Quenched Disorder From Sea-Bosons

Girish S. Setlur
The Institute of Mathematical Sciences
Taramani, Chennai 600113

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Abstract
The degenerate Fermi gas coupled to a random potential is used to study metal-insulator transitions in various dimensions. We first recast the problem in the sea-boson language that allows for an easy evaluation of important physical attributes. We evaluate the dynamical number-number correlation function and from this compute the a.c. conductivity. We find that the d.c. conductivity vanishes in one and two dimensions. For a Hamiltonian that forbids scattering of an electron from within the Fermi surface to another state within the Fermi surface we find that there is no metal-insulator transition in three dimensions either.

1 Introduction
In a series of published works [1] [2], and a recent preprint[3], we showed how to extract the anomalous exponents in case of the Luttinger model using sea-bosons. This paves the way for application of the amended sea-boson theory that now is powerful enough to reproduce most of the exactly known results in 1d, to other systems such as electrons with quenched disorder with and without Coulomb interactions in various dimensions. The relevant literature on this subject is vast and we shall not attempt to be exhaustive in surveying it. Anderson’s pioneering work on localization[4] was followed by the work of Abrahams et.al. [8] and later on a more rigorous formulation of the notion of disorder averaging was given by McKane and Stone[6]. This relates to a single electron in a disordered potential. The classic review of Lee and Ramakrishnan [7] includes many references on the literature concerning the degenerate electron gas in a disordered potential. A more recent review is by Abrahams et. al. [8].
2 Number-Number Correlation Function

Eventually, we would like to compute the a.c. conductivity at absolute zero. Unfortunately this quantity is rather difficult to compute. This is because it involves first calculating the dynamical number-number correlation function. This latter function has proved very difficult to evaluate. Before we evaluate this quantity we would like to say a few words about how the dynamical number-number correlation enters into the picture. It is defined as follows.

\[ N(k; k') = \langle n_k(t)n_{k'}(t') \rangle > - \langle n_k(t) \rangle < n_{k'}(t') \rangle \]  

(1)

Notice that the a.c. conductivity is related to the dynamical total momentum-momentum correlation function. This formula was derived in an earlier preprint[3].

The momentum-momentum correlation function in turn may be related to the dynamical number-number correlation function.

\[ \langle \delta \mathbf{p}(t) \cdot \delta \mathbf{p}(0) \rangle = \sum_{kk'} (k \cdot k') \left[ \langle n_k(t) \rangle n_{k'}(0) - \langle n_k(t) \rangle \langle n_{k'}(0) \rangle \right] \]  

(2)

In a recent preprint[3], we provided some hints as to how might go about computing the number-number correlation function for the interacting system. It involves functional differentiation of the average momentum distribution with respect to sources that couple to the number operator. When this is done carefully we find the following appealing form of the dynamical number-number correlation function.

\[ N(k; k') = (1 - n_F(k))(1 - n_F(k'))S_{AA}(k; k') + n_F(k)n_F(k')S_{BB}(k; k') \]

\[ - (1 - n_F(k))n_F(k')S_{AB}(k; k') - (1 - n_F(k'))n_F(k)S_{BA}(k; k') \]  

(3)

Here the various quantities are defined recursively. First (here \( m, n = A, B \)),

\[ S_{mn}(k; k') = e^{-2<\hat{S}_m(k)>} e^{-2<\hat{S}_n(k')>} S_{mn}^0(k; k') \]  

(4)

One could take the point of view that \( S_{mn}^0(k; k') \) is evaluated by assuming that \( \hat{a}_n(q) \) are canonical bosons, dropping all the square roots and so on. The reason being that the corrections caused by fluctuations in the momentum distributions are included in the exponential prefactors. These quantities are defined recursively.

\[ \hat{S}_A(k, t) = \sum_q A_{k-q/2}^q \langle q, t \rangle A_{k+q/2}(q, t) \]  

(5)

\[ \hat{S}_B(k, t) = \sum_q A_{k+q/2}^q \langle q, t \rangle A_{k-q/2}(q, t) \]  

(6)

\[ S_{mn}^0(k; k') = \langle \hat{S}_m(k, t)\hat{S}_n(k', t') \rangle - \langle \hat{S}_m(k, t) \rangle \langle \hat{S}_n(k', t') \rangle \]  

