On the empty balls of a critical super-Brownian motion

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Abstract

Let \( \{X_t\}_{t \geq 0} \) be a \( d \)-dimensional critical super-Brownian motion started from a Poisson random measure whose intensity is the Lebesgue measure. Denote by \( R_t := \sup\{u > 0 : X_t(\{x \in \mathbb{R}^d : |x| < u\}) = 0\} \) the radius of the largest empty ball centered at the origin of \( X_t \). In this work, we prove that for \( r > 0 \),

\[
\lim_{t \to \infty} \mathbb{P}\left( \frac{R_t}{(1/d)^{1/(3-d)}} \geq r \right) = e^{-A_d(r)},
\]

where \( A_d(r) \) satisfies \( \lim_{r \to \infty} \frac{A_d(r)}{r^{d-2} |d|_{[d-2]}} = C \) for some \( C \in (0, \infty) \) depending only on \( d \).

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1. Introduction and Main results

1.1. Introduction

In this work, we consider a \( d \)-dimensional measure-valued Markov process \( \{X_t\}_{t \geq 0} \), called super-Brownian motion (henceforth SBM). For convenience of the reader, we give a brief introduction to the SBM and some pertinent results needed in this article.

To characterize the SBM, we first introduce some notations. Let \( p > d \) be a constant. Define \( \phi_p(x) := (1 + |x|^2)^{-p/2}, \ x \in \mathbb{R}^d \). Denote by

\[
M_p := \left\{ \mu \text{ is a locally finite measure on } \mathbb{R}^d : \int_{\mathbb{R}^d} \phi_p(x) \mu(dx) < \infty \right\}
\]

the space of \( p \)-tempered measures; see Etheridge [5, p23]. We equip \( M_p \) with the topology such that \( \mu_n \) converges to \( \mu \) in \( M_p \) if and only if

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} f(x) \mu_n(dx) = \int_{\mathbb{R}^d} f(x) \mu(dx), \ \forall f \in \{ g + \alpha \phi_p : \alpha \in \mathbb{R}, g \in C_c(\mathbb{R}^d) \},
\]
where $C_c(\mathbb{R}^d)$ stands for the class of continuous functions of compact support in $\mathbb{R}^d$. In this paper, for a measure $\mu$, we always use $\mathbb{E}_\mu$ to denote the expectation with respect to $\mathbb{P}_\mu$, the probability measure under which the SBM has initial value $X_0 = \mu$. Let $\psi$ be a function of the form

$$\psi(u) = au + bu^2 + \int_{(0,\infty)} (e^{-ru} - 1 + ru) n(dr), \quad u \geq 0,$$

where $a \in \mathbb{R}$, $b \geq 0$ and $n$ is a $\sigma$-finite measure on $(0, \infty)$ such that $\int_{(0,\infty)} (r \wedge r^2) n(dr) < \infty$.

The SBM with initial value $\mu \in M_\rho$ and branching mechanism $\psi$ is a measure-valued process, whose transition probabilities are characterized through their Laplace transforms. For any $\mu \in M_\rho$ and nonnegative continuous function $\phi$ satisfying $\sup_{x \in \mathbb{R}^d} \frac{\phi(x)}{\phi_p(x)} < \infty$, we have

$$\mathbb{E}_\mu \left[ e^{-<X_t,\phi>} \right] = e^{-\int_{\mathbb{R}^d} u(t,x) \mu(dx)}, \tag{1.1}$$

where $<X_t, \phi> := \int_{\mathbb{R}^d} \phi(x) X_t(dx)$ and $u(t, x)$ is the unique positive solution to the following nonlinear partial differential equation:

$$\begin{cases}
\frac{\partial u(t,x)}{\partial t} = \frac{1}{2} \Delta u(t, x) - \psi(u(t, x)), \\
u(0, x) = \phi(x).
\end{cases}$$

In above, $\Delta u(t, x) := \sum_{i=1}^d \frac{\partial^2 u(t,x)}{\partial x_i^2}$ is the Laplace operator. $\{X_t\}_{t \geq 0}$ is called a supercritical (critical, subcritical) SBM if $a < 0$ ($= 0$, $> 0$). Our works only consider a typical critical branching mechanism, called binary branching, which is given by

$$\psi(u) = u^2.$$

In this case, the partial differential equation above is reduced to

$$\begin{cases}
\frac{\partial u(t,x)}{\partial t} = \frac{1}{2} \Delta u(t, x) - u^2(t, x), \\
u(0, x) = \phi(x).
\end{cases} \tag{1.2}$$

Let $\{W_t\}_{t \geq 0}$ be a $d$-dimensional standard Brownian motion. Note that the partial differential equation above is equivalent to the integral equation:

$$u(t, x) + \int_0^t \mathbb{E}_x \left[ u^2(t - s, W_s) \right] ds = \mathbb{E}_x \left[ \phi(W_t) \right], \tag{1.3}$$

where for $x \in \mathbb{R}^d$, $\mathbb{E}_x$ stands for the expectation with respect to the probability $\mathbb{P}_x$, the probability measure under which $\{W_t\}_{t \geq 0}$ starts from $x$. Furthermore, one can also use the martingale problem to characterize the SBM; see Perkins [16, p159]. We refer the reader to Etheridge [5], Perkins [16], Le Gall [11] and Li [13] for a more detailed overview to SBM.

In our work, we consider the SBM starts from the Poisson random measure whose intensity is the Lebeguse measure $\lambda$ on $\mathbb{R}^d$ (PRM($\lambda$) for short). Namely, for any Borel measurable set $A \subset \mathbb{R}^d$,

$$\mathbb{P}_{\text{PRM}(\lambda)}(X_0(A) = k) = \frac{\lambda^k(A)}{k!} e^{-\lambda(A)}, \quad k \geq 0.$$

For ease of notation, we write $\mathbb{P} := \mathbb{P}_{\text{PRM}(\lambda)}$. 
For $u > 0$, let $B(u) := \{x \in \mathbb{R}^d : |x| < u\}$ be the $d$-dimensional ball with radius $u$ and center at the origin. Write

$$R_t := \sup\{u > 0 : X_t(B(u)) = 0\},$$

with the convention $\sup \emptyset = 0$. In other words, $R_t$ is the radius of the largest ball around the origin which does not contain any mass at time $t$ and $B(R(t))$ is the largest empty ball.

This paper aims at showing that after suitable renormalization, $R_t$ converges in distribution to some non-degenerate limit as $t \to \infty$. We note that the renormalization scale depends on the dimension.

The research on the empty ball was first conducted by Révész [19] for the critical branching Wiener process model started from $PRM(\lambda)$. This model, denoted by $\{Z_n\}_{n \geq 0}$, is defined as follows. At time 0, there exist infinite many particles distributed according to $PRM(\lambda)$. Then, these particles move independently according to the standard normal distribution in unit time. Afterwards, each particle produces children independently according to the Bernoulli distribution $\xi$ in an instant, where $P(\xi = 0) = P(\xi = 1) = 1/2$. This forms a random measure at time 1, denoted by $Z_1$. Similarly, for $n \geq 2$, each particle at time $n - 1$, starting from where its parent die, executes a displacement according to the standard normal distribution during time $n - 1$ to time $n$ and afterwards executes a reproduction instantly according to $\xi$. This forms a random measure at time $n$, denoted by $Z_n$.

Let

$$R(n) := \sup\{u > 0 : Z_n(B(u)) = 0\}. \quad (1.4)$$

Révész [19] proved that $R(n)/n$ converges in distribution to an exponential distribution for the case $d = 1$. For $d \geq 2$, he presented following two conjectures (see Révész [19, Conjecture 1]):

(i) If $d = 2$, then for any $r \in (0, \infty)$,

$$\lim_{n \to \infty} \mathbb{P}\left( \frac{R(n)}{\sqrt{n}} \geq r \right) = e^{-F_2(r)} \in (0, 1),$$

where $F_2(r)$ satisfies

$$\lim_{r \to \infty} \frac{F_2(r)}{\pi r^2} = 1;$$

(ii) If $d \geq 3$, then for any $r \in (0, \infty)$,

$$\lim_{n \to \infty} \mathbb{P}(R(n) \geq r) = e^{-F_d(r)} \in (0, 1), \quad (1.5)$$

where $F_d(r)$ satisfies

$$\lim_{n \to \infty} \frac{F_d(r)}{C_d r^{d-2}} = 1, \quad (1.6)$$

and $\lim_{d \to \infty} C_d / \left[\frac{\pi^{d/2}}{\Gamma(d/2 + 1)}\right]^{(d-2)/d} = 1$ ($\Gamma(\cdot)$ is the Gamma function). Later, Hu [8] partially confirmed Révész’s conjecture for $d \geq 3$ by showing that $\lim_{n \to \infty} \mathbb{P}(R(n) \geq r)$ exists in $(0, 1)$. But (1.6) remains unproven.

Our work gives complete weak convergence results of $R_t$, and these results are consistent with Révész’s conjectures. In $d = 1$, we use the modulus of continuity for SBM and Markov property of historical SBM to show that $R_t/t$ converges in law. For $d = 2$, by using the scaling property of
SBM, we obtain \( R_t / \sqrt{t} \) converges in law. For \( d \geq 3 \), we use the mild solution of the PDE (1.2) and Feynman-Kac formula to prove that \( R_t \) converges in law. Moreover, in our recent study, we believe that our results for the SBM model can facilitate us to solve Révész’s conjectures, especially in the case of \( d = 2 \).

