Thermal properties of relativistic Dunkl oscillators

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Abstract For a better understanding of the physical systems, even at the quantum level, the thermal quantities can be investigated. Recently, we realized that the parity of the system can also be examined simultaneously, by substituting the Dunkl operator to the ordinary differential operator in quantum mechanics. In this manuscript, we consider two relativistic Dunkl oscillators and investigate their thermal quantities with well-known statistical methods. Besides, we establish a relationship between the Dunkl–Dirac oscillator and the Dunkl–Anti–Jayne–Cummings model by defining the Dunkl creation and annihilation operators. Therefore, we conclude that our model can be regarded as an appropriate scenario for the theory of an open quantum system coupled to a thermal bath.

1 Introduction

In the middle of the last century, Wigner wrote a very interesting article discussing whether the equations of motion of a quantum mechanical system can determine the commutation relations [1]. Wigner sought the answer to this question on free particle and harmonic oscillator systems. Impressed by Wigner’s question, Yang solved the one-dimensional quantum harmonic oscillator problem the following year by defining an additional reflection operator in the usual Heisenberg algebra [2]. This deformed algebra can be given in the form of [3]

\[ [\hat{x}, \hat{p}] = i(1 + \mu \hat{R}), \]  

(1)

where \( \mu \) is the Wigner parameter, \( \hat{R} \) is the reflection operator and \( \hbar = 1 \). In this deformed algebra, the following relations

\[ \hat{R}\hat{x} = -\hat{x}\hat{R}, \quad \hat{R}\hat{p} = -\hat{p}\hat{R}, \quad \hat{R}^2 = 1, \quad \hat{R}f(x) = f(-x). \]  

(2)

are satisfied. It is worth noting that the position representation of this deformed algebra is not unique. An interesting position representation is written in terms of the Dunkl operator [4] as

\[ \hat{p} = \frac{1}{i} \hat{D} = \frac{1}{i} \left[ \frac{d}{dx} + \frac{\mu}{x} (1 - \hat{R}) \right], \]  

(3)

where \( \hat{p} \) denotes the usual quantum mechanical momentum operator. A Dunkl operator, \( \hat{D} \), can be defined as a linear superposition of differential and difference operators [5] for use in various problems in mathematics [6–8] and theoretical physics [9–14]. Especially in recent years, we observe that this interest has increased in the context of relativistic and non-relativistic quantum mechanical problems [15–26].

In physics, there are various well-known oscillator models, namely Klein–Gordon, Dirac, Duffin–Kemmer–Petiau, Pauli, or harmonic oscillators. They are often treated as toy models in research to describe nature. All these oscillator systems basically differ from each other according to whether the model is relativistic or not and the spins of the particles. However, none of them takes the parity of the particles into account. Since the Dunkl operator carries a parity operator, it can allow obtaining additional information in the classical solutions. According to this motivation, some authors already examined Dunkl–Dirac [15, 17], Dunkl–Klein–Gordon [23, 26], Dunkl-pseudo-harmonic [24] and Dunkl-Duffin–Kemmer–Petiau [25] oscillators under various scenario and dimensions [27].

On the other hand, especially in the last two decades, we see an increase in the number of studies discussing the thermal properties of systems in addition to their quantum structure. For example, thermal properties of one, three, and fractional Dirac oscillators are investigated in [28–30]. Similarly, thermodynamics quantities of Klein–Gordon, Duffin–Kemmer–Petiau, harmonic, and Pauli oscillators are derived in [31–35], respectively. Last year, Dong et al. studied the Dunkl–Schrödinger equation with a
generalized form of Dunkl operator under the effect of the harmonic oscillator potential energy [22]. There, they showed how the thermal quantities of positive and negative parity systems by comparing the thermodynamic functions. To our best knowledge, any similar investigation has not been executed in any Dunkl relativistic systems. With this motivation in this manuscript, we consider one-dimensional Dunkl–Klein–Gordon and Dunkl–Dirac oscillators and intend to obtain their thermodynamic functions.

