COMPLETE LIFTING OF DOUBLE-LINEAR SEMI-BASIC TANGENT VALUED FORMS TO WEIL LIKE FUNCTORS ON DOUBLE VECTOR BUNDLES

WLODZIMIERZ M. MIKULSKI

Abstract. Let $F$ be a product preserving gauge bundle functor on double vector bundles. We introduce the complete lifting $\mathcal{F}\varphi : FK \to \wedge^p T^*FM \otimes TFK$ of a double-linear semi-basic tangent valued $p$-form $\varphi : K \to \wedge^p T^*M \otimes TK$ on a double vector bundle $K$ with base $M$. We prove that this complete lifting preserves the Frolicher–Nijenhuis bracket. We apply the results obtained to double-linear connections.

1. Introduction

We assume that any manifold considered in the paper is Hausdorff, second countable, finite dimensional, without boundary and smooth (i.e. of class $C^\infty$). All maps between manifolds are assumed to be smooth (of class $C^\infty$).

Definition 1.1. An almost double vector bundle is a system $K = (K_r, K_l, E_r, E_l)$ of vector bundles $K_r = (K, \tau_r, E_r), K_l = (K, \tau_l, E_l), E_r = (E_r, \tau_l, M)$ and $E_l = (E_l, \tau_r, M)$ such that $\tau_l \circ \tau_r = \tau_r \circ \tau_l$ (this means that the respective diagram is commutative). We call $M$ the basis of $K$.

If $K' = (K'_r, K'_l, E'_r, E'_l)$ is another almost double vector bundle, an almost double vector bundle map $K \to K'$ is a map $f : K \to K'$ such that there are maps $f_r : E_r \to E'_r, f_l : E_l \to E'_l$ and $f : M \to M'$ such that $(f, f_r) : K_r \to K'_r, (f, f_l) : K_l \to K'_l, (f_r, f) : E_r \to E'_r$ and $(f_l, f) : E_l \to E'_l$ are vector bundle maps. We call $f : M \to M'$ the base map of $f$.

For example, we have the trivial almost double vector bundle $K = (K_r, K_l, E_r, E_l)$, where $K_l = (R^{m_1} \times R^{m_2}, \tau_r, R^{m_1} \times R^{n_2}, \tau_l, R^{n_1} \times R^{m_1} \times R^{n_1}), K_r = (R^{m_1} \times R^{m_2} \times R^{n_1} \times R^{m_2}, \tau_r, R^{m_1} \times R^{m_2}), E_r = (R^{m_1} \times R^{m_2}, \tau_l, R^{m_1})$ and $E_l = (R^{m_1} \times R^{n_1}, \tau_r, R^{m_1})$, and where $\tau_r, \tau_l, \tau_l, \tau_r$ are the obvious projections. We will denote this trivial almost double vector bundle by $R^{m_1,m_2,n_1,n_2}$.

Definition 1.2. A double vector bundle is a locally trivial almost double vector bundle $K$. This means that there are nonnegative integers $m_1, m_2, n_1, n_2$ such

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that for any \( x \in M \) there is an open neighborhood \( \Omega \subset M \) of \( x \) such that \( K|_\Omega = R^{m_1,m_2,n_1,n_2} \) modulo an almost double vector bundle isomorphism.

The tangent bundle
\[
TE = ((TE, \pi^{TE}, E), (TE, T\pi, TM), (E, \pi, M), (TM, \pi^{TM}, M))
\]
of a vector bundle \( E = (E, \pi, M) \) is an example of a double vector bundle.

Any manifold \( M \) can be treated as the double vector bundle \( \mathbf{M} \) with basis \( \mathbf{M} \).

**Definition 1.3.** Let \( K \) be a double vector bundle as above. A **double-linear vector field** on \( K \) is a vector field \( Z \) on \( K \) such that the flow of \( Z \) is formed by (local) double vector bundle isomorphisms.

Any double linear vector field \( Z \) on \( K \) is projectable with respect to the (common) projection \( K \to M \). Thus we have the underlying vector field \( Z \) on \( M \).

**Definition 1.4.** Let \( K \) be a double vector bundle as above with basis \( \mathbf{M} \). A **double-linear semi-basic tangent valued \( p \)-form** on \( K \) is a section \( \varphi : K \to \pi^{TM} \otimes TK \) such that \( \varphi(X_1, \ldots, X_p) \) is a double linear vector field on \( K \) for any vector fields \( X_1, \ldots, X_p \) on the basis \( \mathbf{M} \) of \( K \).

**Definition 1.5.** Let \( K \) be as above. A **double-linear connection** in \( K \) is a double-linear semi-basic tangent valued 1-form \( \Gamma : K \to T^*M \otimes TK \) on \( K \) such that the underlying vector field of \( \Gamma(X) \) is equal to \( X \) for any vector field \( X \) on basis \( \mathbf{M} \).

Let \( \mathcal{DVB} \) denote the category of all double vector bundles and their almost double vector bundle maps, and let \( \mathcal{FM} \) denote the category of fibered manifolds and fibered maps. (In [14], the notation 2-VB instead of \( \mathcal{DVB} \) is used.)

The general concept of (gauge) bundle functors can be found in [7]. We need the following particular case of it.

