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Wenbin Shen, Wei Chen & Rong Sun

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Earth’s Temporal Principal Moments of Inertia and Variable Rotation

SHEN Wenbin  CHEN Wei  SUN Rong

Abstract  Based on the gravity field models EGM96 and EIGEN-GL04C, the Earth’s time-dependent principal moments of inertia $A$, $B$, $C$ are obtained, and the variable rotation of the Earth is determined. Numerical results show that $A$, $B$, and $C$ have increasing tendencies; the tilt of the rotation axis increases $2.1 \times 10^{-8}$ mas/yr; the third component of the rotational angular velocity, $\omega_3$, has a decrease of $1.0 \times 10^{-22}$ rad/s$^2$, which is around 23% of the present observed value. Studies show in detail that both $\theta$ and $\omega_3$ experience complex fluctuations at various time scales due to the variations of $A$, $B$ and $C$.

Keywords  Earth rotation; principal moments of inertia; wavelet analysis

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Introduction

Concerning the study of the Earth’s rotation, man has investigated the causes of the variations of the angular velocity and of the position of the Earth’s rotation axis in space and on Earth (precession, nutations, polar motion, and polar wander). The variation of the Earth’s angular velocity is directly related to the variation of the length-of-day (LOD). Polar motion corresponds to the displacement of the rotation axis in a frame tied to the Earth. It takes place within a square, 15 m long on each side during seasonal or sub-seasonal time scales. At the seasonal time scale, polar motion has two main components: the Chandler wobble with a period of about 435 days, and an annual component. The polar wander is the long-term component of the motion of the pole. The precession is the long-term component of the motion of the pole in space; nutations are the corresponding periodic components (Dehant & de Viron, 2002).

As we know, the mass redistribution within the Earth is an important factor that influences the rotation of the Earth but was poorly determined (Lambeck, 1980). However, at present, high-accuracy gravimetry has been developed and it is possible to obtain the variations of the inertia tensor since the inertia tensor is relevant up to the second-degree potential coefficients (Marchenko & Abrikosov, 2001). Some gravity models, such as EGM96 (1996) and EIGEN-GL04C (2006), provide the ratio of the variation of the low-order coefficients and thus enable us to model the variation of the inertia tensor caused by the mass redistribution.

In this paper we study the Earth’s rotation based on the temporary principal moments of inertia of the Earth determined by the temporary gravity field.
1 The solutions of the Earth’s free wobble

In the Tisserand axes system $r_x y_z$, the rotation of a nonrigid Earth could be described by the Liouville Equation (Lambeck, 1980; Goldstain et al., 2002):

$$L = \frac{\partial}{\partial t}(I\omega) + \omega \times (I\omega)$$

(1)

where $L$ is the external torque; $\omega$ is the angular velocity vector. The Tisserand axes are actually the mean axes of the principal axes $A, B$ and $C$, respectively. We can obtain the time dependent longitude $\lambda$ and latitude $\theta$ ($i = A, B, C$) of axis $i$ according to the variable second-degree potential coefficients. Then the orientations of the axes $T_x, T_y$ and $T_z$ could be expressed as $(\lambda, \theta)(i = A, B, C)$, where $X$ means the mean value of $X$. So the Tisserand axes system is not the Earth’s principal axes system.

In the Earth’s principal axes system, the inertia tensor could be expressed as:

$$I = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

(2)

while in the Tisserand axes system:

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

(3)

where $I$ is symmetric, and meets $I_{ij} = I_{ji}$ ($i, j = 1, 2, 3; i \neq j$). The principal axes system can be transformed to the Earth-fixed coordinate system by the following rotation matrix

$$R_1 = \begin{bmatrix} \cos \lambda_i \cos \theta_i & \sin \lambda_i \cos \theta_i & -\sin \theta_i \\ -\sin \lambda_i & \cos \lambda_i & 0 \\ \cos \lambda_i \sin \theta_i & \sin \lambda_i \sin \theta_i & \cos \theta_i \end{bmatrix}$$

(4)

And the Earth-fixed coordinate system can be transformed to the Tisserand axes system by

$$R_2 = \begin{bmatrix} \cos \lambda_i \cos \theta_i & \sin \lambda_i \cos \theta_i & -\sin \theta_i \\ -\sin \lambda_i & \cos \lambda_i & 0 \\ \cos \lambda_i \sin \theta_i & \sin \lambda_i \sin \theta_i & \cos \theta_i \end{bmatrix}$$

(5)

Then the transformation metric, from the principal axes system to the Tisserand axes system, could be expressed as:

$$P = R_1 R_2$$

(6)

According to the norm of metric transformation, one gets (Huang, et al., 2003)

$$I = P^T \tilde{I} P = P^T \tilde{I} P$$

(7)

On the other hand, $I$ could be decomposed as:

$$I = I_0 + \Delta I = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix} + \begin{bmatrix} \Delta I_{11} & \Delta I_{12} & \Delta I_{13} \\ \Delta I_{21} & \Delta I_{22} & \Delta I_{23} \\ \Delta I_{31} & \Delta I_{32} & \Delta I_{33} \end{bmatrix}$$

(8)

where $I_0$ is time-independent while $\Delta I$ is time-dependent.