A relevant work to our problem is Zhou [21]. Let \( \{ K_t \}_{t \geq 0} \) be a \( d \)-dimensional \((1 + \beta)\)-SBM and \( \tau := \sup\{ t \geq 0 : K_t(B(g(t))) > 0 \} \) be the local extinction time, where \( \beta \in (0, 1] \) and \( g(t) \geq 0 \) is a nondecreasing and right continuous function on \([0, \infty)\). Assume that \( d \beta < 2 \). Zhou proved that

\[
\mathbb{P}_\lambda(\tau < \infty) = \begin{cases} 1, & \text{if } \int_1^\infty g^d(y)y^{-(1+\frac{\beta}{2})}dy < \infty; \\ 0, & \text{otherwise.} \end{cases}
\]

This result implies that in our setting, if \( d = 1 \) and \( X_0 = \lambda \), then the leading order of \( R_t \) is \( t \) as \( t \to \infty \). Conversely, by studying \( R_t \) in the case of \( d \geq 2 \), one can also obtain some results of the local extinction time of \((1 + \beta)\)-SBM in the case of \( d \beta \geq 2 \).

We also mention that in the last few decades, limit theory of SBMs concerning its local behaviours has been studied intensively. For example, Iscoe [9] studied the decay rate of the hitting probabilities \( \mathbb{P}_{\delta_i}(\exists t > 0, X_t(B(r)) > 0) \) as \( |x| \to \infty \). Dawson et al. [2] considered decay rates of the probabilities

\[
\mathbb{P}_m(X_t(B(x, r)) > 0), \quad \mathbb{P}_m(\exists t \geq \delta \text{ s.t. } X_t(B(x, r)) > 0)
\]

and

\[
\mathbb{P}_m(\exists t \geq 0 \text{ s.t. } X_t(B(x, r)) > 0) \text{ as } r \to 0,
\]

where \( \delta, r > 0, x \in \mathbb{R}^d, B(x, r) := \{ y \in \mathbb{R}^d : |y - x| \leq r \} \) and \( m \) is a finite measure on \( \mathbb{R}^d \). Namely, the probabilities above consider the SBM hits an arbitrarily small ball. Note that

\[
\mathbb{P}(R_t \geq r) = 1 - \mathbb{P}(X_t(B(r)) > 0).
\]

Their works can also give some results for \( R_t \). However, since they assumed \( X_0 = m \) is a finite measure, the SBM will die out in finite time. This implies \( R_t = \infty \) for large \( t \). Therefore, to make this question meaningful, we consider the SBM starts from an infinite measure PRM(\( \lambda \)). For the maximum of supercritical SBM, see Kyprianou et al. [10] and Pinsky [18]. Moreover, Ren et al. [17] and Engländer [4] considered the corresponding large deviation probabilities. In addition, Mueller et al. [15] investigated the left tail probability of the density of the SBM. For the local times of SBM, see Sugitani [20], Hong [7, 6] and Dawson et al. [3] and the references therein.

### 1.2. Main Results

We first consider the 1-dimensional super Brownian motion.

**Theorem 1.1.** If \( d = 1 \), then for any \( r \in (0, \infty) \),

\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{R_t}{t} \geq r \right) = e^{-2r}.
\]

Intuitively, the reason for an empty ball \( B(u) \) (\( u > 0 \) and may depend on \( t \)) to form is that an SBM starting away from \( B(u) \) is hard to reach \( B(u) \), while the SBM starting from the set \( B(u) \) of finite measure will die out in finite time. Since in higher dimensions there are more ”particles” in \( B(u) \), the SBM starting from \( B(u) \) will take a longer time to die out. Thus, in higher dimensions, \( R_t \) is smaller.
Theorem 1.2. If \( d = 2 \), then for any \( r \in (0, \infty) \),
\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{R_t}{\sqrt{t}} \geq r \right) = e^{-A_2(r)} \in (0, 1),
\]
where \( A_2(r) := -\log \mathbb{P}_\lambda(X_t(B(r)) = 0) \) satisfying
\[
\lim_{r \to \infty} \frac{A_2(r)}{\pi r^2} = 1.
\]

Remark 1.1. Recall that \( R(n) \) is defined in (1.4). Assume that \( d = 2 \). In [19, Theorem 2], Révész proved that for any \( \varepsilon > 0 \) there exist \( 0 < c(\varepsilon) < C(\varepsilon) < \infty \) such that
\[
c(\varepsilon) e^{-2\pi(1+\varepsilon)r^2} < \liminf_{n \to \infty} \mathbb{P} \left( \frac{R(n)}{\sqrt{n}} \geq r \right) \leq \limsup_{n \to \infty} \mathbb{P} \left( \frac{R(n)}{\sqrt{n}} \geq r \right) \leq C(\varepsilon) e^{-2\pi(1-\varepsilon)r^2}.
\]

From above two theorems, one can see that \( R_t \to \infty \) for \( 1 \leq d < 3 \). This is because the SBM suffers local extinction in low dimensions. However, since in high dimensions \( (d \geq 3) \), the SBM is persistent (see [5, p49]), we don’t need any renormalization for \( R_t \).

Theorem 1.3. If \( d \geq 3 \), then for any \( r \in (0, \infty) \),
\[
\lim_{t \to \infty} \mathbb{P}(R_t \geq r) = e^{-\kappa_d r^{d-2}},
\]
where \( \kappa_d := -\lim_{t \to \infty} \int_{\mathbb{R}^d} \log \mathbb{P}_\delta(X_t(B(1)) = 0) dx \in (0, \infty) \).

Remark 1.2. For \( d \geq 3 \), Révész [19, Theorem 3] proved that there exist constants \( 0 < c_d < C_d < \infty \) such that
\[
e^{-c_d r^{d-2}} \leq \liminf_{n \to \infty} \mathbb{P}(R(n) > r) \leq \limsup_{n \to \infty} \mathbb{P}(R(n) > r) \leq e^{-C_d r^{d-2}}.
\]

Later, Hu [8] proved that \( \lim_{n \to \infty} \mathbb{P}(R(n) > r) \) exists. However, Hu did not give any explicit expression of the limit.

Remark 1.3. Although in all theorems above we consider \( X_0 = \text{PRM}(\lambda) \), it is still true if \( X_0 \) is the Lebesgue measure on \( \mathbb{R}^d \). Moreover, there exists a constant \( T > 0 \) such that \( \mathbb{P}_\lambda(R_t/t \geq r) \) is decreasing (w.r.t. \( t \)) when \( d = 1 \) and \( t \geq T \); \( \mathbb{P}_\lambda(R_t/\sqrt{t} \geq r) = \mathbb{P}_\lambda(R_1 \geq r) \) for \( d = 2 \) and \( t > 0 \); \( \mathbb{P}_\lambda(R_t \geq r) \) is increasing when \( d \geq 3 \) and \( t \geq T \).

The rest of this paper is organized as follows. In Section 2 we collect some properties of the total mass process \( \{X_t(\mathbb{R}^d)\}_{t \geq 0} \) and the historical super-Brownian motion. In Section 3 we use the modulus of continuity for historical SBM to prove that for \( d = 1 \),
\[
\lim_{t \to \infty} \mathbb{P}_\lambda(X_t(B(rt)) = 0) = e^{-2r}.
\]
Then, by (1.3), we argue that, under \( \mathbb{P}_\lambda \) and \( \mathbb{P}_{\text{PRM}(\lambda)} \), \( R_t/t \) have the same limit (in the sense of convergence in distribution). Theorem 1.1 is then proved. In Section 4 we prove Theorem 1.2 by the scaling property of the 2-dimensional SBM. We prove Theorem 1.3 in Section 5. The idea is to use the mild solution of the partial differential equation (1.2) to establish the existence of the limit. Then, using the second moment method and [2, Lemma 3.2], we conclude that the limit is non-degenerate.
2. Preliminaries

The total mass process \( \{X_t(\mathbb{R}^d)\}_{t \geq 0} \) with initial measure \( m \in M_p \) is the so-called continuous state branching process with initial value \( m(\mathbb{R}^d) \). We refer the reader to \([12]\) for a more detailed overview to it. The following lemma considers the Laplace transform and extinction probability of the continuous state branching process, which can be found in \([5, \text{p}22]\).