We prepare the manuscript as follows: In the next section, we derive the solutions of Dunkl–Klein–Gordon and Dunkl–Dirac oscillators. In Sect. 3, we give the methodology of contraction of the partition and thus the thermal quantity functions. As a numerical example, we study the Dunkl–Dirac oscillator. After we obtain the Helmholtz free energy, entropy, internal energy and heat capacity functions, we depict them for the comparison of positive and negative parity systems. We conclude the manuscript with a brief section.

2 One-dimensional relativistic Dunkl oscillators

In this section, we briefly introduce two relativistic oscillators, namely Dunkl–Klein–Gordon and Dunkl–Dirac oscillators, and discuss their solutions in one spatial dimension.

2.1 Dunkl–Klein–Gordon oscillator

One-dimensional stationary Dunkl–Klein–Gordon oscillator is given in the form of

\[ E^2 - \left( \frac{1}{i} \hat{D} + im\omega \right) \left( \frac{1}{i} \hat{D} - im\omega \right) \psi(x) = 0, \]  

where \( m \) and \( \omega \) denote the mass and frequency of the oscillator, respectively. Substituting the Dunkl derivative given in Eq. (3) in Eq. (4), we get

\[ \hat{H} \psi(x) = 0, \]  

with the Dunkl–Klein–Gordon Hamilton operator

\[ \hat{H} = \frac{d^2}{dx^2} + \frac{2\mu}{x} \frac{d}{dx} - \frac{\mu}{x^2} (1 - \hat{R}) - m^2 \omega^2 x^2 + m\omega (1 + 2\mu \hat{R}) + E^2 - m^2. \]  

Since the Dunkl–Hamiltonian commutes with the reflection operator, they should have a common eigenbasis. Therefore, they can be diagonalized simultaneously. So, the eigenfunction, \( \psi(x) \), can be selected to have a definite parity \( \hat{R} \psi(x) = \pm \psi(x) \) with \( s = \pm \). As a result, Eq. (5) appears as

\[ \left[ \frac{d^2}{dy^2} + \frac{2\mu}{s} \frac{d}{dy} - \frac{\mu}{s^2} (1 - s) - m^2 \omega^2 s^2 + m\omega (1 + 2\mu s) + E^2 - m^2 \right] \psi'(y) = 0. \]  

The solution of Eq. (7) is given in many references [11, 12, 18, 24]; therefore, we address it here briefly. At first, we introduce a new variable by \( y = m\omega x^2 \), and then, we follow the ansatz: \( \psi'(y) = y^{s/2} e^{-\frac{y}{2}} \psi(y) \). After straightforward calculation, we find

\[ y \frac{d^2}{dy^2} + \left( 1 - \frac{s}{2} + \mu \right) \frac{d}{dy} - \frac{y}{4} + \frac{[1 + 2\mu s]}{4} \frac{E^2 - m^2}{4m\omega} \]  

\[ \Psi_n(y) = C \text{F} \left( \frac{(2\mu + 1)(1 - s)}{4} - \frac{E^2 - m^2}{4m\omega}, 1 - \frac{s}{2} + \mu; y \right). \]  

Therefore, the wave function reads

\[ \Psi_n(x) = C(m\omega x) \frac{1}{\sqrt{y}} e^{-\frac{m\omega y^2}{4}} \text{F} \left( \frac{(2\mu + 1)(1 - s)}{4} - \frac{E^2 - m^2}{4m\omega}, 1 - \frac{s}{2} + \mu; m\omega x^2 \right). \]  

The confluent hypergeometric function reduces to a polynomial of degree \( n \) in \( y \), if the first argument is equal to a negative integer.

\[ \frac{(2\mu + 1)(1 - s)}{4} - \frac{E^2 - m^2}{4m\omega} = -n. \]  

This fact gives the energy quantization

\[ \frac{E_n}{m} = \pm \sqrt{4nr + 2r \left( \mu + \frac{1}{2} \right)(1 - s) + 1}. \]
Reduced probability densities versus coordinate

Fig. 1 Reduced probability densities versus coordinate

where $r \equiv \omega/\mu$. We see that the energy spectrum, which depends on the Wigner parameter, differs according to the parity values. We note in the absence of $\mu$ and $s$, we obtain the standard result given in [37]. In Fig. 1, we plot the first three excited states’ probability densities versus coordinate for positive and negative parities.

