FRACTIONAL SCHRÖDINGER EQUATIONS WITH POTENTIALS OF HIGHER-ORDER SINGULARITIES

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Abstract. In this paper we consider the space-fractional Schrödinger equation with a singular potential. We show that it has a so-called very weak solutions. The uniqueness and consistency results are proved in an appropriate sense. Numerical simulations are done, and a particles accumulating effect is observed. From the mathematical point of view a "splitting of the strong singularity" phenomena is observed.

1. Introduction

In the present paper we investigate the fractional Schrödinger equation with a singular potential. Namely, we consider the Cauchy problem

\[ \begin{aligned}
    iu_t(t, x) + (-\Delta)^s u(t, x) + p(x)u(t, x) &= 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\
    u(0, x) &= u_0(x),
\end{aligned} \tag{1.1} \]

where \( p \) is assumed to be non-negative, and \( s > 0 \). We use the fractional Laplacian instead of the classical one and prove that the problem has a so-called "very weak solution".

While the study of the Fractional Schrödinger equation is mathematically challenging, from the physical point of view it is a natural extension of the standard Schrödinger equation when the Brownian trajectories in Feynman path integrals are replaced by Levy flights. The FSE was introduced by Laskin in quantum mechanics [Las00], [Las02] and more recently, it was proposed as a model in optics by Longhi [Lon15] and applied to laser implementation. For more general overview about the FSE and its related topics in physics, one can see [Las18].

On the other hand, our intention to consider singular potentials is also natural from a physical point of view. It can describe a particle which is free to move in two regions of space with a barrier between the two regions. For example, an electron can move almost freely in a conducting material, but if two conducting surfaces are put close together, the interface between them acts as a barrier for the electron. One can for instance see [OCV10], [LRSRM13] and the references mentioned there.

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As mentioned before, our aim is to prove the well-posedness of the Cauchy problem (1.1) in the very weak sense. The appearance of the concept of very weak solutions traces back to the paper [GR15], where the authors introduced the concept for the analysis of second order hyperbolic equations with irregular coefficients in time. It was later applied in [MRT19], [RT17a], and [RT17b] for the study of different physical models, and some numerical analysis was done in [ART19]. We want here to apply it for the fractional Schrödinger equation.

In the present paper we will use the following notations:

- \( f \lesssim g \) means that there exists a positive constant \( C \) such that \( f \leq Cg \).
- The fractional Sobolev space \( H^s(\mathbb{R}^d) \) is defined as:
  \[
  H^s(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) : \|u\|_{H^s} := \|u\|_{L^2} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^2} < +\infty \right\}.
  \]
- We will denote by \( \| \cdot \|_k \), for \( k \geq 0 \), the norm defined by
  \[
  \|u(t, \cdot)\|_k := \sum_{l=0}^{k} \|\partial_t^l u(t, \cdot)\|_{L^2} + \|(-\Delta)^{\frac{s}{2}} u(t, \cdot)\|_{L^2},
  \]
  and we will simply denote it by \( \|u(t, \cdot)\| \), when \( k = 0 \). We note that \( \|u(t, \cdot)\| \approx \|u(t, \cdot)\|_{H^s} \).

2. Main results

For \( s > 0 \) and \( T > 0 \), we consider the Cauchy problem

\[
\begin{cases}
  iu_t(t, x) + (-\Delta)^s u(t, x) + p(x)u(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\
  u(0, x) = u_0(x),
\end{cases}
\]

where the potential \( p \) is non-negative and singular. We start by stating the following Lemma for the case when the coefficient \( p \) is a regular function.

**Lemma 2.1.** Let \( p \in L^\infty(\mathbb{R}^d) \) be non-negative and assume that \( u_0 \in H^s(\mathbb{R}^d) \), for \( s > 0 \). Then the estimate

\[
\|u(t, \cdot)\|_{H^s} \lesssim (1 + \|p\|_{L^\infty}) \|u_0\|_{H^s},
\]

holds for the unique solution \( u \in C([0, T]; H^s) \) to the Cauchy problem (2.1).

