Off-diagonal long-range order in a harmonically confined two-dimensional Bose gas

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We investigate the presence of off-diagonal long-range order in a harmonically confined two-dimensional Bose gas. In the noninteracting case, an analytical calculation of the finite-temperature one-particle density matrix provides an exact description of the spatial correlations known to be associated with the existence of a Bose-Einstein condensate below the transition temperature \( T_c^{(0)} \). We treat the effects of repulsive interactions within the semiclassical Hartree-Fock-Bogliubov approximation and find that even though the system remains in the same undecondensed phase for all \( T \geq 0 \), there appears to be a revival of off-diagonal long-range order for temperatures \( T < T_c^{(0)} \). We suggest that this reentrant order is related to a phase transition in the system which is not the BEC state.

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I. INTRODUCTION

Recent advances in the controlled fabrication and manipulation of ultra-low temperature trapped Bose gases \(^{1,2}\) has finally made it possible to experimentally investigate the existence of off-diagonal long-range order (ODLRO), i.e., Bose-Einstein condensation (BEC), in dimensions lower than three. This remarkable experimental achievement has rekindled an interest in the classic theoretical problem concerning the existence of BEC in two-dimensions (2D). Although it is well known that finite-temperature BEC can never occur in a homogeneous 2D Bose gas \(^{3}\), the ideal trapped Bose gas can undergo BEC below some critical temperature \( T_c^{(0)} \). In the case of a 2D isotropic harmonic trap with confining frequency \( \omega_0 \) (unless stated otherwise, we always implicitly assume this confinement geometry), Bagnato and Kleppner have shown that \( 1/\beta_c^{(0)} = \hbar \omega_0 \sqrt{6N/\pi^2} \), where \( \beta = 1/(k_B T) \); Shevchenko later extended this result to include more general power law potentials \(^{4}\). In the thermodynamic limit, viz., \( N \rightarrow \infty, \omega_0 \rightarrow 0 \), such that \( N^{1/2} \omega_0 = \text{constant} \), BEC is also known to theoretically occur with the critical temperature remaining unchanged. Note that this is not the usual thermodynamic limit, which in 2D would demand that \( N \omega_0 = \text{constant} \), resulting in no BEC (see also Ref. \(^{5}\) for an alternative thermodynamic limit in the case of an interacting Bose gas).

If repulsive interactions between bosons are included, (we only consider repulsive interactions in this paper), then a definitive answer to the question of BEC in the trapped 2D gas is not so clear. In fact, there is no \( a \) priori reason for the existence of BEC in a confined two or three-dimensional system—BEC is a purely kinematical phenomenon in the sense that even the ideal Bose gas can undergo the phase transition. In contrast, a transition to the superfluid state is wholly dynamical and \( cannot \) occur without interactions. Therefore, one could sensibly assume that interactions prevent the spatial redistribution of atoms required to achieve Bose gas degeneracy, and therefore destroy the BEC phase transition altogether.

It is only recently that a rigorous proof has been given to show that there is no finite temperature BEC in the interacting trapped 2D Bose gas in the thermodynamic limit \(^{3}\). Of course, in actual experiments, the thermodynamic limit is never fully realized, and it becomes necessary to consider systems consisting of a \( \text{finite} \) number of atoms confined in a trap. In this regard, Lieb and Seiringer \(^{6}\) have proved that the finite, interacting trapped Bose gas does have 100\% BEC at \( T = 0 \). To date however, no irrefutable case has been made (experimentally or theoretically) in support of finite temperature BEC in an interacting finite 2D system.

In a recent paper, Petrov \( et \) \( al. \) \(^{7}\) have suggested that there are in fact two BEC regimes for finite, trapped, (quasi)-2D systems. The general argument supporting their claim hinges on an approximate analytical calculation of the one-particle density matrix from which information about the phase correlations of the condensate can be obtained. For large particle number and sufficiently low temperatures, Petrov \( et \) \( al. \) deduce that the phase fluctuations are governed by \( \sim T \ln(N) \). Thus, at temperatures well below \( T_c \), fluctuations in the phase are suppressed and one has a \( \text{true condensate} \) whereas for \( T < T_c \), the phase fluctuations are enhanced and one has a \( \text{quasi-condensate} \). Similar results have recently been obtained by Bogliubov \( et \) \( al. \), in strictly two-dimensions \(^{8}\).

