FIXED POINT THEOREMS FOR NONCOMMUTATIVE FUNCTIONS

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Abstract. We establish a fixed point theorem for mappings of square matrices of all sizes which respect the matrix sizes and direct sums of matrices. The conclusions are stronger if such a mapping also respects matrix similarities, i.e., is a noncommutative function. As a special case, we prove the corresponding contractive mapping theorem which can be viewed as a new version of the Banach Fixed Point Theorem. This result is then applied to prove the existence and uniqueness of a solution of the initial value problem for ODEs in noncommutative spaces. As a by-product of the ideas developed in this paper, we establish a noncommutative version of the principle of nested closed sets.

1. Introduction
The theory of noncommutative (nc) functions has its origin in the articles of Joseph L. Taylor [21, 22]. It was further developed by D.-V. Voiculescu [23, 24] in his fundamental work in free probability. Based on pioneering ideas of J. L. Taylor, the second author and Victor Vinnikov [13] developed the nc difference-differential calculus and used it for studying various questions of nc analysis, in particular, extending the classical (commutative) theory of analytic functions to a nc setting. A special case of nc rational functions, which is important for applications in optimization and control, where matrices are natural variables and the problems are dimension-independent (see [8, 9] for a detailed discussion), can be studied independently — see [14, 12]. The theory of nc rational functions is motivated by and useful in nc semialgebraic geometry — see, e.g., [2, 7, 10, 4]. We also mention the works of Helton–Klep–McCullough [5, 6], of Popescu [16, 17, 18], and of Muhly–Solel [15] on various aspects of nc function theory. The goal of the present paper is to establish a certain type of fixed point theorems as a useful tool in nc analysis.

We provide the reader with some basic definitions from [13]. Let \( R \) be a unital ring. For a bi-module \( M \) over \( R \), we define the nc space over \( M \),

\[
\mathcal{M}_{nc} := \prod_{n=1}^{\infty} M_{n \times n}.
\]

A subset \( \Omega \subseteq \mathcal{M}_{nc} \) is called a nc set if it is closed under direct sums; that is, denoting \( \Omega_n = \Omega \cap M_{n \times n} \), we have

\[
X \in \Omega_n, Y \in \Omega_m \implies X \oplus Y := \left[ \begin{array}{c|c} X & O_{n \times m} \\ \hline O_{m \times n} & Y \end{array} \right] \in \Omega_{n+m},
\]

where \( O_{p \times q} \) denotes the \( p \times q \) matrix whose all entries are 0.

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Notice that matrices over \( \mathcal{R} \) act from the right and from the left on matrices over \( \mathcal{M} \) by the standard rules of matrix multiplication: if \( T \in \mathcal{R}^{r \times p} \) and \( S \in \mathcal{R}^{p \times s} \), then for \( X \in \mathcal{M}^{p \times p} \) we have
\[
TX \in \mathcal{M}^{r \times p}, \quad XS \in \mathcal{M}^{p \times s}.
\]

In the special case where \( \mathcal{M} = \mathcal{R}^d \), we identify matrices over \( \mathcal{M} \) with \( d \)-tuples of matrices over \( \mathcal{R} \):
\[
(\mathcal{R}^d)^{p \times q} \cong (\mathcal{R}^{p \times q})^d.
\]
Under this identification, for \( d \)-tuples \( X = (X_1, \ldots, X_d) \in (\mathcal{R}^{n \times n})^d \) and \( Y = (Y_1, \ldots, Y_d) \in (\mathcal{R}^{m \times m})^d \), their direct sums have the form
\[
X \oplus Y = \left( \begin{bmatrix} X_1 & O_{n \times m} \\ O_{m \times n} & Y_1 \end{bmatrix}, \ldots, \begin{bmatrix} X_d & O_{n \times m} \\ O_{m \times n} & Y_d \end{bmatrix} \right) \in (\mathcal{R}^{(n+m) \times (n+m)})^d;
\]
and for a \( d \)-tuple \( X = (X_1, \ldots, X_d) \in (\mathcal{R}^{p \times p})^d \) and matrices \( T \in \mathcal{R}^{r \times p}, S \in \mathcal{R}^{p \times s} \),
\[
TY = (TY_1, \ldots, TY_d) \in (\mathcal{R}^{r \times p})^d, \quad XS = (X_1S, \ldots, X_dS) \in (\mathcal{R}^{p \times s})^d;
\]
that is, \( T \) and \( S \) act on \( d \)-tuples of matrices componentwise.

Let \( \mathcal{M} \) and \( \mathcal{N} \) be bi-modules over \( \mathcal{R} \), and let \( \Omega \subseteq \mathcal{M}_{\text{nc}} \) be a nc set. A mapping
\[
f : \Omega \to \mathcal{N}_{\text{nc}}
\]
with the property that \( f(\Omega_n) \subseteq \mathcal{N}^{n \times n}, n = 1, 2, \ldots \), is called a nc function if \( f \) satisfies the following two conditions:
\[
\begin{align*}
& (3) \quad f \text{ respects direct sums: } f(X \oplus Y) = f(X) \oplus f(Y), \quad X, Y \in \Omega; \\
& (4) \quad f \text{ respects similarities: } \text{if } X \in \Omega_n \text{ and } S \in \mathcal{R}^{n \times n} \text{ is invertible} \\
& \quad \text{with } SXS^{-1} \in \Omega_n, \text{ then } f(SXS^{-1}) = Sf(X)S^{-1},
\end{align*}
\]
or, equivalently, satisfies the single condition:
\[
(5) \quad f \text{ respects intertwinings: } \text{if } X \in \Omega_n, Y \in \Omega_m, \text{ and } T \in \mathcal{R}^{n \times m} \\
\quad \text{are such that } XT = TY, \text{ then } f(X)T = Tf(Y).
\]

Condition \( (5) \) was used by J. L. Taylor in the case where \( \mathcal{M} = \mathbb{C}^d \), together with an additional assumption of analyticity of \( f(X) \) as a function of matrix entries \((X_i)_{j,k}, \ i = 1, \ldots, d; \ j, k = 1, \ldots, n, \) for every \( n \in \mathbb{N} \) — see \cite{22}.

Notice a certain discrepancy in our terminology (inherited from \cite{14}): a set is nc if it respects direct sums, while a function is nc if it respects both direct sums and similarities. Nevertheless, we prefer to keep it this way, setting the minimal assumptions under which the theory of nc functions starts revealing its phenomena.

**Example 1.** Let
\[
p = \sum_{w \in \mathcal{F}_d : |w| \leq m} p_w z^w
\]
be a nc polynomial, where \( \mathcal{F}_d \) is the free semigroup on \( d \) generators (an alphabet) \( g_1, \ldots, g_d \), with the unit element \( \emptyset \) (the empty word); for a word \( w = g_{i_1} \cdots g_{i_k} \in \mathcal{F}_d \) and a \( d \)-tuple of noncommuting indeterminates \( z = (z_1, \ldots, z_d) \), we use the notation \( z^w := z_{i_1} \cdots z_{i_k} \) (and \( z^\emptyset := 1 \)); the length of the word \( w = g_{i_1} \cdots g_{i_k} \) is \( |w| := k \) (and \( |\emptyset| := 0 \)); \( p_w \in \mathcal{R} \) for every \( w \in \mathcal{F}_d \). Then \( p \) can be viewed as a nc function.
p: Ω = (R^d)_{nc} → R_{nc}: for a d-tuple \( X = (X_1, \ldots, X_d) \) of \( n \times n \) matrices over \( R \), we set

\[
p(X) := \sum_{w \in F_d: |w| \leq m} p_w X^w,
\]

where \( X^w := X_{i_1} \cdots X_{i_k} \) for \( w = g_{i_1} \cdots g_{i_k} \in F_d \) and \( X^\emptyset := I_n \) (the \( n \times n \) identity matrix).

