Stability integral manifold of the differential equations system in critical case

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Abstract. We consider the stability problem for non-zero integral manifolds of a non-linear finite-dimensional system of ordinary differential equations, where the right-hand side is a periodic vector function with respect to an independent variable and contains a parameter. It is assumed that the studied system has a trivial integral manifold for all values of the parameter, and the corresponding linear subsystem does not have the property of exponential dichotomy. The aim of the paper is to find sufficient conditions of existence in a neighborhood of the system equilibrium state for stable non-zero integral manifold to be lower dimension than the original phase space. For this purpose, based on the classical method of Lyapunov functions and the transforming matrix method operators are constructed, allowing solve the task by finding their fixed points. Due to the specific nature of the considered systems Lyapunov functions method is modified.

1. Introduction

Let the system of ordinary differential equations

\[ \dot{y} = f(v, y, t) \]  

for any \( v, t \) have equilibrium state \( y = 0 \) and in the field of \( \Lambda \) its solutions exist and are unique.

Here and further \( f, v, y \) \((n+m)-\) vectors, \( \dot{y} = \frac{dy}{dt} \), \( f(v, y, t + T) = f(v, y, t) \), \( \Lambda = \Lambda_1 \times \Lambda_2 \), \( \Lambda_1 = \{ y : \|y\| \leq \Delta_1 \} \subset R^{n+m} \), \( \Lambda_2 = \{ v : \|v\| \leq \Delta_2 \} \subset R^m \), \( \Delta_1, \Delta_2 \) \( - \) constants, \( R^d \) standard Euclidean space of dimension \( d \) \( \| \| \) \( - \) the Euclidean norm in \( R^d \).

Suppose that the change of variables

\[ y = \Gamma(\varepsilon, x, \phi, t), \quad v = \xi(\varepsilon) \]  

brings the system (1.1) to the form

\[ \begin{cases} \dot{x} = X(\varepsilon, x, \phi, t) \cdot x, \\ \dot{\phi} = \Phi(\varepsilon, x, \phi, t), \end{cases} \]  

(1.3)
where $\Gamma, \xi, \varepsilon - (n + m)$ - vectors, $X - (n \times n)$ - matrix, $x - n$ - vector, $\phi, \Phi - m$ - vectors, $\Gamma(\varepsilon, x, \phi, t) \neq 0$ at $x \neq 0$, $\Gamma(\varepsilon, 0, \phi, t) \equiv 0$. In addition, we assume that the system

$$\begin{cases}
\dot{x} = X(0, 0, \phi, t) \cdot x,
\phi = \Phi(0, 0, \phi, t),
\end{cases}$$

(1.4)

has $m$ - parametric family of non-zero $kT$ - periodic solutions $x = \tilde{x}(\phi_0, t), \phi = \tilde{\phi}(\phi_0, t)$.

We say that the system (1.1) has an $n$ - dimensional nontrivial periodic integral manifold $\psi(\phi_0, t)$, if for all $\phi_0$ exists such value $v_0 = \xi(\varepsilon_0)$ of parameter $\nu$, at which $\Gamma(\varepsilon_0, \psi(\phi_0, t), \phi^\nu(\phi_0, t), t) \equiv f(\xi(\varepsilon_0), \Gamma(\varepsilon_0, \psi(\phi_0, t), \phi^\nu(\phi_0, t), t), t)$, moreover $\psi(\phi_0, t)$ does not become zero for any values $\phi_0$ and $t$, being $\omega$ - periodic in components $m$ - vector $\phi_0$, $kT$ - periodic in $t$, where $k$ - natural number, $\omega = \text{col}(\omega_1, \omega_2, \ldots, \omega_m)$, $\phi = \phi^\nu(\phi_0, t)$ defines an integral curve on the manifold.

The problem of stable nontrivial periodic integral manifold existence of the system (1.1) near its equilibrium state is solved in this paper.

In applied research the invariant surface having a smaller dimension makes it possible to simplify these models, and also to study their typical properties and obtain additional information about the state of the studied systems. This is especially true for stable manifolds that "attract" trajectories over time. Therefore, the study of integral manifolds, the behavior of trajectories on them and near them, helps to solve many problems in the theory of nonlinear oscillations, dynamics, and problems of control theory, fluid mechanics etc. (see, for example [1, pp 6–9]).

However, the diversity of differential equations systems arising in applied research creates difficulties in obtaining general effective methods for investigation of integral manifolds (especially in critical cases), which determines the relevance of this work.

The most productive methods of integral manifolds study should include the method of a small parameter, point mapping, averaging, and the method of integral manifolds.

The method of integral manifolds developed by N. N. Bogolyubov, Yu. A. Mitropolsky and A. M. Samoilenko [2, 3, 4, 5], consists in constructing the Green's function and is successfully used for many systems of the form (1.3) (see, for example [6, 7, 8, 9]). However, in our case this approach cannot be implemented because the system (1.3) with all the parameter values has zero integral manifolds $x = 0$ and the system (1.4) - $m$ - parametric family of periodic solutions. These conditions can be overcome only by finding the solution to the auxiliary vector (bifurcation) equation and the transition in its neighborhood [10, 11].

