A MEALY MACHINE WITH POLYNOMIAL GROWTH OF IRRATIONAL DEGREE

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Abstract. We consider a very simple Mealy machine (two non-trivial states over a two-symbol alphabet), and derive some properties of the semigroup it generates. It is an infinite, finitely generated semigroup, and we show that the growth function of its balls behaves asymptotically like $n^\alpha$, for $\alpha = 1 + \log 2/\log (1 + \sqrt{5})/2$; that the semigroup satisfies the identity $g^6 = g^4$; and that its lattice of two-sided ideals is a chain.

1. Introduction

Algebraic objects may be defined by universal properties, like the identities they satisfy, or a presentation as a quotient of two free objects. They may also be defined by their action on simpler objects, viz, for an algebra, as endomorphisms of a vector space; for a group, as permutations of a set, and for a semigroup, as mappings of a set.

If the semigroup to be defined is infinite, it naturally must act on an infinite set; extra conditions must be imposed on the action to ensure that it remains describable (by a finite sequence of mathematical symbols, say). A specially interesting class of semigroups appear by requesting that the infinite set be the set $X^*$ of words over a finite alphabet, and that the action be given by a finite-state automaton.

The growth function $\gamma(\ell)$ of a finitely generated semigroup $\Gamma$ — the number of semigroup elements that can be obtained as products of at most $\ell$ generators — is an important invariant of the semigroup. It depends on the chosen generating set, but its asymptotics do not. This function is at most exponential. If it is bounded by a polynomial, then $\Gamma$ is of polynomial growth; following [6], its Gelfand-Kirillov dimension (abbreviated GK dimension) is then defined as the infimum of those $\alpha$ such that $\gamma(\ell)/\ell^\alpha$ does not converge to 0.

The “Bergman gap” [8, Theorem 2.5] asserts that a semigroup may have GK dimension 0, 1, or $\geq 2$; by a result by Warfield [11], there exist semigroups of GK dimension $\beta$ for any $\beta \geq 2$. Belov and Ivanov [3,4] construct finitely presented semigroups with non-integer GK dimension. Shneerson [10] constructs relatively free (i.e., free relative to an identity) semigroups of intermediate growth (asymptotically $n^\log m$). His proof involves Fibonacci numbers, as does the present construction.

This paper describes the semigroup $\Gamma(I)$ generated by the 3-state automaton $I$ in Figure 1 (see [3] for the definition of a semigroup generated by an automaton), and proves that its GK dimension is irrational. A much more precise statement appears in Theorem 2.4.

The main purpose of this paper is to show that a very simple-minded finite-state automaton can produce a semigroup with a highly unusual growth pattern,
asymptotically $n^{\alpha}$ for an irrational $\alpha$. Other exotic types of growth of automata have also been discovered; for example, of type $e^{\sqrt{n}}$ in [2], and of type $n^{\log(n)/2 \log m}$ for integer $m$, in [9]. It is intriguing that in all these cases the determination of the growth function relies on enumerations of partitions with certain constraints.

1.1. Plan. The next section gives an automata-free version of the semigroup studied in this paper, along with a presentation of its main results. Section 3 gives the necessary definitions of semigroups generated by automata. Statements whose proof are too long to fit smoothly in the text are stated as underlined. Their proofs appear in Section 5.

1.2. Notation. All actions are written on the right in this paper. The identity is written $e$. We use $=$ for equality of group elements, and $\equiv$ for graphical (letter-by-letter) equality of words representing these elements. We denote by $X^*$ the free monoid on the set $X$. The length of a word $w$ is denoted by $|w|$; this is also the length of a minimal word representing a semigroup element.

The integers are written $\mathbb{Z}$, and the naturals (containing 0) are written $\mathbb{N}$. Congruence is written $\equiv$, and the ‘mod 2’ operation (remainder after division by 2) is written $n \% 2$ as in the C programming language.

Sequences of the form $f_a f_{a+2} \ldots f_b$, sometimes simply written $f_a \ldots f_b$, appear throughout the paper. They are taken to be $e$ if $a > b$, and $f_a f_{a+2} f_{a+4} \ldots f_b - 2 f_b$ otherwise.

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2. Main results

Consider the following transformations $s$, $f$ of the integers $\mathbb{Z}$:

$$\begin{align*}
x^s &= \begin{cases} 
    x - 1 & \text{if } x \text{ is odd}, \\
    x + 1 & \text{if } x \text{ is even}; 
\end{cases} \\
x^f &= \begin{cases} 
    x - 2^n - 1 & \text{if there exists } n \geq 0 \text{ such that } x \equiv 3 \cdot 2^n \pmod{2^{n+2}}, \\
    x + 3 \cdot 2^n - 1 & \text{if there exists } n \geq 0 \text{ such that } x \equiv 2^n \pmod{2^{n+2}}, \\
    -1 & \text{if } x = 0.
\end{cases}
\end{align*}$$

Note that $f$ is uniquely defined, because every $x \in \mathbb{Z}$ is either 0 or of the form $2^n y$ for unique $n \geq 0$ and odd $y$, with either $y \equiv 1$ or $y \equiv 3$ modulo 4.
2.1. The semigroup. Let $F$ be the semigroup generated by the transformations $s$ and $f$. Define furthermore the elements $f_n$ of $F$ by $f_1 = s$, $f_2 = f$, and inductively $f_n = f_{n-2} f_{n-1}$ for $n \geq 3$. For example, $f_3 = sf$, $f_4 = fsf$, $f_5 = sf^2 sf$. These words are sometimes called the “Fibonacci sequence” — see §3 for the connection.

Recall [5] that a rewriting system for a semigroup $\Gamma$ generated by a set $Q$ is a set of equations, called rules, of the form $\ell \rightarrow r$, with $\ell, r \in Q^*$. An elementary reduction in a word $w \in Q^*$ is the replacement of a subword equal to the left-hand side of a rule by the right-hand side; if no elementary reduction is possible, the word is reduced. A rewriting system is terminating if there is no $w \in Q^*$ to which an infinite sequence of elementary reductions can be applied; and it is confluent if the reduced word obtained after applying as many elementary reductions as possible does not depend on the choice of elementary reductions. It is complete if it is terminating and confluent; the set of reduced words is then in bijection with the semigroup, through the natural evaluation map $Q^* \rightarrow \Gamma$, and is called a normal form for $\Gamma$.

For $n \geq 1$, define words $r_n$ and $r'_n$ over the alphabet $\{s, f\}$ as follows:

\begin{align*}
    r_1 &= s^2, & r'_1 &= e, \\
    r_n &= f_{n+1} f_n^2, & r'_n &= f_n^3 (f_{n+2} f_{n+2} f_{n+7} \cdots f_{n-1} f_{n+1}) f_n.
\end{align*}

In particular, $r_2 = sf^3$ and $r'_2 = sf$, and $r_3 = f_4 f_3^2$ and $r'_3 = f_4$. For $n \geq 3$, we have $\|r'_n\| = \|r\| - 4$.

**Theorem 2.1.** The semigroup $F$ is infinite, and admits as presentation

\[(s, f \mid r_n = r'_n \text{ for all } n \geq 1),\]

Furthermore, after the relations $sw = sw'$ (which occur for even $n$) are replaced by the equivalent relations $w = w'$, this presentation is a complete rewriting system.

2.2. A normal form. Let $\Phi_n$ denote the sequence of Fibonacci numbers, defined by $\Phi_1 = \Phi_2 = 1$ and $\Phi_n = \Phi_{n-2} + \Phi_{n-1}$ for $n \geq 3$. Then

**Theorem 2.2.** Every element $g$ of $F$ admits a unique representation as a word of the form

\[(4) \quad w_g = s f_{i_1} f_{i_2} \cdots f_{i_m} \cdots f_{i_n},\]

for some $n \geq 0$, some $\epsilon \in \{0, 1\}$, and some indices $i_1, \ldots, i_n$ satisfying

\[3 \leq i_1, i_1 + 1 < i_2, \ldots, i_m - 1 + 1 < i_m > i_{m+1} > \cdots > i_n \geq 1.\]

We call $i_m$ the maximal index of $w_g$.

When spelled out in the generating set $\{s, f\}$, this representation of $g$ is essentially minimal: if $i_1$ is even, then $w_g$ is the unique minimal-length representation of $g$, while if $i_1$ is odd, then an initial $s^2$ must be cancelled to obtain the unique minimal representation of $g$. Its length (see §3) is

\[\|g\| = (-1)^i \epsilon + \Phi_{i_1} + \cdots + \Phi_i.\]

By a slight extension of the definition of rewriting system, let us admit rules of the form $\ell \rightarrow r$ that mean that the subword $\ell$ may be replaced in $w$ by the word $r$ if it is a prefix of $w$. We will actually consider the following rewriting system; it is on an infinite generating set, but its rules are much simpler, since their left-hand sides have length at most 3.
**Theorem 2.3.** On the generating set \( \{f_i : i \geq 1\} \), the semigroup \( F \) admits a complete rewriting system with rules

(N1) \( f_1^2 \rightarrow e \)
(N2) \( f_a f_{a+1} \rightarrow f_{a+2} \quad (a \geq 1) \)
(N3) \( f_a^2 \rightarrow f_{a-2} f_{a+1} \quad (a \geq 3) \)
(N4) \( f_a f_2 \rightarrow f_a \quad (a \geq 2) \)
(N5) \( \wedge f_2 \rightarrow f_1 f_3 \)
(N6) \( f_{a+1} f_a f_{a+3} \rightarrow f_{a+3} f_{a+2} \quad (a \geq 1) \)
(N7) \( f_{a+2} f_a f_{a+3} \rightarrow f_{a} f_a f_{a+3} f_{a+2} \quad (a \geq 1) \)
(N8) \( f_{a+p} f_a f_{a+q} \rightarrow f_{a+p-2} \cdots f_a f_2 \cdots f_a f_{a+q} \quad (a \geq 2, p \geq 1, q \geq 2 \text{ even}) \)
(N9) \( f_{a+p} f_a f_{a+q} \rightarrow f_{a+p-2} \cdots f_a f_2 \cdots f_a f_{a+q} \quad (a \geq 1, p \geq 3, q \geq 3 \text{ odd}) \)
(N10) \( f_{1+p} f_1 f_{1+q} \rightarrow f_{1+p} f_2 f_4 \cdots f_q \quad (p \geq 1, q \geq 2 \text{ even}) \).