**Proof.** We multiply the equation in (2.1) by \( u_t \) and by integrating, we get

\[
\text{Re} (\langle i\partial_t u(t, \cdot), \partial_t u(t, \cdot) \rangle_{L^2}) + \langle (-\Delta)^s u(t, \cdot), \partial_t u(t, \cdot) \rangle_{L^2} = 0.
\]

It is easy to see that

\[
\text{Re} (\langle i\partial_t u(t, \cdot), \partial_t u(t, \cdot) \rangle_{L^2}) = 0,
\]

\[
\text{Re} (\langle p(\cdot) u(t, \cdot), \partial_t u(t, \cdot) \rangle_{L^2}) = \frac{1}{2} \partial_t \|p^{\frac{1}{2}}(\cdot) u(t, \cdot)\|_{L^2}^2,
\]

and

\[
\text{Re} (\langle (-\Delta)^s u(t, \cdot), \partial_t u(t, \cdot) \rangle_{L^2}) = \frac{1}{2} \partial_t \|(-\Delta)^{\frac{s}{2}} u(t, \cdot)\|_{L^2}^2.
\]
The last equality is a consequence of the fact that \((-\Delta)^s\) is a self-adjoint operator. Let us denote by

\[ E(t) := \|(-\Delta)^s u(t, \cdot)\|^2_{L^2} + \|p^{\frac{1}{2}}(\cdot) u(t, \cdot)\|^2_{L^2}. \]

It follows from (2.3) that \(\partial_t E(t) = 0\) and thus

\[ E(t) = E(0). \]

Therefore

\[ \|p^{\frac{1}{2}} u(t, \cdot)\|^2_{L^2} \lesssim \|(-\Delta)^s u_0\|^2_{L^2} + \|p\|_{L^\infty} \|u_0\|^2_{L^2}, \]

and

\[ \|(-\Delta)^s u(t, \cdot)\|^2_{L^2} \lesssim \|(-\Delta)^s u_0\|^2_{L^2} + \|p\|_{L^\infty} \|u_0\|^2_{L^2}, \]

where we used that \(\|p^{\frac{1}{2}} u_0\|^2_{L^2}\) can be estimated by

\[ \|p^{\frac{1}{2}} u_0\|^2_{L^2} \lesssim \|p\|_{L^\infty} \|u_0\|^2_{L^2}. \]

Moreover, it follows that

\[ \|(-\Delta)^s u(t, \cdot)\|_{L^2} \lesssim \left(1 + \|p\|_{L^\infty}^{\frac{1}{2}}\right) \|u_0\|_{H^s}. \]

Let us estimate \(u\). After application of the Fourier transformation in (2.1), we get the auxiliary Cauchy problem

\[ i\hat{\dot{u}}(t, \xi) + |\xi|^{2s} \hat{u}(t, \xi) = \hat{f}(t, \xi); \quad \hat{u}(0, \xi) = \hat{u}_0(\xi), \]

where \(\hat{u}, \hat{f}\) denote the Fourier transforms of \(u\) and \(f\) with respect to the spatial variable \(x\) and \(f(t, x) := -p(x)u(t, x)\). Using Duhamel’s principle (see, e.g. [ER18]), we get the following representation of the solution to the Cauchy problem (2.6),

\[ \hat{u}(t, \xi) = \hat{u}_0(\xi) \exp(-i|\xi|^{2s}t) + \int_0^t \exp\left(-i|\xi|^{2s}(t - s)\right) \hat{f}(s, \xi)ds. \]

Taking the \(L^2\) norm in (2.7) and using the fact that \(\exp(-i|\xi|^{2s}t)\) is a unitary operator we get the estimate

\[ \|\hat{u}(\cdot, \cdot)\|_{L^2} \lesssim \|\hat{u}_0\|_{L^2} + \int_0^T \|\hat{f}(s, \cdot)\|_{L^2} ds. \]

Using the Plancherel-Parseval formula, the estimate (2.4) and the fact that \(\|f(t, \cdot)\|_{L^2} = \|p(\cdot) u(t, \cdot)\|_{L^2}\) can be estimated by

\[ \|p(\cdot) u(t, \cdot)\|_{L^2} \lesssim \|p\|_{L^\infty}^{\frac{1}{2}} \|p^{\frac{1}{2}} u(t, \cdot)\|_{L^2}, \]

we arrive at

\[ \|u(t, \cdot)\|_{L^2} \lesssim \left(1 + \|p\|_{L^\infty}^{\frac{1}{2}}\right)^2 \|u_0\|_{H^s}. \]

