F-SYMBOLS, KILLING TENSORS AND CONSERVED BEL-TYPE CURRENTS

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In the framework of the General Relativity we show that from three generalizations of Killing vector fields, namely f-symbols, symmetric Stäckel-Killing and antisymmetric Killing-Yano tensors, some conserved currents can be obtained through adequate contractions of the above mentioned objects with rank four tensors having the properties of Bel or Bel-Robinson tensors in Einstein spaces.

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1. Introduction

The Bel tensor (the superenergy tensor of the gravitational field \[ B_T \]) was first introduced in [1] by exploring an analogy between the gravitational and the electromagnetic fields. A rank four tensor was obtained, an analogous of the energy-momentum tensor for the electromagnetic field but with the Riemann tensor of curvature instead of the electromagnetic field tensor

\[
B_{\mu \nu \lambda \rho} = \frac{1}{2} \left( R^{\mu \alpha \lambda \beta} R_{\alpha \beta}^{\nu \rho} + \ast R_{\alpha \beta}^{\nu \rho} \ast R^\lambda_\alpha \ast R^\rho_\beta \right) .
\]

(1)

where \( R_{\mu \nu \lambda \rho} \) is the Riemann curvature tensor and the various types of Hodge duals are

\[
\ast R_{\mu \nu \alpha \beta} = \frac{1}{2} \varepsilon_{\mu \nu \alpha \beta} R_{\lambda \rho}^\alpha , \quad R_{\mu \nu \lambda \rho} = \frac{1}{2} \varepsilon_{\mu \nu \alpha \beta} R_{\alpha \beta}^\lambda , \quad \ast R_{\mu \nu \alpha \beta} = \frac{1}{2} \varepsilon_{\mu \nu \alpha \beta} \varepsilon_{\lambda \rho \tau} R_{\lambda \rho \tau}^\alpha (\lambda \rho \tau) ,
\]

\( \varepsilon_{\mu \nu \lambda \rho} \) being the canonical volume element of the spacetime. The Bel tensor is traceless \( \nabla_\alpha B^\alpha_{\lambda \rho} = 0 \) and have the following symmetries

\[
\nabla_{\lambda \rho} B_{\mu \nu} = \nabla_{(\mu \nu)} B_{\lambda \rho} = \nabla_{\lambda \rho} B_{\mu \nu} .
\]

(2)

Its divergence is

\[
\nabla_\mu B^{\mu \nu \lambda \rho} = R^\nu_\alpha \lambda \beta J^{\alpha \beta \gamma} + R^{\nu \rho}_{\alpha \beta} J^{\lambda \beta \gamma} - \frac{1}{2} g^{\lambda \rho} R^\nu_\alpha \beta J^{\lambda \beta \gamma} - \frac{1}{2} g^{\rho \nu} R^\lambda_\sigma \gamma J^{\sigma \gamma} \]

where \( J^{\mu \nu \lambda \rho} \equiv \nabla_\mu R_{\nu \lambda \rho} - \nabla_\nu R_{\mu \lambda \rho} , \quad R_{\mu \nu} \) being the Ricci tensor. It is therefore obvious that the Bel tensor is locally conserved in empty or Einstein spacetimes.
In vacuum the Bel tensor is known as Bel-Robinson tensor and it was introduced for the first time in [4]. The Bel-Robinson tensor is defined as
\[
\mathcal{B}^{\mu_\nu \lambda_\rho} = R^{\mu_\alpha \lambda_\beta} R^{\nu_\beta \rho_\alpha} + *R^{\mu_\alpha \lambda_\beta} R^{\nu_\beta \rho_\alpha} \quad \text{(in vacuum)}
\]
and satisfies
\[
\mathcal{B}^{\mu_\nu \lambda_\rho} = \mathcal{B}^{\mu \nu \lambda_\rho}, \quad \mathcal{B}^{\mu \lambda_\rho} = 0.
\]
In vacuum and in Einstein spaces the Bel-Robinson is divergenceless
\[
\nabla_\mu \mathcal{B}^{\mu_\nu \lambda_\rho} = 0.
\]
A tensor with the properties (4) and (5) of the Bel-Robinson tensor can be constructed for any spacetime, irrespective of its emptiness character if in (3) the conformal Weyl tensor is used instead of the Riemann curvature tensor
\[
\mathcal{B}^{\mu_\nu \lambda_\rho} \equiv C^{\mu_\alpha \lambda_\beta} C^{\nu_\rho_\beta} + *C^{\mu_\alpha \lambda_\beta} *C^{\nu_\rho_\beta}.
\]
A study of the Bel-Robinson tensor and of its impact on the evolution of the Universe within the framework of the Bianchi type-I spacetime can be found in [5].

In the noticeable cases when Bel and Bel-Robinson tensors and generally the superenergy tensors of physical fields are divergenceless various conserved currents can be built in the presence of Killing vector fields [2, 6].

