RESIDUALLY FREE GROUPS DO NOT ADMIT A UNIFORM POLYNOMIAL ISOPERIMETRIC FUNCTION

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Abstract. We show that there is no uniform polynomial isoperimetric function for finitely presented subgroups of direct products of free groups, by producing a sequence of subgroups $G_r \leq F_2^{(1)} \times \cdots \times F_2^{(r)}$ of direct products of 2-generated free groups with Dehn function bounded below by $\delta_{G_r}(n) \geq n^r$. The groups $G_r$ are obtained from the examples of non-coabelian subdirect products of free groups constructed by Bridson, Howie, Miller and Short. As a consequence we obtain that residually free groups do not admit a uniform polynomial isoperimetric function.

1. Introduction

Subgroups of direct products of free groups (for short SPF groups) form a special class of residually free groups that have attracted a lot of attention due to their interesting topological finiteness properties [9, 10]. Indeed, the first examples of groups admitting a classifying space with finite $k$-skeleton, but no classifying space with finite $(k+1)$-skeleton, for $k \geq 2$ – the Stallings–Bieri groups – belong to that class [21, 5].

Let us say that a group $G$ is VCA (for virtually coabelian) if $G$ is virtually a co-abelian subgroup of a finite product of groups. It turns out all SPF groups with strong enough finiteness properties are VCA [19, Corollary 3.5]. While not all SPF groups are VCA, they all contain finite index subgroups which are iterated fibre products over nilpotent groups [10].

A way to study topological properties of groups from a quantitative point of view is to estimate their filling invariants, and in particular their Dehn functions. Recall that the Dehn function $\delta_G(n)$ of a finitely presented group $G = \langle X \mid R \rangle$ is defined as the number of conjugates of relations from $R$ needed to detect if a word in $X$ of length at most $n$ represents the trivial word.

The study of Dehn functions of SPF groups was initiated by Gersten who proved that the Dehn function of the Stallings–Bieri groups admits a quintic upper bound [18]. This bound was improved to a cubic bound in [2]. In [6] Bridson argued that in fact the Dehn function of Stallings–Bieri groups is quadratic. There was a flaw in his argument, but it was subsequently proved that the assertion that they are quadratic is correct [15, 11].

Bridson conducted a general study of the Dehn functions of cocyclic subgroups of products of groups [7]. He showed that finitely presented cocyclic subgroups of a product of groups $G_1 \times \cdots \times G_r$ admit a polynomial isoperimetric function, if the $\delta_{G_i}$ are all polynomial [7]. In particular it follows directly from his work and the fact that limit groups are CAT(0) [1] that all cocyclic subgroups of products of limit groups admit a polynomial isoperimetric function. In [13] Dison pursued a systematic study of the Dehn functions of...
coabelian subgroups of direct products of limit groups. He showed that when the group has strong enough finiteness properties, then its Dehn function is polynomial. He also proved that a large class of full subdirect products of limit groups admit a sextic isoperimetric function. More specifically for Bestvina–Brady groups, which are a generalization of the Stallings–Bieri groups to subgroups of Right Angled Artin groups, he obtains a quartic bound on their Dehn functions \[12\]. These results naturally led him to ask the following question:

**Question 1.** \[14\] Is there a uniform polynomial bound \(p(n)\) such that \(\delta_G(n) \leq p(n)\) for all SPF groups \(G\)?

Dison also provides the first example of an SPF group which does not admit a quadratic (or linear) isoperimetric function: he actually shows that his example satisfies a cubic isoperimetric inequality and that it admits a sextic isoperimetric function \[13\].

The purpose of this note is to show that the answer to Question 1 is negative.

**Theorem 1.1.** For every \(r \geq 3\) there is a finitely presented subgroup \(G_r \leq F_2 \times \cdots \times F_2^{(r)}\) with \(\delta_{G_r}(n) \geq n^r\).

The groups \(G_r\) in Theorem 1.1 are explicitly described as quotients of the groups constructed by Bridson, Howie, Miller and Short in \[10, Section 4\], which themselves are conilpotent subgroups of direct products of free groups. As a direct consequence we obtain:

**Corollary 1.2.** The class of finitely presented residually free groups does not admit a uniform polynomial isoperimetric function.

Let us close this introduction with a few natural questions that naturally arise from this result. We first recall Dison’s question:

**Question 2.** \[14\] Does every finitely presented subgroup of a product of free (or limit) groups admit a polynomial isoperimetric function?

