STARSHAPEH COMPACT HYPERSURFACES WITH PRESCRIBED
m–TH MEAN CURVATURE IN HYPERBOLIC SPACE

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1. Introduction

Let $S^n$ be the unit sphere in the Euclidean space $\mathbb{R}^{n+1}$, and let $e$ be the standard metric on $S^n$ induced from $\mathbb{R}^{n+1}$. Suppose that $(u,\rho)$ are the spherical coordinates in $\mathbb{R}^{n+1}$, where $u \in S^n$, $\rho \in [0, \infty)$. By choosing the smooth function $\varphi(\rho) := \sinh^2 \rho$ on $[0, \infty)$ we can define a Riemannian metric $h$ on the set $\{ (u,\rho) : u \in S^n, 0 \leq \rho < \infty \}$ as follows

$$h = d\rho^2 + \varphi(\rho)e.$$  

This gives the space form $\mathbb{R}^{n+1}(-1)$ which is the hyperbolic space $H^{n+1}$ with sectional curvature $-1$. For a smooth hypersurface $M$ in $\mathbb{R}^{n+1}(-1)$, we denote by $\lambda_1, \cdots, \lambda_n$ its principal curvatures with respect to the metric $g := h|_M$. Then, for each $1 \leq k \leq n$, the $k$-th mean curvature of $M$ is defined as

$$H_k = \left( \begin{array}{c} n \\ k \end{array} \right)^{-1} \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$  

Let $\psi(u, \rho)$, $u \in S^n$, $\rho \in (0, \infty)$, be a given positive smooth function satisfying suitable conditions. We are interested in the existence of a smooth hypersurface $M$ embedded in $\mathbb{R}^{n+1}(-1)$ as a graph over $S^n$ so that its $k$-th mean curvature is given by $\psi$. We refer the readers to [7] and [5] for the introductory material and the history of this problem.

It is clear that $M := \{(u, z(u)) : u \in S^n \}$ is an embedded hypersurface in $\mathbb{R}^{n+1}(-1)$ for any smooth positive function $z$ on $S^n$. We call $z$ $k$-admissible if the principal curvatures $(\lambda_1(z(u)), \cdots, \lambda_n(z(u)))$ of $M$ belong to $\Gamma_k$, where $\Gamma_k$ is the connected component of $\{ \lambda \in \mathbb{R}^n : H_k(\lambda) > 0 \}$ containing the positive cone $\{ \lambda \in \mathbb{R}^n : \lambda_1 > 0, \cdots, \lambda_n > 0 \}$.

The main result of this paper is the following

**Theorem 1.** Let $1 \leq k \leq n$, and let $\psi$ be a smooth positive function in the annulus $\Omega : u \in S^n, \rho \in [R_1, R_2], 0 < R_1 < R_2 < \infty$, satisfying the conditions

$$\psi(u, R_1) \geq \coth^k(R_1) \quad \text{and} \quad \psi(u, R_2) \leq \coth^k(R_2) \quad \text{for } u \in S^n$$

and

$$\frac{\partial}{\partial \rho} (\psi(u, \rho) \sinh^k \rho) \leq 0 \quad \text{for all } u \in S^n \text{ and } \rho \in [R_1, R_2].$$

Then there exists a positive smooth $k$-admissible function $z$ on $S^n$ such that the closed hypersurface $M := \{(u, z(u)) : u \in S^n \}$ is in $\Omega \subset \mathbb{R}^{n+1}(-1)$, and its $k$-th mean
curvature is given by $\psi$:

$$H_k(\lambda_1(z(u)), \cdots, \lambda_n(z(u))) = \psi(u, z(u)) \quad \forall \ u \in S^n.$$  

In the Euclidean space $(\mathbb{R}^{n+1}(0))$, such results were obtained in the case $k = 0$ by Bakelman and Kantor [3], [4] and by Treibergs and Wei [19], in the case $k = n$ by Oliker [15], and for general $k$ by Caffarelli, Nirenberg and Spruck [7]. In the elliptic space $(\mathbb{R}^{n+1}(+1))$, such result is the combination of the work of Barbosa, Lira and Oliker [5] and that of Li and Oliker [14]. The $k = n$ case in Theorem 1 was established by Oliker [16]. Our proof of Theorem 1 uses the $C^0$ and $C^1$ a priori estimates obtained in [5] and the arguments in [14] which is based on the degree theory for fully nonlinear elliptic operator of second order developed in [12]. The main work for us to prove Theorem 1 is to give the $C^2$ a priori estimates. In establishing the $C^2$ estimates, we make use Lemma 2, a quantitative version of a theorem of Davis [9] which, to our knowledge, was given in [1]. The theorem in [9] says that a rotationally invariant function on symmetric matrices is concave if and only if it is concave on the diagonal matrices, while Lemma 2 allows the use of this term in making $C^2$ a priori estimates. The use of such a concave term in $C^2$ estimates for solutions of the Monge-Ampère equation has been extensive, see e.g. Calabi [8] and Pogorelov [17]. The use of Lemma 2 in $C^2$ estimates for solutions of more general equations can be found in [1], [2], [11], [18], [20] and [21].

