EFFECTS OF MAGNETIC FIELDS ON THE DISKOSEISMIC MODES OF ACCRETING BLACK HOLES

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ABSTRACT

The origin of the rapid quasi-periodic variabilities observed in a number of accreting black hole X-ray binaries is not understood. It has been suggested that these variabilities are associated with diskoseismic oscillation modes of the black hole accretion disk. In particular, in a disk with no magnetic field, the so-called g-modes (inertial oscillations) can be self-trapped in the inner region of the disk due to general relativistic effects. Real accretion disks, however, are expected to be turbulent and contain appreciable magnetic fields. In this paper, we show that even a weak magnetic field (with magnetic energy much less than the thermal energy) can modify or “destroy” the self-trapping zone of the disk’s g-modes, rendering their existence questionable in realistic black hole accretion disks. The so-called corrugation modes (c-modes) are also strongly affected when the poloidal field approaches equal-partition. On the other hand, acoustic oscillations (p-modes), which do not have vertical structure, are not affected qualitatively by the magnetic field, and therefore may survive in a turbulent, magnetic disk.

Key words: accretion, accretion disks – black hole physics – hydrodynamics – magnetic fields – MHD – X-rays: binaries

Online-only material: color figures

1. INTRODUCTION

In recent years, quasi-periodic variability has been observed for a number of Galactic compact X-ray binary systems. Of particular interest are several accreting black hole (BH) binaries, which show pairs of quasi-periodic oscillations (QPOs) of fixed frequencies having ratios close to 2:3 (for example, GRO J1655-40 shows which show pairs of quasi-periodic oscillations (QPOs) of particular interest are several accreting black hole (BH) binaries, for a number of Galactic compact X-ray binary systems. Of those mentioned above, usually assume that the unperturbed flow is laminar and has no magnetic field. Real accretion disks, on the other hand, are highly turbulent due to the nonlinear development of magnetorotational instability (MRI; see Balbus & Hawley 1998 for a review). The question therefore arises as to how the MRI-driven turbulence affects the oscillation modes obtained from hydrodynamical models and to what extent these trapped modes remain “valid” in a realistic situation. Arras et al. (2006) attempted to address this issue by carrying out MHD simulations in the shearing-box geometry. They showed that axisymmetric standing sound waves give rise to distinct peaks in the temporal power spectrum, while inertial waves do not. The discrete frequencies obtained by them were due to the imposed periodic boundary conditions adopted in the simulations, and not due to any relativistic effect. Arras et al. suggested that their results possess a serious problem for QPO models based on g-modes. Recently, Reynolds & Miller (2008) reported on the results of global simulations of BH accretion disks (using Paczynski–Wiita pseudo-Newtonian potential) and showed that, while axisymmetric g-mode oscillations manifest in the hydrodynamic disk with no magnetic field, they disappear in the magnetic disk where MHD turbulence develops.

In this paper, we analytically study the effects of magnetic fields on the relativistic diskoseismic modes in accretion disks around BHs. We consider both poloidal and toroidal fields, and use local analysis of the full MHD equations to examine how the magnetic field changes the radial wave-propagation diagrams for various modes. We show that the trapping region of g-modes can be easily “destroyed” even when the disk’s field strength is such that the associated Alfvén speed is much smaller than the speed of sound. On the other hand, the propagation characteristics of p-modes (acoustic oscillations) and c-modes are largely unchanged. Note that since we assume laminar flows for our unperturbed disks, we do not directly address the effects of turbulence on disk modes. However, we believe that our work is relevant to this issue, since magnetic fields naturally arise in a turbulent disk.

We summarize the basic MHD equations in Section 2 and review the properties of diskoseismic modes important for our analysis in Section 3. In Sections 4 and 5 we examine the effects of poloidal field and toroidal field on those modes, respectively, and discuss the implications of our result in Section 6.
2. BASIC EQUATIONS

We consider a non-self-gravitating accretion disk, satisfying the usual ideal MHD equations:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1) \]

\[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla \Pi - \nabla \Phi + \frac{1}{\rho} \mathbf{v} \times \mathbf{T}, \quad \text{and} \quad (2) \]

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \quad (3) \]

Here, \( \rho, P, \) and \( \mathbf{v} \) are the fluid density, pressure, and velocity, respectively, \( \Phi \) is the gravitational potential, and

\[ \Pi \equiv P + \frac{B^2}{8\pi}, \quad \mathbf{T} \equiv \frac{1}{4\pi} (\mathbf{B} \cdot \nabla) \mathbf{B}. \quad (4) \]

are the total pressure and the magnetic tension, respectively. The magnetic field \( \mathbf{B} \) also satisfies the equation \( \nabla \times \mathbf{B} = 0 \).

