ON THE UNIQUE UNEXPECTED QUARTIC IN $\mathbb{P}^2$

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Abstract. The computation of the dimension of linear systems of plane curves through a bunch of given multiple points is one of the most classic issues in Algebraic Geometry. However, it is still an open problem. Despite many partial results, a complete solution is not known, even if the fixed points are in general position. The famous Segre-Harbourne-Gimigliano-Hirschowitz conjecture, which predicts the answer, stands unsolved for more than 50 years.

When we consider fixed points in special position, even more interesting situations may occur. Recently Di Gennaro, Ilardi and Valls discovered a special configuration $Z$ of nine points with a remarkable property: a general triple point always fails to impose independent conditions on the ideal of $Z$ in degree 4. The peculiar structure and properties of this kind of unexpected curves were studied by Cook II, Harbourne, Migliore and Nagel. Our contribution to this topic is the classification of low degree unexpected curves. In particular, we prove that the above mentioned configuration $Z$ is the unique one giving rise to an unexpected quartic.

Moreover, we translate the existence of unexpected curves in terms of de Jonquières transformations. This allows us to give some numerical necessary conditions for the existence of such curves, and to find an easy proof of their uniqueness.

1. Introduction

One of the central problems in algebraic geometry is the study of linear systems of hypersurfaces of $\mathbb{P}^n$ with imposed singularities, namely divisors containing a set of given points $P_1, \ldots, P_r$ with multiplicities $m_1, \ldots, m_r$. Interpolation theory addresses the problem of computing the dimension of such systems. This is actually an open problem; in general, naively counting parameters does not always give the correct result.

Let us start with the basic definitions in the projective plane. We work over the field of complex numbers $\mathbb{C}$.

Definition 1.1. Let $R = \mathbb{C}[x, y, z]$ be the coordinate ring of $\mathbb{P}^2$. For $d \in \mathbb{N}$, let $R_d$ be the vector space of degree $d$ forms. Given $P_1, \ldots, P_r \in \mathbb{P}^2$ and their ideals $I_1, \ldots, I_r$, we define the fat point subscheme of $\mathbb{P}^2$ supported at $P_1, \ldots, P_r$ with multiplicities $m_1, \ldots, m_r$ to be the scheme $X = m_1 P_1 + \ldots + m_r P_r$ associated to the ideal

$$I(X) = I_1^{m_1} \cap \ldots \cap I_r^{m_r}.$$  

We will indicate by $I(X)_d$ the homogeneous component of degree $d$ of $I(X)$. The vector space $I(X)_d$ is the linear system of curves of degree $d$ in $\mathbb{P}^2$ containing $X$, that is, having multiplicity at least $m_i$ at $P_i$ for all $i = 1, \ldots, r$.

The expected dimension of such a system is

$$\expdim I(X)_d = \max \left\{ \binom{d+2}{2} - \sum_{i=1}^{r} \binom{m_i+1}{2}, 0 \right\}. \quad (1)$$

In general $\dim I(X)_d \geq \expdim I(X)_d$. If the conditions given by $X$ are either independent or $\dim I(X)_d = 0$, then $\dim I(X)_d = \expdim I(X)_d$ and the system is called nonspecial. Otherwise it is called special.

Classifying the special linear systems is a very hard task, even if the base points are in general position. A conjectural answer comes from the celebrated Segre-Harbourne-Gimigliano-Hirschowitz conjecture (see [16], [11], [10], [14]).

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**Conjecture 1.2** (SHGH Conjecture). Let \( X = m_1P_1 + \ldots + m_rP_r \) be a fat point scheme. Assume that \( P_1, \ldots, P_r \) are in general position. If the linear system \( I(X)_d \) is special, then its general element is non-reduced, namely the linear system has some multiple fixed component.

Mathematicians have been working on this problem for over a century. A nice survey of the known results and the techniques applied to get them is [4]. We also refer to [3].

In [6], Cook II, Harbourne, Migliore and Nagel focused on a subtler problem about special linear systems of plane curves. Namely, they drop the hypothesis of generality of some of the points, and they propose a classification problem analogous to the SHGH Conjecture (see [6, Problem 1.4]), although this problem seems too difficult to be solved in full generality. Following their example, we focus on a simplified version, and we consider degree \( d \) curves containing a general point of multiplicity \( d - 1 \) and a bunch of (not necessarily general) simple points.

**Definition 1.3.** Let \( d \in \mathbb{N} \). We say that a finite set of distinct points \( Z \subset \mathbb{P}^2 \) admits an unexpected curve of degree \( d \) if

\[
\dim I(Z + (d - 1)P)_d > \max \left\{ \dim I(Z)_d - \binom{d}{2}, 0 \right\}
\]

for a general \( P \in \mathbb{P}^2 \).

We want to stress that in the definition of an unexpected curve we do not take into account the number of conditions that \( Z \) imposes on curves of degree \( d \). Compare inequality (2) with equation (1).

In the paper [8] Di Marca, Malara and Oneto present a way to produce families of unexpected curves using supersolvable arrangements of lines. Moreover, in [2] Bauer, Malara, Szemberg and Szpond consider the existence of special linear systems in a higher dimensional projective space, namely in \( \mathbb{P}^3 \), and exhibit there a quartic surface with unexpected postulation properties.

One of the purposes of this paper is to classify all unexpected plane curves in low degrees. For \( d = 3 \) we have the following result.

**Theorem 1.4.** No set of points \( Z \subset \mathbb{P}^2 \) admits an unexpected cubic over \( \mathbb{C} \).

Things become more complicated for \( d = 4 \). In this case there exists a configuration of nine points in \( \mathbb{P}^2 \) which admits an unexpected quartic. It was first presented by Di Gennaro, Ilardi and Vallès in [7] and is discussed by Harbourne in [12, Example 4.1.10].

**Example 1.5** (An unexpected quartic). Let \( Z_1, Z_2, Z_3, Z_4 \in \mathbb{P}^2 \) be four general points. The lines joining any two of them determine three intersection points \( Z_5, Z_6, Z_7 \). Take a line through any two of the points \( Z_5, Z_6 \) and \( Z_7 \) (in Figure 1, through \( Z_6 \) and \( Z_7 \)). Call \( Z_8 \) and \( Z_9 \) the two intersection points with the previous lines and define \( Z = \{ Z_1, \ldots, Z_9 \} \).

Since a set of four points in \( \mathbb{P}^2 \) is unique up to projective equivalence, and since all the points are defined by \( Z_1, \ldots, Z_4 \), the configuration \( Z \) is unique up to projective equivalence as well.

![Figure 1. A configuration of nine points in \( \mathbb{P}^2 \) admitting an unexpected quartic.](image-url)
In this paper we deeply analyse the geometry of this configuration and prove the following result.

