Replacing projection on finitely generated convex cones with projection on bounded polytopes

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October 26, 2020

Abstract

This paper is devoted to the general problem of projection onto a polyhedral convex cone generated by a finite set of generators. This problem is reformulated into projection onto the polytope obtained by simple truncation of the original cone. Then it can be solved with just two closely related projections onto the same bounded polytope. This approach’s computational performance is conditioned by the crucial tool’s efficiency for solving the fundamental problem of finding the least norm element in a convex hull of a given finite set of points. In our numerical experiments, we used for this purpose the specialized finite algorithm implemented in the open-source system for matrix-vector calculations OCTAVE. This algorithm is practically indifferent to the proportions between the number of generators and their dimensionality and significantly outperformed a general-purpose quadratic programming algorithm of the active-set variety built into OCTAVE when the number of points exceeds the dimensionality of the original problem.

Keywords: Orthogonal projection; convex cones; truncation

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Introduction

This paper advocates a simple finite algorithm for solving orthogonal projection problems onto closed convex cones formed by finite numbers of generators. The numerous applications of this problem in restricted statistical inferences [1], machine learning, digital signal processing, linear optimization [8] and other problems keep this problem as one of the most popular research subjects. For instance, it ranks 5th in popularity in Wiki minimization internet resource [13] with more than 480000 visits at the time of writing this note.

This close attention inspired the development of many effective algorithms, especially for isotone projection cones [2]. For simplicial or lattice cones (see f.i. [3] and the references therein) when the number of cone generators is less than the dimensionality of underlying space, the results are less spectacular, even when there are quite encouraging experiments with different heuristics [5].

Here we consider the general case with the arbitrary number of generators which attracted less attention possibly because of combinatorics, involved in the selection of active face on which the solution of the projection problem lies. We are interested in a finite projection algorithm, even if there are also iterative schemes for isotone cones [4], which might be indispensable for huge-dimensional problems.

Great attention is given in the literature to projection onto truncated convex cones as local approximations of polyhedrons. These truncated cones are then replaced by non-truncated simplicial cones to simplify the following projection operations. The simple idea of the current submission is just the opposite. Instead, we suggest to project onto the finitely generated cone by truncating it to the bounded polytope and use the well-established algorithm [11] based on the article [6] with many computational improvements. To justify this approach, we show the simple way to truncate a cone in such a manner that ensures the equivalence of truncated and non-truncated problems. Then we present the polytope projection algorithm and compare its performance with the general-purpose quadratic programming code. Finally, the polytope projection algorithm is applied to solve polyhedral cone projection problems known from the literature. Apart from demonstrating the algorithm performance, it gave a chance to compare it with the popular iterative FISTA algorithm.
1 Notations and preliminaries

Our basic space is a finite-dimensional Euclidean space, denoted as $E$. If necessary, the dimensionality of $E$ is derived as $\text{dim}(E)$. The non-negative orthant of $E$ is denoted as $E^+$. The null vector of $E$ is denoted as $0$ to avoid confusion with the number 0. We also use the notation $1 = (1, 1, \ldots, 1)$ to simplify certain expressions. The real axis is denoted by $\mathbb{R}$. The non-negative part of $\mathbb{R}$ is denoted by $\mathbb{R}^+$.

For $\alpha$ in $\mathbb{R}$ and $A \subset E$ we use the shorthand $\alpha A$ for $\{\alpha a, a \in A\}$. Also we use $A \times B$ as notation for $\{(a, b) \mid a \in A, b \in B\}$. Single-pont set $\{a\}$ with $a \in E$ is denoted simply as $a$ when it does not lead to misunderstanding.

Convexity of subsets of $E$ is defined in a standard way. The convex hull of set $X$ denoted as $\text{co}\{X\}$ or $X^c$ and the conical hull of $X$ is denoted as $\text{Co}\{X\}$. When set $X$ has the certain structure, namely $X = X_1 \cup X_2 \cup \ldots \cup X_k$ we write $\text{co}\{X\} = \text{co}\{X_1, X_2, \ldots, X_k\}$. We also denote as

$$\text{lin}\{X\} = \{z = \sum_{i=1}^{k} \lambda_i x^i, \lambda_i \in \mathbb{R}, x^i \in X, \ i = 1, 2, \ldots, k; \ k = 1, 2, \ldots\}$$

the linear envelope of a set $X$. Of course, due to Karateodory $k$ may be limited to $\text{dim}(E)$, if necessary. The inequality $x \geq y$ with $x, y \in E$ is understood componentwise.

