Approximation bounds on maximum edge 2-coloring of dense graphs

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Abstract

For a graph $G$ and integer $q \geq 2$, an edge $q$-coloring of $G$ is an assignment of colors to edges of $G$, such that edges incident on a vertex span at most $q$ distinct colors. The maximum edge $q$-coloring problem seeks to maximize the number of colors in an edge $q$-coloring of a graph $G$. The problem has been studied in combinatorics in the context of anti-Ramsey numbers. Algorithmically, the problem is NP-Hard for $q \geq 2$ and assuming the unique games conjecture, it cannot be approximated in polynomial time to a factor less than $1 + 1/q$. The case $q = 2$, is particularly relevant in practice, and has been well studied from the viewpoint of approximation algorithms. A 2-factor algorithm is known for general graphs, and recently a $5/3$-factor approximation bound was shown for graphs with perfect matching. The algorithm (which we refer to as the matching based algorithm) is as follows: “Find a maximum matching $M$ of $G$. Give distinct colors to the edges of $M$. Let $C_1, C_2, \ldots, C_t$ be the connected components that results when $M$ is removed from $G$. To all edges of $C_i$ give the $(|M| + i)$th color.”

In this paper, we first show that the approximation guarantee of the matching based algorithm is $(1 + \frac{2}{\delta})$ for graphs with perfect matching and minimum degree $\delta$. For $\delta \geq 4$, this is better than the $\frac{\delta}{\delta - 1}$ approximation guarantee proved in [1]. For triangle free graphs with perfect matching, we prove that the approximation factor is $(1 + \frac{1}{\delta - 1})$, which is better than $5/3$ for $\delta \geq 3$.

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For the general case we show an approximation factor of \((1 + \epsilon)\), where 
\[ \epsilon = \frac{\kappa + 2 - 1}{\delta - 1}. \]
Here \(\delta\) is the minimum degree and \(\kappa = n/|M|\) is the ratio of the number of vertices to the cardinality of the maximum matching in \(G\). When \(\kappa < \delta - 3\), this is better than the previously known 2 factor for the general case. It may be noted that for a graph with maximum degree \(\Delta\) and minimum degree \(\delta\), \(\kappa \leq \frac{\Delta + 2}{\delta}\). We have the following immediate corollaries:

1. For \(d\)-regular graphs, ours is a \((1 + \frac{6}{d-1})\) factor approximation algorithm. This is better than the 2 factor known for the general case \([4, 5, 6]\) when \(d \geq 8\). (Note that all \(d\)-regular graphs need not have a perfect matching and thus 5/3 factor of \([1]\) is not applicable.)

2. For graphs of minimum degree \(h\sqrt{n} + 1\), we have an approximation factor of \((1 + \frac{6}{h^2})\). For \(h > \sqrt{6}\), this is better than the 2 factor known for the general case \([4, 5, 6]\).

We also show that when \(\delta > \frac{n}{\sqrt{2}}\), the matching based algorithm is an optimum algorithm for even \(n\), and an additive 1 factor algorithm for odd \(n\).

1 Introduction

For a graph \(G\) and an integer \(q \geq 2\), the edge \(q\)-coloring problem seeks to maximise the number of colors used to color the edges of \(G\), subject to the constraint that a vertex \(v \in V(G)\), is incident with edges of at most \(q\) different colors. We note that this problem differs from the classical edge coloring, where all edges incident at a vertex have distinct color. The problem has also been found useful in modelling channel assignment in networks equipped with multi-channel wireless interfaces \([8]\).

In a combinatorial setting, the number of colors used in maximum edge \(q\)-coloring relates to anti-Ramsey number. For graphs \(G\) and \(H\), the anti-Ramsey number \(ar(G, H)\) denotes the maximum number of colors \(k\), such that in an edge coloring of \(G\) with \(k\) colors, all subgraphs of \(G\) isomorphic to \(H\) have at least two edges of the same color. Then, it is seen that the number of colors in maximum edge \(q\)-coloring of \(G\) is \(ar(G, K_{1,q+1})\) where \(K_{1,q+1}\) is a star with \(q + 2\) vertices. Further details on anti-Ramsey numbers can be found in Erdős et. al \([3]\). Combinatorial bounds for edge \(q\)-coloring for several classes of graphs are obtained in \([7]\).