We will never use a different notation for a semigroup element and its normal form.

**2.3. Growth.** Using this minimal representation, the (ball) growth function of \( F \),

\[ \gamma(\ell) = \# \{g \in F : \|g\| \leq \ell\}, \]

may be quite precisely estimated; namely

**Theorem 2.4.** There are constants \( C, D > 0 \) such that the growth function of \( F \) satisfies

\[ C \ell^\alpha \leq \gamma(\ell) \leq D \ell^\alpha \]

for all \( \ell \in \mathbb{N} \), where \( \varphi = (1 + \sqrt{5})/2 \), and \( \alpha = 1 + \log 2 / \log \varphi \approx 2.4401 \), and

\[ C = \frac{2\sqrt{5}}{\sqrt{5\varphi^2(2\varphi-1)(2\varphi)}} \quad \text{and} \quad D = \frac{2\sqrt{5}}{\sqrt{5\varphi^2(2\varphi-1)}}. \]

Therefore, the Gelfand-Kirillov dimension of \( F \) is \( \alpha \).

Experimental computations indicate that actually the function \( \gamma(\ell)/\ell^\alpha \) does not converge, but oscillates between \( \approx 0.201 \) and \( \approx 0.205 \), reaching its maxima at Fibonacci numbers.

**2.4. Identities.** We have not determined the complete set of identities satisfied by \( F \); nevertheless, we know

**Theorem 2.5.** The semigroup \( F \) satisfies the identity \( g^6 = g^4 \).

**2.5. The ideal structure of \( F \).** We describe in this subsection the quotient and ideal structure of \( F \). First, we may consider for all \( n \in \mathbb{N} \) the quotient of \( F \), denoted \( W_n \), acting as transformations on \( X^n \). Let us denote, for \( n \in \mathbb{N} \),

\[ z_n = \begin{cases} f_3 f_5 \cdots f_{n+2} & \text{if } n \text{ is odd}, \\ f_4 f_6 \cdots f_{n+2} & \text{if } n \text{ is even}. \end{cases} \]

Then we have...
Theorem 2.6.  
1. The semigroup $W_n$ is presented as follows:
   
   $W_n = \langle s, f_1 r_k = r_k' \mid 1 \leq k \leq n + 2, \text{ and } s z_n = f z_n = z_n, f_{n+2} f_{n+1} = z_n \rangle$.

2. The elements of $W_n$ may be described as all normal forms $f_1 \cdots f_m \cdots f_t$ whose maximal index $i_m$ is $n + 2$, normal forms of the form $z_n f_{i_{m-1}} \cdots f_t$, for $n \geq i_{m-1} > \cdots > i_t$, and those normal forms of maximal index $n + 2$ that include neither $z_n$ nor $f_{n+2} f_{n+1}$.

3. The semigroup $W_n$ has order $\sum_{k=1}^{n} 2^{k+2} \Phi_k - 2^n - 2^{n+1} \Phi_n + 2$.

We identify for the remainder of this section the quotient semigroup $W_n$ with the set of normal forms of its elements, which then form a subset of $F$.

Corollary 2.7. The Hausdorff dimension (in the sense of (3.6) of $F$ is 0.

Proof. Using the count of elements in $W_n$, we have
\[
\#W_n \leq 2 + \sum_{k=1}^{n} 2^{k+2} \Phi_k \leq \frac{4}{\sqrt{5}} \frac{(2\varphi)^{n+1} - 1}{2\varphi - 1} \leq (2\varphi)^{n+1},
\]
so
\[
\text{Hdim}(F) = \liminf_{n \to \infty} \frac{\log \#W_n}{\log \# \text{End}(X^n)} = \liminf_{n \to \infty} \frac{\log(2\varphi)^{n+1}}{\log 2(2^n-1)} = 0. \quad \square
\]

Corollary 2.8. Let $g$ be an arbitrary element of $F$ with maximal index $n \geq 3$.

Then its trace (in the sense of (3.6)) satisfies
\[
\tau(g) = \begin{cases} 2^{2-n} & \text{if } g \text{ includes } z_{n-2} \text{ or } f_n f_{n-1} \text{ in its normal form}, \\ 2^{3-n} & \text{otherwise}. \end{cases}
\]

The ideal $I_n = F f_{n+3} F$ coincides with $F_{2-n}$, as defined in (3.6). They therefore form a unique chain.

Theorem 2.9. All two-sided ideals of $F$ are of the form $F f_n F$, and equivalently of the form $F \xi$ as defined in (3.6). They therefore form a unique chain.

Recall that if $I$ is an ideal of $F$, the quotient $F/I$ is the semigroup whose elements are the equivalence classes of the congruence $I \times I \cup \Delta$, where $\Delta$ is the identity relation (= diagonal) of $F$.

Proposition 2.10. For $n \geq 2$, the quotient $F/I_n$ has order $\sum_{k=1}^{n} 2^{k+2} \Phi_k - 2^{n+1} - 2^{n+1} \Phi_n + 3$. It is obtained from $W_n$ by identifying together all idempotents of rank 1, i.e. all maps $f_p : X^n \to X^n$ defined by $f_p(x_1 \ldots x_n) = p$, for all $p \in X^n$.

Proof. By Theorem 2.6, the quotient $F/I_n$ is a quotient of $W_n$. The transformation $f_{n+3}$ induces a transformation of rank 1 on $X^n$, and its left multiples, which are identified in $F/I_n$, are the maps $f_p$ in the statement of the Proposition. \quad \square

3. Definitions

A finite-state automaton, also called Mealy machine, is comprised of the following data: a set $X$, the alphabet; a set $Q$, the states; a function $\tau : X \times Q \to Q$, the transition; a function $\pi : X \times Q \to X$, the output. In this section we suppose that the alphabet is $X = \{0, \ldots, d\}$; we will later specialize to $d = 1$.

Such an automaton $A$ is usually represented, as in the figure above, as a graph. The states are represented by vertices, and there is an edge from vertex $q$ to vertex $r$, labelled $i \to o$, whenever $\tau(i, q) = r$ and $\pi(i, q) = o$. 

3.1. **Action.** States \( q \in Q \) of \( \mathcal{A} \) yield transformations of \( X^* \), which are defined simultaneously as follows:

\[
(1 \ldots i_n)^g = \pi(i_1, q)(i_2 \ldots i_n)^{\tau(i_1, q)},
\]

These transformations generate a semigroup \( \Gamma(\mathcal{A}) = (Q) \), called the semigroup generated by \( \mathcal{A} \).

3.2. **Decomposition.** Yet another way of describing the automaton \( \mathcal{A} \) is via its semigroup decomposition. Given \( g \in \Gamma \), it yields by (5) a transformation \( \pi_g \) of \( X \) by restriction to length-1 words; and, for all \( i \in X \), a transformation \( iX^* \rightarrow i\pi_g X^* \), again by restriction. By (6), the composition \( X^* \rightarrow iX^* \rightarrow i\pi_g X^* \rightarrow X^* \) is again an element of \( \Gamma \), which we denote by \( g_i \). We write

\[
\phi(g) = \ll g_0, \ldots, g_d \gg \pi_g
\]

for the decomposition of \( g \in \Gamma \), with \( g_i \in \Gamma \) and \( \pi_g : X \rightarrow X \). Multiplication of such decompositions obeys the rule

\[
\ll g_0, \ldots, g_d \gg \pi_g \ll h_0, \ldots, h_d \gg \pi_h = \ll g_0 h \pi_g(0), \ldots, g_d h \pi_g(d) \gg \pi_g \pi_h.
\]

It is therefore sufficient to know the decomposition of generators, and these are determined by the transition and output functions \( \tau, \pi : X \times Q \rightarrow Q \) of \( \mathcal{A} \), by

\[
\phi(q) = \ll \tau(0, q), \ldots, \tau(d, q) \gg (i \mapsto \pi(i, q) \vee i).
\]

3.3. **Metrics.** On the semigroup \( \Gamma \) generated by a set \( Q \), define the “norm”

\[
\| \| g \| = \min \{ \ell \in \mathbb{N} : g = q_1 \cdots q_\ell, \quad q_i \in Q \forall i \}.
\]

(we have \( \| gh \| \leq \| g \| + \| h \| \), which justifies calling it a norm.) The ball growth function is the function \( \gamma : \mathbb{N} \rightarrow \mathbb{N} \) defined by

\[
\gamma(\ell) = \# \{ g \in \Gamma : \| g \| \leq \ell \};
\]

it measures the volume growth of balls in the discrete normed space \( \Gamma \).

3.4. **Contraction.** An automaton is contracting if there are constants \( C \) and \( \eta < 1 \) such that whenever \( \phi(g) = \ll g_0, \ldots, g_d \gg \pi_g \), we have

\[
\| g \| \leq \eta \| g \| + C.
\]

Contracting automata are most studied, in part because contraction gives a natural strategy of proof by induction.

