\tau) i_k)
= i_k(\sum_{n \neq 0, k} t_n i_n - t_0 i_0 - t_\tau i_\tau) i_k = \sum_{n \neq k, 7} t_n i_n - t_k i_k - t_\tau i_\tau.
\]
\[
\Rightarrow \phi_{\gamma_0(e_k e_k)}(p) = \sum_{n \neq j, k} t_n i_n - t_j i_j - t_k i_k;
\]
\[
\Rightarrow \phi_{\gamma_0(e_k)}(i_\tau) = -i_\tau;
\]
\[
\Rightarrow \phi_{\gamma_0(e_k e_k)}(i_\tau) = i_\tau;
\]

\[
\phi_{\gamma_0(e_k)}(p i_\tau) = (i_k i_\tau)(p i_\tau)(i_\tau i_k)
= ((i_k i_\tau)p)(i_\tau(i_\tau i_k))
= -(i_k i_\tau)p i_k
= ((i_\tau i_k)p) i_k
= i_\tau(i_k p i_k)
= i_\tau(i_k(t_0 i_0 + t_1 i_1 + \cdots t_\tau i_\tau) i_k)
= i_\tau(\sum_{n \neq 0, k} t_n i_n - t_0 i_0 - t_k i_k)
= -(\sum_{n \neq k, 7} t_n i_n + t_k i_k + t_\tau i_\tau) i_\tau
\]
\[
\Rightarrow \phi_{\gamma_0(e_k e_k)}(p i_\tau) = (\sum_{n \neq j, k} t_n i_n - t_j i_j - t_k i_k) i_\tau.
\]
\[
\Rightarrow \phi_{\gamma_0(e_k e_k)}(p i_\tau) = \phi_{\gamma_0(e_k)}(p i_\tau).
\]

Similarly for \( k = 5, 4, 3. \)

\[\square\]

**Corollary 3.3.** Let \( \xi^k \) be the universal \( Spin(k) \) bundle, \( k = 2, \cdots, 6. \) Then the tangent bundle along the fibers \( RP^7 \) of the projective bundle

\[ \tilde{\pi} : P(8 - k + \xi^k) \to BSpin(k), \]

admits \((7 - k)\) linearly independent sections

\[ T_F(\xi^k + 8 - k) = \tilde{\pi}^*(\xi^k) + 7 - k. \]

**Proof.** Apply Lemma 3.1. For the standard inclusion \( i_k : BSpin(k) \to BSpin(7) \) one has

\[ i_k^*(\xi^7) = \xi^k + 7 - k, \]

therefore

\[ T_F(i_k^*(\xi^7 + 1)) = i_k^*(\tilde{\pi}^*(\xi^7)) \]
\[ \Leftrightarrow T_F(\xi^k + 8 - k) = \tilde{\pi}^*(\xi^k) + 7 - k. \]

Alternatively one can apply Lemma 3.2 to define \((7 - k)\)-sections of the tangent bundle along the fibers of \( T_F(i_k^*(\xi)). \)

\[\square\]

Let \( Tr^*(i_{m-1}) \) and \( Tr^*(j_{m-1}) \) be the transfer homomorphism of (3.1) and (3.2) respectively. Then by naturality of the transfer \( i_m^* Tr^*(j_m) \) is the transfer homomorphism of (3.6).
Lemma 3.4. Let $2 \leq m \leq 7$. The transfer homomorphism of (3.3) is the sum of three components,

$$i^*_m Tr^*(j_m) = Tr^*(j_{m-1}) + Tr^*(i_{m-1}) + \text{Id},$$

This corresponds to the endpoints and the interior of the orbit type manifold

$$\text{Spin}(m)|\text{Spin}(m+1)|\text{Pin}(m)$$

which is the line segment. The corresponding isotropy groups are: $\text{Spin}(m)$ at one endpoint, $\text{Pin}(m-1)$ at another endpoint, and $\text{Spin}(m-1)$ for the points in the interior.

Proof. Lemma 3.3 coincides with Lemma 1.9 and Lemma 1.10 of [2] for $m = 4$ and $m = 3$ respectively. However for all cases it is convenient to use the octonionic representation of Clifford algebras in Section 2.

By naturality of the transfer map $i^*_m Tr(j_m)$ coincides with transfer homomorphism of (3.5).