**Theorem 1.6.** Up to projective equivalence, the configuration of points $Z \subset \mathbb{P}^2$ in Example 1.5 is the only one which admits an unexpected curve of degree 4.

Our paper is organized as follows. In Section 2 we use plane geometry arguments and Bezout’s Theorem to prove Theorem 1.4. A standard tool to prove that a linear system is empty is to degenerate some of the points to a special position. If the degenerated linear system is empty, then the original one is empty as well. Moreover, we describe in detail the configuration of nine points of Example 1.5 and give a new proof of an unexpected quartic.

Similarly to the proof of Theorem 1.4, in Section 3 we show that if a subset $Z \subset \mathbb{P}^2$ of nine distinct points with four of them collinear admits an unexpected quartic, then it has to be projectively equivalent to the one of Example 1.5. This will conclude the first part of the proof of Theorem 1.6.

In Section 4 we show that a subset $Z \subset \mathbb{P}^2$ of nine distinct points admitting an unexpected quartic has necessarily at least four collinear points. Instead of geometric arguments, here we use the notion of semistability of vector bundles. This argument is clearly more powerful and it can be naturally generalized for higher degrees, even though it needs less elementary machinery.

Finally, in Section 5 we rephrase the problem of finding subsets of points admitting an unexpected curve of degree $d$ from a different point of view, using de Jonqui`eres transformations. Moreover, we provide necessary conditions for the existence of unexpected curves in higher degree, and we use them to complete the proof of Theorem 1.6.

2. **Unexpected cubics and quartics in $\mathbb{P}^2$**

Let us fix the notation. Given two points $A, B \in \mathbb{P}^2$, we denote by $AB$ the line joining them. We call a line simple if it contains only two points of $Z$. A line is $k$-rich, for $k \geq 3$, if it contains exactly $k$ points of $Z$.

Throughout this section and Section 3 we will repeatedly use the following general result.

**Proposition 2.1.** Let $d$ be a positive integer. Let $Z$ be a set of $2d + 1$ distinct points, and $P \in \mathbb{P}^2$ a general point. If there are a $d$-rich line $L_Q$ and a simple line $L_R$ such that $L_Q \cap L_R \neq Z$, then $I(Z + (d - 1)P)_d = 0$.

**Proof.** Assume by contradiction that $I(Z + (d - 1)P)_d \neq 0$. Consider $\{Q_1, \ldots, Q_d\} = Z \cap L_Q$, $\{R_1, R_2\} = Z \cap L_R$ and $\{S_1, \ldots, S_{d-1}\} = Z \setminus (L_Q \cup L_R)$. Let $C$ be the degree $d$ curve defined by a non-zero element of $I(Z + (d - 1)P)_d$.

We specialize the $(d - 1)$-ple point $P$ to a general point on $L_R$. By Bézout’s Theorem, $L_R$ and $L_Q$ are irreducible components of $C$. We are left with a degree $(d - 2)$ curve $C'$ passing through $d - 1$ simple points and a $(d - 2)$-ple point $P$. Again by Bezout’s Theorem, for all $i \in \{1, \ldots, d - 1\}$ the line joining $S_i$ and $P$ is an irreducible component of $C'$. Hence $C'$ is a curve of degree $d - 2$ having $d - 1$ lines as components, which is impossible. \qed

First we address the problem of existence of unexpected cubics. We show that they cannot appear if the ground field is $\mathbb{C}$.

**Proof of Theorem 1.4.** Consider a set of points $Z \subset \mathbb{P}^2$. If $|Z| < 7$, then any unexpected cubic is reducible by Proposition 5.7. By [6, Theorem 5.9], this implies that some subset of $Z$ admits an unexpected conic, but this is impossible by [6, Theorem 1.2] and [6, Corollary 6.8].

Assume now that $|Z| > 7$. Let $W$ be any subset of seven points of $Z$. Observe that $I(W + 2P)_3$ contains $I(Z + 2P)_3$ for every $P \in \mathbb{P}^2$, hence $\dim I(W + 2P)_3 \geq \dim I(Z + 2P)_3$. In particular, in order to conclude we need to prove that no subset $Z$ of seven points admits an unexpected cubic over $\mathbb{C}$. Therefore, for the rest of the proof we assume that $|Z| = 7$.

If an unexpected cubic exists, then [6, Theorem 1.2] implies that $Z$ contains no subset of four or more collinear points. On the other hand, [6, Corollary 6.8] shows that the points of $Z$ cannot be in linearly general position.

Hence suppose that $L$ is a 3-rich line and consider $\{Z_1, Z_2, Z_3\} = Z \cap L$. Let $Z_1$ and $Z_3$ be two points of $Z \setminus L$. By Proposition 2.1, $Z_1Z_3$ must meet $L$ in a point of $Z$, and we assume that $Z_3 = L \cap Z_1Z_2$. Since there cannot be four collinear points, we have that $Z \setminus (L \cup Z_1Z_2) = \{Z_6, Z_7\}$. Again by
Proposition 2.1, the lines \(Z_4Z_6\) and \(Z_5Z_6\) meet \(L\) in a point of \(Z\). Up to relabeling, the only possibility is that \(Z_1 \in Z_4Z_6\) and that \(Z_2 \in Z_5Z_6\). A similar argument is used to show that \(Z_2 \in Z_4Z_7\) and that \(Z_1 \in Z_5Z_7\). Hence, up to projective equivalence, \(Z\) is the configuration described in Figure 2.

![Figure 2. The given configuration.](image)

It is easy to check that in this case \(Z_3 \in Z_6Z_7\) over \(\mathbb{C}\). Since the line \(Z_6Z_7\) is simple and \(L\) is a 3-rich line, \(Z\) does not admit an unexpected cubic by Proposition 2.1. \(\square\)

Actually, a stronger version of Theorem 1.4 holds. In [1], Akesseh proves that unexpected cubics exist only if the characteristic of the ground field is 2.

Now we turn our attention to the case \(d = 4\). The proof of existence of the unexpected quartic in Example 1.5 presented in [6] uses splitting types. Here we give a new, simpler proof.

**Proposition 2.2.** The configuration \(Z\) of nine points of Example 1.5 admits an unexpected curve of degree 4.

**Proof.** Let \(P = [a, b, c]\) be a general point. Up to projective equivalence, we can assume that

\[
Z_1 = [-1, 0, 1], \quad Z_2 = [0, -1, 1], \quad Z_3 = [1, 0, 1], \quad Z_4 = [0, 1, 1].
\]

By construction, the remaining points have coordinates

\[
Z_5 = [0, 0, 1], \quad Z_6 = [1, -1, 0], \quad Z_7 = [1, 1, 0], \quad Z_8 = [0, 1, 0], \quad Z_9 = [1, 0, 0].
\]

Let \(L_1\) be the linear form defining the line \(Z_1Z_3\), let \(L_2\) define \(Z_2Z_4\), and let \(L_3\) define the line \(Z_6Z_7\). Furthermore, for every \(j\) define \(M_j\) to be the linear form defining the line \(PZ_j\).

