Compactness and finite forcibility of graphons

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Abstract

Graphons are analytic objects associated with convergent sequences of graphs. Problems from extremal combinatorics and theoretical computer science led to a study of graphons determined by finitely many subgraph densities, which are referred to as finitely forcible. Following the intuition that such graphons should have finitary structure, Lovász and Szegedy conjectured that the topological space of typical vertices of a finitely forcible graphon is always compact. We disprove the conjecture by constructing a finitely forcible graphon such that the associated space is not compact. In our construction, the space fails to be even locally compact.

1 Introduction

Recently, a theory of limits of combinatorial structures emerged and attracted substantial attention. The most studied case is that of limits of dense graphs initiated by Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi [6, 8, 22, 24], which we also address in this paper. A sequence of graphs is convergent if the density of every graph as a subgraph in the graphs contained in the sequence converges. A convergent sequence of graphs can be associated with an analytic object (graphon) which is a symmetric measurable function from the unit square $[0, 1]^2$ to $[0, 1]$. Graph limits and graphons are also closely related to flag algebras introduced by Razborov [28], which were successfully applied to numerous problems in extremal combinatorics [1–4, 12–17, 26–30]. The development of the graph limit theory is also reflected in a recent monograph by Lovász [19].

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*The work leading to this invention has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement no. 259385.

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In this paper, we are concerned with finitely forcible graphons, i.e., those that are uniquely determined (up to a natural equivalence) by finitely many subgraph densities. Such graphons are related to uniqueness of extremal configurations in extremal graph theory as well as to other problems. For example, the classical result of Chung, Graham and Wilson [9] asserting that a large graph is pseudo-random if and only if the homomorphic densities of $K_2$ and $C_4$ are the same as in the Erdős-Rényi random graph $G_{n,1/2}$ can be cast in the language of graphons as follows: The graphon identically equal to $1/2$ is uniquely determined by homomorphic densities of $K_2$ and $C_4$, i.e., it is finitely forcible. Another example that can be cast in the language of finite forcibility is the asymptotic version of the theorem of Turán [31]: There exists a unique graphon with edge density $\frac{r-1}{r}$ and zero density of $K_{r+1}$, i.e., it is finitely forcible.

The result of Chung, Graham, and Wilson [9] was generalized by Lovász and Sós [20] who proved that any graphon that is a stepfunction is finitely forcible. This result was further extended and a systematic study of finitely graphons was initiated by Lovász and Szegedy [21] who found first examples of finitely forcible graphons that are not stepfunctions. In particular, they observed that every example of a finitely forcible graphon that they found had a somewhat finite structure. Formalizing this intuition, they associated typical vertices of a graphon $W$ with the topological space $T(W) \subseteq L_1[0,1]$ and observed that their examples of finitely forcible graphons $W$ have compact and at most 1-dimensional $T(W)$. This led them to make the following conjectures [21, Conjectures 9 and 10].

**Conjecture 1** (Lovász and Szegedy). If $W$ is a finitely forcible graphon, then $T(W)$ is a compact space.

They noted that they could not even prove that $T(W)$ had to be locally compact. We give a construction of a finitely forcible graphon $W$ such that $T(W)$ fails to be locally compact, in particular, $T(W)$ is not compact.

**Theorem 1.** There exists a finitely forcible graphon $W_R$ such that the topological space $T(W_R)$ is not locally compact.

They also made the following conjecture on the dimension of $T(W)$ of a finitely forcible graphon $W$.

**Conjecture 2** (Lovász and Szegedy). If $W$ is a finitely forcible graphon, then $T(W)$ is finite dimensional.

Lovász and Szegedy noted in their paper that they did not want to specify the notion of dimension they had in mind. Since every non-compact subset of $L_1[0,1]$ has infinite Minkowski dimension, Theorem 1 also provides a partial answer to Conjecture 2. Since we believe that other notions of dimension than the Minkowski dimension (e.g., the Lebesgue dimension) are more appropriate
measures of dimension for $T(W)$, we do not claim to disprove this conjecture in this paper. We discuss Conjecture 2 in more details in Section 6 where we also mention other applications of techniques used in this paper.

## 2 Notation

In this section, we introduce notation related to concepts used in this paper. A graph is a pair $(V, E)$ where $E \subseteq \binom{V}{2}$. The elements of $V$ are called vertices and the elements of $E$ are called edges. The order of a graph $G$ is the number of its vertices and it is denoted by $|G|$. The density $d(H, G)$ of $H$ in $G$ is the probability that $|H|$ randomly chosen distinct vertices of $G$ induce a subgraph isomorphic to $H$. If $|H| > |G|$, we set $d(H, G) = 0$. A sequence of graphs $(G_i)_{i \in \mathbb{N}}$ is convergent if the sequence $(d(H, G_i))_{i \in \mathbb{N}}$ converges for every graph $H$.

We now present basic notions from the theory of dense graph limits as developed in [6][8][22]. A graphon $W$ is a symmetric measurable function from $[0, 1]^2$ to $[0, 1]$. Here, symmetric stands for the property that $W(x, y) = W(y, x)$ for every $x, y \in [0, 1]$. A $W$-random graph of order $k$ is obtained by sampling $k$ random points $x_1, \ldots, x_k \in [0, 1]$ uniformly and independently and joining the $i$-th and the $j$-th vertex by an edge with probability $W(x_i, x_j)$. Since the points of $[0, 1]$ play the role of vertices, we refer to them as to vertices of $W$. To simplify our notation further, if $A \subseteq [0, 1]$ is measurable, we use $|A|$ for its measure. The density $d(H, W)$ of a graph $H$ in a graphon $W$ is equal to the probability that a $W$-random graph of order $|H|$ is isomorphic to $H$. Clearly, the following holds:

$$d(H, W) = \frac{|H|!}{|\text{Aut}(H)|} \int_{[0,1]^{|H|}} \prod_{(i,j) \in E(H)} W(x_i, x_j) \prod_{(i,j) \notin E(H)} (1 - W(x_i, x_j)) \, d\lambda_{|H|},$$

where $\text{Aut}(H)$ is the automorphism group of $H$. One of the key results in the theory of dense graph limits asserts that for every convergent sequence $(G_i)_{i \in \mathbb{N}}$ of graphs with increasing orders, there exists a graphon $W$ (called the limit of the sequence) such that for every graph $H$,

$$d(H, W) = \lim_{i \to \infty} d(H, G_i).$$

Conversely, if $W$ is a graphon, then the sequence of $W$-random graphs with increasing orders converges with probability one and its limit is $W$.

