THE COLORED GLASS CONDENSA TE AND EXTREME QCD\textsuperscript{a}

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The high energy limit of QCD is controlled by the small-$x$ part of a hadron wavefunction. I argue that this part is universal to all hadrons and is composed of a new form of matter: a Colored Glass Condensate. This matter is weakly interacting at very small $x$, but is non-perturbative because of the highly occupied boson states which compose the condensate. Such a matter might be studied in high energy lepton-hadron or hadron-hadron interactions.

1 Introduction: Parton saturation at small-$x$

There has been much activity in the last few years in an attempt to understand the physics of nuclear and hadronic processes in the regime where Bjorken’s $x$ becomes very small \cite{1-13} (and References therein). A remarkable feature about this regime is that the gluon density in the hadron wavefunction is so high that perturbation theory breaks down even for a small coupling constant, because of the strong non-linear effects. This is quite similar to high temperature QCD where the physics at the scale $g^2T$ is non-perturbative because of the large thermal occupation numbers of the soft gluons \cite{14} (and Refs. therein).

The important new phenomenon which is expected in these conditions is parton saturation, which is the fact that the density of partons per unit phase space ($dN/d^2p_{\perp}d\tau d^2x_{\perp}$) cannot grow forever as $x$ becomes arbitrarily small (here, $\tau \equiv \ln(1/x)$). If this growth were to occur, then the cross section for deep inelastic scattering at fixed $Q^2$ would grow unacceptably large and eventually violate unitarity bounds. One rather expects the cross section to approach some asymptotic value at high energy, that is, to saturate. It is also intuitively obvious that this should happen, since if the phase space density becomes too large, repulsive interactions are generated among the gluons, and eventually it will be energetically unfavorable to increase the density anyfurther.

Saturation is expected at a phase-space density of order $1/\alpha_s$, a value typical of condensates \cite{6,7}. This, together with the fact that gluons are massless bosons, leads naturally to the expectation that the saturated gluons form a new form of matter, which is a Bose condensate. Since gluons carry color which is a gauge-dependent quantity, any gauge-invariant formulation will necessarily involve an average over all colors, to restore the invariance. This averaging

\textsuperscript{a}Talk presented at “Strong and Electroweak Matter 2000”, Marseille, June 14–17, 2000.

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procedure bears a formal resemblance to the averaging over background fields done for spin glasses. The new matter is therefore called the Colored Glass Condensate (CGC).

In what follows, I will briefly explain the assumptions which justify this physical picture, and the mathematical formalism which is used to describe it.

2 The Colored Glass Condensate

The CGC picture holds, strictly speaking, only in the infinite-momentum frame, where the hadron propagates almost at the speed of light, and, by Lorentz contraction, appears to the external probe as an infinitesimally thin two-dimensional sheet located at \( z = t \), or \( x^- = 0 \). In what follows, I shall use light-cone (LC) vector notations: for an arbitrary 4-vector \( v^\mu \), I write \( v^\mu = (v^+, v^-, v_\perp) \), with \( v^+ \equiv (1/\sqrt{2})(v^0 + v^3) \), \( v^- \equiv (1/\sqrt{2})(v^0 - v^3) \), and \( v_\perp \equiv (v^1, v^2) \). The dot product reads then: \( p \cdot x = p^+ x^- + p^- x^+ - p_\perp \cdot x_\perp; \)

\( p^- \) and \( p^+ \) are, respectively, the LC energy and longitudinal momentum; correspondingly, \( x^+ \) and \( x^- \) are the LC time and longitudinal coordinate.

In the infinite-momentum frame, the parton interpretation makes sense and deep inelastic scattering (DIS) proceeds via the instantaneous absorption of the virtual photon \( \gamma^* \) (with 4-momentum \( q^\mu \)) by some parton in the hadron. The Bjorken \( x_B \) variable is defined as \( x_B \equiv Q^2 / 2 P \cdot q \), where \( Q^2 \equiv -q^\mu q_\mu \), and \( P^\mu = \delta^\mu_+ P^+ \) with large \( P^+ \) is the hadron 4-momentum. By kinematics, \( x_B \) coincides with the longitudinal momentum fraction \( x \equiv p^+/P^+ \) of the struck parton: \( x_B = x \).