The results presented in this paper are obtained on the basis of the transforming matrix method modification proposed in [1] and Lyapunov function method, application of which allowed us to obtain new sufficient conditions for the existence of a stable local integral manifolds of the system (1.1).

2. Basic symbols and definitions

Let $F(\phi, t) \in \Omega_1, \varepsilon(\phi) \in \Omega_2$ - vector-functions $\omega$ - periodic in components of the vector $\phi$ restricted respectively by numbers $\delta_{10}$ and $\delta_{20}$, satisfying the Lipchitz condition:

$$\|F(\phi, t_1) - F(\phi, t_2)\| \leq \delta_{10} \|\phi_1 - \phi_2\| + \delta_{12} |t_1 - t_2|,$$

(2.1)

$$\|\varepsilon(\phi) - \varepsilon(\phi)_1\| \leq \delta_{20} \|\phi - \phi_2\|,$$

(2.2)
having, respectively, dimension \( n \) and \( l \) \((0<l \leq n+m)\), \( F(\phi,t) = kT \)-periodical in \( t \),

\[
\|F(\phi,t)\| = \left[ \sum_{i=1}^{n} \sup_{\phi \in [0,1]} |F_i(\phi,t)| \right]^{1/2}, \quad \|e(\phi)\| = \left[ \sum_{i=1}^{l} \sup_{\phi \in [0,1]} |e_i(\phi)| \right]^{1/2}.
\]

Note that if we introduce the specified norm for sets then they become convex compacts sets \([1, p 15]\).

To solve the differential equation

\[
\dot{\phi} = \Phi(e(\phi), F(\phi_0, t), \phi, t),
\]

satisfying the initial data \( \phi(0) = \phi_0 \), we take the notation \( \phi^\varepsilon \). Suppose, in addition, \( Y^\varepsilon_{\phi_0}(t) \) – matizant of the equation

\[
\dot{x} = X(e(\phi), F(\phi_0, t), \phi^\varepsilon, t) \cdot x.
\]

Here and further \( F(\phi_0, t) \in \Omega_k \), \( n + m - l \) values of vector components \( e \) taken equal 0, but instead of remaining \( l \) values in the equations of the system (1.3) elements of the function are substituted \( e(\phi_0) \in \Omega_k \).

**Definition 1.** Nonsingular functional \( n \times n \)-matrix \( Q^\varepsilon_{\phi_0} \) with constant determinant, continuous in all its variables and \( \omega \)-periodic in components of the vector \( \phi_0 \), will be called the transforming matrix of the system (1.3), if the matrix

\[
(Y^\varepsilon_{\phi_0}(t) - I_n) \cdot Q^\varepsilon_{\phi_0}
\]

has the column \( q^\varepsilon_{\phi_0} \) that are not traded identically zero. Here \( I_n \) – is the unit \( n \times n \)-matrix.

Denote \( X = \{ x : \|x\| \leq \delta_1 \} \subset \mathbb{R}^n \), \( E = \{ e : \|e\| \leq \delta_2 \} \subset \mathbb{R}^{n+m} \), \( \Theta = \{ \phi : \phi \in [0, \omega] \} \) and let \( \Gamma(\varepsilon, x, \phi, t) \to 0 \) at \( \|x\| \to 0 \) uniformly with respect to variables \( \varepsilon, x, \phi, t \) in the field \( \mathbb{R}^{n+1} \times \mathbb{R} \times E \).

3. **Theorems on the existence and stability of integral manifold**

Here and below, we assume that the right-hand sides of the system (1.3) are \( \omega \)-periodic in the components of the vector \( \phi \) and \( T \)-periodic in the independent variable \( t \in \mathbb{R} \), in addition, they are continuous, assuring existence and uniqueness of the system (1.3) solutions in the field \( \mathbb{R}^{n+1} \times \mathbb{R} \times E \) for sufficiently small \( \delta_1 \) and \( \delta_2 \). This, in particular, means that the substitution (1.2) not only retains the existence and uniqueness properties of the system (1.1) solutions, but also retains \( \omega \)-periodicity in \( \phi \) and \( T \)-periodicity in \( t \).

