Research Article

Linear Diophantine Fuzzy Einstein Aggregation Operators for Multi-Criteria Decision-Making Problems

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Abstract

The linear Diophantine fuzzy set (LDFS) has been proved to be an efficient tool in expressing decision maker (DM) evaluation values in multicriteria decision-making (MCDM) procedure. To more effectively represent DMs' evaluation information in complicated MCDM process, this paper proposes a MCDM method based on proposed novel aggregation operators (AOs) under linear Diophantine fuzzy set (LDFS). A q-Rung orthopair fuzzy set (q-ROFS), Pythagorean fuzzy set (PFS), and intuitionistic fuzzy set (IFS) are rudimentary concepts in computational intelligence, which have diverse applications in modeling uncertainty and MCDM. Unfortunately, these theories have their own limitations related to the membership and nonmembership grades. The linear Diophantine fuzzy set (LDFS) is a new approach towards uncertainty which has the ability to relax the strict constraints of IFS, PFS, and q–ROFS by considering reference/control parameters. LDFS provides an appropriate way to the decision experts (DEs) in order to deal with vague and uncertain information in a comprehensive way. Under these environments, we introduce several AOs named as linear Diophantine fuzzy Einstein weighted averaging (LDFEWA) operator, linear Diophantine fuzzy Einstein ordered weighted averaging (LDFEOWA) operator, linear Diophantine fuzzy Einstein weighted geometric (LDFEWG) operator, and linear Diophantine fuzzy Einstein ordered weighted geometric (LDFEOWG) operator. We investigate certain characteristics and operational laws with some illustrations. Ultimately, an innovative approach for MCDM under the linear Diophantine fuzzy information is examined by implementing suggested aggregation operators. A useful example related to a country’s national health administration (NHA) to create a fully developed postacute care (PAC) model network for the health recovery of patients suffering from cerebrovascular diseases (CVDs) is exhibited to specify the practicability and efficacy of the intended approach.

1. Introduction and Literature Review

The problem of vague and misleading information has become a major issue for decades. Aggregation of data is important for decision-making corporate, administrative, social, medical, technological, psychological, and artificial intelligence fields. Awareness of the alternative has traditionally been seen as a crisp number or linguistic number. However, due to its uncertainty, the data cannot easily be aggregated. Multicriteria decision-making (MCDM) is an approach where feasible objects are accessed by decision experts (DEs) under the multiple criterion. Traditionally, an object’s evaluation is thought of as either a crisp number or a linguistic number. However, owing to the unpredictable existence of real-world problems, classical mathematics cannot solve these complex problems literally. In any real-life problem-solving technique, the complexity characterizes the behavior of an object, whose components interrelate in...
multiple ways and follow different logical rules, meaning there is no fixed rule to handle multiple challenges due to various uncertainties in real-life circumstances. Many scholars from all over the world have apparently studied MCDM management techniques extensively. This effort resulted in a multitude of innovative solutions to complex real concerns. The frameworks for this objective are largely based on a summary of the issues at hand. To deal with uncertainties the researchers have been proposed various mathematical techniques. Zadeh [1] introduced fuzzy sets (FSs), Pawlak [2] developed rough sets, and Molodtsov [3] proposed soft sets. These sets are independent generalizations of the crisp sets. Subsequently, the idea of intuitionistic fuzzy sets (IFSs) is presented by Atanassov [4, 5] as an extension of FSs and the concept of PFSs is introduced by Yager [6, 7] as a generalization of IFSs. A Pythagorean fuzzy number (PFN) developed by Zhang and Xu [8] is significantly superior than the intuitionistic fuzzy number (IFN). Yager [9] introduced the idea of generalized orthopair fuzzy sets which is also known as q-rung orthopair fuzzy set (q-ROFS). Ali [10] proposed two aspects of q-ROFS in terms of L-fuzzy sets and orbits. Ali and Shabir [11] investigated fuzzy soft sets and soft sets to discuss their logic connectives. Zhang [12] introduced bipolar fuzzy sets and relations. Smarandache [13] introduced the ideas of neutrosophic, neutrosophic sets, and neutrosophic probability.

Alcantud et al. [14–16] introduced a technique by utilizing N-soft set approach towards rough sets. They presented some extensions of soft sets and fuzzy sets and their applications in MCDM under incomplete information. Karaslan and Hunu [17] studied neutrosophic [18] soft sets and their applications in decision-making. Peng et al. [19, 20] introduced some results on Pythagorean fuzzy information measures and their applications in MCDM. Naem et al. [21] proposed some novel features of Pythagorean m-polar fuzzy sets with applications. Riaz and Tehrim [22] introduced the idea of bipolar fuzzy soft topology with decision-making application. Riaz et al. [23] presented the notion of soft multirough set topology and its applications to MCDM problems.

Aggregation operators for MCDM have been studied by numerous researchers. Garg [24] presented the idea of generalized Pythagorean fuzzy information aggregation using Einstein operators and its applications to decision-making. Garg [24] proposed applications of Einstein operations under PFS environment. Garg et al. [25] derived new generalized dice similarity measures for complex q-rung orthopair fuzzy sets and established certain properties of suggested information measures. Garg and Arora [26] studied Archimedean t-norm of the intuitionistic fuzzy soft set (IFSS) and developed new generalized Maclaurin symmetric mean aggregation operators for information aggregation. A novel complex q-rung orthopair fuzzy Bonferroni mean was developed by Liu et al. [27] for solving MCDM problems. A robust multiple attribute group decision-making (MAGDM) method was introduced by Liu and Wang [28] by using intuitionistic fuzzy Einstein interactive operations. Hesitant intuitionistic fuzzy linguistic aggregation operators were derived by Liu et al. [29] for MADM. Akram et al. [30] presented an extension of Dombi’s aggregation operators for decision-making under m-polar fuzzy information. An application towards hospital performance measurement was proposed by Yang et al. [31] using the idea of triangular single-valued neutrosophic data. Pythagorean fuzzy Einstein weighted geometric aggregation operator was studied by Rahman et al. [32] for solving MAGDM problems. New interaction power Bonferroni mean aggregation operators were developed by Wang and Li [33] for MADM of real-world problems. Riaz et al. [34–37] proposed q-rung orthopair fuzzy hybrid, Einstein, prioritized, and Einstein prioritized AOs. Einstein operators, studied by Wang and Liu [38], were based on uncertain intuitionistic fuzzy information.

Xu et al. [39–42] proposed various aggregation operators using intuitionistic fuzzy sets and hesitant fuzzy sets for MCDM. Ye [43–45] introduced weighted aggregation operators for MCDM with interval-valued hesitant fuzzy sets, single-valued neutrosophic sets, and simplified neutrosophic sets. Liu et al. [47] introduced linguistic intuitionistic cubic fuzzy AOs, Abdullah et al. [48] presented a novel concept of sine trigonometric picture fuzzy AOs, and Naeem et al. [49] initiated the concept of similarity measures for fractional orthotriple fuzzy sets and their applications towards MCDM. Ashraf et al. [50] initiated the novel concept of spherical fuzzy sets. The idea of linear Diophantine fuzzy set (LDFS) was defined by Riaz and Hashmi [51] as a new concept for modeling uncertainties in MADM problems. A LDFS is a strong model to relax the limitations of MD and NMD due to existence of reference/control parameters. Riaz et al. [52] extended LDFS to the idea of soft rough LDFS sets with application to sustainable material handling equipment. Riaz et al. [53] introduced spherical linear Diophantine fuzzy sets with modeling uncertainties in MCDM.

The rest of this article is as follows. In Section 2, the basic concept of LDFS with its operational laws is quickly analyzed. In Section 3, some LDFS-based t-norm, t-conorm, and Einstein operational laws are defined. In Section 4, several linear Diophantine fuzzy Einstein AOs are developed. In Section 5, a robust MCDM approach with proposed operators is shown by a practical example to demonstrate the effectiveness of LDFSs for a country’s national health administration (NHA) to create fully developed postacute care (PAC) model network to improve the health recovery of patients suffering from cerebrovascular diseases (CVDs). Finally, we conclude the results of this manuscript in Section 6.

1.1. Motivation and Objectives. LDFS is a new approach towards uncertainty and vagueness which is superior to existing approaches of IFSs, PFSs, and q-ROFSs. The eminent characteristic of LDFS is that against each satisfaction and disinchantment degrees, there exists a pair of reference or control parameters, so the valuation area of theoretical knowledge they can describe is superior. Due to the constraints of certain existing approaches and their analogous operators, it may be challenging for decision makers (DMs)
to choose optimal or convincing alternatives. Therefore, this article intends to tackle these difficulties with LDFS-based aggregation operators designated as the LDFEWA operator, LDFEWG operator, and LDFEOWG operator. These operators not only can derive a ranking but also can have a prominent influence in identifying the optimal choice.

The inspiration for the proposed work is addressed in every part of this article. This paper has multiple objectives as follows:

1. The existing models IFSs, PFSs, and q-ROFSs have their strict limitations for truthness/membership grades (MGs) and falsity/nonmembership grades (NMGs). LDFS is an innovative flexible approach to relax these limitations. The decision makers (DMs) can choose these grades in [0, 1] without any restriction. Additionally, the reference or control parameters are used as a weight vector such that the sum of reference parameters is less than unity. These parameters classify the physical sense of the objects and help to deal with the uncertain information about the objects under consideration.

2. Einstein AOs based on t-norm and t-conorm are utilized to assemble the information data into a particular point. These operators remove the inconsistency and irrationality of the operational laws and provide us a broad range for the decision-making patterns.

3. The third and vital intention is to assemble a powerful association of the proposed model with the MCDM obstacles. We develop four novel operators to determine MCDM difficulties under the effect of control parameterizations of the proposed model. A useful example related to a country’s national health administration (NHA) to create a fully developed postcare application (PAC) model network for the health recovery of patients suffering from cerebrovascular diseases (CVDs) is exhibited to demonstrate the practicability and efficacy of the intended approach.

The set of control parameters represents an essential role in decision-making models. They accommodate us to improve the valuation area of satisfaction and dissatisfaction functions and parameterize the model, which gives us a variety of taking alternatives under different physical situations. The deficiency in IFS, PFS, and q-ROFS is that they have no parameterizations. This novel idea enhances the existing methodologies.

Many researchers [9, 10, 29] worked on the novel idea of q-ROFSs which is the annex of IFSs and PFSs with the condition \(0 \leq (\sigma_\eta(\bar{\eta}))^\alpha + (\rho_\eta(\bar{\eta}))^\beta \leq 1\), where \(\sigma_\eta(\bar{\eta})\) and \(\rho_\eta(\bar{\eta})\) represent MG and NMG corresponding to alternative \(\eta\), respectively. For very large values of \(q\), the valuation space of q-ROFS approaches to that of LDFS, but there is deficiency in q-ROFS, and it cannot deal with reference parameters, which is a key factor of LDFS. For small values of \(q\), the valuation space of q-ROFS is smaller than that of LDFS, but this affects MCDM. For the input data in MCDM problems if \(\sigma_\eta(\bar{\eta})\) and \(\rho_\eta(\bar{\eta})\) equal to 1, then q-ROFS fails to deal with this situation because \(1 + 1 > 1\). However, with the suitable choice of reference parameters, we can easily deal these types of values in LDFSs (i.e., \((1)(0.6) + (1)(0.3) < 1\); the choice of numeric values of reference parameters is according to the situation and decision-making problem with the condition that there sum is less than 1). From all the discussion, it is clear that our proposed idea is more suitable and superior to others and contains a variety of reference parameters. We can use this approach in various applications of engineering, medical, artificial intelligence, and MADM methods. In Table 1, we can see the comparison between the proposed approach with the existing concepts. Figure 1 shows the graphical composition of LDFNs with some existing fuzzy numbers.

### 2. Preliminaries

In this part, we recall certain rudiments of LDFSs, some of its operations, and score functions. Throughout the study, we utilize \(\varnothing\) as a universal set.

**Definition 1** (see [51]). A linear Diophantine fuzzy set (LDFS) \(\mathbb{D}\) in \(\varnothing\) is defined as

\[
\mathbb{D} = \left\{ (\bar{\eta}, (\sigma_\eta(\bar{\eta}), \rho_\eta(\bar{\eta})), (\alpha_\eta(\bar{\eta}), \beta_\eta(\bar{\eta})): \bar{\eta} \in \varnothing \right\},
\]

where \(\sigma_\eta(\bar{\eta}), \rho_\eta(\bar{\eta}), \alpha_\eta(\bar{\eta}), \beta_\eta(\bar{\eta}) \in [0, 1]\) are the membership grade (MG), the nonmembership grade (NMG), and the corresponding reference parameters, respectively, which satisfy the basic conditions:

\[
0 \leq \alpha_\eta(\bar{\eta}) + \beta_\eta(\bar{\eta}) \leq 1,
\]

for all \(\bar{\eta} \in \varnothing\). The LDFS

\[
\mathfrak{D}_\eta = \left\{ (\bar{\eta}, (1, 0)), (0, 1)): \bar{\eta} \in \varnothing \right\},
\]

is called the absolute LDFS in \(\varnothing\). The LDFS

\[
\mathfrak{D}_\varnothing = \left\{ (\eta, (0, 1)), (0, 1)): \bar{\eta} \in \varnothing \right\},
\]

called the null LDFS in \(\varnothing\).

The reference or control parameters are valuable for modeling or analyzing a critical system. We can characterize different systems by altering the dynamic representation of these parameters. In addition, \(\pi_\eta(\bar{\eta}) = 1 - (\alpha_\eta(\bar{\eta})\sigma_\eta(\bar{\eta}) + \beta_\eta(\bar{\eta})\rho_\eta(\bar{\eta}))\) is called the indeterminacy degree and its corresponding reference parameter of \(\eta\) to \(\mathbb{D}\).

**Definition 2** (see [51]). A linear Diophantine fuzzy number (LDFN) can be written in the form \(\Delta = (\sigma_\Delta, \rho_\Delta)\),

\[
(\alpha_\Delta, \beta_\Delta),
\]

satisfying the following conditions:

1. \(\sigma_\Delta, \rho_\Delta, \alpha_\Delta, \beta_\Delta \in [0, 1]\)
2. \(0 \leq \alpha_\Delta + \beta_\Delta \leq 1\)
3. \(0 \leq \alpha_\Delta \sigma_\Delta + \beta_\Delta \rho_\Delta \leq 1\)
Table 1: Comparison between LDFS with some existing fuzzy sets.

| Concepts          | Remarks                                                                 |
|-------------------|--------------------------------------------------------------------------|
| Fuzzy sets [1]    | It considers membership grades (MGs), but it does not consider nonmembership grades (NMGs) |
| IFSs [4]          | It cannot be applied if $\sigma_\eta (\tilde{\eta}) + \rho_\eta (\tilde{\eta}) > 1$ for some $\tilde{\eta}$ |
| PFSs [6,7]        | It cannot be applied if $(\sigma_\eta (\tilde{\eta}))^q + (\rho_\eta (\tilde{\eta}))^q > 1$ for some $\tilde{\eta}$  |
| $q$-ROFs [9]      | It cannot be applied for smaller values of “q” with $(\sigma_\eta (\tilde{\eta}))^q + (\rho_\eta (\tilde{\eta}))^q > 1$, or if $\sigma_\eta (\tilde{\eta}) = \rho_\eta (\tilde{\eta}) = 1$ for some $\tilde{\eta}$ |

(1) To deal with the situations when IFS, PFS, and $q$-ROFS cannot be applied; (2) the reference or control parameters are used as the weight vector such that they cannot exceed unity; (3) MGs and NMGs can be chosen freely from $[0, 1]$; (4) the real value of linear combination $\alpha_\eta \sigma_\eta + \beta_\eta \rho_\eta$ always lies in $[0, 1]$.