At \( x \ll 1 \), the gluon density increases faster, and is the driving force towards saturation. The dynamics responsible for this increase is the BFKL evolution, as I briefly recall now (see also Ref. and Refs. therein):

Let me call soft, respectively fast, a parton having small, respectively, large longitudinal momentum (this separation will be made more precise in a moment). A soft gluon, with \( p^+ = xP^+ \ll P^+ \), is a shortlived excitation which is typically radiated by a fast parton (e.g., a valence quark) with a larger longitudinal momentum \( k^+ \gg p^+ \), and thus a longer lifetime. Indeed, by the uncertainty principle, the lifetime of the parton system in Fig. 1.a is:

\[
\Delta x^+ = \frac{1}{\varepsilon_{k-p} + \varepsilon_p \varepsilon_{k-p}} \approx \frac{1}{\varepsilon_p} = \frac{2p^+}{P^2_\perp} \propto x, \tag{1}
\]

where \( \varepsilon_k \equiv k^2_- / 2k^+ \) is the LC energy of the on-shell excitation with momentum \( \vec{k} = (k^+, k_\perp) \), and I have used the fact that, for comparable transverse momenta \( k_\perp \) and \( p_\perp \), \( \varepsilon_p \gg \varepsilon_k, \varepsilon_{k-p} \). To conclude, softer partons have larger energies, and therefore shorter lifetimes.
Figure 1: a) Soft gluon emission by a fast parton; b) a second emission; c) a gluon cascade.

The lowest-order process in Fig. 1.a is amended by radiative corrections enhanced by the large rapidity gap \( \Delta \tau \equiv \ln(k^+/p^+) \sim \ln(1/x) \). (I assume here that \( \ln(1/x) \gg 1 \).) For instance, the probability for the emission of a second gluon with momentum \( k_1^+ \) in the range \( k^+ > k_1^+ > p^+ \) is (cf. Fig. 1.b):

\[
\Delta P \propto \alpha_s \int_{p^+}^{k^+} \frac{dk_1^+}{k_1^+} = \alpha_s \ln \frac{k^+}{p^+} \sim \alpha_s \ln \frac{1}{x},
\]

and becomes exceedingly large as \( x \to 0 \). It is then highly probable that more gluons will be emitted along the way, thus giving birth to the gluon cascade depicted in Fig. 1.c. For a fixed number \( N \) of gluons in this cascade, the largest contribution, of order \( (\alpha_s \ln(1/x))^N \), comes from the kinematical domain where

\[
k^+ \equiv k_0^+ \gg k_1^+ \gg k_2^+ \gg \cdots \gg k_N^+ \equiv p^+.
\]

Other momentum orderings give contributions which are suppressed by, at least, one factor of \( 1/\ln(1/x) \), and thus can be neglected to leading logarithmic accuracy (LLA). For the dominant contribution in eq. (3), the number of radiated gluons increases exponentially with \( \Delta \tau \): \( N(x) \sim \exp\{A\alpha_s \ln(1/x)\} \), with constant \( A \). This is a coherency effect, consequence of the separation of scales in eq. (3), and of the bosonic nature of the gluons:

Indeed, because of its short lifetime, the soft gluon at the lower end of the cascade “sees” the \( N \) previous gluons as a frozen color charge distribution,
with an average color charge \( Q \equiv \sqrt{\langle Q_a Q_a \rangle} \propto N \). Thus, the \( N \)th gluon is emitted \textit{coherently} off the color charge fluctuations of the previously emitted gluons, with a differential probability (compare to eq. (2)):

\[
dP_N \propto \alpha_s N \, d\tau_N,
\]

which implies that \( N(\tau) \sim e^{\Delta \alpha_s \tau} \), as anticipated. Then, also the gluon density \( xG(x, Q^2) \equiv (dN/d\tau) \propto e^{\Delta \alpha_s \tau} \) grows exponentially\(^c\) with \( \tau \equiv \ln(1/x) \).

Thus, the BFKL picture is that of an unstable growth of the color charge fluctuations as \( x \) becomes smaller and smaller. However, this evolution assumes the radiated gluons to behave as free particles, so it ceases to be valid at very low \( x \), where the gluon density becomes so large that the radiated gluons overlap each other in the transverse plane and start interacting. This is the onset of saturation.

This is also the regime where the description in terms of a Colored Glass Condensate becomes appropriate: Because of the hierarchy of scales in eq. (3), the soft gluons “see” the fast partons as an effective color charge which is \textit{static} (i.e., independent of \( x^+ \)), and \textit{localized} near the LC (i.e., at \( x^- = 0 \)), with a \textit{random} density \( \rho_a(x^-, x_\perp) \). (This is random since the soft gluons can belong to different cascades, and the instantaneous configuration of the cascades inside the hadron is random. Of course, the integrated color charge over the whole hadron must vanish, because of confinement.) The spatial correlations of the effective charge \( \rho_a(x^-, x_\perp) \) reflect the quantum dynamics of the fast partons, and are encoded in a statistical weight function \( W[\rho] \).

