Introduction

In this paper we find a representative of each orbit of the adjoint action of a real affine classical group of its Lie algebra. These orbits are not determined by the usual Jordan invariants of eigenvalues and block sizes, but require a noneigenvalue parameter called a modulus. This modulus was discovered by Weyl [4] for the affine general linear group. The cases for the real affine orthogonal and symplectic groups have been treated in [2] and [3], respectively. The arguments in [1] and [2] have been the motivation for the approach used here. Many essential details are new. We treat each real classical affine group as an isotropy group of a complex classical group with an anti-linear automorphism of order 2. This ensures that we do not have to work with vector spaces defined over the quaternions.

Here is a list of the contents of each section. §1 defines a real classical group. §2 defines the notion of type as in [1] and distinguished type as in [2]. Our main result is: every distinguished type is the sum of an indecomposable nilpotent distinguished type and a sum of indecomposable types, which is unique up to reordering of the sumands, see table 2 on page 3. §3 classifies indecomposable nilpotent distinguished types. §4 lists the isotropy groups of the real classical groups that are isomorphic to a real affine classical group.

1 The real classical groups

Let $\tilde{V}$ be a complex vector space of dimension $n$ and let $\text{Gl}(\tilde{V})$ be the group of invertible complex linear mappings of $\tilde{V}$ into itself. Let $\tilde{\tau}$ be a nondegenerate symmetric or alternating complex bilinear form on $\tilde{V}$. Set $G(\tilde{V}, \tilde{\tau}) = \{ g \in \text{Gl}(\tilde{V}) | g^*\tau = \tau \}$. Recall that $(g^*\tau)(\tilde{u}, \tilde{v}) = \tau(g(\tilde{u}), g(\tilde{v}))$ for every $\tilde{u}, \tilde{v} \in \tilde{V}$. Then $G(\tilde{V}, \tau) = O(\tilde{V}, \tilde{\tau})$ if $\tilde{\tau}$ is symmetric and $\text{Sp}(\tilde{V}, \tilde{\tau})$ is $\tilde{\tau}$ is alternating. $G$ is a complex classical group if it is equal to $\text{Gl}(\tilde{V})$, $O(\tilde{V}, \tilde{\tau})$ or $\text{Sp}(\tilde{V}, \tilde{\tau})$.

We now define a real classical group. Suppose that $\tilde{\sigma} : \tilde{V} \rightarrow \tilde{V}$ is an anti-linear map, that is, $\tilde{\sigma}(\tilde{u} + \tilde{v}) = \tilde{\sigma}(\tilde{u}) + \tilde{\sigma}(\tilde{v})$ for every $\tilde{u}, \tilde{v} \in \tilde{V}$ and for
every $\alpha \in \mathbb{C}$ one has $\tilde{\sigma}(\alpha \tilde{v}) = \pi \tilde{\sigma}(\tilde{v})$ for every $\tilde{v} \in \tilde{V}$. Moreover for every $g \in G$ suppose that

either
$$\tilde{\sigma}_\pm(g) = \tilde{\sigma}_\mp^{-1}g\tilde{\sigma}_\mp \text{ for every } g \in G, \text{ where } (\tilde{\sigma})_\pm^2 = \pm \text{id}_{\tilde{V}} \text{ and } \tilde{\sigma}_\pm \tau = \tau;$$
or
$$\tilde{\sigma} : V \to V^* \text{ such that } \tilde{\sigma}^*(g) = \tilde{\sigma}^{-1}g^T \tilde{\sigma}, \text{ where } g^T : V^* \to V^* \text{ is defined by } (g^T(\tilde{u}^*))((\tilde{v})) = \tilde{u}^*(g(\tilde{v})) \text{ for every } \tilde{u}^* \in \tilde{V^*} \text{ and } \forall \tilde{v} \in \tilde{V}. \text{ Here } g \in \text{Gl}(\tilde{V}). \text{ Let } \tau_\pm(\tilde{u}, \tilde{v}) = ((\tilde{\sigma}(\tilde{u}))((\tilde{v})).

Then $\tilde{\sigma}$ is an automorphism of $G$ of order 2. Let $G(\tilde{V}, \tilde{\sigma}, \tilde{\tau}) = \{ g \in G \mid \tilde{\sigma}^*(g) = g \& g^* \tau = \tau \}$. Then $G(\tilde{V}, \tilde{\sigma}, \tilde{\tau})$ is a real classical group. For an explicit description of the anti-linear mappings, see appendix 1.

| Complex group $\tilde{V}$ | Real group $\tilde{\sigma}_\pm$ | Standard notation $	ext{Gl}(n, \mathbb{R})$ |
|--------------------------|------------------------------|------------------------------------------|
| $\text{Gl}(\tilde{V})$  | Gl($\tilde{V}, \tilde{\sigma}_+$) | $\text{Gl}(n, \mathbb{R})$ |
|                          | Gl($\tilde{V}, \tilde{\sigma}_-$), $n$ even | $\text{U}^*(n)$ |
|                          | Gl($\tilde{V}, \tau_\pm(p)$), $0 \leq p \leq \left[ \frac{n}{2} \right]$ | $\text{U}(n - p, p)$ |
| $\text{O}(\tilde{V}, \tilde{\tau})$ | O($\tilde{V}, \tilde{\tau}, \tilde{\sigma}_+$), $0 \leq p \leq \left[ \frac{n}{2} \right]$ | O($n - p, p)$ |
|                          | O($\tilde{V}, \tilde{\tau}, \tilde{\sigma}_-$), $n$ even | O$^*(n)$ |
| $\text{Sp}(\tilde{V}, \tilde{\tau})$ | Sp($\tilde{V}, \tilde{\tau}, \tilde{\sigma}_+$), $n$ even | Sp($n, \mathbb{R}$) |
|                          | Sp($\tilde{V}, \tilde{\tau}, \tilde{\sigma}_-$), $0 \leq p \leq \left[ \frac{n}{4} \right]$ | Sp($n - p, p)$ |

Table 1. The real classical groups. See appendix 1 for the definition of $\tau_\pm(p)$ and $\tilde{\sigma}_\pm(p)$.

A real classical group $G(\tilde{V}, \tilde{\sigma}, \tilde{\tau})$ is a Lie group with Lie algebra $g(\tilde{V}, \tilde{\sigma}, \tilde{\tau}) = \{ X \in \text{gl}(\tilde{V}) \mid \tilde{\sigma}^*X = X \& \tilde{\tau}(X\tilde{u}, \tilde{v}) + \tilde{\tau}(\tilde{u}, \tilde{v}) = 0 \text{ for every } \tilde{u}, \tilde{v} \in \tilde{V} \}$.

## 2 Classification of adjoint orbits

Let $G(\tilde{V}, \tilde{\sigma}, \tilde{\tau})$ be a real classical group. Suppose that $v^0$ is a nonzero vector in $\tilde{V}$, which lies in an eigenspace of the complex linear map $\tilde{\sigma} : \tilde{V} \to \tilde{V}$ defined by $\tilde{\sigma}(\alpha \tilde{v}) = \alpha \tilde{\sigma}(\tilde{v})$ for every $\alpha \in \mathbb{C}$ and every $\tilde{v} \in \tilde{V}$. Note that $\tilde{\sigma}_\pm^2 = \pm \text{id}_{\tilde{V}}$. In addition, suppose that $v^0$ is $\tilde{\tau}$ isotropic, that is, $\tau(v^0, v^0) = 0$. Let $G(\tilde{V}, \tilde{\sigma}, \tilde{\tau})_{v^0} = \{ g \in G(\tilde{V}, \tilde{\sigma}, \tilde{\tau}) \mid g(v^0) = v^0 \}$. The isotropy group $G(\tilde{V}, \tilde{\sigma}, \tilde{\tau})_{v^0}$ is a closed subgroup of the Lie group $G(\tilde{V}, \tilde{\sigma}, \tilde{\tau})$ and hence is a
The classification of indecomposable types is given in [1]. From [1, prop.3 p.343] it follows that an indecomposable type is uniform. Let \( \tau \) be a \( \sigma \) adjoint action of \( G \) on a vector space \( V \) where \( \sigma \) is a Lie group. Its Lie algebra is \( g(V, \sigma, \tau) \). The adjoint action of \( G(V, \sigma, \tau) \) on its Lie algebra \( g(V, \sigma, \tau) \) is given by

\[
G(V, \sigma, \tau) \times g(V, \sigma, \tau) \rightarrow g(V, \sigma, \tau) \quad : (P, X) \mapsto PXP^{-1}.
\]

We begin the classification of adjoint orbits by defining the concept of an indecomposable type and an indecomposable distinguished type. Let \( W \) be a \( \tau \)-nondegenerate \( \sigma \) invariant subspace of the finite dimensional complex vector space \( V \). If \( Y \in g(W, \sigma|W; \tau|W) \), the Lie algebra of \( G(W, \sigma|W; \tau|W) \), then \( (Y, W; \sigma|W, \tau|W) \) is a pair. Two pairs \( (Y, W; \sigma|W, \tau|W) \) and \( (Y', W'; \sigma|W', \tau|W') \) are equivalent if there is a bijective complex linear mapping \( P : W \rightarrow W' \) such that \( Y' = PYP^{-1}, \sigma' = P\sigma P^{-1}, \) and \( P^*\tau' = \tau \). Being equivalent is an equivalence relation on the set of pairs. An equivalence class of pairs is a type, denoted by \( \Delta \), which is represented by the pair \( (Y, W; \sigma|W, \tau|W) \). We define the dimension of \( \Delta \), denoted \( \dim \Delta \), by \( \dim W \) and the index of \( \Delta \), denoted \( \text{ind} \Delta \), by the number of negative eigenvalues of the Gram matrix \( \langle \tau(v_i, v_j) \rangle \) of \( \tau \) with respect to the basis \( \{v_1, \ldots, v_{\dim W}\} \) of \( W \). Let \( Y = S + N \) be the Jordan decomposition of \( Y \) into a semisimple linear map \( S \) and a commuting nilpotent map \( N \), both of which lie in \( g(W, \sigma, \tau) \). To see this we argue as follows. Since \( \sigma N \sigma^{-1} \) and \( \sigma S \sigma^{-1} \) are commuting nilpotent and semisimple linear maps, whose sum is \( \sigma Y \sigma^{-1} = Y \) and the Jordan decomposition of \( Y \) is unique, it follows that \( N = \sigma N \sigma^{-1} \) and \( S = \sigma S \sigma^{-1} \). So \( N \) and \( S \) commute with \( \sigma \). A similar argument shows that \( N^* \tau = \tau \) and \( S^* \tau = \tau \). Thus \( S \) and \( N \) lie in \( g(W, \sigma, \tau) \). There is a unique nonnegative integer \( h \) such that \( N^h \neq 0 \) but \( N^{h+1} = 0 \). We call \( h \) the height of \( \Delta \) and denote it by \( \text{ht}(\Delta) \). \( \text{ht}(\Delta) \) does not depend on the choice of representative of \( \Delta \). We say that the type \( \Delta \), represented by the pair \( (W, \sigma, \tau) \), is uniform if \( NW = \ker N^h W \). Suppose that \( W = W_1 \oplus W_2 \), where \( W_i \) are proper, \( Y \) invariant subspaces of \( V \), which are \( \sigma \) invariant, \( \tau \) nondegenerate, and \( \tau \) orthogonal. Then \( \Delta \) is the sum of two types \( \Delta_i \), represented by \( (Y|W_i, W_i; \sigma|W_i, \tau|W_i) \). We write \( \Delta = \Delta_1 + \Delta_2 \). The type \( \Delta \) is indecomposable if it cannot be written as the sum of two types. In other words, the pair \( (Y, W; \sigma, \tau) \) represents an indecomposable type \( \Delta \) if there is no proper \( Y \) and \( \sigma \) invariant subspace of \( W \) on which \( \tau \) is nondegenerate. From [1] prop.3 p.343 it follows that an indecomposable type is uniform. The classification of indecomposable types is given in [1].

Next we introduce the concept of a triple. Let \( \sigma \) be the complex linear mapping of \( V \) into itself associated to the anti-linear mapping \( \sigma \) defined by setting \( \sigma(\alpha v) = \alpha \sigma(v) \) for every \( v \in V \) and every \( \alpha \in \mathbb{C} \). Again let \( W \) be a \( \sigma \) (and thus \( \sigma \)) invariant subspace of \( V \), which is \( \tau \) nondegenerate. We say that
(Y, W, v^0; σ, τ) is a \textit{triple} if 1) \(v^0\) is a nonzero vector in W. If σ is involved, then \(v^0\) is an eigenvector of \(\bar{σ}\) corresponding to the eigenvalue \(λ\), that is ±1 if \(σ = σ_+\) or ±i if \(σ = σ_-\); 2) if τ or \(τ_0\) is involved, then \(v^0\) is τ isotropic, that is, \(τ(v^0, v^0) = 0\) or \(τ_0(v^0, v^0) = 0\). The linear map \(Y \in g(W; σ, τ)\) and \(Yv^0 = 0\). If \(v\) is a nonzero vector in \(V\) which satisfies condition 1) and 2) of the definition of a triple, then we call \(v\) \textit{special}. Because \(S\) and \(N\) are polynomials in \(Y\) with complex coefficients and \(Yv^0 = 0\), it follows that \(Sv^0 = Nv^0 = 0\). So \(S, N \in g(W, σ, τ)v^0\). We say that two triples \((Y, W, v^0; σ, τ)\) and \((Y', W', (v')^0; σ', τ')\) are \textit{equivalent} if there is a complex bijective linear mapping \(P : W \to W'\) such that \(Y' = PYP^{-1}, P(v^0) = (v')^0, σ' = PσP^{-1}\), and \(P^*τ' = τ\). Note that the vector \((v')^0\) is a special vector in \(V'\), since

\[
\overline{σ'}((v')^0) = σ'(σ(v^0)) = Pσ(P^{-1}(v')^0) = P(σ(v^0)) = P(\overline{σ}(v^0)) = P(λv^0), \quad \text{where} \quad λ = ±1 \text{ or } ±i
\]

and

\[
τ'((v')^0, (v')^0) = τ'(Pv^0, Pv^0) = (P^*τ')(v^0, v^0) = τ(v^0, v^0) = 0.
\]

Being equivalent is an equivalence relation on the set of triples. We call an equivalence class of triples a \textit{distinguished type}, which we denote by \(\Delta\). We say that \(\Delta_1\) is represented by the triple \((Y, W, v^0; σ, τ)\). Suppose that \(W = W_1 \oplus W_2\), where \(W_i\) are proper, \(σ\) and \(Y\) invariant, \(τ\) nondegenerate, and \(τ\) orthogonal subspaces of \(V\) with \(v^0\) a \(τ\) isotropic special vector in \(W_1\). Then \((W_1, Y|W_1, v^0; σ|W_1, τ|W_1)\) is a triple, whose corresponding distinguished type is \(\Delta_1\). Let the pair \((W_2, Y|W_2; σ|W_2, τ|W_2)\) represent the type \(Δ_2\). We say that the distinguished type \(\Delta\) represented by the triple \((Y|W, v^0; σ|W, τ|W)\), is the \textit{sum} of the distinguished type \(\Delta_1\) and the type \(Δ_2\), which we write as \(\Delta = \Delta_1 + Δ_2\). If \(\Delta\) cannot be written as the sum of a distinguished type and any type, then \(\Delta\) is an \textit{indecomposable} distinguished type. In other words, the triple \((Y, W, v^0; σ, τ)\) represents an indecomposable distinguished type if there is no proper, \(Y\) and \(σ\) invariant, \(τ\) nondegenerate subspace of \(W\), which contains \(v^0\).

The goal of the next few sections is to prove

**Theorem 2.1** Every distinguished type is the sum of an indecomposable nilpotent distinguished type and a sum of indecomposable types, which is unique up to reordering of the summands.
The proof of this theorem requires classifying indecomposable distinguished types. Recall that indecomposable types have been classified in [1]. Theorem 3.1 solves the conjugacy class problem for the Lie algebra \( g(V, \sigma, \tau) \) under the adjoint action of \( G(V, \sigma, \tau) \). Indeed, there is a one to one correspondence between equivalence classes triples and orbits of the adjoint action of \( G(V, \sigma, \tau) \) on its Lie algebra \( g(V, \sigma, \tau) \).

Suppose that \( \Delta \) is a distinguished type, which is represented by the triple \( (Y, W, v^0; \sigma, \tau) \). We say that \( \Delta \) has distinguished height \( h \) if there is a special vector \( w \in W \) such that \( Y^h w = v^0 \). We denote the distinguished height of \( \Delta \) by \( \text{dht}(\Delta) \).

