THE LEAST UNRAMIFIED PRIME WHICH DOES NOT SPLIT COMPLETELY

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Abstract. Let $K/F$ be a finite extension of number fields of degree $n \geq 2$. We establish effective field-uniform unconditional upper bounds for the least norm of a prime ideal $p$ of $F$ which is degree 1 over $\mathbb{Q}$ and does not ramify or split completely in $K$. We improve upon the previous best known general estimates due to X. Li when $F=\mathbb{Q}$ and Murty–Patankar when $K/F$ is Galois. Our bounds are the first when $K/F$ is not assumed to be Galois and $F \neq \mathbb{Q}$.

1. Introduction

1.1. History. Let $K/F$ be a finite extension of number fields of degree $n \geq 2$. Define

$$\mathcal{P}(F) = \{p \text{ prime ideal of } F \text{ which is degree 1 over } \mathbb{Q}\},$$

$$P(K/F) = \min\{N^F_p : p \in \mathcal{P}(F) \text{ and } p \text{ does not ramify or split completely in } K\},$$

$$P^*(K/F) = \min\{N^F_p : p \in \mathcal{P}(F) \text{ and } p \text{ does not split completely in } K\}.$$

The focus of this paper is to establish field-uniform upper bounds for $P(K/F)$ and $P^*(K/F)$. The study of these quantities has classical origins and has been explored in a variety of cases. Indeed, when $K = \mathbb{Q}(\sqrt{d})$ is a quadratic field over $F = \mathbb{Q}$, this reduces to the problem of bounding the least quadratic nonresidue. Assuming the Generalized Riemann Hypothesis (GRH), Ankeny [Ank52] proved $P(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \ll (\log |d|)^2$. Much less is known unconditionally and progress is notoriously difficult. Namely,

$$P(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \ll \varepsilon |d|^\frac{1}{16} + \varepsilon$$

for $\varepsilon > 0$. Aside from the factor of $\varepsilon$, this result of Burgess [Bur57, Bur62] from over fifty years ago remains essentially the best known unconditional bound.

More generally, when $K$ is Galois over $F$ of degree $n \geq 2$, V.K. Murty [Mur94] showed under the assumption of GRH for the Dedekind zeta function of $K$ that

$$P(K/F) \ll \left(\frac{1}{n} \log D_K\right)^2,$$

where $D_K = |\text{disc}(K/\mathbb{Q})|$ and the implied constant is absolute. Murty remarks that the same analytic method can yield an unconditional estimate of the form $O_F(D_K^{1/2(n-1)})$. By a different approach involving geometry of numbers, Vaaler and Voloch [VV00] established an explicit variant of such an unconditional estimate for $P^*(K/\mathbb{Q})$ when $K$ is Galois over $\mathbb{Q}$.
If \( K \) is some finite extension of \( \mathbb{Q} \) (not necessarily Galois) then, using an elegant argument, X. Li [Li12] superseded this prior unconditional bound for \( \mathcal{P}_*(K/\mathbb{Q}) \). Namely, he showed that

\[
\mathcal{P}_*(K/\mathbb{Q}) \ll_{\varepsilon} D_K^{\frac{1+\varepsilon}{2(n-1)}},
\]

where

\[
A = A(n) = \sup_{\lambda > 0} \left( \frac{1 - \frac{n}{n-1} e^{-\lambda}}{\lambda} \right) \geq 1 - \sqrt{\frac{2}{n-1}}.
\]

The key innovation of Li was to incorporate methods of Heath-Brown [HB92] for Dirichlet \( L \)-functions to obtain a stronger explicit inequality for the Dedekind zeta function.

Recently, Murty and Patankar [MP15, Theorem 4.1] adapted Li’s argument to obtain the first unconditional field-uniform estimate for \( \mathcal{P}(K/F) \) when \( K \) is Galois over \( F \). To introduce their result, let \( N_F = 16 \) if there is a sequence of fields \( \mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_m = F \) with each \( F_{i+1}/F_i \) normal and \( N_F = 4[F: \mathbb{Q}]! \) otherwise. Define

\[
B_F = \min\{N_F \log D_F, c_1 D_F^{1/|F:\mathbb{Q}|}\},
\]

for some sufficiently small absolute constant \( c_1 > 0 \). Murty and Patankar proved if \( K/F \) is Galois of degree \( n \geq 2 \) then

\[
\mathcal{P}(K/F) \leq C_F D_K^{\frac{4}{n-1}},
\]

where \( C_F = e^{O(|F:\mathbb{Q}|(\log D_F)^2)} + e^{O(B_F)} \) and the implied constants are absolute. Note the constant \( C_F \) in the quoted result (1.6) differs from [MP15, Theorem 4.1] since there seems to be a typo stemming from equation (4.1) therein and its application in their proof. We remark that the dependence on \( F \) in (1.6) is natural given the current status of the effective Chebotarev Density Theorem [LO77] and the Brauer–Siegel theorem [Sta74].

### 1.2. Results

The primary focus of this paper is to improve the exponent of \( D_K^{\frac{1}{n-1}} \) in both (1.3) and especially in (1.6). As a secondary objective, we consider both \( \mathcal{P}_*(K/F) \) and \( \mathcal{P}(K/F) \) for any finite extension \( K/F \) which, in that generality, is new. We also demonstrate that one may take the non-split prime in (1.3) to be unramified in \( K \) with some minor loss.

