Coherent state topological cluster state production

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Abstract. We present results illustrating the construction of three-dimensional (3D) topological cluster states with coherent state logic. Such a construction would be ideally suited for wave-guide implementations of optical quantum information processing. We investigate the use of a deterministic controlled-Z gate, showing that given large enough initial cat states, it is possible to build large 3D cluster states. We model X and Z basis measurements by displaced photon number detections and x-quadrature homodyne detections, respectively. We investigate whether teleportation can aid in cluster state construction and whether this introduction of located loss errors fits within the topological cluster state framework.

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1. Introduction

The recent use of topological techniques in the cluster state quantum computation scheme [1, 2] has led to some very encouraging fault-tolerant threshold predictions. The initial estimate for the computational basis error threshold was 0.75% [1, 2], but it is believed this could reach as high as 1% [3], making this architecture a serious contender for scalable quantum computing [4]. This scheme encodes qubits as defects in a three-dimensional (3D) cluster state, constructing the necessary qubit gates via topological braiding operations on 3D defects through the volume of the cluster state. A remarkable aspect of these 3D topological cluster states is their resilience to loss, with estimates that they could tolerate a located loss error as high as 25% [5], and may even be able to recover from simultaneous computational basis and located loss errors [5].

The bare requirements a physical system must possess in order to implement topological cluster state computation are [6]: (i) state preparations of $|0\rangle_{\text{Logical}}$ and $|0\rangle_{\text{Logical}} + e^{i\theta}|1\rangle_{\text{Logical}}$, where the subscript ‘Logical’ refers to the physical logical qubits that are used to construct the 3D cluster, not encoded qubits; (ii) $X$ and $Z$ basis measurements; and (iii) the controlled-$Z$ entangling gate. An implementation with optical systems has been proposed [4] based on integrated optics and in-line nonlinear elements in the form of single-atom cavity electrodynamics, showing that with the so-called photonic modules, large-scale topological cluster state computation could in principle be implemented. An alternative approach is to move all nonlinearities offline, using techniques of linear optical quantum computing (LOQC) [7].

Unfortunately, LOQC with photonic qubits involves massive amounts of resource recycling, which would hugely complicate the optical circuit. Surprisingly, an alternative LOQC scheme based on coherent-state logic exists [9, 10] that requires no recycling of resource states. In this paper, we consider the construction of 3D topological cluster states using such a coherent state logic, building on a deterministic type of linear optical controlled-$Z$ gate [9]. The deterministic nature of this scheme makes it an ideal candidate for implementation with integrated linear optical quantum circuits [11–14], since, as we will show, given the appropriate integrated optical wave guide, this scheme is only in principle limited by the construction of the initial qubits.

The use of coherent states for the logical encoding of qubits into a specific quantum error correcting code was first proposed by Cochrane et al [8] and the use of coherent states for universal quantum computation was first proposed by Ralph et al [9, 10], where qubits were encoded as $|0\rangle_{\text{Logical}} = |−\alpha\rangle$ and $|1\rangle_{\text{Logical}} = |\alpha\rangle$, requiring Bell state measurements for teleportation and a resource of cat states of the form $(|−\alpha\rangle + |\alpha\rangle)/\sqrt{2}$. Since the qubits were only approximately orthogonal, with $|\langle\alpha|−\alpha\rangle|^2 = e^{-4\alpha^2}$, sufficiently large cat states were required for this scheme to be viable. However, the construction of large optical cat states in the laboratory is difficult, the first attempts at creating small cat states were by subtracting a photon from squeezed vacuum [15, 16]. The largest cat states created to date via ancilla-assisted two-photon subtracted squeezed vacuum [17] have an average number of photons of $1.42 \approx 1.96$ with a fidelity of 0.60, while initial experiments [18] with three-photon-subtracted squeezed vacuum suggest that cat states with an average photon number as high as $1.76^2 \approx 3.1$ could be produced. However, even with the hurdle of constructing large-amplitude cat states, quantum computation with coherent state logic may still be competitive with other optical quantum computation schemes, since the success probability for the basic gates is quite high and coherent state qubit teleportation is deterministic. A recent extension to this coherent state computation scheme by Lund et al [19] showed that universal quantum computation was still possible with small-amplitude coherent states.
In this paper, we address the second two physical requirements for 3D cluster state production with coherent states. In contrast to [10], we define logical qubits as $|\text{Logical} = \text{vac}\rangle \equiv |0\rangle$ and $|\text{Logical} = |\alpha\rangle\rangle$. Both definitions being equivalent up to a displacement in phase space. As in [9, 10], we are confined by the inherent error associated with our non-orthogonal qubit definition, with $(|\alpha\rangle = e^{-|\alpha|^2/2}$, and the feasibility of constructing large-amplitude cat states of the form $|0\rangle + |\alpha\rangle$, the basic states required for cluster state production in this scheme. In section 2, we use a beam splitter as our basic controlled-$Z$ gate [9], showing that it is in principle possible to deterministically construct large 3D cluster states with coherent state logic, finding that coherent cat states with an average number of photons $> 99.6$, meaning cat states with amplitudes larger than $19.96 (|\alpha = 0\rangle + |\alpha = 19.96\rangle)$ will result in a computational error probability (EP) per qubit $< 1\%$. Next, in section 3, we model X and Z basis measurements by displaced photon number detections and $x$-quadrature homodyne detections, respectively, showing that measurement errors per qubit can be made below $1\%$ for amplitudes above $25.17 (|\alpha = 0\rangle + |\alpha = 25.17\rangle)$, that is, coherent cat states with an average number of photons $> 158.4$. In section 4, we make use of teleportation to clean up the cluster states built from the basic scheme, showing that we can produce 3D cluster states from low-amplitude cat states with arbitrarily high fidelity, albeit at the expense of moving away from the completely deterministic nature of the basic controlled-$Z$ gate and introducing a success probability. In section 5, we attempt to capitalize on the topological 3D cluster state code’s ability to deal with simultaneous computational basis and located loss errors, investigating whether cleaning up our 3D cluster states with teleportation can reduce the amplitude size of the initial cat state required by trading off computation error for located loss error. We present preliminary results that suggest that in the most ambitious scenario for topological cluster states, when the average number of photons in the initial cat state is between 28.57 and 31.02 (i.e. $10.69 < \alpha < 11.14$), it is advantageous to use teleportation. Finally, in section 6, we conclude.

