Twin-peak quasiperiodic oscillations as an internal resonance
(Research Note)

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ABSTRACT

Aims. Two inter-related peaks occur in high-frequency power spectra of X-ray lightcurves of several black-hole candidates. We further explore the idea that a non-linear resonance mechanism, operating in strong-gravity regime, is responsible for these quasi-periodic oscillations (QPOs).

Methods. By extending the multiple-scales analysis of Rebusco, we construct two-dimensional phase-space sections, which enable us to identify different topologies governing the system and to follow evolutionary tracks of the twin peaks. This suggests that the original (Abramowicz & Kluzniak) parametric-resonance scheme can be viewed as an ingenuous account of the QPOs model with an internal resonance.

Results. We show an example of internal resonance in a system with up to two critical points, and we describe a general technique that permits to treat other cases in a systematical manner. A separatrix divides the phase-space sections into regions of different topology: inside the libration region the evolutionary tracks bring the observed twin-peak frequencies to an exact rational ratio, whereas in the circulation region the observed frequencies remain off resonance. Our scheme predicts the power should cyclically be exchanged between the two oscillations. Likewise the high-frequency QPOs in neutron-star binaries, also in black-hole sources one expects, as a general property of the non-linear model, that slight detuning pushes the twin-peak frequencies out of sharp resonance.

Key words. Accretion, accretion-discs – Black hole physics – Quasi-periodic oscillations

1. Introduction

Twin peaks occur in high-frequency (~ 50–450 Hz) power spectra of X-ray (~ 2–60 keV) lightcurves of several black-hole candidates (see van der Klis 2006; McClintock & Remillard 2003 for recent reviews of observational properties and theoretical interpretations). This transient phenomenon seems to be connected with the kilohertz quasi-periodic oscillations (QPOs) in neutron star sources, of which more examples are known (about the tens at present). The nature of black hole high-frequency QPOs remains puzzling despite variety of models proposed in the literature. In neutron-star low-mass X-ray binaries these twin QPOs are known to occur often simultaneously and they can be highly coherent (Q ≳ 10^2; e.g. Barret et al. 2005), slowly wandering in frequencies between different observations, whereas in black-hole candidates the QPO coherence appears to be lower (Q ~ 2–10) and the presence of a pair pops up only when a collection of observations is carefully analyzed.

In Abramowicz & Kluzniak (2001) and Kluzniak & Abramowicz (2001) an idea of accretion disc resonance was proposed, which naturally incorporates pairs of frequencies occurring in a ratio of small integer numbers. This scheme predicts the observed frequency ratios in black-hole QPO sources should prefer the 3:2 ratio; it also suggests this could be understood if a non-linear coupling mechanism operates in a black-hole accretion disc, where strong-gravity effects are essential. Indeed, especially in those black-hole candidates where the high-frequency QPOs have been reported, they occur very close to the ratio of small integer numbers, 3:2 in particular (Miller et al. 2001; Strohmayer 2001; Homan et al. 2004; Remillard et al. 2005; Maccarone & Schnittman 2005).

Nowadays, the original account can be viewed as a naïve model with the internal resonance. Various realizations of this scheme have been examined in terms of accretion disc/torus oscillations (e.g. Abramowicz et al. 2003; Bursa et al. 2004; Kato 2004; Li & Narayan 2004; Schnittman & Rezzolla 2005; Zanotti et al. 2005).

So far the “right” model has not yet been identified. However, it has been recognized that fruitful knowledge about common properties of high-frequency QPOs can be gained by investigating a very general resonance scheme, which likely governs matter near a compact accreting body. To this aim, Abramowicz et al. (2003) examined the epicyclic resonances in a nearly-geodesic motion in strong gravity. Rebusco (2004) and Horák (2004), by employing the method of multiple scales (Nayfeh & Mook 1979), have demonstrated that the 3:2 resonance is indeed the most prominent one near horizon of a
central black hole. Only certain frequency combinations are allowed, depending on symmetries which the system exhibits, and only some of the allowed combinations have chance to give rise to a strong resonance.

In the present paper we pursue this approach further and we find tracks that an axially symmetric system with two degrees of freedom, near resonance, should follow in the plane of energy (of the oscillations) versus radius (where the oscillations take place). We show different topologies of the phase space in the way that closely resembles the method of disturbing function, familiar from the studies of the evolution of mean orbital elements in celestial mechanics (Kozai 1962; Lidov 1962). The analogy is very illuminating and it provides a systematic way of distinguishing topologically different states of the system. In particular, one can discriminate regions of phase space where the observed frequency ratio fluctuates around an exact rational number from those regions where this ratio remains always outside the resonance. Our model suggests that even black-hole twin QPOs should vary in frequency and they should not stay at a firmly fixed frequency ratio, albeit the expected variation is very small – certainly less than what has been frequently reported in neutron star binaries and what can be tested with the data available at present.

2. A conservative system with two degrees of freedom

2.1. Non-linear terms in the governing equations

Let us consider an oscillatory system with two degrees of freedom, which is described by coupled differential equations of the form

\[ \dot{\delta \rho} + \omega_{\delta}^{2} \delta \rho = f_{\delta}(\delta \rho, \delta \theta, \dot{\delta \rho}, \dot{\delta \theta}), \]

\[ \dot{\delta \theta} + \omega_{\theta}^{2} \delta \theta = f_{\theta}(\delta \rho, \delta \theta, \dot{\delta \rho}, \dot{\delta \theta}). \]

We assume that the right-hand-side functions are nonlinear (their Taylor expansions start with the second order), and that they are invariant under reflection of time. Clearly, these equations include the case of a nearly circular motion under the influence of a perturbing force: \( \delta \rho \) and \( \delta \theta \) are small perturbations of the position, whereas \( \omega_{\rho}(r) \) and \( \omega_{\theta}(r) \) have a meaning of the radial and the vertical epicyclic frequencies along the circular orbit \( r = r_{0}, \theta = \pi/2 \):

\[ \omega_{\rho}^{2} = \frac{\partial^2 U}{\partial r^2}, \quad \omega_{\theta}^{2} = \frac{1}{r_0} \left( \frac{\partial^2 U}{\partial \theta^2} \right), \]

where the effective potential is

\[ U(r, \theta) = \Phi(r, \theta) + \frac{\ell^2}{2r^2 \sin^2 \theta}. \]

\( \Phi(r, \theta) \) is the gravitational potential, for which axial symmetry and staticity will be assumed. These assumptions make our system qualitatively different from models requiring non-axisymmetric perturbations.

