Ideally embedded space-times

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Due to the growing interest in embeddings of space-time in higher-dimensional spaces we consider a specific type of embedding. After proving an inequality between intrinsically defined curvature invariants and the squared mean curvature, we extend the notion of ideal embeddings from Riemannian geometry to the indefinite case. Ideal embeddings are such that the embedded manifold receives the least amount of tension from the surrounding space. Then it is shown that the de Sitter spaces, a Robertson-Walker space-time and some anisotropic perfect fluid metrics can be ideally embedded in a five-dimensional pseudo-Euclidean space.

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I. INTRODUCTION

In recent years the ideas of Kaluza and Klein have received new attention. Shortly after the publication of the General Theory of Relativity Kaluza proposed to unify gravity and electromagnetism by adding an extra dimension. Klein suggested that this fifth dimension would be compactified and unobservable on experimentally accessible energy scales. This idea of compactifying the extra dimension has dominated the search for a unified theory and lead to the eleven-dimensional supergravity theory and more recent ten-dimensional superstring theory (see Ref. 1 for an overview).

Instead of compactifying the extra dimensions other approaches have been developed. In the Space-Time-Matter (STM) theory² the (3 + 1)-dimensional cosmologies may be recovered from the geometry of (4 + 1)-dimensional, vacuum General Relativity. Matter in four dimensions is induced by the shape of the embedded hypersurface and the five-dimensional Ricci flat geometry. More recently the Randall-Sundrum scenario has gained a lot of support. In Ref. 3,4 they try to solve the hierarchy problem between the observed Planck and weak scales by embedding the three-brane in a non-factorizable five-dimensional metric.

From a mathematical point of view the theory of embeddings starts with the definition of a manifold by Riemann. Shortly after the publication of his famous Habilitationsschrift (see e.g. Ref. 5 for a translation) Schlafli⁶ conjectured that any n-dimensional Riemannian manifold could be locally and isometrically embedded in a d-dimensional Euclidean space with d = n(n + 1)/2. This was proven by Janet and Cartan and extended to manifolds with indefinite metric by Friedman⁷. The Janet-Cartan theorem as it became known implies that we at maximum need ten dimensions to locally and isometrically embed any four-dimensional space-time.

A lesser known theorem by Campbell and Magaard⁸ states that any analytical Riemannian space \( V_n(s, t) \) can be locally and isometrically embedded in a Ricci flat Riemannian space \( V_{n+1}(\tilde{s}, \tilde{t}) \), with \( \tilde{s} = s + 1, \tilde{t} = t \) or \( \tilde{s} = s, \tilde{t} = t + 1 \). This theorem has obvious applications in STM-theory⁹. For further generalizations of the Campbell-Magaard theorem to embedding spaces which are Einstein, scalar field sourced or have nondegenerate Ricci tensor see Ref. 10,11,12.

In applications of the embedding theorems one often starts from a given metric and looks for the embedding space with the minimal dimension or one puts restrictions on the source type¹³,¹⁴,¹⁵. In the following we will take a different approach by putting a restriction on the type of embedding. Using some recently defined intrinsic curvature invariants on a manifold we prove an inequality between intrinsic and extrinsic curvatures of an embedded Lorentzian manifold in a pseudo-Euclidean space. For a proof in the Riemannian case see Ref. 16. An embedding for which the equality holds is called ideal and in this case the shape operators take on specified forms. The space-times which satisfy such an ideal embedding in a five-dimensional space are determined. In the remainder all embeddings are local and isometric.

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II. \textit{Λ-CURVATURES OF CHEN}

Starting from a Lorentzian manifold \((M,g)\) with signature \((m-1(+),1(-))\) isometrically embedded in a pseudo-Euclidean space \((E_n,\eta)\) of signature \((n-1,1)\) or \((m-1,n-m+1)\) we will introduce the intrinsically defined \(Λ\)-curvature invariants of Chen\textsuperscript{17,18}.

We denote the Levi-Civita connection on \(M\) with \(∇\) and on \(E_n\) with \(\nabla\). The covariant derivative in \(E_n\) between two tangent vectors \(X\) and \(Y\) on \(M\) can be decomposed in a tangential and normal part,

\[ \nabla_X Y = ∇_X Y + Ω(X,Y) \, ,\]

with \(Ω : TM \times TM \to N(M)\) the second fundamental form. If we choose an orthonormal basis \(\{ξ_α\}\) in the normal space \(N(M)\) of \(M\) and denote the signature of the basis vectors with \(ε_α = η(ξ_α,ξ_A) = ±1\), we can define \(Ω\) as

\[ Ω(X,Y) = \sum_{A=m+1}^{n} ε_α η(\nabla_X Y,ξ_A) \xi_A \, . \]  

(1)

In the following Greek indices run from 1 to \(m\), Latin indices from 1 to \(n\) and capital indices from \(m+1\) to \(n\), unless otherwise stated.

The integrability conditions for the existence of an embedding are given by the Gauss-Codazzi-Ricci equations\textsuperscript{18,19},

\[ R_{αβγμ} = \sum_A ε_α η(Ω^A α γ Η^A μ, Ω^A α μ Η^A γ) \] 

(2)

\[ ∇_γ Ω^A α β μ - ∇_β Ω^A α γ μ = \sum_B ε_B \{ S^{BA} A γ B - S^{BA} B B γ Ω^B α μ \} \] 

(3)

\[ ∇_β S^{BA} α - ∇_α S^{BA} B = \sum_C ε_C \{ S^{CB} C A B - S^{CB} B C A Ω^C α μ \} + g^{μσ} \{ Ω^B γ μ Ω^A α τ - Ω^B γ τ Ω^A μ α \} \] 

(4)

with \(S^{AB}\) the torsion vector. For an interpretation of this vector as a gauge field in a Kaluza-Klein view of embeddings see Ref.\textsuperscript{20} and as a real connection on space-time see Ref.\textsuperscript{21}.