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### 2. SOME FUNDAMENTAL FORMULAE

Let us define a function $f$ on $\Gamma_k$ by

$$f(\lambda) = \left( \frac{n}{k} \right)^{-1} \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} \right)^{1/k},$$

where $\lambda := (\lambda_1, \cdots, \lambda_n) \in \Gamma_k$. It is well known that $f$ is smooth, positive, concave, and strictly increasing with respect to each variable, see e.g. [6]. Now our problem is equivalent to finding a smooth positive $k$-admissible function $z$ on $S^n$ so that

$$F(B) = \overline{\psi}$$

on $\mathcal{M} := \{(u, z(u)) : u \in S^n\}$, where $\overline{\psi} := \psi^{1/k}$, $B$ is the second fundamental form of $\mathcal{M}$, and $F(B) := f(\lambda(B))$ with $\lambda(B)$ being the eigenvalues of $B$ with respect to the metric $g$ on $\mathcal{M}$.

Suppose now $\mathcal{M}$ is the graph of a smooth positive $k$-admissible function $z$ on $S^n$. Let us recall the formulae given in [5] for the components of $g = (g_{ij})$ and $B = (b_{ij})$ on $\mathcal{M}$ under a local coordinate. Let $\theta^1, \cdots, \theta^n$ be a smooth local coordinate of $S^n$, which
of course gives a local coordinate of \( M \). If we denote by \( \{ e_{ij} \} \) the components of \( e \) under this local coordinate, and set \( z_i = \frac{\partial z}{\partial \theta_i} \) and \( z_{ij} = \frac{\partial^2 z}{\partial \theta_i \partial \theta_j} \), then

\[
g_{ij} = \varphi e_{ij} + z_i z_j \tag{2}
\]

and

\[
g^{ij} = \frac{1}{\varphi} \left[ e^{ij} - \frac{z^i z^j}{\varphi + |\nabla' z|^2} \right], \quad z^i = e^{ij} z_j \tag{3}
\]

where \((g^{ij}) = (g_{ij})^{-1}\), \((e^{ij}) = (e_{ij})^{-1}\), and \( \nabla' \) denotes the Levi-Civita connection on \( S^n \). Moreover, for the second fundamental form we have

\[
b_{ij} = \varphi \sqrt{\varphi^2 + \varphi |\nabla' z|^2} \left\{ -\nabla'_{ij} z + \frac{\partial \ln \varphi}{\partial \rho} z_i z_j + \frac{1}{2} \frac{\partial \varphi}{\partial \rho} e_{ij} \right\}. \tag{4}
\]

We also need the following well-known fundamental equations for a hypersurface \( M \) in \( \mathcal{R}^{n+1}(-1) \):

- **Codazzi equation:** \( \nabla_i b_{jk} = \nabla_j b_{ki} = \nabla_k b_{ij} \) \tag{5}
- **Gauss equation:** \( R_{ijkl} = (b_{ik} b_{jl} - b_{il} b_{kj}) - (g_{ik} g_{jl} - g_{il} g_{kj}) \) \tag{6}
- **Ricci equation:** \( \nabla_i \nabla_k b_{ij} - \nabla_k \nabla_i b_{ij} = b_{ip} g^{pq} R_{qjkl} + b_{jp} g^{pq} R_{qikl} \) \tag{7}

where \( R_{ijkl} \) denotes the Riemannian curvature tensor of \( M \), and \( \nabla_i \) and \( \nabla_j \nabla_j \) the covariant differentiations in the metric \( g \) with respect to some local coordinates on \( M \).

As the preparation for deriving the \( C^2 \)-estimates, let us introduce the following two functions on \( M \)

\[
\tau = \frac{\varphi(z)}{\sqrt{\varphi(z) + |\nabla' z|^2}} \quad \text{and} \quad \eta = -\cosh(z). \tag{8}
\]

We have

**Lemma 1.** For \( \tau \) and \( \eta \) the following equations hold

\[
\nabla_i \tau = -b_{ip} g^{pq} \nabla_q \eta, \tag{9}
\]

\[
\nabla_{ij} \tau = -\nabla_p b_{ij} g^{pq} \nabla_q \eta - \tau b_{ip} g^{pq} b_{qj} - \eta b_{ij}, \tag{10}
\]

\[
\nabla_i \eta = \tau b_{ij} + \eta g_{ij}. \tag{11}
\]

**Proof.** These formulae have been derived in [5] by using another model of \( \mathcal{R}^{n+1}(-1) \). In fact, we can show them directly. Since (10) is an immediate consequence of (9), (11) and the Codazzi equation (5), it suffices to verify (9) and (11). Let \( c(\rho) = \cosh(\rho) \) and \( s(\rho) = \sinh(\rho) \). Then

\[
\nabla_i \tau = \frac{2sc}{\sqrt{\varphi + |\nabla' z|^2}} \nabla_i z_i - \frac{\varphi}{(\varphi + |\nabla' z|^2)^{3/2}} \left( sc z_i + e^{pq} \nabla'_{ip} z^q \nabla'_{ij} z_j \right) = \frac{1}{(\varphi + |\nabla' z|^2)^{3/2}} \left( sc \varphi z_i + 2sc z_i |\nabla' z|^2 - \varphi \nabla'_{ip} z^q \right). \]
Noting that $\nabla_q \eta = -sz_q$, we have from (3) and (4) that
\[ b_{ip} g^{mp} \nabla_q \eta = -\frac{s}{\varphi + |\nabla'z|^2} b_{ip} z^p = -\nabla_i \tau. \]

