GEOMETRIC Riemann SCHEME OF THE Painlevé EQUATIONS

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Abstract. In this paper, we introduce the notion of geometric Riemann scheme of the sixth Painlevé equation, which consists of the pair of accessible singular points and matrix of linear approximation around each singular point on the boundary divisor in the Hirzebruch surface. Giving this in the differential system satisfying certain conditions, we can recover the Painlevé VI system with the polynomial Hamiltonian. We give a generalization of the Painlevé VI system by generalizing the geometric Riemann scheme of the sixth Painlevé equation. This system has movable branch points. Nevertheless, we show that this system has rich birational symmetries. We also consider the case of the Painlevé V, IV and III systems. Finally, we study non-linear ordinary differential systems in dimension two with only simple accessible singular points. We show the existence theorem of these equations, which can be considered as a non-linear version of the existence theorem of Fuchsian differential equations from the viewpoint of geometrical property.

1. Introduction

For a linear differential equation of Fuchs type, we can make a Riemann scheme. This is the pair of singularity and local exponent. Conversely, by giving the Riemann scheme satisfying the Fuchs relation, we can recover a linear differential equation of Fuchs type.

Painlevé equations are second-order non-linear ordinary differential equations. Here, we consider the following problem.

Problem 1.1. Can we construct a generalization of the Riemann scheme for each Painlevé equation?

Since the solutions of the Painlevé equations are transcendental functions, it is difficult to make one in the same way as linear differential equations.

So, we give up an analytic generalization of the Riemann scheme, and we want to aim at a generalization of the Riemann scheme from the viewpoint of its geometrical property.
In this paper, we consider the case of the sixth Painlevé equation. The sixth Painlevé equation is equivalent to the following Hamiltonian system:

\[
\begin{align*}
\frac{dx}{dt} &= \frac{\partial H_{VI}}{\partial y} = \frac{1}{t(t-1)} \{2y(x-t)(x-1)x - (\alpha_0 - 1)(x-1)x - \alpha_3(x-t)x \\
&\quad - \alpha_4(x-t)(x-1)\}, \\
\frac{dy}{dt} &= -\frac{\partial H_{VI}}{\partial x} = \frac{1}{t(t-1)} [\{-(x-t)(x-1) + (x-t)x + (x-1)x\}y^2 + \{(\alpha_0 - 1)(2x-1) \\
&\quad + \alpha_3(2x-t) + \alpha_4(2x-t-1)\}y - \alpha_2(\alpha_1 + \alpha_2)]
\end{align*}
\]

with the polynomial Hamiltonian

\[ H_{VI}(x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{1}{t(t-1)} [y^2(x-t)(x-1)x - \{(\alpha_0 - 1)(x-1)x + \alpha_3(x-t)x \\
&\quad + \alpha_4(x-t)(x-1)\}y + \alpha_2(\alpha_1 + \alpha_2)x] (\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1). \]

Since each right hand side of this system is polynomial with respect to \(x, y\), by Cauchy’s existence and uniqueness theorem of solutions, there exists unique holomorphic solution with initial values \((x, y) = (x_0, y_0) \in \mathbb{C}^2\).

Let us extend the regular vector field defined on \(\mathbb{C}^2 \times B\)

\[ v = \frac{\partial}{\partial t} + \frac{\partial H_{VI}}{\partial y} \frac{\partial}{\partial x} - \frac{\partial H_{VI}}{\partial x} \frac{\partial}{\partial y} \]

to a rational vector field on \(\Sigma_2 \times B\), where \(B = \mathbb{C} - \{0, 1\}\).

Here, we review the Hirzebruch surface \(\Sigma_2\), which is obtained by gluing four copies of \(\mathbb{C}^2\) via the following identification.

\[
U_j \cong \mathbb{C}^2 \ni (z_j, w_j) \ (j = 0, 1, 2, 3)
\]

\[
\begin{align*}
z_0 &= x, \quad w_0 = y, \quad z_1 = \frac{1}{x}, \quad w_1 = -(xy + \alpha_0)x, \\
z_2 &= z_0, \quad w_2 = \frac{1}{w_0}, \quad z_3 = z_1, \quad w_3 = \frac{1}{w_1}.
\end{align*}
\]
We define a divisor $D^{(0)}$ on $\Sigma_2$:

$$D^{(0)} = \{(z_2, w_2) \in U_2|w_2 = 0\} \cup \{(z_3, w_3) \in U_3|w_3 = 0\} \cong \mathbb{P}^1.$$  

The self-intersection number of $D^{(0)}$ is given by

$$\left(D^{(0)}\right)^2 = 2.$$  

In the coordinate system $(z_1, w_1)$ the right hand side of this system is polynomial with respect to $z_1, w_1$. However, on the boundary divisor $D^{(0)} \cong \mathbb{P}^1$ this system has a pole in each coordinate system $(z_i, w_i)$ $i = 2, 3$. By calculating the accessible singular points on $D^{(0)}$, we obtain simple four singular points $z_2 = 0, 1, t, \infty$ (see Definition 2.1).

By rewriting the system at each singular point, this rational vector field has a pole along the divisor $D^{(0)}$, whose order is one.

By resolving all singular points, we can construct the space of initial conditions of the Painlevé VI system (see [4]). This space parametrizes all meromorphic solutions including holomorphic solutions.

Conversely, we can recover the Painlevé VI system by all patching data of its space of initial conditions. At first, we decompose its patching data into the pair of singular points and local index around each singular point. In the next section, we review the notion of accessible singularity and local index.

2. Accessible singularity and local index

Let us review the notion of accessible singularity. Let $B$ be a connected open domain in $\mathbb{C}$ and $\pi: \mathcal{W} \rightarrow B$ a smooth proper holomorphic map. We assume that $\mathcal{H} \subset \mathcal{W}$ is a normal crossing divisor which is flat over $B$. Let us consider a rational vector field $\tilde{v}$ on $\mathcal{W}$ satisfying the condition

$$\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})).$$  

Fixing $t_0 \in B$ and $P \in \mathcal{W}_{t_0}$, we can take a local coordinate system $(x_1, \ldots, x_n)$ of $\mathcal{W}_{t_0}$ centered at $P$ such that $\mathcal{H}_{\text{smooth}}$ can be defined by the local equation $x_1 = 0$. Since $\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H}))$, we can write down the vector field $\tilde{v}$ near $P = (0, \ldots, 0, t_0)$ as follows:

$$\tilde{v} = \frac{\partial}{\partial t} + g_1 \frac{\partial}{\partial x_1} + \frac{g_2}{x_1} \frac{\partial}{\partial x_2} + \cdots + \frac{g_n}{x_1} \frac{\partial}{\partial x_n}.$$  

This vector field defines the following system of differential equations

$$\frac{dx_1}{dt} = g_1(x_1, \ldots, x_n, t), \quad \frac{dx_2}{dt} = \frac{g_2(x_1, \ldots, x_n, t)}{x_1}, \ldots, \quad \frac{dx_n}{dt} = \frac{g_n(x_1, \ldots, x_n, t)}{x_1}.$$  

Here $g_i(x_1, \ldots, x_n, t)$, $i = 1, 2, \ldots, n$, are holomorphic functions defined near $P = (0, \ldots, 0, t_0)$.

**Definition 2.1.** With the above notation, assume that the rational vector field $\tilde{v}$ on $\mathcal{W}$ satisfies the condition

$$(A) \quad \tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})).$$
We say that $\tilde{v}$ has an accessible singularity at $P = (0, \ldots, 0, t_0)$ if
\[ x_1 = 0 \text{ and } g_i(0, \ldots, 0, t_0) = 0 \text{ for every } i, \ 2 \leq i \leq n. \]

If $P \in \mathcal{H}_{\text{smooth}}$ is not an accessible singularity, all solutions of the ordinary differential equation passing through $P$ are vertical solutions, that is, the solutions are contained in the fiber $\mathcal{W}_0$ over $t = t_0$. If $P \in \mathcal{H}_{\text{smooth}}$ is an accessible singularity, there may be a solution of (6) which passes through $P$ and goes into the interior $\mathcal{W} - \mathcal{H}$ of $\mathcal{W}$.

Here we review the notion of local index. Let $v$ be an algebraic vector field with an accessible singular point $\overrightarrow{P} = (0, \ldots, 0)$ and $(x_1, \ldots, x_n)$ be a coordinate system in a neighborhood centered at $\overrightarrow{P}$. Assume that the system associated with $v$ near $\overrightarrow{P}$ can be written as

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_{n-1} \\
  x_n
\end{pmatrix} = \frac{1}{x_1} \begin{pmatrix}
  a_{11} & 0 & 0 & \cdots & 0 \\
  a_{21} & a_{22} & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \cdots & \vdots \\
  a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)(n-1)} & 0 \\
  a_{1n} & a_{2n} & \cdots & a_{(n-1)n} & a_{nn}
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_{n-1} \\
  x_n
\end{pmatrix} + \begin{pmatrix}
  x_1 h_1(x_1, \ldots, x_n, t) \\
  h_2(x_1, \ldots, x_n, t) \\
  \vdots \\
  h_{n-1}(x_1, \ldots, x_n, t) \\
  h_n(x_1, \ldots, x_n, t)
\end{pmatrix},
\end{align*}
\]

where $h_i \in \mathbb{C}(t)[x_1, \ldots, x_n]$, $a_{ij} \in \mathbb{C}(t)$. Here, $h_1$ is a polynomial which vanishes at $\overrightarrow{P}$ and $h_i$, $i = 2, 3, \ldots, n$ are polynomials of order at least 2 in $x_1, x_2, \ldots, x_n$. We call ordered set of the eigenvalues $(a_{11}, a_{22}, \cdots, a_{nn})$ local index at $\overrightarrow{P}$.