(7)

where \( m, n = A, B \).
3 The Toy Hamiltonian

Here we couple the free Fermi gas to a disorder potential and compute the a.c. conductivity. The diagonalization is rendered trivial in the sea-boson language. However, the formula for the dynamical number-number correlation function in terms of the bosons is very nontrivial and can therefore be expected to lead to nontrivial results.

\[
H = \sum_{\mathbf{kq}} \frac{k \cdot q}{m} A^\dagger_k(\mathbf{q}) A_k(\mathbf{q}) + \sum_{\mathbf{q}} \frac{U_{\text{dis}}(\mathbf{q})}{\sqrt{V}} \sum_{\mathbf{k}} \left[ A_k(-\mathbf{q}) + A^\dagger_k(\mathbf{q}) \right] \tag{8}
\]

The above hamiltonian describes electrons close to the Fermi surface interacting with the disorder potential. However, notice that no externally chosen cutoff is needed. A natural smooth cutoff emerges by not linearizing the bare fermion dispersion. In the Fermi language, Eq.( 8) is equivalent to the following hamiltonian.

\[
H = \sum_{\mathbf{k}} \epsilon_k c^\dagger_k c_k + \sum_{\mathbf{k}, \mathbf{q}} \frac{U_{\text{dis}}(\mathbf{q})}{\sqrt{V}} [\Lambda_k(-\mathbf{q}) + \Lambda_k(\mathbf{q})] c^\dagger_{k+\mathbf{q}/2} c_{k-\mathbf{q}/2} \tag{9}
\]

Here \( \Lambda_k(\mathbf{q}) = n_F(\mathbf{k}+\mathbf{q}/2)(1-n_F(\mathbf{k}-\mathbf{q}/2)) \) and \( n_F(\mathbf{k}) = \theta(k_F-|\mathbf{k}|) \). Thus the toy hamiltonian Eq.( 9) describes electrons coupling to the disorder potential near the Fermi surface in such a way that processes that take an electron below the Fermi surface and place it in another state below the Fermi surface or both above the Fermi surface are forbidden. We shall see that in this case there is no metal insulator transition in any dimension. However, we reproduce the results that in one and two dimensions, the d.c. conductivity is zero. This hamiltonian may be trivially diagonalized by the following transformation.

\[
A_k(\mathbf{q}, t) = A^0_k(\mathbf{q}) e^{-\frac{k \cdot q}{m} t} - \frac{U_{\text{dis}}(\mathbf{q})}{\sqrt{V}} \frac{m}{k \cdot q} \tag{10}
\]

Thus we may compute the following quantities,

\[
\langle \hat{S}_A(\mathbf{k}) \rangle = \sum_{\mathbf{q}} \frac{|U_{\text{dis}}(\mathbf{q})|^2}{V} m^2 n_F(\mathbf{k} - \mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2)) \tag{11}
\]

Also for the number fluctuations,

\[
\langle A^\dagger_k(\mathbf{q}, t) A_k(\mathbf{q}, t) A^\dagger_{k'}(\mathbf{q}', t') A_{k'}(\mathbf{q}', t') \rangle - \langle A^\dagger_k(\mathbf{q}, t) A_k(\mathbf{q}, t) \rangle \langle A^\dagger_{k'}(\mathbf{q}', t') A_{k'}(\mathbf{q}', t') \rangle = \frac{|U_{\text{dis}}(\mathbf{q})|^2}{V} \frac{m^2}{(k \cdot q)^2} e^{-\frac{k \cdot q}{m} (t-t')} \delta_{k,k'} \delta_{\mathbf{q}, \mathbf{q}'} n_F(\mathbf{k} - \mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2)) \tag{12}
\]

Thus we may compute the following quantities,

\[
< \hat{S}_A(\mathbf{k}) > = \sum_{\mathbf{q}} \frac{|U_{\text{dis}}(\mathbf{q})|^2}{V} \frac{m^2}{((\mathbf{k} - \mathbf{q}/2) \cdot \mathbf{q})^2} n_F(\mathbf{k} - \mathbf{q})(1-n_F(\mathbf{k})) \tag{13}
\]
\[
< \hat{S}_B(k) > = \sum_q \frac{|U_{dis}(q)|^2}{V} \frac{m^2}{((k + q/2).q)^2} n_F(k)(1 - n_F(k + q))
\]

\[
S_{AA}(k, t; k', t') = \delta_{k,k'} \sum_q \frac{|U_{dis}(q)|^2}{V} \frac{1}{(k.q - \epsilon_q)^2} e^{-i(\frac{\epsilon_q}{m} - \epsilon_q)(t-t')} n_F(k)(1 - n_F(k))
\]

