Thompson Sampling with Unrestricted Delays

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We investigate properties of Thompson Sampling in the stochastic multi-armed bandit problem with delayed feedback. In a setting with i.i.d delays, we establish to our knowledge the first regret bounds for Thompson Sampling with arbitrary delay distributions, including ones with unbounded expectation. Our bounds are qualitatively comparable to the best available bounds derived via ad-hoc algorithms, and only depend on delays via selected quantiles of the delay distributions. Furthermore, in extensive simulation experiments, we find that Thompson Sampling outperforms a number of alternative proposals, including methods specifically designed for settings with delayed feedback.

CCS Concepts: Computing methodologies → Machine learning.

Additional Key Words and Phrases: Adaptive Experiments, Delayed Feedback, Thompson Sampling

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1 INTRODUCTION

The stochastic multi-armed bandit (MAB) problem is a framework for sequential experimentation that has been widely used in a number of application areas, including online advertising and recommendations [8] and medical trials [14, 25]. In the basic stochastic MAB specification, each of \(k = 1, \ldots, K\) available arms has a reward distribution and, when an agent selects an arm, a reward drawn from the corresponding reward distribution is immediately revealed to them [1, 5, 8, 10, 21]. In many real world settings, however, the assumption that rewards are revealed immediately following an action is not applicable. For example, in a clinical trial it may take time to assess whether a patient has responded to the given treatment [6, 11, 33], while in an e-mail marketing campaign, it may take time to see whether a user clicks on an ad [7].

Motivated by this observation, there has been considerable recent interest on methods and theory for MAB problems with delays between when an action is taken, and when the corresponding reward is observed [9, 13, 17, 22–24, 30, 31, 34]. These papers, however, all require either modifying familiar MAB algorithms to account for delays, or propose new, delay-robust algorithms that are likely unfamiliar to practitioners.

The focus of this paper is in understanding how Thompson sampling [29], used out of the box and without any adaptations, behaves under delays (as discussed further below, we consider a specification where the posterior beliefs underlying Thompson sampling are updated whenever new rewards are observed, and otherwise we proceed as usual). Thompson sampling is a robust MAB
algorithm that consistently achieves strong empirical performance across a number of benchmarks, and is popular among practitioners [8, 28].

Our main result is that, using analytic ideas that build on results for both Thompson sampling without delays [2, 3, 18] and recent ideas for accommodating delays [22], we can verify that Thompson sampling admits strong formal guarantees in the setting with unrestricted delays. Specifically we prove a $O \left( \sum_i \log T / q_i + d_i(q_i) \right)$ regret bound for any selected $q_i \in (0, 1)$, where $d_i(q_i)$ is the $q_i$-th quantile of the delay distribution of the $i$-th arm. Meanwhile, our experiments align with observed strong empirical performance of Thompson sampling: In a number of simulation specifications adapted from recent papers that propose new delay-robust MAB algorithms, we find that Thompson sampling matches or outperforms the proposed algorithms. Overall, our results suggest Thompson sampling to be a robust and reliable method for stochastic MAB problems with delays.

1.1 Related Work

Early results on MAB with delays made strong assumptions on delay distributions: For example, Dudik et al. [9] consider a model with constant (deterministic) delays, Mandel et al. [23] assume bounded delays, while Joulani et al. [17] assume that the delay distribution has bounded expectation. More recently, there has been interest in guarantees that are robust to heavy-tailedness in the delay distribution. Gael et al. [13] consider a setting with polynomial tail bounds on the delay distribution while in a recent advance, Lancewicki et al. [22] developed algorithms based on UCB and successive elimination that allow for unrestricted delay distributions. In this paper, we also allow for unrestricted delay distributions, and adapt ideas from Lancewicki et al. [22] in order to do so.

There are a considerable number of results on the behavior of Thompson sampling without delays [1–4, 12, 15, 18, 19, 27, 28]. However, while Thompson sampling is known empirically to be robust to delays [8], we are not aware of formal regret guarantees available in this setting. Joulani et al. [17] propose a meta-algorithm for turning any stochastic MAB algorithm with guarantees in the delay-free setting into one that has guarantees with delays. However, when applied to Thompson sampling, their meta-algorithm would require subtle modifications to Thompson sampling, and their results only apply to delay distributions with bounded expectation. Gur and Momeni [16] show that Thompson sampling has desirable properties in a setting without delays, but where the analyst may sometimes acquire additional information from external sources. At a high level, our paper is aligned with Gur and Momeni [16] in that we both find Thompson sampling to be robust to non-standard information flows. Qin and Russo [26] propose a robust variant of Thompson sampling that, when used for arm selection, is guaranteed never to perform much worse than a uniformly randomized experiment in choosing a good arm to deploy—even in non-stationary environments and under arbitrary delays.

Finally, we also note recent work on sequential learning with delays that go beyond the MAB-based specification considered here. Vernade et al. [30] studied partially observed stochastic Bernoulli bandit where only a reward of 1 can be directly observed with the value of delay. Under the assumption that the delay distribution is known they provided algorithms that have close to optimal asymptotic performance. Vernade et al. [31] further extended this framework to linear bandit where a version of Thompson Sampling was provided but no theoretical analysis was given. Pike-Burke et al. [24] studied the MAB with delays when we only have access to aggregated anonymous feedback. Zhou et al. [34] studied a generalized linear contextual bandit with delays, while Vernade et al. [32] considered the case of non-stationary bandits with delays when intermediate observations are available.
2 PROBLEM SETUP AND BACKGROUND

We consider the following problem setup, which adds the stochastic delay structure to the classical stochastic multi-armed bandit problem. Suppose we have $K$ arms with reward distribution $\nu_1, \ldots, \nu_K$. We assume that all reward distributions are supported on the interval $[0, 1]$, and that there is a unique optimal arm (with the highest mean reward). At each round $t = 1, \ldots, T$, the agent chooses an action $a_t$. The environment samples a reward $r_t$ from $\nu_{a_t}$, a delay $l_t$, and the agent observes the reward at round $t + l_t$. The setup is described in Model 1. We assume for simplicity that delays are supported on $\mathbb{N} \cup \{\infty\}$; this is without loss of generality since the agent only collects feedback and chooses new actions at integer time points. We also note that the values of delay and the original time of the reward are not revealed to the agent.