3.5. **Traces.** Let \( g \) be a transformation of \( X^* \) given by an automaton. Then for all \( n \in \mathbb{N} \) we have \( \#g(X^{n+1}) \leq \#X \#g(X^n) \), so the limit

\[
\tau(g) := \lim_{n \rightarrow \infty} \frac{\#g(X^n)}{\#X^n} \in [0, 1]
\]

exists. We call it the trace of \( g \). The following facts are easily checked:

- \( \tau(gh) \leq \min \{ \tau(g), \tau(h) \} \);
- \( \tau(g) = 1 \) if and only if \( g \) is invertible.

Therefore, if \( \Gamma \) is the semigroup generated by an automaton, we may define for any \( \xi \in [0, 1] \) a two-sided ideal \( \Gamma_\xi = \{ g \in \Gamma : \tau(g) \leq \xi \} \). We have \( \Gamma_\xi = \Gamma \) if and only if \( \xi = 1 \).
3.6. Hausdorff dimension. Let Γ be a semigroup acting on a rooted tree X*. There is another metric on Γ, defined by
\[
d(g, h) = \exp(- \max \{ n : v^g = v^h \text{ for all } v \in X^n \}),
\]
with the convention that \( \exp(-\infty) = 0 \). This turns Γ into a metric space of diameter at most 1. The Hausdorff dimension of Γ is defined as the Hausdorff dimension of this metric space. Let \( \text{End}(X^n) \) denote the set of prefix-preserving maps \( X^n \to X^n \). If \( \#X = d + 1 \), then
\[
\# \text{End}(X^n) = ((d + 1)^{d+1})^{((d+1)^n-1)/d}.
\]
Then the Hausdorff dimension of Γ is given by the formula
\[
\text{Hdim}(\Gamma) = \liminf_{n \to \infty} \frac{\log \# \text{End}(X^n)}{\log \#X^n},
\]
where Γ\_n is the quotient of Γ acting as a transformation semigroup of \( X^n \); see [1] for an analogous definition in the case of profinite groups.

4. The automaton I and the semigroup F

We now consider the automaton I from Figure [1]. This automaton has three states \( f, s, e \) and a two-letter alphabet \( X = \{0, 1\} \).

Consider the following transformations \( \sigma, \zeta \) of \( X \):
\[
\sigma(i) = 1 - i, \quad \zeta(i) = 0.
\]
Then the decomposition of the states are
\[
\phi(f) = (s, f) \cdot \zeta, \quad \phi(s) = (s, e) \cdot \sigma, \quad \phi(e) = (e, e) \cdot (s, f).
\]
Clearly \( e \) acts as the identity transformation, and \( s \) is invertible, of order 2.

4.1. Action on integers. We show now that the semigroup defined in Section [2] by its action on integers is the semigroup generated by an automaton.

**Theorem 4.1.** The semigroups \( F \) and \( \Gamma(I) \) are isomorphic.

**Proof.** The action of \( F = \langle s, f \rangle \) given in [1] extends to a continuous action of \( F \) on the 2-adics \( \mathbb{Z}_2 \). Consider the following bijection \( \Theta \) between \( X^\infty \) and \( \mathbb{Z}_2 \):
\[
\Theta(x_1x_2\ldots) = \sum_{i=1}^{\infty} (1 - x_i)2^{i-1}.
\]
Let us denote temporarily \( \Gamma(I) = \langle \tilde{s}, \tilde{f} \rangle \). The theorem will follow from \( \Theta(x^\tilde{s}) = \Theta(x)^s \) and \( \Theta(x^\tilde{f}) = \Theta(x)^f \) for all \( x \in X^\infty \).

Consider therefore \( x = x_1x_2\ldots \in X^\infty \), and write \( y = x_2x_3\ldots \). If \( x_1 = 0 \), then
\[
\Theta(x)^s = \Theta(0y)^s = (1 + 2\Theta(y))^s = 2\Theta(y) = \Theta(1y) = \Theta(x^\tilde{s}),
\]
while if \( x_1 = 1 \), then
\[
\Theta(x)^s = \Theta(1y)^s = (2\Theta(y))^s = 1 + 2\Theta(y) = \Theta(0y) = \Theta(x^\tilde{s}).
\]
Let next \( n \in \mathbb{N} \cup \{\infty\} \) be maximal such that \( x_1 = \cdots = x_n = 1 \). If \( n = \infty \), then
\[
\Theta(x)^f = \Theta(11\ldots)^f = 0^f = -1 = \Theta(00\ldots) = \Theta(x^\tilde{f});
\]
otherwise, \( x = 1\ldots 10x_{n+2}x_{n+3}\ldots \); write \( y = x_{n+3}x_{n+4}\ldots \). If \( x_{n+2} = 0 \), then
\[
\Theta(x)^f = (2^n + 2^{n+1} + 2^{n+2}\Theta(y))^f = 2^{n+1} - 1 + 2^{n+2}\Theta(y) = \Theta(0\ldots 001y) = \Theta(x^\tilde{f}),
\]
while if \( x_1 = 1 \), then
\[
\Theta(x)^f = (2^n + 2n+2\Theta(y))^f = 2^{n+2} - 1 + 2n+2\Theta(y) = \Theta(0\ldots000y) = \Theta(x^f).
\]

\[\square\]

4.2. Fibonacci sequence. From now on, we identify the semigroup \( F \) with \( \Gamma(I) \) and use the notation \( F \) for both. Recall the definition of the following elements of \( F: f_1 = s, f_2 = f, \) and \( f_n = f_{n-2}f_{n-1} \) for \( n \geq 3 \), and the Fibonacci numbers \( \Phi_n \) defined by \( \Phi_1 = \Phi_2 = 1 \) and \( \Phi_n = \Phi_{n-2} + \Phi_{n-1} \) for \( n \geq 3 \). We note that \( \|f_n\| = \Phi_n \) for all \( n \geq 1 \). Set \( \varphi = (1 + \sqrt{5})/2; \) recall that \( \Phi^n \approx \varphi^n/\sqrt{5} \).

Note that if \( n \) is even and \( \geq 6 \), then \( f_n \) starts with \( fssf \), while if \( n \) is odd and \( \geq 5 \), then \( f_n \) starts with \( sffs \). The following statements are easily proven by induction:

**Lemma 4.2.** (1) For all \( k \leq n-2 \) with \( k \equiv n \pmod{2} \) we have \( f_n = f_k(f_{k+1}f_{k+3}\cdots f_{n-1}) \); in particular, \( f_k \) is a prefix of \( f_n \).

(2) For all \( k > 1 \) we have \( f_{k+1}^2 = f_{k-1}f_{k+2} \).

4.3. Some relations. By (8), we have
\[
\ll g_0, g_1 \gg \zeta \ll h_0, h_1 \gg \zeta = \ll g_0h_0, g_1h_0 \gg \zeta
\]
for all \( g_i, h_i \in F \), a calculation that we will use repeatedly; so by direct computation

**Lemma 4.3.** In \( F \) the following relations hold: \( s^2 = e; f^3 = f \).

**Proof.** If \( x = x_1\ldots x_n \), then \( x^s = ((1 - x_1)x_2\ldots x_n)^s = x \), so \( s^2 = e \). Then
\[
\phi(f^2) = \ll s, f \gg \zeta \ll s, f \gg \zeta = \ll s^2, fs \gg \zeta,
\]
\[
\phi(f^3) = \ll s^2, fs \gg \zeta \ll s, fs \gg \zeta = \ll s^3, fs^2 \gg \zeta = \ll s, fs \gg \zeta = f.
\]

\[\square\]

4.4. Contraction. The proofs will ultimately all rely on some form of induction on the length of representations of semigroup elements as words over \( \{s, f\} \).

**Lemma 4.4.** The semigroup \( F \) is contracting.

**Proof.** Consider \( g \in F \), and write it as a word \( w \) of minimal length \( w_1\ldots w_n \); write also \( \phi(g) = \ll g_0, g_1 \gg \pi_g \). Then, by Lemma 4.3, there cannot be more than two \( f \)'s in a row in \( w \), so every group of three letters \( w_{3k+1}w_{3k+2}w_{3k+3} \) contains at least an \( s \), which will contribute no letter to \( g_0 \) nor \( g_1 \). The other two letters contribute at most one letter each to each of \( g_0 \) and \( g_1 \). In total, \( \|g_0\| \leq \frac{2}{3}(\|g\| + 2) \), and similarly for \( \|g_1\| \).

Note, in fact, that the contraction ratio \( \eta \) may be chosen as \( 1/\varphi \), with a little more care. This becomes apparent in the next result.

