Reparameterisation Invariance Constraints on Heavy Particle Effective Field Theories

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Abstract

Since fields in the heavy quark effective theory are described by both a velocity and a residual momentum, there is redundancy in the theory: small shifts in velocity may be absorbed into a redefinition of the residual momentum. We demonstrate that this trivial reparameterisation invariance has non-trivial consequences: it relates coefficients of terms of different orders in the $1/m$ expansion and requires linear combinations of these operators to be multiplicatively renormalised. For example, the operator $-D^2/2m$ in the effective lagrangian has zero anomalous dimension, coefficient one, and does not receive any non-perturbative contributions from matching conditions. We also demonstrate that this invariance severely restricts the forms of operators which may appear in chiral lagrangians for heavy particles.
1. Introduction

The dynamics of heavy particles at low energies may be described by a heavy particle effective field theory, in which the effective lagrangian is expanded in inverse powers of the heavy particle mass [1]–[7]. The particles in the effective theory are described by velocity dependent fields [6] with velocity \( v \), residual momentum \( k \), and total momentum \( p = mv + k \). There is an ambiguity in assigning a velocity and momentum to a particle when one considers \( 1/m \) corrections to the effective field theory. The same physical momentum may be parameterised by

\[
(v, k) \leftrightarrow (v + \frac{q}{m}, k - q), \quad v^2 = \left(v + \frac{q}{m}\right)^2 = 1,
\]

(1.1)

where \( q \) is an arbitrary four-vector which satisfies \((v + q/m)^2 = 1\). The effective field theory must be invariant under the reparameterisation of the velocity and momentum, Eq. (1.1). This invariance has long been recognised (see, for example, [8]), however what is less well known is that it places constraints on the effective lagrangian, and relates coefficients of terms which are of different order in the \( 1/m \) expansion.

We will first discuss the consequences of reparameterisation invariance for the simple case of a spin-0 field in Sec. 2, and generalise the result to the somewhat more complicated case of particles with spin in Sec. 3. A few sample applications to matching conditions, anomalous dimensions and chiral lagrangians are discussed in Sec. 4. One important result that is obtained in Sec. 4 is that the coefficients of certain \( 1/m \) operators in the effective theory are exactly fixed, and cannot be modified by non-perturbative corrections. Section 5 discusses the consequences of reparameterisation invariance for matrix elements.

2. Reparameterisation Invariance for Scalar Fields

Consider a coloured scalar field [9] with mass \( m \) coupled to gluons, with lagrangian

\[
\mathcal{L} = D_\mu \phi^* D^\mu \phi - m^2 \phi^* \phi.
\]

(2.1)

The low energy effective lagrangian is given in terms of a velocity dependent effective field [6]

\[
\phi_v(x) = \sqrt{2m} \ e^{imv \cdot x} \phi(x),
\]

(2.2)
where $v$ is a velocity four-vector of unit length, $v^2 = 1$. The field $\phi_v$ creates and annihilates scalars with definite velocity $v$, which is a good quantum number in the $m \to \infty$ limit. The effective lagrangian which describes the low-energy dynamics of the full theory Eq. (2.1) is

$$\mathcal{L}_{\text{eff}} = \sum_v \phi_v^* (iv \cdot D) \phi_v + \mathcal{O}\left(\frac{1}{m}\right). \tag{2.3}$$

The reparameterisation transformation corresponding to Eq. (1.1) for the velocity dependent fields is

$$\phi_w(x) = e^{iq \cdot x} \phi_v(x), \quad w = v + \frac{q}{m}, \tag{2.4}$$

under which the effective lagrangian must remain invariant. We will explicitly work out the consequences of reparameterisation invariance up to order $1/m$. The most general effective lagrangian for the scalar field theory up to terms of order $1/m$ is

$$\mathcal{L}_{\text{eff}} = \sum_v \phi_v^* (iv \cdot D) \phi_v - \frac{A}{2m} \phi_v^* D^2 \phi_v, \tag{2.5}$$

where $A$ is a constant, and we have used the lowest order equation of motion to eliminate a term of the form $\phi_v^* (v \cdot D)^2 \phi_v$. Substituting the reparameterisation transformation (2.4) gives

$$\mathcal{L}_{\text{eff}} = \sum_v \phi_w^* \{v \cdot (iD + q)\} \phi_w - \frac{A}{2m} \phi_w^* (D^\mu - iq^\mu)^2 \phi_w. \tag{2.6}$$

