On $p$-adic entropy of some solenoid dynamical systems

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1 Introduction

Let $d$ be a positive integer. For a $\mathbb{Z}^d$-action on a set $X$, the periodic entropy of the $\mathbb{Z}^d$-action on $X$ is defined by

$$h(X) := \lim_{n \to \infty} \frac{1}{[\mathbb{Z}^d : (n\mathbb{Z})^d]} \log |\text{Fix}_{(n\mathbb{Z})^d}(X)|$$

if the limit exists. Here $\text{Fix}_{(n\mathbb{Z})^d}(X)$ is the set of fixed points of the $(n\mathbb{Z})^d$-action on $X$. If $X$ is a compact metrizable abelian group and the $\mathbb{Z}^d$-action on $X$ is continuous, one can define other entropies, topological entropy and measure theoretic entropy (with respect to the normalized Haar measure on $X$). See [10, Chapter V] and [8, Appendix A] for the definition of these entropies.

We consider the action of $\mathbb{Z}^d$ on a discrete abelian group $L_d(\mathbb{Z}) := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ given by

$$\delta \cdot \sum_{v \in \mathbb{Z}^d} a_v t^v = \sum_{v \in \mathbb{Z}^d} a_v t^{v + \delta}$$

(1)

for $\delta \in \mathbb{Z}^d$ and $\sum_{v \in \mathbb{Z}^d} a_v t^v \in L_d(\mathbb{Z})$. Here, for $t = (t_1, \ldots, t_d)$ and $v = (v_1, \ldots, v_d)$, define $t^v = t_1^{v_1} \cdots t_d^{v_d}$. Then the Pontryagin dual $\hat{L}_d(\mathbb{Z})$ is a compact abelian group and the action on $L_d(\mathbb{Z})$ induces the action on $\hat{L}_d(\mathbb{Z})$. For a fixed $f \in L_d(\mathbb{Z})$, the above action induces the $\mathbb{Z}^d$-actions on $L_d(\mathbb{Z})/f L_d(\mathbb{Z})$ and its Pontryagin dual $X_f := (L_d(\mathbb{Z})/f L_d(\mathbb{Z}))^\wedge$. In [8, Theorem 3.1], Lind, Schmidt and Ward showed that the (topological) entropy of $X_f$ is given by

$$h(X_f) = m(f)$$

for $0 \neq f \in L_d(\mathbb{Z})$. Here $m(f)$ is the Mahler measure of $f$, which is defined by

$$m(f) := \frac{1}{(2\pi \sqrt{-1})^d} \int_{T^d} \log |f(z_1, \ldots, z_d)| \frac{dz_1}{z_1} \cdots \frac{dz_d}{z_d} \in \mathbb{R},$$
and
\[ T^d = \{ (z_1, \ldots, z_d) \in \mathbb{C}^d \mid |z_1| = \cdots = |z_d| = 1 \} \] (2)
is the \( d \)-torus. Note that the periodic entropy of \( X_f \) exists and coincides with the topological entropy of \( X_f \) if and only if \( f \) does not vanish on \( T^d \) ([5], [8]).

Let \( K \) be a number field with the ring \( \mathcal{O}_K \) of the integers and \( \mathbb{Q} / f \) equal \( f \in L_d(\mathcal{O}_K) := \mathcal{O}_K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \). In [7], Einsiedler extended this theorem to the case \( X_f := (L_d(\mathcal{O}_K)/fL_d(\mathcal{O}_K))^\wedge \) and proved that
\[ h(X_f) = m(N_{K/\mathbb{Q}}(f)) \]
holds. Here
\[ N_{K/\mathbb{Q}}(f) = \prod_{\tau: K \hookrightarrow \mathbb{C}} \tau(f) \in L_d(\mathbb{Z}). \]

Let \( p \) be a prime, \( \mathbb{C}_p \) be the completion of \( \mathbb{Q}_p \) with \( |p|_p = p^{-1} \) and \( \log_p \) be the \( p \)-adic logarithm with \( \log_p(p) = 0 \). In [4], Deninger introduced the \( p \)-adic entropy as a \( p \)-adic analogue of (periodic) entropy. A simple definition of the \( p \)-adic entropy is the following (see [4] for the detailed definition).

**Definition 1.1.** Assume that \( \mathbb{Z}^d \) acts on a set \( X \). If the limit
\[ h_p(X) := \lim_{N \rightarrow \infty} \frac{1}{N^d} \sum_{n \in \mathbb{Z}^d} \log_p |\text{Fix}_{(n\mathbb{Z})^d}(X)| \]
exists, we call \( h_p(X) \) the \( p \)-adic entropy.