Let $m = 7$. We consider $RP^7$ as $S^7 = S^6 \cap \{x_0 \geq 0\}$ with identified antipodal points in $S^6 = S^7 \cap \{x_0 = 0\}$. Parametrize $S^6$ as

$$v = \cos t \cdot i_0 + \sin t \cdot x, \quad x \in S^6 \subset \text{Im}\mathcal{O}, \quad 0 \leq t \leq \pi/2.$$  

Then as above $i_0$ is invariant under action of $\text{Spin}(7)$ and we have

$$v' = i_k (i_7 (\cos t \cdot i_0 + \sin t \cdot x)) i_k = \cos t \cdot i_0 + \sin t \cdot i_k (i_7 x i_7) i_k.$$  

So the orbit space of the action of $\text{Spin}(7)$ on $RP^7$ is the line segment $[0, \pi/2]$; we have three types of orbits: the endpoint $t = 0$ corresponds to the pole $e_0$, with the isotropy group $\text{Spin}(7)$. The endpoint $t = \pi/2$ corresponds to the orbit $RP^5 = \{ \pm x \}$, its points have the isotropy groups conjugate to $\text{Pin}(6)$, the isotropy group of $\{ \pm i_7 \}$. Each point $t \in (0, \pi/2)$ corresponds to the orbit $\cos t \cdot e_0 + \sin t \cdot x$, the sphere, consisting of points with the isotropy group conjugate to $\text{Spin}(6)$.

Now let $m = 6$ and consider $RP^6$ as $S^6 = S(\text{Im}\mathcal{O}) \cap \{x_7 \geq 0\}$ with identified antipodal points in $S^6 = S^7 \cap \{x_7 = 0\}$. Parametrize $S^6$ as

$$v = \cos t \cdot i_7 + \sin t \cdot x, \quad x \in S^5, \quad 0 \leq t \leq \pi/2.$$  

As above $i_7$ is invariant under action of $i_j i_k \in \text{Spin}(6)$ and we have

$$v' = i_i (i_j (\cos t \cdot i_7 + \sin t \cdot x)) i_j = \cos t \cdot i_7 + \sin t \cdot i_i (i_j x i_k) i_j.$$  

The orbit space of the action of $\text{Spin}(6)$ on $RP^6$ is the line segment $[0, \pi/2]$ again: we have three types of orbits: the endpoint $t = 0$ corresponds to the pole $e_7$, with the isotropy group $\text{Spin}(6)$. The endpoint $t = \pi/2$ corresponds to the orbit $RP^5 = \{ \pm x \}$, its points have the isotropy groups conjugate to $\text{Pin}(5)$, the isotropy group of $\{ \pm i_7 \}$. Each point $t \in (0, \pi/2)$ corresponds to the orbit $\cos t \cdot e_7 + \sin t \cdot x$, the sphere, consisting of points with the isotropy group conjugate to $\text{Spin}(5)$.

The proof for $m = 5, 4, 3$ is identical and is left to the reader.

Consider again the bundles (3.1) and (3.2). Let $\lambda \to P(\xi^{m-1})$ be the canonical real line bundle. $\lambda$ splits off the bundle $j^*_m(\xi^m)$ as the canonical direct summand. Let $f_{m-1}$ be the classifying map of $\lambda$.

Lemma 3.5. One has for the composition of the transfer map $Tr_m$ followed by the classifying map $f_m$ is zero in symplectic cobordism

$$i) \quad i^*_m Tr^*(j_m) f^*_m = Tr^*(j_{m-1}) f^*_m - \text{Id}, \quad 2 \leq m \leq 7;$$  

$$ii) \quad Tr^*(j_m) f^*_m = 0, \quad 2 \leq m \leq 6.$$  

Proof. For i) apply Lemma 3.4. The pullback bundle $i^*_m Tr^*(j_m)$ coincides with (3.5). Then $B\text{Spin}(m)$ is simply connected for $m \geq 2$: this is the consequence of the exact sequence of homotopy groups of the fibration $\text{Spin}(m) \to ES\text{Spin}(m) \to B\text{Spin}(m)$ and the fact that $\text{Spin}(m)$ is path connected for $m \geq 2$ [1]. So that only the first component of the transfer homomorphism in Lemma 3.4 is relevant.
For ii) let \( m = 6 \). By Corollary 3.3 transfer map \( i_6^* \mathcal{T}_r 7 \) of
\[
i_6^*(j_7) = P(\xi^6 \oplus \mathbb{R}^2)
\]
is trivial as its fiber \( \mathbb{R}P^7 \) admits \( \text{Spin}(6) \)-equivariant vector field \( (p, pi) \) and therefore the bundle along the fibers of
\[
P(\xi^6 \oplus \mathbb{R}^2) \to B\text{Spin}(6)
\]
admits a section. Now apply again Lemma 3.4 for \( m = 7 \) and then \( m = 6 \) to complete the proof of ii) for \( m = 6 \).

Then for \( m < 6 \) the transfer homomorphism of \( P(\xi^6 \oplus \mathbb{R}^8-m) \), \( 2 \leq m < 6 \) is trivial as the pullback of \( i_6^* \mathcal{T}_r 7 \) by \( i_m^* \). Apply again Lemma 3.4 for \( m+1 \) and \( m \) to complete the proof of ii).

