FOUR-BODY EFFECTS IN GLOBULAR CLUSTER BLACK HOLE COALESCENCE

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ABSTRACT

In the high density cores of globular clusters, multibody interactions are expected to be common, with the result that black holes in binaries are hardened by interactions. It was shown by Sigurdsson & Hernquist (1993) and others that 10 $M_\odot$ black holes interacting exclusively by three-body encounters do not merge in the clusters themselves. Therefore, there is little chance for the black hole to overcome the core potential. The major problem with black holes in globular clusters is that the mergers occur well away from their host globulars. If the initial mass of a black hole is $>50 M_\odot$, as may result from a high-mass low-metallicity star or rapid merger of main-sequence stars, it has enough inertia to remain in the core and grow by coalescence (Miller & Hamilton 2002). But what if only low-mass black holes are produced?

Here we propose a new mechanism for the coalescence of low-mass black holes in globular clusters, involving binary-binary interactions. Studies of such four-body encounters have been comparatively rare, but have shown that in roughly 20-50% of the interactions the final state is an unbound single star plus a stable hierarchical triple system (Mikkola 1984; McMillan, Hut, & Makino 1991; Rasio, McMillan, & Hut 1995). This allows an important new effect: studies of planetary and stellar systems have shown that if there is a large relative inclination between the orbital planes of the inner binary and the outer object of the triple, then over many orbital periods the relative inclination periodically trades off with the eccentricity of the inner binary, sometimes leading to very high eccentricities (Kozai 1962; Harrington 1968, 1974; Lidov & Ziglin 1976). In turn, this can enhance the gravitational radiation rate enormously, leading to merger without a strong kick and allowing even low-mass binary black holes in globulars to be potential gravitational wave sources.

In § 2 we discuss the principles of this resonance, as derived in the case of three objects of arbitrary mass by Lidov & Ziglin (1976). To their treatment we add, in § 3, a simple term that accounts for general relativistic pericenter precession. We show that although, as expected, this precession decreases the maximum attainable eccentricity for a given set of initial conditions, the decrease is typically minor and thus there is sig-
ificant phase space in which the eccentricity resonance leads to rapid merger. In § 4 we use these results in a simple model for the mergers of black holes, and show that, depending on the fraction of black holes in binaries, this effect can lead to a dramatic increase in the retention of black holes in globulars, and to the growth of $\sim 10^{2-3} M_\odot$ black holes in their cores.

2. PRINCIPLES OF THE KOZAI RESONANCE

When looking for changes in the orbital properties of a three-body system that extend over many orbital periods of both the inner binary and the outer tertiary, it is convenient to average the motion over both these periods, a procedure called double averaging. A general analysis of the double-averaged three-body problem has been performed to quadrupolar order for Newtonian gravity by Lidov & Ziglin (1976) in Hill’s case, in which the eccentricity resonance leads to significant phase space in which the eccentricity resonance leads to rapid merger. In § 4 we use these results in a simple model for the mergers of black holes, and show that, depending on the fraction of black holes in binaries, this effect can lead to a dramatic increase in the retention of black holes in globulars, and to the growth of $\sim 10^{2-3} M_\odot$ black holes in their cores.

\[ \alpha = \epsilon^{1/2} \cos i_1 + \beta \cos i_2 , \quad \beta = \mu_2 \sqrt{M_2} \left( \frac{a_2}{a_1} \right) (1 - e_2^2) . \]

The constant \( \beta \) (a combination of the constants \( a_1, a_2, \) and \( e_2 \)) represents the total angular momentum of the outer binary while \( \alpha \) is the total system angular momentum (with contributions from both the inner and outer binaries). Both \( \alpha \) and \( \beta \) are made dimensionless by dividing by \( L_1 = \mu_1 \sqrt{GM_1 a_1} \), the angular momentum that the inner binary would have if it were on a circular orbit.

The Hamiltonian, \( \mathcal{H} \), itself is constant. For convenience, we define \( \mathcal{H} = -k(W + \frac{\pi}{3}) \), with \( k = 3\mu a_2^2/[8a_2^3(1 - e_2^2)^{3/2}] \) and obtain:

\[ W = -2\epsilon + \epsilon \cos^2 I + 5(1 - \epsilon) \sin^2 \omega \left( \cos^2 I - 1 \right) , \]

which is equation (30) from Lidov & Ziglin (1976). Here \( \omega \) is the argument of pericenter of the inner binary and the scaled angular momenta \( \alpha, \beta, \) and \( \sqrt{\epsilon} \) form a triangle from which the relative inclination \( I = i_1 + i_2 \) can be obtained using the law of cosines:

\[ \cos I = \frac{\alpha^2 - \beta^2 - \epsilon}{2\beta\sqrt{\epsilon}} . \]

