Pathwise Non-uniqueness for the SPDE’s of Some Super-Brownian Motions with Immigration

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Abstract: We prove pathwise non-uniqueness in the stochastic partial differential equations (SPDE’s) of some one-dimensional super-Brownian motions with immigration and zero initial value. In contrast to a closely related case studied in a recent work by Mueller, Mytnik, and Perkins \[14\], the solutions of the present SPDE’s are assumed to be nonnegative and are unique in law. In proving possible separation of solutions, we introduce a novel method, called continuous decomposition, to validate natural immigrant-wise semimartingale calculations for the approximating solutions, which may be of independent interest in the study of superprocesses with immigration.

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1. Introduction

In this work, we construct a pair of distinct nonnegative solutions for any of the stochastic partial differential equations (SPDE’s) associated with some one-dimensional super-Brownian motions with (continuous) immigration and zero initial value. Hence, we resolve the long-standing open problem concerning the pathwise uniqueness of SPDE’s for one-dimensional super-Brownian motions in the negative, when additional immigration is present.

We start with some informal descriptions of the class of super-Brownian motions with immigration considered throughout this work. See \[12\] and \[18\] for super-Brownian motions as well as their connections with branching processes. Imagine that, in the barren territory $\mathbb{R}$, clouds of independent immigrants with infinitesimal initial mass land randomly in space throughout time. The underlying immigration mechanism is time-homogeneous and gives a high intensity of arrivals, so the inter-landing times of the immigrants are infinitesimal. After landing, each of the immigrating processes evolves independently of each other as a super-Brownian motion, obeying an SPDE of the form

$$\frac{\partial X}{\partial t}(x, t) = \frac{\Delta X}{2}(x, t) + X(x, t)^{1/2} \dot{W}(x, t), \quad X \geq 0, \quad (1.1)$$

subject to infinitesimal initial mass, where $W$ is (two-parameter) space-time white noise on $\mathbb{R} \times \mathbb{R}_+$. (See \[10\], \[19\], and Section III.4 of \[18\]. Interpretations of the components of the SPDE (1.1) can be found in Section 1.2 and Section 3.2 of \[4\].) Superposing their masses determines a super-Brownian motion with immigration and zero initial value.

We study in particular the super-Brownian motions with immigration whose density processes

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are characterized by the SPDE’s:
\[
\frac{\partial X}{\partial t}(x, t) = \frac{\Delta X}{2}(x, t) + \psi(x) + X(x, t)^{1/2}\dot{W}(x, t), \quad X \geq 0,
\]
(1.2)
\[
X(x, 0) = 0.
\]
Here, \(\mathcal{C}_c^+(\mathbb{R})\) being the function space of nonnegative continuous functions on \(\mathbb{R}\) with compact support, we can show that the corresponding super-Brownian motion with immigration takes values in the space of continuous functions with compact support (cf. Section III.4 of [22]).

Equipped with the complete separable metric
\[
\|\varphi\|_{\mathcal{C}_c^+} \triangleq \sum_{i=1}^{\infty} \int_{\mathbb{R}} |\varphi(x)|^2 e^{\lambda |x|} dx < \infty, \quad \lambda \in (0, \infty).
\]
(1.4)

\(\mathcal{C}_c^+(\mathbb{R})\) denote the function space of rapidly decreasing functions \(f\):
\[
|f|_\lambda \triangleq \sup_{x \in \mathbb{R}} |f(x)| e^{\lambda |x|} < \infty \quad \forall \lambda \in (0, \infty).
\]
(1.5)

For convenience, we follow the convention in [22] and choose the larger space \(\mathcal{C}_c^+(\mathbb{R})\) for the state space of the process \((X_t)\). Then by saying that \(X = (X_t)\) is a solution to an SPDE of the form (1.2), we require that \(X\) be a nonnegative \((\mathcal{G}_t)\)-adapted continuous process with state space \(\mathcal{C}_c^+(\mathbb{R})\) and satisfy the following weak formulation by regarding \(W\) and \(\dot{W}\) as \((\mathcal{G}_t)\)-space-time white noise, which has the following formal definition by regarding \(W\) as a linear operator from \(L^2(\mathbb{R})\) into a linear space of \((\mathcal{G}_t)\)-Brownian motions: for any \(d \in \mathbb{N}\), \(\phi_1, \ldots, \phi_d \in L^2(\mathbb{R})\), and \(a_1, \ldots, a_d \in \mathbb{R}\),
\[
W \left( \sum_{j=1}^{d} a_j \phi_j \right) = \sum_{j=1}^{d} a_j W(\phi_j) \quad \text{a.s.}
\]

\(W(\phi_1), \ldots, W(\phi_d)\) is a \(d\)-dimensional \((\mathcal{G}_t)\)-Brownian motion with covariance matrix \([\langle \phi_i, \phi_j \rangle_{L^2(\mathbb{R})}]_{1 \leq i,j \leq d}\).

