MATCHING CONDITIONS IN RELATIVISTIC
ASTROPHYSICS

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Abstract

We present an exact electrovacuum solution of Einstein-Maxwell equations with infinite sets of multipole moments which can be used to describe the exterior gravitational field of a rotating charged mass distribution. We show that in the special case of a slowly rotating and slightly deformed body, the exterior solution can be matched to an interior solution belonging to the Hartle-Thorne family of approximate solutions. To search for exact interior solutions, we propose to use the derivatives of the curvature eigenvalues to formulate a $C^3$—matching condition from which the minimum radius can be derived at which the matching of interior and exterior spacetimes can be carried out. We prove the validity of the $C^3$—matching in the particular case of a static mass with a quadrupole moment. The corresponding interior solution is obtained numerically and the matching with the exterior solution gives as a result the minimum radius of the mass configuration.

Keywords: Matching conditions; Astrophytical Compact Objects.
I. INTRODUCTION

In general terms, the group of massive compact objects is defined in astrophysics to contain black holes, neutron stars, white dwarfs, planet-like compact objects, and other exotic dense stars. The main common characteristic of compact objects is that they are small for their mass. In the case of weak gravitational fields, Newtonian gravity provides an adequate physical description of compact objects. However, for the study of very massive objects like black holes and neutron stars it is necessary to take into account relativistic effects. Moreover, exotic objects like pulsars and quasars can be investigated only in the framework of relativistic astrophysics. Furthermore, the accuracy of modern navigation techniques made it necessary to take into account relativistic effects of the gravitational field of the Earth and other planets of the Solar system.

In general relativity, the gravitational field of compact objects must be described by a metric satisfying Einstein’s equations

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} , \]  

in the interior part of the object \((T_{\mu\nu} \neq 0)\) as well as outside in empty space \((T_{\mu\nu} = 0)\). Since at the classical level the most important sources of gravity are the mass and charge distributions, it is clear that the problem of finding a metric that describes the gravitational field of compact objects can be divided into four related problems. The first one consists in finding an exact vacuum solution \((R_{\mu\nu} = 0)\) that describes the field corresponding to the mass distribution. Secondly, the charge distribution must be considered by solving the electrovacuum field equations

\[ R_{\mu\nu} = 8\pi \left( F_{\mu\lambda} F^{\lambda}_{\nu} - \frac{1}{4} g_{\mu\nu} F^{\lambda\tau} F_{\lambda\tau} \right) , \]  

where \(F_{\mu\nu}\) is the Faraday tensor corresponding to a charge distribution \(Q(x^\mu)\). As for the interior region, it is necessary to propose a model to describe the internal gravitational structure of the object. The most popular model is that of a perfect fluid

\[ T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - g_{\mu\nu} p , \]  

with density \(\rho(x^\mu)\), pressure \(p(x^\mu)\), and 4-velocity \(u^\mu\). Finally, the fourth part of the problem consists in considering the mass and charge distribution simultaneously, i.e. one needs an
exact solution to the equations

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}(\rho, p, Q), \]  

where a specific model must be proposed for the energy-momentum tensor. Each of the above problems is very difficult to solve because Einstein’s field equations are highly non-linear. To overcome in part these difficulties it is necessary to assume that the gravitational field is invariant with respect to certain transformations. In particular, one can assume that the field is stationary and axially symmetric, conditions that are in good agreement with observations. In fact, compact objects show in general a rotation that is constant over quite long periods of time, and their shape deviates only slightly from spherical symmetry.

In this work, we review a family of electrovacuum solutions that describe the exterior gravitational and electromagnetic fields of a rotating compact object. This represents a solution to the first two problems mentioned above. It turns out that in the limit of a weak gravitational field, a particular solution of this family can be matched smoothly with an interior approximate solution. This opens the possibility of searching for an exact interior solution that could be matched with the exact exterior solution. As a particular example, we consider the case of a static mass with quadrupole moment, and search for a physically meaningful interior solution. The problem of matching the solutions arises immediately. It is generally accepted that to match an interior and an exterior solution one first should determine a matching surface \( \Sigma \). If the metrics and their second order derivatives coincide on \( \Sigma \), the matching is reached in the sense that the curvature of the interior spacetime region passes smoothly into the exterior region. This can be considered as a \( C^2 \)–matching. In this procedure, however, the determination of \( \Sigma \) remains unclear. In the investigation of the example of an interior solution with quadrupole moment reported in this work, we noticed that at certain distance from the origin of coordinates the curvature eigenvalues present a non-expected behavior in the sense that they change their sign several times passing through points of maximum and minimum local curvature. This behavior could be associated with the existence of a repulsive gravitational potential. It then seems plausible that to avoid repulsive gravity one can cover the region of repulsion with an interior solution. The location of \( \Sigma \) can then be anywhere outside the region of repulsion that, in turn, can be invariantly defined in terms of the derivatives of the curvature eigenvalues. This can be considered as a \( C^3 \)–matching procedure. In this work, we use this procedure to match the exterior field
II. THE EXTERIOR SOLUTION

