Nonparametric Inference for Location Parameters of Veronese Whitney means and antimeans on Kendall Shape Spaces

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June 20, 2018

1 Introduction

To date, Object Data Analysis is the most inclusive type of data analysis, as far as metric sample spaces are concerned. Examples of object spaces are axial spaces (see [2], [7]), spaces of directions (see [21]), Stiefel manifolds (see Hendriks and Landsman(1998)[9]). It extends multivariate data analysis, landmark based shape analysis, and in the infinite dimensional case, it also extends functional data analysis (see Patrangenaru and Ellingson (2015)[12]).

Fréchet (1948)[8] noticed that for higher complexity data, such as the shape of a random contour, numbers or vectors do not provide a meaningful representation. To investigate these kind of data he introduced the notion of elements, which are nowadays called objects. In that paper, he mentioned that object can represent for example “the shape of an egg randomly taken from a basket of eggs”. Fréchet’s visionary concepts, were nevertheless hard to handle computationally during his time. It took many decades, until such data became the bread and butter of modern data analysis. In particular various typed shapes of configurations extracted from digital images were represented as points on projective shape spaces (see [15], [16]), on affine shape spaces(see [17], [20]), or on Kendall shape spaces (see [10], [6]). To analyze the mean and variance of the random object $X$ on a smooth object space $M$ with a metric $\rho$, Fréchet defined what we call the Fréchet function given by

\begin{equation}
\mathcal{F}(p) = \mathbb{E}(\rho^2(p, x)),
\end{equation}
and in $(\mathcal{M}, \rho)$ is complete, the minimizers of the Fréchet function form the Fréchet mean set. If $\rho = \rho_g$ is the geodesic distance associated with a Riemannian structure $g$ on $\mathcal{M}$, there are no necessary and sufficient conditions for the existence of a unique minimizer of $\mathcal{F}$ in (1.1) (see e.g. Patrangenaru and Ellingson (2015)\cite{12}, ch.4), therefore in general, with the possible exception of complete flat Riemannian manifolds, it is preferable to consider only the case when $\rho$ is the “chord” distance on $\mathcal{M}$ induced by the Euclidean distance in $\mathbb{R}^N$ via an embedding $j : \mathcal{M} \to \mathbb{R}^N$. the Fréchet function becomes

\begin{equation}
\mathcal{F}(p) = \int_{\mathcal{M}} \| j(x) - j(p) \|^2_0 Q(dx),
\end{equation}

where $Q = P_X$ is the probability measure on $\mathcal{M}$, associated with $X$. Also, given $X_1, \ldots, X_n$ i.i.d.r.v.'s from $Q$, their extrinsic sample mean (set) is the extrinsic mean (set) of the empirical distribution $\hat{Q}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$, (see e.g. Patrangenaru and Ellingson(2015)\cite{12}, chapter 4).

In this paper we will assume in addition that $(\mathcal{M}, \rho)$ is a compact metric space, therefore the Fréchet function is bounded, and its extreme values are attained at a set of points on $\mathcal{M}$. It makes sense to consider a location parameter for $X$, the extrinsic antimean set, which is the set of maximizers of the Fréchet function in (1.2) (see e.g. Patrangenaru, Guo and Yao (2016)\cite{18}). In case the extrinsic antimean set has one point only, that point is called extrinsic antimean of $X$, and is labeled $\alpha_{\mu_j,E}(Q)$, or simply $\alpha_{\mu,E}$, when $j$ and $Q$ are known.

In this paper after a brief revision of VW-means in Section\ref{sec:VWmeans} which are extrinsic means on real and complex projective spaces, relative to the Veronese-Whitney embeddings, we give two examples of sample VW-means computations on Kendall shape spaces. In Section\ref{sec:antimeans} we derive large sample and pivotal nonparametric bootstrap confidence regions for VW-antimeans, using VW-anti-covariance matrices, and their sample counterparts.

## 2 VW-antimeans on $\mathbb{C}P^q$

Planar direct similarity shapes of $k$-ads (labeled set of $k$ points) in the Euclidean plane, at least two of which are distinct) were introduced by D. G. Kendall (1984)\cite{10}, who showed that the can be represented as points on a complex projective space $\mathbb{C}P^q$, where $q = k - 2$. Since 1992, due to Kent(1992)\cite{11}, shape analysts are using the so called Veronese-Whitney (VW) embedding of $\mathbb{C}P^q$ in the space of $(q - 1) \times (q - 1)$ self adjoint complex matrices. The
corresponding the Fréchet mean set is called the VW-mean set (See Patrangenaru and Ellingson (2015), ch. 3 [12]),
and if there is a unique point in the VW-mean set of \( X \), this point is called the VW-mean of \( X \), and is labeled \( \mu_{E}(X) \) or simply \( \mu_{E} \).

