A lower bound on the Bekenstein-Hawking temperature of black holes

Shahar Hod

The Ruppin Academic Center, Emeq Hefer 40250, Israel
and
The Hadassah Institute, Jerusalem 91010, Israel

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We present evidence for the existence of a quantum lower bound on the Bekenstein-Hawking temperature of black holes. The suggested bound is supported by a gedanken experiment in which a charged particle is dropped into a Kerr black hole. It is proved that the temperature of the final Kerr-Newman black-hole configuration is bounded from below by the relation $T_{\text{BH}} \times r_H > (\hbar/r_H)^2$, where $r_H$ is the horizon radius of the black hole.

I. INTRODUCTION

It is well known [1, 2] that, for mundane physical systems of spatial size $R$, the thermodynamic (continuum) description breaks down in the low-temperature regime $T \sim \hbar/R$ [3]. In particular, these low temperature systems are characterized by thermal fluctuations whose wavelengths $\lambda_{\text{thermal}} \sim \hbar/T$ are of order $R$, the spatial size of the system, in which case the underlying quantum (discrete) nature of the system can no longer be ignored. Hence, for mundane physical systems of spatial size $R$, the physical notion of temperature is restricted to the high-temperature thermodynamic regime [1, 2]

$$T \times R \gg \hbar. \quad (1)$$

Interestingly, black holes are known to have a well-defined notion of temperature in the complementary regime of low temperatures. In particular, the Bekenstein-Hawking temperature of generic Kerr-Newman black holes is given by [4, 5]

$$T_{\text{BH}} = \frac{\hbar(r_+ - r_-)}{4\pi(r_+^2 + a^2)}, \quad (2)$$

where

$$r_{\pm} = M + (M^2 - a^2 - Q^2)^{1/2} \quad (3)$$

are the radii of the black-hole (outer and inner) horizons [6]. The relation (2) implies that near-extremal black holes in the regime $(r_+ - r_-)/r_+ \ll 1$ are characterized by the strong inequality [7]

$$T_{\text{BH}} \times r_+ \ll \hbar. \quad (4)$$

It is quite remarkable that black holes have a well defined notion of temperature in the regime (4) of low temperatures, where mundane physical systems are governed by finite-size (quantum) effects and no longer have a self-consistent thermodynamic description.

One naturally wonders whether black holes can have a physically well-defined notion of temperature all the way down to the extremal (zero-temperature) limit $T_{\text{BH}} \times r_+ / \hbar \rightarrow 0$? In order to address this intriguing question, we shall analyze in this paper a gedanken experiment which is designed to bring a Kerr-Newman black hole as close as possible to its extremal limit. We shall show below that the results of this gedanken experiment provide compelling evidence that the Bekenstein-Hawking temperature of the black holes is bounded from below by the quantum inequality $T_{\text{BH}} \times r_+ \gg (\hbar/r_+)^2$.

II. THE GEDANKEN EXPERIMENT

We consider a spherical body of proper radius $R$, rest mass $\mu$, and electric charge $q$ which is slowly lowered towards a Kerr black hole of mass $M$ and angular momentum $J = Ma$ along the symmetry axis of the black hole [8]. The black-hole spacetime is described by the line element [4, 11]

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} \left[adt - (r^2 + a^2)d\phi\right]^2, \quad (5)$$
where $\Delta \equiv r^2 - 2M r + a^2$ and $\rho^2 \equiv r^2 + a^2 \cos^2 \theta$.

The test-particle approximation implies that the parameters of the body are characterized by the strong inequalities

$$\mu \ll R \ll r_+ .$$

(6)

These relations imply that the particle which is lowered into the black hole has negligible self-gravity (that is, $\mu / R \ll 1$) and that it is much smaller than the geometric length-scale set by the black-hole horizon radius. In addition, the weak (positive) energy condition implies that the radius of the charged body is bounded from below by its classical radius \[12–14\]

$$R \geq R_c \equiv \frac{q^2}{2\mu} .$$

(7)

This inequality ensures that the energy density inside the spherical charged body is positive \[15\].

