Quantum Harmonic Oscillators with Nonlinear Effective Masses

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We study the eigen-energy and eigen-function of a quantum particle acquiring the probability density-dependent effective mass (DDEM) in harmonic oscillators. Instead of discrete eigen-energies, continuous energy spectra are revealed due to the introduction of a nonlinear effective mass. Analytically, we map this problem into an infinite discrete dynamical system and obtain the stationary solutions by perturbation theory, along with the proof on the monotonicity in the perturbed eigen-energies. Numerical results not only give agreement to the asymptotic solutions stemmed from the expansion of Hermite-Gaussian functions, but also unveil a family of peakon-like solutions without linear counterparts. As nonlinear Schrödinger wave equation has served as an important model equation in various sub-fields in physics, our proposed generalized quantum harmonic oscillator opens an unexplored area for quantum particles with nonlinear effective masses.

I. INTRODUCTION

Quantum harmonic oscillator is the most important model system in quantum mechanics, which remarkably exhibits an exact, analytical solution with discrete (quantized) eigen-energies compared to the predictions of classical counterparts [1]. Instead of a given mass, $m_0$, when particles (electrons or holes) move inside a periodic potential or interact with other identical particles, their motions differ from those in a vacuum, resulting in an effective mass [2]. With an effective mass, denoted as $m^*$, the corresponding Schrödinger equation for a quantum particle in a one-dimensional harmonic oscillator, characterized by the spring constant $k$, has the form:

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = \frac{-1}{2m^*(x)} \frac{\partial^2}{\partial x^2} \Psi + \frac{k}{2} x^2 \Psi. \quad (1)$$

Here $\Psi(x,t)$ is the probability amplitude function projected in the spatial coordinate. In particular, with a nonuniform composition in potential or particle distributions, a position-dependent effective mass (PDEM) Schrödinger equation has gained much interest for its applications from semiconductors to quantum fluids [3–7]. Recently, a PDEM Schrödinger equation exhibiting a similar position-dependence for both the potential and mass was exactly solved [8].

With the correspondence between Schrödinger equation and the paraxial wave equation, similar concept of position-dependent effects is also studied in the dispersion management optical fiber link [9]. Moreover, in addition to position-dependence, chromatic dispersion may also have intensity-dependent dispersion (IDD) in the optical domains [10, 11]. IDD, or in general the nonlinear corrections to the chromatic dispersion as a function of the wave intensity, has arisen in a variety of wave phenomena, such as shallow water waves [12, 13], acoustic waves in micro-inhomogeneous media [14], ultrashort coherent pulses in quantum well waveguide structures [15], the saturation of atomic-level population [16], electromagnetically-induced transparency in a chain-Λ configuration [17], or nonlocal nonlinearity mediated by dipole-dipole interactions [18]. Inspired by IDD, in this work, we consider a quantum particle acquiring an probability density-dependent effective mass (DDEM), i.e., $m^*(|\Psi|^2)$, in a harmonic potential described by the following generalized Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = \frac{-1}{2m_0} (1 + b |\Psi|^2) \frac{\partial^2}{\partial x^2} \Psi + \frac{k}{2} x^2 \Psi. \quad (2)$$

Here, the DDEM is approximated by assuming $[m^*(|\Psi|^2)]^{-1} \approx [m_0 (1 - b |\Psi|^2)]^{-1} \approx [m_0]^{-1} (1 + b |\Psi|^2)$, with the parameter $b$ denoting the contribution from the nonlinear effective mass term. As one can see, when the nonlinear effective mass term is zero, i.e., $b = 0$, Eq. (2) is reduced to the well-known scenario for a quantum particle in a parabolic potential.

However, when $b \neq 0$, instead of the discrete energies, continuous energy spectra are revealed due to the introduction of a nonlinear effective mass. Analytical solutions for the corresponding eigen-energy and eigen-function are derived with the help of perturbation theory. Numerical solutions obtained by directly solving Eq. (2) give good agreement to the analytical ones obtained from the expansion of Hermite-Gaussian functions. Moreover, we unveil a family of peakon-like solutions supported by DDEM, which has no counterpart in the linear limit. Our perturbed solutions and numerical results for this generalized quantum harmonic oscillator with nonlinear effective masses opens an unexplored area for quantum particles.