**Definition 1.2.** Let \( f \in \mathbb{C}_p[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \). If the limit
\[ m_p(f) = \lim_{N \rightarrow \infty} \frac{1}{N^d} \sum_{\xi \in \mu_N} \log_p f(\xi) \] (3)
exists, we call \( m_p(f) \) the \( p \)-adic Mahler measure of \( f \). Here
\[ \mu_N = \{ z \in \mathbb{C}_p \mid z^N = 1 \}. \]

Note that the R.H.S. of (3) is a \( p \)-adic analogue of the line integral on the \( p \)-adic \( d \)-torus
\[ T^d_p = \{ (z_1, \ldots, z_d) \in \mathbb{C}_p^d \mid |z_1|_p = \cdots = |z_d|_p = 1 \} \]
(4)
See [1] for details.
Deninger proved Theorem 1.3 as a $p$-adic analogue of Lind-Schmidt-Ward’s theorem.

**Theorem 1.3.** [4, Theorem 1.1] Let $f \in L_d(\mathbb{Z})$ and assume that $f$ does not vanish at any $z \in T_p^d$. Then the $p$-adic entropy $h_p(X_f)$ of the $\mathbb{Z}^d$-action on $X_f$ exists and we have

$$h_p(X_f) = m_p(f).$$

The first aim in this paper is to prove the following theorem as a $p$-adic analogue of Einsiedler’s theorem.

**Theorem 1.4.** Let $K$ be a number field and $O_K$ be the ring of the integers of $K$. Assume that $f \in L_d(O_K)$ and $N_{K/Q}(f)$ does not vanish at any point of the $p$-adic torus $T_p^d$. Then the $p$-adic entropy $h_p(X_f)$ of the $\mathbb{Z}^d$-action on $X_f = (L_d(O_K)/fL_d(O_K))^\wedge$ exists and we have

$$h_p(X_f) = m_p(N_{K/Q}(f)).$$

Here, the action on $X_f$ is induced by the action on $L_d(O_K)$ as (1).

We will prove Theorem 1.4 in Section 2.

In Section 3, we will discuss the notion of the $p$-adic expansiveness, following Bräuer [2], to explain the statement of our main result. The expansiveness of the action on the dynamical system is classical. For example, it is well-known that for an expansive $\mathbb{Z}^d$-action the notions of topological entropy, measure theoretic entropy and periodic entropy coincide [8, Theorem A.1]. Because we do not have $p$-adic analogues of topological entropy nor measure theoretic entropy, the $p$-adically expansiveness seems to be important.

In Section 4, we consider the $p$-adic entropy of solenoidal automorphisms. Let $S \subset P$ a subset of the set of all primes $P$. The solenoid $\Sigma_S$ is defined to be the Pontryagin dual of the discrete abelian group $\mathbb{Z}[1/S]$. In particular, the Pontryagin dual of $\mathbb{Z}[1/P] = \mathbb{Q}$ is called the full solenoid and denoted by $\Sigma$. We fix a matrix $A \in \text{GL}_m(\mathbb{Z}[1/S])$. The $\mathbb{Z}$-action on $\mathbb{Z}[1/S]^m$ defined by $A$ induces the action on $\Sigma_S^m$ (see Section 4 for detail). Lind and Ward showed that the (measure theoretic) entropy of the $\mathbb{Z}$-action on $\Sigma^m$ as above is given by

$$h(\Sigma^m) = \sum_{l \leq \infty} \sum_{|\lambda_l| > 1} \log |\lambda_l| \in \mathbb{R}$$

(5)

where $l \leq \infty$ means that $l$ runs over all places of $\mathbb{Q}$ and $\lambda$ runs over all eigenvalues of $A$ with $|\lambda_l| > 1$ [9, Theorems 1, 2]. Our goal is to obtain a $p$-adic analogue.
of this theorem. However, it is not straightforward because the entropy in Lind-
Ward’s theorem is measure theoretic and we do not have $p$-adic analogue of mea-
sure theoretic entropy. Furthermore, if $S$ is infinite, above $\mathbb{Z}$-action on $\Sigma^m_S$ is neither
expansive nor $p$-adically expansive. Now, we will modify the dynamical syst em to
make the action $p$-adically expansive and prove the following theorem.

**Theorem 1.5.** Let $S$ be a finite set of primes and we fix $A \in \text{GL}_m(\mathbb{Z}[1/S])$ and
embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ for all $p \in S$. We see $\mathbb{Z}[1/S]^m$ as a $\mathbb{Z}[t^\pm]$-module with
the structure induced by above $\mathbb{Z}$-action on $\mathbb{Z}[1/S]^m$. Assume that the eigenvalues
$\lambda_1, \ldots, \lambda_m$ of $A$ satisfy $|\lambda_k|_p \neq 1$ for all $k = 1, \ldots, m$ and all $p \in S$. Then the
following properties hold :

1. As a $\mathbb{Z}[t^\pm]$-module, $\mathbb{Z}[1/S]^m$ is finitely generated and the $\mathbb{Z}$-action on $\Sigma^m_S$ is
   $p$-adically expansive.