Our approach is founded upon Li’s argument blended with ideas of Heath-Brown [HB92] for zero-free regions of Dirichlet \( L \)-functions and their generalization in [Zam16, Zam17] for Hecke \( L \)-functions. Namely, we consider more general sums over prime ideals of \( F \) which depend on a choice of polynomial. To state our main result, we introduce a definition: a polynomial \( P(x) \in \mathbb{R}_{\geq 0}[x] \) is admissible if \( P(0) = 0, P'(0) = 1 \), and

\[
\Re\{P(1/z)\} \geq 0 \quad \text{for } \Re\{z\} \geq 1.
\]

**Theorem 1.1.** Let \( K/F \) be an extension of number fields of degree \( n \geq 2 \). Let \( \varepsilon > 0 \) be fixed and \( P(x) = \sum_{d=1}^{d} a_k x^d \) be a fixed admissible polynomial. There exists a prime ideal \( \mathfrak{p} \) of \( F \) such that \( \mathfrak{p} \) does not split completely in \( K \), \( \mathfrak{p} \) is degree 1 over \( \mathbb{Q} \), and

\[
N_{\mathbb{Q}}^{F} \mathfrak{p} \leq C_F D_K^{\frac{4+\varepsilon}{4(n-1)}},
\]

where \( C_F = e^{O(|F:\mathbb{Q}|(\log D_F)^2)} + e^{O(B_F)} \), \( B_F \) is given by (1.5), and

\[
A = A(n, P) = \sup_{\lambda > 0} \left( \left[ P(1) - \frac{n}{n-1} e^{-\lambda} \sum_{k=1}^{d} a_k \sum_{j=0}^{k-1} \frac{\lambda^{j}}{j!} \right] / \lambda \right).
\]
If $K/F$ is Galois then one may take $p$ to also be unramified in $K$. All implied constants depend at most on $\varepsilon$ and $P$.

Remark.

- While $A$ depends on $n$, it is bounded above and below independent of $n$. In particular, if $P(x) = x + x^2$ then
  \[ A(n, P) \geq 1 - 2n^{-2/3}, \]
  which improves over (1.4) as $n \to \infty$. Moreover, the exponent $\frac{1+\varepsilon}{4A}$ becomes a nearly sixteen-fold improvement over the exponent 4 in (1.6) as $n \to \infty$. With a different choice of $P$, we have by Table 1 that $\frac{1+\varepsilon}{4A} < \frac{5}{12}$ for all $n \geq 2$. This constitutes a nine-fold improvement over (1.6) for all $n \geq 2$.

- If $K/F$ is not assumed to be Galois, then one may still take $p$ to be unramified in $K$ but we show it satisfies the slightly weaker bound
  \[ N_F^p \ll (C_F + n \frac{3P(x)}{4}) D_K^{\frac{1+\varepsilon}{4A(n-1)}}. \]
  By a classical result of Minkowski, recall that $n \leq [K : \mathbb{Q}] \ll \log D_K$, so, unless $n$ is unusually large, this additional factor is negligible compared to $D_K^{\frac{\varepsilon}{n-1}}$.

We restate Theorem 1.1 in the special case $F = \mathbb{Q}$.

**Corollary 1.2.** Let $K$ be a number field of degree $n \geq 2$. Let $\varepsilon > 0$ be fixed and $P(x) = \sum_{d=1}^{d} a_k x^d$ be a fixed admissible polynomial. The least rational prime $p$ which does not split completely in $K$ satisfies
  \[ p \ll D_K^{\frac{1+\varepsilon}{4A(n-1)}}, \]
  where $A = A(n, P)$ is given by (1.8). If $K/\mathbb{Q}$ is Galois then one may also take $p$ to be unramified in $K$. Furthermore, if $P(x) = x + x^2$ then $A \geq 1 - 2n^{-2/3}$. All implied constants depend at most on $\varepsilon$ and $P$.

Choosing a certain admissible polynomial $P(x) = P_{100}(x)$ of degree 100, say, Corollary 1.2 yields savings for every degree $n$ over the special case (1.3) where $P(x) = P_1(x) = x$. For example, if $K/\mathbb{Q}$ is an extension of degree 5 then, by Corollary 1.2 with $P = P_{100}$,
  \[ P(K/\mathbb{Q}) \ll D_K^{1/8.7}, \]
  whereas if $P = P_1$ then $1/8.7$ is replaced by $1/6.1$. See Section 5 and Table 1 for further details on these computations.

Finally, we describe the organization of the paper. Section 2 collects standard estimates related to counting prime ideals in a number field $F$. Section 3 contains an explicit inequality of the Dedekind zeta function and a generalization related to admissible polynomials. Section 4 has the proof of Theorem 1.1 and Section 5 outlines the computation of admissible polynomials and Table 1.

**Notation.** We henceforth adhere to the convention that all implied constants in all asymptotic inequalities $f \ll g$ or $f = O(g)$ are absolute with respect to all parameters and effectively computable. If an implied constant depends on a parameter, such as $\varepsilon$, then we use $\ll_{\varepsilon}$ and $O_{\varepsilon}$ to denote that the implied constant depends at most on $\varepsilon$.