Relabelling the dummy variable $w$ in Eq. (2.6) as $v$ and $v$ as $v - q/m$, gives the modified lagrangian

$$\mathcal{L}_{\text{eff}} = \sum_v \phi_v^* \left\{(v - \frac{q}{m}) \cdot (iD + q)\right\} \phi_v - \frac{A}{2m} \phi_v^* (D - iq)^2 \phi_v. \tag{2.7}$$

Expanding to first order in the (infinitesimal) transformation parameter $q$ gives the change in $\mathcal{L}$,

$$\delta \mathcal{L}_{\text{eff}} = \sum_v \phi_v^* \left\{v \cdot q - \frac{iq \cdot D}{m}\right\} \phi_v + i \frac{A}{m} \phi_v^* (q \cdot D) \phi_v$$

$$= (A - 1) \phi_v^* \frac{q \cdot D}{m} \phi_v + \mathcal{O}\left(\frac{q^2}{m^2}\right), \tag{2.8}$$

using $q \cdot v = \mathcal{O}(q^2/m)$ from (1.1). The lagrangian (2.2) is reparameterisation invariant up to order $1/m$ only if $A = 1$. Thus reparameterisation invariance has fixed the coefficient of one of the $1/m$ terms in the effective lagrangian. The tree-level matching condition of Eq. (2.3) determined $A = 1$, but we now have the stronger result that $A = 1$ is exact.
It is an elementary exercise to determine the most general possible reparameterisation invariant scalar lagrangian. The most general possible lagrangian may be written in the form

\[ \mathcal{L} = \sum_v \mathcal{L}_v(\phi_v(x), v^\mu, iD^\mu), \]  

where \( D^\mu \) represents a covariant derivative acting on the heavy field \( \phi_v \). Substituting the field reparameterisation Eq. (2.4), and replacing the dummy index \( w \) by \( v \) as before gives

\[ \mathcal{L} = \sum_v \mathcal{L}_v(\phi_v(x), (v - \frac{q}{m})^\mu, iD^\mu + q^\mu). \]  

For the lagrangian to be reparameterisation invariant, it is necessary and sufficient that factors of \( v \) and \( D \) occur only in the combination

\[ V^\mu = v^\mu + \frac{iD^\mu}{m}. \]  

This linear combination is precisely \( p^\mu/m \), where \( p^\mu \) is the total momentum of the particle, and is the only quantity which is unambiguously defined at order \( 1/m \).

The results just derived may be easily extended to include scalar fields coupled to an external source. The source is velocity independent, and in the effective theory, it must couple only to reparameterisation invariant combinations of operators in the effective theory. Thus a scalar source coupling \( J^*(x)\phi(x) \) can couple to \( J^*(x)e^{-imv^\mu x_\mu} \phi_v(x) \) as well as higher dimension operators, a vector source can couple to \( e^{-imv^\mu x_\mu}V^\mu \phi_v(x) \), and so forth.

3. Vector and Spinor Fields

The preceding analysis also applies to particles with spin. The only complication which arises is that the effective fields satisfy the velocity dependent constraints

\[ \frac{1 - \dot{\phi}}{2} v^\mu \psi_v = 0 \]  

for a heavy spinor \( \psi_v \), and

\[ v^\mu A^\mu_v = 0 \]  

for a heavy vector field \( A^\mu_v \), which must be preserved by the reparameterisation transformation. This makes the transformation law for the fields somewhat more complicated.
We first consider the case of a heavy vector field $A_v$. The lagrangian must be invariant under the transformation

$$A_{w}^{\mu}(x) = e^{iq \cdot x} R_{\nu}^{\mu}(w,v) A_{v}^{\nu}(x), \quad w = v + \frac{q}{m},$$

(3.3)

where $R_{\nu}^{\mu}(w,v)$ is a Lorentz transformation whose form we must determine. Define the matrix $\Lambda(v', v)$ to be a Lorentz transformation in the $v - w$ plane which rotates $v$ into $v'$, i.e. $v' = \Lambda(w, v) v$. The $\Lambda$ matrix may be written as

$$\Lambda(w, v) = \exp \left[ i J_{\alpha \beta} v'^{\alpha} v^{\beta} \right],$$

(3.4)

where $\theta$ is the boost angle, and

$$[J_{\alpha \beta}]_{\mu \nu} = -i (g_{\alpha \mu} g_{\beta \nu} - g_{\alpha \nu} g_{\beta \mu})$$