We are interested in the case with local index

\[
(1, a_{22}/a_{11}, \ldots, a_{nn}/a_{11}) \in \mathbb{Z}^n.
\]

These properties suggest the possibilities that $a_1$ is the residue of the formal Laurent series:

\[
y_1(t) = \frac{a_{11}}{(t - t_0)} + b_1 + b_2(t - t_0) + \cdots + b_n(t - t_0)^{n-1} + \cdots \quad (b_i \in \mathbb{C}),
\]

and the ratio $(1, a_{22}/a_{11}, \ldots, a_{nn}/a_{11})$ is resonance data of the formal Laurent series of each $y_i(t)$ ($i = 2, \ldots, n$), where $(y_1, \ldots, y_n)$ is original coordinate system satisfying $(x_1, \ldots, x_n) = \langle f_1(y_1, \ldots, y_n), \ldots, f_n(y_1, \ldots, y_n), f_i(y_1, \ldots, y_n) \rangle \in \mathbb{C}(t)[y_1, \ldots, y_n]$.

If each component of $(1, a_{22}/a_{11}, \ldots, a_{nn}/a_{11})$ has the same sign, we may resolve the accessible singularity by blowing-up finitely many times. However, when different signs appear, we may need to both blow up and blow down.

The $\alpha$-test,

\[
t = t_0 + \alpha T, \quad x_i = \alpha X_i, \quad \alpha \to 0,
\]
yields the following reduced system:

(11) \[
\frac{d}{dT} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix} = \frac{1}{X_1} \begin{pmatrix} a_{11}(t_0) & 0 & 0 & \cdots & 0 \\ a_{21}(t_0) & a_{22}(t_0) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(n-1)1}(t_0) & a_{(n-1)2}(t_0) & \cdots & a_{(n-1)(n-1)}(t_0) & 0 \\ a_{n1}(t_0) & a_{n2}(t_0) & \cdots & a_{n(n-1)}(t_0) & a_{nn}(t_0) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix},
\]

where \( a_{ij}(t_0) \in \mathbb{C} \). Fixing \( t = t_0 \), this system is the system of the first order ordinary differential equation with constant coefficient. Let us solve this system. At first, we solve the first equation:

(12) \[
X_1(T) = a_{11}(t_0)T + C_1 \quad (C_1 \in \mathbb{C}).
\]

Substituting this into the second equation in (11), we can obtain the first order linear ordinary differential equation:

(13) \[
\frac{dX_2}{dT} = \frac{a_{22}(t_0)X_2}{a_{11}(t_0)T + C_1} + a_{21}(t_0).
\]

By variation of constant, in the case of \( a_{11}(t_0) \neq a_{22}(t_0) \) we can solve explicitly:

(14) \[
X_2(T) = C_2(a_{11}(t_0)T + C_1)\alpha_{a_{11}(t_0)} + \frac{a_{21}(t_0)(a_{11}(t_0)T + C_1)}{a_{11}(t_0) - a_{22}(t_0)} \quad (C_2 \in \mathbb{C}).
\]

This solution is a single-valued solution if and only if

\[
\frac{a_{22}(t_0)}{a_{11}(t_0)} \in \mathbb{Z}.
\]

In the case of \( a_{11}(t_0) = a_{22}(t_0) \) we can solve explicitly:

(15) \[
X_2(T) = C_2(a_{11}(t_0)T + C_1) + \frac{a_{21}(t_0)(a_{11}(t_0)T + C_1)\Log(a_{11}(t_0)T + C_1)}{a_{11}(t_0)} \quad (C_2 \in \mathbb{C}).
\]

This solution is a single-valued solution if and only if

\[
a_{21}(t_0) = 0.
\]

Of course, \( \frac{a_{22}(t_0)}{a_{11}(t_0)} = 1 \in \mathbb{Z} \). In the same way, we can obtain the solutions for each variables \( (X_3, \ldots, X_n) \). The conditions \( \frac{a_{ij}(t_0)}{a_{11}(t_0)} \in \mathbb{Z}, \ (j = 2, 3, \ldots, n) \) are necessary condition in order to have the Painlevé property.

Now, let us rewrite the system centered at each singular point \( X = 0, 1, t, \infty \).

1. By taking the coordinate system \( (X, Y) = (z_2, w_2) \) centered at the point \( (z_2, w_2) = (0, 0) \), the system is given by

\[
\frac{d}{dt} \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{Y} \left\{ \begin{pmatrix} 2 & -\frac{a_1}{t-1} \\ 0 & \frac{1}{t-1} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \cdots \right\}.
\]
Now, let us make a change of variables $X, Y, t$ with a small parameter $\kappa$

\begin{equation}
X = \kappa Z, \quad Y = \kappa W, \quad t = t_0 + \kappa T \quad (t_0 \in \mathbb{C} - \{0, 1\}).
\end{equation}

Then the system can also be written in the new variables $Z, W, T$. This new system tends to the system as $\kappa \to 0$

\begin{equation}
\frac{d}{dT} \begin{pmatrix} Z \\ W \end{pmatrix} = \frac{1}{W} \left\{ \begin{pmatrix} \frac{2}{t_0 - 1} & -\alpha_4 \\ 0 & \frac{1}{t_0 - 1} \end{pmatrix} \begin{pmatrix} Z \\ W \end{pmatrix} \right\}.
\end{equation}

Fixing $t = t_0$, this system is the system of the first order ordinary differential equation with constant coefficient. Let us solve this system. At first, we solve the second equation:

\begin{equation}
W(T) = \frac{T}{t_0 - 1} + C_1 \quad (C_1 \in \mathbb{C}).
\end{equation}

Substituting this into the first equation in (17), we can obtain the first order linear ordinary differential equation:

\begin{equation}
\frac{dZ}{dT} = \frac{t_0 - 1}{T + C_1(t_0 - 1)} \left( \frac{2}{t_0 - 1} Z - \frac{\alpha_4}{t_0 - 1} \left( \frac{T}{t_0 - 1} + C_1 \right) \right).
\end{equation}

By variation of constant, we can solve explicitly:

\begin{equation}
Z(T) = C_2 \left( T + (t_0 - 1)C_1 \right)^2 + \frac{\alpha_4(T + (t_0 - 1)C_1)}{t_0 - 1} \quad (C_2 \in \mathbb{C}).
\end{equation}

Thus, we can obtain single-valued solutions. For the Painlevé property, this is the necessary condition.

In the same way, we can obtain the following:

2. By taking the coordinate system $(X, Y) = (z_2 - 1, w_2)$ centered at the point $(z_2, w_2) = (1, 0)$, the system is given by

\begin{equation}
\frac{d}{dt} \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{Y} \left\{ \begin{pmatrix} -\frac{2}{t} & -\alpha_3 \\ 0 & -\frac{1}{t} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \cdots \right\}
\end{equation}

3. By taking the coordinate system $(X, Y) = (z_2 - t, w_2)$ centered at the point $(z_2, w_2) = (t, 0)$, the system is given by

\begin{equation}
\frac{d}{dt} \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{Y} \left\{ \begin{pmatrix} 2 & -\alpha_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \cdots \right\}
\end{equation}

4. By taking the coordinate system $(X, Y) = (z_3, w_3)$ centered at the point $(z_3, w_3) = (0, 0)$, the system is given by

\begin{equation}
\frac{d}{dt} \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{Y} \left\{ \begin{pmatrix} \frac{2}{t(t-1)} & -\alpha_1 \\ 0 & \frac{1}{t(t-1)} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \cdots \right\}
\end{equation}

Conversely, we can recover the system by giving accessible singular points and matrix of linear approximation around each point.
The pair of accessible singular point and matrix of linear approximation around each point is called \textit{geometric Riemann scheme}.

3. Recovery of the Painlevé VI system

Let us consider the system of the first order ordinary differential equations of polynomial type.

\[
\begin{cases}
\frac{dx}{dt} = f_1(x, y), \\
\frac{dy}{dt} = f_2(x, y) \quad (f_i \in \mathbb{C}(t)[x, y]).
\end{cases}
\]  

We assume that associated vector field defined on \( \mathbb{C}^2 \times B \)
\[
v = \frac{\partial}{\partial t} + f_1(x, y) \frac{\partial}{\partial x} + f_2(x, y) \frac{\partial}{\partial y}
\]
belongs in
\[
v \in H^0(\Sigma_2, \Theta_{\Sigma_2}(-\log \mathcal{H})(\mathcal{H})).
\]
This condition is equivalent to the following:

(1) Holomorphy in the coordinate system \((x_1, y_1) = (1/x, -(xy + \alpha_2)x)\),
(2) In the coordinate system \((X, Y) = (x, 1/y)\), the differential system (22) must be taken of the form:

\[
\begin{cases}
\frac{dX}{dt} = \frac{F_1(X, Y)}{Y}, \\
\frac{dY}{dt} = F_2(X, Y) \quad (F_i \in \mathbb{C}(t)[X, Y]).
\end{cases}
\]  

\textbf{Proposition 3.1.} Under above assumptions 1 and 2, the system (22) is given by

\[
\begin{cases}
\frac{dx}{dt} = a_1x^3y + a_2x^2y + \frac{1}{2}((3a_1 + 2a_3a_2 - a_4)x^2 + a_5xy + ((a_2 + a_9)a_2 - a_6)x + a_7y + a_8, \\
\frac{dy}{dt} = a_3x^2y^2 + a_9xy^2 + a_{10}y^2 + a_4xy + a_6y + \frac{1}{2}((a_1a_3 + a_4)a_2 \quad (a_i \in \mathbb{C}(t)).
\end{cases}
\]

Here, \(a_i = a_i(t), \ (i = 1, 2, \ldots, 10)\) are undetermined coefficients.
In the coordinate system \((x_1, y_1) = (1/x, -(xy + \alpha_2)x)\), the system (24) can be rewritten as follows:

\[
\frac{dx}{dt} = a_7x_1^4y_1 + a_5x_1^2y_1 + a_2x_1y_1 + a_1x_1y_1 + \alpha_2a_7x_1^3 + (\alpha_2a_5 - a_8)x_1^2 + (a_6 - \alpha_2a_9)x_1 - \frac{1}{2}\alpha_2(a_1 + 2a_3) + \frac{a_4}{2},
\]

\[
\frac{dy}{dt} = -2a_7x_1^2y_1 - (2a_5 + a_{10})x_1^2y_1 - 3\alpha_2a_7x_1^2y_1 - (2a_2 + a_9)x_1y_1^2 - (2a_1 + a_3)y_1^2
\]

\[
- \{\alpha_2(3a_5 + 2a_{10}) - 2a_8\}x_1y_1 - \alpha_2^2a_7x_1 - (\alpha_2a_2 + a_6)y_1 - \alpha_2\{\alpha_2(a_5 + a_{10}) - a_8\}.
\]