As a generic example, let the total gravitational field be given as a superposition of the central potential of a spherical star, \( \Phi_{s}(r) \), plus an axisymmetric term \( \Phi_{a}(r, \theta) \):

\[ \Phi(r, \theta) = \Phi_{s}(r) + \Phi_{a}(r, \theta). \]

For the central field we assume the form

\[ \Phi_{s}(r) = -\frac{GM}{r}, \]

where we set \( r \equiv r \) or \( \tilde{r} = r - r_{s} \) (to adopt the Newtonian or the pseudo-Newtonian approximations; \( r_{s} \equiv 2GM/c^2 \)). We assume a circular ring (mass \( m \), radius \( a \)) as a source of the perturbing potential,

\[ \Phi_{a}(r, \theta) = -\frac{2Gm K(k)}{\pi B^{1/2}}, \]

where \( K(k) \) is a complete elliptical integral of the first kind, \( B(r, \theta) \equiv r^2 + a^2 + 2ar \sin \theta, k(r, \theta) \equiv 4ar \sin \theta / B(r). \)

At this point a remark is necessary regarding the interpretation of the potential. We conceive it as a toy-model for strong-gravity effects and internal resonances that we seek in the system of a black hole and an accretion disc, but the origin of the perturbing potential \( \Phi_{a}(r, \theta) \) is not supposed to be the gravitational field of the accretion disc itself. Of course the inner disc is not self-gravitating in black hole binaries and the ring is not introduced here with the aim of representing the accretion flow gravity. What we imagine is that hydrodynamic and magnetic forces are producing qualitative effects, which can be captured by the ring potential in our equations. On the other hand, our approach is rather general and it is worth remembering that the same formalism can be successfully applied also in systems where the disc self-gravity plays a non-negligible role.

Also in general relativity, weakly perturbed (i.e. nearly-geodesic) motion of gas elements orbiting around a Schwarzschild black hole can be described by effective potential that contains a spherical term arising from the gravitational field of the central black hole, plus a perturbing term, which we assume axially symmetric. Naturally, the interpretation of the perturbing term is more complicated if one would like to derive its particular form from an exact solution of Einstein’s equations. We do not want to enter into complications of a special model here (but see Letelier 2003; Karas et al. 2004; Semerák 2004 and references cited therein for a review and examples of spacetimes that contain a black hole and a gravitating ring in general relativity). Also, one may ask whether the adopted approach can comprehend frame-dragging effects of Kerr spacetime while the black hole rotation is considered as perturbation to the spherical field of a static non-rotating black hole; even if a general answer to that question would be affirmative, the Kerr metric has rather special properties concerning the integrability of the geodesic motion and the form of non-sphericity, which quickly decays with radius. Our current opinion is such that the rotation-related effects of Kerr metric cannot be viewed as the origin of the perturbation required for black-hole high-frequency twin QPOs.

The angular momentum of a test particle orbiting the centre on an equatorial circular orbit is

\[ \ell \equiv \int \left( \frac{\partial \Phi}{\partial r} \right) = GMr \left( \frac{r^2}{p^2} + \mu^{r} (r + a)E(k_{0}) + (r - a)K(k_{0}) \right). \]

where \( E(k) \) is a complete elliptical integrals of the second kind and the right-hand-side terms are evaluated at the orbit radius,
Fig. 1. Angular momentum $\ell(r)$ (thick line) of a test particle on a circular orbit in the combined gravitational field of a spherical body and a ring. The mass of the ring relative to the central mass is $\mu = 0.1$ and the particle mass is set to unity. Left panel: the case of Newtonian central field. Right panel: the pseudo-Newtonian case. Radius has been scaled with respect to the ring radius (here, $a = 9R_0$), and the angular momentum has been scaled by the value of Keplerian angular momentum $\ell_K(a)$. Keplerian angular momentum in the central field is also plotted (thin line). Circular orbits are Rayleigh unstable and the epicyclic approximation is inadequate in regions where the angular momentum decreases with radius.

Fig. 2. The behaviour of epicyclic frequencies (thick lines; $\omega_0$ – top, $\omega_\theta$ – bottom) provides basic clues to the origin of oscillations and the effects expected to occur in strong gravity. Left/right panels refer to the Newtonian/pseudo-Newtonian cases, as in the previous figure. Units of the ordinates are scaled by $\sqrt{GM/a^3}$ to make frequencies dimensionless. The frequencies in absence of the ring ($\mu = 0$) are also indicated for reference (thin lines; $\omega_0(r) = \omega_\theta(r)$ in the Newtonian case).

$$k_0 \equiv k(r_0, \pi/2), \mu \equiv m/M.$$ Figure captures a typical curve of the angular momentum as a function of radius. The corresponding orbital frequency is

$$\Omega^2(r) = \frac{GM}{r^3} \left[ \frac{r^2}{2} + \mu r \frac{(r + a)E(k_0) + (r - a)K(k_0)}{\pi(r^2 - a^2)} \right]. \quad (9)$$

Equation gives the epicyclic frequencies,

$$\omega^2_0(r) = \frac{GM}{r^3} \left[ \frac{r^2}{2} + \frac{2\mu}{\pi} \frac{(r - a)^2K(k_0) - a^2E(k_0)}{r^2(r - a)^2(r + a)} \right], \quad (10)$$

$$\omega^2_\theta(r) = \frac{GM}{r^3} \left[ \frac{r^2}{2} + \frac{2\mu}{\pi} \frac{E(k_0)}{(r - a)^2(r + a)} \right]. \quad (11)$$

The epicyclic frequencies as functions of radius are plotted in Figure (we will drop the index “0” for the sake of brevity).

The difference between the radial epicyclic frequency and the orbital frequency gives the shift of pericentre. The difference of vertical epicyclic and orbital frequencies gives the nodal precession.

Internal resonances can occur in the system . In order to capture this phenomenon, we carry out a multiple-scales expansion (Nayfeh & Mook 1979) in the form

$$\delta p(t, e) = \sum_{n=1}^{4} e^n \rho_n(T_\mu), \quad \delta \theta(t, e) = \sum_{n=1}^{4} e^n \theta_n(T_\mu), \quad (12)$$

where $T_\mu = e^\mu t$ are treated as independent time scales. We will terminate the expansion at the fourth order ($\mu = 0, 1, 2, 3$; the number of time scales is the same as the order at which the expansions are truncated).
Because equations (14)–(15) are conservative, the growth of energy should occur in a more pronounced rate. Because amplitudes are in the ratio of small integers, the periodic exchange of terms in their general form.