Now, Bézout’s theorem implies that \(I(Z)_3\) is nonspecial, and therefore \(I(Z)_4\) is nonspecial as well. In the beginning of this Section, we saw that a general double point always imposes 3 independent conditions, so \(I(Z + 2P)_4\) is nonspecial. One can check that

\[
G_1 = L_1L_2M_6M_7, \quad G_2 = L_1L_3M_2M_4, \quad G_3 = L_2L_3M_1M_3
\]

are linearly independent and thus form a basis of \(I(Z + 2P)_4\). Since each \(G_i\) is singular at \(P\), we have \(G_i(P) = (G_i)_x(P) = (G_i)_y(P) = 0\) for every \(i\). The existence of an unexpected quartic is equivalent to the fact that the three additional conditions that the triple point \(P\) imposes on \(G_1, G_2, G_3\) (given by the three second order partials in \(x\) and \(y\)) are linearly dependent. This means that

\[
\det \begin{pmatrix}
(G_1)_{xx} & (G_2)_{xx} & (G_3)_{xx} \\
(G_1)_{xy} & (G_2)_{xy} & (G_3)_{xy} \\
(G_1)_{yy} & (G_2)_{yy} & (G_3)_{yy}
\end{pmatrix} = 0.
\]

This condition can be directly checked by exploiting the facts that \(G_1, G_2, G_3\) are completely reducible with pairwise common factors, and that \(M_j(P) = 0\) for every \(j\). \(\square\)

3. Geometric conditions on unexpected quartics

We will show that, up to projective equivalence, there is a unique configuration of nine distinct points in \(\mathbb{P}^2\) which admits an unexpected quartic.

Throughout this section, \(Z\) will indicate a set of nine distinct points, and \(P \in \mathbb{P}^2\) a general point.

If an unexpected quartic exists, then [6, Theorem 1.2] shows that \(Z\) does not contain any subset of five or more collinear points. On the other hand, by [6, Corollary 6.8], the points of \(Z\) cannot be in
linearly general position. In this section we aim to provide further necessary conditions for the sets $Z$ giving rise to unexpected quartics.

For instance, the presence of a 4-rich line imposes a precise behaviour on the configuration. The next propositions show how such a line has to intersect the other lines.

**Proposition 3.1.** If there are two 4-rich lines $L_Q, L_R$ such that $L_Q \cap L_R \notin Z$, then $I(Z + 3P)_4 = 0$.

**Proof.** Assume by contradiction that $I(Z + 3P)_4 \neq 0$. By hypothesis there exists a unique point $S \in Z \setminus (L_R \cup L_Q)$. Set $Z \cap L_R = \{R_1, R_2, R_3, R_4\}$. By Proposition 2.1, for any $i \in \{1, 2, 3, 4\}$ the lines $SR_i$ and $L_Q$ meet in a point of $Z$, say $Q_i$ (see Figure 3). Up to projective equivalence, we assume that

$$
S = [0,0,1], \quad R_1 = [1,0,0], \quad R_2 = [0,1,0], \quad Q_3 = [1,1,1].
$$

This choice of coordinates implies that $L_R$ is the line $z = 0$ and that $R_3 = [1,1,0]$. Since $Q_4 \notin L_R$, $Q_4 = [a,b,1]$ for some parameters $a$ and $b$. Therefore $L_Q$ is the line $(1-b)x + (a-1)y + (b-a)z = 0$, $Q_1 = [a-b,0,1-b]$ with $a \neq b$ and $b \neq 1$, $Q_2 = [0,a-b,1-a]$ with $a \neq 1$ and finally $R_4 = [a,b,0]$ with $a \neq 0$ and $b \neq 0$.

Now let $D$ be the quartic defined by a non-zero element of $I(Z + 3P)_4$. We consider three different specializations of $P$ that put constraints on $a$ and $b$ and we show that there is no choice of $a$ and $b$ that satisfies all the constraints simultaneously.

First of all, observe that the lines $R_1Q_2$ and $R_2Q_1$ are simple. If we specialize $P$ to the point $R_1Q_2 \cap R_2Q_1 = [(a-b)/(b-1),(b-a)/(1-a),1]$, then $D$ contains $R_1Q_2, R_2Q_1$ and the singular conic $R_3Q_3 \cup R_4Q_4$. Moreover, $P$ must be on either $R_3Q_3$ or $R_4Q_4$, in which case $b = 2-a$ or $1/a+1/b = 2$ respectively.

Similarly as before, the lines $R_1Q_3$ and $R_3Q_1$ are simple. If we specialize $P$ to the point $R_1Q_3 \cap R_3Q_1 = [(2b-a-1)/(b-1),1,1]$, then $D$ contains $R_1Q_3, R_3Q_1$ and the singular conic $R_2Q_2 \cup R_4Q_4$. The conclusion now is that $P$ must be on either $R_2Q_2$ or $R_4Q_4$. The first case yields the condition $a = 2b - 1$, whereas the second one gives the condition $b = 1/2$.

On one hand, if we assume that $a = 2b - 1$, then the only one compatible constraint between the two provided by the first specialization of $P$ is $1/a+1/b = 2$. This gives the only one possible solution $(a,b) = (-1/2,1/4)$. On the other hand, if we assume that $b = 1/2$, from the first specialization of $P$ we must have that $b = 2-a$. Then another solution is $(a,b) = (3/2,1/2)$.

Finally, we observe that the lines $R_1Q_4$ and $R_4Q_1$ are simple as well. If we specialize $P$ to the point $R_1Q_4 \cap R_4Q_1 = [ab-2a+b,b(b-1),b-1]$, then $D$ contains $R_1Q_4, R_4Q_1$ and the singular conic $R_2Q_2 \cup R_3Q_3$. The conclusion now is that $P$ must be on either $R_2Q_2$ or $R_3Q_3$. Since $R_2Q_2$ and $R_3Q_3$ are the lines $x = 0$ and $x - y = 0$ respectively, we arrive at a contradiction either if $(a,b) = (-1/2,1/4)$ or $(a,b) = (3/2,1/2)$.

The previous result is important because it imposes a tight restriction on the set $Z$ admitting an unexpected quartic. Indeed, there is only one $Z$ having more than one 4-rich line.