By summing (2.5) and (2.9) we get our estimate and the lemma is proved. \(\Box\)
Remark 2.1. Requiring further regularity on the initial data $u_0$, one can prove that the estimate

$$\|u(t, \cdot)\|_k \lesssim (1 + \|p\|_{L^\infty}) \|u_0\|_{H^s(1+2k)},$$

holds for all $k \geq 0$. For this, we use the estimate (2.9) and proceed by induction on $k \geq 1$, on the property that, if $v_k := \partial_t^k u$, where $u$ is the solution to the Cauchy problem (2.1), solves the equation

$$i\partial_t v_k(t, x) + (-\Delta)^s v_k(t, x) + p(x)v_k(t, x) = 0,$$

with initial data $v_k(0, x)$, then $v_{k+1} = \partial_t v_k$ solves the same equation with initial data $v_{k+1}(0, x) = -i(-\Delta)^s v_k(0, x) - ip(x)v_k(0, x)$.

2.1. Existence of very weak solutions. In what follows, we consider the case when the potential $p$ is strongly singular, we have in mind the $\delta$ or the $\delta^2$-functions. As mentioned above, we want to prove the existence of a very weak solution to the Cauchy problem (2.1). We first regularise the coefficient $p$ and the data $u_0$ by convolution with a suitable mollifier $\psi$, we obtain families of smooth functions $(p_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, namely

$$p_\varepsilon(x) = p * \psi_\varepsilon(x) \text{ and } u_{0,\varepsilon}(x) = u_0 * \psi_\varepsilon(x),$$

where

$$\psi_\varepsilon(x) = \varepsilon^{-1}\psi(x/\varepsilon), \varepsilon \in (0, 1],$$

and the function $\psi$ is a Friedrichs-mollifier, i.e. $\psi \in C_0^\infty(\mathbb{R}^d)$, $\psi \geq 0$ and $\int \psi = 1$. The above regularisation works when $p$ is at least a distribution. For more generality, we will make assumptions on the regularisations $(p_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, instead of making them on $p$ and $u_0$. That is, we assume that there exist $N, N_0 \in \mathbb{N}$ such that

$$(2.10) \quad \|p_\varepsilon\|_{L^\infty} \leq C\varepsilon^{-N}$$

and

$$(2.11) \quad \|u_{0,\varepsilon}\|_{H^s} \leq C_0\varepsilon^{-N_0}.$$

We have the following definition.

Definition 1 (Moderateness).

(i) We say that the net of functions $(f_\varepsilon)_\varepsilon$ is $H^s$-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that

$$\|f_\varepsilon\|_{H^s} \leq c\varepsilon^{-N}.$$

(ii) We say that the net of functions $(g_\varepsilon)_\varepsilon$ is $L^\infty$-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that

$$\|g_\varepsilon\|_{L^\infty} \leq c\varepsilon^{-N}.$$

(iii) We say that the net of functions $(u_\varepsilon)_\varepsilon$ from $C([0, T]; H^s)$ is $H^s$-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that

$$\|u_\varepsilon(t, \cdot)\|_{H^s} \leq c\varepsilon^{-N}$$

for all $t \in [0, T]$. 
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Remark 2.2. We see that, \((u_0, \varepsilon)\) and \((p_\varepsilon)\) are moderate by assumption. We also note that such assumptions are natural for distributional coefficients in the sense that regularisations of distributions are moderate. Precisely, by the structure theorems for distributions (see, e.g. [FJ98]), we know that

\[\text{(2.12)}\]

Compactly supported distributions \(\mathcal{E}'(\mathbb{R}^d) \subset \{C^\infty(\mathbb{R}^d) - \text{moderate families}\},\)

and we see from (2.12), that a solution to a Cauchy problem may not exist in the sense of distributions, while it may exist in the set of \(C^\infty\)-moderate functions.

Now, let us introduce the notion of a very weak solution to the Cauchy problem (2.1).

