SOME GEOMETRIC CALCULATIONS ON WASSERSTEIN SPACE

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Abstract. We compute the Riemannian connection and curvature for the Wasserstein space of a smooth compact Riemannian manifold.

1. Introduction

If $M$ is a smooth compact Riemannian manifold then the Wasserstein space $P_2(M)$ is the space of Borel probability measures on $M$, equipped with the Wasserstein metric $W_2$. We refer to [21] for background information on Wasserstein spaces. The Wasserstein space originated in the study of optimal transport. It has had applications to PDE theory [16], metric geometry [8, 19, 20] and functional inequalities [9, 17].

Otto showed that the heat flow on measures can be considered as a gradient flow on Wasserstein space [16]. In order to do this, he introduced a certain formal Riemannian metric on the Wasserstein space. This Riemannian metric has some remarkable properties. Using O’Neill’s theorem, Otto gave a formal argument that $P_2(\mathbb{R}^n)$ has nonnegative sectional curvature. This was made rigorous in [8, Theorem A.8] and [19, Proposition 2.10] in the following sense: $M$ has nonnegative sectional curvature if and only if the length space $P_2(M)$ has nonnegative Alexandrov curvature.

In this paper we study the Riemannian geometry of the Wasserstein space. In order to write meaningful expressions, we restrict ourselves to the subspace $P_\infty(M)$ of absolutely continuous measures with a smooth positive density function. The space $P_\infty(M)$ is a smooth infinite-dimensional manifold in the sense, for example, of [7]. The formal calculations that we perform can be considered as rigorous calculations on this smooth manifold, although we do not emphasize this point.

In Section 3 we show that if $c$ is a smooth immersed curve in $P_\infty(M)$ then its length in $P_2(M)$, in the sense of metric geometry, equals its Riemannian length as computed with Otto’s metric. In Section 4 we compute the Levi-Civita connection on $P_\infty(M)$. We use it to derive the equation for parallel transport and the geodesic equation.

In Section 5 we compute the Riemannian curvature of $P_\infty(M)$. The answer is relatively simple. As an application, if $M$ has sectional curvatures bounded below by $r \in \mathbb{R}$, one can ask whether $P_\infty(M)$ necessarily has sectional curvatures bounded below by $r$. This turns out to be the case if and only if $r = 0$.

There has been recent interest in doing Hamiltonian mechanics on the Wasserstein space of a symplectic manifold [1, 4, 5]. In Section 6 we briefly describe the Poisson geometry.
of $P^\infty(M)$. We show that if $M$ is a Poisson manifold then $P^\infty(M)$ has a natural Poisson structure. We also show that if $M$ is symplectic then the symplectic leaves of the Poisson structure on $P^\infty(M)$ are the orbits of the group of Hamiltonian diffeomorphisms, thereby making contact with [11, 5]. This approach is not really new; closely related results, with applications to PDEs, were obtained quite a while ago by Alan Weinstein and collaborators [10, 11, 22]. However, it may be worth advertising this viewpoint.

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2. Manifolds of measures

In what follows, we use the Einstein summation convention freely.

Let $M$ be a smooth connected closed Riemannian manifold of positive dimension. We denote the Riemannian density by $d\text{vol}_M$. Let $P_2(M)$ denote the space of Borel probability measures on $M$, equipped with the Wasserstein metric $W_2$. For relevant results about optimal transport and the Wasserstein metric, we refer to [8, Sections 1 and 2] and references therein.

Put
\begin{equation}
(2.1) \quad P^\infty(M) = \{\rho \, d\text{vol}_M : \rho \in C^\infty(M), \rho > 0, \int_M \rho \, d\text{vol}_M = 1\}.
\end{equation}

Then $P^\infty(M)$ is a dense subset of $P_2(M)$, as is the complement of $P^\infty(M)$ in $P_2(M)$. We do not claim that $P^\infty(M)$ is necessarily a totally convex subset of $P_2(M)$, i.e. that if $\mu_0, \mu_1 \in P^\infty(M)$ then the minimizing geodesic in $P_2(M)$ joining them necessarily lies in $P^\infty(M)$. However, the absolutely continuous probability measures on $M$ do form a totally convex subset of $P_2(M)$ [12]. For the purposes of this paper, we give $P^\infty(M)$ the smooth topology. (This differs from the subspace topology on $P^\infty(M)$ coming from its inclusion in $P_2(M)$.) Then $P^\infty(M)$ has the structure of an infinite-dimensional smooth manifold in the sense of [7]. The formal calculations in this paper can be rigorously justified as being calculations on the smooth manifold $P^\infty(M)$. However, we will not belabor this point.

Given $\phi \in C^\infty(M)$, define $F_\phi \in C^\infty(P^\infty(M))$ by
\begin{equation}
(2.2) \quad F_\phi(\rho \, d\text{vol}_M) = \int_M \phi \, \rho \, d\text{vol}_M.
\end{equation}

This gives an injection $P^\infty(M) \to (C^\infty(M))^*$, i.e. the functions $F_\phi$ separate points in $P^\infty(M)$. We will think of the functions $F_\phi$ as “coordinates” on $P^\infty(M)$.

Given $\phi \in C^\infty(M)$, define a vector field $V_\phi$ on $P^\infty(M)$ by saying that for $F \in C^\infty(P^\infty(M))$,
\begin{equation}
(2.3) \quad (V_\phi F)(\rho \, d\text{vol}_M) = \frac{d}{d\epsilon}|_{\epsilon=0} F(\rho \, d\text{vol}_M - \epsilon \nabla^i(\rho \nabla_i \phi) \, d\text{vol}_M).
\end{equation}

The map $\phi \to V_\phi$ passes to an isomorphism $C^\infty(M)/\mathbb{R} \to T_{\rho \, d\text{vol}_M}P^\infty(M)$. This parametrization of $T_{\rho \, d\text{vol}_M}P^\infty(M)$ goes back to Otto’s paper [16]; see [2] for further discussion. Otto’s
Riemannian metric on $P^\infty(M)$ is given \cite{[16]} by
\begin{equation}
(2.4) \quad \langle V_{\phi_1}, V_{\phi_2}\rangle(\rho \, d\text{vol}_M) = \int_M \langle \nabla \phi_1, \nabla \phi_2 \rangle \, \rho \, d\text{vol}_M
\end{equation}
\begin{equation}
= - \int_M \phi_1 \nabla(\rho \nabla \phi_2) \, d\text{vol}_M.
\end{equation}
In view of (2.3), we write $\delta_{V_{\phi_1}} \rho = - \nabla(\rho \nabla \phi)$. Then
\begin{equation}
(2.5) \quad \langle V_{\phi_1}, V_{\phi_2}\rangle(\rho \, d\text{vol}_M) = \int_M \phi_1 \delta_{V_{\phi_2}} \rho \, d\text{vol}_M = \int_M \phi_2 \delta_{V_{\phi_1}} \rho \, d\text{vol}_M.
\end{equation}