When we have interacting fields such as in Einstein-Klein-Gordon or Einstein-Maxwell theories, the superenergy currents associated with individual fields are not conserved and an interchange of some superenergy quantities take place between the gravitational and the non-gravitational fields in such a manner that a total, mixed current is conserved. For example, it was shown [2, 7] that for Einstein-Klein-Gordon theory a mixed conserved superenergy current can be constructed whenever there is also a Killing vector field
\[
\nabla_\mu \left( \mathcal{B}^{\mu_\nu \lambda_\rho} + S^{\mu_\nu \lambda_\rho} \xi_\nu \xi_\lambda \xi_\rho \right) = 0,
\]
where $S^{\mu_\nu \lambda_\rho}$ is the superenergy tensor of the scalar field. Moreover, this current reduces to the corresponding Bel conserved current in the absence of the scalar field and to the conserved superenergy current of the scalar field in a flat spacetime if the gravitational field is removed. Such conservation laws makes the superenergy tensors not only mathematically appealing, they opening perspectives in identifying their physical significance.

There are also situations when the conserved currents for individual fields exists under very general circumstances even if the divergenceless property does not hold. In [8] it was shown that independently conserved currents can be constructed from the Bel tensor when there is a hypersurface orthogonal Killing vector or when there are two commuting Killing vectors that acts orthogonally transitive on non-null surfaces. In the first case we have a current proportional with the Killing vector
\[
\nabla_\mu \left( \mathcal{B}^{\mu_\nu \lambda_\rho} \xi_\nu \xi_\lambda \xi_\rho \right) = \gamma \xi_\mu, \quad \nabla_\mu \left( \mathcal{B}^{\mu_\nu \lambda_\rho} \xi_\nu \xi_\lambda \xi_\rho \right) = 0,
\]
while in the second we have four currents lying in the 2-plane generated by the two Killing vectors
\[ T^\mu_{(\nu\lambda\rho)}\xi^\nu_i\xi^\lambda_j\xi^\rho_k = \alpha_{ijk}\xi^1_{\mu} + \beta_{ijk}\xi^2_{\mu}, \quad \nabla^\mu (T^\mu_{(\nu\lambda\rho)}\xi^\nu_i\xi^\lambda_j\xi^\rho_k) = 0. \] (9)

The case of a four-dimensional Einstein-Maxwell theory with a source-free electromagnetic field is discussed in [9, 10]. While in [9] it is shown that for electromagnetic fields inheriting the symmetry of the spacetime the Chevreton superenergy currents are individually conserved if there are hypersurface orthogonal Killing vector fields, in [10] it is treated the case when there is a two-parameter Abelian isometry group that acts orthogonally transitive on non-null surfaces and it is shown that the corresponding superenergy currents lies in the orbits of the group and are conserved.

As can be seen from (7), (8) and (9) a Killing vector field is required for constructing Bel type currents. There are many generalizations of Killing vector fields to higher rank tensors, the most notable being Stackel-Killing tensors, Killing-Yano tensors and f-symbols. In the next section we show that when a spacetime admits one or more of the above geometrical objects generalizing a Killing vector field, some conserved currents can be constructed from Bel or Bel-Robinson type tensors.

2. Conserved currents

We present three types of generalization of Killing vector fields - f-symbols, Killing-Yano and St"ackel-Killing tensors - and use them to construct Bel-type conserved currents.

2.1. f-symbols and conserved currents

f-symbols have been introduced in [11] where they naturally arise in investigation of extra supersymmetries in a pseudo-classical model for spinning particles. A study about their existence for a particular spacetime can be found in [12]. A rank two f-symbol is a tensor field \( f_{\mu\nu} \) with no particular symmetries satisfying
\[ \nabla_\lambda f_{\mu\nu} + \nabla_\nu f_{\mu\lambda} = 0. \] (10)
Eq. (10) is identical with the equation for Killing-Yano tensors, what makes the difference being the antisymmetry property not required for f-symbols. It follows that the divergence in the second index vanishes \( \nabla_\nu f_{\mu\nu} = 0 \) while for the divergence in the first index we have \( \nabla_\mu f_{\mu\nu} = -\partial_\nu f_{\mu\nu} \), i.e. the first index divergence vanishes if the trace of the f-symbol is a constant. In this particular case, since the metric tensor is covariantly constant and trivially satisfies equation (10), we can subtract the trace part from the f-symbol and the remaining trace free part is itself a f-symbol.