**Proposition 3.2.** Assume that $I(Z + 3P)_4 \neq 0$. If there are two 4-rich lines $L_Q, L_R$, then the configuration of the points of $Z$ is the one described in Example 1.5.
Proof. By Proposition 3.1, the 4-rich lines \( L_Q \) and \( L_R \) meet in a point of \( Z \). Then we can suppose that \( L_R \cap Z = \{R_1, R_2, R_3, B\} \) and that \( L_Q \cap Z = \{Q_1, Q_2, Q_3, B\} \). Let \( \{S_1, S_2\} = Z \setminus (L_R \cup L_Q) \) and \( L_S \) be the line containing \( S_1 \) and \( S_2 \). By Proposition 2.1, \( L_S \) meets \( L_Q \) and \( L_R \) in a point of \( Z \).

Assume that \( B \in L_S \). Let \( L_{ij} \) be the line joining \( S_i \) and \( R_j \) for \( i \in \{1, 2\} \) and \( j \in \{1, 2, 3\} \). By Proposition 2.1, each line \( L_{ij} \) meets \( L_Q \) in a point of \( Z \). We show that the two cubics \( C_1 = L_{11} \cup L_{12} \cup L_{13} \) and \( C_2 = L_{21} \cup L_{22} \cup L_{23} \) never coincide when restricted on the line \( L_Q \). Assume that \( B = [1, 0, 0] \) and that the equations of \( L_Q \), \( L_R \) and \( L_S \) are respectively \( y = 0 \), \( z = 0 \) and \( y - z = 0 \). In particular, \( S_i = [s_i, 1, 1] \) and \( R_j = [r_j, 0, 1] \) for some parameters \( s_i \) and \( r_j \) for \( i \in \{1, 2\} \) and \( j \in \{1, 2, 3\} \). With these assumptions, we obtain that \( L_{ij} \) is defined by the linear form \( l_{ij} = x + (r_j - s_i)y - rz = 0 \) for all \( i \) and \( j \). With a bit of work, one can see that \( l_{11}l_{12}l_{13} = l_{21}l_{22}l_{23} \) if and only if \( s_1 = s_2 \), and the latter condition is impossible since \( S_1 \neq S_2 \).

The above argument implies that \( B \notin L_S \). Up to relabeling, we assume that \( Q_2 \in L_S \) and \( R_2 \in L_S \). Let \( M_1 \) be the line containing \( R_1 \) and \( S_2 \) and let \( M_3 \) be the line containing \( R_3 \) and \( S_2 \). By Proposition 2.1, \( M_1 \) and \( M_3 \) meet \( L_Q \) in a point of \( Z \), and up to relabeling we assume that \( Q_1 \in M_1 \) and that \( Q_3 \in M_3 \). Now consider the line \( N_1 \) joining \( S_1 \) and \( R_1 \) and the line \( N_3 \) joining \( S_1 \) and \( R_3 \). Again by Proposition 2.1, \( N_1 \) and \( N_3 \) meet \( L_Q \) in a point of \( Z \). In particular, \( Q_1 \notin N_1 \) for \( i \in \{1, 3\} \) because \( S_1 \neq S_2 \). Moreover, \( Q_2 \notin N_1 \) because \( R_0 \neq R_4 \) for \( i \in \{1, 3\} \). Therefore, the only possibility is that \( Q_3 \in N_1 \) and that \( Q_1 \in N_3 \). Hence the obtained configuration is projectively equivalent to the one described in Example 1.5.

The following further property of 4-rich lines is a further step toward the proof of uniqueness.

**Proposition 3.3.** If there is a 3-rich line \( L_Q \) and exactly one 4-rich line \( L_R \) such that \( L_Q \cap L_R \notin Z \), then \( I(Z + 3P)_4 = 0 \).

**Proof.** Assume by contradiction that \( I(Z + 3P)_4 \neq 0 \). Suppose that \( L_R \cap Z = \{R_1, R_2, R_3, R_4\} \) and that \( L_Q \cap Z = \{Q_1, Q_2, Q_3\} \). By hypothesis there are only two points \( S_1, S_2 \) in \( Z \setminus (L_R \cup L_Q) \). Moreover, by Proposition 2.1 \( L_S = S_1S_2 \) must meet either \( L_R \) or \( L_Q \) in a point of \( Z \).

Suppose that \( L_S \) meets \( L_R \) in a point of \( Z \). If, in turn, \( L_S \) meets \( L_Q \) in a point of \( Z \), then a 4-rich line distinct from \( L_R \) appears, which is not allowed by hypothesis. Hence \( L_S \cap L_Q \cap Z = \emptyset \) and we assume that \( R_4 \in L_S \). Let \( L_{ij} \) be the line joining \( S_i \) and \( Q_j \) for \( i \in \{1, 2\} \) and \( j \in \{1, 2, 3\} \). By Proposition 2.1 and our assumption, \( L_{ij} \) meets \( L_R \) in a point of \( Z \) different from \( R_4 \) for all \( i \) and \( j \). Moreover, \( L_{ij} \) and \( L_{ij'} \) meet \( L_R \) in distinct points for all \( i \) and \( j_1 \neq j_2 \). Hence the two cubics \( C_1 = L_{11} \cup L_{12} \cup L_{13} \) and \( C_2 = L_{21} \cup L_{22} \cup L_{23} \) must coincide when restricted on the line \( L_Q \). With a similar argument of Proposition 3.2, one can check that this happens if and only if \( S_1 = S_2 \), a contradiction.

Now suppose that \( L_S \) meets \( L_Q \) in a point of \( Z \). In particular, suppose that \( L_S \) contains the point \( Q_3 \). By Proposition 2.1, the lines \( Q_1S_1 \) and \( Q_1S_2 \) are not simple, hence up to labeling we assume that \( Q_1S_1 \cap L_R = R_1 \) and \( Q_1S_2 \cap L_R = R_2 \). Regarding the lines \( Q_2S_1 \) and \( Q_2S_2 \), we have three cases to consider:

1. **Figure 4. The cases (1), (2) and (3) of Proposition 3.3.**

   (1) \( Q_2S_1 \cap L_R = R_3 \) and \( Q_2S_2 \cap L_R = R_4 \). Observe that the lines \( Q_1R_3, Q_1R_4 \) and \( Q_2R_2 \) are simple. Let \( X_1 = Q_1R_3 \cap Q_2R_3 \) and \( X_2 = Q_1R_4 \cap Q_2R_2 \). On one hand, we can specialize \( P \) on \( X_1 \). In order for \( D \) to exist, we need a conic having \( L_R \) and \( L_S \) as components. Noting that \( X_1 \notin L_R \), we have that \( X_1 \in L_S \). On the other hand, we can specialize \( P \) on \( X_2 \). In order
for $D$ to exist, we need a conic having $L_R$ and $L_S$ as components and, similarly as before, $X_2 \in L_S$. Hence $Q_3 R_2$ and $L_S$ should coincide, a contradiction.

