A Generalization of Lomax Distribution with Properties, Copula and Real Data Applications

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Abstract
A new generalization of Lomax distribution is derived and studied. Some of its useful properties are derived. A simple clayton copula is used to generate many bivariate and multivariate type models. We performed graphical simulations to assess the finite sample behavior of the estimations. The new model is employed in modelling three real data sets.

Key Words: Lomax Distribution; Copula; Kaplan-Meier; Maximum Likelihood; Simulation; Modeling; Applications; Real Data.

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1. Introduction
The Lomax (Lx) distribution or Pareto type II (PaII) distribution was presented for modeling business failure data by Lomax (1954). The Lx distribution has found wide attention and applications in a variety of fields, for instance, biological sciences, actuarial science, income, wealth inequality, engineering, medical, engineering, lifetime and reliability modeling. It has been applied for modeling data obtained from income and wealth (see Harris (1968) and Asgharzadeh and Valiollahi (2011)), firm size (see Corbellini et al. (2007)), reliability and life testing (see Hassan and Al-Ghamdi (2009)). A random variable (RV) is said to have the Lomax distribution if its survival function (SF) is given by

\[ S_\alpha(y) = 1 - \Pi_\alpha(y) = (1 + y)^{-\alpha} \mid_{y \geq 0}, \tag{1} \]

where \( \Pi_\alpha(y) = 1 - (1 + y)^{-\alpha} \mid_{y \geq 0} \) is the cumulative distribution function (CDF) of the standard Lx model and

\[ h_\alpha(y) = \alpha (1 + y)^{-\alpha - 1} \mid_{y \geq 0}, \tag{2} \]

is the probability density function (PDF) of the standard Lx model, where \( \alpha \) is the shape parameters. The PDF in (2) is a special case from the well-known Burr XII (BXII) model. Many useful details about the Lx model and its relationship with other models can be found in Burr (1942, 1968 and 1973), Lomax (1954), Burr and Cislak (1968), Harris (1968), Rodriguez (1977), Tadikamalla (1980). Cordeiro et al. (2016) investigated a new flexible class of continuous distributions called the generalized odd log-logistic-G (GOLL-G) family with only two extra shape parameters. In the research, we introduce a new version of the BXII model using the GOLL-G family called the generalized odd log-logistic Lx (GOLLLx). For an arbitrary baseline CDF \( \Psi(y) \), the CDF of the GOLL-G family is given by

\[ F_{a,b,\Psi}(y) = \frac{\Psi_{a}(y)^{ab}}{\Psi_{a}(y)^{ab} + [1 - \Psi_{a}(y)]^{b}}, \tag{3} \]

where \( \Psi \) is the parameter vector of the base line model. For \( b = 1 \) we get the OLL-G family (Gleaton and Lynch (2006)). For \( a = 1 \), the OLL-G family reduces to the proportional reversed hazard rate G (PRHR-G) family (Gupta and Gupta (2007)). The CDF of the GOLLLx is given by
\[
F_{\Phi}(y) = \frac{[1-(1+y)^{-\alpha}]^{ab}}{[1-(1+y)^{-\alpha}]^{ab} + [1-(1+y)^{-\alpha}]^{b\alpha-1}}
\]

where \( \Phi = a, b, \alpha \). The PDF corresponding to (4) can be given as

\[
f_{\Phi}(y) = \frac{aab(1 + y)^{a-1}[1-(1+y)^{-\alpha}]^{ab-1}b[1-(1+y)^{-\alpha}]^{b\alpha-1}}{[1-(1+y)^{-\alpha}]^{ab} + [1-(1+y)^{-\alpha}]^{b\alpha-1}}^2.
\]

The hazard rate function (HRF) for the GOLLLx model can be obtained from

\[
h_{\Phi}(y) = f_{\Phi}(y) / \left[ 1 - F_{\Phi}(y) \right].
\]

For \( b = 1 \) we get the OLLLx model. For \( a = 1 \) we get the PRHRLx model. Recently, many useful extensions are presented by Bhatti et al. (2018), Goual Yousof (2019), Ibrahim and Yousof (2020), Karamikabir et al. (2020), Mansour et al. (2020), Ansari et al. (2020), Goual et al. (2020) and Yadav et al. (2020). Based on generalized binomial expansions and after some algebraic processes, the PDF in (6) can be rewritten as

\[
f_{\Phi}(y) = \sum_{\ell_4=0}^{\infty} c_{\ell_4} h_{\alpha^*}(y),
\]

where \( \alpha^* = \alpha(1 + \ell_4) \) and

\[
c_{\ell_4} = \frac{ab}{1 + \ell_4} \sum_{\ell_3 + \ell_2 = 0}^{\infty} \sum_{\ell_3 + \ell_1 = \ell_4} (-1)^{\ell_2+\ell_3+\ell_4} \left( \frac{2}{\ell_1} \right) \left( \frac{2}{\ell_3} \right) \left( \frac{2}{\ell_4} \right) \left( \frac{1 + \ell_3}{\ell_2} \right) \left( \frac{-\alpha(1 + 1)}{\ell_2} \right) \left( \frac{ab(1 + 1 + b\ell_2 - 1)}{\ell_3} \right).
\]

\( h_{\alpha^*}(y) \) is the PDF of the Lx model with parameters \( \alpha^* \). Accordingly, the PDF of the new model can be expressed as a linear mixture of the Lx PDF. So, many properties of the new Lx model can be derived from (6) and those of the standard Lx model. Let \( Y \) be a RV having the Lx distribution (2) with parameter \( a_2 \). For \( m < a_2 \), the \( m \)th ordinary and incomplete moments (lc-Ms) of \( Y \) are, respectively, given by

\[
\mu_m' = aB(\alpha - m, 1 + m) \quad \text{and} \quad \mu_m(y) = aB(t; \alpha - m, 1 + m),
\]

where

\[
B(a_1, a_2) = \int_0^{\infty} (1 + y)^{-(a_1 + a_2)} y^{a_2 - 1} dy \quad \text{and} \quad B(t; a_1, a_2) = \int_0^y (1 + y)^{-(a_1 + a_2)} y^{a_1 - 1} dy.
\]

are the beta and the incomplete beta functions of the second type, respectively.