The symmetric part of a f-symbol \( S_{\mu\nu} = \frac{1}{2}(f_{\mu\nu} + f_{\nu\mu}) \) is a St"ackel-Killing tensor, i.e. it satisfies \( \nabla_\lambda S_{\mu\nu} = 0 \), while for the antisymmetric part \( A_{\mu\nu} = \frac{1}{2}(f_{\mu\nu} - f_{\nu\mu}) \) we have \( \nabla_\nu A_{\mu\lambda} + \nabla_\lambda A_{\mu\nu} = \nabla_\mu S_{\lambda\nu} \), meaning that the antisymmetric part satisfies itself the f-symbol equation (10) (i.e. it is a Killing-Yano tensor) if the symmetric part \( S_{\mu\nu} \) vanishes or is covariantly constant.
By analogy with the integrability equation
\[ \nabla_{\nu} \nabla_{\lambda} \xi_{\mu} = -R_{\nu\lambda\mu}^{\alpha} \xi_{\alpha} = 0 \] (11)
satisfied by any Killing vector field \( \xi_{\mu} \), a similar equation can be written down for an arbitrary \( f \)-symbol. For example, for a rank two \( f \)-symbol, if the defining equation
\[ (\nabla_{\lambda} f_{\nu\mu} + \nabla_{\nu} f_{\lambda\mu} = 0) \] is covariantly derived once again we obtain
\[ \nabla_{\lambda} \nabla_{\rho} f_{\mu\nu} + \nabla_{\lambda} \nabla_{\mu} f_{\rho\nu} = 0. \] (12)
After we rewrite twice the above equation with \( \lambda \rightarrow \rho \rightarrow \mu \rightarrow \lambda \) and add up the resulting equations (the last one with a minus sign) then make use of the Ricci ([\( \nabla_{\lambda}, \nabla_{\rho} \)]\( f_{\mu\nu} = -R_{\lambda\rho\mu}^{\alpha} f_{\alpha\nu} - R_{\lambda\rho\nu}^{\alpha} f_{\mu\alpha} \)) and algebraic Bianchi (\( R_{\mu}^{\alpha\beta\gamma} + R_{\mu}^{\alpha\gamma\beta} + R_{\mu}^{\beta\gamma\alpha} = 0 \)) identities we end up with an integrability condition for a \( f \)-symbol, analogous to the eq. (11) for the Killing vector fields
\[ 2\nabla_{\lambda} \nabla_{\rho} f_{\mu\nu} = 2R_{\lambda\rho\mu}^{\alpha} f_{\alpha\nu} + R_{\nu\lambda\mu}^{\alpha} f_{\rho\alpha} + R_{\nu\rho\mu}^{\alpha} f_{\lambda\alpha}. \] (13)
Observe that the sum of the last three terms is not over the circular permutation of \( \mu, \lambda \) and \( \rho \) indices and since \( f_{\mu\nu} \) have no particular symmetries the above equation can not be further compacted or simplified.

The first current we construct is from a rank-four tensor \( T_{\mu\nu\lambda\rho}^{BR} \) with the symmetry and divergence properties of the Bel-Robinson tensor in vacuum or Einstein spaces
\[ T_{\mu\nu\lambda\rho}^{BR} = T_{(\mu\nu\lambda\rho)}^{BR} \] (14)
\[ \nabla_{\mu} T_{\mu\nu\lambda\rho}^{BR} = 0 \] (15)
\[ T_{\alpha\mu\nu\lambda}^{BR} = 0 \] (16)
and a rank three \( f \)-symbol
\[ \nabla_{\rho} h_{\mu\nu\lambda} + \nabla_{\lambda} h_{\mu\nu\rho} = 0. \] (17)
(The intrinsic algebraic characterization the Bel-Robinson type tensors was obtained in [13], where the necessary and sufficient conditions for a rank four symmetric tensor to be a Bel-Robinson type tensor - i.e. to be the superenergy tensor of a tensor with the same algebraic symmetries as the Weyl tensor - are found. This may be seen as the first Rainich theory result for rank four tensors.)

Contracting the above tensors we obtain the vector field
\[ j^{\mu}(T, h) = T_{\mu\nu\lambda\rho}^{BR} h_{\nu\lambda\rho} \] (18)
whose divergence is
\[ \nabla_{\mu} j^{\mu}(T, h) = \nabla_{\mu} T_{\mu\nu\lambda\rho}^{BR} h_{\nu\lambda\rho} + T_{\mu\nu\lambda\rho}^{BR} \nabla_{\mu} h_{\nu\lambda\rho} = T_{\mu\nu\lambda\rho}^{BR} \nabla_{\mu} h_{\nu\lambda\rho}. \] (19)
Taking into account the defining equation (17), the above divergence becomes
\[ \nabla_{\mu} j^{\mu}(T, h) = -T_{\mu\nu\lambda\rho}^{BR} \nabla_{\rho} h_{\nu\lambda\mu} = -T_{\mu\nu\lambda\rho}^{BR} \nabla_{\rho} h_{\nu\lambda\mu} \] (20)
the last equality being a consequence of the total symmetry property \( T_{\mu
u\lambda\rho}^{\alpha\beta} = T_{(\mu\nu\lambda\rho)}^{\alpha\beta} \). What it was obtained is \( \nabla_\mu j^\mu(T, h) = -\nabla_\mu T_{\mu
u\alpha\beta}^{\alpha\beta} f_{\alpha\beta} \), which means that the current \( j^\mu(T, h) \) defined in (18) is conserved. The Gauss theorem for vector fields can be applied on a covariantly conserved current since from a divergence-free vector field \( \nabla_\mu j^\mu(T, h) = 0 \) (21) we can construct the following current

\[
J^\mu(T, h) = (-g)^{1/2} j^\mu(T, h)
\]

whose ordinary divergence vanishes

\[
\partial_\mu J^\mu(T, h) = 0.
\]

