Interconnections between type II superstrings,
M theory and $\mathcal{N} = 4$ supersymmetric Yang–Mills

Michael B. Green

Department of Applied Mathematics and Theoretical Physics,
Silver Street, Cambridge CB3 9EW, UK

These lecture notes begin with a review of the first nonleading contributions to the derivative expansion of the M theory effective action compactified on a two-torus. The form of these higher-derivative interactions is shown to follow from ten-dimensional type IIB supersymmetry as well as from one-loop quantum corrections to classical eleven-dimensional supergravity. The detailed information concerning D-instanton effects encoded in these terms is related to the problem of evaluating the Witten index for $N$ D-particles in the type IIA theory. Using the AdS/CFT conjecture, it also leads to very specific predictions of multi-instanton contributions in $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang–Mills theory in the limit of strong ’t Hooft coupling.

1Extended version of lectures given at 22nd Johns Hopkins Workshop (Gothenberg, August 20-22 1998); ‘Quantum Aspects of Gauge Theories, Supersymmetry and Unification’, TMR meeting (Corfu, September 20-26 1998); Andrjewski lectures (Berlin, November 1-6 1998).
1. Introduction.

2. M theory on $T^2$ and type II superstrings on $S^1$.

3. Linearized supersymmetry and higher derivative terms.

4. Nonlinear supersymmetry constraints on terms of order $\alpha'^{-1}$.

5. The eleven-dimensional perspective.
   
   5.1 The $R^4$ term from one loop in eleven dimensions.
   
   5.2 One loop with sixteen dilatini.

6. Fourier series and properties of D-instantons.

7. D-particle bound states and the Witten index.
   
   7.1 D-particles and D-instantons.
   
   7.2 The deficit term.

8. The AdS/CFT correspondence and higher derivative terms.
   
   8.1 $\mathcal{N} = 4$ Yang–Mills fields and supercurrents.
   
   8.2 Generalities concerning higher derivative terms.
   
   8.3 Scalar $AdS$ Green functions and the instanton profile.

9. D-instantons and Yang–Mills instantons.
   
   9.1 The classical D-instanton in flat space and in $AdS_5 \times S^5$.
   
   9.2 Instantons in the limit of large $N$.

A. Derivative expansion of the string tree amplitude.

B. Modular covariant derivatives.

C. Some properties of Type IIB Supergravity
   
   C.1 Spinors and gamma matrices.
   
   C.2 The fields and their supersymmetry transformations.

D. Fermions in type IIB on $S^1$ and M theory on $T^2$. 
1 Introduction

“The incalculable in full pursuit of the ineffable”

Over the past few years a number of very interesting nonperturbative aspects of string theory have emerged and are encapsulated in the term ‘M theory’. The precise meaning of this term is open to a multitude of interpretations although a common theme is that M theory represents a framework for describing nonperturbative quantum gravity that reduces to perturbative string theory or classical eleven-dimensional supergravity in various limits. Although there have been attempts to give a microscopic definition of M theory it is not at all obvious that the correct concepts have yet been discovered. Nevertheless, certain fascinating features are emerging that will surely be of lasting significance. Foremost among these is the interplay between quantum gravity and Yang–Mills gauge theory, which has been suggested by the structure of string perturbation theory for many years but has recently come to the fore in the realization of the ‘holographic’ principle in the context of the AdS/CFT correspondence.

The point of view followed in these notes is that it is of interest to unravel the extent to which features of the theory emerge purely as a consequence of its very large symmetries and do not depend on the microscopic model. All present formulations of string theory and its M theory extensions depend in some manner on the background. Obviously this is not a satisfactory state of affairs for a quantum theory of gravity. In practise, this means that the properties of the theory depend on the number of moduli of the particular background chosen. The number of moduli grows with the number of compact dimensions — there are no moduli in eleven dimensions while the ten-dimensional type IIA theory has a single modulus (the dilaton) and the ten-dimensional type IIB theory has two moduli (associated with the complex scalar field). In lower dimensions the number of moduli increases further and there is a corresponding increase in the size of the duality group. In order to avoid the possibility of terminological confusion the term ‘M theory’ will be restricted in these notes to apply to the low energy expansion of eleven-dimensional supergravity compactified on specified backgrounds.

Overview

In section 2 we will begin with a brief review of the classical relationships between M theory compactified on a two-torus, $T^2$, to nine dimensions and the two type II superstring theories compactified on a circle, $S^1$. These are con-
figurations with maximal supersymmetry — there are 32 unbroken components in the supercharges. The nine-dimensional duality group is $SL(2, \mathbb{Z})$ and there are three scalar moduli fields that parameterize the three-dimensional coset space $SL(2, \mathbb{Z}) \backslash SL(2, \mathbb{R})/U(1) \times R$. The scalars may be identified with the volume and the complex structure of $T^2$. The duality properties were originally motivated by comparison of the classical (Einstein–Hilbert) eleven-dimensional action with the classical low energy string type II string actions. Furthermore, the BPS states of the theories also coincide. For example, there are states in which the M2-brane (the membrane solution of eleven-dimensional supergravity) is wrapped around the two-cycles of $T^2$. Such states are interpreted in the string theories as fundamental string states with the string wound around the circle that forms the tenth dimension. This is a simple illustration of the more general network of dualities that relate all the different string perturbation theories and eleven-dimensional supergravity.

The duality that is manifested by the low energy effective actions of eleven-dimensional supergravity and type II string theories is supposed to extend to the complete quantum theories. We will consider the expansion of the quantum effective action in powers of derivatives on the fields. Thus, in string theory language the Wilsonian action will be expanded in inverse powers of the string tension $T_F$:

$$\alpha'^{-4} S = \alpha'^{-4} (S^{(0)} + \alpha' S^{(1)} + \ldots + (\alpha')^n S^{(n)} + \ldots),$$

where, by dimensional analysis, each power of $\alpha'$ implicitly comes with two derivatives on bosonic fields whereas a fermionic field counts as a half power of a derivative, as usual. Obviously an expansion such as (1.1) is of limited value since it does not capture effects that are nonperturbative in $\alpha'$. The leading term, $S^{(0)}$, contains the classical Einstein–Hilbert action and all the terms related by supersymmetry. We will use this notation even in the case of the type IIB theory where there is a well-known difficulty since there is no globally well-defined action for the self-dual five-form field strength, $F_5$. However, for the most part we will only use the notion of an action as a shorthand for packaging the equations of motion, which are well-defined. For such purposes a locally well-defined action, such as that in (1.1), is sufficient. The first nonzero terms beyond $S^{(0)}$ are those in $S^{(3)}$ which has a coefficient $\alpha'^{-1}$. A striking example is the topological term in type IIA ten-dimensional string theory,

$$\frac{1}{(2\pi)^5 \alpha'} \int d^{10}x B \wedge X_8(R),$$

where $X_8(R)$ is an eight-form made out of the curvature $R$ and $B$ is the two-form $B = \frac{1}{2} F_{\mu \nu} \gamma^\mu \gamma^\nu$. Recall that the string tension $T_F$ has dimensions (length)$^{-2}$ and $\alpha' = (2\pi T_F)^{-1}$ has dimension (length)$^2$. 
form Neveu-Schwarz–Neveu-Schwarz (NS ⊗ NS) antisymmetric tensor potential. This term, with its precisely determined overall coefficient, can be discovered directly from a one-loop string calculation in which the stringy regulator plays a crucial rôle in regulating an otherwise quadratically divergent diagram. It can also be discovered by considering the cancellation of chiral gravitational anomalies in the type IIA five-brane. These anomalies arise from the coupling of the world-volume chiral fermions and self-dual antisymmetric three-form field strength to the pull-back of the target space gravitational field. In the presence of a five-brane (1.2) transforms anomalously under local diffeomorphisms and local Lorentz transformations in just the correct manner to cancel the world-volume anomaly. This is an example of the ‘anomaly inflow mechanism’ for cancelling local anomalies [1], which was the method used in [10] to derive the generalization of the term (1.2) to eleven-dimensional supergravity where it is the integral of an eleven-form, $C^{(3)} ∧ X_8$. Both of these arguments for the presence of the term (1.2) are outside the realm of perturbative quantum field theory – the first method invokes a stringy cutoff while the second method requires consistency of nonperturbative solitonic chiral five-branes.

Obviously (1.2) is not the only higher derivative term in the effective action. There will be terms with arbitrarily high numbers of derivatives as well as terms that are nonperturbative in $\alpha'$. As we will see, the large amount of supersymmetry severely constrains the form of certain classes of ‘protected’ terms in the effective action. Whereas the classical type IIB supergravity (the supersymmetrized Einstein theory) is invariant under continuous SL(2,IR) transformations the IIB superstring only possesses a discrete (local) $SL(2,\mathbb{Z})$. This means that the quantum effective action only possesses this discrete symmetry which must be a symmetry order by order in $\alpha'$.

The first corrections to the Einstein–Hilbert term are of order $\alpha'^{-1}$ ($\alpha'^3$ relative to the leading term). One such term is a specific contraction of four Riemann curvatures that only involves the Weyl tensor components (recall that the Weyl tensor is the part of the Riemann tensor that is trace-free). In the type IIB theory this term, which will be denoted by $R^4$, must have the form (in string frame) [1],

$$\int d^{10}x \sqrt{-g} e^{-\phi/2} f^{(0,0)}(\tau, \bar{\tau}) R^4,$$  \hspace{1cm} (1.3)

where $\tau$ is a complex scalar field,

$$\tau \equiv \tau_1 + i \tau_2 = C^{(0)} + i e^{-\phi},$$  \hspace{1cm} (1.4)

and $C^{(0)}$ is the Ramond–Ramond ($\mathcal{R} ⊗ \mathcal{R}$) pseudoscalar field and $\phi$ is the type IIB dilaton field (which will sometimes be written as $\phi^B$ in order to distinguish it from the type IIA dilaton). The power of $e^{-\phi/2}$ in (1.3) cancels after
transformation to the Einstein frame,

\[ g \rightarrow g e^{\phi/2}, \quad (1.5) \]

where \( g_e \) is the Einstein-frame metric. In this frame the curvature is invariant under \( SL(2, \mathbb{Z}) \) so that the function \( f^{(0,0)}(\tau, \bar{\tau}) \) must be a scalar under \( SL(2, \mathbb{Z}) \) transformations which act as

\[ \tau \rightarrow a\tau + b \frac{c\tau + d}{c't + d}, \quad (1.6) \]

with integer \( a, b, c, d : \quad ad - bc = 1 \). The notation in (1.3) has been chosen with later generalizations in mind where we will encounter modular forms, \( f^{(w,\bar{w})}(\tau, \bar{\tau}) \), where the superscripts \((w, \bar{w})\) denote holomorphic and antiholomorphic weights. The string coupling constant is proportional to \( e^\phi \) when \( \phi \) is constant. It follows that the large \( \tau^2 = e^{-\phi} \) expansion of \( f(0,0) \) must reproduce the terms that are known in the perturbative type IIB string expansion \([11, 12]\).

It will also include exponentially suppressed terms that correspond to the presence of arbitrary numbers of D-instantons that will be discussed in subsequent sections.

At the linearized level the physical fields of IIB supergravity package together into a constrained ‘on-shell’ superfield that is a function of a \( SO(9,1) \) Grassmann spinor, \( \theta \) \([13]\). In section 3 this will be used to express the \( R^4 \) term, together with many other interaction terms of the same dimension, in terms of an integral over the sixteen components of \( \theta \), which is half the total number of components of the two type IIB supersymmetries. This makes manifest the fact that these terms of order \( \alpha'^{-1} \) are related by supersymmetry at the linearised level. We will refer to terms such as these that are integrals over a fraction of the superspace as ‘protected’ terms. Generally these interactions violate the \( U(1)_R \) symmetry of the classical type IIB theory. For example, among such terms is one of the form \([14]\)

\[ \int d^{10} x \sqrt{-g} e^{-\phi/2} f^{(12,-12)}(\tau, \bar{\tau}) \lambda^{16}, \quad (1.7) \]

where \( \lambda \) is the complex dilatino which is a sixteen-component chiral \( SO(9,1) \) spinor. The nonholomorphic modular form, \( f^{(12,-12)} \), transforms with a phase under \( SL(2, \mathbb{Z}) \) in a manner that cancels the phase transformation of the dilatino. The characteristic feature of these terms is that the linearized approximation is a good approximation for the D-instanton contributions but not for the perturbative contributions of zero instanton number. The sixteen components of \( \theta \) are identified as the components of supersymmetry that are broken in the D-instanton background \([11, 14]\).

The constraints imposed by the full nonlinear supersymmetry in the type IIB theory will be the subject of section 4 where the terms of order \( \alpha'^{-1} \) in
the derivative expansion of the effective action of type IIB supergravity will be deduced by requiring the closure of the superalgebra at this order \[15\]. This is a kind of Noether procedure that implements supersymmetry up to any given order in \(\alpha'\). In this manner terms we shall determine not only the terms in \(S^{(3)}\) but certain of the modified supersymmetry transformations that relate \(S^{(3)}\) to the classical action \(S^{(0)}\). We will see that supersymmetry requires that the functions of the moduli \(f^{(w,-w)}(\tau, \bar{\tau})\) are eigenfunctions of the Laplace operator on the Poincare upper-half plane with specific eigenvalues that depend on \(w\). Assuming that the theory is invariant under \(SL(2, \mathbb{Z})\) determines the solutions uniquely.

We will see in section 5 that the protected higher-derivative terms, such as \(f^{(0,0)} R^4\) can be deduced by consideration of one-loop Feynman diagrams in the eleven-dimensional theory compactified on \(T^2\) \[16\]. Thus, the \(R^4\) term will be deduced by considering the one-loop scattering of four gravitons. This amplitude can be expressed as a sum over windings around the torus of the world-line of the particle circulating in the loop. The dependence on the moduli of the torus emerges from the sectors of nonzero winding while the ultraviolet (UV) divergence is contained in the zero-winding sector. Since the supergravity field theory is not renormalizable these UV divergences cannot be interpreted in the absence of a microscopic definition of M theory (just as string theory regulates the ultraviolet divergences in ten-dimensional supergravity). However, with any sensible regularization this term is proportional to the volume of the torus and it disappears in the limit in which the torus has zero volume. But this is the limit in which M theory is identified with the ten-dimensional type IIB theory. We will find that the expression for the \(R^4\) term that was deduced from supersymmetry in section 4 is reproduced by this one-loop calculation. More generally, for finite volume, there are volume-dependent terms that are ill-defined due to the UV divergence. However, if the condition of T-duality, that relates the type IIB and IIA string theories in nine dimensions is imposed, then the regularized coefficients are determined uniquely and there are no ambiguities. Having fixed the value of the regularized divergence the eleven-dimensional limit may be recovered with a specific finite coefficient for the \(R^4\) term\[5\].

In similar fashion the other terms of order \(\alpha'^{-1}\) can be obtained from one-loop processes in M theory on \(T^2\). The interaction of sixteen dilatini, \(\lambda^{16}\), will also be considered in detail in section 5. This involves the evaluation of a sixteen-gravitino amplitude in which the gravitini are polarized in appropriate directions with respect to the two-torus so as to be interpreted as spin-1/2 dialtini in the

---

\[5\] For related work in toroidally compactified type II theories see [17, 18] and in type I theories see [19, 20, 21]. \(SL(2, \mathbb{Z})\)-invariant expressions for higher-dimensional terms in the type IIB effective action have also been proposed in [22, 23, 24, 25, 26].
IIB theory \[14\]. In this case there is no eleven-dimensional divergence at all and the loop amplitude is proportional to the modular form \(f^{(12,-12)}\).

The structure of these \(o(\alpha'^{-1})\) interactions will be analyzed further in section 6. The large \(\tau_2\) (small coupling) expansion of any of these terms is determined by writing the modular form, \(f^{(w,-w)}(\tau,\bar{\tau})\), as a Fourier series in powers of \(e^{2\pi i \tau_1}\), which is a series of D-instanton terms. The zero Fourier mode contains the perturbative contributions (powers of \(\tau_2^{-1}\)) which consist of the tree-level and one-loop terms with no higher-order power behaved terms. The absence of perturbative terms beyond one loop points to a perturbative non-renormalization theorem which is a consequence of the fact that the protected interactions can be expressed as integrals over half the superspace. The charge-\(K\) D-instanton contribution is correlated with a phase factor \(e^{2\pi i K \tau_1}\). The significant \(\tau_2\) dependence of the charge-\(K\) D-instanton term is contained in the Bessel function \(K^{-1}(2\pi |K|\tau_2)\) which has a large-\(\tau_2\) expansion that is suppressed by the exponential factor \(e^{-2\pi |K|\tau_2(1+o(\tau_2^{-1}))}\). The coefficient of the leading charge-\(K\) D-instanton contribution will be found to be (with suitable normalization)

\[
\sum_{m|K} \frac{1}{m^2}, \tag{1.8}
\]

where the notation indicates a sum over all the divisors of \(K\). On the other hand, an extension of ideas in \[30\] suggest that this D-instanton measure should be identified with the partition function of the \(SU(K)\) ‘zero-dimensional matrix model’, that is the compactification to zero dimensions of ten-dimensional supersymmetric Yang–Mills theory. In this manner we will be able to deduce the partition function for arbitrary \(K\).

The \(K\) D-instanton measure is related to the problem of evaluating the Witten index for \(K\) D-particles which will be the subject of section 7 \[31\]. The chain of duality relations between string theory and eleven-dimensional supergravity requires that there be precisely one threshold bound state of \(K\) D-particles of charge \(K\) \[1\]. This means that the Witten index for the \(K\) D-particle system must be equal to one. The case of two D-particles (\(K = 2\)) was considered in detail in \[32, 33\] where it was noted that the Witten index has two contributions – the ‘bulk’ term is equal to the zero-dimensional matrix model partition function while the ‘boundary’ term is obtained from the region of moduli in which the two D-particles can be described as identical free particles. We will see how these ideas generalize to arbitrary \(K\). The bulk term in the index is now given by the zero-dimensional matrix model partition function for general \(K\) \[1.8\]. Furthermore, if it is assumed that the boundary term in the

---

\(^{6}\)Some indirect arguments corroborate the validity of this ‘theorem’ \[27, 28, 29\].
index is again given by the free propagation of identical D-particles of any charge less than $K = mn$ then the total index is equal to one as expected.

In section 8 we turn to consider the effect of the higher derivative terms on the conjectured equivalence of type IIB string theory in $AdS_5 \times S^5$ (five-dimensional anti de Sitter space $\times S^5$) and four-dimensional $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang–Mills theory on the boundary of $AdS_5$ (with the $S^5$ appended). According to this correspondence (see also) the higher-derivative interactions of order $\alpha^{-1}$ in the type IIB theory should correspond to contributions of correlation functions of composite operators in the Yang–Mills theory of order $N^{-3/2}$ relative to the leading terms. This leads to a prediction of the precise form of the Yang–Mills instanton contributions to classes of correlation functions that may be obtained from the protected supergravity processes. These exact expressions exhibit the Montonen–Olive duality $SL(2, \mathbb{Z})$ symmetry that is inherited from the supergravity interaction terms discussed in the earlier sections.

Section 9 will focus on the correspondence between the D-instanton contributions to IIB supergravity in $AdS_5 \times S^5$ and the contribution of Yang–Mills instantons in the boundary supersymmetric Yang–Mills theory. The D-instanton is the stringy version of a classical solution to type IIB supergravity that is concentrated around a point in euclidean ten-dimensional space. In the Einstein frame this solution is one in which the metric is flat and $\nabla^2 e^\phi = 0$ and the BPS condition relates the euclidean continuation of $C^{(0)}$ to $e^{-\phi}$.

We will illustrate the AdS/CFT correspondence by comparing the one-instanton contribution to the sixteen-dilatino amplitude in three different ways. Firstly (method (a)), we will extract the leading one D-instanton term in $f^{(12, -12)}$ which is the coefficient of the sixteen-dilatino vertex. The amplitude with specified boundary data is obtained by contracting each leg of this vertex with a dilatino propagator from the vertex to a point on the boundary and integrating the position of the vertex in the bulk $AdS_5 \times S^5$ space. The AdS/CFT correspondence identifies the boundary value of the dilatino as the source of a particular fermionic supercurrent of the $\mathcal{N} = 4$ Yang–Mills theory. It is therefore of interest to compare the sixteen-dilatino amplitude with the correlation function of sixteen of these supercurrents. The second method (method (b)) is the direct determination of the leading one-instanton term in $SU(2)$ $\mathcal{N} = 4$ Yang–Mills theory which we will see has the same form as the supergravity amplitude determined by method (a), up to an undetermined overall coefficient. The third method (method (c)), based on the semiclassical approximation to scattering in a D-instanton background, again gives the same expression. The agreement between the functional form of the correlators in $SU(2)$ Yang–Mills and the bulk amplitudes in IIB supergravity might appear
somewhat surprising since the AdS/CFT correspondence is supposed to relate the leading $\alpha'$ piece of the $R^4$ term to the large-$N$ limit of $SU(N)$ Yang–Mills, rather than to the $N = 2$ case. However, recent work on the large-$N$ limit of the one-instanton contributions of the Yang–Mills theory \[39\] shows that all the $N$-dependence enters in the overall coefficients of these correlation functions and their functional form is independent of $N$, thus explaining the agreement for $N = 2$. Furthermore, this overall coefficient has the predicted $N$-dependence for large $N$. This impressive agreement extends to the behaviour of the charge-$K$ $D$-instanton contribution \[40, 41\] to leading order in the coupling constant. This will also be described (with few details) in section 9.

2 M theory on $T^2$ and string theory on $S^1$

The Einstein–Hilbert action of the eleven-dimensional theory has the form,

$$S_{EH} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-G^{(11)}} R,$$ (2.1)

where the overall normalization has been expressed in terms of the coupling $\kappa_{11} \equiv (2\pi l_P)^{3/2}$ and $l_P$ is the eleven-dimensional Planck length. The additional terms in the fully supersymmetric classical action of \[4\] involve the three-form potential, $C^{(3)}$, and the gravitino, $\psi_{\hat{\mu}}$ ($\hat{\mu}, \hat{\nu} = 0, 1, \ldots, 9, 11$).

It was argued in \[1\] that eleven-dimensional M theory compactified on a circle of radius $R_{11} l_P$ \[1\] should be identified with ten-dimensional type IIA string theory with a coupling constant, $g = e^{\phi_A}$ (where $\phi_A$ is the IIA dilaton), that is related to $R_{11}$. This is seen by parameterizing the eleven-dimensional metric, $G_{\hat{\mu}\hat{\nu}}$,

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu + R_{11}^2 (dx^{11} - C^{(1)}_{\mu} dx^\mu)^2,$$ (2.2)

where $C^{(1)}_{\mu}$ is the $\mathcal{R} \otimes \mathcal{R}$ one-form potential of the type IIA theory. It is easy to check that with the identifications

$$R_{11} = e^{2\phi_A/3},$$ (2.3)

and

$$G_{\mu\nu} = e^{-2\phi_A/3} g_{\mu\nu} \left(\frac{l_P}{l_S}\right)^2,$$ (2.4)

where $g_{\mu\nu}$ is the string-frame metric of type IIA string theory, the Einstein–Hilbert action (2.1) compactified on a circle becomes

$$S_{EH} = \frac{2\pi}{(2\pi l_S)^8} \int d^{10}x \sqrt{-g} e^{-2\phi_A} R.$$ (2.5)

Footnote: We will use a convention in which capital letters denote distances in eleven-dimensional Planck units while lower case letters denote distances in string units.
In these equations \( l_s \) is the string scale which is related to the fundamental string tension, \( T_F \), by

\[
l_s^2 \equiv \alpha' = \frac{1}{2\pi T_F}
\]

and the curvature in (2.5) is expressed in the string metric. The overall coefficient in (2.5) is often denoted by \((2\kappa_{10}^2)^{-1}\) where \(\kappa_{10}^2 = (2\pi l_s)^8/4\pi\). With these conventions the tension in a Dp-brane is given by

\[
T_p = \frac{1}{l_s} (2\pi l_s)^{-p} e^{-\phi}\n\]

and so the tension of the D-string \((p = 1)\) is \(e^{-\phi}T_F\) \[12\]. The identification of the masses of the Kaluza–Klein modes of compactified M theory and the masses of the type IIA D-particles \((p = 0\) in (2.7)) leads to the relation between the scales,

\[
\frac{1}{R_{11} l_p} = e^{-\phi} l_s^{-p},
\]

or

\[
l_p = e^{\phi/3} l_s^p,
\]

which expresses the eleven-dimensional Planck scale in terms of the string scale and coupling constant.

We will now generalize this description and obtain more insight by making use of the fact that in the nine-dimensional theory the \(SL(2,\mathbb{Z})\) symmetry of the IIB string theory can be interpreted as a geometric symmetry of M-theory compactified on a torus \[2, 3\]. Three scalar fields arise from the compactification of the eleven-dimensional theory (with coordinates labelled \(x^0, \ldots, x^9, x^{11}\)) on a two-torus, \(T^2\), oriented in the \((x^9, x^{11})\) directions so that \(G^{09} = R_9\). These scalars correspond to the volume of the torus, \(V\), and the complex structure, \(\Omega\).

Using the ansatz (2.2) for the eleven-dimensional metric we have

\[
\sqrt{-G^{(11)}} = \sqrt{G^{(2)}} \sqrt{-G^{(9)}} = V \sqrt{-G^{(9)}},
\]

where \(V = R_9 R_{11}\) is the volume of \(T^2\). The metric on the two-torus is

\[
G_{IJ}^{(2)} = \frac{V}{\Omega_2} \left( \begin{array}{cc} |\Omega|^2 & \Omega_1 \\ \Omega_1 & 1 \end{array} \right)
\]

\((I, J = 9, 11)\) where the complex structure is given by

\[
\Omega \equiv \Omega_1 + i\Omega_2 = \frac{G_{911}^{(2)} + i\sqrt{G^{(2)}}}{G_{1111}^{(2)}} = C^{(1)} + i\frac{R_9}{R_{11}}.
\]
In type IIA language the radius of the tenth dimension is identified as

\[ r_A = R_9 \frac{l_P}{l_S} = R_9 R_{11}^{1/2} = \sqrt{\frac{1}{2}} \Omega_2^T. \] (2.13)

Now T-duality on this circle can be used to determine the relation with the type IIB theory. This gives

\[ r_B = \frac{1}{r_A} = R_9^{-1} R_{11}^{-1/2}, \quad e^{-\phi_B} = r_A e^{-\phi_A} = \frac{R_9}{R_{11}} \] (2.14)

and \( C^{(0)} = C^{(1)}_9 = \Omega_1 \). These equivalences can be stated in the form,

\[ \Omega = \tau^A \equiv C^{(1)}_9 + ir_A e^{-\phi_A} \]
\[ = \tau^B \equiv C^{(0)} + ie^{-\phi_B}. \] (2.15)

Large diffeomorphisms of the torus are \( SL(2, \mathbb{Z}) \) transformations under which \( \Omega \) transforms by

\[ \Omega \rightarrow \frac{a\Omega + b}{c\Omega + d} \] (2.16)

with \( a, b, c, d \in \mathbb{Z} \) and \( ad - bc = 1 \).

The type IIB Einstein–Hilbert action is obtained by compactification of the type IIA theory on a circle of radius \( r_A \) followed by T-duality, leading to the type IIB expression (now in the type IIB string frame),

\[ S_{EH} = \frac{2\pi}{(2\pi l_S)^8} \int d^{10}x \sqrt{-g^B} e^{-2\phi_B} R, \] (2.17)

where \( g^B_{\mu\nu} \) denotes the IIB string metric. Supersymmetry determines the remaining terms in the complete classical action in any of these parameterizations, subject to the caveat that the IIB theory does not have a globally well-defined action because of the special features of self-dual antisymmetric tensor fields. It follows from simple dimensional analysis that all the terms that only involve bosonic fields have two derivatives. A pair of fermionic fields is dimensionally equivalent to a derivative on a bosonic field, so there are terms with two fermion fields and a single derivative as well as terms with four fermionic fields and no derivatives.

