Dynamical scaling at the quantum Hall transition: Coulomb blockade versus phase breaking

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We argue that the finite temperature dynamics of the integer quantum Hall system is governed by two independent length scales. The consistent scaling description of the transition makes crucial use of two temperature critical exponents, reflecting the interplay between charging effects and interaction-induced dephasing. Experimental implications of the two-scale picture are discussed.

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Scaling treatment of the Anderson metal-to-insulator transition is central to understanding of the integer quantum Hall (QH) effect \cite{polyakov2004}. The plateau transitions are understood as isolated critical points separating two localized phases, so that the localization length $\xi$ only diverges at a discrete set of the critical energies $E_c$. While a reliable analytical theory is sorely missing, the scaling ideas have long served to correlate the results of experiment and of numerical simulation. The observed dynamical scaling, however, still presents a puzzle which has defied a convincing explanation for almost a decade, starting from the very first experiments \cite{pollak1983}. On the experimental side, the scaling has been probed by tuning through the transition at different temperatures (by varying the Landau level filling factor) and observing how fast the critical singularities are rounded off with increasing $T$. The experimental data tell us that the long-distance cutoff $L_h$ scales as $T^{-1/2}$ with the dynamical critical exponent $z = 1$. Specifically, the dissipative dc conductivity $g$ (in units of $e^2/h$) has the scaling form $g = g_c F(L_h/\xi)$, where $F(0) = 1$, $F(\infty) = 0$, and $g_c \sim 1$. The traditional use of $z$ in this context is related to the common belief \cite{polini2010} that at criticality the only relevant temporal scale is $\tau \sim T^{-1}$.

It can be readily seen, however, that despite the simplicity of this experimental picture, it implies the inadegacy, in describing the QH critical point, of the usual theoretical framework \cite{zhang1988} based on the assumption that the system at criticality can be characterized by just one temporal scale $T^{-1}$. Indeed, the peculiarity of the Anderson transition in two dimensions – the non-vanishing of the interaction-induced dephasing length $L_\phi$, one encounters the difficulty in trying to connect the $T^{-1}$ behavior of $L_h$ with the usual dependence $L_\phi \propto T^{-1/2}$, which merely reflects the diffusive character of transport of interacting particles and should be valid at the QH critical point as well. Hence the concept of the quantum-classical crossover controlled by the dephasing length appears to be inadequate to the physics of the QH transition. Note, however, that the discarding of $L_\phi$ is not quite trivial since $L_\phi \ll L_h$ in the low-$T$ limit, which means that the shorter of the two length scales is irrelevant.

In this paper, we attempt to sort out the problem of the dynamical scaling. Our findings are as follows. The scaling description of the integer QH transition for interacting electrons includes two independent length scales, $L_h \propto T^{-1}$ and $L_\phi \propto T^{-1/2}$. They govern the temperature driven scaling outwards and towards the un-
stable fixed point \( \mathcal{I} \), respectively (Fig. 1). Both are related to the corresponding temporal scales \( \tau_h \) and \( \tau_\phi \) via the diffusion law \( (z = 2): \tau_h \sim \Delta L^2_0 \propto T^{-2} \) and \( \tau_\phi \sim D L^2_0 \propto T^{-1} \). The Coulomb interaction therefore does not change the true dynamical exponent \( z \) from 2 to 1; instead, it leads to the emergence of the two different scales. It is only if one uses the usual representation of the length scales in the form \( L_h \propto T^{-1/z_1} \) and \( L_\phi \propto T^{-1/z_2} \) that there appears the dynamical exponent \( z_1 = 1 \), whereas \( z_2 \) remains equal to 2 [3]. The typical energy transfer is \( T \) and the phase-breaking rate \( \tau_\phi^{-1} \) is also of order \( T \); however, the scattering rate \( \tau_h^{-1} \) behaves as \( T^2 \). The corresponding cutoff \( L_h \) has nothing to do with the phase breaking: the temperature smearing of the transition is controlled by charging effects similar to those in the Coulomb blockade regime. The shape of the Coulomb gap in the one-particle DOS at the critical point has no direct relation to either of the dynamical exponents \( z_1 \) or \( z_2 \). Separately, we argue that \( \rho_1(\omega) \) vanishes as \( \exp[-\alpha \ln^2(T_c/|\omega|)] \), where \( \alpha \sim 1 \) and \( T_c \) is a characteristic width of the gap.