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2. Counting prime ideals

Let $F$ be a number field of degree $n_F = [F : \mathbb{Q}]$ with discriminant $D_F = |\text{disc}(F/\mathbb{Q})|$ and ring of integers $\mathcal{O}_F$. Denote $N_F^F$ to be the absolute norm of $F$ over $\mathbb{Q}$. For each integral ideal $n \subseteq \mathcal{O}_F$, define

$$\Lambda_F(n) = \begin{cases} 
\log N_F^F p & \text{if } n \text{ is a power of a prime ideal } p, \\
0 & \text{otherwise.}
\end{cases}$$

**Lemma 2.1.** Let $F$ be a number field and $\eta > 0$ be arbitrary. Define

$$X_0 = X_0(F, \eta) := \exp(10n_F (\log D_F)^2) + \exp(B_F \log(1/\eta)), \tag{2.1}$$

where $B_F$ is defined by (1.5). For $X \geq X_0$,

$$(1 - \eta)X + O\left(\frac{X}{(\log X)^2}\right) \leq \sum_{N_F^F n \leq X} \Lambda_F(n) \leq (1 + \eta)X + O\left(\frac{X}{(\log X)^2}\right). \tag{2.2}$$

All implied constants are absolute.

**Proof.** The effective Chebotarev Density Theorem [LO77] implies that, for $X \geq X_0$,

$$|\sum_{N_F^F n \leq X} \Lambda_F(n) - X| \leq X^\beta + O(X \exp(-cn_F^{-1/2}(\log X)^{1/2})), \tag{2.3}$$

where $c > 0$ is some absolute constant and $\beta > 1/2$ is a real zero of the Dedekind function of $K$, if it exists. By a theorem of Stark [Sta74, Theorem 1'], any real zero $\beta$ of the Dedekind zeta function $\zeta_F(s)$ satisfies

$$\beta < 1 - \frac{1}{B_F},$$

where $B_F$ is given by (1.5). Hence, by (2.1), we have $X^\beta = X \cdot X^{\beta-1} \leq \eta X$. By Minkowski's bound, observe that $n_F \ll \log D_F \ll \sqrt{\log X}$. It follows that $n_F^{-1/2}(\log X)^{1/2} \gg (\log X)^{1/4}$, so the error term in (2.2) is crudely bounded by $O(X/(\log X)^2)$. \qed

**Lemma 2.2.** Let $k \geq 1$ be an integer and $\eta \in (0, 1/2)$ be arbitrary. Let $X \geq Y \geq X_0$ where $X_0 = X_0(F, \eta)$ is defined by (2.1). Denote $E_{k-1}(t) = \sum_{j=0}^{k-1} t^j/j!$. Then

$$\sum_{Y < N_F^F n \leq X} \frac{\Lambda_F(n)}{N_F^F n^\sigma} (\log N_Q^F n)^{k-1} \geq \frac{(k-1)!}{(\sigma - 1)^k} \cdot (1 - \eta) \left(Y^{1-\sigma} - X^{1-\sigma} E_{k-1}((\sigma - 1) \log X)\right) + O_k\left(\frac{1}{(\sigma - 1)^{k-1}}\right)$$

uniformly for $1 < \sigma < 2$.

**Proof.** This is a combination of partial summation and Lemma 2.1. We include the proof for sake of completeness. Define $\psi_F(t) = \sum_{N_F^F n < t} \Lambda_F(n)$ for $t > 1$. By partial summation,

$$\sum_{Y < N_F^F n \leq X} \frac{\Lambda_F(n)}{N_F^F n^\sigma} (\log N_Q^F n)^{k-1} \psi_F(t) = \psi_F(X)X^{-\sigma}(\log X)^{k-1} - \int_Y^X \psi_F(t) \frac{d}{dt} \left[t^{-\sigma} (\log t)^{k-1}\right] dt.$$
By Lemma 2.1, it follows for \( t \geq Y \geq X_0 \) that
\[-\psi_F(t) \frac{d}{dt} [t^{-\sigma}(\log t)^k - 1] \geq (1 - \eta) \sigma t^{-\sigma}(\log t)^{k-1} \{1 + O_k\left(\frac{1}{\log t}\right)\}.\]

Discarding the first term in the previous equation by positivity and using the above inequality, we deduce that
\[
\sum_{Y < N_Q \leq X} \frac{\Lambda_F(n)}{N_Q^\sigma n^\sigma} (\log N_Q n)^{k-1} \geq (1 - \eta) \int_Y^X t^{-\sigma}(\log t)^{k-1} dt + O_k\left(\int_Y^X t^{-\sigma}(\log t)^{k-2} dt\right).
\]
The remaining integrals are computed by parts. One iteration yields:
\[
\int_Y^X t^{-\sigma}(\log t)^{k-1} dt = \frac{Y^{1-\sigma}(\log Y)^{k-1}}{\sigma - 1} - \frac{X^{1-\sigma}(\log X)^{k-1}}{\sigma - 1} + \frac{k - 1}{(\sigma - 1)} \int_Y^X t^{-\sigma}(\log t)^{k-2} dt.
\]
Proceeding by induction, we conclude that
\[
\int_Y^X t^{-\sigma}(\log t)^{k-1} dt = (k - 1)! \sum_{j=0}^{k-1} \frac{Y^{1-\sigma}(\log Y)^{k-1-j}}{(k - 1 - j)! (\sigma - 1)^{j+1}} - \frac{X^{1-\sigma}(\log X)^{k-1-j}}{(k - 1 - j)! (\sigma - 1)^{j+1}}
\]
\[
= \frac{(k - 1)!}{(\sigma - 1)^k} \left(Y^{1-\sigma} E_{k-1}((\sigma - 1) \log Y) - X^{1-\sigma} E_{k-1}((\sigma - 1) \log X)\right).
\]
Substituting this expression in (2.3) and observing \( 1 \leq E_{k-1}(t) \leq e^t \) (in order to simplify the main term and error term involving \( Y \)), we obtain the desired result. \( \square \)

