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Junhua He
gyu@wm.edu

Louis A. Valentin
College of William and Mary

Xiaoyan Yin

Gexin Yu
College of William and Mary, gyu@wm.edu

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Extremal permutations in routing cycles

Junhua He∗
School of Mathematical Sciences
University of Electronic Science and Technology of China
Chengdu, Sichuan, China

Louis A. Valentin
Department of Mathematics
College of William and Mary
Williamsburg, VA, U.S.A.

Xiaoyan Yin†
School of Mathematics and Statistics
Xidian University
Xi’an, Shaanxi, China

Gexin Yu‡§
Department of Mathematics
College of William and Mary
Williamsburg, VA, U.S.A.

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Abstract

Let \( G \) be a graph whose vertices are labeled \( 1, \ldots, n \), and \( \pi \) be a permutation on \( \{1, 2, \ldots, n\} \). A pebble \( p_i \) that is initially placed at the vertex \( i \) has destination \( \pi(i) \) for each \( i \in [n] \). At each step, we choose a matching and swap the two pebbles on each of the edges. Let \( rt(G, \pi) \), the routing number for \( \pi \), be the minimum number of steps necessary for the pebbles to reach their destinations.

Li, Lu and Yang proved that \( rt(C_n, \pi) \leq n - 1 \) for every permutation \( \pi \) on the \( n \)-cycle \( C_n \) and conjectured that for \( n \geq 5 \), if \( rt(C_n, \pi) = n - 1 \), then \( \pi = 23 \cdots n1 \) or its inverse. By a computer search, they showed that the conjecture holds for \( n < 8 \).

We prove in this paper that the conjecture holds for all even \( n \geq 6 \).

Keywords: Routing number, permutation, sorting algorithm, Cayley graphs

1 Introduction

Routing problems occur in many areas of computer science. Sorting a list involves routing each element to the proper location. Communication across a network involves routing...
messages through appropriate intermediaries. Message passing between multiprocessors
requires the routing of signals to correct processors.

In each case, one would like the routing to be done as quickly as possible. Let us
consider the routing model introduced by Alon, Chung, and Graham [2] in 1994. Let
\( G = (V,E) \) be a graph whose vertices are labeled as \( 1,\ldots,n \). For a permutation \( \pi \) on
\( [n] \), a pebble \( p_i \), which has destination \( \pi(i) \), is placed at \( i \) for each \( i \in [n] \). For example,
let \( \pi = 342165 = (1\ 2\ 3\ 4\ 5\ 6\ 3\ 4\ 2\ 1\ 6\ 5) \), then the destination of \( (p_1, p_2, p_3, p_4, p_5, p_6) \) are
\( (3, 4, 2, 1, 6, 5) \).

We wish to move pebbles to their destinations. To do so, at each round we select a
matching of \( G \) and swap the pebbles at the endpoints of each edge, and repeat rounds
until all pebbles are in place.

Let \( rt(G, \pi) \) denote the minimum number of rounds necessary to route \( \pi \) on \( G \). Then,
the **routing number** of \( G \) is defined as:

\[
rt(G) = \max_\pi \{ rt(G, \pi) \}.
\]

Note that when each matching in the routing process consists of one edge, the routing
number corresponds to the diameter of Cayley graph generated by (cyclically) adjacent
transpositions. The diameter problem of Cayley graphs has a rich research literature, and
we refer the reader to the book [3] for a comprehensive survey.

Very few results are known for the exact values of the routing numbers of graphs.

**Theorem 1** (Alon, Chung, and Graham [2]). **The following are true:**

1. \( rt(P_n) = n \), \( rt(K_n) = 2 \) and \( rt(K_{n,n}) = 4 \);
2. \( rt(G) \geq diam(G) \) and \( rt(G) \geq \frac{2}{|C|} \min\{|A|, |B|\} \), where \( diam(G) \) is the diameter of
   \( G \) and \( C \) is a set that cuts \( G \) into parts \( A \) and \( B \);
3. \( rt(G) \leq rt(H) \) and \( rt(T_n) < 3n \), where \( H \) is a spanning subgraph of \( G \) and \( T_n \) is a
   tree on \( n \) vertices;
4. \( rt(G_1 \times G_2) \leq 2rt(G_1) + rt(G_2) \), and \( n \leq rt(Q_n) \leq 2n - 1 \).

Zhang [6] improved the above bound on trees, showing that \( rt(T_n) \leq \left\lfloor \frac{3n}{2} \right\rfloor + O(\log n) \).

Li, Lu, and Yang [5] showed that \( n + 1 \leq rt(Q_n) \leq 2n - 2 \), improving both the previous upper and lower bounds on hypercubes. Among other results, they showed that
\( rt(C_n) = n - 1 \), and made the following conjecture.

**Conjecture 2** (Li, Lu, Yang [5]). **For** \( n \geq 5 \), **if** \( rt(C_n, \pi) = n - 1 \), **then** \( \pi \) **is the rotation
\( 23\cdots n1 \) or its inverse \( n12\cdots (n - 1) \).

The conjecture does not hold for \( n = 4 \); the permutation that transposes two non-
adjacent vertices and fixes the other two serves as a counterexample. They verified the
conjecture for $n < 8$ through a computer search. The conjecture is kind of counter-intuitive: the permutations on the cycle requiring most time to route are the ones that each pebble is very close to (actually at distance one from) its destination.

In this article, we give a proof of the conjecture when $n$ is even.

**Theorem 3.** For even $n \geq 6$, if $rt(C_n, \pi) = n - 1$, then $\pi$ is the rotation $23\cdots n1$ or its inverse.

Some new tools are introduced in the proof, beyond the ideas from [1] by Albert, Li, Strang, and the last author. These tools, introduced in Section 2, enable us to describe precisely the swapping process of each pebble, and determine the pebbles that need $n - 1$ steps to route, see Section 3 for detail. In Section 4, we show that the only permutations that need $n - 1$ steps to route must be the two permutations in the theorem.

## 2 Important notion and tools

Let $G = C_n$ with even $n \geq 6$ and label the vertices of $C_n$ as $1, 2, \ldots, n$ in the clockwise order. Let the clockwise direction be the positive direction and counter clockwise be the negative direction. In the rest of the paper, when list pebbles or sets of pebbles in a row, we always think them to be in the clockwise order on the cycle.

### 2.1 The Odd-Even Routing Algorithm

An odd-even sort or odd-even transposition sort is a classic sorting algorithm (see [4]) used to sort a list of numbers on parallel processors. To describe this algorithm, we may place the $n$ numbers to be sorted on the vertices of the path $P_n = v_1v_2\ldots v_n$. An edge $e = v_iv_{i+1}$ is odd if and only if $i$ is odd. At each odd step (respectively, even step) of the routing process, we select a matching consisting of odd edges (respectively, even edges) whose two numbers have the wrong order and swap the numbers on the endpoints.

We apply a similar algorithm on even cycles, whose edges can be partitioned into two perfect matchings. We shall call edges in one perfect matching to be even and the others to be odd. Thus, once the parity of one edge is specified, the parity of all the edges is determined. During the odd steps (respectively, even steps) we choose a matching consisting of odd edges (respectively, even edges) whose two numbers have the wrong order and swap the numbers on the endpoints.

### 2.2 Spins and Disbursements

There are exactly two paths for pebble $p_i$ to reach its destination, by traveling either in the positive or negative direction. Let $d^+(i, j)$ denote the distance from the vertex $i$ to the vertex $j$ along the positive direction. Then if $i < j$, then $d^+(i, j) = j - i$ if $i < j$, and $d^+(i, j) = j - i + n$ if $i > j$. For simplicity, for pebbles $p_i$ and $p_j$, we define $d^+(p_i, p_j) = d^+(i, j)$, and if $p_i$ and $p_j$ are on the endpoints of an edge, we sometimes call the edge $p_ip_j$. 