We denote the standard inner product in $E$ as $xy$, the Euclidean norm is denoted $\|x\|^2 = xx$ as usual. The unit ball in $E$ is denoted as $U_E = \{x : \|x\| \leq 1\}$ or $U$ if the space is clear from the context. We also make some use of Chebyshev metric $\|x\|_\infty = \max_{i=1,2,\ldots,\text{dim}(E)} |x_i|$ and the correspondent unit ball $U_\infty(E) = \{x : \|x\|_\infty \leq 1\}$ or $U_\infty$ if it clear enough. The non-negative part of Chebyshev unit cube is denoted by $U_\infty^+ = \{x : \|x\|_\infty \leq 1, x \geq 0\}$.

The (orthogonal) projection problem in $E$ for a point $b$ and a closed convex subset $X$ of $E$ is defined as

$$\min_{x \in X} \|x - b\|^2 = \|\Pi_X(b) - b\|^2. \quad (1)$$

It defines the single-valued projection operator $\Pi_X : E \to X$ which is well-defined and has many useful properties such as Lipschitz continuity with unit Lipschitz constant, non-expansion, and others. Because of this, it is widely used in many computational algorithms and theoretical considerations in convex analysis and beyond.
Our primal interest in this paper is the projection problem \( \Pi \) when the set \( X \) is a closed convex cone, so we introduce some additional relevant definitions.

A closed convex cone \( K \) is a closed subset of \( E \), such that \( K = K + K \) and \( \alpha K = K \) for any \( \alpha > 0 \). We assume that further on, all cones are convex and closed.

For \( A \subset E \) a conical hull of \( A \) is defined and denoted as \( \text{Co}\{A\} = \bigcup_{\lambda \geq 0} \lambda A \). If \( K(A) = \text{Co}\{A\} \), \( A = \{a^i, i = 1,2,\ldots,m\} \) then \( K(A) \) is called finitely generated and \( a^i, i = 1,2,\ldots,m \) are called generators. We call \( K(A) \) a pointed cone, if \( 0 \notin \text{co}\{A\} \).

Considering the set \( A \subset E \) as an ordered collection of generators we can think of it as synonym for linear operator (matrix) \( A : E \to E' \) which operates on \( E \) in the following way:

\[
Ax = y, \quad y = (y_1, y_2, \ldots, y_m), \quad y_i = a^i x, \quad i = 1,2,\ldots,m, m = \dim(E').
\]

In this notation a cone \( K(A) \) can also be defined as

\[
K(A) = AE'_+. \tag{2}
\]

Notice that we can also apply an arbitrary scaling \( K(A) = ADE'_+ \) with a diagonal matrix \( D \) with positive entries.

The projection problem

\[
\min_{z \in K(A)} \|b - z\|^2 = \|b - \Pi_K(b)\|^2 = \|b - b^K\|^2 \tag{3}
\]

with solution \( b^K = \Pi_K(b) \) for finitely generated cones with explicitly given generators \( A = \{a^i, i = 1,2,\ldots,m\} \) can be written as the semidefinite quadratic programming (QP) problem

\[
\min \|b - z\|^2 = \min \|b - z\|^2 \quad z - \sum_{i=1}^m u_i a^i = 0, \quad z - Au = 0, \quad u \geq 0 \tag{4}
\]

in \( n + m \) variables \( z_u \) and \( n + m \) linear constrains, where \( n = \dim(E) \). General-purpose QP solvers commonly require positive definiteness of objective and do not use this QP problem’s specific form, which leads to their disadvantage compared with specialized methods. The problem \( \Pi \) can be
somewhat simplified by getting rid of the variable $z$. It leads to the equivalent problem

$$\min_{u \geq 0} \{ uHu - 2bAu \}$$

(5)

in baricentric coordinates $u$ only with $H = A^T A$, where $A^T$ is the transpose of $A$.

Each of these forms has its advantages-disadvantages: (4) is semidefinite in the space of $x,u$ variables, but preserve the sparseness of data, (5) has a smaller number of variables, but the matrix $H$ is often dense and can also be semidefinite if $m > n$.