Definition 2 (Very weak solution). The net \((u_\varepsilon)\) \(\in C([0, T] ; H^s)\) is said to be a very weak solution of order \(s\) to the Cauchy problem (2.1) if there exists an \(L^\infty\)-moderate regularisation of the coefficient \(p\) and \(H^s\)-moderate regularisation of \(u_0\) such that

\[\text{(2.13)}\]

\[
\begin{aligned}
    i\partial_t u_\varepsilon(t, x) + (-\Delta)^s u_\varepsilon(t, x) + p_\varepsilon(x)u_\varepsilon(t, x) &= 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\
    u_\varepsilon(0, x) &= u_{0, \varepsilon}(x),
\end{aligned}
\]

for all \(\varepsilon \in (0, 1]\), and is \(C\)-moderate.

We can now state the following theorem, the proof of which follows immediately from the definitions. In what follows we understand \(p \geq 0\) as its regularisations \(p_\varepsilon\) satisfying \(p_\varepsilon \geq 0\) for all \(\varepsilon \in (0, 1]\). This is clearly the case when \(p\) is a distribution.

Theorem 2.2 (Existence). Let \(p \geq 0\) and \(s > 0\). Assume that the regularisations of the coefficient \(p\) and the Cauchy data \(u_0\) satisfy the assumptions (2.10) and (2.11). Then the Cauchy problem (2.1) has a very weak solution.

Proof. The coefficient \(p\) and the data \(u_0\) are moderate by assumption. To prove that a very weak solution exists, we need to prove that the net \((u_\varepsilon)\), solution to the family of regularized Cauchy problems

\[
\begin{aligned}
    i\partial_t u_\varepsilon(t, x) + (-\Delta)^s u_\varepsilon(t, x) + p_\varepsilon(x)u_\varepsilon(t, x) &= 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\
    u_\varepsilon(0, x) &= u_{0, \varepsilon}(x),
\end{aligned}
\]

is \(C\)-moderate. Indeed, using the assumptions (2.10), (2.11) and the energy estimate (2.2), we arrive at

\[\|u(t, \cdot)\| \lesssim \varepsilon^{-N_0-N}.\]

The net \((u_\varepsilon)\) is then \(C\)-moderate and the existence of a very weak solution is proved.

In order to prove uniqueness and consistency of the very weak solution in the forthcoming theorems, we need the following lemma.

Lemma 2.3. Let \(u_0 \in H^s(\mathbb{R}^d)\) and assume that \(p \in L^\infty(\mathbb{R}^d)\) is non-negative. Then, the energy conservation

\[\text{(2.14)}\]

\[\|u(t, \cdot)\|_{L^2} = \|u_0\|_{L^2},\]
holds for all \( t \in [0,T] \), for the unique solution \( u \in C([0,T];H^s) \) to the Cauchy problem (2.1).

**Proof.** We first multiply the equation in (2.1) by \(-i\), we obtain
\[
u_t(t,x) - i(-\Delta)^s u(t,x) - ip(x)u(t,x) = 0.
\]
Multiplying the last equation by \( u \), integrating over \( \mathbb{R}^d \) and taking the real part, we get
\[
\text{Re} \langle u_t(t,\cdot), u(t,\cdot) \rangle_{L^2} - i \langle (-\Delta)^s u(t,\cdot), u(t,\cdot) \rangle_{L^2} - i \langle p(\cdot)u(t,\cdot), u(t,\cdot) \rangle_{L^2} = 0.
\]
Using similar arguments as in lemma 2.1, it is easy to see that
\[
\text{Re} \langle u_t(t,\cdot), u(t,\cdot) \rangle_{L^2} = \frac{1}{2} \partial_t \| u(t,\cdot) \|_{L^2}^2
\]
and that
\[
\text{Re} \langle (-\Delta)^s u(t,\cdot), u(t,\cdot) \rangle_{L^2} = \text{Re} \langle p(\cdot)u(t,\cdot), u(t,\cdot) \rangle_{L^2} = 0.
\]
Thus, we have energy conservation, i.e. \( \| u(t,\cdot) \|_{L^2} \) is constant for all \( t \in [0,T] \) and the statement is proved. \( \square \)

2.2. **Uniqueness.** We prove the uniqueness of a very weak solution to the Cauchy problem (2.1) in the sense of the following definition.

