Nonlinear reaction-diffusion equations with delay: some theorems, test problems, exact and numerical solutions

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Abstract. The paper deals with nonlinear reaction-diffusion equations with one or several delays. We formulate theorems that allow constructing exact solutions for some classes of these equations, which depend on several arbitrary functions. Examples of application of these theorems for obtaining new exact solutions in elementary functions are provided. We state basic principles of construction, selection, and use of test problems for nonlinear partial differential equations with delay. Some test problems which can be suitable for estimating accuracy of approximate analytical and numerical methods of solving reaction-diffusion equations with delay are presented. Some examples of numerical solutions of nonlinear test problems with delay are considered.

Keywords: reaction-diffusion equations with delay, partial differential equations with delay, exact solutions, nonlinear test problems, numerical solutions

1. Introduction
Differential equations with delay arise in biology, biophysics, biochemistry, chemistry, medicine, control, climate model theory, fluid mechanics, heat conduction and other areas [1-5]. It is noteworthy that these equations occur in the mathematical theory of artificial neural networks, whose results are used for signal and image processing and in image recognition problems [6,7].

The paper deals with nonlinear reaction-diffusion equations with one or several delays, which allow modelling systems and processes when the rate of change of the unknown depends not only on the current state of the system but also on the values of the unknown at certain times in the past.

2. Theorems of generation of exact solutions for equations with one delay
Consider a class of nonlinear reaction-diffusion equations with delay

\[ u_t = a \Delta u + f(u - kw) + ug(u - kw) + wh(u - kw), \quad w = u(x, t - \tau), \]

(1)

where \( f(z), g(z), \) and \( h(z) \) are arbitrary functions and \( \tau > 0, a > 0, \) and \( k \) are free parameters.

Theorem 1. With \( k > 0, \) Eq. (1) admits a solution

\[ u = \theta(x) + e^{c_1 t} \xi(x, t), \quad c_1 = (\ln k)/\tau, \]

where \( \theta(x) \) and \( \xi(x, t) \) are arbitrary functions.
where the function $\theta = \theta(x)$ satisfies the independent stationary equation
\begin{equation}
 a \Delta \theta + f(\zeta) + (g(\zeta) + h(\zeta)) \theta = 0, \quad \zeta = (1-k)\theta;
\end{equation}
and the function $\xi = \xi(x,t)$ is a $\tau$-periodic solution of the equation
\begin{equation}
 \xi_t = a \Delta \xi + \left( g(\zeta) + \frac{1}{k} h(\zeta) - c_1 \right) \xi, \quad \xi(x,t) = \xi(x,t+\tau),
\end{equation}
which is linear with respect to $\xi$.

With $k = 1$, which corresponds to $\zeta = 0$, Eq. (2) becomes linear and inhomogeneous; shifting $\theta$ by a constant, we obtain the Helmholtz equation, whose exact solutions can be found in [8].

With $k \neq 1$, the simplest solution of Eq. (2) is a constant $\theta_0 = \zeta_0/(1-k)$, where $\zeta_0$ is a root of the algebraic (transcendental) equation $(1-k) f(\zeta_0) + \zeta_0 (g(\zeta_0) + h(\zeta_0)) = 0$.

In the one-dimensional case, the general solution of Eq. (2) can be expressed in terms of integrals. For a number of functions $f(\zeta)$, $g(\zeta)$, and $h(\zeta)$, exact solutions of Eq. (2) are presented in [9] (the one-dimensional case) and [10] (the two- and three-dimensional cases).

Eq. (3) admits stationary solutions $\xi_n = \xi_0(x)$, as well as $\tau$-periodic solutions of the form
\begin{equation}
 \xi_n = \varphi_n(x) \cos(\beta_n t) + \psi_n(x) \sin(\beta_n t), \quad \beta_n = \frac{2 \pi n}{\tau}, \quad n = 1, 2, \ldots
\end{equation}

With $g(\zeta) = h(\zeta) \equiv 0$, Eq. (3) becomes a linear equation with constant coefficients and does not depend on $\theta$. Formulas of $\tau$-periodic solutions of one-dimensional equations of type (3) are presented in [11, 12].

**Theorem 2.** With $k < 0$, Eq. (1) admits a solution
\begin{equation}
 u = \theta(x) + c_2 e^{\gamma t}, \quad c_2 = (\ln|k|)/\tau,
\end{equation}
where the function $\theta = \theta(x)$ satisfies Eq. (2); and the function $\eta = \eta(x,t)$ is a $\tau$-aperiodic solution of the equation
\begin{equation}
 \eta_t = a \Delta \eta + \left( g(\zeta) + \frac{1}{k} h(\zeta) - c_2 \right) \eta, \quad \zeta = (1-k)\theta, \quad \eta(x,t) = -\eta(x,t+\tau),
\end{equation}
which is linear with respect to $\eta$.