2 Scaled truncation

To transform a projection problem for polyhedral cone into the equivalent projection problem for the bounded polytope we have to present a suitable way to transform the cone $K(A)$ into the polytope $Q(A)$ in a way that makes (1) and (3) equivalent, that is

$$\min_{x \in K(A)} \|x - b\| = \|b^K - b\| = \min_{x \in Q(A)} \|x - b\| = \|b^Q - b\|, \quad b^K = b^Q.$$ (7)

The simple way to do this is to use the definition (2) and truncate the cone $K(A)$ in the following way:

$$Q(A) = P_{\rho}(A) = \{ z = Au, \; u \in \rho U^+_{\infty}(E') \} = \rho AU^+_{\infty}(E') = \rho P(A)$$

(6)

where $\rho > 0$ is a scaling parameter and $P(A) = \{ z = Au, \; u \in U^+_{\infty}(E') \} = \text{co}\{0,A\}$.

For what follows we introduce the following notations:

- $b^K$ is the solution of the projection problem

$$\min_{x \in K(A)} \|x - b\| = \|b^K - b\| = \alpha$$

(7)

- $b^\rho$ is the solution of the projection problem

$$\min_{x \in \rho A^c} \|x - b\| = \|b^\rho - b\| = \beta$$

(8)
• $b^\circ$ is the solution of the projection problem
\[
\min_{x \in \text{co}\{0, \rho A_c\}} \|x - b\| = \|b^\circ - b\| = \gamma
\]  
(9)

• $a^\circ$ is the solution of the least-norm problem for the set $A_c$
\[
\min_{x \in A_c} \|x\| = \|a^\circ\|.  
\]  
(10)

Of course $\min_{x \in \rho A_c} \|x\| = \rho \|a^\circ\|$.

We assume $\alpha > 0$, $\gamma > 0$ to avoid triviality.

Further on we define the affine function
\[
\Lambda(x) = (x - b^\rho)(b^\rho - b).
\]  
(11)

Notice that $\Lambda(b^\rho) = 0$ and $\Lambda(b) = -\beta^2 < 0$ by construction and $\Lambda(x) \geq 0$ for $x \in \rho A_c$ by optimality conditions in (8).

The following theorem establishes the simple condition for equivalence between truncated and untruncated projection problems.

**Theorem 2.1.** If $\rho > 0$ is such that $\|b^\rho\|^2 > bb^\rho$ then $b^K = b^\circ$.

**Proof.** The idea of the following proof is illustrated on the Fig. 1. By definition $\alpha \leq \beta$. If $\alpha = \beta$ the theorem holds trivially, therefore we assume the strict inequality $\alpha < \beta$. Then $b^K \in b + \alpha U \subset b + \beta U$ where the inclusion is strict. Estimate
\[
\Lambda(b^K) \leq \max_{x \in b + \alpha U} \Lambda(x) = \Lambda(b) + \max_{z \in \alpha U} (b^\rho - b)z = \Lambda(b) + \alpha \|b^\rho - b\| = -\beta^2 + \alpha \beta < 0
\]

It implies $b^K \in K(A) \cap \{x : \Lambda(x) \leq 0\} \subset \text{co}\{0, \rho A_c\}$ and by uniqueness argument $b^K = b^\circ$. 

Apart from its simplicity, the assumption of this theorem is easy to satisfy. It is sufficient to solve the auxiliary least-norm problem $\min_{x \in A_c} \|x\| = \|a^\circ\| = d_A$ and notice that $\|b^\rho\| \geq \rho d_A$ for any $\rho > 0$. By choosing $\rho > \rho_{\text{min}} = \|b\|/d_A$ it can be obtained that $\rho d_A > \|b\|$ and therefore $\|b^\rho\| > \|b\|$ and hence
\[
\|b^\rho\|^2 > \|b\| \|b^\rho\| \geq bb^\rho
\]
Figure 1: The geometry of the proof. The shaded area is the truncated cone $\text{co}\{0, \rho A_c\}$
Figure 2: The geometry of the proof for the condition $\rho > \rho_{\min}$. The shaded area is the set $\rho A_c$. 
as theorem 2.1 assumes.

It can be also shown directly that the condition $\rho > \rho_{\text{min}}$, which is more stringent than the condition of the theorem 2.1, is sufficient to guarantee that $b^K = b^{co}$. The geometry of this demonstration is shown on Fig. 2. The formal proof goes as following: if $\rho > \rho_{\text{min}}$ then

$$\|x\|^2 \geq \rho^2\|a^{co}\|^2 > \|b\|^2 = \|b^K\|^2 + \|b^K - b\|^2 \geq \|b^K\|^2$$

for any $x \in \rho A_c$ by optimality condition of $b^K$.

