Abstract—Compound matrices have found applications in many fields of science including systems and control theory. In particular, a sufficient condition for $k$-contraction is that a logarithmic norm (also called matrix measure) of the $k$-additive compound of the Jacobian is uniformly negative. However, this computation may be difficult to perform analytically and expensive numerically because the $k$-additive compound of an $n \times n$ matrix has dimensions $\binom{n+k}{k} \times \binom{n+k}{k}$. This article establishes a duality relation between the $k$ and $(n-k)$ compounds of an $n \times n$ matrix $\Lambda$. This duality relation is used to derive a sufficient condition for $k$-contraction that does not require the computation of any $k$-compounds. These theoretical results are demonstrated by deriving a sufficient condition for $k$-contraction of an $n$-dimensional Hopfield network that does not require to compute any compounds. In particular, for $k = 2$ this sufficient condition implies that the network is 2-contracting and thus admits a strong asymptotic property: every bounded solution of the network converges to an equilibrium point, that may not be unique. This is relevant, for example, when using the Hopfield network as an associative memory that stores patterns as equilibrium points of the dynamics.

Index Terms—Contracting systems, Hopfield networks, $k$-shifted logarithmic norm, matrix measure, stability.

I. INTRODUCTION

A NONLINEAR dynamical system is called contracting if any two solutions approach one another at an exponential rate [8]. This implies many useful asymptotic properties that resemble those of asymptotically stable linear systems. For example, if the vector field is time-varying and $T$-periodic and the state-space is convex and bounded, then the system admits a unique $T$-periodic solution that is globally exponentially stable [1], [31], [45]. If the periodicity of the vector field represents a $T$-periodic excitation, then this implies that the system entrains to the excitation. In fact, contracting systems have a well-defined frequency response, as shown in [41] in the closely related context of convergent systems [42]. For a sensitivity analysis of this frequency response, see [23].

These properties are important in many applications ranging from the synchronized response of biological processes, to periodic excitations like the cell cycle [32], [45] or the 24-h solar day, to the entrainment of synchronous generators to the frequency of the electric grid.

In particular, if the vector field of a contracting system is time-invariant, then the system admits a globally exponentially stable equilibrium point. Contractivity implies many other useful properties, e.g., a contractive system is input-to-state stable [10], [48].

An important advantage of contraction theory is that there exists a simple sufficient condition for contraction, namely, that a logarithmic norm (also called matrix measure) of the Jacobian of the vector field is uniformly negative. For the $L_1, L_2,$ and $L_\infty$ norms, this sufficient condition is easy to check, and, in particular, does not require explicit knowledge of the solutions of the system.

Contraction theory has found numerous applications in robotics [50], synchronization in multiagent systems [52], the design of observers and closed-loop controllers [2], [16], neural networks and learning theory [54], and more.

However, many systems cannot be studied using contraction theory. For example, if the dynamics admits more than a single equilibrium, then the system is clearly not contracting. Existence of more than a single equilibrium is prevalent in many important mathematical models and real-world systems. Ecological models that include several equilibrium points allow switching between several possible behaviors, e.g., outbreaks [21]. Epidemiological models typically include at least two equilibrium points corresponding to the disease-free steady state and the endemic steady state [24]. Multistability in biochemical and cellular systems allows to transform graded signals into an all-or-nothing response and to “remember” transitory stimuli [5], [27]. Other important examples of systems that are not contractive include systems that are almost globally stable, systems with trajectories contracting at a rate slower than exponential, and more (e.g., see [4]).
There is considerable interest in extending contraction theory to systems that are not contractive in the usual sense; see, e.g., [20] and [34]. Motivated by the seminal work of Muldowney [37], Wu et al. [55] recently introduced the notion of \( k \)-contractive systems. Roughly speaking, the solutions of these systems contract \( k \)-dimensional parallelotopes. For \( k = 1 \), this reduces to standard contraction. However, \( k \)-contraction with \( k > 1 \) can be used to analyze the asymptotic behavior of systems that are not contractive. For example, every bounded solution of a time-invariant \( 2 \)-contractive system converges to an equilibrium that is not necessarily unique [29].

Wu et al. [57] introduced the more general notion of \( \alpha \)-contracting systems, where \( \alpha \) is a real number. Roughly speaking, the dynamics of such systems contracts any set with Hausdorff dimension larger or equal to \( \alpha \). In particular, any attractor of such a system has Hausdorff dimension smaller than \( \alpha \).

A sufficient condition for \( k \)-contraction is that for any \( \alpha \), \( Q \in \mathbb{R}^{n \times n} \) implies that \( Q \) has all the \( \alpha \)-eigenvalues in the open unit disk.

The main contributions of this article include the following. 1) A derivation of new duality relations between \( k \) and \((k-1)\)-contractive systems that are contractive. 2) A derivation of new duality relations between logarithmic norms of additive compounds. 3) Using these duality relations to derive a new sufficient condition for \( k \)-contraction that does not require computing any compounds. This alleviates the computational difficulties in establishing \( k \)-contraction.

We demonstrate our theoretical results by deriving a sufficient condition for \( k \)-contraction in an \( n \)-dimensional Hopfield neural network. This system typically admits more than a single equilibrium and is, thus, not contractive (i.e., not 1-contractive) with respect to (w.r.t.) any norm. Our condition does not require to compute any compounds. For \( k = 2 \), this condition implies a strong asymptotic property: Any bounded solution of the network converges to an equilibrium.

It is worth noting the following: 1) Multistability enables the use of Hopfield neural networks as associative memory models in neuroscience [15, Ch. 7]; 2) The classic stability analysis of these models requires symmetry of the synaptic matrix (in order for a certain energy function to be well defined and decreasing along the network solutions); 3) our approach does not require symmetry of the synaptic matrix.

The rest of this article is organized as follows. The next section reviews known definitions and results that are used later on. Section III derives the duality relations between compounds. Section IV shows how these duality relations can be used to prove \( k \)-contraction without computing any compounds. In Section V, we demonstrate the usefulness of the theoretical results by deriving a sufficient condition for \( k \)-contraction in Hopfield neural networks. Finally, Section VI concludes the article. We use standard notation. Vectors [matrices] are denoted by small [capital] letters. For a matrix \( A, a_{ij} \) is entry \((i,j)\) of \( A \), and \( A^T \) is the transpose of \( A \). For a square matrix \( A \), \( \text{tr}(A) [\text{det}(A)] \) is the trace [determinant] of \( A \). For a symmetric matrix \( Q \in \mathbb{R}^{n \times n} \), we use \( Q > 0 \) \((Q \succeq 0)\) to denote that \( Q \) is positive-definite [positive semidefinite]. A square matrix is called antidiagonal if all its entries are zero except those on the diagonal going from the lower left corner to the upper right corner, known as the antidiagonal.

### II. Preliminaries

This section reviews several known definitions and results that are used later on. Let \( Q(k,n) \) denote all the \( (\alpha,k) \) increasing sequences of \( k \) integers from the set \( \{1,\ldots,n\} \), ordered lexicographically, i.e., for \( \alpha, \beta \in Q(k,n) \), we have that \( \alpha \) precedes \( \beta \) if there exists an index \( \ell \) such that: \( \alpha_i = \beta_i \) for \( i \in \{1,\ldots,\ell\} \) and \( \alpha_{\ell+1} < \beta_{\ell+1} \). For example
\[
Q(3,4) = \{(1,2,3), (1,2,4), (1,3,4), (2,3,4)\}.
\]

Let \( A \in \mathbb{C}^{n \times m} \). Fix \( k \in \{1,\ldots,\min(n,m)\} \). For two sequences \( \alpha \in Q(k,n), \beta \in Q(k,m) \), let \( A[\alpha|\beta] \) denote the \( k \times k \) submatrix obtained by taking the entries of \( A \) in the rows indexed by \( \alpha \) and the columns indexed by \( \beta \). For example
\[
A[(2,4)|(1,2)] = \begin{bmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{bmatrix}.
\]

The minor of \( A \) corresponding to \( \alpha, \beta \) is
\[
A[\alpha|\beta] := \text{det}(A[\alpha|\beta]).
\]

For example, if \( m = n \) then \( Q(n,n) \) includes the single element \( \alpha = (1,\ldots,n) \), so \( A[\alpha|\alpha] = A \), and \( A[\alpha|\alpha] = \text{det}(A) \).

### A. Compound Matrices

The \( k \)-multiplicative compound of a matrix \( A \) is a matrix that collects all the \( k \)-minors of \( A \).

**Definition 1:** Let \( A \in \mathbb{C}^{n \times m} \) and fix \( k \in \{1,\ldots,\min(n,m)\} \). The \( k \)-multiplicative compound of \( A \), denoted \( A^{(k)} \), is the \( (n \times m) \) matrix that contains all the \( k \)-minors of \( A \) ordered lexicographically.

For example, if \( n = m = 3 \) and \( k = 2 \) then
\[
A^{(2)} = \begin{bmatrix} A((1,2)|(1,2)) & A((1,2)|(1,3)) & A((1,2)|(2,3)) \\ A((1,3)|(1,2)) & A((1,3)|(1,3)) & A((1,3)|(2,3)) \\ A((2,3)|(1,2)) & A((2,3)|(1,3)) & A((2,3)|(2,3)) \end{bmatrix}.
\]

In particular, Definition 1 implies that \( A^{(1)} = A \), and if \( n = m \) then \( A^{(n)} = \text{det}(A) \). Note also that by definition \( (A^T)^{(k)} = (A^{(k)})^T \). In particular, if \( A \) is symmetric then \( (A^{(k)})^T = (A^{(k)}) \), so \( A^{(k)} \) is also symmetric.

The term multiplicative compound is justified by the following important result.

**Theorem 1 (Cauchy-Binet Theorem):** Let \( A \in \mathbb{C}^{n \times m} \) and \( B \in \mathbb{C}^{m \times p} \). Then, for any \( k \in \{1,2,\ldots,\min\{n,m,p\}\} \), we have
\[
(AB)^{(k)} = A^{(k)}B^{(k)}.
\]
Note that for \( m = p = k = n \), this reduces to \( \det(AB) = \det(A) \det(B) \).

Theorem 1 implies in particular that if \( A \) is square and invertible, then \( A^{(k)} \) is also invertible, and \( (A^{(k)})^{-1} = (A^{-1})^{(k)} \).

If \( A = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \), then Definition 1 implies that \( A^{(2)} = \text{diag}(\lambda_1 \lambda_2, \lambda_1 \lambda_3, \lambda_2 \lambda_3) \), so every eigenvalue of \( A^{(2)} \) is a product of two eigenvalues of \( A \). More generally, the multiplicative compound has a useful spectral property. Let \( \lambda_1, \ldots, \lambda_n \) denote the eigenvalues of \( A \in \mathbb{C}^{n \times n} \). Then, the \( \binom{n}{k} \) eigenvalues of \( A^{(k)} \) are all the products of \( k \) eigenvalues of \( A \), i.e.,

\[
\prod_{i=1}^{k} \lambda_{a_i}, \text{ for all } \alpha \in Q(k, n). \tag{1}
\]

A similar property also holds for the singular values of \( A^{(k)} \). Equation (1) implies the Sylvester–Franke identity

\[
\det(A^{(k)}) = (\det(A))^{(k-1)} \tag{2}
\]

(see, e.g., [13]).