The maximum \( \epsilon \) (and hence minimum \( e_1 \)) occurs for \( \omega = 0 \), and the minimum \( \epsilon \) (and hence maximum \( e_1 \)) occurs for \( \omega = \pi/2 \); see Lidov & Ziglin (1976). Therefore, given initial values for \( \epsilon_0 \) and \( \omega_0 \), the maximum eccentricity may be derived from conservation of \( W \) at \( \omega = \pi/2 \). The time required to push the system from its minimum to maximum eccentricity, is of order

\[ \tau_{\text{evo}} \approx f \left( \frac{\mu_1}{m_2} \frac{b_2^3}{a_1^2} \right)^{1/2} \left( \frac{b_2^2}{GM_2} \right)^{1/2} \]

(e.g., Innanen et al. 1997), where \( b_2 = a_2 \sqrt{1 - e_2^2} \) is the semimajor axis of the tertiary and typically \( f \sim \text{few} \) for \( I \) near \( 90^\circ \), which is the case of interest here.

3. THE KOZAI RESONANCE WITH GR PRECESSION

Post-Newtonian precession may be included in a couple of equivalent ways. One is to modify the Hamiltonian directly, by changing the gravitational potential to simulate some of the effects of general relativity. The modification of the potential is not unique, and depends on which aspect of general relativity is to be reproduced (see Artemova, Bjornsson, & Novikov 1996). For our purposes it is the precession of pericenter that is important (as opposed to, e.g., the location of the innermost stable circular orbit), and hence the correct lowest-order modification is $-GM/r \to (-GM/r)(1 + 3GM/rc^2)$ (Artemova et al. 1996). Averaging the correction term over the orbits of the tertiary and inner binary, we obtain a correction to the double-averaged Hamiltonian of

\[ \mathcal{H}_{PN} = -\frac{3(GM_0)^2 m_1}{a_1^2 c^2 \epsilon^{1/2}} = -kW_{PN} . \]

This result may also be obtained from the first-order general relativistic precession rate of $d\omega = (6\pi GM_1/[a_1(1 - e_1^2)c^2])$ over one binary period (see Misner, Thorne, & Wheeler 1973, p. 1110) using the equation of motion $d\omega/dt = $
\(-2k\sqrt{\pi/L_1})(\partial W/\partial e)\) derived by Lidov & Ziglin (1976). Substituting and integrating, we find that the first-order post-Newtonian contribution to \(W\) is

\[
W_{\text{PN}} = \frac{8}{\sqrt{c}} \frac{M_1}{m_2} \left( \frac{b_2}{a_1} \right)^3 \frac{GM_1}{a_1c^2} \equiv \theta_{\text{PN}} \epsilon^{-1/2}. \tag{6}
\]

in agreement with equation [5]. We have also checked our expressions with direct numerical three-body integrations; note that equation [5] corrects a factor of two error in equation (19) of Lin et al. (2000).

Adding the new term \(W_{\text{PN}}\) to equation [8] and making use of equation [6], we find

\[
W = -2\epsilon + \epsilon \cos^2 I + 5(1 - \epsilon) \sin^2 \omega (\cos^2 I - 1) + \theta_{\text{PN}}/\epsilon^{1/2}. \tag{7}
\]

As in the previous section, for a given set of initial conditions, one can therefore solve for the minimum \(\epsilon\) (maximum \(e\)), by setting \(\omega = \pi/2\) and using the conservation of \(W\). In general we expect that initially the inner binary will have significant eccentricity caused by perturbations during the four-body encounter, but for simplicity we will assume that the initial eccentricity is small enough that \(\epsilon_0 \approx 1\). In the restricted three-body problem in which \(m_0 > m_2 > m_1\) and the initial relative inclination is \(I_0\), the approximate solution for \(\epsilon_{\text{min}}\) when \(5 \cos^2 I_0 < 3\) (high inclination) and \(\theta_{\text{PN}} < 3\) (weak precession) is

\[
\epsilon_{\text{min}} \approx \frac{1}{6} \left[ \theta_{\text{PN}} + \sqrt{\theta_{\text{PN}}^2 + 60 \cos^2 I_0} \right]. \tag{8}
\]

