A NOTE ON CATALAN NUMBERS ASSOCIATED WITH
\textit{p}-ADIC INTEGRAL ON $\mathbb{Z}_p$

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Abstract. In this paper, we study Catalan numbers which can be represented by the \textit{p}-adic integral on $\mathbb{Z}_p$ and we investigate some properties and formulae related to Catalan numbers and special numbers.

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $\textit{p}$-adic integers, the field of $\textit{p}$-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The $\textit{p}$-adic norm $| \cdot |_p$ is normalized as $|p|_p = \frac{1}{p}$. As is well known, the Euler numbers are defined by the generating function to be

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (\text{see } [1-13]). \quad (1.1)$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers. From (1.1), we note that

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l. \quad (1.2)$$

Recently, $\lambda$-Changhee polynomials are defined by the generating function to be

$$\frac{2}{(1 + t)\lambda + 1} e^x = \sum_{n=0}^{\infty} C_{\lambda,n}(x) \frac{t^n}{n!}, \quad \text{where } \lambda \in \mathbb{Z}_p \quad (\text{see } [5]). \quad (1.3)$$

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By replacing $t$ by $e^{\lambda t} - 1$, we get

\[
\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} Ch_{n,\lambda}(x) \frac{1}{n!} (e^{\lambda t} - 1)^n
\]

\[
= \sum_{n=0}^{\infty} Ch_{n,\lambda}(x) \sum_{m-n}^{\infty} S_2(m,n) \lambda^{-m} \frac{t^m}{m!},
\]

where $S_2(m,n)$ is the stirling number of the second kind. When $x = 0$, $Ch_{n,\lambda} = Ch_{n,\lambda}(0)$ are called the $\lambda$-Changhee numbers. From (1.4), we note that

\[
E_m(x) = \sum_{n=0}^{m} Ch_{n,\lambda}(x) S_2(m,n) \lambda^{-m}, \quad (m \geq 0).
\]

(1.5)

For $n \geq 0$, the stirling number of the first kind is defined as

\[
(x)_n = x(x-1) \cdots (x-n+1) = \prod_{l=1}^{n-1} (x-l) = \sum_{l=0}^{n} S_1(n,l)x^l,
\]

(1.6)

and the stirling number of the second kind is given by

\[
x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (\text{see } [7, 8, 9]).
\]

(1.7)

Let $C(\mathbb{Z}_p)$ be the space of all continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic $p$-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

\[
\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}(x + p^N \mathbb{Z}_p)
\]

\[
= \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad (\text{see } [8]).
\]

(1.8)

From (1.8), we note that

\[
I_{-1}(f_1) + I_{-1}(f) = 2f(0), \quad \text{where } f_1(x) = f(x + 1).
\]

(1.9)

Thus, by (1.9), we get

\[
I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad (\text{see } [8]).
\]

(1.10)
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where \( f_n(x) = f(x + n), \ n \in \mathbb{N} \). It is not difficult to show that

\[
1 - \sqrt{1 - 4t} = \frac{(-1)^n}{2t} \left( \sum_{n=1}^{\infty} \binom{2n}{n} \frac{(-1)^{n-1}}{4^n(2n-1)} (-1)^n 4^n t^n \right)
\]

\[
= \frac{1}{2} \sum_{n=1}^{\infty} \binom{2n}{n} \frac{1}{2n-1} t^{n-1}
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \binom{2n+2}{n+1} \frac{1}{2n+1} t^n
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n+2)(2n+1)(2n) \cdots (n+2)}{(n+1)n!(2n+1)} t^n
\]

\[
= \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} n^n, \quad \text{(see \([1,2,3]\)).}
\]

As is well known, the Catalan number \( C_n \) is defined by \( C_n = \binom{2n}{n} \frac{1}{n+1}, \ (n \geq 0) \). From (1.11), we note that the generating function of Catalan numbers is given by

\[
\frac{2}{1 + \sqrt{1 - 4t}} = \sum_{n=0}^{\infty} n!C_n t^n, \quad \text{(see \([9-13]\)).}
\]

In this paper, we study Catalan numbers associated with p-adic integral on \( Z_p \) and we give Witt’s type formula related Catalan numbers.