The mean curvature vector is defined as

\[ H = \sum_A ε_α g^α β Ω^A β ξ_A \, . \]

Let \(\{e_α\}\) be an orthonormal basis of \(M\). The sectional curvature of a two-plane spanned by the orthonormal vectors \(\{e_α, e_β\}\) is defined by

\[ K(e_α \wedge e_β) = ε_α β g(R(e_α, e_β)e_β, e_α) \, , \]

with \(ε_α β = ε_α ε_β\). The scalar curvature of an \(r\)-plane section \(L\) spanned by the orthonormal vectors \(\{e_1,\ldots,e_r\}\) is defined as

\[ τ(L) = \sum_{α<β} K(e_α \wedge e_β) \, , \, 1 \leq α < β \leq r \, . \]

The scalar curvature of the whole Lorentzian manifold is denoted by \(R\). Denote the constant \(c(n_1,\ldots,n_k)\) by

\[ c(n_1,\ldots,n_k) = \frac{2(m+k - \sum_{j=1}^{k} n_j)}{m+k-1 - \sum_{j=1}^{k} n_j} \, . \]

We are now in a situation to define the \(Λ\)-curvature invariants of Chen in the pseudo-Riemannian case as

\[ Λ(n_1,\ldots,n_k) = \frac{c(n_1,\ldots,n_k) [R - \inf \{ τ(L_1) + \ldots + τ(L_k) \mid L_j \text{ a non-null } n_j \text{-plane section, } L_i \perp L_j \}]}{2} \, , \]

and
\[ \hat{\Lambda}(n_1, \ldots, n_k) = c(n_1, \ldots, n_k) \left[ R - \sup \{ \tau(L_1) + \ldots + \tau(L_k) \mid L_j \text{ a non-null } n_j - \text{plane section}, L_i \perp L_j \} \right] . \]

Note that in our definition the plane sections can be timelike or spacelike. Let \( \{ e_1, \ldots, e_m, \xi_{m+1}, \ldots, \xi_n \} \) be an orthonormal basis of \( E_n \). Because we have space-time applications in mind we take \( M \) to be time-orientable, i.e. there exists a global nowhere-zero timelike vector field which we denote with \( e_m \). From (I) we have

\[ \Omega^A_{m\alpha} = -\eta(e_m, \tilde{\nabla}_e \xi_A) = -\eta(e_m, \tilde{\nabla}_m \xi_A) , \]

with \( A = m + 1, \ldots, n \) and \( \alpha = 1, \ldots, m - 1 \).

**Definition II.1** An embedding \( x : (M, g) \rightarrow (E_{n-1,1}, \eta) \) is called causal-type preserving if \( \tilde{\nabla}_e \xi_A \) is spacelike, \( \forall A = m + 1, \ldots, n \) and \( \forall \alpha = 1, \ldots, m - 1 \).

**Definition II.2** An embedding \( x : (M, g) \rightarrow (E_{m-1,n-m+1}, \eta) \) is called causal-type preserving if \( \tilde{\nabla}_m \xi_A \) is timelike, \( \forall A = m + 1, \ldots, n \).

From the above we see that causal-type preserving embeddings have \( \Omega^A_{m\alpha} = 0, \forall \alpha = 1, \ldots, m - 1 \).

### III. IDEAL EMBEDDINGS

We can now formulate and proof an inequality relating the above intrinsically defined curvature invariants and the square of the extrinsic mean curvature of the embedded manifold.

**Theorem III.1** Let \( x : (M, g) \rightarrow (E_n, \eta) \) be a causal-type preserving embedding of a Lorentzian \( m \)-dimensional manifold in a \( n \)-dimensional pseudo-Euclidean manifold. For any \( k \)-tuple \( (n_1, \ldots, n_k) \) we have that

\[ \|H\|^2 \geq \Lambda(n_1, \ldots, n_k) , \]

if \( (E_n, \eta) \) has signature \( (n-1,1) \) and

\[ \|H\|^2 \leq \hat{\Lambda}(n_1, \ldots, n_k) , \]

if \( (E_n, \eta) \) has signature \( (m-1, m+1) \).

**Proof:**

Starting from the Gauss equation (2) w.r.t. an orthonormal basis \( \{ e_1, \ldots, e_m, \xi_{m+1}, \ldots, \xi_n \} \) we can express the scalar curvature of \( M \) as

\[ 2R = \sum_{\alpha,\beta=1}^{m} \varepsilon_{\alpha\beta} R_{\alpha\beta\alpha\beta} \]

\[ = \sum_{A=m+1}^{n} \varepsilon_A \left( \sum_{\alpha=1}^{m} \varepsilon_{\alpha} \Omega^A_{\alpha\alpha} \right)^2 - \sum_{A=m+1}^{n} \varepsilon_A \sum_{\alpha,\beta=1}^{m} \varepsilon_{\alpha\beta} \left( \Omega^A_{\alpha\beta} \right)^2 \]

\[ = \|H\|^2 - \Omega^2 . \]

If we put, with \( k \geq 1 \),

\[ \phi = 2R - \frac{m+k-1 - \sum_{j=1}^{k} n_j}{m+k - \sum_{j=1}^{k} n_j} \|H\|^2 , \]

\[ \gamma = m+k - \sum_{j=1}^{k} n_j , \]
it is a small calculation to show that
\[ \|H\|^2 = \gamma(\phi + \Omega^2). \]  

We choose \( \xi_{m+1} \) along \( \tilde{H} \) and put \( a_{\alpha} = \varepsilon_{\alpha} \Omega_{\alpha\alpha}^{m+1} \). Equation (8) becomes
\[ \varepsilon_{m+1} \left( \sum_{\alpha=1}^{m} a_{\alpha} \right)^2 = \gamma \left\{ \phi + \varepsilon_{m+1} \sum_{\alpha=1}^{m} a_{\alpha}^2 + \varepsilon_{m+1} \sum_{\alpha \neq \beta=1}^{m} \varepsilon_{\alpha\beta}(\Omega_{\alpha\beta}^{m+1})^2 + \sum_{A=m+2}^{n} \varepsilon_{A} \sum_{\alpha,\beta=1}^{m} \varepsilon_{\alpha\beta}(\Omega_{\alpha\beta}^{A})^2 \right\}. \]  

