Occupation densities for SPDE’s with Reflection

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Abstract

We consider the solution \((u, \eta)\) of the white-noise driven stochastic partial differential equation with reflection on the space interval \([0, 1]\) introduced by Nualart and Pardoux. First, we prove that at any fixed time \(t > 0\), the measure \(\eta([0, t] \times d\theta)\) is absolutely continuous w.r.t. the Lebesgue measure \(d\theta\) on \((0, 1)\). We characterize the density as a family of additive functionals of \(u\), and we interpret it as a renormalized local time at \(0\) of \((u(t, \theta))_{t \geq 0}\). Finally we study the behaviour of \(\eta\) at the boundary of \([0, 1]\). The main technical novelty is a projection principle from the Dirichlet space of a Gaussian process, vector-valued solution of a linear SPDE, to the Dirichlet space of the process \(u\).

1 Introduction

We are concerned with the solution \((u, \eta)\) of the stochastic partial differential equation with reflection of the Nualart-Pardoux type, see [8]:

\[
\begin{equation}
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{1}{2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 W}{\partial t \partial \theta} + \eta(t, \theta) \\
\eta(0, \theta) &= x(\theta), \quad u(t, 0) = u(t, 1) = 0 \\
\eta(t, \theta) &\geq 0, \quad d\eta \geq 0, \quad \int u \, d\eta = 0
\end{aligned}
\end{equation}
\]

(1)
where $u$ is a continuous function of $(t, \theta) \in \overline{\mathcal{O}} := [0, +\infty) \times [0, 1]$, $\eta$ a positive measure on $\mathcal{O} := [0, +\infty) \times (0, 1)$, $x : [0, 1] \mapsto [0, \infty)$ and $\{W(t, \theta) : (t, \theta) \in \overline{\mathcal{O}}\}$ is a Brownian sheet. We denote by $\nu$ the law of a Bessel Bridge $(e_\theta)_{\theta \in [0,1]}$ of dimension 3.

The main aim of this paper is to prove the following properties of the reflecting measure $\eta$:

1. For all $t \geq 0$ the measure $\eta([0, t], d\theta)$ is absolutely continuous with respect to the Lebesgue measure $d\theta$ on $(0, 1)$:
   \[
   \eta([0, t], d\theta) = \eta([0, t], \theta) \, d\theta.
   \] (2)

The process $(\eta([0, t], \theta))_{t \geq 0}$, $\theta \in (0, 1)$, is an Additive Functional of $u$, increasing only on $\{t : u(t, \theta) = 0\}$, with Revuz measure:
   \[
   \frac{1}{2\sqrt{2\pi\theta^3(1-\theta)^3}} \nu(dx \mid x(\theta) = 0).
   \] (3)

2. For all $t \geq 0$:
   \[
   \eta([0, t], \theta) = \frac{3}{4} \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon^3} \int_{0}^{t} 1_{[0, \epsilon]}(u(s, \theta)) \, ds,
   \] (4)
in probability.

3. There exists a family of Additive Functionals of $u$, $(l^a(\cdot, \theta))_{a \in [0, \infty), \theta \in (0, 1)}$, such that $l^a(\cdot, \theta)$ increases only on $\{t : u(t, \theta) = a\}$ and such that the following occupation times formula holds for all $F \in B_b(\mathbb{R})$:
   \[
   \int_{0}^{t} F(u(s, \theta)) \, ds = \int_{0}^{\infty} F(a) \, l^a(t, \theta) \, da, \quad t \geq 0.
   \] (5)

4. For all $t \geq 0$:
   \[
   \eta([0, t], \theta) = \frac{1}{4} \lim_{a \downarrow 0} \frac{1}{a^2} l^a(t, \theta)
   \] (6)
in probability.

5. For all $t \geq 0$ and $a \in (0, 1)$:
   \[
   \lim_{\epsilon \downarrow 0} \sqrt{\epsilon} \int_{0}^{a} \left(1 + \frac{\theta}{\epsilon}\right) \eta([0, t], d\theta) = \sqrt{\frac{2}{\pi}} \, t
   \] (7)
and symmetrically:

\[
\lim_{\epsilon \downarrow 0} \sqrt{\epsilon} \int_{a}^{1} \left( 1 \wedge \frac{1 - \theta}{\epsilon} \right) \eta([0, t], d\theta) = \sqrt{\frac{2}{\pi}} t, \tag{8}
\]

in probability.

Recall that if \( B \) is a linear Brownian Motion and \((X, L)\) is the unique continuous solution of the Skorohod problem:

\[
dX = dB + dL, \quad X(0) = x \geq 0, \quad L(0) = 0,
\]

\( X \geq 0, \quad t \mapsto L(t) \) non-decreasing, \( \int_{0}^{\infty} X(t) dL(t) = 0, \)

then it turns out that \( 2L \) is the local time of \( X \) at 0 and:

\[
L(t) = \frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{0}^{t} 1_{[0,\epsilon]}(X(s)) \, ds. \tag{9}
\]

In the infinite-dimensional equation \([4]\), the reflecting term \( \eta \) is a random measure on space-time. In \([10]\), the following decomposition formula was proved:

\[
\eta(ds, d\theta) = \delta_{r(s)}(d\theta) \eta(ds, (0, 1)), \tag{10}
\]

where \( \delta_{a} \) is the Dirac mass at \( a \in (0, 1) \) and \( r(s) \in (0, 1) \), for \( \eta(ds, (0, 1)) \)-a.e. \( s \), is the unique \( r \in (0, 1) \) such that \( u(s, r) = 0 \). This formula was used in \([10]\) to write equation \([1]\) as the following Skorohod problem in the infinite dimensional convex set \( K_{0} \) of continuous non-negative \( x : [0, 1] \mapsto [0, \infty) \):

\[
du = \frac{1}{2} \frac{\partial^{2} u}{\partial \theta^{2}} \, dt + dW + \frac{1}{2} n(u) \cdot dL,
\]

interpreting the set of \( x \in K_{0} \) having a unique zero in \((0, 1)\) as the boundary of \( K_{0} \), the increasing process \( t \mapsto L_{t} := 2\eta([0, t], (0, 1)) \) as the local time of \( u \) at this boundary and the measure \( n(u) = \delta_{r(s)} \) as the normal vector field to this boundary at \( u(s, \cdot) \).

On the other hand, the absolute-continuity result \([2]\) suggests an interpretation of \( \eta \) as sum of reflecting processes \( t \mapsto \eta([0, t], \theta) \), each depending only on \( (u(t, \theta))_{t \geq 0} \) and increasing only on \( \{ t : u(t, \theta) = 0 \} \). Therefore, by \([2]\) equation \([1]\) can also be interpreted as the following infinite system of
1-dimensional Skorohod problems, parametrized by $\theta \in (0, 1)$ and coupled through the interaction given by the second derivative w.r.t. $\theta$:

$$
\begin{cases}
  u(t, \theta) = x(\theta) + \frac{1}{2} \int_0^t \frac{\partial^2 u}{\partial \theta^2}(s, \theta) \, ds + \frac{\partial W}{\partial \theta}(t, \theta) + \eta([0, t], \theta) \\
  u(t, 0) = u(t, 1) = 0 \\
  u \geq 0, \ \eta(dt, \theta) \geq 0, \ \int_0^\infty u(t, \theta) \eta(dt, \theta) = 0, \ \forall \theta \in (0, 1),
\end{cases}
$$

(11)

see (17) below. This interpretation is reminiscent of the result of Funaki and Olla in [3], where the fluctuations around the hydrodynamic limit of a particle system with reflection on a wall is proved to be governed by the SPDE (1).

By (5), $(u(t, \theta))_{t \geq 0}$ admits for all $a \geq 0$ a local time at $a$, $(l_a(t, \theta))_{t \geq 0}$. However, by (3), the reflecting term $\eta([0, \cdot], \theta)$ which appears in (11) is not proportional to $l^0(\cdot, \theta)$, which in fact turns out to be identically 0, and is rather a renormalized local time. The necessity of such renormalization is linked with the unusual rescaling of (1). These two properties of $\eta$ seem to be significant differences w.r.t. the finite-dimensional Skorohod problems.

The formulae (7) and (8) give informations about the behaviour of $\eta$ near the boundary of $[0, 1]$. In particular, (7) and (8) prove that for any $t > 0$ and any initial condition $x$, the mass of $\eta$ on $[0, t] \times (0, 1)$ is infinite. This solves a problem posed by Nualart and Pardoux in [8]. Notice also that the right hand sides of (7)-(8) are independent of the initial condition $x$.

In [10] it was proved that for all $I \subset \subset (0, 1)$, the process $t \mapsto \eta([0, t] \times I)$, where $\eta$ is the reflecting term of (1), is an Additive Functional of $u$, with Revuz measure:

$$
\frac{1}{2} \int_I \frac{1}{\sqrt{2\pi \theta^3(1 - \theta)^3}} \nu(dx \mid x(\theta) = 0) \, d\theta.
$$

(12)

At a heuristic level, the informations given by the formulae (2), (1), (3), (4) and (8) are already contained in (12) and in the properties of the invariant measure $\nu$ of $u$: for instance, if the limit in the right-hand side of (4) exists for all $\theta \in (0, 1)$, then by the properties of $\nu$ the Revuz-measure of the limit is (3) and therefore (2) holds by (12) and by the injectivity of the Revuz-correspondence.
However, the existence of such limit is not implied by the structure of (12) alone. According to the Theory of Dirichlet Forms, a sufficient condition for the convergence of a family of additive functionals of a Markov process, as for instance in (4), is the convergence in the Dirichlet space of the corresponding 1-potentials: see Chapter 5 of [3]. In our case, this amounts to introduce the potentials:

$$U_\epsilon(x) := \frac{3}{4} \int_0^\infty e^{-t} \frac{1}{\epsilon^3} \mathbb{E} \left[ 1_{[0,\epsilon]}(u(s, \theta)) \right] \, ds,$$

where $x : [0,1] \mapsto [0,\infty)$ is continuous and $u$ is the corresponding solution of (4), and prove that $U_\epsilon$ has a limit as $\epsilon \to 0$ with respect to the Dirichlet Form:

$$\mathcal{E}(\varphi, \psi) := \frac{1}{2} \int \langle \nabla \varphi, \nabla \psi \rangle \, d\nu, \quad \varphi, \psi \in W^{1,2}(\nu),$$

where $\nabla$ and $\langle \cdot, \cdot \rangle$ denote respectively the gradient and the canonical scalar product in $H := L^2(0,1)$. Indeed, as proved in [10], $u$ is the diffusion properly associated with $\mathcal{E}$ in $L^2(\nu)$.

