A SUFFICIENT CONDITION FOR SUBELLIPTICITY OF THE $\bar{\partial}$-NEUMANN OPERATOR

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Abstract. We give a sufficient condition for subelliptic estimates for the $\bar{\partial}$-Neumann operator on smoothly bounded, pseudoconvex domains in $\mathbb{C}^n$. This condition is a quantified version of McNeal's condition ($\tilde{P}$) for compactness of the $\bar{\partial}$-Neumann operator, and it extends Catlin's sufficiency condition for subellipticity as it is less stringent.

1. Introduction

Let $\Omega \subset \subset \mathbb{C}^n$ be a smoothly bounded domain. Suppose that $p \in b\Omega$ is a point in the boundary of $\Omega$, and that $b\Omega$ is pseudoconvex near $p$. We shall show that the existence of a certain family of functions near the boundary point $p$ implies that a subelliptic estimate for the $\bar{\partial}$-Neumann operator holds near that point.

The $\bar{\partial}$-Neumann operator $N_{p,q}$ is the inverse of the complex Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ for $(p,q)$-forms. Establishing the existence of the $\bar{\partial}$-Neumann operator leads to a particular solution of the Cauchy-Riemann equations, but just in the $L^2$-sense. Thus one is not just interested in the existence of such an $L^2$-solution $u$ for given data $f$, but one is also interested in the kind of regularity statements that can be made about $u$ when $f$ is regular; for notation and details on the $\bar{\partial}$-Neumann problem see section 2.

On domains with certain geometric conditions on the boundary, the question of existence of a solution to the $\bar{\partial}$-Neumann problem was settled through the works of Hörmander [Hör], Kohn [Koh1, Koh2] and Morrey [Mor]. In fact, Hörmander's results in [Hör] imply that there exists a bounded operator $N_{p,q}$ on $L^2_{p,q}(\Omega)$, which inverts the complex Laplacian under the assumption that $\Omega$ is a bounded, pseudoconvex domain.

In the following, we will be concerned only with the local regularity question for the $\bar{\partial}$-Neumann problem, i.e. conditions on $\Omega$ which imply that $u := N_{p,q}f$ is smooth wherever $f$ is. A fundamental step concerning this question was done by Kohn and Nirenberg. They showed in [Koh-Nir] that, if a so-called subelliptic estimate of order $\epsilon$ holds for the $\bar{\partial}$-Neumann problem on a neighborhood $V$ of a given point $p$ in $b\Omega$, then $f_{|V} \in H^s_{p,q}(V)$ implies $N_{p,q}f_{|V'} \in H^{s+2\epsilon}_{p,q}(V')$ for $V' \subset \subset V$; here $H^s_{p,q}$ denotes the $L^2$-Sobolev space of order $s$ on $(p,q)$-forms. Thus it is natural to inquire about subelliptic estimates for the $\bar{\partial}$-Neumann problem.

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Denote by $D^{p,q}(V \cap \bar{\Omega})$ the set of smooth $(p,q)$-forms $u$, which are supported in $V \cap \bar{\Omega}$, such that $u$ belongs to the domain of $\bar{\partial}^*$. A subelliptic estimate of order $\epsilon > 0$ near $p \in b\Omega$ is said to hold, if

\begin{equation}
\|u\|^2 \leq C(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2) \quad \text{for all } u \in D^{p,q}(V \cap \bar{\Omega}),
\end{equation}

where the norm on the left hand side is the tangential $L^2$-Sobolev norm of order $\epsilon$.

The most general result concerning subelliptic estimates for the $\bar{\partial}$-Neumann problem was obtained by Catlin [Cat]. He showed that the existence of a certain, uniformly bounded family of functions $\{\lambda_\delta\}$ on a pseudoconvex domain is sufficient for a subelliptic estimate to hold. Moreover, Catlin proved that one can construct such a family of functions on any smoothly bounded, pseudoconvex domain, which is of finite type in the sense of D’Angelo [D’An].

We extend Catlin’s sufficiency result by replacing the boundedness condition on the weight functions $\lambda_\delta$ with that of self-bounded complex gradient, a weaker condition which allows unbounded families of functions. This notion was introduced by McNeal in [McN2].

**Definition 1.2.** Let $\Omega \subset \subset \mathbb{C}^n$ be a smoothly bounded domain. A plurisubharmonic function $\phi \in C^2(\Omega)$ is said to have a self-bounded complex gradient, if there exists a constant $C > 0$ such that

\begin{equation}
\left| \sum_{k=1}^n \frac{\partial \phi}{\partial z_k} (z) \xi_k \right|^2 \leq C \sum_{k,l=1}^n \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} (z) \xi_k \bar{\xi}_l
\end{equation}

holds for all $\xi \in \mathbb{C}^n$, $z \in \Omega$. We write $|\partial \phi|_{i\partial \bar{\partial} \phi} \leq \sqrt{C}$ when we mean \((1.3)\).

Notice that, if $\lambda \in C^2(\Omega)$ is plurisubharmonic and bounded, then $\phi = e^\lambda$ has a self-bounded complex gradient with $C = \sup_{z \in \Omega} e^{\lambda(z)}$. Furthermore, notice the behavior of inequality \((1.3)\) under scaling; replacing $\phi$ by $t\phi$ for $t > 0$, the left hand side of \((1.3)\) is quadratic in $t$, while the right hand side is linear in $t$.

The main result in this paper is the following:

**Theorem 1.4.** Let $\Omega \subset \subset \mathbb{C}^n$ be a smoothly bounded domain. Let $p$ be a given point in $b\Omega$ and suppose that $b\Omega \cap U$ is pseudoconvex, where $U$ is a neighborhood of $p$. Denote by $S_\delta$ the set $\{z \in \Omega \mid -\delta < r(z) < 0\}$, where $r$ is a fixed, smooth defining function of $\Omega$. Assume that for all $\delta > 0$ sufficiently small there exists a plurisubharmonic function $\phi_\delta \in C^2(\bar{\Omega} \cap U)$, such that

(i) $|\partial \phi_\delta|_{i\partial \bar{\partial} \phi_\delta} \leq C$, where the constant $C > 0$ is independent of $\delta$,

(ii) for all smooth $(p,q)$-forms $u$, $z \in S_\delta \cap U$ and for some $\epsilon \in (0, \frac{1}{2})$

\[ \sum_{|I|=p, |J|=q}^{'} \sum_{k,l=1}^n \frac{\partial^2 \phi_\delta}{\partial z_k \partial \bar{z}_l} (z) u_{I,k} \bar{u}_{I,l} \geq c \delta^{-2\epsilon} |u|^2, \]

where the constant $c > 0$ does not depend on $\delta$ or $u$. 

Then there exists a neighborhood $V \subset U$ of $p$ such that a subelliptic estimate of order $\epsilon$ holds.

The only difference between Theorem 1.4 and Catlin’s sufficiency result is that we substituted the uniform boundedness condition on $\{\lambda_{\delta}\}$ by condition (i). The existence of Catlin’s family of functions $\{\lambda_{\delta}\}$ implies the existence of the above family $\{\phi_{\delta}\}$ by setting $\phi_{\delta} = e^{\lambda_{\delta}}$. One reason, however, to generalize the Theorem of Catlin is to establish sharper subelliptic estimates in various geometric situations.

The uniform boundedness of $\{\lambda_{\delta}\}$ is crucial for Catlin’s proof as it lets him transform estimates with weights of the form $e^{-\lambda_{\delta}}$ into unweighted estimates. Families of functions which have a self-bounded complex gradient are in general not uniformly bounded, and so Catlin’s proof does not work. However, McNeal found a duality argument in [McN2], which allows one to pass to unweighted estimates from estimates with weights, when the weight functions have a self-bounded complex gradient.

The paper is structured as follows. In section 2 we review briefly the setting of the $\bar{\partial}$-Neumann problem. In section 3 we derive two weighted $L^2$-inequalities, which are specific for weights having a self-bounded complex gradient. Using those inequalities we obtain two compactness estimates on $\bar{\partial}^{\ast}N_q$ and $\bar{\partial}^{\ast}N_{q+1}$ in section 4. In section 5 we convert these compactness estimates to a family of $L^2$-estimates in terms of the Dirichlet form. With those estimates at hand we complete the proof of Theorem 1.4 in section 6. In the last section we consider an example domain to see how the functions $\{\phi_{\delta}\}$ can be constructed.

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2. Preliminaries

Let $\Omega \subset \subset \mathbb{C}^n$ be a smoothly bounded domain, i.e. $\Omega$ is bounded and there is a smooth function $r$ such that $\Omega = \{z \in \mathbb{C}^n \mid r(z) < 0\}$ and $\nabla r \neq 0$ whenever $r = 0$.

Let $0 \leq p, q \leq n$. We write an arbitrary $(p,q)$-form $u$ as

\begin{equation}
    u = \sum'_{|I|=p, |J|=q} u_{I,J} dz^I \wedge d\bar{z}^J,
\end{equation}

where $I = \{i_1, \ldots, i_p\}$, $J = \{j_1, \ldots, j_q\}$ and $dz^I = dz^{i_1} \wedge \cdots \wedge dz^{i_p}$, $d\bar{z}^J = d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$. Here $\sum'$ means that we only sum over strictly increasing index sets. We define the coefficients $u_{I,J}$ for arbitrary index sets $I$ and $J$, so that the $u_{I,J}$’s are antisymmetric functions of $I$ and $J$.

Let $\Lambda^{p,q}(\Omega)$ and $\Lambda^p_c(\Omega)$ denote the $(p,q)$-forms with coefficients in $C^\infty(\overline{\Omega})$ and $C^\infty_c(\Omega)$, respectively. We use the pointwise inner product $\langle \cdot, \cdot \rangle$ defined by $\langle dz^k, dz^l \rangle = \delta^k_l = \langle d\bar{z}^k, d\bar{z}^l \rangle$. By linearity we extend this inner product
to \((p, q)\)-forms. The global \(L^2\)-inner product on \(\Omega\) is defined by
\[
(u, v)_\Omega = \int_\Omega \langle u, v \rangle dV,
\]
where \(dV\) is the euclidean volume form. The \(L^2\)-norm of a \(u \in \Lambda^{p,q}_c(\Omega)\) on \(\Omega\) is then given by \(\|u\|_\Omega^2 = (u, u)_\Omega\) and we define \(L^2_{p,q}(\Omega)\) to be the completion of \(\Lambda^{p,q}_c(\Omega)\) under the \(L^2\)-norm; we drop the subscript \(\Omega\), when there is no reason for confusion.

If \(\phi \in C^2(\bar{\Omega})\), we denote by \(L^2_{p,q}(\Omega, \phi)\) the space of \((p, q)\)-forms \(u\) such that
\[
\|u\|_{\phi, \Omega}^2 = (u, u)_{\phi, \Omega} = \int_{\Omega} (ue^{-\phi}) dV < \infty.
\]
Notice that the weighted \(L^2\)-space, \(L^2_{p,q}(\Omega, \phi)\), equals \(L^2_{p,q}(\Omega)\).