As a specific example we can adopt an explicit form describing the orbital motion,

\[
\begin{align*}
\frac{d}{dt} & = \sum_{\mu=0}^{4} e^\mu D_\mu, \\
\frac{d^2}{dt^2} & = \sum_{\mu=0}^{4} \sum_{\nu=0}^{4} e^{\mu+\nu} D_\mu D_\nu,
\end{align*}
\]

where \(D_\mu \equiv \partial / \partial T_\mu\). The method tackles the governing equations in their general form.

Because equations (14)–(15) are conservative, the growth of energy in one mode must be balanced by the energy loss in the other mode. Close to resonance radii, where the two epicyclic frequencies are in ratio of small integers, the periodic exchange of energy should occur in a more pronounced rate. Because amplitudes of the oscillations are connected with eccentricity and inclination, the solution alternates between an inclined, almost circular trajectory at certain stages, and an eccentric, almost equatorial case at some other time. We remark that, in a non-inclination, the solution alternates between an inclined, almost circular trajectory at certain stages, and an eccentric, almost equatorial case at some other time. We remark that, in a non-linear system, the eigenfrequencies \(\omega_x\) and \(\omega_y\) are expected to differ from the observed (i.e., predicted) frequencies, which can be revealed e.g. by Fourier analysis of data time series. Relevance of this fact for QPOs was first recognized by Abramowicz et al. 2003 and Rebusco 2004, when they discussed a model for Sco X-1. It was further employed by Horák et al. 2004, who suggested that a connection should exist between the high-frequency QPOs and normal-branched oscillations.

We expand the effective potential derivatives and the functions \(f_\rho\) and \(f_\theta\) into Taylor series, up to the fourth order about a circular orbit. The expansion provides many nonlinear terms containing various derivatives

\[ u_{ij} \equiv \left( \frac{\partial^i \partial^j U}{\partial \rho \partial \theta} \right)_{[n=\pi/2]} . \]

By imposing constraints on the potential and its derivatives, we identify resonances that are expected in a particular system (and we reject those resonances that cannot be realized). In the next subsection, formulation of these constraints is still kept completely general, valid for the arbitrary form of \(U\). Only later, in subsection 2.2, we come to our original motivation from the orbital motion around a black hole and we employ the symmetry of the potential \(U\). We consider the case of potential symmetric with respect to the equatorial plane. This implies the condition \(u_{2k+1} = 0, k \in N\), somewhat reducing the number of terms in the expansions.

Amplitudes of the oscillations are characterized by a small parameter: \(\delta \rho \sim \epsilon, \delta \theta \sim \epsilon\). We impose the solvability constraints and seek a solution in the form (12). To this aim, we first write the explicit form of these constraints in different orders of approximation.

Table 1. Possible resonances and secular terms in the second order of approximation. Only regular secular terms are present in the first/second row corresponding to the resonances, respectively. In each record corresponding to the resonances, the first/second row gives an expression for secular terms in radial/vertical oscillations. To simplify our notation we introduce \(\Lambda_\alpha \equiv C^{(\alpha)}_0\) and \(K_\alpha \equiv c^{(\alpha)}_0\).

| \(\omega_0/\omega_x\) | Secular terms |
|---------------------|--------------|
| Outside resonance   | \(-2i\omega_0 D_1 \tilde{A}_x\) |
| 1:2                 | \(-2i\omega_0 D_1 \tilde{A}_x\) |
| 2:1                 | \(-2i\omega_0 D_1 \tilde{A}_x\) |

2.2. Solvability constraints

In the first order, we obtain equations describing two independent harmonic oscillators,

\[
\begin{align*}
(D_0^2 + \omega_x^2) \rho_1 &= 0, \\
(D_0^2 + \omega_y^2) \theta_1 &= 0.
\end{align*}
\]

The solutions can be expressed in the form

\[
\rho_1 = \tilde{A}_x + \tilde{A}_{-x}, \quad \theta_1 = \tilde{A}_\theta + \tilde{A}_{-\theta},
\]

where we denote \(\tilde{A}_x \equiv A_x e^{i\omega_0 T_0}\), \(\tilde{A}_{-x} \equiv A_x e^{-i\omega_0 T_0}\) with \(x = \rho,\theta\). The complex amplitudes \(A_x\) generally depend on slower scales, \(T_1, T_2, T_3\).

An algorithmic nature of the multiple-scales method allows us to determine the form of solvability conditions in a conservative system with two degrees of freedom. The conditions arise by eliminating the terms that are secular in the fastest variable, \(T_0\). This constraint is imposed, because otherwise the solutions in the form of series would not converge uniformly. In fact, the reason why the same number of scales is required as the order of approximation in the expansion (12) is that one secular term gets eliminated in each order, and therefore the functions \(A_x(T_0)\) are determined by the same number of solvability conditions as the number of their variables.

In the second order, the terms proportional to \(\epsilon^2\) in the expanded left-hand side of the governing equations (11–21)  are

\[
\begin{align*}
\frac{\delta x + \omega_x^2 x}{x} & = (D_0^2 + \omega_x^2) x + 2i \omega_0 D_1 \tilde{A}_x - 2i \omega_0 D_1 \tilde{A}_{-x},
\end{align*}
\]

On the right-hand side, the second-order terms result from the expansion of the nonlinearity \(f_\delta(\delta \rho, \delta \theta, \delta \varphi, \delta \theta)\) with \(D_0 \delta \rho_1\) and \(D_0 \delta \theta\) in place of \(\delta \rho\) and \(\delta \theta\), respectively. They can be expressed as linear combinations of quadratic terms constructed from \(A_{-x}\) and \(\tilde{A}_{-\theta}\):

\[
\left[ f_\delta(\delta \rho, \delta \theta, \delta \rho, \delta \theta) \right] = \sum_{|j|+|k|=2} C^{(1)}_{j,k} \tilde{A}_{-x_j} \tilde{A}_{-\theta_k} \tilde{A}_{-x} \tilde{A}_{-\theta}.
\]
where \( \alpha = (\alpha_1, \ldots, \alpha_d) \) and \(|\alpha| = \alpha_1 + \ldots + \alpha_d\). The constants \( C^{(\alpha)}_a \) are given by angular frequencies of \( \omega_\rho \) and by coefficients of the Taylor expansion of \( f_{\rho} \). Equating the right-hand sides of eqs. (19)–(20), we find

\[
(D_0^2 + \omega_\rho^2)x_2 = -2i\omega_\rho D_1x_3 + 2i\omega_\rho D_1\tilde{A}_{-x} + \sum_{|\alpha|=2} C^{(\alpha)}_a \bar{A}_p^{\alpha_1} \bar{A}_q^{\alpha_2} \bar{A}_r^{\alpha_3} \bar{A}_s^{\alpha_4}.
\]