In terms of the weighted $L^2$-spaces $L^2(M, \rho \, d\text{vol}_M)$ and $\Omega^1_{L^2}(M, \rho \, d\text{vol}_M)$, let $d$ be the usual differential on functions and let $d^*_\rho$ be its formal adjoint. Then (2.4) can be written as
\begin{equation}
(2.6) \quad \langle V_{\phi_1}, V_{\phi_2}\rangle(\rho \, d\text{vol}_M) = \int_M \langle d\phi_1, d\phi_2 \rangle(\rho \, d\text{vol}_M) = \int_M \phi_1 (d^*_\rho d\phi_2 \rho \, d\text{vol}_M).
\end{equation}

We now relate the function $F_\phi$ and the vector field $V_\phi$.

**Lemma 2.7.** The gradient of $F_\phi$ is $V_\phi$.

**Proof.** Letting $\nabla F_\phi$ denote the gradient of $F_\phi$, for all $\phi' \in C^\infty(M)$ we have
\begin{equation}
(2.8) \quad \langle \nabla F_\phi, V_{\phi'} \rangle(\rho \, d\text{vol}_M) = (V_{\phi'} F_\phi)(\rho \, d\text{vol}_M) = - \int_M \phi \nabla(\rho \nabla \phi') \, d\text{vol}_M
\end{equation}
\begin{equation}
= \langle V_\phi, V_{\phi'} \rangle(\rho \, d\text{vol}_M).
\end{equation}
This proves the lemma. \qed

3. LENGTHS OF CURVES

In this section we relate the Riemannian metric (2.4) to the Wasserstein metric. One such relation was given in \cite{[17]}, where it was heuristically shown that the geodesic distance coming from (2.4) equals the Wasserstein metric. To give a rigorous relation, we recall that a curve $c : [0, 1] \to P_2(M)$ has a length given by
\begin{equation}
(3.1) \quad L(c) = \sup_{J \in \mathbb{N}} \sup_{0=t_0 \leq t_1 \leq \ldots \leq t_J=1} \sum_{j=1}^J W_2(c(t_{j-1}), c(t_j)).
\end{equation}
From the triangle inequality, the expression $\sum_{j=1}^J W_2(c(t_{j-1}), c(t_j))$ is nondecreasing under a refinement of the partition $0 = t_0 \leq t_1 \leq \ldots \leq t_J = 1$.

If $c : [0, 1] \to P^\infty(M)$ is a smooth curve in $P^\infty(M)$ then we write $c(t) = (M, \rho(t) \, d\text{vol}_M$ and let $\phi(t)$ satisfy $\frac{\partial}{\partial t} = - \nabla(\rho \nabla \phi)$, where we normalize $\phi$ by requiring for example that $\int_M \phi (\rho \, d\text{vol}_M = 0$. If $c$ is immersed then $\nabla \phi(t) \neq 0$. The Riemannian length of $c$, as computed using (2.4), is
\begin{equation}
(3.2) \quad \int_0^1 \langle c'(t), c'(t) \rangle^\frac{1}{2} \, dt = \int_0^1 \left( \int_M |\nabla \phi(t)|^2(m) \, \rho(t) \, d\text{vol}_M \right)^\frac{1}{2} \, dt.
\end{equation}
The next proposition says that this equals the length of $c$ in the metric sense.

**Proposition 3.3.** If $c : [0, 1] \to P^\infty(M)$ is a smooth immersed curve then its length $L(c)$ in the Wasserstein space $P_2(M)$ satisfies

$$L(c) = \int_0^1 \langle c'(t), c'(t) \rangle^{\frac{1}{2}} dt.$$ \hspace{1cm} (3.4)

**Proof.** We can parametrize $c$ so that $\int_M |\nabla \phi(t)|^2 \rho(t) \, d\text{vol}_M$ is a constant $C > 0$ with respect to $t$.

Let $\{S_t\}_{t \in [0, 1]}$ be the one-parameter family of diffeomorphisms of $M$ given by

$$\frac{\partial S_t(m)}{\partial t} = (\nabla \phi(t))(S_t(m))$$ \hspace{1cm} (3.5)

with $S_0(m) = m$. Then $c(t) = (S_t)_* (\rho(0) \, d\text{vol}_M)$.

Given a partition $0 = t_0 \leq t_1 \leq \ldots \leq t_J = 1$ of $[0, 1]$, a particular transference plan from $c(t_{j-1})$ to $c(t_j)$ comes from the Monge transport $S_{t_j} \circ S_{t_{j-1}}^{-1}$. Then

$$W_2(c(t_{j-1}), c(t_j))^2 \leq \int_M d(m, S_{t_j} (S_{t_{j-1}}^{-1}(m)))^2 \rho(t_{j-1}) \, d\text{vol}_M$$

$$= \int_M d(S_{t_{j-1}}(m), S_{t_j}(m))^2 \rho(0) \, d\text{vol}_M$$

$$\leq \int_M \left( \int_{t_{j-1}}^{t_j} |\nabla \phi(t)|(S_t(m)) \, dt \right)^2 \rho(0) \, d\text{vol}_M$$

$$\leq (t_j - t_{j-1}) \int_M \int_{t_{j-1}}^{t_j} |\nabla \phi(t)|^2(S_t(m)) \, dt \rho(0) \, d\text{vol}_M$$

$$= (t_j - t_{j-1}) \int_{t_{j-1}}^{t_j} \int_M |\nabla \phi(t)|^2(m) \rho(t) \, d\text{vol}_M \, dt,$$

so

$$W_2(c(t_{j-1}), c(t_j)) \leq (t_j - t_{j-1})^{\frac{1}{2}} \left( \int_{t_{j-1}}^{t_j} \int_M |\nabla \phi(t)|^2(m) \rho(t) \, d\text{vol}_M \, dt \right)^{\frac{1}{2}}$$ \hspace{1cm} (3.6)

$$= (t_j - t_{j-1}) \left( \int_M |\nabla \phi(t'_j)|^2(m) \rho(t'_j) \, d\text{vol}_M \right)^{\frac{1}{2}}$$