Model 1: Stochastic multi-armed bandit with delays

\[
\text{for } t \in [T] \text{ do}
\begin{align*}
\text{Agent picks an action } & a_t \in [K]. \\
\text{Environment samples } & r_t \sim \nu_{a_t} \text{ and } l_t. \\
\text{Agent get a reward } & r_t, \text{ which is not immediately revealed.} \\
\text{The set } & \{(a_s, r_s) : t = s + l_s\} \text{ is revealed to the agent, which contains values of action, reward pairs from previous rounds.}
\end{align*}
\text{end}
\]

Our setup leaves the delay structure unspecified. In this paper, we mainly focus on one particular form of delay, i.i.d delays. In this setting, each arm has a separate delay distribution and $l_t$ is sampled independently of everything else from the delay distribution of arm $a_t$. We will evaluate our agent by the expected regret (called regret from hereon) which under Model 1 can be expressed as

\[
R_T = T\mu_{i^*} - \sum_{t=1}^{T} \mathbb{E}[r_t] = \sum_{i=1}^{K} \Delta_i \mathbb{E}[m_T(i)],
\]

where $\mu_i$ is the mean of distribution $\nu_i$ and $i^*$ denotes the optimal index. $\Delta_i = \mu_{i^*} - \mu_i$ is the gap between the optimal arm and the arm $i$ and $m_T(i)$ is the number of time we pull arm $i$ by time $T$.

2.1 Thompson Sampling under Delay

Thompson sampling [29] is an adaptive Bayesian method that chooses the actions at each round according to the current posterior probability that the action maximizes expected reward; see Russo et al. [28] for a recent review. We consider a Bernoulli bandit algorithm that starts with a uniform prior (Beta(1, 1)) on the mean parameter $\mu_i$ of each arm; see Algorithm 2 for details. We note that our analysis below extends naturally to reward distributions with bounded support following the argument in Agrawal and Goyal [1]. The main idea here is that the algorithm emerging from Beta-Bernoulli bandits can in fact be applied (and has good regret properties) for any setting with bounded outcomes. However, we do note that the analysis below does not accommodate reward distributions with unbounded support.

3 THEORETICAL RESULTS

In this section we present formal regret guarantees of Algorithm 2 applying to Bernoulli bandit with i.i.d delays. We make the following assumptions on top of Model 1.

• Assumption 1: The distributions $\nu_1, \ldots, \nu_K$ are Bernoulli distributions with means $\mu_1 > \cdots > \mu_K$. 

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ALGORITHM 2: Thompson Sampling for Bernoulli Bandits under Delays

for $i = 1, \ldots, K$ do
  Set counters $S_i = 0$, $F_i = 0$.
end

for $t \in [T]$ do
  For $i \in [K]$, sample $\theta_i(t)$ from Beta($S_i + 1, F_i + 1$).
  Play $a_t = \text{argmax}_i \theta_i(t)$.
  for Revealed observation $(a_s, r_s)$ with $s + l_s = t$ do
    $S_{a_s} = S_{a_s} + r_s$
    $F_{a_s} = F_{a_s} + 1 - r_s$
  end
end

- Assumption 2: Each arm has a delay distribution $D_1, \ldots, D_K$ supported on non-negative integers and $\infty$ and $l_t$ is sampled from $D_{a_t}$, which is independent of rewards and past delays.

We use the following notation throughout:

- #actions taken: $m_t(i) = \sum_{s=1}^{t} 1\{a_s = i\}$.
- #available observations: $n_t(i) = \sum_{\{s: s + l_s \leq t\}} 1\{a_s = i\}$.
- Delay quantile: $d_i(q) = \inf\{d : \Pr[l_t \leq d \mid A = i] \geq q\}$.

We also write $\theta_i(t)$ for the posterior sample of arm $i$ at time $t$.

3.1 Two-arm case

For simplicity, we start by considering the case with $K = 2$, as this enables us to present a proof with less notational overhead. Recall that, by assumption, $\mu_1 > \mu_2$; and we write the arm gap as $\Delta = \mu_1 - \mu_2$. In this setting, we show the following.

**THEOREM 3.1.** Under assumption 1 and 2 with $K = 2$, suppose further we have i.i.d delays for each arm. Then the regret $R_T$ of Algorithm 2 is bounded by

$$\min_{q_1, q_2 \in [0, 1]} \frac{48 \log T}{q_2 \Delta} + \frac{6}{\Delta} \left( \frac{32 \log T}{q_1 \Delta} + d_1(q_1) \Delta + \Delta \right)$$

$$+ d_2(q_2) \Delta + O\left( \frac{1}{\Delta} + \frac{1}{\Delta^3} \right)$$

In other words, the regret of a Bernoulli bandit with delays and $K = 2$ arms is bounded to order

$$R_T = O\left( \frac{\log T}{\Delta} \left( \frac{1}{q_1} + \frac{1}{q_2} \right) + (d_1(q_1) + d_2(q_2))\Delta \right)$$

for any choice of $q_1$ and $q_2$. For example, if we set $q_1 = q_2 = 0.5$, then the above bound depends on the medians of the delay distributions.

As discussed above, most earlier results on MAB with delays—using Thompson sampling or other algorithms—made further assumptions on the delay distributions (e.g., Joulani et al. [17] assumed bounded expectations for delays) and so our results are not directly comparable to them.
Only recently, Lancewicki et al. [22] obtained bounds that hold with unrestricted delays for some variants of UCB and successive elimination. With $K = 2$, their bound is

$$R_T \leq \min_{q_1, q_2} \frac{40 \log T}{\Delta} \left( \frac{1}{q_1} + \frac{1}{q_2} \right) + \log(2)(d_1(q_1) + d_2(q_2))\Delta$$

(3)

for a variant of the Successive Elimination algorithm of Even-Dar et al. [10] adapted to the setting with delays. We see both bounds have an extra term involving linear combination of the quantiles of the delay distribution and both are of the same big-O order. In particular in the case of a constant delay of value $d$, both bounds have an additive $O(d)$ term compared to the regret bound without delay.

We note that the similar appearance of the bounds (2) and (3) is not a coincidence: As seen below, our proof involves incorporating some key ideas from Lancewicki et al. [22] into a study of Thompson sampling that builds on Agrawal and Goyal [1].

**Proof of Theorem 3.1.** When $K = 2$, regret measures how often we pull the second arm in expectation,

$$R_T = \Delta \mathbb{E}[m_T(2)],$$

(4)

and so to bound regret at $T$ we need to bound $m_T(2)$. To this end, we decompose the problem as follows:

- Let $Y_j$ be the number of times the 2nd arm is pulled between $j$-th and $(j + 1)$-st draws of the first arm, $Y_j = |\{t : m_T(1) = j\}| - 1$.
- Let $\tau_2 = \inf \{t : n_T(2) \geq 24\Delta^{-2}\log(T)\}$.
- Let $j_0 = m_T(1)$.

Because the agent must pull the 2nd arm each time they don’t pull the 1st, we see that

$$m_T(2) \leq m_T(2) + \sum_{j=j_0}^{m_T(1)} Y_j.$$  

(5)

To proceed, we then posit the following events,

$$F_1 = \left\{ \exists t \leq T, i : m_T(i) \geq \frac{24 \log(T)}{\Delta}, n_T + d_i(q_i)(i) < \frac{q_i}{2} m_T(i) \right\},$$

$$F_2 = \left\{ \exists t \leq T : \theta_2(t) > \mu_2 + \frac{\Delta}{2}, n_2(t) \geq \frac{24 \log T}{\Delta^2} \right\},$$

(6)

and argue that these two events are rare. Our definition of the events $F_1$ and $F_2$ is motivated by ideas used in both Agrawal and Goyal [1] and Lancewicki et al. [22].