**Definition 1.6.** A **gauge bundle functor** on \( \mathcal{DVB} \) is a covariant functor \( F : \mathcal{DVB} \to \mathcal{FM} \) sending any double vector bundle \( K \) with basis \( \mathbf{M} \) into a fibered manifold \( F(K) \to \mathbf{M} \) over \( \mathbf{M} \), and any double vector bundle map \( f : K \to K' \) with the base map \( f : \mathbf{M} \to \mathbf{M}' \) into a fibered map \( Ff : F(K) \to F(K') \) over \( f : \mathbf{M} \to \mathbf{M}' \), and satisfying the following conditions:

(i) **Localization condition:** For every double vector bundle \( K \) with basis \( \mathbf{M} \) and any open subset \( U \subset \mathbf{M} \) the inclusion map \( i_{K|U} : K|U \to K \) induces a diffeomorphism \( F(i_{K|U}) : F(K|U) \to F(K) \).

(ii) **Regularity condition:** \( F \) transforms smoothly parametrized families of \( \mathcal{DVB} \)-maps into smoothly parametrized families of \( \mathcal{FM} \)-maps.

A gauge bundle functor \( F \) on \( \mathcal{DVB} \) is product preserving (ppgb-functor) if \( F(K_1 \times K_2) = F(K_1) \times F(K_2) \) for any \( \mathcal{DVB} \)-objects \( K_1 \) and \( K_2 \). Product preserving gauge bundle functors can be also called Weil like functors, because the product preserving bundle functors on manifolds are the usual Weil functors.

A simple example of a ppgb-functor on \( \mathcal{DVB} \) is the tangent functor \( T \) sending any \( \mathcal{DVB} \)-object \( K \) into the tangent bundle \( TK \) (over \( \mathbf{M} \)) and any \( \mathcal{DVB} \)-map \( f : K \to K' \) into the tangent map \( Tf : TK \to TK' \).
By [14], the ppgb-functors \( F \) on \( \mathcal{DV}B \) are in bijection with the \( \mathcal{A}^F \)-bilinear maps \( \circ^F : U^F \times V^F \to W^F \), where \( \mathcal{A}^F \) are Weil algebras and \( U^F, V^F, W^F \) are finitely dimensional (over \( \mathbb{R} \)) \( \mathcal{A}^F \)-modules. Moreover, the ppgb-functors \( F \) on \( \mathcal{DV}B \) have values in \( \mathcal{DV}B \). For any such \( F \), if \( K \) is a \( \mathcal{DV}B \)-object with basis \( M \), then \( FK \) is a \( \mathcal{DV}B \)-object with basis \( FM = T^A M \); see [14].

Let \( F \) be a ppgb-functor on \( \mathcal{DV}B \) and let \( \circ^F : U^F \times V^F \to W^F \) be the corresponding \( \mathcal{A}^F \)-bilinear map. Let \( K \) be a \( \mathcal{DV}B \)-object. Then any double-linear vector field \( Z \) on \( K \) can be lifted to the double-linear vector field \( \mathcal{F}Z \) on \( FK \) via \( F \)-prolongation of flow. By [14], for any \( a \in \mathcal{A}^F \) we have the affinor \( \operatorname{aff}(a) : TFK \to TFK \) on \( FK \). We have \( \operatorname{aff}(a_1 a_2) = \operatorname{aff}(a_1) \circ \operatorname{aff}(a_2) \) and \( \operatorname{aff}(1) \) is the identity affinor. If \( f : K \to K_1 \) is a \( \mathcal{DV}B \)-map, then \( TFK \circ \operatorname{aff}(a) = \operatorname{aff}(a) \circ TFK \).

The main result of the paper is the following one (see Theorem 4.5):

Next we study the complete lifting \( \mathcal{F} \). We prove that \( \mathcal{F} \) commutes with the Frolicher–Nijenhuis bracket (see Theorem 5.1) and apply this fact to double-linear connections \( \Gamma : K \to T^* M \otimes TK \) in \( K \) (see Theorem 6.3).

By the local description of double vector bundles, presented in [8], the notion of double vector bundles in the sense of the present paper is equivalent to the one in the book [11]. Product preserving (gauge) bundle functors are studied in [1] [6] [7] [9] [10] [12] [13] [14] [16] [17] [18]. Liftings of vector fields to product preserving (gauge) bundle functors are studied in [5] [10] [14]. Complete lifting of general connections on fibered manifolds to Weil functors is studied in [7]. Complete lifting of semi-basic tangent valued \( p \)-forms on fibered manifolds to Weil functors is studied in [2] [3]. Complete lifting of linear semi-basic tangent valued forms to product preserving gauge bundle functors on vector bundles is studied in [15]. The Frolicher–Nijenhuis bracket on projectable tangent valued forms is studied in [4].

2. PRELIMINARIES

Let \( K \) be a double vector bundle. Let \( M \) be the basis of \( K \) and \( \pi : K \to M \) be the projection.

**Lemma 2.1.** Let \( Z, Z_1 \) be double-linear vector fields on \( K \), \( \alpha \) a real number and \( f : M \to \mathbb{R} \) a map. Then \( Z + Z_1, \alpha Z, f \circ \pi \cdot Z \) and \( [Z, Z_1] \) are double linear vector fields on \( K \).