2. Deterministic scheme

One of the requirements a physical system must possess to be suitable for 3D topological state quantum computing is the ability to perform a controlled-$Z$ gate between neighbouring qubits. We model our basic controlled-$Z$ gate as a symmetric beam splitter, where we define the beam splitter by the following transformations on incident creation operators $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$:

\[
\hat{a}_1^\dagger \rightarrow \cos(\theta) \hat{a}_1^\dagger + e^{-i\phi} \sin(\theta) \hat{a}_2^\dagger ,
\]

\[
\hat{a}_2^\dagger \rightarrow -e^{i\phi} \sin(\theta) \hat{a}_1^\dagger + \cos(\theta) \hat{a}_2^\dagger
\]

and note that a beam splitter does not entangle incident coherent states since

\[
|\alpha\rangle|\beta\rangle \rightarrow \text{BS} |\alpha\rangle \cos(\theta) - e^{i\phi} \beta \sin(\theta)\rangle |e^{-i\phi} \alpha \sin(\theta) + \beta \cos(\theta)\rangle.
\]

If we consider two cat states of the form

\[
\frac{1}{\mathcal{N}} \left( |0\rangle + |\alpha\rangle \right) \otimes \left( |0\rangle + |\alpha\rangle \right),
\]

incident on a symmetric beam splitter of reflectivity $\theta = \pi/2\alpha^2$, $\phi = -\pi/2$ [9], shown in figure 1, the output is given by

\[
\frac{1}{\mathcal{N}} \left( |00\rangle + i|\alpha \sin(\theta)\rangle |\alpha \cos(\theta)\rangle + |\alpha \cos(\theta)\rangle |\alpha \sin(\theta)\rangle + |e^{i\theta} \alpha\rangle |e^{i\theta} \alpha\rangle \right),
\]

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\[ |0\rangle + |\alpha\rangle \]

\[ |0\rangle + |\alpha\rangle \]

**Figure 1.** Deterministic controlled-Z gate modelled as a beam splitter of reflectivity \( \theta_1 = \frac{\pi}{2\alpha^2} \), \( \phi_1 = -\frac{\pi}{2} \) \[9\]. Note, for clarity, the input cat states are un-normalized.

**Figure 2.** Unit cell for the 3D topological cluster state \([1, 2]\). The solid black lines between coloured qubit circles represent controlled-Z gates.

where \( N = 2(1 + e^{-\alpha^2/2}) \) is the normalization. When \( \alpha \gg 1 \), equation (3) is approximately

\[ \frac{1}{2} \left( |00\rangle + |0\alpha\rangle + |\alpha 0\rangle - |\alpha \alpha\rangle \right). \]  \hspace{1cm} (4)

This can be seen from the inner product of two coherent states of different amplitudes \([20]\):

\[ \langle \beta | \alpha \rangle = \exp\left\{ -\frac{1}{2} \left( |\alpha|^2 + |\beta|^2 \right) + \alpha \beta^* \right\}. \]  \hspace{1cm} (5)

The approximate \( |11\rangle \) logical state picks up a \( \pi \) phase:

\[ \langle \alpha, \alpha | e^{i\theta} \alpha, e^{i\theta} \alpha \rangle = \exp[-2\alpha^2(1 - e^{i\theta})] \approx \exp[2i\theta \alpha^2] = -1, \]  \hspace{1cm} (6)

while the approximate \( |01\rangle \) logical and \( |10\rangle \) logical states do not pick up any phase factors:

\[ \langle 0, \alpha, | i \alpha \sin \theta, \alpha \cos \theta \rangle = \exp[\alpha^2(1 - \cos \theta)]1, \]

\[ \approx 1, \]  \hspace{1cm} (7)

where since \( \theta = \pi/2\alpha^2 \) and \( \alpha \gg 1 \), we can assume that \( \theta \ll 1 \).

In order to build the 3D topological cluster state with coherent state qubits, we first consider the construction of the elementary building blocks for the cluster state unit cell \([1, 2]\), shown in figure 2. These progressive building blocks are shown in figure 3.

To calculate the fidelity for each of the cluster states in figure 3, we make a comparison with a hypothetical cluster state made with ideal coherent state controlled-Z gates of the form given in equation (4). These fidelities are shown in figure 4. Note that as the amplitude size of
Figure 3. The progressive basic building blocks for the 3D topological cluster state unit cell in figure 2: (a) the two-qubit cluster; (b) the three-qubit cluster; (c) the five-qubit linear cluster; (d) the five-qubit star cluster; and (e) the eight-qubit loop cluster. Each vertical/horizontal line corresponds to a controlled-Z gate constructed from the beam splitter shown in figure 1. Each coloured qubit circle is initially in the cat state \((|0\rangle + |\alpha\rangle) / \sqrt{N}\).

the initial cat state increases, the fidelity approaches 1. Also, as expected, the fidelity decreases as the complexity of the cluster state increases.

We gauge the capability of the deterministic controlled-Z gate to construct 3D cluster states by calculating the EP per qubit/gate as a function of initial cat state amplitude \(\alpha\) for each of the fidelity curves in figure 4. This is done by first considering the construction of the building block cluster states in figure 3 from ideal orthogonal qubit states that are affected by a specific error model, leading to the resulting cluster states having a non-unit fidelity. We compare this fidelity with the fidelity curves in figure 4 and then generate the corresponding EP per qubit/gate as a function of the size of initial cat states \(\alpha\). When considering the construction from ideal orthogonal qubits, we use the stabilizer formalism with a pseudo-random number generator to generate fidelities (and visibilities in section 3).

Since it is unclear what errors the approximate controlled-Z gate in figure 1 introduces, we consider two error models for the ideal orthogonal qubit case: the case of an ideal controlled-Z followed by each output mode going through a single-qubit depolarizing channel of the form [21]

\[
\varepsilon(\rho) = (1 - p_1)\rho + \frac{p_1}{3} (X\rho X + Y\rho Y + Z\rho Z)
\]

and the case of an ideal controlled-Z followed by the output modes going through a two-qubit depolarizing channel of the form [21]

\[
\varepsilon(\rho) = (1 - p_2)\rho + \frac{p_2}{15} \sum_{ij} (A_i \otimes B_j)\rho (A_i \otimes B_j),
\]
Figure 4. The fidelity of constructing the progressive basic building blocks from figure 3 with the beam splitter controlled-\(Z\) gate in figure 1. In red the two-qubit cluster (figure 3(a)) fidelity is shown; in magenta the three-qubit cluster (figure 3(b)) fidelity is shown; in green the five-qubit linear cluster (figure 3(c)) fidelity is shown; in blue the five-qubit star cluster (figure 3(d)) fidelity is shown; in cyan the eight-qubit loop cluster (figure 3(e)) fidelity is shown.

Figure 5. The fidelity EP per qubit (\(p_1\), solid lines) and per controlled-\(Z\) gate (\(p_2\), dashed lines) for each of the unit cell building blocks in figure 3. In red, the two-qubit cluster (figure 3(a)) EP is shown; in magenta the three-qubit cluster (figure 3(b)) EP is shown; in green the five-qubit linear cluster (figure 3(c)) EP is shown; in blue the five-qubit star cluster (figure 3(d)) EP is shown; in cyan the eight-qubit loop cluster (figure 3(e)) EP is shown.

where the summation is over the 15 combinations of Pauli matrices \((A_i \otimes B_j) \in \{(I \otimes X), (I \otimes Y), \ldots, (Z \otimes Z)\}\). This two-qubit depolarizing channel case is the worst case scenario and will lead to a lower bound for the estimation of the EP per qubit.