Similarly, the angular velocity $\omega$ could be expressed as:

$$\omega = \Omega \begin{bmatrix} m_1 \\ m_2 \\ 1 + m_3 \end{bmatrix}$$

(9)

where $m_3$ meets (Lambeck, 1980)

$$m_3 = \frac{L_3}{C \Omega} - \frac{\Delta I_{33}}{C}$$

(10)

In this paper, we investigate the free rotation of the Earth. In this case, $L = 0$, and Eq.(10) reduces to

$$m_3 = \frac{\Delta I_{33}}{C} = \frac{I_{33} - C}{C}$$

(11)

2 The Earth’s inertia moment tensor and its temporal variation

In order to determine the Earth’s inertia moment tensor, we need to solve the Earth’s principal axes and moments of inertia which correspond to its principal axes. The principal axes and moments of inertia are usually determined based on the harmonic analysis of the gravity field using the second-order harmonic coefficients. The solution of the principal moments of inertia can be expressed as (Marchenko and Abrikosov, 2001)

$$A = K(e_3 + 2He_1)$$

$$B = K[(1 - 2H)e_3 + 2He_1]$$

$$C = 3Ke_3$$

(12)

where

$$K = -\frac{\sqrt{15} M a^2}{6H}$$

(13)
\[
\begin{align*}
\epsilon_1 &= 2 \sqrt{\frac{p}{3}} \sin\left(\frac{\phi + \pi}{3}\right) \\
\epsilon_2 &= -2 \sqrt{\frac{p}{3}} \sin\left(\frac{\phi}{3}\right) \\
\epsilon_3 &= 2 \sqrt{\frac{p}{3}} \sin\left(\frac{\phi - \pi}{3}\right)
\end{align*}
\]

\[
\begin{align*}
p &= \mathcal{C}_{2,0}^2 + \mathcal{C}_{2,1}^2 + \mathcal{S}_{2,1}^2 + \mathcal{S}_{2,2}^2 \\
q &= \frac{2\mathcal{C}_{2,0}^2}{3\sqrt{3}} + \frac{\mathcal{C}_{2,0}^3}{\sqrt{3}} (\mathcal{C}_{2,1}^2 + \mathcal{S}_{2,1}^2 - 2\mathcal{C}_{2,2}^2 - 2\mathcal{S}_{2,2}^2) \\
\varphi &= \arcsin\left(\frac{3\sqrt{3}}{2} p^{-3/2} q\right) \quad (-\frac{\pi}{2} < \varphi < \frac{\pi}{2})
\end{align*}
\]

In Eqs. (12)–(15), \(M\) and \(a\) denote the Earth's mass and semi-major axis respectively; \(\mathcal{C}_m, \mathcal{S}_m\) are the fully normalized harmonic coefficients and

\[H = 1 - \frac{A + B}{2C}\]

is the dynamic flattening of the Earth.

The solution of the principal axes can be expressed as (Marchenko and Abrikosov, 2001)

\[r_i^*= [l_i, m_i, n_i]^T\]

where

\[
\begin{align*}
u_i &= \frac{\mathcal{C}_{2,1} \mathcal{S}_{2,2} + \mathcal{S}_{2,1} (\epsilon_i - 2\mathcal{C}_{2,0}/\sqrt{3})}{\mathcal{S}_{2,1} \mathcal{S}_{2,2} + \mathcal{C}_{2,1} (\epsilon_i + \mathcal{C}_{2,2} + \mathcal{S}_{2,0}/\sqrt{3})} \\
l_i &= \frac{n_i}{\mathcal{C}_{2,1}} \left(\epsilon_i - \frac{2\mathcal{C}_{2,0}^2}{3\sqrt{3}} - \mathcal{S}_{2,1} \nu_i\right) \\
m_i &= u_i n_i \\
n_i &= \frac{1 + u_i + \left(\frac{\mathcal{S}_{2,1} \nu_i - \epsilon_i + 2\mathcal{C}_{2,0}/\sqrt{3}}{\mathcal{C}_{2,1}}\right)^{3/2}}{4}
\end{align*}
\]