2. Catalan numbers associated with p-adic integral on \( Z_p \)

For \( t \in \mathbb{C}_p \) with \( |t|_p < p^{-\frac{1}{p-1}} \), we observe that

\[
\int_{\mathbb{Z}_p} (1 + t)^x d\mu_{-1}(x) = \frac{2}{1 + \sqrt{1 + t}} = \sum_{n=0}^{\infty} C h_n \frac{t^n}{n!}.
\]

From (1.9), we note that

\[
\int_{\mathbb{Z}_p} e^{tx} d\mu_{-1}(x) = \frac{2}{e^t + 1} \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}
\]

(2.2)
On the other hand,
\[
\int_{\mathbb{Z}} (1 + t)^{\frac{x}{2}} d\mu_{-1}(x) = \sum_{m=0}^{\infty} \int_{\mathbb{Z}} x^m d\mu_{-1}(x) \frac{1}{2^m m!} \left( \log(1 + t) \right)^m
\]
\[
= \sum_{m=0}^{\infty} E_m 2^{-m} \sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} 2^{-m} E_m S_1(n, m) \right) \frac{t^n}{n!}.
\]
(2.3)

Therefore, by (2.1) and (2.3), we obtain the following theorem.

**Theorem 1.** For \( n \geq 0 \), we have
\[
\sum_{m=0}^{n} 2^{-m} E_m S_1(n, m) = Ch_{n, \frac{1}{2}}.
\]

By replacing \( t \) by \( -4t \) in (2.1), we get
\[
\int_{\mathbb{Z}} (1 - 4t)^{\frac{x}{2}} d\mu_{-1}(x) = \frac{2}{1 + \sqrt{1 - 4t}} = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} t^n
\]
\[
= \sum_{n=0}^{\infty} C_n t^n.
\]
(2.4)

On the other hand,
\[
\int_{\mathbb{Z}} (1 - 4t)^{\frac{x}{2}} d\mu_{-1}(x) = \sum_{n=0}^{\infty} Ch_{n, \frac{1}{2}} \frac{(-4)^n t^n}{n!}.
\]
(2.5)

Therefore, by (2.4) and (2.5), we obtain the following theorem.

**Theorem 2.** For \( n \geq 0 \), we have
\[
C_n = (-1)^n 4^n Ch_{n, \frac{1}{2}}.
\]

From (2.4), we have
\[
\sum_{n=0}^{\infty} (-1)^n 4^n \int_{\mathbb{Z}} \frac{x}{n} d\mu_{-1}(x) t^n = \sum_{n=0}^{\infty} C_n t^n
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} t^n.
\]
(2.6)

Therefore, by comparing the coefficients on the both sides of (2.6), we obtain the following theorem.
Theorem 3. For \( n \geq 0 \), we have

\[
\int_{\mathbb{Z}_p} \left( \frac{x}{n} \right) d\mu_{n-1}(x) = \frac{(-1)^n}{4^n} C_n = \frac{(-1)^n}{4^n(n+1)} \binom{2n}{n}.
\]

From (2.3), we have the following equation:

\[
\int_{\mathbb{Z}_p} \left( \frac{x}{n} \right) d\mu_{n-1}(x) = \frac{1}{n!} \sum_{m=0}^{n} 2^{-m} E_m S_1(n, m) \tag{2.7}
\]

Therefore, by Theorem 3 and (2.7), we obtain the following theorem.

Theorem 4. For \( n \geq 0 \), we have

\[
C_n = \frac{(-1)^n}{n!} \sum_{m=0}^{n} 2^{2n-m} E_m S_1(n, m).
\]

Now, we observe that

\[
\sqrt{1 + t} = (1 + t)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{n}{2} t^n = \sum_{n=0}^{\infty} \frac{n}{n!} t^n
\]

\[
= \sum_{n=0}^{\infty} \frac{1(\frac{1}{2} - 1)(\frac{1}{2} - 2) \cdots (\frac{1}{2} - n + 1)}{n!} t^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^{n-1} \cdot 3 \cdot 5 \cdots (2n - 3)}{n! 2^n} t^n \tag{2.8}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^{n-1} \cdot 2 \cdot 3 \cdot 4 \cdots (2n - 3)(2n - 2)(2n - 1)(2n)}{n! 2^n 4 \cdots (2n - 2)(2n - 1)(2n)} t^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^{n-1} (2n)!}{n! 4^n (2n - 1)!} t^n = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^{n-1}}{4^n (2n - 1)} t^n.
\]
By (2.1) and (2.8), we get

\[
2 = \left( \sum_{n=0}^{\infty} C_{n, \frac{1}{2}} \frac{t^n}{n!} \right) (1 + \sqrt{1 + t})
\]

\[
= \sum_{n=0}^{\infty} C_{n, \frac{1}{2}} \frac{t^n}{n!} + \sum_{k=0}^{\infty} C_{k, \frac{1}{2}} \frac{t^k}{k!} \left( \sum_{m=0}^{\infty} \left( \frac{2m}{m} \right) \frac{(-1)^{m-1}}{4^m (2m-1)} t^m \right)
\]

\[
= \sum_{n=0}^{\infty} C_{n, \frac{1}{2}} \frac{t^n}{n!} + \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} C_m \frac{(m+1)(-1)^{m-1}}{4^m (2m-1)} \frac{m! n!}{(n-m)! m!} C_{n-m, \frac{1}{2}} \right) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} C_{n, \frac{1}{2}} \frac{t^n}{n!} + \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} C_m \frac{(m+1)(-1)^{m+1}}{4^m (2m-1)} \frac{n!}{(n-m)! m!} C_{n-m, \frac{1}{2}} \right) \frac{t^n}{n!}
\]

