A Universal Operator Theoretic Framework for Quantum Fault Tolerance

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In this paper we introduce a universal operator theoretic framework for quantum fault tolerance. This incorporates a top-down approach that implements a system-level criterion based on specification of the full system dynamics, applied at every level of error correction concatenation. This leads to more accurate determinations of error thresholds than could previously be obtained. This is demonstrated both formally and with an explicit numerical example. The basis for our approach is the Quantum Computer Condition (QCC), an inequality governing the evolution of a quantum computer. We show that all known coding schemes are actually special cases of the QCC. We demonstrate this by introducing a new, operator theoretic form of entanglement assisted quantum error correction, which incorporates as special cases all known error correcting protocols, and is itself a special case of the QCC.

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INTRODUCTION

The theory of quantum computational fault tolerance allows for successful quantum computation despite faults in the implementation of the quantum computing machine. To achieve fault tolerance, quantum information is encoded into logical qubits comprised of a suitable number of physical qubits, and quantum gates are implemented in such a way that a single error affects only one of a set of encoded qubits. Quantum error correcting techniques are then applied, making use of syndrome measurements, to correct errors that may have occurred. To reduce the probability of multiple errors occurring during any single gate operation, quantum error correction methods (or error avoidance methods, such as decoherence free subspaces or noiseless subsystems) are concatenated, thus allowing for the possibility of fault-tolerant computation.

In order to successfully implement fault-tolerant quantum computer operation, it is essential to have accurate values of error thresholds. Thus, the determination of these values has been an important research focus over the past decade. The standard treatment of fault-tolerance thus far presented in the literature may be characterized as a “bottom-up” approach. In this standard, bottom-up approach, an error model for primitive gates is assumed and a corresponding method of quantum error correction or avoidance is chosen. Fault tolerance conditions are derived through a worst-case analysis of error propagation, or by making assumptions that limit error propagation.

In this paper we propose a very different “top-down” approach that starts from the fault tolerance design objective, and develops specifications for lower level functionality. Our methods enable selection of optimal quantum error correction techniques by avoiding overly pessimistic worst case analysis, or unnecessary assumptions limiting error propagation. Our approach imposes a system-level criterion based on specification of the full system dynamics at each level of concatenation, thus allowing for the systematic determination of optimal error correction, avoidance and fault-tolerance solutions. As we will show, a crucial advantage of our top-down approach is that it leads to more accurate determinations of error thresholds than can be obtained using the standard, bottom-up approach. Our top-down approach is based on an operator theoretic measure of implementation inaccuracy, which compares the full system dynamics of the actually implemented (i.e., noisy) quantum computation directly with the ideally defined (i.e., noiseless), quantum computation. This comparison is expressed by the Quantum Computer Condition (QCC), an explicit mathematical constraint which requires that the difference between the actually implemented and ideally defined computations be bounded by a parameter with a prescribed value. The QCC provides a universal framework for quantum computational fault tolerance that, concomitant to providing a means to more accurately determine error threshold values than are obtained with standard quantum fault tolerance, allows the determination of optimal error correction codes (or error avoidance methods), obviating the need to make ad hoc choices. This is related to the fact that, as we show, the QCC incorporates, as special cases, all known methods of quantum error correction and avoidance. In the process of demonstrating the latter we introduce a new family of error correction codes and avoidance methods.
A UNIVERSAL FRAMEWORK FOR QUANTUM ERROR CORRECTION AND AVOIDANCE

In this section we show that the known methods of error correction and avoidance are all special cases of a universal framework provided by the QCC. To do this we first briefly describe and review the known methods of error correction and avoidance, and then introduce a new, operator theoretic formulation of entanglement assisted error correction which contains the previously known entanglement assisted error correction, operator quantum error correction and standard quantum error correction protocols as special cases. The entanglement assisted operator error correction presented here supports a new family of error correction codes and error avoidance methods. We then show that the entanglement assisted operator error correction introduced here is a special case of the QCC. After summarizing the known methods of quantum error correction and avoidance we review the definition of the QCC \[ \text{QCC} \] and show that all of the methods of error correction and avoidance described here are subsumed under the universal, operator theoretic framework provided by the QCC.