**Lemma 3.2.** Under the conditions of Theorem 3.1, $\mathbb{P}[F_1] \leq \frac{1}{T}$ and $\mathbb{P}[F_2] \leq \frac{2}{T}$.

**Proof.** See section A of the appendix. □

The upshot is that we can now define a “good” event $G = \neg F_1 \cap \neg F_2$, and note that by the union bound, $\mathbb{P}[G] \geq 1 - 3/T$. Thus,

$$\mathbb{E}[m_T(2)] \leq \mathbb{P}[G]\mathbb{E}[m_T(2) \mid G] + \mathbb{P}[\neg G]\mathbb{E}[m_T(2) \mid \neg G] \leq \mathbb{E}[m_T(2) \mid G] + 3.$$  

(7)
We now plug (5) into (7). Our goal at this point is to prove that
\[
\mathbb{E}[m_{r_2}(2) \mid G] \leq \frac{48 \log T}{q_2 \Delta^2} + d_2(q_2), \text{ and}
\]
\[
\mathbb{E}\left[\sum_{j=0}^{m(T)} Y_j \mid G\right] \leq O\left(\frac{1}{\Delta^2} + \frac{1}{\Delta^4}\right) + \left(\frac{32 \log T}{q_1 \Delta^2} + d_1(q_1) + 1\right) \frac{6}{\Delta}
\]
(8)
(9)
Combining (4), (5), (7), and the equations above then yields the desired result.

Now, to check (8), we note that on \(G\), if \(m_{r_2-d_1}^j(q_2) \geq \frac{24 \log(T)}{q_2}\) then
\[
n_{r_2}^j(q_2) = n_{r_2-d_1}^j(q_2) + d_2(q_2) \geq \frac{q_2}{2} m_{r_2-d_1}^j(q_2) \geq \frac{q_2}{2} m_{r_2-d_1}^j(q_2).
\]
This implies
\[
m_{r_2-d_1}^j(q_2) \leq \max\left\{\frac{48 \log T}{q_2 \Delta^2}, \frac{24 \log(T)}{q_2}\right\} = \frac{48 \log T}{q_2 \Delta^2}\text{ by our assumption } \Delta \in (0, 1]
\]
Hence, we have
\[
m_{r_2}^j(q_2) = m_{r_2-d_1}^j(q_2) + m_{r_2}^j(q_2) - m_{r_2-d_1}^j(q_2) \leq \frac{48 \log T}{q_2 \Delta^2} + d_2(q_2),
\]
and so in fact the inequality (8) holds almost surely conditionally on \(G\).

Next, to check (9), we bound \(\mathbb{E}\left[\sum_{j=1}^{T} Y_j \mid G\right]\) instead. Let \(t_j = \inf\{t : m_t(1) = j\}\) be the time we pull arm 1 the \(j\)-th time. Let \(X(j,s,y)\) denote the number of trials before a Beta\((s + 1, j - s + 1)\) exceeds \(y\) as in Agrawal and Goyal [1]. We proceed by proving the following lemma.

**Lemma 3.3.** Let \(s(j)\) be the number of successes among the \(j\) observed rewards from arm 1. Then for \(j \geq j_0\),
\[
\sum_{j=1}^{T} \mathbb{E}\left[Y_j \mid G\right] \leq \sum_{j=1}^{T} \mathbb{E}\left[\min\{X(n_{t_j}(1), s(n_{t_j}(1)), \mu_2 + \frac{\Delta}{2}), T\}\right]
\]
\[
+ \sum_{k=1}^{T} \mathbb{E}\left[\min\{X(k, s(k)), \mu_2 + \frac{\Delta}{2}, T\}\right] \tag{10}
\]

**Proof.** See section B of the appendix.

To bound right side of (10), we use the following lemma from Agrawal and Goyal [4] and its corollary.

**Lemma 3.4.** For any \(i \neq 1\), let \(y_i = \mu_i + \frac{\Delta_i}{2}\). Let \(D_i\) denote the KL-divergence between \(\mu_1\) and \(y_i\), i.e. \(D_i = y_i \log \frac{y_i}{\mu_1} + (1 - y_i) \log \frac{1 - y_i}{1 - \mu_1}\). Then
\[
\mathbb{E}\left[\min\{X(k, s(k), y_i), T\}\right] \leq \begin{cases} 
\frac{6}{\Delta_i}, & k < \frac{6}{\Delta_i} \\
O\left(e^{-\Delta_i^2 k/8} + \frac{4}{(k + 1)\Delta_i^2} e^{-D_i k} + \frac{1}{e^{\Delta_i^2 k/16} - 1}\right), & k \geq \frac{6}{\Delta_i}
\end{cases}
\]
\[k < \frac{6}{\Delta_i}\]
\[k \geq \frac{6}{\Delta_i}\]
In particular, when $k \geq \frac{16\log T}{\Delta_i^2}$, $\mathbb{E} \left[ \min \{X(k, s(k), y_i), T\} \right] = O\left(\frac{1}{T}\right)$

Corollary 3.5. Under the assumption of Lemma 3.4, we further have the following bound

$$\sum_{k=1}^{T} \mathbb{E} \left[ \min \{X(k, s(k), y_i), T\} \right] \leq O\left(\frac{1}{\Delta_i^2} + \frac{1}{\Delta_i^2 D_i} + \frac{1}{\Delta_i^4}\right)$$

Now we bound $n_{ij}(1)$. Condition on $G$, since $G \subset \neg F_1$, we know if $m_r(1) \geq \frac{32\log T}{q_i\Delta^2}$ then $n_{t+d_i(q_i)}(1) \geq \frac{q_i}{2} m_r(1) \geq \frac{16\log T}{\Delta^2}$. Hence, let $l = \frac{32\log T}{q_i\Delta^2}$, we have $m_{t_i}(1) \geq \frac{32\log T}{q_i\Delta^2}$, and $n_{t_i}(1) \geq n_{t+di(q_i)}(1) \geq \frac{16\log T}{\Delta^2}$.