For the VW-embedding of the complex projective space, we have the following theorem for VW-antimeans

**THEOREM 2.1.** Let \( Q \) be a probability distribution on \( \mathbb{C}P^{k-2} \) and let \( \{ [Z_r], \| Z_r \| = 1 \}_{r=1}^{n} \) be a random sample from \( Q \). (a) \( Q \) is VW-nonfocal iff \( \lambda \), the smallest eigenvalue of \( E[Z_1Z_1^*] \) is simple and in this case \( \alpha_{\mu_J, E} = [m] \), where \( m \) is an eigenvector of \( E[Z_1Z_1^*] \) corresponding to \( \lambda \), with \( \| m \| = 1 \). (b) The sample VW antimean \( \alpha_{X_E} = [m] \), where \( m \) is an eigenvector of norm 1 of \( J = \frac{1}{n} \sum_{i=1}^{n} Z_i Z_i^* \), \( \| Z_i \| = 1 \), \( i = 1, \ldots, n \), corresponding to the smallest eigenvalue of \( J \).

2.1 Simulation

We ran a simulation using an example of the VW-embedding of a complex projective space (a Kendall shape space) to compare VW means and VW antimeans for a data set of landmarks configuration. In this context we ran a non-parametric bootstrap for sample VW-means and sample VW-antimeans. The objective of our simulations was to see if the bootstrap distributions of the sample VW means (respectively sample VW-antimeans) is concentrated or not. For this simulations, the data represents coordinates of \( k = 11 \) landmarks, and it has \( N = 100 \) observations. The data are displayed in figure 1. Note that the corresponding shape variable is valued in \( \mathbb{C}P^{9} \) (real dimension = 18).

The figure 2 is a representative of the vw sample mean of the coordinates of landmarks of the mean shape. It is easy to note that the pattern in figure 1 is close to the extrinsic mean in figure 2 after a rotation, translation and scaling. Here closeness is in the sense of small distance relative to the diameter of the object space. We computed the nonpivotal bootstrap distribution of the sample VW means using MATLAB, that we ran for 500 random re-samples. An icon of the spherical representation of the bootstrap distribution of the sample VW means is displayed in figure 2. Note that the sample VW mean bootstrap distribution is very concentrated around the sample VW mean.

As for the sample VW-antimean shape, its representative is shown in figure 4. The sample VW-antimean is far from the given sample, since according to the general definitions afore mentioned, on average, the square chord distance
between the sample VW antimean and the sample observations in the ambient space is maximized. The relative location of the landmarks in the icon of the sample VW antimean shape looks also different when compared with the original landmarks configuration, after the registration process. We also computed the nonpivotal bootstrap sample VW-antimeans distribution using MATLAB, that we ran again on 500 random resamples. An affine representation of the spherical representation of the bootstrap distribution of the sample VW antimeans is displayed in figure 5. Note that the bootstrap distribution of the bootstrap sample VW anti-means is fairly wide spread around the sample VW
From Theorem 2.1 we know that the sample VW-antimean is represented by an eigenvector of norm 1 of $J = \frac{1}{n} \sum_{i=1}^{n} z_i z_i^*$, $\| z_i \| = 1$, $i = 1, \ldots, n$, corresponding to the smallest eigenvalue of $J$, where $[z_i] \in \mathbb{C}P^9$ are obtained by applying the submatrix of the last 10 rows of the Helmert matrix (see Mardia et al. [14], p. 461) to the centered normalized data point $x_i + y_i \in \mathbb{C}^{11}$. The smallest eigenvalue of $J$ is very close to zero, since data is fairly concentrated, explaining the pattern in figure 5.

2.2 Application

We would be interested to find out whether the bootstrap distribution of the sample VW antimeans is also spread out for real data, such as landmark data extracted from medical imaging outputs. Our data consists of midface positions of a group of eighth landmarks on the skulls of eight and fourteen year-old North American children. The data used is the so called University School data. The data set represents coordinates of anatomical landmarks, whose names and position on the skull are given in Bookstein ([1997][5]). The registered coordinates are displayed in figure 6. The shape variable is valued in a planar shape space $\mathbb{C}P^6$ (real dimension = 12).

In figure 7 is displayed an icon of the sample VW mean of the landmarks coordinates. As with simulated data,
it is easy to note that after a rotation, translation and scaling of the Helmertized spherical coordinates, the landmark configurations in figures 6 and 7 look very close to the sample VW mean shape configuration.