This color charge acts as a source for the soft gluons which, because of their large occupation numbers, can be treated in the \textit{classical approximation}. This suggests the following \textit{effective theory} for the soft gluon correlators, originally proposed by McLerran and Venugopalan\(^d\): First, one solves the classical Yang-Mills equations with the source \( \rho_a \), and in the LC-gauge \( A^+ = 0 \):

\[
[D_\nu, F^{\nu\mu}] = \delta^{\mu+} \rho_a(x^-, x_\perp).
\]

(5)

In the saturation regime, \( A \sim \rho \sim 1/g \) (see Sect. 3.2 below), and eq. (5) must be solved \textit{exactly}: the classical problem is fully non-perturbative. The non-linear effects correspond to interactions among the soft gluons, to all orders. The corresponding solution \( A^\mu \equiv A^\mu[\rho] \) will be constructed in Sect. 3.1. Then, the soft correlation functions of interest are obtained as:

\[
\langle A_a^\mu(x) A_b^\nu(y) \cdots \rangle_\Lambda = \int \mathcal{D}\rho \ W_\Lambda[\rho] \, A_a^\mu(x) A_b^\nu(y) \cdots.
\]

(6)

\(^c\)A more refined treatment, using the BFKL equation\(^b\), gives \( A = 4N_c \ln 2/\pi \).
\(^d\)This gauge choice will be motivated in Sect. 3.2.
Note the scale $\Lambda \equiv \Lambda^+$ in eq. (6): this is the separation scale between fast ($p^+ > \Lambda^+$) and soft ($p^+ < \Lambda^+$) degrees of freedom. Clearly, the structure and the correlations of $\rho$ will depend upon $\Lambda$ (e.g., $\rho$ has support at $x^- \lesssim 1/\Lambda^+$, and $W[\rho] \equiv W_\Lambda[\rho]$). Moreover, the soft correlations must be evaluated at longitudinal momenta $p^+$ which are not too small as compared to $\Lambda^+$: otherwise, the quantum corrections due to the “semi-fast” gluons with $k^+$ momenta in the range $p^+ < k^+ < \Lambda^+$ would be relatively large, of $O(\alpha_s \ln(\Lambda^+/p^+))$, since enhanced by the large rapidity interval $\Delta \tau = \ln(\Lambda^+/p^+) \gg 1$.

Remarkably, the equations (5)–(6) are those for a glass (here, a colored glass): There is a source, and the source is averaged over. This is entirely analogous to what is done for spin glasses when averaging over stochastic magnetic fields. By computing the two-point function as above, one has found a saturation regime where $\langle A_i^a A_i^a \rangle_\Lambda \propto 1/\alpha_s$, a value typical for a condensate (see also Sect. 3.2 below). We thus conclude that the matter which describes the small $x$ part of a hadron wavefunction is a Colored Glass Condensate.

3 Saturation in the McLerran-Venugopalan model

In this section, I present the solution to eq. (6) and explore its consequences for the saturation of gluons.

3.1 The non-Abelian Weizsäcker-Williams field

Consider first the Abelian version of eq. (6), as a warming up:

$$\partial_\nu F^{\nu\mu} = \delta^{\mu+} \rho(\vec{x}), \quad (7)$$

where $A^+ = 0$, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and $\vec{x} \equiv (x^-, x^\perp)$. Eq. (7) can be easily solved in momentum space, with the result that $A^- = 0$ and

$$A^i(p) = -\frac{p^i}{p^+} \frac{\rho(p^+, p^\perp)}{p^2_\perp} \cdot (8)$$

Here, one needs a prescription to handle the pole at $p^+ = 0$: different prescriptions correspond to different boundary conditions in $x^-$. For instance, with the “retarded” prescription $1/(p^+ + i\varepsilon)$, one obtains

$$A^i(x^-, x^\perp) = \int_{-\infty}^{x^-} dy^- \partial^i \alpha(y^-, x^\perp), \quad (9)$$

$^c$Note that the “retardation” refers here to $x^-$, and not to the LC time $x^+$. 

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with \( \alpha \) satisfying \(-\nabla^2 \alpha(x) = \rho(x)\).