**Theorem 1.** Let transforming matrix of the system (1.3) be constructed so that there exist a number \( l \) \((0<l \leq n+m)\) and such column \( q^\varepsilon_{\phi_0} \), for which in order to find the system solution

\[
\begin{align*}
q^\varepsilon_{\phi_0} = 0,
\int_0^{T} \Phi(e(\phi_0), F(\phi_0, t), \phi^\varepsilon, t) \, dt = 0.
\end{align*}
\]

it is sufficient to find the solution to some, generally speaking, distinct from (3.1) system of \( l \) equations

\[
S^\varepsilon_{\phi_0} = 0
\]
having for each function $F(\phi_0,t) \in \Omega_2$ the unique solution $\varepsilon^F(\phi_0)$ from the set of $\Omega_2$. In addition, suppose that for $t \in [0; kT]$ executed:

$$\left\|Y^F_\varepsilon(\phi_0, t') \cdot Q^F_\varepsilon(\phi_0) \right\| \leq r_0,$$  \hspace{1cm} (3.3)

$$\left\|Y^F_\varepsilon(\phi_0, t') \cdot Q^F_\varepsilon(\phi_0) - Y^F_\varepsilon(\phi_0, t) \cdot Q^F_\varepsilon(\phi_0) \right\| \leq r_1 \left\|\phi_0 - \phi_0\right\| + r_2 \left|\varepsilon^F - t\right|.$$  \hspace{1cm} (3.4)

Then for any vector $\phi_0 \in \mathbb{R}^n$ you can specify the value of the parameter $\nu$, that the system (1.1) will have a nonzero integral manifold $\psi(\phi_0, t) \in \Omega_4$ in a neighborhood of the equilibrium state $y=0$.

**Proof.** Since $\varepsilon = \varepsilon^F(\phi_0)$ is a solution to the system (3.2), then at $\varepsilon = \varepsilon^F(\phi_0)$ the expression (3.1) also becomes the identity. Consequently, the differential equation (2.4) for each function $F(\phi_0, t) \in \Omega_4$ has $kT$–periodic solution

$$x^F(\phi_0, t) = Y^F_\varepsilon(\phi_0, t) \cdot Q^F_\varepsilon(\phi_0) \cdot C,$$  \hspace{1cm} (3.5)

where all elements of the constant $n$–vector $C$ are zero, except for the element corresponding to the column number $Q^F_\varepsilon(\phi_0)$, which is equal to an arbitrary constant $c$. Non-singularity of the transforming matrix ensures non-triviality $x^F(\phi_0, t)$.

In accordance with the conditions (3.3) and (3.4), $x^F(\phi_0, t)$ limited to $r_0 \cdot |c|$, satisfies the Lipschitz condition with a constant $r_1 \cdot |c|$ in the variable $\phi_0$ and $r_2 \cdot |c|$ at $t$. So by reducing $c$ it is always possible to achieve $x^F(\phi_0, t) \in \Omega_4$. Thus, we have constructed the operator defined by equations (3.2) and (3.5), to which, because of the uniqueness values for $\varepsilon^F(\phi_0)$ each function $F(\phi_0, t) \in \Omega_4$, we can apply theorem [1, p 26]. Consequently, this operator has a fixed point

$$\Psi(\phi_0, t) = Y^F_\varepsilon(\phi_0, t) \cdot Q^F_\varepsilon(\phi_0) \cdot C.$$  \hspace{1cm}

It is clear that at $\varepsilon = \varepsilon^F(\phi_0)$ function $\Psi(\phi_0, t)$, $\phi_0^*$ a family of nonzero $kT$–periodic solutions to the system (1.3) is defined. Indeed, in order to verify this, it is sufficient to take into account appeal of equation (3.1) to the identity and to differentiate $\Psi(\phi_0, t)$, given that the function $\phi_0^*$ satisfies the equation (2.3).

To complete the proof of the theorem we should return to the system (1.1) by means of the substitution (1.2). So, $\Psi(\phi_0, t)$ – the required $n$–dimensional nontrivial periodic integral manifold of the system (1.1).

The theorem is proved.

Note that the proof of Theorem 1 generally follows the proof of the theorem on the existence of integral manifolds in [12, p 79] for systems of differential equations, which are not solved with respect to derivatives.

The technique of constructing a transforming matrix demonstrates theorems, establishing the existence of integral manifolds by properties of the right-hand sides of system (1.3). The case when the problem of the system (1.1) local integral manifold existence is solved using linear terms of the matrix $X(\varepsilon, 0, \phi, t)$ with respect to components of the vector $\varepsilon$ is discussed in detail in [1, 13]. Therefore, we consider the case when the linear terms in components of the vector $\varepsilon$ are either absent in the matrix $X(\varepsilon, 0, \phi, t)$ (Theorem 2), or they do not allow us to solve the problem of the local integral manifold existence (Theorem 3).
Theorem 2. Suppose that in the field $R^{n 	imes n} \times X \times E$ at sufficiently small $\delta_1$ and $\delta_2$, $X(\varepsilon,x,\phi,t), \Phi(\varepsilon,x,\phi,t)$ - Lipchitz-continuous in all their variables,

\[
X(\varepsilon,x,\phi,t) = \text{diag}(X_1(\varepsilon,x,\phi,t), X_2(\varepsilon,x,\phi,t)), \quad \det\left(\exp\left(X_1(0,0,\phi,kT) - I_{n,\phi}\right)\right) \neq 0,
\]