**Lemma 2.1.** Let \( \theta > 0, d \geq 1 \). Then

\[
\mathbb{E}_m \left[ e^{-\theta X_t(\mathbb{R}^d)} \right] = e^{-\frac{\theta m(\mathbb{R}^d)}{\theta + d}}
\]

and

\[
\mathbb{P}_m(X_t(\mathbb{R}^d) = 0) = e^{-\frac{\theta m(\mathbb{R}^d)}{\theta + d}},
\]

where by convention, if \( m(\mathbb{R}^d) = \infty \), then

\[
\mathbb{E}_m \left[ e^{-\theta X_t(\mathbb{R}^d)} \right] = 0.
\]

In the remaining of this section, we always assume \( d = 1 \). We first give a brief introduction to the historical SBM. For an explicit definition and a more elaborated discussion on it, we refer the reader to \([16, \text{p}187]\). Let \( D(\mathbb{R}_+) \) be the space of càdlàg paths from \( \mathbb{R}_+ \) to \( \mathbb{R} \) with the Skorokhod topology. For \( y \in D(\mathbb{R}_+) \) and \( t \geq 0 \), let \( y_t' = y_{A_t} \). Define the stopped function space

\[ \hat{\mathcal{E}} := \{(t, y_t') : t \geq 0, \ y \in D(\mathbb{R}_+)\} \]

with the subspace topology it inherits from \( \mathbb{R}_+ \times D(\mathbb{R}_+) \). Then \( \hat{\mathcal{E}} \) is a Polish space. Define an \( \hat{\mathcal{E}} \)-valued process \( \{\hat{W}_t\}_{t \geq 0} \):

\[ \hat{W}_t := (t, W_{A_t}). \]

Let \( D(\mathbb{R}_+, \hat{\mathcal{E}}) \) be the space of càdlàg paths from \( \mathbb{R}_+ \) to \( \hat{\mathcal{E}} \). For \( x \in \mathbb{R} \) and \( A \in \mathcal{B}(D(\mathbb{R}_+, \hat{\mathcal{E}})) \), we define

\[ \hat{\mathcal{P}}_x(\hat{W}_t \in A) := \mathbb{P}_x((t, W) \in A). \]

Then \( \{(\hat{W}_t)_{t \geq 0}, (\hat{\mathcal{P}}_x)_{x \in \mathbb{R}}\} \) is an \( \hat{\mathcal{E}} \)-valued Borel strong Markov process. For a space \( E \), we use \( M_F(E) \) to denote the space of finite measures on \( E \). Thus, for \( m \in M_F(\mathbb{R}) \), we can construct an \( M_F(\hat{\mathcal{E}}) \)-valued superprocess \( \{Y_t\}_{t \geq 0} \) with spatial motion \( \{\hat{W}_t\}_{t \geq 0} \), binary branching and initial value \( m \) (by identifying \( \mathbb{R} = \{(0, y'_{0}) : y \in D(\mathbb{R}_+)\} \subset \hat{\mathcal{E}} \), so \( M_F(\mathbb{R}) \subset M_F(\hat{\mathcal{E}}) \)). Let \( Q_m \) be the corresponding probability measure. The historical SBM \( \{H_t\}_{t \geq 0} \) (with respect to the original SBM \( \{X_t\}_{t \geq 0} \) with initial value \( m \)) is defined by

\[ H_t(A) := Y_t(\Pi^{-1}(A)), A \subset D_t(\mathbb{R}_+), \]

where \( D_t(\mathbb{R}_+) := \{y \in D(\mathbb{R}_+) : y = y_t'\} \) and \( \Pi((t, y')) = y_t' \) is the projection map from \( \hat{\mathcal{E}} \) to \( D(\mathbb{R}_+) \). Moreover, \( \{X_t\}_{t \geq 0} \) can be obtained through

\[ X_t(A) := H_t(\{y : y \in D_t(\mathbb{R}_+) : y_t \in A\}) \text{ for } A \in \mathcal{B}(\mathbb{R}). \]

Let \( S(H_t) \) be the closed support of the random measure \( H_t \) and \( C(\mathbb{R}_+) \) be the space of continuous functions from \( \mathbb{R}_+ \) to \( \mathbb{R} \). The following lemma gives a uniform modulus of continuity for all the paths in \( S(H_t), t \geq 0; \) see \([16, \text{p}195]\).
Lemma 2.2. Let $c > 2$ be a constant. There exists a random variable $\Delta$ such that $Q_m$ almost surely, for all $t \geq 0$,

$$S(H_t) \subset \{ y. \in C(\mathbb{R}_+) : |y_r - y_s| \leq c |r - s| \log(r - s)|^{1/2}, \forall r, s > 0, |r - s| \leq \Delta \}.$$ 

Moreover, there are constants $\rho > 0$ depending only on $c$ and $\kappa > 0$ depending only on $d, c$ such that

$$Q_m(\Delta \leq r) \leq \kappa m(1) r^\rho \text{ for } r \in [0, 1].$$

For $t \geq 0$, define

$$M'_f(D(\mathbb{R}_+)) := \{ \mu \in M_f(D(\mathbb{R}_+)) : \mu(\{ y. \in D(\mathbb{R}_+) : y \neq y. \}) = 0 \}.$$ 

In fact, $\{ H_t \}_{t \geq 0}$ is an inhomogeneous Borel strong Markov process. Furthermore, $H_t \in M'_f(D(\mathbb{R}_+))$. For $\tau \geq 0$ and $\mu \in M'_f(D(\mathbb{R}_+))$, write $Q_{\tau,\mu}(H_{\tau+t} \in A) := Q_m(H_{\tau+t} \in A | H_{\tau} = \mu)$ for measurable set $A \subset M'_f(D(\mathbb{R}_+))$. In other words, under $Q_{\tau,\mu}$, the historical SBM starts at time $\tau$ with its initial value $\mu$. The following lemma is borrowed from [16, p194].

Lemma 2.3. If $A$ is a Borel subset of $D(\mathbb{R}_+)$, and $\mu \in M'_f(D(\mathbb{R}_+))$, then for any $t > \tau$,

$$Q_{\tau,\mu}(H_{\tau}(y. \in D(\mathbb{R}_+) : y. \in A)) = 0, \forall s \geq t = e^{-\frac{2m(A)}{r^2}}.$$

3. Proof of Theorem [1.1]

This section is devoted to prove Theorem [1.1]. We first present a proposition concerning the empty ball of SBM starting from Lebesgue measure $\lambda$. The proof is mainly inspired by Zhou [21, Lemma 2.2]. As usual, for a measure $\mu$ on some space $E$, we write $\mu(1) := \mu(E)$ for convenience.

Proposition 3.1. If $d = 1$, then for any $r > 0$,

$$\lim_{t \to \infty} P_{\lambda}(X_t(B(rt)) = 0) = e^{-2r}.$$ 

Proof. For a Borel measurable set $A \subset \mathbb{R}$, we denote by $\lambda|A := \lambda(\cdot \cap A)$ the Lebesgue measure restricted to $A$. Fix $\delta \in (1/2, 1)$. Observe that

$$P_{\lambda}(X_t(B(rt)) > 0) \leq P_{\lambda|B(rt + r^\delta)}(X_t(B(rt)) > 0) + P_{\lambda|B(rt + r^\delta)}(X_t(B(rt)) > 0).$$

For the first term on the r.h.s. of (3.1), by Lemma 2.1

$$P_{\lambda|B(rt + r^\delta)}(X_t(B(rt)) > 0) \leq P_{\lambda|B(rt + r^\delta)}(X_t(1) > 0)$$

$$= 1 - \exp \left\{ -\frac{1}{t} \lambda(B(rt + r^\delta)) \right\}$$

$$= 1 - \exp \left\{ -\frac{2}{r}(rt + r^\delta) \right\}. \quad (3.2)$$

Thus,

$$\lim_{t \to \infty} \sup P_{\lambda|B(rt + r^\delta)}(X_t(B(rt)) > 0) \leq 1 - e^{-2r}. \quad (3.3)$$
Next, we use the continuity modulus of historical SBM to prove that the second term on the r.h.s. of (3.1) tends to 0. For \( j = 0, 1, \ldots \), let

\[
\lambda_j := \lambda \left( B(rt + 1 - t^\delta + t^\delta(j+1)) - B(rt + 1 - t^\delta(j)) \right).
\]

So, \( \lambda |B'(rt + t^\delta) = \sum_{j \geq 0} \lambda_j \). Recall that under \( Q_m \), \( \{H_t\}_{t \geq 0} \) stands for the historical super-Brownian of \( \{X_t\}_{t \geq 0} \) starting from measure \( m \in \mathcal{M}_f(\mathbb{R}) \). It is simple to see

\[
\mathbb{P}_{\lambda |B'(rt + t^\delta)} (X_t(B(rt)) > 0) \leq \sum_{j \geq 0} \mathbb{P}_{\lambda_j} (\exists s \in [0, t], X_s(B(rt)) > 0) \\
\leq \sum_{j \geq 0} Q_{\lambda_j} (\exists s \in [0, t], H_s(A(s, rt)) > 0),
\]

where

\[
A(u, v) := \{y \in C(\mathbb{R}_+) : \inf_{s \leq u} |y_s| \leq v\} \text{ for } u, v \geq 0.
\]

