On the rigidity of small domains

A. Crachiola
Department of Mathematics
Wayne State University
Detroit, MI 48202, USA
crach@math.wayne.edu

L. Makar-Limanov*
Department of Mathematics
Wayne State University
Detroit, MI 48202, USA
Department of Mathematics & Computer Science
Bar-Ilan University
Ramat-Gan 52900, Israel
lml@math.wayne.edu, lml@macs.biu.ac.il

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Abstract

Let \( k \) be an algebraically closed field of arbitrary characteristic. Let \( A \) be an affine domain over \( k \) with transcendence degree 1 which is not isomorphic to \( k[x] \), and let \( B \) be a domain over \( k \). We show that the AK invariant distributes over the tensor product of \( A \) by \( B \). As a consequence, we obtain a generalization of the cancellation theorem of S. Abhyankar, P. Eakin, and W. Heinzer.

Keywords: AK invariant, cancellation problem, locally nilpotent derivation, locally finite iterative higher derivation

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1 Introduction

All rings in this article are commutative with identity. For a ring \( A \), let \( A^{[n]} \) denote the polynomial ring in \( n \) indeterminates over \( A \). Let \( k \) be a field of arbitrary characteristic. Consider the well known

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Zariski cancellation problem. Let $V_1$ and $V_2$ be affine varieties over $k$ such that $V_1 \times k^n \cong V_2 \times k^n$ for some positive integer $n$. Does it follow that $V_1 \cong V_2$?

Significant results on this problem were published in 1972. S. Abhyankar, P. Eakin, and W. Heinzer [AEH] answered the question affirmatively for affine curves using algebraic methods (see Corollary 3.2). Also, M. Hochster [Hoc] gave a negative answer by constructing a 4-dimensional counterexample over the real numbers. Given this counterexample, it is natural to seek some restriction on $V_1$ and $V_2$ under which we may solve the problem. Because the example given by Hochster requires the formally real property of the real numbers, a natural restriction is that $k$ be algebraically closed. However, in 1989 W. Danielewski [Dan] provided a 2-dimensional counterexample over the complex numbers which lead to a class of similar counterexamples [Fie, Wil]. Another classical restriction on the Zariski cancellation problem is that $V_2$ be affine space. An affirmative answer to this case for surfaces was given by T. Fujita, M. Miyanishi, and T. Sugie [Fuj, MS] for characteristic 0 fields, and P. Russell [Rus] extended their result to fields of arbitrary characteristic. Also, T. Fujita and S. Iitaka [FI] solved the problem affirmatively for varieties $V_i$ of any dimension over $C$ in the case when the logarithmic Kodaira dimension of $V_i$ is not $-\infty$. Beyond this, the problem is still open.

In this article, we consider another perspective. The Zariski cancellation problem can be posed algebraically as follows. Let $A_1$ and $A_2$ be affine domains over $k$. Does $A_1[n] \cong A_2[n]$ imply $A_1 \cong A_2$? Viewing these polynomial rings as tensor products $A_1 \otimes_k k^n$, we can pose a more general cancellation question. If $B$ is an algebra over $k$ such that $A_1 \otimes_k B \cong A_2 \otimes_k B$, under what conditions can we conclude that $A_1 \cong A_2$? Of course we still have the counterexamples due to Danielewski and Hochster. However, in light of the positive result of Abhyankar, Eakin, and Heinzer, in this article we shall study the “small” 1-dimensional case of this more general cancellation question. For us this means that $A_1$ and $A_2$ have transcendence degree 1 over $k$.

A fruitful approach to understanding cancellation is to study additive group actions on a variety. Over characteristic 0 fields, this means studying locally nilpotent derivations on the ring of regular functions. Over prime characteristic fields, we can analogously consider locally finite iterative higher derivations. One tool which has been found beneficial in the characteristic 0 setting is the AK invariant, defined for a variety as the subring of regular functions which remain invariant under all additive group actions on the variety. The main goal of this article is to prove the following theorem, which has immediate consequences on the general question of cancellation, including the theorem of Abhyankar, Eakin, and Heinzer.

**Theorem.** Let $k$ be an algebraically closed field. Let $A$ be an affine domain over $k$ with $\text{trdeg}_k(A) = 1$ which is not isomorphic to $k^1$. Let $B$ be a domain over $k$. Then

$$\text{AK}(A \otimes_k B) = \text{AK}(A) \otimes_k \text{AK}(B).$$

Aside from the important geometric motivation behind this result, it is valu-
able from the ring theoretic perspective as a means for studying the structure of certain rings. Because isomorphisms of rings restrict to isomorphisms of their AK invariants, we can use the AK invariant as a probe into the automorphism group of a ring. For any ring $A$, to say that $\text{AK}(A) = A$ is to say that there are no (nontrivial) exponential maps on $A$. (This notion will be explained in the sequel.) This in turn tells us that $A$ is lacking a certain type of automorphism. In case $A$ is a $k$-algebra, it means there are no nontrivial actions of the additive group $k^+$ on $A$. We will call a ring $A$ rigid if $\text{AK}(A) = A$. In fact, all domains with transcendence degree 1 over $k$ are rigid, with the exception of $k[[1]]$ (see Lemma 2.3). So the main theorem of this article is a statement on the rigidity of such domains. We see, for instance, that a tensor product of two rigid transcendence degree 1 domains will remain rigid. Also, if $A \otimes_k B$ is not rigid, where $A$ and $B$ are as in the theorem, the exponential maps on this tensor product leave $A$ fixed like an anchor around which elements of $B$ are moved.