The energy \[16\] of the charged body in the near-horizon black-hole spacetime is given by \[15, 17\]

$$E(r) = \mu \sqrt{\frac{r_0^2 - 2Mr_0 + a^2}{r_0^2 + a^2}} + \frac{Mq^2}{2(r_0^2 + a^2)} ,$$

(8)

where $r_0$ is the radial coordinate of the body’s center of mass in the black-hole spacetime. The first term on the r.h.s of (8) represents the energy associated with the rest mass $\mu$ of the body red-shifted by the black-hole gravitational field \[4, 18\]. The second term on the r.h.s of (8) represents the self-energy of the charged body in the curved black-hole spacetime \[15, 17, 19, 20\].

The proper height $l$ of the body’s center of mass above the black-hole horizon is related by the integral relation \[4\]

$$l(r_0) = \int_{r_+}^{r_0} \sqrt{\frac{r^2 + a^2}{r^2 - 2Mr + a^2}} dr$$

(9)

to the Boyer-Lindquist radial coordinate $r_0$. In the near-horizon $l \ll r_+$ region one finds the relation

$$r_0(l) - r_+ = (r_+ - r_-) \frac{l^2}{4\alpha} [1 + O(l^2/r_+^2)] ,$$

(10)

where $\alpha \equiv r_+^2 + a^2$. Taking cognizance of Eqs. \[8\] and \[10\], one finds

$$E(l) = \frac{(r_+ - r_-)\mu l + M q^2}{2\alpha} \cdot [1 + O(l^2/r_+^2)]$$

(11)

for the energy of the body in the near-horizon $l \ll r_+$ region.

Suppose now that the charged object is slowly lowered towards the black hole until its center of mass lies a proper height $l_0$ (with $l_0 \geq R$) above the black-hole horizon. The object is then released to fall into the black hole. The assimilation of the charged body by the black hole produces a final Kerr-Newman black-hole configuration whose physical parameters (mass, charge, and angular momentum) are given by

$$M \to M_{\text{new}} = M + E(l_0) ; \quad a \to a_{\text{new}} = a[1 - E(l_0)/M + O(E^2/M^2)] ; \quad Q = 0 \to Q_{\text{new}} = q .$$

(12)

The change in the black-hole temperature caused by the assimilation of the charged body can be quantified by the dimensionless physical function

$$\Theta(\bar{a}) \equiv \frac{\Delta T_{\text{BH}}}{T_{\text{BH}}} ,$$

(13)

where $\bar{a} \equiv a/M$ is the dimensionless angular momentum of the black hole \[21\].

Our goal is to bring the black hole as close as possible to its extremal (zero-temperature) limit. Thus, we would like to minimize the value of the dimensionless physical parameter $\Theta$. In particular, we would like to examine whether $\Theta(\bar{a})$, the dimensionless change in the black-hole temperature, can be made negative all the way down to the extremal $\bar{a} \to 1$ (zero-temperature, $T_{\text{BH}} \to 0$) limit.

We shall henceforth consider black holes in the regime

$$\bar{a} \geq \sqrt{2\sqrt{3} - 3} ,$$

(14)
in which case a minimization of the energy delivered to the black hole also corresponds to a minimization of the
Bekenstein-Hawking temperature of the final black-hole configuration [22]. The fact that the energy \( E(l_0) \) of the
charged particle in the black-hole spacetime is an increasing function of the dropping height \( l_0 \) [see Eq. (11)] implies
that, in order to minimize the physical parameter \( \Theta(\bar{a}) \) in the regime (14), one should release the body to fall into
the black hole from a point whose proper height above the black-hole horizon is as small as possible. We therefore
face the important question: How small can the dropping height \( l_0 \) be made?

As pointed out by Bekenstein [4], the expression (11) for the energy of our charged spherical object in the black-hole
spacetime is only valid in the restricted regime \( l_0 \geq R \), where every part of the body is still outside the horizon. This
fact implies, in particular, that the adiabatic (slow) descent of the charged spherical body towards the black hole
must stop when its center of mass lies a proper height \( l_0 \rightarrow R^+ \) above the horizon. At this point the bottom of the
body is almost swallowed by the black hole and the body [having a minimized (red-shifted) energy \( E(l_0 \rightarrow R) \)] should
then be released to fall into the black hole [4]. In addition, remembering that the weak (positive) energy condition
sets the lower bound (14) on the proper radius of the charged spherical body, one finds the relation (15)

\[
\frac{l_0}{R} = \frac{q^2}{2\mu}
\]

for the optimal dropping point of the charged body.