**Definition 3 (Uniqueness).** We say that the Cauchy problem (2.1) has a unique very weak solution, if for all families of regularisations \((p_\varepsilon)_\varepsilon\), \((\tilde{p}_\varepsilon)_\varepsilon\), \((u_0,\varepsilon)_\varepsilon\) and \((\tilde{u}_0,\varepsilon)_\varepsilon\) of the coefficient \( p \) and the Cauchy data \( u_0 \), satisfying
\[
\| p_\varepsilon - \tilde{p}_\varepsilon \|_{L^\infty} \leq C_k \varepsilon^k \text{ for all } k > 0
\]
and
\[
\| u_0,\varepsilon - \tilde{u}_0,\varepsilon \|_{L^2} \leq C_l \varepsilon^l \text{ for all } l > 0,
\]
we have
\[
\| u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot) \|_{L^2} \leq C_N \varepsilon^N
\]
for all \( N > 0 \), where \((u_\varepsilon)_\varepsilon\) and \((\tilde{u}_\varepsilon)_\varepsilon\) are the families of solutions to the corresponding regularized Cauchy problems.

**Theorem 2.4 (Uniqueness).** Let \( p \geq 0 \) and \( u_0 \in H^s(\mathbb{R}^d) \) and assume that they satisfy the assumptions (2.10) and (2.11). Then, the Cauchy problem (2.1) has a unique very weak solution.

**Proof.** Let \((p_\varepsilon)_\varepsilon\), \((\tilde{p}_\varepsilon)_\varepsilon\) and \((u_0,\varepsilon)_\varepsilon\), \((\tilde{u}_0,\varepsilon)_\varepsilon\), regularisations of \( p \) and \( u_0 \), satisfying
\[
\| p_\varepsilon - \tilde{p}_\varepsilon \|_{L^\infty} \leq C_k \varepsilon^k \text{ for all } k > 0
\]
and
\[
\| u_0,\varepsilon - \tilde{u}_0,\varepsilon \|_{L^2} \leq C_l \varepsilon^l \text{ for all } l > 0,
\]
and let us denote by \( U_\varepsilon(t, x) := u_\varepsilon(t, x) - \tilde{u}_\varepsilon(t, x) \), where \((u_\varepsilon)_\varepsilon\) and \((\tilde{u}_\varepsilon)_\varepsilon\) are the families of solutions to the regularized Cauchy problems, corresponding to the families \((p_\varepsilon, u_\varepsilon)_\varepsilon\) and \((\tilde{p}_\varepsilon, \tilde{u}_0)_\varepsilon\). Then, \( U_\varepsilon \) solves the Cauchy problem
\[
\begin{cases}
  i\partial_t U_\varepsilon(t, x) + (-\Delta)^s U_\varepsilon(t, x) + p_\varepsilon(x) U_\varepsilon(t, x) = f_\varepsilon(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\
  U_\varepsilon(0, x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x),
\end{cases}
\]
where
\[
f_\varepsilon(t, x) = (\tilde{p}_\varepsilon(x) - p_\varepsilon(x)) \tilde{u}_\varepsilon(t, x).
\]
Let \((V_\varepsilon)_\varepsilon\) and \((W_\varepsilon)_\varepsilon\), the families of solutions to the auxiliary Cauchy problems
\[
\begin{cases}
  i\partial_t V_\varepsilon(x, t; s) + (-\Delta)^s V_\varepsilon(x, t; s) + p_\varepsilon(x) V_\varepsilon(x, t; s) = 0, \\
  V_\varepsilon(x, s; s) = f_\varepsilon(s, x),
\end{cases}
\]
and
\[
\begin{cases}
  i\partial_t W_\varepsilon(x, t; s) + (-\Delta)^s W_\varepsilon(x, t; s) + p_\varepsilon(x) W_\varepsilon(x, t; s) = 0, \\
  W_\varepsilon(0, x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x).
\end{cases}
\]
Using Duhamel’s principle, \( U_\varepsilon \) is given by
\[
U_\varepsilon(t, x) = W_\varepsilon(t, x) + \int_0^t V_\varepsilon(x, t - s; s)ds.
\]
Taking the \(L^2\) norm in (2.16) and using (2.14) to estimate \( V_\varepsilon \) and \( W_\varepsilon \), we get
\[
\|U_\varepsilon(t, \cdot)\|_{L^2} \leq \|W_\varepsilon(t, \cdot)\|_{L^2} + \int_0^T \|V_\varepsilon(\cdot, t - s; s)\|_{L^2}ds \\
\lesssim \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} + \int_0^T \|f_\varepsilon(s, \cdot)\|_{L^2}ds \\
\lesssim \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} + \|\tilde{p}_\varepsilon - p_\varepsilon\|_{L^\infty} \int_0^T \|\tilde{u}_\varepsilon(s, \cdot)\|_{L^2}ds.
\]
From the one hand, we have that \( \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_\varepsilon\varepsilon^l \), for all \( l > 0 \). On the other hand, \((u_\varepsilon)_\varepsilon\) as a very weak solution to the Cauchy problem (2.1) is moderate and \( \|\tilde{p}_\varepsilon - p_\varepsilon\|_{L^\infty} \leq C_k \varepsilon^k \) for all \( k > 0 \). Therefore,
\[
\|U_\varepsilon(t, \cdot)\|_{L^2} = \|u_\varepsilon(t, \cdot) - \tilde{u}_\varepsilon(t, \cdot)\|_{L^2} \lesssim \varepsilon^N,
\]
for all \( N > 0 \), which means that the very weak solution is unique.