Let us now verify (11) for any fixed $\bar{u} \in S^n$. Noting that the both sides of (11) are tensorial, we may assume that the local coordinates are chosen such that $\frac{\partial y_{jk}}{\partial \theta^i} = 0$ at $\bar{u}$. Then from (2) we have
\[ \frac{\partial e_{lj}}{\partial \theta^i} = -\frac{2sc}{\varphi^2} (g_{lj} - z_l z_j) z_i - \frac{1}{\varphi} (z_l z_j + z_l z_{ij}). \]

Thus the corresponding Christoffel symbols of $S^n$ are given by
\[ \Gamma^k_{ij} = \frac{1}{2} \left\{ \frac{\partial e_{lj}}{\partial \theta^i} + \frac{\partial e_{li}}{\partial \theta^j} - \frac{\partial e_{ij}}{\partial \theta^l} \right\} = -\frac{sc}{\varphi} \left\{ e^{kl} g_{lj} z_i + e^{kl} g_{li} z_j - g_{ij} z^k - z_l z_j z_i \right\} - \frac{1}{\varphi} z_{ij} z^k. \] (12)

This, together with (2), gives
\[ \nabla'_{ij} z = z_{ij} - \Gamma^k_{ij} z_k \]
\[ = \frac{\varphi + |\nabla'z|^2}{\varphi} z_{ij} + \frac{sc}{\varphi^2} \left\{ g_{lj} z_i z^l + g_{li} z_j z^l - g_{ij} |\nabla'z|^2 - z_l z_j |\nabla'z|^2 \right\} \]
\[ = \frac{\varphi + |\nabla'z|^2}{\varphi} z_{ij} + \frac{sc}{\varphi} \left( 2z_i z_j - |\nabla'z|^2 e_{ij} \right). \]

Noting that $\nabla_{ij} z = z_{ij}$ at $\bar{u}$, we thus have
\[ \nabla_{ij} z = \frac{1}{\varphi + |\nabla'z|^2} \left( \varphi \nabla'_{ij} z - 2sc z_i z_j + sc |\nabla'z|^2 e_{ij} \right). \] (13)

Therefore
\[ \nabla_{ij} \eta = -cz_i z_j - s \nabla_{ij} z = -cz_i z_j - \frac{s}{\varphi + |\nabla'z|^2} \left( \varphi \nabla'_{ij} z - 2sc z_i z_j + sc |\nabla'z|^2 e_{ij} \right). \]

But from (2) and (4) we can see that the right hand side of the above equation is exactly $\tau b_{ij} + \eta g_{ij}$. $\blacksquare$

3. $C^2$-estimates

Now we are in a position to derive the $C^2$ estimates for any smooth positive $k$-admissible solutions of (1) in $\mathcal{R}^{n+1}(-1)$. Let us set
\[ f_i = \frac{\partial f}{\partial \lambda_i} \quad F^{ij} = \frac{\partial F}{\partial b_{ij}} \quad \text{and} \quad F^{ij,kl} = \frac{\partial^2 F}{\partial b_{ij} \partial b_{kl}}. \]

We will achieve our aim by choosing suitable test function and making full use of the terms involving $F^{ij,kl}$. In particular, we need the following

Lemma 2. ([1]) For any symmetric matrix $(\eta_{ij})$ there holds
\[ F^{ij,kl} \eta_{ij} \eta_{kl} = \sum_{i,j} \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \eta_{ii} \eta_{jj} + \sum_{i \neq j} \frac{f_i - f_j}{\lambda_i - \lambda_j} \eta_{ij}^2. \]
where the second term on the right-hand side must be interpreted as a limit whenever \( \lambda_i = \lambda_j \).

This result was, to our knowledge, first stated in [1]; for proofs one may consult [11, 2].

**Theorem 2.** Let \( 1 \leq k \leq n \) and let \( \psi \) be a positive \( C^2 \) function in the annulus \( \Omega : u \in S^n, \rho \in [R_1, R_2], 0 < R_1 < R_2 < a \). Let \( z \in C^4(S^n) \) be a positive \( k \)-admissible solution of (1) in \( \mathbb{R}^{n+1}(-1) \) satisfying

\[
R_1 \leq z \leq R_2 \quad \text{and} \quad |\nabla' z| \leq C_0 = \text{constant on } S^n.
\]

Then

\[
\|z\|_{C^2(S^n)} \leq C,
\]

where the constant \( C \) depends only on \( k, n, R_1, R_2, C_0 \) and \( \|\psi\|_{C^2(\Omega)} \).