Fix \( \delta \in (0, 2\delta - 1) \). For \( t > 1 \), let \( l_j := \left| t e^{\delta(j+1)} \right| \), \( j \geq 0 \). Define \( \mathcal{F}_t := \sigma(H_s, s \in [0, t]) \). By the Markov property of \( \{H_t\}_{t \geq 0} \) and Lemma [2.3]

\[
\begin{align*}
\mathbb{Q}_\lambda \left[ H_{(i+1)/l_j}(A(it/l_j, rt + 1)) = 0 \right] &= \mathbb{Q}_\lambda \left[ \mathbb{Q}_{it/l_j, H_{it/l_j}} \left[ H_{(i+1)/l_j}(A(it/l_j, rt + 1)) = 0 | \mathcal{F}_{it/l_j} \right] \right] \\
&= \mathbb{Q}_\lambda \left[ \mathbb{Q}_{it/l_j, H_{it/l_j}} \left[ H_{(i+1)/l_j} \left\{ y \in C(\mathbb{R}_+) : y^{it/l_j} \in A(it/l_j, rt + 1) \right\} \right] = 0 | \mathcal{F}_{it/l_j} \right] \\
&= \mathbb{Q}_\lambda \left[ \exp \left\{ -2H_{it/l_j} \left( A(it/l_j, rt + 1) \right) \right\} \right].
\end{align*}
\]

Thus, for any \( 1 \leq i \leq l_j - 1 \), we have

\[
\begin{align*}
\mathbb{Q}_\lambda \left[ H_{(i+1)/l_j}(A(it/l_j, rt + 1)) > 0 \right] &= 1 - \mathbb{Q}_\lambda \left[ \exp \left\{ -2t/l_j H_{it/l_j} \left( A(it/l_j, rt + 1) \right) \right\} \right] \\
&\leq \frac{2l_j}{t} \mathbb{Q}_\lambda \left[ H_{it/l_j}(A(it/l_j, rt + 1)) \right] \\
&= \frac{2l_j}{t} \int_{\mathbb{R}} \mathbb{Q}_\delta \left[ H_{it/l_j}(A(it/l_j, rt + 1)) \right] \lambda_j(dx) \\
&= \frac{2l_j}{t} \int_{\mathbb{R}} \mathbb{P}_x \left( \inf_{s \leq t/l_j} |W_s| \leq rt + 1 \right) \lambda_j(dx) \\
&\leq \frac{2l_j}{t} \int_{\mathbb{R}} \mathbb{P}_0 \left( \sup_{s \leq t/l_j} |W_s| > rt^\delta - rt^\delta + 1 \right) \lambda_j(dx)
\end{align*}
\]

\[
\text{(3.6)}
\]


where the third equality follows from the fact that the mean measure of $H_r$ is the Wiener measure stopped at time $t$ (i.e. one moment formula of $H_r$; see [16, p191, II.8.5]). It is simple to see that for any $t > 4$ (thus $t^\delta > 2$) and $j \geq 0$,

$$
\mathbb{P}_0 \left( \sup_{s \leq t/l_j} |W_s| > t^\delta - 2 + t^\delta j \right)
= \mathbb{P}_0 \left( \sup_{s \leq t/l_j} W_s > t^\delta - 2 + t^\delta j \text{ or } \inf_{s \leq t/l_j} W_s > t^\delta - 2 + t^\delta j \right)
\leq 2 \mathbb{P}_0 \left( \sup_{s \leq t/l_j} W_s > t^\delta - 2 + t^\delta j \right)
= 2 \mathbb{P}_0 \left( \left| W_{t/l_j} \right| > t^\delta - 2 + t^\delta j \right)
= 4 \mathbb{P}_0 \left( \frac{W_{t/l_j}}{\sqrt{t/l_j}} > \frac{t^\delta - 2 + t^\delta j}{\sqrt{t/l_j}} \right)
= 4 \mathbb{P}_0 \left( \frac{W_{t/l_j}}{\sqrt{t/l_j}} > \left(\delta(j^{(j+1)}) - \frac{1}{2}\right)/2 \right)
\leq \frac{8}{\sqrt{2\pi}} e^{-t^2 (\delta(j^{(j+1)}) - \frac{1}{2})^2} \exp \left\{ - \frac{t^2 (\delta(j^{(j+1)}) - \frac{1}{2})^2}{8} \right\},
$$

(3.7)

where the second equality follows from the reflection property of the Brownian motion (see [14, p45]) and the last inequality comes from the following classical estimate for standard normal random variable $X$:

$$
\mathbb{P}(X > x) \leq \frac{1}{x \sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x > 0.
$$

Plugging (3.7) into (3.6) yields that there exists $C_1 > 0$ depending only on $r$, $\delta$ and $\bar{\delta}$ such that for any $t > C_1$ and $j \geq 0$,

$$
\mathbb{Q}_{\lambda_j} \left( H_{(i+1)t/l_j}(A((i+1)t/l_j, rt + 1)) > 0 \right)
\leq \frac{16l_j}{t \sqrt{2\pi}} \lambda_j(1) t^{-\delta(j^{(j+1)} - \frac{1}{2})} \exp \left\{ - \frac{t^2 (\delta(j^{(j+1)}) - \frac{1}{2})^2}{8} \right\}
\leq \frac{16 \lambda_j(1)}{\sqrt{2\pi}} 3(r + 1) t^{-\delta(j^{(j+1)} + 1)} \exp \left\{ - \frac{t^2 (\delta(j^{(j+1)}) - \frac{1}{2})^2}{8} \right\}
\leq \exp \left\{ - \frac{t^2 (\delta(j^{(j+1)}) - \frac{1}{2})^2}{9} \right\}.
$$

(3.8)

Fix $j \geq 0$. Consider $\{X_i\}_{i \geq 0}$ starts from $\lambda_j$. Write

$$
T_j := \inf\{s \geq 0 : X_s(B(rt)) > 0\}
$$

with the convention that $\inf 0 = +\infty$. Since under the event $\{\exists s \in [0, t], H_r(A(s, rt)) > 0\}$, we have $0 < T_j \leq t$, there exists an integer $i \in [0, 1, \ldots, l_j - 1]$ such that $T_j \in (it/l_j, (i + 1)t/l_j)$. For $i \geq 1$, if the SBM $\{X_i\}_{i \geq 0}$ has charged the set $B(rt + 1)$ at time $(i - 1)t/l_j$ then $H_{(i+1)t/l_j}(A((i+1)t/l_j, rt + 1)) > 0$. Otherwise, it has not charge $B(rt + 1)$, then the support process for $\{X_i\}_{i \geq 0}$ has to travel a distance of at least 1 on time interval $[(i-1)t/l_j, T_j] \subset [(i-1)t/l_j, (i+1)t/l_j]$, which implies $\Delta < 2t/l_j$ (use the modulus continuity and the fact $2t/l_j \to 0$). For $i = 0$, since

$$
X_0 = \lambda_j = \lambda \left( B(rt + t^\delta - 1 + t^\delta(j+1)) - B(rt + t^\delta - 1 + t^\delta j) \right),
$$


the support process for \( \{X_t\}_{t \geq 0} \) has to travel a distance of at least \( t^j - 2 + t^j \) on time interval \([0, t/l_j]\), which also implies \( \Delta < 2t/l_j \). Putting these together, there exists \( C_2 > 0 \) depending only

\[ \mathbb{Q}_{l_j}(\exists s \in [0, t], H_s(A(s, rt)) > 0) \]

\[ \leq \sum_{i=1}^{l_j-1} \mathbb{Q}_{l_j}[H_{i(l+1)/l_j}(A(it/l_j, rt + 1)) > 0] + \mathbb{Q}_{l_j}(\Delta \leq 2t/l_j) \]

\[ \leq l_j \exp \left\{ -\frac{t^{2(j/1)-1}}{9} + \lambda_j(1)\kappa 2\rho \exp \{-\rho t^{\hat{d}(j/1)}\} \right\} \]

\[ \leq t e^{\rho t^{\hat{d}(j/1)}} \exp \left\{ -\frac{t^{2(j/1)-1}}{9} \right\} + 3(r + 1)t^{2\hat{d}(j/1)+1} \kappa 2\rho \exp \{-\rho t^{\hat{d}(j/1)}\} \]

\[ \leq \exp \left\{ -\rho t^{\hat{d}(j/1)/2} \right\}, \quad (3.9) \]

where the second inequality follows from \((3.8)\) and Lemma \(2.2\). Plugging above into \((3.4)\) yields that

\[ \mathbb{P}_{d,B^c(\sqrt{rt+\rho})}(X_t(B(\sqrt{rt})) > 0) \leq \sum_{j \geq 2} \exp \left\{ -\rho t^{\hat{d}j/2} \right\} + 2 \exp \left\{ -\rho t^{\hat{d}}/2 \right\} \]

\[ \leq \int_1^\infty \exp \left\{ -\rho t^{\hat{d}x} \right\} dx + 2 \exp \left\{ -\rho t^{\hat{d}}/2 \right\} \]

\[ = \frac{1}{\delta \log t} \int_{\rho t^{\hat{d}}/2}^\infty e^{-u} du + 2 \exp \left\{ -\rho t^{\hat{d}}/2 \right\} \]

\[ \leq \left[ \frac{2}{\rho t^{\hat{d}} \delta \log t} + 2 \right] \exp \left\{ -\rho t^{\hat{d}}/2 \right\}. \quad (3.10) \]

Thus,

\[ \lim_{t \to \infty} e^{\rho t^{\hat{d}/3} \mathbb{P}_{d,B^c(\sqrt{rt+\rho})}(X_t(B(\sqrt{rt})) > 0) = 0. \quad (3.11) \]

This, combined with \((3.1)\) and \((3.3)\), gives

\[ \liminf_{t \to \infty} \mathbb{P}_{d}(X_t(B(\sqrt{rt})) = 0) \geq e^{-2r}. \quad (3.12) \]