**Lemma 4.5.** The decomposition of \( f_n \) satisfies
\[
\phi(f_1) = \ll e, e \gg \sigma, \quad \phi(f_2) = \ll s, f \gg \zeta,
\]
\[
\phi(f_3) = \ll f, s \gg \zeta, \quad \phi(f_4) = \ll f_3, f^2 \gg \zeta,
\]
\[
\phi(f_{2n}) = \ll f_{2n-1}, f sf_{n-2} \cdots \gg \zeta \quad \text{if} \ n \geq 3,
\]
\[
\phi(f_{2n+1}) = \ll f_{2n}, f_2(sf_7 \cdots f_{2n-1}) \gg \zeta \quad \text{if} \ n \geq 2.
\]
5. Relations. Recall that for \( n \geq 1 \) we defined in \([2,3]\) words \( r_n \) and \( r'_n \) over the alphabet \( \{s,f\} \) by \( r_1 = s^2, r'_1 = e \), and

\[
\begin{align*}
r_n &= f_{n+1}f_n^2 = f_{n-1}f_n^3 \quad (\equiv f_{n+1}f_{n-2}f_{n+1} \text{ if } n \geq 3), \\
r'_n &= f_{n\%2+1}(f_{n\%2+5}f_{n\%2+7} \cdots f_{n-1}f_{n+1})f_n.
\end{align*}
\]

Note that \( r'_n \), for odd \( n \geq 3 \), can be obtained from \( r_n \) by removing the first four letters; indeed

\[
(9) \quad sfsrc'_n = f_4f_1 \cdot f_2(f_6 \cdots f_{n+1})f_n = f_4f_3(f_6f_8 \cdots f_{n+1})f_n = f_4f_5^2(f_8 \cdots f_{n+1})f_n
\]

\[
= f_6f_5(f_8 \cdots f_{n+1})f_n = \cdots = f_{n-3}f_{n-2}f_{n+1}f_n = f_{n-1}f_n = r_n.
\]

Similarly, \( r'_n \) for even \( n \geq 4 \) can be obtained from \( r_n \) by removing the first three letters and replacing them by \( s \); indeed, writing \( r'_n = sr'_n \),

\[
(10) \quad sfsrc'_n = f_3f_2(f_5 \cdots f_{n+1})f_n = f_3f_2(f_5f_7 \cdots f_{n+1})f_n = f_3f_5^2(f_7 \cdots f_{n+1})f_n
\]

\[
= f_5f_6^2(f_9 \cdots f_{n+1})f_n = \cdots = f_{n-3}f_{n-2}f_{n+1}f_n = f_{n-1}f_n = r_n.
\]

Let \( e_n \) be the word obtained from \( f_n \) by deleting its first two symbols.

**Lemma 5.1.** If \( n \geq 4 \), then we have \( \phi(f_n) = \langle s f s r'_n, e_{n-1} \rangle \zeta \).

**Proof.** Assume first that \( n \) is even. By Lemma \([1,2]\) we have \( f_{n-1} = f_3f_4f_6 \cdots f_{n-2} \), so \( e_{n-1} = f_4f_6 \cdots f_{n-2} \), and the Lemma holds by Lemma \([1,5]\). Similarly, if \( n \) is odd then \( e_{n-1} = f_2f_5 \cdots f_{n-2} \) by Lemma \([1,2]\) and again the Lemma holds by Lemma \([1,6]\). \( \square \)

**Lemma 5.2.** In \( F \) the relations \( r_n = r'_n \) hold for all \( n \geq 1 \).

**Proof.** The cases \( n \leq 2 \) are covered in \([2,3]\) since we have \( r_2 = sf^3 = sf = r_2' \). Let us therefore assume \( n \geq 3 \). We follow the notation of Lemma \([5,1]\). We have the decomposition

\[
\phi(r_n) = \langle \varepsilon n f_n^2 \rangle \zeta = \langle s f r'_n \rangle \zeta.
\]

Suppose first that \( n \) is even. Then by \([3]\) we have

\[
\phi(r'_n) = \phi(f_1(f_5f_7 \cdots f_{n+1})f_n) = \langle e_4, f_4 \rangle \zeta = \langle f_4, s \rangle \zeta = \langle f_n \rangle \zeta = \langle f_n, \varepsilon \rangle \zeta = \langle \varepsilon n f_n \rangle \zeta = \langle \varepsilon n f_n \rangle \zeta.
\]

By the previous Lemma, \( \phi(r'_n) = \phi(r_n) \), as desired. \( \square \)
where * stands for an element of $F$ that is not relevant to the calculation. If $n$ is odd, then similarly

$$
\phi(r'_n) = \phi(f_2f_6 \cdots f_{n+1}f_n) = \phi f_1 f_2 \cdots f_5 \cdots f_n \cdots f_{n-1} \cdots f_{r'_n-1} \cdots f_{r'_n} f_{r'_n-1} \cdots f_n \cdots f_5 \cdots f_2 \cdots f_1 \cdot f_{n+1} \cdot f_n \cdot f_{n-1} \cdots f_{r'_n-1} \cdots f_{r'_n} f_{r'_n-1} \cdots f_n \cdots f_5 \cdots f_2 \cdots f_1.
$$

We have $r_{n-1} = r'_n$ by induction, so $r_n = r'_n$.\hfill\Box

### 5.1. More relations.

We will ultimately show that $\{r_n = r'_n\}$ is a complete set of relations for $F$; however, we will first describe more relations, that are consequences of these but allow much faster simplifications.

**Lemma 5.3.** Consider $a, b \in \mathbb{N}$ with $a \geq 3$ and $a \geq b + 2$. Then

$$f_1f_2^b = f_a.$$

**Proof.** We proceed by induction on $b$. Since $a \geq b + 2$, the word $f_a$ ends with $f_{b+2}$; it therefore suffices to show that $f_{b+2}f_2^b = f_{b+2}$. We also note that the statement holds for $b \leq 2$, since then $f_3f_2^2 = f_3$ and $f_4f_2^2 = f_4$ by Lemma 5.3. Then for $b \geq 3$ we have

$$f_{b+2}f_2^b = f_b f_{b+1}f_2^b = f_b r_b = f_b r'_b
= f_b f_1 + b^2 f_5 + b^2 f_7 + b^2 \cdots f_{b-1} f_{b+1} f_b$$

$$= f_b f_2 + b^2 f_4 + b^2 f_6 + b^2 \cdots f_{b-1} f_{b+1} f_b$$

$$= f_b f_3 + b^2 f_5 + b^2 f_7 + b^2 \cdots f_{b-1} f_{b+1} f_b$$

$$= \cdots = f_b f_{b-3} f_{b+1} f_b = f_b f_{b-3} f_{b-2} f_{b-1} f_b = f_b f_{b-1} f_b = f_{b+2}.\hfill\Box
$$

**Lemma 5.4.** For all $n \geq 2$ we have

$$r'_n = (f_1^2 f_2^2 f_3^2 \cdots f_{n-3} f_{n-2}) f_{n+1}.$$

**Proof.** If $n = 2$ then $r'_2 = f_3$ and the Lemma holds. If $n \geq 3$, then

$$r'_n = f_1 + n^2 f_5 + n^2 \cdots f_{n-1} f_{n+1} f_n$$

$$= f_1 + n^2 f_2 + n^2 f_3 + n^2 \cdots f_{n-1} f_{n+1} f_n$$

$$= \cdots = f_1 + n^2 f_2 + n^2 f_3 + n^2 \cdots f_{n-3} f_{n-2} f_{n-1} f_n.\hfill\Box
$$

**Lemma 5.5.** The equality $f_n^4 = f_n^2$ holds in $F$ for $n \leq 4$, and $f_n^5 = f_n^3$ holds in $F$ for all $n \geq 1$.

**Proof.** The first statement follows from the definition of $r_n$. It suffices to check the second one for $n \geq 5$. Using Lemmata 5.3 and 5.4, we have

$$f_n^5 = f_n(f_{n-2} f_{n-1}) f_n^4 = f_n f_n - 2r_n = f_n f_n - 2 r'_n = f_n f_n - 4 f_1 f_2 f_3 f_4 \cdots f_{n-3} f_{n-2} f_{n+1}$$

$$= \cdots = f_n f_n - 2 f_{n-3} f_{n-2} f_{n+1} = f_n f_n - 2 f_{n-3} f_{n-2} f_{n+1} = f_n f_n - 2 f_{n-3} f_{n-2} f_{n+1}$$

$$= \cdots = f_n f_n - 2 f_{n+1} = f_n f_n - 2 f_{n+1} = f_n^3.\hfill\Box
$$

**Lemma 5.6.** For $n \geq 8$ we have

$$f_{n-2} f_{n-3} f_{n-5} f_{n-7} \cdot f_{n-2} f_n = f_{n-2} f_{n-3} f_{n-5} \cdot f_{n-3} f_n$$

$$= f_{n-2} f_{n-3} \cdot f_{n-4} f_n = f_{n-2} \cdot f_{n-5} f_n = f_n.$$
Proof. At first we prove by induction that for \( n \geq 1 \) we have
\[
 f_{n+1}^2 f_{n+2} = f_{n+3}.
\]
The induction starts with two cases: for \( n = 1 \) we have \( f_2^2 f_3 = f_2 f_3 = f_4 \), and for \( n = 2 \) we have \( f_3^2 f_4 = f_3 f_4 = f_5 \). Using twice Lemma 5.3 and the induction hypothesis, we have for \( n \geq 3 \)
\[
 f_{n+1}^2 f_{n+2} = f_{n+2}^3 f_{n+1}^2 = f_{n+2}^2 f_{n+1} f_{n+1} = f_{n+1} f_{n+2} f_{n+1} = f_{n+1} f_{n+2} = f_{n+3}.
\]

Using the arguments above, we carry out the following transformations
\[
 f_{n-2} f_{n-3} f_{n-5} f_{n-7} \cdot f_{n-2} f_{n} =
\]
\[
 = f_{n-2} f_{n-3} f_{n-5} f_{n-7} \cdot f_{n-6} f_{n-5} f_{n-3} f_{n}
\]
\[
 = f_{n-2} f_{n-3} f_{n-5} \cdot f_{n-6} f_{n-5} = f_{n-2} f_{n-3} f_{n-5} \cdot f_{n-5} f_{n-4} f_{n}
\]
\[
 = f_{n-2} f_{n-3} \cdot f_{n-4} f_{n} = f_{n-2} f_{n-3} f_{n-4} f_{n-3} f_{n-1}
\]
\[
 = f_{n-2} \cdot f_{n-5} f_{n-4} f_{n-3} f_{n-1} = f_{n-2} \cdot f_{n-5} f_{n}
\]
\[
 = f_{n-2} f_{n-3} f_{n-1} = f_{n}.
\]

5.2. A normal form for \( F \). We first show that every semigroup element can be written in the form of Theorem 2.2. We write \( \mathcal{N} \) for the set of words described there.