Hence $b^K \notin \rho A_c$ and moreover

$$\|b^K\| < \theta\|\rho a^{co}\| \leq \theta\|x\|$$

for $\theta \geq 1$ and any $x \in \rho A_c$.

Therefore $b^K \notin \{Au, u \geq 1\}$ or $b^K \in \text{co}\{0, \rho A_c\}$. Then

$$\|b^K - b\| = \min_{x \in K(A)} \|x - b\| \leq \min_{x \in \text{co}\{0, \rho A_c\}} \|x - b\| \leq \|b^K - b\|$$

and $b^K = b^{co}$ which terminates this demonstration.

The following Algorithm is based on this truncation. Using the estimate $\rho > \rho_{\text{min}} = \|b\|/\|b^{min}\|$ we can solve cone projection problem (3) by solving 2 closely related projection problems specified in Algorithm 1.

Notice that (12) and (13) are both projections on polytopes which differ by one vertex only so we can use advanced feasible basis in one problem to start the solution process for the other which make this approach even more attractive.

### 3 Numerical experiments

This section describes our experience with the suggested approach to solve projection problems for polyhedral cones by reducing them to polytope projections. For this approach to be useful in a practical sense, we have to be first of all sure that the polytope projection problems are solved efficiently for polytopes of different proportions. We use in our numerical experiments the in-house OCTAVE [12] implementation of the algorithm [9] which we consider now sufficiently reliable and efficient. We substantiate this claim by computational experiments which compare the performance of this routine with of-the-shelf OCTAVE general-purpose quadratic optimization function QP of active-set variety.
Data: The set \( A = \{a^i, i = 1, 2, \ldots, m\} \) of generators of a cone \( K(A) \), the vector \( b \) to be projected on the cone \( K(A) \).

Result: The solution vector \( b^K \) of the problem (3).

Phase 1. Compute a suitable value for the scaling parameter \( \rho \) by solving the auxiliary polytope projection problem

\[
\min_{z \in A_c} \| z \|^2 = \| b^{\text{min}} \|^2,
\]

and setting an appropriate lower estimate for the scaling parameter as \( \rho_{\text{min}} = \| b \| / \| b^{\text{min}} \| \).

Phase 2. Choose any \( \rho \geq \rho_{\text{min}} \) and solve the projection problem

\[
\min_{z \in \text{co}\{0, \rho A_c\}} \| z - b \|^2 = \| b^\rho - b \|^2 = \| b^K - b \|^2,
\]

according to Lemma 2.1.

Algorithm 1: The Cone Truncated to Polytope (CTP) algorithm

These experiments are conducted with the test problems, inspired by P.Wolfe’s “pancake” random sets [14]. These sets are determined by two scalar parameters \( 0 < \delta < \Delta \) and consist of, say, \( m \) random points uniformly distributed in a \( n \)-dimensional box-like set \( B_{\delta, \Delta} \) which is a composition of \((n-1)\)-dimensional cube, scaled by \( \Delta \Delta[-1, +1]^{n-1} \) and the interval \([-1, +1]\) scaled by \( \delta \) and shifted up by \( 2\delta \).

The simple expression, where \( U^k_\infty \) is the \( k \)-dimensional Chebyshev unit ball

\[
B_{\delta, \Delta} = \Delta U^{n-1}_\infty \times \delta(U^1_\infty + 2) \subset E^n
\]

can formally describe this set and Fig. 3 demonstrates the geometry of this data-set for 3-dimensional case. The least-norm problem for such sets can be made computationally rather difficult either by choosing extreme values of \( \delta, \Delta \) and/or by populating \( B_{\delta, \Delta} \) with a large number of points. If \( m \gg n \) and \( n \) is sufficiently large the random points tend to distribute along the sides of \( B_{\delta, \Delta} \) which makes the least-norm problem computationally ill-conditioned for \( \delta \ll \Delta \).

The following experiments, described in subsection 3.2 were conducted with the polyhedral cones, generated by the sets of generators, produced by the same expression [14]. The special attention in these massive experiments
was given to the peculiarities in solving projection problems \[12, 13\] for different total sizes of the data-sets. Here again, the polytope projection routine demonstrated its reliability and effectiveness.