If we use the notation
\[ \tilde{a}_1 = a_1, \]
\[ \tilde{a}_2 = a_2 + \ldots + a_{n_1}, \]
\[ \tilde{a}_3 = a_{n_1+1} + \ldots + a_{n_1+n_2}, \]
\[ \vdots \]
\[ \tilde{a}_{k+1} = a_{n_1+\ldots+n_{k-1}+1} + \ldots + a_{n_1+\ldots+n_k}, \]
\[ \tilde{a}_{k+2} = a_{n_1+\ldots+n_k+1}, \]
\[ \vdots \]
\[ \tilde{a}_\gamma = a_{m-1}, \]
\[ \tilde{a}_{\gamma+1} = a_m, \]
we have
\[ \left( \sum_{\alpha=1}^{\gamma+1} \tilde{a}_{\alpha} \right)^2 = \left( \sum_{\alpha=1}^{m} a_{\alpha} \right)^2, \]
and
\[ \sum_{\alpha=1}^{\gamma+1} (\tilde{a}_{\alpha})^2 = \sum_{\alpha=1}^{m} (a_{\alpha})^2 + \sum_{2 \leq \alpha_1 \neq \alpha_2 \leq n_1} a_{\alpha_1} a_{\beta_1} + \sum_{\alpha_2 \neq \beta_2 \in Q_2} a_{\alpha_2} a_{\beta_2} + \ldots + \sum_{\alpha_k \neq \beta_k \in Q_k} a_{\alpha_k} a_{\beta_k}, \]
with \( Q_1 = \{1, \ldots, n_1\}, Q_2 = \{n_1+1, \ldots, n_1+n_2\}, \ldots, Q_k = \{n_1+\ldots+n_{k-1}+1, \ldots, n_1+\ldots+n_k\} \). Equation (9) becomes
\[ \varepsilon_{m+1} \left( \sum_{\alpha=1}^{\gamma+1} \tilde{a}_{\alpha} \right)^2 = \gamma \left\{ \phi + \varepsilon_{m+1} \sum_{\alpha=1}^{\gamma+1} (\tilde{a}_{\alpha})^2 + \varepsilon_{m+1} \sum_{\alpha \neq \beta=1}^{m} \varepsilon_{\alpha\beta}(\Omega_{\alpha\beta}^{m+1})^2 + \sum_{A=m+2}^{n} \varepsilon_{A} \sum_{\alpha,\beta=1}^{m} \varepsilon_{\alpha\beta}(\Omega_{\alpha\beta}^{A})^2 \right\}. \]  

We need the following algebraic lemma,
Lemma III.1 If $\bar{a}_1, ..., \bar{a}_n, c$ are $n + 1$ ($n \geq 2$) real numbers such that
\[
\left( \sum_{i=1}^{n} \bar{a}_i \right)^2 = (n - 1) \left( \sum_{i=1}^{n} (\bar{a}_i)^2 + c \right),
\]
we have that $2\bar{a}_1\bar{a}_2 \geq c$ and equality holds iff $\bar{a}_1 + \bar{a}_2 = \bar{a}_3 = ... = \bar{a}_n$.

Two separate cases appear. We first look at the case when $\bar{H}$ is spacelike, i.e. $\varepsilon_{m+1} = 1$. Using the above lemma equation (10) becomes
\[
\bar{a}_1\bar{a}_2 \geq \frac{1}{2} \phi + \frac{1}{2} \sum_{\alpha \neq \beta=1}^{m} \varepsilon_{\alpha \beta} (\Omega_{\alpha \beta}^{m+1})^2 + \frac{1}{2} \sum_{A=m+2}^{n} \varepsilon_{A} \sum_{\alpha, \beta=1}^{m} \varepsilon_{\alpha \beta} (\Omega_{\alpha \beta}^{A})^2 - \frac{1}{2} \sum_{2 \leq \alpha_1 \neq \beta_1 \leq n_1} a_{\alpha_1} a_{\beta_1} - ... - \frac{1}{2} \sum_{\alpha_k \neq \beta_k \in Q_k} a_{\alpha_k} a_{\beta_k}.
\]

Because
\[
\sum_{\alpha_j \neq \beta_j} a_{\alpha_j} a_{\beta_j} = 2 \sum_{\alpha_j < \beta_j} a_{\alpha_j} a_{\beta_j},
\]
we have
\[
\sum_{j=1}^{k} \sum_{\alpha_j < \beta_j \in Q_j} a_{\alpha_j} a_{\beta_j} \geq \frac{1}{2} \phi + \frac{m}{2} \varepsilon_{\alpha \beta} (\Omega_{\alpha \beta}^{m+1})^2 + \frac{1}{2} \sum_{A=m+2}^{n} \varepsilon_{A} \sum_{\alpha, \beta=1}^{m} \varepsilon_{\alpha \beta} (\Omega_{\alpha \beta}^{A})^2.
\]

Let $L_j$ be a $n_j$-dimensional subspace of $T_{p}M$ such that
\[
L_j = \text{span}\{\varepsilon_{n_1+...+n_{j-1}+1}, ..., \varepsilon_{n_1+...+n_j}\}.
\]

The scalar curvature of the plane section is given by
\[
\tau(L_j) = \sum_{\alpha_j < \beta_j \in Q_j} \varepsilon_{\alpha_j \beta_j} \sum_{A=m+1}^{n} \varepsilon_{A} \left[ \Omega_{\alpha_j \alpha_j}^{A} \Omega_{\alpha_j \beta_j}^{A} - (\Omega_{\alpha_j \beta_j}^{A})^2 \right].
\]

Then using the above notation we find
\[
\tau(L_1) + ... + \tau(L_k) = \sum_{j=1}^{k} \sum_{\alpha_j < \beta_j \in Q_j} a_{\alpha_j} a_{\beta_j}
\]
\[
- \sum_{j=1}^{k} \sum_{\alpha_j < \beta_j \in Q_j} \varepsilon_{\alpha_j \beta_j} (\Omega_{\alpha_j \beta_j}^{m+1})^2 + \sum_{j=1}^{k} \sum_{\alpha_j < \beta_j \in Q_j} \varepsilon_{\alpha_j \beta_j} \sum_{A=m+2}^{n} \varepsilon_{A} \left[ \Omega_{\alpha_j \alpha_j}^{A} \Omega_{\alpha_j \beta_j}^{A} - (\Omega_{\alpha_j \beta_j}^{A})^2 \right].
\]

