Strategies for linear rewriting systems:
link with parallel rewriting and involutive divisions

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Abstract

We study rewriting systems whose underlying set of terms is equipped with a vector space structure over a given field. We introduce parallel rewriting relations, which are rewriting relations compatible with the vector space structure, as well as rewriting strategies, which consist in choosing one rewriting step for each reducible basis element of the vector space. Using these notions, we introduce the S-confluence property and show that it implies confluence. We deduce a proof of the diamond’s lemma, based on strategies. We illustrate our general framework with rewriting systems over rational Weyl algebras, that are vector spaces over a field of rational functions. In particular, we show that involutive divisions induce rewriting strategies over rational Weyl algebras, and using the S-confluence property, we show that involutive sets induce confluent rewriting systems over rational Weyl algebras.

Keywords: confluence, parallel rewriting, rewriting strategies, involutive divisions.

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1 Introduction

Rewriting systems are computational models given by a set of syntactic expressions and transformation rules used to simplify expressions into equivalent ones. Since rewriting theory is applicable to different problems of mathematics and computer science, it was developed for many syntaxes of terms, e.g., strings, (Σ−, higher-order, infinitary) terms, graphs, (commutative, noncommutative, vectors of) polynomials, (linear combinations of) trees, (higher-dimensional) cells. Abstract rewriting theory unifies these contexts and provides universal formulations of rewriting properties, such as termination, normalisation and (local) confluence. Newman’s lemma is one of the most famous results of abstract rewriting and asserts that under termination hypothesis, local confluence implies confluence.

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In the context of rewriting over algebraic structures, the Newman’s lemma is used in conjunction with the critical pairs lemma to algorithmically prove confluence. This is something fundamental since confluent rewriting systems provide methods for solving decision problems, computing (linear, homotopy) bases, Hilbert series, or free resolutions [10, 11, 13, 20]. From these methods, one get constructive proofs of theoretical results, such as embedding, coherence or homological theorems [3, 4, 8, 12, 15, 19], but also applications to problems coming from topics modelled by algebra, such as cryptography, analysis of (ordinary differential, partial derivative, time-delay) equations or control theory. For instance, many informations of functional equations may be read over free resolutions: integrability conditions, parametrization of solutions, existence of autonomous curves [6, 17].

When one considers algebraic structures with underlying vector space operations, the conjunction of Newman’s lemma and the critical pairs lemma is traditionally known under the name of the diamond’s lemma. In practice, this lemma is used to test if a generating set of a polynomial ideal is a Gröbner basis, since confluent linear rewriting systems are usually induced by Gröbner bases or one of their numerous adaptations to different classes of algebras or operads [3, 5, 7, 13, 16]. As an illustration of theses classes, let us mention polynomial Weyl algebras that are models of differential operators with polynomial coefficients. These algebras are composed of polynomials over two sets of variables, the state variables $x_1, \ldots, x_n$ and the vector field variables $\partial_1, \ldots, \partial_n$, and submitted to the commutation rules

$$\forall 1 \leq i \neq j \leq n : \quad x_i x_j = x_j x_i, \quad \partial_i \partial_j = \partial_j \partial_i, \quad \partial_i x_j = x_j \partial_i, \quad \partial_i x_i = x_i \partial_i + 1.$$  

These relations represent classical rules from differential calculus: the second one means that second order derivatives of smooth functions commute, the third one means that $x_j$ is constant for differentiation with respect to $x_i$ and the last one represents the Leibniz’s rule for differentiation with respect to $x_i$, that is, $\partial_i (x_i f) = x_i \partial_i (f) + f$, for any smooth function $f$. Rewriting over vector spaces requires to introduce a notion of well-formed rewriting step, also called positive rewriting step in [10], that is, each step reduces only one basis element together with its coefficient in a given vector. Typically, the vector is a polynomial and the reduced basis element is a monomial that appears in this polynomial with a nonzero coefficient. Doing so avoids pathological situations, e.g., $v_1 \rightarrow v_2$ implies that the rewriting step $v_2 = v_2 - v_1 + v_1 \rightarrow v_2 - v_2 + v_1 = v_1$ is not well-formed. On the other hand, well-formed rewriting steps are not compatible with vector space operations since as soon as two different basis elements are rewritten in the well-formed rewriting steps $u_1 \rightarrow u_2$ and $v_1 \rightarrow v_2$, then $u_1 + v_1 \rightarrow u_2 + v_2$ is not well-formed.

The notion of well-formed rewriting step is specific to rewriting systems over vector spaces and, as mentioned above, it is not compatible with the underlying algebraic operations. This lack of compatibility makes the theory of linear rewriting rather painful, for instance the proof of the critical pair lemma is more involved than for string rewriting (see [10 Theorem 4.2.1]). In our view, these observations call for the development of a theory of linear rewriting free from well-formed rewriting steps. In the long run, we hope that this will contribute to bridge the gap between abstract and linear rewriting, for instance to find a common proof of Newman’s lemma and of the diamond lemma.

**Our results**

In the present paper, we introduce an alternative approach to rewriting theory over vector spaces, which does not use the notion of well-formed rewriting step. Instead, our rewriting steps only depend on vector spaces operations, and they may reduce many basis elements at once, still avoiding pathological situations. Moreover, our framework is valid in every vector space and can be applied to the case where the coefficients do not commute with basis elements. We illustrate this last point with rewriting systems over rational Weyl algebras, and from this, we show that so-called involutive bases [9] induce confluent rewriting systems.

Given a set of rewriting rules of the form $e \rightarrow v$, where $e$ is a basis element and $v$ is a vector, we introduce the notion of parallel rewriting relation with rewriting steps $v_1 \rightarrow v_2$ that consists in
replacing each left-hand side of a rule $e$ occurring in $v_1$ by the corresponding right-hand side $v$ to get $v_2$. As mentioned above, this definition is purely internal to the category of vector spaces and does not require any notion of well-formed rewriting step. Our approach is related to the classical one through the notion of *strategy*, which means that the rewriting preorder induced by $\rightarrow$ is terminating. Following the ideas of [10], and contrary to rewriting with Gröbner bases, rewriting with strategies does not require any monomial order but uses the order induced by the rewriting process itself. The link with classical rewriting over vector spaces is given by a confluence criterion. Indeed, the initial rewriting rules induce a rewriting relation $\rightarrow_R$ involving well-formed rewriting steps only, and we say that $\rightarrow_R$ is *S-confluent* if for every rewriting rule $e \rightarrow v, e$ and $v$ are joinable using $\rightarrow$. In Theorem 2.9 we show that the $S$-confluence property implies confluence of $\rightarrow_R$, and we construct a basis of the vector space quotiented by the equivalence relation of $\rightarrow_R$ in terms of normal forms for $\rightarrow$. Moreover, we also show that the $S$-confluence property is characterised by a decreasingness property, which enables us to provide a new proof of the diamond’s lemma in Theorem 2.15.

Since our approach works for arbitrary vector spaces, it may be declined in different classes of algebras over fields, including rational Weyl algebras. The later extend polynomial Weyl algebras presented above, since they are composed of differential operators with coefficients in the field of rational functions. Unlike polynomial Weyl algebras, that are vector spaces over the field of constants and modules over the ring of polynomials, rational Weyl algebras are vector spaces over the field of rational functions. Notice that because of the Leibniz’s rule, rational functions do not commute with operators. Rewriting-like methods in this context yield applications to formal analysis of linear systems of ordinary differential or partial derivative equations as mentioned above. In particular, involutive divisions, such as Janet, Thomas and Pommaret divisions, provide deterministic techniques to rewrite differential operators. By determinism, we mean that each differential operator admits at most one involutive divisor, that is, it may be rewritten into at most one other differential operator. This determinism is strongly related to our notion of strategy and parallel reductions. In particular, we show in Theorem 3.12 that involutive bases induce $S$-confluent rewriting systems. Finally, we show how most of the axioms of involutive divisions may be formalised in purely rewriting language using rewriting strategies.

**Organisation.** In Section 2 we present our general framework and general results for rewriting systems over vector spaces. In Section 2.1 we introduce parallel rewriting relations, rewriting strategies, and a normalisation operator associated to each strategy. In Section 2.2 we introduce the $S$-confluence property, and show that it implies confluence property for well-formed rewriting relations and that bases of quotient vector spaces may be constructed in terms of normal forms for parallel rewriting relations. We also show that $S$-confluence is characterised by a decreasingness property, from which we deduce a proof of the diamond’s lemma based on strategies. In Section 3 we illustrate our general framework by rewriting systems over rational Weyl algebras. In Section 3.1 we introduce well-formed rewriting systems over rational Weyl algebras. In Section 3.2 we recall the definition of an involutive division and of involutive sets of operators and show that involutive divisions define rewriting strategies, and that involutive sets $S$-confluent rewriting relations.