**Lemma 2.2** Suppose that \( \Delta = \Delta' + \Delta \). Then \( \text{dht}(\Delta) = \text{dht}(\Delta') \).

**Proof.** First we show that \( \text{dht}(\Delta) = \text{dht}(\Delta') \). Suppose that \( (Y, W, v^0; \sigma, \tau) \) is a triple which represents \( \Delta \) and that \( W = W_1 \oplus W_2 \), where \( W_i \) are proper, \( Y \) and \( \sigma \) invariant, \( \tau \) nondegenerate, \( \tau \) orthogonal subspaces of \( W \) with \( v^0 \in W_1 \). The triple \( (Y^1 W_1, W_1, v^0; \sigma, \tau) \) represents \( \Delta' \) and the pair \( (Y^1 W_2, W_2; \sigma, \tau) \) represents the type \( \Delta \). Suppose that \( \text{dht}(\Delta') = h' \). Then there is a special vector \( w' \in W_1 \) such that \( Y^h w' = v^0 \). Since \( W_1 \subseteq W \) we get \( \text{dht}(\Delta) \geq h' \). Because \( \text{dht}(\Delta) = h \), there is a special vector \( w \in W \) such that \( Y^h w = v^0 \). But \( W = W_1 \oplus W_2 \). So we may write \( w = w_1 + w_2 \), where \( w_i \in W_i \) and \( w_i \) is a special vector. Since \( W_i \) is \( Y \) invariant, one has \( v^0 = Y^h w = Y^h w_1 + Y^h w_2 \), where \( Y^h w_i \in W_i \). By construction \( v^0 \in W_1 \). Therefore \( Y^h w_1 = v^0 \). Consequently, \( h \leq \text{dht}(\Delta') = h' \). So \( h = h' \). Note that \( \dim \Delta' < \dim \Delta \).

Let \( \Delta \) be a distinguished type of distinguished height \( h \), which is represented by the triple \( (Y, W, v^0; \sigma, \tau) \). If \( \tau \) is not involved in the triple representing \( \Delta \), let \( \mu(\Delta) \) be the set of all \( \lambda \in \mathbb{C} \setminus \{0\} \) such that \( \lambda Y^h w = v^0 \) for some special vector \( w \in W \). If \( \tau \) is involved, let \( \mu(\Delta) \) be the set of all \( \tau(w, v^0) \in \mathbb{C} \) such that \( w \) is a special vector in \( W \) with \( Y^h w = v^0 \). We call \( \mu(\Delta) \) the set of parameters of the distinguished type \( \Delta \).

**Lemma 2.3** If \( \Delta = \Delta' + \Delta \) is a distinguished type of distinguished height \( h \), then \( \mu(\Delta) = \mu(\Delta') \).

**Proof.** Suppose that \( \tau \) is not involved in the triple \( (Y, W, v^0; \sigma, \tau) \) representing \( \Delta \), and that \( \lambda \in \mu(\Delta') \). Then there is a special vector \( w \in W_1 \) such that \( \lambda Y^h w = v^0 \), since the distinguished height of \( \Delta' \), represented by the triple \( (Y_1, W_1, v^0; \sigma|W_1, \tau|W_1) \), is \( h \) by lemma 2.2. But \( W_1 \subseteq W \). So \( \lambda \in \mu(\Delta) \), by definition. Hence \( \mu(\Delta') \subseteq \mu(\Delta) \). Next we prove the reverse inclusion. If \( \lambda \in \mu(\Delta) \), there is a special vector \( w \in W \) such that \( \lambda Y^h w = v^0 \). But
Δ = Δ′ + Δ. So there are Y and σ invariant subspaces \( W_i \) which are \( \tau \) orthogonal such that \( W = W_1 \oplus W_2 \) with \( v^0 \in W_1 \). Write \( w = w_1 + w_2 \). Then \( w_i \) are special vectors in \( W_i \) with \( \lambda Y^h w = \lambda Y^h w_1 + \lambda Y^h w_2 \). But \( Y^h w_i \in W_i \), since \( W_i \) are Y invariant and \( \lambda Y^h w = v^0 \). So \( \lambda Y^h w_1 = v_0 \) since \( v^0 \in W_1 \) by hypothesis, that is, \( \lambda \in \mu(\Delta') \). Thus \( \mu(\Delta) \subseteq \mu(\Delta') \), which shows that \( \mu(\Delta') = \mu(\Delta) \) as desired.

Now suppose that \( \tau \) is involved in the triple \( (Y, W, v^0; \sigma, \tau) \) representing \( \Delta \). Since \( W_1 \subseteq W \) it follows from the definition of the set of parameters that \( \mu(\Delta') \subseteq \mu(\Delta) \). Suppose that there is a special vector \( w \in W \) with \( Y^h w = v^0 \) such that \( \tau(w, v^0) \notin \mu(\Delta') \). Write \( w = w_1 + w_2 \) with \( w_i \in W_i \), which are special vectors. Then \( Y^h w_1 = v^1 \). Since \( \Delta = \Delta' + \Delta \) the subspace \( W_1 \) is \( \tau \) orthogonal to the subspace \( W_1 \) and \( v^0 \in W_1 \). So

\[ \tau(w, v^0) = \tau(w_1, v^0) + \tau(w_2, v^0) = \tau(w_1, v^0). \]

But \( \tau(w_1, v^0) \in \mu(\Delta') \). This is a contradiction. Hence \( \mu(\Delta') = \mu(\Delta) \). □

Lemma 2.4 Suppose that \( \Delta \) is a distinguished type. Then we may write \( \Delta = \Delta' + \Delta \), where the distinguished type \( \Delta' \) is indecomposable and nilpotent.

Proof. If \( \Delta \) is not indecomposable, we find another distinguished type \( \Delta'' \) of the same distinguished height and set of parameters as \( \Delta \) and a type \( \Delta \), where \( \dim \Delta > 0 \), such that \( \Delta = \Delta'' + \Delta \). Because \( \dim \Delta > \dim \Delta'' \) after a finite number of repetitions, we obtain a distinguished type \( \Delta' \), which we can not write as the sum of a distinguished type and a type \( \Delta \), namely, \( \Delta = \Delta' + \Delta \). In other words, \( \Delta' \) is an indecomposable distinguished type, which by lemmas 2.2 and 2.3 has the same distinguished height and set of parameters as \( \Delta \).

We now show that the indecomposable distinguished type \( \Delta' \), represented by the triple \( (Y|W, W, v^0; \sigma, \tau) \) is nilpotent. Let \( W_0 \) be the generalized eigenspace of \( Y|W \) corresponding to the eigenvalue 0. Then \( v^0 \in W_0 \), because \( Yv^0 = 0 \). Moreover, \( W_0 \) is Y and \( \sigma \) invariant and is \( \tau \) nondegenerate. On \( W_0 \) the linear map \( Y \) is nilpotent. Since \( \Delta' \) is indecomposable, it follows that the triple \( (Y|W, W, v^0; \sigma, \tau) \) is equal to the triple \( (Y|W_0, W_0, v^0; \sigma, \tau) \). Hence the indecomposable distinguished type \( \Delta' \) is nilpotent. □

Lemma 2.4 proves theorem 2.1 except for the uniqueness assertion. To prove uniqueness we need the classification of indecomposable nilpotent distinguished types given in the next section.
3 Classification of nilpotent indecomposable distinguished types

In this section we classify nilpotent indecomposable distinguished types.

First we prove

**Proposition 3.1** Let \( \Delta \) be an indecomposable nilpotent distinguished type of distinguished height \( h \) and set of parameters \( \mu(\Delta) \). Suppose that \( \Delta \) is represented by the triple \( (Y, W, v^0; \sigma, \tau) \), where \( Y \in g(W, \sigma, \tau) \) is nilpotent and \( Y v^0 = 0 \). Then \( \Delta \) is uniform and \( \dim \ker Y = 1 \) or 2.

**Proof.** Suppose that \( \tau \) is not involved in the triple defining \( \Delta \). Since \( \text{dht}(\Delta) = h \), there is a special vector \( w \in W \) and \( \lambda \in \mu(\Delta) \) such that \( \lambda Y^h w = v^0 \). Look at the subspace \( \widetilde{W} = \text{span}_C \{w, Yw, \ldots, Y^h w = v^0\} \).

Since \( Y^{h+1} W = 0 \), the subspace \( \widetilde{W} \) is \( Y \) invariant and contains the vector \( v^0 \). \( \widetilde{W} \) is also \( \sigma \) invariant, because \( w \), being a special vector, is an eigenvector of the complex linear map \( \sigma \), which commutes with \( Y \). Hence \( \widetilde{W} \) is \( \sigma \) invariant and hence \( \sigma \) invariant. But \( \Delta \) is indecomposable. So \( \widetilde{W} = W \). Hence \( \Delta \) is uniform because \( Y \) has only one Jordan block on \( W \). Thus \( \dim \ker Y = 1 \).

Now suppose that \( \tau \) is involved in the triple defining \( \Delta \). There are two cases.

**Case 1.** \( \mu(\Delta) \neq \{0\} \). Since \( \text{dht}(\Delta) = h \) and \( \mu(\Delta) \neq \{0\} \), there is a special vector \( w \in W \) such that \( Y^h w = v^0 \) and \( \tau(w, v^0) = \mu \neq 0 \). The subspace \( \widetilde{W} = \text{span}_C \{w, Yw, \ldots, Y^h w = v^0\} \) contains \( v^0 \) and is both \( Y \) and \( \sigma \) invariant. Let \( T \) be the \( (h + 1) \times (h + 1) \) matrix, whose \( ij \) entry is \( \tau(Y^i w, Y^j w) \). \( T \) has nonzero entries on its anti-diagonal since

\[
\tau(Y^{i-1} w, Y^{j-1} w) = (-1)^{i-1} \tau(w, Y^{i+j-2} w), \quad \text{since } Y \in g(W, \sigma, \tau)
\]

\[
= (-1)^{i-1} \tau(w, Y^h w), \quad \text{since } i + j = h + 2 \text{ on the anti-diagonal}
\]

\[
= (-1)^{i-1} \tau(w, v^0) = (-1)^{i-1} \mu \neq 0.
\]

\( T \) has zero entries below the anti-diagonal because \( Y^\ell w = 0 \) when \( \ell > h \). Thus \( \det T \neq 0 \). In other words, \( \tau \) is nondegenerate on \( \widetilde{W} \). But \( \Delta \) is indecomposable. Hence \( \widetilde{W} = W \). Consequently, \( \Delta \) is uniform, because \( Y \) is one Jordan block. Hence \( \dim \ker Y = 1 \).

**Case 2.** \( \mu(\Delta) = \{0\} \). By the results of [B] p.343 the type \( \Delta \), represented by the pair \( (Y, W; \sigma, \tau) \), which underlies the distinguished type \( \Delta \), is the sum \( \Delta_1 + \cdots + \Delta_k \) of indecomposable types. Here \( \Delta_\ell \) has height \( h_\ell \) and is represented by the pair \( (Y|W_\ell, \tau|W_\ell, \sigma|W_\ell, \tau|W_\ell) \) for \( 1 \leq \ell \leq k \). Without
loss of generality we may assume that $h_1 \leq h_2 \leq \cdots \leq h_k \leq h$. Then $W = \sum_{\ell}^k \oplus W_{\ell}$, where $W_{\ell}$ are $Y$ and $\sigma$ invariant and pairwise $\tau$ orthogonal with $\tau|W_{\ell}$ nondegenerate. Write $v^0 = \sum_{\ell}^k v^0_{\ell}$ with $v^0_{\ell} \in W_{\ell}$. Let $j$ be the smallest integer with $1 \leq j \leq k$ such that $v^0_j \neq 0$. Then $Y v^0_j = 0$, because $0 = Y v^0 = \sum_{\ell}^k Y v^0_{\ell}$ and $Y v^0_{\ell} \in W_{\ell}$ for all $1 \leq \ell \leq k$. There is a vector $z \in W_j$ such that $\tau(z, v^0_j) \neq 0$, because $\tau|W_j$ is nondegenerate. The following argument shows that we may assume that $z$ is a special vector. Since the subspace $W_j$ is $\sigma$, and hence $\sigma$, invariant, we have $W_j = W_j^1 \oplus W_j^2$, where $W_j^1$ is the eigenspace of $\sigma|W_j$ corresponding to the nonzero eigenvalue $\lambda$ and $W_j^2$ is its eigenspace corresponding to the eigenvalue $-\lambda$. Recall that $\lambda = \pm 1$ or $\pm i$, since $(\sigma|W_j)^2 = \pm 1$. Write $z = z_1 + z_2$, where $z_i \in W_j^i$. Suppose that $\tau(z_i, v^0_j) = 0$ for $i = 1$ and $2$. Then $\tau(z, v^0_j) = \tau(z_1, v^0_j) + \tau(z_2, v^0_j) = 0$, which contradicts the definition of the vector $z$. Suppose that $\tau(z_1, v^0_j) \neq 0$. Otherwise interchange $z_1$ and $z_2$ and rename $z_1$ as $z$. Since $W_j$ is $\tau$ orthogonal to $W_{\ell}$ for every $\ell \neq j$, it follows that $\tau(z, v^0_j) = 0$. Moreover, $Y^h z \neq 0$, for suppose that $Y^n z = 0$, then

$$0 = \tau(Y^h z, w) = (-1)^h \tau(z, Y^h w) = (-1)^h \tau(z, v^0) \neq 0.$$  

Suppose that $h_j < h$. Then $W' = \sum_{\ell}^k \oplus W_{\ell} \neq \{0\}$. By hypothesis that $\Delta$ has distinguished height $h$, there is a special vector $w' \in W'$ such that $Y^h w' = v^0$. Since $W'$ is $Y$ invariant, $Y^h w' \in W'$. So $v^0 \in W'$. This implies $v^0_0 = 0$, because $v^0 = v^0_j + v'$ for some $v' \in W'$. This contradicts the definition of the vector $v^0_j$. Hence $h_j = h$. Since $\Delta$ has distinguished height $h$, there is a special vector $w \in W$ such that $Y^h w = v^0$. Consider the subspace

$$\tilde{W} = \text{span}_C \{w, Yw, \ldots, Y^h w = v^0; z, Yz, \ldots, Y^h z\}.$$  

Clearly $\tilde{W}$ is $Y$ invariant and contains the vector $v^0$. $\tilde{W}$ is also $\sigma$ invariant, because the vectors $z$ and $w$ are special. From the definition of the vectors $z$ and $w$ it follows that $\tau(z, Y^h w) = \tau(z, v^0) \neq 0$ and $\tau(w, Y^h w) = \tau(w, v^0) = 0$, since $\mu(\Delta) = \{0\}$. Look at the $2(h + 1) \times 2(h + 1)$ matrix $T = \left(\frac{\tau(Y^i-1 w, Y^j-1 w)}{\tau(Y^i-1 z, Y^j-1 z)}\right)$, where $i + j = h + 2$ and $1 \leq i, j \leq h + 1$. Then

$$T = \begin{pmatrix}
* & 0 & 0 & * & + \\
0 & 0 & + & 0 & \\
* & + & * & * & \\
+ & 0 & * & * & \\
+ & 0 & * & 0 & 
\end{pmatrix}.$$  

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where the entries below the anti-diagonal of each block of $T$ vanish, because 
$\tau(Y u, v) = -\tau(u, Y v)$ for every $u, v \in \mathring{W}$ and $Y^\ell | W = 0$ for every $\ell > h$.
The entries on the anti-diagonal of the upper right hand and lower left hand block are nonvanishing because 
$\tau(Y^{i-1}z, Y^{j-1}w) = (-1)^{i-1}\tau(z, Y^{i+j-2}w) = (-1)^{i-1}\tau(z, Y^h w) \neq 0$ and 
$\tau(Y^{i-1}w, Y^{j-1}z) = (-1)^{j-1}\tau(Y^h w, z) \neq 0$. The entries on the anti-diagonal of the upper left hand block vanish, because 
$\tau(w, Y^h w) = 0$. To compute the determinant of $T$ we expand by minors of 
the $(h + 2)^{rd}$ row and $(h + 1)^{st}$ column. Removing this row and column gives 
a matrix of the same form as $T$ with one fewer row and column. Since 
the entries on the anti-diagonal of the lower left hand block of $T$ are nonzero, 
we find that the determinant of $T$ is the product of these entries and the 
determinant of the upper right hand block. Hence det $T$ is nonzero. Thus 
$\tau|\mathring{W}$ is nondegenerate. Since $\Delta$ is indecomposable, it follows that 
$\mathring{W} = W$. Consequently, $\Delta$ is uniform, because $Y|W$ is the sum of two Jordan blocks 
of the same size, and dim ker $Y = 2$. 
\end{proof}

By proposition 3.1 the nilpotent indecomposable distinguished type $\Delta$ of 
distinguished height $h$ and set of parameters $\mu(\Delta)$ is uniform. Thus the 
reduced type $\overline{\Delta}$, represented by the tuple $(\overline{v}, \overline{w} = W/W(Y^h|W); \overline{\sigma}, \overline{\tau})$ exists. 
One has $v^0 \in NW$, because $v^0 \in \ker Y$ and $NW = \ker Y^h|W$. Hence 
$\overline{(v^0)} = 0$. Since $NW = \ker Y^h|W$ and $Y^{h+1}|W = 0$, the induced map 
$$
\overline{Y}: \mathring{W} \rightarrow \overline{W}: \overline{w} = w + Y W \mapsto \overline{Y} \overline{w} = Yw + YW = YW,
$$
vanishes identically. Thus the reduced type $\overline{\Delta}$ is semisimple and is represented by the pair $(0, \overline{W}; \overline{\sigma}, \overline{\tau})$. Here $\overline{\sigma}: \overline{W} \rightarrow \overline{W}$ is the anti-linear 
map induced by $\sigma$ with $\overline{\sigma}^2 = \pm 1$ and $\overline{\tau}: \overline{W} \rightarrow \overline{W}$ is the bilinear form 
$\overline{\tau}(\overline{u}, \overline{v}) = \tau(u, Y^h v)$ for $u, v \in W$. The form $\overline{\tau}$ is well defined, because for 
$\overline{u} = u + NW$ and $\overline{v} = v + NW$

$$
\overline{\tau}(u + Y W, v + Y W) = \tau(u + Y W, Y^h v), \text{ since } Y^{h+1}|W = 0
$$

$$
= (-1)^h \tau(Y^h (u + Y W), v), \text{ since } Y \in g(W, \sigma, \tau)
$$

$$
= (-1)^h \tau(Y^h u, v) = \tau(u, Y^h v).
$$

Also $\overline{\tau}$ is nondegenerate, for if $0 = \overline{\tau}(\overline{u}, \overline{v})$ for every $\overline{u} \in \overline{W}$, then $0 = \tau(u, Y^h v)$ for every $u \in W$. So $Y^h v = 0$, since $\tau$ is nondegenerate on $W$. Hence $\overline{v} = 0$. Note that $\overline{\tau}$ is symmetric if $h$ is even and $\tau$ is symmetric or $h$ is odd and $\tau$ is alternating. Otherwise $\overline{\tau}$ is alternating.