We can break this question into the two following more specific ones:

**Question 3.** Does a finitely presented subgroup of a product of \(r\) free groups have isoperimetric function \(n^r\)?

**Question 4.** Is there a uniform isoperimetric function if we restrict to subdirect products of a fixed conilpotency class?

2. **Background**

Let \(G\) be a finitely presented group, let \(\mathcal{P} = \langle X \mid R \rangle\) be a finite presentation. For a word \(w(X) = x_1 \ldots x_n\) with \(x_i \in X^{\pm 1}\), let \(l(w(X)) = n\) be its word length, and for an element \(g \in G\) let \(l(g) = \text{dist}_{\text{Cay}(G,X)}(1,g)\) be its distance from the identity in the Cayley graph \(\text{Cay}(G,X)\). We say that a word \(w(X)\) is *null-homotopic* if it represents the trivial element in \(G\). The area of a null-homotopic word is

\[
\text{Area}_{G,\mathcal{P}}(w(X)) = \min \left\{ k \mid w(X) =_{\text{Free}(X)} \prod_{i=1}^{k} \theta_i(X)r_i\theta_i(X)^{-1}, r_i \in R^{\pm 1}, \theta_i(X) \text{ words in } X \right\}
\]

The Dehn function of \(G\) (with respect to the presentation \(\mathcal{P}\)) is then defined as

\[
\delta_G(n) = \max \{ \text{Area}(w(X)) \mid w(X) \text{ null-homotopic, } l(w(X)) \leq n \}.
\]
For non-decreasing functions \( f, g : \mathbb{N} \to \mathbb{R}_{\geq 0} \) we say that \( f \) is asymptotically bounded by \( g \) and write \( f \leq g \) if there is a constant \( C \geq 1 \) such that \( f(n) \leq Cg(Cn) + Cn + C \) for all \( n \in \mathbb{N} \). We say that \( f \) is asymptotically equal to \( g \) and write \( f \asymp g \) if \( f \leq g \asymp f \). Note that the Dehn function of \( G \) is well-defined up to asymptotic equivalence, with respect to changes of presentation. This justifies the notation \( \delta \leq 1/\text{divid} \simeq t_0 \) and \( C \ll 1/\text{divid} \simeq t_0 \) this fact without explicit mention when working with \( G \).

Definition 2.1. For \( r \geq 1 \), \( G_1 \times \cdots \times G_r \) a direct product of \( r \) groups, \( r \geq k \geq 1 \), and \( 1 \leq i_1 < \cdots < i_k \leq r \), let \( p_{i_1,\ldots,i_k} : G_1 \times \cdots \times G_r \to G_{i_1} \times \cdots \times G_{i_k} \) be the canonical projection.

We say that a subgroup \( H \leq G_1 \times \cdots \times G_r \) has the VSP property (virtual surjection to pairs property) if \( p_{i_1,i_2}(H) \leq G_{i_1} \times G_{i_2} \) is a finite index subgroup for all \( 1 \leq i_1 < i_2 \leq r \).

We further say that \( H \leq G_1 \times \cdots \times G_r \) is subdirect if \( p_i(H) = G_i \) for all \( 1 \leq i \leq r \) and full if \( H \cap G_i := (H \cap (1 \times \cdots \times 1 \times G_i \times 1 \times \cdots \times 1)) \neq 1 \) for all \( 1 \leq i \leq r \).

A group \( G \) is called a limit group (or fully residually free) if for every finite subset \( S \subset G \) there is a homomorphism \( \phi : G \to F_2 \) such that the restriction \( \phi|_S \) is injective. We call \( G \) residually free if for every \( g \in G \setminus \{1\} \) there is a homomorphism \( \phi : G \to F_2 \) with \( \phi(g) \neq 1 \).

A finitely generated group \( G \) is residually free if and only if \( G \) admits an embedding in a finite product of limit groups [4] (see also [10]), emphasizing the importance of studying subgroups of products of limit groups.

For a group \( G \) we denote by \( \gamma_c(G) \) the \( c \)-th term of the lower central series of \( G \). A key result on finitely presented subgroups of direct products of limit groups is the following:

Theorem 2.2. Let \( H \leq G_1 \times \cdots \times G_r \) be a full subdirect product that satisfies the VSP property. Then \( H \) is finitely presented and there are finite index subgroups \( G_{i,0} \leq G_i \) such that \( \gamma_{c-1}(G_{i,0}) \leq G_i \).

