Optimally Dense Packings for Fully Asymptotic Coxeter Tilings by Horoballs of Different Types

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Abstract

The goal of this paper is to determine the optimal horoball packing arrangements and their densities for all four fully asymptotic Coxeter tilings (Coxeter honeycombs) in hyperbolic 3-space $\mathbb{H}^3$. Centers of horoballs are required to lie at vertices of the regular polyhedral cells constituting the tiling. We allow horoballs of different types at the various vertices. Our results are derived through a generalization of the projective methodology for

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hyperbolic spaces. The main result states that the known Böröczky–Florian density upper bound for "congruent horoball" packings of $H^3$ remains valid for the class of fully asymptotic Coxeter tilings, even if packing conditions are relaxed by allowing for horoballs of different types under prescribed symmetry groups. The consequences of this remarkable result are discussed for various Coxeter tilings.

1 Introduction

Local optimal ball packings for regular tilings have been studied extensively in the literature. Of special interest are tilings in hyperbolic $n$-space; Böröczky and Florian [B–F64] gave the universal density upper bound for all congruent ball packings of the 3-dimensional hyperbolic space $H^3$ without any symmetry assumptions. This classical result provides the density upper bound realized by a regular horoball packing of $(3, 3, 6)$ in $H^3$, as shown in Section 2. The optimal density is related to the Dirichlet-Voronoi cell of every ball, as follows:

$$s_0 = (1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - + + \ldots)^{-1} \approx 0.85327609.$$  

This limit is achieved by 4 horoballs centered at the vertices of a regular ideal simplex, tangent to each other at the "midpoints" of the edges, i.e. as the projection of the simplex center into any edge of the simplex.

Papers [Be], [Bo–R], [G–K–K], [K91], [R06] introduce recent novel developments in the classical topic of the ball (or sphere) packings of $H^3$. Locally dense (optimal) ball, horoball packings in $H^3$ are of great significance, as important information regarding crystal structures can be obtained using locally optimal ball and horoball arrangements.

The present work is based on the projective interpretation of the hyperbolic geometry, proposed in [M93], [M97]. In subsequent works the second author studied a class of face transitive tilings [D–H–M], the so-called generalized Lambert-cube tilings in [Sz03-1] and [Sz05-1], where an algorithmic approach for determining volumes of hyperbolic polyhedra was developed and implemented. Using this novel approach, the locally optimal ball packings were found for the configurations in which the ball and horoball centers lie either within the Lambert-cubes or at the vertices of the cubes, respectively, and the optimal packing densities of the corresponding tilings were computed [Sz05-1]. Optimal ball and horoball packings of the regular Coxeter honeycombs in $H^d$, $(d \geq 3)$ with one horoball
In related work, $d$-dimensional ($d \geq 3$) hyperbolic prism honeycombs generated by "inscribed hyperspheres" were investigated in [Sz06-1] and [Sz06-2]. The optimal hyperball packings of infinitely many 3-dimensional prism tilings (mosaics) together with their metric data were determined in [Sz06-2]. In hyperbolic 4-space $\mathbb{H}^4$ there are only 2 honeycombs with metric data corresponding to their 3-dimensional counterparts. The densities of the optimal hyperball packings in 4-space are determined in [Sz06-2]. In $\mathbb{H}^5$ there are 3 types of such mosaics, and the corresponding problems are extensively studied. In $\mathbb{H}^d$ ($d > 5$) there are no longer any regular prism tilings.

In this paper, we study locally optimal ball and horoball packings in the four fully asymptotic hyperbolic tilings of $\mathbb{H}^3$, while allowing different types of horoballs to be centered at the vertices of the honeycombs. In Section 2, we provide preliminaries on the $d$-dimensional honeycombs. In Section 3, we introduce the projective model [Sz05-1] to determine the densities of the optimally dense horoball packings in hyperbolic space $\mathbb{H}^d$. In Section 4 we determine the optimal packing densities in $\mathbb{H}^3$ for various honeycombs, when horoballs of various types are allowed. We find that the densest possible packings yield density values identical to that of the Böröczky–Florian bound [B–F64]. In all studied configurations, the optimal densities never surpass the Böröczky–Florian upper bound, even when replacing the "congruency" constrains with regularity constraints. We finish the paper with conclusions and directions for future research.

2 Overview on $d$-dimensional hyperbolic honeycombs

Hyperbolic geometry is based on the principles of Bolyai-Lobachevsky geometry [P06]. A $d$-dimensional honeycomb $\mathcal{P}$, also referred to as a solid tessellation or tiling, is an infinite collection of congruent polyhedra (polytopes) that fit together face-to-face to fill the entire geometric space ($\mathbb{H}^d$ ($d \geq 2$)) exactly once. We take the cells to be congruent regular polyhedra. A honeycomb with cells congruent to a given regular polyhedron $P$ exists if and only if the dihedral angle of $P$ is a submultiple of $2\pi$ (in the hyperbolic plane zero angles are also permissible). A complete classification of honeycombs with bounded cells was first given by SCHLEGEL in 1883. The classification was completed by including the polyhedra
with unbounded cells, namely the fully asymptotic ones by Coxeter in 1954 [C56]. Such honeycombs exist only for \( d \leq 5 \) in hyperbolic \( d \)-space \( \mathbb{H}^d \). In this paper Coxeter honeycombs or Coxeter tilings refer to tilings described in Table 1.

An alternative approach to describing honeycombs involves analysis of their symmetry groups. If \( \mathcal{P} \) is a Coxeter honeycomb, then any rigid motion moving one cell into another maps the entire honeycomb onto itself. The symmetry group of a honeycomb is denoted by \( \text{Sym}\mathcal{P} \). The characteristic simplex \( \mathcal{F} \) of any cell \( P \in \mathcal{P} \) is a fundamental domain of the symmetry group \( \text{Sym}\mathcal{P} \) generated by reflections in its facets which are \( (d-1) \)-dimensional hyperfaces.