We assume that the fluid obeys the barotropic equation of state \( P = P(\rho) \).

We adopt the cylindrical coordinates \( (r, \phi, z) \), which are centered on the central BH and have the \( z \)-axis in the direction perpendicular to the disk plane. The unperturbed background flow is assumed to be axisymmetric with a velocity field \( \mathbf{v} = r\Omega(r) \hat{\phi} \), and a magnetic field \( \mathbf{B} = B_\phi(r) \hat{\phi} + B_z \hat{z}, \) i.e., \( B_\phi \) is constant while \( B_z \) has a radial dependence. Force balance in the unperturbed imply

\[ \mathbf{G} \equiv \frac{1}{\rho} \nabla \Pi - \frac{1}{\rho} \mathbf{T} = \Omega^2 r \hat{r} - \nabla \Phi. \quad (5) \]

Consider perturbations of the form \( e^{i m \phi - i \omega t} \). The linearized fluid equations are

\[ -i \omega \delta \rho + \frac{1}{r} \frac{\partial}{\partial r} (\rho r \delta v_r) + \frac{\partial}{\partial z} (\rho \delta v_z) = 0, \quad (6) \]

\[ -i \omega \delta v_r - 2\Omega \delta v_\phi = G_\phi \frac{\partial}{\partial r} \delta \Pi + \frac{1}{\rho} (\partial \mathbf{T})_\phi, \quad (7) \]

\[ -i \omega \delta v_\phi + \frac{k^2}{2\Omega} \delta v_r = -\frac{im}{\rho} \delta \Pi + \frac{1}{\rho} (\partial \mathbf{T})_\phi, \quad (8) \]

\[ -i \omega \delta v_z = G_z \frac{\partial}{\partial r} \delta \Pi + \frac{1}{\rho} (\partial \mathbf{T})_z, \quad (9) \]

\[ -i \omega \delta B_r = \left( \frac{im B_\phi}{r} + B_z \frac{\partial}{\partial z} \right) \delta v_r + \frac{d \Omega}{dr} \delta B_r, \quad (10) \]

\[ -i \omega \delta B_\phi = \frac{\partial}{\partial r} (B_\phi \delta v_r) + B_z \frac{\partial}{\partial z} \delta v_\phi - B_\phi \frac{\partial}{\partial z} \delta v_r + r \frac{d \Omega}{dr} \delta B_\phi, \quad (11) \]

\[ -i \omega \delta B_z = -\frac{B_z}{r} \frac{\partial}{\partial r} (r \delta v_r) - \frac{im B_z}{r} \delta v_\phi + \frac{im B_\phi}{r} \delta v_z, \quad (12) \]

where

\[ \tilde{\omega} = \omega - m \Omega \quad (13) \]

is the comoving wave frequency, and

\[ \kappa^2 \equiv \frac{2\Omega d (r^2 \Omega)}{dr} \quad (14) \]

is the radial epicyclic frequency. In the above equations, \( \delta \rho, \delta \Pi, \delta \mathbf{v} \) and \( \delta \mathbf{B} \) are Eulerian perturbations, while \( \rho, \) and \( \mathbf{B} \) refer to the unperturbed flow variables. In addition, for barotropic fluid, we have

\[ \delta \rho = \frac{1}{c_s^2} \delta P = \frac{1}{c_s^2} \left( \delta \Pi - \frac{1}{4\pi} \mathbf{B} \cdot \delta \mathbf{B} \right), \quad (15) \]

where \( c_s \) is the speed of sound.