The slogan for this article is therefore, “small rigid domains stay rigid”. It remains to study the question of rigidity for larger domains.

## 2 Exponential maps and the AK invariant

Let us review some relevant notions with a view towards the definition of the AK invariant.

Let $k$ be a field of arbitrary characteristic and let $A$ be a $k$-algebra. Suppose $\varphi : A \rightarrow A[t]$ is a $k$-algebra homomorphism. We write $\varphi = \varphi_t : A \rightarrow A[t]$ if we wish to emphasize an indeterminate $t$. We say that $\varphi$ is an exponential map on $A$ if it satisfies the following two additional properties.

i. $\varepsilon_0 \varphi_t$ is the identity on $A$, where $\varepsilon_0 : A[t] \rightarrow A$ is evaluation at $t = 0$.

ii. $\varphi_s \varphi_t = \varphi_{s+t}$, where $\varphi_s$ is extended by $\varphi_s(t) = t$ to a homomorphism $A[t] \rightarrow A[s,t]$.

(When $A$ is the coordinate ring of an affine variety $\text{Spec}(A)$ over $k$, the exponential maps on $A$ correspond to algebraic actions of the additive group $k^+$ on $\text{Spec}(A)$ [Ess, §9.5].)

Given an exponential map $\varphi : A \rightarrow A[t]$, set $\varphi(t) = t$ to obtain an automorphism of $A[t]$ with inverse $\varphi_{-t}$. Consider the map $\varepsilon_1 \varphi : A \rightarrow A$, where $\varepsilon_1 : A[t] \rightarrow A$ is evaluation at $t = 1$. One can check that $\varepsilon_1 \varphi$ is an automorphism of $A$ with inverse $\varepsilon_1 \varphi_{-1}$.

Define

$$A^\varphi = \{ a \in A \mid \varphi(a) = a \},$$

a subalgebra of $A$ called the ring of $\varphi$-invariants.

For each $a \in A$ and each natural number $i$, let $D^i(a)$ denote the $t^i$-coefficient of $\varphi(a)$. Let $D = \{ D^0, D^1, D^2, \ldots \}$. To say that $\varphi$ is a $k$-algebra homomorphism is equivalent to saying that the sequence $\{ D^i(a) \}$ has finitely many nonzero
elements for each \( a \in A \), that \( D^n : A \to A \) is \( k \)-linear for each natural number \( n \), and that the Leibniz rule

\[
D^n(ab) = \sum_{i+j=n} D^i(a)D^j(b)
\]

holds for all natural numbers \( n \) and all \( a, b \in A \). The above properties (i) and (ii) of the exponential map \( \varphi \) translate into the following properties of \( D \).

i. \( D^0 \) is the identity map.

ii. (iterative property) For all natural numbers \( i, j \),

\[
D^i D^j = \binom{i+j}{i} D^{i+j}.
\]

Due to all of these properties, the collection \( D \) is called a \textit{locally finite iterative higher derivation on} \( A \). More generally, a \textit{higher derivation on} \( A \) is a collection \( D = \{ D^i \} \) of \( k \)-linear maps on \( A \) such that \( D^0 \) is the identity and the above Leibniz rule holds. The notion of higher derivations is due to H. Hasse and F.K. Schmidt [HS]. Every higher derivation on \( A \) has a unique extension to a higher derivation on any given localization of \( A \), determined through extension of the Leibniz rule to fractions. [Mat, §27] When the characteristic of \( A \) is 0, each \( D^i \) is determined by \( D^1 \), which is a locally nilpotent derivation on \( A \).

Let \( \text{EXP}(A) \) denote the set of all exponential maps on \( A \). We define the \textit{AK invariant}, or \textit{ring of absolute constants of} \( A \) as

\[
\text{AK}(A) = \bigcap_{\varphi \in \text{EXP}(A)} A^\varphi.
\]

This is a subalgebra of \( A \) which is isomorphism preserved. Indeed, any isomorphism \( f : A \to B \) of \( k \)-algebras restricts to an isomorphism \( f : \text{AK}(A) \to \text{AK}(B) \). To understand this, observe that if \( \varphi \in \text{EXP}(A) \) then \( f\varphi f^{-1} \in \text{EXP}(B) \). We say that \( A \) is \textit{rigid} if \( \text{AK}(A) = A \). That is, the only exponential map on \( A \) is the standard inclusion \( \varphi(a) = a \) for all \( a \in A \).