**Example 2.** Let \( R = \mathbb{C} \), \( M = \mathbb{C}^d \), \( N = \mathbb{C} \). Let \( \Omega \subseteq M_{nc} \) be the nc set of \( d \)-tuples \( X = (X_1, \ldots, X_d) \) of square matrices of the same size such that the spectral radius of \( X_1 + \cdots + X_d \) is strictly less than 1. Then \( f: \Omega \rightarrow N_{nc} \) defined by

\[
f(X) := (I_n - X_1 - \cdots - X_d)^{-1} = \sum_{j=0}^{\infty} (X_1 + \cdots + X_d)^j = \sum_{j=0}^{\infty} \sum_{w \in F_d: |w| = j} X^w
\]
is a nc function.

In the next two examples, we present some mappings of matrices which respect matrix sizes, but fail to be nc functions because one of the two conditions, (3) or (4), is not satisfied.

**Example 3.** Let \( R \) be commutative, and let \( f: \Omega = R_{nc} \rightarrow R_{nc} \) be defined by

\[
f(X) := (\det X)I_n, \quad X \in R^{n \times n}, \quad n \in \mathbb{N}.
\]

Then, clearly, \( f \) respects the matrix size and similarities, i.e., \( f(\Omega_n) \subseteq R^{n \times n} \), \( n \in \mathbb{N} \), and (4) holds. However, \( f \) does not respect direct sums, i.e., (3) does not always hold. Thus, \( f \) is not a nc function. Notice that replacing \( \det X \) by \( \text{trace} X \), i.e., defining

\[
f(X) := (\text{trace} X)I_n, \quad X \in R^{n \times n}, \quad n \in \mathbb{N},
\]

would bring the same conclusions.

**Example 4.** Let \( R = M = N = F \), where \( F = \mathbb{R} \) or \( F = \mathbb{C} \). For every scalar \( t \), define \( f_t: M_{nc} \rightarrow N_{nc} \) by

\[
f_t(X) := t\hat{X},
\]

where \( \hat{X} \) is a square matrix over \( F \) which has the same size and the same main diagonal as \( X \), and whose all off-diagonal entries are zeros. Then, clearly, \( f_t \) respects direct sums, i.e., (3) holds, however \( f_t \) does not respect similarities, i.e., (4) does not always hold, for \( t \neq 0 \). For example, let

\[
X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.
\]

Then

\[
SXS^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \neq \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = S\hat{X}S^{-1}.
\]

Thus, \( f_t \) is not a nc function unless \( t = 0 \).
2. The results

A part of our main results is valid under much more general assumptions on the sets of matrices and the corresponding functions. If $S$ is a set, then we will write $S^{n \times n}$ for a set of $n \times n$ matrices with entries from $S$, where we consider matrices just as arrays, i.e., not assuming any algebraic structure on them. We will also extend the notation of (1) to an arbitrary set $S$ in the place of $M$.

Fix some element $O \in S$. We define direct sums of matrices from $S$ in the same way as in (2) except that $O_{p \times q}$ is understood now as the $p \times q$ matrix whose all entries are equal to $O$.

Theorem 5. Let $S$ be a set, $O \in S$, and let $\Omega \subseteq S_{nc}$ respect direct sums of matrices. Define

$\supp \Omega := \{ n \in \mathbb{N} : \Omega_n \neq \emptyset \}$. Let $f : \Omega \rightarrow \Omega$ satisfy

- $f(\Omega_n) \subseteq \Omega_n$, $n \in \supp \Omega$;
- $f$ respects direct sums: $f(X \oplus Y) = f(X) \oplus f(Y)$, $X, Y \in \Omega$;
- For every $n \in \supp \Omega$ the mapping $f|_{\Omega_n}$ has a unique fixed point, $X^*_n$.

Let $d = \gcd\{ n : n \in \supp \Omega \}$. Then

1. There exists $X_* \in S^{d \times d}$ such that

$$X^*_n = \bigoplus_{\alpha=1}^{n/d} X_*, \quad n \in \supp \Omega.$$  

2. If, moreover, $S = M$ is a bi-module over $R$, $O = 0 \in M$, and $f$ is a nc function, then there exists a nc set $\tilde{\Omega} \supseteq \Omega$ with $\supp \tilde{\Omega} = \mathbb{N}d$ and a nc function $\tilde{f} : \tilde{\Omega} \rightarrow \Omega$ such that

- $\tilde{f}|_{\Omega} = f$;
- For every $n \in \mathbb{N}d$ the mapping $\tilde{f}|_{\tilde{\Omega}_n}$ has a unique fixed point

$$X^*_n = \bigoplus_{\alpha=1}^{n/d} X_*.$$  

Theorem 5 establishes the following general principle. Given a mapping on matrices which respects matrix sizes and direct sums of matrices, and given that this mapping has a unique fixed point in each matrix dimension, all these fixed points arise as multiple copies of a single matrix $X_*$. Moreover, if the mapping is a nc function, then it can be extended to a nc function whose domain includes the matrix $X_*$. This principle can be used further to generalize any fixed point theorem where the fixed point is unique to the noncommutative setting; in Theorem 8 we will present a nc version of the Banach Fixed Point Theorem.

Remark 6. A statement analogous to part 2 of Theorem 5 for a mapping $f$ which respects direct sums but not necessarily respects similarities is false, as the following example shows.

Example 7. Let $\mathcal{M} = \mathcal{R} = \mathbb{Z}/(\mathbb{Z}/2\mathbb{Z})$, and let $\Omega$ be defined as follows:

$\supp \Omega = \{ 2n + 5m, \quad n \in \mathbb{N}, m \in \mathbb{N} \}$,

$\Omega_2 = \{ O_{2 \times 2}, I_2 \}$, $\Omega_5 = \{ O_{5 \times 5}, I_5, \text{diag}[0, 0, 1, 0, 0] \}$, and $\Omega_{n+5m}$ is a finite set of matrices which are direct sums, in any possible order, of matrices from $\Omega_2$ and $\Omega_5$. Clearly, $\Omega$ is a nc set over $\mathbb{Z}_2$. Define

$f(O_{2 \times 2}) = I_2 = f(I_2)$,
Theorem 8. Let \( \Omega \) be a set, \( \Omega \subseteq S_{nc} \) respect direct sums of matrices. Suppose that \( \Omega \) is a complete metric space with respect to a metric \( \rho_n \) for every \( n \in \text{supp} \Omega \). Let \( f : \Omega \to \Omega \) satisfy

- \( f(\Omega_n) \subseteq \Omega_n, n \in \text{supp} \Omega \);
- \( f \) respects direct sums: \( f(X \oplus Y) = f(X) \oplus f(Y), X, Y \in \Omega \);

