FENCHEL–MOREAU IDENTITIES ON SELF-DUAL CONES

HONG-BIN CHEN AND JIAMING XIA

ABSTRACT. A pointed convex cone in a Hilbert space naturally induces a partial order, and further a notion of nondecreasingness for functions. We consider extended real-valued functions defined on the cone. Monotone conjugates for these functions can be defined in an analogous way to the standard convex conjugate. The only difference is that the supremum is taken over the cone instead of the entire space. The main result is the corresponding Fenchel–Moreau biconjugation identity for proper, convex, lower semicontinuous and nondecreasing functions provided the cone is perfect. Essentially, a cone is called perfect if it is self-dual and each face of the cone is self-dual in its own span. When the cone is merely self-dual, the Fenchel–Moreau identity still holds for such a function under an additional assumption that its domain has nonempty interior.

1. INTRODUCTION

The classical Fenchel–Moreau identity can be stated as \( f = f^{**} \) for convex \( f : \mathcal{H} \to (-\infty, +\infty] \) satisfying a few additional regularity conditions. Here \( \mathcal{H} \) is a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and the convex conjugate is given by

\[
 f^*(x) = \sup_{y \in \mathcal{H}} \{ \langle y, x \rangle - f(y) \}, \quad \forall x \in \mathcal{H}. 
\]

Note that the supremum is taken over the entire space \( \mathcal{H} \).

On the other hand, it is well-known (c.f. [13, Theorem 12.4]) that if \( f : [0, +\infty)^d \to (-\infty, +\infty] \) is convex with extra usual assumptions and, in addition, is nondecreasing in the sense that

\[
 f(x) \geq f(y), \quad \text{if } x - y \in [0, \infty)^d,
\]

then we also have \( f = f^{**} \). Here * stands for the monotone conjugate defined by

\[
 f^*(x) = \sup_{y \in [0, \infty)^d} \{ \langle y, x \rangle - f(y) \}, \quad \forall x \in [0, \infty)^d.
\]

The inner product appearing above is the standard one in \( \mathbb{R}^d \). The nonnegative orthant \( [0, \infty)^d \) is a cone in \( \mathbb{R}^d \) and the nondecreasingness can be formulated with respect to the partial order induced by this cone. Compared with the convex conjugate, the supremum above is taken over the cone.

Recently, in [9], to study a certain Hamilton–Jacobi equation with spatial variables in the set of \( n \times n \) (symmetric) positive semidefinite (p.s.d.) matrices denoted by \( S_n^+ \), a version of the Fenchel–Moreau identity on \( S_n^+ \) is needed to verify that the unique solution admits a variational formula. The derivation of such formulae for Hamilton–Jacobi equations on entire Euclidean spaces are known and can be seen, for instance, in [1, 11]. On \( S_n^+ \), [9, Proposition B.1] proves that \( f = f^{**} \) holds.

2010 Mathematics Subject Classification. 46N10, 52A07.

Key words and phrases. Fenchel–Moreau identity, monotone conjugate, self-dual cone.
if \( f : S^n_+ \to (-\infty, +\infty] \) is convex with some usual regularity assumptions and is nondecreasing in the sense that
\[
f(x) \geq f(y), \quad \text{if } x - y \in S^n_+.
\]
Accordingly, here * stands for the monotone conjugate with respect to \( S^n_+ \) given by
\[
f^*(x) = \sup_{y \in S^n_+} \{ \langle y, x \rangle - f(y) \}, \quad \forall x \in S^n_+.
\]
The inner product is the Frobenius inner product for matrices. Again, in this case, \( S^n_+ \) can be viewed as a cone in \( S^n \), the space of \( n \times n \) real symmetric matrices.

In view of these two examples, it is natural to pursue a generalization to an arbitrary (convex) cone \( C \) in a Hilbert space \( H \). More precisely, we want to show \( f = f^{**} \) for proper, lower semicontinuous and convex \( f : C \to (-\infty, +\infty] \) which is also nondecreasing in the sense that
\[
f(x) \geq f(y), \quad \text{if } x - y \in C,
\]
where
\[
f^*(x) = \sup_{y \in C} \{ \langle y, x \rangle - f(y) \}, \quad \forall x \in C.
\]
The first thing to notice is that \( C \) has to be self-dual in order for this to hold. Indeed, suppose that \( f \) is smooth, then the supremum in the definition of \( f^*(x) \) is achieved at \( y \) such that \( x = \nabla f(y) \). By the nondecreasingness of \( f \), we must have that \( \nabla f(y) \) is in the dual cone of \( C \). Since \( x \) can vary in \( C \) and \( \nabla f(y) \) in the dual of \( C \), in order for the identity to hold for all \( f \) on the entire \( C \), it is necessary that \( C \) is equal to its dual.

Assuming \( C \) is self-dual, we can show \( f = f^{**} \) on the entire \( C \) in Proposition 2.3, however, only for those \( f \) whose (effective) domain has nonempty interior. Things can go wrong when the domain resides entirely in the boundary of \( C \), as there is not much control from the self-duality on the “gradient” of \( f \). To tackle this problem, we need additional conditions on \( C \) to ensure that its boundary behaves nicely. One candidate is the notion of perfect cones first introduced in [3] in the setting of Euclidean spaces. Heuristically, a perfect cone has the following property. If \( F \) is an intersection of some subspace of \( H \) with the boundary of \( C \), then \( F \) is self-dual in the linear space spanned by \( F \). Hence, the structure of self-duality is still present after zooming into a portion of the boundary.

The nonnegative orthant \( [0, \infty)^d \) and the set of p.s.d. matrices \( S^n_+ \) are both perfect cones. The former is easy to see using Definition 2.1 and the latter will be proved in Lemma 5.1. An example of an infinite-dimensional perfect cone is given in Lemma 5.3. Classical references for properties of cones and self-dual cones in Euclidean spaces or Hilbert spaces include [2, 4, 5, 7, 12].

The rest of the paper is organized as follows. We introduce definitions and state main results in Section 2. These results will be proved in Section 3 and Section 4. Lastly, examples of perfect cones in finite dimensions and infinite dimensions are given in Section 5.

Acknowledgements. HBC thanks Jean-Christophe Mourrat and Tim Hoheisel for helpful discussions.
2. Definitions and main results

Let $\mathcal{H}$ be a real Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$ and the associated norm $| \cdot |$. We refer to an element in $\mathcal{H}$ sometimes as a vector, though $\mathcal{H}$ can be possibly infinite-dimensional. We denote the interior, the closure and the boundary relative to $\mathcal{H}$ by int, cl and bd, respectively.

2.1. Definitions related to cones. Let $\mathcal{C}$ be a cone in $\mathcal{H}$. In this work, for simplicity, we require all cones to be convex and contain the origin. Hence, $\mathcal{C}$ is a cone if and only if it satisfies
\[ \alpha x + \beta y \in \mathcal{C}, \quad \forall x, y \in \mathcal{C}, \quad \forall \alpha, \beta \geq 0. \]
Naturally, $\mathcal{C}$ induces a preorder $\preceq$ on $\mathcal{H}$ given by
\[ x \preceq y \quad \text{if and only if} \quad y - x \in \mathcal{C}. \]
We also write $x \succeq y$ if $y \preceq x$. When $\mathcal{C}$ is pointed, namely $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$, this preorder becomes a partial order. The dual of $\mathcal{C}$ with respect to $\mathcal{H}$ is given by
\[ \mathcal{C}^\circ = \{ x \in \mathcal{H} : \langle x, y \rangle \geq 0, \forall y \in \mathcal{C} \}. \] (2.1)

The cone $\mathcal{C}$ is said to be self-dual provided $\mathcal{C} = \mathcal{C}^\circ$. It is clear that a self-dual cone is closed and pointed.