The above discussion of exponential maps, locally finite iterative higher derivations, and the AK invariant makes sense more generally for any (not necessarily commutative) ring. However, we will not need this generality.

Given an exponential map \( \varphi \) on a domain \( A \) over \( k \), we can define the \( \varphi \)-\textit{degree} of an element \( a \in A \) by \( \deg_\varphi(a) = \deg_\varphi(\varphi(a)) \) (where \( \deg_0(0) = -\infty \)). Note that \( A^\varphi \) consists of all elements of \( A \) with non-positive \( \varphi \)-degree. The function \( \deg_\varphi \) is a degree function on \( A \), i.e. it satisfies these two properties for all \( a, b \in A \).

i. \( \deg_\varphi(ab) = \deg_\varphi(a) + \deg_\varphi(b) \).

ii. \( \deg_\varphi(a+b) \leq \max\{\deg_\varphi(a), \deg_\varphi(b)\} \).
Equipped with these notions, we now collect some useful facts.

**Lemma 2.1.** Let \( \varphi \) be an exponential map on a domain \( A \) over \( k \). Let \( D = \{ D^i \} \) be the locally finite iterative higher derivation associated to \( \varphi \).

- a. If \( a, b \in A \) such that \( ab \in A^p \setminus 0 \), then \( a, b \in A^p \).
- b. \( A^p \) is algebraically closed in \( A \).
- c. For each \( a \in A \), \( \deg_{\varphi}(D^i(a)) \leq \deg_{\varphi}(a) - i \). In particular, if \( a \in A \setminus 0 \) and \( n = \deg_{\varphi}(a) \), then \( D^n(a) \in A^p \).

**Proof.**

- a: We have \( 0 = \deg_{\varphi}(ab) = \deg_{\varphi}(a) + \deg_{\varphi}(b) \), which implies that \( \deg_{\varphi}(a) = \deg_{\varphi}(b) = 0 \).
- b: If \( a \in A \setminus 0 \) and \( c_n a^n + \cdots + c_1 a + c_0 = 0 \) is a polynomial relation with minimal possible degree \( n \geq 1 \), where each \( c_i \in A^p \) with \( c_0 \neq 0 \), then \( a(c_n a^{n-1} + \cdots + c_1) = -c_0 \in A^p \setminus 0 \). By part a, \( a \in A^p \).
- c: Use the iterative property of \( D \) to check that \( D^j(D^i(a)) = 0 \) whenever \( j > \deg_{\varphi}(a) - i \). \( \square \)

**Lemma 2.2.** Let \( \varphi \) be a nontrivial exponential map (i.e. not the standard inclusion) on a domain \( A \) over \( k \) with \( \text{char}(A) = p \geq 0 \). Let \( x \in A \) with minimal positive \( \varphi \)-degree \( n \).

- a. \( D^i(x) \in A^p \) for each \( i \geq 1 \). Moreover, \( D^i(x) = 0 \) whenever \( i \geq 1 \) is not a power of \( p \).
- b. If \( a \in A \setminus 0 \), then \( n \) divides \( \deg_{\varphi}(a) \).
- c. Let \( c = D^n(x) \). Then \( A \) is a subalgebra of \( A^p[c^{-1}][x] \), where \( A^p[c^{-1}] \subseteq \text{Frac}(A^p) \) is the localization of \( A^p \) at \( c \).

**Proof.** In proving parts a and b we will utilize the following fact. If \( p \) is prime and \( i = p^j q \) for some natural numbers \( i, j, q \), then \( \left( \frac{1}{p^j} \right) \equiv q \pmod{p} \) [Sato Lemma 5.1].

- a: By part c of Lemma 2.1, \( D^i(x) \in A^p \) for all \( i \geq 1 \). If \( p = 0 \) then \( n = 1 \), for given any element in \( A \setminus A^p \) we can find an element with \( \varphi \)-degree 1 by applying the locally nilpotent derivation \( D^1 \) sufficiently many times. In this case, the second statement is immediate. Suppose now that \( p \) is prime and that \( i > 1 \) is not a power of \( p \), say \( i = p^j q \), where \( j \) is a nonnegative integer and \( q \geq 2 \) is an integer not divisible by \( p \). Then \( D^{i-p^j}(x) \in A^p \) and

\[
0 = D^{p^j}D^{i-p^j}(x) = \left( \frac{i}{p^j} \right) D^i(x) = q D^i(x).
\]

We can divide by \( q \) to conclude that \( D^i(x) = 0 \).