**Terminology and conventions of rewriting theory.** Throughout the paper, we use the standard terminology and conventions of rewriting theory [2]. An *abstract rewriting system* is a pair $(A, \rightarrow)$, where $A$ is a set and $\rightarrow$ is binary relation on $A$, called rewriting relation. An element $(a, b) \in \rightarrow$ is written $a \rightarrow b$ and is called a rewriting step. A *normal form* for $\rightarrow$ is an element $a \in A$ such that there is no $b \in A$ such that $a \rightarrow b$. We denote by $\rightarrow^*$ (respectively, $\leftrightarrow^*$) the closure of $\rightarrow$ under transitivity and reflexivity (respectively, and symmetry). The equivalence class of $a \in A$ modulo the equivalence relation $\leftrightarrow^*$ is written $\left[a\right]_{\leftrightarrow^*}$, and the set of all equivalence classes is written $A/\leftrightarrow^*$. When $a \rightarrow^* b$, that is, there exists a (possibly empty) finite sequence of rewriting steps from $a$ to $b$, we say that $a$ *rewrites* $b$.

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1 A module is like a vector space, except that its scalars form a ring instead of a field.
into $b$. The rewriting relation $\rightarrow$ is said to be confluent if whenever $a \rightarrow^* b$ and $a \rightarrow^* c$, then $b$ and $c$ are joinable, that is, there exists $d$ such that $b \rightarrow^* c$ and $c \rightarrow^* d$. The confluence property is equivalent to the Church-Rosser property that asserts that whenever $a \leftrightarrow b$, then $a$ and $b$ are joinable.

## 2 Parallel rewriting relations over vector spaces

In this section, we introduce parallel rewriting relations and rewriting strategies over vector spaces. We deduce a confluence criterion in terms of strategies as well as a method based on strategies to construct linear bases of vector spaces. We also obtain a new proof of the diamond’s lemma.

Throughout the section, we fix a ground field $\mathbb{K}$, a $\mathbb{K}$-vector space $V$, and a basis $\mathcal{B}$ of $V$. We say vectors and basis elements for elements of $V$ and $\mathcal{B}$, respectively. Every vector $v$ admits a unique finite decomposition with respect to the basis $\mathcal{B}$ and coefficients in the ground field:

$$v = \sum \lambda_i e_i, \quad \lambda_i \neq 0.$$  

(1)

The set of basis elements which appear in the decomposition (1) is called the support of $v$ and is written $\text{supp}(v)$.

### 2.1 Rewriting strategies over vector spaces

In this section, we introduce parallel rewriting relations and rewriting strategies over vector spaces, and we construct a normalisation operator associated to a strategy. This normalisation operator is used in Section 2.2 to formulate our confluence criterion and to provide a rewriting method for constructing linear bases of vector spaces.

We first recall the traditional approach of rewriting over vector spaces, which consists in reducing one basis element at each step. We fix a subset $R$ of $\mathcal{B} \times V$, whose elements are called rewriting rules. A rewriting rule is denoted by $e \rightarrow_R v$, where $e$ and $v$ are left and right-hand sides of this rule, i.e., its images through the natural projections of $R$ on $\mathcal{B}$ and $V$, respectively. We extend these rules into a rewriting relation on $V$, still written $\rightarrow_R$, with rewriting steps are of the following form:

$$\lambda e + u \rightarrow_R \lambda v + u,$$

(2)

where $e \rightarrow_R v \in R$ is a rewriting rule, $\lambda$ is a nonzero scalar and $u$ is a vector such that $e$ does not belong to its support. A normal form for $\rightarrow_R$ is called an $R$-normal form. In the sequel, we refer $\rightarrow_R$ as the rewriting relation induced by the rewriting rules $R$. The relation $\rightarrow_R$ is not stable under vector space operations, that is, $u_1 \rightarrow_R u_2, v_1 \rightarrow_R v_2$, and $\mu \in \mathbb{K}$ generally do not imply $\mu u_1 + v_1 \rightarrow_R \mu u_2 + v_2$. In contrast, the following proposition shows that $\leftrightarrow_R$ is compatible with these operations.

**Proposition 2.1.** The quotient set $V/ \leftrightarrow_R$ is a vector space and the projection $V \rightarrow V/ \leftrightarrow_R$ is a linear map.

**Proof.** The proof is an adaptation of [10] Lemma 3.1.3]. The statement of the proposition precisely means that $u_1 \leftrightarrow_R u_2, v_1 \leftrightarrow_R v_2$, and $\mu \in \mathbb{K}$ imply $\mu u_1 + v_1 \leftrightarrow_R \mu u_2 + v_2$. Let us first show the following implication

$$u_1 \rightarrow_R u_2 \Rightarrow \mu u_1 + v_1 \leftrightarrow_R \mu u_2 + v_2.$$ 

(3)

By definition of $\rightarrow_R$, we have $u_1 = \lambda e + u$ and $u_2 = \lambda v + u$, where $e \rightarrow_R v$ is a rewriting rule, $\lambda$ is a scalar and $e$ does not belong to $\text{supp}(u)$. Let $\nu$ be the coefficient of $e$ in $v_1$, so that we may write $v_1 = \nu e + v'_1$, where $e$ does not belong to $\text{supp}(v'_1)$. Since $\mu u_1 + v_1 = (\mu \lambda + \nu)e + \mu u + v'_1$ and $\mu u_2 + v_2 = \nu e + \mu \nu v + \mu u + v'_1$, we have $\mu u_1 + v_1 \leftrightarrow_R (\mu \lambda + \nu)v + \mu u + v'_1 \leftrightarrow_R \mu u_2 + v_1$, which proves (3). If $u_1 \leftrightarrow_R u_2$, using (3), an induction on the length of the path $u_1 \leftrightarrow_R u_2$ shows that $\mu u_1 + v_1 \leftrightarrow_R \mu u_2 + v_1$, and by an analogous argument, we have $\mu u_2 + v_1 \leftrightarrow_R \mu u_2 + v_2$. Hence, we have $\mu u_1 + v_1 \leftrightarrow_R \mu u_2 + v_2$, which concludes the proof. \[\square\]
In the following definition, we introduce parallel rewriting relations as being compatible with vector space operations.

**Definition 2.2.** A parallel rewriting relation on $V$ is a rewriting relation $\rightarrow$ on $V$ such that for every rewriting steps $u \rightarrow u'$ and $v \rightarrow v'$ and for every scalar $\lambda \in \mathbb{K}$, there exists a rewriting step $u + \lambda v \rightarrow u' + \lambda v'$.

The term parallel means that $u$ and $v$ may be reduced at once using $\rightarrow$, and justify the double head arrows. Rewriting strategies that we introduce in Definition 2.4 induce parallel rewriting relations with pairwise distinct left-hand sides. The following proposition provides a characterisation of parallel head arrows. The term parallel means that $u$ and $v$ may be reduced at once using $\rightarrow$, and justify the double head arrows. Rewriting strategies that we introduce in Definition 2.4 induce parallel rewriting relations with pairwise distinct left-hand sides. The following proposition provides a characterisation of parallel head arrows. The two associations between parallel rewriting relations with distinct left-hand sides, let $\rightarrow_S$ be the parallel rewriting relation that extends $S$: based on Proposition 2.3, $\rightarrow_S$ corresponds to the function $r_S : \mathcal{B} \rightarrow V$ defined by $r_S(e) = v$ if there exists a rewriting rule $e \rightarrow_R v$ in $S$ and $r_S(e) = e$, otherwise. In the sequel, the unique endomorphism of $V$ that extends $r_S$ is also written $r_S$. Let us denote by $\prec_S$ the rewriting preorder on $\mathcal{B}$ induced by $\rightarrow_S$, that is, $\prec_S$ is the transitive closure of the relation: there exists a vector $v$ such that $e' \rightarrow_S v$ and $e \in \text{supp}(v)$. In other words, $e \prec_S e'$ if and only if $e \in \text{supp}(r_S^n(e'))$, for some nonnegative integer $n$.