However, Bhaduri \( et \) \( al. \) \(^{10}\) have suggested that there is no BEC phase transition in a finite, interacting 2D Bose gas (see also Refs. \(^{11, 12, 13}\) for similar conclusions). In their investigation, a semiclassical approximation to the Hartree-Fock-Bogliubov (HFB) mean-field theory was used to investigate the thermodynamics of the trapped 2D Bose gas. Although the semiclassical HFB approach could not shed any light on the question of phase fluctuations, it was shown that the HFB equations could be solved self-consistently all the way down to \( T = 0 \) provided one \( does \) \( not \) invoke the presence of a condensate. This finding seems to indicate that that no \( a \) priori assumption of a condensate is made (as it should be), the system remains in the same \( \text{uncondensed phase} \) at all temperatures. (Note that applying
of the absence of a condensate order-parameter however, the density distribution for the uncondensed phase was found to clearly exhibit a large spatial accumulation of atoms near the center of the trap for \( T < T_c^{(0)} \), similar to what is found in systems with a condensate. This suggests that there may be some sort of “revival” of ODLRO in the system, even though it does not lead to the formation of a condensate. Since the off-diagonal one-particle density matrix can provide information about phase correlations and ODLRO in the system, an investigation of the off-diagonal density matrix within the HFB approximation needs to be performed. This is indeed the central motivation for the present work.

The rest of our paper is organized as follows. In Sec. II we investigate the noninteracting 2D gas in a harmonic trap where an exact expression for the finite temperature correlation function can be obtained. In Sec. III, we calculate the correlation function for the interacting system within the semiclassical HFB approximation and investigate its temperature dependence both above and below \( T_c^{(0)} \). Finally, in Sec. IV we present our concluding remarks.

II. NONINTERACTING GAS

In this section, we wish to pay special attention to the off-diagonal, one-particle density matrix \( \rho^{(0)}(\mathbf{r}, \mathbf{r}'; \beta) \) of the noninteracting, confined 2D Bose gas. Our motivation lies in the fact that ODLRO in the one-particle density matrix and BEC are interrelated \(^{14}\) (see below for details). For the isotropic HO, \( \rho^{(0)}(\mathbf{r}, \mathbf{r}'; \beta) \) can be calculated analytically, and we have an exact result from which we can quantitatively test the validity of the semiclassical approximation. We will be particularly interested how well the semiclassical approximation describes the phase correlations of the gas below the BEC critical temperature \( T_c^{(0)} \). To the best of our knowledge, this is the first such study in two-dimensions.

By definition, the off-diagonal one-particle density matrix is given by \(^{13}\)

\[
\rho^{(0)}(\mathbf{r}, \mathbf{r}'; \beta) = \sum_j \psi_j^{(0)}(\mathbf{r}) \psi_j^{(0)}(\mathbf{r}') \beta_j^{(0)},
\]

where \( \psi_j^{(0)}(\mathbf{r}) \) are exact eigenfunctions of the \( d \)-dimensional harmonic oscillator (HO) and \( n_j^{(0)} \) is the Bose distribution function. It is most convenient to evaluate \(^{13}\) for the 1D case, and then use the separability of the HO potential to obtain the general result. Explicitly, we have

\[
\rho^{(0)}(x, x'; \beta) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} e^{-\frac{1}{2}(x^2+x'^2)} \frac{H_n(x)H_n(x')}{2^n n!} \frac{1}{e^{\beta(\varepsilon_n-\mu)}-1},
\]

where \( \varepsilon_n \) are the energy levels of the HO, \( \mu \) is the chemical potential, and we have scaled all energies and lengths by \( \hbar \omega_0 \) and \( r_0 = \sqrt{\hbar / m \omega_0} \), respectively. We now set the zero of energy to correspond to the lowest trap level, viz., \( n = 0 \), and obtain from \(^{13}\)

\[
\rho^{(0)}(x, x'; \beta) = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{\infty} e^{j \beta \mu} \sum_{n=0}^{\infty} e^{-\frac{1}{2}(x^2+x'^2)} \frac{H_n(x)H_n(x')}{2^n n!} e^{-j \beta n},
\]

where we have used \(^{13}\)

\[
\sum_{j=1}^{\infty} e^{j \beta \mu} e^{-j \beta n} = \frac{1}{e^{\beta(\varepsilon_n-\mu)}-1}.
\]