Since a generic immigration function under consideration has compact support, it can be shown that the corresponding super-Brownian motion with immigration satisfies the following formal definition by regarding \(W\) as a linear operator from \(L^2(\mathbb{R})\) into a linear space of \((\mathcal{G}_t)\)-Brownian motions: for any \(d \in \mathbb{N}\), \(\phi_1, \ldots, \phi_d \in L^2(\mathbb{R})\), and \(a_1, \ldots, a_d \in \mathbb{R}\),
\[
W \left( \sum_{j=1}^{d} a_j \phi_j \right) = \sum_{j=1}^{d} a_j W(\phi_j) \quad \text{a.s.}
\]

and \(W(\phi_1), \ldots, W(\phi_d)\) is a \(d\)-dimensional \((\mathcal{G}_t)\)-Brownian motion with covariance matrix \([\langle \phi_i, \phi_j \rangle_{L^2(\mathbb{R})}]_{1 \leq i,j \leq d}\).}

The fundamental question for the SPDE (1.2) of a super-Brownian motion with immigration concerns its uniqueness theory. This calls for nontrivial investigations, since the SPDE has a non-Lipschitz diffusion coefficient. Uniqueness in law for the SPDE (1.2) holds and can be proved by the
duality method (cf. Section 1.6 of [5]) via Laplace transforms. In fact, it holds even if we impose general nonnegative initial conditions for the SPDE (1.1) of super-Brownian motions and the SPDE’s (1.2) under consideration. Nonetheless, duality methods for more general SPDE’s of the form
\[
\frac{\partial X}{\partial t}(x,t) = \frac{\Delta X}{2}(x,t) + b(X(x,t)) + \sigma(X(x,t))\dot{W}(x,t)
\]  
(1.8)
up to now seem only available when \( b \) and \( \sigma \) are of rather special forms, and hence are non-robust. (See [15] for the duality method for the case \( b = 0 \) and \( \sigma(x) = x^p \), where \( p \in (1/2, 1) \) and nonnegative solutions are assumed.) After all, duality is based on exactness and thus can be destroyed by even slight changes of coefficients in the context of SPDE’s.

Under the classical theory of stochastic differential equations (SDE’s), uniqueness in law in an SDE is a consequence of pathwise uniqueness of its solutions (cf. Theorem IX.1.7 of [20]). The strength of the classical method for pathwise uniqueness of solutions is that it has emphasis only on the range of the Hölder exponents of coefficients, instead of on the particular forms of the coefficients. It is then natural to consider circumventing the duality method by proving pathwise uniqueness in (1.8). Here, pathwise uniqueness in an SPDE ensures that any two solutions subject to the same space-time white noise and initial value always coincide almost surely. Our objective in the present work is to settle the question of pathwise uniqueness in the particular SPDE’s (1.2).

Let us discuss some results on pathwise uniqueness of various SDE’s and SPDE’s which are closely related to the SPDE’s (1.2). We focus on the role of non-Lipschitz diffusion coefficients in determining pathwise uniqueness.

For one-dimensional SDE’s with Hölder-\( p \)-diffusion coefficients, the famous Girsanov example (see Section V.26 of [21]) showed the necessity of the condition \( p \geq \frac{1}{2} \) for pathwise uniqueness of solutions. The sufficiency was later confirmed in the seminal work [24] by Yamada and Watanabe as far as the cases with sufficiently regular drift coefficients are concerned. In fact, the work [24] showed that a finite-dimensional SDE defined by
\[
dX(t) = b_i(X_t)dt + \sigma_i(X_t)dB_t, \quad 1 \leq i \leq d,
\]  
(1.9)

enjoys pathwise uniqueness as long as all \( b_i \)’s are Lipschitz continuous and each \( \sigma_i \) is Hölder \( p \)-continuous, for any \( p \geq \frac{1}{2} \).

In view of these complete results for SDE’s and the strong parallels between (1.8) and (1.9), it had been hoped for decades that pathwise uniqueness would also hold in (1.8) if the diffusion coefficient \( \sigma \) is Hölder-\( p \)-continuous whenever \( p \geq \frac{1}{2} \). It was shown in [16] that this is the case if \( \sigma \) is Hölder-\( p \) for \( p > \frac{3}{4} \), but in [2] and [14] that pathwise uniqueness in
\[
\frac{\partial X}{\partial t}(x,t) = \frac{\Delta X}{2}(x,t) + \|X(x,t)\|^p\dot{W}(x,t),
\]
(1.10)
fails for \( p < \frac{3}{4} \). Here, a non-zero solution to (1.10) exists and, as 0 is obviously another solution, both pathwise uniqueness and uniqueness in law fail. All these results point to the general conclusion that pathwise uniqueness of solutions holds for Hölder-\( p \) diffusion coefficients \( \sigma \) for \( p > \frac{3}{4} \) but can fail for \( p \in (0, \frac{3}{4}) \). (See also [17] for the case of coloured noises)

In this work, we confirm pathwise non-uniqueness of solutions to the SPDE’s (1.2). We stress that by definition, only nonnegative solutions are considered in this regard and hence are unique in law by the duality argument mentioned above. Our main result is given by the following theorem.

**Theorem (Main result).** For any nonzero immigration function \( \psi \in C([0, \infty) \times \mathbb{R}, \mathbb{R}^+) \), there exists a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) which accommodates a \((\mathcal{F}_t)\)-space-time white noise \( W \) and two solutions \( X \) and \( Y \) of the SPDE (1.2) with respect to \((\mathcal{F}_t)\) such that \( \mathbb{P}\left( \sup_{t \in [0, 2]} \|X_t - Y_t\|_{\text{rap}} > 0 \right) > 0 \).
A comparison of diffusion coefficients may suggest that the construction in [14] of a nonzero signed solution of the particular case
\[
\frac{\partial X}{\partial t}(x,t) = \frac{\Delta X}{2}(x,t) + |X(x,t)|^{1/2} \dot{W}(x,t),
\]
for (1.10) should be closely related to our case (1.2). Indeed, many features of our construction will follow that in [14], but several new problems arise due to the assumed nonnegativity of our solutions and the fact that in our setting uniqueness in law does hold. The latter means that we will be dealing with two nonzero solutions which have the same law and are nontrivially correlated through the shared white noise.