**Proof.** We will estimate the maximal principal curvature of \( \mathcal{M} \). Since \( z \) is \( k \)-admissible, this estimate, together with the \( C^0 \) and \( C^1 \) bounds of \( z \) and the equation (4), implies an estimate for \( \|z\|_{C^2(S^n)} \). Consider the function

\[
\widetilde{W}(u, \xi) = B(\xi, \xi) \exp \left[ \Phi(\tau) - \beta \eta \right],
\]

where \( u \in S^n, \xi \) is a unit tangent vector of \( \mathcal{M} \) at \((u, z(u))\), \( \tau \) and \( \eta \) are defined as in (8), and the function \( \Phi \) and the constant \( \beta > 0 \) will be determined later. Suppose the maximum of \( \widetilde{W} \) is attained at some point \( \bar{u} \in S^n \) in the unit tangential direction \( \bar{\xi} \) of \( \mathcal{M} \) at \((\bar{u}, z(\bar{u}))\). We may choose the local coordinates \( \theta^1, \cdots, \theta^n \) around \( \bar{u} \) such that

\[
g_{ij} = \delta_{ij} \quad \text{and} \quad \frac{\partial g_{ij}}{\partial \theta^k} = 0 \text{ at } \bar{u}.
\]

Moreover, since \( \bar{\xi} \) is the maximal principal direction of \( \mathcal{M} \) at \((\bar{u}, z(\bar{u}))\), such coordinates can be chosen so that \( \{b_{ij}\} \) is diagonal at \( \bar{u} \) and \( b_{11}(\bar{u}) = B(\bar{\xi}, \bar{\xi}) \).

Consider the local function \( Z = b_{11}/g_{11} \). By direct calculation we have at \( \bar{u} \) that

\[
\nabla_i Z = \frac{\partial b_{11}}{\partial \theta^i} = \nabla_i b_{11}
\]

and

\[
\nabla_i \nabla_j Z = \frac{\partial^2 b_{11}}{\partial \theta^i \partial \theta^j} - b_{11} \frac{\partial^2 g_{11}}{\partial \theta^i \partial \theta^j} = \frac{\partial^2 b_{11}}{\partial \theta^i \partial \theta^j} - 2 \frac{\partial \Gamma^1_{ij}}{\partial \theta^i} b_{11} = \nabla_i \nabla_j b_{11}.
\]

It is clear that the function

\[
W(u) = Z(u) \exp \left[ \Phi(\tau) - \beta \eta \right],
\]

has a local maximum at \( \bar{u} \). Thus at \( \bar{u} \)

\[
0 = \nabla_i (\log W) = \frac{\nabla_i Z}{Z} + \Phi \nabla_i \tau - \beta \nabla_i \eta = \frac{\nabla_i b_{11}}{b_{11}} + \Phi' \nabla_i \tau - \beta \nabla_i \eta \quad (14)
\]
and the matrix
\[
\{\nabla_{ij}(\log W)\} = \left\{\frac{\nabla_i\nabla_j Z}{Z} - \frac{\nabla_i Z \nabla_j Z}{Z^2} + \Phi' \nabla_i \tau + \Phi'' \nabla_i \tau \nabla_j \tau - \beta \nabla_i \eta\right\}
\]
\[
= \left\{\frac{\nabla_i \nabla_j b_{i1}}{b_{i1}} - \frac{\nabla_i b_{i1} \nabla_j b_{i1}}{b_{i1}^2} + \Phi' \nabla_i \tau + \Phi'' \nabla_i \tau \nabla_j \tau - \beta \nabla_i \eta\right\}
\]
is negative semi-definite. Therefore
\[
0 \geq F^{ij} \nabla_{ij}(\log W) = \frac{1}{b_{i1}} F^{ij} \nabla_i \nabla_j b_{i1} - \frac{1}{b_{i1}^2} F^{ij} \nabla_i b_{i1} \nabla_j b_{i1} + \Phi' F^{ij} \nabla_i \tau \nabla_j \tau
\]
\[
+ \Phi'' F^{ij} \nabla_i \tau \nabla_j \tau - \beta F^{ij} \nabla_i \eta. \quad (15)
\]
Since \(\{b_{ij}\}\) is diagonal at \(\bar{u}\), \(\{F^{ij}\}\) is also diagonal there and \(F^{ii} = f_i\). For simplicity, we let \(\lambda_i = b_{ii}(\bar{u})\) and assume \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\), moreover we may assume \(\lambda_1 \geq 1\). Then, see lemma 2 in [10] or lemma A.2 in [13], we have \(f_1 \leq f_2 \leq \cdots \leq f_n\). It follows from (15) that
\[
0 \geq \frac{1}{\lambda_i} \sum_i f_i \nabla_i \nabla_i b_{i1} - \frac{1}{\lambda_i^2} \sum_i f_i |\nabla_i b_{i1}|^2 + \Phi' \sum_i f_i \nabla_i \tau
\]
\[
+ \Phi'' \sum_i f_i |\nabla_i \tau|^2 - \beta \sum_i f_i \nabla_i \eta. \quad (16)
\]
Now we take the covariant differentiation on (1) to get
\[
F^{ij} \nabla_1 \nabla_1 b_{ij} + F^{ij,kl} \nabla_1 b_{ij} \nabla_1 b_{kl} = \nabla_1 \bar{\psi}. \quad (17)
\]
\(\bar{\psi}\)From (5), (6) and (7) it follows that
\[
\nabla_1 \nabla_1 b_{ii} = \nabla_i \nabla_i b_{ii} = \nabla_i b_{ii} + \sum_k b_{ik} R_{kii} + \sum_k b_{ik} R_{k1l}
\]
\[
= \nabla_i b_{ii} + b_{ii} b_{ii} - b_{ii} b_{ii} - (b_{ii} b_{ii} + b_{ii} - b_{ii} b_{ii} + i \delta_{ii} + b_{ii} - b_{ii} \delta_{ii}).
\]
This shows that
\[
F^{ij} \nabla_1 \nabla_1 b_{ij} = \sum_i f_i \nabla_i \nabla_i b_{i1} + \lambda_i \sum_i f_i \lambda_i - \lambda_i^2 \sum_i f_i \lambda_i + \lambda_i \mathcal{T} - \sum_i f_i \lambda_i,
\]
where \(\mathcal{T} := \sum_i f_i\). Since the degree one homogeneity of \(f\) implies \(\sum_i f_i \lambda_i = \bar{\psi}\), the above equation together with (17) gives
\[
\sum_i f_i \nabla_i \nabla_i b_{i1} = -F^{ij,kl} \nabla_1 b_{ij} \nabla_1 b_{kl} + \nabla_1 \bar{\psi} + \lambda_i^2 \bar{\psi} + \bar{\psi} - \lambda_i \sum_i f_i \lambda_i^2 - \lambda_i \mathcal{T}.
\]
Plugging this into (16), noting that \(\bar{\psi} \geq c_0 > 0\) and \(\lambda_1 \geq 1\), we therefore obtain
\[
0 \geq c_0 \lambda_1 - \frac{1}{\lambda_1} F^{ij,kl} \nabla_1 b_{ij} \nabla_1 b_{kl} + \frac{\nabla_1 \bar{\psi}}{\lambda_1} - \sum_i f_i \lambda_i^2 - \mathcal{T} - \frac{1}{\lambda_i^2} \sum_i f_i |\nabla_i b_{i1}|^2
\]
\[
+ \Phi' \sum_i f_i \nabla_i \tau + \Phi'' \sum_i f_i |\nabla_i \tau|^2 - \beta \sum_i f_i \nabla_i \eta. \quad (18)
\]
From (10) and (11) we have
\[
\beta \sum_i f_i \nabla_i \eta = \beta \tau \sum_i f_i \lambda_i + \beta \eta T = \beta \tau \psi + \beta \eta T.
\] (19)
and
\[
\Phi' \sum_i f_i \nabla_i \tau = \Phi' \left\{ - \sum_p \nabla_p \eta \left( \sum_i f_i \nabla_p b_{ii} \right) - \tau \sum_i f_i \lambda_i^2 - \eta \sum_i f_i \lambda_i \right\}
\leq -C|\Phi'| - \Phi' \tau \sum_i f_i \lambda_i^2.
\] (20)