Figure 1: Plots for the new PDF and its corresponding HRF for some selected parameter values.

2. Properties
2.1 Asymptotics
Let \( \varepsilon = \inf\{y | n_{d(y)} > 0\} \), the asymptotics of the CDF, PDF and HRF as \( Y \to \varepsilon \) are given by

\[
F_{\Phi}(y) \big|_{y \to \varepsilon} \sim [1 - (1+y)^{-\alpha}]^{ab}, \quad f_{\Phi}(y) \big|_{y \to \varepsilon} \sim \frac{aab(1 + y)^{a-1}}{[1-(1+y)^{-\alpha}]^{ab+1}}
\]

and

\[
h_{\Phi}(y) \big|_{y \to \varepsilon} \sim \frac{aab(1 + y)^{-\alpha-1}}{[1-(1+y)^{-\alpha}]^{-ab+1}}.
\]
The asymptotics of CDF, PDF and HRF as \( Y \to \infty \) are given by

\[
1 - F_\Psi(y)\big|_{y\to\infty} \sim \frac{b^a}{(1+y)^{a\alpha}}, \quad f_\Psi(y)\big|_{y\to\infty} \sim \frac{ab^a\alpha}{(1+y)^{a\alpha+1}}, \quad \text{and} \quad h_\Psi(y)\big|_{y\to\infty} \sim \frac{aa}{1+y}.
\]

### 2.2 Ordinary moment

The \( m \)-th ordinary moment of \( Y \) is given by

\[
\mu'_m = E(Y^m) = \sum_{\ell_4=0}^{\infty} c_{\ell_4} \int_0^{m} y^m h_\Psi(y)dy.
\]

Then,

\[
\mu'_m = E(Y^m) = \sum_{\ell_4=0}^{\infty} c_{\ell_4} \alpha^\beta B(\alpha^\beta - m, 1 + m)\big|_{(m<\alpha^\beta)}.
\]

Setting \( m = 1 \) in (7), we have the mean of \( Y \). The effects of the parameters \( a, b, \alpha \) on the mean ( \( \mu'_1 \) ), variance (\( \nu'(Y) \)), skewness (\( S(Y) \)) and kurtosis (\( K(Y) \)) for given values are listed in Table 1. Form Tables 1 and 2 we note that the new additional shape parameters and has an effect on \( \mu'_1, \nu(Y), S(W) \) and \( K(Y) \). For the GOLLLx model, \( S(Y) \) can range in the interval \((-183.1, 7514.7)\). However, for the \( Lx \) model, \( S(Y) \) can range in the interval \((-0.4104, 4.6476)\). For the GOLLLx model, \( K(Y) \) can range in the interval \((-1531.11, 56479275)\). However, for the \( Lx \) model, \( K(Y) \) can range in the interval \((0.93244, 73.8)\).

**Table 1: Numerical results for \( \mu'_1, \nu(Y), S(Y), K(Y) \) for the GOLLLx.**

| a   | b   | \( \alpha \) | \( \mu'_1 \)  | \( \nu(Y) \)  | \( S(Y) \)  | \( K(Y) \) |
|-----|-----|-------------|-------------|-------------|-------------|-------------|
| 1   | 10  | 5           | 0.8598479   | 0.326637100 | 3.6522880   | 50.47191   |
| 2   | 0.7540910 |       | 0.55488670  | 1.6301880   | 9.990193    |
| 5   | 0.7232649 |       | 0.007758169 | 0.6661556   | 5.150165    |
| 10  | 0.7187439 |       | 0.001897504 | 0.3352351   | 4.442170    |
| 25  | 0.7174704 |       | 0.000301688 | 0.1343592   | 4.239094    |
| 40  | 0.7173224 |       | 0.000117759 | 0.0839946   | 4.215275    |
| 50  | 0.7172882 |       | 7.535265×10^{-5} | 0.0672000   | 4.209720    |
| 60  | 4.871488×10^{-6} | | 3.725648×10^{-5} | 125.43350   | 15750.29    |
| 75  | 2.314250×10^{-6} | | 1.777331×10^{-5} | 576.40470   | 332410.3    |
| 80  | 8.330471×10^{-7} | | 6.403259×10^{-7} | 961.01880   | 923925.1    |
| 90  | 1.071033×10^{-7} | | 8.242558×10^{-8} | 2681.3500   | 7191418     |
| 100 | 1.364333×10^{-8} | | 1.050770×10^{-8} | 7514.6940   | 5647926     |

