FIXED POINTS OF $p$-TORAL GROUPS ACTING ON PARTITION COMPLEXES

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Abstract. We consider the action of $p$-toral subgroups of $U(n)$ on the unitary partition complex $L_n$. We show that if $H \subseteq U(n)$ has noncontractible fixed points on $L_n$, then the image of $H$ in the projective unitary group $U(n)/S^1$ is an elementary abelian $p$-group.

1. Introduction

Let $n$ denote the set $\{1, \ldots, n\}$ and let $P_n$ denote the nerve of the poset of proper, nontrivial partitions of $n$, ordered by coarsening. In [ADL2], Arone, Dwyer, and Lesh compute the Bredon homology of $P_n$ for appropriately constrained $p$-local Mackey functors on the category of $\Sigma_n$-sets. The calculation is part of a program to obtain a proof of the Whitehead Conjecture and the collapse of the homotopy spectral sequence of the Goodwillie tower of the identity functor for $S^1$ by using the Bousfield-Kan cosimplicial resolution of $S^1$. A key element in the calculation of [ADL2] is understanding which $p$-subgroups $H \subseteq \Sigma_n$ can have noncontractible fixed point sets $(P_n)^H$. It turns out that if $H \subseteq \Sigma_n$ is a $p$-group and $(P_n)^H$ is not contractible, then $H$ is elementary abelian (Proposition 6.6 of [ADL2]).

In this paper, we consider the corresponding question in the unitary context, following analogies set up by Arone and Lesh in [A] and [AL]. Let $L_n$ denote the nerve of the (topological) poset of proper partitions of $\mathbb{C}^n$ into orthogonal subspaces, where “proper” means that we exclude the partition consisting of the single subspace $\mathbb{C}^n$ itself. The action of the unitary group $U(n)$ on $\mathbb{C}^n$ induces an action on $L_n$. The complex $L_n$ with its $U(n)$ action is strongly related to the $bu$-analogues
of symmetric powers of spheres and to the Weiss tower for the functor $V \mapsto BU(V)$ (see [AL] Theorem 9.5 and [A] Theorems 2 and 3).

In moving from finite group theory to compact Lie groups, one replaces the notion of a $p$-group with that of a $p$-toral group, i.e., an extension of a finite $p$-group by a torus. In this paper we study the action of $p$-toral subgroups $H \subseteq U(n)$ on $\mathcal{L}_n$ in order to find out when the fixed point set $(\mathcal{L}_n)^H$ is contractible. We show that the answer is “most of the time”: the condition that $(\mathcal{L}_n)^H$ is not contractible puts considerable group-theoretic restrictions on the $p$-toral subgroup $H$.

Recall that the center of $U(n)$ is $S^1$, and that the projective unitary group is defined as $PU(n) = U(n)/S^1$. A subgroup $H$ of $U(n)$ is called projective elementary abelian if its image in $PU(n)$ is elementary abelian. Our main theorem is the following.

**Theorem 1.1.** Suppose that $H$ is a $p$-toral subgroup of $U(n)$ and $(\mathcal{L}_n)^H$ is not contractible. Then $H$ is a projective elementary abelian $p$-subgroup of $U(n)$.

Proposition 2.2 shows that Theorem 1.1 is group-theoretically sharp, in that we produce a projective elementary abelian 2-subgroup of $U(2)$ whose fixed points on $\mathcal{L}_2$ are not contractible. However, the same subgroup, embedded in $U(3)$, has contractible fixed points on $\mathcal{L}_3$. (Remark 2.4). In future work [BJLSW], we plan to establish further restrictions on the projective abelian $p$-subgroups of $U(n)$ that can have noncontractible fixed point sets by considering their representation theory in greater detail.

The organization of the paper is as follows. Before delving into $\mathcal{L}_n$ in general, in Section 2 we study the first non-trivial example ($n = 2$) directly and in detail. In Section 3 we study conditions on $H$ that imply contractibility of $(\mathcal{L}_n)^H$, and in Section 4 we use these conditions to prove Theorem 1.1.

**Notation and Terminology**

We generally do not distinguish notation for a category and its nerve, and we trust to context to make clear which is being discussed.

By “subgroup,” we always mean a closed subgroup. If $G$ is a group, we write $Z(G)$ for the center of $G$, and $Z_p(G)$ for the elements of $Z(G)$ of order $p$.