![FIG. 1. Scaling with lowering \( T \) outwards and towards the unstable fixed point is governed by different length scales with temperature exponents \( z_1 = 1 \) and \( z_2 = 2 \), respectively.](image)

Our basic point in the description of the dynamical scaling is that the QH system at the critical point is metallic (in contrast to the critical system at a conventional Anderson transition in three dimensions) and it makes perfect sense to treat it as an ordinary dirty metal with \( g \sim 1 \). We therefore begin with the effect of electron-electron scattering on the quantum interference of diffusons \( \mathcal{I} \) in a weakly disordered metal \( (g \gg 1) \) with completely broken time-reversal symmetry. To the best of our knowledge, this has not been spelled out clearly in the literature. The diffusion propagator \( D_{\omega q}^{\phi} \) for interacting electrons is a function of two frequencies – only in the absence of interactions \( D_{\omega q}^{\phi} \propto \delta(\omega) \). It is convenient to choose the mixed representation \( D_{\omega q}^{\phi \nu} = \int \frac{d\omega}{2\pi} \exp(-i\omega t_0)D_{\omega q}^{\phi} \) and regard the delay time \( t_0 \) as a parameter. The Dyson’s equation assumes then the algebraic form \( [D_{\omega q}^{t_0}]^{-1} = [D_{\omega q}^{(0)}]^{-1} - \Sigma_{\omega q}^{(0)} \), where the bare propagator \( D_{\omega q}^{(0)} = 1/(i\omega + D_{\omega q}^2) \). We define the diffusion decay rate \( 1/\tau_{\omega q}^{D}(t_0) = -\Re\Sigma_{\omega q}^{(0)} \) as a function of \( t_0 \) (assuming that the weak interaction does not renormalize \( D_{\omega q}^{(0)} \) on the microscopic scale). Particle number conservation dictates that \( 1/\tau_{\omega q}^{D}(0) = 0 \), since the dynamical part of the density-density correlator \( \langle \rho\rho \rangle_{\omega q} \) is expressed in terms of the integral \( \int \frac{d\omega}{2\pi} D_{\omega q}^{\phi} \). Thus, in contrast to the more familiar Cooperon, \( D_{\omega q}^{\phi} \) cannot be characterized by a single phase-breaking time (this should also be contrasted with the cutoff of the full diffusion propagator by a constant \( \tau_{D}^{\nu} \), cf. [10]). To calculate \( 1/\tau_{\omega q}^{D}(t_0) \), we use the method [11], within the framework of which the electron-electron interaction is mediated by thermal fluctuations of a classical \( (\omega \ll T) \) electromagnetic field with the correlator \( \langle VV \rangle_{\omega q} = 4\pi^2 v_T/\varepsilon(\omega^2 + v_s^2 q^2) \), where \( v_s = (e^2/\varepsilon h) \) is the charge-spreading velocity, \( \varepsilon \) the bare dielectric constant (Nyquist noise). We transform to real space by writing the equation for the diffusion in the form

\[
\frac{\partial}{\partial t} - D \frac{d^2}{d\mathbf{r}^2} + \frac{i}{\hbar} \left( V(\mathbf{r}, t - \frac{t_0}{2}) - V(\mathbf{r}, t + \frac{t_0}{2}) \right) \right] \mathcal{D}_{\omega q}^{(0)}(\mathbf{r}, t) = \delta(\mathbf{r})\delta(t) .
\]

