Chow Rings of $\overline{M}_{p,2}(N, d)$ and $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$ and Gromov-Witten Invariants of Projective Hypersurfaces of Degree 1 and 2

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Abstract

In this paper, we prove formulas that represent two-pointed Gromov-Witten invariant $\langle O_h^a O_h^b \rangle_{0,d}$ of projective hypersurfaces with $d = 1, 2$ in terms of Chow ring of $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$, the moduli spaces of stable maps from genus 0 stable curves to projective space $\mathbb{P}^{N-1}$. Our formulas are based on representation of the intersection number $w(O_h^a O_h^b)_{0,d}$, which was introduced by Jinzenji, in terms of Chow ring of $\overline{M}_{p,2}(N, d)$, the moduli space of quasi maps from $\mathbb{P}^{1}$ to $\mathbb{P}^{N-1}$ with two marked points. In order to prove our formulas, we use the results on Chow ring of $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$, that were derived by A. Mustaţă and M. Mustaţă. We also present explicit toric data of $\overline{M}_{p,2}(N, d)$ and prove relations of Chow ring of $\overline{M}_{p,2}(N, d)$.

1 Introduction

1.1 Our Aim.

In computing Gromov-Witten invariants, we usually use classical mirror symmetry or fixed point localization theorem ([2] or [11], etc.) especially when we are dealing with basic examples such as projective hypersurfaces. Furthermore, typical proofs of classical Mirror theorem for toric complete intersections were done by using fixed point localization technique ([2], [7], [12]). Since localization technique does not need detailed structure of Chow ring of the corresponding moduli space, it is still unclear how Gromov-Witten invariants are written in terms of Chow ring of the moduli space. In this paper, we prove formulas that represent genus 0 Gromov-Witten invariants of projective hypersurfaces of degree 1 and 2 in terms of Chow ring of the moduli space of stable maps $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$.

1.2 The Chow Rings of Moduli Spaces of Stable Maps.

In order to accomplish our program, we need to know detailed structure of the Chow ring of $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$. We mainly refer to Mustaţă's results [13][14], and Cox's results [4] to obtain information we need.

In [4], Cox computed Chow ring of $\overline{M}_{0,2}(\mathbb{P}^{1}, 2)$. Its structure is described by using natural basis $D_0, D_1, D_2, H_1, H_2, \psi_1, \psi_2$. $H_1$ and $H_2$ are pullbacks of hyperplane class with respect to evaluation maps $ev_1, ev_2: \overline{M}_{0,2}(\mathbb{P}^{1}, 2) \rightarrow \mathbb{P}^{1}$. $\psi_1$ and $\psi_2$ are so called $\psi$-classes of universal curve $C \rightarrow \overline{M}_{0,2}(\mathbb{P}^{1}, 2)$. $D_i$'s are classes that correspond to loci which parametrize stable maps from nodal curves. The stable maps that belong to each $D_i$ are represented by the following graphs.
1.3 Our Motivation: Comparing $\overline{M}_{0,1}(\mathbb{P}^{N-1}, d)$ in general case. They constructed intermediate moduli spaces $\overline{M}_{0,1}(\mathbb{P}^{N-1}, d, k)$ and their substra $\overline{M}_{I}^{k}$, and computed extended Chow rings $B^{*}(\overline{M}_{0,1}(\mathbb{P}^{N-1}, d, k))$. $\overline{M}_{I}^{k}$ is defined for integer $k$ $(0 \leq k \leq d)$ and nested set $I \subset \mathcal{P}\setminus\{0, D\}$, and it parametrizes $k$-stable maps of $I$-split type (where $\mathcal{P}$ is a power set of $D = \{1, 2, \ldots, d\}$). The extended Chow ring $B^{*}(\overline{M}_{0,1}(\mathbb{P}^{N-1}, d))$ is generated by classes associated with $\overline{M}_{I}^{k}$'s, and $A^{*}(\overline{M}_{0,1}(\mathbb{P}^{N-1}, d))$ is given as a subring of $B^{*}(\overline{M}_{0,1}(\mathbb{P}^{N-1}, d))$ that are invariant under action of symmetric group $S_{d}$. In [13], they extended their strategy to compute $A^{*}(\overline{M}_{0,m}(\mathbb{P}^{N-1}, d))$. For example, generators of $B^{*}(\overline{M}_{0,2}(\mathbb{P}^{N-1}, 2))$ are given by $H, \psi, T_{(1D)}, T_{(2D)}, T_{(1D,2D)}$, and $T_{(1D,2M)}$.

Our study is influenced by concept of the moduli space $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$ of quasi-maps from $\mathbb{P}^{1}$ with two marked points to projective space $\mathbb{P}^{N-1}$, which was introduced by Jinzenji ([8]). This moduli space was also constructed rigorously by Fontaine and Kim [1]. In [8], he presented outline of construction of the moduli space $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$, implied that it is a toric variety and conjectured generators and relations of its Chow ring. Although he mentioned the fact that the moduli space $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$ is a toric variety, no explicit proof was given. So, in Section 2 of this paper, we prove the following proposition:

**Proposition 1.1** The space $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$ is a toric variety, and its Chow ring is isomorphic to a quotient ring of polynomial ring $\mathbb{C}[H_{0}, H_{1}, \ldots, H_{d}]$ modulo an ideal generated by

\[
H_{0}^{N}, \ H_{1}^{N}(H_{0} - 2H_{1} + H_{2}), \ H_{2}^{N}(H_{1} - 2H_{2} + H_{3}), \ldots, \ H_{d-1}^{N}(H_{d-2} - 2H_{d-1} + H_{d}), \ H_{d}^{N}. \quad (1.1)
\]

We prove it by giving a concrete toric data and using standard theory of toric variety.

From early stage of our study, we have been interested in bad loci of $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$. Although $\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)$ parametrizes degree $d$ holomorphic maps from $\mathbb{P}^{1}$ to $\mathbb{P}^{N-1}$ with two
marked points (see section 2 of this paper, or [8]), it has some loci which do not correspond to holomorphic maps. It is given as follows: let us consider a “rational map” \( p: \mathbb{P}^1 \to \mathbb{P}^{N-1} \) given by

\[
p([s : t]) = \left[ \sum_{j=0}^{d} a_j^0 s^{d-j} t^j : \ldots : \sum_{j=0}^{d} a_j^N s^{d-j} t^j \right].
\]

If all the polynomials \( \sum_{j=0}^{d} a_j^j s^{d-j} t^j \) are divisible by one polynomial \( f(s, t) \), then image of \( p \) at zero points of \( f(s, t) \) cannot be defined. However, we thought that \( \overline{\mathcal{M}}_{0,2}(\mathbb{P}^{N-1}, d) \) can be constructed from \( \overline{\mathcal{M}}_{0,2}(N, d) \) by successive blow up along these bad loci. In the \( d = 1 \) case, it is well known that \( \overline{\mathcal{M}}_{0,2}(\mathbb{P}^{N-1}, 1) \) is isomorphic to blow-up of \( \mathbb{P}^{N-1} \times \mathbb{P}^{N-1} \) along its diagonal subset \( \Delta \). Moreover, it can be shown that \( \overline{\mathcal{M}}_{0,2}(N, 1) \) is isomorphic to \( \mathbb{P}^{N-1} \times \mathbb{P}^{N-1} \), and its “bad loci” is given by the diagonal set \( \Delta \). In this case, \( \overline{\mathcal{M}}_{0,2}(\mathbb{P}^{N-1}, 1) \) is blow-up of \( \overline{\mathcal{M}}_{0,2}(N, 1) \) along its bad locus.

Also in general degree \( d \), similar claim may be true, but we haven’t obtained rigorous proof of this kind of result. However, we show that Chow ring of \( \overline{\mathcal{M}}_{0,2}(N, 2) \) is closely related to Chow ring of \( \overline{\mathcal{M}}_{0,2}(\mathbb{P}^{N-1}, 2) \). To state our result, we transform the basis of \( A^\ast(\overline{\mathcal{M}}_{0,2}(\mathbb{P}^{N-1}, 2)) \) as follows:

**Definition 1.1 (Key transformation)**

\[
h_0 := H, \quad h_1 := H + \psi, \quad h_2 := H + 2\psi + S_2.
\]

Then, we prove the following lemma:

**Lemma 1.1** In the ring \( A^\ast(\overline{\mathcal{M}}_{0,2}(\mathbb{P}^{N-1}, 2)) \), the following relations hold.

\[
h_0^N = 0, \quad h_1^N(h_0 - 2h_1 + h_2) = 0, \quad h_2^N = 0.
\]

These relations are nothing but the relations of the Chow ring of \( \overline{\mathcal{M}}_{0,2}(N, 2) \)!

### 1.4 Our Motivation 2: Computing GW invariants.

For degree \( k \) projective hypersurface \( M_N^k \subset \mathbb{P}^{N-1} \), Jinzenji introduced intersection number \( w(O_{h^0}O_{h^1})_{0,d} \) of \( \overline{\mathcal{M}}_{0,2}(N, d) \) defined as follows.