Example 1 (reimbursement cases for medical treatment). Medical expenses must be primarily to relieve or check a physical or mental sickness, illness, or disorder. They do not include expenses that are for cosmetic purposes or are merely beneficial to general health. Usually, the reimbursement is organized directly between the establishments involved, so you need not to pay for the treatment. However, in many cases, the treatment involves highly qualified doctors at well-established hospitals. So, people take their treatment by self-payment and tend to reimbursement cases. These cases will be sent to national health authority (NHA) for their approval and releasing expenses. However, the decision experts (DEs) will decide the amount of expenses to be paid. The DEs also provide particular list of medicine that are, and are not, eligible for reimbursement. Since medicines are compounds or chemicals products that are used to cure, ease symptoms, prevent disease, or help in the treatment of illnesses, so it is critical to decide which medicine is really used, or not used, for treatment and which is used for other purposes.

Let $\tilde{\mathbb{E}} = \{\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3, \tilde{\eta}_4, \tilde{\eta}_5\}$ be the list some lifesaving medicine. Two or more drugs can be combined in an appropriate ratio during preparation of medicines to get a high impact of such medicines. If the reference or control parameter is considered as

$$
\begin{align*}
\sigma_\Delta &= \text{"Membership grades or favorable in health recovery,"} \\
\rho_\Delta &= \text{"Non-membership grades or not favorable in health recovery,"} \\
\alpha_\Delta &= \text{"Less side effect safer surgeries,"} \\
\beta_\Delta &= \text{"More side effect safer surgeries,"}
\end{align*}
$$

then the LDFS is given in Table 2.

Definition 3 (see [51]). Let $\Delta = \left( \rho_\Delta, \sigma_\Delta \right)$ be a LDFN; then, we denote the score function of LDFN as $\mathcal{I} (\Delta)$, and we define it by

$$
\mathcal{I} (\Delta) = \frac{1}{2} \left[ (\sigma_\Delta - \rho_\Delta) + (\alpha_\Delta - \beta_\Delta) \right].
$$

Here, LDFN ($\tilde{\mathbb{E}}$) is an assemblage of LDFNs on $\tilde{\mathbb{E}}$. The range of $\mathcal{I} (\Delta)$ is given as $\mathcal{I} (\Delta) : LDFN (\tilde{\mathbb{E}}) \rightarrow [-1, 1]$. 

Figure 1: Comparison view of IFNs, PFFNs, $q$-ROFNs, and LDFNs.
Theorem 4 (see [51]). Let $A = \left(\langle \alpha \_1, \rho \_1 \rangle, \langle \alpha \_2, \beta \_2 \rangle \right)$ be a LDFN; then, we denote the accuracy function of LDFN as $\psi(A)$, and we define it by

$$\psi(A) = \frac{1}{2} \left(\frac{\alpha \_1 + \rho \_1}{2} + (\alpha \_2 + \beta \_2)\right). \quad (7)$$

Definition 5 (see [51]). Let $A_1$ and $A_2$ be two LDFNs. By utilizing the score and accuracy functions, we can efficiently associate these two LDFNs as

(i) If $\mathfrak{F}(A_1) < \mathfrak{F}(A_2)$, then $A_1 < A_2$
(ii) If $\mathfrak{F}(A_1) > \mathfrak{F}(A_2)$, then $A_1 > A_2$
(iii) If $\mathfrak{F}(A_1) = \mathfrak{F}(A_2)$, then $A_2 = A_1$

Definition 6 (see [51]). Let $A_i = \left(\langle \alpha \_i, \rho \_i \rangle, \langle \alpha \_i, \beta \_i \rangle \right)$ be a LDFN and $\mathfrak{W} > 0$. Then,

(i) $A \_i \_i = (\langle \rho \_i, \sigma \_i \rangle, \langle \beta \_i, \alpha \_i \rangle)$
(ii) $\mathfrak{W} A_1 = (\left(1 - (1 - \sigma \_i)^\mathfrak{W}, \rho \_i^\mathfrak{W}\right), \left(1 - (1 - \alpha \_i)^\mathfrak{W}, \rho \_1^\mathfrak{W}\right))$
(iii) $A \_1 \_2 = (\langle \sigma \_1^\mathfrak{W}, 1 - (1 - \rho \_1^\mathfrak{W}), \langle \alpha \_1^\mathfrak{W}, 1 - (1 - \beta \_1^\mathfrak{W})\rangle)$

Definition 7 (see [51]). Let $A_{i} = \left(\langle \sigma \_i, \rho \_i \rangle, \langle \alpha \_i, \beta \_i \rangle \right)$ be two LDFNs with $i = 1, 2$. Then,

(i) $A_1 < A_2 \iff \sigma_1 \leq \sigma_2, \rho_1 \leq \rho_2, \alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2$
(ii) $A_1 = A_2 \iff \sigma_1 = \sigma_2, \rho_1 = \rho_2, \alpha_1 = \alpha_2, \beta_1 = \beta_2$
(iii) $A_1 \oplus A_2 = (\langle \sigma_1 + \sigma_2 - \sigma_1 \rho_1 \rho_2, \alpha_1 + \alpha_2 - \alpha_1 \rho_1 \rho_2 \rangle, \langle \beta_1 + \beta_2 - \beta_1 \rho_1 \rho_2 \rangle)$
(iv) $A_1 \otimes A_2 = (\langle \sigma_1 \rho_1 + \rho_1 \rho_2, \alpha_1 \rho_1 + \rho_1 \rho_2 \rangle, \langle \beta_1 \rho_1 + \rho_1 \rho_2 \rangle)$

Definition 8 (see [51]). Let $A_1 = \left(\langle \sigma \_i, \rho \_i \rangle, \langle \alpha \_i, \beta \_i \rangle \right)$ be a collection of LDFNs with $i \in \Delta$. Then,

(i) $\cup_{i \in \Delta} A_i = (\langle \sup_{i \in \Delta} \sigma_i, \sup_{i \in \Delta} \rho_i \rangle, \langle \inf_{i \in \Delta} \alpha_i, \inf_{i \in \Delta} \beta_i \rangle)$
(ii) $\cap_{i \in \Delta} A_i = (\langle \inf_{i \in \Delta} \sigma_i, \sup_{i \in \Delta} \rho_i \rangle, \langle \inf_{i \in \Delta} \alpha_i, \sup_{i \in \Delta} \beta_i \rangle)$

Example 2. Let $A_1 = (\langle 0.81, 0.47 \rangle, \langle 0.52, 0.39 \rangle)$ and $A_2 = (\langle 0.91, 0.36 \rangle, \langle 0.64, 0.27 \rangle)$ be two LDFNs. Then, it is clear that $A_1 \subseteq A_2$. One can verify that

(i) $A \_1 \_1 = (\langle 0.47, 0.81 \rangle, \langle 0.39, 0.52 \rangle)$
(ii) $A \_1 \cup A \_2 = (\langle 0.91, 0.36 \rangle, \langle 0.64, 0.27 \rangle) = A \_2$

(iii) $A_1 \cap A_2 = (\langle 0.81, 0.47 \rangle, \langle 0.52, 0.39 \rangle) = A_1$
(iv) $A \_1 \_\oplus A \_2 = (\langle 0.9829, 0.1692 \rangle, \langle 0.8272, 0.1053 \rangle)$
(v) $A \_1 \_\otimes A \_2 = (\langle 0.7371, 0.6608 \rangle, \langle 0.3328, 0.5547 \rangle)$

In addition, if $\mathfrak{W} = 0.1$, then we have

(i) $\mathfrak{W} A_1 = (\langle 0.1530, 0.9272 \rangle, \langle 0.0707, 0.9101 \rangle)$
(ii) $A_1^\mathfrak{W} = (\langle 0.9791, 0.0615 \rangle, \langle 0.9366, 0.0482 \rangle)$

3. Einstein Operational Laws for LDFNs

Within this section, the t-norm and t-conorm are given as an outline, and we shall provide examples of these concepts. Some of the characteristics of Einstein’s operations are given to LDFNs.

For the very first time, triangular norms have indeed been introduced in the form of probabilistic metric spaces, as we are using them currently. It also plays an essential role in statistics, decision-making, and cooperative games. Most parameterized t-norm groups are renowned for their functional equation solutions. T-norms have been used for fuzzy set theory at the intersection of two fuzzy sets. T-conorms are used for modeling disjunction or union. Triangular norms and conorms are operations that generalize logical conjunction and logical disjunction to fuzzy logic. These are basic descriptions of the conjunction and disjunction in mathematical fuzzy logic semantics which are used in the MCDM to integrate the prerequisite.

The t-norm is expressed by the binary operation $\bar{\Pi}$ that meets the required constraints at the interval $[0, 1]$:

(i) $\bar{\Pi} \left(\bar{T}, \bar{\sigma} \right) = \bar{T} \left(\bar{T}, \bar{\sigma} \right)$
(ii) $\bar{\Pi} \left(\bar{T}, \bar{\Pi} \left(\bar{T}, \bar{\sigma} \right) \right) = \bar{\Pi} \left(\bar{T}, \bar{\Pi} \left(\bar{T}, \bar{\sigma} \right) \right)$
(iii) If $\bar{\Pi} \leq \bar{\Pi}$ and $\bar{\Pi} \leq \bar{\Pi}$, then $\bar{\Pi} \left(\bar{T}, \bar{\Pi} \right) \leq \bar{\Pi} \left(\bar{T}, \bar{\Pi} \right)$
(iv) $\bar{\Pi} \left(\bar{T}, 1 \right) = 1$ (neutral element) and $\bar{\Pi} \left(0, 0 \right) = 0$

A t-conorm is a binary operation $\bar{\Pi}$ that meets specific constraints at the interval $[0, 1]$:

(i) $\bar{\Pi} \left(\bar{T}, \bar{T} \right) = \bar{\Pi} \left(\bar{T}, \bar{T} \right)$
(ii) $\bar{\Pi} \left(\bar{T}, \bar{\Pi} \left(\bar{T}, \bar{T} \right) \right) = \bar{\Pi} \left(\bar{T}, \bar{\Pi} \left(\bar{T}, \bar{T} \right) \right)$
(iii) If $\bar{\Pi} \leq \bar{\Pi}$ and $\bar{\Pi} \leq \bar{\Pi}$, then $\bar{\Pi} \left(\bar{T}, \bar{\Pi} \right) \leq \bar{\Pi} \left(\bar{T}, \bar{\Pi} \right)$
(iv) $\bar{\Pi} \left(\bar{T}, 0 \right) = 0$ (neutral element 0) and $\bar{\Pi} \left(1, 1 \right) = 1$

Definition 9. Let $A_1 = \left(\langle \sigma \_i, \rho \_i \rangle, \langle \alpha \_i, \beta \_i \rangle \right)$ be two LDFNs with $i = 1, 2$ and $\bar{\mathfrak{W}} > 0$ be any real number; then, Einstein operations for LDFNs is defined as follows:

(i) $A \_i \_i = (\langle \rho \_1 \_1, \sigma \_i \_i \rangle, \langle \beta \_i \_i, \alpha \_i \_i \rangle)$
(ii) $A \_1 \_\lor A \_2 = (\langle \max \{\sigma_1, \sigma_2\}, \min \{\rho_1, \rho_2\} \rangle, \langle \max \{\alpha_1, \alpha_2\}, \min \{\beta_1, \beta_2\} \rangle)$
(iii) $A \_1 \_\lor A \_2 = (\langle \min \{\sigma_1, \sigma_2\}, \max \{\rho_1, \rho_2\} \rangle, \langle \min \{\alpha_1, \alpha_2\}, \max \{\beta_1, \beta_2\} \rangle)$
(v) \( \Delta_1 \otimes_{\varepsilon} \Delta_2 = (\langle \sigma_1 + \sigma_2 / 1 + \sigma_1^* \varepsilon \sigma_2, \rho_1^* \varepsilon \rho_2 / 1 + (1 - \rho_1) \rangle \cdot \varepsilon (1 - \rho_2))\), \(\langle \alpha_1 + \alpha_2 / 1 + \alpha_1^* \varepsilon \alpha_2, \beta_1^* \varepsilon \beta_2 / 1 + (1 - \beta_1) \rangle \cdot \varepsilon (1 - \beta_2)\)

(vi) \( \overline{\Delta}_1 \otimes_{\varepsilon} \overline{\Delta}_2 = (\langle (1 + \sigma_1) \overline{\varepsilon} - (1 - \sigma_1) \overline{\varepsilon} / (1 + \sigma_1) \overline{\varepsilon} + (1 - \sigma_1) \overline{\varepsilon}, 2 \rho_1 \overline{\varepsilon} / (2 - \rho_1) \overline{\varepsilon} + \rho_1 \overline{\varepsilon}, (1 + \alpha_1) \overline{\varepsilon} - (1 - \alpha_1) \overline{\varepsilon} / (1 + \alpha_1) \overline{\varepsilon} + (1 - \alpha_1) \overline{\varepsilon}, 2 \partial_1 \overline{\varepsilon} / (2 - \beta_1) \overline{\varepsilon} + \beta_1 \overline{\varepsilon} + \beta_1 \overline{\varepsilon} \rangle \cdot \varepsilon (1 - \beta_2)\)

(vii) \( \overline{\Delta}_1^{\overline{e}} = (\langle 2 \sigma_1^{\overline{e}} / (2 - \sigma_1)^{\overline{e}} + \sigma_1^{\overline{e}}, (1 + \rho_1)^{\overline{e}} - (1 - \rho_1)^{\overline{e}} / (1 + \rho_1)^{\overline{e}} + (1 - \rho_1)^{\overline{e}}, 2 \partial_1^{\overline{e}} / (2 - \beta_1)^{\overline{e}} + \beta_1^{\overline{e}} + \beta_1^{\overline{e}} \rangle \cdot \varepsilon (1 - \beta_2)\)

Theorem 1. Let \( \Delta_i = (\langle \sigma_i, \rho_i \rangle, \langle a_i, \beta_i \rangle) \) be two LDFNs with \( i = 1, 2 \) and \( \overline{\varepsilon} > 0 \) be any real number; then,

(i) \( \Delta_1 \otimes_{\varepsilon} \Delta_2 = \Delta_2 \otimes_{\varepsilon} \Delta_1 \)

(ii) \( \Delta_1 @ \varepsilon \Delta_2 = \Delta_2 @ \varepsilon \Delta_1 \)

(iii) \( \langle \Delta_1 @ \varepsilon \Delta_2 \rangle^{\overline{e}} = \overline{\Delta}_1^{\overline{e}} @ \overline{\Delta}_2^{\overline{e}} \)

(iv) \( \overline{\Delta}_1^{\overline{e}} @ \overline{\Delta}_2^{\overline{e}} = \overline{\Delta}_1^{\overline{e}} @ \overline{\Delta}_2^{\overline{e}} \)

\[
\begin{align*}
\frac{\sigma_1 \cdot \sigma_2}{1 + (1 - \sigma_1) \cdot (1 - \sigma_2)} &= \frac{\sigma_1 \cdot \sigma_2}{2 - \sigma_1 - \sigma_2 + \sigma_1 \cdot \sigma_2} = \frac{2 \sigma_1 \cdot \sigma_2}{4 - 2 \sigma_1 - 2 \sigma_2 + 2 \sigma_1 \cdot \sigma_2} = \frac{2 \sigma_1 \cdot \sigma_2}{2 - \sigma_1} \cdot \frac{2 - \sigma_2}{(2 - \sigma_2) + \sigma_1 \cdot \sigma_2}, \\
\frac{\alpha_1 \cdot \alpha_2}{1 + (1 - \alpha_1) \cdot (1 - \alpha_2)} &= \frac{\alpha_1 \cdot \alpha_2}{2 - \alpha_1 - \alpha_2 + \alpha_1 \cdot \alpha_2} = \frac{2 \alpha_1 \cdot \alpha_2}{4 - 2 \alpha_1 - 2 \alpha_2 + 2 \alpha_1 \cdot \alpha_2} = \frac{2 \alpha_1 \cdot \alpha_2}{2 - \alpha_1} \cdot \frac{2 - \alpha_2}{(2 - \alpha_2) + \alpha_1 \cdot \alpha_2}
\end{align*}
\]