When \(60 \cos^2 I_0 > \theta_{\text{PN}}\) this reduces to the Newtonian solution, in which the maximum eccentricity is \(\epsilon_{\text{max}} = \sqrt{1 - (5/3) \cos^2 I_0}\) (Innanen et al. 1997). If instead \(I_0 \approx \pi/2\) so that \(60 \cos^2 I_0 < \theta_{\text{PN}}\), then \(\epsilon_{\text{max}} \approx 1 - \theta_{\text{PN}}^2/9\). More generally, for any set of masses, if \(e \rightarrow 1\) is allowed in the Newtonian problem then \(\epsilon_{\text{max}} \sim 1 - O(\theta_{\text{PN}}^2)\) when general relativistic precession is included. Numerically, for \(M_1 = M_\odot\) and \(a_1 = 1\,\text{AU}, \theta_{\text{PN}} = 8 \times 10^{-18} \left( M_1/m_2 \right) \left( b_2/a_1 \right)^3\). Equation [6] shows that in the restricted three-body problem the maximum possible eccentricity (minimum \(\epsilon_{\text{min}}\)) is attained for the initial condition \(I_0 = \pi/2\) (initially perpendicular circular orbits). If \(m_1\) has non-negligible mass, so that \(m_2\) dominates the total angular momentum less, then the critical \(I_0\) increases (Lidov & Ziglin 1976). Figure 1 shows the critical inclination in the Newtonian case (\(\theta_{\text{PN}} = 0\)) for several mass ratios and semimajor axes.

We want to know whether this process can cause the inner binary to reach a high enough eccentricity that it merges by gravitational radiation before the next encounter with a star in the globular cluster (which will typically alter the eccentricities and inclinations significantly). Encounters with black holes in globular clusters are usually dominated by gravitational focusing instead of the pure geometrical cross section; this is true within \(100\,\text{AU}\) of a 10 \(M_\odot\) black hole, where we have assumed a velocity dispersion of \(10\,\text{km}\,\text{s}^{-1}\) for the interlopers (see Miller & Hamilton 2002). In this limit the encounter time is

\[
\tau_{\text{enc}} \approx 6 \times 10^7 n_0^{-1} (1\,\text{AU}/a_2)(10\,M_\odot/M_2) \,\text{yr}, \tag{9}
\]

where the number density of stars in the core of the globular is \(10^3 n_0\,\text{pc}^{-3}\). Note that it is the semimajor axis of the outermost object, \(m_2\), that sets the encounter time scale, because in a stable hierarchical triple \(a_2\) must be a factor of several greater than \(a_1\).

The timescale for merger by gravitational radiation for a high eccentricity orbit is (Peter 1964)

\[
\tau_{\text{GR}} \approx 5 \times 10^{11} \left( \frac{M_2}{M_\odot} \right)^4 \left( \frac{a_1}{1\,\text{AU}} \right)^4 \epsilon_{\text{min}}^{3/2} \,\text{yr}. \tag{10}
\]

The steep dependence on eccentricity means that shrinkage of the orbit is dominated by the time spent near maximum eccentricity. Assuming that \(\tau_{\text{GR}} \gg \tau_{\text{enc}}\) so that orbital decay occurs over many Kozai oscillation cycles, one finds that the fraction of time spent near \(\epsilon_{\text{max}}\) is of order \(\epsilon_{\text{min}}\) (Innanen et al. 1997, equation [5]), so that \(\tau_{\text{GR}} \approx 5 \times 10^{11} \left( \frac{M_2}{M_\odot} \right)^4 \epsilon_{\text{min}}^{3/2} \,\text{yr}\). The condition for merger before an encounter is then simply \(\tau_{\text{GR}} < \tau_{\text{enc}}\).

Note that in the Newtonian case \(\theta_{\text{PN}} = 0\), all systems with the same masses, \(b_2/a_1\), and \(I_0\) are dynamically identical, in that the maximum eccentricity does not depend on the individual values of \(b_2\) and \(a_1\). The introduction of post-Newtonian precession breaks this scaling. If \(b_2/a_1\) is fixed, then \(\theta_{\text{PN}} \propto a_1^{-1}\) and therefore the maximum eccentricity attained is given by \(\epsilon_{\text{min}} \propto a_1^{-2}\) (cf. equation [8] for the restricted problem). The merger time is \(\tau_{\text{GR}} \propto a_1^{-1} \epsilon_{\text{min}}^{-1/2} \propto a_1^{-2}\). That is, a wider binary can be pushed to higher eccentricities, and actually merge faster, than a closer binary. Note, however, that the solid angle for this orientation is proportional to \(\theta_{\text{PN}} \propto a_1^{-1}\), because the optimum angle is usually close to \(\pi/2\), so the solid angle is proportional to the cosine of the inclination. Therefore, if binary-binary interactions leave the binary and tertiary inclinations randomly oriented with respect to each other then a smaller fraction of wide binaries will fall into the optimal orientation. Qualitatively this means that as the binary is hardened by various interactions, every time a triple is formed it has a chance to push the eccentricity high enough that the binary
merges before the next encounter. The smaller the system, the larger the probability of such an orientation, because both the solid angle and the encounter time are larger.