The rhs of eq. (21) contains one secular term independently of the eigenfrequencies \( \omega_\rho \) and \( \omega_\theta \). However, additional secular terms may appear in the resonance. For example, when \( \omega_\rho \approx 2\omega_\theta \), the terms proportional to \( \tilde{A}_\rho^2 \) in the \( \rho \)-equation \( x \to \rho \) and \( \tilde{A}_\rho \tilde{A}_- \) in the \( \theta \)-equation \( x \to \theta \) become secular and they should be also included to the solvability conditions. The similar situation happens when \( \omega_\rho \approx \omega_\theta/2 \). These are internal resonances, which show a qualitatively different behaviour: the corresponding terms are secular only for special (resonant) combinations of \( \omega_\rho \) and \( \omega_\theta \), contrary to the terms that appear always and are referred to as regular secular terms. Possible resonances in the second order of approximation and the corresponding secular terms in eq. (21) are listed in Table 1. Let us assume, for a moment, that the system is far from any resonance. Then

\[
D_1\tilde{A}_x = 0.
\]

The frequencies and amplitudes are constant and the behaviour of the system is almost identical as what one finds in the linear approximation. The only difference is the presence of higher harmonics oscillating with frequencies \( 2\omega_\rho, 2\omega_\theta \) and \( |\omega_\rho \pm \omega_\theta| \). They are given by a particular solution of equation (21) after eliminating the secular term,

\[
x_2 = \sum_{|\alpha|=2} Q^{(\alpha)}_a \bar{A}_p^{\alpha_1} \bar{A}_q^{\alpha_2} \bar{A}_r^{\alpha_3} \bar{A}_s^{\alpha_4},
\]

(23)

Under the assumption of time-reflection invariance, constants \( Q^{(\alpha)}_a \) are real and their relation to constants \( C^{(\alpha)}_a \) becomes obvious by substituting \( x_2 \) into equation (21). We find

\[
Q^{(\alpha)}_{klnm} = \frac{C^{(\alpha)}_{klnm}}{\omega_\rho^2 - (k-l)\omega_\rho + (m-n)\omega_\theta}.
\]

(24)

In the third order, the discussion is analogous in many respects. The terms proportional to \( e^{i\theta} \), which appear on the lhs of the governing equations, are given by

\[
\left[ \delta^2 + \omega_\rho^2 \right] x_3 = \left( D_0^2 + \omega_\rho^2 \right) x_3 + 2i\omega_\rho D_1x_3 + 2i\omega_\rho D_1\tilde{A}_{-x}.
\]

(25)

The terms containing \( D_1x_1 \) and \( D_1x_2 \) vanish in consequence of the solvability condition (22). The rhs contains now cubic terms of the Taylor expansion. We obtain

\[
(D_0^2 + \omega_\rho^2)x_3 = -2i\omega_\rho D_1\tilde{A}_x + 2i\omega_\rho D_1\tilde{A}_{-x} + \sum_{|\alpha|=3} C^{(\alpha)}_a \bar{A}_p^{\alpha_1} \bar{A}_q^{\alpha_2} \bar{A}_r^{\alpha_3} \bar{A}_s^{\alpha_4},
\]

(26)

where constants \( C^{(\alpha)}_a \) are real. The secular terms are summarized in Table 2 together with resonances possible in the third order of approximation. Again, far from any resonance we eliminate the terms that are secular independently of \( \omega_\rho \), \( \omega_\theta \). The resulting solvability conditions take the form

\[
D_2A_\rho = -\frac{i}{2\omega_\rho} \left[ K_{1200} |A_\rho|^2 + K_{0111} |A_\rho|^2 \right] A_\rho,
\]

(27)

\[
D_2A_\theta = -\frac{i}{2\omega_\theta} \left[ K_{1110} |A_\rho|^2 + K_{0012} |A_\theta|^2 \right] A_\theta.
\]

(28)

A particular solution of eq. (26) is given by a linear combination of cubic terms constructed from \( \tilde{A}_{\alpha\rho} \) and \( \tilde{A}_{\alpha\theta} \).

\[
x_3 = \sum_{|\alpha|=3} \sum_{|\beta|=3} J^{(\alpha\beta)}_a \bar{A}_p^{\alpha_1} \bar{A}_q^{\alpha_2} \bar{A}_r^{\alpha_3} \bar{A}_s^{\alpha_4},
\]

(29)

where all coefficients \( J^{(\alpha\beta)}_a \) are now real.

Finally, in the fourth order of the approximation,

\[
\left[ \delta^4 + \omega_\rho^2 \right] x_4 = \left( D_0^2 + \omega_\rho^2 \right) x_4 + 2D_1D_0x_1 + 2D_1D_2 x_2.
\]

(30)

The operator \( D_0D_2 \) acts on \( x_2 \), given by eq. (25). The resulting form is found by employing the solvability conditions (27) and (28):

\[
2D_0D_2 x_2 = \omega_\rho^2 \sum_{|\alpha|=1} J^{(\alpha)}_a \bar{A}_p^{\alpha_1} \bar{A}_q^{\alpha_2} \bar{A}_r^{\alpha_3} \bar{A}_s^{\alpha_4},
\]

(31)

where \( J^{(\alpha)}_a \) are real constants. By expanding the rhs we arrive at the governing equation

\[
(D_0^2 + \omega_\rho^2) x_4 = -2i\omega_\rho D_1\tilde{A}_x + 2i\omega_\rho D_1\tilde{A}_{-x} + \sum_{|\alpha|=4} C^{(\alpha)}_a \bar{A}_p^{\alpha_1} \bar{A}_q^{\alpha_2} \bar{A}_r^{\alpha_3} \bar{A}_s^{\alpha_4},
\]