Exact $\tau$-aperiodic solutions of Eq. (4) are sought in the form
\begin{equation}
 \eta_n = \varphi_n(x) \cos(\beta_n t) + \psi_n(x) \sin(\beta_n t), \quad \beta_n = \frac{\pi (2n-1)}{\tau}, \quad n = 1, 2, \ldots
\end{equation}

With $g(\zeta) = h(\zeta) \equiv 0$, Eq. (4) becomes a linear equation with constant coefficients and does not depend on $\theta$. Formulas of $\tau$-aperiodic solutions of one-dimensional equations of type (4) are presented in [11, 12].

**Theorem 3.** With $k = 1$, Theorem 1 is valid. In addition, Eq. (1) admits a solution
\begin{equation}
 u = t \varphi(x) + \psi(x),
\end{equation}
where the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ satisfies the system of stationary equations
\begin{equation}
 a \Delta \varphi + \varphi(g(\tau \varphi) + h(\tau \varphi)) = 0, \\
a \Delta \psi + \psi(g(\tau \varphi) + h(\tau \varphi)) = \varphi - f(\tau \varphi) + \tau \varphi h(\tau \varphi).
\end{equation}

In the one-dimensional case, the general solution of the first equation in (6) can be expressed in terms of integrals for arbitrary $g(z)$ and $h(z)$. For the two-dimensional and three-dimensional cases, some exact solutions are presented in [10].

**Remark 1.** Exact solutions for one-dimensional PDEs of type (1) and other reaction-diffusion equations including those of hyperbolic type can be found in [11–15].
3. Exact solutions for a class of equations with several delays
Consider a class of more complex nonlinear reaction-diffusion equations with several delays
\[ u_t = a \Delta u_{xx} + bw_0 + f(u - w_1, \ldots, u - w_n), \quad w_j = u(x, t - \tau_j), \quad (7) \]
where \( f \) is an arbitrary function; \( \tau_j \geq 0, j = 0, 1, \ldots, n; a > 0. \)

**Exact solution 1.** Eq. (7) admits an additive separable solution
\[ u = \varphi(x) + \psi(t), \]
where the function \( \varphi(x) \) satisfies the Helmholtz equation
\[ a \Delta \varphi(x) + b \varphi(x) = 0; \quad (8) \]
and the function \( \psi(t) \) is a solution of an ordinary differential equation with several delays
\[ \psi'(t) = b\psi(t - \tau_0) + f(\psi(t - \psi(t - \tau_1), \ldots, \psi(t - \psi(t - \tau_n))). \]

Exact solutions of the Helmholtz equation can be found in [8].

**Remark 2.** Exact solutions for some ODEs are presented in [15]. Some numerical methods for ODEs and related issues are studied in [16–18].

**Exact solution 2.** Eq. (7) admits a solution of the form (5), where the function \( \varphi = \varphi(x) \) satisfies Eq. (8); and \( \psi = \psi(x) \) is a solution of the inhomogeneous Helmholtz equation
\[ a \Delta \psi + b \psi = \varphi - f(\tau_1 \varphi, \ldots, \tau_n \varphi). \]

4. Principles of construction and use of test problems
The following principles are useful in the process of construction, selection and use of test problems, designed for estimating accuracy of approximate analytical and numerical methods of solving nonlinear partial differential equations with delay.

1. Test problems are constructed on the basis of exact solutions of a considered or wider class of nonlinear partial differential equations with delay.
2. Test problems and their solutions should contain a number of free parameters, which can be varied within wide limits.
3. It is better to choose simple test problems that have solutions in elementary functions.
4. One should use several test problems to estimate accuracy of numerical methods.
5. It is helpful to consider test problems with larger gradients of the unknowns in the initial data or boundary conditions (for example, rapidly oscillating initial data).
6. It is useful to consider rapidly growing solutions for sufficiently large times.
7. It is desirable to test numerical methods near the values of the parameters that determine the region of instability (ill-posed problems).

Well-chosen test problems make it possible to compare and improve efficient numerical methods and to filter out unusable ones. The theorems described in Section 2 allow obtaining exact solutions, which afterwards are used to construct corresponding test problems for reaction-diffusion equations with delay.

5. Test problems and their exact solutions
Consider a one-dimensional equation of type (1) with a nonlinear kinetic function
\[ u_t = au_{xx} + bu(1 - s(u - kw)), \quad w = u(x, t - \tau). \quad (9) \]
Eq. (9) contains five parameters: $a > 0$, $b$, $s$, $k$, and $\tau > 0$. In a degenerate case, with $k = 0$ or $\tau = 0$, Eq. (9) is referred to as the Fisher’s equation.