**Definition 1.2**

\[
w(O_{h^0}O_{h^1})_{0,d} := \int_{\overline{\mathcal{M}}_{0,2}(N,d)} e_{1}^{\ast}(h^0) \wedge e_{2}^{\ast}(h^1) \wedge c_{top}(E^k_d),
\]

where \( E^k_d \) is an orbi-bundle on \( \overline{\mathcal{M}}_{0,2}(N,d) \), which corresponds to the condition that the images of quasi maps are contained in degree \( k \) hypersurface \( M_N^k \). This orbi-bundle is constructed in [8]. In [8], he obtained the result that mirror map of hypersurface \( M_N^k \) used in mirror computation of Gromov-Witten invariants is reconstructed as generating function of these intersection numbers, and generalized this framework to the case of toric manifolds with two Kähler forms. He also obtained a formula that represents \( w(O_{h^0}O_{h^1})_{0,d} \) in terms of Chow ring of \( \overline{\mathcal{M}}_{0,2}(N,d) \). Let \( e_k(x,y) = \prod_{j=0}^{k} (jx + (k-j)y) \). Then \( w(O_{h^0}O_{h^1})_{0,d} \) has the following expression.

\[
w(O_{h^0}O_{h^1})_{0,d} = \int_{\overline{\mathcal{M}}_{0,2}(N,d)} H_0^d H_d^{d-1} \prod_{j=1}^{d-1} \prod_{l=1}^{d} e_{k}(H_{i}H_{i+1}) \prod_{j=1}^{d-1} (kH_{j}) .
\]

(1.2)

In [10], he proved mirror formulas that express Gromov-Witten invariant \( w(O_{h^0}O_{h^1})_{0,d} \) of hypersurface \( M_N^k \) in terms \( w(O_{h^0}O_{h^1})_{0,f} \) (\( f \leq d \)) and Gromov-Witten invariants of degree lower than \( d \) in the \( d = 1, 2, 3 \) cases. For example, in \( d = 1, 2 \) cases, GW invariants \( w(O_{h^0}O_{h^1})_{0,d} \) of degree \( k \) hypersurface \( M_N^k \subset \mathbb{P}^{N-1} \) have the following expression.

\[
\langle O_{h^0}O_{h^1} \rangle_{0,1} = w(O_{h^0}O_{h^1})_{0,1} - w(O_{h^0}O_{h^0})_{0,1},
\]

(1.3)

where \( a, b \geq 0, a + b = 2N - k - 3 \),

\[
(\text{where } a, b \geq 0, a + b = 2N - k - 3).
\]
\[
\langle \mathcal{O}_{h \times \mathcal{O}_{h^k}} \rangle_{0,2} = w(\mathcal{O}_{h \times \mathcal{O}_{h^k}})_{0,2} - w(\mathcal{O}_{h^{a+k} \times \mathcal{O}_{h^k}})_{0,2}
- \frac{1}{k} \langle \mathcal{O}_{h \times \mathcal{O}_{h^k}} \mathcal{O}_{h^{1+k-N}} \rangle_{0,1} w(\mathcal{O}_{h^{a+k-N} \times \mathcal{O}_{h^k}})_{0,1},
\]
(1.4)

We use these formulas to prove our main theorems.

We expected that the equation (1.2) can be extended to formulas that express the Gromov-Witten invariant \(\langle \mathcal{O}_{h \times \mathcal{O}_{h^k}} \rangle_{0,d} \) in terms of Chow ring of \(\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)\), because Chow rings of \(\overline{M}_{0,2}(\mathbb{P}^{N-1}, d)\) and \(\mathcal{M}_{0,2}(N, d)\) are closely related to each other, as can be seen in Lemma 1.3.

### 1.5 Main Results.

In Section 4 of this paper, we first review the structure of the Chow ring \(A^*\left(\overline{M}_{0,2}(\mathbb{P}^{N-1}, 1)\right)\) presented in [14], and show that three classes \(h_0, h_1, t\) are generators of \(A^*\left(\overline{M}_{0,2}(\mathbb{P}^{N-1}, 1)\right)\). Then we prove the following theorem:

**Theorem 1.1** For the Gromov-Witten invariant \(\langle \mathcal{O}_{h \times \mathcal{O}_{h^k}} \rangle_{0,1} \) of a hypersurface \(M^k_N\), the following formula holds:

\[
\int_{\overline{M}_{0,2}(\mathbb{P}^{N-1}, 1)} h_0^a h_1^b e^k(h_0, h_1 + t) = \langle \mathcal{O}_{h \times \mathcal{O}_{h^k}} \rangle_{0,1}.
\]

In Section 5, we review the structure of the Chow ring \(A^*(\overline{M}_{0,2}(\mathbb{P}^{N-1}, 2))\) presented in [14] again, and show that it is generated by five classes \(h_0, h_1, h_2, S_0, S_1\) of degree 2 and a class \(P_1\) of degree 4. Furthermore, by using the ring structure in detail, we prove the following:

**Theorem 1.2** For the Gromov-Witten invariants \(\langle \mathcal{O}_{h \times \mathcal{O}_{h^k}} \rangle_{0,2} \) of hypersurface \(M^k_N\), the following formula holds:

\[
\int_{\overline{M}_{0,2}(\mathbb{P}^{N-1}, 2)} (h_0^a - (h_1 + S_0)^a) h_2^b e^k(h_0, h_1 + S_0) e^k(h_1 + S_0, h_2 + 2S_0 + S_1) = \langle \mathcal{O}_{h \times \mathcal{O}_{h^k}} \rangle_{0,2}.
\]

Theorem 1.1 seems to be satisfying to us but Theorem 1.2 includes somewhat strange factor \(-(h_1 + S_0)^a\). In fact, \(\int_{\overline{M}_{0,2}(\mathbb{P}^{N-1}, 2)} h_0^a h_1^b e^k(h_0, h_1 + S_0) e^k(h_1 + S_0, h_2 + 2S_0 + S_1)\) gives us the value quite close to \(\langle \mathcal{O}_{h \times \mathcal{O}_{h^k}} \rangle_{0,2}\), but does not coincide with it. In this case, there might be some possibilities to obtain more satisfying formulas.

### 1.6 Organizations of this paper.

In Section 2, we construct toric data of \(\overline{M}_{0,2}(N, d)\) and compute its Chow ring. We use some standard theory of toric varieties presented in literatures, e.g. [5], etc. In Section 3 following [8, 9, 10], we derive explicit formulas of Gromov-Witten invariants of projective hypersurface \(M^k_N\) in the case of \(d = 1, 2\). We use these formulas to prove our main results. In the Section 4 and 5 we prove Theorem 1.1 and 1.2. The section 5 contains a proof of Lemma 1.3.

### 1.7 Notation.

All varieties considered in this paper are varieties over complex number field. \(M^k_N\) is a hypersurface of degree \(k\) in projective space \(\mathbb{P}^{N-1}\). \(M_{m,n}(A)\) is a set of \(m \times n\)-matrices over ring \(A\).

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2 The Moduli Space $\overline{M}_{p_{0,2}}(N, d)$

In this section we construct toric data of $\overline{M}_{p_{0,2}}(N, d)$ and prove Proposition 1.1. First, we discuss what $\overline{M}_{p_{0,2}}(N, d)$ is (see Section 2.1.1. of [3]).

$\overline{M}_{p_{0,2}}(N, d)$ is a compactified moduli space that parametrizes degree $d$ polynomial maps from 2-pointed $\mathbb{P}^1$ to $\mathbb{P}^{N-1}$. A degree $d$ polynomial map is given by,

$$p([s : t]) = [a_0 s^d + a_1 s^{d-1} t + \ldots + a_d t^d],$$

where $a_i$’s are vectors in $\mathbb{C}^N$. Then we introduce uncompactified moduli space $M_{p_{0,2}}(N, d)$ defined by,

$$M_{p_{0,2}}(N, d) := \{(a_0, a_1, \ldots, a_d) \mid a_0, a_d \neq 0\}/(\mathbb{C}^*)^2.$$

We set the 2-marked points in $\mathbb{P}^1$ as $0 := [1 : 0]$ and $\infty := [0 : 1]$. Then the condition $a_0, a_d \neq 0$ comes from requirement that images of these two marked points are well-defined in $\mathbb{P}^{N-1}$. The $(\mathbb{C}^*)^2$ action is induced from the automorphisms of $\mathbb{P}^1$ which fixes 0 and $\infty$, and equivalence relation used in the definition of projective space $\mathbb{P}^{N-1}$, and it is explicitly written as follows.

$$(\mu, \nu) \cdot (a_0, a_1, \ldots, a_d) = (\mu a_0, \mu \nu a_1, \ldots, \mu \nu^{d-1} a_{d-1}, \mu \nu^d a_d).$$

If we use the $(\mathbb{C}^*)^2$ action to turn $a_0$ and $a_d$ into the points in $\mathbb{P}^{N-1}$, $M_{p_{0,2}}(N, d)$ can be described as follows:

$$M_{p_{0,2}}(N, d) = \{([a_0], a_1, \ldots, a_{d-1}, [a_d]) \in \mathbb{P}^{N-1} \times (\mathbb{C})^{N(d+1)} \times \mathbb{P}^{N-1}/\mathbb{Z}_d.\)

The $\mathbb{Z}_d$-action is given by,

$$\zeta_d \cdot ([a_0], a_1, \ldots, a_{d-1}, [a_d]) = ([a_0], \zeta_d a_1, \ldots, \zeta_d^{d-1} a_{d-1}, [a_d]),$$

where $\zeta_d$ is the $d$-th primitive root of unity.