The \( k \)-multiplicative compound can be used to study the evolution of the volume of \( k \)-dimensional parallelepipeds under a differential equation. The parallelepiped with vertices \( x^1, \ldots, x^k \in \mathbb{R}^n \) (and the zero vertex) is

\[
P(x^1, \ldots, x^k) := \left\{ \sum_{i=1}^{k} c_i x^i : c_i \in [0, 1] \right\}.
\]

Let \( X := \left[ x^1 \quad \ldots \quad x^k \right]^{(k)} \). Note that \( X \) has dimensions \( \binom{n}{k} \times 1 \), i.e., it is a column vector. It is well known (see, e.g., [6]) that the volume of \( P(x^1, \ldots, x^k) \) satisfies

\[
\text{volume}(P(x^1, \ldots, x^n)) = \left| \left[ x^1 \quad \ldots \quad x^k \right]^{(k)} \right|_2 \tag{3}
\]

where \( \cdot \mid_2 \) denotes the \( L_2 \) norm. For example, if \( n = 3 \) and \( x^i = a_i e^i \), \( i = 1, 2 \), where \( e^i \) is the \( i \)th canonical vector in \( \mathbb{R}^3 \), then \( P(x^1, x^2) \) is the parallelepiped with vertices \( 0 0 0, [a_1 \ 0 \ 0]^T, [0 \ a_2 \ 0]^T, [0 \ 0 \ a_3]^T \), and (3) gives

\[
\text{volume}(P(x^1, x^2)) = \left| \begin{array}{cc} a_1 e_1 & a_2 e_2 \\ 0 & 0 \end{array} \right|_2^{(2)} = \left| \begin{array}{cc} a_1 a_2 \ 0 \ 0 \end{array} \right|_2 = |a_1 a_2|.
\]

For the particular case \( k = n \), (3) becomes the well-known formula

\[
\text{volume}(P(x^1, \ldots, x^n)) = |\text{det}(x^1, \ldots, x^n)|.
\]

To study the time evolution of such \( k \)-volumes under the action of a differential equation requires the \( k \)-additive compound of a square matrix.

**Definition 2:** Let \( A \in \mathbb{C}^{n \times n} \). The \( k \)-additive compound matrix of \( A \) is the \( \binom{n}{k} \times \binom{n}{k} \) matrix defined by

\[
A^{[k]} := \frac{d}{d\varepsilon} (I_n + \varepsilon A)^{(k)} |_{\varepsilon=0}. \tag{4}
\]

The derivative here is well defined, as every entry of \( (I_n + \varepsilon A)^{(k)} \) is a polynomial in \( \varepsilon \). Note that this definition implies that

\[
A^{[k]} = \frac{d}{d\varepsilon} (\exp(A\varepsilon))^{(k)} |_{\varepsilon=0} \tag{5}
\]

and also the Taylor series expansion

\[
(I_n + \varepsilon A)^{(k)} = I_r + \varepsilon A^{[k]} + o(\varepsilon) \tag{6}
\]

where \( r := \binom{n}{k} \).

For example, if \( A = \text{diag}(a_1, a_2, a_3) \), then \( \exp(A\varepsilon) = \text{diag}(\exp(\varepsilon a_1), \exp(\varepsilon a_2), \exp(\varepsilon a_3)) \), so \( \exp(A\varepsilon) \) is a product of \( \binom{n}{k} \) eigenvalues of \( A \). Hence, the \( \binom{n}{k} \) eigenvalues of \( A^{[k]} \) are

\[
\sum_{i=1}^{k} \lambda_{a_i}, \text{ for all } \alpha \in Q(k, n). \tag{7}
\]

Also, (2) implies that

\[
\text{tr}(A^{[k]}) = \binom{n-1}{k-1} \text{tr}(A). \tag{8}
\]

The next result provides a useful explicit formula for \( A^{[k]} \) in terms of the entries \( a_{ij} \) of \( A \). Recall that any entry of \( A^{[k]} \) is a minor \( A(a(i,j)) \). Thus, it is natural to index the entries of \( A^{(k)} \) and \( A^{[k]} \) using \( \alpha, \beta \in Q(k, n) \).

**Proposition 2:** [46] Fix \( \alpha, \beta \in Q(k, n) \) and let \( \alpha = \{i_1, \ldots, i_k\} \) and \( \beta = \{j_1, \ldots, j_k\} \). Then, the entry of \( A^{[k]} \) corresponding to \( (\alpha, \beta) \) is equal to the following.

1. \( \sum_{\ell=1}^{k} a_{i_\ell j_\ell} \), if \( i_\ell = j_\ell \) for all \( \ell \in \{1, \ldots, k\} \).
2. \((-1)^{k+\ell} a_{i_\ell j_\ell} \), if all the indices in \( \alpha \) and \( \beta \) agree, except for a single index \( i_\ell \neq j_\ell \).
3. 0, otherwise.

Note that the first case in the proposition corresponds to the diagonal entries of \( A^{[k]} \). Also, the proposition implies in particular that \( A^{[1]} = A \), and \( A^{[2]} = \sum_{\ell=1}^{n} a_{\ell \ell} = \text{tr}(A) \).

**Example 1:** For \( A \in \mathbb{R}^{3 \times 3} \) and \( k = 2 \), Proposition 2 yields

\[
A^{[2]} = \begin{pmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{pmatrix}.
\]

where the indexes \( \alpha \in Q(2,3) \) \( \beta \in Q(2,3) \) are marked on right-hand [top] side of the matrix. For example, the entry in the second row and third column of \( A^{[3]} \) corresponds to \( (\alpha, \beta) = ((1,3),(2,3)) \). As \( \alpha \) and \( \beta \) agree in all indices except for the index \( i_1 = 1 \) and \( j_1 = 2 \), this entry is equal to \( (-1)^{1+2} a_{12} = a_{12} \).

**Proposition 2** implies that the mapping \( A \to A^{[k]} \) is linear and, in particular

\[
(A + B)^{[k]} = A^{[k]} + B^{[k]}, \text{ for any } A, B \in \mathbb{C}^{n \times n}.
\]

This justifies the term additive compound.
It is useful to consider the additive compound of a matrix under a coordinate transformation. Let \( A \in \mathbb{C}^{n \times n} \). Fix \( k \in \{1, \ldots, n\} \), and an invertible matrix \( T \in \mathbb{C}^{n \times n} \). Then
\[
(TAT^{-1})^{[k]} = T^{[k]}A^{[k]}(T^{-1})^{[k]}.
\]  
(9)

To explain the use of the additive compound to study \( k \)-contraction in nonlinear dynamical systems, we briefly review logarithmic norms (also called matrix measures and Lozinskii measures).

**B. Logarithmic Norms**

A norm \( | \cdot : \mathbb{R}^n \to \mathbb{R}_+ \) induces a matrix norm \( \| \cdot \| : \mathbb{R}^{n \times n} \to \mathbb{R}_+ \) defined by \( \| A \| := \max_{|x|=1} |Ax| \), and a logarithmic norm \( \mu : \mathbb{R}^{n \times n} \to \mathbb{R} \) defined by
\[
\mu(A) := \lim_{\varepsilon \to 0} \frac{\| I + \varepsilon A \| - 1}{\varepsilon}.
\]

It is well known that the solution of \( \dot{x} = AX \) satisfies
\[
\frac{d}{dt} \log(|x(t)|) \leq \mu(A)
\]
where the derivative here is the upper right Dini derivative.

Logarithmic norms play an important role in numerical linear algebra and in contraction theory (see e.g., [1, 49, 51]). For the \( L_1, L_2 \), and \( L_{\infty} \) norms, there exist closed-form expressions for the induced logarithmic norms, namely
\[
\mu_1(A) = \max_i \left( a_{ii} + \sum_{j=1, i \neq j}^n |a_{ij}| \right),
\]
\[
\mu_2(A) = \lambda_1 \left( A + A^T \right),
\]
\[
\mu_{\infty}(A) = \max_i \left( a_{ii} + \sum_{j=1, i \neq j}^n |a_{ij}| \right)
\]
where \( \lambda_i(S) \) denotes the \( i \)-th largest eigenvalue of the symmetric matrix \( S \), i.e., \( \lambda_1(S) \geq \lambda_2(S) \geq \ldots \geq \lambda_n(S) \). Using Proposition 2, this can be generalized to closed-form expressions for the induced logarithmic norms of the additive compounds of a matrix.

**Proposition 3:** (see e.g., [37]) Let \( A \in \mathbb{R}^{n \times n} \), and fix \( k \in \{1, \ldots, n\} \). Then
\[
\mu_1(A^{[k]}) = \max_{\alpha \in Q(k,n)} \left( a_{ii} + \sum_{j \notin \alpha} |a_{ij}| \right),
\]
\[
\mu_2(A^{[k]}) = \sum_{i=1}^{k} \lambda_i \left( A + A^T \right),
\]
\[
\mu_{\infty}(A^{[k]}) = \max_{\alpha \in Q(k,n)} \left( a_{ii} + \sum_{j \notin \alpha} |a_{ij}| \right).
\]

It is straightforward to verify that if \( H \in \mathbb{R}^{n \times n} \) is invertible, then the scaled norm \(| \cdot |_H \), defined by \(|x|_H := |Hx| \), induces the logarithmic norm
\[
\mu_H(A) = \mu(HAH^{-1}).
\]  
(10)

\( L_p \) norms are invariant under permutations and sign changes, i.e., if \( P \) is a permutation [signature] matrix, then \(|x|_p = |PSx|_p \) for any \( x \). This yields the following result.

**Lemma 4:** Let \( \mu_p(\cdot) \) denote the logarithmic norm induced by \(| \cdot |_p \). If \( U \in \mathbb{R}^{n \times n} \) is the product of a permutation matrix and a signature matrix, then
\[
\mu_{p,U}(A) = \mu_p(A), \quad \text{for any } A \in \mathbb{R}^{n \times n}.
\]

We also require a duality result for the logarithmic norm that follows from a well-known relation between an \( L_p \) norm and its dual norm.

**Lemma 5:** Let \( p, q \in [1, \infty] \) such that \( p^{-1} + q^{-1} = 1 \). Then
\[
\mu_p(A) = \mu_q(A^T), \quad \text{for any } A \in \mathbb{R}^{n \times n}.
\]

For the sake of completeness, we include a proof of Lemma 5 in the Appendix.

**C. \( k \)-Contraction**

Motivated by the seminal work of Muldowney [37], Wu et al. [55] recently introduced the notion of \( k \)-contractive systems. Roughly speaking, the solutions of these systems contract \( k \)-dimensional parallelohedrons. For \( k = 1 \), this reduces to standard contraction.