| 3   | 0.015 | 5           | 1.3328×10^{-5} | 5.2882×10^{-8} | 565.0617   | 617625.9   |
| 0.1 | 0.001966 |       | 6.1593×10^{-5} | 2.2140007    | 15.80801   |
| 0.5 | 0.070440 |       | 0.0025435    | 2.140407     | 15.80801   |
| 1   | 0.160977 |       | 0.0056581    | 1.655923     | 9.734984   |
| 5   | 0.519958 |       | 0.0163016    | 1.153708     | 7.108161   |
| 10  | 0.733841 |       | 0.0226245    | 1.097702     | 6.769463   |
| 20  | 0.984661 |       | 0.0306119    | 1.070302     | 6.555797   |
| 50  | 1.378768 |       | 0.0448314    | 1.054058     | 6.589766   |
| 100 | 1.730548 |       | 0.0594514    | 1.048677     | 6.568111   |
| 150 | 1.960502 |       | 0.0700359    | 1.046878     | 6.560931   |
| 500 | 2.765278 |       | 0.1136273    | 1.044384     | 6.550913   |
| 1000| 3.324861 |       | 0.1500068    | 1.043484     | 6.547871   |
| 3000| 4.387358 |       | 0.2328648    | 1.043491     | 6.547344   |
| 5000| 4.966800 |       | 0.2856750    | 1.043419     | 6.547059   |
| 7000| 5.382125 |       | 0.3268405    | 1.043389     | 6.546937   |
| 9000| 5.711093 |       | 0.3614103    | 1.043372     | 6.546869   |
| 10000| 5.854005 |       | 0.3769692    | 1.043366     | 6.546845   |

| 4   | 1.5  | 5           | 2.5216120    | 0.590997000 | 2.0270130   | 17.48084   |
| 3   | 0.8666202 |       | 0.0373406600 | 1.1463170    | 7.326794   |
| 5   | 0.4524331 |       | 0.0078818440 | 0.8951852    | 5.88929    |
| 10  | 0.2046185 |       | 0.0013274300 | 0.7275170    | 5.165168   |
| 15  | 0.1320247 |       | 0.0005176865 | 0.6747110    | 4.972186   |
| 20  | 0.0974268 |       | 0.0002728258 | 0.6488175    | 5.133086   |
2.3 Moment generating function

The moment generating function (MGF) of $Y$, say $M_Y(t) = E[exp(tY)]$, can be obtained from (6) as

$$M_Y(t) = \sum_{\ell_4=0}^{\infty} c_{[\ell_4]} M_{\alpha^*}^\ell(t),$$

where $M_{\alpha^*}(t)$ is the MGF of the Lx distribution with parameters $\alpha^*$. Then, we have

$$M_Y(t) = \sum_{\ell_4, r=0}^{\infty} \frac{t^\ell}{r!} c_{[\ell_4]} B(\alpha^* - m, 1 + m) |_{m=\alpha^*}.$$  

2.4 Ic-Ms

The $s^{th}$ Ic-M, say $I_s(q)$, of the GOLLLx distribution is given by $I_s(q) = \int_0^q y^s f(y)dy$. Then, from equation (7), we have $I_s(t) = \sum_{\ell_4=0}^{\infty} c_{[\ell_4]} \int_0^t y^s h_{\alpha^*}(y)dy$ and using the lower incomplete gamma function, we obtain

$$I_s(t) = \sum_{\ell_4=0}^{\infty} c_{[\ell_4]} B(t^\alpha; \alpha^* - s, 1 + s).$$

The first Ic-M of $Y$, referred to $I_1(t)$, is just determined from the above equation by setting $s = 1$. The first Ic-M has main applications related to the Bonferroni and Lorenz curves and the mean residual life and the mean waiting time. Moreover, the amount of scattering in a population is clearly measured, to some extent, by the totality of deviations from the mean and median. The mean deviations, about the mean and about the median of $Y$ depend on $I_1(t)$. The Bonferroni $[B_{Y,F}(y),t]$ and Lorenz $[L_{Y,F}]$ curves have many applications especially in deconomics.

| $\alpha$ | $\mu_1^*$ | V(Y) | S(Y) | K(Y) |
|---------|-----------|------|------|------|
| 5       | 0.2500000 | 0.1041667 | 4.647580 | 73.8000 |
| 10      | 0.1111111 | 0.0154321 | 2.811057 | 17.82857 |
| 20      | 0.0526316 | 0.0030779 | 2.343806 | 12.13015 |
| 50      | 0.0204082 | 0.0004339 | 2.126365 | 10.06002 |
| 75      | 0.0135135 | 0.0001876 | 2.08269 | 9.684103 |
| 100     | 0.0101010 | 0.0001041 | 2.059443 | 9.522699 |
| 200     | 0.0050251 | 2.608252x10^{-5} | 1.945802 | 8.349871 |
| 300     | 0.0033445 | 1.038924x10^{-5} | 2.956477 | 10.32769 |
| 400     | 0.0025063 | 4.541270x10^{-6} | 4.316849 | 22.29550 |
| 500     | 0.0020040 | 2.703376x10^{-6} | 3.090030 | 26.71821 |
| 750     | 0.0013351 | 2.323909x10^{-6} | -0.410385 | 3.747853 |
| 1000    | 0.0010010 | 2.108303x10^{-6} | -0.195204 | 0.932436 |
| 1500    | 0.0006671 | 1.152765x10^{-6} | 0.659860 | 1.406983 |
| 2000    | 0.0005003 | 4.810577x10^{-7} | 2.153126 | 4.878395 |