If we use the inequality (11) and the notation
\[
Q_{k+1} = \{n_1 + ... + n_k + 1, ..., m\},
\]
\[
Q = Q_1 \cup ... \cup Q_k \cup Q_{k+1},
\]
\[
Q^2 = (Q_1 \times Q_1) \cup ... \cup (Q_k \times Q_k) \cup (Q_{k+1} \times Q_{k+1}),
\]
\[
\nabla^2 = (Q \times Q)/Q^2,
\]
we have
\[ \tau(L_1) + \ldots + \tau(L_k) \geq \frac{1}{2} \phi + \frac{1}{2} \sum_{A=m+1}^{n} \varepsilon_A \sum_{(\alpha,\beta) \in \nabla^2} \varepsilon_{\alpha\beta} (\Omega^A_{\alpha\beta})^2 + \frac{1}{2} \sum_{A=m+2}^{n} \varepsilon_A \sum_{k=1}^{k} \left( \sum_{\alpha \in Q_j} \varepsilon_{\alpha} \Omega^A_{\alpha\alpha} \right)^2. \] (12)

The signature of the embedding space \( E_n \) is chosen to be \((n - 1, 1)\) such that all \( \varepsilon_A = 1 \) and the condition of causal-type preserving ensures that the terms with possible minus signs appearing on the righthand side vanish. We have

\[ \tau(L_1) + \ldots + \tau(L_k) \geq \frac{1}{2} \phi. \]

This holds for all mutually orthogonal subspaces \( L_j \), in particular for the infimum,

\[ \|H\|^2 \geq \Lambda(n_1, \ldots, n_k). \] (13)

The case when \( \vec{H} \) is timelike is analogous and we find instead of (12),

\[ \tau(L_1) + \ldots + \tau(L_k) \leq \frac{1}{2} \phi. \]

This holds again for all mutually orthogonal subspaces, in particular for the supremum,

\[ \|H\|^2 \leq \hat{\Lambda}(n_1, \ldots, n_k). \] (14)

It remains to show the inequality when \( k = 0 \). Starting from (12) and again choosing \( \xi_{m+1} \) along \( \vec{H} \) we find

\[ 2R = \|H\|^2 - \varepsilon_{m+1} \sum_{\alpha=1}^{m} (a_\alpha)^2 - \varepsilon_{m+1} \sum_{\alpha \neq \beta}^{m} \varepsilon_{\alpha\beta} (\Omega^{m+1}_{\alpha\beta})^2 - \sum_{A=m+2}^{n} \varepsilon_A \sum_{\alpha,\beta=1}^{m} \varepsilon_{\alpha\beta} (\Omega^A_{\alpha\beta})^2, \] (15)

with \( a_\alpha = \varepsilon_{\alpha} \Omega^{m+1}_{\alpha\alpha} \). We have

\[ \sum_{\alpha=1}^{m} (a_\alpha)^2 = \left( \sum_{\alpha=1}^{m} a_\alpha \right)^2 - 2 \sum_{\alpha < \beta} a_\alpha a_\beta \]

\[ = \varepsilon_{m+1} \|H\|^2 + \sum_{\alpha < \beta} (a_\alpha - a_\beta)^2 - (m - 1) \sum_{\alpha=1}^{m} (a_\alpha)^2 \]

\[ m \sum_{\alpha=1}^{m} (a_\alpha)^2 \]

\[ \geq \varepsilon_{m+1} \|H\|^2. \]

If \( \vec{H} \) is spacelike, (15) with the above inequality becomes
\[ 2R \leq \frac{m-1}{m} \|H\|^2 - \sum_{\alpha \neq \beta=1}^{m} \varepsilon_{\alpha\beta} (\Omega_{\alpha\beta}^{m+1})^2 - \sum_{A=m+2}^{n} \varepsilon_A \sum_{\alpha, \beta=1}^{m} \varepsilon_{\alpha\beta} (\Omega_A^{\alpha\beta})^2. \]

The signature of the embedded space is chosen to be \((n-1, 1)\) and because of the condition of causal-type preserving, we find

\[\|H\|^2 \geq \frac{2m}{m-1} R = \Lambda(0). \tag{16}\]

The proof for the timelike case is similar. \hfill \Box

Notice that due to our choice of signature for the embedded space \(E_n\), \(H\) is always non-null. So we exclude the case of quasi-minimal embeddings.

If there is equality we can determine the form of the second fundamental forms.

\textbf{Corollary III.1} There is equality in (5) or (6) at a point \(p \in M\) iff there exists an orthonormal basis at \(p\) such that the second fundamental forms take the form

\[
\Omega_{m+1} = \begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{pmatrix},
\]

with \(a_1 + \ldots + a_n = a_{n1} + \ldots + a_{n1+n2} = \ldots = a_{n1+\ldots+n_{k-1}+1} + \ldots + a_{n1+\ldots+n_k} = a_m\) and

\[
\Omega_r = \begin{pmatrix}
A_{r1} \\
A_{r2} \\
\vdots \\
A_{rk} \\
0 \\
\vdots \\
0
\end{pmatrix},
\]

with \(\text{Trace}(A_{rj}) = 0, r = m+2, \ldots, n, j = 1, \ldots, k.\)

As in Ref. \[17\] we have the following

\textbf{Definition III.1} An isometric embedding \(x : (M, g) \to (E_n, \eta)\) is called an ideal embedding if and only if there exists a \(k\)-tuple \((n_1, \ldots, n_k)\) such that in a neighbourhood \(U\) of a point \(p \in M\) there is equality in (5) or (6) respectively.

If the pseudo-Euclidean embedding space has signature \((n-1, 1)\) an ideally embedded manifold \(M\) means that the squared mean curvature of \(M\) is minimal. Because \(\bar{H}\) measures the tension on \(M\) from the surrounding space an ideal embedding in \(E_{n-1,1}\) can be considered as a best way of living in a best world for the neighbourhood \(U_{\bar{H}}\). In the case of an embedding space with signature \((n-m+1, m-1)\) the situation is reversed. An ideally embedded manifold receives the maximum possible amount of tension from the surrounding space at each point of \(M\). Although this situation is not ideal we reserve the notation for both occasions.

\textbf{IV. IDEALLY EMBEDDED SPACE-TIMES}

Using the above notion of ideal embedding gives us a natural set of second fundamental forms to consider. Notice that this is the reverse situation usually adopted in the literature. There one often starts from a given metric and looks for the minimal embedding, i.e. with the least extra dimensions, or one puts some constraints on the curvature tensor through the choice of matter and/or Petrov type (Although see Ref. \[22\] for a different approach).
We will restrict our manifold $M$ to be a four-dimensional space-time embedded in a five-dimensional pseudo-Euclidean space. The torsion vector is zero in this case, so (4) is trivially satisfied and (3) simplifies significantly. We further only study those cases when there is equality for a k-tuple with only spacelike plane sections. The case with a timelike plane section in the k-tuple will be considered separately.