- b: Again if \( p = 0 \) then \( n = 1 \) and the claim is obvious. Assume that \( p \) is prime. By part a we have \( n = p^m \) for some integer \( m \geq 0 \). If \( m = 0 \), the claim is immediate. Assume that \( m > 0 \). Let \( d = \deg_{\varphi}(a) \). Suppose that
$p$ does not divide $d$. By part (3) of Lemma 2.1, $\text{deg}_{\varphi}(D^{d-1}(a)) \leq 1$. Now, $D^1 D^{d-1}(a) = dD^d(a) \neq 0$. So $\text{deg}_{\varphi}(D^{d-1}(a)) = 1 < n$, contradicting the minimality of $n$. Hence we can write $d = p^k d_1$ with $k \geq 1$ and $d_1$ not divisible by $p$. Making a similar computation, $D^p D^{d-p^k}(a) = d_1 D^d(a) \neq 0$. This implies that $\text{deg}_{\varphi}(D^{d-p^k}(a)) = p^k$. Since $n = p^m$ is minimal, we must have $k \geq m$, and so $n$ divides $d$.

(3): Let $a \in A \setminus 0$. By part (1) we can write $\text{deg}_{\varphi}(a) = ln$ for some natural number $l$. If $l = 0$ then $a \in A^\varphi$ and we are done. We use induction on $l > 0$. Elements $c^i a$ and $D^{i n}(a)x^l$ both have $\varphi$-degree $ln$. Let us check that $D^i n(c^i a) = D^i n(D^{i n}(a)x^l)$. First, $D^{i n}(c^i a) = c^i D^{i n}(a)$ by the Leibniz rule and since $c^i$ is $\varphi$-invariant. Secondly, since $D^{i n}(x^l) = D^n(x^l) = c^i$ and $D^{i n}(a)$ is $\varphi$-invariant, we see that $D^{i n}(D^{i n}(a)x^l) = c D^{i n}(a)$ as well. (Remark: Though the equality $D^{i n}(x^l) = D^n(x^l)$ does follow from the Leibniz rule, it may be more immediately observed as follows. $D^n(x)$ is the leading $t$-coefficient of $\varphi(x)$, and $\psi$ is a homomorphism. Hence the leading $t$-coefficient of $\varphi(x^l)$ is also that of $\varphi(x^l)^1$. Therefore, the element $y = c^i a - D^{i n}(a)x^l \varphi$-degree less than $ln$ and hence less than or equal to $(l-1)n$. By the inductive hypothesis, $y \in A^\varphi[c^{-1}][x]$. So $a = c^{-l}(y + D^{i n}(a)x^l) \in A^\varphi[c^{-1}][x]$.

\begin{proof}
To see that $\text{AK}(k[X]) = k$ (where $X$ is an indeterminate), observe that $\psi(X) = X + t$ defines an exponential map on $k[X]$ with ring of $\psi$-invariants $k$. Suppose $\text{AK}(A) \neq A$. Let $\varphi \in \text{EXP}(A)$ be nontrivial. Part (3) of Lemma 2.1 implies that $A^\varphi = k$. By part (1) of Lemma 2.2, $A \subseteq k[x]$ for some $x \in A$ with minimal positive $\varphi$-degree. So $A = k[x]$.

Thus $k[1]$ is the only transcendence degree 1 domain which is not rigid. By considering exponential maps of the form $\varphi_i(X_j) = X_j + \delta_{ij} t$, where $\delta_{ij}$ is the Kronecker delta, one can see that $\text{AK}(k[1]) = k$ for each natural number $n$. However, if $A$ is a domain with transcendence degree $n \geq 2$ over $k$, then $\text{AK}(A) = k$ does not imply that $A \cong k[1]$. One example will be given in the next section.

3 The main result and corollaries

For most of our statements, tensor products are of $k$-algebras over $k$, and transcendence degrees are taken over $k$. So we write $\otimes$ and $\text{trdeg}$ rather than $\otimes_k$ and $\text{trdeg}_k$. If we need to specify a different field, we will decorate the notation.

We can extend any $\varphi \in \text{EXP}(A)$ to an exponential map on $A \otimes B$ by defining $\varphi(b) = b$ for all $b \in B$. In other words, we set $\varphi(\sum a_i \otimes b_i) = \sum_i \varphi(a_i) \otimes b_i$. Any element of $\text{AK}(A \otimes B)$ must be invariant under such exponential maps, and so $\text{AK}(A \otimes B) \subseteq \text{AK}(A) \otimes B$. We can interchange the roles of $A$ and $B$ in that argument and further conclude that $\text{AK}(A \otimes B) \subseteq \text{AK}(A) \otimes \text{AK}(B)$. 
Theorem. Let $k$ be an algebraically closed field. Let $A$ be an affine domain over $k$ with $\text{trdeg}(A) = 1$ which is not isomorphic to $k[1]$. Let $B$ be a domain over $k$. Then

$$\text{AK}(A \otimes B) = \text{AK}(A) \otimes \text{AK}(B).$$

Of course, this can also be written as $\text{AK}(A \otimes B) = A \otimes \text{AK}(B)$. The conclusion is false when $A \cong k[1]$, as discussed below. This theorem has some immediate corollaries.