Quantum error correction. Quantum error correction (QEC) is a method of detecting and correcting errors that affect quantum information [2]. The quantum information to be protected is stored in a subspace of a quantum system, known as the code subspace. Errors that can be corrected by a quantum error correction code take the state out of the code subspace. Measurement of the error syndrome identifies the required recovery operation without affecting the encoded quantum information. Thus, the proper correction scheme can be applied without causing disturbance. We can write the general error correction condition of a QEC as

\[
V_{dec} \mathcal{R} \mathcal{V}_{enc} \rho = \rho
\]

for all density matrices \( \rho \), where \( \rho \) contains the quantum information to be protected. For simplicity, we take \( \rho \) to be defined on the space of \( k \) logical qubits. The map \( \mathcal{V}_{enc} \) adjoins an \( s \)-qubit ancilla state to \( \rho \) and performs the encoding into the code subspace of the full Hilbert space of \( n = k + s \) qubits. The dynamical map \( \varepsilon \) describes the errors that affect the encoded state. \( \mathcal{R} \) is the syndrome measurement and recovery operation. The map \( V_{dec} \) is the decoding operation corresponding to \( \mathcal{V}_{enc} \), which yields the final state consisting of \( k \) logical qubits. Error correction is thus successful if this final state is the same as the initial state \( \rho \).

Operator quantum error correction. Operator quantum error correction (OQEC) \[ \text{OQEC} \] is a superoperator formalism that generalizes QEC, incorporating error avoidance techniques, including decoherence free subspaces and noiseless subsystems, into a single framework. Following \[ \text{OQEC} \], OQEC partitions the system’s Hilbert space as \( H = A \otimes B \oplus K \). Information is stored on subsystem \( A \) in the encoded state \( \rho_A = \mathcal{V}_{enc} \rho \), where \( \rho \) is the state of \( k \) logical qubits and \( \rho_A \) is the state of \( n = k + s \) qubits as for QEC. The full OQEC state defined on the complete Hilbert space, \( H \), is obtained via \( W_{\mathcal{R}}(\rho_A) = \rho_A \otimes \rho_B \oplus \mathcal{0}_K \), where \( \rho_B \) is an arbitrary density matrix defined on space \( B \). Due to error avoidance techniques, certain errors may only affect \( \rho_B \). Errors that are not thus protected will affect \( \rho_A \), and are described by error map \( \varepsilon \). These errors are deemed correctable if there exists a dynamical map \( R \) that reverses the action of \( \varepsilon \) on \( \rho_A \), up to a transformation on space \( B \). We can write the general error correction condition of OQEC as

\[
V_{dec} \mathcal{R} \mathcal{V}_{enc} \rho = \rho
\]

for all \( \rho \), where \( F_{\mathcal{R}B} \) is a projection of \( H \) onto \( A \otimes B \) and \( \mathcal{R}B \) is the partial trace over \( B \). Standard QEC codes are thus a special case of OQEC where the dimension of \( B \) is 1.

Entanglement assisted quantum error correction. Entanglement assisted quantum error correction (EAQEC) is another generalization of QEC \[ \text{QEC} \] for error correction in general pertains to the protection of quantum information as it traverses a quantum channel. We will refer to the transmitter and receiver, respectively, as Alice and Bob. Alice would like to encode \( k \) logical qubits into \( n \) physical qubits which will be sent to Bob via a noisy channel, \( \varepsilon \). In EAQEC Alice and Bob are presumed to share \( c \) maximally entangled pairs of qubits. Alice encodes her \( k \) qubits into \( n = k + c + s \) qubits via the operation \( \mathcal{V}_{enc} \), where \( s \) is the number of required ancilla qubits, and we have replaced the symbol \( \mathcal{V}_{enc} \) used in QEC and OQEC with the symbol \( \mathcal{V}_{enc} \) to indicate that the size of the target space has been increased from \( k + s \) to \( k + c + s \). Bob measures an error syndrome using his portion of the initially shared ebits in addition to the \( n \) qubits he received from Alice. He then carries out any transformations required to recover from errors. We denote the syndrome measurement followed by the recovery with the symbol \( \mathcal{R} \), replacing the symbol \( R \) used for QEC and OQEC, to reflect the fact that the operation now involves the additional shared ebits. Finally, he performs the decoding operation \( \mathcal{V}_{dec} \) to obtain a final state of \( k \) logical qubits. Error correction is successful if this is the same as Alice’s initial state of \( k \) qubits. Following \[ \text{EOQEC} \], we say that an [\([n, k; c]\)] EAQEC code consists of an encoding operation \( \mathcal{V}_{enc} \) and a decoding operation \( \mathcal{V}_{dec} \). The error correction condition can then be written as

\[
\mathcal{V}_{dec} \mathcal{R} \mathcal{V}_{enc} \rho = \rho .
\]