The paper is organized as follows: in Session II, we introduce the quantum harmonic oscillator into this generalized Schrödinger equation with nonlinear effective mass and reduce Eq. (2) into an infinite dynamical system. Then, by perturbation theory and with the help of the eigen-solutions of quantum harmonic oscillator, we study the corresponding eigen-energy with the introduction of DDEM, as a function of the parameter $b$. The monotonicity of the perturbed eigen-energy is also proved. In Section III, explicitly, we derive the analytical solutions of eigen-energies and the corresponding wavefunctions for the ground and second-order excited...
states in the asymptotical limit The comparison between analytical solutions and numerical results is illustrated in Section IV, demonstrating good agreement on the solutions with a smooth profile, stemmed from the expansion of Hermite-Gaussian wavefunctions. A new family of peakon-like solutions with a discontinuity in its first-order derivative is also unveiled, which has no linear counterparts. Finally, we summarize this work with some perspectives in Conclusion.

II. QUANTUM HARMONIC OSCILLATOR WITH DDEM

Without loss of generality, in the following, we set \( \hbar = 1 \), \( k > 0 \), \( m_0 = 1 \) for the simplicity in tackling Eq. (2). Here, lookin for the stationary solutions \( \Psi(x, t) = \psi(x) e^{-iE t} \), we consider a family of differential equations parametrized by a continuous DDEM parameter \( b \neq 0 \) of the form

\[
E \psi + \frac{1}{2} (1 + b |\psi|^2) \frac{\partial^2 \psi}{\partial x^2} - \frac{k}{2} x^2 \psi = 0, \tag{3}
\]

where \( E \) is the corresponding eigen-energy, \( x \) denotes a real variable for the coordinate, and \( \psi(x) \) is a square integrable function. This stationary Schrödinger wave equation can be seen as a generalized quantum harmonic oscillator.

When \( b = 0 \) and \( k = 1 \), Eq. (3) becomes the well-known equation for the quantum harmonic oscillator, which supports eigen-function of the \( n \)-th order excited state in the position representation reads [19]:

\[
\phi_n(x) = \mu_n e^{-x^2/2} H_n(x), \tag{4}
\]

where \( \mu_n = (2^n n! \sqrt{\pi})^{-1/2} \) and \( H_n(x) \) is the \( n \)-th order Hermite polynomial. The corresponding eigen-values are equal to \( E_n = n + \frac{1}{2} \), for any \( n \in \mathbb{N} \). We are interested in finding pairs \( (\psi_n, E_n)_b \) fulfilling Eq. (3) for a set \( b \neq 0 \).

A. Perturbation Theory for Eigen-Energies and Eigen-Functions

To investigate Eq. (2) with \( b \neq 0 \) (but keeping \( k \neq 1 \) first), we apply the perturbation theory based on the expansion of the solution on the eigen-function \( \phi_n(x) \). That is,

\[
\Psi(x, t; E) = \sum_{n=0}^{\infty} B_n(t) \phi_n(x). \tag{5}
\]

By plugging this expansion into Eq. (2), one has

\[
\begin{align*}
\sum_{n=0}^{\infty} dB_n(t) \phi_n(x) + \frac{1}{2} \sum_{n=0}^{\infty} B_n(t) \frac{d^2 \phi_n(x)}{dx^2} + \frac{1}{2} \sum_{p=0, q=0} B_p(t) \overline{B_q(t)} \phi_n(x) \phi_q(x) [b \sum_{n=0}^{\infty} B_n(t) \frac{d^2 \phi_n(x)}{dx^2} - k \frac{1}{2} x^2 \sum_{n=0}^{\infty} B_n(t) \phi_n(x)], \\
+ \sum_{p=0, q=0} B_p(t) \overline{B_q(t)} \phi_n(x) \phi_q(x) [b \sum_{n=0}^{\infty} B_n(t) \frac{d^2 \phi_n(x)}{dx^2} - k \frac{1}{2} x^2 \sum_{n=0}^{\infty} B_n(t) \phi_n(x)] - \frac{k}{2} x^2 \sum_{n=0}^{\infty} B_n(t) \phi_n(x), \\
= i \sum_{n=0}^{\infty} dB_n(t) \phi_n(x) + \left( \frac{1}{2} - b \right) \sum_{n=0}^{\infty} x^2 B_n(t) \phi_n(x) - \sum_{n=0}^{\infty} E_n B_n(t) \phi_n(x) \\
- b \sum_{n=0, p=0, q=0} \sum_{n=0}^{\infty} E_n B_n(t) B_p(t) \overline{B_q(t)} \phi_n(x) \phi_p(x) \phi_q(x) + \frac{b}{2} x^2 \sum_{n=0, p=0, q=0} \sum_{n=0}^{\infty} B_n(t) B_p(t) \overline{B_q(t)} \phi_n(x) \phi_p(x) \phi_q(x) = 0. \tag{6}
\end{align*}
\]