However, due to the strong irregularity of the reflecting measure $\eta$ in (4), a direct computation of the norm of the gradient of $U_\epsilon$ seems to be out of reach. In order to overcome this difficulty, we take advantage of a connection between equation (4) and the following $\mathbb{R}^3$-valued linear SPDE with additive white-noise:

$$\begin{cases}
\partial z_3 / \partial t = \frac{1}{2} \partial^2 z_3 / \partial \theta^2 + \partial^2 \overline{W}_3 / \partial t \partial \theta \\
z_3(t, 0) = z_3(t, 1) = 0 \\
z_3(0, \theta) = \overline{x}(\theta)
\end{cases}
$$

(13)

where $\overline{x} \in H^3$ and $\overline{W}_3$ the is the $\mathbb{R}^3$-valued Gaussian process whose components are 3 independent copies of $W$. The process $z_3$ is also called the $\mathbb{R}^3$-valued random string (see [4] and [7]), and is the diffusion properly associated with the Dirichlet Form in $L^2(\mu_3)$:

$$\Lambda^3(F,G) := \int_{H^3} \langle \nabla F, \nabla G \rangle_{H^3} \, d\mu_3, \quad F, G \in W^{1,2}(\mu_3),$$

where $\mu_3$ is the law in $H^3$ of a standard $\mathbb{R}^3$-valued Brownian Bridge, and $\nabla F : H^3 \mapsto H^3$ is the gradient of $F$ in $H^3$. Then, in [10] it was noticed
that the Dirichlet Form $\mathcal{E}$ is the image of $\Lambda^3$ under the map $\Phi_3 : H^3 \mapsto H$, $\Phi_3(y)(\theta) := |y(\theta)|_{\mathbb{R}^3}$, i.e. $\nu$ is the image $\mu_3$ under $\Phi_3$ and:

$$W^{1,2}(\nu) = \{ \varphi \in L^2(\nu) : \varphi \circ \Phi_3 \in W^{1,2}(\mu_3) \},$$

$$\mathcal{E}(\varphi, \psi) = \Lambda^3(\varphi \circ \Phi_3, \psi \circ \Phi_3) \quad \forall \varphi, \psi \in W^{1,2}(\nu).$$

This connection involves directly the Dirichlet Forms $\mathcal{E}$ and $\Lambda^3$, but not the corresponding processes. In particular, it does not imply that $u$ is equal in law to $|z_3|$. Nevertheless, in this paper we prove that this connection gives a useful projection principle from $W^{1,2}(\mu_3)$ onto $W^{1,2}(\nu)$ and that, in particular, the convergence in $W^{1,2}(\mu_3)$ of the 1-potentials of $z_3$:

$$U_\epsilon(\pi) := \frac{3}{4} \int_0^\infty e^{-t} \frac{1}{\epsilon^3} \mathbb{E} \left[ 1_{[0,\epsilon]}(|z_3(s,\theta)|) \right] ds,$$

as $\epsilon \to 0$, implies the convergence of the 1-potentials $U_\epsilon$ of $u$ in $W^{1,2}(\nu)$, and therefore that (1) holds. Also the formulae (4), (7) and (8) are proved similarly. Therefore, precise and non-trivial informations about $u$ can be obtained from the study of the Gaussian process $z_3$.

We recall that an analogous connection has been proved in [11] to hold between the $\mathbb{R}^d$-valued solution of a linear white-noise driven SPDE, $d \geq 4$, and the solution of a real-valued non-linear white-noise driven SPDE with a singular drift.

The paper is organized as follows. Section 2 contains the main definitions and the preliminary results on potentials of the random string in dimension 3. In section 3 the occupation densities and the occupation times formula (5) are obtained for the SPDE with reflection (1). The main results, together with some corollaries, are then proved in section 4.

2 The 3-dimensional random string

We denote by $(g_t(\theta, \theta') : t > 0, \theta, \theta' \in (0, 1))$ the fundamental solution of the heat equation with homogeneous Dirichlet boundary condition, i.e.:

$$\begin{cases}
\frac{\partial g}{\partial t} = \frac{1}{2} \frac{\partial^2 g}{\partial \theta^2} \\
g_t(0, \theta') = g_t(1, \theta') = 0 \\
g_0(\theta, \cdot) = \delta_\theta
\end{cases}$$
where $\delta_a$ is the Dirac mass at $a \in (0, 1)$. Moreover, we set $H := L^2(0, 1)$ with the canonical scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, $K_0 := \{ x \in H : x \geq 0 \}$,

$$C_0 := C_0(0, 1) := \{ c : [0, 1] \mapsto \mathbb{R} \text{ continuous, } c(0) = c(1) = 0 \},$$

$A : D(A) \subset H \mapsto H$, $D(A) := W^{2,2} \cap W^{1,2}_0(0, 1)$, $A := \frac{1}{2} \frac{d^2}{d\theta^2}$,

and for all Fréchet differentiable $F : H \mapsto \mathbb{R}$ we denote by $\nabla F : H \mapsto H$ the gradient in $H$. We set $\mathcal{O} := [0, +\infty) \times (0, 1)$ and $\overline{\mathcal{O}} := [0, +\infty) \times [0, 1]$. We denote by $(e^{tA})_{t \geq 0}$ the semigroup generated by $A$ in $H$, i.e.:

$$e^{tA}h(\theta) := \int_0^1 g_t(\theta, \theta') h(\theta') \, d\theta', \quad h \in H.$$ 

Let $W$ be a two-parameter Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. a Gaussian process with zero mean and covariance function

$$\mathbb{E} [W(t, \theta) W(t', \theta')] = (t \wedge t')(\theta \wedge \theta'), \quad (t, \theta), (t', \theta') \in \overline{\mathcal{O}}.$$ 

Let $\overline{W}_3 := (\overline{W}_3)_{i=1,2,3}$ be a $\mathbb{R}^3$-valued process, whose components are three independent copies of $W$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathcal{F}_t$ the $\sigma$-field generated by the random variables $(W(s, \theta) : (s, \theta) \in [0, t] \times [0, 1])$.

We set for $\overline{x} \in H^3 = L^2((0, 1); \mathbb{R}^3)$:

$$w_3(t, \theta) := \int_0^t \int_0^1 g_{t-s}(\theta, \theta') \overline{W}_3(ds, d\theta')$$

$$z_3(t, \theta) := e^{tA} \overline{x}(\theta) + w_3(t, \theta), \quad Z_3(t, \overline{x}) := z_3(t, \cdot).$$

Then $z_3$ is the unique solution of the following $\mathbb{R}^3$-valued linear SPDE with additive white-noise:

$$\begin{cases}
\frac{\partial z_3}{\partial t} = \frac{1}{2} \frac{\partial^2 z_3}{\partial \theta^2} + \frac{\partial^2 \overline{W}_3}{\partial t \partial \theta} \\
z_3(t, 0) = z_3(t, 1) = 0 \\
z_3(0, \theta) = \overline{x}(\theta)
\end{cases} \quad (14)$$
where \( \tilde{\tau} \in H^3 \). The process \( z_3 \) is also called the \( \mathbb{R}^3 \)-valued random string; see \[4\] and \[7\]. Recall that the law of \( Z_3(t, \tilde{\tau}) \) is the Gaussian measure \( \mathcal{N}(e^{tA}\tilde{\tau}, Q_t) \) on \( H^3 \), with mean \( e^{tA}\tilde{\tau} \) and covariance operator \( Q_t : H^3 \mapsto H^3 : Q_t h(\theta) = \int_0^t \int_0^1 q_t(\theta, \theta') h(\theta') d\theta' ds, \) for all \( t \in [0, \infty], \theta, \theta' \in (0, 1) \), \( h \in H^3 \), where:

\[
q_t(\theta, \theta') := \int_0^t g_{2s}(\theta, \theta') ds, \quad t \in [0, \infty], \theta, \theta' \in (0, 1). \tag{16}
\]

We denote by \( (\tilde{\beta}(\theta))_{\theta \in [0,1]} \) a 3-dimensional standard Brownian Bridge, and by \( \mu_3 \) the law of \( \tilde{\beta} \). Recall that \( \mu_3 \) is equal to the Gaussian measure \( \mathcal{N}(0, Q_\infty) \) on \( H^3, Q_\infty = (-2A)^{-1}, \) and:

\[
q_\infty(\theta, \theta') = \theta \wedge \theta' - \theta\theta'. \tag{17}
\]

We set also for all \( t \in [0, \infty], \theta, \theta' \in (0, 1) \):

\[
q^t(\theta, \theta') := \int_t^\infty g_{2s}(\theta, \theta') ds = q_\infty(\theta, \theta') - q_t(\theta, \theta'). \tag{18}
\]

It is well known that \( Z_3 \) is the Markov process associated with the Dirichlet Form in \( L^2(\mu_3) \):

\[
\Lambda^3(F, G) := \int_{H^3} \langle \nabla F, \nabla G \rangle_{H^3} d\mu_3, \quad F, G \in W^{1,2}(\mu_3),
\]

where \( \nabla F : H^3 \mapsto H^3 \) is the gradient of \( F \) in \( H^3 \). For all \( f : H^3 \mapsto \mathbb{R} \) bounded and Borel and for all \( \tilde{\tau} \in H^3 \) we set:

\[
R_3(1) f(\tilde{\tau}) := \int_0^\infty e^{-t} \mathbb{E}[f(Z_3(t, \tilde{\tau}))] dt.
\]