(3.5)

are the Lorentz generators in the spin-1 representation. The Lorentz boost matrix is computed in Appendix A. Consider an external state in the full theory with polarisation vector $\epsilon$, satisfying $p \cdot \epsilon = 0$. In the effective theory, the polarisation vectors are given by

$$\epsilon_{w} = \Lambda(v, p/m) \epsilon, \quad \epsilon_{w} = \Lambda(w, p/m) \epsilon,$$

(3.6)

so the appropriate reparameterisation transformation for spin-1 fields is

$$\epsilon_{w} = \Lambda(p/m, w)^{-1} \Lambda(p/m, v) \epsilon_{v}.$$  

(3.7)

Note that because the Lorentz group is non-Abelian, this is not the same as the (incorrect) transformation

$$\epsilon_{w} = \Lambda(w, v) \epsilon_{v}.$$  

(3.8)

Eqs. (3.7) and (3.8) differ by a Thomas precession term proportional to $q^{[\alpha} k^{\beta]}/m^2$, the area of the spherical triangle on $S^3$ with vertices at $v$, $w$ and $p/m$. It is not possible to make a reparameterisation invariant lagrangian using the transformation law of Eq. (3.8).

The lagrangian may be made invariant at order $1/m$ using (3.8), but at order $1/m^2$, there are terms which are antisymmetric in $D^\mu q^\nu$, which cannot be cancelled by the variation of any term of order $1/m^2$ in $L_{\text{eff}}$.

The transformation (3.7) is defined for polarisation vectors. To find the corresponding field redefinition, $p/m$ should be replaced by the operator

$$p^\mu / m \rightarrow U^\mu, \quad U^\mu = \mathcal{V}^\mu / |\mathcal{V}|$$

(3.9)
in Eqs. (3.6) and (3.4), where \( \mathcal{V} \) is defined in Eq. (2.11). The reparameterisation transformation Eq. (3.7) can be written as

\[
A_w(x) = e^{ig \cdot x} \Lambda \left( \frac{v^\mu + iD^\mu}{m}, w \right) A_v(x) \nonumber \rightarrow \Lambda \left( \frac{v^\mu + iD^\mu}{m}, v \right) A_v(x), \tag{3.10}
\]

since

\[
\left( \frac{w^\mu + iD^\mu}{m} \right) e^{ig \cdot x} = e^{ig \cdot x} \left( \frac{v^\mu + iD^\mu}{m} \right). \tag{3.11}
\]

Thus the only operator transformation that is required is of the form

\[
\Lambda \left( \frac{v^\mu + iD^\mu}{m}, v \right) \tag{3.12}
\]

where the same velocity \( v \) occurs in both arguments. There is an operator ordering ambiguity in the transformation Eq. (3.12) at order \( 1/m^2 \), since

\[
[\mathcal{V}^\mu, \mathcal{V}^\nu] = ig \frac{F^{\mu\nu}}{m}, \tag{3.13}
\]

which produces an ordering ambiguity in the reparameterisation transformation Eq. (3.7) at order \( 1/m^3 \). However, different orderings just differ by powers of the field strength \( F^{\mu\nu} \) times \( A_v \), and correspond to field redefinitions in the effective theory. Thus one can pick a particular ordering in the definition of \( \Lambda \) in Eq. (3.12) and use it consistently. To order \( 1/m \), the field \( A_v \) that appears in the effective lagrangian is

\[
\mathcal{A}_v^\mu = A_v^\mu - v^\mu \frac{iD \cdot A_v}{m} + O \left( \frac{1}{m^2} \right), \tag{3.14}
\]

using Eq. (A.6) and \( v \cdot A_v = 0 \).

To construct the most general lagrangian invariant under (3.3), it is convenient to introduce the field

\[
\mathcal{A}_v^\mu(x) = \Lambda^\mu_{\nu}(p/m, v) A_v^\nu(x) \tag{3.15}
\]

which simply picks up a phase under reparameterisation

\[
\mathcal{A}_w^\mu(x) = e^{ig \cdot x} \mathcal{A}_v^\mu(x) \tag{3.16}
\]
and satisfies
\[ p^\mu A_\mu(x) = 0. \] (3.17)

The most general reparameterisation invariant lagrangian may now be written in the form
\[ \mathcal{L} = \sum_v \mathcal{L}_v(A_v(x), \mathcal{V}^\mu) = \sum_v \mathcal{L}_v(\Lambda^\mu_\nu(p/m, v)A^\nu_v(x), \mathcal{V}^\mu), \] (3.18)
using the same argument as for scalar fields.