Recently, the maximum edge \(q\)-coloring problem has also been studied from an algorithmic perspective. Anna and Popa \([1]\), proved that the problem is NP-Hard for every \(q \geq 2\). Moreover, they proved that, assuming the unique games conjecture, it cannot be approximated within a factor less than \(1 + 1/q\) for every \(q \geq 2\) and assuming just \(P \neq NP\), it cannot be approximated within a factor less than \(1 + (q - 2)/(q - 1)^2\) for every \(q \geq 3\) \([2]\). A 2-factor approximation algorithm (cf. Algorithm \([1]\)) for the maximum edge 2-coloring for general graphs was given by Feng et. al. in \([4, 5, 6]\). The algorithm of Feng et. al. is simple and intuitive and is given below (Algorithm \([1]\)). Hereafter we refer to this algorithm as the matching based algorithm. The same authors also showed that the problem is polynomial time solvable for trees and complete graphs. Adamaszek and Popa \([1]\) showed that the matching based algorithm of Feng et. al. improves to a 5/3-factor algorithm for graphs with perfect matching.
Algorithm 1 Matching Based Algorithm for edge 2-coloring [1]

Let $G$ be the input graph.
Let $M$ be a maximum matching in $G$.
Assign distinct color to each edge of $M$.
Assign a new color to each component of $G\setminus M$.

1.1 Our Results:

Our contribution in this paper is to show that the approximation guarantee of the matching based algorithm (Algorithm [H]) improves significantly for graphs with large minimum degree. Specifically, we obtain the following results:

**Result 1:** For graphs with perfect matching, the matching based algorithm has an approximation guarantee of $(1 + \frac{2}{\delta})$ where $\delta$ is the minimum degree. Recall that Adamaszek and Popa [1] proved an approximation guarantee of $\frac{5}{3}$ for this case. The approximation guarantee proved by us is better than that of Adamaszek and Popa for $\delta \geq 4$, and equals to theirs for $\delta = 3$.

**Comment:** For graphs $G$ with $\delta \geq \lceil n/2 \rceil$, Algorithm [H] is almost optimal. In fact it is an additive 1 approximation algorithm. If we further assume that $n$ is even, for $\delta > \lceil n/2 \rceil$ the matching based algorithm is optimal. (This follows from result 1, noting that such graphs always have a perfect matching.)

**Result 2:** For triangle free graphs with perfect matching, we show a better approximation guarantee, namely $(1 + \frac{1}{\delta} - \frac{1}{\delta})$. This is better than that of [1] for $\delta \geq 3$.

**Result 3:** For the general case we show that the matching based algorithm is a $(1 + \frac{\kappa + 2 - \delta}{\delta} - \frac{1}{\delta})$-factor approximation algorithm. Here $\kappa = n/|M|$ where $n$ is the number of vertices in $G$ and $M$ is a maximum matching in $G$. Considering the attempts of Adamaszek and Popa to get a better approximation guarantee for graphs with perfect matching, it is natural to consider this ratio. This factor is better than the previous known 2 factor for the general case [4, 5, 6], when $\kappa < 2\sqrt{\frac{\delta}{n}} - 3$.

From the technical looking approximation factor $(1 + (\kappa + 2)/(\delta - 1))$, we can easily get the following corollaries for interesting special cases:

- **Corollary 1 of Result 3:** It is easy to see that $\kappa \leq \frac{2(\Delta + \delta)}{\delta}$ where $\Delta$ is the maximum degree of $G$. (Consider the set of vertices $V(M)$ spanned by the edges of $M$. Since due to maximality of $M$, all edges of $G$ should be adjacent to at least one vertex in $V(M)$, we get $|M|\Delta \geq (n - 2|M|)\delta$.) Therefore for $d$-regular graphs, the approximation guarantee is $(1 + \frac{d}{\delta - 1})$. This is better than the 2 factor known for the general case [4, 5, 6], when $k \geq 8$. (Note that all $d$-regular graphs need not have a perfect matching and thus 5/3 factor of [1] or the $(1 + 2/\delta)$ factor from our result 1, is not applicable.)