The EP per qubit (\(p_1\)) and per controlled-\(Z\) gate (\(p_2\)) for each of the building blocks in figure 3 as a function of coherent amplitude \(\alpha\) is shown in figure 5. Considering the error model to be an ideal controlled-\(Z\) gate followed by a two-qubit depolarizing channel places more stringent restrictions on the size of coherent cat states required than when the error model
Table 1. The parameters required to fit the solid curves for the probability of error per qubit in figure 5 to an exponentially damped quadratic of the form $b(a_0 + a_1\alpha + a_2\alpha^2) \exp(-c_1\alpha)$.

| Cluster | $a_0$ | $a_1$ | $a_2$ | $b$ | $c_1$ |
|---------|-------|-------|-------|-----|-------|
| Figure 3(a) | 257.12 | -16.79 | 0.5414 | $3.203 \times 10^{-4}$ | 0.1328 |
| Figure 3(b) | 224.85 | -14.76 | 0.4687 | $4.441 \times 10^{-4}$ | 0.1300 |
| Figure 3(c) | 140.91 | -9.889 | 0.3333 | $8.903 \times 10^{-4}$ | 0.1389 |
| Figure 3(d) | 260.34 | -16.56 | 0.5245 | $5.156 \times 10^{-4}$ | 0.1297 |
| Figure 3(e) | 197.93 | -13.17 | 0.4286 | $6.268 \times 10^{-4}$ | 0.1335 |

Table 2. The parameters required to fit the dashed curves for the probability of error per controlled-Z gate in figure 5 to an exponentially damped quadratic of the form $b(a_0 + a_1\alpha + a_2\alpha^2) \exp(-c_1\alpha)$.

| Cluster | $a_0$ | $a_1$ | $a_2$ | $b$ | $c_1$ |
|---------|-------|-------|-------|-----|-------|
| Figure 3(a) | 956.85 | -52.95 | 1.546 | $1.577 \times 10^{-4}$ | 0.1177 |
| Figure 3(b) | 1756.71 | -93.02 | 2.624 | $9.022 \times 10^{-5}$ | 0.1130 |
| Figure 3(c) | 2104.96 | -109.00 | 2.989 | $7.601 \times 10^{-5}$ | 0.1101 |
| Figure 3(d) | 5847.52 | -279.91 | 7.387 | $3.009 \times 10^{-5}$ | 0.1044 |
| Figure 3(e) | 3242.62 | -163.98 | 4.444 | $4.957 \times 10^{-5}$ | 0.1079 |

Table 3. The size of the coherent states ($\alpha$) required to achieve a single-qubit depolarizing EP (1QDE) and a two-qubit depolarizing EP (2QDE) of 1 and 0.1%, respectively.

| Cluster | 1QDE 1% | 1QDE 0.1% | 2QDE 1% | 2QDE 0.1% |
|---------|---------|-----------|---------|-----------|
| Figure 3(a) | 11.14 | 35.28 | 17.60 | 49.31 |
| Figure 3(b) | 12.45 | 39.14 | 18.88 | 52.26 |
| Figure 3(c) | 13.03 | 39.45 | 19.40 | 53.69 |
| Figure 3(d) | 14.69 | 43.16 | 21.87 | 58.31 |
| Figure 3(e) | 13.60 | 40.94 | 19.96 | 55.07 |

is just an ideal controlled-Z gate followed by each output undergoing a single-qubit depolarizing channel, as can be seen in figure 5 with the dashed curves all to the right of the solid curves.

We consider the scaling relationship of the curves in figure 5 as a function of $\alpha$, fitting each curve to an exponentially damped quadratic of the form $b(a_0 + a_1\alpha + a_2\alpha^2) \exp(-c_1\alpha)$. The results of these scaling relationships can be found in tables 1 and 2. We found that the root mean square error (RMSE) for each of the exponentially damped quadratic fits in these tables is always less than $7.8 \times 10^{-5}$.

We can read off the size of the coherent state amplitudes required for each of the cluster state building blocks to have an EP per qubit/gate below 1% from figure 5. This is summarized in table 3. We note that for the five-qubit star cluster state, we would require $\alpha$ to be between 14.69 and 21.87, whereas for the eight-qubit loop cluster, we would require $\alpha$ to be between 13.60 and 19.96. This suggests that the construction of star clusters places more stringent
restrictions on the size of initial cat states than the construction of loop cluster states. Also in table 3 we use the scaling relationships from tables 1 and 2 to estimate the size of initial cat states required for an EP per qubit/gate to be one order of magnitude below the predicted 1% threshold.

3. Measurements

Another physical requirement a physical system must possess to be suitable for 3D topological cluster state quantum computing is the ability to measure qubits in the X and Z basis.

We model X basis measurements by first displacing our state by \(-\frac{a}{2}\), followed by a photon number measurement, effectively implementing a photon number parity measurement. That is, given the phase resulting from displacing a coherent state and the projection of a coherent state into the number basis \(|\hat{a}^\dagger \hat{a}| = \frac{1}{\sqrt{2}} (c_0 + c_1)|X^+\rangle + \frac{1}{\sqrt{2}} (c_0 - c_1)|X^-\rangle\). In terms of coherent state logic, measuring the corresponding state gives

\[
\langle X\text{-basis meas.} \mid (c_0|0\rangle + c_1|\alpha\rangle) = \frac{e^{-\alpha^2/2} \left(\frac{\alpha}{2}\right)^n}{\sqrt{n!}} \left[(-1)^n c_0 + c_1\right].
\]

When \(n\) is even, we detect \(|X^+\rangle\); when \(n\) is odd, we detect \(|X^-\rangle\).