Corollary 3.1. Let $k$ be an algebraically closed field. Let $A_1$ and $A_2$ be affine domains over $k$ with $\text{trdeg}(A_i) = 1$, $i = 1, 2$. Let $B$ be a domain over $k$ such that $\text{AK}(B) = k$. If $A_1 \otimes B \cong A_2 \otimes B$, then $A_1 \cong A_2$.

Proof. Suppose neither $A_1$ nor $A_2$ is isomorphic to $k[1]$. Applying AK to both sides gives us $A_1 \otimes k \cong A_2 \otimes k$, so that $A_1 \cong A_2$. If $A_1 \cong k[1]$ but $A_2 \not\cong k[1]$, then $A_2 \cong \text{AK}(A_2 \otimes B) \cong \text{AK}(A_1 \otimes B) \subseteq \text{AK}(A_1) \otimes \text{AK}(B) \cong k$.

But this is absurd. \qed

As a special case, take $B = k^n$. The theorem implies that $\text{AK}(A[n]) = \text{AK}(A)$ for any affine domain with transcendence degree 1 over $k$. Since any isomorphism $f : A \rightarrow B$ restricts to an isomorphism $f : \text{AK}(A) \rightarrow \text{AK}(B)$, we recover the cancellation theorem of S. Abhyankar, P. Eakin, and W. Heinzer.

Corollary 3.2 (see [AEH]). Let $k$ be an algebraically closed field. Let $A_1$ and $A_2$ be affine domains over $k$ with $\text{trdeg}(A_i) = 1$, $i = 1, 2$. If $A_1 \cong A_2$, then $A_1 \cong A_2$. Moreover, if $f : A_1 \rightarrow A_2$ and $A_1 \not\cong k[1]$, then $f$ restricts to an isomorphism of $A_1$ onto $A_2$.

We feel compelled to again mention the geometric content of Corollary 3.1.

Corollary 3.3. Let $k$ be an algebraically closed field. Let $V_1$ and $V_2$ be affine curves over $k$. If $V_1 \times k^n \cong V_2 \times k^n$, then $V_1 \cong V_2$.

Proof. In algebraic terms it means $A_1^n \cong A_2^n$, where $V_i = \text{Spec}(A_i)$, $i = 1, 2$, and we must check that $A_1 \cong A_2$. \qed

Remarks. (1) The conclusion of Corollary 3.1 is still true for some more general choices of $B$. For example, if $\text{AK}(B) \cong k[1]$ and $A_1 \otimes B \cong A_2 \otimes B$, then we can apply AK twice to find that $A_1 \cong A_2$. (There are several surfaces known to have AK invariant $k[1]$ or $k[2]$.) For any choice of $B$ with finite transcendence degree, we can apply AK to $A_i \otimes B$ several times in an attempt to show cancellation. Through each application of AK, the transcendence degree of the second factor of the tensor product will decrease, unless it is rigid. So $B$ must be rigid in any cancellation counterexample with minimal dimension.

(2) As mentioned earlier, Corollary 3.2 is false when we increase the transcendence degree of $A_1$ and $A_2$. The following example is due to W. Danielewski [Dan]. Let $A_n$ be the coordinate ring of the surface $x^n y = z^2 - 1$ over the
complex numbers $\mathbb{C}$. Then $A_1 \not\cong A_2$ while $A_1^{[1]} \cong A_2^{[1]}$. In fact, $A_i \not\cong A_j$ whenever $i \neq j$, but $A_i^{[1]} \cong A_j^{[1]}$ for all $i, j$. These domains also provide a counterexample to the formula of our main result when $A \cong \mathbb{C}^{[1]}$. One can prove that $\text{AK}(A_1) = \mathbb{C}$ and $\text{AK}(A_2) = \mathbb{C}[x]$ [ML2 ML3]. Now, $\text{AK}(A_2^{[1]}) \cong \text{AK}(A_1^{[1]}) \subseteq \text{AK}(A_1) = \mathbb{C}$. Thus $\text{AK}(\mathbb{C}^{[1]} \otimes A_2) = \mathbb{C}$ while $\text{AK}(\mathbb{C}^{[1]} \otimes \text{AK}(A_2) = \mathbb{C}[x])$.

(3) In the special case $B = k^{[1]}$, we can extend the theorem (and hence its corollaries) to some non-algebraically closed fields. Suppose $F$ is a perfect field. (In particular, $F$ could be any characteristic 0 field.) Let $k$ be an algebraic closure of $F$. Let $A$ be an affine domain over $F$ with $\text{trdeg}_F(A) = 1$. It is known that if $A \otimes_F k \cong k^{[1]}$ then $A \cong F^{[1]}$ [Asa]. Using this fact we can easily check that $\text{AK}(A^{[1]}) = \text{AK}(A)$ by considering the extension of scalars $A \otimes_F k$ and applying the theorem.