Next, we deal with the upper bound. By the first moment formula of SBM (see \([11, p38]\)), we have for any \( \varepsilon \in (0, r) \),

\[ \mathbb{E}_{d,B^c(\sqrt{rt+\rho})} \left[ \frac{1}{t} X_t(B^c(\sqrt{rt})) \right] = \frac{1}{t} \int_{|x| \leq rt+\rho} \mathbb{P}_0(|x + W_t| > rt)dx \]

\[ = \frac{1}{t} \int_{|x| \leq (r-\varepsilon)t} \mathbb{P}_0(|W_t| > \varepsilon t)dx + \frac{1}{t} \int_{(r-\varepsilon)t < |x| \leq rt+\rho} 1dx \]

\[ \leq 2\mathbb{P}_0(|W_t| > \varepsilon t)(r-\varepsilon) + 2(\varepsilon t + \rho^2)/t. \]

By the law of large numbers,

\[ \limsup_{t \to \infty} \mathbb{E}_{d,B^c(\sqrt{rt+\rho})} \left[ \frac{1}{t} X_t(B^c(\sqrt{rt})) \right] \leq 2\varepsilon. \]
Letting $\varepsilon \to 0$ yields

$$\lim_{t \to \infty} \mathbb{E}_{\lambda(B(rt + \rho^p))} \left[ \frac{1}{t} X_t(B'(rt)) \right] = 0. \quad (3.13)$$

By the branching property of SBM, for any $\eta > 0$,

$$\lim_{t \to \infty} \mathbb{E}_{\lambda} \left[ e^{-\eta t^{-1} X_t(B(rt))} \right] = \lim_{t \to \infty} \mathbb{E}_{\lambda(B'(rt + \rho^p))} \left[ e^{-\eta t^{-1} X_t(B(rt))} \right] = \lim_{t \to \infty} \mathbb{E}_{\lambda[B(rt + \rho^p)]} \left[ e^{-\eta t^{-1} X_t(B(rt))} \right],$$

where the last inequality is because by (3.11),

$$\lim_{t \to \infty} t^{-1} X_t(B(rt)) = 0, \quad \mathbb{P}_{\lambda(B'(rt + \rho^p))}-\text{in distribution.}$$

On the other hand, from (3.13), \( \frac{1}{t} X_t(B'(rt)) \) converges in distribution to 0 (under $\mathbb{P}_{\lambda(B(rt + \rho^p))}$). Furthermore, by Lemma 2.1, we have \( t^{-1} X_t(1) \) converges in distribution (under $\mathbb{P}_{\lambda(B(rt + \rho^p))}$). Therefore, under $\mathbb{P}_{\lambda(B(rt + \rho^p))}$, \( t^{-1} X_t(1) - t^{-1} X_t(B'(rt)) \) converges in distribution. Hence,

$$\lim_{t \to \infty} \mathbb{E}_{\lambda} \left[ e^{-\eta t^{-1} X_t(B(rt))} \right] = \lim_{t \to \infty} \mathbb{E}_{\lambda[B(rt + \rho^p)]} \left[ e^{-\eta t^{-1} X_t(B(rt))} \right] \leq \lim_{t \to \infty} \mathbb{E}_{\lambda} \left[ e^{-\eta t^{-1} X_t(B(rt))} \right] = e^{-\frac{2\eta}{\lambda}}. \quad (3.14)$$

Thus, for any $\eta > 0$

$$\lim_{t \to \infty} \sup_{\lambda} \mathbb{P}_{\lambda}(X_t(B(rt)) = 0) = \lim_{t \to \infty} \sup_{\lambda} \mathbb{P}_{\lambda}(t^{-1} X_t(B(rt)) = 0) \leq \lim_{t \to \infty} \mathbb{E}_{\lambda} \left[ e^{-\eta t^{-1} X_t(B(rt))} \right] = e^{-\frac{2\eta}{\lambda}}. \quad (3.15)$$

Finally,

$$\lim_{t \to \infty} \sup_{\lambda} \mathbb{P}_{\lambda}(X_t(B(rt)) = 0) \leq \lim_{\eta \to \infty} e^{-\frac{2\eta}{\lambda}} \leq e^{-2r}.$$

This, combined with (3.12), concludes the proposition. \( \square \)

Now, we are ready to present the proof of Theorem 1.1. Namely, we are going to prove

$$\lim_{t \to \infty} \mathbb{P} \left( \frac{R_t}{t} \geq r \right) = e^{-2r} \text{ for } r > 0.$$

The idea of the proof is to use the integral equation (1.5) to argue that $\mathbb{P} \left( \frac{R_t}{t} \geq r \right)$ and $\mathbb{P}_{\lambda} \left( \frac{R_t}{t} \geq r \right)$ have the same asymptotics. We then use Proposition 3.1 to conclude the theorem.
Proof of Theorem 1.1} For a point measure $F$, write $u \in F$ if $u$ is an atom of $F$. Recall that $X_0 = \sum_{u \in X_0} \delta_u$ is a Poisson random measure with intensity measure $\lambda$. Let $\{X^u_t\}_{t \geq 0}$ be the SBM started from $\delta_u$ (i.e. a single particle at position $u$). By the branching property,

$$
P \left( \frac{R_t}{t} \geq r \right) = P \left( X_t(B(tr)) = 0 \right) = P \left( \forall u \in X_0, X^u_t(B(tr)) = 0 \right) = E \left[ \prod_{u \in X_0} P_{\delta_u} \left( X_t(B(tr)) = 0 \right) \right] = E \left[ e^{\sum_{u \in X_0} \log P_{\delta_u} \left( X_t(B(tr)) = 0 \right)} \right] = e^{-\int_0^t \log P_{\delta_u} \left( X_t(B(tr)) = 0 \right) d\lambda(u)} = e^{-\int_0^t \log P_{\delta_u} \left( X_t(B(tr)) = 0 \right) dx},$$

where the last equality follows from the Laplace functional formula of Poisson random measures (see [1], p19, (2.17)).

For $\theta > 0$, let

$$u_\theta(t, x) := -\log E_{\delta_x} \left[ e^{-<X, \theta 1_{B(t)}>} \right], \quad t \geq 0, \quad x \in \mathbb{R}.$$

Since $u_\theta(t, x)$ is increasing w.r.t. $\theta$, $u(t, x) := \lim_{\theta \to \infty} u_\theta(t, x)$ exists. Therefore,

$$P_{\delta_x} \left( X_t(B(tr)) = 0 \right) = \lim_{\theta \to \infty} E_{\delta_x} \left[ e^{-<X, \theta 1_{B(t)}>} \right] = \lim_{\theta \to \infty} e^{-u_\theta(t, x)} = e^{-u(t, x)}.$$

From (1.3), we have

$$u_\theta(t, x) + \int_0^t E_x \left[ u_\theta^2(t - s, W_s) \right] ds = E_x \left[ \theta 1_{B(t)}(W_t) \right].$$

Integrating w.r.t. $x$ gives

$$\int_{\mathbb{R}} u_\theta(t, x) dx + \int_0^t \int_{\mathbb{R}} u_\theta^2(t - s, y) dy ds = 2r\theta t.$$

Set

$$G_\theta(t) := \int_{\mathbb{R}} u_\theta(t, x) dx = 2r\theta t - \int_0^t \int_{\mathbb{R}} u_\theta^2(s, y) dy ds.$$

From Lemma 2.1, for any $\theta, \ t > 0$,

$$u_\theta(t, x) < \frac{1}{t}.$$

By (1.1),

$$P_{\lambda} \left( X_t(B(tr)) = 0 \right) = \lim_{\theta \to \infty} E_{\lambda} \left[ e^{-<X, \theta 1_{B(t)}>} \right] = \lim_{\theta \to \infty} e^{-\int_0^t u_\theta(t, x) dx} = e^{-\int_0^t u(t, x) dx},$$

(3.19)
where the last inequality follows from Lévy’s monotone convergence theorem. Thus, by Proposition 3.1 there exists some $C_3 > 1$ depending only on $r$ such that for $t > C_3$ and $\theta > 0$,

$$\int_{\mathbb{R}} u_{\theta}(t, x) dx \leq \int_{\mathbb{R}} u(t, x) dx < 3r.$$ 

Then, for $\theta > \frac{1}{2}$ and $t > C_3$,

$$G'_{\theta}(t) = 2r\theta - \int_{\mathbb{R}} u_{\theta}^2(t, y) dy \geq 2r\theta - \frac{1}{t} \int_{\mathbb{R}} u_{\theta}(t, y) dy \geq 2r\theta - \int_{\mathbb{R}} u(t, y) dy \geq 2r\theta - 3r > 0,$$

where the first inequality follows from (3.18). Thus, for any $t_2 > t_1 > C_3$ and $\theta > \frac{3}{2}$,

$$\int_{\mathbb{R}} u_{\theta}(t_1, x) dx < \int_{\mathbb{R}} u_{\theta}(t_2, x) dx.$$ 