Assuming that astrophysical compact objects satisfy the conditions of stationarity and axial symmetry, their gravitational and electromagnetic fields can be described by the line element

\[ ds^2 = f(dt - \omega d\varphi)^2 - \frac{\sigma^2}{f} \left[ e^{2\gamma}(x^2 - y^2) \left( \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2) d\varphi^2 \right] , \]  

where \((t, \varphi, x, y)\) are prolate spheroidal coordinates, and the functions \(f, \omega\) and \(\gamma\) depend on \(x\) and \(y\) only. In general terms, we can say that a solution of the Einstein-Maxwell field equations can be considered as a candidate to describe the exterior field of a compact object if the metric functions can be written with the functional dependence

\[ f = f(x, y, \tilde{P}_n), \quad \omega = \omega(x, y, \tilde{P}_n), \quad \gamma = \gamma(x, y, \tilde{P}_n), \]

where \(\tilde{P}_n = (M_n, J_n, Q_n, H_n)\), \(n = 0, 1, 2, \ldots\), represent invariant sets of multipole moments associated with all possible sources of gravitation, namely, \(M_n\) are the gravitoelectric, \(J_n\) the gravitomagnetic, \(Q_n\) the electric, and \(H_n\) the magnetic moments. A family of solutions satisfying this property was derived in Ref. [2].

It was obtained by using the solution generating techniques which are based upon the symmetries of the field equations in the Ernst representation

\[ (\xi \xi^* - \mathcal{F} \mathcal{F}^* - 1) \nabla^2 \xi = 2(\xi^* \nabla \xi - \mathcal{F}^* \nabla \mathcal{F}) \nabla \xi , \]

\[ (\xi \xi^* - \mathcal{F} \mathcal{F}^* - 1) \nabla^2 \mathcal{F} = 2(\xi^* \nabla \xi - \mathcal{F}^* \nabla \mathcal{F}) \nabla \mathcal{F}, \]

where the gravitational potential \(\xi\) and the electromagnetic \(\mathcal{F}\) Ernst potentials are defined as

\[ \xi = \frac{1 - f - \Omega}{1 + f + \Omega} , \quad \mathcal{F} = 2 \frac{\Phi}{1 + f + \Omega} , \]

with \(\sigma(x^2 - 1)\Omega_x = f^2 \omega_y\), and \(\sigma(1 - y^2)\Omega_y = -f^2 \omega_x\). Moreover, the potential \(\Phi\) is determined uniquely once the electromagnetic potentials \(A_t\) and \(A_\varphi\) are given. Finally, \(\nabla\) represents the gradient operator in prolate spheroidal coordinates.

The explicit form of the solution can be written as

\[ \xi = \frac{(a_+ + ib_+) e^{2b_0} + a_- + ib_-}{(a_+ + ib_+) e^{2b_0} - a_- - ib_-} (1 - c_0^2 + g_0^2)^{1/2} , \quad \Phi = \frac{e_0 + ig_0}{1 + \xi} , \]
where

\[ \hat{\psi} = \sum_{n=1}^{\infty} (-1)^n q_n P_n(y) Q_n(x) \]  

\[ a_\pm = (x \pm 1)^{\delta-1} [x(1 - \lambda \mu) \pm (1 + \lambda \mu)] , \]  

\[ b_\pm = (x \pm 1)^{\delta-1} [y(\lambda + \mu) \mp (\lambda - \mu)] , \]

with

\[ \lambda = \alpha_1 (x^2 - 1)^{1-\delta} (x+y)^{2\delta-2} e^{2\delta \sum_{n=1}^{\infty} (-1)^n q_n \beta_n} , \]  

\[ \mu = \alpha_2 (x^2 - 1)^{1-\delta} (x-y)^{2\delta-2} e^{2\delta \sum_{n=1}^{\infty} (-1)^n q_n \beta_n} , \]

and

\[ \beta_{n\pm} = (\pm 1)^n \left[ \frac{1}{2} \ln \left( \frac{x \mp y}{x^2 - 1} \right) - Q_1(x) \right] + P_n(y) Q_{n-1}(x) \]

\[ - \sum_{k=1}^{n-1} (\pm 1)^k P_{n-k}(y) [Q_{n-k+1}(x) - Q_{n-k-1}(x)] . \]

Here \( P_n(y) \) and \( Q_n(x) \) represent the Legendre polynomials and functions of second kind, respectively. The constant parameters \( e_0, g_0, \sigma, \alpha_1, \alpha_2, q_n, \) and \( \delta \) determine the gravitational and electromagnetic multipole moments. The metric functions \( f \) and \( \omega \) can be obtained from the definitions of the Ernst potentials, whereas the function \( \gamma \) can be calculated by quadratures once \( f \) and \( \omega \) are known. In general, this solution is asymptotically flat and free of singularities along the axis of symmetry, \( y = 1 \), outside certain region situated close to the origin of coordinates. The sets of infinite multipole moments can be chosen in such a way as to reproduce the shape of ordinary axially symmetric compact objects.

One of the most interesting solutions contained in this family is the one with non-vanishing parameters \( q_0 = 1, q_2 = q, \delta, \alpha_1 = \alpha_2 = (\sigma - m)/a \), where \( m \) and \( a \) are new constants. Consequently, the solution possesses the following independent parameters: \( m, a, \delta, \) and \( q \).