The scheme of a regular polytope \( P \) is a weighted graph (diagram) characterizing \( P \subset \mathbb{H}^d \) up to congruence. The nodes of the scheme, numbered by \( 0, 1, \ldots, d \), correspond to the bounding hyperplanes of \( \mathcal{F} \). Two nodes are joined by an edge if the corresponding hyperplanes are non-orthogonal. Let the set of weights \( (n_1, n_2, n_3, \ldots, n_{d-1}) \) be the Schlafli symbol of \( P \), and \( n_d \) be the weight describing the dihedral angle of \( P \), such that the dihedral angle is equal to \( \frac{2\pi}{n_d} \). In this case \( \mathcal{F} \) is the Coxeter simplex with the scheme:

\[
\begin{array}{cccccc}
0 & n_1 & n_2 & \ldots & n_{d-1} & n_d \\
1 & 2 & \ldots & d-2 & d-1 & d
\end{array}
\]

![Figure 1: Coxeter-Schlafli simplex scheme](image)

The Schlafli symbol of the honeycomb \( \mathcal{P} \) is the ordered set \( (n_1, n_2, n_3, \ldots, n_{d-1}, n_d) \) above. A \( (d+1) \times (d+1) \) symmetric matrix \( (b^{ij}) \) is constructed for each scheme in the following manner: \( b^{ii} = 1 \) and if \( i \neq j \in \{0, 1, 2, \ldots, d\} \) then \( b^{ij} = -\cos \frac{2\pi}{n_{ij}} \). For all angles between the facets \( i, j \) of \( \mathcal{F} \) holds then \( n_k = n_{k-1,k} \).

Reversing the numbering of the nodes of scheme \( \mathcal{P} \) while keeping the weights, leads to the scheme of the dual honeycomb \( \mathcal{P}^* \) whose symmetry group coincides with \( \text{Sym}\mathcal{P} \).

In this paper we investigate regular Coxeter honeycombs and their optimal horoball packings in the hyperbolic space \( \mathbb{H}^3 \), where the horoballs are allowed to be of different types. \( \text{Sym}\mathcal{P} \) denotes the symmetry group of the honeycomb \( \mathcal{P}_{n_1,n_2\ldots,n_d} \), thus

\[
P_{n_1n_2\ldots n_d} = \bigcup_{\gamma \in \text{Sym}\mathcal{P}_{n_1n_2\ldots n_{d-1}}} \gamma(\mathcal{F}_{n_1n_2\ldots n_d}).
\]

In order to calculate the packing density, we relate each ball or horoball, respectively, to its regular polytope \( P_{n_1n_2\ldots n_d} \) in which it is contained. These polytopes
are not necessarily assumed to be Dirichlet-Voronoi cells.

Table 1: Classification of 3-dimensional Coxeter tilings

| No. | Description                                           | Schläfli symbol \((p, q, r)\) |
|-----|-------------------------------------------------------|-------------------------------|
| 1.  | Cells having proper centers and vertices              | \((3,5,3), (4,3,5), (5,3,4), (5,3,5)\) |
| 2.  | Fully asymptotic cells                                | \((3,3,6), (3,4,4), (4,3,6), (5,3,6)\) |
| 3.  | Infinite centers and proper or ideal vertices          | \((3,6,3), (4,4,4), (6,3,6)\)\((4,4,3), (6,3,3), (6,3,4), (6,3,5)\) |

As listed in Table 1, Coxeter tilings with parameters in row 1 include cells having proper centers and vertices. The polyhedra of honeycombs with Schläfli symbols in row 3 of Table 1, have infinite centers and proper or ideal vertices. The polyhedra of tilings in row 2 of Table 1 are called fully asymptotic; moreover their centers are proper and their vertices lie on the absolute of the hyperbolic space i.e. they are ideal vertices.

3 The projective model

3.1 Basic Notions

Let \(X\) denote one of either the \(d\)-dimensional sphere \(S^d\), the \(d\) dimensional Euclidean space \(E^d\), or the hyperbolic space \(H^d\), \(d \geq 2\). For \(H^d\) we use the projective model in Lorentz space \(E^{1,d}\) of signature \((1, d)\), i.e. \(E^{1,d}\) is the real vector space \(V^{d+1}\) equipped with the bilinear form of signature \((1, d)\)

\[
\langle x, y \rangle = -x^0 y^0 + x^1 y^1 + \cdots + x^d y^d \tag{3.1}
\]

where the non-zero vectors

\[
x = (x^0, x^1, \ldots, x^d) \in V^{d+1} \quad \text{and} \quad y = (y^0, y^1, \ldots, y^d) \in V^{d+1},
\]

are determined up to real factors and they represent points in \(P^d(\mathbb{R})\). \(H^d\) is represented as the interior of the absolute quadratic form

\[
Q = \{ [x] \in P^d | \langle x, x \rangle = 0 \} = \partial H^d \tag{3.2}
\]

in real projective space \(P^d(\mathbb{V}^{d+1}, V_{d+1})\). All proper interior point \(x \in H^d\) are characterized by \(\langle x, x \rangle < 0\).
The points on the boundary $\partial \mathbb{H}^d$ in $\mathcal{P}^d$ represent the absolute points at infinity of $\mathbb{H}^d$. Points $y$ with $\langle y, y \rangle > 0$ lie outside $\partial \mathbb{H}^d$ and are called outer points of $\mathbb{H}^d$. Let $P([x]) \in \mathcal{P}^d$; a point $[y] \in \mathcal{P}^d$ is said to be conjugate to $[x]$ relative to $Q$ when $\langle x, y \rangle = 0$. The set of all points conjugate to $P([x])$ form a projective (polar) hyperplane

$$pol(P) := \{ [y] \in \mathcal{P}^d | \langle x, y \rangle = 0 \}.$$ (3.3)

Hence the bilinear form $Q$ by (3.1) induces a bijection (linear polarity $V_{d+1} \rightarrow V_{d+1}$) from the points of $\mathcal{P}^d$ onto its hyperplanes.

Point $X[x]$ and the hyperplane $\alpha[a]$ are called incident if the value of the linear form $a$ on the vector $x$ is equal to zero; i.e., $xa = 0 \ (x \in V_{d+1} \setminus \{0\}, \ a \in V_{d+1} \setminus \{0\})$. Straight lines in $\mathcal{P}^d$ are characterized by the 2-subspaces of $V_{d+1}$ or $(d-1)$-spaces of $V_{d+1}$ [M97].

Let $P \subset \mathbb{H}^d$ denote a polyhedron bounded by hyperplanes $H^i$, which are characterized by unit normal vectors $b^i \in V_{d+1}$ directed inwards with respect to $P$:

$$H^i := \{ x \in \mathbb{H}^d | \langle x, b^i \rangle = 0 \} \text{ with } \langle b^i, b^i \rangle = 1.$$ (3.4)

We always assume $P$ to be an acute-angled polyhedron and the vertices to be proper points or to lie at infinity.