2.3. Consistency. Now we give the consistency result, which means that the very weak solution to the Cauchy problem (2.1) converges in an appropriate norm, to the classical solution, when the latter exists.

**Theorem 2.5** (Consistency). Let \( p \in L^\infty(\mathbb{R}^d) \) be non-negative. Assume that \( u_0 \in H^s(\mathbb{R}^d) \) for \( s > 0 \), and let us consider the Cauchy problem
\[
\begin{cases}
  iu_t(t, x) + (-\Delta)^s u(t, x) + p(x) u(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\
  u(0, x) = u_0(x).
\end{cases}
\]
Let denote by \( w \) and let \((u_\varepsilon)_\varepsilon \) be a very weak solution of \((2.17)\). Then for any regularising families of the coefficient \( p \) and the Cauchy data \( u_0\), the net \((u_\varepsilon)_\varepsilon \) converges in \( L^2 \) as \( \varepsilon \to 0 \) to the unique classical solution of the Cauchy problem \((2.17)\).

**Proof.** Let \( u \) be the classical solution to

\[
\begin{align*}
&\left\{ \begin{array}{ll}
iu_t(t,x) + (-\Delta)^s u(t,x) + p(x)u(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d, \\
u(0,x) = u_0(x),
\end{array} \right.
\]

and let \((u_\varepsilon)_\varepsilon \) its very weak solution. It satisfies

\[
\begin{align*}
&\left\{ \begin{array}{ll}
i\partial u_\varepsilon(t,x) + (-\Delta)^s u_\varepsilon(t,x) + p_\varepsilon(x)u_\varepsilon(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d, \\
u_\varepsilon(0,x) = u_{0,\varepsilon}(x).
\end{array} \right.
\]

Let denote by \( W_\varepsilon(t,x) := u(t,x) - u_\varepsilon(t,x) \). It solves

\[
\begin{align*}
&\left\{ \begin{array}{ll}
i\partial W_\varepsilon(t,x) + (-\Delta)^s W_\varepsilon(t,x) + p_\varepsilon(x)W_\varepsilon(t,x) = \eta_\varepsilon(t,x), & (t,x) \in [0,T] \times \mathbb{R}^d, \\
W_\varepsilon(0,x) = (u_0 - u_{0,\varepsilon})(x),
\end{array} \right.
\]

where \( \eta_\varepsilon(t,x) := (p_\varepsilon(x) - p(x))u(t,x) \). Using Duhamel’s principle and similar arguments as in Theorem \((2.4)\), we get the estimate

\[
\|W_\varepsilon(t,\cdot)\|_{L^2} \lesssim \|u_0 - u_{0,\varepsilon}\|_{L^2} + \int_0^T \|\eta_\varepsilon(s,\cdot)\|_{L^2}ds.
\]

When \( \varepsilon \to 0 \), the right hand side of the last inequality tends to 0, since \( \|p_\varepsilon - p\|_{L^\infty} \to 0 \) and \( \|u_0 - u_{0,\varepsilon}\|_{L^2} \to 0 \). It follows that the very weak solution converges to the classical one in \( L^2 \). \( \square \)