(2) $Q_3 S_1 \cap L_R = R_2$ and $Q_3 S_2 \cap L_R = R_3$. Note that the lines $Q_1 R_3, Q_1 R_4$ and $Q_2 R_1$ are simple. Now repeat the argument from case (1).

(3) $Q_3 S_1 \cap L_R = R_2$ and $Q_3 S_2 \cap L_R = R_1$. Up to projective equivalence, we assume that

$$R_1 = [1, 0, 0], \quad R_2 = [0, 1, 0], \quad S_1 = [0, 0, 1], \quad S_2 = [1, 1, 1].$$

This choice of coordinates implies that $L_R$ and $L_S$ are the lines $z = 0$ and $x - y = 0$ respectively. Moreover, we obtain that $Q_1 = [1, 0, 1], Q_2 = [0, 1, 1]$. Therefore, $L_Q$ is the line $x + y - z = 0$ and $Q_3 = [1, 1, 2]$. Now we assume that $R_3 = [1, a, 0]$ and $R_4 = [1, b, 0]$ for some parameters $a$ and $b$ such that $a \neq 0, b \neq 0$ and $a \neq b$. Moreover, we exclude the case $\{a, b\} = \{-1, 1\}$, which is not allowed by hypothesis and in particular coincides with the configuration of Example 1.5. Let $D$ be the quartic defined by a non-zero element of $I(Z + 3P)_4$. We follow the same argument used in Proposition 3.1. First of all, observe that, with the given constraints on the parameters $a$ and $b$, the lines $R_3 S_1, R_3 Q_2, R_3 Q_3$ and $R_4 S_2$ are simple.

If we specialize $P$ to the point $Y_1 = R_3 S_1 \cap R_4 S_2 = [b - 1, a(b - 1), -b]$, then $D$ contains $R_3 S_1, R_3 S_2$ and the singular conic $L_Q \cup L_R$. Moreover, $Y_1$ must be on either $L_R$ or $L_Q$, in which case $a = b$ (impossible by our assumption) or $ab = 1$ respectively. Hence, we can rewrite $R_1 = [a, 1, 0]$.

If we specialize $P$ to the point $Y_2 = R_3 Q_2 \cap R_4 S_2 = [1, 1 + a - a^2, 1 - a^2]$, then $D$ contains $R_3 Q_2, R_3 S_2$ and the singular conic $Q_3 R_2 \cup Q_1 R_1$. The conclusion now is that $Y_2$ must be on either $Q_3 R_2 : 2x - z = 0$ or $Q_1 R_1 : y = 0$. The first case yields the condition $a^2 + 1 = 0$, whereas the second one gives the condition $a^2 - a - 1 = 0$.

Finally, if we specialize $P$ to the point $Y_3 = R_3 Q_3 \cap R_4 S_2 = [a + 2, 2a + 1, 2(a + 1)]$, then $D$ contains $R_3 Q_3, R_3 S_2$ and the singular conic $Q_1 R_1 \cup Q_2 R_2$. Hence $Y_3$ must be on either $Q_1 R_1 : y = 0$ or $Q_2 R_2 : x = 0$, so either $a = -1/2$ or $a = -2$. Since neither $a = -1/2$ nor $a = -2$ is a root of either of the polynomials $a^2 - a - 1$ or $a^2 - a + 2$, we have that the initial constraints on $a$ and $b$ prevent the configuration from admitting an unexpected quartic. 

**Proposition 3.4.** If there is exactly one 4-rich line $L$, then $I(Z + 3P)_4 = 0$.

**Proof.** Suppose there is a 4-rich line $L$, with $L \cap Z = \{Z_1, Z_2, Z_3, Z_4\}$. Let $R = \{R_1, \ldots, R_5\}$ denote the remaining five points of $Z$. First of all, by Propositions 3.2 and 3.3, if three of the points of $R$ are collinear, then $I(Z + 3P)_4 = 0$. For $i \in \{1, \ldots, 5\}$, let $L_i$ be the line containing $Z_i$ and $R_i$. Since $|R|$ is odd and none of the lines $L_i$ can contain more than 2 points of $R$, one of the lines $L_i$ must contain exactly one of the points of $R$. So say $L_1$ contains only $R_1$. For $j \in \{2, \ldots, 5\}$, define $M_k$ to be the line $R_1 R_k$. By Proposition 2.1, the line $M_2$ intersects $L$ in $\{Z_1, Z_2, Z_3, Z_4\}$. Hence up to relabelings we may assume that $M_2 \cap L = Z_2$. Now the point $R_3$ cannot belong to the line $M_2$ by hypothesis, so $M_2 \neq M_3$ and we may assume that $M_3 \cap L = Z_3$. By the same argument we obtain that $M_2, M_3, M_4$ are distinct lines and $M_4 \cap L = Z_4$. Then the line $M_5$ contains two points of $Z$ and $M_5 \cap L \cap Z = \emptyset$. Proposition 2.1 implies that $I(Z + 3P)_4 = 0$. 

**Corollary 3.5.** The configuration of Example 1.5 is the only configuration of nine points in $\mathbb{P}^2$ containing a 4-rich line which admits an unexpected quartic.

Corollary 3.5 concludes the first part of the proof of Theorem 1.6. In the next section we will show that if a configuration $Z$ of nine points admits an unexpected quartic, then $Z$ has a 4-rich line.

4. Unexpected curves and stability conditions

In this section $Z \subset \mathbb{P}^2$ will denote any finite set of distinct points. For us, the stability (respectively, the semistability) of $Z$ is the stability (respectively, the semistability) of its dual line arrangement $A_Z$. The latter is defined in [6, Section 6] in terms of the derivation bundle of $A_Z$, but what we actually need are the following properties. The first one follows from [6, Proposition 6.4].

**Lemma 4.1.** If $Z \subset \mathbb{P}^2$ is semistable or stable, then $Z$ admits no unexpected curve.

The next results are proven in [6, Lemma 6.5] and [6, Proposition 6.7].
Lemma 4.2. Let $Z \subset \mathbb{P}^2$ be a set of points and $P \in Z$. Consider $Z' = Z \setminus \{P\}$ and the line arrangement $A = \{PQ \mid Q \in Z'\}$. We define the set $Z'' = \mathbb{P}^2$ to be the dual of $A$. Then

(1) if $|Z|$ is odd, $Z'$ is stable and $|Z''| > \frac{|Z|+1}{2}$, then $Z$ is stable,
(2) if $|Z|$ is odd and $Z'$ is stable, then $Z$ is semistable,
(3) if $|Z|$ is even, $Z'$ is semistable and $|Z''| > \frac{|Z|}{2}$, then $Z$ is stable,
(4) if $|Z|$ is even and $Z'$ is stable, then $Z$ is stable.