\[
X_1(0,0,\phi,kT) - \text{const}, \quad X_2(\varepsilon,x,\phi,t) = \text{diag}(\alpha_1(\varepsilon,x,\phi,t), \alpha_2(\varepsilon,x,\phi,t), \ldots, \alpha_p(\varepsilon,x,\phi,t)).
\]

\[
\Phi(\varepsilon,x,\phi,t) = \text{coln}(\Phi_1(\varepsilon,x,\phi,t), \Phi_2(\varepsilon,x,\phi,t), \ldots, \Phi_m(\varepsilon,x,\phi,t), \ldots, \Phi_{\rho+i}(\varepsilon,x,\phi,t)),
\]

\[
\alpha_i(\varepsilon,0,\phi,t) = \varepsilon_i^2 - \alpha_i(\varepsilon,0,\phi,t).
\]

integrals $\int_0^T \alpha_i(\varepsilon,0,\phi,t) \, dt$ are positive definite forms at all $\phi$ with $\varepsilon$ not lower than the fourth degree, $\int_0^T \alpha_i(0,0,\phi,t) \, dt \equiv 0$, $\alpha_i(\varepsilon,x,\phi,t) > 0$ at sufficiently small $x \neq 0$, where

\[
X = \{x: \|x\| \leq \delta_1\} \subset R^n, \quad E = \{\varepsilon: \|\varepsilon\| \leq \delta_2\} \subset R^{n \times m}.
\]

Here and further $X_1(\varepsilon,0,\phi,t), X_2(\varepsilon,0,\phi,t)$ - respectively $((n-p) \times (n-p))$ - and $(p \times p)$ - matrix, $\alpha_i(\varepsilon,x,\phi,t)$ - scalar functions, $I_{n,\phi}$ - unit $((n-p) \times (n-p))$ - matrix.

Then, for any vector $\phi_0 \in R^n$ one can specify the value of the parameter $\nu$ that the system (1.1) will have a nonzero integral manifold $\psi(\phi_0,t) \in \Omega$ in a neighborhood of the equilibrium state $y=0$.

Proof. Based on the properties of the right-hand sides of the system (1.3), the solution $\phi_i^T$ to the equation (2.3) is restricted to satisfy the condition of Lipchitz in all its variables for $t \in [0,kT]$ and $\omega$-periodic in the initial data $\phi_0$. And then matrizant $Y_i^T(\phi_0,t)$ of the equation (2.4) has exactly the same properties (see, for example, [1, p 29]).

It can be shown (see, for example, [1, p 34]) that for the matrizant of equation (2.4) the representation $Y_i^T(\phi_0,t) = \tilde{Y}_i^T(\phi_0,t) + \bar{Y}_i^T(\phi_0,t)$, where $\tilde{Y}_i^T(\phi_0,t)$ is a matrizant of the equations system $\dot{x} = X(\varepsilon(\phi_0),0,\phi^T,t) \cdot x$, and $\bar{Y}_i^T(\phi_0,t)$ is limited to satisfy Lipchitz condition with respect to all of its variables for $t \in [0,kT]$ with constant, which can be made arbitrarily small by decreasing the number of $\delta_{i0}$ ($\|F(\phi_0,t)\| \leq \delta_{i0}$).

Since the matrix $X(\varepsilon,x,\phi,t)$ is quasi-diagonal, then $Y_i^T(\phi_0,t) = \text{diag}(Y_i^T(\phi_0,t), Y_2^T(\phi_0,t), \ldots, Y_p^T(\phi_0,t))$, and $\bar{Y}_i^T(\phi_0,t) = \text{diag}(\bar{y}_i^T(\phi_0,t), \bar{y}_2^T(\phi_0,t), \ldots, \bar{y}_p^T(\phi_0,t))$.

We’ll define by the symbol $J_p$ the following $(p \times p)$ - matrix

\[
J_p = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 1 \\
0 & 1 & 0 & \ldots & 0 & 1 \\
& & & \ddots & & \\
0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}.
\]

By choosing the transforming matrix of the system (1.3) in quasi-diagonal form $Q_i^T(\phi_0) = \text{diag}(I_{n,\phi}, J_p)$ we’ll fulfill (3.3) and (3.4).
In this case each nonzero component of the last column \( q^T (\phi_h) \) of the matrix (2.5) is a function \( e_i^2 (\phi_h) \cdot kT - f_i^T (e(\phi_h),\phi_h) \), in which by reducing constants \( \delta_1 \) and \( \delta_2 \), we can achieve the inequality \( f_i^T (e(\phi_h),\phi_h) > 0 \), and at \( F(\phi_h,t) = 0 \) the function \( f_i^T (e(\phi_h),\phi_h) \) is a positive definite form at \( e(\phi_h) \) not lower than the fourth degree. The vector \( \int_0^T \Phi (e(\phi_h), F(\phi_h,t), \phi_h, t) \) has similar properties.