Let \(u \in \Lambda^{p,q}(\bar{\Omega})\), then the \(\bar{\partial}\)-operator is defined as
\[
\bar{\partial}_{p,q} u = \bar{\partial} u := \sum_{|I|=p, |J|=q} \sum_{k=1}^n \bar{\partial}_k u_{I,J} d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J,
\]
where \(\bar{\partial}_k := \frac{\partial}{\partial z_k}\), and \(u\) is expressed as in \((2.1)\). Observe that \(\bar{\partial}^2 = 0\). We extend the differential operator \(\bar{\partial}\), still denoted by \(\bar{\partial}\), to act on non-smooth forms in the sense of distributions. Then, by restricting the domain of \(\bar{\partial}\) to those forms \(g \in L^2_{p,q}(\Omega)\), where \(\bar{\partial}g\) in the distributional sense belongs to \(L^2_{p,q+1}(\Omega)\), \(\bar{\partial}\) becomes an operator on Hilbert spaces at each form level. Note that \(\bar{\partial}\) is a densely defined operator on \(L^2_{p,q}(\Omega)\), because differentiation is a continuous map in the distributional sense. Thus we can define the Hilbert space adjoint, \(\bar{\partial}^*\), to \(\bar{\partial}\) with respect to the \(L^2\)-inner product on the appropriate form level in the usual way:
we say that \(u \in L^2_{p,q+1}(\Omega)\) belongs to the domain of \(\bar{\partial}^*\), i.e. \(u \in \text{Dom}(\bar{\partial}^*)\), if there exists a constant \(C > 0\) so that
\[
|\langle \bar{\partial} w, u \rangle| \leq C \|w\| \text{ holds for all } w \in \text{Dom}(\bar{\partial}).
\]
By the Riesz representation theorem it follows, that, if \(u \in \text{Dom}(\bar{\partial}^*)\), there exists a unique \(v \in L^2_{p,q}(\Omega)\), such that
\[
(w, v) = \langle \bar{\partial} w, u \rangle
\]
holds for all \(w \in \text{Dom}(\bar{\partial})\); we write \(\bar{\partial}^* u\) for \(v\). This reveals that certain boundary conditions must hold on any smooth \((p, q+1)\)-form, which belongs to \(\text{Dom}(\bar{\partial}^*)\). In fact, one can show that \(u \in D^{p,q+1}(\Omega) := \text{Dom}(\bar{\partial}^*) \cap \Lambda^{p,q+1}(\Omega)\) holds if and only if
\[
\sum_{k=1}^n u_{I,kJ} \frac{\partial r}{\partial z_k} = 0 \text{ on } b\Omega
\]
for all \(I\) and \(J\) which are strictly increasing index sets of length \(p\) and \(q\), respectively. Here, \(r\) is a defining function of \(\Omega\).
The Hilbert space adjoint, $\partial^\ast$, to $\bar{\partial}$ with respect to the $L^2(\Omega, \phi)$-inner product is defined by $\bar{\partial}^\ast = e^\phi \partial^\ast e^{-\phi}$. In view of (2.2) it is easy to see that $\text{Dom}(\bar{\partial}^\ast) = \text{Dom}(\partial^\ast)$ holds.

Now we are ready to formulate the $\bar{\partial}$-Neumann problem. It is the following: given $f \in L^2_{p,q}(\Omega)$, find $u \in L^2_{p,q}(\Omega)$ such that the following holds

\[
\begin{cases}
(\bar{\partial}\bar{\partial}^\ast + \bar{\partial}^\ast\bar{\partial})u = f \\
\bar{\partial}u \in \text{Dom}(\bar{\partial}^\ast), \bar{\partial}^\ast u \in \text{Dom}(\bar{\partial})
\end{cases}
\]

(2.3)

The complex Laplacian, $\Box_{p,q} := \bar{\partial}\bar{\partial}^\ast + \bar{\partial}^\ast\bar{\partial}$, is itself elliptic, but the boundary conditions, which are implied by membership to $\text{Dom}(\bar{\partial}^\ast)$, are not. The ellipticity of $\Box_{p,q}$ implies that Gårding’s inequality holds in the interior of $\Omega$, i.e.

\[
\|u\|_2^2 \lesssim \|\bar{\partial}u\|^2 + \|\bar{\partial}^\ast u\|^2 \quad \text{for } u \in \Lambda_{\omega}^{p,q}(\Omega),
\]

where $\|\|_1$ denotes the usual $L^2$-Sobolev 1-norm. We remark, though, (2.4) does not hold for general $u \in \mathcal{D}^{p,q}(\Omega)$. However, a substitute estimate, below, does hold for $u \in \mathcal{D}^{p,q}(\Omega)$.

Let $p \in b\Omega$. We may choose a neighborhood $U$ of $p$ and a local coordinate system $(x_1, \ldots, x_{2n-1}, r) \in \mathbb{R}^{2n-1} \times \mathbb{R}$, such that the last coordinate is a local defining function of the boundary. Call $(U, (x, r))$ a special boundary chart. We shall denote the dual variable of $x$ by $\xi$, and define $\langle x, \xi \rangle := \sum_{j=1}^{2n-1} x_j \xi_j$.

For $f \in C_c^\infty(U \cap \Omega)$ we define the tangential Fourier transform of $f$ by

\[
\hat{f}(\xi, r) := \int_{\mathbb{R}^{2n-1}} e^{-2\pi i \langle x, \xi \rangle} f(x, r) dx.
\]

Via the tangential Bessel potential $\Lambda_\omega^s$ of order $s$,

\[
(\Lambda_\omega^s f)(x, r) := \int_{\mathbb{R}^{2n-1}} e^{2\pi i \langle x, \xi \rangle} (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi, r) d\xi,
\]

we can define the tangential $L^2$-Sobolev norm of $f$ of order $s$ by

\[
\|f\|_s^2 := \|\Lambda_\omega^s f\|^2 = \int_{-\infty}^0 \int_{\mathbb{R}^{2n-1}} (1 + |\xi|^2)^s |\hat{f}(\xi, r)|^2 d\xi dr.
\]

A subelliptic estimate of order $\epsilon > 0$ holds if there exists $C > 0$ such that

\[
\|u\|_\epsilon^2 \leq C\|\bar{\partial}u\|^2 + \|\bar{\partial}^\ast u\|^2
\]

for $u \in \mathcal{D}^{p,q}(\Omega)$ supported near the boundary point $p$.

From here on, we restrict our considerations to $(0, q)$-forms. The system (2.4) does not see the $dz$’s and the general case for $(p, q)$-forms can be derived easily. For notational ease we shall write $u_{j}$ instead of $u_{0,j}$, for the components of a $(0, q)$-form $u$. We shall denote the Dirichlet form associated to $\Box_{1,q}$ as usual by $Q(., .)$, i.e. $Q(u, v) := (\bar{\partial}u, \bar{\partial}v) + (\bar{\partial}^\ast u, \bar{\partial}^\ast v)$ for $u, v \in \mathcal{D}^{0,q}(\Omega)$. 
For quantities $A$ and $B$ we use the notation $|A| \lesssim |B|$ to mean $|A| \leq C|B|$ for some constant $C > 0$, which is independent of relevant parameters. It will be specifically mentioned or clear from the context, what those parameters are. Furthermore, we call the elementary inequality $|AB| \leq \eta A^2 + \frac{1}{4\eta}B^2$ for $\eta > 0$ the (sc)-(lc) inequality.

3. Basic estimates

In this section, we derive two basic weighted inequalities for forms in $\mathcal{D}^{0,q}(\Omega)$. We will make extensive use of these inequalities in our proof of subellipticity. Our starting point is the following Proposition 3.1, which has been derived by McNeal in [McN2].

**Proposition 3.1.** Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded, pseudoconvex domain, and suppose that $\phi \in C^2(\overline{\Omega}) \cap PSH(\Omega)$. If $|\partial \phi|_{\partial \partial \phi} \leq 1$, then

$$\frac{1}{2} \sum_{|I|=q-1} \int_{\Omega} \sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial z_k \partial z_l} u_{kI} \overline{u}_{lI} e^{-2\phi} dV \leq \|\partial u\|_{2\phi}^2 + 3\|\partial^* u\|_{2\phi}^2,$$

(3.2)

holds for all $u \in \Lambda^{0,q}(\Omega)$.

We remark that inequality (3.2) is one of the key points leading to the subelliptic estimate. In fact, this inequality will be used in section 4 enabling us to obtain “good” estimates near the boundary. In the following, we derive a Gårding-like weighted inequality. This inequality is also crucial as it will give us “good” estimates in the interior.

**Proposition 3.3.** Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded, pseudoconvex domain, and suppose that $\phi \in C^2(\overline{\Omega}) \cap PSH(\Omega)$ satisfies $|\partial \phi|_{\partial \partial \phi} \leq \frac{1}{\sqrt{24}}$. Then for all $u \in \Lambda^{0,q}_{c}(\Omega)$, it holds that

$$\|ue^{-\phi}\|_1^2 \lesssim \|\partial u\|_{2\phi}^2 + \|\partial^* u\|_{2\phi}^2,$$

(3.4)

where $\| \cdot \|$ denotes the $L^2$-Sobolev 1-norm on $\Omega$.

For the proof of Proposition 3.3 we need to introduce the Hodge-Star Operator $\ast$, that is the map

$$\ast : \Lambda^{p,q}(\overline{\Omega}) \longrightarrow \Lambda^{n-p,n-q}(\overline{\Omega})$$

defined by $\psi \wedge \ast \varphi = \langle \psi, \varphi \rangle dV$ for $\psi, \varphi \in \Lambda^{p,q}(\overline{\Omega})$. The basic properties of the Hodge-Star Operator are summarized in the following lemma.

**Lemma 3.5.**

(i) $\ast \ast = (-1)^{p+q}$ id on $\Lambda^{p,q}(\overline{\Omega})$,

(ii) $|\varphi| = |\ast \varphi|$ for $\varphi \in \Lambda^{p,q}(\overline{\Omega})$, where $|\varphi|^2 = \langle \varphi, \varphi \rangle$,

(iii) $\partial^* = -\ast \partial \ast$ on $\Lambda_c^{p,q}(\overline{\Omega})$.

A proof of Lemma 3.5 can be found in [Che-Sha], chapter 9.
Proof of Proposition 3.5. Let $u \in \Lambda_c^{0,q}(\bar{\Omega})$. By Gårding’s inequality \cite{24}, we have
\[
\|ue^{-\phi}\|_1^2 \leq \|\bar{\partial}(ue^{-\phi})\|^2 + \|\bar{\partial}^*(ue^{-\phi})\|^2 = \|\bar{\partial}(ue^{-\phi})\|^2 + \|\bar{\partial}^*_u u\|^2_{2\phi}.
\]
Thus we just need to consider the term $\|\bar{\partial}(ue^{-\phi})\|^2$. For that define $v \in \Lambda_c^{n,n-q}(\bar{\Omega})$ by $v = su$. Here we denote the coefficients of $v$ by $v_J$ for $|J| = n - q$. Then, by Lemma \ref{lem:3.5} and commuting, it follows
\[
\|\bar{\partial}(ue^{-\phi})\|^2 = \|\bar{\partial}^*(ve^{-\phi})\|^2 \leq \|\bar{\partial}^*_v v\|^2_{2\phi} + \|\bar{\partial}^*,\phi\|v\|^2_{2\phi},
\]
where the last step follows from $\phi$ having a self-bounded complex gradient. Note that $v \in D^{n,n-q}(\Omega)$, since $v$ is identically zero on the boundary of $\Omega$. Hence we can apply inequality \eqref{eq:3.2}:
\[
\sum' \int_{|J|=n-q-1} \sum_k \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} v_{k,l} \bar{v}_{l,J} e^{-2\phi} dV \leq 2\|\bar{\partial} v\|^2_{2\phi} + 6\|\bar{\partial}^*_v v\|^2_{2\phi}.
\]
Since $|\partial \phi|\partial \bar{\partial} \phi \leq \frac{1}{\sqrt{24}}$, it follows that
\[
\|\bar{\partial}^*_v v\|^2_{2\phi} \leq 2\|\bar{\partial}^*_v v\|^2_{2\phi} + 2\|\bar{\partial}^*,\phi\|v\|^2_{2\phi} = 2\|\bar{\partial}^*_v v\|^2_{2\phi} + 2\sum' \int_{|J|=n-q-1} \sum_k \frac{\partial \phi}{\partial z_l} v_{k,l} \bar{v}_{l,J} e^{-2\phi} dV \leq \frac{1}{12} \sum_k \frac{\partial \phi}{\partial z_l} v_{k,l} \bar{v}_{l,J} e^{-2\phi} dV.
\]
Thus we obtain
\[
\sum' \int_{|J|=n-q-1} \sum_k \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} v_{k,l} \bar{v}_{l,J} e^{-2\phi} dV \leq 4\|\bar{\partial} v\|^2_{2\phi} + 24\|\bar{\partial}^*_v v\|^2_{2\phi} = 4\|\bar{\partial}^*_u u\|^2_{2\phi} + 24\|\bar{\partial} u\|^2_{2\phi}
\]
where the second line holds by Lemma \ref{lem:3.5}. So we are left with estimating the term $\|\bar{\partial}^*_u u\|^2_{2\phi}$. As before, we just need to commute:
\[
\|\bar{\partial}^*_u u\|^2_{2\phi} \leq \|\bar{\partial}^*_u u\|^2_{2\phi} + \|\bar{\partial}^*,\phi\|u\|^2_{2\phi} = \|\bar{\partial}^*_u u\|^2_{2\phi} + \sum' \int_{|J|=q-1} \sum_k \frac{\partial \phi}{\partial z_l} u_{k,l} \bar{u}_{l,J} e^{-2\phi} dV,
\]
To finish we use inequality (3.2) again, that is
\[ \sum_{|t|=q-1}^n \int_{\Omega} \sum_{k,l=1}^n \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} u_{kl} \bar{u}_t e^{-2\phi} dV \lesssim \|ar{\partial} u\|_{2\phi}^2 + \|ar{\partial}_\phi^* u\|_{2\phi}^2. \]