3.1 Specialized versus General-Purpose

The tables \[1, 2\] demonstrate the results of tests with polytopes of different proportions between numbers of points in the bundles and the dimensionality of the underlying space. The common notations in these tables: \texttt{ptp} — the solution time (sec) for the PTP routine, \texttt{qp} — the same for the QP function, \texttt{obj-\texttt{ptp}}, \texttt{obj-\texttt{qp}} — objective values for PTP and QP respectively, \texttt{PTP/QP} — the time-ratio for these algorithms. First of all it can be concluded from these experiments that PTP routine is more robust than QP solver. Solution time for PTP grows approximately linear in these cases, when for QP solver it radically increases when the number of points becomes larger than their dimension. Highly likely that it is the result of possible semi-definiteness of the quadratic form in (5). The PTP routine is slower on simple problems when \( m < n \), but even in these cases, its performance is acceptable, and solution time is still low.

It can be concluded from these experiments that PTP might be a function of choice for the cases when the number of points in a dataset is either greater than their dimensionality or varies on a large scale.
Table 1: The test with the fixed number of points $m = 600$ in the bundle, $n$ — dimensionality of the space.

| $n$ | ptp | qp | obj-pto | obj-qp | ptp/qp |
|-----|-----|----|---------|--------|--------|
| 100 | 1.03| 408.58 | $6.18602490e-03$ | $6.18602490e-03$ | 0.003 |
| 200 | 2.42| 420.37 | $7.42018810e-03$ | $7.42018810e-03$ | 0.006 |
| 300 | 4.86| 410.15 | $8.75037778e-03$ | $8.75037778e-03$ | 0.012 |
| 400 | 8.11| 267.24 | $9.51469830e-03$ | $9.51469830e-03$ | 0.030 |
| 500 | 9.42| 163.81 | $1.03756037e-02$ | $1.03756037e-02$ | 0.058 |
| 600 | 5.93| 2.78 | $1.09087458e-02$ | $1.09087458e-02$ | 2.136 |
| 700 | 5.79| 0.55 | $1.13905009e-02$ | $1.13905009e-02$ | 10.585 |
| 800 | 5.94| 0.55 | $1.15418384e-02$ | $1.15418384e-02$ | 10.811 |
| 900 | 6.10| 0.56 | $1.11699504e-02$ | $1.11699504e-02$ | 10.840 |
| 1000| 6.21| 0.56 | $1.12752146e-02$ | $1.12752146e-02$ | 11.057 |

Table 2: Test with the fixed dimensionality $n = 600$ of points in the bundle, $m$ — the number of points.

| $m$ | ptp | qp | obj-pto | obj-qp | ptp/qp |
|-----|-----|----|---------|--------|--------|
| 100 | 0.29| 0.17 | $1.10150924e-02$ | $1.10150924e-02$ | 1.699 |
| 200 | 0.68| 0.06 | $1.09582666e-02$ | $1.09582666e-02$ | 12.369 |
| 300 | 0.98| 0.13 | $1.13598866e-02$ | $1.13598866e-02$ | 7.553 |
| 400 | 1.92| 0.23 | $1.10932105e-02$ | $1.10932105e-02$ | 8.525 |
| 500 | 3.32| 0.37 | $1.13554032e-02$ | $1.13554032e-02$ | 8.889 |
| 600 | 6.04| 3.10 | $1.09087458e-02$ | $1.09087458e-02$ | 1.946 |
| 700 | 12.43| 295.61 | $1.02334609e-02$ | $1.02334609e-02$ | 0.042 |
| 800 | 19.23| 657.93 | $9.91185272e-03$ | $9.91185272e-03$ | 0.029 |
| 900 | 25.13| 1394.20 | $9.36404467e-03$ | $9.36404467e-03$ | 0.018 |
| 1000| 32.36| 2050.40 | $9.03549357e-03$ | $9.03549357e-03$ | 0.016 |
3.2 Numerical experiments with the cones

We performed three series of computational experiments to demonstrate peculiarities of computational efficiency of the CTP algorithm. Each series consisted in several experiments with conic projection problems with dimensionality \( m \) and number of vectors \( n \) such that the size \( m \cdot n \) of the matrix \( A \) is approximately the same, up to the integrality of \( m \) and \( n \). These series were determined by three values of the product \( m \cdot n = 10^6, 2 \cdot 10^6 \) and \( 3 \cdot 10^6 \) and in each of these series the dimensionality \( m \) was increasing by 50 starting with the initial value 50 while there was a corresponding \( n \geq 500 \) such that the product \( m \cdot n \) does not exceed and is as close as possible to \( 10^6, 2 \cdot 10^6 \) or \( 3 \cdot 10^6 \). All in all there were solved 237 such problems with \( \delta = 0.01 \) and \( \Delta = 50 \). The computational complexity of a solution for every problem instance was measured in terms of the number of iterations of underlying projection subroutines. The results of these experiments are shown in the aggregated form in Fig. 3.2. The iteration complexity of the CTP algorithm is demonstrated as the function of the relative dimensionality of the underlying space of variables. The projection iterations and dimensionality were scaled to \([0, 1] \) intervals to make them comparable. The actual ranges of these values are demonstrated in Table 3.2.