Eq. (23) can be regarded as a conservation law for a fluid whose density is \( (-g)^{1/2} j^0(T, h) \) and whose motion is encapsulated in \( (-g)^{1/2} j^i(T, h), i = 1, 2, 3 \). The Gauss theorem can then be used to obtain an integral form and if there is no flow through the boundary of the integration domain we have

\[
\int (-g)^{1/2} j^0(T, h) d^3x = ct.
\]

Contracting the rank four tensor \( T_{\mu
u\alpha\beta}^{\alpha\beta} f_{\alpha\beta} \mu \) with the symmetry, divergence and trace properties of the Bel-Robinson tensor from eq. (14), (15) and (16) with a rank two f-symbol \( f_{\mu\nu} \) we can obtain conserved currents, but only if the spacetime admits symmetries generated by Killing or conformal Killing vector fields. Thus we start with the rank four tensor \( T_{\mu
u\alpha\beta}^{\alpha\beta} f_{\alpha\beta} \mu \) with the above mentioned properties and a rank two f-symbol \( f_{\mu\nu} \). Because of the total symmetry properties of the Bel-Robinson tensor there are only one independent contraction with the f-symbol \( f_{\mu\nu} \) namely

\[
T_{\mu
u}^{\alpha\beta} = T_{\mu\nu\alpha\beta}^{\alpha\beta} f_{\alpha\beta}
\]

which is evidently symmetric and easy to see that is also divergence- and trace-free

\[
T_{\mu\nu} = T_{\nu\mu}, \quad \nabla_\mu T_{\mu\nu} = 0, \quad T_{\mu\nu} = 0.
\]

Even if the property \( \nabla_\mu T_{\mu\nu} = 0 \) from (26) represents a local conservation law, an integral form cannot be constructed via Gauss theorem since the integrand would be a vector field and we cannot add vectors at different points when we deal with curved spacetimes. However, if the spacetime admit isometries (generated by Killing vector fields) or conformal symmetries (generated by conformal Killing vector fields) we can go further and construct a divergenceless vector field for which the Gauss theorem can be applied to obtain an integral conservation law.

Thus if \( \xi_\mu \) is a Killing vector field, than the current \( j^\mu(T, f, \xi) \) given by

\[
j^\mu(T, f, \xi) = T_{\mu\nu}^{\alpha\beta} \xi_\nu = T_{\mu\nu\alpha\beta}^{\alpha\beta} f_{\alpha\beta} \xi_\nu
\]
is covariantly conserved
\[ \nabla_\mu j^\mu (T, f, \xi) = 0. \] (28)

Demonstration is simple, deriving from symmetry and divergence property of \( T^{\mu \nu} \) and also taking into account the Killing vectors equation.

Since \( T^{\mu \nu} = T^{R \mu \nu \alpha \beta} f_{\alpha \beta} \) is also traceless (in four dimensions the Bel-Robinson tensor is traceless), a conserved current can be constructed as
\[ j^\mu (T, f, \zeta) = T^{\mu \nu} \zeta_\nu = T^{R \mu \nu \alpha \beta} f_{\alpha \beta} \zeta_\nu \] (29)

where \( \zeta_\mu \) satisfies
\[ \nabla_\nu \zeta_\mu + \nabla_\mu \zeta_\nu = \chi g_{\mu \nu} \] (30)
i.e. \( \zeta_\mu \) is a conformal Killing vector field. Since \( \nabla_\mu T^{\mu \nu} = 0 \) and \( T^{\mu \nu} = T^{\nu \mu} \) we have for the divergence of (29)
\[ \nabla_\mu j^\mu (T, f, \zeta) = T^{\mu \nu} \nabla_\nu \zeta_\mu = T^{\mu \nu} \nabla_\nu \zeta_\mu. \] (31)

Taken then into account the defining equation of conformal Killing vector fields (30) we have
\[ \nabla_\mu j^\mu (T, f, \zeta) = T^{\mu \nu} \zeta_\nu - T^{\mu \nu} \nabla_\nu \zeta_\mu \] (32)
which because of the traceless property \( T^{\mu \nu} g_{\mu \nu} = 0 \) lead to
\[ \nabla_\mu j^\mu (T, f, \zeta) = -T^{\mu \nu} \nabla_\nu \zeta_\mu = -\nabla_\mu j^\mu (T, f, \zeta) = 0. \] (33)

From a tensor \( T^{\mu \nu \lambda \rho} \) with the symmetry and divergence properties of the Bel tensor in vacuum or Einstein spaces
\[ T^{\mu \nu \lambda \rho} = T^{R (\mu \nu \lambda \rho)} = T^{R \lambda \mu \nu \rho} \] (34)
\[ \nabla_\mu T^{\mu \nu \lambda \rho} = 0 \] (35)
\[ T^{R \alpha \mu \nu \lambda} = 0 \] (36)
and a rank three f-symbol \( h_{\mu \nu \lambda} \) we can form a locally conserved current \( j^\mu (T, h) \).

