THE SINGULARITY PROBLEM FOR
SPACE-TIMES WITH TORSION

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Summary. - The problem of a rigorous theory of singularities in space-times with torsion is addressed. We define geodesics as curves whose tangent vector moves by parallel transport. This is different from what other authors have done, because their definition of geodesics only involves the Christoffel connection, though studying theories with torsion. We propose a preliminary definition of singularities which is based on timelike or null geodesic incompleteness, even though for theories with torsion the paths of particles are not geodesics. The study of the geodesic equation for cosmological models with torsion shows that the definition has a physical relevance. It can also be motivated, as done in the literature, remarking that the causal structure of a space-time with torsion does not get changed with respect to general relativity. We then prove how to extend Hawking’s singularity theorem without causality assumptions to the space-time of the ECSK theory. This is achieved studying the generalized Raychaudhuri equation in the ECSK theory, the conditions for the existence of conjugate points and properties of maximal timelike geodesics. Hawking’s theorem can be generalized, provided the torsion tensor obeys some conditions. Thus our result can also be interpreted as a no-singularity theorem if these additional conditions are not satisfied. In other words, it turns out that the occurrence of singularities in closed cosmological models based on the ECSK theory is less generic than in general relativity. Our work is to be compared with previous papers in the literature. There are some relevant differences, because we rely on a different definition of geodesics,
we keep the field equations of the ECSK theory in their original form rather than casting
them in a form similar to general relativity with a modified energy momentum tensor, and
we emphasize the role played by the full extrinsic curvature tensor, which now contains
torsion.

1. - Introduction.

The singularity problem is still of vital importance in modern cosmology [1,2] and the
aim of this paper is the study of singularities for classical theories of gravitation. Indeed,
the singularity theorems of Penrose, Hawking and Geroch [3-7] show that Einstein’s general
relativity leads to the occurrence of singularities in cosmology in a rather generic way. On
the other hand, much work has also been done on alternative theories of gravitation. In
particular, it is by now well known that, when we describe gravity as the gauge theory of
the Poincaré group, this naturally leads to theories with torsion [8,9]. Thus it is natural
to address the question: is there a rigorous theory of singularities in a space-time with
torsion? To this purpose, at first we present in sect. 2 and 3 a brief review of the
space-time manifold and of the definition of singularities in general relativity. We then
define in sect. 4 geodesics in space-times with torsion, and we discuss their relevance for
cosmology. In sect. 5, we prove how to extend Hawking’s singularity theorem without
causality assumptions to the space-time of the Einstein-Cartan-Sciama-Kibble (hereafter
referred to as ECSK) theory. A comparison with previous important work appeared in
[10] is also made. Finally, our approach and our results are summarized in sect. 6.

2. - The space-time manifold.

A space-time \((M, g)\) is the following collection of mathematical entities [1,11]:
1) A connected four-dimensional Hausdorff \(C^\infty\) manifold \(M\);
2) A Lorentz metric $g$ on $M$, namely the assignment of a nondegenerate bilinear form $g_p: T_p M \times T_p M \to \mathbb{R}$ with diagonal form $(-, +, +, +)$ to each tangent space. Thus $g$ has signature $+2$ and is not positive-definite;

3) A time orientation, given by a globally defined timelike vector field $X: M \to TM$. A timelike or null tangent vector $v \in T_p M$ is said to be future (respectively, past) directed if $g(X(p), v) < 0$, (respectively, $g(X(p), v) > 0$).

Some important remarks are now in order:

a) Condition (1) can be formulated for each number of space-time dimensions $\geq 2$.

b) Also the convention $(+, -, -, -)$ for the diagonal form of the metric can be chosen. This convention seems to be more useful in the study of spinors, and can also be adopted in using tensors as Penrose does so as to avoid a change of conventions. The definitions of timelike and spacelike will then become: $X$ is timelike if $g(X(p), X(p)) > 0$, $\forall p \in M$, and $X$ is spacelike if $g(X(p), X(p)) < 0 \forall p \in M$.

c) The pair $(M, g)$ is only defined up to equivalence. Two pairs $(M, g)$ and $(M', g')$ are equivalent if there is a diffeomorphism $\alpha: M \to M'$ such that: $\alpha_* g = g'$. Thus we are really dealing with an equivalence class of pairs [1].