We now outline our construction of the distinct solutions and the argument for their separation. The distinct solutions are obtained by approximation, and each \( \varepsilon \)-approximating pair, still denoted by \( (X,Y) \) but now subject to \( \mathbb{P}_\varepsilon \), consists of super-Brownian motions with intermittent immigration and subject to the same space-time white noise. By intermittent immigration, we mean the informal description that immigrants land after intervals of deterministic lengths and with initial masses centered at i.i.d. targets, and then, along with their offspring, evolve independently as true super-Brownian motions. More precisely, the initial masses of the immigrant processes associated with the \( \varepsilon \)-approximating solutions are of the form \( \psi(1) J^\varepsilon_x(\cdot) \) with \( x \) identified as the target, where
\[
J^\varepsilon_x(z) \equiv \varepsilon^{1/2} J((x-z)\varepsilon^{-1/2}), \quad z \in \mathbb{R},
\]
for an even \( C_c^\pm(\mathbb{R}) \)-function \( J \) which is bounded by 1, has topological support contained in \([-1,1]\), and satisfies \( \int_{\mathbb{R}} J(z)dz = 1 \). In addition, the immigration events occur at alternating times
\[
s_i = \left( i - \frac{1}{2} \right) \varepsilon \quad \text{and} \quad t_i = i \varepsilon \quad \text{for } i \in \mathbb{N},
\]
and the targets associated with the immigrants of \( X \) and \( Y \) are given by the i.i.d. spatial variables \( x_i \) and \( y_i \) at \( s_i \) and \( t_i \), respectively, where
\[
\mathbb{P}_\varepsilon(x_i \in dx) = \mathbb{P}_\varepsilon(y_i \in dx) \equiv \frac{\psi(x)dx}{\psi(1)}.
\]

The weak existence of the approximating solutions subject to these landing times and landing targets follows from the usual Peano’s approximation argument. (See Section 2 for more details.) We remark that, despite the above informal description, it seems difficult to refine this argument for a construction of the immigration clusters, because the \( \varepsilon \)-approximating solutions \( X \) and \( Y \), as their sums, have to be subject to the same space-time white noise.

A simple heuristic argument by passing \( \varepsilon \downarrow 0 \) suggests that these approximating solutions converge to true solutions of the SPDE (1.2) (cf. (2.4)). The rigorous proof takes some routine, but a bit tedious, work. (See Proposition 5.6 for the precise result, and Section 3.9 of [4] for the proof.) We remark that alternative constructions of super-Brownian motions with immigration can be done through Poisson point processes; see, for example, [3].

We notice that a similar construction of approximating solutions also appears in [14] for the equation (1.11). Each one is constructed from its “positive part” and “negative part” as two super-Brownian motions with intermittent immigration but now subject to pairwise annihilation upon collision. In this case, the explicit construction of the immigrant processes for both parts is possible, and all of them are subject to independent space-time white noises and are consistently adapted, that is obeying their defining properties with respect to the same filtration. In fact, these features are at the heart of making immigrant-wise semimartingale calculations possible in [14].

For the above approximating solutions of (1.2), we overcome the restriction from Peano’s approximation and employ a novel method, called continuous decomposition, to elicit the immigrant
processes from the approximating solutions such that they are consistently adapted. There seems no obvious reason why the resulting clusters should satisfy this property, if the general theory of coupling is applied to globally decompose the approximating solutions. In contrast, our method focuses on their local decompositions over time intervals of infinitesimal lengths but still gives the global decompositions:

\[ X = \sum_{i=1}^{\infty} X^i \quad \text{and} \quad Y = \sum_{i=1}^{\infty} Y^i. \]

The resulting clusters associated with each approximating solution are independent and are adapted to a common enlargement of the underlying filtration obtained by “continuously” joining \( \sigma \)-fields of independent variables which validate the local decompositions for both approximating solutions. Moreover, the latter property makes all natural immigrant-wise semimartingale calculations possible. See Section 3 for the details of this discussion.

We now explain why the approximating solutions are uniformly separated. We switch to the conditional probability measure under which the total mass process of a generic cluster, say \( X^i \), from the above continuous decomposition of the \( \varepsilon \)-approximating solution \( X \), hits 1. Let us call this conditional probability measure \( Q^i_{\varepsilon} \) from now on. The motivation of this approach is that by independence of the immigrants \( X^i \), there must be an immigrant cluster associated with \( X \) whose total mass hits 1, so we should be able to carry the conditional separation under a generic \( Q^i_{\varepsilon} \), whenever it exists, to the separation of the approximating solutions under \( P_{\varepsilon} \).

The readers may notice that our argument for separation under \( P_{\varepsilon} \) is reminiscent of similar ones in the studies of SDE’s and SPDE’s on pathwise uniqueness of solutions by excursion theory (cf. [1] and [2]), except that in the present context, the excursions of immigrant clusters can overlap in time without waiting until the earlier ones die out. Nonetheless, as in [14], we can still use some inclusion-exclusion arguments to get separation of our approximate solutions which is uniformly in \( \varepsilon \). (See Section 5.)

