Research Article

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Comparison estimates on the first eigenvalue of a quasilinear elliptic system

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Abstract: We study a system of quasilinear eigenvalue problems with Dirichlet boundary conditions on complete compact Riemannian manifolds. In particular, Cheng comparison estimates and the inequality of Faber–Krahn for the first eigenvalue of a \((p, q)\)-Laplacian are recovered. Lastly, we reprove a Cheeger-type estimate for the \(p\)-Laplacian, \(1 < p < \infty\), from where a lower bound estimate in terms of Cheeger’s constant for the first eigenvalue of a \((p, q)\)-Laplacian is built. As a corollary, the first eigenvalue converges to Cheeger’s constant as \(p, q \to 1, 1\).

Keywords: \(p\)-Laplacian, eigenvalue problems, Cheng-type estimates, Faber–Krahn inequality, Cheeger constant

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1 Preliminaries and main results

1.1 Introduction

Let \(\Omega\) be a bounded domain in an \(N\)-dimensional Riemannian manifold \((M, g)\). We prove some comparison estimates of Cheng type, Cheeger type and Faber–Krahn type on the principal Dirichlet eigenvalue for the following quasilinear elliptic system:

\[
\begin{align*}
\Delta_p u + \lambda |u|^{a-1} |v|^{b-1} v &= 0 & \text{in } \Omega \subseteq M, \\
\Delta_q v + \lambda |u|^{a-1} |v|^{b-1} v &= 0 & \text{in } \Omega \subseteq M, \\
u &= 0, & \text{on } \partial \Omega, \\
(u, v) &\in W^{1, p}_0(\Omega) \times W^{1, q}_0(\Omega),
\end{align*}
\]

(1.1)

where \(1 < p, q < \infty\) and \(a, \beta > 0\) are real numbers satisfying \(a/p + \beta/q = 1\).

The principal eigenvalue of (1.1) denoted by \(\lambda_{1, p, q}(\Omega)\) is the least positive real number for which the system has a nontrivial solution \((u, v)\) called an eigenvector in the product Sobolev space \(W^{1, p}_0(\Omega) \times W^{1, q}_0(\Omega)\) with \(u \neq 0\) and \(v \neq 0\). The differential operator in (1.1) is the so called \(p\)-Laplacian, that is,

\[\Delta_p u = \text{div}(|du|^{p-2} du) \quad \text{for } u \in W^{1, p}_0,\]

where \(\text{div}\) and \(d\) are respectively the divergence and gradient operators. When \(p = 2\), then \(\Delta_p\) is the usual Laplace–Beltrami operator. The \(p\)-Laplace operator arises in problems from pure mathematics such as in the theories of quasiregular and quasiconformal mappings as well as in modelling problems of physical phenom-
ena in non-Newtonian fluids, nonlinear elasticity, glaciology, petroleum extraction, porous media flows and reaction-diffusion processes; see for instance [26] for a detailed description of the p-Laplace operator.

### 1.2 Eigenvalue problem for the p-Laplacian

The nonlinear eigenvalue problem for the p-Laplacian is the following:

\[
\begin{aligned}
\Delta_p f + \lambda |f|^{p-2} f &= 0 \quad \text{in } \Omega, \\
f &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  

(1.2)

with \( p \in [1, \infty) \). In the local coordinates system, the p-Laplacian is written as

\[
\Delta_p f = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} |\nabla|x|^{p-2} \frac{\partial}{\partial x^i} f \right),
\]

where \( |g| = \det(g_{ij}) \) and \( g^{ij} = (g_{ij})^{-1} \) is the inverse metric. The first p-eigenvalue \( \lambda_{1,p}(\Omega) \) of the p-Laplacian is the smallest nonzero number \( \lambda \) for which the Dirichlet problem (1.2) has a nontrivial solution \( f \in W^{1,p}_0(\Omega) \), where the Sobolev space \( W^{1,p}_0(\Omega) \) is the completion of \( C^\infty_0(\Omega) \) with respect to the Sobolev norm

\[
\|f\|_{1,p} = \left( \int_{\Omega} (|f|^p + |\nabla|^p) \, d\mu \right)^{\frac{1}{p}}
\]

and \( d\mu \) is the Riemannian volume element of \((M, g)\). The first p-eigenvalue can be variationally characterized by

\[
\lambda_{1,p}(\Omega) = \inf_{f} \left\{ \int_{\Omega} |\nabla|^p \, d\mu : f \neq 0, f \in W^{1,p}_0(\Omega) \right\}
\]  

(1.3)

satisfying the following constraint:

\[
\int_{\Omega} |f|^{p-2} f \, d\mu = 0.
\]

The corresponding eigenfunction is the minimizer of (1.3) and satisfies the Euler–Lagrange equation

\[
\int_{\Omega} |\nabla|^p \langle df, d\phi \rangle \, d\mu - \lambda \int_{\Omega} |f|^{p-2} \langle f, \phi \rangle \, d\mu = 0
\]

for \( \phi \in C^\infty_0(\Omega) \) in the sense of distribution. Here and in the remainder of the paper, \( \langle \cdot, \cdot \rangle \) is the inner product with respect to the metric \( g \). We know that (1.2) has weak solutions with only partial regularity in general [20, 26, 33].

There have been many interesting geometric results on \( \lambda_{1,p} \) in the recent years; see, for example, the references [1, 22, 25, 27, 28, 32]. In particular, the first author, Abolarinwa [1], Takeuchi [32], Matei [28], Mao [27] and Lima, Montenegro and Santos [25] obtained the classical estimates of Faber–Krahn, Cheeger, Mackean and Cheng-type inequalities on \( \lambda_{1,p} \). For evolving manifolds, see [2–6].