(2.9)

By comparing the coefficients on the both sides of (2.9), we get

\[
C_{n, \frac{1}{2}} + \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) C_m C_{n-m, \frac{1}{2}} (m+1)! (-1)^{m+1} 4^m (2m-1) \frac{1}{4^m (2m-1)} = \begin{cases} 2, & \text{if } n = 0 \\ 0, & \text{if } n > 0. \end{cases}
\]

(2.10)

Therefore, by (2.10), we obtain the following theorem.

**Theorem 5.** For \( n \in \mathbb{N} \), we have

\[
C_{n, \frac{1}{2}} = \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) C_m C_{n-m, \frac{1}{2}} (m+1)! (-1)^{m+1} \frac{1}{4^m (2m-1)}.
\]

and

\[
C_{0, \frac{1}{2}} = 1.
\]

By replacing \( t \) by \( \frac{1}{4} (1 - e^{2t}) \) in (1.12), we get

\[
\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} C_n \frac{1}{4^n} (1 - e^{2t})^n
\]

\[
= \sum_{n=0}^{\infty} C_n \frac{(-1)^n}{4^n} (e^{2t} - 1)^n
\]

\[
= \sum_{n=0}^{\infty} C_n \frac{(-1)^n}{4^n} n! \sum_{m=n}^{\infty} S_2(m, n) \frac{2^m t^m}{m!}
\]

\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} C_n \frac{(-1)^n 2^m 2^{-n} n! S_2(m, n)}{m!} \right) \frac{t^m}{m!}.
\]

(2.11)
Therefore, by (1.1) and (2.11), we obtain the following theorem.

**Theorem 6.** For \( m \geq 0 \), we have

\[
E_m = \sum_{n=0}^{m} C_n (-1)^n 2^{m-2n} n! S_2(m, n).
\]

Now, we observe

\[
(1 + t)^{\frac{x}{2}} = \sum_{m=0}^{\infty} \left( \frac{x}{2} \right)^m \frac{\log(1 + t)}{m!} = \sum_{m=0}^{\infty} \left( \frac{x}{2} \right)^m \sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \left( \frac{x}{2} \right)^m S_1(n, m) \right) \frac{t^n}{n!}.
\]  

We consider the \( \frac{1}{2} \)-Changhee polynomials which are given by the generating function to be

\[
\int_{\mathbb{Z}_p} (1 + t)^{\frac{x}{2}} d\mu(y) = \frac{2}{1 + \sqrt{1 + t}} \sqrt{(1 + t)^{\frac{x}{2}}}
\]

\[
= \sum_{n=0}^{\infty} Ch_{n, \frac{1}{2}}(x) \frac{t^n}{n!}.
\]  

When \( x = 0 \), we note that \( Ch_{n, \frac{1}{2}} = Ch_{n, \frac{1}{2}}(0) \). By (2.11), (2.12) and (2.13), we get

\[
\sum_{n=0}^{\infty} Ch_{n, \frac{1}{2}}(x) \frac{t^n}{n!} = \left( \frac{2}{1 + \sqrt{1 + t}} \right) ((1 + t)^{\frac{x}{2}})
\]

\[
= \left( \sum_{k=0}^{\infty} Ch_{k, \frac{1}{2}} \frac{t^k}{k!} \right) \left( \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} \left( \frac{x}{2} \right)^j S_1(m, j) \right) \frac{t^m}{m!} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \sum_{j=0}^{m} \left( \frac{x}{2} \right)^j S_1(m, j) Ch_{n-m, \frac{1}{2}} \left( \begin{array}{c} n \\ m \end{array} \right) \right) \frac{t^n}{n!}.
\]  

Therefore, by (2.14), we obtain the following theorem.