The words in \( \mathcal{N} \) are those words that can be written as a product \( f_{i_1} \cdots f_{i_m} \cdots f_{i_n} \), where the sequence \( i_1, \ldots, i_m \) is increasing by steps of at least two, and the sequence \( i_m, \ldots, i_n \) is decreasing by steps of at least one.

An arbitrary word \( w \in \{s,f\}^* \) is put into normal form by replacing subwords that are left-hand sides of the rules (N1)-(N10) by their right-hand side, until no such substitution is possible. For example, consider the possible reductions that can be performed on the word \( f_4 f_3^2 f_4 \):

\[
 f_4 f_3^2 f_4 \quad \text{N3} : f_4^2 f_1 f_4 \quad f_4 f_3 f_5 \quad \text{N8} : p = 1, q = 2 \quad f_2 f_5 \quad \text{N5} : f_2 \rightarrow f_1 f_3 \quad f_1 f_3 f_5.
\]

\[
 f_4 f_1 f_4 \quad \text{N3} : f_1^2 \rightarrow f_2 f_1 \quad f_4 f_1 f_2 f_5 \quad \text{N2} : f_1 f_2 \rightarrow f_3 \quad f_4 f_1 f_2 f_5 \quad \text{N3} : f_2^2 \rightarrow f_2 f_5 \quad f_4 f_1 f_2 f_5 \quad \text{N10} : p = 3, q = 3
\]

Lemma 5.7. The rewriting system in Theorem 2.2 is terminating.

Proof. We define a function \( \eta = (\eta_1, \eta_2) : \{f_1, f_2, \ldots\}^* \rightarrow \mathbb{N}^2 \) with the lexicographic ordering, such that if \( w \rightarrow w' \) is obtained by an elementary reduction then
\[ \eta(w) > \eta(w'). \] Since \( \mathbb{N}^2 \) is well ordered, this will prove that the rewriting system is terminating.

Consider therefore \( w \in \{f_1, f_2, \ldots \}^\ast \), say \( w = f_{i_1} \ldots f_{i_n} \). Define

\[
\eta_1(w) = \begin{cases} 
\sum_{j=1}^n \Phi_{i_j} - 3 & \text{if } n \geq 2 \text{ and } i_1 = 1 \text{ and } i_2 \geq 3 \text{ is odd}, \\
\sum_{j=1}^n \Phi_{i_j} & \text{otherwise};
\end{cases}
\]

\[
\eta_2(w) = \sum_{j=1}^n (n + 1 - j)i_j.
\]

We may then check that \( \eta_1(w) > \eta_1(w') \) if \( w' \) is obtained from \( w \) using one of the rules \( (N1,4–5,8–10) \), while \( \eta_1(w) = \eta_1(w') \) and \( \eta_2(w) > \eta_2(w') \) if \( w' \) is obtained from \( w \) using one of the rules \( (N2–3,6–7) \).

For example, in rule \( (N9) \), only relation \( r_{\alpha+2} \to r'_{\alpha+2} \) is used, and it shortens the length (as a word over the alphabet \{s, f\}) of \( w \) by 4; note that the length of \( w \) is precisely \( \sum \Phi_{i_j} \). Write \( w' = f_{i_1'} \ldots f_{i_{n'}'} \); then

\[
\eta_1(w) \geq \sum_{j=1}^n \Phi_{i_j} - 3 > \sum_{j=1}^n \Phi_{i_j} - 4 = \sum_{j=1}^{n'} \Phi_{i_j'} \geq \eta_1(w').
\]

Consider, as another example, rule \( (N6) \). We have \( \Phi_{a+1} + \Phi_a + \Phi_{a+3} = \Phi_{a+3} + \Phi_{a+2} \), so \( \eta_1(w) = \eta_1(w') = \sum \Phi_{i_j} \). Say that the left-hand side of the rule appears at positions \( n - j \), \( n - j + 1 \) and \( n - j + 2 \) in \( w \); then

\[
\eta_2(w) - \eta_2(w') \geq (j + 1)(a + 1) + ja + (j - 1)(a + 3) - j(a + 3) - (j - 1)(a + 2)
\]

\[
= (j + 1)(a - 1) + 1 > 0.
\]

All other rules are proven to decrease \( \eta \) in a similar fashion. \( \square \)

The following calculations will be useful in the proof of Lemma 5.9. Let \( u = f_{i_1} \ldots f_{i_m} \) be an arbitrary semigroup word written in normal form such that \( n \geq 1 \), and \( i_j, i_m \geq 1 \), and \( i_m \) is the maximal index. If \( i_1 = 1 \) or \( i_m = 1 \), the argument below should be applied to the word obtained from \( u \) by removing these instances of \( f_1 \), adapting the statements accordingly. If \( i_1 = 3 \), let \( k \) be the largest integer such that \( i_r = 2r + 1 \) for all \( r < k \); if \( i_1 = 4 \), let \( k \) be the largest integer such that \( i_r = 2r + 2 \) for all \( r < k \); if \( i_1 \geq 5 \), then \( k \) need not be defined. Then, by Lemma 4.5, the element \( u \) affords the decomposition \( \phi(u) = \ll u_0, u_1 \rr \zeta \), where

\[
(11) \quad u_0 = \begin{cases} 
fi_1 \cdots fin - 1 \cdots f_{i_m} & \text{if } i_1 > 3, \\
fi_1 \cdots fi_{2k-1}fi_{k-1} \cdots fi_{i_m} & \text{if } i_1 = 3,
\end{cases}
\]

\[
(12) \quad u_1 = \begin{cases} 
(f_i f_3 \cdots fi_{i_1-2})fi_{i_2-1} \cdots fin & \text{if } i_1 \text{ is odd and } i_2 \neq i_1 - 1, \\
(f_i f_6 \cdots fi_{i_1-1})fi_{i_2-1} \cdots fin & \text{if } i_1 \text{ is odd and } i_2 = i_1 - 1, \\
(f_i f_6 \cdots fi_{i_1-2})fi_{i_2-1} \cdots fin & \text{if } i_1 \text{ is even and } i_2 \neq i_1 - 1, \\
(f_i f_3 \cdots fi_{i_1-1})fi_{i_2-1} \cdots fin & \text{if } i_1 \text{ is even and } i_2 = i_1 - 1, \\
fi_1 \cdots fi_{2k-1}fi_{k-1} \cdots fi_{i_m} & \text{if } i_1 = 4, i_k \leq 2k - 2, \\
fi_{i_2-1}fi_{i_3-1} \cdots fin & \text{if } i_1 = 4, i_k = 2k - 1 \text{ or } i_k \geq 2k + 3.
\end{cases}
\]

The maximal index of \( u_1 \) is always less than \( i_m \), but the maximal index of \( u_0 \) is equal to \( i_m \) if and only if \( i_1 = 3 \) and \( i_k \leq 2k - 2 \).
As an example, consider \( u = f_1 f_3 f_8 \) and \( v = f_3 f_5 f_7 \). For \( u \), we first compute \( \phi(f_3 f_8) = \langle f_1 f_3 f_7, f_1 f_4 f_6 \rangle \zeta \) using the first lines in (11) and (12) and \( k = 2 \); so
\[
\phi(u) = \langle f_1 f_7, f_1 f_3 f_7 \rangle \zeta.
\]
Similarly, \( \phi(v) = \langle f_1 f_7, f_1 f_4 f_6 \rangle \zeta \), using the first lines in (11) and (12) and \( k = 4 \). Note that \( u_0 = v_0 \) while \( u \neq v \); this happened because one of \( u, v \) started with \( f_1 \).

**Lemma 5.8.** The equations \( sf sf^2 x = fx \) and \( sf y = sf y \) have no solution in \( F \).

**Proof.** We will show that if the first equation had a solution \( x \), then the second equation would have a solution \( y \) that is not longer than \( x \); and if the second equation had a solution \( y \), then the first equation would have a solution \( x \) that is shorter than \( y \).

Assume therefore that \( x \) is a solution to the first equation, and write \( \phi(x) = \langle x_0, x_1 \rangle \pi \). Then \( \phi(sf sf^2 x) = \langle sf x_0, sf^2 x_0 \rangle \pi = \langle sf x_0, f x_0 \rangle \pi \), so in particular, by considering the second coordinate, \( y = sx_0 \) is a solution to the second equation, and is not longer than \( x \).

Assume then that \( y \) is a solution to the second equation, and write \( \phi(y) = \langle y_0, y_1 \rangle \rho \). Then \( \langle f y_0, s y_0 \rangle \rho = \langle sf y_1, f y_1 \rangle \zeta \rho \), so \( y_0 = sf y_1 = sf s(y_1) = sf y_0 \) and similarly \( y_1 = sf s y_1 \). Clearly \( y \notin \langle s \rangle \); so \( y = sf z \) for some \( i \in \mathbb{N} \) and \( z \in F \). If \( i \) is odd then \( y_0 \) begins with \( f \), while if \( i \) is even then \( y_1 \) begins with \( f \). In all cases one of \( y_0, y_1 \) begins with \( f \), say \( y_1 = fx \); then \( x \) is a solution to the first equation, and is shorter than \( y \).

**Lemma 5.9.** All elements of \( N \) are distinct in \( F \).

**Proof.** The elements \( e \) and \( f_1 \) are distinct since they act differently on \( X \). In addition, they differ from all other elements of \( F \) since they are invertible.

Consider now two distinct normal forms
\[
u = f_1^i f_{i_1} \cdots f_{i_m} f_{j_n}, \quad v = f_1 f_{j_1} \cdots f_{j_p} f_{j_n},
\]
with \( i_m, j_p \geq 3 \) and \( i_m \geq j_p \).