### 1.3 Eigenvalue problem for a (p, q)-Laplacian

Here we say that \( \lambda \) is an eigenvalue of the system (1.1) provided \( u \in W^{1,p}_0(\Omega) \) and \( v \in W^{1,q}_0(\Omega) \) satisfy the system of Euler–Lagrange equations

\[
\begin{aligned}
\int_{\Omega} |u|^{p-2} \langle du, d\phi \rangle \, d\mu - \lambda \int_{\Omega} |u|^p |\nabla|^{q-2} \langle \nabla u, \phi \rangle \, d\mu &= 0, \\
\int_{\Omega} |v|^{q-2} \langle dv, d\psi \rangle \, d\mu - \lambda \int_{\Omega} |v|^q |\nabla|^{p-2} \langle \nabla v, \psi \rangle \, d\mu &= 0
\end{aligned}
\]
for $\phi \in W_0^{1,p}(\Omega)$ and $\psi \in W_0^{1,q}(\Omega)$. The pair $(u, v)$, $u > 0$, $v > 0$, consists of the corresponding eigenfunctions. In a similar manner to the first $p$-eigenvalue, the principal $(p, q)$-eigenvalue is variationally characterized as

$$\lambda_{1,p,q}(\Omega) = \inf \{ A(u, v) : (u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega), B(u, v) = 1 \},$$

where

$$A(u, v) = \frac{a}{p} \int_\Omega |u|^p \, d\mu + \frac{\beta}{q} \int_\Omega |v|^q \, d\mu \quad \text{and} \quad B(u, v) = \int_\Omega |u|^{q-1} |v|^q \, d\mu$$

for

$$a > 0, \quad \beta > 0, \quad \frac{a}{p} + \frac{\beta}{q} = 1.$$

The existence, simplicity, stability and some other properties of $\lambda_{1,p,q}(\Omega)$ have been studied in [7, 11, 17–21]; see also the references therein. Indeed, $\lambda_{1,p,q}(\Omega)$ has been proved to be positive and simple for bounded and unbounded domains in $\mathbb{R}^N$. Recently, the second author [9] applied the approach of symmetrization and co-area formula used by the first author [1] to obtain some geometric results of Faber–Krahn and Cheeger inequalities for $\lambda_{1,p,q}(\Omega)$. This shows that the classical approaches work well for system (1.1) without much difficulty involved.

### 1.4 Main results

The major aim of this paper is to prove Cheng-type comparison estimates [15, 16], a Faber–Krahn-type inequality and Cheeger-type estimates [14, 28] for $\lambda_{1,p,q}(\Omega)$. Precisely, let $B(x_0, r_0)$ be the open geodesic ball of radius $r_0$ centered at $x_0$ in $M$ and let $V_N(k, r_0)$ be a geodesic ball of the same radius $r_0$ in an $N$-dimensional space form $M_k$ of constant sectional curvature $k$. Denote the first eigenvalue of (1.1) on $\overline{B}(x_0, r_0)$ by $\lambda_{1,p,q}(B(x_0, r_0))$ and on $V_N(k, r_0)$ by $\lambda_{1,p,q}(V_N(k, r_0))$. Then, in Section 2, we prove the following theorem.

**Theorem 1.1.** Let $M$ be an $N$-dimensional complete Riemannian manifold such that for its Ricci curvature $\text{Ric}(M) \geq (N - 1)k$, $k \in \mathbb{R}$, holds. Then for any $x_0 \in M$ and $r_0 \in (0, d_M)$, where $d_M$ denotes the diameter of $M$, we have

$$\lambda_{1,p,q}(B(x_0, r_0)) \leq \lambda_{1,p,q}(V_N(k, r_0)). \quad (1.4)$$

Equality holds if and only if $B(x_0, r_0)$ is isometric to $V_N(k, r_0)$.

In a simple language, Cheng’s eigenvalue comparison estimate says that when a domain is large, its first Dirichlet eigenvalue is small and the size of the domain accounts for its curvature. A natural consequence of Theorem 1.1 is the following.

**Corollary 1.2.** Suppose that $M$ is an $N$-dimensional compact Riemannian manifold with Ricci curvature $\text{Ric}(M) \geq (N - 1)k$, $k \in \mathbb{R}$. Then

$$\lambda_{1,p,q}(M) \leq \lambda_{1,p,q}(V_N(k, \frac{d_M}{2})),$$

where $d_M$ denotes the diameter of $M$.

In Section 3, we consider the case where $M$ is compact with positive Ricci curvature $\text{Ric}(M) \geq (N - 1)k$, $k > 0$. We first prove the Faber–Krahn-type inequality for any domain $\Omega$ in a complete, simply connected Riemannian manifold of constant sectional curvature.

**Theorem 1.3.** Let $\Omega$ be a domain and $B(x_0, R)$ be the geodesic ball of radius $R > 0$, both in a complete, simply connected Riemannian manifold $M_k$ of constant sectional curvature $k$ such that $\text{Vol}(\Omega) = \text{Vol}(B(x_0, R))$. Then the following inequality holds:

$$\lambda_{1,p,q}(\Omega) \geq \lambda_{1,p,q}(B(x_0, R)). \quad (1.5)$$

The equality in (1.5) holds if and only if $\Omega$ is the geodesic ball $B(x_0, R)$. 

Theorem 1.3 says in particular that the geodesic balls are with smallest $\lambda_{1,p,q}$ among all domains of a given volume. A generalization of the Faber–Krahn inequality for $\Delta_p$ on a ball in the Euclidean $N$-sphere with radius $1/k^2$ has been established by Matei [28, Theorem 2.1] (case $p = 2$ is due to Berard and Meyer [13]). Finally, Matei’s result for the case of the $(p, q)$-Laplacian will be discussed at the end of the proof of Theorem 1.3. In this case, $\Omega$ will be a domain in a compact Riemannian manifold $M$ with positive Ricci curvature $\text{Ric}(M) \geq (N - 1)k$, $k > 0$.