Therefore, any of \( l = p + m \) equations of the system (3.1) can be represented as

\[
e_i(\phi_h) = \pm \frac{1}{kT} \int f_i^T (e(\phi_h),\phi_h).\]

Applying the operator (3.2) defined by equations \( e_i(\phi_h) = \frac{1}{kT} \int f_i^T (e(\phi_h),\phi_h) \), and, following the technique of [1], it is easy to prove that for sufficiently small \( \delta_h \) (see (2.1) and (2.2)), this operator is contractive and for each function \( F(\phi_h,t) \in \Omega_1 \) puts the space \( \Omega_2 \) to \( \Omega_2 \). This confirms that all conditions of the Theorem 1 are fulfilled, and, therefore, the local non-zero integral manifold of the system (1.1) exists.

The theorem is proved.

**Theorem 3.** Suppose that in the field \( R^{m+1} \times X \times E \) at sufficiently small \( \delta_1 \) and \( \delta_2 \), \( X(e,x,\phi_t), \Phi(e,x,\phi_t) \) - Lipschitz-continuous in all their variables, \( X(e,x,\phi_t) \) has a quasitriangular form

\[
X(e,x,\phi,t) = \left[ \begin{array}{ccc} X_1(e,x,\phi,t) & X_2(e,x,\phi,t) & \Xi \\ \Xi & \Xi & \Xi \\ X_2(e,x,\phi,t) & \Xi & \Xi \end{array} \right].
\]

\( X(e,0,\phi,t) = \text{diag} \left( X_1(e,0,\phi,t), X_2(e,0,\phi,t) \right), \) \( \text{det} \left( \exp \left( X_1(e,0,\phi, kT) - I_{n,p} \right) \right) \neq 0, \)

\[
X_1(0,0,\phi,kT) - \text{const}, \quad X_2(e,x,\phi,t) = \text{diag} \left( \alpha_1(e,x,\phi,t), \alpha_2(e,x,\phi,t), ..., \alpha_p(e,x,\phi,t) \right),
\]

\( \alpha_1(e,0,\phi,t) = \alpha_i(e) \) at \( i = p \),

\[
\Phi(e,x,\phi,t) = \text{col} \left( \Phi_1(e,x,\phi,t), \Phi_2(e,x,\phi,t), ..., \Phi_n(e,x,\phi,t) \right), \quad \Phi_i(e,0,\phi,t) = \alpha_{p+i}(e),
\]

at \( i = p \) done \( \alpha_i(e) = a_1 \cdot e_1 + a_2 \cdot e_2 + ... + a_i \cdot e_i + \alpha_i(e), \quad \alpha_i(e) = o(\|e\|), \)

\[
\alpha_i(e,0,\phi,t) = e_i^2 - \alpha_i(e,0,\phi,t),
\]

integral \( \int_0^T \bar{\alpha}_p(e,0,\phi,t) \) at all \( \phi \) is a positively determined form at \( e \) no lower than the fourth degree, \( \int_0^T \bar{\alpha}_p(0,0,\phi,t) = 0, \quad \bar{\alpha}_p(e,x,\phi,t) > 0 \) at sufficiently small \( x \neq 0 \), where \( X = \{ x : \| x \| \leq 2 \delta \} \subset R^n, \quad E = \{ e : \| e \| \leq \delta_2 \} \subset R^{n+m}. \)

Here and further \( X_1(e,0,\phi,t), X_2(e,0,\phi,t) \) - respectively \( ((n-p) \times (n-p)) \) - and \( (p \times p) \) - matrix, elements of the constant \( (p \times (n-p)) \) - matrix \( \Xi \) are zero, \( \alpha_1(e,x,\phi,t) \) - scalar functions, \( I_{n,p} \) - unit \( ((n-p) \times (n-p)) \) - matrix, \( l = p + m \), determinant of \( ((l-1) \times (l-1)) \) - matrix
is different from zero.

Then, for any vector \( \phi_0 \in R^m \) one can specify the value of the parameter \( \nu \) that the system (1.1) will have a nonzero integral manifold \( \psi(\phi_0, t) \in \Omega \) in a neighborhood of the equilibrium state \( y = 0 \).

**Proof.** Similar to the proof of the Theorem 2 we see that the solution \( \phi^t \) and matrixizant \( Y^t(\phi_0, t) \) are limited satisfying the Lipschitz condition in all of its variables for \( t \in [0,kT] \) and \( \omega \)-periodic in initial data \( \phi_0 \). Additionally, \( Y^t(\phi_0, t) = \hat{Y}^t(\phi_0, t) + \hat{Y}^t(\phi_0, t) \) and \( \hat{Y}^t(\phi_0, t) \) have the same properties as in theorem 2.