Consider a routing process of a permutation \( \pi \) on \( C_n \) with pebble set \( P = \{ p_1, \ldots, p_n \} \). For each pebble \( p_i \), let \( s(p_i) \), the spin of \( p_i \), represent the displacement for \( p_i \) to reach its destination from its current position. So, \( s(p_i) \in \{ d^+(i, \pi(i)), d^+(i, \pi(i)) - n \} \). Note that the spin of a pebble changes with its movement.

A sequence \( B = (s(p_1), s(p_2), \ldots, s(p_n)) \) is called a valid disbursement of \( \pi \) if the spins can be realized by a routing process on \( \pi \). Not all combinations of spins give valid disbursements. The following lemma gives a necessary and sufficient condition for a set of spins to be a valid disbursement.

**Lemma 4.** Let \( B = (s(p_1), s(p_2), \ldots, s(p_n)) \) be an assignment of the spins to the pebbles. It is a valid disbursement if and only if \( \sum_{i=1}^{n} s(p_i) = 0 \).

**Proof.** To see the necessity, we observe that when two pebbles are swapped, one moves forward one step and one moves backward one step, so the sum of spins remains invariant. Since \( B \) is a valid disbursement, the final spins are all zeroes, so the sum is also zero.

For sufficiency, we can move the pebbles one by one along their assigned directions. \( \Box \)

From this lemma, a valid disbursement of a non-identity permutation \( \pi \) must contain both positive and negative spins. Let \( s(p_i) > 0 \) and \( s(p_j) < 0 \) in a valid disbursement \( B \) of \( \pi \). By flipping the spins of \( p_i \) and \( p_j \), we change the spins of \( p_i \) and \( p_j \) to \( s(p_i) - n \) and \( s(p_j) + n \), respectively. Clearly, after one flip, we obtain a new valid disbursement.

A valid disbursement \((s(p_1), \ldots, s(p_n))\) is minimized if \( \sum_{p \in P} |s(p)| \) is minimized. The following simple fact is very important.

**Lemma 5.** If a valid disbursement is minimized, then \( s(p_i) - s(p_j) \leq n \) for all \( i, j \in [n] \).

**Proof.** For otherwise, one can make the sum smaller by flipping the spins of \( p_i \) and \( p_j \). \( \Box \)

### 2.3 An order relation

Once a valid disbursement \( B = (s(p_1), s(p_2), \ldots, s(p_n)) \) of \( \pi \) is given, some restrictions are placed on the routing processes realizing \( B \). For example, if \( s(p_i) - s(p_j) > d^+(p_i, p_j) \), then \( p_i \) and \( p_j \) must swap at some round in the routing processes. In other words, each valid disbursement is associated with an order relation on the pebbles.

**Definition 6.** Let \( B = (s(p_1), s(p_2), \ldots, s(p_n)) \) be a valid disbursement. We call \( p_i \succ p_j \) if \( s(p_i) - s(p_j) > d^+(p_i, p_j) \).

Note that the order relation is transitive. To see that, let \( p_i \succ p_j \) and \( p_j \succ p_k \). Then \( s(p_i) - s(p_j) > d^+(p_i, p_j) \) and \( s(p_j) - s(p_k) > d^+(p_j, p_k) \). It follows that \( s(p_i) - s(p_k) > d^+(p_i, p_j) + d^+(p_j, p_k) \geq d^+(p_i, p_k) \), which implies \( p_i \succ p_k \).

As two pebbles have different destinations, \( s(p_i) - s(p_j) \neq d^+(p_i, p_j) \), so if \( p_i \succ p_j \) is not true, then \( s(p_i) - s(p_j) < d^+(p_i, p_j) \). When \( p_i \succ p_j \), we say that \( p_i \) and \( p_j \) are comparable, or more precisely, \( p_i \) is bigger than \( p_j \) and \( p_j \) is smaller than \( p_i \). If \( p_i \) is neither bigger nor smaller than \( p_j \), we call them incomparable. If each pebble in set \( P_1 \) is bigger than every pebble in set \( P_2 \), we also write \( P_1 \succ P_2 \).

The following lemma provides a convenient way to determine order relations.
Lemma 7. Let $x, y, z$ be three pebbles in the clockwise order sitting on the cycle. If $x > z$, then $x > y$ or $y > z$. Furthermore, if $x > z$, then $y$ is not smaller than $z$ and not bigger than $x$.

Proof. For otherwise, $s(x) - s(y) < d^+(x, y)$ and $s(y) - s(z) < d^+(y, z)$. It follows that $s(x) - s(z) < d^+(x, y) + d^+(y, z) = d^+(x, z)$. Then $x$ is not bigger than $z$, a contradiction.

For the furthermore part, if $x > z$ and $z > y$, then $s(x) - s(z) \geq d^+(x, z)$ and $s(z) - s(y) \geq d^+(z, y)$, and it follows that $s(x) - s(y) \geq d^+(x, z) + d^+(z, y) > n$, a contradiction. Likewise, if $x > z$ and $y > x$, then $s(x) - s(z) \geq d^+(x, z)$ and $s(y) - s(x) \geq d^+(y, x)$, and it follows that $s(y) - s(z) \geq d^+(x, z) + d^+(y, x) > n$, a contradiction. \hfill $\square$

The following lemma says that we only need to swap comparable pebbles to route the permutation.

Lemma 8. Let $B$ be a minimized disbursement of $\pi$. If a pebble $p$ is incomparable with all other pebbles, then $s(p) = 0$, i.e., the pebble $p$ is at its destination vertex.

Proof. Suppose that $s(p) \neq 0$. By symmetry, let $s(p) > 0$.

Let $\pi = \Pi_{\pi_1}$ be a cycle decomposition of $\pi$, where $\pi_i = (i_1, \cdots, i_{r_i})$. Then the pebble placed at $i_k$, which we call pebble $i_k$ to save symbols, has destination $i_{k+1}$ for all $k \leq r_i$, with $i_{r_i+1} = i_1$. Let $P_i$ be the set of pebbles on $\pi_i$. We say that $\pi_i$ is the orbit of the pebbles in $P_i$. We may assume that $p = i_1$.

We claim that for each $j$, $\sum_{q \in P_j} s(q) = an$ for some integer $a$. To see this, we note that $s(jk) \in \{d^+(jk, jk+1), d^+(jk, jk+1) - n\}$. Thus, if the spins are all positive, the sum equals $bn$ for some positive integer $b$. However, each switch of a spin from positive to negative would cause a change of $-n$ in the sum. So the sum of spins remains a multiple of $n$.

We further claim that all pebbles in $P_i$ have positive spins, which also implies that $\sum_{q \in P_j} s(q) = bn$ for some positive integer $b$. Note that $s(i_1) = s(p) > 0$. Let $j$ be the smallest integer so that $s(i_j) < 0$. Then $s(i_j) = d^+(i_j, i_{j+1}) - n \leq -d^+(i_1, i_j)$. So $s(i_1) > 0 \geq s(i_j) + d^+(i_1, i_j)$. It follows that $s(i_1) - s(i_j) > d^+(i_1, i_j)$, that is, $p = i_1 \succ i_j$, a contradiction.

As the sum of all spins is zero, there exists some orbit $\pi_j$ with spin sum $cn$ for some integer $c < 0$. In particular, there exists a pebble $q \in P_j$ such that $q$ passes $p = i_1$ in the negative direction to arrive its destination. So $s(q) + d^+(p, q) < 0 < s(p)$ and it follows that $p \succ q$, a contradiction. \hfill $\square$

Note that the order of pebbles is always associated with the current disbursement, which may not be the same as the initial disbursement. The following lemma says that whether or not two pebbles swap is determined by the initial disbursement. So we will not keep tracking of the spins, but just see whether necessary swaps are performed.

Lemma 9. If $p_i$ and $p_j$ are incomparable, then in the sorting process, they will always be incomparable. If $p_i \succ p_j$, then $p_i$ and $p_j$ are incomparable after the swap of $p_i$ and $p_j$. 