Here we used the facts $|\nabla_p \eta| \leq C$ and $|\nabla_p \psi| \leq C$ at $\bar{u}$ which can be demonstrated as follows. Since $g_{ij} = \delta_{ij}$ at $\bar{u}$, it follows from (2) that $(z_p)^2 \leq 1$ at $\bar{u}$. Note that $\nabla_p \eta = -\sinh(z)z_p$. Therefore $|\nabla_i \eta| \leq C$ at $\bar{u}$. For $|\nabla_p \psi|$, we note that $\nabla_p \psi = \bar{\psi}_p + \bar{\psi}_z z_p$. Thus, by using (3), we have at $\bar{u}$ that
\[
|\nabla_p \psi|^2 \leq C \left( 1 + |\bar{\psi}_p|^2 \right) \leq C \left( 1 + g^{ij} \bar{\psi}_i \bar{\psi}_j \right) \leq C \left( 1 + \varphi^{-1} e^{ij} \bar{\psi}_i \bar{\psi}_j \right)
= C \left( 1 + \varphi^{-1} |\nabla \bar{\psi}|^2 \right) \leq C.
\]

One can show that
\[
\frac{\nabla_{11} \bar{\psi}}{\lambda_1} \geq -C \quad \text{at } \bar{u}. \tag{21}
\]
To see this, note that $\frac{\partial g_{ij}}{\partial x^k} = 0$ at $\bar{u}$, we have
\[
\nabla_{11} \bar{\psi} = \bar{\psi}_{11} + 2 \bar{\psi}_{z1} z_1 + \bar{\psi}_{zz} (z_1)^2 + \bar{\psi}_z z_{11}.
\]
Similar to the above argument we can show $|\bar{\psi}_{z1}| \leq C$ at $\bar{u}$. Therefore at $\bar{u}$
\[
|\nabla_{11} \bar{\psi}| \leq C \left( 1 + |\bar{\psi}_{11}| + |z_{11}| \right).
\]
Let us estimate $|\bar{\psi}_{11}|$. It follows from (3) that
\[
|\nabla_{11} \bar{\psi}|^2 \leq g^{ik} g^{jl} \nabla_{ij} \bar{\psi} \nabla_{kl} \bar{\psi}
\leq \varphi^{-2} e^{ik} e^{jl} \bar{\psi} \nabla_{ij} \bar{\psi} + \varphi^{-2} (\varphi + |\nabla' z|^2)^{-2} (z^i z^j \nabla_{ij} \bar{\psi})^2
\leq \varphi^{-2} |\nabla' \bar{\psi}|^2 + \varphi^{-2} (\varphi + |\nabla' z|^2)^{-2} |\nabla' \bar{\psi}|^2 |\nabla' z| \nabla' \bar{\psi}
\leq \varphi^{-2} |\nabla' \bar{\psi}|^2 + \varphi^{-2} (\varphi + |\nabla' z|^2)^{-2} |\nabla' \bar{\psi}|^2 |\nabla' z|^4
\leq C.
\]
By using (2) we obtain at $\bar{u}$ that
\[
\sum_i |z_i|^2 = g_{ij} z^i z^j = \varphi \left| \nabla' z \right|^2 + |\nabla' z|^4 \leq C.
\]
This together with (3) then implies $|e^{iy}| \leq C$. Thus it follows from (12) that $|\Gamma_{11}^{ik}| \leq C(1 + |z_{11}|)$ at $\bar{a}$. Since $\bar{\psi}_{11} = \nabla'_{11} \bar{\psi} + \Gamma_{11}^{ik} \bar{\psi}_k$, we therefore have $|\bar{\psi}_{11}| \leq C(1 + |z_{11}|)$. Since $z_{11} = \nabla_{11}z$ at $\bar{a}$, from (13) and (4) we finally obtain
\[
|\nabla_{11}\bar{\psi}| \leq C(1 + |z_{11}|) \leq C(1 + \lambda_1)
\]
which gives (21).