for some $t'_j \in [t_{j-1}, t_j]$. It follows that

$$L(c) \leq \int_0^1 \langle c'(t), c'(t) \rangle^{\frac{1}{2}} dt.$$ \hspace{1cm} (3.8)
Next, from \[8, \text{Lemma A.1}],

\[
(t_j - t_{j-1}) \left| \int_M \phi(t_{j-1}) \rho(t_j) \, d\text{vol}_M - \int_M \phi(t_{j-1}) \rho(t_{j-1}) \, d\text{vol}_M \right|^2 \leq W_2(c(t_{j-1}), c(t_j))^2 \int_{t_{j-1}}^{t_j} \int_M |\nabla \phi(t_{j-1})|^2 \, d\mu_t \, dt,
\]

where \(\{\mu_t\}_{t \in [t_{j-1}, t_j]}\) is the Wasserstein geodesic between \(c(t_{j-1})\) and \(c(t_j)\). Now

\[
\int_M \phi(t_{j-1}) \rho(t_j) \, d\text{vol}_M - \int_M \phi(t_{j-1}) \rho(t_{j-1}) \, d\text{vol}_M = - \int_M \int_{t_{j-1}}^{t_j} \phi(t_{j-1}) \nabla_i (\rho(t) \nabla_i \phi(t)) \, dt \, d\text{vol}_M = \int_{t_{j-1}}^{t_j} \int_M \langle \nabla \phi(t_{j-1}), \nabla \phi(t) \rangle \rho(t) \, d\text{vol}_M \, dt,
\]

so (3.9) becomes

\[
(t_j - t_{j-1}) \left( \int_{t_{j-1}}^{t_j} \int_M \langle \nabla \phi(t_{j-1}), \nabla \phi(t) \rangle \rho(t) \, d\text{vol}_M \, dt \right)^2 \leq W_2(c(t_{j-1}), c(t_j))^2 \int_{t_{j-1}}^{t_j} \int_M |\nabla \phi(t_{j-1})|^2 \, d\mu_t \, dt.
\]

Thus

\[
L(c) \geq \sum_{j=1}^{J} \frac{\int_{t_{j-1}}^{t_j} \int_M \langle \nabla \phi(t_{j-1}), \nabla \phi(t) \rangle \rho(t) \, d\text{vol}_M \, dt}{\sqrt{\frac{1}{t_j-t_{j-1}} \int_{t_{j-1}}^{t_j} \int_M |\nabla \phi(t_{j-1})|^2 \, d\mu_t \, dt}} (t_j - t_{j-1}).
\]

As the partition of [0, 1] becomes finer, the term \(\frac{\int_{t_{j-1}}^{t_j} \int_M \langle \nabla \phi(t_{j-1}), \nabla \phi(t) \rangle \rho(t) \, d\text{vol}_M \, dt}{t_j-t_{j-1}}\) uniformly approaches the constant \(C\).
The Wasserstein geodesic \( \{\mu_t\}_{t \in [t_{j-1}, t_j]} \) has the form \( \mu_t = (F_t)_*\mu_{t_{j-1}} \) for measurable maps \( F_t : M \to M \) with \( F_{t_{j-1}} = \text{Id} \) [12]. Then

\[
(3.13) \quad \left| \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \int_M |\nabla\phi(t_{j-1})|^2 \, d\mu_t \, dt - C \right| = \\
\left| \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \left( \int_M |\nabla\phi(t_{j-1})|^2 \, d\mu_t - \int_M |\nabla\phi(t_{j-1})|^2 \, d\mu_{t_{j-1}} \right) \, dt \right| = \\
\left| \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \int_M (|\nabla\phi(t_{j-1})|^2 \circ F_t - |\nabla\phi(t_{j-1})|^2) \, d\mu_{t_{j-1}} \, dt \right| \leq \\
\frac{1}{t_j - t_{j-1}} \| \nabla|\nabla\phi(t_{j-1})|^2 \|_\infty \int_{t_{j-1}}^{t_j} \int_M (d(m, F_t(m)) \, d\mu_{t_{j-1}}(m) \, dt \leq \\
\frac{1}{t_j - t_{j-1}} \| \nabla|\nabla\phi(t_{j-1})|^2 \|_\infty \int_{t_{j-1}}^{t_j} \sqrt{\int_M d(m, F_t(m))^2 \, d\mu_{t_{j-1}}(m) \, dt} = \\
\frac{1}{t_j - t_{j-1}} \| \nabla|\nabla\phi(t_{j-1})|^2 \|_\infty \int_{t_{j-1}}^{t_j} W_2(\mu_{t_{j-1}}, \mu_t) \, dt \leq \\
\| \nabla|\nabla\phi(t_{j-1})|^2 \|_\infty W_2(c(t_{j-1}), c(t_j)).
\]

Now continuity of a 1-parameter family of smooth measures in the smooth topology implies continuity in the weak-* topology, which is metricized by \( W_2 \) (as \( M \) is compact). It follows that as the partition of \([0, 1]\) becomes finer, the term \( \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \int_M |\nabla\phi(t_{j-1})|^2 \, d\mu_t \, dt \) uniformly approaches the constant \( C \). Thus from (3.12),

\[
(3.14) \quad L(c) \geq \sqrt{C} = \int_0^1 \langle c'(t), c'(t) \rangle^{\frac{1}{2}} \, dt.
\]

This proves the proposition. \( \Box \)

Remark 3.15. Let \( X \) be a finite-dimensional Alexandrov space and let \( R \) be its set of nonsingular points. There is a continuous Riemannian metric \( g \) on \( R \) so that lengths of curves in \( R \) can be computed using \( g \) [15]. (Note that in general, \( R \) and \( X - R \) are dense in \( X \).) This is somewhat similar to the situation for \( P^\infty(M) \subset P_2(M) \).

In fact, there is an open dense subset \( O \subset X \) with a Lipschitz manifold structure and a Riemannian metric of bounded variation that extends \( g \) [18]. We do not know if there is a Riemannian manifold structure, in some appropriate sense, on an open dense subset of \( P_2(M) \). Other approaches to geometrizing \( P_2(M) \), with a view toward gradient flow, are in [2] [3]; see also [14].

4. Levi-Civita connection, parallel transport and geodesics

In this section we compute the Levi-Civita connection of \( P^\infty(M) \). We derive the formula for parallel transport in \( P^\infty(M) \) and the geodesic equation for \( P^\infty(M) \).
We first compute commutators of our canonical vector fields \( \{V_\phi\}_{\phi \in C^\infty(M)} \).