Consider the time-varying nonlinear system
\[
\dot{x} = f(t,x)
\]  
(11)
where \( x \in \Omega \subseteq \mathbb{R}^n \), and \( \Omega \) is a convex set. We assume that \( f \) is \( C^1 \), and denote its Jacobian with respect to \( x \) by \( J(t,x) := \frac{\partial}{\partial x} f(t,x) \). A sufficient condition for \( k \)-contraction with rate \( \eta > 0 \) is that
\[
\mu(J^{[k]}(t,x)) \leq -\eta < 0, \quad \text{for all } x \in \Omega, t \geq 0.
\]  
(12)
For \( k = 1 \), this reduces to the standard sufficient condition for contraction. However, for \( k > 1 \) this condition is weaker than the one required for 1-contraction. As a simple example, a matrix \( A \in \mathbb{R}^{n \times n} \) is Hurwitz iff there exists \( P \) such that \( PA + A^T P < 0 \), i.e., iff \( \mu_{2,1}(A) < 0 \) [1]. This implies that \( A^{[2]} \) is contractive w.r.t. some scaled \( L_2 \) norm iff \( A^{[2]} \) is Hurwitz, i.e., iff the sum of any two eigenvalues of \( A \) has a negative real part. Note that this spectral property implies that any bounded solution of \( \dot{x} = Ax \) converges to an equilibrium.

\( k \)-contraction has several important implications. First, every bounded solution of a 2-contractive time-invariant nonlinear dynamical system converges to an equilibrium (that may not

\[1\] A signature matrix is a diagonal matrix whose diagonal elements are plus or minus 1.
be unique) [29]. Second, for \( n \)-dimensional LTV systems \( k \)-contraction implies the existence of an \((n - k + 1)\)-dimensional stable subspace.

**Proposition 6:** [37] Suppose that the LTV system \( \dot{x}(t) = A(t)x(t) \), where \( A : [t_0, \infty) \to \mathbb{R}^{n \times n} \) is a continuous matrix function, is uniformly stable. Let \( x(t_0) = x_0 \) denote an initial condition at time \( t_0 \). Fix \( k \in \{1, \ldots, n\} \). The following two conditions are equivalent.

1) The LTV system admits an \((n - k + 1)\)-dimensional subspace \( \mathcal{X}(t_0) \subseteq \mathbb{R}^n \) such that

\[
\lim_{t \to \infty} x(t, t_0, x_0) = 0 \quad \text{for any } x_0 \in \mathcal{X}(t_0).
\]

2) Every solution of

\[
\dot{y}(t) = A^{[k]}(t)y(t)
\]

converges to the origin as \( t \to \infty \).

Note that if \( \mu(A^{[k]}(t)) \leq -\eta < 0 \), then clearly condition 2) holds, and thus, 1) holds.

**Example 2:** Consider the LTV \( \dot{x}(t) = A(t)x(t) \), \( x(t_0) = x_0 \), with

\[
A(t) = (1/2) \begin{bmatrix}
-3 + 3\cos^2(t) & 2 - 3\cos(t)\sin(t) \\
-2 - 3\cos(t)\sin(t) & -3 + 3\sin^2(t)
\end{bmatrix}.
\]

It can be verified that \( x(t) = \Phi(t, t_0)x_0 \), where the transition matrix \( \Phi(t, t_0) \) is

\[
\Phi(t, t_0) = \begin{bmatrix}
\cos(t) & \sin(t) \\
-\sin(t) & \cos(t)
\end{bmatrix} \text{diag}(1, \exp(-3(t - t_0)/2))
\]

\[
\times \begin{bmatrix}
\cos(t_0) & -\sin(t_0) \\
\sin(t_0) & \cos(t_0)
\end{bmatrix}.
\]

This implies that the LTV is uniformly stable, but not contractive. Here

\[
A^{[2]}(t) = \text{tr}(A(t)) = -3/2
\]

so the LTV is 2-contraction, and Proposition 6 implies that the LTV admits a 1-D linear subspace \( \mathcal{X} \subseteq \mathbb{R}^2 \) such that (13) holds.

Indeed, it follows from (15) that \( \text{span} \left( \begin{bmatrix} \sin(t_0) \\ \cos(t_0) \end{bmatrix} \right) \) is such a subspace.

In principle, verifying that (12) holds can be done by computing \( \mu(J^{[k]}(t, x)) \). However, in practice this is nontrivial, as \( J^{[k]} \) is an \( (n \times n) \) matrix. The results in this article allow to verify \( k \)-contraction without computing the \( k \)-compounds. The next two sections include our main results.

**III. Duality Relations**

From here on we fix an integer \( n > 0 \), and an integer \( k \in \{1, \ldots, n - 1\} \). Let \( r := \binom{n}{k} \). Note that the matrices \( A^{(k)} \), \( A^{(n-k)} \), \( A^{[k]} \), and \( A^{[n-k]} \) all have the same dimensions, namely, \( r \times r \). We begin by deriving certain duality relations between these matrices. These use an antidiagonal matrix \( U(k, n) \) that we now define.

**A. Matrix \( U(k, n) \)**

Denote the lexicographically ordered sequences in \( Q(k, n) \) by \( \alpha^1, \ldots, \alpha^6 \). The signature of \( \alpha^j \) is \( s(\alpha^j) := (-1)^{\alpha^j_1 + \cdots + \alpha^j_k} \), and the complement of \( \alpha^j \) is

\[
\overline{\alpha^j} := \{1, \ldots, n\} \setminus \alpha^j.
\]

For simplicity, we use set notation here, but we always assume that the entries of \( \overline{\alpha^j} \) are arranged in the lexicographic order.

**Definition 3:** Let \( U = U(k, n) \in \{\{1, \ldots, n\} \setminus \alpha^j\}^{r \times r} \) be the antidiagonal matrix with entries

\[
u_{ij} = \begin{cases}
s(\alpha^j), & \text{if } i + j = r + 1, \\
0, & \text{otherwise}.
\end{cases}
\]

**Example 3:** For \( n = 4 \) and \( k = 2 \), we have \( \alpha^1 = (1, 2) \), \( \alpha^2 = (1, 3) \), \( \alpha^3 = (1, 4) \), \( \alpha^4 = (2, 3) \), \( \alpha^5 = (2, 4) \), and \( \alpha^6 = (3, 4) \), so \( s(\alpha^1) = -1 \), \( s(\alpha^2) = 1 \), \( s(\alpha^3) = -1 \), \( s(\alpha^4) = 1 \), \( s(\alpha^5) = 1 \). Thus

\[
U(2, 4) = \begin{bmatrix}
0 & 0 & 0 & s(\alpha^5) \\
0 & 0 & 0 & 0 \\
0 & 0 & s(\alpha^3) & 0 \\
0 & 0 & 0 & 0 \\
s(\alpha^1) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{bmatrix}.
\]

**Example 4:** For \( k = 1 \), we have \( Q(1, n) = ((1), (2), \ldots, (n)) \), so \( s(\alpha^j) = (-1)^j \), and thus, in this particular case, the definition of \( U \) yields

\[
u_{ij} = \begin{cases}
(-1)^j, & \text{if } i + j = r + 1, \\
0, & \text{otherwise}.
\end{cases}
\]

However, Example 3 shows that this expression does not hold in general.

Note that \( U^T U = UU^T = I_r \).

The next result will be useful in proving duality relations between matrix compounds.

**Lemma 7:** For any \( B \in \mathbb{C}^{r \times r} \) and any \( i, j \in \{1, \ldots, r\} \), we have

\[
(U^T BU)_{ij} = s(\alpha^j)(s(\alpha^i) b_{r+1-i, r+1-j}.
\]

**Proof:** Let \( z := (U^T BU)_{ij} \). Then, \( z \) is the product of row \( i \) of \( U^T B \) and column \( j \) of \( U \), i.e.,

\[
z = \sum_{k=1}^{r}(U^T B)_{i,k} u_{k,j}.
\]

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and applying (16) gives \( z = (U^T B)_{i,r+1-j}s(\alpha^j) \). Now \((U^T B)_{i,r+1-j}\) is the product of \(U^T\) and column \(r+1-j\) of \(B\), so

\[
z = s(\alpha^j) \sum_{k=1}^{r} u_{k,i} b_{k,r+1-j}
\]

and applying (16) again gives

\[
z = s(\alpha^j) u_{r+1-i,i} b_{r+1-i,r+1-j} = s(\alpha^j) s(\alpha^i) b_{r+1-i,r+1-j}.
\]

\[\blacksquare\]

**B. Duality Between Multiplicative Compounds**

The next result describes a duality relation between the two multiplicative compound matrices \(A^{(k)}\) and \(A^{(n-k)}\).

**Theorem 8:** Fix \(A \in C^{n \times n}\), and let \(U \in \{1, 0, 1\}^{T \times r}\) be the antidiagonal matrix defined in Definition 3. Then

\[
(A^{(k)})^T U^T A^{(n-k)} U = \det(A) I_r.
\]

(19)

In other words, \(U^T A^{(n-k)} U\) is the adjugate matrix of \((A^{(k)})^T\). For \(k = 1\), \(U\) becomes the matrix in (17), and (19) becomes \(A^T U^T A^{(n-1)} U = \det(A) I_n\), which is just \(\text{adj}(A) A = \det(A) I_n\). Formulas that are equivalent to (19) are known, see e.g., [19, p. 29] (where it appears without a proof), but without the explicit expression of the matrix \(U\).

The proof of Theorem 8 uses two auxiliary results. The first result describes a duality relation between \(Q(k,n)\) and \(Q(n-k,n)\).

**Lemma 9:** If \(Q(k,n) = (\alpha^1, \ldots, \alpha^r)\) then

\[
Q(n-k,n) = (\overline{\alpha^1}, \ldots, \overline{\alpha^r}).
\]

(20)

**Proof:** It is clear that \(Q(n-k,n)\) includes the sequences \(\overline{\alpha^i}\), \(i \in \{1, \ldots, r\}\). The ordering in (20) follows from the fact that the lexicographic ordering of \(Q(k,n)\) is \(\alpha^1, \ldots, \alpha^r\), and the definition of the lexicographic ordering. \[\blacksquare\]

**Example 5:** For \(n = 4\) and \(k = 3\), we have \(r = 4\), and

\[
Q(3,4) = ((1,2,3),(1,2,4),(1,3,4),(2,3,4)).
\]

Clearly

\[
Q(1,4) = ((1),(2),(3),(4))
\]

and this agrees with (20).

**Lemma 10:** For any \(i, j \in \{1, \ldots, r\}\), we have

\[
(U^T A^{(n-k)} U)_{ij} = s(\alpha^j) s(\alpha^i) A(\overline{\alpha^i} \overline{\alpha^j}).
\]

(21)

**Proof:** By Lemma 9

\[
(A^{(n-k)})_{pq} = A(\alpha^p 1 + p | \alpha^q 1 + q)
\]

for any \(p, q \in \{1, \ldots, r\}\). Combining this with (18) yields (21).