2.2 Dunkl–Dirac oscillator

Next, we consider one-dimensional stationary Dunkl–Dirac oscillator, which is given in the form of

$$\hat{H}_D \psi = E^s \psi^s,$$

where

$$\hat{H}_D = \left[ \alpha_s \left( \frac{1}{i} \hat{D} - i \beta \mu \omega \hat{x} \right) + \beta m \right],$$

is the Dunkl–Dirac Hamilton operator, $\psi^s$ is a two-component spinor, $\psi = \begin{pmatrix} \Phi \\ f \end{pmatrix}$, and $\alpha_s$ and $\beta$ are the Dirac matrices. Here, we take the following form

$$\alpha_s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and obtain the following decoupled set of equations out of Eq. (14)

$$\left( \frac{1}{i} D + i m \omega \right) \Phi = (E^s - m) \Phi,$$

$$\left( \frac{1}{i} D - i m \omega \right) f = (E^s + m) f.$$ (17)

After performing a simple algebra between Eqs. (16) and (17), we find the stationary Dirac equation for the upper spinor component as:

$$\left[ (D - m \omega)(D + m \omega) + E^{s2} - m^2 \right] \Phi = 0.$$ (18)

Since Eq. (18) is similar to Eq. (4), we immediately can write the solution for the upper component of the spinor as

$$\Phi^s = N_1(m \omega x)^{1-s/2} e^{-m \omega x^2/2} F\left(-n, 1 - \frac{s}{2}, \mu; m \omega x^2 \right).$$ (19)

Thus, the spinor solution reads

$$\psi^s_n = N_1(m \omega x)^{1-s/2} e^{-m \omega x^2/2} \left( \frac{d}{dx} + \frac{(\mu + \frac{1}{2})(1-s)}{x} \right) F\left(-n, 1 - \frac{s}{2}, \mu; m \omega x^2 \right).$$ (20)

The energy quantization leads to the same energy function given in Eq. (12). We depict the energy spectrum function versus the node numbers in Fig. 2. We observe that the energy spectrum of even and odd parity particles differs relatively more in the ground states. It is worth noting that for $\mu = s = 0$, the energy function reduces to the one given in [38].
In order to establish a connection between the Dunkl–Dirac oscillator and quantum optics, we introduce the Dunkl creation \( \hat{a}_D^\dagger \) and annihilation \( \hat{a}_D \) operators

\[
\hat{a}_D^\dagger = \frac{1}{\sqrt{2m\hbar\omega}} (m_0\hat{x} - \hbar\hat{D}), \\
\hat{a}_D = \frac{1}{\sqrt{2m\hbar\omega}} (m_0\hat{x} + \hbar\hat{D}),
\]

which satisfy

\[
\left[ \hat{a}_D^\dagger, \hat{a}_D \right] = 1 + 2\mu \hat{R}.
\]

By substituting Eqs. (21) and (22) into Eq. (13), we express the Dunkl–Dirac Hamiltonian in the form of

\[
\hat{H}_D = g \left( \sigma^\tau \hat{a}_D + \sigma^\tau \hat{a}_D^\dagger \right) + m\sigma_z,
\]

where \( \sigma^\tau = \frac{1}{2} (\sigma_x \mp \sigma_y) \). It is worth noting that the mapped Hamiltonian corresponds to the Dunkl–Anti–Jaynes-Cummings (DAJC) model, and it reduces to the ordinary Anti–Jaynes–Cummings (AJC) model when the Wigner parameter is taken as zero.

3 Thermodynamics of relativistic Dunkl oscillators

In this section, we examine the thermal properties of the relativistic Dunkl oscillators. Since the energy spectrum function of Klein–Gordon and Dirac oscillators is the same, their thermal properties are also similar [31]. However, in the Dunkl case, the relativistic energy spectra of odd and even parity particles differ from each other; therefore, their partition functions, thus, thermal properties differ.