Notice the crucial difference between this equation and that for the Cooperon (cf. [11]): in the latter case the times \( t \) and \( t_0 \) are interchanged in the argument of the effective potential; as a result, \( t_0 \) becomes a “mute variable” – the averaged Cooperon does not depend on \( t_0 \) and this is why it is characterized by the single time \( \tau_{\omega q}^{C} \). Calculating the correlator of the potential in Eq. (1), we observe that \( \tau_{\omega q}^{C}(t_0) \) can be obtained similarly to \( \tau_{\omega q}^{C} \) by introducing the effective interaction \( \langle VV \rangle_{\omega q}^{C} = \langle VV \rangle_{\omega q}(1 - \cos \omega t_0) \). It follows immediately that in the limit \( t_0 \gg \tau_{\omega q}^{C}(t_0) \), where the oscillating term \( \cos \omega t_0 \) can be safely ignored, the particle-hole and particle-particle propagators decay in the same way: \( \tau_{\omega q}^{D}(\infty) = \tau_{\omega q}^{C} \). The difference shows up at smaller \( t_0 \): one gets with logarithmic accuracy the equation for the decay rate of \( \mathcal{D}_{\omega q}^{t_0} \):

\[
\frac{1}{\tau_{\omega q}^{D}(t_0)} = \frac{T}{g} \ln \frac{T}{\max\{q^2, (D_{t_0})^{-1}, [\nu_s \tau_{\omega q}^{D}(t_0)]^{-2}\} .
\]

This formula tells us that for \( q \sim [D_{\omega q}^{t_0}(t_0)]^{-1/2} \), which are relevant in the calculation of the conductivity, the decay rate starts to fall off as \( \ln(T_{t_0}) \) at \( t_0 \ll \tau_{\omega q}^{D}(\infty) \). In the extreme of small \( t_0 \ll T^{-1} \) the quasiclassical treatment is no longer accurate, but an estimate can be readily obtained by cutting off the frequency integration at \( \omega \sim T \), – it follows that the dephasing rate vanishes algebraically at zero \( t_0 \): \( 1/\tau_{\omega q}^{D}(t_0) \sim (T/g)(T_{t_0})^2 \).

Now let us look at the effect of the interaction on the quantum interference of diffusons. In the unitary limit, the leading weak-localization correction is given by the
familiar expression $\delta g^D \sim g^{-1} \ln(L/l)$, where $l$ is the mean free path (or the Larmor radius, when it is smaller), $L$ an inelastic scattering length. However, the mechanism of the infrared cutoff in the high-$B$ limit deserves comment, since the dephasing time $\tau_D^\phi(t_0)$ tends to infinity as $t_0 \to 0$. The quasiclassical treatment of the Coulomb interaction allows to calculate first the contribution to $g$ from diffusions $D^\omega(\mathbf{r}, t)$ moving in a given (as if externally applied) Nyquist potential. The Gaussian average over the thermal electromagnetic fluctuations $\langle\ldots\rangle$ below can then be safely performed. For the leading correction, this gives $\delta g^D = g^{-1} \int_0^\infty dt \langle A(t) \rangle$, where $A = A_2 + A_3$ is a sum of two- and three-diffusion terms $\langle\ldots\rangle$ (a proper cutoff to the ballistic scale is assumed). Consider the simplest two-diffusion contribution

$$A_2(t) = 2D^2 \int_0^t dt'D^{t'-t}(0, t')D^t(0, t-t'), \quad (3)$$

which already reveals the peculiarity of the dephasing in the unitary case. Though one could have expected that $\langle A_2(t) \rangle$ would decay exponentially at $t \gg \tau_D^\phi(\infty)$, it can be readily seen from Eq. (3) that $\langle A_2(t) \rangle$ remains singular on the scale of $\tau_D^\phi(\infty)$. The phase coherence is preserved because of the vanishing of the dephasing rate at $t' = 0$ and $t' = t$. A similar “breakdown” of the dephasing occurs in $\langle A_3(t) \rangle$. However, adding all the pieces, we find that the total contribution to $\delta g^D$, $\langle A(t) \rangle \propto \exp[-t/\tau_D^\phi(\infty)]$, decays on the scale of the shortest dephasing time. This proves that the interaction-induced cutoff for $\delta g^D$ is given by the phase-breaking length related to $\tau_D^\phi(\infty)$ (which contrasts with the result of Ref. 12, where the inelastic cutoff in the unitary limit was identified with a much longer energy-relaxation length).