This differs from the QEC condition (eq.\[ \text{QEC} \]) in that the encoding and decoding operations \( \mathcal{V}_{enc} \) and \( \mathcal{V}_{dec} \), as well as \( \mathcal{R} \), act on the initially shared ebits in addition to the ancilla. We obtain QEC as a special case of EAQEC by eliminating the ebits, that is, by setting \( c = 0 \).
Entanglement assisted operator quantum error correction. We now generalize the EAQEC of [10] by introducing and defining an operator theoretic generalization of it that we shall refer to as entanglement assisted operator quantum error correction (EAOQEC), that includes EAQEC, as well as QEC, and hence QEC, as special cases. We then show that EAOQEC, and thus all known methods of quantum error correction and avoidance, are special cases of the QCC. The EAOQEC coding and decoding scheme is summarized in the following steps: (1) $V_{enc}$ encodes Alice’s state $\rho$ to $\rho_A$ using the $c$ shared ebits and $s$ ancilla qubits. (2) Alice encodes $\rho_A$ as a generalized noiseless subsystem into a larger system $W_{\rho_B}(\rho_A) = \rho_A \otimes \rho_B \oplus 0_K$, replacing the symbol $W_{\rho_B}$ used for OQEC, reflecting the presence of shared ebits. (3) Alice sends the qubits through a noisy channel, $\varepsilon$. (4) Bob performs syndrome measurement and recovery operation $R$. (5) $F_{AB}$ projects the entire Hilbert space onto $A \otimes B$, replacing the symbol $F_{AB}$ used for OQEC to reflect the fact that the projection applies to an enlarged space involving the additional shared ebits. (6) The noisy part of the encoded system is traced out as $Tr_B(\rho_A \otimes \rho_B) = \rho_A$. (7) $V_{dec}$, the final decoding, recovers the initial state. The EAOQEC formulation presented here allows for new error correction and avoidance protocols beyond those already incorporated in QEC, OQEC, and EAQEC. The error correction condition of EAOQEC is given by

$$V_{dec} Tr_B F_{AB} R \varepsilon W_{\rho_B} V_{enc} \rho = \rho. \quad (4)$$

We see that EAQEC codes arise as a special case of EAOQEC, i.e., eq. (4) reduces to eq. (3), when $\dim B = 1$. We see that OQEC emerges as a special case of EAQEC, i.e., eq. (4) reduces to eq. (2), when $c = 0$ (note that $\lim_{c \to 0} V_{enc} = \lim_{c \to 0} F_{AB} = F_{AB}$, $\lim_{c \to 0} R = R$ and $\lim_{c \to 0} W_{\rho_B} = W_{\rho_B}$). Finally, we see that we obtain QEC as a special case of EAOQEC, i.e., reduce eq. (4) to eq. (1), when $\dim B = 1$ and $c = 0$. In the next sub-section of the paper, we show that EAOQEC is itself a special case of the QCC.

The Quantum Computer Condition

The QCC compares the dynamics of the initial quantum state $\rho$ under the unitary evolution of an ideal (noiseless) quantum computer, $U$, to the evolution of the state under the actual implementation with a real (noisy) quantum computer $P$, by taking their difference under the Schatten-1 norm:

$$\|M_{\{\cdot\}}(P \cdot (M_{\{\cdot\}}(\rho))) - U \rho U^\dagger\|_1 \leq \alpha. \quad (5)$$

The left hand side, which we call the implementation inaccuracy, is the norm of the difference between the actual (implemented) and desired (ideal) final quantum states. It is bounded by $\alpha$, a prescribed accuracy required of the quantum computation. Because the ideal (unitary) operator $U$ acts on logical qubits and the actual (positive dynamical map $P$) implementation $P$ acts on the computational or physical qubits, many of which may be needed to encode one logical qubit, we introduce the pair of linking maps, $M_{\{\cdot\}}$ and $M_{\{\cdot\}}$ to connect the logical and computational Hilbert spaces. Note that these maps do not represent the physical processes which encode and decode qubits. All such physical processes are, in principle, subject to noise and are therefore properly part of the dynamics represented by $P$. It is convenient to define $\hat{P} \equiv M_{\{\cdot\}} \cdot P \cdot M_{\{\cdot\}},$ in terms of which the QCC becomes

$$\|\hat{P}(\rho) - U \rho U^\dagger\|_1 \leq \alpha. \quad (6)$$

We take note of the fact that, in considering the properties of the implementation inaccuracy (i.e., the left-hand-side of either eq. (5) or (6)), its origin in an inequality that must hold for all states, $\rho$, implicitly ascribes the operation of taking the supremum over the complete set of density matrices.