Because of the symmetries of the Bel tensor there are a unique independent contraction in three indices between \( T^{R \mu \nu \lambda \rho} \) and \( h_{\mu \nu \lambda} \), namely
\[ j^\mu (T, h) = T^{\mu \nu \lambda \rho} h_{\nu \lambda \rho} \] (37)
whose divergence is
\[ \nabla_\mu j^\mu (T, h) = \left( \nabla_\mu T^{R \mu \nu \lambda \rho} \right) h_{\lambda \rho \nu} + T^{R \mu \nu \lambda \rho} \nabla_\mu h_{\lambda \rho \nu} \] (38)
Because of (35) and using the defining equation (17) of a rank three f-symbol we have

\[ \nabla_{\mu} j^\mu(T, h) = -B^\mu_{\nu\lambda\rho} \nabla_{\nu} h_{\lambda\rho\mu} = -T^\mu_{\nu\lambda\rho} \nabla_{\nu} h_{\lambda\rho\mu} = -T^\mu_{\nu\lambda\rho} \nabla_{\mu} h_{\lambda\rho\nu} \] (39)

where we make use only of the symmetry \( B^\mu_{\nu\lambda\rho} = B^\mu_{(\nu\lambda\rho)} \) and then rename the '\( \mu \)' and '\( \nu \)' indices. We have obtain that \( \nabla_{\mu} j^\mu(T, h) = -\nabla_{\mu} j^\mu(T, h) \) which means the \( j^\mu(T, h) \) current is divergenceless. We don’t make use of the symmetries of the Bel tensor related with the pairs of indices interchange.

Because of the symmetries of the Bel tensor there are only two independent contractions between the Bel-type tensor and a rank two f-symbol \( f_{\mu\nu} \)

\[ T^{(1)\mu\nu} = B^\mu_{\nu\lambda\rho} f_{\lambda\rho} \] (40)
\[ T^{(2)\mu\nu} = B^\mu_{\lambda\nu\rho} f_{\lambda\rho} \] (41)

The first contraction \( T^{(1)\mu\nu} \) is obvious symmetric but has nonzero divergence while the for the second contraction - which is not symmetric - we have \( \nabla_\mu T^{(2)\mu\nu} = 0 \) and \( \nabla_\nu T^{(2)\mu\nu} \neq 0 \). Both of them are useless in forming conserved current through contractions with Killing or conformal Killing vector fields.

**Ex: The Robertson-Walker spacetime**

We do not write down the currents but only mention that a metric which have all the ingredients to construct a current like (27) is the Robertson-Walker metric which in spherical coordinates have the expression

\[ ds^2 = dt^2 - a^2(t) \left[ \frac{1}{1 - k r^2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \] (42)

and when \( k \neq 0 \) and \( a(t) = ct \) admits three rank two f-symbols [14]

\[ f^{(1)}_{tt} = 1 \] (43)
\[ f^{(2)}_{rt} = \frac{1}{\sqrt{1 - k r^2}} \cos \theta \]
\[ f^{(2)}_{\theta t} = -r \sqrt{1 - k r^2} \sin \theta \] (44)
\[ f^{(3)}_{\phi t} = r^2 \sin^2 \theta \] (45)
and seven Killing vector fields

\[ \xi^{(1)} = \cos \theta \frac{1}{\sqrt{1 - kr^2}} \, dr - r \sin \theta \sqrt{1 - kr^2} \, d\theta \] (46)

\[ \xi^{(2)} = \sin \theta \cos \phi \frac{1}{\sqrt{1 - kr^2}} \, dr + r \cos \theta \cos \phi \sqrt{1 - kr^2} \, d\theta - r \sin \theta \sin \phi \sqrt{1 - kr^2} \, d\phi \] (47)

\[ \xi^{(3)} = \sin \theta \sin \phi \frac{1}{\sqrt{1 - kr^2}} \, dr + r \cos \theta \sin \phi \sqrt{1 - kr^2} \, d\theta + r \sin \theta \cos \phi \sqrt{1 - kr^2} \, d\phi \] (48)

\[ \xi^{(4)} = r^2 \cos \phi \, d\theta - r^2 \sin \theta \cos \theta \sin \phi \, d\phi \] (49)

\[ \xi^{(5)} = r^2 \sin \phi \, d\theta + r^2 \sin \theta \cos \phi \, d\phi \] (50)

\[ \xi^{(6)} = r^2 \sin^2 \theta \, d\phi \] (51)

\[ \xi^{(7)} = dt \] (52)