Let $M = \lceil \frac{32\log T}{q_i\Delta^2} + d_i(q_i) \rceil$, $y = \mu_2 + \frac{\Delta}{2}$ and $D = y \log \frac{y}{\mu_i} + (1 - y) \log \frac{1-y}{1-\mu_i}$ we then have

$$\mathbb{E} \left[ \sum_{j=1}^{T} Y_j \mid G \right]$$

$$\leq \sum_{j=1}^{T} \mathbb{E} \left[ \min \{X(n_{t_j}(1), s(n_{t_j}(1)), \mu_2 + \frac{\Delta}{2}), T\} \mid G \right] +$$

$$+ \sum_{k=1}^{T} \mathbb{E} \left[ \min \{X(k, s(k), \mu_2 + \frac{\Delta}{2}), T\} \mid G \right]$$

$$\leq \sum_{j=1}^{M} \mathbb{E} \left[ \min \{X(n_{t_j}(1), s(n_{t_j}(1)), \mu_2 + \frac{\Delta}{2}), T\} \mid G \right] +$$

$$+ \sum_{j=M+1}^{T} \mathbb{E} \left[ \min \{X(n_{t_j}(1), s(n_{t_j}(1)), \mu_2 + \frac{\Delta}{2}), T\} \mid G \right] +$$

$$O\left(\frac{1}{\Delta^2} + \frac{1}{\Delta^2 D} + \frac{1}{\Delta^4}\right)$$

$$\leq M \max_n \mathbb{E} \left[ \min \{X(n, s(n), \mu_2 + \frac{\Delta}{2}), T\} \mid G \right] + \frac{16}{T} T$$

$$+ O\left(\frac{1}{\Delta^2} + \frac{1}{\Delta^2 D} + \frac{1}{\Delta^4}\right)$$

$$\leq \frac{6M}{\Delta} + O\left(\frac{1}{\Delta^2} + \frac{1}{\Delta^2 D} + \frac{1}{\Delta^4}\right)$$

\[ \square \]

3.2 Multi-arm case

Now we present the results in the case of more than 2 arms. In this case we have the following result.
Theorem 3.6. Under assumption 1 and 2 with $K > 2$, suppose further we have i.i.d delays for each arm. Then the regret (1) of Algorithm 2 is bounded by

$$
\min_{q_i \in (0, 1]} \sum_{i=2}^{K} 48 \log \frac{T}{q_i \Delta_i} + d_i(q_i) \Delta_i 
+ \sum_{i=2}^{K} O \left( \frac{1}{\Delta_i} + \frac{1}{\Delta_i^3} \right) \left( \frac{32 \log T}{q_i \Delta_i} + d_i(q_i) \Delta_i + \Delta_i \right) \frac{6}{\Delta_i} 
+ \sum_{i=2}^{K} O \left( \frac{1}{\Delta_i} + \frac{1}{\Delta_i^3} \right) \left( \sum_{i=2}^{K} \frac{48 \log T}{q_i \Delta_i^2} + d_i(q_i) \right) + 4(K - 1)
$$

Proof. See section C of the appendix.

Corollary 3.7. The regret of Algorithm 2 when $K > 2$ is of order

$$
O \left( \sum_{i=1}^{K} \frac{\log T}{\Delta_i} \left( \frac{1}{q_1} + \frac{1}{q_i} \right) + (d_i(q_1) + d_i(q_i)) \Delta_i \right)
$$

Remark: The bound in Lancewicki et al. [22] in this case is

$$
\min_{q_i \neq 1} \sum_{i=1}^{K} 40 \log \frac{T}{\Delta_i} \left( \frac{1}{q_1} + \frac{1}{q_i} \right) + \log(K) \max_{i \neq 1} \{(d_i(q_1) + d_i(q_i)) \Delta_i \}. \tag{11}
$$

If we consider the number of arms $K$ to be fixed then this bound has the same scaling as ours. However, if $K$ can grow, then the cost of delays in the bound of Lancewicki et al. [22] has a better $K$-scaling than ours (logarithmic as opposed to linear). It would be an interesting topic for further work to see if an alternative proof could establish log($K$)-scaling for the cost of delays in Thompson sampling, or if there exists a separation in results one can get from Thompson sampling versus the algorithms considered in Lancewicki et al. [22]. In any case, we note that in our experiments we consider many settings with a moderately large number of arms, $K = 20$, and the good behavior of Thompson sampling there suggests that it is at least moderately robust to the large-$K$ setting.

4 NUMERICAL EXPERIMENTS

We have shown that in stochastic multi-armed bandit with i.i.d. delays, Thompson Sampling can achieve comparable regret bounds as a variant based on Successive Elimination, which constructs upper confidence bounds. However, the advantage of Thompson Sampling under delays goes far beyond achieving good theoretical bounds. In this section, we demonstrate through extensive experiments that Thompson Sampling can often outperform a number of UCB variants under various delay structures. Specifically, we will compare Thompson Sampling with baselines under a number of delay settings considered in prior work, as well as in some new settings (including ones with non-i.i.d. delays). Note that our implementation of Thompson Sampling does not change depending on the assumptions of delays, but the UCB variants used as baselines change from setting to setting in order to accommodate different delay distributions. We use the Bernoulli bandit setting for all the experiments.

Table 1 gives a summary of all settings we consider. We explicitly list the type of delays as well as whether the setup is considered in previous work.

4.1 Methods

We consider the following methods for all experiments and also include methods designed to target certain settings in previous work if there is any.
Table 1. Summary of all experimental settings we consider

| I.I.D? | Delay Type       | Reference            |
|--------|------------------|----------------------|
| Yes    | Fixed            | Lancewicki et al. [22] |
| Yes    | $\alpha$-Pareto  | Gael et al. [13]     |
| Yes    | Packet-loss      | Lancewicki et al. [22] |
| Yes    | Geometric        | Vernade et al. [30]  |
| Yes    | Uniform          | $\times$            |
| No     | Queue-based      | $\times$            |

- Delayed-UCB1, straightforward adaptation of UCB1 [5] in delay case outlined in Joulani et al. [17]. We employ random tie breaking.
- Successive Elimination with delays (SE), which is shown to achieve great theoretical and empirical performance with unrestricted delays in Lancewicki et al. [22]. We show the algorithm details from their work in Algorithm 3. We use the same radius when constructing upper confidence bounds as Lancewicki et al. [22].
- Thompson Sampling with delays (TS), detailed in Algorithm 2.

Furthermore, whenever we reuse simulations from prior work (see Table 1), we also consider as baselines algorithms proposed in the corresponding papers. Specifically, for $\alpha$-Pareto delays we also consider the PatientBandit (PB) algorithm of Gael et al. [13], for the packet-loss setting we also consider the Phased Successive Elimination (PSE) algorithm of Lancewicki et al. [22], and for geometric delays we also consider the UD-UCB algorithm of Vernade et al. [30].

The main algorithms we consider, namely Delayed-UCB1, SE and TS, are all agnostic to the delay distribution, i.e., the algorithm itself doesn’t explicitly depend on what we assume about delays. In contrast, the other baselines we use may depend on the delay distributions—and when this is the case we let the algorithm use oracle information about the delays (thus making our comparison potentially too favorable towards these baselines).