The QCC concisely incorporates a complete specification of the full dissipative, decohering dynamics of the actual, practical device used as the quantum computer, a specification of the ideally-defined quantum computation intended to be performed by the computer, and a quantitative criterion for the accuracy with which the computation must be executed given the inevitability of residual errors surviving even after error correction has been applied. Thus, the QCC provides a rigorous universal, high level framework allowing for the top-down analysis of fault tolerant quantum computation.

As a universal framework for quantum computation we now show that the QCC incorporates EAOQEC, and thus OQEC, EAQEC, and QEC as well, as special cases. (In the following section, we demonstrate that the QCC goes beyond quantum error correction and provides a superoperator approach to quantum fault-tolerance.) The
error correction condition of EAOQEC is given by:
\[ \mathcal{V}_{dec} Tr_B \mathcal{F}_{AB} \mathcal{R} \mathcal{W}_{pB} \mathcal{V}_{enc} \rho = \rho \] (7)
This can be rewritten as:
\[ \| \mathcal{V}_{dec} Tr_B \mathcal{F}_{AB} \mathcal{R} \mathcal{W}_{pB} \mathcal{V}_{enc} \rho - \rho \|_1 = 0, \] (8)
which is a special case of the QCC with \( \alpha = 0, U = I \) (we note that quantum error correction in general protects quantum information that traverses a quantum communications channel, and that such a channel can be regarded as a quantum computer that implements the identity operation), and \( \tilde{P} = \mathcal{V}_{dec} Tr_B \mathcal{F}_{AB} \mathcal{R} \mathcal{W}_{pB} \mathcal{V}_{enc} \).
(Note that the noiseless linking maps \( \mathcal{M}_{\{c\rightarrow\bar{c}\}} \) and \( \mathcal{M}_{\{1\rightarrow\bar{c}\}} \) are subsumed in the definitions of \( \mathcal{V}_{enc} \) and \( \mathcal{V}_{dec} \).
Thus EAOQEC is a special case of the QCC, and, given the discussion immediately following eq.(4), we see that all known protocols for quantum computational error correction and avoidance therefore arise as special cases of the QCC as well. The relationship among the different categories of error correction is depicted in Figure 1.

**OPERATOR QUANTUM FAULT TOLERANCE (OQFT): A UNIVERSAL FRAMEWORK**

In the previous section we showed that the QCC provides a universal framework that encompasses all of the known approaches to quantum error correction and avoidance. In this section we apply the QCC to the further problem of quantum fault tolerance. We will show that the use of the QCC leads naturally to a top-down approach to the problem of quantum computational fault tolerance. In this top-down approach, the success criterion for fault tolerance is expressed in terms of operator norms that characterize the difference between the quantum state produced by the actual quantum computer and the desired quantum state which would be produced by a perfect, noiseless quantum computer. We refer to this top-down approach as Operator Quantum Fault Tolerance (OQFT) by virtue of the fact that the approach is based on operator theory. OQFT thus generalizes standard quantum fault tolerance theory in a manner analogous to the generalization of QEC by OQEC. Our approach is to be contrasted with the standard, bottom-up approach in which the success criterion is expressed in terms of derived error probabilities, without reference to a quantification of the effects of the errors on the quantum state itself. An advantage of the top-down approach is that it allows for more accurate computation of error-thresholds.

The quantum error correction and avoidance techniques discussed in the previous section are designed to protect quantum information as it traverses a quantum channel. Thus, in the statement of the QCC, the desired unitary operation is simply the identity \( U = I \). Quantum fault tolerance applies to the more general case of quantum computation, where \( U \neq I \) in general. Quantum fault tolerance also allows for the possibility that errors may arise during the execution of the quantum error correction/avoidance algorithm itself. The presence of a non-zero \( \alpha \) in the right hand side of the QCC corresponds to the realization that, despite our encoding efforts, there is always some residual error in the final state of the quantum computation. Thus, the QCC provides a proper framework for the study of quantum fault tolerance, the procedure of producing the (nearly) correct final quantum state.