We can now prove Theorem 8. We assume that \(A\) is nonsingular. The general case follows by a continuity argument. Denote \(Z := (A^{(k)})^T U^T A^{(n-k)} U\). Fix \(i, j \in \{1, \ldots, r\}\). Then, \(z_{ij}\) is the product of row \(i\) of \((A^{(k)})^T\) and column \(j\) of \(U^T A^{(n-k)} U\),

\[
\begin{align*}
\begin{pmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\tilde{a}_3
\end{pmatrix}
\end{align*}
\]

and combining this with Lemma 10 gives

\[
z_{ij} = \sum_{\ell=1}^{r} A(\alpha^\ell | \alpha^i) s(\alpha^\ell) s(\alpha^j) A(\overline{\alpha^\ell} \overline{\alpha^i}).
\]

(22)

Jacobi’s identity [36, p. 166] asserts that for any \(p, q\)

\[
A(\alpha^p | \alpha^q) = \det(A) s(\alpha^p) s(\alpha^q) B(\overline{\alpha^p} \overline{\alpha^q})
\]

where \(B := A^{-1}\). In particular

\[
A(\alpha^j | \alpha^i) = \det(A) s(\alpha^j) s(\alpha^i) B(\alpha^j | \alpha^i)
\]

and substituting this in (22) yields

\[
z_{ij} = \det(A) \sum_{\ell=1}^{r} A(\alpha^\ell | \alpha^i) s(\alpha^\ell) s(\alpha^j) s(\alpha^j) B(\alpha^j | \alpha^i)
\]

\[
= \det(A) \sum_{\ell=1}^{r} A(\alpha^\ell | \alpha^i) B(\alpha^j | \alpha^i).
\]

In other words, \(z_{ij} / \det(A)\) is the product of row \(j\) of \((A^{-1})^{(k)}\) with column \(i\) of \(A^{(k)}\), i.e.,

\[
Z^T = \det(A)(A^{-1})^{(k)} A^{(k)}
\]

\[
= \det(A)(A^{-1} A)^{(k)}
\]

\[
= \det(A) I_r
\]

and this completes the proof of Theorem 8.

**Example 6:** Consider the case \(n = 3\), \(k = 2\), and \(A = \text{diag}(a_1, a_2, a_3)\). Then, \(A^{(2)} = \text{diag}(a_1 a_2, a_1 a_3, a_2 a_3)\)

\[
U = \begin{bmatrix}
0 & 0 & -1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{bmatrix}
\]

(23)

and \(U^T A U = \text{diag}(a_3, a_2, a_1)\). Thus

\[
(A^{(2)})^T U^T A U = (a_1 a_2 a_3) I_3
\]

(24)

and this agrees with (19).

To provide a geometric intuition for (24), let \(e^i\), \(i = 1, 2, 3\), denote the \(i\)th canonical vector in \(\mathbb{R}^3\). Assume for simplicity that \(a_i \geq 0\), \(i = 1, 2, 3\). Then, \((A^{(2)})_{11}\) is the volume of a parallelepiped with vertices \(a_1 e^1, a_2 e^2, a_3 e^3\), and \((U^T A U)_{11}\) is the volume of the parallelepiped (in fact, line) with vertex \(a_3 e^3\). The product of these two volumes is the volume of a parallelepiped with vertices \(a_1 e^1, a_2 e^2, a_3 e^3\), i.e., \(\det(A)\) (see Fig. 1). \[\blacksquare\]
If \( n \) is even, then taking \( k = n/2 \) in Theorem 8 yields the following result.

**Corollary 11:** Let \( A \in \mathbb{C}^{n \times n} \), with \( n \) even. Let \( r := (n/2) \). Then

\[
(UA^{(n/2)})^T A^{(n/2)} U = \det(A) I_r. \tag{25}
\]

**C. Duality Between Additive Compounds**

The next result describes a duality relation between the additive compound matrices \( A^{[k]} \) and \( A^{[n-k]} \).

**Theorem 12:** Let \( A \in \mathbb{C}^{n \times n} \). Then

\[
(A^{[k]})^T + U^T A^{[n-k]} U = \det(A) I_r. \tag{26}
\]

**Proof:** Fix \( \varepsilon > 0 \). Theorem 8 yields

\[
(I_n + \varepsilon A)^{(k)} U^T (I_n + \varepsilon A)^{(n-k)} U = \det(I + \varepsilon A) I_r. \tag{27}
\]

By (6), the term on the left-hand side of (27) is

\[
(I_r + \varepsilon A^{[k]})^T U^T (I_r + \varepsilon A^{[n-k]}) U + o(\varepsilon)
= I_r + \varepsilon \left( A^{[k]} \right)^T + \varepsilon U^T A^{[n-k]} U + o(\varepsilon).
\]

The term on the right-hand side of (27) is

\[
\det(I + \varepsilon A) I_r = (1 + \varepsilon \text{tr}(A) + o(\varepsilon)) I_r.
\]

We conclude that

\[
\varepsilon \left( A^{[k]} \right)^T + \varepsilon U^T A^{[n-k]} U + o(\varepsilon) = (\varepsilon \text{tr}(A) + o(\varepsilon)) I_r.
\]

Dividing both sides by \( \varepsilon \), and taking \( \varepsilon \to 0 \) completes the proof. \( \Box \)

**Example 7:** Consider the case \( n = 3, k = 2, \) and \( A = \text{diag}(a_1, a_2, a_3) \). Then \( A^{[2]} = \text{diag}(a_1 + a_2, a_1 + a_3, a_2 + a_3) \), \( U \) is as in (23), and \( U^T A U = \text{diag}(a_2, a_2, a_1) \). Thus

\[
\left( A^{[2]} \right)^T + U^T A^{[n-k]} U = (a_1 + a_2 + a_3) I_3.
\]

and this agrees with (26). \( \Box \)

**Remark 1:** The unpublished work [38] includes a result that is similar to Theorem 12. However, the result there uses a different matrix \( U \), and is in fact wrong. A counterexample to the result as stated in [38] is, for example, the case \( n = 4 \) and \( k = 2 \).

**Remark 2:** It follows from (26) that \( A^{[n-1]} = \text{tr}(A) I_n - U^T A U \). This special case already appeared in [46], and has been used in the analysis of \((n-1)\)-positive matrices [53].

One implication of Theorem 12, that will be used below, is the following.

**Corollary 13:** The matrices \((A^{[k]})^T\) and \(U^T A^{[n-k]} U\) commute.

The proof of this result is placed in the Appendix.

If \( n \) is even, then taking \( k = n/2 \) in Theorem 12 yields the following result.

**Corollary 14:** Let \( A \in \mathbb{C}^{n \times n} \), with \( n \) even. Let \( r := (n/2) \). Then

\[
(A^{[n/2]})^T + U^T A^{[n/2]} U = \det(A) I_r. \tag{28}
\]

**D. Duality Between Multiplicative Compounds of the Matrix Exponential**

The next result uses the duality relations above to derive a duality relation for the multiplicative compounds of the exponential of a matrix.

**Theorem 15:** Let \( A \in \mathbb{C}^{n \times n} \). Then

\[
\left( \exp(A) \right)^{(k)} = \exp(\text{tr}(A) U^T \exp(-A))^{(n-k)} U. \tag{29}
\]

**Proof:** This provides an expression for all the 2-minors of \( \exp(A) \) that does not require to compute any minors. \( \Box \)

**Remark 2:** We proved Theorems 8 and 12 for \( k \in \{1, \ldots, n-1\} \), and it is not necessarily obvious how to define \( A^{0} \) and \( A^{[0]} \). However, these results can now be used to define the 0-compounds in a consistent manner. Substituting \( k = 0 \) in Theorem 8 yields \( r = (0) = 1 \), and

\[
\det(A) I_1 = (A^{(0)})^T U^T A^{(n)} U
= A^{(0)} \det(A)
\]

so it is natural to define \( A^{(0)} := 1 \). A similar argument using Theorem 12 suggests the definition \( A^{[0]} := 0 \).

**E. Duality Between Logarithmic Norms of Additive Compounds**

The next result uses the duality relations above to relate \( \mu(A^{[k]}) \) and \( \mu(A^{[n-k]}) \).

**Proposition 16:** Let \( A \in \mathbb{R}^{n \times n} \). Fix \( k \in \{1, \ldots, n-1\} \), and let \( \varepsilon = \left( \frac{k}{n} \right) \). Let \( U \in \{-1, 0, 1\}^{n \times n} \) be the matrix defined in (16). Then, for any logarithmic norm \( \mu : \mathbb{R}^{n \times r} \to \mathbb{R} \), we have

\[
\mu(A^{[k]}) = \text{tr}(A) + \mu(U^T \left(-A^{[n-k]}\right)^T). \tag{30}
\]

**Proof:** By (26)

\[
A^{[k]} = \text{tr}(A) I_r - U^T \left(A^{[n-k]}\right)^T U.
\]

Applying \( \mu \) on both sides of this equation and using the fact that \( \mu(c I + B) = c + \mu(B) \) for any scalar \( c \) (see, e.g., [11]) yields \( \mu(A^{[k]}) = \text{tr}(A) + \mu(-U^T (A^{[n-k]} U^T)) \), and this completes the proof. \( \Box \)

**Corollary 17:** Fix \( A \in \mathbb{R}^{n \times n} \), \( p \in \{1, 2, \infty\} \), and \( k \in \{1, \ldots, n\} \). Let \( T \in \mathbb{R}^{n \times n} \) be invertible, and let \( q \) be defined by \( p^{-1} + q^{-1} = 1 \). Then

\[
\mu_{p, T^{(k)}} \left(A^{[k]}\right) = \text{tr}(A) + \mu_{q, T^{(n-k)}} \left(-A^{[n-k]}\right). \tag{31}
\]

**Proof:** Using (10) and (9) yields

\[
\mu_{p, T^{(k)}} \left(A^{[k]}\right) = \mu_p(T^{(k)} A^{[k]}(T^{(k)})^{-1})
= \mu_p(T A^{[k]} T^{-1})
= \mu_p((T A T^{-1})^{[k]})
= \mu_q((T A T^{-1})^{[n-k]})
= \mu_{q, T^{(n-k)}} \left(-A^{[n-k]}\right)
= \mu_{q, T^{(n-k)}} \left(-A^{[n-k]}\right).
\]
where we used the duality relation for logarithmic norms in Lemma 5. Retracing the steps of the proof of Proposition 16 and taking into account that \( \text{tr}(TAT^{-1}) = \text{tr}(A) \) we have
\[
\mu_{p,T(k)}(A^{[k]}) = \text{tr}(A) + \mu_q\left(-U^T(TAT^{-1})^{[n-k]}U\right).
\]
Finally, using Lemma 4 yields
\[
\mu_{p,T(k)}(A^{[k]}) = \text{tr}(A) + \mu_q\left(-\left(TAT^{-1}\right)^{[n-k]}\right) = \text{tr}(A) + \mu_q(T^{(n-k)}(-A^{[n-k]}))
\]
which completes the proof. \(\square\)

**Corollary 18:** Let \( A \in \mathbb{R}^{n \times n} \), with \( n \) even. Let \( r := \binom{n}{n/2} \). Then, for any logarithmic norm \( \mu : \mathbb{R}^{r \times r} \to \mathbb{R} \), we have
\[
\mu\left(A^{[n/2]}\right) = \text{tr}(A) + \mu_t\left(-\left(A^{[n/2]}\right)^T\right).
\]
In particular, for any logarithmic norm \( \mu_p \) induced by an \( L_p \) norm, we have
\[
\mu_p\left(A^{[n/2]}\right) = \text{tr}(A) + \mu_q\left(-\left(A^{[n/2]}\right)\right)
\]
and for the logarithmic norm induced by the \( L_2 \) norm
\[
\mu_2\left(A^{[n/2]}\right) = \text{tr}(A) + \mu_2\left(-\left(A^{[n/2]}\right)\right).
\]

**Example 9:** Consider \( A \in \mathbb{R}^{(2^k) \times (2^k)} \), with \( A = \text{diag}(\lambda_1, \ldots, \lambda_{2^k}) \), and \( \lambda_1 \geq \cdots \geq \lambda_{2^k} \). Then, \( \text{tr}(A) = \sum_{i=1}^{2^k} \lambda_i \), and \( \mu_2(-A^{[n/2]}) = -\sum_{i=1}^{2^k} \lambda_i \), so clearly \( (34) \) holds. \(\square\)

The formulas in Proposition 3 are not always easy to use. Indeed, the cardinality of the set \( Q(k,n) \) is \( \binom{n}{k} \) and this may be very large. In the next section, we use the duality relation \( (30) \) to derive a sufficient condition for \( k \)-contraction of nonlinear dynamical systems, which does not require calculating compounds of the Jacobian.