The Robertson-Walker metric is conformally flat so the Bel-Robinson tensor \( T_{\mu \nu \lambda \rho} \) being identically zero cannot be used to construct conserved currents. However this metric have the special feature that the Bel tensor \( \mathcal{B}_{\mu \nu \lambda \rho} \) is divergenceless in every index, this property making it a potential candidate for the construction of the conserved currents by contracting it with the rank two f-symbols and then with the killing vector fields. Since we need a totally symmetric and divergenceless tensor and since an operation of total symmetrization does not destroy the divergenceless properties of the Bel tensor, what we finally use to construct the conserved currents is the totally symmetrized Bel tensor \( \mathcal{B}^{(\mu \nu \lambda \rho)} \) and the conserved currents can be constructed as

\[ j^{(ij)}_{\mu} = \mathcal{B}^{(\mu \nu \lambda \rho)} f^{(i)}_{\lambda \rho} \xi^{(j)}_{\nu} \] (53)

**Ex: The Minkowski spacetime**

In Minkowski spacetime the superenergy tensor associated with a massive or mass-less scalar field \( \Phi \) given by [2]

\[ S_{\mu \nu \lambda \rho} = \nabla_\mu \nabla_\lambda \Phi \nabla_\rho \nabla_\nu \Phi + \nabla_\mu \nabla_\rho \Phi \nabla_\nu \nabla_\lambda \Phi - g_{\mu \nu} \nabla_\lambda \nabla^\alpha \Phi \nabla_\rho \nabla_\sigma \Phi - g_{\lambda \rho} \nabla_\mu \nabla^\alpha \Phi \nabla_\nu \nabla_\sigma \Phi + \frac{1}{2} g_{\mu \nu} g_{\lambda \rho} \nabla_\alpha \nabla_\beta \Phi \nabla^\alpha \nabla^\beta \Phi \] (54)

have the following symmetries

\[ S_{\mu \nu \lambda \rho} = S_{(\mu \nu)(\lambda \rho)} = S_{\lambda \rho \mu \nu} \] (55)

and is divergenceless

\[ \nabla_\alpha S^\alpha_{\nu \lambda \rho} = 0. \] (56)
It is easy to see that Minkowski spacetime admits the rank two f-symbols

\[ f_{\mu\nu} = A_{\mu\nu\alpha} x^\alpha + B_{\mu\nu} \]  

(57)

where \( A_{\mu\nu\lambda} \) and \( B_{\mu\nu} \) are arbitrary constant matrices except the antisymmetry condition \( A_{\mu\nu\lambda} = A_{\mu\nu[\lambda} \) and the rank three f-symbols

\[ h_{\mu\nu\lambda} = A_{\mu\nu\lambda\alpha} x^\alpha + B_{\mu\nu\lambda} \]  

(58)

where \( A_{\mu\nu\lambda\rho} \) and \( B_{\mu\nu\lambda} \) are arbitrary constant matrices except the condition \( A_{\mu\nu\lambda\rho} = A_{\mu\nu[\lambda\rho]} \), and also admit the maximum number of ten Killing vector fields. We can thus construct conserved currents using either rank three f-symbols as in (37)

\[ j^\mu(S, h) = S^{\mu\nu\lambda\rho}[A_{\nu\lambda\rho\alpha} x^\alpha + B_{\nu\lambda\rho}] \]  

(59)

either rank two f-symbols as in (27) (with the difference that we use \( S^{(\mu\nu\lambda\rho)} \) instead of \( S^{\mu\nu\lambda\rho} \) since this operation preserves divergence free property)

\[ j^\mu(S, f, \xi) = S^{(\mu\nu\lambda\rho)}[A_{\lambda\rho\alpha} x^\alpha + B_{\lambda\rho}] \xi_{\nu} \]  

(60)

where \( \xi_{\nu} \) is one of the ten Killing vector fields in Minkowski spacetime.

### 2.2. Killing tensors and conserved Bel-type currents

The generalization of Killing vectors to symmetric tensor fields lead to the concept of Stäckel-Killing tensor also known simply as Killing tensors. A rank \( r \) tensor \( K_{\mu_1...\mu_r} \) is a Stäckel-Killing tensor if it is totally symmetric

\[ K_{\mu_1...\mu_r} = K_{\mu_1...\mu_r} \]

and satisfy

\[ \nabla_{(\nu} K_{\mu_1...\mu_r)} = 0. \]

Rank two Stäckel-Killing tensors lead to geodesic constants of motion quadratic in momenta \[15\] and generally higher rank Stäckel-Killing tensors lead to a corresponding degree polynomial constants of motion. The existence of Stäckel-Killing tensor fields lead to the integrability of geodesic motion and also to the separability of the Hamilton-Jacobi \[16\] and the Klein-Gordon \[17\] equations.

