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Critical and injective modules over skew polynomial rings

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Abstract

Let $R$ be a commutative local $k$-algebra of Krull dimension one, where $k$ is a field. Let $\alpha$ be a $k$-algebra automorphism of $R$, and define $S$ to be the skew polynomial algebra $R[\theta; \alpha]$. We offer, under some additional assumptions on $R$, a criterion for $S$ to have injective hulls of all simple $S$-modules locally Artinian - that is, for $S$ to satisfy property $(\diamond)$. It is easy and well known that if $\alpha$ is of finite order, then $S$ has this property, but in order to get the criterion when $\alpha$ has infinite order we found it necessary to classify all cyclic (Krull) critical $S$-modules in this case, a result which may be of independent interest. With the help of the above we show that $\hat{S} = k[[X]][\theta, \alpha]$ satisfies $(\diamond)$ for all $k$-algebra automorphisms $\alpha$ of $k[[X]]$.

Keywords: Injective module, Noetherian ring, simple module, skew polynomial ring.

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1 Introduction

1.1 Motivating context

A Noetherian ring $S$ is said to satisfy property $(\diamond)$ if the injective hull $E_S(V)$ of every simple right or left $S$-module $V$ is locally Artinian, meaning that every finitely generated submodule of $E_S(V)$ is Artinian. This paper forms part of a project whose ultimate aim is to determine when the skew polynomial ring $S := R[\theta; \alpha]$ satisfies $(\diamond)$ when $R$ is commutative Noetherian and $\alpha$ is an automorphism of $R$ (so that $\theta r = \alpha(r)\theta$ for $r \in R$). In our earlier paper [3] we showed that the key to $(\diamond)$ for such skew polynomial rings is the case where $S$ is primitive. Necessary and sufficient conditions for primitivity are known [12], and recalled in Theorem 2.2 below. In particular it is easy to see that $R[\theta; \alpha]$ is never primitive when $|\alpha| < \infty$. For the primitive case we proved in [3, Theorem 5.4]:

**Theorem 1.1.** Let $R$ be a commutative Noetherian ring, $\alpha$ an automorphism of $R$. Set $S := R[\theta; \alpha]$ as above, and assume that $S$ is primitive.

1. If $R$ is Artinian then $S$ satisfies $(\diamond)$.

2. Suppose that $R$ contains an uncountable field. If the Krull dimension of $R$ is at least 2, or if Spec($R$) is uncountable, then $S$ does not satisfy $(\diamond)$.

It’s clear that a crucial case resting in the gap between (1) and (2) of Theorem 1.1 occurs when $R$ is a local Noetherian domain of Krull dimension 1 and $\alpha$ has infinite order. This is the situation we address in the present paper.
1.2 Critical modules

Let $k$ be an arbitrary field. Let $R$ be a discrete valuation ring which is a $k$-algebra with maximal ideal $M = XR$, with residue field $R/M = k$. Let $\alpha$ be a $k$-algebra automorphism of $R$, so $\alpha(X) = qX$ for some unit $q$ of $R$. Set $S = R[\theta; \alpha]$, and assume that $|\alpha| = \infty$. Recall that $S$ has Krull dimension 2 [13, Proposition 6.5.4(i)]. Thus to determine the validity or otherwise of ($\diamond$) for $S$ it is first necessary to study the simple and the 1-critical $S$-modules. This is done in §§3 and §§5, using the fact that the localisation $T := S(X^{-1})$ of $S$ is a principal right and left ideal domain, coupled with work of Bavula and Van Oystaeyen [2]. The following result gives an abbreviated version of what we discover.

The definitions of the sets (A), (B) and (C), of irreducible elements of $S$, can be found in §4.2. Additionally, set (F) := \{ $f \in k[\theta] : f$ monic irreducible $\}$, which is a subset of (C). Recall that a nonzero module $M$ with Krull dimension $Kdim(M)$ is critical if $Kdim(N) < Kdim(M)$ for all proper factors $N$ of $M$; every nonzero module with Krull dimension contains a critical submodule, [13, Lemma 6.2.10]. We say that two critical $S$-modules $M$ and $N$ are hull similar if they contain a nonzero isomorphic submodule. The following result summarises parts of Theorems 5.5 and 5.10.

**Theorem 1.2.** The irreducible elements of $S$ can be divided into 3 disjoint sets (A), (B) and (C) with the following properties.

1. There are bijective correspondences between these sets, together with (F), and the equivalence classes of finitely generated critical torsion right $S$-modules as follows.
   - (F) $\longleftrightarrow$ \{ $V : V$ simple, $dim_k(V) < \infty$ $\}/\sim$.
   - (A) $\longleftrightarrow$ \{ $V : V$ unfaithful finitely generated 1-critical $\}/\sim$.
   - (B) $\longleftrightarrow$ \{ $V : V$ simple, $dim_k(V) = \infty$ $\}/\sim$.
   - (C) $\longleftrightarrow$ \{ $V : V$ faithful finitely generated 1-critical $\}/\sim$.

2. The equivalence relation $\sim$ is isomorphism for (F), (A) and (B) and hull similarity for (C).

3. In cases (A), (B) and (C) the map from left to right takes the irreducible element $c$ to the module $S/cS$. In case (F) the element $f$ of (F) is sent to the module $S/(XS + fS)$.

1.3 Monoid commutativity

Armed with the classification of Theorem 1.2 we use elementary homological algebra and analysis of non-split extensions of critical $S$- and $T$-modules to deduce a criterion for $S$ to satisfy ($\diamond$) in terms of what we call the monoid commutativity of the sets (B) and (C) of irreducible elements of $S$. The following result is part of Theorem 6.7.

**Theorem 1.3.** Let $S$ be as in 1.2. Then the following are equivalent.

1. $S$ satisfies ($\diamond$) for right modules.

2. Given irreducible elements $b$ and $c$ of $S$, respectively of types (B) and (C), there exist irreducible elements $b'$ and $c'$ of $S$, respectively of types (B) and (C), such that $cb = b'c'$.

Naturally there is a parallel result for left modules.
1.4 (⋄) for \( \hat{S} \)

In \[6.3\] we specialise from the above setup by taking \( R \) to be \( k[[X]] \), continuing to assume that \( \alpha \) is an arbitrary \( k \)-algebra automorphism of infinite order. Write \( \hat{S} := k[[X]][\theta; \alpha] \). We prove a type of converse Eisenstein criterion for \( \hat{S} \) (Lemma \[6.8\]), which provides a sufficient condition for an element of \( \hat{S} \) to be reducible and which may be of independent interest. This permits us to confirm that \( \hat{S} \) satisfies the condition stated in Theorem \[1.3(2)\], thus yielding one of the main results of the paper:

**Theorem 1.4.** \( \hat{S} \) satisfies (⋄).

2 Background results and notation

2.1 Primitivity of skew polynomial algebras

We begin by recalling the following definition from \[12\]:

**Definition 2.1.** Given a ring \( R \) and \( \alpha \in \text{Aut}(R) \), \( R \) is \( \alpha \)-special if there is an element \( a \) of \( R \) such that the following conditions are satisfied.

(a) For all \( n \geq 1 \), \( N^\alpha_n(a) := \alpha a \) \( \cdots \alpha \) \( \alpha^{n-1}(a) \neq 0 \).

(b) For every non-zero \( \alpha \)-stable ideal \( I \) of \( R \), there exists \( n \geq 1 \) such that \( N^\alpha_n(a) \in I \).

When this occurs, the element \( a \) is called an \( \alpha \)-special element.

Here is the resulting characterisation of primitivity for skew polynomial rings:

**Theorem 2.2.** \[12, \text{Theorem 3.10}\] Let \( R \) be a commutative Noetherian ring and let \( \alpha \in \text{Aut}(R) \). Then \( R[\theta; \alpha] \) is primitive if and only if \( R \) is \( \alpha \)-special and \( \alpha \) has infinite order.

From the definition it follows easily that an \( \alpha \)-special ring is \( \alpha \)-prime. Clearly, an \( \alpha \)-simple ring is \( \alpha \)-special, with 1 as \( \alpha \)-special element in this case. Consider, however, the algebra \( R \) defined in \[1.2\]. Clearly \( < X > \) is a proper \( \alpha \)-ideal of \( R \), so that \( R \) is not \( \alpha \)-simple; but \( X \) is an \( \alpha \)-special element of \( R \), so \( R \) is \( \alpha \)-special. Since \( \alpha \) is by hypothesis of infinite order, it therefore follows from Theorem 2.2 that

\[ S = R[\theta; \alpha] \text{ is primitive.} \]

In fact one can easily confirm by explicit construction that \( S \) is primitive, and - even better - the simple \( S \)-modules can be completely described using work of Bavula and Van Oystaeyen \[2\], as we shall explain in \[\S\S 5.3 \text{ and } 5.4\].

2.2 Property (⋄)

Let \( R \) be a commutative Noetherian ring and \( \alpha \) an automorphism of \( R \). In \[3\] the following question was addressed:

\[ \text{For which } R \text{ and } \alpha \text{ does } R[\theta; \alpha] \text{ satisfy (⋄)?} \]  \hspace{1cm} (2)

It is not difficult to show that the key to this question is to answer it when \( R[\theta; \alpha] \) is primitive and \( R \) is a domain; see \[3\] Corollary 3.5 and \[\S 4.2\]. Bringing Theorem 2.2 to bear on this situation, we found the following:

**Proposition 2.3.** \[3\] Proposition 5.3 Let \( R \) be a commutative Noetherian domain and \( \alpha \) an automorphism of \( R \). Suppose that \( R[\theta; \alpha] \) is primitive, with \( \alpha \)-special element \( a \). Let \( A \) denote the multiplicative subsemigroup of \( R \{0\} \) generated by \( \{\alpha^i(a) : i \in \mathbb{Z}\} \), so that \( A \) is an \( \alpha \)-invariant multiplicatively closed subset of \( R \{0\} \) and hence satisfies the Ore condition in \( R[\theta; \alpha] \). Suppose that \( R \infty A^{-1} \) is not a field. Then \( R[\theta; \alpha] \) does not satisfy (⋄).
Notice that Proposition 2.3 fails to determine whether the primitive domains $S = R[\theta; \alpha]$ of §1.2 satisfy $(\diamondsuit)$. More precisely, we were able to obtain the following necessary and sufficient conditions in [3]; the result was stated as Theorem 1.1, but is repeated here for the reader’s convenience.