In the following we will be concerned with higher-derivative corrections to the lowest-order action that necessarily arise in the quantum theory. Certain of these terms are known from tree-level and one-loop string perturbation theory. For example, the exact expression for the four-graviton scattering amplitude in either of the type II superstring theories has a simple expansion in powers
of \( \alpha' = l_s^2 \). The expression, which is reproduced in appendix A (A.1), has an overall kinematic factor of \( \tau^2 \tilde{K} \) where

\[
\tilde{K} = t^{\mu_1 \ldots \mu_8}_8 t^{\nu_1 \ldots \nu_8}_8 \prod_{r=1}^{4} \zeta_{\mu_r \nu_r} k^{(r)}_{\mu_r} k^{(r)}_{\nu_r},
\]  

and the eighth-rank tensor \( t_8 \) arises in open superstring amplitudes and is defined in [45]. The factor, \( \tilde{K} \), is eighth-order in the external momenta and is the linearized approximation to \( R^4 \). The leading term in the expansion of the amplitude in powers of \( \alpha' \) in (A.1) summarizes the sum of the tree-level Feynman diagrams for on-shell four-graviton scattering in either of the ten-dimensional low energy type II supergravity theories. This consists of the sum of pole terms together with a four-graviton contact interaction with two derivatives that arise from the Einstein–Hilbert term. The next term, with coefficient \( \zeta(3) \) is a contact interaction with eight powers of momenta (contained in \( \tilde{K} \)) that arises from the \( R^4 \) term (which was deduced in this manner in [47] and from a four-loop sigma model term in [48]). The one-loop contribution to the four-graviton amplitude has coefficient \( \tilde{K} \) with no dilaton dependence. It does not possess massless particle poles and its leading low energy contribution is also to the \( R^4 \) term [45]. The effect of the tree-level \( R^4 \) term on Calabi–Yau compactifications has recently been reconsidered in [49, 50] and its effect on black hole solutions in \( \text{AdS}_5 \times S^5 \) has been analyzed in [51] and [52].

After compactification to nine dimensions on a circle of radius \( r_A \) in the IIA units (or \( r_B = 1/r_A \) in the IIB units) the one-loop term gets a simple \( 1/r_A^2 \) correction due to the winding of the string around the compact tenth dimension. Putting these terms together with the appropriate coefficients leads to the expressions for the nine-dimensional \( R^4 \) term in coordinates appropriate to the type IIA, IIB or M-theory parameterizations [12],

\[
S_{R^4} = \frac{1}{3 \cdot (4\pi)^7 l_S} \int d^9x \sqrt{-g^{A(9)}} R^4 R_A \left[ 2\zeta(3)(\tau_A^2)^2 + \frac{2\pi^2}{3} \left( 1 + \frac{1}{r_A^2} \right) + \cdots \right]
\]

\[
= \frac{1}{3 \cdot (4\pi)^7 l_S} \int d^9x \sqrt{-g^{B(9)}} R^4 r_B \left[ 2\zeta(3)(\tau_B^2)^2 + \frac{2\pi^2}{3} \left( 1 + \frac{1}{r_B^2} \right) + \cdots \right]
\]

\[
= \frac{1}{3 \cdot (4\pi)^7 l_P} \int d^9x \sqrt{-g^{(9)}} R^4 R_9 R_{11} \left[ 2\zeta(3) \frac{1}{R_{11}^3} + \frac{2\pi^2}{3} + \frac{2\pi^2}{3 R_9^2 R_{11} + \cdots} \right],
\]  

(2.19)

where \( g^{A(9)} \), \( g^{B(9)} \) are the nine-dimensional metrics in the IIA and IIB theories. An important feature of the first two expressions is that they are related by

---

*Here and in the following we will ignore the Gauss-Bonnet term which also arises in the one-loop amplitude and is proportional to the product of two eight-dimensional Levi–Civita symbols.*
T-duality. This is trivial for the leading terms but less so at one loop where the winding term of one of the type II theories (e.g., the $2\pi^2/3r_A^2$ term in the first expression) is interchanged with the nonwinding term of the other (e.g., the $2\pi^2/3$ term in the second expression).

The ellipsis in (2.13) indicate other terms that are yet to be determined. As we will see there are in fact no further perturbative terms (higher inverse powers of $\tau_2$) but there will be exponentially suppressed terms that are powers of $e^{2\pi i r_A^2} = e^{2\pi i r_B^2} = e^{2\pi i \Omega}$. Such contributions are associated with D-instantons.

Generally, a $Dp$-brane is a stringy soliton extended in $p \geq 0$ spatial directions while the D-instanton corresponds to the value $p = -1$ and is associated with a point-like space-time event rather like a small Yang–Mills instanton. The classical D-instanton is a solution of euclidean type IIB supergravity in which $e^{\phi_B}$ and $C^{(0)}$ have nontrivial profiles while the other fields (including the Einstein-frame metric) are trivial. The action for a D-instanton with charge $K$ is $2\pi|K|/g$, where $g$ is the asymptotic value of $\tau_2^{-1}$ so that the D-instanton contributions in (2.19) should be characterized by terms proportional to

$$e^{-2\pi(|K|/g + i KC^{(0)})},$$

where the $-$ sign refers to a D-instanton and $+$ to an anti D-instanton. This classical configuration will be reviewed in section 9 where we will also see a direct correspondence between Yang–Mills instantons and D-instantons in the context of the AdS/CFT conjecture. String scattering amplitudes in a D-instanton background are described by world-sheets with boundaries on which the target-space coordinates satisfy Dirichlet boundary conditions. Such point-like closed string boundary conditions have a significant short distance effect on string scattering amplitudes – a fact that led to the suggestion that they may be relevant in the formulation of a string theory of hadrons.

An explicit calculation of the leading effect of a single D-instanton was carried out in [11] by considering four-graviton scattering in the presence of Dirichlet world-sheet boundaries. This established the presence of the charge-one D-instanton term in the ten-dimensional type IIB effective action that arises in the $r_B \rightarrow \infty$ limit of $S^{IIB}_{R^4}$. However, in order to ensure the $SL(2,\mathbb{Z})$ duality invariance of the effective action the expressions in (2.19), there must be an infinite number of terms with D-instantons of arbitrarily high charge. In order to exhibit this modular invariance it is instructive to write the last line of (2.19) in terms of the complex structure as

$$S_{R^4} = \frac{1}{3 \cdot (4\pi)^4 l_P} \int d^9x \sqrt{-G^{(0)}} R^4 \left\{ \frac{1}{4} \left[ 2\zeta(3) \Omega_2^{3/2} + \frac{2\pi^2}{3} \Omega_2^{-1/2} + \cdots + \frac{2\pi^2}{3} \chi \right] + \frac{2\pi^2}{3} \chi \right\}.$$
We have here identified the expression in parentheses with the expansion of an as yet undetermined function, \( f^{(0,0)}(\Omega, \bar{\Omega}) \), that must transform as a scalar under \( SL(2, \mathbb{Z}) \) transformations of the complex structure, \( \Omega \). This translates into a function \( f^{(0,0)}(\tau^B, \bar{\tau}^B) \) in the IIB theory and \( f^{(0,0)}(\tau^A, \bar{\tau}^A) \) in the IIA theory.

The fact that this function must match the two known perturbative coefficients and that it must contain powers of \( e^{2\pi i \tau^B} \) motivated the conjecture \([11]\) that the complete coefficient of the \( R^4 \) term in (1.3) is the modular function

\[
f^{(0,0)}(\tau, \bar{\tau}) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^{3/2}}{|m + n\tau|^3} = 2\zeta(3) E_4(\tau),
\]

where the notation \( E_4 \) denotes a nonholomorphic Eisenstein series (as defined, for example, in \([57]\)). The overall normalization in (2.22) has been chosen so that the leading term for large \( \tau_2 \) is \( 2\zeta(3)\tau_2^{3/2} \). This coincides with the tree-level expression in parentheses in the second line of (2.19) (after allowing for the extra factor of \( \tau_2^{1/2} \) due to the transformation between the string frame and Einstein frame). This expression for \( f^{(0,0)} \) was further motivated by the study of the one-loop corrections to eleven-dimensional supergravity on \( T^2 \) \([16]\) (see section 5) and will be derived from supersymmetry in section 4.

In the meantime it is worth noting that the perturbative contributions to the decompactification limits that give the ten-dimensional type IIA theories and eleven-dimensional M-theory are easily obtained from (2.19) and (2.21). In the ten-dimensional IIA limit, \( r_A \to \infty \) in the first line in (2.19), the D-instanton terms vanish exponentially since \( \tau^A \sim i r_A e^{-\phi^A} \to i\infty \), so \( e^{2\pi i \tau^A} \to 0 \) in this limit. This leaves only the perturbative terms. If we now consider the further decompactification to eleven dimensions \( e^{\phi^A} = R_4^{3/2} \to \infty \) the tree-level IIA contribution vanishes and the one-loop term gives a finite contribution. This eleven-dimensional limit coincides with the limit \( V \to \infty \) in (2.21). In this limit the \( V^{-1/2} \) term vanishes and the finite eleven-dimensional term comes entirely from the term proportional to \( V \). It is important to note if there had been other terms of higher order in the type IIA coupling constant, \( e^{\phi^A} = R_4^{3/2} \), the eleven-dimensional limit would have been singular order by order in IIA perturbation theory and the limit would not have been uniform. However, if \( f^{(0,0)} \) is given by the expression (2.22) there are no further perturbative terms and the eleven-dimensional limit is simply

\[
S_{R^4} = \frac{1}{18 \cdot (4\pi)^7 \cdot l_P^2} \int d^{11}x \sqrt{-G^{(11)}} R^4
\]
\[ \int d^{11}x \sqrt{-G^{(11)}} R^4. \quad (2.23) \]

As argued in [12] the coefficient of this term is in precise agreement with supersymmetry which relates the \( R^4 \) term to the eleven-form \( C^{(3)} \wedge X_8 \).

### 3 Linearized supersymmetry and higher derivative terms

The existence of a large number of interactions in the IIB theory that are of the same dimension as the \( R^4 \) interaction can be motivated very simply by using linearized supersymmetry. This can be implemented by packaging the physical fields or their field strengths into a constrained superfield \( \Phi (x^\mu - i \bar{\theta} \gamma^\mu \theta, \theta) \) where \( \theta^a (a = 1, \ldots, 16) \) is a complex Grassmann coordinate that transforms as a Weyl spinor of \( SO(9,1) \). This superfield is taken to satisfy the holomorphic condition \[ D^* \Phi = 0, \quad (3.1) \]

and the constraints, \[ D^4 \Phi = D^{*4} \Phi^*, \quad (3.2) \]

where \[ D_A = \frac{\partial}{\partial \theta^A} + 2i(\gamma^\mu \theta^A) \partial_\mu, \quad D_A^* = -\frac{\partial}{\partial \theta^* A} \quad (3.3) \]

are the holomorphic and anti-holomorphic covariant derivatives that anticommute with the rigid supersymmetries \[ Q_A = \frac{\partial}{\partial \theta^A}, \quad Q^*_A = -\frac{\partial}{\partial \theta^* A} + 2i(\bar{\theta} \gamma^\mu) \partial_\mu. \quad (3.4) \]

The constraints \((3.1)\) and \((3.2)\) ensure that the field \( \Phi \) has an expansion in powers of \( \theta \) (but not \( \theta^* \)), that terminates after the \( \theta^8 \) term and contains the 256 fields in an ‘on-shell’ supermultiplet.\(^9\)

\[ \Phi = \tau_0 (1 + \Delta) \]

\[ = \tau_0 (1 + a + i \bar{\theta} \gamma^\lambda \theta + \bar{\theta} \gamma^{\mu \nu \rho} \theta + \bar{\theta} \gamma^{\mu \nu} \theta \gamma^\rho \partial_\rho \psi + \bar{\theta} \gamma^{\mu \nu \rho} \theta \gamma^\sigma \partial_\sigma \psi + \bar{\theta} \gamma^{\mu \nu} \theta \gamma^\rho \gamma^\sigma \theta + \cdots + \theta^8 \partial^4 \bar{\tau}) \]

\[ \equiv \sum_{r=0}^{8} \Phi^{(r)} \theta^r, \quad (3.5) \]

\(^9\)We are using the usual convention that \( \gamma^{\mu_1 \cdots \mu_p} \) is the antisymmetrized product of \( p \) gamma matrices, normalized so that \( \gamma^{\mu_1 \cdots \mu_p} \equiv \gamma^\mu \cdots \gamma^\mu \) when \( \mu_1 \neq \ldots \neq \mu_p \).

15
where $\tau_0 \Delta$ is the linearized fluctuation around a flat background with a constant scalar, $\tau_0 = \tau - \tau_0 a = C_0^{(0)} + ig^{-1}$. The normalizations that will not concern us here. The symbol $\hat{G}_{\mu \nu \rho} (\mu, \nu, \rho = 0, \ldots, 9)$ denotes the ‘supercovariant’ combination of $G$ and fermion bilinears defined in appendix C, where $G_{\mu \nu \rho}$ and $G^*_{\mu \nu \rho}$ are complex combinations of the field strengths of the $R \otimes R$ and $NS \otimes NS$ two-form potentials. The $\theta^4$ terms are $R$, the Weyl curvature, and $\hat{F}_{51 \cdots 5}$, which is the field strength of the fourth-rank $R \otimes R$ potential (where the hat again denotes the supercovariant combination defined in appendix C). The fermionic field $\lambda$ is the dilatino and $\psi_\mu$ is the gravitino. The gamma matrices with world indices are defined by $\gamma^\mu = e^\mu_m \gamma^m$, where $m = 0, \ldots, 9$ is the $SO(9,1)$ tangent-space index and $e^\mu_m$ is the inverse zehnbein. A barred Weyl spinor, such as $\bar{\theta}_a$, is defined by $\bar{\theta}_a \equiv \theta^b \gamma_0 \delta_{ba}$. (3.6)

The terms indicated by $\cdots$ in (3.5) fill in the remaining members of the ten-dimensional $N = 2$ chiral supermultiplet, comprising (in symbolic notation) $\partial \psi_\mu^*', \partial^2 G_{\mu \nu \sigma}^*, \partial^3 \lambda^*$ and $\partial^4 \tau^*$. The $U(1)$ R-symmetry charge $u_r$ of any of the component fields $\Phi^{(r)}$ in the expansion (3.5) can easily be determined since they are correlated with the powers of $\theta$. Assigning a charge $-1/2$ to $\theta$ and an overall charge 2 to the superfield leads to the charge for the field with $r$ powers of $\theta$,

$$u_r = 2 - \frac{r}{2}$$ (3.7)

(as in [58, 59]). For example, $u_\partial = 2$; $u_\lambda = 3/2$; $u_G = 1$; $u_\psi = 1/2$; $u_R = u_{F_5} = 0$; $\ldots$. Although the linearized theory cannot capture the full structure of the terms in the effective action it can be used to relate various terms in the limit of weak coupling, $\text{Im} \tau_0 = g^{-1} \to \infty$ (where $g = e^{\phi_0}$ is the string coupling constant). The linearized approximations to the complete interactions are those that arise by integrating a function of $\Phi$ over the sixteen components of $\theta$,

$$S^{(3)}_{\text{linear}} = \int d^{10}x d^{16}\theta F[\Phi] + \text{c.c.,}$$ (3.8)

which is manifestly invariant under the rigid supersymmetry transformations, (3.4). The various component interactions contained in (3.8) are obtained from the $\theta^{16}$ term in the expansion,

$$F[\Phi] = F(\tau_0) + \Delta \frac{\partial}{\partial \tau_0} F(\tau_0) + \frac{1}{2} \Delta^2 \left( \frac{\partial}{\partial \tau_0} \right)^2 F(\tau_0) + \cdots$$ (3.9)
Using the expression for $\Delta$ in (3.5) and substituting into (3.8) leads to all the possible interactions at this order in $\alpha'$ [14, 25],

$$S^{(3)} = \int d^{10}x \det e \left(f^{(12,-12)}\lambda^{16} + f^{(11,-11)}\hat{G}\lambda^{14} + \ldots + f^{(8,-8)}\hat{G}^8 + \ldots + f^{(0,0)}R^4 + \ldots + f^{(-12,12)}\lambda^{*16}\right),$$

where $\det e = \det e^m_\mu$ is the determinant of the vierbein.

The $R^4$ interaction comes from the $\Delta^4$ term while the $\lambda^{16}$ comes from the $\Delta^{16}$ term. We see, in particular, that the tensor structure of the contractions between the four Riemann tensors in the $R^4$ term is summarized by the Grassmann integration [60],

$$R^4 = \int d^{16}\theta (R_{\theta^4})^4,$$

where

$$R_{\theta^4} = \bar{\theta}\gamma^{\mu\nu\sigma} \theta \bar{\theta} \gamma_{\rho\tau} \theta R_{\mu\nu\rho\tau}.$$ (3.11)

It follows from a standard Fierz transformation that all possible contractions of the Riemann tensor appearing in $R_{\theta^4}$ vanish, so that only the trace-free part — the Weyl tensor — survives.

The superscripts that label the coefficients $f^{(w,-w)}$ are related to the violation of the $U(1)$ charge. Thus, the linearized form of the general term in (3.10) contains a product of $p$ fields,

$$\int d^{10}x \det e f^{(w,-w)} \prod_{k=1}^{p} \Phi^{(r_k)},$$

which violates the $U(1)$ charge by

$$2w = \sum_{k=1}^{p} u_{r_k} = 2p - 8$$

units, where we have used [13.7] and the fact that the total power of $\theta$ must be $\sum_{k=1}^{p} r_k = 16$. For example the $R^4$ term ($w = 0$) conserves the $U(1)$ charge while the $\lambda^{16}$ term ($w = 12$) violates the $U(1)$ charge by 24 and there are many other terms that violate the charge by any even number.

In the linearized approximation, $g \to 0$ ($\tau_2 \to \infty$), the coefficients $f^{(w,-w)}$ are constants that are related to each other by use of the Taylor expansion, (3.9). For example, the $R^4$ term has coefficient $\partial^4_{\tau_2} F$ while the $\lambda^{16}$ term has coefficient $\partial^{16}_{\tau_2} F$ so that, at the linearized level,

$$f^{(12,-12)} \sim \left(\tau_2 \frac{\partial}{\partial \tau_2}\right)^{12} f^{(0,0)},$$

(3.15)
where for the moment we are not concerned about the overall constant. In writing this we have used the fact that the linearized approximation is valid only if the inhomogeneous term in the modular covariant derivative, $D$ (defined in appendix B), is negligible which requires that

$$2\tau_2 \partial_{\tau_0} f^{(w,-w)} >> wf^{(w,-w)} \quad (3.16)$$

since only in this case does the modular covariant derivative reduce to the ordinary derivative. This inequality is obviously not satisfied by terms in the expansion of $f^{(w,-w)}$ that are powers of $\tau_2$, such as the perturbative tree and one-loop terms in (2.19) (with $\tau_2 \rightarrow \tau_0$). However, when acting on a factor such as $\tau_0^n e^{-2\pi i |K| \tau_0}$ (where $n$ is any constant) which is characteristic of a charge-$K$ D-instanton, the inhomogeneous term may be neglected in the limit $\tau_0 \rightarrow \infty$ and the covariant derivative linearizes. Therefore, a linearized superspace expression such as (3.8) should contain the exact leading multi-instanton contributions to the $R^4$ and related terms. These leading instanton terms arise by substituting the expression

$$F_K = c_K e^{2\pi i |K| \Phi} \quad (3.17)$$

into (3.8).

In the exact theory that will be considered in detail in the next section, the $SL(2,\mathbb{Z})$ symmetry of the IIB theory requires that the $f^{(w,-w)}(\tau, \bar{\tau})$ are modular forms with holomorphic and anti-holomorphic weights as indicated in the superscripts. In that case the derivative $\tau_0 \partial_{\tau_0}$ in (3.15) will be replaced by the covariant derivative, $D_w$, that maps a modular form of weight $(w, \hat{w})$ into a modular form of weight $(w + 1, \hat{w} - 1)$

$$D_w F^{(w,\hat{w})} = i \left( \tau_2 \frac{\partial}{\partial \tau} - \frac{i w}{2} \right) F^{(w+1,\hat{w}-1)} = F^{(w+2,\hat{w})}. \quad (3.18)$$

The covariant version of (3.15) is

$$f^{(12,-12)}(\tau, \bar{\tau}) = D^{12} f^{(0,0)}(\tau, \bar{\tau}) \equiv D_{11} D_{10} \ldots D_0 f^{(0,0)}(\tau, \bar{\tau}). \quad (3.19)$$

We will take this relation to define the relative normalizations of $f^{(0,0)}$ and $f^{(12,-12)}$.

---

10 Appendix B summarizes some relevant properties of modular covariant derivatives and laplacians.
4 Nonlinear supersymmetry constraints on terms of order $\alpha'^{-1}$

In this section, which is based on [15] where further details may be found, the constraints imposed by the full nonlinear supersymmetry of the type IIB theory will be used to derive the detailed expressions for all the terms in $S^{(3)}$ — the terms in the derivative expansion of the action of the type IIB theory that are of order $\alpha'^{-1}$.

The procedure will be to impose supersymmetry order by order in $\alpha'$ by expressing the supersymmetry transformations on an arbitrary field $\Psi$ as the series,

$$\delta_{\epsilon} \Psi = \left( \delta^{(0)} + \alpha' \delta^{(1)} + \ldots + \alpha'^n \delta^{(n)} + \ldots \right) \Psi,$$

while the effective action has the expansion (1.1). In principle, the action can be constructed by imposing the conditions,

$$\left( \sum_{m=0}^{\infty} \alpha'^m \delta^{(m)} \right) \sum_{n=0}^{\infty} \alpha'^n S^{(n)} = 0,$$

order by order in $\alpha'$. In addition, it is important to impose closure of the supersymmetry algebra on all the fields. In theories of this type the supersymmetry algebra only closes on a field $\Phi$ up to the equations of motion and also includes specific local symmetry transformations, $\delta_{\text{local}} \Phi$. The generic structure of the superalgebra is therefore of the form,

$$(\delta_1 \delta_2 - \delta_2 \delta_1) \Phi = \xi^\mu D_\mu \Phi + \Phi \text{ equations of motion} + \delta_{\text{local}} \Phi,$$

where the first term is the usual translation term with parameter

$$\xi^\mu = -2 \text{Im} \bar{\epsilon}_2 \gamma^\mu \epsilon_1.$$

Imposing the supersymmetry conditions (4.2) together with closure of the superalgebra (4.3) simultaneously determines both the action and the supersymmetry transformations on the fields. The $\alpha'^0$ ($n = m = 0$) term in (4.2) corresponds to the supersymmetry of the classical theory. These terms in the action and in the supersymmetry transformations were determined in [59], ignoring terms that are quartic in fermion fields. Some of these zeroth order supersymmetry transformations and action are reviewed in appendix C. One particular term quartic in the fermion fields is needed in the following (the term $L_1^{(0)}$ below) with a coefficient that is derived in [15].

\footnote{As stated in the introduction, it is really the equations of motion rather than the action that will be considered in the following.}
There are no \( n = 1 \) or \( n = 2 \) terms at tree-level or one-loop, and there is strong evidence that supersymmetry precludes the presence of any such terms in \( \mathcal{S}^{(3)} \), which are of order \((\alpha')^3\) relative to \( \mathcal{S}^{(0)} \). These terms are eighth order in derivatives.

The procedure of imposing supersymmetry and closure can become very complicated unless one makes a judicious choice of starting point. We will start by selecting two special terms in \( \mathcal{L}^{(3)} \) that involve sixteen fermionic fields \( \hat{G} \). The choice is motivated by the fact that these two terms mix with each other and with no other terms under the classical supersymmetry transformations. The two terms are the sixteen-fermion terms contained in

\[
\mathcal{L}_1^{(3)} = \det e^{\lambda_{16}^* \psi_\mu^* \epsilon} \left( f^{(12,-12)}(\tau, \bar{\tau}) \lambda_{16} + f^{(11,-11)}(\tau, \bar{\tau}) \hat{G} \lambda_{14} \right)
\]

where the ellipsis represents other terms in \( \hat{G} \) which do not affect the subsequent argument \( \mathcal{D} \) (details of the spinor algebra that leads to the precise coefficients in this and following equations can be found in appendix C and \( \mathcal{L} \)).

From the lowest order supersymmetry transformations given in appendix C we obtain terms that transform (4.3) into

\[
\delta_1 \mathcal{L}_1^{(3)} = -i \det e^{\lambda_{16}^* \psi_\mu^* \epsilon} \left( \bar{\epsilon}^* \gamma^\mu \hat{G} \lambda_{14} \right)
\]

where we have only kept terms proportional to \( \lambda_{16}^* \psi_\mu^* \epsilon \). It is important to check whether there could also be a contribution of the same form as (4.3) arising from a \((\alpha')^3 \mathcal{D}^{(3)}\) variation of the fields in the lowest order action \( \mathcal{S}^{(0)} \). However, it is easy to see by inspection that no term with \( \lambda_{16}^* \psi_\mu^* \epsilon \) can arise from the variation of any term in \( \mathcal{S}^{(0)} \). This means that we must require \( \delta_1 \mathcal{L}_1^{(3)} = 0 \), which implies that

\[
D_{11} f^{(11,-11)} = -4 \cdot \frac{3}{144} f^{(12,-12)}.
\]

This condition is consistent with the modular weights assigned to the functions \( f^{(w,-w)} \).

We now consider the term in the variation of (4.3) that is proportional to \( \det e^{\lambda_{16}^* \lambda^* \epsilon^*} \),

\[
\delta_2 \mathcal{L}_1^{(3)} = -2i \det e^{\lambda_{16}^* (\bar{\epsilon}^* \lambda^*)} \left( \bar{D}_{-12} f^{(12,-12)} + 3 \cdot 144 \cdot 15 \cdot \frac{2}{f^{(11,-11)}} \right) + \ldots,
\]

\(\text{12}\) Our notation is chosen so that \((\alpha')^n \int d^{10}x \mathcal{L}^{(n)} = S^{(n)}\).

\(\text{13}\) The overall normalization of the action does not affect the arguments of this section.
where we have made explicit only the terms containing $\lambda^6 \lambda^* e^*$. In this case, there is another contribution of the same form as $\delta_2^{(0)} L_1^{(3)}$ that arises from the $(\alpha')^3 \delta^{(3)}$ variation of terms in the lowest order IIB Lagrangian $L^{(0)}$. Even though the complete set of interactions in the classical theory is not tabulated explicitly in the literature (it is implicit in the superspace formulation [13]), it is easy to convince oneself that the only possible term that can vary into $\delta_2^{(0)} L_1^{(3)}$ is a term of the form,

$$L_1^{(0)} = \frac{1}{256} \det e \lambda \gamma^\mu \rho \lambda^* \lambda^* \gamma_{\mu \nu \rho} \lambda,$$

(4.9)

which is the unique tensor structure containing $\lambda^2 \lambda^*$. The coefficient of this term was determined by the lowest order supersymmetry transformations in [15] by considering the mixing of $L_1^{(0)}$ with other terms in the classical action.