Let $\Delta$ be the uniform nilpotent type of height $h$, represented by the pair

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(Y, W; σ, τ), whose reduced type is the semisimple type \( \overline{\Delta} \), represented by the pair \((0, \overline{W}; \overline{\sigma}, \overline{\tau})\).

**Corollary 3.1A** The nilpotent type \( \Delta \) associated to the indecomposable nilpotent distinguished type \( \overline{\Delta} \) of distinguished height \( h \) and set of parameters \( \mu(\overline{\Delta}) \) is indecomposable.

**Proof.** By [1] prop.5 p.343] we can write \( \overline{\Delta} = \overline{\Delta}_1 + \cdots + \overline{\Delta}_k \), where \( \overline{\Delta}_\ell \) is an indecomposable semisimple type represented by the pair \((0, \overline{W}_\ell; \overline{\sigma}|W_\ell, \overline{\tau}|W_\ell)\) for \( 1 \leq \ell \leq k \), see appendix 3. Let \( \Delta_\ell \) be the nilpotent type determined by the semisimple type \( \overline{\Delta}_\ell \). \( \Delta_\ell \) is represented by the pair \((Y|W_\ell, W_\ell; \sigma|W_\ell, \tau|W_\ell)\), see appendix 2. By [1] prop.3 p.343] the type \( \Delta_\ell \) is uniquely determined by \( \overline{\Delta}_\ell \) and the height \( h_\ell \) and is indecomposable. Hence \( \Delta = \Delta_1 + \cdots + \Delta_k \), where \( W = \sum_{\ell=1}^k \oplus W_\ell \). Since \( \Delta \) has distinguished height \( h \), there is a special vector \( w \in W \) such that \( Y^h w = v^0 \). Write \( w = \sum_{\ell=1}^k w_\ell \) with \( w_\ell \in W_\ell \) and \( v^0 = \sum_{\ell=1}^k \ell_\ell^0 \) with \( \ell_\ell^0 \in W_\ell \). Then \( \sum_{\ell=1}^k \ell_\ell^0 = v^0 = Y^h w = \sum_{\ell=1}^k Y^h \ell_\ell \). Since the subspace \( W_\ell \) is \( Y \) invariant, it follows that \( \ell_\ell^0 = Y^h \ell_\ell \) for all \( 1 \leq \ell \leq k \). Since \( w \) is a special vector in \( W \), there is a nonzero \( \lambda \in \mathbb{C} \) such that the vector \( w \) is an eigenvector of the complex linear map \( \overline{\sigma} \) corresponding to the eigenvalue \( \lambda \).

\[
\sum_{\ell=1}^k \lambda \ell_\ell = \lambda w = \overline{\sigma}(w) = \sum_{\ell=1}^k \overline{\sigma}(\ell_\ell).
\]

But \( \overline{\sigma}(\ell_\ell) \in W_\ell \) for all \( 1 \leq \ell \leq k \), since \( W_\ell \) is a subspace of \( W \) which is \( \sigma \) invariant and hence \( \overline{\sigma} \) invariant. In other words, \( \ell_\ell \) is a special vector in \( W_\ell \) for \( 1 \leq \ell \leq k \). Let \( j \) be the smallest integer \( 1 \leq j \leq k \) such that \( \ell_j^0 \neq 0 \). Let \( W' = \sum_{\ell=j+1}^k \oplus W_\ell \).

Suppose that \( \tau \) is not involved in the triple defining the distinguished type \( \overline{\Delta} \). If \( j = k \), then \( Y^h \ell_k = \ell_k^0 = v^0 \). Set \( \ell'' = \ell_k \) and go to equation \( \overline{\Pi} \). Suppose that there is a nonzero vector \( \ell' \in W' \) such that \( \ell' = \ell_j' + \ell' \). Let \( w' = \sum_{\ell=j+1}^k \ell_\ell \). Then \( w' \) is a special vector in \( W' \) and

\[
Y^h \ell' = \sum_{\ell=1}^k Y^h \ell_\ell = \sum_{\ell=j+1}^k \ell_\ell = \sum_{\ell=j}^k \ell_\ell - \ell_j^0 = v^0 - \ell_j^0 = \ell'.
\]

Let \( \ell'' = \ell_j + \ell' \). Then \( \ell'' \) is a special vector in \( W_j \oplus W' \) such that \( Y^h \ell'' = Y^h \ell_j + Y^h \ell' = \ell_j^0 + \ell' = \ell'' \). Let

\[
\hat{W} = \text{span}_\mathbb{C}\{w'', Yw'', \ldots, Y^h \ell'' = v^0\}.
\]
Then \( \hat{W} \subseteq W_j \oplus W' \) is a \( Y \) and \( \sigma \) invariant subspace of \( W \), which contains \( v^0 \). Because \( \Delta \) is indecomposable, \( \hat{W} = W \). But this contradicts the fact that \( W_j \oplus W' \) is a proper subspace of \( W \). Thus \( v' = 0 \), which implies \( W' = \{0\} \) and thus \( W = W_j \). So \( \Delta = \Delta_j \), which is indecomposable.

Suppose that \( \tau \) is involved in the triple representing the distinguished type \( \hat{\Delta} \). Suppose that \( \tau(w'', Y^h w'') = \mu \neq 0 \). The \((h + 1) \times (h + 1)\) matrix 
\[
(\tau(Y^i w'', Y^j w''))
\]

of \( \tau \) with respect to the basis \( \{w'', Y w'', \ldots, Y^h w''\} \) of \( \hat{W} \) is
\[
T = \begin{pmatrix}
* & -\mu \\
-\mu & \mu \\
\mu & 0 \\
\end{pmatrix},
\]

since \( \tau(Y^i w'', Y^j w'') = 0 \) if \( (i - 1) + (j - 1) > h \). But \( \det T \neq 0 \). So \( \tau|\hat{W} \) is nondegenerate. Hence \( \hat{W} = W \), because the distinguished type \( \Delta \) is indecomposable. This contradicts the fact that \( \hat{W} \) is a proper subspace of \( W \). Hence the vector \( v' = 0 \). So \( \Delta = \Delta_j \), which is indecomposable.

Suppose that \( \tau(w'', Y^h w'') = 0 \). Then either \( \tau \) is symmetric or \( \tau = \tau_s \) and \( h \) is odd, or \( \tau \) is alternating and \( h \) is even. So \( \hat{W} \) is a \( \tau \) isotropic subspace of \( W \). Here \( \tau \) is nondegenerate bilinear form on \( W \), because for \( 1 \leq \ell \leq k \) one has \( \tau|W_\ell \) is nondegenerate, since \( \Delta_\ell \) is a type, and the subspaces \( W_\ell \) are pairwise \( \tau \) orthogonal. Hence there is a nonzero vector \( z \in W_j \oplus W' \) such that \( \tau(z, Y^h w'') = \mu \neq 0 \) and \( \tau(z, Y^h z) = \nu \neq 0 \). We may suppose that the vector \( z \) is special, for we may write \( z = z_1 + z_2 \), where \( z_i \in (W_j \oplus W') \cap W^i \) and \( W^i \) is an eigenspace of \( \sigma|W \). Now \( 0 \neq \tau(z, Y^h w) = \tau(z_1, Y^h w) + \tau(z_2, Y^h w) \) implies that one of the summands is not zero, say \( \tau(z_1, Y^h w) \neq 0 \). Otherwise interchange \( z_1 \) and \( z_2 \) and rename \( z_1 \) to be \( z \). Then \( z \) is a special vector. Consider the subspace
\[
W^\vee = \text{span}\{z, Y z, \ldots, Y^h z; w'', Y w'', \ldots, Y^h w'' = v^0\} \subseteq W_j \oplus W'.
\]

Then \( W^\vee \) is \( Y \) invariant. \( W^\vee \) is \( \sigma \) invariant because the vectors \( w'' \) and \( z \) are special. Also we have
\[
\tau(Y^i z, Y^j z) = \begin{cases} 
(-1)^j \nu, & \text{if } i + j = h \\
0, & \text{if } i + j > h, 
\end{cases}
\]
\[
\tau(Y^i w'', Y^j w'') = \begin{cases} 
(-1)^j \mu, & \text{if } i + j = h \\
0, & \text{if } i + j > h, 
\end{cases}
\]

and \( \tau(Y^i w'', Y^j w'') = 0 \). Thus the matrix \( T \) of \( \tau \) on \( W^\vee \) has nonvanishing entries on the antidiagonal of each block, zero entries below the antidiagonal
of each block, and a zero lower right hand block, that is,

$$T = \begin{pmatrix}
* & + & * & + & 0 \\
+ & 0 & + & 0 & 0 \\
* & + & 0 & 0 & 0 \\
+ & 0 & 0 & 0 & 0
\end{pmatrix}$$

Because all the entries of $T$ below the main anti-diagonal vanish and all the entries on the main anti-diagonal are nonzero the determinant of $T$ is nonvanishing. So $\tau|W^\vee$ is nondegenerate. Hence $W^\vee = W$, because $\Delta$ is indecomposable. This contradicts the fact that $W^\vee$ is a proper subspace of $W$. Thus the vector $v' = 0$. So $W' = \{0\}$, that is, $W = W_j$. Hence $\Delta = \Delta_j$, which is indecomposable. □

We now classify nilpotent indecomposable distinguished types $\Delta$ of distinguished height $h$ and set of parameters $\mu(\Delta)$. Suppose that $\Delta$ is represented by the triple $(Y, W, v_0; \sigma, \tau)$. From proposition 3.1 it follows that the dimension of the reduced semisimple type $\overline{\Delta}$ is 1 or 2. The results of this classification are given in table 2. We argue case by case.

| Lie algebra | $\overline{\Delta}$ | $\Delta$ | dim $W$ | Conditions |
|-------------|----------------------|----------|---------|------------|
| $\text{gl}(V, \sigma_+)_{v^0}$ | $\Delta_0(0)$ | $\Delta_h(0), \lambda \in \mathbb{R} \setminus \{0\}$ | 1 | A |
| $\text{gl}(V, \sigma_-)_{v^0}$ | $\Delta_0(0,0)$ | $\Delta_h(0,0)$ | 2 | B |
| $\text{gl}(V, \tau_+)_v$ | $\Delta_0^\beta(0)$ | $\Delta_h^\beta(0), \lambda \in \mathbb{R}_{>0}$ | 1 | |
| $\text{gl}(V, \tau_-)_v$ | $\Delta_0^\alpha(0) + \Delta_0^\beta(0)$ | $\Delta_h^\alpha(0) + \Delta_h^\beta(0)$ | 2 | |
| $\text{o}(V, \sigma_+, \tau)_v$ | $\Delta_0^\varepsilon(0)$ | $\Delta_h^\varepsilon(0), \lambda \in \mathbb{R}_{>0}, h$ even | 1 | A |
| $\text{o}(V, \sigma_+, \tau)_v$ | $\Delta_0^\varepsilon(0) + \Delta_0^\beta(0)$ | $\Delta_h^\varepsilon(0) + \Delta_h^\beta(0), h$ even | 2 | A |
| $\text{o}(V, \sigma_-, \tau)_v$ | $\Delta_0(0,0)$ | $\Delta_h(0,0), h$ even | 2 | B |
| $\text{o}(V, \sigma_-, \tau)_v$ | $\Delta_0^\varepsilon(0)$ | $\Delta_h^\varepsilon(0), \lambda \in \mathbb{R}_{>0}, h$ odd | 1 | A |
| $\text{sp}(V, \sigma_+, \tau)_v$ | $\Delta_0(0,0)$ | $\Delta_h(0,0), h$ even | 2 | A |
| $\text{sp}(V, \sigma_+, \tau)_v$ | $\Delta_0(0,0)$ | $\Delta_h(0,0), h$ odd | 2 | A |
| $\text{sp}(V, \sigma_-, \tau)_v$ | $\Delta_0(0,0)$ | $\Delta_h(0,0), h$ even | 2 | B |

Table 2. List of indecomposable nilpotent distinguished types. Here $\varepsilon \in \mathbb{R}$ with $\varepsilon^2 = 1$. Condition A is: $\sigma_+(v^0) = v^0$ and condition B is: $v^0 \in V_{\sigma_-}$ with $\sigma_-(v^0) = \pm iv^0$ and $\tau(v^0, v^0) = 0$. 12
Case 1. $\dim \overline{W} = 1$.

Suppose that $\tau$ is not involved in the triple defining $\Delta$ and $\sigma = \sigma_+$.

Let $Y \in \text{gl}(W; \sigma_+)$ with $(0, \overline{W}; \overline{\sigma}_+^0) \in \Delta_0(0)$ and $(Y, W; \sigma_+) \in \Delta_h(0)$. The distinguished type $\overline{\Delta}$ has distinguished height $h$ and set of parameters $\mu(\overline{\Delta})$ with $v^0 \in W_{\sigma_+} = \{ w \in W \mid \sigma_+(w) = w \}$. The case where $\overline{\sigma}_+(v^0) = -v^0$ is handled by renaming the $-1$ eigenspace of $\sigma_+$ as $W_{\sigma_+}$. There is a nonzero vector $w \in W_{\sigma_+}$ and a nonzero $\lambda \in \mu(\overline{\Delta})$ such that $v^0 = \lambda Y^h w$. Since $\text{dht}(\Delta) = h$ and $Y \in \text{gl}(W; \sigma_+)$, it follows that $\overline{W} = \text{span}_C \{ w, Yw, \ldots, Y^h w \}$ is a subspace of $W_{\sigma_+}$, which is $Y$ and $\sigma_+$ invariant. Moreover, $v^0 \in \overline{W}$. Since $\Delta$ is indecomposable, $\overline{W} = W$. Now

$$\overline{\Delta} Y^h w = \sigma_+(\lambda Y^h w), \quad \text{since } \sigma_+ \text{ is anti-linear and } Y^h w \in W_{\sigma_+}$$

$$= \sigma_+(v^0) = v^0, \quad \text{because } (Y, W, v^0; \sigma_+) \text{ is a triple}$$

$$= \lambda Y^h w.$$

Thus $\overline{\lambda} = \lambda$, that is, $\lambda \in \mathbb{R}$. Denote $\overline{\Delta}$ by $\Delta_h(0), \lambda \neq 0$. Here $\lambda$ is a real nonzero modulus.

The case $\text{gl}(W; \sigma_-)$ does not occur because the anti-linear mapping $\overline{\sigma}_-$ is only defined when $\overline{W}$ is even dimensional.

Suppose that $\tau_*$ or $\tau$ occurs in the triple representing the distinguished type $\overline{\Delta}$ and that the induced bilinear form on $\overline{W}$ is hermitian or symmetric, see appendix 1. The alternatives are

| Lie algebra   | $\overline{\Delta}$ | $\Delta$ |
|---------------|----------------------|----------|
| $\text{gl}(W; \tau_*)$ | $\overline{\Delta}_0(0)$ | $\Delta_h(0)$ |
| $\text{o}(W; \sigma_+, \tau)$ | $\overline{\Delta}_0(0)$ | $\Delta_h(0)$, $h$ even |
| $\text{sp}(W; \sigma_+, \tau)$ | $\overline{\Delta}_0(0)$ | $\Delta_h(0), h$ odd |

First we look at the case where $\overline{\Delta}$ is a distinguished type, represented by the triple $(Y, W, v^0; \tau_*)$, of distinguished height $h$, where $\tau_*$ is hermitian. Hence $h$ is even. Then there is a nonzero vector $w \in W$ such that $Y^h w = v^0$. Thus the subspace $\overline{W} = \text{span}_C \{ w, Yw, \ldots, Y^h w = v^0 \}$ of $W$ is $Y$ invariant and contains the vector $v^0$. Since $\overline{\tau}_*$ is nondegenerate on $\overline{W}$ and $\dim \overline{W} = 1$, it follows that $\overline{\tau}_*(\overline{w}, \overline{w}) \in \mathbb{R} \setminus \{0\}$, because $\overline{\tau}_*$ is hermitian and $\overline{w} \neq 0$ since $w \notin YW$. Thus we may assume that $\overline{\tau}_*(\overline{w}, \overline{w}) = \varepsilon \overline{\lambda}$, where $\lambda > 0$ and $\varepsilon^2 = 1$.