**Theorem 2.4.** [3, Theorem 5.4] Let $R$ be a commutative Noetherian ring, $\alpha$ an automorphism of $R$. Suppose that $R[\theta; \alpha]$ is primitive.

(a) If $R$ has Krull dimension 0 then $R[\theta; \alpha]$ satisfies $(\diamondsuit)$.

(b) Suppose that $R$ contains an uncountable field. Suppose also that either $R$ has Krull dimension at least 2, or $\text{Spec}(R)$ is uncountable. Then $R[\theta; \alpha]$ does not satisfy $(\diamondsuit)$.

Thus the rings $S$ of §1.2 lie in the gap between the two listed families (a) and (b), and so are fundamental to understanding which skew polynomial extensions of commutative Noetherian rings $R$ satisfy $(\diamondsuit)$.

### 2.3 Notation

For the reader’s convenience we repeat here some notation listed earlier. Let $k$ be an arbitrary field. Henceforth $R$ stands for a discrete valuation ring which is a $k$-algebra with maximal ideal $M = XR$, with residue field $R/M = k$. Let $\alpha$ be a $k$-algebra automorphism of $R$, so $\alpha(X) = qX$ for some unit $q$ of $R$. Set $S = R[\theta; \alpha]$, and assume that $|\alpha| = \infty$; that is, there does not exist $n \in \mathbb{N}$ such that $N_n(q) := q\alpha(q) \cdots \alpha^{n-1}(q) = 1$.

We write

$$\mathcal{X} := \{X^n : n \geq 0\},$$

so $\mathcal{X}$ is an Ore subset of $S$, and we denote by $T$ the localisation of $S$ at $\mathcal{X}$; that is,

$$T := S\mathcal{X}^{-1} = S(X^{-1}) = Q[\theta; \alpha],$$

where $Q = Q(R) = R(X^{-1})$ is the quotient field of $R$. Taking into account that $X$ is a normal element of $S$, elements of $T$ will either be written as $X^{-n}s$ or $s_0X^{-n}$ for some $n \in \mathbb{N}$ and $s, s_0 \in S$. Throughout the paper, “module” will mean “right module” unless otherwise indicated.

### 3 Injective hulls of uniform $S$-modules

The class of simple $S$–modules, and - more generally - the class of uniform $S$–modules, splits into two families, as outlined in the following two subsections.

#### 3.1 $\mathcal{X}$–tension $S$–modules

Recall that $\mathcal{X}$ denotes the Ore set $\{X^n : n \geq 0\}$ in $S$.

**Lemma 3.1.** Let $E$ be an indecomposable injective $S$–module, and suppose that the $\mathcal{X}$–tension submodule of $E$ is non-zero. Then $E$ is $\mathcal{X}$–tension and one of the following cases pertains:

1. $E = ES(S/\mathcal{X})$, and this module is an infinite tower of copies of $k(\theta)$.
2. $E = ES((\mathcal{X} + p(\theta)S))$ for some irreducible polynomial $p(\theta) \in k(\theta)$, and $E$ is locally finite dimensional.
Proof. By hypothesis there exists a non-zero element $v$ of $E$ with $vX = 0$. Thus $vS$ is a factor of $S/XS \cong k[\theta]$. Suppose first that $vS \cong k[\theta]$. Then

$$E_{S/XS}(vS) \cong E_{k[\theta]}(k[\theta]) \cong k(\theta).$$

Now $X$ is a normal element of $S$ and therefore the ideal $XS$ has the Artin-Rees property [13 Proposition 4.2.6]. This implies that

$$E_S(vS) = \bigcup_{n \geq 1} \text{Ann}_{E_S(vS)}(X^n S).$$

One can now show by induction that $\text{Ann}_{E_S(vS)}(X^n S)$ is a tower of $n$ copies of $k(\theta)$, the case $n = 1$ being given by (1). The induction step follows using the map from $\text{Ann}_{E_S(vS)}(X^{n+1} S)$ to $\text{Ann}_{E_S(vS)}(X^n S)$ given by multiplication by $X$, which preserves the submodule structure and whose kernel is $\text{Ann}_{E_S(vS)}(XS) = k(\theta)$.

Suppose on the other hand $vS$ is a proper factor of $k[\theta]$. Then $vS$ is finite dimensional and hence contains a simple submodule of the form $S/(XS + p(\theta)S)$ for some irreducible polynomial $p(\theta)$. Thus $E = E_S(S/(XS + p(\theta)S))$, as stated in (2). Now $E_{S/XS}(S/(XS + p(\theta)S))$ is Artinian by Matlis’s theorem [14], and the final part of (2) follows from this and the fact that $XS$ has the Artin-Rees property [13 Proposition 4.2.6], arguing in a similar way to the previous paragraph.

### 3.2 $X$–torsion free uniform $S$–modules

The following lemma shows that Lemma 3.1(2) encompasses all the unfaithful simple $S$–modules.

**Lemma 3.2.** Let $V$ be a simple $S$–module. Then:

1. $V$ is faithful if and only if $V$ is $X$–torsion free.
2. If $V$ is $X$–torsion free, then $V$ admits a structure as a (necessarily simple) $T$–module.

**Proof.**

(1) Suppose $V$ is faithful. Since $V$ is simple and $X$ is normal, if, for any $n \geq 1$, $X^n$ kills a nonzero element of $V$ then $VX^n = 0$, a contradiction. So $V$ is $X$–torsion free.

For the reverse implication, suppose that $V$ is a $X$–torsion free simple $S$–module, and suppose that $P := \text{Ann}_S(V) \neq \{0\}$. Then $P \cap R = \{0\}$, so $PT$ is a proper prime ideal of $T$. But $T(\theta^{-1})$ is a simple ring by [13] Theorem 1.8.5, and so $\theta \not\in P$. Since $P$ is a primitive ideal of $S$, this forces $P$ to be $\theta S + XS$, contradicting the fact that $V$ is $X$–torsion free. Therefore $V$ is faithful.

(2) Since $X$ is normal and $VX \neq 0$, the $S$–submodule $VX$ must equal $V$, so $V$ admits a structure as a $T$–module since $T = S(X^{-1})$.

**Proposition 3.3.**

1. Let $U$ be an $X$–torsion free $S$–module. Then

$$E_S(U) = E_T(U \otimes_S T).$$

2. Let $V$ be a faithful simple $S$–module. Then $E_S(V) = E_T(V)$.

**Proof.**

(1) If $0 \neq u \otimes_S t = u \otimes_S sX^{-n} \in U \otimes_S T$, with $u \in U$, $s \in S$, $t \in T$ and $n \in \mathbb{N}$, then $0 \neq u \otimes_t X^n \in U$, so that $U \otimes_S T$ is an essential extension of $U$ as $S$–modules. The result will therefore follow if we show that

$$E_S(U \otimes_S T) = E_T(U \otimes_S T).$$

(5)
A similar argument to that just given shows that \( E_T(U \otimes_S T) \) is an essential extension of \( U \otimes_S T \) as \( S \)-modules, so that it remains only to show that \( E_T(U \otimes_S T) \) is an injective \( S \)-module. For this, let \( 0 \neq I \triangleleft S \) and \( f \in \text{Hom}_S(I, E_T(U \otimes_S T)) \). Define \( \overline{f} : IT \rightarrow E_T(U \otimes_S T) \) such that \( \overline{f}(it) := f(i)t \) for \( i \in I \) and \( t \in T \). It is easy to check that \( \overline{f} \) is well-defined, with \( \overline{f}|_I = f \). So \( \overline{f} \) extends to a \( T \)-homomorphism from \( T \) to \( E_T(U \otimes_S T) \), and the restriction of this map to \( S \) gives the required extension of \( f \) to \( S \). This proves (1).

(2) Let \( V \) be a faithful simple \( S \)-module. By Lemma 3.2 the \( S \)-module structure of \( V \) extends to a \( T \)-module structure, with \( V = VX \). Hence, \( V = V \otimes_S T \), and so (2) is a special case of (1).

Let \( V \) be a faithful simple \( S \)-module. Since \( T \) is a prime Noetherian ring of Krull dimension 1, \( E_T(V) \) satisfies (\( \diamond \)) as a \( T \)-module (see for instance [15, Proposition 5.5]). Regarding (\( \diamond \)) for \( S \), in the light of Proposition 3.3(2) and Lemma 3.1(2), the remaining issue therefore is:

**Question 3.4.** Given a faithful simple \( S \)-module \( V \), a simple \( T \)-module \( W \) which occurs as a subfactor of \( E_T(V) \), and a finitely generated \( S \)-submodule \( W_0 \) of \( W \), does \( W_0 \) have a finite composition series?

Theorem 1.4 gives a positive answer to this question in case \( R = k[[X]] \).

## 4 Irreducible elements

### 4.1 Elementary lemmas

We have seen in [5] that the injective hulls of simple \( S \)- and simple \( T \)-modules are closely connected. Recall that \( T \) is a principal right ideal ring by [13, Theorem 1.2.9(ii)]. Hence, to better understand the representation theory of \( S \) and \( T \) we need to study the irreducible elements of these algebras, where, by definition, an irreducible element \( s \) of a ring \( S \) is a non-zero non-unit of \( S \) such that, whenever \( s = ab \) with \( a,b \in S \), then either \( a \) or \( b \) is a unit of \( S \).

**Lemma 4.1.** Let \( S \) be a noetherian domain. Then every non-zero non-unit of \( S \) can be written as a finite product of irreducible elements.

**Proof.** [5, Proposition 0.9.3].

**Lemma 4.2.**

(1) Let \( z \in S \setminus XS \). Then \( z \) is irreducible in \( S \) if and only if \( z \) is irreducible in \( T \).

(2) Let \( W \) be a simple \( T \)-module. Then there is an element \( z \) of \( S \) such that \( z \) is irreducible in both \( S \) and \( T \), with \( W \cong T/zT \).