2. The $p$-adic entropy $h_p(\Sigma^m_S)$ of the $\mathbb{Z}$-action exists for all $p \in S$ and we have
   $$h_p(\Sigma^m_S) = \sum_{l \in S} \sum_{|\lambda_l|_p > 1} \log_p |\lambda_l|_l + \sum_{|\lambda_l|_p > 1} \log_p \lambda_l.$$ 

Here, we see $\lambda_k \in \mathbb{C}_p$ under the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ for all $p \in S$.

**Notation:** In this paper, let $p$ be a prime, $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$ with the
norm $|p|_p = p^{-1}$ and $\log_p$ be the $p$-adic logarithm with $\log_p p = 0$. For a positive
integer $d$ and a commutative ring $A$, we write $L_d(A) = A[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. For a locally
compact abelian group $M$, we denote its Pontryagin dual by $\hat{M}$ or $M^\wedge$.

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## 2 Proof of Theorem 1.4

In this section, we will prove Theorem 1.4.

**Lemma 2.1.** Let $L/K$ be a finite extension of fields and $V$ be a finite dimensional
$L$-vector space. For any $f \in \text{End}_L(V)$, we have

$$\det_K f = N_{L/K}(\det_L f).$$
The proof of Lemma 2.1 is given by an elementary linear algebra and hence omitted.

**Corollary 2.2.** Let \( K \) be a number field and \( F_1 \) (resp. \( F_2 \)) be the fractional field of \( L_d(\mathbb{Q}) \) (resp. \( L_d(K) \)). For any \( f \in \text{GL}_d(F_2) \), we have

\[
N_{K/\mathbb{Q}}(\det_{F_2} f) = \det_{F_1} f.
\]

Here, for \( \text{Hom}_\mathbb{Q}(K, \mathbb{Q}) = \{ \tau_1, \ldots, \tau_r \} \) and \( P = \sum_{v \in \mathbb{Z}^d} a_v t^v \in F_2 \), we define \( \tau_i(P) = \sum_{v \in \mathbb{Z}^d} \tau_i(a_v) t^v \in F_2 \), \( N_{K/\mathbb{Q}}(P) = \prod_{i=1}^r \tau_i(P) \).

**Proof.** We may identify \( \text{Hom}_{\mathbb{Q}}(F_2, \overline{F_1}) \) and \( N_{F_2/F_1} \) with \( \text{Hom}_\mathbb{Q}(K, \overline{\mathbb{Q}}) \) and \( N_{K/\mathbb{Q}} \) respectively. Using Lemma 2.1, we get

\[
\det_{F_1} f = \det_{F_1}(\det_{F_2} f) = N_{K/\mathbb{Q}}(\det_{F_2} f).
\]

\( \square \)

**Theorem 2.3.** [4, Theorem 3.2] Assume that \( f \in M_d(L_d(\mathbb{Z})) \) and \( \det f \) does not vanish at any point of the \( p \)-adic \( d \)-torus \( T^d_p \) given by (4). Then the \( p \)-adic entropy \( h_p(X_f) \) of the \( \mathbb{Z}^d \)-action on \( X_f := (L_d(\mathbb{Z})^t / f L_d(\mathbb{Z})^t)^\wedge \) exists and we have

\[
h_p(X_f) = m_p(\det f).
\]

Here, the \( \mathbb{Z}^d \)-action on \( X_f \) is induced by the \( \mathbb{Z}^d \)-action on \( L_d(\mathbb{Z})^t \) given by

\[
\delta \left( \sum_{v \in \mathbb{Z}^d} a_v^{(1)} t^v, \ldots, \sum_{v \in \mathbb{Z}^d} a_v^{(r)} t^v \right) = \left( \sum_{v \in \mathbb{Z}^d} a_v^{(1)} t^{v+\delta}, \ldots, \sum_{v \in \mathbb{Z}^d} a_v^{(r)} t^{v+\delta} \right)
\]

and \( h_p(X_f) \) is given by Definition 1.1.

Theorem 1.4 follows immediately from the following theorem.