The Gram matrix $G(P) := (\langle b^i, b^j \rangle)\ i, j \in \{0, 1, 2 \ldots d\}$ of normal vectors $b^i$ associated with $P$ is an indecomposable symmetric matrix of signature $(1, d)$ with entries $\langle b^i, b^i \rangle = 1$ and $\langle b^i, b^j \rangle \leq 0$ for $i \neq j$, having the following geometrical meaning

$$\langle b^i, b^j \rangle = \begin{cases} 0 & \text{if } H^i \perp H^j, \\ - \cos \alpha^{ij} & \text{if } H^i, H^j \text{ intersect on } P \text{ at angle } \alpha^{ij}, \\ -1 & \text{if } H^i, H^j \text{ are parallel in hyperbolic sense}, \\ - \cosh l^{ij} & \text{if } H^i, H^j \text{ admit a common perpendicular of length } l^{ij}. \end{cases}$$

**Definition 3.1** [K91, B–H] An orthoscheme $\mathcal{O}$ in $X$ is a simplex bounded by $d+1$ hyperplanes $H^0, \ldots, H^d$ such that

$$H^i \perp H^j, \text{ for } j \neq i - 1, i, i + 1.$$ 

A plane orthoscheme is a right-angled triangle, the area of which can be expressed by the defect formula. For orthoschemes we denote the $(d-1)$-hyperface opposite to the vertex $A_i$ by $H^i \ (0 \leq i \leq d)$. An orthoscheme $\mathcal{O}$ has $d$ dihedral
angles different from right angles. Let $\alpha^{ij}$ denote the dihedral angle of $O$ between the faces $H^i$ and $H^j$. Then we have

$$\alpha^{ij} = \frac{\pi}{2}, \text{ if } 0 \leq i < j - 1 \leq d.$$  

The remaining $d$ dihedral angles $\alpha^{i,i+1}$, $(0 \leq i \leq d-1)$ are called the essential angles of $O$. The initial vertex $A_0$ and final vertex $A_d$ of the orthogonal edge-path $igcup_{i=0}^{d-1} A_i A_{i+1}$ are called principal vertices of the orthoscheme.

In this work, the characteristic simplex $F$ of any honeycomb $P$ with Schl"afli symbol $(n_1, n_2, n_3, \ldots, n_d)$ is an orthoscheme.

The matrix $(b^{ij}) = G(P)$ is the so called Coxeter-Schl"afli matrix of the orthoscheme $F$ with parameters $n_1, n_2, n_3, \ldots, n_d$:

$$
(b^{ij}) := \begin{pmatrix}
1 & -\cos \frac{\pi}{n_1} & 0 & \ldots & 0 \\
-\cos \frac{\pi}{n_1} & 1 & -\cos \frac{\pi}{n_2} & \ldots & 0 \\
0 & -\cos \frac{\pi}{n_2} & 1 & \ldots & 0 \\
0 & 0 & -\cos \frac{\pi}{n_3} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & -\cos \frac{\pi}{n_d} & 1
\end{pmatrix}. \quad (3.5)
$$

Inverting the Coxeter-Schl"afli matrix $(b^{ij})$ (3.5) of an orthoscheme gives the matrix $(a_{ij})$, which can be used to express distances between two vertices through the formula [Sz06-2]:

$$
\cosh \frac{d_{ij}}{k} = -\frac{a_{ij}}{\sqrt{a_{ii}a_{jj}}}. \quad (3.6)
$$

In this paper we set the sectional curvature of $\mathbb{H}^d$, $K = -k^2$, to be $k = 1$. The distance $s$ of two proper points $(x)$ and $(y)$ is calculated by the formula:

$$
\cosh \frac{s}{k} = -\frac{\langle x, y \rangle}{\sqrt{\langle x, x \rangle \langle y, y \rangle}}. \quad (3.7)
$$
3.2 Characterization of horoballs in Hyperbolic Space $\mathbb{H}^3$

**Definition 3.2** A horosphere in the hyperbolic geometry is the surface orthogonal to the set of parallel lines, passing through the same point on the absolute quadratic surface (simply absolute) of the hyperbolic space.

We represent hyperbolic space $\mathbb{H}^3$ in the Cayley-Klein ball model. We introduce a projective coordinate system using vector basis $b_i \ (i = 0, 1, 2, 3)$ for $\mathcal{P}^3$ where the coordinates of center of the model is $A_2(1, 0, 0, 0)$. We pick an arbitrary point at infinity to be $A_3(1, 0, 0, 1)$.

As it is known, the equation of a horosphere with center $A_3(1, 0, 0, 1)$ through point $S(1, 0, 0, s)$ is derived using the surface pencil of the absolute sphere and a plane tangent to the sphere at point $A_3(1, 0, 0, 1)$. The equation of the absolute sphere is $-x^0x^0 + x^1x^1 + x^2x^2 + x^3x^3 = 0$. The equation of a plane tangent to the absolute of our model at point $A_3(1, 0, 0, 1)$ is $x^0 - x^3 = 0$.

The general equation of the horosphere is

$$0 = \lambda(-x^0x^0 + x^1x^1 + x^2x^2 + x^3x^3) + \mu(x^0 - x^3)^2.$$

This passes through point $S(1, 0, 0, s)$ so we may write

$$\lambda(-1 + s^2) + \mu(-1 + s)^2 = 0 \Rightarrow \frac{\lambda}{\mu} = \frac{(s - 1)^2}{1 - s^2}$$

If $s \neq 1$, then

$$\frac{(s - 1)^2}{1 - s^2}(-x^0x^0 + x^1x^1 + x^2x^2 + x^3x^3) + (x^0 - x^3)^2 = 0 \Leftrightarrow (s - 1)(-x^0x^0 + x^1x^1 + x^2x^2 + x^3x^3) - (1 + s)(x^0 - x^3)^2 = 0$$

This way we obtain the following equation for the horosphere in our Cayley-Klein model of $\mathbb{H}^3$:

$$-2sx^0x^0 - 2x^3x^3 + 2(s + 1)(x^0x^3) + (s - 1)(x^1x^1 + x^2x^2) = 0 \quad (3.8)$$