4 Proof of the main result

Let $k$ be an algebraically closed field. Let $A$ be an affine domain over $k$ with $\text{trdeg}(A) = 1$ which is not isomorphic to $k^{[1]}$. By Lemma 2.3, $\text{AK}(A) = A$. Let $B$ be a domain over $k$. We will view $A$ and $B$ as subalgebras of $A \otimes B$ in the natural way. It is well known that the tensor product of two affine domains is again an affine domain [Har]. Now suppose $z \in A \otimes B$ is a zero divisor. Write $z = \sum_{i=1}^m a_i \otimes b_i$. Then $z$ belongs to the affine domain $A \otimes k[b_1, \ldots, b_m]$. By Theorem 36, Chapter III, of Zariski and Samuel [ZS], $z$ is a zero divisor of this subdomain, and so $z = 0$. Therefore, $A \otimes B$ is a domain, and the lemmas of the previous section apply to it.

Let us note that this next lemma does not require that $A$ have transcendence degree 1. It still holds true for an affine domain $A$ of any (necessarily finite) transcendence degree.

**Lemma 4.1.** If $A \subseteq \text{AK}(A \otimes B)$ then $\text{AK}(A \otimes B) = \text{AK}(A) \otimes \text{AK}(B)$.

**Proof.** We need to show that $\text{AK}(B) \subseteq \text{AK}(A \otimes B)$. Let $b \in \text{AK}(B)$ and suppose that $\varphi(b) \not\neq b$ for some $\varphi \in \text{Exp}(A \otimes B)$. Let $f \in A \otimes B$ denote the leading $t$-coefficient of $\varphi(b)$. Write $f = \sum_{m} a_m \otimes b_m$, where the set $\{b_m\}$ is linearly independent over $k$. Let $\{g_1, \ldots, g_n\}$ be a finite generating set of $A$ over $k$. Since $k$ is infinite, there exists a choice of values $c_i \in k$ for each $g_i$ such that evaluation of $g_i$ at $c_i$ is a well-defined homomorphism whose kernel does not include the element $a_1$. (In other words, there exists a point $(c_1, \ldots, c_n) \in \text{Spec}(A)$ which is not a zero of the regular function $a_1$.) Let $\sigma : A \otimes B \to B$ denote the map which sends $g_i \in A$ to $c_i$, $i = 1, \ldots, n$, leaving all elements of $B$ fixed. Let $\psi = \sigma \varphi$. We claim that $\psi \in \text{Exp}(B)$. It is clear that $\psi$ is a $k$-homomorphism and that $\varepsilon_{\varphi} \psi$ is the identity on $B$. Thus the $t\text{'}$-coefficients define a locally finite higher derivation $\{\sigma D^i\}$ on $B$, and it remains to check that this derivation is iterative. This follows routinely from the iterative property of the higher derivation associated to $\varphi$ along with the fact that $\varphi(a) = a$ for all $a \in A$. We
choose the value $c$ the product $k$ choice results in a $g$ and hence $i$ for each $t$ is algebraic over $k$. In order to embed $k$ of the coefficients of $A$ generated by all of these coefficients. Then $A$ is a subalgebra of $\mathbb{E}[x]$ which is not contained in $E$. Since $E$ is finitely generated over $k$, we can write $E = k[t_1, \ldots, t_k][\alpha_1, \ldots, \alpha_l]$ where elements $t_1, \ldots, t_k$ transcendental over $k$, $\alpha_1$ is algebraic over $k[t_1, \ldots, t_k]$, and $\alpha_i$ is algebraic over $k[t_1, \ldots, t_k][\alpha_1, \ldots, \alpha_{i-1}]$ for each $i = 2, \ldots, l$. We will choose values in $k$ for $t_1, \ldots, t_k$ and $\alpha_1, \ldots, \alpha_l$ in order to embed $A$ in $k[x]$. For this specialization process to work successfully, we must choose these values so that certain bad situations do not occur. First, we must insure that the relations satisfied by each $\alpha_i$ is not sent to zero. Next, we must insure that the relations satisfied by each $\alpha_i$ do not become contradictory (e.g. $0 = 1$). Finally, we do not want all elements of $A$ to become elements of $k$. Let us now outline how to avoid these bad situations.

Each $g_i$ is a polynomial in $x$ with coefficients in $E$. Consider these coefficients as fractions with numerators in $k[t_1, \ldots, t_k][\alpha_1, \ldots, \alpha_l]$ and denominators in $k[t_1, \ldots, t_k]$. Let $d$ be a common denominator for all of these fractions. We will require $d$ to not become 0.