Similarly,

\[
\begin{align*}
\frac{\rho_1 + \rho_2}{1 + \rho_1 \cdot \rho_2} &= \frac{(1 + \rho_1) \cdot (1 + \rho_2) - (1 - \rho_1) \cdot (1 - \rho_2)}{(1 + \rho_1) \cdot (1 + \rho_2) + (1 - \rho_1) \cdot (1 - \rho_2)}, \\
\frac{\beta_1 + \beta_2}{1 + \beta_1 \cdot \beta_2} &= \frac{(1 + \beta_1) \cdot (1 + \beta_2) - (1 - \beta_1) \cdot (1 - \beta_2)}{(1 + \beta_1) \cdot (1 + \beta_2) + (1 - \beta_1) \cdot (1 - \beta_2)}
\end{align*}
\]

So, we write \( (\Delta_1 @ \varepsilon \Delta_2) \) equivalent to

\[
(\Delta_1 @ \varepsilon \Delta_2) = \left( \begin{array}{c}
\frac{2 \sigma_1 \cdot \sigma_2}{2 - \sigma_1} \cdot \frac{2 - \sigma_2}{(2 - \sigma_2) + \sigma_1 \cdot \sigma_2} \\
\frac{2 \alpha_1 \cdot \alpha_2}{2 - \alpha_1} \cdot \frac{2 - \alpha_2}{(2 - \alpha_2) + \alpha_1 \cdot \alpha_2}
\end{array} \right)
\]
Assume \( \mathcal{A} = (1 + \rho_1) \cdot \triangeleft (1 + \rho_2) \), \( \mathcal{B} = (1 - \rho_1) \cdot \triangeleft (1 - \rho_2) \), \( \mathcal{C} = (\sigma_1) \cdot \triangeleft (\sigma_2) \), and \( \mathcal{D} = (2 - \sigma_1) \cdot \triangeleft (2 - \sigma_2) \), \( \mathcal{A}' = (1 + \beta_1) \cdot \triangeleft (1 + \beta_2) \), \( \mathcal{B}' = (1 - \beta_1) \cdot \triangeleft (1 - \beta_2) \), \( \mathcal{C}' = (a_1) \cdot \triangeleft (a_2) \), and \( \mathcal{D}' = (2 - a_1) \cdot \triangeleft (2 - a_2) \), then

\[
\Delta_1 \otimes \Delta_2 = \left( \begin{array}{c}
\frac{2 \cdot 2 \cdot \mathcal{C}' \cdot \mathcal{A} - \mathcal{B}'}{2 \cdot 2 \cdot \mathcal{D} + \mathcal{C}' \cdot \mathcal{A} + \mathcal{B}'} \\
\frac{2 \cdot 2 \cdot \mathcal{C}' \cdot \mathcal{A} - \mathcal{B}'}{2 \cdot 2 \cdot \mathcal{D} + \mathcal{C}' \cdot \mathcal{A} + \mathcal{B}'}
\end{array} \right)
\]

\[
(\Delta_1 \otimes \Delta_2)^* = \left( \begin{array}{c}
\frac{2 \cdot 2 \cdot \mathcal{C}' \cdot \mathcal{A} - \mathcal{B}'}{2 \cdot 2 \cdot \mathcal{D} + \mathcal{C}' \cdot \mathcal{A} + \mathcal{B}'} \\
\frac{2 \cdot 2 \cdot \mathcal{C}' \cdot \mathcal{A} - \mathcal{B}'}{2 \cdot 2 \cdot \mathcal{D} + \mathcal{C}' \cdot \mathcal{A} + \mathcal{B}'}
\end{array} \right)^*
\]

\[
= \left( \begin{array}{c}
\frac{2 \cdot 2 \cdot \mathcal{C} \cdot \mathcal{A} - \mathcal{B}}{2 \cdot 2 \cdot \mathcal{D} + \mathcal{C} \cdot \mathcal{A} + \mathcal{B}} \\
\frac{2 \cdot 2 \cdot \mathcal{C} \cdot \mathcal{A} - \mathcal{B}}{2 \cdot 2 \cdot \mathcal{D} + \mathcal{C} \cdot \mathcal{A} + \mathcal{B}}
\end{array} \right)^*
\]

\[
(\Delta_1 \otimes \Delta_2)^* = \left( \begin{array}{c}
\frac{2 \cdot 2 \cdot \mathcal{C} \cdot \mathcal{A} - \mathcal{B}}{2 \cdot 2 \cdot \mathcal{D} + \mathcal{C} \cdot \mathcal{A} + \mathcal{B}} \\
\frac{2 \cdot 2 \cdot \mathcal{C} \cdot \mathcal{A} - \mathcal{B}}{2 \cdot 2 \cdot \mathcal{D} + \mathcal{C} \cdot \mathcal{A} + \mathcal{B}}
\end{array} \right)
\]

Now, suppose \( \mathcal{A}' = (1 + \rho_1) \cdot \triangeleft \), \( \mathcal{B}' = (1 - \rho_1) \cdot \triangeleft \), \( \mathcal{C}' = (\sigma_1) \cdot \triangeleft \), \( \mathcal{D}' = (2 - \sigma_1) \cdot \triangeleft \), \( \mathcal{A}_1 = (1 + \rho_1) \cdot \triangeleft \), \( \mathcal{B}_1 = (1 - \rho_1) \cdot \triangeleft \), \( \mathcal{C}_1 = (\sigma_1) \cdot \triangeleft \), \( \mathcal{D}_1 = (2 - \sigma_1) \cdot \triangeleft \), \( \mathcal{A}_1' = (1 + \beta_1) \cdot \triangeleft \), \( \mathcal{B}_1' = (1 - \beta_1) \cdot \triangeleft \), \( \mathcal{C}_1' = (a_1) \cdot \triangeleft \), \( \mathcal{D}_1' = (2 - a_1) \cdot \triangeleft \). On right-hand sides,

\[
\Delta_1 = \left( \begin{array}{c}
\frac{2 \cdot \mathcal{C} \cdot \mathcal{A} - \mathcal{B}}{2 \cdot \mathcal{D} + \mathcal{C} \cdot \mathcal{A} + \mathcal{B}} \\
\frac{2 \cdot \mathcal{C} \cdot \mathcal{A} - \mathcal{B}}{2 \cdot \mathcal{D} + \mathcal{C} \cdot \mathcal{A} + \mathcal{B}}
\end{array} \right)
\]

\[
\Delta_2 = \left( \begin{array}{c}
\frac{2 \cdot \mathcal{C} \cdot \mathcal{A} - \mathcal{B}}{2 \cdot \mathcal{D} + \mathcal{C} \cdot \mathcal{A} + \mathcal{B}} \\
\frac{2 \cdot \mathcal{C} \cdot \mathcal{A} - \mathcal{B}}{2 \cdot \mathcal{D} + \mathcal{C} \cdot \mathcal{A} + \mathcal{B}}
\end{array} \right)
\]

\[
\Delta_1 \otimes \Delta_2 = \left( \begin{array}{c}
\frac{2 \cdot \mathcal{C} \cdot \mathcal{A} - \mathcal{B}}{2 \cdot \mathcal{D} + \mathcal{C} \cdot \mathcal{A} + \mathcal{B}} \\
\frac{2 \cdot \mathcal{C} \cdot \mathcal{A} - \mathcal{B}}{2 \cdot \mathcal{D} + \mathcal{C} \cdot \mathcal{A} + \mathcal{B}}
\end{array} \right)
\]

\[
(\Delta_1 \otimes \Delta_2)^* = \left( \begin{array}{c}
\frac{2 \cdot \mathcal{C} \cdot \mathcal{A} - \mathcal{B}}{2 \cdot \mathcal{D} + \mathcal{C} \cdot \mathcal{A} + \mathcal{B}} \\
\frac{2 \cdot \mathcal{C} \cdot \mathcal{A} - \mathcal{B}}{2 \cdot \mathcal{D} + \mathcal{C} \cdot \mathcal{A} + \mathcal{B}}
\end{array} \right)^*
\]
\[
\begin{align*}
\Delta^{-\pi} \otimes E^{\bar{\pi}} &= \left( \left( \frac{2C_1}{\mathcal{D}_1 + C_1', \mathcal{A}_1 + B_1} \right) \otimes \left( \frac{2C_1'}{\mathcal{A}_1' + B_1'} \right) \right) \otimes E \left( \left( \frac{2C_2}{\mathcal{D}_2 = C_2', \mathcal{A}_2 + B_2} \right) \left( \frac{2C_2'}{\mathcal{D}_2' + C_2', \mathcal{A}_2' + B_2'} \right) \right) \\
&= \left( \left( \frac{2((C_1' + C_1))(D_1 + C_1')}{1 + (1 - 2C_1/D_1 + C_1')} \right) \left( \frac{2D_1 - B_1}{1 + (1 - 2C_1/D_1 + C_1') \left( 1 - 2C_1/D_1 + C_1' \right) \left( 1 + D_1 + B_1 + D_1 - B_1/D_1 + B_1' \right) \right) \right) \\
&= \left( \left( \frac{2((C_1' + C_1)}{(D_1' + C_1')} \right) \left( \frac{2D_1 - B_1}{1 + (1 - 2C_1/D_1 + C_1') \left( 1 - 2C_1/D_1 + C_1' \right) \left( 1 + D_1 + B_1 + D_1 - B_1/D_1 + B_1' \right) \right) \right) \\
&= \left( \left( \frac{2C_1}{\mathcal{D}_1 + C_1', \mathcal{A}_1 + B_1} \right) \otimes \left( \frac{2C_1'}{\mathcal{A}_1' + B_1'} \right) \right) \otimes E \left( \left( \frac{2C_2}{\mathcal{D}_2 = C_2', \mathcal{A}_2 + B_2} \right) \left( \frac{2C_2'}{\mathcal{D}_2' + C_2', \mathcal{A}_2' + B_2'} \right) \right).
\end{align*}
\]

Hence, we prove

\[
(\Delta_{1} \otimes \varphi \Delta_{2})^{\bar{\pi}} = \Delta^{-\pi} \otimes E^{\bar{\pi}}.
\]
Assume $\mathcal{A} = (1 + \sigma_1)r_{\mathcal{E}}(1 + \sigma_2)$, $\mathcal{B} = (1 - \sigma_1)r_{\mathcal{E}}$
$(1 - \sigma_2)$, $\mathcal{C} = (\rho_1)r_{\mathcal{E}}(\rho_2)$, and $\mathcal{D} = (2 - \rho_1)r_{\mathcal{E}}$
$(2 - \rho_2)$, $\mathcal{A}' = (1 + \alpha_1)r_{\mathcal{E}}(1 + \alpha_2)$, $\mathcal{B}' = (1 - \alpha_1)r_{\mathcal{E}}$
$(1 - \alpha_2)$, $\mathcal{C}' = (\beta_1)r_{\mathcal{E}}(\beta_2)$, and $\mathcal{D}' = (2 - \beta_1)r_{\mathcal{E}}$
$(2 - \beta_2)$; then,

\[
\Delta_1 \Phi \Delta_3 = \left\langle \frac{\mathcal{A} - \mathcal{B}}{\mathcal{A} + \mathcal{B}} \cdot 2 \mathcal{C} \mathcal{E}, \left\langle \frac{\mathcal{A}' - \mathcal{B}'}{\mathcal{A}' + \mathcal{B}'} \cdot 2 \mathcal{C}' \mathcal{E} \right\rangle \right\rangle
\]

\[
\mathcal{E} \cdot \mathcal{G} (\Delta_1 \Phi \Delta_3) = \left(\frac{\mathcal{A} - \mathcal{B}}{\mathcal{A} + \mathcal{B}} \cdot 2 \mathcal{C} \mathcal{E}, \left\langle \frac{\mathcal{A}' - \mathcal{B}'}{\mathcal{A}' + \mathcal{B}'} \cdot 2 \mathcal{C}' \mathcal{E} \right\rangle \right) \mathcal{E}
\]

\[
\begin{aligned}
\mathcal{E} \cdot \mathcal{G} (\Delta_1 \Phi \Delta_3) &= \left\langle 1 + (\mathcal{A} - \mathcal{B} | \mathcal{A} + \mathcal{B}) \mathcal{E} - (1 - (\mathcal{A} - \mathcal{B} | \mathcal{A} + \mathcal{B}) \mathcal{E}) \right\rangle, \\
&= \left\langle \frac{1 - (\mathcal{A} - \mathcal{B} | \mathcal{A} + \mathcal{B}) \mathcal{E}}{\mathcal{A} + \mathcal{B} \mathcal{E} + \mathcal{C} \mathcal{F}} + \frac{2 \cdot (2 \mathcal{C} | \mathcal{D} + \mathcal{E}) \mathcal{F}}{(2 - 2 \mathcal{C} | \mathcal{D} + \mathcal{E}) \mathcal{F} + (2 \mathcal{C} | \mathcal{D} + \mathcal{E}) \mathcal{F}} \right\rangle
\end{aligned}
\]

\[
\mathcal{E} \cdot \mathcal{G} (\Delta_1 \Phi \Delta_3) = \left\langle \frac{\mathcal{A} - \mathcal{B}}{\mathcal{A} + \mathcal{B}} \cdot 2 \mathcal{C} \mathcal{E}, \left\langle \frac{\mathcal{A}' - \mathcal{B}'}{\mathcal{A}' + \mathcal{B}'} \cdot 2 \mathcal{C}' \mathcal{E} \right\rangle \right\rangle
\]

\[
\begin{aligned}
\mathcal{E} \cdot \mathcal{G} (\Delta_1 \Phi \Delta_3) &= \left\langle \left(1 + \sigma_1\right)\mathcal{E} - \left(1 - \sigma_1\right)\mathcal{E}, \left(1 + \sigma_2\right)\mathcal{E} \right\rangle, \\
&= \left\langle \left(1 + \sigma_1\right)\mathcal{E} - \left(1 - \sigma_1\right)\mathcal{E}, \left(1 + \sigma_2\right)\mathcal{E} \right\rangle
\end{aligned}
\]