Heavy fermions in the effective theory are described by velocity dependent spinor fields \( \psi_v \) that satisfy the constraint
\[ \not{\partial} \psi_v = \psi_v \] (3.19)
(we treat here only the case of fermions; the arguments are easily generalised to heavy anti-fermions, which satisfy \( \not{\partial} \psi_v = -\psi_v \)). A consistent reparameterisation transformation for spinor fields is defined by analogy with the vector transformation, Eq. (3.7),
\[ \psi_w(x) = e^{iq \cdot x} \tilde{\Lambda}(w, p/m) \tilde{\Lambda}(v, p/m)^{-1} \psi_v(x), \quad w = v + \frac{q}{m}, \] (3.20)
where \( \tilde{\Lambda} \) are the Lorentz boosts in the spinor representation. The spinor lagrangian may be written in the form
\[ \mathcal{L} = \sum_v \mathcal{L}_v(\Psi_v(x), \mathcal{V}^\mu), \] (3.21)
where the reparameterisation covariant spinor field
\[ \Psi_v(x) \equiv \tilde{\Lambda}(p/m, v)\psi_v(x), \] (3.22)
transforms as
\[ \Psi_w(x) = e^{iq \cdot x} \Psi_v(x). \] (3.23)
The field \( \Psi \) may be written using the explicit form for \( \tilde{\Lambda} \) in Appendix A, and choosing a particular operator ordering for the covariant derivatives. At order \( 1/m \),
\[ \Psi_v(x) = \left( 1 + \frac{iD}{2m} \right) \psi_v(x). \] (3.24)
The terms in the effective lagrangian are bilinears in the Fermi fields. The reparameterisation invariant combinations of the standard fermion bilinears are

\[ \Psi_v \Psi_v = \bar{\psi}_v \psi_v, \]

\[ \Psi_v \gamma_5 \Psi_v = 0, \]

\[ \Psi_v \gamma^\mu \Psi_v = \bar{\psi}_v \left( v^\mu + \frac{iD^\mu}{m} \right) \psi_v + \mathcal{O}\left(1/m^2\right), \] \hspace{1cm} (3.25)

\[ \Psi_v \gamma^\mu \gamma_5 \Psi_v = \bar{\psi}_v \left( \gamma^\mu \gamma_5 - v^\mu \frac{iD}{m} \gamma_5 \right) \psi_v + \mathcal{O}\left(1/m^2\right), \]

\[ \Psi_v \sigma^{\alpha\beta} \Psi_v = \epsilon^{\alpha\beta\lambda\sigma} \bar{\Psi}_v \gamma^\sigma \gamma_5 \gamma_5 \nu_\lambda \psi_v. \]

4. Applications

Reparameterisation invariance constrains terms in the effective lagrangian. As a simple example, we have already seen that the kinetic term in the effective theory must have the form

\[ v \cdot iD + \frac{(iD)^2}{2m}, \] \hspace{1cm} (4.1)

a result which was proved in Sec. 2 for scalar fields, but can also be seen to be true for vector and spinor fields using the results of Secs. 3–4. The coefficient of the the \((iD)^2\) operator in the effective theory is fixed to be \(1/2m\), and is not renormalised. This agrees with a one loop computation of the anomalous dimension \[11\]. More importantly, this result is a non-perturbative non-renormalisation theorem. It has recently been suggested that there may be non-perturbative corrections in the heavy quark theory \[12\] at order \(1/m\) that modify the matching condition for the operator \(D^2/m\). This cannot be true if the effective theory is regulated to preserve reparameterisation invariance.*

As another example, the leading spin dependent term in the heavy quark effective theory is

\[ \frac{gC}{2m} \bar{\psi}_v \sigma^{\alpha\beta} F_{\alpha\beta} \psi_v = \frac{gC}{2m} \epsilon^{\alpha\beta\lambda\sigma} \bar{\psi}_v \nu_\lambda \gamma^\sigma \gamma_5 F_{\alpha\beta} \psi_v, \] \hspace{1cm} (4.2)

where \(C = 1\) at tree level. This operator is not related to the kinetic term by reparameterisation invariance, so \(C\) is not protected from radiative corrections. Using the results