**Theorem 2.4.** Let \( K \) be a number field and \( O_K \) be the ring of integers of \( K \). Assume that \( f \in M_d(L_d(\mathbb{Q})) \) and \( N_{K/\mathbb{Q}}(\det f) \) does not vanish at any point of the \( p \)-adic \( d \)-torus \( T^d_p \). Then the \( p \)-adic entropy \( h_p(X_f) \) of the \( \mathbb{Z}^d \)-action on \( X_f := (L_d(O_K)^t / f L_d(\mathbb{Q})^t)^\wedge \) exists and we have

\[
h_p(X_f) = m_p(N_{K/\mathbb{Q}}(\det f)).
\]

Here, the \( \mathbb{Z}^d \)-action on \( X_f \) is induced by the \( \mathbb{Z}^d \)-action on \( L_d(\mathbb{Z})^t \) as (6).
Proof. Let \([K : \mathbb{Q}] = r\). For \(f \in M_{r}(L_{d}(O_{K}))\), we define \(L_{d}(\mathbb{Z})\)-linear homomorphism

\[\varphi_{f} : L_{d}(O_{K})^{s} \to L_{d}(O_{K})^{s}\]

by

\[x \mapsto fx\]

and let \(A_{f} \in M_{rs}(L_{d}(\mathbb{Z}))\) be its matrix representation. Since

\[L_{d}(O_{K})^{s}/f \cdot L_{d}(O_{K})^{s} = \text{Coker}(\varphi_{f}) \cong \text{Coker}(A_{f}) = L_{d}(\mathbb{Z})^{rs}/A_{f}L_{d}(\mathbb{Z})^{rs},\]

we get

\[\left|\text{Fix}_{(a \mathbb{Z})^{s}} \left(L_{d}(O_{K})^{s}/f \cdot L_{d}(O_{K})^{s}\right)\right| = \left|\text{Fix}_{(a \mathbb{Z})^{s}} \left(L_{d}(\mathbb{Z})^{rs}/A_{f}L_{d}(\mathbb{Z})^{rs}\right)\right|\].

Theorem 2.3 and Corollary 2.2 imply

\[h_{p}(X_{f}) = h_{p}(X_{A_{f}}) = m_{p}(\det A_{f}) = m_{p}(N_{K/\mathbb{Q}}(\det f)).\]

\[\square\]

Example 2.5. Let \(K = \mathbb{Q}(\sqrt{2})\) and \(f = 3t + \sqrt{2} \in O_{K}[t^{\pm 1}]\). Since the equation

\[N_{K/\mathbb{Q}}(f) = 9t^{2} - 2 = 0\]

has the roots \(t = \pm \sqrt{2}/3\), we have

\[\left|\frac{\sqrt{2}}{3}\right|_{p} = \begin{cases} 1 & (p \neq 2, 3) \\ 2^{-\frac{1}{2}} & (p = 2) \\ 3 & (p = 3). \end{cases}\]

If \(p = 2, 3\), there exists the \(p\)-adic entropy of \(f\) and we have

\[h_{2}(X_{f}) = m_{2}(9t^{2} - 2) = \log_{2} 9 \in \mathbb{C}_{2},\]

\[h_{3}(X_{f}) = m_{3}(9t^{2} - 2) = \log_{3} 2 \in \mathbb{C}_{3}.\]

Note that

\[h(X_{f}) = m(9t^{2} - 2) = \log 9 \in \mathbb{R}.\]
3 \textit{p-adically expansiveness}

In this section, we will recall the (classical) expansiveness and explain the \textit{p-adically expansiveness}, which is introduced by Br"auer. Let $\Gamma$ be a countable discrete group in this section.

\textbf{Definition 3.1.} Let $X$ be a compact metrizable topological space and assume that $\Gamma$ acts on $X$. A continuous action of $\Gamma$ on $X$ is expansive if the following holds: There is a metric $d$ defining the topology of $X$ and some $\epsilon > 0$ such that for every pair of distinct points $x \neq y$ in $X$ there exists an element $\gamma \in \Gamma$ with $d(\gamma x, \gamma y) \geq \epsilon$.

It is known that if the action of $\Gamma$ on $X$ is expansive, for every cofinite normal subgroup $\Lambda$ of $\Gamma$, $\text{Fix}_\Lambda(X)$ is finite. Moreover, for the case $\Gamma = \mathbb{Z}^d$ the following holds.

\textbf{Theorem 3.2.} \cite[Theorem A.1]{8} Let $X$ be a compact metrizable abelian group and assume that $\mathbb{Z}^d$ acts expansively on $X$. Then, topological entropy, measure theoretic entropy and periodic entropy coincide.

To describe the \textit{p-adically expansiveness}, we consider equivalent conditions to the expansiveness.