Every graphon can be assigned a topological space corresponding to its typical vertices [23]. For a graphon $W$, define for $x \in [0, 1]$ a function $f^W_x(y) = W(x, y)$. For an open set $A \subseteq L_1[0, 1]$, we write $A^W$ for $\{x \in [0, 1], f^W_x \in A\}$. Let $T(W)$ be the set formed by the functions $f \in L_1[0, 1]$ such that $|U^W| > 0$ for every neighborhood $U$ of $f$. The set $T(W)$ inherits topology from $L_1[0, 1]$. The vertices $x \in [0, 1]$ with $f^W_x \in T(W)$ are called typical vertices of a graphon $W$. Notice that almost every vertex is typical [23].
Two graphons $W_1$ and $W_2$ are weakly isomorphic if $d(H,W_1) = d(H,W_2)$ for every graph $H$. If $\varphi : [0,1] \to [0,1]$ is a measure preserving map, then the graphon $W^{\varphi}(x,y) := W(\varphi(x),\varphi(y))$ is always weakly isomorphic to $W$. The opposite is true in the following sense [5]: if two graphons $W_1$ and $W_2$ are weakly isomorphic, then there exist measure measure preserving maps $\varphi_1 : [0,1] \to [0,1]$ and $\varphi_2 : [0,1] \to [0,1]$ such that $W_1^{\varphi_1} = W_2^{\varphi_2}$ almost everywhere.

A graphon $W$ is finitely forcible if there exist graphs $H_1, \ldots, H_k$ such that every graphon $W'$ satisfying $d(H_i,W') = d(H_i,W)$ for $i \in \{1, \ldots, k\}$ is weakly isomorphic to $W$. For example, the result of Diaconis, Homes, and Janson [10] asserts that the half graphon $W_\Delta(x,y)$ defined as $W_\Delta(x,y) = 1$ if $x+y \geq 1$, and $W_\Delta = 0$, otherwise, is finitely forcible. Also see [21] for further results.

When dealing with a finitely forcible graphon, we usually give a set of equality constraints that uniquely determines $W$ instead of specifying the finitely many subgraphs that uniquely determine $W$. A constraint is an equality between two density expressions where a density expression is recursively defined as follows: a real number or a graph $H$ are density expressions, and if $D_1$ and $D_2$ are two density expression, then the sum $D_1 + D_2$ and the product $D_1 \cdot D_2$ are also density expressions. The value of the density expression is the value obtained by substituting for every subgraph $H$ its density in the graphon. Observe that if $W$ is a unique (up to weak isomorphism) graphon that satisfies a finite set $C$ of constraints, then it is finitely forcible. In particular, $W$ is the unique (up to weak isomorphism) graphon with densities of subgraphs appearing in $C$ equal to their densities in $W$. This holds since any graphon with these densities satisfies all constraints in $C$ and thus it must be weakly isomorphic to $W$.

We extend the notion of density expressions to rooted density expressions following the ideas from the concept of flag algebras from [28]. A subgraph is rooted if it has $m$ distinguished vertices labeled with numbers $1, \ldots, m$. These vertices are referred to as roots while the other vertices are non-roots. Two rooted graphs are compatible if the subgraphs induced by their roots are isomorphic through an isomorphism mapping the roots with the same label to each other. Similarly, two rooted graphs are isomorphic if there exists an isomorphism that maps the $i$-th root of one of them to the $i$-th root of the other.

A rooted density expression is a density expression such that all graphs that appear in it are mutually compatible rooted graphs. We will also speak about compatible rooted density expressions to emphasize that the rooted graphs in all of them are mutually compatible. The value of a rooted density expression is defined in the next paragraph.

Fix a rooted graph $H$. Let $H_0$ be the graph induced by the roots of $H$, and let $m = |H_0|$. For a graphon $W$ with $d(H_0,W) > 0$, we let the auxiliary function $c : [0,1]^m \to [0,1]$ denote the probability that an $m$-tuple $(x_1, \ldots, x_m) \in [0,1]^m$.
induces a copy of $H_0$ in $W$ respecting the labeling of vertices of $H_0$:

$$c(x_1, \ldots, x_m) = \left( \prod_{(i,j) \in E(H_0)} W(x_i, x_j) \right) \cdot \left( \prod_{(i,j) \notin E(H_0)} (1 - W(x_i, x_j)) \right).$$

We next define a probability measure $\mu$ on $[0, 1]^m$. If $A \subseteq [0, 1]^m$ is a Borel set, then:

$$\mu(A) = \frac{\int_A c(x_1, \ldots, x_m) d\lambda_m}{\int_{[0, 1]^m} c(x_1, \ldots, x_m) d\lambda_m}.$$

When $x_1, \ldots, x_m \in [0, 1]$ are fixed, then the density of a graph $H$ with root vertices $x_1, \ldots, x_m$ is the probability that a random sample of non-roots yields a copy of $H$ conditioned on the roots inducing $H_0$. Noticing that an automorphism of a rooted graph has all roots as fixed points, we obtain that this is equal to

$$\frac{|H| - m)!}{|\text{Aut}(H)|} \int_{[0, 1]^{|H| - m}} \prod_{(i,j) \in E(H) \setminus E(H_0)} W(x_i, x_j) \prod_{(i,j) \notin E(H) \cup \{h_0\}} (1 - W(x_i, x_j)) d\lambda_{|H| - m}.$$

For different choices of $x_1, \ldots, x_m$, we obtain different values. The value of a rooted density expression is a random variable determined by the choice of the roots according to the probability distribution $\mu$.

We now consider a constraint such that both left and right hand sides $D$ and $D'$ are compatible rooted density expression. Such a constraint should be interpreted to mean that it holds $D - D' = 0$ with probability one. It can be shown (see, e.g., [28]) that the expected value of a rooted density expression $D$ with roots inducing $H_0$ is equal to $[D] / d(H_0, W)$, where $[D]$ is an ordinary density expression independent of $W$. Observe that if $D$ and $D'$ are compatible rooted density expressions, then a graphon satisfies $D = D'$ if and only if it satisfies the (ordinary) constraint $[(D - D') \times (D - D')] = 0$. Since this allows us to express constraints involving rooted density expressions as ordinary constraints, we will not distinguish between the two types of constraints in what follows.