Proof. If \( p_i \) and \( p_j \) are incomparable, then in the sorting process, \((s(p_i) - s(p_j)) - d^+(p_i, p_j)\) does not change: if a pebble swaps with both \( p_i \) and \( p_j \), then it must be bigger than or smaller than both \( p_i \) and \( p_j \), thus the distance from \( p_i \) to \( p_j \) does not change and \( s(p_i) - s(p_j) \) does not change; If a pebble swaps only with \( p_i \), then \( s(p_i) \) increases by one and \( d^+(p_i, p_j) \) increases by one; If a pebble swaps only with \( p_j \), then \( s(p_j) \) increases by one and \( d^+(p_i, p_j) \) decreases by one.

Now let \( p_i > p_j \). We only need to show that they become incomparable right after the swap of \( p_i \) and \( p_j \), by what we just proved. Since \( p_i > p_j \), \( n > s(p_i) - s(p_j) \geq d^+(p_i, p_j) + 1 \geq 2 \). Let \( s'(p_i) \) and \( s'(p_j) \) respectively be the new spins of \( p_i \) and \( p_j \) after the swap of \( p_i \) and \( p_j \). Note that \( p_i \) and \( p_j \) are adjacent to each other only if the pebbles on the segment from \( p_i \) to \( p_j \) along the positive direction have swapped with \( p_i \) (for those smaller than \( p_i \)) or \( p_j \) (for those bigger than \( p_j \)). Then \( s'(p_i) - s'(p_j) = s(p_i) - s(p_j) - d^+(p_i, p_j) - 2 \). Therefore, \( s'(p_j) - s'(p_i) < 0 \), and \( p_j \) cannot be bigger than \( p_i \). Also, \( s'(p_i) - s'(p_j) \leq n - 2 < n - 1 \), thus after the swap, \( p_i \) cannot be bigger than \( p_j \).

When \( B \) is minimized and two pebbles \( p_i \) and \( p_j \) satisfy \( s(p_i) - s(p_j) = n \), we still get a minimized disbursement after flipping the spins of \( p_i \) and \( p_j \). The following lemma tells us how the order relation changes when we do such a flip.

**Lemma 10.** Let \( B \) be a minimized disbursement of \( \pi \), and \( s(p_i) - s(p_j) = n \). Let \( B' \) be the disbursement after flipping the spins of \( p_i \) and \( p_j \). Let \( k \notin \{i, j\} \). Then

1. \( p_j > p_i \) under \( B' \), and the order relation remains unchanged for pebbles other than \( p_i \) and \( p_j \);
2. \( p_k > p_i \) under \( B' \) if \( p_i \) and \( p_k \) are incomparable under \( B \); and \( p_i \) and \( p_k \) are incomparable under \( B' \) if \( p_i > p_k \) under \( B \);
3. \( p_j > p_k \) under \( B' \) if \( p_j \) and \( p_k \) are incomparable under \( B \); and \( p_j \) and \( p_k \) are incomparable under \( B' \) if \( p_k > p_j \) under \( B \).

**Proof.** Let \( s'(p_i) = s(p_i) - n \) and \( s'(p_j) = s(p_j) + n \).

(1) Clearly \( p_j > p_i \) under \( B' \), since \( s'(p_j) - s'(p_i) = 2n - (s(p_i) - s(p_j)) = n > d^+(p_j, p_i) \). For pebbles not in \( \{p_i, p_j\} \), the spins and distance do not change from \( B \) to \( B' \), so their order relation does not change as well.

(2) For \( k \notin \{i, j\} \), we know that

\[
s(p_k) - s'(p_i) - d^+(p_k, p_i) = s(p_k) - (s(p_i) - n) - (n - d^+(p_i, p_k)) = s(p_k) - s(p_i) + d^+(p_i, p_k).
\]

Therefore, \( s(p_k) - s'(p_i) > d^+(p_k, p_i) \) if and only if \( s(p_i) < s(p_k) + d^+(p_i, p_k) \). Note that \( p_k \) cannot be bigger than \( p_i \) under \( B \), for otherwise, \( s(p_k) - s'(p_j) > s(p_i) - s(p_j) = n \). It follows that \( p_k > p_i \) under \( B' \) if and only if \( p_k \) and \( p_i \) are incomparable under \( B \).

By flipping the spins of \( p_i \) and \( p_j \) in \( B' \), we get \( B \). So we have the other part as well.

(3) Since \( s'(p_j) - s'(p_i) = n \) under \( B' \), and one gets \( B \) after flipping the spins of \( p_i \) and \( p_j \) in \( B' \), these two statements follow from (2).

\[\square\]
2.4 The window of a pebble

Let $B$ be a minimized disbursement of $\pi$ with associated order $\succ$. For an arbitrary pebble $p_0$, let
\[
U = \{p \in P : p \succ p_0\} \quad \text{and} \quad W = \{p \in P : p_0 \succ p\}.
\]
By Lemma 8, the routing process ends when no pebble has a bigger or smaller pebble. So we have the following equation, which is heavily used to determine the spins of the pebbles in our later proofs.
\[
s(p_0) = |W| - |U|.
\]  
(1)

By Lemma 7, there are no $u \in U, w \in W$ such that $u, w, p_0$ or $p_0, u, w$ on $C_n$. So if $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_t\}$, then we may assume that the pebbles in $U \cup W$ and $p_0$ are ordered as $u_r, \ldots, u_1, p_0, w_1, \ldots, w_t$ on $C_n$. We denote the set of pebbles incomparable to $p_0$ between $p_0$ and $w_t$ (between $u_r$ and $p_0$ resp.) by $X$ ($Y$ resp.).

A segment is a sequence of consecutive pebbles. An $U$-segment is a segment whose pebbles are all in $U$, and likewise, we have $W$-segments, $X$-segments and $Y$-segments. So we can group the pebbles between $u_r$ and $w_t$ along the positive direction as
\[
\text{win}(p_0) = (U_k, Y_k, U_{k-1}, \ldots, U_1, Y_1, p_0, X_1, W_1, \ldots, X_l, W_l),
\]
where $X_1, Y_1$ may be empty, and win$(p_0)$ is called the initial window of $p_0$. So in the window win$(p_0)$ of the pebble $p_0$, from the leftmost (the $U$-segment) to $p_0$, the segments are alternatively $U$- and $Y$-segments, and from the rightmost (the $W$-segment) to $p_0$, the segments are alternatively $W$- and $X$-segments. We shall use this notation without further notice.

We denote the set of all other pebbles as $Z$. So sometimes we write $\pi$ as
\[
\pi = (Z, U_k, Y_k, U_{k-1}, \ldots, U_1, Y_1, p_0, X_1, W_1, \ldots, X_l, W_l).
\]
By transitivity, we have $u_i \succ w_j$ since $u_i \succ p_0 \succ w_j$ for all $1 \leq i \leq r$ and $1 \leq j \leq t$, and in particular, $u_r \succ w_t$, hence $n \geq s(u_r) - s(w_t) > d^+(u_r, w_t)$. By Lemma 7, if $i \geq j$, then $u \succ y$ for all $u \in U_i$, $y \in Y_j$; If $k \geq l$, then $w \prec x$ for all $w \in W_k$, $x \in X_l$.

2.5 Two important lemmas

A nice property of the odd-even routing algorithm is the following

Lemma 11. Let $p$ be a pebble and $Q$ be a segment of pebbles. If $p \succ Q$ or $Q \succ p$, then once $p$ starts to swap with a pebble in $Q$ in an odd-even routing algorithm, $p$ will not stop swapping until $p$ swaps with all pebbles in $Q$ (in the following $|Q| - 1$ or more steps).

Proof. By symmetry, we let $p \succ Q$. An enlargement of $Q$ is a segment obtained from $Q$ by mixing some pebbles that are smaller than $p$. So $p \succ Q'$ if $Q'$ is an enlargement of $Q$.