Collecting all our estimates, we obtain
\[ \|ue^{-\phi}\|_1^2 \lesssim \|ar{\partial} u\|_{2\phi}^2 + \|ar{\partial}_\phi^* u\|_{2\phi}^2 \quad \text{for } u \in \Lambda_{c,q}^0(\Omega). \]

Since the \(L^2\)-Sobolev 1-norm dominates the \(L^2\)-norm, \(3.3\) implies that
\[ \|ue^{-\phi}\|_1^2 \lesssim \|ar{\partial} u\|_{2\phi}^2 + \|ar{\partial}_\phi^* u\|_{2\phi}^2 \]
holds for all \(u \in \Lambda_{c,q}^0(\Omega)\). In the following, we show that this inequality is in fact true for all \(u \in \mathcal{D}^{0,q}(\Omega)\).

**Proposition 3.6.** Let \(\Omega \subset \subset \mathbb{C}^n\) be a smoothly bounded, pseudoconvex domain, and suppose that \(\phi \in C^2(\overline{\Omega}) \cap \text{PSH}(\Omega)\) satisfies \(|\partial \phi|_{i\partial} \leq \frac{1}{\sqrt{2}}\). Then for \(u \in \mathcal{D}^{0,q}(\Omega)\) it holds that
\[ \|u\|_{2\phi}^2 \lesssim \|ar{\partial} u\|_{2\phi}^2 + \|ar{\partial}_\phi^* u\|_{2\phi}^2. \]

**Proof.** Set \(\psi_t(z) = \phi(z) + t|z|^2\) for \(t > 0\). Then \(\psi_t\) is strictly plurisubharmonic, since for \(\xi \in \mathbb{C}^n\), \(z \in \Omega\) it holds
\[ \sum_{k,l=1}^n \frac{\partial^2 \psi_t}{\partial z_k \partial \bar{z}_l}(z)\xi_k \bar{\xi}_l = \sum_{k,l=1}^n \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l}(z)\xi_k \bar{\xi}_l + t|\xi|^2. \]

Moreover, we observe that
\[ \left| \sum_{k=1}^n \frac{\partial \psi_t}{\partial z_k}(z)\xi_k \right|^2 \leq 2 \sum_{k=1}^n \left| \frac{\partial \phi}{\partial z_k}(z)\xi_k \right|^2 + 2t^2 |z|^2 |\xi|^2. \]

Since \(\Omega\) is a bounded domain, we can choose a \(t > 0\), such that \(24t|z|^2 \leq 1\) holds for all \(z \in \Omega\). Then \(|\partial \psi_t|_{i\partial} \psi_t \leq 1\), and thus inequality \(3.2\) holds for \(\psi_t\). That is
\[ \frac{1}{2} \sum_{|t|=q-1}^n \sum_{k,l=1}^n \frac{\partial^2 \psi_t}{\partial z_k \partial \bar{z}_l} u_{kl} \bar{u}_t e^{-2\psi_t} dV \leq \|ar{\partial} u\|_{2\psi_t}^2 + 3\|ar{\partial}_\phi^* u\|_{2\psi_t}^2. \]

Note that \(e^{-2|z|^2}\) is bounded from above by 1 and that \(\phi\) is plurisubharmonic on \(\Omega\). Hence it follows that
\begin{align*}
\frac{1}{2} \int_{\Omega} t|u|^2 e^{-2\psi} dV & \leq \|ar{\partial} u\|_{2\phi}^2 + 3\|ar{\partial}_\phi^* u\|_{2\psi_t}^2 \\
& \leq \|ar{\partial} u\|_{2\phi}^2 + 6\|ar{\partial}_\phi^* u\|_{2\phi}^2 + 6\|ar{\partial}_\phi^*, (t|z|^2) u\|_{2\psi_t}^2.
\end{align*}
By our choice of \( t \) we can estimate the last term

\[
6\| [\bar{\partial}^*, (t|z|^2)]u \|_{2\psi_t}^2 = 6 \sum'_{|l| = q-1} \| \sum_k^n \frac{\partial(t|z|^2)}{\partial z_k} u_{k\ell} \|_{2\psi_t}^2 \leq \frac{1}{4} t \| u \|_{2\psi_t}^2.
\]

Therefore it holds that

\[
\frac{1}{4} \int_{\Omega} t|u|^2 e^{-2\psi_t} \leq \| \bar{\partial}u \|_{2\phi}^2 + 6\| \bar{\partial}^* u \|_{2\phi}^2.
\]

Since \( e^{-t|z|^2} \) is bounded from below on \( \Omega \), our claim follows. \( \square \)

4. Estimates for \( \bar{\partial}^* N_q \)

By a compactness estimate for \( \bar{\partial}^* N_q \) we mean the following: for all \( \eta > 0 \) there exists a \( C(\eta) > 0 \) such that

\[
\| \bar{\partial}^* N_q \alpha \| \lesssim \eta \| \alpha \| + C(\eta) \| \alpha \|_{-s}
\]

for all \( \alpha \in L^2_{0,q}(\Omega) \). Here \( \| . \|_{-s} \), \( s > 0 \), denotes the \( L^2 \)-Sobolev norm of order \( -s \). The constant in \( \lesssim \) does depend on \( s \) but not on \( \alpha \), \( \eta \) or \( C(\eta) \). The family of estimates (4.1) is equivalent to \( \bar{\partial}^* N_q \) being a compact operator from \( L^2_{0,q}(\Omega) \) to \( L^2_{0,q-1}(\Omega) \); for a proof see for instance \( \text{McN2} \). We remark that for compactness of \( \bar{\partial}^* N_q \) it is sufficient to establish (4.1) for \( \bar{\partial}^- \)-closed forms \( \alpha \in L^2_{0,q}(\Omega) \), see \( \text{McN2} \).

In this section, we derive with the aid of our weighted estimates from section 3 two versions of compactness estimates for \( \bar{\partial}^* N_q \). We start out with a quantified version of (4.1), i.e. we describe \( C(\eta) \) for each \( \eta \).

Since the weight functions \( \{ \phi_\delta \} \) are just defined on \( \Omega \cap U \), where \( U \) is a neighborhood of a given \( p \in b\Omega \) (see hypotheses in Theorem 1.4), we need to restrict our considerations to an approximating subdomain of \( \Omega \), which lies in \( U \).

Proposition 4.2. Suppose that \( \Omega \subset \mathbb{C}^n \) is a smoothly bounded domain. Let \( p \) be a point in \( b\Omega \) and suppose that \( b\Omega \cap U \) is pseudoconvex, where \( U \) is a neighborhood of \( p \). Then there exists a smoothly bounded, pseudoconvex domain \( \Omega_a \subset \Omega \cap U \) with \( \Omega_a \subset \subset U \) satisfying the following properties

1. \( b\Omega \cap b\Omega_a \) contains a neighborhood of \( p \) in \( b\Omega \),
2. all points in \( b\Omega_a \setminus b\Omega \) are strongly pseudoconvex.

A proof of Proposition 4.2 can be found in \( \text{McN1} \). We call such a domain \( \Omega_a \) an approximating subdomain associated to \( (\Omega, p, U) \). The crucial feature, for our current purposes, of such an approximating subdomain \( \Omega_a \) is that it is a smoothly bounded, pseudoconvex domain. Therefore we can apply the inequalities (3.2), (3.4) and (3.7) on \( \Omega_a \) using the \( \phi_\delta \)'s as weight functions. We remark that for using these inequalities a rescaling of the \( \phi_\delta \)'s might be necessary, so that \( |\partial \phi_\delta|_{\bar{\partial} \bar{\partial} \phi_\delta} \leq \frac{1}{\sqrt{24}} \) holds for all \( \delta > 0 \) sufficiently small.
Theorem 4.3. Assume the hypotheses of Theorem 4.4. Let \( \Omega_\alpha \) be an approximating subdomain associated to \( (\Omega,p,U) \). Then there exists a neighborhood \( V \subset \subset U \) of \( p \), such that for \( \alpha \in L^2_{\Omega,q}(\Omega_\alpha) \), \( \bar{\partial} \)-closed and supported in \( V \cap \Omega_\alpha \), the following estimate holds:

\[
\| \bar{\partial}^* N^\Omega_\alpha \alpha \|^2_{\Omega_\alpha} \lesssim \delta^{2e} \| \alpha \|^2_{\Omega_\alpha} + \delta^{-2+2e} \| \alpha \|^2_{-1,\Omega_\alpha}.
\]

The constant in \( \lesssim \) neither depends on \( \alpha \) nor \( \delta \).

Proof. For notational ease we shall write \( \| \cdot \| \) for \( \| \cdot \|_{\Omega_\alpha} \) and \( N_q \) for \( N^\Omega_\alpha \).

Let \( W \subset \subset U \) be a neighborhood of \( p \), such that \( W \cap \Omega \subset \Omega_a \) and \( \overline{W} \cap b\Omega_a \subset b\Omega \). Also, let \( V \subset \subset W \) be a neighborhood of \( p \) and \( \alpha \in L^2_{\Omega,q}(\Omega_\alpha) \) be a \( \bar{\partial} \)-closed form, which is supported in \( V \cap \Omega_\alpha \). Define the functional 

\[
F : (\{ e^{-\frac{\phi}{e}} \bar{\partial}_{\phi}^* u \mid u \in D^0,q(\Omega_a) \}, \| \cdot \|_{\phi_\delta}) \rightarrow \mathbb{C}
\]

We start with showing that \( F \) satisfies the following estimate

\[
|F(e^{-\frac{\phi}{e}} \bar{\partial}_{\phi}^* u)| \leq \| e^{-\frac{\phi}{e}} \bar{\partial}_{\phi}^* u \|_{\phi_\delta} (\delta^e \| \alpha \| + \delta^{-1+e} \| \alpha \|_{-1}).
\]

Recall that \( S_\delta = \{ z \in \Omega_a \mid -\delta < r(z) < 0 \} \), where \( r \) is the fixed defining function of \( \Omega \). Let \( \chi \in C^\infty_c(W) \) such that \( \chi \equiv 1 \) on \( V \) and \( \chi \geq 0 \). Recall that the support of \( \alpha \) is in \( V \). Then, by the generalized Cauchy-Schwarz inequality, we obtain

\[
|F(e^{-\frac{\phi}{e}} \bar{\partial}_{\phi}^* u)| = |(u,\alpha)_{\phi_\delta}|
\]

\[
\lesssim \| u e^{-\phi_\delta} \|^2_{W \cap S_\delta} \| \alpha \| + \| e^{-\phi_\delta} \chi u \|^2_{\Omega_\alpha \setminus S_\delta} \| \alpha \|_{-1}.
\]

In view of our claim \( 3.5 \), we need to estimate the terms \( \| u e^{-\phi_\delta} \|^2_{W \cap S_\delta} \) and \( \| e^{-\phi_\delta} \chi u \|^2_{\Omega_\alpha \setminus S_\delta} \) appropriately.

