MG13 proceedings: Construction of gauge-invariant variables for linear-order metric perturbations on an arbitrary background spacetime

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An outline of a proof of the decomposition of the linear metric perturbation into gauge-invariant and gauge-variant parts on an arbitrary background spacetime is discussed through an explicit construction of gauge-invariant and gauge-variant parts. Although this outline is incomplete, yet, due to our assumptions, we propose a conjecture which states that the linear metric perturbation is always decomposed into its gauge-invariant and gauge-variant parts. If this conjecture is true, we can develop the higher-order gauge-invariant perturbation theory on an arbitrary background spacetime.

Keywords: higher-order perturbations, gauge-invariance, arbitrary background spacetime

1. Introduction

As well-known, general relativity is based on general covariance and the “gauge degree of freedom”, which is unphysical degree of freedom of perturbations, arises due to this general covariance. Furthermore, gauge-transformation rules for higher-order perturbations are very complicated. So, it is worthwhile to investigate higher-order gauge-invariant perturbation theory from a general point of view.

According to this motivations, we have been formulating the higher-order general-relativistic gauge-invariant perturbation theory. These works are based on the single assumption that we already know the procedure to find gauge-invariant variables for linear-order metric perturbations. (Conjecture 2.1 in this article) and our formulation is well-defined except for this assumption.

The main purpose of this article is to give a brief outline of a proof of this assumption.

2. Perturbations in general relativity and gauge-invariant variables

Here, we concentrate on the second-kind gauge in perturbation theories with general covariance. In perturbation theories, we always treat two spacetime manifolds. One is the physical spacetime $\mathcal{M}_\lambda$ which is our nature itself and another is the background spacetime $\mathcal{M}_0$ which is prepared by hand for perturbative analyses. The gauge choice of the second kind is the point identification map $X_\lambda : \mathcal{M}_0 \rightarrow \mathcal{M}_\lambda$. The gauge transformation of the second kind is a change $X_\lambda \rightarrow Y_\lambda$ of this identification.

Once we specify a gauge choice $X_\lambda$, we can define perturbations of a physical variable $\bar{Q}_\lambda$ using the pulled-back $X_\lambda^* \bar{Q}$ of $\bar{Q}_\lambda$. $X_\lambda^* \bar{Q}$ is expanded as

$$XQ := X_\lambda^* \bar{Q}_\lambda |_{\mathcal{M}_0} = Q_0 + \lambda (X_\lambda^* \bar{Q}^{(1)}) + \frac{1}{2} \lambda^2 (X_\lambda^* \bar{Q}^{(2)}) + O(\lambda^3).$$

(1)
Here, \( (1)_X^Q \) and \( (2)_X^Q \) are the first-order (second-order) perturbation of \( \tilde{Q}_\lambda \).

The diffeomorphism \( \Phi_\lambda := (X_\lambda)^{-1} \circ Y_\lambda \) is the map \( \Phi_\lambda : \mathcal{M}_0 \to \mathcal{M}_0 \) and does change the point identification. So, \( \Phi_\lambda \) is the gauge transformation \( \Phi_\lambda : \mathcal{X}_0 \to Y_\lambda \) and the induced pull-back operates as \( \gamma^0 Q_\lambda = \Phi_\lambda^* \tilde{Q}_\lambda \). The generic Taylor expansion leads the order-by-order gauge-transformation rules for the perturbations as

\[
(1)_X^Q - (1)_X^Q = \mathcal{L}_{\xi^{(1)}} Q_0, \quad (2)_X^Q - (2)_X^Q = 2 \mathcal{L}_{\xi^{(1)}} (1)_X^Q + \left\{ \mathcal{L}_{\xi^{(2)}} + \mathcal{L}_{\xi^{(1)}}^2 \right\} Q_0.
\]

where \( \xi^{(1)} \) and \( \xi^{(2)} \) are the gauge-invariant variables for higher-order perturbations.

We call the \( k \)th-order perturbation \( (k)_X^Q \) is gauge invariant iff \( (k)_X^Q = (k)_X^Q \) for any gauge choice \( X_\lambda \) and \( Y_\lambda \).

Through these setup, we first consider the metric perturbation to construct gauge-invariant variables for higher-order perturbations. The pulled-back metric \( \gamma^0 \tilde{g}_{ab} \) is expanded as Eq. (1): \( \gamma^0 \tilde{g}_{ab} = g_{ab} + \lambda \gamma_{ab} + (\lambda^2/2) \gamma_{ab} + O^3(\lambda) \), where \( g_{ab} \) is the metric on \( \mathcal{M}_0 \). Our starting point of the construction of gauge-invariant variables is the following assumption for \( h_{ab} \):

**Conjecture 2.1.** If there is a symmetric tensor field \( h_{ab} \) of the second rank, whose gauge transformation rule is \( \gamma h_{ab} = X_a \gamma h_{ab} = \mathcal{L}_{\xi^{(1)}} g_{ab} \), then there exist a tensor field \( \mathcal{H}_{ab} \) and a vector field \( X^a \) such that \( h_{ab} \) is decomposed as \( h_{ab} =: \mathcal{H}_{ab} + \mathcal{L}_X g_{ab} \), where \( \mathcal{H}_{ab} \) and \( X^a \) are transformed as \( \gamma \mathcal{H}_{ab} = -\mathcal{H}_{ab} = 0 \), \( \gamma X^a = -\mathcal{X}^a \), respectively.