The summation over trap levels can be performed exactly \(^{17}\), and we finally obtain in any dimension

\[
\rho^{(0)}(\mathbf{r}, \mathbf{r}'; \beta) = \sum_{j=1}^{\infty} \frac{e^{j \beta \mu}}{\pi (1-e^{-2j \beta})^{d/2}} e^{-1/4||\mathbf{r}+\mathbf{r}'||^2 \tanh(j \beta /2)+||\mathbf{r}-\mathbf{r}'||^2 \coth(j \beta /2))}.
\]

Putting \( d = 3 \) in the above equation reproduces the result obtained by Barnett et al., \(^{18}\). The chemical potential is obtained by enforcing that the total particle number \( N \), remain fixed, viz.,

\[
N = \sum_{j=1}^{\infty} \frac{e^{j \beta \mu}}{(1-e^{-j \beta})^d}.
\]
FIG. 1: Exact quantum-mechanical densities for temperatures (starting from the lowest curve) $T/T_c^{(0)} = 1.2, 1.0, 0.9, 0.8, 0.6$. The figure inset shows the fractional occupancy $N_0/N$ of the lowest oscillator level for $N = 10^4$ atoms (solid curve) and the thermodynamic limit (dashed line) as given by Eq. (7).

The diagonal part of the one-particle density matrix yields the exact quantum-mechanical finite temperature density, $\rho^{(0)}(r)$, of the trapped gas. The densities for various temperatures are shown in Fig. 1 for $d = 2$ and $N = 10^4$ atoms. Figure 1 illustrates that even though we are not in the thermodynamic limit, the single-particle density distribution exhibits an enormous accumulation of atoms near the center of the trap for $T < T_c^{(0)}$. This type of behaviour is consistent with the confined gas having undergone a phase transition to the BEC state. The usual approach for confirming that this phase transition is indeed BEC is to examine the temperature dependence of the fractional occupancy of the lowest oscillator level, $N_0/N$. The inset to Fig. 1 shows that below $T/T_c^{(0)} \approx 1$, the ground state of the system becomes macroscopically occupied, with already 50% of the atoms in the lowest level at $T/T_c^{(0)} \approx 0.7$. Remarkably, even at $N = 10^4$ atoms, the fractional occupancy closely follows the quadratic temperature dependence obtained strictly in the thermodynamic limit (dashed line in figure inset), viz.,

$$\frac{N_0}{N} = 1 - \left( \frac{T}{T_c^{(0)}} \right)^2.$$

Although $N_0/N$ gives us a strong indication that the finite system has a BEC transition at $T/T_c^{(0)} \approx 1$, it does not provide us with a definitive test for the presence of ODLRO in the system.

It is well known since the early work of Penrose and Onsager \cite{14} that for a homogeneous system, the phenomenon of BEC is intimately related to the presence of ODLRO in the off-diagonal one-particle density matrix, namely,

$$\lim_{|r - r'| \to \infty} \rho^{(0)}(r, r'; \beta) \neq 0.$$

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In fact, the low temperature behaviour of $\rho^{(0)}(\mathbf{r},\mathbf{r}')$ is directly related to the phase fluctuations of the condensate and the non-zero limiting value of Eq. (8) yields the value of the order parameter (i.e., condensate fraction) in the system. For finite inhomogeneous systems however, the above criterion has to be modified to account for the fact that ODLRO cannot extend beyond the physical size of the system. This is accomplished by defining a normalized correlation function (sometimes called the first-order coherence function) by

$$g^{(0)}(\mathbf{r},\mathbf{r}';\beta) = \frac{\rho^{(0)}(\mathbf{r},\mathbf{r}';\beta)}{\sqrt{\rho^{(0)}(\mathbf{r};\beta)}\sqrt{\rho^{(0)}(\mathbf{r}';\beta)}} ,$$

such that

$$\lim_{|\mathbf{r}-\mathbf{r}'|\to \xi} g^{(0)}(\mathbf{r},\mathbf{r}';\beta) \neq 0 .$$