1. Estimating \( \| u e^{-\phi_\delta} \|^2_{W \cap S_\delta} \): Recall that \( \phi_\delta \) has a self-bounded complex gradient on \( \Omega_a \subset U \cap \Omega \) by hypothesis (i). Hence inequality \( 3.2 \) holds, and the plurisubharmonicity of \( \phi_\delta \) implies then, that

\[
\sum_{|L|=q-1} \int_{W \cap S_\delta} \sum_{k,l=1}^n \frac{\partial^2 \phi_\delta}{\partial z_k \partial \overline{z}_l} u_{k,l} u_{\ell} e^{-2\phi_\delta} dV \lesssim \| \bar{\partial} u \|^2_{2\phi_\delta} + \| \bar{\partial}_{\phi_\delta} u \|^2_{2\phi_\delta}
\]

holds uniformly for all \( \delta > 0 \) small. Invoking hypothesis (ii) and noting that \( W \subset U \) yields

\[
\| u \|^2_{W \cap S_\delta} \lesssim \delta^e (\| \bar{\partial} u \|^2_{2\phi_\delta} + \| \bar{\partial}_{\phi_\delta} u \|^2_{2\phi_\delta}).
\]

2. Estimating \( \| e^{-\phi_\delta} \chi u \|^2_{\Omega_\alpha \setminus S_\delta} \): Let \( h_\delta : \mathbb{R}^+_0 \rightarrow [0,1] \) be a smooth function with \( h_\delta(x) = 0 \) for \( x \in [0, \frac{\delta}{2}] \) and \( h_\delta(x) = 1 \) for \( x \geq \delta \). We can choose \( h_\delta \) such that \( |h_\delta'| \lesssim \delta^{-1} \). Define \( \zeta_\delta \in C^\infty(\Omega_a) \) by \( \zeta_\delta(z) = h_\delta(-r(z)) \), where \( r \) is the fixed defining function of \( \Omega \). Note that

\[
\frac{\partial \zeta_\delta}{\partial x_j} \lesssim \delta^{-1} \frac{\partial r}{\partial x_j} \lesssim \delta^{-1}
\]
holds on $\Omega_\delta$ for all $j \in \{1, \ldots, 2n\}$. Clearly, we have

$$\|e^{-\phi_\delta \chi}u\|_{\Omega_\delta \setminus S_\delta} \leq \|e^{-\phi_\delta \zeta_\delta \chi}u\|_1.$$  

Since $\zeta_\delta \cdot \chi$ is identically zero near the boundary of $\Omega_\delta$, we can use our weighted Gårding’s inequality \[\text{(4.7)}\] to start estimating

$$\|e^{-\phi_\delta \zeta_\delta \chi}u\|_1^2 \lesssim \|\partial(\zeta_\delta \chi u)\|^2_{2\phi_\delta} + \|\partial^*_{\phi_\delta} (\zeta_\delta \chi u)\|^2_{2\phi_\delta}$$

$$\lesssim \|\partial u\|^2_{2\phi_\delta} + \|\partial^*_{\phi_\delta} u\|^2_{2\phi_\delta} + \sum_{j=1}^{2n} \left( \|\partial \zeta_\delta \chi u\|^2_{2\phi_\delta} + \|\partial \chi u\|^2_{2\phi_\delta} \right)$$

$$\lesssim \|\partial u\|^2_{2\phi_\delta} + \|\partial^*_{\phi_\delta} u\|^2_{2\phi_\delta} + \sum_{j=1}^{2n} \max_{z \in \Omega_\delta} \left| \frac{\partial \zeta_\delta}{\partial x_j} \right|^2 (\|u\|_{W^{n,S_\delta}}^2 + \|u\|^2_{2\phi_\delta}).$$

The last estimate holds since $\chi$ is supported in $W$ and $\frac{\partial \zeta_\delta}{\partial x_j} = 0$ on $\Omega_\delta \setminus S_\delta$. By the inequalities \[\text{(4.7)}\] and \[\text{(1.7)}\], it follows

$$\|e^{-\phi_\delta \zeta_\delta \chi}u\|_1^2 \lesssim \|\partial u\|^2_{2\phi_\delta} + \|\partial^*_{\phi_\delta} u\|^2_{2\phi_\delta} + \delta^{-2} (\|u\|_{W^{n,S_\delta}}^2)$$

for all $\delta > 0$ small enough. Using the estimate \[\text{(1.6)}\] for $\|u\|_{2\phi_\delta}$, we obtain

$$\|e^{-\phi_\delta \zeta_\delta \chi}u\|_1^2 \lesssim \delta^{-2+2\epsilon} (\|\partial u\|^2_{2\phi_\delta} + \|\partial^*_{\phi_\delta} u\|^2_{2\phi_\delta}),$$

thus we can conclude

$$\|e^{-\phi_\delta \zeta_\delta \chi}u\|_1^2 \lesssim \delta^{-2+2\epsilon} (\|\partial u\|^2_{2\phi_\delta} + \|\partial^*_{\phi_\delta} u\|^2_{2\phi_\delta}).$$

Write $u = u_1 + u_2$, where $u_1 \in \ker \tilde{\partial}$ and $u_2 \perp_{\phi_\delta} \ker \tilde{\partial}$. Note that $u_1 \in \mathcal{D}^{0,q}(\Omega_\delta)$. Thus, since $\alpha \in \ker \tilde{\partial}$, we get, using the estimates \[\text{(1.6)}\] and \[\text{(1.8)}\],

$$|(u, \alpha)_{\phi_\delta}| = |(u_1, \alpha)_{\phi_\delta}| \lesssim \|\partial^*_{\phi_\delta} u_1\|_{2\phi_\delta} (\delta^\epsilon \|\alpha\| + \delta^{-1+\epsilon} \|\alpha\|_{-1}).$$

However, $u_2 \perp_{\phi_\delta} \ker \tilde{\partial}$, therefore we get $\|\partial^*_{\phi_\delta} u\|_{2\phi_\delta} = \|\partial^*_{\phi_\delta} u_2\|_{2\phi_\delta}$. Hence our claimed inequality \[\text{(1.8)}\] holds:

$$|F(e^{-\phi_\delta \zeta_\delta \chi}u)| = |(u, \alpha)_{\phi_\delta}| \lesssim \|e^{-\phi_\delta \zeta_\delta \chi}u\|_{\phi_\delta} (\delta^\epsilon \|\alpha\| + \delta^{-1+\epsilon} \|\alpha\|_{-1}).$$

That is, $F$ is a bounded linear functional on $(\{e^{-\phi_\delta \zeta_\delta \chi}u \mid u \in \mathcal{D}^{0,q}(\Omega_\delta)\}, \|\cdot\|_{\phi_\delta})$, which is a subset of $L^2_{0,q-1}(\Omega_\delta, \phi_\delta)$. By the Hahn-Banach Theorem, $F$ extends to a bounded linear functional on $L^2_{0,q-1}(\Omega_\delta, \phi_\delta)$ with the same bound. The Riesz representation theorem yields, that there exists a unique $v \in L^2_{0,q-1}(\Omega_\delta, \phi_\delta)$ such that for all $g \in L^2_{0,q-1}(\Omega_\delta, \phi_\delta)$

$$F(g) = (g, v)_{\phi_\delta},$$

$$\|v\|^2_{\phi_\delta} \lesssim \delta^2 \|\alpha\|^2 + \delta^{-2+2\epsilon} \|\alpha\|^2_{-1}.$$  

In particular, we get for all $u \in \mathcal{D}^{0,q}(\Omega_\delta)$

$$(u, \tilde{\partial} e^{-\phi_\delta \zeta_\delta \chi}u)_{\phi_\delta} = (e^{-\phi_\delta \zeta_\delta \chi}u, v)_{\phi_\delta} = (u, \alpha)_{\phi_\delta},$$
Note that $\mathcal{D}^{0,q}(\Omega_a)$ is dense in $L^2_{(0,q)}(\Omega_a, \phi_\delta)$. Hence, setting $s = e^{-\frac{\phi_\delta}{2}} v$, it follows that $\bar{\partial} s = \alpha$ in the distributional sense and

$$\|s\|^2 \lesssim \delta^2 \|\alpha\|^2 + \delta^{-2+2\varepsilon} \|\alpha\|_{-1}^2.$$  

But the minimal $L^2(\Omega_a)$-solution, $\bar{\partial}^* N_q \alpha$, to the $\bar{\partial}$-problem for $\alpha$ on $\Omega_a$ must also satisfy this estimate; that is

(4.9)  

$$\|\bar{\partial}^* N_q \alpha\|^2 \lesssim \delta^2 \|\alpha\|^2 + \delta^{-2+2\varepsilon} \|\alpha\|_{-1}^2.$$  

Remark. Observe that the only point where the form level $q$ of the $(0,q)$-forms comes into play, is in hypothesis (ii) of Theorem 1.4. Notice that this condition on the complex hessian of $\phi_\delta$ near the boundary also holds for $(0,q+1)$-forms. Thus by a proof analogous to the above, we obtain the following: there exists a neighborhood $V \subset \subset U$ of $p$ such that for all $\beta \in L^0_{0,q+1}(\Omega_a)$, which are $\bar{\partial}$-closed and supported in $V \cap \bar{\Omega}_a$, the following estimate holds

(4.10)  

$$\|\bar{\partial}^* N_{q+1} \alpha\|^2 \lesssim \delta^2 \|\beta\|^2 + \delta^{-2+2\varepsilon} \|\beta\|_{-1,\Omega_a}^2.$$  

These families of estimates, (4.9) and (4.10), are the heart of the matter for our proof of subellipticity. But to convert these estimates on $\bar{\partial}^* N_{q+1}$ to usable estimates on $\mathcal{D}^{0,q}(\Omega)$, we shall need exact regularity of the operator $\bar{\partial}^* \bar{\partial} N_q$. By exact regularity we mean that $\bar{\partial}^* \bar{\partial} N_q$ preserves the $L^2$-Sobolev spaces.

Kohn showed in [Koh3], that exact regularity of $\bar{\partial}^* \bar{\partial} N_q$ follows from compactness of $N_q$ in $L^0_{0,q}(\Omega)$, if $\Omega$ is a smoothly bounded, pseudoconvex domain. It is an easy consequence of the formula

$$N_q = (\bar{\partial} N_{q-1})(\bar{\partial}^* N_q) + (\bar{\partial}^* N_{q+1})(\bar{\partial} N_q),$$

that compactness of the operators $\bar{\partial}^* N_q$ and $\bar{\partial}^* N_{q+1}$ implies compactness of $N_q$.

The estimates (4.9) and (4.10) do not imply compactness as they do not hold for all $\bar{\partial}$-closed forms in $L^2_{0,q}(\Omega_a)$ and $L^2_{0,q+1}(\Omega_a)$, respectively. However, we show below that $N_q^{\Omega_a}$ is a compact operator on $L^2_{0,q}(\Omega_a)$ by using a proof similar to the one of Theorem 1.4. The crucial property of the approximating subdomain $\Omega_a$ for this argument is that $\Omega_a$ is strongly pseudoconvex off the boundary of $\Omega$. In particular, we use Kohn’s result that near a point in the boundary of strong pseudoconvexity a subelliptic estimate of order $\frac{1}{2}$ holds.

Proposition 4.11. Assume that the hypotheses of Theorem 1.4 hold. Let $\Omega_a$ be an approximating subdomain associated to $(\Omega, p, U)$. Then the $\bar{\partial}$-Neumann operator $N_q^{\Omega_a}$ is a compact operator on $L^2_{0,q}(\Omega_a)$.