**Algorithm 3**: Successive Elimination with Delays

**Input**: Number of rounds $T$, number of arms $K$

**Initializations**: $S \leftarrow [K]$, $t \leftarrow 1$

while $t < T$ do

- Pull each arm $i \in S$
- Observe any incoming feedback
- Set $t \leftarrow t + |S|$
- Update lower and upper confidence bounds where the radius is $\sqrt{\frac{2}{\max\{n_t(i), 1\}}}$
- Remove from $S$ all arms $i$ such that exists $j$ with $UCB_t(i) < LCB_t(j)$

end

4.2 I.I.D Delays

In this part we mainly focus on experiments with i.i.d delays considered in Section 2.

**Fixed Delays**. In this setting, all the delays have a fixed value. As in Lancewicki et al. [22], we set this value to be 250. We fix the number of arms to be $K = 20$ and the means of each arm are generated uniformly from the interval $[0.25, 0.75]$. The maximum round is $T = 20000$. All results are averaged cross 100 replications. Figure 1 shows the resulting regret plot. We see that TS
significantly outperforms the UCB style algorithms and the cumulative regret plot plateaus long before the other two methods.

Fig. 1. Regret of Delayed-UCB1, SE, TS for fixed delay 250, averaged over 100 replications. Error bars are displayed via shading.

We also experimented with increasing and decreasing delays. As expected, the gap between UCB style algorithms and TS is not significant when the delay is 0 and widens as we increase the delay.

**α-Pareto delays.** These delays are distributions with heavy tails and infinite expectations with $\alpha \leq 1$. The parameter $\alpha$ controls the tail behavior with heavier tails for smaller $\alpha$’s. We employ the experimental setup in Gael et al. [13] which has $K = 2$ arms and $T = 3000$ rounds. We fix the mean parameters to be $\mu_1 = 0.4$ and $\mu_2 = 0.45$ to make the problem instance reasonably difficult. We further let $\alpha_1 = 1$, which controls the tail of the delay distribution of the first arm. We vary the value of $\alpha_2$ and let it change in the set $\{0.2, 0.5, 0.8\}$. In case of $\alpha$-Pareto Delays, Gael et al. [13] proposed the PatientBandit (PB) algorithm which requires an input of oracle $\alpha$ that captures the tail decay of the delay distributions to construct upper confidence bounds. PB is a robust algorithm that, as shown in Gael et al. [13], can also handle a partially observed setting where a feedback of 0 can mean either a reward of 0 or the delay has not passed. Figure 2 shows the resulting cumulative regret of all four algorithms averaged over 300 replications.

We see that SE only outperforms PB when $\alpha$ is small, which means the delay distribution, i.e. when the delay distribution has a very heavy tail. This is partly due to the fact that PB algorithm uses a conservative upper confidence bound since it assumes a uniform $\alpha$ controlling the tails of all delay distributions. We also see that TS and Delayed-UCB1 are significantly better than SE and PB. In addition, TS has much smaller regret compared to Delayed-UCB1.

**Packet-loss.** This setting refers to a delay with infinite expectation. Specifically the delay will have a value of 0 with probability $p$ and infinity otherwise. In Lanczewski et al. [22], the Phased Successive Elimination (PSE) algorithm was provided specifically for this setting, which balances the observed reward counts of each arm during each phase and then does elimination based on upper and lower confidence bound. We run our experiments with $K = 20$ arms and the means of each arm are sampled uniformly from interval $[0.25, 0.75]$. We sample the probabilities $p$ of the packet loss uniformly from interval $[0, 1]$. We average our results across 200 replications and run
Fig. 2. Regret of UCB without log factor, SE, TS and PB for Pareto distributed delays with varying $\alpha$ for the optimal arm 2, averaged over 300 replications for $T = 10000$ rounds in each replication. We use the same PSE algorithm details as in Lancewicki et al. [22].

For $T = 10000$ rounds in each replication. We use the same PSE algorithm details as in Lancewicki et al. [22].

Figure 3 shows the resulting regret plot. We see that TS and Delayed-UCB stand out as the clear winners and TS performs slightly better. The reason that Delayed-UCB is comparable to TS is that in the parameter setting we use to run the experiments, the optimal arm has a small probability ($p = 0.108$) in packet loss. This means the probability of having infinite delay is close to 0.9 for the optimal arm. So the information about the optimal arm is very limited. In fact as we increase this
probability we see a widening gap between the two. As we do not know in practice the optimal arms, we think even in this setting TS stands out as a safe choice to use.

**Geometric Delays.** In this setting, the delays are Geometrically distributed, which means they can still be arbitrarily long but the expectation is finite. Vernade et al. [30] proposed an algorithm which assumes a known delay distribution and used geometric delays in the simulation. Specifically, the algorithm uses CDF functions of the delay distribution to form a conditionally unbiased estimator. Then an upper confidence bound is formed to select which arm to pull at each round. We include this algorithm, which is called UD-UCB in Vernade et al. [30] into our comparison. As in Vernade et al. [30], we ran for $T = 10000$ rounds and let the means of three arms be $(0.5, 0.4, 0.3)$. We use $p = 0.01$ to sample the delay and average across 200 replications.

Figure 4 shows the resulting regret plot. We see that TS performs much better than the other three methods and interestingly is also much better than UD-UCB even if UD-UCB knows the delay distribution and uses this extra information in the algorithm. This again shows the robustness of Thompson Sampling to various delays.

Fig. 4. Regret of UCB, SE, UD-UCB, TS algorithms under delays sampled from Geometric$(0.01)$. Results are averaged over 200 runs. Error bars are displayed via shading.

**Uniformly-distributed Delays.** Finally we consider a uniform delay distribution. We will use the same setup as experiments for fixed delays, namely $K = 20$ arms and uniformly sampled means. However, instead of having a fixed value 250 we will sample delay uniformly from the integers in interval $[150, 300]$. In this setting we only consider the original three methods as we did not find any algorithm in the literature specifically designed to target such delay. We average across 100 replications. Figure 5 shows the resulting regret plot. We see again that TS performs significantly better than the other methods.

4.3 **Non i.i.d Delays**

In our formal analysis we only considered the behavior of Thompson sampling under i.i.d delays. Here, however, we seek to empirically validate its behavior with non-i.i.d. delays and find that, although corresponding theory still remains to be developed, Thompson Sampling still appears to achieve reasonable performance when facing such delays.
Queue-based Delay. We describe a queue-based delay mechanism here which is inspired by classical models for queuing system [20]. The basic idea is once we select an arm $i$, the current action goes to the queue for arm $i$. If there is no other actions in the queue, i.e. the queue is empty then the reward is revealed immediately. Otherwise, the reward will be revealed when the other actions in the queue are cleared. The time to clear an action for each arm will be an exponential distribution with rate 0.1. Clearly from our description, this is a non i.i.d scenario. We let $K = 5$, sample means uniformly from $[0.25, 0.75]$ and average over 200 runs. Figure 6 shows the resulting regret plot. We see that even under this non i.i.d delay scenario, Thompson Sampling works much better than the other two methods.
5 DISCUSSION

In this paper we presented a regret analysis of Thompson Sampling under i.i.d unrestricted delays which could have potentially unbounded expectations. The regret bound is of the same order as the UCB-based algorithms proposed in the literature. In addition to theoretical findings, we present a comprehensive empirical study of existing methods under various delay distributions, including unbounded ones with infinite expectations, and further consider a non i.i.d delay mechanism based on queues. In all cases, we find Thompson sampling to be a robust and performant algorithm that does not require any problem-dependent tuning.

