Stability of de Sitter solutions sourced by dark spinors

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Cosmology of ELKO and Lorentz Invariant NSS has been investigated using dynamical system method starting from the proposal of Boehmer et al [arXiv:1003.3858]. Some different results have been obtained by a different approach. The exact solutions described by fixed points of the dynamical system and their stability have been discussed. Some stable solutions corresponding to de Sitter universes have been obtained.

I. INTRODUCTION

Recently a new class of spinors referred to as dark spinors or ELKO has been found [1] and has received quite some attention [2]. Their effects in cosmology have been investigated [3,4,5]. Their dynamics is more general than that of Dirac or Majorana spinors, even when self-interactions are taken into account. This leads to a more general and thus more interesting cosmological behavior than that exhibited by normal spinors, including for instance the existence of non-trivial de Sitter solutions. However, ELKOs spinors are not Lorentz invariant and their definition requires a preferred space-like direction. In a recent article[6], a new class of theories of non-standard spinors (NSS) has been constructed. By providing a general treatment of class of NSS models based on an action principle a Lorentz invariant, ghost-free, non-local spinor field theory has been developed. The cosmological applications of both the original ELKO and the Lorentz invariant NSS have been examined respectively. Especially, and the existence of non-trivial de Sitter solutions in each case has been discussed.

The cosmological dynamics of the effective scalar degree of freedom in both ELKO and Lorentz invariant NSS cosmology show a large number of very interesting properties. The cosmological evolution of the NSS energy density exhibits a much wider range of behavior than that seen with Dirac spinors. The existence of stable de Sitter solutions means that NSS could represent an alternative to scalar field inflation or dark energy.

In this paper we investigate the evolution of a model of the Friedmann-Robertson-Walker (FRW) universe in both ELKO and Lorentz invariant NSS cosmology using the dynamical systems approach [7, 8, 9]. Some results different from [6] will be given.

II. ELKO COSMOLOGY

Consider a setting in which we only have ELKO spinor fields coupled minimally to gravity, this means we neglect all possible interaction terms. We begin by focusing on the background of a flat FRW spacetime with line element:

$$d s^2 = dt^2 - a^2(t) d x^2.$$ 

A ELKO spinor can be defined by \(\psi = \varphi(t)\xi\), with \(\xi\xi = 1\), where \(\xi\) is a constant spinor, and then \(\varphi\) can be treated as the only dynamical variable cosmological. The energy density \(\rho_\varphi\), and pressure \(p_\varphi\), are, respectively [6],

$$\rho_\varphi = \left[ \frac{1}{2} \dot{\varphi}^2 + V(\varphi^2) \right] + \frac{3}{8} H^2 \varphi^2,$$

$$p_\varphi = \left[ \frac{1}{2} \dot{\varphi}^2 - V(\varphi^2) \right] - \frac{3}{8} H^2 \varphi^2 - \frac{1}{4} H \varphi^2 - \frac{1}{2} H \dot{\varphi} \varphi,$$ (1)

where \(H = \dot{a} / a\) is the Hubble scalar. The conservation equation \(\dot{\rho}_\varphi + 3H(\rho_\varphi + p_\varphi) = 0\) implies that the field equation for \(\varphi\) is

$$\ddot{\varphi} + \frac{dV}{d\varphi} - \frac{3}{4} H^2 \dot{\varphi} + 3H \dot{\varphi} = 0.$$ (2)

The Friedman equation now reads
\[ H^2 = \frac{8 \pi G}{3(1 - \pi G \varphi^2)} \left[ \frac{1}{2} \varphi^2 + V(\varphi^2) \right], \tag{3} \]

and the Raychaudhuri equation is
\[ H = -\frac{4 \pi G}{1 - \pi G \varphi^2} \left( \varphi^2 - \frac{1}{2} H \varphi \dot{\varphi} \right). \tag{4} \]