The vector potential (9) is static, \( \partial^\mu A^\mu \equiv (\partial A^i / \partial x^+) = 0 \), and defines a two-dimensional pure gauge: \( F^{ij} = 0 \). But, of course, this is not a pure gauge in four dimensions, since the electric field \( F^{i+} = -\partial^+ A^i = \partial \alpha \) is non-zero. The above solution, also known as the Weizsäcker-Williams field, is the analog of the Coulomb field in the infinite momentum frame.

It turns out that the corresponding non-Abelian solution has a similar structure: Specifically, the solution \( A^\mu \) to eq. (5) in the LC gauge (\( A^+ = 0 \)) can be chosen such as \( A^− = 0 \), and \( A^i \) is static and a two-dimensional pure gauge: \( F^{ij} = 0 \). This can be written as follows (for retarded boundary conditions):

\[
A^i(x^−, x_⊥) = (i/g) U^\dagger(x^−, x_⊥) \partial^i U^\dagger(x^−, x_⊥) \quad (10)
\]

\[
U^\dagger(x^−, x_⊥) = P \exp \left\{ ig \int_{-\infty}^{x^−} dz^- \alpha(z^−, x_⊥) \right\}, \quad (11)
\]

\[- \nabla^2_⊥ \alpha(x) = \tilde{\rho}(x) \equiv U^\dagger(x) \rho(x) U(x), \quad (12)\]

where \( P \) is the usual path-ordering operator, and \( \tilde{\rho}(x) \) is the value of the classical source in the covariant gauge \( \partial_\mu A^\mu = 0 \).

Note that, while \( \alpha \) is linearly related to \( \tilde{\rho} \), its relation to the LC-gauge source \( \rho \) is more complicated, since the gauge rotations \( U \) and \( U^\dagger \) are themselves functionals of \( \alpha \), cf. eq. (11). Thus, in order to perform the average in eq. (6), it is more convenient to use \( \tilde{\rho} \) (rather than \( \rho \)) as the independent variable. This is possible because the measure and the weight function in eq. (6) are gauge invariant; e.g., \( W_\Lambda[\rho] = W_\Lambda[\tilde{\rho}] \). Then, the non-Abelian solution \( A^i[\tilde{\rho}] \) is known explicitly, via eqs. (10)–(12), and eq. (6) is replaced by

\[
\langle A^i(x^+, x_\perp) A^j(x^+, y_\perp) \cdots \rangle_\Lambda = \int D\tilde{\rho} \ W_\Lambda[\tilde{\rho}] \ A^i_\Lambda[\tilde{\rho}] A^j_\Lambda[\tilde{\rho}] \cdots. \quad (13)
\]

In particular, this equation shows that only the equal-time \((x^+ = y^+)\) correlators of the soft transverse \((\mu = i, i = 1, 2)\) fields can be computed in this model; but these are precisely the gluon correlators which are probed in DIS.

Recall finally that the source \( \tilde{\rho} \) is created by the fast partons with \(|p^+| > \Lambda^+\), so its support is restricted to \(|x^-| \lesssim 1/\Lambda^+\). The small-\(x\) external probe, on the other hand, is sensitive only to the gross features of the color fields \( \mathcal{A}^i \) over large distances \(|x^-| \gg 1/\Lambda^+\), where one can replace eq. (11) with

\[
U^\dagger(x^-, x_\perp) \approx \theta(x^-) P \exp \left\{ ig \int_{-\infty}^{\infty} dz^- \alpha(z^-, x_\perp) \right\} \equiv \theta(x^-) v^\dagger(x_\perp), \quad (14)
\]
and therefore (cf. eq. (15)):

\[ A_i^+ \approx \theta(x^-) \frac{i}{g} v(\partial^i v^+) \equiv \theta(x^-) A_{i\infty}^+(x_\perp), \quad \mathcal{F}^{i+} \approx -\delta(x^-) A_{i\infty}^+(x_\perp), \quad (15) \]

where, strictly speaking, the \( \delta \)-function is localized at \( x^- \lesssim 1/\Lambda^+ \).