Lemma 4.3. If $Z \subset \mathbb{P}^2$ is a set of at least four points in linearly general position, then $Z$ is stable.

There are some configurations of points which will be useful for us.

Definition 4.4. (1) For $n \geq 3$, the factors of the polynomial

$$(x^n - y^n)(x^n - z^n)(y^n - z^n) \in \mathbb{C}[x, y, z]$$

define the Fermat arrangement of $3n$ lines in $\mathbb{P}^2$. Its dual is a configuration of $3n$ points in $\mathbb{P}^2$, called the dual Fermat configuration and denoted by $F_n$.

(2) We will indicate by $S$ the configuration of six points in $\mathbb{P}^2$ given by the intersection points of four general lines.

See [12, Remark 1.1.4.(2)] for a figure and some comments on the Fermat arrangements.

Proposition 4.5. If $Z \subset \mathbb{P}^2$ is a set of nine points containing no $k$-rich lines for $k \geq 4$, then either $Z = F_3$ or $Z$ is semistable.

Proof. Let $Z \subset \mathbb{P}^2$ be a set of nine points different from $F_3$ and with no $k$-rich lines for $k > 3$. Our idea is to reduce gradually to the study of smaller subsets of $Z$.

Since $Z$ is not $F_3$, there exists a simple line $L$. Assume $L \cap Z = \{Z_8, Z_9\}$. By Lemma 4.2(2), in order to conclude it is enough to show that there exists a stable subset of $Z$ with eight points. Since there are no 4-rich lines, the set $\{Z_j \mid j \in \{1, \ldots, 8\}\}$ has at least five distinct elements $L, L_1, \ldots, L_4$. Up to relabeling, we can assume $Z_1 \in L_1, \ldots, Z_4 \in L_4$. Let $A = \{Z_5, Z_6, Z_7\}$ and set $W_8 = Z \setminus \{A\}$. All we need to do is to prove that $W_8$ is stable. In order to do that, we want to apply Lemma 4.2(3).

Define $W_2 = W_8 \setminus \{Z_0\}$. We will prove that $W_7$ is semistable. In turn, by Lemma 4.2(2) it is enough to check that there exists a stable subset $W_6 \subset W_7$ with six elements.

Now we want to show that $W_7$ contains at least subset $W_6$ of six elements which is different from the configuration $S$ of Definition 4.4. Consider one of the subsets of six points of $W_7$. If it is not $S$, then we are done. If it is $S$, then the seventh point of $W_7$ does not lie on any of the four 3-rich lines of $S$, because our hypothesis guarantees that $Z$ has no $k$-rich lines for $k \geq 4$. Therefore, if we replace one of the points of $S$ with the seventh one, the resulting subset of $W_7$ is not $S$. Call this subset $W_6$.

Since $W_6$ is not $S$, there exists a subset $W_5 \subset W_6$ of five elements with at most one 3-rich line. By Lemma 4.2(4), it is enough to prove that $W_5$ is stable. If $W_5$ has no 3-rich lines, then it is stable by Lemma 4.3. Otherwise $W_5$ has exactly one 3-rich line. Up to projective equivalence, we assume that $W_5 = \{B_1, B_2, B_3, B_4, B_5\}$, where

$$B_1 = [1, 0, 0], \quad B_2 = [0, 1, 0], \quad B_3 = [0, 0, 1], \quad B_4 = [1, 1, 1], \quad B_5 = [1, a, 0]$$

for some parameter $a$. Using the Macaulay2 lines one can verify that also in this case $W_5$ is stable.

Note that in this case $|W_5| = 5$, and in the above code the invariant $m_{W_5}$ is the multiplicity index defined in [6, Definition 3.1].

The last result of this section is an important step toward the proof of Theorem 1.6.
Lemma 4.6. Up to projective equivalence, the only configuration of nine points \( Z \subset \mathbb{P}^2 \) admitting an unexpected quartic is the one presented in Example 1.5.

Proof. First assume that \( Z \) admits no 4-rich lines. Since the configuration \( F_3 \) does not admit an unexpected curve by [6, Section 6], Proposition 4.5 and Lemma 4.1 imply that \( Z \) does not admit an unexpected quartic.

If \( Z \) has a 4-rich line, then we conclude by Corollary 3.5. \( \square \)

Remark 4.7. It is interesting to point out that if \( n \geq 5 \) then the configuration \( F_n \) admits unexpected curves of degrees \( n+2, \ldots, 2n-3 \) by [6, Proposition 6.12].

5. De Jonquières Transformations and Unexpected Curves

Unexpected curves do not need to be irreducible. However, as [6, Theorem 5.9] shows, they always have only one nontrivial irreducible component. For this reason, all unexpected curves we consider from now on will be assumed to be irreducible.

In this Section we look at the peculiar behaviour that unexpected curves have under the action of a special class of Cremona transformations of \( \mathbb{P}^2 \), called de Jonquières transformations. This observation will easily lead to some numerical conditions on the degree of an unexpected curve.

First, recall that a degree \( d \) divisor in \( \mathbb{P}^n \) is a monoid if it has a singular point of order \( d-1 \).

Definition 5.1. Let \( P, Q_1, \ldots, Q_{2d-2} \in \mathbb{P}^2 \). The de Jonquières transformation of degree \( d \) with centers \( P, Q_1, \ldots, Q_{2d-2} \) is the rational map associated to the linear system

\[
I((d-1)P + Q_1 + \cdots + Q_{2d-2})_d
\]

of degree \( d \) monoids singular at \( P \) and containing the simple points \( Q_1, \ldots, Q_{2d-2} \).

Remark 5.2. Let \( S \subset \mathbb{P}^n \) be a hypersurface, and let \( \mathcal{L} \) be a linear system on \( \mathbb{P}^n \). The restriction map \( \rho : \mathcal{L} \to \mathcal{L}|_S \) induces the so called Castelnuovo exact sequence

\[
0 \to \ker(\rho) \to \mathcal{L} \to \mathcal{L}|_S,
\]

where \( \ker(\rho) \) is the linear system \( \{0\} \cup \{D \in \mathcal{L} \mid D \supset S\} \).

If both \( \ker(\rho) \) and \( \mathcal{L}|_S \) are nonspecial with non-negative virtual dimension, then their first cohomology groups vanish. Therefore, by Castelnuovo exact sequence, \( H^1\mathcal{L} = 0 \) and \( \mathcal{L} \) is nonspecial as well.