To state the last result, we let $h(\Omega)$ be Cheeger’s constant defined by

$$h(\Omega) := \inf_{\Omega'} \frac{\text{Vol}_{N-1}(\partial \Omega')}{\text{Vol}_{N}(\Omega')},$$

where the infimum is taken over all open submanifolds $\Omega'$ with compact closure in $\Omega$ and smooth boundary $\partial \Omega'$. Here $\text{Vol}_{N-1}(\partial \Omega')$ and $\text{Vol}_{N}(\Omega')$ denote the $(N - 1)$-dimensional and $N$-dimensional Riemannian volumes on $\partial \Omega'$ and $\Omega'$, respectively.

**Proposition 1.4.** Let $\Omega$ be a bounded domain with smooth boundary in a complete Riemannian manifold. Then

$$\lambda_{1,p,q}(\Omega) \geq \frac{a}{p} \left( \frac{h(\Omega)}{p} \right)^p \|u\|^p_p + \frac{\beta}{q} \left( \frac{h(\Omega)}{q} \right)^q \|v\|^q_q,$$

where the pair $(u, v)$ consists of the corresponding eigenfunctions to $\lambda_{1,p,q}(\Omega)$ and $\|u\|_r$ is the $L^r(\Omega)$-norm

$$\|u\|_{L^r(\Omega)} = \left( \int_{\Omega} |u|^r \, d\mu(x) \right)^{1/r}.$$

Moreover, $\lambda_{1,p,q}(\Omega)$ converges to Cheeger’s constant as $p \to 1$ and $q \to 1$.

## 2 Proofs of Theorem 1.1 and Corollary 1.2

### 2.1 Proof of Theorem 1.1

**Proof.** Let there exist a first pair of eigenfunctions $(\bar{u}, \bar{v})$, $\bar{u} > 0$, $\bar{v} > 0$, of the $(p, q)$-Laplacian on $\overline{V_N(k, r_0)}$ with Dirichlet boundary with $(\bar{u}, \bar{v}) \in W^{1,p}_0(\overline{V_N(k, r_0)}) \times W^{1,q}_0(\overline{V_N(k, r_0)})$. Then $\bar{u}$, $\bar{v}$ are radial (since $V_N(k, r_0)$ is a ball in a simply connected space form which is two-points homogeneous). Let $r$ be the distance function on $M$ with respect to the point $x_0$. Then

$$(\bar{u} \circ r, \bar{v} \circ r) \in W^{1,p}_0(B(x_0, r_0)) \times W^{1,q}_0(B(x_0, r_0))$$

satisfy the boundary conditions. Therefore, by definition,

$$\lambda_{1,p,q}(B(x_0, r_0)) \leq \frac{a}{p} \int_{B(x_0, r_0)} |d(\bar{u} \circ r)|^p \, d\mu + \frac{\beta}{q} \int_{B(x_0, r_0)} |d(\bar{v} \circ r)|^q \, d\mu,$$

with

$$\int_{B(x_0, r_0)} |u|^{p-1} |v|^{q-1} \langle \bar{u}, \bar{v} \rangle \, d\mu = 1.$$

Define a $C^\infty$-map $\theta : (0, r_\xi) \times S^{N-1} \to M$ by $\theta(t, \xi) := \exp_{x_0}(t, \xi)$, where $S^{N-1}$ is the $(N - 1)$-sphere in $T_{x_0}M$, $\exp_{x_0}$ is a local diffeomorphism from a neighborhood of $x_0$ in $M$ and

$$r_\xi = r_\xi(x_0) := \sup\{t > 0 : \exp_{x_0}(s, \xi) \text{ is the unique minimal geodesic from } x_0\}.$$

Since $M$ is complete,

$$B(x_0, r_0) = \{\exp_{x_0}(t, \xi) : \xi \in S^{N-1} \text{ and } t \in [0, a(\xi)]\},$$
where \( a(\xi) := \min\{r_\xi, r_0\} \). Then integration over \( B(x_0, r_0) \) can be pulled back to the tangent space using geodesic polar coordinates. Hence

\[
\begin{aligned}
& \int_{\theta(x_0, r_0)} \int\ t^{N-1} \theta(t, \xi) dt, \\
& \int_{\theta(x_0, r_0)} \int\ t^{N-1} \theta(t, \xi) dt,
\end{aligned}
\]

where \( dS^{N-1} \) is the canonical measure of \( S^{N-1} = S^{N-1}_{r_0} \),

\[
\theta(t, \xi) \times t^{N-1} = \sqrt{\det(g_{ij})}
\]

is the volume density induced by \( \exp_{x_0} \) and \( a(\xi) \leq r_0 \) such that \( \exp_{x_0}(a(\xi), \xi) \) is the cut point of \( x_0 \) along the geodesic \( t \rightarrow \exp_{x_0}(t, \xi) \).