One has

$$\tau_*(w, v^0) = \tau_*(w, Y^h w) = \overline{\tau}_*(\overline{w}, \overline{w}) = \varepsilon \lambda,$$

which shows that $\varepsilon \lambda \in \mu(\overline{\Delta})$. Moreover, we have

$$\tau_*(Y^{i-1} w, Y j^1 w) = \begin{cases} (-1)^{i-1} \varepsilon \lambda, & \text{if } (i - 1) + (j - 1) = h \\ 0, & \text{otherwise} \end{cases}$$
So the bilinear form $\tau_w$ on $\tilde{W}$ is nondegenerate, since the matrix $T$ whose $ij^{th}$ entry is $((\tau_w(Y_i^{-1}w,Y_j^{-1}w))$ has nonzero entries on its anti-diagonal and zero entries below its anti-diagonal, which implies $\det T \neq 0$. Hence $W = \tilde{W}$, because $\Delta$ is indecomposable. Denote $\Delta$ by $\Delta_h^\epsilon(0)$, $\lambda > 0$, $h$ is even. It has a positive modulus $\lambda$. Suppose that $\overline{\tau}_w$ is alternating on $\overline{W}$, which implies that $h$ is odd, since $\tau_w$ is hermitian. Then $\overline{\tau}_w = i\overline{\tau}_w$ is hermitian. Thus we may assume that $\overline{\tau}_w$ is a nondegenerate hermitian form on $\overline{W}$. Using the same argument as above, we find that the distinguished type $\Delta$ is $\Delta_h^\epsilon(0)$, $\lambda > 0$, $h$ is odd. It has a positive modulus $\lambda$.

The cases $o(W,v^0;\sigma_+)$ with $h$ odd and $sp(W,v^0;\sigma_+)$ with $h$ even do not occur because $\overline{\tau}_w$ is nondegenerate and alternating, which implies that $\dim \overline{W}$ is even. Also the cases $o(W,v^0;\sigma_+)$ and $sp(W,v^0;\sigma_+)$ do not occur, because $\dim \overline{W}$ must be even for $\overline{\tau}_w$ to be defined.

**Case 2.** $\dim \overline{W} = 2$.

Suppose $\tau$ is not involved in the triple defining the distinguished type $\Delta$.

We look at the case of $\mathfrak{gl}(W;\sigma_+)$, represented by the pair $\overline{(Y,0)} = (\overline{W};\sigma_+)$ is not indecomposable. The following argument shows that $\Delta$ is not indecomposable. Consequently, this case is excluded. The reduced type $\Delta$ is the sum of the two indecomposable types: $\Delta_0(0)$ and $\Delta_h(0)$. There are special vectors $\overline{z}$ and $\overline{w}$ in $\overline{W}$ such that $\{\overline{z},\overline{w}\}$ is a basis of $\overline{W}$. By [1] prop.3 p.343 there are $z,w \in W$ with $\overline{z} = z + YW \in \overline{W}$ and $\overline{w} = w + YW \in \overline{W}$ such that $\{z,Yz,,\ldots,Y^h z; w, Yw,\ldots,Y^h w\}$ is a basis of $W$. Hence $Y^h z \neq 0$. Since $W = W^1 \oplus W \cap W^2$, where $W^i$ are eigenspaces of $\tau_w$, we may write $z = z_1 + z_2$, where $z_i \in W \cap W^i$ for $i = 1, 2$. Now $0 \neq Y^h z = Y^h z_1 + Y^h z_2$ implies $Y^h z_i$ are not both zero. Suppose that $Y^h z_1 \neq 0$. Otherwise interchange $z_1$ and $z_2$ and rename $z_1$ as $z$. Then $z$ is a special vector. A similar argument shows that we may assume that $w$ is a special vector. We now show that we can choose a basis of $W$ so that $Y^h w = v^0$. Since $\ker Y = \text{span}\{Y^h z, Y^h w\}$ and $v^0 \in \ker Y$ by hypothesis, there are $\alpha, \beta \in \mathbb{C}$ not both zero such that $v^0 = \alpha Y^h z + \beta Y^h w$. If $\alpha \neq 0$ let $z' = \alpha^{-1} z$ and $w' = \alpha w + \beta z$; while if $\alpha = 0$ and $\beta \neq 0$ let $z' = \beta z$ and $w' = -\beta^{-1} z$. Then $W' = \text{span}_{\mathbb{C}}\{Y^h z', Y^{h-1} z', \ldots, z', w', Yw',\ldots,Y^h w'\}$ is a $Y$ and $\sigma$ invariant subspace of $W$, since $z'$ and $w'$ are special vectors. Moreover, $Y^h w' = v^0$. Since the distinguished type $\Delta$ represented by the triple $(Y,W,v^0;\sigma_+)$ is indecomposable, $W = W'$. But $\overline{\Delta}$ is the sum of the distinguished type $\Delta'$, represented by the triple $(\overline{Y},\overline{W'},\overline{v^0};\sigma_+)$, where $\overline{W'} = \text{span}\{w', Yw',\ldots,Y^h w' = v^0\}$, and the type $\Delta$, represented by the pair
(Y, W^i; \sigma_\pm), where W^\dagger = \text{span}_\mathbb{C}\{z', Yz', \ldots, Y^hz'\}. In other words, \Delta is decomposable.

We now look at the case gl(W; \sigma_\pm). The reduced type \overline{\Delta}, represented by the pair (\overline{Y} = 0, \overline{W}; \overline{\sigma}_\pm) is indecomposable. There is a basis \{\overline{z}, \overline{w}\} of special vectors of the vector space \overline{W} such that \overline{z} + \overline{\sigma}_\pm(\overline{w}) is a basis of the vector space \overline{W_{\sigma_\pm}} over the quaternions, see appendix 1. By [I, prop. 3 p.343] there are vectors \overline{z}, \overline{w} \in W with \overline{z} = z + YW and \overline{w} = w + YW such that \{z, Yz, \ldots, Y^hz; w, Yw, \ldots, Y^hw\} is a basis of W. Because Y^hz \neq 0 and Y^hw \neq 0, we may assume that the vectors z and w are special. Arguing as in the preceding paragraph, we may choose z and w so that Y^hw = v_0. We denote the distinguished type \Delta by \Delta_h(0,0). There is no modulus.

Suppose that \tau is involved in the triple defining the distinguished type \Delta and that the bilinear form on \overline{W} is alternating. Suppose that \tau_s is hermitian. Then h is odd, because \overline{\tau} is alternating. Hence \overline{\tau}_s = i\overline{\tau}_s is a nondegenerate symmetric hermitian form on \overline{W}, where the reduced type \overline{\Delta} is represented by the triple (\overline{Y}, \overline{\sigma}, \overline{\tau}). Using the same argument as in case 1, we find that distinguished type \Delta is the sum of an indecomposable distinguished type \Delta_h^1(0), \lambda_1 > 0 and an indecomposable type \Delta_h^2(0). Hence \Delta is not indecomposable. This contradicts our hypothesis. Thus the case gl(W, \tau_s), h odd does not occur.

The remaining alternatives are listed below.

| Lie algebra  | \overline{\Delta} | \Delta       |
|--------------|-------------------|--------------|
| o(W; \sigma_+, \tau) | \Delta_0(0,0)      | \Delta_h(0,0), h odd |
| o(W; \sigma_-, \tau) | \Delta_0^\dagger(0,0) | \Delta^\dagger_h(0,0), h odd |
| sp(W; \sigma_+, \tau) | \Delta_0(0,0)      | \Delta_h(0,0), h even |
| sp(W; \sigma_-, \tau) | \Delta_0^\dagger(0,0) | \Delta^\dagger_h(0,0), h even. |

Suppose that \sigma = \sigma_+ and that the reduced type \overline{\Delta} is represented by the pair (\overline{Y} = 0, \overline{W}; \overline{\sigma}_+, \overline{\tau}). There are nonzero vectors \overline{z} and \overline{w} in \overline{W} such that \{\overline{z}, \overline{w}\} is a basis for \overline{W}_{\sigma_+} with \overline{\tau}(\overline{z}, \overline{z}) = 0 = \overline{\tau}(\overline{w}, \overline{w}) and \overline{\tau}(\overline{z}, \overline{w}) = \mu \neq 0. By [I prop.3 p.343] there are vectors z, w \in W so that \overline{z} = z + YW \in \overline{W} and \overline{w} = w + YW \in \overline{W} such that \overline{W} = \text{span}_\mathbb{C}\{z, Yz, \ldots, Y^hz; w, Yw, \ldots, Y^hw\} is a subspace of W. Since Y^hz and Y^hw are both nonzero, we may assume that z, w \in W_{\sigma_+}. Moreover,

\[
\tau(Y^{i-1}z, Y^{j-1}w) = \begin{cases} 
(-1)^j \mu, & \text{if } (i-1) + (j-1) = h \\
0, & \text{if } i + j > h,
\end{cases}
\]
\( \tau(Y^{-1}z, Y^{-1}z) = 0 = \tau(Y^{-1}w, Y^{-1}w) \). We now show that we can choose a basis of \( W_\sigma \) so that \( Yh w = v^0 \). There are \( \alpha, \beta \in \mathbb{C} \) not both zero such that \( v^0 = \beta Yh z + \alpha Yh w \), since \( v^0 \in \ker Yh \) and \( \ker Yh = \text{span}\{Yh z, Yh w\} \). Let \( (z') = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) \), where \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \beta \end{pmatrix} \) if \( \alpha \neq 0 \) and \( \begin{pmatrix} a & 0 \\ 0 & \beta ^{-1} \end{pmatrix} \) if \( \beta \neq 0 \). Then \( \{z', Yz', \ldots, Y^h z'; w', Yw', \ldots, Y^h w'\} \) is spans \( \tilde{W} \) because \( \tau(Y^{-1}z', Y^{-1}z') = 0 = \tau(Y^{-1}w', Y^{-1}w') \) and

\[
\tau(Y^j z', Y^h j w') = (-1)^j \tau(az + bw, Y^h(cz + dw)) \\
= ac \tilde{\tau}(z, z) + bd \tilde{\tau}(w, w) + (ad - bc) \tilde{\tau}((z, w)) \\
= \tilde{\tau}(z, w), \quad \text{since} \quad \tilde{\tau} \text{ is alternating and} \quad ad - bc = 1 \equiv \mu;
\]

while \( \tau(Y^{-1}z', y^{-1}j w') = 0 \) if \( (i - 1) + (j - 1) > h \). Thus the matrix

\[
T = \begin{pmatrix} \tau(Y^{-1}z', Y^{-1}z') & \tau(Y^{-1}z', Y^{-1}w') \\ \tau(Y^{-1}w', Y^{-1}z') & \tau(Y^{-1}w', Y^{-1}w') \end{pmatrix}
\]

on \( \tilde{W} \) is

\[
\begin{pmatrix} 0 & * & + \\ + & * & 0 \\ + & 0 & 0 \end{pmatrix}.
\]

So \( \det T \neq 0 \), since its antidiagonal entries are all nonzero. Thus \( \tau \) is nondegenerate on \( \tilde{W} \). By construction \( v^0 = Y^h w' \). But \( \tilde{W} \) is a \( Y \) invariant subspace of \( W \), which contains the vector \( v^0 \). It is also \( \sigma_\pm \) invariant, since the vectors \( z' \) and \( w' \) lie in \( W_\sigma \) and thus are special. Hence \( \tilde{W} = W \), since \( \Delta \) is indecomposable. We denote \( \Delta \) by \( \Delta_h (0,0) \), when \( h \) is odd for \( o(W, v^0; \sigma_+, \tau) \) or when \( h \) is even for \( \text{sp}(W, v^0; \sigma_+, \tau) \). There is no modulus.

Suppose that \( \sigma = \sigma_- \) and that the reduced type \( \underline{\Delta} \) is represented by the pair \( (\overline{\psi} = 0, \overline{\sigma \_ - \psi}) \), where \( \overline{\psi} \) is alternating. There is a basis \( \{\overline{z}, \overline{w}\} \) of eigenvectors of \( \sigma_- \) on \( \overline{W} \) such that \( \overline{y} = \overline{z} + \overline{\sigma_- (w)} j \) is a basis for the 1 dimensional quaternionic vector space \( \overline{W}_\sigma \) with a nondegenerate hamiltonian alternating form \( \tau \), see appendix 1. Thus \( \tau(\overline{z}, \overline{y}) = \lambda \varepsilon j \), where \( \lambda \in \mathbb{R}_{>0} \) and \( \varepsilon^2 = 1 \). Let \( \overline{z}' = \lambda^{-1/2} \overline{z} \) and \( \overline{w}' = \lambda^{-1/2} \overline{w} \), where \( \overline{z} \) and \( \overline{w} \) are special vectors. Then with \( \overline{y}' = \overline{z}' + \overline{\sigma_- (w') j} \) one has \( \tau(\overline{y}', \overline{y}') = \varepsilon j \). Since \( \tau(z', Yh w') = \overline{\tau}(\overline{z'}, \overline{w'}) \neq 0 \) for \( z' \in \overline{z}' \) and \( w' \in \overline{w}' \) with \( Yh z' \) and \( Yh w \) both nonzero, an argument shows that we may assume that \( z' \) and \( w' \) are special vectors in \( W \) and that \( W' = \text{span}_C \{z', \ldots, Y^h z'; w, \ldots, Y^h w'\} \) contains the vector \( v^0 \). Since the distinguished type \( \underline{\Delta} \) represented by the triple
(Y, W, v^0; σ, τ) is indecomposable, it follows that W′ = W and there is no modulus. Hence the distinguished type \( \Delta \) is \( \Delta^+_h(0, 0) \) with \( h \) odd, when \( \tau \) is symmetric and \( h \) is even when \( \tau \) is alternating.

Suppose that \( \tau \) or \( \tau_\ast \) is involved in the triple defining the distinguished type \( \Delta \) and that the bilinear form on \( \overline{W} \) is symmetric or hermitian. The alternatives are listed below.

| Lie algebra | \( \overline{\Delta} \) | \( \Delta \) |
|-------------|----------------|---------|
| \( \text{gl}(W; \tau_\ast) \) | \( \Delta^+_\mu(0) + \Delta^-_\mu(0) \) | \( h \) even |
| \( \text{o}(W; \sigma_+, \tau) \) | \( \Delta^+_\mu(0) + \Delta^-_\mu(0) \) | \( h \) even |
| \( \text{o}(W; \sigma_-, \tau) \) | \( \Delta_0(0, 0) \) | \( h \) even |
| \( \text{sp}(W; \sigma_+, \tau) \) | \( \Delta^+_\mu(0) + \Delta^-_\mu(0) \) | \( h \) odd |
| \( \text{sp}(W; \sigma_-, \tau) \) | \( \Delta_0(0, 0) \) | \( h \) odd. |

We treat the case when the bilinear form on \( \overline{W} \) is the hermitian form \( \tau_\ast \). Suppose that \( \overline{\tau}_\ast(\overline{z}, \overline{w}) \neq 0 \) for every nonzero \( \overline{z} \in \overline{W} \). Then there is a basis \( \{\overline{z}, \overline{w}\} \) of \( \overline{W} \), which is \( \overline{\tau}_\ast \) orthogonal, that is, \( \overline{\tau}_\ast(\overline{z}, \overline{w}) = \mu \neq 0 \), \( \overline{\tau}_\ast(\overline{w}, \overline{w}) = \lambda \neq 0 \), and \( \overline{\tau}_\ast(\overline{w}, \overline{w}) = 0 \). Then there are vectors \( z, w \in W \) with \( \overline{z} = z + YW \) and \( \overline{w} = w + YW \) such that \( \{z, yz, \ldots, Y^h z; w, Yw, \ldots, Y^h w\} \) forms a \( \tau_\ast \) orthogonal basis of \( W \). We can choose \( \overline{w} \) so that \( Y^h w = v^0 \). Thus \( \overline{\Delta} \) is the sum of the distinguished type \( \Delta_1 \), represented by the triple \( (Y|W_1, W_1, v^0; \tau_\ast|W_1) \), where \( W_1 = \text{span}\{w, Yw, \ldots, Y^h w = v^0\} \), and a type \( \Delta_2 \), represented by the pair \( (Y|W_2, W_2; \tau_\ast|W_2) \), where \( W_2 = \text{span}\{z, yz, \ldots, Y^h z\} \). This contradicts our hypothesis that \( \Delta \) is indecomposable. Thus there is a nonzero vector \( \overline{z} \in \overline{W} \) such that \( \overline{\tau}_\ast(\overline{z}, \overline{z}) = 0 \). Since \( \overline{\tau}_\ast \) is nondegenerate on \( \overline{W} \), there is a vector \( \overline{y} \in \overline{W} \) such that \( \overline{\tau}_\ast(\overline{z}, \overline{y}) = \eta \neq 0 \). Let \( \overline{w} = \overline{z} - \overline{\tau}_\ast(\overline{z}, \overline{y}) \overline{y} \). Then \( \overline{w} \) is a \( \overline{\tau}_\ast \) isotropic vector in \( \overline{W} \). Thus the matrix of \( \overline{\tau}_\ast \) with respect to the basis \( \{\overline{z}, \overline{w}\} \) is \( \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} \), because \( \overline{\tau}_\ast(\overline{z}, \overline{w}) = \overline{\tau}_\ast(\overline{z}, \overline{y}) = \eta \). Hence the semisimple reduced type \( \overline{\Delta} \) is equal to \( \Delta^+_\mu(0) + \Delta^-_\mu(0) \). Using [1, prop.2 p.343] there are vectors \( z, w \in W \) with \( \overline{z} = z + YW \) and \( \overline{w} = w + YW \) such that \( \{Y^h z, Y^{h-1} z, \ldots, z; w, Yw, \ldots, Y^h w\} \) is a basis of \( W \) where \( \tau_\ast(Y^i z, Y^j w) = \begin{cases} \lambda^{i+j} & \text{if } i + j = h \\ 0 & \text{otherwise} \end{cases} \) and \( \tau_\ast(Y^i z, Y^j z) = 0 = \tau_\ast(Y^i w, Y^j w) \).