In the limiting case \( \alpha = 0, a = 0, q = 0 \) and \( \delta = 1 \), the only independent parameter is \( m \) and the Ernst potential \( g_0 \) determines the Schwarzschild spacetime. Moreover, for \( \alpha = a = 0 \) and \( q = 0 \) we obtain the Ernst potential of the Zipoy-Voorhees (ZV) \( [4, 5] \) static solution which is characterized by the parameters \( m \) and \( \delta \). Furthermore, for \( \alpha = a = 0 \) and \( \delta = 1 \), the resulting solution coincides with the Erez-Rosen (ER) static spacetime \( [6] \). The Kerr metric is also contained as a special case for \( q = 0 \) and \( \delta = 1 \). The physical significance of the parameters entering this particular solution can be established in an invariant manner by calculating the relativistic Geroch–Hansen \( [7, 8] \) multipole moments. We use here the
procedure formulated in Ref. [9] which allows us to derive the gravitoelectric \( M_n \) as well as the gravitomagnetic \( J_n \) multipole moments. A lengthy but straightforward calculation yields

\[
M_{2k+1} = J_{2k} = 0, \quad k = 0, 1, 2, \ldots
\]  

\[
M_0 = m + \sigma(\delta - 1)
\]  

\[
M_2 = \frac{2}{15} \sigma^3 \delta q - \frac{1}{3} \sigma^3 (\delta^3 - 3\delta^2 - 4\delta + 6) - m\sigma^2 \delta(\delta - 2) - 3m^2 \sigma(\delta - 1) - m^3,
\]

\[
J_1 = ma + 2a\sigma(\delta - 1),
\]

\[
J_3 = \frac{4}{15} a\sigma^3 \delta q
\]

\[
- a \left[ \frac{2}{3} \sigma^3 (\delta^3 - 3\delta^2 - \delta + 3) + m\sigma^2 (3\delta^2 - 6\delta + 2) + 4m^2 \sigma(\delta - 1) + m^3 \right].
\]

The even gravitomagnetic and the odd gravitoelectric multipoles vanish identically because the solution possesses and additional reflection symmetry with respect to the hyperplane \( y = 0 \) which can be interpreted as the equatorial plane. Higher odd gravitomagnetic and even gravitoelectric multipoles can be shown to be linearly dependent since they are completely determined in terms of the parameters \( m, a, q \) and \( \delta \). From the above expressions we see that the ZV parameter \( \delta \) enters explicitly the value of the total mass \( M_0 \) as well as the angular momentum \( J_1 \) of the source. The mass quadrupole \( M_2 \) can be interpreted as a nonlinear superposition of the quadrupoles corresponding to the ZV, ER and Kerr spacetimes. A generalization of the Kerr metric which includes an arbitrary quadrupole moment is obtained by imposing the condition \( \delta = 1 \). The resulting multipoles are

\[
M_{2k+1} = J_{2k} = 0, \quad k = 0, 1, 2, \ldots
\]

\[
M_0 = m, \quad M_2 = -ma^2 + \frac{2}{15} qm^3 \left(1 - \frac{a^2}{m^2}\right)^{3/2}, \ldots
\]

\[
J_1 = ma, \quad J_3 = -ma^3 + \frac{4}{15} qm^3 a \left(1 - \frac{a^2}{m^2}\right)^{3/2}, \ldots
\]

It is interesting to note that this particular exact solution in the limit \( a \to m \) leads to the spacetime of an extreme Kerr black hole, regardless of the value of the quadrupole parameter \( q \).

In the limiting static case of the ZV metric the only non-vanishing parameters are \( m = \sigma \) and \( \delta \) so that all gravitomagnetic multipoles vanish and we obtain

\[
M_0 = m\delta, \quad M_2 = \frac{1}{3} \delta m^3 (1 - \delta^2),
\]
for the leading gravitoelectric multipoles. Consequently, the ZV solution represents the exterior gravitational field of a static deformed body.

In all the above special solutions the electromagnetic field vanishes identically. One can easily obtain the corresponding electrovacuum generalizations by assuming that $e_0 \neq 0$ and $g_0 \neq 0$. The computation of the respective electromagnetic multipole moments can be performed in an invariant manner and the result can be expressed as

$$E_n = e_0 M_n , \quad H_n = g_0 J_n .$$

(25)

This means that the charge distribution resembles the mass distribution. The electric moments vanish identically if no mass distribution exists. This result is in accordance with our physical intuitive interpretation of a charge distribution. The magnetic moments turn out to be proportional to the gravitomagnetic multipoles, with no magnetic monopole. This is a physical reasonable result in the sense that the magnetic field is generated by the motion of the charge distribution, in the present case, by the rotation of the compact object.

It is worth noticing that in all the above special solutions we assumed that $\alpha_1 = \alpha_2$ and obtained generalizations of the Kerr metric with arbitrary quadrupole moment. More general solutions can be obtained by relaxing this condition. Consider, for instance, the special solution with $\delta = 1$, $q_0 = 1$, $q_i = 0$ for $i = 1, 2, \ldots$, and

$$\alpha_1 = \frac{\sigma - m}{a + \frac{l}{\eta}} , \quad \alpha_2 = \frac{\sigma - m}{a - \frac{l}{\eta}} , \quad \sigma^2 = \frac{m^2 + l^2}{\eta^2} - a^2 , \quad \eta = \frac{1}{\sqrt{1 - e_0^2}} .$$

(26)

The resulting potential corresponds to the charged Kerr-Taub-NUT spacetime with total charge $Q_0 = me_0$, where $l$ is the Taub-NUT parameter.