The field equations can be formulated as an autonomous system of two differential equations by defining
\[ u = \pi G \dot{\varphi}, \]
\[ v = \sqrt{\pi G \varphi}, \]
\[ h = \sqrt{\pi G H(t)}, \]
and
\[ f(v) = (\pi G)^2 V(\varphi^2), \]
and introducing a new time coordinate
\[ \tau = \frac{t}{\sqrt{\pi G}}. \]
The definition \( v = \sqrt{\pi G \varphi} \) here is different from the one given in [6], which leads to different results from [6]. Then, the field equations can be rewritten as
\[ u_{\tau} = -\frac{df(v)}{dv} + \frac{u^2 + 2f(v)}{1 - v^2} v - 2\sqrt{3} \sqrt{\frac{u^2 + 2f(v)}{1 - v^2}} u, \tag{5} \]
\[ v_{\tau} = u, \tag{6} \]
\[ h = \frac{2}{\sqrt{3}} \sqrt{\frac{u^2 + 2f(v)}{1 - v^2}}, \tag{7} \]
and
\[ h_{\tau} = -\frac{4}{(1 - v^2)} \left( u^2 - \sqrt{\frac{u^2 + 2f(v)}{3(1 - v^2)}} u v \right), \tag{8} \]
where \( u_{\tau} = du/d\tau \). These equations (5,6) of \( u \) and \( v \) decouple from \( h \) and then define a dynamical flow in a large phase volume, the behavior can be analyzed qualitatively by standard techniques from the theory of ordinary differential equations. Of particular interest are the exact solutions which define the fixed points of the system and their stability around these fixed points. The fixed points of the system can be obtained by solving \( u_{\tau} = 0 \) and \( v_{\tau} = 0 \) for \( u \) and \( v \):
\[ \frac{df(v)}{dv} = \frac{2vf(v)}{(1 - v^2)}, \]
\[ u = 0, \tag{9} \]
which only corresponds to the equation (5.16) in [6]. The equation \( \frac{df(v)}{dv} = 2f(v)v/(1 - v^2) \) can in principle have infinitely many solutions. For example, \( f(v) = c/(1 - v^2) \) solve this equation for all values of \( v \) and in that case we would encounter a fixed line.

The Jacobian of the system (5,6) is
\[ \mathcal{M} = \begin{pmatrix} \frac{\partial u_{\tau}}{\partial u} & \frac{\partial u_{\tau}}{\partial v} \\ \frac{\partial v_{\tau}}{\partial u} & \frac{\partial v_{\tau}}{\partial v} \end{pmatrix}. \tag{10} \]
where

\[ \frac{\partial u_r}{\partial u} = \frac{2uv}{(1-v^2)} - 2\sqrt{3}\sqrt{\frac{u^2 + 2f(v)}{1-v^2}} - \frac{2\sqrt{3}}{\sqrt{(1-v^2)(u^2 + 2f(v))}}u^2, \]
\[ \frac{\partial u_r}{\partial v} = -\frac{d^2 f(v)}{dv^2} + \frac{2}{(1-v^2)}\frac{df(v)}{dv} + \frac{u^2 + 2f(v)}{(1-v^2)^2} - \frac{2u^2 + 4f(v)}{(1-v^2)^2}v^2, \]
\[ \frac{\partial v_r}{\partial u} = 1, \frac{\partial v_r}{\partial v} = 0. \]  

At the fixed point (9) we have
\[ \frac{\partial u_r}{\partial u} = -2\sqrt{3}\sqrt{\frac{2f(v)}{1-v^2}}, \frac{\partial u_r}{\partial v} = 0, \]
\[ \frac{\partial v_r}{\partial u} = 1, \frac{\partial v_r}{\partial v} = 0. \]

The Jacobian \( \mathcal{M} \) has the eigenvalues: \(-2\sqrt{3}\sqrt{\frac{2f(v)}{1-v^2}}, 0\). For this nonhyperbolic fixed point the linearization theorem does not yield any information about the stability of it and therefore, the center manifold theorem is needed. The theorem shows that the qualitative behavior in a neighborhood of a nonhyperbolic fixed point \( p \) is determined by its behavior on the center manifold near \( p \). Since the dimension of the center manifold is generally smaller than the dimension of the dynamical system, this greatly simplifies the problem.