**Lemma 2.3.** Let \( K \) be a finite extension of \( F \). Let \( V(K/F) \) be the set of places \( v \) of \( F \) which ramify in \( K \) and \( p_v \) be the prime ideal of \( F \) attached to \( v \). Unconditionally,
\[
\sum_{v \in V(K/F)} \log N_Q^{F_p} p_v \leq \log D_K.
\]

If \( K/F \) is Galois then
\[
\sum_{v \in V(K/F)} \frac{\log N_Q^{F_p} p_v}{N_Q^{F_p} p_v} \leq \sqrt{2[F : \mathbb{Q}] [K : F]} \log D_K.
\]

**Proof.** The unconditional inequality follows from the well-known formula
\[
\log D_K = [K : F] \log D_F + \log N_Q^{F} \mathfrak{d}_{K/F},
\]
where \( \mathfrak{d}_{K/F} = N_F^{K} \mathfrak{D}_{K/F} \) and \( \mathfrak{D}_{K/F} \) is the relative different ideal of \( K/F \). If \( K/F \) is Galois then, by Cauchy-Schwarz and [Ser81] Proposition 5, Section I.3,
\[
\sum_{v \in V(K/F)} \frac{\log N_Q^{F_p} p_v}{N_Q^{F_p} p_v} \leq \left( \sum_{v \in V(K/F)} \log N_Q^{F_p} p_v \right)^{1/2} \left( \sum_{v \in V(K/F)} \frac{\log N_Q^{F_p} p_v}{N_Q^{F_p} p_v} \right)^{1/2}
\]
\[
\leq \left( \frac{2}{[K : F]} \log D_K \right)^{1/2} \left( |F : \mathbb{Q}| \sum_p \frac{\log p}{p^2} \right)^{1/2}
\]
\[
\leq \sqrt{\frac{2[F : \mathbb{Q}]}{[K : F]} \log D_K},
\]
as desired. In the above, we used that there are at most \([F : \mathbb{Q}]\) prime ideals \(p\) of \(F\) above a given rational prime \(p\) and \(\sum_p \frac{\log p}{p^2} < 1\).

## 3. Polynomial explicit inequality

Let \(K\) be a number field with \(D_K = |\text{disc}(K/\mathbb{Q})|\) and let \(\zeta_K(s)\) be the Dedekind zeta function of \(K\). Our starting point is a variant of the classical explicit formula.

**Proposition 3.1** (Thorner–Z). Let \(K\) be a number field and \(0 < \varepsilon < 1/8\) be arbitrary. There exists \(\delta = \delta(\varepsilon) > 0\) such that

\[-\text{Re}\{\frac{\zeta_K'(s)}{\zeta_K(s)}\} \leq \left(\frac{1}{4} + \varepsilon\right) \log D_K + \text{Re}\left\{\frac{1}{s - 1}\right\} - \sum_{|1 + \alpha - \rho| < \delta} \text{Re}\left\{\frac{1}{s - \rho}\right\} + O_{\varepsilon}(\mathbb{Q})^{[K : \mathbb{Q}]},\]

uniformly for \(s = \sigma + it\) with \(1 < \sigma < 1 + \varepsilon\) and \(|t| \leq 1\).

**Remark.** The value \(1/4\) is derived from the convexity bound for \(\zeta_K(s)\) in the critical strip.

**Proof.** This follows from [TZ17, Proposition 2.6]; similar variants appear in [Li12, KN12]. See [Zam17, Proposition 3.2.3] for details. □

We would like to analyze more general sums over prime ideals by considering higher derivatives of the logarithmic derivative \(-\frac{\zeta_K'(s)}{\zeta_K(s)}\). This generalization (Proposition 3.2) is motivated by the work of Heath-Brown [HB92, Section 4].

Given a polynomial \(P(x) \in \mathbb{R}_{\geq 0}[x]\) of degree \(d\) with \(P(0) = 0\), write

\[P(x) = \sum_{k=1}^d a_k x^k\]

and define

\[S(\sigma) = S_K(\sigma; P) := \sum_{n \subseteq \mathcal{O}_K} \frac{\Lambda_K(n)}{N_n^\sigma} \sum_{k=1}^d a_k \frac{(\sigma - 1) \log N_n)^{k-1}}{(k-1)!}\]

for \(\sigma > 1\). Recall the definition of an admissible polynomial from (1.7). Note the condition \(P'(0) = 1\) is imposed for normalization purposes since it implies \(a_1 = 1\).

**Proposition 3.2.** Let \(0 < \varepsilon < 1/8\) and \(\lambda > 0\) be arbitrary. If \(P(x) = \sum_{k=1}^d a_k x^k\) is an admissible polynomial of degree \(d\) then

\[S(\sigma) = S_K(\sigma, P) \leq \left(\frac{1}{4} + \varepsilon\right) \log D_K + \frac{P(1)}{\sigma - 1} + O_{\varepsilon, P, \lambda}(\mathbb{Q})^{[K : \mathbb{Q}]},\]

uniformly for

\[1 < \sigma \leq 1 + \min\left\{\varepsilon, \frac{\lambda K : \mathbb{Q}}{\log D_K}\right\}.\]