Proof. As before, we write $N_q$ for $N_q^{\Omega_a}$, and $\|\|_\Omega$ for $\|\|_{\Omega_a}$. We start out with showing that $\bar{\partial}^* N_q$ is a compact operator. By the remark following (4.1) we
obtain compactness of $\bar{\partial}^* N_q$, if we can show that for all $\eta > 0$ there exists a $C(\eta) > 0$ such that

$$\| \bar{\partial}^* N_q \alpha \| \lesssim \eta \| \alpha \| + C(\eta) \| \alpha \|^{-\frac{1}{2}}$$

holds for all $\bar{\partial}$-closed $\alpha \in L^2_{\bar{\partial}q}(\Omega_a)$.

Let $\eta > 0$ be given. By our hypotheses there exists a function $\phi_{\eta} \in C^2(\bar{\Omega}_a) \cap PSH(\Omega_a)$ which has a self-bounded complex gradient and satisfies

$$\sum_{|I|=q-1} \sum_{k,l=1}^{n} \frac{\partial^2 \phi_{\eta}}{\partial z_k \partial \bar{z}_l}(z) u_{k,l} \bar{u}_{l} \geq \eta^{-2} |u|^2 \text{ for } u \in \Lambda^{0,q}(\Omega_a)$$

on a strip $S_{\eta'} = \{ z \in \Omega_a \cap \Omega | - \eta' < r(z) < 0 \}$ for some $\eta' > 0$ chosen small enough, depending on $\eta$. Here $r$ is the fixed defining function of $\Omega$.

Let $\alpha$ be a $\bar{\partial}$-closed $(0,q)$-form with coefficients in $L^2(\Omega_a)$. Define the linear functional $F : \{ e^{-\frac{\phi_{\eta}}{2}} \bar{\partial}^* u \mid u \in \mathcal{D}^{0,q}(\Omega_a) \}, \| \cdot \|_{\phi_{\eta}} \rightarrow \mathbb{C}$ by

$$F(e^{-\frac{\phi_{\eta}}{2}} \bar{\partial}^* u) = (u, \alpha)_{\phi_{\eta}}.$$ 

We shall show that $F$ is a bounded functional satisfying

$$|F(e^{-\frac{\phi_{\eta}}{2}} \bar{\partial}^* u)| \lesssim \| e^{-\frac{\phi_{\eta}}{2}} \bar{\partial}^* u \|_{\phi_{\eta}} \| \phi_{\eta} \| \| \alpha \| + C(\eta) \| \alpha \|^{-\frac{1}{2}}$$

for some $C(\eta) > 0$. For that let $\chi \in C^\infty(\Omega_a)$ be a non-negative function such that $\chi = 1$ on $\Omega_a \setminus S_{\eta'}$ and $\chi = 0$ on $S_{\eta'}$. Then

$$|F(e^{-\frac{\phi_{\eta}}{2}} \bar{\partial}^* u)| = |(u, \alpha)_{S_{\eta'}}| + |(u, \alpha)_{\Omega_a \setminus S_{\eta'}}| \lesssim \| ue^{-\phi_{\eta}} \|_{S_{\eta'}} \| \alpha \| + \| \chi ue^{-\phi_{\eta}} \|_{S_{\eta'}} \| \alpha \|^{-\frac{1}{2}},$$

where the second line follows by the generalized Cauchy-Schwarz inequality. In view of our claimed inequality (4.13), we need to get control of the terms $\| ue^{-\phi_{\eta}} \|_{S_{\eta'}}$ and $\| \chi ue^{-\phi_{\eta}} \|_{S_{\eta'}}$.

Since $\phi_{\eta} \in C^2(\Omega_a) \cap PSH(\Omega_a)$ has a self-bounded complex gradient and $\Omega_a$ is pseudoconvex, we can use inequality (3.2) to estimate $\| ue^{-\phi_{\eta}} \|_{S_{\eta'}}$:

$$\sum_{|I|=q-1} \int_{S_{\eta'}} \sum_{k,l=1}^{n} \frac{\partial^2 \phi_{\eta}}{\partial z_k \partial \bar{z}_l}(u_{k,l} \bar{u}_{l} e^{-2\phi_{\eta}}) dV \lesssim \| \bar{\partial} u \|_{2\phi_{\eta}}^2 + |\bar{\partial}^* u_{\phi_{\eta}} u |_{2\phi_{\eta}}^2.$$ 

By inequality (4.12) it follows

$$|F(e^{-\frac{\phi_{\eta}}{2}} \bar{\partial}^* u)|^2 \lesssim \eta(\| \bar{\partial} u \|_{2\phi_{\eta}}^2 + |\bar{\partial}^* u_{\phi_{\eta}} u |_{2\phi_{\eta}}^2).$$
In order to estimate \( \| \chi u e^{-\phi} \|_{L^2} \), note that \( \text{supp} \chi \cap b\Omega_a \subset b\Omega_a \setminus b\Omega \) and recall that, by our choice of \( \Omega_a \), we have that \( b\Omega_a \setminus b\Omega \) is strongly pseudoconvex. Thus an subelliptic estimate of order \( \frac{1}{2} \) holds for \( \chi u e^{-\phi} \):

\[
\| \chi u e^{-\phi} \|_{L^2}^2 \lesssim \| \bar{\partial}(\chi u e^{-\phi}) \|_{L^2}^2 + \| \bar{\partial}^* (\chi u e^{-\phi}) \|_{L^2}^2 \\
\lesssim C(\eta)^2 (\| \bar{\partial} u \|_{L^2}^2 + \| \bar{\partial}^* u \|_{L^2}^2 + \| u \|_{L^2}^2)
\]

where the last line follows by inequality (5.7).

Now we are set up for proving inequality (4.13). Write \( u = u_1 + u_2 \), where \( u_1 \in \ker \bar{\partial} \) and \( u_2 \perp_{\phi_n} \ker \bar{\partial} \). Thus, since \( \alpha \in \ker \bar{\partial} \), we get, using our above estimates for the terms \( \| u e^{-\phi} \|^2 \) and \( \| \chi u e^{-\phi} \|^2 \):

\[
|F(e^{-\frac{\alpha}{2}} \bar{\partial}^* u)| = |(u_1, \alpha)_{\phi_n}| \lesssim \| u_1 e^{-\phi} \|^2 \| \alpha \|_{L^2} + \| \chi u_1 e^{-\phi} \|_{L^2} \| \alpha \|_{L^2}^{-\frac{1}{2}}
\]

Recall that \( \| \bar{\partial}^* u \|_{L^2} \) holds, since \( u_2 \perp_{\phi_n} \ker \bar{\partial} \). This implies our claimed inequality (4.13). By arguments analogous to the ones in the proof of Theorem 4.3 it follows that

\[
\| \bar{\partial}^* N_q \alpha \| \lesssim \eta \| \alpha \| + C(\eta) \| \alpha \|_{L^2}^{-\frac{1}{2}}
\]

holds for all \( \bar{\partial} \)-closed forms \( \alpha \in L^2_{0,q}(\Omega_a) \). Thus \( \bar{\partial}^* N_q \) is a compact operator from \( L^2_{0,q}(\Omega_a) \) to \( L^2_{0,q-1}(\Omega_a) \). A similar proof yields the compactness of \( \bar{\partial}^* N_{q+1} \). Therefore \( N_q \), the \( \bar{\partial} \)-Neumann operator on \( \Omega_a \), is a compact operator on \( L^2_{0,q}(\Omega_a) \).

5. Estimates on \( D^{0,q}(\Omega) \)

In this section we convert the families of estimates, (4.9) and (4.10), obtained in section 4 to estimates for forms in \( D^{0,q}(\Omega) \). As already mentioned in section 4 we need exact regularity to hold for operators related to \( N_q \). We begin with a result of Kohn.

**Proposition 5.1.** Suppose \( \Omega \subset \subset \mathbb{C}^n \) is a smoothly bounded, pseudoconvex domain, such that its \( \bar{\partial} \)-Neumann operator, \( N_q \), is compact on \( L^2_{0,q}(\Omega) \). Let \( s > 0 \), then the following holds

1. if \( \beta \in H^s_{0,q}(\Omega) \), then \( \| \bar{\partial}^* \bar{\partial} \beta \|_s \lesssim \| \beta \|_s \),
2. if \( \beta \in H^s_{0,q-1}(\Omega) \), then \( \| N_q \bar{\partial} \beta \|_s \lesssim \| \beta \|_s \).

Here, the constants in \( \lesssim \) depend on \( s \) but not on \( \beta \).

A proof of Proposition 5.1 is contained in [Koh3]. An easy consequence of Proposition 5.1 is the exact regularity of the \( L^2 \)-adjoint operators of \( \bar{\partial}^* \bar{\partial} N_q \) and \( N_q \bar{\partial} \) in the \( L^2 \)-Sobolev spaces of negative order. In particular, the following holds.
Lemma 5.2. Suppose $\Omega \subset \subset \mathbb{C}^n$ is a smoothly bounded, pseudoconvex domain, such that its $\bar{\partial}$-Neumann operator, $N_q$, is compact on $L^2_{0,q}(\Omega)$. Then, if $\alpha \in \Lambda^{0,q}(\Omega)$, it follows that

\begin{align}
\|\bar{\partial}^* N_q \alpha\|_{-1} & \lesssim \|\alpha\|_{-1}, \tag{5.3} \\
\|\bar{\partial}\bar{\partial}^* N_q \alpha\|_{-1} & \lesssim \|\alpha\|_{-1}. \tag{5.4}
\end{align}

Proof. Let $\alpha \in \Lambda^{0,q}(\Omega)$. Then

$$\|\bar{\partial}^* N_q \alpha\|_{-1} = \sup\{\|\bar{\partial}^* N_q \alpha, \beta\| \mid \beta \in H^1_{0,q-1}(\Omega), \|\beta\|_1 \leq 1\}. $$

Since $\beta \in H^1_{0,q-1}(\Omega)$ is in Dom($\bar{\partial}$), we obtain

$$(\bar{\partial}^* N_q \alpha, \beta) = (N_q \alpha, \bar{\partial} \beta) = (\alpha, N_q \bar{\partial} \beta) \lesssim \|\alpha\|_{-1} \|N_q \bar{\partial} \beta\|_1,$$

by the generalized Cauchy-Schwarz inequality. Proposition 5.1, part (2), yields exact regularity for $N_q \bar{\partial}$, in particular $\|N_q \bar{\partial} \beta\|_1 \lesssim \|\beta\|_1$ holds for all $\beta \in H^1_{0,q-1}(\Omega)$. Thus we have

$$\|\bar{\partial}^* N_q \alpha\|_{-1} \lesssim \sup\{\|\alpha\|_{-1} \|\beta\|_1 \mid \beta \in H^1_{0,q-1}(\Omega), \|\beta\|_1 \leq 1\} = \|\alpha\|_{-1},$$

which proves (5.3).

The proof of (5.4) is very similar. Since $\alpha = (\bar{\partial}\bar{\partial}^* N_q + \bar{\partial}^* \bar{\partial} N_q)\alpha$, it holds that

$$\|\bar{\partial}\bar{\partial}^* N_q \alpha\|_{-1} = \|\alpha - \bar{\partial}^* \bar{\partial} N_q \alpha\|_{-1} \leq \|\alpha\|_{-1} + \|\bar{\partial}^* \bar{\partial} N_q \alpha\|_{-1},$$

where

$$\|\bar{\partial}^* \bar{\partial} N_q \alpha\|_{-1} = \sup\{\|\bar{\partial}^* \bar{\partial} N_q \alpha, \beta\| \mid \beta \in H^1_{0,q}(\Omega), \|\beta\|_1 \leq 1\}. $$

As before, note that $\beta \in H^1_{0,q}(\Omega)$ is in Dom($\bar{\partial}$). Moreover, since $\alpha \in \Lambda^{0,q}(\Omega)$, it holds that $\bar{\partial} N_q \alpha = N_{q+1} \bar{\partial} \alpha$. Thus we obtain

$$(\bar{\partial}^* \bar{\partial} N_q \alpha, \beta) = (N_{q+1} \bar{\partial} \alpha, \bar{\partial} \beta) = (\alpha, \bar{\partial}^* N_{q+1} \bar{\partial} \beta) \lesssim \|\alpha\|_{-1} \|\bar{\partial}^* \bar{\partial} N_q \beta\|_1.$$ Part (1) of Proposition 5.1 tells us that $\|\bar{\partial}^* \bar{\partial} N_q \beta\|_1 \lesssim \|\beta\|_1$ holds for all $\beta \in H^1_{0,q}(\Omega)$. Hence it follows

$$\|\bar{\partial}^* \bar{\partial} N_q \alpha\|_{-1} \lesssim \sup\{\|\alpha\|_{-1} \|\beta\|_1 \mid \beta \in H^1_{0,q}(\Omega), \|\beta\|_1 \leq 1\} = \|\alpha\|_{-1},$$

which proves (5.4). \qed

Recall that we showed in Proposition 4.11 that the $\bar{\partial}$-Neumann operator, $N^{\Omega_\alpha}$, associated to the approximating subdomain $\Omega_\alpha$ is compact. Therefore, the exact regularity results (5.3) and (5.4) hold for $N^{\Omega_\alpha}_q$. Now we are ready to derive estimates for forms in $\mathcal{D}^{0,q}(\Omega)$.