In order to compactify $M_{p_{0,2}}(N, d)$, we should add infinite loci corresponding to $a_i = \infty$, ($1 \leq i \leq d - 1$). In [3], divisor coordinates $u_i$’s are introduced where the locus $a_i = \infty$ is described as zero locus of $u_i$. Let $N, d$ be positive integers. The compactified moduli space $M_{p_{0,2}}(N, d)$ is defined as follows:

**Definition 2.1**

$$\overline{M}_{p_{0,2}}(N, d) := \{(a_0, \ldots, a_d, u_1, \ldots, u_{d-1}) \in \mathbb{C}^{N(d+1)} \times \mathbb{C}^{d-1} \mid a_0 \neq 0, (a_i, u_i) \neq 0 (1 \leq i \leq d - 1), a_d \neq 0\}/(\mathbb{C}^*)^{d+1},$$

where the $(\mathbb{C}^*)^{d+1}$-action is given by

$$(\lambda_0, \ldots, \lambda_d) \cdot (a_0, \ldots, a_d, u_1, \ldots, u_{d-1})$$

$$= (\lambda_0 a_0, \lambda_1 a_1, \ldots, \lambda_d a_d, \lambda_0^{-1} \lambda_2^{-1} u_1, \lambda_1^{-1} \lambda_3^{-1} u_2, \ldots, \lambda_{d-2}^{-1} \lambda_d^{-1} u_{d-1}).$$

(2.1)

2.1 Construction of Toric data of $\overline{M}_{p_{0,2}}(N, d)$

Let

$$p_1, p_2, \ldots, p_N \in \mathbb{Z}^{N-1}$$

be column vectors which are the 1-skelton of the fan of $\mathbb{P}^{N-1}$, i.e.

$$(p_1, p_2, \ldots, p_{N-1}, p_N) = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & -1 \\ 0 & 1 & 0 & \ldots & 0 & -1 \\ 0 & 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \end{pmatrix} \in M_{N-1,N}(\mathbb{Z}).$$

Next, we introduce $(d + 1)$ column vectors

$$v_0', v_1', \ldots, v_d' \in \mathbb{Z}^{d-1},$$
for $k$

$$
(v_0', v_1', \ldots, v_{d-1}', v_d') = \begin{pmatrix}
-1 & 2 & -1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & 0 \\
0 & 0 & -1 & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\
0 & 0 & 0 & 0 & \cdots & -1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 2 & -1
\end{pmatrix} \in M_{d-1,d+1}(\mathbb{Z}).
$$

The $(d-1) \times (d-1)$-submatrix in center of this matrix is Cartan matrix $A_{d-1}$. Finally, we define column vectors,

$$
v_{i,j} \quad (0 \leq i \leq d, 1 \leq j \leq N), \quad u_k \quad (1 \leq k \leq d - 1)
$$

as follows:

for $j \neq N$, 

$$
v_{i,j} = \begin{pmatrix} 0_{N-1} \\ \vdots \\ p_j \\ \vdots \\ 0_{N-1} \\ v_1' \end{pmatrix} \leftarrow i \quad \in \mathbb{Z}^{(d+1)(N-1)+(d-1)},
$$

for $j = N$, 

$$
v_{i,N} = \begin{pmatrix} 0_{N-1} \\ \vdots \\ p_N \\ \vdots \\ 0_{N-1} \\ v_1' \end{pmatrix} \leftarrow i \quad \in \mathbb{Z}^{(d+1)(N-1)+(d-1)}
$$

and for $k = 1, \ldots, d - 1$, 

$$
u_k = \begin{pmatrix} 0_{N-1} \\ \vdots \\ 0_{N-1} \\ -e_k \end{pmatrix} \in \mathbb{Z}^{(d+1)(N-1)+(d-1)}
$$

where $0_{N-1}$ (resp. $0_{d-1}$) is the zero vector in $\mathbb{Z}^{N-1}$ (resp. $\mathbb{Z}^{d-1}$) and $e_k$ is the $k$-th standard basis of $\mathbb{Z}^{d-1}$. With this set-up, we define the following toric data:

**Definition 2.2** $\Sigma_{N,d}$ is the fan generated by $\{v_{i,j}\}_{i=0,\ldots,d, \ j=1,\ldots,N}$ and $\{u_k\}_{k=1,\ldots,d-1}$.

**Example 2.1** ($N = 1, \ d = 2$)

$$
v_{0,1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_{0,2} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad v_{1,1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_{1,2} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \quad
$$

$$
v_{2,1} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_{2,2} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \quad u_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.
$$
Toric variety $X_\Sigma$ defined by the fan $\Sigma = \Sigma_{N,d}$ realizes the variety $\overline{M}_{0,2}(N,d)$. To see it, we use a standard fact of general theory of toric variety, which describes $X_\Sigma$ as a quotient space of an open subset of $\mathbb{C}^{\Sigma(1)}$. We review here some facts on general fan $\Sigma$ in lattice set $N = \mathbb{Z}^n$ (see Chapter 3 of [2] or [5]). Let $\Sigma(1)$ be a set of 1-skeletons of $\Sigma$. Although $\rho \in \Sigma(1)$ is a ray on $N_\mathbb{R} := N \otimes \mathbb{R} = \mathbb{R}^n$, we identify this ray as generator vector $v_\rho$ of a semi-group $N \cap \rho$. Let $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. We have the following exact sequence for the Chow group $A_{n-1}(X_\Sigma)$:

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(X_\Sigma) \to 0,$$

(2.2)

where $M \to \mathbb{Z}^{\Sigma(1)}$ is defined by $m \mapsto ((m \cdot v_\rho)_{\rho \in \Sigma(1)})$ ( $(m \cdot v_\rho)$ is standard inner product), and $\mathbb{Z}^{\Sigma(1)} \to A_{n-1}(X_\Sigma)$ is defined by $(m_\rho) \mapsto \sum_{\rho} m_\rho [D_\rho]$ ([D_\rho] is a divisor class associated to $\rho \in \Sigma(1)$). Next, we define a closed subset $Z(\Sigma)$ of $\mathbb{C}^{\Sigma(1)}$. The primitive collection $S$ is a subset of $\Sigma(1)$ which is not the set of 1-dimensional cones of some cone $\sigma \in \Sigma$ but every proper subset of $S$ is contained in some cone in the fan. Then, let

$$Z(\Sigma) := \bigcup_{S : \text{prim.coll.}} V(S),$$

where $V(S) = \{x \in \mathbb{C}^{\Sigma(1)} \mid x_\rho = 0, \rho \in S\} \subset \mathbb{C}^{\Sigma(1)}$. Finally, let $G := \text{Hom}_{\mathbb{Z}}(A_{n-1}(X_\Sigma), \mathbb{C}^*)$. The group $G$ acts on $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ as $g \cdot (x_\rho)_{\rho \in \Sigma(1)} := (g([D_\rho])x_\rho)$. Then the following theorem holds:

**Theorem 2.1** If the 1-dimensional cones of $\Sigma$ span $N_\mathbb{R}$, then:

1. $X_\Sigma$ is the categorical quotient of $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ by $G$;
2. $X_\Sigma$ is the geometric quotient of $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ by $G$ if and only if $X_\Sigma$ is simplicial.

See [3] for more detail.

We use this theorem in the case of $\Sigma = \Sigma_{N,d}$. Note that $N = M = \mathbb{Z}^{(d+1)(N-1)+(d-1)}$ in this case. First, we write down some relations among the vectors in $\Sigma(1)$ to determine the subset $Z(\Sigma)$. Since we have the relations,

$$p_1 + p_2 + \cdots + p_N = 0_{N-1},$$

and

$$v'_0 = -e_1,$$
$$v'_1 = -2(-e_1) + (-e_2),$$
$$v'_i = (-e_{i-1}) - 2(-e_i) + (-e_{i+1}) \ (2 \leq i \leq d - 2),$$
$$v'_{d-1} = -e_{d-2} - 2(-e_{d-1}),$$
$$v'_d = -e_{d-1},$$

we obtain the following relations among the vectors in $\Sigma(1)$:

\[
\begin{align*}
\Sigma^N_{j=1} v_{0,j} & = u_1, \\
\Sigma^N_{j=1} v_{1,j} & = -2u_1 + u_2, \\
\Sigma^N_{j=1} v_{i,j} & = u_{i-1} - 2u_i + u_{i+1} \ (2 \leq i \leq d - 2), \\
\Sigma^N_{j=1} v_{d-1,j} & = u_{d-2} - 2u_{d-1}, \\
\Sigma^N_{j=1} v_{d,j} & = u_{d-1}.
\end{align*}
\]

(2.3)

These relations determine the primitive collections of the fan $\Sigma = \Sigma_{N,d}$ as follows:

\[
\begin{align*}
\{v_{0,1}, v_{0,2}, \ldots, v_{0,N}\}, \\
\{v_{1,1}, v_{1,2}, \ldots, v_{1,N}, u_1\}, \\
\{v_{i,1}, v_{i,2}, \ldots, v_{i,N}, u_1\} \ (2 \leq i \leq d - 2), \\
\{v_{d-1,1}, v_{d-1,2}, \ldots, v_{d-1,N}, u_{d-1}\}, \\
\{v_{d,1}, v_{d,2}, \ldots, v_{d,N}\}.
\end{align*}
\]

We introduce the following notation,

$$\mathbb{C}^{\Sigma(1)} = \{x = (a_0, a_1, \ldots, a_d, u_1, u_2, \ldots, u_{d-1}) \mid a_i \in \mathbb{C}^N, u_i \in \mathbb{C}\},$$

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where \(a_i\) and \(u_i\) represent \((x_{v_{i,1}}, \ldots, x_{v_{i,N}})\) and \(x_u\), respectively. Then we can easily see that

\[
Z(\Sigma) = \{ \mathbf{x} \in \mathbb{C}^{\Sigma(1)} \mid a_0 = 0, a_d = 0, (a_i, u_i) = 0 (1 \leq i \leq d - 1) \}.
\]

Let us check the \(G = (\mathbb{C}^*)^{d+1}\)-action on \(\mathbb{C}^{\Sigma(1)}\) (note that \(|\Sigma(1)| - \text{rank}(M) = d + 1\). Then we can see \(\text{rank}(A_{\dim(X_S)}(X_S)) = d + 1\) from the exact sequence (2.2). Let \([D_{i,j}]\) (resp. \([U_k]\)) be a divisor class that corresponds to 1-skeleton \(v_{i,j}\) (resp. \(u_k\)). By the exact sequence (2.2) and definition of \(v_{i,j}\) and \(u_k\), we obtain the following relations on a Chow group \(A_{\dim(X_S)}(X_S)\):

\[
[D_{i,1}] = [D_{i,2}] = \cdots = [D_{i,N}] (0 \leq i \leq d),
[D_k] = -[D_{k-1,N}] + 2[D_k,N] - [D_{k+1,N}] (1 \leq k \leq d - 1).
\]

Let \(\lambda_i := g([D_{i,1}]) (g \in G)\). The above relation tells us that \(g([U_k]) = \lambda_{k-1}^{-1}\lambda_k^2\lambda_{k+1}^{-1}\), and the \((\mathbb{C}^*)^{d+1}\)-action turns out to be,

\[
(\lambda_0, \ldots, \lambda_d) \cdot (a_0, a_1, \ldots, a_d, u_1, u_2, \ldots, u_{d-1}) = (\lambda_0 a_0, \lambda_1 a_1, \ldots, \lambda_d a_d),
\]

\[
\lambda_0^{-1}\lambda_1^2\lambda_2^{-1}u_1, \ldots, \lambda_{d-2}^{-1}\lambda_{d-1}^2\lambda_d^{-1}u_{d-1}).
\]

This action is the same as the \((\mathbb{C}^*)^{d+1}\)-action (2.1) in Definition 2.1.