We write $\mathcal{L}_n$ for the (topological) category of proper partitions of $\mathbb{C}^n$, where morphisms are given by coarsenings. An object $\lambda$ in $\mathcal{L}_n$ is a partition of $\mathbb{C}^n$ into nonzero orthogonal subspaces $v_1, \ldots, v_m$, and by *proper* we mean that $m > 1$. We often think of $\lambda$ as given by
the equivalence classes of an equivalence relation \( \sim_\lambda \), and accordingly speak of the set of classes \( \text{cl}(\lambda) := \{v_1, \ldots, v_m\} \).

The action of the unitary group \( U(n) \) on \( \mathbb{C}^n \) induces an action of \( U(n) \) on \( \mathbb{L}_n \), and if \( H \) is a subgroup of \( U(n) \), we write \( (\mathbb{L}_n)^H \) for fixed points of the action of \( H \) on \( \mathbb{L}_n \). We say that an object \( \lambda \) in \( (\mathbb{L}_n)^H \) is weakly fixed by \( H \), since \( \text{cl}(\lambda) \) is stabilized as a set, but \( H \) may permute the elements of \( \text{cl}(\lambda) \) nontrivially.

We say \( \lambda \) is strongly fixed by \( H \), or strongly \( H \)-fixed, if each individual element of \( \text{cl}(\lambda) \) is stabilized by \( H \), that is, if the action of \( H \) on \( \text{cl}(\lambda) \) is trivial. We denote the subcategory of proper strongly \( H \)-fixed partitions of \( \mathbb{C}^n \) by \( (\mathbb{L}_n)_{\text{st}}^H \).

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2. Low degree examples

To provide the reader with some intuition about the spaces \( \mathbb{L}_n \), in this section we work out the homeomorphism type of \( \mathbb{L}_2 \), which is the smallest interesting example of a unitary partition complex. (Observe that \( \mathbb{L}_1 \) is empty, since \( \mathbb{C} \) has no proper partition.)

To begin, we observe that a proper partition of \( \mathbb{C}^2 \) can only be a partition into two orthogonal lines. Since there are no proper coarsenings or refinements of such a partition, the poset category of partitions of \( \mathbb{C}^2 \) has no nonidentity morphisms, and \( \mathbb{L}_2 \) is simply the space of objects.

Proposition 2.1. The space \( \mathbb{L}_2 \) is homeomorphic to \( \mathbb{R}P^2 \).

Proof. A partition of \( \mathbb{C}^2 \) is an unordered pair consisting of a line in \( \mathbb{C}^2 \) and its orthogonal complement. The space of lines in \( \mathbb{C}^2 \) is the projective space \( \mathbb{C}P^1 \). Because the pair is unordered, \( \mathbb{L}_2 \) is the quotient of \( \mathbb{C}P^1 \) by the action of the involution that interchanges a line and its orthogonal complement.

More explicitly, note that the line spanned by \((0, 1)\) has orthogonal complement spanned by \((1, 0)\) (a special case), and in general the line in \( \mathbb{C}^2 \) spanned by \((1, z)\) with \( z \in \mathbb{C}\{0\} \) has orthogonal complement spanned by \((1, -1/z)\). Thus \( \mathbb{L}_2 \) is homeomorphic to the quotient of \( S^2 \cong \mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\} \) by the involution \( z \mapsto -1/z \) (and \( 0 \leftrightarrow \infty \)). The involution exchanges points in the region \( \|z\| > 1 \) with those in the region \( \|z\| < 1 \), so we only need to consider the quotient of the unit
disk $\|z\| \leq 1$ by the action on the boundary circle. When $\|z\| = 1$, we can write $z = e^{i\varphi}$ and $-1/\overline{z} = -e^{i\varphi}$ for some $\varphi \in \mathbb{R}$, whence the transformation is the antipodal map on the boundary of the unit disk. We conclude that $\mathcal{L}_2$ is homeomorphic to the quotient space obtained from the disk $\|z\| \leq 1$ by identifying antipodal points on the boundary circle, namely $\mathbb{R}P^2$.