In order to deduce the conditions required for fault-tolerance, we proceed by examining the consequences of concatenating a quantum error-correction code or an error-avoidance method, labeling the successive levels of concatenation by the index \( (i) \). At each level of concatenation, the implementation inaccuracy is given by
\[ \| \tilde{P}^{(i)} \rho - U \rho U^\dagger \|_1 . \] (9)
We may now write down the condition that comprises OQFT. Fault tolerance is deemed successful at a given level of concatenation if the implementation inaccuracy at concatenation level \( i + 1 \) is less than the implementation inaccuracy at level \( i \), that is, if
\[ \sup_\rho \| \tilde{P}^{(i+1)} \rho - U \rho U^\dagger \|_1 < \sup_\rho \| \tilde{P}^{(i)} \rho - U \rho U^\dagger \|_1 < 1, \] (10)
where the supremum operation arises in accordance with the discussion immediately following eq.(6). The OQFT condition, the inequality appearing in (10), is a completely general, operator-theoretic condition that must be satisfied in order to achieve fault tolerant operation of a quantum computer. This is a top-down condition, obtained within the framework provided by the QCC, that allows for more accurate determination of error-thresholds than is possible in the standard, bottom-up approach, as we will see in the following example. The example also illustrates that the standard bottom-up approach to fault tolerance arises as a special case of the OQFT condition given in (10).

**Application of the QCC to Fault Tolerance**

We illustrate the application of eq.(10) to fault tolerance with the following example. We first introduce an operator, \( \Upsilon \), presumed to act on the computational Hilbert space, that faithfully implements the effect of the unitary operator \( U \) (which acts on the smaller, logical Hilbert space):
\[ \mathcal{M}_{\{c\rightarrow\bar{c}\}} \left[ \Upsilon \left( \mathcal{M}_{\{1\rightarrow\bar{c}\}} \rho \right) \Upsilon^\dagger \right] = U \rho U^\dagger. \] (11)
We now introduce a dynamical model for the evolution of the state of the quantum computer, at a given level
of concatenation \(i\), including computational errors, given by:

\[
P^{(i)} \left( \mathcal{M}_{(1\dot{\longrightarrow}c)} \rho \right) = \left( 1 - \epsilon^{(i)} \right) \mathcal{Y} \left( \mathcal{M}_{(1\dot{\longrightarrow}c)} \rho \right) \mathcal{Y}^\dagger + \epsilon^{(i)} Q^{(i)} \left( \mathcal{M}_{(1\dot{\longrightarrow}c)} \rho \right),
\]

where \(\epsilon^{(i)}\) depends on the probability for the occurrence of errors and \(Q^{(i)}\) is a superoperator that represents the effects of the errors in the computational Hilbert space. Note that the superoperator, \(P^{(i)}\), is a linear combination of the operator, \(\mathcal{Y}\), and the superoperator, \(Q^{(i)}\). The above model used in the present example is sufficiently general to include, as special cases, local stochastic noise and locally correlated stochastic noise, the models that define standard quantum fault-tolerance \cite{7}. The implementation inaccuracy is then

\[
\| \hat{P}^{(i)} \rho - U \rho U^\dagger \|_1 = \left\| \left( 1 - \epsilon^{(i)} \right) \mathcal{M}_{(c\dot{\longrightarrow}i)} \left[ \mathcal{Y} \left( \mathcal{M}_{(1\dot{\longrightarrow}c)} \rho \right) \mathcal{Y}^\dagger \right] + \epsilon^{(i)} \hat{Q}^{(i)} \rho - U \rho U^\dagger \right\|_1,
\]

where we have absorbed the linking maps into \(\hat{Q}^{(i)} = \mathcal{M}_{(c\dot{\longrightarrow}i)} Q^{(i)} \mathcal{M}_{(1\dot{\longrightarrow}c)}\). With \(\| \cdot \|_1\) the implementation inaccuracy then reduces to the simple form

\[
\| \hat{P}^{(i)} \rho - U \rho U^\dagger \|_1 = \epsilon^{(i)} \| \hat{Q}^{(i)} \rho - U \rho U^\dagger \|_1.
\]

The OQFT success criterion given by eq.\(\text{[10]}\) then becomes

\[
\frac{\epsilon^{(i+1)}}{\epsilon^{(i)}} \cdot \frac{\sup_p \| \hat{Q}^{(i+1)} \rho - U \rho U^\dagger \|_1}{\sup_p \| \hat{Q}^{(i)} \rho - U \rho U^\dagger \|_1} < 1.
\]

Note that this differs in an important way from the criterion used in standard treatments of fault tolerance, which is based solely on a comparison of error probabilities:

\[
\frac{\epsilon^{(i+1)}}{\epsilon^{(i)}} < 1.
\]

Equation \(\text{[15]}\) takes explicitly into account the fact that the error dynamics associated with residual errors in general differs at each level of concatenation, namely, that \(Q^{(i+1)} \neq Q^{(i)}\). If the dynamical effects of the errors tend to become smaller at higher levels of concatenation, quantum fault tolerance can be achieved at error thresholds larger than those predicted by standard fault tolerance theory. This improvement in error threshold values arises due to the correction provided by the ratio of dynamical factors derived from the implementation inaccuracy.