We see that $L_1^{(0)}$ can vary into the same form as $\delta_2^{(0)} L_1^{(3)}$ if we assume a higher-order variation of $\lambda^*$ of the form,

$$\delta^{(3)} \lambda^*_a = -\frac{1}{6} i g(\tau, \bar{\tau}) (\lambda^{14})_{cd} (\gamma^{\mu \nu \rho} \gamma^0)_{de} (\gamma_{\mu \nu \rho} e^*)_a,$$

(4.10)

where $g(\tau, \bar{\tau})$ is another function to be determined. Substituting in (4.9) gives a contribution,

$$\delta^{(3)} L_1^{(0)} = -\frac{1}{768} i \det e g(\tau, \bar{\tau}) \lambda \gamma^{\mu \nu \rho} \gamma_{\rho_1 \rho_2 \rho_3} e^* (\lambda^{14})_{cd} (\gamma^{\rho_1 \rho_2 \rho_3} \gamma^0)_{de} \lambda^* \gamma_{\mu \nu \rho} \lambda = -30 i \det e g(\tau, \bar{\tau}) \lambda (\xi \lambda^*).$$

(4.11)

Comparing with (4.8) we see that in order for the total contribution to $\delta L_1$ to vanish at order $(\alpha')^3$, there must be a linear relation between the function $g$ and the functions $f^{(11, -11)}$ and $f^{(12, -12)}$.

$$D_{-12} f^{(12, -12)} + 3 \cdot 144 \cdot \frac{15}{2} f^{(11, -11)} + 15 g = 0.$$

(4.12)

At this stage we have two equations ((4.7) and (4.12)) relating the three unknown functions, $f^{(12, -12)}$, $f^{(11, -11)}$ and $g$. A further constraint on these functions is obtained by requiring the closure of the superalgebra. We need to consider closure of the supersymmetry transformations on the field $\lambda^*$. First, in the classical theory (and keeping only the terms linear in $\lambda$ derivatives) it was found in eq. (4.5) of [52] that

$$(\delta_1^{(0)} \delta_2^{(0)} - \delta_2^{(0)} \delta_1^{(0)}) \lambda^* = \xi^\mu D_\mu \lambda^* - \frac{3}{8} i [\bar{\epsilon}_2 \gamma^\rho e_1 - (1 \leftrightarrow 2)] \gamma^\rho \gamma^\mu D_\mu \lambda^* - \frac{1}{96} i [\bar{\epsilon}_2 \gamma^{\rho_1 \rho_2 \rho_3} e_1 - (1 \leftrightarrow 2)] \gamma^{\rho_1 \rho_2 \rho_3} \gamma^\mu D_\mu \lambda^*$$

(4.13)
This has the form of (4.3). The first term on the right-hand-side is of the form expected for the commutator of two supersymmetry transformations. The other terms that have been exhibited explicitly are proportional to the linear terms in the $\lambda^*$ equation of motion. Many other terms that are not needed are indicated by $\ldots$. These terms contribute to the commutator to complete the low-energy $\lambda^*$ field equation as well as generating local transformations of $\lambda^*$ [5].

The higher order terms in $L^{(3)}$ modify the equations of motion and this should also be apparent by considering the closure of the algebra. Therefore, we now consider terms that enter at order $\alpha'^3$ from the commutator of a $\delta^{(0)}$ with a $\delta^{(3)}$. More precisely, we shall consider terms in the commutator involving only $\epsilon^*_{2}$ and $\epsilon_1$,

\[
\left( \delta^{(0)}_{\epsilon_1} \delta^{(3)}_{\epsilon_2} - \delta^{(3)}_{\epsilon_2} \delta^{(0)}_{\epsilon_1} \right) \lambda^* = - \frac{1}{3} \left( \tau_2 \frac{\partial}{\partial \tau} - i \frac{45}{8} \right) i g (\bar{\epsilon}^*_1 \gamma_5 \lambda) (\lambda^{14})_{cd} (\gamma^{\mu\nu\rho} \gamma^0)_{dc} (\gamma_{\mu\nu\rho} \epsilon^*_2)^a \lambda^a \\
= 32 D_{11} g \lambda^*_b \left[ \frac{3}{8} \bar{\epsilon}_2 \gamma^\mu \epsilon_1 (\gamma_\mu)_{ba} + \frac{1}{96} \bar{\epsilon}_2 \gamma^{\mu\nu\rho} \epsilon_1 (\gamma_{\mu\nu\rho})_{ba} \right] + \delta_{\tilde{\epsilon}} \lambda^*. \tag{4.14}
\]

In the last line, we have separated a local supersymmetry transformation, $\delta_{\tilde{\epsilon}} \lambda^*$, which is to be identified with a supersymmetry transformation of the form (4.11) with a particular field dependent coefficient, $\tilde{\epsilon} = \frac{4}{9} \epsilon^*_2 (\bar{\epsilon}^*_1 \lambda)$.

Combining (4.13) and (4.14) (including the powers of $\alpha'$) we see that in order for the right-hand side of the commutator to vanish the $\lambda^*$ field equation must be of the form,

\[
i \gamma^\mu D_\mu \lambda^* - (\alpha')^3 32 D_{11} g \lambda^{15} + \ldots = 0, \tag{4.15}
\]

where the ellipsis indicates terms with different structure that we have not considered. This equation has to be identified with the appropriate sum of terms in the $\lambda^*$ equation of motion that is obtained by varying the action with respect to $\lambda$. At the same order in $\alpha'$ this is given by,

\[
i \gamma^\mu D_\mu \lambda^* - (\alpha')^3 f^{(12,-12)} \lambda^{15} + \ldots = 0, \tag{4.16}
\]

where we have only made explicit the term that is proportional to $\lambda^{15}$. Comparing (4.15) and (4.16) gives the relation,

\[
32 D_{11} g = f^{(12,-12)}. \tag{4.17}
\]

Substituting (4.17) into (4.17) gives,

\[
g = - \frac{3 \cdot 144}{128} f^{(11,-11)}. \tag{4.18}
\]
There is no ambiguity in this relation between $g$ and $f^{(11,-11)}$ because there is no solution to $D_{11}g = 0$. Substituting (4.18) into (4.12) gives,

$$
\bar{D}_{-12} f^{(12,-12)} = 3 \cdot 144 \left( -\frac{15}{2} + \frac{45}{64} \right) f^{(11,-11)}.
$$

(4.19)

The two simultaneous first-order differential equations, (4.19) and (4.7) are simply reduced to the independent second-order eigenvalues equations,

$$
\nabla^2_{(-12)} f^{(12,-12)} = 4D_{11} \bar{D}_{-12} f^{(12,-12)} = \left( -132 + \frac{3}{4} \right) f^{(12,-12)}
$$

(4.20)

and

$$
\nabla^2_{(+11)} f^{(11,-11)} = 4\bar{D}_{-12} D_{11} f^{(11,-11)} = \left( -132 + \frac{3}{4} \right) f^{(11,-11)},
$$

(4.21)

where the laplacians are defined in appendix B.

By applying $\bar{D}_{12}$ to (4.20) we can define the function,

$$
\tilde{f}^{(0,0)}(\tau, \bar{\tau}) \equiv \bar{D}_{12} f^{(12,-12)}(\tau, \bar{\tau}).
$$

(4.22)

Applying $D_{12}$ to this equation and using (4.20) together with properties of covariant derivatives in appendix B leads to

$$
f^{(12,-12)}(\tau, \bar{\tau}) = kD^{12}\tilde{f}^{(0,0)},
$$

(4.23)

where $k$ is an irrelevant constant. Therefore, from (3.19), we can make the identification

$$
\tilde{f}^{(0,0)} = k^{-1}f^{(0,0)}.
$$

(4.24)

Putting these relations together we see that the Laplace eigenvalue equation for $f^{(12,-12)}$ (4.20) implies

$$
\nabla^2 f^{(0,0)}(\tau, \bar{\tau}) \equiv 4\bar{\tau}^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \bar{\tau}} f^{(0,0)}(\tau, \bar{\tau}) = \frac{3}{4} f^{(0,0)}(\tau, \bar{\tau}).
$$

(4.25)

It is an important fact that we will not prove here (see, for example, exercise 2 in section 3.5 of [57]) that (4.25) has the unique solution (2.22) provided it is assumed to transform as a scalar under $SL(2, \mathbb{Z})$ and has polynomial growth as $\tau_2 \to \infty$ (which is the perturbative regime).

Substituting (4.23) in (4.23) and (4.24) determines $f^{(12,-12)}$ to be

$$
f^{(12,-12)}(\tau, \bar{\tau}) = \frac{1}{2^{12}} \frac{\Gamma(\frac{27}{2})}{\Gamma(\frac{11}{2})} \sum_{(m,n) \neq (0,0)} \tau_2^{3/2} \frac{(m+n\bar{\tau})^{24}}{|m+n\tau|^{27}}.
$$

(4.26)
Similarly, (4.21) gives a unique expression for the modular form \( f^{(11, -11)} \). The expressions for all the coefficients, \( f^{(w, -w)} \), in (4.10) should also emerge from a more detailed application of the Noether procedure that considers all the possible mixing of terms in \( S^{(3)} \) with arbitrary \( U(1) \) charges. These are given by applying \( w \) covariant derivatives to \( f^{(0, 0)} \) which results in the expression

\[
f^{(w, -w)}(\tau, \bar{\tau}) = \frac{1}{2^w} \frac{\Gamma \left( \frac{w + \frac{3}{2}}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} \sum_{(m, n) \neq (0, 0)} \frac{\tau_2^{3/2}}{|m + n\tau|^3} \left( \frac{m + n\bar{\tau}}{m + n\tau} \right)^w
\]

This transforms under \( SL(2, \mathbb{Z}) \) transformations by a phase

\[
f^{(w, -w)}(\tau, \bar{\tau}) \rightarrow \left( \frac{c\tau + d}{c\bar{\tau} + d} \right)^w f^{(w, -w)}(\tau, \bar{\tau}), \quad (4.28)
\]

as expected for a \((w, -w)\) modular form.

The Noether procedure rapidly escalates in complication when applied at higher orders in the derivative expansion and it is probably not a practical procedure for terms with many more derivatives. It may just about be feasible to deduce terms of order \( \alpha' \) that are of order \( \alpha'^5 \) beyond the Einstein–Hilbert term. For example, this may be a method of determining the higher derivative terms suggested in [26] which were motivated by exactly known perturbative contributions [61, 62].

The dimensions of the terms of yet higher order in \( \alpha' \) suggested in [22, 26] are so high that it is not obvious why they should be protected by supersymmetry. Nevertheless, they probably are protected based on the evidence for the exact form of these terms presented in these references.

There is also a suggestion [17, 63] for how the structure of the protected \( o(\alpha'^{-1}) \) terms generalizes to compactifications to lower dimensions, where there are more moduli. The suggestion is that the functions \( f^{(w, -w)} \) are generalized Eisenstein series for the appropriate duality groups.

5 The eleven-dimensional perspective

We shall now review the manner in which the \( o(\alpha'^{-1}) \) terms in the type II string actions can be deduced from eleven-dimensional supergravity. This was the subject of [11] where it was pointed out that the complete \( R^4 \) term arises from the one-loop quantum contribution to four-graviton scattering in eleven-dimensional supergravity compactified on \( T^2 \). Similarly, the \( \lambda^{16} \) interaction
can be deduced from the one-loop scattering amplitude of sixteen gravitini in eleven-dimensional supergravity \[14\]. The type IIB dilatino, $\lambda$, is identified with the gravitino of eleven-dimensional supergravity with its polarizations in appropriate directions. These calculations and generalizations were reviewed in \[14\] where some further details may be found, including a discussion of higher derivative contributions (which were also the subject of \[22\]).

### 5.1 The $R^4$ term from one loop in eleven dimensions

We will begin with the $R^4$ term which will be determined from the scattering amplitude for four gravitons at one loop in eleven-dimensional supergravity \[16\]. A fully covariant calculation would involve the sum of loop diagrams with the various component fields (the graviton, $g_{\hat{\mu}\hat{\nu}}$, the third-rank antisymmetric potential, $C_{\hat{\mu}\hat{\nu}\hat{\rho}}^{(3)}$, and the gravitino, $\Psi_{\hat{\mu}}^a$) and their ghosts circulating around the loop. An efficient method for summing all these contributions is to use a manifestly supersymmetric formalism based on the eleven-dimensional point particle \[16\] (details of the superparticle formalism outlined below have yet to appear \[65\]).

This method is modeled on the method used to construct string one-loop amplitudes in the light-cone gauge which is the parameterization in which

$$X_+^\pm(t) = p_+^\pm t + x_+^\pm,$$  \hspace{1cm} (5.1)

where $t$ labels the world-line of the superparticle, $X_+^\pm(t) = (X_0^0(t) \pm X_8(t))/\sqrt{2}$ and $p_+^\pm$ is a constant (the directions $x_9, x_{11}$ will continue to be the $T^2$ directions). The physical states are described in terms of the transverse $SO(9)$ representations (instead of the $SO(8)$ that is relevant for the transverse space in the case of the superstring). The quantum states are constructed in terms of fermionic sixteen-component spinors, $S^A$ ($A = 1, \ldots, 16$), satisfying the anticommutation relations,

$$\{S^A, S^B\} = \delta^{AB}. \hspace{1cm} (5.2)$$

The 32 components of the supercharges decompose the components that are realized linearly in the light-cone gauge,

$$Q^A = \sqrt{p^+} S^A, \hspace{1cm} (5.3)$$

and those that are nonlinearly realized,

$$\tilde{Q}^A = \frac{1}{\sqrt{p^+}}(\Gamma^I \cdot \dot{{X}} S)^A, \hspace{1cm} (5.4)$$

where $\Gamma_{AB}^I$ are the $16 \times 16$ $SO(9)$ gamma matrices and $X^I(t)$ are the transverse coordinates of the superparticle.
The vertex operators that describe the linearized interactions can be derived by imposing the conditions that the $g$, $\psi$ and $C^{(3)}$ vertices transform into each other under supersymmetry in the appropriate manner \[65\] which mimics the way in which light-cone superstring vertices can be derived. The resulting light-cone graviton vertex operator is given by

$$V_{g}^{(r)} = \zeta_{IJ}^{(r)}(\dot{X}^{I}\dot{X}^{J} - 2\dot{X}^{I}S_{\gamma}^{JL}S_{k}^{L} + 2S_{\gamma}^{IL}S_{k}^{L}S_{\gamma}^{JM}S_{k}^{M})e^{ik\cdot X},$$ (5.5)

where, as usual in the light-cone gauge operator formalism, a special frame has been chosen in which $k^{+} = 0$ and $\zeta_{IJ}^{(r)}$ is the transverse polarization tensor of the eleven-dimensional graviton ($I = 1, \cdots, 7, 9, 11$). This expression reduces to the graviton vertex in the type IIA superparticle theory when reduced on a circle to ten dimensions \[65\]. In addition to the cubic graviton vertex higher-order contact interactions are necessary in order to reproduce general gauge-invariant amplitudes. Such contact terms compensate for short distance singularities in products of $V_{g}^{(r)}$’s that involve factors such as $\langle \dot{X}(\tau)\dot{X}(\tau') \rangle$ but are not needed for the amplitudes we are interested in.

The loop amplitude compactified on a two-torus is given by

$$A_{R^4} = \frac{1}{V} \prod_{r=1}^{4} dt_{r} d^{9}p \sum_{m,n} \text{tr}_{S}(V_{g}^{(1)}(k_{1})V_{g}^{(2)}(k_{2})V_{g}^{(3)}(k_{3})V_{g}^{(4)}(k_{4})), \quad (5.6)$$

where $\text{tr}_{S}$ denotes the trace over the fermionic variables. The integers $m, n$ are the Kaluza–Klein charges in the $T^2$ directions ($x^9, x^{11}$). For convenience we will only consider the situation in which all the external polarizations and momenta are oriented in the directions $x^1, \ldots, x^7$ which are transverse to the light-cone as well as the $T^2$ directions. The fermionic trace is saturated by the term with sixteen $S$ factors, which picks out the last term in \[22\] (the term with four $S$’s) so that no contractions of $\dot{X}$’s in the prefactor of the vertices. This picks out an overall kinematic factor which can be shown to be the same as the overall factor in the ten-dimensional one-loop superstring amplitude defined in \[2.18\] and can be expressed in a compact manner as an integral over a sixteen-component Grassmann spinor, as described in section 3.

After extracting the overall kinematic factor the remaining bosonic factor is simply the loop amplitude for the four-point amplitude in scalar $\phi^3$ field theory. The full momentum dependence of this amplitude was discussed in \[22, 33\]. Expanding in a power series in the momenta leads to higher derivatives acting on the fourth power of the curvature. However, here we are only interested in the leading momentum dependence. Therefore, after extracting the kinematic factor the external momenta can be set equal to zero in the remainder of the
expression. The result can be written as

\[ A_{R^4} = \frac{1}{V} \tilde{K} \int_{\hat{t}}^{\infty} dt^4 \int d^9 p \sum_{l_1, l_2} \exp \left( -t(p^2 + G^{IJ} l_I l_J) \right), \]  
(5.7)

where \( G^{IJ} \) is the inverse metric on \( T^2 \) and \( l_I = (m, n) \) \((I, J = 1, 2)\). Substituting for the inverse metric,

\[ G^{IJ} l_I l_J = \frac{1}{V} |m + n\Omega|^2, \]  
(5.8)

and performing a double Poisson resummation converts the sum over Kaluza-Klein charges \((m, n)\) to a sum over the windings \((\hat{m}, \hat{n})\) of the world-line of the particle circulating in the loop around the two independent cycles of \( T^2 \). After the straightforward gaussian integration the result is

\[ V A_{R^4} = \tilde{K} V \int_{\hat{t}}^{\infty} dt^{\hat{t}} \sum_{\hat{m}, \hat{n}} e^{-V i(\hat{m} + \hat{n})^2/\Omega^2} \]  
(5.9)

where \( \hat{t} = t^{-1} \). The cubic ultraviolet divergence in (5.9) is contained in the zero winding term, \( \hat{m} = \hat{n} = 0 \), which has the divergent coefficient \( C \). The second equality in (5.9) gives the first two terms in the expansion for large \( \Omega^2 \) (this expansion will be discussed in detail in section 6). The amplitude (5.9) comes from a term in the effective action of the form (2.21) with \( f^{(0,0)} \) again of the form (2.22). As remarked earlier, in the limit \( V \to 0 \) the regularized \( VC \) term vanishes and the amplitude has a finite limit in type IIB coordinates, i.e., \( r_B \to \infty \) with \( e^\phi \) held constant. Substituting \( \Omega \) by \( \tau = C^{(0)} + i e^{-\phi} \) leads to precisely the same expression for the \( R^4 \) term in the type IIB theory that we deduced earlier from supersymmetry.

It is easy to argue on dimensional grounds that the finite part of the coefficient of the \( R^4 \) term can only come from one loop in eleven dimensions. Higher loops lead to finite contributions that have extra derivatives acting on \( R^4 \) [64]. However, the systematics of the divergent parts is more obscure and the coefficient \( C \) has contributions from divergences of diagrams with arbitrary numbers of loops. These divergences must be regularized by microscopic effects in the eleven-dimensional theory that lie outside the realm of perturbation theory. Although we do not know how to fix the regularized value of \( C \) directly it is precisely determined by symmetry considerations. Keeping \( V \) nonzero and
comparing the leading terms of the large-$\Omega_2$ expansion in (5.4) with (2.21) shows that these expressions only agree if $C$ has the value

$$C = \frac{2\pi^2}{3}. \quad (5.10)$$

We saw that the coefficients in (2.21) are consistent with T-duality that relates the IIA and IIB string perturbation expansions in nine dimensions. This means that only for the value of $C$ in (5.10) is (5.9) consistent with T-duality. This leads to the same value of $C$ as the argument based on supersymmetry that was mentioned at the end of section 2.

### 5.2 One loop with sixteen dilatini

In a similar manner all the interactions that are related by linearized supersymmetry to the $R^4$ term can be deduced by considering one-loop processes in eleven-dimensional supergravity on $T^2$ \cite{14}. As we saw in section 3 the generic terms of this type violate the $U(1)$ R-symmetry charge of the type IIB theory that is conserved classically. An extreme example of this is the interaction between sixteen dilatini. Since the dilatino $\lambda$ carries a $U(1)$ charge of $3/2$ the interaction $\lambda^{16}$ violates 24 units of charge.

As in the case of four-graviton scattering the calculation of the general loop amplitude that describes the scattering of sixteen external dilatinis is facilitated by using the light-cone superspace description of the eleven-dimensional superparticle. However, the $\lambda^{16}$ term of interest can be extracted from the zero-momentum process and this can be calculated without using the general gravitino vertex operators. The zero-momentum loop amplitude has the structure (ignoring an overall constant)

$$A_{\lambda^{16}} = \frac{1}{V} \sum_{m,n} \int d^9p \int \frac{d\tau}{\tau} \tau^{16} \text{tr}_S(V^{(1)}(0) \cdots V^{(16)}(0)) e^{-\tau(p^2 + m\Omega^2 + n\Omega^2)}, \quad (5.11)$$

where $V^{(r)}(0)$ is the zero-momentum vertex for the $\lambda$ component of the $r$th gravitino. The trace in the above expression is over the fermionic operators in the vertices.

The fermions of the IIB theory compactified on $S^1$, the complex fields $\psi$ and $\lambda$, are related to specific projections of the eleven-dimensional gravitino of M theory on $T^2$. These relations are described in appendix D (based on \cite{14}). It follows that the vertex $V_\lambda$ is a specific projection of the vertex operator for the eleven-dimensional gravitino, $V_\varphi$. The gravitino vertex can again be deduced from eleven-dimensional supersymmetry in the light-cone gauge \cite{14}. The zero-
momentum vertex is very simply expressed in terms of the 32-component eleven-
dimensional supercharge, \( Q \), by the covariant expression

\[
V_\Psi = \bar{\zeta}_\mu Q p^\mu, \tag{5.12}
\]

where \( \zeta_\mu \) is the zero-momentum wave function, \( p^\mu \) is the momentum of the particle circulating around the loop and \( \bar{\zeta}_\mu \equiv \zeta_\mu \Gamma^0 \) (with no complex conjugation). Since the zero momentum vertex is independent of the proper time around the loop the volume of integration over the proper times of the vertices gives the factor of \( \tau^{16} \) in (5.11).

We now want to consider the components of \( \Psi_\mu \) containing \( \lambda \). Inverting the relations in equation (D.12) in appendix D gives,

\[
\Psi_z = \mathcal{P}_z \Gamma_z \chi + \mathcal{P}_z \Gamma_\bar{z} \lambda, \quad \Psi_{\bar{z}} = \mathcal{P}_\bar{z} \Gamma_{\bar{z}} \chi^* + \mathcal{P}_\bar{z} \Gamma_z \lambda^*. \tag{5.13}
\]

from which it follows that

\[
V_\lambda = -\bar{\lambda}^* \Gamma_\bar{z} \mathcal{P}_\bar{z} Q p_z = -\bar{\lambda}^* \Gamma_z q p_{\bar{z}}, \tag{5.14}
\]

where \( q = \mathcal{P}_z Q \) is a projected supercharge that satisfies the anticommutation relations

\[
\{ q^A, q^B \} = \mathcal{P}^{AC}_z \mathcal{P}^{BD}_z \{ Q^C, Q^D \} = \Gamma^{AB}_z p_z. \tag{5.15}
\]

The momentum dependence can be scaled out by changing to \( \hat{q}^A = q^A (p_z)^{-\frac{1}{2}} \), which satisfies \( \{ \hat{q}^A, \hat{q}^B \} = \Gamma^{AB}_z \). In the chirally projected subspace \( \Gamma^{AB}_z \propto \delta^{AB} \) and the commutation relation does not depend on \( \bar{z} \).

Substituting the vertex (5.14) into the loop amplitude (5.11) (contact terms are again not needed in this spin-\( \frac{1}{2} \) process) and integrating over the loop momentum gives an expression of the form

\[
\mathcal{V}_{A\lambda^{16}} = \hat{K} \sqrt{\frac{4}{\pi}} f(12, -12)(\Omega, \bar{\Omega}), \tag{5.16}
\]

where the kinematic prefactor,

\[
\hat{K} = \bar{\lambda}^{(1)s}_{A_1} \cdots \bar{\lambda}^{(16)s}_{A_{16}} \text{tr}(\hat{q}^{A_1} \cdots \hat{q}^{A_{16}}), \tag{5.17}
\]

is manifestly antisymmetric under permutations of the (commuting) fermion wave functions due to the anticyclic property of the trace. It can be rewritten (up to an overall constant) as

\[
\hat{K} = \bar{\lambda}^{(1)s}_{A_1} \cdots \bar{\lambda}^{(16)s}_{A_{16}} \varepsilon^{A_1 \cdots A_{16}}_{16}, \tag{5.18}
\]

where \( \varepsilon_{16} \) is the rank sixteen epsilon tensor with spinor indices.
The rest of the expression (5.16) depends on the circumference \( r_B \) through the factor \( V^{-\frac{1}{2}} \). The \( \Omega \) and \( \bar{\Omega} \) dependence comes from the loop integration and is given by

\[
f^{(\Omega, \bar{\Omega})} = c \sum_{m,n} \int dt \frac{t^{23/2}}{t} \left( \frac{1}{\sqrt{\Omega_2}} (m + n\Omega) \right)^{24} \exp \left( -t \frac{1}{\Omega_2} |m + n\Omega|^2 \right)
\]

(5.19)

(\text{where } c \text{ is an overall constant}). This expression is superficially cubically divergent due to the sum over the Kaluza-Klein charges. However, this does not take into account the phase dependence which, for generic \( \Omega \), can lead to a cancellation between the growing terms in the sum. Indeed, for asymptotically large \( m \) and \( n \) the sum over discrete momenta \( p_z \) and \( p_{\bar{z}} \) can be replaced by integrals and the result is proportional to

\[
\int dp_z dp_{\bar{z}} p_z^{21 \frac{1}{2}} (p_z)^{-23/2},
\]

(5.20)

which vanishes if \( |p_z| \) is regularized. Equivalently, it vanishes if the phase integration is carried out before the integration over \( |p_z| \). This is the regularization prescription that will be used in the following, motivated by the fact that it leads to an \( SL(2, \mathbb{Z}) \) invariant answer.

In order to extract the finite result it is useful, as before, to use a double Poisson summation to transform from the discrete momentum sum to a sum over windings of the loop around the torus. Writing

\[
f^{(\Omega, \bar{\Omega})} = c \sum_{\hat{m}, \hat{n}} \int dt \frac{t^{23/2}}{t} \exp \left( -t |\hat{m} + \hat{n}\Omega|^2 \right)
\]

the Poisson resummation equates this to the manifestly finite expression,

\[
f^{(-12, 12)} = \frac{c}{\sqrt{\Omega_2}} \left( \frac{\partial}{\partial \alpha} \right)^{24} \sum_{\hat{m}, \hat{n}} \frac{\pi}{\hat{\Omega}_2} \int dt \frac{t^{21/2}}{t} \exp \left( -\frac{\pi^2}{\hat{\Omega}_2^2} |\hat{m} + \hat{n}\Omega|^2 + \frac{i\pi\alpha}{\hat{\Omega}_2} (\hat{m} + \hat{n}\bar{\Omega}) \right) \bigg|_{\alpha = 0}
\]

(5.21)

\[
f^{(-12, 12)} = \frac{c}{\sqrt{\Omega_2}} \Gamma(27/2) \Omega_2^{3/2} \sum_{(\hat{m}, \hat{n}) \neq (0,0)} (\hat{m} + \hat{n}\bar{\Omega})^{24} |\hat{m} + \hat{n}\Omega|^{27},
\]

(5.22)

where \( \hat{m} \) and \( \hat{n} \) are winding numbers.

After translating from the M-theory coordinates (\( \Omega, V \)) to the string-frame IIB coordinates (\( \tau, r_B \)) the expression (5.22) is identical to (1.26) up to an overall
normalization that can be absorbed in $c$. The amplitude $A_{\lambda^16}$ corresponds to the $\lambda^16$ term in the IIB effective action, (3.10), in the limit, $r_B \to \infty$, of ten decompactified dimensions. In fact, in this case there is no divergent term at all even at finite $r_B$. The eleven-dimensional limit of $A_{\lambda^16}$ is not only finite but it vanishes. The nine-dimensional effective action in the IIB parameterization contains the term

$$S^{(3)}_{\lambda^16} = \frac{1}{3 \cdot (4\pi)^7 l_s} \int d^9 x r_B \det e^{-\phi/2} \lambda^{16} f(12,-12)(\tau, \bar{\tau}).$$

(5.23)

This expression has been written in the string frame. The transformation from Einstein frame to string frame includes the rescaling $\lambda \to \lambda e^{-\phi/8}$ in addition to the rescaling of the zehnbein, $e^\mu_a \to e^\mu_a e^{\phi/4}$.