### III. MATCHING WITH AN APPROXIMATE INTERIOR SOLUTION

In general relativity, the gravitational field of a compact object is described by a Riemannian manifold which must be well defined in the entire spacetime. Usually, the exterior metrics are characterized by the presence of curvature singularities in a very specific region of spacetime. A possibility to eliminate those singularities is to find a physically meaningful regular interior solution that covers the singular region. If the two solutions can be matched smoothly on some hypersurface $\Sigma$, one can say that the entire manifold is well defined. In this section we will see that a particular solution contained in the above family determines a well-defined manifold in the entire spacetime of a particular rotating compact object.
Consider the solution (9) with the special choice
\[ e_0 = g_0 = 0, \quad q_0 = 1, \quad q_1 = 0, \quad q_2 = q, \quad \delta = 1 - q, \]
(27)
i.e., we consider a vacuum solution with a ZV parameter given in terms of the quadrupole parameter. Moreover, the angular momentum parameters are chosen as
\[ \alpha_1 = \alpha_2 = \frac{\sigma - m}{a}, \quad \sigma^2 = m^2 - a^2. \]
(28)
Furthermore, let us assume that the condition for a slowly rotating slightly deformed compact object is satisfied, i.e.,
\[ |q_j| << |q|, \quad j = 3, 4, \ldots, \quad \frac{a^2}{m^2} << 1. \]
(29)
Expanding the resulting potential up to the first order in \( q \) and up to the second order in \( a/m \), and introducing spherical-like coordinates \( R = R(x, y) \) and \( \Theta = \Theta(x, y) \), a stationary axisymmetric spacetime is obtained whose line element can be written as
\[
ds^2 = \left(1 - \frac{2M}{R}\right) \left[1 + 2k_1 P_2(\cos \Theta) + 2 \left(1 - \frac{2M}{R}\right)^{-1} \frac{J^2}{R^4} (2 \cos^2 \Theta - 1)\right] dt^2
- \left(1 - \frac{2M}{R}\right)^{-1} \left[1 - 2k_2 P_2(\cos \Theta) - 2 \left(1 - \frac{2M}{R}\right)^{-1} \frac{J^2}{R^4}\right] dR^2
- R^2 [1 - 2k_3 P_2(\cos \Theta)] (d\Theta^2 + \sin^2 \Theta d\phi^2) + \frac{4J}{R} \sin^2 \Theta dt d\phi,
\]
(30)
where
\[ k_1 = \frac{J^2}{MR^3} \left(1 + \frac{M}{R}\right) + \frac{5Q - J^2/\mathcal{M}^2}{8} \frac{2}{\mathcal{M}} Q^2_2 \left(\frac{R}{\mathcal{M}} - 1\right), \]
\[ k_2 = k_1 - \frac{6J^2}{R^4}, \]
\[ k_3 = k_1 + \frac{J^2}{R^4} - \frac{5Q - J^2/\mathcal{M}}{4} \left(1 - \frac{2M}{R}\right)^{-1/2} \frac{2}{\mathcal{M}} Q^1_2 \left(\frac{R}{\mathcal{M}} - 1\right). \]
Here \( Q^m_l \) are the associated Legendre functions of the second kind
\[ Q^1_2(x) = (x^2 - 1)^{1/2} \left[\frac{3x^2 - 2}{x^2 - 1} - \frac{3}{2} x \ln \frac{x + 1}{x - 1}\right], \]
(31)
\[ Q^2_2(x) = \frac{3}{2} (x^2 - 1) \ln \frac{x + 1}{x - 1} + \frac{5x - 3x^3}{x^2 - 1}. \]
(32)
The metric (30) represents the exterior gravitational field of a slowly and rigidly rotating compact object in which the deviation from spherical symmetry has been taken into account.
only up to the first order in the quadrupole parameter $q$. The main gravitational multipoles are the total mass $\mathcal{M}$, the total angular momentum $J$ and the quadrupole moment $Q$ which reduce to

$$\mathcal{M} = m(1 - q), \quad J = -ma, \quad Q = \frac{J^2}{m} - \frac{4}{5} m^3 q . \quad (33)$$