Then \( f \) is uniquely determined. Indeed, elements of \( \Omega \) are diagonal matrices, and the sequence 0, 0, 1, 0, 0 appears on the diagonal of a matrix \( X \) from \( \Omega \) if and only if this sequence belongs to a copy of the matrix \( A = \text{diag}[0, 0, 1, 0, 0] \in \Omega_5 \), i.e. \( X = Y \oplus A \oplus Z \), with \( Y, Z \in \Omega \). Then

\[
\begin{align*}
\tilde{f}(X) &= f(Y) \oplus f(A) \oplus f(Z) = f(Y) \oplus \mathbf{O}_{5 \times 5} \oplus f(Z) \quad &\text{if } Y \text{ and } Z \text{ have no copies of } A \text{ on the diagonal, then} \\
\tilde{f}(X) &= \mathbf{I} \oplus \mathbf{O}_{5 \times 5} \oplus \mathbf{I} \quad &\text{otherwise, applying the same argument for } Y \text{ (and } Z) \text{ in the place of } X, \text{ we eventually obtain that } \tilde{f}(X) \text{ is uniquely determined and equal a diagonal matrix with } \mathbf{O}_{5 \times 5} \text{ at the same block diagonal spots as copies of } A \text{ in } X, \text{ and } 1 \text{ at the other diagonal spots. Suppose there exists a } \text{nc set } \Omega \supseteq \Omega \text{ such that } \text{supp } \Omega = \mathbb{N} \text{ (clearly, we have } d = \gcd(2, 5) = 1), \text{ and a mapping } \tilde{f} : \Omega \to \tilde{\Omega} \text{ satisfying}
\end{align*}
\]

- \( \tilde{f}(\Omega_n) \subseteq \tilde{\Omega}_n \);
- \( \tilde{f} \) respects direct sums: \( \tilde{f}(X \oplus Y) = \tilde{f}(X) \oplus \tilde{f}(Y), X, Y \in \tilde{\Omega} \);
- \( \tilde{f}|_\Omega = f \);
- For every \( n \in \mathbb{N} \) the mapping \( \tilde{f}|_{\Omega_n} \) has a unique fixed point

\[
X_{s_n} = \bigoplus_{n=1}^{N} X_s.
\]

Then \( \tilde{f}(\text{diag}[0, 0, 1, 0, 0]) = \tilde{f}(\text{diag}[0, 0, 1, 0, 0]) = \mathbf{O}_{5 \times 5} \). On the other hand, since \( f(\mathbf{I}_2) = \mathbf{I}_2 \), we must have \( 1 = \mathbf{I}_1 \in \tilde{\Omega}_1 \), \( \tilde{f}(1) = 1 \), and

\[
\begin{align*}
\tilde{f}(\text{diag}[0, 0, 1, 0, 0]) &= \tilde{f}(\mathbf{O}_{2 \times 2} \oplus 1 \oplus \mathbf{O}_{2 \times 2}) = \tilde{f}(\mathbf{O}_{2 \times 2}) \oplus \tilde{f}(1) \oplus \tilde{f}(\mathbf{O}_{2 \times 2}) \\
&= f(\mathbf{O}_{2 \times 2}) \oplus \tilde{f}(1) \oplus f(\mathbf{O}_{2 \times 2}) = \mathbf{I}_2 \oplus 1 \oplus \mathbf{I}_2 = \mathbf{I}_5,
\end{align*}
\]
i.e. we obtain a contradiction.

We will need the following definition of a complex (real) operator space (see [H 20]) which gives rise to a natural topology on a \( \text{nc} \) space. Let \( \mathbb{F} \) be a field, \( \mathbb{F} = \mathbb{C} \) or \( \mathbb{F} = \mathbb{R} \). A vector space \( \mathbb{V} \) over \( \mathbb{F} \) is called an \textit{operator space} if a sequence of Banach space norms \( \| \cdot \|_n \) on \( \mathbb{V}^{n \times n} \), \( n = 1, 2, \ldots \) is defined so that the following two conditions hold:

- For every \( n, m \in \mathbb{N}, X \in \mathbb{V}^{n \times n} \text{ and } Y \in \mathbb{V}^{m \times m}, \)
  \[
  \|X \oplus Y\|_{n+m} = \max\{\|X\|_n, \|Y\|_m\};
  \]
- For every \( n \in \mathbb{N}, X \in \mathbb{V}^{n \times n} \text{ and } S, T \in \mathbb{F}^{n \times n}, \)
  \[
  \|SXT\|_n \leq \|S\| \|X\|_n \|T\|,
  \]
  where \( \| \cdot \| \) denotes the \( (2, 2) \) operator norm on \( \mathbb{F}^{n \times n} \).

**Theorem 8.** Let \( S \) be a set, \( \Omega \subseteq S_{nc} \) respect direct sums of matrices. Suppose that \( \Omega \) is a complete metric space with respect to a metric \( \rho_n \) for every \( n \in \text{supp} \Omega \). Let \( f : \Omega \to \Omega \) satisfy

- \( f(\Omega_n) \subseteq \Omega_n, n \in \text{supp} \Omega \);
- \( f \) respects direct sums: \( f(X \oplus Y) = f(X) \oplus f(Y), X, Y \in \Omega \);
For every \( n \in \text{supp} \, \Omega \) there exists \( c_n : 0 \leq c_n < 1 \) so that
\[
\rho_n(f(X), f(Y)) \leq c_n \rho_n(X, Y), \quad X, Y \in \Omega_\eta.
\]
Let \( d = \gcd\{n : n \in \text{supp} \, \Omega \} \). Then:

1. There exists \( X_\ast \in S^{d \times d} \) such that for every \( n \in \text{supp} \, \Omega \) the mapping \( f|_{\Omega_\eta} \)
   has a unique fixed point \( X_{\ast n} = \bigoplus_{\alpha=1}^{n/d} X_\ast \).

2. Suppose additionally that \( F \) is a field, \( F = \mathbb{C} \) or \( F = \mathbb{R} \), \( S = \mathcal{V} \) is an
   operator space over \( F \), so that for every \( n \in \text{supp} \, \Omega \) one has \( \rho_n(X, Y) = \|X - Y\|_n \), \( X, Y \in \Omega_\eta \), \( O = 0 \in \mathcal{V} \), and that \( f \) is a nc function. Then the
   conclusions of Theorem 8, part 2, hold; moreover \( \hat{\Omega} \) and \( \hat{f} \) can be chosen
   such that for every \( n \in \mathbb{N}d \):
   - \( \hat{\Omega}_n \) is a complete metric space with respect to the metric
     \( \rho_n(X, Y) = \|X - Y\|_n \), \( X, Y \in \hat{\Omega}_n \)
     (which extends the metric \( \rho_n \) for \( n \in \text{supp} \, \Omega \));
   - There exists \( \hat{c}_n : 0 \leq \hat{c}_n < 1 \) (obviously, \( \hat{c}_n \geq c_n \) for \( n \in \text{supp} \, \Omega \)) such that
     \[
     \|\hat{f}(X) - \hat{f}(X_{\ast n})\|_n \leq \hat{c}_n \|X - X_{\ast n}\|_n, \quad X \in \hat{\Omega}_n;
     \]
   - For an arbitrary \( X^0 \in \hat{\Omega}_n \), define \( X^{j+1} = f(X^j), \ j = 0, 1, \ldots \). Then
     \( X_{\ast n} = \lim_{j \to \infty} X^j \). Moreover,
     \[
     \|X^j - X_{\ast n}\|_n \leq (\hat{c}_n)^j \|X^0 - X_{\ast n}\|_n, \quad j = 1, 2, \ldots.
     \]

Remark 9. We do not know whether one can find \( \hat{\Omega} \) and \( \hat{f} \) such that for every
\( n \in \mathbb{N}d \) there exists \( c_n^\ast : 0 \leq c_n^\ast < 1 \) satisfying
\[
\|\hat{f}(X) - \hat{f}(Y)\|_n \leq c_n^\ast \|X - Y\|_n, \quad X, Y \in \hat{\Omega},
\]
and leave this as an open question.