### 3 Partitioned graphons

In this section, we introduce partitioned graphons. Some of the methods presented in this section are analogous to those used by Lovász and Sós in [20] and Norine [25] (see the construction in Section 6). In particular, they used similar types of arguments to specialize their constraints to parts of graphons they were forcing as we do in this section. However, since it is hard to refer to any particular lemma in their paper instead of presenting a full argument, we decided to give all details.
A degree of a vertex $x \in [0, 1]$ of a graphon $W$ is equal to

$$
\int_{[0,1]} W(x, y) dy .
$$

Note that the degree is well-defined for almost every vertex of $W$. A graphon $W$ is partitioned if there exist $k \in \mathbb{N}$ and positive reals $a_1, \ldots, a_k$ with $\sum_i a_i = 1$ and distinct reals $d_1, \ldots, d_k \in [0, 1]$ such that the the set of vertices of $W$ with degree $d_i$ has measure $a_i$. We will often speak just about partitioned graphons when having in mind fixed values of $k, a_1, \ldots, a_k$, and $d_1, \ldots, d_k$. Having a fixed partition can be finitely forced as given in the next lemma.

**Lemma 2.** Let $a_1, \ldots, a_k$ be positive real numbers summing to one and let $d_1, \ldots, d_k$ be distinct reals between zero and one. There exists a finite set of constraints $C$ such that any graphon $W$ satisfying $C$ also satisfies the following:

The set of vertices of $W$ with degree $d_i$ has measure $a_i$. In other words, such $W$ must be a partitioned graphon with parts of sizes $a_1, \ldots, a_k$ and degrees $d_1, \ldots, d_k$.

**Proof.** The graphon is forced by the following set of constraints:

$$
\prod_{i=1}^k (e_1 - d_i) = 0 , \text{ and } \prod_{i=1, i \neq j}^k (e_1 - d_i) = a_j \prod_{i=1, i \neq j}^k (d_j - d_i) \text{ for every } j, 1 \leq j \leq k,
$$

where $e_1$ is an edge with one root and one non-root. The first constraint says that the degree of almost every vertex is equal to one of the numbers $d_1, \ldots, d_k$. For $j \leq k$, the left hand side of the second constraint before applying the $[\cdot]$-operator is non-zero only if the degree of the root is $d_j$. Hence, the left hand side is equal to

$$
\prod_{i=1, i \neq j}^k (d_j - d_i)
$$

in that case. Therefore, the measure of vertices of degree $d_j$ is forced to be $a_j$. \qed

Assume that $W$ is a partitioned graphon. We write $A_i$ for the set of vertices of degree $d_i$ for $i, 1 \leq i \leq k$ and identify $A_i$ with the interval $[0, a_i)$ (note that the measure of $A_i$ is $a_i$). This will be convenient when defining partitioned graphons. For example, we can use the following when defining a graphon $W$: $W(x, y) = 1$ if $x \in A_1, y \in A_2$ and $x \geq y$. 


A graph $H$ is decorated if its vertices are labeled with parts $A_1, \ldots, A_k$. The density of a decorated graph $H$ is the probability that randomly chosen $|H|$ vertices induce a subgraph isomorphic to $H$ with its vertices contained in the parts corresponding to the labels. For example, if $H$ is an edge with vertices decorated with parts $A_1$ and $A_2$, then the density of $H$ is the density of edges between $A_1$ and $A_2$, i.e.,

$$d(H,W) = \int \int_{A_1 A_2} W(x,y) \, dx \, dy .$$

Similarly as in the case of non-decorated graphs, we can define rooted decorated subgraphs. A constraint that uses (rooted or non-rooted) decorated subgraphs is referred to as decorated.

The next lemma shows that decorated constraints are not more powerful than non-decorated ones, and therefore they can be used to show that a graphon is finitely forcible. We will always apply this lemma after forcing a graphon $W$ to be partitioned using Lemma 2. Before proving it, we introduce convention for drawing density expressions: edges of graphs are always drawn solid, non-edges dashed, and if two vertices are not joined, then the picture represents the sum over both possibilities. If a graph contains some roots, the roots are depicted by square vertices, and the non-root vertices by circles. If there are more roots from the same part, then the squares are rotated to distinguish the roots. Decorations of vertices are always drawn inside vertices.

**Lemma 3.** Let $k \in \mathbb{N}$, and let $a_1, \ldots, a_k$ be positive real numbers summing to one and let $d_1, \ldots, d_k$ be distinct reals between zero and one. If $W$ is a partitioned graphon with $k$ parts formed by vertices of degree $d_i$ and measure $a_i$ each, then any decorated (rooted or non-rooted) constraint can be expressed as a non-decorated constraint, i.e., $W$ satisfies the decorated constraint if and only if it satisfies the non-decorated constraint.

**Proof.** By the argument analogous to the non-decorated case, it is enough to show that the density of a non-rooted decorated subgraph can be expressed as a combination of densities of non-decorated subgraphs. Let $H$ be a non-rooted

![Figure 1: The graph $H_1$ from the proof of Lemma 3 if $H$ is an edge with roots decorated with $A_2$ and $A_3$.](image-url)
decorated subgraph with vertices \(v_1, \ldots, v_n\) such that \(v_i\) is labeled with a part \(A_{\ell_i}\). Let \(H_i\) be the sum of all rooted non-decorated graphs on \(n + 1\) vertices with \(n\) roots such that the roots induce \(H\) with the \(j\)-th vertex being \(v_j\) for \(j = 1, \ldots, n\) and the only non-root is always adjacent to \(v_i\) (an example is given in Figure 1). We claim that the density of \(H\) is equal to the following:

\[
\frac{|H|!}{|\text{Aut}(H)|} \left[ \prod_{i=1}^{n} \prod_{j=1, j \neq \ell_i}^{k} \frac{H_i - d_j}{d_{\ell_i} - d_j} \right].
\]  

(1)

Indeed, if the \(n\) roots are chosen on a copy of \(H\) such that the \(i\)-th root is not from \(A_{\ell_i}\), then the second product of the above expression is zero. Otherwise, the second product is one, possibly except for a set of measure zero. Hence, the value of (1) is exactly the probability that randomly chosen \(n\) vertices induce a labeled copy of \(H\) such that the \(i\)-th vertex belong to \(A_{\ell_i}\). 

Since decorated constraints are not more powerful than non-decorated ones, we will not distinguish between decorated and non-decorated constraints in what follows.