Conversely, if \( G_1,\ldots,G_r \) are finitely generated limit groups and \( H \leq G_1 \times \cdots \times G_r \) is a finitely presented full subdirect product then \( H \) has the VSP property.

Proof. This is a direct consequence of Theorem A, Theorem D and Proposition 3.2 in [10]. □

3. LARGE DISTORTION OR LARGE DEHN FUNCTION

In this section we want to explain a generalisation of Proposition 4.2 in [20] to asymmetric fibre products in a product of non-hyperbolic groups. The asymmetric 0-1-2 Lemma says that if \( P \leq G_1 \times G_2 \) is the fibre product of two homomorphism \( \phi_1 : G_1 \to Q \) and \( \phi_2 : G_2 \to Q \), with \( G_1 \) and \( G_2 \) finitely generated and \( Q \) finitely presented, then \( P \) is finitely generated (see for instance [3] Lemma 2.1) for the symmetric 0-1-2 Lemma).

We will now assume that \( G_1 \) and \( G_2 \) are also finitely presented. We equip these groups with presentations \( G_1 = (X_1 \mid R_1) \), \( G_2 = (X_2 \mid S_2) \) and \( Q = (X_Q \mid T_Q) \) such that the families \( X_1 = \{a_1,b_1,\ldots\} \), \( X_2 = \{a_2,b_2,\ldots\} \) and \( X_Q = \{a_Q,b_Q,\ldots\} \) are in bijection with a given family \( X = \{a,b,\ldots\} \). We shall assume that these bijections induce morphisms \( j_1 : F_X \to G_1 \), \( j_2 : F_X \to G_2 \), \( j_Q : F_X \to G_Q \) satisfying \( \phi_1 \circ j_1 = \phi_2 \circ j_2 : F_X \to G_Q \). Using the identification of the generating families \( X_1, X_2 \) and \( X_Q \) with \( X \), we define (in the obvious way) the subsets \( R, S \) and \( T \) of \( F_X \). Note that since \( Q \) is a quotient of both \( G_1 \) and \( G_2 \), we can assume on increasing \( T \) if necessary that it contains \( R \) and \( S \). Denote by
We assume that there is a constant $K$ and since $q$ we let $v$ in $F$. Proof. By moving all letters of $\Pi$ deduce that $\Box$ are done.

The main result of this section is the following generalisation of \cite[Proposition 4.2]{10}. We will combine this result with the existence of certain conilpotent subgroups of products of $r \geq 3$ free groups proved in \cite{11} to deduce Theorem 1.1.

**Proposition 3.1.** Let $G_1 = \langle X_1 \mid R_1 \rangle$, $G_2 = \langle X_2 \mid S_2 \rangle$, $Q = \langle X_Q \mid T_Q \rangle$ and $P = \langle X_{\Delta} \cup T_1 \rangle$ be as above. Let $h = (g, 1) \in P \cap (G_1 \times \{1\})$. Let $v \in F_X$ such that $g = j_1(v)$, and let $C = \max\{|r|_X; r \in R\}$. Then

\[
\text{Area}_Q(v) \leq |h|_P + \delta G_2(|h|_P) + \delta G_1(C|h|_P + |v|_{F_X}).
\]

**Proof of Proposition 3.1.** We let $w = w(X, T') \in F_{X \cup T'}$ be a reduced word of length $|h|_P$ such that $h = w(X_{\Delta}, T_1)$, where $w(X_{\Delta}, T_1)$ means the element of $P$ obtained by substituting letters of $w$ in $X$ and $T'$ by the corresponding elements in $X_{\Delta}$ and $T_1$.

**Claim 1.** The element $w(X, 1) \in F_X$ is a product of conjugates of elements of $S$ whose number $k$ is at most $\delta G_2(|h|_P)$.

**Proof.** Projecting $w$ in $P$, and using the product structure in $G_1 \times G_2$, we obtain

\[
w(X_{\Delta}, T_1) = w(X_1, T_1)w(X_2, 1).
\]

So by projecting to $G_2$, we deduce that $w(X_2, 1)$ maps to the trivial element in $G_2$, so we are done.