For the internal structure of the compact object one can assume a perfect fluid model, satisfying a one-parameter equation of state, $p = p(\rho)$. The fact that we are limiting ourselves to the case of slow rotation simplifies the problem. In fact, one can first solve the non-rotating case and then linearize the field equations with respect to the angular velocity $\Omega$ which is assumed to be uniform. The 4-velocity $u^\mu$ can be identified with the velocity of an observer moving with the particles of the fluid so that $u^R = u^\Theta = 0$, and $u^\varphi = \Omega u^t$. The explicit integration of Einstein’s equations depends on the particular choice of the density function $\rho(R)$, and on the equation of state. For the sake of simplicity, we consider here only the simplest case with $\rho = \text{const}$ so that the pressure $p(R)$ must be determined by the field equations. Then, in the limiting case of a non-rotating object the total mass is $\mathcal{M} = 4\pi \rho \mathcal{R}^3 / 3$, where $\mathcal{R}$ is the radius of the body. Under these conditions, the resulting line element can be written as

$$ds^2 = (1 + 2\Phi) dt^2 - \left[ 1 + 2R \frac{d\Phi_0(R)}{dR} + \Phi_2(R) P_2(\cos \Theta) \right] dR^2$$

$$- R^2 [1 + 2\Phi_2(R) P_2(\cos \Theta)] [d\Theta^2 + \sin^2 \Theta (d\varphi - \tilde{\omega} dt)^2], \quad (34)$$

where

$$\Phi = \Phi_0(R) + \Phi_2(R) P_2(\cos \Theta), \quad (35)$$

is the interior Newtonian potential. Here $\Phi_0$ is the interior Newtonian potential for the non-rotating configuration and $\Phi_2(R)$ is the perturbation due to the rotation. Moreover, $\tilde{\omega} = \Omega - \bar{\omega}$ is the angular velocity of the local inertial frame, where $\Omega = \text{const}$ is the angular velocity of the fluid and $\bar{\omega} = \bar{\omega}(R)$ is the angular velocity of the fluid relative to the inertial frame.

It can be shown that the interior solution (34) can be matched smoothly on the surface $R = \mathcal{R}$ with the exterior solution (30) in the special case of an unperturbed configuration with $\Phi_2 = 0$ and

$$\Phi_0(R) = -2\pi \rho \left( \mathcal{R}^2 - \frac{R^2}{3} \right), \quad (36)$$
where $R$ is the radius of the non-rotating configuration. In the general case of a rotating deformed body, the matching can be performed only numerically. In fact, the solutions of the field equations for the inner distribution of mass are calculated using the matching conditions as boundary conditions. In this manner, it is possible to find realistic solutions that describe the interior and exterior gravitational field of slowly rotating and slightly deformed mass distributions. Several examples of this procedure were originally presented by Hartle and Thorne [12, 13].

The above result shows that in the case of a slowly and rigidly rotating compact object that slightly deviates from spherical symmetry, the solution presented here is physically meaningful and determines a well-defined manifold in the entire spacetime. We interpret this result as a strong indication that in general the solution (9) describes the exterior gravitational field of rotating astrophysical compact bodies with arbitrary sets of gravitational and electromagnetic multipole moments.

IV. MATCHING WITH AN EXACT INTERIOR SOLUTION

Rather few exact stationary solutions that involve a matter distribution in rotation are to be found in the literature. In particular, the interior solution for the rotating Kerr solution is still unknown. In fact, the quest for a realistic exact solution, representing both the interior and exterior gravitational field generated by a self-gravitating axisymmetric distribution of a perfect fluid mass in stationary rotation is considered as a major problem in general relativity. We believe that the inclusion of a quadrupole in the exterior and in the interior solutions adds a new physical degree of freedom that could be used to search for realistic interior solutions. We will study in this section the entire Riemannian manifold corresponding to the simple case of a static exterior solution with only quadrupole moment.

The simplest generalization of the Schwarzschild spacetime which includes a quadrupole parameter can be obtained from the Zipoy–Voorhees solution with $\delta = 1 - q$. The corresponding line element in spherical-like coordinates can be represented as

$$\begin{align*}
    ds^2 &= \left(1 - \frac{2m}{r}\right)^{1-q} dt^2 \\
    &\quad \quad - \left(1 - \frac{2m}{r}\right)^{q} \left[ \left(1 + \frac{m^2 \sin^2 \theta}{r^2 - 2mr}\right)^{q(2-q)} \left(\frac{dr^2}{1 - \frac{2m}{r}} + \frac{r^2 d\theta^2}{r^2} + r^2 \sin^2 \theta d\phi^2\right) \right].
\end{align*}$$

\[37\]
This solution is axially symmetric and reduces to the spherically symmetric Schwarzschild metric in the limit $q \to 0$. It is asymptotically flat for any finite values of the parameters $m$ and $q$. Moreover, in the limiting case $m \to 0$ it can be shown that the metric is flat. This means that, independently of the value of $q$, there exists a coordinate transformation that transforms the resulting metric into the Minkowski solution. From a physical point of view this is an important property because it means that the parameter $q$ is related to a genuine mass distribution, i.e., there is no quadrupole moment without mass. To see this explicitly, we calculate the multipole moments of the solution by using the invariant definition proposed by Geroch [7]. The lowest mass multipole moments $M_n$, $n = 0, 1, \ldots$ are given by

$$M_0 = (1 - q)m, \quad M_2 = \frac{m^3}{3}q(1 - q)(2 - q), \quad (38)$$

whereas higher moments are proportional to $mq$ and can be completely rewritten in terms of $M_0$ and $M_2$. This means that the arbitrary parameters $m$ and $q$ determine the mass and quadrupole which are the only independent multipole moments of the solution. In the limiting case $q = 0$ only the monopole $M_0 = m$ survives, as in the Schwarzschild spacetime. In the limit $m = 0$, with $q \neq 0$, all moments vanish identically, implying that no mass distribution is present and the spacetime must be flat. This is in accordance with the result mentioned above for the metric (38). Furthermore, notice that all odd multipole moments are zero because the solution possesses an additional reflection symmetry with respect to the equatorial plane.