To perform local (WKB) analysis, we consider perturbations with spatial dependence \( e^{ik_r r + ik_\phi \phi} \). In the leading-order approximation, we keep only the radial gradient of \( \tilde{\Omega}(r) \) and \( B_\phi(r) \) while assuming that the variation scales of all the other background quantities are much larger than the wavelength of the perturbation, i.e., \( k_r, k_\phi \gg 1/r \). The linearized MHD equations then reduce to

\[ -\frac{i \tilde{\omega}}{\rho c_s^2} \delta \Pi + ik_r \delta v_r + ik_\phi \delta v_\phi + ik_z \delta v_z + i \tilde{\omega} B_\phi \delta B_\phi + \frac{4\pi \rho}{\rho c_s^2} \delta B_z = 0, \quad (16) \]

\[ -\frac{i k_r}{\rho} \delta \Pi + i \tilde{\omega} \delta v_r + 2\Omega \delta v_\phi + \left( \frac{ik_r B_z}{4\pi \rho} + \frac{ik_\phi B_\phi}{4\pi \rho} \right) \delta B_r + \frac{B_\phi}{2\pi \rho} \delta B_\phi = 0, \quad (17) \]

\[ -\frac{i k_\phi}{\rho} \delta \Pi - \frac{k^2}{2\Omega} \delta v_r + i \tilde{\omega} \delta v_\phi + \left( \frac{ik_r B_z}{4\pi \rho} + \frac{ik_\phi B_\phi}{4\pi \rho} \right) \delta B_z = 0, \quad (18) \]

\[ -\frac{i k_z}{\rho} \delta \Pi + i \tilde{\omega} \delta v_r + \left( \frac{ik_r B_z}{4\pi \rho} + \frac{ik_\phi B_\phi}{4\pi \rho} \right) \delta B_\phi = 0, \quad (19) \]

\[ ik_r B_\phi \delta v_r - ik_z B_\phi \delta v_\phi + ik_\phi B_\phi \delta v_z - p \Omega B_\phi \delta B_r - i \tilde{\omega} B_\phi = 0, \quad (20) \]

\[ ik_r B_\phi \delta v_r + ik_\phi B_\phi \delta v_\phi - ik_z B_\phi \delta v_z = 0, \quad (21) \]

where \( k_r \equiv m/r \). We have assumed \( B_\phi \sim r^q \) and \( p \equiv d \ln \Omega / d \ln r \). Note that in deriving Equations (16)–(22), we have dropped the terms proportional to \( G_\phi \) and \( G_z \) in Equations (7) and (9)—since \( G_\phi \equiv \left( \Omega^2 - \Omega_k^2 \right) r \sim \Omega_k^2 r \Omega/\Omega' \) and \( G_z \sim \Omega_k^2 z \) (where \( \Omega_k \) is the Keplerian frequency, i.e., the angular frequency in the absence of pressure force), \( G_\phi \) is much smaller than the other terms in Equation (7) provided that \( k_r, k_\phi \gg 1 + \nu_{\phi}^2/c_s^2 \), and \( G_z \) is also negligible if we focus on the mid-plane of the disk.

3. HYDRODYNAMIC LIMIT: DISKOSEISMIC MODES

In the absence of magnetic fields, for \( k_r, k_\phi \gg k_\phi \), the perturbed MHD Equations (16)–(22) lead to the dispersion relation:

\[ \tilde{\omega}^2 - \kappa^2 \equiv (\omega^2 - k^2 c_s^2) = k_r^2 c_s^2 \tilde{\omega}^2. \quad (23) \]

For \( k_z = 0 \) (or \( \omega^2 \gg k^2 c_s^2 \)), this becomes \( \tilde{\omega}^2 = k^2 c_s^2 + \kappa^2 \), the usual dispersion relation for spiral density wave; for \( \omega^2 \ll k^2 c_s^2 \), this becomes \( \tilde{\omega} = \pm k c_s \sqrt{(k^2 c_s^2 + \kappa^2)^{1/2}} \), describing inertial oscillations (e.g., Goodman 1993).
For an accretion disk with scale height $H \ll r$, the vertical dependence of the perturbation is not well described by the plane wave $e^{ikz}$ unless $k, H \gg 1$. Okazaki et al. (1987) showed that for a thin disk, the perturbation equations can be approximately separated in $r$ and $z$ (see also Nowak & Wagoner 1991, 1992; Ipser 1994). For example, for vertically isothermal disks with constant scale height $H$, one finds $\delta P(r, z), \delta v_r(r, z), \delta v_z(r, z) \propto H_\epsilon(z/H)$, while $\delta v_r(r, z) \propto H'_\epsilon(z/H)$, where $H'_\epsilon(Z)$ (with $n = 0, 1, \ldots$) are the Hermite polynomials and $H'_\epsilon(Z) = d H_\epsilon(Z)/dZ$. With this separation of variables, Okazaki et al. (1987) obtained the dispersion relation for a given $n$:

$$ (\tilde{\omega}^2 - \kappa^2)(\tilde{\omega}^2 - n^2\Omega^2_\perp) = k_z^2 c_s^2 \tilde{\omega}^2, \quad (24) $$

where $\Omega_\perp$ is the vertical epicyclic frequency and is related to $H$ by $H = c_0/\Omega_\perp$. An important property of relativistic disks around BHs is that $\kappa$ is nonmonotonic. Three types of trapped modes can be identified (see Figure 1; see also Wagoner 1999, Kato 2001, and Ortega-Rodríguez et al. 2008 for reviews):

1. P-modes. For $n = 0$, waves can propagate in the region where $\tilde{\omega}^2 > \kappa^2$. These are acoustic waves (modified by disk rotation) and have also been termed inertial-acoustic modes. If waves can be reflected at the disk’s inner radius ($r_{\text{ISCO}}$, the innermost stable circular orbit), discrete p-modes can be self-trapped at the innermost region of the disk (see Figures 1(a) and 1(c)).

2. G-modes. For $n \geq 1$, waves can propagate in the region where $\tilde{\omega}^2 < \kappa^2 < n^2\Omega^2_\perp$ or $\tilde{\omega}^2 > \Omega^2_\perp > \kappa^2$ (note that $\kappa < \Omega_\perp$ in GR). The former specifies the g-mode propagation zone; self-trapped g-modes can be maintained in the region where $\kappa$ peaks (for $m = 0$: see Figure 1(b)) or in the region where $\Omega - \kappa/m < \omega/m < \Omega + \kappa/m$ (for $m \neq 0$: see Figure 1(d)). Because these discrete, self-trapped modes do not require special boundary conditions (e.g., wave reflection at $r = r_{\text{ISCO}}$), they have been the focus of most studies of relativistic diskoseismology. Note that although we call these g-modes (following the terminology of Kato 2001 and Wagoner 1999), they have no relation to gravity waves, which are driven by buoyancy. Instead, these modes describe inertial oscillations and have also been termed inertial modes (or inertial-gravity modes).

3. C-modes. For $n \geq 1$ and $m \geq 1$, the wave-propagation condition $\tilde{\omega}^2 > \Omega^2_\perp > \kappa^2$ leads to an additional wave-trapping region, where $\omega/m < \Omega - \sqrt{n}\Omega_\perp/m$ (see Figure 1(d)). Note that for spinning BHs, $\Omega_\perp < \Omega$. Clearly, these modes exist only when $\Omega - \sqrt{n}\Omega_\perp/m > 0$ and wave reflection occurs at $r = r_{\text{ISCO}}$. Following the previous works (e.g., Kato (1990) and Silbergleit et al. (2001), who focused on the “fundamental” $n = m = 1$ mode, corresponding to the Lense–Thirring precession of the inner disk), we call these (corrigation) c-modes.

Comparing Equations (23) and (24), we see that we can obtain the radial dispersion relation of different modes by adopting the vertical “quantization” condition $k_z = \sqrt{n}\Omega/H$ in Equation (23), with $k_z = 0$ specifying p-modes. In a generic disk (e.g., when the disk is not vertically isothermal), the same “quantization” condition would not hold, but we still expect $k_z \sim 1/H \sim \Omega_\perp/c_s$ for the (vertically) lowest-order g-mode or c-mode. In the following sections, we will adopt $k_z = \sqrt{n}\Omega_\perp/c_s$, with $\eta$ of order unity, when we study how magnetic fields modify low-order g-modes and c-modes.