A subset $\mathcal{F}$ of $\mathcal{C}$ is a face of $\mathcal{C}$ if $\mathcal{F}$ is a cone and satisfies that if $0 \preceq x \preceq y$ and $y \in \mathcal{F}$, then $x \in \mathcal{F}$.
\[ \text{if} \quad 0 \preceq x \preceq y \quad \text{and} \quad y \in \mathcal{F}, \quad \text{then} \quad x \in \mathcal{F}. \] (2.2)

Denote by $\mathcal{F}^\vee$ the dual cone of $\mathcal{F}$ in the space $\text{span} \mathcal{F}$. Here $\text{span}$ stands for the closed linear span. The following definition is a generalization of [3, Definition 3] from Euclidean spaces to Hilbert spaces.

**Definition 2.1.** A cone $\mathcal{C} \subset \mathcal{H}$ is said to be **perfect** if it is self-dual and every face $\mathcal{F}$ of $\mathcal{C}$ satisfies
\[ \begin{array}{l}
(1) \quad \mathcal{F}^\vee = \mathcal{F}; \\
(2) \quad \mathcal{F} \text{ has nonempty interior with respect to } \text{span} \mathcal{F}.
\end{array} \]

In other words, $\mathcal{F}^\vee = \mathcal{F}$ means $\mathcal{F}$ is self-dual in its own closed span. Since $\mathcal{C}$ is itself a face and $\text{span} \mathcal{C} = \mathcal{H}$ due to self-duality, a perfect cone must have nonempty interior. In finite-dimensions, a self-dual cone always has nonempty interior (c.f. [6, Exercise 6.15]). Hence, if $\mathcal{H}$ is finite-dimensional, then (2) automatically follows from (1). Compared with [3, Definition 3] where only (1) is imposed, condition (2) is added to ensure this non-degeneracy in infinite dimensions. In Section 5, we give two examples of perfect cones, a finite-dimensional one and an infinite-dimensional one.

2.2. Definitions related to functions. The domain of a function $f : \mathcal{C} \to (-\infty, +\infty]$ is defined as
\[ \text{dom } f = \{ x \in \mathcal{C} : f(x) < \infty \}. \] (2.3)
A function $f : \mathcal{C} \to (-\infty, +\infty]$ is said to be **C-nondecreasing** provided
\[ f(x) \geq f(y), \quad \forall x \succeq y \succeq 0. \]
For any $f : \mathcal{C} \to (-\infty, +\infty]$, we define the **monotone conjugate** of $f$ with respect to $\mathcal{C}$ by
\[ f^*(y) = \sup_{z \in \mathcal{C}} \{ \langle z, y \rangle - f(z) \}, \quad \forall y \in \mathcal{C}. \] (2.4)
Lastly, $f$ is said to be proper if $f$ is not identically equal to $+\infty$. We denote by $\Gamma_{>}(C)$ the collection of functions on $C$ with values in $(-\infty, +\infty]$ that are proper, convex, lower semicontinuous (l.s.c.) and $C$-nondecreasing.

2.3. Main results.

**Theorem 2.2.** Suppose that $C$ is a perfect cone. Let $f : C \to (-\infty, +\infty]$ be proper. Then, $f = f^{**}$ if and only if $f \in \Gamma_{>}(C)$.

If $f = f^{**}$, then it is easy to see $f \in \Gamma_{>}(C)$ necessarily. The nontrivial part is the sufficient condition for $f = f^{**}$. Theorem 2.2 shows that, when $C$ is perfect, no additional assumption on the function $f$ is needed. When $C$ is merely self-dual, the assumption int dom $f \neq \emptyset$ is added to ensure $f = f^{**}$.

**Proposition 2.3.** Suppose that $C$ is self-dual. Let $f : C \to (-\infty, +\infty]$ be proper and satisfy int dom $f \neq \emptyset$. Then $f = f^{**}$ if and only if $f \in \Gamma_{>}(C)$.

If int dom $f = \emptyset$, then dom $f$ lies in bd $C$, where things can go wrong. In view of Definition 2.1, beyond self-duality, a perfect cone has much nicer boundary.

3. Proof of Proposition 2.3

In the first part of this section, we state some basic results that are needed throughout this work. In the second part, we prove Proposition 2.3.

3.1. Preliminary results. For $a \in H$ and $\nu \in \mathbb{R}$, we define the affine function $L_{a, \nu}$ with slope $a$ and translation $\nu$ by

$$L_{a, \nu}(x) = \langle a, x \rangle + \nu, \quad \forall x \in H.$$  \hfill (3.1)

For a function $f : E \to (-\infty, +\infty]$ defined on a subset $E \subset H$, we can extend it in the standard way to $f : H \to (-\infty, +\infty]$ by setting $f(x) = \infty$ for $x \notin E$. For $f : H \to (-\infty, +\infty]$, we define its domain by

$$\text{dom } f = \{ x \in H : f(x) < \infty \}.$$  

Note that by the standard extension, the above definition is equivalent to (2.3) where only functions defined on $C$ are considered. Henceforth, we shall not distinguish functions defined on $C$ from their standard extensions to $H$. Denote by $\Gamma_{0}(E)$ the collection of proper, convex and l.s.c. functions from $E \subset H$ to $(-\infty, [\infty]$. In particular, when $C$ is closed which holds if $C$ is self-dual, the collection $\Gamma_{>}(C) \subset \Gamma_{0}(C)$ can be viewed as a subcollection of $\Gamma_{0}(H)$.

For $f : H \to (-\infty, +\infty]$ and each $x \in H$, we define the subdifferential of $f$ at $x$ by

$$\partial f(x) = \{ u \in H : f(y) \geq f(x) + \langle y - x, u \rangle, \forall y \in H \}. \hfill (3.2)$$

The effective domain of $\partial f$ is defined to be

$$\text{dom } \partial f = \{ x \in H : \partial f(x) \neq \emptyset \}.$$  

We now list some lemmas needed in our proofs.

**Lemma 3.1.** For a convex set $A \subset H$, if $y \in \text{cl } A$ and $y' \in \text{int } A$, then $\lambda y + (1 - \lambda) y' \in \text{int } A$ for all $\lambda \in [0,1)$.

**Lemma 3.2.** For $f \in \Gamma_{0}(H)$, it holds that int dom $f \subset \text{dom } \partial f \subset \text{dom } f$.

**Lemma 3.3.** Let $f \in \Gamma_{0}(H)$, $x \in H$ and $y \in \text{dom } f$. For every $\alpha \in (0,1)$, set $x_\alpha = (1 - \alpha)x + \alpha y$. Then $\lim_{\alpha \to 0} f(x_\alpha) = f(x)$.
Lemma 3.4. Suppose that $C$ is closed. Let $f \in \Gamma_0(C)$, $x \in C$ and $u \in C$. If $u \in \partial f(x)$, then $f^*(u) = \langle x, u \rangle - f(x)$.

Lemma 3.5. For $f \in \Gamma_0(C)$ and $x \in C$, we have

$$f^{**}(x) = \sup L_{a,\nu}(x)$$

where the supremum is taken over the collection \{$L_{a,\nu} : a \in C, \nu \in \mathbb{R}, L_{a,\nu} \leq f$\}.

Lemma 3.1, 3.2 and 3.3 can be derived from [6, Proposition 3.35], [6, Proposition 16.21] and [6, Proposition 9.14], respectively. For completeness, let us quickly prove Lemma 3.4 and Lemma 3.5.

Proof of Lemma 3.4. By the standard extension, we have $f \in \Gamma_0(H)$. Invoking [6, Theorem 16.23], it is classically known that

$$\sup_{z \in H} \{ \langle z, u \rangle - f(z) \} = \langle x, u \rangle - f(x).$$

By assumption, we know $x \in \text{dom } \partial f$. Hence, Lemma 3.2 implies $x \in \text{dom } f$ and thus the right hand side of the above equation is finite. Then, the supremum on the left must also be finite. On the other hand, by the extension, we have $f(z) = \infty$ if $z \notin C$, which yields

$$\sup_{z \in H} \{ \langle z, u \rangle - f(z) \} = \sup_{z \in C} \{ \langle z, u \rangle - f(z) \} = f^*(u).$$

Proof of Lemma 3.5. By the definition of the monotone conjugate in (2.4), we have

$$(3.3) \quad f^{**}(x) = \sup_{y \in C} \{ \langle y, x \rangle - f^*(y) \}, \quad \forall x \in C.$$ 

Note that, for each $y \in C$,

$$L_{y,-f^*(y)}(x) = \langle y, x \rangle - f^*(y), \quad \forall x \in C.$$ 

is an affine function with slope $y \in C$. Moreover, using (2.4) again, we can see that $L_{y,-f^*(y)} \leq f$ on $C$. Therefore, we have $f^{**}(x) \leq \sup L_{a,\nu}(x)$ for all $x \in C$.