Table 2: Numerical results for $\mu_1^*$, V(Y), S(Y), K(Y) for the Lx.
demography, insurance, reliability, medicine where \( L_{Y,t} = \frac{\int_{(0,Y)(t)} f_{\Psi}(x)}{F(Y)} \left( f_{\Psi}(x)dz \right) \) and \( B_{Y,F}(y,t) = \frac{\int_{(0,Y)(t)} f_{\Psi}(x)}{(Y)F(Y)} = \frac{L_{Y,t}}{F(Y)} \) where \( F(y) = F_{\Psi}(y) \). Then, we have
\[
L_{Y,t} = \sum_{\ell=0}^{\infty} \frac{\alpha \beta B(t; \alpha^*, s, 1 + s)}{\sum_{\ell=0}^{\infty} \alpha \beta B(s, 1 + s)} |_{s < \alpha \alpha^*}
\]
and
\[
B_{Y,F}(y,t) = \left( 1 - \frac{\left( 1 - \left( 1 + Y \right) - a \right) / \alpha}{\alpha} \right) \frac{\sum_{\ell=0}^{\infty} \alpha \beta B(t; \alpha^*, s, 1 + s)}{\sum_{\ell=0}^{\infty} \alpha \beta B(s, 1 + s)} |_{s < \alpha \alpha^*}.
\]

2.5 Residual and reversed residual life functions
The \( m^{th} \) moment of the residual life (RL), denoted by \( u_m(t) = E[(Y - t)^m] \), \( y > t, m = 1, 2, \ldots \). The \( m^{th} \) moment of the residual life of \( Y \) is given by \( u_m(t) = \frac{\int_{(y-t)^m} f_{\Psi}(y)dy}{\int_{y} f_{\Psi}(y)dy} \). Then, we can write
\[
u_m(t) = \frac{1}{\int_{y} f_{\Psi}(y)dy} \sum_{i=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(-1)^{m-i+1} m! mt^{m-i}}{l ! (m-i) !} \sum_{\ell=0}^{\infty} \alpha \beta B(t; \alpha^*, s, 1 + s, m + 1 + m).
\]
The \( m^{th} \) moment reversed residual life, say \( U_m(t) = E[(t - Y)^m] \), \( t > 0, Y \leq t, m = 1, 2, \ldots \). Then, \( U_m(t) \) is defined by \( U_m(t) = \frac{\int_{y} f_{\Psi}(y)dy}{\int_{y} f_{\Psi}(y)dy} \). The \( m^{th} \) moment of the reversed residual life of \( Y \)
\[
u_m(t) = \frac{1}{\int_{y} f_{\Psi}(y)dy} \sum_{i=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(-1)^{m+i+1} m! mt^{m-i}}{l ! (m-i) !} \sum_{\ell=0}^{\infty} \alpha \beta B(t; \alpha^*, s, 1 + s, m + 1 + m).
\]

3. Maximum likelihood method
Suppose that \((y_1, \ldots, y_m)\) is a random sample (rs) from the GOLLLx model with parameter vector \( \Psi \). The log-likelihood function \( \ell_m(\Psi) \) for \( \Psi \) is given by
\[
\ell_m(\Psi) = m \log(ab) - (\alpha + 1) \log(Y_1 + 1) + (ab - 1) \sum_{i=1}^{m} \log(1 - z_i^{-\alpha})
\]
\[
+ (a - 1) \sum_{i=1}^{m} \log(1 - z_i^b) - 2 \sum_{i=1}^{m} \log(z_i^{ab} + (1 - z_i^b)^a).
\]

where \( s_i = (y_i + 1) \) and \( z_i = [1 - s_i^{-\alpha}] \). The above \( \ell_m(\Psi) \) can be maximized numerically via SAS (PROC NLMixed) or R (optim) or Ox program (via sub-routine MaxBFGS), among others.

4. Copula
We derive some new bivariate type GOLLLx (BGOLLLx) model using “Farlie Gumbel Morgenstern” (FGM) Copula (see Morgenstern (1956), Gumbel (1958) and Gumbel (1960)),” Clayton Copula,” “modified FGM” and “Renyi's entropy” (Pogaza and Djaferi (2011)). However, future works may be allocated to study these new models. First, we consider the joint CDF of the FGM family, where \( C_{\alpha}(u, d) = u \psi(1 + \Omega u d) \), where the marginal function \( u = F_1, d = F_2, \Omega (\alpha) \in (-1, 1) \) is a dependence parameter and for every \( u, \psi \in (0, 1) \), \( C(u, 0) = C(0, \psi) = 0 \) which is "grounded minimum" and \( C(u, 1) = u \) and \( C(1, d) = d \) which is "grounded maximum". \( C(u_1, d_1) = C(u_2, d_2) - C(u_1, d_2) - C(u_2, d_1) \geq 0 \).

4.1 BGOLLLx type via FGM Copula
A Copula is continuous in \( u \) and \( d \); actually, it satisfies the "stronger Lipschitz condition", where
\[
|C(u_2, d_2) - C(u_1, d_1)| \leq |u_2 - u_1| + |d_2 - d_1|
\]
For \( 0 \leq u_1 \leq u_2 \leq 1 \) and \( 0 \leq \psi_1 \leq \psi_2 \leq 1 \), we have
\[
C(u_1, d_1) + C(u_2, d_2) - C(u_1, d_2) - C(u_2, d_1) = Pr(u_1 \leq U \leq u_2, d_1 \leq D \leq d_2) \geq 0.
\]
Then, setting \( u^* = 1 - F_{\Psi_1}(x_1) \mid x_1 \geq 0 \) and \( d^* = 1 - F_{\Psi_2}(x_2) \mid x_2 \geq 0 \) we can easily get the joint CDF of the FGM family. The joint PDF can then derived from \( c_{\alpha}(u, \psi) = 1 + \Omega u d \mid u \geq 1 - 2u \) or from \( f(x_1, x_2) = C(F_{\Psi_1}(x_1), F_{\Psi_2}(x_2))f_{\Psi_1}(x_1) \), \( f_{\Psi_2}(x_2) \).
4.2 BGOLLLx and MGOLLLx type via Clayton Copula