We shift the fixed point to \((0, 0)\) by setting

\[ v = \bar{v} + v_c, v_c = -2\sqrt{3}\sqrt{\frac{2f(v)}{1-v^2}}. \]

and write the equation (5,6) as, up to second order,

\[ u_r = \frac{2\sqrt{3}\sqrt{2f(v_c)}}{\sqrt{(1-v_c^2)}} u + \frac{v_c}{(1-v_c^2)} u^2 - \frac{4\sqrt{6}f(v_c)v_c}{\sqrt{(1-v_c^2)^3}} u\bar{v} - \frac{12v_c(1+v_c^2)f(v_c)}{(1-v_c^2)^3}\bar{v}^2, \]
\[ \bar{v}_r = u, \]  

where
\[ \frac{df(v)}{dv} = \frac{2vf(v)}{(1-v^2)}. \]

has been used.

The Jacobi decomposition

\[ \mathcal{M} = \begin{pmatrix} -2\sqrt{3}\sqrt{\frac{2f(v_c)}{1-v_c^2}} & 0 \\ 1 & 0 \end{pmatrix} \]
\[ = \begin{pmatrix} 0 & -\frac{1}{2\sqrt{3}\sqrt{\frac{2f(v_c)}{1-v_c^2}}} \\ \frac{1}{2\sqrt{3}\sqrt{\frac{2f(v_c)}{1-v_c^2}}} & -\frac{1}{2\sqrt{3}\sqrt{\frac{2f(v_c)}{1-v_c^2}}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 2\sqrt{3}\sqrt{\frac{2f(v_c)}{1-v_c^2}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2\sqrt{3}\sqrt{\frac{2f(v_c)}{1-v_c^2}} \\ 0 & 0 \end{pmatrix} \]
\[ = SJS^{-1}, \]

gives the transformation

\[ u = y, \]
\[ \bar{v} = \frac{1}{2\sqrt{3}\sqrt{\frac{2f(v_c)}{1-v_c^2}}} x - \frac{1}{2\sqrt{3}\sqrt{\frac{2f(v_c)}{1-v_c^2}}} y, \]  

(13)
\begin{align*}
x &= u + 2\sqrt{3\frac{2f(v_c)}{1-v_c^2}}v, \\
y &= u, \\
\end{align*}
and then
\begin{align*}
x_\tau &= u_\tau + 2\sqrt{3\frac{2f(v_c)}{1-v_c^2}}\nu, \\
y_\tau &= u_\tau.
\end{align*}

The system (12) can be written in diagonal form
\begin{align*}
x_\tau &= F(x, y), \\
y_\tau &= \lambda x + G(x, y),
\end{align*}
where
\begin{align*}
F(x, y) &= -\frac{v_c}{2(1-v_c^2)}x^2 - \frac{v_c(1-v_c^2)}{2(1-v_c^2)^2}xy - \frac{v_c}{2(1-v_c^2)}y^2, \\
G(x, y) &= -\frac{v_c}{2(1-v_c^2)}x^2 - \frac{v_c(1-v_c^2)}{2(1-v_c^2)^2}xy - \frac{v_c}{2(1-v_c^2)}y^2,
\end{align*}
\begin{equation}
(x, y) \in \mathbb{R} \times \mathbb{R}, \quad \lambda = -\frac{2\sqrt{3\frac{2f(v_c)}{1-v_c^2}}}{\sqrt{1-v_c^2}}
\end{equation}
is a negative eigenvalues of the matrix $\mathcal{J}$ and $F, G$ vanish at the origin $(0,0)$ and have vanishing derivatives at $(0,0)$. The center manifold theorem asserts that there exists a 1-dimensional invariant local center manifold $W^c(0)$ of (16) tangent to the center subspace (the $y = 0$ space) at 0. Moreover, $W^c(0)$ can be represented as
\begin{equation}
W^c(0) = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = y(x), |x| < \delta\} : y(0) = 0, y'(0) = 0,
\end{equation}
for $\delta$ sufficiently small (see [9], p. 155) where $y'(x) = \frac{dy}{dx}$. The function $y(x)$ that defines the local center manifold satisfies
\begin{equation}
y'(x)F(x, y(x)) - \lambda y(x) - G(x, y(x)) = 0,
\end{equation}
for $|x| < \delta$; and the flow on the center manifold $W^c(0)$ is defined by the equation
\begin{equation}
x_\tau = F(x, y(x)),
\end{equation}
for all $x$ with $|x| < \delta$.