Proposition 5.5. Assume the hypotheses of Theorem 1.2. Then there exists a neighborhood $W \subset \subset U$ of $p$, such that for all sufficiently small $\delta > 0$ and $\eta > 0$

$$\|u\|_{L^2_W} \lesssim \frac{\delta^{2\alpha}}{\eta} (\|\bar{\partial} u\|_{L^2_W} + \|\bar{\partial}^* u\|_{L^2_W} + \delta^{-2} \|\bar{\partial} u\|_{-1,\Omega}^2 + \eta \delta^{-2} \|u\|_{-1,\Omega}^2)$$
holds for \( u \in \mathcal{D}^{0,q}(\Omega) \) supported in \( W \cap \bar{\Omega} \). Here, the constant in \( \lesssim \) does depend on \( \eta \) but not on \( \delta \).

**Proof.** Recall that Theorem 4.3 and the following remark say that if \( \Omega_a \) is an approximating subdomain associated to \((\Omega,p,U)\), then there exists a neighborhood \( V \subset U \) of \( p \) such that

\[
\| \partial^* N_{q}^{\Omega_a} \alpha \|_{\Omega_a}^2 \lesssim \delta^{2q} \| \alpha \|_{\Omega_a}^2 + \delta^{-2q} \| \alpha \|_{-1,\Omega_a}^2, 
\]

\[
\| \partial^* N_{q+1}^{\Omega_a} \beta \|_{\Omega_a}^2 \lesssim \delta^{2q} \| \beta \|_{\Omega_a}^2 + \delta^{-2q} \| \beta \|_{-1,\Omega_a}^2
\]

hold for all \( \alpha \in L^2_{0,q}(\Omega_a) \) and \( \beta \in L^2_{0,q+1}(\Omega_a) \), which are \( \partial \)-closed and supported in \( V \cap \bar{\Omega}_a \). For notational ease we denote the \( L^2 \)-norm on \( \Omega_a \) by \( \| \cdot \| \) and write \( N_q \) for the \( \partial \)-Neumann operator on \( \Omega_a \).

Recall that \( V \) in Theorem 4.3 was chosen such that \( V \cap b\Omega_a \subset \subset b\Omega \). Let \( W \subset V \) be a neighborhood of \( p \), and \( \zeta \in C^\infty_c(V) \), \( \zeta \geq 0 \) and \( \zeta \equiv 1 \) on \( W \). Let \( u \in \mathcal{D}^{0,q}(\Omega) \) be supported in \( W \cap \bar{\Omega} \). Then it follows that \( u \in \mathcal{D}^{0,q}(\Omega_a) \). Since we can write

\[
u = \zeta u = \zeta \partial N_{q-1} \partial^* u + \zeta \partial^* N_{q+1} \partial u,
\]

we obtain the estimate

\[
\| u \|^2 \lesssim \| \zeta \partial N_{q-1} \partial^* u \|^2 + \| \partial^* N_{q+1} \partial u \|^2.
\]

Because \( \partial u \) is a \( \partial \)-closed \((0,q+1)\)-form supported in \( W \subset V \), we can use inequality (5.7) to estimate the last term in the above inequality, i.e.

\[
\| u \|^2 \lesssim \| \zeta \partial N_{q-1} \partial^* u \|^2 + \delta^q \| \partial u \|^2 + \delta^{-2q} \| \partial u \|_{-1}^2
\]

(5.8)

So we are left with estimating \( \| \zeta \partial N_{q-1} \partial^* u \|^2 \):

\[
\| \zeta \partial N_{q-1} \partial^* u \|^2 = \| [\zeta^2, \partial] N_{q-1} \partial^* u, \partial N_{q-1} \partial^* u \| + \| \partial \zeta^2 N_{q-1} \partial^* u, \partial N_{q-1} \partial^* u \|
\]

\[
= \| [\zeta^2, \partial] N_{q-1} \partial^* u, u - \partial^* N_{q+1} \partial u \| + \| \partial \zeta^2 N_{q-1} \partial^* u, N_q \partial^* u \|
\]

since \( \partial N_{q-1} \partial^* u = N_q \partial \partial^* u \) for \( u \in \mathcal{D}^{0,q}(\Omega_a) \). By our choice of the cut-off function \( \zeta \) it follows, that the supports of \([\zeta^2, \partial] N_{q-1} \partial^* u \) and \( u \) are disjoint. Therefore

\[
\| \zeta \partial N_{q-1} \partial^* u \|^2 \lesssim \| \zeta^2, \partial \partial^* N_q u \| \| \partial^* N_{q+1} \partial u \| + \| \partial^* N_q (\partial \zeta^2 N_{q-1} \partial^* u) \| \| \partial^* u \|
\]

Using the (sc)-(lc) inequality, we get

\[
(A) \lesssim \eta \| \zeta^2, \partial \partial^* N_q u \|^2 + \frac{1}{\eta} \| \partial^* N_{q+1} \partial u \|^2
\]

for \( \eta > 0 \). Recall that \( \partial^* N_{q+1} \) is a bounded map from \( L^2_{(0,q+1)}(\Omega_a) \) to \( L^2_{(0,q)}(\Omega_a) \), and also note that \([\zeta^2, \partial] \) is a differential operator of order zero. Using inequality (5.7) again, we obtain

\[
(A) \lesssim \eta \| u \|^2 + \frac{1}{\eta} (\delta^q \| \partial u \|^2 + \delta^{-2} \| \partial u \|_{-1}^2).
\]
To estimate term (B) note that $\bar{\partial}\bar{\partial}^* u$ is a $\bar{\partial}$-closed $(0,q)$-form, which is supported in $V$. Thus, by our estimate (5.3) on $\bar{\partial}^* N_q$, it follows

$$\|\bar{\partial}^* N_q(\bar{\partial}\bar{\partial}^* u)\| \lesssim \delta^\epsilon \|\bar{\partial}\bar{\partial}^* u\| + \delta^{-1-\epsilon} \|\bar{\partial}\bar{\partial}^* u\|_1. \tag{B_1}$$

By commuting $\bar{\partial}$ and $\zeta^2$, we obtain for (B1):

$$(B_1) \leq \|\zeta^2 \bar{\partial}\bar{\partial}^* N_q u\| + \|\zeta^2, \bar{\partial}\bar{\partial}^* N_q u\| \lesssim \|\bar{\partial}\bar{\partial}^* N_q u\| + \|\bar{\partial}^* N_q u\| \lesssim \|u\|.$$  

The last step holds, since $\bar{\partial}\bar{\partial}^* N_q$ is a bounded operator on $L^2_{0,q}(\Omega_a)$ and $\bar{\partial}^* N_q$ is a bounded operator from $L^2_{0,q}(\Omega_a)$ to $L^2_{0,q-1}(\Omega_a)$.

For estimating (B2) commute $\bar{\partial}$ and $\zeta^2$ again, that is

$$(B_2) \leq \|\zeta^2 \bar{\partial}\bar{\partial}^* N_q u\|_1 + \|\zeta^2, \bar{\partial}\bar{\partial}^* N_q u\|_1 \lesssim \|\bar{\partial}\bar{\partial}^* N_q u\|_1 + \|\bar{\partial}^* N_q u\|_1 \lesssim \|u\|_1.$$  

by (5.3) and (5.4). Combining our estimates for (B1) and (B2), we get

$$(B) \lesssim (\delta^\epsilon \|u\| + \delta^{-1-\epsilon} \|u\|_1)\|\bar{\partial}\bar{\partial}^* u\| \lesssim \eta(\|u\|^2 + \delta^{-2} \|u\|_1^2) + \frac{\delta^{2\epsilon}}{\eta} \|\bar{\partial}\bar{\partial}^* u\|^2,$$

where the last step, again, follows by the (sc)-(lc) inequality, and $\eta > 0$.

Recall that we need the above estimates on (A) and (B) to get control on the term $\|\zeta^2 \bar{\partial}\bar{\partial}^* N_q u\|$. We now have

$$\|\zeta^2 \bar{\partial}^* N_q u\|^2 \lesssim \frac{\delta^{2\epsilon}}{\eta} (\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 + \delta^{-2} \|\bar{\partial} u\|_1^2) + \eta(\|u\|^2 + \delta^{-2} \|u\|_1^2).$$

Combining this last estimate with inequality (5.8), it follows that

$$\|u\|^2 \lesssim \frac{\delta^{2\epsilon}}{\eta} (\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 + \delta^{-2} \|\bar{\partial} u\|_1^2) + \eta(\|u\|^2 + \delta^{-2} \|u\|_1^2)$$

holds uniformly for all $\eta > 0$. Finally, for all sufficiently small $\eta > 0$ we can absorb the term $\eta\|u\|^2$ into the left hand side and obtain

$$\|u\|^2 \lesssim \frac{\delta^{2\epsilon}}{\eta} (\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 + \delta^{-2} \|\bar{\partial} u\|_1^2) + \eta\delta^{-2} \|u\|_1^2.$$  

Recall that here $\|\|\|$ denotes the $L^2$-norm on $\Omega_a$. However, $\Omega_a \subset \Omega$ and $u \in D^{0,\delta}(\Omega)$ is supported in $W \cap \Omega_a$. Thus we can conclude

$$\|u\|_{\Omega_a} \lesssim \frac{\delta^{2\epsilon}}{\eta} (\|\bar{\partial} u\|^2_{\Omega_a} + \|\bar{\partial}^* u\|^2_{\Omega_a} + \delta^{-2} \|\bar{\partial} u\|_{1,\Omega_a}^2) + \eta\delta^{-2} \|u\|_{1,\Omega_a}^2,$$

for all $\eta > 0$ sufficiently small. □
6. Subelliptic estimate

In this section we show how to derive subelliptic estimates from the family of estimates obtained in Proposition 5.5. We begin with stating the main result of this section.

**Theorem 6.1.** Let $\Omega \subset \subset \mathbb{C}^n$ be a smoothly bounded domain, $p$ a point on the boundary of $\Omega$. Let $V$ be a special boundary chart near $p$ such that $V \cap b\Omega$ is pseudoconvex. Suppose that

$$\|u\|^2 \lesssim \frac{\delta^2 e}{\eta} (Q(u,u) + \delta^{-2}\|\bar{\partial}u\|_{-1}^2) + \eta \delta^{-2}\|u\|_{-1}^2$$

holds for all $u \in \mathcal{D}^{0,q}(\Omega)$ supported in $V \cap \overline{\Omega}$, and for all $\eta, \delta > 0$ sufficiently small. Let $W \subset \subset V$ be a neighborhood of $p$. Then

$$\|u\|^2 \lesssim Q(u,u)$$

holds for all $u \in \mathcal{D}^{0,q}(\Omega)$ which are supported in $W \cap \overline{\Omega}$.