There is a dual relation between the geometry of a space admitting a rank two Stäckel-Killing tensor field and the geometry of a space whose metric is that Killing tensor \[19\ 20\ 21\ 22\].

Stäckel-Killing tensors can be also used in conjunction with Bel and Bel-Robinson type tensors in order to obtain conserved currents. Thus, instead of a rank three f-symbol a rank three Killing tensor \( L_{\mu\nu\lambda} \) can be used in \[18\] to obtain the current

\[ j^\mu(B_R, T, L) = B_{R\mu\nu\lambda\rho} T^{\mu\nu\lambda\rho} L_{\nu\lambda\rho} \]  

(61)

whose divergence is

\[ \nabla_{\mu} j^\mu(B_R, T, L) = B_{R\mu\nu\lambda\rho} \nabla_{\mu} L_{\nu\lambda\rho}. \]  

(62)
Since the Killing tensors equation $\nabla (\mu L_{\nu \lambda \rho}) = 0$ implies $\nabla (\mu L_{\nu \lambda \rho}) = 0$, where the curly brackets stands for summation over the cyclical permutations of the enclosed indices, we have

$$\nabla \mu^{BR}(T, L) = T^{\mu \nu \lambda \rho}(-\nabla _\rho L_{\mu \nu \lambda} - \nabla _\lambda L_{\rho \mu \nu} - \nabla _\nu L_{\lambda \rho \mu}) = -3 T^{\mu \nu \lambda \rho} \nabla \mu L_{\nu \lambda \rho}. \quad (63)$$

For the last equality we use the fact that $T^{\mu \nu \lambda \rho}$ being totally symmetric all its contractions with an arbitrary tensor are the same. From eq. (62) and (63) we evidently have

$$\nabla \mu^{BR}(T, h) = 0.$$  

Again, as for the above presented rank three case, instead of or rank two f-symbol a rank two Killing tensor can be used to produce a conserved current via a subsequent contraction with a Killing or conformal Killing vector field. Thus the contraction between a Bel-Robinson type tensor $T^{\mu \nu \alpha \beta}$ and a rank two Killing tensor $P_{\mu \nu}$ lead to

$$T^{\mu \nu} = T^{\mu \nu \alpha \beta} K_{\alpha \beta} \quad (64)$$

which is evidently symmetric and trace free. It’s divergence is

$$\nabla \mu T^{\mu \nu} = T^{\mu \nu \alpha \beta} \nabla \mu K_{\alpha \beta} \quad (65)$$

The defining properties $K_{\mu \nu} = K_{(\mu \nu)}$ and $\nabla (\lambda K_{\mu \nu}) = 0$ of the Killing tensors leads to $\nabla (\lambda K_{\mu \nu}) = 0$ which allows us to write for the above equation

$$\nabla \mu T^{\mu \nu} = -T^{\mu \nu \alpha \beta} \nabla \beta K_{\mu \alpha} - T^{\mu \nu \alpha \beta} \nabla \alpha K_{\beta \mu} = -2 T^{\mu \nu \alpha \beta} \nabla \mu K_{\alpha \beta} \quad (66)$$

which compared with (65) means that $\nabla \mu T^{\mu \nu} = 0$. Having a symmetric, tracefree and divergenceless rank two tensor field, a conserved current can be constructed as in the case of f-symbols by contractions with Killing or even conformal Killing vector fields.

Also, because of the tracefree property of the Bel-Robinson tensor in four dimensions a rank two conformal Killing tensor, i.e. a rank two symmetric tensor $P_{\alpha \beta}$ satisfying

$$\nabla \mu P_{\alpha \beta} + \nabla \alpha P_{\beta \mu} + \nabla \beta P_{\mu \alpha} = \chi_{\alpha} g_{\beta \mu} + \chi_{\beta} g_{\mu \alpha} + \chi_{\mu} g_{\alpha \beta} \quad (67)$$

can be use instead of a Stäckel-Killing one. We have for the divergence of the symmetric tensor $T^{\mu \nu} = T^{\mu \nu \alpha \beta} P_{\alpha \beta}$

$$\nabla \mu T^{\mu \nu} = T^{\mu \nu \alpha \beta} \nabla \mu P_{\alpha \beta} \quad (68)$$

Since $\nabla \mu P_{\alpha \beta} = -\nabla \alpha P_{\beta \mu} - \nabla \beta P_{\mu \alpha} + \chi_{(\alpha} g_{\beta \mu)}$ and because of the total symmetry of $T^{\mu \nu \lambda \rho}$ we have

$$\nabla \mu T^{\mu \nu} = -2 T^{\mu \nu \alpha \beta} \nabla \mu P_{\alpha \beta} + T^{\mu \nu \alpha \beta} \chi_{(\alpha} g_{\beta \mu)} \quad (69)$$
As the last term $B^\mu\nu\alpha\beta \chi_{\alpha\beta\mu}$ is null because implies various traces of $T^\mu\nu\alpha\beta$ which are null by definition, what remains is

$$\nabla_\mu T^{\mu\nu} = -2 B^R_{\mu\nu\alpha\beta} \nabla_\mu P_{\alpha\beta}$$  \hspace{1cm} (70)

From eq. (68) and (70) we evidently have $\nabla_\mu T^{\mu\nu} = 0$ q.e.d.