**Proof.** This is essentially [Zam16, Proposition 5.2] with Proposition 3.1 used in place of [Zam16, Lemma 4.3]. Our argument proceeds similarly but we exhibit a different range
of $\sigma$ which is more suitable for our purposes. For simplicity, denote $L = \log D_K$ and $n_K = [K : \mathbb{Q}]$. Define

$$P_2(x) := \sum_{k=2}^{d} a_k x^k = P(x) - a_1 x.$$  

From the functional equation of $\zeta_K(s)$, it follows by [Zam16, Lemma 2.6] that

$$\frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \left( - \frac{\zeta_K}{\zeta_K}(s) \right) = \frac{1}{(s-1)^k} - \sum_{\rho} \frac{1}{(s-\rho)^k} + \frac{1}{\sigma^k} - \frac{(-1)^k}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \left( \frac{\gamma_K}{\gamma_K}(s) \right)$$  

$$= \frac{1}{(s-1)^k} - \sum_{\rho} \frac{1}{(s-\rho)^k} + O(n_K)$$

for $\text{Re}\{s\} > 1$. On the other hand, from the Euler product of $\zeta_K(s)$ one can verify that

$$\frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \left( - \frac{\zeta_K}{\zeta_K}(s) \right) = \sum_{n \leq O_K} \frac{A_K(n)}{Nn^s}(\log Nn)^{k-1}$$

for $\text{Re}\{s\} > 1$. Comparing these two expressions at $s = \sigma$ with (3.1) and taking real parts, we deduce that

$$S(\sigma; P_2) = \frac{1}{\sigma-1} \sum_{k=2}^{d} a_k \text{Re}\left\{ 1 - \sum_{\rho} \left( \frac{\sigma-1}{\sigma-\rho} \right)^{k-1} \right\} + O_p(n_K)$$

for $\sigma > 1$. We wish to restrict the sum over zeros $\rho$ in (3.2) to $|1-\rho| < \delta$ for $\delta = \delta(\varepsilon) > 0$ given by Proposition 3.1. Observe by [LO77, Lemma 5.4] that

$$\sum_{\rho = \beta + i\gamma \atop |1-\rho| \geq \delta} \text{Re}\left\{ \left( \frac{\sigma-1}{\sigma-\rho} \right)^{k-1} \right\} \ll_{\varepsilon,k} (\sigma-1)^{k-1} \sum_{T=0}^{\infty} \frac{1}{1+t^2}$$

$$\ll_{\varepsilon,k,\lambda} \left( \frac{n_K}{L} \right)^{k-1} \sum_{T=0}^{\infty} \frac{L + n_K \log(T+3)}{1+T^2}$$

$$\ll_{\varepsilon,k,\lambda} n_K,$$

since $k \geq 2$, $\sigma < 1 + \frac{\lambda n_K}{L}$, and $n_K \ll L$. Now, consider the linear polynomial $P_1(x) = a_1 x = x$ as $P'(0) = 1$. By Proposition 3.1 we find that

$$S(\sigma; P_1) \leq (\frac{1}{4} + \varepsilon)L + a_1 \text{Re}\left\{ \frac{1}{\sigma-1} - \sum_{|1-\rho| < \delta} \frac{1}{\sigma-\rho} \right\} + O_\varepsilon(n_K).$$

\footnote{This is redundant as the expression is already real, but clarifies the later use of admissibility of $P$.}
Notice \( S(\sigma; P) = S(\sigma; P_1) + S(\sigma; P_2) \) by linearity in the second argument. Hence, we may combine the above with (3.2) and (3.3) yielding

\[
S(\sigma; P) \leq \left( \frac{1}{4} + \epsilon \right) \mathcal{L} + \frac{1}{\sigma - 1} \sum_{k=1}^{d} a_k \text{Re}\left\{ 1 - \sum_{|1 - \rho| < \delta} \left( \frac{\sigma - 1}{\sigma - \rho} \right)^{k-1} \right\} + O_{\epsilon,P,\lambda}(n_K)
\]

\[
\leq \left( \frac{1}{4} + \epsilon \right) \mathcal{L} + \frac{1}{\sigma - 1} P(1) - \frac{1}{\sigma - 1} \sum_{|1 - \rho| < \delta} \text{Re}\left\{ P\left( \frac{\sigma - 1}{\sigma - \rho} \right) \right\} + O_{\epsilon,P,\lambda}(n_K)
\]

\[
\leq \left( \frac{1}{4} + \epsilon \right) \mathcal{L} + \frac{P(1)}{\sigma - 1} + O_{\epsilon,P,\lambda}(n_K).
\]

In the last step, we noted \( \text{Re}\left\{ P\left( \frac{\sigma - 1}{\sigma - \rho} \right) \right\} \geq 0 \) by admissibility of \( P \).

\[ \square \]

4. PROOF OF THEOREM 1.1

We will deduce Theorem 1.1 from the following result.

**Theorem 4.1.** Let \( K/F \) be an extension of number fields of degree \( n \geq 2 \) and \( X \geq Y \). Assume one of the following holds:

(A1) Every prime ideal \( p \) of \( F \) which is degree 1 over \( \mathbb{Q} \) with \( Y < N_F^p \leq X \) splits completely in \( K \).

(A2) Every unramified prime ideal \( p \) of \( F \) which is degree 1 over \( \mathbb{Q} \) with \( Y < N_F^p \leq X \) splits completely in \( K \).

(A3) Assumption (A2) holds and \( K/F \) is Galois.

Let \( 0 < \epsilon < \frac{1}{8} \) be arbitrary and \( P(x) = \sum_{k=1}^{d} a_k x^d \) be an admissible polynomial. For \( M = M(\epsilon, P) \) sufficiently large, define

\[
Y_0 = \begin{cases} 
X_0 & \text{if (A1) or (A3) hold,} \\
X_0 + Mn & \text{if (A2) holds,}
\end{cases}
\]

where \( X_0 = X_0(F, \eta) \) is given by (2.1) and \( \eta = \eta(\epsilon, P) \) is sufficiently small. If \( X \geq Y \geq Y_0 \) then

\[
(1 - \epsilon) A \log X \leq \left( \frac{1}{4} + \epsilon \right) \log D_K + \frac{n}{n-1} P(1) \log Y + O_{\epsilon,P}(F: \mathbb{Q}),
\]

where \( A = A(n, P) \) is given by (1.8).