\[
S_{BB}(k, t; k', t') = \delta_{k,k'} \sum_q \frac{|U_{dis}(q)|^2}{V} \frac{1}{(k.q + \epsilon_q)^2} e^{-i(\frac{\epsilon_q}{m} + \epsilon_q)(t-t')} n_F(k)(1 - n_F(k+q))
\]

\[
S_{AB}(k, t; k', t') = \frac{|U_{dis}(k - k')|^2}{V} \frac{1}{(\epsilon_k - \epsilon_{k'})^2} e^{-i(\epsilon_k - \epsilon_{k'})(t-t')} n_F(k')(1 - n_F(k))
\]

\[
S_{BA}(k, t; k', t') = \frac{|U_{dis}(k' - k)|^2}{V} \frac{1}{(\epsilon_{k'} - \epsilon_k)^2} e^{-i(\epsilon_{k'} - \epsilon_k)(t-t')} n_F(k)(1 - n_F(k'))
\]

In an earlier preprint we showed that the real part of the a.c. conductivity may be written as,

\[
Re [\sigma(\omega; U_{dis})] = \left( \frac{\pi e^2}{m^2 V} \right) \frac{1}{\omega} \sum_{k,k'} \sum_{i,j} (k.k') \tilde{N}(k, \epsilon_i, \epsilon_j; k', 0) \delta(\omega - \epsilon_i + \epsilon_j)
\]

where,

\[
N(k, t; k', 0) \equiv \langle n_k(t) n_{k'}(0) \rangle - \langle n_k(t) \rangle \langle n_{k'}(0) \rangle = \sum_{i,j} e^{-i(\epsilon_i - \epsilon_j)t} \tilde{N}(k, \epsilon_i, \epsilon_j; k', 0)
\]
\[ S_{BB}(k; t, t') = \delta_{k, k'} \sum_{i, j} \delta_{k_j, k} \frac{|U_{dis}(k_i - k_j)|^2}{V} \frac{1}{(\epsilon_i - \epsilon_j)^2} e^{-i(\epsilon_i - \epsilon_j)(t-t')} n_F(k_j)(1-n_F(k_i)) \]  

(24)

\[ \tilde{N}(k, \epsilon_i, \epsilon_j; k', 0) = e^{-4S_A^0(k)} \delta_{k, k'} \delta_{k_i, k} \frac{|U_{dis}(k_i - k_j)|^2}{V} \frac{1}{(\epsilon_i - \epsilon_j)^2} n_F(k_j)(1-n_F(k_i)) \]

\[ -e^{-2S_A^0(k)} e^{-2S_B^0(k')} \delta_{k_j, k'} \delta_{k_i, k} \frac{|U_{dis}(k_i - k_j)|^2}{V} \frac{1}{(\epsilon_i - \epsilon_j)^2} n_F(k_j)(1-n_F(k_i)) \]

\[ -e^{-2S_B^0(k)} e^{-2S_A^0(k')} \delta_{k_j, k'} \delta_{k_i, k} \frac{|U_{dis}(k_i - k_j)|^2}{V} \frac{1}{(\epsilon_i - \epsilon_j)^2} n_F(k_j)(1-n_F(k_i)) \]

\[ +e^{-4S_A^0(k)} \delta_{k, k'} \delta_{k_i, k} \frac{|U_{dis}(k_i - k_j)|^2}{V} \frac{1}{(\epsilon_i - \epsilon_j)^2} n_F(k_j)(1-n_F(k_i)) \]  

(25)

### 4 A.C. Conductivity

The disorder averaged a.c. conductivity for Gaussian disorder may be written as

\[ \sigma(\omega) \equiv \Re \left[ \langle \sigma(\omega; U_{dis}) \rangle \right] \]

\[ = \left( \frac{\pi e^2}{m^2 V} \right) \frac{1}{\omega} \sum_{i, j} \left( k_i^2 e^{-4S_A^0(k_i)} + k_j^2 e^{-4S_B^0(k_j)} \right) \frac{\Delta^2}{V} \frac{1}{(k^2 - 2m^2)^2} n_F(k_j)(1-n_F(k_i)) \delta(\omega - \frac{k_i^2}{2m} + \frac{k_j^2}{2m}) \]  