### 2.2 The Chow ring of \(\overline{M}_{0,2}(N,d)\).

In this subsection, we prove Proposition 2.1.

The Chow ring of \(\overline{M}_{0,2}(N,d)\) is computed by its toric data. To illustrate the recipe of computation, we review general theory of toric varieties (Chap.3 of [2] or [5], etc.). Let \(\Sigma\) be a simplicial complete fan on a lattice set \(N\), and

\[
\Sigma(1) := \{ \rho_1, \rho_2, \ldots, \rho_r \}
\]

be an 1-skeleton (i.e. a collection of 1-dimensional cones of \(\Sigma\)). We identify \(\rho_i\) with the generator of semi-group \(\rho_i \cap N\) in the same way as the previous subsection, and we denote it by \(v_i\). We define two ideals of \(\mathbb{C}[x_1,x_2,\ldots,x_r]\). First one is given by,

\[
I(\Sigma) := \left( \sum_{i=1}^{r} (m, v_i) \cdot x_i \mid m \in M \right).
\]

The other one is the ideal called **Stanley-Reisner ideal** of the corresponding fan \(\Sigma\) and is defined by,

\[
SR(\Sigma) := (x_{i_1}x_{i_2} \cdots x_{i_s} \mid \{v_{i_1}, v_{i_2}, \ldots, v_{i_s}\} \in \mathcal{S}),
\]

where \(\mathcal{S}\) is the primitive collection of the fan \(\Sigma\) (see the previous subsection). Chow ring of a toric variety \(X_\Sigma\) is isomorphic to the following quotient ring:

\[
A^*(X_\Sigma) \cong \mathbb{C}[x_1,x_2,\ldots,x_r]/(I(\Sigma) + SR(\Sigma)).
\]

This isomorphism enables us to prove Proposition 2.1.

**Proposition 2.1** \(A^*(\overline{M}_{0,2}(N,d)) \cong \mathbb{C}[H_0, H_1, \ldots, H_d]/\mathcal{I}\),
where \(\mathcal{I} = (H_0^N, H_d^N, H_k^N (-H_{k-1} + 2H_k - H_{k+1}))\).

**Proof.** Let \(x_{i,j}\) (resp. \(y_k\)) be a variable corresponding to a cone \(v_{i,j}\) (resp. \(u_k\)) of the fan \(\Sigma_{N,d}\). Let \(\{e_\alpha\}_{\alpha=1}^{dN+N-2}\) be standard basis of \(\mathbb{Z}^{dN+N-2}\). When \(1 \leq \alpha \leq N - 1\), we have,

\[
\langle e_\alpha, v_{i,j} \rangle = \begin{cases} 
1 & (i = 0, j = \alpha) \\
-1 & (i = 0, j = N) \\
0 & \text{(otherwise)}
\end{cases}
\]

\[
\langle e_\alpha, u_k \rangle = 0 \quad \text{(for all } k).\]
Hence we obtain \( x_{0, \alpha} - x_{0, \beta} \in I(\Sigma) \), and we can identify \( x_{0, \alpha} \) with \( x_{0, \beta} \) in \( A^*(\overline{M}_{0,2}(N, d)) \). In the same manner, when \( \ell(N - 1) + 1 \leq \alpha \leq (\ell + 1)(N - 1) \) (where \( 0 \leq \ell \leq d \)), we have,

\[
\langle e_\alpha, v_{i,j} \rangle = \begin{cases} 
1 & (i = \ell, j = \alpha - \ell(N - 1)) \\
-1 & (i = \ell, j = N) \\
0 & \text{(otherwise)}
\end{cases}
\]

\[
\langle e_\alpha, u_k \rangle = 0 \text{ (for all \( k \))}.
\]

Therefore, \( x_{\ell, \alpha - \ell(N - 1)} = \varphi_{x, \alpha} \in A^*(\overline{M}_{0,2}(N, d)) \) for \( 0 \leq \ell \leq d, \ell(N - 1) + 1 \leq \alpha \leq (\ell + 1)(N - 1) \). Consequently, if we denote \( x_{\ell, \beta} \) by \( X_\ell \), we obtain the relation \( x_{\ell, \beta} = X_\ell \) for \( 0 \leq \ell \leq d, 1 \leq j \leq N - 1 \). If \( (d + 1)(N - 1) + 1 \leq \alpha \leq (d + 1)(N - 1) + (d - 1) \), we have,

\[
\langle e_\alpha, v_{i,j} \rangle = \begin{cases} 
-1 & (i = \alpha - (d + 1)(N - 1) - 1, j = N) \\
2 & (i = \alpha - (d + 1)(N - 1), j = N) \\
-1 & (i = \alpha - (d + 1)(N - 1) + 1, j = N) \\
0 & \text{(otherwise)}
\end{cases}
\]

\[
\langle e_\alpha, u_k \rangle = \begin{cases} 
-1 & (k = \alpha - (d + 1)(N - 1)) \\
0 & \text{(otherwise)}
\end{cases}
\]

Hence we obtain the relation \(-H_{k-1} + 2H_k - H_{k+1} - y_k = 0\), which yields \( y_k = -H_{k-1} + 2H_k - H_{k+1} \) \((1 \leq k \leq d - 1)\). Accordingly, we have shown that

\[
\mathbb{C}[x_\rho \mid \rho \in \Sigma_{N,d}(1)]/I(\Sigma_{N,d}) = \mathbb{C}[H_0, H_1, \ldots, H_d].
\]

Next, we compute the Stanley-Reisner ideal \( SR(\Sigma_{N,d}) \). It is clear that \( SR(\Sigma_{N,d}) \) coincides with \( I = (H_0^N, H_1^N, H_2^N(\frac{H_{k-1}}{(d+1)} + 2H_k - H_{k+1})) \) since we have obtained the primitive collection \( S \) in the previous subsection. \( \square \)

In this paper, we use a class:

\[
d \cdot H_0^{N-1}H_1^NH_2^N \cdots H_{d-1}^NH_d^{N-1}
\]

as the volume form of \( \overline{M}_{0,2}(N, d) \), i.e.

\[
\int_{\overline{M}_{0,2}(N, d)} d \cdot H_0^{N-1}H_1^NH_2^N \cdots H_{d-1}^NH_d^{N-1} = 1.
\]

### 3 The Gromov-Witten Invariants of \( M_N^k \).

In this section, following [8] [10], we derive numerically explicit formulas of the two pointed Gromov-Witten invariants of degree 1 and 2 of \( M_N^k \). In the beginning, we define constants which play important roles in the remaining part of this paper.

**Definition 3.1** Let \( \ell_i^k \) be the coefficient of \( x^k y^{i+1} \) in \( e^k(x, y) := \prod_{i=0}^{k}(jx + (k - j)y) \).

Then, \( e^k(x, y) = \sum_{i=0}^{k-1} \ell_i^k x^{k-i} y^{i+1} \). Note that \( \ell_i^k = \ell_{k-1-i}^k \) and \( \ell_i^k = 0 \) for \( i < 0, k \leq i \). We describe \( \langle \mathcal{O}_h, \mathcal{O}_h \rangle \) as a polynomial of \( \ell_i^k \) in order to prove our main theorems with the aid of the formula [1.2] and Proposition [1.1].

#### 3.1 The Case of \( d = 1 \).

Note that \( \overline{M}_{0,2}(N, 1) \cong \mathbb{P}^{N-1} \times \mathbb{P}^{N-1} \). We can compute degree 1 GW invariants of \( M_N^k \) with the equation [1.20] in Section 1. We can compute \( w(\mathcal{O}_h, \mathcal{O}_h)_{0,1} \) by using [1.2].

\[
w(\mathcal{O}_h, \mathcal{O}_h)_{0,1} = \int H_0^a H_1^b e^k(H_0, H_1) = \int H_0^a H_1^b \sum_{i=0}^{k-1} \ell_i^k H_0^{a+1} H_1^{b-i}
\]

\[
= \int \sum_{i=0}^{k-1} \ell_i^k H_0^{a+1+b} H_1^{b-a} = \ell_k^{N-a-2} H_0^{N-1} H_1^{N-1}.
\]

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Lemma 3.1

We first present the following formula:

\[ w(O_{h^a} O_{h^b})_{0,1} = \ell_{N-\alpha-\gamma-2} = \ell_{k-N+1} \]

(where we used \( a + b = 2N - k - 3 \), and shortened \( \int_{M_{0,2}(N,1)} \) to \( \int \)). Therefore, we have,

\[
\langle O_{h^a} O_{h^b} \rangle_{0,1} = w(O_{h^a} O_{h^b})_{0,1} - w(O_{h^{a+b}})_{0,1}
\]

\[
= \ell_{N-\alpha-2} - \ell_{k-N+1}.
\]

(3.1)

3.2 The Case of \( d = 2 \).

We first present the following formula:

Lemma 3.1

\[
\int_{M_{0,2}(N,2)} H_0^\alpha H_2^\beta H_2^\gamma = \begin{cases} \frac{1}{2} \binom{\beta-\gamma}{N-\alpha-1} & \text{for } N \leq \beta \leq 3N - 2, 0 \leq \alpha, \gamma \leq N - 1 \\ 0 & \text{otherwise} \end{cases}
\]

We omit the proof of it because it is easily done by using the relation \( H_1^N (H_0 - 2H_1 + H_2) = 0 \iff H_1^{N+1} = \frac{1}{2} H_1^N (H_0 + H_2) \) and \( H_0^N = H_2^N = 0 \).

\[
w(O_{h^a} O_{h^b})_{0,2} \text{ and } w(O_{h^{a+b}})_{0,2} \text{ are computed as follows:}
\]

\[
w(O_{h^a} O_{h^b})_{0,2} = \int_{M_{0,2}(N,2)} H_0^\alpha H_2^\beta H_2^\gamma = \begin{cases} \frac{1}{2} \binom{\beta-\gamma}{N-\alpha-1} & \text{for } N \leq \beta \leq 3N - 2, 0 \leq \alpha, \gamma \leq N - 1 \\ 0 & \text{otherwise} \end{cases}
\]

Therefore, we have,

\[
w(O_{h^{a+b}})_{0,1} = \ell_{k-N+1}.
\]

The latter can be computed by using Theorem 1 in [1] which computes \( n \)-pointed degree 1 GW invariants of \( M_k^N \). If we apply this formula to the case of \( n = 3 \), then

\[
\langle O_{h^a} O_{h^b} O_{h^{a+b+k-N}} \rangle_{0,1}
\]

\[
= \int_{M_{0,2}(N,1)} (H_1 - H_0) \cdot e^k(H_0, H_1) \cdot H_0^b H_0^b H_0^b - H_1^b H_1^b H_0^{k-N} - H_1^{1+k-N} / H_0 - H_1
\]

\[
= \left( H_1^{1+k-N} - H_0^{1+k-N} \right) H_0^b \sum_{j=0}^{k-1} \ell_j H_0^{k-j} H_1^{j+1} \cdot \sum_{i=0}^{b-1} H_0^i H_1^{1-i}
\]

\[
= \left( H_1^{1+k-N} - H_0^{1+k-N} \right) \sum_{j=0}^{k-1} \sum_{i=0}^{b-1} \ell_j H_0^a H_0^{a+k-i-j} H_1^{i+b} - \sum_{j=0}^{k-1} \sum_{i=0}^{b-1} \ell_j H_0^a H_0^{a+2k-N+i-j+1} H_1^{b-i+j}
\]

\[
= \sum_{i=0}^{b-1} \left( \frac{\ell_i}{a+k-N+1} - \frac{\ell_i}{a+b-N+1} \right)
\]

\[
= \sum_{i=0}^{b-1} \left( \frac{\ell_i}{a+k-N+1} - \frac{\ell_i}{a+b-N+1} \right).
\]
In the last line, we used the symmetry relation \( \ell_i = \ell_{k-i} \) and inverted the order of terms. Combining these results, we obtain the following formula.

\[
\langle \mathcal{O}_h \mathcal{O}_{h^v} \rangle_{0,2} = w(\mathcal{O}_h \mathcal{O}_{h^v})_{0,2} - \frac{1}{k} \langle \mathcal{O}_h \mathcal{O}_{h^v} \mathcal{O}_{h^v+k-N} \rangle_{0,1} w(\mathcal{O}_{h^v+k-N} \mathcal{O}_{h^v})_{0,1}
\]

\[= \frac{1}{k} \sum_{i,j} \ell_i^k \ell_j^k \frac{1}{2^{i+j-N+2}} \left( \binom{i+j-N+1}{N-a-k+i-1} - \binom{i+j-N+1}{N-k+j-1} \right)
\]

\[= \frac{1}{k} \sum_{i,j} \ell_i^k \ell_j^k \frac{1}{2^{i+j-N+2}} \left( \binom{i+j-N+1}{N-a-k+i-1} - \binom{i+j-N+1}{N-k+j-1} \right)
\]

\[- \frac{1}{k} \ell_i^k \ell_{i+k-N+1} \sum_{i=0}^{b-1} (\ell_i^k - \ell_{i+k-N+1}) \]

\[= \frac{1}{k} \sum_{i,j} \ell_i^k \ell_j^k \frac{1}{2^{i+j-N+2}} \left( \binom{i+j-N+1}{N-a-k+i-1} - \binom{i+j-N+1}{N-k+j-1} \right)
\]

\[- \frac{1}{k} \ell_i^k \ell_{i+k-N+1} \sum_{i=0}^{b-1} (\ell_i^k - \ell_{i+k-N+1}) \]

\[= \frac{1}{k} \sum_{i,j} \ell_i^k \ell_j^k \frac{1}{2^{i+j-N+2}} \left( \binom{i+j-N+1}{N-a-k+i-1} - \binom{i+j-N+1}{N-k+j-1} \right)
\]

\[- \frac{1}{k} \ell_i^k \ell_{i+k-N+1} \sum_{i=0}^{b-1} (\ell_i^k - \ell_{i+k-N+1}) \] (3.2)

4 Proof of Theorem 1.1.

In this section, we will prove Theorem 1.1 of this paper. From now on, an integer \( n \) is sometimes used as \( N - 1 \):

\[ n := N - 1. \]

In order to prove the theorem, we use the results on Chow ring of moduli space \( \overline{M}_{0,n}(\mathbb{P}^n, d) \) of stable maps of degree \( d \) from genus 0 stable curve to projective space \( \mathbb{P}^n \).

4.1 Review of the structure of \( A^*(\overline{M}_{0,n}(\mathbb{P}^n, d)) \).

Theorem 4.1 (Theorem 5.1. of [14]) Let \( M = \{1_M, 2_M, \ldots, m_M\}, D = \{1_D, \ldots, d_D\}, M' = M \setminus \{1_M\}, D' = M' \cup D \) and \( d' = |D'| = d + m - 1 \).

\( B^*(\overline{M}_{0,n}(\mathbb{P}^n, d)) \) is a \( \mathbb{Q} \)-algebra generated by divisors

\[ H, \psi, T_h \]

for all \( h \subset D' \) such that \( h \neq \emptyset \) or \( \{i_M\} \) for \( i_M \in M' \). Let \( T_{b} = 1 \).

The ideal of relations is generated by:

1. \( H^n + 1 \);
2. \( T_h T_{h'} \) unless \( h \cap h' = \emptyset \), or \( \emptyset \neq h \subsetneq h' \) or \( \emptyset \neq h' \subsetneq h \);
3. \( m \geq 1 \) \( T_h T_{h'}(\psi + \sum_{h \cup h' \subseteq h, i_M \cap h} T_{h'}) \) for all \( h \neq h' \) nonempty;
4. \( m \geq 2 \) \( T_h (\psi + \sum_{h \cup \{i_M\} \subseteq h} T_{h'}) \) for all \( h \neq \emptyset \) and \( i_M \in M' \setminus h \);
5. \( m \geq 3 \) \( \psi + \sum_{i_M, j_M \subseteq h} T_h \) for all \( i_M, j_M \in M' \);
6. \( m > 1 \) \( (H + d\psi + \sum_{i_M \subseteq h} |h \cap M'| T_{h'})^{n+1} \);
7. \( T_h (\sum_{h' \neq h} P(t_{h'}) |_{t_{h'}=0}^{1}) + \psi^{-1}(H + c|_{h_D} |\psi|^{n+1}) \) for all \( h \),

where

\[ P(t_{h'}) = (\psi + \sum_{h' \supseteq h} T_{h'} + t_{h'})^{-1} \]

\[ (c|_{h_D} \psi + \sum_{h' \supseteq h} |h'_D \setminus h_D| T_{h'} + |h'_D \setminus h_D|^n t_{h'})^{n+1} \]

\[ (H + c|_{h_D} \cap c|_{h_D} \psi + \sum_{h' \supseteq h} |h'_D \setminus (h_D \cup h'_D)| T_{h'})^{n+1} \].

Here for any \( h \subset D' \), \( h_D := h \cap D \) and \( c|_{h_D} := D \setminus h \).
In this subsection, we prove the following theorem on degree 1 Gromov-Witten invariants:

**Theorem 4.2**

\[
\langle \mathcal{O}_h \mathcal{O}_{h^2} \rangle_{0,2} = \int_{\overline{M}_{0,2}(\mathbb{P}^n, 1)} h_0^a h_1^b e^k(h_0, h_1 + T),
\]

where \(e^k(x, y) := \prod_{j=0}^{k} (jx + (k - j)y) := \sum_{i=0}^{k-1} \ell_i^k x^{k-i} y^{i+1} \).

First, we prove the following lemma.

**Lemma 4.1**

\[
\int_{\overline{M}_{0,2}(\mathbb{P}^n, 1)} h_0^a h_1^b (h_1 + T)^i = \begin{cases} 
1 & (\alpha = n, \gamma \neq n) \\
-1 & (\alpha \neq n, \gamma = n) \\
0 & \text{(otherwise)}
\end{cases}
\]

where \(\alpha, \beta, \gamma \geq 0, \alpha + \beta + \gamma = 2n\).
Proof of Lemma [4.1]

Note that \( \int h_0^\alpha h_1^n = 1 \). If \( \alpha = n \) and \( \gamma \neq n \), then \( \gamma = n - \beta < n \) and \( \beta = n - \gamma > 0 \). Hence \( \alpha + \beta > n \). By the relation \( T(h_1 - h_0) = 0 \), the terms that contain \( T \) in \( h_0^\alpha h_1^n (h_1 + T)^\gamma \) vanish. Therefore, \( h_0^\alpha h_1^n (h_1 + T)^\gamma = h_0^\alpha h_1^n \). If \( \alpha \neq n \) and \( \gamma = n \), then \( \beta + \gamma > n \) as above. Then, \( h_0^\alpha h_1^n (h_1 + T)^\gamma = h_0^\alpha h_1^n T^n = h^n T^n \). We obtain the relation \( h_0^\alpha h_1^n + h^n T^n = 0 \) by multiplying the relation (4.2) by \( h_0^n \), and \( h_0^\alpha h_1^n (h_1 + T)^\gamma = h_0^\alpha h_1^n + h^n T^n = 0 \).