We let $q$ be the number of letters from $R$ in $w$. We let $\pi : F_{X \cup T'} \to F_X$ be the group morphism mapping $T'$ to $T$.

**Claim 2.** Then $\pi(w)$ is a product of $q + k$ conjugates of elements of $T$, from which we deduce that $\text{Area}_Q(w) \leq q + k$.

**Proof.** By moving all letters of $T'$ to the right, we can write $w(X, T') = w(X, 1)w'$ in $F_{X \cup T'}$, where $w'$ is a product of $q$ conjugates of letters in $T'$. On the other hand by Claim 1 $w(X, 1)$ can be written as a product of $k$ conjugates of elements of $S \subset T$. \hfill $\Box$

We now consider the element $v\pi(w)^{-1} \in F_X$, which by construction is mapped to the neutral element of $G_1$. Note that its length is $\leq C|h|_P + |v|_{F_X}$, so that we can write it in $F_X$ as a product of $l \leq \delta G_1(C|h| + |v|_{F_X})$ conjugates of elements of $R$. Finally, writing $v = (v\pi(w)^{-1})\pi(w)$ we observe that

\[
\text{Area}_Q(v) \leq \text{Area}_Q(\pi(w)) + l,
\]

which combined with Claim 2 implies that

\[
\text{Area}_Q(v) \leq q + k + l.
\]

Therefore,

\[
\text{Area}_Q(v) \leq q + \delta G_2(|h|_P) + \delta G_1(C|h|_P + |v|_{F_X}),
\]

and since $q \leq |h|_P$, we are done. \hfill $\Box$

**Corollary 3.2.** We let $G_1$, $G_2$ and $Q$ be finitely presented groups, and for $i = 1, 2$, $\phi_i : G_i \to Q$ be surjective morphisms. We also let $X$ be a finite alphabet and $j_1 : F_X \to G_1$ be a surjective morphism. Let $h_n = (g_n, 1) \in P \cap (G_1 \times 1)$, and $v_n \in F_X$ such that $j_1(v_n) = g_n$. We assume that there is a constant $K \geq 1$ such that
(1) \( \frac{1}{n} \leq |g_n|_{G_1}, |v_n|_{F_X} \leq Kn; \)
(2) \( \text{Area}_Q(v_n) \geq \delta_Q(n/K). \)

Then there is \( C \geq 1 \) such that
\[
\delta_Q(n/K) \leq \delta_{G_1}(C|h_n|_P) + \delta_{G_2}(C|h_n|_P) + |h_n|_P.
\]

Proof. Since \( P \) projects to \( G_1 \), we deduce that
\[
|g_n|_{G_1} \leq |h_n|_P, \tag{3.1}
\]
Combining (3.1) with Assumption (1), we deduce
\[
|v_n|_{F_X} \leq |h_n|_P \tag{3.2}
\]
Applying Proposition 3.1 and (3.2) to \( h_n \) and \( v_n = v_n(X) \) we obtain that there is a constant \( C \geq 1 \) such that
\[
\text{Area}_Q(v_n) \leq |h_n|_P + \delta_{G_2}(C|h_n|_P) + \delta_{G_1}(C|h_n|_P). \tag{3.3}
\]
Since by Assumption (2) \( \text{Area}_Q(v_n) \geq \delta_Q(n/K) \), this completes the proof. \( \square \)

Remark 3.3. Note that linearity of Dehn functions of hyperbolic groups allows us to simplify the conclusion of Corollary 3.2 to \( \delta_{G_2}(C|h_n|_P) + |h_n|_P \geq \delta_Q(n/K) \).

We shall need the following special case of Corollary 3.2, where we assume that \( G_1 \) is free.

Corollary 3.4. We let \( G_1, G_2 \) and \( Q \) be finitely presented groups, and for \( i = 1, 2 \), \( \phi_i : G_i \to Q \) be surjective morphisms. We let \( X = X' \sqcup X'' \) be a finite alphabet, and we assume that \( G_1 = F_{X'} \), so that \( j_1 \) is the natural projection from \( F_X \) to \( F_{X'} \). Let \( h_n = (g_n, 1) \in P \cap (G_1 \times 1) \), and \( v_n \in F_{X'} \), such that \( j_1(v_n) = g_n \). We assume that there is a constant \( K \geq 1 \) such that
(1) \( \frac{1}{n} \leq |v_n|_{F_X} \leq Kn; \)
(2) \( \text{Area}_Q(v_n) \geq \delta_Q(n/K). \)