### 3.2 Gluon distribution function and saturation

Consider the simple approximation where the weight function for \( \tilde{\rho} \), which is not yet known, is taken to be a Gaussian:

\[ W_\Lambda[\tilde{\rho}] \simeq \exp \left\{ -\frac{1}{2} \int d^3x \frac{\tilde{\rho}^2(x)}{\xi^2_\Lambda(x)} \right\}, \quad (16) \]

where \( \xi^2_\Lambda \) is the total color charge density squared of the partons with \( p^+ > \Lambda^+ \). By using this approximation, it has been possible to compute the gluon distribution function \( xG(x, Q^2) \) in the MV model, as I recall now:

By definition, \( G(x, Q^2) \) is the number of gluons in the hadron wavefunction having longitudinal momentum \( k^+ = xP^+ \), and transverse momentum less than \( Q \). In the LC gauge, this is simply related to the Fock-space gluon distribution function, and therefore to the gluon two-point function:

\[ G(x, Q^2) \equiv \int \frac{d^2k_\perp}{(2\pi)^2} \Theta(Q^2 - k_\perp^2) \int \frac{dk^+}{2\pi} 2k^+ \delta \left( x - \frac{k^+}{P^+} \right) \times \left\langle \mathcal{A}_a^i(x^+, k^+, k_\perp)\mathcal{A}_a^i(x^+, -k^+, -k_\perp) \right\rangle, \quad (17) \]

where the brackets denote the average over the hadron wavefunction. In the LC-gauge gauge, \( F^{i+}(k) = ik^+ A^i(k) \), and therefore (with \( k^+ = xP^+ \)):

\[ xG(x, Q^2) = \frac{1}{\pi} \int \frac{d^2k_\perp}{(2\pi)^2} \Theta(Q^2 - k_\perp^2) \left\langle F_a^{i+}(x^+, \vec{k})F_a^{i+}(x^+, -\vec{k}) \right\rangle. \quad (18) \]

If the integration over \( k_\perp \) were unrestricted, this quantity would be manifestly gauge-invariant. But even for a finite \( Q^2 \), this has a gauge-invariant meaning when evaluated on the non-Abelian Weizsäcker-Williams field in Sect. 3.1.\(^7\)

In this approximation, \( F^{i+}(x^+, \vec{k}) \approx \mathcal{F}^{i+}(k_\perp) = -A_{i\infty}^+(k_\perp) \) (cf. eq. (14)), and

\[ xG(x, Q^2) \approx R^2 \int \frac{d^2k_\perp}{(2\pi)^2} \int d^2x_\perp e^{-ik_\perp \cdot x_\perp} \left\langle A_{i\infty}^{ia}(0) A_{i\infty}^{ia}(x_\perp) \right\rangle, \quad (19) \]

\(^7\)Of course, the Fock-space gluon distribution can be defined in any gauge; but it is only in the LC-gauge that the definition (14) can be given a gauge invariant meaning.
where \( R \) is the hadron radius (I have assumed homogeneity in the transverse plane, for simplicity), and the average is to be understood in the sense of eq. (13) with \( \Lambda^+ = xP^+ \). Thus, the r.h.s. of eq. (19) is still dependent on \( x \), but only via the respective dependence of the weight function for \( \tilde{\rho} \).

With the Gaussian weight function (16), and the non-linear classical solution in Sect. 3.1, the gluon distribution (19) can be computed exactly:

\[
\left\langle A^a_{\infty}(0) A^a_{\infty}(x_\perp) \right\rangle = \frac{N_c^2 - 1}{\pi\alpha_s N_c} \frac{1 - e^{-x_\perp^2 \ln(x_\perp^2 \Lambda_{QCD}^2) Q_s^2 / 4}}{x_\perp^2},
\]

(20)

where \( N_c \) is the number of colors, and \( Q_s \propto \alpha_s \xi_\Lambda \) is the saturation momentum and is a function of \( \Lambda^+ \), that is, of Bjorken’s \( x \). This equation displays saturation: the vector potential never becomes larger than \( A^i \sim 1/g \). This is the maximal occupation number permitted for a classical field, since larger occupation numbers are blocked by repulsive interactions of the gluon field.

This interpretation can be made sharper by going to momentum space: If \( N(k_\perp) \) is the Fourier transform of (20) [this is the same as \( (dN/d^2k_\perp d\tau d^2x_\perp) \)], the gluon density per unit rapidity and unit transverse phase-space, then:

\[
N(k_\perp) \propto \alpha_s \left( Q_s^2 / k_\perp^2 \right) \quad \text{for} \quad k_\perp^2 \gg Q_s^2,
\]

(21)

which is the normal perturbative behavior, but\(^9\)

\[
N(k_\perp) \propto \frac{1}{\alpha_s} \ln \left( \frac{k_\perp^2}{Q_s^2} \right) \quad \text{for} \quad \Lambda_{QCD}^2 \ll k_\perp^2 \ll Q_s^2,
\]

(22)

which shows a much slower increase, i.e., saturation, at low momenta.