**Definition 2.4.** A pre-strategy for $R$ is a subset $S$ of $R$ with pairwise distinct left-hand sides. A strategy for $R$ is a pre-strategy such that the rewriting preorder $\prec_R$ of $\rightarrow_S$ is terminating.

Notice that by definition of the rewriting preorder, if $S$ is a strategy for $R$, then $\rightarrow_S$ is terminating. Moreover, if the rewriting preorder $\prec_R$ is terminating, then any pre-strategy is a strategy since $e \prec_R e'$ implies $e \prec_R e'$.

**Example 2.5.** Let us illustrate (pre-)strategies with the following (counter-)examples.

1. Consider the 4-dimensional vector space $V$ with basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ and let $\rightarrow_R$ be the rewriting relation induced by the set of rewriting rules

\[ R = \{e_1 \rightarrow_R e_2, \ e_2 \rightarrow_R e_3 + e_4, \ e_3 \rightarrow_R e_2 - e_4\}. \]

Notice that $\rightarrow_R$ is not terminating since $e_2 \rightarrow_R e_3 + e_4 \rightarrow_R (e_2 - e_4) + e_4 = e_2$. Considering the pre-strategy

\[ S = \{e_1 \rightarrow_R e_2, \ e_2 \rightarrow_R e_3 + e_4\} \subset R, \]
the preorder \( <_S \) is terminating since we have \( e_1 >_S e_2 >_S e_3, e_4 \). Hence, \( S \) is a strategy and \( \rightarrow_S \) is terminating. Notice that any \( v = \lambda_1 e_1 + \cdots + \lambda_4 e_4 \) admits a unique normal form for \( \rightarrow_S \) that is computed as follows:

\[
v \rightarrow_S \lambda_1 e_2 + (\lambda_2 + \lambda_3) e_3 + (\lambda_2 + \lambda_4) e_4 \rightarrow_S (\lambda_1 + \lambda_2 + \lambda_3) e_3 + (\lambda_1 + \lambda_2 + \lambda_4) e_4.
\]

Anticipating the discussion of the paragraph following the example, this unique normal form is written \( S\text{-NF}(v) \):

\[
S\text{-NF}(v) = (\lambda_1 + \lambda_2 + \lambda_3) e_3 + (\lambda_1 + \lambda_2 + \lambda_4) e_4.
\]

2. Let us consider the 3-dimensional vector space \( V \) with basis \( \mathcal{B} = \{e_1, e_2, e_3\} \), let \( \rightarrow_R \) induced by

\[
R = \{ e_1 \rightarrow_R e_2 + e_3, \; e_2 \rightarrow_R e_1, \; e_3 \rightarrow_R -e_1 \},
\]

and \( S = R \). Then, \( S \) is not a strategy since the preorder \( <_S \) is not terminating. In fact, \( <_S \) is cyclic since from the rewriting sequence \( e_2 \rightarrow_S e_1 \rightarrow_S e_2 + e_3 \), we get \( e_2 >_S e_3 \), so that \( e_1 >_S e_2 >_S e_3 >_S e_1 >_S \cdots \). Notice however that \( \rightarrow_S \) is terminating since for every vector \( v = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \), we have

\[
\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \rightarrow_S (\lambda_2 - \lambda_3) e_1 + \lambda_1 e_2 + \lambda_1 e_3 \rightarrow_S (\lambda_2 - \lambda_3) e_2 + (\lambda_2 - \lambda_3) e_3 \rightarrow_S 0.
\]

3. Let \( V \) be the vector space with basis \( \mathcal{B} = \mathbb{N} \), and consider the set of rewriting rules

\[
R = \{ n \rightarrow_R n + 1 : \; n \in \mathbb{N} \}.
\]

Then, a pre-strategy \( S \) corresponds to a subset \( E \) of \( \mathbb{N} \). It is a strategy if and only if for all \( n \in E \), there exists \( k \in \mathbb{N} \) such that \( n + k \notin E \).

Let \( S \) be a strategy for \( R \). In the sequel, we denote elements of \( S \) in the form \( e \rightarrow_S r_S(e) \) and a normal form for \( \rightarrow_S \) is called an \textit{S-normal form}. We denote rewriting rules of \( S \) with double head arrows to emphasize that we will work with the parallel rewriting relation induced by \( S \) rather than the non-parallel one. Since \( S \) is a strategy, \( \rightarrow_S \) is terminating and it is deterministic in the sense that for every vector \( v \), there is at most one \( v' \) such that \( v \rightarrow_S v' \). Hence, each \( v \) admits exactly one \( S \)-normal form that we denote by \( S\text{-NF}(v) \). This defines a map \( S\text{-NF} : V \rightarrow V \). In the following proposition and corollary, we establish results that we use in Section 2.2.

**Proposition 2.6.** Let \( S \) be a strategy for \( R \). Then, the following hold.

1. For any vector \( v \), we have \( v \xrightarrow{\ast} R r_S(v) \) and this rewriting sequence can be chosen so that each intermediate rewriting step belongs to \( S \). In particular, we have the inclusion:

\[
\rightarrow_S \subseteq \xrightarrow{\ast} R.
\]

2. The map \( S\text{-NF} \) is a linear projector.

**Proof.** Let us show Point 1. Since \( S \) is a strategy, the rewriting preorder \( <_S \) is a terminating order. This preorder induces the following terminating order, still written \( <_S \), on \( V \): we let \( v <_S v' \) if \( \text{supp}(v) \) is smaller than \( \text{supp}(v') \) for the multiset order of \( <_S \). We show the statement by induction along \( <_S \). If \( v \) is minimal, then \( r_S(v) = v \) and hence \( v \xrightarrow{\ast} R r_S(v) \). Suppose now that \( v \) is not minimal. Then \( v \) may be uniquely written in the form

\[
v = \sum_{i=1}^{n} \lambda_i e_i + v'
\]

where the basis elements \( e_i \) are the elements of \( \text{supp}(v) \) that are maximal for the rewriting preorder \( <_S \), and the \( \lambda_i \)'s are their coefficients in \( v \). In particular, we have \( v' <_S v \) and by induction, we have
$v' \xrightarrow{*} R r_S(v')$ using rewriting steps of $S$, only. By definition of $<_S$, the rewriting rules that are involved in this rewriting sequence have left-hand sides not greater than $e_i$’s, so that $v \xrightarrow{*} R \sum \lambda_i e_i + r_S(v')$. Moreover, since the $e_i$’s are not comparable for $<_S$, for each indices $i$ and $j$, $e_i$ does not belong to supp$(r_S(e_j))$. Hence, we may reduce successively each $e_i$ into $r_S(e_i)$ and finally have

$$v \xrightarrow{*} R \sum \lambda_i e_i + r_S(v') \xrightarrow{*} R \sum \lambda_i r_S(e_i) + r_S(v') = r_S(v).$$

Let us show Point 2. By definition of $S$-NF, for every vector $v$, there exists an integer $n_v$ such that $S$-NF$(v) = r_S^n(v)$, for every $n \geq n_v$. Let us consider a linear combination $v_3 = \lambda v_1 + v_2$ of two vectors and let $n$ be any integer greater than $n_v$’s. Since a composition of linear maps is a linear map, we have $r_S^n(v_3) = \lambda r_S^n(v_1) + r_S^n(v_2)$, which proves linearity of $S$-NF. Let us show that $S$-NF is a projector. A linear combination of $S$-normal forms is still an $S$-normal form, and for every vector $v$, $S$-NF$(v)$ is an $S$-normal form. From these two facts, for every vector $v$, we have $S$-NF($S$-NF$(v)$) = $S$-NF$(v)$, that is, $S$-NF is a projector.

The conclusion of Point 1 of Proposition 2.6 concerns the binary relation $\xrightarrow{*}_R$. However, in what follows, we do not use this conclusion since one wishes to work as much as possible with relations that are compatible with vector space operations, such as $\xrightarrow{\ast}_R$, see Proposition 2.1. So, instead of using Point 1 of Proposition 2.6, we work with its following consequence, that concerns $\xrightarrow{\ast}_R$.

**Corollary 2.7.** For any vectors $u, v$, if $u \xrightarrow{\ast}_S v$ then $u \xrightarrow{\ast}_R v$. In addition, it is possible to choose this equivalence path such that each intermediate rewriting step belongs to $S$.