A concept which will be very useful in sect. 5 is the one of Lorentzian arc length. Let $\Omega_{pq}$ be the space of all future directed nonspacelike curves $\gamma: [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. Given $\gamma \in \Omega_{pq}$ we choose a partition of $[0, 1]$ such that $\gamma$ restricted to $[t_i, t_{i+1}]$ is smooth $\forall i = 0, 1, ..., n - 1$. The Lorentzian arc length is then defined as [11]

$$L(\gamma) \equiv \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \sqrt{-g(\gamma'(t), \gamma'(t))} \, dt .$$

(2.1)

So as to avoid confusion in comparing with the convention used in [2,8-10], we wish to emphasize that we use the term Riemannian geometry for the case of positive-definite metrics (see [12,13]), whereas general relativity is more properly called a Lorentzian theory [1, 11, 14].
3. - The definition of singularities in general relativity.

The singularity theorems in general relativity [1] were proved using a definition of singularities based on the \( g \)-boundary. Namely, one defines a topological space, the \( g \)-boundary, whose points are equivalence classes of incomplete nonspacelike geodesics. The points of the \( g \)-boundary are then the singular points of space-time. As emphasized for example in [15], this definition has two basic drawbacks:

1) it is based on geodesics, whereas in [16] it was proved there are geodesically complete space-times with curves of finite length and bounded acceleration;

2) there are several alternative ways of forming equivalence classes and defining the topology.

Schmidt’s method is along the following lines [15]. Connections are known to provide a parallelization of the bundle \( L(M) \) of linear frames. This parallelization can be used to define a Riemannian metric, which has the effect of making a connected component of \( L(M) \) into a metric space. This connected component \( L'(M) \) is dense in a complete metric space \( L'_c(M) \). One defines \( \overline{M} \) as the set of orbits of the transformation group on \( L'_c(M) \), and the \( b \)-boundary \( \partial M \) of \( M \) is then defined as: \( \partial M \equiv \overline{M} - M \). Finally, singularities of \( M \) are defined as points of the \( b \)-boundary \( \partial M \) which are contained in the \( b \)-boundary of any extension of \( M \).

However, also Schmidt’s definition has some drawbacks [17]. In fact in a closed Friedmann-Robertson-Walker (hereafter referred to as FRW) universe the initial and final singularities form the same single point of the \( b \)-boundary [18], and in the FRW and Schwarzschild solutions the \( b \)-boundary points are not Hausdorff separated from the corresponding space-time [19]. A fully satisfactory improvement of Schmidt’s definition is still an open problem. Unfortunately, a recent attempt appeared in [17] was not correct. In the next sections we will not use this formal apparatus. But we thought it was worth summarizing this kind of mathematical results in a paper devoted to the study of the singularity problem.
4. - Space-times with torsion and their geodesics.

A space-time with torsion (hereafter referred to as $U_4$ space-time) is defined adding the following fourth requirement to the ones in sect. 2:

4) Given a linear $C^r$ connection $\tilde{\nabla}$ which obeys the metricity condition, a nonvanishing tensor

$$S(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y],$$

(4.1)

where $X$ and $Y$ are arbitrary $C^r$ vector fields and the square bracket denotes their Lie bracket. The tensor $\frac{1}{2}S$ is then called the torsion tensor (compare with [1]).