**Remark 3.3** We have obtained the equation of the horosphere in the Cartesian coordinate system: $(x := \frac{x^1}{x^3}, \ y := \frac{x^2}{x^3}, \ z := \frac{x^3}{x^3})$

$$\frac{2(x^2 + y^2)}{1 - s} + \frac{4(z - \frac{s + 1}{2})}{(1 - s)^2} = 1 \quad (3.9)$$
Remark 3.4 It is useful for visualization purposes to convert the horosphere equation into polar coordinates. By multiplying the polar coordinate form by rotation matrices we can easily obtain horospheres around arbitrary points at infinity in the model. The polar form is by parameters $\phi \in [0, 2\pi)$, $\theta \in [0, \pi]$

$$
x = \sqrt{\frac{1 - s}{2}} \sin \theta \cos \phi \quad y = \sqrt{\frac{1 - s}{2}} \sin \theta \sin \phi
$$

$$
z = \frac{1 + s}{2} + \frac{1 - s}{2} \cos \theta. \tag{3.10}
$$

3.3 Volumes of horoball sectors

The length $l(x)$ of a horospheric arc of a chord segment $x$ is determined by the classical formula due to J. Bolyai:

$$
l(x) = k \sinh \frac{x}{k} \quad \text{(at present } k = 1). \tag{3.11}
$$

The intrinsic geometry of the horosphere is Euclidean, therefore, the area $A$ of a horospherical triangle is computed by the formula of Heron. The volume of the horoball pieces can be calculated using another formula by J. Bolyai. If the area of a domain on the horosphere is $A$, the volume determined by $A$ and the aggregate of axes drawn from $A$ is equal to

$$
V = \frac{1}{2} k A \quad \text{(we assume that } k = 1 \text{ in this paper).} \tag{3.12}
$$

4 Horoball packings for totally asymptotic Coxeter honeycombs

4.1 Basic results on packing in $\mathbb{H}^3$

In this section we determine optimal horoball packings for the four totally asymptotic Coxeter tilings $P_{pqr}$ with Schläfli symbols $(p, q, r) = (3, 3, 6), (3, 4, 4), (4, 3, 6), (5, 3, 6)$. Vertices of a regular cell $P_{pqr}$ are denoted by $E_i$, $(i = 0, 1, 2, 3, 4 \ldots)$, and lie on the absolute of $\mathbb{H}^3$, hence these vertices are centers of horoballs. The number of the vertices of $P_{pqr}$ is denoted by $N_{pqr}$ and we write $B_i$ for the horoball centered at $E_i$. We require a horoball to lie at every ideal vertex of the honeycomb, and we vary the touching types of the horoballs.
Remark 4.1 For example, if \((p, q, r) = (3, 3, 6)\) then we obtain the “tetrahedral case” where \(N_{336} = 4\). For parameter \((p, q, r) = (3, 4, 4)\) we get the “octahedral case” and \(N_{344} = 6\).

The type of a horoball is allowed to expand until either the horoball comes into contact with other horoballs or a non-adjacent side of the honeycomb. These conditions are satisfactory to ensure that the balls form a non-overlapping horoball arrangement, as such the collection of all horoballs is a well defined packing in \(\mathbb{H}^3\), denoted by \(B_{pqr}\).

Definition 4.2 The density of a horoball packing in Coxeter honeycomb \(\mathcal{P}_{pqr}\) is defined as

\[
\delta(B_{pqr}) = \frac{\sum_{i=1}^{N_{pqr}} Vol(B_i \cap \mathcal{P}_{pqr})}{Vol(\mathcal{P}_{pqr})}.
\]

The aim of this section is to determine the optimal packing densities for the four totally asymptotic tilings (see Table 1) in 3-dimensional hyperbolic space \(\mathbb{H}^3\).

We will make heavy use of the following Lemma [Sz05-1]:

Lemma 4.3 Let there \(B_1(x)\) and \(B_2(x)\) denote two horoballs with centers \(C_1\) and \(C_2\) respectively. Take \(\tau_1\) and \(\tau_2\) to be two congruent trihedra, with vertices \(C_1\) and \(C_2\). Assume these horoballs \(B_1\) and \(B_2\) are tangent at point \(I(x) \in C_1C_2\) to be a common edge of the two trihedra \(\tau_1\) and \(\tau_2\). We define the point of contact \(I(0)\) such that the following equality holds for volumes of horoball sectors:

\[
V(0) := 2Vol(B_1(0) \cap \tau_1) = 2Vol(B_2(0) \cap \tau_2).
\]

If \(x\) denotes the hyperbolic distance between \(I(0)\) and \(I(x)\), then the function

\[
V(x) := Vol(B_1(x) \cap \tau_1) + Vol(B_2(x) \cap \tau_2)
\]

strictly increases as \(x \to \pm \infty\).

Proof: Let \(\mathcal{L}\) and \(\mathcal{L}'\) be parallel horocycles with centre \(C\) and let \(A\) and \(B\) be two points on the curve \(\mathcal{L}\) and \(A' := CA \cap \mathcal{L}', B' := CB \cap \mathcal{L}'\). By the classical formula of J. Bolyai

\[
\frac{\mathcal{H}(A'B')}{{\mathcal{H}(AB)}} = e^{\frac{x}{k}},
\]

where the horocyclic distance between \(A\) and \(B\) is denoted by \(\mathcal{H}(A, B)\).
Then by the above formulas we obtain the following volume function:

\[ V(x) = \text{Vol}(B_1(x) \cap \tau_1) + \text{Vol}(B_2(x) \cap \tau_2) = \]
\[ = \frac{1}{2} V(0) \left( e^{\frac{2x}{k}} + e^{-\frac{2x}{k}} \right) = V(0) \cosh \left( \frac{2x}{k} \right). \]

It is well known that this function strictly increases in the interval \((0, \infty)\). This Lemma is illustrated in Fig. 2. □

**Corollary 4.4** The statement of the Lemma holds for horoballs intersecting the arbitrary two congruent frames of rays joining \(C_1\) and \(C_2\) above, respectively.

**Proof:** Follows from the Lemma 4.3, by dividing the frames of rays into congruent trihedra. □

### 4.2 The \((3, 3, 6)\) tetrahedral tiling

The \((3, 3, 6)\) Coxeter tiling is a three dimensional honeycomb with cells comprised of fully asymptotic regular tetrahedra. We arbitrarily select one such tetrahedron \(E_0E_1E_2E_3\) (see Fig. 3), and place the horoball centers at vertices \(E_0, \ldots, E_3\). We vary the types of the horoballs so that they satisfy our constraints of non-overlap. The packing density is obtained by Definition 4.2.