Here, \( \overline{B_n} \) means the complex conjugate of \( B_n \). Then, by multiplying Eq. (7) with \( \phi_m(x) \) and using the orthonormal property of \( \phi_m(x) \), we obtain

\[
\begin{align*}
b & \sum_{n=0, p=0, q=0} \sum_{n=0}^{\infty} [-E_n V_{m, n, p, q} + \frac{1}{2} W_{m, n, p, q}] B_n(t) B_p(t) \overline{B_q(t)} \\
+ & i \frac{dB_m(t)}{dt} - E_m B_m(t) + \left( \frac{1}{2} - b \right) \sum_{n=0}^{\infty} \Gamma_{m, n} B_n(t) = 0. \tag{8}
\end{align*}
\]

where \( \Gamma_{m, n}, V_{m, n, p, q}, \) and \( W_{m, n, p, q} \) are defined as:

\[
\begin{align*}
\Gamma_{m, n} & = \int_{-\infty}^{\infty} \phi_m(x) \phi_n(x) dx, \\
V_{m, n, p, q} & = \int_{-\infty}^{\infty} \phi_m(x) \phi_n(x) \phi_p(x) \phi_q(x) dx, \\
W_{m, n, p, q} & = \int_{-\infty}^{\infty} x^2 \phi_m(x) \phi_n(x) \phi_p(x) \phi_q(x) dx.
\end{align*}
\]

As one can see from Eq. (8), now, we reduce the original partial differential equation into the infinite discrete dynamical system [20]. With the help of the recursive relation of
B. Monotonicity in the perturbed eigen-energy

Given $b > 0$ ($b < 0$), to ensure solutions with linear limit $\phi(x;E) \approx \sqrt{P(E)} \phi_n(x)$ to exist only if $E \geq E_n = n + \frac{1}{2}$ ($E \leq E_n = n + \frac{1}{2}$), we prove that the two integrals inside the square brackets in Eq. (13) is monotonic, i.e.,

$$\frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-x^2} H_{2n(x)}(x)^4 dx - (2n + \frac{1}{2}) \int_{-\infty}^{\infty} e^{-x^2} H_{2n(x)}(x)^4 dx < 0;$$

or equivalently

$$W_{2n,2n,2n} = (4n + 1) V_{2n,2n,2n} < 0.$$  (18)

for $n \geq 0$. In Appendix, the proof on the monotonicity for Eq. (17) and Eq. (18) is given in details.

By using the upper and lower solution method developed in the variational calculus [21], we can further prove the existence of a positive solution (node-less state) through the corresponding Lagrangian for Eq. (2), i.e.,

$$\mathcal{L} = \int_{-\infty}^{\infty} \left[ \left( -\frac{\partial}{\partial t} + \frac{kx^2}{2b} \right) \ln |1 + b|\psi|^2| + \frac{1}{2} |\psi_x|^2 \right] dx,$$

where $Z(E) \equiv \int_{-\infty}^{\infty} \left[ \left( \frac{kx^2}{2b} \right) \ln |1 + b|\psi|^2| + \frac{1}{2} |\psi_x|^2 \right] dx - E Q(E),$  (19)

and $Q \equiv \int_{-\infty}^{\infty} \ln |1 + b|\psi|^2| dx.$  (20)

Here, we also introduce the probability factor $Q$ for this quantum harmonic oscillator with DDEM, by defining

$$Q \equiv \frac{1}{b} \int_{-\infty}^{\infty} \ln |1 + b|\psi|^2| dx.$$  (21)

As the original generalized Schrödinger equation given in Eq. (2) preserves the U(1) symmetry, i.e., $\psi \to \exp[i\theta] \psi,$
the conserved density for this model equation can be derived from Noether theorem [24]. It is noted that Eq. (21) is only applicable when \( b \neq 0 \). When \( b \ll 1 \), this probability factor \( Q \) can be approximated as

\[
\lim_{b \to 0} Q \approx \int_{-\infty}^{\infty} |\psi|^2 \, dx,
\]

which is reduced to the standard definition of probability for quantum wavefunctions. For \( b = 0 \), the corresponding Lagrangian density given in Eq. (19), as well as the conserved density given in Eq. (21), both go to infinity.