**Lemma 4.1.** Given \( \phi_1, \phi_2 \in C^\infty(M) \), the commutator \( [V_{\phi_1}, V_{\phi_2}] \) is given by

\[
[[V_{\phi_1}, V_{\phi_2}]](\rho \ dvol_M) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F(\rho \ dvol_M - \epsilon \nabla_i \left[ \rho \left( (\nabla^i \nabla^j \phi_2) \nabla_j \phi_1 - (\nabla^i \nabla^j \phi_1) \nabla_j \phi_2 \right) \right] \ dvol_M)
\]

for \( F \in C^\infty(P^\infty(M)) \).

**Proof.** We have

\[
[[V_{\phi_1}, V_{\phi_2}]](\rho \ dvol_M) = (V_{\phi_1}(V_{\phi_2} F)) (\rho \ dvol_M) - (V_{\phi_2}(V_{\phi_1} F)) (\rho \ dvol_M) =
\]

\[
\left. \frac{d}{d\epsilon_1} \right|_{\epsilon_1=0} (V_{\phi_2} F)(\rho \ dvol_M - \epsilon_1 \nabla^i(\rho \nabla_i \phi_1) \ dvol_M) - \left. \frac{d}{d\epsilon_2} \right|_{\epsilon_2=0} (V_{\phi_1} F)(\rho \ dvol_M - \epsilon_2 \nabla^i(\rho \nabla_i \phi_2) \ dvol_M) =
\]

\[
\left. \frac{d}{d\epsilon_1} \right|_{\epsilon_1=0} \left. \frac{d}{d\epsilon_2} \right|_{\epsilon_2=0} F((\rho - \epsilon_1 \nabla^i(\rho \nabla_i \phi_1)) \ dvol_M - \epsilon_2 \nabla^j((\rho - \epsilon_1 \nabla^i(\rho \nabla_i \phi_1)) \nabla_j \phi_2) \ dvol_M) =
\]

\[
\left. \frac{d}{d\epsilon_2} \right|_{\epsilon_2=0} \left. \frac{d}{d\epsilon_1} \right|_{\epsilon_1=0} F((\rho - \epsilon_2 \nabla^i(\rho \nabla_i \phi_2)) \ dvol_M - \epsilon_1 \nabla^j((\rho - \epsilon_2 \nabla^i(\rho \nabla_i \phi_2)) \nabla_j \phi_1) \ dvol_M) =
\]

\[
\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F(\rho \ dvol_M + \epsilon \nabla^i((\rho \nabla_i \phi_1) \nabla_j \phi_2) \ dvol_M - \epsilon \nabla^j((\rho \nabla_i \phi_2) \nabla_j \phi_1) \ dvol_M).
\]

One can check that

\[
\nabla^i((\rho \nabla_i \phi_1) \nabla_j \phi_2) - \nabla^i((\rho \nabla_i \phi_2) \nabla_j \phi_1) = -\nabla_i \left[ \rho \left( (\nabla^i \nabla^j \phi_2) \nabla_j \phi_1 - (\nabla^i \nabla^j \phi_1) \nabla_j \phi_2 \right) \right],
\]

from which the lemma follows. \(\square\)

We now compute the Levi-Civita connection.

**Proposition 4.5.** The Levi-Civita connection \( \nabla \) of \( P^\infty(M) \) is given by

\[
((\nabla_{V_{\phi_1}} V_{\phi_2}) F)(\rho \ dvol_M) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F(\rho \ dvol_M - \epsilon \nabla_i \left( \rho \nabla_j \phi_1 \nabla^i \nabla^j \phi_2 \right) \ dvol_M)
\]

for \( F \in C^\infty(P^\infty(M)) \).

**Proof.** Define a vector field \( D_{V_{\phi_1}} V_{\phi_2} \) by

\[
((D_{V_{\phi_1}} V_{\phi_2}) F)(\rho \ dvol_M) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F(\rho \ dvol_M - \epsilon \nabla_i \left( \rho \nabla_j \phi_1 \nabla^i \nabla^j \phi_2 \right) \ dvol_M)
\]

for \( F \in C^\infty(P^\infty(M)) \). We also write

\[
\delta_{D_{V_{\phi_1}} V_{\phi_2}} \rho = -\nabla_i \left( \rho \nabla_j \phi_1 \nabla^i \nabla^j \phi_2 \right).
\]

It is clear from Lemma 4.1 that

\[
D_{V_{\phi_1}} V_{\phi_2} - D_{V_{\phi_2}} V_{\phi_1} = [V_{\phi_1}, V_{\phi_2}].
\]
Next,

\[ (V_{\phi_1}(V_{\phi_2}, V_{\phi_3}) \rho \, \text{dvol}_M) = -\int_M \nabla^i \phi_2 \nabla_i \phi_3 \nabla^j (\rho \nabla_j \phi_1) \, \text{dvol}_M \]

\[ = \int_M \nabla_j \phi_1 \nabla^i \phi_2 \nabla_i \phi_3 \rho \, \text{dvol}_M + \int_M \nabla_j \phi_1 \nabla^i \phi_3 \nabla_i \phi_2 \rho \, \text{dvol}_M \]

\[ = -\int_M \phi_3 \nabla_i (\rho \nabla_j \phi_1 \nabla^i \phi_2) \, \text{dvol}_M - \int_M \phi_2 \nabla_i (\rho \nabla_j \phi_1 \nabla^i \phi_3) \, \text{dvol}_M \]

\[ = \int_M \phi_3 \delta_{DV_{\phi_1} V_{\phi_2}} \rho \, \text{dvol}_M + \int_M \phi_2 \delta_{DV_{\phi_1} V_{\phi_3}} \rho \, \text{dvol}_M \]

\[ = \langle DV_{\phi_1} V_{\phi_2}, V_{\phi_3} \rangle (\rho \, \text{dvol}_M) + \langle V_{\phi_2}, DV_{\phi_1} V_{\phi_3} \rangle (\rho \, \text{dvol}_M). \]

Thus

\[ (4.11) \quad V_{\phi_1}(V_{\phi_2}, V_{\phi_3}) = \langle DV_{\phi_1} V_{\phi_2}, V_{\phi_3} \rangle + \langle V_{\phi_2}, DV_{\phi_1} V_{\phi_3} \rangle. \]