Define the orthoscheme \(A_0A_1A_2A_3\) as follows: \(A_0 = E_0\) and \(A_3 = E_3\) are two vertices of the tetrahedra (see Fig. 3); \(A_2\) is the center of the triangle \(E_0E_1E_2\) opposite the vertex \(A_3\), and \(A_1\) is the footpoint of \(A_3\) on the edge \(E_0E_1\). One tetrahedral cell is decomposable into 6 such congruent orthoschemes. The Schlafli
symbol of orthoscheme \( A_0A_1A_2A_3 \) is \((3, 6, 3)\), and the orthoscheme is labeled by \( \mathcal{O}_{(3,6,3)} \).

\( B_0 \) and \( B_3 \) are two horoballs centered at \( E_0 \) and \( E_3 \), i.e., the two vertices of the tetrahedra common with the orthoscheme. The density of the \((3, 3, 6)\) Coxeter tiling is obtained using Definition 4.2:

\[
\delta(B_{336}) = \frac{\text{Vol}(B_0 \cap \mathcal{O}_{(3,6,3)}) + \text{Vol}(B_3 \cap \mathcal{O}_{(3,6,3)})}{\text{Vol}(\mathcal{O}_{(3,6,3)})}
\]

**Proposition 4.5** The packing density obtained in \( \mathcal{O}_{(3,6,3)} \) can be extended to tetrahedron \( P_{336} \) and therefore to the entire \( \mathbb{H}^3 \).

**Proof:** We consider the following steps:

1. In an optimally dense packing, at least two horoballs must touch each other in the tetrahedron, otherwise the density could be improved by blowing up any one horoball until it touches a neighboring horoball.

2. If two horoballs touch at the “midpoint” of edge \( A_0A_3 \) as projection of the simplex centre on it, then by blowing up the remaining two horoballs, they will also touch at “midpoints”, due to symmetry considerations. Note, that this case is the arrangement in the Böröczky–Florian density upper bound using the same horoballs.

3. Given two horoballs tangent at a non-midpoint of an edge, then the horoball, having the midpoint in its interior, will contain the “midpoint” of all 3 edges extending from its center. As a result the remaining 3 horoballs should be of the same touching type. The “small horoballs” each is tangent only to the large horoball.

We just showed that the “largest horoball” determines the configuration of all other horoballs, and as a consequence the packing density. One parameter corresponding to the “largest horoball” suffices to determine the packing density for all candidates of optimal density.

Due to symmetry, it is enough to consider cases within orthoscheme \( \mathcal{O}_{(3,6,3)} \), where the horoball at \( E_0 \) expands from the midpoint until it becomes tangent to the side of the cell opposite to it. Assume no balls cover the midpoint along \( E_0E_3 \). Then the horoball \( B_0 \) can be expanded until the midpoint. In this case, the horoball \( B_3 \) is contained within \( \mathcal{O}_{(3,6,3)} \). Finally, the packing density can be
varied by expanding the horoball $B_0$ while keeping $B_3$ tangent to it. If we expand $B_0$ until it touches $A_2$, all candidates for optimal packings within the orthoscheme are considered. The densities obtained from the orthoscheme $O_{(3,6,3)}$ will cover all candidates for optimally dense packings of honeycomb $(3,3,6)$. Densities determined within the orthoschemes can be generalized to the entire $\mathbb{H}^3$ by the symmetries of $P_{336}$. □

In the rest of this section, we prove the basic theorem on the optimal packing density in $(3,3,6)$. First, we define the tangent point $I(0) \in A_0A_3$ of horoballs $B_0$ and $B_3$ so that the following equality holds for the volumes of the horoball sectors

$$V(0) := 2Vol(B_0(0) \cap O_{(3,6,3)}) = 2Vol(B_3(0) \cap O_{(3,6,3)}).$$

Consider point $I(x)$ on the edge $A_0A_3$ where the horoballs $B_i(x)$, $(i = 0, 3)$ are tangent at point $I(x) \in A_0A_3$. as the hyperbolic distance between $I(0)$ and $I(x)$. Function $V(x)$ is defined as follows:

$$V(x) := Vol(B_0(x) \cap O_{(3,6,3)}) + Vol(B_3(x) \cap O_{(3,6,3)}).$$

By Lemma 4.3 it follows that function $V(x)$ strictly increases as $I(x) (x \in [-\arctanh(1/2), \arctanh(1/2)])$ moves away from $I(0)$ along $A_0A_3$. That implies that the density function

$$\delta(B_{336})(x) = \frac{V(x)}{Vol(O_{(3,6,3)})} = \frac{Vol(B_0(0) \cap O_{(3,6,3)})e^{2x} + Vol(B_3(0) \cap O_{(3,6,3)})e^{-2x}}{Vol(O_{(3,6,3)})}$$

attains its maximum at the two endpoints of the interval $[-\arctanh(1/2), \arctanh(1/2)]$.

**Definition 4.6** A coordinate system is assigned to the orthoscheme $A_0A_1A_2A_3$; let $A_2 := (1, 0, 0, 0)$ be the origin, $A_0 := (1, 0, 1, 0)$, $A_3 := (1, 0, 0, 1)$ and $A_1 = (1, \sqrt{3}/4, 1/4, 0)$ where $A_1$ is the “midpoint” of the edge $E_0E_1$ (see Fig. 3).

In the orthoscheme $A_0A_1A_2A_3$, the horoball centers are located at points $A_0$ and $A_3$, and the horoballs meet along side $A_0A_3$ (See Fig. 3 (i)). In order to determine the properties of the packing, we calculate the five points of intersection of the horoballs with the edges of the orthoscheme. The distances of these five points of intersection determine the area of the horospheric triangles, hence the volume of the horoball sectors through Bolyai’s formulas (3.11-12).
In order to simplify the computations and visualizations of horospheres we introduce a new parameter $s$. We define $s$ to be the Euclidean parameter of the origin $A_2$ and point $S(1, 0, 0, s)$ along segment $A_2A_3$. Based on the proof of Proposition 4.4, $s$ is the only parameter that determines the density of the packing.

All horoball configurations that yield valid candidates for the optimal packing density occur while we continuously vary the horoball parameter $s \in [0, 1/2]$. In order to determine the optimal packing, we proceed as follows. We express the density $\delta(B_{336})$ as the function of $s$, study its behavior and determine its extremal points. If $s = 1/2$, all horoballs are in same type, thus we achieve the packing arrangement $B_{336}^1$, which is the Böröczky–Florian case with the known packing density. On the other hand, for $s = 0$, we find a different horoball arrangement $B_{336}^2$ (see Fig. 3) with the same density as for $s = 1/2$. Finally, from Lemma 4.3 it follows that all other densities with horoball parameter $s \in (0, 1/2)$ are smaller.