The main result of this section is the following:

**Proposition 2.1**
1. For all $\theta \in (0, 1)$, $a \in \mathbb{R}^3$, the function $U_{3}^{\theta, a} : H^3 \mapsto \mathbb{R}$:

$$U_{3}^{\theta, a}(x) := \int_{0}^{\infty} e^{-t} \frac{1}{(2\pi q_t(\theta, \theta))^{3/2}} \exp \left(-\frac{|e^{tA}x(\theta) - a|^2}{2q_t(\theta, \theta)}\right) dt,$$

(19)

is well defined and belongs to $C_b(H^3) \cap W^{1,2}(\mu_3)$. If $(a_n, \theta_n) \to (a, \theta) \in \mathbb{R}^3 \times (0, 1)$, then:

$$\lim_{n \to \infty} \int_{H^3} \left| U_{3}^{\theta_n, a_n} - U_{3}^{\theta, a} \right|^2 + \left| \nabla U_{3}^{\theta_n, a_n} - \nabla U_{3}^{\theta, a} \right|^2 \, d\mu_3 = 0.$$  

(20)

Moreover $(\theta^{3/2}(1 - \theta)^{3/2}U_{3}^{\theta, a})_{\theta \in (0, 1), a \in \mathbb{R}^3}$ is uniformly bounded, i.e.:

$$\sup_{\theta \in (0, 1), a \in \mathbb{R}^3} \theta^{3/2}(1 - \theta)^{3/2} U_{3}^{\theta, a}(x) < \infty.$$  

(21)

2. Set $\bar{\varphi}(x) := \|x(\theta)\|/\sqrt{\theta}$, $x \in (C_0)^3$. Then $\Gamma^0_{3} := R_3(1)\bar{\varphi}$ converges to $\sqrt{8/\pi}$ in $W^{1,2}(\mu_3)$ as $\theta \to 0$ or $\theta \to 1$.

**Proof.** Let $\omega_3 := 4\pi/3$. If $\lambda \in \mathbb{R}$, we denote by $\lambda \cdot I$ the linear application $\mathbb{R}^3 \ni \alpha \mapsto \lambda \cdot \alpha \in \mathbb{R}^3$.

**Step 1.** Let $\bar{x} \in H^3$ be fixed. Notice that $z_3(t, \theta)$ has law $N(e^{tA}\bar{x}(\theta), q_t(\theta, \theta) \cdot I)$, where $q_t(\theta, \theta)$ is defined as in (16). We denote by $(G_t(a, b) : t, a, b > 0)$ the fundamental solution of the heat equation on $(0, +\infty)$ with homogeneous Dirichlet boundary condition. By the reflection principle we have the explicit representation:

$$G_t(a, b) = \frac{1}{\sqrt{2\pi t}} \exp \left(-\frac{(a - b)^2}{2t}\right) \left(1 - \exp \left(-\frac{2ab}{t}\right)\right).$$

We set $\tau_\theta := \inf \{t > 0 : \theta + B_t \in \{0, 1\}\}$, $\theta \in (0, 1)$. Then we have:

$$g_t(\theta, \theta') = G_t(\theta, \theta') - \mathbb{E} \left[1_{\{t > \tau_{\theta'}, \theta' + B_{\tau_{\theta'}} = 1\}} G_t - \tau_{\theta'}(\theta, 1)\right].$$

Let $c_0 := 1 - \exp(-1) \in (0, 1)$. Then for all $t > 0$ and $a \geq 0$:

$$\frac{c_0}{\sqrt{2\pi t}} \left(1 \wedge \frac{2a^2}{t}\right) \leq G_t(a, a) \leq \frac{1}{\sqrt{2\pi t}} \left(1 \wedge \frac{2a^2}{t}\right).$$
Let now $\theta \in [0, 1/2]$. Then:

$$
\int_0^t E \left[ 1_{(2s>T_\theta+\theta+B_{T_\theta}=1)} G_{2s-T_\theta}(\theta, 1) \right] ds
= E \left[ \int_0^{(2t-T_\theta)^+} \frac{1_{(\theta+B_{T_\theta}=1)}}{2\sqrt{2\pi r}} \exp \left( -\left( \frac{\theta - 1}{2r} \right)^2 \left( 1 - \exp \left( -\frac{2\theta}{r} \right) \right) \right) dr \right]
\leq E \left[ 1_{(\theta+B_{T_\theta}=1)} \right] \int_0^{2t} \frac{1}{2\sqrt{2\pi r}} \exp \left( -\left( \frac{\theta - 1}{2r} \right)^2 \left( 1 - \exp \left( -\frac{2\theta}{r} \right) \right) \right) dr
\leq \frac{c_1}{\sqrt{\pi}} t \theta^2, \quad c_1 := \sup_{r>0} \frac{1}{\sqrt{2r^3}} \exp \left( -\frac{1}{8r} \right) < \infty.
$$

For all $t > 0$ and $\theta \in [0, 1/2]$ we obtain:

$$
q_t(\theta, \theta) \geq \int_0^t G_{2s}(\theta, \theta) ds - \frac{c_1}{\sqrt{\pi}} t \theta^2 \geq \int_0^t \frac{c_0}{2\sqrt{\pi s}} \left( 1 \wedge \theta^2 \right) ds - \frac{c_1}{\sqrt{\pi}} t \theta^2
= \frac{c_0}{\sqrt{\pi}} \left( 1_{(t\leq\theta^2)} \sqrt{t} + 1_{(\theta^2\leq t)} \left( 2\theta - \frac{\theta^2}{\sqrt{t}} \right) \right) - t \frac{c_1}{\sqrt{\pi}} \theta^2
\geq \frac{1}{\sqrt{\pi}} \left( c_0 1_{(t\leq\theta^2)} \sqrt{t} + c_0 1_{(\theta^2\leq t)} \theta - c_1 t \theta^2 \right)
$$

Let $t_0 := (c_0/2c_1) \wedge (c_0/2c_1)^2$. If $t \geq t_0$, then $q_t(\theta, \theta) \geq q_{t_0}(\theta, \theta)$. If $t \leq t_0$ then:

$$
\frac{\theta}{q_t(\theta, \theta)} \leq \sqrt{\pi} \left( \frac{\theta}{\sqrt{t} (c_0 - c_1 \sqrt{t} \theta^2)} 1_{(t\leq\theta^2)} + \frac{\theta}{\theta (c_0 - c_1 t \theta)} 1_{(\theta^2\leq t)} \right)
\leq \frac{2\sqrt{\pi}}{c_0} \left( \frac{1}{\sqrt{t}} + 1 \right)
$$

By symmetry, we obtain that there exists $C_0 > 0$ such that for all $\theta \in (0, 1)$:

$$
\left( \frac{\theta (1 - \theta)}{q_t(\theta, \theta)} \right)^{3/2} \leq C_0 \left( \frac{1}{t^{3/4}} \wedge 1 \right), \quad t > 0.
$$

**Step 2.** Fix $\theta \in (0, 1)$. By (22), $U_3^{\theta,a}$ is well defined and in $C_0(H^3)$. Moreover for all $\varpi \in H^3$:

$$
\theta^{3/2}(1 - \theta)^{3/2} U_3^{\theta,a}(\varpi) = \int_0^\infty e^{-t} \left( \frac{\theta (1 - \theta)}{2q_t(\theta, \theta)} \right)^{3/2} \exp \left( -\frac{|e^{tA}(\theta) - a|^2}{2q_t(\theta, \theta)} \right) dt
$$
\begin{align*}
\leq \frac{C_0}{(2\pi)^{3/2}} \int_0^\infty e^{-t} \left( \frac{1}{t^{3/4}} \wedge 1 \right) dt < \infty,
\end{align*}
so that (21) is proved. For all \( \epsilon > 0 \) we set:
\[
\bar{f}(\bar{y}) := \frac{1}{\omega_3 \epsilon^3} 1_{\|y(\theta) - \bar{a}\| \leq \epsilon}, \quad \bar{y} \in (C_0)^3.
\]
Let \( \bar{x} \in H^3 \). Then:
\[
R_3(1) \bar{f}(\bar{x}) = \int_0^\infty e^{-t} \frac{1}{\omega_3 \epsilon^3} \int_{\mathbb{R}^3} 1_{(|\alpha| \leq \epsilon)} \mathcal{N} \left( e^{tA}\bar{x}(\theta) - a, q_t(\theta, \theta) \cdot I \right) (d\alpha) dt
\]
\[
= \int_0^\infty dt e^{-t} \frac{1}{\omega_3 \epsilon^3} \int_{(|\alpha| \leq \epsilon)} \frac{1}{(2\pi q_t(\theta, \theta))^{3/2}} \exp \left( -\frac{|\alpha - e^{tA}\bar{x}(\theta) + a|^2}{2q_t(\theta, \theta)} \right) d\alpha
\]
\[
= \frac{1}{\omega_3 \epsilon^3} \int_{(|\alpha| \leq \epsilon)} \left[ \int_0^\infty e^{-t} \frac{1}{(2\pi q_t(\theta, \theta))^{3/2}} \exp \left( -\frac{|\alpha - e^{tA}\bar{x}(\theta) + a|^2}{2q_t(\theta, \theta)} \right) dt \right] d\alpha
\]
\[
= \frac{1}{\omega_3 \epsilon^3} \int_{(|\alpha| \leq \epsilon)} U_{3,a+\alpha}^0(\bar{x}) d\alpha.
\]
(23)