We now consider a low-dimensional fixed point set. Let $\tau \in U(2)$ be represented by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, generating $\mathbb{Z}/2 \subseteq U(2)$. The following proposition shows that Theorem 1.1 is group-theoretically sharp, in that there exist projective elementary abelian $p$-groups with noncontractible fixed point sets. However, other projective elementary abelian $p$-groups have contractible fixed point sets (see Remark 2.4). In [BJLSW], we will establish further restrictions of a representation-theoretic nature on $p$-toral subgroups of $U(n)$ that can have noncontractible fixed point sets on $\mathcal{L}_n$.

**Proposition 2.2.** The fixed-point space $(\mathcal{L}_2)^\tau$ is homeomorphic to the space $S^1 \sqcup \ast$.

To prove Proposition 2.2, we set up a little notation. Let $L_z$ denote the line in $\mathbb{C}^2$ spanned by $(1, z)$, and let $L_\infty$ denote the line spanned by $(0, 1)$. As a set, $\mathcal{L}_2$ consists of pairs $\{L_z, L_{-1/\overline{z}}\}$ where $z \in \mathbb{C}\{0\}$, together with one extra point $\{L_0, L_\infty\}$. Recall that $\tau \in U(2)$ is the linear transformation that exchanges the standard basis vectors of $\mathbb{C}^2$. If $z \in \mathbb{C}\{0\}$, then $\tau(L_z) = L_{1/z}$ and $\tau$ exchanges $L_0$ and $L_\infty$.

**Lemma 2.3.** As a set, the fixed points of the action of $\tau$ on $\mathcal{L}_2$ consist of the point $\{L_1, L_{-1}\}$, the point $\{L_0, L_\infty\}$, and the set of points $\{L_{ir}, L_{-i/r}\}$ where $r \in \mathbb{R}\{0\}$.

**Proof.** Direct computation establishes that the points of $\mathcal{L}_2$ in the statement of the lemma are in fact fixed by $\tau$.

Points in $\mathcal{L}_2$ besides $\{L_0, L_\infty\}$ have the form $\{L_z, L_{-1/\overline{z}}\}$ where $z \in \mathbb{C}\{0\}$. If such a point is fixed by $\tau$, then either each line in the pair is fixed by $\tau$ (the partition is strongly fixed), or else the lines are interchanged by $\tau$ (the partition is only weakly fixed). In the first case, since $\tau(L_z) = L_{1/z}$, we must have $z = 1/z$, so $z = \pm 1$, corresponding to the point $\{L_1, L_{-1}\}$. In the second case, we must have $1/z = -1/\overline{z}$, meaning $\overline{z} = -z$, so $z$ is purely imaginary, say $z = ir$ for $r \in \mathbb{R}\{0\}$. Thus $\{L_z, L_{-1/\overline{z}}\}$ has the form $\{L_{ir}, L_{-i/r}\}$. □
**Proof of Proposition 2.2.** To determine the fixed point set of the action of $\tau$ on $L_2$ as a topological space, and not just as a set (as in Lemma 2.3), we recall from the proof of Proposition 2.1 that $L_2$ can be identified as the quotient space of the disk $\|z\| \leq 1$ in $\mathbb{C}^1$ by the antipodal action on the boundary circle $\|z\| = 1$. According to Lemma 2.3, the fixed points of $\tau$ correspond to the points of the unit disk that lie on the imaginary axis, $\{ir \mid r \in [-1, 1] \subseteq \mathbb{R}\}$, together with the real points 1 and $-1$. The points 1 and $-1$ are identified by passing to $L_2$, as are the points $i$ and $-i$, which gives $S^1 \sqcup *$ as the fixed point set of $\tau$ acting on $L_2$.

\[\square\]

**Remark 2.4.** Considering the embedding of $\tau$ in $U(3)$ as the transformation $\tau' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ gives a rather different result for the fixed points of $\tau'$ on $L_3$. Objects of $L_3$ consist either of three orthogonal complex lines, or of one two-dimensional subspace and one line. The fixed points $(L_3)^{\tau'}$ have a terminal object, consisting of the two-dimensional subspace generated by the first two standard basis vectors of $\mathbb{C}^3$ and the line generated by the third. Therefore the nerve of $(L_3)^{\tau'}$ is contractible.

3. CONDITIONS FOR CONTRACTIBILITY OF THE FIXED POINT SETS

In this section, we turn to $L_n$ for a general $n$ and establish preliminary criteria for the action of a subgroup $H \subseteq U(n)$ on $L_n$ to have a contractible fixed point set on $L_n$.