### 6 Fourier series and properties of D-instantons

We have seen that the terms in the ten-dimensional type IIB effective action of order $\alpha'^{-1}$ have the form (in string frame) [66],

$$\frac{1}{\alpha'} \int d^{10} x \det e^{-\phi/2} f^{(w,-w)}(\tau, \bar{\tau}) \mathcal{O}_{\{u_r\}},$$

(6.1)

where $\mathcal{O}_{\{u_r\}}$ is a function of the fields that reduces, at the linearized level to a monomial of $p$ fields with $U(1)$ R-symmetry charges $u_1, u_2, \ldots, u_p$ (see (3.13)). The total $U(1)$ charge of the interaction is $2w$ where $w = p - 4$ (see (3.14)) and the coefficient $f^{(w,-w)}$ is given by (4.27). The coefficient functions can be expressed as Fourier series,

$$f^{(w,-w)} = \sum_{K=-\infty}^{\infty} F_{K,w} e^{2\pi i K \tau_1},$$

(6.2)

where the Fourier coefficients, $F_{K,w}$, with nonzero $K$ are the D-instanton terms when $K > 0$ and anti D-instanton terms when $K < 0$. This expansion is readily derived by a generalization of the method that leads to the analogous expansion of the Eisenstein series, $E_s$ (see, for example, [57]). First one performs the sum over all the terms in (4.27) with $n = 0$ and $m \neq 0$. This leads to the ‘tree-level’ term of the form $2\zeta(3)e^{-2\phi}$. For the terms with $n \neq 0$ one may perform a Poisson resummation that converts the sum over $m$ into a sum over a conjugate integer, $\hat{m}$. The sum over all $n \neq 0$ with $\hat{m} = 0$ gives the ‘one-loop’ term that is independent of $\tau_2$ while the remaining sum (over all $n \neq 0$, $\hat{m} \neq 0$) gives the D-instanton contributions with $K = \hat{m} n$. 

31
The result is \( \frac{66}{24} \)

\[ f(w, -w) = 2\zeta(3)\tau_2^3 + \frac{2\pi^2}{3}\tau_2^{-\frac{1}{2}}c_w + \sum_{K=1}^{\infty} \left( \mathcal{F}_{K,w}e^{2\pi i K\tau_1} + \mathcal{F}_{K,-w}e^{-2\pi i K\tau_1} \right), \]  

where

\[ c_w = \frac{(-1)^w\pi}{4}\frac{1}{\Gamma\left(\frac{3}{2} + w\right)\Gamma\left(\frac{3}{2} - w\right)}. \]  

The first two terms in (6.3) have the interpretation of the tree-level and one-loop string terms while the instanton and anti-instanton terms are contained in

\[ \mathcal{F}_{K,w} = 4\pi^{1/2}(2\pi|K|)^{1/2}Z_K\sum_{k=0}^{\infty} \left( \frac{c_{w,k}}{(2\pi K\tau_2)k-w}e^{2\pi i K\tau_2} \right), \]  

where

\[ c_{w,k} = \frac{(-1)^w\Gamma(3/2)}{2^{k-w}k!\Gamma(-w+3/2)\Gamma(-k+w+1/2)} \]  

and

\[ Z_K = \sum_{m|K} \frac{1}{m^{\tau_2^2}}. \]  

The infinite series in (6.3) begins with the power \( \tau_2^2 \) for a D-instanton \((K > 0)\) while the series of corrections to the anti D-instanton \((K < 0)\) starts with the power \( \tau_2^{-w} \).

A particularly simple example of this formula is the expansion of the coefficient \( f^{(0,0)} \) of the \( \mathcal{R}_4 \) term which is given by

\[ f^{(0,0)}(\tau, \bar{\tau}) = 2\zeta(3)\tau_2^3 + \frac{2\pi^2}{3}\tau_2^{-\frac{1}{2}} + 8\pi\tau_2^\frac{3}{2} \sum_{K\neq 0} |K| Z_K K^{-1}(2\pi|K|\tau_2)e^{2\pi i K\tau_1}, \]  

\[ = 2\zeta(3)\tau_2^3 + \frac{2\pi^2}{3}\tau_2^{-\frac{1}{2}} + 4\pi \sum_{K=1}^{\infty} |K|^{1/2}Z_K \]  

\[ \times \left( e^{2\pi i K\tau} + e^{-2\pi i K\bar{\tau}} \right) \left( 1 + \sum_{k=1}^{\infty} (4\pi K\tau_2)^{-k} \frac{\Gamma(k-1/2)}{\Gamma(-k-1/2)k!} \right), \]  

where the asymptotic expansion of the Bessel function \( K^{-1}(2\pi|K|\tau_2) \) (which is useful for small coupling, or large \( \tau_2 \)) has been used in the second equality.

The expressions (6.3) and (6.8) reproduce precisely the perturbative tree-level and one-loop terms and demonstrate that there is a perturbative non-renormalization theorem beyond one loop. The absence of higher-order perturbative corrections must be related to the fact that the terms in (6.1) are given by integrals over half the on-shell superspace although a direct proof of the
absence of higher-order perturbative corrections has not been given (see however \[28\]). The non-perturbative terms in \(6.8\) are contributions of D-instantons (and anti-D-instantons) of arbitrary charge \(K\) together with an infinite sequence of perturbative fluctuations around each instanton configuration.

The D-instanton sum has a simple origin from the point of view of the loop integrals considered in section 5. There, the circulating particle can be considered to be a Kaluza–Klein mode of charge \(n\) of M theory compactified on the first circle of circumference \(R_{11}\). For \(n \neq 0\) this is a charge-\(n\) D-particle. The world-line of this D-particle winds \(\hat{m}\) times around the second circle of radius \(r_A\) (in IIA string units). The action for such a configuration is given by the Dirac–Born–Infeld action for the D-particle and is equal to

\[
2\pi|\hat{m}|(r_A e^{\phi_A} \mp iC^{(1)})
\]

(where the \(\mp\) distinguishes instantons and anti-instantons). After performing a T-duality on the euclidean circle \([11]\), the real part of this expression is identified with the charge-\(K\) D-instanton action,

\[
S_K=\hat{m} = 2\pi|K| e^{\phi_B}
\]

(6.10)

and the \(R \otimes R\) potential \(C^{(1)}\) is identified with \(C^{(0)}\).

From \(6.5\) we see that the leading contribution to the \(N\) D-instanton contribution in \(F_{K,w}(\tau, \bar{\tau})\) is

\[
\tau_2^{\frac{1}{2}} F_{K,w} \sim Z_K S_K^{-7/2+p} e^{-\left(S_K - 2\pi i KC^{(0)}\right)} \left(1 + o(K e^\phi)\right),
\]

(6.11)

where \(p = w + 4\) and we have factored out the measure factor \(Z_K\) \([6.7]\), which depends on \(K\) but not on \(S_K\) and is normalized so that \(Z_1 = 1\). The power \(-7/2\) in \(6.11\) arises from the combination of ten bosonic zero modes and sixteen fermionic zero modes while the factor of \(S_K\) for each external state comes from the normalization of the external states.

### 7 D-particle bound states and the Witten index

One of the central planks of the web of dualities that relate string theory and M theory is the identification of D-particles (D0-branes) of type IIA superstring theory with Kaluza–Klein modes of eleven-dimensional supergravity compactified on \(S^1\) \([4]\). There is a single Kaluza–Klein mode of a given charge, \(N\), with mass \(N\) times the mass of charge-one D-particle. This implies that there must be precisely one threshold bound state (a bound state with zero binding energy) of \(N\) D-particles. The interaction of \(N\) D-particles is described by the
one-dimensional matrix model' — a quantum mechanical system that is isomorphic to the dimensional reduction of ten-dimensional $U(N)$ supersymmetric Yang-Mills theory to one (time) dimension \[30\]. It is notoriously difficult to analyze the possible threshold bound states of this theory. One approach used in \[32, 33\] in the case $N = 2$ is to demonstrate that the Witten index is equal to one. This shows that there is at least one bound state. Here we will show how the results of the last section indicate that the Witten index is equal to one for arbitrary $N$ (this section is based on \[31\]) \[14\].

### 7.1 D-particles and D-instantons

The Witten index for the system of $N$ D-particles was defined in \[32, 33\] by

$$I(N) = \int d^9x \lim_{\beta \to \infty} \text{tr}(-1)^F e^{-\beta H}(x,x)$$

$$= \lim_{R \to \infty} \lim_{\beta \to 0} \left\{ \int_{|x| < R} d^9x \text{tr}(-1)^F e^{-\beta H} + \frac{1}{2} \int_{|x| = R} d^9x \int_{\beta}^{\infty} d\beta' \text{tr}e_n(-1)^F Q e^{-\beta' H} \right\},$$

(7.2)

where the first term defines a bulk contribution $I_{bulk}^{(N)}$ (which in general is not an integer) and the second term a deficit contribution $I_{def}^{(N)}$ (which corrects the bulk term). The separation of the integer $I(N)$ into these two parts is dependent on the order of limits. We shall always consider the infinite volume limit to be taken before the limit $\beta \to 0$.

The trace in (7.2) is defined over gauge invariant states of the Hilbert space, which may be written as a trace over all states if a projector onto gauge invariant states (states satisfying the constraints of Gauss' law, $C| = 0$) is inserted. For the bulk term this gives

$$I_{bulk}^{(N)} = \frac{1}{\text{Vol}(SU(N)/Z_N)} \lim_{\beta \to 0} \int_{SU(N)} d\eta \int d^9x \text{tr}(-1)^F e^{i\eta^a C} e^{-\beta H}(x,x),$$

(7.3)

where $\eta^a$ is the gauge parameter. Using the heat kernel approximation for the propagator in the hamiltonian $H$, which is valid in the limit $\beta \to 0$, it was observed in \[32\] and \[33\] that the exponent of the bulk part of the index can be written in a $SO(10)$ invariant form. This is possible because the parameter $\eta$ turns into a tenth bosonic coordinate, $x^9$.

The resulting integrals are seen to be equivalent to the $SU(N)$ integral in the partition function for the zero-dimensional matrix model. This is the model

\[14\] See \[67\] for a very different approach to the threshold bound state problem.
that is obtained by compactification of supersymmetric Yang–Mills theory from ten (euclidean) dimensions to zero dimensions that describes dynamics in the presence of $N$ D-instantons [30, 68, 69]. The situation with D-instantons is somewhat distinct from that of other D$p$-branes since $p = -1$ and the world-volume reduces to a space-time point. The configuration space of $N$ D-instantons is determined by bosonic and fermionic $U(N)$ matrices $A_\mu$ and $\psi_\alpha$, where $\mu$ and $\alpha$ denote $SO(10)$ vector and spinor indices, respectively. The reduction of the supersymmetric Yang–Mills action to a point is given by

$$S = \frac{1}{4} \text{tr}([A_\mu, A_\nu]^2) + \frac{i}{2} \text{tr}(\bar{\psi} \Gamma^\mu [A_\mu, \psi]).$$  \tag{7.4}$$

The ‘center of mass’ degrees of freedom are associated with the element of the Cartan subalgebra of $U(N)$ proportional to the unit matrix and do not appear in the action (7.4). They do, however, enter as shift symmetries in the supersymmetry transformations,

$$\delta A_\mu = i \bar{\eta} \Gamma_\mu \psi,$$

$$\delta \psi = [A_\mu, A_\nu] \Gamma^{\mu\nu} \eta + \epsilon.$$  \tag{7.5}$$

The sixteen-dimensional spinor $\eta$ parameterizes the dimensionally reduced supersymmetry of the $SU(N)$ YM theory and the spinor $\epsilon$ acts as a constant shift on the fermions in the ‘center of mass’ degrees of freedom. Hence the $U(1)$ part of $U(N)$ fermionic fields $\psi$ plays the role of the sixteen fermionic collective coordinates associated with the charge-$N$ D-instanton.

After factoring out the centre of mass coordinates the $N$ D-instanton partition function is given by

$$Z = \int d^{10} y \int d^{16} \epsilon Z_N,$$  \tag{7.6}$$

where

$$Z_N = \frac{1}{\text{Vol}(SU(N)/Z_N)} \int_{SU(N)} d\psi dA \exp(-S_{YM}[A, \psi]).$$  \tag{7.7}$$

The integration over $y^\mu$ and $\epsilon^A$ is the integral over the overall bosonic and fermionic collective coordinates of the collection of D-instantons of total charge $N$. The integration measure in (7.7) is the group invariant Haar measure which defines the appropriate normalization (as discussed in [70, 71]). A non-zero value for $Z_N$ can only arise from configurations in which there are no extra fermionic zero modes (in addition to $\epsilon$) which is characteristic of a single multiply-charged D-instanton. According to [30] the partition function of the $SU(N)$ zero-dimensional matrix model should be identified with the D-instanton measure, $Z_N$ [6.7]. The integral (7.7) was evaluated for the case $N = 2$ in
The sub-integral over configurations in which the elements of the Cartan subalgebra, $A^\mu, \psi^3$, are fixed was evaluated in \cite{72} and corresponds to two D-instantons at a fixed separation. More recently, the integral has been explicitly evaluated for general $N$ by deforming the integrand into a topological density \cite{73} and numerical estimates have been made in \cite{70}.

The fact that the integral (7.7) is the same as the bulk integral which appears in the calculation of the Witten index in the bound state problem of D-particles is the statement of T-duality. The radius $\beta$ of the euclidean circle in (7.3) shrinks to zero size and the insertion of $(-1)^F$ enforces supersymmetric (periodic) boundary conditions on the fermionic fields. The conjectured form for the measure, $Z_N$ given by (6.7) therefore suggests that the value of the bulk integral is given, for arbitrary $N$, by

$$I_{\text{bulk}}(N) = \sum_{\hat{m} | N} \frac{1}{\hat{m}^2} = Z_N,$$

where an overall constant has been fixed by noting that $I_{\text{bulk}}^{(1)} = 1$. Note that for $N = 2$ the value of $I_{\text{bulk}}^{(2)} = 5/4$ in (7.8) agrees with the explicit calculation of \cite{32, 33}.

### 7.2 The deficit term

The appearance of a deficit term is related to the presence of the boundary terms in (7.2). For the $SU(2)$ case a heuristic evaluation of this term was given in \cite{32} and a more rigorous argument in \cite{33}. It was argued that since the boundary term for the system of two D-particles arises from the region in which the particles are separated it is obtained by treating the particles as identical free particles with moduli space $R^9/Z_2$. Since the Witten index vanishes for free particles the deficit term, $I_{\text{def}}^{(2)}$, must cancel the bulk term for the free system, $I_{\text{bulk}}^{(2)}$, where the subscript 0 indicates the free theory. But $I_{\text{def}}^{(2)}$ is easily evaluated and is equal to 1/4. In the following \cite{31} we shall assume that this prescription generalizes to $N > 2$ D-particles so that, for prime values of $N$,

$$I_{\text{def}}^{(N)} = -I_{\text{bulk}}^{(N)},$$

where $I_{\text{bulk}}^{(N)}$ is the bulk index for $N$ identical free particles moving on $R^{9(N-1)}/S_N$. For non-prime values, $N = \hat{m} \hat{n}$, the generalization will take into account the regions of moduli space of $\hat{m}$ free charge-$\hat{n}$ particles on $R^{9(\hat{m}-1)}/S_{\hat{m}}$.

In the $N = 2$ case considered in \cite{32, 33} the only configuration that contributed to the trace over the free two-particle states was one with an odd permutation of the two particles. If the trace is expressed as a functional integral
this includes only configurations in which the two D-particles are described by a single euclidean world-line that winds twice around the compact $\beta$ direction. In order to generalize this to arbitrary $N$ we need to consider the action of $S_N$, which is the Weyl group of $SU(N)$. This can be parameterized by matrices $M_{ab}$ acting on the positions of the $N$ particles modded out by the overall translation invariance, $X^i_a$ ($i = 1, \ldots, 9$), and the fermions $\psi^a_\alpha$. The index $a = 1, \ldots, N - 1$ labels the different $U(1)$’s in the Cartan subalgebra. The action of an element in $S_N$ is given by

$$X^i_a \rightarrow M^b_a X^i_b, \quad \psi^a_\alpha \rightarrow M^b_a \psi^b_\alpha,$$

(7.10)

where the vector index runs over $i = 1, \ldots, 9$ and the spinor index $\alpha = 1, \ldots, 16$.

The fermion fields $\psi$ satisfy the following anticommutation relations,

$$\{\psi^a_\alpha, \psi^\beta_b\} = \delta_{ab} \delta^\alpha_\beta.$$

(7.11)

It is convenient to build up the fermionic Hilbert space by defining fermionic creation and annihilation operators $\hat{\psi}^a_\alpha, \hat{\psi}^{\dagger a}_\alpha$ by

$$\hat{\psi}^a_\alpha = \frac{1}{\sqrt{2}} (\psi^a_\alpha - i \psi^a_{\alpha - 1}), \quad \hat{\psi}^{\dagger a}_\alpha = \frac{1}{\sqrt{2}} (\psi^a_\alpha + i \psi^a_{\alpha - 1}), \quad \alpha = 1, \ldots, 8,$$

(7.12)

which satisfy $\{\hat{\psi}^{\dagger a}_\alpha, \hat{\psi}^b_\beta\} = \delta_{ab} \delta^\alpha_\beta$. A general wave function $|\Psi\rangle$ can be expanded as

$$|\Psi\rangle = \left(\Psi^{(0)}(X_a) + \Psi^{(1)}(X_a) \hat{\psi}^{\dagger a}_1 + \frac{1}{2} \Psi^{(2)}(X_a) \hat{\psi}^{\dagger a}_1 \hat{\psi}^{\dagger a}_2 + \cdots \right) |0\rangle,$$

(7.13)

where the vector and spinor indices are suppressed and the highest term has $8 \times N$ fermionic creation operators acting on the Fock space vacuum.

The expression for the bulk contribution to the Witten index for $N$ free D-particles (with $N$ a prime) is given by

$$I^{(N)}_{0\ bulk} = \lim_{\beta \rightarrow 0} \tr(-1)^F e^{-\beta H}.$$

(7.14)

The trace is taken over gauge invariant states in the Hilbert space and hence one has to insert a projector on states which are invariant under the Weyl permutation group,

$$\mathcal{P} = \frac{1}{N!} \sum_{\pi \in S_N} M_\pi.$$

(7.15)

The matrix $M_\pi$ representing $\pi \in S_N$ act as in (7.10) on the coordinates $X_a$ and the fermions $\hat{\psi}_a$ in the wave function $\Psi$ given in (7.13). The bulk index for the free theory is therefore given by

$$I^{(N)}_{0\ bulk} = \lim_{\beta \rightarrow 0} <\Psi|(-1)^F e^{-\beta H} \mathcal{P}|\Psi>.$$

(7.16)
The factor of \((-1)^F\) counts bosons (even number of \(\hat{\psi}\)) with +1 and fermions (odd number of \(\hat{\psi}\)) with −1. Because the fermions transform under the symmetric group matrices \(M\), the introduction of the projector \(P\) can give a non-vanishing contribution.

The action of \(M_\pi\) on \(\hat{\psi}_a\) and \(X_a\) in (7.16) factorizes into a trace over the Hilbert space built from the fermionic creation operators and a bosonic gaussian integral coming from the heat kernel approximation for the free propagator in the limit \(\beta \to 0\) [32].

\[
\mathcal{I}^{(N)}_{0\text{bulk}} = \lim_{\beta \to 0} \frac{1}{N!} \sum_{\pi \in S_N} \text{tr}_\psi \left( (-1)^F M_\pi \right) \int \prod_{i=1}^{N-1} \prod_{a=1}^{N-1} dX_a^{i} \frac{e^{-\frac{(X - M_\pi X)^2}{2\beta}}}{(2\pi \beta)^{(N-1)g/2}}. \tag{7.17}
\]

For all values of \(\alpha\) the fermionic trace is given by

\[
\text{tr}_\psi \left( (-1)^F M_\pi \right) = \langle 0 | 0 \rangle - \sum_a \langle 0 | \hat{\psi}_a M_\pi ab \hat{\psi}_b^\dagger | 0 \rangle + \cdots
\]

\[
= 1 - \text{tr}(M_\pi) + \frac{1}{2} \text{tr}(M_\pi^2) - \frac{1}{2} \text{tr}(M_\pi)^2 + \cdots \tag{7.18}
\]

so that the trace over all eight fermion components gives a factor of \(\det(1 - M_\pi)^8\).

The gaussian integration over the coordinates \(X_a^{i}\) similarly gives a factor of \(\det(1 - M_\pi)^{-9}\).

The determinant is easily evaluated by using an explicit representation for the matrices \(M_\pi\). Recall that these represent the action of \(S_N\) on the elements of the Cartan subalgebra. The roots of \(SU(N)\) are the vectors in \(R^N\),

\[
e_i - e_j, \quad i \neq j; \quad i, j = 1, \cdots, N, \tag{7.19}
\]

where \(e_i\) the \(i\)th unit vector of \(R^N\). All the roots lie in a \((N-1)\)-dimensional subspace \(\mathcal{R}_\omega\) orthogonal to the vector \(\omega = e_1 + e_2 + \cdots + e_N = (1,1,\cdots,1)\) and an element \(\pi \in S_N\) acts as a permutation, \(\pi : e_i \to e_{\pi(i)}\).

Two classes of permutations need to be distinguished:

(a) Cyclic permutations

These can be represented up to conjugation by the \(N \times N\) matrix

\[
M_c = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \tag{7.20}
\]
which has eigenvalues

$$\lambda_k = e^{\frac{2\pi i k}{N}}, \quad k = 0, 1, \cdots, N - 1. \quad (7.21)$$

The matrix $M_c$ does not leave any of the roots (7.19) invariant and $\omega$ is the unique eigenvector with eigenvalue $\lambda_0 = 1$. We are interested in evaluating $\det(1 - M_\pi) = \det'(1 - M_c)$, where the prime indicates the omission of the zero eigenvalue. This is the determinant in the space orthogonal to $\omega$ and is given by the product of non-zero eigenvalues,

$$\det'(1 - M_c) = \prod_{k=1}^{N-1} (1 - \lambda_k) = 2^{N-1} \prod_{k=1}^{N-1} \sin \frac{\pi k}{N} = N. \quad (7.22)$$

(b) Non-cyclic permutations

These can be represented by

$$M_{nc} = \begin{pmatrix}
M_1 & 0 & 0 & \cdots & 0 \\
0 & M_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & M_p
\end{pmatrix} \quad (7.23)$$

where $M_1, M_2, \cdots, M_p$ represent cyclic permutations of subsets of elements and can be written in the form of (7.20). The matrix $M_{nc}$ has $p > 1$ unit eigenvalues so there are $p - 1$ eigenvectors $w_j \in R_\omega$ for which $(1 - M_{nc})w_j = 0$ ($j = 1, \cdots, p - 1$). As a result, some elements of the Cartan subalgebra are left invariant by $M_\pi$ and

$$\det(1 - M_\pi) = \det'(1 - M_{nc}) = 0, \quad (7.24)$$

where the prime again indicates the omission of the zero eigenvalue associated with $\omega$.

To be more precise, a zero eigenvalue of the bosonic determinant $\det(1 - M_{nc})$ should be interpreted as the inverse volume, $R^{-1}$ in the limit $R \to \infty$, whereas the fermionic determinant vanishes identically. This means that only the cyclic permutations contribute to the index and the expression (7.17) reduces to

$$I_{0 \text{ bulk}}^{(N)} = \sum_{c \in S_N} \frac{1}{N!}(\det'(1 - M_c))^{-1}. \quad (7.25)$$

The non-vanishing determinant $\det'(1 - M_c) = N$ simply counts the winding number of a fundamental D-particle world-line and the result is the obvious generalization of the $N = 2$ case. The non-cyclic permutations do not contribute to the index but have the interpretation of $p$ disconnected D-particle world-lines
winding around the euclidean circle. This is a multi D-instanton configuration and the vanishing of the determinant is due to the appearance of extra fermionic zero modes. Since each of the \((N - 1)!\) cyclic elements of \(S_N\) gives the same contribution to (7.25) the result is

\[
I_{0}^{(N)}_{\text{bulk}} = \frac{1}{N^2} = - I_{\text{def}}^{(N)}. \tag{7.26}
\]

Combining this with our earlier ansatz that \(I_{\text{bulk}}^{(N)} = Z_N = 1 + 1/N^2\) leads to the conclusion that

\[
I^{(N)} = I_{\text{bulk}}^{(N)} + I_{\text{def}}^{(N)} = 1, \tag{7.27}
\]

for prime values of \(N\). This is consistent with the presence of at least one bound state for every prime value of \(N\).

Now consider the non-prime charge sector in which \(N = \hat{m}n\) with \(\hat{m}, n > 1\). It is useful to recall the interpretation of the D-instanton of charge \(N\) as a single wrapped euclidean D-particle world-line. The contributions to the instanton measure labelled by \(n\) were associated with the world-line of a charge-\(n\) D-particle winding \(\hat{m} = N/n\) times. In the case when \(N\) is prime the only contribution to the deficit term comes from \(n = 1, \hat{m} = N\), which corresponds to \(\hat{m}\) windings of a charge-\(n = 1\) D-particle world-line. The other possibility, \(n = N, \hat{m} = 1\), gives the net contribution of one to the Witten index of the threshold bound state. It is therefore very compelling to assume that the contribution to the Witten index in the non-prime case, \(N = \hat{m}n\), of a subset of \(\hat{m}\) D-particles of charge \(n\) can be calculated in the same way as that of \(N\) fundamental \((n = 1)\) D-particles. In other words, we assume a generalization of the property that was strongly motivated in the \(N = 2\) case in \([32, 33]\) that the deficit term is determined by free supersymmetric particle dynamics with 16 supercharges. This means that the space of free-particle states that enters in the calculation of \(I_{0}^{\text{bulk}}\) should be enlarged to include the infinite tower of charge-\(n\) threshold bound states. In that case a cyclic permutation of \(\hat{m}\) identical charge-\(n\) particles contributes a term

\[
I_{0}^{(\hat{m})}_{\text{bulk}} = \frac{1}{\hat{m}^2}, \tag{7.28}
\]

which leads to a total deficit in the charge-\(N\) sector of

\[
I_{\text{def}}^{(N)} = - \sum_{\substack{m|N \\hat{m} > 1}} I_{0}^{(\hat{m})}_{\text{bulk}} = - \sum_{\substack{m|N \\hat{m} > 1}} \frac{1}{\hat{m}^2}, \tag{7.29}
\]

consistent with the Witten index \(I^{(N)} = 1\) for all integer \(N\).

The success of these assumptions points to the systematics that needs to emerge from a more precise treatment of the integration over the boundary.
term when \( N \) is not prime. This suggests that there are regions of moduli space in which the non-abelian integration leads to \( \hat{m} \) charge-\( n \) threshold bound states which behave as free particles. Related issues arise in the considerations of \([19, 20] \) concerning D-string instantons in \( d = 8 \) type I string theories.

The systematics of the ten-dimensional D-particle problem is consistent with the expected features of D-particle quantum mechanics with less supersymmetry. The clearest examples are in six and four dimensions. The six-dimensional case is realized in the compactification of IIA on \( K3 \) by wrapping D2-branes around a two-cycle in \( K3 \) and scaling to the limit in which the two-cycle vanishes. Using duality with the heterotic string on \( T^4 \) identifies the resulting D-particles as massless gauge particles which have no threshold bound states. The four-dimensional case is realized by wrapping the type IIB D3-brane around a three-cycle in a Calabi–Yau threefold and again scaling to the limit of zero area as in \([74]\). The successful resolution of the conifold singularity in terms of light states again requires the absence of threshold bound states. The absence of threshold bound states in these cases implies that the Witten index should vanish. In both cases the bulk term in the N-particle index can be evaluated explicitly (as in \([73]\) or extracted from the T-dual D-instanton problem (as in \([70]\) and has the form

\[
I_{\text{bulk}}^{(N) D=4,6} = \frac{1}{N^2}.
\]  

Furthermore, it is easy to see that in these cases the deficit term is given by

\[
I_{\text{def}}^{(N) D=4,6} = -1/N^2 \quad \text{(assuming, as before, that it is given by the free-particle index)}.
\]

The extra factor of \( N \) here compared to (7.26) comes about because in these lower-dimensional cases the number of fermions is equal to the number of bosons. This systematics suggests that

\[
I^{(N) D=4,6} = I_{\text{bulk}}^{(N) D=4,6} + I_{\text{def}}^{(N) D=4,6} = 0,
\]

as expected.