Substituting (15) into (11), one finds the remarkably simple (and universal) expression

\[
E_{\text{min}}(\bar{a}) = \frac{q^2}{4M}
\]

for the minimal energy delivered to the black hole by the charged body. Taking cognizance of Eqs. (2), (12), (13),
and (16), one finds the universal expression [26–28]

\[
\Theta_{\text{min}}(\bar{a}) = -\frac{q^2}{4M^2}
\]

for the smallest possible (most negative) value of the dimensionless physical parameter \( \Theta(\bar{a}) \) which quantifies the
change in the black-hole temperature caused by the assimilation of the charged body. Interestingly, one finds from
(17) the characteristic inequality

\[
\Theta_{\text{min}}(\bar{a}) < 0 ,
\]

which is valid for all values \( \bar{a} \in [0, 1) \) of the black-hole rotation parameter. The simple inequality (18) implies that, by
absorbing charged particles, the black hole can approach arbitrarily close to the extremal (zero-temperature) \( T_{\text{BH}} \rightarrow 0 \)
limit.

It is important to emphasize again that this conclusion is based on the assumption [4] that the charged body can be
lowered adiabatically (slowly) until its bottom almost touches the black-hole horizon [29]. In the next section we shall
show, however, that Thorne’s famous hoop conjecture [30] implies that, for near-extremal black holes, the charged
body cannot be lowered adiabatically all the way down to the horizon of the black hole.

### III. THE HOOP CONJECTURE AND THE LOWER BOUND ON THE BLACK-HOLE TEMPERATURE

In the previous section we have seen that, by absorbing a charged particle, a black hole can approach arbitrarily
close to the extremal (zero-temperature) \( T_{\text{BH}} \rightarrow 0 \) limit. As we have emphasized above, this interesting conclusion
rests on the assumption that the charged body can be lowered slowly all the way down to the horizon of the black hole
[29]. In the present section we shall show, however, that Thorne’s famous hoop conjecture [30] implies that, for near-extremal black holes, the charged
body cannot be lowered adiabatically all the way down to the horizon of the black hole.

The Thorne hoop conjecture [30] asserts that a physical system of total mass (energy) \( M \) forms a black hole if its
circumference radius \( r_c \) is equal to (or smaller than) the corresponding radius \( r_{\text{sch}} = 2M \) of the Schwarzschild black
hole. It is worth emphasizing that the validity of this version of the hoop conjecture is supported by several studies
[31]. However, it is also important to emphasize the fact that there are known spacetime solutions of the Einstein
field equations which provide explicit counterexamples to this version of the hoop conjecture [32, 33].

A weaker (and therefore a more robust) version of the hoop conjecture for spacetimes with no angular momentum
was suggested in [34, 35]. Here we would like to generalize this weaker version of the hoop conjecture to the generic
case of spacetimes which possess angular momentum and electric charge. In particular, we conjecture that: A physical
system of mass $M$, angular momentum $J$, and electric charge $Q$ forms a black hole if its circumference radius $r_c$ is equal to (or smaller than) the corresponding Kerr-Newman black-hole radius $r_{KN} = M + \sqrt{M^2 - (J/M)^2 - Q^2}$. That is, we conjecture that

$$r_c \leq M + \sqrt{M^2 - (J/M)^2 - Q^2} \implies \text{Black-hole horizon exists .} \quad (19)$$

In the context of our gedanken experiment, this weaker version of the hoop conjecture implies that a new (and larger) horizon is formed if the charged body reaches the radial coordinate $r_0 = r_{\text{hoop}}$, where $r_{\text{hoop}}(\mu, q)$ is defined by the Kerr-Newman functional relation [see Eq. (1)]

$$r_{\text{hoop}} = M + \mathcal{E}(r_{\text{hoop}}) + \sqrt{[M + \mathcal{E}(r_{\text{hoop}})]^2 - \left\{J/[M + \mathcal{E}(r_{\text{hoop}})]\right\}^2 - (Q + q)^2} . \quad (20)$$

Substituting $(8)$ into $(20)$, and assuming $r_{\text{hoop}} - r_+ \ll r_+ - r_- \ll r_+$, one finds

$$r_{\text{hoop}} - r_+ = \frac{2\beta^2 \mu^2}{r_+ - r_-} \quad (21)$$

for the radius of the new horizon, where

$$\beta = 1 + \sqrt{1 - \frac{q^2}{8\mu^2}} \cdot \tau \quad \text{with} \quad \tau = \frac{r_+ - r_-}{r_+} . \quad (22)$$