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* We thank Mark Wise for discussions on this point.
of Eq. (3.25), one finds that the reparameterisation invariant generalisation of Eq. (4.2) to order $1/m^2$ is
\begin{equation}
\frac{gC}{2m} \epsilon^{\alpha\beta\lambda\sigma} \bar{\psi}_v F_{\alpha\beta} \left( v_\lambda + \frac{iD_\lambda}{m} \right) \gamma_\sigma \gamma_5 \psi_v
= \frac{gC}{2m} \bar{\psi}_v \sigma^{\alpha\beta} \left( F_{\alpha\beta} + 2F_{\sigma\alpha} \frac{iD_\beta}{m} v^\sigma \right) \psi_v.
\end{equation}

(4.3)

A similar analysis applies to external currents in the effective theory. For example, the weak current $J_\mu = \bar{c}_v \Gamma_\mu b_{v'}$, where $\Gamma_\mu = \gamma_\mu$ or $\gamma_\mu \gamma_5$, and $c_v$ and $b_{v'}$ are heavy $c$ and $b$ quark fields is written in reparameterisation-invariant form as
\begin{equation}
J_\mu = \bar{c}_v \Gamma_\mu b_{v'} - \frac{1}{2m_c} \bar{c}_v i \not\! \partial \Gamma_\mu b_{v'} + \mathcal{O} \left( \frac{1}{m_c^2} \right)
+ \frac{1}{2m_b} \bar{c}_v \Gamma_\mu \not\! D b_{v'} + \mathcal{O} \left( \frac{1}{m_b^2} \right),
\end{equation}

(4.4)

This agrees with the results in [13], in which the $\mathcal{O}(\alpha_s)$ matching of the operators $\bar{c}_v \Gamma_\mu b_{v'}$ and $(-i/2m_c)\bar{c}_v \not\! \partial \Gamma_\mu b_{v'}$ were found to be identical. It also agrees with [14] where it was found that the operators $\bar{c}_v \Gamma_\mu b_{v'}$ and $\bar{c}_v \not\! D \Gamma_\mu b_{v'}$ have the same anomalous dimension in the effective theory. Furthermore, it extends this result to additional operators at all orders in $1/m$. Note that this does not mean that (4.4) is the complete expression for the current in the effective theory. There will be other terms whose coefficients are unrelated to the zeroth order coefficient by reparameterisation invariance, just as the quark magnetic moment operator is not determined from the zeroth order kinetic term in the effective lagrangian.

Finally, reparameterisation invariance also provides useful information for chiral perturbation theory for heavy matter fields [13]–[20]. In this case, one cannot compute the matching conditions explicitly, so the operator coefficients are undetermined constants. Reparameterisation invariance eliminates a large number of operators in the chiral expansion, or determines their coefficients, thus considerably reducing the number of free parameters in the computation. As a simple example, consider a theory with a heavy scalar $T_v$ and a heavy vector $B_\mu^v$. The effective lagrangian could contain a term of the form
\begin{equation}
T_v iD_\mu B_\mu^v.
\end{equation}

(4.5)

Under the reparameterisation transformation, this term has a variation of the form
\begin{equation}
T_v q_\mu B_\mu^v.
\end{equation}

(4.6)
which cannot be cancelled by any term in the effective lagrangian which is of order one
(or of higher order in $1/m$). This is easily seen by writing the lagrangian in terms of the
fields in (3.18), where (4.5) could only arise from

$$T_v \mathcal{Y}_\mu B^\mu_v$$

which is zero by (3.17). Thus the term $T_v iD_\mu B^\mu_v$ cannot occur in the chiral lagrangian

5. Matrix Elements

The discussion has focused on the applications of reparameterisation invariance to
the effective lagrangian; in this section we discuss some of the applications to matrix
elements in the heavy particle effective field theory. As might be expected, the only
constraint it places on matrix elements is entirely trivial. Labelling states with both
velocity and residual momentum increases the number of possible form factors allowed;
imposing reparameterisation invariance simply reduces these back to the usual number of
form factors.