### 2.2 Confluence relative to a strategy

In this section, we introduce the $S$-confluence property for a set of rewriting rules $R$. We show that $S$-confluence implies confluence of $\rightarrow_R$ and guarantees that $S$-normal forms basis elements form a basis of $V/\xrightarrow{\ast}_R$. We also show that $S$-confluence is characterised in terms of a decreasingness property. We finish by a new proof of the diamond’s lemma, based on $S$-confluence.

**Definition 2.8.** Given a strategy $S$ for $R$, we say that $\rightarrow_R$ is $S$-confluent if for every rewriting rule $e \rightarrow_R v$ in $R$, we have $S$-NF$(e - v) = 0$.

The following theorem illustrates how $S$-confluence is related to confluence of $\rightarrow_R$ and the vector space $V/\xrightarrow{\ast}_R$.

**Theorem 2.9.** Let $R$ be a set of rewriting rules and let $S$ be a strategy for $R$. If the rewriting relation $\rightarrow_R$ is $S$-confluent, then it is confluent and we have a vector spaces isomorphism

$$V/\xrightarrow{\ast}_R \cong \text{im}(S\text{-NF}).$$

In particular, $\{[e]_{\xrightarrow{\ast}_R} : S\text{-NF}(e) = e\}$ is a basis of $V/\xrightarrow{\ast}_R$.

**Proof.** Since the map $S$-NF is a linear projector, the vector space $V$ admits a direct sum decomposition $V = \ker(S\text{-NF}) \oplus \text{im}(S\text{-NF})$. Hence, the isomorphism (5) means that $\xrightarrow{\ast}_R$ is the equivalence relation induced by $\ker(S\text{-NF})$, that is, $v_1 \xrightarrow{\ast}_R v_2$ if and only if $v_1 - v_2 \in \ker(S\text{-NF})$. By Point 1 Proposition 2.6 we have $v_1 \xrightarrow{\ast}_R r_S(v_i)$, which yields $v_1 \xrightarrow{\ast}_R S\text{-NF}(v_i)$ by induction. Hence, $S\text{-NF}(v_1 - v_2) = 0$ implies that $v_1 \xrightarrow{\ast}_R v_2$. The converse also implies confluence of $\rightarrow_R$ since if $v_1 \xrightarrow{\ast}_R v_2$ is equivalent to $S$-NF$(v_1 - v_2) = 0$, the previous argument shows that $\rightarrow_R$ has the Church-Rosser property. From the $S$-confluence hypothesis, $v_1 \rightarrow_R v_2$ implies $S\text{-NF}(v_1 - v_2) = 0$: indeed, with the notations of (2), for every rewriting step $v_1 = \lambda e + u \rightarrow_R v_2 = \lambda v + u$, we have $S\text{-NF}(v_1 - v_2) = \lambda S\text{-NF}(e - v) = 0$. Since the binary relation defined by $S$-NF$(v_1 - v_2) = 0$ is closed under transitivity, reflexivity, and symmetry, it contains $\xrightarrow{\ast}_R$, that is, $v_1 \xrightarrow{\ast}_R v_2$ implies $v_1 - v_2 \in \ker(S\text{-NF})$. Finally, since $S$-NF is a projector, its image admits as a basis the set of basis elements $e$ such that $S$-NF$(e) = e$. That concludes the proof.
Note that $S$-confluence is a sufficient but not a necessary condition for confluence. Indeed, with $\mathbb{Z}$ the set of integers and the rewriting rules $n \to_R n + 1$ as in Point 3 of Example 2.5 there is no strategy such that $\to_R$ is confluent relative to this strategy.

**Example 2.10.** Let us continue Point 1 of Example 2.5. The following identities hold:

\[
S\text{-NF}(e_1) = e_3 + e_4 = S\text{-NF}(e_2), \quad S\text{-NF}(e_2) = e_3 + e_4 = S\text{-NF}(e_3 + e_4)
\]

\[
S\text{-NF}(e_3) = e_3 = S\text{-NF}(e_2 - e_4),
\]

so that $\to_R$ is $S$-confluent, and hence confluent. Notice that if we replace the rule $e_3 \to_R e_2 - e_4$ by $e_3 \to_R e_3$ and $S\text{-NF}(e_2) = e_3 + e_4$, so $\to_R$ is not $S$-confluent anymore.

In Proposition 2.12, we show that $S$-confluence is characterized in terms of the decreasingness property that we introduce in Definition 2.11. The latter is stated in terms of the relation $\rightarrow^*_R$, since here again we want to take into account compatibility with vector space operations. In particular, in the proof of Proposition 2.12, we apply of Corollary 2.7 instead of Point 1 of Proposition 2.6, as mentioned above. Moreover, in Definition 2.11 we make use of the following notation:

**Definition 2.11.** Given a strategy $S$ for $R$ and a terminating order $\prec$ on $R$, we say that $R$ is decreasing w.r.t. $(S, \prec)$ if for every rewriting rule $e \to_R v$, we have a diagram:

\[
e \xrightarrow{S} r_S(e) \\
\downarrow \\
\prec
\]

where each rewriting rule occurring in the dotted arrows is strictly smaller than the rewriting rule $e \to_R v$ relative to $\prec$.

**Proposition 2.12.** Let $S$ be a strategy for $R$. The following assertions are equivalent.

1. The set of rewriting rules $R$ is $S$-confluent.

2. There exists a terminating order $\prec$ on $R$ such that $R$ is decreasing w.r.t. $(S, \prec)$.

**Proof.** $(1) \Rightarrow (2)$: We define the order $\prec$ on $R$ by $e \to_R r_S(e) < e' \to_R v$ whenever $e' \to_R v \notin S$. This order is terminating since each chain of strictly decreasing elements has length 2. Let $e \to_R v$ be a rewriting rule and let us construct a decreasing diagram as in Definition 2.11. We distinguish three cases.

- If $e$ is an $S$-normal form, then we have $S\text{-NF}(e) = e$ and $e \to_R v$ does not belong to $S$. Moreover, since $\to_R$ is $S$-confluent, we have $S\text{-NF}(e) = S\text{-NF}(v)$, and from Corollary 2.7 we get the following diagram

\[
e \xrightarrow{S} e \\
\downarrow \\
v \xrightarrow{\rightarrow^*_R} e
\]

where the equivalence path $v \leftrightarrow^*_R e$ is such that each intermediate rewriting step belongs to $S$. By definition of $\prec$, this diagram is decreasing w.r.t. $(S, \prec)$.
• If \( e \rightarrow_R v = e \rightarrow_S r_S(e) \) belongs to \( S \), then we have the following diagram:

\[
\begin{array}{c}
e \\
\downarrow s \\
\end{array}
\begin{array}{c}
r_S(e) \\
\downarrow s \\
r_S(e) \\
\end{array}
\]

Since no rewriting step occurs in the right and bottom faces, this diagram obviously is decreasing w.r.t. \((S, \prec)\).

• In the last case, \( e \) is not an \( S \)-normal form and \( e \rightarrow_R v \) does not belong to \( S \). In particular, there exists a rewriting rule of the form \( e \rightarrow_S r_S(e) \) in \( S \). By definition of the map \( S\text{-NF} \) and of the \( S \)-confluence property, we have the equalities

\[
S\text{-NF}(r_S(e)) = S\text{-NF}(e) = S\text{-NF}(v).
\]

By Corollary 2.7, we get the following diagram:

\[
\begin{array}{c}
e \\
\downarrow s \\
\end{array}
\begin{array}{c}
r_S(e) \\
\downarrow s \\
v \xleftarrow{s} S\text{-NF}(e).
\end{array}
\]

where the rewriting steps occurring in \( v \xleftarrow{s} S\text{-NF}(e) \) and \( r_S(e) \xleftarrow{s} S\text{-NF}(e) \) belong to \( S \). Hence, the diagram is decreasing w.r.t. \((S, \prec)\).