In this conjecture, \( \mathcal{H}_{ab} \) and \( X^a \) are gauge-invariant and gauge-variant parts of the perturbation \( h_{ab} \), respectively. If we accept Conjecture 2.1, we can recursively define gauge-invariant variables for higher-order perturbations.

### 3. An outline of a proof of Conjecture 2.1

To prove Conjecture 2.1, we assume that the background spacetimes \( \mathcal{M}_0 \) admit ADM decomposition, whose metric is given by \( g_{ab} = -\alpha^2 (dt)_a (dt)_b + q_{ij} (dx^i + \beta^i dt)_a (dx^j + \beta^j dt)_b \). We decompose of the components \( \{ h_{ij}, h_{ij} \} \) of \( h_{ab} \) as

\[
h_{ii} =: D_i h_{(V)L} + h_{(V)i} - \frac{2}{\alpha} (D_i \alpha - \beta^k K_{ik}) (h_{(V)L}) - \Delta^{-1} D^k \partial_k h_{(TV)k},
\]

\[
h_{ij} =: \frac{1}{n} q_{ij} h_{(L)} + D_i h_{(TV)j} + D_j h_{(TV)i} - \frac{2}{n} q_{ij} D^k h_{(TV)k} + h_{(TT)ij} + \frac{1}{\alpha} K_{ij} (h_{(V)L}) - \Delta^{-1} D^k \partial_k h_{(TV)k} - \frac{2}{\alpha} K_{ij} \beta^k h_{(TV)k},
\]

\[
D^i h_{(V)i} = 0, \quad q^{ij} h_{(TT)ij} = 0 = D^i h_{(TT)i},
\]

where \( M^i_j \) is defined by \( M^i_j := -\alpha^2 K^i_j + \beta^i \beta^k K_{ki} - \beta^j D_i \alpha + \alpha D_i \beta^j \). Here, \( K_{ij} \) is the extrinsic curvature and \( D_i \) is the covariant derivative associate with the metric \( q_{ij} \).
Here, we assumed the existence of Green functions of the elliptic derivative operators $\Delta := D^iD_i$ and $\mathcal{F} := \Delta - \frac{4}{\alpha} \left(D_i\alpha - \beta^j K_{ij}\right)D^i - 2 D^i \left\{ \frac{1}{\alpha} \left(D_i\alpha - \beta^j K_{ij}\right) \right\}$, and the existence and the uniqueness of the solution $A_i$ to the equation

$$D_j^k A_k + D^m \left[ \frac{2}{\alpha} \tilde{K}_{mj} \left\{ \mathcal{F}^{-1} D^k \left( \frac{2}{\alpha} M_k^l A_l - \partial_l A_k \right) - \beta^k A_k \right\} \right] = L_j$$

for given a vector field $L_j$. We note that the relations (3)–(5) are invertible if we accept these three assumptions. These assumptions also imply that we have ignored perturbative modes which belong to the kernel of the above derivative operators and trivial solutions to Eq. (6). We call these modes as zero modes. The issue on the treatments of these zero modes is called zero-mode problem, which is a remaining problem in our formulation.

Due to Eqs. (3)–(5), the gauge-transformation rule $\gamma h_{ab} - \chi h_{ab} = E_{\xi,ij}g_{ab}$ leads

$$
\begin{align*}
\gamma h_{VL} - \chi h_{VL} &= \xi_t + \Delta^{-1} D^k \partial_k \xi_k, \\
\gamma h_{(V)L} - \chi h_{(V)L} &= \Delta^{-1} D^k \partial_k \xi_k,
\end{align*}
$$

These yield the gauge-variant part $X_a$ in Conjecture 2.1 is given by $X_a = X_t(\partial_t)a + X_i(dx^i) a$ with $X_t := h_{(VT)ij}$ and $X_i := h_{(V,L)} - \Delta^{-1} D^k \partial_k h_{(VT)kj}$. Using the variables $X_t$ and $X_i$, we can construct gauge-invariant variables for $h_{ab}$ as

$$-2\Phi := h_{tt} + \frac{2}{\alpha} \left( \partial^t \alpha + \beta^i D_i \alpha - \beta^j \beta^i K_{ij} \right) X_t - 2 \partial_t X_t$$

$$+ \frac{2}{\alpha} \left( \beta^j K_{kj} - \beta^i \alpha + \alpha q^j \partial_i \beta_j + \alpha^2 D_k \alpha - \alpha \beta^j D^l \beta_k - \beta^j \beta^i D_i \alpha \right) X_i,$$

$$-2n \Psi := h_{(L)} - 2 D^i X_i, \quad \nu_i := h_{(V)L} - \partial_i X_i + D_i \Delta^{-1} D^k \partial_k X_k, \quad \chi_{ij} := h_{(TT)ij},$$

where $n$ is the dimension of the $t = \text{const}$ hypersurface. The representations of the original components of $h_{ab}$ in terms of these gauge-invariant variables, $X_t$, and $X_i$ yield the assertion of Conjecture 2.1.$\Box$

4. Discussion

Due to the above proof of Conjecture 2.1 we almost completed our formulation of general-relativistic higher-order gauge-invariant perturbation theories. This indicates the possibility of the wide applications of our formulation. Although our arguments do not include zero modes and these also have their physical meaning[3] we propose Conjecture 2.1 as an conjecture.

References

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