Since \( v^0 \in \ker Y = \text{span}\{Y^h z, Y^h w\} \), there are \( \alpha, \beta \in \mathbb{C} \) not both zero such that \( v^0 = \alpha Y^h z + \beta Y^h w \). We now show that we can find a basis of \( W \) such that \( v^0 = Y^h w \). If \( \beta \neq 0 \) set \( z' = \beta^{-1} z \) and \( w' = az + \beta w \); while if \( \alpha \neq 0 \) and \( \beta = 0 \), set \( w' = \alpha z \) and \( z' = -\alpha^{-1} w \). In either case \( v^0 = Y^h w' \) and \( \{Y^h z', -Y^{h-1} z', \ldots, z', w', Yw', \ldots, Y^h w'\} \) is a basis of \( W \), which is \( Y \) invariant, and with respect to which the matrix of \( \tau_\ast \) is \( \begin{pmatrix} 0 & \eta^{h+1} \\ \eta^{h+1} & 0 \end{pmatrix} \). The
indecomposable nilpotent distinguished type \( \Delta \) is denoted \( \Delta^+_h(0) + \Delta^-_h(0) \). There is no modulus.

Suppose that \( \overline{\tau} \) is symmetric and \( \sigma = \sigma_+ \). In the preceding paragraph replace \( \overline{\tau}_* \) by \( \overline{\tau}_+ = \overline{\tau}W\sigma_+ \), \( \tau_* \) by \( \tau_+ \), and \( W \) by \( W\sigma_+ \). Arguing as before, we find that \( \overline{\Delta} \) is \( \Delta^+_h(0) + \Delta^-_h(0) \), when \( h \) is even and \( \tau \) is symmetric or \( \tau \) is alternating and \( h \) is odd. There is no modulus.

Suppose that \( \overline{\tau} \) is a symmetric bilinear form on \( \overline{W} \) and \( \sigma = \sigma_- \). Here the triple \((W,v^0;\sigma_-,\tau)\) represents the indecomposable nilpotent distinguished type \( \Delta \) of distinguished height \( h \), where \( h \) even when \( \tau \) is symmetric or \( h \) is odd when \( \tau \) is alternating. The pair \((\overline{\tau} = 0,\overline{W};\overline{\sigma}_-,\overline{\tau})\) represents the reduced type \( \overline{\Delta} \). There is a basis \( \{\overline{z},\overline{w}\} \) of special vectors of \( \overline{W} \) such that \( \overline{y} = \overline{x} + \overline{\sigma}_-(\overline{w}) \) is a basis of the 1 dimensional quaternionic vector space \( \overline{W}_{\sigma_-} \) with hamiltonian symmetric form \( \tau \) such that \( \tau(\overline{x},\overline{y}) = \lambda \), where \( \lambda \in \mathbb{R} > 0 \). Let \( \overline{z}' = \lambda^{-1/2} \overline{z}, \overline{w}' = \lambda^{-1/2} \overline{w} \), and \( \overline{y}' = \lambda^{-1/2} \overline{y} \). Then \( \overline{\tau}(\overline{z}',\overline{w}') = 1 = \overline{\tau}(\overline{w}',\overline{w}') \) and \( \overline{\tau}(\overline{z}',\overline{w}') = 0 \) and the reduced type \( \overline{\Delta} \) is \( \Delta_0(0,0) \). Using [I] prop.2 p.343, we find that there are vectors \( z, w \in W \) with \( \overline{z} = z + YW \) and \( \overline{w} = w + YW \) such that \( \{Y^hz, Y^{h-1}z, \ldots, z; w, Yw, \ldots, Y^hw\} \) is a basis of \( W \) such that \( \tau(Y^iz,Y^jw) = \begin{cases} \ (-1)^i\lambda, & \text{if } i + j = h \\ 0, & \text{otherwise} \end{cases} \), where \( \lambda \in \mathbb{R} \setminus \{0\} \), and \( \tau(Y^iz,Y^jz) = 0 = \tau(Y^iw,Y^jw) \). Since \( \tau(z,Y^hw) \neq 0 \), we may assume that \( z \) and \( w \) are special vectors. Then \( W \) is \( \sigma_- \) invariant. We now show that we can choose a \( \tau \) orthogonal basis of \( W \) such that \( Y^hw = v^0 \). Since \( v^0 \in \ker Y^h = \text{span}\{Y^hz, Y^hw\} \), there are \( \alpha, \beta \in \mathbb{C} \) not both zero such that \( v^0 = \alpha Y^hz + \beta Y^hw \). If \( \beta \neq 0 \), set \( z' = \beta^{-1}z \) and \( w' = \alpha z + \beta w \). If \( \alpha \neq 0 \) and \( \beta = 0 \), set \( z' = -\alpha^{-1}w \) and \( w' = \alpha z \). In either case \( Y^hw' = v^0 \) and \( \{Y^hz', -Y^{h-1}z', \ldots, z'; w', Yw', \ldots, Y^hw'\} \) is a basis of \( W \) such that the matrix of \( \tau \) is \( \left( \begin{array}{c} \lambda_{m+1}^0 \\ \vdots \\ \lambda_0^{m+1} \end{array} \right) \). The nilpotent distinguished type \( \Delta \) is denoted \( \Delta_h(0,0) \). There is no modulus.

This completes the classification of nilpotent indecomposable distinguished special types and proves

**Proposition 3.2** An indecomposable nilpotent distinguished type \( \Delta \) is uniquely determined by its distinguished height \( h \), an element of its set of parameters \( \mu(\Delta) \), if nonempty, and the dimension of its reduced type \( \overline{\Delta} \).

**Proof of theorem 2.1** We prove the uniqueness of the decomposition of a distinguished type \( \Delta \) into a sum of an indecomposable nilpotent distinguished type \( \Delta' \) and a type \( \Delta \), which is the sum of indecomposable types \( \sum_{\ell=1}^k \Delta_\ell \). From the fact that \( \Delta' \) has the same distinguished type and set of
parameters as $\Delta$, using proposition 3.2 it follows that $\Delta'$ is unique. From the theorem [11, p.343] it follows that the decomposition of the type $\Delta$ into a sum of indecomposable types is unique up to reordering of the summands. Thus the decomposition $\Delta = \Delta' + \sum_{\ell=1}^k \Delta_\ell$ is unique. \hfill $\square$

## 4 The real affine classical groups

Let $\tilde{V}$ be a complex vector space of dimension $n$ with an anti-linear mapping $\tilde{\sigma}$ of $\tilde{V}$ into itself such that $\tilde{\sigma}^2 = \pm \text{id}_{\tilde{V}}$ and a nondegenerate bilinear form $\tau : \tilde{V} \times \tilde{V} \to \mathbb{C}$ that is symmetric or alternating and satisfies $\tilde{\sigma}^* \tau = \overline{\tau}$. $\tilde{V}$ may also have a nondegenerate hermitian form $\tilde{\tau}$, see appendix 1.

Let $G(\tilde{V}, \tilde{\sigma}, \tilde{\tau})$ be a real classical group, see table 1. In other words, $g \in G(\tilde{V}, \tilde{\sigma}, \tilde{\tau})$ if and only if $g \in \text{Gl}(\tilde{V})$ such that $\tilde{\sigma}(g) = g$ and $g^* \tilde{\tau} = \tilde{\tau}$. The affine real classical group $\text{Aff}G(\tilde{V}, \tilde{\sigma}, \tilde{\tau})$ is the set of complex affine mappings $\text{aff}g : \tilde{V} \to \tilde{V} : \tilde{v} \mapsto g(\tilde{v}) + \tilde{u}$, where $g \in G(\tilde{V}, \tilde{\sigma}, \tilde{\tau})$ and $\tilde{u}, \tilde{v} \in \tilde{V}$, with group multiplication given by composition of affine mappings. For a nonzero vector in $\tilde{V}$ the isotropy group $G(\tilde{V}, \tilde{\sigma}, \tilde{\tau})_{v^0}$ is the set of all $g \in G(\tilde{V}, \tilde{\sigma}, \tilde{\tau})$ such that $g(v^0) = v^0$.

In this section we prove

**Theorem 4.1** Every real affine classical group $\text{Aff}G(\tilde{V}, \tilde{\sigma}, \tilde{\tau})$ is isomorphic to an isotropy group of a real classical group.

The proof proceeds case by case.

**Case 1.** $\text{Aff}G(\tilde{V}, \tilde{\sigma}_+)$. Let $V^\vee = \tilde{V} \times \mathbb{C}$. Set

$$\sigma_+^\vee : V^\vee \to V^\vee : v^\vee = (\tilde{v}, z) \mapsto \sigma_+^\vee(v^\vee) = (\tilde{\sigma}_+(\tilde{v}), \tau),$$

where

$$\tilde{\sigma}_+ : \tilde{V} \to \tilde{V} : (z_1, \ldots, z_n)^T \mapsto (z_1, \ldots, z_n)^T,$$

see appendix 1. Then $\sigma_+^\vee$ is an anti-linear mapping of $V^\vee$ into itself with $(\sigma_+^\vee)^2 = \text{id}_{V^\vee}$. Let $V^\vee_{\sigma_+^\vee} = \{v^\vee \in V^\vee \mid \sigma_+^\vee(v^\vee) = v^\vee\}$. Then $V^\vee_{\sigma_+^\vee}$ is an $n + 1$ dimensional real vector space with basis $e^\vee = \{e_1, \ldots, e_n; e_{n+1}\}$, where $\tilde{e} = \{e_1, \ldots, e_n\}$ is a basis of $\tilde{V}$. For a nonzero vector $(v^\vee)^0 \in V^\vee_{\sigma_+^\vee}$, which is special by definition, let $G(V^\vee, \sigma_+^\vee)_{(v^\vee)^0}$ is the set of all $g \in \text{Gl}(V^\vee)$ such that $\sigma_+^\vee g = g \sigma_+^\vee$ and $g((v^\vee)^0) = (v^\vee)^0$. Choose $(v^\vee)^0 = e_{n+1}$. Then $(v^\vee)^0$ is a special vector and

$$G(V^\vee, \sigma_+^\vee)_{(v^\vee)^0} = \left\{ \begin{pmatrix} \tilde{A} & 0 \\ \tilde{A}^T & 1 \end{pmatrix} \in \text{Gl}(V^\vee_{\sigma_+^\vee}) \mid \tilde{A} \in \text{Gl}(\tilde{V}_{\sigma_+}) & \tilde{d} \in \tilde{V}_{\sigma_+} \right\}$$

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is isomorphic to $\text{Aff}G(\tilde{V}, \tilde{\sigma}_+)$.

**Case 2.** $\text{Aff}G(\tilde{V}, \tilde{\sigma}_+, \tilde{\tau})$. Let $\tilde{V} = \mathbb{C} \times \tilde{V} \times \mathbb{C}$. For $0 \leq p \leq \left[ \frac{n}{2} \right]$ set

$$\tilde{\sigma}_+ = \tilde{\sigma}_+^{(p)} : \tilde{V} \to \tilde{V} : \tilde{v} = (z_0, \tilde{v}, z_{n+1})^T \to \tilde{\sigma}_+(\tilde{v}) = (\tilde{z}_0, \tilde{\sigma}_+^{(p)}(\tilde{v}), \tilde{z}_{n+1})^T,$$

where

$$\tilde{\sigma}_+ = \tilde{\sigma}_+^{(p)} : \tilde{V} \to \tilde{V} : \tilde{v} = (z_1, \ldots, z_n)^T \to \left( -I_{n-p} 0 \atop 0 I_p \right) \left( \begin{array}{c} \tilde{z}_1 \\ \vdots \\ \tilde{z}_n \end{array} \right) ;$$

see appendix 1. Then $\tilde{\sigma}_+$ is an anti-linear mapping of $\tilde{V}$ into itself such that $(\tilde{\sigma}_+)^2 = \text{id}_{\tilde{V}}$. Set $\tilde{V}_{\tilde{\sigma}_+} = \{ \tilde{v} \in \tilde{V} | \tilde{\sigma}_+(\tilde{v}) = \tilde{v} \}$. Then $\tilde{V}_{\tilde{\sigma}_+}$ is an $n + 2$ dimensional real vector space. Let $\tau : \tilde{V} \times \tilde{V} \to \mathbb{C}$ be the complex valued bilinear form given by

$$\tau(\tilde{u}, \tilde{v}) = \begin{cases} w_0 z_0 + w_0 z_{n+1} + w_n + 1 z_n + 1, & \text{if } \tilde{\tau} \text{ is symmetric} \\ w_0 z_{n+1} + w_0 z_0 + \tilde{\tau}(\tilde{u}, \tilde{v}) - w_n + 1 z_{n+1}, & \text{if } \tilde{\tau} \text{ is alternating} \end{cases}.$$

The bilinear form $\tau$ is nondegenerate and symmetric if $\tilde{\tau}$ is symmetric and alternating if $\tilde{\tau}$ is alternating. It is easy to check that $\tilde{\tau}(\tilde{\sigma}_+(\tilde{w}), \tilde{\sigma}_+(\tilde{v})) = \tilde{\tau}(\tilde{w}, \tilde{v})$.

Suppose that $\tau$ is symmetric. Let $\tau_+ = \tau|_{\tilde{V}_{\tilde{\sigma}_+}}$, where $\tilde{V}_{\tilde{\sigma}_+} = \{ \tilde{v} \in \tilde{V} | \tilde{\sigma}_+(\tilde{v}) = \tilde{v} \}$. Then $\tau_+$ is a nondegenerate real valued symmetric bilinear form on $\tilde{V}_{\tilde{\sigma}_+}$. For $0 \leq p \leq \left[ \frac{n}{2} \right]$ suppose that $\tau_+ = \tau_+^{(p)}$ is a nondegenerate, symmetric bilinear form on $\tilde{V}$ that has index $n - p$. Then the form $\tau_+$ on $\tilde{V}_{\tilde{\sigma}_+}$ is nondegenerate and symmetric with index $n - p + 1$. The matrix of $\tau_+$ with respect to the basis $\tilde{e} = \{ e_0; e_1; \ldots; e_n; e_{n+1} \}$ of $\tilde{V}_{\tilde{\sigma}_+}$ is $T_+ = \begin{pmatrix} 0 \\ 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, where $T_+ = I_{n-p, p} = \begin{pmatrix} -I_{n-p} & 0 \\ 0 & I_p \end{pmatrix}$ is the matrix of $\tilde{\tau}$ with respect to the basis $\tilde{e} = \{ e_1, \ldots, e_n \}$ of $\tilde{V}$. Let $\tilde{v}^0$ be a $\tau_+$ isotropic vector in $\tilde{V}_{\tilde{\sigma}_+}$, that is, $\tau_+(\tilde{v}^0, \tilde{v}^0) = 0$. Choose $\tilde{v}^0 = e_{n+1}$. By definition $\tilde{v}^0$ is a special vector. A calculation shows that the isotropy group

$$G(\tilde{V}, \tilde{\sigma}_+, \tau)_{\tilde{v}^0} =$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \tilde{d} & \tilde{A} & 0 \\ -\tilde{d}^T \tilde{T}_+ \tilde{A} & -\tilde{d}^T \tilde{T}_+ \tilde{A} & 1 \end{pmatrix} \in \text{O}(\mathbb{R}^{n+2}, T_+) \in \text{O}(\mathbb{R}^n, \tilde{T}_+) \right\}$$

$$= \text{O}(\mathbb{R}^{n+2}, T_+)_{e_{n+1}}.$$

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The mapping

\[ G(\tilde{V}, \tilde{\sigma}_+, \tau)_{\tilde{\sigma}_0} \rightarrow G(\tilde{V}, \tilde{\sigma}_+, \tau) \times \tilde{V} : \begin{pmatrix} 1 & d & 0 \\ -\frac{1}{2} & \tilde{A} & 0 \\ -\tilde{d}^2 \tilde{T}_+ \tilde{A} & -\tilde{d}^2 \tilde{T}_+ \tilde{A} & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ d & \tilde{A} \end{pmatrix} \]

is a group isomorphism. The semidirect product \( G(\tilde{V}, \tilde{\sigma}_+, \tau) \times \tilde{V} \) is isomorphic to the real affine symplectic group \( \text{AffO}(\tilde{V}, \tilde{\sigma}_+, \tau) \) of real affine orthogonal mappings of \((\mathbb{R}^n, \tilde{T}_+)\) into itself.