According to Theorem 3.2.2 in [10], if the origin $x = 0$ of (20) is stable (resp. unstable) then the origin of (16) is also stable (resp. unstable). Therefore, we have to find the local center manifold, i.e., the problem reduces to the computation of $y(x)$.

The condition (18) allows for an approximation of $y(x)$ by a Taylor series at $x = 0$. Since $y(0) = 0, y'(0) = 0$, it is obvious that $y(x)$ commences with quadratic terms. We substitute
\begin{equation}
y(x) = y_2x^2 + y_3x^3 + y_4x^4 + \cdots
\end{equation}
into (18) and set the coefficients of like powers of $x$ equal to zero to find the unknowns
\begin{align*}
y_2 &= -\frac{A}{\lambda} = -\frac{v_c}{4\sqrt{6f(v_c)(1-v_c)^3}}, \\
y_3 &= (B-2A)\frac{A}{\lambda^2} = -\frac{v_c}{96(1-v_c^2)^3}f(v_c), \\
y_4 &= (2B-C)\frac{A^2}{\lambda^3} + (5AB-6A^2-B^2)\frac{A}{\lambda^3} = -\frac{v_c}{384\sqrt{3f(v_c)}\sqrt{2f(v_c)}(1-v_c^2)^5}. \\
\end{align*}
Therefore, (20) yields

\[ x_\tau = -\frac{v_c (1 + v_c^2)}{2 (1 - v_c^2)^2} x^2 + \frac{v_c^2 (1 - 3v_c^2) (1 + v_c^2)}{8 (1 - v_c^2)^3 \sqrt{6f(v_c) (1 - v_c^2)}} x^3 + \frac{v_c^3 (3 - 11v_c^4) (1 + v_c^2)}{96f(v_c) (1 - v_c^2)^5} x^4 + \ldots \]

means that this fixed point is stable when \( v_c = 0 \) or a saddle-node, otherwise. It is important to note that this stability is independent of the potential function \( f(v) = (\pi G)^2 V(\varphi^2) \). The equation (8) indicate the whole fixed (point) line (9) corresponds to de Sitter solutions, however only the point \( u = 0, v = 0 \) is stable.

### III. COSMOLOGY OF LORENTZ INVARIANT NSS

Now we turn to cosmology of the Lorentz invariant NSS spinor \( \psi \) introduced in [6]. By defining

\[
\begin{align*}
\tilde{\psi} & = a^{3/2} \psi, \\
\tilde{\Phi} & = \frac{\psi \tilde{\psi}}{\psi \tilde{\psi}}, \\
\tilde{\Psi} & = \tilde{\Phi} \tilde{\psi} = \tilde{\psi} \tilde{\psi}
\end{align*}
\]

and

\[ \Phi = a^{-3} \tilde{\Phi}, \Psi = a^{-3} \tilde{\Psi}. \]

the energy density \( \rho_\psi \) and pressure \( p_\psi \) can be written as

\[
\begin{align*}
\rho_\psi & = \Psi + V(\Phi), \\
p_\psi & = V'(\Phi) \Phi - V(\Phi),
\end{align*}
\]

where \( V'(\Phi) = dV/d\Phi \). Then \( \Phi \) and \( \Psi \) can be treated as the dynamical variables cosmological. The field equations of them are

\[
\begin{align*}
\ddot{\Phi} & = 2 \left[ \tilde{\Psi} - V'(\Phi) \tilde{\Phi} \right], \\
\ddot{\Psi} & = -V'(\Phi) \dot{\Phi},
\end{align*}
\]