We start by assuming the system is in thermal equilibrium with a thermal bath at the temperature \( T \). Then, we write the canonical ensemble partition function as

\[
Z_s = \sum_{n=0}^\infty e^{-\frac{E_n^s}{K_B T}},
\]

where \( K_B \) is the Boltzmann constant, and \( E_0^s \) is the ground-state energy corresponding to \( n = 0 \). Here, we have to restrict ourselves with the stationary state’s positive energy solutions, since the partition function does not converge with the negative energy solutions in canonical ensemble. As our initial evaluation, let us test the convergence of the series given in Eq. (25) by using the integral formula

\[
\int_0^{+\infty} e^{-\frac{1}{\sqrt{ax+b}}} dx = \frac{2\tau^2}{a} \left( 1 + \sqrt{\tau} \right) e^{-\frac{1}{\sqrt{\tau}}}, \quad \text{for} \quad \tau > 0.
\]

This implies that the partition function is convergent. In order to evaluate this function, we use the Euler–MacLaurin formula defined with [39]

\[
\sum_{n=0}^{\infty} f(n) = f(0) + \int_0^{+\infty} f(x)dx - \sum_{p=1} B_{2p} (2p)! f^{(2p-1)}(0),
\]

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where $B_{2p}$ are the Bernoulli numbers, i.e., $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, and $B_6 = \frac{1}{42}$. Therefore, Eq. (27) reads

$$\sum_{n=0} f(n) = f(0) + \int_0^{+\infty} f(x)dx - \frac{1}{12} f^{(1)}(0) + \frac{1}{720} f^{(3)}(0) - \frac{1}{30240} f^{(5)}(0) + \cdots \tag{28}$$

Here, $f^{(n)}$ is the derivative of order $n$. In our case, the determination of the partition function can only be carried out on numerical methods. We find the partition function in the form of

$$Z^s = \frac{1}{2} + \sqrt{\alpha_s} \tau + \frac{\tau^2}{2r} + \frac{r}{6\sqrt{\alpha_s} \tau} + O\left(\frac{1}{\tau^3}\right) \tag{29}$$

Here, we use the following dimensionless parameters,

$$\alpha_s \equiv 2r \left(\frac{1}{2} + \mu\right)(1-s) + 1, \quad \tau \equiv \frac{K_BT}{mT_0}, \quad r \equiv \frac{\omega}{m}, \tag{30}$$

where $T_0 = \frac{m}{\hbar^2}$ is the characteristic temperature that splits the range of temperature to very low temperature, $T << T_0$, and very high temperature, $T >> T_0$, regions [40]. We observe that only in the odd case the Wigner parameter value takes a role. Also, it is worth mentioning that in the high-temperature regime, $T \to \infty$, the Dunkl parameter’s contribution remains negligible, and the partition functions become unique and indistinguishable.

$$Z^+ = Z^- = \frac{\tau^2}{2r}. \tag{31}$$

Before we derive the thermal quantities, we depict the Dunkl partition functions versus reduced temperature in Fig. 3. It is worth noting that in the low-temperature region, $(T < T_0)$, the higher-order terms neglected in Eq. (29) will come to dominate. Therefore, we have to show the characteristic behavior of the thermal quantities in the interval of $\tau \geq 1$. Hereafter, we take $K_B = m = 1$ in the rest of the manuscript, and we examine three different oscillator frequencies, $r = 1, r = 1.5, r = 2$. Furthermore, in the odd partition function, we assume $\mu = 1/2$. We observe that the partition function of even and odd parity systems increases monotonically with temperature, and for a fixed temperature, the partition function decreases when the oscillator frequency grows. Moreover, we notice that the partition function of even and odd parity systems differs at relatively small temperatures.

After determining the partition function, we derive the Helmholz free energy, entropy, mean energy and specific heat functions of the relativistic Dunkl oscillators. At first, we use

$$F^s = -\tau \ln Z^s,$$

to derive the Dunkl–Helmholtz free energy. We find

$$F^s = -\tau \ln \left[\frac{1}{2} + \frac{1}{2r}\left(\tau^2 + \tau \sqrt{\alpha_s}\right) + \frac{r}{6\tau \sqrt{\alpha_s}}\right]. \tag{32}$$

In Fig. 4, we demonstrate the plots of the Dunkl–Helmholtz free energy functions of the relativistic oscillators for the considered oscillator frequencies. In both systems, when the oscillator has greater frequency, then system’s thermal function gets greater values.