We computed the bootstrap distribution of the sample VW mean shapes using MATLAB, that we ran for 500 random resamples. An icon of the Helmertized spherical representation of the bootstrap distribution of the sample
VW means is displayed in figure 8. Note that the bootstrap distribution of the sample VW means is very concentrated around the sample VW mean.

As for the sample VW antimean, its representative is shown in figure 9. The sample VW antimean is on average far from the data, since the squared chord distance between the sample VW antimean and the data in the ambient spaces is maximized. The relative location of the landmarks in the sample antimean icon looks very different from the one in the original shape data.

We also computed the bootstrap distribution using MATLAB, that we ran for 500 random resamples. A Helmer-tized spherical representation of the bootstrap distribution of the sample VW antimeans is displayed in figure 10. Here again, the bootstrap distribution of the sample VW mean is fairly spread out at each landmark location.

3 Extrinsic Anti-Covariance Matrices and confidence regions for VW Antimeans

In this section we will discuss the asymptotic distribution of sample antimeans in axial data analysis and in planar shape analysis, after a review of a Central Limit Theorem for extrinsic sample antimeans.
3.1 Central Limit Theorem for Extrinsic Sample Antimeans

In preparation, we are using the large sample distribution for extrinsic sample antimeans given in Patrangenaru et al (2016 [18]).

Assume \( j \) is an embedding of a \( d \)-dimensional manifold \( \mathcal{M} \) such that \( j(\mathcal{M}) \) is closed in \( \mathbb{R}^k \), and \( Q = P_X \) is an \( \alpha j \)-nonfocal probability measure on \( \mathcal{M} \) such that \( j(Q) \) has finite moments of order 2. Let \( \mu \) and \( \Sigma \) be the mean and covariance matrix of \( j(Q) \) regarded as a probability measure on \( \mathbb{R}^k \). Let \( \mathcal{F} \) be the set of \( \alpha j \)-focal points of \( j(\mathcal{M}) \), and let \( P_{F,j} : \mathcal{F}^c \to j(\mathcal{M}) \) be the projection on \( j(\mathcal{M}) \). \( P_{F,j} \) is differentiable at \( \mu \) and has the differentiability class of \( j(\mathcal{M}) \) around any \( \alpha j \) nonfocal point.

A local frame field \( x \to (f_1(x), \ldots, f_p(x)) \) on an open subset of \( \mathcal{M} \) such that for each \( x \in \mathcal{M} \), \((d_x j(f_1(x)), \ldots, d_x j(f_d(x)))\) are orthonormal vectors in \( \mathbb{R}^k \). A local frame field \( p \to (e_1(p), \ldots, e_k(p)) \), defined on an open neighborhood \( U \subseteq \mathbb{R}^k \) is adapted to the embedding \( j \) if it is an orthonormal frame field and \( \forall x \in j^{-1}(U), e_r(j(x)) = d_x j(f_r(x)), r = 1, \ldots, d. \)

Let \( e_1, \ldots, e_k \) be the canonical basis of \( \mathbb{R}^k \) and assume \((e_1(p), \ldots, e_k(p))\) is an adapted frame field around \( P_{F,j}(\mu) = j(\alpha \mu_E) \). Then \( d_\mu P_{F,j}(e_b) \in TP_{F,j(\mu)}j(\mathcal{M}) \) is a linear combination of \( e_1(P_{F,j}(\mu)), \ldots, e_d(P_{F,j}(\mu)) \):

\[
(3.1) \quad d_\mu P_{F,j}(e_b) = \sum_{a=1}^{d} (d_\mu P_{F,j}(e_a)) \cdot e_a(P_{F,j}(\mu))e_a(P_{F,j}(\mu))
\]

By the delta method, \( n^{1/2}(P_{F,j}(\bar{j}(X)) - P_{F,j}(\mu)) \) converges weakly to \( N_k(0_k, \alpha \Sigma_\mu) \), where \( \bar{j}(X) = \frac{1}{n} \sum_{i=1}^{n} j(X_i) \) and

\[
(3.2) \quad \alpha \Sigma_\mu = \sum_{a=1}^{d} d_\mu P_{F,j}(e_b) \cdot e_a(P_{F,j}(\mu))e_a(P_{F,j}(\mu))_{b=1,\ldots,k} \\
\times \sum_{a=1}^{d} d_\mu P_{F,j}(e_b) \cdot e_a(P_{F,j}(\mu))e_a(P_{F,j}(\mu))_{b=1,\ldots,k}^T
\]

here \( \Sigma \) is the covariance matrix of \( j(X_1) \) w.r.t the canonical basis \( e_1, \ldots, e_k \).