We model Z basis measurements by homodyne detection in the \(x\)-quadrature: \(\langle x|\beta\rangle\). Since the basic controlled-Z gate transforms a small portion of the initially real cat states into the complex direction, as can be seen in equation (3), we need to consider the \(x\)-projection of a general complex coherent state, keeping track of all possible phase terms. We do this by considering both \(\langle x|\hat{a}|\beta\rangle\) and \(\langle x|\hat{a}^\dagger|\beta\rangle\). By solving the ODE resulting from

\[
\langle x|\hat{a}|\beta\rangle = \beta \langle x|\beta\rangle
\]

and

\[
\langle x|\hat{a}^\dagger|\beta\rangle = (2\pi)^{-1/4} \exp \left\{i\phi + 2i \text{Re}[\beta] \text{Im}[\beta] - (\text{Im}[\beta])^2 - \frac{1}{2} (x - 2\beta)^2 \right\},
\]

where we determine the as yet undetermined phase \(\phi\) from \(\langle x|\hat{a}^\dagger|\beta\rangle\). Given the \(x\)-projection of a number state and two properties of Hermite polynomials:

\[
\sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!} = e^{2xt - t^2},
\]

\[
H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x),
\]

we find

\[
D(\alpha) D(\beta) = \exp[i \text{Im}[\alpha \beta^*]] D(\alpha + \beta),
\]

\[
\langle n|\beta\rangle = \exp \left\{- \frac{|\beta|^2}{2} \right\} \beta^n \sqrt{n!},
\]

we find

\[
\langle n|D(-\frac{a}{2})|\beta\rangle = \langle n| \exp \left\{- \frac{i\alpha}{2} \text{Im}[\beta^*] \right\} \frac{|\beta - \alpha|}{2} \right\}
\exp \left\{- \frac{i\alpha}{2} \text{Im}[\beta^*] - \frac{1}{2} |\beta|^2 + \frac{\alpha}{2} \text{Re}[\beta] - \frac{\alpha^2}{8} \right\} \frac{(\beta - \frac{a}{2})^n}{\sqrt{n!}}.
\]

We gain some insight into how equation (11) models an X basis measurement by considering the X basis measurement of an arbitrary qubit state \(c_0|0\rangle_{\text{Logical}} + c_1|1\rangle_{\text{Logical}} = \frac{1}{\sqrt{2}} (c_0 + c_1)|X^+\rangle + \frac{1}{\sqrt{2}} (c_0 - c_1)|X^-\rangle\). In terms of coherent state logic, measuring the corresponding state gives

\[
\langle X\text{-basis meas.} \mid (c_0|0\rangle + c_1|\alpha\rangle) = \frac{e^{-\alpha^2/2} \left(\frac{\alpha}{2}\right)^n}{\sqrt{n!}} \left[(-1)^n c_0 + c_1\right].
\]
Figure 6. The measurement outcome for the detection of \( c_0|0\rangle + c_1|\alpha\rangle \) in the \( Z \) basis. In red \( \langle x|0\rangle \) is plotted and in green \( \langle x|\alpha = 3.75\rangle \) is plotted.

we find \( \langle x|\hat{a}^\dagger|\beta\rangle \)

\[
= (2\pi)^{-\frac{1}{4}} (x - \beta) \exp\left\{-\frac{|\beta|^2}{2} - \frac{x^2}{4} + x\beta - \frac{\beta^2}{2}\right\}.
\] (15)

Next, if we consider the ODE associated with \( \langle x|\hat{a}^\dagger|\beta\rangle \) given in equation (15), we find \( \phi = -\text{Re}[\beta] \text{Im}[\beta] \). The \( Z \) basis measurement is then found to be

\[
\langle x|\beta\rangle = (2\pi)^{-\frac{1}{4}} \exp\left\{i\text{Re}[\beta] \text{Im}[\beta] - (\text{Im}[\beta])^2 - \frac{1}{4} (x - 2\beta)^2\right\}.
\] (16)

We gain some insight into how equation (16) models a \( Z \) basis measurement by considering the \( Z \) basis measurement of an arbitrary coherent logic state \( c_0|0\rangle + c_1|\alpha\rangle \):

\[
\langle \text{Z-basis meas.}|(c_0|0\rangle + c_1|\alpha\rangle)\rangle = (2\pi)^{-\frac{1}{4}} \left[c_0 e^{-\frac{x^2}{2}} + c_1 e^{-\frac{1}{4}(x - 2\alpha)^2}\right].
\] (17)

When we detect \( |Z^+\rangle \), we expect a homodyne outcome centred around \( x = 0 \), and when we detect \( |Z^-\rangle \), we expect a homodyne outcome centred around \( x = 2\alpha \), as shown in figure 6.

We evaluate the effectiveness of modelling \( X \) and \( Z \) basis measurements by displaced photon number and \( x \)-homodyne detections by looking at the visibility [20]:

\[
V = \frac{P_{\text{max}} - P_{\text{min}}}{P_{\text{max}} + P_{\text{min}}}.
\] (18)

Since coherent states do not entangle on beam splitters, it is straightforward to calculate the visibility. For example, consider measuring the two-qubit cluster state in figure 3(a) in the \( XZ \) basis. The initial state for this cluster state is given by

\[
\frac{1}{N} \sum_{m_1,m_2 \in \{0,\alpha\}} |m_1, m_2\rangle.
\] (19)

After the beam splitter controlled-Z gate, this state becomes

\[
|\psi_{\text{IBS}}\rangle = \frac{1}{N} \sum_{m_1,m_2 \in \{0,\alpha\}} |p_1, p_2\rangle,
\] (20)
where

\[ p_1 = m_1 \cos \theta + i m_2 \sin \theta, \]

\[ p_2 = i m_1 \sin \theta + m_2 \cos \theta. \]

The probability of a given measurement is then given by

\[ P_{1BS} = |\langle X\text{-basis}| \langle Z\text{-basis} | \psi_{1BS} \rangle|^2 = \left| \langle n | D(\frac{\pi}{2}) (x) | \psi_{1BS} \rangle \right|^2. \]

By writing the ideal two-qubit cluster state with the first mode in the X basis and the second mode in the Z basis:

\[ \left( |X^+\rangle + |X^-\rangle \right), \]

we can calculate \( P_{\text{max}} \) and \( P_{\text{min}} \) for the two-qubit cluster:

\[ P_{\text{max}} = \left( \int_{-\infty}^{\alpha} dx \sum_{n \text{ even}} + \int_{\alpha}^{\infty} dx \sum_{n \text{ odd}} \right) P_{1BS} \]

and

\[ P_{\text{min}} = \left( \int_{-\infty}^{\alpha} dx \sum_{n \text{ odd}} + \int_{\alpha}^{\infty} dx \sum_{n \text{ even}} \right) P_{1BS}. \]

The reason we integrate from either \( \alpha \) to \( \infty \) or \( -\infty \) to \( \alpha \) is to minimize the errors resulting from having non-orthogonal qubits. That is, the midway point between the Z basis outcomes is \( x = \alpha \), as can be seen in figure 6. This method can be extended to calculate the visibility for larger cluster states.

To determine the performance of our measurement model, we measure certain stabilizers of the 3D cluster state. We will first endeavour to measure the operator \( ZZXXZ \), since this is a local stabilizer for the 3D topological cluster state and can also be used to initiate a faulty cluster state into the correct state [4]. Applying the stabilizer \( ZZXXZ \) to the six faces of the unit cell in figure 2 results in the Z measurements cancelling each other; we next endeavour to measure \( X \) on the six faces of the unit cell.