**Proof.** Using \( \mathcal{DV}B \)-charts, we may assume \( K = \mathbb{R}^{m_1, m_2, n_1, n_2} \). Let \( x^1, \ldots, x^{m_1}, u^1, \ldots, u^{m_2}, v^1, \ldots, v^{n_1}, w^1, \ldots, w^{n_2} \) be the usual coordinates. A map \( f : K \to K \)
is a $\mathcal{VB}$-map if and only if it is of the form

$$x^i \circ f = \alpha^i(x), \quad i = 1, \ldots, m_1,$$

$$w^j \circ f = \sum_{j_1 = 1}^{m_2} \beta_{j_1}^j(x) w^{j_1}, \quad j = 1, \ldots, m_2,$$

$$v^k \circ f = \sum_{k_1 = 1}^{n_1} \gamma_{k_1}^k(x) v^{k_1}, \quad k = 1, \ldots, n_1,$$

$$w^l \circ f = \sum_{l_1 = 1}^{n_2} \gamma_{l_1}^l(x) w^{l_1} + \sum_{j_1 = 1}^{m_2} \sum_{k_1 = 1}^{n_1} \sigma_{j_1k_1}^l(x) w^{j_1} v^{k_1}, \quad l = 1, \ldots, n_2,$$

where $x = (x_1, \ldots, x_{m_1})$. Consequently, a vector field $Z$ on $K$ is double linear if and only if it is of the form

$$Z = \sum_{i = 1}^{m_1} a^i(x) \frac{\partial}{\partial x^i} + \sum_{j,j_1 = 1}^{m_2} b_{j_1}^j(x) w^{j_1} \frac{\partial}{\partial w^j} + \sum_{k,k_1 = 1}^{n_1} c_k^k(x) v^k \frac{\partial}{\partial v^k} + \sum_{l,l_1 = 1}^{n_2} e_{l_1}^l(x) w^{l_1} \frac{\partial}{\partial w^l} + \sum_{j_2 = 1}^{m_2} \sum_{k_2 = 1}^{n_1} \sum_{l_2 = 1}^{n_2} f_{j_2k_2}^{l_2}(x) w^{j_2} v^{k_2} \frac{\partial}{\partial w^{l_2}}, \quad (2.1)$$

The lemma is now clear. $\Box$

Now, we treat $K$ as a fibered manifold over $M$ or (generally) let $\pi : K \to M$ be an arbitrary fibered manifold.

**Definition 2.2.** A projectable semi-basic tangent valued $p$-form on $K$ is a section $\varphi : K \to \bigwedge^p T^* M \otimes TK$ such that $\varphi(X_1, \ldots, X_p)$ is a projectable vector field on $K$.

Given a projectable semi-basic tangent valued $p$-form $\varphi : K \to \bigwedge^p T^* M \otimes TK$ we have the underlying tangent valued $p$-form $\varphi : M \to \bigwedge^p T^* M \otimes TM$ on $M$ such that $\varphi(X_1, \ldots, X_p)$ is the underlying vector field of the projectable vector field $\varphi(X_1, \ldots, X_p)$ for any vector fields $X_1, \ldots, X_p$ on $M$.

The following fact is well known; see e.g. [3][4].

**Lemma 2.3.** Given a projectable semi-basic tangent-valued $p$-form $\varphi : K \to \bigwedge^p T^* M \otimes TK$ on $K$ and a projectable semi-basic tangent valued $q$-form $\psi : K \to \bigwedge^q T^* M \otimes TK$ on $K$ there exists a (unique) projectable semi-basic tangent valued
\((p + q)\)-form \([[\varphi, \psi]] : K \to \wedge^{p+q}T^*M \otimes TK\) on \(K\) such that
\([[\varphi, \psi]](X_1, \ldots, X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma} \text{sgn} \cdot [\varphi(X_{\sigma_1}, \ldots, X_{\sigma_p}), \psi(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)})]
+ \frac{-1}{p!(q-1)!} \sum_{\sigma} \text{sgn} \cdot \varphi([\varphi(X_{\sigma_1}, \ldots, X_{\sigma_p}), X_{\sigma(p+1)}], X_{\sigma(p+2)}, \ldots)
+ \frac{(-1)^{pq}}{(p-1)!q!} \sum_{\sigma} \text{sgn} \cdot \varphi([\varphi(X_{\sigma_1}, \ldots, X_{\sigma_q}), X_{\sigma(q+1)}], X_{\sigma(q+2)}, \ldots)
+ \frac{(-1)^{p-1}}{(p-1)!(q-1)!^2} \sum_{\sigma} \text{sgn} \cdot \varphi(\varphi([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3}, \ldots), X_{\sigma(p+2)}, \ldots)
+ \frac{(-1)^{p-1}q}{(p-1)!(q-1)!^2} \sum_{\sigma} \text{sgn} \cdot \varphi(\varphi([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3}, \ldots), X_{\sigma(q+2)}, \ldots)
\end{equation}
for any vector fields \(X_1, \ldots, X_{p+q}\) on \(M\), where sums are over all permutations \(\sigma : \{1, \ldots, p + q\} \to \{1, \ldots, p + q\}\) and \(\text{sgn} \sigma\) is the signum of \(\sigma\).

The underlying tangent valued \((p + q)\)-form of \([[\varphi, \psi]]\) is \([[\varphi, \psi]]\).

**Definition 2.4.** The bracket \([[–, –]]\) is called the Frolicher–Nijenhuis bracket.

**Proposition 2.5.** Let \(K\) be a double vector bundle with basis \(M\). Let \(\varphi : K \to \wedge^pT^*M \otimes TK\) be a double-linear (then projectable) semi-basic valued \(p\)-form on \(K\) and let \(\psi : K \to \wedge^qT^*M \otimes TK\) be a double-linear semi-basic tangent valued \(q\)-form on \(K\). Then the Frolicher–Nijenhuis bracket \([[\varphi, \psi]] : K \to \wedge^{p+q}T^*M \otimes TK\) is a double-linear semi-basic tangent valued \((p + q)\)-form on \(K\).

**Proof.** It follows from formula (2.2), Lemma 2.1 and Definition 1.4 \(\square\)

We end this section with the \(\text{DVB}\)-version of the well-known fact of the simplicity of vector fields.