The asymptotic distribution \( N_k(0_k, \alpha \Sigma_\mu) \) is degenerate and the support of this distribution is on \( TP_{F,j}(\mathcal{M}) \), since the range of \( d_\mu P_{F,j} \) is \( TP_{F,j(\mu)}j(\mathcal{M}) \). Note that \( d_\mu P_{F,j}(e_b) \cdot e_a(P_{F,j}(\mu)) = 0 \) for \( a = d + 1, \ldots, k \).

The tangential component \( tan(v) \) of \( v \in \mathbb{R}^k \), w.r.t the basis \( e_a(P_{F,j}(\mu)) \in TP_{F,j(\mu)}j(\mathcal{M}), a = 1, \ldots, d \) is given
by

\begin{equation}
\tan(v) = [e_1(P_{F,j}(\mu))^T v, \ldots, e_d(P_{F,j}(\mu))^T v]^T
\end{equation}

Then the random vector 
\((d_{\alpha \mu E,j})^{-1}(\tan(P_{F,j}(\overline{(j(X)})) - P_{F,j}(\mu))) = \sum_{a=1}^d X^a j f_a\) has the following covariance matrix w.r.t the basis \(f_1(\alpha \mu E), \ldots, f_d(\alpha \mu E)\):

\begin{equation}
\alpha \Sigma_{j, E} = e_a(P_{F,j}(\mu))^T \alpha \Sigma_{\mu} e_b(P_{F,j}(\mu)) \sum_{a=1, b \leq d}^d
\end{equation}

which is the anti-covariance matrix of the random object \(X\). Similarly, given i.i.d. r.o.'s \(X_1, \cdots, X_n\) from \(Q\), we define the sample anti-covariance matrix \(aS_{j, E, n}\) as the anti-covariance matrix associated with the empirical distribution \(\hat{Q}\).

### 3.2 VW anti-covariance in \(\mathbb{R}P^{N-1}\) and \(\mathbb{C}P^{k-2}\)

We first consider the case when \(\mathcal{M} = \mathbb{R}P^{N-1}\), the real projective space which can be identified with the sphere \(S^{N-1} = \{x \in \mathbb{R}^N ||x||^2 = 1\}\) with antipodal points identified (see Mardia and Jupp (2009) [13]). Here the points in \(\mathbb{R}^N\) are regarded as \(N \times 1\) vectors. \(\mathbb{R}P^{N-1}\) can be identified with the quotient space \(S^{N-1}/\{x, -x\}\); it is a compact homogeneous space, with the group \(SO(N)\) acting transitively on \((\mathbb{R}P^{N-1}, \rho_0)\), where the distance \(\rho_0\) on \(\mathbb{R}P^{N-1}\) is induced by the chord distance on the space \(S(N, \mathbb{R})\) of symmetric \(N \times N\) and the embedding \(j\) that is compatible with two transitive group actions of \(SO(N)\) on \(\mathbb{R}P^{N-1}\), respectively on \(j(\mathbb{R}P^{N-1})\), that is

\begin{equation}
\omega(T \cdot [x]) = T \otimes j([x]), \forall T \in SO(N), \forall [x] \in \mathbb{R}P^{N-1}
\end{equation}

where \(T \cdot [x] = [Tx]\).

Such an embedding is said to be equivariant (See Kent 1992 [11]). The equivariant embedding of \(\mathbb{R}P^{N-1}\) that was used so far in the axial data analysis literature is the Veronese Whitney (VW) embedding \(j : \mathbb{R}P^{N-1} \rightarrow S_+ (N, \mathbb{R})\), that associates to an axis the matrix of the orthogonal projection on this axis (See Patrangenaru and Ellingson 2015 [12], chapter 3)

\begin{equation}
j([x]) = xx^T, ||x|| = 1
\end{equation}
Here $S_+(N, \mathbb{R})$ is the set of nonnegative definite symmetric $N \times N$ matrices, and in this case

\[(3.7) \quad T \otimes A = TAT^T, \quad \forall T \in SO(N), \quad \forall A \in S_+(N, \mathbb{R})\]

**DEFINITION 3.1.** A random object $[X] = Y$ on $\mathbb{R}^{P^{N-1}}$ is $\alpha$VW-nonfocal if it is $\alpha j$-nonfocal w.r.t. the VW-embedding in (3.6).

Then we have the following proposition from Patrangenaru et al (2016) [19].

**PROPOSITION 3.1.** A random object $[X] = Y$ on $\mathbb{R}^{P^{N-1}}$, $X^T X = 1$ is $\alpha$VW-nonfocal iff the smallest eigenvalue of $E(XX^T)$ is positive and has multiplicity 1.

Now we consider the anti-covariance on $\mathbb{R}^{P^{N-1}}$.