**Example: OQFT condition with five-qubit code**

As a concrete example of Eqs. 10 and 15 we analyze a case where we attempt to store one qubit of information, and thus have \(U = I\), using an error correction code. We assume a general environment with two independent noise generators: one causing bit-flips with probability \(p_x\), and the other phase-flips with probability \(p_z\). For simplicity we assume perfect readout and recovery operations. We note that the case of unbalanced noise generators (i.e., biased noise) is physically important, and achieving fault tolerance in such an environment has been the subject of recent analysis \cite{12}. The final state of the one qubit of information after encoding, occurrence of errors, and any attempts at recovery, and decoding if necessary, will be:

\[
\hat{P}^{(i)} \rho = \left( 1 - \epsilon^{(i)} \right) \rho + \epsilon^{(i)} \left( \eta^x_k \rho_x + \eta^y_k \rho_y + \eta^z_k \rho_z \right),
\]

where \(\rho_k = \sigma_k \rho \sigma_k^\dagger\) for \(k = x, y, z\), the \(\sigma_k\) are the Pauli operators, \(\eta^x_k = \epsilon_k^x / \epsilon^{(i)}\), \(\epsilon^y_k = \epsilon_k^y / \epsilon^{(i)}\), and the \(\epsilon_k^i\) are functions of the concatenation level, \(i\), the noise generators represented by \(p_x\) and \(p_z\), the error correcting code, and the recovery operation. We emphasize the dependence of \(\epsilon_k^i\) on the concatenation level, noting that if the concatenation procedure is working properly, the quantity \(\epsilon^{(i)}\) should decrease with increasing \(i\).

We now calculate the supremum of the implementation inaccuracy at the \(i\)th concatenation level using Eq. \(\text{[17]}\)

\[
\sup_p \| \hat{P}^{(i)} \rho - \rho \|_1 = \epsilon^{(i)} \sup_p \| \eta^x_k \rho_x + \eta^y_k \rho_y + \eta^z_k \rho_z - \rho \|_1.
\]

The OQFT success criterion, eq.\(\text{[10]}\), for applying concatenation up to level \(i + 1\), is now

\[
\frac{\epsilon^{(i+1)}}{\epsilon^{(i)}} \cdot \frac{\sup_p \| \eta^x_k \rho_x + \eta^y_k \rho_y + \eta^z_k \rho_z - \rho \|_1}{\sup_p \| \eta^x_k \rho_x + \eta^y_k \rho_y + \eta^z_k \rho_z - \rho \|_1} < 1.
\]

It is important to note that the quantities \(\eta_k\) appearing in eq.\(\text{[19]}\) do not in general factor out of the norm. The first factor on the left-hand-side of eq. \(\text{[19]}\) is analogous to that of standard quantum fault tolerance (cf eq.\(\text{[16]}\)), while the second factor on the left-hand-side of eq. \(\text{[19]}\) is the correction factor that arises upon imposing the condition of operator quantum fault tolerance. For simplicity, we will refer to the ratio comprising the entire left-hand-side of eq. \(\text{[19]}\) as the OQFT \(P\)-ratio (cf eq.\(\text{[11]}\)). We will refer to the ratio of norms on the left-hand-side of eq. \(\text{[19]}\) that appears to the right of the dot as the OQFT \(Q\)-ratio (cf eq.\(\text{[13]}\)). We will refer to the ratio appearing in eq. \(\text{[19]}\) as the (standard) quantum fault tolerance (QFT) ratio.

Proceeding with the analysis, we now assume an environment which causes bit flips to occur with probability \(p_x\), and phase flips with probability \(p_z\). We will employ the \([5,1,3]\) code to protect one qubit of quantum information and compare one, and two, concatenations of the code, making use of the OQFT condition (eq.\(\text{[10]}\) or eq.\(\text{[17]}\)) as the criterion for fault tolerance. The numerical results of this analysis for a region in \(\{p_x, p_z\}\)
The OQFT condition yields a threshold advantage of about 15% compared to the result obtained from standard QFT.