We set \( Y^t(\phi_0,kT) - I_n = \begin{pmatrix} Y_{11}(\epsilon,F,\phi_0) & Y_{12}(\epsilon,F,\phi_0) \\ Y_{21}(\epsilon,F,\phi_0) & Y_{22}(\epsilon,F,\phi_0) \end{pmatrix} \), where matrix \( Y_{11}(\epsilon,F,\phi_0) \), \( Y_{12}(\epsilon,F,\phi_0) \), \( Y_{21}(\epsilon,F,\phi_0) \), and \( Y_{22}(\epsilon,F,\phi_0) \) has dimensions \( ((n-p) \times (n-p)) \), \( ((n-p) \times p) \), \( (p \times (n-p)) \), and \( (p \times p) \), respectively, and \( Y_{22}(\epsilon,F,\phi_0) \approx \Xi \) (as \( X(\epsilon,x,f,t) \) has a quasitriangular form). Then, due to execution of inequality \( \det(\exp(X_{1}(0,0,\phi,kT)-I_{n-p})) \neq 0 \), by reducing values of constants \( \delta_1 \) and \( \delta_2 \) we can achieve equality \( \det Y_{11}(\epsilon,F,\phi_0) = a_{0j} \neq 0 \). This means that there is an inverse matrix \( Y_{11}(\epsilon,F,\phi_0)^{-1} \) to matrix \( Y_{11}(\epsilon,F,\phi_0) \), which is also restricted, \( \omega \)-periodic in \( \phi_0 \) and satisfies the Lipschitz condition in \( \phi_0 \).

Let the vector \( q = \text{colon}(q_1,q_2,\ldots,q_{n-p}) \) determine by the equality \( q = -\left( Y_{11}(\epsilon,F,\phi_0) \right)^{-1} \cdot Y_{12}(\epsilon,F,\phi_0) \cdot J_1 \), where elements of the constant \( (p \times 1) \)-vector \( J_1 \) are equal to one.

Then the transforming matrix of the system (1.3) we will choose in the form \( Q^t(\phi_0) = \begin{pmatrix} I_{n-p} & \overline{Q} \\ \Xi & J_{p} \end{pmatrix} \).

Here among the elements of \( ((n-p) \times p) \)-matrix \( \overline{Q} \) are only nonzero elements \( q_1,q_2,\ldots,q_{n-p} \) of the last column.

Since elements of the vector \( q \) are linear combinations of matrix elements \( \{ Y_{11}(\epsilon,F,\phi_0) \}^{-1} \) and \( Y_{12}(\epsilon,F,\phi_0) \), then \( Q_{11}(\epsilon,F,\phi_0) \) is bounded, \( \omega \)-periodic in \( \phi_0 \) and satisfies the Lipschitz condition with respect to \( \phi_0 \) and hence, (3.3) and (3.4) will be executed.

Because of the transforming matrix choice \( Q^t(\phi_0) \) the last column \( q_{n}^t(\phi_0) \) of the matrix (2.5) will contain exactly \( p \) the non-zero component. Indeed, the first \( (n-p) \)-coordinate of the column \( q_{n}^t(\phi_0) \) will take the form of \( Y_{11}(\epsilon,F,\phi_0) \cdot q + Y_{12}(\epsilon,F,\phi_0) \cdot J_1 \equiv 0 \), and the rest of \( p \) coordinates will be written as follows \( Y_{22}(\epsilon,F,\phi_0) \cdot J_1 \). The diagonal form of the matrix \( X^t(\epsilon,x,F,t) \) also provides diagonality of the matrix \( Y_{22}(\epsilon,F,\phi_0) \).
In this case, the $p$ component of the last column $q_k^p(\phi_i)$ of the matrix (2.5) is a function $e_p^2(\phi_i) \cdot kT - f_p^p(\varepsilon(\phi_i),\phi_i)$, in which by reducing constants $\xi$ and $\delta$, it is possible to achieve inequality $f_p^p(\varepsilon(\phi_i),\phi_i) > 0$, and the function $f_p^p(\varepsilon(\phi_i),\phi_i)$ is a positively definite form at $\varepsilon(\phi_i)$ no lower than the fourth degree. Therefore, the equation number $p$ of the system (3.1) can be represented as $e_p(\phi_i) = \frac{1}{kT} \sqrt{f_p^p(\varepsilon(\phi_i),\phi_i)}$.

Suppose that the operator $S_e^p(\phi_i)$ is given by $l-1$ the non-zero equation of the system (3.1) (without the equation number $p$), and instead of equation number $p$ of the system (3.1) to specify the operator $S_e^p(\phi_i)$ we use equation $e_p(\phi_i) = \frac{1}{kT} \sqrt{f_p^p(\varepsilon(\phi_i),\phi_i)}$. Hence, the system of equations (3.2) can be represented in the form $A \cdot (\varepsilon(\phi_i)) = (\varepsilon(\phi_i),\phi_i)$, where for $p \neq i$ and for elements $a_{ij}$ of $(l \times l)$-matrix $A = (a_{ij})$ is performed $a_{pi} = 0$ and $a_{pp} = 1$. In addition, the function $e^p(\varepsilon(\phi_i),\phi_i)$ is $\omega$-periodic in $\phi_i$ and limited, satisfying the Lipschitz condition in $\phi_i$ with constants, which can be made arbitrarily close to zero by decreasing numbers $\delta_i$.