**Test problem 1.** Eq. (9) with the following initial and boundary conditions:

$$
u(x, t) = e^{ct}(\cos(\pi x/2) + 2 \sin(\pi x/2)), \quad c = (\ln k)/\tau, \quad 0 \leq x \leq 1, \quad -\tau \leq t \leq 0;$$

$$
u(0, t) = e^{ct}, \quad t \geq 0;$$

$$
u(1, t) = 2e^{ct}, \quad t \geq 0. \quad (10)$$

With $b = (\ln k)/\tau + a\pi^2/4$ and $k > 0$, test problem (9), (10) has the exact solution

$$
u(x, t) = e^{ct}(\cos(\pi x/2) + 2 \sin(\pi x/2)), \quad c = (\ln k)/\tau, \quad 0 \leq x \leq 1, \quad t > 0. \quad (11)$$

**Test problem 2.** Eq. (9) with the following initial and boundary conditions:

$$
u(x, t) = \cosh^{-2}(x) + e^{ct} \cosh^3(x), \quad c = (\ln k)/\tau, \quad 0 \leq x \leq 1, \quad -\tau \leq t \leq 0;$$

$$
u(0, t) = 1 + e^{ct}, \quad t \geq 0;$$

$$
u(1, t) = \cosh^{-2}(1) + e^{ct} \cosh^3(1), \quad t \geq 0. \quad (12)$$

With $b = -4a$, $k = e^{5\sigma}$, and $s = \frac{3}{2(1-k)}$, test problem (9), (12) has the exact solution

$$
u(x, t) = \cosh^{-2}(x) + e^{ct} \cosh^3(x), \quad c = (\ln k)/\tau, \quad 0 \leq x \leq 1, \quad t > 0. \quad (13)$$

6. Numerical solutions of test problem (10), (11) based on the method of lines

The idea of the method of lines is to reduce a partial differential equation to a system of ordinary differential equations by approximation of the spatial derivatives by finite differences [19, 20].

Let us define spatial mesh points as $x_m = mh$, where $m = 0, 1, \ldots, M$ and $h = 1/M$ is a step size. Then we proceed with discretization of the initial-boundary value problem and get a system of $M - 1$ regular ordinary differential equations and two boundary conditions. The resulting problem with the initial conditions can be solved by well-known numerical methods, such as the implicit Runge-Kutta methods.

The resulting discretized system for test problem (9), (10) has the form

$$
\begin{align*}
(u_m)'_t &= a\delta_{xx}u_m + bu_m(1 - s(u_m - kw_m)), \quad m = 1, \ldots, M - 1, \quad 0 \leq t \leq T; \\
u_0(t) &= e^{ct}, \quad 0 \leq t \leq T; \\
u_M(t) &= 2e^{ct}, \quad 0 \leq t \leq T; \\
u_m(t) &= e^{ct}(\cos(\pi x_m/2) + 2 \sin(\pi x_m/2)), \quad m = 0, 1, \ldots, M, \quad -\tau \leq t \leq 0,
\end{align*}
$$

where $\delta_{xx}$ is a finite difference operator, $\delta_{xx}u_m = (u_{m+1} - 2u_m + u_{m-1})/h^2$; $T$ is a time interval of calculations. The discretized system for test problem (9), (12) is obtained in the same way.

The numerical experiment has been carried out in the computer algebra system Mathematica. The system has been solved by the predefined function NDSolve, which is a general numerical differential equation solver of Mathematica. The option Method of the function NDSolve allows choosing the implicit Runge-Kutta method with the Lobatto IIIC coefficients and the second order of local accuracy. This helps to solve stiff systems that naturally occur after discretization of a partial differential equation with delay. Information about initialization of differential equations with delay, implementation of the method of lines, as well as description of the function NDSolve can be found on the webpages listed in the footnote$^1$.

$^1$ http://reference.wolfram.com/language/tutorial/NDSolveDelayDifferentialEquations.html
http://reference.wolfram.com/language/tutorial/NDSolveMethodOfLines.html
The calculations have been carried out from \( t = 0 \) to \( t = 50\tau \) with the following values of parameters: \( a = 1, \quad k = 0.5, \quad s = 0.2 \).

The numerical solutions of system (13) for \( M = 50 \) and delay times \( \tau = 0.05 \) and \( \tau = 0.5 \) together with exact solutions (11) of test problem (9), (10) are shown in logarithmic scale on Figure 1. The difference between the exact and numerical solutions is visible only when both of them are practically zero.

![Figure 1](image)

**Figure 1.** Graphs in logarithmic scale of the exact solution (solid line) and numerical solution with \( M = 50 \) (circles) of problem (9), (10) for \( a = 1, \quad k = 0.5, \quad s = 0.2 \) and a) \( \tau = 0.05 \), b) \( \tau = 0.5 \). The upper, middle and lower lines correspond to \( t_1 = 2\tau, \quad t_2 = 10\tau, \quad \) and \( t_3 = 20\tau \) respectively.