**4.1. Proof of Theorem 1.1 from Theorem 4.1.** Without loss, assume \( \epsilon \in (0, \frac{1}{8}) \). Taking \( Y = Y_0 \) and rescaling \( \epsilon > 0 \) appropriately in Theorem 4.1 yields

\[
A \log X \leq \left( \frac{1}{4} + \epsilon \right) \log D_K + 3P(1) \log Y_0 + O_{\epsilon,P}(F: \mathbb{Q}).
\]
By considering cases arising from (4.1) and fixing \( \varepsilon \) and \( P \), this yields the desired bound for \( X \) in all cases. Moreover, if \( P(x) = x + x^2 \) and \( \lambda > 0 \) then

\[
A(n, P) \geq \frac{2 - n}{\lambda(n)} e^{-\lambda(2 + \lambda)} \geq \frac{2}{(n - 1)\lambda} + \frac{n}{n - 1} - \frac{n\lambda^2}{6(n - 1)}
\]

\[
= \frac{n}{n - 1} \left( 1 - \frac{2}{n\lambda} - \frac{\lambda^2}{6} \right)
\]

\[
= \frac{n}{n - 1} \left( 1 - \frac{6^{2/3}}{2n^{2/3}} \right) \geq 1 - \frac{2}{n^{2/3}},
\]

upon setting \( \lambda = \sqrt[3]{6/n} \).

4.2. Proof of Theorem 4.1 Let \( 0 < \lambda < \lambda(\varepsilon, P) \) where \( \lambda(\varepsilon, P) \) is some sufficiently large constant and let \( \sigma = 1 + \frac{\lambda}{\log X} \). One can verify \( A = A(n, P) \geq A(2, P) \gg \nu(1) \) and \( A \ll P(1) \) from (1.8). Thus, by (4.2), we may assume without loss that \( X \geq e^{\lambda(\varepsilon, P)/\varepsilon} \) and \( X \geq D_{K/F}^{1/4(n-1)} \). This implies that \( 1 < \sigma < 1 + \min \{ \varepsilon, \frac{4\lambda(\varepsilon, P)K/\log K}{D_{K/F}} \} \). Now, letting \( \mathfrak{D}_{K/F} \) be the relative different of \( K/F \), consider

\[
S := \sum_{Y < N_{K/F}^{\mathfrak{D}_{K/F}} = 1} \sum_{\mathfrak{N}^k \subseteq X} \frac{A_K(\mathfrak{N})}{N_{K/F}^{\mathfrak{D}_{K/F}}} \sum_{k=1}^d a_k \frac{((\sigma - 1) \log N_{K/F}^{\mathfrak{D}_{K/F}})^{k-1}}{(k-1)!}.
\]

By the positivity of the terms and Proposition 3.2 it follows that

\[
S \leq S(\sigma; P) \leq \frac{P(1)}{\sigma - 1} + \left( \frac{1}{4} + \frac{\varepsilon}{2} \right) \log D_K + O_{\varepsilon, P}(\log [K : \mathbb{Q}]).
\]

On the other hand, by any of (A1), (A2), or (A3), each unramified prime ideal of \( F \) splits completely into \( [K : F] \) prime ideals. Hence, denoting \( \mathfrak{d}_{K/F} = N_{F/K}^{\mathfrak{D}_{K/F}} \), we have that

\[
S = [K : F] \sum_{Y < N_{K/F}^{\mathfrak{D}_{K/F}} = 1} \sum_{\mathfrak{N}^k \subseteq X} \frac{A_F(\mathfrak{n})}{N_{K/F}^{\mathfrak{D}_{K/F}}} \sum_{k=1}^d a_k \frac{((\sigma - 1) \log N_{K/F}^{\mathfrak{D}_{K/F}})^{k-1}}{(k-1)!} \geq [K : F] \sum_{k=1}^d a_k (S_k - R_k - T_k),
\]

where

\[
S_k = \frac{(\sigma - 1)^{k-1}}{(k-1)!} \sum_{Y < N_{K/F}^{\mathfrak{D}_{K/F}} = 1} \frac{A_F(\mathfrak{n})}{N_{K/F}^{\mathfrak{D}_{K/F}}} (\log N_{K/F}^{\mathfrak{D}_{K/F}})^{k-1},
\]

\[
R_k = \frac{(\sigma - 1)^{k-1}}{(k-1)!} \sum_{Y < N_{K/F}^{\mathfrak{D}_{K/F}} = 1} \frac{A_F(\mathfrak{n})}{N_{K/F}^{\mathfrak{D}_{K/F}}} (\log N_{K/F}^{\mathfrak{D}_{K/F}})^{k-1},
\]

\[
T_k = \frac{(\sigma - 1)^{k-1}}{(k-1)!} \sum_{Y < N_{K/F}^{\mathfrak{D}_{K/F}} = 1} \frac{A_F(\mathfrak{n})}{N_{K/F}^{\mathfrak{D}_{K/F}}} (\log N_{K/F}^{\mathfrak{D}_{K/F}})^{k-1}.
\]
Here $\sum'$ indicates a restriction to ideals $n = p^j$ where $p$ is of degree $\geq 2$ over $\mathbb{Q}$ and $j \geq 1$. We estimate each $S_k$ using Lemma 2.2 with $\eta = \eta(\varepsilon, P)$ sufficiently small to deduce that

$$\sum_{k=1}^{d} a_k S_k \geq \frac{1 - \eta}{\sigma - 1} \sum_{k=1}^{d} a_k (Y^{1-\sigma} - X^{1-\sigma} E_{k-1}((\sigma - 1) \log X)) + O_P(1).$$