- **Corollary 2 of Result 3:** For graphs of minimum degree $h\sqrt{n} + 1$, we have an approximation factor of $(1 + \frac{h}{\delta})$. To see this note that $\kappa \leq \frac{\Delta + \delta}{\delta} \leq \frac{\delta}{2} + 1$. Now substituting $\Delta \leq n$ and $\delta \geq h\sqrt{n} + 1$, we see that the approximation factor is at most $(1 + \frac{h}{\delta})$. For $h > \sqrt{6}$, this is better than the 2 factor known for the general case [4, 5, 6].
2 Notation and Preliminaries

Throughout this paper, we consider connected graphs with minimum degree \( \delta \geq 3 \). An edge 2-coloring of a graph \( G \) with \( c \) colors is a map \( C : E(G) \rightarrow [c] \), such that a vertex is incident with edges of at most two colors. Let \( \text{ALG}(G) \) denote the number of colors in the coloring returned by Algorithm 1 for the graph \( G \), and let \( \text{OPT}(G) \) be the number of colors in the maximum edge 2-coloring of \( G \).

2.1 Characteristic subgraph

Let \( C \) be an edge 2-coloring of the graph \( G \). A subgraph of \( G \) containing exactly one edge of each color in \( C \) is called a characteristic subgraph of \( G \) with respect to coloring \( C \). Note that the maximum degree of a characteristic graph is at most two.

Lemma 2.1. For a graph \( G \) and an edge 2-coloring \( C \) of \( G \), there exists a characteristic graph \( \chi \) which is disjoint union of paths.

Proof. Let \( \chi \) be a characteristic graph with minimum number of cycles. For sake of contradiction, suppose \( \chi \) has a cycle. Let \( u \) be one of the vertices in the cycle, and \( v, w \) be its two neighbors in the cycle. Since \( \delta \geq 3 \), there is neighbor \( z \) of \( u \) in \( G \), which is not incident with an edge in \( \chi \). Now the edge \( uz \) must have the same color as \( uv \) or \( uw \), say \( uv \). Then \( \chi - uv + uz \) is a characteristic subgraph with smaller number of cycles. Hence, \( \chi \) did not have any cycles. □

In the remainder of the paper, we will tacitly assume that characteristic subgraphs do not contain cycles. The components of the characteristic subgraphs (which are paths) will be called characteristic paths.

3 The Case of Graphs with perfect matching

In this section, we derive some useful bounds for graphs with perfect matching. The proofs here also help to illustrate key ideas in simpler setting, which will be extended further in the proof of Theorem 4.1.

Theorem 3.1. Let \( G \) be an \( n \)-vertex graph with perfect matching, where \( \delta(G) \geq 3 \). Then, we have:

(i) \( \text{OPT}(G) \leq (1 + 2/\delta) \cdot \text{ALG}(G) \). Furthermore, if \( G \) is triangle-free then \( \text{OPT}(G) \leq (1 + 1/(\delta - 1)) \cdot \text{ALG}(G) \).

(ii) \( \text{OPT}(G) \leq \text{ALG}(G) + 1 \) for \( \delta \geq \lceil n/2 \rceil \).

The above theorem yields the approximation factor of 5/3 given by Anna and Popa in [1] for graphs with perfect matching (under added assumption of \( \delta \geq 3 \)). When graphs are also triangle-free, we get an improved approximation factor of 3/2.