(26)

where,

\[ S_A^0(k) = \sum_q \frac{\Delta^2}{V} \frac{m^2}{(k.q - q^2/2)^2} n_F(k - q)(1-n_F(k)) \]  

(27)

\[ S_B^0(k) = \sum_q \frac{\Delta^2}{V} \frac{m^2}{(k.q + q^2/2)^2} n_F(k)(1-n_F(k + q)) \]  

(28)

In other words,

\[ \sigma(\omega) \equiv \Re \left[ \langle \sigma(\omega; U_{dis}) \rangle \right] \]

\[ = \left( \frac{2\pi e^2 \Delta^2}{m^3} \right) \frac{1}{\omega^3} \int_{\epsilon_F - \omega}^{\epsilon_F} d\epsilon \ D(\omega + \epsilon) D(\epsilon) \left( (\omega + \epsilon) e^{-4S_A^0(\omega + \epsilon)} + \epsilon e^{-4S_B^0(\epsilon)} \right) \]  

(29)

\[ S_A^0(\epsilon) = \Delta^2 \int_{\epsilon_F}^{\epsilon_F} d\epsilon' D(\epsilon') \frac{1}{(\epsilon - \epsilon')^2} \]

(30)

\[ S_B^0(\epsilon) = \Delta^2 \int_{\epsilon_F}^{\infty} d\epsilon' D(\epsilon') \frac{1}{(\epsilon' - \epsilon)^2} \]  

(31)
\[ D(\epsilon) = \frac{m}{(2\pi)^d} (2m \epsilon)^{\frac{d-2}{2}} \] (32)

Using Mathematica\textsuperscript{TM} we find,

\[ S_A^0(\epsilon) = \Delta^2 \frac{m}{(2\pi)^d} (2m)^{\frac{d-2}{2}} \left( \frac{2 \epsilon^{d/2}}{d \epsilon^2} \right) H2F1[2, d/2, 1 + d/2, \frac{\epsilon}{\epsilon_F}] \] (33)

\[ S_B^0(\epsilon) = \Delta^2 \frac{m}{(2\pi)^d} (2m)^{\frac{d-2}{2}} \left( \frac{-2 \epsilon^{(4-d)/2}}{-4 + d} \right) H2F1[2, 2 - d/2, 3 - d/2, \frac{\epsilon}{\epsilon_F}] \] (34)

In one dimension, further simplification is not possible. Hence we write,

\[ S_A^0(\epsilon) = \Delta^2 \frac{m}{(2\pi)^d} (2m)^{\frac{1}{2}} \left( \frac{1}{\epsilon_F - \epsilon} - \frac{1}{\epsilon} \right) \] (35)

\[ S_B^0(\epsilon) = \Delta^2 \frac{m}{(2\pi)^d} \frac{1}{\epsilon_F - \epsilon} \] (36)

In two spatial dimensions we have,

\[ S_A^0(\epsilon) = \Delta^2 \frac{m}{(2\pi)^d} (2m)^{\frac{1}{2}} \left( \frac{1}{\epsilon_F - \epsilon} - \frac{1}{\epsilon} \right) \] (37)

\[ S_B^0(\epsilon) = \Delta^2 \frac{m}{(2\pi)^d} \frac{1}{\epsilon_F - \epsilon} \] (38)

In three spatial dimensions we have,

\[ S_A^0(\epsilon) = \Delta^2 \frac{m}{(2\pi)^d} (2m)^{\frac{1}{2}} \left( \frac{1}{\epsilon_F - \epsilon} - \frac{1}{\epsilon} \right) \] (39)

\[ S_B^0(\epsilon) = \Delta^2 \frac{m}{(2\pi)^d} \frac{1}{\epsilon_F - \epsilon} \] (40)

In two dimensions we have,

\[ \sigma(\omega) = \left( \frac{2 \pi e^2 \Delta^2}{m} \right) \frac{1}{\omega^2} \frac{m^2}{(2\pi)^4} \int_{\epsilon_F - \omega}^{\epsilon_F} d\epsilon \left( (\omega + \epsilon) e^{-4 \Delta^2 \frac{m}{(2\pi)^2} \left( \frac{1}{\epsilon_F - \epsilon} \frac{\epsilon^2}{\epsilon} \right) + \epsilon} e^{-4 \Delta^2 \frac{m}{(2\pi)^2} \left( \frac{1}{\epsilon_F - \epsilon} \frac{\epsilon^2}{\epsilon} \right) + \epsilon} \right) \] (41)

This may be approximately evaluated as follows.