**Proof.** (1) \( \Rightarrow \): Let \( z \in S \setminus XS \), and suppose that \( z \) is irreducible in \( S \). Let

\[
z = (aX^{-\ell})(bX^{-m})
\]

be a factorisation of \( z \) in \( T \), with \( a,b \in S \setminus XS \) and \( \ell, m \in \mathbb{Z}_{\geq 0} \). Writing \( b = \sum_{i=0}^{\ell} b_i \theta^i \) with \( b_i \in R \), this yields

\[
z = a(\sum_i b_i N_i^a(q^i \theta^i))X^{-(\ell+m)},
\]

with \( a \) and \( \sum_i b_i N_i^a(q^i \theta^i) \) in \( S \setminus XS \). Thus

\[
zX^{(\ell+m)} = a(\sum_i b_i N_i^a(q^i \theta^i)),
\]

(6)
where the factors appearing in the above equation are all in $S$, and all except $X^{(\ell+m)}$ are not in $XS$. Since $XS$ is a completely prime ideal of $S$, this forces $\ell + m = 0$ and hence $\ell = m = 0$.

Thus (6) is a factorisation of $z$ in $S$, so one of $a, b$ is a unit in $S$, as required.

$\iff$ Let $z \in S \setminus XS$, and suppose that $z$ is irreducible in $T$. Suppose that $z = ab$ where $a$ and $b$ are non-zero non-units of $S$. Thus $a$ or $b$ must be a unit of $T$, say $a$ is such. So

$$a \in Q \cap S = R,$$

where $Q = R(X^{-1})$ is the quotient field of $R$. But $a$ is not a unit of $S$, thus $a \in XR$. This forces $z \in XS$, a contradiction. Therefore no such factorisation of $z$ exists, and $z$ is irreducible in $S$. If $b$ is a unit in $T$ the argument is similar. $\square$

4.2 Taxonomy

Consider the following taxonomy of the irreducible elements of $S$. Since $S/XS \cong k[\theta]$, an irreducible element $z$ of $S$ can be uniquely written in the form

$$z = f + Xs, \quad \text{where } f \in k[\theta], s \in S.$$

After normalising by multiplying $z$ by a suitable unit in $S$ - that is, by a suitable element from $R \setminus XR$ - there are the following three mutually exclusive possibilities for $z$:

(A) $z = X$ or $z = \theta$, \quad $f = 0, s = 1$ or $f = \theta, s = 0$;
(B) $z = 1 + Xs$, \quad $s \in S \setminus R$;
(C) $z = f + Xs$, \quad $f \in k[\theta] \setminus k$, $f$ monic, $s \in S, z \neq \theta$.

Note that we forbid $s \in R$ in type (B) in order to exclude units from the list. We’ll make use of these labels below, where we aim to develop our understanding of the simple and the 1-critical $S$-modules. We first need an easy lemma:

Lemma 4.3. Let $z \in S \setminus XS$. Then

(1) $S \cap zT = zS$.
(2) $S/zS$ is a $X$-torsion free $S$-module.

Proof. (1) Let $t \in T$ with $zt = s_0 \in zT \cap S$, with $t = sX^{-n}$ for some $s \in S$ and $n \in \mathbb{Z}_{\geq 0}$. Thus

$$zs = s_0X^n. \quad (7)$$

Suppose that $n \geq 1$. Since $XS$ is a completely prime ideal of the domain $S$ and $z \notin XS$, (7) shows that $s \in X^nS = SX^n$. Write $s = \hat{s}X^n$ for some $\hat{s} \in S$. Then $t = sX^{-n} = \hat{s}X^nX^{-n} = \hat{s} \in S$, as required.

(2) Given $s \in S$, if $sX^n \in zS$ for some $n \in \mathbb{N}$, then $s \in S \cap zT = zS$ by (1). $\square$

Remark 4.4. Lemma 4.3 applies to a product $z$ of type (B) or (C) irreducibles in $S$. 

7
5 Simple and critical $S$–modules and $T$–modules

5.1 Recap on critical modules

Denote the (Gabriel-Rentschler) Krull dimension of a module $M$ by $\text{Kdim}(M)$, and recall that our main rings of interest, $S$ and $T$, have

$$\text{Kdim}(S) = 2 \quad \text{and} \quad \text{Kdim}(T) = 1,$$

by [13, Theorem 6.5.4(i)]. Given a ring $R$ with Krull dimension and a non-negative integer $n$, an $R$-module $M$ is $n$-critical if $\text{Kdim}(M) = n$ and $\text{Kdim}(N) < n$ for every proper factor $N$ of $M$. For background on critical modules, see for example [13, §6.2].

**Definition 5.1.** Let $R$ be a right noetherian ring, and $M$ a finitely generated (right) $R$-module. A critical composition series of $M$ is a finite chain

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of submodules of $M$ such that

- $(•)$ $M_i/M_{i−1}$ is critical for $i = 1, \ldots, n$;
- $(•)$ $\text{Kdim}(M_i/M_{i−1}) \leq \text{Kdim}(M_{i+1}/M_i)$ for all $i = 1, \ldots, n − 1$.

Finitely generated critical $R$-modules $V$ and $W$ are called hull-similar if their injective hulls are isomorphic. Critical modules are uniform, and so critical modules $V$ and $W$ are hull-similar if and only if they share an isomorphic non-zero submodule. It is clear that hull-similarity is an equivalence relation on the class of finitely generated critical $R$-modules. The generalisation of the Jordan-Holder theorem to this setting is as follows:

**Theorem 5.2.** ([13, Proposition 6.2.20]) Let $R$ be a right noetherian ring, and $M$ a finitely generated (right) $R$-module. Then:

1. $M$ has a critical composition series.
2. Any two critical composition series of $M$ have the same length, and after a suitable permutation the composition factors are pairwise hull-similar.

Since $\text{Kdim}(S) = 2$, every finitely generated $S$–module $M$ has $\text{Kdim}(M) \leq 2$ by [13, Lemma 6.2.5]. It’s easy to see that the finitely generated 2-critical $S$–modules are its non-zero right ideals [13, Proposition 6.3.10], and they form a single hull-similarity class since $S$ is a domain satisfying the Ore condition. The classification of 0-critical (that is, simple) and 1-critical $S$–modules runs parallel to the classification of simple $T$–modules, and can therefore exploit the fact that the localisation $T$ of $S$ is a principal left and right ideal domain, [13, Theorem 1.2.9(ii)]. We therefore recall in §5.2 some classical results about the latter.

5.2 Principal left and right ideal domains

**Definition 5.3.** (Ore [16]) Let $D$ be a left and right PID and let $a, b \in D$.

(a) $a$ is (right) similar to $b$ if there exists $u \in D$ such that $1$ is the highest common left factor of $u$ and $b$, (that is, $uD + bD = D$), and $ua$ is the least common right multiple of $u$ and $b$, (that is, $uaD = uD \cap bD$).

(b) Left similarity is defined analogously.

The following results can now be proved by straightforward calculation:

**Theorem 5.4.** Let $D$ be a left and right PID and let $a, b \in D$.

1. (Ore, [16, Theorem 18], [10, pages 33-34]) $a$ and $b$ are right similar if and only if they are left similar (so we may drop the adjectives left and right).
The following are equivalent:

(i) \( a \) and \( b \) are similar.
(ii) \( D/aD \cong D/hD \).
(iii) \( D/Da \cong D/Db \).

In particular, of course, elements \( a \) and \( b \) of \( D \) are similar if \( a = ub \) or \( a = bu \) for a unit \( u \) of \( D \), and this is equivalent to similarity if \( D \) is commutative, but not in general.

5.3 Towards a classification of simple \( S \)-modules

Recall the definition of type (B) irreducible elements from §4.2.

Theorem 5.5. Let \( S \) and \( T \) be as defined in §2.3.

(1) Let \( V \) be a faithful simple \( S \)-module. Then there exists an element \( z \) of \( S \) satisfying the following properties (a) and (b).

(a) \( z \) is irreducible in both \( S \) and \( T \);
(b) As \( S \)-modules, \( V \cong S/zS \cong T/zT \).

Moreover, any element \( z \) of \( S \) satisfying (b) also satisfies

(c) \( z \) is a type (B) irreducible.

(2) Conversely, if \( z \in S \) satisfies (1)(a) and (1)(c), then \( S/zS \cong T/zT \) is a faithful simple \( S \)-module.

(3) Let \( z \in S \) satisfy (1)(a) and (1)(c), and let \( y \in S \setminus XS \). Then the following are equivalent:

(a) \( S/zS \cong S/yS \);
(b) \( T/zT \cong T/yT \) as \( T \)-modules;
(c) \( z \) and \( y \) are similar in \( T \).

Moreover, when these equivalent conditions hold \( y \) is a type (B) irreducible element of \( S \).

(4) The simple \( S \)-modules which are not faithful are the \( X \)-torsion simple modules

\[
V_f := \frac{S}{XS + zS}, \quad f \in k[\theta], \ f \ a \ monic \ irreducible \ polynomial,
\]

with \( V_f \cong V_g \) if and only if \( f = g \).

Proof. (1): Let \( V \) be a faithful simple \( S \)-module. Then \( V \) is a simple \( T \)-module by Lemma 3.2. By Lemma 4.2 there is an element \( z \in S \), not equal to \( X \) or \( \theta \) and irreducible in both \( S \) and \( T \), such that \( V \cong T/zT \). Since \( S + zT/zT \) is a non-zero \( S \)-submodule of \( V \), it must equal \( T/zT \), and so it follows using Lemma 4.3 that, as \( S \)-modules,

\[
T/zT = S + zT/zT \cong S/S \cap zT = S/zS.
\]

This proves (1)(a) and (1)(b).

Suppose now that \( z \) is an element of \( S \) satisfying (1)(b). Then \( T/zT \) is a simple \( T \)-module, so \( z \) must be irreducible in \( T \). Similarly, since \( V \) is by hypothesis a simple \( S \)-module, \( z \) is irreducible in \( S \) - equivalently, \( X \) is not a factor of \( z \) in \( S \). To prove (1)(c), we now apply [2, Theorem 8.1] (or [1, Theorem 7.4]). This result, specialised to the present setting, states that for an element \( z \) of \( S \) which is irreducible in \( T \), \( T/zT \) is a \( X \)-torsion free simple \( S \)-module if and only if a condition labelled (CO) in [2] is satisfied, namely

\[
S = XS + zS.
\]
Clearly, (8) fails when \( z \) is type (A) or (C), as in these cases \( z \) is not congruent to a unit \( \text{mod} \, XS \). This proves the remainder of (1)(c).