\textbf{Definition 3.3.} If $M$ is a discrete left $\mathbb{Z}[\Gamma]$-module, the $\Gamma$-action on $M$ induces the action on the Pontryagin dual $X := \hat{M}$. Conversely, if $X$ is a compact $\Gamma$-module, the $\mathbb{Z}[\Gamma]$-module structure of $M := \hat{X}$ is induced. In particular, for $f \in \mathbb{Z}[\Gamma]$ and $M = \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f$, we write $X = X_f := \hat{M}$.

\textbf{Theorem 3.4.} \cite[Theorem 3.2]{6} Let $f \in \mathbb{Z}[\Gamma]$. Then, the $\Gamma$-action on $X_f$ is expansive if and only if $f \in L^1(\Gamma)^\times$. Here, $L^1(\Gamma)$ is an algebra given by

$$L^1(\Gamma) = \left\{ w = (w_\gamma)_{\gamma \in \Gamma} \in \prod_{\Gamma} \mathbb{R} \left| \sum_{\gamma \in \Gamma} |w_\gamma| < \infty \right. \right\}.$$

\textbf{Theorem 3.5.} \cite[Theorem 3.1]{3} Let $X$ be a compact $\Gamma$-module. Then, the $\Gamma$-action on $X$ is expansive if and only if $M = X$ is a finitely generated $\mathbb{Z}[\Gamma]$-module satisfying $L^1(\Gamma) \otimes_{\mathbb{Z}[\Gamma]} M = 0$.

In the case $\Gamma = \mathbb{Z}^d$, the following holds.

\textbf{Theorem 3.6.} \cite[Corollary 3.2]{3}, \cite[Theorem 6.5]{10} Let $X$ be a compact metrizable $\mathbb{Z}^d$-module and assume that $M = \hat{X}$ is a finitely generated $L_d(\mathbb{Z}) = \mathbb{Z}[t^\pm_1, \ldots, t^\pm_d] = \mathbb{Z}[\mathbb{Z}^d]$-module. Then, the following conditions are equivalent:

1. The $\mathbb{Z}^d$-action on $X$ is expansive.
2. For every $p \in \text{Ass}(M)$, we have $V_C(p) \cap T^d = \emptyset$.

3. The module $M$ is $S_\infty$-torsion, where $S_\infty \subset L_d(\mathbb{Z})$ is the multiplicative system $S_\infty = L_d(\mathbb{Z}) \cap L^1(\mathbb{Z}^d)^\times$.

Here, $T^d$ is the $d$-torus given by (2) and define
\[
\text{Ass}(M) = \{ p \in \text{Spec}L_d(\mathbb{Z}) | p = \text{ann}(a) \text{ for some } a \in M \},
\]
and for $p \in \text{Spec}L_d(\mathbb{Z})$ and for $K = \mathbb{C}$ or $\overline{Q}_p$,
\[
V_K(p) = \{ z \in (K^\times)^d | f(z) = 0 \text{ for every } f \in p \}.
\]

We will summarize the properties of the $p$-adically expansiveness by considering these theorems.

**Definition 3.7.** The algebra $c_0(\Gamma)$ is defined by
\[
c_0(\Gamma) := \left\{ \sum_{\gamma \in \Gamma} x_\gamma \gamma \in \mathbb{Q}_p[[\Gamma]] \left| [x_\gamma]_p \to 0 \text{ as } \gamma \to \infty \right. (\ast) \right\},
\]
where (\ast) means that for any $\epsilon > 0$ there exists a finite set $S \subset \Gamma$ such that $[x_\gamma]_p < \epsilon$ holds for every $\gamma \in \Gamma \setminus S$. For two elements $\sum_{\gamma \in \Gamma} x_\gamma \gamma$ and $\sum_{\gamma \in \Gamma} y_\gamma \gamma \in c_0(\Gamma)$, the product is given by
\[
\sum_{\gamma \in \Gamma} x_\gamma \gamma \cdot \sum_{\gamma \in \Gamma} y_\gamma \gamma := \sum_{\gamma \in \Gamma} \left( \sum_{\delta \in \Gamma} x_\delta y_{\delta - \gamma} \right) \gamma \in c_0(\Gamma).
\]
The algebra $c_0(\Gamma)$ is equipped with a norm
\[
\| \cdot \| : c_0(\Gamma) \to \mathbb{R}_{\geq 0} ; \sum_{\gamma \in \Gamma} x_\gamma \gamma \mapsto \max_{\gamma \in \Gamma} [x_\gamma]_p.
\]