Our approach in this paper is based on Newtonian theory. The GR effect can be incorporated into our analysis by using the Paczynski–Witta pseudo-Newtonian potential, $\Phi = -M/(r - 2M)$. Alternatively, we could simply replace the Newtonian $\Omega$, $\Omega_\perp$, and $\kappa$ by their exact GR counterparts (e.g., Okazaki et al. 1987):

$$ \Omega = \sqrt{Mr^3}/(1 + a\sqrt{Mr^3}) \quad (25) $$

$$ \Omega_\perp = \Omega \left[1 - \frac{4aM^{1/2}}{r^{3/2}} + \frac{3a^2}{r^2}\right]^{1/2}, \quad (26) $$

$$ \kappa = \left[\left(M\left(r^2 - 6Mr + 8aM^{1/2}r^{1/2} - 3a^2\right)\right)^{1/2}/r^{3/2} + aM^{1/2}\right]^2 \quad (27) $$

(in geometric units such that $G = c = 1$), where $a$ is the spin parameter of the black hole. In general, $\Omega_\perp > \Omega > \kappa$. In the case of a Schwarzschild BH, $\Omega = \Omega_\perp > \kappa$, with $\kappa$ peaking at $r = 8M$ and becoming zero at $r_{\text{ISCO}} = 6M$. This nonmonotonic behavior of the radial epicyclic frequency is preserved for Kerr BHs and, as discussed above, is the key ingredient for the existence of trapped diskoseismic modes.

4. EFFECT OF POLOIDAL FIELDS

First, we consider the case of a pure poloidal field, with $B_\phi = 0$. For $k_z, k_r \gg k_\phi$, Equations (16)–(22) then lead to the dispersion relation:

$$ \tilde{\omega}^6 - \frac{(k^2 + k^2_\phi)^2 (c_s^2 + v_{A_\phi}^2) + k_z^2 v_{A_\phi}^2 + \kappa^2}{k^2_\phi c_s^2} \tilde{\omega}^4 $$

$$ + \left\{k^2_\phi v_{A_\phi}^2 + k^2_\phi (2c_s^2 + v_{A_\phi}^2) + \frac{d\Omega^2}{dn} \right\} \tilde{\omega}^2 $$

$$ - k^4_\phi v_{A_\phi}^2, \quad \frac{d\Omega^2}{dn} = 0, \quad (28) $$

Note that nonaxisymmetric g-modes with $\omega/m < \Omega(r_{\text{ISCO}})$ contain corotation resonance in the wave zone, leading to strong damping of the mode (Kato 2003; Li et al. 2003; Zhang & Lai 2006). On the other hand, modes with $\Omega(r_{\text{ISCO}}) < \omega/m < \max(\Omega + \kappa/m)$ do not suffer corotational damping, and are therefore of great interest.
where $v_{Az} \equiv B_z/\sqrt{4\pi \rho}$. Within the incompressible limit, this reduces to the dispersion relation found in, e.g., Balbus & Hawley (1991). For a given $k = (k_r, k_\theta, k_z)$, Equation (28) admits three branches, corresponding to fast and slow magnetosonic waves, and Alfvén waves, all modified by differential rotation. For $k_z \gg k_r$, the Alfvén branch can become unstable when $k_z^2 v_{Az}^2 \ll -d\Omega^2/dr$. This is the well-known MRI (e.g., Balbus & Hawley 1998).

4.1. P-modes

If $k_z = 0$, Equation (28) reduces to

$$\tilde{\omega}^2 = \kappa^2 + k_r^2 (c_s^2 + v_{Az}^2).$$

(29)

This is almost the same expression as in the pure hydrodynamic case ($\tilde{\omega}^2 = \kappa^2 + k_r^2 c_s^2$), the only difference being that the speed of sound is replaced by the fast magnetosonic wave speed, $\sqrt{c_s^2 + v_{Az}^2}$. Thus, the basic property of p-modes is not affected by poloidal magnetic fields.