As

\[ (4.12) \quad 2\langle \nabla_{V_{\phi_1}} V_{\phi_2}, V_{\phi_3} \rangle = V_{\phi_1} \langle V_{\phi_2}, V_{\phi_3} \rangle + V_{\phi_2} \langle V_{\phi_3}, V_{\phi_1} \rangle - V_{\phi_3} \langle V_{\phi_1}, V_{\phi_2} \rangle + \langle V_{\phi_3}, [V_{\phi_1}, V_{\phi_2}] \rangle - \langle V_{\phi_2}, [V_{\phi_1}, V_{\phi_3}] \rangle - \langle V_{\phi_1}, [V_{\phi_2}, V_{\phi_3}] \rangle, \]

substituting (4.9) and (4.11) into the right-hand side of (4.12) shows that

\[ (4.13) \quad \langle \nabla_{V_{\phi_1}} V_{\phi_2}, V_{\phi_3} \rangle = \langle DV_{\phi_1} V_{\phi_2}, V_{\phi_3} \rangle \]

for all \( \phi_3 \in C^\infty(M) \). The proposition follows. \( \square \)

**Lemma 4.14.** The connection coefficients at \( \rho \text{ dvol}_M \) are given by

\[ (4.15) \quad \langle \nabla_{V_{\phi_1}} V_{\phi_2}, V_{\phi_3} \rangle = \int_M \nabla_i \phi_1 \nabla_j \phi_3 \nabla^i \phi_2 \rho \, \text{dvol}_M. \]

**Proof.** This follows from (2.5) and (4.6). \( \square \)

Let \( G_\rho \) be the Green’s operator for \( d_\rho^*d \) on \( L^2(M, \rho \text{ dvol}_M) \). (More explicitly, if \( \int_M f \rho \text{ dvol}_M = 0 \) and \( \phi = G_\rho f \) then \( \phi \) satisfies \(-\frac{1}{\rho} \nabla^i (\rho \nabla_i \phi) = f \) and \( \int_M \phi \rho \text{ dvol}_M = 0 \), while \( G_\rho 1 = 0 \).)

Let \( \Pi_\rho \) denote orthogonal projection onto \( \text{Im}(d) \subset \Omega^1_{\text{L2}}(M, \rho \text{ dvol}_M). \)

**Lemma 4.16.** At \( \rho \text{ dvol}_M \), we have \( \nabla_{V_{\phi_1}} V_{\phi_2} = V_{\phi} \), where \( \phi = G_\rho d_\rho^* (\nabla_i \nabla_j \phi_2 \nabla^j \phi_1 \, dx^i) \).
**Proof.** Given \( \phi_3 \in C^\infty(M) \), we have

\[
\langle V_{\phi_3}, V_\phi \rangle (\rho \, d\text{vol}_M) = \int_M \langle d\phi_3, dG_\rho \rho^* (\nabla_i \nabla_j \phi_2 \nabla_j \phi_1 \, dx^i) \rangle \, \rho \, d\text{vol}_M \\
= \int_M \langle d\phi_3, \Pi_\rho (\nabla_i \nabla_j \phi_2 \nabla_j \phi_1 \, dx^i) \rangle \, \rho \, d\text{vol}_M \\
= \int_M \langle d\phi_3, \nabla_i \nabla_j \phi_2 \nabla_j \phi_1 \, dx^i \rangle \, \rho \, d\text{vol}_M = \langle V_{\phi_3}, \nabla_{V_{\phi_1}} V_{\phi_2} \rangle (\rho \, d\text{vol}_M).
\]

The lemma follows. \( \square \)

To derive the equation for parallel transport, let \( c : (a, b) \to \mathcal{P}_\infty(M) \) be a smooth curve. As before, we write \( c(t) = \rho(t) \, d\text{vol}_M \) and define \( \phi(t) \in C^\infty(M) \), up to a constant, by \( \frac{dc}{dt} = V_{\phi(t)} \). Let \( V_{\eta(t)} \) be a vector field along \( c \), with \( \eta(t) \in C^\infty(M) \). If \( \{ \phi_\alpha \}_{\alpha=1}^\infty \) is a basis for \( C^\infty(M)/\mathbb{R} \) then \( \{ V_{\phi_\alpha} \}_{\alpha=1}^\infty \) is a global basis for \( TP^\infty(M) \) and we can write \( \eta(t) = \sum_\alpha \eta_\alpha(t) V_{\phi_\alpha} \big|_{c(t)} \). The condition for \( V_\eta \) to be parallel along \( c \) is

\[
\sum_\alpha \frac{d\eta_\alpha}{dt} \big|_{c(t)} V_{\phi_\alpha} + \sum_\alpha \eta_\alpha(t) \nabla_{V_{\phi(t)}} V_{\eta_\alpha} \big|_{c(t)} = 0,
\]

or

\[
\frac{d\eta}{dt} + \nabla_{V_{\phi(t)}} V_{\eta} = 0.
\]

**Proposition 4.20.** The equation for \( V_\eta \) to be parallel along \( c \) is

\[
\nabla_i \left( \rho \left( \nabla^j \frac{\partial \eta}{\partial t} + \nabla_j \phi \nabla^j \eta \right) \right) = 0.
\]

**Proof.** This follows from (2.3), (4.6) and (4.19). \( \square \)

As a check on equation (4.21), we show that parallel transport along \( c \) preserves the inner product.

**Lemma 4.22.** If \( V_{\eta_1} \) and \( V_{\eta_2} \) are parallel vector fields along \( c \) then \( \int_M \langle \nabla \eta_1, \nabla \eta_2 \rangle \, \rho \, d\text{vol}_M \) is constant in \( t \).
Proof. We have
\begin{align}
\frac{d}{dt} \int_M \langle \nabla \eta_1, \nabla \eta_2 \rangle \rho \, d\text{vol}_M &= \int_M \nabla^i \frac{\partial \eta_1}{\partial t} \nabla_i \eta_2 \rho \, d\text{vol}_M + \int_M \nabla_i \eta_1 \nabla^i \frac{\partial \eta_2}{\partial t} \rho \, d\text{vol}_M - \\
&\quad \int_M \nabla_i \eta_1 \nabla^i \eta_2 \nabla^j (\rho \nabla_j \phi) \, d\text{vol}_M \\
&= \int_M \nabla^i \frac{\partial \eta_1}{\partial t} \nabla_i \eta_2 \rho \, d\text{vol}_M + \int_M \nabla_i \eta_1 \nabla^i \frac{\partial \eta_2}{\partial t} \rho \, d\text{vol}_M + \\
&\quad \int_M \left( \nabla^i \nabla_j \eta_1 \nabla_i \eta_2 + \nabla_i \eta_1 \nabla^i \nabla_j \eta_2 \right) \nabla_j \phi \rho \, d\text{vol}_M \\
&= - \int_M \nabla_2 \nabla_i \left( \rho \left( \nabla^i \frac{\partial \eta_1}{\partial t} + \nabla_j \phi \nabla^i \nabla^j \eta_2 \right) \right) \, d\text{vol}_M - \\
&\quad \int_M \eta_2 \nabla_i \left( \rho \left( \nabla^i \frac{\partial \eta_2}{\partial t} + \nabla_j \phi \nabla^i \nabla^j \eta_2 \right) \right) \, d\text{vol}_M \\
&= 0.
\end{align}

This proves the lemma. \qed

Finally, we derive the geodesic equation.