Note, however, that the above argument is not rigorous, since the local Gaussian form for \( W[\rho] \) in eq. (16) is valid only at sufficiently large transverse momentum scales so that the effects of high gluon density are small. It is therefore important to verify if saturation comes up similarly with a more realistic form for the weight function, as obtained after including the quantum evolution in \( x \). This will be discussed now.

4 The non-linear evolution equation

Because the separation of scales is only logarithmic, the effective theory (5)–(6) with scale \( \Lambda^+ \) applies only to gluon correlations at a scale \( p^+ \) slightly below \( \Lambda^+ \). If one is interested in correlations at the softer scale \( b\Lambda^+ \) with \( b \ll 1 \),

\(^9\)In relativistic heavy ion collisions, one expects \( Q_s \sim 1 \text{ GeV} \) at RHIC, and \( Q_s \sim 2 - 3 \text{ GeV} \) at LHC. Thus, at least at LHC, the kinematical window in eq. (22) should be non-negligible.
then, to LLA, one has to include also the corrections of order $\alpha_s \ln(1/b)$ due to the semi-fast quantum fluctuations with longitudinal momenta in the strip

$$b\Lambda^+ < |k^+| < \Lambda^+. \tag{23}$$

Together with the dynamical information already contained in the effective theory at scale $\Lambda^+$ (“Theory I”), these additional corrections will determine the effective theory at the softer scale $b\Lambda^+$ (“Theory II”). This suggests an iterative construction of the effective theory where the quantum fluctuations are integrated out in layers of $k^+$, down to the physical scale of interest. At each step in this procedure, the quantum corrections must be computed to leading order in $\alpha_s \ln(1/b)$ (LLA), but to all orders in the strong background field $A^i \sim 1/g$ produced by the color source $\rho$ at the previous step.

To compute quantum corrections, one needs the quantum generalization of the McLerran-Venugopalan model. This is obtained by replacing eq. (6) with (below, the gauge condition $A^+ = 0$ is implicit):

$$\langle T A^\mu(x) A^\nu(y) \rangle = \int D\rho W_\Lambda[\rho] \left\{ \frac{\int_{\Lambda}^\Lambda D A A^\mu(x) A^\nu(y) e^{iS[A,\rho]}}{\int_{\Lambda}^\Lambda D A e^{iS[A,\rho]}} \right\}, \tag{24}$$

which involves two functional integrals: a quantum path integral over the soft ($k^+ < \Lambda^+$) gluon fields $A^\mu$, which defines quantum expectation values at fixed $\rho$, and a classical average over $\rho$, with weight function $W_\Lambda[\rho]$. (This double averaging is similar to the one performed for spin systems at finite temperature and in a random external magnetic field.) Unlike eq. (13), the 2-point function given by eq. (24) is independent of the arbitrary separation scale $\Lambda^+$: the cutoff dependence of the quantum loops cancels against the corresponding dependence of the classical weight function $W_\Lambda[\rho]$.

The action $S[A,\rho]$ is chosen such as to be gauge-invariant and reproduce the classical equations of motion (3) in the saddle point approximation $\delta S/\delta A^\mu = 0$. It reads:

$$S = S_{YM} + S_W,$$

where $S_{YM} = \int d^4x (-F_{\mu\nu}^2/4)$ is the Yang-Mills action, and $S_W$ is a gauge-invariant generalization of the eikonal vertex $\int d^4x \rho A^-$ (with $T$ denoting time-ordering of color matrices):

$$S_W \equiv \frac{i}{gN_c} \int d^3\vec{x} \text{Tr} \left\{ \rho(\vec{x}) T \exp\left( ig \int dx^+ A^- (x^+, \vec{x}) \right) \right\}. \tag{25}$$