The interesting feature of these graphs is that in all three series they demonstrated the highest complexity somewhere close to each other in the range of lower dimensional problems with larger number of generators, how-
Table 3: Maximal/minimal values for dimensionality ($m$), number of generators ($n$), iterations ($it$) for 2 phases of the algorithm

|      | phase-1          | phase-2          |
|------|------------------|------------------|
|      | $m$   | $n$   | $it$   | $m$   | $n$   | $it$   |
| A    | 2950/50 | 20000/513 | 4557/382 | 1950/50 | 20000/513 | 2749/392 |
| B    | 3950/50 | 40000/506 | 7366/407 | 3950/50 | 40000/506 | 4263/398 |
| C    | 5950/50 | 60000/504 | 10138/450 | 5950/50 | 60000/504 | 5314/411 |

ever for smallest dimensions with highest number of points the complexity was decreasing again. The OCTAVE-script which implements this algorithm can be found at ResearchGate by following the reference [10].

### 3.3 Quadratic optimization

To show the difference in performance of the finite algorithm given in this paper and the popular iterative algorithm FISTA [15], we applied our algorithm to solve the quadratic optimization problem described there as one of the tests. The latter is the trivial quadratic $n$-dimensional problem

$$\min_{x \in \mathbb{E}^n} \|Ax\|^2 = \min_{z \in \text{lin}(A)} \|z\|^2,$$

where $A$ is the Laplacian operator

$$A = \begin{bmatrix} 2 & -1 & \cdots & -1 \\ -1 & 2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 2 & -1 \\ -1 & 2 \end{bmatrix}_n$$

with $n = 201$ and $\text{lin}(A)$ is the linear envelope of the matrix $A$, considered as the set of its columns.

To convert the problem (15) into conical projection, notice that $\text{lin}(A) = \text{Co}\{-b, A\} = K_A$, where $b$ is such vector that $Ab \geq 1$. As the columns of $A$ are linear independent, such vector exists and can be obtained as a solution
of the least-norm problem

$$\min_{z \in \text{co}\{A\}} \|z\|^2 = \|b\|^2.$$  \hfill (16)

After this the problem (15) is solved by the Algorithm by projecting 0 on the cone $K_A$. Due to the particular nature of this problem, it is not a great surprise that both these problem were solved in $n$ and $n + 1$ iterations. The details of these two solution processes are shown on Fig. 3.3. It is interesting to notice the great similarity between these two processes. It can be said that both of them, for the most iterations, mainly collected information on the structure of the set of generators without a substantial decrease in violation of optimality conditions. When the algorithm collected this information, the projections are computed at the single final step of the processes practically up to machine accuracy.

For comparison, we present in Fig. 6 the results from [16] relating to the same quadratic problem. These graphs show the convergence of the whole family of FISTA-type algorithms. The convergence of these algorithms is demonstrated there from the point of view of two criteria: the distance $\|x^k - x^*\|$ of the $k$-th iteration $x^k$ from the solution $x^* = 0$ and deviation of the value of objective function $\Phi(x^k) = \|x^k\|^2$ from the optimum $\Phi(x^*) = \Phi(0) = 0$. One can see the significant progress in acceleration of these algorithms.

Unfortunately, there are no run-time measurements for these codes in [16], so there is no direct way to compare the practical effectiveness of these algorithms with the CTP routine. It can only be said that even the fastest
of the FISTA-algorithms by the number of iterations is about 3-orders of magnitude slower than the finite algorithm presented here. Of course, FISTA-algorithms have a wider applicability area and smaller memory footprint, but it looks like that for quadratic problems of the medium size, the exact finite algorithm is preferable.

Conclusions

As a conclusion, it may be stated that the result of this notice paves the way for robust and practical approaches for solving projection problems on polyhedral cones independently of the proportions between their dimensionality and number of generators. The computational efficiency of the described algorithm can be improved by providing intelligent restart procedures for both phases of the algorithm or by other means, which will be a subject of further investigations.

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