For the proof of Theorem 6.1 we use a method from [Cat]. That is, we introduce a sequence of pseudo-differential operators, which represent a partition of unity in the tangential Fourier transform variables:

Let $\{p_k(t)\}_{k=0}^{\infty}$ be a sequence of functions on $\mathbb{R}$ satisfying the following conditions:

1. $\sum_{k=0}^{\infty} p_k^2(t) = 1$ for all $t \in \mathbb{R}$,
2. $p_0(t) = 0$ for all $t \geq 2$, and $p_k(t) = 0$ for all $t \notin (2^{k-1},2^{k+1})$, $k \geq 1$.

We can choose the $p_k$’s such that $|p_k'(t)| \leq C2^{-k}$ holds for all $k \in \mathbb{N}_0$, $t \in \mathbb{R}$ for some $C > 0$. Let $\mathcal{S}(\mathbb{R}^{2n})$ be the class of Schwartz functions on $\mathbb{R}^{2n}$. Denote by $\mathbb{R}^{2n}_-$ the set $\{(x_1,\ldots,x_{2n-1},r) \mid r \leq 0\}$ and $\mathcal{S}(\mathbb{R}_-^{2n})$ be the restriction of $\mathcal{S}(\mathbb{R}^{2n})$ to $\mathbb{R}^{2n}_-$.

For $f \in \mathcal{S}(\mathbb{R}^{2n})$ define the operators $P_k$ by

$$\tilde{P}_k f(\xi,r) := p_k(|\xi|) \tilde{f}(\xi,r),$$

where $\tilde{f}$ is the tangential Fourier transform, that is

$$\tilde{f}(\xi,r) = \int_{\mathbb{R}^{2n-1}} e^{-2\pi i (x,\xi)} f(x,r) dx.$$

On $(0,q)$-forms we define the $P_k$’s to act componentwise.

One of the crucial features of such operators $P_k$ is that it makes the tangential Sobolev $s$-norm of a function $f \in \mathcal{S}(\mathbb{R}^{2n}_-)$ comparable to a series involving $L^2$-norms of $P_k f$. In general, we have:

**Lemma 6.3.** For $f \in \mathcal{S}(\mathbb{R}^{2n}_-)$ and $s = s_1 + s_2$ it holds that

$$\|f\|_s^2 \approx \sum_{k=0}^{\infty} 2^{2ks_1} \|P_k f\|_{s_2}^2.$$
Proof. Let $f \in \mathcal{S}(\mathbb{R}^{2n})$, $s = s_1 + s_2$. From the definition of the tangential Sobolev $s$-norm and since $\sum_{k=0}^{\infty} p_k^2 = 1$ holds, it follows that

$$\|f\|_s^2 = \int_{-\infty}^{\infty} \int_{\mathbb{R}^{2n-1}} (1 + |\xi|^2)^s (\sum_{k=0}^{\infty} p_k^2(|\xi|)) |\tilde{f}(\xi, r)|^2 d\xi dr.$$ 

Since $(1 + |\xi|^2)^{s_1} \approx 2^{2k s_1}$ as long as $|\xi|$ is in the support of $p_k$, we obtain

$$\|f\|_s^2 \approx \sum_{k=0}^{\infty} 2^{2k s_1} \int_{-\infty}^{0} \int_{\mathbb{R}^{2n-1}} (1 + |\xi|^2)^{s_2} p_k(|\xi|) |\tilde{f}(\xi, r)|^2 d\xi dr$$

$$= \sum_{k=0}^{\infty} 2^{2k s_1} \|P_k f\|_{s_2}^2.$$ 

\[\square\]

Suppose $u = \sum'_{|I|=q} v_J dz^I$ is in $\mathcal{D}^{0,q}(\Omega)$ and supported in $V \cap \Omega$, where $V$ is a special boundary chart near a boundary point $p$. Then we can write

$$u = \sum'_{|I|=q} u_I dx^I,$$

where $I = \{i_1, \ldots, i_q\}$ with $1 \leq i_l \leq 2n$. The operator $P_k$ acting on a $(0,q)$-form $u$ means the following:

$$P_k u = \sum'_{|I|=q} (P_k u_I) dx^I.$$ 

We remark that $u \in \mathcal{D}^{0,q}(\Omega)$ if and only if $u_I(x', 0) = 0$ for $x' \in \mathbb{R}^{2n-1}$ whenever $2n \in I$. This leads to another crucial property of the operator $P_k$, that is: $P_k u \in \mathcal{D}^{0,q}(\Omega)$ whenever $u \in \mathcal{D}^{0,q}(\Omega)$. However, the $P_k$’s do not see the support of $u$, i.e. if $u$ is compactly supported, we cannot conclude the same for $P_k u$. Thus inequality (6.2) does not hold for $P_k u$ in general. We shall introduce an appropriately chosen cut-off function $\chi$ and consider $\chi P_k u$. To be able to deal with certain error terms arising from inequality (6.2) applied to $\chi P_k u$, we collect a few facts in the following lemmata.

**Lemma 6.4.** If $f, g \in \mathcal{S}(\mathbb{R}^{2n})$ and $\sigma \in \mathbb{R}$, then

$$\sum_{k=0}^{\infty} 2^{2k \sigma} \| [P_k, f] g \|_2^2 \lesssim \|g\|_{\sigma-1}^2,$$

where the constant in $\lesssim$ does not depend on $g$.

The proof of Lemma 6.4

**Lemma 6.5.** Let $D$ be any differential operator of first order with coefficients in $\mathcal{C}^{\infty}(\mathbb{R}^{2n})$ acting on smooth $q$-forms, let $\chi \in \mathcal{S}(\mathbb{R}^{2n})$ and $\sigma > 0$. 

\[ \sum_{k=0}^{\infty} 2^{2k\sigma} \| D(\chi P_k u) \|_{-\sigma}^2 \lesssim \| Du \|^2 + \| u \|^2 + \sum_{|I|=q} \left\| \partial_{x^I} u \right\|_{2^2}^2. \]

holds for all \( q \)-forms \( u \) with coefficients in \( S(\mathbb{R}^{2n}) \). Here, the constant in \( \lesssim \) does not depend on \( u \).

**Proof.** Recall that \( \Lambda_t^{-\sigma} \) denotes the tangential Bessel potential of order \(-\sigma\). We obtain
\[
\sum_{k=0}^{\infty} 2^{2k\sigma} \| D(\chi P_k u) \|_{-\sigma}^2 \leq \sum_{k=0}^{\infty} 2^{2k\sigma} \| D(\chi P_k u) \|_{-\sigma}^2 = \sum_{k=0}^{\infty} 2^{2k\sigma} \| \Lambda_t^{-\sigma} D(\chi P_k u) \|^2
\]
\[
\lesssim \sum_{k=0}^{\infty} 2^{2k\sigma} \| \chi \Lambda_t^{-\sigma} D P_k u \|^2 + \sum_{k=0}^{\infty} 2^{2k\sigma} \| [\Lambda_t^{-\sigma} D, \chi] P_k u \|^2,
\]
where the last step follows by commuting. We note that \([\Lambda_t^{-\sigma} D, \chi]\) is of tangential order \(-\sigma\) and of normal order 0. Therefore, invoking Lemma 6.3 we get
\[
\sum_{k=0}^{\infty} 2^{2k\sigma} \| [\Lambda_t^{-\sigma} D, \chi] P_k u \|^2 \lesssim \sum_{k=0}^{\infty} 2^{2k\sigma} \| P_k u \|_{-\sigma}^2 \approx \| u \|^2.
\]

Similarly, we obtain by commuting
\[
\sum_{k=0}^{\infty} 2^{2k\sigma} \| \chi \Lambda_t^{-\sigma} D P_k u \|^2 \lesssim \sum_{k=0}^{\infty} 2^{2k\sigma} \| \chi D \Lambda_t^{-\sigma} P_k u \|^2 + \sum_{k=0}^{\infty} 2^{2k\sigma} \| P_k u \|_{-\sigma}^2
\]
\[
\lesssim \sum_{k=0}^{\infty} 2^{2k\sigma} \| P_k (\chi D \Lambda_t^{-\sigma} u) \|^2 + \sum_{k=0}^{\infty} 2^{2k\sigma} \| [\chi D \Lambda_t^{-\sigma}, P_k] u \|^2 + \| u \|^2
\]
\[
\lesssim \| \chi D \Lambda_t^{-\sigma} u \|^2_{\sigma} + \sum_{k=0}^{\infty} 2^{2k\sigma} \| [\chi D \Lambda_t^{-\sigma}, P_k] u \|^2 + \| u \|^2,
\]
where the last line follows again by Lemma 6.3. We write
\[ \chi D = \sum_{|I|=q}^{2n} \sum_{j=1}^{2n} a^I_j \frac{\partial}{\partial x_j}, \]
and estimate term (A) by commuting:
\[
(A) = \| \chi D \Lambda_t^{-\sigma} u \|^2_{\sigma} \lesssim \| \Lambda_t^{-\sigma} \chi D u \|^2_{\sigma} + \| [\chi D, \Lambda_t^{-\sigma}] u \|^2_{\sigma}
\]
\[
\lesssim \| Du \|^2 + \sum_{|I|=q}^{2n} \sum_{j=1}^{2n} \| [a^I_j \frac{\partial}{\partial x_j}, \Lambda_t^{-\sigma}] u_I \|^2_{\sigma}.
\]
Since $\frac{\partial}{\partial x_j}$ and $\Lambda_t^{-\sigma}$ commute, it follows that
\[
(A) \lesssim \|Du\|^2 + \sum_{|l|=q}^2 \sum_{j=1}^{2n} \|[a_j^l, \Lambda_t^{-\sigma}]\|_{\sigma_1}^2 \frac{\partial u_l}{\partial x_j} \|\sigma_1^2.
\]
\[
\lesssim \|Du\|^2 + \|u\|^2 + \sum_{|l|=q}^2 \left\| \frac{\partial u_l}{\partial x_{2n}} \right\|_{-1}^2.
\]

Here, the last estimate holds since $[a_j^l, \Lambda_t^{-\sigma}]$ is of tangential order $-\sigma - 1$ and $\frac{\partial}{\partial x_j}$ is a tangential derivative if $j \in \{1, \ldots, 2n - 1\}$. We are left with estimating the terms $(B_k)$. We first notice that
\[
(B_k) \lesssim \sum_{|l|=q}^2 \sum_{j=1}^{2n} \left\|[a_j^l, P_k] \frac{\partial}{\partial x_j} \Lambda_t^{-\sigma} u_l \right\|_{\sigma_1}^2.
\]
Lemma 6.3 implies now
\[
\sum_{k=0}^{\infty} 2^{2k\sigma} (B_k) \lesssim \sum_{|l|=q}^2 \sum_{j=1}^{2n} \left\| \frac{\partial}{\partial x_j} \Lambda_t^{-\sigma} u_l \right\|_{\sigma_1}^2 \lesssim \|u\|^2 + \sum_{|l|=q}^2 \left\| \frac{\partial u_l}{\partial x_{2n}} \right\|_{-1}^2.
\]
Combining all our estimates we end up with the claimed inequality.
\[
\sum_{k=0}^{\infty} 2^{2k\sigma} \|D(\chi P_k u)\|_{\sigma_1}^2 \lesssim \|Du\|^2 + \|u\|^2 + \sum_{|l|=q}^2 \left\| \frac{\partial u_l}{\partial x_{2n}} \right\|_{-1}^2.
\]

Having collected the basic facts concerning the $P_k$’s, we are ready to prove Theorem 6.1.