One can check that $(c_0(\Gamma), \| \cdot \|)$ is a $p$-adic Banach algebra over $\mathbb{Q}_p$. See [4, Section 2] for the definition of the $p$-adic Banach algebra and details of $c_0(\Gamma)$. Note that we see $c_0(\Gamma)$ as a $p$-adic analogue of $L^1(\Gamma)$ and we have the correspondence
\[
L^1(\Gamma) \leftrightarrow c_0(\Gamma) \subset \subset C[\Gamma] \leftrightarrow \mathbb{Q}_p[\Gamma]
\]
between the archimedian algebra and non-archimedian one. We replace $L^1(\Gamma)$ with $c_0(\Gamma)$ in Theorem 3.4, Theorem 3.5 and Theorem 3.6 as follow.
**Definition 3.8.** Let $X$ be an abelian group and $p$ be a prime. Then $X$ is said to have bounded $p$-torsion if there exists an integer $i_0 \geq 0$ such that $\text{Ker}(p^i : X \to X) = \text{Ker}(p^{i_0} : X \to X)$ for all $i \geq i_0$.

**Theorem 3.9.** [5, Theorem 14] Let $f \in \mathbb{Z}[\Gamma]$. The following conditions are equivalent:

1. The abelian group $X_f = (\mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f)^\wedge$ has bounded $p$-torsion.

2. There exists an element $g \in c_0(\Gamma)$ such that $gf = 1$.

3. For $M_f = \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f$, we have $c_0(\Gamma) \otimes_{\mathbb{Z}[\Gamma]} M_f = 0$.

In this case, $\text{Fix}_N(X_f)$ is finite for any cofinite normal subgroup $N \subset \Gamma$.

**Remark 3.10.** The condition 2 in Theorem 3.9 implies that $f \in c_0(\Gamma)^\times$. In other words, if $fg = 1$ for $f, g \in c_0(\Gamma)$, then we have $gf = 1$. This answers a question raised by Deninger [5, Section 3]. Indeed, for $f = \sum_{\gamma \in \Gamma} x_\gamma \gamma, g = \sum_{\gamma \in \Gamma} y_\gamma \gamma$, since $fg = \sum_{\gamma \in \Gamma} x_\gamma \gamma \cdot \sum_{\gamma \in \Gamma} y_\gamma \gamma = \sum_{\gamma \in \Gamma} \left( \sum_{\delta \in \Gamma} x_\delta y_{\delta^{-1}} \gamma \right) \gamma = 1,$

it follows that

\[
\begin{align*}
\sum_{\delta \in \Gamma} x_\delta y_{\delta^{-1}} &= \begin{cases} 1 & (\gamma = 1) \\
0 & (\gamma \neq 1). \end{cases} 
\end{align*}
\]  

(7)

Now, since

\[
gf = \sum_{\gamma \in \Gamma} \left( \sum_{\delta \in \Gamma} y_\delta x_{\delta^{-1}} \gamma \right) \gamma = \sum_{\gamma \in \Gamma} \left( \sum_{\delta \in \Gamma} x_\delta y_{\gamma \delta^{-1}} \gamma \right),
\]

it is enough to show that

\[
\begin{align*}
\sum_{\delta \in \Gamma} x_\delta y_{\gamma \delta^{-1}} &= \begin{cases} 1 & (\gamma = 1) \\
0 & (\gamma \neq 1). \end{cases} \end{align*}
\]  

(8)

The first equation in (8) follows from the first one in (7). We obtain the second equation in (8) by substituting $\delta y_{\delta^{-1}}(\neq 1)$ for $\gamma$ in the second one in (7).

**Definition 3.11.** [2] The $\Gamma$-action on $X_f$ is $p$-adically expansive if either condition 1-3 in Theorem 3.9 is satisfied.
The following theorem is a \( p \)-adic analogue of Theorem 3.6.

**Theorem 3.12.** [2, Proposition 4.19, Proposition 4.22] Let \( X \) be a compact \( \mathbb{Z}^d \)-module and assume that \( M = \hat{X} \) is a finitely generated \( L_d(\mathbb{Z}) \)-module. Then, the following conditions are equivalent:

1. The abelian group \( X \) has bounded \( p \)-torsion.
2. For every \( p \in \text{Ass}(M) \), we have \( V_{\mathbb{Z}_p}(p) \cap T_p^d = \emptyset \).
3. The module \( M \) is \( S \)-torsion, where \( S = L_d(\mathbb{Z}) \cap c_0(\mathbb{Z}^d)^\times \).

Here, \( T_p^d \) is the \( p \)-adic torus given by (4).

**Definition 3.13.** [2] The \( \mathbb{Z}^d \)-action on \( X \) is \( p \)-adically expansive if either condition 1-3 in Theorem 3.12 is satisfied. This is compatible with Definition 3.11.