Now, suppose $\mathcal{A}_1 = (1 + \sigma_1)\mathcal{E}$, $\mathcal{B}_1 = (1 - \sigma_1)\mathcal{E}$,
$\mathcal{C}_1 = (\sigma_1)\mathcal{E}$, $\mathcal{D}_1 = (2 - \sigma_1)\mathcal{E}$, $\mathcal{A}_2 = (1 + \sigma_2)\mathcal{E}$, $\mathcal{B}_2$
$= (1 - \sigma_2)\mathcal{E}$, $\mathcal{C}_2 = \rho_2\mathcal{E}$, $\mathcal{D}_2 = (2 - \rho_2)\mathcal{E}$, $\mathcal{A}_1' = (1 + \alpha_1)\mathcal{E}$, $\mathcal{B}_1' = (1 - \alpha_1)\mathcal{E}$, $\mathcal{C}_1' = \beta_1\mathcal{E}$, $\mathcal{D}_1' = (2 - \beta_1)\mathcal{E}$, $\mathcal{A}_2' = (1 + \alpha_2)\mathcal{E}$, $\mathcal{B}_2' = (1 - \alpha_2)\mathcal{E}$, $\mathcal{C}_2' = \beta_2\mathcal{E}$, and $\mathcal{D}_2' = (2 - \beta_2)\mathcal{E}$. On right-hand sides,
\[
\begin{align*}
\Pi_{\Phi D_1} &= \left( \begin{pmatrix} (1 + \sigma_1) \frac{\pi}{\pi} - (1 - \sigma_1) \frac{\pi}{\pi} \\ (1 + \sigma_1) \frac{\pi}{\pi} + (1 - \sigma_1) \frac{\pi}{\pi} \end{pmatrix}, \begin{pmatrix} 2\rho_1 \frac{\pi}{\pi} \\ (1 + \alpha_1) \frac{\pi}{\pi} + (1 - \alpha_1) \frac{\pi}{\pi} \end{pmatrix} \right) \\
&\quad \left( \begin{pmatrix} 2\beta_1 \frac{\pi}{\pi} \\ (1 + \alpha_1) \frac{\pi}{\pi} + (1 - \alpha_1) \frac{\pi}{\pi} \end{pmatrix}, \begin{pmatrix} (1 + \sigma_1) \frac{\pi}{\pi} - (1 - \sigma_1) \frac{\pi}{\pi} \\ (1 + \sigma_1) \frac{\pi}{\pi} + (1 - \sigma_1) \frac{\pi}{\pi} \end{pmatrix} \right) 
\end{align*}
\]
\[
\Pi_{\Phi D_1} = \left( \begin{pmatrix} \mathcal{A}_1 - \mathcal{B}_1 \\ \mathcal{A}_1 + \mathcal{B}_1 \mathcal{D}_1 + \mathcal{C}_1 \end{pmatrix}, \begin{pmatrix} \mathcal{A}_1' - \mathcal{B}_1' \\ \mathcal{A}_1' + \mathcal{B}_1' \mathcal{D}_1' + \mathcal{C}_1' \end{pmatrix} \right).
\]

\[
\begin{align*}
\Pi_{\Phi D_1} &= \left( \begin{pmatrix} (1 + \sigma_2) \frac{\pi}{\pi} - (1 - \sigma_2) \frac{\pi}{\pi} \\ (1 + \sigma_2) \frac{\pi}{\pi} + (1 - \sigma_2) \frac{\pi}{\pi} \end{pmatrix}, \begin{pmatrix} 2\rho_2 \frac{\pi}{\pi} \\ (1 + \alpha_2) \frac{\pi}{\pi} + (1 - \alpha_2) \frac{\pi}{\pi} \end{pmatrix} \right) \\
&\quad \left( \begin{pmatrix} 2\beta_2 \frac{\pi}{\pi} \\ (1 + \alpha_2) \frac{\pi}{\pi} + (1 - \alpha_2) \frac{\pi}{\pi} \end{pmatrix}, \begin{pmatrix} (1 + \sigma_2) \frac{\pi}{\pi} - (1 - \sigma_2) \frac{\pi}{\pi} \\ (1 + \sigma_2) \frac{\pi}{\pi} + (1 - \sigma_2) \frac{\pi}{\pi} \end{pmatrix} \right) 
\end{align*}
\]
\[
\Pi_{\Phi D_1} = \left( \begin{pmatrix} \mathcal{A}_2 - \mathcal{B}_2 \\ \mathcal{A}_2 + \mathcal{B}_2 \mathcal{D}_2 + \mathcal{C}_2 \end{pmatrix}, \begin{pmatrix} \mathcal{A}_2' - \mathcal{B}_2' \\ \mathcal{A}_2' + \mathcal{B}_2' \mathcal{D}_2' + \mathcal{C}_2' \end{pmatrix} \right).
\]

\[
\begin{align*}
\Pi_{\Phi D_1} \oplus \Pi_{\Phi D_2} &= \left( \begin{pmatrix} (1 + \sigma_1) \frac{\pi}{\pi} - (1 - \sigma_1) \frac{\pi}{\pi} \\ (1 + \sigma_1) \frac{\pi}{\pi} + (1 - \sigma_1) \frac{\pi}{\pi} \end{pmatrix}, \begin{pmatrix} 2\rho_1 \frac{\pi}{\pi} \\ (1 + \alpha_1) \frac{\pi}{\pi} + (1 - \alpha_1) \frac{\pi}{\pi} \end{pmatrix} \right) \\
&\quad \left( \begin{pmatrix} 2\beta_1 \frac{\pi}{\pi} \\ (1 + \alpha_1) \frac{\pi}{\pi} + (1 - \alpha_1) \frac{\pi}{\pi} \end{pmatrix}, \begin{pmatrix} (1 + \sigma_1) \frac{\pi}{\pi} - (1 - \sigma_1) \frac{\pi}{\pi} \\ (1 + \sigma_1) \frac{\pi}{\pi} + (1 - \sigma_1) \frac{\pi}{\pi} \end{pmatrix} \right) 
\end{align*}
\]
\[
\Pi_{\Phi D_1} \oplus \Pi_{\Phi D_2} = \left( \begin{pmatrix} \mathcal{A}_1' - \mathcal{B}_1' \\ \mathcal{A}_1' + \mathcal{B}_1' \mathcal{D}_1' + \mathcal{C}_1' \end{pmatrix}, \begin{pmatrix} \mathcal{A}_2' - \mathcal{B}_2' \\ \mathcal{A}_2' + \mathcal{B}_2' \mathcal{D}_2' + \mathcal{C}_2' \end{pmatrix} \right).
\]

(16)
Hence, we prove
\[ \overline{\varphi_\pi (\varphi_\pi \Lambda) \varphi_\pi \Lambda} = \overline{\varphi_\pi \Lambda_1 \varphi_\pi \varphi_\pi \varphi_\pi \Lambda} \]  \hspace{1cm} (17)

\[ \overline{\varphi_\pi \Delta} = \left( \begin{array}{c}
\left( (1 + \sigma_1) \overline{\varphi_\pi} - (1 - \sigma_1) \overline{\varphi_\pi} \right) \\
(1 + \sigma_1) \overline{\varphi_\pi} + (1 - \sigma_1) \overline{\varphi_\pi} \\
(2 - \rho) \overline{\varphi_\pi} + \rho \overline{\varphi_\pi}
\end{array} \right),
\left( \begin{array}{c}
(1 + \sigma_1) \overline{\varphi_\pi} - (1 - \sigma_1) \overline{\varphi_\pi} \\
(1 + \sigma_1) \overline{\varphi_\pi} + (1 - \sigma_1) \overline{\varphi_\pi} \\
(2 - \beta) \overline{\varphi_\pi} + \beta \overline{\varphi_\pi}
\end{array} \right) \right)
\]  \hspace{1cm} (18)

where \( \sigma_1 = (1 + \rho \overline{\varphi_\pi}), \ \beta_1 = (1 - \sigma_1 \overline{\varphi_\pi}), \ C_1 = (\rho \overline{\varphi_\pi}), \ D_1 = (2 - \rho \overline{\varphi_\pi}), \ A_1 = ((1 + \alpha \overline{\varphi_\pi}), \ \beta_1 = ((1 - \alpha \overline{\varphi_\pi}), \ C'_1 = (\beta \overline{\varphi}), \ \text{and } D'_1 = (2 - \beta \overline{\varphi}). \) Therefore,

\[ \overline{\varphi_\pi \Delta \Delta_1 \overline{\varphi_\pi \varphi_\pi \varphi_\pi \varphi_\pi \Delta} = \left( \begin{array}{c}
(\overline{\varphi_\pi \Delta_1 \overline{\varphi_\pi}} \overline{\varphi_\pi} - \overline{\varphi_\pi \Delta_1 \overline{\varphi_\pi}} \overline{\varphi_\pi} \\
\overline{\varphi_\pi \Delta_1 \overline{\varphi_\pi}} \overline{\varphi_\pi} + \overline{\varphi_\pi \Delta_1 \overline{\varphi_\pi}} \overline{\varphi_\pi} \\
(2 - \rho) \overline{\varphi_\pi} + \rho \overline{\varphi_\pi}
\end{array} \right),
\left( \begin{array}{c}
(1 + \sigma_1) \overline{\varphi_\pi} - (1 - \sigma_1) \overline{\varphi_\pi} \\
(1 + \sigma_1) \overline{\varphi_\pi} + (1 - \sigma_1) \overline{\varphi_\pi} \\
(2 - \beta) \overline{\varphi_\pi} + \beta \overline{\varphi_\pi}
\end{array} \right) \right) \]  \hspace{1cm} (19)

(viii) Take \( \overline{\varphi_\pi}, \overline{\varphi_\pi} > 0: \)
Hence, we prove $\tilde{\Xi}_{1,2}^i \Delta \otimes \tilde{\Xi}_{1,2} \Delta = (\tilde{\Xi}_{1} + \tilde{\Xi}_{2})$.

Theorem 2. Let $\Delta = (\langle \alpha_i, \rho_i \rangle, \langle \alpha_i, \beta_i \rangle)$ be two LDFNs with $i = 1, 2$; then,

(i) $\Delta^i \vee \rho^i \Delta^2 = (\Delta \wedge \rho \Delta)$

(ii) $\Delta^i \wedge \rho^i \Delta^2 = (\Delta \vee \rho \Delta)$

(iii) $\Delta^i \otimes \rho^i \Delta^2 = (\Delta \oplus \rho \Delta)$

(iv) $\Delta^i \vee \rho^i \Delta^2 = (\Delta \wedge \rho \Delta)$

(v) $(\Delta \wedge \rho \Delta)_{\rho^i} \Delta^2 = (\Delta \vee \rho \Delta)$

Proof. This is a trivial case. Hence, we omit the proof.

Theorem 3. Let $\Delta = (\langle \alpha_i, \rho_i \rangle, \langle \alpha_i, \beta_i \rangle)$ be two LDFNs with $i = 1, 2, 3, 4$; then,

(i) $\Delta^i \vee \rho^i \Delta^2 = (\Delta \wedge \rho \Delta)$

(ii) $\Delta^i \wedge \rho^i \Delta^2 = (\Delta \vee \rho \Delta)$

(iii) $\Delta^i \otimes \rho^i \Delta^2 = (\Delta \oplus \rho \Delta)$

(iv) $\Delta^i \vee \rho^i \Delta^2 = (\Delta \wedge \rho \Delta)$

(v) $(\Delta \wedge \rho \Delta)_{\rho^i} \Delta^2 = (\Delta \vee \rho \Delta)$

Proof. This is a trivial case. Hence, we omit the proof.

4. Linear Diophantine Fuzzy Einstein Aggregation (LDFEA) Operators

In this segment, we propose certain fresh Einstein operators for LDFNs, namely, LDFEA operator, LDFEWA operator, LDFEWG operator, and LDFEOWG operator with some of their properties.

4.1. LDFEA Operator

Definition 10. Let $\Delta = (\langle \alpha_i, \rho_i \rangle, \langle \alpha_i, \beta_i \rangle)$ be a collection of LDFNs and $\Sigma = (\overline{\Xi}_1, \overline{\Xi}_2, \ldots, \overline{\Xi}_n)$ be the weight vector (WV) with $\sum_\alpha^\Sigma \Xi = 1$. Then, $\Sigma$: LDFN (\Sigma) $\rightarrow$ LDFN (\Sigma) is called linear Diophantine fuzzy Einstein weighted average (LDFEA) operator and defined as

$$\text{LDFEWA}(\Delta_1, \Delta_2, \ldots, \Delta_n) = \sum_\alpha^\Sigma \Xi_\alpha \Delta_\alpha = \sum_\alpha^\Sigma \Xi_\alpha \Delta_\alpha \otimes \Sigma \Xi_\alpha \Delta_\alpha \Delta_\alpha \Delta_\alpha \Delta_\alpha$$

In LDFEA operator, we use $\Sigma$ as a WV and $\Delta_\alpha$ are the LDFNs, where $\alpha = 1, 2, \ldots, n$. LDFN (\Sigma) is the collection of all LDFNs.

Theorem 4. Let $\Delta = (\langle \alpha_i, \rho_i \rangle, \langle \alpha_i, \beta_i \rangle)$ be a collection of LDFNs and $\Sigma = (\overline{\Xi}_1, \overline{\Xi}_2, \ldots, \overline{\Xi}_n)$ be the WV with $\sum_\alpha^\Sigma \Xi = 1$. Then, the mapping $\Sigma$: LDFN (\Sigma) $\rightarrow$ LDFN (\Sigma) is called LDFEA operator and can be written as

$$\text{LDFEWA}(\Delta_1, \Delta_2) = \sum_\alpha^\Sigma \Xi_\alpha \Delta_\alpha \otimes \Sigma \Xi_\alpha \Delta_\alpha \Delta_\alpha \Delta_\alpha$$

Proof. By using the process of Mathematical induction, we prove this result.