The Friedman equation and the Raychaudhuri equation now read, respectively,

\[ H^2 = \frac{\kappa}{3} [\Psi + V(\Phi)], \]

and

\[ 3H^2 + 2 \dot{H} = \kappa [V(\Phi) - V'(\Phi) \Phi]. \]

In contrast to [6], we do not search for the de Sitter solutions directly using the condition \( p_\psi = -\rho_\psi \). Instead, we will discuss the exact solutions of the equations (22) and (23) given by fixed points of a dynamical system and then obtain de Sitter solutions from (24). Using

\[ H = \frac{\dot{a}}{a}, \dot{\Phi} = a^3 \dot{\Phi}, \ddot{\Phi} = a^3 \ddot{\Phi}, \]

and (24), (25) the equations (22) and (23) can be rewritten as

\[
\begin{align*}
\ddot{\Phi} & = -6 \Phi \sqrt{\frac{\kappa}{3} [\Psi + V(\Phi)] - 3\kappa V(\Phi) \Phi - \frac{3}{2} \alpha \Phi + \left( \frac{3}{2} \alpha \Phi - 2 \right) V'(\Phi) \Phi - \left( \frac{3}{2} \alpha \Phi - 2 \right) \Psi}, \\
\ddot{\Psi} & = -3 \Psi \sqrt{\frac{\kappa}{3} [\Psi + V(\Phi)] - \dot{\Phi} V'(\Phi) - 3 \Phi V'(\Phi) \sqrt{\frac{\kappa}{3} [\Psi + V(\Phi)]},
\end{align*}
\]

(26)
which have decoupled from $H$. (25) becomes

$$\dot{H} = -\frac{\kappa}{2} [\Psi + V'(\Phi)\Phi] .$$  

(27)

Let

$$\Theta = \dot{\Phi} ,$$

then we obtain a dynamical system

$$\begin{align*}
\dot{\Theta} &= -2\sqrt{3\kappa} \Theta \sqrt{\Psi + V(\Phi)} - 3\kappa V(\Phi)\Phi + \left(\frac{3}{2} \kappa \Phi - 2\right) V'(\Phi)\Phi - \left(\frac{3}{2} \kappa \Phi - 2\right) \Psi , \\
\dot{\Phi} &= \Theta \\
\dot{\Psi} &= -\Theta V'(\Phi) - \sqrt{3\kappa} V'(\Phi) \sqrt{\Psi + V(\Phi)} - \sqrt{3\kappa} \sqrt{\Psi + V(\Phi)} ,
\end{align*}$$

(28)

with fixed points

1) A:

$$-\left(\frac{3}{2} \kappa \Phi + 2\right) V(\Phi) + \left(\frac{3}{2} \kappa \Phi - 2\right) V'(\Phi)\Phi = 0 ,$$

$$\Theta = 0 ,$$

$$\Psi = -V(\Phi) ,$$

2) B:

$$-3\kappa V(\Phi) + 3\kappa V'(\Phi)\Phi - 4V'(\Phi) = 0 ,$$

$$\Theta = 0 ,$$

$$\Psi = -\Phi V'(\Phi) ,$$

3) C:

$$\Phi = 0 ,$$

$$\Theta = 0 ,$$

$$\Psi = 0 .$$

The equation (27) indicates that the points B and C correspond to the de Sitter solution

$$\dot{H} = 0 .$$

The stability of these fixed points is described by the Jacobian

$$\mathcal{M} = \begin{pmatrix}
\frac{\partial \Theta}{\partial \Theta} & \frac{\partial \Theta}{\partial \Phi} & \frac{\partial \Theta}{\partial \Psi} \\
\frac{\partial \Phi}{\partial \Theta} & \frac{\partial \Phi}{\partial \Phi} & \frac{\partial \Phi}{\partial \Psi} \\
\frac{\partial \Psi}{\partial \Theta} & \frac{\partial \Psi}{\partial \Phi} & \frac{\partial \Psi}{\partial \Psi}
\end{pmatrix} ,$$