**Remark 3.14.** Does there exist \( p \)-adic entropy if the action is \( p \)-adically expansive? This is not true in general. An example can be found in [2, Example 7.1]. However there exists an example of a \( p \)-adically expansive \( \mathbb{Z}^d \)-action whose \( p \)-adic entropy exists [4, Theorem 1.1]. It is an open problem to define a better notion of \( p \)-adic entropy or \( p \)-adically expansiveness. See [2] and [5] for detail.

### 4 \( p \)-adic entropy of solenoids

In this section, we will prove Theorem 1.5. First, we will explain the dynamical system which we consider. Let \( m \in \mathbb{N} \). For a subset \( S \) of primes, let \( M = \mathbb{Z}[1/S]^m, X = \Sigma_S^m (\Sigma_S = \mathbb{Z}[1/S]^\times \) is a solenoid\) and fix \( A \in \text{GL}_m(\mathbb{Z}[1/S]) \). The \( \mathbb{Z} \)-action on \( M \) given by

\[
 n \cdot x := A^n x
\]

induces the \( \mathbb{Z} \)-action on \( X \). As mentioned in Section 3, \( M \) also has the \( \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}] \)-module structure defined by

\[
 f \cdot x := f(A)x.
\]

We will consider a \( p \)-adic analogue of Lind-Ward’s theorem (see (5)) by modifying the dynamical system and prove Theorem 1.5.

**Lemma 4.1.** Let \( S \) be a finite set of primes and \( A \in M_m(\mathbb{Z}[1/S]) \). Assume that \( \det A \neq 0 \). Then we have

\[
 \left| \mathbb{Z} \left[ \frac{1}{S} \right]^m / A\mathbb{Z} \left[ \frac{1}{S} \right]^m \right| = \prod_{l \in S} |\det A_l| : |\det A|.
\]
We may assume that \( e_1, \ldots, e_m \in \mathbb{N} \) and \( e_i \) is prime to \( p \) for all \( 1 \leq i \leq m \) and all \( p \in S \). Because

\[
\mathbb{Z} \left[ \frac{1}{S} \right]^m / A \mathbb{Z} \left[ \frac{1}{S} \right]^m = \prod_{i=1}^{m} \mathbb{Z} \left[ \frac{1}{S} \right] / e_i \mathbb{Z} \left[ \frac{1}{S} \right] \cong \prod_{i=1}^{m} \mathbb{Z} / e_i \mathbb{Z},
\]

we have

\[
\left| \mathbb{Z} \left[ \frac{1}{S} \right]^m / A \mathbb{Z} \left[ \frac{1}{S} \right]^m \right| = \prod_{i=1}^{m} e_i = \prod_{l \in S} p^i_0 \cdot |\det A|,
\]

for some \( a_l \in \mathbb{Z} \). On the other hand, since \( e_i \) is prime to \( p \) for all \( 1 \leq i \leq m \) and all \( p \in S \), we see that

\[
v_l(p^i_0 \cdot \det A) = v_l \left( \prod_{l \in S} p^i_0 \cdot |\det A| \right) = v_l \left( \prod_{i=1}^{m} e_i \right) = 0,
\]

where \( v_l \) is the \( l \)-adic valuation. This implies \( a_l = -v_l(\det A) \). \( \Box \)

**Proof of Theorem 1.5.** 1. We write \( M = \mathbb{Z} \left[ 1/S \right]^m \) and \( R = \mathbb{Z}[t^{\pm 1}] \). Let \( e_k \) be the standard basis for \( M \). We will show that

\[
M = Re_1 + \cdots + Re_m,
\]

but it is enough to show that for all \( 1 \leq k \leq m \) and \( \{jl\}_{l \in S} \subset \prod_{l \in S} \mathbb{Z}_{\geq 0} \), \( \left( \prod_{l \in S} l^{-j_l} \right) e_k \) is contained in the R.H.S. of (9). Put \( \chi_A(u) = \det(uI - A) = \sum_{i=0}^{m} \alpha_i u^i \in \mathbb{Z} \left[ 1/S \right][u] \). Then the assumption is equivalent to the condition that for all \( p \in S \) the Newton polygon of \( \chi_A \) does not have the segments with slope 0. This means that there exists exactly one index which maximizes \( \{ |\alpha_m|_p, |\alpha_{m-1}|_p, \ldots, |\alpha_0|_p \} \). Denote the index which maximizes \( \{ |\alpha_m|_p, |\alpha_{m-1}|_p, \ldots, |\alpha_0|_p \} \) by \( i_0 \) and we write \( \alpha_{i_0} = p^{-\epsilon_0} \beta_{i_0} \), where \( \beta_{i_0} \in \mathbb{Z} \left[ 1/S \right], |\beta_{i_0}|_p = 1 \) and \( e_{i_0} \in \mathbb{Z}_{\geq 0} \). Using the Cayley-Hamilton theorem, we get