We start by measuring smaller operators on the building block cluster states required to construct the unit cell, first measuring the operator \( XZ \) on the two-qubit cluster (figure 7(a)), next measuring \( ZXZ \) on the five-qubit linear cluster (figure 7(b)), then measuring \( ZZXXZ \) for the five-qubit cluster.

Figure 7. Detect \( XZ \) for the two-qubit cluster. (b) Detect \( ZXZ \) for the five-qubit linear cluster. (c) Detect the stabilizer \( ZZXXZ \) for the five-qubit cluster. (d) Detect the stabilizer \( ZZXXZ \) for the 17 qubit cluster. Each vertical/horizontal line corresponds to a controlled-Z gate.
Figure 8. Detect $XX$ for the three-qubit cluster. (b) Detect $XXX$ for the five-qubit linear cluster. (c) Detect the stabilizer $XXXX$ for the eight-qubit loop cluster. (d) Detect the stabilizer $XXXXXX$ for the 14 qubit cluster. (e) Detect the stabilizer $XXXXXX$ for the 18 qubit unit cell. Each vertical/horizontal line corresponds to a controlled-$Z$ gate.

We next focus on measuring only $X$ operators. In a similar way to the $ZZXZZ$ case, we progress by considering the measurement of operators on the building blocks of the unit cell. We first measure $XX$ on the three-qubit cluster (figure 8(a)), next measuring $XXX$ in the five-qubit linear cluster (figure 8(b)), then measuring $XXXX$ on the eight-qubit loop cluster (figure 8(c)), next measuring $XXXXXX$ on the 14 qubit cluster (figure 8(d)) and finally measuring $XXXXXX$ on the 18 qubit unit cell (figure 8(e)).

We plot the visibility as a function of the amplitude size of the initial cat state in figure 9. It is worth noting that the visibility for measuring $ZXZ$ for the five-qubit linear cluster state is identical to measuring $ZXZ$ for the three-qubit cluster state, while measuring $ZXZZZ$ for the 17 qubit cluster state is identical to measuring $ZXZZZ$ for the five-qubit star cluster state. This shows that only the qubits that are actually measured influence the calculation of the visibility.
Figure 9. The visibility for the detection of $XZ$ for the two-qubit cluster is shown in red. The visibility for the detection of $ZXZ$ for the five-qubit linear cluster is shown in green; this line also corresponds to detecting $ZXZ$ for the three-qubit cluster. The visibility for the detection of $ZZXZZ$ for the 17 qubit cluster is shown in blue; this line also corresponds to detecting $ZZXZZ$ for the five-qubit star cluster state. In the top left-hand corner, we show the visibility for the ideal 2, 5 and 17 qubit cluster states. The visibility for the detection of $XX$ for the three-qubit cluster is shown in yellow. The visibility for the detection of $XXX$ for the five-qubit linear cluster is shown in orange. The visibility for the detection of $XXXX$ for the eight-qubit loop cluster is shown in brown. The visibility for the detection of $XXXXX$ for the 14 qubit cluster is shown in cyan. The visibility for the detection of $XXXXXX$ for the 18 unit cell cluster is shown in magenta. Since these curves required a full density matrix analysis, the generated data points for the 14 and 18 qubit cases are explicitly shown.

for the stabilizer $ZZXZZ$. This is not the case when we measure $X$ operators alone. As we can see in figure 9, as the initial cat state amplitude is increased, the visibility increases as well. It is also worth noting that even if we had a perfect controlled-$Z$ gate for cat state logic, since the qubits are intrinsically non-orthogonal, the visibility is not automatically 1, as also shown in figure 9.

As in section 2, to judge how well our measurement model fits within the thresholds for the 3D topological cluster state code, we calculate the EP per qubit/gate as a function of initial cat state amplitude $\alpha$ for each of the visibility curves in figure 9. This is again done by first considering the construction of the building block cluster states in figures 7 and 8 from ideal orthogonal qubit states that are either affected by the single-qubit depolarizing channel (equation (8)) or the two-qubit depolarizing channel (equation (9)), resulting in non-unit visibility when we detect the desired operator. We then compare this visibility with the visibility curves in figure 9, generating the corresponding EP per qubit/gate as a function of the size of the initial cat states $\alpha$. The two-qubit depolarizing channel case will once again give a lower bound on the EP per qubit estimation since it is the worst possible case scenario.

We plot the EP per qubit $p_1$ as a function of initial cat state amplitude when we consider the error model as an ideal controlled-$Z$ gate followed by two single-qubit depolarizing channels in figure 10 for various cluster states. As the complexity of the cluster state increases, the size of
Figure 10. The operator detection EP $p_1$ per qubit when we consider the error model to be an ideal controlled-$Z$ gate followed by two single-qubit depolarizing channels. In red we consider detecting $XZ$ on the two-qubit cluster (figure 7(a)); in green we consider detecting $ZXZ$ on the five-qubit linear cluster (figure 7(b)); in blue we consider detecting $ZZXZZ$ on the five-qubit star cluster (figure 7(c)); in orange we consider detecting $ZZXZZ$ on the five-qubit star cluster (figure 7(d)); in brown we consider detecting $XXXXX$ on the eight-qubit loop cluster (figure 8(c)); in cyan we consider detecting $XXXXX$ on the 14 qubit cluster (figure 8(d)); in magenta we consider detecting $XXXXX$ on the 18 qubit unit cell (figure 8(e)). Since these curves required a full density matrix analysis, the generated data points for the 14 and 18 qubit cases are explicitly shown.

the initial coherent amplitude cat state also slowly increases. This suggests that this error model may not be sufficient for the controlled-$Z$ modelled as a beam splitter.

We plot the EP per controlled-$Z$ gate $p_2$ as a function of initial cat state amplitude when we consider the error model as an ideal controlled-$Z$ gate followed by two-qubit depolarizing channels in figure 11 for various cluster states. In general, as the complexity of the cluster state increases, the size of the initial coherent amplitude cat state also slowly increases. However, some of the more complex cluster state curves overlap with less complex cluster state curves, such as the 18 qubit unit cell curve overlapping the five-qubit star cluster curve. This suggests that modelling the error as an ideal controlled-$Z$ gate followed by two-qubit depolarizing channels may accurately describe the error associated with modelling the controlled-$Z$ as a beam splitter.

As in section 2, we consider the scaling relationship of the curves in figures 10 and 11 as a function of $\alpha$ by fitting each curve to an exponentially damped quadratic of the form $b(a_0 + a_1\alpha + a_2\alpha^2 + \exp(-c_1\alpha))$. The results of these scaling relationships are shown in tables 4 and 5. We found that the RMSE for each of the exponentially damped quadratic fits in these tables is always less than $7.0 \times 10^{-5}$. However, as indicated by the * in these tables, it was not always possible to fit an exponentially damped quadratic; in three cases a higher order polynomial fit was required.