Under \( Q^i_{\varepsilon} \), an application of Girsanov’s theorem shows that \( X^i(1) \) is a constant multiple of a 4-dimensional Bessel squared process near its birth time and hence has a known growth rate. To obtain conditional separation of the approximating solutions under \( Q^i_{\varepsilon} \), we will show that the local growth rate of \( Y \) near \( (x_i,s_i) \) is smaller than this known growth rate of \( X^i(1) \), which in turn will be smaller than the local growth rate of \( X \) near \( (x_i,s_i) \). The latter step will use the known modulus of continuity of the support of super-Brownian motion.

To introduce the appropriate local growth rate from the above scheme, we need to identify a growing space-time region starting at \( (x_i,s_i) \) and then identify the sub-collection of immigrant clusters from \( Y \) which can possibly invade this region in small time. For the former, we envelope the support processes of \( Y^j \) and \( X^i \) by approximating parabolas of the form

\[ \mathcal{P}_{\beta}^{(x,s)}(t) = \left\{ (z,r) \in \mathbb{R} \times [s,t]; |x-z| \leq \varepsilon^{1/2} + (r-s)^{\beta} \right\} \]  

(1.14)

for \( \beta \) close to \( 1/2 \) and consider the propagation of these parabolas instead of that of the support processes. The almost-sure growth rate of the support process of super-Brownian motion shows, for example, that

\[ \text{supp}(X^i) \cap (\mathbb{R} \times [s_i,t]) \subseteq \mathcal{P}_\beta^{(x_i,s_i)}(t) \quad \text{for } t-s_i \text{ small}, \]

where \( \text{supp}(X^i) \) is the space-time support of the random function

\[ (x,s) \mapsto X^i(x,s). \]

(See Section 4.2 and Proposition 5.5.) On the other hand, the \( Q^i_{\varepsilon} \)-probability that one of the \( Y^j \) clusters born before \( X^i \) invades the territory of \( X^i \) can be made relatively small by taking \( t \) with \( t-s_i \) small, which follows from an argument similar to Lemma 8.4 of [14] (see Proposition 5.5).
These \( Y_j \) clusters can henceforth be excluded from our consideration. As a result, the tractable geometry of the approximating parabolas (1.14) yields the space-time rectangle

\[
\mathcal{R}^i(t) = \left[ x_1 - 2(\varepsilon^{1/2} + (t-s_i)^\beta), x_1 + 2(\varepsilon^{1/2} + (t-s_i)^\beta) \right] \times [s_i, t]
\]

so that the immigrant processes \( Y_j \) landing inside \( \mathcal{R}^i(t) \) are the only possible candidates which can invade the “territory” of \( X^i \) by time \( t \). This identifies a family of clusters, say, \( \{ Y_j; j \in \mathcal{J}^i(t) \} \). Furthermore, we can classify these clusters \( Y_j \) into critical clusters and lateral clusters. In essence, the critical clusters are born near the territory of \( X^i \) so the interactions between these clusters and \( X^i \) are significant. In contrast, the lateral clusters must evolve for relatively larger amounts of time before they start to interfere with \( X^i \).

Up to this point, the framework we set for investigating conditional separation of approximating solutions is very similar to that in [14]. The interactions between the approximating solutions considered in both cases are, however, very different in nature. For example, the covariance between \( X^i \) and \( Y^j \) under \( \mathbb{Q}_i \) from Girsanov’s theorem is the main source of difficulty in our case, as is not in [14]. For this reason, our case calls for a new analysis in many aspects. Our result for the conditional separation can be captured quantitatively by saying: for arbitrarily small \( \delta > 0 \),

\[
\begin{align*}
\text{with high } \mathbb{Q}_i\text{-probability, } X^i_1(1) &\geq \text{constant } \cdot (t-s_i)^{1+\delta} \quad \text{and} \\
\sum_{j \in \mathcal{J}^i(t)} Y^j_1(1) &\leq \text{constant } \cdot (t-s_i)^{2-\delta}, \text{ for } t \text{ close to } s_i^+.
\end{align*}
\]

(1.15)

Here, the initial behavior of \( X^i(1) \) under \( \mathbb{Q}_i \) as a constant multiple of a 4-dimensional Bessel squared process readily gives the first part of (1.15). (See Section 4.1.) On the other hand, the extra order, which is roughly \( (t-s_i)^{1/2} \), for the sum of the (potential) invaders \( Y^j \) can be seen as the result of using spatial structure, which is not available for the SDE’s discussed above.

In fact, the above framework needs to be further modified in a critical way due to a technical difficulty which arises in our setting (but not in [14]). We must consider a slightly looser definition for critical clusters, and a slightly more stringent definition for lateral clusters. It will be convenient to consider this modified classification for the \( Y^j \) clusters, still indexed by \( j \in \mathcal{J}^i(t) \) for convenience, landing inside a slightly larger rectangle in place of \( \mathcal{R}^i(t) \). Write

\[
\mathcal{J}^i(t) = \mathcal{C}^i(t) \cup \mathcal{L}^i(t),
\]

where \( \mathcal{C}^i(t) \) and \( \mathcal{L}^i(t) \) are the random index sets associated with critical clusters and lateral clusters, respectively. (See Section 4.2 for the precise classification.)