Proof of Theorem 3.1. Let \( C \) be an optimal edge 2-coloring of \( G \), and let \( \chi \) be a characteristic subgraph of \( G \) with respect to coloring \( C \). Let \( N_i, i \in \{0, 1, 2\} \)
denote the vertices of $G$ with degree $i$ in $\chi$. Let $n_i = |N_i|$. Clearly $n_0 + n_1 + n_2 = n$. The number of colors $c$ in $C$ which is same as the number of edges in $\chi$ is given by $c = \frac{2n_2 + n_1}{2}$. For $v \in N_2$, let $N'(v)$ denote the neighbors of $v$ through edges not in $\chi$. Clearly $|N'(v)| \geq \delta - 2$ for all $v \in N_2$.

**Claim:** For $u, v \in N_2$, $v \notin N'(u)$. Let $C_u$ and $C_v$ be the colors incident at vertices $u$ and $v$ respectively. If $v \in N'(u)$ then the color of $uv$ belongs to $C_u \cap C_v$. By the definition of characteristic graph, if $uv$ is not an edge in $\chi$, then $C_u \cap C_v = \emptyset$. Thus edge $uv$ cannot get a color if $v \in N'(u)$. We infer that $v \notin N'(u)$.

From the above claim, it follows that $N'(v) \subseteq N_0 \cup N_1$ for $v \in N_2$. Consider the bipartite graph $H$ with bipartition $N_2 \cup (N_0 \cup N_1)$ and edge set of $H$, $E(H)$ given by $E(H) := \cup_{v \in N_2} E(v, N'(v))$. We show that $d_H(u) \leq 4$ for $u \in N_0$ and $d_H(u) \leq 2$ for $u \in N_1$. Assume $u \in N_0$. Now, there are at most two colors incident at $u$ in $C$, say colors $a$ and $b$. If $w$ is a neighbor of $u$ in $H$, we must have $a \in C_w$ or $b \in C_w$. Since there are at most two vertices $w \in N_2$ such that $a \in C_w$, it follows that $u$ has at most 4 neighbors in $H$. For $u \in N_1$, let $z$ be the unique neighbor of $u$ in $\chi$. Let $a$ be the color of edge $uz$. Notice that $uz \notin E(H)$ since it is an edge of $\chi$. Also $a \notin C_w$ for $w \in N_2 \{z\}$. Thus $u$ is not incident with $a$-colored edges in $H$. Thus edges incident on $u$ in $H$ must have the same color, and by the previous arguments, there can be at most two of them. Thus $d_H(u) \leq 2$ for $u \in N_1$. Counting the edges across the bipartition in two ways we have:

$$n_2(\delta - 2) \leq 4n_0 + 2n_1 \tag{1}$$

The result now follows from some algebra, as shown below.

$$c = \frac{2n_2 + n_1}{2} = \frac{(n_0 + n_1 + n_2) + (n_2 - n_0)}{2} = \frac{n + n_2 - n_0}{2} \tag{2}$$

From (1), using $n_0 + n_1 + n_2 = n$ we see that $n_2\delta \leq 2n + 2n_0$. Therefore we have $(n_2 - n_0)\delta \leq n_2\delta - 2n_0 \leq 2n$.

Substituting, we get

$$c \leq \frac{n}{2} \left(1 + \frac{2}{\delta}\right). \tag{3}$$

This proves the claimed approximation factor in part (i) of the theorem for graphs with perfect matching. If the graph is further assumed to be triangle-free, we prove that a vertex $u \in N_0 \cup N_1$ can have almost one edge of a given color incident on it in $H$. Let $u \in N_0 \cup N_1$ and let $a$ be one of the colors incident at $u$ in $C$. Then, the only possible $a$-colored edges incident at $u$ in $H$ are edges $uv$ where $w \in N_2$ with $a \in C_w$. Let $x, y \in N_2$ be such that $a \in C_x \cap C_y$. Then $xy$ is an edge in $\chi$. As $G$ is triangle-free we conclude at most one of $\{x, y\}$ is a neighbor of $u$. Thus $u$ is incident with at most one edge of color $a$ in $H$. Now it follows that, in this case, we have $d_H(u) \leq 2$ for $u \in N_0$ and $d_H(u) \leq 1$ for $u \in N_1$. The Equation (1) can now be written as:

$$n_2(\delta - 2) \leq 2n_0 + n_1. \tag{4}$$
It follows that \( n_2(\delta - 1) \leq n + n_0 \). Making similar substitutions in Equation (2), we get

\[
\frac{c}{n^2} \leq \frac{1}{\delta - 1} + 1.
\]

(5)

This proves the approximation factor claimed for triangle-free graphs in part (i) of the theorem.