We finish this section with two lemmas that are straightforward corollaries of Lemma 3. The first one says that we can finitely force a finitely forcible graphon on a part of a partitioned graphon.

**Lemma 4.** Let \(W_0\) be a finitely forcible graphon. Then for every choice of \(k \in \mathbb{N}\), positive reals \(a_1, \ldots, a_k\) summing to one, distinct reals \(d_1, \ldots, d_k\) between zero and one, and \(\ell \leq k\), there exists a finite set of constraints \(\mathcal{C}\) such that the graphon induced by the \(\ell\)-th part of every graphon \(W\) that is a partitioned graphon with \(k\) parts \(A_1, \ldots, A_k\) of measures \(a_1, \ldots, a_k\) and degrees \(d_1, \ldots, d_k\), respectively, and that satisfies \(\mathcal{C}\) is weakly isomorphic to \(W_0\). More precisely, there exist measure preserving maps \(\varphi\) and \(\varphi'\) from \(A_\ell\) to itself such that \(W_0(\varphi(x)/|A_\ell|, \varphi(y)/|A_\ell|) = W(\varphi'(x), \varphi'(y))\) for almost every \(x, y \in A_\ell\).

**Proof.** Assume that \(W_0\) is forced by constraints of the form

\[d(H_i, W) = d_i\]

for \(i \in [m]\). The set \(\mathcal{C}\) is then formed by constraints of the form

\[d(H_i', W) = a_{\ell}^{H_i}d_i,\]

where \(H_i'\) is the graph \(H_i\) with all vertices decorated with \(A_\ell\). 

The second lemma asserts finite forcibility of pseudorandom bipartite graphs between different parts of a partitioned graphon.
Lemma 5. For every choice of $k \in \mathbb{N}$, positive reals $a_1, \ldots, a_k$ summing to one, distinct reals $d_1, \ldots, d_k$ between zero and one, $\ell, \ell' \leq k$, $\ell \neq \ell'$, and $p \in [0, 1]$, there exists a finite set of constraints $C$ such that every graphon $W$ that is a partitioned graphon with $k$ parts $A_1, \ldots, A_k$ of measures $a_1, \ldots, a_k$ and degrees $d_1, \ldots, d_k$, respectively, and that satisfies $C$ also satisfies that $W(x, y) = p$ for almost every $x \in A_\ell$ and $y \in A_{\ell'}$.

Proof. Let $H$ be a rooted edge with the root decorated with $A_\ell$ and the non-root decorated with $A_{\ell'}$, let $H_1$ be a triangle with two roots such that the roots are decorated with $A_\ell$ and the non-root with $A_{\ell'}$, and let $H_2$ be a cherry (a path on three vertices) with two roots on its non-edge such that the roots are decorated with $A_\ell$ and the non-root with $A_{\ell'}$. The set $C$ is formed by three constraints: $H = p$, $H_1 = p^2$, and $H_2 = p^2$ (also see Figure 2). These constraints imply that

$$\int_{A_\ell} W(x, y) \, dy = a_{\ell'} p \quad \text{and} \quad \int_{A_{\ell'}} W(x, y) \cdot W(x', y) \, dy = a_{\ell'} p^2$$

for almost every $x, x' \in A_\ell$. Following the reasoning given in [21, proof of Lemma 3.3], the second equation implies that

$$\int_{A_{\ell'}} W^2(x, y) \, dy = a_{\ell'} p^2$$

for almost every $x \in A_\ell$. Cauchy-Schwarz inequality yields that $W(x, y) = p$ for almost every $x \in A_\ell$ and $y \in A_{\ell'}$. \hfill \Box

4 Rademacher Graphon

In this section, we introduce a graphon $W_R$ which we refer to as Rademacher graphon. The name comes from the fact that the adjacencies between its parts $A$ and $C$ resembles Rademacher system of functions (such adjacencies also appear in [19, Example 13.30]). We establish its finite forcibility in the next section.

The graphon $W_R$ has eight parts. Instead of using $A_1, \ldots, A_8$ for its parts, we use $A$, $A'$, $B$, $B'$, $B''$, $C$, $C'$ and $D$. All the parts except for $C$ have the same size $a = 1/9$; the size of $C$ is $2a = 2/9$. 

![Figure 2: The constraints used in the proof of Lemma 5.](image-url)
Figure 3: Rademacher graphon $W_R$. 
Table 1: The degrees of vertices in the nine parts of Rademacher graphon $W_R$.

For $x \in [0, 1)$, let us denote by $[x]$ the smallest integer $k$ such that $x + 2^{-k} < 1$. The graphon $W_R$ is then defined as follows (also see Figure 3). Let $x$ and $y$ be two of its vertices. The value $W_R(x, y)$ is equal to 1 in the following cases:

- $x, y \in A$ and $[x/a] \neq [y/a]$,
- $x, y \in A'$ and $[x/a] \neq [y/a]$,
- $x \in A$, $y \in A'$ and $[x/a] = [y/a]$,
- $x \in A$, $y \in B$ and $x + y \leq a$,
- $x \in A$, $y \in B''$ and $x + y \geq a$,
- $x \in A'$, $y \in B'$ and $x + y \leq a$,
- $x \in A'$, $y \in B''$ and $y \leq x$,
- $x, y \in B$ and $x + y \geq a$,
- $x, y \in B'$ and $x + y \geq a$,
- $x \in A$, $y \in C$ and $\lfloor \frac{y}{2a} \cdot 2^{[x/a]} \rfloor$ is even,
- $x \in A'$, $y \in C'$ and $1 - 2^{-[x/a]} - x/a)2^{[x/a]} + y/a \leq 1$,
- $x, y \in C'$ and $x + y \geq a$.

If $x, y \in C$ and $\lfloor \frac{y}{2a} \cdot 2^{[x/a]} \rfloor$ is even, then $W_R(x, y) = (1 - 2^{-[x/a]} - x/a)2^{[x/a]}$. If $x, y \in C$, then $W_R(x, y) = 3/4$ if $x + y \geq 2a$. If $y \in D$, then

$$W_R(x, y) = \begin{cases} 
0.2 & \text{if } x \in A' \text{ or } x \in B', \\
0.4 & \text{if } x \in B'', \text{ and} \\
0.8 & \text{if } x \in C'.
\end{cases}$$

Finally, $W_R(x, y) = 0$ if neither $(x, y)$ nor the symmetric pair fall in any of the described cases.

The degrees of vertices in the eight parts of Rademacher graphon $W_R$ are routine to compute and they are given in Table 1.