We will show that the de Jonquières transformation is birational whenever there are not too many base points on a line.

Lemma 5.3. Let \( d \geq 2 \), and let \( Y = \{Y_1, \ldots, Y_{2d-2}\} \subset \mathbb{P}^2 \) such that \( Y \) does not contain \( d+1 \) collinear points. Let \( P \in \mathbb{P}^2 \) be a general point and let \( \mathcal{L} = I((d-1)P + Y)_d \). Then \( \mathcal{L} \) is nonspecial of projective dimension \( 2 \) and \( \mathcal{L}^2 = 1 \), that is, the associated rational map \( \varphi \) is birational.

Proof. We argue by induction on \( d \). For \( d = 2 \), \( \mathcal{L} = I(P + Y_1 + Y_2)_2 \) and therefore the associated rational map is the standard Cremona transformation. Assume that \( d \geq 3 \).

Observe that \( Y \) contains at most one \( d \)-rich line. Let \( L \) be a line such that \( |L \cap Y| \) is maximum and let \( T \in L \cap Y \). Let \( R \) be the line joining \( T \) and \( P \). Since \( P \) in general, \( R \cap Y = \{T\} \). The restriction \( \rho \) to \( R \) induces the Castelnuovo exact sequence

\[
0 \to \ker(\rho) \to \mathcal{L} \to \mathcal{L}|_R,
\]

First note that both \( \ker(\rho) \) and \( \mathcal{L}|_R \) have non-negative expected dimension. It is easy to see that \( \mathcal{L}|_R \) is nonspecial. We aim to show that \( \ker(\rho) \) is nonspecial as well.

Claim 1. \( \ker(\rho) \) is nonspecial.

Proof of Claim 1. Let \( Y' = Y \setminus \{T\} \). Then \( Y' \) does not contain \( d+1 \) collinear points and \( \ker(\rho) = I((d-2)P + Y')_{d-1} \). By induction hypothesis, the multiple point and the first \( 2d-4 \) simple points of \( Y' \) give independent conditions to \( \ker(\rho) \). Let \( T' \) be the last simple point, \( Y'' = Y' \setminus \{T'\} \) and \( \mathcal{L} := I((d-2)P + Y'')_{d-1} \). Assume by contradiction that \( T' \) does not give an independent condition.
Then $T'$ is a base point for $\tilde{L}$. Hence the base locus $B$s $\tilde{L}$ contains at least a $(d - 2)$-ple point and $2d - 3$ simple points, so

$$\tilde{L}^2 \leq (d - 1)^2 - (d - 2)^2 - 2d + 3 = 0.$$ 

But $\tilde{L}^2 = 1$ by hypothesis, contradiction.

Since $\ker(\rho)$ is nonspecial, $\mathcal{L}$ is nonspecial as well. In particular, $\varphi$ maps $\mathbb{P}^2$ to $\mathbb{P}^2$. Now we focus on the self-intersection of $\mathcal{L}$.

**Claim 2.** The general element of $\mathcal{L}$ is irreducible. In particular, $\mathcal{L}$ has no fixed components and $\mathcal{L}^2 > 0$.

**Proof of Claim 2.** Let $D \in \mathcal{L}$ be general, and assume by contradiction

$$D = A + B \text{ with } \deg A = a \text{ and } \deg B = b.$$ 

Since $D$ is a monoid, up to order we have that $A = L_1 + \cdots + L_a$ is a union of lines and $B$ is an irreducible monoid. Moreover, $D$ is singular at $A \cap B$, so $A \cap B \subset Bs \mathcal{L}$ by Bertini’s Theorem.

First we want to prove that $L_i \cap Y \neq \emptyset$ for every $i \in \{1, \ldots, a\}$. Indeed, assume by contradiction that $L_i \cap Y = \emptyset$. Then we can replace $L_1$ with any line $L \ni P$ and get a new element of $\mathcal{L}$. By choosing $L$ to be the line joining $P$ and $Q \in B$, we see that every $Q \in B$ is a base point. Then $B \subset Bs \mathcal{L}$ and therefore

$$\mathcal{L} \cong \mathcal{L}_{2,a}(a, 1^s) \cong \mathcal{L}_{1,a}(1^s),$$ 

where $s = |Y \setminus B|$. Then $3 = \dim \mathcal{L} = \dim \mathcal{L}_{1,a}(1^s) = a - s$, hence $s = a - 3$. Up to reordering, we may assume that $L_1, L_2, L_3$ contain $P$ and no point of $Y$, while each $L_4, \ldots, L_a$ contain a point of $Y$. That point is in both $A$ and $B$, so it is a singular point for $D$ and therefore $L_1 \subset Bs \mathcal{L}$. Set $Y_B = Y \cap B$.

By construction

$$|Y_B| = 2d - 2 - (a - 3) = 2b + a + 1 > 2b - 2.$$ 

Since $B$ is irreducible, if $Y_B$ contains $b + 1$ collinear points, then $B$ is a line. Since $|A \cap Y| \leq d - 3$ by construction, $B$ contains more than $d + 1$ points and therefore $Y$ contains $d + 1$ collinear points, a contradiction. So $Y_B$ contains no $b + 1$ collinear points, therefore the first $2b - 2$ points of $Y_B$ give independent conditions on $I((b - 1)P + Y_B)_h$, and moreover $(I((b - 1)P + Y_B)_b)^2 = 1$. Hence the remaining points of $Y_B$ have to be base points, but that would decrease the self-intersection of $I((b - 1)P + Y_B)_b$, a contradiction. Thus every $L_i$ contains at least a point of $Y$, and the generality of $P$ implies it is exactly one point. By construction, $B$ has multiplicity $b - 1$ at $P$, so $B$ meets $L_1$ in exactly one point $X_1$ outside $P$. There are two possibilities.

(1) Assume that $X_1 \in Y$. Since both $A$ and $B$ pass through $X_1$, it is a singular point for $D$ and therefore $L_1 \subset Bs \mathcal{L}$.

(2) Assume that $X_1 \notin Y$. In this case $L_1 \cdot D \geq d - 1 + 1 + 1 > d$ and therefore $L_1 \subset Bs \mathcal{L}$.

Anyway we have $A \subset Bs \mathcal{L}$. Since $P$ is general, $|A \cap Y| = a$ and therefore

$$|Y_B| = 2d - 2 - a = 2b + a - 2 > 2b - 2.$$ 

We argue as before to find a contradiction. □

On one hand, we just checked that $\mathcal{L}^2 \geq 1$. On the other hand, $\mathcal{L}$ has a $(d - 1)$-ple point and $2d - 2$ simple points, so

$$1 \leq \mathcal{L}^2 \leq d^2 - (d - 1)^2 - 2d + 2 = 1 \Rightarrow \mathcal{L}^2 = 1.$$ 

Keeping in mind Lemma 5.3, we can take another look to Example 1.5.