Theories I and II are defined as in eq. (24), but with separation scales equal to $\Lambda^+$, and $b\Lambda^+$, respectively. The difference $\Delta W \equiv W_{b\Lambda} - W_\Lambda$ can be obtained by matching calculations of gluon correlations at the scale $p^+ \lesssim b\Lambda^+$ in the two theories. In Theory II, and to lowest order in $\alpha_s$, these correlations...
are found at tree level, i.e., in the classical approximation. In Theory I, and
to the same accuracy, they involve also the logarithmically enhanced quantum
corrections due to the “semi-fast” fluctuations defined by eq. (23). The result
can be expressed as an evolution equation\(^{11}\) for \( W_\tau[\rho] \) (with \( \tau \equiv \ln(P^+/\Lambda^+) \))
with respect to variations in \( \tau \), originally derived in Ref. \( 9 \). This reads:

\[
\frac{\partial W_\tau[\rho]}{\partial \tau} = \alpha_s \left\{ \frac{1}{2} \frac{\delta^2}{\delta \rho_x \delta \rho_y} [W_\tau \chi_{xy}] - \frac{\delta}{\delta \rho_x} [W_\tau \sigma_x] \right\}.
\] (26)

in condensed notations where (with \( x^- = 1/\Lambda^+ = e^\tau x_0^- \)), e.g.,

\[
\frac{\delta}{\delta \rho_x} [W_\tau \sigma_x] = \int d^2 x_\perp \frac{\delta}{\delta \rho_a(x^-, x_\perp)} [W_\tau \sigma_a(x^-, x_\perp)],
\] (27)

and the functions \( \sigma_a(x_\perp) \) and \( \chi_{ab}(x_\perp, y_\perp) \) are the one- and two-point correlators of the color charge \( \delta \rho_a(x) \) of the semi-fast quantum fluctuations:

\[
\alpha_s \ln \frac{1}{b} \sigma_a(x_\perp) \equiv \int dx^- \langle \delta \rho_a(x^+, x^-, x_\perp) \rangle_\rho,
\] (28)

\[
\alpha_s \ln \frac{1}{b} \chi_{ab}(x_\perp, y_\perp) \equiv \int dx^- \int dy^- \langle \delta \rho_a(x^+, x^-, x_\perp) \delta \rho_b(x^+, y^-, y_\perp) \rangle_\rho.
\]

Thus, \( \sigma \) and \( \chi \) are (generally non-linear) functionals of \( \rho \) obtained by averaging over the semi-fast quantum fluctuations in the presence of the tree-level fields and sources, \( A^i \) and \( \rho \). To the order of interest, the semi-fast fields can be integrated out in the Gaussian approximation; thus, the Feynman graphs contributing to \( \sigma \) and \( \chi \) involve, at most, one loop. Some typical contributions
to lowest order in \( \rho \) are shown in Fig. 2. As one can see on these figures, \( \sigma \)
is a one-loop virtual correction to the tree-level source and describes vacuum
polarisation, while \( \chi \) is a tree diagram which describes the real emission of a
semi-fast gluon (cf. Fig 1.b).

The functional equation (26) is equivalent to an infinite hierarchy of or-
dinary equations for the correlators of the charge density. For instance, by
multiplying eq. (26) with \( \rho_x \rho_y \) and functionally integrating over \( \rho \), one obtains
an evolution equation for the two-point function:

\[
\frac{d}{dt} \langle \rho_x \rho_y \rangle_\tau = \alpha_s \langle \chi_{xy} + \rho_x \sigma_y + \sigma_x \rho_y \rangle_\tau,
\] (29)

which in general involves also the higher \( n \)-point functions, via \( \sigma \) and \( \chi \). But
in the limit of a weak source, i.e., with \( \sigma \) and \( \chi \) computed to lowest order\(^h\)

\^This equation insures that correlation functions like (24) are independent of \( \Lambda^+ \).
in $\rho$, this becomes a closed equation for $\langle \rho \rho \rangle$ which coincides with the BFKL equation, as necessary on physical grounds. This is a very non-trivial check of the effective theory in eqs. (24)–(25). Specifically, the linear (in $\rho$) contribution to $\sigma$ depicted in Fig. 2.a is evaluated as:

$$\langle \delta \rho_a(x) \rangle = \frac{g^2 N_c}{2\pi} F(x^-) \int \frac{d^2 p_\perp}{(2\pi)^2} \int \frac{d^2 k_\perp}{(2\pi)^2} e^{i (p_\perp - k_\perp) \cdot x_\perp} \frac{p_\perp \cdot k_\perp}{p_\perp^2 k_\perp^2} \rho^a(q_\perp),$$

with the “form factor”

$$F(x^-) \equiv \theta(x^-) \frac{e^{-i\Lambda^+ x^-} - e^{-i\Lambda^+ x^-}}{x},$$

which has support at $1/\Lambda^+ \lesssim x^- \lesssim 1/b\Lambda^+$, and yields the expected logarithmic enhancement (cf. eq. (28)) after the integration over $x^-:

$$\int dx^- F(x^-) = \ln \frac{1}{b}. \quad (32)$$

The kernel of the two-dimensional integral in eq. (30) can be recognized as the virtual part of the BFKL kernel; the corresponding real part is generated by $\chi$ when evaluated to lowest order in $\rho$ (cf. Fig. 2.b).