We conclude that the above metric describes the exterior gravitational field of a static deformed mass. The deformation is described by the quadrupole moment $M_2$ which is positive for a prolate mass distribution and negative for an oblate one. Notice that in order to avoid the appearance of a negative total mass $M_0$ the condition $q < 1$ must be satisfied.

A. Matching conditions

In this subsection we analyze several approaches which could be used to determine the matching hypersurface $\Sigma$. Instead of presenting a rigorous analysis, we will present an intuitive method based on the behavior of the curvature and the motion of test particles.

To investigate the structure of possible curvature singularities, we consider the
Kretschmann scalar $K = R_{\mu\nu\lambda\tau}R^{\mu\nu\lambda\tau}$. A straightforward computation leads to
\[
K = \frac{16m^2(1-q)^2(r^2 - 2mr + m^2\sin^2\theta)^{2q^2 - 4q - 1}}{r^4(2-2q+q^2)(1-2m/r)^2(q^2-q+1)}L(r, \theta),
\]
with
\[
L(r, \theta) = 3(r - 2m + qm)^2(r^2 - 2mr + m^2\sin^2\theta) - q(2 - q)\sin^2\theta[q^2 - 2q + 3(r - m)(r - 2m + qm)].
\]
In the limiting case $q = 0$, we obtain the Schwarzschild value $K = 48m^2/r^6$ with the only singularity situated at the origin of coordinates $r \to 0$. In general, one can show that the singularity at the origin, $r = 0$, is present for any values of $q$. Moreover, an additional singularity appears at the radius $r = 2m$ which, according to the metric (38), is also a horizon in the sense that the norm of the timelike Killing tensor vanishes at that radius. Outside the hypersurface $r = 2m$ no additional horizon exists, indicating that the singularities situated at the origin and at $r = 2m$ are naked. Moreover, for values of the quadrupole parameter within the interval
\[
q \in \left(1 - \sqrt{3}/2, 1 + \sqrt{3}/2\right) \setminus \{0\}
\]
a singular hypersurface appears at a distance
\[
r_{\pm} = m(1 \pm \cos\theta)
\]
from the origin of coordinates. This type of singularity is always contained within the naked singularity situated at the radius $r = 2m$, and is related to a negative total mass $M_0$ for $q > 1$. Nevertheless, in the interval $q \in (1 - \sqrt{3}/2, 1] \setminus \{0\}$ the singularity is generated by a more realistic source with positive mass. This configuration of naked singularities is schematically illustrated in Fig. 1.

The analysis of singularities is important to determine the matching hypersurface $\Sigma$. Indeed, in the case under consideration it is clear that $\Sigma$ cannot be situated inside the sphere defined by the radius $r = 2m$. To eliminate all the singularities it is necessary to match the above solution (38) with an interior solution which covers completely the naked hypersurface $r = 2m$.

Another important aspect related to the presence of naked singularities is the problem of repulsive gravity. In fact, it now seems to be established that naked singularities can
FIG. 1: Structure of naked singularities of a spacetime with quadrupole parameter $q$. Plot (a) represents the limiting case of a Schwarzschild spacetime ($q = 0$) with a singularity at the origin of coordinates surrounded by the horizon (dashed curve) situated at $r = 2m$. Once the quadrupole parameter $q$ is included, the horizon transforms into a naked singularity (solid curve) and the central singularity becomes naked as well. This case is illustrated in plot (b). For values of the quadrupole parameter within the interval $q \in \left(1 - \sqrt{3/2}, 1 + \sqrt{3/2}\right) \backslash \{0\}$, two additional naked singularities appear as depicted in plot (c).

appear as the result of a realistic gravitational collapse [14] and that naked singularities can generate repulsive gravity. Currently, there is no invariant definition of repulsive gravity in the context of general relativity, although some attempts have been made by using invariant quantities constructed with the curvature of spacetime [15–17]. Nevertheless, it is possible to consider an intuitive approach by using the fact that the motion of test particles in stationary axisymmetric gravitational fields reduces to the motion in an effective potential. This is a consequence of the fact that the geodesic equations possess two first integrals associated with stationarity and axial symmetry. The explicit form of the effective potential depends also on the type of motion under consideration.

In the case of a massive test particle moving along a geodesic contained in the equatorial plane ($\theta = \pi/2$) of the Zipoy–Voorhees spacetime (38), one can show that the effective potential reduces to

$$V_{\text{eff}}^2 = \left(1 - \frac{2m}{r}\right)^{1-q} \left[1 + \frac{L^2}{r^2} \left(1 - \frac{2m}{r}\right)^{-q}\right],$$

(43)

where $L$ is constant associated to the angular momentum of the test particle as measured by a static observer at rest at infinity. This expression shows that the behavior of the effective potential strongly depends on the value of the quadrupole parameter $q$. This behavior is illustrated in Fig. 2.
FIG. 2: The effective potential for the motion of timelike particles. Plot (a) shows the typical behavior of the effective potential of a black hole configuration with \( q = 0 \). The case of a naked singularity with \( q = 1/2 \) is depicted in plot (b).