(29)

where

$$\frac{\partial \dot{\Theta}}{\partial \Theta} = -2\sqrt{3\kappa} \sqrt{\Psi + V(\Phi)} ,$$

$$\frac{\partial \dot{\Theta}}{\partial \Phi} = -\sqrt{3\kappa} \Theta \frac{V'(\Phi)}{\sqrt{\Psi + V(\Phi)}} - 3\kappa V(\Phi) - 2V'(\Phi) + \left(\frac{3}{2} \kappa \Phi - 2\right) V''(\Phi)\Phi - \frac{3}{2} \kappa \Psi ,$$

$$\frac{\partial \dot{\Theta}}{\partial \Psi} = -\frac{\sqrt{3\kappa} \Theta}{\sqrt{\Psi + V(\Phi)}} + \left(2 - \frac{3}{2} \kappa \Phi\right) ,$$

$$\frac{\partial \dot{\Phi}}{\partial \Theta} = 0 ,$$

$$\frac{\partial \dot{\Phi}}{\partial \Phi} = 1 ,$$

$$\frac{\partial \dot{\Phi}}{\partial \Psi} = 0 ,$$

$$\frac{\partial \dot{\Psi}}{\partial \Theta} = 0 ,$$

$$\frac{\partial \dot{\Psi}}{\partial \Phi} = 0 ,$$

$$\frac{\partial \dot{\Psi}}{\partial \Psi} = 0 .$$
\[
\begin{aligned}
\frac{\partial \Psi}{\partial \Theta} &= -V'(\Phi), \\
\frac{\partial \Psi}{\partial \Phi} &= -\Theta V''(\Phi) - \sqrt{3\kappa (\Phi V''(\Phi) + V'(\Phi))} \sqrt{\Psi + V(\Phi)} - \frac{\sqrt{3\kappa} \Phi (V'(\Phi))^2 + \Psi V'(\Phi)}{2 \sqrt{\Psi + V(\Phi)}}, \\
\frac{\partial \Psi}{\partial \Psi} &= -\sqrt{3\kappa} \sqrt{\Psi + V(\Phi)} - \frac{\sqrt{3\kappa} \Phi V'(\Phi) + \Psi}{2 \sqrt{\Psi + V(\Phi)}}.
\end{aligned}
\]

At B:

\[
-3\kappa V(\Phi_c) + 3\kappa V'(\Phi_c)\Phi_c - 4V'(\Phi_c) = 0,
\]

\[
\Theta = 0,
\]

\[
\Psi_c = -\Phi_c V'(\Phi_c),
\]

the Jacobian of \( M \) has the eigenvalues:

\[
-4\sqrt{-V'(\Phi_c)}, \quad -\sqrt{-V'(\Phi_c)} - \sqrt{-V'(\Phi_c) - V''(\Phi_c)\Phi_c (2 - \frac{3}{2}\kappa \Phi_c)}, \quad -\sqrt{-V'(\Phi_c) + \sqrt{-V'(\Phi_c) - V''(\Phi_c)\Phi_c (2 - \frac{3}{2}\kappa \Phi_c)}}.
\]

i) In the case \( V'(\Phi_c) < 0 \),

\[
V'(\Phi_c) < 0, \quad (30)
\]

when

\[
0 < \Phi_c < \frac{4}{3\kappa}, \quad V''(\Phi_c) > 0, \quad (31)
\]

or

\[
\Phi_c > \frac{4}{3\kappa}, \quad V''(\Phi_c) < 0, \quad (32)
\]

the point B is stable, otherwise or \( \Phi_c < 0 \), it is unstable. However, if

\[
V''(\Phi_c) = 0,
\]

or

\[
\Phi_c = 0,
\]

the eigenvalues are \(-4\sqrt{-V'(\Phi_c)}, -2\sqrt{-V'(\Phi_c)}, 0\), and then B is a saddle-node i.e. it behaves like a saddle or an attractor depending on the direction from which the orbit approaches.

ii) In the case

\[
V'(\Phi_c) = 0,
\]

the eigenvalues are 0, \(-\sqrt{-V''(\Phi_c)\Phi_c (2 - \frac{3}{2}\kappa \Phi_c)}, \sqrt{-V''(\Phi_c)\Phi_c (2 - \frac{3}{2}\kappa \Phi_c)}\).