For $\Lambda = 2$,
Then,
\[
\text{LDFEWA}(\Delta_1, \Delta_2) = \sum_{i=1}^{n_1} \Phi_{\Delta_i} \sum_{j=1}^{n_2} \Phi_{\Delta_j}
\]
\[
= \left( \frac{(1 + \sigma_1) \sigma_i - (1 - \sigma_1) \sigma_i}{(1 + \sigma_1 + (1 - \sigma_1) \sigma_i)} \right) \left( \frac{2(\rho_1) \rho_i}{(2 - \rho_1 + (\rho_1) \rho_i)} \right) + \left( \frac{(1 + \alpha_1) \alpha_i - (1 - \alpha_1) \alpha_i}{(1 + \alpha_1 + (1 - \alpha_1) \alpha_i)} \right) \left( \frac{2(\beta_1) \beta_i}{(2 - \beta_1 + (\beta_1) \beta_i)} \right)
\]
\[
\Phi_{\Delta_i} \left( \frac{(1 + \sigma_2) \sigma_i - (1 - \sigma_2) \sigma_i}{(1 + \sigma_2 + (1 - \sigma_2) \sigma_i)} \right) \left( \frac{2(\rho_2) \rho_i}{(2 - \rho_2 + (\rho_2) \rho_i)} \right) + \left( \frac{(1 + \alpha_2) \alpha_i - (1 - \alpha_2) \alpha_i}{(1 + \alpha_2 + (1 - \alpha_2) \alpha_i)} \right) \left( \frac{2(\beta_2) \beta_i}{(2 - \beta_2 + (\beta_2) \beta_i)} \right)
\]
\[
= \left( \frac{(1 + \sigma_1) \sigma_i - (1 - \sigma_1) \sigma_i}{(1 + \sigma_1 + (1 - \sigma_1) \sigma_i)} \right) \left( \frac{2(\rho_1) \rho_i}{(2 - \rho_1 + (\rho_1) \rho_i)} \right) + \left( \frac{(1 + \alpha_1) \alpha_i - (1 - \alpha_1) \alpha_i}{(1 + \alpha_1 + (1 - \alpha_1) \alpha_i)} \right) \left( \frac{2(\beta_1) \beta_i}{(2 - \beta_1 + (\beta_1) \beta_i)} \right)
\]
\[
\cdot \left( \frac{1 + (1 + \sigma_1) \sigma_i - (1 - \sigma_1) \sigma_i}{1 + (1 + \sigma_1 + (1 - \sigma_1) \sigma_i)} \cdot \left( \frac{2(\rho_1) \rho_i}{2 - \rho_1 + (\rho_1) \rho_i} \right) \cdot \left( 1 - 2(\rho_1) \rho_i + (2 - \rho_1 + (\rho_1) \rho_i) \right) \right)
\]
\[
= \left( \frac{(1 + \alpha_1) \alpha_i - (1 - \alpha_1) \alpha_i}{(1 + \alpha_1 + (1 - \alpha_1) \alpha_i)} \right) \left( \frac{2(\beta_1) \beta_i}{2 - \beta_1 + (\beta_1) \beta_i} \right) + \left( \frac{(1 + \alpha_2) \alpha_i - (1 - \alpha_2) \alpha_i}{(1 + \alpha_2 + (1 - \alpha_2) \alpha_i)} \right) \left( \frac{2(\beta_2) \beta_i}{2 - \beta_2 + (\beta_2) \beta_i} \right)
\]
\[
\cdot \left( \frac{1 + (1 + \alpha_1) \alpha_i - (1 - \alpha_1) \alpha_i}{1 + (1 + \alpha_1 + (1 - \alpha_1) \alpha_i)} \cdot \left( \frac{2(\beta_1) \beta_i}{2 - \beta_1 + (\beta_1) \beta_i} \right) \cdot \left( 1 - 2(\beta_1) \beta_i + (2 - \beta_1 + (\beta_1) \beta_i) \right) \right)
\]
\[
= \left( \frac{(1 + \sigma_1) \sigma_i - (1 - \sigma_1) \sigma_i}{(1 + \sigma_1 + (1 - \sigma_1) \sigma_i)} \right) \left( \frac{2(\rho_1) \rho_i}{(2 - \rho_1 + (\rho_1) \rho_i)} \right) + \left( \frac{(1 + \alpha_1) \alpha_i - (1 - \alpha_1) \alpha_i}{(1 + \alpha_1 + (1 - \alpha_1) \alpha_i)} \right) \left( \frac{2(\beta_1) \beta_i}{(2 - \beta_1 + (\beta_1) \beta_i)} \right)
\]
\[
\cdot \left( \frac{1 + (1 + \sigma_1) \sigma_i - (1 - \sigma_1) \sigma_i}{1 + (1 + \sigma_1 + (1 - \sigma_1) \sigma_i)} \cdot \left( \frac{2(\rho_1) \rho_i}{2 - \rho_1 + (\rho_1) \rho_i} \right) \cdot \left( 1 - 2(\rho_1) \rho_i + (2 - \rho_1 + (\rho_1) \rho_i) \right) \right)
\]
\[
\cdot \left( \frac{(1 + \alpha_1) \alpha_i - (1 - \alpha_1) \alpha_i}{(1 + \alpha_1 + (1 - \alpha_1) \alpha_i)} \right) \left( \frac{2(\beta_1) \beta_i}{2 - \beta_1 + (\beta_1) \beta_i} \right) + \left( \frac{(1 + \alpha_2) \alpha_i - (1 - \alpha_2) \alpha_i}{(1 + \alpha_2 + (1 - \alpha_2) \alpha_i)} \right) \left( \frac{2(\beta_2) \beta_i}{2 - \beta_2 + (\beta_2) \beta_i} \right)
\]
\[
\cdot \left( \frac{1 + (1 + \alpha_2) \alpha_i - (1 - \alpha_2) \alpha_i}{1 + (1 + \alpha_2 + (1 - \alpha_2) \alpha_i)} \cdot \left( \frac{2(\beta_2) \beta_i}{2 - \beta_2 + (\beta_2) \beta_i} \right) \cdot \left( 1 - 2(\beta_2) \beta_i + (2 - \beta_2 + (\beta_2) \beta_i) \right) \right)
\]
\[
\cdot \left( \frac{(1 + \alpha_2) \alpha_i - (1 - \alpha_2) \alpha_i}{(1 + \alpha_2 + (1 - \alpha_2) \alpha_i)} \right) \left( \frac{2(\beta_2) \beta_i}{(2 - \beta_2 + (\beta_2) \beta_i)} \right)
\]
\[
\cdot \left( \frac{1 + (1 + \alpha_2) \alpha_i - (1 - \alpha_2) \alpha_i}{1 + (1 + \alpha_2 + (1 - \alpha_2) \alpha_i)} \cdot \left( \frac{2(\beta_2) \beta_i}{2 - \beta_2 + (\beta_2) \beta_i} \right) \cdot \left( 1 - 2(\beta_2) \beta_i + (2 - \beta_2 + (\beta_2) \beta_i) \right) \right)
\]
\[
(24)
\]
We proved that \( \Lambda = 2 \). Presume that the result for \( \Lambda = k \) is correct, and we have obtained

LDFEWA(\( \Delta_1, \Delta_2, \ldots, \Delta_k \))
\[
= \left( \frac{\prod_{\Delta_i=1}^{k} (1 + \sigma_i) \sigma_i - \prod_{\Delta_i=1}^{k} (1 - \sigma_i) \sigma_i}{\prod_{\Delta_i=1}^{k} (1 + \sigma_i) \sigma_i + \prod_{\Delta_i=1}^{k} (1 - \sigma_i) \sigma_i} \right) \left( \frac{2 \prod_{\Delta_i=1}^{k} (\rho_i) \rho_i}{\prod_{\Delta_i=1}^{k} (2 - \rho_i) \rho_i + \prod_{\Delta_i=1}^{k} (\rho_i) \rho_i} \right)
\]
\[
\cdot \left( \frac{\prod_{\Delta_i=1}^{k} (1 + \alpha_i) \alpha_i - \prod_{\Delta_i=1}^{k} (1 - \alpha_i) \alpha_i}{\prod_{\Delta_i=1}^{k} (1 + \alpha_i) \alpha_i + \prod_{\Delta_i=1}^{k} (1 - \alpha_i) \alpha_i} \right) \left( \frac{2 \prod_{\Delta_i=1}^{k} (\beta_i) \beta_i}{\prod_{\Delta_i=1}^{k} (2 - \beta_i) \beta_i + \prod_{\Delta_i=1}^{k} (\beta_i) \beta_i} \right)
\]
\[
(25)
\]
Now, we are going to prove \( n = k + 1: \)

\[
\text{LDFEWA}\left(\Delta_1, \Delta_2, \ldots, \Delta_k, \Delta_{k+1}\right) = \text{LDFEWA}\left(\Delta_1, \Delta_2, \ldots, \Delta_k\right)^{\sigma_{k+1}} \Delta_{k+1}
\]

\[
= \left( \frac{\prod_{\lambda=1}^{k} (1 + \sigma_{\lambda})^{\tau_{\lambda}} - \prod_{\lambda=1}^{k} (1 - \sigma_{\lambda})^{\tau_{\lambda}}}{\prod_{\lambda=1}^{k} (1 + \sigma_{\lambda})^{\tau_{\lambda}} + \prod_{\lambda=1}^{k} (1 - \sigma_{\lambda})^{\tau_{\lambda}}} \right) \left( \frac{2 \prod_{\lambda=1}^{k} \rho_{\lambda}^{\tau_{\lambda}}}{\prod_{\lambda=1}^{k} (2 - \rho_{\lambda})^{\tau_{\lambda}} + \prod_{\lambda=1}^{k} (\rho_{\lambda})^{\tau_{\lambda}}} \right)
\]

\[
\cdot \left( \frac{\prod_{\lambda=1}^{k+1} (1 + \alpha_{\lambda})^{\tau_{\lambda}} - \prod_{\lambda=1}^{k+1} (1 - \alpha_{\lambda})^{\tau_{\lambda}}}{\prod_{\lambda=1}^{k+1} (1 + \alpha_{\lambda})^{\tau_{\lambda}} + \prod_{\lambda=1}^{k+1} (1 - \alpha_{\lambda})^{\tau_{\lambda}}} \right) \left( \frac{2 \prod_{\lambda=1}^{k+1} \beta_{\lambda}^{\tau_{\lambda}}}{\prod_{\lambda=1}^{k+1} (2 - \beta_{\lambda})^{\tau_{\lambda}} + \prod_{\lambda=1}^{k+1} (\beta_{\lambda})^{\tau_{\lambda}}} \right)
\]

\[
\Phi\left( \frac{(1 + (\sigma_{k+1}))^{\tau_{k+1}} - (1 - (\sigma_{k+1}))^{\tau_{k+1}}}{(1 + (\sigma_{k+1}))^{\tau_{k+1}} + (1 - (\sigma_{k+1}))^{\tau_{k+1}}} \right) \left( \frac{2 (\rho_{k+1})^{\tau_{k+1}}}{(2 - (\rho_{k+1}))^{\tau_{k+1}} + ((\rho_{k+1}))^{\tau_{k+1}}} \right)
\]

\[
\cdot \left( \frac{(1 + (\alpha_{k+1}))^{\tau_{k+1}} - (1 - (\alpha_{k+1}))^{\tau_{k+1}}}{(1 + (\alpha_{k+1}))^{\tau_{k+1}} + (1 - (\alpha_{k+1}))^{\tau_{k+1}}} \right) \left( \frac{2 (\beta_{k+1})^{\tau_{k+1}}}{(2 - (\beta_{k+1}))^{\tau_{k+1}} + ((\beta_{k+1}))^{\tau_{k+1}}} \right)
\]

\[
\Phi\left( \frac{(1 + (\sigma_{k+1}))^{\tau_{k+1}} - (1 - (\sigma_{k+1}))^{\tau_{k+1}}}{(1 + (\sigma_{k+1}))^{\tau_{k+1}} + (1 - (\sigma_{k+1}))^{\tau_{k+1}}} \right) \left( \frac{2 (\rho_{k+1})^{\tau_{k+1}}}{(2 - (\rho_{k+1}))^{\tau_{k+1}} + ((\rho_{k+1}))^{\tau_{k+1}}} \right)
\]

\[
\cdot \left( \frac{(1 + (\alpha_{k+1}))^{\tau_{k+1}} - (1 - (\alpha_{k+1}))^{\tau_{k+1}}}{(1 + (\alpha_{k+1}))^{\tau_{k+1}} + (1 - (\alpha_{k+1}))^{\tau_{k+1}}} \right) \left( \frac{2 (\beta_{k+1})^{\tau_{k+1}}}{(2 - (\beta_{k+1}))^{\tau_{k+1}} + ((\beta_{k+1}))^{\tau_{k+1}}} \right)
\]

(26)

Result holds for \( \Lambda = k + 1. \) In this way, we have completed the proof.

\( \Box \)

**Theorem 5.** Let \( \Delta_{\Lambda} = (\langle \sigma_{\lambda}, \rho_{\lambda}\rangle, \langle \alpha_{\lambda}, \beta_{\lambda}\rangle) \) be a collection of LDFNs. Then,

1. **Idempotency:** if \( \Delta_{\Lambda} = (\langle \sigma_{\lambda}, \rho_{\lambda}\rangle, \langle \alpha_{\lambda}, \beta_{\lambda}\rangle) = \Lambda = (\langle t, f \rangle, \langle \alpha, \beta \rangle) \) (\( \Lambda = 1, 2, \ldots, n \)) for all \( \Lambda, \) then

\[
\text{LDFEWA}\left(\Delta_1, \Delta_2, \ldots, \Delta_n\right) = \Delta. \tag{27}
\]

2. **Monotonicity:** if \( \Delta_{\Lambda} = (\langle \sigma_{\lambda}, \rho_{\lambda}\rangle, \langle \alpha_{\lambda}, \beta_{\lambda}\rangle) \) and \( \Delta_{\Lambda} = (\langle \sigma_{\lambda}', \rho_{\lambda}'\rangle, \langle \alpha_{\lambda}', \beta_{\lambda}'\rangle) \) (\( \Lambda = 1, 2, \ldots, n \)) are two sets of LDFNs, \( t_{\Lambda} \geq t_{\Lambda}', f_{\Lambda} \leq f_{\Lambda}' \) and \( \alpha_{\Lambda} \geq \alpha_{\Lambda}', \beta_{\Lambda} \leq \beta_{\Lambda}' \) for all \( \Lambda, \) then

\[
\frac{\prod_{\lambda=1}^{3} (1 + \sigma_{\lambda})^{\tau_{\lambda}} - \prod_{\lambda=1}^{3} (1 - \sigma_{\lambda})^{\tau_{\lambda}}}{\prod_{\lambda=1}^{3} (1 + \sigma_{\lambda})^{\tau_{\lambda}} + \prod_{\lambda=1}^{3} (1 - \sigma_{\lambda})^{\tau_{\lambda}}} = 0.7177
\]

\[
\frac{\prod_{\lambda=1}^{3} (1 + \alpha_{\lambda})^{\tau_{\lambda}} - \prod_{\lambda=1}^{3} (1 - \alpha_{\lambda})^{\tau_{\lambda}}}{\prod_{\lambda=1}^{3} (1 + \alpha_{\lambda})^{\tau_{\lambda}} + \prod_{\lambda=1}^{3} (1 - \alpha_{\lambda})^{\tau_{\lambda}}} = 0.5269,
\]

**Proof.** We can easily prove this theorem by definition, so we omit the proof.

\( \Box \)

**Example 3.** Let \( \Delta_1 = (\langle 0.83, 0.38 \rangle, \langle 0.12, 0.69 \rangle), \) \( \Delta_2 = (\langle 0.51, 0.98 \rangle, \langle 0.36, 0.51 \rangle), \) and \( \Delta_3 = (\langle 0.79, 0.36 \rangle, \langle 0.26, 0.71 \rangle) \) be three LDFNs with \( W^\Delta = (0.3, 0.4, 0.3)^T. \) Then, we have

\[
\text{LDFEWA}\left(\Delta_1, \Delta_2, \ldots, \Delta_n\right) \geq \text{LDFEWA}\left(\Delta_1', \Delta_2', \ldots, \Delta_n'\right).
\]
LDFEWA\( (\Delta_1, \Delta_2, \Delta_3) = (\langle 0.7177, 0.5269 \rangle, \langle 0.2606, 0.5238 \rangle) \). (29)

4.2. LDFEOWA Operator

Definition 11. Let \( \Delta_\Lambda = (\langle \sigma_\Lambda, \rho_\Lambda \rangle, \langle \alpha_\Lambda, \beta_\Lambda \rangle) \) be an assembly of LDFNs and \( \overline{\Xi} = (\Xi_1, \Xi_2, \ldots, \Xi_n) \) be the WV with \( \sum_{\Lambda=1}^{n} \Xi_\Lambda = 1 \). Then, U: LDFN \( (\overline{\Xi})^n \rightarrow \text{LDFN}(\overline{\Xi}) \) is called linear Diophantine fuzzy Einstein ordered weighted average (LDFEOWA) operator and defined as

\[
\text{LDFEOWA}(\Delta_1, \Delta_2, \ldots, \Delta_n) = \sum_{\Lambda=1}^{n} \Xi_\Lambda \Delta_\Lambda = \overline{\Xi}
\]

\[
\begin{align*}
\varphi_{\text{LDFEOWA}(1)} \overline{\Xi}_1 \varphi_{\text{LDFEOWA}(2)} \overline{\Xi}_2 \cdots \varphi_{\text{LDFEOWA}(n)} \overline{\Xi}_n \varphi_{\text{LDFEOWA}(n)}
\end{align*}
\]

(30)

In LDFEOWA operator, we use \( \overline{\Xi} \) as a WV and \( \Delta_\Lambda \) are the LDFNs, where \( \Lambda = 1, 2, \ldots, n \). LDFN \( (\overline{\Xi}) \) is the collection of all LDFNs and \( E(1), E(2), \ldots, E(n) \) is a permutation of \( (1, 2, \ldots, n) \) such that \( \Delta_{E(i-1)} \geq \Delta_{E(i)} \).