These two terms, \( Z(E) \) and \( Q(E) \), shown in Eq. (20), correspond to the Lagrangian of our generalized harmonic oscillator and the conserved quantity, respectively. As the DDEM parameter \( b \to 0 \), the Lagrangian shown in Eq. (20) can be reduced to

\[
\int_{-\infty}^{\infty} (-E |\psi|^2 + \frac{k}{2} x^2 |\psi|^2 + \frac{1}{2} |\psi_x|^2) dx,
\]

which is the Lagrangian for the linear equation, i.e., \(-\frac{1}{2} \psi_{xx} + \frac{k}{2} x^2 \psi = E \psi\). By following the same concept in tackling weak non-linearity [25], the perturbation theory based on the expansion of the Hermite-Gaussian functions to deal with the DDEM ensures that when \( E \to E_n \), one has \( Q(E) \to 0 \).

III. EIGEN-ENERGIES AND EIGEN-FUNCTIONS OBTAINED FROM PERTURBATION

A. The Ground State

Now with the analytical formula given in Eq. (13), we explicitly give the perturbed eigen-energy \( E_0^b \) and eigen-function \( \psi^b_0(x) \) for the ground state in our generalized quantum harmonic oscillator with a given DDEM parameter \( b \). For the ground state, we can assume that \( B_0 \gg B_{2n} \), \( n = 1, 2, 3, \ldots \). Then, from Eq. (11), one has

\[
B_0^2 \approx \frac{E_0^b - E_0}{b(-\frac{1}{2} W_{0,0,0,0,0} + \frac{1}{2} V_{0,0,0,0,0})},
\]

and

\[
B_{2n} \approx \frac{b B_0^2 (-\frac{1}{2} W_{2n,0,0,0,0} + E_0 V_{2n,0,0,0,0})}{(E_0^b - E_{2n})},
\]

where \( \Gamma_{m,n}, V_{m,n,p,q}, \) and \( W_{m,n,p,q} \) have the values:

\[
W_{0,0,0,0,0} = \frac{1}{4\sqrt{2\pi}},
\]

\[
V_{0,0,0,0} = \frac{1}{\sqrt{2\pi}},
\]

\[
V_{2n,0,0,0} = \frac{(-1)^n}{\sqrt{\pi 2^{2n+1} (2n)!}} (2n - 1)!!,
\]

\[
W_{2n,0,0,0} = \frac{(-1)^{n+1}}{\sqrt{\pi 2^{2n+5} (2n)!}} (2n - 1)!! (2n - 1),
\]

for \( n \geq 1 \). Therefore, from Eqs. (23) and (24), explicitly we have, noting that \( E_0 = \frac{1}{2}, E_n = n + \frac{1}{2} \).

\[
B_0^2 \approx \frac{8\sqrt{2\pi}}{3b} (E_0^b - \frac{1}{2}),
\]

\[
B_{2n} \approx \frac{b B_0^2 (-1)^n (\frac{1}{4} n + \frac{3}{8} ) (2n - 1)!!}{\sqrt{\pi 2^{2n+1} (2n)!} (E_0^b - 2n - \frac{1}{2})},
\]

\[
\approx \frac{1}{8} b B_0^2 (-1)^{n+1} (2n - 1)!!, \quad \text{as} \quad n \to \infty.
\]

With the coefficients above, the perturbed solution of \( \psi^b_0(x) \) can be conducted immediately as

\[
\psi^b_0(x) \approx B_0 \phi_0(x) + B_2 \phi_2(x) + B_4 \phi_4(x) + \cdots.
\]

We notice that \( \psi^b_0(x) \to \phi_0(x) \) as \( E_0^b \to E_0 = \frac{1}{2} \). Again, with the orthonormality of \( \phi_{2n}(x) \), in the asymptotical limit, \( n \to \infty \), the probability factor \( Q \) defined in Eq. (21) becomes:

\[
Q(E_0^b) = b \int_{-\infty}^{\infty} \ln |1 + b |\psi^b_0(x)|^2 dx,
\]

\[
\approx \int_{-\infty}^{\infty} |\psi^b_0(x)|^2 dx = B_0^2 + B_2^2 + B_4^2 + \cdots.
\]

\[
\approx \frac{8\sqrt{2\pi}}{3b} (E_0^b - \frac{1}{2}) [1 + \frac{1}{2} (E_0^b - \frac{1}{2})^2 \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2 (2n)!}],
\]

It is noted that the identity \((2n - 1)!! = \frac{(2n)!}{2^n n!}\) is applied. Therefore, we see that \( Q(E_0^b) \to 0 \) as \( E_0^b \to E_0 = \frac{1}{2} \). As one can see from Eq. (28), the probability factor \( Q(E) \) is linearly proportional in the leading order, to the eigen-energy \( E \), but with the coefficient inversely proportional to the DDEM parameter \( b \).