8 The AdS/CFT correspondence and higher derivative terms

Up to now the discussion of the higher derivative terms has been in the context of the low energy effective action of string theory or M theory compactified on a circle or two-torus. We now turn to consider the effect of the higher derivative terms when the IIB string propagates in an \( AdS_5 \times S^5 \) background (much of this section and section 9 is based on \([73, 74]\)). For our purposes it will be sufficient
to consider the euclidean version of $AdS_5$, which has a boundary with topology $S^4$ (or $R^4$). The metric will be parameterized by

\[ ds^2 = \frac{L^2}{\rho^2} (dx \cdot dx + d\rho^2) + d\omega_5^2 = \frac{L^2}{\rho^2} (dx \cdot dx + dy \cdot dy), \]

(8.1)

where $(x^\mu, y^i)$ $(\mu = 0, \cdots, 3, i = 1, \cdots, 6)$ are ten-dimensional cartesian coordinates with $\rho^2 = y^2$ and $d\omega_5^2$ is the spherically-symmetric constant curvature metric on $S^5$ with radius $L$. In these coordinates the $AdS$ boundary is located at $\rho = 0$. The second equality in (8.1) makes it obvious that the metric is conformally equivalent to the flat ten-dimensional metric. This will prove to be important later. The only nonvanishing curvature components in this background are

\[ R_{MNPQ} = -\frac{1}{L^2} (g_{MP} g_{NQ} - g_{MQ} g_{NP}) \quad R_{mn} = +\frac{1}{L^2} (g_{mp} g_{nq} - g_{mq} g_{np}) \]

(8.2)

(upper case Latin indices, $M, N, \ldots = 0, 1, 2, 3, 5$, label the $AdS_5$ coordinates and lower case Latin indices, $m, n, \ldots = 1, 2, 3, 4, 5$ label the $S^5$ coordinates). The non-vanishing components of the Ricci tensor are

\[ R_{MN} = -\frac{4}{L^2} g_{MN} \quad R_{mn} = +\frac{4}{L^2} g_{mn}, \]

(8.3)

so that the scalar curvature vanishes. The dilaton and $C^{(0)}$ are arbitrary constants and the only other field that is nonvanishing in the $AdS_5 \times S^5$ background is $F_5$, which has the solution,

\[ F_{MNPQR} = \frac{1}{L^3} \varepsilon_{MNPQR}, \quad F_{mnpqr} = \frac{1}{L} \varepsilon_{mnpqr}, \]

(8.4)

which corresponds to a configuration in which the total $\mathcal{R} \otimes \mathcal{R}$ charge associated with $F_5$ is equal to $\alpha'^{-2} L^4 e^{-\phi}$, which must be an integer, $N$. The isometry of this space is the product $SO(4,2) \times SU(4)$. These factors package together with fermionic symmetries into the supergroup $OSp(4,2|4)$ which also acts as a superconformal symmetry group on the boundary.

This background is maximally supersymmetric (just like the Minkowski vacuum) and there are 32 conserved supercharges that transform as a complex chiral spinor of the tangent-space group, $SO(4,1) \times SO(5)$. In the basis where the ten dimensional $\Gamma_\Lambda$ matrices\[\]
are given by $\Gamma_M = \sigma_1 \otimes \gamma_M \otimes I$ and $\Gamma_m = \sigma_2 \otimes I \otimes \gamma_m$, the supersymmetries are generated by the Killing spinors that satisfy

\[ D_\Lambda \epsilon - \frac{1}{2L} (\sigma_1 \otimes I \otimes I) \Gamma_\Lambda \epsilon = 0, \]

(8.5)

\[\]

In this section and the next the ten-dimensional gamma matrices will be denoted by upper case $\Gamma$ while lower case $\gamma$ will be reserved for four-dimensional gamma matrices.
which follows from the requirement that the gravitino supersymmetry transformation should vanish. In this basis the complex chiral supersymmetry parameter reads
\[ \epsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \zeta \otimes \kappa, \]
(8.6)
where \( \zeta \) is a complex four-component \( SO(4,1) \) spinor and \( \kappa \) is a complex four-component \( SO(5) \) spinor. The Killing spinor equation (8.5) has components
\[ D_M \zeta_{\pm} - \frac{1}{2L} \gamma_M \zeta_{\pm} = 0, \]
(8.7)
\[ D_m \kappa_{\pm} + \frac{1}{2L} \gamma_m \kappa_{\pm} = 0. \]
(8.8)
The Killing spinors \( \kappa_{\pm} \) on \( S^5 \) may be used to construct the Kaluza-Klein excitations of all the fields in the IIB gauged supergravity starting from the modes of the massless complex singlet scalar. We will later make use of the Euclidean continuation of the Killing spinors on \( AdS_5 \), which requires strategically placed factors of \( i \).

The metric (8.1) can be obtained from the geometry of a stack of coincident D3-branes, where each D3-brane carries a unit charge associated with \( F_5 = dC^4 \). The effective world-volume theory of \( N \) coincident D3-branes is described by \( U(N) \) gauge theory, where the integer \( N \) is again the total \( R \otimes R \) charge. In terms of the low energy supergravity theory the classical D3-brane geometry interpolates from the Minkowski region far away from the horizon to the near-horizon geometry, which is \( AdS_5 \times S^5 \). The transition region grows with \( N \) and the \( N \to \infty \) limit considered in [77, 34] is one in which the horizon region fills all space. This leads to the AdS/CFT conjecture [34] which suggests that the bulk string theory is equivalent to superconformal \( \mathcal{N} = 4 \) SU(\( N \)) Yang–Mills theory on the boundary [6] (see also [35, 36]). The supergroup \( OSp(4,2|4) \) is that associated with the superconformal \( \mathcal{N} = 4 \) Yang–Mills theory.

According to this idea the effective action of type IIB supergravity, evaluated on a solution of the equations of motion with prescribed boundary conditions, is equated with the generating functional of connected gauge-invariant correlation functions in the Yang–Mills theory. The parameters of the \( \mathcal{N} = 4 \) Yang–Mills theory and the IIB superstring on \( AdS_5 \times S^5 \) are related by the dictionary,
\[ g = \frac{g^2_{Yd}}{4\pi}, \quad 2\pi \hat{C}^{(0)} = \theta_{YM}, \quad \frac{L^2}{\alpha'} = \sqrt{g^2_{YM}/N}, \]
(8.9)
where \( g = e^\phi \) and \( \hat{C}^{(0)} \) are the arbitrary constant scalar string fields, \( g_{YM} \) is the Yang-Mills coupling and \( \theta_{YM} \) is the vacuum angle. The complex Yang–Mills

\[ \text{It has been argued in [30, 32] that the theory should be } SU(N) \text{ and not } U(N). \]
coupling is therefore identified with the constant boundary value of the complex scalar field of the IIB superstring,

$$S = \frac{\theta_{\nu M}}{2\pi} + \frac{4\pi i}{g_{\nu M}^2} = \tilde{C}(0) + \frac{i}{g}.$$  

This explicit connection between the bulk theory and the boundary theory can be expressed symbolically in terms of the generating functions \[36, 35, 79\],

$$\exp(-S_{IIB}[\Phi_m(J)]) = \int DA \exp(-S_{YM}[A] + O_\Delta[A]J).$$  

The left-hand side is the generating function for the amplitudes in IIB superstring theory or its low energy supergravity limit. The effective action \(S_{IIB}\) is evaluated in terms of the ‘massless’ supergravity fields and their Kaluza–Klein descendents, that we have generically indicated with \(\Phi(z; \omega)\), where \(\omega\) are the coordinates on \(S^5\) and \(z^M \equiv (x^\mu, \rho)\) \((M = 0, 1, 2, 3, 5\) and \(\mu = 0, 1, 2, 3)\) are the AdS\(5\) coordinates. The notation in \(8.11\) indicates that this action depends on the boundary data, \(J(x)\), of the bulk fields. The right-hand side of \(8.11\) is the generating function of the correlation functions of the gauge-invariant composite operators, \(O(A)\), to which \(J\) couples in the boundary \(\mathcal{N} = 4\) supersymmetric Yang–Mills, where the fluctuating Yang–Mills potential is denoted by \(A\).

For general values of the dimensionless ratio, \(\alpha'/L^2\), an expansion in powers of \(\alpha'\) is not necessarily a good approximation to the theory. Only for \(AdS_5\) scales that are large compared to the string scale, i.e. \(\alpha' \ll L^2\), is the curvature small and the \(\alpha'\) expansion can still be interpreted as a derivative expansion in the nontrivial background. It follows from \(8.9\) that in this regime classical supergravity is related to the Yang–Mills limit in which \(g_{YM}^2 N \rightarrow \infty\). This requires \(N \rightarrow \infty\) (since \(SL(2, \mathbb{Z})\) duality can always be used to map the large string coupling to a value that is not large). Since, according to 't Hooft \[80\] the quantity \(g_{YM}^2 N\) is the coupling constant of large-\(N\) Yang–Mills theory we see that the AdS/CFT conjecture relates the strong coupling limit of one theory to the weak coupling limit of the other. Several tests of this conjecture have been made at the level of the two-point and three-point correlations of currents \[81\] based on the semiclassical approximation to the bulk supergravity \((g << 1)\) which is valid in the limit \(\alpha'/L^2 << 1\), which corresponds to the large-\(N\) limit with \(g_{YM}^2 N\) fixed at a large value.

The spectrum of IIB supergravity compactified on \(AdS_5 \times S^5\) is described in \[82\]. Here we are most interested in the identification of the lowest lying modes that form the supergravity supermultiplet. While the massless dilaton is associated with the constant mode on \(S^5\), i.e. with the scalar spherical harmonic \(Y_\ell\) with \(\ell = 0\), the other scalars in the supermultiplet are associated with excitations on the 5-sphere. In particular the real scalars \(Q^{ij}\) \((i, j, k = 1, 2, \ldots, 6)\),
with mass $m^2 = -4/L^2$ in the $20_R$ of the $SO(6)$ isometry group of $S^5$ result from a combination of the trace of the internal metric and the self-dual $R \otimes R$ five-form field, $F^{(5)}$, with $\ell = 2$ ($Q^{ij}$ are quadrupole moments of $S^5$). Similarly the complex scalars $E^{AB}$ with mass $m^2 = -3/L^2$ and their conjugates (in the $10$) are associated with the pure two-form fluctuations with $\ell = 1$ of the complexified antisymmetric tensor in the internal directions. The $15$ massless vectors $V^{(i)}_M$ that gauge the $SO(6)$ isometry group are in one-to-one correspondence with the Killing vectors of $S^5$ and result from a linear combination with $\ell = 1$ of the mixed components of the metric and the internal three-form components of the $R \otimes R$ four-form potential, $C^{(4)}$. The $6$ complex antisymmetric tensors $B^{ABM}$ with $m^2 = 1/L^2$, which have first order equations of motion, result from scalar spherical harmonics with $\ell = 1$. The analysis of the fermions is similar. The $4$ dilatini $\Lambda^A$ with mass $m = -3/(2L)$ are proportional to the internal Killing spinors $\kappa^+$. The $20_c$ spinors $\chi^{A}_{BC}$ with mass $m = -1/(2L)$ correspond to internal components of the gravitino with $\ell = 1$. Finally the supergravity multiplet is completed by the massless $4^*$ gravitinos $\Psi_{MA}$ which are proportional to the internal Killing spinors $\kappa^-$. The above fields are those that act as sources for the superconformal currents in the boundary Yang–Mills theory. The higher Kaluza–Klein modes on $S^5$, which have higher values of $\ell$, are bulk fields of dimension $\Delta = \ell$. The boundary value of any of these fields couples to a composite gauge invariant operator in the Yang–Mills theory that can be expressed as a rank $l$ symmetric traceless tensor of $SO(6)$ (the isometry group of $S^5$) made from a product of $l$ superfields of the boundary theory. These operators will not concern us in the following but are a vital part of the complete story.

Whereas the duality group for the type IIB theory compactified on $T^6$ is $E_{6(+6)}$, the duality group for the $AdS_5 \times S^5$ background is $SL(2, Z)$, which is inherited from ten dimensions. The local symmetry group is $U(4)$ where the $U(1)$ factor is a remnant of the local symmetry of flat ten-dimensional classical supergravity. However, there is no corresponding $U(1)$ symmetry in the $\mathcal{N} = 4$ Yang–Mills boundary theory where the R-symmetry is $SU(4)$. This fits with the fact that in IIB string theory the classical $SL(2, \mathbb{R})$ is replaced by (local) $SL(2, \mathbb{Z})$ and the continuous $U(1)$ symmetry is not present.

### 8.1 $\mathcal{N} = 4$ Yang–Mills fields and supercurrents

The $\mathcal{N} = 4$ four-dimensional Yang–Mills theory is classically invariant under superconformal transformations as well as under global $SU(4)$ transformations,
which form the R-symmetry group of automorphisms of the \( N = 4 \) supersymmetry algebra. The physical fields comprise six real scalars, four Weyl spinors, and one gauge vector potential, all in the adjoint representation of the gauge group, which may be arbitrary in general but should be \( SU(N) \) for the most straightforward application of the AdS/CFT correspondence. The real scalars, \( \varphi^i \) \((i = 1, \ldots, 6)\), form a 6 of the \( SU(4) \) R symmetry group and can be written as antisymmetric tensors in the defining representation of \( SU(4) \),

\[
\varphi^i_{AB} = \frac{1}{2} \xi_{ABCD} \varphi^{CD} \quad \text{with} \quad \varphi^{AB} = -\varphi^{BA}.
\]

The spinors, \( \lambda^A, \bar{\lambda}^B \), transform as 4 and \( 4^\ast \), respectively, and the vector \( A_\mu \) is a singlet of \( SU(4) \). The classical Minkowski signature action is given by

\[
S = \int d^4x \, \text{Tr} \left\{ (D_\mu \varphi^{AB})(D^\mu \varphi_{AB}) - \frac{1}{2} i(\lambda^A \delta \gamma \cdot D_{\alpha\dot{\alpha}} \bar{\lambda}_A) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} - g_{YM} (\partial_\mu \varphi^{AB}) [\varphi^{BC}, \varphi^{DA}] + 2 g_{YM}^2 [\varphi^{AB}, \varphi^{CD}] [\varphi_{AB}, \varphi_{CD}],
\]

where \( \text{Tr} \) denotes a trace over the \( SU(N) \) colour indices and \( D_{\mu} = \delta_{\mu} - i g_{YM} f^{abc} A_{\mu c} \) is the covariant derivative in the adjoint representation. We are here using a standard chiral basis for the spinors with the \( 2 \times 2 \) Pauli matrices \( \sigma^{\mu} \) coupling the complex chiral spinors.

The supersymmetry transformations on the component fields with parameters \( \eta^A_\alpha \) and \( \bar{\eta}^A_{\dot{\alpha}} \) are given by,

\[
\delta \varphi^{AB} = \frac{1}{2} \lambda^A \eta^A_\alpha - \lambda^B \eta^B_\alpha + \frac{1}{2} \xi_{ABCD} \eta^C_D \bar{\lambda}_A, \\
\delta \lambda^A = -\frac{1}{2} F_{\mu\nu} \sigma^{\mu\nu} = 4 i (\lambda^A \delta \gamma \cdot D_{\alpha\dot{\alpha}} \bar{\lambda}_A) - 8 g_{YM} [\varphi^{AB}, \varphi^{CD}] [\varphi_{AB}, \varphi_{CD}], \\
\delta A_\mu = -i \lambda^A \sigma^{\mu}_{\alpha\dot{\alpha}} \bar{\eta}^A_{\dot{\alpha}} - i \eta^A_\alpha \sigma^{\mu}_{\alpha\dot{\alpha}} \bar{\lambda}_A.
\]

These are ordinary global supersymmetries when the 16 components of \( \eta^A_\alpha \) and \( \bar{\eta}^A_{\dot{\alpha}} \) are constant but the symmetry extends to the thirty-two component superconformal symmetry when \( \eta \) and \( \bar{\eta} \) are linear functions of \( x^\mu \),

\[
\eta^A_\beta \rightarrow \frac{1}{\sqrt{\rho_0}} (\rho_0 \eta^A_\beta + (x - x_0)_\mu \sigma^{\mu}_{\beta\dot{\beta}} \bar{\zeta}_{\beta A}) \equiv \left( \frac{1 - \gamma^5}{2} \right) \frac{1}{\sqrt{\rho_0}} \gamma_\mu z^\mu \zeta^{(0)},
\]

where \( \zeta \) is the Grassmann spinor associated with the special supersymmetries which is packaged together with \( \eta \) into a four-component spinor in the second equality where

\[
\zeta^{(0)} = \left( \eta^A_\alpha \bar{\zeta}_{\alpha A} \right),
\]

\[18\] The superscripts \( A, B = 1, \ldots, 4 \) label 4’s of \( SU(4) \) while subscripts label the 4’s.
\( z^M \equiv (x^\mu - x_0^\mu, \rho) \) and \( \gamma^M \equiv (\gamma^\mu, \gamma^5) \). Equation (8.13) is just a chiral projection on a spinor of \( \text{AdS}_5 \) (with tangent-space group \( SO(4,1) \)). The normalization factor of \( \rho_0^{-1/2} \) in (8.14) has been chosen for later convenience.

The \( \beta \) function for this theory vanishes which means that the quantum theory is expected to share the superconformal invariance of the classical theory. The Noether currents associated with these superconformal transformations, as well as the chiral \( SU(4) \) transformations, form a supermultiplet with 256 component currents, including the energy-momentum tensor \[83, 84\]. For example, the lowest member of the supermultiplet is the scalar current that couples to \( Q_{ij}, Q_{ij} = \phi_i \phi_j - \delta_{ij} \phi_k \phi_k / 6 \).

Equation (8.16) only involves the bilinear terms in the current, but it is important that the full nonabelian currents also contain higher-order contributions \[76\] which are crucial in verifying the AdS/CFT correspondence \[85\]. The current that couples to the boundary value of the dilatino is

\[
\hat{A}_\alpha^A = -\sigma^{\mu \nu \alpha \beta} F_{\mu \nu} \lambda_\beta^A
\]

(again in the linearized approximation). The complete list of currents in the supermultiplet is given in \[84\]. These superconformal currents are in one to one correspondence with the fields of the supergravity multiplet.

From the AdS/CFT conjecture the supersymmetrized Einstein–Hilbert action gives the leading large-\( g_s^2 \) \( N \) contributions to the correlation functions of the superconformal currents. For example, a correlation function of \( M \) stress tensors,

\[
\langle T_{\mu_1 \nu_1}(x_1) T_{\mu_2 \nu_2}(x_2) \ldots T_{\mu_r \nu_r}(x_r) \ldots T_{\mu_M \nu_M}(x_M) \rangle,
\]

(8.18) corresponds to bulk superstring amplitudes in which the boundary values of the gravitons have polarizations, \( h_{\mu_r \nu_r} \), which are oriented in the four-dimensional Minkowski directions, \( \mu_r = M_r, \nu_r = N_r \) where \( M_r, N_r = 0, 1, 2, 3 \). Since the coefficient of the Einstein–Hilbert term is \( e^{-2\phi} \alpha'^{-4} = L^8 N^2 \) the leading behaviour of the correlation functions obtained in this manner is of order \( N^2 \). In general there is little known about such Yang–Mills correlation functions at large 't Hooft coupling. However, the free-field value of the correlation functions of two superconformal currents, such as \( \langle T_{\mu_1 \nu_1}(x_1^\mu) T_{\mu_2 \nu_2}(x_2^\mu) \rangle \), is known to be exact \[86, 87\] by virtue of the relation of this correlation function to the \( R \)-symmetry anomaly, which is not renormalized due to the Adler–Bardeen theorem. An analogous Ward identity prevents the three-point correlation function of superconformal currents (in the \( N = 4 \) theory, but not for \( N < 4 \)) from receiving renormalizations beyond those of the free field theory \[88\]. These free-field correlators of two and three superconformal currents are both proportional to

\[
\text{dim } (SU(N)) = N^2 - 1.
\]
The leading factor of $N^2$ corresponds to the coefficient of $\alpha'^{-4} g^{-2}$ in the Einstein–Hilbert action. This would be the exact $N$-dependence if the gauge group were $U(N)$ but (8.13) shows that there must be one other contribution to the Einstein–Hilbert action in the $AdS_5 \times S^5$ background that arises at order $N^{-2}$ (order $\alpha'^4$) relative to the leading term. Such a term might come from a one string loop effect which has not yet been elucidated. Since the terms that concern us here are of order $\alpha'^3$ relative to the leading term they dominate over this effect.

8.2 Generalities concerning higher derivatives terms

We wish to consider the effects of the $o(\alpha'^{-1})$ terms on the AdS/CFT conjecture. Since these terms are suppressed by $\alpha'^3$ relative to the leading terms they will correspond to $O(N^{-3/2})$ corrections to the leading behaviour. The interesting feature of these terms is that they manifest the $SL(2, \mathbb{Z})$ duality in a very nontrivial fashion.

We saw from the definition of the $R^4$ term, (3.11), that the only components of the curvature which contribute are those of the Weyl tensor. We can expand $R_{\theta^4}$ (defined in (3.12)) in fluctuations around its classical value, $R_{\theta^4}^{(0)}$,

$$R_{\theta^4} = R_{\theta^4}^{(0)} + \hat{R}_{\theta^4},$$  

(8.20)

where $\hat{R}$ is the fluctuation\footnote{A more complete discussion, given in \cite{89}, shows that there is a field definition in which the off-shell fluctuations of the metric can be restricted to those that only give fluctuations to the Weyl components of the curvature.}. Since $AdS_5 \times S^5$ is conformally flat the Weyl tensor vanishes and it follows that

$$R_{\theta^4}^{(0)} = 0.$$  

(8.21)

This leads immediately to several consequences:

1. $R^4 = 0$ in the $AdS_5 \times S^5$ background, which means that the dilaton equation of motion is unchanged in this background.

2. $\delta R^4/\delta g_{\mu\nu} = 0$ (since the differential is proportional to $(R_{\theta^4}^{(0)})^3$), which means that the Einstein equation is also unaffected. It is also easy to see that none of the other $o(\alpha'^3)$ terms which are related to $R^4$ by supersymmetry contribute to the equations of motion. This means that the $AdS_5 \times S^5$ background is unaltered by the presence of this term, so this background is a solution of the effective equations of motion of string theory through $o(\alpha'^5)$, to all orders in $g$ (a more general superspace argument
leads to the statement that this background is exact to all orders in $\alpha'$. 

3. Equation (8.21) also implies that

$$
\frac{\delta}{\delta g_{\mu\nu}} \frac{\delta}{\delta g_{\rho\sigma}} R^4 \bigg|_{AdS_5 \times S^5} = 0 = \frac{\delta}{\delta g_{\mu\nu}} \frac{\delta}{\delta g_{\rho\sigma}} \frac{\delta}{\delta g_{\tau\omega}} R^4 \bigg|_{AdS_5 \times S^5}. \tag{8.22}
$$

This shows that there is no renormalization of the graviton two-point or three-point functions in the $AdS_5 \times S^5$ background at $o(\alpha'^3)$.

However, there is a non-zero four-graviton contribution from the $R^4$ term which is obtained by differentiating four times with respect to the metric. This gives a contribution to the amplitude which adds to the term which arises from the Einstein–Hilbert action, so that up to $o(\alpha'^3)$ the four-graviton amplitude is proportional to

$$
\alpha'^{-4} e^{-2\phi} A_{\text{String}}^{(1)} + k\alpha'^{-1} e^{-\phi/2} f(0,0)(\tau,\bar{\tau}) A_{\text{String}}^{(2)}, \tag{8.23}
$$

where $A_{\text{String}}^{(1)}$ is the classical amplitude obtained from the Einstein–Hilbert action while $A_{\text{String}}^{(2)}$ is the contribution from the $R^4$ term and $k$ is a well-defined constant.

For example, the contribution of the $R^4$ term to four-graviton scattering is obtained by contracting the linearized $R^4$ vertex with four graviton propagators that propagate from the interaction point at $(x^\mu_0, \rho_0)$ to the boundary at points $(x^\mu_r, \rho_0, \tau = 0)$, where $r$ labels the position of the $r$th boundary field. The linearized curvature is

$$
R_{\mu_1 \nu_1 \tau_1 \tau_1} = D_{\nu_1} D_{\tau_1} h_{\mu_1 \nu_1}, \tag{8.24}
$$

where $h_{\mu\nu}$ is the linearized fluctuation of the metric around its value in $AdS_5 \times S^5$ and $D$ is the $AdS_5 \times S^5$ covariant derivative. The bulk-to-boundary graviton propagator in $AdS_5 \times S^5$ is an obvious generalization of the scalar and spin-1/2 propagators that will appear in later sections. The four-graviton interaction vertex can be expressed in the form (2.18) where the tensors $t_s$ in the kinematic factor $\vec{K}$ have the form of the product of four inverse $AdS_5 \times S^5$ metrics summed over various permutations of their indices.

Using the AdS/CFT conjecture the $R^4$ term gives a contribution to the correlation functions of stress tensors in the boundary Yang–Mills field theory that are suppressed by a factor of $N^{-3/2}$ relative to the leading free-field contribution. The vanishing of the graviton one-point function, translates into the statement that the one-point function of the stress tensor $\langle T_{\mu\nu} \rangle$ vanishes, which follows from conformal invariance. Similarly, the vanishing of the $\alpha'^{3}R^4$ contribution...
to the two-graviton and three-graviton S-matrix elements in IIB supergravity translates into the statements that the correlation functions of two and of three stress tensors in \( \mathcal{N} = 4 \) Yang–Mills theory is not renormalized from the free-field value at this order in \( \mathcal{N}^{-1} \). This is in accord with the exact results [86, 87, 88] cited earlier.

It is only when we come to the four-graviton amplitude that the \( R^4 \) term contributes and therefore the correlation function of four stress tensors gets a new contribution. The supergravity calculations give the following expression for the momentum-space correlation function of four stress tensors,

\[
A_{YM}^4 = \langle T_{\mu_1 \nu_1}(1) T_{\mu_2 \nu_2}(2) T_{\mu_3 \nu_3}(3) T_{\mu_4 \nu_4}(4) \rangle = N^2 A_4^{(1)} + \tilde{k} N^{1/2} f^{(0,0)}(S, \bar{S}) A_4^{(2)} + \ldots,
\]  

(8.25)

where \( A_4^{(1)} \) is the contribution at leading order in \( g^2_{YM} N \) and \( A_4^{(2)} \) is the correction arising from the \( R^4 \) term (and an irrelevant constant has been absorbed into \( \tilde{k} \)). The term \( A_4^{(1)} \) comes from the four-graviton amplitude computed from the Einstein action in \( AdS_5 \times S^5 \), while \( A_4^{(2)} \) corresponds to the four-graviton vertex in the \( R^4 \) term. No further terms are necessary at this order in the \( \alpha' \) expansion. The scalar field \( \tau \) is now interpreted as the complex coupling constant, \( S \).

The second term in (8.25) has a remarkable amount of information concerning the Yang–Mills theory. It is the first non-leading term in the \( 1/\mathcal{N} \) expansion but is an exactly known function of the (complex) coupling. As we have seen, the factor \( f^{(0,0)} \) contains two terms that are perturbative in string theory and an infinite number of D-instanton terms. Therefore, in the limit \( g_{YM} \to 0 \) with \( g^2_{YM} N \) fixed and large, the expression (8.25) takes the form

\[
N^{-2} A_{YM}^4 = A_4^{(1)} + \tilde{k} A_4^{(2)} \left[ 2\zeta(3) \left( \frac{g^2_{YM} N}{4\pi} \right)^{-3/2} + \frac{2\pi^2}{3N^2} \left( \frac{g^2_{YM} N}{4\pi} \right)^{1/2} \right] + (4\pi)^{3/2} \sum_{K=1}^{\infty} Z_K K^{1/2} \left( e^{-K(8\pi^2 g^2_{YM} + i\theta)} + e^{-K(8\pi^2 g^2_{YM} - i\theta)} \right) \left( 1 + o(g^2_{YM}/K) \right),
\]  

(8.26)

which includes an infinite series of instanton corrections. We will return to a discussion of these corrections later. An important property of this expression is its invariance under \( SL(2, \mathbb{Z}) \) (which is interpreted as Montonen–Olive electromagnetic duality). This transformation, which maps the small coupling regime to large coupling, is manifest in (8.23) but not in (8.26), which is the ’t Hooft expansion, and is only valid when \( g_{YM} \) is very small. The presence of noninteger powers of \( g^2_{YM} N \) in this expansion is one indication of how far removed the strong coupling expansion is from weakly coupled perturbation theory.
8.3 The scalar $\text{AdS}$ Green function and the instanton profile.