First we calculate the five intersections of the edges of $O_{(3,6,3)}$ and the two horoballs $B_0$ and $B_3$ for $s = 0$. Recall that these are all a function of the “type” of the large horoball, and depend on the parameter $s$. The length of the sides of the horospherical triangle, with vertices $X_i, Y_i, Z_i \in B_i, i \in \{0, 3\}$ are given by

$$a_i := 2 \sinh \frac{d(X_i, Y_i)}{2}, b_i := 2 \sinh \frac{d(Y_i, Z_i)}{2}, c_i := 2 \sinh \frac{d(Z_i, X_i)}{2}.$$

Based on the side lengths, Heron’s formula can be used to obtain the area $A_i(s)$ of horospherical triangles $X_iY_iZ_i$. Using Bolyai’s volume formula (3.12) for horoball pieces, the Definition 4.2, and setting $k = 1$, the density of the pack-
ing can be expressed as a function of $s$ as well

$$\delta(B_{336})(s) = \frac{1}{2}(A_0(s) + A_3(s)) \frac{Vol(O_{(3,6,3)})}{V}, \quad s \in (0,1/2).$$

From Lemma 4.3 it follows, that the optimal densities are realized on the endpoints of the interval $[0,1/2]$. In order to determine the highest packing density, we calculate the density for horoball pieces with $s = 0$. The volumes of orthoscheme $O_{(3,6,3)}$ in $\mathbb{H}^3$ are calculated using Lobachevsky’s volume formula (see [S03-1]). All other densities with $0 \leq s \leq 1/2$ can be evaluated using the corresponding volume formula. The results are displayed in Fig. 4.a, while Fig. 4.b and Fig. 4.c illustrate the ball arrangements $B_{336}^i$ ($i = 1, 2$) for $s = 1/2$ and $s = 0$, respectively. Note that $B_{336}^1$ and $B_{336}^2$ are two different cases. In the first case the horoballs are of same type and they touch each other pairwise, while for $s = 0$ we have balls with 2 distinct types, and the “small balls” do not meet. As a result, we have just proved:

**Theorem 4.7** There are two distinct optimally dense horoball arrangements $B_{336}^i$, ($i = 1, 2$) for the tetrahedral Coxeter tiling $(3, 3, 6)$ with the same density: $\delta(B_{336}^2) \approx 0.85327609$.

### 4.3 The $(3, 4, 4)$ octahedral tiling

In this section we consider horoball packings with centers located at ideal vertices of the octahedral honeycomb $(3, 4, 4)$. Our approach is as in the previous section for the $(3, 3, 6)$ Coxeter tiling. We again allow the horoballs to be of different
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The $(3, 4, 4)$ Coxeter tiling decomposes the 3-dimensional hyperbolic space into congruent cells consisting of regular fully asymptotic octahedra. Four octahedra meet along each edge. As in the case of $(3, 3, 6)$, we again choose one cell of the tiling in order to perform our density calculations. Again, we vary the types of the horoballs centered in the honeycomb vertices. The packing density obtained from this cell is again extended to $\mathbb{H}^3$ by $\text{Sym} P_{344}$.

We define the orthoscheme $A_0A_1A_2A_3$ of an octahedron for the calculations. Let $A_0$ and $A_3$ be two adjacent vertices of the octahedron $E_0E_1E_2E_3E_4E_5$ (see Fig. 5. a-b-c), $A_2$ be the center of the octahedra, and take $A_1$ as the midpoint, in the Euclidean sense, of the edge extending from $A_0$ sharing a common facet with $A_3$. This orthoscheme is $O_{444}$ and has Schlӓfli symbol $(4, 4, 4)$.

**Definition 4.8** The following coordinate system is defined for orthoscheme $A_0A_1A_2A_3$ (see Fig. 6):

$$A_2 = (1, 0, 0, 0), A_0 = (1, 0, 1, 0), A_3 = (1, 0, 0, 1), A_1 = (1, \frac{1}{2}, \frac{1}{2}, 0).$$

Using Lobachevsky volume formula for orthoschemes, we obtain the volume of one octahedron [Sz05-2]:

$$\text{Vol}(P_{344}) = 16 \cdot \text{Vol}(O_{444}) \approx 3.66384.$$  (4.1)

Applying the definition of the packing density for the case of tiling $(3, 4, 4)$, we obtain:

$$\delta(B_{344}) = \frac{\sum_{i=1}^{6} \text{Vol}(B_i \cap P_{344})}{\text{Vol}(P_{344})},$$

where $B_i \cap P_{344}$ denotes the 6 horoball sectors, one in each vertex of the octahedron $P_{344}$, and we assume that the horoballs $B_i$ form a horoball packing in $\mathbb{H}^3$.

We consider the following three basic horoball configurations $B_i^j_{344}$, $(i = 1, 2, 3)$:

1. All 6 horoballs are of the same type and the adjacent horoballs touch each other at the ”midpoints” of each edge. We define the point of tangency of two horoballs $B_0$ and $B_3$ on side $A_0A_3$ to be $\text{I}(0)$ so that the following equality holds:

$$V(0) := 6 \cdot \text{Vol}(B_0(0) \cap P_{344}) = 6 \cdot \text{Vol}(B_3(0) \cap P_{344}) = 6 \cdot V_0.$$  (4.1)

In this case $V_0 := \text{Vol}(B_i \cap P_{344}) = 0.5$ $(i = 1, \ldots, 6)$ (see Fig. 5.a, Fig. 6.c).
2. Two "larger horoballs" with centers at $E_3$ and $E_5$ are tangent at the center of the octahedron, while horoballs at the remaining six vertices touch both "larger" horoballs. The point of tangency of the above two horoball types on segment $I(0)E_0$ is denoted by $I(x_1)$ where $x_1 = \log(2)/2$ is the hyperbolic distance between $I(0)$ and $I(x_1)$. In this case $V_1 := Vol(B_i \cap P_{344}) = 1$ $(i = 3, 5)$ and $V_2 := Vol(B_i \cap P_{344}) = 0.25$ $(i = 0, 1, 2, 4)$ (see Fig. 5.b, Fig. 6.b).