### 3. Numerical experiments

In this Section, we do some numerical experiments. Let us analyse our problem by regularising a distributional potential \( p(x) \) by a parameter \( \varepsilon \). We define \( p_\varepsilon(x) := (p \ast \varphi_\varepsilon)(x) \), as the convolution with the mollifier \( \varphi_\varepsilon(x) = \frac{1}{\varepsilon^5} \varphi(x/\varepsilon) \), where

\[
\varphi(x) = \begin{cases} 
    c \exp \left( \frac{1}{x^2 - 1} \right), & |x| < 1, \\
    0, & |x| \geq 1,
\end{cases}
\]

with \( c \approx 2.2523 \) to have \( \int_{-\infty}^{\infty} \varphi(x)dx = 1 \). Then, instead of \((2.1)\) we consider the regularised Cauchy problem for the 1D Schrödinger equation

\[(3.1)\]

\[
i\partial u_\varepsilon(t,x) - \partial_x^2 u_\varepsilon(t,x) + p_\varepsilon(x)u_\varepsilon(t,x) = 0, \quad (t,x) \in [0,T] \times \mathbb{R},
\]

with the initial data \( u_\varepsilon(0,x) = u_0(x) \), for all \( x \in \mathbb{R} \). Here, we put

\[
u_0(x) = \begin{cases} 
    \exp \left( \frac{1}{(x-5)^2 - 0.25} \right), & |x - 5| < 0.5, \\
    0, & |x - 5| \geq 0.5.
\end{cases}
\]

Note that \( \text{supp} \ u_0 \subset [4.5, 5.5] \).
Figure 1. In these plots, we analyse behaviour of the solution of the Schrödinger equation (3.1) with a δ-like potential. In the top left plot, the graphic of the position density of particles at the initial time is given. In the further plots, we draw the position density function $|u|^2$ at $t = 0.0428, 0.1070, 0.1391, 0.2140, 0.2996$ for $\varepsilon = 0.05$. Here, a δ-like function with the support at point 3 is considered.

Figure 2. In these plots, we analyse the time evolution of the position density $|u|^2$ for different regular potentials. Here, the cases of the potentials with $p(x) = 0$, $p(x) = 1$, and $p(x) = (x - 5)^2$ are considered.

Here, we consider the following cases when potential is a regular function: $p(x) = 0$, $p(x) = 1$, and $p(x) = (x - 5)^2$; when potential is a singular function: $p(x) = \frac{1}{30} \delta(x - 3)$
Figure 3. In these plots, we analyse behaviour of the solution of the Schrödinger equation (3.1) with a $\delta$-like potential for different values of the parameter $\varepsilon$. Here, we compare the position density function of particles $|u|^2$ at $t = 0.214$ for $\varepsilon = 0.035, 0.080, 0.300, 0.800$. Here, the case of the potential with a $\delta$-like function behaviour with the support at point 3 is considered.

Figure 4. In these plots, we compare the energy function $E(t)$ of the Schrödinger equation (3.1) corresponding to the $\delta$-potential case for $\varepsilon = 0.05, 0.11, 0.49$.

with $p_\varepsilon(x) = \frac{1}{3\varepsilon^2} \varphi_\varepsilon(x - 3)$ and $p(x) = \frac{1}{3\varepsilon^2} \delta^2(x - 3)$ in the sense $p_\varepsilon(x) = \frac{1}{3\varepsilon^2} \varphi_\varepsilon^2(x - 3)$, where $\delta$ denoting the standard Dirac’s delta-distribution.

In Figure 1, we analyse behaviour of the solution of the Schrödinger equation (3.1) with a $\delta$-like potential. In the top left plot, the graphic of the position density of
Figure 5. In these plots, we analyse the solution of the Schrödinger equation (3.1) with a $\delta^2$-like potential. In the top left plot, we study the position density function $|u(t, x)|^2$ at $t = 0.0000, 0.0214, 0.0428, 0.0642$ for $\varepsilon = 0.05$. In further plots, we compare the energy function $E(t)$ of the Schrödinger equation (3.1) corresponding to the $\delta^2$-potential case for $\varepsilon = 0.05, 0.15, 0.25, 0.50$. In the right-bottom plot, we compare the energy function for $\varepsilon = 0.15, 0.25, 0.50$.