Hereinafter, we consider cases when \( h_0^\alpha h_1^n (h_1 + T)^\gamma \) vanishes. If \( \alpha = \gamma = n \) and \( \beta = 0 \), then \( h_0^\alpha h_1^n (h_1 + T)^\gamma = h_0^\alpha h_1^n \). If \( \gamma < n \) and \( n - \beta + 1 \leq \alpha \leq n - 1 \), then \( \alpha + \beta \geq n + 1 \). Hence \( h_0^\alpha h_1^n (h_1 + T)^\gamma = h_0^\alpha h_1^n + h^n T^n \). However, \( \beta + \gamma = 2n - \alpha > n + 1 \), and it vanishes. Finally, we have to consider a case of \( \gamma > n \). We multiply the relation (4.1) by \( h_1 - h_0 + T \), and obtain the relation \( (h_1 + T)^{n+1} = 0 \). Therefore \( h_0^\alpha h_1^n (h_1 + T)^\gamma = 0 \). ⊗

Proof of Theorem [1.1]

Then, the relations are given as follows:

\[
\begin{align*}
H^{n+1}, & \\
S_0 U_1, & \quad S_0 U_2, \quad U_1 U_2, \\
T_1 T_2 (\psi + S_0), & \quad T_1 U_2 \psi, \quad T_2 U_1 \psi, \\
T_1 (\psi + U_1), & \quad T_2 (\psi + U_2), \quad S_0 \psi, \\
(H + 2 \psi + U_1 + U_2)^{n+1}, &
\end{align*}
\]

and the relations (5) in Theorem [4.1] We take a close look at the relation (5) individually since they have quite long expression.

The case of \( h = 0 \):

\[
\begin{align*}
& \frac{(H + 2 \psi + 2 S_0 + U_1 + T_1)^{n+1} - (H + \psi + S_0)^{n+1}}{(H + 2 \psi + 2 S_0 + U_1)^{n+1} - (H + \psi + S_0)^{n+1}} \\
& + \frac{(H + 2 \psi + 2 S_0 + U_2 + T_2)^{n+1} - (H + \psi + S_0)^{n+1}}{(H + 2 \psi + 2 S_0 + U_2)^{n+1} - (H + \psi + S_0)^{n+1}} \\
& + \frac{(H + 2 \psi + U_1)^{n+1} - (H + \psi)^{n+1}}{(H + 2 \psi + U_1)^{n+1} - (H + \psi)^{n+1}} \\
& + \frac{(H + 2 \psi + U_2)^{n+1} - (H + \psi)^{n+1}}{(H + 2 \psi + U_2)^{n+1} - (H + \psi)^{n+1}} \\
& + \frac{(H + 2 \psi + 2 S_0)^{n+1} - H^{n+1}}{(H + 2 \psi + 2 S_0)^{n+1} - H^{n+1}} - \frac{(H + 2 \psi)^{n+1} - (H + \psi)^{n+1}}{(H + 2 \psi)^{n+1} - (H + \psi)^{n+1}} \\
& + \frac{(H + 2 \psi)^{n+1} - H^{n+1}}{(H + 2 \psi)^{n+1} - H^{n+1}} + \frac{(H + 2 \psi)^{n+1}}{(H + 2 \psi)^{n+1}}.
\end{align*}
\]
The cases of \( h = \{1_D\} \) and \( h = \{2_D\} \):

\[
T_1 \left( \frac{(H + \psi + S_0 + U_2 + T_2)^{n+1} - H^{n+1}}{\psi + S_0 + U_2 + T_2} - \frac{(H + \psi + S_0 + U_2)^{n+1} - H^{n+1}}{\psi + S_0 + U_2} + \frac{(H + \psi + U_2)^{n+1} - H^{n+1}}{\psi + U_2} - \frac{(H + \psi)^{n+1} - H^{n+1}}{\psi} + \frac{(H + \psi)^{n+1}}{\psi} \right).
\]

\[
T_2 \left( \frac{(H + \psi + S_0 + U_1 + T_1)^{n+1} - H^{n+1}}{\psi + S_0 + U_1 + T_1} - \frac{(H + \psi + S_0 + U_1)^{n+1} - H^{n+1}}{\psi + S_0 + U_1} + \frac{(H + \psi + U_1)^{n+1} - H^{n+1}}{\psi + U_1} - \frac{(H + \psi)^{n+1} - H^{n+1}}{\psi} + \frac{(H + \psi)^{n+1}}{\psi} \right).
\]

The cases of \( h = \{1_D, 2_D\} \) and \( h = \{2_D, 2_M\} \):

\[
U_1 \left( \frac{(H + \psi + S_0 + U_2 + T_2)^{n+1} - H^{n+1}}{\psi + S_0 + U_2 + T_2} - \frac{(H + \psi + S_0 + U_2)^{n+1} - H^{n+1}}{\psi + S_0 + U_2} + \frac{(H + \psi + U_2)^{n+1} - H^{n+1}}{\psi + U_2} - \frac{(H + \psi)^{n+1} - H^{n+1}}{\psi} + \frac{(H + \psi)^{n+1}}{\psi} \right).
\]

\[
U_2 \left( \frac{(H + \psi + S_0 + U_1 + T_1)^{n+1} - H^{n+1}}{\psi + S_0 + U_1 + T_1} - \frac{(H + \psi + S_0 + U_1)^{n+1} - H^{n+1}}{\psi + S_0 + U_1} + \frac{(H + \psi + U_1)^{n+1} - H^{n+1}}{\psi + U_1} - \frac{(H + \psi)^{n+1} - H^{n+1}}{\psi} + \frac{(H + \psi)^{n+1}}{\psi} \right).
\]

The relation of \( h = \{1_D, 2_D\} \) is trivial.

We can simplify these relations by using the relations from (1) to (4). We write down the simplest form:

\[
\sum_{i=0}^{n} (H + \psi + S_0)^{n-i}((H + 2\psi + 2S_0 + U_1 + T_1)^i + (H + 2\psi + 2S_0 + U_2 + T_2)^i)
- (H + 2\psi + 2S_0 + U_1)^i - (H + 2\psi + 2S_0 + U_2)^i
+ \sum_{i=0}^{n} (H + \psi)^{n-i}((H + 2\psi + S_0)^i - (H + 2\psi)^i)
+ 2\sum_{i=0}^{n} H^{n-i}(H + 2\psi + 2S_0)^i, \quad (5.6)
\]

\[
T_1 \sum_{i=0}^{n} H^{n-i}((H + U_2 + T_2)^i + (H + \psi + S_0)^i - H^i), \quad (5.7)
\]

\[
T_2 \sum_{i=0}^{n} H^{n-i}((H + U_1 + T_1)^i + (H + \psi + S_0)^i - H^i), \quad (5.8)
\]

\[
U_1 \sum_{i=0}^{n} H^{n-i}((H + T_2)^i - H^i + (H + \psi)^i), \quad (5.9)
\]

\[
U_2 \sum_{i=0}^{n} H^{n-i}((H + T_1)^i - H^i + (H + \psi)^i). \quad (5.10)
\]
Generators of $\mathcal{A}^*(\overline{M}_{0,2}(\mathbb{P}^n,2))$ are given by,

$$H, \, \psi, \, S_0, \, S_1 := T_1 + T_2, \, S_2 := U_1 + U_2, \quad P_1 := T_1T_2, \, P_2 := U_1U_2, \, P_3 := T_1U_2 + T_2U_1.$$ 

But we don’t have to use $P_2$ and $P_3$. $P_2 = U_1U_2$ vanishes by the relation (5.2). On the other hand, by using the relation (5.4), we obtain,

$$0 = T_1(\psi + U_1) + T_1(\psi + U_1) = S_1\psi + S_1S_2 - P_3.$$ 

Hence $P_3$ is represented in terms of $\psi, S_1$ and $S_2$.

We can derive some useful relations from them by applying the following transformation:

**Definition 5.1 (Key transformation)**

$$h_0 := H, \, h_1 := H + \psi, \, h_2 := H + 2\psi + S_2.$$ 

Then the generators of $\mathcal{A}^*(\overline{M}_{0,2}(\mathbb{P}^n,2))$ turn into $h_0, h_1, h_2, S_0, S_1$ and $P_1$.

**Lemma 5.1**

$h_0^{n+1} = 0, \, h_1^{n+1} = 0, \, h_1^{n+1}(h_0 - 2h_1 + h_2) = 0$.

**Proof.** The first two are clear from (5.1) and (5.5). If we multiply by $\psi$ the relation (5.9) and (5.10), and add the resulting relations, then we obtain the third relation as follows:

$$\psi \cdot (5.9) + \psi \cdot (5.10)$$

$$= \psi U_1 \sum_{i=0}^{n} H^{n-i}(H + \psi)^i + \psi U_2 \sum_{i=0}^{n} H^{n-i}(H + \psi)^i \quad \text{(by (5.3))}$$

$$= S_2((H + \psi)^{n+1} - H^{n+1})$$

$$= (h_0 - 2h_1 + h_2)h_1^{n+1}. \quad \square$$

These relations are interesting since they are the same as the ones of $\mathcal{A}^*(\overline{M}_{0,2}(N,2))$. Hence we use the same volume form as the one of $\overline{M}_{0,2}(N,2)$.

$$\int_{\overline{M}_{0,2}(\mathbb{P}^n,2)} 2h_0^n h_1^{n+1} h_2^n = 1.$$ 

If there is no room for misunderstanding, we abbreviate $\int_{\overline{M}_{0,2}(\mathbb{P}^n,2)}$ as $\int$. We write down the relations from (5.1) to (5.5) in terms of new generators.