Remark 10. We now give a useful interpretation of part 2 of Theorem 8. First of all, in a general setting of part 1 of Theorem 8 we define a relation \( \sim \) on a \( \Omega \subseteq S_{\text{nc}} \) as follows: \( X \sim Y \) if both \( X \) and \( Y \) are direct sums of several copies of the
same matrix over \( S \). Observe that \( \sim \) is an equivalence relation on \( \Omega \), and define the quotient set \( \hat{\Omega} := \Omega/\sim \). We will call the equivalence class \( \hat{X} \in \hat{\Omega} \) of \( X \in \Omega \) a
noncommutative singleton. Since \( f : \Omega \to \Omega \) preserves the equivalence \( \sim \), it gives rise to the quotient mapping \( \hat{f} : \hat{\Omega} \to \hat{\Omega} \). The conclusion of part 1 of Theorem 8 means that the fixed points \( X_{\ast n} \) are equivalent, and therefore the mapping \( \hat{f} \) has
a unique fixed point. However, the assumptions of part 1 are too general to have a
good interpretation in terms of the mapping \( \hat{f} \). If one strengthen them as in part 2, then one defines a function \( \rho : \Omega \times \Omega \to \mathbb{R}_+ \) as follows. For any \( n, m \in \text{supp} \, \Omega \), \( X \in \Omega_n \), \( Y \in \Omega_m \), set
\[
\rho(X, Y) = \left\| \bigoplus_{\alpha=1}^{m} X - \bigoplus_{\beta=1}^{n} Y \right\|_{nm}.
\]
Since \( \mathcal{S} = \mathcal{V} \) is an operator space, \( \rho \) extends \( \rho_n \) for every \( n \in \text{supp} \, \Omega \). Clearly, \( \rho \) is
a pseudometric on \( \hat{\Omega} \), and \( \rho(X, Y) = 0 \) if and only if \( X \sim Y \). Observe that \( \hat{\Omega} \) is a
metric space with respect to the quotient metric \( \hat{\rho} \) defined by
\( \hat{\rho}(\hat{X}, \hat{Y}) := \rho(X, Y) \). Moreover, for a nc set \( \Omega \supseteq \hat{\Omega} \) there is a natural embedding \( \hat{\Omega} \hookrightarrow \hat{\Omega} \), and \( \hat{f} \) admits
an extension $\hat{f}$ to $\hat{\Omega}$ which also has a unique fixed point, the equivalence class of a matrix $X_0 \in \mathbb{M}_{d\times d}$, such that $X_{n+1} = \bigoplus_{\alpha=1}^{n/d} X_0$ for every $n \in \text{supp} \Omega$.

Even though part 2 of Theorem 8 can be interpreted as a unique fixed point theorem in a metric space, it does not follow directly from the classical Banach Fixed Point Theorem for two reasons. First, the possibility that the smallest possible Lipschitz constant for $\hat{f}$, that is $\sup_{n \in \text{supp} \Omega} c_n$, is equal to 1, is not ruled out. Second, the metric space $\hat{\Omega}$ may be incomplete, as the following example shows.

**Example 11.** Let $\Omega = \mathbb{C}_{nc}$, with the $(2,2)$ operator norm topology on $\mathbb{C}^{n\times n}$, $n = 1, \ldots$. Define recursively the following sequence in $\Omega$:

$$X^0 := 1 \in \Omega_1, \quad X^{j+1} := \left[ \begin{array}{c} X_j & \frac{1}{2} I_2 \\ \frac{1}{2} I_2 & X_j \end{array} \right] \in \Omega_{2^{j+1}}. $$

This is a Cauchy sequence in pseudometric $\rho$. Indeed,

$$\rho(X^{j+k}, X^j) \leq \|X^{j+k} - X^{j+k-1}\|_{2^{j+k}} + \cdots + \|X^{j+1} - X^j\|_{2^{j+1}} = \frac{1}{2^{j+k-1}} + \cdots + \frac{1}{2^j} \to 0$$

as $j, k \to \infty$. Correspondingly, the sequence $\{\hat{X}^j\}_{j=1,2,\ldots} \in \hat{\Omega}$ is Cauchy in metric $\hat{\rho}$:

$$\hat{\rho}(\hat{X}^{j+k}, \hat{X}^j) = \rho(X^{j+k}, X^j) \to 0$$

as $j, k \to \infty$. Since matrices $X^j$ have infinitely increasing number of nonzero entries in the first row, and the sequence $X_{j+k}^j$ is eventually constant for every fixed $k$, and the entries $X_{jk}^j$ continuously depend on matrices $X^j$, the representatives of the class $\hat{X} := \lim_{j \to \infty} \hat{X}^j$ (provided the limit exists) would have infinitely increasing number of nonzero entries in the first row as the matrix size of these representatives increases. This is, however, impossible for elements of $\hat{\Omega}$, thus the sequence $\{\hat{X}^j\}_{j=1,2,\ldots}$ does not converge, and the metric space $\hat{\Omega}$ is incomplete.

**Example 12.** In the setting of Example 11, restrict $f_t$ to the set $\Omega$ of all matrices of the $(2,2)$ operator norm at most 1. Then $\text{supp} \Omega = \mathbb{N}$, and the function $f_t$ is a self-mapping of $\Omega$ for all $t$: $|t| \leq 1$. For every $t$: $|t| < 1$, the function $f_t$ satisfies the assumptions of Theorem 8 (with the metric $\rho_n$ induced by the $(2,2)$ operator norm and with $c_n = t$ for all $n$), and we obtain the conclusion of the theorem with $X_0 = O_{1\times 1}$ (and since $d = 1$, part 2 of the theorem becomes trivial). The function $f_t$ does not satisfy the contractivity assumption with any $c_n < 1$, and $f_1$ has a plenty of fixed points: every diagonal matrix is a fixed point.

Our proof of Theorem 9 is based on the following lemma (whose name is explained in Remark 10).