**Proposition 4.24.** The geodesic equation for $c$ is
\begin{equation}
\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 = 0,
\end{equation}
modulo the addition of a spatially-constant function to $\phi$.

**Proof.** Taking $\eta = \phi$ in (4.21) gives
\begin{equation}
\nabla_i \left( \rho \nabla^i \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) \right) = 0.
\end{equation}

Thus $\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2$ is spatially constant. Redefining $\phi$ by adding to it a function of $t$ alone, we can assume that (4.25) holds. \qed

**Remark 4.27.** Equation (4.25) has been known for a while, at least in the case of $\mathbb{R}^n$, to be the formal equation for Wasserstein geodesics. For general Riemannian manifolds $M$, it was formally derived as the Wasserstein geodesic equation in [17] by minimizing lengths of curves. For $t > 0$, it has the Hopf-Lax solution
\begin{equation}
\phi(t, m) = \inf_{m' \in M} \left( \phi(0, m') + \frac{d(m, m')^2}{2t} \right).
\end{equation}

Given $\mu_0, \mu_1 \in P^\infty(M)$, it is known that there is a unique minimizing Wasserstein geodesic $\{\mu_t\}_{t \in [0,1]}$ joining them. It is of the form $\mu_t = (F_t)_* \mu_0$, where $F_t \in \text{Diff}(M)$ is given by $F_t(m) = \exp_m(-t \nabla_m \phi_0)$ for an appropriate Lipschitz function $\phi_0$ [12]. If $\phi_0$ happens to be smooth then defining $\rho(t)$ by $\mu_t = \rho(t) \, d\text{vol}_M$ and defining $\phi(t) \in C^\infty(M) / \mathbb{R}$
as above, it is known that $\phi$ satisfies (4.25), with $\phi(0) = \phi_0$ [21, Section 5.4.7]. In this way, (4.25) rigorously describes certain geodesics in the Wasserstein space $P^2(M)$.

5. Curvature

In this section we compute the Riemannian curvature tensor of $P^\infty(M)$. Given $\phi, \phi' \in C^\infty(M)$, define $T_{\phi\phi'} \in \Omega^1_{L^2}(M)$ by

\begin{equation}
T_{\phi\phi'} = (I - \Pi_\rho) \left( \nabla^i \phi \nabla_i \nabla_j \phi' \, dx^j \right).
\end{equation}

(The left-hand side depends on $\rho$, but we suppress this for simplicity of notation.)

Lemma 5.2. $T_{\phi\phi'} + T_{\phi'\phi} = 0$.

Proof. As

\begin{equation}
\nabla^i \phi \nabla_i \nabla_j \phi' \, dx^j + \nabla^i \phi' \nabla_i \nabla_j \phi \, dx^j = d\langle \nabla \phi, \nabla \phi' \rangle,
\end{equation}
and $I - \Pi_\rho$ projects away from $\text{Im}(d)$, the lemma follows. \hfill \Box

Theorem 5.4. Given $\phi_1, \phi_2, \phi_3, \phi_4 \in C^\infty(M)$, the Riemannian curvature operator $\bar{R}$ of $P^\infty(M)$ is given by

\begin{equation}
\langle \bar{R}(V_{\phi_1}, V_{\phi_2})V_{\phi_3}, V_{\phi_4} \rangle = \int_M \langle R(\nabla \phi_1, \nabla \phi_2)\nabla \phi_3, \nabla \phi_4 \rangle \rho \, d\text{vol}_M - 2\langle T_{\phi_1\phi_2}, T_{\phi_3\phi_4} \rangle + \langle T_{\phi_2\phi_3}, T_{\phi_1\phi_4} \rangle - \langle T_{\phi_1\phi_3}, T_{\phi_2\phi_4} \rangle,
\end{equation}

where both sides are evaluated at $\rho \, d\text{vol}_M \in P^\infty(M)$.

Proof. We use the formula

\begin{equation}
\langle \bar{R}(V_{\phi_1}, V_{\phi_2})V_{\phi_3}, V_{\phi_4} \rangle = V_{\phi_1} \langle \nabla_{V_{\phi_2}} V_{\phi_3}, V_{\phi_4} \rangle - \langle \nabla_{V_{\phi_2}} V_{\phi_4}, \nabla_{V_{\phi_1}} V_{\phi_3} \rangle - V_{\phi_2} \langle \nabla_{V_{\phi_1}} V_{\phi_3}, V_{\phi_4} \rangle + \langle \nabla_{V_{\phi_1}} V_{\phi_4}, \nabla_{V_{\phi_2}} V_{\phi_3} \rangle - \langle \nabla_{[V_{\phi_1}, V_{\phi_2}]} V_{\phi_3}, V_{\phi_4} \rangle.
\end{equation}

First, from (2.3) and (4.14),

\begin{equation}
V_{\phi_1} \langle \nabla_{V_{\phi_2}} V_{\phi_3}, V_{\phi_4} \rangle = -\int_M \nabla_i \phi_2 \nabla_j \phi_4 \nabla^i \nabla_j \phi_3 \nabla^k (\rho \nabla_k \phi_1) \, d\text{vol}_M
\end{equation}

\begin{align*}
&= \int_M \nabla^k \nabla_i \phi_2 \nabla_j \phi_4 \nabla^i \nabla_j \phi_3 \nabla_k \phi_1 \rho \, d\text{vol}_M + \\
&\quad + \int_M \nabla_i \phi_2 \nabla^k \nabla_j \phi_4 \nabla^i \nabla_j \phi_3 \nabla_k \phi_1 \rho \, d\text{vol}_M + \\
&\quad + \int_M \nabla_i \phi_2 \nabla_j \phi_4 \nabla^k \nabla^i \nabla_j \phi_3 \nabla_k \phi_1 \rho \, d\text{vol}_M.
\end{align*}
Similarly,

\begin{equation}
V_{\phi_2} \langle \nabla_{V_{\phi_1}} V_{\phi_3}, V_{\phi_4} \rangle = \int_M \nabla^k \nabla_i \phi_1 \nabla_j \phi_4 \nabla^i \nabla^j \phi_3 \nabla k \phi_2 \rho \ \mathrm{dvol}_M + \int_M \nabla_i \phi_1 \nabla^k \nabla_j \phi_4 \nabla^i \nabla^j \phi_3 \nabla k \phi_2 \rho \ \mathrm{dvol}_M + \int_M \nabla_i \phi_1 \nabla_j \phi_4 \nabla^k \nabla^i \phi_3 \nabla k \phi_2 \rho \ \mathrm{dvol}_M .
\end{equation}