We denote the orthonormal basis of an ideally embedded space-time $M$ for which the second fundamental forms take their special forms as $\{ e_\alpha \} = \{ \vec{w}, \vec{v}, \vec{q}, \vec{u} \}$ with $u_\alpha u^\alpha = -1$. From the above corollary we have three possible cases:

i) equality with $k = 0$,

$$\Omega_{\alpha \beta} = \mu g_{\alpha \beta},$$

ii) equality with $k = 1, n = 2$,

$$\Omega_{\alpha \beta} = (\mu - \lambda) w_\alpha w_\beta + \lambda v_\alpha v_\beta + \mu q_\alpha q_\beta - \mu u_\alpha u_\beta,$$

iii) equality with $k = 1, n = 3$,

$$\Omega_{\alpha \beta} = (\mu - \lambda - \nu) w_\alpha w_\beta + \lambda v_\alpha v_\beta + \nu q_\alpha q_\beta - \mu u_\alpha u_\beta.$$

Before we determine the metrics which can be ideally embedded with one of the above second fundamental forms we mention two results which limit the possible outcomes.

**Theorem IV.1** (23) No nonflat vacuum metric can be embedded in a 5-dimensional pseudo-Euclidean space.

**Theorem IV.2** (13) There are no embedding class one solutions of the Einstein-Maxwell equations with a non-null electromagnetic field.

A. Case i:

If we take as shape operator

$$\Omega_{\alpha \beta} = \mu g_{\alpha \beta},$$

i.e. the embedding is umbilical, the Codazzi equations (9) become

$$g_{\alpha \beta} \nabla_\gamma \mu = g_{\alpha \gamma} \nabla_\beta \mu,$$

or contracting over $\alpha$ and $\beta$ gives $\nabla_\gamma \mu = 0$. The Gauss equations (2) give

$$R_{\alpha \beta \gamma \delta} = 2\varepsilon \mu^2 g_{\alpha \gamma} g_{\beta \delta},$$

with $\mu$ a constant. The space-time is a space of constant curvature, a de Sitter space if $\varepsilon = 1$ or an anti de Sitter space if $\varepsilon = -1$ (Ref. 15 p103). Due to our assumption of time-orientability the space obtained from the de Sitter space in which points are identified by reflection through the origin of the embedding space is excluded (Ref. 24 p130).

B. Case ii:

With respect to the orthonormal basis $\{ w^\alpha, v^\alpha, q^\alpha, u^\alpha \}$, $w^\alpha$ timelike, the second fundamental form becomes,

$$\Omega_{\alpha \beta} = -\mu u_\alpha u_\beta + \mu q_\alpha q_\beta + \lambda v_\alpha v_\beta + (\mu - \lambda) w_\alpha w_\beta.$$

If we decompose the covariant derivatives,

$$\nabla_\beta u_\alpha = w_\alpha A_\beta + v_\alpha B_\beta + q_\alpha C_\beta,$$

$$\nabla_\beta w_\alpha = u_\alpha A_\beta + v_\alpha D_\beta + q_\alpha E_\beta,$$

$$\nabla_\beta v_\alpha = u_\alpha B_\beta - w_\alpha D_\beta + q_\alpha F_\beta,$$

$$\nabla_\beta q_\alpha = u_\alpha C_\beta - w_\alpha E_\beta - v_\alpha F_\beta.$$
the Codazzi equations give

\[
\begin{align*}
\lambda A_\alpha &= \nabla_w u_\alpha + \lambda A_v v_\alpha - \nabla_v \lambda w_\alpha, \\
(\lambda - \mu) B_\alpha &= -\nabla_v u_\alpha - \nabla_u \lambda v_\alpha - \lambda A_v w_\alpha, \\
(2\lambda - \mu) D_\alpha &= \lambda A_v u_\alpha + (2\lambda - \mu) D_q q_\alpha - \nabla_w \lambda v_\alpha - \nabla_v (\mu - \lambda) w_\alpha, \\
\lambda E_\alpha &= -\nabla_w \mu q_\alpha + (2\lambda - \mu) D_q v_\alpha + \nabla_q \lambda w_\alpha, \\
(\lambda - \mu) F_\alpha &= \nabla_v \mu q_\alpha + \nabla_q \lambda v_\alpha - (2\lambda - \mu) D_q w_\alpha,
\end{align*}
\]

and

\[
\nabla_u \mu = \nabla_q \mu = 0,
\]

with \( A_v, D_q \) scalars and \( u^\alpha \nabla_\alpha = \nabla_u \), etc. There is no equation for \( C_\alpha \).

The Ricci identities \( 2\nabla_{[\gamma} \nabla_{\beta]} z_\alpha = z^\sigma R_{\sigma \alpha \beta \gamma} \), with \( z^\alpha \) one of the basis vectors, give

\[
\begin{align*}
\nabla_{[\alpha A_{\beta]} - D_{[\alpha B_{\beta]} - E_{[\alpha C_{\beta]} &= \varepsilon \mu (\mu - \lambda) u_{[\beta w_\alpha]}, \\
\nabla_{[\alpha A_{\beta]} + D_{[\alpha B_{\beta]} - F_{[\alpha C_{\beta]} &= \varepsilon \mu \lambda u_{[\beta v_\alpha]}, \\
\nabla_{[\alpha C_{\beta]} + E_{[\alpha A_{\beta]} + F_{[\alpha B_{\beta]} &= \varepsilon \mu^2 u_{[\beta q_\alpha]}, \\
\nabla_{[\alpha D_{\beta]} + B_{[\alpha A_{\beta]} - F_{[\alpha E_{\beta]} &= \varepsilon \mu (\mu - \lambda) w_{[\beta v_\alpha]}, \\
\nabla_{[\alpha E_{\beta]} + C_{[\alpha A_{\beta]} + F_{[\alpha D_{\beta]} &= \varepsilon \mu (\mu - \lambda) w_{[\beta q_\alpha]}, \\
\nabla_{[\alpha F_{\beta]} + C_{[\alpha B_{\beta]} - E_{[\alpha D_{\beta]} &= \varepsilon \mu \lambda v_{[\beta q_\alpha]}.
\end{align*}
\]