The general, non-linear, expressions for $\sigma$ and $\chi$ valid in the saturation regime will be presented somewhere else. (See also Ref. 10 for an alternative calculation, with different results though.) Here, I would like only to emphasize the longitudinal structure of the quantum corrections, which is already
manifest on the linear approximation (30)–(31): Since generated by modes with \( k^+ \) momenta in the strip (23), the induced source \( \langle \delta \rho \rangle \) is localized at \( 1/\Lambda^+ \lesssim x^- \lesssim 1/b \Lambda^+ \), that is, on top of the tree-level source \( \rho \) (which has support at \( 0 \lesssim x^- \lesssim 1/\Lambda^+ \)). Because of that, the functional derivatives in eqs. (26)–(27) are to be evaluated at \( x^- = x^- \equiv 1/\Lambda^+ \): the evolution from \( W_\tau [\rho] \) to \( W_{\tau + d\tau} [\rho] \) is due to changes in \( \rho \) within the rapidity interval \( (\tau, \tau + d\tau) \).

In fact, the precise longitudinal picture depends upon the \( i\epsilon \) prescription for the “axial” pole \( 1/p^+ \) in the LC-gauge propagator of the semi-fast gluons. The result in eq. (31) has been obtained \(^{16}\) by using the retarded prescription \( 1/(p^+ + i\epsilon) \), for consistency with the classical solution (10)–(12). If the advanced prescription \( 1/(p^+ - i\epsilon) \) was used instead, the corresponding \( \langle \delta \rho_\alpha (\vec{x}) \rangle \) would be located at negative \( x^- \), but its integral over \( x^- \) — which defines \( \sigma (x_\perp) \), cf. eq. (28) — would nevertheless be the same. That is, the BFKL equation is obtained independently of the axial prescription.

This is, however, specific to the linearized (or BFKL) limit of the evolution equation. In general, the non-linear effects depend upon the gauge condition, as shown by the explicit calculations of \( \sigma \) and \( \chi \) using different prescriptions.\(^{10,12,13}\) The simplest results \(^{16}\) are obtained by using the retarded, or the advanced, prescriptions alluded to before. With a retarded prescription, both the classical mean field \( A^+_\alpha \), eq. (15), and the induced source \( \langle \delta \rho_\alpha \rangle \), eq. (30), sit at \( x^- > 0 \), or \( z < t \); that is, the soft gluons fields are behind their source, the hadron (which is located at \( z = t \)). It is intuitively plausible that within this picture there are no initial state gluon interactions. This is the counterpart of the conclusion of Mueller and Kovchegov\(^{17}\), who used an advanced prescription \( 1/(p^+ - i\epsilon) \), and showed that all the final state gluon interactions disappeared.

These two prescriptions are simple in that one can either put gluon interactions in the final or initial state. Other gauge conditions such as Leibbrandt-Mandelstam or principle value do not have this simple feature, and lead to other complications for computations as well.\(^{12}\) Of course, gauge invariant quantities must come out the same whatever gauge is used for their computation. But the classical color charge \( \rho_\alpha (\vec{x}) \) is not a physical observable by itself, but just a convenient tool to summarize the (generally, gauge-variant) correlations inherited by the soft gluons from the fast partons. The discrepancies between the results for \( \sigma \) and \( \chi \) found in Refs. \(^{12,13}\) and Ref. \(^{13}\) may be ultimately attributed to the different methods used to fix the gauge. In fact, our results \(^{16}\) are consistent with some previous results by Balitskii\(^3\) and Kovchegov\(^4\), to which they reduce in the large \( N_c \) limit.

To conclude, the non-linear evolution equation (26) together with the expressions for \( \sigma \) and \( \chi \) to be presented in Ref. \(^{16}\) provide a theoretical framework.

\(^1\)This prescription is necessary for fixing the residual gauge freedom in the LC-gauge.
to perform calculations in the saturation regime. Further studies within this framework should lead to a quantitative understanding of unitarity in high energy scattering, and of the structure of final states and multiparticle production. This would have important applications to the theory of deep inelastic scattering, hadronic interactions and ultrarelativistic nuclear collisions.

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