(2) Suppose that \( z \in S \) satisfies (1)(a) and (1)(c). Then (8) holds, so \( T/zT \) is a \( X \)-torsion free simple \( S \)-module by [2, Theorem 8.1]. This proves (1)(b).

(3) Let \( z \) and \( y \) be as stated. By part (2), \( S/zS \cong T/zT \) is a faithful simple \( S \)-module.

(4) The first part of (4) follows from Lemma 3.1. If \( f \neq g \) are monic irreducible polynomials in \( k[\theta] \) then \( V_f \) and \( V_g \) have different annihilator ideals, so are not isomorphic.

Remark 5.6. There is an obvious missing component in Theorem 5.5 as a classification of the simple \( S \)-modules - namely, it gives no answer to:

**Question 5.7.** Given type (B) irreducible elements \( a \) and \( b \) of \( S \), is there a method to determine whether \( a \) and \( b \) are similar in \( T \)?

Beyond the rather cumbersome definition of similarity we note only the observation of Jacobson [10, page 36] that, for skew polynomial algebras of automorphism type whose coefficient ring is a division ring, such as \( T = R[\langle X^{-1} \rangle; \alpha] \), similar polynomials have the same degree in \( \theta \), since this degree determines the dimension as an \( R[\langle X^{-1} \rangle] \)-vector space of the associated cyclic factor module.

The following easy lemma will be needed in the discussion of 1-critical modules in §6.

**Lemma 5.8.** Let \( V \) be a faithful simple \( S \)-module and \( U \) an unfaithful simple \( S \)-module. Then \( \text{Ext}_S^1(U, V) = 0 \).

**Proof.** Let \( V \) and \( U \) be as stated, and suppose that \( Y \) is an indecomposable extension of \( V \) by \( U \). By Theorem 5.5(4) there exists a monic irreducible polynomial \( f \in k[\theta] \) such that \( U = V_f = S/(XS + fS) \). In particular, there exists \( y \in V \setminus U \) such that \( yX \in V \). Moreover, \( yX \neq 0 \), since \( V \) is \( X \)-torsion free by Lemma 3.2 and \( V \) is by hypothesis an essential submodule of \( Y \), so \( Y \) is also \( X \)-torsion free.

But the \( S \)-action on \( V \) extends to a structure as \( T \)-module, by Lemma 3.2. Therefore

\[
y = (yX)X^{-1} \in V,
\]

a contradiction. So no such module \( Y \) exists.

### 5.4 Hull similarity classes of 1-critical \( S \)-modules

In this section we state and prove an analogue of Theorem 5.5 for 1-critical \( S \)-modules. As preparation for this we need to study certain 1-critical cyclic modules, as follows:
Lemma 5.9. Let $S$ and $T$ be as defined in §2.3 and let $z$ be a type (C) irreducible element of $S$.

(1) Every simple subfactor $V$ of $S/zS$ is finite dimensional;

(2) $S/zS$ is a faithful cyclic 1-critical $S$-module.

Proof. (1) Write $W$ for $S/zS$. Since $z$ is irreducible in $T$ by Lemma 4.2(1), $T/zT = W \otimes_S T$ is a simple $T$-module. Since $z$ has type (C), Lemma 4.3 shows that $W$ is $X$-torsion free.

Let $B$ be a non-zero submodule of $W$, and define $M := W/B$. Since $T$ is a flat left $S$-module, the short exact sequence $0 \rightarrow B \rightarrow W \rightarrow M \rightarrow 0$ yields the exact sequence of $T$-modules

$$0 \rightarrow B \otimes_S T \rightarrow W \otimes_S T \rightarrow M \otimes_S T \rightarrow 0.$$  

By (9), $B \otimes_S T \neq 0$, so that $B \otimes_S T = W \otimes_S T$ since $W \otimes_S T$ is simple. Therefore, by exactness of (10), $M \otimes_S T = 0$; in other words, $M$ is a finitely generated $X$-torsion $S$-module. Since $M$ is finitely generated and $X$ is a normal element of $S$, there is a positive integer $\ell$ such that $MX^\ell = 0$.

(11) In particular, if $M$ is non-zero we can choose $0 \neq m \in M$ with $mX^\ell = 0$, say $m = w + B$ for some $w \in W$. Let $I := \text{Ann}_S(w)$, so $I \neq 0$ since $K\dim(W) \leq 1$ by [13, Lemma 6.3.9]. We claim that $I \not\subseteq SX$.

For, choose $0 \neq \beta \in I$ and write $\beta = rX^r$ for some $r \geq 0$, with $r \not\in SX$. (We can find $r$ since $\bigcap_n(SX)^n = 0$.) If $w \tau \neq 0$ then the equation $w\tau X^r = 0$ shows that there exists a non-zero $X$-torsion element of $W$, contradicting (9). This proves (12). Thus

$$m(I + XS) = 0,$$

and hence, by (12), $mS$ is a non-zero finite dimensional submodule of $M$. Repeating this argument in the factor of $W$ by $B_1 := B + wS$, and so on, the fact that $W$ is noetherian forces

$$\dim_k(W/B) < \infty.$$  

We claim next that

$$\text{socle}(W) = 0.$$  

To see this, suppose that $U$ is a simple submodule of $W$. By (9) and Theorem 5.5(4), $\dim_k(U) = \infty$. Notice that, by Theorem 5.5(1),(2) and (3) and the fact that $z$ has type (C), $W$ itself cannot be simple, so that $U \not\subseteq W$. We can therefore apply (13) with $B = U$ to conclude that $W/U$ contains a submodule $A/U$ with $\dim_k(A/U) < \infty$. However Lemma 5.8 now implies that $A$ splits,

$$A \cong U \oplus C,$$

where $C$ is a finite-dimensional simple submodule. Since $CX = 0$, this contradicts (9). This proves (14).

Now let $B \subseteq A$ be submodules of $W$ with $A/B$ a simple module. Then $B$ must be non-zero by (14). Therefore (13) implies that $\dim_k(A/B) < \infty$, proving part (1).

(2) It is immediate from (14) and (13) that $S/zS$ is a cyclic 1-critical $S$-module. If $S/zS$ is not faithful, then it contains a nonzero submodule killed by either $XS$ or by $\theta S$, since these are the only height one primes of $S$. From (9) it is clear that the first of these
possibilities is not possible. Suppose that $S/zS$ has a nonzero submodule killed by $\theta S$. Since $S/zS$ is critical, it is uniform. Therefore, since $\theta S$ has the Artin-Rees property by [13, Proposition 4.2.6], there exists $n \geq 1$ such that

$$(S/zS)\theta^n = 0. \quad (15)$$

Let $z = f + Xs$, where $s \in S$ and $0 \neq f \in k[\theta]$ is monic, with $f \neq \theta$. Then (15) implies that

$$\theta^n = (f + Xs)\gamma$$

for some $\gamma \in S$, and a quick calculation shows that this is impossible. Hence $S/zS$ is faithful, as required.

Here is the promised 1-critical analogue of Theorem 5.5. We repeat Lemma 5.9(2) as part (2) of the theorem, to emphasise it in conjunction with part (1).

**Theorem 5.10.** Let $S$ and $T$ be as defined in §1.

(1) Let $M$ be a finitely generated 1-critical $S$-module.

(a) If $M$ is unfaithful, then $M \cong S/XS$ or $M \cong S/\theta S$.

(b) If $M$ is faithful, then $M$ is hull similar to $S/zS$ for a type (C) irreducible element $z$ of $S$. More precisely, $S/zS$ is isomorphic to a submodule of $M$.

(2) Let $z$ be a type (C) irreducible element of $S$. Then $S/zS$ is a faithful cyclic 1-critical $S$-module.

(3) Let $z$ and $w$ be irreducible elements of $S$ with $z$ of type (C). Then the following are equivalent:

(a) $S/zS$ and $S/wS$ are hull-similar.

(b) $T/zT$ and $T/wT$ are isomorphic simple $T$-modules.

(c) $z$ and $w$ are similar in $T$.

When these equivalent statements hold $w$ is of type (C).

**Proof.** (1)(a) Suppose that $M$ is unfaithful. The only height one primes of $S$ are $XS$ and $\theta S$, so $M$ has a non-zero submodule $A$ with $A(XS) = 0$ or $A(\theta S) = 0$. Only one of these possibilities can occur for $M$, as otherwise $M$, being uniform, would contain a non-zero submodule killed by $XS + \theta S$, contradicting the fact that $M$ is 1-critical.

So let us suppose that there exists $0 \neq A \subseteq M$ with $A(XS) = 0$; so $\text{Ann}_S(A) = XS$ since $A$ is 1-critical. Since $XS$ satisfies the Artin-Rees property by [13, Proposition 4.2.6], there exists $t \geq 1$ such that $MX^t = 0 \neq MX^{t-1}$. We claim that

$$t = 1. \quad (16)$$

Suppose that (16) is false. Then we can choose $m \in M$ with

$$mX^2 = 0 \neq mX;$$

moreover, since $mS$ is a non-zero submodule of $M$, it is 1-critical and hull similar to $S/XS$.

Now $mS/mSX$ is Artinian and annihilated by $XS$. Hence there exists $0 \neq f \in k[\theta]$ such that $mSf \subseteq mSX$. That is, for some non-zero element $f$ of $k[\theta]$,

$$mSfX = 0 = mSX\hat{f}, \quad (17)$$

where the second equality follows from the relation $\theta X = qX\theta$. But now $mSX$ is a non-zero submodule of $M$ which is killed by the co-Artinian ideal $XS + \hat{f}S$ of $S$, contradicting the fact that $M$ is 1-critical.
Thus (16) is proved. Hence $M$ is a finitely generated torsion-free 1-critical (and so torsion-free and uniform) $k[\theta]$-module. So it is isomorphic to a non-zero ideal of $k[\theta]$, that is to $k[\theta]$ itself.

If $M$ on the other hand contains a non-zero element annihilated by $\theta$, then the argument is similar.