Since $X \geq Y$, $\sigma = 1 + \frac{\lambda}{\log X}$, and $e^{-t} \geq 1 - t$ for $t > 0$, we have that $Y^{1-\sigma} \geq 1 - (\sigma - 1) \log Y$. The above equation therefore implies that

$$(4.6) \quad \frac{1}{1 - \eta} \sum_{k=1}^{d} a_k S_k \geq \left( \frac{P(1) - e^{-\lambda}}{\lambda} \sum_{k=1}^{d} a_k E_{k-1}(\lambda) \right) \log X - P(1) \log Y + O_P(1)$$

To estimate $R_k$, we claim that

$$(4.7) \quad \sum_{k=1}^{d} a_k R_k \leq \varepsilon P_{[\mathbb{K} : \mathbb{F}] \log D_{\mathbb{K}}} + O_{\varepsilon, P}([\mathbb{F} : \mathbb{Q}]).$$

We divide into cases according to assumptions (A1), (A2), and (A3).

- If (A1) holds then $R_k = 0$ for all $k$ which trivially yields the claim.

- If (A2) holds then, as $\lambda < \lambda(\varepsilon, P)$ and $\sigma = 1 + \frac{\lambda}{\log X}$,

$$\sum_{k=1}^{d} a_k R_k \ll \varepsilon_P \sum_{\substack{Y < N_{\mathbb{F}}p \leq X \\
(n, [\mathbb{K} : \mathbb{F}] \neq 1}} \frac{A_{\mathbb{F}}(n)}{N_{\mathbb{Q}}n^\sigma} \ll \varepsilon_P \sum_{\substack{N_{\mathbb{Q}}p > 1 \\
p \nmid [\mathbb{K} : \mathbb{F}]}} \frac{\log N_{\mathbb{Q}}p}{N_{\mathbb{Q}}p}.$$ 

Since $Y \geq Y_0 \geq M[\mathbb{K} : \mathbb{F}]$ from (4.1) and $M = M(\varepsilon, P)$ is sufficiently large, it follows by Lemma 2.3 that

$$\sum_{k=1}^{d} a_k R_k \leq \frac{\varepsilon}{2[\mathbb{K} : \mathbb{F}]} \log D_{\mathbb{K}}.$$

- If (A3) holds then we argue as above and apply Lemma 2.3 in the $\mathbb{K}/\mathbb{F}$ Galois case to deduce that

$$\sum_{k=1}^{d} a_k R_k \ll \varepsilon_P \sqrt{\frac{[\mathbb{F} : \mathbb{Q}]}{[\mathbb{K} : \mathbb{F}]} \log D_{\mathbb{K}}}.$$ 

By AM-GM, claim (4.7) follows.

This proves the claim in all cases. Finally, to estimate $T_k$, we similarly observe that

$$\sum_{k=1}^{d} a_k T_k \ll \varepsilon_P \sum_{\substack{Y < N_{\mathbb{F}}p \leq X \\
(n, [\mathbb{K} : \mathbb{F}] \neq 1}} \frac{\log N_{\mathbb{Q}}p}{N_{\mathbb{F}}p^\sigma} \ll \varepsilon_P \sum_p \frac{\log p}{p^{2\sigma}} \ll \varepsilon_P [\mathbb{F} : \mathbb{Q}].$$ 

Combining (4.4), (4.5), (4.6), (4.7), and the above, it follows that

$$(4.8) \quad (n - 1)a(\lambda) \log X - \eta mb(\lambda) \log X \leq (\frac{1}{4} + \varepsilon) \log D_{\mathbb{K}} + n P(1) \log Y + O_{\varepsilon, P}([\mathbb{K} : \mathbb{Q}]).$$
where $n = [K : F]$,

$$a(\lambda) = a(\lambda; n, P) = \left( P(1) - \frac{n}{n-1} e^{-\lambda} \sum_{k=1}^{d} a_k E_{k-1}(\lambda) \right),$$

$$b(\lambda) = b(\lambda; P) = \left( P(1) - e^{-\lambda} \sum_{k=1}^{d} a_k E_{k-1}(\lambda) \right).$$

One can verify that the supremum of $b(\lambda)$ over $\lambda > 0$ exists and $A = A(n, P) = \sup_{\lambda > 0} a(\lambda)$ is bounded independent of $n$. By taking $\eta = \eta(\varepsilon, P)$ sufficiently small, we may therefore assume that $\eta b(\lambda) < \frac{n-1}{n} \varepsilon A$. Hence, (4.8) implies

$$(n-1)[a(\lambda) - \varepsilon A] \log X \leq (\frac{1}{4} + \varepsilon) \log D_K + nP(1) \log Y + O_{\varepsilon,P}([K:F]).$$

Dividing both sides by $n-1$ and taking the supremum over $0 < \lambda < \lambda(\varepsilon, P)$ yields the desired result, except for the range of $\lambda$ in definition of $A$. By straightforward calculus arguments, the supremum of $a(\lambda)$ occurs at $\lambda = \lambda_{n,P} > 0$ and one can verify that $\lambda_{n,P}$ is bounded above independent of $n$. Hence, for $\lambda(\varepsilon, P)$ sufficiently large,

$$\sup_{0 < \lambda < \lambda(\varepsilon,P)} a(\lambda) = \sup_{\lambda > 0} a(\lambda) = A.$$

This completes the proof. \hfill \Box

5. Admissible polynomials with large values

Here we outline the computation of admissible polynomials $P(x)$ such that $P(1)$ is large which leads to large values for $A(n, P)$ in Theorem 1.1. The key lemma for our calculations follows from arguments in [HB92, Section 4] based on the maximum modulus principle.