If \( i_n = j_q = 1 \), then \( 0^o = 1 \) while \( 0^o = 0 \) so they are distinct; if \( i_n \neq 1 \) and \( j_q = 1 \) the same argument applies; if \( i_n = j_q = 1 \) then we may cancel both from the normal form and proceed. Let us therefore suppose \( i_n, j_q > 1 \).

We proceed by induction on the maximal index \( i_m \), to prove that all normal forms with at most that maximal index are distinct in \( F \).

The induction starts with the elements \( f_3, f_5 f_2, f_1 f_3 \) and \( f_1 f_5 f_2 \). They afford the following decompositions:
\[
\phi(f_3) = \langle f_1 f_3, f_1 \rangle \zeta, \quad \phi(f_5 f_2) = \langle f_1 f_5 f_1, e \rangle \zeta, \\
\phi(f_1 f_3) = \langle f_1, f_1 f_3 \rangle \zeta, \quad \phi(f_1 f_5 f_2) = \langle e, f_1 f_5 f_1 \rangle \zeta.
\]
The first coordinates of these decompositions are different by the remarks above. Let us therefore suppose \( i_m \geq 4 \).

If \( \delta = \epsilon = 1 \): up to left-multiplying \( u \) and \( v \) by \( f_1 \), we may replace \( \delta \) and \( \epsilon \) by 0 and proceed.

If \( \delta = \epsilon = 0 \): then
\[
\phi(u) = \langle f_{i_1-1} \cdots f_{i_m-1} f_{i_n-1}, u_1 \rangle \zeta, \\
\phi(v) = \langle f_{j_1-1} \cdots f_{j_p-1} f_{j_q-1}, v_1 \rangle \zeta,
\]
where \( u_1 \) and \( v_1 \) are defined by (12).
If $i_1, j_1 > 3$, then the first coordinates of $\phi(u)$ and $\phi(v)$ are in normal form, and are distinct by assumption, so by induction they are distinct in $F$. Similarly, if $i_1 = j_1 = 3$ then the second coordinates equal $f_1 f_{i_2-1} \cdots f_{i_n-1}$ and $f_1 f_{j_2-1} \cdots f_{j_p-1} \cdots f_{j_q-1}$, and they are in normal form, so by induction $u$ and $v$ are distinct in $F$.

We consider therefore the case $i_1 = 3, j_1 > 3$. Let $k$ be maximal such that $i_r = 2r + 1$ for all $r < k$. Then the first coordinate of $\phi(v)$ is in normal form and starts by $f_{j_1-1}$ while the first coordinate of $\phi(u)$, when put in normal form, is $f_1 f_{2k-1} f_{i_k-1} \cdots f_{i_n-1}$. By the next item or by the induction hypothesis these elements are distinct in $F$, whence $u$ and $v$ are distinct in $F$, too.

The case $i_1 = 3, j_1 > 3$ is considered similarly.

**If $\delta \neq \epsilon$:** by symmetry, we may assume $\delta = 0$ and $\epsilon = 1$. Then 

$$\phi(v) = \langle e_{j_1-1} f_{j_2-1} \cdots f_{j_p-1} \cdots f_{j_q-1} \cdots f_{j_k-1} \cdots f_{j_{n-1}} \rangle.$$ 

Assume for contradiction that $u, v$ are equal in $F$. Then, in $F$, we have 

$$u_0 = f_{i_1-1} \cdots f_{i_{n-1}} = v_1 = e_{j_1-1} f_{j_2-1} \cdots f_{j_p-1} \cdots f_{j_q-1},$$

$$u_1 = e_{i_1-1} f_{i_2-1} \cdots f_{i_{n-1}} = v_0 = f_{j_1-1} \cdots f_{j_k-1} \cdots f_{j_{n-1}},$$

so these words, when put in normal form, must be equal. It follows from Lemma 5.1 that $u_0 = sf u_1$ if $i_1$ is odd and $u_0 = f s u_1$ if $i_1$ is even. Similarly, we have $v_0 = sf v_1$ or $v_0 = f s v_1$, depending on the parity of $j_1$.

We consider in turn all possibilities for the parity of $i_1$ and $j_1$. If $i_1$ is even and $j_1$ is odd, then $u_0 = sf v_0 = sf s v_1 = v_1$; it follows from (12) that when put in normal form, $v_1$ starts with $f$. Using Lemma 5.8, we reach the contradiction. The same argument holds, by symmetry, if $i_1$ is odd and $j_1$ is even.

If $i_1$ and $j_1$ are both even, then we find $(sf)^2 u_0 = u_0$; now by assumption $u_0$ is of the form $f_k \cdots$ for an odd $k$, so its normal form starts with $f$; this contradicts Lemma 5.8.

Finally, if $i_1$ and $j_1$ are both odd, then we find $(fs)^2 u_0 = u_0$, and $u_0$ is of the form $f_k \cdots$ for an even $k$, so its normal form starts with $s$; again this contradicts Lemma 5.8.

**Proof of Theorem 2.3.** We saw in Lemma 5.2.7 that the rewriting system is terminating. On the other hand, consider a word $w \in \{f_1, f_2, \ldots \}^*$ and two reduced words $w', w''$ that are gotten from $w$ by applying two different sequences of elementary reductions. The reduced forms of the rewriting system correspond precisely to the normal form described in Theorem 2.2 and they define equal elements in $F$, so by Lemma 5.3.4 they are graphically equal. This proves that the rewriting system is confluent. 

**Proof of Theorem 2.2.** Every semigroup element can be represented by a word, and this word can be put in reduced form by applying the rules from Theorem 2.3. As noted above, this reduced form coincides with the normal form; therefore the natural map $N \to F$ is onto. It is one-to-one by Lemma 5.2.9.

Finally, all the rules (N1-10) derive from the rules $s^2 = 1$ (N1) and $r_n \to r'_n$ for $n \geq 2$ (N4,8-10), and from graphical equalities (N2-3,5-7). Each of the rules $r_n \to r'_n$ is strictly length-shortening, so $N$ is precisely the set of irreducible words
with respect to the rules, and consists of minimal representative words. In other words, if there were a shorter representation \( w' \) of some \( w \in \mathcal{N} \), then this shorter representation could be reduced to a word \( w'' \in \mathcal{N} \) using the rules \( r_n \mapsto r'_n \), still yielding a word shorter, and therefore different, from \( w \). This contradicts Lemma 5.9.

Now \( \| f_n \| = \Phi_n \), and the initial \( f_1 \) should cancel with the initial \( s \) of \( f_1 \) if \( i_1 \) is odd; this explains the length formula. \( \square \)

**Proof of Theorem 2.1.** The relations \( r'_n = r_n \) for \( n \geq 1 \) hold in \( F \) by Lemma 5.2. They are also sufficient; otherwise, there would be two distinct words \( w, w' \) that are unequal in the abstract semigroup \( \langle s, f | r'_n = r_n \rangle \) but are equal in \( F \); by Theorem 2.2 the normal form representation of \( w \) and \( w' \) are equal; but then \( w \) and \( w' \) can be transformed into each other using only the relations \( r_n = r'_n \), contradicting our initial assumption. \( \square \)

### 5.3. Growth.

**Proof of Theorem 2.4.** Consider a typical element in the ball of radius \( \ell \) in \( F \). It will have a minimal normal form as given by Theorem 2.2, whence \( \Phi_i \leq \ell \). The maximal length of a normal form for that value of \( i_m \) is \( \Phi_{i_m} = \Phi_{i_m-1} + 2\Phi_{i_m-2} + \Phi_{i_m-3} + 2\Phi_{i_m-4} + \cdots < 2\Phi_{i_m+1} \), so it follows that \( \ell < 2\Phi_{i_m+1} \). Now since \( \Phi_i = \frac{\varphi^i}{\sqrt{5}} \), we obtain

\[
\frac{\varphi^{i_m}}{\sqrt{5}} \leq \ell < 2/\sqrt{5}\varphi^{i_m+1},
\]

so

\[
\frac{\log(\sqrt{5}\ell/(2\varphi))}{\log \varphi} \leq i_m \leq \frac{\log(\sqrt{5}\ell)}{\log \varphi}.
\]

Given any value of \( i_m \), there are \( 2^{i_m} \) choices of \( i_j \) to include in the normal form with \( j > m \), and \( 2\Phi_{i_m-2} \) choices for the \( i_j \) to include with \( j < m \): the 2 counts the possible values of \( i_1 \), and the \( 2\Phi_{i_m-2} \) counts the possible values of \( (i_1, \ldots, i_{m-2}) \in \{0, 1\}^{m-4} \) with no two consecutive 1’s. It follows that the ball growth function \( \gamma(\ell) \) is given by

\[
\gamma(\ell) = \sum_{k=1}^{i_m} 2^k 2\Phi_{k-2} \approx \frac{2}{\sqrt{5}\varphi^2} \left( \frac{2\varphi}{2\varphi - 1} \right)^{i_m} - 1,
\]

which gives the upper and lower bounds

\[
\frac{2}{\sqrt{5}\varphi^2(2\varphi - 1)} \left( \frac{\sqrt{5}\ell}{2\varphi} \right)^{\alpha} \leq \gamma(\ell) \leq \frac{2}{\sqrt{5}\varphi^2(2\varphi - 1)} (\sqrt{5}\ell)^{\alpha},
\]

with \( \alpha = \log(2\varphi)/\log \varphi \). \( \square \)

### 5.4. Identities.

**Proof of Theorem 2.3.** More generally, we will show that all of the equalities \( g(wg)^5 = g(wg)^5 \), for all \( w \in \{1, s, f, fs, f^2, f^2s\} \), hold in \( F \). We prove this by considering simultaneously all these equalities, and proceed by induction on \( \| g \| \).