Combining (18), (19), (20) and (21), we thus obtain
\[
0 \geq c_0 \lambda_1 - C(1 + |\Phi'|) - (1 + \beta \eta)T - \beta \tau \bar{\psi} - (\Phi' \tau + 1) \sum_i f_i \lambda_i^2 + \Phi'' \sum_i f_i |\nabla_i \tau|^2
\]
\[
- \frac{1}{\lambda_1^2} \sum_i f_i |\nabla_i b_{11}|^2 - \frac{1}{\lambda_1} F^{ijkl} \nabla_i b_{ij} \nabla_k b_{kl}.
\]
(22)

Now we will use Lemma 2, similar to the way used in [18].

\textbf{Case 1.} $\lambda_n < -\theta \lambda_1$ for some positive constant $\theta$ (to be chosen later).

In this case, using the concavity of $F$ we may discard the last term on the right hand side of (22) since it is nonnegative. Also from (14) we have for any $\varepsilon > 0$
\[
\frac{1}{\lambda_1^2} \sum_i f_i |\nabla_i b_{11}|^2 = \sum_i f_i |\Phi' \nabla_i \tau - \beta \nabla_i \eta|^2
\]
\[
\leq (1 + \varepsilon^{-1}) \beta^2 \sum_i f_i |\nabla_i \eta|^2 + (1 + \varepsilon) (\Phi')^2 \sum_i f_i |\nabla_i \tau|^2.
\]

Therefore, from (22) it yields
\[
0 \geq c_0 \lambda_1 - C(1 + |\Phi'|) - [(1 + \beta \eta) + C(1 + \varepsilon^{-1}) \beta^2] T - \beta \tau \bar{\psi} - (\Phi' \tau + 1) \sum_i f_i \lambda_i^2
\]
\[
+ [\Phi'' - (1 + \varepsilon)(\Phi')^2] \sum_i f_i |\nabla_i \tau|^2.
\]
(23)

Using (9) we have
\[
\sum_i f_i |\nabla_i \tau|^2 = \sum_i f_i \lambda_i^2 |\nabla_i \eta|^2 \leq c_1 \sum_i f_i \lambda_i^2
\]
for some constant $c_1 > 0$. If we can choose $\Phi$ such that $\Phi'' - (1 + \varepsilon)(\Phi')^2 \leq 0$, then from (23) we have
\[
0 \geq c_0 \lambda_1 - C(1 + |\Phi'|) - [(1 + \beta \eta) + C(1 + \varepsilon^{-1}) \beta^2] T - \beta \tau \bar{\psi}
\]
\[
+ [-(\Phi' \tau + 1) + c_1 (\Phi'' - (1 + \varepsilon)(\Phi')^2)] \sum_i f_i \lambda_i^2.
\]
(24)

In order to choose $\Phi$, let $a > 0$ be a positive number such that $\tau \geq 2a$ which is guaranteed by our assumption. Then we define
\[
\Phi(\tau) = -\log(\tau - a).
\]
It is easy to check that $\Phi'' - (1 + \varepsilon)(\Phi')^2 < 0$. Moreover, for $\varepsilon = \frac{a^2}{2c_1}$ we have
\[
-(\Phi' \tau + 1) + c_1 (\Phi'' - (1 + \varepsilon)(\Phi')^2) = \frac{a}{\tau - a} - \frac{c_1 \varepsilon}{(\tau - a)^2} \geq \frac{a^2}{2(\tau - a)^2} \geq c_2 > 0.
\]
Therefore we get from (24) that

\[ 0 \geq c_0 \lambda_1 - C - C T + c_2 \sum_i f_i \lambda_i^2. \]

Since \( \lambda_n \leq -\theta \lambda_1 \) and \( f_n \geq \frac{1}{n} T \), we have \( \sum_i f_i \lambda_i^2 \geq f_n \lambda_n^2 \geq \frac{1}{n} \theta^2 T \lambda_1^2 \). Hence

\[ 0 \geq c_0 \lambda_1 - C - C T + \frac{c_2 \theta^2}{n} T \lambda_1^2. \]

This clearly implies \( \lambda_1 \) is bounded from above.