(1)(b) Suppose that $M$ is a finitely generated faithful 1-critical $S$-module. We show first that

$$M \text{ is } X \text{-torsion free.} \tag{18}$$

If (18) is false then there exists $0 \neq m \in M$ with $mX = 0$. But $M$, being critical, is uniform, and is therefore an essential extension of $mS$. Since $(mS)(XS) = 0$ and the invertible ideal $XS$ has the Artin-Rees property \cite[Proposition 4.2.6]{13}, $MX^\ell = 0$ for some positive integer $\ell$. This contradicts the fact that $M$ is faithful, so (18) is proved.

Now choose $0 \neq m \in M$ and $z \in S \setminus XS$ with $mz = 0$ and $z$ irreducible in $S$ and $T$. Note that we can do this thanks to (18) coupled with Lemmas 4.1 and 4.2(1). Moreover $z$ is not type (A) since if it were then $mX = 0$ or $m\theta = 0$, and in both cases the Artin-Rees property applied to the essential extension $mS \subseteq M$ implies that $M$ is unfaithful, a contradiction. Nor is $z$ type (B), since if it were then $S/zS$ would be simple by Theorem 5.5(2), contradicting the 1-criticality of $M$. Therefore

$$z \text{ has type (C).} \tag{19}$$

Now $zS \subseteq \text{Ann}_S(m)$, so that $mS$ is a factor of $S/zS$. But $S/zS$ is 1-critical by (19) and Lemma 5.9(2), and the socle of $M$ is $\{0\}$ since $M$ is 1-critical by hypothesis. So we must have

$$S/zS \cong mS \subseteq M. \tag{20}$$

Thus $E_S(S/zS) \cong E_S(M)$ since $M$ is uniform, so $M$ is hull-similar to $S/zS$ as claimed.

(2) This is Lemma 5.9(2).

(3) Statements (b) and (c) are equivalent by Theorem 5.4(2).

$(a) \iff (b)$: Note first that $S/zS$ is $X$-torsion free by Lemma 4.3(2). Now we have the following chain of equivalences,

$$S/wS \text{ is hull-similar to } S/zS \iff E_S(S/wS) \cong E_S(S/zS) \text{ as } S \text{-modules} \iff E_T(S/wS \otimes_S T) \cong E_T(S/zS \otimes_S T) \text{ as } T \text{-modules} \iff E_T(T/wT) \cong E_T(T/zT) \text{ as } T\text{-modules} \iff T/wT \cong T/zT \text{ as } T\text{-modules},$$

where the first equivalence is by definition of hull-similarity, the second by Proposition 3.3(1), and the last equivalence holds because $T/wT$ and $T/zT$ are simple modules due to $T$ being a principal right ideal domain.

Finally, if (a) holds, then $S/wS$ is faithful and has Krull dimension 1 by part (2) of the theorem applied to $S/zS$. So $w$ cannot be type (A) or type (B) by statement (1) and Theorem 5.5(2). Hence $w$ must be type (C).

In view of the above result we introduce the obvious terminology: a finitely generated critical $S$-module is type (C) 1-critical if it is hull-similar to $S/zS$ where $z$ is a type (C) irreducible in $S$. 

\[13\]
6 (商贸) for $S$ via the representation theory of $T$.

6.1 Injective hulls of type (B) $T$-modules

In this subsection we make use of the analysis of §§5.3 and 5.4 to study injective hulls and extensions of simple $S$-modules.

We begin by examining the injective hull of a faithful simple $S$-module $V$, bearing in mind that $V$ admits a structure as $T$-module by Lemma 3.2 and its $S$- and $T$-hulls are one and the same by Proposition 3.3(2).

**Proposition 6.1.** Let $S$ and $T$ be as in §2.3 and let $V$ be a faithful simple $S$-module. Let $C := \{1 + Xs : s \in S\}$.

1. $C$ is an $\alpha$-invariant Ore set in $S$.
2. The set $W$ of $C$-torsion elements of $E_T(V)$ is a non-zero $T$-submodule of $E_T(V)$.
3. Let $c \in C$, and let $c = c_1 \cdots c_t$ be a factorisation of $c$ as a product of irreducible elements of $S$. Then $c_i$ has type (B) for all $i$.
4. As an $S$-module, $W$ is locally Artinian, with all composition factors type (B).
5. If $E_T(V)/W$ is nonzero then it has the following properties.
   (i) $E_T(V)/W$ is a $C$-torsion-free injective $T$-module.
   (ii) As $S$-module $E_T(V)/W$ is injective, a direct sum of injective hulls of 1-critical type (C) $S$-modules.
   (iii) The decomposition (ii) is also a decomposition of $E_T(V)/W$ as a direct sum of injective hulls of simple $T$-modules.

**Proof.** (1) Since $X$ is normal in $S$, $XS$ satisfies the Artin-Rees property by [13, Proposition 4.2.6]. Hence, by [13, Proposition 4.2.9(i)], $C$ is an Ore set in $S$. It’s clear from the definitions of $C$ and $\alpha$ that $C$ is $\alpha$-invariant.

(2) That $W$ is an $S$-submodule of $E_T(V)$ is a standard consequence of the Ore condition. By Theorem 5.5(1) $V \cong T/zT$ for an irreducible element $z$ of $C$, so that $0 \neq 1 + zT \in W$.

Let $y \in W$ and $c \in C$ be such that $yc = 0$. One can easily see that there is a $\overline{t} \in C$ such that $Xc = \overline{t}X$. Then

$$0 = yc = (yX^{-1})(\overline{t}X)$$

and $yX^{-1}\overline{t} = 0$. Hence $WX^{-1} \subseteq W$ and the result follows.

(3) This is a straightforward check.

(4) Clearly it is enough to prove that if $w \in W$ then $wS$ is Artinian with all composition factors type (B). But such an element $w$ satisfies $wc = 0$ for some $c \in C$, so that $wS$ is a factor of $S/cS$. The result follows since $S/cS$ has the desired form by part (3), Lemma 4.1 and Theorem 5.5(2).

(5) That $E_T(V)/W$ is $C$-torsion-free is an immediate consequence of the fact that $C$ is multiplicatively closed. It is $T$-injective since it is a factor of an injective $T$-module and $T$ is hereditary (being a principal right ideal ring). Since $E_T(V)/W$ is a $T$-module it is $X$-torsion free. An easy lemma shows that injective $T$-modules are injective as $S$-modules, so $E_T(V)/W$ is an injective $S$-module. Since $S$ is noetherian, every injective $S$-module is a direct sum of injective hulls of critical $S$-modules. So to confirm the claimed structure of $E_T(V)/W$ as $S$-module we have to confirm that each finitely generated critical $S$-submodule $U$ of $E_T(V)/W$ contains a cyclic critical of type (C). First note that such a $U$ must be 1-critical: for $U$ is $X$-torsion free since $E_T(V)/W$ is a $T$-module, so $U$ cannot be finite dimensional; and if $U$ is an infinite dimensional simple $S$-module then it is type (B), and hence $C$-torsion, which contradicts the fact that $E_T(V)/W$ is $C$-torsion free. But
now Theorem 5.10(1) shows that $U$ is either $\hat{S}/\theta \hat{S}$ or it contains a submodule $S/zS$ for a type (C) irreducible $z$.

The first of these possibilities is ruled out by an argument similar to that used in part (2). Namely, if $U \cong S/\theta S$ then there is an element $e$ of $E_T(V) \setminus W$ with $e\theta \in W$. But then $e\theta d = 0$ for an element $d$ which is a product of type (B) irreducibles. It is easy to check that $\theta d = d'\theta$ where $d' \in \mathcal{C}$. Since $E_T(V)$ is an essential extension of the faithful simple $S$-module $V$, it contains no non-zero elements killed by $\theta$. Hence $ed' = 0$, so $e \in W$, a contradiction.

The final claim in (5) follows immediately from Proposition 3.3(1).

Corollary 6.2. Continue with the notation of Proposition 6.1. Then the following are equivalent.

(1) $E_S(V)$ is a locally Artinian $S$-module.
(2) $W = E_S(V)$.
(3) $W = E_T(V)$.

Proof. (2) $\iff$ (3): By Proposition 3.3.
(2) $\iff$ (1): By Proposition 6.1(4),(5).

6.2 Equivalent conditions for $\diamond$ for $S$

We summarise the equivalences between the nature of Ext-spaces of critical modules for $S$ and $T$ and the validity of property $\diamond$ for $S$ in Theorem 6.6, which is a consequence of the following well-known result.

Proposition 6.3. [17, Corollary 3.2] Let $U$ be a Noetherian domain, $a, b$ nonzero elements of $U$. Then the following are equivalent:

(a) $\text{Ext}^1_U(U/aU, U/bU) = 0$;
(b) $\text{Ext}^1_U(U/Ub, U/Ua) = 0$;
(c) $Ua + bU = U$.

Before applying Proposition 6.3 to the analysis of $\diamond$ for $S$ we record the following corollary, which we will need below and which can be deduced either from the Artin-Rees property of $\theta T$ together with (a) $\iff$ (b), or by elementary calculations in $T$ using criterion (c) along with our taxonomy in §4.2 of canonical irreducible elements.

Corollary 6.4. Let $c$ be an irreducible element of $T$ which is not an associate of $\theta$. Then

$$\text{Ext}^1_T(T/c, T/T\theta) = \text{Ext}^1_T(T/T\theta, T/c) = 0,$$

and similarly for right modules.

The following lemma is needed for the proof of Theorem 6.6

Lemma 6.5. Let $w$ and $z$ be irreducible elements of $S$, with $w$ of type (B) and $z$ of type (C). Then $\text{Ext}^1_S(S/zS, S/wS) \neq 0$ if and only if there is a short exact sequence of $S$-modules

$$0 \rightarrow S/wS \rightarrow A \rightarrow S/zS \rightarrow 0$$

with $A$ uniform.
Proof. The implication from right to left is clear from basic properties of Ext. For the reverse implication, set $W := S/wS$ and $U := S/zS$, and suppose that $\Ext^1_S(U, W) \neq 0$. Then there exists a module extension

$$0 \rightarrow W \xrightarrow{\alpha} A \xrightarrow{\pi} U \rightarrow 0 \quad (21)$$

in which $A$ does not split as the direct sum of $U$ and $W$. We suppose for a contradiction that $A$ is not uniform. Thus $\alpha(W)$ is not essential in $A$, so we can choose a non-zero submodule $B$ of $A$ maximal such that

$$B \cap \alpha(W) = 0. \quad (22)$$

We now obtain from (21) the exact sequence

$$0 \rightarrow W \xrightarrow{\alpha} A/B \rightarrow U/\pi(B) \rightarrow 0, \quad (23)$$

Note that, in (23),

$$U/\pi(B) \neq 0 \neq \pi(B), \quad (24)$$

due to (22) and the assumption that (21) is non-split.