Since $\int_{\mathbb{R}} u(t, x) dx = \lim_{\theta \to \infty} \int_{\mathbb{R}} u_{\theta}(t, x) dx$, we have

$$\int_{\mathbb{R}} u(t_1, x) dx \leq \int_{\mathbb{R}} u(t_2, x) dx.$$ 

So $\int_{\mathbb{R}} u(t, x) dx$ is increasing on $(C_3, \infty)$. Thus, by Proposition 3.1

$$\lim_{t \to \infty} e^{-\frac{1}{t} \int_{\mathbb{R}} u_{\theta}(t, x) dx} = \lim_{t \to \infty} \mathbb{P}(X_{t}(B(tr)) = 0) \downarrow e^{-2r}.$$ 

(3.21)

Since $x - \frac{x^2}{2} < 1 - e^{-x} < x$ for $x > 0$, we have

$$\int_{\mathbb{R}} u(t, x) dx \geq \int_{\mathbb{R}} 1 - e^{-u(t,x)} dx \geq \int_{\mathbb{R}} u(t, x) - \frac{u^2(t, x)}{2} dx \geq \left(1 - \frac{1}{2t}\right) \int_{\mathbb{R}} u(t, x) dx.$$ 

(3.22)

where the last inequality uses the fact that $u(t, x) = \lim_{\theta \to \infty} u_{\theta}(t, x) \leq \frac{1}{t}$ (see 3.18). This, combined with (3.21), yields that

$$\lim_{t \to \infty} \int_{\mathbb{R}} 1 - e^{-u(t,x)} dx = \lim_{t \to \infty} \int_{\mathbb{R}} u(t, x) dx = 2r.$$ 

(3.23)

So, by (3.16) and (3.17),

$$\lim_{t \to \infty} \mathbb{P}\left(\frac{R_t}{t} \geq r\right) = \lim_{t \to \infty} e^{-\frac{1}{t} \int_{\mathbb{R}} 1 - e^{-u(t,x)} dx} = e^{-2r}.$$ 

We have completed the proof of Theorem 1.1. □
4. Proof of Theorem 1.2

In this section, we are going to prove Theorem 1.2. Namely, if \( d = 2 \), then for any \( r \in (0, \infty) \),
\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{R_t}{\sqrt{t}} \geq r \right) = e^{-A_2(r)} \in (0, 1),
\]
where \( A_2(r) = -\log \mathbb{P}_\lambda(X_1(B(r)) = 0) \) satisfying
\[
\lim_{r \to \infty} \frac{A_2(r)}{\pi r^2} = 1.
\]
The idea of the proof can be divided into 4 steps:

**Step 1.** By the scaling property, we have
\[
\mathbb{P}_\lambda \left( X_t \left( B(\sqrt{tr}) \right) = 0 \right) = \mathbb{P}_\lambda(X_1(B(r)) = 0) \text{ for } t > 0;
\]

**Step 2.** Use the Feynman-Kac representation to give the desired lower bound of \( \mathbb{P}_\lambda(X_1(B(r)) = 0) \);

**Step 3.** From the observation
\[
\mathbb{P}_\lambda(X_1(B(r)) = 0) \leq \mathbb{P}_\lambda|B_r(X_1(B(r)) = 0)
\]
and the extinction probability of \( \{X_t(1)\}_{t \geq 0} \), we get the desired upper bound of \( \mathbb{P}_\lambda(X_1(B(r)) = 0) \);

**Step 4.** Similar to the proof of Theorem 1.1, we obtain
\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{R_t}{\sqrt{t}} \geq r \right) = \lim_{t \to \infty} \mathbb{P}_\lambda \left( X_t \left( B(\sqrt{tr}) \right) = 0 \right).
\]

**Proof of Theorem 1.2.** **Step 1.** From [5, p51], we have the following scaling property for 2-dimensional SBM \( \{X_t\}_{t \geq 0} \) started from Lebesgue measure: for any \( \eta, t > 0 \) and \( A \in B(\mathbb{R}^2) \),
\[
X_t(A) \overset{\text{Law}}{=} \frac{1}{\eta^2} X_{\eta^2 t}(\eta A).
\]
Let \( \eta = \frac{1}{\sqrt{t}} \) and \( A = B(\sqrt{tr}) \), then
\[
X_t(B(\sqrt{tr})) \overset{\text{Law}}{=} \frac{1}{t} X_1(B(r)).
\]
Thus,
\[
\mathbb{P}_\lambda \left( X_t \left( B(\sqrt{tr}) \right) = 0 \right) = \mathbb{P}_\lambda(X_1(B(r)) = 0).
\]

**Step 2.** Similar to the case of \( d = 1 \), let
\[
u(t, x) := -\log \mathbb{P}_\delta(X_1(B(r)) = 0), t > 0, x \in \mathbb{R}^2.
\]
Due to the same reason as (3.19), we have
\[
\mathbb{P}_\lambda(X_1(B(r)) = 0) = e^{-\int_{\mathbb{R}^2} \nu(1, x) dx}. \quad (4.1)
\]
Thus, to finish Step 2, it suffices to give an upper bound of \( \nu(1, x) \). Fix \( \delta > 0 \), we are going to prove
\[
M_\delta(r) := \sup_{|x| > (1 + \delta) r, t > 0} \nu(t, x) < \infty. \quad (4.2)
\]
Suppose that \((4.2)\) is not true. Then, by the fact that \(u(t, x) < 1/t\), there exist some \(|x_n| > (1 + \delta)r\), \(n \geq 1\) and \(t_n \to 0\) such that \(u(t_n, x_n) \to \infty\). Thus,

\[
\lim_{n \to \infty} \mathbb{P}_{\delta_{x_n}}(X_n(B(r)) > 0) = 1 - \lim_{n \to \infty} \mathbb{P}_{\delta_{x_n}}(X_n(B(r)) = 0) = 1 - \lim_{n \to \infty} e^{-\alpha(t_n, x_n)} = 1.
\]

(4.3)

On the other hand, since \(B(r) \subset B'(x_n, \delta r)\), we have

\[
\lim \sup_{n \to \infty} \mathbb{P}_{\delta_{x_n}}(X_n(B(r)) > 0) \leq \lim \sup_{n \to \infty} \mathbb{P}_{\delta_{x_n}}(X_n(B(x_n, \delta r)^c) > 0) = \lim \sup_{n \to \infty} \mathbb{P}_{\delta_0}(X_n(B(0, \delta r)^c) > 0) = 0,
\]

(4.4)

where the last equality follows from the continuity modulus of the historical SBM. Thus, \((4.4)\) contradicts \((4.3)\), and therefore \((4.2)\) holds. Furthermore, since \(\delta\) and \(r\) are arbitrary positive constants, we have

\[
M(r) := \sup_{|x| \geq r, t > 0} u(t, x) < \infty.
\]

(4.5)

Let \(\phi(x) = 1_{B(r)}(x)\). For \(\theta > 0\), let

\[
u_\theta(t, x) := -\log \mathbb{E}_x \left[ e^{-\langle X_t, \theta \phi \rangle} \right].
\]

By the Feynman-Kac representation \([16, \text{p170}]\),

\[
u_\theta(t, x) = \mathbb{E}_x \left[ \theta \phi(W_t) e^{-\int_0^t \nu_\theta(s, W_s) \, ds} \right].
\]

Let \(T_r := \inf\{t > 0 : |W_t| = r\}\). Then, by the strong Markov property of Brownian motion, for \(|x| \geq (1 + \delta)r\),

\[
u_\theta(t, x) = \mathbb{E}_x \left[ 1_{\{T_r < t\}} e^{-\int_0^{T_r} \nu_\theta(s, W_s) \, ds} \mathbb{E}_{W_{T_r}} \left[ \theta \phi(W_{t-T_r}) e^{-\int_0^{t-T_r} \nu_\theta(s-T_r, W_s) \, ds} \right] \right]
\leq \mathbb{E}_x \left[ 1_{\{T_r < t\}} \nu_\theta(t - T_r, W_{T_r}) \right]
\leq M(r) \mathbb{P}_0 \left( \inf_{s \in (0, t)} |x + W_s| \leq r \right),
\]

(4.6)

where the first equality follows from the fact that \(\phi(W_t) = 0\) for \(T_r > t\) and the last inequality follows from \((4.5)\). Let \(\{W_t^1\}_{t \geq 0}\) be the one-dimensional standard-Brownian motion. Since \(\sup_{s \in (0, t)} W_s^1 \overset{\text{Law}}{=} |W_t^1|\), we have

\[
\mathbb{P}_0 \left( \inf_{s \in (0, t)} |x + W_s| \leq r \right) \leq \mathbb{P}_0 \left( |x| - r \leq \sup_{s \in (0, t)} |W_s| \right)
\leq 2 \mathbb{P}_0 \left( \sup_{s \in (0, t)} W_s^1 \geq (|x| - r)/\sqrt{2} \right)
= 2 \mathbb{P}_0 \left( |W_t^1|/\sqrt{t} \geq (|x| - r)/\sqrt{2t} \right).
\]
\[ \leq \frac{4 \sqrt{t}}{\sqrt{\pi(|x| - r)}} e^{- \frac{(|x| - r)^2}{4t}}. \quad (4.7) \]

Plugging above into (4.6) yields that
\[ u_\theta(t, x) \leq M(r) \frac{4 \sqrt{t}}{\sqrt{\pi(|x| - r)}} e^{- \frac{(|x| - r)^2}{4t}}. \]