The “Clayton Copula” can be considered as $C_{\Theta}(d_1, d_2) = [(1/d_1)^{\alpha} + (1/d_2)^{\alpha} - 1]^{-\alpha^{-1}}$. Setting $d_1 = F_{\Psi_1}(t)$ and $d_2 = F_{\Psi_2}(x)$. Then, the BGOLLLx type can be derived from $C(d_1, d_2) = C(F_{\Psi_1}(t), F_{\Psi_2}(x))$. Similarly, the MGOLLLx (m-dimensional extension) from the above can be derived from $C(d_1) = \left[ \sum_{i=1}^{m} d_i^{-\alpha} + 1 - m \right]^{-\alpha^{-1}}$.

4.3 BGOLLLx type via Renyi’s entropy

Using the theorem of Pougaza and Djafari (2011) where $R(u, d) = x_2u + x_1d - x_1x_2$. Then, the associated BGOLLLx will be $R(u, d) = R(F_{\Psi_1}(x_1), F_{\Psi_2}(x_2))$.

4.4 BGOLLLx type via modified FGM Copula

The modified version of the bivariate FGM copula defined as (Rodriguez-Lallena and Ubeda-Flores (2004)) $C_{\Theta}(u, d) = ud + \Theta \frac{O(u)}{u} \frac{\Psi(d)}{d}$, where $O(u) = u\overline{O}(u)$, and $\Psi(d) = d\overline{\Psi}(d)$, where $O(u)$ and $\Psi(v)$ are two continuous functions on $(0,1)$ where $O(0) = O(1) = \Psi(0) = \Psi(1) = 0$. Let

\[ b = \inf \left\{ \frac{\partial}{\partial u} \overline{O}(u) : \mathcal{H}_1(u) < 0, \right\} \quad \alpha = \sup \left\{ \frac{\partial}{\partial u} \overline{O}(u) : \mathcal{H}_1(u) < 0, \right\} \\
\[ c = \inf \left\{ \frac{\partial}{\partial d} \overline{\Psi}(d) : \mathcal{H}_2(d) > 0, \right\} \quad s = \sup \left\{ \frac{\partial}{\partial d} \overline{\Psi}(d) : \mathcal{H}_2(d) > 0, \right\}. 

Then, $\min(ba, cs) \geq 1$, where $\frac{\partial}{\partial u} \overline{O}(u) = O(u) + u \frac{\partial}{\partial u} O(u), \mathcal{H}_1(u) = \left\{ u \in (0,1) : \frac{\partial}{\partial u} \overline{O}(u) \text{ exists} \right\}$ and $\mathcal{H}_2(d) = \left\{ d \in (0,1) : \frac{\partial}{\partial d} \overline{\Psi}(d) \text{ exists} \right\}$.

4.4.1 BGOLLLx-FGM (Type I)

The BvGOLLLx-FGM (Type-I) copula can be obtained directly from $C_{\Theta}(u, v) = uv + \Theta \phi(u) \psi(v)$.

4.4.2 BGOLLLx-FGM (Type II)

Consider the following functional form for both $O(u)$ and $\Psi(\psi)$ which satisfy all the conditions stated earlier where $O(u)(\alpha_1=0) = u^{a_1}(1-u)^{-1-a_1}$ and $\Psi(d)(\alpha_2=0) = d^{a_2}(1-d)^{-1-a_2}$. The corresponding bivariate copula (henceforth, BGOLLLx-FGM (Type-II) copula) can be derived from $C_{\alpha_1, \alpha_2}(u, d) = ud[1 + \Theta u^{a_1}d^{a_2}(1-u)^{1-a_1}(1-d)^{1-a_2}]$.

4.4.3 BGOLLLx-FGM (Type III)

Consider the following functional form for both $O(u)$ and $\Psi(\psi)$ which satisfy all the conditions stated earlier where $O^*(u) = u[log(1 + \overline{u})]$ and $\Psi^*(d) = d[log(1 + \overline{d})]$. Then, the associated CDF of the BvGOLLLx-FGM (Type-III) $C_{\Theta}(u, d) = ud + [1 + \Theta O^*(u) \Psi^*(d)]$

4.4.4 BGOLLLx-FGM (Type IV)

The BGOLLLx-FGM (Type-IV) model can be derived from $C(u, d) = dF^{-1}(u) + F^{-1}(u) - F^{-1}(u)F^{-1}(d)$ where $F^{-1}(u)$ and $F^{-1}(d)$ can be easily derived.

5. Graphical simulations

To assess the finite sample behavior of the MLEs, consider the following algorithm:

1) use $\gamma_u = 1 - \left( \frac{u^{\frac{1}{\alpha(a-\beta)}}}{1+(\frac{u^{\frac{1}{\alpha(a-\beta)}}})^\frac{1}{\alpha(a-\beta)}} \right) - 1$ to generate 1000 samples of size $m$ from the GOLLLx distribution;

2) compute the MLEs for the 1000 samples

3) compute the SEs of the MLEs for the 1000 samples. The standard errors (SEs) were computed by inverting the observed information matrix;

4) compute the biases $(B_h(m))$ and mean squared errors (MSEs) given for $h = \Phi$. We repeated these steps for $m = 50, 100, \ldots, 1000$ with $\alpha = 1$, $\beta = 1$, $m = 1$, $h = 1$ so computing biases, mean squared errors (MSE$_h(m)$) for $\Phi$ and $m = 50, 100, \ldots, 1000$.
Figure 2: Biases and MSEs for $a, b, \alpha$ and $m = 50, 100, \ldots, 1000$ for the GOLLLx model.