We claim that $\alpha(W)$ is essential in $A/B$. Suppose that this is false, and let $C$ be a non-zero submodule of $A/B$. Writing $C$ as $D/B$ for a submodule $D$ of $A$ strictly containing $B$, this is equivalent to

$$\alpha(W) \cap D \subseteq \alpha(W) \cap B = 0,$$

which contradicts the maximality of $B$. Thus $\pi(W)$ is indeed essential in $A/B$. Equivalently, since $W$ is simple, $A/B$ is uniform, so the exact sequence (23) shows that

$$\Ext^1_S(U/\pi(B), W) \neq 0. \quad (25)$$

But the module $U/\pi(B)$ is finite dimensional by Lemma 5.9, using (24) and since $z$ is a type $(C)$ irreducible element. This means that (25) contradicts Lemma 5.8. Therefore the module $A$ in (21) is uniform, as required.

To state Theorem 6.6 precisely it is necessary to consider, here and in Theorem 6.7, one-sided versions of property $(\diamond)$ - thus we say that an algebra $A$ satisfies right $(\diamond)$ if the injective hulls of its simple right modules are locally Artinian; and similarly for left $(\diamond)$.

Theorem 6.6. (1) Let $w$ and $z$ be irreducible elements of $S$, with $w$ of type $(B)$ and $z$ of type $(C)$. Then the following are equivalent:

- (a) $\Ext^1_S(S/zS, S/wS) = 0$;
- (b) $\Ext^1_S(S/Sw, S/Sz) = 0$;
- (c) $\Ext^1_T(T/zT, T/wT) = 0$;
- (d) $\Ext^1_T(T/Tw, T/Tz) = 0$.

(2) $S$ satisfies right $(\diamond)$ if and only if the equivalent statements in (1) hold for all such elements $z$ and $w$.

Proof. (1) That (a) is equivalent to (b) and (c) is equivalent to (d) is immediate from Proposition 6.3.

To see that (a) $\iff$ (c), set $W := S/wS$ and $U := S/zS$, so that $W = T/wT$ and $E_T(W) = E_T(W)$ by Proposition 3.3(2) and Theorem 5.5(2). Suppose first that $\Ext^1_S(U, W) \neq 0$. Then by Lemma 6.5 there exists a cyclic uniform $S$–module $V$ which is an extension of $W$ by $U$. Therefore, by Lemma 4.3(2) and Proposition 5.3(1), $V \otimes_ST$ is a cyclic uniform $T$–module, which is an essential extension of its simple $T$–submodule
$W \otimes_S T = W$, where the equality follows from Lemma 3.2. Exactness of the functor $- \otimes_S T$ and the fact that $U$ is $X$-torsion free guarantee that

$$(V \otimes_S T)/W \cong T/zT \not= 0.$$ 

This proves that $\text{Ext}^1_T(T/zT,T/wT) \not= 0$, so that (c)$\implies$(a). The converse is proved by a similar but easier argument.

(2) Suppose that right $(\circ)$ holds for $S$. This means that, for all type (B) irreducible elements $w$ of $S$, $E_S(S/wS)$ is locally artinian. Transferring this fact to the $T$–modules $E_T(T/wT)$ by means of Theorem 5.5 and Proposition 3.3(2), we see that none of these injective $T$–modules $E_T(T/wT)$ contains a simple $T$-subfactor $T/zT$ with $z$ of type (C), proving (c).

Conversely, if right $(\circ)$ fails for $S$ then by Corollary 5.2 there exists a faithful simple right $S$–module $V$ for which, in the notation of Proposition 6.1, $W \subseteq E_S(V)$. In view of Theorem 5.10 and Proposition 6.1(5)(ii) there exists a type (C) irreducible $d$ of $S$ such that $S/dS \subseteq E_S(V)/W$. Take $e \in E_S(V)$ such that $(eS + W)/W$ is isomorphic to $S/dS$. Then $eS$ is a uniform extension of $V$, and $eS \cap W$ has a finite composition series whose factors are all faithful simple $S$–modules, by Proposition 6.1(4). Since $eS/eS \cap W \cong S/dS$, an application of the long exact sequence of Ext shows that

$$\text{Ext}^1_S(S/dS,Y) \not= 0$$ 

for some composition factor $Y$ of $eS \cap W$. Hence (a) fails to hold, as required. 

The following consequence of the above result reduces the issue of the validity of right $(\circ)$ for $S$ to a “monoidal commutativity” condition on the irreducible elements of $T$, as in (c).

**Theorem 6.7.** The following statements are equivalent.

(a) Right $(\circ)$ holds for $S$.

(b) There does not exist a uniserial right $T$-module with composition length 2 whose socle is a type (B) simple and whose simple image is type (C).

(c) There does not exist a uniserial left $T$-module with composition length 2 whose socle is a type (C) simple and whose simple image is type (B).

(d) There does not exist a cyclic 1-critical left $S$–module with an infinite dimensional simple image.

(e) Let $b,c \in S$ be irreducible, with $b$ type (B) and $c$ type (C). Then there exist $b',c'$ in $S$, respectively irreducibles of types (B) and (C) and respectively similar to $b$ and to $c$, with

$$cb = b'c'.$$

**Proof.** (a)$\iff$(b): This follows from Theorem 6.6(2) with (c) of Theorem 6.6(1).

(b)$\iff$(d): This is the equivalence of Theorem 6.6(c) and (d).

(c)$\iff$(d): Suppose that (c) holds, but that a left $S$-module $M$ of the form described in (d) exists. That is, $M$ is cyclic and uniform, and is the extension of a 1-critical $S$-module $A$ by an infinite dimensional simple $S$-module. Notice that $A$ is faithful: for, if this is not the case $A$ is annihilated by $X^jS$ for some $i,j \geq 0$, and since this ideal satisfies the Artin-Rees property by [13] Proposition 4.2.6, $(X^jS)^tM = 0$ for some $t \geq 1$. This is impossible since $M$ has a faithful factor by hypothesis. Hence, by (the left-side version of) Theorem 5.10(1)(b), $A$ contains a copy of $S/Sz$ for a type (C) irreducible element $z$.
of $S$. Now $M$ is $X$-torsion free, since its essential submodule $A$ is, and so $T \otimes_S M$ is an essential extension of $T/Tz$ and maps onto $T/Tw$ for a type (B) simple critical element $w$ of $S$. Therefore, since $T \otimes_S M$ has finite composition length and cannot contain any subfactor isomorphic to $T/T\theta$ thanks to Corollary 6.4, an application of the long exact sequence of Ext shows that it must contain as a subfactor a non-split extension of a type (C) simple module by a type (B) one, contradicting (c).

$(d) \implies (c)$: Conversely, suppose that (d) holds, but that a left $T$-module $N$ exists as in (c), so $N = Ty$ is a non-split $T$-module of composition length two, with unique composition series

$$0 \rightarrow F \rightarrow N \rightarrow P \rightarrow 0,$$

with $F$ and $P$ respectively type (C) and type (B) simple $T$-modules. More precisely, let $p \in P$ be such that the epimorphism $\pi : N \rightarrow P$ maps $y$ to $p$, and by Theorem 5.5(1),(2),

$$P = Sp \cong S/Sb$$

for a type (B) irreducible element $b$ of $S$. By restriction, we obtain an exact sequence

$$0 \rightarrow F_0 \rightarrow Sy \rightarrow S/Sb \rightarrow 0,$$

(26)

where $F_0 := F \cap Sy$. Since $N$ is a uniform $T$-module, it’s easy to show that $Sy$ is a uniform $S$-module, and since $b$ is type (B) irreducible, $S/Sb$ is a simple $S$-module. Consider now any non-zero element $u$ of $F_0$. Because $F$ is a simple $T$-module,

$$T \otimes_S Su = Tu = F,$$

(27)

so $cu = 0$ for a type (C) irreducible element $c$ of $S$. Thus $F_0$ is hull-similar to its type (C) 1-critical submodule

$$C := Su \cong S/Sc.$$

Now $F/Su$ is $X$-torsion by (27). Therefore every element of the finitely generated $S$-module $F_0/Su$ is killed by both a power of $X$ and by an irreducible element of type (C). Filtering $F_0/Su$ by the elements killed by $X$, then the elements killed by $X^2$ and so on, we therefore see that $F_0/Su$ is a finite dimensional $S$-module. From the exact sequence (26) we obtain the exact sequence

$$0 \rightarrow F_0/Su \rightarrow Sy/Su \rightarrow S/Sb \rightarrow 0,$$

(28)

By Lemma 3.1 the $S$-injective hull of $F_0/Su$ is locally finite dimensional. Since $S/Sb$ is an infinite dimensional simple $S$-module, $Sy/Su$ does not embed in $E_S(F_0/Su)$ and the sequence is split exact. We may thus choose an element $w \in Sy$ whose image modulo $Su$ generates a $S$-module isomorphic to $S/Sb$. Set $W := Sw$. We have thus found a non-split exact sequence

$$0 \rightarrow F_0 \cap W \rightarrow W \rightarrow S/Sb \rightarrow 0,$$

(29)

contradicting (d) and so completing the proof.

$(c) \implies (e)$: Assume $(c)$, and let $b$ and $c$ be irreducibles in $S$, respectively of types (B) and (C). Then $T/Tcb$ has composition length 2, so it is either semisimple or has essential socle. By (c), $T/Tcb$ splits; that is,

$$T/Tcb = A \oplus V,$$

where $A = Tb/Tcb \cong T/Tc$ and $V \cong T/Tb$ via the $T$-module projection $\psi : T/Tcb \rightarrow V$ with $\ker \psi = A$. Set $\psi(1 + Tcb) := v$, so there exists $a \in A$ such that $1 + Tcb = (a, v)$. Let $M = T(a, v)$. 

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Since $A = Ta$, there exists a type (C) irreducible element $c'$, which we can choose to be in $S$ by Lemma 1.2 with 

$\text{Ann}_T(a) = Tc'$.