(32)

with \( C^{(\alpha)}_a \) real constants. Only one secular term independent of \( \omega_\rho \) and \( \omega_\theta \) appears on the rhs: \(-2i\omega_\rho D_1\tilde{A}_x\). The sum contains only terms that become secular near a resonance. These terms and the solvability conditions are listed in Table 3.
A notable feature of internal resonances $k:l$ is that $k\omega_r$ and $l\omega_r$ need not be infinitesimally close to each other, as might be expected from the linear analysis. Consider, for example, an internal resonance $1:2$, i.e. $\omega_1 = 2\omega_r$. By eliminating the secular terms, we obtain solvability conditions (see Tab. [I])

\begin{align}
-2i\omega_r D_1 \tilde{A}_r + K_{1001} \tilde{A}_r \tilde{A}_\theta &= 0, \\
-2i\omega_r D_1 \tilde{A}_\theta + \Lambda_{0200} \tilde{A}_r^3 &= 0.
\end{align}

In each of these equations the first term is regular, while the second term is nearly secular (resonant) one. The solvability conditions give us the long-term behaviour of the amplitudes and phases of oscillations. Suppose now that the system departs from the sharp ratio by small (first-order) deviations $\omega_2 = 2\omega_r + \epsilon \sigma$, where $\sigma$ is the detuning parameter. The terms proportional to $\tilde{A}_r \tilde{A}_\theta$ and $\tilde{A}_r^3$ still remain secular with respect to the variable $T_0$. This can be demonstrated from $\tilde{A}_r \tilde{A}_\theta = A_r^* A_\theta^* e^{i(\omega_1 - \omega_2) T_0} = A_r^* A_\theta^* e^{i\theta_1} e^{i\sigma T_0}$. Similar relation holds for $\tilde{A}_r^3$.

### 2.3. Solution of the solvability constraints

By comparing the coefficients with same powers of $\epsilon$ on both sides of the Taylor-expanded governing equations [I]–[2], we obtain relations for functions $\rho_j(T_j)$ and $\theta_j(T_j)$ that can be solved successively. After rearranging to a “canonical” form,

\begin{align}
[D_0 + \omega_r^2] \rho_n &= \sum K_{ijkl} \tilde{A}_r^i \tilde{A}_\theta^j \tilde{A}_r^k \tilde{A}_\theta^l, \\
[D_0 + \omega_r^2] \theta_n &= \sum \Lambda_{ijkl} \tilde{A}_r^i \tilde{A}_\theta^j \tilde{A}_r^k \tilde{A}_\theta^l,
\end{align}

where $n$ is the order of approximation. In this way we identify constants $K_{ijkl}$ and $\Lambda_{ijkl}$. The studied gravitational potential is symmetric with respect to the equatorial plane, therefore, the series [33]–[34] cannot contain terms proportional to odd derivatives of the effective potential with respect to $\theta$. Hence, contrary to a general case, only specific resonances occur here: $\omega_1: \omega_2 = 1:2, 1:1, 3:2$ and $1:4$. These are all possible combinations that may occur within the given order of approximation (the first three cases were originally identified by Rebusco [2004] although she does not mention the fourth possible combination). A general argument of nonlinear analysis suggests that the dominant resonances are those ones which correspond to ratios of small natural numbers, although not every conceivable resonant combination comes up in a given physical system. Indeed, it appears that the $3:2$ ratio is the most important case when the high-frequency QPO pairs are debated, however, the true role of this resonance has not yet been understood; see also the discussion in Bursa [2005] and Lasota [2005]. Proceeding further to higher-order terms of the expansions reveals even more resonances, but these are expected to be very weak (recently various kinds of weird combinations have been examined by Török et al. [2005]). Hereafter we concentrate on the first three combinations.

### The case of 1:2 resonance

The solvability conditions take the form (cp. Tab. [I] and Horák [2004])

\begin{align}
D_1 \tilde{A}_r &= -\frac{i}{2\omega_r} K_{0002} \tilde{A}_r^2, \\
D_1 \tilde{A}_\theta &= -\frac{i}{2\omega_r} \Lambda_{0110} \tilde{A}_r \tilde{A}_\theta.,
\end{align}

The coefficients of the resonant terms are given by

\begin{align}
K_{0002} &= -\omega_1^2 - \frac{u_{12}}{2\tau_0}, \\
\Lambda_{0110} &= -2\omega_2^2 - \frac{u_{12}}{\tau_0},
\end{align}

satisfying a mutual relation $2K_{0002} = \Lambda_{0110}$.

### The case of 1:1 resonance

The solvability conditions for the first order, $D_1 \tilde{A}_r = D_1 \tilde{A}_\theta = 0$, imply that the complex amplitudes $\tilde{A}_r$ and $\tilde{A}_\theta$ depend only on the second scale $T_2$. The 1:1 ($\omega_r \approx \omega_0$) resonance is the only epicyclic resonance of the system with reflection symmetry, which occurs in the third order of approximation. The dependence on $T_2$ implies a slower behaviour. The solvability conditions are

\begin{align}
D_2 \tilde{A}_r &= -\frac{i}{2\omega_r} \left[ K_{1200} |A_r|^2 \tilde{A}_r^2 + K_{0111} |A_\theta|^2 \tilde{A}_\theta + K_{1002} \tilde{A}_r \tilde{A}_\theta^2 \right], \\
D_2 \tilde{A}_\theta &= -\frac{i}{2\omega_r} \left[ \Lambda_{1101} |A_r|^2 \tilde{A}_r + \Lambda_{0012} |A_\theta|^2 \tilde{A}_\theta + \Lambda_{0210} \tilde{A}_r^3 \right],
\end{align}

and the coefficients of the resonant terms are given by

\begin{align}
K_{1200} &= \frac{5 u_{22}^2}{6 \omega_0} - \frac{1}{3} u_{40}, \\
K_{0111} &= \frac{1}{3} \left(-10 \omega_2^2 + \frac{2u_{12}^2}{\tau_0^2 \omega_r^2} - 3 u_{22}\right).
\end{align}
where the resonant coefficients are
\[
K_{2002} = -\frac{9}{8} \frac{u_{22}}{r_{0}^{2}} + \frac{81}{16} \frac{u_{22}^{2}}{r_{0}^{4} \omega_{p}^{2}} - \frac{27}{16} \frac{u_{22}}{r_{0} \omega_{p}} - \frac{27}{16} \frac{u_{30}}{r_{0} \omega_{p}} - \frac{227}{16} \frac{u_{22}^{2}}{r_{0}^{2} \omega_{p}^{2}} - \frac{243}{16} \frac{u_{12}^{3}}{r_{0}^{4} \omega_{p}^{4}} - \frac{243}{16} \frac{u_{12}^{3}}{r_{0}^{4} \omega_{p}^{4}}.
\]
(55)

\[
\Lambda_{3010} = \frac{3}{2} K_{2002}.
\]
(56)