The above correlation function describes a local measure of coherence in the system. For example, the ability of two BEC’s, initially separated by $|\mathbf{r} - \mathbf{r}'|$, to form an interference pattern is quantified by $g^{(0)}(\mathbf{r},\mathbf{r}')$ [19]. In this sense, the correlation function also characterizes local fluctuations of the phase of the condensate order-parameter. The quantity $\xi$ is the large-distance length-scale over which the ODLRO of the finite system is manifested. At zero temperature, $\xi = l_0$, whereas for $T \neq 0$, $\xi$ is taken to be the spatial extent of the single-particle density $\rho^{(0)}(\mathbf{r})$. To simplify our calculations, we fix one of the coordinate, $\mathbf{r}'$ to be at the center of the trap. Note that in Eq. (9), dividing the off-diagonal density matrix by the square root of the single-particle densities removes any effects of the local-density phase fluctuations are very small, and the density distribution corresponds to that of a “true condensate”. At higher temperature, we see an abrupt jump in the large-distance decay of the spatial correlations. A direct application of Eq. (10) yields condensate fractions of $N_0/N = 0.02, 0.47, 0.71, 0.84, 0.93$, for temperatures $T/T_c^{(0)} = 1.0, 0.8, 0.6, 0.4, 0.2$, respectively. Thus, below $T_c^{(0)}$, the finite noninteracting 2D Bose gas contains a non-zero order-parameter (i.e., a BEC).

It also proves very useful to compare the spatial extent of the single-particle density, $\rho^{(0)}(\mathbf{r})$, to the spatial decay of the correlation function $g^{(0)}(\mathbf{r},\mathbf{r}').$ This information is also presented in Fig. 2 where the exact quantum mechanical densities are represented by dashed lines. These densities have been scaled by their value at $\mathbf{r} = 0$ to allow for a comparison of the length scales between $\rho^{(0)}(\mathbf{r})$ and $g^{(0)}(\mathbf{r},\mathbf{r}').$ From right to left, the dashed lines correspond to densities at $T/T_c^{(0)} = 1.0, 0.9, 0.8, 0.6.$ Densities at lower temperatures are almost indistinguishable on the scale of the figure. We note that at low temperatures, the correlation function is practically a constant $g^{(0)}(\mathbf{r},\mathbf{r}) = 1$ over the entire region for which the density (i.e., physical size of the system) of the gas is nonzero. This means that phase fluctuations are very small, and the density distribution corresponds to that of a “true condensate”. At higher temperatures (i.e., $0.9 < T/T_c^{(0)} < 1.0$) the single-particle density can become comparable to that of $g^{(0)}(\mathbf{r},\mathbf{r})$, and the gas can be said to be a “quasi-condensate” in the sense that the phase correlations do not extend to all of the atoms in the trap. For $T/T_c^{(0)} > 1.0$, the spatial extent of the single-particle density greatly exceeds characteristic length of the correlations, and system is in the normal state.

Having obtained the exact expression for $\rho^{(0)}(\mathbf{r},\mathbf{r}';\beta)$, we now wish to examine its semiclassical approximation, namely, its behaviour when $k_B T \gg \hbar \omega_0$. It is easy to show that Eq. (5) reduces to

$$\rho^{(0)}(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi\beta)^{d/2}} \sum_{j=1}^{\infty} \frac{e^{j\beta\mu}}{j^{d/2}} \exp \left[ -\beta [V(\mathbf{r}) + V(\mathbf{r}')] / 2 \right] \exp \left[ -|\mathbf{r} - \mathbf{r}'|^2 / 2 \beta \right]^{1/j} ,$$

or

$$\rho^{(0)}(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi\beta)^{d/2}} g_d(\exp [\beta(\mu - V(\mathbf{r}) + V(\mathbf{r}')] / 2] ; \exp [-|\mathbf{r} - \mathbf{r}'|^2 / 2\beta] ) ,$$

where $V(\mathbf{r}) = 1/2r^2$, and as in Ref. [20] we have introduced the generalized Bose function

$$g_\alpha(x, y) = \sum_{j=1}^{\infty} x^j y^{1/j} j^\alpha .$$

For $\mathbf{r} = \mathbf{r}'$ and $d = 2$, Eq. (12) becomes

$$\rho^{(0)}(\mathbf{r}) = \frac{1}{(2\pi)^2} \int \frac{d^2\mathbf{p}}{\exp \left[ (p_x^2 + \frac{1}{2}r^2 + -\mu) / \beta \right] - 1} ,$$

and
FIG. 2: Exact off-diagonal correlation function (solid curves) for temperatures (from left to right) $T/T_c^{(0)} = 1.0, 0.9, 0.8, 0.4, 0.2, 0.01$. The dashed lines correspond to the scaled exact quantum mechanical densities at temperatures (from right to left) $T/T_c^{(0)} = 1.0, 0.9, 0.8, 0.6$. The filled circles are the semiclassical approximation to the off-diagonal density matrix as given by Eq. (16).