To prove part (ii), observe that by Dirac’s theorem, a graph \( G \) with \( \delta \geq \lfloor n/2 \rfloor \) has a hamilton cycle, and hence a maximum matching of size \( \lfloor n/2 \rfloor \). Thus \( \text{ALG}(G) \geq \lfloor n/2 \rfloor + 1 \). Further from part (i), we have \( c \leq n/2 + n/\delta < \lfloor n/2 \rfloor + 3 \). Thus since \( c \) is an integer we get \( c \leq \lfloor n/2 \rfloor + 2 \leq \text{ALG}(G) + 1 \). Note that if we assume that \( n \) is even, and \( \delta > n/2 \), this proves that the algorithm is optimal.

4 Result for general graphs

In this section, we prove our main result, which is the following:

**Theorem 4.1.** Let \( G \) be an \( n \)-vertex connected graph with \( \delta(G) \geq 3 \). Let \( M \) be a maximum matching of \( G \). Then \( \text{OPT}(G) \leq (1 + \varepsilon) \cdot \text{ALG}(G) \) where \( \varepsilon = \frac{\kappa + 2}{\delta - 2} \), with \( \kappa = n/|M| \) being the ratio of number of vertices to the size of maximum matching of the graph.

**Proof.** Let \( C \) be an optimal coloring of \( G \) using \( c \) colors. Let \( \chi \) be a characteristic subgraph of \( G \) with respect to coloring \( C \), with the maximum number of characteristic paths. We will say a vertex \( v \) is an internal vertex of \( \chi \), if it is an internal vertex of one of the characteristic paths. Similarly, a vertex will be called a terminal vertex of \( \chi \) if it is a terminal vertex of one of the characteristic paths. In the proof of Theorem 3.1, we bound the number of edges in the characteristic subgraph (which is same as the number of colors in the coloring) by \( n/2 + (n_2 - n_0)/2 \). In the case of graphs with perfect matching, \( n/2 + 1 \) is a lower bound on the number of colors returned by the Algorithm 1, and thus intuitively, the term \( (n_2 - n_0)/2 \) was the “excess”. We tried to upper bound this excess in the previous proof. However in the general case, \( n/2 \) could be a gross over-estimate of the maximum matching, so the previous strategy does not work. Instead, we consider the following excess: We pick a matching \( M' \) from within \( \chi \) by selecting alternate edges in each characteristic path, starting with the first edge in each path. Let \( t \) be the number of unselected edges. Then \( c = |M'| + t \leq |M| + t \).

The remainder of the proof attempts to upper bound the excess term \( t \). In fact we show that \( t \leq |M| \cdot ((\kappa + 2)/(\delta - 1)) \) from which the theorem immediately follows. Let \( T \) be the set of vertices consisting of left endpoint of each unselected edge (see Figure 4). Thus we have \( |T| = t \). First we note that \( T \) is an independent set in \( G \). This is because vertices in \( T \) are mutually non-adjacent internal vertices of \( \chi \), and hence have mutually disjoint incident colors. For each \( v \in T \), choose a set of \( \delta - 2 \) edges incident at \( v \), which are not present in \( \chi \) (this is possible, as each vertex has at most two neighbors in \( \chi \)). Let us call these edges as special edges. Let \( H_0, H_1 \) and \( H_2 \) be sets of vertices of \( V \setminus T \) which are incident with 0, 1 and 2 of these special edges. Let \( h_i = |H_i| \). Since, the vertices in \( T \) are...
incident with mutually disjoint sets of colors, a vertex in $V \setminus T$ is incident with at most two special edges emanating from $T$. Thus $V \setminus T = H_0 \uplus H_1 \uplus H_2$, or $h_0 + h_1 + h_2 = n - t$. Counting the special edges across the bipartition $(T, V \setminus T)$ we have:

$$t(\delta - 2) = 2h_2 + h_1$$

Moreover, as $h_0 + h_1 + h_2 = n - t$, we can rewrite the above equality as $t(\delta - 2) = n - t + (h_2 - h_0)$, and thus,

$$t(\delta - 1) \leq n + h_2.$$ 

Figure 1: Characteristic paths are shown on the left. The unselected edges are marked with $\times$, and corresponding vertices in $T$ are indicated by a box. The special edges are indicated with dashed lines. The set $H_2$ is indicated by the ellipse.

To obtain a bound on $h_2$, we consider the neighbors of vertices in $H_2$ which are not in $T$. Note that each vertex $v \in H_2$ has at least $\delta - 2$ neighbors outside $T$. Let $A$ be the set of terminal vertices of the characteristic subgraph $\chi$ and $B$ be the set of internal vertices of $\chi$, which are not in $T$. Let $O$ denote the set of vertices which are not incident with an edge in $\chi$. Note that $H_2 \subseteq O$. This is because a vertex incident with an edge in $\chi$ can only receive special edges of at most one more color. But then it has at most one neighbor in $T$, and hence it is not a vertex in $H_2$. We show the following:

Claim (a): Let $a$ be a color present in a characteristic path of length at least two in $\chi$. Then there is no $a$-colored edge between two vertices of $O$. We prove by contradiction. Let $uw$ be an edge in a characteristic path of length at least two, and let $a$ be color of $uv$. Assume that $u$ is not a terminal vertex of $\chi$. If possible let $x$ and $y$ be two vertices in $O$ with edge $xy$ having color $a$. Then $\chi - uw + xy$ is a characteristic subgraph with more characteristic paths than $\chi$, a contradiction. The claim follows.

Claim (b): Let $a$ be a color present in a characteristic path of length at least two in $\chi$. Then there is no $a$-colored triangle formed by the $a$-colored edge in $\chi$ and a vertex from $O$. To prove, again assume that $uw$ is an $a$-colored edge in $\chi$ where $u$ is not a terminal vertex. If possible let $uw$ be an $a$-colored triangle with $w \in O$. Again, $\chi - uw + vw$ is a characteristic subgraph with more characteristic paths than $\chi$, contradicting the choice of $\chi$. The claim follows.

Claim (c): The neighbors of vertices in $H_2$, which are not in $T$, lie in $A$. To prove, we observe that the colors incident on vertices in $H_2$ appear in a characteristic path of length at least two (as vertices in $T$ are on such paths). Then
as $H_2 \subseteq O$, from Claim (a), we have that a vertex $v \in H_2$ is not incident with a vertex in $O$. Further, $v$ is not incident with a vertex in $B$, as the only internal vertices in $\chi$ that are incident with colors at $v$ are its neighbors in $T$. Hence, all the neighbors of $v$, not in $T$ are in $A$.

Consider the bipartite graph $X$ with bipartition $(H_2, A)$ with edge set consisting of $A$-$H_2$ edges in $G$. Clearly $d_X(v) \geq \delta - 2$ for $v \in H_2$. We now prove that $d_X(u) \leq \delta - 2$ for $u \in A$. Let $u \in A$, and let $uz$ be the edge of $\chi$ incident at $u$, where $a$ is the color of $uz$. Let $v$ be a neighbor of $u$ in $H_2$. We show that $uv$ is not of color $a$. For sake of contradiction suppose that color of $uv$ is $b$. Then as $v \in H_2$, it has a neighbor $w$ in $T$ which is incident with edge of color $a$. But then $w = z$ as $z$ is the only internal vertex in $\chi$ incident with edge of color $a$. Now $uwz$ form an $a$-colored triangle, which contradicts Claim (b). Thus, all the edges of $X$ incident at $v$ must have the color different from $a$, and hence must have the same color (say $b$). Clearly all the vertices in $H_2$ incident with edge of color $b$, must be incident with a $b$-colored special edge from a vertex in $T$. As vertices in $T$ are incident with mutually disjoint colors, we infer that all such vertices are incident with $b$-colored special edges from a single vertex $w \in T$ (see Figure 2). Since there are exactly $\delta - 2$ special edges emanating from a vertex in $T$, we conclude there are at most $\delta - 2$ vertices in $H_2$ incident with color $b$. Hence $d_X(u) \leq \delta - 2$ for $u \in A$. Now, we have $h_2(\delta - 2) \leq |E(H_2, A)| \leq (\delta - 2) \cdot |A|$. Finally, observe that $|A| \leq 2|M|$ and hence $h_2 \leq 2|M|$. Substituting in Equation (7), we have:

$$t(\delta - 1) \leq n + 2|M|$$

$$t \leq |M| \cdot \left(\frac{\kappa + 2}{\delta - 1}\right).$$

where $\kappa = n/|M|$. The claimed approximation now follows from the inequality $c \leq |M| + t$.  

\[\square\]

5 Tightness of Result 3

In this section we give a construction to show that the approximation factor in Theorem 4.1 is tight.

**Theorem 5.1.** Given $\kappa$ and $\delta$, where $\kappa < \delta - 3$, and a sufficiently large $t$ with respect to $\delta$ (say $t > \frac{(\delta - 1)^2}{\kappa}$), there exists a graph with number of vertices $n = \kappa t$, and minimum degree $\delta$ such that it is possible to get a 2-edge coloring for this graph that uses at least $t(1 + \frac{\kappa^2}{\delta - 1})$ colors.

**Proof.** It is enough to construct a graph with the cardinality of maximum matching equals to $t$, $n = \kappa t$ minimum degree $\delta$ and demonstrate a 2-edge coloring for this graph that uses at least $t(1 + \frac{\kappa^2}{\delta - 1})$ colors. Construct a bipartite graph $G$ with set of vertices $S$ on one side and $T$ on the other side (see Figure 3 for schematic representation). Let $S$ be the disjoint union of two sets of vertices $S_1$ and $S_2$. Also let $T$ be the disjoint union of two sets of vertices $A$ and $B$. Let $|A| = |S| = t$, and $|S_1| = \delta$.

In $G$, we will first add a matching between $A$ and $S$. In our coloring of $G$, the edges of this matching will get distinct colors, say from $1$ to $t$. All the
vertices in $T = A \cup B$ are made adjacent to all the vertices in $S_1$, so that all the vertices in $T \cup S_1$ have degree at least $\delta$. These edges between $T$ and $S_1$ other than the matching edges will be given the color $t + 1$.

Let $t$ be chosen such that $h = t(\delta + 1 - \kappa)/(\delta - 1) - (\delta - 1)$, is a positive integer. (We can assume that $t > (\delta + 1 - \kappa)/(\delta - 1)$, so that $h > 0$.) Let $S_2 = u_1, u_2, \ldots, u_{t-\delta}$, and let $B = \{v_1, v_2, \ldots, v_\alpha\}$, where $\alpha = (t - \delta - h + 1)(\delta - 1)$. We make each $u_i, 1 \leq i \leq h$ adjacent to $\{v_1, v_2, \ldots, v_\delta\}$, so that the degree of $u_i$ is $\delta$, for $1 \leq i \leq h$. All the edges from $u_i$ to $v_j, 1 \leq i \leq h$ and $1 \leq j \leq \delta - 1$ is given the same color, say $t + 2$.

For $h < i \leq t - \delta$, make $u_i$ adjacent to $v_{(i-h)(\delta-1)+1}, \ldots, v_{(i-h+1)(\delta-1)}$. Thus each of these vertices $u_i$, $h < i \leq t - \delta$ gets an exclusive neighborhood of $\delta - 1$ vertices each in $B$. All the edges from $u_{h+j}, 1 \leq j \leq t - h - \delta$ to the vertices of $B$, will be given the color $t + 2 + j$.