**Proof of Proposition 3.1.**

(i) Degree of polynomials \(f_i(x, y)\) with respect to \(y\)

If the system (22) belongs in \(H^0(\Sigma_2, \Theta_{\Sigma_2}( - \log \mathcal{H})(\mathcal{H}))\), in the coordinate system \((X, Y) = (x, 1/y)\) this system must be taken of the form:

\[
\left\{ \begin{array}{l}
\frac{dX}{dt} = \frac{F_1(X, Y)}{Y}, \\
\frac{dY}{dt} = F_2(X, Y) \quad (F_i \in \mathbb{C}(t)[X, Y]).
\end{array} \right.
\]

By this condition, we see that the system (22) must be taken of the form:

\[
\left\{ \begin{array}{l}
\frac{dx}{dt} = b_1(x) + b_2(x)y, \\
\frac{dy}{dt} = b_3(x) + b_4(x)y + b_5(x)y^2 \quad (b_i \in \mathbb{C}(t)[x]).
\end{array} \right.
\]

Here, the degree of each \(b_i\) with respect to \(x\) is given by

\[
\text{deg}(b_1) = l, \quad \text{deg}(b_2) = m, \quad \text{deg}(b_3) = n, \quad \text{deg}(b_4) = p, \quad \text{deg}(b_5) = r,
\]

where \(l, m, n, p, r \in \mathbb{N}\).

(ii) Holomorphy in the coordinate system \((x_1, y_1) = (1/x, -(xy + \alpha_2)x)\)

In the coordinate system \((x_1, y_1) = (1/x, -(xy + \alpha_2)x)\), the first equation of the system (27) is given by

\[
\frac{dx_1}{dt} = -x_1^2 \left\{ b_1 \left( \frac{1}{x_1} \right) + b_2 \left( \frac{1}{x_1} \right) \left( -x_1^2y_1 - \alpha_2x_1 \right) \right\}.
\]

Since the right hand side of this system must be polynomial with respect to \(x_1\), we compare two terms

\[
\left\{ \begin{array}{l}
b_1 \left( \frac{1}{x_1} \right) = \frac{b_1^{(0)}}{x_1^l} + \cdots, \\
b_2 \left( \frac{1}{x_1} \right) = -\alpha_2 \frac{b_2^{(m)}}{x_1^{m-1}} + \cdots.
\end{array} \right.
\]
Since \( b_1^{(l)} \neq 0 \) and \( b_2^{(m)} \neq 0 \), we can obtain

(31) \[ l = m - 1 \]

Next, we compare the term involving \( y_1 \):

(32) \[-x_1^4 b_2 \left( \frac{1}{x_1} \right) y_1 = -x_1^4 \left( \frac{b_2^{(m)}}{x_1^m} + \frac{b_2^{(m-1)}}{x_1^{m-1}} + \cdots \right) y_1 \quad \text{for} \quad (b_2^{(j)} \in \mathbb{C}(t)).\]

If this becomes polynomial with respect to \( x_1, y_1 \),

(33) \[ m = 4 \]

In the same way, we can obtain

(34) \[ \deg(b_1) = 3, \quad \deg(b_2) = 4, \quad \deg(b_3) = 1, \quad \deg(b_4) = 2, \quad \deg(b_5) = 3 \]

Finally, by comparing undetermined coefficients, we can obtain the conclusion.

For the system (24), by giving the following geometric Riemann scheme we can recover the Painlevé VI system with the polynomial Hamiltonian \( H_{VI} \).

**Theorem 3.2.** For the system (24), we give the following geometric Riemann scheme:

(35) \[
\begin{pmatrix}
X = 0 \\
X = 1 \\
X = t \\
X = \infty
\end{pmatrix}
\begin{pmatrix}
f_0 \left( \begin{array}{cc} 2 & -\alpha_4 \\ 0 & 1 \end{array} \right) \\
f_1 \left( \begin{array}{cc} 2 & -\alpha_3 \\ 0 & 1 \end{array} \right) \\
f_2 \left( \begin{array}{cc} 2 & -\alpha_0 \\ 0 & 1 \end{array} \right) \\
f_3 \left( \begin{array}{cc} 2 & -\alpha_1 \\ 0 & 1 \end{array} \right)
\end{pmatrix}
\]

Here, \( X = 0, 1, t, \infty \) are accessible singular points, \( f_i \in \mathbb{C}(t) \) and \( \alpha_i \) are constant parameters. Then, this system coincides with the Painlevé VI system with the polynomial Hamiltonian \( H_{VI} \).

**Proof of Theorem 3.2.** At first, we can rewrite the system (24) in the coordinate system \((X, Y) = (x, 1/y)\) centered at \((X, Y) = (0, 0)\)

(36) \[
\begin{cases}
\frac{dX}{dt} = \frac{a_1 X^3 + a_2 X^2 + a_5 X + a_7}{Y} + \frac{1}{2} \{(3a_1 + 2a_3)\alpha_2 - a_4\} X^2 + \{(a_2 + a_9)\alpha_2 - a_6\} X + a_8, \\
\frac{dY}{dt} = -a_{10} - a_9 X - a_3 X^2 - a_4 X Y - a_6 Y - \frac{1}{2}(a_1 \alpha_2 + a_4)\alpha_2 Y^2 \quad (a_i \in \mathbb{C}(t)).
\end{cases}
\]

By Definition 2.1 we can calculate the accessible singular points

(37) \[ Y = 0, \quad a_1 X^3 + a_2 X^2 + a_5 X + a_7 = 0. \]

By the assumption, \( X = Y = 0 \) is a solution of the system (37). Thus, we obtain the condition

\[ a_7 = 0. \]
By the assumption, the matrix of linear approximation around $X = 0$ is given by

$$f_0 \begin{pmatrix} 2 & -\alpha_4 \\ 0 & 1 \end{pmatrix}.$$  

So, we obtain

$$a_5 = -2a_{10}, \quad a_8 = \alpha_4 a_{10}.$$  

In the same way, we can obtain the conditions at each singular point $X = 1, t, \infty$. Thus, we have completed the proof of Theorem 3.2.

4. A Generalization of the Painlevé VI System

By generalizing the eigenvalues $(2,1)$ to $(n_i, 1) \ (i = 1, 2, 3, 4)$ in (35), we will construct a generalization of the Painlevé VI system.

**Theorem 4.1.** For the system (24), we give the following geometric Riemann scheme:

$$
\begin{pmatrix}
X = 0 & X = 1 & X = t & X = \infty \\
\begin{pmatrix} n_1 & \alpha_4 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} n_2 & \alpha_3 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} n_3 & \alpha_0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} n_4 & \alpha_1 \\ 0 & 1 \end{pmatrix}
\end{pmatrix}.
$$

Here, $X = 0, 1, t, \infty$ are accessible singular points, $f_i \in \mathbb{C}(t), \ n_i \in \mathbb{C}$ and $\alpha_i$ are constant parameters. Then, this system coincides with

$$
\delta t (t-1) \frac{dx}{dt} = n_1 n_2 n_3 x (x - 1) (t - x) y + \left\{ (2n_1 n_2 n_3 - n_1 n_2 - n_1 n_3 - n_2 n_3) \alpha_1 - n_1 n_2 n_3 \alpha_2 \right\} x^2 \\
+ \left\{ -(2n_1 n_2 n_3 - n_1 n_2 - n_1 n_3 - n_2 n_3) \alpha_1 + n_1 n_2 n_3 \alpha_2 + n_1 n_3 \alpha_3 (t - 1) \\
+ n_2 n_3 \alpha_4 t \right\} x - n_2 n_3 \alpha_4 t,
$$

$$
\delta t (t-1) \frac{dy}{dt} = [(n_1 n_2 + n_2 n_3 + n_1 n_3) x^2 - \left\{ n_1 n_2 + n_2 n_3 + (n_1 + n_2) n_3 t \right\} x + n_2 n_3 t] y^2 \\
+ \left\{ 2(n_1 n_2 n_3 - n_1 n_2 - n_1 n_3 - n_2 n_3) \alpha_1 \\
+ (-2n_1 n_2 - 2n_1 n_3 - 2n_2 n_3 + n_1 n_2 n_3) \alpha_2 \right\} x \\
+ (-n_1 n_2 - n_1 n_3 - n_2 n_3 + 2n_1 n_2 n_3) \alpha_1 \\
+ \left\{ (n_1 n_2 - n_1 - n_2) n_3 t - n_1 n_2 - n_2 n_3 \right\} \alpha_2 \\
- n_1 n_3 \alpha_3 (t - 1) - n_2 n_3 \alpha_4 t \right\} y - \alpha_2 [(-n_1 n_2 - n_1 n_3 - n_2 n_3 + 2n_1 n_2 n_3) \alpha_1 \\
+ (-n_1 n_2 - n_1 n_3 - n_2 n_3 + n_1 n_2 n_3) \alpha_2].
$$

Here, $\delta := n_1 n_2 \alpha_0 + (2n_1 n_2 n_3 - n_1 n_2 - n_1 n_3 - n_2 n_3) \alpha_1 - n_1 n_2 n_3 \alpha_2 + n_1 n_3 \alpha_3 + n_2 n_3 \alpha_4$.