4.2. G-modes

For a fixed $k_z = \sqrt{\eta}/H = \sqrt{\eta} \Omega_\perp/c_s$ (see Section 3), we can rewrite Equation (28) as an expression for $k_r^2$:

$$(c_s^2 + v_{Az}^2)k_r^2 = \frac{\tilde{\omega}^2 - \omega_1^2}{\tilde{\omega}^2 - \omega_2^2} (\tilde{\omega}^2 - \omega_3^2).$$

(30)

The five critical frequencies are given by

$$\omega_1^2 = \eta (\Omega_\perp)^2,$$

(31)

$$\omega_2^2 = \frac{1}{2} [k_r^2 + 2\eta (\Omega_\perp)^2 b^2 + \sqrt{k_r^4 + 4\eta (\Omega_\perp)^4 b^4}],$$

(32)

$$\omega_3^2 = \eta (\Omega_\perp)^2 b^2,$$

(33)

$$\omega_4^2 = \frac{\eta (\Omega_\perp)^2 b^2}{1 + b^2},$$

(34)

$$\omega_5^2 = \frac{1}{2} [k_r^2 + 2\eta (\Omega_\perp)^2 b^2 - \sqrt{k_r^4 + 4\eta (\Omega_\perp)^4 b^4}],$$

(35)

where $b \equiv v_{Az}/c_s$.

Equation (30) allows us to identify various wave-propagation regions ($k_r^2 > 0$). First, we consider subthermal fields, with $b < 1$. When $b \lesssim 0.4$ (and with $\eta = 1$ for the lowest-order g-modes), the five critical frequencies satisfy $\omega_1^2 > \omega_2^2 > \omega_3^2 > \omega_4^2 > \omega_5^2 > 0 > \omega_2^2$ in the inner region of the disk. Thus, there are three wave-propagation regions:

Region I : $\tilde{\omega}^2 > \omega_1^2$,

(36)

Region II : $\omega_3^2 < \omega_2^2 < \omega_2^2$, and

(37)

Region III : $\omega_2^2 < \omega_3^2$.

(38)

Region II corresponds to the original g-mode cavity modified by the magnetic field; in the zero-field limit, $\omega_3 = 0, \omega_2 = \kappa$, and Equation (37) reduces to $\tilde{\omega}^2 < \kappa^2$. Figure 2 depicts the critical frequencies $\omega_2$ and $\omega_3$ for several values of $b$. This also serves as the propagation diagram for $m = 0$ g-modes (wave can propagate in a region where $\omega_3 < \omega < \omega_2$). We see that as the magnetic field increases, the g-mode self-trapping zone gradually shrinks and disappears even when the magnetic field is still very subthermal (for a Schwarzschild BH, this occurs for $b \gtrsim 0.08$). More precisely, the g-mode cavity can still exist for large $b$, but it now requires a reflection boundary at $r_{\text{ISCO}}$. This behavior can be easily understood by inspecting Equation (32): while $\kappa$ peaks at some radius $r_{\text{max}}$, $\omega_{\perp}$ and $\Omega_{\perp}$ both increase monotonically with decreasing $r$. Since $\Omega_{\perp}$ and $\Omega_{\perp}$ are much larger than $\kappa$ in the inner region of the disk, the $2(\Omega_{\perp} b^2)$ term or the $4\Omega_{\perp} \Omega_{\perp} b$ term can dominate over $\kappa^2$ even when $b$ is small, therefore making the self-trapping zone disappear. Roughly, this occurs at $b \gg b_{\text{crit}} \sim (\kappa^2/2\Omega_{\perp} b^2)_{\text{max}}$.

For nonaxisymmetric perturbations ($m \neq 0$), the wave-propagation region II is determined by (i) $\Omega - \omega_2/m < \omega < \Omega + \omega_2/m$, and (ii) $\omega/m > \Omega + \omega_3/m$ or $\omega/m < \Omega - \omega_3/m$. Figure 3 shows the propagation diagram. As mentioned before (see Footnote 1), for $b = 0$, only the modes with $\omega > m\Omega_{\text{ISCO}}$ are of interest, since otherwise there is a corotation resonance in the wave zone, leading to strong mode damping (Kato 2003; Li et al. 2003; Zhang & Lai 2006). Thus, self-trapped g-modes reside around the radius where $\Omega + \omega_2/m$ is the maximum (and this maximum arises because $\kappa$ depends nonmonotonically on $r$). We see from Figure 3 that this g-mode self-trapping region
disappears as \( b \) increases. The larger the \( m \) is, the more fragile is the cavity. For example, the \( m = 1 \) cavity disappears for \( b \gtrsim 0.015 \), while for \( m = 2 \), this occurs for \( b \gtrsim 0.005 \).