3. One horoball of the "maximally large" type centered at $E_3$. The large horoball is tangent to all non-neighboring sides of the octahedron and it determines the other five horoballs touching the "large horoball". The point of tangency of the two horoballs along segment $I(x_1)E_0$ is denoted by $I(x_2)$ where $x_2 = -\log(2)/2$ is the hyperbolic distance between $I(0)$ and $I(x_2)$. In this case $V_3 := Vol(B_3 \cap P_{344}) = 0.25$, $V_5 := Vol(B_5 \cap P_{344}) = 0.0625$ and $V_i := Vol(B_i \cap P_{344}) = 0.03125$ $(i = 0, 1, 2, 4)$ (see Fig. 5.c, Fig. 6.a).

Due to symmetry considerations it is sufficient to consider the cases when one horoball extends from the "midpoint" of an edge until it touches the opposite side of the cell. Here we give an analogous argument as in the proof of Proposition 4.4. Assume that none of the horoballs covers a "midpoint". Then the packing density may be improved until at least one ball reaches a midpoint. Moreover, consider that point $I(x)$ is on edge $A_0A_3 = E_0E_3$. This point is where the horoballs $B_i(x)$, $(i = 0, 3)$ are tangent at point $I(x) \in A_0I(0)$. Then $x$ is the hyperbolic distance between $I(0)$ and $I(x)$, and it is analogous to the previous section. It
is easy to see that we have to study two different cases to determine the optimal horoball arrangement:

1. \( x \in [0, x_1] \), horoballs \( B_3 \) and \( B_5 \) touch horoballs \( B_i \) \((i = 1, 2, 3, 4)\).

2. \( x \in [x_1, x_2] \), horoball \( B_3 \) touches horoballs \( B_i \) \((i = 1, 2, 3, 4, 5)\).

In the **first case** the function \( V(x) \) can be computed by the following formula

\[
V(x) := 4 \cdot Vol(B_0(x) \cap P_{344}) + 2 \cdot Vol(B_3(x) \cap P_{344}) \quad x \in [0, x_1].
\]

Similarly to the Lemma 4.3, we can prove the following Lemma:

**Lemma 4.9**

\[
V(x) := 4 \cdot Vol(B_0(x) \cap P_{344}) + 2 \cdot Vol(B_3(x) \cap P_{344}) = V_0(2 \cdot e^{2x} + 4 \cdot e^{-2x}), \quad x \in [0, x_1],
\]

and the maxima of function \( V(x) \) are realized in points \( I(0) \) and \( I(x_1) \).

In the **second case**, similarly to Lemma 4.6, the volume function \( V(x) \) is given by the following formula:

**Lemma 4.10**

\[
V(x) := 4 \cdot V_1e^{-2(x-x_1)} + V_2 \cdot (e^{2(x-x_1)} + e^{-2(x-x_1)}), \quad x \in [x_1, x_2],
\]

and the maximum of function \( V(x) \) is achieved at points \( I(x_1) \) and \( I(x_2) \).

By Definition 4.2 and Remark 4.1, as well as by Lemmas 4.5-4.6, it follows that the densest horoball packing is realized in three distinct primary horoball arrangements \( B_{344}^i \) \((i = 1, 2, 3)\). These optimal horoball packings belong to horoball parameters \( s = -1/3, s = 0, s = 1/3 \) (see Section (3.1)) and are illustrated in Fig. 6.a-b-c.

The maximal density can be computed by the method described in Section 4.2. Thus we have proven

**Theorem 4.11** *Three different optimally dense horoball arrangements \( B_{344}^i \) \((i = 1, 2, 3)\) exist for the octahedral Coxeter tiling \( (3, 4, 4) \), which share the optimum density \( \delta(B_{344}^i) \approx 0.818808 \).*
a. \( s = -1/3 \)  

b. \( s = 0 \)  

c. \( s = 1/3 \)

Figure 6: Optimal horoball packings of Coxeter honeycomb \((3, 4, 4)\).

4.4 The \((4, 3, 6)\) Cubic Tiling

The optimal packing densities for the cubic Coxeter tiling \((4, 3, 6)\) can be obtained by similar approaches as in the previous two sections for \((3, 3, 6)\) and \((3, 4, 4)\).

We consider horoball packings with centers located at the ideal vertices of the cube honeycomb \((4, 3, 6)\).

Analogous to the above cases we introduce a projective coordinate system, by an orthogonal vector basis with signature \((-1, 1, 1, 1)\), with the following coordinates of the vertices of the infinite regular cube (see Fig. 7), in the Cayley-Klein ball model:

\[
\begin{align*}
E_0(1, -\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, \frac{1}{3}), & \quad E_1(1, -\frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{3}, -\frac{1}{3}), & \quad E_2(1, 0, 2\frac{\sqrt{2}}{3}, -\frac{1}{3}), \\
E_3(1, 0, 0, 1), & \quad E_4(1, \frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{3}, -\frac{1}{3}).
\end{align*}
\]

Using Lobachevsky volume formula for orthoschemes, we obtain the volume of one cubic cell [Sz03-2]: \( Vol(P_{436}) = 48 \cdot Vol(O_{(4,3,6)}) \approx 0.507471 \). Applying the definition of the packing density for the case of tiling \((4, 3, 6)\), we obtain:

\[
\delta(B_{436}) = \frac{\sum_{i=1}^{8} Vol(B_i \cap P_{436})}{Vol(P_{436})}, \tag{4.7}
\]

where \( B_i \cap P_{436} \) \((i = 1, \ldots, 8)\) denote the 8 horoball sectors, one in each vertex of the cube \( P_{436} \) and we assume that the horoballs \( B_i \) form a packing in \( \mathbb{H}^3 \). We consider the following four main horoball configurations \( B_{436}^i \) \((i = 1, 2, 3, 4)\):
1. All 8 horoballs are of the same type and the adjacent horoballs touch each other at the "midpoints" of each edge (see Fig. 8.a). The density of this packing: $\delta(B_{1436}^1) \approx 0.682621$.

2. Two "larger horoballs" with centers at $E_3$ and $E_7$ are tangent at the center of the cube, while horoballs at the remaining six vertices touch both "larger" horoballs (see Fig. 8.b). The density of this packing: $\delta(B_{1436}^2) \approx 0.682621$.

3. One large horoballs in same types touch each other in the "midpoints" of the "diagonals" of cube faces. The other four horoballs touch the adjacent "large horoballs" (see Fig. 9.a). The density of this packing: $\delta(B_{1436}^3) \approx 0.853276$.

4. One horoball of the "maximally large" type centered at $E_3$. The large horoball is tangent to all non-neighboring sides of the cube and it determines the other five horoballs touching the "large horoball" (see Fig. 9.b). The density of this packing: $\delta(B_{1436}^4) \approx 0.853276$.