We finish this section with establishing that Rademacher graphon, assuming its finite forcibility, yields Theorem 1.
Proposition 6. The topological space $T(W_R)$ is not locally compact.

Proof. We understand the interval $[0, 1]$ to be partitioned by the intervals $A, A', B, B', B'', C, C'$ and $D$. Let $g : [0, 1] \to [0, 1]$ be the function defined as follows:

$$g(x) = \begin{cases} 1 & \text{if } x \in A' \cup B'' \cup C', \\ 0.2 & \text{if } x \in D, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Further, let $g_{i, \delta} : [0, 1] \to [0, 1]$ for $i \in \mathbb{N}$ and $\delta \in (0, 1)$ be defined as follows:

$$g_{i, \delta}(x) = \begin{cases} 1 & \text{if } x \in A \text{ and } \lfloor x/a \rfloor = i, \\ 1 & \text{if } x \in A' \text{ and } \lfloor x/a \rfloor \neq i, \\ 1 & \text{if } x \in B' \text{ and } x \leq (1 + \delta)2^{-i}, \\ 1 & \text{if } x \in B'' \text{ and } x \leq 1 - (1 + \delta)2^{-i}, \\ \delta & \text{if } x \in C \text{ and } \lfloor 2^i \cdot x/2a \rfloor \text{ is even}, \\ 1 & \text{if } x \in C' \text{ and } x/a \leq 1 - \delta, \\ 0.2 & \text{if } x \in D, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $W_R(x, 2/9 - (1 + \delta)2^{-i}) = g_{i, \delta}(x)$ for every $i \in \mathbb{N}$, $\delta \in (0, 1)$, and $x \in [0, 1]$. The following two estimates on the distances between $g$ and $g_{i, \delta}$ are straightforward to obtain:

$$\|g_{i, \delta} - g\|_1 = \frac{(4 + 2\delta)2^{-i} + 2\delta}{9}$$

$$\|g_{i, \delta} - g_{i', \delta'}\|_1 > \frac{\delta + \delta'}{18} \quad \text{for } i \neq i'.$$

Hence, since $g_{i, \delta} \in T(W_R)$ for every $i \in \mathbb{N}$ and $\delta \in (0, 1)$, we obtain that $g \in T(W_R)$. However, for every $\varepsilon > 0$, all the functions $g_{i, \varepsilon}$ with $i > \log_2 \varepsilon^{-1}$ are at $L_1$-distance at most $\varepsilon$ from $g$ and the $L_1$-distance between any pair of them is at least $\varepsilon/9$. We conclude that no neighborhood of $g$ in $T(W)$ is compact.

5 Forcing

In this section, we prove that Rademacher graphon $W_R$ is finitely forcible. We first describe the set of constraints. We give names to the different kinds of these constraints to refer to them later. The whole set of constraints is denoted by $\mathcal{C}_R$ in what follows.

- The partition constraints forcing the existence of eight parts of sizes as in $W_R$ and with vertex degrees as in $W_R$ (the existence of such constraints follows from Lemma 2),
The monotonicity constraints.

- the zero constraints setting the edge density inside $B''$ and $D$ to zero as well as setting the edge density between the following pairs of parts to zero: $A$ and $C'$, $A$ and $D$, $A'$ and $B$, $B$ and $B'$, $B$ and $B''$, $B$ and $C$, $B$ and $C'$, $B$ and $D$, $B'$ and $B''$, $B'$ and $C$, $B'$ and $C'$, $B''$ and $C$, $B''$ and $C'$, and $C$ and $D$.

- the triangular constraints forcing the half graphons on $B$, $B'$, $C$, and $C'$ with densities $1$, $1$, $1$ and $3/4$ (see Lemma 4 and [21, Corollaries 3.15 and 5.2] for their existence), respectively,

- the pseudorandom constraints forcing the pseudorandom bipartite graph between $D$ and the parts $A'$, $B'$, $B''$, and $C'$ with densities $0.2$, $0.2$, $0.4$, and $0.8$, respectively (see Lemma 5 for their existence),

- the monotonicity constraints depicted in Figure 4

- the split constraints depicted in Figure 5

- the infinitary constraints depicted in Figure 6 and

- the orthogonality constraints depicted in Figure 7

The existence of the corresponding monotonicity, split, infinitary, and orthogonality constraints as ordinary constraints follows from Lemma 3. Also note that the first five monotonicity constraints imply that the graphon has values zero and one almost everywhere between the corresponding parts (also see [21, Lemma 3.3] for further details).

**Theorem 7.** If $W$ is a graphon satisfying all constraints in $C_R$, then there exist measure preserving maps $\varphi, \psi : [0,1] \to [0,1]$ such that $W^\varphi$ and $W_R^\psi$ are equal almost everywhere.
\begin{figure}[h]
\centering
\begin{subfigure}{0.25\textwidth}
\centering
\begin{align*}
A + A' &= \frac{1}{9} \\
A' + B' &= \frac{1}{9} \\
A + A' &= \frac{1}{9} \\
A' + A' &= \frac{1}{9}
\end{align*}
\caption{(a) (b) (c) (d)}
\end{subfigure}
\begin{subfigure}{0.25\textwidth}
\centering
\begin{align*}
A' + A' &= 0 \\
A' - A' &= 0 \\
C + \frac{3}{2} &= \frac{1}{6}
\end{align*}
\caption{(e) (f) (h)}
\end{subfigure}
\caption{The split constraints.}
\end{figure}

\begin{figure}[h]
\centering
\begin{subfigure}{0.25\textwidth}
\centering
\begin{align*}
A' &= \frac{1}{243} \\
A &= \frac{1}{243}
\end{align*}
\caption{(a) (c)}
\end{subfigure}
\begin{subfigure}{0.25\textwidth}
\centering
\begin{align*}
A - B &= A - B
\end{align*}
\caption{(b)}
\end{subfigure}
\begin{subfigure}{0.25\textwidth}
\centering
\begin{align*}
A' &= \frac{1}{243} \\
A' &= \frac{1}{243}
\end{align*}
\caption{(c) (d)}
\end{subfigure}
\caption{The infinitary constraints.}
\end{figure}
Proof. Since $W$ satisfies the partition constraints contained in $C_R$, Lemma 2 yields that the interval $[0, 1]$ can be partitioned into eight parts all but one having measure $1/9$ and the remaining one with measure $2/9$ such that almost all vertices in the parts have degrees as those in the corresponding parts of $W_R$. In particular, there exists a measure preserving map $\varphi : [0, 1] \rightarrow [0, 1]$ such that the subintervals of $[0, 1]$ corresponding to the parts of $W_R$ are mapped to the corresponding parts of $W$. From now on, we use $A, A', B, B', B'', C, C', \text{ and } D$ for the subintervals of $[0, 1]$ corresponding to the parts.