1. If \( \lambda = \mu \neq 0 \):

From the Codazzi equations we find \( \nabla_v \mu = A_v = D_q = 0 \) and

\[
\nabla_{[\beta w_\alpha} = \nabla_u \ln \lambda (u_\alpha u_\beta - v_\alpha v_\beta - q_\alpha q_\beta)
\]

Let us denote the projection operator on the timelike hypersurface orthogonal to \( w^\alpha \) by \( h_\alpha^\beta = \delta_\alpha^\beta - w_\alpha w^\beta \). From the Gauss equations we find that

\[
w^\alpha R_{\alpha \beta \gamma \delta} = 0,
\]

and so \( w^\alpha \) is a constant vector field (see Ref. 13 p553), i.e. \( \nabla_{[\beta w_\alpha} = 0 \) or \( \lambda = \mu = \text{constant} \). If we denote with \( 3R_{\alpha \beta \gamma \delta} \) the Riemann tensor of the timelike hypersurface, the Gauss equations give

\[
3R_{\alpha \beta \gamma \delta} = R_{\alpha \beta \gamma \delta} = 2\varepsilon \lambda^2 h_{\alpha [\gamma} h_{\delta] \beta}.
\]

The timelike 3-space is a space of constant curvature. We can then choose coordinates such that the metric reads

\[
ds^2 = dz^2 + \frac{dy^2 + dx^2 - dt^2}{[1 + \frac{1}{2} \varepsilon \lambda^2 (y^2 + x^2 - t^2)]^2},
\]

with \( \lambda = \text{constant} \). Because the embedding is quasi-umbilical (i.e. there exist functions \( \phi \) and \( \psi \) such that \( \Omega_{\alpha \beta} = \phi g_{\alpha \beta} + \psi w_\alpha w_\beta \)) the metric is conformally flat\(^{24}\). The Ricci tensor is

\[
R_{\alpha \beta} = 2\varepsilon \lambda^2 h_{\alpha \beta},
\]

with Segré type A1, [1(11.1)], and the energy-momentum tensor does not satisfies any of the known energy conditions\(^{24}\). Due to the observation that the Universe is accelerating, cosmological models with such a strange equation of state are recently under investigation.
2. If $\lambda = 0$, $\mu \neq 0$:

From the Codazzi equations we find $\nabla w \mu = D_q = 0$ and

$$\nabla_\beta v_\alpha = \nabla_\nu \ln \mu \left(u_\alpha u_\beta - w_\alpha w_\beta - q_\alpha q_\beta\right).$$

This is the previous case with the roles of $v^\alpha$ and $w^\alpha$ interchanged.

3. If $\mu = 2\lambda \neq 0$:

The Codazzi equations give $A_v = \nabla v \lambda = \nabla w \lambda = 0$. Then $A_\alpha = B_\alpha = D_\alpha = E_\alpha = F_\alpha = 0$. The Ricci identity (18) gives $\lambda = 0$, so we must take $\mu \neq 2\lambda$.

4. If $\lambda \neq 0$, $\mu - \lambda \neq 0$ and $\mu - 2\lambda \neq 0$:

Let $p^\beta_\alpha = \delta^\beta_\alpha - v_\alpha v^\beta - w_\alpha w^\beta$ be the projection operator on the 2-space $V_2$ orthogonal to $v^\alpha$ and $w^\alpha$. The second fundamental forms of the embedding of $V_2$ in the space-time $(M, g)$ are

$$\Omega^v_{\alpha\beta} = p^\gamma_\sigma \nabla^\gamma v^\sigma = \frac{\nabla^\nu \mu}{\lambda - \mu} p_{\alpha\beta},$$

and

$$\Omega^w_{\alpha\beta} = - \frac{\nabla w \mu}{\lambda} p_{\alpha\beta}.$$  

Using the Gauss equations we find for the Riemann tensor of the timelike 2-space $V_2$,

$$2R_{\alpha\beta\gamma\delta} = 2 \left\{ \varepsilon \mu^2 + \left( \frac{\nabla w \mu}{\lambda - \mu} \right)^2 + \left( \frac{\nabla w \mu}{\lambda} \right)^2 \right\} p_{[\alpha\beta]p_{\gamma\delta]}.$$  

(25)

It is a small calculation to show that the coefficient has zero-derivative in the $u$ and $q$ directions. The 2-space $V_2$ is a space of constant curvature. We can choose coordinates such that

$$w_\alpha = (e^{\phi(y,z)}, 0, 0, 0), \quad v_\alpha = (0, e^{\xi(y,z)}, 0, 0),$$

and the metric reads

$$ds^2 = e^{2\phi(y,z)} dz^2 + e^{2\xi(y,z)} dy^2 + Y^2(y,z) \{ dx^2 - \Sigma^2(x,k) dt^2 \}.$$  

(26)

with $\Sigma(x,k) = \sin(x)$, $x$ or $\sinh(x)$ if $k = 1$, 0 or $-1$ and

$$kY^2 = \varepsilon \mu^2 + \left( \frac{\nabla w \mu}{\lambda - \mu} \right)^2 + \left( \frac{\nabla w \mu}{\lambda} \right)^2.$$  

These metrics have a group $G_3$ working on the two-surface of constant curvature and therefore have Petrov type D or O. Because the two-surface is timelike the energy-momentum content cannot be a perfect fluid, a null electromagnetic field or pure radiation (see Ref. 15 ch.15) and due to theorems 11.2 and 11.2 also vacuum and an electromagnetic non-null field are not possible. We can however interpret this space-time as filled with an anisotropic perfect fluid satisfying the strong energy condition if and only if the extra dimension is timelike ($\varepsilon = -1$) and $\mu$ and $\lambda$ satisfy any of the following conditions:

1) $\lambda > 0$, $\mu > \lambda$,

2) $\lambda > 0$, $-\lambda \leq \mu \leq \frac{1}{2} \lambda$,

3) $\lambda < 0$, $\frac{1}{2} \lambda \leq \mu \leq -\lambda$,

4) $\lambda < 0$, $\mu < \lambda$. 
C. Case iii:

With respect to an orthonormal tetrad \( \{ w^\alpha, v^\alpha, q^\alpha, u^\alpha \} \) the shape operator takes the form

\[
\Omega_{\alpha \beta} = -\mu u_\alpha u_\beta + \nu q_\alpha q_\beta + \lambda v_\alpha v_\beta + (\mu - \lambda - \nu) w_\alpha w_\beta .
\] (27)

If we use the same decompositions of the covariant derivatives as in the previous case, the Codazzi equations give

\[
\begin{align*}
(\lambda + \nu) A_\alpha &= \nabla w_\mu u_\alpha + \nabla u_\mu (\mu - \lambda - \nu) w_\alpha + (\lambda + \nu) A_\nu v_\alpha + (\lambda + \nu) A_\nu q_\alpha , \\
(\mu - \lambda) B_\alpha &= -\nabla v_\mu u_\alpha + (\lambda + \nu) A_\nu w_\alpha + \nabla u_\lambda v_\alpha + (\mu - \lambda) B_\nu q_\alpha , \\
(\mu - \nu) C_\alpha &= -\nabla q_\mu u_\alpha + (\lambda + \nu) A_\nu w_\alpha + (\mu - \lambda) B_\nu v_\alpha + \nabla w_\nu q_\alpha , \\
(\mu - 2\lambda - \nu) D_\alpha &= -(\lambda + \nu) A_\nu u_\alpha + \nabla v_\nu (\mu - \lambda - \nu) w_\alpha + \nabla w_\nu v_\alpha + (\mu - 2\lambda - \nu) D_\nu q_\alpha , \\
(\mu - \lambda - 2\nu) E_\alpha &= -(\lambda + \nu) A_\nu u_\alpha + \nabla q_\nu (\mu - \lambda - \nu) w_\alpha + (\mu - 2\lambda - \nu) D_\nu q_\alpha + \nabla w_\nu q_\alpha , \\
(\lambda - \nu) F_\alpha &= -(\lambda - \mu) B_\nu u_\alpha + (\mu - 2\lambda - \nu) D_\nu w_\alpha + \nabla q_\nu v_\alpha + \nabla w_\nu q_\alpha ,
\end{align*}
\]

with \( A_\nu, A_q, B_q, D_q \) scalars. The Ricci identities are

\[
\begin{align*}
\nabla_{[\alpha} A_{\beta]} - D_{[\alpha} B_{\beta]} - E_{[\alpha} C_{\beta]} &= \varepsilon \mu (\mu - \lambda - \nu) u_{[\beta} w_{\alpha]} , \\
\nabla_{[\alpha} B_{\beta]} + D_{[\alpha} A_{\beta]} - F_{[\alpha} C_{\beta]} &= \varepsilon \mu \lambda u_{[\beta} v_{\alpha]} , \\
\nabla_{[\alpha} C_{\beta]} + E_{[\alpha} A_{\beta]} + F_{[\alpha} B_{\beta]} &= \varepsilon \mu \nu u_{[\beta} q_{\alpha]} , \\
\nabla_{[\alpha} D_{\beta]} + B_{[\alpha} A_{\beta]} - F_{[\alpha} E_{\beta]} &= \varepsilon \mu \lambda - \nu u_{[\beta} q_{\alpha]} , \\
\nabla_{[\alpha} E_{\beta]} + C_{[\alpha} A_{\beta]} + F_{[\alpha} D_{\beta]} &= \varepsilon \nu \mu - \nu w_{[\beta} q_{\alpha]} , \\
\nabla_{[\alpha} F_{\beta]} + C_{[\alpha} B_{\beta]} - E_{[\alpha} D_{\beta]} &= \varepsilon \nu \lambda - \nu w_{[\beta} q_{\alpha]} .
\end{align*}
\] (28) (29) (30) (31) (32) (33)

Using the Gauss equations we find the Ricci tensor,

\[
R_{\alpha \beta} = \varepsilon \left\{ -\mu^2 u_\alpha u_\beta + \nu (2\mu - \nu) q_\alpha q_\beta + \lambda (2\mu - \lambda) v_\alpha v_\beta + (\mu - \lambda - \nu) (\mu + \lambda + \nu) w_\alpha w_\beta \right\} .
\] (34)

In the generic case the Segré type is A1, [111,1]. We will restrict the calculations in the following to perfect fluid space-times. This means \( \mu, \lambda \) and \( \nu \) must satisfy one of the following conditions

\[
A) \ \mu = 3\lambda , \ \nu = \lambda , \\
B) \ \mu = -\lambda , \ \nu = \lambda , \\
C) \ \mu = -\nu , \ \lambda = -3\nu , \\
D) \ \mu = -\lambda , \ \nu = -3\lambda .
\]

The cases \( B, C \) and \( D \) are the same with the roles of the spacelike vectors interchanged. Before we study the above cases in detail we give first the decomposition of the covariant derivative of the timelike direction \( u^\alpha \) into its irreducible parts if \( \lambda + \nu \neq 0 \), \( \mu - \lambda \neq 0 \) and \( \mu - \nu \neq 0 \).

The acceleration reads,

\[
\dot{u}_\alpha = \frac{\nabla q_\mu u_\alpha}{\mu - \nu q_\alpha} + \frac{\nabla v_\mu w_\alpha}{\mu - \lambda} - \frac{\nabla w_\mu}{\lambda + \nu} w_\alpha ,
\] (35)

the expansion,

\[
\theta = \frac{\nabla u (\mu - \lambda - \nu)}{\lambda + \nu} + \frac{\nabla u \lambda}{\mu - \lambda} + \frac{\nabla u \nu}{\mu - \nu},
\] (36)

the shear,
\[ \sigma_{\alpha\beta} = \left\{ \frac{2\nabla_u(\mu - \lambda - \nu)}{3(\lambda + \nu)} - \frac{\nabla_u\lambda}{3(\mu - \lambda)} - \frac{\nabla_u\nu}{3(\mu - \nu)} \right\} w\alpha w\beta + \frac{\mu + \nu}{\mu - \lambda} A_\nu w(\alpha v\beta) \\
\quad + \frac{\mu + \nu}{\mu - \nu} A_\nu w(\alpha q\beta) + \left\{ - \frac{\nabla_u(\mu - \lambda - \nu)}{3(\lambda + \nu)} + \frac{2\nabla_u\lambda}{3(\mu - \lambda)} - \frac{\nabla_u\nu}{3(\mu - \nu)} \right\} v\alpha v\beta \\
\quad + \frac{2(\mu - \lambda - \nu)}{\mu - \nu} B_q v(\alpha q\beta) + \left\{ - \frac{\nabla_u(\mu - \lambda - \nu)}{3(\lambda + \nu)} - \frac{\nabla_u\lambda}{3(\mu - \lambda)} + \frac{2\nabla_u\nu}{3(\mu - \nu)} \right\} q\alpha q\beta , \]