In future work, it would be of considerable interest to study the behavior of Thompson Sampling beyond the reward-independent i.i.d setting. One example is to analyze queue-based delays we introduced in Section 4.3, where we have already empirically established that Thompson Sampling is quite robust. Another example is to consider a reward-dependent setting where the delay and the reward are sampled jointly from a distribution that allows for dependence (e.g., where small rewards come with longer delays). In our experiments, we have found some reward-dependent settings where applying Algorithm 2 directly results in a large regret; and it would be interesting to investigate variants of Thompson Sampling that may achieve better performance here.

REFERENCES

[1] Shipra Agrawal and Navin Goyal. 2012. Analysis of Thompson Sampling for the Multi-armed Bandit Problem. In Proceedings of the 25th Annual Conference on Learning Theory. 39.1–39.26.
[2] Shipra Agrawal and Navin Goyal. 2013. Further Optimal Regret Bounds for Thompson Sampling. In Proceedings of the Sixteenth International Conference on Artificial Intelligence and Statistics. 99–107.
[3] Shipra Agrawal and Navin Goyal. 2013. Thompson Sampling for Contextual Bandits with Linear Payoffs. In Proceedings of the 30th International Conference on Machine Learning. 127–135.
[4] Shipra Agrawal and Navin Goyal. 2017. Near-Optimal Regret Bounds for Thompson Sampling. J. ACM 64, 5, Article 30 (Sep 2017), 24 pages.
[5] P. Auer, N. Cesa-Bianchi, and P. Fischer. 2004. Finite-time Analysis of the Multiarmed Bandit Problem. Machine Learning 47 (2004), 235–256.
[6] Chunyan Cai, Suyu Liu, and Ying Yuan. 2014. A Bayesian design for phase II clinical trials with delayed responses based on multiple imputation. Statistics in Medicine 33, 23 (2014), 4017–4028.
[7] Olivier Chapelle. 2014. Modeling Delayed Feedback in Display Advertising. In Proceedings of the 20th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD ’14). 1097–1105.
[8] Olivier Chapelle and Lihong Li. 2011. An Empirical Evaluation of Thompson Sampling. In Advances in Neural Information Processing Systems, Vol. 24.
[9] Miroslav Dudík, Daniel Hsu, Satyen Kale, Nikos Karampatziakis, John Langford, Lev Reyzin, and Tong Zhang. 2011. Efficient Optimal Learning for Contextual Bandits. In Proceedings of the Twenty-Seventh Conference on Uncertainty in Artificial Intelligence (UAI ’11). AUAI Press, Arlington, Virginia, USA, 169–178.
[10] Eyal Even-Dar, Shie Mannor, and Yishay Mansour. 2002. PAC Bounds for Multi-Armed Bandit and Markov Decision Processes. In Proceedings of the 15th Annual Conference on Computational Learning Theory (COLT ’02). Springer-Verlag, Berlin, Heidelberg, 255–270.
[11] Piero Ferolla, Maria Pia Brizzi, Tim Meyer, Wasiat Mansoor, Julien Mazières, Christine Do Cao, Hervé Léna, Alfredo Berruti, Vincenzo Damiano, Wienke Buikhuisen, Henning Grenbæk, Catherine Lombard-Bohas, Christian Grohé, Vincenzo Minotti, Marcello Tiseo, Javier De Castro, Nicholas Reed, Gabriella GiSLimberti, Neha Singh, Miona Stankovic, Kjell Öberg, and Eric Baudin. 2017. Efficacy and safety of long-acting pasireotide or everolimus alone or in combination in patients with advanced carcinoids of the lung and thymus (LUNA): an open-label, multicentre, randomised, phase 2 trial. The Lancet Oncology 18, 12 (2017), 1652–1664.
[12] Kris Johnson Ferreira, David Simchi-Levi, and He Wang. 2018. Online Network Revenue Management Using Thompson Sampling. Operations Research 66, 6 (2018), 1586–1602.
[13] Manegueu Anne Gael, Claire Vernade, Alexandra Cantepientier, and Michal Valko. 2020. Stochastic bandits with arm-dependent delays. In Proceedings of the 37th International Conference on Machine Learning. 3348–3356.
[14] John C. Gittins. 1979. Bandit processes and dynamic allocation indices. Journal of the Royal Statistical Society (Series B) 41 (1979), 148–164.
[15] Aditya Gopalan, Shie Mannor, and Yishay Mansour. 2014. Thompson Sampling for Complex Online Problems. In Proceedings of the 31st International Conference on Machine Learning. 100–108.

[16] Yonatan Gur and Ahmadreza Momeni. 2022. Adaptive sequential experiments with unknown information flows. Manufacturing & Service Operations Management forthcoming (2022).

[17] Poornia Joulani, Andras Gyorgy, and Csaba Szepesvari. 2013. Online Learning under Delayed Feedback. In Proceedings of the 30th International Conference on Machine Learning. 1453–1461.

[18] Emilie Kaufmann, Nathaniel Korda, and Rémi Munos. 2012. Thompson Sampling: An Asymptotically Optimal Finite-Time Analysis. In Proceedings of the 23rd International Conference on Algorithmic Learning Theory (ALT’12). Springer-Verlag, Berlin, Heidelberg, 199–213.

[19] Jaya Kawale, Hing H Bui, Branislav Kveton, Long Tran-Thanh, and Sanjay Chawla. 2015. Efficient Thompson Sampling for Online Matrix-Factorization Recommendation. In Advances in Neural Information Processing Systems, Vol. 28.

[20] F. P. Kelly. 2011. Reversibility and Stochastic Networks. Cambridge University Press.

[21] Tze Leung Lai and Herbert E. Robbins. 1985. Asymptotically efficient adaptive allocation rules. Advances in Applied Mathematics 6 (1985), 4–22.

[22] Tal Lancewicki, Shahar Segal, Tomer Koren, and Yishay Mansour. 2021. Stochastic Multi-Armed Bandits with Unrestricted Delay Distributions. In Proceedings of the 38th International Conference on Machine Learning. 5969–5978.

[23] Travis Mandel, Yun-En Liu, Emma Brunskill, and Zoran Popovic. 2015. The Queue Method: Handling Delay, Heuristics, Prior Data, and Evaluation in Bandits. Proceedings of the AAAI Conference on Artificial Intelligence 29, 1 (Feb. 2015).