Whereas the effective potential of a black corresponds to the typical potential of an attractive field, the effective potential of a naked singularity is characterized by the presence of a barrier which acts on test particles as a source of repulsive gravity. Although this result is very intuitive, the disadvantage of this analysis is that it is not invariant. In fact, a coordinate transformation can be used to arbitrarily change the position of the barrier of repulsive gravity. Moreover, the identification of the spatial coordinate \( r \) as a radial coordinate presents certain problems in the case of metrics with quadrupole moments \[18\]. To avoid this problem we investigate a set of scalars that can be constructed from the curvature tensor and are linear in the parameters that enter the metric, namely, the eigenvalues of the Riemann tensor. Let us recall that the curvature of the Zipoy–Voorhees metric belongs to type I in Petrov’s classification. On the other hand, type I metrics possess three different curvature eigenvalues whose real parts are scalars \[19\]. The explicit calculation of the curvature eigenvalues for this metric shows \[20\] that all of them are real and, consequently, they behave as scalars under arbitrary diffeomorphisms. The resulting analytic expressions are rather cumbersome. For this reason we performed a numerical analysis and found out the main differences between black holes and naked singularities. The results are illustrated in Fig. 3.

We took a particular eigenvalue which represents the qualitative behavior of all the eigenvalues. In the case of a black hole, the eigenvalue diverges near the origin of coordinates,
FIG. 3: Behavior of the curvature eigenvalue on the equatorial plane ($\theta = \pi/2$) of the Zipoy-Voorhees metric. Plot (a) corresponds to a black hole solution with $q = 0$. Plot (b) illustrates the behavior in case of a naked singularity with $q = -2$.

where the curvature singularity is situated, and it decreases rapidly as $r$ increases, tending to zero at spatial infinity. In the case of a naked singularity the situation changes drastically. The eigenvalue vanishes at spatial infinity and then increases as the value of the radial coordinate decreases. At a specific radius $r = r_{min}$, the eigenvalue reaches a local maximum and then rapidly decreases until it vanishes. This oscillatory behavior becomes more frequent as the origin of coordinates is approached. It seems plausible to interpret this peculiar behavior as an invariant manifestation of the presence of repulsive gravity. On the other hand, if one would like to avoid the effects of repulsive gravity, one would propose $r_{min}$ as the minimum radius where the matching with an interior solution should be carried out. If we denote the eigenvalue as $\lambda$, then $r_{min}$ can be defined invariantly by means of the equation

$$\frac{\partial \lambda}{\partial r} \bigg|_{r=r_{min}} = 0.$$  \hfill (44)

Then, the radius $r_{min}$ determines the matching hypersurface $\Sigma$ and one could interpret condition (44) as a $C^3$—matching condition. In concrete cases, one must calculate all possible eigenvalues $\lambda_i$ and all possible points satisfying the matching condition $\partial \lambda_i / \partial r = 0$. The radius $r_{min}$ corresponds then to the first extremum that can be found when approaching the origin of coordinates from infinity. In the next section we will show that this approach can be successfully carried out in the case of the Zipoy–Voorhees metric.
B. An interior solution

In the search for an interior solution that could be matched to the exterior solution with quadrupole moment given in Eq. (38), we found that an appropriate form of the line element can be written as

\[ ds^2 = f \, dt^2 - \frac{e^{2\gamma_0}}{f} \left( \frac{dr^2}{h} + d\theta^2 \right) - \frac{h^2}{f} \, d\varphi^2 , \tag{45} \]

where

\[ e^{2\gamma_0} = (r^2 - 2mr + m^2 \cos^2 \theta) e^{2\gamma(r,\theta)} , \tag{46} \]

and \( f = f(r,\theta), \ h = h(r), \) and \( \mu = \mu(r,\theta) \). This line element preserves axial symmetry and staticity.

The inner structure of the mass distribution with a quadrupole moment can be described by a perfect fluid energy–momentum tensor (3). In general, in order to solve Einstein’s equations completely, pressure and energy must be functions of the coordinates \( r \) and \( \theta \). However, if we assume that \( \rho = \text{const} \), the resulting system of differential equations is still compatible. The assumption of constant density drastically reduces the complexity of the problem. Then, the corresponding field equations reduce to

\[ p_r = -\frac{1}{2}(p + \rho)\frac{f_r}{f}, \quad p_\theta = -\frac{1}{2}(p + \rho)\frac{f_\theta}{f}, \tag{47} \]

\[ \mu_{rr} = -\frac{1}{2h} \left( 2\mu_{\theta\theta} + h_r \mu_r - 32\pi p \frac{\mu e^{2\gamma_0}}{f} \right) , \tag{48} \]

\[ f_{rr} = \frac{f_r^2}{f} - \left( \frac{h_r}{2h} + \frac{\mu_r}{\mu} \right) f_r + \frac{f_\theta^2}{hf} - \frac{\mu_\theta f_r}{\mu h} - \frac{f_\theta^2}{h} + 8\pi \frac{(3p + \rho) e^{2\gamma_0}}{h}. \tag{49} \]

Moreover, the function \( \gamma \) turns out to be determined by a set of two partial differential equations which can be integrated by quadratures once \( f \) and \( \mu \) are known. The integrability condition of these partial differential equations turns out to be satisfied identically by virtue of the remaining field equations.