Now, it is well known that the curve $\gamma$ is defined to be a geodesic curve if its tangent vector moves by parallel transport, so that $\nabla_X X$ is parallel to $\left(\frac{\partial}{\partial t}\right)_\gamma$ (see, however, comment before our definition of singularities). A new parameter $s(t)$, called affine parameter, can always be found such that, in local coordinates, this condition is finally expressed by the equation

$$\frac{d^2 x^a}{ds^2} + \Gamma^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = 0.$$

(4.2)

The geodesic equation (4.2) will now contain the effect of torsion through the symmetric part of the connection coefficients:

$$\Gamma^a_{(bc)} = \left\{ \begin{array}{c} a \\ b \\ c \end{array} \right\} + 2S^a_{(bc)},$$

where $S^a_{(bc)}$ is not to be confused with the vanishing $S_{(bc)}^a$. It is very useful to study this equation in a case of cosmological interest. For example, in a closed FRW universe the only nonvanishing components of the torsion tensor are the ones given in [20]

$$S_{m0}^m = Q(t), \quad \forall m = 1, 2, 3,$$

so that (4.2) yields

$$\frac{d^2 x^0}{ds^2} + a \frac{da}{ds} \frac{ds}{dt} c_{ii} \left(\frac{dx^i}{ds}\right)^2 = 0,$$

(4.3)
In (4.3), \( c_{ii} \) are the diagonal components of the unit three-sphere metric, and we are summing over all \( i = 1, 2, 3 \). In (4.4), we used the result of [20] according to which

\[
\Gamma^m_{0m} = \frac{\dot{a}}{a} - 2Q, \quad \Gamma^m_{m0} = \frac{\dot{a}}{a}, \quad \forall m = 1, 2, 3.
\] (4.5)

Of course, \( \dot{a} \) denotes \( \frac{da}{ds} \). One can use the relation \( c_{ij} = \text{diag}(1, (\sin \chi)^2, (\sin \chi)^2(\sin \theta)^2) \) to compute the connection coefficients \( \Gamma^m_{ij} \) and write down a more explicit form of (4.3), (4.4). However, we can already get a qualitative understanding of what happens from (4.3), (4.4). In fact, setting \( s_0 = s(t = 0) \), if the field equations are such that both \( \frac{1}{a} \frac{da}{ds} \frac{ds}{dt} \) and \( Q \) remain finite at \( s_0 \) (and similarly for \( s_f = s(t = t_f) \), where \( t_f \) is the time at which the torsion-free model reaches the singularity in the future), a solution to (4.3), (4.4) will exist for all values of \( s \) and the model will be nonspacelike geodesically complete. This qualitative argument seems to suggest that, whatever the physical source of torsion is (spin or theories with quadratic Lagrangians etc.), nonspacelike geodesic completeness is a concept of physical relevance even though test particles may not move along geodesics.

An important comment is now in order. We have defined geodesics exactly as one does in general relativity (see [1], p. 33), for reasons which will become even more clear studying maximal timelike geodesics in sect. 5. However, our definition differs from the one adopted in [10] (p. 1068). In that paper, our geodesics are just called autoparallel curves, whereas the authors interpret as geodesics the curves of extremal length whose tangent vector is parallely transported according to the Christoffel connection.

Now, in view of the fact that the definition of timelike, null and spacelike vectors is not affected by the presence of torsion, the whole theory of causal structure [1] remains unchanged. Combining this remark (also made in [10]) with the qualitative argument concerning the geodesic equation, we here give the following preliminary definition:

**Definition** A \( U_4 \) space-time is singularity-free if it is timelike and null geodesically complete, where geodesics are defined as curves whose tangent vector moves by parallel transport with respect to the full \( U_4 \) connection.
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This definition differs from the one given in [10] because we rely on a different definition of geodesics, and it has the drawbacks already illustrated in the beginning of sect. 3. However, it seems to have the following advantages:

1) it is a preliminary definition which allows a direct comparison with the corresponding situation in general relativity;

2) it is generic in that it does not depend on the specific physical theory which is the source of torsion;

3) it has physical relevance as we have shown before looking at a closed FRW model and at the causal structure [10].