2.3. Killing-Yano tensors and conserved currents

The last generalization of Killing vectors we consider refers to Killing-Yano tensors [23]. A rank two Killing-Yano tensor is an antisymmetric tensor $Y^{\mu\nu} = Y^{\nu\mu}$ which satisfies

$$\nabla_\lambda Y^{\mu\nu} + \nabla_\nu Y^{\mu\lambda} = 0.$$  \hspace{1cm} (71)

Killing-Yano tensors have major impact in physics. They play an important role in the existence of the geodesic constants of motion [17] in curved spacetimes. There is a deep connection between Killing-Yano tensors, supersymmetries in pseudo-classical spinning particle models [11, 24, 25] (models which besides usual spacetimes coordinates includes a number of anticommuting ones to describe the spin degrees of freedom) and Dirac-type operators on curved spacetimes [26, 27, 28, 29]. Killing-Yano tensors are directly related with the absence of gravitational quantum anomalies [30, 27, 31] (conservation laws valid at classical level ceasing to be true at quantum level). Interesting applications of Killing-Yano tensors like their relations with Nambu tensors and superintegrability can be found in [32, 33].

There are two conserved currents that can be obtained from a Killing-Yano tensor $Y^{\mu\nu}$ and a divergenceless Bel-Robinson type tensor $T^R_{\mu\nu\lambda\rho}$. The first one have the expression

$$j^{(1)}_\mu(T, Y) = R^R_{\mu\alpha_1\alpha_2\alpha_3} \prod_{i=1}^{3} \varepsilon^{\alpha_1\beta\gamma\delta} \nabla_\delta Y^{\beta\gamma}$$  \hspace{1cm} (72)

while for the second we have

$$j^{(2)}_\mu(T, Y) = R^R_{\mu\alpha_1\alpha_2\alpha_3} \prod_{i=1}^{3} Y^{\alpha_1}_\gamma Y^{\alpha_2}_\beta \varepsilon^{\beta\delta\sigma\tau} \nabla_\tau Y^{\delta\sigma}.$$  \hspace{1cm} (73)

Neither $j^{(1)}_\mu(T, Y)$ nor $j^{(2)}_\mu(T, Y)$ are new conserved currents since if a spacetime admit a Killing-Yano tensor $Y^{\mu\nu}$ than

$$v^\mu = \varepsilon^{\mu\alpha\beta\gamma} \nabla_\gamma Y_{\alpha\beta}$$  \hspace{1cm} (74)

and

$$w^\mu = Y^\mu_\alpha Y_{\beta}^\alpha v^\beta$$  \hspace{1cm} (75)

are both Killing vector fields [34].
It is therefore evident that instead of the product \( \prod_{i=1}^{3} \varepsilon^{\alpha_i \beta \gamma \delta} \nabla_\delta Y_{\beta \gamma} \) in (72) we can use an expression like \((\varepsilon^{\alpha_1 \beta \gamma \delta} \nabla_\delta Y_{\beta \gamma})(\varepsilon^{\alpha_2 \beta \gamma \delta} \nabla_\delta Y_{\beta \gamma})(\varepsilon^{\alpha_3 \beta \gamma \delta} \nabla_\delta Y_{\beta \gamma})\) since each parenthesis represents in fact a different Killing vector fields. A similar statement stands true for eq. (73). Also, in (72) and (73) a symmetrized Bel type tensor \( T^{(\mu \nu \lambda \rho)} \) can be used instead of \( T^{\mu \nu \lambda \rho} \) because a Bel type tensor have only the partial symmetry \( T^{\mu \nu \lambda \rho} = T^{\lambda \rho (\mu \nu)} \) while a symmetry in at least three indices is required to obtain a conserved current.

3. Conclusions

We use three types of generalizations of Killing vector fields - a symmetrical one which lead to Stäckel-Killing tensors, an antisymmetrical one leading to the Killing-Yano tensors and a generalization that do not required a definite symmetry leading to f-symbols - to construct conserved Bel type currents by suitable contractions with tensors having the symmetry and divergence properties of the Bel and Bel-Robinson tensors in Einstein spacetimes. Such a tensor could be for example the basic superenergy tensor of the scalar field in the Minkowski spacetime. When the rank four tensor is also trace free a conformal Killing tensor can be use instead of a Stäckel-Killing one. The conserved currents derived by using Killing-Yano tensors are not new, they could be obtained directly from some Killing vector fields. As examples we mention Robertson-Walker and Minkowski spacetimes. We also found the rank two and three f-symbols for the Minkowski spacetime and write down an integrability condition for rank two f-symbols.

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