Theorem 6. Let \( \Delta_\Lambda = (\langle \sigma_\Lambda, \rho_\Lambda \rangle, \langle \alpha_\Lambda, \beta_\Lambda \rangle) \) be an assembly of LDFNs and \( \overline{\Xi} = (\Xi_1, \Xi_2, \ldots, \Xi_n) \) be the WV with \( \sum_{\Lambda=1}^{n} \Xi_\Lambda = 1 \). Then, U: LDFN \( (\overline{\Xi})^n \rightarrow \text{LDFN}(\overline{\Xi}) \) is called LDFEOWA and can be written as

\[
\text{LDFEOWA}(\Delta_1, \Delta_2, \ldots, \Delta_n) = \left( \frac{2 \prod_{\Lambda=1}^{n} \rho_\Lambda^{\alpha_\Lambda}}{\prod_{\Lambda=1}^{n} (2 - \beta_\Lambda)^{\alpha_\Lambda} + \prod_{\Lambda=1}^{n} (\beta_\Lambda)^{\alpha_\Lambda}} \right)
\]

(31)

where \( E(1), E(2), \ldots, E(n) \) is a permutation of \( (1, 2, \ldots, n) \) such that \( \Delta_{E(i-1)} \geq \Delta_{E(i)} \).

Proof. The proof is identical to that of Theorem 4. \( \square \)

Theorem 7. Let \( \Delta_\Lambda = (\langle \sigma_\Lambda, \rho_\Lambda \rangle, \langle \alpha_\Lambda, \beta_\Lambda \rangle) \) be a collection of LDFNs. Then,

1. Idempotency: if \( \Delta_\Lambda = (\langle \sigma_\Lambda, \rho_\Lambda \rangle, \langle \alpha_\Lambda, \beta_\Lambda \rangle) = \Delta = (\langle t_\Lambda, f_\Lambda \rangle, \langle \alpha_\Lambda, \beta_\Lambda \rangle) (\Lambda = 1, 2, \ldots, n) \) for all \( \Lambda \), then

\[
\text{LDFEOWA}(\Delta_1, \Delta_2, \ldots, \Delta_n) = \Delta.
\]

(32)

Proof. We can easily proof this theorem by definition, so we omit the proof. \( \square \)

Example 5. Let \( \Delta_\Lambda = (\langle 0.83, 0.38 \rangle, \langle 0.12, 0.69 \rangle), \Delta_\Lambda = (\langle 0.51, 0.98 \rangle, \langle 0.36, 0.51 \rangle), \) and \( \Delta_\Lambda = (\langle 0.79, 0.36 \rangle, \langle 0.26, 0.71 \rangle) \) be three LDFNs with WV \( \overline{\Xi} = (0.3, 0.4, 0.3)^t \). Then, we have
\( \mathcal{F} - (\Delta_1) = -0.06, \mathcal{F} - (\Delta_2) = -0.31, \mathcal{F} - (\Delta_3) = -0.01, \) (34)

where \( \Delta_{E(1)} = \Delta_1, \Delta_{E(2)} = \Delta_2, \) and \( \Delta_{E(3)} = \Delta_3. \) Then, we have

\[
\frac{\prod_{\lambda=1}^3 \left(1 + \sigma_{E(\lambda)}\right) \bar{\varepsilon}_\lambda - \prod_{\lambda=1}^3 \left(1 - \sigma_{E(\lambda)}\right) \bar{\varepsilon}_\lambda}{\prod_{\lambda=1}^3 \left(1 + \sigma_{E(\lambda)}\right) \bar{\varepsilon}_\lambda + \prod_{\lambda=1}^3 \left(1 - \sigma_{E(\lambda)}\right) \bar{\varepsilon}_\lambda} = 0.7467, \\
\frac{2 \prod_{\lambda=1}^3 \bar{\sigma}_{E(\lambda)} \bar{\varepsilon}_\lambda}{\prod_{\lambda=1}^3 \left(1 - \rho_{E(\lambda)}\right) \bar{\varepsilon}_\lambda + \prod_{\lambda=1}^3 \left(\rho_{E(\lambda)}\right) \bar{\varepsilon}_\lambda} = 0.4688, \\
\frac{\prod_{\lambda=1}^3 \left(1 + \alpha_{E(\lambda)}\right) \bar{\varepsilon}_\lambda - \prod_{\lambda=1}^3 \left(1 - \alpha_{E(\lambda)}\right) \bar{\varepsilon}_\lambda}{\prod_{\lambda=1}^3 \left(1 + \alpha_{E(\lambda)}\right) \bar{\varepsilon}_\lambda + \prod_{\lambda=1}^3 \left(1 - \alpha_{E(\lambda)}\right) \bar{\varepsilon}_\lambda} = 0.2366, \\
\frac{2 \prod_{\lambda=1}^3 \bar{\beta}_{E(\lambda)} \bar{\varepsilon}_\lambda}{\prod_{\lambda=1}^3 \left(2 - \beta_{E(\lambda)}\right) \bar{\varepsilon}_\lambda + \prod_{\lambda=1}^3 \left(\beta_{E(\lambda)}\right) \bar{\varepsilon}_\lambda} = 0.5399,
\]

LDFEOWA \((\Delta_1, \Delta_2, \Delta_3) = \left(\prod_{\lambda=1}^3 \left(1 + \sigma_{E(\lambda)}\right) \bar{\varepsilon}_\lambda - \prod_{\lambda=1}^3 \left(1 - \sigma_{E(\lambda)}\right) \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^3 \left(1 + \sigma_{E(\lambda)}\right) \bar{\varepsilon}_\lambda + \prod_{\lambda=1}^3 \left(1 - \sigma_{E(\lambda)}\right) \bar{\varepsilon}_\lambda\right), \),

LDFEWA \((\Delta_1, \Delta_2, \Delta_3) = \langle (0.7467, 0.4688), (0.2366, 0.5399) \rangle.

4.3. LDFEWG Operator

Definition 12. Let \( \Delta = \langle \langle \sigma_\lambda, \rho_\lambda \rangle, \langle \alpha_\lambda, \beta_\lambda \rangle \rangle \) be a collection of LDFNs and \( \overline{\sigma} = (\overline{\sigma}_1, \overline{\sigma}_2, \ldots, \overline{\sigma}_n) \rangle \) be the WV with \( \sum_{\lambda=1}^n \overline{\sigma}_\lambda = 1. \) Then, the mapping \( \mathcal{U}: \text{LDFN}(\overline{\sigma})^n \rightarrow \text{LDNW}(\overline{\sigma}) \) is called linear Diophantine fuzzy Einstein weighted geometric (LDFEWG) operator and defined as

LDFEWG \((\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n) = \prod_{\lambda=1}^n \overline{\sigma}_\lambda \Delta_\lambda = \overline{\sigma}_1 \ast \overline{\sigma}_2 \ast \overline{\sigma}_3 \ast \ldots \ast \overline{\sigma}_n \ast \Delta_n.
\]

In LDFEWG operator, we use \( \overline{\sigma} \) as a WV and \( \Delta_\lambda \) are the LDFNs, where \( \lambda = 1, 2, \ldots, n. \) LDFN(\( \overline{\sigma} \)) is the collection of all LDFNs.

Theorem 8. Let \( \Delta = \langle \langle \sigma_\lambda, \rho_\lambda \rangle, \langle \alpha_\lambda, \beta_\lambda \rangle \rangle \) be the assemblage of LDFNs and \( \overline{\sigma} = (\overline{\sigma}_1, \overline{\sigma}_2, \ldots, \overline{\sigma}_n) \rangle \) be the WV with \( \sum_{\lambda=1}^n \overline{\sigma}_\lambda = 1. \) Then, \( \mathcal{U}: \text{LDFN}(\overline{\sigma})^n \rightarrow \text{LDNW}(\overline{\sigma}) \) is called LDFEWG operator and can be written as

LDFEWG \((\Delta_1, \Delta_2, \ldots, \Delta_n) = \left(\prod_{\lambda=1}^n \sigma_\lambda \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(1 + \rho_\lambda\right) \bar{\varepsilon}_\lambda - \prod_{\lambda=1}^n \left(1 - \rho_\lambda\right) \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(1 + \alpha_\lambda\right) \bar{\varepsilon}_\lambda + \prod_{\lambda=1}^n \left(1 - \alpha_\lambda\right) \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(2 - \beta_\lambda\right) \bar{\varepsilon}_\lambda + \prod_{\lambda=1}^n \left(\beta_\lambda\right) \bar{\varepsilon}_\lambda\right), \)

LDFEWG \((\Delta_1, \Delta_2, \ldots, \Delta_n) = \left(\prod_{\lambda=1}^n \alpha_\lambda \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(1 + \beta_\lambda\right) \bar{\varepsilon}_\lambda - \prod_{\lambda=1}^n \left(1 - \beta_\lambda\right) \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(2 - \alpha_\lambda\right) \bar{\varepsilon}_\lambda + \prod_{\lambda=1}^n \left(\alpha_\lambda\right) \bar{\varepsilon}_\lambda\right), \)

LDFEWG \((\Delta_1, \Delta_2, \ldots, \Delta_n) = \left(\prod_{\lambda=1}^n \sigma_\lambda \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(1 + \rho_\lambda\right) \bar{\varepsilon}_\lambda - \prod_{\lambda=1}^n \left(1 - \rho_\lambda\right) \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(1 + \alpha_\lambda\right) \bar{\varepsilon}_\lambda + \prod_{\lambda=1}^n \left(1 - \alpha_\lambda\right) \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(2 - \beta_\lambda\right) \bar{\varepsilon}_\lambda + \prod_{\lambda=1}^n \left(\beta_\lambda\right) \bar{\varepsilon}_\lambda\right), \)

LDFEWG \((\Delta_1, \Delta_2, \ldots, \Delta_n) = \left(\prod_{\lambda=1}^n \alpha_\lambda \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(1 + \beta_\lambda\right) \bar{\varepsilon}_\lambda - \prod_{\lambda=1}^n \left(1 - \beta_\lambda\right) \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(2 - \alpha_\lambda\right) \bar{\varepsilon}_\lambda + \prod_{\lambda=1}^n \left(\alpha_\lambda\right) \bar{\varepsilon}_\lambda\right), \)

LDFEWG \((\Delta_1, \Delta_2, \ldots, \Delta_n) = \left(\prod_{\lambda=1}^n \sigma_\lambda \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(1 + \rho_\lambda\right) \bar{\varepsilon}_\lambda - \prod_{\lambda=1}^n \left(1 - \rho_\lambda\right) \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(1 + \alpha_\lambda\right) \bar{\varepsilon}_\lambda + \prod_{\lambda=1}^n \left(1 - \alpha_\lambda\right) \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(2 - \beta_\lambda\right) \bar{\varepsilon}_\lambda + \prod_{\lambda=1}^n \left(\beta_\lambda\right) \bar{\varepsilon}_\lambda\right), \)

LDFEWG \((\Delta_1, \Delta_2, \ldots, \Delta_n) = \left(\prod_{\lambda=1}^n \alpha_\lambda \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(1 + \beta_\lambda\right) \bar{\varepsilon}_\lambda - \prod_{\lambda=1}^n \left(1 - \beta_\lambda\right) \bar{\varepsilon}_\lambda, \prod_{\lambda=1}^n \left(2 - \alpha_\lambda\right) \bar{\varepsilon}_\lambda + \prod_{\lambda=1}^n \left(\alpha_\lambda\right) \bar{\varepsilon}_\lambda\right), \)
Proof. By using the process of Mathematical induction, we prove this result.

For $\Lambda = 2$,

$$\text{LDFEWG}(\Delta_1, \Delta_2) = \Xi_{1, \gamma} \Delta_1 \otimes \Xi_{2, \gamma} \Delta_2.$$  \hspace{1cm} (38)

Then,

$$\text{LDFEOWA}(\Delta_1, \Delta_2) = \Xi_{1, \gamma} \Delta_1 \otimes \Xi_{2, \gamma} \Delta_2$$

$$= \Xi_{1, \gamma} \Delta_1 \otimes \Xi_{2, \gamma} \Delta_2 \quad \ast \quad \ast \quad \ast \quad \ast$$

As we know that both $\Xi_{1, \gamma} \Delta_1$ and $\Xi_{2, \gamma} \Delta_2$ are LDFNs and also $\Xi_{1, \gamma} \Delta_1 \otimes \Xi_{2, \gamma} \Delta_2$ is LDFN, then

$$\Xi_{1, \gamma} \Delta_1 = \left\langle \frac{2(\sigma_1)^{\Xi_1} (1 + \rho_1)^{\Xi_1} - (1 - \rho_1)^{\Xi_1}}{(2 - \sigma_1)^{\Xi_1} + (\sigma_1)^{\Xi_1}} \right\rangle \left\langle \frac{2(\sigma_1)^{\Xi_1} (1 + \rho_1)^{\Xi_1} - (1 - \rho_1)^{\Xi_1}}{(2 - \sigma_1)^{\Xi_1} + (\sigma_1)^{\Xi_1}} \right\rangle,$$

$$\Xi_{2, \gamma} \Delta_2 = \left\langle \frac{2(\sigma_2)^{\Xi_2} (1 + \rho_2)^{\Xi_2} - (1 - \rho_2)^{\Xi_2}}{(2 - \sigma_2)^{\Xi_2} + (\sigma_2)^{\Xi_2}} \right\rangle \left\langle \frac{2(\sigma_2)^{\Xi_2} (1 + \rho_2)^{\Xi_2} - (1 - \rho_2)^{\Xi_2}}{(2 - \sigma_2)^{\Xi_2} + (\sigma_2)^{\Xi_2}} \right\rangle.$$  \hspace{1cm} (39)

$$\text{LDFEW}(\Delta_1, \Delta_2) = \Xi_{1, \gamma} \Delta_1 \otimes \Xi_{2, \gamma} \Delta_2.$$  \hspace{1cm} (40)
We proved that $\Lambda = 2$.