Note that $c'$ is similar to $c$ by Theorem 5.10(5), because

$$T/Tc' \cong A \cong T/Tc.$$ 

Now $c'(a,v) = (0,v')$, and $v' \neq 0$ because $M = T(a,v)$ cannot be a factor of the simple module $T/Tc'$. Thus 

$$\text{Ann}_T(v') = Tb''$$

for an irreducible element $b''$, again chosen to be in $S$ thanks to Lemma 1.2 and which is similar to $b$ by Theorem 5.5(3). Therefore

$$Tb''c' \subseteq \text{Ann}_T((a,v)) = \text{Ann}_T(1 + Tcb) = Tcb,$$

so that

$$Tb''c' = Tcb$$

since the factors by both these left ideals have composition length 2. By Lemma 4.3(1) and Remark 1.4, 

$$Sb''c' = Tb''c' \cap S = Tcb \cap S = Scb.$$

There is therefore a unit $u$ of $S$ such that 

$$ub''c' = pb.$$

Since $u = 1 + Xw$ for some $w \in R$, $b' := ub''$ is a type (B) irreducible in $S$, similar to $b''$ which is similar to $b$, and so (e) is proved.

$(e) \implies (c)$: Assume (e), and let $M$ be as in (c), namely a uniserial left $T$-module which is an extension of a type (C) simple $T$-module $A$ by a type (B) simple module. Being uniserial of length 2, $M$ is cyclic, say $M = Tm$. By hypothesis, $\text{Ann}_T(m + A) = Tb$ for some irreducible type (B) element, which we can choose to be in $S$ by Lemma 1.2. Since $M$ is not simple, $0 \neq bm \in A$, so $A = Tbm$ and there is a type (C) irreducible element $c$ of $S$ such that $c(bm) = 0$. Therefore $M$ is a factor of $T/Tc$, and, comparing composition lengths, we see that 

$$M \cong T/Tc.$$

Now apply (e) to obtain irreducibles $b'$ and $c'$ in $S$, respectively of types (B) and (C), with $cb = b'c'$. So

$$b'c'm = cbm = 0,$$

but $c'm \neq 0$ as $M = Tm$ is not a simple $T$-module. Thus

$$T(c'm) \cong T/Tb'$$

is a simple type (B) $T$-submodule of $M$, contradicting the fact that $M$ is uniserial with type (C) socle. Therefore no such uniserial module $M$ exists, and (c) is proved. \qed
6.3 Proof of (⋆) for \( \hat{S} \)

Our aim is to prove that \( \hat{S} = k[[X]][\theta; \alpha] \) satisfies (⋆) by showing that it satisfies the monoidal commutativity criterion of Theorem 6.7 (c). For this, we need some additional information about the irreducibility of polynomials in the skew polynomial ring \( \hat{S} \), as provided by the next lemma. Note that in it we do not need to assume that \( q \) is not a root of unity and the observation it contains seems to be of independent interest even when \( q = 1 \), that is when \( \hat{S} \) is a commutative ring. We write \( q := \sum_{p \geq 0} q_p X^p \), where \( q_p \in k \) and \( q_0 \neq 0 \) as \( q \) is a unit of \( k[[X]] \). Consider \( z = \sum_{t=0}^{m} z_t \theta^t \in \hat{S} \) with \( z_t \in k[[X]] \) for all \( i \). If \( X \) divides all the \( z_t \), except for \( z_0 \) and \( X^2 \) does not divide \( z_m \), then the generalised Eisenstein criterion, [9] §1, Theorem, tells us that \( z \) is irreducible. The following lemma is in the direction of a converse to this. Its formulation is rather technical due to the requirements of its proof, but the underlying idea is quite simple. It shows that if there is an integer \( n \) with \( 0 < n < m \) such that \( X \) does not divide \( z_n \) and \( X \) divides \( z_i \) for all \( i > n \), then \( z \) is necessarily reducible.

**Lemma 6.8.** Retain the notation and definition of \( \hat{S} \) as above. Let \( 1 \leq n < m \) and let

\[
   z = \sum_{i=0}^{n} f_i \theta^i + \sum_{i=n+1}^{m} g_i \theta^i \in \hat{S},
\]

where \( f_i, g_i \in k[[X]] \) and \( g_m \neq 0 \). Suppose that

1. \( f_n \) is invertible (that is, \( X \not| f_n \)), and
2. \( g_i \) is not invertible (that is, \( g_i \in Xk[[X]] \)), for all \( i = n + 1, \ldots, m \).

Then \( z \) is divisible on the right by a monic polynomial in \( \theta \) of degree \( n \). In particular, \( z \) is a reducible polynomial.

**Proof.** Let \( z \in \hat{S} \) be as above. As \( f_n \) is invertible in \( \hat{S} \), we can without loss of generality replace \( z \) by \( f_n^{-1} z \) and thus assume henceforth that \( f_n = 1 \). We claim that there exist \( h_0, \ldots, h_{n-1} \in k[[X]] \) such that

\[
   z \in \hat{S}(h_0 + h_1 \theta + \ldots + h_{n-1} \theta^{n-1} - \theta^n). \quad (30)
\]

Note that the lemma is immediate from (30).

Let \( h_0, h_1, \ldots, h_{n-1} \) be elements of \( k[[X]] \) which remain to be determined, and denote the left ideal \( \hat{S}(h_0 + h_1 \theta + \ldots + h_{n-1} \theta^{n-1} - \theta^n) \) of \( \hat{S} \) by \( I \). Throughout this proof we denote congruence (mod \( I \)) by \( \equiv \). Thus we are aiming to choose elements \( h_i \) of \( k[[X]] \) such that

\[
   z \in I. \quad (31)
\]

By definition of \( I \),

\[
   \theta^n \equiv h_0 + h_1 \theta + \ldots + h_{n-1} \theta^{n-1}. \quad (32)
\]

Multiplying (32) on the left by \( \theta \), we deduce that

\[
   \theta^{n+1} \equiv \theta \left( \sum_{i=0}^{n-1} h_i \theta^i \right)
   \equiv \sum_{i=0}^{n-2} \alpha(h_i) \theta^{i+1} + \alpha(h_{n-1}) \theta^n
   \equiv \sum_{j=1}^{n-1} \alpha(h_{j-1}) \theta^j + \alpha(h_{n-1}) \left( \sum_{j=0}^{n-1} h_j \theta^j \right)
   \equiv \alpha(h_{n-1}) h_0 + \sum_{j=1}^{n-1} (\alpha(h_{j-1}) + \alpha(h_{n-1}) h_j) \theta^j.
\]
This yields by induction: for all \( \ell \geq 1 \),
\[
\theta^{n+\ell} = \sum_{i=0}^{n-1} y_{i,\ell} \theta^i,
\]  
(33)
where for each \( i \) and \( \ell \), \( y_{i,\ell} \in k[[X]] \) is a \( k \)-linear combination of products of elements of the set \( \{ \alpha'(h_j) : j \in \{0, \ldots, n-1\}, r \in \{0, \ldots, \ell\} \} \).

Write \( h_i = \sum_{p \geq 0} h_{i,p} X^p \) for \( h_{i,p} \in k \) and similarly present the elements \( f_i, g_j \) and \( y_{i,j-n} \) of \( k[[X]] \). Note that \( \alpha(h_j) = \sum_p h_{j,p}(qX)^p \), for any \( r \in \mathbb{N} \), \( \alpha'(h_j) = \sum_p h_{j,p}(\alpha'(X))^p \) and that \( \alpha'(X) = N^p(q)X \), so \( \alpha'(h_j) \) is a sum of products of \( h_{j,p} \) for \( p \leq p \) with some coefficients of \( q \), namely \( q_{p^*} \) for \( p^* \leq p \). This means that, for all \( r, p \in \mathbb{N} \) and \( j \in \{0, \ldots, n-1\} \),
\[
\alpha'(h_j)_p \text{ is an explicit } k \text{-linear combination of } h_{j,p^*} \text{ for } p^* \leq p.
\]  
(34)
(In case \( q \in k \), we actually have \( \alpha'(h_j)_p = q^p h_{j,p} \), a \( k \)-linear combination of \( h_{j,p} \).)

Since \( f_n = 1 \), from (32) and (33) we have
\[
z = \sum_{i=0}^{n-1} f_i \theta^i + \sum_{i=0}^{n-1} h_i \theta^i + \sum_{j=n+1}^{m} \sum_{i=0}^{n-1} g_j y_{i,j-n} \theta^i.
\]  
(35)
So \( z \in \hat{S}(h_0 + h_1 \theta + \ldots + h_{n-1} \theta^{n-1} - \theta^n) \) if and only if for each \( i \in \{0, \ldots, n-1\} \)
\[
f_i + h_i + \sum_{j=n+1}^{m} g_j y_{i,j-n} = 0
\]  
(36)
in \( k[[X]] \).

We claim that the system of equations (36) in the unknowns \( h_0, \ldots, h_{n-1} \) does indeed have a solution in \( k[[X]] \).

Recall that \( X \mid g_j \) for all \( j \in \{n+1, \ldots, m\} \). Therefore, looking at coefficients of terms of degree \( p = 0 \) and \( p = 1 \) in \( X \) in (36), we get
\[
h_{i,0} = -f_{i,0}, \quad \forall i \in \{0, \ldots, n-1\}
\]  
(37)
\[
h_{i,1} = -f_{i,1} - \sum_{j=n+1}^{m} g_j y_{i,j-n,0}, \quad \forall i \in \{0, \ldots, n-1\}
\]  
(38)
where, by (33) and (34), \( y_{i,j-n,0} \) is a \( k \)-linear combination of products of elements from the set
\[
\{ \alpha'(h_j,0) : j \in \{0, \ldots, n-1\}, r \in \{0, \ldots, m-n\} \}.
\]
Assume now that \( t \) is a positive integer and that \( h_{i,p} \) are known for all \( i \in \{0, \ldots, n-1\} \) and \( p \leq t \). By (36), we require, for all \( i \in \{0, \ldots, n-1\} \),
\[
h_{i,t+1} = -f_{i,t+1} - \sum_{j=n+1}^{m} \sum_{s=0}^{t} g_j y_{i,j-n,s}
\]  
(39)
where, by (33) and (34), for each \( i \in \{0, \ldots, n-1\} \) and \( s \in \{0, \ldots, t\} \), \( y_{i,j-n,s} \) is a \( k \)-linear combination of products of elements from
\[
\{ \alpha'(h_j,s) : j \in \{0, \ldots, n-1\}, r \in \{0, \ldots, i\}, \ell \in \{0, \ldots, s\} \}.
\]
So each \( h_{i,t+1} \) exists, and hence \( h_0, \ldots, h_{n-1} \) exist also.