Figure 2 (left panel) shows how the three biases vary with respect to $m$. Figure 2 (right panel) shows how the three MSEs vary with respect to $m$. The broken lines in Figure 2 corresponds to the biases being 0. From Figure 2, the biases for each parameter decrease to zero as $m \to \infty$, the MSEs for each parameter decrease to zero as $m \to \infty$.

6. Comparing models
To illustrate the flexibility of the GOLLLx model, we provide three applications. The 1st data set called breaking stress data. This data set consists of 100 observations of breaking stress of carbon fibres (in Gba) given by Nichols and Padgett (2006). The 2nd data set called survival times. In this application, we work with the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, originally observed and reported by Bjerkedal, T. (1960). The 3rd data set called taxes revenue data. The actual taxes revenue data (in 1000 million Egyptian pounds) given in Altun et al. (2018a, b). For all data sets, we compare the GOLLLx distribution with the standard Lx, the exponentiated Lx (ExpLx), the Burr XII (BXII), the Marshall-Olkin Burr XII (MOBXII), the Topp-Leone Burr XII (TLBXII), the Zografos-Balakrishnan Burr XII (ZBBXII), the five-parameters beta Burr XII (FBBXII), beta Burr XII (BBXII), beta exponentiated Burr XII (BEBXII), the five-parameters Kumaraswamy Burr XII (FKumBXII) and Kumaraswamy Burr XII (KumBXII) distributions. All competitive models are given in Yousof et al. (2019) and Altun et al. (2018 a, b).
We consider the well-known G-O-F statistics: the Akaike Information Criterion (CA), Bayesian Information Criterion (CBayes), Hannan-Quinn Information Criterion (CHQ), Consistent Akaike Information Criterion (CA) respectively. Tables 6, 7, and 8 give CA, CBayes, CHQ and CA values for the data set I, II, III respectively. Based on the values in Tables 6, 7, 8 and Figures 4-8 the GOLLlx model has the lowest CA, CBayes, CA, CHQ respectively.

For the Egyptian tax’s revenue data: CA = 384.12, CHQ = 386.55, CA = 384.56. From 1, 2 and 3 we conclude that the new model has the lowest CA, CBayes, CA, CHQ for all data sets. Figure 3 gives the TTT plots. Based on Figure 3, the HRF of the three real data sets are increasing. Figure 4 gives the estimated PDFs. Figure 5 gives the estimated CDFs. Figure 6 gives the estimated HRFs. Figure 7 gives the P-P plots. Figure 8 gives Kaplan-Meier survival Plot. Based on Figures 4-8 the GOLLlx model has a good fit to the empirical functions. Based on the values in Tables 6, 7, 8 and Figures 4-8 the GOLLlx model has the best fits as compared to BXII other models in the three applications with small values for CA, CBayes, CA, CHQ. Many useful real data can be found in Mansour et al. (2017 and 2018 and 2019) and Merovci (2017).

Table 3: MLEs, SEs and CIs for the data set I

| Model   | MLEs      | SEs   | CI   |
|---------|-----------|-------|------|
| Lx      | ---,---, ---, 0.8025,--- | ---,---, (0.08025),--- | ---,---, (0.64,0.96),--- |
| ExpLx   | ---,---, 0.6437,0.767,--- | ---,---, (0.0603),(0.094),--- | ---,---, (0.48, 0.72),(0.59),--- |
| BXII    | ---,---, 5.941,0.187,--- | ---,---, (1.279),(0.044),--- | ---,---, (3.43,8.45),(0.10,0.27),--- |
| MOBXII  | ---,---, 1.192,4.834,838.73 | ---,---, (0.952),(4.896),(229.34) |
| TLBXII  | ---,---, 1.350,1.061,13.728 | ---,---, (0.378),(0.384),(8.400) |
| KumBXII | 48.103 ,79.516 ,0.351 ,2.730 ,--- | 19.348 ,58.186 ,0.098 ,1.077 ,--- |
| BBXII   | 359.683 ,260.097 ,0.175 ,1.123 ,--- | (10.18,86.03),(0.1935),(0.62,4.84),--- |
| BEXII   | 0.381, 11.949, 0.937, 33.402, 1.705 | (0.078), (4.635), (0.267), (6.287),(0.478) |
| FBBXII  | 0.421, 0.834, 6.111, 1.674, 3.450 | (0.23,0.53),(2.86,21),(0.41,1.5),(21.45),(0.8,2.6) |
| FKumBXII| 0.542,4.223,5.313,0.411,4.152  | (0.137),(1.882),(2.318),(0.497),(1.995) |
| ZBBXII  | 123.101,---,0.368,139.247,--- | (0.3, 0.8),(0.53,3.7),(0.9, 0.1),(0.7),(0.8) |
| GOLLLx  | 5.013,1.036,0.576,--- | (3.137),(1.046),(0.569),--- |