Let us now bound the sum of the total masses \( Y^j_1(1), j \in \mathcal{J}^i(t) \), under \( \mathbb{Q}_i \). As in [14], this part plays a key role in this work. The treatment of the sum is through an analysis of its first-moment. The emphasis is on the covariation process between \( X^i \) and \( Y^j \) under \( \mathbb{Q}_i \) for \( j \in \mathcal{J}^i(t) \) resulting from Girsanov’s theorem.

For the critical clusters \( Y^j \), their covariation processes with \( X^i \) have absolute bounds given by

\[
\int_{t_j}^t \frac{[Y^j_1(1)]^{1/2}}{[X^i_1(1)]^{1/2}} ds
\]

(1.16)

for \( t \) sufficiently close to \( t_j^+ \) (cf. Lemma 4.2 below), so only the total masses of the clusters need to be handled. In this direction, we use an improved modulus of continuity of the total mass processes \( Y^j(1) \) and the lower bound of \( X^i(1) \) in (1.15) to give deterministic bounds for the integrands in (1.16). The overall effect is a Riemann-sum bound for the sum of the total masses \( Y^j_1(1), j \in \mathcal{C}^i(t) \), which has growth similar to that in the second part of (1.15). See Section 4.4.

The lateral clusters pose an additional difficulty here which is not present in [14] due to the correlations between these clusters and \( X^i \). The question is still whether or not conditioning on the
nearby $X^i$ can pull along the nearby $Y^j$’s at a greater rate, even though any of these $Y^j$ does not interfere with $X^i$ right at the beginning. In order to help bound the contributions of these clusters, we argue that a lateral cluster $Y^j$ is independent of $X^i$ until these clusters collide (cf. Lemma 4.16 and Proposition 4.17). This allows us to adapt the arguments for the critical clusters and furthermore bound the growth rate of the sums of the total masses $Y^j_t(\mathbb{1})$, $j \in \mathcal{L}^i(t)$, by the desired order. See the discussion in Section 4.5 for more on this issue.

We close our discussion in this section by making an immediate corollary for the SPDE (1.2) in which $\psi(\mathbb{1})$ is small and the initial value is replaced by a nonzero nonnegative $\mathcal{C}^{\text{rap}}(\mathbb{R})$-function. In this case, pathwise non-uniqueness remains true. This follows from the Markov property of super-Brownian motions with immigration and the recurrence of Bessel-squared processes for small dimensions. More precisely, we can run a copy of such a super-Brownian motion with immigration until its total mass first hits zero, and then concatenate this piece with the separating solutions obtained by our main theorem to construct two separating solutions.

This paper is organized as follows. In Section 2, we give the precise definition of the pairs of approximating solutions considered throughout this paper and state their existence in Theorem 2.1. In Section 3, we outline the idea behind our continuous decomposition of a super-Brownian motion with intermittent immigration and then give the rigorous proof for the continuous decompositions of the approximation solutions. Some properties of the resulting clusters from the decompositions are discussed at the end of Section 3. In Section 4, we state some basic results and then proceed to the setup for obtaining the conditional separation of the approximating solutions. Due to the complexity, the main two lemmas of Section 4 are proved in Section 4.4 and Section 4.5 respectively, with some preliminaries set in Section 4.3. In Section 5, we show the uniform separation of approximating solutions under $\mathbb{P}_\varepsilon$, which completes the proof of our main theorem.

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2. Super-Brownian motions with intermittent immigration

In this section, we describe in more detail the pairs of approximating solutions to the SPDE (1.2) considered throughout this paper. Recall that, for any locally integrable function $f$ on $\mathbb{R}$, we use the identification (1.7). We will further write

$$f(\Gamma) = f(\mathbb{1}_\Gamma), \quad \Gamma \in \mathcal{B}(\mathbb{R}),$$

whenever the right-hand side makes sense.

For $\varepsilon \in (0, 1]$, the $\varepsilon$-approximating solution $X$ is a nonnegative càdlàg $\mathcal{C}^{\text{rap}}(\mathbb{R})$-valued process and, moreover, continuous within each $[s_i, s_{i+1})$. Its time evolution is given by

$$X_t(\phi) = \int_0^t X_s \left( \frac{\Delta}{2} \phi \right) ds + \int_{(0,t]} \int_{\mathbb{R}} \phi(x) dA_X(x,s)$$
$$+ \int_0^t \int_{\mathbb{R}} X(x,s)^{1/2} \phi(x) dW(x,s), \quad \phi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

(2.1)
In (2.1), the nonnegative measure $A^X$ on $\mathbb{R} \times \mathbb{R}_+$ is contributed by the initial masses of the immigrant processes associated with $X$ and is defined by

$$A^X(\Gamma \times [0, t]) \triangleq \sum_{i: 0 < s_i \leq t} \psi(\mathbb{1}) J_{\varepsilon i}^x(\Gamma)$$  \hspace{1cm} (2.2)

(recall our notation $J_{\varepsilon i}^x$ in (1.12) and the i.i.d. spatial random points $\{x_i\}$ with individual law (1.13)), and $W$ is a space-time white noise.