Suppose that \( \tilde{\tau}_- = \tilde{\tau}|\tilde{V}_{\tilde{\sigma}_+} \) is nondegenerate and alternating. Then \( n = \dim \tilde{V} \) is even, say \( 2m \). The alternating bilinear form \( \tilde{\tau}_- = \tilde{\tau}|\tilde{V}_{\tilde{\sigma}_+} \), with respect to the basis \( \tilde{e} = \{e_0; e_1, \ldots, e_m, f_1, \ldots, f_m; e_{n+1}\} \) of \( \tilde{V}_{\tilde{\sigma}_+} \) has matrix \( \tilde{T}_- = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \), where \( \tilde{T}_- = \begin{pmatrix} 0 & l_m \\ -l_m & 0 \end{pmatrix} \) is the matrix of \( \tilde{\tau}_- \) on \( \tilde{V}_{\tilde{\sigma}_+} \), which has a basis \( \{e_1, \ldots, e_m, f_1, \ldots, f_m\} \). Let \( \tilde{v}^0 \) be a nonzero vector in \( \tilde{V}_{\tilde{\sigma}_+} \). Since \( \tilde{\tau}_- \) is alternating on \( \tilde{V}_{\tilde{\sigma}_+} \), the vector \( \tilde{v}^0 \) is \( \tilde{\tau} \)-isotropic and hence is a special vector. We may choose \( \tilde{v}^0 = e_{n+1} \). Consider the isotropy group

\[ G(\tilde{V}, \tilde{\sigma}_+, \tau)_{\tilde{v}^0} = \left\{ g \in G(\tilde{V}, \tilde{\sigma}_+, \tau) \mid g\tilde{\sigma}_+ = \tilde{\sigma}_+ g, g^* \tau = \tau \& g(\tilde{v}^0) = \tilde{v}^0 \right\}. \]

A calculation gives

\[ G(\tilde{V}, \tilde{\sigma}_+, \tau)_{\tilde{v}^0} = \left\{ \begin{pmatrix} 1 & d & 0 \\ -\frac{1}{2} & \tilde{A} & 0 \\ -\tilde{d}^2 \tilde{T}_- \tilde{A} & -\tilde{d}^2 \tilde{T}_- \tilde{A} & 1 \end{pmatrix} \in \text{Sp}(\mathbb{R}^{2(m+1)}, \tilde{T}_-) \mid \tilde{d} \in \mathbb{R}^m \& \tilde{A} \in \text{Sp}(\mathbb{R}^m, \tilde{T}_-) \right\} = \text{Sp}(\mathbb{R}^{2(m+1)}, \tilde{T}_-)_{\tilde{v}^0}. \]

The mapping

\[ G(\tilde{V}, \tilde{\sigma}_+, \tau)_{\tilde{v}^0} \rightarrow G(\tilde{V}, \tilde{\sigma}_+, \tau) \times \tilde{V} : \begin{pmatrix} 1 & d & 0 \\ -\frac{1}{2} & \tilde{A} & 0 \\ -\tilde{d}^2 \tilde{T}_- \tilde{A} & -\tilde{d}^2 \tilde{T}_- \tilde{A} & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ d & \tilde{A} \end{pmatrix} \]

is a group isomorphism. The semidirect product \( G(\tilde{V}, \tilde{\sigma}_+, \tau) \times \tilde{V} \) is isomorphic to the real affine symplectic group \( \text{AffSp}(\mathbb{R}^{2m}, \tilde{T}_-) \) of real affine symplectic mappings of \((\mathbb{R}^{2m}, \tilde{T}_-)\) into itself.

This completes the discussion of the cases where \( \tilde{\sigma} = \tilde{\sigma}_+ \).
CASE 3. $\text{Aff}(\tilde{V}, \tilde{\sigma}_-)$. Let $V^\vee = \tilde{V} \times \mathbb{C}^2$, where $\tilde{V}$ is a complex vector space of dimension $n = 2m$. Set

$$\sigma_\vee^{-1} : V^\vee \to V^\vee :$$

$$((\tilde{u}, \tilde{w}), (u_{m+1}, w_{m+1}))^T \mapsto (\tilde{\sigma}_-(\tilde{u}, \tilde{w})^T, (\overline{w}_{m+1} - \overline{u}_{m+1})), \quad (2)$$

where

$$\tilde{\sigma}_- : \tilde{V} \to \tilde{V} : (\tilde{u}, \tilde{w})^T \mapsto (\overline{w}, -\overline{u})^T, \quad (3)$$

see appendix 1. Then $\sigma_\vee^{-1}$ is an anti-linear mapping of $V^\vee$ into itself such that $(\sigma_\vee^{-1})_\vee = -\text{id}_{V^\vee}$. Let $\tilde{\sigma}_\vee^{-1} : V^\vee \to V^\vee$ be the complex linear mapping associated to $\sigma_\vee^{-1}$. In other words, $\tilde{\sigma}_\vee^{-1}(\alpha v^\vee) = \alpha \sigma_\vee^{-1}(v^\vee)$ for every $\alpha \in \mathbb{C}$ and every $v^\vee \in V^\vee$. With respect to the basis $\epsilon^\vee = \{e_1, \ldots, e_m, f_1, \ldots, f_m, e_{m+1}, f_{m+1}\}$ of $V^\vee$ the matrix of $\tilde{\sigma}_\vee^{-1}$ is

$$\begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
-i & 0 & \ldots & 0 & 0 \\
0 & i & \ldots & 0 & 0 \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix},$$

using equations (2) and (3).

Let $A \in \text{Gl}(V^\vee)$. Then $A$ commutes with $\tilde{\sigma}_\vee^{-1}$ if and only if its $2(m+1) \times 2(m+1)$ complex matrix with respect to the basis $\epsilon^\vee$ is of the form

$$\begin{pmatrix}
A & B \\
-B & A \\
c^T & -dT \\
d^T & c^T
\end{pmatrix} \begin{pmatrix} a & b \\
b & a \\
e & f \\
f & e
\end{pmatrix}, \quad \text{where } A, B \in \text{gl}(\mathbb{C}^m); \ a, b, c, d \in \mathbb{C}^m; \ e, f \in \mathbb{C}.$$

Hence $A \in \text{Gl}(V^\vee, \sigma_\vee^{-1})$ if and only if its $2(m+1) \times 2(m+1)$ complex matrix is of the form

$$\begin{pmatrix}
A & B \\
-B & A \\
-\overline{c}^T & \overline{d}^T \\
-\overline{d}^T & -\overline{c}^T
\end{pmatrix} \begin{pmatrix} a & b \\
b & \overline{a} \\
e & f \\
f & \overline{e}
\end{pmatrix}.$$

Turn the $2(m+1)$ dimensional complex vector space $V^\vee$ into an $m+1$ dimensional quaternionic vector space $V^\vee_{\sigma_\vee^{-1}}$ by defining scalar multiplication $\cdot$ as $(\alpha + \beta j) \cdot v^\vee = \alpha v^\vee + \beta \sigma_\vee^{-1}(v^\vee)$, where $\alpha, \beta \in \mathbb{C}$ (and thus $\alpha + \beta j \in \mathbb{H}$) and $v^\vee \in V^\vee$. The complex linear isomorphism

$$V^\vee_{\sigma_\vee^{-1}} \to V^\vee_{\sigma_\vee^{-1}} :$$

$$v^\vee = ((\tilde{u}, \tilde{w}), (u_{m+1}, w_{m+1}))^T \mapsto v^\vee + \sigma_\vee^{-1}(v^\vee)$$

$$= ((\tilde{u} + \tilde{w}j, \tilde{w} - \tilde{u}j), (u_{m+1} + w_{m+1}j, w_{m+1} - u_{m+1}j))$$

induces the quaternionic isomorphism

$$\rho^\vee : V^\vee_{\sigma_\vee^{-1}} \to (V^\vee_{\sigma_\vee^{-1}})^\mathbb{H} = \mathbb{H}^{m+1} :$$

$$\rho^\vee = ((\tilde{u}, \tilde{w}), (u_{m+1}, w_{m+1}))^T \mapsto (\tilde{u}^T + \tilde{w}^Tj, u_{m+1} + w_{m+1}j).$$

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The isomorphism $\rho^\vee$ gives rise to the group isomorphism

$$\tilde{\Sigma} : G(V^\vee, \sigma^\vee) \to \text{Gl}(\langle V_{\sigma^\vee}\rangle^\mathbb{H}) :$$

$$A = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \begin{pmatrix} e^T & d^T \\ -\overline{e}^T & -\overline{d}^T \end{pmatrix} \rightarrow \rho^\vee A (\rho^\vee)^{-1} = \mathcal{A} = \begin{pmatrix} \frac{\overline{A}}{\overline{a}} & \overline{c} \\ \frac{\overline{A}}{\overline{a}} & \overline{e} \end{pmatrix},$$

(4)

where $\overline{A}^T = A^T + B^T j \in \text{Gl}(\mathbb{H}^m) ; \overline{a}^T = a^T + b^T j, \overline{c} = c^T + d^T j \in \mathbb{H}^m$; and $\overline{e}^T = e + f j \in \mathbb{H}$. Note all vectors in $\mathbb{H}^m$ are row vectors and the matrices in $\text{Gl}(\mathbb{H}^m)$ operate on the right on $\mathbb{H}^m$. Moreover, multiplication in $\text{Gl}(\mathbb{H}^m)$ is $\mathcal{A} \mathcal{B}$, where $\mathcal{A} \in \text{Gl}(\mathbb{H}^{m+2})$ is applied first followed by $\mathcal{B} \in \text{Gl}(\mathbb{H}^{m+2})$. For a special vector $(v^\vee)^0 \in V_{\sigma^\vee}$ let $G(V^\vee, \sigma^\vee)(v^\vee)^0$ be the isotropy group consisting of the elements of $G(V^\vee, \sigma^\vee)$ which leave the vector $(v^\vee)^0$ fixed. Using the basis $e^\vee$ of $V_{\sigma^\vee}$, choose $(v^\vee)^0 = \epsilon_{m+1} + i \phi_{m+1}$. Then

$$\sigma^\vee(-\epsilon_{m+1} + i \phi_{m+1}) = -\phi_{m+1} + i \epsilon_{m+1} = i(\epsilon_{m+1} + i \phi_{m+1}).$$

So $(v^\vee)^0$ is a special vector. Moreover,

$$\mathcal{A}((v^\vee)^0) = (a + ib, -\overline{b} + i\overline{a} \in G(V^\vee, \sigma^\vee)(v^\vee)^0, \epsilon + if, -\overline{f} + i\overline{e})^T = (0, 0 \mid 1, i)^T = (v^\vee)^0,$$

if and only if $a = 0, b = 0$ and $e = 1, f = 0$. Hence

$$G(V^\vee, \sigma^\vee)(v^\vee)^0 = \begin{pmatrix} \frac{A}{\overline{A}} & \frac{B}{\overline{B}} \\ \overline{B} & \overline{A} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in G(V^\vee, \sigma^\vee)(v^\vee)^0 \in \text{Gl}(\mathbb{C}^2) \& c, d \in \mathbb{C}^m.$$

The isomorphism $\tilde{\Sigma}$ restricts to the isomorphism

$$\Sigma : G(V^\vee, \sigma^\vee)(v^\vee)^0 \to \text{Aff}(\langle V_{\sigma^\vee}\rangle^\mathbb{H}) :$$

$$\mathcal{A} = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \begin{pmatrix} e^T & d^T \\ -\overline{e}^T & -\overline{d}^T \end{pmatrix} \rightarrow \rho^\vee A (\rho^\vee)^{-1} = \mathcal{A} = \begin{pmatrix} \frac{\overline{A}}{\overline{a}} & \overline{c} \\ \frac{\overline{A}}{\overline{a}} & \overline{e} \end{pmatrix},$$

CASE 4. $\text{Aff}G(V, \sigma^\vee, \tau)$. Let $\tilde{V} = \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$, where $\dim \tilde{V} = 2m$. Define

$$\tilde{\sigma} : \tilde{V} \to \tilde{V} :$$

$$(z_0, w_0; \overline{v}, z_{m+1}, w_{m+1})^T \mapsto (\overline{w}_0, -\overline{z}_0; \tilde{\sigma}(-\overline{v}), \overline{w}_{m+1}, \overline{z}_{m+1})^T,$$

(5)

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where
\[ \tilde{\sigma}_- : \tilde{V} \rightarrow \tilde{V} : \tilde{v} = (\tilde{u}, \tilde{w})^T \mapsto (\tilde{w}, -\tilde{u})^T, \]
see appendix 1. Then \( \tilde{\sigma}_- \) is an anti-linear mapping of \( \tilde{V} \) into itself with \( (\tilde{\sigma}_-)^2 = -\text{id}_\tilde{V} \). Let \( \tilde{\sigma}_- : \hat{V} \rightarrow \hat{V} \) be the complex linear mapping associated to \( \tilde{\sigma}_- \). With respect to the basis \( \tilde{e} = \{e_0, f_0; e_1, \ldots, e_m, f_1, \ldots, f_m; e_{m+1}, f_{m+1}\} \) of \( \tilde{V} \) the matrix of \( \tilde{\sigma}_- \) is
\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & I_m & 0 \\
0 & 0 & 0 & I_m \\
\end{pmatrix},
\]
using equation (5) and (6). Let \( \mathcal{A} \in \text{Gl}(\hat{V}) \). \( \mathcal{A} \) commutes with \( \tilde{\sigma}_- \) if and only if its \( 2(m+2) \times 2(m+2) \) complex matrix with respect to the basis \( \tilde{e} \) is of the form
\[
\begin{pmatrix}
a & b^T \\
d & A \\
f & g^T \\
h & e \\
\end{pmatrix},
\]
where \( \tilde{A} = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \in \text{gl}(\hat{V}) \) with \( A, B \in \text{gl}(\mathbb{C}^m) \); \( b, d, e, g \in \mathcal{M}_{2m,2} = \{ (\begin{smallmatrix} 1 \end{smallmatrix})^{1 \times 2} \in \mathcal{M}_{2m,2} \mid x_1, x_2 \in \mathbb{C}^m \} ; a, c, f, h \in \mathcal{M}_{2,2} = \{ (\begin{smallmatrix} 1 \\mathbf{1}_n \end{smallmatrix})^{1 \times 2} \in \text{gl}(\mathbb{C}^2) \mid y_1, y_2 \in \mathbb{C} \} \). Here \( \mathcal{M}_{2m,2} \) is the set of \( 2m \times 2 \) complex matrices. Hence \( \mathcal{A} \in \text{Gl}(\hat{V}, \tilde{\sigma}_-) \) if and only if \( \mathcal{A} \) is of the form (7), where \( A = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \in \text{gl}(\mathbb{C}^{2m}) \) with \( A, B \in \text{gl}(\mathbb{C}^m) \); \( b, d, e, g \in \mathcal{M}_{2m,2} = \{ (\begin{smallmatrix} 1 \\mathbf{1}_n \end{smallmatrix})^{1 \times 2} \in \mathcal{M}_{2m,2}(\mathbb{C}) \mid x_1, x_2 \in \mathbb{M}_{2m}(\mathbb{C}) \} ; a, c, f, h \in \mathcal{M}_{2,2} = \{ (\begin{smallmatrix} 1 \\mathbf{1}_n \end{smallmatrix})^{1 \times 2} \in \text{gl}(\mathbb{C}^2) \mid y_1, y_2 \in \mathbb{C} \} \).

Turn the \( 2(m+2) \) dimensional complex vector space \( \hat{V} \) into an \( m+2 \) dimensional quaternionic vector space \( \hat{V}_{\tilde{\sigma}_-} \) by defining scalar multiplication \( \cdot \) as \( (\alpha + \beta j) \cdot \tilde{v} = \alpha \tilde{v} + \beta \tilde{\sigma}_-(\tilde{v}) \), where \( \alpha, \beta \in \mathbb{C} \) and \( \tilde{v} \in \hat{V} \). The complex linear isomorphism
\[ \hat{V} \rightarrow \hat{V}_{\tilde{\sigma}_-}^\mathbb{H} = \mathbb{H}^{m+2} : \tilde{v} = ((u_0, w_0), (\tilde{u}, \tilde{w}), (u_{m+1}, w_{m+1}))^T \mapsto ((\tilde{w}, -\tilde{u}), (\tilde{u}, \tilde{w}), u_{m+1} + w_{m+1}, u_{m+1} + w_{m+1})^T \]
induces the quaternionic isomorphism
\[ \hat{\rho} : \hat{V}_{\tilde{\sigma}_-} \rightarrow \hat{V}_{\tilde{\sigma}_-}^\mathbb{H} = \mathbb{H}^{m+2} : \tilde{v} = ((u_0, w_0), (\tilde{u}, \tilde{w}), (u_{m+1}, v_{m+1}))^T \mapsto ((u_0 + w_0 j, -w_0 - u_0 j), (\tilde{u}^T + \tilde{w}^T j, \tilde{w}^T - \tilde{u}^T j), (u_{m+1} + w_{m+1} j, u_{m+1} + w_{m+1} j))^T. \]
Suppose that the bilinear form $\tau$ on $\hat{V}$ is symmetric. With $\hat{u} = ((u_0, w_0), (u_{m+1}, w_{m+1}))^T$ and $\hat{u}' = ((u'_0, w'_0), (u'_{m+1}, w'_{m+1}))^T \in \hat{V}$ define the bilinear form

$$\tau : \hat{V} \times \hat{V} : (\hat{u}, \hat{u}') \mapsto u_0 u'_m + w_0 w'_m + \tau(\hat{u}, \hat{u}') + u_0 u_{m+1} + w_0 w_{m+1}.$$  

With respect to the basis $e' = \{e'_0, f'_0; e'_1, \ldots, e'_m, f'_1, \ldots, f'_m; e'_{m+1}, f'_{m+1}\}$ of $\hat{V}$ the matrix $T_+$ of $\tau$ is \( \left( \begin{array}{cc} 0 & I_m \\ I_m & 0 \end{array} \right) \), where $T_+ = (t^m_0 \begin{array}{c} 0 \\ I_m \end{array})$ is the matrix of the bilinear form $\tau$ on $\hat{V}$ with respect to the basis $e'$. Note that for every $\hat{u}, \hat{u}' \in \hat{V}$ one has $\tau(\hat{\sigma}_-(\hat{u}), \hat{\sigma}_-(\hat{u}')) = \tau(\hat{u}, \hat{u}').$ On $\hat{V}_\sigma$ define a quaternion valued form $\tau_+ : \hat{V}_\sigma \times \hat{V}_\sigma \rightarrow \mathbb{H}$ by

$$\tau_+ (\hat{u} + \hat{v}j, \hat{w} + \hat{z}j) = (\hat{u} + \hat{v}j) \left( \begin{array}{cc} 0 & I_m \\ I_m & 0 \end{array} \right) (\hat{w}^T + \hat{z}^T j)^\sigma.$$

The matrix of $\tau_+$ with respect to the basis

$$\eta = \left\{ \frac{\sqrt{2}}{2}(e_0 + f_0 j); \frac{\sqrt{2}}{2}(e_1 + f_1 j), \ldots, \frac{\sqrt{2}}{2}(e_m + f_m j); \frac{\sqrt{2}}{2}(e_{m+1} + f_{m+1} j) \right\}$$

of $\hat{V}_\sigma$ is $\left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$. Observe that the isomorphism $\hat{\rho}$ \( \left[ \begin{array}{c} \rho \end{array} \right] \) pulls back the hamiltonian symmetric form $\tau_+$ on $\hat{V}_\sigma$ to $\mathbb{H}^{m+2}$ to the symmetric form $\tau_+$ on $\hat{V}_\sigma$.