**Lemma 5.2**

$$P_3 = S_1(h_2 - h_1), \, P_1(h_2 - h_0) = 0, \, P_1(h_1 - h_0 + S_0) = 0,$$

$$S_0(h_1 - h_0) = 0, \, S_0(h_1 - h_2) = 0, \, S_1(h_1 - h_0)(h_1 - h_2) = 0.$$ 

We have to compute the following intersection number in order to prove of Theorem 1.2.

$$\int h_0^n (h_1 + S_0)^3 (h_2 + 2S_0 + S_1)^3 h_2^3. \quad (5.11)$$

For this purpose, we set

$$g_0 := h_0, \, g_1 := h_1 + S_0, \, g_2 := h_2 + 2S_0 + S_1,$$

and derive relations among $g_0, g_1,$ and $g_2$.

**Lemma 5.3**

$$g_0^{n+1} = 0, \, g_1^{n+1}(g_0 - 2g_1 + g_2) = 0, \, (g_1 - g_0) \sum_{i=0}^{n} (g_1^i + g_2^i) g_1^{n-i} = 0, \, (g_1 - g_0)g_2^{n+1} = 0.$$
Proof. The first relation is trivial. The second relation is equivalent to
\[ S_1(h_1 + S_0)^{n+1} = 0 \]
by lemma 5.1. It is shown by adding (5.7) and (5.8), and by multiplying the resulting expression by \( (\psi + S_0) \).
\[
0 = (\psi + S_0)(T_1)^i + T_2
\]
where we use the relations,
\[
0 = (\psi + S_0)(T_1)^i + T_2
\]
by (5.3).
\[
S_1((H + \psi + S_0)^{n+1} - H^{n+1})
\]
where we use the relations, \( T_1(\psi + S_0)T_2 = 0, T_1(\psi + S_0)U_2 = 0 \) and \( T_2(\psi + S_0)U_1 = 0 \). The third relation is obtained from multiplying the relation (5.8) by \( (\psi + S_0) \).
\[
0 = (\psi + S_0) \cdot \sum_{i=0}^{n} (H + \psi + S_0)^{n-i}((H + 2\psi + 2S_0 + S_1 + S_2)^i - (H + 2\psi + 2S_0 + S_2)^i)
\]
\[
+ (\psi + S_0) \cdot \sum_{i=0}^{n} (H + \psi)^{n-i}((H + 2\psi + S_2)^i - (H + 2\psi)^i)
\]
\[
+ 2(\psi + S_0) \cdot \sum_{i=0}^{n} H^{n-i}(H + 2\psi + 2S_0)^i
\]
\[
= (\psi + S_0) \cdot \sum_{i=0}^{n} (H + \psi + S_0)^{n-i}(H + 2\psi + 2S_0 + S_1 + S_2)^i - (H + 2\psi + 2S_0 + S_2)^i)
\]
\[
+ (\psi + S_0) \cdot \sum_{i=0}^{n} (H + \psi + S_0)^{n-i}((H + 2\psi + 2S_0 + S_2)^i - (H + 2\psi + 2S_0)^i)
\]
\[
+ 2(\psi + S_0) \cdot \sum_{i=0}^{n} H^{n-i}(H + 2\psi + 2S_0)^i
\]
\[
= (\psi + S_0) \cdot \sum_{i=0}^{n} (H + \psi + S_0)^{n-i}(H + 2\psi + 2S_0 + S_1 + S_2)^i
\]
\[
- (\psi + S_0) \cdot \sum_{i=0}^{n} (H + \psi + S_0)^{n-i}(H + 2\psi + 2S_0)^i
\]
\[
+ 2(\psi + S_0) \cdot \sum_{i=0}^{n} H^{n-i}(H + 2\psi + 2S_0)^i
\]
\[
= (\psi + S_0) \cdot \sum_{i=0}^{n} (H + \psi + S_0)^{n-i}(H + 2\psi + 2S_0 + S_1 + S_2)^i
\]
\[
+ (H + \psi + S_0)^{n+1} - H^{n+1}
\]
\[
= (\psi + S_0) \cdot \sum_{i=0}^{n} (H + \psi + S_0)^{n-i}((H + 2\psi + 2S_0 + S_1 + S_2)^i + H^i)
\]
\[
= (g_1 - g_0) \sum_{i=0}^{n} g_1^{n-i}(g_2^i + g_0).
\]
On the second equation, we used \( (\psi + S_0)(U_1 + T_1)(U_2 + T_2) = 0 \) which comes from (5.3), and \( U_1U_2 = 0 \). On the third equation, we inserted \( S_0 \) to \( (H + \psi)^{n-i} \) since \( (H + 2\psi + S_2)^i - (H + 2\psi)^i \) was divisible by \( S_2 \), and \( S_0S_2 = 0 \).

The last relation is shown by multiplying the second relation by \( (g_1 - g_2) \) and by applying the first relation to the resulting expression.
\[
(g_1 - g_0)(g_1 - g_2) \sum_{i=0}^{n} (g_0 + g_2^i)g_1^{n-i} = (g_1 - g_0)(g_1^{n+1} - g_2^{n+1} + (g_1 - g_2) \sum_{i=0}^{n} g_0^ig_1^{n-i})
\]
\[
= (g_1 - g_0)g_1^{n+1} - (g_1 - g_0)g_2^{n+1} + (g_1 - g_2)g_1^{n+1}
\]
\[= - (g_1 - g_0)g_2^{n+1}. \square \]
Next, we compute some intersection numbers on $\mathcal{M}_{0,2}(\mathbb{P}^n, 2)$.

**Lemma 5.4**

$$
\int S_1^n h_0^n h_1^n h_2^n = 0 \quad \text{(for } a + b + c + d = 3n + 1, \ 0 < a < n),
$$

$$
\int S_1^n h_1^n h_0^n h_2^n = -1.
$$

**Proof.** Let us discuss the first equation. If $c = 0$, then $b + d = 3n + 1 - a > 2n + 1$. Hence $S_1^n h_0^n h_2^n = 0$ by the relation $h_0^{n+1} = h_2^{n+1} = 0$. If $c \neq 0$, then thanks to the relation $S_1(h_1 - h_0)(h_1 - h_2) = 0$ in Lemma 5.2, we have only to consider the case of $c = 1$. Then $b + d = 3n - a > 2n$, and $S_1^n h_0^n h_2^n = 0$ as above. For the second equation, we compute $H^n(H + 2\psi + S_2)^n \cdot (5.9) + (5.10)$:

\[
0 = H^n(H + 2\psi + S_2)^n \cdot (5.9) + (5.10)
\]

\[
= H^n(H + 2\psi + S_2)^n(U_1 T_2^n + U_2 T_1^n + (U_1 + U_2)(H + \psi)^n) \quad \text{(by } H^{n+1} = 0)
\]

\[
= h_0^n h_2^n(U_1 T_2^n + U_2 T_1^n) + (h_0 - 2h_1 + h_2) h_1^n
\]

\[
= h_0^n h_2^n(U_1 T_2^n + U_2 T_1^n) - 2h_0^n h_1^{n+1} h_2^n.
\]

It is clear that $U_1 T_2^n + U_2 T_1^n - S_1^{n-1} P_3 = U_1 T_2^n + U_2 T_1^n - (T_1 + T_2) h_0^{n-1} (T_1 + T_2) U_1$ is divisible by $P_1 = T_1 T_2$. Then by using the relation $P_1(h_2 - h_0) = 0$ of Lemma 5.2 and $h_0^{n+1} = h_2^{n+1} = 0$, we obtain $h_0^n h_2^n(U_1 T_2^n + U_2 T_1^n) = h_0^n h_2^n S_1^{n-1} P_3$. Since $P_3 = S_1(h_2 - h_1)$, we obtain the following equation:

\[-S_1^n h_0^n h_1^n h_2^n - 2h_0^n h_1^{n+1} h_2^n = 0.\]

Now, we compute intersection numbers of the type given in 5.11.

**Lemma 5.5** Let $\alpha, \beta, \gamma, \delta$ be nonnegative integers which satisfies $\alpha + \beta + \gamma + \delta = 3n + 1$.

If $1 \leq \delta \leq n$, then

\[
\int g_0^n g_1^\beta g_2^n g_2^n h_2^n = \begin{cases} 
\frac{1}{2(n-1)} \{ (\delta-n-1) - (\beta-n-1) \}, & (\gamma < n \text{ or } \gamma = n, \ n-\delta + 1 > \alpha) \\
-1, & (\gamma = n, \ n-\delta + 1 \leq \alpha \leq n) \\
0, & (0 \leq \alpha \leq n, \ n+1 \leq \gamma \leq 2n-\delta).
\end{cases}
\]

If $\delta = 0$, $0 \leq \alpha \leq n$, $0 \leq \gamma \leq 2n$, then it is zero.

**Proof.** First, we consider the case of $\delta = 0$, $0 \leq \alpha \leq n$ and $0 \leq \gamma \leq 2n$. If $n + 1 \leq \gamma \leq 2n$, then $\alpha + \beta \geq n + 1$. Hence it is zero since there is the relations $(g_1 - g_0) g_2^{n+1} = 0$ and $g_0^{n+1} = 0$ of Lemma 5.3. If $0 \leq \gamma \leq n$, then we have only to consider the case of $g_0^n g_1^{n+1} g_2^n$ since there is the relation $g_1^{n+1}(g_0 - 2g_1 + g_2) = 0$ and $g_0^{n+1} = 0$. To prove $g_0^n g_1^{n+1} g_2^n = 0$, we should multiply $g_0^n g_1^n$ to the relation $(g_1 - g_0) \sum_{i=0}^{n-1}(g_0 + g_2) g_1^{i-1} = 0$.