We next construct a measure preserving map $\psi$ consisting of measure preserving maps on the intervals $A, A', B, B', B'', C$ and $C'$. We choose these maps such that there exist decreasing functions $f_A : A \rightarrow [0, 1]$ and $f_{A'} : A' \rightarrow [0, 1]$, and increasing functions $f_B : B \rightarrow [0, 1]$, $f_{B'} : B' \rightarrow [0, 1]$, $f_{B''} : B'' \rightarrow [0, 1]$, $f_C : C \rightarrow [0, 1]$ and $f_{C'} : C' \rightarrow [0, 1]$ such that the following holds almost everywhere (the existence of such maps and functions follows from Monotone Reordering Theorem):

$$
\forall x \in A \quad f_A(\psi(x)) = \int_B W^\varphi(x, y)dy \\
\forall x \in A' \quad f_{A'}(\psi(x)) = \int_{B'} W^\varphi(x, y)dy \\
\forall x \in B \quad f_B(\psi(x)) = \int_B W^\varphi(x, y)dy \\
\forall x \in B' \quad f_{B'}(\psi(x)) = \int_{B'} W^\varphi(x, y)dy \\
\forall x \in B'' \quad f_{B''}(\psi(x)) = \int_{B''} W^\varphi(x, y)dy \\
\forall x \in C \quad f_C(\psi(x)) = \int_C W^\varphi(x, y)dy \\
\forall x \in C' \quad f_{C'}(\psi(x)) = \int_{C'} W^\varphi(x, y)dy
$$
In the rest of the proof, we establish that $W^\varphi$ and $W^\psi_R$ are equal almost everywhere.

The pseudorandom and zero constraints in $C_R$ imply that $W^\varphi$ and $W^\psi_R$ agree almost everywhere on $D \times [0,1]$ and $[0,1] \times D$. The zero and triangular constraints and the choice of $\psi$ on $B$, $B'$, $C$, and $C'$ yield the same conclusion for $(B \cup B' \cup B'' \cup C \cup C')^2$, $A \times B'$, $A' \times B$, $B \times A'$, and $B' \times A$.

Let us now introduce some additional notation. If $x$ is a vertex and $Y$ is one of the parts, let $N_Y(x)$ denote the set of $y \in Y$ such that $W^\varphi(x,y) > 0$. If $x$ and $y$ belong to the same part, then we write $x \preceq y$ iff $\psi(x) \leq \psi(y)$. Observe that the monotonicity constraint (a) from Figure 3 and the choice of $\psi$ implies the existence of a set $Z$ of measure zero such that $N_B(x') \setminus N_B(x)$ has measure zero for $x, x' \in A \setminus Z$ if and only if $x \preceq x'$. Since the degree of every vertex in $B$ is $1/9$, this yields that the graphons $W^\varphi$ and $W^\psi_R$ agree almost everywhere on $A \times B$. The same reasoning applies to $A' \times B'$ and $(B \cup B') \times (A \cup A')$.

We now apply the same reasoning using the monotonicity constraint (b) and the split constraints (b) to deduce the existence of a zero measure set $Z$ such that $N_{B''}(x) \setminus N_{B'}(x')$ has measure zero if and only if $x \preceq x'$ for $x, x' \in A \setminus Z$. The monotonicity constraint also imply that $W^\varphi$ has only values zero and one almost everywhere on $A \times B''$. Since the measure of $N_B(x) \cup N_{B''}(x)$ is $1/9$ for almost all $x \in A$ by the split constraint (b), the choice of $\psi$ on $B''$ implies that the graphons $W^\varphi$ and $W^\psi_R$ agree almost everywhere on $A \times B''$. The degree regularity in $B''$, the split constraint (d), and the monotonicity constraint (d), which yields that $W^\varphi$ has values zero and one almost everywhere on $A' \times B''$, yield the agreement almost everywhere on $A' \times B''$. Symmetrically, they agree almost everywhere on $B'' \times (A \cup A')$.

We now focus on the graphon $W^\varphi$ on $A^2$. Observe first that the measure of $N_B(x)$ is equal to $\psi(x)$ for almost all $x \in A$. The monotonicity constraints (f) and (h) from Figure 4 imply that there exists a set $Z$ of measure zero such that every point $x \in A \setminus Z$ can be associated with a unique open interval $J_x \subseteq A$ such that $W^\varphi(x, x') = 0$ for almost every $x' \in \psi^{-1}(J_x)$, and $W^\varphi(x, x') = 1$ for almost every $x' \in A \setminus \psi^{-1}(J_x)$. The interval $J_x$ can be empty for some choice of $x$. Recall that $|J_x|$ is the measure of the interval $J_x$, and let $\mathcal{J}$ be the set of all intervals $J_x$, $x \in A$, with $|J_x| > 0$. Since the intervals in $\mathcal{J}$ are disjoint, the set $\mathcal{J}$ is equipped with a natural linear order.

Let us now focus on the infinitary constraint (b) from Figure 6. Fix three vertices (two from $A$ and one from $B$) as in the figure and let $x$ be the left vertex from $A$. Observe that if $x \in A$ is fixed, then the set of choices of the other two vertices has non-zero measure unless $\psi(x) = \sup J_x$. The left hand side of the constraint is equal to the measure of $J_x$, i.e., $\sup J_x - \inf J_x$. The right hand side is equal to $1/9 - \sup J_x$. We conclude that $\inf J_x = 1/9 - 2|J_x|$. This implies that the set $\mathcal{J}$ is well-ordered and countable.
Let us write $J_k$ for the $k$-th interval contained in $\mathcal{J}$. Furthermore, for $k \geq 1$, define
\[
\beta_k = \frac{2(1 - 9 \inf J_{k+1})}{1 - 9 \inf J_k} = \frac{2|J_{k+1}|}{|J_k|},
\]
and let $\beta_0$ be equal to $1 - 9 \inf J_1$. Note that by the observations made in the last paragraph and since $\inf J_{k+1} \geq \sup J_k$, we obtain $\beta_k \leq 1$ for every $k \geq 0$. In case that $\mathcal{J}$ is finite, we define $\beta_k = 0$ for $k > |\mathcal{J}|$. We can now express the density of non-edges with both end-vertices in $A$ as
\[
\sum_{J \in \mathcal{J}} |J|^2 = \sum_{k=1}^{\infty} \left( \frac{1}{9} \cdot \prod_{k'=0}^{k-1} \beta_{k'} \right)^2.
\]
Since the sum is forced to be $1/243$ by the infinitary constraint (a), we get that $\beta_k = 1$ for every $k$. This implies that for every $k$, $J_k = \left(\frac{1 - 2^{-k+1}}{9}, \frac{1 - 2^{-k}}{9}\right)$. In particular, the graphons $W^\varphi$ and $W^\psi_R$ agree almost everywhere on $A^2$.