The meaning of the remark 1) is that one can now try to make the same (and eventually additional) assumptions which lead to singularity theorems in general relativity, and check whether one gets timelike and/or null geodesic incompleteness. Indeed, the extrinsic curvature tensor and the vorticity which appears in the Raychaudhuri equation will now explicitly contain the effects of torsion, and it is not a priori clear what is going to happen. Namely, if one adopts the above definition as a preliminary definition of singularities in a $U_4$ space-time, the main unsolved issues seem to be:

1) How can we explain from first principles that a space-time which is nonspacelike geodesically incomplete may become nonspacelike geodesically complete in the presence of torsion? And is the converse possible?

2) What happens in a $U_4$ space-time [10] under the assumptions which lead to the theorems of Penrose, Hawking and Geroch?

Question 1) should not seem trivial in view of the FRW example discussed before. In fact one should study the singularity problem in a generic space-time. This is why we shall partially study question 2) in the next section.
5. - A singularity theorem without causality assumptions for $U_4$ space-times.

In this section we shall denote by $R(X,Y)$ the four-dimensional Ricci tensor with scalar curvature $R$, and by $K(X,Y)$ the extrinsic curvature tensor of a spacelike three-surface. The energy-momentum tensor will be written as $T(X,Y)$, so that the Einstein equations are

$$R(X,Y) - \frac{1}{2}g(X,Y)R = T(X,Y). \quad (5.1)$$

In so doing, we are absorbing the $8\pi G$ factor into the definition of $T(X,Y)$. A linear torsion-free connection will be denoted by $\nabla$, so as to avoid confusion with $\tilde{\nabla}$ appearing in (4.1). For the case of general relativity, it was proved in [5] that singularities must occur under certain assumptions, even though no causality requirements are made. In fact, Hawking’s result [1, 5] states that space-time cannot be timelike geodesically complete if

1) $R(X,X) \geq 0$ for any nonspacelike vector $X$ (which can also be written in the form: $T(X,X) \geq g(X,X)\frac{T}{2}$),

2) there exists a compact spacelike three-surface $\Sigma$ without edge,

3) the trace $K$ of the extrinsic curvature tensor $K(X,Y)$ of $\Sigma$ is either everywhere positive or everywhere negative.

We are now going to study the following problem: is there a suitable generalization of this theorem in the case of a $U_4$ space-time? Indeed, a careful examination of Hawking’s proof (see [1], p. 273) shows that the arguments which are to be modified in a $U_4$ space-time are the ones involving the Raychaudhuri equation and the results which prove the existence or the nonexistence of conjugate points. We are now going to examine them in detail.

1) Raychaudhuri equation.

The generalized Raychaudhuri equation in the ECSK theory of gravity has been derived in [21, 22] (see also [23, 24]). It turns out that, denoting by $\tilde{\omega}_{ab}$ and $\sigma_{ab}$, respectively,
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the vorticity and the shear tensors, the expansion $\theta$ for a timelike congruence of curves obeys the equation

$$\frac{d\theta}{ds} = -(R(U, U) + 2\sigma^2 - 2\tilde{\omega}^2) - \frac{\theta^2}{3} + \tilde{\nabla}_a (\dot{U})^a.$$  \hfill (5.2)

In (5.2), $U$ is the unit timelike tangent vector, and we have set

$$2\sigma^2 \equiv \sigma_{ab}\sigma^{ab}, \quad 2\tilde{\omega}^2 \equiv \left(\omega_{ab} + \frac{1}{2}\tilde{S}_{ab}\right)\left(\omega^{ab} + \frac{1}{2}\tilde{S}^{ab}\right),$$  \hfill (5.3)

where $\omega_{ab}$ is the vorticity tensor for the torsion-free connection $\nabla$, and $\tilde{S}_{bc}$ is obtained from the spin tensor $\sigma^a_{bc}$ through a relation usually assumed to be of the form [22, 25]

$$\sigma^a_{bc} = \tilde{S}_{bc}U^a.$$  \hfill (5.4)

II) Existence of conjugate points.