By (22) and the Dominated Convergence Theorem, we have that for all \( (\theta, a) \in (0, 1) \times \mathbb{R}^3 \):
\[
\lim_{\epsilon \to 0} R_3(1) \bar{f}(\bar{x}) = U_{3,a}^0(\bar{x}), \quad \forall \bar{x} \in H^3,
\]
(24)
uniformly for \( \bar{x} \) in bounded sets of \( H^3 \), and by (22):
\[
|R_3(1) \bar{f}(\bar{x})| \leq \int_0^\infty e^{-t} \frac{1}{(2\pi q_t(\theta, \theta))^{3/2}} dt < \infty.
\]
(25)

**Step 3.** Notice that by the Dominated Convergence Theorem the map \( \mathbb{R}^3 \times (0, 1) \ni (a, \theta) \mapsto U_{3,a}^\theta \in L^2(\mu_3) \) is continuous. We want to prove now that \( U_{3,a}^\theta \) is in \( W^{1,2}(\mu_3) \): to this aim we shall prove that \( R_3(1) \bar{f} \) converges to \( U_{3,a}^\theta \) in \( W^{1,2}(\mu_3) \). We define for all \( \bar{x} \in (C_0)^3, a \in \mathbb{R}^3 \setminus \{\mathcal{T}(\theta)\} \):
\[
U_{\bar{x}}^{\theta,a}(\bar{x}) := -\int_0^\infty e^{-t} \frac{e^{tA}\bar{x}(\theta)}{(2\pi)^{3/2}(q_t(\theta, \theta))^{3/2}} \psi \left( (e^{tA}\bar{x}(\theta) - a)/\sqrt{q_t(\theta, \theta)} \right) dt,
\]
where: \( \psi: \mathbb{R}^3 \mapsto \mathbb{R}^3 \), \( \psi(a) := a \exp \left( -\frac{|a|^2}{2} \right) \).

Recall now that for all \( \bar{h} \in H^3 \) and \( \theta \in (0, 1) \):

\[
|e^{tA}\bar{h}(\theta)| \leq e^{tA}||\bar{h}(\theta)|| = \int_0^1 |\bar{h}(\theta')| g_t(\theta, \theta') \, d\theta' \leq \int_0^1 |\bar{h}(\theta')| G_t(\theta, \theta') \, d\theta' \\
\leq \frac{1 \wedge (2\theta/t)}{\sqrt{2\pi t}} \int_0^1 |\bar{h}(\theta')| \exp \left( -\frac{|\theta - \theta'|^2}{2t} \right) \, d\theta' \leq \frac{1 \wedge (2\theta/t)}{t^{1/4}} ||\bar{h}||. \tag{27}
\]

so that:

\[
\left| \sup_{||\bar{h}||=1} U_{\bar{h}}^{\theta,a}(\mathcal{F}) \right| \leq \int_0^\infty \frac{e^{-t}}{(q_t(\theta, \theta))^{2t^{1/4}}} \left| \psi \left( (e^{tA}\bar{F}(\theta) - a) / \sqrt{q_t(\theta, \theta)} \right) \right| \, dt.
\]

Since \( \bar{\beta} \) has law \( \mu_3 = \mathcal{N}(0, Q_\infty) \), then \( e^{tA}\bar{\beta} \) has law \( \mathcal{N}(0, e^{tA}Q_\infty e^{tA}) = \mathcal{N}(0, Q_\infty - Q_t) \). Then, by (22), and since \( |\psi| \leq 1 \):

\[
\left( \mathbb{E} \left( \sup_{||\bar{h}||=1} U_{\bar{h}}^{\theta,a}(\mathcal{F}) \right)^2 \right)^{1/2} \leq \int_0^\infty \frac{e^{-t}}{(q_t(\theta, \theta))^{2t^{1/4}}} \cdot \left( \int_{\mathbb{R}^3} |\psi \left( \alpha / \sqrt{q_t(\theta, \theta)} \right)|^2 \mathcal{N}(a, q^1(\theta, \theta) \cdot I) \, (d\alpha) \right)^{1/2} \, dt \\
\leq \int_0^\infty \frac{e^{-t}}{(q_t(\theta, \theta))^{3/4t^{1/4}}} \left\{ 1 \wedge \left[ \left( \frac{q_t(\theta, \theta)}{2\pi q^1(\theta, \theta)} \right)^{3/4} ||\psi||_{L^2(\mathbb{R}^3)} \right] \right\} \, dt \\
\leq \frac{1}{(q^1(\theta, \theta))^{3/4}} \int_0^1 \frac{1}{(q_t(\theta, \theta))^{5/4t^{1/4}}} \, dt + \frac{1}{(q^1(\theta, \theta))^2} \int_1^\infty e^{-t} \, dt < \infty.
\]

Therefore, setting for \( \mu_3 \)-a.e. \( \mathcal{F} \):

\[
U^{\theta,a} := - \int_0^\infty e^{-t} \frac{g_t(\theta, \cdot)}{(2\pi)^{3/2}(q_t(\theta, \theta))^2} \psi \left( (e^{tA}\bar{F}(\theta) - a) / \sqrt{q_t(\theta, \theta)} \right) \, dt,
\]

we have that \( U^{\theta,a} \in L^2(H^3, \mu_3; H^3) \), and \( \langle U^{\theta,a}, \bar{h} \rangle = U_{\bar{h}}^{\theta,a} \) in \( L^2(\mu_3) \), for all \( \bar{h} \in H^3 \). Arguing analogously we have:

\[
\left( \mathbb{E} \left[ ||U^{\theta,a+\alpha}(\mathcal{F}) - U^{\theta,a}(\mathcal{F})||^2 \right] \right)^{1/2} \leq \int_0^\infty \frac{e^{-t}}{(q_t(\theta, \theta))^{2t^{1/4}}}.
\]

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\[
\left\{ 1 \wedge \left[ \left( \frac{q_t(\theta, \theta)}{2\pi q^t(\theta, \theta)} \right)^{3/4} \left\| \psi \left( \cdot + \alpha/\sqrt{q_t(\theta, \theta)} \right) - \psi \right\|_{L^2(\mathbb{R}^3)} \right] \right\} dt \tag{28}
\]
which tends to 0 as \( \alpha \to 0 \). Therefore we can differentiate under the integral sign in (23) and obtain:
\[
\nabla R_3(1) f = \frac{1}{\omega_3 \epsilon^3} \int_{|\alpha| \leq \epsilon} U^{\theta,\alpha} d\alpha, \quad \text{in} \ L^2(H^3, \mu_3; H^3).
\]
Therefore by (28):
\[
\int_{H^3} \left\| \nabla R_3(1) f - U^{\theta,\alpha} \right\|^2 d\mu_3 \leq \frac{1}{\omega_3 \epsilon^3} \int_{|\alpha| \leq \epsilon} \int_{H^3} \left\| U^{\theta,\alpha+\alpha} - U^{\theta,\alpha} \right\|^2 d\mu_3 d\alpha \to 0
\]
as \( \epsilon \to 0 \). Therefore, \( R_3(1) f \) converges to \( U^{\theta,\alpha}_3 \) in \( L^2(\mu_3) \) and \( \nabla R_3(1) f \) converges to \( U^{\theta,\alpha}_3 \) in \( L^2(H^3, \mu_3; H^3) \) as \( \epsilon \to 0 \). Since \( W^{1,2}(\mu_3) \) is complete, then \( U^{\theta,\alpha}_3 \in W^{1,2}(\mu_3) \), \( \nabla U^{\theta,\alpha}_3 = U^{\theta,\alpha}_3 \) in \( L^2(H^3, \mu_3; H^3) \) and \( R_3(1) f \) converges to \( U^{\theta,\alpha}_3 \) in \( W^{1,2}(\mu_3) \) as \( \epsilon \to 0 \). Moreover, by (28), (20) is proved.

**Step 4.** We prove now the last assertion. By symmetry, it is enough to consider the case \( \theta \to 0 \). Recall that \( \Gamma^\theta_3(\overline{x}) = |\overline{x}(\theta)|/\sqrt{\theta}, \overline{x} \in (C_0)^3 \). Then:
\[
\Gamma^\theta_3(\overline{x}) := R_3(1) \overline{\Gamma}^\theta(\overline{x}) = \frac{1}{\sqrt{\theta}} \int_0^\infty e^{-t} \int_{\mathbb{R}^3} |\alpha| \mathcal{N} \left( e^{tA} \overline{x}(\theta), q_t(\theta, \theta) \cdot I \right) (d\alpha) dt
\]
\[
= \int_0^\infty e^{-t} \sqrt{\frac{q_t(\theta, \theta)}{\theta}} \int_{\mathbb{R}^3} |\alpha| \mathcal{N} \left( e^{tA} \overline{x}(\theta)/\sqrt{q_t(\theta, \theta)}, I \right) (d\alpha) dt,
\]
and:
\[
\nabla \Gamma^\theta_3(\overline{x}) = - \int_0^\infty e^{-t} \sqrt{\frac{q_t(\theta, \theta)}{\theta}} q_t(\theta, \overline{x}) \cdot
\]
\[
\cdot \int_{\mathbb{R}^3} |\alpha| \left( \alpha - e^{tA} \overline{x}(\theta)/\sqrt{q_t(\theta, \theta)} \right) \mathcal{N} \left( e^{tA} \overline{x}(\theta)/\sqrt{q_t(\theta, \theta)}, I \right) (d\alpha) dt,
\]
for all \( \overline{x} \in (C_0)^3 \). By (27) and by Schwartz’s inequality:
\[
|\nabla \Gamma^\theta_3(\overline{x})| \leq \sqrt{3} \int_0^\infty e^{-t} \frac{1}{t^{1/4}} \left( 1 \wedge \frac{2\theta}{t} \right) \sqrt{3 + \frac{|e^{tA} \overline{x}(\theta)|^2}{q_t(\theta, \theta)}} dt.
\]
By the sub-additivity of the square-root, by (22) and since \( q'(\theta, \theta) \leq \theta(1 - \theta) \):

\[
\left[ \mathbb{E} \left( \left\| \nabla \Gamma_3^\theta (\bar{\beta}) \right\|^2 \right) \right]^{1/2} \leq \sqrt{3} \int_0^\infty \frac{e^{-t}}{t^{1/4}} \left( 1 + \frac{2\theta}{t} \right) \left( \sqrt{3} + \sqrt{\frac{q'(\theta, \theta)}{q_0(\theta, \theta)}} \right) dt
\]

\[
\leq \sqrt{3} \int_0^\infty e^{-t} \left( 1 + \frac{2\theta}{t} \right) \left( \sqrt{3} + \frac{(C_0)^{1/3}}{t^{3/4}} \right) dt \to 0
\]

as \( \theta \to 0 \). Since \( \mu_3 \) is invariant for \( z_3 \), we have

\[
\mathbb{E} \left( \Gamma_3^\theta (\bar{\beta}) \right) = \frac{1}{\sqrt{\theta}} \mathbb{E} \left( |\bar{\beta}(\theta)| \right) =: c_\theta.
\]