**IV. Compound-Free Sufficient Condition for \( k \)-Contraction**

We begin by defining a new matrix operator. As we will see below, this operator can be used to bound the logarithmic norm of the \( k \) additive compound of a Jacobian matrix.

**A. \( k \)-Shifted Logarithmic Norm**

**Definition 4:** Given an integer \( n \geq 1 \), \( k \in \{1, \ldots, n\} \), \( p \in \{1, 2, \infty\} \), and an invertible matrix \( T \in \mathbb{R}^{n \times n} \), the \( k \)-shifted logarithmic norm \( \tau_{p,k,T} : \mathbb{R}^{n \times n} \to \mathbb{R} \) is defined by
\[
\tau_{p,k,T}(A) := \text{tr}(A) + (n-k)\mu_q(T(-A))
\]
where \( q \) is such that \( q^{-1} + p^{-1} = 1 \).

Note that the terminology is justified by the equality \( \tau_{p,k,T}(A) = \mu_q(T\text{tr}(A)I_n + (n-k)(-A)) \). Note also that the matrix \( U \) does not appear in the definition of \( \tau_{p,k,T} \). As we will see below, this is due to Lemma 4.

We can now state the main result in this section.

**Theorem 19:** For any \( A \in \mathbb{R}^{n \times n} \), \( k \in \{1, \ldots, n\} \), \( p \in \{1, 2, \infty\} \), and an invertible matrix \( T \in \mathbb{R}^{n \times n} \), we have
\[
\mu_{p,T(k)}(A^{[k]}) \leq \tau_{p,k,T}(A).
\]

In other words, \( \tau_{p,k,T}(A) \) provides an upper bound on \( \mu_{p,T(k)}(A^{[k]}) \) that does not require to compute any compounds. Thus, if \( \tau_{p,k,T}(A) \leq -\eta < 0 \) then \( x = Ax \) is \( k \)-contracting with rate \( \eta \) w.r.t. the scaled \( L_p \) norm with weight matrix \( T(k) \). Similarly, if the Jacobian \( J(t,x) \) of \( (11) \) satisfies
\[
\tau_{p,k,T}(J(t,x)) \leq -\eta < 0, \quad \text{for all } t \geq 0, x \in \Omega
\]
then \( (11) \) is \( k \)-contracting with rate \( \eta \) w.r.t. the scaled \( L_p \) norm with weight matrix \( T(k) \).

Note that the upper bound \( (36) \) in Theorem 19 provides a sufficient condition for \( k \)-contraction that does not require computing any compounds, and only requires computing \( \mu_q(T(-A)) \), that is, the same computation that is needed in establishing standard contraction. \(^2\)

For \( k = 1, (36) \) yields a nonstandard sufficient condition for contraction w.r.t. \( L_p \), namely
\[
\text{tr}(J(t,x)) + (n-1)\mu_q(-J(t,x)) \leq -\eta < 0
\]
for all \( t \geq 0, x \in \Omega \).

The rest of this section is devoted to the proof of Theorem 19. This requires the following auxiliary result that may be of independent interest.

**Proposition 20:** Fix \( A \in \mathbb{R}^{n \times n} \), \( p \in \{1, 2, \infty\} \), and \( k, \ell \in \{1, \ldots, n\} \) with \( \ell \leq k \). Let \( T \in \mathbb{R}^{n \times n} \) be invertible. Then
\[
\frac{1}{k} \mu_{p,T(k)}(A^{[k]}) \leq \frac{1}{\ell} \mu_{p,T(\ell)}(A^{[\ell]})
\]
For example, for \( \ell = 1 \) this gives \( \mu_{p,T(1)}(A^{[1]}) \leq k \mu_p(T(A)) \) for any \( k \geq 1 \).

**Proof:** We will use the following easy to verify fact. If \( \alpha_1 \geq \cdots \geq \alpha_n \) and \( k \in \{1, \ldots, n\} \) then
\[
\frac{1}{k} \sum_{i=1}^{k} \alpha_i - \frac{1}{k+1} \sum_{i=1}^{k+1} \alpha_i \geq \frac{\alpha_k - \alpha_{k+1}}{k+1} \geq 0.
\]

We begin by proving \( (37) \) for \( T = I_n \). We first consider the case \( p = 2 \). Let \( \lambda_1 \geq \cdots \geq \lambda_n \) denote the eigenvalues of \( (A + A^T)/2 \). Then
\[
\mu_2(A^{[k]}) \leq \mu_2(A^{[k+1]}) = \frac{1}{k} \sum_{i=1}^{k} \lambda_i - \frac{1}{k+1} \sum_{i=1}^{k+1} \lambda_i \geq 0.
\]

We now consider the case \( p = 1 \). For any \( \alpha \in Q(k,n) \), let \( c_{\alpha,i} := a_{ii} + \sum_{j \neq i} a_{ij}, \) with \( i \in \{1, \ldots, n\} \). Let \( \beta := \text{argmax}_{\alpha \in Q(k,n)} \sum_{i \in \alpha} c_{\alpha,i} \). Let \( i_1, \ldots, i_k \in \{1, \ldots, n\} \) be such that \( c_{\beta,i_i} \geq \cdots \geq c_{\beta,i_k} \). Using Proposition 3 and \( (38) \) gives
\[
k^{-1} \mu_1(A^{[k]}) = \max_{\alpha \in Q(k,n)} k^{-1} \sum_{i \in \alpha} c_{\alpha,i}
\]
\(^2More precisely, it requires to compute only the trivial compounds \( J = J^{[1]} \) and \( tr(J) = J^{[n]} \).
\[
\begin{align*}
&\ell^{-1}(c_{\beta,i_1} + \cdots + c_{\beta,i_k}) \\
&\leq \ell^{-1}(c_{\beta,i_1} + \cdots + c_{\beta,i_k}) \\
&\leq \max_{\tau \in \mathbb{Q}_{(\ell,n)}} \ell^{-1} \sum_{i \in \gamma} c_{\gamma,i} \\
&= \ell^{-1} \mu_1(A^{[\ell]}).
\end{align*}
\]

The proof for the case \( p = \infty \) is similar, and thus, omitted. We conclude that
\[
\frac{1}{k} \mu_p(A^{[k]}) \leq \frac{1}{\ell} \mu_p(A^{[\ell]}).
\]

To complete the proof of (37), fix an invertible matrix \( T \). Using (9) and the fact that \( \mu_T(A) = \mu(TAT^{-1}) \), we have that
\[
k^{-1} \mu_{p,T}(A^{[k]}) = k^{-1} \mu_p((TAT^{-1})^{[k]}) \\
\leq \ell^{-1} \mu_p((TAT^{-1})^{[\ell]}) \\
= \ell^{-1} \mu_{p,T}(A^{[\ell]})
\]

and this completes the proof.

**Example 10:** Let \( A = I_n \) and fix \( k \in \{1, \ldots, n\} \). Then, \( A^{[k]} = kI_r \), so for any monotonic norm\(^3\) and any invertible matrix \( T \), we have \( \mu_{p,T}(A^{[k]}) = k \). Thus, inequality (37) holds with an equality, implying that the bound cannot be improved in general.

**Remark 4:** Proposition 20 implies in particular that if the system (11) satisfies the infinitesimal condition for \( \ell \)-contraction with rate \( \eta \) in (12) w.r.t. an \( L_p \) norm scaled by \( T^{(\ell)} \), with \( p \in \{1, 2, \infty\} \), then for any \( k \in \{1, \ldots, n\} \), the system is \( k \)-contracting with rate \( \frac{k}{\ell} \eta \) w.r.t. the same norm scaled by \( T^{(k)} \) (see also [4], [55]). More generally, if a system is \( \alpha \)-contracting, with \( \alpha > 0 \) real, then it is also \( \alpha (\varepsilon) \)-contracting for any \( \varepsilon \geq 0 \) [57].

We can now prove Theorem 19.

**Proof of Theorem 19:** Using Proposition 20 with \( A \) replaced by \(-A, k \) by \( n - k \), and \( \ell \) by one, gives
\[
\mu_{p,T^{(n-k)}}(-A^{[n-k]}) \leq (n-k) \mu_{p,T}(-A).
\]

Combining this with Corollary 17 gives
\[
\mu_{p,T^{(k)}}(A^{[k]}) = \text{tr}(A) + \mu_{q,T^{(n-k)}}(-A^{[n-k]}) \\
\leq \text{tr}(A) + (n-k) \mu_{q,T}(-A)
\]

and this completes the proof.

**Remark 5:** Suppose that the system (11) satisfies the infinitesimal condition for \( k \)-contraction w.r.t. to \( L_p \) for some \( p \in \{1, 2, \infty\} \). Then, the system is also \( n \)-contractive, i.e., \( \text{tr}(J(x)) < 0 \) for all \( x \in \Omega \). Suppose now that either \( p = 1 \) or \( p = \infty \). Since there exists at least one diagonal entry of \( J(x) \) that is negative
\[
\mu_p(-J(x)) > 0, \text{ for all } x \in \Omega.
\]

For \( p = 2 \), we have \( \mu_2(-J(x)) = \lambda_1(-J(x) + JT(x))/2 \) and since
\[
\text{tr}(-J(x) + JT(x))/2 = \text{tr}(-J(x)) > 0
\]

the formula for the \( L_2 \) logarithmic norm implies that (39) holds also when \( p = 2 \). Thus, the sufficient condition for \( k \)-contraction in Theorem 19 is a tradeoff between the negativity of \( \text{tr}(J(x)) \) and the positivity of \( (n-k) \mu_{p}(-J(x)) \). In this case, if the sufficient condition for \( k \)-contraction holds, i.e.,
\[
\text{tr}(J) + (n-k) \mu_{p}(-J) \leq -\eta < 0
\]

then clearly
\[
\text{tr}(J) + (n-k+1) \mu_{p}(-J) \leq -\eta < 0
\]

so the sufficient condition for \((k+1)\)-contraction also holds.

The next result summarizes several properties of the \( k \)-shifted logarithmic norm \( \tau_{p,k} \). For the sake of simplicity, we take \( T = I_n \) and write \( \tau_{p,k} \) for \( \tau_{p,k,1} \). The proof follows from Definition 4, linearity of the trace operator, and known properties of logarithmic norms (see, e.g., [11]).