**Proposition 4.2.** By using the Painlevé $\alpha$-method, we see that the system (40) has movable branch points.
Proposition 4.3. The eigenvalues $n_i$ satisfy the following relation:

\[ \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} = 2. \]

Proof of Proposition 4.3 For the system (36), we put $f := a_1 X^3 + a_2 X^2 + a_5 X + a_7$. Since the cubic equation $f = 0$ has the solutions $X = 0, 1, t$, from the relation between solution and coefficient we obtain

\[ -\frac{a_2}{a_1} = 0 + 1 + t = t + 1, \]
\[ \frac{a_5}{a_1} = 0 \times 1 + 0 \times t + 1 \times t = t. \]

We summarize that

\[ a_2 = -(t + 1)a_1, \]
\[ a_5 = ta_1. \]

The equation $f$ is given by

\[ f = a_1 (X^2 - (t + 1)X + t)X = a_1 (X - 0)(X - 1)(X - t). \]

Thus, we can obtain

\[
\begin{align*}
\frac{dX}{dt} &= \frac{a_1 (X - 0)(X - 1)(X - t)}{Y} + \frac{1}{2} \{(3a_1 + 2a_3)\alpha_2 - a_4\} X^2 + \{(a_2 + a_9)\alpha_2 - a_6\} X + a_8, \\
\frac{dY}{dt} &= -a_{10} - a_9 X - a_3 X^2 - a_4 XY - a_6 Y - \frac{1}{2} (a_1 \alpha_2 + a_4) \alpha_2 Y^2 \quad (a_i \in \mathbb{C}(t)).
\end{align*}
\]

Next, by giving the eigenvalues of the matrix of linear approximation around each point $X = 0, 1, t$, we can obtain

\[
\begin{align*}
ta_1 &= n_1 (-a_{10}), \\
(1 - t) a_1 &= n_2 (-a_{10} - a_9 - a_3), \\
t(t - 1) a_1 &= n_3 (-a_{10} - ta_9 - t^2 a_3).
\end{align*}
\]

From the first equation, we obtain $a_{10} = -\frac{t}{n_1} a_1$. Next, substituting this into the second and the third equations, we obtain

\[
\begin{align*}
(1 - t) a_1 &= n_2 \left( \frac{t}{n_1} a_1 - a_9 - a_3 \right), \\
(t - 1) a_1 &= n_3 \left( \frac{1}{n_1} a_1 - a_9 - ta_3 \right).
\end{align*}
\]

By calculating $n_3 \times (46) - n_2 \times (47)$, we obtain

\[ (n_2 + n_3)(1 - t) a_1 = \frac{n_2 n_3}{n_1} (t - 1) a_1 - n_2 n_3 (1 - t) a_3. \]
We summarize that

\[(n_1 n_2 + n_1 n_3 + n_2 n_3) a_1 = -n_1 n_2 n_3 a_3.\]

Solving on \(a_3\), we obtain

\[a_3 = \frac{-n_1 n_2 + n_1 n_3 + n_2 n_3}{n_1 n_2 n_3} a_1.\]

Next, in the coordinate system \((z_3, w_3)\) we see that \(X = \infty\) is a singular point. By giving the eigenvalues of the matrix of linear approximation around \(X = \infty\), we can obtain

\[n_0(2a_1 + a_3) = a_1.\]

Substituting (50) into this equation, we can obtain the relation of the eigenvalues \(n_i\):

\[\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} = 2.\]

We have completed the proof of Proposition 4.3.

The system (40) has the following birational symmetries.

**Theorem 4.4.** The system (40) is invariant under the following transformations: with the notation \((*) = (x, y, t; n_1, n_2, n_3, n_4; \alpha_0, \alpha_1, \ldots, \alpha_4)\).

\[s : (*) \rightarrow (x + \frac{\alpha_2}{y}, y, t; n_1, n_2, n_3, n_4; \alpha_0 + \alpha_2 - n_3 \alpha_2,\]

\[\frac{(-n_1 n_2 - n_1 n_3 - n_2 n_3 + 2n_1 n_2 n_3) \alpha_1 + (-n_1 n_2 - n_1 n_3 - n_2 n_3 + n_1 n_2 n_3) \alpha_2}{-n_1 n_2 - n_1 n_3 - n_2 n_3 + 2n_1 n_2 n_3},\]

\[-\alpha_2, \alpha_3 + \alpha_2 - n_2 \alpha_2, \alpha_4 + \alpha_2 - n_1 \alpha_2),\]

\[\pi_1 : (*) \rightarrow (1 - x, -y, 1 - t; n_2, n_1, n_3, n_4; \alpha_0, \alpha_1, \alpha_2, \alpha_4, \alpha_3),\]

\[\pi_2 : (*) \rightarrow \left(\frac{t - x}{t - 1}, -(t - 1)y, \frac{t}{t - 1}; n_3, n_2, n_1, n_4; \alpha_4, \alpha_1, \alpha_2, \alpha_3, \alpha_0\right),\]

\[\pi_3 : (*) \rightarrow \left(\frac{1}{x}, -(yx + \alpha_2)x, \frac{1}{t}; n_4, n_2, n_3, n_1; \alpha_0, \frac{\alpha_4 n_2 n_3 n_4}{n_1 (2n_1 n_2 n_3 - n_1 n_2 - n_1 n_3 - n_2 n_3)}\right),\]

\[\alpha_2, \alpha_3, \frac{\alpha_1 (2n_1 n_2 n_3 - n_1 n_2 - n_1 n_3 - n_2 n_3)}{n_1 n_2 n_3}.\]

All transformations satisfy the relation: \(s^2 = \pi_j = 1\). The transformations \(\pi_j\) change the eigenvalues \(n_1, n_2, n_3, n_4\) in addition to some parameter’s changes.

The transformation \(s\) is a generalization of the Euler transformation of the Painlevé VI system. The transformations \(\pi_j\) correspond to the permutation of the singular points \(0, 1, t, \infty\). The transformations on sign change of exponents can not be found.

We remark that all transformations coincide with the ones in the case of Painlevé VI system when \(n_1 = n_2 = n_3 = n_4 = 2\).

For the system (40), we consider the following problem.
Problem 4.5. When does the system (40) have the Painlevé property?

In order to have no movable branch points, the eigenvalues \( n_i \) must belong to \( \mathbb{Z} \) (see Section 2). At first, let us classify the natural number solutions for the equation (41).

Proposition 4.6. For the equation (41) the natural number solutions
\[
\{(n_1, n_2, n_3, n_4) \in \mathbb{N}^4 | 1 \leq n_1 \leq n_2 \leq n_3 \leq n_4 \}
\]
can be classified into four types:
\[
\{(n_1, n_2, n_3, n_4) = (1, 2, 3, 6), (1, 2, 4, 4), (1, 3, 3, 3), (2, 2, 2, 2)\}.
\]

We remark that from the symmetry of the equation (41), we can set
\[
1 \leq n_1 \leq n_2 \leq n_3 \leq n_4.
\]
The type of \((n_1, n_2, n_3, n_4) = (2, 2, 2, 2)\) is the case of Painlevé VI.

Proof of Proposition 4.6.
(i) The case of \( n_1 \geq 3 \)
By assumption, we see that \( \frac{1}{n_i} \leq \frac{1}{3} \). Then, we see that
\[
\frac{1}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \geq \sum_{i=1}^{4} \frac{1}{n_i}.
\]
This contradicts the equation (41).

(ii) The case of \( n_1 = 2 \)
In this case, we consider
\[
\frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} = \frac{3}{2}.
\]
(ii-1) The case of \( n_2 = 2 \)
In this case, we consider
\[
\frac{1}{n_3} + \frac{1}{n_4} = 1.
\]
Since \( n_j \geq 2 \), we see that \( \frac{1}{n_j} \leq \frac{1}{2} \). Then, we obtain
\[
n_3 = n_4 = 2.
\]
Consequently, we can obtain \((n_1, n_2, n_3, n_4) = (2, 2, 2, 2)\).

(ii-1) The case of \( n_2 \geq 3 \)
Since \( n_j \geq 3 \), we see that \( \frac{1}{n_j} \leq \frac{1}{3} \). Then, we obtain
\[
\sum_{i=2}^{4} \frac{1}{n_i} \leq 1.
\]
This contradicts the equation (54).
(iii) The case of \( n_1 = 1 \)
In this case, we consider

\begin{equation}
\frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} = 1.
\end{equation}

\text{(iiI-1) The case of } n_2 = 1

In this case, we consider

\begin{equation}
\frac{1}{n_3} + \frac{1}{n_4} = 0.
\end{equation}

Since \(n_j \geq 1\), this is contradiction.

\text{(iiI-2) The case of } n_2 = 2

In this case, we consider

\begin{equation}
\frac{1}{n_3} + \frac{1}{n_4} = \frac{1}{2}.
\end{equation}

Since \(n_3 = 2, n_4 \geq 2\). This is contradiction. So, \(n_3 \geq 3\).

\begin{itemize}
  \item If \(n_3 = 3\), we obtain \(n_4 = 6\). Consequently, we can obtain \((n_1, n_2, n_3, n_4) = (1, 2, 3, 6)\).
  \item If \(n_3 = 4\), we obtain \(n_4 = 4\). Consequently, we can obtain \((n_1, n_2, n_3, n_4) = (1, 2, 4, 4)\).
\end{itemize}

Now, if \(5 \leq n_3 \leq n_4\), then \(\frac{1}{n_j} \leq \frac{1}{5}\). We obtain

\begin{equation}
\frac{1}{n_3} + \frac{1}{n_4} \leq \frac{2}{5} < \frac{1}{2}.
\end{equation}

This contradicts the equation (56).

\text{(iiI-3) The case of } 3 \leq n_2 \leq n_3 \leq n_4

In this case, from \(\frac{1}{n_j} \leq \frac{1}{3}\), we obtain

\begin{equation}
\frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} \leq 1.
\end{equation}

From the equation (55), we obtain \(n_2 = n_3 = n_4 = 3\). Consequently, we can obtain \((n_1, n_2, n_3, n_4) = (1, 3, 3, 3)\).

\begin{itemize}
  \item If \(4 \leq n_2 \leq n_3 \leq n_4\), then we obtain
    \begin{equation}
    \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} \leq \frac{3}{4}.
    \end{equation}
\end{itemize}

This contradicts the equation (55). Thus, we have completed the proof of Proposition 4.6.