(2) \( \Rightarrow \) (1): Let \( e \rightarrow_R v \) be a rewriting rule and let us assume by induction that for every rewriting step \( e' \rightarrow_R v' \) smaller than \( e \rightarrow_R v \) for \( \prec \), we have \( S\text{-NF}(e') = S\text{-NF}(v') \). Consider a decreasing diagram:

\[
\begin{array}{c}
e \\
\downarrow s \\
\end{array}
\begin{array}{c}
r_S(e) \\
\downarrow s \\
v \xleftarrow{s} S\text{-NF}(e).
\end{array}
\]

Using our induction hypothesis and adapting the argument of the proof of Theorem 2.9, we have \( S\text{-NF}(v) = S\text{-NF}(u) = S\text{-NF}(r_S(e)) \). Hence, since \( S\text{-NF}(e) = S\text{-NF}(r_S(e)) \) by definition of the map \( S\text{-NF} \), we have \( S\text{-NF}(e) = S\text{-NF}(v) \). The order \( \prec \) being terminating, this inductive argument proves that \( S\text{-NF}(e) = S\text{-NF}(v) \) for every rewriting rule \( e \rightarrow_R v \), that is, \( R \) is \( S \)-confluent.

\begin{remark}
In the case where the rewriting system comes from a set-theoretic rewriting system (that is, the right-hand sides of the rewriting rules are elements of the basis), the fact that local \( S \)-confluence implies that \( \rightarrow_R \) is confluent is a special case of Van Oostroom’s decreasing diagrams [21]. More precisely, based on Proposition 2.12 local \( S \)-confluence implies that the pair of rewriting relations \((\rightarrow_S, \rightarrow_R)\) is decreasing with respect to conversions (see [21] Definition 3)], using the order \( \prec \) on \( R \) and the discrete order on \( \rightarrow_S \). By [21] Theorem 3], this implies that the relations \((\rightarrow_S, \rightarrow_R)\) commute. Using the fact that \( \rightarrow_S \subseteq \star \rightarrow_R \), one can then recover that \( \rightarrow_R \) is confluent.
\end{remark}

\begin{example}
Let us illustrate Proposition 2.12 with Example 2.10. Let us consider the following order \( \prec \) on rewriting rules:

\[
(e_1 \rightarrow_R e_2) \prec (e_3 \rightarrow_R e_2 - e_4), \quad (e_2 \rightarrow_R e_3 + e_4) \prec (e_3 \rightarrow_R e_2 - e_4).
\]

\end{example}
This choice is guided by the heuristic that rules advancing towards an $S$-normal form should be favored over rules that do not: here $e_3$ is an $S$-normal form, so the rule that rewrites it should be larger for the order $\prec$. The decreasing diagrams are the following:

We finish this section by showing how the diamond’s lemma fits as a particular case of our setup.

**Theorem 2.15 ([3]).** Let $R$ be a set of rewriting rules such that the rewriting preorder $<_R$ of $\rightarrow_R$ is terminating and for every $e \in \mathcal{B}$ such that $e \rightarrow_R v$ and $e \rightarrow_R v'$, $v$ and $v'$ are joinable. Then, $\rightarrow_R$ is confluent.

**Proof.** For every basis element $e$ that is reducible by $\rightarrow_R$, we select exactly one arbitrary rewriting rule with left hand-side $e$. Then, let $S$ be the pre-strategy composed of these selected rewriting rules. Since $<_R$ is terminating, $<_S$ is also terminating, so that $S$ is a strategy for $R$. Let us show that $\rightarrow_R$ is $S$-confluent using the criterion of Proposition 2.12. For that, we define the terminating order $\prec$ on $R$ by letting $(e \rightarrow_R v) \prec (e' \rightarrow_R v')$ whenever $e <_R e'$. If $e \rightarrow_R v$ is a rewriting rule, then $e$ is not an $R$-normal form and $e \not\rightarrow_S r_S(e)$ means $e \rightarrow_S r_S(e)$. Using the confluence hypothesis of the theorem, we have a diagram

Each rewriting rule appearing in the right and bottom faces is strictly smaller than $e \rightarrow_R r$ for $\prec$ by definition of this rewriting preorder. Hence, this diagram is decreasing, so that $R$ is $S$-confluent. From Theorem 2.9 $\rightarrow_R$ is confluent.

Notice that in the proof of the diamond’s lemma, we select for $<_R$ another order than the one given in the proof of Proposition 2.12 (the latter asserts that each rule of $S$ is smaller than each rule which is not in $S$ and there is no other comparison). This is a good illustration of the flexibility of the characterisation of $S$-confluence given in Proposition 2.12.

### 3 rewriting strategies over rational Weyl algebras

In this section, we investigate rewriting systems over rational Weyl algebras and relate involutive divisions to rewriting strategies for such systems. In particular, we show that involutive sets in rational Weyl algebras induce confluent rewriting systems.

Throughout the section, we fix a set $X = \{x_1, \cdots, x_n\}$ of indeterminates and the field of fractions of the polynomial algebra $\mathbb{Q}[x_1, \cdots, x_n]$ over $\mathbb{Q}$ is denoted by $\mathbb{Q}(X) := \mathbb{Q}(x_1, \cdots, x_n)$, it is the set of rational functions in the indeterminates $X$. We fix another set of variables $\Delta = \{\partial_1, \cdots, \partial_n\}$ that model partial derivative operators, see Example 3.2. We denote by $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ the monomial over $\Delta$ with multi-exponent $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$. Finally, let $\text{Mon}(\Delta)$ be the set of monomials over $\Delta$:

$$\text{Mon}(\Delta) := \{\partial^\alpha : \alpha \in \mathbb{N}^n\}.$$  

In what follows, we keep the terminology monomials only for elements of $\text{Mon}(\Delta)$ and not for elements in $\text{Mon}(X)$.
3.1 Rewriting systems over rational Weyl algebras

In this section, we recall the definition of the rational Weyl algebra and introduce rewriting systems on the rational Weyl algebra induced by monic operators.

Definition 3.1. The rational Weyl algebra over \( \mathbb{Q}(X) \) is the set of polynomials \( \mathbb{Q}(X)[\Delta] \) with coefficients in \( \mathbb{Q}(X) \) and indeterminates \( \Delta \). The multiplication of this \( \mathbb{Q} \)-algebra is induced by the commutation laws \( \partial_i \partial_j = \partial_j \partial_i \) and

\[
\partial_i f = f \partial_i + \frac{d}{dx_i}(f), \quad f \in \mathbb{Q}(X), \quad 1 \leq i \leq n,
\]

where \( d/dx_i : \mathbb{Q}(X) \to \mathbb{Q}(X) \) is the partial derivative operator with respect to \( x_i \). This algebra is denoted by \( B_n(\mathbb{Q}) \).

Notice that \( B_n(\mathbb{Q}) \) is a \( \mathbb{Q}(X) \)-vector space and that the monomial set \( \text{Mon}(\Delta) \) is a basis of \( B_n(\mathbb{Q}) \). Elements of \( B_n(\mathbb{Q}) \) should be thought of as differential operators whose coefficients are rational functions, and for this reason, a generic element of this algebra is denoted by \( D \) and is called a differential operator. In the following example, we illustrate how these operators provide an algebraic model of linear systems of ordinary differential (in the case \( n = 1 \)) and partial derivative equations (in the case \( n \geq 2 \)) with one unknown function.

Example 3.2.

1. The linear ordinary differential equation \( y'(x) = xy(x) \) is written in the form \( (Dy)(x) = 0 \), where the operator \( D := \partial - x \) belongs to \( B_1(\mathbb{Q}) = \mathbb{Q}(x)[\partial] \).

2. Consider Janet’s example \([17]\), that is, the linear system of partial derivative equations with 3 variables, one unknown function, and the two equations \( y_{33}(x) = x_2 y_{11}(x) \) and \( y_{22}(x) = 0 \), where \( y_{ij}(x) \) denotes the second order derivative of the unknown function \( y(x) \) with respect to the variables \( x_i \) and \( x_j \). Then, these equations are written \( (D_1 y)(x) = 0 \) and \( (D_2 y)(x) = 0 \), where \( D_1, D_2 \in B_3(\mathbb{Q}) \) are defined as follows:

\[
D_1 := \partial_3^2 - x_2 \partial_1^2, \quad D_2 := \partial_2^2.
\]

Remark 3.3. In \([1]\) of Example \([8,2]\) we implicitly used that every \( f \in \mathbb{Q}(X) \) induces a unique multiplication operator \( y(x) \mapsto f(x)y(x) \).

The next step before introducing rewriting systems over rational Weyl algebras is to recall the definition of monic operators. We fix a monomial order \( \prec \) on \( \text{Mon}(\Delta) \), that is, a terminating total order which is admissible, i.e., \( \partial^\alpha \prec \partial^\beta \) implies \( \partial^{\alpha+\gamma} \prec \partial^{\beta+\gamma} \), for every \( \alpha, \beta, \gamma \in \mathbb{N}^n \). Given an operator \( D \), we denote by \( \text{lm}(D) \) the leading monomial of \( D \) with respect to \( \prec \), that is, \( \text{lm}(D) \) is the greatest element of \( \text{supp}(D) \), where the support is defined w.r.t. the basis \( \text{Mon}(\Delta) \).