Since we have assumed that \( \bar{u} \) is everywhere nonnegative and \( \bar{u}(r_0) = 0 \), we have \( \frac{\bar{u}}{dt}(r_0) < 0 \) in the \( r_0 \)-neighborhood. We have \( \frac{\bar{u}}{dt} < 0 \) in \( (0, r) \) (see Proposition 2.1 below). Integrating by parts then yields

\[
\begin{aligned}
& \int_{t=0}^{a(\xi)} \left( -\left( \frac{d\bar{u}}{dt} \right)^{p-1} t^{N-1} \theta(t, \xi) \right) dt \\
= & \left. (-\bar{u}) \left( -\left( \frac{d\bar{u}}{dt} \right)^{p-1} t^{N-1} \theta(t, \xi) \right) \right|_{0}^{a(\xi)} - \int_{t=0}^{a(\xi)} \frac{d\bar{u}}{dt} \left( \left( -\frac{d\bar{u}}{dt} \right)^{p-1} t^{N-1} \theta(t, \xi) \right) dt.
\end{aligned}
\]

By a straightforward computation, we have

\[
\frac{1}{t^{N-1} \theta(t, \xi)} \frac{d}{dt} \left( N^{N-1} \theta(t, \xi) \left( -\frac{d\bar{u}}{dt} \right)^{p-1} \right) = -\left( \frac{d\bar{u}}{dt} \right)^{p-2} \left[ (p-1) \frac{d^2\bar{u}}{dt^2} + \left( \frac{N-1}{t} + \frac{1}{\theta(t, \xi)} \frac{d\theta(t, \xi)}{dt} \right) \frac{d\bar{u}}{dt} \right].
\]

Then, using the facts that \( \frac{d\bar{u}}{dt}(0) = 0 \) and

\[
\left( -\bar{u} \left( \frac{d\bar{u}}{dt} \right)^{p-1} t^{N-1} \theta(t, \xi) \right) a(\xi) \leq 0,
\]

we obtain

\[
\begin{aligned}
& \int_{0}^{a(\xi)} \left| \frac{d\bar{u}}{dt} \right|^{p-1} t^{N-1} \theta(t, \xi) dt \\
& \int_{0}^{a(\xi)} \left| \frac{d\bar{u}}{dt} \right|^{p-2} \left[ (p-1) \frac{d^2\bar{u}}{dt^2} + \left( \frac{N-1}{t} + \frac{1}{\theta(t, \xi)} \frac{d\theta(t, \xi)}{dt} \right) \frac{d\bar{u}}{dt} \right] t^{N-1} \theta(t, \xi) dt.
\end{aligned}
\]

Notice that

\[
\Delta_{\theta^p} = \text{div}(\left| \frac{d\bar{u}}{dt} \right|^{p-2} d\bar{u} + (p-2) \left| \frac{d\bar{u}}{dt} \right|^{p-2} d\bar{u})
\]

Since \( \bar{u} \) is radial, writing \( \Delta \) in geodesic polar coordinates at \( k \), we have

\[
\Delta \bar{u} = \frac{d^2\bar{u}}{dt^2} + \left( \frac{N-1}{t} + \frac{1}{\theta_k(t, \xi)} \frac{d\theta_k(t, \xi)}{dt} \right) \bar{u},
\]

where \( \theta_k(t, \xi) \) is the corresponding volume density on \( V_{N}(k, r_0) \) viewed through the exponential map of \( M_k \).

Hence

\[
\Delta p \bar{u} = \left| \frac{d\bar{u}}{dt} \right|^{p-2} \left[ \frac{d^2\bar{u}}{dt^2} + \left( \frac{N-1}{t} + \frac{1}{\theta_k(t, \xi)} \frac{d\theta_k(t, \xi)}{dt} \right) \frac{d\bar{u}}{dt} + (p-2) \frac{d^2\bar{u}}{dt^2} \right]
\]

\[
= \left| \frac{d\bar{u}}{dt} \right|^{p-2} \left[ (p-1) \frac{d^2\bar{u}}{dt^2} + \left( \frac{N-1}{t} + \frac{1}{\theta_k(t, \xi)} \frac{d\theta_k(t, \xi)}{dt} \right) \frac{d\bar{u}}{dt} \right].
\]

By the assumption on the Ricci curvature and the classical Bishop's comparison theorem,

\[
\frac{d}{dt} \left( \frac{\theta(t, \xi)}{\theta_k(t, \xi)} \right) \leq 0,
\]

(2.5)
which implies
\[ \frac{d\dot{u}}{dt} \cdot \frac{1}{\theta(t, \xi)} \frac{d\theta(t, \xi)}{dt} \geq \frac{d\dot{u}}{dt} \cdot \frac{1}{\theta_k(t, \xi)} \frac{d\theta_k(t, \xi)}{dt}. \]

Hence, by (2.3) we have
\[ -\dot{u} \left| \frac{d\dot{u}}{dt} \right|^{p-2} \left[ (p-1) \frac{d^2\dot{u}}{dt^2} + \left( \frac{N-1}{t} + \frac{1}{\theta(t, \xi)} \frac{d\theta(t, \xi)}{dt} \right) \frac{d\dot{u}}{dt} \right] \leq -\dot{u} \left| \frac{d\dot{u}}{dt} \right|^{p-2} \left[ (p-1) \frac{d^2\dot{u}}{dt^2} + \left( \frac{N-1}{t} + \frac{1}{\theta_k(t, \xi)} \frac{d\theta_k(t, \xi)}{dt} \right) \frac{d\dot{u}}{dt} \right] = -\dot{u} \Delta_p \dot{u} \]

by using definition (2.5). Therefore, combining (2.3) and (2.6) yields
\[ \int_0^a \left| \frac{d\dot{u}}{dt} \right|^p \times t^{N-1} \theta(t, \xi) \, dt \leq - \int_0^a \dot{u} \Delta_p \dot{u} \, dt \bigg|_{V_N(k, r_0)}, \]

and by (2.2) we have
\[ \int_{B(x_0, r_0)} |d(\dot{u} \circ r)|^p \, d\mu \leq - \int_{\xi \in S^{N-1}} \dot{u} \Delta_p \dot{u} \, dt = - \int_{V_N(k, r_0)} \dot{u} \Delta_p \dot{u} \, d\mu = \int_{V_N(k, r_0)} |d(\dot{u} \circ r)|^p \, d\mu. \]

Similarly,
\[ \int_{B(x_0, r_0)} |d(\dot{v} \circ r)|^q \, d\mu \leq \int_{V_N(k, r_0)} |d(\dot{v} \circ r)|^q \, d\mu. \]