In order to make use of (8.11) to perform explicit detailed calculations of Yang–Mills correlation functions we need to evaluate the bulk-to-boundary Green functions for the various fields of the bulk theory. These are defined as specific normalized limits of bulk-to-bulk Green functions [35, 36, 88] when one point is taken to the $\text{AdS}$ boundary. The precise forms of these propagators depend on the spin and mass of the field. For example, the normalized bulk-to-boundary Green function for the $\text{AdS}$ Laplace operator for a dimension $\Delta$ scalar field is given by

$$G_\Delta(x, \rho; x', 0, \omega) = c_\Delta K_\Delta(x^\mu, \rho; x'^\mu, 0), \tag{8.27}$$

which is independent of $\omega$ and where $c_\Delta = \frac{\Gamma(\Delta)}{(\pi^2 \Gamma(\Delta - 2))}$ and

$$K_\Delta(x^\mu, \rho; x'^\mu, 0) = \frac{\rho^\Delta}{(\rho^2 + (x - x')^2)^{\Delta/2}}. \tag{8.28}$$

The expression (8.27) is appropriate for an ‘S-wave’ process in which there are no excitations in the directions of the five-sphere, $S^5$. In terms of $K_\Delta$ the bulk field

$$\Phi_m(z; J) = c_\Delta \int d^4x' K_\Delta(x, \rho; x', 0)J_\Delta(x') \tag{8.29}$$

satisfies the boundary condition as $\rho \to 0$,

$$\Phi_m(x, \rho; J) \approx \rho^{4-\Delta} J_\Delta(x), \tag{8.30}$$

since $\rho^{\Delta-4} K_\Delta$ reduces to a $\delta$-function on the boundary. The conformal dimension of the operator is related to the $\text{AdS}$ mass of the corresponding bulk field by $(mL)^2 = \Delta(\Delta - 4)$, so that $\Delta_\pm = 2 \pm \sqrt{4 + (mL)^2}$ and only the positive branch, $\Delta = \Delta_+$, is relevant for the lowest-‘mass’ supergravity multiplet. In the case of a massless scalar field ($\Delta_+ = 4$) the propagator reduces to $\delta(4)(x^\mu - x'^\mu)$ in the limit $\rho \to 0$. In comparing the D-instanton contributions of the bulk theory with the Yang–Mills instanton contributions to the boundary theory it is of significance, as we will see in section 9, that the Green function $K_4$ defined in (8.28) has the same form (up to an additive constant) as the profile of $e^\phi$ in the D-instanton solution centered on the point $(x^\mu, \rho)$ in $\text{AdS}_5 \times S^5$ and evaluated at the boundary point $(x'^\mu, 0)$.

We now turn to the Yang–Mills instanton. Recall that that the instanton solution for the $SU(2)$ Yang–Mills potential gives a nontrivial nonabelian field strength of the form

$$F_{(0)\mu\nu}^{a}(x^\mu; x_0^\mu, \rho_0) = -\frac{4}{g_{YM}} \frac{\eta_{a}\rho_0^2}{(\rho_0^2 + (x - x_0)^2)^{3/2}}. \tag{8.31}$$
where $\eta_{\mu \nu}$ is the standard 't Hooft symbol, $\rho_0$ is the arbitrary scale and $x_0^\mu$ the position of the instanton on $S^4$. Squaring $F^-$ and using properties of $\eta$ leads to the equality

$$\left( F^-(0) \right)^2 = \frac{4}{g_{YM}^2} K_4(x_0^\mu, \rho_0; x^\mu, 0), \quad (8.32)$$

which is again equal to the scalar Green function for the $\Delta = 4$ case where the fifth coordinate $\rho$ is now identified with the instanton scale. This is a key observation in identifying D-instanton effects of the bulk theory with Yang–Mills instanton effects in the boundary theory. It is related to the fact that the moduli space of a Yang–Mills instanton on $S^4$ has an $AdS_5$ factor.

9 D-instantons and Yang–Mills instantons

We now want to compare the leading order one-instanton contribution to the supersymmetric Yang–Mills correlation functions with the amplitudes of the IIB superstring theory with appropriate boundary conditions. For illustrative purposes we will only consider the sixteen-$\Lambda$ amplitude in the $AdS_5 \times S^5$ background and the corresponding Yang–Mills sixteen-$\Lambda$ correlation function [76]. From either perspective the leading instanton contribution arises from the product of sixteen factors each carrying one single fermionic zero mode. Correlation functions of other superconformal currents can be calculated in analogous fashion. For certain of these it is possible to express the result very explicitly in terms of dilogarithms [76, 92].

From the bulk type IIB point of view we are interested in the leading D-instanton contribution to amplitudes, which may either be extracted directly from the exactly known $o(\alpha'^{-1})$ terms in the effective action (method (a) below) or deduced by integration over the fluctuations around the classical $AdS_5 \times S^5$ D-instanton solution (method (c)). From the Yang–Mills perspective we are interested in the instanton contributions to supercurrent correlators of the $\mathcal{N} = 4$ theory (method (b)).

Method (a). Expansion of $\alpha'^{-1} f^{(12,12)} \Lambda^{16}$ term in bulk IIB supergravity

The first method for obtaining this amplitude is to expand the function $f^{(12,−12)}$ in order to extract the one-instanton term. The leading behaviour comes from the leading power of $g_{12} = \tau_2^{-12}$ in $\mathcal{F}_{1,12}$ defined by (6.3) and (6.5). We want to consider the situation in which all sixteen fermions propagate to specific values on the boundary. The Dirac operator acting on spin-1/2 fields in $AdS_5$ was given in [93, 94] as

$$\gamma \cdot D \Lambda = \epsilon^M \frac{\gamma^L}{L} \left( \partial_M + \frac{1}{4} \omega^M_{MN} \gamma_M \gamma_N \right) \Lambda = (\rho \gamma^5 \partial_5 + \rho \gamma^\mu \partial_\mu - 2 \gamma^5) \Lambda, \quad (9.1)$$
where $e^M_L$ is the vielbein, $\omega^N_M$ the spin connection (hatted indices refer to the tangent space) and $\gamma^\mu$ are the four-dimensional Dirac matrices. Equation (9.1) leads to the normalized bulk-to-boundary propagator of the fermionic field $\Lambda$ of mass $m = -3/2L$, associated with the composite operator $\hat{\Lambda}$ of dimension $\Delta = 7/2$,

$$K^{F}_{7/2}(\rho_0, x_0; x) = K_4(\rho_0, x_0; x) \frac{1}{\sqrt{\rho_0}} (\rho_0 \gamma^5 + (x_0 - x)^\mu \gamma_\mu), \quad (9.2)$$

which, suppressing all spinor indices, leads to

$$\Lambda_J(x_0, \rho_0) = \int d^4x K^{F}_{7/2}(\rho_0, x_0; x) J_{\Lambda}(x), \quad (9.3)$$

where $J_{\Lambda}(x)$ is a left-handed boundary value of $\Lambda$ and acts as the source for the composite operator $\hat{\Lambda}$ in the boundary $\mathcal{N} = 4$ Yang–Mills theory satisfying $\hat{\gamma}^5 J_{\Lambda} = J_{\Lambda}$. As a result, the classical action for the operator $\Lambda^{16}$ in the $AdS_5 \times S^5$ supergravity action is

$$S_{\Lambda}[J] = (\text{const.}) e^{-2\pi(\frac{1}{4} - i C^{(0)})} g^{-12} V_{S^5} \int \frac{d^4 x_0 d\rho_0}{\rho_0^5} \varepsilon_{16} \prod_{p=1}^{16} \left[ K_4(\rho_0, x_0; x_p) \frac{1}{\sqrt{\rho_0}} \left( \rho_0 \gamma^5 + (x_0 - x_p)^\mu \gamma_\mu \right) J_{\Lambda}(x_p) \right], \quad (9.4)$$

where $J_{\Lambda}(x_p)$ is the wave-function of the dilatino evaluated at the boundary point $(x_p, 0)$ and we have set $e^\phi = g$ and $C^{(0)} = G^{(0)}$ (since the scalar fields are taken to be constant in the $AdS_5 \times S^5$ background) and $V_{S^5} = \pi^4$ is the $S^5$ volume. The spinor indices on the $J$'s are contracted with the rank-sixteen antisymmetric tensor, $\varepsilon_{16}^{1a_2...16}$. We have not kept track of the overall constant normalization in this expression. The final integration over the moduli $x_0^0, \rho_0$ has not been carried out explicitly in (9.4). For certain of the other correlation functions the integration over the instanton moduli can be evaluated in terms of standard functions [24, 22]. According to (8.11) the sixteen-$\Lambda$ Yang–Mills correlation function is extracted from (9.4) by differentiating with respect to all the $J_{\Lambda}$'s.

**Method (b). Sixteen-$\Lambda$ correlator in the boundary Yang–Mills theory**

We will now consider the correlation function of sixteen fermionic superconformal current operators in a charge-one instanton background,

$$G_{\hat{\Lambda}^{16}}(x_p) = \prod_{p=1}^{16} g_{\gamma^{(1)}}^{2} \hat{\Lambda}^{A_p}(x_p)_{\kappa=1}, \quad (9.5)$$

53
where \( \hat{\Lambda} \) is defined in (8.17). The \( SU(2) \) Yang–Mills instanton has sixteen supermoduli which correspond to the fermionic zero modes that are induced by the broken supersymmetries. The special feature of the product of fields inside the correlator (9.5), as well as the others that are related by supersymmetry, is that they provide precisely the sixteen fermionic zero modes that are needed to give a nonzero result in the instanton background. In particular, it is easy to see that \( \hat{\Lambda} \) is linear in fermionic zero modes so a total of sixteen factors is needed for a non-zero result. To leading order in \( g_{YM} \), \( G_{\hat{\Lambda}16} \) does not receive contribution from anti-instantons.

The fermionic zero modes can be deduced by applying the broken components of the supersymmetry transformations to the fields. The bilinear \( \hat{\Lambda}^A_{\alpha} = -\sigma^{\mu\nu\beta}_{\alpha} F^{-}_{\mu
u} \lambda^A_{\beta} \) is proportional to the zero mode of \( \lambda^A_{\alpha} \) which will be denoted \( \lambda^{(0)}_{\alpha} \) and can be deduced from the second line of (8.13),

\[
\lambda^{(0)}_{\alpha} \equiv \delta\lambda^A_{\alpha} = \frac{1}{2} F^{-}_{(0)\mu\nu} \sigma^{\mu\nu\beta}_{\alpha} \left( \rho_0 \eta^A_{\beta} + (x - x_0)_\mu \sigma^\mu_{\beta\bar{\gamma}} \tilde{\xi}_A^{\bar{\gamma}} \right).
\]

The leading term in \( \hat{\Lambda} \) in the one-instanton sector is simply obtained by using the instanton profile \( F^{-}_{(0)\mu\nu} \) (8.31) and substituting each \( \lambda^A_{\alpha} \) by the corresponding zero mode, \( \lambda^{(0)}_{\alpha} \). The resulting correlation function has the form

\[
G_{\hat{\Lambda}16}(x_p) = (\text{const.}) g_{YM}^8 e^{i2\pi x_p + i\theta_{YM}} \int d^4x_0 \frac{d\rho_0}{\rho_0^9} \int d^8\eta d^8\xi \prod_{p=1}^{16} \left[ K_4(x_p^\mu, \rho_0; x_0^\mu, 0) \frac{1}{\sqrt{\rho_0}} (\rho_0 \eta^A_{\alpha} + (x_p - x_0)_\mu \sigma^\mu_{\alpha\bar{\gamma}} \tilde{\xi}_A^{\bar{\gamma}}) \right].
\]

Integration over the fermion zero modes leads to the sixteen-index invariant tensor \( \varepsilon_{16} \). Converting to four-component spinor notation and using (8.11) this result coincides with (9.4), up to the undetermined overall constant.

One is entitled to ask why such a comparison of a D-instanton effect with a \( SU(2) \) Yang–Mills instanton effect is being made in the first place. In method (a) we used the expression for the \( o(\alpha'^{-1}) \) interaction in the \( AdS_5 \times S^5 \) background which should only be a good approximation to strongly coupled \( SU(N) \) Yang–Mills in the limit \( N \to \infty \). On the other hand method (b) computed the effect of an \( SU(2) \) Yang–Mills instanton. However, as will be described in the final subsection, the \( N \)-dependence of these correlation functions was shown in [39] to only affect the overall coefficient of the correlation function and not its functional form. Furthermore, the coefficient has the expected large-\( N \) limit, \( N^{1/2} \). This will be reviewed in the last subsection along with the results in [40, 41] which demonstrate that the large-\( N \) limit of the Yang–Mills calculation agrees perfectly with the leading D-instanton contribution even in the multi-instanton sectors.
9.1 The classical D-instanton in flat space and in $AdS_5 \times S^5$

Later in this subsection we will evaluate the single D-instanton terms in the sixteen-dilatino amplitude in $AdS_5 \times S^5$ by semi-classical quantization in the background of a D-instanton solution of type IIB supergravity (this is method (c)). First we need to recall some properties of the classical D-instanton solution.

In flat ten-dimensional euclidean space the charge-$K$ D-instanton solution is a finite-action euclidean supersymmetric (BPS–saturated) solution in which the metric is trivial ($g_{\mu\nu} = \eta_{\mu\nu}$ in the Einstein frame) but the complex scalar $\tau = C^{(0)} + ie^{-\phi}$ has a nontrivial profile with a singularity at the position of the D-instanton. The (euclidean) $\mathcal{R} \otimes \mathcal{R}$ scalar is related to the dilaton by the BPS condition $\partial_{\Sigma} C^{(0)} = \pm i \partial_{\Sigma} e^{-\phi}$, while the dilaton solution is the harmonic function (correcting an error in [53])

$$e^{\tilde{\phi}}(10) = g + \frac{3K\alpha'^4}{\pi^4|X - X_0|^8}. \quad (9.8)$$

This is the classical solution of the ten-dimensional Laplace equation, $\partial^2 e^{\phi} = 0$, outside an infinitesimal sphere centered on the point $X_0^\Lambda$ (where $X^\Lambda$ is the ten-dimensional coordinate and $X_0^\Lambda$ is the location of the D-instanton$^{20}$), $g$ is the asymptotic value of the string coupling and the normalization of the second term has a quantized value by virtue of a condition analogous to the Dirac–Nepomechie–Teitelboim condition that quantizes the charge of an electrically charged $p$-brane and of its magnetically charged $p'$-brane dual$^{20}$. It is notable that the solution in (9.8) is simply the Green function for a scalar field to propagate from $X_0$ to $X$ subject to the boundary condition that $e^{\phi} = g$ at $|X| \to \infty$ or $|X_0| \to \infty$.

We are now interested in solving the equations of motion of the IIB theory in euclidean $AdS_5 \times S^5$. The BPS condition for a D-instanton in this background again requires $\partial_{\Sigma} e^{-\phi} = \pm i \partial_{\Sigma} C^{(0)}$ that leads to

$$g^{\Lambda\Sigma} \nabla_{\Lambda} \nabla_{\Sigma} e^{\phi} = 0, \quad (9.9)$$

and (in the Einstein frame) the Einstein equations are unaltered by the presence of the D-instanton so that $AdS_5 \times S^5$ remains a solution. Equation (9.9) is identical to the equation for the Green function of a massless scalar propagating between the location of the D-instanton $(x_0^\mu, y_0^i)$ and the point $(x^\mu, y^i)$, which is the bulk-to-bulk propagator (subject to the boundary condition that it is constant in the limits $\rho \to 0$ and $\rho \to \infty$). This is easy to solve using the

$^{20}$Having run out of many conventional options we are here using upper case Greek letters to signify the ten-dimensional indices.
conformal flatness of $AdS_5 \times S^5$ which implies that the solution for the dilaton is of the form
\[ e^{\hat{\phi}} = g + \frac{\rho_0^4 \rho^4}{L^8} (e^{\hat{\phi}(10)} - g), \tag{9.10} \]
where $\rho_0 = |y_0|$ and $e^{\hat{\phi}(10)}$ is the harmonic function that appeared in the flat ten-dimensional case, (9.8). In evaluating D-instanton dominated amplitudes we will only be interested in the case in which the point $(x^i, y^i)$ approaches the boundary ($\rho \equiv |y| \to 0$), in which case it is necessary to rescale the dilaton profile (just as it is necessary to rescale the scalar bulk-to-bulk propagator, [35, 36]) so that the combination
\[ \rho^{-4} (e^{\hat{\phi}} - g) = \frac{3K(\alpha')^4}{L^8 \pi^4} \frac{\rho_0^4}{((x-x_0)^2 + \rho_0^2)^4}, \tag{9.11} \]
is of relevance in the $\rho \to 0$ limit.

As mentioned earlier, the correspondence with the Yang–Mills instanton follows from the fact that $\rho_0^4/((x-x_0)^2 + \rho_0^2)^4 = K_4$ is proportional to the instanton number density, $(F^{(0)})^2$, in the $\mathcal{N} = 4$ Yang–Mills theory. Strikingly, the scale size of the Yang–Mills instanton is replaced by the distance $\rho_0$ of the D-instanton from the boundary. This is another indication of how the geometry of the Yang–Mills theory is encoded in the IIB superstring. Note, in particular, that as the D-instanton approaches the boundary $\rho_0 \to 0$, the expression for $\rho^{-4} e^{\hat{\phi}}$ reduces to a $\delta$ function that corresponds to a zero-size Yang–Mills instanton.

The BPS condition implies that we can write the solution for the $R \otimes R$ scalar as
\[ \tilde{C}^{(0)} = \tilde{C}^{(0)} + if(x, y), \tag{9.12} \]
where $\tilde{C}^{(0)}$ is the constant real part of the field (which corresponds to $\theta_{YM}/2\pi$) and
\[ f = A - \frac{1}{g} + e^{-\hat{\phi}}. \tag{9.13} \]
Since the action is independent of constant shifts of $C^{(0)}$ it does not depend on the arbitrary constant, $A$. In a manner that follows closely the flat ten-dimensional case considered in the appendix of [11] the action for a single D-instanton of charge $K$ can be written as
\[ S_K = \frac{L_{10}}{\alpha'^4} \int \frac{d\rho d^4x d^5\omega}{\rho^5} g^{\Lambda\Sigma} \nabla_\Lambda (e^{2\hat{\phi}} f \partial_\Sigma f), \tag{9.14} \]
which reduces to an integral over the boundaries of $AdS_5 \times S^5$ and the surface of an infinitesimal sphere centered on the D-instanton at $x = x_0, y = y_0$. With the
choice $A = 0$ in (9.13) the entire D-instanton action comes from the boundary of the infinitesimal sphere. Substituting for $f$ from (9.13) gives

$$S_K = \frac{2\pi |K|}{g},$$

(9.15)

which is the same answer as in the flat ten-dimensional case. On the other hand, with the choice $A = 1/g$ in (9.13) the expression (9.14) reduces to an integral over the boundary at $\rho = 0$ but the total action remains the same as $S_K$ in (9.15). Remarkably, in this case the boundary integrand is identical to the action density of the standard four-dimensional Yang–Mills instanton. It is crucial for this agreement that the integral (9.14) includes an integral over the $S^5$ factor even though the classical moduli space of the Yang–Mills instanton does not include an $S^5$. This is a concrete manifestation of the holographic principle whereby the physics of the bulk is encoded on the boundary [97, 98].

Whereas the $AdS_5 \times S^5$ metric remains unchanged by the presence of the D-instanton in the Einstein frame it is radically altered in the string frame where the instanton is manifested as a space-time wormhole (as in the flat ten-dimensional case [53]). For finite values of $K$ the dilaton becomes large in the Planck-scale neck and the classical solution is not reliable in that region. However, for very large instanton number, the neck region becomes much larger than the Planck scale so, by analogy with the D-brane examples studied in [34], it should be very interesting to study the implications of the modified $AdS_5 \times S^5$ geometry in the large-$K$ limit of the large-$N$ theory.

**Method (c). Fluctuations around a classical D-instanton**

The D-instanton contribution to the amplitude with sixteen external dilatini, $\Lambda^4_A$, may now be obtained directly by semi-classical quantization around the classical D-instanton solution in $AdS_5 \times S^5$. The leading instanton contribution can be determined by applying supersymmetry transformations to the scalar field which has an instanton profile given by (9.11). Since the D-instanton background breaks half the supersymmetries the relevant transformations are those in which the supersymmetry parameter corresponds to the Killing spinors for the sixteen broken supersymmetries. These Killing spinors have $U(1)$ charge $1/2$ and are defined by a modified version of (8.5) that includes the non-trivial composite $U(1)$ connection, $Q_M$ [59], that is made from the IIB scalar field [53],

$$D_M \xi \equiv (D_M - \frac{i}{2} Q_M) \xi = \frac{1}{2L} \gamma_M \xi. \quad (9.16)$$

Substituting the euclidean D-instanton solution into the expression for the composite connection gives

$$Q_M = \frac{i}{2} e^{-\hat{\phi}} \partial_M e^{\hat{\phi}}. \quad (9.17)$$
with \( \hat{\phi} \) defined by (9.11). The solution of (9.16) is
\[
\zeta_{\pm} = e^{-\hat{\phi}/4} \frac{z^M}{\sqrt{\rho_0}} \zeta^{(0)}_{\pm},
\]
(9.18)
where \( \zeta^{(0)}_{\pm} \) is a constant spinor satisfying \( \gamma_5 \zeta^{(0)}_{\pm} = \pm \zeta^{(0)}_{\pm} \).

The sixteen broken supersymmetry transformations associated with \( \zeta^{(0)}_{\pm} \) give rise to the dilatino zero-modes,
\[
\Lambda_{(0)} = \delta \Lambda = (\gamma^M \hat{P}_M) \zeta_{-},
\]
(9.19)
where \( \hat{P}_M \) is the expression for \( P_M = i \partial_M \tau^* / 2 \tau_2 \) in the D-instanton background [11],
\[
\hat{P}_M = e^{-\hat{\phi}} \partial_M e^{\hat{\phi}}.
\]
(9.20)
Using the Killing spinor equation and the D-instanton equation \( D^M \hat{P}_M = 0 \) it is easy to check (recalling that \( P_M \) has \( U(1) \) charge 2) that
\[
\gamma^M D_M \Lambda_{(0)} = -\frac{3}{2 \mathcal{L}} \Lambda_{(0)},
\]
(9.21)
so that \( \Lambda_{(0)} \) is a solution of the appropriate massive Dirac equation. We will be interested in amplitudes with external states located on the boundary, in which case we may use the fact that for \( \rho \sim 0 \),
\[
\hat{P}_M \sim \frac{1}{g} \partial_M e^{\phi}
\]
(9.22)
in (9.19), which leads to
\[
\Lambda_{(0)} \sim \frac{4}{g} (e^{\phi} - g) \zeta_{-}.
\]
(9.23)
This means that near \( \rho = 0 \) the dilatino profile in the D-instanton background is proportional to \( \rho^4 K_4(x_0, \rho_0; x, 0) \).

As a result the leading contribution to the sixteen-dilatino amplitude again reproduces the corresponding sixteen-current correlator in \( \mathcal{N} = 4 \) supersymmetric Yang–Mills theory. Explicitly, the D-instanton approximation to the amplitude with sixteen external dilatinis, \( \Lambda^A_{\alpha} \), at points on the \( \rho = 0 \) boundary is (up to an overall constant factor)
\[
\langle \prod_{p=1}^{16} \Lambda^A_{\alpha_p}(x_p, 0) \rangle_J = (\text{const.}) \ g^{-12} e^{-2\pi K^{(4)}(\frac{1}{g} + ic^{(0)})} V_{S^5} \int \frac{d^4 x_0 d\rho_0}{\rho_0^5} \int d^{16} \zeta^{(0)}_{-} \]
\[
\prod_{p=1}^{16} K_4(x_0, \rho_0; x_p) \frac{1}{\sqrt{\rho_0}} \left( \rho_0 \eta^A_{\alpha_p} + (x_p - x_0)_\mu \sigma^\mu_{\alpha_p \alpha_r} \bar{\xi}_{\alpha_r A_p} \right) J_A(x_p) \right].
\]
(9.24)
58
The overall factor of \((\text{const.}) g^{-12}\) in this expression should be derived from the normalization of the bosonic and fermionic zero modes but it has been inserted by hand. Up to this overall constant factor, the amplitude (9.24) agrees with (9.7) and therefore with (9.4).

In similar manner the instanton profiles of all the fields in the supergravity multiplet follow by applying the broken supersymmetries to \(P_M\) any number of times, just as they do in the flat ten-dimensional case [11]. The single D-instanton contributions to any correlation function can then be determined.

9.2 Instantons in the limit of large \(N\)

We now return briefly to the issue of comparing the D-instanton and Yang–Mills instanton effects for gauge groups \(SU(N)\) in the limit \(N \to \infty\). The string computation was valid only in the region of large \(g^2N\) since we kept only the first term in the \(\alpha'\) expansion that contributes to \(\mathcal{R}^4\). In the classical \(\mathcal{N} = 4\) Yang–Mills theory with a gauge group \(SU(N)\) an instanton background has \(8N\) fermion zero modes, \(2N\) for each of the 4 adjoint Weyl fermions. It is a special feature of the \(\mathcal{N} = 2\) case that the number of fermionic zero modes coincides with the number of broken supersymmetries but for larger values of \(N\) there are more fermionic zero modes in the classical D-instanton background than the number of broken supersymmetries. However, most of these modes receive perturbative corrections that gives them nonzero energy and the only zero modes which are protected are the sixteen which correspond to broken supersymmetries. In other words, most of the classical zero modes are not protected by symmetries and do not cause the correlation functions that we have been discussing (such as the \(\hat{\Lambda}^{16}\) correlator) to vanish.

The measure for a Yang–Mills instanton for the group \(SU(N)\) was considered a long time ago in nonsupersymmetric theories. The measure is defined as the jacobian for the transformation from bosonic zero-modes to the collective coordinates and for pure \(SU(N)\) Yang–Mills it has the form [9]

\[
\frac{(2\pi)^{4N}}{(N-1)!(N-2)!} \left( \frac{\rho_0}{g_{YM}} \right)^{4N} \frac{d\rho_0 d^4x_0}{\rho_0^5} \tag{9.25}
\]

(where \(\rho_0\) is an arbitrary scale), which accounts for the number of ways of embedding the \(SU(2)\) instanton solution in \(SU(N)\) and the coupling constant dependence is due to the eight bosonic zero modes [10]. In supersymmetric theories this coefficient is modified due to the gaussian fluctuations of the other fields. In [39] the measure for a single Yang–Mills instanton in \(SU(N)\) \(\mathcal{N} = 4\) supersymmetric Yang–Mills was computed, including the integration over fermionic zero modes. The result is that the factor (9.25) is replaced by
So that the overall power of the coupling is the same as in the $SU(2)$ case with $\mathcal{N} = 4$ supersymmetry that we considered earlier but there is now an explicit $N$-dependent coefficient. The large-$N$ limit of this expression can be extracted by using Stirling’s approximation for the gamma functions with the result that the overall measure at large $N$ is proportional to

$$g_{YM}^8 N^{1/2} \sim \alpha'^{-1} \tau^7/2,$$

which agrees with the $K = 1$ case of the expression deduced from the D-instanton measure in (6.11).