**Example 5.4.** Let $P$ be a general point and let $Z = \{Z_1, \ldots, Z_9\} \subset \mathbb{P}^2$ be a set of nine points, not containing five collinear points. Let $\varphi$ be the degree 4 de Jonquières transformation with centers $P$ and $Z_1, \ldots, Z_6$, and let

$$\begin{array}{c}
\mathbb{P}^n \\
\Phi \downarrow \\
\varphi \circ \Phi \downarrow \\
X \\
\end{array}$$
be the resolution of indeterminacy of \( \varphi \). By Lemma 5.3, \( \varphi : \mathbb{P}^2 \to \mathbb{P}^2 \) is birational. This means that a quartic \( D \) triple in \( P \) and containing \( Z_1, \ldots, Z_9 \) becomes a line \( \varphi(D) \) through \( \Phi(Z_7), \Phi(Z_8), \Phi(Z_9) \).

It follows that if we consider \( Z \) as in Example 1.5, whenever we pick any six points of \( Z \), the corresponding de Jonquières transformation sends the last three points to collinear points.

An immediate consequence is that no degree \( d \) irreducible unexpected curve can exist if \( d \) is too large compared with \( |Z| \).

**Proposition 5.5.** Let \( d \in \mathbb{N} \) and let \( Z \subset \mathbb{P}^2 \) be a set of \( t \) points not containing \( d + 1 \) collinear points. If \( t \leq 2d - 2 \), then \( Z \) admits no irreducible unexpected curve of degree \( d \).

**Proof.** Let \( P \in \mathbb{P}^2 \) be a general point. In order to disprove the existence of an unexpected curve, it is enough to observe that the linear system \( I((d - 1)P + Z)_d \) is nonspecial by Lemma 5.3.

Lemma 5.3 allows us to give an equivalent definition of unexpected curve.

**Corollary 5.6.** Let \( d \in \mathbb{N} \) and let \( Z = \{Z_1, \ldots, Z_t\} \subset \mathbb{P}^2 \) be a set of \( t > 2d - 2 \) points not containing \( d + 1 \) collinear points. Then the following conditions are equivalent.

1. \( Z \) admits an irreducible unexpected curve of degree \( d \).
2. For any choice of \( Z_1, \ldots, Z_{2d-2} \in Z \) and for every general point \( P \in \mathbb{P}^2 \), the de Jonquières transformation with centers \( P \) and \( Z_1, \ldots, Z_{2d-2} \) maps \( Z_{2d-1}, \ldots, Z_t \) to collinear points.

On one hand, these simple observations give a numerical necessary condition on the degree of any unexpected curve. On the other hand, they provide an elementary geometric way to show the uniqueness of an unexpected curve (see [6, Corollary 5.5] for a different proof).

**Proposition 5.7.** Let \( d \in \mathbb{N} \) and let \( Z \subset \mathbb{P}^2 \) be a set of \( t \) points.

1. If \( t \leq 2d \), then \( Z \) does not admit an irreducible unexpected curve of degree \( d \).
2. If \( t \geq 2d + 1 \) and \( Z \) admits an irreducible unexpected curve, then this curve is unique.

**Proof.** We can assume that \( Z \) does not have \( d + 1 \) collinear points. By Proposition 5.5, we can also assume \( t \geq 2d - 1 \). Let \( \mathcal{L} = I((d - 1)P + Z)_d \). Fix \( Z_1, \ldots, Z_{2d-2} \in Z \) and let \( \varphi \) be the de Jonquières transformation with centers a general point \( P \) and \( Z_1, \ldots, Z_{2d-2} \). As before, let \( \Phi \) be the resolution of indeterminacy of \( \varphi \) and let \( Q = \{\Phi(Z_{2d-1}), \ldots, \Phi(Z_t)\} \). We already observed that \( \mathcal{L} = \varphi^*(I(Q)_1) \).

1. Write \( t = 2d - e \), with \( e \in \{0, 1\} \). On one hand,

\[
\expdim \mathcal{L} = \left( \frac{d + 2}{2} \right) - \left( \frac{d}{2} \right) - (2d - e) = 1 + e.
\]

On the other hand, \( \mathcal{L} \) is the pullback of the system of lines through \( t = 2 - e \) base points. Since

\[
-2 - e \in \{1, 2\}, \quad \varphi^*(I(Q)_1)
\]

has dimension \( 1 + e \). It follows that \( \mathcal{L} \) is nonspecial and therefore \( Z \) does not admit an unexpected curve.

2. \( Q \) consists of \( t - 2d + 2 \) points. By hypothesis and Corollary 5.6, these points are collinear. However, there is exactly one line through \( t - 2d + 2 \) collinear points, so \( \dim \mathcal{L} = 1 \).

Basically, Proposition 5.7 implies that an irreducible unexpected curve can appear only if the expected dimension is 0, and in this case the actual dimension is necessarily 1.

As a consequence, we can now complete the proof of our main result.

**Proof of Theorem 1.6.** Thanks to Lemma 4.6, we know that thesis holds if \( |Z| = 9 \). If \( |Z| < 9 \), then the unexpected curve is reducible by Proposition 5.7. By [6, Theorem 5.9], this implies that some subset of \( Z \) admits an unexpected cubic, and this contradicts Theorem 1.4.

Assume now that \( |Z| > 9 \). Let \( W \) be any subset of nine points of \( Z \). Observe \( I(W + 3P)_4 \) contains \( I(Z + 3P)_4 \) for every \( P \in \mathbb{P}^2 \), hence \( \dim I(W + 3P)_4 \geq \dim I(Z + 3P)_4 \). The latter equals 1 by Proposition 5.7. Since \( I(W + 3P)_4 \) is expected to be empty, \( W \) admits an unexpected quartic too, so \( \dim I(W + 3P)_4 = 1 \) for the same reason. It follows that \( I(V + 3P)_4 = I(Z + 3P)_4 \) for every \( V \subset Z \) such that \( 9 \leq |V| \leq |Z| \). In particular, we consider a set of 10 points. This \( V \) enjoys a peculiar property: if we remove any point from it, we get a subset \( W \) admitting an unexpected quartic. By Lemma 4.6, this means that every time we remove a point from \( V \), we get a configuration equivalent to Example 1.5. Such configuration has three 4-rich lines. It order to preserve this property, if we
remove $Z_9$ (see Figure 1), the tenth point of $V$ should lie in the intersection of two 3-rich lines, and this is not possible. □

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