Then the required inequality in (1.4) follows from (2.1), that is,
\[ \lambda_{1,p,q}(B(x_0, r_0)) \leq \frac{a}{p} \int_{B(x_0, r_0)} |d(\dot{u} \circ r)|^p \, d\mu + \frac{\beta}{q} \int_{B(x_0, r_0)} |d(\dot{v} \circ r)|^q \, d\mu \]
\[ \leq \frac{a}{p} \int_{V_N(k, r_0)} |d(\dot{u} \circ r)|^p \, d\mu + \frac{\beta}{q} \int_{V_N(k, r_0)} |d(\dot{v} \circ r)|^q \, d\mu = \lambda_{1,p,q}(V_N(k, r_0)). \]

In conclusion, the equality
\[ \lambda_{1,p}(B(x_0, r_0)) = \lambda_{1,p}(V_N(k, r_0)) \]
holds when there is equality in (2.3) and (2.6). Then we see that \( a(\xi) \equiv r_0 \) for almost all \( \xi \) and by continuity for all \( \xi \). Hence \( \theta(t, \xi) = \theta_k(t, \xi) \), which implies equality in Bishop’s inequality (2.5). This then proves that \( B(x_0, r_0) \) is isometric to \( V_N(k, r_0) \).

The Cheng comparison result is valid regardless of the cut locus since the Lebesgue measure of the cut locus is 0 with respect to the \( N \)-dimensional Lebesgue measure of the manifold, which implies that integration over the cut locus vanishes.

**Proposition 2.1** ([27, Proposition 3.1]). Let \( \phi(s) \) be any solution of
\[ [ |\phi'(s)|^{p-2} \theta(s)^{N-1} \phi'(s)]' + \lambda \theta(s)^{N-1} |\phi(s)|^{p-2} \phi(s) = 0, \quad 1 < p < \infty, \quad (2.7) \]
where \( \theta(s) > 0 \) on \( (0, a) \). Then \( \phi'(s) < 0 \) on \( (0, a) \) whenever \( \phi(s) > 0 \) on \( (0, a) \) and \( \lambda > 0 \). Here \( t = \frac{d}{ds} \).

**Proof.** Integrating (2.7) from 0 to \( s \) yields
\[ |\phi'(s)|^{p-2} \theta(s)^{N-1} \phi'(s) = -\lambda \int_0^s \theta(s)^{N-1} |\phi(s)|^{p-2} \phi(s) \, ds. \]

The claim of the proposition follows since \( \theta(s) > 0 \) on \( (0, a) \).
Remark 2.2. By using (2.4), functions $\bar{u}$ and $\bar{v}$ which are radial satisfy

$$|u'|^p - 2\left[(p - 1)u'' + \left(\frac{N - 1}{t} + \frac{d\theta_k^p(t, \xi)}{\theta_k^p(t, \xi)}\right)u'\right] + \lambda|u|^{a-1}|u|^{\beta-1}u = 0,$$

$$|v'|^q - 2\left[(q - 1)v'' + \left(\frac{N - 1}{t} + \frac{d\theta_k^p(t, \xi)}{\theta_k^p(t, \xi)}\right)v'\right] + \lambda|v|^{a-1}|v|^{\beta-1}v = 0.$$

Notice that each equation in the last system is of the form (2.7); this can be clearly seen by putting $p = q$.

### 2.2 Proof of Corollary 1.2

We can mimic the steps in the proof of [28, Corollary 1.1] (see also [16, Theorem 2.1]) to establish the proof of Corollary 1.2.

**Proof.** Let $x_i$ be a point in $M$ such that $B(x_i, \frac{d\mu}{t^2})$, where $i = 1, 2, \ldots, m$ are pairwise disjoint. Let $r_i$ be the distance function with respect to $x_i$ and $\varphi_i = \varphi \circ r_i$, $\psi_i = \psi \circ r_i$, where $(\varphi, \psi)$ is the first pair of radial eigenfunctions of $V_N(k, \frac{d\mu}{t^2})$. Then, by Theorem 1.1, we have

$$\frac{\alpha}{p} \int_{B(x_i, \frac{d\mu}{t^2})} |d\varphi_i|^p \ d\mu + \frac{\beta}{q} \int_{B(x_i, \frac{d\mu}{t^2})} |d\psi_i|^q \ d\mu \leq \lambda_{1, p, q} \left(V_N \left(\frac{k}{t^2}, \frac{d\mu}{t^2} \right)\right),$$

with

$$\int_{B(x_i, \frac{d\mu}{t^2})} |\varphi_i|^{a-1}|\psi_i|^{\beta-1} \langle \varphi_i, \psi_i \rangle \ d\mu = 1.$$

We can extend $\varphi_i$ and $\psi_i$ to be zero outside $B(x_i, \frac{d\mu}{t^2})$. Then, by elementary linear algebra, there exist constants $C_i, i = 1, 2, \ldots, m$, not all equal zero such that

$$\int_M \left(\sum_{i=1}^m C_i \varphi_i\right)^{a-1} \left(\sum_{i=1}^m C_i \psi_i\right)^{\beta-1} \ d\mu = 0.$$

Since $B(x_i, \frac{d\mu}{t^2})$ are pairwise disjoint,

$$\sum_{i=1}^m C_i \varphi_i \not= 0 \quad \text{and} \quad \sum_{i=1}^m C_i \psi_i \not= 0.$$