The same reasoning as for $A^2$ yields that the graphons $W^\varphi$ and $W^\psi_R$ agree almost everywhere on $A^2$. Let $\mathcal{J}'$ be the corresponding set of intervals for $A'$ and let $J'_1, J'_2, \ldots$ be their ordering. The split constraints (e) and (f) from Figure 5 imply that for almost every $x \in A$ with $|N_{A'}(x)| > 0$, there exists $J' \in \mathcal{J}'$ such that $N_{A'}(x) \Delta \psi^{-1}(J')$ has measure zero and $W^\varphi(x, y) = 1$ for almost every $y \in \psi^{-1}(J')$.

Let $x \in \psi^{-1}(J_k)$. The split constraint (a) from Figure 5 yields that $|N_{A'}(x)| = \frac{1}{2^{k-2}}$. Consequently, $N_{A'}(x) \Delta \psi^{-1}(J'_k)$ has measure zero for almost every $x \in \psi^{-1}(J_k)$ and $W(x, x') = 1$ for almost every $x \in \psi^{-1}(J_k)$ and $x' \in \psi^{-1}(J'_k)$. We conclude that the graphons $W^\varphi$ and $W^\psi_R$ agree almost everywhere on $A \times A'$ and $A' \times A$.

The orthogonality constraints (a) and (b) from Figure 7 yield that there exist measurable subsets $I_k \subseteq C$ with $|I_k| = 1/9$ for every $k \geq 1$ such that it holds for almost every $x \in \psi^{-1}(J_k)$ that $N_C(x)$ differs from $I_k$ on a set of measure zero and $W^\varphi(x, y) = 1$ for almost every $y \in I_k$. The construction of $\psi$ and the split constraint (h) from Figure 5 imply that $|N_{A}(x)| = 1/9 - \psi(x)/2$ for almost every $x \in C$. Since $\psi^{-1}(J_1) \setminus N_{A}(x)$ has measure zero for almost every $x \in I_1$, we get that $|J_1| \leq |N_{A}(x)|$ for almost every $x \in I_1$. This implies that $I_1$ and $\psi^{-1}([0, 1/9])$ differ on a set of measure zero (also see Figure 8). Since $\psi^{-1}(J_2) \setminus N_{A}(x)$ has measure zero for almost every $x \in I_2$ and $J_1 \cap J_2$ has measure zero, we get that $|J_1| + |J_2| \leq |N_{A}(x)|$ for almost every $x \in I_1 \cap I_2$ and that $|J_2| \leq |N_{A}(x)|$ for almost every $x \in I_2 \setminus I_1$. This implies that $I_2$ and $\psi^{-1}([0, 1/18] \cup [1/9, 1/6])$ differ on a set of measure zero. Iterating the argument, we obtain that $I_k$ differs from the preimage with respect to $\psi$ of the set
\[
\bigcup_{i=1}^{2^{k-1}} \left[ \frac{2i - 2}{9 \cdot 2^{k-1}}, \frac{2i - 1}{9 \cdot 2^{k-1}} \right].
\]
Figure 8: Illustration of the argument used in the proof of Theorem 7 to establish that the graphons $W^\varphi$ and $W_\psi^R$ agree almost everywhere on $A \times C$.

on a set of measure zero for every $k \in \mathbb{N}$. This yields that the graphons $W^\varphi$ and $W_\psi^R$ agree almost everywhere on $A \times C$.

The orthogonality constraint (c) from Figure 7 implies that $(C \setminus N_C(x)) \cap N_C(x')$ has measure zero for every $k$, almost every $x \in A \setminus \psi^{-1}(J_k)$, and almost every $x' \in \psi^{-1}(J'_k)$. In particular, almost every $x' \in \psi^{-1}(J'_k)$ satisfies that $N_C(x') \setminus I_k$ has measure zero, i.e., $W^\varphi(x', y) = 0$ for almost every $x' \in \psi^{-1}(J'_k)$ and $y \notin I_k$.

We now interpret the orthogonality constraint (d) from Figure 7. Fix an integer $k \geq 1$ and a typical vertex $x' \in \psi^{-1}(J'_k)$. The left term in the product on the left hand side of the constraint is equal to the square of $\int_C W^\varphi(x', y)dy = \int I_k W^\varphi(x', y)dy$. The right term in the product is equal to the square of $|J'_k| = 2^{-\frac{\psi(x')}{a}}/9$. The term on the right hand side is equal to the probability that randomly chosen $x''$ and $y$ satisfy $x'' \in A'$, $y \in B'$, $x'' \in \psi^{-1}(J'_k)$, and $\psi(x') \leq \psi(y) < \psi(x'')$. This is equal to

$$\frac{(1 - 2^{-\frac{\psi(x')}{a}} - \psi(x')/a)^2}{2 \cdot 9^2}.$$  

We deduce that almost every $x' \in \psi^{-1}(J'_k)$ satisfies

$$\int_{I_k} W^\varphi(x', y)dy = \frac{1 - 2^{-\frac{\psi(x')}{a}} - \psi(x')/a}{9 \cdot 2^{-\frac{\psi(x')}{a}}}.$$  

(2)

We apply the same reasoning to the orthogonality constraint (e) from Figure 7 and deduce that almost every pair of vertices $x', x'' \in \psi^{-1}(J'_k)$ satisfies

$$\frac{9^2}{4} \cdot \left( \int_{I_k} W^\varphi(x', y)W^\varphi(x'', y)dy \right)^2 \cdot \left( 2^{-\frac{\psi(x')}{a}} \right)^4 = \frac{(1 - 2^{-\frac{\psi(x')}{a}} - \psi(x')/a)^2}{2} \cdot \left( 1 - 2^{-\frac{\psi(x'')}{a}} - \psi(x'')/a \right)^2.$$  