States in the effective theory have a velocity $v$ and a residual momentum $k$, with total
momentum $p = mv + k$. Thus there is also a reparameterisation invariance transformation
on the physical states which redefines $v$ and $k$, but keeps $p$ fixed. Consider the matrix
element of the vector current between two spinless particles,

$$\langle v, k' | j^\mu | v, k \rangle = f_1 v^\mu + f_2 (k^\mu + k'^\mu) + f_3 (k^\mu - k'^\mu), \quad (5.1)$$

where $f_i$ are three independent form factors, and

$$j^\mu = \overline{\psi}_v \left( v^\mu + \frac{iD^\mu}{m} \right) \psi_v. \quad (5.2)$$

It is well known that this matrix element should have only two independent form factors,
$f_+$ and $f_-$. The reparameterisation invariance on the states may be used to show that one
can eliminate one of the form factors, and write Eq. (5.1) in the form

$$\langle v, k' | j^\mu | v, k \rangle = f_1 \left( v^\mu + \frac{k^\mu + k'^\mu}{2m} \right) + f_3 (k^\mu - k'^\mu), \quad (5.3)$$

where $f_1$ and $f_3$ are functions of $v + k/m$ and $v' + k/m$. This is equivalent to the $f_\pm$ form
factor decomposition, and is a trivial application of reparameterisation invariance; there
are redundant variables in the effective theory which lead to redundant form factors which can then be eliminated.

Finally, one can easily see that the formulæ of Secs. 2–3 can be applied to external states with velocity \( v \), residual momentum \( k \), and spin, by replacing \( p/m \) by \( v + k/m \). There is no operator ordering ambiguity because the residual momentum \( k \) for external states is a number. The redundant form factors for particles with spin can be eliminated using the methods used above for the form factors of spinless particles.

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**Appendix A. Lorentz Boosts**

The Lorentz boost

\[
\Lambda(w, v, \theta) = \exp \left[ iJ_{\alpha\beta} w^\alpha v^\beta \theta \right], \quad [J_{\alpha\beta}]_{\mu\nu} = -i \left( g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu} \right), \quad (A.1)
\]

is a Lorentz boost in the \( w - v \) plane with boost parameter \( \theta \). To compute \( \Lambda(w, v, \theta) \) explicitly, define the matrix

\[
N_{\alpha\beta} = w^\alpha v^\beta - v^\alpha w^\beta, \quad \Lambda(w, v, \theta) = e^{\theta N}, \quad (A.2)
\]

A straightforward computation by expanding the exponential in a power series gives

\[
\Lambda(w, v, \theta)^\alpha_\beta = g^\alpha_\beta + \left( \frac{1 - \cosh \lambda \theta}{\lambda^2} \right) (w^\alpha w_\beta + v^\alpha v_\beta) + \frac{\sinh \lambda \theta}{\lambda} (w^\alpha v_\beta - v^\alpha w_\beta) + (w \cdot v) \left( \frac{\cosh \lambda \theta - 1}{\lambda^2} \right) (w^\alpha v_\beta + v^\alpha w_\beta), \quad (A.3)
\]

where

\[
\lambda^2 = (w \cdot v)^2 - 1. \quad (A.4)
\]

To obtain the boost matrix \( \Lambda(w, v) \) which rotates \( v \) into \( w \), the boost parameter \( \theta \) must have the value

\[
\sinh \lambda \theta = \lambda, \quad (A.5)
\]
so that
\[
\Lambda(w, v)^{\alpha \beta} = g^{\alpha \beta} - \frac{1}{1 + v \cdot w} (w^\alpha w_\beta + v^\alpha v_\beta) + (w^\alpha v_\beta - v^\alpha w_\beta) + \frac{v \cdot w}{1 + v \cdot w} (w^\alpha v_\beta + v^\alpha w_\beta).
\] (A.6)

The corresponding transformations \(\tilde{\Lambda}(w, v, \theta)\) and \(\tilde{\Lambda}(w, v)\) in the spinor representation may be obtained by using Eq. (A.1), and replacing the Lorentz generators \(J^{\alpha \beta}\) by their values in the spinor representation,
\[
J^{\alpha \beta} w_\alpha v_\beta = -\frac{1}{2} \sigma^{\alpha \beta} w_\alpha v_\beta = -\frac{i}{4} [\hat{w}, \hat{v}] .
\] (A.7)

The exponential is evaluated explicitly using the identity
\[
[\hat{w}, \hat{v}]^2 = 4 \lambda^2 ,
\] (A.8)
to give
\[
\tilde{\Lambda}(w, v, \theta) = \cosh \left(\frac{\lambda \theta}{2}\right) + \frac{1}{2 \lambda} [\hat{w}, \hat{v}] \sinh \left(\frac{\lambda \theta}{2}\right) .
\] (A.9)

For the transformation that rotates \(v\) into \(w\), \(\theta\) has the value Eq. (A.5), so that
\[
\tilde{\Lambda}(w, v) = \frac{1 + \hat{w} \hat{v}}{\sqrt{2 (1 + v \cdot w)}},
\] (A.10)
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