We first claim that if $x, y \succ Q$, then after swapping with some pebbles in $Q$, $x$ and $y$ can never be on an edge unless they have swapped with all pebbles in $Q$. Suppose that
the first time such an edge occurs at step $s$. That is, after step $s - 1$, we have $q, x, y, q'$, where $x, y \succ \{q, q'\}$. Now, at step $s - 1$, $xq$ and $yq'$ are among the chosen edges. So, after step $s - 1$, we should have $q, x, q', y, a$ contradiction.

Note that for a pebble $q$ between $p$ and $Q$ along the positive direction, by Lemma 7, either $p \succ q$ or $q \succ Q$; in the former case, $q$ either becomes a pebble in an enlargement of $Q$ or $p, q$ swap before $p$ meets a pebble in $Q$. So we may assume that all pebbles from $p$ to $Q$ along the positive direction are bigger than $Q$. Now by the claim, $p$ never meets a pebble incomparable with $p$ before it finishes swapping with $Q$. So $p$ moves continuously.

**Lemma 12** (Rotation Lemma). Let $q$ be an integer with $-\frac{n}{2} < q \leq \frac{n}{2}$, and $\pi$ be the permutation that satisfies $\pi(a) = a + q \pmod{n}$ for each $a \in [n]$. Then, $rt(C_n, \pi) = n - \lfloor |q| \rfloor$.

**Proof.** By symmetry we only consider the case when $q > 0$. For each pebble $p$, the spin of $p$ is either $q$ or $q - n$. Since the sum of spins is zero, there must be exactly $n - q$ pebbles with the positive spin and $q$ pebbles with the negative spin. So, $n - q \leq rt(C_n, \pi)$.

Now, we show that $rt(C_n, \pi) \leq n - q$. Let the pebbles be $p_1, p_2, \ldots, p_n$ on $C_n$. We assign spins to the pebbles so that $s(p_{2i-1}) = q - n$ for $1 \leq i \leq q$ and each of the remaining $n - q$ pebbles has the spin $q$. We use an odd-even routing algorithm with odd edge $p_1p_2$. As $q \leq n/2$, no two pebbles with spin $q - n$ are adjacent. In the routing process, $p_1$ will be paired with $p_2, p_4, \ldots, p_{2q}, p_{2q+1}, \ldots, p_n$ in the first $n - q$ steps, thus reach its destination, and similar things occur for all other pebbles with negative spins. When all pebbles with negative spins reach their destinations, there is no comparable pebbles, so each pebble will be in place. Thus, $\pi$ is routed in $n - q$ steps. $\square$

### 3 Extremal Windows

Now we count the steps needed for an arbitrary pebble, say $p_0$, to swap with all comparable pebbles. Let the initial window of $p_0$ be

$$win(p_0) = (U_k, Y_k, U_{k-1}, \ldots, U_1, Y_1, p_0, X_1, W_1, \ldots, X_l, W_l).$$

As we only consider how $p_0$ swaps with other pebbles, we ignore the swaps between pebbles inside each of the segments, and regard them to be incomparable for now.

By Lemma 11, when we apply the odd-even routing algorithm, $p_0$ will meet a segment $S \in \{U_1, \ldots, U_k, W_1, \ldots, W_l\}$ and swap with all the pebbles in $S$ in the following $|S|$ steps. Assume that $p_0$ meets the segments in the order $S_1, S_2, \ldots, S_{k+l}$, where $S_i \in \{U_1, \ldots, U_k, W_1, \ldots, W_l\}$.

For $i = 1, 2, \ldots, k + l - 1$, let $\omega_i$ be the waiting time between $S_{i-1}$ and $S_i$, that is, the number of steps that $p_0$ waits between swapping with the last pebble of $S_{i-1}$ and swapping with the first pebble of $S_i$. Let $\alpha$ be the largest index such that $\omega_\alpha \neq 0$.

By symmetry, we assume that $S_\alpha = W_l$. Because of the parity, a swap of $p_0$ and $W$ (or $p_0$ and $U$) cannot be followed by a swap of $p_0$ and $U$ (or $p_0$ and $W$). Therefore, as $\omega_{\alpha+1} = \cdots = \omega_{k+l} = 0$, $\bigcup_{i \geq \alpha} S_i = \bigcup_{i=1}^{l} W_i$ and $p_0$ will swap with them continuously until it reaches its destination. Let $w$ be the pebble in $W_l$ next to $X_l$. Since we ignore the swaps
between pebbles in $W$, $w$ only moves in one direction (counter-clockwise). Then the steps for $p_0$ to be in place are the steps for $p_0$ and $w$ to meet plus $| \cup_{i=t}^k W_i |$.

To meet $p_0$, $w$ has to swap with pebbles in $\cup_{j=1}^t X_j$ and $\cup_{i=1}^k U_i$. We also note that, $w$ is always paired with a comparable pebble starting from the first or the second step, depending on the parity of the first edge with which $w$ is incident. Thus the total number of steps for $p_0$ to be in place is:

$$
\sum_{j=1}^t |X_j| + \sum_{j=t}^l |W_j| + \sum_{i=1}^k |U_i| + \delta,
$$

where $\delta = 0$ if $w \in W_t$ is paired with an $X$-pebble in the first step, and $\delta = 1$ otherwise.

By symmetry, if $S_\alpha = U_t$, then the number of steps for $p_0$ to be in place is

$$
\sum_{j=1}^t |Y_j| + \sum_{j=t}^k |U_j| + \sum_{i=1}^l |W_i| + \delta, \text{ where } \delta \in \{0,1\}.
$$

Therefore, every permutation that takes $n - 1$ steps to route must contain a pebble $p_0$ such that (when $S_\alpha = W_t$)

$$
\sum_{j=1}^k |Y_j| + \sum_{j=t+1}^l |X_j| + \sum_{j=1}^{t-1} |W_j| + |Z| = \delta, \text{ where } \delta \in \{0,1\}.
$$

or (when $S_\alpha = U_t$)

$$
\sum_{j=1}^l |X_j| + \sum_{j=t+1}^k |Y_j| + \sum_{j=1}^{t-1} |U_j| + |Z| = \delta, \text{ where } \delta \in \{0,1\}.
$$

Now we are ready to determine the extreme windows that need $n - 1$ steps to route.

**Lemma 13.** Every permutation that takes $n - 1$ steps to route must contain a pebble $p_0$ whose window is one of the following

(i) $|\text{win}(p_0)| = n$ and $\text{win}(p_0) = (p_0, X, W)$ (or $\text{win}(p_0) = (U, Y, p_0)$).

(ii) $|\text{win}(p_0)| = n - 1$ and $\text{win}(p_0) = (U, p_0, X, W)$ (or $\text{win}(p_0) = (U, Y, p_0, W)$).

(iii) $|\text{win}(p_0)| = n$, and $\text{win}(p_0) = (p_0, X_1, W_1, X_2, W_2)$ and $\min(|W_1|, |X_2|) = 1$ (or $\text{win}(p_0) = (U_2, Y_2, U_1, Y_1, p_0)$ and $\min(|Y_2|, |U_1|) = 1$).

In other words, if the window of a pebble is not one of the above ones, then in $n - 2$ steps, the pebble will be in place.
Proof. By symmetry, we may assume that (4) holds. As \( \delta = 0 \) or \( 1 \), all the terms in the left-hand side of (4) are zeros or ones.

We first claim that when \( |\text{win}(p_0)| = n \), \( U_k \) or \( W_l \) must be empty. For otherwise, let \( u_p \in U_k \) and \( w_q \in W_l \) be the furthest \( U \)-pebble and \( W \)-pebble to \( p_0 \), respectively. As \( |\text{win}(p_0)| = n \), no pebble is bigger than \( u_p \) and no pebble is smaller than \( w_q \) by Lemma 7, so \( s(u_p) \geq 1 + |Y| + |W| \) and \( s(w_q) \leq -(1 + |X| + |U|) \), and it follows that \( s(u_p) - s(w_q) \geq n + 1 \), a contradiction.