### 4.3. C-modes

For \( m \neq 0 \) and \( \eta \sim 1 \), Equation (30) also describes trapped c-modes. When \( b \lesssim 0.4 \), \( \omega_c^2 \) is the largest among all the critical frequencies and the c-mode propagation zone corresponds to region I (see Equation (36)). Note that since \( \omega_c \) is not affected by the magnetic field, the trapping region is determined by \( \omega_c/m < \Omega - \Omega_1/m = \Omega - \Omega_\perp/m \) (for \( \eta = 1 \), see the upper panel of Figure 4; see Figure 1(d)). When \( b \gtrsim 0.4 \), the ordering between \( \omega_1 \) and \( \omega_2 \) switches and c-modes propagate in the region where \( \omega_c^2/\omega_1^2 > \omega_c^2/\omega_2^2 \), with the trapping zone determined by \( \omega_c/m < \Omega - \Omega_2/m \) (see the bottom panel of Figure 4). Thus, in the presence of a reflection boundary at \( r_{\text{ISCO}} \), c-modes are not affected by the poloidal magnetic field when \( b \lesssim 0.4 \), but can be appreciably modified when \( b \gtrsim 0.4 \).

From Equation (30) we can identify other wave-propagation zones (see Equation (38)). Figure 5 gives an example, for \( m = 2 \) and \( b = 0.7 \). Note that, except for the c-mode trapping zone discussed above, all the propagation zones are bounded by at least one “singular point” (where \( k_r \to \infty \)). Unlike the turning point \( (k_r \to 0) \) associated with wave reflection, wave absorption is expected to occur at these singular points (see Zhang & Lai 2006; Tsang & Lai 2008a and references therein). Thus, the new wave-trapping regions given by Equation (38) will not lead to interesting global oscillation modes (note that in the case of \( b = 0.7 \), the ordering of five critical frequencies is different from the one described in Section 4.2.) However, our conclusion still holds true, i.e., there is no chance to form a wave-trapping
zone bounded by two reflection points other than the c-mode oscillation region, which is bounded by a reflection point and the ISCO.

5. EFFECT OF TOROIDAL FIELDS

In this section, we consider the effect of a pure toroidal field, with $B_\phi = 0$. Various instabilities may exist for such field geometry, depending upon the rotation profile $\Omega(r)$ and the magnetic field profile $B_0(r)$ (e.g., Acheson & Gibbsons 1978; Terquem & Papaloizou 1996). Here we focus on how $B_\phi$ affects the diskoseismic modes.

5.1. P-modes

With $k_z = 0$, Equations (16)–(22) reduce to

$$\ddot{\omega}^2 = \omega^2 + k_r^2 (c_s^2 + v_{A\phi}^2),$$

where $v_{A\phi} \equiv B_\phi/\sqrt{4\pi \rho}$. Thus, the toroidal field affects p-modes in the same way as the poloidal field does (see Section 4.1).

5.2. G-modes

Since the general dispersion relation for $m \neq 0$ is quite complicated, here we focus on axisymmetric perturbations. With $m = 0$, Equations (16)–(22) lead to

$$\omega^4 - \left[ k_r^2 + \left( c_s^2 + v_{A\phi}^2 \right) \left( \omega^2 + k_r^2 (c_s^2 + v_{A\phi}^2) \right) \right] \omega^2 + 2(1 - q)v_{A\phi}^2 c_s^2 k_r^2/r^2 = 0,$$

where $q = d \ln B_\phi/d \ln r$. Solving for $k_r^2$, we have

$$k_r^2 = \frac{(\omega^2 - \omega_0^2)(\omega^2 - \omega_2^2)}{(c_s^2 + v_{A\phi}^2)\omega^2},$$

with the two critical frequencies given by

$$\omega_{0,2} = \frac{\kappa^2 + \eta \Omega_l^2 (1 + b_\phi^2)}{2} \pm \frac{1}{2} \sqrt{\left[ k_r^2 - \eta \Omega_l^2 (1 + b_\phi^2) \right]^2 - 8(1 - q)\eta v_{A\phi}^2 \Omega_l^2 / r^2},$$

where $b_\phi \equiv v_{A\phi}/c_s$ and we have used $k_z = \sqrt{\eta}/H = \sqrt{\eta} \Omega_l/c_s$ as in Section 4. Clearly, for $b_\phi = 0$, Equation (42) reduces to Equation (24).