\[
\sum_{0 \leq i \leq m} \alpha_i A^i + p^{-\epsilon_0} \beta_{i_0} A^{i_0} = 0.
\]
Here, we identify $A^0$ with the identity matrix $I$. Multiplying this by some rationals and $A^{-i_0}$, we obtain

$$p^{-1} \gamma_{i_0} I = - \sum_{i \neq i_0} c_i A^{i-i_0},$$

where $\gamma_{i_0} = \beta_{i_0} \prod_{l \in S}^{-f_i} c_i = p^{e_{i_0} - \alpha_{i_0}} \prod_{l \in S}^{-f_i}$ for all $i \neq i_0$ and $f_i = \min \{ \nu_l(\alpha_i), 0 \}$ for all $l \in S$. We can check that $\gamma_{i_0}, c_i \in \mathbb{Z}$ and $\gamma_{i_0}$ is prime to $p$. Since $\gamma_{i_0}$ is prime to $p$, we can take $x_{i_0}, y_{i_0} \in \mathbb{Z}$ satisfying $\gamma_{i_0} x_{i_0} + py_{i_0} = 1$. Thus we get

$$p^{-1} I = p^{-1}(\gamma_{i_0} x_{i_0} + py_{i_0}) I$$

$$= -x_{i_0} \sum_{i \neq i_0} c_i A^{i-i_0} + y_{i_0} I.$$

We take $g_p(t) = -x_{i_0} \sum_{i \neq i_0} c_i t^{i-i_0} + y_{i_0} \in \mathbb{Z}[t^{\pm 1}]$ and then it follows that

$$\left( \prod_{p \in S} g_p(A)^{j_p} \right) e_k = \left( \prod_{l \in S} \Gamma_l^{-h_l} \right) e_k$$

2. Using Lemma 4.1, we compute

$$| \text{Fix}_{\mathbb{Z}}(X) | = | \ker (I - A^n) |$$

$$= | (\text{Coker}(I - A^n))^\wedge |$$

$$= \left| \mathbb{Z} \left[ \frac{1}{S} \right] / (I - A^n) \mathbb{Z} \left[ \frac{1}{S} \right] \right|$$

$$= \prod_{l \in S} \Gamma_l^{-\nu_l(\det(I - A^n))} \cdot \prod_{k=1}^m (1 - A_k^n)$$

and

$$\frac{1}{n} \log_p \left( \prod_{l \in S} \Gamma_l^{-\nu_l(\det(I - A^n))} \cdot \prod_{k=1}^m (1 - A_k^n) \right)$$

$$= - \sum_{l \in S, l \neq p} \frac{1}{n} \nu_l(\det(I - A^n)) \log_p l + \sum_{k=1}^m \frac{1}{n} \log_p (1 - A_k^n).$$
We can check that the second term of (11) is convergent to \( \sum_{|\lambda_k|>1} \log_p \lambda_k \) as \( n \to \infty \).

In the first term of (11), since
\[
\begin{cases}
  v_l(1 - \lambda_k^n) = 0 & \text{if } |\lambda_k| < 1 \\
  v_l(1 - \lambda_k^n) = v_l(\lambda_k^n) = n \cdot v_l(\lambda_k) & \text{if } |\lambda_k| > 1,
\end{cases}
\]

it follows that

the first term of (11) = \(- \sum_{l \in S} \sum_{k=1}^m v_l(1 - \lambda_k^n) \frac{1}{n} \log_p l \)

= \(- \sum_{l \in S} \sum_{|\lambda_k|>1} v_l(\lambda_k) \log_p l \)

= \sum_{l \in S} \sum_{|\lambda_k|>1} \log_p |\lambda_k|.

\( \square \)

**Example 4.2.** Let \( S = \{2, 3\} \) and
\[
A = \begin{pmatrix} \frac{1}{6} & 1 \\ -1 & 0 \end{pmatrix} \in \text{GL}_2 \left( \mathbb{Z} \left[ \frac{1}{6} \right] \right).
\]

The eigenvalues of \( A \) are \( \lambda_{\pm} = \frac{1 \pm \sqrt{-143}}{12} \). Since \( |\lambda_{\pm}|_p = p^{\pm 1} \) for \( p = 2, 3 \), \( p \)-adic entropies are as follow:

\[
\begin{align*}
  h_2(A; X) &= \log_2 3 + \log_2 \lambda_+ \in \mathbb{C}_2, \\
  h_3(A; X) &= \log_3 2 + \log_3 \lambda_+ \in \mathbb{C}_3.
\end{align*}
\]

Note that
\[
h(A; X) = \log 2 + \log 3 = \log 6 \in \mathbb{R}.
\]

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