B. The Second Order Excited State

In addition to the ground state with \( n = 0 \), in general, all the perturbed eigen-energy \( E_0^b \), and eigen-function \( \psi^b_2(x) \) can be written explicitly. Here, we illustrate the solutions for the second order excited state, \( E_2^b \) and \( \psi^b_2(x) \), by assuming \( B_2 \gg B_{2n} \), \( n = 0, 2, 3, \cdots \). Again, with Eqs. (9) and (11), one can directly obtain:

\[
B_2^2 \approx \frac{E_2^b - E_2}{b||\sqrt{\frac{\sqrt{2\pi V_0,2,2,2,2}}{2}} - (\frac{2}{3} - E_2)V_2,2,2,2,2 - \frac{\sqrt{3}}{2} V_4,2,2,2,2\}}{b||\sqrt{\frac{\sqrt{2\pi V_0,2,2,2,2}}{2}} - (\frac{2}{3} - E_2)V_2,2,2,2,2 - \frac{\sqrt{3}}{2} V_4,2,2,2,2\}}
\]

and

\[
B_{2n} \approx \frac{b B_2^2 (-\frac{1}{2} W_{2n,2,2,2,2} + E_2 V_{2n,2,2,2,2})}{(E_2^b - E_2)}
\]

with

\[
V_{2n,2,2,2,2} = \frac{(-1)^{n-3} (2n - 1)!! (8n^3 - 60n^2 + 94n - 1)}{\sqrt{\pi 2^{2n+10} (2n)!}}.
\]

As a result, we have
\[ B_2^2 \approx \frac{256\sqrt{2\pi}}{3276}(E_b^2 - E_2), \] (32)
and
\[ B_{2n}^2 \approx \frac{b^2B_2^2(2n)![(16n^4 - 112n^3 + 32n^2 + 476n + 23)^2}{2^{4n+16}\pi(n!)^2(E_b^2 - E_2)^2}, \]
\[ \approx \frac{b^2B_2^2(2n)!n^6}{2^{4n+10}\pi(n!)^2}, \] as \( n \to \infty. \) (33)

Then, the perturbation of \( \psi_b^0(x) \) can be constructed by collection the coefficients above, i.e., \( \psi_b^0(x) \approx B_2\phi_2(x) + B_4\phi_4(x) + B_6\phi_6(x) + \cdots. \) It is noted that here, the expansion starts from \( n = 2 \) as \( B_0 = 0. \) Again, we have \( \psi_b^0(x) \to \phi_2(x) \) as \( E_b^2 \to E_2 = \frac{5}{2}. \) Moreover, the resulting probability factor \( Q(E_b^2) \) in the asymptotical limit, \( n \to \infty \) has the form:
\[ Q(E_b^2) = \frac{1}{b} \int_{-\infty}^{\infty} \ln |1 + b|\psi_b^0(x)|^2|dx, \]
\[ \approx \int_{-\infty}^{\infty} |\psi_b^0|^2 |dx = B_2^2 + B_4^2 + B_6^2 + \cdots. \]
\[ \approx \frac{512\sqrt{2\pi}}{4356}(E_b - \frac{5}{2}) \times \]
\[ [1 + \frac{2}{2} \times \frac{512}{4356}(E_b - \frac{5}{2})^2 \sum_{n=1}^{\infty} \frac{(2n)!n^6}{(n!(2n+5)^2)]. \]

Here, again, we see that \( Q(E_b^2) \to 0 \) as \( E_b^2 \to E_2 = \frac{5}{2}. \)

In addition to the ground and second order excited states, for all the even number of \( n, \) the perturbed eigen-energy \( E_{2n}^b \) and eigen-function \( \psi_{2n}^b(x), \) as well as the corresponding probability factor \( Q(E_{2n}^b) \), can be derived explicitly, with the help of Eqs. (11), (13) and (21), respectively. As for the odd number of \( n, \) Eqs. (11) and (12) provide the required conditions to have the eigen-energy and eigen-function with introduction of the DDEM parameter \( b. \)

**IV. NUMERICAL RESULTS BY DIRECT SIMULATIONS**

**A. The Ground State**

In order to verify the validity of our analytical solutions obtained by the perturbation theory, we also perform the numerical calculations for Eq. (3) directly without applying any approximation. To maintain some level of formal rigor and mathematical correctness, we shall talk about finding solutions of differential equations [26]. To find the solutions of the eigen-value problem with the nonlinear term, we connect with a quantum harmonic oscillator by solving Eq. (3) with Fourier spectral method. Using the matrix elements, we diagonalize the matrix numerically and perform the iteration to ensure the truncated Fourier basis having the eigen-value converged. For low energy states, already the smallest basis of 512 elements gives more than sufficient accuracy.