We turn now to the interaction-induced dephasing at the integer QH transition. We assume that the interaction is weak enough not to break down the integer QH effect, i.e. $e^2/\lambda \ll \Gamma$, where $\lambda$ is the magnetic length, $\Gamma$ the width of the disorder-broadened Landau level. It is then legitimate to repeat the above analysis of the phase breaking right at the QH metallic point by endowing the diffusion coefficient with a strong dispersion at $Dq^2/\omega \gtrsim 1$. The power-law dispersion at large $q^2/\omega$ signals that the QH metal starts to develop the critical eigenfunction correlations. However, as follows from the calculation with constant $D$, this does not change the dependence of $L$ on $T$, since the relevant $Dq^2/\omega$ are of order unity. Specifically, an estimate can be readily obtained by setting $g \sim 1$ in Eq. (2), which gives $T\tau_D^\phi(\infty) \sim 1$ and $L \sim (D/T)^{1/2}$ ($z_2 = 2$). Notice that when the Fermi energy coincides with $E_c$, the localization effects can be neglected at all $\omega \ll \Gamma$, since $\xi \gg (D/\omega)^{1/2}$ within the energy band of width $\omega$ around $E_c$. In sum, the scale on which the dephasing occurs at the critical point is certainly $(D/T)^{1/2} \ll L$. We are led to conclude that while the phase breaking controls the temperature scaling of $g_c$ right at the critical point, it does not control the observed metal-insulator crossover.

The reason for the strong increase of the cutoff $L_h$ as compared to $L_\phi$ is that away from the critical point transport is governed by charging effects: the Coulomb blockade on the scale of $\xi$ drastically narrows the crossover region. Indeed, one can identify two characteristic energies on the scale of $\xi$: the charging energy $U_c \sim e^2/\xi \Delta$ and the “on-site” energy spacing $\Delta \sim 1/(\partial n/\partial \mu)\xi^2$. Near the transition $U_c \gg \Delta$. The naive description of scaling in terms of $L_\phi/\xi$ amounts to the assumption that the QH system shows crossover at $T/\Delta \sim 1$. It is evident, however, that the system behaves as a metal only if $T$ exceeds $U_c$, otherwise the scattering is blocked as in the usual Coulomb blockade regime. The QH system at given $E_F$ can thus be modeled as a dense array of quantum dots of size $\xi$ coupled via the tunneling integral $\sim \Delta$. The scaling form of $g$ then reads

$$g = g_0 F(U_c/T), \quad (4)$$

or, equivalently, $g = g_0 F(L_h/\xi)$ with $L_h \sim e^2/\xi T$, so that $z_1 = 1$ (these arguments parallel those in [4], where $F(x)$ was argued to fall off at $x \to \infty$ as $\ln F \sim -x^{1/2}$). Hence, the scaling around the unstable fixed point indeed necessitates dealing with two scales, $L_h$ and $L_\phi$ (Fig. 1). Also, while the typical energy transfer and the dephasing rate are both $\sim T$, the scattering rate $\tau_D^{-1} \sim DL_h^{-2}$ is much smaller:

$$1/\tau_h \sim T^2/\tau_c, \quad \tau_c \sim e^4/\xi^2 D. \quad (5)$$

To test the two-scale picture with $z_1 \neq z_2$ experimentally, we suggest to measure the temperature dependent correction to the critical conductivity $\delta g_c(T)$. Specifically, according to numerical simulations [4,5], the finite-size correction to $g_c$ scales as $L^{-y}$ with $y \approx 0.4+0.5$ (in fact, it can be shown analytically [5] that $y$ is not an independent exponent, namely there exists the nontrivial relation $y = \eta$, where $\eta \approx 0.4$ is the usual critical exponent of eigenfunction correlations $z_2 = 2$). We predict that, while the smearing of the transition is controlled by $L_h$ ($z_1 = 1$), the critical conductivity scales with $L_\phi$ ($z_2 = 2$), i.e. $\delta g_c \propto T^{y/2}$. Another possible test is based on the fact that $L_\phi \ll L_h$. Naively, one may well think that when $L_h$ becomes larger, as $T \to 0$, than the system size $L$, there must appear strong mesoscopic fluctuations (say of the height of the conductivity peak). However, our approach suggests that this is not true, since in the range $L_\phi \ll L \ll L_h$ the width of the critical region is already $T$ independent but the mesoscopic fluctuations are still suppressed (at $\tau_D^{-1} \sim T$, the only parameter that governs the amplitude of the fluctuations is $L^2/T(D)$). The absence of the fluctuations at $L_h \gtrsim L$ would give a strong experimental support to the two-scale picture.