**Theorem 7.** For \( n \geq 0 \), we have

\[
Ch_{n, \frac{1}{2}}(x) = \sum_{m=0}^{n} \sum_{j=0}^{m} \left( \frac{x}{2} \right)^j S_1(m, j) Ch_{n-m, \frac{1}{2}} \left( \begin{array}{c} n \\ m \end{array} \right).
\]
By replacing $t$ by $-4t$ in (2.13), we define Catalan polynomials which are given by the generating function to be

$$\int_{\mathbb{Z}_p} (1 - 4t)^{\frac{x+y}{2}} d\mu_{-1}(y) = \frac{2}{1 + \sqrt{1 - 4t}} \sqrt{(1 - 4t)^x}$$

$$= \sum_{n=0}^{\infty} C_n(x)t^n. \quad (2.15)$$

From (1.12) and (2.12), we note that

$$\sum_{n=0}^{\infty} C_n(x)t^n = \left( \frac{2}{1 + \sqrt{1 - 4t}} \right) \left( 1 - 4t \right)^{\frac{x}{2}}$$

$$= \left( \sum_{k=0}^{\infty} C_k t^k \right) \left( \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} \left( \frac{x}{2} \right)^j S_1(m,j) \right) \frac{(-4)^m t^m}{m!} \right) \quad (2.16)$$

$$= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \sum_{j=0}^{m} \left( \frac{x}{2} \right)^j S_1(m,j) \frac{(-4)^m}{m!} C_{n-m} \right) t^n.$$

Therefore, by comparing the coefficients on the both sides of (2.16), we obtain the following theorem.

**Theorem 8.** For $n \geq 0$, we have

$$C_n(x) = \sum_{m=0}^{n} \sum_{j=0}^{m} \left( \frac{x}{2} \right)^j S_1(m,j) \frac{(-4)^m}{m!} C_{n-m}.$$  

Remark. From (2.8) and (2.12), we note that

$$\sum_{n=0}^{\infty} \left( \begin{array}{c} 2n \\ n \end{array} \right) \frac{(-1)^{n-1}}{4^n(2n-1)} t^n = \sum_{n=0}^{\infty} \left( \frac{1}{m!} \sum_{m=0}^{n} \left( \frac{1}{2} \right)^m S_1(n,m) \right) t^n \quad (2.17)$$

By (2.17), we get

$$\frac{1}{n+1} \left( \begin{array}{c} 2n \\ n \end{array} \right) \frac{(-1)^{n-1} n+1}{4^n(2n-1)} = \frac{1}{n!} \sum_{m=0}^{n} \left( \frac{1}{2} \right)^m S_1(n,m) \quad (2.18)$$

Thus, from (2.18), we have

$$C_n = \frac{4^n(2n-1)}{(n+1)!} (-1)^{n-1} \sum_{m=0}^{n} \left( \frac{1}{2} \right)^m S_1(n,m), \quad (n \geq 0).$$

**Corollary 9.** For $n \geq 0$, we have

$$C_n = (-1)^{n+1} \frac{4^n(2n-1)}{(n+1)!} \sum_{m=0}^{n} \left( \frac{1}{2} \right)^m S_1(n,m).$$
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References

1. H. W. Gould, Sums and convolved sums of Catalan numbers and their generating functions, Indian J. Math. 46 (2004), no. 2-3, 137-160.
2. R. Hampel, On the problem of Catalan, (Polish) Prace Mat. 4 (1960), 11-19.
3. S. Hyyro, On the Catalan problem, (Finnish) Arkhimedes 1963 (1963), no. 1, 53-54.
4. K. Inkeri, On Catalan’s problem, Acta Arith. 9 (1964), 285-290.
5. D. S. Kim, T. Kim, Some identities of Korobov-type polynomials associated with p-adic integrals on Zp, Advances in Difference Equations (2015) 2015:282 DOI 10.1186/s13662-015-09602-8.
6. T. Kim, D. S. Kim, A note on nonlinear Changhee differential equations, Russ. J. Math. Phys. 23 (2016), 88-92.
7. T. Kim D. S. Kim, J.-J. Seo, H.-I. Kwon, Differential equations associated with λ-Changhee polynomials, J. Nonlinear Sci. Appl. 9 (2016), 3098-3111.
8. T. Kim, A study on the q-Euler numbers and the fermionic q-integral of the product of several type q-Bernstein polynomials on Zp, Adv. Stud. Contemp. Math. (Kyungshang) 23 (2013), no. 1, 5-11.
9. J. Morgado, Some remarks on an identity of Catalan concerning the Fibonacci numbers, Special issue in honor of António Monteiro. Portugal. Math. 39 (1980), no. 1-4, 341-348 (1985).
10. A. Natucci, Ricerche sistematiche intorno al “teorema di Catalan”, (Italian) Giorn. Mat. Battaglini. (5) 2 (82) (1954), 297-300.
11. R. Rangarajan, P. Shashikala, A pair of classical orthogonal polynomials connected to Catalan numbers, Adv. Stud Contemp. Math. (Kyungshang) 23 (2013), no. 2, 323-335.
12. A. Rotkiewicz, Sur le problème de Catalan, (French) Elem. Math. 15 (1960), 121-134.
13. A. D. Sands, On generalised Catalan numbers, Discrete Math. 21 (1978), no. 2, 219-221.

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