The eight-qubit case required a fourth-order polynomial fit given by $b(a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4 + \exp(-c_1\alpha))$, where $a_0 = -2534.23$, $a_1 = 461.99$, $a_2 = -63.81$, $a_3 = 3.571$, $a_4 = -0.1004$, $b = -2.052 \times 10^{-4}$, $c_1 = 0.2714$ for the single-qubit depolarizing channel case.

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Figure 11. The operator detection EP $p_2$ per controlled-$Z$ gate when we consider the error model to be an ideal controlled-$Z$ gate followed by a two-qubit depolarizing channel. In red we consider detecting $XZ$ on the two-qubit cluster (figure 7(a)); in green we consider detecting $ZXZ$ on the five-qubit linear cluster (figure 7(b)); in blue we consider detecting $ZZXZZ$ on the five-qubit star cluster (figure 7(c)); in orange we consider detecting $ZZZXX$ on the five-qubit star cluster (figure 7(d)); in brown we consider detecting $XXZXX$ on the 17 qubit cluster (figure 7(e)); in cyan we consider detecting $XXZXX$ on the 18 qubit linear cluster (figure 7(f)); in magenta we consider detecting $XXZXX$ on the 18 qubit unit cell (figure 7(g)). Since these curves required a full density matrix analysis, the generated data points for the 14 and 18 qubit cases are explicitly shown.

Table 4. The parameters required to fit the curves for the probability of error per qubit in figure 10 to an exponentially damped quadratic of the form $b(a_0 + a_1 \alpha + a_2 \alpha^2) \exp(-c_1 \alpha)$. The $\ast$ indicates that an exponentially damped quadratic could not be fitted to these cases; a higher order polynomial fit was required, as described in the text.

| Cluster | $a_0$ | $a_1$ | $a_2$ | $b$ | $c_1$ |
|---------|-------|-------|-------|-----|-------|
| Figure 7(a) | 185.95 | −12.18 | 0.3823 | 4.830 × 10$^{-4}$ | 0.1294 |
| Figure 7(b) | 238.14 | −15.22 | 0.4909 | 3.208 × 10$^{-4}$ | 0.1303 |
| Figure 7(c) | 551.96 | −30.74 | 0.9019 | 2.350 × 10$^{-4}$ | 0.1168 |
| Figure 7(d) | 185.41 | −11.52 | 0.3647 | 3.352 × 10$^{-4}$ | 0.1256 |
| Figure 8(c) | $\ast$ | $\ast$ | $\ast$ | $\ast$ | $\ast$ |
| Figure 8(d) | $\ast$ | $\ast$ | $\ast$ | $\ast$ | $\ast$ |
| Figure 8(e) | 93.32 | −6.882 | 0.3656 | 3.028 × 10$^{-3}$ | 0.1767 |

and $a_0 = -2587.92$, $a_1 = 452.60$, $a_2 = -52.73$, $a_3 = 2.593$, $a_4 = -0.06227$, $b = -3.348 \times 10^{-4}$, $c_1 = 0.2365$ for the two-qubit depolarizing channel case. This led to an RMSE of 6.0 × 10$^{-5}$. The 14 qubit case required a sixth-order polynomial fit given by $b(a_0 + a_1 \alpha + a_2 \alpha^2 + a_3 \alpha^3 + a_4 \alpha^4 + a_5 \alpha^5 + a_6 \alpha^6) \exp(-c_1 \alpha)$, where $a_0 = -18853.09$, $a_1 = 25744.26$, $a_2 = -14898.16$, $a_3 = 3783.90$, $a_4 = -488.70$, $a_5 = 30.73$, $a_6 = -0.7853$, $b = -1.407 \times 10^{-4}$, $c_1 = 0.5785$. This led to an RMSE of 2.5 × 10$^{-4}$.

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Table 5. The parameters required to fit the curves for the probability of error per controlled-Z gate in figure 11 to an exponentially damped quadratic of the form $b(a_0 + a_1 \alpha + a_2 \alpha^2) \exp(-c_1 \alpha)$. The * indicates that an exponentially damped quadratic could not be fitted to this case; a higher-order polynomial fit was required, as described in the text.

| Cluster       | $a_0$    | $a_1$    | $a_2$    | $b$           | $c_1$   |
|---------------|----------|----------|----------|---------------|---------|
| Figure 7(a)   | 1332.25  | -69.01   | 1.897    | $1.120 \times 10^{-4}$ | 0.1110  |
| Figure 7(b)   | 346.15   | -20.91   | 0.6374   | $3.721 \times 10^{-4}$ | 0.1238  |
| Figure 7(c)   | 37726.04 | -1443.81 | 30.81    | $4.274 \times 10^{-6}$ | 0.0839  |
| Figure 7(d)   | 234.36   | -14.52   | 0.4614   | $4.157 \times 10^{-4}$ | 0.1264  |
| Figure 8(c)   | *        | *        | *        | *             | *       |
| Figure 8(d)   | 33.65    | -3.036   | 0.1814   | 0.01862       | 0.2056  |
| Figure 8(e)   | 121.22   | -9.027   | 0.4938   | $4.379 \times 10^{-3}$ | 0.1790  |

Table 6. The size of the coherent states ($\alpha$) required to achieve a single-qubit depolarizing EP (1QDE) and two-qubit depolarizing probability (2QDE) of 1 and 0.1%.

| Cluster       | 1QDE 1% | 1QDE 0.1% | 2QDE 1% | 2QDE 0.1% |
|---------------|---------|-----------|---------|-----------|
| Figure 7(a)   | 11.80   | 37.60     | 18.62   | 52.06     |
| Figure 7(b)   | 10.97   | 35.41     | 15.15   | 44.79     |
| Figure 7(c)   | 16.43   | 47.93     | 26.19   | 71.08     |
| Figure 7(d)   | 10.11   | 34.74     | 12.95   | 40.55     |
| Figure 8(c)   | 15.33   | 34.67     | 21.36   | 43.89     |
| Figure 8(d)   | 17.62   | 28.03     | 22.86   | 40.08     |
| Figure 8(e)   | 18.96   | 40.30     | 25.17   | 44.95     |

We can read off the size of the coherent state amplitude required for each of the cluster states considered in figures 10 and 11 to have an EP below 1%. This is summarized in table 6. We note that for the 18 qubit unit cell we would require $\alpha$ to be between 18.96 and 25.17. Also in table 6, we use the scaling relationships in tables 4 and 5 to estimate the size of the initial cat states required for an EP to be one order of magnitude below the predicted 1% threshold. As noted in section 1, the best cat states produced in the laboratory to date have an average of 3.1 photons [18], which is equivalent to a cat state of the form $|0\rangle + |3.52\rangle$. In order to achieve an EP below a threshold of 1% we would require substantially larger cat states. However, these results illustrate the resources required to construct 3D cluster state with a nonlinearity in the form of initial cat states in conjunction with an integrated linear optical network.