**Lemma 2.6.** Let \(Z\) be a double linear vector field on a double vector bundle \(K\) such that the underlying vector field \(Z\) on basis \(M\) is nonzero at a point \(x_o \in M\). Then there exists a local \(\text{DVB}\)-coordinate system \((x^1, \ldots)\) on \(K\) with centrum \(x_o\) such that \(Z = \frac{\partial}{\partial x^1}\).

**Proof.** The proof is quite similar to that of the manifold case. We may assume that \(K = \mathbb{R}^{m_1, m_2, n_1, n_2}, x_0 = 0\) and \(Z|_0 = \frac{\partial}{\partial x^1}|_0\). Let \(\{\varphi_t\}\) be the flow of \(Z\). Then \(\Phi : K \to K\) given by \(\Phi(x^1, \ldots) = \varphi_{x_1}(0, x^2, \ldots)\) is a local \(\text{DVB}\)-isomorphism sending \(\frac{\partial}{\partial x^1}\) to \(Z\). \(\square\)

3. **On the Complete Lifting of Double-Linear Vector Fields to PPGB-Functors on Double Vector Bundles**

Let \(F : \text{DVB} \to \mathcal{M}\) be a ppgb-functor. We know that \(F : \text{DVB} \to \text{DVB}\). Let \(Z\) be a double-linear vector field on a double vector bundle \(K\).

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Definition 3.1. The complete lift of $Z$ to $F$ is the double-linear vector field $FZ$ on $FK$ corresponding to the flow $\{F\varphi_t\}$, where $\{\varphi_t\}$ is the flow of $Z$.

Lemma 3.2. If $\varphi : K \to K_1$ is a (locally defined) $DVB$-isomorphism, then $F(\varphi_*Z) = (F\varphi)_*FZ$.

Proof. The flow of $\varphi_*Z$ is $\{\varphi \circ \varphi_t \circ \varphi^{-1}\}$. Then the flow of $F(\varphi_*Z)$ is $\{F\varphi \circ F\varphi_t \circ (F\varphi)^{-1}\}$. The last flow is the one of $(F\varphi)_*FZ$. 

□

Lemma 3.3. If $\alpha$ is a real number, then $F(\alpha Z) = \alpha FZ$. Consequently, $F(\alpha Z + \alpha_1 Z_1) = \alpha FZ + \alpha_1 FZ_1$ for any real numbers $\alpha$ and $\alpha_1$ and any double linear vector fields $Z$ and $Z_1$ on $K$.

Proof. If $\{\varphi_t\}$ is the flow of $Z$, then $\{\varphi_{\alpha t}\}$ is the flow of $\alpha Z$. So, $\{F\varphi_{\alpha t}\}$ is the flow of $F(\alpha Z)$ and of $\alpha FZ$. Hence, $F$ is $R$-linear because of the homogeneous function theorem and the nonlinear Peetre theorem [7]. □

Let $\diamond^F : U^F \times V^F \to W^F$ be the $A^F$-bilinear map corresponding to $F$.

Lemma 3.4. Let $Z$ be a double linear vector field on a double vector bundle $K$ with basis $M$ and let $a \in A^F$. Then $af(a) \circ FZ$ is a double linear vector field on $FK$.

Proof. We may assume that the underlying vector field $Z$ is nowhere vanishing. Then using $DVB$-charts and Lemma 2.6 we may assume that $Z = \frac{\partial}{\partial x^1}$ and $K = R^{m_1,n_2,n_1,n_2}$. Then $FK = (A^F)^{m_1} \times (U^F)^{m_2} \times (V^F)^{n_1} \times (W^F)^{n_2}$ and $af(a) \circ FZ$ can be treated as a vector field on $(A^F)^{m_1}$ (and consequently as a double linear vector field on $FK$).

By Lemma 2.1 if $Z$ and $Z_1$ are double linear vector fields on $K$ then so is $[Z,Z_1]$.

Proposition 3.5. For any double linear vector fields $Z$ and $Z_1$ on $K$ and any $a,a_1 \in A^F$ we have

$$[af(a) \circ FZ, af(a_1) \circ FZ_1] = af(aa_1) \circ F([Z,Z_1]).$$

(3.1)

Proof. We may assume that $K = R^{m_1,n_2,n_1,n_2}$, $Z = \frac{\partial}{\partial x^1}$ and $Z_1 = f(x^1,\ldots,x^{m_1})Z_2$, where $Z_2 \in \{\frac{\partial}{\partial x^1},u^j\frac{\partial}{\partial u^1},v^k\frac{\partial}{\partial v^1},w^l\frac{\partial}{\partial w^1},v^j\frac{\partial}{\partial v^1}\}$.

If $Z_2 = \frac{\partial}{\partial v^1}$, then the formula is the well-know one for usual Weil functors on manifolds. For other values of $Z_2$, using formula 3.2 (below) and the known formula $aFZ(a_1Ff) = aa_1F(Z(f))$ for usual Weil functors on manifolds, we get $[af(a) \circ FZ, af(a_1) \circ F(fZ_2)] = [a \cdot FZ,a_1Ff \cdot FZ_2] = aFZ(a_1Ff) \cdot FZ_2 = aa_1F(Z(f)) \cdot FZ_2 = aa_1 \cdot F(Z(f)Z_2) = af(aa_1) \circ F([Z,Z_1])$. □

Lemma 3.6. Let $Z$ be a double linear vector field on $K$ and let $f : M \to R$ be a map. Then

$$F(f \circ \pi \cdot Z) = Ff \circ F\pi \cdot FZ,$$

(3.2)

where $\pi : K \to M$ is the projection (we treat $M$ as a $DVB$-object and $\pi$ as a $DVB$-map in the obvious way) and $Ff : FM \to FR = A^F$. Here (in the right of the formula) $a \cdot y := af(a)(y)$ for $a \in A^F$ and $y \in TFK$. 