**Case 1.** \( \delta = 0 \) or \( \delta = |Z| = 1 \). Then \( \sum_{j=1}^{k} |Y_j| = \sum_{j=t+1}^{l} |X_j| = \sum_{j=1}^{l-1} |W_j| = 0 \). It follows that \( Y = \emptyset \), \( t = 1 = l \). So \( \text{win}(p_0) = (U, p_0, X, W) \). By the above claim, when \( |\text{win}(p_0)| = n \), \( U \) or \( W \) must be empty, so we have (i) or (ii) in the lemma, where \( X \) (or \( Y \)) could be empty.

**Case 2.** \( \delta = 1 \) and \( |Z| = 0 \). Then \( |\text{win}(p_0)| = n \), and one of the following holds:

- \( \sum_{j=1}^{k} |Y_j| = 1 \), and \( \sum_{j=t+1}^{l} |X_j| = \sum_{j=1}^{l-1} |W_j| = 0 \). Then \( |Y| = 1 \), \( t = l = 1 \). So there are at most two \( U \)-sets, \( U_1 \) and \( U_2 \), and when there are two, \( Y_1 = \emptyset \) and \( |Y_2| = 1 \). Because of the above claim, we have \( \text{win}(p_0) = (U_2, y, U_1, p_0) \).
- \( \sum_{j=t+1}^{l} |X_j| = 1 \) and \( \sum_{j=1}^{k} |Y_j| = \sum_{j=1}^{l-1} |W_j| = 0 \). Then \( Y = \emptyset \) and \( t = |X_2| = 1 \). Because of the above claim, \( \text{win}(p_0) = (p_0, X_1, W_1, x_2, W_2) \).
- \( \sum_{j=1}^{l-1} |W_j| = 1 \) and \( \sum_{j=1}^{k} |Y_j| = \sum_{j=t+1}^{l} |X_j| = 0 \). Then \( Y = \emptyset \), \( t = 2 \) and \( |W_1| = 1 \), and \( X_i = \emptyset \) for \( i \geq 3 \). Because of the above claim, \( \text{win}(p_0) = (p_0, X_1, w_1, X_2, W_2) \).

So we have the desired extremal windows in the lemma.

**4 Proof of Theorem 3**

In this section, we show how to deal with the extremal situations in Lemma 13.

For each of the extreme windows, we will decompose it into blocks of the following kinds. Let \( q_1, q_2, \ldots, q_s \) be a segment of pebbles. It is called a block with head \( q_1 \) if \( q_1 \succ q_i \) for \( i \geq 2 \) and the other pebbles are incomparable; it is called a block with tail \( q_s \) if \( q_i \succ q_s \) for \( i < s \) and the other pebbles are incomparable; it is called an isolated block if none of the pebbles is comparable.

We start with a minimized disbursement \( B \) of \( \pi \), the permutation that cannot be routed in \( n - 2 \) steps. By Lemma 13, \( \pi \) should contain a pebble with one of the extreme windows. We shall determine \( \pi \) explicitly and alter the disbursement and/or the odd-even routing algorithm to show that it can be routed in \( n - 2 \) steps.

**4.1 Extremal window type 1: \( \text{win}(p_0) = (p_0, X, W) \) and \( |\text{win}(p_0)| = n \)**

**Lemma 14.** If permutation \( \pi \) needs \( n - 1 \) steps to route and some pebble \( p_0 \) in \( \pi \) has \( \text{win}(p_0) = (p_0, X, W) \) and \( |\text{win}(p_0)| = n \), then \( \pi \) is \( 23 \ldots n1 \) or its inverse.
Proof. Let \(X = x_1 x_2 \ldots x_a\) and \(W = w_1 w_2 \ldots w_b\). Consider the spins of \(p_0\) and \(w_b\). Note that \(\text{spin}(p_0) = |W|\) and \(s(w_b) \leq -(1 + |X|)\) since no pebble is smaller than \(w_b\) (by Lemma 7) and \(\{p_0\} \cup X \succ w_b\). So \(s(p_0) - s(w_b) \geq n\) and it follows that \(s(w_b) = -1 - |X|\) and \(w_b\) is only comparable with \(X \cup \{p_0\}\). Now repeat the argument for \(w_{b-1}, \ldots, w_1\) successively, we have \(s(w_i) = -1 - |X|\) for \(1 \leq i \leq b\).

Consider the spin of \(x_1\). It is clear that \(s(x_1) \geq |W|\) since \(x_1 \succ W\) and no pebble is bigger than \(x_1\) (by Lemma 7). Then \(s(x_1) - s(w_b) \geq n\). It follows that \(s(x_1) = |W|\) and \(x_1 \succ W\) is the only order relation involving \(x_1\). Inductively we have \(s(x_i) = |W|\) for all \(x_i \in X\) and \(\{p_0\} \cup X \succ W\) is the only order relation in the permutation.

So along the positive direction every pebble is \(|W|\) steps away from its destination. So \(\pi\) is a rotation. By Lemma 12, \(\pi\) must be \(23 \cdots n1\) or its inverse. \(\square\)

4.2 Extremal window type 2: \(\text{win}(p_0) = (U, p_0, X, W)\) and \(|\text{win}(p_0)| = n - 1\)

Lemma 15. If a permutation \(\pi\) contains a pebble \(p_0\) such that \(\text{win}(p_0) = (U, p_0, X, W)\) and \(\pi = (\{z\}, U, p_0, X, W)\), where \(U, W \neq \emptyset\), then \(U\) and \(W\) are isolated blocks, and \(X\) can be partitioned into \(X_1, \ldots, X_r\) such that \(X_i\) is either an isolated block or a block with tail \(x_i\). Furthermore, \(s(z) = c \leq 0\), and if \(c < 0\), then \(X_r\) is an isolated block with \(-c\) pebbles, and the order relations are

\[
U \cup \{p_0\} \cup X \succ W, \quad X_r \succ \{z\}, \quad X_i - x_i \succ x_i \text{ for some } 1 \leq i \leq r.
\]

Proof. Let \(U = u_1 u_2 \ldots u_p, X = x_1 x_2 \ldots x_a\) and \(W = w_1 w_2 \ldots w_b\). Consider the spins of \(u_1\) and \(w_b\). As no pebble is bigger than \(u_1\) and \(u_1 \succ \{p_0\} \cup W\), \(s(u_1) \geq 1 + |W| = 1 + b\). Similarly, no pebble is smaller than \(w_b\) and \(U \cup \{p_0\} \cup X \succ w_b\), so \(s(w_b) \leq -(1 + |U| + |X|) = -(1 + a + p)\). So \(s(u_1) - s(w_b) \geq 1 + b + 1 + a + p = n\), and the equality must hold. So \(s(u_1) = 1 + b, s(w_b) = -(1 + a + p)\) and the only order relation involving \(u_1\) and \(w_b\) are \(u_1 \succ \{p_0\} \cup W\) and \(U \cup \{p_0\} \cup X \succ w_b\). Inductively we can consider \(u_2\) and \(w_{b-1}\) and all pebbles in \(U\) and \(W\) and conclude that \(U \cup \{p_0\} \cup X \succ W\) is the only order relation involving \(U\) and \(W\).

Now consider the spins of pebbles in \(X\). As \(s(w_b) = -(1 + a + k)\) and \(s(x) - s(w_b) \leq n\) for each \(x \in X\), we have \(s(x) \leq b + 1\). Note that \(z\) cannot be bigger than any pebble in \(X\), for otherwise \(z \succ W\) and contradict to what we just concluded. But \(z\) may be smaller than some pebbles in \(X\), thus \(s(z) \leq 0\).