Consequently, the included angles of the Earth’s principal axes and the geocentric coordinate system’s axes respectively are \((\alpha_i, \beta_i, \gamma_i)\):

\[
\begin{align*}
\alpha_i &= \arccos \left[\frac{l_i}{\sqrt{l_i^2 + m_i^2}}\right] \\
\beta_i &= \arccos \left[\frac{m_i}{\sqrt{l_i^2 + m_i^2}}\right] \\
\gamma_i &= \arcsin \left[\frac{n_i}{\sqrt{l_i^2 + m_i^2}}\right]
\end{align*}
\]

Hence the longitude and co-latitude of the Earth’s principal axes are \((\lambda_i, \theta_i, \varphi_i)\):

Thus, the Earth’s principal axes and moments of inertia are determined.

In fact, the harmonic coefficients of the gravity model are time-dependent (see Table 1 for relevant

| Parameter | Value | Ratio of variation \(\times 10^{11}\) yr\(^{-1}\) |
|-----------|-------|-----------------------------------|
| Geo-gravitational constant \(GM\) | 3.986 004 415 \times 10^{14} m^3 s^{-2} | \n|
| Semi-major axis of the Earth \(a\) | 6378 136 3 m | \n|
| EGM96 (Epoch 1996) Harmonic coefficients | \n|
| \(\mathcal{C}_{2,0}\) | \((- 48 416 537.173 6 \pm 3.561 063 5) \times 10^{11}\) | \n|
| \(\mathcal{C}_{2,1}\) | \((- 18.987 635 0 00 000 000) \times 10^{11}\) | \n|
| \(\mathcal{S}_{2,1}\) | \((- 119.528 012 \pm 0.00 000 000) \times 10^{11}\) | \n|
| \(\mathcal{C}_{2,2}\) | \((- 243 914 352 398 \pm 5.373 915 4) \times 10^{11}\) | \n|
| \(\mathcal{S}_{2,2}\) | \((- 140 016 683 654 \pm 5.435 326 9) \times 10^{11}\) | \n|
| Geo-gravitational constant \(GM\) | 3.986 004 415 \times 10^{14} m^3 s^{-2} | \n|
| Semi-major axis of the Earth \(a\) | 6378 136 46 m | \n|
| EIGEN-GL04C (Epoch 2005) Harmonic coefficients | \n|
| \(\mathcal{C}_{2,0}\) | \((- 48 416 522.709 4 \pm 2.507) \times 10^{11}\) | \n|
| \(\mathcal{C}_{2,1}\) | \((- 25.521 499 816 2 \pm 1.618) \times 10^{11}\) | \n|
| \(\mathcal{S}_{2,1}\) | \((119.528 012 \pm 0.000 000 0) \times 10^{11}\) | \n|
| \(\mathcal{C}_{2,2}\) | \((243 914.352.398 \pm 5.373 915.4) \times 10^{11}\) | \n|
| \(\mathcal{S}_{2,2}\) | \((140 016.683 654 \pm 5.435 326 9) \times 10^{11}\) | \n|
| Dynamic flattening of the Earth \(H\) (Epoch 1997) | \((- 3.273 \pm 3.2) \times 10^{-9}\) | \n|
| Gravitational constant \(G\) | \((- 6.674 2 \times 10^{-11} m^3 kg^{-1} s^{-2}\) | \n|
parameters of the gravity model EGM96 and EIGEN-GL04C, so the Earth’s principal axes and moments of inertia vary with time (see Tables 2 and Table 3).

### Table 2: The Earth’s principal moments of inertia and temporal variations

| Model        | Principal moment of inertia Value (Epoch 2006) / (10^{27} \text{ kg m}^2) | Ratio of variation (1996–2006) / (10^{11}\text{yr}^{-1}) |
|--------------|--------------------------------------------------------------------------|----------------------------------------------------------|
| A            | (8.008 085 ± 0.000 012)                                                  | 9.052 686                                                |
| B            | (8.008 262 ± 0.000 009)                                                  | 9.052 473                                                |
| C            | (8.034 476 ± 0.000 008)                                                  | 9.026 752                                                |
| EGM96        |                                                                          |                                                          |
| B            | (8.008 083 ± 0.000 008)                                                  | 9.052 727                                                |
| C            | (8.034 474 ± 0.000 008)                                                  | 9.026 797                                                |