When $q = 1$ (i.e., $B_\phi \propto r$), Equation (42) gives $\omega_2^2 = \eta(\Omega_l)^2(1 + b_\phi^2)$, and $\omega_0^2 = \kappa^2$. Since $\omega_0^2$ is independent of $B_\phi$, the g-mode propagation zone is unaffected no matter how strong the field is. When $q \neq 1$, as long as $v_{A\phi} \ll \Omega_l r$, which is valid in most disk situations, the $8(1 - q)\eta v_{A\phi}^2 \Omega_l^2 / r^2$ term in Equation (42) represents only a small correction, i.e., $\omega_2^2$ is still very close to $\kappa^2$. Thus for a general toroidal field satisfying $v_{A\phi} \ll \Omega_l r$, the axisymmetric g-mode propagation zone is not affected by the magnetic field.

6. SUMMARY AND DISCUSSION

In this paper, we have studied the effects of both poloidal and toroidal magnetic fields on the diskoseismic modes in BH accretion disks. Previous works by Kato, Wagener, and others have been based on hydrodynamic disks with no magnetic field. The key finding of our paper is that the g-mode self-trapping zone (which arises from the GR effect) disappears when the disk contains even a small poloidal magnetic field, corresponding to $v_{A\phi} / c_s = 0.01–0.1$ (see Figures 2 and 3; $v_{A\phi}$ is the Alfvén speed and $c_s$ is the speed of sound). It is well known that the combination of a weak poloidal field and differential rotation gives rise to MRI, making real astrophysical disks turbulent. Earlier numerical simulations indicated that the magnetic field grows as MRI develops, until it saturates at $v_{A\phi} / c_s \sim 0.1–1$, with the toroidal field stronger than the poloidal field by a few factors (see, e.g., Hawley et al. 1996; Balbus & Hawley 1998). Recent simulations showed that the turbulent state strongly depends on the net magnetic flux through the disk (e.g., Fromang & Papaloizou 2007; Simon et al. 2008). In any case, it is likely that the magnetic field in a turbulent disk is large enough to “destroy” the g-mode self-trapping zone.

Thus, the g-mode properties (including the frequencies and excitations) derived from hydrodynamical models are unlikely to be applicable to real BH accretion disks. The disappearance of the g-mode trapping zone might also explain why Arras et al. (2006) and Reynolds & Miller (2008) did not see any global g-modes in their MHD simulations.

As mentioned in Section 1, g-mode oscillations have been considered to be a promising candidate to explain QPOs in BH X-ray binaries. Theoretically, these modes are appealing because in hydrodynamic disks their existence depends on the GR effect and does not require special disk boundary conditions. Our analytical results presented in this paper, together with recent numerical simulations (Arras et al. 2006; Reynolds & Miller 2008), suggest that magnetic fields and turbulence associated with real accretion disks can significantly change this picture.

While g-modes can be easily modified or “destroyed” by magnetic fields, our analysis shows that p-modes are not affected qualitatively. The magnetic field simply changes the speed of sound to the fast magnetosonic wave speed and leaves the p-mode propagation diagram unchanged. We also showed that a weak poloidal field ($v_{A\phi} / c_s \ll 1$) does not affect the c-mode propagation zone, although a stronger field modifies it. Our results therefore suggest that global p-mode oscillation is robust and may exist in real BH accretion disks, provided that partial wave reflection at the disk’s inner edge can be achieved. Of particular interest is the nonaxisymmetric p-modes, since they may be excited by instabilities associated with corotation resonance (Lai & Tsang 2008; Tsang & Lai 2008a, 2008b).

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