By the Poincaré inequality for \( \Lambda^3 \), see [1], there exists \( C > 0 \) such that:

\[
\mathbb{E} \left( \Gamma_3^\theta (\bar{\beta}) - c_\theta \right)^2 \leq \frac{1}{C} \mathbb{E} \left( \left\| \nabla \Gamma_3^\theta (\bar{\beta}) \right\|^2 \right) \to 0,
\]

as \( \theta \to 0 \), and since:

\[
c_\theta = \frac{4\pi}{\sqrt{\theta}} \int_0^\infty \frac{1}{(2\pi\theta(1 - \theta))^{3/2}} r^3 \exp \left( -\frac{r^2}{2\theta(1 - \theta)} \right) dr
\]

\[
= \sqrt{1 - \theta} \sqrt{\frac{2}{\pi}} \int_0^\infty r^3 \exp \left( -\frac{r^2}{2} \right) dr = \sqrt{1 - \theta} \sqrt{\frac{8}{\pi}} \to \sqrt{\frac{8}{\pi}}
\]

we obtain that \( \Gamma_3^\theta \) converges to \( \sqrt{8/\pi} \) in \( W^{1,2}(\mu_3) \) as \( \theta \to 0 \). \( \square \)

3 Occupation densities

Following [8], we set the:

**Definition 3.1** A pair \( (u, \eta) \) is said to be a solution of equation (4) with initial value \( x \in K_0 \cap C_0 \), if:

(i) \( \{u(t, \theta) : (t, \theta) \in \mathcal{O}\} \) is a continuous and adapted process, i.e. \( u(t, \theta) \) is \( \mathcal{F}_t \)-measurable for all \( (t, \theta) \in \mathcal{O} \), a.s. \( u(\cdot, \cdot) \) is continuous on \( \mathcal{O} \), \( u(t, \cdot) \in K_0 \cap C_0 \) for all \( t \geq 0 \), and \( u(0, \cdot) = x \).

(ii) \( \eta(dt, d\theta) \) is a random positive measure on \( \mathcal{O} \) such that \( \eta([0, T] \times [\delta, 1 - \delta]) < +\infty \) for all \( T, \delta > 0 \), and \( \eta \) is adapted, i.e. \( \eta(B) \) is \( \mathcal{F}_t \)-measurable for every Borel set \( B \subset [0, t] \times (0, 1) \).
(iii) For all $t \geq 0$ and $h \in D(A)$
\[\langle u(t, \cdot), h \rangle - \langle x, h \rangle - \int_0^t \langle u(s, \cdot), Ah \rangle \, ds\]
\[= - \int_0^1 h'(\theta) W(t, \theta) \, d\theta + \int_0^t \int_0^1 h(\theta) \eta(ds, d\theta), \quad \text{a.s..} \quad (29)\]

(iv) $\int_\Omega u \, d\eta = 0$.

In [8], existence and uniqueness solutions of equation (1) were proved.

We denote by $(e(\theta))_{\theta \in [0,1]}$ the 3-Bessel Bridge between 0 and 0, see [9], and by $\nu$ the law on $K_0$ of $e$. We recall the following result, proved in [11].

**Theorem 3.1** Let $\Phi_3 : H^3 = L^2(0,1; \mathbb{R}^3) \hookrightarrow K_0$, $\Phi_3(y)(\theta) := |y(\theta)|_{\mathbb{R}^3}$.

1. The process $u$ is a Strong-Feller Markov process properly associated with the symmetric Dirichlet Form $\mathcal{E}$ in $L^2(\nu)$:
\[\frac{1}{2} \int_{K_0} \langle \nabla \varphi, \nabla \psi \rangle \, d\nu, \quad \varphi, \psi \in W^{1,2}(\nu).\]

2. The Dirichlet Form $\mathcal{E}$ is the image of $\Lambda^3$ under the map $\Phi_3$, i.e. $\nu$ is the image $\mu_3$ under $\Phi_3$ and:
\[W^{1,2}(\nu) = \{\varphi \in L^2(\nu) : \varphi \circ \Phi_3 \in W^{1,2}(\mu_3)\},\]
\[\mathcal{E}(\varphi, \psi) = \Lambda^3(\varphi \circ \Phi_3, \psi \circ \Phi_3) \quad \forall \varphi, \psi \in W^{1,2}(\nu). \quad (30)\]

We refer to [3] and [4] for all basic definitions in the Theory of Dirichlet Forms. Notice that by point 1 in Theorem [3,4] and by Theorem IV.5.1 in [3], the Dirichlet Form $\mathcal{E}$ is quasi-regular. In particular, by the transfer method stated in VI.2 of [3] we can apply several results of [3] in our setting.

We recall the definition of an Additive Functional of the Markov process $u$. We denote by $(\mathbb{P}_x : x \in K_0)$ the family the of laws of $u$ on $E := C([0, \infty); K_0)$ and the coordinate process on $K_0$ by: $X_t : E \hookrightarrow K_0$, $t \geq 0$, $X_t(e) := e(t)$. By a Positive Continuous Additive Functional (PCAF) in the strict sense of $u$, we mean a family of functions $A_t : E \hookrightarrow \mathbb{R}^+$, $t \geq 0$, such that:
(A.1) $(A_t)_{t \geq 0}$ is adapted to the minimum admissible filtration $(\mathcal{N}_t)_{t \geq 0}$ of $u$, see Appendix A.2 in [3].

(A.2) There exists a set $\Lambda \in \mathcal{N}_\infty$ such that $\mathbb{P}_x(\Lambda) = 1$ for all $x \in K_0$, $\theta_t(\Lambda) \subseteq \Lambda$ for all $t \geq 0$, and for all $\omega \in \Lambda$: $t \mapsto A_t(\omega)$ is continuous non-decreasing, $A_0(\omega) = 0$ and for all $t, s \geq 0$:

$$A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s \omega), \quad (31)$$

where $(\theta_s)_{s \geq 0}$ is the time-translation semigroup on $E$.

Two PCAFs in the strict sense $A^1$ and $A^2$ are said to be equivalent if

$$\mathbb{P}_x(A^1_t = A^2_t) = 1, \forall t > 0, \forall x \in K_0.$$ 

If $A$ is a linear combination of PCAFs in the strict sense of $u$, then the Revuz-measure of $A$ is a Borel signed measure $m$ on $K_0$ such that:

$$\int_{K_0} \varphi \, dm = \int_{K_0} \mathbb{E}_x \left[ \int_0^1 \varphi(X_t) \, dA_t \right] \nu(dx), \quad \forall \varphi \in C_b(K_0).$$

Moreover $U \in D(\mathcal{E})$ is the 1-potential of a PCAF $A$ in the strict sense with Revuz-measure $m$, if:

$$\mathcal{E}_1(U, \varphi) = \int_{K_0} \varphi \, dm, \quad \forall \varphi \in D(\mathcal{E}) \cap C_b(K_0),$$

where $\mathcal{E}_1 := \mathcal{E} + (\cdot, \cdot)_{L^2(\nu)}$. We introduce the following notion of convergence of Positive Continuous Additive Functionals in the strict sense of $X$.

**Definition 3.2** Let $(A_n(t))_{t \geq 0}$, $n \in \mathbb{N} \cup \{\infty\}$, be a sequence of PCAF’s in the strict sense of $u$. We say that $A_n$ converges to $A_\infty$, if:

1. For all $\epsilon > 0$ and for all $x \in K_0 \cap C_0$:

$$\lim_{n \to \infty} A_n(t + \epsilon) - A_n(\epsilon) = A_\infty(t + \epsilon) - A_\infty(\epsilon), \quad (32)$$

uniformly for $t$ in compact sets of $[0, \infty)$, $\mathbb{P}_x$-almost surely.

2. For $\mathcal{E}$-q.e. $x \in K_0 \cap C_0$:

$$\lim_{n \to \infty} A_n(t) = A_\infty(t), \quad (33)$$

uniformly for $t$ in compact sets of $[0, \infty)$, $\mathbb{P}_x$-almost surely.
Lemma 3.1 Let \((A_n(t))_{t \geq 0}, n \in \mathbb{N} \cup \{\infty\}\), be a sequence of PCAF’s in the strict sense of \(X\), and let \(U_n\) be the 1-potential of \(A_n\), \(n \in \mathbb{N} \cup \{\infty\}\). If \(U_n \to U_\infty\) in \(D(\mathcal{E})\), then \(A_n\) converges to \(A_\infty\) in the sense of Definition 3.2.