**Lemma 13 (Noncommutative singleton lemma).** Let $S$ be a set, $O \in S$. Let $\Omega \subseteq S_{nc}$ respect direct sums of matrices and be of the form $\Omega = \{X_n\}_{n \in \text{supp} \Omega}$ with $X_n \in \Omega_n$. Then there exists $X \in S_{d\times d}$, with $d = \gcd\{n: n \in \text{supp} \Omega\}$, such that

$$X_n = \bigoplus_{\alpha=1}^{n/d} X, \quad n \in \text{supp} \Omega. \tag{9}$$

We also apply Lemma 13 to obtain a nc version of another important principle of analysis.
Theorem 14 (The principle of nested nc sets). Let \( S \) be a set, \( O \in S \), and let \( \Omega \subseteq S_{nc} \) respect direct sums of matrices. Suppose that \( \Omega_n \) is a complete metric space with respect to a metric \( \rho_n \) for every \( n \in \text{supp} \Omega \). Given a sequence of sets
\[
\Omega^j = \prod_{n \in \text{supp} \Omega} \Omega_n^j \subseteq \Omega, \quad j = 1, \ldots
\]
such that
- \( \Omega^j \) respects direct sum of matrices, for every \( j = 1, \ldots; \)
- \( \Omega_n^j \) is a non-empty closed subset of \( \Omega_n \), for every \( n \in \text{supp} \Omega \), \( j = 1, \ldots; \)
- \( \Omega_1 \supseteq \Omega_2 \supseteq \cdots \)
- \( \text{diam} \Omega_n^j := \sup_{X,Y \in \Omega_n^j} \rho_n(X,Y) \to 0 \) as \( j \to \infty \), for every \( n \in \text{supp} \Omega \),
there exists a unique \( X* \in S^{d \times d} \), with \( d = \gcd \{ n : n \in \text{supp} \Omega \} \), such that
\[
\bigcap_{j=1}^{\infty} \Omega^j = \left\{ \bigoplus_{\alpha=1}^{n/d} X_* \right\}_{n \in \text{supp} \Omega}.
\]

Remark 15. Similarly to Remark 10, we can interpret the conclusion of Theorem 14 in terms of quotient sets of nc sets. Namely, in the assumptions of Theorem 14, the sequence of nested quotient sets \( \hat{\Omega}^j \) has a nonempty intersection consisting of a single point. Again, the assumptions on the metrics \( \rho_n \) are too general to have a good interpretation in terms of quotient sets. If we assume, as in part 2 of Theorem 8 that \( S = V \) is an operator space and the metrics \( \rho_n \) are norm-induced, then we can define the corresponding pseudometric \( \hat{\rho} \) on \( \Omega \) as in Remark 10. Clearly, the sequence \( \text{diam} \hat{\Omega}^j := \sup_{X,Y \in \hat{\Omega}^j} \hat{\rho}(\hat{X},\hat{Y}) \) is non-increasing. However, it may happen that it does not converge to 0. Also, the sets \( \hat{\Omega}^j \) are not necessarily complete metric spaces. So, both the assumptions of the classical principle of nested closed sets may fail in our case, as confirmed by the following example.

Example 16. Let \( \Omega = C_{nc} \), with the \((2,2)\) operator norm topology on \( C^{n \times n} \), \( n = 1, \ldots \). Let
\[
\Omega_n^j := \left\{ X \in C^{n \times n} : 0 \leq \|X\|_n \leq \frac{n}{n+j} \right\}, \quad n, j = 1, \ldots
\]
It is easy to see that the sets \( \Omega_n^j \) satisfy the conditions of Theorem 14 and, for a fixed \( n \), \( 0_{n \times n} \) is the unique common point of the sets \( \Omega_n^j \). Define
\[
X_n^j := \frac{n}{n+j} I_n \in \Omega_n^j, \quad Y_n^j := -X_n^j \in \Omega_n^j, \quad n, j = 1, \ldots
\]
We have \( \|X_n^j - Y_n^j\|_n \to 2 \) as \( n \to \infty \), hence \( \text{diam} \hat{\Omega}^j = 2 \) for every \( j \), and \( \lim_{j \to \infty} \text{diam} \hat{\Omega} \neq 0 \). The sequence \( \{X_n^j\}_{n=1,\ldots} \) is Cauchy in pseudometric \( \rho \) for every fixed \( j \), and so is the sequence \( \{X_n^j\}_{n=1,\ldots} \) in metric \( \hat{\rho} \). However, the latter has no limit in \( \hat{\Omega} \), since such a limit would be a class whose representative are of norm 1, which is impossible.

We now present an application of Theorem 8 to initial value problems for ODEs in nc spaces. The following theorem is a nc counterpart of (a version of) the existence and uniqueness theorem for solutions of ODEs in the classical setting [11, Theorem 3.7].
Theorem 17. Let $I$ be an interval in $\mathbb{R}$ and let $t_0$ be a point in the interior of $I$. Let $V$ be a (real or complex) operator space and let $\Xi \subseteq V$ be a nc set, with $\Xi_n$ a closed subspace of the Banach space $V^{n \times n}$ for every $n \in \text{supp} \Xi$. Let $X_{0n} \in \Xi_n$, $n \in \text{supp} \Xi$, and let $\{X_{0n}\}_{n \in \text{supp} \Xi}$ be a nc set. Suppose that $g: I \times \Xi \to \Xi$ satisfies the conditions:

- $g(t, \cdot)$ maps $\Xi_n$ to itself, $n \in \text{supp} \Xi$, and respects direct sums of matrices, for every $t \in I$;
- $g_n := g|_{I \times \Xi_n}$ is continuous for every $n \in \text{supp} \Xi$;
- There is a constant $C > 0$ such that
  \[ \|g(t, X) - g(t, Y)\|_n \leq C\|X - Y\|_n \]
  for every $t \in I$, $n \in \text{supp} \Xi$, and $X, Y \in \Xi_n$.

Then

1. There is a matrix $X_0 \in V^{d \times d}$, with $d = \gcd\{n: n \in \text{supp} \Xi\}$, such that
   \[ X_{0n} = \bigoplus_{\alpha=1}^{n/d} X_0, \quad n \in \text{supp} \Xi. \]  
(10)

2. There is a continuously differentiable function $X_\ast: I \to V^{d \times d}$ such that, for every $n \in \text{supp} \Xi$,
   \[ X_{\ast n} = \bigoplus_{\alpha=1}^{n/d} X_\ast: I \to \Xi_n \]
   is a unique solution of the initial value problem for the first-order ODE
   \[ \dot{X} = g_n(t, X), \quad X(t_0) = X_{0n}. \]  
(12)

3. Suppose that, in addition, $g(t, \cdot)$ respects similarities of matrices, thus is a nc function, for every $t \in I$. Then there exist a nc set $\bar{\Xi}(t) \supseteq \Xi$ with $\text{supp} \bar{\Xi}(t) = N_d$, $t \in I$, so that for every $n \in N_d$ one has a fiber bundle $\Psi_n$ with the total space
   \[ \bar{\Xi}_n = \coprod_{t \in I} \bar{\Xi}(t)_n \subseteq I \times V^{n \times n}, \]
   the base space $I$ and the projection $\pi_n: \bar{\Xi}_n \to I$ defined by $\pi_n: \bar{\Xi}(t) \to t$; a map $\bar{g}$ of the set $\coprod_{t \in I} \bar{\Xi}(t)$ to itself such that $\bar{g}_n := \bar{g}|_{\Xi_n}$ is a continuous bundle endomorphism for every $n = N_d$, that extends the function $g$ (where we identify all copies of $\Xi_n$ in $\bar{\Xi}(t)_n$, $t \in I$); and, for every $n \in N_d$, a unique continuously differentiable cross-section of the fiber bundle $\Psi_n$,
   \[ X_{\ast n} = \bigoplus_{\alpha=1}^{n/d} X_\ast: I \to \bar{\Xi}_n, \]
(13)
which is a solution of the initial value problem for the first-order ODE
   \[ \dot{X} = \bar{g}_n(t, X), \quad X(t_0) = \bigoplus_{\alpha=1}^{n/d} X_0. \]  
(14)
3. The proofs