with the normalization condition

$$N = \int \rho^{(0)}(\mathbf{r}) \, d^2 \mathbf{r}. \quad (15)$$

If one attempts to solve Eqs. (14,15) for all $T$, it is readily shown that there exists a temperature $T^*$ below which the equations can no longer be satisfied [10, 11]. In fact, the temperature $T^*$ is identical to the the critical temperature $T_c^{(0)}$ obtained by Bagnato and Kleppner [4]. Thus, in order to obtain solutions below $T_c^{(0)}$, a condensate order parameter is required. We introduce the macroscopic occupation of the ground state in the one-particle density matrix according to the ansatz

$$\rho^{(0)}(\mathbf{r}, \mathbf{r}') = N_0 \psi^{(0)}_0(\mathbf{r}) \psi^{(0)}_0(\mathbf{r}') + \frac{1}{(2\pi)^{d/2} g_d/2} \int \frac{d\mathbf{p}}{\exp \left[ \beta (\mu - [V(\mathbf{r}) + V(\mathbf{r}')] / 2 \right] \exp \left[ -|\mathbf{r} - \mathbf{r}'|^2 / 2\beta \right]}, \quad (16)$$

where $\psi^{(0)}_0(\mathbf{r})$ is the zero temperature ground-state eigenfunction of the HO and

$$N_0 = \frac{e^{\beta \mu}}{1 - e^{\beta \mu}}. \quad (17)$$

The 2D density is then given by Eq. (16) with $\mathbf{r} = \mathbf{r}'$,

$$\rho^{(0)}(\mathbf{r}) = N_0 |\psi^{(0)}_0(\mathbf{r})|^2 + \frac{1}{(2\pi)^2} \int \frac{d^2 \mathbf{p}}{\exp \left( \frac{p^2}{2} + \frac{1}{2} r^2 + -\mu \right) \beta^{-1}}. \quad (18)$$
The semiclassical densities obtained from Eq. (18) are indistinguishable from the exact densities obtained from Eq. (5). The semiclassical correlation function, Eq. (16), at various temperatures is represented in Fig. 2 as filled circles. The superb agreement between the exact (solid curves) and semiclassical results shows that the semiclassical approximation is valid well below the critical temperature provided one takes into account the macroscopic occupation of the ground state separately. If the atoms had remained in the uncondensed phase, Eqs. (14,15) would have been sufficient to describe the gas for all \( T \geq 0 \).

### III. INTERACTING GAS

The thermodynamic properties of the interacting Bose gas are most commonly described by the self-consistent mean-field HFB equations. In most calculations (see Ref. [21, 22] and references therein for details), the semiclassical approximation is applied to the discrete set of coupled Bogliubov equations, leading to the considerably simplified semiclassical HFB theory. The semiclassical HFB approximation is well-known to be an excellent tool for describing the interacting 3D Bose gas [21, 22]. The 2D version of the semiclassical HFB theory is formally identical to the 3D version and has already been examined by Mullin [23]. We present only the essentials of the model here for completeness.

In the semiclassical HFB approach, the condensed and uncondensed atoms are described by a set of coupled, self-consistent equations. The macroscopic condensate wavefunction, \( \psi_0(r) \), is characterized by the finite-temperature Gross-Pitaevskii equation which is given by

\[
\left[ -\frac{1}{2} \nabla^2 + \frac{1}{2} r^2 - \mu + 2\gamma \rho(r) - \gamma \rho_0(r) \right] \psi_0(r) = 0 ,
\]

where \( \gamma \) is the coupling constant associated with the assumed repulsive zero-range two-body pseudo-potential. The total density, \( \rho(r) \), is made up of a condensate \( \rho_0(r) \equiv |\psi_0(r)|^2 \), and a noncondensate density,

\[
\rho_T(r) = \int \left\{ [u^2(p, r) + v^2(p, r)] f(p, r) + v^2(p, r) \right\} d^2 p ,
\]