Since \( M(r) \) is decreasing (thus bounded) and \( \lim_{\theta \to \infty} u_\theta(t, x) = u(t, x) \), there exists some constant \( C_4 > 0 \) such that for any \(|x| \geq (1 + \delta)r\),
\[ u(1, x) \leq M(r) \frac{4 \sqrt{t}}{\sqrt{\pi(|x| - r)}} e^{- \frac{(|x| - r)^2}{4t}} < \frac{C_4}{|x| - r} e^{- \frac{(|x| - r)^2}{4t}}. \]

Thus,
\[
\int_{\mathbb{R}^2} u(1, x) dx \leq \int_{|x| \leq (1 + \delta)r} 1 dx + \int_{|x| > (1 + \delta)r} \frac{C_4}{|x| - r} e^{- \frac{(|x| - r)^2}{4t}} dx
\leq \pi (1 + \delta)^2 r^2 + \int_{|x| > (1 + \delta)r} \frac{C_4}{|x| - r} e^{- \frac{(|x| - r)^2}{4t}} dx
\leq \pi (1 + \delta)^2 r^2 + \int_{w > \delta r} \frac{C_4}{w} e^{\frac{w^2}{2}} w^2 dw
= \pi (1 + \delta)^2 r^2 + C_4 e^{-\delta^2 r^2/2}.
\]

Taking limits yields that
\[
\lim_{r \to \infty} \frac{\int_{\mathbb{R}^2} u(1, x) dx}{\pi r^2} \leq 1 + \delta.
\]

Let \( \delta \to 0 \), then plugging it into (4.1) yields
\[
\lim_{r \to \infty} \frac{-\log \mathbb{P}_A(X_1(B(r)) = 0)}{\pi r^2} \leq 1.
\quad (4.8)
\]

**Step 3.** Fix \( \bar{\delta} \in (0, 1) \). It is simple to see that
\[
\mathbb{P}_A(X_1(B(\bar{\delta} r)) = 0) = \exp \left\{ \int_{\mathbb{R}^2} \log \mathbb{P}_{\bar{\delta}^2}(X_1(B(r)) = 0) dx \right\}
\leq \exp \left\{ \int_{|x| \leq (1 - \bar{\delta})r} \log \mathbb{P}_{\bar{\delta}^2}(X_1(B(r)) = 0) dx \right\}
= \exp \left\{ \int_{|x| \leq (1 - \bar{\delta})r} \log \mathbb{P}_{\bar{\delta}^2}(X_1(B(x, r)) = 0) dx \right\}
\leq \exp \left\{ \int_{|x| \leq (1 - \bar{\delta})r} \log \mathbb{P}_{\bar{\delta}^2}(X_1(B(\bar{\delta} r)) = 0) dx \right\}
= \exp \left\{ \pi (1 - \bar{\delta})^2 r^2 \log \mathbb{P}_{\bar{\delta}^2}(X_1(B(\bar{\delta} r)) = 0) \right\}. \quad (4.9)
\]

By the dominated convergence theorem and Lemma 2.1,
\[
\lim_{r \to \infty} \log \mathbb{P}_{\bar{\delta}^2}(X_1(B(\bar{\delta} r)) = 0) = \log \mathbb{P}_{\bar{\delta}^2}(X_1(\mathbb{R}^2) = 0) = -1.
\]
Therefore,
\[
\liminf_{r \to \infty} \frac{-\log \mathbb{P}_\lambda(X_1(B(r)) = 0)}{\pi r^2} \geq (1 - \delta)^2.
\]
The desired lower bound follows by letting \( \delta \to 0 \).

Step 4. By similar arguments as in the proof of (3.16)-(3.23), we obtain that
\[
\lim_{t \to \infty} \mathbb{P}_\lambda \left( \frac{R_t}{\sqrt{t}} \geq r \right) = \lim_{t \to \infty} e^{-\int_0^r 1 - e^{u(t,x)}dx} = \lim_{t \to \infty} e^{-\int_0^r u(t,x)dx}.
\]
In fact, the mainly changes are to replace \( t, \mathbb{R} \), Proposition 3.1 with \( \sqrt{t}, \mathbb{R}^2, (4.8) \), respectively. So, we feel free to omit its details here. Putting all steps together, we get that
\[
\lim_{t \to \infty} \mathbb{P}_\lambda \left( \frac{R_t}{\sqrt{t}} \geq r \right) = \lim_{t \to \infty} e^{-A_2(r)}.
\]
We have completed the proof of Theorem 1.2.

5. Proof of Theorem 1.3

In this section, we are going to prove Theorem 1.3. Namely, if \( d \geq 3 \), then for any \( r \in (0, \infty) \),
\[
\lim_{t \to \infty} \mathbb{P}(R_t \geq r) = e^{-\kappa_d r^{d-2}} \in (0, 1).
\]
The proof will be divided into 4 steps:
Step 1. Using the semigroup property of \( u(t,x) \) and the mild form of the PDE (1.2) to show that
\[
\lim_{t \to \infty} \mathbb{P}_\lambda(R_t \geq r) \text{ exists};
\]
Step 2. By the scaling property of \( u(t,x) \), we obtain
\[
\lim_{t \to \infty} \mathbb{P}(R_t \geq r) = \lim_{t \to \infty} \mathbb{P}_\lambda(R_t \geq r) = e^{-\kappa_d r^{d-2}};
\]
Step 3. Applying [2, Lemma 3.2], we prove \( \kappa_d > 0 \).
Step 4. By the second moment method, we show \( \kappa_d < \infty \).

Proof of Theorem 1.3 Step 1. In this step, we show that \( \lim_{t \to \infty} \mathbb{P}_\lambda(R_t \geq r) \) exists. Let \( \psi(x) \in C^2_b(\mathbb{R}^d) \) be a non-negative radially symmetric function such that \( \{ x : \psi(x) > 0 \} = \{ x : |x| \leq r \} \).
Note that
\[
\mathbb{P}_\lambda(X_t(B(r)) = 0) = \lim_{\theta \to \infty} \mathbb{E}_{\theta^r} [e^{-\langle X_t, \theta^r 1_{B(r)} \rangle}] \\
= \lim_{\theta \to \infty} \mathbb{E}_{\theta^r} [e^{-\langle X_t, \theta^r \phi(x) \rangle}] \\
=: \lim_{\theta \to \infty} e^{-A_2(t,x)} \\
=: e^{-A(t,x)},
\]
where \( u^{(\phi)}(t, x) \) is the unique positive solution to the equation:

\[
\begin{cases}
\frac{\partial u(t, x)}{\partial t} = \frac{1}{2} \Delta u(t, x) - u^2(t, x), \\
u(0, x) = \theta \psi(x).
\end{cases}
\]  \hspace{1cm} (5.2)

Fix \( t_1 \in (0, t) \). From (1.3), we have

\[
u^{(\phi)(t_1, \cdot)}(t - t_1, x) + \int_0^{t-t_1} \mathbb{E}_x \left[ \left( \nu^{(\phi)(t_1, \cdot)}(t - s, W_s) \right)^2 \right] ds = \mathbb{E}_x \left[ \nu^{(\phi)}(t_1, W_{t-t_1}) \right].
\]

By the semigroup property \( u^{(\phi)(t_1, \cdot)}(t - t_1, x) = u^{(\phi)}(t, x) \) (see [11, p32]), we get that

\[
u^{(\phi)}(t, x) + \int_0^{t-t} \mathbb{E}_x \left[ \left( \nu^{(\phi)}(t - s, W_s) \right)^2 \right] ds = \mathbb{E}_x \left[ \nu^{(\phi)}(t_1, W_{t-t_1}) \right].
\]

Since \( u^{(\phi)}(t, x) < \frac{1}{7} \), by the dominated convergence theorem,

\[
u(t, x) + \int_0^{t-t} \mathbb{E}_x \left[ (u(t - s, W_s))^2 \right] ds = \mathbb{E}_x [ \nu(t_1, W_{t-t_1}) ].
\]

We write above into its mild form:

\[
u(t, x) = P_{t-t_1}(t_1, x) - \int_{t_1}^t P_{t-s} u^2(s, x) ds.
\]

Integrating w.r.t. \( x \) and making use of Fubini’s theorem, we obtain

\[
\int_{\mathbb{R}^d} \nu(t, x) dx = \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} P_{t-t_1}(x, y) dx \right] u(t_1, y) dy - \int_{t_1}^t \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} P_{t-s}(x, y) dx u^2(s, y) dy \right] ds
\]

\[
= \int_{\mathbb{R}^d} u(t_1, y) dy - \int_{t_1}^t \int_{\mathbb{R}^d} u^2(s, y) dy ds.
\]

Thus, \( \int_{\mathbb{R}^d} \nu(t, x) dx \) is decreasing w.r.t. \( t \). This, together with (3.19), implies

\[
\lim_{t \to \infty} \mathbb{P}_\lambda (R_t \geq r) = \lim_{t \to \infty} e^{-\frac{1}{2} \int_{\mathbb{R}^d} \nu(t, x) dx}
\]

exists.