It is still an open question whether we classify all integer solutions for the equation (41).

5. THE CASE OF PAINLEVÉ V SYSTEM

In this section, we give the geometric Riemann scheme of the Painlevé V system. In this case, the accessible singular point \(X = Y = 0\) has multiplicity of order 2. By making two times blowing-ups, this accessible singular point transforms into simple singular point. For this simple point, we give a matrix of linear approximation around this point.
In Proposition 5.2, we will show that the condition of the double point $X = 0$ (2) is equivalent to the pair of a simple accessible singular point and a matrix of degenerate type as linear approximation around this point.

**Theorem 5.1.** For the system (24), we give the following geometric Riemann scheme:

\[
\begin{pmatrix}
X' \\
Y'
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-t & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
2\alpha_3 & n_3
\end{pmatrix} \begin{pmatrix}
X \\
0
\end{pmatrix} + \begin{pmatrix}
(n_1 & \alpha_0) \\
0 & 1
\end{pmatrix} \begin{pmatrix}
f_0(n_1 & \alpha_0) \\
0 & 1
\end{pmatrix} \begin{pmatrix}
f_1(n_1 & \alpha_0) \\
0 & 1
\end{pmatrix} \begin{pmatrix}
f_2(n_2 & \alpha_1) \\
0 & 1
\end{pmatrix} \begin{pmatrix}
X \\
Y
\end{pmatrix}
\]

Here, $X = 0, 1, \infty$ are accessible singular points, $f_i \in \mathbb{C}(t), n_i \in \mathbb{C}, t \in \mathbb{C} - \{0\}$ and $\alpha_i$ are constant parameters. The symbol $X = 0$ (2) means that the point $X = Y = 0$ has multiplicity of order 2, and $(X', Y') = (x, x^2y)$. Then, this system coincides with

\[
\begin{aligned}
\delta \frac{dx}{dt} &= 2n_1n_2x^3y - 2n_1n_2x^2y - 2n_1(\alpha_1 - n_2\alpha_2)x^2 + \{2n_2\alpha_0 + n_1\alpha_1 - 2n_1n_2\alpha_2 + (n_1 + n_2)t\}x \\
&\quad - (n_1 + n_2)t, \\
\delta \frac{dy}{dt} &= -2n_1(2n_2 - 1)x^2y^2 + 2(2n_1n_2 - n_1 - n_2)xy^2 + 2n_1\{2\alpha_1 - (3n_2 - 2)\alpha_2\}xy \\
&\quad - \{2n_2\alpha_0 + n_1\alpha_1 - 2(2n_1n_2 - n_1 - n_2)\alpha_2 + (n_1 + n_2)t\}y + 2n_1\alpha_2(\alpha_1 + \alpha_2 - n_2\alpha_2),
\end{aligned}
\]

where $\delta := t\{2n_2\alpha_0 + n_1\alpha_1 - 2(n_1 + n_2)\alpha_2 + 2(2n_1n_2 - n_1 - n_2)\alpha_3 - (n_1 - n_2)t\}$.

This system can be considered as a generalization of the Painlevé V system. The case of $(n_1, n_2, n_3) = (2, 2, 2)$ is equivalent to the Painlevé V system.

**Proof of Theorem 5.1.** We only consider the case of multiplicity of order 2. At first, we can rewrite the system (24) in the coordinate system $(X, Y) = (x, 1/y)$ centered at $(X, Y) = (0, 0)$

\[
\begin{aligned}
dX/dt &= a_1X^3 + a_2X^2 + a_5X + a_7 + \frac{1}{2}\{(3a_1 + 2a_3)\alpha_2 - a_4\}X^2 + \{(a_2 + a_9)\alpha_2 - a_6\}X + a_8, \\
dY/dt &= -a_{10} - a_9X - a_3X^2 - a_4XY - a_6Y - \frac{1}{2}(a_1\alpha_2 + a_4)\alpha_2Y^2 = (a_i \in \mathbb{C}(t)).
\end{aligned}
\]

By Definition 2.7, we can calculate the accessible singular points

\[
Y = 0, \quad a_1X^3 + a_2X^2 + a_5X + a_7 = 0.
\]

By the assumption, $X = Y = 0$ is a solution of the system (37). Thus, we obtain the condition

\[a_7 = 0.\]

Moreover, since this singular point has multiplicity of order 2, we need the condition

\[a_5 = 0.\]

This condition is necessary condition for multiplicity of order 2.
Next, let us resolve the multiplicity of this point by making two times blowing-ups.

**Step 1.** We blow up at the point $X = Y = 0$:

$$X_1 = X, \quad Y_1 = \frac{Y}{X}.$$  \hfill (61)

Since $X_1 = Y_1 = 0$ must be a singular point, we need the condition

$$a_{10} = 0.$$  

We summarize that the singular point $X = Y = 0$ has multiplicity of order 2 if and only if

$$a_5 = a_7 = a_{10} = 0.$$  \hfill (62)

**Step 2.** We blow up at the point $X_1 = Y_1 = 0$

$$X_2 = X_1, \quad Y_2 = \frac{Y_1}{X_1}.$$  \hfill (63)

Here, in order to take a suitable coordinate system we make a change of variables:

$$X_3 = X_2, \quad Y_3 = \frac{1}{Y_2}.$$  \hfill (64)

We see that the patching data between $(X_3, Y_3)$ and $(x, y)$ is given by $(X_3, Y_3) = (x, x^2y)$. 

---

**Figure 2.** Resolution of multiplicity of order 2
In the coordinate system \((X_3, Y_3)\) we rewrite the system (24):

\[
\begin{align*}
    \frac{dX_3}{dt} &= a_8 + \frac{1}{2} \{(3a_1 + 2a_3)\alpha_2 - a_4\} X^2_3 + a_1 X_3 Y_3 + \{(a_2 + a_9)\alpha_2 - a_6\} X_3 + a_2 Y_3, \\
    \frac{dY_3}{dt} &= \frac{2a_8 Y_3}{X_3} + \frac{(2a_2 + a_9) Y^2_3}{X_3} + \frac{1}{2} \alpha_2 (\alpha_2 a_1 + a_4) X^2_3 + (2a_1 + a_3) Y^2_3 + \alpha_2 (3a_1 + 2a_3) X_3 Y_3 \\
    &\quad + \{2\alpha_2 (a_2 + a_9) - a_6\} Y_3.
\end{align*}
\]

By the assumption, \((X_3, Y_3) = (0, -t)\) is a simple singular point. So, we obtain the condition

\[
a_8 = \frac{1}{2} t(2a_2 + a_9).
\]

Finally, by the assumption, the matrix of linear approximation around \((X_3, Y_3) = (0, -t)\) is given by

\[
(66) \quad f_0 \begin{pmatrix} 1 & 0 \\ 2\alpha_3 & n_3 \end{pmatrix}.
\]

So, we obtain

\[
a_2 = -\frac{1}{4} (n_3 + 2) a_9, \quad a_1 = -\frac{2 + 2t^2 a_3 + t\{2a_6 + (n_3 - 2)\alpha_2 a_9 - 2\alpha_3 a_9\}}{4t^2}.
\]

For the remaining singular points \(X = 1\) and \(X = \infty\), we can obtain the conditions in the same way of Painlevé VI case. Thus, we have completed the proof of Theorem 5.1.

By using the conditions (62) in the proof of Theorem 5.1, we easily see the following

**Proposition 5.2.** The condition \(X = 0\) (2) is equivalent to the following:

\[
(67) \quad \begin{pmatrix} X = 0 \\ f \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix},
\]

where \(f, * \in \mathbb{C}(t)\).

**Remark 5.3.** The condition (67) means that \(X = 0\) is a singular point, and the matrix of linear approximation around this point is given by

\[
(68) \quad f \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.
\]

**Proposition 5.4.** The eigenvalues \(n_i\) satisfy the following relation:

\[
(69) \quad 2n_1 n_2 n_3 - (n_1 + n_2) n_3 - 2(n_1 + n_2) = 0.
\]

We see that the case of \((n_1, n_2, n_3) = (2, 2, 2)\) is equivalent to the Painlevé V system.
Proposition 5.5. For the equation (69) the natural number solutions
\[ \{(n_1, n_2, n_3) \in \mathbb{N}^3 | n_1 \geq n_2 \} \]
can be classified into six types:
\[ \{(n_1, n_2, n_3) = (2, 1, 6), (2, 2, 2), (3, 1, 4), (3, 3, 1), (5, 1, 3), (6, 2, 1) \} \]

We remark that from the symmetry of the equation (69), we can set
\[ n_1 \geq n_2. \]
The type of \((n_1, n_2, n_3) = (2, 2, 2)\) is the case of Painlevé V.

It is still an open question whether we classify all integer solutions for the equation (69).

Finally, we show that the system (58) has the following birational symmetries.

Theorem 5.6. The system (58) is invariant under the following transformations: with the notation \((*) = (x, y, t; n_1, n_2, n_3; \alpha_0, \alpha_1, \alpha_2, \alpha_3)\),
\[ s : (*) \rightarrow (x + \frac{\alpha_2}{y}, y, t; n_1, n_2, n_3; \alpha_0 + \alpha_2 - n_1 \alpha_2, \alpha_1 + \alpha_2 - n_2 \alpha_2, -\alpha_2, \frac{2(\alpha_2 + \alpha_3)n_1 n_2 - (3\alpha_2 + \alpha_3)n_1 - (3\alpha_2 + \alpha_3)n_2}{2n_1 n_2 - n_1 - n_2}), \]
\[ \pi : (*) \rightarrow \left( \frac{x}{x - 1}, -(x - 1)((x - 1)y + \alpha_2), -t; n_2, n_1, n_3; \alpha_1, \alpha_0, \alpha_2, \alpha_3 \right). \]

All transformations satisfy the relation: \(s^2 = \pi^2 = 1\). The transformation \(\pi\) changes the eigenvalues \(n_1, n_2, n_3\) in addition to some parameter’s changes.