where

\[
f(p, r) = \frac{1}{e^{\beta \varepsilon} - 1} ,
\]

\[
u^2(p, r) = \frac{\Lambda + \varepsilon}{2\varepsilon} ,
\]

\[
v^2(p, r) = \frac{\Lambda - \varepsilon}{2\varepsilon} ,
\]

\[
\Lambda = \frac{p^2}{2} + \frac{1}{2} r^2 - \mu + 2\gamma \rho(r) ,
\]

and

\[
\varepsilon(p, r) = \sqrt{\Lambda^2 - (\gamma \rho_0(r))^2} .
\]

The total number of particles satisfies

\[
N = \int [\rho_0(r) + \rho_T(r)] d^2 r.
\]

In the absence of a condensate, it is easy to show that the above set of HFB equations reduce to

\[
\rho_T(r) = \frac{1}{2\pi} \int \frac{d^2 p}{\exp \left[ \left( \frac{u^2}{2} + \frac{1}{2} r^2 + 2\gamma \rho_T(r) - \mu \right) \beta \right] - 1} ,
\]
FIG. 3: Semiclassical densities for $N = 10^4$ atoms with $\gamma = 0.16$ (solid curves) and $\gamma = 0.31$ (dashed curves). The lowest lying curves correspond to $T/T_c^{(0)} = 1.1$ and the higher lying curves to $T/T_c^{(0)} = 0.8$. The figure inset shows the chemical potential as a function of temperature. The long-dashed-short-dashed curve corresponds to $\gamma = 0$. The value $\gamma = 0.31$ is consistent with an effective 2D interaction strength for Rb [10].

with

$$N = \int \rho_T(r) \, d^2r.$$  \hspace{1cm} (28)

Equations (27,28) are equivalent to the so-called self-consistent Thomas-Fermi approximation used in Refs. [10,11]. Putting $\gamma = 0$ in Eq. (27) yields an expression that is identical to the semiclassical approximation of the exact diagonal density matrix, viz., Eq. (14), found in the previous section; recall from Sec. II that below $T_c^{(0)}$, Eqs. (27,29) cannot be solved self-consistently. However, if one puts any finite $\gamma > 0$ in Eq. (27), self-consistent solutions to the HFB equations can be obtained all the way down to $T = 0$ [10,11,12]. To illustrate this, we show in Fig. 3 the density and chemical potential (figure inset) of the gas for $\gamma = 0$ (long-dash-short-dashed line), $\gamma = 0.16$ (solid line), and $\gamma = 0.31$ (dashed line). In contrast to the noninteracting 2D gas (where a condensate order-parameter has to be introduced below $T_c^{(0)}$), the interacting system remains in the uncondensed phase for all $T \geq 0$.

Although no condensation occurs, it is interesting to compare the densities of the noninteracting (Fig. 1) and interacting gases (Fig. 3) just above and below $T_c^{(0)}$. The essential point to be made from this comparison is that in both cases, there is an increase in the density of atoms near the center of the trap for $T < T_c^{(0)}$. While the noninteracting density shows a much larger enhancement at $r = 0$, the interacting gas also displays a similar (albeit diminished) behaviour. Since an increase in the density of atoms near the center of the trap is associated with ODLRO in the noninteracting gas, it is worthwhile investigating the off-diagonal density matrix for the interacting 2D gas within the semiclassical HFB approximation.
FIG. 4: The semiclassical HFB approximation to the correlation function with $\gamma = 0.16$. The curves correspond to (from left to right) $T/T_c^{(0)} = 1.0, 0.8, 0.6, 0.4$. 

The 2D semiclassical HFB off-diagonal density matrix can be obtained from the leading order term in the Wigner-Kirkwood semiclassical expansion of the quantum mechanical off-diagonal density matrix [24]. The noncondensed particles are treated as bosons in an effective potential, and we obtain

$$\rho(r, r'; \beta) = \frac{1}{2\pi\beta} g_1(\exp[\beta(\mu - [V_{\text{eff}}(r) + V_{\text{eff}}(r')]/2)], \exp[-|r - r'|^2/2\beta]),$$

(29)

where

$$V_{\text{eff}}(r) = \frac{1}{2}r^2 + 2\gamma\rho(r).$$

(30)

Since there is no condensate for $\gamma > 0$, we have not included the first term in Eq. (16). Of course, for $r = r'$, Eq. (29) reduces to Eq. (27).