Hence

$$\lambda_{1, p, q}(M) \leq \frac{\alpha}{p} \int_M \left|d \sum_{i=1}^m C_i \varphi_i\right|^p \ d\mu + \frac{\beta}{q} \int_M \left|d \sum_{i=1}^m C_i \psi_i\right|^q \ d\mu$$

$$= \frac{\alpha}{p} \left( \int_{B(x_i, \frac{d\mu}{t^2})} |C_1 d\varphi_1|^p \ d\mu + \cdots + \int_{B(x_n, \frac{d\mu}{t^2})} |C_m d\varphi_m|^p \ d\mu \right)$$

$$+ \frac{\beta}{q} \left( \int_{B(x_i, \frac{d\mu}{t^2})} |C_1 d\psi_1|^q \ d\mu + \cdots + \int_{B(x_n, \frac{d\mu}{t^2})} |C_m d\psi_m|^q \ d\mu \right)$$

$$= \frac{\alpha}{p} \int_{B(x_i, \frac{d\mu}{t^2})} \left|d \sum_{i=1}^m C_i \varphi_i\right|^p \ d\mu + \frac{\beta}{q} \int_{B(x_i, \frac{d\mu}{t^2})} \left|d \sum_{i=1}^m C_i \psi_i\right|^q \ d\mu$$

$$\leq \lambda_{1, p, q} \left(V_N \left(\frac{k}{t^2}, \frac{d\mu}{t^2} \right)\right),$$

which completes the proof. □
Remark 2.3. Suppose $M$ has nonnegative Ricci curvature $\text{Ric}(M) \geq 0$. Then the above inequality reads

$$\lambda_{1,p,q}(M) \leq \lambda_{1,p,q}(V_N(0, \frac{d_M}{2})).$$

Thus an upper bound can be explicitly found for $\lambda_{1,p,q}(M)$ by estimating $\lambda_{1,p,q}(V_N(0, \frac{d_M}{2}))$. The case $p, q = 2$ for $\Delta_p$ is found as

$$\lambda_1(M) \leq 2m^2 N(N + 4) (d_M^2), \quad m = 1,$$

by Cheng in [16, Corollary 2.2].

3 Faber–Krahn-type inequality

The main tools that will be employed in the proof of Theorem 1.3 are symmetrization procedure and the inequalities of Pólya–Szegö and Hardy–Littlewood. We first recall the definition of the symmetrization of a function and its properties

Definition 3.1. Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$. Let $f$ be a nonnegative measurable function in $\Omega$ which vanishes on the boundary. The set $\{x \in \Omega : f(x) > t, t > 0\}$ is called the level set of $f$. Let $\Omega^*$ be the ball centered at the origin in $\mathbb{R}^N$ with the same volume as $\Omega$. The function $f^* : \Omega^* \to \mathbb{R}^+$ is the nonincreasing symmetric rearrangement of $f$ with

$$\text{Vol}\{x \in \Omega^* : f^*(x) > t\} = \text{Vol}\{x \in \Omega : f(x) > t\}.$$ 

For more details on symmetrization, see [24, 30, 31]. Now using the above symmetrization, we have the following lemma.

Lemma 3.2. Let $\Omega$ be a compact domain. Let $f, g : \Omega \to \mathbb{R}^+$ be nonnegative measurable functions and let $f^*, g^* : B(x, R) \to \mathbb{R}^+$ be radially nonincreasing functions such that $\text{Vol}(\Omega) = \text{Vol}(B(x, R))$. Then the following assertions hold:

(i) Equimeasurability of level sets:

$$\int f = \int_{B(x, R)} f^*.$$ 

(ii) Pólya–Szegö inequality:

$$\int_{\Omega} |df|^p \geq \int_{B(x, R)} |df^*|^p \quad \text{for } p > 1.$$ 

(iii) Hardy–Littlewood inequality:

$$\int_{\Omega} fg \leq \int_{B(x, R)} f^* g^*.$$ 

The Pólya–Szegö inequality says the Dirichlet integral $\int_{\Omega} |df|^p$ decreases under the influence of symmetrization. This inequality can be realized by combining the co-area formula and Hölder’s inequality. A version of the proof of (i) and (ii) is in Aubin [8, Proposition 2.17]. The Hardy–Littlewood inequality’s proof is contained in [10, Theorem 2.2, p. 44]. Though, the nonincreasing rearrangement does not preserve the product of functions in general, the equality in (iii) is attainable for suitable functions; see [10] for details.

3.1 Proof of Theorem 1.3

Proof. Let $u^*$ and $v^*$ be the nonincreasing rearrangement of $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$, respectively. Let $(u, v)$ be the minimizing eigenfunction satisfying $\int_{\Omega} |u|^{p-1} |v|^{q-1} (u, v) \, d\mu = 1$ for $\alpha, \beta > 0$. We know that $u > 0$.
and \( \nu > 0 \). It follows from the Pólya–Szegö inequality that
\[
\int_{\Omega} |du|^p \, d\mu \geq \int_{B(x_0, R)} |du^*|^p \, d\mu \quad \text{and} \quad \int_{\Omega} |dv|^q \, d\mu \geq \int_{B(x_0, R)} |dv^*|^q \, d\mu.
\]
Then
\[
\lambda_{1, p, q}(\Omega) = \frac{\alpha}{p} \int_{\Omega} |du|^p \, d\mu + \frac{\beta}{q} \int_{\Omega} |dv|^q \, d\mu \\
\geq \frac{\alpha}{p} \int_{B(x_0, R)} |du^*|^p \, d\mu + \frac{\beta}{q} \int_{B(x_0, R)} |dv^*|^q \, d\mu.
\]