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Proof. By Lemma 2.1, \( f \circ \pi \cdot Z \) is double linear. So, both sides of (3.2) make sense. By the linearity of \( \mathcal{F} \), we may assume that \( Z \) is not \( \pi \)-vertical. Then by Lemma 2.6, we may assume that \( K = \mathbb{R}^{m_1,m_2,n_1,n_2} \) and \( Z = \frac{\partial}{\partial z} \). Then we may additionally assume that \( K = M \) is a manifold, \( Z \) is a vector field on \( M \) and \( F \) is a Weil functor on manifolds. Then our lemma is the (well known for Weil functors on manifolds) formula \( \mathcal{F}(fZ) = Ff \cdot \mathcal{F}Z \). □

4. On the Complete Lifting of Double-linear Semi-Basic Tangent valued \( p \)-forms to ppgb-functors on Double Vector Bundles

For a moment, let \( F \) be a ppgb-functor (Weil functor) on manifolds. Let \( \omega \in \Omega_p(M) \) be a \( p \)-form on a manifold \( M \). Then \( \omega : TM \times_M \ldots \times_M TM \to \mathbb{R} \) is a fiber skew \( p \)-linear map. Applying \( F \), we get the fiber skew \( p \)-linear (over \( A^F \)) map \( F\omega : FTM \times_{FM} \ldots \times_{FM} FTM \to A^F \) (this is a well-known fact for Weil functors on manifolds). Then applying the exchange isomorphism \( \eta_M : TFM \to FTM \), which is a vector bundle isomorphism (this is also a well-known fact for Weil functors on manifolds), we obtain the \( A^F \)-valued \( p \)-form

\[
\mathcal{F} \omega := F\omega \circ (\eta_M \times \ldots \times \eta_M) : TFM \times_{FM} \ldots \times_{FM} TFM \to A^F
\]

over \( FM \).

Lemma 4.1. \( \mathcal{F} \omega \) is the unique \( A^F \)-valued \( p \)-form on \( FM \) such that

\[
\mathcal{F} \omega(af(a_1) \circ \mathcal{F} X_1, \ldots, af(a_p) \circ \mathcal{F} X_p) = a_1 \cdot \ldots \cdot a_p \cdot F(\omega(X_1, \ldots, X_p)) \tag{4.1}
\]

for any vector fields \( X_1, \ldots, X_p \) on \( M \) and any \( a_1, \ldots, a_p \in A^F \).

Proof. The uniqueness is a consequence of the well-known fact for Weil functors on manifolds that the vector fields \( af(a) \circ \mathcal{F} X \) generate over \( C^\infty(M) \) the vector space \( \mathcal{X}(FM) \). Formula (4.1) follows from the well-known (for Weil functors on manifolds) equalities \( \mathcal{F} X = \eta_M^{-1} \circ \mathcal{F} X \) and \( \eta_M \circ af(a) = a \cdot \eta_M \). □

Definition 4.2. The \( A^F \)-valued \( p \)-form on \( FM \) satisfying (4.1) is called the complete lift of \( \omega \) to \( F \).

For the rest of this section, let \( F : \mathcal{DVB} \to FM \) be a ppgb-functor.

Let \( x^1, \ldots, x^{m_1}, u^1, \ldots, u^{m_2}, v^1, \ldots, v^{n_1}, w^1, \ldots, w^{n_2} \) be the usual coordinates on \( \mathbb{R}^{m_1,m_2,n_1,n_2} \).

Because of the local expression (2.1) of double-linear vector fields and of the Definition 1.4 of double-linear semi-basic tangent valued \( p \)-forms, any double-linear semi-basic tangent valued \( p \)-form \( \varphi \) on \( \mathbb{R}^{m_1,m_2,n_1,n_2} \) is of the form

\[
\varphi = \sum_{i=1}^{m_1} \varphi^i \otimes_{\mathbb{R}} \frac{\partial}{\partial x^i} + \sum_{j,j_1=1}^{m_2} \psi^j_{j_1} \otimes_{\mathbb{R}} w^{j_1} \frac{\partial}{\partial w^j} + \sum_{k,k_1=1}^{n_1} \chi^k_{k_1} \otimes_{\mathbb{R}} u^{k_1} \frac{\partial}{\partial u^k} + \sum_{l,l_1=1}^{n_2} \xi^l_{l_1} \otimes_{\mathbb{R}} w^{l_1} \frac{\partial}{\partial w^l} + \sum_{j=1}^{m_2} \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} \rho_{j,k}^l \otimes_{\mathbb{R}} u^j v^k w^l \frac{\partial}{\partial w^l}.
\]
for unique p-forms \( \varphi^i, \psi^j, \chi^k, \xi^l, \rho^m \) on \( \mathbb{R}^m \), where \( (\omega \otimes_{\mathbb{R}} Z)(X_1, \ldots, X_p) := \omega(X_1, \ldots, X_p) \circ \pi \cdot Z \).