Presume that the result for $\Lambda = k$ is correct, and we have obtained

\[
\text{LDFEWG}(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) = \left( \begin{array}{c}
2 \prod_{\alpha=1}^{k} \alpha_{\Lambda}^{-1} \prod_{\alpha=1}^{k} \alpha_{1}^{-1} (1 + \rho_{\Lambda})^{-1} - \prod_{\alpha=1}^{k} (1 - \rho_{\Lambda})^{-1} \\
(1 - \rho_{\Lambda})^{-1} + \prod_{\alpha=1}^{k} (1 - \rho_{\Lambda})^{-1}
\end{array} \right)
\]

Now, we are going to prove $n = k + 1$:

\[
\text{LDFEWG}(\Lambda_1, \Lambda_2, \ldots, \Lambda_{k+1}) = \text{LDFEWA}(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) \oplus \mathbb{Z}_{k+1} \ast \Delta_{k+1}
\]

Result holds for $\Lambda = k + 1$. In this way, we complete proof. \(\Box\)

**Theorem 9.** Let $\Lambda_{\Lambda} = \{(\{\sigma_{\Lambda}, \rho_{\Lambda}\}, \langle\alpha_{\Lambda}, \beta_{\Lambda}\rangle): \Lambda = 1, 2, 3, \ldots, n\}$ be a collection of LDFNs. Then,

1. **Idempotency:** if $\Lambda_{\Lambda} = \{(\{\sigma_{\Lambda}, \rho_{\Lambda}\}, \langle\alpha_{\Lambda}, \beta_{\Lambda}\rangle) = \Lambda \in \{(t, f), \langle\alpha_{\Lambda}, \beta_{\Lambda}\rangle) (\Lambda = 1, 2, \ldots, n\}$ for all $\Lambda$, then

\[
\text{LDFEWG}(\Lambda_1, \Lambda_2, \ldots, \Lambda_n) = \Lambda:
\]

2. **Monotonicity:** if $\Lambda_{\Lambda} = \{(\{\sigma_{\Lambda}, \rho_{\Lambda}\}, \langle\alpha_{\Lambda}, \beta_{\Lambda}\rangle) and

\Lambda_{\Lambda} = \{(t_{\Lambda}, f_{\Lambda}), \langle\alpha_{\Lambda}, \beta_{\Lambda}\rangle) (\Lambda = 1, 2, \ldots, n\}$ are two sets of LDFNs, $t_{\Lambda} \geq t_{\Lambda}', f_{\Lambda} \leq f_{\Lambda}'$ and $\alpha_{\Lambda} \geq \alpha_{\Lambda}', \beta_{\Lambda} \leq \beta_{\Lambda}'$ for all $\Lambda$, then

\[
\text{LDFEWG}(\Lambda_1, \Lambda_2, \ldots, \Lambda_n) \geq \text{LDFEWG}(\Lambda_1', \Lambda_2', \ldots, \Lambda_n').
\]

**Proof.** We can easily prove this theorem by definition, so we omit the proof. \(\Box\)
Example 6. Let $\Delta_1 = (\langle 0.67, 0.81 \rangle, \langle 0.52, 0.32 \rangle)$, $\Delta_2 = (\langle 0.76, 0.17 \rangle, \langle 0.71, 0.11 \rangle)$, and $\Delta_3 = (\langle 0.56, 0.89 \rangle, \langle 0.18, 0.37 \rangle)$ be three LDFNs with WV $\Xi = (0.2, 0.5, 0.3)^T$. Then, we have

\[
2\prod_{\lambda=1}^{3} \sigma_{\lambda}^{2} - 2\prod_{\lambda=1}^{3} \sigma_{\lambda}^{2} + \prod_{\lambda=1}^{3} \alpha_{\lambda}^{2} = 0.577962 \prod_{\lambda=1}^{3} (1 + \rho_{\lambda}) - 2\prod_{\lambda=1}^{3} (1 - \rho_{\lambda}) = 0.627824,
\]

\[
2\prod_{\lambda=1}^{3} \alpha_{\lambda}^{2} - 2\prod_{\lambda=1}^{3} \alpha_{\lambda}^{2} + \prod_{\lambda=1}^{3} \alpha_{\lambda}^{2} = 0.397039 \prod_{\lambda=1}^{3} (1 + \beta_{\lambda}) - 2\prod_{\lambda=1}^{3} (1 - \beta_{\lambda}) = 0.233681,
\]

\[
\text{LDFEOWG}(\Delta_1, \Delta_2, \Delta_3) = (0.577962, 0.627824), (0.397039, 0.233681).
\]

4.4. LDFEOWG Operator

Definition 13. Let $\Delta_\Lambda = (\langle \alpha_{\Lambda}, \rho_{\Lambda} \rangle, \langle \beta_{\Lambda}, \alpha_{\Lambda} \rangle)$ be a collection of LDFNs and $\Xi = (\Xi_1, \Xi_2, \ldots, \Xi_n)^T$ be the WV with $\sum_{\lambda=1}^{n} \Xi_\lambda = 1$. Then, $U: \text{LDFN}(\Xi)^n \rightarrow \text{LDFN}(\Xi)$ is called linear Diophantine fuzzy Einstein ordered weighted geometric (LDFEOWG) operator and defined as

\[
\text{LDFEOWG}(\Xi_1, \Xi_2, \ldots, \Xi_n) = \prod_{\lambda=1}^{n} \Xi_\lambda \Delta_\Lambda = \Xi_1 \otimes \Xi_2 \otimes \Xi_3 \otimes \Xi_n.
\]

In the LDFEOWG operator, we use $\Xi$ as a WV and $\Delta_\Lambda$ are the LDFNs, where $\Lambda = 1, 2, \ldots, n$. LDFN($\Xi$) is the collection of all LDFNs and $E(1), E(2), \ldots, E(n)$ is a permutation of $(1, 2, \ldots, n)$ such that $\Delta_{E(1)} \geq \Delta_{E(2)}$. Theorem 10. Let $\Delta_\Lambda = (\langle \alpha_{\Lambda}, \rho_{\Lambda} \rangle, \langle \beta_{\Lambda}, \alpha_{\Lambda} \rangle)$ be a collection of LDFNs and $\Xi = (\Xi_1, \Xi_2, \ldots, \Xi_n)^T$ be the WV with $\sum_{\lambda=1}^{n} \Xi_\lambda = 1$. Then, $U: \text{LDFN}(\Xi)^n \rightarrow \text{LDFN}(\Xi)$ is called the LDFEOWG operator and can be written as

\[
\text{LDFEOWG}(\Delta_1, \Delta_2, \ldots, \Delta_n) = \left\langle \prod_{\lambda=1}^{n} \alpha_{\lambda}^{2} - \prod_{\lambda=1}^{n} \alpha_{\lambda}^{2} + \prod_{\lambda=1}^{n} \alpha_{\lambda}^{2}, \prod_{\lambda=1}^{n} (1 + \rho_{\lambda}) - \prod_{\lambda=1}^{n} (1 - \rho_{\lambda}) \right\rangle,
\]

where $E(1), E(2), \ldots, E(n)$ is a permutation of $(1, 2, \ldots, n)$ such that $\Delta_{E(1)} \geq \Delta_{E(2)}$. The proof is identical to that of Theorem 10. 

\[
\text{LDFEOWG}(\Delta_1, \Delta_2, \ldots, \Delta_n) = \left( \prod_{\lambda=1}^{n} \alpha_{\lambda}^{2} - \prod_{\lambda=1}^{n} \alpha_{\lambda}^{2} + \prod_{\lambda=1}^{n} \alpha_{\lambda}^{2}, \prod_{\lambda=1}^{n} (1 + \rho_{\lambda}) - \prod_{\lambda=1}^{n} (1 - \rho_{\lambda}) \right)
\]

\[
\text{Theorem 11. Let } \Delta_\Lambda = (\langle \alpha_{\Lambda}, \rho_{\Lambda} \rangle, \langle \beta_{\Lambda}, \alpha_{\Lambda} \rangle) \text{ be a collection of LDFNs. Then,}
\]

(1) Idempotency: if $\Delta_\Lambda = (\langle \alpha_{\Lambda}, \rho_{\Lambda} \rangle, \langle \beta_{\Lambda}, \alpha_{\Lambda} \rangle) = \Delta = (\langle t, f \rangle, \langle \alpha, \beta \rangle)(\Lambda = 1, 2, \ldots, n)$ for all $\Lambda$, then
5. MCDM to Hospital-Based PAC-CVD

In this portion, the first case study regarding the selection of hospitals in the PAC-CVD program is presented. We set some attributes related to the selected list of alternatives for the appropriate decision. Then, we establish a novel algorithm based on LDFEWA and LDFEWG operators to deal with this MCDM problem. A numerical example is shown to verify the consistency, versatility, and superiority of our suggested aggregation operators and algorithms.

5.1. Case Study for Hospital-Based PAC-CVD. A country’s national health administration (NHA) is deciding to create a fully developed postacute care (PAC) model system to improve the quality of recovery of patients with cerebrovascular diseases (CVDs). Cerebrovascular accident (CVA) is usually known as stroke. Stroke results in weakness of limbs, one-sided weakness, and whole body weakness means paralysis. It may cause paralysis of tongue and inability to speak even due to blockage of vessels supplying specific area of brain (thrombosis of vessels) in which case it is called infarctive stroke, and in other case, it may be due to rupture of vessels supplying specific area of the brain in which case it is called haemorrhagic stroke, weakness, or paralysis and depends on type of vessel damaged that causes brain tissue damage, and ultimately, the part of the body that is controlled by that area of the brain is paralyzed. There are some types such as stenosis which is due to narrowing of blood vessels supplying the brain and results in decreased blood supply to the brain causing different effects on the body depending on vessel, and it may include carotid stenosis and vertebral stenosis. Also, vascular malformations may be congenital (present at time of birth) and acquired (that are caused by different diseases at any stage of life). The management flowchart of PAC-CVD program is given in Figure 2.

Improving the efficiency of hospital-based PAC in emergency stroke treatment, assessment, and selection of proper medical care facilities is crucial in the program. However, the selection of pilot hospitals is an extremely confused and MCDM problem, due to the complexity of NHA in the country. To illustrate the workability and applicability of the suggested operators in the medical and health care domains, an interpretive problem concerning the assortment of pilot hospitals in the PAC-CVD program is implemented.

Participants considered for PAC-CVD must include a network of health care organizations at different levels, directed by a primary institution. A pilot hospital must
accommodate restoration services (including physical therapy, occupational therapy, and speech therapy) and develop a PAC team of doctors of PAC-CVD qualifications including neurologists, neurosurgeons, internists, and family physicians. The risk factors for this problem are given in Figure 3. It is important to take serious decision about this problem. The death rate chart per 100,000 population due to CVD is given in Figure 4.

The criteria needed for pilot hospital recruitment are set out in Table 3.

A clear and sufficient budget was given for the PAC-CVD program. Five function-related groups (FRGs) were designed for reasonably practicable patient care pathways. Groups are shown in Table 4.

5.2. Proposed Methodology. We discuss MCDM concern with the use of proposed operators. Take into account the set of alternatives \( \mathcal{A} = \{A_1, A_2, \ldots, A_m\} \) with \( m \) elements, and \( \mathcal{C} = \{C_1, C_2, \ldots, C_n\} \) is the set of criteria with \( n \) elements. Decision makers have their specific opinion matrix \( D = (\mathcal{P}_{ij})_{m \times n} \) where \( \mathcal{P}_{ij} \) is given for the alternatives \( A_j \in \mathcal{A} \) with respect to the criteria \( C_j \in \mathcal{C} \) by the decision maker in the form of LDFNs. You can also address the MCDM concern in terms of a LDFNs decision matrix provided by \( D = (\mathcal{P}_{ij})_{m \times n} = (\langle \sigma_{ij}, \rho_{ij} \rangle, \langle \alpha_{ij}, \beta_{ij} \rangle)_{m \times n} \).

The proposed operators will apply to the MCDM which will include the following points.

Step 1: obtain the judgement matrix \( D = (\mathcal{P}_{ij})_{m \times n} \) in the format of LDFNs from the decision maker (Table 5).

Step 2: the parameters in the judgement matrix are characterized by two categories, such as the cost form criteria \( (\tau_c) \) and the benefit form criteria \( (\tau_b) \). If all parameters are from the same form, normalization is not needed, so there are various forms of parameters in the MCDM; in this case, the \( D \) matrix has been converted into a normalization matrix using the normalization formula \( Y(k_{ij}) = (\langle \sigma_{ij}, \rho_{ij} \rangle, \langle \alpha_{ij}, \beta_{ij} \rangle) : \)

\[
k_{ij} = \begin{cases} \mathcal{P}^e_{ij}; & j \in \tau_c, \\
\mathcal{P}^\beta_{ij}; & j \in \tau_b,
\end{cases}
\]

where \( \mathcal{P}^e_{ij} \) shows the compliment of \( \mathcal{P}_{ij} \).

Step 3: using one of the mentioned operators to evaluate combined evaluations of the options,

\[
k_i = LDFEWA(k_{1i}, k_{2i}, \ldots, k_{mi}), \quad (i = 1, 2, 3, \ldots, m),
\]

or \( k_i = LDFEWG(k_{1i}, k_{2i}, \ldots, k_{mi}), \quad (i = 1, 2, 3, \ldots, m). \)

Step 4: determine the score value through all combined alternatives evaluations.

Step 5: alternative options are sorted first by score function and the best alternative can be selected.

The pictorial view of the algorithm is shown in Figure 5.

5.3. Numerical Example. All participating hospitals had the right to financial rewards, surcharges for outpatient diagnosis and treatment, and requests for reimbursement of other types of medical expenses. The suggested approach was used to assist NHA in selecting the best set of hospitals. Here, we use LDFN, \( \mathcal{A}_\Lambda = \{\langle \rho_\Lambda, \sigma_\Lambda \rangle, \langle \beta_\Lambda, \alpha_\Lambda \rangle\} : \Lambda = 1, 2, 3, \ldots, n \} \), in this approach if we consider the control parameters as \( \alpha = \) highly effect on patients and \( \beta = \) not highly effect on patients.

5.3.1. By Using LDFEWA

Step 1: consider a set of alternatives \( \mathcal{A} = \{A_1, A_2, A_3, A_4, A_5\} \) given in Table 4, and \( \mathcal{C} = \{C_1, C_2, C_3, C_4\} \),
Figure 3: Risk factors.

Figure 4: Death rate chart per 100,000 population.

Table 3: Requirements for selecting the appropriate option in the PAC-CVD.

| Criterions           | Definition                                                                                                                                 |
|----------------------|---------------------------------------------------------------------------------------------------------------------------------------------|
| Ω₁                   | “Health care systems combine” Organizing and managing a team of different health care institutions to ensure consistent service delivery and the transformation from acute to long-term care |
| Ω₂                   | “Manpower allocation” About the areas of expertise, number, and job status of all health care specialists; the proportion of doctor to patient in each medical unit |
| Ω₃                   | “Comprehensiveness of equipment” Completeness of the facilities relating to the PAC department with respect to the number of hospital beds |
| Ω₄                   | “Health quality assurance” Quality control of health care received by each member organization and the ability to handle complications |
| Ω₅                   | “Reliability” This is the ability of the system to function on the basis of the situations |
| Ω₆                   | “Operational cost” This criterion examines operational funds provided to function-related groups (FRGs) |
The finite set of criterions, given in Table 3, is the finite set of criterions. The decision maker provides a matrix of their own opinion
\[ D = (P_{ij})_{5 \times 6} \]
given in Table 6.

Step 2: we get the normalized decision matrix given in Table 7 by taking complement of cost-type criterions given in Table 3, where \( \Sigma_6 \) is operational cost is the cost-type criterion.