Next, using (2.4), Lemma 4.16 and (5.1),

\begin{equation}
\langle \nabla_{V_{\phi_2}} V_{\phi_3}, \nabla_{V_{\phi_1}} V_{\phi_4} \rangle = \langle dG^*_\rho \nabla_i \nabla_j \phi_3 \nabla^j \phi_2 \ n_i \chi \rangle_{L^2} = \langle \Pi_\rho (\nabla_i \nabla_j \phi_3 \nabla^j \phi_2 \ n_i \chi) \rangle_{L^2} = \langle \nabla_i \nabla j \phi_3 \nabla^j \phi_2 \ n_i \chi, \nabla k \nabla l \phi_4 \nabla^l \phi_1 \rangle_{L^2} = \langle T_{\phi_2 \phi_3}, T_{\phi_1 \phi_4} \rangle
\end{equation}

Similarly,

\begin{equation}
\langle \nabla_{V_{\phi_2}} V_{\phi_3}, \nabla_{V_{\phi_1}} V_{\phi_4} \rangle = \int_M \nabla_i \nabla j \phi_3 \nabla^j \phi_1 \nabla^i \nabla l \phi_4 \nabla^l \phi_2 \rho \ \mathrm{dvol}_M - \langle T_{\phi_2 \phi_3}, T_{\phi_1 \phi_4} \rangle .
\end{equation}

Finally, we compute \( \langle \nabla_{[V_{\phi_1}, V_{\phi_2}]} V_{\phi_3}, V_{\phi_4} \rangle \). From (1.2), we can write \( [V_{\phi_1}, V_{\phi_2}] = V_\phi \), where

\begin{equation}
\phi = G_\rho d^*_\rho \left( \nabla_i \nabla j \phi_2 \nabla^j \phi_1 \ n_i \chi \right).
\end{equation}

Then from (4.15),

\begin{equation}
\langle \nabla_{[V_{\phi_1}, V_{\phi_2}]} V_{\phi_3}, V_{\phi_4} \rangle = \int_M \nabla_i \phi \nabla j \phi_4 \nabla^i \nabla^j \phi_3 \rho \ \mathrm{dvol}_M = \langle d\phi, \nabla i \phi_4 \nabla j \phi_3 \ n_i \chi \rangle_{L^2} = \langle dG^*_\rho \nabla i \phi_4 \nabla j \phi_3 \ n_i \chi \rangle_{L^2} = \langle \Pi_\rho \left( \nabla_i \nabla j \phi_2 \nabla^j \phi_1 \ n_i \chi - \nabla_i \nabla j \phi_1 \nabla^i \phi_2 \ n_i \chi \right) \rangle_{L^2} = \int_M \left( \nabla_i \nabla j \phi_2 \nabla^j \phi_1 - \nabla_i \nabla j \phi_1 \nabla^i \phi_2 \right) \nabla k \phi_4 \nabla^i \nabla^k \phi_3 \rho \ \mathrm{dvol}_M - \langle T_{\phi_1 \phi_2}, T_{\phi_4 \phi_3} \rangle + \langle T_{\phi_2 \phi_1}, T_{\phi_3 \phi_4} \rangle
\end{equation}

The theorem follows from combining equations (5.6)-(5.12).
the 2-plane spanned by \(V_{\phi_1}\) and \(V_{\phi_2}\) is
\[
\mathcal{R}(V_{\phi_1}, V_{\phi_2}) = \int_M K(\nabla \phi_1, \nabla \phi_2) \left( |\nabla \phi_1|^2 |\nabla \phi_2|^2 - \langle \nabla \phi_1, \nabla \phi_2 \rangle^2 \right) \rho \, d\text{vol}_M + 3 |T_{\phi_1, \phi_2}|^2,
\]
where \(K(\nabla \phi_1, \nabla \phi_2)\) denotes the sectional curvature of the 2-plane spanned by \(\nabla \phi_1\) and \(\nabla \phi_2\).

**Corollary 5.15.** If \(M\) has nonnegative sectional curvature then \(P^\infty(M)\) has nonnegative sectional curvature.

**Remark 5.16.** One can ask whether the condition of \(M\) having sectional curvature bounded below by \(r \in \mathbb{R}\) implies that \(P^\infty(M)\) has sectional curvature bounded below by \(r\). This is not the case unless \(r = 0\). The reason is one of normalizations. The normalizations on \(\phi_1\) and \(\phi_2\) are \(\int_M |\nabla \phi_1|^2 \rho \, d\text{vol}_M = \int_M |\nabla \phi_2|^2 \rho \, d\text{vol}_M = 1\) and \(\int_M (\nabla \phi_1, \nabla \phi_2) \rho \, d\text{vol}_M = 0\). One cannot conclude from this that \(\int_M (|\nabla \phi_1|^2 |\nabla \phi_2|^2 - \langle \nabla \phi_1, \nabla \phi_2 \rangle^2) \rho \, d\text{vol}_M \geq 1\) or \(\leq 1\).

More generally, if \(M\) has nonnegative sectional curvature then \(P_2(M)\) is an Alexandrov space with nonnegative curvature [8, Theorem A.8], [19, Proposition 2.10(iv)]. On the other hand, if \(M\) does not have nonnegative sectional curvature then one sees by an explicit construction that \(P_2(M)\) is not an Alexandrov space with curvature bounded below [19, Proposition 2.10(iv)].

**Remark 5.17.** The formula [5.5] has the structure of the O’Neill formula for the sectional curvature of the base space of a Riemannian submersion. In the case \(M = \mathbb{R}^n\), Otto argued that \(P^\infty(\mathbb{R}^n)\) is formally the quotient space of \(\text{Diff}(\mathbb{R}^n)\), with an \(L^2\)-metric, by the subgroup that preserves a fixed volume form [16]. As \(\text{Diff}(\mathbb{R}^n)\) is formally flat, it followed that \(P^\infty(\mathbb{R}^n)\) formally had nonnegative sectional curvature.