### Table 3: The Earth’s principal axes and their temporal variations (Epoch 2006, \(i = A, B, C\))

| Model        | Principal axis | Longitude \(\lambda\) / (°) | Annual variation / (°) | Co-latitude \(\phi\) / (°) | Annual variation / (°) |
|--------------|----------------|------------------------------|------------------------|----------------------------|------------------------|
| A            | 345.071 218    | 0.000 033 27                |                        |                           |                        |
| EGM96        | 75.071 218     | -4.558 131 \times 10^5     | -3.473 326 \times 10^{18} |                          |                        |
| B            | -7.229 872 \times 10^6 | 1.068 731 \times 10^{16} |                         |                           |                        |
| C            | 279.261 221    | 89.999 917                 |                         |                           |                        |
| A            | A              | 155.437 211                 | -9.464 631 \times 10^5  | 0.000 042 07              |                        |
| EIGEN-GL04C  | 345.071 287    | -3.175 868 \times 10^4     | -4.711 728 \times 10^{17} |                        |                        |
| B            | 75.071 287     | 0.000 091                  |                         |                           |                        |
| C            | -6.849 252 \times 10^5 | 6.075 444 \times 10^{16} |                         |                           |                        |
|               | 280.405 679    | 89.999 900                 |                         |                           |                        |
|               | -1013.263030   | 1.036 087 \times 10^4      |                         |                           |                        |

### Table 4: Variation information of relevant parameters during 1996–2005

| Parameter | max | min | mean |
|-----------|-----|-----|------|
| \(\theta\) / (rad) | 1.569 612 307 173 966 \times 10^6 | 1.569 612 304 947 410 \times 10^6 | 1.569 612 306 164 200 \times 10^6 |

3 Variations in rotation rate and polar motion

We use the time-dependent values, obtained from the last section, solving Eqs.(7)–(11), and we obtain the variations of \(\omega_3\) and polar motion (see Fig.1).

![Fig.1](image_url)

**Fig.1** Variation of \(\omega_3\) during 1990–2010

From Fig.1, \(\omega_3\) has a secular decreasing of 1.04 \times 10^{-22} \text{ rad/s}^2 (fitting result), approximately accounting for 23% of the total value \(-4.5 \times 10^{-22} \text{ rad/s}^2\) provided by Groten (2004). Fig.2 shows that \(\omega_3\) has very complex variations during recent decades, which implies that the mass redistribution can influence \(\omega_3\) at various time scales: \(d_{12}, d_{13}, d_{11}\) show \(\omega_3\) have periods of 22, 14, and 8 years (known as the decadal variation), as well as other short-period variations ranging from a few days to two or three years. Thus we can conclude that the mass redistribution should be partly responsible for the decadal oscillation of the length of day (LOD).

Conventionally, the material slowing down of the Earth’s rotation is attributed to tidal frictions. However, our results show that the increase of the Earth’s inertial moment also contributes to the deceleration. As to the free rotation, the Earth’s system should hold the conservation of the angular moment. Then the deceleration of the Earth’s rotation is a result of the increasing inertial moment. From this point of view, our results seem to be reliable.

On the other hand, the amplitude of Chandler wobble, \(\theta\), also has complex fluctuations. In 20 years, \(\theta\) seems to have an increasing tendency of \(2.06 \times 10^{-8} \text{ mas/yr}\) (a fitting result). Here we still adapt the coif5 wavelet to decompose \(\theta\) into 12 layers, and the results are shown in Fig.4. Investigating Fig.4, it can be concluded that increasing \(A, B, C\) would lead to the polar motions with periods ranging 24 (\(d_{12}\)), 14 (\(d_{13}\)), 8 (\(d_{11}\)), 3 (\(d_{10}\)) and 2 (\(d_9\)) years; and the annual and sub-annual polar motions also appear. These values are in good agreement with the observations (Lambeck, 1980).

### Table 4: Variation information of relevant parameters during 1996–2005

| Parameter | max | min | mean |
|-----------|-----|-----|------|
| \(\theta\) / (rad) | 1.569 612 307 173 966 \times 10^6 | 1.569 612 304 947 410 \times 10^6 | 1.569 612 306 164 200 \times 10^6 |
4 Conclusions

Based on the gravity models EGM96 and EIGEN-GL04C, we first calculate the temporal principal moments of inertia of the Earth, and then obtain the numerical solutions to Liouville’s equations. Calculations show that the Earth’s principal moments of inertia are increasing, which might be a clue of the Earth’s expansion (Shen, et al, 2007); both $\omega_3$ and $\theta$ have short-period fluctuations but have secular increasing and decreasing tendency. Our results show that variations of $\omega_3$ and $\theta$ in all scales are relevant to the increase of $A$, $B$ and $C$. Thus we should not treat the Earth’s principal moments of inertia as constants as man traditionally did, because they vary with time and could contribute to the observed variations in the Earth’s rotation.

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Fig. 4  Wavelet analysis of $\theta$ during 1990–2010

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