Proof. Since \(U_n \to U_\infty\) in \(D(\mathcal{E})\), by Corollary 5.2.1 in [3], we have point 2 of Definition 3.2, i.e. there exists an \(\mathcal{E}\)-exceptional set \(V\) such that (33) holds for all \(x \in K_0 \setminus V\). By the Strong Feller property of \(X\), \(P_x\)-a.s. \(X_t \in E \setminus V, \) for all \(t > 0\) and for all \(x \in K_0\), and by the additivity property (32) holds for all \(x \in K_0\). □

Remark 3.1 We recall that if \((A, \mathcal{A})\) is a measurable space, \((\Omega, \mathcal{F}, \mathbb{P})\) a probability space and \(X_n\) is a sequence of \(\mathcal{A} \otimes \mathcal{F}\)-measurable random variables, such that \(X_n(a, \cdot)\) converges in probability for every \(a \in A\), then there exists a \(\mathcal{A} \otimes \mathcal{F}\)-measurable random variable \(X\), such that \(X(a, \cdot)\) is the limit in probability of \(X_n(a, \cdot)\) for every \(a \in A\).

We can now state the main result of this section:

Theorem 3.2 Let \(\theta \in (0, 1), a \geq 0\).

1. For all \((\theta, a) \in (0, 1) \times [0, \infty)\), there exists a PCAF in the strict sense of \(u\), \((l^a(t, \theta))_{t \geq 0}\), such that \((l^a(\cdot, \theta))_{\theta \in (0, 1), a \in [0, \infty)}\) is continuous in the sense of Definition 3.2 and jointly measurable, and such that for all \(a \geq 0\):

\[
l^a(t, \theta) = \lim_{\epsilon \searrow 0} \frac{1}{\epsilon} \int_0^t 1_{[a, a+\epsilon]}(u(s, \theta)) \, ds, \quad t \geq 0,
\]

in the sense of Definition 3.2.

2. The Revuz measure of \(l^a(\cdot, \theta)\) is:

\[
\sqrt{\frac{2}{\pi \theta^3(1-\theta)^3}} \, a^2 \exp \left( -\frac{a^2}{2\theta(1-\theta)} \right) \nu(dx \mid x(\theta) = a), \quad a \geq 0,
\]

and in particular \(l^0(\cdot, \theta) \equiv 0\). Moreover, \(l^a(\cdot, \theta)\) increases only on \(\{t : u(t, \theta) = a\}\).

3. The following occupation times formula holds for all \(\theta \in (0, 1)\):

\[
\int_0^t F(u(s, \theta)) \, ds = \int_0^\infty F(a) \, l^a(t, \theta) \, da, \quad F \in B_b(\mathbb{R}), \quad t \geq 0. \quad (34)
\]
For an overview on existence of occupation densities see [2].

We set \( \Lambda^3_1 := \Lambda^3 + (\cdot, \cdot)_{L^2(\mu_3)} \), \( \mathcal{E}_1 := \mathcal{E} + (\cdot, \cdot)_{L^2(\nu)} \). For all \( f : H \mapsto \mathbb{R} \) bounded and Borel and for all \( x \in K_0 \cap C_0 \) we introduce the 1-resolvent of \( u \):

\[
R(1)f(x) = \int_0^\infty e^{-t} \mathbb{E}_x [f(X_t)] \, dt,
\]

where \( \mathbb{E}_x \) denotes the expectation w.r.t. the law of the solution \( u \) of (1) with initial value \( x \). The next Lemma gives the projection principle from the Dirichlet space \( W^{1,2}(\mu_3) \), associated with the Gaussian process \( z_3 \), to the Dirichlet space \( W^{1,2}(\nu) \) of the solution \( u \) of the SPDE with reflection (1).

**Lemma 3.2** There exists a unique bounded linear operator \( \Pi : W^{1,2}(\mu_3) \mapsto W^{1,2}(\nu) \), such that for all \( F, G \in W^{1,2}(\mu_3) \) and \( f \in W^{1,2}(\nu) \):

\[
\Lambda^3_1(F, f \circ \Phi_3) = \mathcal{E}_1(\Pi F, f),
\]

\[
\Lambda^3_1((\Pi F) \circ \Phi_3, G) = \Lambda^3_1(F, (\Pi G) \circ \Phi_3).
\]

In particular, we have that for all \( \varphi \in L^2(\nu) \) and \( F \in W^{1,2}(\mu_3) \):

\[
R(1)\varphi = \Pi (R_3(1) [\varphi \circ \Phi_3]),
\]

\[
\|\Pi F\|_{\mathcal{E}_1} \leq \|F\|_{\Lambda^3_1}.
\]

Finally, \( \Pi \) is Markovian, i.e. \( \Pi 1 = 1 \) and:

\[
F \in W^{1,2}(\mu_3), \quad 0 \leq F \leq 1 \implies 0 \leq \Pi F \leq 1.
\]

**Proof.** Let \( \mathcal{D} := \{ \varphi \circ \Phi_3 : \varphi \in W^{1,2}(\nu) \} \subset W^{1,2}(\mu_3) \). Let \( W^{1,2}(\mu_3) \) be endowed with the scalar product \( \Lambda^3_1 \); then, by (39), \( \mathcal{D} \) is a closed subspace of \( W^{1,2}(\mu_3) \). Therefore there exists a unique bounded linear projector \( \bar{\Pi} : W^{1,2}(\mu_3) \mapsto \mathcal{D} \), symmetric with respect to the scalar product \( \Lambda^3_1 \). For all \( F \in W^{1,2}(\mu_3) \) we set \( \Pi F := f \) where \( f \) is the unique element of \( W^{1,2}(\nu) \) such that \( f \circ \Phi_3 = \bar{\Pi} F \). Then (35) and (36) are satisfied by construction. Let now \( \varphi, \psi \in W^{1,2}(\nu) \). Then by (30):

\[
\mathcal{E}_1(R(1)\varphi, \psi) = \int_{K_0} \varphi \psi \, d\nu = \int_{H^3} (\varphi \circ \Phi_3)(\psi \circ \Phi_3) \, d\mu_3
\]

\[
= \Lambda^3_1(R_3(1) [\varphi \circ \Phi_3], \psi \circ \Phi_3) = \Lambda^3_1(\bar{\Pi} R_3(1) [\varphi \circ \Phi_3], \psi \circ \Phi_3)
\]

\[
= \mathcal{E}_1(\Pi R_3(1) [\varphi \circ \Phi_3], \psi),
\]
which implies (37). Then, since \( \hat{\Pi} \) is a symmetric projector:
\[
\| \Pi F \|_{\mathcal{E}_1} = \| \hat{\Pi} F \|_{\Lambda_3^1} \leq \| F \|_{\Lambda_3^1},
\]
so that (38) is proved. Notice now that 1 \( \in \mathcal{D} \), so that obviously \( \Pi 1 = 1 \).
Moreover, recall that \( \hat{\Pi} F \) is characterized by the property:
\[
\hat{\Pi} F \in \mathcal{D}, \quad \Lambda_3^1(F - \hat{\Pi} F, G) = 0, \quad \forall G \in \mathcal{D}.
\]
Let \( F \in W^{1,2}(\mu_3) \) such that \( F \geq 0 \). Since \( \mathcal{E} \) is a Dirichlet Form, then \( (\hat{\Pi} F)^+ := (-\hat{\Pi} F) \vee 0 \) still belongs to \( \mathcal{D} \), and since \( \Lambda_3^1 \) is a Dirichlet Form:
\[
0 = \Lambda_3^1(F - \hat{\Pi} F, (\hat{\Pi} F)^+) = \Lambda_3^1(F, (\hat{\Pi} F)^+) + \| (\hat{\Pi} F)^+ \|_{\Lambda_3^1}^2 \geq \| (\hat{\Pi} F)^+ \|_{\Lambda_3^1}^2,
\]
so that \( \hat{\Pi} F \geq 0 \), and (39) follows. \( \square \)

**Proof of Theorem 3.2.** Let \( a \geq 0 \). For all \( \epsilon > 0 \) we set:
\[
f^\epsilon(y) := \frac{1}{\epsilon} 1_{[a,a+\epsilon]}(y(\theta)), \quad y \in K_0 \cap C_0.
\]
By Lemma 3.2, we have that:
\[
R(1)f^\epsilon = \Pi (R_3(1)[f^\epsilon \circ \Phi_3]) = \frac{1}{\epsilon} \int_{[a \leq |\alpha| \leq a+\epsilon]} \Pi \left( U_{3}^{\theta,a+a} \right) d\alpha
\]
\[
= \frac{1}{\epsilon} \int_{a}^{a+\epsilon} r^2 dr \int_{S^2} \Pi \left( U_{3}^{\theta,r,a} \right) \mathcal{H}^2(dn),
\]
where \( \mathcal{H}^2 \) is the 2-dimensional Hausdorff measure, \( U_{3}^{\theta,a,a} \) is the 1-potential in \( W^{1,2}(\mu_3) \) defined by (19) and \( \Pi \) is the operator defined in Lemma 3.2. By (20) above, the map \( r \mapsto U_{3}^{\theta,r,a} \in W^{1,2}(\mu_3) \) is continuous. Let \( U^{\theta,a} \in W^{1,2}(\nu) \) be defined by:
\[
U^{\theta,a} := a^2 \int_{S^2} \Pi \left( U_{3}^{\theta,a} \right) \mathcal{H}^2(dn), \quad a \geq 0.
\]
By (38) we have that \( R(1)f^\epsilon \) converges to \( U^{\theta,a} \) in \( W^{1,2}(\nu) \) as \( \epsilon \to 0 \). For all \( \epsilon > 0 \) and \( \varphi \in W^{1,2}(\nu) \cap C_b(K_0) \) we have:
\[
\mathcal{E}_1(R(1)f^\epsilon, \varphi) = \int_{K_0} f^\epsilon \varphi d\nu = \frac{1}{\epsilon} \mathbb{E} \left[ \varphi(\epsilon) 1_{[\epsilon,\epsilon+\epsilon]}(\epsilon(\theta)) \right],
\]
where the law of $e$ is $\nu$ and $\mathcal{E}_1 = \mathcal{E} + (\cdot, \cdot)_{L^2(\nu)}$. Letting $\epsilon \to 0$ we get:

$$
\mathcal{E}_1(U^{\theta,a}, \varphi) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E} \left[ \varphi(e) \cdot 1_{[a,a+\epsilon]}(e(\theta)) \right] \\
= \sqrt{\frac{2}{\pi\theta^3(1-\theta)^3}} \cdot \frac{a^2}{2\theta(1-\theta)} \cdot \mathbb{E} \left[ \varphi(e) \mid e(\theta) = a \right].
$$