Next, we consider the case of $1 \leq \delta \leq n - 1$, $0 \leq \alpha \leq n$, $n+1 \leq \gamma \leq 2n - \delta$. In this case, the left hand side is zero since $\alpha + \beta \geq n + 1$, and there are relations $(g_1 - g_0) g_2^{n+1} = 0$ and $g_0^{n+1} = 0$.

If $1 \leq \delta \leq n$, $\gamma = n$ and $n-\delta + 1 \leq \alpha \leq n$, then

\[
h_0^n (h_1 + S_0)^{\beta}(h_2 + 2S_0 + S_1)^{n} h_2^n
\]

\[
= h_0^n h_1^n (h_2 + S_1)^{\gamma} h_2^n
\]

\[
= S_1^n h_0^n h_1^n h_2^n
\]

(by Lemma 5.4, $n + \delta > n$)

\[
= S_1^n h_0^n h_1^n h_2^n
\]

(by Lemma 5.3, $S_1(h_1 - h_0)(h_1 - h_2) = 0$).

Therefore, $\int g_0^n g_1^\beta g_2^n g_2^n h_2^n = -1$ by Lemma 5.4.

Now, we prove the top case. If $2 \leq \delta \leq n$, $n+1-\delta \leq \alpha \leq n$ and $n+1-\delta \leq \gamma \leq n-1$, then the right hand side is zero since $\beta - n - 1 = 2n - \alpha - \gamma - \delta < n - \alpha, n - \gamma$. On the left hand side, it is

\[
h_0^n (h_1 + S_0)^{\beta}(h_2 + 2S_0 + S_1)^{n} h_2^n
\]

\[
= h_0^n h_1^n (h_2 + S_1)^{\gamma} h_2^n
\]

(by $\alpha + \delta > n$, Lemma 5.1, 5.2)

\[
= h_0^n h_1^n h_2^{n+\delta}
\]

(by Lemma 5.3)

\[
= 0
\]

(by $\gamma + \delta > n$).
If \( \alpha = n, \beta = n + 1, \gamma = n - \delta \) and \( 1 \leq \delta \leq n \), then the right hand side is \( \frac{1}{2} \). On the left hand side, it is
\[
\frac{h_0^n}{h_1^n} (h_1 + S_0)^{n+1} (h_2 + 2S_0 + S_1)^{n-\delta} h_2^\delta
= \frac{h_0^n}{h_1^n} h_2^{n+1}
\]
Therefore, \( \int g_0^n g_1^{n+1} g_2^{n-\delta} h_2^\delta = \frac{1}{2} \).

If \( \alpha = n - \delta, \beta = n + 1, \gamma = n \) and \( 1 \leq \delta \leq n \), then the right hand side is \( -\frac{1}{2} \). On the left hand side, we consider \( 0 = g_0^n g_2^{n-\delta} h_2^\delta \cdot (g_1 - g_0) \sum_{i=0}^n (g_0^i + g_2^i) g_1^{n-i} \) from Lemma 5.3
\[
0 = g_0^n g_2^{n-\delta} h_2^\delta \cdot (g_1 - g_0) \sum_{i=0}^n (g_0^i + g_2^i) g_1^{n-i}
= g_0^n g_2^{n-\delta} h_2^\delta \cdot (g_1 - g_0) (2g_1^n + \sum_{i=1}^\delta g_1 g_1^{n-i}) \quad \text{(by Lemma 5.3)}
\]
Here, if \( 0 < i < \delta \), then
\[
\int g_0^n g_2^{n-\delta} h_2^\delta (g_1 - g_0) g_1 g_1^{n-i}
= \int g_0^n g_2^{n-\delta} g_1^{n+i} h_2^\delta - g_0^n g_1^{n+i+1} h_2^\delta - g_0^n g_2^{n-\delta + 1} g_1^{n+i} h_2^\delta
= (-1) - (-1) \quad \text{(by the above case)}
= 0
\]
Therefore,
\[
0 = \int g_0^n g_2^{n-\delta} h_2^\delta \cdot (g_1 - g_0) (2g_1^n + g_0^\delta g_1^{n-\delta})
= \int (2g_0^n g_2^{n-\delta} g_1^{n+1} h_2^\delta - 2g_0^n g_1^{n+1} h_2^\delta + g_0^n g_2^{n-\delta + 1} g_1^{n+i} h_2^\delta)
= 2g_0^n g_1^{n+1} g_2^{n-\delta} - 2 \cdot (-1) + (-1)
= 2g_0^n g_1^{n+1} g_2^{n-\delta} + 1.
\]
Accordingly, \( \int g_0^n g_1^{n+1} g_2^{n-\delta} = -\frac{1}{2} \).

Finally, we use induction along \( \beta \geq n + 1 \) to prove the remaining part of this lemma. To execute induction, the following equation is needed: if \( 1 \leq \delta \leq n - 1, 0 \leq \alpha \leq n \) and \( n + 1 \leq \gamma \leq 2n + 1 - \delta \), then
\[
g_0^{\alpha} g_1^{\beta} g_2^{\gamma} h_2^\delta = 0. \tag{5.12}
\]

It is already proved in the above.

If \( \beta = n + 1 \), then Lemma 5.5 is true as above. If \( \beta > n + 1 \), by using the relation \( g_1^{n+1} (g_0 - 2g_1 + g_2) = 0 \), we obtain,
\[
\int g_0^{\alpha} g_1^{\beta} g_2^{\gamma} h_2^\delta = \frac{1}{2} \frac{1}{2} \int g_0^{\alpha+1} g_1^{\beta-1} g_2^{\gamma+1} h_2^\delta + \frac{1}{2} \int g_0^{\beta} g_1^{\gamma} h_2^\delta
= \frac{1}{2} \frac{1}{2} \left\{ \frac{(\beta - n - 2)}{n - \alpha - 1} - \frac{(\beta - n - 2)}{n - \gamma} \right\}
+ \frac{1}{2} \frac{1}{2} \left\{ \frac{(\beta - n - 2)}{n - \alpha} - \frac{(\beta - n - 2)}{n - \gamma - 1} \right\}
= \frac{1}{2} \left\{ \frac{(\beta - n - 1)}{n - \alpha} - \frac{(\beta - n - 1)}{n - \gamma} \right\},
\]
where we use the law of Pascal’s triangle in the last line. The equation 5.12 is used in the case of \( \gamma = n \) in the above induction. \( \square \)

In this Lemma, we do not compute the intersection numbers in the cases of \( 0 \leq \delta \leq n \) and \( \gamma > 2n - \delta \). Although the above result is not complete in this sense, it is sufficient to prove Theorem 1.2.
Proof of Theorem 1.2

Let $a$, $b$ be nonnegative integers satisfying $a + b = 3n - 2k$, and $k$ be a positive integer. First, we compute intersection number $\int h_0^a h_2^b e^k(g_0, g_1)e^k(g_1, g_2)/(k g_1)$:

$$
\int h_0^a h_2^b e^k(g_0, g_1)e^k(g_1, g_2)_{k g_1} = \int h_0^a h_2^b \frac{1}{k} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \ell_i^k \ell_j^k g_0^{-i} g_1^{-j+1} g_2^{-k-j} = \int \frac{1}{k} \sum_{i,j} \ell_i^k \ell_j^k g_0^{-i} g_1^{-j+1} g_2^{-k-j} h_2^b.
$$

Here we note that it takes a form to which we can apply Lemma 5.5. Indeed, if $k$, which is the maximum degree of $g_2$ in this summation, is greater than $2n - b$, then the total degree $a + b + 2k + 1$ becomes greater than $3n + 1$, and the intersection number vanishes. We also note that the three cases in Lemma 5.5 are disjoint with each other. Therefore, we obtain,

$$
\int h_0^a h_2^b e^k(g_0, g_1)e^k(g_1, g_2)_{k g_1} = \frac{1}{k} \sum_{i,j} \ell_i^k \ell_j^k \frac{1}{2^{i+j-1}} \left( \begin{array}{c} i+j-n \\ n-a-k+i \end{array} \right) - \left( \begin{array}{c} i+j-n \\ n-k+j \end{array} \right)
$$

Finally, we compute $\int g_1^a h_2^b e^k(g_0, g_1)e^k(g_1, g_2)/(k g_1)$:

$$
\int g_1^a h_2^b e^k(g_0, g_1)e^k(g_1, g_2)_{k g_1} = \int \frac{1}{k} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \ell_i^k \ell_j^k g_0^{-i} g_1^{-j+1} g_2^{-k-j} h_2^b.
$$

At this stage, we use the fact that the expression in Lemma 5.5, which contains binomial coefficients, is anti-symmetric under interchange of $\alpha$ and $\gamma$. Then we obtain,

$$
\int g_1^a h_2^b e^k(g_0, g_1)e^k(g_1, g_2)_{k g_1} = \frac{1}{k} \ell_k^b \sum_{i=0}^{k-n} \ell_i^k \ell_i^k = \frac{1}{k} \ell_k^b \sum_{i=n-k}^{b-1} \ell_i^k \ell_i^k.
$$

Combining these results, we reach the final expression:

$$
\int (h_0^a - g_1^a) h_2^b e^k(g_0, g_1)e^k(g_1, g_2)_{k g_1} = \frac{1}{k} \sum_{i,j} \ell_i^k \ell_j^k \frac{1}{2^{i+j-1}} \left( \begin{array}{c} i+j-n \\ n-a-k+i \end{array} \right) - \left( \begin{array}{c} i+j-n \\ n-k+j \end{array} \right)
$$

which coincides with the r.h.s. of (3.2). □
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