By introducing the detuning parameter,
\[
\sigma \equiv 3 \frac{\omega_{r}}{\omega_{p}} - 2 = e^{2} \delta_{2} + e^{3} \delta_{3},
\]
(57)

The case of 3:2 resonance

The solvability conditions involve both the third and the fourth orders. Hence, the amplitudes \(A_{p}, A_{0}\) are functions of both time scales \(T_{3}\) and \(T_{4}\). The elimination of regular secular terms in the third order \((3\omega_{r} \approx 2\omega_{p})\); see Tab. 2 gives

\[
D_{2} \tilde{A}_{p} = -\frac{i}{2 \omega_{p}} \left[K_{1200} \left|A_{p}\right|^{2} \tilde{A}_{p} + K_{0111} \left|A_{0}\right|^{2} \tilde{A}_{p}\right],
\]
(48)

\[
D_{2} \tilde{A}_{0} = -\frac{i}{2 \omega_{p}} \left[K_{0111} \left|A_{p}\right|^{2} \tilde{A}_{0} + K_{0012} \left|A_{0}\right|^{2} \tilde{A}_{0}\right]
\]
(49)

with the coefficients

\[
K_{1200} = \frac{\rho_{0}}{r_{0}^{2}} \left(\frac{15 u_{30}^{2}}{8 \omega_{p}} - \frac{7 u_{12}^{3} r_{0}^{2}}{16 \omega_{p}^{2}} + \frac{9 u_{12}^{2}}{4 r_{0} \omega_{p}} - 4 u_{22}ight),
\]
(50)

\[
K_{0111} = \frac{1}{4} \left(-15 \omega_{p}^{2} + \frac{9 u_{12}^{2}}{4 r_{0}^{2} \omega_{p}^{2}} - 4 u_{22}ight),
\]
(51)

\[
K_{0012} = -\frac{u_{04}}{2 \rho_{0}} + \frac{135 u_{12}^{2}}{64} \frac{u_{01} u_{02}}{r_{0} \omega_{p}^{2}} - 153 \frac{u_{12}}{16} \frac{u_{04}}{r_{0} \omega_{p}} + 135 \omega_{p}^{2},
\]
(52)

\[
K_{1200} = \frac{1}{4} \left(-15 \omega_{p}^{2} + \frac{9 u_{12}^{2}}{4 r_{0}^{2} \omega_{p}^{2}} - 4 u_{22}ight),
\]
(53)

The elimination of the resonant terms gives the solvability condition in the fourth order (Tab. 3),

\[
D_{3} \tilde{A}_{p} = -\frac{i}{2 \omega_{p}} K_{2002} \tilde{A}_{p} \tilde{A}_{p}^{*}, \quad D_{3} \tilde{A}_{0} = -\frac{i}{2 \omega_{p}} \Lambda_{3010} \tilde{A}_{p} \tilde{A}_{p}^{*},
\]
(54)

where the resonant coefficients are

\[
K_{2002} = -\frac{9}{8} \frac{u_{22}}{r_{0}^{2}} + \frac{27}{16} \frac{u_{22}^{2}}{r_{0}^{2} \omega_{p}^{2}} - \frac{27}{16} \frac{u_{22}}{r_{0} \omega_{p}} - \frac{27}{16} \frac{u_{30}}{r_{0} \omega_{p}} - \frac{27}{16} \frac{u_{22}^{2}}{r_{0}^{2} \omega_{p}^{2}} - \frac{27}{16} \frac{u_{22}^{2}}{r_{0}^{2} \omega_{p}^{2}} - \frac{27}{16} \frac{u_{22}^{2}}{r_{0}^{2} \omega_{p}^{2}}.
\]
(55)

\[
\Lambda_{3010} = \frac{3}{2} K_{2002}.
\]
(56)

By introducing the detuning parameter,

\[
\sigma \equiv 3 \frac{\omega_{r}}{\omega_{p}} - 2 = e^{2} \delta_{2} + e^{3} \delta_{3},
\]
(57)

3. The system evolution near the 3:2 resonance

3.1. The integrals of motion

The case of 3:2 resonance is particularly relevant for the high-frequency QPOs, both on observational and theoretical grounds.
where the condition |\( F(u) | \geq | G(u) | \) is satisfied.

(cf. Abramowicz & Kluzniak 2001; Kluzniak et al. 2004 for arguments in favour of 3:2 ratio in high-frequency QPOs and for further references). We therefore discuss this case in more detail, however, similar discussion could be presented also for other resonances (Horák 2005). Reintroducing single physical time \( t \), the equations for the second and the third order can be combined together. Time derivatives are then given by:

\[
\frac{d}{dt} = e^2 D_2 + e^3 D_3.
\]

Amplitudes and phases of the oscillations are governed by equations:

\[
\begin{align*}
\dot{a}_r &= \frac{1}{4} \beta \omega_0 a_r^2 a_0 \sin \gamma, \\
\dot{a}_\theta &= -\frac{1}{4} \beta \omega_0 a_\theta^3 a_0 \sin \gamma, \\
\dot{\gamma} &= -\sigma \omega_0 + \frac{a_0}{4} \left[ \mu \dot{a}_r^2 + \mu_0 a_0^2 + \frac{a_0}{2} (a a_\theta^2 - \beta a_0^2) \cos \gamma \right],
\end{align*}
\]

where \( \sigma = \epsilon a_\mu, \dot{a}_\theta = \dot{a}_0 \) and \( \mu_\sigma, \mu_\theta \) and \( \beta \) are defined by relations \( \Lambda_{0310} = \frac{4}{3} K_{2002} = \beta a_0^2, k_{1200} - k_{1101} = \omega_0^2 \mu_\sigma, \) and \( k_{0111} - k_{0012} = \omega_0^2 \mu_\theta \). The amplitudes and phases are not mutually independent; the equations (66–67) imply that the quantity

\[
\mathcal{E} = a_r^2 + \frac{a_\theta^2}{4} = \text{const}
\]

remains conserved during the system evolution. Clearly, \( \mathcal{E} \) is proportional to the total energy of the oscillations. The existence of this integral is a general property of a conservative system. Naturally, it is not limited to the particular form of the perturbing potential (7), which we consider here, and it holds in Newtonian, pseudo-Newtonian as well as general-relativity versions of the equations of motion (the pseudo-Newtonian case was examined, in detail, by Abramowicz et al. 2003 and Horák 2004). One can watch the accuracy to which \( \mathcal{E} \) is conserved in order to verify the analytical approach against the exact numerical solution (we show such comparison in Figure 4) for the pseudo-Newtonian case of Abramowicz et al. 2003.