Substituting the radial coordinate $(21)$ into Eq. $(10)$, one finds

$$l(r_{\text{hoop}}) = \frac{4\beta \mu}{\tau} . \quad (23)$$

Taking cognizance of Eqs. $(15)$ and $(23)$ one realizes that, in the regime

$$l(r_{\text{hoop}}) > R_{\text{min}} = \frac{q^2}{2\mu} , \quad (24)$$

a new (and larger) horizon is formed before the spherical charged body touches the horizon of the original black hole. Thus, in the regime $(24)$, one should take

$$l_0^{\text{min}} = l(r_{\text{hoop}}) \quad (25)$$

in Eq. $(11)$ in order to minimize the energy delivered to the black hole by the charged body. This implies

$$\mathcal{E}^{\text{min}}(\bar{a}) = \frac{4\beta \mu^2 + q^2}{4r_+} \quad (26)$$

for the smallest possible energy delivered by the charged particle to the black hole in the regime $(24)$. Taking cognizance of Eqs. $(2)$, $(12)$, $(13)$, and $(26)$, one finds the relation

$$\Theta^{\text{min}}(\bar{a}) = \frac{8\mu^2 - q^2}{2r_+ \sqrt{1 - a^2}} \quad (27)$$

in the regime $(24)$.

Interestingly, one finds from $(27)$ that the black-hole-charged-body system is characterized by the inequality

$$\Theta^{\text{min}}(\bar{a}) > 0 \quad (28)$$

in the regime $(24)$. Note, in particular, that the inequality $(28)$ is satisfied by near-extremal black holes whose dimensionless temperature $\tau$ is characterized by the relation [see Eqs. $(22)$ and $(23)$]

$$\tau < \frac{8\mu^2}{q^2} . \quad (29)$$

Taking cognizance of Eqs. $(28)$ and $(29)$ one realizes that, in our gedanken experiment, the Bekenstein-Hawking temperature of the black holes cannot be lowered below the critical value

$$T_{\text{BH}} \times r_+ = \frac{\hbar}{\pi} \frac{\mu^2}{q^2} , \quad (30)$$

where $\mu$ and $q$ are the proper mass and electric charge of the absorbed particle, respectively.
IV. THE QUANTUM BUOYANCY EFFECT AND THE LOWER BOUND ON THE BLACK-HOLE TEMPERATURE

Thus far, we have analyzed the gedanken experiment at the classical level. It is important to emphasize, however, that the well-known quantum buoyancy effect \cite{43} in the black-hole spacetime should also be taken into account in the present gedanken experiment. This quantum buoyancy effect stems from the fact that the slowly lowered object interacts with the quantum thermal atmosphere of the black-hole spacetime \cite{43, 44}.

In particular, as shown by Bekenstein \cite{44}, the quantum buoyancy effect shifts the optimal dropping point \cite{24} of the object from \( l_{0}^{\text{min}} = R \) [see Eq. (15)] to a slightly higher point whose proper radial distance from the black-hole horizon is given by \cite{44}

\[
l_{0}^{\text{min}} = (1 + \epsilon) \cdot R ,
\]

where the dimensionless factor \( \epsilon \) is given by \cite{44, 45}

\[
\epsilon \equiv \sqrt{\frac{N}{120\pi}} \cdot \frac{\hbar}{\mu R}.
\]

and \( N \) is the effective number of quantum radiation species \cite{44}. The quantum shift (increase) \( \epsilon R \) [see Eq. (31)] in the radial proper distance of the optimal dropping point results in a quantum increase \( \epsilon \cdot (r_+ - r_-) \mu R/\alpha \) \cite{44} in the energy delivered to the black hole. Taking into account this quantum buoyancy increase in the energy delivered to the black hole, one finds that the classical expression \cite{17} for the dimensionless function \( \Theta(\bar{a}) \) acquires a positive quantum correction term. In particular, for near-extremal black holes the quantum-mechanically corrected expression for \( \Theta(\bar{a}) \) is given by \cite{37, 39}

\[
\Theta_{\text{min}}(\bar{a} \to 1) = -\frac{q^2}{4M^2} \cdot \left( 1 - \epsilon \cdot \frac{8r_+}{r_+ - r_-} \right).
\]