A similar characterization applies to the other approximating solution $Y$. It is a nonnegative càdlàg $\mathcal{C}_{\text{rap}}(\mathbb{R})$-valued process satisfying

$$Y_t(\phi) = \int_0^t Y_s \left( \frac{\Delta \phi}{2} \right) ds + \int_{(0,t]} \phi(x) dA^Y(x, s)$$

$$\hspace{2cm} + \int_0^t \int_\mathbb{R} Y(x, s)^{1/2} \phi(x) dW(x, s), \quad \phi \in \mathcal{C}_c^\infty(\mathbb{R}),$$

and is continuous over each $[t_j, t_{j+1})$. The nonnegative measure $A^Y$ on $\mathbb{R} \times \mathbb{R}_+$ is now defined by

$$A^Y(\Gamma \times [0, t]) \triangleq \sum_{j: 0 < t_j \leq t} \psi(\mathbb{1}) J_{\varepsilon j}^y(\Gamma).$$

We stress that here $X$ and $Y$ are subject to the same space-time white noise $W$.

The existence of these pairs of $\varepsilon$-approximation solutions follows by considering the so-called mild forms of solutions of SPDE's and then using the classical Peano’s existence argument as in Theorem 2.1. of [22]. The precise result is summarized in the following theorem. Here and throughout this paper, we use the notation “$\mathcal{G} \perp \mathcal{F}$” to mean that the $\sigma$-field $\mathcal{G}$ and the random element $\mathcal{F}$ are independent, and analogous notation applies to other pairs of objects which allow probabilistic independence in the usual sense.

**Theorem 2.1.** Fix an immigration function $\psi \in \mathcal{C}_c^+(\mathbb{R}) \setminus \{0\}$. For any $\varepsilon \in (0, 1]$, we can construct a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_\varepsilon)$, with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions, on which there exist the following random elements:

1. an $(\mathcal{F}_t)$-space-time white noise $W$,
2. two nonnegative $(\mathcal{F}_t)$-adapted $\mathcal{C}_{\text{rap}}(\mathbb{R})$-valued processes $X$ and $Y$ satisfying (2.1) and (2.3) with respect to $W$ with paths which are càdlàg on $\mathbb{R}_+$ and continuous within each $[s_i, s_{i+1})$ and $[t_j, t_{j+1})$, respectively,
3. and i.i.d. random variables $x_i, y_i$ with law given by (1.13), taking values in the topological support of $\psi$, and satisfying the property that

$$\forall \ i \in \mathbb{N}, \quad \sigma(X_{s_i}, Y_{s_i} ; s < s_i) \perp x_i \text{ and } \sigma(X_{s_i}, Y_{s_i} ; s < s_i) \perp y_i.$$

Both $X$ and $Y$ are genuine approximating solutions to the SPDE (1.2) with respect to the same white noise. More precisely, for every sequence $(\varepsilon_n) \subseteq (0, 1]$ with $\varepsilon_n \searrow 0$, the sequence of laws

$$\mathbb{P}_{\varepsilon_n}(X \in \cdot, Y \in \cdot), \quad n \in \mathbb{N},$$

as probability measures on $D(\mathbb{R}_+, \mathcal{C}_{\text{rap}}(\mathbb{R})) \times D(\mathbb{R}_+, \mathcal{C}_{\text{rap}}(\mathbb{R}))$ is tight, and the limit of every convergent subsequence is the joint law of a pair of solutions to (1.2) with respect to the same space-time white noise. Although the proof of this result concerning weak convergence is somewhat lengthy (see Section 3.9 of [4]), the readers should be convinced immediately upon considering the limiting behaviour of the random measures $A^X$: for any $t \in (0, \infty)$,

$$\mathbb{P} \lim_{\varepsilon \searrow 0^+} \int_{[0,t]} \int_\mathbb{R} \phi(x) dA^X(x, s) = \mathbb{P} \lim_{\varepsilon \searrow 0^+} \psi(\mathbb{1}) \sum_{i=1}^{\lfloor \varepsilon^{-1} t \rfloor} \phi(x_i) = t \langle \psi, \phi \rangle$$  \hspace{1cm} (2.4)
for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$, by the law of large numbers. Here, $\mathbb{P}$-lim denotes convergence in probability, and $\lceil t \rceil$ is the greatest integer less than or equal to $t$.

**Notation 2.2.** The following convention will be in force throughout this paper unless otherwise mentioned. We continue to suppress the dependence on $\varepsilon$ as above whenever the context is clear. In this case, we only use the probability measure $\mathbb{P}_\varepsilon$ to emphasize the dependence on $\varepsilon$. The subscript $\varepsilon$ of $\mathbb{P}_\varepsilon$ is further omitted in cases where there is no ambiguity, although in this context we will remind the readers of this practice.

## 3. Continuous decompositions of approximating processes

For every $\varepsilon \in (0, 1]$, we have two approximating solutions $X$ and $Y$, as stated in Theorem 2.1, to the SPDE (1.2) of a super-Brownian motion with immigration and zero initial condition. From their informal descriptions, we expect to decompose them into

$$X = \sum_{i=1}^{\infty} X^i \quad \text{and} \quad Y = \sum_{i=1}^{\infty} Y^i,$$

(3.1)

where the summands $X^i$ and $Y^i$ are super-Brownian motions started at $s_i$ and $t_i$ and with starting measures $J_{X}^{\varepsilon_i}$ and $J_{Y}^{\varepsilon_i}$, respectively, for each $i$, and each of the families $\{X^i\}$ and $\{Y^i\}$ consists of independent random elements.