and the vorticity,
\[ \omega_{\alpha\beta} = \frac{\mu - 2\lambda - \nu}{\mu - \lambda} w[\alpha v\beta] + \frac{\mu - \lambda - 2\nu}{\mu - \nu} w[\alpha q\beta] + \frac{\lambda - \nu}{\mu - \nu} B_q v[\alpha q\beta] . \tag{37} \]

1. If \( \mu = -\lambda \) and \( \nu = \lambda \neq 0 \):

From the Codazzi equations we find \( B_q = D_q = 0 \) and \( \nabla_u \lambda = \nabla_q \lambda = 0 \). Projecting the Ricci identity \( 29 \) on \( u^\alpha v^\beta \) and \( 30 \) on \( u^\alpha q^\beta \) gives \( A_v^2 = A_q^2 \). If we further project \( 29 \) on \( u^\alpha q^\beta \) we find \( A_v A_q = 0 \), so
\[ A_v = A_q = 0 . \]
Combining \( 28 \), \( 29 \) and \( 31 \) gives
\[ (\nabla_u \ln \lambda)^2 = 4(\nabla_u \ln \lambda)^2 , \tag{38} \]
\[ \nabla_u \nabla_u \ln \lambda = \frac{3}{2} \nabla_u \ln \lambda \nabla_u \ln \lambda , \tag{39} \]
and \( 29 \) then becomes
\[ 2\nabla_u \nabla_u \ln \lambda - 3(\nabla_u \ln \lambda)^2 - 4\epsilon \lambda^2 = 0 . \tag{40} \]
If we differentiate \( 38 \) in the direction of \( u^\alpha \) and use \( 39 \) and \( 40 \) we find \( \lambda = 0 \). This case does not lead to ideally embedded perfect fluid space-times.

2. If \( \mu = 3\lambda \) and \( \nu = \lambda \):

From the Codazzi equations we find \( A_v = A_q = B_q = 0 \) and \( \nabla_u \lambda = \nabla_v \lambda = \nabla_q \lambda = 0 \). We find that \( u^\alpha \) is geodesic, hypersurface orthogonal and shearfree. The expansion of the timelike congruence with tangent \( u^\alpha \) is given by \( \theta = \frac{1}{2} \nabla_u \ln \lambda \). From the Ricci identities we have the equation
\[ 2\nabla_u \nabla_u \ln \lambda + (\nabla_u \ln \lambda)^2 - 12\epsilon \lambda^2 = 0 . \tag{41} \]
It follows that if \( \theta = 0 \), \( \lambda = 0 \) and space-time is flat. Therefore we take \( \theta \neq 0 \). We then choose coordinates adapted to the timelike vector, \( u_\alpha = (0, 0, 0, u_4) \). The metric becomes
\[ ds^2 = h_{ij} dx^i dx^j - (u_4)^2 dt^2 , \]
with \( i, j = 1, 2, 3 \). Then \( \theta = \theta(t) \), \( \lambda = \lambda(t) \) and \( u_4 = u_4(t) \). The second fundamental form of the embedding of the spacelike hypersurface orthogonal to \( u^\alpha \) in \((M, g)\) is
\[ \Omega^u_{ij} = \frac{1}{3} \theta h_{ij} . \]
The Riemann tensor of the 3-space reads

\[ R_{ijkl} = 2 \left\{ \varepsilon \lambda^2 - \frac{1}{9} \theta^2 \right\} h_{i[k} h_{j]} , \]

the spacelike hypersurface is a space of constant curvature. The metric can be written, after a coordinate transformation \( u_4(t) dt \to dt \), as

\[ ds^2 = a^2(t) \left\{ dr^2 + \Sigma^2(r, k) (d\phi^2 + \sin^2(\phi) d\psi^2) \right\} - dt^2 , \]

with

\[ ka^{-2} = \varepsilon \lambda^2 - \frac{1}{9} \theta^2 , \]

and \( \Sigma(r, k) = \sin(r), r \) or \( \sinh(r) \) if \( k = 1, 0 \) or \(-1\). This metric is a Robertson-Walker metric (see Ref. 26 for the first results on the embedding of R-W models in flat 5-dimensional spaces). Combining (41) and (43) \( a(t) \) must be a solution of

\[ (\partial_t a)^2 = \varepsilon c^2 a^6 - k , \]

with \( c = \text{constant} \) and \( \lambda = c a^2 \). From the expression of the Ricci tensor (34) and the Einstein equations we can write the energy \( \rho \) and pressure \( p \) of the perfect fluid as

\[ \kappa \rho = 3 c^2 a^4 \text{ and } \kappa p = -7 c^2 a^4 , \]

with \( \varepsilon = +1 \).

V. CONCLUSION

In the study of embeddings of a space-time in some higher-dimensional space attention has focused primarily on intrinsic properties of the submanifold (e.g. the source type or Petrov type). But the fact that we embed our space-time metric in a greater space gives us the opportunity to consider also extrinsic properties of our model. From this viewpoint an ideal embedding seems to be the most natural and simple type of embedding to study. Ideally embedded space-times receive the least amount of tension from the surrounding space. We found that ideally embedded hypersurfaces in a pseudo-Euclidean space contain the de Sitter spaces and a Robertson-Walker model. Embeddings of the de Sitter and Robertson-Walker models were already considered by Ponce de Leon.\textsuperscript{27} It was later realized that his 5-dimensional embedding space was flat\textsuperscript{2,28} and this was used in e.g. Ref. 29 to study the structure of the Big Bang.

Furthermore a class of anisotropic perfect fluid models containing a timelike two-surface of constant curvature has also been shown to be ideally embedded. Because the non-flat vacuum models were excluded from our study due to Theorem IV.1 we will study them in a following paper.

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