(40)

By Lemma 3.2, $\Pi$ is a Markovian operator and by (21) in Proposition 2.1 the family $(U_{3,n}^{\theta,a} : n \in \mathbb{S}^2)$ is uniformly bounded in the supremum-norm. Therefore, $U^{\theta,a}$ is bounded, and by (40) $U^{\theta,a}$ is the 1-potential of a non-negative finite measure. By Theorem 5.1.6 in [3], there exists a PCAF $(l^a(t,\theta))_{t \geq 0}$ in the strict sense of $u$, with 1-potential equal to $U^{\theta,a}$ and with Revuz-measure given by (40). Notice now that $R(1)f^t$ is the 1-potential of the following PCAF in the strict sense of $u$:

$$
t \mapsto \frac{1}{\epsilon} \int_0^t 1_{[a,a+\epsilon]}(u(s,\theta)) \, ds, \quad t \geq 0.
$$

Therefore, points 1 and 2 of Theorem 3.2 are proved by (20), Lemma 3.1 and Remark 3.1. To prove the last assertion of point 2, just notice that the following PCAF of $u$:

$$
t \mapsto \int_0^t |u(s,\theta) - a| \, l^a(ds,\theta),
$$

has Revuz measure:

$$
\sqrt{\frac{2}{\pi\theta^3(1-\theta)^3}} \cdot \frac{a^2}{2\theta(1-\theta)} \cdot |x(\theta) - a| \, \nu(dx \mid x(\theta) = a) \equiv 0.
$$

To prove point 3 it is enough to notice that the PCAF of $u$ in the left hand-side of (34) has 1-potential $R(1)F_\theta$, where $F_\theta(y) := F(y(\theta))$, $y \in K_0 \cap C_0$, and the PCAF in the right hand side has 1-potential:

$$
\int_0^\infty r^2 F(r) \, dr \int_{\mathbb{S}^2} \Pi \left( U_{3,r,n}^{\theta,r,n} \right) \, \mathcal{H}^2(da) = \Pi \left( \int_{\mathbb{R}^3} F(|x|) \, U_{3,x}^{\theta,x} \, d\alpha \right)
$$

$$
= \Pi \left( R_{3,1}[F_\theta \circ \Phi_3] \right) = R(1)F_\theta.
$$

Since $R(1)F_\theta$ is bounded, then, arguing like in Theorem 5.1.6 of [3], the two processes in (34) coincide as PCAF’s in the strict sense. □
4 The reflecting measure $\eta$

Recall that $\eta$ is the reflecting measure on $\mathcal{O} = [0, \infty) \times (0, 1)$ which appears in equation (1). The main result of this section is the following:

**Theorem 4.1** Let $\theta \in (0, 1)$, $a \geq 0$.

1. For all $\theta \in (0, 1)$, there exists a PCAF in the strict sense $(l(t, \theta))_{t \geq 0}$ of $u$, such that $(l(\cdot, \theta))_{\theta \in (0,1)}$ is continuous in the sense of Definition 3.2 and jointly measurable, and such that:

   $$l(t, \theta) = \lim_{\epsilon \downarrow 0} \frac{3}{\epsilon^3} \int_0^t 1_{[0,\epsilon]}(u(s, \theta)) \, ds,$$

   in the sense of Definition 3.2.

2. The PCAF $(l(t, \theta))_{t \geq 0}$ has Revuz measure:

   $$\sqrt{\frac{2}{\pi \theta^3 (1 - \theta)^3}} \nu(dx \mid x(\theta) = 0),$$

   and increases only on $\{t : u(t, \theta) = 0\}$.

3. We have:

   $$l(t, \theta) = \lim_{a \downarrow 0} \frac{1}{a^2} l^a(t, \theta)$$

   in the sense of Definition 3.2.

4. For all $t \geq 0$ and $x \in K_0$, $\eta([0, t], d\theta)$ is absolutely continuous w.r.t. the Lebesgue measure $d\theta$ and:

   $$\eta([0, t], d\theta) = \frac{1}{4} l(t, \theta) \, d\theta. \quad (41)$$

5. For all $a \in (0, 1)$:

   $$\lim_{\epsilon \downarrow 0} \sqrt{\epsilon} \int_0^a \left( 1 \wedge \frac{\theta}{\epsilon} \right) \eta([0, t], d\theta) = \sqrt{\frac{2}{\pi}} t,$$

   $$\lim_{\epsilon \downarrow 0} \sqrt{\epsilon} \int_a^1 \left( 1 \wedge \frac{1 - \theta}{\epsilon} \right) \eta([0, t], d\theta) = \sqrt{\frac{2}{\pi}} t$$

   in the sense of Definition 3.2.
Proof. For all $\epsilon > 0$ we set:

$$g^\epsilon(y) := \frac{3}{\epsilon^3} 1_{[0,\epsilon]}(y(\theta)), \quad y \in K_0 \cap C_0.$$  

By Lemma 3.2, we have that:

$$R(1)g^\epsilon = \Pi \left( R_3(1) [g^\epsilon \circ \Phi_3] \right) = \frac{3}{\epsilon^3} \int_{|\alpha| \leq \epsilon} \Pi \left( U_3^0, \alpha \right) d\alpha$$  

$$= \frac{3}{\epsilon^3} \int_0^\epsilon r^2 dr \int_{S^2} \Pi \left( U_3^{0, r \cdot n} \right) H^2(dn).$$

By Lemma 3.2, $R(1)g^\epsilon$ converges in $W^{1,2}(\nu)$ as $\epsilon \to 0$ to:

$$U^\theta := 4\pi \Pi \left( U_3^{0, 0} \right), \quad a \geq 0.$$  

For all $\epsilon > 0$ and $\varphi \in W^{1,2}(\nu) \cap C_b(K_0)$ we have:

$$\mathcal{E}_1(R(1)g^\epsilon, \varphi) = \int_{K_0} g^\epsilon \varphi d\nu = \frac{3}{\epsilon^3} \mathbb{E} \left[ \varphi(e) 1_{[0,\epsilon]}(e(\theta)) \right],$$

and letting $\epsilon \to 0$ we get:

$$\mathcal{E}_1(U^\theta, \varphi) = \lim_{\epsilon \to 0} \frac{3}{\epsilon^3} \mathbb{E} \left[ \varphi(e) 1_{[0,\epsilon]}(e(\theta)) \right]$$

$$= \sqrt{\frac{2}{\pi \theta^3(1-\theta)^3}} \mathbb{E} \left[ \varphi(e) \mid e(\theta) = 0 \right]. \quad (42)$$

Since $\Pi$ is Markovian, by (21) $U^\theta$ is bounded, and by (12) $U^\theta$ is the 1-potential of a non-negative finite measure. By Theorem 5.1.6 in [3], there exists a PCAF $(l(t, \theta))_{t \geq 0}$ in the strict sense of $u$, with 1-potential equal to $U^\theta$ and with Revuz-measure given by (42). Since $R(1)g^\epsilon$ is the 1-potential of the following PCAF of $u$:

$$t \mapsto \frac{3}{\epsilon^3} \int_0^t 1_{[0,\epsilon]}(u(s, \theta)) ds, \quad t \geq 0,$$

then, points 1 and 2 of Theorem 4.1 are proved by (20), Lemma 3.1 and Remark 3.1. To prove the last assertion of point 2, just notice that the following PCAF of $u$:

$$t \mapsto \int_0^t u(s, \theta) l(ds, \theta),$$

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has Revuz measure:
\[
\sqrt{\frac{2}{\pi \theta^3 (1 - \theta)^3}} \cdot x(\theta) \nu(\,dx \mid x(\theta) = 0) \equiv 0.
\]

From the Proof of Theorem 3.2, we know that the 1-potential of \( l^a(\cdot, \theta) \) is:
\[
U^{\theta, a} = a^2 \int_{S^2} \Pi \left( U_3^{\theta, a, n} \right) \mathcal{H}^2(dn), \quad a \geq 0.
\]
Then \( U^{\theta, a}/a^2 \) converges as \( a \to 0 \) to \( U^{\theta} \) in \( W^{1, 2}(\nu) \). Since \( U^{\theta, a}/a^2 \) is the 1-potential of \( l^a(\cdot, \theta)/a^2 \), by Lemma 3.1 point 3 of Theorem 4.1 is proved.

Let now \( I \subset (0, 1) \) be Borel. Notice that the following PCAF in the strict sense of \( u \):
\[
t \mapsto \frac{1}{4} \int_I l(t, \theta) \, d\theta
\]
has Revuz measure:
\[
\frac{1}{2} \int_I \frac{1}{\sqrt{2\pi \theta^3 (1 - \theta)^3}} \nu(\,dx \mid x(\theta) = 0) \, d\theta,
\]
and 1-potential equal to:
\[
\frac{1}{4} U^I := \frac{1}{4} \int_I U^\theta \, d\theta.
\]
On the other hand, it was proved in Theorem 7 of [10] that the PCAF in the strict sense of \( u \):
\[
t \mapsto \eta([0, t] \times I)
\]
has Revuz measure equal to (44). Therefore, by Theorem 5.1.6 in [3], the two PCAFs of \( u \) in (43) and (45) coincide, and since \( U^I \) is a bounded 1-potential then they coincide as PCAFs in the strict sense. Therefore point 4 is proved.