Although we have imposed several physical conditions which simplify the form of the field equations, we were unable to find analytic solutions. However, it is possible to perform a numerical integration by imposing appropriate initial conditions. In particular, we demand that the metric functions and the pressure are finite at the axis. Then, it is possible to plot all the metric functions and thermodynamic variables. In particular, the pressure behaves as shown in Fig. [4]
FIG. 4: Plot of the inner pressure as a function of the spatial coordinates.

It can be seen that the pressure is finite in the entire interior domain, and tends to zero at certain hypersurface $R(r, \theta)$ which depends on the initial value of the pressure on the axis. Incidentally, it turns out that by increasing the value of the pressure on the axis, the “radius function” $R(r, \theta)$ can be reduced. Furthermore, if we demand that the hypersurface $R(r, \theta)$ coincides with the origin of coordinates, the value of the pressure at that point diverges. From a physical point of view, this is exactly the behavior that is expected from a physically meaningful pressure function.

This solution can be used to calculate numerically the corresponding Riemann tensor and its eigenvalues. As a result we obtain that the solution is free of singularities in the entire region contained within the radius function $R(r, \theta)$. In particular, one of the eigenvalues presents on the equatorial plane the behavior depicted in Fig. 5. All the eigenvalues have a finite value at the symmetry axis and decrease as the boundary surface is approached.

To apply the $C^3$—matching procedure proposed above we compare the behavior of the eigenvalue plotted in Fig. 5 with the corresponding eigenvalue plotted in Fig. 6 using the same scale in both graphics. The result is illustrated in Fig. 6. It then becomes clear that the first possible point where the matching can be performed is exactly at $r_{\text{min}}$ which in this particular case corresponds to $r_{\text{min}} \approx 5 M_0$. This fixes the initial value of the pressure on the axis which is then used to attack the problem of matching the interior and exterior.
FIG. 5: Behavior of the curvature eigenvalue on the equatorial plane \( (\theta = \pi/2) \) of the interior solution.

FIG. 6: Curvature eigenvalues of the interior solution and of the exterior solution with the same scale.

metric functions. In all the cases we analyzed, we obtained a reasonable matching, within the accuracy of the numerical calculations. We repeated the same procedure for different values of the angular coordinate \( (\theta = \pi/4 \text{ and } \theta = 0) \), and obtained that the matching can always be reached by fixing in an appropriate manner the arbitrary constants that enter the metric functions \( f \) and \( \mu \).
V. CONCLUSIONS

In this work, we presented an exact electrovacuum solution of Einstein-Maxwell equations which contains four different sets of multipole moments. An invariant calculation shows that they can be interpreted as the gravitoelectric, gravitomagnetic, electric and magnetic multipole moments. The solution is asymptotically flat and is free of singularities in a region situated around the origin of coordinates. The rotating Kerr metric is contained as a special case. The NUT parameter can also be included by a suitable choice of the arbitrary constants which enter the Ernst potentials. We conclude that this solution can be used to describe the exterior gravitational field a charged rotating mass distribution.

In the particular case of slowly rotating and slightly deformed mass distribution we obtained the explicit form of the metric, and showed that it can be matched with an interior solution which is contained within the class of Hartle-Thorne solutions. This reinforces the conclusion that the solution represents the interior as well as the exterior gravitational field of astrophysical compact objects.

We study the problem of matching the interior and exterior spacetimes. We propose a $C^3$—matching which consists in demanding that the derivatives of a particular curvature eigenvalue are smooth on the matching hypersurface. To prove the validity of this approach we derived an interior solution for the simplest case of a static mass with an arbitrary quadrupole moment, represented by the Zipoy–Voorhees vacuum solution. The numerical integration of the corresponding field equations shows that interior perfect fluid solutions exist which are characterized by a constant density profile with a variable pressure. Fixing the value of the angular coordinate $\theta$, we performed numerically the $C^3$—matching. As a result we obtain a minimum radius at which the matching can be carried out and a fixed value for the pressure on the symmetry axis. These values are then used to reach the smooth matching of the interior and exterior metric functions. In all the cases analyzed in this manner we obtained a reasonable numerical matching.

The idea of using the $C^3$—matching condition to determine the minimum radius, at which an interior solution can be matched with an exterior one, has been proved also in a particular case where analytical methods can be applied, namely, in the case of the Kerr-Newman class of solutions. The obtained results are reasonable and compatible with other results obtained by analyzing the motion of test particles [21]. These results indicate that
it should be possible to determine the minimum radius of an astrophysical compact object by using the idea of the $C^a$–matching presented here. To prove this conjecture in general, it will be necessary to use more powerful methods related to the mathematical behavior of geodesics and curvature. This problem is currently under investigation. An important application of this analysis would be to relate the minimum size of a compact object with its binding energy. As a result we would obtain the maximum binding energy which is physically allowed for an astrophysical compact object.

Acknowledgments

I would like to thank D. Bini, A. Geralico, R. Kerr, O. Luongo, and R. Ruffini for helpful comments. I also thank ICRANet for support.

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