The following example shows that the above lemma does not hold when the base ring is replaced by \( k[[X]](X) \).
Example 6.9. For \( q \in k^* \), let \( S = k[X]/(X; \alpha) \) where \( \alpha(X) = qX \). The element \( r = 1 + X + \theta + X \theta^2 \in S \) is irreducible in \( S \). To check this note that if \( a, b \in k[X] \) are such that \( b \not< X > \), then

\[
r = \alpha(b^{-1})b^{-1}[bX\theta - X\alpha(a) + \alpha(b)][(a + b\theta) + 1 + X - b^{-1}a + Xb^{-1}\alpha(b^{-1})\alpha(a)a].
\]

So \( r \in S(a + b\theta) \) if and only if \( \alpha(b)(b - a) = -X(\alpha(b)a + \alpha(a)a) \). Comparing the degrees on the RHS and LHS of the last equation we conclude that \( r \) is not divisible on the right by a polynomial of the form \( a + b\theta \) for any \( a, b \in k[X] \) with \( b \not< X > \).

Making use of the above for \( S_\alpha = k[X]/(X; \alpha^{-1}) \) we see that also, with the same constraints on \( a \) and \( b \), \( r \not\in (a + b\theta)S \), proving that \( r \) is irreducible.

A very useful consequence of Lemma 6.8 provides a canonical form for each type (C) similarity class of irreducible elements of \( S \):

Corollary 6.10. Let \( c \) be a type (C) irreducible element of \( \hat{S} \), of degree \( n \geq 1 \) as a polynomial in \( \theta \). Then there is a unit \( u \) of \( S \) such that \( \hat{c} := uc \) is an irreducible element of \( \hat{S} \), so \( \hat{c} \) is similar to \( c \), with \( \hat{c} \) having form

\[
\hat{c} = h_0 + \sum_{i=1}^{n-1} h_i \theta^i + \theta^n,
\]

when \( n > 1 \), with \( h_0 \neq 0 \) and \( h_i \in k[[X]] \) for all \( i \); and \( \hat{c} \) having form

\[
\hat{c} = h_0 + \theta
\]

with \( 0 \neq h_0 \in k[[X]] \) when \( n = 1 \).

Proof. Let \( c = \sum_{i=0}^{n} r_i \theta^i \), where \( r_i \in k[[X]] \) for all \( i \) and \( r_n \neq 0 \). Then

\[
r_0 \neq 0,
\]

since otherwise \( c = u\theta \) for some unit \( u \) of \( \hat{S} \) as \( c \) is irreducible, so that \( c \) has type (A), which is a contradiction. Moreover

\[
n > 0,
\]

because if \( n = 0 \) then \( c \in k[[X]] \), which is again a contradiction to \( c \) being type (C).

There exists at least one \( i \) for which \( X \not| r_i \), because otherwise, as \( c \) is irreducible, \( c = Xu' \) for a unit \( u' \) of \( \hat{S} \), once more contradicting the fact that \( c \) is type (C). Let \( m \) be the greatest value of \( i \) such that

\[
X \not| r_m.
\]

If \( m = 0 \) then, by (40), \( r_0 \) is a unit in \( \hat{S} \) and so

\[
\hat{c} := r_0^{-1}c = 1 + X\tau
\]

for some \( \tau \in \hat{S} \), so that \( \hat{c} \) is type (B), a contradiction. So

\[
m > 0.
\]

Suppose that \( m < n \). Then

\[
c = \sum_{i=0}^{m} r_i \theta^i + \sum_{i=m+1}^{n} r_i \theta^i, \quad r_m \not\in (X), \quad r_i \in (X) \forall i > m, \quad r_n \neq 0.
\]

But then, noting (43), Lemma 6.8 applies to the expression (44) for \( c \), and tells us that \( c \) is reducible, a contradiction. Therefore \( m = n \), so that \( r_n \) is a unit in \( k[[X]] \). That is, we can define

\[
\hat{c} := r_n^{-1}c,
\]

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so that
\[ \hat{c} = \sum_{i=0}^{n-1} h_i \theta^i + \theta^n, \]  
(45)

with all \( h_i \in k[[X]] \), \( h_0 \neq 0 \) and \( n > 0 \). Finally, by \[5.2\] \( \hat{c} \) is similar to \( c \).

We now apply Lemma \[6.8\] and Corollary \[6.10\] to prove that the monoidal commutativity condition of Theorem \[6.7\] (c) is satisfied, hence deducing that \( \hat{S} \) satisfies (\( \diamond \)).

**Proposition 6.11.** Let \( b \) and \( c \) be irreducible elements of \( \hat{S} \), respectively of type (B) and (C). Then there exist irreducible elements \( b' \) and \( c' \) of \( \hat{S} \), respectively similar to \( b \) and \( c \), such that \( cb = b'c' \).

**Proof.** Note that if \( c \) is replaced by \( \hat{c} := uc \) for a unit \( u \) of \( \hat{S} \), and we prove that \( \hat{c}b = \hat{bc} \) with \( \hat{b} \) and \( c' \) irreducibles respectively of types (B) and (C), then \( cb = u^{-1}\hat{c}b = (u^{-1}\hat{b})c' = b'c' \), with \( b' \) and \( c' \) irreducibles in \( \hat{S} \) respectively of types (B) and (C).

Thus Corollary \[6.10\] allows us to assume that \( c \) has the form
\[ c = \sum_{i=0}^{n} h_i \theta^i \]  
(46)

with \( h_i \in k[[X]] \), \( h_0 \neq 0 \), \( n > 0 \) and \( h_n = 1 \).

Let \( b = 1 + Xs \), where \( s = \sum_{j=0}^{\ell} s_j \theta^j \) with \( s_j \in k[[X]] \) for all \( j \). Since \( b \) is irreducible and so not invertible in \( \hat{S} \),
\[ \ell > 0. \]  
(47)

Thus, using \[46\],
\[ cb = \left( \sum_{i=0}^{n} h_i \theta^i \right) \left( 1 + X \sum_{j=0}^{\ell} s_j \theta^j \right) = \sum_{i=0}^{n} h_i \theta^i + X \sum_{i=0}^{n} \sum_{j=0}^{\ell} q^i h_i \alpha^i (s_j) \theta^{i+j} . \]

In the above expression for \( cb \), observe that \( n > 0 \) and \( h_n = 1 \) by \[41\] and \[46\], while the highest power of \( \theta \) occurring is \( n + \ell \), with \( n + \ell > n \) by \[47\]. Since the coefficient of \( \theta^{n+\ell} \) is divisible by \( X \), the hypotheses of Lemma \[6.8\] are satisfied by \( cb \). Therefore we can conclude that \( cb \) is divisible on the right by a monic polynomial \( c' \) of degree \( n \); that is,
\[ cb = b'c' . \]  
(48)

Comparing degrees on the right and left of \[48\],
\[ \deg_{\theta}(b') = \ell > 0 \]
by \[47\], so that neither \( c' \) nor \( b' \) are units in \( \hat{S} \). Comparing the two factorisations provided by \[48\] in the PID \( T \), both must have length 2 by the Jordan-Holder theorem, since
Theorem 6.7(d) again, the fact that of Theorem 5.10(2), and hence (50) follows. But $N_t$ exists.

Let $M$ be a cyclic faithful 1-critical left $S$-module. By the left module version of Theorem 5.10(4)(b), $M$ is hull-similar to $\hat{S}/\hat{S}c$ for a type (C) irreducible element $c$ of $\hat{S}$. In particular,

\[ M \text{ is } \mathcal{X} \text{-torsion free.} \quad (49) \]

The equivalent statements (a) - (e) of Theorem 6.7 all hold, by Corollary 6.12. By Theorem 6.7(d), every proper factor of $S$ modules from that given by Theorem 5.10(4)(a) and (5), as follows:

Corollary 6.13. Let $I$ be a left [resp. right] ideal of $\hat{S}$ with $\hat{S}/I$ faithful and 1-critical. Then $I = \hat{S}z$ [resp. $I = z\hat{S}$] for an irreducible element $z$ of $\hat{S}$ (which necessarily will have type (C)).

Proof. We prove the version for left ideals; the right hand version is obtained by interchanging left and right throughout. Let $M = \hat{S}/I$ be a cyclic faithful 1-critical left $\hat{S}$-module. By the left module version of Theorem 5.10(1)(b), $M$ is hull-similar to $\hat{S}/\hat{S}c$ for a type (C) irreducible element $c$ of $\hat{S}$. In particular,

\[ M \text{ is } \mathcal{X} \text{-torsion free.} \quad (49) \]

For, let $N$ be a non-zero $T$-submodule of $T/TI$. Then $N \cap M \neq 0$ by (49), so that, by Theorem 6.7(d) again,

\[ \dim_k(M/N \cap M) < \infty. \quad (51) \]

Let $y \in T \otimes \hat{S} M$. There thus exists $n \geq 1$ such that $X^ny \in M$ and then, by (51), there exists $t \geq 1$ such that

\[ X^t(X^ny) \in N. \]

But $N$ is a $T$-submodule of $T \otimes \hat{S} M$, so $y = X^{-(t+n)}X^{t+n}y \in N$. Thus $N = T \otimes \hat{S} M$ and (50) follows.

In view of (50), $TI = Tz$ for an irreducible element $z$, which we can choose to be in $\hat{S}$ by Lemma 4.2. That is, $z \in TI \cap \hat{S} = I$, and $z$ has type (C) since otherwise it has type (B) and then $\hat{S}/\hat{S}z$ is simple by the left module version of Theorem 5.10(2), contradicting the fact that $\hat{S}/I$ is 1-critical. Therefore $\hat{S}/\hat{S}z$ is itself 1-critical by the left module version of Theorem 5.10(2), and hence $\hat{S}z = I$, as required. \qed
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