The transformation \(\pi\) corresponds to the permutation of the singular points 1 and \(\infty\). The transformations on sign change of exponents can not be found.

We remark that all transformations coincide with the ones in the case of Painlevé V system when \(n_1 = n_2 = n_3 = 2\).

6. The case of Painlevé IV system

In this section, we give the geometric Riemann scheme of the Painlevé IV system. In this case, the accessible singular point \(X = Y = 0\) has multiplicity of order 3. By making three times blowing-ups, this accessible singular point transformes into a simple singular point. For this simple point, we give a matrix of linear approximation around this point. In Proposition 6.2, we will show that the condition of the triple point \(X = 0\) (3) is equivalent to the pair of a simple accessible singular point and the eigenvalues of two matrices for the expansion (see Proposition 6.2) around this point.
Theorem 6.1. For the system (24), we give the following geometric Riemann scheme:

\[
\begin{pmatrix}
X = 0 \quad \text{(3)} \\
Y = 0
\end{pmatrix}

\quad \begin{pmatrix}
X = \infty
\end{pmatrix}

\begin{pmatrix}
X' \\
Y'
\end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}, \quad f_0 \begin{pmatrix} 1 & 0 \\ 2t & n_2 \end{pmatrix}, \quad f_1 \begin{pmatrix} n_1 & \alpha_1 \\ 0 & 1 \end{pmatrix}.
\]

Here, $X = 0, \infty$ are accessible singular points, $f_i \in \mathbb{C}(t)$, $n_i \in \mathbb{C}$, $t \in \mathbb{C}$ and $\alpha_i$ are constant parameters. The symbol $X = 0 \ (3)$ means that the point $X = Y = 0$ has multiplicity of order 3, and $(X', Y') = (x, x^3y)$. Then, this system coincides with

\[
\begin{align*}
\frac{dx}{dt} &= a(t) \left(x^3 y + \frac{(n_1 \alpha_2 - \alpha_1)x^2}{n_1} + \frac{(2n_1 - 1)tx}{3n_1} + \frac{n_1 + 1}{6n_1}\right), \\
\frac{dy}{dt} &= a(t) \left(-\frac{(2n_1 - 1)x^2 y^2}{n_1} + \frac{(2\alpha_1 - (3n_1 - 2)\alpha_2)}{n_1} xy - \frac{(2n_1 - 1)ty}{3n_1} + \frac{\alpha_2(\alpha_1 - (n_1 - 1)\alpha_2)}{n_1}\right),
\end{align*}
\]

where $a(t) \in \mathbb{C}(t)$.

This system can be considered as a generalization of the Painlevé IV system. The case of $(n_1, n_2) = (2, 3)$ and $a(t) = 4$ is equivalent to the Painlevé IV system.

Proof of Theorem 6.1. We only consider the case of multiplicity of order 3. At first, we can rewrite the system (24) in the coordinate system $(X, Y) = (x, 1/y)$ centered at $(X, Y) = (0, 0)$

\[
\begin{align*}
\frac{dX}{dt} &= a_1 X^3 + a_2 X^2 + a_5 X + a_7 + \frac{1}{2} \{(3a_1 + 2a_3)\alpha_2 - a_4\}X^2 + \{(a_2 + a_9)\alpha_2 - a_6\}X + a_8, \\
\frac{dY}{dt} &= -a_9 X - a_3 X^2 - a_4 XY - a_6 Y - \frac{1}{2}(a_1 \alpha_2 + a_4)\alpha_2 Y^2 \quad (a_i \in \mathbb{C}(t)).
\end{align*}
\]
By Definition 2.1 we can calculate the accessible singular points
\[(74)\]
\[Y = 0, \quad a_1X^3 + a_2X^2 + a_5X + a_7 = 0.\]
By the assumption, \(X = Y = 0\) is a solution of the system \[(37)\]. Thus, we obtain the condition
\[a_7 = 0.\]
Moreover, since this singular point has multiplicity of order 3, we need the conditions
\[a_2 = a_5 = 0.\]
This condition is necessary condition for multiplicity of order 3.

Next, let us resolve the multiplicity of this point by making three times blowing-ups.

**Step 1.** We blow up at the point \(X = Y = 0\):
\[(75)\]
\[X_1 = X, \quad Y_1 = \frac{Y}{X}.\]
Since \(X_1 = Y_1 = 0\) must be a double singular point, we need the condition
\[a_{10} = 0.\]

**Step 2.** We blow up at the point \(X_1 = Y_1 = 0\)
\[(76)\]
\[X_2 = X_1, \quad Y_2 = \frac{Y_1}{X_1}.\]
Since \(X_2 = Y_2 = 0\) must be a singular point, we need the condition
\[a_9 = 0.\]
We summarize that the singular point \(X = Y = 0\) has multiplicity of order 3 if and only if
\[(77)\]
\[a_2 = a_5 = a_7 = a_9 = a_{10} = 0.\]

**Step 3.** We blow up at the point \(X_2 = Y_2 = 0\)
\[(78)\]
\[X_3 = X_2, \quad Y_3 = \frac{Y_2}{X_2}.\]
Here, in order to take a suitable coordinate system we make a change of variables:
\[(79)\]
\[X_4 = X_3, \quad Y_4 = \frac{1}{Y_3}.\]
We see that the patching data between \((X_4, Y_4)\) and \((x, y)\) is given by \((X_4, Y_4) = (x, x^3y)\).

By doing the same argument in the proof of Theorem 5.1, we can obtain some conditions.

For the remaining singular point \(X = \infty\), we can obtain the conditions in the same way of Painlevé VI case. Thus, we have completed the proof of Theorem 6.1. \(\square\)

By using the conditions \[(77)\] in the proof of Theorem 6.1 we easily see the following
Proposition 6.2. The condition \( X = 0 \) (3) is equivalent to the following conditions: \( X = 0 \) is a singular point, and the eigenvalues \( a_2, a_5, a_9, a_{10} \) for the following expansion of the system (73) are given by

\[
a_2 = a_5 = a_9 = a_{10} = 0,
\]

where (80)

\[
\frac{d}{dt} \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{Y} \left\{ \begin{pmatrix} a_5 & * \\ 0 & -a_{10} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} a_2 & * \\ 0 & -a_9 \end{pmatrix} \begin{pmatrix} X^2 \\ XY \end{pmatrix} + \begin{pmatrix} a_1 & * \\ 0 & -a_3 \end{pmatrix} \begin{pmatrix} X^3 \\ X^2Y \end{pmatrix} + \cdots \right\},
\]

where \( * \in \mathbb{C}(t) \).

Proposition 6.3. The eigenvalues \( n_i \) satisfy the following relation:

\[
2n_1n_2 - 3n_1 - n_2 - 3 = 0.
\]

We see that the case of \((n_1, n_2) = (2, 3)\) is equivalent to the Painlevé IV system.

Proposition 6.4. For the equation (81) the natural number solutions can be classified into three types:

\[
\{(n_1, n_2) = (1, 6), (2, 3), (5, 2)\}.
\]

The type of \((n_1, n_2) = (2, 3)\) is the case of Painlevé IV.

Proof of Proposition 6.4. At first, we rewrite the equation (81) as follows:

\[
n_1 = \frac{n_2 + 3}{2n_2 - 3}.
\]

(i) The case of \( n_1 \geq 6 \)

By assumption, we see that \( 6 \leq \frac{n_2 + 3}{2n_2 - 3} \). Then, we see that

\[
n_2 \leq \frac{21}{11} < 1.
\]

This contradicts the condition \( n_2 \geq 1 \). Thus, we see that \( n_1 \leq 5 \).

The remaining cases can be solved by the same argument in the proof of Proposition 4.6. \( \square \)

It is still an open question whether we classify all integer solutions for the equation (81).

Finally, we show that the system (72) has the following birational symmetry.

Theorem 6.5. The system (72) is invariant under the following transformation: with the notation \((*) = (x, y, t; n_1, n_2; \alpha_0, \alpha_1, \alpha_2)\),

\[
s : (* \rightarrow \left( x + \frac{\alpha_2}{y}; y, t; n_1, n_2; \alpha_1 + \alpha_2 - n_1\alpha_2, -\alpha_2 \right).
\]
The transformation satisfies the relation: $s^2 = 1$. The transformations on sign change of exponents can not be found.

We remark that the transformation $s$ coincides with the one in the case of Painlevé IV system when $(n_1, n_2) = (2, 3)$.

7. The case of Painlevé III system

In this section, we give the geometric Riemann scheme of the Painlevé III system. In this case, two accessible singular points have multiplicity of order 2.

**Theorem 7.1.** For the system (24), we give the following geometric Riemann scheme:

\[
\begin{pmatrix}
X' \\
Y'
\end{pmatrix} =
\begin{pmatrix}
0 \\
-1
\end{pmatrix},
\quad
f_0
\begin{pmatrix}
1 & 0 \\
2\alpha_0 & n_1
\end{pmatrix}
\]
\[
\begin{pmatrix}
X'' \\
Y''
\end{pmatrix} =
\begin{pmatrix}
0 \\
-1
\end{pmatrix},
\quad
f_1
\begin{pmatrix}
1 & 0 \\
2\alpha_1 & n_2
\end{pmatrix}
\]

Here, $X = 0, \infty$ are accessible singular points, $f_i \in \mathbb{C}(t)$, $n_i \in \mathbb{C}$, $t \in \mathbb{C} - \{0\}$ and $\alpha_i$ are constant parameters, and $(X', Y') = (x, x^2y)$ and $(X'', Y'') = \left(\frac{1}{x}, -(\frac{x+y+\alpha_2}{x})\right)$. Then, this system coincides with

\[
\begin{align*}
\delta t \frac{dx}{dt} &= -(n_1 + 2)x^2y + 2x^2 + 2(n_1\alpha_1 - 2\alpha_2)x - n_1t, \\
\delta t \frac{dy}{dt} &= 4xy^2 - 4xy - (2n_1\alpha_1 + (n_1 - 6)\alpha_2)y - 2\alpha_2,
\end{align*}
\]

where $\delta := 4\alpha_0 + 2n_1\alpha_1 - (n_1 + 2)\alpha_2$.