Pursuing this further, Figure 3 shows the various ratios of interest for a case where we have set $p_z$ to .06, which would correspond to a situation of practical experimental interest. In this figure we highlight the effect of the Q-ratio on the threshold value. Specifically, using the standard QFT ratio ($\text{cf. eq.}(10)$), concatenation would have been deemed unsuccessful for any $p_z \gtrsim .0785$ since at that point the QFT-ratio becomes greater than 1. The correction factor of OQFT relaxes this value, and we see that, in fact, concatenation to the second level is successful for $p_z$ values as large as $p_z \lesssim .0905$ (at which point the $P$-ratio becomes greater than one).

CONCLUSIONS

In this paper we introduced operator quantum fault tolerance (OQFT), a universal, operator-theoretic framework for quantum fault tolerance that includes previously known, standard quantum fault tolerance as a special case. OQFT comprises a top-down approach that employs a system-level criterion based on specification of the full system dynamics. OQFT allows us to calculate more accurate values of error correction thresholds than is possible using previous approaches to fault tolerance. This was demonstrated both formally and with an explicit numerical example. OQFT is based on the quantum computer condition (QCC) which was demonstrated to provide a universal organizing structure for all known methods of error correction and error avoidance. We did this by introducing a new, operator theoretic form of entanglement assisted error correction, which includes all the previously known methods of error correction and avoidance, and is itself a special case of the QCC.

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**APPENDIX**

After the present paper was completed, we were apprised of an independent formulation of EAOQEC presented by Hsieh, *et al.* in [16]. The connection between the two formulations can be made transparent by considering an equation that depicts both:

\[ \mathcal{V}_{\text{dec,kin}} \left( \rho \cdot \mathcal{R} \cdot \mathcal{V}_{\text{dec,dyn}} \right) = \mathcal{V}_{\text{enc,dyn}} \mathcal{V}_{\text{enc,kin}} \cdot \rho = \rho. \quad (20) \]

Here the upper line within curly brackets corresponds to the EAOQEC of the present paper, and the lower line corresponds to that of [16]. In this equation, we have factored the encoding and decoding operations into separate kinematic and dynamic parts. The kinematic factors append or remove any necessary ancilla qubits and ebits (*i.e.*, increase or decrease, as required, the Hilbert space dimension), and the dynamic factors represent unitary evolution of the qubits (*i.e.*, implement the physical encoding and decoding operations) [17].

The principal difference between the EAOQEC formulation presented in the present paper and that presented in [16] is seen in the placement of the unitary part of the decoding operation relative to the measurement and recovery operation. In the present paper the unitary dynamical part of the decoding operation, \( V_{\text{dec,dyn}} \), is performed *after* a (general) measurement and recovery operation, \( \mathcal{R} \), whereas in [16] the unitary decoding operation, \( V_{\text{dec,dyn}} \) (represented by the symbol \( U \) in [16]), is performed *before* a (specifically defined) measurement and recovery operation, \( \mathcal{R} \) (represented by the symbol \( \mathcal{D}_0 \) in [16]). Thus, the relative order of the unitary decoding operation and the measurement/recovery operation differs between the two formulations of EAOQEC. We note that the unitary decoding operation and the measurement/recovery operation do not in general commute. We further note that the formulation of EAOQEC given in [16] is defined within the stabilizer formalism, whereas the formulation of EAOQEC given in the present paper allows for non-stabilizer codes, such as non-additive codes, in addition to stabilizer codes.

Finally, we recapitulate that we have shown that all forms of error correction and error avoidance are in fact special cases of the more fundamental QCC (with \( \alpha = 0 \) and \( U = I \) in either eq. (5) or (6)), and that the QCC (with \( \alpha > 0 \) and \( U \) arbitrary in either eq. (4) or (5)) provides a universal operator theoretic framework for quantum fault tolerance, as demonstrated in the present paper.