**Proposition 21:** Fix \( A, B \in \mathbb{R}^{n \times n}, k \in \{1, \ldots, n\} \), and \( p \in \{1, 2, \infty\} \). Then
1. \( \tau_{p,k}(0) = 0 \).
2. \( |\tau_{p,k}(A) - \tau_{p,k}(B)| \leq |\text{tr}(A - B)| \|A - B\|_{q,T} \).
3. \( \tau_{p,k}(A + B) \leq \tau_{p,k}(A) + \tau_{p,k}(B) \).
4. \( \tau_{p,k}(cA) = c \tau_{p,k}(A) \), for any \( c \in \mathbb{R^+} \).
5. \( \tau_{p,k}(A + cI_n) = \tau_{p,k}(A) + kc \), for any \( c \in \mathbb{R} \). In particular
6. \( \tau_{p,k}(I_n) = k \), \( \tau_{p,k}(-I_n) = -k \).
7. \( \tau_{p,k}(rA + (1-r)B) \leq r \tau_{p,k}(A) + (1-r) \tau_{p,k}(B) \), for any \( r \in [0, 1] \).

In particular, \( \tau_{p,k} \) is continuous, subadditive, positively homogeneous of degree one, and convex. The latter property implies that it is possible to verify the sufficient condition for \( k \)-contraction in Theorem 19 for a polytope of dynamical systems by checking only the vertices of this polytope.

**V. APPLICATIONS**

We now describe several applications of Theorem 19. In particular, we show how it can be used to prove \( k \)-contraction in \( n \)-dimensional nonlinear systems without computing any \( k \)-compounds. However, it is instructive to begin with LTI systems.

**A. \( k \)-Contraction in LTI Systems**

Consider the LTI system
\[
\dot{x}(t) = Ax(t)
\]
with \( A \in \mathbb{R}^{n \times n} \).

\(^3\)Recall that a norm \(|| \cdot || : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) is monotonic if \(|y_i| \leq |x_i|, i = 1, \ldots, n\), implies that \(|y| \leq |x| \). All \( L_p \) norms are monotonic; see [7].
Suppose first that \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \), with
\[
\lambda_1 \geq \cdots \geq \lambda_n. 
\] 
(41)

Fix \( p \in \{1, 2, \infty\} \). Then
\[
\tau_{p,k}(A) = \text{tr}(A) + (n-k)\mu_q(-A) 
= \lambda_1 + \cdots + \lambda_n - (n-k)\lambda_n 
= -(n-k-1)\lambda_n + \lambda_1 + \cdots + \lambda_{n-1}. 
\]
Combining this with (41) implies that \( \tau_{p,k}(A) < 0 \) if and only if
\[
\lambda_1 + \cdots + \lambda_{n-1} < (n-k-1)\lambda_n < 0. 
\] 
(42)
If \( k = n-1 \), then this is equivalent to \( \lambda_1 + \cdots + \lambda_{n-1} < 0 \), which is indeed a necessary and sufficient condition for \((n-1)\)-contraction of (40). For \( k < n-1 \), condition (42) requires that the sum of the first \( n-1 \) eigenvalues is “negative enough” to guarantee \( k \)-contraction.

Now assume that \( A \) is not necessarily diagonal. Let \( \lambda_1, \ldots, \lambda_n \) denote the eigenvalues of \( A \), ordered such that \( \text{Re}(\lambda_1) \geq \cdots \geq \text{Re}(\lambda_n) \). Then, (35) with \( T = I_n \) yields
\[
\tau_{p,k}(A) = \text{tr}(A) + (n-k)\mu_q(-A). 
\]
Using the bound \( \mu(-A) \geq \text{Re}(-\lambda_n) \) gives
\[
\tau_{p,k}(A) \geq \sum_{i=1}^{n} \text{Re}(\lambda_i) + (n-k)\text{Re}(-\lambda_n) 
\geq n \text{Re}(\lambda_n) - (n-k)\text{Re}(\lambda_n) 
= k \text{Re}(\lambda_n) 
\]
so in particular if \( \tau_{p,k}(A) < 0 \) then we must have that \( \text{Re}(\lambda_n) < 0 \). In other words, if the sufficient \( k \)-contraction condition holds then \( A \) is not “too unstable” in the sense that it has at least one eigenvalue with a negative real part.

Recall that for the LTI system (40), \( k \)-contraction implies that every sum of \( k \) eigenvalues of \( A \) has a negative real part [55]. Combining this with Theorem 19 yields the following result.

**Corollary 22:** Let \( A \in \mathbb{R}^{n \times n} \). Suppose that there exist an invertible matrix \( T \in \mathbb{R}^{n \times n}, k \in \{1, \ldots, n\} \), and \( q \in \{1, 2, \infty\} \) such that
\[
\text{tr}(A) + (n-k)\mu_{q,T}(-A) < 0. 
\]
Then, every sum of \( k \) eigenvalues of \( A \) has a negative real part.

The next result shows how Theorem 19 can be used to derive a “\( k \)-trace dominance condition” guaranteeing \( k \)-contraction.

**Corollary 23:** Fix \( k \in \{1, \ldots, n\} \). If there exist \( d_1, \ldots, d_n > 0 \) such that
\[
-(n-k-1)a_{ii} + \sum_{j \neq i} (a_{jj} + (n-k)\frac{d_j}{d_i}a_{ji}) \leq -\eta < 0
\] 
(43)
for all \( i \in \{1, \ldots, n\} \), then the LTI system (40) is \( k \)-contractive with rate \( \eta \) w.r.t. the scaled norm \( \| \cdot \|_{\infty,D} \), where \( D := \text{diag}(d_1, \ldots, d_n) \).

Note that condition (43) involves \( n \) equations, and each equation includes \( n \) terms, whereas determining \( \mu(A^{[k]}) \) requires calculating an \( \binom{n}{k} \times \binom{n}{k} \) matrix.

Note that for \( k = n \) condition (43) becomes
\[
\text{tr}(A) \leq -\eta < 0
\]
whereas for \( k = n-1 \) and \( D = I_n \) it becomes
\[
\text{tr}(A) - a_{pp} + \sum_{j \neq p} |a_{jp}| \leq -\eta < 0, \text{ for all } p \in \{1, \ldots, n\}. 
\]

**Proof:** We prove Corollary 23 for the case \( D = I_n \). The general case follows by replacing \( A \) with \( DAD^{-1} \). Consider
\[
\tau_{\infty,k}(A) = \text{tr}(A) + (n-k)\mu_1(-A) 
= \sum_{i=1}^{n} a_{ii} + (n-k) \max(c_1, \ldots, c_n)
\]
where \( c_i := -a_{ii} + \sum_{j \neq i} |a_{ji}| \), i.e., the sum of the entries in column \( i \) of \((-A)\), with off-diagonal entries taken with absolute value. For concreteness, assume that \( \max(c_1, \ldots, c_n) = c_1 \), i.e.,
\[
-a_{11} + \sum_{j \neq 1} |a_{j1}| \geq -a_{1\ell} + \sum_{j \neq \ell} |a_{j\ell}|, \text{ for all } \ell \geq 1.
\]
Then
\[
\tau_{\infty,k}(A) = \sum_{i=1}^{n} a_{ii} + (n-k) \left(-a_{11} + \sum_{j \neq 1} |a_{j1}|\right) 
= -(n-k-1)a_{11} + \sum_{j \neq 1} (a_{jj} + (n-k)|a_{j1}|).
\]

Comparing this with (43) completes the proof.

### B. \( k \)-Contraction in the LTV Systems

Applying Theorem 19 to prove \( k \)-contraction requires bounding \( \mu_{q,T}(-(J(t,x))) \) from above. For the particular case of an LTV system and the \( L_2 \) norm (i.e., \( q = 2 \)), we can apply a useful bound that was derived in [47].

**Lemma 24:** [47] Consider the matrix LTV system
\[
\dot{X}(t) = A(t)X(t), \quad X(0) = I
\] 
(44)
with \( A : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n} \) continuous. Suppose that there exist \( Q > 0 \) and a continuous function \( \theta : \mathbb{R}_+ \rightarrow \mathbb{R} \) such that
\[
A^T(t)Q + QA(t) + 2\theta(t)Q \succeq 0,
\]
for all \( t \geq 0 \). Let \( P > 0 \) be such that \( P^2 = Q \). Then \( \mu_{2,P}(-A(t)) \leq \theta(t) \), for all \( t \geq 0 \).

The proof follows from multiplying \( (45) \) by \( P^{-1} \) on the left- and right-hand sides.

**Example 11:** For \( Q = I_n \), (45) becomes
\[
(-A(t) - A^T(t))/2 \leq \theta(t)I_n, \text{ for all } t \geq 0.
\]
This implies that
\[
\mu_2(-A(t)) \leq \mu_2(\theta(t)I_n) = \theta(t), \text{ for all } t \geq 0.
\]
\( \square \)
Combining the bound (46) with Theorem 19 yields the following result.

**Proposition 25:** Consider the matrix LTV system (44) and suppose that the conditions in Lemma 24 hold. Fix \( k \in \{1, \ldots, n\} \). If

\[
\text{tr}(A(t)) + (n - k)\theta(t) \leq -\eta < 0, \quad \text{for all } t \geq 0
\]

then the LTV system is \( k \)-contracting with rate \( \eta \) w.r.t. the scaled \( L_2 \) norm \( \| \cdot \|_{2,p} \).

**Proof:** Consider

\[
\tau_{2,k}(A(t)) = \text{tr}(A(t)) + (n - k)\mu_{2,p}(-A(t)).
\]

Combining this with (46) gives

\[
\tau_{2,k}(A(t)) \leq \text{tr}(A(t)) + (n - k)\theta(t)
\]

and applying Theorem 19 completes the proof. \( \square \)

**Example 12:** Consider the LTI (40), but with an uncertainty in the matrix \( A \). A standard approach for modeling this is to assume that \( A \) is constant, unknown, and belongs to the convex hull of a set of \( s \) known matrices \( A_1, \ldots, A_s \). We assume that all the matrices have the same trace

\[ r := \text{tr}(A_i), \quad i = 1, \ldots, s. \]

This is a typical case, for example, in modeling biological interaction networks (also known as chemical reaction networks), see, e.g., [4]. Proposition 25 implies that if we can find \( Q > 0 \) and \( \theta \in \mathbb{R} \) such that

\[
A_i^T Q + QA_i + 2\theta Q \succeq 0, \quad \text{for all } i \in \{1, \ldots, s\}
\]

and

\[
r + (n - k)\theta \leq -\eta < 0
\]

then the uncertain LTI is \( k \)-contractive. We emphasize again that this does not require computing any compounds. \( \square \)

### C. \( k \)-Contraction in \( n \)-Dimensional Hopfield Neural Networks

Consider the Hopfield neural network [18]

\[
\dot{x}_i(t) = -\frac{x_i(t)}{r_i} + \sum_{j=1}^{n} \omega_{ij} \phi_j(x_j(t)) + u_i, \quad i = 1, \ldots, n
\]

(47)

where \( r_i > 0, u_i \) is a constant input to neuron \( i \), \( \phi_j : \mathbb{R} \to \mathbb{R} \) is the activation function of neuron \( j \), and \( W = \{\omega_{ij}\}_{i,j=1}^{n} \) is the network connection matrix. We assume that every \( \phi_j \) is \( C^1 \).