We give an elementary discussion on obtaining the decompositions in (3.1). Later on, we will require additional properties of the decompositions. Now, it follows from the uniqueness in law of super-Brownian motions and the defining equation (2.1) that $X$ is a (time-inhomogeneous) Markov process and, for each $i \in \mathbb{N}$, $(X_t^{i})_{t \in [s_i, s_{i+1})}$ defines a super-Brownian motion with initial distribution $X_{s_i}$. (Cf. the proof of Theorem IV.4.2 in [6] and the martingale problem characterization of super-Brownian motion in [18].) Hence, each of the equalities in (3.1) holds in the sense of being identical in distribution. We then recall the following general theorem (see Theorem 6.10 in [8]).

**Theorem 3.1.** Fix any measurable space $E_1$ and Polish space $E_2$, and let $\xi \overset{(d)}= \bar{\xi}$ and $\eta$ be random elements taking values in $E_1$ and $E_2$, respectively. Here, we only assume that $\xi$ and $\eta$ are defined on the same probability space. Then there exists a measurable function $F \colon E_1 \times [0, 1] \rightarrow E_2$ such that for any random variable $\bar{U}$ uniformly distributed over $[0, 1]$ with $\bar{U} \perp \xi$, the random element $\bar{\eta} = F(\bar{\xi}, \bar{U})$ solves $(\xi, \eta) \overset{(d)}= (\bar{\xi}, \bar{\eta})$.

By the preceding discussions and Theorem 3.1, we can immediately construct the summands $X^i$ and $Y^i$ by introducing additional independent uniform variables and validate the equalities (3.1) as almost-sure equalities. Such decompositions, however, are too crude because, for example, we are unable to say that all the resulting random processes perform their defining properties with respect to the same filtration. This difficulty implies in particular that we cannot do semimartingale calculations for them. A finer decomposition method, however, does yield a solution to this problem, and the result is stated in Theorem 3.2 below. See also Figure 3 for a sketch of the decomposition of $X$ along $x$ in the support of $\psi$.

**Theorem 3.2 (Continuous decomposition).** Fix $\varepsilon \in (0, 1]$, and let $(X, Y, W)$ be as in Theorem 2.1. By changing the underlying probability space if necessary, we can find a filtration $(\mathcal{G}_t)$ satisfying the usual conditions and two families $\{X^i\}$ and $\{Y^i\}$ of nonnegative $\mathcal{F}_{\text{cadp}}(\mathbb{R})$-valued processes, such that the followings are satisfied:

(i) The processes $X^i$, $i \in \mathbb{N}$, are independent.

(ii) The equality in (3.1) involving $X$ and $X^i$ holds almost surely.
(iii) Each $X_i^t$, $t \in [s_i, \infty)$ has sample paths in $C([s_i, \infty), \mathcal{C}_{rap}(\mathbb{R}))$ and is a $(\mathcal{F}_t)_{t \geq s_i}$-super-Brownian motion started at time $s_i$ with starting measure $\psi(x)J_{s_i}^1$. Also, $X_i^t \equiv 0$ for every $t \in [0, s_i)$.

(iv) The processes $Y_i$, $i \in \mathbb{N}$, satisfy properties analogous to (i)–(iii) with the roles of $X$ and $\{(X^i, x_i, s_i)\}$ replaced by $Y$ and $\{(Y^j, y_i, t_j)\}$, respectively.

(v) The conditions (i) and (ii) of Theorem 2.1 hold with $(\mathcal{F}_t)$ replaced by $(\mathcal{G}_t)$, and the condition (iii) of the same theorem is replaced by the stronger independent landing property:

$$\forall i \in \mathbb{N}, \quad \sigma \left( X_i^t, Y_i^t : s < s_i, \; j \in \mathbb{N} \right) \perp x_i$$

$$\quad \sigma \left( X_i^t, Y_i^t : s < t_i, \; j \in \mathbb{N} \right) \perp y_i.$$  \hspace{1cm} (3.2)

Due to the length of the proof of Theorem 3.2, we first outline the informal idea for the convenience of readers. Recall that the first immigration event for $X$ and $Y$ occurs at $s_1 = \varepsilon/2$. Take a grid of $[\varepsilon/2, \infty)$ containing all the points $s_i$ and $t_i$ for $i \in \mathbb{N}$ and with “infinitesimal” mesh size. Here, the mesh size of a grid is the supremum of the distances between consecutive grid points. The key observation in this construction is that, over any subinterval $[t, t + \Delta t] \subseteq [s_i, s_{i+1})$ from this grid, $(X_r, r \in [t, t + \Delta t])$ has the same distribution as the sum of $i$ independent super-Brownian motions started at $t$ over $[t, t + \Delta t]$, whenever the sum of the initial conditions of these independent super-Brownian motions has the same distribution as $X_t$.

This fact allows us to inductively decompose $X$ over the intervals of infinitesimal lengths from this grid, such that the resulting infinitesimal pieces of super-Brownian motions can be concatenated in the natural way to obtain the desired super-Brownian motions. More precisely, we apply Theorem 3.1 by bringing in independent uniform variables as “allocators” to obtain these infinitesimal pieces. A similar method applies to continuously decompose $Y$ into the desired independent super-Brownian motions by using another family of independent allocators.

Finally, because the path regularity of these concatenated processes and $W$ allows us to characterize their laws over the entire time horizon $\mathbb{R}_+$ by their laws over $[0, \varepsilon/2]$ and their probabilistic transitions on this grid with infinitesimal mesh size, the filtration obtained by sequentially adding the $\sigma$-fields of the independent allocators will be the desired one. The time evolutions of these stochastic processes are now consistent with the “progression” of the enlarged filtration.