We recall the following terminology from [GW98, Prop. 3.1.6]. Let $V$ be a complex representation of a group $G$; then $V$ splits into irreducibles as

$$V \cong \bigoplus_k V_k^{\oplus m_k}$$

where for different $k$’s, the corresponding irreducible representations $V_k$ are non-isomorphic. This decomposition is not canonical; however, if we group all the isomorphic irreducibles together into $W_k = V_k^{\oplus m_k}$, the decomposition

$$V \cong \bigoplus_k W_k$$

is canonical. It is called the *isotypic* decomposition of $V$, and we call the subspaces $W_k$ the *isotypic components* of $V$. If $V$ has only one isotypic component, we say it is an *isotypic representation*; otherwise, we say it is *polytypic*. 
Often we are in a situation where a representation $V$ of $U(n)$ is fixed, and we will restrict it to subgroups $H$ of $U(n)$. In this case we say that $H$ acts isotypically/polytypically or that $H$ is isotypic/polytypic if $V$ is such as an $H$-representation. When $V$ is the standard representation of $U(n)$ on $\mathbb{C}^n$ we denote it simply by $\mathbb{C}^n$.

**Definition 3.1.** Let $(\mathcal{L}_n)^H_{iso}$ denote strongly $H$-fixed partitions of $\mathbb{C}^n$ such that $H$ acts isotypically on each block of the partition. We call such partitions isotypic.

**Proposition 3.2.** Let $H \subseteq U(n)$ be connected and polytypic. Then $(\mathcal{L}_n)^H$ is contractible.

**Proof.** The action of $H$ on $\text{cl}(\lambda)$ for $\lambda \in (\mathcal{L}_n)^H$ defines a continuous map

$$H \to \Sigma_{\text{cl}(H)}$$

from $H$ to the symmetric group on $\text{cl}(\lambda)$. Since $H$ is connected and $\Sigma_{\text{cl}(H)}$ is discrete, this map is trivial. Thus any weakly $H$-fixed partition must be strongly fixed, and it is sufficient to prove that $(\mathcal{L}_n)^H_{st}$ is contractible.

A strongly $H$-fixed partition is a decomposition of $\mathbb{C}^n$ into representations of $H$. Each of these representations can in turn be decomposed into its isotypic components, defining a functor

$$\phi : (\mathcal{L}_n)^H_{st} \to (\mathcal{L}_n)^H_{iso},$$

together with a natural transformation from the postcomposition of $\phi$ with the inclusion $(\mathcal{L}_n)^H_{iso} \hookrightarrow (\mathcal{L}_n)^H$ to the identity functor on $(\mathcal{L}_n)^H_{st}$. It follows that $(\mathcal{L}_n)^H_{st}$ is homotopy equivalent to $(\mathcal{L}_n)^H_{iso}$. However, $H$ is polytypic, and the decomposition of $\mathbb{C}^n$ into isotypic components is a proper partition of $\mathbb{C}^n$. This partition is a terminal object of $(\mathcal{L}_n)^H_{iso}$, whence $(\mathcal{L}_n)^H_{iso}$ is contractible. \qed

We now give restrictions on the behavior of the action of nonconnected $p$-toral groups $H$ on $\mathcal{L}_n$. To do so, we need to pick out certain elements of order $p$.

**Definition 3.3.** For a topological group $H$, let $H/p$ denote the quotient of $H$ by its normal subgroup generated by commutators, $p$-th powers and the connected component of the identity.

Note that $H/p$ is an elementary abelian $p$-group, and the map $H \to H/p$ is initial among homomorphisms from $H$ to finite elementary abelian $p$-groups. Recall that $Z(H)$ denotes the center of $H$ and $Z_p(H)$ denotes elements of $Z(H)$ that have order $p$. Precomposing
map \( H \to H/p \) with the inclusion of \( Z_p(H) \to H/p \) gives a canonical map

\[ Z_p(H) \to H/p \]

Lemma 3.4. Let \( H \subseteq U(n) \) be a subgroup, and suppose there exists \( V \cong \mathbb{Z}/p \) such that

\[ V \subseteq \ker (Z_p(H) \to H/p) \]

Then \( V \) does not act transitively on \( \text{cl}(\lambda) \) for any weakly \( H \)-fixed partition \( \lambda \) of \( \mathbb{C}^n \).