The matrix $A \in G(\hat{V}, \hat{\sigma})$ \( \left[ \begin{array}{c} \hat{A} \end{array} \right] \) fixes the vector $(\hat{v})^0 = e_{m+1} + if_{m+1} \in \hat{V}_\sigma$, which special since $\hat{\sigma}_-((\hat{v})^0) = i(\hat{v})^0$ and $\tau((\hat{v})^0, (\hat{v})^0) = 0$, if and only if $A = \left( \begin{array}{cc} a & b^T \\ d & g^T - i_2 \end{array} \right)$. Here $a, f; b, d, g$ and $\hat{A}$ satisfy the conditions

\[ a + \beta_j = \alpha - \beta_j. \]

\[ For \alpha + \beta_j \in \mathbb{H} \text{ one has } (\alpha + \beta_j)^g = \alpha - \beta_j. \]
following equation (7). A calculation shows that $\mathcal{A} \in G(\hat{V}, \hat{\sigma}_-)_G$ preserves the symmetric bilinear form $\tau_+$ on $\hat{V}$ if and only if

$$
\mathcal{A} = \begin{pmatrix}
I_2 & 0 & 0 \\
\frac{1}{2} \tilde{d}^T \tilde{A} \tilde{d} & -\tilde{d}^T \tilde{A} & 0 \\
-\frac{1}{2} \tilde{d}^T \tilde{A} & \tilde{d} & I_2
\end{pmatrix},
$$

where $\tilde{d} = \begin{pmatrix} \tilde{d}_1 & \tilde{d}_2 \end{pmatrix}$ with $\tilde{d}_1, \tilde{d}_2 \in \mathbb{C}^m$ and $\tilde{A} \in G(\hat{V}, \hat{\sigma}_-, \hat{\tau}_+)$, that is, $\tilde{A} \in \text{GL}(\hat{V})$ such that $\tilde{A} \hat{\sigma}_- = \hat{\sigma}_- \tilde{A}$ and $\tilde{T}_+ = \tilde{A}^T \tilde{T}_+ \tilde{A}$. The isomorphism $\tilde{\rho}$ gives rise to the group isomorphism

$$
\tilde{\Sigma} : G(\hat{V}, \hat{\sigma}_-, \tau_+) \to G(\hat{V}_{\tilde{\rho}}^\mathbb{H}, \tau_+)
$$

$$
\tilde{\Sigma} = \begin{pmatrix}
a & b^T & c \\
d & A & e \\
f & g^T & h^T
\end{pmatrix} \mapsto \tilde{\rho} \mathcal{A} \tilde{\rho}^{-1} = \mathcal{A} = \begin{pmatrix}
a^T & d^T & f^T \\
b & A & g \\
c^T & e^T & h^T
\end{pmatrix},
$$

(9)

where $a^T = a_1 + a_2 j$, $c^T = c_1 + c_2 j$, $f^T = f_1 + f_2 j$, and $h = h_1 + h_2 j \in \mathbb{H}$; $b = b_1^T + b_2^T j$, $d^T = d_1^T + d_2^T j$, $e^T = e_1^T + e_2^T j$, and $g = g_1^T + g_2^T \in \mathbb{H}^m$; $A^T = A^T + B^T j$ with $A, B \in \text{gl}(\mathbb{C}^m)$. Moreover, $(\tilde{\Sigma})^* \tau_+ = \tau_+$. The map $\tilde{\Sigma}$ restricts to the group isomorphism

$$
\Sigma : G(\hat{V}, \hat{\sigma}_-, \tau_+)_G \to \text{Aff}(\mathbb{H}^m, \tau_+)
$$

$$
\begin{pmatrix}
I_2 & 0 & 0 \\
\frac{1}{2} \tilde{d}^T \tilde{A} \tilde{d} & -\tilde{d}^T \tilde{A} & 0 \\
-\frac{1}{2} \tilde{d}^T \tilde{A} & \tilde{d} & I_2
\end{pmatrix} \mapsto \begin{pmatrix}
\tilde{A}^T & \tilde{d}^T \\
0 & 1
\end{pmatrix},
$$

where $\tilde{A} = A^T + B^T j \in G(\mathbb{H}^m, \tau_+)^m$ and $\tilde{d} = \tilde{d}_1^T + \tilde{d}_2^T \in \mathbb{H}$.

Suppose that $\tilde{\tau}$ is a nondegenerate alternating bilinear form on $\hat{V}$. With $\tilde{u} = (\tilde{z}, \tilde{w})$, $\tilde{u}' = (\tilde{z}', \tilde{w}') \in \hat{V}$ let

$$
\tau : \hat{V} \times \hat{V} \to \mathbb{C} : ((\tilde{z}, \tilde{w}), (\tilde{z}', \tilde{w}')) \mapsto \tilde{z}_0 w'_{m+1} - \tilde{w}_0 z'_{m+1} + \tilde{\tau}(\tilde{u}, \tilde{u}') + \tilde{w}_0 z_{m+1} - \tilde{z}_0 w_{m+1}.
$$

Then $\tau$ is a nondegenerate alternating bilinear form on $\hat{V}$. With respect to the basis $\{e_0, f_0; e_1, \ldots, e_m, f_1, \ldots, f_m; e_{m+1}; f_{m+1}\}$ of $\hat{V}$ the matrix of $\tau$ is $T_- = \begin{pmatrix} 0 & J_2 \\
\frac{1}{2} & 0 \end{pmatrix}$, where $J_2 = \begin{pmatrix} 0 & 1 \\
0 & 0 \end{pmatrix}$ is the matrix of the nondegenerate alternating form $\tilde{\tau}$ on $\hat{V}$ with respect to the basis $\{e_1, \ldots, e_m; f_1, \ldots, f_m\}$ and $J_2 = \begin{pmatrix} 0 & 1 \\
0 & 0 \end{pmatrix}$. Note that $\tau(\tilde{\sigma}_-(\tilde{u}), \tilde{\sigma}_-(\tilde{v})) = \tau(\tilde{u}, \tilde{v})$. Let $\tau_- = \tau|_{\hat{V}_{\tilde{\sigma}_-}}$. Then $\tau_-$ is a nondegenerate alternating bilinear
form on $\tilde{V}_\sigma$. On $\tilde{V}_\sigma^\mathbb{H} = \mathbb{H}^{m+2}$ for each $0 \leq p/2 \leq m$ define a hamiltonian alternating quaternion valued form $\tau_- : \mathbb{H}^{m+2} \times \mathbb{H}^{m+2} \rightarrow \mathbb{H}$ by

$$
\tau_- (\hat{u} + \hat{v} j, \hat{w} + \hat{z} j) = (\hat{u} + \hat{v} j) \begin{pmatrix} 0 & I_{m-p/2, p/2} \\ \frac{-I_{m-p/2, p/2}}{j} & 0 \end{pmatrix} ((\hat{w} \hat{w})^T + (\hat{z} \hat{z})^T) q,
$$

where $I_{m-p/2, p/2} = \begin{pmatrix} -I_{m-p/2} & 0 \\ 0 & I_{p} \end{pmatrix}$. The matrix of $\tau_-$ with respect to the basis $\eta = \{ \frac{1}{\sqrt{2}} (e_0 + f_0) j, \frac{1}{\sqrt{2}} (e_1 + f_1) j, \ldots, \frac{1}{\sqrt{2}} (e_m + f_m) j, \frac{1}{\sqrt{2}} (e_{m+1} + f_{m+1}) j \}$ of $\tilde{V}_\sigma^\mathbb{H}$ is $\begin{pmatrix} 0 & \tilde{b}^{\rightarrow} \\ \tilde{a}^{\rightarrow} & 0 \end{pmatrix}$. Observe that the map $\tilde{\rho}$ pulls back the hamiltonian alternating form $\tau_-$ on $\tilde{V}_\sigma^\mathbb{H}$ to the alternating form $\tau_-$ on $\tilde{V}$. Let $(\tilde{v})^o = e_m + i f_{m+1} \in \tilde{V}_\sigma^-$. The vector $(\tilde{v})^o$ is special since $\vec{\sigma}_-(\tilde{v})^o = i(\tilde{v})^o$ and $\tau((\tilde{v})^o, (\tilde{v})^o) = 0$, because $\tau$ is alternating. An element $\tilde{A}$ of the isotropy group $G(\vec{V}, \vec{\sigma}_-, \vec{\tau})$ preserves the alternating form $\tau_-$ on $\tilde{V}_\sigma^-$ if and only if $\tilde{A} = \begin{pmatrix} \tilde{I}_2 & 0 \\ \tilde{d} & \tilde{A} \end{pmatrix}$, where $\tilde{d} = (\tilde{d}_1, \tilde{d}_2) \in \mathbb{C}^{m_1}$, and $\tilde{A} = \tilde{G}(\vec{V}, \vec{\sigma}_-, \vec{\tau})$, that is, $\tilde{A} \in \mathrm{GL}(\vec{V})$, $\vec{\sigma}_- \tilde{A} = \tilde{A} \vec{\sigma}_-$, and $\tilde{T}_- = \tilde{A}^T \tilde{T}_- \tilde{A}$. The isomorphism $\tilde{\rho}$ gives rise to the group isomorphism

$$
\tilde{\Sigma} : G(\vec{V}, \vec{\sigma}_-, \vec{\tau}_-) \rightarrow G(\tilde{V}_\sigma^\mathbb{H}, \tau_-) : \quad \mathcal{A} = \begin{pmatrix} a & b^T \\ d & \tilde{A} \end{pmatrix} \rightarrow \tilde{\rho} \mathcal{A} \tilde{\rho}^{-1} = \mathcal{A} = \begin{pmatrix} a^T & d^T \\ b & \tilde{A}^T \end{pmatrix} \quad (10)
$$

where $a = a_1 + a_2 j$, $b = b_1 + b_2 j$, $c = c_1 + c_2 j$, $f = f_1 + f_2 j$, and $h = h_1 + h_2 j \in \mathbb{H}$; $b^T = b_1^T + b_2^T j$, $d^T = d_1^T + d_2^T j$, $e^T = e_1^T + e_2^T j$, and $g^T = g_1^T + g_2^T \in \mathbb{H}^m$; $\tilde{A} = \tilde{A}^T + \tilde{B}^T j$ with $A, B \in \mathrm{gl}(\mathbb{H}^m)$. Moreover, $(\tilde{\rho})^{*} \vec{\tau}_- = \tau_-$. The map $\tilde{\Sigma}$ restricts to the group isomorphism

$$
\Sigma : G(\vec{V}, \vec{\sigma}_-, \vec{\tau}_-) \rightarrow \mathrm{AffG}(\mathbb{H}^m, \tau_-) : \quad \mathcal{A} = \begin{pmatrix} I_2 \\ -\frac{1}{2} \tilde{J}_2 \tilde{d}^T \tilde{T}_- \tilde{A} \end{pmatrix} \rightarrow \mathcal{A} = \begin{pmatrix} A^T & d^T \\ \tilde{A} & 0 \end{pmatrix},
$$

where $\tilde{A} = \tilde{A}^T + B^T j \in G(\mathbb{H}^m, \tau_-)$ and $\tilde{d} = \tilde{d}_1^T + \tilde{d}_2^T \in \mathbb{H}^m$.

This completes case 4 and the proof of theorem 4.1. \qed

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Appendix 1. Classification of anti-linear mappings

Two anti-linear mappings $\tilde{\sigma}$ and $\tilde{\sigma}'$ of the complex vector space $\tilde{V}$ into itself are *equivalent* if and only if $\tilde{\sigma}' = \alpha k^{-1} \tilde{\sigma} k$ for some $k \in G$ and some $\alpha \in \mathbb{C} \setminus \{0\}$. Note that $\tilde{\sigma}$ and $-\tilde{\sigma}$ are equivalent. We may suppose that $\tilde{\sigma}^2 = \pm \text{id}_{\tilde{V}}$. Equivalence classes of anti-linear mappings for $G = G(\tilde{V}, \tilde{\sigma}, \tilde{\tau})$ are determined in [1, app 1, p.357]. We list explicit representatives below.

**Case 1.** $\tilde{\sigma} = \tilde{\sigma}_+$. Suppose that $\tilde{V} = \mathbb{C}^n$, where $z = (z_1, \ldots, z_n)^T \in \mathbb{C}^n$. Let $\tilde{\sigma} : \mathbb{C}^n \to \mathbb{C}^n : z \mapsto \tilde{z} = (\bar{z}_1, \ldots, \bar{z}_n)^T$. Set $\tilde{\sigma}_+ = \tilde{\sigma}$. Since

$$
\tilde{\sigma}_+(\alpha z + \beta z') = \overline{\alpha z + \beta z'} = \overline{\alpha} \overline{z} + \overline{\beta} \overline{z}' = \overline{\alpha} \tilde{\sigma}_+(z) + \overline{\beta} \tilde{\sigma}_+(z'),
$$

the mapping $\tilde{\sigma}_+$ is anti-linear. Also $\tilde{\sigma}_+^2 = \text{id}_{\mathbb{C}^n}$, since $\tilde{\sigma}_+^2(z) = \tilde{\sigma}_+(\overline{z}) = z$. This handles the case when $G = \text{Gl}(\tilde{V}, \tilde{\sigma}_+)$. Suppose that on $\mathbb{C}^n$ we have a nondegenerate complex valued bilinear form $\tilde{\tau} : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$.

If $\tilde{\tau}$ is *symmetric*, we may assume that its matrix with respect to the standard basis of $\mathbb{C}^n$ is $I_n$, that is, $\tilde{\tau}(z, w) = w^T I_n z$. Let

$$
\tilde{V}_{\tilde{\sigma}_+} = \{ z \in \mathbb{C}^n | \tilde{\sigma}_+(z) = z \} = \{ x \in \mathbb{R}^n | z = x + iy \} = \mathbb{R}^n
$$

and set $\tilde{\tau}_+ = \tilde{\tau}|_{\tilde{V}_{\tilde{\sigma}_+}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Then $\tilde{\tau}_+$ is a nondegenerate real valued symmetric bilinear form on $\mathbb{R}^n$. Replacing $\tilde{\sigma}_+$ by $-\tilde{\sigma}_+$, if necessary,
we may assume that \( \tau_+ \) has signature \((n - p, p)\) for \(0 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor\). We may suppose that the matrix of \( \tau_+ \) with respect to the standard basis of \( \mathbb{R}^n \) is 
\[
I_{n-p,p} = \begin{pmatrix} -I_{n-p} & 0 \\ 0 & I_p \end{pmatrix}.
\]
Then \( \tau_+(x, y) = y^T I_{n-p,p}x \). Let
\[
\tilde{\sigma}_+^{(p)} : \mathbb{C}^n \to \mathbb{C}^n : z \mapsto I_{n-p,p}z.
\]
Then
\[
\tilde{\tau}(\tilde{\sigma}_+^{(p)}(z), \tilde{\sigma}_+^{(p)}(w)) = (I_{n-p,p}w)^T I_n(I_{n-p,p}z) = \tilde{w}^T(I_{n-p,p})^T I_n(I_{n-p,p})z = \tilde{w}^T I_n\tilde{z} = \tilde{\tau}(z, w).
\]
This handles the case when \( G = O(\tilde{V}, \tilde{\sigma}_+^+, \tilde{\tau}) \).

If \( \tilde{\tau} \) is alternating, then \( n \) is even, say \( 2m \), since \( \tilde{\tau} \) is nondegenerate. We may assume that the matrix of the nondegenerate real valued alternating form \( \tau_+ : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R} \) with respect to the standard basis of \( \mathbb{R}^n = \mathbb{R}^{2m} \times \mathbb{R}^{2m} \) is
\[
J_{2m} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},
\]
that is, \( \tau_+ \left( \begin{pmatrix} z \\ z' \end{pmatrix}, \begin{pmatrix} w \\ w' \end{pmatrix} \right) = (x'y')^T J_{2m} \begin{pmatrix} x \\ y \end{pmatrix} \). Now
\[
\tilde{\tau}(\tilde{\sigma}_+(z, w), \tilde{\sigma}_+(z', w')) = ((\tilde{z}'^T, (\tilde{w}'^T)J_{2m}) (\tilde{z}), (\tilde{w})).
\]
This handles the case when \( G = \text{Sp}(\tilde{V}, \tilde{\sigma}_+, \tilde{\tau}) \).