**Step 2.** In this step, we show that \( \lim_{t \to \infty} \mathbb{P} (R_t \geq r) = e^{-kr^d} \). Let \( \phi(x) \in C^2_b(\mathbb{R}^d) \) be a non-negative radially symmetric function such that \( \{ x : \phi(x) > 0 \} = \{ x : |x| \leq 1 \} \). Let \( \phi_r(x) = \phi(x/r) \). It follows that

\[
\mathbb{P}_{\delta_t} (X_t(B(r)) = 0) = \lim_{\theta \to \infty} \mathbb{E}_{\delta_t} \left[ e^{-<X_t, \theta \delta(B(r))>} \right]
\]

\[
= \lim_{\theta \to \infty} \mathbb{E}_{\delta_t} \left[ e^{-<X_t, \theta \phi_r(x)>} \right]
\]

\[
= \lim_{\theta \to \infty} e^{-\theta \nu(t, x)}
\]

\[
= e^{-\nu(t, x)}.
\]  \hspace{1cm} (5.3)

In this step, we write \( u'(t, x) := u(t, x) \) to emphasize that \( u(t, x) \) depends on \( r \). So,

\[
\mathbb{P}_{\delta_t} (X_t(B(r)) = 0) = e^{-u'(t, x)}.
\]  \hspace{1cm} (5.4)
Thus, to finish this step it suffices to give an upper bound of \( u'(t, x) \). Note that \( u'(t, x) \) is the unique solution of
\[
\begin{cases}
\frac{\partial u(t, x)}{\partial t} = \frac{1}{2} \Delta u(t, x) - u^2(t, x), \\
u(0, x) = \theta \phi_r(x).
\end{cases}
\tag{5.5}
\]
Therefore, we have the following scaling property of \( u(t, x) \). For \( \theta, \varepsilon > 0 \),
\[
u_\theta'(t, x) = \varepsilon^{-2} \nu_{\theta \varepsilon}^{1/\varepsilon}(t \varepsilon^{-2}, x \varepsilon^{-1}).
\]
This yields that
\[
P_\lambda(X_t(B(r)) = 0) = \lim_{\theta \to \infty} e^{-\int_{\mathbb{R}^d} \nu_\theta'(t, x) dx}
= \lim_{\theta \to \infty} e^{-\int_{\mathbb{R}^d} r^2 \nu_\theta(t, x) dx}
= e^{-\int_{\mathbb{R}^d} r^2 \nu(t, x) dx}
= e^{-\kappa_d r^{d-2}}.
\tag{5.6}
\]
Hence, using the monotonicity of \( \int_{\mathbb{R}^d} u(t, x) dx \) and (5.1), we have
\[
\lim_{t \to \infty} P_\lambda(X_t(B(r)) = 0) = e^{-\kappa_d r^{d-2}}.
\]
This, combined with (3.22), yields that
\[
\lim_{t \to \infty} P(R_t \geq r) = \lim_{t \to \infty} e^{-\int_{\mathbb{R}^d} u(t, x) dx}
= \lim_{t \to \infty} \int_{\mathbb{R}^d} u(t, x) dx
= \lim_{t \to \infty} P_\lambda(R_t \geq r)
= \lim_{t \to \infty} P_\lambda(X_t(B(r)) = 0)
= e^{-\kappa_d r^{d-2}}.
\tag{5.7}
\]

**Step 3.** In this step, we show that \( \kappa_d < \infty \). From [2, Lemma 3.2], there exists a constant \( C(d) \) depending only on \( d \) such that for all \( t > 1 \) and \( x \in \mathbb{R}^d \)
\[
u^1(t, x) < C(d) p(t + 1, x),
\]
where \( p(t, x) := \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} \) is the density function of the \( d \)-dimensional Brownian motion. Hence,
\[
\kappa_d = \lim_{t \to \infty} \int_{\mathbb{R}^d} u^1(t, x) dx \leq C(d) < \infty.
\]

**Step 4.** In this step, we show that \( \kappa_d > 0 \). In fact, this has been proved in [2, Lemma 3.3]. Nevertheless, here we use a different method to prove it. Since
\[
P(R_t \geq r) = e^{-\int_{\mathbb{R}^d} \delta(X_t(B(r)) > 0) dx}, \tag{5.8}
\]
To prove \( \kappa_d > 0 \), it suffices to get a lower bound of \( P_{\delta}(X_t(B(r)) > 0) \).
By the Paley-Zygmund inequality,

\[
P_{\delta_t}(X_t(B(r)) > 0) \geq \frac{\mathbb{E}_0^2[X_t(B(r))] - \mathbb{E}_0[X_t^2(B(r))] - (P_t1_{B(r)}(x))^2}{(P_t1_{B(r)}(x))^2 + 2 \int_0^t P_s [P_{t-s}1_{B(r)}(x)]^2 ds},
\]

(5.9)

where the equality follows from the moments formula of SBM (see [11, p.38-39]).

In the next, we are going to give a lower bound of \(P_t1_{B(r)}(x)\). In the following, we assume \(t \geq r^2\).

Observe that

\[
e^{-\frac{2r^2}{2r^2}} \geq \begin{cases} e^{-\frac{|t|^2}{2r^2}}, & |x| \geq r; \\ e^{-1}, & |x| < r. \end{cases}
\]

Thus,

\[
e^{-\frac{2r^2}{2r^2}} \geq e^{-1} e^{-\frac{|t|^2}{r^2}}.
\]

(5.10)

Observe that if \(|y| \leq r\), then

\[
|y - x|^2 \leq (|x| + |y|)^2 \leq |x|^2 + 2r|x| + r^2.
\]

Let \(v_d(r)\) be the volume of \(d\)-dimensional ball with radius \(r\). Through simple calculations, we have

\[
P_t1_{B(r)}(x) = \mathbb{P}_0(|W_t + x| \leq r)
\]

\[
= \int_{|y| \leq r} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|y|^2}{2t}} dy
\]

\[
\geq \int_{|y| \leq r} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|y|^2}{2t}} e^{-\frac{r^2}{2t}} dy
\]

\[
\geq \int_{|y| \leq r} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|y|^2}{2t}} e^{-1/2} e^{-\frac{2r^2}{2t}} dy
\]

\[
\geq \int_{|y| \leq r} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|y|^2}{2t}} e^{-3/2} e^{-\frac{r^2}{t}} dy
\]

\[
\geq e^{-3/2} \int_{|y| \leq r} 3^{-d/2} \frac{1}{(2\pi t/3)^{d/2}} e^{\frac{|y|^2}{2t}} dy
\]

\[
\geq e^{-3/2} 3^{-d/2} v_d(1) r^d p(t/3, x),
\]

(5.11)

where the third inequality follows from (5.10).

In the next, we give an upper bound of \(\int_0^t P_s [P_{t-s}1_{B(r)}(x)]^2 ds\). Note that for \(s \in (0, t)\),

\[
P_{t-s}1_{B(r)}(x) = \mathbb{P}_0(|x + W_{t-s}| \leq r)
\]

\[
= \int_{|y| \leq r} \frac{1}{(2\pi(t-s))^{d/2}} e^{-\frac{|y|^2}{2(t-s)}} dy
\]

\[
\leq \frac{v_d(1) r^d}{(t-s)^{d/2}}.
\]
Thus, by the semigroup property of \( \{P_t\}_{t \geq 0} \), we have
\[
P_s [P_{t-s}1_{B(r)}(x)]^2 \leq \left[ \frac{v_d(1)r^d}{(t-s)^{d/2}} \wedge 1 \right] P_t1_{B(r)}(x),
\]
which yields
\[
\int_0^t P_s [P_{t-s}1_{B(r)}(x)]^2 \, ds \leq P_t1_{B(r)}(x) \left[ \int_0^{t-r^2} \frac{r^d}{(t-s)^{d/2}} \, ds + \int_{t-r^2}^t 1 \, ds \right]
= P_t1_{B(r)}(x) \left[ \int_{r^2}^{t} \frac{r^d}{16^{d/2}} \, du + r^2 \right]
= P_t1_{B(r)}(x) \left[ \frac{2}{d-2} (r^2 - t^{1-\frac{d}{2}}r^d) + r^2 \right]
\leq 3r^2 P_t1_{B(r)}(x). \tag{5.12}
\]
Plugging (5.11) and (5.12) into (5.9) yields that
\[
P_{\delta_x}(X_t(B(r)) > 0) \geq \frac{P_t1_{B(r)}(x)}{P_t1_{B(r)}(x) + 6r^2}
\geq \frac{e^{-3/2}3^{-d/2}v_d(1)r^d p(t/3, x)}{7r^2}
= c(d) r^{d-2} p(t/3, x), \tag{5.13}
\]
where \( c(d) := e^{-3/2}3^{-d/2}v_d(1)/7 \) and the second inequality follows from the fact that for \( t \) large enough,
\[
P_t1_{B(r)}(x) = \int_{|y| \leq r} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|y|^2}{2t}} \, dy
\leq \frac{1}{(2\pi t)^{d/2}} v_d(1)r^d
< r^2.
\]
Plugging (5.13) into (5.8) yields that for \( t \) large enough,
\[
P(R_t \geq r) \leq e^{-c(d)r^{d-2}} \int_x p(t/3, x) \, dx = e^{-c(d)r^{d-2}}.
\]
Thus, \( \kappa_d \geq c_d > 0. \]

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