Proof. Let \( \lambda \) be in \( (L_n)^H \). Since \( V \subseteq H \), it follows that \( V \) permutes the vector spaces comprising \( \lambda \). We claim this permutation is not transitive. Assume the contrary. As there is more than one vector space comprising \( \lambda \), it follows that \( | \text{cl}(\lambda) | = p \). We choose a bijection \( \text{cl}(\lambda) \cong \{1, 2, \ldots, p\} \) such that the image of \( V \) under the resulting map \( H \to \Sigma_p \) is generated by the \( p \)-cycle \( (1, 2, \ldots, p) \). Since \( V \) is in the center of \( H \), the image of the map \( H \to \Sigma_p \) must be contained in the centralizer of the permutation \( (1, 2, \ldots, p) \). This centralizer is the subgroup generated by \( (1, 2, \ldots, p) \). Thus we obtain a map

\[ H \to \langle (1, 2, \ldots, p) \rangle \subseteq \Sigma_p \]

with \( V \) mapping non-trivially. But this gives a map \( H \to \mathbb{Z}/p \) and so factors through \( H \to H/p \), contradicting the assumption \( V \subseteq \ker (Z_p(H) \to H/p) \).

We conclude that \( V \) does not act transitively on the vector spaces comprising \( \lambda \), thus proving the lemma. \( \square \)

Recall that any partition \( \lambda \) in \( L_n \) corresponds to an equivalence relation on points of \( \mathbb{C}^n \), and we write \( x \sim_\lambda y \) if \( x \) and \( y \) are in the same subspace of the partition \( \lambda \). We now define another equivalence relation which incorporates the group action.

Definition 3.5. Let \( J \subseteq U(n) \) be a subgroup of the unitary group, and let \( \lambda \) be an element of \( L_n \) corresponding to the relation \( \sim_\lambda \). Define \( (\lambda/J) \) to be the partition, called the \( J \)-orbit of \( \lambda \), of \( \mathbb{C}^n \) associated to the equivalence relation generated by identifying \( x \) and \( y \) when there is an element \( j \) of \( J \) such that \( x \sim_\lambda jy \).

Lemma 3.6. Let \( J \) be a normal subgroup of \( H \), \( \lambda \in (L_n)^H \), and \( (\lambda/J) \) the \( J \)-orbit of \( \lambda \). Assume that \( (\lambda/J) \) is not the trivial partition of \( \mathbb{C}^n \). Then \( (\lambda/J) \in (L_n)^H \).

Proof. We want to show that \( (\lambda/J) \) is fixed by \( H \). Suppose that \( x \sim_\lambda (\lambda/J) y \), and let \( h \in H \). By definition of \( (\lambda/J) \), there exists some \( j \in J \) such that \( x \sim_\lambda jy \). Since \( \lambda \) is fixed by \( H \), we have \( hx \sim_\lambda hjy \).
Since $J$ is normal in $H$, the element $hjh^{-1}$ is in $J$. The equivalence $hx \sim_{\lambda} (hjh^{-1})hy$ therefore shows that $hx \sim_{(\lambda/J)} hy$. \hfill \Box

We now bring these results together to give conditions under which $(L_n)^H$ is contractible.

**Theorem 3.7.** Let $H \subset U(n)$, and let $J$ be a normal subgroup of $H$ such that for every $\lambda \in (L_n)^H$, $(\lambda/J)$ is not the trivial partition of $\mathbb{C}^n$. If $J$ is polytypic, then

$$(L_n)^H \simeq (L_n)^H \cap (L_n)_\text{iso}^J$$

and $(L_n)^H$ is contractible.