For the other direction, for $L_{a,\nu}$ satisfying $a \in C$ and $L_{a,\nu} \leq f$, we have

$$\langle a, y \rangle + \nu \leq f(y), \quad \forall y \in C.$$ 

Rearrange and take supremum in $y \in C$ to get $f^*(a) \leq -\nu$. This yields

$$L_{a,\nu}(x) \leq \langle a, x \rangle - f^*(a) \leq f^{**}(x),$$

which implies $\sup L_{a,\nu}(x) \leq f^{**}(x)$. \qed

3.2. Proof of Proposition 2.3. Let $f : C \to (-\infty, +\infty]$ be proper and satisfy $\text{int} \text{ dom } f \neq \emptyset$. It is obvious from the definition of the monotone conjugate in (2.4) that $f^{**}$ is convex and l.s.c. In addition, by the self-duality of $C$, we also see that $f^{**}$ is $C$-nondecreasing. Therefore, assuming $f = f^{**}$ and that $f$ is proper, we have $f \in \Gamma_\mathcal{K}(C)$.

From now on, we assume $f \in \Gamma_\mathcal{K}(C)$ and prove the converse. For convenience, we write $\Omega = \text{dom } f$. The plan is to prove the identity $f = f^{**}$ first on $\text{int } \Omega$, then on $\text{cl } \Omega$, and finally on the entire $C$. 
3.2.1. **Analysis on** \(\text{int} \Omega\). Let \(x \in \text{int} \Omega\). By Lemma 3.2, we know \(\partial f(x)\) is not empty. For each \(v \in C\), there is \(\epsilon > 0\) small so that \(x - \epsilon v \in \Omega\). For each \(u \in \partial f(x)\), by the definition of subdifferentials and nondecreasingness, we have

\[
\langle v, u \rangle \geq \frac{1}{\epsilon} \left( f(x) - f(x - \epsilon v) \right) \geq 0.
\]

By self-duality of \(C\), this implies

\[
\emptyset \neq \partial f(x) \subset C, \quad \forall x \in \text{int} \Omega.
\]

Invoking Lemma 3.4, from (3.4) we can deduce

\[
f(x) \leq \sup_{y \in C} \left\{ \langle y, x \rangle - f^*(y) \right\} = f^{**}(x).
\]

On the other hand, by the formula (3.3), it is easy to see from the definition of \(f^*\) in (2.4) that

\[
f(z) \geq f^{**}(z), \quad \forall z \in C.
\]

Hence, we obtain

\[
f(x) = f^{**}(x), \quad \forall x \in \text{int} \Omega.
\]

3.2.2. **Analysis on** \(\text{cl} \Omega\). Let \(x \in \text{cl} \Omega\) and we distinguish two cases:

\[
\lambda x \in \text{cl} \Omega, \quad \exists \lambda > 1; \tag{3.6}
\]

\[
\lambda x \notin \text{cl} \Omega, \quad \forall \lambda > 1. \tag{3.7}
\]

First we consider the case in (3.6). Fix \(\lambda > 1\) that satisfies (3.6). Since \(\text{int} \Omega \neq \emptyset\), we can find \(x' \in \text{int} \Omega\). Now, we consider

\[
y = \gamma \lambda x + (1 - \gamma)x'.
\]

By Lemma 3.1, we have \(y \in \text{int} \Omega\) for all \(\gamma \in [0, 1)\). Since \(\lambda > 1\), we can choose \(\gamma \in [0, 1)\) close to 1 such that \(\gamma \lambda > 1\). By this, set \(w = y - x\) and we have

\[
w \in C, \quad x + w \in \text{int} \Omega
\]

The second term along with Lemma 3.1 implies

\[
x_\alpha = x + \alpha w \in \text{int} \Omega, \quad \forall \alpha \in (0, 1].
\]

Lemma 3.3 yields

\[
\lim_{\alpha \to 0} f(x_\alpha) = f(x).
\]

Since \(x_1 \in \Omega\) and \(x_1 \geq x \geq 0\), by \(\mathcal{C}\)-nondecreasingness, we also have

\[
x \in \Omega. \tag{3.10}
\]

By (3.4) and (3.8), for each \(\alpha \in (0, 1]\), there is \(x^*_\alpha \in \partial f(x_\alpha) \cap \mathcal{C}\). Recall the definition of affine functions in (3.1) , set

\[
\mathcal{L}_\alpha = L_{x^*_\alpha} f(x_\alpha) - \langle x_\alpha, x^*_\alpha \rangle
\]

By the definition of subdifferentials in (3.2), we have

\[
\mathcal{L}_\alpha \leq f, \quad \forall \alpha \in (0, 1]. \tag{3.11}
\]

For \(\alpha \in (0, 1/2)\), we also have

\[
f(x_{\alpha + \frac{1}{2}}) \geq f(x_\alpha) + \frac{1}{2} \langle w, x^*_\alpha \rangle,
\]
which along with $\mathcal{C}$-nondecreasingness gives
\[
\langle w, x^*_\alpha \rangle \leq 2(f(x_1) - f(x)) = C, \quad \forall \alpha \in (0, 1/2).
\]
The last constant in the above display is nonnegative and finite due to (3.8) and (3.10). Using this, (3.11) becomes
\[
f(x) \geq \mathcal{L}_\alpha(x) = \alpha \langle w, x^*_\alpha \rangle + f(x_\alpha) \geq -C\alpha + f(x_\alpha), \quad \forall \alpha \in (0, 1/2).
\]
By (3.9), sending $\alpha$ to $0$, we can see that
\[
f(x) = \lim_{\alpha \to 0} \mathcal{L}_\alpha(x).
\]
Since $\mathcal{L}_\alpha$ is affine and satisfies (3.11), the above display together with Lemma 3.5 yields
\[
f(x) \leq f^{**}(x).
\]
This along with (3.5) implies $f(x) = f^{**}(x)$ for all $x \in \text{cl} \Omega$ satisfying (3.6).