Particles at the initial time is given. In the further plots, we draw the position density function $|u|^2$ at $t = 0.0428, 0.1070, 0.1391, 0.2140, 0.2996$ for $\varepsilon = 0.05$. Here, a $\delta$-like function with the support at point 3 is considered. We observe that a delta-function potential causing an accumulating of particles phenomena in the place of the support of the singularity.
In Figure 2, we analyse the time evolution of the position density for different regular potentials. Here, the cases of the potentials with $p(x) = 0$, $p(x) = 1$, and $p(x) = (x - 5)^2$ are considered.

In Figure 3, we analyse behaviour of the solution of the Schrödinger equation (3.1) with a $\delta$-like potential for different values of the parameter $\varepsilon$. Here, we compare the position density function of particles $|u|^2$ at $t = 0.214$ for $\varepsilon = 0.035, 0.080, 0.300, 0.800$. Here, the case of the potential with a $\delta$-like function behaviour with the support at point 3 is considered. Here, we can see that the numerical simulations of the regularised equation (3.1) are stable under the changing of the values of the parameter $\varepsilon$.

In Figure 4, we compare the energy function

$$E(t) = \|\nabla u(t, \cdot)\|_{L^2}^2 + \|p^\frac{1}{2}(\cdot)u(t, \cdot)\|_{L^2}^2. \tag{3.2}$$

of the Schrödinger equation (3.1) corresponding to the $\delta$-potential case for different values of the parameter $\varepsilon$. Simulations show that $E(t) \approx E(0)$ for $t > 0$.

In Figure 5, we analyse the solution of the Schrödinger equation (3.1) with a $\delta^2$-like potential. In the left plot, we study the position density function $|u(t, x)|^2$ at $t = 0.0000, 0.0214, 0.0428, 0.0642$ for $\varepsilon = 0.05$. In the right plot, we compare the energy function $E(t)$ of the Schrödinger equation (3.1) corresponding to the $\delta^2$-potential case for $\varepsilon = 0.05, 0.15, 0.25, 0.50$. From these plots we conclude that thanks to the concept of the very weak solution the studying of the processes in physics are possible despite the impossibility of the multiplication of the distributions in the theory of distributions.

**Remark 3.1.** By analysing these cases, from Figures 4 and 5 we see that the energy function $E(t)$ given by (3.2) satisfies $E(t) \approx E(0)$ for $t > 0$. Moreover, it is observed that $E(t)$ depends on $\varepsilon$ by confirming the theory, that is, $E(t) = E_\varepsilon(t)$. From the bottom plots of Figure 5 we observe that the energy $E(t)$ of the Schrödinger equation (3.1) with a $\delta^2$-like potential corresponding to the case $\varepsilon = 0.5$ is increased around 200 times as $\varepsilon$ is decreased 10 times by justifying the theoretical part.

**Remark 3.2.** From the behaviours of the density function $|u(t, x)|^2$ of the Schrödinger equation (3.1) corresponding to the cases of $\delta$-like and $\delta^2$-like potentials, namely, from the left plot of Figure 3 and the upper–left plot of Figure 5 we observe a ”splitting of the strong singularity” effect. Explanation of this phenomena is still an open question from the theoretical point of view.

A second order in time and in space Crank-Nicolson scheme is used for the numerical analysis of the equation (3.1). All numerical computations are made in C++ by using the sweep method. In above numerical simulations, we use the Matlab R2018b. For all simulations we take $\Delta t = 0.0107$, $\Delta x = 0.01$.

**3.1. Conclusion.** The theoretical and numerical analysis conducted in this paper showed that numerical methods work well in situations where a rigorous mathematical formulation of the problem is difficult in the framework of the classical theory of distributions. The ideology of very weak solutions eliminates this difficulty in the
case of the terms with multiplication of distributions. In particular, in the case of the Schrödinger equation, we see that a delta-function potential causing an effect of accumulating particles in the place of the support of the singularity.

Numerical simulations have shown that the idea of very weak solutions suit nice to numerical modelling. Moreover, using the theory of very weak solutions, we are able to deal with the uniqueness of numerical solutions of partial differential equations with coefficients of higher order singularity in some appropriate sense.

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