**Proof.** Under our assumptions, the assignment $\lambda \mapsto (\lambda/J)$ defines a functor

$$(L_n)^H \to (L_n)^H \cap (L_n)_\text{st}^J.$$  

Since $(\lambda/J)$ is a coarsening of $\lambda$, there is a natural transformation from the identity to the composite

$$(L_n)^H \to (L_n)^H \cap (L_n)_\text{st}^J \leftarrow (L_n)^H,$$

showing that the induced map on classifying spaces is homotopic to the identity. Since for $\lambda$ in $(L_n)^H \cap (L_n)_\text{st}^J$, we know $(\lambda/J) = \lambda$, it follows that the map on classifying spaces induced by the functor $\lambda \mapsto (\lambda/J)$ is a deformation retraction, giving a homotopy equivalence

$$(L_n)^H \simeq (L_n)^H \cap (L_n)_\text{st}^J.$$  

Next, we show that $(L_n)^H \cap (L_n)_\text{st}^J$ is homotopy equivalent to $(L_n)^H \cap (L_n)_\text{iso}^J$. Suppose that $\lambda \in (L_n)^H \cap (L_n)_\text{st}^J$, and construct a new partition $\tilde{\lambda}$ by taking each $v \in \text{cl}(\lambda)$, and refining it into its $J$-isotypic components. We claim that $\tilde{\lambda}$ is weakly fixed by $H$. Indeed, suppose the refinement $\tilde{\lambda}$ of $\lambda$ has $\text{cl}(\tilde{\lambda}) = \{v_i\}_{i \in I}$ with $\oplus_i v_i = v$ and each $v_i$ irreducible. Let $h \in H$ and $j \in J$ be arbitrary, and let $x$ be an element of $v_{ik} \in \text{cl}(\tilde{\lambda})$. Then

$$jhx = hh^{-1}jhx = hj'x,$$

for some $j'$ in $J$. But $j'$ is an element of $v_{ik}$. Thus we conclude that $hv_{ik}$ is a representation of $J$.

Since $v_{ik}$ is a $J$-representation, there is a corresponding map $\rho_{ik} : J \to \text{GL}(v_{ik})$ from $J$ to the linear automorphisms of $v_{ik}$. Define $\rho_{ik}^h : J \to \text{GL}(v_{ik})$ by $\rho_{ik}^h(j) = \rho_{ik}(h^{-1}jh)$. Since

$$jhx = h(h^{-1}jh)x = h\rho_{ik}^h(j)x,$$
the map \( x \mapsto hx \) defines an isomorphism from the representation determined by \( \rho_{i_k}^h \) to \( hv_{i_k} \). Since \( \rho_{i_k} \) is irreducible, so is \( \rho_{i_k}^h \). Thus \( hv_{i_k} \) is irreducible. Moreover, if \( v_{i_k} \) is isomorphic to \( v_{i_{k'}} \), then the representations corresponding to \( \rho_{i_k}^h \) and \( \rho_{i_{k'}}^h \) are isomorphic as well, whence \( hv_{i_k} \) is isomorphic to \( hv_{i_{k'}} \). It follows that \( hv_i = \oplus_k hv_{i_k} \) is \( J \)-isotypic.

Since \( h \) fixes \( \lambda \), we have that \( h \) permutes the elements of \( \text{cl}(\lambda) \). Let \( W \in \text{cl}(\lambda) \), and let \( hW \in \text{cl}(\lambda) \) denote its image. The previous paragraph shows that \( h \) maps the isotypic components of \( W \) to the isotypic components of \( hW \). It follows that \( H \) fixes the refinement of \( \lambda \) into its \( J \)-isotypic components as claimed.

Hence, we have a functor

\[
(\mathcal{L}_n)^H \cap (\mathcal{L}_n)^J_{\text{st}} \rightarrow (\mathcal{L}_n)^H \cap (\mathcal{L}_n)^J_{\text{iso}}.
\]

The induced map on classifying spaces is a deformation retraction as above, because the image of a partition under this functor is always a refinement of the original partition. This establishes the first part of the theorem.

It remains to show that \((\mathcal{L}_n)^H \cap (\mathcal{L}_n)^J_{\text{iso}}\) is contractible.

Since we are assuming that \( J \) is polytypic, the decomposition \( \mu \) of \( \mathbb{C}^n \) into the isotypic components of \( J \) is an element of \( \mathcal{L}_n \) and \( \mu \) is terminal in \((\mathcal{L}_n)^J_{\text{iso}}\). We would like to prove that \( \mu \in (\mathcal{L}_n)^H \).

We showed above that for any \( \lambda \), the refinement of \( \lambda \) into its \( J \)-isotypic components was weakly fixed by \( H \). This proof did not use that \( \lambda \) was a proper partition, whence it also applies to \( \lambda = \mathbb{C}^n \), showing that \( \mu \in (\mathcal{L}_n)^H \).