Interestingly, one finds from (33) that the black-hole-charged-body system is characterized by the inequality

\[
\Theta_{\text{min}}(\bar{a}) > 0
\]

in the regime

\[
\tau < 8\epsilon .
\]

The relations (31) and (35) imply that, due to the quantum buoyancy effect \cite{43, 44}, the Bekenstein-Hawking temperature of the black holes cannot be lowered below the critical value \cite{42}

\[
T_{\text{BH}}^c \times r_+ = \epsilon \cdot \frac{\hbar}{\pi}.
\]

Interestingly, the quantum lower bound \cite{36} becomes stronger than the classical lower bound \cite{30} in the regime \( \epsilon > \mu^2/q^2 \), which corresponds to charged objects \cite{39} in the regime \( q > (360\pi/N\hbar)^{1/2}\mu^2 \).

V. SUMMARY AND DISCUSSION

We have analyzed a gedanken experiment in which a spherical charged particle is lowered into a Kerr black hole. It was shown that if the charged particle can be lowered slowly all the way down to the horizon of the black hole, then the Bekenstein-Hawking temperature of the final black-hole configuration can approach arbitrarily close to the extremal (zero-temperature) \( T_{\text{BH}} \to 0 \) limit.

However, we have shown that Thorne’s famous hoop conjecture \cite{30} [and also its weaker (and more robust) generalization \cite{19}] implies that, for near-extremal black holes in the regime \cite{29}, a new (and larger) horizon is already formed before the charged particle touches the horizon of the original black hole. The hoop conjecture therefore implies that, in our gedanken experiment, the temperature of the final black-hole configuration cannot approach arbitrarily close to zero \cite{47}. In particular, we have proved that the Bekenstein-Hawking temperature of the black holes is an irreducible quantity in the near-extremal regime \( T_{\text{BH}} < T_{\text{BH}}^c \) determined by the critical temperature \cite{30}.

It is worth emphasizing that we have provided in this paper only one specific example, not a general proof, to the fact that the black-hole temperature cannot approach arbitrarily close to zero. Nevertheless, this intriguing conclusion...
of our gedanken experiment makes it tempting to conjecture that the Bekenstein-Hawking temperature of black holes is bounded from below by the simple universal relation [see Eq. (30)]

\[ T_{BH} \times r_+ \gg \left( \frac{\hbar}{r_+} \right)^2. \]  

(37)

We believe that it would be highly important to test the general validity of the conjectured lower bound on the Bekenstein-Hawking temperature of the black holes.

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[6] Here \( M, J \equiv M_a \), and \( Q \) are respectively the mass, angular momentum, and electric charge of the Kerr-Newman black hole.
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[21] We recall that the parameters \( \{ M, \alpha \} \) are the mass and angular momentum per unit mass of the original Kerr black hole.
[22] That is, Kerr black holes in the regime are characterized by the relation \( (\partial T_{BH}/\partial M)_{T,Q} > 0 \) [see Eqs. (2) and (3)].
[23] It is important to emphasize that our assumption \( r_0^{\text{min}} \ll r_+ \) [see Eq. (10)] corresponds to charged particles in the regime \( q^2 \ll \mu r_+ \).
[24] That is, the dropping point for which the energy delivered to the black hole, and thus also the physical parameter \( \Theta(\bar{a}) \), are minimized.
[25] The expression (10) for the minimal energy delivered to the black hole by the charged body is universal in the sense that it is independent of the black-hole rotation parameter \( \bar{a} \).
[26] The expression (17) for the dimensionless physical quantity \( \Theta^{\text{min}}(\bar{a}) \) is universal in the sense that it is independent of the black-hole rotation parameter \( \bar{a} \).
[27] It is worth noting that, for generic values of the dropping height \( l_0 \) and in the regime \( q^2 \ll r_+(r_+ - r_-) \), one finds from Eqs. (2), (11), (12), and (13), the relation \( \Theta(l_0; \bar{a}) = \frac{(1+\bar{a}^2-2\sqrt{1-\bar{a}^2})}{2a\sqrt{1-\bar{a}^2}} \frac{2l_0 - (2-\sqrt{1-\bar{a}^2})q^2}{2a\sqrt{1-\bar{a}^2}} \) for the dimensionless physical parameter \( \Theta \) which quantifies the change in the black-hole temperature caused by the absorbed particle. This expression implies that, in the regime (14), the dimensionless physical parameter \( \Theta(l_0; \bar{a}) \) is an increasing function of the dropping height \( l_0 \) (see [22]).
[28] Note that the relation \( q^2 = 2\mu R \ll r_+^2 \) for our charged spherical object [see Eqs. (9) and (7)] implies \( |\Delta T_{BH}^{\text{min}}| \ll T_{BH} \). Here \( \Delta T_{BH}^{\text{min}} \) denotes the most negative value which is physically allowed for the change \( \Delta T_{BH} \) in the black-hole temperature in our gedanken experiment.
That is, the simple inequality (15) is based on the assumption (4) that the proper distance of the body’s center of mass from the black-hole horizon at the dropping point can approach arbitrarily close to the limiting value \( l_0 \to R^{\min} = q^2/2\mu \) [see Eq. (15)].