From nonsingularity of the matrix $A_0$ it follows that $\det A \neq 0$ and (3.1) is reduced to the form $e(\phi_i) = A^{-1} \cdot e^p(\varepsilon(\phi_i),\phi_i)$.

By reducing $\delta_i$ it is easy to verify that the operator defining the equation (3.6) is contractive and for each function $F(\phi_i,t) \in \Omega_3$ transfers the space $\Omega_2$ into $\Omega_2$. This means that all conditions of the Theorem 1 are fulfilled, and, hence, the local non-zero integral manifold of the system (1.1) exists.

The theorem is proved.

Suppose now that conditions of the Theorem 1 are fulfilled further and, therefore, the system (1.1) for certain values of the parameter $v_0 = \xi(\epsilon_0)$ has an integral manifold $\psi(\phi_i,t) \in \Omega_2 \cdot \psi(\phi_i,0) = \psi_0$. We denote by $x = x(x_0,\phi_i,t), \psi = \phi(x_0,\phi_i,t)$ the solution to the system (1.3) satisfying the initial data $x(x_0,\phi_i,0) = x_0, \phi(x_0,\phi_i,0) = \phi_i$.

**Lemma 1.** If $\epsilon = \epsilon_0$ for any number $\delta > 0$ there exists a number $\Delta > 0$, that the fulfillment of the inequality $\|x_0 - \psi_0\| < \Delta$ for any $\phi_i \in \Theta$ implies $t \geq 0$ the inequality $\|x(t,\phi_i,t) - \psi(\phi_i,0)\| < \delta$ for all $\phi_i \in \Theta$, then with $v_0 = \xi(\epsilon_0)$ the integral manifold $\psi(\phi_i,t)$ of the system (1.1) is stable.

**Proof.** The validity of the Lemma 1 follows from the identity $\Gamma(\epsilon,0,\phi_i,t) \equiv 0$ and uniform convergence in the field $R^{n+1} \times X \times E$ of function $\Gamma(\epsilon,x,\phi_i,t)$ to 0 with $\|x\| \rightarrow 0$.

**Theorem 4.** If there exists a sign-definite function $V(x,t)$, for which function $\frac{dV}{dt} = \nabla \cdot \nabla V(x,t) - \psi(\phi_i,t)$ has constant sign, opposite to $V(x,t)$, then with $v_0 = \xi(\epsilon_0)$ integral manifold $\psi(\phi_i,t)$ of the system (1.1) is stable.

**Proof.** Let $\varepsilon = \varepsilon_0$ and $V(x,t)$ be positive. Then for any number $\delta > 0$ there exist such numbers $\Delta > 0$ and $\tilde{\Delta} > 0$, for which with $\|x\| = \Delta \leq \delta, t \geq 0$ will be done $V(x,t) \geq \tilde{\Delta}$, besides $\Delta \leq \delta_i$.

We now choose $x_0$ so that
\[ V(x_0 - \psi_0, 0) < \tilde{t} \]  
(3.7)

and \( \|x_0 - \psi_0\| \leq \Delta < \Delta \) for any \( \psi_0 \in \Theta \). Due to the fact that

\[ \frac{dV}{dt} \leq 0, \]  
(3.8)

inequality \( \|x(x_0, \phi_0, t) - \psi(\phi_0, t)\| < \Delta \leq \delta \) correct for all \( \phi_0 \in \Theta \) with \( t \geq 0 \). Indeed, if this were not so, there would be such constant \( \phi_0^* \in \Theta \) and \( t^* \geq 0 \) for which \( \|x(x_0, \phi_0^*, t^*) - \psi(\phi_0^*, t^*)\| = \Delta \) was executed and so \( V(x(x_0, \phi_0^*, t^*) - \psi(\phi_0^*, t^*), t^*) \geq \tilde{t} \), which contradicts to the simultaneous execution of relations (3.7) and (3.8).

The proof of Theorem 4 is completed by applying Lemma 1.

The theorem is proved.

As an illustration of the use of Theorem 4, we consider the simplest systems of the form (1.3), to which none of the sufficient criteria for the existence of stable integral manifolds can be applied [2, 3, 5, 10].

**Example 1.** Let in the system of two differential equations

\[
\begin{align*}
\dot{x} &= (\varepsilon_1 + \varepsilon_2 - x)_3 \cdot x, \\
\phi &= \varepsilon_2 + \cos 2t
\end{align*}
\]  
(3.9)

all variables are scalar.