**Lemma 5.1** (Heath-Brown). A polynomial $P(x) \in \mathbb{R}_{\geq 0}[x]$ satisfying $P(0) = 0$ and $P'(0) = 1$ is admissible provided

$$\Re\left\{ P\left( \frac{1}{1+iy} \right) \right\} \geq 0 \quad \text{for all } y \geq 0.$$  

For each integer $d \geq 1$, write $P(x) = \sum_{k=1}^{d} a_k x^k$ where $a_k \geq 0$ and $a_1 = 1$. We wish to determine $a_2, \ldots, a_d$ such that $P(1) = 1 + a_2 + \cdots + a_d$ is maximum. From Lemma 5.1 it suffices to verify that for all $y \geq 0$,

$$\sum_{k=1}^{d} a_k \Re\left\{ \frac{(1-iy)^k}{(1+y^2)^k} \right\} \geq 0, \quad \text{or equivalently,} \quad \sum_{k=1}^{d} a_k (1+y^2)^{d-k} \sum_{0 \leq j \leq k/2} (-1)^j \binom{k}{2j} y^{2j} \geq 0.$$

Expanding the above as a polynomial in $y$, let $a = (a_1, a_2, a_3, \ldots, a_d)$ and $C_{2j}^{(d)} = C_{2j}^{(d)}(a)$ be the coefficient of $y^{2j}$ for $0 \leq j \leq d-1$; all other coefficients are zero. As $a_1 = 1$, one can see that $C_0^{(d)} = 1 + a_2 + \cdots + a_d = P(1)$. Therefore, $P(x)$ is admissible if the remaining $d-1$ coefficients $C_{2j}^{(d)}$ for $1 \leq j \leq d-1$ are non-negative. Notice $C_{2j}^{(d)}$ are linear expressions in $a_2, \ldots, a_d$. Thus, one may apply the simplex method to maximize the objective function $P(1) = 1 + a_2 + \cdots + a_d$ given the system of linear inequalities $\{C_{2j}^{(d)}(a) \geq 0\}^{d-1}_{j=1} \cup \{a_j \geq 0\}^{d-1}_{j=2}$. Based on computational evidence for $1 \leq d \leq 100$, the maximum of this linear system occurs precisely when $C_{2j}^{(d)}(a) = 0$ for all $1 \leq j \leq d-1$. We suspect this scenario is always the case, but we did not seriously investigate it as that is not our aim.
\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
$n$ & $4A(n, P_{100}) \geq \lambda = \lambda(n, P_{100})$ & $4A(n, P_1) \geq \lambda = \lambda(n, P_1)$ \\
\hline
2 & 2.444 & 21.68 & 1.493 \\
3 & 2.734 & 17.63 & 1.827 \\
4 & 2.904 & 15.50 & 2.039 \\
5 & 3.021 & 14.11 & 2.193 \\
6 & 3.108 & 13.10 & 2.311 \\
7 & 3.176 & 12.33 & 2.406 \\
8 & 3.231 & 11.70 & 2.485 \\
9 & 3.277 & 11.19 & 2.553 \\
10 & 3.316 & 10.75 & 2.611 \\
20 & 3.530 & 8.340 & 2.951 \\
50 & 3.720 & 6.043 & 3.293 \\
100 & 3.814 & 4.763 & 3.483 \\
200 & 3.878 & 3.764 & 3.625 \\
500 & 3.931 & 2.764 & 3.757 \\
1000 & 3.956 & 2.191 & 3.826 \\
2000 & 3.971 & 1.737 & 3.876 \\
5000 & 3.984 & 1.279 & 3.921 \\
10000 & 3.990 & 1.015 & 3.944 \\
\hline
\end{tabular}
\end{center}

Table 1. Values of $A = A(n, P_d)$ when $d = 100$ versus $d = 1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Plot of $f(n) = 4A(n, P_d)$ for $2 \leq n \leq 100$ with $d = 1$ (red circles) below and $d = 100$ (blue diamonds) above.}
\end{figure}
Thus, for each integer \( d \geq 1 \), let \( P_d(x) \) be the polynomial associated to the unique solution \( a \) (if it exists) satisfying \( C_{2j}^{(d)}(a) = 0 \) for \( 1 \leq j \leq d - 1 \). For example,

\[
P_1(x) = x, \quad P_2(x) = x + x^2, \quad P_3(x) = x + x^2 + \frac{2}{3}x^3, \quad P_4(x) = x + x^2 + \frac{4}{5}x^3 + \frac{2}{5}x^4.
\]

These are the same polynomials exhibited in [HB92, Section 4]. Estimate (1.3) is based on the choice of \( P_1(x) = x \). Setting \( d = 100 \), we may compute \( P_{100}(x) \) and subsequently \( A(n,P_{100}) \) in Table 1 for fixed values of \( n \). One can compare \( A(n,P_{100}) \) with \( A(n,P_1) \) in Table 1 and Figure 1 above to observe the savings afforded by Corollary 1.2 over (1.3).

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