Equations (65–67) can be merged in a single equation by introducing the following parameterization:

\[
\begin{align*}
\sigma &= \frac{\partial}{\partial \rho} \mathcal{E}, \\
\rho^2 &= \frac{4}{9} \left( 1 - \xi^2 \right) \mathcal{E}.
\end{align*}
\]

Then the oscillations are described by two equations for \( \xi(t) \) and \( \gamma(t) \):

\[
\begin{align*}
\dot{\xi} &= \frac{1}{16} \beta \omega_0 \xi^2 (1 - \xi^2) \mathcal{E}^{3/2} \sin \gamma, \\
\dot{\gamma} &= -\sigma \omega_0 + \frac{4}{3} \omega_0 \mathcal{E} \left[ \mu \dot{\xi}^2 + \frac{4}{3} \beta \mu_0 (1 - \xi^2) \mathcal{E}^{1/2} \cos \gamma \right],
\end{align*}
\]

which satisfy the identity

\[
\xi \dot{\gamma} - \dot{\xi} \gamma = 0.
\]

Substituting for \( \dot{\xi} \) and \( \dot{\gamma} \) from eqs. (71)–(72), eq. (73) implies that

\[
\mathcal{F} \equiv 8 \left( 1 - \xi^2 \right) \sigma + \mathcal{E} \left[ \mu \dot{\xi}^2 + \frac{4}{3} \mu_0 (1 - \xi^2)^2 \right] + \beta \mathcal{E}^{3/2} \xi^3 (1 - \xi^2) \cos \gamma
\]

is a second integral of motion. For a given value of energy \( \mathcal{E} \), the system follows \( \mathcal{F} = \text{const} \). Curves in the other words, projection of the solution onto \( (\gamma, \xi) \)-plane satisfies

\[
\mathcal{F}(\gamma, \xi) = \text{const}.
\]

This allows us to construct two-dimensional phase-space sections in which the system evolution takes place.

### 3.2. Stationary points and the phase-plane topology

Stationary points are given by the condition \( \dot{\xi} = \dot{\gamma} = 0 \). According to eq. (71), \( \gamma \)-coordinate of these points satisfies \( \sin \gamma = 0 \), and therefore \( \gamma = \kappa \pi \) with \( k \) being an integer. Substituting \( \dot{\gamma} = 0 \) and \( \cos \gamma = \pm 1 \) in eq. (72), we find a cubic equation:

\[
-4\sigma + \left[ \mu \xi^2 + \frac{4}{3} \mu_0 (1 - \xi^2) \right] \mathcal{E} \pm \beta \mathcal{E}^{3/2} (3 - 5 \xi^2) \mathcal{E}^{3/2} = 0.
\]
The solution determines \( \xi \)-coordinate of the stationary points. In the case of small oscillations (\( \mathcal{E} \ll 1 \)), the solution can be approximated by keeping only terms up to the linear one in \( \mathcal{E} \) in eq. (76). We obtain

\[
\xi^2 = \frac{\partial r - \mu_\theta}{\frac{3}{2} \mu_\sigma - \mu_\theta}, \tag{77}
\]

where \( \tilde{\sigma} \equiv \sigma / \mathcal{E} \). The first correction to this solution is of the order of \( \mathcal{E}^{1/2} \). Deviations between \( \xi \)-coordinates of stationary points at odd and even multiples of \( \pi \) are of the same order.

The solution (77) lies within the allowed range provided that \( -\frac{1}{2} \mu_\sigma \leq \tilde{\sigma} \leq \frac{1}{2} \mu_\sigma \) with the denominator \( D \equiv \frac{3}{2} \mu_\sigma - \mu_\theta \leq 0 \), respectively. This can be expressed in terms of energy \( \mathcal{E} \); given the detuning parameter \( \sigma \), stationary points appear in the \((\gamma, \xi)\) plane if the energy of oscillations satisfies

\[
\frac{\sigma}{\mu_\theta} \leq \mathcal{E} \leq 4 \frac{\sigma}{\mu_\sigma} \quad \text{for} \quad D \geq 0. \tag{78}
\]

The examination of phase-plane topology near critical points leads to the equation

\[
\left( \frac{\partial \dot{\xi}}{\partial \xi} - \lambda \right) \left( \frac{\partial y}{\partial y} - \lambda \right) - \frac{\partial \dot{y}}{\partial y} \frac{\partial y}{\partial \xi} = 0 \tag{79}
\]

for eigenvalues \( \lambda \) of the system of linearized equations (77) and (72). Evaluating the partial derivatives at the critical point and keeping only the terms of the lowest order in \( \mathcal{E}^{1/2} \), we obtain

\[
\lambda^2 = \pm \frac{\sqrt{\mathcal{E}}}{\beta \omega_0} \left( 1 - \xi^2 \right) D \mathcal{E}^{5/2}. \tag{80}
\]

Examining the sign of \( \lambda^2 \) demonstrates that the critical points of central topology alternate with those of saddle topology.

### 3.3. The time dependence

The equation for \( \xi(t) \) can be derived by combining eqs. (71) and (74). Eliminating \( \cos \gamma \), we arrive at the relation

\[
\mathcal{K}u^2 = F^2(u) - G^2(u), \tag{81}
\]

where we introduced a new variable \( u(t) \equiv \xi^2 \). The constant \( \mathcal{K} \) and functions \( F(u) \) and \( G(u) \) are defined by

\[
\mathcal{K} \equiv \frac{1}{\mathcal{E}^{1/2}} \left( \frac{8}{\sqrt{\omega_0} \beta} \right)^2, \tag{82}
\]

\[
F(u) \equiv u^{3/2}(1 - u), \tag{83}
\]

\[
G(u) \equiv \frac{1}{\beta \mathcal{E}^{5/2}} \left[ F - 8 \sigma(1 - u) - \mu_\sigma u^2 + 4 \frac{\mu_\theta}{\mu_\sigma} \mathcal{E}(1 - u)^2 \right]. \tag{84}
\]

The motion is allowed only for \( u^2 \geq 0 \), and so the condition \( \pm F(u) = G(u) \) gives us two turning points, \( u_1 \) and \( u_2 \), between which the evolutionary path oscillates. The functions \( \pm F(u) \) and \( G(u) \) are plotted in Figure 4.