For any such \( \varphi \) we define its complete lift \( F\varphi \) by

\[
F\varphi := \sum_{i=1}^{m_1} F\varphi^i \otimes_{A^F} F \frac{\partial}{\partial x^i} + \sum_{j,j_1=1}^{m_2} F\psi^j_{j_1} \otimes_{A^F} F \left( w^{j_1} \frac{\partial}{\partial w^j} \right) \\
+ \sum_{k,k_1=1}^{n_1} F\chi^k_{k_1} \otimes_{A^F} F \left( \psi^{k_1} \frac{\partial}{\partial \psi^k} \right) + \sum_{l,l_1=1}^{n_2} F\xi^l_{l_1} \otimes_{A^F} F \left( \rho^{l_1} \frac{\partial}{\partial \rho^l} \right) \\
+ \sum_{j=1}^{m_2} \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} F\rho^l_{jk} \otimes_{A^F} F \left( w^l v^k \frac{\partial}{\partial w^l} \right) \quad (4.2)
\]

where \( (F\omega \otimes_{A^F} FZ)(Y_1, \ldots, Y_p) := F\omega(Y_1, \ldots, Y_p) \circ F\pi \cdot FZ \) for \( Y_1, \ldots, Y_p \in \mathcal{A}(FR_{m_1}) \).

**Proposition 4.3.** The complete lift \( F\varphi \) as in (4.2) is the unique double-linear semi-basic tangent valued p-form on \( FR_{m_1,m_2,n_1,n_2} \) such that

\[
F\varphi(af(a_1) \circ FX_1, \ldots, af(a_p) \circ FX_p) = af(a_1 \cdot \ldots \cdot a_p) \circ F(\varphi(X_1, \ldots, X_p)) \quad (4.3)
\]

for any \( a_1, \ldots, a_p \in A^F \) and any \( X_1, \ldots, X_p \in \mathcal{A}(\mathbb{R}^{m_1}) \).

**Proof.** The uniqueness is clear because the vector fields \( af(a) \circ FX \) for \( a \in A^F \) and \( X \in \mathcal{A}(\mathbb{R}^{m_1}) \) generate (over \( C^\infty(\mathbb{R}^{m_1}) \)) the vector space \( \mathcal{A}(FR_{m_1}) \). This is a well-known fact for Weil functors on manifolds.

Now, we prove (4.3). Since both sides of (4.3) are linear in \( \varphi \), we may assume that \( \varphi = \omega \otimes_{\mathbb{R}} Z \), where \( \omega \in \Omega^p(\mathbb{R}^{m_1}) \) and \( Z \in \left\{ \frac{\partial}{\partial x^i}, w^1 \frac{\partial}{\partial w^1}, v^k \frac{\partial}{\partial v^k}, w^l \frac{\partial}{\partial w^l}, u^j v^k \frac{\partial}{\partial w^l} \right\} \).

Then by (4.2), (4.1) and (3.2), we have

\[
F\varphi(af(a_1) \circ FX_1, \ldots, af(a_p) \circ FX_p) \\
= F(\omega \otimes_{\mathbb{R}} Z)(af(a_1) \circ FX_1, \ldots, af(a_p) \circ FX_p) \\
= (F\omega \otimes_{A^F} FZ)(af(a_1) \circ FX_1, \ldots, af(a_p) \circ FX_p) \\
= F\omega(af(a_1) \circ FX_1, \ldots, af(a_p) \circ FX_p) \circ F\pi \cdot FZ \\
= a_1 \cdot \ldots \cdot a_p \cdot F(\omega(X_1, \ldots, X_p)) \circ F\pi \cdot FZ \\
= a_1 \cdot \ldots \cdot a_p \cdot F(\varphi(X_1, \ldots, X_p)) \\
= af(a_1 \cdot \ldots \cdot a_p) \circ F(\varphi(X_1, \ldots, X_p)). \quad \square
\]

**Lemma 4.4.** For any (local) double vector bundle isomorphism \( f : \mathbb{R}^{m_1,m_2,n_1,n_2} \rightarrow \mathbb{R}^{m_1,m_2,n_1,n_2} \) and any double-linear semi-basic tangent valued p-form \( \varphi \) on the double vector bundle \( \mathbb{R}^{m_1,m_2,n_1,n_2} \), we have \( (Ff)_* F\varphi = F(f_* \varphi) \).

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Proof. We have
\[(Ff)\mathcal{F}\varphi(af(a_1) \circ FX_1, \ldots, af(a_p)FX_p)\]
\[= \mathcal{F}\varphi(Ff_s^{-1}(af(a_1) \circ FX_1), \ldots, Ff_s^{-1}(af(a_p) \circ FX_p))\]
\[= \mathcal{F}\varphi(af(a_1) \circ (f_s^{-1}X_1), \ldots, af(a_p) \circ (f_s^{-1}X_p))\]
\[= af(a_1 \cdots a_p) \circ \mathcal{F}(f_s^{-1}X_1, \ldots, f_s^{-1}X_p)\]
\[= af(a_1 \cdots a_p) \circ \mathcal{F}(\varphi(f_s^{-1}X_1, \ldots, f_s^{-1}X_p))\]
\[= af(a_1 \cdots a_p) \circ \mathcal{F}((f_s\varphi)(X_1, \ldots, X_p))\]
\[= \mathcal{F}(f_s\varphi)(af(a_1) \cdot FX_1, \ldots, af(a_p) \cdot FX_p).\]

Now, applying the uniqueness case of Proposition 4.3 (or, better, the sentence of the proof of the uniqueness case of Proposition 4.3) we end the proof. \(\Box\)

We are now in a position to prove the following result.