### 6. Poisson Structure

Let \(M\) be a smooth connected closed manifold. We do not give it a Riemannian metric. In this section we describe a natural Poisson structure on \(P^\infty(M)\) arising from a Poisson structure on \(M\). If \(M\) is a symplectic manifold then we show that the symplectic leaves in \(P^\infty(M)\) are orbits of the action of the group \(\text{Ham}(M)\) of Hamiltonian diffeomorphisms acting on \(P^\infty(M)\). We recover the symplectic structure on the orbits that was considered in [15].

Let \(M\) be a smooth manifold and let \(p \in C^\infty(\wedge^2 TM)\) be a skew bivector field. Given \(f_1, f_2 \in C^\infty(M)\), one defines the Poisson bracket \(\{f_1, f_2\} \in C^\infty(M)\) by \(\{f_1, f_2\} = p(df_1 \otimes df_2)\). There is a skew trivector field \(\partial p \in C^\infty(\wedge^3 TM)\) so that for \(f_1, f_2, f_3 \in C^\infty(M)\),
\[
(\partial p)(df_1, df_2, df_3) = \{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\}.
\]

One says that \(p\) defines a Poisson structure on \(M\) if \(\partial p = 0\). We assume hereafter that \(p\) is a Poisson structure on \(M\).
Definition 6.2. Define a skew bivector field \( P \in C^\infty(\Lambda^2 T^\infty(M)) \) by saying that its Poisson bracket is \( \{ F_{\phi_1}, F_{\phi_2} \} = F_{\{\phi_1, \phi_2\}} \), i.e.

\[
\{ F_{\phi_1}, F_{\phi_2} \}(\mu) = \int_M \{ \phi_1, \phi_2 \} \, d\mu
\]

for \( \mu \in P^\infty(M) \).

The map \( \phi \mapsto dF_\phi \bigg|_{\mu} \) passes to an isomorphism \( C^\infty(M)/\mathbb{R} \to T^*_\mu P^\infty(M) \). As the right-hand side of (6.3) vanishes if \( \phi_1 \) or \( \phi_2 \) is constant, equation (6.3) does define an element of \( C^\infty(\Lambda^2 T^\infty(M)) \).

Proposition 6.4. \( P \) is a Poisson structure on \( P^\infty(M) \).

Proof. It suffices to show that \( \partial P \) vanishes. This follows from the equation

\[
(\partial P)(dF_{\phi_1}, dF_{\phi_2}, dF_{\phi_3}) = \{ \{ F_{\phi_1}, F_{\phi_2} \}, F_{\phi_3} \} + \{ \{ F_{\phi_2}, F_{\phi_3} \}, F_{\phi_1} \} + \{ \{ F_{\phi_1}, F_{\phi_3} \}, F_{\phi_2} \} = 0.
\]

A finite-dimensional Poisson manifold has a (possibly singular) foliation with symplectic leaves [6]. The leafwise tangent vector fields are spanned by the vector fields \( W_f \) defined by \( W_f h = \{ f, h \} \). The symplectic form \( \Omega \) on a leaf is given by saying that \( \Omega(W_f, W_g) = \{ f, g \} \).

Suppose now that \( (M, \omega) \) is a closed 2\( n \)-dimensional symplectic manifold. Let \( \text{Ham}(M) \) be the group of Hamiltonian symplectomorphisms of \( M \) [13, Chapter 3.1].

Proposition 6.6. The symplectic leaves of \( P^\infty(M) \) are the orbits of the action of \( \text{Ham}(M) \) on \( P^\infty(M) \). Given \( \mu \in P^\infty(M) \) and \( \phi_1, \phi_2 \in C^\infty(M) \), let \( \tilde{H}_{\phi_1}, \tilde{H}_{\phi_2} \in T^*_\mu P^\infty(M) \) be the infinitesimal motions of \( \mu \) under the flows generated by the Hamiltonian vector fields \( H_{\phi_1}, H_{\phi_2} \) on \( M \). Then \( \Omega(\tilde{H}_{\phi_1}, \tilde{H}_{\phi_2}) = \int_M \{ \phi_1, \phi_2 \} \, d\mu \).

Proof. Write \( \mu = \rho \omega^n \). We claim that \( (W_{F_{\phi'}} \tilde{F})(\mu) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \tilde{F}(\mu - \epsilon \{ \phi, \rho \} \omega^n) \) for \( \tilde{F} \in C^\infty(\mu P^\infty(M)) \). To show this, it is enough to check it for each \( \tilde{F} = F_{\phi'} \), with \( \phi' \in C^\infty(M) \). But

\[
(W_{F_{\phi'}} F_{\phi'})(\mu) = F_{\{\phi, \phi'\}}(\mu) = \int_M \{ \phi, \phi' \} \rho \omega^n = -\int_M \phi' \{ \phi, \rho \} \omega^n,
\]

from which the claim follows. This shows that \( W_{F_{\phi'}} = \tilde{H}_\phi \).

Next, at \( \mu \in P^\infty(M) \) we have

\[
\Omega(\tilde{H}_{\phi_1}, \tilde{H}_{\phi_2}) = \Omega(W_{F_{\phi_1}}, W_{F_{\phi_2}}) = \{ F_{\phi_1}, F_{\phi_2} \}(\mu) = \int_M \{ \phi_1, \phi_2 \} \, d\mu.
\]

This proves the proposition. \( \square \)
Remark 6.9. As a check on Proposition 6.6, suppose that \( \phi_2 \in C^\infty(M) \) is such that \( \widehat{H}_{\phi_2} \) vanishes at \( \mu = \rho \omega^n \). Then \( \{ \phi_2, \rho \} = 0 \), so by our formula we have
\[
(6.10) \quad \Omega(\widehat{H}_{\phi_1}, \widehat{H}_{\phi_2}) = \int_M \{ \phi_1, \phi_2 \} d\mu = \int_M \{ \phi_1, \phi_2 \} \rho \omega^n = \int_M \phi_1 \{ \phi_2, \rho \} \omega^n = 0.
\]

Remark 6.11. The Poisson structure on \( P^\infty(M) \) is the restriction of the Poisson structure on \( (C^\infty(M))^* \) considered in [10, 11, 22]. Here the Poisson structure on \( (C^\infty(M))^* \) comes from the general construction of a Poisson structure on the dual of a Lie algebra, considering \( C^\infty(M) \) to be a Lie algebra with respect to the Poisson bracket on \( C^\infty(M) \). The cited papers use the Poisson structure on \( (C^\infty(M))^* \) to show that certain PDE’s are Hamiltonian flows.

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