Table 1: MLEs, SEs and CIs for the data set I

| Model   | MLEs      | SEs   | CI   |
|---------|-----------|-------|------|
| Lx      | ---,---, ---, 0.8025,--- | ---,---, (0.08025),--- | ---,---, (0.64,0.96),--- |
| ExpLx   | ---,---, 0.6437,0.767,--- | ---,---, (0.0603),(0.094),--- | ---,---, (0.48, 0.72),(0.59),--- |
| BXII    | ---,---, 5.941,0.187,--- | ---,---, (1.279),(0.044),--- | ---,---, (3.43,8.45),(0.10,0.27),--- |
| MOBXII  | ---,---, 1.192,4.834,838.73 | ---,---, (0.952),(4.896),(229.34) |
| TLBXII  | ---,---, 1.350,1.061,13.728 | ---,---, (0.378),(0.384),(8.400) |
| KumBXII | 48.103 ,79.516 ,0.351 ,2.730 ,--- | 19.348 ,58.186 ,0.098 ,1.077 ,--- |
| BBXII   | 359.683 ,260.097 ,0.175 ,1.123 ,--- | (10.18,86.03),(0.1935),(0.62,4.84),--- |
| BEXII   | 0.381, 11.949, 0.937, 33.402, 1.705 | (0.078), (4.635), (0.267), (6.287),(0.478) |
| FBBXII  | 0.421, 0.834, 6.111, 1.674, 3.450 | (0.23,0.53),(2.86,21),(0.41,1.5),(21.45),(0.8,2.6) |
| FKumBXII| 0.542,4.223,5.313,0.411,4.152  | (0.137),(1.882),(2.318),(0.497),(1.995) |
| ZBBXII  | 123.101,---,0.368,139.247,--- | (0.3, 0.8),(0.53,3.7),(0.9, 0.1),(0.7),(0.8) |
| GOLLLx  | 5.013,1.036,0.576,--- | (3.137),(1.046),(0.569),--- |
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Table 4: MLEs, SEs and CIs for the data set II.

| Model | a, b, α, θ, γ | Lx     | MLEs     | ---, ---, ---, 1.0244, --- |
|-------|--------------|--------|----------|-----------------------------|
|       |              | SEs    | ---, ---, (0.1207), --- |
|       |              | CIs    | ---, ---, (0.1292), --- |
| ExpLx | MLEs         | ---, ---, 0.8257, 0.9739, --- |
|       |              | SEs    | ---, (0.093), (0.1292), --- |
|       |              | CIs    | ---, (0.62, 0.98), (0.64, 1.16), --- |
| BXII  | MLEs         | ---, ---, 3.102, 0.465, --- |
|       |              | SEs    | ---, (0.538), (0.077), --- |
|       |              | CIs    | ---, (2.05, 4.16), (0.31, 0.62), --- |
| MOBXII| MLEs         | ---, ---, 2.059, 1.533, 6.760 |
|       |              | SEs    | ---, (0.864), (0.907), (4.587) |
|       |              | CIs    | ---, (0.57, 3.95), (0.33), (0.1575) |
| TLBXII| MLEs         | ---, ---, 0.78, 1.26, --- |
|       |              | SEs    | ---, ---, (0.78), (1.26), --- |
|       |              | CIs    | ---, ---, (0.62, 0.98), (0.64, 1.16), --- |
| KumBXII| MLEs       | 14.105, 7.424, 0.525, 2.274, --- |
|       |              | SEs    | (10.805), (11.850), (0.990), (4.587) |
|       |              | CIs    | (0.3528), (0.3065), (0.107), (0.33), (4.21) |
| BBXII | MLEs         | 2.555, 6.058, 1.800, 0.294, --- |
|       |              | SEs    | (1.859), (10.391), (0.955), (0.466), --- |
|       |              | CIs    | (0.628), (0.2642), (0.367), (0.121) |
| BBXII | MLEs         | 1.876, 2.991, 1.780, 1.341, 0.572 |
|       |              | SEs    | (0.094), (1.731), (0.702), (0.816), (0.325) |
|       |              | CIs    | (1.7, 2.06), (0.64), (0.40, 3.2), (0.29), (0.121) |
| FBBXII| MLEs         | 0.621, 0.549, 3.838, 1.381, 1.665 |
|       |              | SEs    | (0.541), (1.011), (2.785), (2.312), (0.436) |
|       |              | CIs    | (0.17), (0.25), (0.93), (0.59), (0.8), (4.5) |
| FKumBXII| MLEs     | 0.558, 0.308, 3.999, 2.131, 1.475 |
|       |              | SEs    | (0.442), (0.314), (2.082), (1.833), (0.361) |
|       |              | CIs    | (0.14), (0.9), (3.1), (0.57), (0.76), (2.2) |
| GOLLLx| MLEs         | 4.226, 0.663, 0.464, ---, --- |
|       |              | SEs    | (3.177), (0.701), (0.641), ---, --- |
|       |              | CIs    | (0, 10.4), (0, 2.06), (0, 1.66), ---, --- |

Table 5: MLEs, SEs and CIs for the data set III.