Each $\alpha_i$ satisfies a polynomial $P_i$ with coefficients in $k[t_1, \ldots, t_k]$. Let $\overline{P_i}$ denote the polynomial with coefficients in $k$ that will be obtained by specializing the coefficients of $P_i$. We will choose the values of $t_1, \ldots, t_k$ so that the coefficients of $\overline{P_i}$ are nonzero. Then we can chose a specialization value for $\alpha_i$ to be a root of $\overline{P_i}$ (since $k$ is algebraically closed). In this way, the relations satisfied by each $\alpha_i$ will not become contradictions. Let $p_i$ denote the product of the coefficients of $P_i$ $(i = 1, \ldots, l)$. As long as $p_i$ does not become zero, none of the coefficients of $P_i$ can become zero.

We can assume that the generator $g_1 \in E[x]$ has $x$-degree at least 1. We want $g_1$ to specialize to a nonconstant polynomial in $k[x]$. The leading $x$-coefficient of $g_1$ satisfies a polynomial $P_0$ with coefficients in $k[t_1, \ldots, t_k]$. Let $p_0$ be the constant coefficient of $P_0$. We can assume $p_0 \neq 0$. If we specialize in a way that $p_0$ does not become 0, then the leading $x$-coefficient of $g_1$ will not become 0, and hence $g_1$ will specialize to a nonconstant polynomial.

Since $k$ is infinite, there exists a $k$-tuple $(c_1, \ldots, c_k)$ which is not a zero of the product $d p_0 \cdots p_i \in k[t_1, \ldots, t_k]$, and hence not a zero of any of its factors. Choose the value $c_i$ for $t_i$, $i = 1, \ldots, k$. By virtue of the above discussion, this choice results in a $k$-homomorphism $A \rightarrow k[x]$ which sends $g_1$ to a nonconstant
polynomial. Since the image of this map has transcendence degree 1 over \( k \), the kernel must be trivial. \( \square \)

Viewing \( A \) as a subalgebra of \( k[x] \), Liouville's theorem \( \text{[isa]} \) implies that the fraction field \( \text{Frac}(A) \) of \( A \) is a simple extension of \( k \). Combining this with the above lemma, we can choose the generator of \( \text{Frac}(A) \) over \( k \), say \( y \), to be a polynomial in \( x \). From this it follows that \( A \subseteq k[y] \). We will carry this assumption through the remainder of the proof.

**Lemma 4.3.** There exists a nonzero element \( u \in A \) with the following properties:

i. The ideal \( k[y]u \) of \( k[y] \) is contained in \( A \).

ii. If \( a \in A \) such that \( k[y]a \subseteq A \), then \( u \) divides \( a \) in \( k[y] \).

iii. If \( f \in A \otimes B \) such that the ideal \( (k[y] \otimes B)f \) of \( k[y] \otimes B \) is contained in \( A \otimes B \), then \( u \) divides \( f \) in \( k[y] \otimes B \).

**Proof.** Write \( y = gh^{-1} \) where \( g, h \in A \) and \( h \) is a monic polynomial in \( y \) with degree \( n \). Let us check that \( k[y]h^{n-1} \) is contained in \( A \). For \( m = 0, \ldots, n-1 \) we have \( y^m h^{n-1} = g^m h^{n-m-1} \in A \). Now let \( m \geq n \) and suppose that \( y^n h^{n-1} \in A \) for \( 0 \leq l < m \). Since \( h = y^n + (\text{terms of degree less than } n) \), we can write \( y^m = y^{m-n} h + p_m(y) \), where \( p_m(y) \) has degree at most \( m-1 \). Then \( y^m h^{n-1} = y^{m-n} h^n + p_m(y) h^{n-1} \in A \) by assumption. By induction, \( y^m h^{n-1} \in A \) for all \( m \geq 0 \). So the ideal \( k[y]h^{n-1} \) is contained in \( A \). Let \( \mathfrak{a} \) be the (nontrivial) ideal generated by all ideals of \( k[y] \) that are contained in \( A \). Let \( u \) be the generator of \( \mathfrak{a} \). It is clear that \( u \) has properties (i) and (ii). Suppose \( f \in A \otimes B \) and \( (k[y] \otimes B)f \subseteq A \otimes B \). Write \( f = \sum_i a_i \otimes b_i \) for some \( a_i \in A \), \( b_i \in B \), where the set \( \{b_i\} \) is linearly independent over \( k \). Let \( q \in k[y] \). Since \( qf = \sum_i (qa_i) \otimes b_i \in A \otimes B \), we have \( qa_i \in A \) for each \( i \). Element \( q \) was arbitrary, and so property (ii) implies that \( u \) divides each \( a_i \) in \( k[y] \), and thus \( u \) divides \( f \) in \( k[y] \otimes B \). \( \square \)

An exponential map \( \varphi : A \otimes B \to (A \otimes B)[t] \) is uniquely extended to a homomorphism \( \varphi : \text{Frac}(A) \otimes B \to \text{Frac}((A \otimes B)[t]) \) by setting \( \varphi(a^{-1}) = \varphi(a)^{-1} \) for all \( a \in A \setminus 0 \). Remark that this extension of \( \varphi \) retains the property that \( \varepsilon_0 \varphi \) is the identity map, where \( \varepsilon_0 \) is evaluation at \( t = 0 \).