By using Lemma 4.6 and its corollaries, similarly to the above cases we may again write the volume function of the horoball pieces to prove the optimality of two of the four limiting cases. Finally, we obtain the following
Figure 8: Locally optimal packings of Coxeter honeycomb (4, 3, 6)

Figure 9: Optimal horoball packings of Coxeter honeycomb (4, 3, 6).

**Theorem 4.12** Two different optimally dense horoball arrangements $B_i^{136}$, \(i = 3, 4\) exist for the cubic Coxeter tiling (4, 3, 6), which share the optimum density $\delta(B_i^{136}) \approx 0.85327609$.

5 The (5, 3, 6) Dodecahedral Tiling

The optimal packing densities for the dodecahedral Coxeter tiling (5, 3, 6) is obtained through a similar approach as in the previous three section, hence we omit the details of the derivation.

We consider horoball packings with centers of horoballs located at ideal vertices of the dodecahedral honeycomb (5, 3, 6).
Analogous to the previous cases we introduce the projective coordinate system, by an orthogonal vector basis with signature \((-1, 1, 1, 1)\), with the following coordinates of the vertices of the infinite regular dodecahedron, in the Cayley-Klein ball model. The dodecahedron contains a cubic sub-lattice with coordinates adopted from the previous section, as well as 12 other vertices obtainable through rotation about these vertices.

Using Lobachevsky volume formula for orthoschemes, we obtain the volume of one dodecahedron [Sz05-2]:

\[
Vol(P_{5,3,6}) = 120 \cdot Vol(O_{5,3,6}) \approx 20.580199 \ldots
\]

Applying the definition of the packing density for the case of tiling \((5, 3, 6)\), we obtain:

\[
\delta(B_{5,3,6}) = \sum_{i=1}^{20} \frac{Vol(B_i \cap P_{5,3,6})}{Vol(P_{5,3,6})},
\]

where \(B_i \cap P_{5,3,6} (i = 1, \ldots, 20)\) denote the 20 horoball sectors, one in each vertex of the dodecahedron \(P_{5,3,6}\). By using that the dodecahedral tiling has a cubic sublattice, we consider the following five main horoball configurations \(B_{5,3,6}^i (i = 1, \ldots, 5)\):

1. All 20 horoballs are of the same type and adjacent horoballs are tangent at the "midpoints" of each connecting edge. The density of this packing: \(\delta(B_{5,3,6}^1) \approx 0.550841\).

2. Two types of horoballs occur in this packing configuration. Eight larger horoballs are centered at the lattice points of the dodecahedra making up a cubic sublattice as in packing \(B_{4,3,6}^1\). Twelve smaller types of horoballs are located at the remaining 12 lattice points. The density of this packing: \(\delta(B_{5,3,6}^2) \approx 0.70309\).

3. This packing configuration contains horoballs of 4 types. The cubic sublattice within the dodecahedral lattice has the same ball configuration as \(B_{4,3,6}^2\), and the two types of the balls on the cubic lattice points uniquely determine two types of neighboring horoballs. The density of this packing: \(\delta(B_{5,3,6}^3) \approx 0.78725\).

4. There are three types of horoballs in this packing. The cubic sublattice within the dodecahedral lattice has the same ball configuration as \(B_{4,3,6}^3\), and the larger horoball uniquely determines the type of the remaining 12 horoballs. The density of this packing: \(\delta(B_{5,3,6}^4) \approx 0.78481\).
5. This limiting case is an extension of packing $B_{536}^5$. We inflate the horoball located at $(1, 0, 0, 1)$ until it touches the non-adjacent side. This packing consists of 6 horoball types, 4 of which are on the cubic sublattice and are a non-limiting case of the cubic tiling. These 4 horoballs uniquely determine two horoball types of the remaining 12 vertices of the dodecahedral lattice. The density of this packing: $\delta(B_{536}^5) \approx 0.71246$.

![Image of two tangent horoballs](image.png)

Figure 10: Two tangent horoballs in a packing of the $(5, 3, 6)$ honeycomb.

By using Lemma 4.6 and its corollaries, similarly to the above three cases we may again write the volume function of the horoball pieces to prove the optimality of two of the four limiting cases, leading to the following theorem:

**Theorem 5.1** The optimally dense horoball arrangement $B_{536}^4$ for the dodecahedral Coxeter tiling $(5, 3, 6)$ has optimal density $\delta(B_{536}^4) \approx 0.787251$.

## 6 Conclusion

Locally optimal horoball packings were studied in this paper, in which different types of horoballs were placed at the lattice points of fully asymptotic Coxeter honeycombs. We proved that the value obtained by Böröczky and Florian as the universal density upper bound for all congruent ball packings in hyperbolic 3-space remains the upper bound even if horoballs of different types are considered.
Moreover, there are two distinct optimally dense packings for the \((3, 3, 6)\) Coxeter tiling. We again encounter the Böröczky – Florian density upper bound as upper limits when varying the types of horoballs at the lattice points of the \((4, 3, 6)\) cubic Coxeter tilings, with two distinct realizations of optimality. For the \((3, 4, 4)\) octahedral Coxeter tiling, the optimal packing density is less than the maximal value, and there are three distinct configurations of balls yielding the same optimal value. The case if the \((5, 3, 6)\) dodecahedral tiling we have obtained some interesting horoball arrangements with less densities. Table 2 contains a summary of the optimal packing densities under our constraints.
Table 2. Optimal packing densities for the four fully asymptotic Coxeter tilings

| Schläfli symbol | Optimal density |
|-----------------|---------------|
| (3, 3, 6)       | 0.853276*     |
| (3, 4, 4)       | 0.818808      |
| (4, 3, 6)       | 0.853276*     |
| (5, 3, 6)       | 0.787251      |

*These values are identical to the Böröczky and Florian limit.

In the future it will be interesting to investigate tilings given various uniform conditions on the configuration of the balls in $\mathbb{H}^3$ as well as higher dimensional hyperbolic spaces. These studies may show the existence of multiple optimal configurations for given tilings, similarly as we have observed in $\mathbb{H}^3$. To the knowledge of the authors the solution of the above problem is still open.

Optimal sphere packings in other homogeneous Thurston geometries represent another huge class of open mathematical problems. For these non-Euclidean geometries only very few results are known [Sz07-2], [Sz10-1]. Detailed studies are the objective of ongoing research.

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