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Interestingly, it was shown in (34) that the (weaker version (34) of) the hoop conjecture must be invoked in order to guarantee the validity of the generalized second law of thermodynamics (1) in a gedanken experiment in which an entropy bearing object is lowered slowly into a near-extremal black hole.

We shall henceforth assume that the original Kerr black hole is a near-extremal one.

Here we have used the approximated relation (37) on the Bekenstein-Hawking temperature of the black holes.

As discussed above, this new (and larger) horizon is expected to be formed according to the original hoop conjecture (30).

It is worth noting that, in the regime (14), a minimization of the energy (11) which is delivered to the black hole also corresponds to a minimization of the dimensionless physical parameter \( \Theta \) which quantifies the change in the black-hole temperature (see (22) and (27)).

Here we have used the approximated relation \( T_{\text{BH}} \times r_+ \simeq \tau h/8\pi \) for the near-extremal Kerr-Newman black holes with \( \tilde{a} \simeq 1 \).

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As shown by Bekenstein (44), the relations (41) and (42) are valid for macroscopic and mesoscopic bodies in the regime \( R \gg h/\mu \). This strong inequality corresponds to the regime \( \epsilon \ll 1 \).

That is, after the assimilation of the charged particle by the black hole.

It is worth noting that, had we used the original hoop conjecture (30) instead of its weaker version (19), we would have found that the Bekenstein-Hawking temperature of the final Kerr-Newman black-hole configuration (after the assimilation of the charged particle) is higher than the temperature of the original Kerr black hole in the entire regime \( \tilde{a} \in [0,1] \).

That is, the fact that, in our gedanken experiment, the Bekenstein-Hawking temperature of the black holes is an *irreducible* quantity in the near-extremal regime \( T_{\text{BH}} < T_{\text{BH}}^\text{crit} \) determined by the critical temperature (30).

Here we have used the inequality \( \mu^2/q^2 = \mu/2R \) [see Eq. (7)] for our charged massive particle. In addition, we have used the inequalities \( h/\mu \leq \mu \ll r_+ \) [see Eq. (9)] which characterize the physical parameters of the captured particle. (As emphasized by Bekenstein (4), the inequality \( R \geq h/\mu \) reflects the fact that the proper radius of the particle is bounded from below by its Compton length (4)).

It is interesting to note that the suggested lower bound (37) on the Bekenstein-Hawking temperature of the black holes is *universal* in the sense that it is *independent* of the physical parameters (proper mass and electric charge) of the captured particle which was used in our gedanken experiment in order to infer the bound.

It is worth noting that, taking cognizance of Eq. (36) and using the strong inequalities \( \mu \ll r_+ \) and \( R \ll r_+ \) [see Eq. (6)], one can obtain the stronger lower bound \( T_{\text{BH}} \times r_+ \gg h^{3/2}/r_+ \) on the Bekenstein-Hawking temperature of the black holes. Note, however, that this bound, which is a direct consequence of the quantum buoyancy effect, is probably of no relevance if, instead of being lowered slowly towards the black hole, the charged particle splits off from a larger body which falls freely (and thus experiences no buoyant force) towards the black hole (note that, in order to deliver as small as possible energy to the black hole, the splitting of the larger body into two particles should take place in the near-horizon region and, in addition, the second particle should escape the black hole). We therefore believe that the relation (37) should be regarded as the more fundamental bound on the Bekenstein-Hawking temperature of the black holes (that is, a generic bound which is *independent* of the manner in which the charged object arrives at the near-horizon region).