The period of energy exchange can be found by integrating eq. (61):

\[
T \approx \frac{16\pi}{\beta \omega_0} \mathcal{E}^{-3/2} \int_{u_1}^{u_2} \frac{du}{\sqrt{F^2(u) - G^2(u)}}. \tag{85}
\]

where \( T \) can be roughly approximated as

\[
T \sim \frac{16\pi}{\beta \omega_0} \mathcal{E}^{-3/2}. \tag{86}
\]

Notice that this time-scale is longer than the period of individual oscillations and it says how fast the system swaps itself between the radial and vertical oscillation modes. The simple estimate (86) is quite precise in most parts of the phase space, although it becomes inaccurate near stationary points, where the rate of energy exchange slows down.

Finally, we illustrate our results with a simple case, which we already introduced at the beginning of the paper (sect. 2.1): the gravitational field generated by a pseudo-Newtonian star and a narrow circular ring. The resonant condition \( \omega_\theta / \omega_r = 3 / 2 \) is now fulfilled at three different radii, two of them lying between the star and the ring, and the third one outside the ring. The resonances occurring at first two radii are called the inner resonances, whereas the latter is the outer resonance (not to be confused with ‘internal resonance’, which they are all). We restrict ourselves to the inner resonances, for which we find \( \sigma, \mu_\sigma, \mu_\theta \) and \( \beta \) as functions of \( r \). For a fixed radius, inequalities (78) give us the energy range of the oscillations. The result is shown in Figure 5 where we identify three different phase-plane topologies in the \((r, \mathcal{E})\)-section. These can be distinguished by the number of critical points and the shape of separatrices. The topology change is evident in Figure 6 where two-dimensional plots are constructed for the integral of motion \( F(\xi, \gamma) \).

#### 3.4. Frequencies of the resonant oscillations

Equations (63)–(65) give the shift of actual (observed) frequencies of oscillations, \( \omega_r^* \) and \( \omega_\theta^* \), with respect to the eigenfrequencies \( \omega_r \) and \( \omega_\theta \):

\[
\omega_r^* = \omega_r + \phi_r, \quad \omega_\theta^* = \omega_\theta + \phi_\theta. \tag{87}
\]

These relations can be combined to find

\[
2 \omega_\theta^* - 3 \omega_r^* = 2 \omega_\theta - 3 \omega_r + 2 \phi_\theta - 3 \phi_r = \gamma. \tag{88}
\]

The observed frequencies are in exact 3:2 ratio if (and only if) the time-derivative of the phase function \( \gamma \) vanishes. An immediate implication for the frequencies of stationary oscillations with constant amplitudes is that they stay in exact 3:2 ratio, even if the eigenfrequencies may depart from it. Outside stationary points, it is evident from Fig. 5 that \( \dot{\gamma} = 0 \) represents turning points on libration tracks (those ones, which are encircled by the separatrix curve). Hence, eq. (88) discriminates between librating and circulating trajectories in the \((\gamma, \xi, \gamma)\)-plane. Circulating trajectories span the full range of \( -\pi \leq \gamma < \pi \) and they do not contain any turning point; \( \dot{\gamma} \) remains nonzero and the twin frequencies never cross the exact 3:2 ratio in the region of circulation. On the other hand, there are two points \( \gamma = 0 \) on each librating trajectory. In such state the ratio of observed frequencies slowly fluctuates about 3:2.
4. Conclusions

We have discussed the resonance scheme for high-frequency QPOs via multiple-scales analysis, assuming an axisymmetric conservative system with two degrees of freedom. This approach provides a useful insight into general properties that are common to different conceivable mechanisms driving the oscillations, although it does not address the question how the observed signal is actually formed and modulated. In our scenario, amplitudes and phases of the oscillations are mutually connected and they follow tracks in the phase space with distinct topologies. The particular form was assumed to couple the oscillation modes via the non-spherical terms in the gravitational field of a ring. We consider this to be a toy-model for rather general behaviour, which should replace in any system governed by equations of type (1–2).

We assumed the Newtonian (or the pseudo-Newtonian) description of the central gravitational field with a perturbation by an aligned ring as an example. The adopted form is not essential for general conclusions. In fact, equations (1–2) cover also the nearly-geodesic motion around a Schwarzschild black hole. Compared with the pseudo-Newtonian case, general relativity does not bring qualitatively new features, as long as the system is conservative (additional terms will arise in the expansions, which then translate to slightly different value of the resonance radius and to different duration of time intervals in physical units). A natural question arises whether the gravitational field of a rotating black hole could provide the perturbation required for the internal resonance in a surrounding disc. We considered this possibility, however, it is unlikely that Kerr metric could be itself sufficient: in the weak-field limit non-spherical terms seem to be incapable of creating separatrices in the phase-space sections discussed above, whereas in the full (exact, vacuum) Kerr metric the special mathematical properties of the spacetime ensure the integrability of the geodesic motion, and hence prevent the occurrence of internal resonances. Therefore, the problem of a specific mechanism launching and maintaining the oscillations remains unanswered.

Various options for the generalisation of our scheme could be motivated by papers of other authors who proposed specific models including non-gravitational forces (see Pétri [2005] for a recent exposition of the problem and for references). As a next step towards an astrophysically realistic scheme, one should take dissipative and non-potential forces into account, as well as non-axisymmetric perturbations. These will allow our system to migrate across contours in the phase-plane and to undergo transitions when crossing separatrices. Such additional terms could also supplement the influence of external forcing and initiate the oscillations of the system. The internal resonance would then define the actual frequencies that are excited; this way the strong gravity unmask itself.

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Fig. 6. Different topologies of the phase-space sections are shown in $(\xi, \gamma)$-plane. The system evolution follows the contour lines $\mathcal{F}(\xi, \gamma) = \text{const.}$ Their shape determines a possible range of the frequency change of the QPOs. Separatrix curve (thick line; cases B and C) divides the regions of circulating tracks from the regions of libration. Parameters and notation as in previous figure: A (left; $r = 0.6a$, $E = 0.02$), B (middle; $r = 0.65a$, $E = 0.02$), C (right; $r = 0.55a$, $E = 0.05$).
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