The total number of colors used is clearly $2t - h - \delta + 2$. Note that by this construction the degree of each $u_i$ is $\delta$. Thus the minimum degree of $G$ is $\delta$. Moreover the number of vertices in $G$ is $2t + \alpha = 2t + (t - \delta - h + 1)(\delta - 1) = t(\delta + 1) - h(\delta - 1) - (\delta - 1)^2$. Since $G$ is a bipartite graph with $t$ vertices on one side, the cardinality of the maximum matching is $t$. It is easy to verify that $n/t = \kappa$, by substituting $h = t(\delta + 1 - \kappa)/(\delta - 1) - (\delta - 1)$ in the expression for $n$.

The number of colors used is $2t - (\delta - 2) - h \geq 2t - t\frac{\delta+1-k}{\delta-1} = t(1 + \frac{k-\delta}{\delta-1})$, as required.
6 Tight example for Result 1

Let $G$ be any $d$-regular graph on $n$ vertices such that it can be properly edge colored using $d$ colors. (Clearly such graphs exist, for example all $d$-regular bipartite graphs are $d$-edge colorable). Properly edge color $G$ using colors 1 to $d$. We then construct a new graph $G'$ from $G$ as follows: Replace each vertex $v$ of $G$ by a clique $K_v$ of $d$ vertices $v_1, \ldots, v_d$. Thus $G'$ has $n'$ vertices. Note that $n'$ is an even number.

Add an edge $u_i v_i$ in $G'$ if $uv \in E(G)$ and $uv$ is colored with color $i$ in the edge coloring of $G$. Clearly, $G'$ is $d$-regular and has a perfect matching: The set of all edges $M$ of $G'$ which are not part of any clique $K_v, v \in V(G)$ clearly form a perfect matching of $G'$. More precisely, $M = \bigcup_{1 \leq i \leq d} \{u_i v_i \in E(G') : uv \in E(G) \text{ and } uv \text{ is colored } i \in G\}$. But it is easy to see that removing all the $\frac{n'}{2}$ edges of the perfect matching $M$ of $G'$ leaves $\frac{n'}{2} = n$ connected components, namely $\{K_v : v \in V(G)\}$. Coloring the $\frac{n'}{2}$ edges in the matching $M$ with $\frac{n'}{2}$ distinct colors and coloring the edges of each of the $\frac{n'}{2}$ components with a new color yields a 2-coloring using $\frac{n'}{2} + \frac{n'}{2} = \frac{n'}{2}(1 + 2/d)$ colors.

On the other hand, it is easy to see that $M$ is not the only perfect matching available in $G'$. Suppose $d$ is even. Then another simple way to get a perfect matching of $G'$ is as follows: From each clique $K_v, v \in V(G)$ pick a matching of size $d/2$. The union of all these matchings clearly is a perfect matching of $G'$. Let us name this perfect matching as $M_1$. Note that the matching based algorithm picks up an arbitrary perfect matching, colors its edges with distinct colors and then gives new colors to the connected component that results when that perfect matching is removed from the graph: one new color per component. Suppose the matching based algorithm picks up $M_1$ instead of $M$ to start with. It is obvious that if $M_1$ is removed from $G'$, the resulting graph has only one connected component. Therefore the matching based algorithm (Algorithm 1) yields a 2-coloring of size $\frac{n'}{2} + 1$, where $n'$ is the number of vertices of $G'$. (To
deal with the case when $d$ is odd, we can assume that the graph $G$ was a $d$-regular bipartite graph such that there exists a perfect matching $F$ in it such that $G \setminus F$ is still connected. When properly edge coloring $G$ we can make sure that $F$ forms the set of edges colored $d$. Now to get $M_1$ we pick a $\frac{d-1}{2}$ sized matching of the first $d-1$ vertices from each clique $K_v$, and the set of edges $\{u_d v_d : uv \in F\}$. Clearly $G' \setminus M_1$ remains connected.)

Thus for $G'$, $OPT(G')/ALG(G') = \left(1 + \frac{d}{2} - \frac{1+\frac{d}{2}}{n}\right)$ which is very close to $(1 + \frac{2}{d})$ for large $n$.

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