The large-$N$ limit of the contribution of $K$ instantons to the protected processes that we have been considering was derived in two interesting recent papers [40, 41]. Making extensive use of the ADHM construction this paper showed that the large-$N$ limit is dominated by a saddle point which constrains the $K$ $SU(2)$ instantons to a very special point in the multi-instanton moduli space. This is a point in the subspace of the full moduli space at which the instantons are in $K$ commuting $SU(2)$ subgroups of $SU(N)$. Clearly, this can only be precisely true for $N \to \infty$. For finite $N$ such configurations may dominate for $K \ll N$ but when $K > N/2$ no such configurations exist. Furthermore, it was shown in [40] that at the dominant saddle point the instantons coincide in space and have equal scale sizes — they behave as a single charge-$K$ instanton. Even more impressively, the evaluation of the large-$N$ limit of the fluctuations of the fields around the coincident instantons induces five new effective moduli that parameterize the five-sphere. Finally, the overall coefficient is identified in [40, 41] with the partition function of the zero-dimensional matrix model, $Z_K$. The result is that the effective moduli space of the leading contribution of $K$ Yang–Mills instantons to the protected processes is $AdS_5 \times S^5$. The resulting measure coincides precisely with the leading term in the D-instanton measure in (6.11) for general $K \geq 1$.

These results apply to the instanton corrections to the protected processes at leading power of the Yang–Mills coupling constant. However, the full $SL(2, \mathbb{Z})$-invariant structure of the Yang–Mills correlation functions suggested by the AdS/CFT correspondence in [22] is very much richer. We have seen that in the limit $g_{YM}^2 N \to \infty (\alpha' \ll L^2)$ not only must each instanton contribution to a correlation function be accompanied by an infinite number of perturbative corrections but there are vital non-instanton terms which correspond to the tree-level and one-loop contributions in the string theory. These are the terms
in (8.26) that have fractional powers of $g_{YM}$ and are not visible in weakly-coupled Yang–Mills theory and therefore cannot be obtained by matching the semiclassical approximations of the bulk and boundary theories. Explaining the nature of these terms from the Yang–Mills perspective would appear to be of some interest.

Acknowledgments:
I am very grateful to Pierre Vanhove and Constantin Bachas for useful comments concerning the material in this article.

A Derivative expansion of the string tree amplitude

The tree amplitude for four-graviton scattering in either of the type II superstring theories is given by

$$A_{4}^{\text{tree}} = \alpha'^{-4} e^{-2\phi} \frac{\tilde{K}}{stu} \frac{\Gamma(1 - \alpha's)\Gamma(1 - \alpha't)\Gamma(1 - \alpha'u)}{\Gamma(1 + \alpha's)\Gamma(1 + \alpha't)\Gamma(1 + \alpha'u)} (A.1)$$

where the second equality follows from elementary properties of Γ functions. The overall kinematic factor $\tilde{K}$, defined in (2.18), is the linearized expansion of $R^4$ and $s,t,u$ are the Mandelstam invariants in string frame (and satisfy $s + t + u = 0$). These are related to the Mandelstam invariants in M-theory coordinates (which we will denote $S, T$ and $U$) by

$$s = \frac{S}{R_{11}}, \quad t = \frac{T}{R_{11}}, \quad u = \frac{U}{R_{11}}. \quad (A.2)$$

Every term in the exponent of (A.1) can be expressed as a polynomial in $s$ and $t$ multiplied by $stu$, as can be seen from the identity,

$$s^{2n+1} + t^{2n+1} + u^{2n+1} = stu \sum_{r=1}^{n} \sum_{q=0}^{2n-2r} \frac{(2n + 1)!}{r!(2n + 1 - r)!} (-1)^q s^{2n-1-r} t^r u^{q-1}, \quad (A.3)$$

where $n \geq 1$. This means that the massless poles only contribute to the first term in the expansion of the exponential.

When expressed in terms of the Mandelstam invariants in the M-theory metric the expression (A.1) has the low-energy expansion,

$$A_{4}^{\text{tree}} \sim \tilde{K} \left( \frac{1}{STU} + \frac{2\zeta(3)}{R_{11}^3} + \frac{2\zeta(5)}{R_{11}^5} (S^2 + ST + T^2) + \frac{2\zeta(3)^2}{R_{11}^9} STU \right)$$

61
\[
+ \frac{2\zeta(7)}{R_{11}^7}(S^4 + 2S^3T + 3S^2T^2 + 2ST^3 + T^4) + \frac{2\zeta(3)\zeta(5)}{R_{11}^8}STU(S^2 + ST + T^2) \\
+ \frac{2\zeta(9)}{R_{11}^9}(S^6 + 4S^5T + \cdots + T^6) + \frac{4}{3R_{11}^9}\zeta(3)^3S^2T^2U^2 + \cdots
\]  

(A.4)

The first term in this expansion combines with the kinematic factor, \(\tilde{K}\), to give the tree-level amplitude that is described by the Einstein–Hilbert action of eleven-dimensional supergravity. The term with coefficient \(\zeta(3)\) can be identified (section 5 of these notes) with a one-loop effect in four-graviton scattering in eleven-dimensional supergravity compactified on a circle of radius \(R_{11} = e^{2\phi/3}\). This explains the corresponding term in (A.1) when \(A_{44}^{tree}\) is interpreted as a type IIA tree amplitude. When (A.1) is interpreted as a IIB tree amplitude the \(\zeta(3)\) term comes from the zero-volume limit of M theory on \(T^2\). Similarly, there is some evidence [64] that the term with coefficient \(\zeta(5)\) in (A.4) can be identified with a two-loop effect in eleven-dimensional supergravity. The higher-order terms in (A.4) give terms in the effective action with derivatives acting on \(R^4\) and other higher-derivative terms.

B Modular covariant derivatives.

The various coefficient functions in the effective action are \((w, \hat{w})\) forms, where \(w\) refers to the holomorphic modular weight and \(\hat{w}\) to the anti-holomorphic modular weight. A nonholomorphic modular form \(F(w, \hat{w})\) transforms as,

\[
F(w, \hat{w}) \rightarrow F(w, \hat{w}) (c\tau + d)^w(c\bar{\tau} + d)^{\hat{w}},
\]

under the \(SL(2, \mathbb{Z})\) transformation (1.6). Equation (B.1) describes a \(U(1)\) transformation when \(\hat{w} = -w\).

The modular covariant derivative,

\[
D_w = i \left( \frac{\partial}{\partial \tau} - i \frac{w}{2\tau_2} \right),
\]

maps \(F(w, \hat{w})\) into \(F(w+1, \hat{w}-1)\) while the anti-holomorphic covariant derivative, \(\bar{D}_{\hat{w}} = \bar{D}_w\), maps \(F(w, \hat{w})\) into \(F(w, \hat{w}+1)\). It is more convenient for our purposes to define the covariant derivatives,

\[
D_w = \tau_2D = i \left( \tau_2 \frac{\partial}{\partial \tau} - i \frac{w}{2} \right), \quad \bar{D}_{\hat{w}} = \tau_2\bar{D} = -i \left( \tau_2 \frac{\partial}{\partial \bar{\tau}} + i \frac{\hat{w}}{2} \right) \tag{B.3}
\]

which change the \(U(1)\) charge of \(F\) by two units,

\[
D_w F(w, \hat{w}) = F(w+1, \hat{w}+1), \quad \bar{D}_{\hat{w}} F(w, \hat{w}) = F(w-1, \hat{w}+1). \tag{B.4}
\]
The Laplace operator on the fundamental domain of $SL(2, \mathbb{Z})$ is defined to be,
\[
\nabla_0^2 \equiv \nabla^2 = 4\tau^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \bar{\tau}},
\]
when acting on $(0,0)$ forms. More generally, we shall be interested in the Laplacian acting on $(w,-w)$ forms. There are two such Laplacians which are defined by,
\[
\nabla^2_(-) w = 4D_{w-1} \bar{D}_{-w} = 4\tau^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \bar{\tau}} - 2iw\tau_2 \left( \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \bar{\tau}} \right) - w(w-1),
\]
and
\[
\nabla^2_+ w = 4\bar{D}_{w-1} D_{-w} = 4\tau^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \bar{\tau}} - 2iw\tau_2 \left( \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \bar{\tau}} \right) - w(w+1),
\]
\[
\nabla^2_(-) w - 2w.
\]

Now consider a $(w,-w)$ form that is an eigenfunction of the Laplace operator $\nabla^2_(-) w$ with eigenvalue $\sigma_\text{w},$
\[
\nabla^2_(-) w F^{(w,-w)} = 4D_{w-1} \bar{D}_{-w} F^{(w,-w)} = \sigma_w F^{(w,-w)}.
\]
Applying $\bar{D}_{-w}$ to this equation gives,
\[
\nabla^2_+ w^{-1,-w+1} F^{(w,-w)} = \sigma_w F^{(w-1,-w+1)}.
\]
It is also easy to see that,
\[
\nabla^2_(-) w^{-1,-w+1} F^{(w-1,-w+1)} = 4D_{w-2} \bar{D}_{-w+1} F^{(w-1,-w+1)},
\]
\[
\nabla^2_(-) w^{-1,-w+1} F^{(w-1,-w+1)} = (\sigma_w + 2w - 2) F^{(w-1,-w+1)}
\]
where $F^{(w-1,-w+1)} = \bar{D}_{-w} F^{(w,-w)}.$ Repeating this for $m$ steps gives
\[
\nabla^2_(-) w^{-m,-w+m} F^{(w-m,-w+m)} = 4D_{w-m-1} \bar{D}_{-w+m+1} F^{(w-m,-w+m)},
\]
\[
\nabla^2_(-) w^{-m,-w+m} F^{(w-m,-w+m)} = (\sigma_w + 2mw - m^2 - m) F^{(w-m,-w+m)}
\]
Similarly,
\[
\nabla^2_+ w^{-m} F^{(w-m,-w+m)} = (\sigma_w + 2mw - 2w - m^2 + m) F^{(w-m,-w+m)}.
\]
This relation between eigenvalue equations will be useful in analyzing the equations that are satisfied by the modular forms that enter in $S^{(3)}$.

In section 4 it is proved that $f^{(12,-12)}$ satisfies
\[
\nabla^2_(-) 12 f^{(12,-12)} = \left(-132 + \frac{3}{4}\right) f^{(12,-12)}.
\]
This is (B.11) with \( w = 12, m = 0 \) and \( \sigma_{12} = -132 + 3/4 \). It is also argued in section 3 that \( f^{(12,-12)} \) is related to \( f^{(0,0)} \) by
\[
f^{(12,-12)} = D^{12} f^{(0,0)} \equiv D_{11} \cdots D_1 D_0 f^{(0,0)}.
\] (B.14)
This equation together with (B.13) imply that \( f^{(0,0)} \) satisfies (4.25).

More generally, let us denote a solution of the scalar Laplace equation with eigenvalue \( \sigma = s(s-1) > 1/4 \) by \( E_s(\tau) \) \[57\],
\[
\nabla^2 E_s = s(s-1) E_s.
\] (B.15)
We can express \( E_s(\tau) \) in terms of the nonholomorphic Eisenstein series,
\[
E_s(\tau) = \frac{1}{2} \tau_2^s \sum_{(m,n)=1} |m\tau + n|^{-2s},
\] (B.16)
where \((m,n) = 1\) indicates that \( m \) and \( n \) are coprime. The eigenfunctions \( E_s(\tau) \) are singled out by their power law behavior near the boundary of the moduli space, which agrees with the known tree-level and perturbative contributions to the interactions that we are considering. The solution to the Laplace equation (B.15) is unique for a given \( s \) assuming that \( E_s \) is a modular function. Therefore, if we assume that \( f^{(0,0)} \) is well-defined on the fundamental domain of \( SL(2, \mathbb{Z}) \) the solution is unique and its behaviour near the \( \tau \to \infty \) boundary agrees with the known tree-level and one-loop contributions. It follows from (B.14) that \( f^{(12,-12)} \) is also determined uniquely by its Laplace equation (B.13). In this case there are no string calculations with which to compare the perturbative terms.

C Some properties of Type IIB Supergravity

C.1 Spinors and gamma matrices

The spinors that enter into the IIB theory are complex Weyl spinors. The gravitino \( \psi_\mu \) and dilatino \( \lambda \) have opposite chiralities and the supersymmetry parameter \( \epsilon \) has the same chirality as the gravitino. The complex conjugate of the product of a pair of spinors is defined by
\[
(\lambda_a \rho_b)^* = -\lambda_a^* \rho_b^*.
\] (C.1)
The conjugate of any spinor is defined by, \( \bar{\lambda} = \lambda^* \gamma^0 \). We will choose our metric to be space-like and the \( \gamma \) matrices to be real and satisfy the Clifford algebra,
\[
\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.
\] (C.2)
Noting that,
\[ \gamma^0 \gamma^\mu = - (\gamma^\mu)^T \gamma^0, \] (C.3)
it follows that two complex chiral spinors of the same chirality, \( \lambda_1 \) and \( \lambda_2 \), satisfy the relations,
\[ \bar{\lambda}_1 \gamma^\mu \lambda_2 = - \bar{\lambda}_2^s \gamma^\mu \lambda_1^s, \] (C.4)
while two chiral spinors of opposite chiralities, \( \lambda \) and \( \epsilon \), satisfy,
\[ \bar{\lambda} \epsilon = \bar{\epsilon}^* \lambda^*, \] (C.5)
The Fierz identity for ten-dimensional complex Weyl spinors can be expressed as,
\[ \lambda_1^s \bar{\lambda}_2^s = - \frac{1}{16} \bar{\lambda}_2 \gamma^\mu \lambda_1 \gamma^\nu_{ab} + \frac{1}{96} \bar{\lambda}_2 \gamma^{\mu\nu\rho} \lambda_1 \gamma^{\nu\rho}_{ab} - \frac{1}{3840} \bar{\lambda}_2 \gamma_{\rho_1 ... \rho_5} \lambda_1 \gamma_{\rho_1 ... \rho_5}^{ab}, \] (C.6)
where \( \lambda_1 \) and \( \lambda_2 \) are two chiral spinors of the same chirality.
An additional useful identity is,
\[ \gamma^{\rho_1 ... \rho_5} \lambda_1 \bar{\lambda}_2 \gamma_{\rho_1 ... \rho_5} \lambda_3 = 0, \] (C.7)
where \( \lambda_1 \), \( \lambda_2 \) and \( \lambda_3 \) are three chiral spinors of the same chirality.
Some gamma matrix identities that are useful in proving the various relationships in the text are,
\[ \text{tr}(\gamma_{\mu\nu\rho} \gamma^{\rho_1 ... \rho_5}_{\mu\nu\rho}) = -16 \left( \delta_{\mu}^{\rho_1} \delta_{\nu}^{\rho_2} \delta_{\rho}^{\rho_3} - \delta_{\mu}^{\rho_2} \delta_{\nu}^{\rho_3} \delta_{\rho}^{\rho_1} + \delta_{\mu}^{\rho_3} \delta_{\nu}^{\rho_1} \delta_{\rho}^{\rho_2} - \delta_{\mu}^{\rho_1} \delta_{\nu}^{\rho_2} \delta_{\rho}^{\rho_3} \right). \] (C.8)
\[ \gamma^\mu \gamma_\sigma \gamma_\mu = -8 \gamma_\sigma, \]
\[ \gamma^\mu \gamma_{\sigma_1 \sigma_2 \sigma_3} \gamma_\mu = -4 \gamma_{\sigma_1 \sigma_2 \sigma_3}, \]
\[ \gamma^\mu \gamma_{\sigma_1 ... \sigma_5} \gamma_\mu = 0, \]
\[ \gamma^{\mu\nu\rho} \gamma_\sigma \gamma_{\mu\nu\rho} = -288 \gamma_\sigma, \] (C.9)
\[ \gamma^{\mu\nu\rho} \gamma_{\sigma_1 \sigma_2 \sigma_3} \gamma_{\mu\nu\rho} = -48 \gamma_{\sigma_1 \sigma_2 \sigma_3}, \]
\[ \gamma^{\mu\nu\rho} \gamma_{\sigma_1 ... \sigma_5} \gamma_{\mu\nu\rho} = -14 \gamma_{\sigma_1 ... \sigma_5}. \]
In section 4 the first two terms in parentheses on the right-hand-side of (3.10) are considered in detail. The precise notation used in those terms uses the following definitions

\[(\lambda^r)_{a_{r+1}\ldots a_{16}} \equiv \frac{1}{r!} \epsilon_{a_1\ldots a_{16}} \lambda^{a_1} \ldots \lambda^{a_r}, \tag{C.10}\]

so that,

\[\lambda^{16} = \frac{1}{16!} \epsilon_{a_1\ldots a_{16}} \lambda^{a_1} \ldots \lambda^{a_{16}}, \tag{C.11}\]

and

\[\hat{G}\lambda^{14} \equiv \hat{G}_{\mu
u\rho}(\gamma^{\mu
u\rho}\gamma^0)_{a_{15}a_{16}}(\lambda^{14})_{a_{15}a_{16}}, \tag{C.12}\]

The following identities follow very simply from (C.10),

\[(\lambda^{14})_{ab} \lambda^c = (\lambda^{15})_b \delta^c_a - (\lambda^{15})_a \delta^c_b, \tag{C.13}\]

and

\[(\lambda^{14})_{ab} \lambda_c \lambda_d = \lambda^{16} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}). \tag{C.14}\]

\section*{C.2 The fields and their supersymmetry transformations}

Here we will review various features of type IIB supergravity that are useful in the body of the paper. Most of this material can be found in \cite{58,59} in a form that is adapted to the field definitions in which the global symmetry is \(SU(1, 1)\) and the scalar fields parameterize the coset space \(SU(1, 1)/U(1)\), which is the Poincaré disk. It is simple to transform this to our parameterization in which the global symmetry is \(SL(2, \mathbb{R})\) and the scalars parameterize the coset space \(SL(2, \mathbb{R})/U(1)\), or the upper half plane.

The theory is then defined in terms of the following fields: the scalar fields can be parameterized by the frame field,

\[
V \equiv \begin{pmatrix} V^1_+ & V^1_- \\ V^2_+ & V^2_- \end{pmatrix} = \frac{1}{\sqrt{-2i\tau_2}} \begin{pmatrix} \tau e^{-i\phi} & \tau e^{i\phi} \\ e^{-i\phi} & e^{i\phi} \end{pmatrix}, \tag{C.15}\]

where \(V^\alpha_\beta (\alpha = 1, 2)\) is a \(SL(2, \mathbb{Z})\) matrix that transforms from the left by the global \(SL(2, \mathbb{R})\) and from the right by the local \(U(1)\). Note that we are using a complex basis for convenience. A general transformation is then written as,

\[
(V^\alpha_+, V^\alpha_-) \rightarrow U^\alpha_\beta (V^\beta_+ e^{i\Theta}, V^\beta_- e^{-i\Theta}), \tag{C.16}\]
where $U$ is a $SL(2, \mathbb{R})$ matrix and $\Sigma$ is the $U(1)$ phase. An appropriate choice of $\Sigma$ fixes the gauge and eliminates the scalar field $\phi$. We will make the gauge choice $\phi = 0$. Since this gauge is not maintained by generic symmetry transformations, it is necessary to compensate a symmetry transformation with an appropriate local $U(1)$ transformation to maintain the gauge. In particular, the local supersymmetry transformations require compensating local $U(1)$ transformations. The supersymmetry and $U(1)$ transformations of $V^\alpha$ are given by,

$$\delta^{(0)}V^\alpha = iV^\alpha \epsilon^\alpha - i\Sigma V^\alpha.$$  \hspace{1cm} (C.17)

This choice ensures that the gauge $\phi = 0$ is maintained if a local supersymmetry transformation is accompanied by a $U(1)$ transformation with parameter,

$$\Sigma = \frac{1}{2}(\bar{\epsilon} \lambda^* - \bar{\epsilon}^* \lambda).$$  \hspace{1cm} (C.18)

The $SL(2, \mathbb{R})$ singlet expression,

$$Q_\mu = -i \epsilon_{\alpha\beta} V^\alpha_+ \partial_\mu V^\beta_-, $$  \hspace{1cm} (C.19)

is the composite $U(1)$ connection and transforms as $Q \to Q + \partial_\mu \Sigma$ under infinitesimal local $U(1)$ transformations, while the $SL(2, \mathbb{R})$ singlet expression

$$P_\mu = -\epsilon_{\alpha\beta} V^\alpha_+ \partial_\mu V^\beta_-,$$  \hspace{1cm} (C.20)

transforms with $U(1)$ charge $u_P = 2$. In the gauge $\phi = 0$, the expression for $P_\mu$ takes the simple form,

$$P_\mu = \frac{i}{2} \bar{\rho} \gamma_2,$$  \hspace{1cm} (C.21)

while

$$Q_\mu = -i \epsilon_{\alpha\beta} V^\alpha_+ \partial_\mu V^\beta_- = -\frac{1}{2} \bar{\rho} \rho_1.$$  \hspace{1cm} (C.22)

The fermions comprise the complex chiral gravitino, $\psi^a_\mu$, which has $U(1)$ charge $u_\psi = 1/2$, and the dilatino, $\lambda^a$, with $U(1)$ charge $u_\lambda = 3/2$. These two fields have opposite chiralities. The graviton is a $U(1)$ and $SL(2, \mathbb{R})$ singlet as is the antisymmetric fourth-rank potential, $C^{(4)}$, which has a field strength $F_5 = dC^{(4)}$. As is well known, this field strength has an equation of motion that is expressed by the self-duality condition $F_5 = *F_5$, which cannot be obtained from a globally well-defined Lagrangian. For this reason, our considerations are restricted to statements concerning the on-shell properties of the theory where the fields satisfy the equations of motion.

The two antisymmetric second-rank potentials, $B_{\mu\nu}$ and $C^{(2)}_{\mu\nu}$, have field strengths $F^1$ ($NS \otimes NS$) and $F^2$ ($R \otimes R$) that form an $SL(2, \mathbb{R})$ doublet, $F^\alpha$. It is very natural to package them into the $SL(2, \mathbb{R})$ singlet fields,

$$G = -\epsilon_{\alpha\beta} V^\alpha_+ F^\beta, \quad G^\star = -\epsilon_{\alpha\beta} V^\alpha_- F^\beta,$$  \hspace{1cm} (C.23)
which carry $U(1)$ charges $u_G = +1$ and $u_{G^*} = -1$, respectively.

In a fixed $U(1)$ gauge, a global $SL(2, \mathbb{R})$ transformation which acts on $\tau$ by

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad (C.24)$$

with $ad - bc = 1$, induces a $U(1)$ transformation on the fields that depends on their charge. Thus, a field $\Phi$ with $U(1)$ charge $u_\Phi$ transforms as,

$$\Phi \rightarrow \Phi \left( \frac{c\bar{\tau} + d}{c\tau + d} \right)^{u_\Phi/2}, \quad (C.25)$$

The higher derivative terms of interest to us only respect the $SL(2, \mathbb{Z})$ subgroup of $SL(2, \mathbb{R})$ for which $a, b, c, d$ are integers and the continuous $U(1)$ symmetry is broken.

The supersymmetry of the action is naturally described in terms of combinations of bosonic fields and fermion bilinears which are ‘supercovariant’, which means that they do not contain derivatives of the supersymmetry parameter $\epsilon$ in their transformations. These combinations are,

$$\hat{G}_{\mu\nu \rho} = G_{\mu\nu \rho} - 3\bar{\psi}_{[\mu} \gamma_{\nu \rho]} \lambda - 6i\bar{\psi}_{[\mu} \gamma_{\nu} \psi_{\rho]},
\hat{P}_\mu = P_\mu - \bar{\psi}^* \lambda,
\hat{F}_{5, \mu_1, ..., \mu_5} = F_{5, \mu_1, ..., \mu_5} - 5\bar{\psi}_{[\mu_1} \gamma_{\mu_2 \mu_3 \mu_4} \psi_{\mu_5]} - \frac{1}{16} \bar{\lambda} \gamma_{\mu_1, ..., \mu_5} \lambda. \quad (C.26)$$

We will now present the lowest-order supersymmetry transformations, suitably adapted from those given in [59] to the $SL(2, \mathbb{R})$ parameterization. From (C.17) and (C.18), it follows that

$$\delta^{(0)} \tau = 2\tau_2 \bar{\epsilon}^* \lambda, \quad \delta^{(0)} \bar{\tau} = -2\tau_2 \epsilon \lambda^*. \quad (C.27)$$

It follows from the definition of $Q_\mu$ and the transformations of $\tau$ and $\bar{\tau}$ that

$$\delta^{(0)} Q_\mu = -\bar{\epsilon} \lambda^* P_\mu + c.c. \quad (C.28)$$

Also, the supersymmetry transformation of the zehnbein is given by,

$$\delta^{(0)} e^m_\mu = i(\bar{\epsilon} \gamma^m \psi_\mu + \epsilon^* \gamma^m \psi^*_\mu). \quad (C.29)$$

The transformation of the dilatino is given, in the fixed $U(1)$ gauge, by

$$\delta^{(0)} \lambda = i\gamma^\mu \epsilon^* \hat{P}_\mu - \frac{1}{24} i\gamma^{\mu \nu \rho} \epsilon \hat{G}_{\mu \nu \rho} + \delta^{(0)} \Sigma \lambda
= i\gamma^\mu \epsilon^* \hat{P}_\mu + \frac{i}{8} \gamma^{\mu \nu \tau} \epsilon \left( \bar{\psi}_{[\mu} \gamma_{\nu \tau]} \lambda \right) - i\gamma^\mu \epsilon^* \left( \bar{\psi}^*_\mu \lambda \right) + \delta^{(0)} \Sigma \lambda + \ldots \quad (C.30)$$
where we have only kept the terms that are needed in the body of this paper in the second line. The $\delta \Sigma$ arises from the compensating $U(1)$ gauge transformation,

$$\delta^{(0)} \Sigma_a = \frac{3}{2} i \Sigma \lambda_a = \frac{3}{4} i \lambda_a (\epsilon^* \lambda) - \frac{3}{4} i \lambda_a (\bar{\epsilon} \lambda).$$  \hfill (C.31)

The gravitino transformation is given by,

$$\delta^{(0)} \psi_{\mu} = D_{\mu} \epsilon + \frac{1}{480} \gamma_{\rho_1 \cdots \rho_5} \gamma_{\mu} \bar{F}_{\rho_1 \cdots \rho_5} + \frac{1}{96} \left( \gamma_{\mu} \gamma^{\rho} \hat{G}_{\nu \rho \lambda} - 9 \gamma_{\rho} \hat{G}_{\mu \rho \lambda} \right) \epsilon^* -\frac{7}{16} \left( \gamma_{\rho} \lambda \bar{\psi}_{\mu} \gamma^\rho \epsilon^* - \frac{1}{1680} \gamma_{\rho_1 \cdots \rho_5} \lambda \bar{\psi}_{\mu} \gamma^{\rho_1 \cdots \rho_5} \epsilon^* \right) +\frac{1}{32} i \left[ \left( \frac{9}{4} \gamma_{\rho} \gamma^\rho + 3 \gamma^\rho \gamma_{\rho} \right) \epsilon \bar{\lambda} \gamma_{\rho} \lambda \right. \right.$$  

$$\left. - \left( \frac{1}{24} \gamma_{\rho_1 \cdots \rho_5} \gamma_{\rho_1 \cdots \rho_5} \gamma_{\mu} \bar{\lambda} \gamma_{\rho_1 \cdots \rho_5} \lambda + \frac{1}{960} \gamma_{\mu} \gamma^{\rho_1 \cdots \rho_5} \epsilon \bar{\lambda} \gamma_{\rho_1 \cdots \rho_5} \lambda \right) \right] + \delta^{(0)} \Sigma (\psi_{\mu}),$$  \hfill (C.32)

where the compensating $U(1)$ transformation is given by

$$\delta^{(0)} \Sigma (\psi_{\mu}) = \frac{1}{2} i \Sigma = \frac{1}{4} i \psi_{\mu} (\epsilon^* \lambda) - \frac{1}{4} i \psi_{\mu} (\bar{\epsilon} \lambda).$$  \hfill (C.33)

By using (C.6) and (C.32) extensively we may manipulate the variation of $\gamma^\mu \psi^*_\mu$ into the form,

$$\delta^{(0)} (\gamma^\mu \psi^*_\mu) = -\frac{3}{4} i \lambda_a (\epsilon^* \lambda) + \frac{1}{1920} i (\gamma^{\rho_1 \cdots \rho_5} \epsilon^*)^a \left( \bar{\lambda} \gamma_{\rho_1 \cdots \rho_5} \lambda \right) + \ldots,$$  \hfill (C.34)

where we have only kept the terms bilinear in $\lambda, \lambda^*$. This implies the relation,

$$\left( \lambda \right)^{15} \delta^{(0)} (\gamma^\mu \psi^*_\mu) = -15 i \lambda^{16} (\bar{\epsilon}^*) + \ldots,$$  \hfill (C.35)

which we use in the body of the text.