Conjugate points are defined as in general relativity [1], but bearing in mind that now the Riemann tensor is the one obtained from the connection $\tilde{\nabla}$ appearing in (4.1):

$$R(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z.$$  \hfill (5.5)

In general relativity, if one assumes that at $s_0$ one has $\theta(s_0) = \theta_0 < 0$, and $R(U, U) \geq 0$, everywhere, then one can prove there is a point conjugate to $q$ along $\gamma(s)$ between $\gamma(s_0)$ and $\gamma\left(s_0 - \frac{3}{\theta_0}\right)$, provided $\gamma(s)$ can be extended to $\gamma\left(s_0 - \frac{3}{\theta_0}\right)$. This result is then extended to prove the existence of points conjugate to a three-surface $\Sigma$ along $\gamma(s)$ within a distance $\frac{3}{\theta'}$ from $\Sigma$, where $\theta'$ is the initial value of $\theta$ given by the trace $K$ of $K(X, Y)$, provided $K < 0$ and $\gamma(s)$ can be extended to that distance (see propositions 4.4.1 and 4.4.3 in [1]). This is achieved studying an equation of the kind (5.2) where $\tilde{\omega}^2 = \omega^2$ is vanishing because $\omega_{ab}$ is constant and initially vanishing and the last term on the right-hand side vanishes as well. However, in the ECSK theory, $\tilde{\omega}^2$ will still contribute in view of (5.3). Thus the inequality

$$\frac{d\theta}{ds} \leq -\frac{\theta^2}{3}$$
can only make sense if we assume that
\[ R(U, U) - 2\sigma^2 \geq 0 , \] (5.6)
where we do not strictly need to include $2\sigma^2$ on the left-hand side of (5.6) because $\sigma^2$ is positive [1, 24]. If (5.6) holds true, we can write (see (5.2) and set there $\tilde{\nabla}_a (\dot{U})^a = 0$)
\[ \int_{\theta_0}^{\theta} y^{-2} dy \leq -\frac{1}{3} \int_{s_0}^{s} dx , \] (5.7)
which implies
\[ \theta \leq \frac{3}{s - (s_0 - \frac{3}{\theta_0})} , \] (5.8)
where $\theta_0 < 0$. Thus $\theta$ becomes infinite and there are conjugate points for some $s \in [s_0, s_0 - \frac{3}{\theta_0}]$. However, (5.6) is a restriction on the torsion tensor. In fact, the equations of the ECSK theory are given by (5.1) plus another one more suitably written in component language in the form [25]
\[ S^a_{bc} - \delta^a_b S^d_{dc} - \delta^a_c S^d_{bd} = \sigma^a_{bc} . \] (5.9)
In following [25] we temporarily choose (up to (5.12)) a convention opposite to the one of sec. 4, working with a torsion tensor $S^a_{bc} = -S^a_{cb}$, rather than $S^a_{bc} = -S^a_{cb}$.

In (5.9) we have absorbed the $8\pi G$ factor into the definition of $\sigma^a_{bc}$, whereas this is not done in (5.4). Setting $\epsilon = g(U, U) = -1$, $\rho = 8\pi G$, the insertion of (5.4) into (5.9) and the multiplication by $U_a$ yields
\[ \tilde{S}_{bc} = \frac{1}{\rho \epsilon} \left( U_a S^a_{bc} - U_b S^d_{dc} - U_c S^d_{bd} \right) , \] (5.10)
which implies, defining
\[ f(\omega, \omega S) \equiv \omega_{ab} \bar{\omega}^{ab} + \frac{1}{2} \omega_{ab} \bar{S}^{ab} + \frac{1}{2} \bar{S}_{ab} \omega^{ab} \]
\[ = \omega_{ab} \bar{\omega}^{ab} + \frac{\omega_{ab}}{2\rho \epsilon} \left( U_f S^{f}_{ab} - U^a S^h_{ah} - U^b S^h_{ha} \right) \]
\[ + \frac{\omega^{ab}}{2\rho \epsilon} \left( U_f S^f_{ab} - U_a S^h_{hb} - U_b S^h_{ah} \right) \] (5.11)
and using (5.3) and (5.6), that