| Model | a, b, α, θ, γ | Lx     | MLEs     | ---, ---, 0.392, --- |
|-------|--------------|--------|----------|-----------------------------|
|       |              | SEs    | ---, ---, (0.051), --- |
|       |              | CIs    | ---, ---, (0.29, 0.49), --- |
| ExpLx | MLEs         | ---, ---, 0.332, 0.604, --- |
|       |              | SEs    | ---, ---, (0.0405), (0.122), --- |
|       |              | CIs    | ---, ---, (0.25, 0.41), (0.36, 0.84), --- |
| BXII  | MLEs         | ---, ---, 5.615, 0.072, --- |
|       |              | SEs    | ---, ---, (15.048), (0.194), --- |
|       |              | CIs    | ---, ---, (0.3511), (0.45), --- |
| MOBXII| MLEs         | ---, ---, 8.017, 0.419, 70.359 |
|       |              | SEs    | ---, ---, (22.083), (0.312), (63.831) |
|       |              | CIs    | ---, ---, (0.5129), (0.103), (0.19547) |
| TLBXII| MLEs         | ---, ---, 91.320, 0.012, 141.073 |
|       |              | SEs    | ---, ---, (15.071), (0.002), (70.028) |
|       |              | CIs    | ---, ---, (61.78, 120.86), (0.008), (0.02), (3.82, 278.33) |
| KumBXII| MLEs       | 18.130, 6.857, 10.694, 0.081, --- |
|       |              | SEs    | (3.689), (1.035), (1.166), (0.012), --- |
|       |              | CIs    | (10.89, 25.36), (4.83, 8.89), (8.41), (12.98), (0.06), (0.10), --- |
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**Table 6:** C-Al, C-Bayes, C-CA, C-HQ values for the data I.

| Model        | C-Al, C-Bayes, C-CA, C-HQ |
|--------------|---------------------------|
| Lx           | 495.25, 497.86, 495.295, 496.31 |
| ExpLx        | 470.39, 475.59, 470.51, 472.49 |
| BXII         | 382.94, 388.15, 383.06, 385.05 |
| MOBXII       | 305.78, 313.60, 306.03, 308.96 |
| TLBXII       | 323.52, 331.35, 323.77, 326.70 |
| KumBXII      | 303.76, 314.21, 304.18, 308.00 |
| BBXII        | 305.64, 316.06, 306.06, 309.85 |
| BEBXII       | 305.82, 318.84, 306.46, 311.09 |
| FBBXII       | 304.26, 317.31, 304.89, 309.56 |
| FKumBXII     | 305.50, 318.55, 306.14, 310.80 |
| GOLLLx       | 301.44, 309.26, 301.69, 304.61 |

**Table 7:** C-Al, C-Bayes, C-CA, C-HQ values for the data II.

| Model        | C-Al, C-Bayes, C-CA, C-HQ |
|--------------|---------------------------|
| Lx           | 283.18, 285.45, 283.23, 284.08 |
| ExpLx        | 282.06, 286.61, 282.23, 283.87 |
| BXII         | 209.60, 214.15, 209.77, 211.40 |
| MOBXII       | 209.74, 216.56, 210.09, 212.44 |
| TLBXII       | 211.80, 218.63, 212.15, 214.52 |
| KumBXII      | 208.76, 217.86, 209.36, 212.38 |
| BBXII        | 210.44, 219.54, 211.03, 214.06 |
| BEBXII       | 212.10, 223.50, 213.00, 216.60 |
| FBBXII       | 206.80, 218.20, 207.71, 211.30 |
| FKumBXII     | 206.50, 217.90, 207.41, 211.00 |
| GOLLLx       | 204.05, 210.88, 206.77, 204.405 |

**Table 7:** C-Al, C-Bayes, C-CA, C-HQ values for the data III.

| Model        | C-Al, C-Bayes, C-CA, C-HQ |
|--------------|---------------------------|
| Lx           | 531.53, 533.61, 531.60, 532.34 |
| ExpLx        | 397.92, 402.08, 398.13, 399.01 |
| BXII         | 518.46, 522.62, 518.67, 520.08 |
| MOBXII       | 387.22, 389.38, 387.66, 389.68 |
| TLBXII       | 385.94, 392.18, 386.38, 388.40 |
| KumBXII      | 385.58, 393.90, 386.32, 388.86 |
| BBXII        | 385.56, 394.10, 386.30, 389.10 |
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| Distribution | Values 1  | Values 2  | Values 3  | Values 4  |
|--------------|-----------|-----------|-----------|-----------|
| BEBXII       | 387.04, 397.42, 388.17, 391.09 |
| FBBXII       | 386.74, 397.14, 387.87, 390.84 |
| FKumBXII     | 386.96, 397.36, 388.09, 391.06 |
| GOLLLx       | 384.12, 390.35, 386.55, 384.56 |

**Figure 3:** TTT plots.

**Figure 4:** Estimated PDFs.
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Figure 5: Estimated CDFs.

Data I. Data II. Data III.

Figure 6: Estimated HRFs.

Data I. Data II. Data III.

Figure 7: P-P plots.

Data I. Data II. Data III.
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7. Conclusions
A new generalization of Lomax distribution is derived and studied. The new extension has only three parameters. Some of its useful mathematical properties are derived. We performed graphical simulations to assess the finite sample behavior of the estimations. The effects of all parameters on the mean, variance, skewness and kurtosis for given values are studied. We note that the new additional shape parameters and has an effect on the mean, variance, skewness and kurtosis. For the new Lx model, skewness can range in the interval (−183.1, 7514.7). However, for the standard Lx model, skewness can range in the interval (−0.4104, 4.6476). For the GOLLLx model, kurtosis can range in the interval (0.93244, 73.8). The new model is employed in modelling three real data sets. For all data sets, we compared the new Lx distribution with the standard Lx, the exponentiated Lx, the Burr XII, beta Burr XII, the Marshall-Olkin Burr XII, the Topp-Leone Burr XII, the Zografos-Balakrishnan Burr XII, beta exponentiated Burr XII, the five-parameters beta Burr XII, the five-parameters Kumaraswamy Burr XII and Kumaraswamy Burr XII distributions. The new Lx distribution is a useful alternative for the above-mentioned models in modeling breaking stress data, survival times of guinea pig’s data and the Egyptian taxes revenue data.

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