Step 3: evaluate combined evaluations of alternatives utilizing the LDFEWA operator:

\[ k_i = \text{LDFEWA}(k_{i1}, k_{i2}, \ldots, k_{in}) \]

For \( i = 1 \),

\[ k_1 = \text{LDFEWA}(k_{11}, k_{12}, \ldots, k_{1n}) \]

\[ = (0.6427, 0.479081, 0.287157, 0.491374). \]

(56)

For \( i = 2 \),

\[ k_2 = \text{LDFEWA}(k_{21}, k_{22}, \ldots, k_{2n}) \]

\[ = (0.6104, 0.276825, 0.285648, 0.464307). \]

(57)

For \( i = 3 \),

\[ k_3 = \text{LDFEWA}(k_{31}, k_{32}, \ldots, k_{3n}) \]

\[ = (0.397313, 0.576326, 0.250777, 0.387601). \]

(58)

For \( i = 4 \),

\[ k_4 = \text{LDFEWA}(k_{41}, k_{42}, \ldots, k_{4n}) \]

\[ = (0.499624, 0.510399, 0.415691, 0.330775). \]

(59)

For \( i = 5 \),

\[ k_5 = \text{LDFEWA}(k_{51}, k_{52}, \ldots, k_{5n}) \]

\[ = (0.6427, 0.479081, 0.287157, 0.491374). \]

(60)

Table 4: Function-related groups.

| Group name | Description                  |
|------------|------------------------------|
| \( \Delta_1 \) | FRG1 “PAC and rehabilitative services of high intensity” |
| \( \Delta_2 \) | FRG2 “CAP and medium-intensity rehabilitative services” |
| \( \Delta_3 \) | FRG3 “Organizational care and regaining consciousness” |
| \( \Delta_4 \) | FRG4 “Oncology medical services and rehabilitation mode” |
| \( \Delta_5 \) | FRG5 “Home-based health care” |

Table 5: The judgement matrix in terms of LDFNs.

| \( \Sigma_1 \) | \( \Sigma_2 \) | \( \Sigma_n \) |
|----------------|----------------|----------------|
| \( \Delta_1 \) | \( \langle \sigma_{11}, \rho_{11}, \alpha_{11}, \beta_{11} \rangle \) | \( \langle \sigma_{12}, \rho_{12}, \alpha_{12}, \beta_{12} \rangle \) | \( \ldots \) | \( \langle \sigma_{1n}, \rho_{1n}, \alpha_{1n}, \beta_{1n} \rangle \) |
| \( \Delta_2 \) | \( \langle \sigma_{21}, \rho_{21}, \alpha_{21}, \beta_{21} \rangle \) | \( \langle \sigma_{22}, \rho_{22}, \alpha_{22}, \beta_{22} \rangle \) | \( \ldots \) | \( \langle \sigma_{2n}, \rho_{2n}, \alpha_{2n}, \beta_{2n} \rangle \) |
| \( \Delta_n \) | \( \langle \sigma_{m1}, \rho_{m1}, \alpha_{m1}, \beta_{m1} \rangle \) | \( \langle \sigma_{m2}, \rho_{m2}, \alpha_{m2}, \beta_{m2} \rangle \) | \( \ldots \) | \( \langle \sigma_{mn}, \rho_{mn}, \alpha_{mn}, \beta_{mn} \rangle \) |

Figure 5: Pictorial view of proposed algorithm.

\[ \Sigma_4, \Sigma_5, \Sigma_6 \], given in Table 3, is the finite set of criterions. The decision maker provides a matrix of their own opinion \( D = (P_{ij})_{5 \times 6} \) given in Table 6.
| $\Delta_1$  | $\Delta_2$  | $\Delta_3$  | $\Delta_4$  | $\Delta_5$  |
|------------|------------|------------|------------|------------|
| (0.81, 0.47, 0.52, 0.39) | (0.56, 0.27, 0.37, 0.41) | (0.45, 0.37, 0.56, 0.37) | (0.81, 0.31, 0.81, 0.13) | (0.45, 0.37, 0.56, 0.37) |
| (0.71, 0.56, 0.15, 0.79) | (0.81, 0.16, 0.31, 0.71) | (0.71, 0.52, 0.56, 0.21) | (0.71, 0.22, 0.67, 0.29) | (0.89, 0.32, 0.56, 0.33) |
| (0.36, 0.73, 0.31, 0.63) | (0.27, 0.67, 0.11, 0.32) | (0.27, 0.81, 0.45, 0.35) | (0.69, 0.23, 0.67, 0.13) | (0.93, 0.63, 0.53, 0.21) |
| (0.79, 0.36, 0.16, 0.56) | (0.37, 0.56, 0.36, 0.67) | (0.56, 0.79, 0.23, 0.51) | (0.93, 0.63, 0.53, 0.21) | (0.83, 0.21, 0.76, 0.13) |

Table 6: LDF decision matrix provided by the decision maker.
Table 7: Normalized LDF decision matrix.

|   | \(\varnothing_1\) | \(\varnothing_2\) | \(\varnothing_3\) | \(\varnothing_4\) | \(\varnothing_5\) | \(\varnothing_6\) |
|---|------------------|------------------|------------------|------------------|------------------|------------------|
| \(\Delta_1\) | \((0.81, 0.47), (0.52, 0.39)\) | \((0.47, 0.81), (0.32, 0.51)\) | \((0.56, 0.71), (0.79, 0.15)\) | \((0.36, 0.73), (0.31, 0.63)\) | \((0.79, 0.36), (0.16, 0.56)\) | \((0.43, 0.36), (0.31, 0.57)\) |
| \(\Delta_2\) | \((0.56, 0.27), (0.37, 0.41)\) | \((0.56, 0.31), (0.25, 0.61)\) | \((0.16, 0.81), (0.71, 0.31)\) | \((0.27, 0.67), (0.11, 0.32)\) | \((0.37, 0.56), (0.36, 0.67)\) | \((0.71, 0.37), (0.21, 0.37)\) |
| \(\Delta_3\) | \((0.32, 0.56), (0.11, 0.81)\) | \((0.23, 0.73), (0.31, 0.75)\) | \((0.79, 0.33), (0.11, 0.21)\) | \((0.19, 0.39), (0.32, 0.56)\) | \((0.71, 0.81), (0.26, 0.76)\) | \((0.63, 0.56), (0.32, 0.27)\) |
| \(\Delta_4\) | \((0.16, 0.79), (0.14, 0.76)\) | \((0.45, 0.37), (0.56, 0.37)\) | \((0.52, 0.71), (0.21, 0.56)\) | \((0.27, 0.81), (0.45, 0.35)\) | \((0.56, 0.79), (0.23, 0.51)\) | \((0.53, 0.71), (0.21, 0.37)\) |
| \(\Delta_5\) | \((0.81, 0.31), (0.81, 0.13)\) | \((0.71, 0.22), (0.67, 0.29)\) | \((0.32, 0.89), (0.33, 0.56)\) | \((0.69, 0.23), (0.67, 0.13)\) | \((0.93, 0.63), (0.53, 0.21)\) | \((0.21, 0.83), (0.13, 0.76)\) |
Table 8: LDF decision matrix provided by the decision maker.

| $Δ_1$   | $Δ_2$   | $Δ_3$   | $Δ_4$   | $Δ_5$   |
|---------|---------|---------|---------|---------|
| $Ω_1$   | $Ω_2$   | $Ω_3$   | $Ω_4$   | $Ω_5$   |
| (0.81, 0.47, 0.52, 0.39) | (0.47, 0.81, 0.32, 0.51) | (0.71, 0.56, 0.15, 0.79) | (0.36, 0.73, 0.31, 0.63) | (0.79, 0.36, 0.16, 0.56) |
| $Δ_2$   | $Δ_3$   | $Δ_4$   | $Δ_5$   |
| (0.56, 0.27, 0.37, 0.41) | (0.56, 0.31, 0.25, 0.61) | (0.81, 0.16, 0.31, 0.71) | (0.27, 0.67, 0.11, 0.32) |
| $Δ_3$   | $Δ_4$   | $Δ_5$   |
| (0.32, 0.56, 0.11, 0.81) | (0.23, 0.73, 0.31, 0.75) | (0.33, 0.79, 0.21, 0.11) |
| $Δ_4$   |
| (0.16, 0.79, 0.14, 0.76) | (0.45, 0.37, 0.56, 0.37) | (0.71, 0.52, 0.56, 0.21) |
| $Δ_5$   |
| (0.81, 0.31, 0.81, 0.13) | (0.71, 0.22, 0.67, 0.29) | (0.89, 0.32, 0.56, 0.33) | (0.69, 0.23, 0.67, 0.13) | (0.93, 0.63, 0.53, 0.21) | (0.83, 0.21, 0.76, 0.13) |
Table 9: Normalized LDF decision matrix.

|   | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ | $\Omega_6$ |
|---|---|---|---|---|---|---|
| $\Delta_1$ | $(0.81, 0.47, 0.52)$, $(0.52, 0.39)$ | $(0.47, 0.81, 0.32)$, $(0.32, 0.51)$ | $(0.56, 0.71, 0.79)$, $(0.79, 0.15)$ | $(0.36, 0.73, 0.31)$, $(0.31, 0.63)$ | $(0.79, 0.36, 0.16)$, $(0.16, 0.56)$ | $(0.43, 0.36, 0.31)$, $(0.31, 0.57)$ |
| $\Delta_2$ | $(0.56, 0.27, 0.37)$, $(0.37, 0.41)$ | $(0.56, 0.31, 0.25)$, $(0.25, 0.61)$ | $(0.16, 0.81, 0.71)$, $(0.71, 0.31)$ | $(0.27, 0.67, 0.11)$, $(0.11, 0.32)$ | $(0.37, 0.56, 0.36)$, $(0.36, 0.67)$ | $(0.71, 0.37, 0.21)$, $(0.21, 0.37)$ |
| $\Delta_3$ | $(0.32, 0.56, 0.11)$, $(0.11, 0.81)$ | $(0.23, 0.73, 0.31)$, $(0.31, 0.75)$ | $(0.79, 0.33, 0.11)$, $(0.11, 0.21)$ | $(0.19, 0.39, 0.32)$, $(0.32, 0.56)$ | $(0.71, 0.81, 0.26)$, $(0.26, 0.76)$ | $(0.63, 0.56, 0.32)$, $(0.32, 0.27)$ |
| $\Delta_4$ | $(0.16, 0.79, 0.14)$, $(0.14, 0.76)$ | $(0.45, 0.37, 0.56)$, $(0.56, 0.37)$ | $(0.52, 0.71, 0.21)$, $(0.21, 0.56)$ | $(0.27, 0.81, 0.45)$, $(0.45, 0.35)$ | $(0.56, 0.79, 0.23)$, $(0.23, 0.51)$ | $(0.53, 0.71, 0.21)$, $(0.21, 0.37)$ |
| $\Delta_5$ | $(0.81, 0.31, 0.81)$, $(0.81, 0.13)$ | $(0.71, 0.22, 0.67)$, $(0.67, 0.29)$ | $(0.32, 0.89, 0.33)$, $(0.33, 0.56)$ | $(0.69, 0.23, 0.67)$, $(0.67, 0.13)$ | $(0.93, 0.63, 0.53)$, $(0.53, 0.21)$ | $(0.21, 0.83, 0.13)$, $(0.13, 0.76)$ |
5.3.2. By Using LDFEWG

Step 1: consider a set of alternatives \( \Lambda = \{ \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5 \} \), given in Table 4, and \( \mathcal{O} = \{ \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5, \mathcal{O}_6 \} \), given in Table 3, is the finite set of criterions. The decision maker provides a matrix of their own opinion \( D = (D_{ij})_{5 \times 6} \) given in Table 8.

Step 2: we get the normalized decision matrix given in Table 9 by taking compliment of cost type criterions given in Table 3, where \( \mathcal{O}\_6 = \) operational cost is the cost type criterion.

Step 3: determine cumulative assessments of the alternatives by using the LDFEWG operator:

\[
k_i = \text{LDFEWG}(k_{i1}, k_{i2}, \ldots, k_{in}).
\]

For \( i = 1 \),

\[
k_1 = \text{LDFEWG}(k_{11}, k_{12}, \ldots, k_{1n})
= (0.513386, 0.606629, 0.22999, 0.606982).
\]

For \( i = 2 \),

\[
k_2 = \text{LDFEWG}(k_{21}, k_{22}, \ldots, k_{2n})
= (0.479685, 0.359972, 0.240935, 0.577022).
\]

For \( i = 3 \),

\[
k_3 = \text{LDFEWG}(k_{31}, k_{32}, \ldots, k_{3n})
= (0.303617, 0.700602, 0.212904, 0.591719).
\]

For \( i = 4 \),

\[
k_4 = \text{LDFEWG}(k_{41}, k_{42}, \ldots, k_{4n})
= (0.379724, 0.643451, 0.313625, 0.42978).
\]

For \( i = 5 \),

\[
k_5 = \text{LDFEWG}(k_{51}, k_{52}, \ldots, k_{5n})
= (0.644385, 0.413937, 0.473322, 0.313688).
\]

Step 4: the scoring values pertaining to any \( k_i \) are

\[
k_5 = \text{LDFEWG}(k_{51}, k_{52}, \ldots, k_{5n})
= (0.800931, 0.307604, 0.612316, 0.234647).
\]

(60)

\[
\mathbf{3}^{-}(k_i) = -0.020299, \mathbf{3}^{-}(k_2) = 0.077458, \mathbf{3}^{-}(k_3) = -0.157918, \mathbf{3}^{-}(k_4) = 0.0370705, \mathbf{3}^{-}(k_5) = 0.435498.
\]

(61)

\[
k_5 \succeq k_4 \succeq k_3 \succeq k_1.
\]

(62)
\[ \tilde{\mathcal{S}}(k_1) = -0.235117, \]
\[ \tilde{\mathcal{S}}(k_2) = -0.108187, \]
\[ \tilde{\mathcal{S}}(k_3) = -0.3879, \]
\[ \tilde{\mathcal{S}}(k_4) = -0.189941, \]
\[ \tilde{\mathcal{S}}(k_5) = 0.195041. \]

Step 5: rank the value obtained in Step 4 using the score function:
\[ k_5 \succeq k_2 \succeq k_4 \succeq k_1 \succeq k_3. \]

5.4. Comparison Analysis. In this section, we compare proposed AOs with some existing AOs. The uniqueness of our proposed operators is that they both yield the same result. We equate our results by solving the information data with some preexisting operators and arriving at the same optimal decision. This demonstrates the robustness and validity of our proposed models. Because of their reference parameterizations, the presented techniques on LDFNs are more effective and superior to some existing theories. The beauty of this structure is that it creates independence between membership and nonmembership grades and creates categorization criteria due to parameterizations. The comparison of presented aggregation operators with some existing operators is given in Table 10.

6. Conclusion

Multi-criteria decision-making (MCDM) is an important real decision problem, and its most basic and most important research direction is how to express these uncertain information. The IFSs, PFSs, and q-ROFSs are all a good way to deal with fuzzy information. However, LDFs are more general; their outstanding ability is to relax the strict constraints of IFS, PFS, and q–ROFS by considering reference/control parameters. MCDM is an important branch of operation research. The procedures extended for this assignment essentially depend on the nature of problem under judgement. Our daily-life circumstances are unpredictable, imprecise, and obscure. The existing structures were assembled below the hypothesis that decision makers (DMs) consider certain constraints while evaluating multiple alternatives and attributes. However, this kind of state becomes difficult for DMs so that they will assign MGs and NMGs under various restrictions. In order to relax these restrictions, LDFS is a new approach towards uncertainty and decision-making problems which incorporate pair of reference or control parameters against MGs and NMGs. We have utilized LDFSs for evaluating the integrity of DMS knowledge in the fundamental framework and to eliminate any deformation in the decision analysis. The influential privilege of computing the control parameters into the examination is to decrease the feasibility of errors that are created by the theoretical knowledge depending on fundamental evaluations of MGs and NMGs. Additionally, we have developed several aggregation operators named as LDFEWA operator, LDFEOWA operator, LDFEWG operator, and LDFEOWG operator. Many interesting characteristics of the proposed operators are studied and their illustration is well proved. A brief discussion regarding LDFS-based t-norm and t-conorm is expressed. A practical application of the proposed method for MCDM to the national health administration (NHA) for the creation of a fully developed postacute care (PAC) model network is given. Comparative studies with some existing approaches demonstrate the feasibility and reliability of the proposed operators. The superiority of the proposed work has been justified with the validity test. Thus, we conclude that the proposed approach presents a better and easier way to solve the uncertainties of real-life problems.

The proposed work exhibits a broad scope of potential applications. For further research, recognizing the perfection of novel LDFSs, it may be extended to any other aggregation operators, such as prioritized AOs, power mean AOs, Dombi’s AOs, Bonferroni mean AOs, and Heronian mean AOs. We hope that our research results will be successful for researchers working in the fields of information aggregation, statistical techniques, intelligent systems, machine learning, neural networks, and psychiatric disorder.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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