Proof of Lemma 13. First we prove that there exist $k \in \mathbb{N}, n_1, \ldots, n_k \in \text{supp } \Omega$ such that
\[ d = \gcd\{n_1, \ldots, n_k\}. \]
Order elements of \text{supp } \Omega increasing. We have
\[ n_1 \geq \gcd\{n_1, n_2\} \geq \gcd\{n_1, n_2, n_3\} \geq \ldots \geq (\geq d). \]
There is at most a finite number of strict inequalities in this chain of inequalities.
Let $k$ be the first integer satisfying
\[ \gcd\{n_1, \ldots, n_k\} = \gcd\{n_1, \ldots, n_k, n_{k+1}\} = \ldots. \]
Then
\[ \gcd\{n_1, \ldots, n_k\} = \gcd\{n_1, \ldots, n_k, \ldots\} = \gcd\{n : n \in \text{supp } \Omega\} = d \]
because every $n \in \text{supp } \Omega$ is divisible by $\gcd\{n_1, \ldots, n_k\}$.
Let $s = \text{lcm}\{n_1, \ldots, n_k\}$. Then $s \in \text{supp } \Omega$ and for every $j = 1, \ldots, k$, the matrix
\[ \bigoplus_{\alpha=1}^{s/n_j} X_n \]
is in $\Omega$. By the assumption, this matrix must coincide with $X_s$. Let us use the convention that the rows and columns of a $n \times n$ matrix $M$ over $\mathcal{S}$ are enumerated from 0 to $n-1$. Define the diagonal shift $S$ which acts on such matrices as follows:
\[ (SM)_{ij} = M_{(i+1) \mod n, (j+1) \mod n}, \quad i, j = 0, \ldots, n-1. \]
Clearly, the inverse shift is given by
\[ (S^{-1}M)_{ij} = M_{(i-1) \mod n, (j-1) \mod n}, \quad i, j = 0, \ldots, n-1. \]
Since $X_s$ is a block diagonal (in particular, block circulant) matrix, we have
\[ S^{\pm n_j} X_s = X_s. \]
Since there exist $m_1, \ldots, m_k \in \mathbb{Z}$ such that $d = m_1n_1 + \ldots + m_kn_k$ (see, e.g., [23, Problem 1.1]), we obtain from (15) that
\[ S^{d} X_s = X_s. \]
Since $X_s$ has the form (13), all off-diagonal $d \times d$ block entries equal $O_{d \times d}$, and it follows from (16) that all $d \times d$ block diagonal entries of $X_s$ are equal, say to $X \in S^{d \times d}$. Therefore, $X_s = \bigoplus_{\alpha=1}^{s/n_j} X$. Comparing this with $X_s = \bigoplus_{\beta=1}^{s/n_j} X_n$, we obtain $X_n = \bigoplus_{\gamma=1}^{n/d} X$, for every $j = 1, \ldots, k$.
For every $n \in \text{supp } \Omega$, we have
\[ \bigoplus_{\alpha=1}^{n_1} X_n = X_{n1} = \bigoplus_{\beta=1}^{n_1} X_{n1} = \bigoplus_{\gamma=1}^{n_1/d} X = \bigoplus_{\alpha=1}^{n_1 \div d} \bigoplus_{\delta=1}^{n_1 / d} X \in \Omega_{n_1}. \]
Therefore, (14) holds.

Proof of Theorem 1. Observe that $\{X_s\}_{n \in \text{supp } \Omega}$ is a nc set. Indeed, since $f$ respects direct sums, for any $n, m \in \text{supp } \Omega$ one has
\[ f(X_{sn} \oplus X_{sm}) = f(X_{sn}) \oplus f(X_{sm}) = X_{sn} \oplus X_{sm} \in \Omega_{n+m}. \]
Since $X_{s(n+m)}$ is the only fixed point of $f$ in $\Omega_{n+m}$, one must have $X_{sn} \oplus X_{sm} = X_{s(n+m)}$, so that the set $\{X_s\}_{n \in \text{supp } \Omega}$ respects direct sums of matrices. By Lemma 13 there exists $X_s \in S^{d \times d}$ such that (14) holds.
2. If \( d \in \text{supp} \Omega \), then there is nothing to prove: we just set \( \tilde{\Omega} = \Omega, \tilde{f} = f \).

Let \( d \notin \text{supp} \Omega \). We define nc extensions of \( \Omega \) and \( f \) as follows. Set \( \Omega_d = \{ X_\ast \} \),

\[
\tilde{\Omega}_{kd} := \Omega_{kd} \cup \bigcup_{k', k'' : k' + k'' = k} (\tilde{\Omega}_{k'd} \oplus \tilde{\Omega}_{k''d}), \quad k = 2, 3, \ldots,
\]

where \( \Omega_{kd} = \emptyset \) if \( kd \notin \text{supp} \Omega \). By the construction, \( \tilde{\Omega} \) is an nc set. Notice that \( \tilde{\Omega} \) consists of matrices which are obtained as direct sums of matrices from \( \Omega \) and copies of \( X_\ast \) in every possible order, and that such direct sum decompositions are not necessarily unique. Let

\[
X = \bigoplus_{\alpha=1}^{m} X_\alpha, \quad Y = \bigoplus_{\beta=1}^{n} Y_\beta
\]

for some \( m, n \in \mathbb{N} \), \( X_\alpha \in \Omega_{j_\alpha} \), \( j_\alpha \in \text{supp} \Omega \), or \( X_\alpha = X_\ast \), and \( Y_\beta \in \Omega_{k_\beta} \), \( k_\beta \in \text{supp} \Omega \), or \( Y_\beta = X_\ast \). Define

\[
\tilde{f}(X) := \bigoplus_{\alpha=1}^{m} \tilde{f}(X_\alpha)
\]

where \( \tilde{f}(X_\alpha) := f(X_\alpha) \) if \( X_\alpha \in \Omega_{j_\alpha} \), and \( \tilde{f}(X_\alpha) := X_\ast \) if \( X_\alpha = X_\ast \). Define

\[
\tilde{f}(Y) := \bigoplus_{\beta=1}^{n} \tilde{f}(Y_\beta)
\]

where \( \tilde{f}(Y_\beta) := f(Y_\beta) \) if \( Y_\beta \in \Omega_{k_\beta} \), and \( \tilde{f}(Y_\beta) := X_\ast \) if \( Y_\beta = X_\ast \). We will show that \( \tilde{f} \) is correctly defined and is a nc function. Suppose we have \( SY = XS \) for some matrix

\[
S \in \mathcal{R}^{(j_1 + \cdots + j_m) \times (k_1 + \cdots + k_n) \times d}
\]

where we set \( j_\alpha = 1 \) if \( X_\alpha = X_\ast \), and \( k_\beta = 1 \) if \( Y_\beta = X_\ast \). We may view \( S \) as a \( m \times n \) block matrix with blocks \( S_{\alpha \beta} \in \mathcal{R}^{j_\alpha \times k_\beta \times d} \). Then

\[
(SY)_{\alpha \beta} = S_{\alpha \beta} Y_\beta = X_\ast S_{\alpha \beta} = (XS)_{\alpha \beta}.
\]

We have four cases:

Case 1. \( X_\alpha \in \Omega_{j_\alpha d} \), \( Y_\beta \in \Omega_{k_\beta d} \). Since \( f \) is a nc function, we have \( S_{\alpha \beta} f(Y_\beta) = f(X_\alpha) S_{\alpha \beta} \), i.e. \( S_{\alpha \beta} \tilde{f}(Y_\beta) = \tilde{f}(X_\alpha) S_{\alpha \beta} \).