Recall from [7] that any \( \text{Spin}(5) \)-bundle is \( \text{MSp} \)-orientable. So is \( \xi^m \oplus \mathbb{R}^8-m \) for \( m \leq 5 \). Therefore Lemma 3.1 implies that \( T_F(P(\xi^m \oplus \mathbb{R}^8-m)) \) is \( \text{MSp} \)-orientable. Recall from [3] that in our situation the transfer homomorphism is expressed by Boardman’s ”Unkern” map which is zero because of zero Euler class.

4. Proof of Theorem 1.1

Here we follow the notations of [2]. The proof of Theorem 1.1 is organized as follows.

The tensor square of the canonical \( \text{Sp}(1) \)-bundle \( \zeta \to B\text{Sp}(1) \) has a trivial summand
\[
\zeta \otimes_H \zeta^* = \Lambda + 1,
\]
where \( \Lambda \to B\text{Sp}(1) \) is the canonical \( \text{Spin}(3) \)-bundle.

Let \( N \) be the normalizer of the torus \( S^1 = U(1) \) in \( S^3 = \text{Sp}(1) = \text{Spin}(3) \). Clearly the bundle
\[
p : BN \to B\text{Sp}(1) = B\text{Spin}(3)
\]
is the projective bundle of \( \Lambda \). The quotient map \( N/U(1) = \mathbb{Z}/2 \) induces the map
\[
f : BN \to B\mathbb{Z}/2,
\]
the classifying map of the canonical real line bundle
\[
(4.1) \quad \lambda \to BN, \quad \lambda \otimes 2 = 1.
\]
The pullback of \( \Lambda \) splits canonically over \( BN \)
\[
(4.2) \quad p^*(\Lambda) = \lambda + \mu.
\]

Lemma 3.5 case \( m = 2 \) implies
\[
(4.3) \quad Tr^* f^* = 0
\]
in symplectic cobordism, where \( Tr \) is the transfer map of \( p \).

Then it turns out [7], [2] p.4394, that \( \Lambda \) is \( \text{MSp} \)-orientable and the Thom class can be chosen in such a way that its restriction to the zero section is equal to
\[
(4.4) \quad \tilde{e}(\Lambda) = \theta_1 + \sum_i \phi_i \pi_i, \quad x = pf_i^*(\zeta).
\]

Now let \( \Lambda_i \) be the pullback of \( \Lambda \) induced by projection on \( i \)-th factor \( B\text{Sp}(1)^4 \to B\text{Sp}(1) \) and \( \lambda \) be as above. Then ([2], Lemma 4.5)
\[
(4.5) \quad \lambda \otimes \mathbb{R} \sum_i 4 \Lambda_i \to B\mathbb{Z}/2 \times B\text{Sp}(1)^4 \text{ is } \text{MSp-orientable},
\]
\[
(4.6) \quad \lambda \otimes \mathbb{R} \sum_i 2 \Lambda_i \to B\mathbb{Z}/2 \times B\text{Sp}(1)^2 \text{ is } \text{SC-orientable}.
\]
Because of (4.2) and (4.1) the pullback of (4.5) over 

$$(f, p) \times 1 : BN \times BSp(1)^3 \to B\mathbb{Z}/2 \times BSp(1)^4$$

has a trivial summand and therefore zero $MSp$-orientation Euler class.

Similarly for the pullback of (4.6) over 

$$(f, p) \times 1 : BN \times BSp(1)^3 \to B\mathbb{Z}/2 \times BSp(1)^4.$$  

Thus (2), Lemma 4.6) one has in 

$MSp^*(BN \times BSp(1)^3)$

\begin{equation}
0 = \prod_{s=1}^{4} (\theta_1 + \sum_{r \geq 1} \phi_r x_s^{2r}) + \sum_{m,n,p,q \geq 0} f^*(\gamma_{mnpq}) x_1^m x_2^n x_3^p x_4^q
\end{equation}

\begin{equation}
= \sum_{i,j,k,l \geq 1} \phi_i \phi_j \phi_k \phi_l x_1^{2i} x_2^{2j} x_3^{2k} x_4^{2l} + \sum_{m,n,p,q \geq 0} f^*(\gamma_{mnpq}) x_1^m x_2^n x_3^p x_4^q.
\end{equation}

Here in (4.7) we use the relation $\theta_1 \phi_i \phi_j = 0$ of [6].

Similarly one has in 

$SC^*(BN \times BSp(1))$

\begin{equation}
0 = \sum_{i,j \geq 1} \phi_i \phi_j x_1^{2i} x_2^{2j} + \sum_{m,n \geq 0} f^*(\gamma_{mn}) x_1^m x_2^n.
\end{equation}

Finally to complete the proof of Theorem 1.1 i) apply (4.8) and (4.3).

Similarly apply (4.9) and (4.3) to complete the proof of Theorem 1.1 ii).

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