Now, we turn to the case where $x \in \text{cl} \Omega$ satisfies (3.7). For each $\lambda \in (0, 1)$, we set $x_\lambda = \lambda x$. Since $f$ is $\mathcal{C}$-nondecreasing and $f$ is proper, we must have $0 \in \Omega$. By the convexity of $\text{cl} \Omega$, we must have $x_\lambda \in \text{cl} \Omega$ for all $\lambda \in (0, 1)$. Note that, every $x_\lambda$ satisfies the condition (3.6). In the above, we have deduced (3.10) and $f = f^{**}$ for the case (3.6). Hence, here we have $x_\lambda \in \Omega$ and
\[
f(x_\lambda) = f^{**}(x_\lambda) < \infty.
\]
By Lemma 3.3, we have
\[
\lim_{\lambda \to 1} f(x_\lambda) = f(x).
\]
Also, Lemma 3.5 implies that there are $a_\lambda \in \mathcal{C}$ and $\nu_\lambda \in \mathbb{R}$ such that
\[
\begin{align}
(3.13) & \quad f \geq L_{a_\lambda, \nu_\lambda}; \\
(3.14) & \quad f(x_\lambda) \leq L_{a_\lambda, \nu_\lambda}(x_\lambda) + 1 - \lambda.
\end{align}
\]
We can rewrite $L_{a_\lambda, \nu_\lambda}(x)$ as
\[
L_{a_\lambda, \nu_\lambda}(x) = \left( \lambda \langle a_\lambda, x \rangle + \nu_\lambda \right) + (1 - \lambda) \langle a_\lambda, x \rangle
\]
\[
= L_{a_\lambda, \nu_\lambda}(x_\lambda) + (1 - \lambda) \langle a_\lambda, x \rangle \geq L_{a_\lambda, \nu_\lambda}(x_\lambda).
\]
Here, the inequality follows from $\langle a_\lambda, x \rangle \geq 0$ due to the self-duality of $\mathcal{C}$. Using (3.14), we can see
\[
L_{a_\lambda, \nu_\lambda}(x_\lambda) \geq f(x_\lambda) + \lambda - 1.
\]
By sending $\lambda$ to $1$, this along with (3.12) implies
\[
\liminf_{\lambda \to 1} L_{a_\lambda, \nu_\lambda}(x) \geq f(x).
\]
Using (3.13) and Lemma 3.5, we obtain $f^{**}(x) \geq f(x)$. Lastly, by (3.5), combining the result from the first case (3.6), we conclude that
\[
(3.15) \quad f(x) = f^{**}(x), \quad \forall x \in \text{cl} \Omega.
\]
3.2.3. Analysis on $C$. Due to (3.15), we only need to consider vectors outside $\overline{cl}\Omega$. Let $x \in C \setminus \overline{cl}\Omega$, and we have $f(x) = \infty$. Since $f$ is proper and $C$-nondecreasing, we must have $0 \in \Omega$. By this, $x \notin \overline{cl}\Omega$ and the convexity of $\overline{cl}\Omega$, we must have
\[
\lambda' = \sup\{\lambda \in [0, \infty) : \lambda x \notin \overline{cl}\Omega\} < 1.
\]
Set $x' = \lambda' x$. Then, we have that $x' \in \overline{cl}\Omega \setminus \text{int}\Omega$ and $\lambda x' \notin \overline{cl}\Omega$ for all $\lambda > 1$.

We need to discuss two cases: $x' \in \Omega$ or not.

In the second case where $x' \notin \Omega$, we have $f(x') = \infty$. We can apply Lemma 3.5 and (3.15) to see that there exists a sequence of affine functions $\{L_{a_n, \nu_n}\}_{n=1}^\infty$ satisfying $a_n \in C$,
\[
f \geq L_{a_n, \nu_n}, \quad \forall n; \quad \lim_{n \to \infty} L_{a_n, \nu_n}(x') = f(x') = \infty.
\]
By the definition of $x'$, we have
\[
L_{a_n, \nu_n}(x) = L_{a_n, \nu_n}(x') + (1 - \lambda')(a_n, x) \geq L_{a_n, \nu_n}(x')
\]
Therefore, we obtain $f(x) = \lim_{n \to \infty} L_{a_n, \nu_n}(x) = \infty$ and Lemma 3.5 gives us $f(x) = f^{**}(x)$ for such $x$.

We now consider the case where $x' \in \Omega$. For every $y \in H$, the outer normal cone to $\Omega$ at $y$ is defined by
\[
n(y) = \{z \in H : \langle z, y' - y \rangle \leq 0, \forall y' \in \Omega\}.
\]
We need the following result.

**Lemma 3.6.** Assume $\text{int}\Omega \neq \emptyset$. For every $y \in \Omega \setminus \text{int}\Omega$ satisfying $\lambda y \notin \overline{cl}\Omega$ for all $\lambda > 1$, there is $z \in n(y) \cap C$ such that $\langle z, y \rangle > 0$.

**Proof.** Fix $y$ satisfying the condition. For every open ball $B \subset H$ centered at $y$, there is some $\lambda > 1$ such that $y' = \lambda y \in C \cap (B \setminus \overline{cl}\Omega)$. Due to $\text{int}\Omega \neq \emptyset$ and $y \in \Omega$, by Lemma 3.1, there is some $y'' \in B \cap \text{int}\Omega \subset \text{int}C$. For $\rho \in [0, 1]$, we set $y_\rho = \rho y' + (1 - \rho)y'' \in B$.

Let us set
\[
\rho_0 = \sup\{\rho \in [0, 1] : y_\rho \in \text{int}\Omega\}.
\]
Since $y' \notin \overline{cl}\Omega$, we must have $\rho_0 < 1$. It can be seen that $y_{\rho_0} \in \overline{cl}\Omega \setminus \text{int}\Omega$ and thus $y_{\rho_0} \in B \cap \text{bd}\Omega$. Due to $y' \in C$, $y'' \in \text{int}C$ and Lemma 3.1, we have $y_{\rho_0} \in \text{int}C$. In summary, we obtain $y_{\rho_0} \in B \cap \text{bd}\Omega \cap \text{int}C$.

By this construction and varying the size of the open balls centered at $y$, we can find a sequence $\{y_n\}_{n=1}^\infty$ such that
\[
y_n \in \text{int}C,
\]
\[
y_n \in \text{bd}\Omega,
\]
\[
\lim_{n \to \infty} y_n = y.
\]
Fix any $n$. By (3.18), there is $\delta > 0$ such that
\[
y_n + B(0, 2\delta) \subset C.
\]
Here, for \( r > 0 \), we write \( B(0, r) = \{ z \in \mathcal{H} : |z| < r \} \). For each \( \epsilon \in (0, \delta) \), due to (3.19), we can also find \( y_{n,\epsilon} \) such that
\[
(3.22) \quad y_{n,\epsilon} \in \Omega, \\
(3.23) \quad |y_{n,\epsilon} - y_n| < \epsilon.
\]
This and (3.21) imply that
\[
(3.24) \quad y_{n,\epsilon} - a \in \mathcal{C}, \quad \forall \epsilon \in (0, \delta), \quad a \in B(0, \delta).
\]
By \( \mathcal{C} \)-nondecreasingness, (3.22) and (3.24), we can see
\[
(3.25) \quad \langle z_n, y_{n,\epsilon} - a - y_n \rangle \leq 0,
\]
which along with (3.23) implies
\[
\langle z_n, a \rangle \geq -|z_n|\epsilon.
\]
Sending \( \epsilon \to 0 \) and varying \( a \in \mathcal{C} \cap B(0, \delta) \), by the self-duality of \( \mathcal{C} \), we conclude that \( z_n \in \mathcal{C} \). Hence, we just showed
\[
\{0\} \subset n(y_n) \subset \mathcal{C}, \quad \forall n.
\]
Now for each \( n \), rescale \( z_n \in n(y_n) \cap \mathcal{C} \) to get \( |z_n| = 1 \). Since \( \mathcal{C} \cap \text{cl} B(0, 1) \) is convex, closed, and bounded, by passing to a subsequence, we can assume that there is \( z \in \mathcal{C} \) such that \( z_n \) converges weakly to \( z \). Since we know \( z_n \in n(y_n) \), we get
\[
(3.26) \quad \lim_{n \to \infty} \langle z_n, w - y_n \rangle = \langle z, w - y \rangle, \quad \forall w \in \Omega.
\]
The above two displays yield \( z \in n(y) \cap \mathcal{C} \).