One way to quantify the probability of merger through the increase of eccentricity is to plot, as a function of the semimajor axis of the inner binary, the range of relative inclinations such that merger occurs before the next encounter of a field black hole with the tertiary (which, being on a wide orbit, will interact before the inner binary will on average). In Figure 2, we assume three $10 M_{\odot}$ black holes, with a given $a_1$ and $a_2$. From $a_2$ and an assumed number density of stars in the cluster ($n = 10^6$ pc$^{-3}$), we compute the average time $\tau_{enc}$ to the next encounter within a distance $a_2$ of the system. We then determine the range of initial inclinations $I$ such that $\tau_{GR} < \tau_{enc}$, by solving for $\epsilon_{min}$ using equation (4) with the initial conditions $e_1 = e_2 = 0.01$ and $\omega = 0$. Note that for wider tertiary orbits, the total angular momentum of the system is dominated more by the tertiary (larger $\beta$), and hence the relative inclination that gives the smallest possible $\epsilon_{min}$ is closer to $90^\circ$ (see Figure 1). If a single Kozai oscillation cycle is longer than $90^\circ$ (see Figure 1), then the triple system becomes unstable, normally by ejecting its least massive member. Suppose that there are typically $\sim 2$ encounters before the triple is disrupted in this way, and that each encounter of the tertiary that does not create an unstable triple produces a new relative inclination $I$ that is drawn from a uniform distribution in cos $I$. Suppose also that every time the inner binary interacts strongly its semimajor axis is decreased by $\sim 20\%$ (typical for strong interactions of three equal-mass objects; see, e.g., Heggie 1975; Sigurdsson & Phinne 1993). Then, in an $n = 10^6$ pc$^{-3}$ cluster there is a $\approx 50\%$ chance that the inner binary will merge before it hardens to $a_1 \approx 0.2$ AU, at which point the binary recoil velocity $v_{recoil}$ exceeds the $\sim 50$ km s$^{-1}$ escape speed typical of the cores of globulars (Webbink 1985). In an $n = 10^5$ pc$^{-3}$ cluster, encounters are less frequent and the fraction rises to $\approx 70\%$.

Thus, depending on the binary fraction and other properties of black holes in globulars, the majority of black holes could merge before being ejected, and growth of intermediate-mass black holes in globulars may proceed naturally even if no black hole is formed with $M > 10 M_{\odot}$. This could influence stellar dynamics in the core, and the gravitational wave signals from globulars, and should be included in future simulations.

This work was supported in part by NASA grant NAG 5-9756 and by NSF grant 5-23467.

4. CONCLUSIONS

The level of importance of the Kozai mechanism depends on several factors including: i) details of the interactions between two binaries, ii) details of the interactions between a triple, and either a binary or a single star, and iii) the fraction of black holes in binaries, which in turn relies on the iv) dynamics of the cluster itself. Understanding these interactions statistically will require extensive long-term simulations. However, the Kozai mechanism has the potential to be the dominant process in the interactions of stellar-mass black holes in globulars, if most such black holes are in binaries. When only three-body interactions are considered, very few black holes are retained by the clusters (only $8\%$ in the simulations of Portegies Zwart & McMillan 2000). This occurs because the same processes that harden a binary toward an eventual merger also impart velocity kicks on the binary that ultimately eject it from the globular before it can merge. In contrast, the majority of black holes can be retained if binary-binary interactions dominate.

For example, suppose that a third of those interactions produce stable triples. Subsequent interactions of the tertiary with field stars will change its eccentricity and semimajor axis. If the pericenter distance of the tertiary is less than a few times $a_1$, then the triple system becomes unstable, normally by ejecting its least massive member. Suppose that there are typically $\sim 2$ encounters before the triple is disrupted in this way, and that each encounter of the tertiary that does not create an unstable triple produces a new relative inclination $I$ that is drawn from a uniform distribution in cos $I$. Suppose also that every time the inner binary interacts strongly its semimajor axis is decreased by $\sim 20\%$ (typical for strong interactions of three equal-mass objects; see, e.g., Heggie 1975; Sigurdsson & Phinne 1993). Then, in an $n = 10^6$ pc$^{-3}$ cluster there is a $\approx 50\%$ chance that the inner binary will merge before it hardens to $a_1 \approx 0.2$ AU, at which point the binary recoil velocity $v_{recoil}$ exceeds the $\sim 50$ km s$^{-1}$ escape speed typical of the cores of globulars (Webbink 1985). In an $n = 10^5$ pc$^{-3}$ cluster, encounters are less frequent and the fraction rises to $\approx 70\%$.

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This work was supported in part by NASA grant NAG 5-9756 and by NSF grant 5-23467.

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