\[ 8\tilde{\omega}^2 = 4f(\omega, S) + g^{bl}g^{cm}\tilde{S}_{bc}\tilde{S}_{lm} \]

\[ = 4f(\omega, S) + \rho^{-2}\left(U_aS_{alm} - U_dS_{d^m} - U^mS_{d^l}^d\right)\left(U_fS_{lf}^m - U_fS_{fm} - U_mS_{lf}^f\right) \]

\[ \leq 4R(U, U). \quad (5.12) \]

Indeed some cases have been studied (see for example [21]) where \( \omega_{ab} \) is vanishing. However, we here prefer to write the equations in general form. Moreover, in extending (5.8) so as to prove the existence of conjugate points to spacelike three-surfaces, the assumption \( K < 0 \) on the trace \( K \) of \( K(X, Y) \) also implies another condition on the torsion tensor. In fact, denoting by \( \chi(X, Y) \) the tensor obtained from the metric and from the lapse and shift functions as the extrinsic curvature in general relativity, in a \( U_4 \) space-time one has

\[ K(X, Y) = \chi(X, Y) + \lambda(X, Y) , \quad (5.13) \]

where the symmetric part of \( \lambda(X, Y) \) (the only one which contributes to \( K \) ) is given by

\[ \lambda_{(ab)} = -2n^\muS_{(a\mu b)} . \quad (5.14) \]

In (5.14) we have changed sign with respect to [26] because that convention for \( K(X, Y) \) is opposite to Hawking’s convention, and we are here following Hawking so as to avoid confusion in comparing theorems. Thus the condition \( K < 0 \) implies the following restriction on torsion:

\[ \lambda = -2g^{ab}n^\muS_{(a\mu b)} < -\chi . \quad (5.15) \]

When (5.6) and (5.15) hold true, one follows exactly the same technique which leads to (5.8) in proving there are points conjugate to a spacelike three-surface.

III) Maximal timelike geodesics.

In general relativity, it is known (proposition 4.5.8 in [1]) that a timelike geodesic curve \( \gamma \) from \( q \) to \( p \) is maximal, if and only if there is no point conjugate to \( q \) along \( \gamma \) in \( (q, p) \). At the risk of boring the expert reader, we are now going to sum up how this result is proved and then extended so as to rule out the existence of points conjugate to
three-surfaces. This last step will then be enlightening in understanding what changes in a $U_4$ space-time.

We shall here follow the conventions of subsect. 4.5 of [1], denoting by $L(Z_1, Z_2)$ the second derivative of the arc length defined in (2.1), by $V$ the unit tangent vector $\frac{\partial}{\partial s}$ and by $T_\gamma$ the vector space consisting of all continuous, piecewise $C^2$ vector fields along the timelike geodesic $\gamma$ orthogonal to $V$ and vanishing at $q$ and $p$. We are here just interested in proving that, if the timelike geodesic $\gamma$ from $q$ to $p$ is maximal, this implies there is no point conjugate to $q$. The idea is to suppose for absurd that $\gamma$ is maximal but there is a point conjugate to $q$. One then finds that $L(Z, Z) > 0$, which in turn implies that $\gamma$ is not maximal, against the hypothesis. This is achieved taking a Jacobi field $W$ along $\gamma$ vanishing at $q$ and $r$, and extending it to $p$ putting $W = 0$ in the interval $[r, p]$. Moreover, one considers a vector $M \in T_\gamma$ so that $g(M, \frac{\partial}{\partial s}W) = -1$ at $r$. In what follows, we shall just say that $M$ is suitably chosen, in a way which will become clear later. One then defines