Then, we show \( \langle z, y \rangle > 0 \). Since \( \text{int} \Omega \neq \emptyset \), there is some \( x_0 \in \text{int} \Omega \subset \text{int} \mathcal{C} \). Therefore, there is some \( \epsilon > 0 \) such that
\[
x_0 - \epsilon u \in \mathcal{C}, \quad \forall u \in \text{cl} B(0, 1).
\]
This along with \( |z_n| = 1 \) implies
\[
0 \leq \epsilon z_n \leq x_0, \quad \forall n.
\]
The \( \mathcal{C} \)-nondecreasingness of \( f \) implies that
\[
\epsilon z_n \in \Omega, \quad \forall n.
\]
Replace \( w \) by \( \epsilon z_n \) in (3.25) to obtain
\[
\langle z_n, y_n \rangle \geq \epsilon, \quad \forall n.
\]
Since \( z_n \) converges to \( z \) weakly and \( y_n \) converges to \( y \) strongly, by passing to the limit in the above display, we get
\[
\langle z, y \rangle \geq \epsilon > 0
\]
completing the proof. □
We now go back to our main proof and apply Lemma 3.6 to \( x' \in \Omega \). Hence, there is \( z \in \mathbf{n}(x') \cap C \) such that
\[
\langle z, w - x' \rangle \leq 0, \quad \forall w \in \Omega,
\]
(3.26)
\[
\langle z, x' \rangle > 0.
\]
(3.27)

By (3.15) and Lemma 3.5, there is an affine function \( L_{a,\nu} \) with \( a \in C \) and \( \nu \in \mathbb{R} \) such that \( f \geq L_{a,\nu} \). For each \( \rho \geq 0 \), define
\[
L_{\rho} = L_{a,\nu} + \rho \langle z, x' \rangle.
\]
Due to (3.26), we can see that
\[
L_{\rho}(w) = L_{a,\nu}(w) + \rho \langle z, w - x' \rangle \leq L_{a,\nu}(w) \leq f(w), \quad \forall w \in \Omega.
\]
(3.28)

On the other hand, recall \( x' = \lambda' x \) given below (3.16). Evaluating \( L_{\rho} \) at \( x \), we have
\[
L_{\rho}(x) = L_{a,\nu}(x) + \rho \langle z, x - x' \rangle = L_{a,\nu}(x) + \rho (\lambda'^{-1} - 1) \langle z, x' \rangle.
\]
By (3.16) and (3.27), we obtain
\[
\lim_{\rho \to \infty} L_{\rho}(x) = \infty = f(x).
\]
This along with (3.28), Lemma 3.5 and (3.5) implies
\[
f(x) = f^{**}(x) \quad \forall x \in C \setminus \text{cl}\Omega.
\]
In view of this and (3.15), we have completed the proof of Proposition 2.3.

4. PROOF OF THEOREM 2.2

We devoted this section to the proof of Theorem 2.2. As commented in the beginning of the proof of Proposition 2.3 in Section 3.2, assuming \( f = f^{**} \), we have \( f \in \Gamma_{>}(\mathcal{C}) \).

Now, let \( \mathcal{C} \) be a perfect cone and \( f : \mathcal{C} \to (-\infty, +\infty] \) be proper. Assuming \( f \in \Gamma_{>}(\mathcal{C}) \), we want to prove \( f = f^{**} \). Again, we write \( \Omega = \text{dom} f \) which is a nonempty subset of \( \mathcal{C} \). Let us introduce
\[
\mathcal{F}_{\Omega} = \{ \lambda y : \lambda \geq 0, y \in \Omega \}.
\]
We will first show that \( f = f^{**} \) holds on \( \mathcal{F}_{\Omega} \) and then on \( \mathcal{C} \).

4.1. Identity on \( \mathcal{F}_{\Omega} \). We prove \( f = f^{**} \) on \( \mathcal{F}_{\Omega} \). The idea is to show \( \Omega \) has nonempty interior relative to \( \mathcal{F}_{\Omega} \) and apply Proposition 2.3 to \( f \) restricted to \( \mathcal{F}_{\Omega} \). Some properties of \( \mathcal{F}_{\Omega} \) are needed and they are stated and proved in the two lemmas below.

Lemma 4.1. The set \( \mathcal{F}_{\Omega} \) is a face of \( \mathcal{C} \).

Proof. Recall the definition of a face above Definition 2.1. Since in this work, we require cones to be convex, to show \( \mathcal{F}_{\Omega} \) is a face, we start by checking it is convex. Note that for any \( x_1, x_2 \in \mathcal{F}_{\Omega} \), there are \( \lambda_1, \lambda_2 \geq 0 \) and \( y_1, y_2 \in \Omega \) such that \( x_i = \lambda_i y_i \) for \( i = 1, 2 \). We can choose \( \mu > 0 \) large enough so that \( \frac{\lambda_i}{\mu} y_i \leq y_i \) for all \( i \). Hence, by
the $\mathcal{C}$-nondecreasingness of $f$, we have $\frac{\lambda_i}{\mu} y_i \in \Omega$ for all $i$. Then, for each $\alpha \in [0, 1]$, it holds that

$$\alpha x_1 + (1 - \alpha)x_2 = \mu \left( \alpha \frac{\lambda_1}{\mu} y_1 + (1 - \alpha) \frac{\lambda_2}{\mu} y_2 \right).$$

By the convexity of $\Omega$, we have $\alpha \frac{\lambda_1}{\mu} y_1 + (1 - \alpha) \frac{\lambda_2}{\mu} y_2 \in \Omega$. Hence, we conclude that $\alpha x_1 + (1 - \alpha)x_2 \in \mathcal{F}_\Omega$, which implies that $\mathcal{F}_\Omega$ is convex. Then, it is easy to see $\mathcal{F}_\Omega$ is a cone.

Now let $0 \leq x \leq y$ and $y \in \mathcal{F}_\Omega$. By definition, there is $\mu > 0$ such that $\mu y \in \Omega$. We can deduce that $0 \leq \mu x \leq \mu y$. Again, the $\mathcal{C}$-nondecreasingness implies $\mu x \in \Omega$ and thus $x \in \mathcal{F}_\Omega$.

**Lemma 4.2.** Given the perfectness of $\mathcal{C}$, the subset $\Omega$ has nonempty interior with respect to the space $\text{span} \mathcal{F}_\Omega$.

**Proof.** For positive integers $m, n \in \mathbb{N}_+$, we set $E_{m,n} = \{ mx \in \mathcal{H} : f(x) \leq n \}$ which is the level set $\{ f \leq n \}$ scaled by $m$. We want to show

$$\mathcal{F}_\Omega = \bigcup_{m,n \in \mathbb{N}_+} E_{m,n}. \tag{4.1}$$

For each $x \in \mathcal{F}_\Omega$, there is $\mu > 0$ such that $y = \mu x \in \Omega$. Then, there is $n \in \mathbb{N}_+$ such that $y \in \{ f \leq n \}$. Choose $m \in \mathbb{N}$ to satisfy $m \mu \geq 1$. Since $f$ is $\mathcal{C}$-nondecreasing and $0 \leq \frac{1}{m \mu} y \leq y$, it yields that $\frac{1}{m \mu} y \in \{ f \leq n \}$, which implies that $x \in E_{m,n}$. The other direction is easy by the definition of $\mathcal{F}_\Omega$. Therefore, we have verified (4.1).

Since $f$ is l.s.c., we know that every $E_{m,n}$ is closed. As a closed subspace of $\mathcal{H}$, the space $\text{span} \mathcal{F}_\Omega$ is complete. On the other hand, by the perfectness of $\mathcal{C}$ and Definition 2.1, the face $\mathcal{F}_\Omega$ also has nonempty interior in $\text{span} \mathcal{F}_\Omega$. Hence, invoking the Baire category theorem (c.f. [14]) and taking into account (4.1), we can deduce that there is a pair $m, n$ such that $E_{m,n}$ has nonempty interior in $\text{span} \mathcal{F}_\Omega$. This implies that the interior of $\{ f \leq n \} \subset \Omega$ relative to $\text{span} \mathcal{F}_\Omega$ is nonempty. Hence, we conclude that $\Omega$ has nonempty interior.