\[ \sigma(\omega) \approx \left( \frac{2 \pi e^2 \Delta^2}{m} \right) \frac{1}{\omega^2} \frac{m^2}{(2\pi)^4} \left( 2 \epsilon_F e^{-4 \Delta^2 \frac{m}{(2\pi)^2} \left( \frac{1}{\epsilon_F - \epsilon} \frac{\epsilon^2}{\epsilon} \right) + \epsilon} \right) \] (42)

It can be seen that the zero frequency limit of the above expression is zero since the integral vanishes exponentially fast \( \sim e^{-\epsilon_F/\omega^2} \). Thus the d.c. conductivity
of a two dimensional system is zero and the frequency dependence is rather nontrivial. Similarly we may expect that in one dimension the d.c. conductivity vanishes. Unfortunately for a similar reason we find that the d.c. conductivity in three dimensions also vanishes. This means we have to include terms beyond what Eq. (9) does. Perhaps the reader can do this or at least offer to collaborate with the author. Please contact me at gsetlur@imsc.res.in

5 Some Technical Musings

It appears that the mathematical literature on the subject of quantum particles in random potentials is vast\cite{9}. It is possible, indeed likely that many mathematically rigorous results are known regarding this problem. But this does not prevent the authors from making some remarks that more knowledgeable readers may choose to critique. In particular, the author is uncomfortable with the notion of disorder averaging. Nature chooses its potentials based on the distribution of impurities, defects and so on. This potential is fixed and well-defined for a particular distribution of these imperfections. The physicists’ ignorance of the precise nature of this potential is not a license to average over these potentials. Nature does not average, people do. But are people justified in averaging? In other words can averaging simplify the problem without washing out essential physics? In order to answer this question we have to make the following conjectures.

\textbf{Defn0 :} Let $U_d$ be the set of all potentials $U(x)$ in a fixed spatial dimension $d$.

\textbf{Defn1 :} Let $F_d$ be the set of all potentials $U(x)$ in a fixed spatial dimension $d$ that has the following property. They all lead to the same exponent $\delta$ for the frequency dependence of the a.c. conductivity. In other words, each of these potentials predicts that $\text{Re}[\sigma(\omega)] \sim \omega^\delta$ (in some region of $\omega$ with possibly some additive part independent of $\omega$) with the same $\delta$.

\textbf{Conjecture 1 :} $F_d$ is dense in $U_d$.

If \textbf{Conjecture1} is valid, then one may average over all these ‘sufficiently erratic’ potentials and expect to extract $\delta$ which is all that physicists care about. It is possible that $\delta$ may be extracted from a numerical solution of the Schrodinger equation using a specific $U$ that belongs to the set $F_d$. But this would involve using the computer for more than checking one’s email, and not everyone likes that.

\textbf{Defn2 :} Let $M_3$ be the set of all potentials $U(x)$ in spatial dimension $d = 3$ that has the following property. They all lead to the same exponent $\beta$ for the mobility edge exponent. In other words, each of these potentials predict that $\sigma_{d.c} \sim (E_F - E_c)^\beta \theta(E_F - E_c)$ with the same $\beta$. However for different potentials, $E_c$ - the mobility edge, may be different.

\textbf{Conjecture 2 :} $F_3$ is dense in $U_3$.

If \textbf{Conjecture2} is valid, then one may average over all these ‘sufficiently
erratic\' potentials and expect to extract $\beta$.

Thus the validity of the process of averaging over potentials rests crucially it seems, on all these sufficiently erratic potentials predicting the same exponents and on these sufficiently erratic potentials spanning nearly all possible potentials.

If both these are satisfied then one may average over all potentials and extract the exponents, or, if one is better at programming, choose a particular potential from this set, numerically solve the Schrodinger equation and extract the exponent from there. In either case we should get the same answer. A final conjecture seems appropriate.

**Conjecture 3** : Let $M_3$ have an exponent $\beta'$ and $F_4$ have an exponent $\delta'$, then $\beta = \beta'$ and $\delta = \delta'$. In other words, these exponents are unique.

With powerful computers now available, purely analytical methods such as this work may seem passé, but a closed formula for the a.c. conductivity that one can stare at (and one that is hopefully right) and admire has a charm that a cold data file on the hard disk is unable to duplicate. Besides, with Coulomb interaction, the problem becomes intractable numerically, however, one may expect to combine the sea-boson method with the present one to extract the exponents analytically.

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