[24] Ciara Pike-Burke, Shipra Agrawal, Csaba Szepesvari, and Steffen Grunewalder. 2018. Bandits with Delayed, Aggregated Anonymous Feedback. In Proceedings of the 35th International Conference on Machine Learning. 4105–4113.

[25] William H. Press. 2009. Bandit solutions provide unified ethical models for randomized clinical trials and comparative effectiveness research. Proceedings of the National Academy of Sciences 106, 52 (2009), 22387–22392.

[26] Chao Qin and Daniel Russo. 2022. Adaptivity and Confounding in Multi-Armed Bandit Experiments. arXiv preprint arXiv:2202.09036 (2022).

[27] Daniel Russo and Benjamin Van Roy. 2016. An Information-Theoretic Analysis of Thompson Sampling. Journal of Machine Learning Research 17, 68 (2016), 1–30.

[28] Daniel Russo, Benjamin Van Roy, Abbas Kazerouni, and Ian Osband. 2018. A Tutorial on Thompson Sampling. Found. Trends Mach. Learn. 11 (2018), 1–96.

[29] W. R. Thompson. 1933. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. Biometrika 25 (1933), 285–294.

[30] Claire Vernade, Olivier Cappé, and Vianney Perchet. 2017. Stochastic Bandit Models for Delayed Conversions. In Proceedings of the Thirty-Third Conference on Uncertainty in Artificial Intelligence, UAI 2017, Sydney, Australia, August 11–15, 2017, Gal Elidan, Kristian Kersting, and Alexander T. Ihler (Eds.). AUAI Press.

[31] Claire Vernade, Alexandra Carpentier, Tor Lattimore, Giovanni Zappella, Beyza Ermis, and Michael Brückner. 2020. Linear bandits with Stochastic Delayed Feedback. In Proceedings of the 37th International Conference on Machine Learning. 9712–9721.

[32] Claire Vernade, Andras Gyorgy, and Timothy Mann. 2020. Non-Stationary Delayed Bandits with Intermediate Observations. In Proceedings of the 37th International Conference on Machine Learning. 9722–9732.

[33] Jiajing Xu and Guosheng Yin. 2014. Two-stage adaptive randomization for delayed response in clinical trials. Journal of the Royal Statistical Society. Series C (Applied Statistics) 63, 4 (2014), 559–578.

[34] Zhengyuan Zhou, Renyuan Xu, and Jose Blanchet. 2019. Learning in Generalized Linear Contextual Bandits with Stochastic Delays. In Advances in Neural Information Processing Systems, Vol. 32.
A PROOF OF LEMMA 3.2

Proof. We use the following lemma in Lancewicki et al. [22].

**Lemma A.1.** At time $t$, for any arm $i$ and quantile $q \in (0, 1]$, it holds that

$$P\{n_{t+d_i(q)} < \frac{q}{2} m_t(i)\} \leq \exp\left(-\frac{q}{8} m_t(i)\right)$$

We have

$$P(F_1) = P\left(\exists t \leq T, i : m_t(i) \geq \frac{24 \log(T)}{q_i}, n_{t+d_i(q_i)}(i) < \frac{q_i}{2} m_t(i)\right)$$

$$\leq \sum_i \sum_{t: m_t(i) \geq \frac{24 \log(T)}{q_i}} P\left(n_{t+d_i(q_i)}(i) < \frac{q_i}{2} m_t(i)\right)$$

$$\leq \sum_i \sum_{t: m_t(i) \geq \frac{24 \log(T)}{q_i}} \exp\left(-\frac{q_i}{8} m_t(i)\right)$$

$$\leq TK \frac{1}{T^3}$$

$$\leq \frac{1}{T}$$

For the second part we have from Lemma 6 in Agrawal and Goyal [1] that

$$\forall t, P\left(\theta_2(t) > \mu_2 + \frac{\Delta}{2}, n_2(t) \geq \frac{24 \log T}{\Delta^2}\right) \leq \frac{2}{T^2},$$

taking a union bound over $t$ gives us the result. 

B PROOF OF LEMMA 3.3

Proof.

$$\sum_{j=1}^{T} \mathbb{E}\left[Y_j \mid G\right]$$

$$\leq \sum_{j=1}^{T} \mathbb{E}\left[\max\{X(n_{t_j}(1), s(n_{t_j}(1)), \mu_2 + \frac{\Delta}{2}), ..., X(n_{t_j+1}(1), s(n_{t_j+1}(1)), \mu_2 + \frac{\Delta}{2})\} \mid G\right]$$

$$\leq \sum_{j=1}^{T} \mathbb{E}\left[\sum_{k=n_{t_j}}^{n_{t_j+1}} X(k, s(k), \mu_2 + \frac{\Delta}{2})\right]$$

$$\leq \sum_{j=1}^{T} \mathbb{E}\left[\min\{X(n_{t_j}(1), s(n_{t_j}(1)), \mu_2 + \frac{\Delta}{2}), T\}\right] + \sum_{k=1}^{T} \mathbb{E}\left[\min\{X(k, s(k), \mu_2 + \frac{\Delta}{2}), T\}\right]$$

The first inequality holds because the fact that posterior distribution of arm 1 changes according to how many rewards have arrived and the time until the sample exceeds a threshold is bounded by the maximum of using fixed posterior. The second inequality holds because the maximum is bounded by the sum. The third inequality holds because each index $k$ enters exactly once except the starting index $n_{t_j}$ which could have repetitions.

□
C PROOF OF THEOREM 3.6

Proof. As in Agrawal and Goyal [1], we define arm \( i \neq 1 \) to be saturated if the number of observed rewards is at least \( \frac{24 \log T}{\Delta_i^2} \) and let the set of saturated arms at time \( t \) be \( C(t) \). Define the following two events:

\[
F_1 = \left\{ \exists t, i : m_t(i) \geq \frac{24 \log(T)}{q_i}, n_{t+d_i(q_i)}(i) < \frac{q_i}{2} m_t(i) \right\}
\]

\[
F_2 = \left\{ \exists t \leq T, i \in C(t) : \theta_i(t) \notin [\mu_i - \frac{\Delta_i}{2}, \mu_i + \frac{\Delta_i}{2}] \right\}
\]

Then we have the following results adapting proofs from Lancewicki et al. [22] and Agrawal and Goyal [1] respectively as in the 2-arm case.

Lemma C.1. \( \Pr(F_1) \leq \frac{1}{4} \) and \( \Pr(F_2) \leq \frac{4(K-1)}{T} \).

Let \( G = \neg F_1 \cap \neg F_2 \). Similar to the two arm case we focus on the clean event \( G \). As in 2-arm case we only need to bound the regret condition on \( G \). We ignore condition on \( G \) for simplicity in the following proof. Define

\[
\tau_i = \inf \{ t : n_t(i) \geq 24 \Delta_i^{-2} \log(T) \}
\]

which means the first time arm \( i \) is in the saturated set. We bound the regret in terms of playing saturated arms and non-saturated arms. To bound the regret due to non-saturated arms, note that condition \( G \), we have

Lemma C.2. \( m_{\tau_i}(i) \leq \frac{48 \log T}{q_i \Delta_i^2} + d_i(q_i) \).