Finally, we discuss briefly the behavior of the one-particle DOS at the critical point $\rho_{1c}(\omega)$. It is a popular
misconception that the reduction $z_1 \to 1$ signifies the linear vanishing of $\rho_1(\omega) \propto |\omega|$ (see, e.g., [1]). In fact, several aspects require comment. First, as argued above, the true dynamical exponent is related to $g_\varepsilon$ and the thermodynamic DOS $\partial n/\partial \mu$, so that it is equal to 2 at the QH transition (merely reflecting the Einstein relation). Second, away from the critical point, the quasiparticle DOS $\rho_0$ that appears in the hopping exponent $\mathbf{13}$ indeed behaves as $\rho_0 \sim |\omega|^z e^{2/\nu}$ at $|\omega| \lesssim U_c$; however, $\rho_0$ does not coincide with $\rho_1$ unless the system is classical and electrons can be treated as point charges. The difference is due to the fact that in the classical treatment of the Coulomb gap $[\mathbf{15}]\Delta/|\omega|$ is sent to $\infty$, whereas near the critical point $\Delta$ is the smallest energy scale. As a result, the rate of the charge spreading becomes a crucial factor in the suppression of $\rho_1$ in the metallic phase. The width of the interaction-induced gap in a metal grows with decreasing $g$ as $\exp[-2(\pi g)^{1/2}]$ [17]. To calculate $\rho_1(\omega)$, we use the elegant quasiclassical method suggested in [18], which works well in the conducting phase even if $g \sim 1$. Adjusting it to the high-$B$ limit (in our case the screening length $D/v_s$ is larger than the Larmor radius), we obtain at $g_\varepsilon \sim 1$

$$\rho_1(\omega) = (\partial n/\partial \mu) \exp[-S(\omega)], \ S \simeq \alpha \ln^2(T_c/|\omega|), \ (6)$$

where the numerical coefficient $\alpha \sim 1$, and the width of the gap $T_c$ is defined by Eq. (5). It is worth noticing that the localization-induced dispersion of the diffusion coefficient at large $q^2/\omega$, which is the only peculiarity of the QH critical point as compared to the Drude metal, is of little importance here (in contrast to the conventional two-dimensional metal, where the localization effects get in the way of the method [18] at $\omega \to 0$). Note also the shape of the gap at the transition, $-\rho_1$ vanishes faster than any power of $\omega$. This should be contrasted with both the power-law behavior of $\rho_1$ at the Anderson transition in $2 + \epsilon$ dimensions [14] and the naive power counting at the QH transition [13]. This result also brings up the question whether the Hartree-Fock method [24], within the framework of which a linear vanishing of $\rho_1$ was observed numerically, captures all the essential physics. Away from the critical point, the “log-normal” suppression of the DOS saturates with decreasing $\omega$ at $|\omega| \sim U_c$ (which means that the charge spreading stops on the scale of $\epsilon$). In the insulating phase, the linear [16] vanishing of $\rho_1$ should be expected at $\omega = 0$, but with a slope suppressed by the factor of $\exp[-S(U_c)]$.

To summarize, we have argued that the temperature driven scaling at the integer QH transition is governed by two independent length scales with the temperature exponents $z_1 = 1$ and $z_2 = 2$. The smearing of the transition is controlled by charging effects ($z_1 = 1$), whereas the interaction-induced phase breaking ($z_2 = 2$) is responsible for corrections to the critical conductivity. We suggested experimental tests of the two-scale picture.

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