**Theorem 4.5.** Let \(F\) be a ppgb-functor on \(DVB\). Let \(\varphi : K \to \wedge^pT^*M \otimes TK\) be a double-linear semi-basic tangent valued \(p\)-form on a double vector bundle \(K\) with basis \(M\). Then there exists one and only one double-linear semi-basic tangent valued \(p\)-form \(\mathcal{F}\varphi : FK \to \wedge^pT^*FM \otimes TFK\) on \(FK\) such that
\[\mathcal{F}\varphi(af(a_1) \circ FX_1, \ldots, af(a_p) \circ FX_p) = af(a_1 \cdots a_p) \circ \mathcal{F}(\varphi(X_1, \ldots, X_p))\] (4.4)
for any vector fields \(X_1, \ldots, X_p\) on \(M\) and any \(a_1, \ldots, a_p \in AF\).

**Proof.** Using \(DVB\)-charts on \(K\), we spread the complete lifting of double-linear semi-basic tangent valued \(p\)-forms on \(\mathbb{R}^{m_1,m_2,m_1,m_2}\) to the one on \(K\). This is possible because of Lemma 4.4 \(\Box\)

5. THE COMPLETE LIFTING OF DOUBLE-LINEAR SEMI-BASIC TANGENT VALUED \(p\)-FORMS PRESERVES THE FROLICHER–NIJENHUIS BRACKET

Let \(F\) be a ppgb-functor on \(DVB\). Then \(F : DVB \to DVB\).

Let \(\varphi : K \to \wedge^pT^*M \otimes TK\) be a double-linear semi-basic tangent valued \(p\)-form on \(K\) and let \(\psi : K \to \wedge^qT^*M \otimes TK\) be a double-linear semi-basic tangent valued \(q\)-form on \(K\). We can lift \(\varphi\) and \(\psi\) to \(FK\) and obtain a double-linear semi-basic tangent valued \(p\)-form \(\mathcal{F}\varphi\) on \(FK\) and a double-linear semi-basic tangent valued \(q\)-form \(\mathcal{F}\psi\) on \(FK\). Then we can produce the Frolicher–Nijenhuis bracket \([\mathcal{F}\varphi, \mathcal{F}\psi]\). By Proposition 2.5 this bracket is a double-linear semi-basic tangent valued \((p + q)\)-form on \(FK\).

On the other hand, by Proposition 2.5 the Frolicher–Nijenhuis bracket \([\varphi, \psi]\) is a double-linear semi-basic tangent valued \((p + q)\)-form on \(K\). So, we can lift it and obtain a double-linear semi-basic tangent valued \((p + q)\)-form \(\mathcal{F}([[\varphi, \psi]])\) on \(FK\).

**Theorem 5.1.** We have
\[\mathcal{F}([[\varphi, \psi]]) = [[\mathcal{F}\varphi, \mathcal{F}\psi]].\] (5.1)
Proof. For any \(a_1, \ldots, a_{p+1} \in A^F\) and vector fields \(X_1, \ldots, X_{p+q}\) on \(M\) we have

\[
[\mathcal{F}\varphi(af(a_1) \circ \mathcal{F}X_1, \ldots, af(a_p) \circ \mathcal{F}X_p),
\mathcal{F}\psi(af(a_{p+1}) \circ \mathcal{F}X_{p+1}, \ldots, af(a_{p+q}) \circ \mathcal{F}X_{p+q})] = af(a_1 \cdot \ldots \cdot a_{p+q}) \circ \mathcal{F}([\varphi(X_1, \ldots, X_p), \psi(X_{p+1}, \ldots, X_{p+q})]).
\]

Indeed, applying formulas (4.4) and (3.1) we easily get

\[
[\mathcal{F}\varphi(af(a_1) \circ \mathcal{F}X_1, \ldots, af(a_p) \circ \mathcal{F}X_p),
\mathcal{F}\psi(af(a_{p+1}) \circ \mathcal{F}X_{p+1}, \ldots, af(a_{p+q}) \circ \mathcal{F}X_{p+q})] = [af(a_1 \cdot \ldots \cdot a_{p+q}) \circ \mathcal{F}(\varphi(X_1, \ldots, X_p), \psi(X_{p+1}, \ldots, X_{p+q}))].
\]

Similarly, we have

\[
\mathcal{F}\psi(af(a_1) \circ \mathcal{F}X_1, \ldots, af(a_p) \circ \mathcal{F}X_p),
af(a_{p+2}) \circ \mathcal{F}X_{p+2}, \ldots, af(a_{p+q}) \circ \mathcal{F}X_{p+q}) = af(a_1 \cdot \ldots \cdot a_{p+q}) \circ \mathcal{F}(\psi([\varphi(X_1, \ldots, X_p), X_{p+1}], X_{p+2}, \ldots, X_{p+q})).
\]

and

\[
\mathcal{F}\psi(\mathcal{F}\varphi([af(a_1) \circ \mathcal{F}X_1, af(a_2) \circ \mathcal{F}X_2], af(a_3) \circ \mathcal{F}X_3, \ldots, af(a_{p+1}) \circ \mathcal{F}X_{p+1}),
af(a_{p+2}) \circ \mathcal{F}X_{p+2}, \ldots, af(a_{p+q}) \circ \mathcal{F}X_{p+q}) = af(a_1 \cdot \ldots \cdot a_{p+q}) \circ \mathcal{F}(\psi([\varphi([X_1, X_2], X_3, \ldots, X_{p+1}], X_{p+2}, \ldots, X_{p+q}))).
\]