This system can be considered as a generalization of the Painlevé III system. The case of $(n_1, n_2) = (2, 2)$ is equivalent to the Painlevé III system.

By the same way of Painlevé V case, we can prove Theorem 7.1. $\square$

**Proposition 7.2.** The eigenvalues $n_i$ satisfy the following relation:

\[
n_1n_2 = 4.
\]

We see that the case of $(n_1, n_2) = (2, 2)$ is equivalent to the Painlevé III system.

**Proposition 7.3.** For the equation (85) the natural number solutions can be classified into two types:

\[
\{(n_1, n_2) = (2, 2), (4, 1)\}.
\]

The type of $(n_1, n_2) = (2, 2)$ is the case of Painlevé III.

It is still an open question whether we classify all integer solutions for the equation (85).

Finally, we show that the system (84) has the following birational symmetries.
Theorem 7.4. The system (84) is invariant under the following transformations: with
the notation \(((\ast)) = (x, y, t; n_1, n_2; \alpha_0, \alpha_1, \alpha_2),\)
\[
\begin{align*}
  s : (\ast) & \rightarrow \left( x + \frac{\alpha_2}{y}, y, t; n_1, n_2; \alpha_0 + \alpha_2 - n_1\alpha_2, \alpha_1 + \alpha_2 - n_2\alpha_2, -\alpha_2 \right), \\
  \pi : (\ast) & \rightarrow \left( \frac{t}{x}, -\frac{x(xy + \alpha_2)}{t}, t; n_2, n_1; \alpha_1, \alpha_0, \alpha_2 \right).
\end{align*}
\]
All transformations satisfy the relation: \(s^2 = \pi^2 = 1\). The transformation \(\pi\) changes
the eigenvalues \(n_1, n_2\) in addition to some parameter’s changes.
The transformation \(\pi\) corresponds to the permutation of the singular points 0 and \(\infty\).
The transformations on sign change of exponents can not be found.
We remark that all transformations coincide with the ones in the case of Painlevé III
system when \(n_1 = n_2 = 2\).

8. Existence theorem of non-linear ordinary differential systems in
dimension two with only simple accessible singular points

For a linear differential equation of Fuchs type, it is well-known that

Theorem 8.1. Let us consider the \(n\)-th order linear ordinary differential equations:
\[
\frac{d^n x}{dt^n} + a_1(t) \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_{n-1}(t) \frac{dx}{dt} + a_n(t)x = 0,
\]
where \(a_i(t)\) are meromorphic functions defined in a domain in the Riemann sphere \(\mathbb{P}^1\).
There exists an ordinary differential equation with \(n\)-th order satisfying the assumptions
\((F1), (F2)\) and \((F3)\).
\(F1\) This equation has only \((m+1)\) points \(x = c_j\) on the Riemann sphere \(\mathbb{P}^1\) as its
regular singular points.
\(F2\) Its local exponent at each singular point \(c_j\) coincides with \(\rho_{jl} \in \mathbb{C}\) \((j = 1, 2, \ldots, m+1, l = 1, 2, \ldots, n)\).
\(F3\) \(\rho_{jl} (j = 1, 2, \ldots, m+1, l = 1, 2, \ldots, n)\) given in \((F2)\) satisfies the Fuchs’ relation:
\[
\sum_{j=1}^{m+1} \sum_{l=1}^{n} \rho_{jl} = \frac{(m-1)n(n-1)}{2}.
\]
Let us consider the following problem.

Problem 8.2. Can we construct a non-linear ordinary differential system in dimension
two satisfying similar conditions of \((F1), (F2)\) and \((F3)\) from the viewpoint of geometrical
property?
In this section, let us consider a system of the first-order ordinary differential equations in dimension two:

\[
\begin{align*}
\frac{dx}{dt} &= f_1(x, y), \\
\frac{dy}{dt} &= f_2(x, y) \quad (f_i \in \mathbb{C}(t)[x, y]).
\end{align*}
\]

We assume that the regular vector field associated with the system \((89)\) defined on \(\mathbb{C}^2 \times B\)

\[
v = \frac{\partial}{\partial t} + \frac{dx}{dt} \frac{\partial}{\partial x} + \frac{dy}{dt} \frac{\partial}{\partial y}
\]
to a rational vector field \(\tilde{v}\) on \(\Sigma_n \times B\)

\[
\tilde{v} \in H^0(\Sigma_n \times B, \Theta_{\Sigma_n \times B}(-\log D^{(0)})(D^{(0)})),
\]

where \(B\) is a domain in \(\mathbb{C}\).

Here, we review the algebraic surface \(\Sigma_n\), which is obtained by gluing four copies of \(\mathbb{C}^2\) via the following identification.

\[
U_j \cong \mathbb{C}^2 \ni (z_j, w_j) \quad (j = 0, 1, 2, 3)
\]

\[
\begin{align*}
z_0 &= x, \quad w_0 = y, \\
z_1 &= \frac{1}{x}, \quad w_1 = -x^n y - \alpha x, \\
z_2 &= z_0, \quad w_2 = \frac{1}{w_0}, \\
z_3 &= z_1, \quad w_3 = \frac{1}{w_1},
\end{align*}
\]

where \(\alpha\) is a complex constant parameter.

We define a divisor \(D^{(0)}\) on \(\Sigma_n\):

\[
D^{(0)} = \{(z_2, w_2) \in U_2 | w_2 = 0\} \cup \{(z_3, w_3) \in U_3 | w_3 = 0\} \cong \mathbb{P}^1.
\]

The self-intersection number of \(D^{(0)}\) is given by

\[
(D^{(0)})^2 = n.
\]

The condition \((91)\) is equivalent to the following:

1. Holomorphy in the coordinate system \((z_1, w_1) = (1/x, -x^n y - \alpha x)\),
2. In the coordinate system \((X, Y) = (x, 1/y)\), the differential system \((89)\) must be taken of the form:

\[
\begin{align*}
\frac{dX}{dt} &= \frac{F_1(X, Y)}{Y}, \\
\frac{dY}{dt} &= F_2(X, Y) \quad (F_i \in \mathbb{C}(t)[X, Y]).
\end{align*}
\]

In the coordinate system \((z_1, w_1)\) the right hand side of this system is polynomial with respect to \(z_1, w_1\). However, on the boundary divisor \(D^{(0)} \cong \mathbb{P}^1\) this system has a pole in each coordinate system \((z_i, w_i)\) \(i = 2, 3\). By rewriting the system at each singular point, this rational vector field has a pole along the divisor \(D^{(0)}\), whose order is one.

In this section, we consider the case of simple accessible singular points.
The following theorem can be considered as a non-linear version of Theorem 8.1 from the viewpoint of geometrical property.

| Category       | Linear                              | Non-linear                             |
|----------------|-------------------------------------|----------------------------------------|
| Condition 1    | regular type                        | simple accessible type (see Section 2) |
| Condition 2    | local exponent                       | local index (see Section 2)            |
| Condition 3    | Fuchs’ relation                      | relation (96)                          |

**Theorem 8.3.** Let us consider an ordinary differential system in dimension two satisfying the condition (91). There exists an ordinary differential system of this type satisfying the assumptions (A1), (A2) and (A3).

(A1) This system has only \((n + 2)\) points \(c_1, c_2, \ldots, c_n, t, \infty\) on the boundary divisor \(D^{(0)} \times B\) as its simple accessible singular points, where \(c_i \in \mathbb{C}\) and \(t \in B\).

(A2) The ratio of its local index at each accessible singular point \(c_i\) coincides with \(m_i \in \mathbb{C} - \{0\}\).

(A3) \(m_i\) \((i = 1, 2, \ldots, n + 2)\) given in (A2) satisfies the relation:

\[
\sum_{i=1}^{n+2} \frac{1}{m_i} = n.
\]

We note that the simple accessible singular point \(P\) means that \(P\) is an accessible singular point and has its multiplicity of order 1.

In this paper, we find

\[
\begin{aligned}
\frac{dx}{dt} &= a_1(t)(x - c_1)(x - c_2) \ldots (x - c_n)(x - t)y + b_1[x], \\
\frac{dy}{dt} &= -\frac{a_1(t)}{m_1m_2 \ldots m_{n+1}} y^2\{m_2m_3 \ldots m_{n+1}(x - c_2)(x - c_3) \ldots (x - c_n)(x - t) \\
&\quad + m_1m_3 \ldots m_{n+1}(x - c_1)(x - c_3) \ldots (x - c_n)(x - t) \\
&\quad + \ldots \\
&\quad + m_1m_2 \ldots m_n(x - c_1)(x - c_2) \ldots (x - c_{n-1})(x - c_n)\} + b_2[x]y + b_3[x],
\end{aligned}
\]

where \(a_1(t) \in \mathbb{C}(t)\) and \(b_i[x] \in \mathbb{C}(t)[x]\) satisfy certain conditions in order to become a polynomial class in the coordinate system \((z_1, w_1)\).

Before we will show that this system satisfies the assumptions (A1), (A2) and (A3), we can obtain the conditions of vector field.

(i) Degree of polynomials \(f_i(x, y)\) with respect to \(y\)

If the system (89) belongs in \(H^0(\Sigma_n \times B, \Theta_{\Sigma_n \times B}(- \log D^{(0)})(D^{(0)}))\), in the coordinate system \((X, Y) = (x, 1/y)\) this system must be taken of the form:

\[
\begin{aligned}
\frac{dX}{dt} &= F_1(X, Y) \\
\frac{dY}{dt} &= F_2(X, Y) \quad (F_i \in \mathbb{C}(t)[X, Y]).
\end{aligned}
\]
By this condition, we see that the system \((89)\) must be taken of the form:

\[
\begin{align*}
\frac{dx}{dt} &= b_1(x) + b_2(x)y, \\
\frac{dy}{dt} &= b_3(x) + b_4(x)y + b_5(x)y^2 \quad (b_i \in \mathbb{C}(t)[x]).
\end{align*}
\]

Here, the degree of each \(b_i\) with respect to \(x\) is given by

\[
\text{deg}(b_1) = l, \text{deg}(b_2) = m, \text{deg}(b_3) = p, \text{deg}(b_4) = q, \text{deg}(b_5) = r,
\]

where \(l, m, n, p, r \in \mathbb{N}\).