Suppose that \( \tilde{\sigma}^2 = -\text{id}_{\tilde{V}} \). Let \( \mathbb{H} = \{ \alpha + \beta j \mid \alpha, \beta \in \mathbb{C}; j^2 = -1; \alpha j = -j\alpha \} \) be the quaternions with anti-involution \( (\alpha + \beta j)^q = \alpha - j\beta \). Note that if \( x \) and \( y \) \in \( \mathbb{H} \), then \( (xy)^q = y^qx^q \). Let \( \tilde{V} = \mathbb{C}^n \) with \( n = 2m \). Consider the anti-linear mapping
\[
\tilde{\sigma} : \mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^n : (z, w)^T \mapsto (-\overline{w}, \overline{z})^T.
\]
Then \( \tilde{\sigma}^2 = -\text{id}_{\mathbb{C}^n} \). Set \( \tilde{\sigma}_- = \tilde{\sigma} \). Turn \( \mathbb{C}^n \) into a vector space \( \tilde{V}_{\tilde{\sigma}_-} \) over the quaternions by defining scalar multiplication \( \cdot \) as \( (\alpha + \beta j) \cdot (z, w) = \alpha (z, w) + \beta \tilde{\sigma}_-(z, w) \). The map \( \rho : \tilde{V}_{\tilde{\sigma}_-} \to \mathbb{H}^m : (z, w)^T \mapsto z^T + w^T j \) is an isomorphism of quaternionic vector spaces, since
\[
\rho((\alpha + \beta j) \cdot (z, w)^T) = \rho(\alpha(z, w)^T + \beta \tilde{\sigma}_-(z, w)^T)) = (\alpha z^T - \beta \overline{w} T + (\alpha w^T + \beta \overline{\sigma}_T)^T) j = (\alpha + \beta j)(z^T + w^T j).
\]
This takes care of the case when \( G = \text{Gl}(\tilde{V}, \tilde{\sigma}_-) \).

Suppose that \( \tilde{\tau} \) is a nondegenerate complex valued bilinear form on \( \mathbb{C}^n \). On \( \tilde{V}_{\tilde{\sigma}_-} \) define the quaternion valued bilinear form
\[
\tilde{\tau}_- : \tilde{V}_{\tilde{\sigma}_-} \times \tilde{V}_{\tilde{\sigma}_-} \to \mathbb{H} : (\tilde{u}, \tilde{v}) \mapsto \tilde{\tau}(\tilde{u}, \tilde{v}) + \tilde{\tau}(\tilde{u}, \tilde{\sigma}_-(\tilde{v})) j
\]
Using the fact that $\tau(\bar{\sigma}_-(\bar{u}), \bar{\sigma}_-(\bar{v})) = \overline{\tau(u, v)}$, which implies $\tau(\bar{\sigma}_-(\bar{u}), \bar{v}) = -\tau(\bar{u}, \bar{\sigma}_-(\bar{v}))$. A straightforward calculation shows that for every $\lambda \in \mathbb{H}$

$$\tau_-(\lambda \cdot \bar{u}, \bar{v}) = \lambda \overline{\tau(\bar{u}, \bar{v})}$$ (11a)

and for every $\mu \in \mathbb{H}$

$$\tau_-(\bar{u}, \mu \cdot \bar{v}) = \overline{\tau(\bar{u}, \bar{v})} \mu^q.$$ (11b)

Suppose that $\tau$ is symmetric. We may assume that the matrix of $\tau$ with respect to the standard basis of $\mathbb{C}^m \times \mathbb{C}^m = \mathbb{C}^n$ is $(I_m \ 0 \ 0 \ I_m)$, that is, $\tau(\bar{u}, \bar{v}) = \bar{v}^T (I_m \ 0 \ I_m) \bar{u}$. Now

$$\tau(\bar{\sigma}_-(\bar{u}), \bar{\sigma}_-(\bar{v})) = \left( \left( \begin{array}{cc} 0 & -I_m \\ I_m & 0 \\ \end{array} \right) \bar{v} \right)^T \left( \begin{array}{cc} I_m & 0 \\ 0 & I_m \\ \end{array} \right) \left( \begin{array}{cc} I_m & 0 \\ 0 & I_m \\ \end{array} \right) \bar{u}$$

$$= \bar{v}^T (I_m \ 0 \ I_m) \bar{u} = \overline{\tau(\bar{u}, \bar{v})},$$

which implies

$$\tau(\bar{u}, \bar{\sigma}_-(\bar{v})) = -\tau(\bar{\sigma}_-(\bar{u}), \bar{v}) = -\overline{\tau(\bar{u}, \bar{v})}.$$ 

So

$$\tau_-(\bar{u}, \bar{v})^q = (\tau(\bar{u}, \bar{v}) + \tau(\bar{u}, \bar{\sigma}_-(\bar{v}))j)^q$$

$$= \tau(\bar{u}, \bar{v}) - \tau(\bar{u}, \bar{\sigma}_-(\bar{v}))j = \tau(\bar{u}, \bar{v}) + \tau(\bar{\sigma}_-(\bar{u}), \bar{v})j$$

$$= \tau(\bar{v}, \bar{u}) + \tau(\bar{\sigma}_-(\bar{u}), \bar{v})j = \tau_-(\bar{v}, \bar{u}),$$

that is, $\tau_-$ is hamiltonian symmetric. Let $\epsilon = \left\{ \frac{1}{\sqrt{2}}(e_\ell, f_\ell) \right\}_{\ell=1}^m$ be a $\tau$ orthogonal basis of $\bar{V}_{\sigma_-(\bar{\bar{\epsilon}})}$. Then

$$\tau_-(\frac{1}{\sqrt{2}}(e_\ell, f_\ell), \frac{1}{\sqrt{2}}(e_k, f_k)) = \frac{1}{2} \tau_-( (e_\ell, f_\ell), (e_k, f_k)) + \tau_-( (e_\ell, f_\ell), \bar{\sigma}_-(e_k, f_k))j$$

$$= \frac{1}{2} \tau_-( (e_\ell, f_\ell), (e_k, f_k)) + \tau_-( (e_\ell, f_\ell), (-f_k, e_k))j$$

$$= \delta_{\ell k}.$$ 

In other words, with respect to the basis $\epsilon$ of $\mathbb{H}^m$, one has

$$\tau_-(u + vj, z + wj) = (u^T + v^Tj)I_n(z + wj)^q.$$ 

This takes care of the case when $G = O(\bar{V}, \bar{\sigma}_-, \tau)$. 

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Suppose that $\tilde{\tau}$ is alternating. We may assume that the matrix of $\tilde{\tau}$ with respect to the standard basis of $\mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^m$ is $J_n = \left( \begin{array}{cc} 0 & I_m \\ -I_m & 0 \end{array} \right)$, that is, $\tilde{\tau}(z, w)^T, (z', w')^T = (z')^T J_n (w)$. For $0 \leq p \leq m$ let

$$\tilde{\sigma}^{(p)} : \mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^m \rightarrow \mathbb{C}^n : (z, w)^T \mapsto (-I_{m-p,p} \bar{w}, I_{m-p-p} \bar{z})^T.$$

Clearly $\tilde{\sigma}^{(p)}$ is anti-linear and $(\tilde{\sigma}^{(p)})^2 = -\text{id}_{\mathbb{C}^n}$. Replacing $\bar{\sigma}$ by $-\bar{\sigma}$, we may assume that $0 \leq p \leq m$. Let $\tilde{u} = (z, w)^T$ and $\tilde{v} = (z', w')^T$. Then

$$\tilde{\tau} (\tilde{\sigma}^{(p)}(\tilde{u}), \tilde{\sigma}^{(p)}(\tilde{v})) =$$

$$= \left( \begin{array}{cc} 0 & -I_{m-p,p} \\ I_{m-p,p} & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ I_m \end{array} \right) \left( \begin{array}{c} 0 \\ -I_{m-p,p} \end{array} \right) \tilde{u}$$

$$= \tilde{v}^T \left( \begin{array}{cc} 0 & I_m \\ -I_m & 0 \end{array} \right) \tilde{u} = \tilde{\tau}(\tilde{u}, \tilde{v}),$$

which implies

$$\tilde{\tau}^{(p)}(\tilde{u}, \tilde{\sigma}^{(p)}(\tilde{v})) = -\tilde{\tau}^{(p)}(\tilde{\sigma}^{(p)}(\tilde{u}), \tilde{\sigma}^{(p)}(\tilde{v})) = -\tilde{\tau}^{(p)}(\tilde{\sigma}^{(p)}(\tilde{u}), \tilde{v}).$$

So

$$(\tilde{\tau}^{(p)}(\tilde{u}, \tilde{v}))^q = (\tilde{\tau}^{(p)}(\tilde{u}, \tilde{v}) + \tilde{\tau}^{(p)}(\tilde{u}, \tilde{\sigma}^{(p)}(\tilde{v}))) = \tilde{\tau}^{(p)}(\tilde{u}, \tilde{v}) = \tilde{\tau}^{(p)}(\tilde{\sigma}^{(p)}(\tilde{u}), \tilde{v})$$

that is, $\tilde{\tau}^{(p)}$ is hamiltonian alternating. Let $\epsilon' = \left( \frac{1}{\sqrt{2}}(e_\ell, f_\ell) \right)_{\ell=1}^m$ be a basis for $\tilde{V}_{\sigma^{(p)}}$. Then

$$\tilde{\tau}^{(p)}(\frac{1}{\sqrt{2}}(e_\ell, f_\ell), \frac{1}{\sqrt{2}}(e_k, f_k)) =$$

$$= -\frac{1}{2} \tilde{\tau}^{(p)}((e_\ell, f_\ell), (e_k, f_k)) - \frac{1}{2} \tilde{\tau}^{(p)}((e_\ell, f_\ell), \tilde{\sigma}^{(p)}(e_k, f_k))$$

$$= -\frac{1}{2} \tilde{\tau}^{(p)}((e_\ell, f_\ell), (e_k, f_k)) - \frac{1}{2} \tilde{\tau}^{(p)}((e_\ell, f_\ell), \delta^{m-p-p}_k (f_k, e_k))$$

where $\delta^{m-p-p}_k = \{ -1, \text{ if } 1 \leq k \leq m-p \\ 1, \text{ if } m-p+1 \leq k \leq n \}$

$$= \delta^{m-p-p}_k \delta_{k,l} \epsilon' \cdot$$

In other words, with respect to the basis $\epsilon'$ of $\tilde{V}_{\sigma^{(p)}}$ one has

$$\tilde{\tau}^{(p)}(u + vj, w + z) = (u^T + v^T j)(j I_{m-p,p})(z + wj)^q.$$
This takes care of the case \( \text{Sp}(\tilde{V}, \tilde{\sigma}_-, \tilde{\tau}(p)) \).

**Case 2.**

Let \( \tilde{\sigma} : \tilde{V} \to \tilde{V}^* \) be an anti-linear mapping. Two such mappings \( \tilde{\sigma} \) and \( \tilde{\sigma}' \) are equivalent if and only if \( \tilde{\sigma}' = \alpha k^* \tilde{\sigma} k \) for some \( \alpha \in \mathbb{C} \setminus \{0\} \) and some \( k \in \text{Gl}(\tilde{V}) \). Here \( k^* : \tilde{V}^* \to \tilde{V}^* \) is defined by \( k^*(\tilde{w}^*)(\tilde{v}) = \tilde{v}^*(k(\tilde{w})) \) for every \( \tilde{v}^* \in \tilde{V}^* \) and every \( \tilde{w} \in \tilde{V} \). Let \( \tilde{V} = \mathbb{C}^n \). Set \( \tilde{\sigma}(p) : \mathbb{C}^n \to (\mathbb{C}^n)^* : z \mapsto (I_{n-p,p} \tilde{\sigma})^T \). Then \( \tilde{\sigma}(p) \) is an anti-linear mapping, since for every \( z \in \mathbb{C}^n \) and every \( \alpha \in \mathbb{C} \) one has \( \tilde{\sigma}(p)(\alpha z) = (\overline{\alpha} z)^T I_{n-p,p} = \overline{\alpha}(\overline{z}^T I_{n-p,p}) = \overline{\alpha} \tilde{\sigma}(p)(z) \). Let \( \tilde{\tau}_*(p) : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} : (z, w) \mapsto (\tilde{\tau}(p)(z))(w) = \overline{z}^T I_{n-p,p} w \). Then \( \tilde{\tau}_*(p) \) is a hermitian form, since

\[
\tilde{\tau}_*(p)(z, w) = \overline{\tilde{\tau}(p)(w)}(z) = \overline{w^T I_{n-p,p} z} = w^T I_{n-p,p} \overline{z} = \overline{z}^T I_{n-p,p} w = \tilde{\tau}_*(p)(w)(z) = \tilde{\tau}_*(p)(w, z).
\]

This takes care of the case \( G = \text{Gl}(\tilde{V}, \tilde{\tau}_*(p)) \).

**Appendix 2. Indecomposable nilpotent types**

In this appendix we give a table which lists all the indecomposable nilpotent types for the real nonexceptional Lie algebras.

| Lie algebra | indecomposable type | index |
|-------------|---------------------|-------|
| \( \text{gl}(V, \sigma_+) \) | \( \Delta_h(0) \) |       |
| \( \text{gl}(V, \sigma_-) \) | \( \Delta_h(0, 0) \) |       |
| \( \text{gl}(V, \tau_*) \) | \( \Delta^{\varepsilon}_h(0), h \text{ even} \) | \( \frac{1}{2} (h + 1 - \delta) \) |
| | \( \Delta^1_h(0), h \text{ odd} \) | \( \frac{1}{2} (h + 1) \) |
| \( o(V, \sigma_+ , \tau) \) | \( \Delta^{\varepsilon}_h(0), h \text{ even} \) | \( \frac{1}{2} (h + 1 - \delta) \) |
| | \( \Delta^1_h(0, 0), h \text{ odd} \) | \( \frac{1}{2} (h + 1) \) |
| \( o(V, \sigma_- , \tau) \) | \( \Delta_h(0, 0), h \text{ even} \) |       |
| | \( \Delta^1_h(0, 0), h \text{ odd} \) |       |
| \( \text{sp}(V, \sigma_+ , \tau) \) | \( \Delta_h(0, 0), h \text{ even} \) |       |
| | \( \Delta^1_h(0), h \text{ odd} \) |       |
| \( \text{sp}(V, \sigma_- , \tau) \) | \( \Delta^{\varepsilon}_h(0), h \text{ even} \) | \( \frac{1}{2} (2(h + 1) - \delta) \) |
| | \( \Delta^1_h(0, 0), h \text{ odd} \) | \( h + 1 \) |

Table A2. List of indecomposable nilpotent types. Here \( \delta = (-1)^{h/2} \varepsilon \) and \( \varepsilon^2 = 1 \).

Table A2 is taken from Table II p.349 of [1].
Appendix 3. Indecomposable semisimple types

In this appendix we give a table which lists all the indecomposable semisimple types with $S = 0$ for the real affine nonexceptional Lie algebras, see [1, Table A p.361].

| Lie algebra | indecomposable type | comments |
|-------------|---------------------|----------|
| $\text{gl}(V)$ | $\Delta_0(0)$ |           |
| $\text{gl}(V, \sigma_+)$ | $\Delta_0(0)$ | (c)       |
| $\text{gl}(V, \sigma_-)$ | $\Delta_0(0,0)$ | (b)       |
| $\text{gl}(V, \tau_+)$ | $\Delta_0(0)$ |           |
| $\text{o}(V, \tau)$ | $\Delta_0(0)$ |           |
| $\text{o}(V, \sigma_+, \tau)$ | $\Delta_0(0)$ |           |
| $\text{o}(V, \sigma_-, \tau)$ | $\Delta_0(0)$ |           |
| $\text{sp}(V, \tau)$ | $\Delta_0(0,0)$ |           |
| $\text{sp}(V, \sigma_+, \tau)$ | $\Delta_0(0,0)$ | (c)       |
| $\text{sp}(V, \sigma_-, \tau)$ | $\Delta_0^0(0,0)$ | ($\Delta_0^0(0,0))^c = \Delta_0(0,0)$ |

Table 3. List of indecomposable semisimple types with $S = 0$.

Let $\Delta$ be an indecomposable semisimple type for $g(W, \sigma, \tau)$, represented by the pair $(W, S; \sigma, \tau)$. Then $(W, S; \tau) \in \Delta^c$ for $S \in g(W; \tau)$. Suppose that $\Delta^c_i$ is an indecomposable summand of $\Delta^c$, which is represented by the pair $(W_1, S; \tau)$. Since $\sigma^* \tau = \tau$, the pair $(\sigma(W_1), S; \sigma^* \tau)$ is well defined. Denote its type by $\sigma(\Delta^c_i)$. Note that either $\sigma(W_1) = W_1$ or $W_1 \cap \sigma(W_1) = \{0\}$. Since $\sigma^2 = \pm 1$, $S = 0$, and $\Delta$ is indecomposable, there are two possible cases:

(b) $\Delta^c = \Delta^c_i + \sigma(\Delta^c_i)$ and $\Delta^c_i = \sigma(\Delta^c_i)$;
(c) $\Delta^c = \Delta^c_i$ and $\Delta^c_i = \sigma(\Delta^c_i)$.  

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