Then there is \( C \geq 1 \) such that
\[
\delta_Q(n/K) \leq \delta_{G_2}(C|h_n|_P) + |h_n|_P.
\]

4. Proof of Theorem 1.1

Bridson, Howie, Miller and Short constructed the first examples of subgroups of direct products of free groups which are conilpotent, but not virtually coabelian, that is, do not have a finite index subgroup which is isomorphic to the kernel of a homomorphism from a direct product of free groups to a free abelian group. The class of examples in their work is of rather general nature, but for simplicity we shall restrict ourselves to a specific subfamily of examples that will suffice for our purposes. However our arguments directly generalize to all of their examples.

For \( r \geq 3 \) let \( F_2^{(1)}, \ldots, F_2^{(r)} \) be 2-generated free groups with generating sets \( F_2^{(i)} = \text{Free}(\{a_i, b_i\}) \). Choose finite normal generating sets \( Y_i = Y_i(a_i, b_i) \) of the \((r-1)\)-th term of the lower central series \( \{Y_i\} = \gamma_{r-1}(F_2^{(1)}) \leq F_2^{(r)} \) and define elements
\[
z_{1,r} = (a_1, a_2, \ldots, a_r), \quad z_{2,r} = (b_1, b_2, \ldots, b_r), \\
z_{3,r} = (a_1^2, a_2^2, \ldots, a_r^2), \quad z_{4,r} = (b_1^2, b_2^2, \ldots, b_r^2).
\]
Denote $Z_r = \{z_{1,r}, z_{2,r}, z_{3,r}, z_{4,r}\}$ and define the finitely generated subgroup
\[ H_r = (X_r) \leq F_2^{(1)} \times \cdots \times F_2^{(r)}, \]
generated by $X_r = Y_1 \cup \cdots \cup Y_r \cup Z_r$.

Then $H_r$ has the following properties:

**Theorem 4.1** ([10] Section 4). The group $H_r$ is a finitely presented full subdirect product and $H_r \cap F_2^{(i)} = \gamma_{r-1}(F_2^{(i)})$ for $1 \leq i \leq r$.

As a consequence the group $H_r$ is a fibre product of the group $F_2^{(1)} = p_1(H_r)$ and the projection $G_{2,r} = p_2, \ldots , r(H_r) \leq F_2^{(2)} \times \cdots \times F_2^{(r)}$ over the 2-generated free nilpotent group $Q_r = F_2^{(1)}/\gamma_{r-1}(F_2^{(1)})$ of class $r-2$. Denote by $\phi_1 : F_2^{(1)} \to Q_r$ and $\phi_2 : G_{2,r} \to Q_r$ the projections defining $H_r$ as a fibre product. Note that we have $\delta_{F_2^{(1)}}(n) \asymp n$ and $\delta_{Q_r}(n) \asymp n^{r-1}$ [13, 17]. The group $G_{2,r}$ is finitely presented by Theorem 4.2 since $H_r$ has the VSP property and therefore the same holds for $G_{2,r}$.

We are now ready to state our main result, whose proof will occupy the rest of this section (observe that Theorem 4.2 is an immediate consequence of Theorem 4.1).

**Theorem 4.2.** For $r \geq 3$, the Dehn function of the group $G_{2,r} = p_2, \ldots , r(H_r) \leq F_2^{(2)} \times \cdots \times F_2^{(r)}$ satisfies $\delta_{G_{2,r}}(n) \asymp n^{r-1}$.

**Proof.** By [10] Proof of Theorem 5.3, the $(r-1)$-fold iterated commutators
\[ w_n(a_1, b_1) = [a_1^n, [a_1^n, \ldots, [a_1^n, b_1^n]\ldots]] \]
satisfy $\text{Area}_{Q_r}(w_n(a_1, b_1)) \asymp n^{r-1} \asymp \delta_{Q_r}(n)$.