Proof. If \( m_{\tau_i-d_i(q_i)}(i) \leq \frac{24 \log T}{q_i} \) then the conclusion obviously holds. If not since we are in \( G \), we know

\[
n_{\tau_i-d_i(q_i)+d_i(q_i)}(i) \geq \frac{q_i}{2} m_{\tau_i-d_i(q_i)}(i)
\]

which implies

\[
m_{\tau_i-d_i(q_i)}(i) \leq \frac{48 \log T}{q_i \Delta_i^2}.
\]

Hence,

\[
m_{\tau_i}(i) \leq \frac{48 \log T}{q_i \Delta_i^2} + d_i(q_i)
\]

\( \square \)

Hence the regret due to unsaturated arms is bounded by

\[
\sum_{i \neq 1} \frac{48 \log T}{q_i \Delta_i^2} + d_i(q_i) \Delta_i.
\]

To bound the regret due to playing saturated arms, we follow Agrawal and Goyal [1] and incorporate delays into the arguments. Specifically let us use the notations they developed. Let \( I_j \) denote the interval between (excluding) \( t_j \) and \( t_{j+1} \). Define the following event

\[
M(t) = \{ \theta_i(t) > \max_{i \in C(t)} (\mu_i + \frac{\Delta_i}{2}) \}
\]
and assume $M(t)$ holds if $C(t)$ is empty. Note that under $G$ all saturated arms are concentrated so essentially $M(t)$ denotes a pull of unsaturated arm. Now let $y_j = \{t \in I_j : M(t) = 1\}$ and let $I_j(l)$ denotes the sub-interval of $I_j$ between $(l-1)$-th and $l$-th occurrences of event $M(t)$ in $I_j$. Finally let

$$V_{j}^{l,a} = \{t \in I_j(l) : \mu_a = \max_{i \in C(t)} \mu_i\}$$

which segments the interval $I_j(l)$ by which saturated arm to pull. Let $R^S(I_j)$ be the regret pulling saturated arms in interval $I_j$. We then have the following crucial lemma bounding regret due to playing saturated arms from Agrawal and Goyal [1].

**Lemma C.3.** \(\sum_{j=0}^{T-1} \mathbb{E} \left[ R^S(I_j) \right] \leq \sum_{j=0}^{T-1} \mathbb{E} \left[ \sum_{a=2}^{K} 3\Delta_a V_{j}^{l,a} \right] + 4(K - 1).\)

We then have

$$\sum_{j=0}^{T-1} \mathbb{E} \left[ \sum_{a=2}^{K} 3\Delta_a V_{j}^{l,a} \right] = \sum_{a=2}^{K} 3\Delta_a \sum_{j=0}^{T-1} \mathbb{E} \left[ V_{j}^{l,a} \right] \tag{12}$$

Now as in non-delay case, by definition, $V_{j}^{l,a}$ is the number of of steps in $I_j(l)$ for which $a$ is the best arm in saturated set and $M(t)$ does not hold. This is the steps until our Beta posterior of arm 1 has a sample exceeding $\mu_a + \frac{\Delta_a}{2}$ or an arm different than $a$ becomes the best or we reach end of round $T$. Hence this is stochastically dominated by steps until the Beta posterior sample exceeding $\mu_a + \frac{\Delta_a}{2}$. Unlike non-delayed case, the posterior distribution is changing, but it could still be bounded as the following because $V_{j}^{l,a}$ is bounded by a sum of all $X$ terms using all possible posterior distributions, i.e.

$$V_{j}^{l,a} \leq \sum_{k=n_{j+1}}^{n_{j+1}} X(k, s(k), \mu_a + \frac{\Delta_a}{2})$$

Hence, we have

$$(12) \leq \sum_{a=2}^{K} 3\Delta_a \mathbb{E} \left[ \sum_{j=1}^{T} y_j + \sum_{k=n_{j+1}(1)}^{n_{j+1}(1)} X(k, s(k), \mu_a + \frac{\Delta_a}{2}) \right]$$

$$\leq \sum_{a=2}^{K} 3\Delta_a \mathbb{E} \left[ \sum_{j=1}^{T} \sum_{k=n_{j+1}(1)}^{n_{j+1}(1)} X(k, s(k), \mu_a + \frac{\Delta_a}{2}) \right]$$

$$+ \sum_{a=2}^{K} 3\Delta_a \mathbb{E} \left[ \sum_{j=1}^{T} y_j \max_{j} \sum_{k=n_{j+1}(1)}^{n_{j+1}(1)} X(k, s(k), \mu_a + \frac{\Delta_a}{2}) \right]$$

Note that $\sum_{j=1}^{T} y_j$ is bounded by the total number of pulls of unsaturated arm, which is $\sum_{i=2}^{K} 48 \log \mathcal{E} + d_i(q_i)$. Also note that the maximum term is bounded by the sum over all $k$, hence the above is
further bounded by
\[
\leq \sum_{a=2}^{K} 3\Delta_a \mathbb{E} \left[ \sum_{j=1}^{T} \sum_{k=n_{j+1}(1)}^{n_{j+1}(1)} X(k, s(k), \mu_a + \frac{\Delta_a}{2}) \right] + \left( \sum_{i=2}^{K} \frac{48 \log T}{q_i \Delta_i^2} + d_i(q_i) \right) \sum_{a=2}^{K} 3\Delta_a \mathbb{E} \left[ \sum_{k=0}^{T} X(k, s(k), \mu_a + \frac{\Delta_a}{2}) \right]
\]

By the proof of lemma 3.3 the first term is bounded by
\[
\sum_{a=2}^{K} 3\Delta_a \left( \sum_{j=1}^{T} \mathbb{E} \left[ \min \{X(n_{j+1}(1), s(n_{j+1}(1)), \mu_a + \frac{\Delta_a}{2}), T\} \right] + \sum_{k=1}^{T} \mathbb{E} \left[ \min \{X(k, s(k), \mu_a + \frac{\Delta_a}{2}), T\} \right] \right)
\]
which is further bounded by the following using the result from 2-arm case
\[
\sum_{a=2}^{K} O \left( \frac{1}{\Delta_a} + \frac{1}{\Delta_a^3} \right) + \left( \frac{32 \log T}{q_i \Delta_i} + d_i(q_i) \Delta_a + \Delta_a \right) \frac{6}{\Delta_a}
\]
The second term according to corollary 3.5 is bounded by
\[
\left( \sum_{i=2}^{K} \frac{48 \log T}{q_i \Delta_i^2} + d_i(q_i) \right) \sum_{a=2}^{K} O \left( \frac{1}{\Delta_a} + \frac{1}{\Delta_a^3} \right)
\]
Combining all terms gives the resulting bound. □