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1. P.W. Shor, *Proceedings, 37th Annual Symposium on Foundations of Computer Science*, pp. 56-65, IEEE Press, Los Alamitos, CA (1996); M. Nielsen, I. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, U.K. (2000).
2. P.W. Shor, Phys. Rev. A 52 R2493 (1995); A.M. Steane, Phys. Rev. Lett. 77, 793 (1996); A. R. Calderbank and P. W. Shor, Phys. Rev. A 54, 1098 (1996); A.R. Calderbank, E.M. Rains, P.M. Shor and N.J.A. Sloane, IEEE Trans. Inf. Th. 44, 1369 (1998); A.R. Calderbank, E.M. Rains, P.W. Shor, and N.J.A. Sloane, Phys. Rev. Lett. 78, 405 (1997).
3. P. Zanardi, M. Rasetti, Phys. Rev. Lett., 79, 3306, (1997); L-M. Duan, G-C. Guo, Phys. Rev. Lett., 79, 1953, (1997).
4. E. Knill, R. Laflamme, and L. Viola, Phys. Rev. Lett. 84, 2525 (2000); L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. 85, 3520 (2000); P. Zanardi, Phys. Rev. A 63, 012301 (2000).
5. E. Knill and R. Laflamme, arXiv:quant-ph/9608012 (1996).
6. D. Aharonov and M. Ben-Or, Proc. 29th Ann. ACM Symp. on Theory of Computing p. 176, New York, ACM (1998); D. Aharonov and M. Ben-Or, arXiv:quant-ph/9906129 (1999); A.Yu. Kitaev, Russian Math. Surveys 52, 1191 (1997); E. Knill, R. Laflamme, and W.H. Zurek, Proc. Roy. Soc. Lond. A 454, 365 (1998); J. Preskill, Proc. Roy. Soc. Lond. A 454, 385 (1998); D. Gottesman, “Stabilizer codes and quantum error correction,” Caltech Ph.D. thesis (1997), arXiv:quant-ph/9705052; B.W. Reichardt, arXiv:quant-ph/0509203 (2005).
7. B.M. Terhal and G. Burkard, Phys. Rev. A 71, 012336; D. Aharonov, A. Kitaev, and J. Preskill, Phys. Rev. Lett. 96, 050504 (2006); P. Aliferis, D. Gottesman, and J. Preskill, Quant. Inf. Comp. 6, 97 (2006); B.M. Terhal and P. Aliferis, Quant. Inf. Comp. 7, 139 (2007); P. Aliferis, D. Gottesman, and J. Preskill, arXiv:quant-ph/0703264 (2007).
8. G. Gilbert, M. Hamrick and Y.S. Weinstein, arXiv/0707.0008.
9. D. Kribs, R. Laflamme, and D. Poulin, Phys. Rev. Lett. 94, 180501 (2005).
10. T. Brun, I. Devetak and M. Hsieh, Science, 314, 436 (2006).
11. In addition to EAQEC, Ref. 10 introduces a protocol based on EAQEC called catalytic operator quantum error correction. It is straightforward to show that catalytic operator quantum error correction is also a special case of the EAQEC protocol that we introduce in the current paper, and hence is a special case as well of the QCC.
12. The QCC holds for the general case of positive dynamical maps, of which completely-positive dynamical maps are a special case.
13. This can be illustrated with the example of the 7-qubit Steane code, for which the computational Hilbert space consists of 7 computational qubits for every logical qubit. We stress that the linking maps \( M_{[l=\cdots]} \) and \( M_{[\ell=\cdots]} \) do not represent transformations carried out by physical devices but simply increase or decrease the Hilbert space dimension, thus going from the logical to computational
Hilbert space and *vice-versa*. For many purposes it is not necessary to explicitly write out the linking maps. Nevertheless, it is necessary to do so in our discussion of quantum fault-tolerance, *cf* eqs. (11) and (12).

[14] We note that the universal QCC-based framework for error correction, error avoidance and fault tolerance presented in this paper applies equally to circuit-based, *i.e.*, so-called two-way, and graph state-based (including cluster state-based), *i.e.*, so-called one-way, quantum computation. This is because in our approach the specification of the actual quantum computation occurs in the content of the positive dynamical map, $P$, that is part of the implementation inaccuracy, *i.e.*, the left-hand-side of eq. (5). For either a circuit-based or graph state-based quantum computation, the dynamical evolution of the state will be described by some positive dynamical map, $P$, and hence the framework presented in this paper will in general apply.

[15] P. Aliferis and J. Preskill, arXiv:0710.1301 (2007).

[16] M.H. Hsieh, I. Devetak, and T. Brun, arXiv:0708.2142 (2007).

[17] In eq. (2), as described in the text, we have essentially adopted the notation used in [9] (this notation with modifications is inherited in eq. (4) as well). In re-casting eq. (4) as the upper line of eq. (20) we have collected all kinematical operations in $V_{\text{dec, kin}}$, thus conforming with the presentation given in [10] ($V_{\text{dec, kin}}$ is not represented by a symbol in [10], but is referred to as “discarding the unwanted systems.”).