**Lemma 4.3.** There exists $k \geq 1$ such that the following holds: Let $g_n \in F_2^{(1)} \cap H_r = \gamma_{r-1}(F_2^{(1)})$ be the element represented by the word $w_{kn}(a_1, b_1)$ and let $h_n = (g_n, 1, \ldots, 1) \in H_r$. Then
\[ |h_n|_{H_r} \leq n \]

**Proof.** We use that the group $H_r$ satisfies the VSP property. Thus, there is a finite index subgroup $\Lambda_1 \leq F_2^{(1)}$ such that $\Lambda_1 \times \{1\} \leq p_{1,j}(H_r)$ for $2 \leq j \leq r$. Note that there is $k \geq 1$ such that $a_1^k, b_1^k \in \Lambda_1$. Thus, we can choose elements $x_{1,j}, y_{1,j} \in H_r$ satisfying
\[ p_1(x_{1,j}) = a_1^k, \quad p_1(y_{1,j}) = 1, \quad p_j(x_{1,j}) = 1, \quad p_j(y_{1,j}) = 1, \]
for $2 \leq j \leq r$. Denote $U = \{x_{1,2}, y_{1,2}, \ldots, x_{1,r}, y_{1,r}\}$.

It is easy to see that the identity
\[ [x_{1,2}^n, \ldots, [x_{1,r-1}^n, y_{1,r}^n]\ldots] = (w_{nk}(a_1, b_1), 1, \ldots, 1) = h_{kn} \in H_r \]
holds. The word $u_n = [x_{1,2}^n, \ldots, [x_{1,r-1}^n, y_{1,r}^n]\ldots]$ having length $\leq 2^{2(r-1)n}$, this proves the lemma.

We are now ready to finish the proof of Theorem 4.2. We shall apply Corollary 3.3 with:

- $X = X_r = X' \cup X''$, where $X' = Y_1$ and $X'' = Y_2 \cup \cdots \cup Y_r \cup Z_r$;
- $P = H_r, G_1 = F_2^{(1)}, G_2 = G_{2,r}$ and $Q = Q_r$;
- $v_n = w_{kn}$.
As already observed the word $v_n$ have length $\lesssim n$, but in order to apply Corollary 3.4 we need to show that $|v_n|_{F_{X'}} \approx n$. This immediately results from the following lemma.

**Lemma 4.4.** Let $F_2 = \langle a, b \rangle$ be the 2-generated free group equipped with its standard word length $\cdot |_{F_2}$ and let $g \in F_2 \setminus \langle a \rangle$. Then the commutator $[a^n, g]$ satisfies

$$[[a^n, g]]_{F_2} \geq 2n + 2.$$

**Proof.** Since $g \notin \langle a \rangle$ every freely reduced word $w(a, b)$ representing $g$ can be decomposed as $w(a, b) = a^k b^{l+1} u(a, b) a^l$ for some $k, l \in \mathbb{Z}$ and $u(a, b)$ either the trivial word or a freely reduced word ending in $b^{\pm 1}$. Thus, we have

$$[a^n, w(a, b)] = a^{n+k} b^{l+1} u(a, b) a^{-n} u(a, b)^{-1} b^{\mp 1} a^{-k}$$

and it follows immediately that

$$[[a^n, g]]_{F_2} \geq 2(n + k + l(u(a, b)) + 1) \geq 2n + 2.$$

So deduce from Corollary 3.4 that

$$n^{r-1} \leq \delta_{G_2}(C|h_n|_{H_r}) + |h_n|_{H_r}.$$

By Lemma 1.3 there exists $C' \geq 1$ such that

$$n^{r-1} \leq \delta_{G_2}(C' n) + n,$$

so $\delta_{G_2}(n) \approx n^{r-1}$, as required.

**Remark 4.5.** We want to remark that we do produce a finitely presented full subdirect product of a product of three 2-generated free groups which admits a cubical lower bound on its Dehn function. In particular, this group is virtually the kernel of a homomorphism from a product of three free groups onto a free abelian group, meaning that we obtain a similar result to [14, Theorem 1.1]. However, we do not know if our group $G_{2,3}$ is commensurable to the example in [14].

More generally, if we could write the groups in [14] and [20] as quotients of finitely presented subgroups of products of four free groups (respectively surface groups) which are conilpotent of class two, then this would give an alternative approach to proving the cubic lower bounds obtained in these works. However, it is not clear that this would significantly simplify the proofs, as it requires the construction of such subgroups and the only known construction of finitely presented conilpotent, non-coabelian subdirect products of free groups is the one of Bridson, Howie, Miller and Short. While their work provides us with some control on the quotients, it does not provide us with the tools to do such a construction.

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