The absolute errors of the numerical solutions of problem (9), (10) for \( M = 10, \quad M = 50, \quad \) and \( M = 100 \) and delay times \( \tau = 0.05, \quad \tau = 0.1, \quad \) and \( \tau = 0.5 \) are presented in Table 1. The percent errors for this case are not relevant due to the solution approaches zero thus the program deals with very small numbers.

| \( \tau \) | \( M = 10 \) | \( M = 50 \) | \( M = 100 \) |
|-----|-----|-----|-----|
| 0.05 | \( 2.8 \cdot 10^{-4} \) | \( 1.3 \cdot 10^{-5} \) | \( 4.5 \cdot 10^{-6} \) |
| 0.1 | \( 4.6 \cdot 10^{-4} \) | \( 2.0 \cdot 10^{-5} \) | \( 6.1 \cdot 10^{-6} \) |
| 0.5 | \( 9.9 \cdot 10^{-4} \) | \( 4.1 \cdot 10^{-5} \) | \( 1.2 \cdot 10^{-5} \) |

7. Numerical solutions of the problem with homogeneous boundary conditions and stationary initial data based on the method of lines

Consider Eq. (9) with the following initial and boundary conditions:

\[
\begin{align*}
    u(0, t) &= u(1, t) = 0, \quad t > 0; \\
    u(x, t) &= 10 \sin(\pi x), \quad 0 \leq x \leq 1, \quad -\tau \leq t \leq 0.
\end{align*}
\]

(14)

The calculations have been carried out from \( t = 0 \) to \( t = 10 \) for two qualitatively different cases: \( \tau = 0 \) and \( \tau = 0.5 \). The numerical solutions of problem (9), (14) with \( M = 50 \) are shown.
on Figure 2. The parameters have the following values: \( a = 1, \ b = 10, \ k = 0.5, \ s = 0.2 \). In case without delay, the solution approaches zero faster than in case with delay. The maximum absolute difference between the two solutions is 3.1 and it is achieved at \( t \approx 0.51 \) and \( x = 0.5 \).

Figure 2. Graphs of the numerical solutions of problem (9), (14) for \( a = 1, \ b = 10, \ k = 0.5, \ s = 0.2 \) with a) \( \tau = 0 \) and b) \( \tau = 0.5 \). The dashed, dotted and solid lines correspond to \( t_1 = 0, \ t_2 = 0.5, \) and \( t_3 = 5 \) respectively.

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References

[1] Wu J 1996 Theory and Applications of Partial Functional Differential Equations (New York: Springer)
[2] Huang J and Zou X 2002 J. Math. Anal. Appl. 271 455-66
[3] Wang X and Li Z 2007 Nonlinear Anal. Theory Methods Appl. 67 2699-711
[4] Polyanin A D and Zhurov A I 2013 Int. J. Non-Linear Mech. 54 115-26
[5] Polyanin A D and Zhurov A I 2013 Int. J. Non-Linear Mech. 57 116-22
[6] Wang L and Gao Y 2006 Phys. Lett. A 350 342-48
[7] Lu J G 2008 Chaos Solitons Fractals 35 116-25
[8] Polyanin A D and Nazarkinetskii V E 2016 Handbook of Linear Partial Differential Equations for Engineers and Scientists, Second Edition (Boca Raton: CRC Press)
[9] Polyanin A D and Zaitsev V F 2003 Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition (Boca Raton: Chapman & Hall/CRC Press)
[10] Polyanin A D and Zaitsev V F 2012 Handbook of Nonlinear Partial Differential Equations (Boca Raton: CRC Press)
[11] Polyanin A D and Zhurov A I 2014 Commun. Nonlinear. Sci. Numer. Simul. 19 417-30
[12] Polyanin A D and Zhurov A I 2014 Int. J. Non-Linear Mech. 62 33-40
[13] Polyanin A D and Zhurov A I 2014 Commun. Nonlinear. Sci. Numer. Simul. 19 409-16
[14] Polyanin A D, Sorokin V G and Vyazmin A V 2015 Theor. Found. Chem. Eng. 49 622-35
[15] Polyanin A D and Sorokin V G 2015 Appl. Math. Lett. 46 38-43
[16] Bellen A and Zennaro M 2003 Numerical Methods for Delay Differential Equations (Oxford: Oxford university press)
[17] Kuang J and Cong Y 2005 Stability of Numerical Methods for Delay Differential Equations (Beijing: Science Press)
[18] Shampine L F and Thompson S 2009 Delay Differential Equations: Recent Advances and New Directions (New York: Springer) chapter 9 pp 245-71
[19] Van Der Houwen P J, Sommeijer B P and Baker C T H 1986 IMA J. Num. Anal. 6 1-23
[20] Rihan F A 2010 Numer. Methods Partial Differ. Equ. 26 1556-71