The stability of (47) has been studied extensively, e.g., via Lyapunov analysis in, [14], [22], and [35]. The work in [12] seems to be the first application of logarithmic norms to analyze Hopfield neural networks; later works include [9] and [43], on contractivity w.r.t. non-Euclidean norms, and [25] and [44], on contractivity w.r.t. Euclidean norms. However, in many applications the network admits more than a single equilibrium. For example, in using a Hopfield network as an associative memory [18], [26], every stored pattern corresponds to an equilibrium. Thus, if the network stores more than a single memory, then it cannot be contractive w.r.t. any norm.

The Jacobian of (47) is

\[
J(x) = -\text{diag}(r_1^{-1}, \ldots, r_n^{-1}) + W \text{diag}(\phi'_1(x_1), \ldots, \phi'_n(x_n))
\]

(48)

where \( \phi'_j(x) := \frac{d}{dx}\phi_j(x)|_{x=x_i} \). Theorem 19 allows to derive a sufficient condition for \( k \)-contraction of the Hopfield network without computing compounds. One possibility is to assume that the \( \phi'_j \)s are bounded, and then apply the same approach as in Corollary 23.

**Proposition 26:** Consider the Hopfield network (47). Assume that the neuron activation functions satisfy

\[
0 \leq m_i \leq |\phi'_i(z)| \leq M_i, \quad \text{for all } z \in \mathbb{R} \text{ and } i \in \{1, \ldots, n\}.
\]

(49)

If there exist \( d_1, \ldots, d_n > 0 \) such that

\[
-(n - k - 1)(-r_i^{-1} - m_i|w_{ji}|) + \sum_{j \neq i} \left( -r_j^{-1} + M_j |w_{ji}| + (n - k) \frac{d_j}{d_i} M_i |w_{ji}| \right) \leq -\eta < 0
\]

(50)

for all \( i \in \{1, \ldots, n\} \), then (47) is \( k \)-contractive with rate \( \eta \) w.r.t. the scaled norm \( \| \cdot \|_{\infty,D} \), with \( D := \text{diag}(d_1, \ldots, d_n) \).

The proof is placed in the Appendix.

A common choice for the activation functions in neural network models is \( \phi_i(z) = a_i \tanh(b_i z) \), and then clearly condition (49) holds for \( m_i = 0 \) and \( M_i = |a_i b_i| \). Note also that if we set \( r_i = \cdots = r_n = r \) then (50) will hold for any \( r \) sufficiently small. This makes sense, as a smaller \( r \) makes (47) “more stable.” We emphasize again that condition (50) does not require to compute any compounds of the Jacobian \( J(x) \) in (48).

**Example 13:** Consider (47) with \( |w_{ij}| = 1 \) for all \( i, j \) (i.e., a binary weight matrix), \( r_i = r \) and \( \phi_i(z) = \tanh(z) \) for all \( i \), i.e.,

\[
\dot{x}_i = -\frac{x_i}{r} + \sum_{j=1}^{n} (\pm 1) \tanh(x_j) + u_i, \quad i = 1, \ldots, n
\]

(51)

where \( \pm 1 \) indicates a value that can be either \(-1 \) or \( +1 \). For large \( r \), this system may have more than a single equilibrium and may not be contractive w.r.t. any norm. We apply Proposition 26 with \( D = I_n \) to find a sufficient condition for \( k \)-contraction. In this case, \( m_i = 0 \) and \( M_i = 1 \) for all \( i \), and (50) becomes

\[
-k r^{-1} + (n - 1)(n - k + 1) \leq -\eta < 0
\]

Thus, a sufficient condition for \( k \)-contraction is

\[
r < \frac{k}{(n - 1)(n - k + 1)}. \tag{52}
\]

Note also that this did not require to compute and analyze \( J^{[k]}(x) \) which in this case is an \( \binom{n}{k} \times \binom{n}{k} \) state-dependent matrix.

For example, if we require \((n - 1)\)-contraction then (52) becomes the condition \( r < 1/2 \). As a specific example, take \( n = 3 \), \( w_{ij} = 1 \) for all \( i, j, r_i = 0.49 \) and \( f_i(z) = \tanh(z) \) for all \( i \), i.e.,

\[
\dot{x}_i = -\frac{x_i}{0.49} + \sum_{i=1}^{3} \tanh(x_i), \quad i = 1, 2, 3. \tag{53}
\]
This system admits at least three equilibrium points

\[ e^1 = 0, \quad e^2 \approx 1.2447 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T, \quad e^3 = -e^2 \]  

so it is not contractive w.r.t any norm. It satisfies the sufficient condition for two-contraction (namely, \( r < 1/2 \)), so the results of Muldowney and Li [29, 37] imply that every bounded solution converges to an equilibrium point. It is clear from (53) that all trajectories of the system are bounded. Fig. 2 depicts several trajectories of this system from random initial conditions. It may be seen that all the trajectories indeed converge to either \( e^2 \) or \( e^3 \). Note that in this example, \( n = 3 \) and \( k = 2 \), so we can easily compute and analyze \( J^{[2]}(x) \) directly, but our goal is merely to demonstrate the bound derived in Theorem 19. \( \square \)

**D. Local Stability of an Equilibrium of a Nonlinear Dynamical System**

Li and Wang [30] proved that a matrix \( A \in \mathbb{R}^{n \times n} \) is Hurwitz if and only if the following two conditions hold:

\[ A^{[2]} \text{ is Hurwitz and } (-1)^n \det(A) > 0. \]  

(55)

The proof is based on the fact that every eigenvalue of \( A^{[2]} \) is the sum of two eigenvalues of \( A \). This result was applied to prove the local stability of the endemic equilibrium \( e \) in an SEIR model with a varying total population size [30] by verifying that \( J(e) \), the Jacobian of the vector function evaluated at the equilibrium, is Hurwitz. In this case, \( J(e) \in \mathbb{R}^{3 \times 3} \) depends on various parameters of the model and verifying that \( J(e) \) is Hurwitz using the Routh–Hurwitz stability criterion is nontrivial. However, in general, verifying that (55) holds requires computing \( A^{[2]} \) which is a matrix of dimensions \( \binom{n}{2} \times \binom{n}{2} \). The next result uses the operator \( \tau_{p,k,T} \) and does not require to compute 2-compounds.

**Corollary 27:** Let \( e \in \mathbb{R}^n \) be an equilibrium of the system \( \dot{x} = f(x) \), with \( f \in C^1 \). Let \( J(x) := \frac{\partial}{\partial x} f(x) \). If there exist \( p \in \{1, 2, \infty\} \) and an invertible matrix \( T \in \mathbb{R}^{n \times n} \) such that

\[ \tau_{p,2,T}(J(e)) = \text{tr}(J(e)) + (n-2)\mu_q,T(-J(e)) < 0 \]  

(56)

where \( p^{-1} + q^{-1} = 1 \), and

\[ (-1)^n \det(J(e)) > 0 \]  

(57)

then \( e \) is locally asymptotically stable.

**Proof:** By Theorem 19, (56) implies that \( \mu((J(e))^{[2]}) < 0 \), and combining this with (57) implies that \( J(e) \) is Hurwitz. \( \blacksquare \)

Note that conditions (56) and (57) do not require to compute \( (J(e))^{[2]} \).

As a simple example consider again the Hopfield network in (53) and the equilibrium points in (54). We already know that condition (56) holds at any point in the state-space, so we only need to check condition (57), i.e.,

\[ \det(J(e)) < 0. \]  

(58)

Using (48) gives

\[ J(x) = -\text{diag}(1/0.49, 1/0.49, 1/0.49) \]

\[ + \begin{bmatrix} 1 - \tanh^2(x_1) & 1 - \tanh^2(x_2) & 1 - \tanh^2(x_3) \\ 1 - \tanh^2(x_1) & 1 - \tanh^2(x_2) & 1 - \tanh^2(x_3) \\ 1 - \tanh^2(x_1) & 1 - \tanh^2(x_2) & 1 - \tanh^2(x_3) \end{bmatrix}. \]

It follows that \( \det(J(x)) = (-1/0.49)^2((-1/0.49) + 3 - \sum_{i=1}^3 \tanh^2(x_i)) \). In particular, \( \text{sgn}(\det(J(x))) > 0 \) and \( \text{sgn}(\det(J(x^2))) < 0 \). Thus, \( e^2 \) is locally asymptotically stable, and \( e^1 \) is not locally asymptotically stable.

**VI. CONCLUSION**

Contraction theory plays an important role in systems and control theory. However, many systems cannot be analyzed using contraction theory. For example, systems with more than a single equilibrium are not contractive w.r.t. any norm.

The notion of \( k \)-contraction provides a useful geometric generalization of contraction theory, but the standard sufficient condition for \( k \)-contraction of \( n \)-dimensional systems may be difficult to verify, as it is based on a compound matrix with dimensions \( \binom{n}{k} \times \binom{n}{k} \). We derived duality relations between compound matrices, and used these to develop a sufficient condition for \( k \)-contraction that does not require to compute any compounds. We demonstrated our approach by deriving a sufficient condition for \( k \)-contraction of a Hopfield neural network. In the particular case where \( k = 2 \), this implies that every bounded solution of the network converges to an equilibrium, which is a useful property when using the network as an associative memory [26]. We believe that the sufficient conditions for \( k \)-contraction derived here will prove useful in more applications.

Several results in this article are proved for \( L_p \) norms, with \( p \in \{1, 2, \infty\} \). It may be of interest to try and generalize the proofs to any \( L_p \) norm. Another interesting direction for future research is to extend the tools described here to control synthesis. In other words, to systematically design a controller such that the closed-loop system satisfies the sufficient condition for \( k \)-contraction.

An analysis of the proof of the bound in (36) shows that it is based on replacing the arithmetic mean of \( (n-k) \) decreasing numbers \( \lambda_1 \geq \cdots \geq \lambda_{n-k} \) by \( \lambda_1 \). An important question is how
conservative is this bound and under what conditions can it be improved.

**APPENDIX: PROOFS**

**Proof of Lemma 5:** We begin by proving a duality for the induced matrix norm. Using the definition of the induced matrix norm and the dual norm of \( L_p \) norms, we have

\[
\|A\|_p = \max_{|x|_p=1} |Ax|_p
\]

\[
= \max_{|x|_p=1} \max_{|y|_q=1} |(Ax)^T y|
\]

\[
= \max_{|x|_p=1} \max_{|y|_q=1} |x^T (A^T y)|
\]

\[
\leq \max_{|x|_p=1} |x|_p |A^T y|_q
\]

\[
= \|A^T\|_q
\]

where we used Hölder’s inequality. Since \((A^T)^T = A\), this implies that \(\|A\|_p = \|A^T\|_q\). Thus

\[
\mu_p(A) = \lim_{h \to 0^+} \frac{\|I_n + hA\|_p - 1}{h}
\]

\[
= \lim_{h \to 0^+} \frac{\|I_n + hA^T\|_q - 1}{h}
\]

\[
= \mu_q(A^T)
\]

and this completes the proof.