$$Z \equiv \epsilon M + \epsilon^{-1}W,$$  \hspace{1cm} (5.16)

where $\epsilon$ is positive and constant. Thus, the general formula for $L(Z_1, Z_2)$ implies that (see lemma 4.5.6 of [1])

$$L(Z, Z) = \epsilon^2 L(M, M) + 2L(W, M) + \epsilon^{-2}L(W, W) = \epsilon^2 L(M, M) + 2,$$  \hspace{1cm} (5.17)

which implies that $L(Z, Z)$ is $> 0$ if $\epsilon$ is suitably small, as we anticipated. The same method is also used in proving there cannot be points conjugate to a three-surface $\Sigma$ if the timelike geodesic $\gamma$ from $\Sigma$ to $p$ is maximal. However, as proved in lemma 4.5.7 of [1], in the case of a three-surface $\Sigma$, the formula for $L(Z_1, Z_2)$ is of the kind

$$L(Z_1, Z_2) = F(Z_1, Z_2) - \chi(Z_1, Z_2),$$  \hspace{1cm} (5.18)

where $\chi(X, Y)$ is the extrinsic curvature tensor of $\Sigma$. But we know that in a $U_4$ space-time $\chi(X, Y)$ gets replaced by the nonsymmetric tensor $K(X, Y)$ defined in (5.13), (5.14), which can be completed with the relation for the antisymmetric part of $\lambda(X, Y)$:

$$\lambda_{[ab]} = -n^\mu S_{b\mu}.$$  \hspace{1cm} (5.19)
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Thus in $U_4$ theory the splitting (5.16) leads to a formula of the kind (5.17) where the requirement

$$L(W, M) + L(M, W) = c > 0$$

will involve torsion because (5.18) gets replaced by

$$L(Z_1, Z_2) = \tilde{F}(Z_1, Z_2) - K(Z_1, Z_2).$$

Namely, the left-hand side of (5.20) will contain $K(W, M) + K(M, W)$. Condition (5.20) also clarifies how to suitably choose $M$ in a $U_4$ space-time. It is worth emphasizing that only $\lambda_{(ab)}$ contributes to (5.20) because the contributions of $\lambda_{[ab]}$ coming from $K(M, W)$ and $K(W, M)$ add up to zero. In proving (5.21), the first step is the generalization of lemma 4.5.4 of [1] to a $U_4$ space-time. This is achieved remarking that the relation

$$\frac{\partial}{\partial u} g \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = 2g \left( \frac{D}{\partial u} \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right)$$

is also valid in a $U_4$ space-time, where now $\frac{D}{\partial u}$ denotes the covariant derivative along the curve with respect to the full $U_4$ connection. In fact, denoting by $X$ the vector $\frac{\partial}{\partial t}$ and using the definition of covariant derivative along a curve one finds

$$\frac{\partial}{\partial u} g(X, X) = 2g \left( \frac{D}{\partial u} X, X \right) + X^a X^b \frac{D}{\partial u} g_{ab},$$

where $\frac{D}{\partial u} g_{ab}$ is vanishing if the connection obeys the metricity condition, which is also assumed in a $U_4$ space-time (see sect. 4 and [8]). In other words, the key role is played by the connection which obeys the metricity condition, and $\frac{\partial}{\partial u} g(X, X)$ will implicitly contain the effects of torsion because of the relation

$$\frac{DX^a}{\partial u} \equiv \frac{\partial X^a}{\partial u} + \Gamma^a_{bc} \frac{dx^b}{du} X^c.$$  

Although this point seems to be elementary, it plays a vital role in leading to (5.21). This is why we chose to greatly emphasize it.
If we now compare the results discussed or proved in I)-III) with p. 273 of [1], we are led to state the following singularity theorem:

**Theorem.** The $U_4$ space-time of the ECSK theory cannot be timelike geodesically complete if

1) $R(U,U) - 2\tilde{\omega}^2 \geq 0$ for any nonspacelike vector $U$;
2) $L(W,M) + L(M,W) = c > 0$ as defined in (5.20), (5.21) and before;
3) there exists a compact spacelike three-surface $S$ without edge;
4) the trace $K$ of the extrinsic curvature tensor $K(X,Y)$ of $S$ is either everywhere positive or everywhere negative.

Conditions 1), 2) and 4) will then involve the torsion tensor defined through (4.1). Indeed, condition 2) can be seen as a prerequisite, but we have chosen to insert it into the statement of the theorem so as to present together all conditions which involve the extrinsic curvature tensor $K(X,Y)$. The compatibility of 1) with the field equations of the ECSK theory is expressed by (5.12) whenever (5.4) makes sense. Otherwise, (5.12) is to be replaced by a different relation. It is worth emphasizing that if we switch off torsion, condition 1) becomes the one required in general relativity because, as explained at p. 96-97 of [1], the vorticity of the torsion-free connection vanishes wherever a $3 \times 3$ matrix which appears in the Jacobi fields is nonsingular. Finally, if $\tilde{\nabla}_a \left( \dot{U} \right)^a$ is not vanishing as we assumed so far (see (5.2) and comment before (5.7)) following [21, 22], condition 1) of our theorem is to be replaced by

$$(1') \ R(U,U) - 2\tilde{\omega}^2 - \tilde{\nabla}_a \left( \dot{U} \right)^a \geq 0 \text{ for any nonspacelike vector } U.$$ 

6. - Concluding remarks.

At first we have taken the point of view according to which nonspacelike geodesic incompleteness can be used as a preliminary definition of singularities also in space-times
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with torsion. We have finally been able to show under which conditions Hawking’s singularity theorem without causality assumptions can be extended to the space-time of the ECSK theory. However, when we assume (5.4) and we require consistency of the additional condition (5.6) with the equations of the ECSK theory, we end up with the relation (5.12) which explicitly involves the torsion tensor on the left-hand side (of course, the torsion tensor is also present in \( R(U, U) \) through the connection coefficients, but this is an implicit appearance of torsion, and it is better not to make this splitting). Also the conditions (5.15) and (5.20) involve the torsion tensor in an explicit way if one uses the formula (5.13). This is why we interpret our result as an indication of the fact that the presence of singularities in the ECSK theory is less generic than in general relativity. Our result is to be compared with [10]. In increasing order of importance, the differences between our work and their work are:

1) They look at the singularity theorem of Hawking and Penrose in the ECSK theory, whereas we look at the singularity theorem without causality assumptions in the ECSK theory.

2) We rely on a different definition of geodesics, as explained in sect. 4.

3) We emphasize the role played by the full extrinsic curvature tensor and by the variation formulae in \( U_4 \) theory, a remark which is absent in [10].

4) We keep the field equations of the ECSK theory in their original form, whereas the authors in [10] cast them in a form analogous to general relativity, but with a modified energy-momentum tensor which contains torsion. We think this technique is not strictly needed. Moreover, from a Hamiltonian point of view, the splitting of the Riemann tensor into the one obtained from the Christoffel symbols plus the one explicitly related to torsion does not seem to be in agreement with the choice of the full connection as a canonical variable. In fact, if we look for example at models with quadratic Lagrangians in \( U_4 \) theory, the frame and the full connection are to be regarded as independent variables (see [20]), and this choice of canonical variables has also been made for the ECSK theory [27-29].
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Problems to be studied for further research are the generalization to $U_4$ space-times of the other singularity theorems in [1] using our approach, and of the results in [15] and [17] that we outlined in sect. 3.

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