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This implies (similarly as in the proof of Lemma 5) that almost every \( x' \in \psi^{-1}(J'_k) \) satisfies:

\[
\left( \int_{I_k} W^\varphi(x', y)^2 \, dy \right)^{1/2} = \frac{1 - 2^{-[\psi(x')/a]} - \psi(x')/a}{3 \cdot 2^{-[\psi(x')/a]}}.
\] (3)

Using Cauchy-Schwartz Inequality, we deduce from (2) and (3) (recall that \(|I_k| = 1/9\)) that the following holds for almost every \( x' \in \psi^{-1}(J'_k) \) and \( y \in I_k, \)

\[
W^\varphi(x', y) = \frac{1 - 2^{-[\psi(x')/a]} - \psi(x')/a}{2^{-[\psi(x')/a]}}.
\]

In other words, \( W^\varphi(x', y) \) is constant almost everywhere on \( I_k \) for almost every \( x' \in \psi^{-1}(J'_k) \) and its value linearly decreases from one to zero almost everywhere inside \( \psi^{-1}(J'_k) \). Hence, the graphons \( W^\varphi \) and \( W^\psi_R \) agree almost everywhere on \( A' \times C \) and \( C \times A' \) (recall that \( W^\varphi(x', y) = 0 \) for almost every pair \( x' \in \psi^{-1}(J'_k) \) and \( y \notin I_k \)).

The monotonicity constraint (e) from Figure 4 yields that at least one of the sets \( N_{C'}(x) \setminus N_{C'}(x') \) or \( N_{C'}(x') \setminus N_{C'}(x) \) has measure zero for every \( k \) and almost every pair \( x, x' \in A' \) and the graphon \( W^\varphi \) has values zero and one almost everywhere on \( A' \times C'' \). This and the regularity on \( A' \) imply that the graphons \( W^\varphi \) and \( W^\psi_R \) agree almost everywhere on \( A' \times C'' \). Since the graphon \( W^\varphi \) is zero almost everywhere on \( A \times C'' \) by one of the zero constraints, we have shown that the graphons \( W^\varphi \) and \( W^\psi_R \) agree almost everywhere on \( (A \cup A') \times (C \cup C'') \) and \( (C \cup C'') \times (A \cup A') \). Since these were the last subsets of their domains that remained to be analyzed, we proved that the graphon \( W^\varphi \) is equal to \( W^\psi_R \) almost everywhere.

Theorem 7 immediately yields the following.

**Corollary 8.** The graphon \( W_R \) is finitely forcible.

### 6 Conclusion

It is quite clear that the construction of Rademacher graphon can be modified to yield other graphons \( W \) with non-compact \( T(W) \). Some of these modifications can yield such graphons with a smaller number of parts at the expense of making the argument that the graphon is finitely forcible less transparent.

In [21], Lovász and Szegedy considered finite forcibility inside two classes of functions. Conjecture 4, which we addressed in this paper, relates to the class they refer to as \( \mathcal{W}_0 \). This class consists of symmetric measurable functions from \([0, 1]^2 \) to \([0, 1]\). A larger class referred to as \( \mathcal{W} \) in [21] is the class containing all symmetric measurable functions from \([0, 1]^2 \) to \( \mathbb{R} \). It is not hard to see that Rademacher graphon \( W_R \) is also finitely forcible inside this larger class. Also
note that stronger constraints involving multigraphs were used in [21] but we have used only constraints involving simple graphs in this paper.

In [19], an analogue of the space $T(W)$ with respect to the following metric is also considered. If $f, g \in L^1[0, 1]$, then

$$d_W(f, g) := \int_0^1 \left| \int_0^1 W(x, y)(f(y) - g(y))dy \right| dx.$$ 

However, the appropriate closure of $T(W)$ always form a compact space [19, Corollary 13.28].

As mentioned in Section 1, Rademacher graphon $W_R$ also provides a partial answer to [21, Conjecture 10] in the sense that the Minkowski dimension of $T(W_R)$ is infinite. However, the dimension is finite when several other notions of dimension are considered. For instance, its Lebesgue dimension is only one.

In [11], the first two authors and Klímošová disprove Conjecture 2 in a more convincing way: they construct a finitely forcible graphon $W$ such that a subspace of $T(W)$ is homeomorphic to $[0, 1]^\infty$. The construction is also based on partitioned graphons used in this paper.

We finish by presenting a construction of a finitely forcible graphon $W_d$ with a part of $T(W_d)$ positive measure isomorphic to $[0, 1]^d$; the construction is analogous to one found earlier by Norine [25]. Fix a positive integer $d$. We construct a graphon $W_d$ with $2d + 2$ parts $A, B_1, \ldots, B_{2d}$, and $C$, each of size $(2d + 2)^{-1}$. If $x, y \in B_i$, then $W_d(x, y) = 1$ if $x + y \geq (2d + 2)^{-1}$, i.e., $W_d$ is the half graphon on each $B_i^2$. If $x \in B_i$ and $y \in C$, then $W_d(x, y) = W_d(y, x) = i/4d$. Fix now a measure preserving map $\varphi$ from $[0, 1]$ to $[0, 1]^d$. If $x \in A$ and $y \in B_i$, $i \leq d$, then $W_d(x, y) = W_d(y, x) = 1$ if $\varphi((2d + 2)x)_i \geq (2d + 2)y$. Finally, if $x \in A$ and $y \in B_i$, $i \geq d + 1$, then $W_d(x, y) = W_d(y, x) = 1$ if $1 - \varphi((2d + 2)x)_i \geq (2d + 2)y$. The graphon $W_d$ is equal to zero for other pairs of vertices. Clearly, $W_d$ is a partitioned graphon with $2d + 2$ parts with vertices inside each part having the same degree and vertices in different parts having different degrees. Using the techniques presented in this paper and generalizing arguments from [18], one can show that $W_d$ is finitely forcible. Since the subspace of $T(W_d)$ formed by typical vertices from $A$ is homeomorphic to $[0, 1]^d$, the Lebesgue dimension of $T(W_d)$ is at least $d$. This shows that finitely forcible graphons can have arbitrary large finite dimension.

**Acknowledgements**

The authors would like to thank Jan Hladký, Tereza Klímošová, Serguei Norine, and Vojtěch Tůma for their comments on the topics discussed in the paper.
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