Case 2. \( X_\alpha = X_\ast, Y_\beta \in \Omega_{k_\beta d} \). Then \( S_{\alpha \beta} Y_\beta = X_\ast S_{\alpha \beta} \) implies

\[
\begin{pmatrix}
S_{\alpha \beta} Y_\beta \\
\vdots \\
S_{\alpha \beta} Y_\beta
\end{pmatrix}
= \begin{pmatrix}
S_{\alpha \beta} \\
\vdots \\
S_{\alpha \beta}
\end{pmatrix}
Y_\beta
= \begin{pmatrix}
X_\ast \\
\vdots \\
X_\ast
\end{pmatrix}
\begin{pmatrix}
S_{\alpha \beta} \\
\vdots \\
S_{\alpha \beta}
\end{pmatrix}
= \begin{pmatrix}
X_\ast S_{\alpha \beta} \\
\vdots \\
X_\ast S_{\alpha \beta}
\end{pmatrix}
\]
for any \( k \) such that \( kd \in \text{supp} \Omega \). Then since

\[
\begin{bmatrix}
X_* \\
\vdots \\
X_*
\end{bmatrix} \in \Omega_{kd}
\]

is a fixed point of \( f|_{\Omega_{kd}} \), we have

\[
\begin{pmatrix}
S_{\alpha\beta} f(Y_\beta) \\
\vdots \\
S_{\alpha\beta} f(Y_\beta)
\end{pmatrix} =
\begin{pmatrix}
S_{\alpha\beta} \\
\vdots \\
S_{\alpha\beta}
\end{pmatrix}
\begin{pmatrix}
X_* \\
\vdots \\
X_*
\end{pmatrix}
\begin{pmatrix}
S_{\alpha\beta} \\
\vdots \\
S_{\alpha\beta}
\end{pmatrix}
= \begin{pmatrix}
X_* \\
\vdots \\
X_*
\end{pmatrix}
\begin{pmatrix}
S_{\alpha\beta} \\
\vdots \\
S_{\alpha\beta}
\end{pmatrix}
= \begin{pmatrix}
X_* S_{\alpha\beta} \\
\vdots \\
X_* S_{\alpha\beta}
\end{pmatrix}
\]

and \( S_{\alpha\beta} f(Y_\beta) = X_* S_{\alpha\beta} \), i.e. \( S_{\alpha\beta} \tilde{f}(Y_\beta) = \tilde{f}(X_\alpha) S_{\alpha\beta} \).

**Case 3.** \( X_\alpha \in \Omega_{j_\alpha d}, Y_\beta = X_* \). Then \( S_{\alpha\beta} X_* = X_* S_{\alpha\beta} \) implies

\[
\begin{pmatrix}
S_{\alpha\beta} X_* \ldots S_{\alpha\beta} X_* 
\end{pmatrix} =
\begin{pmatrix}
S_{\alpha\beta} \ldots S_{\alpha\beta}
\end{pmatrix}
\begin{pmatrix}
X_* \\
\vdots \\
X_*
\end{pmatrix}
= \begin{pmatrix}
X_* \ldots X_*
\end{pmatrix}
= \begin{pmatrix}
X_* \ldots X_*
\end{pmatrix}
\begin{pmatrix}
S_{\alpha\beta} \ldots S_{\alpha\beta}
\end{pmatrix}
= \begin{pmatrix}
X_* S_{\alpha\beta} \ldots X_* S_{\alpha\beta}
\end{pmatrix}
\]

for any \( k \) such that \( kd \in \text{supp} \Omega \). Then since

\[
\begin{bmatrix}
X_* \\
\vdots \\
X_*
\end{bmatrix} \in \Omega_{kd}
\]

is a fixed point of \( f|_{\Omega_{kd}} \), we have

\[
\begin{pmatrix}
S_{\alpha\beta} X_* \ldots S_{\alpha\beta} X_* 
\end{pmatrix} =
\begin{pmatrix}
S_{\alpha\beta} \ldots S_{\alpha\beta}
\end{pmatrix}
\begin{pmatrix}
X_* \\
\vdots \\
X_*
\end{pmatrix}
= \begin{pmatrix}
X_* \ldots X_*
\end{pmatrix}
\begin{pmatrix}
S_{\alpha\beta} \ldots S_{\alpha\beta}
\end{pmatrix}
= \begin{pmatrix}
X_* S_{\alpha\beta} \ldots X_* S_{\alpha\beta}
\end{pmatrix}
\]

and \( S_{\alpha\beta} X_* = f(X_\alpha) S_{\alpha\beta} \), i.e. \( S_{\alpha\beta} \tilde{f}(Y_\beta) = \tilde{f}(X_\alpha) S_{\alpha\beta} \).

**Case 4.** If \( X_\alpha = X_*, Y_\beta = X_* \), then \( S_{\alpha\beta} X_* = X_* S_{\alpha\beta} \) means

\[
S_{\alpha\beta} \tilde{f}(Y_\beta) = \tilde{f}(X_\alpha) S_{\alpha\beta}.
\]

Since we have in all these cases that

\[
S_{\alpha\beta} \tilde{f}(Y_\beta) = \tilde{f}(X_\alpha) S_{\alpha\beta}, \quad \alpha = 1, \ldots, m, \quad \beta = 1, \ldots, n,
\]

we obtain that \( S \tilde{f}(Y) = \tilde{f}(X) S \).

In the case where

\[
X = Y, \quad S = I_{(j_1 + \ldots + j_n)d} = I_{(k_1 + \ldots + k_n)d},
\]

...
we obtain that \( \tilde{f}(X) = \tilde{f}(Y) \), i.e., the definition of \( \tilde{f} \) is independent of the decomposition of a matrix from \( \tilde{\Omega} \) into a direct sum of matrices from \( \Omega \) and copies of \( X_* \). Thus, \( \tilde{f} \) is a correctly defined nc function extending \( f \) and, clearly, \( \tilde{f}|_{\tilde{\Omega}_n} \) has a unique fixed point

\[ X_{*n} = \bigoplus_{\alpha=1}^{n/d} X_* \, . \]  

\[ \square \]

**Proof of Theorem 8.** 1. By the classical Banach Fixed Point Theorem, a.k.a. the contractive mapping principle (see, e.g., [19, pp. 216–217] or [11, Chapter 3]), for every \( n \in \text{supp} \, \Omega \), there exists a unique \( X_{*n} \in \Omega_n \) which is a fixed point of \( f|_{\Omega_n} \). By Theorem 5, there exists \( X_* \in S^{d \times d} \) such that \( X_{*n} = \bigoplus_{\alpha=1}^{n/d} X_* \) for every \( n \in \text{supp} \, \Omega \).