**D Fermions in M theory on $T^2$**

The massless fermions of the type II string theories on $S^1$ are identified with particular projections of the eleven-dimensional gravitino compactified on $T^2$. In order to calculate the $\lambda^{16}$ loop amplitude we need to identify the particular projection that isolates the spin-half dilatino of the type IIB theory.

Compactification of the IIB theory on a circle $S^1$ of circumference $r_B$ (in string frame) in the direction $x^9$ breaks the $SO(9,1)$ Lorentz symmetry to $SO(8,1)$. The complex chiral spin-$\frac{1}{2}$ fermion $\lambda$ simply becomes a complex
spinor of $SO(8,1)$. The gravitino decomposes into a nine-dimensional gravitino $\psi_\alpha$ (where the $SO(8,1)$ vector index $\alpha = 0,1,\cdots,8$) together with a second complex spin-$\frac{1}{2}$ fermion,

$$\chi^A = r_B \psi_9^A,$$

where the factor of $r_B$ comes from the component $e_{99}$ of the zweibein. The nine-dimensional gravitino is defined by shifting $\psi_a$,

$$\hat{\psi}_a = \psi_a + \frac{1}{7} \Gamma_a \Gamma^9 \chi$$

(D.2)

(where $a$ labels the nine-dimensional tangent space, $a = 0,1\cdots,8$) so that the kinetic term is diagonal.

We will now identify the components of the $T^2$ compactification of eleven-dimensional gravitino, $\Psi_{\hat{\mu}}$ ($\hat{\mu} = 0,\cdots,9,11$), that correspond to the IIB fermions on $S^1$. Compactification on $T^2$ breaks the local Lorentz symmetry from $SO(10,1)$ to $SO(8,1) \times SO(2)$. The nine-dimensional fermions can then be organized into eigenstates of $SO(2) \equiv U(1)$, which is related to the $U(1)$ in the denominator of the coset space of the IIB theory. The world indices split into the compact directions $\sigma = 9,11$ and the noncompact directions $\alpha = 0,\cdots,8$.

Making a block diagonal ansatz, the elfbein $e_{\hat{\mu}}$, is written as

$$e_{\hat{\mu}} = \left( \begin{array}{cc} e^s_\sigma & 0 \\ 0 & e^a_\alpha \end{array} \right)$$

(D.3)

where $a$ again labels the nine-dimensional tangent space and $s$ the two-dimensional tangent space ($s = 1,2$). The zweibein, $e^s_\sigma$, of $T^2$ may be chosen, in a special Lorentz frame, to be

$$e^s_\sigma = \sqrt{\frac{V}{\Omega^2}} \begin{pmatrix} \Omega_2 & \Omega_1 \\ 0 & 1 \end{pmatrix}$$

(D.4)

Symplectic reparametrizations of the torus act as $SL(2,\mathbb{R})$ matrices from the left and local Lorentz transformations act as $SO(2)$ transformations from the right. The condition that the zweibein remains in the frame (D.4) leads to the standard $SL(2,\mathbb{Z})$ transformation of the complex structure of the torus (which is (C.24) with $\tau$ replaced by $\Omega$) and induces a specific $\Omega$-dependent $U(1)$ transformation on the fermions.

The spin-$\frac{1}{2}$ components of the compactified gravitino are written in a complex basis $z = x_{11} + i x_9$, $\bar{z} = x_{11} - i x_9$, as $\Psi_z = \Psi_{11} + i \Psi_9$ and $\Psi_{\bar{z}} = \Psi_{11} - i \Psi_9$. To relate these 32-component spinors to $\lambda$ and $\chi$ of the IIB theory we shall first organize them into eigenstates of the $U(1)$ rotations of the compact $T^2$ generated by $i \Gamma^{11} \Gamma^9 / 2 + j_{vec}$, where $j_{vec}$ acts on the vector index so that $j_{vec} \Psi_z = \Psi_{\bar{z}}$ and

70
$j_{vec} \Psi \bar{z} = - \Psi \bar{z}$. Those components of the spin connection that have tangent-space indices in the compact directions are given by

$$\omega_{z,\alpha z} = \frac{i}{2} \frac{\partial_\alpha \Omega}{\Omega_2}, \quad \omega_{\bar{z},\alpha \bar{z}} = \frac{i}{2} \frac{\partial_\alpha \Omega}{\Omega_2}, \quad \omega_{z,\alpha \bar{z}} = \omega_{\bar{z},\alpha z} = \frac{1}{V} \partial_\alpha V.$$  

(D.5)

The type IIB complex scalar field is identified with $\Omega$ so the quantities $P_\alpha = \omega_{z,\alpha z}$ and $\bar{P}_\alpha = \omega_{\bar{z},\alpha \bar{z}}$ correspond to the fields with the same symbol defined in appendix C. The third real scalar field in the nine-dimensional theory is $\omega_{z,\alpha \bar{z}} = -\frac{4}{\Omega_2} \partial_\alpha \ln \frac{E_B}{3}$. We shall make use of projectors made out of the eleven-dimensional $\Gamma$ matrices. The complex combinations, $\Gamma_z = \frac{1}{2}(\Gamma_{11} + i \Gamma_9)$ and $\Gamma_{\bar{z}} = \frac{1}{2}(\Gamma_{11} - i \Gamma_9)$, obey

$$\Gamma_z^2 = \Gamma_{\bar{z}}^2 = 0, \quad \{\Gamma_z, \Gamma_{\bar{z}}\} = 1.$$  

(D.6)

so that the products

$$P_z = (P_z)^T = \Gamma_{\bar{z}} \Gamma_z = \frac{1}{2}(1 - i \Gamma_9 \Gamma_{11}), \quad P_{\bar{z}} = (P_{\bar{z}})^T = \Gamma_z \Gamma_{\bar{z}} = \frac{1}{2}(1 + i \Gamma_9 \Gamma_{11})$$  

(D.7)

(where the superscript $T$ indicates the transpose) are projectors, satisfying

$$P_z^2 = P_z, \quad P_{\bar{z}}^2 = P_{\bar{z}}, \quad P_z P_{\bar{z}} = 0.$$  

(D.8)

The action of these projectors on a 32-component real spinor is to replace it with a complex sixteen component spinor which is an eigenstate of the $U(1)$ spin, $i \Gamma_9 \Gamma_{11}/2$. Therefore, we may identify the four spin-1/2 fermions of the nine-dimensional compactified theory with the components,

$$P_z \Psi_z, \quad P_{\bar{z}} \Psi_{\bar{z}},$$  

(D.9)

$$P_z \Psi_{\bar{z}}, \quad P_{\bar{z}} \Psi_z.$$  

(D.10)

The components $P_z \Psi_z$ have $U(1)$ charges $+\frac{3}{2}$ and $-\frac{1}{2}$, respectively, while the components $P_{\bar{z}} \Psi_{\bar{z}}$ are a mixture of the states with $U(1)$ charges $\pm \frac{1}{2}$. These fields are simply related to the spin-$\frac{1}{2}$ fermions of the IIB theory by converting $\lambda$ and $\chi$ (and their complex conjugates) from eigenstates of $\Gamma^{11}$ to eigenstates of $i \Gamma_9 \Gamma^{11}$. This leads to the identifications,

$$P_z \Psi_z = \Gamma_{\bar{z}} \lambda, \quad P_{\bar{z}} \Psi_{\bar{z}} = \Gamma_z \chi$$  

(D.11)

which gives the eigenstates of $\Gamma^{11}$,

$$\lambda = (1 + \Gamma_{\bar{z}}) P_z \Psi_z, \quad \chi = (1 + \Gamma_z) P_{\bar{z}} \Psi_{\bar{z}}.$$  

(D.12)
In terms of the real components
\[
\lambda_1 = \psi_{11}^+ - \Gamma^9 \psi_9^- , \quad \lambda_2 = \psi_9^+ + \Gamma^9 \psi_{11}^- , \tag{D.13}
\]
where the superscripts \( \pm \) indicate the chirality (the value of \( \Gamma^{11} \)). Similarly, writing \( \chi = \chi_1 + i\chi_2 \) and multiplying (D.11) by \((1 + \Gamma^{11})\) leads to
\[
\chi_1 = \Gamma^9 \psi_9^+ - \psi_{11}^- , \quad \chi_2 = -\psi_9^- - \Gamma^9 \psi_{11}^+ . \tag{D.14}
\]

The remaining components, \( \Psi_a \), form the nine-dimensional gravitino after a shift similar to (D.2),
\[
\hat{\Psi}_a = \Psi_a + \frac{1}{7} \Gamma_a (\Gamma_z \Psi_{\bar{z}} + \Gamma_{\bar{z}} \Psi_z) = \Psi_a + \frac{i}{7} \Gamma_a \Gamma^9 (P_z \chi - P_{\bar{z}} \chi^*) . \tag{D.15}
\]

The relations between the M-theory fields and the IIB fields can be confirmed by comparing the way they behave under supersymmetry transformations.

For completeness the IIB fermions can be related to the IIA fermions in a straightforward manner. The spin-1/2 fermions in the IIA theory follow by direct dimensional reduction of eleven-dimensional supergravity [101]. The nine-dimensional fermions that arise from the spin-\( \frac{3}{2} \) fields in ten dimensions are
\[
\lambda_A = \psi_{11} + \Gamma^9 \Gamma^{11} \psi_9 , \tag{D.16}
\]
which decomposes into the chiral components,
\[
\lambda^+_A = \psi_{11}^+ - \Gamma^9 \psi_9^- , \quad \lambda^-_A = \psi_9^+ + \Gamma^9 \psi_{11}^- . \tag{D.17}
\]
Similarly,
\[
\chi_A = \psi_9 + \Gamma^9 \Gamma^{11} \psi_{11} , \tag{D.18}
\]
which has chiral components
\[
\chi^-_A = \psi_9^- + \Gamma^9 \psi_{11}^+ , \quad \chi^+_A = \psi_{11}^+ - \Gamma^9 \psi_9^- . \tag{D.19}
\]

Comparing (D.13) with (D.17) and (D.14) with (D.19) gives the identification of fields of IIA and IIB in nine dimensions,
\[
\lambda_1 = \lambda^+_A , \quad \lambda_2 = \Gamma^9 \lambda^-_A , \quad \chi_2 = -\chi^-_A , \quad \chi_1 = \Gamma^9 \chi^+_A , \tag{D.20}
\]
which are in agreement with the world sheet T-duality rules.

References

[1] E. Witten, *String theory dynamics in various dimensions*, [hep-th/9503124](http://arxiv.org/abs/hep-th/9503124), Nucl. Phys. B443 (1995) 85.
[2] P. Aspinwall, Some Relationships Between Dualities in String Theory, in Proceedings of ‘S-duality and mirror symmetry’, Trieste 1995, hep-th/9508154; Nucl. Phys. Proc. 46 (1996) 30.

[3] J.H. Schwarz, Lectures on Superstring and M-theory dualities, hep-th/9607201; J.H. Schwarz, An SL(2, \Z) multiplet of type IIb superstrings, hep-th/9508143; Phys. Lett. 360B (1995) 13.

[4] E. Cremmer, B. Julia and J. Scherk, Supergravity theory in eleven dimensions, Phys. Lett. 76B (1978) 409.

[5] C.M. Hull and P.K. Townsend, Unity of superstring dualities, hep-th/9410167; Nucl. Phys. B438 (1995) 109; P.K. Townsend, The eleven-dimensional supermembrane revisited, hep-th/9501068; Phys. Lett. B350 (1995) 184.

[6] G. Dall’Agata G.K. Lechner, Dmitri Sorokin, Covariant actions for the bosonic sector of d = 10 IIB supergravity, hep-th/9707044; Class. Quant. Grav. 14 (1997) L195.

[7] G. Dall’Agatai, K. Lechner, M. Tonin, Action for IIB supergravity in 10-dimensions, 2nd Conference on Quantum Aspects of Gauge Theories, Supersymmetry and Unification, Corfu, Greece, 21-26 Sep 1998; hep-th/9812170; D=10, N = IIB supergravity: Lorentz invariant actions and duality, hep-th/9806140; JHEP 9807 (1998).

[8] C. Vafa and E. Witten, A one-loop test of string-string duality, hep-th/9505053; Nucl. Phys. B447 (1995) 261.

[9] C.G. Callan and J.A. Harvey, Anomalies and fermion zero modes on strings and domain walls, Nucl. Phys. B250 (1985) 427.

[10] M.J. Duff, J.T. Liu and R. Minasian, Eleven-dimensional origin of string-string duality: a one loop test, hep-th/9506124; Nucl. Phys. 452B (1995)261.

[11] M.B. Green and M. Gutperle, Effects of D-instantons, hep-th/9701093; Nucl. Phys. B498 (1997)195.

[12] M.B. Green and P. Vanhove, D-instantons, Strings and M-theory, hep-th/9704145; Phys. Lett. B408 (1997) 122.

[13] P.S. Howe and P.C. West, The complete N = 2, d = 10 supergravity, Nucl. Phys. B238 (1984) 181.
[14] M.B. Green, M. Gutperle and H. Kwon, *Sixteen Fermion and Related Terms in M theory on $T^2$*, hep-th/9710151; Phys. Lett. **B421** (1998) 149.

[15] M.B. Green and S. Sethi, *Supersymmetry constraints on type IIB supergravity*, hep-th/9808061.

[16] M.B. Green, M. Gutperle and P. Vanhove, *One Loop in Eleven-Dimensions*, hep-th/9706175; Phys. Lett. **B409** (1998) 149.

[17] E. Kiritsis and B. Pioline, *On $R^4$ threshold corrections in IIB string theory and (p,q) string instantons*, hep-th/9707018; Nucl. Phys. **B508** (1997) 509.

[18] E. Kiritsis and B. Pioline, *U-duality and D-brane Combinatorics*, hep-th/9700078; Phys. Lett. **418B** (1998) 61.

[19] C. Bachas, C. Fabre, E. Kiritsis, N.A. Obers and P. Vanhove, *Heterotic / type I duality and D-brane instantons*, hep-th/9707126; Nucl. Phys. **B509** (1998) 33.

[20] C. Bachas, *Heterotic versus Type I*, Talk at Strings’97 (Amsterdam, 1997), hep-th/9710102.

[21] E. Kiritsis and N.A. Obers, *Heterotic/Type-I Duality in D<10 Dimensions, Threshold Corrections and D-Instantons*, hep-th/9709058; JHEP **9710** (1997) 004.

[22] J. Russo and A.A. Tseytlin, *One-loop four-graviton amplitude in eleven-dimensional supergravity*, hep-th/970713; Nucl. Phys. **B508** (1997) 245.

[23] J.G. Russo, *An ansatz for a non-perturbative four-graviton amplitude in type IIB superstring theory*, hep-th/9707241; Phys. Lett. **B417** (1998) 253.

[24] A. Kehagias and H. Partouche, *D-Instanton Corrections as (p,q)-String Effects and Non-Renormalization Theorems*, hep-th/9712164; Int. J. Mod. Phys. **A13** (1998) 5075.

[25] A. Kehagias and H. Partouche, *The exact quartic effective action for the type IIB superstring*, hep-th/9710023; Phys. Lett. **B422** (1998) 109.

[26] N. Berkovits and C. Vafa, *Type IIB $R^4H^{4g-4}$ Conjectures*, hep-th/9803143; Nucl. Phys. **B533** (1998).

[27] I. Antoniadis, B. Pioline and T.R. Taylor, *Calculable $e^{-1/\lambda}$ effects*, hep-th/9707222; Nucl. Phys. **B512** (1998) 61.
[28] N. Berkovits, *Construction of $R^4$ terms in $N=2$ $D=8$ superspace*, hep-th/9709116; Nucl. Phys. B514 (1998) 191.

[29] B. Pioline, *A note on non-perturbative $R^4$ couplings*, hep-th/9804023; Phys. Lett. B431 (1998) 73.

[30] E. Witten, *Bound States of Strings and p-branes*, hep-th/9510135; Nucl. Phys. B460 (1996) 335.

[31] M.B. Green and M. Gutperle, *D-particle bound states and the D-instanton measure*, hep-th/9711107; JHEP 9801 (1998) 005.

[32] P. Yi, *Witten index and threshold bound states of D-branes*, hep-th/9704098; Nucl. Phys. B505 (1997) 307.

[33] S. Sethi and M. Stern, *D-brane bound states redux*, hep-th/9705046; Comm. Math. Phys. 194 (1998) 675.

[34] J. Maldacena, *The large $N$ limit of superconformal field theories and supergravity*, hep-th/9711200; Adv. Theor. Math. Phys. 2 (1998) 231.

[35] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, hep-th/9802109; Phys. Lett. B428 (1998) 105.

[36] E. Witten, *Anti de Sitter space and holography*, hep-th/9802150; Adv. Theor. Math. Phys. 2 (1998) 253.

[37] D.I. Olive and C. Montonen, *Magnetic monopoles as gauge particles?*, Phys. Lett. 72B (1977) 117.

[38] A. Sen, *Dyon - monopole bound states, selfdual harmonic forms on the multi - monopole moduli space, and SL(2,Z) invariance in string theory*, hep-th/9402032.

[39] N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *Multiinstantons and Maldacena’s conjecture*, hep-th/9810243.

[40] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, *Yang-Mills instantons in the large N limit and the $AdS/CFT$ correspondence*, hep-th/9808157; Phys. Lett. B329 (1994) 217.

[41] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, *Multi-Instanton Calculus and the $AdS/CFT$ Correspondence in $N=4$ Superconformal Field Theory*, hep-th/9901128.
[42] J. Polchinski, *Tasi lectures on D-branes*, hep-th/9611050.

[43] N. Marcus and J.H. Schwarz, *Field theories that have no manifestly Lorentz invariant formulation*, Phys. Lett. 115B (1982) 111.

[44] E. Witten, *Five-brane effective action in M theory*, hep-th/9610234; J. Geom. Phys. 22 (1997) 103.

[45] M.B. Green and J.H. Schwarz, *Supersymmetrical string theories*, Phys. Lett. 109B (1982) 444.

[46] M.B. Green and J.H. Schwarz, *Supersymmetric dual string theory (III). Loops and renormalization*, Nucl. Phys. 198B (1982) 441.

[47] D.J. Gross and E. Witten, *Superstring modifications of Einstein’s equations*, Nucl. Phys. B277 (1986) 1.

[48] M.T. Grisaru, A.E.M Van de Ven and D. Zanon, *Two-dimensional supersymmetric sigma models on Ricci flat Kähler manifolds are not finite*, Nucl. Phys. B277 (1986) 388; *Four loop divergences for the N=1 supersymmetric nonlinear sigma model in two-dimensions*, Nucl. Phys. B277 (1986) 409.

[49] I. Antoniadis, S. Ferrara, R. Minasian and K.S. Narain, *R^4 couplings in M and type II theories on Calabi-Yau spaces*, hep-th/9707013; Nucl. Phys. B507 (1997) 571.

[50] A. Strominger, *Loop corrections to the universal hypermultiplet*, hep-th/9706193; Phys. Lett. B421 (1998) 139.

[51] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, *String theory and classical absorption by three-branes*, hep-th/9703040.

[52] Pawelczyk and S. Theisen, *AdS_5 x S^5 black hole metric at O(α'^3)*, hep-th/9808126; JHEP 9809 (1998) 010.

[53] G.W. Gibbons, M.B. Green and M.V. Perry, *Instantons and seven-branes in type IIB superstring theory*, Phys. Lett. B370 (1996) 37, hep-th/9511080.

[54] J. Polchinski, *Combinatorics of boundaries in string theory*, hep-th/9407031; Phys. Rev. D50 (1994) 6041.

[55] M.B. Green *A Gas of D instantons*, hep-th/9504108; Phys. Lett. B354 (1995) 271.

[56] M.B. Green, *Pointlike structure and off-shell dual strings*, Nucl.Phys. B124 (1977) 461.
[57] A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications I*, Springer-Verlag (New York) 1985.

[58] J.H. Schwarz and P.C. West, *Symmetries and transformations of chiral N = 2 D = 10 supergravity*, Phys. Lett. **126B** (1983) 301.

[59] J.H. Schwarz, *Covariant field equations of chiral N = 10, D = 10 supergravity*, Nucl. Phys. **B226** (1983) 269.

[60] B.E.W. Nilsson and A. Tollsten, *Supersymmetrization of ζ(3)R_{µνρσ} in superstring theories*, Phys. Lett **181B** (1986) 63.

[61] N. Berkovits and C. Vafa, *N=4 Topological Strings*, hep-th/9407190. Nucl. Phys.**B433** (1995) 123.

[62] H. Ooguri and C. Vafa, *All Loop N=2 String Amplitudes*, hep-th/9505183. Nucl. Phys. **B451** (1995) 121.

[63] N.A. Obers and B. Pioline, *U duality and M theory*, hep-th/9809039. Phys. Rep. (to be published).

[64] M.B. Green, *Connections between M-theory and superstrings*, Talk at Strings’97, hep-th/9712195. Nucl. Phys. B, Proc. Suppl. 68 (1998) 242-251.

[65] M.B. Green, M. Gutperle and H. Kwon, (in preparation).

[66] M.B. Green and M. Gutperle, *D-instanton partition functions*, hep-th/9804123. Phys.Rev. **D58** (1998) 46007.

[67] M. Porrati and Rozenberg, *Bound states at threshold in supersymmetric quantum mechanics*, hep-th/9802119. Nucl. Phys. **B515** (1998) 184.

[68] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large N reduced model as superstring*, hep-th/9612115. Nucl. Phys. **B498** (1997) 467.

[69] V. Periwal, *Matrices on a point as the theory of everything*, hep-th/9611103. Phys. Rev. **D55** (1977) 1711.

[70] W. Krauth, H. Nicolai, M. Staudacher, *Monte Carlo approach to M theory*, hep-th/9803117. Phys. Lett. **B431** (1998) 31.

[71] I.K. Kostov and P. Vanhove, *Matrix string partition functions*, hep-th/9809130. Phys. Lett. **B444** (1998) 196.

[72] M.B. Green and M. Gutperle, *Configurations of two D-instantons*, hep-th/9612127. Phys. Lett. **398** (1997) 69.
[73] G. Moore, N. Nekrasov, S. Shatashvili, D particle bound states and generalized instantons, hep-th/9803265.

[74] A. Strominger, Massless black holes and conifolds in string theory, hep-th/9504090. Nucl. Phys. B451 (1995) 96.

[75] T. Banks and M.B. Green, Nonperturbative effects in $AdS_5 \times S^5$ string theory and $d = 4$ SUSY Yang-Mills, hep-th/9804170. JHEP 9805 (1998) 2.

[76] M. Bianchi, M.B. Green, S. Kovacs and G.C. Rossi, Instantons in supersymmetric Yang-Mills and D-instantons in IIB superstring theory, hep-th/9807033; JHEP 9808 (1998) 13.

[77] S.S. Gubser, I.R. Klebanov, and A.W. Peet, Entropy and temperature of black 3-branes, hep-th/9602135. Phys. Rev. D54 (1996) 3915.

[78] O. Aharony and E. Witten, Anti-de Sitter space and the center of the gauge group, hep-th/9807205.

[79] S. Ferrara, C. Fronsdal and A. Zaffaroni, On $N = 8$ supergravity in $AdS_5$ and $N = 4$ superconformal Yang–Mills theory, hep-th/9802203.

[80] G. 't Hooft, A planar diagram theory for strong interactions, Nucl. Phys. B72 (1974) 461.

[81] W. Mück and K.S. Viswanathan, Conformal field theory correlators from classical field theory on anti-de Sitter space, I and II, hep-th/9804033 and hep-th/9805145. G. Chalmers, H. Nastase, K. Schalm and R. Siebelink, R-current correlators in N=4 Super Yang–Mills theory from anti-de Sitter supergravity, hep-th/9805103.

[82] H.J. Kim, L.J. Romans, and P. van Nieuwenhuizen, Mass spectrum of chiral ten-dimensional $N = 2$ supergravity on $S^2$, Phys. Rev. D32 (1985) 389; M. Gunaydin and N. Marcus, The unitarity supermultiplet of $N=8$ conformal superalgebra involving fields of spin $\leq 2$, Class. Quant. Grav. 2 (1985) L11-17.

[83] P. Howe, K.S. Stelle and P.K. Townsend, Supercurrents, Nucl. Phys. B192 (1981) 332.

[84] E. Bergshoeff, M. de Roo and B. de Wit, Extended conformal supergravity, Nucl. Phys. B182 (1981) 173.

[85] D.Z. Freedman, S.D. Mathur, A. Matusis, L. Rastelli, Comments on 4 point functions in the CFT / AdS correspondence hep-th/9808006.
[86] J. Erdmenger and H. Osborn, *Conformally covariant differential operators: Symmetric tensor fields*, gr-qc/9708040; Class. Quantum Grav. 15 (1998) 273.

[87] S.S. Gubser and I.R. Klebanov, *Absorption by branes and Schwinger terms in the world volume theory*, hep-th/9708005; Phys. Lett. B413 (1997) 41.

[88] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Correlation Functions in the CFT(D)/ADS(D+1) Correspondence*, hep-th/9804058.

[89] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, *Coupling Constant Dependence in the Thermodynamics of N=4 Supersymmetric Yang-Mills Theory*, hep-th/9805156; Nucl. Phys. B534 (1998) 202.

[90] R. Kallosh and A. Rajaraman, *Vacua of M-theory and string theory*, hep-th/9805041; Phys. Rev. D58 (1998) 125003.

[91] H. Liu and A.A. Tseytlin, *D = 4 superYang-Mills, D = 5 gauged supergravity, and D = 4 conformal supergravity*, hep-th/9804083.

[92] J.H. Brodie and M. Gutperle, *String corrections to four point functions in the AdS / CFT correspondence*, hep-th/9809067.

[93] M. Henningson and K. Sfetsos, *Spinors in the AdS/CFT correspondence*, hep-th/9803251; Phys. Lett. B431 (1998) 63.

[94] A.M. Ghezelbash, K. Kaviani, S. Parvizi and A.H. Fatollahi, *Interacting spinors-scalars and AdS/CFT correspondence*, hep-th/9805162; Phys. Lett. B435 (1998) 291.

[95] R.I. Nepomechie, *Magnetic monopoles from antisymmetric tensor gauge fields*, Phys. Rev. D31 (1985) 1921.

[96] C. Teitelboim, *Gauge invariance for extended objects*, Phys. Lett. 167B (1986) 63; *Monopoles of higher rank*, Phys. Lett. B167 (1986) 69.

[97] G. ’t Hooft, *Dimensional reduction in quantum gravity*, Essay dedicated to Abdus Salam. Salamfest 1993:284-296; hep-th/9310026.

[98] L. Susskind, *The world as a hologram*, hep-th/9409081; J. Math. Phys. 36 (1995) 6377.

[99] C. Bernard, *Gauge zero modes, instanton determinants, and QCD calculations*, Phys. Rev. D19 (1979) 3013.
[100] G. ’t Hooft, *Computation of quantum effects due to a four dimensional pseudoparticle*, Phys. Rev. D14 (1976) 3432; erratum: ibid. D18 (1978) 2199.

[101] I.C.G. Campbell and P.C. West, *N=2 D = 10 Nonchiral supergravity and its spontaneous compactification*, Nucl. Phys. B243 (1984) 112.