In Fig. 1, we show the corresponding lowest eigen-mode of the generalized quantum harmonic oscillator described in Eq. (3), in the plot of probability factor versus eigen-energy \( Q(E). \) Starting from \( E_0 = 0.5, \) i.e., the eigen-energy of ground state in the standard quantum harmonic oscillator with \( b = 0, \) now the eigen-energy is no long a discrete value, but a continuous function due to the introduction of DDEM, i.e., \( b \neq 0. \) Here, the initial guess solution has a single-hump profile, i.e., a Gaussian function stemmed from the zero-th order \( H_n(x). \) With a positive value of \( b, \) such as \( b = 1 \) and \( b = 2, \) shown in Blue- and Red-colored curves in Fig. 1, the corresponding probability factor \( Q(E) \) presents an almost linear function of the eigen-energy \( E. \) Now, all the eigen-energy \( E_b^0 \) are larger than that of \( E_0. \) Compared to the analytical formula of \( Q(E_b^0) \) obtained in Eq. (28), the dashed-curves give agreement to the numerical ones, not only on the slope of \( Q(E) \) curves but also on the inversely proportional dependence on \( b. \)

Moreover, the corresponding wavefunction \( \psi_b^0(x) \) is depicted in Fig. 2(a), which shares a similar Gaussian profile with that in the linear case \( b = 0. \) For example, at the marked eigen-energy \( E_A = 0.75, \) the eigen-functions \( \psi_b^0(x) \) have similar Gaussian shapes both for \( b = 1 \) and \( b = 2. \) But with a larger value in the DDEM parameter, such as \( b = 2, \) the amplitude, as well as the width, becomes smaller in the corresponding eigen-functions, as the Red-colored curves shown in Fig. 2(a). The analytical solutions obtained by perturbed theory, depicted in dashed-curves in Fig. 2(a), also reflect this similarity.

However, when \( b \) is negative, there are two distinct re-
FIG. 2. The wavefunction for (a–c) the ground state $\psi(x)$ similar to the peakon-like solutions has a discontinuity in their first-order derivative, $E_{Q}$ region. As shown in Fig. 2(c) for the marked eigen-energy $E_{B}$, any even when $E < E_{Q}$, there exist two singularities, denoted as $E_{Q}^{s,1}$ and $E_{Q}^{s,2} = 0$. When the eigen-energy is smaller than the first singular energy $E_{Q}^{s,1}$, the peakon-like solutions are also already found in the IDD setting for optical waves, even without the introduction of harmonic oscillators [10, 11]. As our perturbation theory starts from the eigen-basis of Hermite-Gaussian functions, it is not applicable to this family of peak-like solutions.

B. The Excited States

In addition to the ground state, the founded second order excited states, both numerically and analytically, are also depicted in Figs. 2(d–f) in solid- and dashed-curves, respectively. Again, we also have three different regions in characterizing the wavefunction profiles. Smooth profiles with the DDEM $b > 0$ and $b < 0$ are shown in Figs. 2(d) and (e) for the marked eigen-energies $E_{D} = 2.75 > E_{2} = 2.5$ and $E_{E} = 2.25 < E_{2}$ in Fig. 3, respectively. As shown in Figs. 2(d) and (e), the two solutions, $\psi_{Q}(x)$, have three humps in their profiles and share the similar profile as the 2nd order Hermite-Gaussian function. By comparing the solid- and dashed-curves, corresponding to our numerical results and analytical solutions, respectively, one can see nearly perfect agreement for the solutions around the eigen-energy $E_{2}$. Nevertheless, when $b < 0$ and $E < E_{Q}^{s,1} \approx 0.8398$, a discontinuous profile emerges due to the singularity happened in the $Q-E$ curve. Unlike the $Q-E$ curves for the ground state, there exist two singularities, denoted as $E_{Q}^{s,1}$ and $E_{Q}^{s,2} = 0$. When the eigen-energy is smaller than the first singular energy $E_{Q}^{s,1}$ but larger than the second singular energy $E_{Q}^{s,2}$, i.e., $E_{Q}^{s,2} < E_{b} < E_{Q}^{s,1}$, for example $E_{F} = 0.75$, the peakon-like solution illustrated in Blue-color in Fig. 2(f), has a pro-
file of \( \exp(-|x|) \) in two of the humps in the sidebands. It is noted that the profile in the central hump remains a smooth one. Nevertheless, when the eigen-energy is smaller the value at the second singularity \( E_{2n}^2 \), for example \( E_G = -0.25 \), the corresponding eigen-function has discontinuities in all the three humps, as the Red-colored curve depicted in Fig. 2(f).