**Proof of Theorem 6.1** Let $V$ be a special boundary chart near $p$ such that inequality (6.2) holds, that is
\[
\|u\|^2 \lesssim \frac{\delta^{2k}}{\eta} (Q(u, u) + \delta^{-2} \|\partial u\|^2_{-1}) + \eta \delta^{-2} \|u\|^2_{-1}
\]
holds for all $u \in D^{\theta, q}(\Omega)$ supported in $V \cap \bar{\Omega}$. Let $W \subset V$ be a neighborhood of $p$, and $u \in D^{\theta, q}(\Omega)$ supported in $W \cap \bar{\Omega}$. Let $\chi \in C_c^\infty(V)$ such that $\chi = 1$ on $W$ and $\chi \geq 0$. Then it follows by Lemma 6.3 and by commuting
\[
\|u\|_{\epsilon}^2 = \|\chi u\|_{\epsilon}^2 \lesssim \sum_{k=0}^{\infty} 2^{2k\epsilon} \|\chi P_k u\|^2 + \sum_{k=0}^{\infty} 2^{2k\epsilon} \|[P_k, \chi] u\|^2
\]
\[
\lesssim \sum_{k=0}^{\infty} 2^{2k\epsilon} \|\chi P_k u\|^2 + \|u\|_{\epsilon}^2_{-1},
\]
where the last step follows by Lemma 6.4. Since $\epsilon \leq \frac{1}{2}$ holds, we obtain
\[
\|u\|_{\epsilon}^2 \lesssim \sum_{k=0}^{\infty} 2^{2k\epsilon} \|\chi P_k u\|^2 + \|u\|^2.
\]
Now inequality (6.2) comes into play. Since $\chi P_k u \in D^{0,q}(\Omega)$ is supported in $V \cap \Omega$, it follows that
\[
\|\chi P_k u\|^2 \lesssim \frac{\delta^{2k}}{\eta} (Q(\chi P_k u, \chi P_k u) + \delta^{-2}\|\tilde{\partial}(\chi P_k u)\|_{-1}^2) + \eta \delta^{-2}\|\chi P_k u\|_{-1}^2
\]
holds uniformly for all $k \in \mathbb{N}_0$, for all positive $\delta < \delta_0$ and $\eta < \eta_0$. Let $k_0 \in \mathbb{N}$ such that $2^{-k_0} \leq \delta_0$. Then we obtain for all $k \geq k_0$
\[
2^{2k} \|\chi P_k u\|^2 \lesssim \frac{1}{\eta} (Q(\chi P_k u, \chi P_k u) + 2^{2k}\|\tilde{\partial}(\chi P_k u)\|_{-1}^2) + \eta 2^{2k(1+\varepsilon)}\|\chi P_k u\|_{-1}^2.
\]
Observe that
\[
\sum_{k=0}^{k_0-1} 2^{2k} \|\chi P_k u\|^2 \lesssim \sum_{k=0}^{k_0-1} 2^{2k} \|u\|^2 \lesssim \|u\|^2.
\]
Thus we can sum up over $k \in \mathbb{N}_0$, obtaining
\[
\sum_{k=0}^{\infty} 2^{2k} \|\chi P_k u\|^2 \lesssim \frac{1}{\eta} \sum_{k=0}^{\infty} (Q(\chi P_k u, \chi P_k u)) + \frac{1}{\eta} \sum_{k=0}^{\infty} 2^{2k} \|\tilde{\partial}(\chi P_k u)\|_{-1}^2
\]
\[
+ \eta \sum_{k=0}^{\infty} 2^{2k(1+\varepsilon)}\|\chi P_k u\|_{-1}^2 + \|u\|^2
\]
Using Lemma 6.3, we have
\[
\sum_{k=0}^{\infty} 2^{2k(1+\varepsilon)}\|\chi P_k u\|_{-1}^2 \lesssim \sum_{k=0}^{\infty} 2^{2k(1+\varepsilon)}\|P_k u\|_{-1}^2 \lesssim \|u\|^2.
\]
Furthermore, applying Lemma 6.5 with $\sigma = 0$ and $\sigma = 1$ resp., we get
\[
\sum_{k=0}^{\infty} Q(\chi P_k u, \chi P_k u) + \sum_{k=0}^{\infty} 2^{2k}\|\tilde{\partial}(\chi P_k u)\|_{-1}^2 \lesssim Q(u, u) + \|u\|^2 + \sum_{|I|=q} \|\frac{\partial u_I}{\partial x_{2n}}\|_{-1}^2.
\]
Note that $\frac{\partial u_I}{\partial x_{2n}}$ can be expressed as a linear combination of the $\frac{\partial}{\partial x_j}$’s and a tangential vector field $T$. Then
\[
\|\frac{\partial u_I}{\partial x_{2n}}\|_{-1} \lesssim \sum_{j=1}^{n} \|\frac{\partial u_I}{\partial x_j}\|_{-1}^2 + \|Tu_I\|_{-1} \lesssim \sum_{j=1}^{n} \|\frac{\partial u_I}{\partial x_j}\|_{-1}^2 \sim \|\tilde{\partial} u\|^2 + \|\tilde{\partial}^* u\|^2 + \|u\|^2 \lesssim Q(u, u).
\]
Thus, by combining our estimates, we obtain
\[
\|u\|^2 \lesssim \sum_{k=0}^{\infty} 2^{2k} \|\chi P_k u\|^2 + \|u\|^2 \lesssim \frac{1}{\eta} Q(u, u) + \eta \|u\|^2.
\]
Choosing $\eta > 0$ small enough, we can absorb the term $\eta \|u\|^2$ into the left hand side and it follows $\|u\|^2 \lesssim Q(u, u)$. \qed
7. An Example

Consider the domain \( D = \{ w \in \mathbb{C}^3 \mid \rho(w) := \text{Re } w_3 + |w_1^2 - w_2w_3|^2 + |w_2|^2 < 0 \} \) near the origin. The 1-type (in the sense of D'Angelo) at \((0,0,0)\) is 4, but at any boundary point of the form \((0, 0, i\epsilon)\), \( \epsilon > 0 \), the 1-type is 8. In the following we show that a subelliptic estimate of order \( \frac{1}{8} - \eta \) holds for any \( \eta > 0 \) near the origin. Instead of constructing the \( \{ \phi_\delta \} \) on \( D \), we consider \( \Omega = \{ z \in \mathbb{C}^3 \mid r(z) < 0 \} \), where

\[
r(z) = |z_3|^2 - 1 + |(1 + z_3)z_1^2 - z_2(z_3 - 1)|^2 + |1 + z_3|^2 |z_2|^4 < 0,
\]

in a neighborhood \( U \) of the boundary point \( p = (0, 0, 1) \). Notice that \( D \) near the origin is biholomorphic to \( \Omega \) via the transformation \( z_1 = w_1 \), \( z_2 = w_2 \) and \( z_3 = \frac{w_2 + 1}{1 - w_3} \). We claim that

\[
\phi_\delta(z) = -\log(-r(z) + \delta) - \log(-\log(|z_1|^2 + \delta^{\frac{1}{2}})) - \log(-\log(|z_2|^2 + \delta^{\frac{1}{2}} + \eta)) = \psi_0(z) + \psi_1(z) + \psi_2(z)
\]

satisfies the hypotheses of Theorem 1.4 on \( \overline{\Omega} \cap U \) with \( \epsilon = \frac{1}{8} - \frac{\eta}{2} \) for \( \eta > 0 \). A straightforward computation shows that \( \phi_\delta \) is plurisubharmonic and has a self-bounded complex gradient near \( p \). In the following we show that

\[
i\partial \bar{\partial} \phi_\delta(z)(\xi, \xi) \geq C\delta^{-\frac{1}{8} + \eta}||\xi||^2
\]

holds for all \( \xi \in \mathbb{C}^3 \) and \( z \in S_8 \cap U \). One computes

\[
i\partial \bar{\partial} r(\xi, \xi) = \frac{4|1 + z_3|^2 |z_1|^2 |\xi_1|^2 + (|z_3 - 1|^2 + 4|z_2|^2 |1 + z_3|^2) |\xi_2|^2 + (1 + |z_1|^2 - z_2|^2) |\xi_3|^2}{r(z) + \delta} + 2 \text{Re}(2(1 + z_3)^2 z_2 \bar{z}_2 - (z_3 - 1)(\bar{z}_3 - \bar{z}_2)) \xi_2 \bar{\xi}_3) + 4 \text{Re}(1 + z_3 z_1 \xi_1 ((z_2^2 - z_2) \bar{\xi}_3 - (z_3 - 1) \bar{\xi}_2)).
\]

Denote the last term on the right hand side by \( (I) \). Estimating \( (I) \) we obtain

\[
(I) \geq -4|1 + z_3|^2 |z_1|^2 |\xi_1|^2 - |z_3 - 1|^2 |\xi_2|^2 - |z_1 - z_2|^2 |\xi_3|^2 + 2 \text{Re}((z_3 - 1) \xi_2 (z_2^2 - z_2) \bar{\xi}_3).
\]

It follows easily that

\[
i\partial \bar{\partial} r(z)(\xi, \xi) \geq |z_2|^2 |\xi_2|^2 + \frac{1}{2} |\xi_3|^2.
\]

This estimate implies that if \( z \in S_\delta \cap U \), then

\[
i\partial \bar{\partial} \psi_0(z)(\xi, \xi) \geq \frac{|z_2|^2 |\xi_2|^2 + \frac{1}{2} |\xi_3|^2}{r(z) + \delta} \geq \frac{1}{4} (\delta^{-\frac{1}{2}} |\xi_2|^2 + \delta^{-1} |\xi_3|^2),
\]

where the first estimate on the right hand side only holds if \( |z_2|^2 \geq \delta^{\frac{1}{2}} \). If \( |z_2|^2 \leq \delta^{\frac{1}{2}} \), then

\[
i\partial \bar{\partial} \psi_2(z)(\xi, \xi) \geq \frac{\delta^{\frac{1}{2} + \eta} |\xi_2|^2}{-\log(|z_2|^2 + \delta^{\frac{1}{2} + \eta})(|z_2|^2 + \delta^{\frac{1}{2} + \eta})^2} \geq \delta^{-\frac{1}{2}} |\xi_2|^2.
\]
Similarly, we obtain \( i\partial\bar{\partial}v_1(z)(\xi,\xi) \geq \frac{1}{2} \delta^{\frac{3}{2}+\eta}|\xi_1|^2 \) for \( |z_1|^2 \leq \delta^{\frac{3}{4}} \) for all \( \delta > 0 \) sufficiently small. Thus it remains to show that (7.1) holds also in the directions involving \( \xi_1 \) for \( z \in S_\delta \cap U \) with \( |z_1|^2 \geq \delta^{\frac{3}{4}} \). For that we shall use a different estimate for the complex Hessian of \( r \), that is

\[
(7.3) \quad i\partial\bar{\partial}r(z)(\xi,\xi) \geq \frac{1}{2}|z_1|^2|\xi_1|^2 - 4|z_3 - 1|^2|\xi_2|^2. 
\]

Then, if \( z \in S_\delta \) and \( |z_1|^2 \geq \delta^{\frac{3}{4}} \), we obtain by using (7.3) and (7.2)

\[
(\delta^{\frac{3}{4}+\eta} + \frac{1}{2})i\partial\bar{\partial}v_0(z)(\xi,\xi) \geq \frac{C_0}{\delta}(\delta^{\frac{3}{4}+\eta}|\xi_1|^2 + (|z_2|^2 - 16\delta^{\frac{3}{4}+\eta}|z_3 - 1|^2)|\xi_2|^2).
\]

Thus we obtain (7.1) for all \( z \in S_\delta \) as long as \( |z_2|^2 \geq 16\delta^{\frac{3}{4}+\eta}|z_3 - 1|^2 \). If the latter inequality is not true, then we can assume that \( |z_2|^2 \leq \delta^{\frac{3}{4}+\eta} \).

However, in that case

\[
\frac{1}{2}i\partial\bar{\partial}v_2(z)(\xi,\xi) + 16\delta^{\frac{3}{4}+\eta}|z_3 - 1|^2|\xi_2|^2 \geq 0,
\]

which completes the proof of (7.1).

With a construction similar to the above one obtains for the domains

\[ D_{k,l,m,n} = \{ w \in \mathbb{C}^3 \mid \text{Re}w_3 + |w_1^k - w_2^l w_3^m|^2 + |w_2^m|^2 < 0 \}, \quad k, l, m, n \in \mathbb{N} \]

a subelliptic estimate of order \( \frac{1}{M} - \eta \), \( \eta > 0 \), where \( M \) is the maximum 1-type near the origin.

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