**PROPOSITION 3.2.** Assume $[X_r], \|X_r\| = 1, r = 1, \ldots, n$ is a random sample from a $\alpha j$-nonfocal probability measure $Q$ on $\mathbb{R}^{P^{N-1}}$. Then the sample VW anti-covariance matrix $aS_{j,E,n}$ is given by

\[(3.8) \quad (aS_{j,E,n})_{ab} = n^{-1}(\eta_a - \eta_1)^{-1}(\eta_b - \eta_1)^{-1} \times \sum_{r} (m_a \cdot X_r)(m_b \cdot X_r)(m_1 \cdot X_r)^2\]

where $\eta_a, a = 1, \ldots, N$, are eigenvalues of $K := \frac{1}{n} \sum_{r=1}^{n} X_r X_r^T$ in increasing order and $m_a, a = 1, \ldots, N$, are corresponding linearly independent unit eigenvectors.

The proof is along the lines of a similar result from VW sample covariance on $\mathbb{R}^{P^{N-1}}$ (see Bhattacharya and Patrangenaru (2005) [3]). Since the map $j$ is equivariant, w.l.o.g. one may assume that $j(aX_{j,E}) = P_{F,j}(j(X))$ is a diagonal matrix, $aX_{j,E} = [m_1] = [e_1]$ and the other unit eigenvectors of $j(X) = D$ are $m_a = e_a, \forall a = 2, \ldots, N$.

We evaluate $d_D P_{F,j}$ based on this description of $T_{[x]} \mathbb{R}^{P^{N-1}}$, one can select in $T_{P_{F,j}}(D)j(\mathbb{R}^{P^{N-1}})$ the orthonormal frame $e_a(P_{F,j}(D)) = d[e_1,j(e_a)]$. Note that $S(N, R)$ has the orthobasis $F^b_a, b \leq a$, where, for $a < b$, the matrix $F^b_a$ has all entries zero except for those in the positions $(a, b), (b, a)$ that are equal to $2^{-1/2}$; also $F^b_a = j([e_a])$.

A straightforward computation shows that if $\eta_a, a = 1, \ldots, N$ are the eigenvalues of $D$ in their increasing order. Then $d_P F_{j,F}(F^b_a) = 0, \forall a, b, 2 \leq b < a \leq N$ and $d_P (F^b_a) = (\eta_b - \eta_1)^{-1} \eta_a (P_{F,j}(D))$; from this equation it follows that, if $j(X)$ is a diagonal matrix $D$, then the entry $(aS_{j,E,n})_{ab}$ is given by.

\[(aS_{j,E,n})_{ab} = n^{-1}(\eta_b - \eta_1)^{-1}(\eta_a - \eta_1)^{-1} \sum_{r=1}^{n} X_r^a X_r^b (X_r^T)^2\]
Taking \( j(X) \) to be a diagonal matrix and \( m_a = e_a \)

Note that \( \alpha \mu_{j,E} = [\nu_1] \), where \( (\nu_a), a = 1, \ldots, N \), are unit eigenvectors of \( E(XX^t) = E(j(Q)) \) corresponding to eigenvalues in their increasing order.

Let \( T([\nu]) = n\|aS_{E,n}^{-\frac{1}{2}}\tan(P_{F,j}(j(X)) - P_{F,j}(E(j(Q))))\|^2 \) (see formula (27) from Patrangenaru et al 2016 [18]). We can derive now the following theorem.

**THEOREM 3.1.** Assume \( j \) is the Veronese-Whitney embedding of \( \mathbb{R}P^{N-1} \) and let \( \{X_r, \|X_r\| = 1, r = 1, \ldots, n \} \) be a random sample from an \( \alpha_j \)-nonfocal distribution \( Q \). Then \( T([\nu]) \) is given by

\[
T([\nu]) = n\nu^t[aS_{E,n}^{-1}[(\nu_a)_{a=2,\ldots,N}]^t]\nu,
\]

and, asymptotically, \( T([\nu]) \) has a \( \chi^2_{N-1} \) distribution.

Proof: Since \( \alpha j \) is an isometric embedding and the tangent space \( T_{[\nu_1]} \mathbb{R}P^{N-1} \) has the orthobasis \( \nu_1, \ldots, \nu_{N-1} \), if we select the first elements of the adapted moving frame (See Bhattacharya and Patrangenaru 2015 [4]) to be \( e_a(P_{F,j}(\nu_{E,j})) = (d_{[\nu_1]}(\nu_a), then the \( a \)th tangential component of \( P_{F,j}(j(X)) - P_{F,j}(\nu) \) w.r.t. this basis of \( T_{P_{F,j}(E(j(Q)))_j}(\mathbb{R}P^{N-1}) \) equals up to a sign the \( a \)th component of \( m - \nu_1 \) w.r.t. the orthobasis \( \nu_2, \ldots, \nu_N \) in \( T_{[\nu_1]} \mathbb{R}P^{N-1} \), namely \( \nu_a^t m \).