**Proof of Corollary 13:** It follows from Theorem 12 that

\[
(A^{[k]})^T = \text{tr}(A) I_r - U^T A^{[n-k]} U
\]

so

\[
(A^{[k]})^T U^T A^{[n-k]} U = (\text{tr}(A) I_r - U^T A^{[n-k]} U) U^T A^{[n-k]} U
\]

\[
= U^T A^{[n-k]} U (\text{tr}(A) I_r - U^T A^{[n-k]} U)
\]

\[
= U^T A^{[n-k]} U (A^{[k]})^T
\]

and this completes the proof.

**Proof of Theorem 15:** Let \( D := (\exp(A))^{(k)} U^T \) \((\exp(A))^{(n-k)} U\). Using the identity \((\exp(A))^{(k)} = \exp(A^{[k]})\) (see, e.g., [37]) and the fact that \( U^T = U^{-1} \) yields

\[
U^T (\exp(A))^{(n-k)} U = U^T \exp(A^{[n-k]} U) = \exp(U^T A^{[n-k]} U).
\]

Thus, \( D = (\exp(A^{[k]}))^{(k)} U^T \) \(\exp(A^{[n-k]} U)\). Therefore, applying Corollary 13 gives

\[
D = \exp((A^{[k]})^T + U^T A^{[n-k]} U)
\]

\[
= \exp(\text{tr}(A) I_r)
\]

\[
= \exp(\text{tr}(A) I_r)
\]

and this completes the proof.

**Proof of Proposition 26:** Consider

\[
\tau_{\infty,k}(J(x))(J(x)) = \text{tr}(J(x)) + (n-k) \mu_1(-J(x))
\]

\[
= \sum_{i=1}^{n} (w_{ii} \phi_i'(x_i) - r_i^{-1})
\]

\[
+ (n-k) \max(c_1(x), \ldots, c_n(x))
\]

where

\[
c_i(x) := r_i^{-1} - w_{ii} \phi_i' + \sum_{j \neq i} |w_{ij} \phi_j'|.
\]

For concreteness, assume that \(\max(c_1(x), \ldots, c_n(x)) = c_1(x)\). Then

\[
\tau_{\infty,k}(J(x)) = -(n-k-1)(w_{ii} \phi_i'(x_i) - r_i^{-1})
\]

\[
+ \sum_{i \neq 1} (w_{ii} \phi_i'(x_i) - r_i^{-1}) + (n-k) \sum_{j \neq 1} |w_{ij} \phi_j'|.
\]

Applying (49) gives

\[
\tau_{\infty,k}(J(x)) \leq -(n-k-1)(w_{ii} m_i - r_i^{-1})
\]

\[
+ \sum_{i \neq 1} (w_{ii} m_i - r_i^{-1}) + (n-k) \sum_{j \neq 1} |w_{ij} M_i|
\]

for all \(x\), and this completes the proof.

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**REFERENCES**

[1] Z. Aminzare and E. D. Sontag, “Contraction methods for nonlinear systems: A brief introduction and some open problems,” in Proc. IEEE 53rd Conf. Decis. Control, Los Angeles, CA, USA, 2014, pp. 3835–3847.

[2] V. Andrieu, B. Jayawardhana, and L. Praly, “Transverse exponential stability and applications,” IEEE Trans. Autom. Control, vol. 61, no. 11, pp. 3396–3411, Nov. 2016.

[3] D. Angeli, M. Banaji, and C. Pantea, “Combinatorial approaches to Hopf bifurcations in systems of interacting elements,” Commun. Math. Sci., vol. 12, no. 6, pp. 1101–1133, 2014.

[4] D. Angeli, M. Ali Al-Radhawi, and E. Sontag, “A robust Lyapunov criterion for non-oscillatory behaviors in biological interaction networks,” IEEE Trans. Autom. Control, vol. 67, no. 7, pp. 3305–3320, Jul. 2022.

[5] D. Angeli, J. E. Ferrell, and E. D. Sontag, “Detection of multistability, bifurcations, and hysteresis in a large class of biological positive-feedback systems,” Proc. Nat. Acad. Sci., vol. 101, no. 7, pp. 1822–1827, 2004.

[6] E. Bar-Shalom, Q. Dulin, and M. Margaliot, “Compound matrices in systems and control theory: A tutorial,” Math. Control, Signals, Syst., vol. 35, pp. 467–521, 2023.

[7] F. L. Bauer, J. Stoer, and C. Witzgall, “Absolute and monotonic norms,” Numer. Math., vol. 3, pp. 257–264, 1961.

[8] F. Bullo, Contraction Theory for Dynamical Systems, 1.1 ed. Seattle, WA, USA: Kindle Direct Publishing, 2023. [Online]. Available: http://motion. me.ucsb.edu/book-ctds

[9] A. Davydov, A. Proskurnikov, and F. Bullo, “Non-Euclidean contrac-

[10] C. Desoer and H. Haneda, “The measure of a matrix as a tool to analyze circuit analysis,” IEEE Trans. Circuits Syst. I. Fundam. Theory Appl., vol. 19, no. 5, pp. 480–486, Sep. 1972.

[11] C. A. Desoer and M. Vidyasagar, Feedback Synthesis: Input-Output Properties, Philadelphia, PA, USA: SIAM, 2009.

[12] Y. Fang and T. G. Kincaid, “Stability analysis of dynamical neural net-

[13] E. Feron, “A note on the Sylvester-Franke theorem,” Amer. Math. Monthly, vol. 101, no. 8, pp. 543–545, 1995.

[14] M. Forti and A. Tesi, “New conditions for global stability of neural sys-

[15] A. Davydov, A. Proskurnikov, and F. Bullo, “Non-Euclidean contrac-

[16] E. Feron, “A note on the Sylvester-Franke theorem,” Amer. Math. Monthly, vol. 101, no. 8, pp. 543–545, 1995.
[15] W. Gerstner, W. M. Kistler, R. Naud, and L. Paninski, *Neuronal Dynamics: From Single Neurons To Networks and Models of Cognition*. Cambridge, U.K.: Cambridge Univ. Press, 2014. [Online]. Available: https://neuraldynamics.epfl.ch

[16] M. Giacugli, V. Andrieu, S. Tarbouriech, and D. Astolfi, “Infinite gain margin, contraction and optimality: An LMI-based design,” *Eur. J. Control*, vol. 68, 2022, Art. no. 100685.

[17] C. Grussler and R. Sepulchre, “Variation diminishing linear time-invariant systems,” *Automatica*, vol. 136, 2022, Art. no. 109985.

[18] J. J. Hopfield, “Neural networks and physical systems with emergent collective computational abilities,” *Proc. Nat. Acad. Sci.*, vol. 79, no. 8, pp. 2554–2558, 1982.

[19] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 2013.

[20] S. Jafarpour, P. Cisneros-Velarde, and F. Bullo, “Weak and semi-contraction for network systems and diffusively coupled oscillators,” *IEEE Trans. Autom. Control*, vol. 67, no. 3, pp. 1285–1300, Mar. 2022.

[21] J. Jiang and J. Shi, “Bistability dynamics in structured ecological models,” in *Spatial Ecology*, S. Cantrell, C. Cosner, and S. Ruan, Eds. London, U.K.: Chapman and Hall, 2009, pp. 33–62.

[22] E. Kaszkurewicz and A. Bhaya. “On a class of globally stable neural circuits,” *IEEE Trans. Circuits Syst. I. Fundam. Theory Appl.*, vol. 41, no. 2, pp. 171–174, 1994.

[23] R. Katz, T. Krieberger, L. Grüne, and M. Margaliot, “On the gain of entrainment in a class of weakly contractive bilinear control systems with applications to the master equation and the ribosome flow model,” 2023. [Online]. Available: https://arxiv.org/abs/2307.03568

[24] I. Z. Kiss, J. C. Miller, and P. L. Simon, *Mathematics of Epidemics on Networks: From Exact to Approximate Models*. Berlin, Germany: Springer, 2017.

[25] L. Kozachkov, M. Ennis, and J-J. E. Slotine, “RNNs of RNNs: Recursive construction of stable assemblies of recurrent neural networks,” in *Proc. Adv. Neural Inf. Process. Syst.*, 2022, pp. 3052–3057. [Online]. Available: https://apensemanticscholar.org/CorpusID:249282160

[26] D. Krotov and J. J. Hopfield, “Dense associative memory for pattern recognition,” in *Proc. Adv. Neural Inf. Process. Syst.*, 2016, pp. 1180–1188.

[27] M. Laurent and N. Kellershohn, “Multistability: A major means of differentiation and evolution in biological systems,” *Trends Biochem. Sci.*, vol. 24, no. 11, pp. 418–22, 1999.

[28] M. Y. Li, J. R. Graef, L. Wang, and J. Karsai, “Global dynamics of a SEIR model with varying total population size,” *Math. Biosciences*, vol. 160, no. 2, pp. 191–213, 1999.

[29] M. Yang and J-S. Maldovan, “On R. A. Smith’s autonomous convergence theorem,” *Rocky Mountain J. Math.*, vol. 25, no. 1, pp. 365–378, 1995.

[30] M. Y. Li and L. Wang, “A criterion for stability of matrices,” *J. Math. Anal. Appl.*, vol. 225, no. 1, pp. 249–264, 1998.

[31] W. Lohmiller and J-J. E. Slotine, “On contraction analysis for non-linear systems,” *Automatica*, vol. 34, pp. 683–696, 1998.

[32] M. Margaliot, E. D. Sontag, and T. Tuller, “Entrainment after small transients,” *Automatica*, vol. 67, pp. 178–184, 2016.

[33] A. N. Michel, J. A. Farrell, and W. Porod, “Qualitative analysis of neural networks,” *IEEE Trans. Circuits Syst.*, vol. 36, no. 2, pp. 229–243, Feb. 1989.

[34] T. Mui, *A Treatise on the Theory of Determinants*, Revised ed. London, U.K.: Longmans, Feb. 1994.

[35] J. S. Muldowney, “Compound matrices and ordinary differential equations,” *Rocky Mountain J. Math.*, vol. 20, pp. 857–872, 1990.

[36] J. S. Maldovan, “Compound matrices and applications,” *Universidad de los Andes, Bogotá*, Colombia, 1998. [Online]. Available: https://sites.uabberta.miami/USER/PISEP/PISEP_2019/lecture_notes/compound_uia.pdf

[37] R. Ofir and M. Margaliot, “The k-component of a difference-algebraic system,” *Automatica*, to be published.

[38] R. Ofir, M. Margaliot, Y. Levron, and J-J. Slotine, “A sufficient condition for k-contraction of the series connection of two systems,” *IEEE Trans. Autom. Control*, vol. 67, no. 9, pp. 4904–5001, Sep. 2022.

[39] A. Pavlov, N. van de Wouw, and H. Nijmeijer, “Frequency response functions and Bode plots for nonlinear convergent systems,” in *Proc. IEEE 45th Conf. Decis. Control*, 2006, pp. 3765–3770.
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