4. Teleportation

The results from the previous sections suggest that initial cat states with an amplitude greater than 20 are required for this scheme to be considered for 3D cluster state production. However, such large cat states are particularly prone to loss. In this section, we consider the use of teleportation to possibly reduce the size of the required cat states, leading to a trade-off between loss susceptibility and gate fidelity. Teleportation will also circumvent the low fidelities achieved by using a beam splitter as a controlled-Z gate shown in figure 4 by attempting to clean up the
Figure 12. Teleportation of the deterministic controlled-Z gate (dashed box):
\[ \theta_1 = \frac{\pi}{2\sqrt{2}}, \phi_1 = -\frac{\pi}{2}, \theta_2 = \frac{\pi}{4}, \phi_2 = 0, \phi_3 = \pi. \] The detectors \( n_1, n_2, n_3 \) and \( n_4 \) are photon number detectors.

Figure 13. The success probability as a function of the detection outcome for the teleporter in figure 12 when \( \alpha = \sqrt{2} \times 4 \approx 5.66 \). In this case, we examine detector outcomes of the form \( \{n_1, n_2, n_3, n_4\} = \{n_a, n_b, n_a, n_b\} \).

cluster states. We base our teleportation protocol, shown in figure 12, on a similar teleportation idea from Ralph et al [10]. We teleport our basic two-qubit cluster state using two copies of the Bell state \( |0,0\rangle + |\alpha,\alpha\rangle \). The fact that we also need to generate this Bell state places a more stringent restriction on the size of initial cat states we need to implement controlled-Z gates—instead of just the state \( |0\rangle + |\alpha\rangle \), we now also need a supply of the larger-amplitude cat states \( |0\rangle + |\sqrt{2}\alpha\rangle \), since the Bell state \( |00\rangle + |\alpha\alpha\rangle \) is the output of \( |0\rangle + |\sqrt{2}\alpha\rangle \) incident on one port of a symmetric beam splitter.

In figures 13 and 14, we examine the detection outcomes for this teleportation scheme. As can be seen from these figures and when one closely examines the output of the teleporter, there
Figure 14. The normalized fidelity as a function of the detection outcome for the teleporter in figure 12 when $\alpha = \sqrt{2} \times 4 \approx 5.66$. In this case, we examine detector outcomes of the form $\{n_1, n_2, n_3, n_4\} = \{n_a, n_b, n_a, n_b\}$.

are four dominant detection sequences:

$$\{n_1, n_2, n_3, n_4\} = \{n_a, 0, n_b, 0\}, \{n_a, 0, 0, n_b\}, \{0, n_a, n_b, 0\}, \{0, n_a, 0, n_b\},$$

(21)

where $n_1, n_2, n_3, n_4$ refer to the photon number detection outcomes in figure 12. We also see from figures 13 and 14 that the maximum success probability and normalized fidelity is centred around $\alpha^2/2$, in this case around $\alpha = 8$.

In contrast to [10], the fidelity in figure 14 only reaches values close to one when the success probability in figure 13 is maximum. This is due to the delicacy of teleporting the phase information hidden in the approximate controlled-$Z$ state in equation (3)—any measurement outcome that is not one of the four sequences in equation (21) removes all phase information on the output, which effectively becomes the identity state. However, since the probability is close to 0 in the regions where the fidelity is less than ideal, this does not have a dominant affect on the success of the teleporter.

Given that we accept that the teleporter detections are of the form given in equation (21), we can examine the success probability and normalized fidelity as a function of $n_a$ and $n_b$, as shown in figures 15 and 16. We again notice that the normalized fidelity in figure 16 is close to 1 only when the success probability in figure 15 is maximum.

As with standard qubit teleportation [24], there are single-qubit corrections necessary on the output of figure 12. We make the assumption that these corrections can be performed at a later time, or we could explicitly avoid applying these corrections altogether by keeping track of the necessary corrections, staying in the so-called Pauli frame [25], compensating for these corrections in subsequent measurements.
Figure 15. The success probability as a function of the detection outcome for the teleporter in figure 12 when $\alpha = \sqrt{2} \times 4 \approx 5.66$. In this case, we examine detector outcomes of the form $\{n_1, n_2, n_3, n_4\} = \{n_a, 0, 0, n_b\}$.

Figure 16. The normalized fidelity as a function of the detection outcome for the teleporter in figure 12 when $\alpha = \sqrt{2} \times 4 \approx 5.66$. In this case, we examine detector outcomes of the form $\{n_1, n_2, n_3, n_4\} = \{n_a, 0, 0, n_b\}$. 
Figure 17. The maximum fidelity the controlled-\(Z\) teleporter in figure 12 can produce as a function of \(\alpha\). Note that \(\alpha\) has been scaled to take the preparation of the Bell state cat states into account, that is, \(\alpha \rightarrow \sqrt{2} \alpha\).

The normalized fidelity of the teleporter in figure 12 can be made arbitrarily close to 1, given the correct detector firing. For an initial cat state amplitude as low as \(\sqrt{2} \times 2 ≈ 2.83\), the maximum normalized fidelity can be made as high as 0.999. This is shown in figure 17. The trade-off for having a high fidelity is the introduction of a success probability. We can see how the success probability of the teleporter scales given that we demand a certain average fidelity for the output. The average fidelity is given by [10]

\[
F_{av} = \frac{1}{P_{det}} \sum_{(n_a, n_b) \in S} \text{Prob}(n_a, n_b) F(n_a, n_b),
\]

where \(P_{det} = \sum_{(n_a, n_b) \in S} \text{Prob}(n_a, n_b)\) is the probability that the detection outcomes are in the set \(S\) and \(F(n_a, n_b)\) is the fidelity between the normalized output of figure 12 and equation (4).

In figure 18, we calculate the success probability of the teleporter given that we demand the output must have an average fidelity > 99%. As can be seen, for \(\alpha\) large enough, the success probability approaches 1.

In the next section we will examine whether teleporting the basic controlled-\(Z\) state is advantageous, that is, whether using a teleporter can reduce the amplitude of the initial cat states.

5. Discussion

In this section, we discuss the performance of incorporating teleportation into our 3D cluster state production. To determine this, we examine whether introducing located loss errors through teleportation can reduce the computational basis errors to a level that is correctable by the 3D topological cluster state code [1, 2]. We consider two possible simultaneous computational basis error and detectable loss error thresholds for the 3D topological cluster state code. We present preliminary results in which we consider the teleportation of a two-qubit cluster. The teleportation of larger cluster states will be the subject of a future paper.