Let us set $\mathcal{C}' = \mathcal{F}_\Omega$, $\mathcal{H}' = \text{span} \mathcal{F}_\Omega$ and $f'$ be the restriction of $f$ to $\mathcal{F}_\Omega$. Since $\Omega \subset \mathcal{F}_\Omega$, it is immediate that $\text{dom} f' = \Omega \subset \mathcal{C}'$. Also, due to $f \in \Gamma^\prime \mathcal{C}$, we have $f' \in \Gamma^\prime \mathcal{C}'$. Lemma 4.1, the perfectness of $\mathcal{C}$ and Definition 2.1 imply that $\mathcal{C}'$ is self-dual in $\mathcal{H}'$. Lemma 4.2 guarantees that $\text{dom} f'$ has nonempty interior in $\mathcal{C}'$. Therefore, invoking Proposition 2.3, we obtain

$$f'(x) = f^{*\prime \prime'}(x), \quad \forall x \in \mathcal{C}'. \tag{4.2}$$

Here, $\ast'$ is the monotone conjugate with respect to $\mathcal{C}'$ given by

$$g^{*'}(y) = \sup_{z \in \mathcal{C}'} \{ (z, y) - g(z) \}, \quad \forall y \in \mathcal{C}'. \tag{4.3}$$

Due to $\Omega \subset \mathcal{C}'$, we have $f(z) = \infty$ for all $z \not\in \mathcal{C}'$. Recall (2.4). Hence, we have

$$f^{*'}(y) = f^{*'}(y) = f^{**}(y), \quad \forall y \in \mathcal{C}'. \tag{4.4}$$

By the formula for $f^{**}$ in (3.3) where the supremum is taken over $\mathcal{C}$ which contains $\mathcal{C}'$, the above display implies that

$$f^{**}(x) \geq f^{*\prime \prime'}(x), \quad \forall x \in \mathcal{C}'. \tag{4.5}$$
This along with (4.2) and \( f = f' \) on \( C' \) yields \( f^{**} \geq f \) on \( C' \). Lastly, from (3.5), we conclude that
\[
(4.3) \quad f(x) = f^{**}(x), \quad \forall x \in \mathcal{F}_\Omega.
\]

4.2. Identity on \( C \). Due to (4.3), we only need to show \( f(x) = f^{**}(x) \) for \( x \in C \setminus C' \).

To start, we record useful properties for faces in the ensuing two lemmas. Note that from the discussion below Definition 2.1 we have \( \text{int}C \neq \emptyset \) if \( C \) is perfect.

**Lemma 4.3.** Let \( F \) be a face of a cone \( C \subset H \). If \( \text{int}C \neq \emptyset \) and \( F \neq C \), then \( \text{int}F = \emptyset \) and thus \( F \subset \text{bd}C \).

**Proof.** Let us argue by contradiction and suppose that there is \( x \in \text{int}F \subset C \). Then for every \( y \in C \), we can find \( \epsilon > 0 \) small so that \( x + \epsilon y \in F \). Since \( x \in C \), we have \( 0 \leq \epsilon y \leq x + \epsilon y \). Then, the definition of faces implies that \( \epsilon y \in F \). Lastly, since \( F \) is a cone and \( \epsilon > 0 \), we obtain \( y \in F \) which implies \( C \subset F \) and thus \( C = F \). We arrive at a contradiction to the assumption that \( F \neq C \). As it is clear that \( \text{int}F \subset \text{int}C \), we also have \( F \subset \text{bd}C \). \( \square \)

**Lemma 4.4.** Suppose that \( C \) is a perfect and that \( F \) is a face of \( C \). For every \( x \in C \setminus F \), there is \( v \in C \) such that \( \langle v, x \rangle > 0 \) and
\[
\langle v, y \rangle = 0, \quad \forall y \in F.
\]

**Proof.** We take \( F' \) to be the intersection of all faces of \( C \) containing both \( F \) and \( x \). It can be checked that \( F' \) is again a face of \( C \). Hence, \( F' \) is the minimal face containing both \( F \) and \( x \). Let us write \( H' = \text{span}F' \) and denote by \( \bar{F}' \) the interior of \( F' \) with respect to \( H' \). Since \( C \) is perfect, we have \( \bar{F}' \neq \emptyset \) and that \( F' \) is self-dual in \( H' \) and thus closed. Since \( F \) is clearly a face of \( F' \), Lemma 4.3 applied to \( F \subset F' \) yields \( F \cap \bar{F}' = \emptyset \).

By the Hahn–Banach separation theorem (c.f. [8, Theorem 1.6]), there are \( \alpha \in \mathbb{R} \) and a nonzero vector \( v \in H' \) such that
\[
(4.4) \quad \langle v, y \rangle \leq \alpha, \quad \forall y \in F,
\]
\[
(4.5) \quad \langle v, z \rangle \geq \alpha, \quad \forall z \in \bar{F}'.
\]
Since \( F' \) is closed and convex, and \( \bar{F}' \neq \emptyset \), by [6, Proposition 3.36 (iii)], we have that the closure of \( \bar{F}' \) is \( F' \). Hence, (4.5) becomes
\[
(4.6) \quad \langle v, z \rangle \geq \alpha, \quad \forall z \in F'.
\]
By (2.2), we have \( 0 \in F \). Due to this and \( F \subset F' \), using (4.4) and (4.6), we must have \( \alpha = 0 \) and
\[
(4.7) \quad \langle v, y \rangle = 0, \quad \forall y \in F.
\]
Then, (4.6) is turned into
\[
(4.8) \quad \langle v, z \rangle \geq 0, \quad \forall z \in F'.
\]
Since \( C \) is perfect and \( F' \) is a face of \( C \), we know that \( F' \) is self-dual in \( H' \). Due to (4.8), the vector \( v \) belongs to the dual of \( F' \) which yields
\[
(4.9) \quad v \in F' \subset C.
\]

Now, we consider the null space of the linear map \( y \mapsto \langle v, y \rangle \) given by
\[
(4.10) \quad \mathcal{E} = \{ y \in H : \langle v, y \rangle = 0 \}.
\]
We want to show $E \cap F'$ is a face of $C$. It is clear that $E \cap F'$ is a cone. For $y \in E \cap F'$ and $z \in C$ satisfying $0 \leq z \leq y$, by (4.9) and the self-duality of $C$, we obtain
\[
\langle v, y - z \rangle \geq 0,
\]
\[
\langle v, z \rangle \geq 0.
\]
Due to $y \in E$, the above two displays yield $\langle v, z \rangle = 0$ which implies that $z \in E$. Since $F'$ is a face, by $0 \leq z \leq y$ and $y \in F'$, we also have $z \in F'$. Hence, we have $z \in E \cap F'$ and thus verified that $E \cap F'$ is a face of $C$. Due to (4.9) and $v \neq 0$, we can deduce
\[
E \cap F' \neq F'.
\]

To conclude, we argue that
\[
(4.12) \quad x \notin E.
\]
Otherwise, since $F'$ contains $x$ by the definition of $F'$, we have $x \in E \cap F'$. From (4.7) and (4.10), we can deduce that $F \subset E$ and thus $F \subset E \cap F'$. Therefore, $E \cap F'$ is a face containing both $x$ and $F$. However, this together with (4.11) contradicts the fact that $F'$ is chosen to be the minimal face containing $x$ and $F$.

Therefore, by contradiction, we conclude that (4.12) must hold. By $x \in C$ and (4.9), we must have $\langle v, x \rangle > 0$. In view of this, (4.7) and (4.9), the vector $v$ satisfies all the desired properties.

With these results, we resume the proof of $f = f^{**}$ on $C \setminus F_\Omega$. Fix any $x \in C \setminus F_\Omega$. For each $\rho > 0$, we set
\[
\mathcal{L}_\rho = L_{\rho v, f(0)},
\]
with $v$ given in Lemma 4.4 corresponding to this $x$ and $F = F_\Omega$. This lemma implies that $v$ is perpendicular to $F_\Omega$ and thus
\[
\mathcal{L}_\rho(y) = \rho \langle v, y \rangle + f(0) = f(0), \quad \forall y \in F_\Omega.
\]
Then, the $C$-nondecreasingness of $f$ implies that
\[
f(y) \geq \mathcal{L}_\rho(y), \quad \forall y \in F_\Omega.
\]
Since we know $f = \infty$ on $C \setminus F_\Omega$, we obtain
\[
f \geq \mathcal{L}_\rho, \quad \forall \rho > 0.
\]
On the other hand, due to $\langle v, x \rangle > 0$ in Lemma 4.4, we have
\[
\lim_{\rho \to \infty} \mathcal{L}_\rho(x) = \infty = f(x).
\]
Hence, by the above two displays, (3.5) and Lemma 3.5, we conclude that $f(x) = f^{**}(x)$ for $x \in C \setminus F_\Omega$. This together with (4.3) completes the proof of Theorem 2.2.