On the other hand, it is clear from the Hardy–Littlewood inequality in Lemma 3.2 that
\[
\int_{\Omega} u^\alpha \nu^\beta \, d\mu \leq \int_{B(x_0, R)} u^{\alpha \beta} \nu^{\beta} \, d\mu
\]
for real numbers \( \alpha, \beta > 0 \), where \( u^* \in W^{1,p}(B(x_0, R)) \) and \( \nu^* \in W^{1,q}(B(x_0, R)) \). Note also that one can identify \( \int_{\Omega} u^\alpha \nu^\beta \, d\mu \) with \( \int_{\Omega} |u|^{\alpha-1} |\nu|^{\beta-1} \langle u, \nu \rangle \, d\mu \). Clearly, since \( u > 0, \nu > 0 \), we obtain
\[
B(u, \nu) = \int_{\Omega} |u|^{\alpha-1} |\nu|^{\beta-1} \langle u, \nu \rangle \, d\mu \\
= \int_{\Omega} u^\alpha \nu^\beta \, d\mu \\
\leq \int_{B(x_0, R)} u^{\alpha \beta} \nu^{\beta} \, d\mu \\
= \int_{B(x_0, R)} |u|^{\alpha-1} |\nu|^{\beta-1} \langle u^*, \nu^* \rangle \, d\mu \\
= B(u^*, \nu^*).
\]

We therefore conclude that
\[
\frac{\alpha}{p} \int_{B(x_0, R)} |du^*|^p \, d\mu + \frac{\beta}{q} \int_{B(x_0, R)} |dv^*|^q \, d\mu \\
\geq \inf \{ \langle u^*, \nu^* \rangle : (u^*, \nu^*) \in W^{1,p}_0(B(x, R)) \times W^{1,q}_0(B(x, R)), B(u^*, \nu^*) = 1 \}
\]
\[= \lambda_{1, p, q}(B(x, R)).
\]

Equation (1.5) is therefore proved.

Matei’s result [28] for the case of the \((p, q)\)-Laplacian can be stated as follows.

**Theorem 3.3.** Let \( M \) be an \( N \)-dimensional compact Riemannian manifold with \( \text{Ric}(M) \geq (N - 1)k \) and let \( B(R) \) be a geodesic ball of Radius \( R > 0 \) in the Euclidean \( N \)-sphere \( \mathbb{S}^N_k \) with constant sectional curvature \( k \) such that for a bounded domain \( \Omega \in M \),
\[
\frac{\text{Vol}(\Omega)}{\text{Vol}(M)} = \frac{\text{Vol}(B(R))}{\text{Vol}(\mathbb{S}^N_k)}.
\]

Then
\[
\lambda_{1, p, q}(\Omega) \geq \gamma \lambda_{1, p, q}(B(R)), \tag{3.1}
\]
where \( \gamma = \text{Vol}(M)/\text{Vol}(\mathbb{S}^N_k) \). Equality holds if and only if there is an isometry which sends \( \Omega \) to \( B(R) \subset \mathbb{S}^N_k \).

Inequality (3.1) reduces to (1.5) of Theorem 1.3 if \( M = \mathbb{S}^N_k \).
4 Lower bound estimates (Cheeger-type)

In this section, we want to give a lower bound for \( \lambda_{1,p,q}(\Omega) \) in terms of the so-called Cheeger’s constant and a lower bound estimate on the first eigenvalue of the \( p \)-Laplacian.

**Definition 4.1.** Cheeger’s constant \( h(\Omega) \) of a domain \( \Omega \) is defined to be

\[
h(\Omega) := \inf_{\Omega'} \frac{\text{Vol}_{N-1}(\partial \Omega')}{\text{Vol}_{N}(\Omega')},
\]

where \( \Omega' \) ranges over smooth subdomains of \( \Omega \) with compact closure in \( \Omega \) with smooth boundary \( \partial \Omega' \), and \( \text{Vol}_{N-1}(\partial \Omega') \) and \( \text{Vol}_{N}(\Omega') \) denote the volumes of \( \partial \Omega' \) and \( \Omega' \), respectively.

Let \( D \) vary over all smooth subdomains of \( \Omega \) whose boundary \( \partial D \) does not touch \( \partial \Omega \). Then the quantity \( \Omega(D) := \frac{\text{Vol}(\partial D)}{\text{Vol}(D)} \) is called Cheeger’s quotient of \( D \). Any subdomain \( E \subset \Omega \) which realizes the infimum in (4.1) is referred to as Cheeger domain in \( \Omega \) while \( \Omega \) is called self-Cheeger if it is a minimizer. Problems involving Cheeger’s constant/domains are very interesting in geometric analysis. For existence, (non-)uniqueness and regularity of Cheeger domains, see [23]. For an introductory survey and some physical applications of Cheeger’s constant, see [29] and the references therein, and see [12] for further results in the manifold setting.

**Theorem 4.2** (Cheeger-type estimate). Let \( \Omega \) be a bounded domain with smooth boundary in a complete Riemannian manifold. Then

\[
\lambda_{1,p}(\Omega) \geq \frac{h(\Omega)}{p}, \quad 1 < p < \infty.
\]

The above theorem was originally proved by Cheeger [14] for \( p = 2 \) in the case of manifolds without boundary, and an extension for the general \( p \) was given by Matei [28]; see also [1, 23, 27, 32] for the general manifolds and \( p > 1 \). For completeness and the importance of Theorem 4.2 to the proof of Proposition 1.4, we repeat the proof here.