Now we consider the VW-embedding \( j: \mathbb{C}P^{k-2} \to S(k-1, \mathbb{C}) \), where \( S(k-1, \mathbb{C}) \) is the space of \( (k-1) \times (k-1) \) self adjoint complex matrices, given by

\[
j([z]) = zz^*, \quad z^* z = 1
\]

where \( j \) is a \( SU(k-1) \) equivariant embedding. Here \( SU(k-1) \) is the special unitary group \( (k-1) \times (k-1) \) matrices of determinant 1. We say that a random object in \( \mathbb{C}P^{k-2} \) is \( \alpha \)VW-nonfocal if it is \( \alpha j \)-nonfocal w.r.t. the embedding in (3.9)

Similarly with proposition [3.1] we get the following proposition.

**PROPOSITION 3.3.** The random object \( Y = [z] \) on \( \mathbb{C}P^{k-1} \) is \( \alpha \)VW-nonfocal iff the smallest eigenvector of \( E\frac{zz^*}{z^*z} \), is positive and has multiplicity 1.
Similar asymptotic results can be obtained for the large sample distribution of VW means of planar shapes, as following. Recall that the planar shapes space \( M = \Sigma^k_2 \) of an ordered set of \( k \) points in \( \mathbb{C} \) at least two of which are distinct, can be identified in different ways with the complex projective space \( \mathbb{C}P^{k-2} \) (see Bhattacharya and Patrangenaru (2005) [3], and Balan and Patrangenaru (2005) [1]). Here we regard \( \mathbb{C}P^{k-2} \) as a set of equivalence classes \( \mathbb{C}P^{k-2} = S^{2k-3}/S^1 \) where \( S^{2k-3} \) is the space of complex vectors in \( \mathbb{C}^{k-1} \) of norm 1, and the equivalence relation on \( S^{2k-3} \) is by multiplication with scalars in \( S^1 \) (complex numbers of modulus 1). A complex vector \( z = (z_1, z_2, \ldots, z_{k-1}) \) of norm 1 corresponding to a given configuration of \( k \) landmarks, with the identification described in Bhattacharya and Patrangenaru (2005) [3], can be displayed in the Euclidean plane (complex line) with the superscripts as labels.

A random variable \( X = [Z], ||Z|| = 1 \), valued in \( \mathbb{C}P^{k-2} \) is \( \alpha_j \)-nonfocal if the smallest eigenvalue of \( E[ZZ^*] \) is simple, and then the extrinsic antimean of \( X \) is \( \alpha_j^\mu_E = [\nu] \), where \( \nu \in \mathbb{C}^{k-1}, ||\nu|| = 1 \), is an eigenvector corresponding to this eigenvalue (See Bhattacharya and Patrangenaru (2003) [3]). The extrinsic sample antimean \( \alpha^\mu_{[z],E} \) of a random sample \( [z_r] = [(z^1_r, \cdots, z^{k-1}_r)], ||z_r|| = 1, r = 1, \cdots, n, \) from such a nonfocal distribution exists with probability converging to 1 as \( n \to \infty \), and is the same as that given by

\[
(3.10) \quad a^\mu_{[z],E} = [m],
\]

where \( m \) is the smallest unit eigenvector of

\[
(3.11) \quad K := \frac{1}{n} \sum_{r=1}^{n} z_r z_r^*.
\]

**PROPOSITION 3.4.** Assume \( X_r = [Z_r], ||Z_r|| = 1, r = 1, \ldots, n \) is a random sample from a \( \alpha_j \)-nonfocal probability measure \( Q \) with a nondegenerate \( \alpha_j \)-extrinsic anti-covariance matrix on \( \mathbb{C}P^{k-2} \). Then the \( \alpha_j \)-extrinsic sample anti-covariance matrix \( \alpha S_{E,n} \) as a complex matrix has the entries

\[
(3.12) \quad (\alpha S_{E,n})_{ab} = n^{-1}(\eta_a - \eta_1)^{-1}(\eta_b - \eta_1)^{-1} \times \sum_{r=1}^{n} (m_a \cdot Z_r)(m_b \cdot Z_r)^* |m_1 \cdot Z_r|^2
\]

where \( \eta_a, a = 2, \cdots, k - 1 \) are eigenvalues of \( K \) in (3.11) in their increasing order and \( m_a, a = 2, \cdots, k - 1 \) are corresponding linearly independent unit eigenvectors.
The proof is similar to that given for Proposition 3.2 and is based on the equivarance of the Veronese-Whitney map \( j \) w.r.t. the actions of \( SU(k-1) \) on \( \mathbb{C}P^{k-2} \) and on the set \( S_+(k-1, \mathbb{C}) \) of non negative semi definite self-adjoint \((k-1) \times (k-1)\) complex matrices (see Bhattacharya and Patrangenaru(2003) [3]). Without loss of generality we may assume that \( K \) in (3.1) is a diagonal matrix and the smallest eigenvalue of \( K \) is a simple root of the characteristic polynomial over \( \mathbb{C} \), with with \( m_1 = e_1 \) as a corresponding complex eigenvector of norm 1.