Definition 3.4. Let \( \prec \) be a monomial order on \( \text{Mon}(\Delta) \). A differential operator \( D \in B_n(\mathbb{Q}) \) is said to be \( \prec \)-monic if the coefficient of \( \text{lm}(D) \) on \( D \) is equal to 1. Moreover, given a monic differential operator \( D \), we denote by \( r(D) := \text{lm}(D) - D \).

Since the monomial order \( \prec \) is fixed, me simply say monic instead of \( \prec \)-monic. Given a set of monic operators \( \Theta \subseteq B_n(\mathbb{Q}) \), let us consider the rewriting relation on \( B_n(\mathbb{Q}) \) induced by the following rewriting rules:

\[
R_\Theta := \left\{ \partial^\alpha \text{lm}(D) \to_{R_\Theta} \partial^\alpha r(D) : D \in \Theta, \partial^\alpha \in \text{Mon}(\Delta) \right\}.
\]

(6)

For simplicity, we write \( D \to_{\Theta} D' \) instead of \( D \to_{R_\Theta} D' \). The rewriting relation \( \to_{\Theta} \) is terminating since the rewriting rules reduce a monomial into a combination of strictly smaller monomials w.r.t. the
terminating order \(\prec\). Moreover, notice that in the case where the coefficient \(\text{lc}(\mathcal{D}) \in \mathbb{Q}(X)\) of \(\text{lm}(\mathcal{D})\) in \(\mathcal{D}\) is not constant, the situation is much harder. Indeed, in this case, the left-hand sides of the rewriting rules are of the form \(\partial^\alpha(\text{lc}(\mathcal{D})\text{lm}(\mathcal{D}))\) and due to commutation laws, these elements are not monomials. In particular, we are not in the situation of our general approach developed in Section 3 anymore.

We finish this section with some comments on \(\rightarrow_\Theta\). Let us consider the linear system of ordinary differential or partial derivative equations with unknown function \(y\) given by

\[
\{(\mathcal{D}y) = 0 : \mathcal{D} \in \Theta\}. \tag{7}
\]

Let \(y(x)\) be an arbitrary solution to this system. Then, for every operator \(\partial^\alpha\) and every \(\mathcal{D} \in \Theta\), we also have \((\partial^\alpha \mathcal{D} y)(x) = 0\), or equivalently, \((\partial^\alpha \text{lm}(\mathcal{D}) y)(x) = (\partial^\alpha r(\mathcal{D}) y)(x)\). Hence, if there is a rewriting path \(\mathcal{D}_1 \rightarrow_\Theta \mathcal{D}_2\), then the solution \(y(x)\) of (7) satisfies \((\mathcal{D}_1 y)(x) = (\mathcal{D}_2 y)(x)\). This remark has deep applications in the formal theory of partial differential equations, for instance for finding integrability conditions or computing dimensions of solution spaces, see [17]. Moreover, notice that since \(\text{Mon}(\Delta)\) is a commutative set, there is another possible choice for rewriting the monomial \(\partial^\alpha \text{lm}(\mathcal{D})\) in (7). Indeed, we could swap \(\partial^\alpha\) and \(\text{lm}(\mathcal{D})\) to get the new rule \(\text{lm}(\mathcal{D}) \partial^\alpha \rightarrow_\Theta r(\mathcal{D}) \partial^\alpha\). This rule is simpler in the sense that it does not require to apply any commutation law to its right-hand side in contrast with (7). However, we do not take this rule into account since it would break the algebraic model of partial derivative equations. Indeed, if \(y(x)\) is a solution of (7), then the relation \((\text{lm}(\mathcal{D}_1) \partial^\alpha y)(x) = (r(\mathcal{D}_1) \partial^\alpha y)(x)\) does not hold in general, as illustrated in (8) of the following example.

**Example 3.5.** We continue Example 3.2

1. Let \(\Theta := \{\mathcal{D}\}\) where \(\mathcal{D} := \partial - x \in B_1(\mathbb{Q})\). Since \(\partial\) is greater than 1 for every monomial order, \(\rightarrow_\Theta\) is induced by the rewriting rules \(\partial^n \rightarrow_\Theta \partial^{n-1} x\), where \(n\) is a strictly positive integer. In particular, we have the following rewriting sequence:

\[
\partial^2 \rightarrow_\Theta \partial x = x \partial + 1 \rightarrow_\Theta x^2 + 1.
\]

In terms of the corresponding differential equation \(y'(x) = xy(x)\), this rewriting sequence has the following meaning. First, notice that the space of solutions of this equation is the one-dimensional \(\mathbb{R}\)-vector space spanned by the function \(e^{x^2/2}\). Moreover, the second order derivative of a solution \(y(x) = Ce^{x^2/2}\), for an arbitrary constant \(C\), is given by the formula \(y''(x) = (x^2 + 1)Ce^{x^2/2}\), which reads \((\partial^2 y)(x) = (x^2 + 1)y(x)\) in terms of operators. Notice that if we allow to reduce the left \(\partial\) in \(\partial^2\), then we get \(\partial^2 \rightarrow_\Theta x^2\), which is false in terms of operators since \(y''(x)\) is not equal to \(x^2 y(x)\).

2. Let \(\Theta := \{\mathcal{D}_1, \mathcal{D}_2\}\), where \(\mathcal{D}_1 := \partial_3^2 - x_2 \partial_1^2\) and \(\mathcal{D}_2 := \partial_2^2\) correspond to the two equations of the Janet example. We define \(\prec\) as being the deg-lex order on \(\text{Mon}(\partial_1, \partial_2, \partial_3)\) induced by \(\partial_1 \prec \partial_2 \prec \partial_3\), so that \(\rightarrow_\Theta\) is induced by the rewriting rules \(\partial_3^2 \rightarrow_\Theta x_2 \partial_1^2\) and \(\partial_2^2 \rightarrow_\Theta 0\). Then, \(\rightarrow_\Theta\) is not confluent since:

\[
\begin{array}{ccc}
\partial_3^2 \partial_3^2 & \rightarrow_\Theta & \partial_2^2(x_2 \partial_1^2) \\
\downarrow & & \downarrow \\
0 & & 2\partial_1^2 \partial_2
\end{array} \tag{8}
\]

The right arrow is an application of the rule \(\partial_2^2 \rightarrow_\Theta 0\), made possible since \(\partial_2^2(x_2 \partial_1^2)\) is equal to \(\partial_2^2 \partial_2 + x_2 \partial_1^2 \partial_2^2\) (to see this, it suffices to apply twice the commutation law \(\partial_2 x_2 = x_2 \partial_2 + 1\)). We deduce from (8) that any solution \(y(x)\) of the equations \((\mathcal{D}_i y)(x) = 0\) has to verify the new integrability condition \(y_{112}(x) = 0\).
3.2 Involutive divisions and strategies

In this section, we interpret involutive divisions in terms of strategies for the rewriting relation induced by a set of monic differential operators. From this, we show that the rewriting system induced by an involutive set of operators is confluent.