In Fig. 3, we plot all the founded eigen-energies, up to \( n = 4 \), by depicting the solution family with the same number of humps in the eigen-functions \( \psi^b_{n}(x) \) in the same colors. One can see clearly that, all the \( Q-E \) curves start from the eigen-energies \( E_n = n + \frac{1}{2} \) of a standard quantum harmonic oscillator, i.e., \( b = 0 \). Around these energy values, \( E_n \), our perturbation theory works perfectly, giving the linear dependence of \( Q(E) \) on the eigen-energy, along with the inversely proportional relation to the DDEM parameter \( b \). In particular, as depicted in the Black dashed-curves, our analytical solutions given in Eq. (34) also illustrate good agreement to the numerical solutions obtained by direct simulations, the validity of our analytical formula in the asymptotic limit, in terms of the probability factor as a function of the eigen-energy, \( Q(E) \), can be easily verified, in particular for the solutions stemmed from the expansion of Hermite-Gaussian functions. However, the nonlinear effective mass also introduces a new family of peakon-like solutions with a discontinuity in their first-order derivative, which definitely deserves further studies.

It has been well studied with the nonlinear Schrödinger wave equation, or the Gross-Pitaevskii equation in general, where the nonlinear terms come from Kerr-effect, or the mean-field interaction. With the eigen-energy and eigen-function illustrated in this work, our proposed generalized quantum harmonic oscillator opens an unexplored area for quantum particles with nonlinear effective masses. A number of promising applications and directions for further exploration may be identified when particles accessing nonlinear correction to their effective mass. Similar models related to our proposed generalized quantum harmonic oscillators, but in more complicated settings involve off-resonant self-induced transparency (SIT) solitons [27, 28] spatially-periodic refractivity doped with two-level systems (TLS) [29, 30], electromagnetically-induced transparency (EIT) via via resonant dipole-dipole interactions [31, 32], and the continuum limit of the Salerno model [33].

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**APPENDIX**

Here, we give the details to prove the inequality shown in Eq. (17) and Eq. (18).

First of all, from Eq. (9), one can see that

\[
W_{2n,2n,2n,2n} - 4(n + 1)V_{2n,2n,2n,2n} = \frac{\sqrt{2n(2n-1)}}{2}V_{2n-2,2n,2n,2n} - (2n + 1/2)V_{2n,2n,2n,2n} + \frac{(2n + 1)(2n + 2)}{2}V_{2n+2,2n,2n,2n},
\]

\[
< nV_{2n-2,2n,2n,2n} - (2n + 1/2)V_{2n,2n,2n,2n} + (n + 1)V_{2n+2,2n,2n,2n},
\]