Consider \(x_1\). By Lemma 7, no pebble is bigger than \(x_1\). As \(x_1 \succ W\), \(s(x_1) \geq |W| = b\). So \(s(x_1) \in \{b, b + 1\}\). Let \(s(x_1) = b + 1\). Then the order relations involving \(x_1\) are \(x_1 \succ W \cup \{x_1\}\) for some \(2 \leq i \leq a\) or \(x_1 \succ W \cup \{z\}\); if \(x_1 \succ z\), then \(x_j \succ z\) for \(1 \leq j \leq a\) by Lemma 7 and we inductively conclude \(s(x_j) = b + 1\), thus \(X\) is an isolated block and \(X \succ z\); if \(x_1 \succ x_i\) for some \(2 \leq i \leq a\), then \(x_j \succ x_i\) for \(1 \leq j < i\) by Lemma 7, and no other pebble in \(X\) is smaller than \(x_i\), for otherwise it would be smaller than \(x_1\) which contradicts what we just concluded. So \(x_1 x_2 \ldots x_a\) is a block with tail \(x_1\). Now we similarly consider \(x_{i+1}\) and get a block partition of \(X\). Now let \(s(x_1) = b\). Then \(x_1 \succ W\) is the only order relation involving \(x_1\), and we will inductively consider \(x_2\) and get a block partition of \(X\). \(\square\)
Now we are ready to show that such permutations can be routed in \( n - 2 \) steps.

**Lemma 16.** If a permutation \( \pi \) contain a pebble \( p_0 \) such that \( \text{win}(p_0) = (U, p_0, X, W) \) and \( \pi = (\{z\}, U, p_0, X, W) \), where \( U, X, W \neq \emptyset \), then \( \pi \) can be routed in at most \( n - 2 \) steps.

**Proof.** First we assume that \( X \neq \emptyset \). Let \( \pi = z u_1 \ldots u_k p_0 x_1 \ldots x_a w_1 \ldots w_b \), with \( u_i \in U, x_i \in X \) and \( w_i \in W \). We use an odd-even routing algorithm so that \( x_a w_1 \) is an odd edge. The order relations are shown in Lemma 15.

By Lemma 11, \( x_a \) swaps with \( w_1 \) in the first step, thus swaps with \( W \) in the following \( |W| - 1 \) steps, so \( w_b \) meets (i.e., is paired with a pebble in) \( U \cup \{p_0\} \cup X \) after \( |W| - 1 \) steps, then \( w_b \) would swap with \( U \cup \{p_0\} \cup X \) in the following \( |U \cup \{p_0\} \cup X| \) steps, so it takes \(|W| - 1 + |U \cup \{p_0\} \cup X| = n - 2 \) steps for \( w_b \) to be in place. As a pebble in \( U \cup \{p_0\} \cup X \) has to pass \( W - w_b \) to meet \( w_b \), all pebbles in \( W \) would be in place after \( n - 2 \) steps.

For \( z \), its window \( \text{win}(z) = (X, r, W, z) \), so \( |\text{win}(z)| = n - |U| - 1 - |X - X_r| \leq n - 2 \), thus, \( z \) will be in place after at most \( n - 2 \) steps, by Lemma 13. For a \( x_j \in X \) that is in a block with tail \( x_i \), its window \( \text{win}(x_j) = (x_j, \{x_{j+1}, \ldots, x_{i-1}\}, x_i, \{x_{i+1}, \ldots, x_a\}, W) \), so \( |\text{win}(x_j)| \leq n - 2 - |U| \leq n - 3 \), thus \( x_j \) will be in place after at most \( n - 2 \) steps, by Lemma 13.

So after \( n - 2 \) steps, there are no comparable pebbles, as each order relation involves a pebble in \( W \cup \{z\} \) or a pebble in a block with a head in \( X \). By Lemma 8, \( \pi \) is routed in at most \( n - 2 \) steps.

**Lemma 17.** If a permutation \( \pi \) contain a pebble \( p_0 \) such that \( \text{win}(p_0) = (U, p_0, W) \) and \( \pi = (\{z\}, U, p_0, W) \), where \( U, W \neq \emptyset \), then \( \pi \) can be routed in at most \( n - 2 \) steps.

**Proof.** Let \( \pi = z u_1 \ldots u_k p_0 w_1 \ldots w_b \). By Lemma 15, \( s(u) = 1 + b \) and \( s(w) = -1 - k \) for \( u \in U, w \in W \) and the only order relation is \( U \cup \{p_0\} \succ W \).

First we consider the case when \( k = 1 \) or \( b = 1 \). By symmetry, let \( k = 1 \). As \( n \geq 6, b \geq 4 \). Flip the spins of \( u \) and \( w_2 \). By Lemma 8, the order relations under the new disbursement are \( p_0 \succ W - w_2, w_2 \succ (W - w_2) \cup \{z, u\} \). We use an odd-even routing algorithm with odd edge \( p_0 w_1 \). Then \( w_2 \) is paired with a smaller pebble in each step, thus will be in place after \( n - 2 \) steps; similarly, \( p_0 \) is paired with a smaller pebble in each of the first \( n - 3 \) steps, thus will be in place after \( n - 3 \) steps. Therefore, after \( n - 2 \) steps, there are no comparable pebbles since each order relation involves \( p_0 \) or \( w_2 \). By Lemma 8, \( \pi \) is routed in \( n - 2 \) steps.

Now, let \( k, b \geq 2 \). We first flip the spins of \( u_1 \) and \( w_b \). By Lemma 10, the order relations under the new disbursement are

\[
U - u_1 \succ \{p_0\} \cup (W - w_b), \quad w_b \succ (W - w_b) \cup \{z, u_1\}, \quad (U - u_1) \cup \{z, w_b\} \succ u_1.
\]

The window for \( p_0 \) is \( \text{win}(p_0) = (U - u_1, p_0, W - w_b) \) and the window for \( z \) is \( \text{win}(z) = (w_b, z, u_1) \), so \( |\text{win}(p_0)| = n - 3 \) and \( |\text{win}(z)| = 3 \leq n - 3 \) (as \( n \geq 6 \)), so by Lemma 13, \( p_0 \) and \( z \) will be in place after \( n - 2 \) steps. The window for \( u_1 \) is \( \text{win}(u_1) = (U - u_1, W - w_b) \). By
$w_b + p_0, \{w_b, z\}, u_1$. As $b \geq 2$, it is not one of the extreme windows in Lemma 13, so $u_1$ will be in place in at most $n - 2$ steps.

Now we show that all pebbles in $W - w_b$ are in place after $n - 2$ steps, which by Lemma 8 implies that all pebbles are in place since each order relation involves a pebble in $(W - w_b) \cup \{p_0, z, u_1\}$. Note that for $1 \leq i \leq b - 1$, $\text{win}(w_i) = (w_b, \{z, u_1\}, U - u_i + p_0, \{w_1, \ldots, w_{i-1}\}, w_i)$. Since $k \geq 2$, $\text{win}(w_i)$ is not one of the extreme windows in Lemma 13; thus, $w_i$ will be in place after $n - 2$ steps.

4.3 Extremal window type 2a: $\text{win}(p_0) = (p_0, X, W)$ and $|\text{win}(p_0)| = n - 1$

This is the case of $\text{win}(p_0) = (U, p_0, X, W)$ with $U = \emptyset$. In this case, $W$ is not necessarily an isolated block. The following lemma tells the possible structures in $\pi$.

**Lemma 18.** If a permutation $\pi$ has a pebble $p_0$ with $\text{win}(p_0) = (p_0, X, W)$ and $\pi = \{\{z\}, p_0, X, W\}$, then one of the following must be true (note that $X$ could be empty)

1. if $s(z) = c > 0$, then $X$ is an isolated block and $W$ can be partitioned into isolated blocks and blocks with heads, say $W_1, W_2, \ldots, W_r$, so that $W_1$ is isolated with $c$ pebbles and $z \succ W_1$, and the only other order relation is $\{p_0\} \cup X \succ W$.