5. Examples of perfect cones

We show that the set of positive semidefinite matrices is a perfect cone, and that an infinite-dimensional circular cone is perfect.
5.1. **Positive semidefinite matrices.** Let \( n \in \mathbb{N} \setminus \{0\} \) and denote by \( S^n \) the set of all \( n \times n \) symmetric matrices, by \( S^n_+ \) the set of all \( n \times n \) positive semidefinite matrices, and by \( S^n_{++} \) the set of all \( n \times n \) positive definite matrices. On \( S^n \), we define the inner product by

\[
\langle x, y \rangle = \text{tr}(x^ty), \quad \forall x, y \in S^n,
\]

where \( \text{tr} \) is the trace of a matrix and \( x^t \) is the transpose of \( x \). Hence, \( S^n \) is a Hilbert space with dimension \( n(n+1)/2 \). The goal is the following.

**Lemma 5.1.** For each positive integer \( n \), the set \( S^n_+ \) is a perfect cone in \( S^n \).

To start, it is well-know that \( S^n_+ \) is self-dual, which is attributed often to Fejér (see, e.g. [10, Theorem 7.5.4]). For completeness of presentation, we prove it below.

**Lemma 5.2.** Let \( x \in S^n \). Then, \( x \in S^n_+ \) if and only if \( \langle x, y \rangle \geq 0 \) for every \( y \in S^n_+ \).

**Proof.** If \( x \in S^n_+ \), then for any \( y \in S^n_+ \) we have \( \langle x, y \rangle = \text{tr}(\sqrt{y}x\sqrt{y}) \geq 0 \). For the other direction, by choosing an orthonormal basis, we may assume that \( x \) is diagonal. Testing by \( y \in S^n_+ \), we can show that all diagonal entries in \( x \) are nonnegative and thus \( x \in S^n_+ \). \( \square \)

**Proof of Lemma 5.1.** Given the above lemma, we only need to verify the conditions on the faces of \( S^n_+ \) stated in Definition 2.1.

The cases \( \mathcal{F} = \{0\} \) and \( \mathcal{F} = S^n_0 \) are trivial, so we assume \( \mathcal{F} \setminus \{0\} \neq \emptyset \) and \( \mathcal{F} \neq S^n_+ \). Lemma 4.3 implies \( \mathcal{F} \subset \text{bd} S^n_+ = S^n_0 \setminus S^n_{++} \). Set

\[
(5.1) \quad m = \max \{\text{rank}(z) : z \in \mathcal{F}\},
\]

where \( \text{rank}(z) \) is the rank of the matrix \( z \). By our assumption on \( \mathcal{F} \), we must have \( 1 \leq m < n \). For each \( k \in \mathbb{N} \setminus \{0\} \), we denote by \( 0_k \) the \( k \times k \) zero matrix. Due to (5.1), there is \( x \in \mathcal{F} \) with \( \text{rank}(x) = m \). By fixing a suitable orthonormal basis, we may assume

\[
(5.2) \quad x = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m, 0_{n-m}),
\]

where \( \lambda_j > 0 \) for all \( 1 \leq j \leq m \).

Let us consider the following set

\[
(5.3) \quad \mathcal{E} = \{ \text{diag}(y^\circ, 0_{n-m}) : y^\circ \in S^n_+ \} \subset S^n_+.
\]

We now show \( \mathcal{E} = \mathcal{F} \). First, we want to prove \( \mathcal{F} \subset \mathcal{E} \). In other words, we claim that for every \( y \in \mathcal{F} \), there is \( y^\circ \in S^n_+ \) such that

\[
(5.4) \quad y = \text{diag}(y^\circ, 0_{n-m}).
\]

Let us argue by contradiction. Suppose that (5.4) does not hold for all \( y \in \mathcal{F} \), then we can find \( y \in \mathcal{F} \) with \( y_{jk} \neq 0 \) for some \( j > m \) or \( k > m \). Due to this and \( y \in S^n_+ \), there must be some \( i > m \) such that \( y_{ii} > 0 \). By reordering the basis, we may assume \( i = m+1 \), and thus

\[
(5.5) \quad y_{m+1,m+1} > 0.
\]

Let \( \hat{y} = (y_{ij})_{i \leq j \leq m+1} \in S^{m+1}_+ \), and we define \( \hat{x} \) similarly. Then, we want to show \( \text{rank}(\hat{x} + \hat{y}) = m+1 \). Let \( v \in \mathbb{R}^{m+1} \setminus \{0\} \). If \( v_j \neq 0 \) for some \( 1 \leq j \leq m \), then we have

\[
v^t(\hat{x} + \hat{y})v \geq v^t\hat{x}v > 0.
\]
The last inequality follows from (5.2). If \( v_j = 0 \) for all \( 1 \leq j \leq m \), then due to \( v \neq 0 \), we must have \( v_{m+1} \neq 0 \), and by (5.5), we get
\[
v^T(\hat{x} + \hat{y})v \geq v^T\hat{y}v = y_{m+1,m+1}v_{m+1}^2 > 0.
\]
In conclusion, we obtain \( v^T(\hat{x} + \hat{y})v > 0 \), which implies that \( \hat{x} + \hat{y} \in S_{++}^{n+1} \) and thus \( \text{rank}(x + y) \geq \text{rank}(\hat{x} + \hat{y}) = m + 1 \). Since \( \mathcal{F} \) is a cone, we have \( x + y \in \mathcal{F} \). But this contradicts the maximality of \( m \) as in (5.1). Hence, every \( y \in \mathcal{F} \) satisfies (5.4), and thus we verified \( \mathcal{F} \subset \mathcal{E} \).

Now, we turn to the proof of \( \mathcal{E} \subset \mathcal{F} \). For every \( y \) of the form (5.4), due to (5.2), there exists a small \( \epsilon > 0 \) such that \( x \geq \epsilon y \geq 0 \). Here, this partial order \( \geq \) is induced by the cone \( S_{+}^n \). Recall the definition of faces above Definition 2.1. Since \( \mathcal{F} \) is a face, we must have \( \epsilon y \in \mathcal{F} \) and thus \( y \in \mathcal{F} \). Hence, we conclude \( \mathcal{E} \subset \mathcal{F} \).

Now, we have \( \mathcal{F} = \mathcal{E} \). In view of (5.3), we can identify \( \mathcal{F} \) with \( S_{+}^n \) and \( \text{span} \mathcal{F} \) with \( S^n \). We know that \( S_{+}^n \) is self-dual by Lemma 5.2. In addition, in finite dimensions, self-dual cones always have nonempty interior (c.f. [6, Exercise 6.15]). Therefore, all conditions on \( \mathcal{F} \) in Definition 2.1 are verified. \( \square \)

5.2. An infinite-dimensional circular cone. Let \( \mathcal{H} = l^2(\mathbb{N}) \) where the elements in \( l^2(\mathbb{N}) \) are precisely \( x = (x_0, x_1, x_2, \ldots) \) with \( \sum_{i=0}^{\infty} x_i^2 < \infty \). The inner product on \( \mathcal{H} \) is given by
\[
\langle x, y \rangle = \sum_{i=0}^{\infty} x_i y_i, \quad \forall x, y \in \mathcal{H}.
\]
We denote by \( |\cdot| \) the associated norm. For each \( x \in \mathcal{H} \), we write \( x_{\geq 1} = (0, x_1, x_2, \ldots) \in \mathcal{H} \). We consider the cone \( \mathcal{C} \) defined by
\[
\mathcal{C} = \{ x \in \mathcal{H} : x_0 \geq 0, |x_{\geq 1}| \leq x_0 \}.
\]
The desired result is stated below.

Lemma 5.3. The cone \( \mathcal{C} \) defined in (5.6) is perfect in \( \mathcal{H} \).

To prove perfectness, we need information about its faces. The next lemma classifies all faces of \( \mathcal{C} \). The definition of faces are given above Definition 2.1.

Lemma 5.4. Under the above setting, if \( \mathcal{F} \) is a face of \( \mathcal{C} \), then either \( \mathcal{F} = \mathcal{C} \) or there is \( x \in \text{bd} \mathcal{C} \) such that \( \mathcal{F} = \{ \lambda x : \lambda \geq 0 \} \).