**Lemma 4.4.** Let \( \varphi : A \otimes B \to (A \otimes B)[t] \) be an exponential map. Let \( \varphi \) also denote the unique extension \( \text{Frac}(A) \otimes B \to \text{Frac}((A \otimes B)[t]) \). Then \( \varphi(k[y] \otimes B) \subseteq (k[y] \otimes B)[t] \).

**Proof.** It suffices to show that \( \varphi(y) \in (k[y] \otimes B)[t] \). Let \( D \) be the locally finite iterative higher derivation associated to \( \varphi \). As mentioned earlier, \( D \) has a unique extension to a higher derivation on any given localization of \( A \otimes B \). In particular, there is a unique way to define the derivation on \( y \). Write \( y = gh^{-1} \) for some \( g, h \in A \). In any extension of \( D \), each \( D^i(y) \) is found as some expression of elements from \( \{D^i(g)\} \) and \( \{D^i(h)\} \), divided by some power of \( h \). Therefore, each \( D^i(y) \) belongs to \( k(y) \otimes B \).
Since $y$ is integral over $A$, $\varphi(y)$ must be integral over $(A \otimes B)[t]$. By viewing $(A \otimes B)[t]$ as a subring of $B[y, t]$, we see that $\varphi(y)$ must belong to $\text{Frac}(B)[y, t]$, i.e. $\varphi(y) \in (k[y] \otimes \text{Frac}(B))[t]$. Thus we can restrict the codomain of our extension of $\varphi$, so that we have $\varphi : \text{Frac}(A) \otimes B \to (k[y] \otimes \text{Frac}(B))[t]$. This extension of $\varphi$ defines an extension of $D$ with each $D^j(y)$ belonging to $k[y] \otimes \text{Frac}(B)$. Combining this with our previous observation, we have $D^j(y) \in k[y] \otimes B$ for each $j$. Hence $\varphi(y) \in (k[y] \otimes B)[t]$.

**Lemma 4.5.** Let $u \in A$ be as in the statement of Lemma 4.3. Suppose $\varphi \in \text{EXP}(A \otimes B)$ and let $D = \{D^i\}$ be the locally finite iterative higher derivation associated to $\varphi$. Then $u$ divides $D^n(u)$ in $k[y] \otimes B$ for each natural number $n$.

**Proof.** We will use induction to show that $(k[y] \otimes B)D^n(u) \subseteq A \otimes B$ and then appeal to Lemma 4.3. If $n = 0$ then $D^n(u) = u$ and the result is true by our choice of $u$. Suppose the result is true for $D^i(u)$, where $0 \leq i < n$. Let $q \in k[y] \otimes B$. By Lemma 4.3, we can extend $\varphi$ to a homomorphism $k[y] \otimes B \to (k[y] \otimes B)[t]$. Thus we can write $\varphi(q) = \sum_i D^i(q)t^i$, where $D^i(q) \in k[y] \otimes B$ for all $i \geq 0$ and $D^0(q) = q$. Now $u, uq \in A \otimes B$, and so $D^i(u), D^i(uq) \in A \otimes B$ for all $i \geq 0$. The Leibniz rule yields

$$D^n(uq) = \sum_{i=0}^{n-1} D^i(u)D^{n-1-i}(q) + D^n(u)q.$$  

By the induction hypothesis, each term in the summation belongs to $A \otimes B$. Thus $D^n(uq) \in A \otimes B$ as well. We have now verified that $(k[y] \otimes B)D^n(u) \subseteq A \otimes B$ for each natural number $n$. The claim follows from Lemma 4.3. □

We are now in position to complete the proof. Since $A \not\subseteq \text{AK}(A \otimes B)$, there exists $\varphi \in \text{EXP}(A \otimes B)$ for which $A \cap (A \otimes B)^\varphi = k$. Let $u \in A$ be as in the statement of Lemma 4.3. Let $n = \deg_\varphi(u)$. Then $D^n(u) \in (A \otimes B)^\varphi \setminus 0$ by part (a) of Lemma 2.1. By Lemma 4.3, we have $D^n(u) = uq$ for some nonzero $g \in k[y] \otimes B$. Both $u$ and $ug^2$ are nonzero elements of $A \otimes B$, and $u \cdot ug^2 = D^2(u)g^2 \in (A \otimes B)^\varphi \setminus 0$. By part (a) of Lemma 2.1, $u \in (A \otimes B)^\varphi$. Therefore, $u \in k$. Since $k[y]u \subseteq A$, we have $A = k[y]$. This contradicts our assumption on $A$.

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