Obviously, when \( \varepsilon_1 = \tilde{\varepsilon} > 0, \varepsilon_2 = 0 \) system (3.9) has a stable integral manifold \( \psi(\phi, t) = \tilde{\varepsilon} \), and function \( \phi = \phi_0 + 0.5 \sin 2t \) determines a periodic solution. In particular, to prove this fact, one can apply Theorem 3 by choosing \( V(x, t) = x^2 \). Then

\[ \frac{dV(x - \tilde{x})}{dt} = -2(x - \tilde{x})^4 \cdot x \leq 0 \]  
with \( \|x - \tilde{x}\| < 0.25 \cdot \tilde{\varepsilon} \).

**Example 2.** For a system of four scalar differential equations

\[
\begin{align*}
\dot{x}_1 &= -x_2 + \varepsilon_1 x_1^2 - x_3^3 \cdot \sin \phi \\
\dot{x}_2 &= x_1 + \varepsilon_2 x_2 - x_3^3 \cdot \sin \phi \\
\dot{x}_3 &= \varepsilon_3 - \varepsilon_1 x_2 - x_3^3 \cdot (1 + \cos^2 \phi) \\
\phi &= \varepsilon_4 + x_1 \cdot \sin \phi
\end{align*}
\]

it is also possible to construct a function \( V(x) = x_1^2 + x_2^2 + x_3^2 \), satisfying all conditions of the Theorem 3. Indeed, the system (3.1) for each function \( F(\phi_0, t) \in \Omega \) has a solution \( e^x(\phi_0) \), where

\[ e_1(\phi_0) = -\frac{1}{2\pi} \int_0^{2\pi} F_1(\phi_0, t) \cdot \sin \phi \cdot d\phi . \]  

Then the integral manifold \( \psi(\phi_0, t) \) is given by \( \psi(\phi_0, t) \) = \( \text{colon}(c \cdot \cos t, c \cdot \sin t, 0) \), where \( c \neq 0 \), and function \( \phi = \phi^*(\phi_0, t) \) determines on it a periodic curve. If now considered \( x = x(x_0, \phi_0, t) \) as a solution close to \( \psi(\phi_0, t) \) \( (\|x(x_0, \phi_0, t) - \psi(\phi_0, t)\| < 0.5 \cdot |\varepsilon|) \), then
\[ \frac{dV}{dt} = -2x_1^2 \cdot \left(1 + \cos^2 \phi''(\phi_1, t)\right) - 2x_1x_3^5 \cdot \sin \phi''(\phi_1, t) - 2x_1x_3^7 \cdot \sin^2 \phi''(\phi_1, t) \leq 0 \]

at small \( c \) and \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0 \).

4. Conclusions

New sufficient conditions of existence of nonzero local stable integral manifold of the system (1.1) are formulated in Theorem 1-4. The critical case (linear subsystem \( \dot{x} = X(0,0,0,t) \cdot x \) has a zero of characteristic indicators) is considered. The results obtained on the basis of the Lyapunov functions method and the transforming matrix method, the application of which allowed to reduce the solution of this problem to searching of nonlinear operators fixed points. The problem of stability of integral manifold of the system (1.1) with respect to all variables \((y)\) is solved, taking into account only characteristics a part of the variables \((x)\). The examples of finding the transforming matrices and Lyapunov functions for the various systems of the form (1.3) are made in this research paper.

References

[1] Kuptsov M I 1997 The existence of integral manifolds and periodic solution of system of ordinary differential equations: PhD dissertation (Mathematics) (Izhevsk: Udmurt State University) p 133
[2] Bogolyubov N N 1945 About some statistical methods in mathematical physics (Lvov: Academy of Sciences of the Ukrainian Soviet Socialist Republic) p 139
[3] Mitropol’skii Yu A and Lykova O B 1973 Integral manifolds in nonlinear mechanics (Moscow: Nauka) p 512
[4] Samoilenko A M and Dvorak A V 1978 Ukrainian Mathematical Journal 29(4) 427–431
[5] Samoilenko A M 1987 The elements of mathematical theory of multi frequency vibrations. Invariant tori (Moscow: Nauka) p 301
[6] Kurbanshoev S Z and Nusairiev M A 2014 Reports of the Academy of Sciences of the Republic of Tajikistan 57(11–12) 807–812
[7] Samoilenko A M, Petryshyn R I and Sopronyuk T M 2003 Ukrainian Mathematical Journal 55(5) 773–800
[8] Shchetinina E V 2010 Vestnik of Samara State University 6 93–105
[9] Sobolev V 2016 Journal of Physics: Conference Series 727(1) 012017
[10] Bibikov Yu N 1991 Multi frequent nonlinear vibrations and their bifurcation (Leningrad: Leningrad State University) p 142
[11] Volkov Yu D 1988 Vestnik Leningrad University 1(2) 102–103
[12] Kuptsov M I, Terekhin M T and Tenyaev V V 2017 Middle-Volga Mathematical Society Journal 19(2) 76–84
[13] Kuptsov M I 1998 Differential equations 34(7) 1005–1007