As in Sec. II, the normalized correlation function for the finite sized system is given by

$$g(r, 0; \beta) = \frac{\rho(r, 0; \beta)}{\sqrt{\rho(r) \rho(0)}},$$

(31)

where again, we have fixed $r' = 0$. The temperature dependence of $g(r, 0; \beta)$ is shown in Figs. 4, 5 for coupling strengths $\gamma = 0.16, 0.31$, respectively. Above $T_c^{(0)}$, the interacting and noninteracting (see Fig. 2) systems have similar correlations. However, as we cool the gas to temperatures just below $T_c^{(0)}$, there is an drastic change in the behaviour of the interacting off-diagonal density matrix, which to the best of our knowledge, has never been noticed.
before. In particular, we observe that the decay of the correlations in the interacting system are qualitatively different from the noninteracting gas. For the interacting gas, values of $r > \sim \ell_0$ lead to the tail of the correlation function decreasing slower than the noninteracting case. In contrast, for $r \lesssim \ell_0$, the interacting correlation function has a much faster spatial decay than the noninteracting system. We have already discussed the fact that the near constancy of the noninteracting correlation function, $g^{(0)}(r, r')$, over the size of the sample ($r \approx \ell_0$) is indicative of negligible phase fluctuations in the condensate [14]. Thus, the re-appearance of ODLRO in $g(r, r')$ below $T_c^{(0)}$ suggests that there may be larger fluctuations in the phase as compared to the noninteracting gas. If we apply Eq. (10) to $g(r, r')$ (with $\gamma = 0.16$), we obtain 0.002, 0.25, 0.58, 0.70 at $T/T_c^{(0)} = 1.0, 0.8, 0.6, 0.4$ respectively, for the asymptotic behaviour of the correlation function. Even though we may not be able to interpret these numbers as condensate fractions, it is tempting to conjecture that they represent a nonzero value of an order-parameter in the system.

We can be more quantitative about the nature of the correlations displayed by $g(r, r')$ below $T_c^{(0)}$ as follows. In the interacting homogeneous 2D Bose gas, Popov [25] has already calculated the low temperature correlation function:

$$g(r, r'; \beta) \approx R^{-\alpha},$$

(32)

where

$$R \equiv \frac{|r - r'|}{\beta \sqrt{\gamma \rho_s}}, \quad \alpha = \frac{1}{2\pi \beta \rho_s},$$

(33)

and $\rho_s$ is the so-called superfluid density. The algebraic ODLRO exhibited by $g(r, r')$ implies that at low temperature, correlations can extend over macroscopic (i.e., size of the system) distances, although phase fluctuations will be nonzero. This is reminiscent of the quasi-condensate described in Ref. [8], and it is only identically at $T = 0$ that the system will attain a “true condensate.”
If the system is finite (i.e., in a 2DHO potential), then Bogliubov et al. \cite{9}, have shown that

\[ R = \frac{|r - r'|}{\beta \sqrt{\rho(0)}}, \quad \alpha = \frac{1}{2\pi \rho(S)}, \quad (34) \]

where \( S = (r + r')/2 \). Eq. (34) is formally identical to (33) except for the slowly varying factor \( \rho(S) \) which is the inhomogeneous boson density. This additional spatial dependence results in a deviation from the strict power law dependence of the uniform gas. We have found that at low temperatures, the power law dependence in Eq. (34) agrees well with the observed numerical behaviour of \( g(r, r'; \beta) \). This suggests that at low temperatures, the interacting system exhibits algebraic ODLRO similar to that of the superfluid state in the uniform Bose gas. At exactly \( T = 0 \), the gas will have a true condensate.

IV. CONCLUSIONS

We have investigated the finite-temperature correlations in both the ideal and interacting trapped, 2D Bose gas. Our study of the noninteracting gas provided an exact framework from which we could compare the validity of the semiclassical approximation over a wide temperature range. Our results indicate that the semiclassical approach works extremely well, even below the critical temperature, provided the macroscopic occupations of the ground state is treated separately. A careful examination of the correlation function reveals that, as in Ref. \cite{8}, there are two BEC regimes for the noninteracting gas. Namely, for temperatures \( 0.9 < T/T_c(0) < 1.0 \), the system is a quasi-condensate in the sense that the phase correlations are comparable to the spatial extent of the single-particle density. On the other hand, when the temperature is lowered to \( T/T_c(0) \lesssim 0.9 \), phase fluctuations become negligible, and the gas is a true condensate. It is straightforward to apply our results to study the finite \( N \) dependence on the coherence properties of trapped gases in dimensions \( d = 1, 2, 3 \). (see e.g., Ref. \cite{13} for work relating to 3D).