The induction is easily checked if \( \| g \| \leq 2 \), by considering in turn the cases \( g = 1, s, f, sf, fs, f^2 \). For longer \( g \), depending on whether \( g \) starts or ends with an \( s \), we have \( \phi(g) = \ll fh, sh \gg \zeta \) or \( \ll sh, fh \gg \zeta \) or \( \ll fh, sh \gg \zeta \sigma \) or \( \ll sh, fh \gg \zeta \sigma \); for
Proof of Theorem 2.6. Let us prove by induction on $h$. Ideal structure.

\[ \phi(gg^5) = \langle fh(fh)^5, sh(fh)^5 \rangle \zeta = \phi(gg^3), \]
\[ \phi(g(sg)^5) = \langle fh(sh)^5, sh(sh)^5 \rangle \zeta = \phi(g(sg)^3), \]
\[ \phi(g(fg)^5) = \langle fh(f^2h)^5, sh(f^2h)^5 \rangle \zeta = \phi(g(fg)^3), \]
\[ \phi(g(fsg)^5) = \langle fh(fsh)^5, sh(fsh)^5 \rangle \zeta = \phi(g(fsg)^3), \]
\[ \phi(g(f^2g)^5) = \langle fh(f^3h)^5, sh(f^3h)^5 \rangle \zeta = \langle fh(fh)^5, sh(fh)^5 \rangle \zeta = \phi(g(f^2g)^3), \]
\[ \phi(g(f^2sg)^5) = \langle fh(f^3sh)^5, sh(f^3sh)^5 \rangle \zeta = \phi(g(f^2sg)^3) . \]

5.5. Ideal structure.

Proof of Theorem 2.6. Let us prove by induction on $n$ that $z_n(u) = 0^n$ for any $u \in X^n$, and that $f_{n+2}f_{n+1} = z_n$ holds in $W_n$, i.e. that $z_n$ and $f_{n+2}f_{n+1}$ are the left-hand zeroes over $X^n$. For $n = 1$ it follows from the definition of $f_3$ that $f_3 : X^1 \to \{0\}$, whence the relations $f_1f_3 = f_2f_3 = f_3$ and $f_3f_2 = f_3$ hold.

Choose $n \geq 2$. The element $z_n$ has the following decomposition:

\[ z_n = \langle f_2f_1 \cdots f_{n+1}, f_1f_4f_6 \cdots f_{n+1} \rangle \zeta = \langle f_1f_3z_{n-1}, f_1z_{n-1} \rangle \zeta \]
if $n$ is odd, and

\[ z_n = \langle f_3f_5 \cdots f_{n+1}, f_2f_2f_5f_7 \cdots f_{n+1} \rangle \zeta = \langle z_{n-1}, f_1f_3f_1z_{n-1} \rangle \zeta \]
if $n$ is even. Clearly both coordinates of decompositions end with $z_{n-1}$, which is a left-hand zero over $X^{n-1}$ by the induction hypothesis. As $z_n(x) = 0$ for $x \in X$, we have $z_n : X^n \to \{0^n\}$.

Similarly, the element $f_{n+2}f_{n+1}$ is written in the following way:

\[ f_{n+2}f_{n+1} = \langle f_{n+1}f_n, f_1f_3 \cdots f_{n}f_n \rangle \zeta = \langle f_{n+1}f_n, f_4 \cdots f_{n-1}f_{n+1} \rangle \zeta = \langle f_{n+1}f_n, z_{n-1} \rangle \zeta \]
if $n$ is odd, and

\[ f_{n+2}f_{n+1} = \langle f_{n+1}f_n, f_4f_6 \cdots f_{n}f_n \rangle \zeta = \langle f_{n+1}f_n, f_3f_1f_3 \cdots f_{n-1}f_{n+1} \rangle \zeta = \langle f_{n+1}f_n, f_1z_{n-1} \rangle \zeta \]
if $n$ is even. By the induction hypothesis and the proof above for $z_n$, the element $f_{n+2}f_{n+1}$ is a left-hand zero over $X^n$.

As $z_n, f_{n+2}f_{n+1} : X^n \to \{0^n\}$, the relations $f_1z = f_2z = z$ and $f_{n+2}f_{n+1} = z$ hold in $W_n$. We show by induction on the maximal index of a semigroup element that $W_n$ includes no elements with maximal index $> n + 2$. It follows from the definition of $f_n$ that \[ f_{n+3} = f_3f_4f_6f_8 \cdots f_{n+2} = f_3f_2f_n = z_n . \]

For any $k > n + 3$ the equality $f_k = f_{k-2}f_{k-1}$ holds and by the induction hypothesis $f_k = f_{k-2}z_n = z_n$.

It is necessary to prove that the other semigroup elements define different transformations of $X^n$. Once again, we use induction on $n$. For $n = 1$ the four semigroup elements $e, f_1, f_3$ and $f_3f_1$ define different automatic transformations over $X^1$. 

Fix $n \geq 2$. Let us write

$$S_n = \{ g \in F \text{ with maximal index } n \},$$
$$Z_n = \{ g \in S_{n+2} : g \text{ includes } z_n \},$$
$$A_n = \{ g \in S_{n+2} : g \text{ includes neither } z_n \text{ nor } f_{n+2}f_{n+1} \},$$
$$B_n = S_{n+2} \setminus (Z_n \cup A_n).$$

The semigroup $W_n$ is constructed as $W_{n-1} \cup B_{n-1} \cup Z_n \cup A_n$. It follows from the proof above that elements with maximal index $n+2$ act as left-hand zeroes on $X^{n-1}$. Then elements of $Z_n$ and $A_n$ coincide with elements $Z_{n-1}$ over $X^{n-1}$. On the other hand, let $g$ be an arbitrary element from $S_{n+2}$. It follows from (12) that any input symbol decreases the maximal index of at least one coordinate in the decomposition of $g$. Hence, there exists the input word $u$ of length $n-1$ such that it transforms $g$ into the element with maximal index 3. As elements from $S_3$ can be distinguished over $X^2$, elements from $S_{n+2}$ cannot act as a left-hand zero over $X^{n+1}$. Therefore elements from $Z_n$ can be distinguished from elements of $B_{n-1} \cup W_{n-1}$ over $X^n$.

Let $g$ be an arbitrary element from $A_n$. As $g$ does not include $z_n$ and $f_{n+2}f_{n+1}$, the second (or the first, if $g$ starts with $f_1$) coordinate of its decomposition either contains $z_{n-2}$ and has maximal index $n$, or it does not include $z_{n-1}$ and has maximal index $n+1$. In both cases the second coordinate does not act as a left-hand zero over $X^{n-1}$ and therefore $g$ isn’t a left-hand zero over $X^n$. Consequently, $A_n \cap Z_n = \emptyset$, and all elements of $W_n$ define different transformations of $X^n$.

Now we count the elements in $W_n$. An arbitrary element with maximal index $k \geq 3$ can be written as

$$\underbrace{f_1^{\alpha_1}}_{\Phi_{k-2} \text{ possibilities}}\underbrace{f_3^{\alpha_3} \cdots f_{k-2}^{\alpha_{k-2}}}_{2^{k-1} \text{ possibilities}} \underbrace{f_k f_{k-1}^{\beta_{k-1}} \cdots f_2^{\beta_2} f_1^{\beta_1}}_{\Phi_k \text{ possibilities}},$$

where $\beta_j \in \{0, 1\}$ for all $j = 1, 2, \ldots, k-1$, and $\alpha_i \alpha_{i+1} = 0$, $i = 3, \ldots, k-2$. Therefore there are $2 \cdot \Phi_{k-2} \cdot 2^{k-1}$ elements with maximal index $k$. The semigroup $W_n$ includes all elements with maximal index that is $< n+2$, semigroup elements of the form $z f_{k-1}^{\beta_{k-1}} \cdots f_2^{\beta_2} f_1^{\beta_1}$, and semigroup elements with maximal index $(n+2)$ that include neither $z_n$ nor $f_{n+2}f_{n+1}$. Therefore,

$$\#W_n = 2 + \sum_{k=3}^{n+1} 2^k \Phi_{k-2} + 2^n + \left(\frac{1}{2} 2^{n+2} \Phi_n - 2^{n+1} \right)$$
$$= \sum_{k=1}^{n} 2^{k+2} \Phi_k - 2^n - 2^{n+1} \Phi_n + 2. \quad \square$$

To prove Corollary 2.8 we begin by a Lemma:
Lemma 5.10. In the semigroup $F$, we have

$$f_n f_{n-1} = \begin{cases} ssf s f_{n+1} & \text{if } n \geq 4 \text{ is even}, \\ ss f s f_{n+1} & \text{if } n \geq 5 \text{ is odd}; \end{cases}$$

$$z_n = \begin{cases} ssf s f_{n+3} & \text{if } n \geq 2 \text{ is even}, \\ ss f s f_{n+3} & \text{if } n \geq 3 \text{ is odd}; \end{cases}$$

$$f_n = \begin{cases} ssf s f_{n-1} f_{n-2} = f z_{n-3} & \text{if } n \geq 6 \text{ is even}, \\ ss f s f_{n-1} f_{n-2} = ss z_{n-3} & \text{if } n \geq 5 \text{ is odd}; \end{cases}$$

Proof. Induction on $n$, the basis of the induction being easily checked by applying the reduction algorithm. For example, take $n \geq 6$ even: then $ss f s f_{n+1} = ss f s f_{n-1} f_n = f_{n-2} f_{n-3} f_n$ by induction. Then $f_{n-2} f_{n-3} f_n = f_{n-2} f_{n-3} f_{n-2} f_{n-1} = f_{n-2} f_{n-1} = f_n f_{n-1}$. All other cases are treated similarly.