**Case 2.** \( \lambda_n \geq -\theta \lambda_1 \).

We now have \( \lambda_i \geq -\theta \lambda_1 \) for all \( i = 1, \cdots, n \). Let us partition \( \{1, \cdots, n\} \) into two parts: \( I = \{ j : f_j \leq 4 f_1 \} \) and \( J = \{ j : f_j > 4 f_1 \} \). Using (14) we have for \( i \in I \) that

\[
\frac{1}{\lambda_i^2} f_i |\nabla_i b_{11}|^2 = f_i |\Phi' \nabla_i \tau - \beta \nabla_i \eta|^2 \\
\leq (1 + \varepsilon)(\Phi')^2 f_i |\nabla_i \tau|^2 + (1 + \varepsilon^{-1}) \beta^2 f_i |\nabla_i \eta|^2 \\
\leq (1 + \varepsilon)(\Phi')^2 f_i |\nabla_i \tau|^2 + C(1 + \varepsilon^{-1}) \beta^2 f_i.
\]

Therefore it follows from (22) that

\[
0 \geq c_0 \lambda_1 - C(1 + |\Phi'|) - \beta \tau \bar{\psi} - (1 + \beta \eta) T - (\Phi' \tau + 1) \sum_i f_i \lambda_i^2 \\
+ [\Phi'' - (1 + \varepsilon)(\Phi')^2] \sum_i f_i |\nabla_i \tau|^2 - C(1 + \varepsilon^{-1}) \beta^2 f_1 \\
- \frac{1}{\lambda_i^2} \sum_{j \in J} f_j |\nabla_j b_{11}|^2 - \frac{1}{\lambda_1} F^{ij,kl} \nabla_i b_{ij} \nabla_1 b_{kl}.
\]

Proceeding exactly as before we have

\[
-(\Phi' \tau + 1) \sum_i f_i \lambda_i^2 + [\Phi'' - (1 + \varepsilon)(\Phi')^2] \sum_i f_i |\nabla_i \tau|^2 \geq c_2 \sum_i f_i \lambda_i^2
\]

if we choose \( \varepsilon = \frac{\theta^2}{2c_1} \). So

\[
0 \geq c_0 \lambda_1 - C(1 + |\Phi'|) - \beta \tau \bar{\psi} - (1 + \beta \eta) T + c_2 \sum_i f_i \lambda_i^2 - C(1 + \varepsilon^{-1}) \beta^2 f_1 \\
- \frac{1}{\lambda_i^2} \sum_{j \in J} f_j |\nabla_j b_{11}|^2 - \frac{1}{\lambda_1} F^{ij,kl} \nabla_i b_{ij} \nabla_1 b_{kl}. \tag{25}
\]

By using Lemma 2 and noting \( 1 \not\in J \) we have

\[
-\frac{1}{\lambda_1} F^{ij,kl} \nabla_i b_{ij} \nabla_1 b_{kl} \geq -\frac{2}{\lambda_1} \sum_{j \in J} \frac{f_i - f_j}{\lambda_i - \lambda_j} |\nabla_j b_{ij}|^2 = -\frac{2}{\lambda_1} \sum_{j \in J} \frac{f_i - f_j}{\lambda_i - \lambda_j} |\nabla_j b_{11}|^2.
\]

Therefore

\[
0 \geq c_0 \lambda_1 - C(1 + |\Phi'|) - \beta \tau \bar{\psi} - (1 + \beta \eta) T - C(1 + \varepsilon^{-1}) \beta^2 f_1 + c_2 \sum_i f_i \lambda_i^2 \\
- \frac{2}{\lambda_1} \sum_{j \in J} \frac{f_i - f_j}{\lambda_i - \lambda_j} |\nabla_j b_{11}|^2 - \frac{1}{\lambda_1^2} \sum_{j \in J} f_j |\nabla_j b_{11}|^2. \tag{26}
\]
We claim that
\[-\frac{2}{\lambda_1} \frac{f_1 - f_j}{\lambda_1 - \lambda_j} \geq \frac{1}{\lambda_1^2} f_j, \quad \forall j \in J.\]
This is equivalent to showing $2f_1 \lambda_1 \leq f_j \lambda_1 + f_j \lambda_j$. Since $j \in J$, we have $f_j > 4f_1$. If $\lambda_j \geq 0$, this is obviously true. If $\lambda_j < 0$, then $-\theta \lambda_1 \leq \lambda_j < 0$, and hence
\[f_j \lambda_1 + f_j \lambda_j \geq (1 - \theta) f_j \lambda_1 \geq 4(1 - \theta) f_1 \lambda_1 \geq 2f_1 \lambda_1\]
if we choose $\theta = \frac{1}{2}$. From this claim and (26) we obtain
\[0 \geq c_0 \lambda_1 - C - \beta \tau \psi - (1 + \beta \eta) T + c_2 \sum_i f_i \lambda_i^2 - C(1 + \varepsilon^{-1}) \beta^2 f_1.\]
Recall the definition of $\eta$, we have $-c_3 \leq \eta \leq -c_4$ for two positive constants $c_3$ and $c_4$. Choose $\beta$ to be sufficiently large so that $-(1 + \beta \eta) \geq 0$. Then we get
\[0 \geq -C + c_0 \lambda_1 + c_2 f_1 \lambda_1^2 - C f_1.\]
This clearly implies an upper bound for $\lambda_1$.

\section{4. Proof of main result}

Since the proof of Theorem 1 essentially follows the lines in [14], only the sketch will be given below.