When

\[
\Phi_c > \frac{4}{3\kappa}, \quad V''(\Phi_c) > 0, \quad -V''(\Phi_c)\Phi_c \left( 2 - \frac{3}{2}\kappa \Phi_c \right) > 0,
\]

or

\[
\frac{4}{3\kappa} > \Phi_c > 0, \quad V''(\Phi_c) < 0, \quad -V''(\Phi_c)\Phi_c \left( 2 - \frac{3}{2}\kappa \Phi_c \right) > 0,
\]

the point B is unstable, while

\[
\Phi_c > \frac{4}{3\kappa}, \quad V''(\Phi_c) < 0, \quad -V''(\Phi_c)\Phi_c \left( 2 - \frac{3}{2}\kappa \Phi_c \right) < 0,
\]

or
or
\[
\frac{4}{3\kappa} > \Phi_c > 0, V''(\Phi_c) > 0, -V''(\Phi_c)\Phi_c \left(2 - \frac{3}{2}\kappa\Phi_c\right) < 0,
\]
it is a center.

When
\[
\Phi_c < 0, V''(\Phi_c) > 0,
\]
B is unstable, while
\[
\Phi_c < 0, V''(\Phi_c) < 0,
\]
it is a center.

When
\[
\Phi_c = 0,
\]
the point B is a center.

iii) In the case
\[
V'(\Phi_c) > 0,
\]
when
\[
-V'(\Phi_c) > V''(\Phi_c)\Phi_c \left(2 - \frac{3}{2}\kappa\Phi_c\right),
\]
B is unstable, while
\[
-V'(\Phi_c) \leq V''(\Phi_c)\Phi_c \left(2 - \frac{3}{2}\kappa\Phi_c\right),
\]
it is a center. These results indicate that the stability of the fixed point B is dependent on \(\Phi_c, V'(\Phi_c)\) and \(V''(\Phi_c)\) but independent of \(V(\Phi_c)\).

At C:
\[
\Phi_c = 0, \Theta_c = 0, \Psi_c = 0,
\]
the Jacobian has the eigenvalues: 
\[
-\sqrt{3\kappa V(0)}, -\sqrt{\kappa V(0)} - 2\sqrt{-V'(0)}, -\sqrt{\kappa V(0)} + 2\sqrt{-V'(0)}.
\]
i) In the case
\[
V(0) > 0,
\]
when
\[
V'(0) \geq 0,
\]
C is stable. When
\[
V'(0) < 0, -\sqrt{3\kappa V(0)} + 2\sqrt{-V'(0)} < 0,
\]
C is stable, while
\[
V'(0) < 0, -\sqrt{3\kappa V(0)} + 2\sqrt{-V'(0)} > 0,
\]
it is unstable. When
\[
V'(0) < 0, -\sqrt{3\kappa V(0)} + 2\sqrt{-V'(0)} = 0,
\]
the Jacobin has a zero eigenvalue: $-\sqrt{3}\kappa V(0), -2\sqrt{3}\kappa V(0), 0$, and then $C$ is a saddle-node.

ii) In the case

$$V(0) < 0,$$

when

$$V'(0) \geq 0,$$

$C$ is a center, while

$$V'(0) < 0,$$

it is unstable.

iii) In the case

$$V(0) = 0,$$

when

$$V'(0) \geq 0,$$

$C$ is a center, while

$$V'(0) < 0,$$

$C$ is unstable. The stability of the fixed point $C$ depends on $V(0)$ and $V'(0)$.

## IV. CONCLUSIONS

Cosmology of ELKO and Lorentz Invariant NSS has been investigated using dynamical system method. The exact solutions described by fixed points of the dynamical system and their stability have been discussed. Some stable solutions corresponding to de Sitter universe have been obtained. In the cosmology of ELKO there exists a stable fixed point corresponding to a de Sitter universe while in the cosmology of Lorentz Invariant NSS we encounter some stable fixed lines corresponding to de Sitter solutions. These results and conclusions, especially the de Sitter solutions and their behavior are different from the ones in [6].

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