We prove now the last assertion. For all \( \epsilon \in (0, 1/2) \) set \( h_\epsilon : [0, 1] \mapsto [0, 1] \):
\[
h_\epsilon(\theta) := \sqrt{\epsilon} \left( \left( 1 \wedge \frac{\theta}{\epsilon} \right) 1_{[0, 1/2]}(\theta) + 4\theta (1 - \theta) 1_{[1/2, 1]}(\theta) \right).
\]
Then \( h_\epsilon \) is concave and continuous on \([0, 1] \), with:
\[
h''_\epsilon(d\theta) = -\frac{1}{\sqrt{\epsilon}} \delta_\epsilon(d\theta) - \sqrt{\epsilon} 8 1_{[1/2, 1]}(\theta) \, d\theta,
\]
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where $\delta_\epsilon$ is the Dirac mass at $\epsilon$. Moreover $h_\epsilon(0) = h_\epsilon(1) = 0$ and $h_\epsilon \to 0$ uniformly on $[0,1]$ as $\epsilon \to 0$. By (28) we have then:

$$\lim_{\epsilon \to 0} \left( \frac{1}{2\sqrt{\epsilon}} \int_0^t u(s, \epsilon) \, ds - \sqrt{\epsilon} \int_0^{1/2} \left( 1 \wedge \frac{\theta}{\epsilon} \right) \eta([0,t], d\theta) \right) = 0.$$  \hspace{1cm} (46)

Recall the definition of $\overline{\gamma}\theta$ given in point 2 of Proposition 2.1. We set $\gamma_\epsilon : K_0 \cap C_0 \mapsto \mathbb{R}$, $\gamma_\epsilon(x) := x(\epsilon) / \sqrt{\epsilon}$. Then, by Lemma 3.2 we have that $R(1) \gamma_\epsilon = \Pi \left( R_3(1) \overline{\gamma}\theta \right)$. By point 2 of Proposition 2.1 and by Lemma 3.2, we obtain that $R(1) \gamma_\epsilon$ converges to $\sqrt{8/\pi}$ in $W^{1,2}(\nu)$. Therefore, by Lemma 3.1:

$$\lim_{\epsilon \to 0} \frac{1}{2\sqrt{\epsilon}} \int_0^t u(s, \epsilon) \, ds = \sqrt{\frac{2}{\pi}} t,$$

in the sense of Definition 3.2, and by (46) point 5 is proved. \hfill \Box

**Corollary 4.1** For all $x \in K_0 \cap C_0$, a.s. the set:

$$S := \{ s > 0 : \exists \theta \in (0,1), u(s, \theta) = 0 \}$$

is dense in $\mathbb{R}^+$ and has zero Lebesgue measure.

**Proof.** By Point 5 in Theorem 4.1, for all $x \in K_0 \cap C_0$, a.s. for all $t > 0$ we have $\eta([0,t] \times (0,1)) = +\infty$, so that in particular $\eta([0,t] \times (0,1)) > 0$. By (iv) in Definition 3.1 the support of $\eta$ is contained in the set $\{u = 0\}$, so that for all $t > 0$ there exists $s \in (0,t) \cap S$. By the Markov property, for all $q \in \mathbb{Q}$ and all $t > q$, there exists $s \in (q,t) \cap S$, which implies the density of $S$ in $\mathbb{R}^+$. To prove that $S$ has zero Lebesgue measure, recall that the law of $u(t, \cdot)$ is absolutely continuous w.r.t. $\nu$ for all $t > 0$, and $\nu(x : \exists \theta \in (0,1), x(\theta) = 0) = 0$. Then, if $\mathcal{H}^1$ is the Lebesgue measure on $\mathbb{R}$:

$$\mathbb{E}_x \left[ \mathcal{H}^1(S) \right] = \int_0^\infty \mathbb{E}_x [1_S(t)] \, dt = \int_0^\infty \mathbb{P}(\exists \theta \in (0,1), u(t, \theta) = 0) \, dt = 0.$$ \hfill \Box

Notice now that, by Points 2 and 4 of Theorem 4.1, equation (1) can be formally written in the following form:

\[
\begin{align*}
  u(t, \theta) &= x(\theta) + \frac{1}{2} \int_0^t \frac{\partial^2 u}{\partial \theta^2}(s, \theta) \, ds + \frac{\partial W}{\partial \theta}(t, \theta) + \frac{1}{4} l(t, \theta) \\
  u(t, 0) &= u(t, 1) = 0 \\
  u \geq 0, \quad l(dt, \theta) \geq 0, \quad \int_0^\infty u(t, \theta) l(dt, \theta) = 0, \quad \forall \theta \in (0,1),
\end{align*}
\]  \hspace{1cm} (47)
where, as usual, the first line is rigorously defined after taking the scalar product in $H$ between each term and any $h \in D(A)$. Formula (47) allows to interpret $(u(\cdot, \theta), l(\cdot, \theta))_{\theta \in (0, 1)}$ as solution of a system of 1-dimensional Skorohod problems, parametrized by $\theta \in (0, 1)$. This fact is reminiscent of the result of Funaki and Olla who proved in [5] that the stationary solution of a certain system of 1-dimensional Skorohod problems converges under a suitable rescaling to the stationary solution of (1).

Finally, we show that $u$ satisfies a closed formula and that equation (1) is related to a fully non-linear equation. Let $(w(t, \theta))_{t \geq 0, \theta \in [0,1]}$ be the Stochastic Convolution:
\[
    w(t, \theta) := \int_0^t \int_0^1 g_{t-s}(\theta, \theta') W(ds, d\theta'),
\]
solution of:
\[
    \begin{cases}
        w(t, \theta) = \frac{1}{2} \int_0^t \frac{\partial^2 w}{\partial \theta^2}(s, \theta) \, ds + \frac{\partial W}{\partial \theta}(t, \theta) \\
        w(t, 0) = w(t, 1) = 0
    \end{cases}
\]
(48)

Subtracting the first line of (47) and the first line of (48), we obtain that:
\[
    (t, \theta) \mapsto \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \int_0^t (u(s, \theta) - w(s, \theta)) \, ds
\]
is in $L^1_{\text{loc}}((0, 1); C([0, T]))$ for all $T > 0$, i.e. admits a measurable version which is continuous in $t$ for all $\theta \in (0, 1)$ and such that the sup-norm in $t \in [0, T]$ is locally integrable in $\theta$. Then, we can write:
\[
    \begin{cases}
        u(t, \theta) = x(\theta) + w(t, \theta) + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \int_0^t (u-w)(s, \theta) \, ds + \frac{1}{4} l(t, \theta) \\
        u(t, 0) = u(t, 1) = 0 \\
        u \geq 0, \ l(dt, \theta) \geq 0, \ \int_0^\infty u(t, \theta) l(dt, \theta) = 0, \ \forall \theta \in (0, 1),
    \end{cases}
\]
(49)

where every term is now well-defined and continuous in $t$, and we can apply Skorohod’s Lemma (see Lemma VI.2.1 in [4]) for fixed $\theta \in (0, 1)$, obtaining:
\[
    \frac{1}{4} l(t, \theta) = \sup_{s \leq t} \left[ - \left( x(\theta) + w(s, \theta) + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \int_0^s (u-w)(r, \theta) \, dr \right) \right] \vee 0,
\]
(50)
for all $t \geq 0$, $\theta \in (0,1)$. Therefore we have the following:

**Corollary 4.2** For all $x \in K_0 \cap C_0$, a.s. $u$ satisfies the closed formula:

\[
\begin{align*}
    u(t, \theta) &= x(\theta) + w(t, \theta) + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \int_0^t (u(s, \theta) - w(s, \theta)) \, ds \\
    &\quad + \sup_{s \leq t} \left[ - \left( x(\theta) + w(s, \theta) + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \int_0^s (u(r, \theta) - w(r, \theta)) \, dr \right) \right] \vee 0,
\end{align*}
\]

for all $t \geq 0$, $\theta \in (0,1)$. In particular $v$, defined by:

\[
v(t, \theta) := \int_0^t (u(s, \theta) - w(s, \theta)) \, ds,
\]

is solution of the following fully non-linear equation:

\[
\begin{align*}
\frac{\partial v}{\partial t} &= \frac{1}{2} \frac{\partial^2 v}{\partial \theta^2} + x(\theta) + \sup_{s \leq t} \left[ - \left( x(\theta) + w(s, \theta) + \frac{1}{2} \frac{\partial^2 v}{\partial \theta^2} (s, \theta) \right) \right] \vee 0 \\
    v(0, \cdot) &= 0, \quad v(t, 0) = v(t, 1) = 0,
\end{align*}
\]

unique in $\mathcal{V} := \{ v' : \overline{\Omega} \mapsto \mathbb{R} \text{ continuous:} \ \partial v'/\partial t \text{ continuous,} \ \partial^2 v'/\partial \theta^2 \in L^1_{\text{loc}}((0,1); C([0,T])) \text{ for all } T > 0 \}$.

The uniqueness of solutions of equation (52) in $\mathcal{V}$ is a consequence of the pathwise uniqueness of solutions of equation (1), proved in [8]: indeed, if $v' \in \mathcal{V}$ is a solution of (52), then setting

\[
u'(t, \theta) := \frac{\partial v'}{\partial t}(t, \theta) + w(t, \theta),
\]

\[
\eta'(dt, d\theta) := dt \left\{ \sup_{s \leq t} \left[ - \left( x(\theta) + w(s, \theta) + \frac{1}{2} \frac{\partial^2 v'}{\partial \theta^2} (s, \theta) \right) \right] \vee 0 \right\} \, d\theta
\]

and repeating the above arguments backwards, we obtain that $(u', \eta')$ is a weak solution of (1), so that $u' = u$ and therefore $v' = v$.

Notice that, by point 5 in Theorem 4.1, by (11)-(20)-(22) and by the continuity of $\partial v/\partial t$ on $\overline{\Omega}$, then, for all $t > 0$, $\partial^2 v/\partial \theta^2(t, \cdot)$ is not in $L^1(0,1)$, so that by the uniqueness a $C^{1,2}(\overline{\Omega})$ solution of (52) does not exist.
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