Thus \( \mu \) is a terminal element in \((\mathcal{L}_n)^H \cap (\mathcal{L}_n)^J_{\text{iso}}\) showing that this category has contractible classifying space, as desired. \( \Box \)

Combining the Theorem 3.7 with Lemma 3.4, we obtain the following result.

**Corollary 3.8.** Let \( H \subseteq U(n) \) be a subgroup such that there exists

\[
V \subseteq \ker (Z_p(H) \rightarrow H/p)
\]

with \( V \cong \mathbb{Z}/p \) polytypic. Then \((\mathcal{L}_n)^H\) is contractible.

**Proof.** By Lemma 3.4 we know that \( V \) does not act transitively on the vector spaces comprising any \( \lambda \) in \((\mathcal{L}_n)^H\). Thus \((\lambda/V)\) is not the trivial partition of \( \mathbb{C}^n \) and it is in \((\mathcal{L}_n)^H\) by Lemma 3.6.

By Theorem 3.7 it follows that \((\mathcal{L}_n)^H\) is contractible. \( \Box \)

In the remainder of this section, we consider subgroups of the projective unitary group \( PU(n) = U(n)/S^1 \).
**Lemma 3.9.** Let \( \mathbb{Z}/p \) be a subgroup of \( PU(n) \) and let \( I \) be its inverse image in \( U(n) \). Then \( I \) is polytypic. In fact, a generator of \( \mathbb{Z}/p \) lifts to an element of order \( p \) in \( U(n) \) and \( I \cong S^1 \times \mathbb{Z}/p \).

**Proof.** Let \( A \in U(n) \) be such that its image \( \overline{A} \in PU(n) \) generates \( \mathbb{Z}/p \). Then \( A^p \) is an element of the central \( S^1 \subseteq U(n) \). Thus \( A^p \) is a diagonal matrix and all its entries are equal. Let \( \alpha \) be some \( p \)th root of the diagonal entry of \( A^p \). Define \( B = \alpha^{-1}A \), so \( B^p = \text{Id} \) and \( \overline{B} = \overline{A} \). To show that \( I \cong S^1 \times \mathbb{Z}/p \), consider 
\[
1 \rightarrow S^1 \rightarrow I \rightarrow \mathbb{Z}/p \rightarrow 1,
\]
which is split by the map \( \mathbb{Z}/p \rightarrow I \) given by \( \overline{A}^i \mapsto B^i \). Hence, \( I \) is the semi-direct product of \( S^1 \) and \( \mathbb{Z}/p \). Therefore, the homomorphism \( S^1 \times \mathbb{Z}/p \rightarrow I \) given by \( (\rho, \overline{A}) \mapsto \rho B \) is an isomorphism.

It remains to show that \( I \) is polytypic. Note that \( B \in U(n) \) is diagonalizable, and any element of \( I \) is of the form \( \rho B \). Thus, if \( I \) were isotypic, \( B \) would have only one eigenvalue, and would therefore be of the form \( B = \zeta \text{Id} \in S^1 \), contradicting the fact that the image of \( B \) in \( PU(n) \) is nontrivial. Hence, \( I \) is polytypic. \( \square \)

**Proposition 3.10.** Let \( H \) be a subgroup of \( U(n) \) with \( \overline{H} \) its projection to \( PU(n) \). If there exists \( \mathbb{Z}/p \) contained in the center of \( \overline{H} \) such that \( \mathbb{Z}/p \subseteq \ker(\overline{H} \rightarrow \overline{H}/p) \), then \( (\mathcal{L}_n)^H \) is contractible.

**Proof.** By Lemma 3.9, the inverse image \( I \subseteq U(n) \) of \( \mathbb{Z}/p \) is polytypic; by Lemma 3.4, \( I \) does not act transitively on \( \text{cl}(\lambda) \) for any \( \lambda \in (\mathcal{L}_n)^H \). It follows that \( (\lambda/I) \) is non-trivial. Hence \( I \) and \( H \) satisfy the conditions of Theorem 3.7, so \( (\mathcal{L}_n)^H \) is contractible. \( \square \)

### 4. Proof of the main theorem

We now restrict our attention to \( p \)-toral subgroups of \( U(n) \). After establishing some preliminary results, we combine them with the results of the previous section to prove Theorem 1.1.