2. Define \( \tilde{\Omega} \) and \( \tilde{f} \) as in the proof of part 2 of Theorem 5. Since for every \( n \in \mathbb{N} \), \( \tilde{\Omega}_{nd} = \Omega_{nd} \cup \bigcup_{n' + n'' = n} (\tilde{\Omega}_{n'd} \oplus \tilde{\Omega}_{n''d}) \) we obtain by induction that \( \tilde{\Omega}_{nd} \) is closed in \( V_{nd} \times V_{nd} \) (because the direct sum or the union of a finite number of closed sets is closed), and therefore is a complete metric space with respect to the metric \( \tilde{\rho}_{nd} \) induced by the norm \( \| \cdot \|_{nd} \) on the Banach space \( V_{nd} \times V_{nd} \). We note that a subspace of a complete metric space is closed if and only if it is complete itself. Next, since every \( X \in \tilde{\Omega} \) is a direct sum of matrices from \( \Omega \) and copies of \( X_* \), and \( \tilde{f} \) respects direct sums, we have (7) with

\[ \tilde{c}_n = \max_{k \in \text{supp} \, \Omega : k \leq n} c_k \, . \]

The estimate (8) is obtained from (7) by iteration, so that the convergence of \( X^j \) to \( X_{*n} \) follows. \( \square \)

**Proof of Theorem 14.** For every \( n \in \text{supp} \, \Omega \), the sets \( \Omega^j_n \), \( j = 1, \ldots, \), are nested and satisfy the conditions of the classical theorem on nested closed sets (see, e.g., [19, Page 195]), hence there exists a unique matrix \( X_{*n} \in \bigcap_{j=1}^{\infty} \Omega^j_n \). Since \( \Omega^j \) are nc sets, so is

\[ \bigcap_{j=1}^{\infty} \Omega^j = \bigcap_{j=1}^{\infty} \bigoplus_{n \in \text{supp} \, \Omega} \Omega^j_n = \bigcap_{n \in \text{supp} \, \Omega} \bigoplus_{j=1}^{\infty} \Omega^j_n = \{ X_{*n} \}_{n \in \text{supp} \, \Omega} \, . \]

By Lemma 13 there exists a unique \( X_* \in S^{d \times d} \) such that \( X_{*n} = \bigoplus_{\alpha=1}^{n/d} X_* \) for every \( n \in \text{supp} \, \Omega \), and the conclusion of the theorem follows. \( \square \)

**Proof of Theorem 17.** 1. This part follows by Lemma 13.

2. For every fixed \( n \in \text{supp} \, \Xi \), the initial value problem (12) can be reformulated as an integral equation

\[ X(t) = X_{0n} + \int_{t_0}^{t} g_n(s, X(s)) \, ds \, . \]

By the fundamental theorem of calculus, a continuous solution of (17) is a continuously differentiable solution of (12). Fix any \( \delta < 1/C \). For every \( n \in \text{supp} \, \Xi \), let \( \Omega_n \) be the metric space of continuous functions \( X_n: [t_0, t_0 + \delta] \to \Xi_n \) with \( X_n(t_0) = X_{0n} \) and with the norm-induced metric. Since \( \Xi_n \) is a closed subspace of the Banach space \( V^{m \times n} \), it is Banach itself. Since \( \Omega_n \) is a closed subset in the Banach space \( C([t_0, t_0 + \delta], \Xi_n) \), it is a complete metric space. The space
Observe that \( f \) and \( \tilde{\omega} \) are continuous functions. Clearly, \( f \) and \( \tilde{\omega} \) have a trivial fiber bundle, with the total space \( \Im(\tilde{\omega}) \) and \( \tilde{\omega} \) does for every \( t \in I \) and \( \tilde{\omega} \) extends \( f \) and has a unique fixed point, \( X_s = \bigoplus_{n=1}^{\infty} X_s \), in each \( \Omega_n \), where \( X_s \) is a continuously differentiable function on \([t_0,t_0 + \delta]\). Moreover, \( X_s = \bigoplus_{n=1}^{\infty} X_s \) for every \( n \in \mathbb{N} \), and by covering \( I \) with overlapping intervals of length \( \delta \), we can extend \( X_s \) to a continuously differentiable function on \( I \), so that \( X_s \) is a unique solution of \( (11) \).

3. If \( d \in \supp \Xi \), we just define \( \tilde{\Xi}(t) := \Xi, t \in I \), so that for every \( n \in \mathbb{N} d \) we have a trivial fiber bundle, with the total space \( I \times \Xi_n \). The initial value problem \( (14) \) is then identified with \( (12) \), and its solution \( (13) \) is identified with \( (11) \).

If \( d \notin \supp \Xi \), then applying the argument in our proof of part 2 of this theorem for the line segment \([t_0,t_0 + \delta]\), we can apply part 2 of Theorem 5 to the contractive mapping \( f : \Omega \to \Omega \) (which respects matrix similarities, as \( g(t, \cdot) \) does for every \( t \in I \)) and obtain a nc set \( \Omega \supseteq \hat{\Omega}, \) with supp \( \hat{\Omega} = \mathbb{N} d \), and a nc function \( \hat{f} : \Omega \to \hat{\Omega} \) which extends \( f \) and has a unique fixed point, \( X_{s_n} = \bigoplus_{n=d}^{\infty} X_s, \) in each \( \Omega_n \), \( n \in \mathbb{N} d \).

Using our construction \( \hat{\Omega} \) as in the proof of Theorem 5,

\[
\hat{\Omega}_d := \{ X_s \}, \quad \hat{\Omega}_{kd} := \Omega_{kd} \cup \bigcup_{k', k'' : k' + k'' = k} (\hat{\Omega}_{k'd} \oplus \hat{\Omega}_{k''d}), \quad k = 2, 3, \ldots,
\]

we obtain that for every \( n \in \mathbb{N} d \), the set \( \hat{\Omega}_n \) consists of continuous \( \mathbb{V}_{n + n} \)-valued functions on \([t_0,t_0 + \delta]\) that are equal to \( X_{0n} = \bigoplus_{n=1}^{d} X_0 \) at \( t_0 \); moreover, for any \( t \in [t_0,t_0 + \delta] \), the values of these functions at \( t \) lie in the set \( \hat{\Xi}(t)_n \) defined recursively by

\[
\hat{\Xi}(t)_d := \{ X_s(t) \}, \quad \hat{\Xi}(t)_{kd} := \Xi_{kd} \cup \bigcup_{k', k'' : k' + k'' = k} (\hat{\Xi}(t)_{k'd} \oplus \hat{\Xi}(t)_{k''d}), \quad k = 2, 3, \ldots.
\]

Clearly, \( \hat{\Xi}(t)_n \) is a complete metric space with respect to the norm-induced metric, and \( \hat{\Xi}(t) \) is a nc set which contains \( \Xi \), for every \( t \in [t_0,t_0 + \delta] \). Covering \( I \) by
the intervals of length $\delta$, we can extend this construction of $\Xi(t)$ to all $t \in \mathcal{I}$. Next, for every $t \in \mathcal{I}$, we define $\tilde{g}(t, \cdot): \tilde{\Xi}(t) \rightarrow \tilde{\Xi}(t)$ as follows. Given an arbitrary $X = \bigoplus_{\alpha=1}^{m} X_{\alpha} \in \tilde{\Xi}(t)$, where $X_{\alpha} \in \Xi$ or $X_{\alpha} = X_{*}(t)$ (as in the proof of part 2 of Theorem, every element of $\tilde{\Xi}(t)$ must be such a direct sum of matrices), we define $\tilde{g}(t, X) = \bigoplus_{\alpha=1}^{m} \tilde{g}(t, X_{\alpha})$,

where $\tilde{g}(t, X_{\alpha}) = g(t, X_{\alpha})$ when $X_{\alpha} \in \Xi$, or $\tilde{g}(t, X_{\alpha}) = X_{*}(t)$ when $X_{\alpha} = X_{*}(t)$. As in the proof of part 2 of Theorem, we can show that $\tilde{g}(t, \cdot)$ is independent of the representation of $X$ as a direct sum of matrices and that $\tilde{g}(t, \cdot)$ is a nc function. The remaining part of the proof is straightforward, and we leave it to the reader.  

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