We may assume that neither $z(u) \equiv R_1$ nor $z(u) \equiv R_2$ is a solution of (1); otherwise we are done. Let us fix some $\overline{R}$ such that $R_1 < \overline{R} < R_2$ and define a family of functions
\[\overline{\psi}(u, \rho) := t \psi(u, \rho) + (1 - t) A^\varepsilon \coth^{1+\varepsilon}(\rho), \quad t \in [0, 1],\]
where $\varepsilon > 0$ is a positive constant and $A = \coth^{-1}(\overline{R})$. Fix $0 < \alpha < 1$, and denote by $C^{4,\alpha}(S^n)$ the subset of functions from $C^{4,\alpha}(S^n)$ which is $k$-admissible. We define a family of operators $\Psi(\cdot, t) : C^{4,\alpha}(S^n) \to C^{2,\alpha}(S^n)$ by
\[\Psi(z(u), t) \equiv F(B) - \overline{\psi}(u, z(u)), \quad u \in S^n,\]
where $z \in C^{4,\alpha}(S^n)$ and $B$ is the second fundamental form of $M := \{(u, z(u)) : u \in S^n\}$. Consider the family of equations
\[\Psi(z, t) \equiv 0.\tag{27}\]
One can show that neither $z(u) \equiv R_1$ nor $z(u) \equiv R_2$ is a solution of (27) for any $t \in [0, 1]$. Therefore, by the strong maximum principle, any solution $z \in C^{4,\alpha}(S^n)$ of (27) satisfying $R_1 \leq z(u) \leq R_2$ for all $u \in S^n$ must satisfy the strict inequalities
\[R_1 < z(u) < R_2 \quad \text{for all} \quad u \in S^n.\tag{28}\]
By using the $C^1$-estimates in [5], Theorem 2, the result of Evans and Krylov, and Schauder theory for second order uniformly elliptic equations one can obtain
\[\|z\|_{C^{4,\alpha}(S^n)} < C\tag{29}\]
for any solution \( z \in C^{4,\alpha}_a(S^n) \) of (27) satisfying (28), where \( C \) is a constant depending only on \( k, n, R_1, R_2 \) and \( \|\psi\|_{C^{2,\alpha}(\Omega)} \).

We can choose a constant \( \delta > 0 \) depending on \( k, n, R_1, R_2 \) and \( C \) such that
\[
\delta \leq \frac{\psi}{u, z(u)} \leq 1 \quad \text{for} \quad u \in S^n,
\]
where \( 0 \leq t \leq 1 \) and \( z \in C^{4,\alpha}(S^n) \) satisfying (28) and (29). Consequently, we can find an open set \( V \) of \( \Gamma_k \) satisfying \( \nabla \subset \Gamma_k \) such that \( \lambda(B) \in V \) for any \( z \in C^{4,\alpha}_a(S^n) \) satisfying (28), (29) and \( \delta \leq F(B) \leq \delta^{-1} \). Now we define an open bounded subset \( O^* \) of \( C^{4,\alpha}(S^n) \) by
\[
O^* := \{ z \in C^{4,\alpha}(S^n) : z \text{ satisfies } (28), (29) \text{ and } \lambda(B) \in V \}
\]
One can show that
\[
\Psi(\cdot, t)^{-1}(0) \cap \partial O^* = \emptyset \quad \text{for } 0 \leq t \leq 1
\]
when \( \Psi(\cdot, t) \) are viewed as maps from \( \overline{O^*} \subset C^{4,\alpha}(S^n) \) to \( C^{2,\alpha}(S^n) \). Therefore, the degree \( \text{deg}(\Psi(\cdot, t), O^*, 0) \) is defined for all \( 0 \leq t \leq 1 \) and is independent of \( t \); see [12].

Comparing a solution with spheres \( z \equiv \text{constant} \) and using the maximum principle as usual, we know that \( z_0(u) \equiv \overline{R} \) is the unique solution in \( O^* \) of the equation \( \Psi(z, 0) = 0 \). Clearly, the linearized operator \( \Psi_*(z_0, 0) \) is of the form
\[
\Psi_*(z_0, 0) = -a^{ij}(u)\nabla_{ij} + b^i(u)\nabla_i + c(u),
\]
where \( (a^{ij}(u)) \) is positive definite. Since
\[
\Psi(sz_0, 0) = \coth(sz_0) - A^\epsilon \coth^{1+\epsilon}(sz_0),
\]
we have, in view of \( A\coth(\overline{R}) = 1, \)
\[
\overline{R}_c(u) = \Psi_*(z_0, 0)(z) = \left. \frac{d}{ds} \right|_{s=1} \Psi(sz_0, 0) = \left. \frac{d}{d\rho} \right|_{\rho=\overline{R}} \left[ \coth(\rho) - A^\epsilon \coth^{1+\epsilon}(\rho) \right]
\]
\[
= -\left. \frac{d}{d\rho} \right|_{\rho=\overline{R}} \coth(\rho) > 0.
\]
Thus \( \Psi_*(z_0, 0) \) is an invertible operator from \( C^{4,\alpha}(S^n) \) to \( C^{2,\alpha}(S^n) \). It follows, as in [14], that
\[
\text{deg}(\Psi(\cdot, 1), O^*, 0) = \text{deg}(\Psi(\cdot, 0), O^*, 0) \neq 0.
\]
Therefore, the equation
\[
\Psi(z, 1) = 0, \quad z \in O^*
\]
has at least one solution. This completes the proof of Theorem 1.

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