The eigenvectors over \( \mathbb{R} \) corresponding to the larger eigenvalues are given by \( m_a = e_a, m'_a = i e_a, a = 2, \cdots, k-1 \), and yield an orthobasis for \( T_{[m_{k-1}]}j(\mathbb{C}P^{k-2}) \). For any \( z \in S^{2k-1} \) which is orthogonal to \( m_1 \) in \( \mathbb{C}^{k-1} \) w.r.t. the real scalar product, then we can define a path \( \gamma_z(t) = [\cos tm_1 + \sin tz] \). Then at \( t = 0, T_{P_{F,j}(K)}j(\mathbb{C}P^{k-2}) \) is generated by the vectors tangent to such paths \( \gamma_z(t) \). Such a vector, has the form \( zm_1^* + m_1z^* \), as a matrix in \( S(k-1, \mathbb{C}) \).

In particular, since the eigenvectors of \( K \) are orthogonal w.r.t. the complex scalar product, one may take \( z = m_a, a = 2, \cdots, k-1 \), or \( z = im_a, a = 2, \cdots, k-1 \), and thus get an orthobasis in \( T_{P_{F,j}(K)}j(M) \). To obtain the orthonormal frame we norm these vectors to have unit lengths

\[
ee_a(P_{F,j}(K)) = d_{[m_1]}j(m_a) = 2^{-1/2}(m_a m_1^* + m_1 m_a^*),
\]

\[
e'_a(P_{F,j}(K)) = d_{[m_1]}j(m_a) = i2^{-1/2}(m_a m_1^* + m_1 m_a^*).
\]

As we assume \( K \) is diagonal. In this case \( m_a = e_a, e_a(P_{F,j}(K)) = 2^{-1/2}E^a \) and \( e'_a(P_{F,j}(K)) = 2^{-1/2}F^a \), where \( E^b_a \) has all entries zero except for those in the positions \((a, b)\) and \((b, a)\) that are equal to 1, and \( F^b_a \) is a matrix with all entries zero except for those in the positions \((a, b)\) and \((b, a)\) that are equal to \( i \), respectively \(-i\). That we have \( d_K P_{F,j}(E^b_a) = d_K P_{F,j}(F^b_a) = 0, \forall 1 < a \leq b \leq k-1, \) and

\[
d_K P_{F,j}(E^b_1) = (\eta_a - \eta_1)^{-1} e_a(P_{F,j}(K)),
\]

\[
d_K P_{F,j}(F^a_1) = (\eta_a - \eta_1)^{-1} e'_a(P_{F,j}(K)).
\]

We evaluate the extrinsic sample anti-covariance matrix \( aS_{E,n} \) in formula (25) in Patrangenaru et al (2016 [13]) using the real scalar product in \( S(k-1, \mathbb{C}) \), namely, \( U \cdot V = ReTr(UV^*) \). Note that,

\[
d_K P_{F,j}(E^b_1) \cdot e_a(P_{F,j}(K)) = (\eta_a - \eta_1)^{-1} \delta_{ba},
\]

\[
d_K P_{F,j}(E^b_1) \cdot e'_a(P_{F,j}(K)) = 0
\]
and

\[ d_K P_{F,j}(F_1^b) \cdot e_a(P_{F,j}(K)) = (\eta_a - \eta_1)^{-1} \delta_{ba}, \]

\[ d_K P_{F,j}(F_1^b) \cdot e_a(P_{F,j}(K)) = 0 \]

Thus we may regard \(aS^E_{E,n}\) as a complex matrix noting that in this case we get

\[ (aS^E_{E,n})_{ab} = n^{-1}(\eta_a - \eta_1)^{-1}(\eta_b - \eta_1)^{-1} \sum_{r=1}^{n} (e_a \cdot Z_r)(e_b \cdot Z_r)^* |m_1 \cdot Z_r|^2 \]

Thus proving (3.12) when \(K\) is diagonal. The general case follows by equivariance.