2. if $s(z) = c < 0$, then $W$ is isolated and $X$ can be partitioned into isolated blocks and blocks with tails, say $X_1, \ldots, X_r$, so that $X_r$ is isolated with $|c|$ pebbles and $X_r \succ z$, and the only other order relation is $\{p_0\} \cup X \succ W$.

3. if $s(z) = 0$, then either $X$ can be partitioned into isolated blocks and blocks with tails and $W$ is an isolated block, or $W$ can be partitioned into isolated blocks and blocks with heads and $X$ is an isolated block, and the only other order relation is $\{p_0\} \cup X \succ W$.

The proof of this lemma is very similar to Lemma 15, and for completeness, we include a proof below.

**Proof.** Let $\pi = z \sigma_0 x_1 x_2 \ldots x_n w_1 w_2 \ldots w_b$, where $x_i \in X$ and $w_i \in W$. Clearly, $s(p_0) = |W| = b$. By Lemma 7, $X \succ W$. As $z$ is incomparable with $p_0$, no pebble in $W$ is bigger than $z$.

Consider $w_b$. Since $\{p_0\} \cup X \succ w_b$, and no pebble is smaller than $w_b$ by Lemma 7, $s(w_b) \leq -(|X| + 1) = -(a + 1)$. On the other hand, $s(p_0) - s(w_b) \leq n$, for otherwise the flip of spins of $p_0$ and $w_b$ gives a smaller disbursement, so $s(w_b) \geq -(a + 2)$. Therefore $s(w_b) \in \{-(a + 1), -(a + 2)\}$. Clearly, if $s(w_b) = -(a + 1)$, then $\{p_0\} \cup X \succ w_b$ is the only order relation involving $w_b$, thus $w_b$ is in an isolated block. We shall move to consider $w_{b-1}$.

Now let $s(w_b) = -(a + 2)$. Then exactly one pebble in $\{z\} \cup (W - w_b)$ is bigger than $w_b$. If $z \succ w_b$, then by Lemma 7, $z \succ w_j$ for each $w_j \in W - w_b$; thus, inductively we conclude that the only order relation involving $W$ is $\{z, p_0\} \cup X \succ W$, and $W$ is an isolated block.

If $w_i \succ w_b$ for some $i < b$, then for $i < j < b$, $w_j$ is not comparable with $w_b$ and $w_i \succ w_j$ by Lemma 7. Note that no pebble $w_l$ with $l < i$ could be bigger than $w_i$, as
otherwise it would be bigger than \( w_b \) which is impossible. Now inductively we conclude that \( w_{b-1}, \ldots, w_{i+1} \) all have spin \(-a+2\) and are only smaller than \( \{p_0, w_i\} \cup X \). That is, \( \{w_i, w_{i+1}, \ldots, w_k\} \) is a block with head \( w_i \).

We now repeat the above argument for \( w_{k-1} \) (if \( s(w_b) = -(a+1) \)) or \( w_{k-1} \) (if \( w_b \) is in a block with head \( w_i \), and eventually \( W \) can be partitioned into isolated blocks and/or blocks with heads. In particular, if \( z \succ w_r \), then \( w_r \) must be in an isolated block, and by Lemma 7, \( z \succ \{w_1, w_2, \ldots, w_r\} \), that is, \( w_1, w_2, \ldots, w_r \) are in an isolated block.

Note that \( z \) is not bigger than a pebble in \( X \). For otherwise, let \( z \succ x \) for some \( x \in X \). Then \( z \succ W \). Thus, \( s(z) \geq 1 + b \) and \( s(w_b) = -(a + 2) \), which implies that \( s(z) - s(w_b) \geq n + 1 \). Now a flip of the spins of \( z \) and \( w_b \) gives a smaller disbursement, a contradiction.

We claim that if \( s(w) = -(a + 2) \) for some \( w \in W \), then the only order relation involving \( X \) is \( X \succ W \) (thus \( X \) is an isolated block). Consider \( x_1 \). Since no pebble is bigger than \( x_1 \), thus \( s(x_1) \geq |W| = b \). Since \( s(x_1) - s(w) \leq n, s(x_1) \leq b \). So \( s(x_1) = b \) and the only order relation involving \( x_1 \) is \( x_1 \succ W \). Now we consider \( x_2, x_3, \ldots, x_a \) successively and similarly to get the conclusion. This means also that if \( s(z) = c \geq 0 \), then the isolated block \( w_1 w_2 \ldots w_i \) has \( c \) pebbles.

Similar to the analysis of the structure in \( W \), when the order relation involving \( W \) is \( \{p_0\} \cup X \succ W \), we get the partition and structure in \( X \).

\( \square \)

**Lemma 19.** If a permutation \( \pi \) contains a pebble \( p_0 \) with \( \text{win}(p_0) = (p_0, X, W) \) and \( \pi = (\{x\}, p_0, X, W) \), then \( \pi \) can be routed in at most \( n - 2 \) steps or is \( 2 \ldots n1 \) or its inverse.

**Proof.** We only consider the case \( s(z) = c \geq 0 \). By Lemma 18, we assume that \( X = x_1 x_2 \ldots x_a \) is an isolated block, and \( W = w_1 \ldots w_b \) has the block decomposition \( W_1, \ldots, W_k \) so that \( W_0 \) is an isolated block with \( c \) pebbles and \( z \succ W_1 \), and for \( i > 0 \), \( W_i \) is either an isolated block or a block with a head (say \( w_i \)).

If \( c = b \), then the spins of \( \{z, p_0\} \cup X \) are all \( b \) and the spins of \( X \) are all \(-a+2\), and \( \pi \) is a rotation. By Lemma 12, if \( \pi \) needs \( n - 1 \) steps to route, \( \pi \) must be one of the two extremal permutations. So we assume \( c < b \).

When \( c = s(z) = 0 \), we may assume that \( W \) is not a block with head \( w_1 \) (that is, \( w_1 \succ W - w_1 \)). Suppose otherwise. We flip the spins of \( p_0 \) and \( w_b \), by Lemma 10, \( w_b \succ \{z, p_0\} \cup (W - w_b) \), and \( X \cup \{z, w_b\} \succ p_0 \). Now we use the odd-even sorting algorithm so that \( w_b z \) is an odd edge. By Lemma 11, \( w_b \) will be in place in \( n - 2 \) steps, \( w_1 \) swaps from the second step and takes \( n - 4 \) steps to be in place, and \( p_0, z \) will be in place in 3 steps. So in at most \( n - 2 \) steps all pebbles will be in place.

Now consider the rest of the cases. We use the odd-even routing algorithm so that \( x_a w_1 (p_0 w_1 \text{ if } X = \emptyset) \) is an odd edge. By Lemma 11, \( x_i \) with \( 1 \leq i \leq a \) meets \( W \) after \( a - i \) steps and swaps with \( W \) in the following \(|W|\) steps, so it will be in place after \( a - i + |W| = a + b - i = n - 2 - i \leq n - 2 \) steps; \( p_0 \) could be regarded as \( x_0 \), so takes at most \( n - 2 \) steps; \( z \) meets \( W_1 \) after \( a + 1 \) steps and takes \( c \) swaps, so will be in place in at most \( a + 1 + c < a + b + 1 \leq n - 2 \) steps. The head \( w \) in block \( W_1 \) swaps with \( W_i \) at the first step, or the second step (if it is not adjacent to \( x \) and is not incident with an
odd edge in the first step), or after \(a + 2\) steps (if \(w = w_1\)), and in the former two cases it will be in place after at most \(1 + (|W_1| - 1) + a + 1 = a + |W_1| + 1 \leq a + b = n - 2\) steps. Now consider the last case. If \(z > w_1\), then \(w_1\) is in an isolated block in \(W\), thus will be in place after \(a + 2 = n - b \leq n - 2\) steps (as \(b > c \geq 1\)). If \(w_1\) is not smaller than \(z\), then \(s(z) = 0\), and \(w_1\) will be in place after at most \((a + 1) + |W_1| - 1 < a + 1 + |W| = n - 1\) steps. So after \(n - 2\) steps, the above pebbles are in place, thus, all pebbles will be in place by Lemma 8, since each order relation involves one of the pebbles. 