We first consider the most stringent and thorough threshold, predicted by Barrett and Stace [5], in which it is estimated that 3D topological cluster states could correct for sole
computational basis errors as high as 0.63% and sole located loss errors as high as 24.9%, with a close to linear relationship for simultaneous loss and computation errors between these two bounds. We next consider a more optimistic outlook for the capabilities of these topological codes, by relaxing the computation basis error bound to 1%, a value that is believed to be reachable [3], and keeping the 24.9% located loss error bound.

We wish to examine the trade-off between the located loss errors introduced by teleportation and the computational basis errors resulting from non-orthogonal basis states, and to determine whether it is worthwhile to teleport the state in equation (3). This will be established by calculating both the loss error rate due to teleportation and the resulting computational error rate due to non-orthogonal qubits for specific initial cat state amplitudes and asking whether these loss and computational error rates could be corrected by the 3D topological code. That is, for a specific value of $\alpha$, does teleportation allow a located loss error rate and computational error rate that is below either the threshold calculated by Barrett and Stace or the optimistic threshold described earlier?

In figure 19, we examine the relationship between computational basis and located loss error rates for teleportation. Before teleportation, each $\alpha$ has an intrinsic computation error rate associated with it due to non-unit fidelity, as shown in figure 5. The output state from the teleportation circuit in figure 12 will have a reduced computation error rate at the expense of having a located loss error introduced. We wish to keep track of the detection outcomes from the teleporter that give the highest output fidelity. By counting the detection outcomes, we generate the curves shown in figure 19.

For example, consider the rightmost curve, corresponding to $\alpha = \sqrt{2} \times 3.0 = 4.24$. Initially we only consider a few detection outcomes centred around $(\alpha/2)^2$, leading to a small teleportation success probability, which means a large located loss error. However, since these detections correspond to high-fidelity outputs, the computational error rate is low, which is why the curve begins at the bottom right of the graph. As we consider more detection outcomes, the teleportation success probability increases, reducing the located loss error rate, at the expense of

Figure 18. The success probability of the teleporter in figure 12 given that the output must have an average fidelity > 99% is shown in blue and the fidelity for the beam splitter controlled-Z gate from figure 1 is shown in green. Note that $\alpha$ has been scaled for the teleporter probability curve to take the preparation of the Bell state cat states into account, that is, $\alpha \rightarrow \sqrt{2} \alpha$. 

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allowing more low-fidelity output states, therefore reducing the computational basis error rate. Consequently, the $\alpha = 4.24$ curve moves from the bottom right to the top left.

The number of detection outcomes considered for each curve in figure 19 increased as a function of $\alpha$. For low values of $\alpha$, there are not many teleportation detection outcomes that result in a computational basis error rate less than 1%, which is why these initial curves are not smooth.

The leftmost dot-dashed curve in figure 19 shows that teleporting coherent cat states with an amplitude of $\alpha = \sqrt{2} \times 9.65 = 13.65$ intersects the Barrett–Stace [5] threshold. The no located loss threshold for this case is equivalent to a computation basis error rate of 0.63%, which corresponds to the output from the basic controlled-$Z$ gate in figure 5 with $\alpha = 14.10$, where we consider the error model that gives the lowest $\alpha$ value, in this case the solid curves in figure 5. This means teleportation would allow initial cat states with $(14.10/2)^2 - (13.65/2)^2 = 3.14$ less photons. These results are summarized in the second column of table 7.

The rightmost dot-dashed curve in figure 19 shows that teleporting coherent cat states with an amplitude of $\alpha = \sqrt{2} \times 7.56 = 10.69$ intersects the more optimistic threshold. The no located loss threshold for this case is equivalent to a computation basis error rate of 1.0%, which corresponds to the output from the basic controlled-$Z$ gate in figure 5 with $\alpha = 11.14$, where we again consider the error model that gives the lowest $\alpha$ value, in this case also the solid curves in figure 5. This means teleportation would allow initial cat states with $(11.14/2)^2 - (10.69/2)^2 = 2.46$ less photons. These results are summarized in the third column of table 7.

The results of figure 19 that are summarized in table 7 suggest that, depending on which threshold is ultimately correct for 3D topological cluster state codes, teleportation will always
result in less stringent restriction being put on the amplitude size of the initial cat state, albeit in some cases only a reduction of 2.46 photons on average. However, it should be noted that a full analysis for the teleportation of a multi-qubit cluster state is necessary before we can claim that teleportation allows for a reduction in coherent amplitude when compared to the completely deterministic scheme.

It is worth exploring the possibility of not being penalized for using Bell states of the form $|00\rangle + |\alpha\alpha\rangle$ for teleportation. If this was conceivable, then teleportation would offer much greater benefits, allowing initial cat states with up to 26 less photons on average. The results of table 7 are re-analysed for such a case in table 8. Possible ways of avoiding these penalties would be to have a reliable source of $|00\rangle + |\alpha\alpha\rangle$ states, such as the entangled cat state produced by Ourjoumtsev et al [26]. However, we could then consider the more complicated entangled cat states required for quantum gate teleportation in Lund et al [19] and proceed with cluster state production purely via teleportation.

6. Conclusion

We have shown that deterministically constructing 3D topological cluster states with coherent logic qubits is possible provided the initial cat states are large enough. Since the entangling
gates for this scheme only involve passive linear optical elements, the use of integrated quantum optical circuits would be ideally suited for implementing this scheme. We found that for cat states with an average number of photons \( \alpha > 158.4 \) (\(|\alpha = 0\rangle + |\alpha = 25.17\rangle\)), the EP per gate for the full 18 qubit unit cell was less than 1\%, a computational error fault-tolerant threshold that is believed the 3D topological cluster states will satisfy.

We have also shown that teleportation could be used to clean up the cluster states produced from deterministic gates and that teleportation might provide a method to trade-off computational basis errors for located loss errors. Preliminary results for the teleportation of two-qubit cluster states suggest that initial cat states with an amplitude of \( \alpha = \sqrt{2} \times 7.56 = 10.69 \) (an average photon number of 28.57) would result in a combined located loss error rate and computational basis error rate that is below an optimistic threshold for 3D topological cluster state codes.

In this paper, we have assumed that all errors arise from the 3D cluster state construction in the form of computational basis errors via the deterministic controlled-Z gate and located loss errors via teleportations to clean up the deterministic controlled-Z gate. In reality there will be other sources of error, such as measurement errors, errors resulting from actual computation and storage errors. Loss errors other than those induced by teleportation will be a particular problem in constructing 3D cluster states using this scheme. Since the \( X \)-basis measurement scheme presented in section 3 is a photon number parity measurement, it would be particularly susceptible to loss. In addition to this, the first physical requirement for topological cluster state computation, the need for state preparations of \(|0\rangle_{\text{Logical}} \) and \(|0\rangle_{\text{Logical}} + e^{i\theta}|1\rangle_{\text{Logical}}\), also needs to be addressed for coherent state logic.

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