**Proof of Theorem 4.2.** Suppose \( \varphi \in C_0^\infty(\Omega) \) is a positive function and let

\[
A(t) := \{ x \in \Omega : \varphi(x) > t \} \quad \text{and} \quad \partial A(t) := \{ x \in \Omega : \varphi(x) = t \}.
\]

By using the co-area formula

\[
\int_\Omega |d\varphi| \, d\mu = \int_{-\infty}^\infty \left( \int_{A(t)} dA(t) \right) dt = \int_{-\infty}^\infty \text{Vol}_{N-1}(\partial A(t)) \, dt
\]

\[
= \int_{-\infty}^\infty \frac{\text{Vol}_{N-1}(\partial A(t))}{\text{Vol}_{N}(A(t))} \cdot \text{Vol}_{N}(A(t)) \, dt
\]

\[
\geq \inf_{\Omega' \subset \subset \Omega} \frac{\text{Vol}_{N-1}(\partial \Omega')}{\text{Vol}_{N}(\Omega')} \int_{-\infty}^\infty \text{Vol}_{N}(A(t)) \, dt
\]

\[
= h(\Omega) \int_\Omega \varphi(x) \, d\mu,
\]

the above condition also holds for \( \varphi \in W^{1,1}_0(\Omega) \) since \( C_0^\infty(\Omega) \) is dense in \( W^{1,1}_0(\Omega) \). Now for any \( p > 1 \) and \( u \in W^{1,p}_0(\Omega) \), define \( \Phi(u) := |u|^p \). Then, by Hölder’s inequality,

\[
\int_\Omega |d\Phi(u)| \, d\mu = \int_\Omega |du|^p \, d\mu \leq p \left( \int_\Omega |u|^p \, d\mu \right)^{(p-1)/p} \left( \int_\Omega |du|^p \, d\mu \right)^{1/p}.
\]

(4.2)
Letting \( \phi = u^p \), by (4.2) we have
\[
\begin{align*}
    h(\Omega) & \leq \int_{\Omega} |d\phi| \, d\mu \\
    & = \int_{\Omega} |d\phi^p| \, d\mu \\
    & = \int_{\Omega} |u^p| \, d\mu \\
    & \leq \left( \int_{\Omega} |u^p| \, d\mu \right)^{1/p} \left( \int_{\Omega} |u^p| \, d\mu \right)^{1/p} \\
    & = p \left( \int_{\Omega} |d\phi^p| \, d\mu \right)^{1/p}.
\end{align*}
\]

Since \( u \in W^{1,p}_0(\Omega) \) was arbitrary, we arrive at
\[
    \frac{h(\Omega)}{p} \leq \lambda_{1,p}(\Omega),
\]
which concludes the result.

**Corollary 4.3** ([23]). The first eigenvalue of the \( p \)-Laplacian \( \lambda_{1,p}(\Omega) \) converges to Cheeger’s constant \( h(\Omega) \) as \( p \to 1 \).

The proof is in [23] and we omit it here. This corollary simply implies that if we take
\[
    \lambda_{1,1}(\Omega) := \limsup_{p \to 1} \lambda_{1,p}(\Omega) = h(\Omega),
\]
then one asks for the solvability of the limiting problem
\[
    -\text{div} \left( \frac{du}{|du|} \right) = \lambda_{1,1}(\Omega) \quad \text{in} \ \Omega,
\]
\[
    u = 0 \quad \text{on} \ \partial \Omega.
\]

The proof of Proposition 1.4 is based on the proof of Theorem 4.2; it is therefore summarized below.

### 4.1 Proof of Proposition 1.4

Let \((u, v) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)\) be the pair of eigenfunctions corresponding to \( \lambda_{1,p,q}(\Omega) \) with \( u > 0, v > 0 \) by definition. Then
\[
    \lambda_{1,p,q}(\Omega) = \frac{\alpha}{p} \int_{\Omega} |du|^p \, d\mu + \frac{\beta}{q} \int_{\Omega} |dv|^q \, d\mu \quad \text{with} \quad \int_{\Omega} |u|^{a-1}v^{\beta-1} \langle u, v \rangle \, d\mu = 1.
\]

Using Theorem 4.2, we have
\[
    \lambda_{1,p}(\Omega) = \frac{\int_{\Omega} |du|^p \, d\mu}{\int_{\Omega} |u|^p \, d\mu} \geq \left( \frac{h(\Omega)}{p} \right)^p,
\]
which implies
\[
    \int_{\Omega} |du|^p \, d\mu \geq \left( \frac{h(\Omega)}{p} \right)^p \|u\|_{p,\Omega}^p \quad \text{and} \quad \int_{\Omega} |dv|^q \, d\mu \geq \left( \frac{h(\Omega)}{p} \right)^q \|v\|_{q,\Omega}^q
\]
since \( u \) and \( p \) in Theorem 4.2 were arbitrary and for the compact embedding of \( W^{1,p}_0(\Omega) \to L^p(\Omega) \). Therefore,
\[
    \lambda_{1,p,q}(\Omega) = \frac{\alpha}{p} \int_{\Omega} |du|^p \, d\mu + \frac{\beta}{q} \int_{\Omega} |dv|^q \, d\mu \geq \frac{\alpha}{p} \left( \frac{h(\Omega)}{p} \right)^p \|u\|_{p,\Omega}^p + \frac{\beta}{q} \left( \frac{h(\Omega)}{p} \right)^q \|v\|_{q,\Omega}^q.
\]
which proves (1.6). The next thing to do is to obtain the limiting behavior of $\lambda_{1,p,q}$ as $p \to 1$ and $q \to 1$. Heuristically, as $p \to 1$ and $q \to 1$, we obtain $a/p + \beta/q \to a + \beta = 1$. Taking $|u|_{r} = 1$, i.e., we normalize the eigenfunction of the $r$-Laplacian and then observe that the lower bound in the last inequality converges to $h(\Omega)$:

$$\limsup_{p \to 1, q \to 1} \lambda_{1,p,q}(\Omega) = h(\Omega).$$

It therefore suffices to obtain a finite bound for $\lambda_{1,p,q}(\Omega)$ as $p \to 1$ and $q \to 1$. To obtain a suitable upper bound for $\lambda_{1,r}(\Omega)$ (resp. $\int_{\Omega} |Du|^{r} \ d\mu$, $r > 1$), we can follow the proof of [23, Corollary 6] and then conclude that

$$\lambda_{1,1,1}(\Omega) := \limsup_{p \to 1, q \to 1} \lambda_{1,p,q}(\Omega) = h(\Omega).$$

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