Proof. It is clear that \( \mathcal{C} \) is a face of itself. Now we consider the case \( \mathcal{F} \neq \mathcal{C} \). If \( \mathcal{F} = \{0\} \), then there is nothing to prove. Hence, let us further assume that there is a nonzero \( x \in \mathcal{F} \subset \mathcal{C} \). In particular, due to (5.6), we have \( x_0 > 0 \). Lemma 4.3 implies \( \mathcal{F} \subset \text{bd} \mathcal{C} \). By (5.10) to be proved later and the definition of \( \mathcal{C} \), the vector \( x \) satisfies
\[
x_{\geq 1} = x_0 > 0.
\]
By definition of faces, \( \mathcal{F} \) is a cone. Due to this and \( x \in \mathcal{F} \), we have
\[
\mathcal{F} \supset \{ \lambda x : \lambda \geq 0 \}.
\]
Suppose that the inclusion is strict and thus there is \( y = \mathcal{F} \setminus \{ \lambda x : \lambda \geq 0 \} \) which is not a multiple of \( x \). It is clear that \( y \) is nonzero and thus we have \( y_0 > 0 \) similarly. Rescaling if needed, we may assume \( y_0 = x_0 \). Recall that in this work, convexity is
built into the definition of cones. Set \( z = \frac{1}{2}(x + y) \). Using Jensen’s inequality, by (5.7) and an analogous one for \( y \), we obtain
\[
|z_{\geq 1}|^2 = \sum_{i=1}^{\infty} \left( \frac{x_i + y_i}{2} \right)^2 \leq \sum_{i=1}^{\infty} \frac{x_i^2 + y_i^2}{2} = \frac{1}{2} (x_0^2 + y_0^2) = z_0^2.
\]
Note that the equality holds only if \( x_i = y_i \) for all \( i \), so the above inequality must be strict. Hence, we have \(|z_{\geq 1}| < z_0\). By (5.10), we thus have \( F \cap \text{int} C \neq \emptyset \). This contradicts the fact that \( F \subseteq \text{bd} C \). Therefore, we must have \( F \setminus \{ \lambda x : \lambda \geq 0 \} = \emptyset \). In view of (5.8), this implies \( F = \{ \lambda x : \lambda \geq 0 \} \) and whence the proof is complete. 

\( \square \)

**Proof of Lemma 5.3.** We first show that \( C \) is self-dual. Recall that the dual cone is defined in (2.1) and denoted by \( C^\circ \). Let \( y \in C^\circ \) and we have
\[
\langle x, y \rangle \geq 0, \quad \forall x \in C.
\]
Since \((1,0,0,\ldots) \in C\), we get \( y_0 \geq 0 \). We consider two cases depending on whether \( y_0 = 0 \) or not. Suppose \( y_0 = 0 \), for any fixed \( i \geq 1 \), we construct \( x' \) in the following way. Set \( x'_0 = 1 \), set \( x'_i = -1 \) if \( y_i \geq 0 \) and \( x'_i = 1 \) if \( y_i < 0 \), and lastly set all other entries of \( x' \) to be zero. Inserting this \( x' \) into (5.9) and varying \( i \), we can see \( y = 0 \) and thus \( y \in C \). Now we consider the case where \( y_0 > 0 \). If \( |y_{\geq 1}| = 0 \), then this immediately implies \( y \in C \). If \(|y_{\geq 1}| \neq 0\), then we set \( \gamma = |y_{\geq 1}|^{-1} > 0 \) and consider \( x' \) given by
\[
x'_0 = y_0; \quad x'_i = -\gamma y_i y_0.
\]
Plugging \( x' \) into (5.9) and using \( y_0 > 0 \), we obtain \( y_0 \geq |y_{\geq 1}| \) and thus \( y \in C \). Hence, \( C \) is self-dual.

Next, we show \( C \) has nonempty interior and more precisely
\[
\text{int} C = \{ x \in H : \ x_0 > 0, \ |x_{\geq 1}| < x_0 \}.
\]
Let \( y \) belong to the set on the right hand side of the above display. Choose \( \epsilon > 0 \) such that \( y_0 - |y_{\geq 1}| > 2\epsilon \). Then, we want to show that, for all \( x \in H \) satisfying \(|x - y| < \epsilon\), we have \( x \in C \). We can see that
\[
(x_0 - y_0)^2 + |x_{\geq 1} - y_{\geq 1}|^2 = |x-y|^2 < \epsilon^2.
\]
This yields \(|x_0 - y_0| < \epsilon \) and \(|x_{\geq 1} - y_{\geq 1}| < \epsilon \). Now, using these, the property of \( \epsilon \) and the triangle inequality, we get
\[
|x_{\geq 1}| \leq |y_{\geq 1}| + \epsilon < (y_0 - 2\epsilon) + \epsilon = y_0 - \epsilon \leq x_0.
\]
Hence, we have \( x \in C \) and can deduce that the right side of (5.10) is contained in \( \text{int} C \). For the other direction, let \( y \in C \) with \( |y_{\geq 1}| = y_0 \). It is easy to see that every neighborhood of \( y \) contains a point not in \( C \). Therefore, we conclude that (5.10) holds.

To show \( C \) is perfect, it remains to check the conditions on the faces of \( C \) stated in Definition 2.1. By (5.10), we have \( \text{int} C \neq \emptyset \) and \( \text{span} C = H \). We have already shown \( C \) is self-dual. Hence, if \( F = C \), then \( F \) is self-dual and has nonempty interior with respect to \( \text{span} F \). Now if \( F \neq C \), then Lemma 5.4 implies \( F = \{ \lambda x : \lambda \geq 0 \} \), which is one-dimensional. We can identify \( \text{span} F \) with \( \mathbb{R} \) and \( F \) with \([0, \infty) \) in an isometric way. Now, it is easy to see that \( F \) is self-dual and has nonempty interior with respect to \( \text{span} F \). By Definition 2.1, we conclude that \( C \) given in (5.6) is perfect.

\( \square \)
REFERENCES

[1] M. Bardi and L. C. Evans. On Hopf's formulas for solutions of Hamilton–Jacobi equations. Nonlinear Analysis: Theory, Methods & Applications, 8(11):1373–1381, 1984.
[2] G. P. Barker. Faces and duality in convex cones. Linear and Multilinear Algebra, 6(3):161–169, 1978.
[3] G. P. Barker. Perfect cones. Linear Algebra and its Applications, 22:211–221, 1978.
[4] G. P. Barker. Theory of cones. Linear Algebra and its Applications, 39:263–291, 1981.
[5] G. P. Barker and J. Foran. Self-dual cones in Euclidean spaces. Linear Algebra and its Applications, 13(1-2):147–155, 1976.
[6] H. H. Bauschke, P. L. Combettes, et al. Convex analysis and monotone operator theory in Hilbert spaces, volume 408. Springer, 2011.
[7] J. Bellissard, B. Iochum, and R. Lima. Homogeneous and facially homogeneous self-dual cones. Linear Algebra and its Applications, 19(1):1–16, 1978.
[8] H. Brezis. Functional analysis, Sobolev spaces and partial differential equations. Springer Science & Business Media, 2010.
[9] H.-B. Chen and J. Xia. Hamilton–Jacobi equations for inference of matrix tensor products. arXiv preprint arXiv:2009.01678, 2020.
[10] R. A. Horn and C. R. Johnson. Matrix analysis. Cambridge university press, 2012.
[11] P.-L. Lions and J.-C. Rochet. Hopf formula and multitime Hamilton–Jacobi equations. Proceedings of the American Mathematical Society, 96(1):79–84, 1986.
[12] R. C. Penney. Self-dual cones in Hilbert space. Journal of Functional Analysis, 21(3):305–315, 1976.
[13] R. T. Rockafellar. Convex analysis. Princeton university press, 1970.
[14] H. Royden and P. Fitzpatrick. Real Analysis. Prentice Hall, 2010.

(Hong-Bin Chen) COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, NEW YORK, NEW YORK, USA

Email address: hbchen@cims.nyu.edu

(Jiaming Xia) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA, USA

Email address: xiajiam@sas.upenn.edu