We consider now the statistic

\[ T((X)_E, \alpha \mu_E) = n \| (aS^E_{E,n})^{1/2} \tan(P_{F,j}(J(X)) - P_{F,j}(\alpha \mu_E)) \|^2 \]

given in formula (27) from Patrangenaru et al 2016 [13] in our context of random objects valued in a complex projective space to get:

**THEOREM 3.2.** Let \(X_r = [Z_r], \|Z_r\| = 1, r = 1, \cdots, n\), be a random sample from a \(\alpha\)-nonfocal probability measure \(Q\) on \(\mathbb{C}P^{k-2}\). Then the random variable given by

\[ T([m], [\nu]) = n([m \cdot \nu_a]_{a=2,\cdots,k-1})^{-1}[(m \cdot \nu_a)_{a=2,\cdots,k-1}]^* \]

has asymptotically a \(\chi^2_{2k-4}\) distribution.

**Proof.** The tangent space \(T_{[\nu_1]} \mathbb{C}P^{k-2}\) has the orthobasis \(\nu_2, \cdots, \nu_{k-1}, \nu_2^*, \cdots, \nu_{k-1}^*\). Note that since \(j\) is an isometric embedding, we may select the first elements of the adapted moving frame to be \(e_a(P_j(\mu)) = (d_{[\nu_1]j})(\nu_a)\), followed by \(e^*_a(P_j(\mu)) = (d_{[\nu_1]j})(^*)\). Then the \(a\)th tangential component of \(P_j(J(X)) - P_j(\mu)\) w.r.t. this basis of \(T_{[\nu_1]} \mathbb{C}P^{k-2}\) equals up to a sign the component of \(m - \nu_1\) w.r.t. the orthobasis \(\nu_2, \cdots, \nu_{k-1}\) in \(T_{[\nu_1]} \mathbb{C}P^{k-2}\), which is \(\nu_1^* m\); and the \(a^*\)th tangential components are given by \(\nu_1^* m\), and together(in complex multiplication) they yield the complex vector \([m \cdot \nu_a]_{a=2,\cdots,k-1}\).

Then we may derive the following large sample confidence regions from this.

**COROLLARY 3.1.** Assume \(X_r = [Z_r], \|Z_r\| = 1, r = 1, \cdots, n,\) is a random sample from a \(\alpha\)-nonfocal probability measure \(Q\) on \(\mathbb{C}P^{k-2}\). An asymptotic \((1 - \alpha)\)-confidence region for \(\alpha \mu^2_E(Q) = [\nu]\) is given by
\[ R_\alpha(X) = \{ [\nu] : T([m], [\nu]) \leq \chi_{2k-4,\alpha}^2 \}, \text{ where } T([m], [\nu]) \text{ is given in (3.14). If } Q \text{ has a nonzero absolutely continuous component w.r.t. the volume measure on } \mathbb{C}P^{k-2}, \text{ then the coverage error of } R_\alpha(X) \text{ is of order } O(n^{-1}). \]

For small samples the coverage error could be quite large, and a bootstrap analogue of Theorem 3.2 is preferable.

**THEOREM 3.3.** Let \( j \) be the Veronese embedding of \( \mathbb{C}P^{k-2} \), and let \( X_r = [Z_r] \), \( \|Z_r\| = 1 \), \( r = 1, \cdots, n \), be a random sample from a \( \alpha_j \)-nonfocal distribution \( Q \) on \( \mathbb{C}P^{k-2} \) having a nonzero absolutely continuous component w.r.t. the volume measure on \( \mathbb{C}P^{k-2} \). Assume in addition that the restriction of the covariance matrix of \( j(Q) \) to \( T([\nu])j(\mathbb{C}P^{k-2}) \) is nondegenerate. Let \( \alpha \mu_E(Q) = [\nu] \) be the extrinsic antimean of \( Q \). For a resample \( \{Z_r^*\}_{r=1}^n \) from the sample consider the matrix \( K^* := n^{-1} \sum Z_r^*Z_r^{**} \). Let \( (\eta_{a}^*)_{a=1,\cdots,k-1} \) be the eigenvalues of \( K^* \) in their increasing order, and let \( (m_{a}^*)_{a=1,\cdots,k-1} \) be the corresponding unit complex eigenvectors. Let \( (aS_{E,n})^* \) be the matrix obtained from \( aS_{E,n} \) by substituting all the entries with \( * \)-entires. Then the bootstrap distribution function of

\[(3.15) \quad T([m]^*, [m]) = n[(m_{1}^* \cdot m_{a}^*)_{a=2,\cdots,k-1}](aS_{E,n})^{-1}[m_{1}^* \cdot m_{a}^*)_{a=2,\cdots,k-1}]^* \]

approximates the true distribution function of \( T([m], [\nu]) \) given in Theorem ?? with an error of order \( O_p(n^{-2}) \).

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