Next, we derive the reduced Dunkl entropy function via

$$S^s = \ln Z^s + \tau \frac{\partial}{\partial \tau} \ln Z^s.$$

![Fig. 3 Dunkl partition functions versus reduced temperature](image-url)
For even and odd parity systems, we obtain
\[
S^s = \ln \left[ \frac{1}{2} + \frac{1}{2r} \left( \tau^2 + \tau \sqrt{\alpha_s} \right) + \frac{r}{6\tau \sqrt{\alpha_s}} \right] + \tau \left[ \frac{\sqrt{\alpha_s} + 2\tau}{2r} - \frac{r}{6\tau^2 \sqrt{\alpha_s}} \right] \left[ \frac{1}{2} + \frac{r}{6\tau \sqrt{\alpha_s}} + \frac{\tau \sqrt{\alpha_s} + \tau^2}{2r} \right]^{-1}.
\] (33)

Then, we demonstrate the Dunkl entropy functions in Fig. 5. We observe that in both systems, the Dunkl entropy function saturates at lower values for greater oscillator frequencies.

Next, we examine the mean energy function by using
\[
U^s = \tau^2 \frac{\partial}{\partial \tau} \ln Z^s.
\]

We find
\[
U^s = \tau^2 \left[ \frac{\sqrt{\alpha_s} + 2\tau}{2r} - \frac{r}{6\tau \sqrt{\alpha_s}} \right] \left[ \frac{1}{2} + \frac{r}{6\tau \sqrt{\alpha_s}} + \frac{\tau \sqrt{\alpha_s} + \tau^2}{2r} \right]^{-1}.
\] (34)

In Fig. 6, we present the characteristic behavior of the Dunkl mean energy functions. We observe that at high temperatures, Dunkl mean energy functions tend to the same value, \( U^s \rightarrow 2T \), in the even and odd relativistic cases.

Finally, we derive the reduced Dunkl heat capacity function by using
\[
C^s = 2\tau \frac{\partial}{\partial \tau} \ln Z^s + \tau^2 \frac{\partial^2}{\partial \tau^2} \ln Z^s.
\]

We obtain
\[
C^s = 2\tau \left[ \frac{\sqrt{\alpha_s} + 2\tau}{2r} - \frac{r}{6\sqrt{\alpha_s} \tau} \right] + \tau^2 \left[ \frac{\sqrt{\alpha_s} + 2\tau}{2r} + \frac{r}{6\sqrt{\alpha_s} \tau} + \frac{1}{2} \right] + \frac{1}{2} - \tau^2 \left( \frac{\sqrt{\alpha_s} + 2\tau}{2r} - \frac{r}{6\sqrt{\alpha_s} \tau} \right)^2 \left( \frac{\sqrt{\alpha_s} + 2\tau}{2r} + \frac{r}{6\sqrt{\alpha_s} \tau} + \frac{1}{2} \right)^2.
\] (35)
Then, we depict the Dunkl heat functions versus reduced temperature in Fig. 7. We see that all relativistic Dunkl heat capacity functions converge to 2 at high temperatures. On the other hand, at low temperatures, the Dunkl heat capacity functions behave differently for the even and odd relativistic cases.

### 4 Conclusion

In this manuscript, we examine the thermal properties of relativistic Dunkl oscillators. To this end, at first, we review the solutions of the Dunkl–Klein–Gordon and Dunkl–Dirac oscillators in one dimension. Then, we take the odd and even parity systems in thermal equilibrium and express the canonical partition functions. After that, we derive the reduced Dunkl–Helmholtz free energy, Dunkl entropy, Dunkl mean energy and Dunkl heat capacity functions. In even and odd parity systems, we demonstrate all these thermal quantities versus the reduced temperature.

Besides all, we mapped the Dunkl–Dirac oscillator Hamiltonian onto Dunkl–Anti–Jaynes–Cummings model of quantum optics. Quantum optics theory offers one of the first test fields of open quantum systems. Basically, an open quantum system deals with the problems of dephasing and damping in the considered system with the claim that all real systems are open and interact with the surroundings [33, 34]. Therefore, the Dunkl–Dirac oscillator can be regarded as an appropriate scenario for the theory of an open quantum system coupled to a thermal bath.

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