**Definition 4.1.** A \( p \)-toral group \( H \) is an extension of a finite \( p \)-group by a torus. In other words, there exists a short exact sequence
\[
1 \rightarrow T \rightarrow H \rightarrow V \rightarrow 1,
\]
where \( T \) is a torus and \( V \) is a finite \( p \)-group.

**Lemma 4.2.** If \( H \) is a \( p \)-toral subgroup of \( U(n) \), then its image \( \overline{H} \) in \( PU(n) \) is also \( p \)-toral.
Proof. Since $H$ is $p$-toral, we have a short exact sequence

$$1 \rightarrow T \rightarrow H \rightarrow V \rightarrow 1,$$

where $T$ is a torus and $V$ is a $p$-group. Let $Z \cong S^1 \subseteq U(n)$ be the center of $U(n)$. We obtain a morphism of short exact sequences

$$
\begin{array}{cccccc}
1 & \longrightarrow & T & \longrightarrow & H & \longrightarrow & V & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & T/(Z \cap T) & \longrightarrow & H/(Z \cap H) & \longrightarrow & Q & \longrightarrow & 1.
\end{array}
$$

The middle arrow is surjective, so the Snake Lemma tells us that $V \rightarrow Q$ is surjective and $Q$ is a $p$-group. Further, $T/(Z \cap T)$ is a compact connected abelian Lie group, and so must be a torus. Since $\overline{\mathcal{H}} = H/(Z \cap H)$, the lemma follows.

Lemma 4.3. Let $P$ be an infinite $p$-toral group, and let $P_0$ denote the identity component of $P$. Then there exists $\mathbb{Z}/p \subseteq \mathbb{Z}/p(H) \cap P_0$.

Proof. Let $(P_0)_p$ denote the group of elements of $P_0$ that have order $p$. The conjugation action of $P$ on itself preserves $(P_0)_p$, while $P_0$ acts trivially on $(P_0)_p$, so we get an action of $P/P_0$ on $(P_0)_p$.

Since $P$ is infinite, $P_0$ is non-trivial, so $\text{rank}(P_0) > 0$. Thus the set $(P_0)_p$ has $p^{\text{rank}(P_0)} > 1$ elements. The $p$-group $P/P_0$ fixes the identity element in $(P_0)_p$, so by the orbit decomposition, there exist at least $p$ elements fixed by $P/P_0$. Choose one which is not the identity; it generates the desired $\mathbb{Z}/p$.

Proposition 4.4. Let $H$ be a $p$-toral subgroup of $U(n)$ such that $S^1 = Z(U(n)) \subseteq H$. If $\overline{\mathcal{H}}$ is infinite, then $(\mathcal{L}_n)^H$ is contractible.

Proof. By Lemmas 4.2 and 4.3 there exists $\mathbb{Z}/p \subseteq \mathbb{Z}/p(\overline{\mathcal{H}}) \cap \overline{\mathcal{H}}_0$. Because $\overline{\mathcal{H}}_0$ is $p$-divisible, $\overline{\mathcal{H}}_0/p$ is trivial, so we have $\mathbb{Z}/p \subseteq \ker(\mathbb{Z}/p(\overline{\mathcal{H}}) \rightarrow \overline{\mathcal{H}}/p)$, and the result follows from Proposition 3.10.

We conclude with the proof of our main theorem.

Proof of Theorem 1.1. Let $H$ be a $p$-toral subgroup of $U(n)$. We want to prove that if $\overline{\mathcal{H}}$ is not an elementary abelian $p$-group, then $(\mathcal{L}_n)^H$ is contractible. If $\overline{\mathcal{H}}$ is infinite, then $(\mathcal{L}_n)^H$ is contractible by Proposition 4.4. So we may assume that $\overline{\mathcal{H}}$ is a finite $p$-group that is not elementary abelian. Then by Lemma 6.5 of [ADL2] there exists $\mathbb{Z}/p \subseteq \ker(\mathbb{Z}/p(\overline{\mathcal{H}}) \rightarrow \overline{\mathcal{H}}/p)$, and the theorem follows by applying Proposition 3.10.
Remark 4.5. Proposition 3.2 and Corollary 3.8 are not used in the proof of Theorem 1.1. We include them for their own interest.

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