and the same formulas with \(\varphi\) replaced by \(\psi\) and vice versa, and the same formulas with indices \(1,\ldots,p+q\) replaced by \(\sigma(1),\ldots,\sigma(p+q)\). Now, using the above formulas and formula (4.4) for \([\varphi, \psi]\) instead of \(\varphi\) and formula (2.2) on the Frolicher–Nijenhuis bracket \([[\varphi, \psi]]\) and formula (2.2) with \(\varphi\) and \(\psi\) replaced by \(\mathcal{F}\varphi\) and \(\mathcal{F}\psi\), and the \(\mathbb{R}\)-linearity of the complete lifting of vector fields (Lemma 3.3),
we get
\[
\mathcal{F}([[\varphi, \psi]])(af(a_1) \circ FX_1, \ldots, af(a_{p+q}) \circ FX_{p+q})
= af(a) \circ \mathcal{F}([[\varphi, \psi]](X_1, \ldots, X_{p+q}))
= \frac{1}{p!q!} \sum_{\sigma} \sgn \sigma \cdot af(a) \circ \mathcal{F}([[\varphi(X_{\sigma_1}, \ldots, X_{\sigma_p}), \psi(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)})]])
\]
\[
+ \frac{(-1)^{pq}}{(p-1)!q!} \sum_{\sigma} \sgn \sigma \cdot af(a) \circ \mathcal{F}([[\varphi(X_{\sigma_1}, \ldots, X_{\sigma_p}), X_{\sigma(p+1)}, X_{\sigma(p+2)}, \ldots]])
\]
\[
+ \frac{(-1)^{p-1}}{(p-1)!(q-1)!!} \sum_{\sigma} \sgn \sigma \cdot af(a) \circ \mathcal{F}([[\varphi([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3}, \ldots, X_{\sigma(p+2)}, \ldots]])
\]
\[
+ \frac{(-1)^{q-1}}{(p-1)!(q-1)!!} \sum_{\sigma} \sgn \sigma \cdot af(a) \circ \mathcal{F}([[\varphi([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3}, \ldots, X_{\sigma(q+2)}, \ldots]])
\]
\[
= \frac{1}{p!q!} \sum_{\sigma} \sgn \sigma \cdot [\mathcal{F} \varphi(af(a_1) \circ FX_1, \ldots), \mathcal{F} \psi(af(a_{p+1}) \circ FX_{p+1}), \ldots]]
\]
\[
+ \frac{(-1)^{pq}}{(p-1)!q!} \sum_{\sigma} \sgn \sigma \cdot \mathcal{F} \psi([[\mathcal{F} \varphi(af(a_1) \circ FX_1, \ldots), af(a_{p+1}) \circ FX_{p+1}], \ldots]]
\]
\[
+ \frac{(-1)^{p-1}}{(p-1)!(q-1)!!} \sum_{\sigma} \sgn \sigma \cdot \mathcal{F} \varphi([[\mathcal{F} \psi(af(a_1) \circ FX_1, \ldots), af(a_{p+1}) \circ FX_{p+1}], \ldots]]
\]
\[
+ \frac{(-1)^{q-1}}{(p-1)!(q-1)!!} \sum_{\sigma} \sgn \sigma \cdot \mathcal{F} \varphi([[\mathcal{F} \psi(af(a_1) \circ FX_1, \ldots), af(a_{p+1}) \circ FX_{p+1}], \ldots]]
\]
\[
= [[\mathcal{F} \varphi, \mathcal{F} \psi]](af(a_1) \circ FX_1, \ldots, af(a_{p+q}) \circ FX_{p+q}),
\]
for any vector fields \(X_1, \ldots, X_{p+q}\) on \(M\) and any \(a_1, \ldots, a_{p+q} \in A^F\), where \(a := a_1 \cdot \ldots \cdot a_{p+q}\). Then, since the vector fields \(af(a) \circ FX\) generate (over \(C^\infty(FM)\)) the space \(\mathcal{X}(FM)\), formula (5.1) holds. \(\square\)

6. An Application to Double-linear General Connections

Let \(F\) be a ppgb-functor on \(DVB\).

In Definition 1.5, we introduced the concept of double-linear connections \(\Gamma\) in a double vector bundle \(K\).

**Lemma 6.1.** Given a double linear connection \(\Gamma\) in \(K\), its complete lift \(\mathcal{F} \Gamma\) is a double-linear connection in \(FK\).

**Proof.** Since \(\Gamma(X)\) is a double-linear vector field on \(K\) with the underlying vector field equal to \(X\), we have that \(\mathcal{F} \Gamma(af(a) \circ FX) = af(a) \cdot \mathcal{F}(\Gamma(X))\) is a double-linear vector field with the underlying vector field equal to \(af(a) \circ FX\). Consequently, for any vector field \(Y \in \mathcal{X}(FM)\), \(\mathcal{F} \Gamma(Y)\) is a double linear vector field with the underlying vector field equal to \(Y\). \(\square\)
Definition 6.2. A curvature of a double linear connection $\Gamma$ in a double vector bundle $K$ is $R_\Gamma := \frac{1}{2}[[\Gamma, \Gamma]] : K \to \wedge^2 T^*M \otimes VK$ (i.e., $R_\Gamma(X, Y) = [\Gamma(X), \Gamma(Y)] - \Gamma([X, Y])$).

Theorem 6.3. We have

$$R_{\mathcal{F}} = \mathcal{F}(R_\Gamma).$$

Proof. It is clear because of $\mathcal{F}([[\Gamma, \Gamma]]) = [[\mathcal{F}\Gamma, \mathcal{F}\Gamma]]$. □

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Włodzimierz M. Mikulski
Faculty of Mathematics and Computer Science UJ, ul. Łojasiewicza 6, 30-348, Cracow, Poland
Wlodzimirz.Mikulski@im.uj.edu.pl

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