(ii) Holomorphy in the coordinate system

In the coordinate system:

\[
(x_1, y_1) = (1/x, -yx^n - g_{n-1}x^{n-1} - \cdots - g_1x),
\]

the first equation of the system \((99)\) is given by

\[
\frac{dx_1}{dt} = -x_1^2 \left\{ b_1 \left( \frac{1}{x_1} \right) + b_2 \left( \frac{1}{x_1} \right) \left( -y_1x^n - g_1x_1^{n-1} - \cdots - g_{n-1}x_1 \right) \right\}.
\]

Since the right hand side of this system must be polynomial with respect to \(x_1\), we compare two terms

\[
\begin{align*}
\left\{ b_1 \left( \frac{1}{x_1} \right) &= b_1^{(l)} + \cdots, \\
\left( b_2 \left( \frac{1}{x_1} \right) &= -g_{n-1} \frac{b_2^{(m)}}{x_1^{m-1}} + \cdots.
\end{align*}
\]

Since \(b_1^{(l)} \neq 0\) and \(b_2^{(m)} \neq 0\), we can obtain

\[
l = m - 1
\]

Next, we compare the term involving \(y_1\):

\[
x_1^{n+2}b_2 \left( \frac{1}{x_1} \right) y_1 = x_1^{n+2} \left( \frac{b_2^{(m)}}{x_1^{m}} + \frac{b_2^{(m-1)}}{x_1^{m-1}} + \cdots \right) y_1 \quad (b_2^{(j)} \in \mathbb{C}(t)).
\]

If this becomes polynomial with respect to \(x_1, y_1\),

\[
m = n + 2.
\]

In the same way, we can obtain

\[
\text{deg}(b_1) = n + 1, \text{deg}(b_2) = n + 2, \text{deg}(b_3) = n - 1, \text{deg}(b_4) = n, \text{deg}(b_5) = n + 1.
\]

At first, we remark that

\textbf{Proposition 8.4.} These systems \((10)\) and \((1)\) satisfy the assumptions \((A1), (A2)\) and \((A3)\).
Next, in general case we consider

\[
\begin{aligned}
\frac{dx}{dt} &= a_1(t)(x - c_1)(x - c_2) \ldots (x - c_n)(x - t)y + b_1[x], \\
\frac{dy}{dt} &= -a_1(t) \frac{m_1m_2 \ldots m_{n+1}}{m_1m_2 \ldots m_{n+1}} y^2\{m_2m_3 \ldots m_{n+1}(x - c_2)(x - c_3) \ldots (x - c_n)(x - t) \\
&\quad + m_1m_3 \ldots m_{n+1}(x - c_1)(x - c_3) \ldots (x - c_n)(x - t) \\
&\quad + \ldots \\
&\quad + m_1m_2 \ldots m_n(x - c_1)(x - c_2) \ldots (x - c_{n-1})(x - c_n)\} + b_2[x]y + b_3[x],
\end{aligned}
\]

(108)

where \( a_1(t) \in \mathbb{C}(t) \) and \( b_i[x] \in \mathbb{C}(t)[x] \) satisfy certain conditions in order to become a polynomial class in the coordinate system \((z_1, w_1)\).

In the coordinate system \((z_1, w_1)\), the system (108) can be rewritten as follows:

\[
\begin{aligned}
\frac{dz_1}{dt} &= a_1(t)z_1(1 - c_1z_1)(1 - c_2z_1) \ldots (1 - c_nz_1)(1 - tz_1)w_1 + c_1[z_1], \\
\frac{dw_1}{dt} &= -na_1(t)(1 - c_1z_1)(1 - c_2z_1) \ldots (1 - c_nz_1)(1 - tz_1)w_1^2 \\
&\quad - \frac{a_1(t)}{m_1m_2 \ldots m_{n+1}} w_1^2\{m_2m_3 \ldots m_{n+1}(1 - c_2z_1)(1 - c_3z_1) \ldots (1 - c_nz_1)(1 - tz_1) \\
&\quad + m_1m_3 \ldots m_{n+1}(1 - c_1z_1)(1 - c_3z_1) \ldots (1 - c_nz_1)(1 - tz_1) \\
&\quad + \ldots \\
&\quad + m_1m_2 \ldots m_n(1 - c_1z_1)(1 - c_2z_1) \ldots (1 - c_{n-1}z_1)(1 - c_nz_1)\} + c_2[z_1]w_1 + c_3[z_1],
\end{aligned}
\]

(109)

where \( c_i[z_1] \in \mathbb{C}(t)[z_1] \) satisfy certain conditions in order to become a polynomial class in the coordinate system \((x, y) = (1/z_1, -z_1^{n}w_1 - \alpha z_1^{n-1})\).

**Proposition 8.5.** The system (108) satisfies the assumptions (A1), (A2) and (A3).

**Proof.** In the coordinate system \((X, Y) = (x, 1/y)\) the system (108) can be rewritten as follows:

\[
\begin{aligned}
\frac{dX}{dt} &= a_1(t)(X - c_1)(X - c_2) \ldots (X - c_n)(X - t) \frac{Y}{Y} + b_1[X], \\
\frac{dY}{dt} &= \frac{a_1(t)}{m_1m_2 \ldots m_{n+1}}\{m_2m_3 \ldots m_{n+1}(X - c_2)(X - c_3) \ldots (X - c_n)(X - t) \\
&\quad + m_1m_3 \ldots m_{n+1}(X - c_1)(X - c_3) \ldots (X - c_n)(X - t) \\
&\quad + \ldots \\
&\quad + m_1m_2 \ldots m_n(X - c_1)(X - c_2) \ldots (X - c_{n-1})(X - c_n)\} - b_2[X]Y - b_3[X]Y^2.
\end{aligned}
\]

(110)

By Definition 2.1 we can calculate its accessible singular points

\[
Y = 0, \quad (X - c_1)(X - c_2) \ldots (X - c_n)(X - t) = 0.
\]

(111)
We obtain

\[(112) \quad X = c_1, c_1, \ldots, c_n, t.\]

Next, let us calculate its local index at each point. At first, in the coordinate system \((X_1, Y_1) = (X - c_1, Y)\) the system (110) can be rewritten as follows:

\[(113) \quad \begin{cases} 
    \frac{dX_1}{dt} = a_1(t)X_1(X_1 + c_1 - c_2)(X_1 + c_1 - c_3)\cdots(X_1 + c_1 - c_n)(X_1 + c_1 - t) + b_1[X_1 + c_1], \\
    \frac{dY_1}{dt} = \frac{a_1(t)}{m_1m_2\ldots m_{n+1}} \times \\
    \{m_2m_3\ldots m_{n+1}(X_1 + c_1 - c_2)(X_1 + c_1 - c_3)\cdots(X_1 + c_1 - c_n)(X_1 + c_1 - t) \\
    + m_1m_2\ldots m_{n+1}X_1(X_1 + c_1 - c_3)\cdots(X_1 + c_1 - c_n)(X_1 + c_1 - t) \\
    + \ldots \\
    + m_1m_2\ldots m_nX_1(X_1 + c_1 - c_2)\cdots(X_1 + c_1 - c_n)(X_1 + c_1 - t) \\
    - b_2[X_1 + c_1]Y_1 - b_3[X_1 + c_1]Y_1^2. \}
\]

The matrix of linear approximation around \((X_1, Y_1) = (0, 0)\) is given by

\[(114) \quad \begin{pmatrix} 
    (a_1(t)(c_1 - c_2)(c_1 - c_3)\cdots(c_1 - c_n)(c_1 - t) \\
    0 \end{pmatrix} \begin{pmatrix} 
    * \\
    m_1 \end{pmatrix}, \]

where \(* \in \mathbb{C}(t)\). We see that the local index at \((X_1, Y_1) = (0, 0)\) is given by

\[(115) \quad \begin{pmatrix} 
    a_1(t)(c_1 - c_2)(c_1 - c_3)\cdots(c_1 - c_n)(c_1 - t), \frac{a_1(t)(c_1 - c_2)(c_1 - c_3)\cdots(c_1 - c_n)(c_1 - t)}{m_1} \end{pmatrix}. \]

The ratio of this local index is \(m_1\).

For the remaining accessible singular points, we can discuss in the same way of this case.

In the coordinate system \((X_2, Y_2) = (z_1, 1/w_1)\) we see that the system (109) admits \(X_2 = Y_2 = 0\) as its accessible singular points. The local index at \((X_2, Y_2) = (0, 0)\) is given by

\[(116) \quad \left( a_1(t), na_1(t) + \frac{a_1(t)}{m_1m_2\ldots m_{n+1}}(m_1m_3\ldots m_nm_{n+1} + m_2m_3\ldots m_nm_{n+1} + \ldots + m_1m_2\ldots m_{n-1}m_n) \right). \]

The ratio of this local index is given by

\[(117) \quad \frac{a_1(t)}{na_1(t) + \frac{a_1(t)}{m_1m_2\ldots m_{n+1}}(m_1m_3\ldots m_nm_{n+1} + m_2m_3\ldots m_nm_{n+1} + \ldots + m_1m_2\ldots m_{n-1}m_n)} = \frac{1}{n + \frac{1}{m_1} + \frac{1}{m_2} + \ldots + \frac{1}{m_{n+1}}}. \]
Setting

\[ m_{n+2} = \frac{1}{n + \frac{1}{m_1} + \frac{1}{m_2} + \ldots + \frac{1}{m_{n+1}}} \]

we can obtain the relation (96).

Thus, we have completed the proof of Proposition 8.5 and Theorem 8.3.

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