Proof of Corollary 2.8. The proof follows from Theorem 2.6. Assume first that $g$ includes $z_{n-2}$ or $f_n f_{n-1}$. Then it acts as a left-hand zero on $X^{-n}$, whence $\tau(g) \leq 2^{n-2}$. On the other hand, $g$ belongs to $W_{n-1}$ and doesn’t act as a left-hand zero on $X^{n+1}$; therefore $\tau(s) = 2^{-(n-2)}$.

Now assume $g$ includes neither $z_{n-2}$ nor $f_n f_{n-1}$. Then $g$ belongs to $W_{n-2}$, but acts as a left-hand zero on $X^{-n+3}$. Therefore $g(X^*) = x_{\beta_1} x_{\beta_2} \cdots x_{\beta_{n-3}} X^*$, whence $\tau(g) = 2^{-(n-3)}$.

Note next that the generator of the ideal $I_n$ has trace $2^{-n}$. By the proof above, every element of trace $2^{-n}$ is a multiple $f_{n+3}$, of $f_{n+2} f_{n+1}$, or of $z_n$. By Lemma 5.10 the ideals generated by these three elements coincide.

Proof of Theorem 2.3. Let $g$ be an arbitrary element with maximal index $n \geq 3$, generating the ideal $I = F g F$. We show that there exist $\ell(g), r(g) \in F$ such that

$$\ell(g) \cdot g \cdot r(g) = \begin{cases} f_{n+1} & \text{if } g \text{ includes } z_{n-2} \text{ or } f_n f_{n-1} \text{ in its normal form}, \\ f_n & \text{otherwise.} \end{cases}$$

Therefore $I = F g F = F f_k F = F_{2-k+3}$ by Corollary 2.8 where $k = n$ or $k = n + 1$ depending on the cases above.

Let us assume that $g = f_{n \beta_{n-1}} \cdots f_{2 \beta_2} f_1$, where $\beta_i \in \{0, 1\}$. Then we claim that for the element $g_2 = f_{1 \beta_1} f_{2 \beta_2} \cdots f_{n-2 \beta_{n-2}}$ the equality

$$g_2 = f_{n \beta_{n-1}} \cdots f_{2 \beta_2} f_1 = f_{n \beta_{n-1}} \cdots f_{2 \beta_2} f_1 = f_{n \beta_{n-1}} \cdots f_{2 \beta_2} = f_{n \beta_{n-1}}$$

holds. Applying $r_2 = r_2'$ or Lemma 5.3 if necessary,

$$f_{n \beta_{n-1}} \cdots f_{2 \beta_2} f_1 = f_{n \beta_{n-1}} \cdots f_{2 \beta_2} f_1 = f_{n \beta_{n-1}} \cdots f_{2 \beta_2} = f_{n \beta_{n-1}} \cdots f_{3 \beta_2}.$$
Now let \( g = f_3^{a_3} \cdots f_{n-2}^{a_{n-2}} f_n \) with \( \alpha_i\alpha_{i+1} = 0, \alpha_i \in \{0,1\} \), be such that \( g \) does not start with \( f_3 f_5 \) nor \( f_4 f_6 \).

At first we show that for any \( m \geq 6 \) and \( k \leq m - 2 \), with \((k,m) \neq (4,6)\), there exists \( w = w(k,m) \in F \) such that \( w f_k f_m = f_m \). If \( k = 3 \) and \( m = 6 \) then \( w = f_4 f_3 f_1 \). If \( m = 7 \) then all cases of \( k \) are covered by the equalities
\[
\begin{align*}
  f_5 f_4 f_1 \cdot f_5 f_7 &= f_5 f_4 f_2 \cdot f_4 f_7 = f_5 f_4 \cdot f_3 f_7 \\
  &= f_5 f_4 f_3 f_4 f_6 = f_5 f_2 f_6 = f_1 f_5 f_4 f_6 = f_1^2 f_5 f_4 f_5 = f_7.
\end{align*}
\]

Using Lemma 5.6 the element \( w(k,m) \) for \( m \geq 8 \) may be constructed as
\[
w(k,m) = f_{m-2} f_{m-3} f_{m-5} f_{m-7} \cdots f_{k-1} f_k f_m.
\]

Indeed,
\[
w(k,m) \cdot f_k f_m = f_{m-2} f_{m-3} f_{m-5} f_{m-7} \cdots f_{k-1} f_k f_m f_k f_{k-1} \cdots f_k f_m \\
= f_{m-2} f_{m-3} f_{m-5} f_{m-7} \cdots f_{k-1} f_k f_m f_k f_{k+1} f_m \\
= \cdots = f_{m-2} f_{m-3} f_{m-5} f_{m-7} \cdots f_{m-2} f_m = f_m.
\]

Let \( 3 \leq i_1 < i_2 < \cdots < i_k < n - 1 \) be the indices such that \( \alpha_{i_j} = 1 \), with the other \( \alpha_j = 0 \). Let us consider the element
\[
g_1 = w(i_k, n) w(i_{k-1}, i_k) \cdots w(i_2, i_3) w(i_1, i_2).
\]

It follows from Lemma 5.6 that
\[\tag{17} g_1 \cdot g = w(i_k, n) w(i_{k-1}, i_k) \cdots w(i_2, i_3) w(i_1, i_2) \cdot f_3^{a_3} \cdots f_{n-2}^{a_{n-2}} f_n = f_n.\]

Now we construct the elements \( \ell(g) \) and \( r(g) \). Choose
\[
g = f_1^{a_1} f_3^{a_3} \cdots f_{n-2}^{a_{n-2}} f_n f_{n-1}^{\beta_{n-1}} \cdots f_2^{a_2} f_1^{a_1}.
\]

If \( g \) includes \( z_{n-2} \) then it has normal form \( f_1^{a_1} z_{n-2} f_{n-1}^{\beta_{n-1}} \cdots f_2^{a_2} f_1^{a_1} \) and we set
\[
\ell(g) = f_3^{a_3} p_{n-2} f_1^{a_1} \quad \text{and} \quad r(g) = g_2 f_{n-1}^{\beta_{n-1}}. \]

According to the arguments above, we have
\[
\ell(g) \cdot g \cdot r(g) = f_{n+1} f_{n-1}^{2 \beta_{n-1}} = f_{n+1}.
\]

Otherwise \( g \) does not include \( z_{n-2} \). Assume first that \( g \) starts neither with \( f_3 f_5 \) nor with \( f_4 f_6 \). We set \( \ell(g) = g_1 f_1^{a_1} \) and \( r(g) = g_2 \). It follows from (16) and the arguments above that
\[
\ell(g) \cdot g \cdot r(g) = f_n f_{n-1}^{\beta_{n-1}}.
\]

If \( \beta_{n-1} = 0 \), then we are done. If \( \beta_{n-1} = 1 \) it follows from Lemma 5.10 that it suffices to take \( \ell(g) = f_s f_s \cdot g_1 f_1^{a_1} \) if \( n \) is even, and \( \ell(g) = s f s \cdot g_1 f_1^{a_1} \) if \( n \) is odd; then \( \ell(g) \cdot g \cdot r(g) = f_{n+1} \).

The last case is when \( g \) starts with \( f_3^{a_3} f_5 f_3 f_5 \cdots f_m \) or \( f_4^{a_4} f_6 f_4 f_6 \cdots f_m \), with \( 6 \leq m < n - 2 \). We may consider respectively the elements \( f_2 f_1^{a_1} g \) or \( f_3 f_1^{a_1} g \), which satisfy the conditions of the previous paragraph.

\[\square\]

**References**

[1] Alexander G. Abercrombie, *Subgroups and subrings of profinite rings*, Math. Proc. Cambridge Philos. Soc. 116 (1994), no. 2, 209–222.

[2] L. Bartholdi, I. I. Reznykov, and V. I. Sushchansky, *The smallest Mealy automaton of intermediate growth*, J. Algebra 295 (2006), no. 2, 387–414, available at [arXiv:math.GR/0407312](http://arxiv.org/abs/math.GR/0407312) MR 2194959 (2006i:68090)

[3] A. Ya. Belov and I. A. Ivanov, *Construction of semigroups with some exotic properties*, Acta Appl. Math. 85 (2005), no. 1-3, 49–56. MR 2128898 (2005m:20198)
[4] ________, *Construction of semigroups with some exotic properties*, Comm. Algebra **31** (2003), no. 2, 673–696. MR **1968920** (2004a:20064)

[5] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston, *Word processing in groups*, Jones and Bartlett Publishers, 1992.

[6] Israel M. Gel’fand and Alexander A. Kirillov, *On fields connected with the enveloping algebras of Lie algebras*, Dokl. Akad. Nauk SSSR **167** (1966), 503–505.

[7] Rostislav I. Grigorchuk, *On Burnside’s problem on periodic groups*, Функционал. Анал. и Прилож. **14** (1980), no. 1, 53–54. English translation: Functional Anal. Appl. **14** (1980), 41–43.

[8] Günter R. Krause and Thomas H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Revised edition, Graduate Studies in Mathematics, vol. 22, American Mathematical Society, Providence, RI, 2000.

[9] I. I. Reznykov and V. I. Sushchansky, *On the 3-state Mealy automata over an m-symbol alphabet of growth order \([n^{\log n/2\log m}]\)*, J. Algebra **304** (2006), 712–754.

[10] Lev M. Shneerson, *Relatively free semigroups of intermediate growth*, J. Algebra **235** (2001), no. 2, 484–546.

[11] Robert B. Warfield Jr., *The Gelfand-Kirillov dimension of a tensor product*, Math. Z. **185** (1984), no. 4, 441–447. MR **733766** (85f:17006)