\[
= n(V_{2n-2,2n,2n,2n} - V_{2n,2n,2n,2n}) + n(V_{2n+2,2n,2n,2n} - V_{2n,2n,2n,2n} + (V_{2n+2,2n,2n,2n} - \frac{1}{2}V_{2n,2n,2n,2n}).
\]
Then, with the formula
\[
\Gamma(h + \frac{1}{2}) = \left( h + \frac{1}{2} \right) h! \sqrt{\pi} = \frac{(2h - 1)!!}{2^h h!} h! \sqrt{\pi}, \tag{A2}
\]
one can have
\[
V_{2n,2n,2n,2n} = \frac{1}{\sqrt{2\pi}} \sum_{\nu=0}^{2n} \left( \frac{\nu - \frac{1}{2}}{\nu} \right) \left( \frac{2n - \nu - \frac{1}{2}}{2n - \nu} \right)^2, \tag{A3}
\]
and
\[
V_{2(n-1),2n,2n,2n} = \frac{1}{\sqrt{2\pi}} \sum_{\nu=0}^{2(n-1)} \left( \frac{\nu - \frac{1}{2}}{\nu} \right) \left( \frac{2n - \nu - \frac{1}{2}}{2n - \nu} \right)^2 \frac{(2n - \nu)(2n - \nu - 1)}{(2n - \nu - \frac{1}{2})^2}.
\]
As the inequality \( \frac{(2n - \nu)(2n - \nu - 1)}{(2n - \nu - \frac{1}{2})^2} < 1 \) is hold, we can know that
\[
V_{2(n-1),2n,2n,2n} < \frac{1}{\sqrt{2\pi}} \sum_{\nu=0}^{2(n-1)} \left( \frac{\nu - \frac{1}{2}}{\nu} \right) \left( \frac{2n - \nu - \frac{1}{2}}{2n - \nu} \right)^2, \tag{A4}
\]
as well as
\[
V_{2(n+1),2n,2n,2n} = \frac{1}{\sqrt{2\pi}} \sum_{\nu=1}^{2n} \left( \frac{\nu - \frac{1}{2}}{\nu} \right) \left( \frac{2n - \nu - \frac{1}{2}}{2n - \nu} \right)^2 \frac{2}{2\nu - 1} \frac{(2n - \nu + \frac{1}{2})^2}{(2n - \nu + 1)(2n - \nu + 2)}, \tag{A5}
\]
Moreover, as the inequality \( \frac{(2n - \nu + \frac{1}{2})^2}{(2n - \nu + 1)(2n - \nu + 2)} < 1 \) is also hold, we can have
\[
\sum_{\nu=1}^{2n} \left( \frac{\nu - \frac{1}{2}}{\nu} \right) \left( \frac{2n - \nu - \frac{1}{2}}{2n - \nu} \right)^2 \frac{2}{2\nu - 1} \frac{(2n - \nu + \frac{1}{2})^2}{(2n - \nu + 1)(2n - \nu + 2)} = \pi^{3/2} \left[ \frac{(4n - 5)!!}{2^{2n}(2n)!} \right]^2 \frac{2n(4n - 1)^2(4n - 3)^2}{2n + 1} + \frac{4n(2n - 1)^2(4n - 3)^2}{2n - 1},
\]
\[
< \pi^{3/2} \left[ \frac{(4n - 5)!!}{2^{2n}(2n)!} \right]^2 [(4n - 1)^2(4n - 3)^2 + 3(2n - 1)^2(4n - 3)^2],
\]
\[
= \pi^{3/2} \left[ \frac{(4n - 5)!!}{2^{2n}(2n)!} \right]^2 (448n^4 - 992n^3 + 796n^2 - 276n + 36). \tag{A6}
\]
Then, with the fact that
\[
\sum_{\nu=0}^{2n} \left( \frac{\nu - \frac{1}{2}}{\nu} \right) \left( \frac{2n - \nu - \frac{1}{2}}{2n - \nu} \right)^2 = \pi^{3/2} \left[ \frac{(4n - 5)!!}{2^{2n}(2n)!} \right]^2 [576n^4 - 896n^3 + 472n^2 - 96n + 9], \tag{A7}
\]
from Eqs. (A6) and (A7), one can reach at the following inequality:
\[
\sum_{\nu=1}^{2n} \left( \frac{\nu - \frac{1}{2}}{\nu} \right) \left( \frac{2n - \nu - \frac{1}{2}}{2n - \nu} \right)^2 \frac{2}{2\nu - 1} \frac{(2n - \nu + \frac{1}{2})^2}{(2n - \nu + 1)(2n - \nu + 2)} < \sum_{\nu=0}^{2n} \left( \frac{\nu - \frac{1}{2}}{\nu} \right) \left( \frac{2n - \nu - \frac{1}{2}}{2n - \nu} \right)^2, \tag{A8}
\]
when \( n \geq 1 \). Consequently, combining Eqs. (A6) and (A8), we have
\[
V_{2(n+1),2n,2n,2n} < \frac{1}{2} V_{2n,2n,2n,2n} + \frac{1}{2\sqrt{2\pi}} \sum_{\nu=0}^{2n} \left( \frac{\nu - \frac{1}{2}}{\nu} \right) \left( \frac{2n - \nu - \frac{1}{2}}{2n - \nu} \right)^2. \tag{A9}
\]
With the results obtained in Eqs. (A2), (A5) and (A9), the inequality shown in Eq. (18) can be reached

\[
W_{2n,2n,2n,2n} - (4n + 1)V_{2n,2n,2n,2n} < -\frac{n}{2\sqrt{2\pi}} \sum_{\nu=2n-1}^{2n} \left( \frac{\nu - 1}{\nu} \right) \left( \frac{2n - \nu - 1}{2n - \nu} \right)^2 - \frac{n}{2\sqrt{2\pi}} \sum_{\nu=3}^{2n} \left( \frac{\nu - 1}{\nu} \right) \left( \frac{2n - \nu - 1}{2n - \nu} \right)^2 \]

(A10)

It is noted that the last two terms shown in Eq. (A10) is negative when \( n \geq 2 \). This completes the proof.

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