We first recall from [1] the definition of involutive divisions and associated notions that are involutive divisors, multiplicative variables, and autoreducibility. For that, we temporarily work with monomials instead of operators and denote these monomials with Latin letters $u, m$ instead of $\partial^a$. Then, we will reuse the operator notation for monomials when we will consider rewriting systems over rational Weyl algebras. An involutive division $L$ on $\text{Mon}(\Delta)$ is defined by a binary relation $\mid_L$ on $U \times \text{Mon}(\Delta)$, for every finite subset $U \subset \text{Mon}(\Delta)$, satisfying for every $u, u' \in U$ and every $m, m' \in \text{Mon}(\Delta)$, the following relations:

\begin{enumerate}  
    \item $u \mid_L m \Rightarrow u \mid m,$  
    \item $u \mid_L u,$  
    \item $u \mid_L um$ and $u \mid_L um'$ if and only if $u \mid_L umm',$  
    \item $u \mid_L m$ and $u' \mid_L m$ implies $u \mid_L u'$ or $u' \mid_L u,$  
    \item $u \mid_L u'$ and $u' \mid_L m$ implies $u \mid_L m,$  
    \item for every $V \subseteq U$ and every $v \in V, v \mid_L m$ implies $v \mid_L m.$
\end{enumerate}

In the sequel, we write $\mid_L$ instead if $\mid_L$ when the context is clear. We say that $u \in U$ is an $L$-involutive divisor of a monomial $m$ if $u \mid_L m$. The variable $\partial_i$ is said to be $L$-multiplicative for $u$ w.r.t. $U$ if $u$ is an $L$-involutive divisor of $\partial_i u$. Notice that $u \mid_L m$ if and only if $m = m'u$, where $m'$ contains only $L$-multiplicative variables for $u$ w.r.t. $U$. Notice also that an involutive division is entirely determined by the list of multiplicative variables w.r.t. each finite set $U$ such that conditions [1], [2] and [3] are fulfilled. We say that $U$ is $L$-autoreduced if every $u \in U$ admits only $u$ as $L$-involutive divisor, i.e., $u' \mid_L u$ implies $u' = u$. Notice that if $U$ is $L$-autoreduced, then every monomial $m$ admits at most one $L$-involutive divisor. We finish this discussion on involutive divisions with two classical examples. Before, let us introduce the following notation: given a monomial $m = \partial^a \in \text{Mon}(\Delta)$, let us denote by $d_k(m) := \alpha_k$ the degree of $m$ w.r.t. the variable $\partial_k$.

Example 3.6. We fix a finite set of monomials $U \subset \text{Mon}(\Delta)$. The Janet, Thomas and Pommaret divisions are the involutive divisions $\mid_J, \mid_T, \text{ and } \mid_P$ such that the variable $\partial_i$, where $1 \leq i \leq n$, is $J, L$ or $P$-multiplicative for $u$ w.r.t. $U$ if and only if

\begin{itemize}  
    \item for $\mid_J$: $d_i(u) = \text{max}\{d_i(u') : u' \in U \text{ and } d_j(u') = d_j(u), \forall j < i \leq n\},$  
    \item for $\mid_T$: $d_i(u) = \text{max}\{d_i(u') : u' \in U\},$  
    \item for $\mid_P$: for every $1 \leq j \leq i$, we have $d_j(u) = 0.$
\end{itemize}

Now, we return to differential operators and we fix a monomial order $\prec$ on $\text{Mon}(\Delta)$. Given a finite set $\Theta \subset B_n(\mathbb{Q})$ of $\prec$-monic differential operators, all the theory of monomial sets can be applied to the case where $U$ is the set of leading monomials of elements of $\Theta$:

$$\text{lm}(\Theta) := \{\text{lm}(D) : D \in \Theta\} \subset \text{Mon}(\Delta)$$

Hence, we may extend the autoreducibility property for monomial sets w.r.t. an involutive division to sets of differential operators.

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Definition 3.7. Let \( \Theta \subset B_n(\mathbb{Q}) \) be a finite set of \(<\)-monic differential operators, let \(<\) be a monomial order, and let \( L \) be an involutive division on \( \text{Mon}(\Delta) \). We say that \( \Theta \) is **left \( L \)-autoreduced** if \( \text{lm}(\Theta) \) is \( L \)-autoreduced.

The adjective ”left” is here to emphasis that it may exist \( D, D' \in \Theta \) such that \( \text{lm}(D) \) is an \( L \)-involutive divisor of a monomial \( \partial^n \in \text{supp}(r(D')) \).

Example 3.8. We can now apply the involutive divisions of Example 3.6 to find the multiplicative variables associated to the differential operators of Example 3.5.

1. Take \( \Theta = \{D\} \), where \( D = \partial - x \in B_1(\mathbb{Q}) \). Then, \( \text{lm}(D) = \partial \), and \( \partial \) is a multiplicative variable for \( D \) for the Janet and Thomas divisions, but not for the Pommaret one. This means that \( \partial \mid \partial^n \) and \( \partial \notmid \partial^n \) for all \( n > 0 \), but that \( \partial \mid \partial^3 \partial^n \), unless \( n = 1 \). In addition, since \( \Theta \) is a singleton, it is trivially left-autoreduced for all three involutive divisions.

2. Take now \( \Theta = \{D_1, D_2\} \), where \( D_1 = \partial_1^2 - x_2 \partial_1^2 \) and \( D_2 = \partial_2^3 \). The following table gives the multiplicative variables for \( D_1 \) and \( D_2 \) w.r.t. \( \Theta \) for all three involutive divisions:

|         | Janet | Thomas | Pommaret |
|---------|-------|--------|----------|
| \( D_1 \) | \( \partial_1, \partial_2, \partial_3 \) | \( \partial_1, \partial_2 \) | \( \emptyset \) |
| \( D_2 \) | \( \partial_1, \partial_2 \) | \( \partial_2 \) | \( \partial_1 \) |

Once again, the leading monomials of elements of \( \Theta \) do not divide each others, so \( \Theta \) is left-autoreduced for all three involutive divisions.

From now on, we fix a set \( \Theta \) of monic (the order being fixed, we drop it in \(<\)-monic) differential operators. Let \( R_{\Theta} \) be the set of rewriting rules of the form \( \partial^n \text{lm}(D) \rightarrow_{\Theta} \partial^m r(D) \), such as in (6). Since \( \text{lm}(\Theta) \) is the only monomial set we will work with, we omit it in the symbol of the involutive division: we write \( \text{lm}(D) \mid_L \partial^n \text{lm}(D) \) when \( \partial^n \) contains only \( L \)-multiplicative variables for \( \text{lm}(D) \) w.r.t. \( \text{lm}(\Theta) \). Finally, we let

\[
S_{\Theta,L} := \left\{ \text{lm}(D) \rightarrow_{\Theta,L} \partial^n r(D) : D \in \Theta, \text{lm}(D) \mid_L \partial^n \text{lm}(D) \right\}.
\]

(9)

Here again, we choose to write \( \partial^n \text{lm}(D) \rightarrow_{\Theta,L} \partial^m r(D) \) instead of \( \partial^n \text{lm}(D) \rightarrow S_{\Theta,L} \partial^m r(D) \) in order to simplify notations.

Proposition 3.9. Let \( L \) be an involutive division on \( \text{Mon}(\Delta) \) such that \( \Theta \) is left \( L \)-autoreduced. Then \( S_{\Theta,L} \) is a strategy for \( R_{\Theta} \).

Proof. If the set \( \Theta \) is left \( L \)-autoreduced, then every monomial admits at most one \( L \)-involutive divisor. Moreover, every left-hand side \( \partial^n \text{lm}(D) \) of a rewriting rule of \( S_{\Theta,L} \) is \( L \)-divisible by \( \text{lm}(D) \). Hence, left-hand sides of \( S_{\Theta,L} \) are pairwise distinct, which means that \( S_{\Theta,L} \) is a pre-strategy for \( R_{\Theta} \). Finally, if \( \prec_{\Theta} \) denotes the rewriting preorder of \( \rightarrow_{\Theta} \), then \( \partial^n \prec_{\Theta} \partial^m \) implies that \( \partial^n \prec \partial^m \), so that \( \prec_{\Theta} \) is terminating. Hence, the rewriting preorder of \( \rightarrow_{S_{\Theta,L}} \) is also terminating, and \( S_{\Theta,L} \) is a strategy for \( R_{\Theta} \). \( \square \)

From Proposition 3.9, any involutive division \( L \) such that \( \Theta \) is left \( L \)-autoreduced induces a strategy \( S_{\Theta,L} \) for \( R_{\Theta} \). Hence, we get a well-defined normalisation operator \( S_{\Theta,L} - \text{NF} \) corresponding to this strategy. The following definition is an adaptation of the notion of involutive bases for polynomial ideals \([9]\) to the case of sets of monic differential operators.

Definition 3.10. Let \( \Theta \subset B_n(\mathbb{Q}) \) be a finite set of differential operators, let \(<\) be a monomial order on \( \text{Mon}(\Delta) \) such that each element of \( \Theta \) is monic, and let \( L \) be an involutive division on \( \text{Mon}(\Delta) \) such that \( \Theta \) is left \( L \)-autoreduced. We say that \( \Theta \) is an **\( L \)-involutive set** if for every \( D \in \Theta \) and every \( \partial^n \in \text{Mon}(\Delta) \), we have \( S_{\Theta,L} - \text{NF}(\partial^n D) = 0 \).
Example 3.11. Let us continue Example 3.8.