In the interacting gas, we found that even though the system does not contain a condensate, there is a revival of ODLRO for \( T/T_c(0) \lesssim 1 \). The low temperature behaviour of the interacting correlation function was found to be consistent with an algebraic (i.e., power law) decay in the phase correlations, similar to what is found in the uniform 2D superfluid Bose gas. Whether or not this re-entrant ODLRO is to the superfluid state is presently being investigated.

Finally, it is interesting to note that Margo and Ceperely \cite{26} have found conclusions analogous to ours for the uniform 2D charged Bose fluid (CBF) with \( \ln(r) \) interactions. Specifically, they found that even though the 2D CBF does not undergo a condensation, the noncondensed fluid exhibits a power-law decay of the one-particle density matrix similar to what is found in the present work. It is not immediately clear what relevance their findings may have on our results for the interacting, inhomogeneous 2D Bose gas.

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\begin{thebibliography}{10}

[1] A. Görlitz et al., Phys. Rev. Lett. 87, 130402 (2001).
[2] I. Bloch, T. W. Hänsch and T. Esslinger, Nature 403, 166 (2000).
[3] P. C. Hohenberg, Phys. Rev. 158, 383 (1967).
[4] V. Bagnato and D. Kleppner, Phys. Rev. A 44, 7439 (1991).
[5] S. I. Shevchenko, Sov. J. Low Temp. Phys. 16, 64 (1990).
[6] E.H. Lieb and R. Seiringer, preprint math-ph/0112032.
[7] W. J. Mullin, J. Low Temp. Phys. 106, 615 (1997).
[8] D. S. Petrov, M. Holzmann, and G. V. Shlyapnikov, Phys. Rev. Lett. 84, 2551 (2000).
[9] N. M. Bogliubov, R. K. Bullough, V. S. Kapitonov, C. Malyshiev, and J. Timonen, Europhys. Lett. 55, 755 (2001).
[10] R. K. Bhaduri et al., J. Phys. B 33, 3895 (2000).
[11] B. P. van Zyl, R. K. Bhaduri and J. Sigetich, J. Phys. B 35, 1251 (2002).
[12] J. P. Fernández and W. Mullin, cond-mat/0203174 (2002).
[13] S. I. Shevchenko, Sov. J. Low Temp. Phys. 18, 223 (1992).
\end{thebibliography}
[14] O. Penrose and L. Onsager, Phys. Rev. 104, 576 (1956).
[15] A. L. Fetter and J. D. Walecka, Quantum Theory of Many-Particle Systems, McGraw-Hill Inc., New York, 1995.
[16] W. Ketterle and N. J. van Druten, Phys. Rev. A 54, 656 (1996).
[17] P. M. Morse and H. Feshback, Methods of Theoretical Physics, (2 vols.), New York, McGraw-Hill, 1953.
[18] S. M. Barnett, S. Franke-Arnold, A. S. Arnold and C. Baxter, J. Phys. B: At. Mol. Opt. Phys. 33, 4177 (2000).
[19] E. Hecht, Optics, 3rd ed. (Addison-Wesley, Reading, 1998).
[20] M. Naraschewski and R. J. Glauber, Phys. Rev. A 59, 4595 (1999).
[21] S. Giorgini, L. P. Pitaevskii and S. Stringari, J. Low. Temp. Phys. 109, 309 (1997).
[22] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).
[23] W. J. Mullin, J. Low. Temp. Phys. 110, 167 (1998).
[24] M. Brack and R. K. Bhaduri, Semiclassical Physics, Addison-Wesley, part of the Frontiers in Physics vol. 96 (1997).
[25] V. N. Popov, Chapter 5 in Functional Integrals and Collective Excitations, Cambridge University Press, 1987.
[26] W. R. Margo and D. M. Ceperley, Phys. Rev. Lett. 73, 826 (1994).