4.4 Extremal window type 3: \(\text{win}(p_0) = \pi = (p_0, X_1, W_1, x, W_2)\) or \(\text{win}(p_0) = \pi = (p_0, X_1, w, X_2, W_2)\)

**Lemma 20.** If a permutation \(\pi\) contains a pebble \(p_0\) with \(\text{win}(p_0) = \pi = (p_0, X_1, W_1, x, W_2)\) or \(\text{win}(p_0) = \pi = (p_0, X_1, w, X_2, W_2)\), then \(X_1\) and \(W_2\) are isolated blocks, and

- if \(\text{win}(p_0) = (p_0, X_1, W_1, x, W_2)\), then \(W_1\) can be partitioned into isolated blocks and blocks with heads. Furthermore, \(c := s(x) - |W_2| \geq 0\), and if \(c > 0\), then the block \(W_0\) in \(W_1\) next to \(X_1\) is an isolated block with \(c\) elements and are all smaller than \(x\); and the only other order relation between segments are \(\{p_0\} \cup X_1 \succ W_1 \cup W_2, x \succ W_2 \cup W_0\).

- if \(\text{win}(p_0) = (p_0, X_1, w, X_2, W_2)\), then \(X_2\) can be partitioned into isolated blocks and blocks with tails. Furthermore, \(c := s(w) + (1 + |X_1|) \leq 0\), and if \(c < 0\), then the block \(X_0\) in \(X_2\) next to \(W_2\) is an isolated block with \(-c\) elements and are all bigger than \(w\); and the order relation between segments are \(\{p_0\} \cup X_1 \succ \{w\} \cup W_2\) and \(X_0 \succ w\).

**Proof.** We only prove the case when \(\text{win}(p_0) = (p_0, X_1, W_1, x, W_2)\), and the other case is very similar. Let \(W_1 = w_1w_2 \ldots w_a\) and \(W_2 = w_{a+1}w_2 \ldots w_b\). Consider the spins of \(p_0\) and \(w_b\). Since \(s(p_0) = b\) and by Lemma 7, no pebble is smaller than \(w_b\), thus \(s(w_b) \leq -(2 + |X_1|) = -(n - b)\); furthermore, as \(s(p_0) - s(w_b) \leq n\), we have \(s(w_b) \geq -(n - b)\), so \(s(w_b) = -(n - b)\), and the order relation involving \(w_b\) is \(\{p_0\} \cup X \succ w_b\). Now we can repeat the argument for \(w_{b-1}, w_{b-2}, \ldots, w_{a+1}\) successively and get \(s(w_1) = -(n - b)\) for all \(w_i \in W_2\). Similarly, by comparing the spins of pebbles in \(X_1\) to that of \(w_b\), we have \(s(x_i) = b\) for all \(x_i \in X_1\). So \(X_1\) and \(W_2\) are isolated blocks. Since \(s(p_0) = s(x)\) for each \(x \in X_1\), let \(X'_1 = X_1 \cup \{p_0\}\).

Now we consider the spins of pebbles in \(W_1\).

We note that no pebble in \(W_1\) is bigger than \(x\), for otherwise \(p_0\) would be bigger than \(x\) as \(p_0 \succ W_1\). Consider \(w_a\). The pebbles in \(X'_1\) are bigger than \(w_a\), and by Lemma 7, no pebble is smaller than \(w_a\), so \(s(w_a) \leq -|X'_1| = b - n + 1\). On the other hand, \(s(p_0) - s(w_a) \leq n\) and \(s(p_0) = b\) implies \(s(w_a) \geq b - n\). That is, \(s(w_a) \in \{b - n, b - n + 1\}\), and at most one pebble other than those in \(X'_1\) is bigger than \(w_a\).

If \(s(w_a) = b - n + 1\), then \(\{p_0\} \cup X_1 \succ w_a\) is the only order relation involving \(w_a\), and we turn to consider \(w_{a-1}\). If \(s(w_a) = b - n\), then \(w_a\) is smaller than \(x\) or some pebble \(w_i \in W_1\); in the former case, all pebbles in \(W_1\) are smaller than \(x\) and inductively one can show that they are incomparable and thus \(W_1\) is an isolated block; in the latter case,
Lemma 7. Let \( w_i \succ w_j \) for \( i + 1 \leq j \leq a \) and \( w_i \) is the only such pebble other than those in \( X_1' \), so \( w_i w_{i+1} \ldots w_a \) is a block with head \( w_i \). Inductively one can have a partition of \( W_1 \) into blocks, as desired.

We observe that if \( x \succ w_i \in W_1 \) then \( w_i \) is in an isolated block and \( x \succ \{w_1, w_2, \ldots, w_i\} \) by Lemma 7. Let \( c := s(x) - |W_2| \). Then \( W_0 := (w_1, \ldots, w_c) \) is an isolated block, \( x \succ W_2 \cup W_0 \) is the only order relation involving \( x \), as desired.

Lemma 21. If a permutation \( \pi \) contains a pebble \( p_0 \) such that \( \text{win}(p_0) = \pi = (p_0, X_1, W_1, x, W_2) \) or \( \text{win}(p_0) = \pi = (p_0, X_1, w, X_2, W_2) \), then \( \pi \) can be routed in \( n - 2 \) steps.

Proof. Again we only consider the case \( \text{win}(p_0) = (p_0, X_1, W_1, x, W_2) \), as the other one is very similar. Let \( \text{win}(p_0) = (p_0x_1x_2 \ldots x_k w_1 \ldots w_a x w_{a+1} \ldots w_b) \) with \( x_i \in X_1 \) and \( w_i \in W_1 \cup W_2 \). By Lemma 20, \( X_1, W_2 \) are isolated blocks, and \( W_1 = \cup_{i=0}^r W_i \), where \( x \succ W_1^0 \), and \( W_i^0 \) for \( i > 0 \) is an isolated block or a block with head \( w_i^1 \). We flip the spins of \( p_0 \) and \( w_b \), and then use an odd-even routing algorithm with odd edge \( x_k w_1 \) to route the permutation.

Now by Lemma 10 and Lemma 20, the order relations under the new disbursement are

\[
X_1 \cup \{w_b\} \succ W_1 \cup (W_2 - w_b) \cup \{p_0\}, \quad x \succ (W_2 - w_b) \cup W_1^0 \cup \{p_0\}, \quad w_i \succ W_i^1 - w_i^1 \text{ for some } i.
\]

We list the windows for the pebbles in \( X_1 \cup \{w_b, x, w_i : 1 \leq i \leq r\} \):

\[
\begin{align*}
\text{win}(x_i) &= (x_i, X_1 - \{x_j : 1 \leq j \leq i\}, W_1, x, W_2 - w_b, p_0), \\
\text{win}(w_i) &= (w_b, p_0, X_1, \cup_{j=0}^{i-1} W_j, w_i, W_1^0 - w_i^1), \\
\text{win}(x) &= (x, W_2 - w_b, w_b, p_0, X_1, W_1^0), \\
\text{win}(w_b) &= (w_b, p_0, X_1, W_1, x, W_2 - w_b).
\end{align*}
\]

Then \( |\text{win}(x_i)| = n - (i - 1) \), \( |\text{win}(w_i)| \leq n - 1 - |W_2| \leq n - 2 \), \( |\text{win}(x)| = n - |W_1 - W_1^0| \) and \( |\text{win}(w_b)| = n \), and none of the windows is among the extreme windows in Lemma 13. Therefore, in at most \( n - 2 \) steps, the pebbles in \( X_1 \cup \{w_b, x, w_i : 1 \leq i \leq r\} \) will be in place. However, each order relation on \( \pi \) involves one of the pebbles, so by Lemma 8, after \( n - 2 \) steps, there are no comparable pebbles, that is, \( \pi \) is routed.

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