1. In the case $\Theta = \{D\}$, with $D = \partial - x$. For the Pommaret division, we have seen that $D$ admits no multiplicative variable, so the strategy $S_{\Theta, P}$ is reduced to the rule $\partial \rightarrow_{\Theta, P} x$. As a result we get:

$$\partial D = \partial^2 - \partial x = \partial^2 - x\partial - 1 \rightarrow_{\Theta, P} \partial^2 - x^2 - 1.$$  

This last term is a normal form for $\rightarrow_{\Theta, P}$, hence $S_{\Theta, P} \cdot \text{NF}(\partial D) \neq 0$ and so $\Theta$ is not $P$-involutive. On the other hand for the Janet and Thomas divisions, $S_{\Theta, J}$ and $S_{\Theta, T}$ coincide, and contain the rules $\partial^{n+1} \rightarrow_{\Theta, L} \partial^n x$, where $L = J, T$. This yields:

$$\partial D = \partial^2 - \partial x = \partial^2 - x\partial - 1 \rightarrow_{\Theta, L} \partial x - x^2 - 1 = x\partial - x^2 \rightarrow_{\Theta, L} 0.$$  

So we get $S_{\Theta, L} \cdot \text{NF}(\partial D) = 0$, and more generally $S_{\Theta, L} \cdot \text{NF}(\partial^n D) = 0$: $\Theta$ is both $J$- and $T$-involutive.

2. In the case $\Theta = \{D_1, D_2\}$, with $D_1 = \partial^3_2 - x_2\partial^1_2$ and $D_2 = \partial^2_1$, $\Theta$ will not be involutive for either of the three involutive divisions of Example 3.8. In the case of the Janet division for example, we have:

$$\partial^3_2 D_2 = \partial^2_1 \partial^1_2 \rightarrow_{\Theta, J} \partial^2_1 (x_2 \partial^1_2) = x_2 \partial^2_1 \partial^1_2 - 2\partial^2_1 \partial_2 \rightarrow_{\Theta, J} 2\partial^2_1 \partial_2.$$  

This last term is a normal form for $S_{\Theta, J}$, so we get $S_{\Theta, J} \cdot \text{NF}(\partial^3_2 D_2) = 2\partial^1_2 \partial_2 \neq 0$: $\Theta$ is not $J$-involutive.

The astute reader may remark that the last computation of the previous example is closely related to the diagram appearing in Example 3.5 which shows that $\rightarrow_{\Theta}$ fails to be confluent. This relationship between confluence and $L$-involutivity is actually a very general one, as shown by the following theorem.

Theorem 3.12. Let $\Theta \subset B_n(\mathbb{Q})$ be a finite set of differential operators, let $\prec$ be a monomial order on $\text{Mon}(\Delta)$ such that each element of $\Theta$ is monic, and let $L$ be an involutive division on $\text{Mon}(\Delta)$ such that $\Theta$ is left $L$-autoreduced. If $\Theta$ is $L$-involutive, then the rewriting relation $\rightarrow_{\Theta}$ is confluent.

Proof. Let $S_{\Theta, L}$ be the strategy for $R_{\Theta}$ defined such as in (9). Since rewriting rules of $R_{\Theta}$ are of the form $\partial^n \text{Im}(D) \rightarrow_{\Theta} \partial^n R(D)$, where $D \in \Theta$ and $\partial^n \in \text{Mon}(\Delta)$, the assumption that $\Theta$ is $L$-involutive means that $\rightarrow_{\Theta}$ is $S_{\Theta, L}$-confluent. By Theorem 2.9 $\rightarrow_{\Theta}$ is confluent. \[\square\]

Remark 3.13. As for term rewriting systems or Gröbner bases theory, there exists a completion procedure in the situation of differential operators, which corresponds to Knuth-Bendix or Buchberger procedures. In the case of the Janet example, it turns out that after a finite number of steps, this procedure yields the following involutive set, see [17]:

$$\overline{\Theta} = \{D_1, D_2, \partial^1_1 \partial_2, \partial^2_2 \partial_3, \partial^1_1, \partial^2_1 \partial_2 \partial_3, \partial^1_1 \partial_2 \partial_3\}.$$  

The end of this section aims to show that axioms (a)–(c) in the definition of an involutive division may be formulated in a purely rewriting language using strategies. We fix a strategy $S$ for $\rightarrow_{\Theta}$. For every $D \in \Theta$, we say that $\text{Im}(D)$ $S$-divides the monomial $\partial^n \in \text{Mon}(\Delta)$ if $S$ contains a rewriting rule of the form $\partial^n \text{Im}(D) \rightarrow_{S} \partial^n r(D)$ and we say that the variable $\partial_i \in \Delta$ is $S$-multiplicative for $D$ if $\partial_i \text{Im}(D)$ is $S$-divisible by $\text{Im}(D)$.

Definition 3.14. A strategy $S$ for $R_{\Theta}$ is said to be involutive if for every left-hand side $\partial^n \text{Im}(D)$ of a rewriting rule in $S$, then $\partial^n$ contains only $S$-multiplicative variables of $D$. 

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Proposition 3.15. If the strategy \( S \) is involutive, then the \( S \)-division satisfies axioms \([a]\)–\([e]\) of the definition of an involutive division. Moreover, if \( L \) is an involutive division on \( \text{Mon}(\Delta) \) such that \( \Theta \) is left \( L \)-autoreduced, then the \( S_{\Theta,L} \)-division is the restriction of \( L \) to \( \text{lm}(\Theta) \).

Proof. Let us show the first assertion. Axioms \([a]\)–\([d]\) and \([e]\) hold since \( S \) is a strategy for \( R_\Theta \). Indeed, left-hand sides of \( R_\Theta \) are of the form \( \partial^\alpha \text{lm}(D) \), hence \([a]\) and left-hand sides of elements of \( S \) are pairwise distinct, hence \([d]\) and \([e]\). Moreover, axioms \([b]\) and \([c]\) hold by definition of an involutive strategy.

Let us show the second assertion. By definition of the strategy \( S_{\Theta,L} \) and of the \( S_{\Theta,L} \)-division, \( \text{lm}(D) \) has the same set of multiplicative variables for \( L \) and for the \( S_{\Theta,L} \)-division. Hence, a monomial \( \partial^\alpha \) is \( L \)-divisible by \( \text{lm}(D) \), with \( D \in \Theta \), if and only if it is \( S_{\Theta,L} \)-divisible by \( \text{lm}(D) \). That proves the assertion.

4 Conclusion and perspectives

In this paper, we considered rewriting systems over vector spaces, where we proposed an alternative approach to the traditional one, since we used parallel rewriting steps. We also established some links with the traditional approach, by giving a confluence criterion as well as a proof of the diamond’s lemma, based on strategies. Finally, we showed that our general framework may be adapted to rational Weyl algebras, where coefficients do not commute with monomials. In particular, we proved that an involutive set in a rational Weyl algebra induces a confluent rewriting system on it. We now present some possible extensions of our work.

A first research direction is to investigate the so-called standardisation properties \([15]\) associated to a rewriting strategy. Indeed, the choice of strategy is nothing but the choice for every vector of a preferred rewriting sequence starting at this vector. Moreover, we have shown in Point 1 of Proposition 2.6 that each elementary rewriting step for a strategy has same source and target points than a rewriting sequence involving rules that do not belong to the strategy. The proof of this fact is based on complete developments of residuals, that play a central role in standardisation results, corresponding to left-hand sides of the strategy. As a particular case, we expect to interpret Janet bases in terms of standardisation.

A second research direction is to extend our work to other algebraic structures than vector spaces. This looks promising since we do away with the notion of well-formed rewriting step, specific to the vector space case. More generally, we hope to be able to extend our results to an arbitrary category \( C \) (satisfying some suitable condition), recovering abstract rewriting in the case where \( C \) is the category of sets, and linear rewriting as presented in this work in the case where \( C \) is the category of vector spaces. Instead of having a set or a vector space of terms to be rewritten, one would then have an object of terms, which would be an object of \( C \).

A last research direction consists in applying rewriting systems over rational Weyl algebras to the formal analysis of systems of partial differential equations. As mentioned above, this topic covers many kinds of problems and many techniques coming from rewriting theory and algebra may be used in this context. We may focus on using rewriting methods applied to the \textit{Spencer cohomology} \([17]\), which, roughly speaking, provides intrinsic properties, namely \( 2 \)-acyclicity and \textit{formal integrability}, that guarantee existence of normal form power series solutions.

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