From now on, we use the notation $\Delta Z_s = Z_s - Z_{s-}$ with $Z_0 = 0$ for a càdlàg process $Z$ taking values in a Polish space.
Proof of Theorem 3.2. Fix $\varepsilon \in (0, 1]$ and we shall drop the subscript $\varepsilon$ of $\mathbb{P}_\varepsilon$. Throughout the proof, we take for each $m \geq 1$ a countable grid $D_m$ of $[\frac{1}{2}, \infty)$ which contains $s_i$ and $t_i$ for any $i \geq 1$ and satisfies $\#(D_m \cap K) < \infty$ for any compact subset $K$ of $\mathbb{R}^+$. We further assume that $D_{m+1} \subseteq D_m$ for each $m$, between any two points $s_i$ and $t_i$ there is another point belonging to $D_1$ and hence to each $D_m$, and the mesh size of $D_m$ goes to zero as $m \to \infty$. In addition, we will write for convenience $\{\SBM_t(\mu_t, dv); t \in \mathbb{R}^+\}$ for the semigroup of super-Brownian motion on $\mathbb{R}$. When the density of the super-Brownian motion on $\mathbb{R}$ started at time $s$ and with starting measure $f(x)dx$ for a nonnegative $\mathcal{C}_{rap}(\mathbb{R})$-function $f$ is concerned, we write $\SBM_{f,[s,t]}$ for the law of its $\mathcal{C}([s, \infty), \mathcal{C}_{rap}(\mathbb{R}))$-valued density restricted to the time interval $[s, t]$.

(Step 1). Fix $m \in \mathbb{N}$ and write $\frac{1}{2} = \tau_0 < \tau_1 < \cdots$ as the consecutive points of $D_m$. Assume, by an enlargement of the underlying probability space where the triplet $\{X, Y, W\}$ lives if necessary, the existence of i.i.d. variables $\{U^X_j, U^Y_j; j \in \mathbb{N}\}$ with

$$U^X_1 \text{ is uniformly distributed over } [0, 1] \quad \text{and} \quad \{U^X_j, U^Y_j; j \in \mathbb{N}\} \perp \mathcal{F}. \quad (3.3)$$

In this step, we will decompose $X$ and $Y$ based on the grid $D_m$ into the random elements

$$X^m = (X^{m,1}, X^{m,2}, \cdots) \quad \text{and} \quad Y^m = (Y^{m,1}, Y^{m,2}, \cdots),$$

respectively. Here,

$$X^{m,i} \in C([s_i, \infty), \mathcal{C}_{rap}(\mathbb{R})) \quad \text{and} \quad Y^{m,i} \in C([t_i, \infty), \mathcal{C}_{rap}(\mathbb{R}))$$

with $X^{m,i} \equiv 0$ on $[0, s_i]$ and $Y^{m,i} \equiv 0$ on $[0, t_i)$, so we will only need to specify $X^{m,i}$ over $[s_i, \infty)$ and $Y^{m,i}$ over $[t_i, \infty)$.

We consider the construction of $X^m$ first. The decomposition of $X$ over $[s_1, s_2]$ should be self-evident. Over this interval, set $X^{m,1} \equiv X$ on $[s_1, s_2)$ with $X^{m,1}_{s_2} = X_{s_2}$ and

$$X^{m,2}_s = \begin{cases} 0, & s \in [s_1, s_2), \\ \psi(1)^j & = \Delta X_{s_2}, \quad s = s_2. \end{cases}$$

From now on, we shall define $X^m$ over $[s_2, \tau_j]$ by an induction on integers $j \geq j^X_*$, where $j^X_* \in \mathbb{N}$ satisfies $s_2 = \tau_j$, such that

$$(X^{m,i}_s; 0 \leq s \leq \tau_k, \quad i \in \mathbb{N}) \in \sigma (X_s; s \leq \tau_k) \vee \sigma (U^X_1, 1 \leq i \leq k), \quad \forall k \in \{0, \cdots, j\}, \quad (3.5)$$

with $\sigma (U^X_1, 1 \leq i \leq 0)$ understood to be the trivial $\sigma$-field $\{\Omega, \emptyset\}$, the laws of $X^{m,i}$ obey

$$\begin{cases} (a) & \mathcal{L} (X^{m,i}_s; s \in [s_i, \tau_j]) \sim \SBM_{\psi(1)^j, [s_i, \tau_j]} \text{ if } s_i \leq \tau_j, \\
(b) & (X^{m,i}_s; s \in [s_i, \tau_j]), \text{ for } i \text{ satisfying } s_i \leq \tau_j, \text{ are independent,} \end{cases} \quad (3.6)$$

and finally

$$X_s = \sum_{i=1}^{\infty} X^{m,i}_s, \quad \forall s \in [0, \tau_j] \quad \text{a.s.} \quad (3.7)$$

By the foregoing identification of $X^m$ over $[s_1, s_2]$, we have obtained the case that $j = j^X_*$, that is, the first step of our inductive construction.
Assume that $X^m$ has been defined up to time $\tau_j$ for some integer $j \geq j^X$ such that (3.5)–(3.7) are all satisfied. We now aim to extend $X^m$ over $[\tau_j, \tau_{j+1}]$ so that all of (3.5)–(3.7) hold with $j$ replaced by $j + 1$. First, consider the case that

$$[\tau_j, \tau_{j+1}] \subseteq [s_k, s_{k+1})$$

for some $k \geq 2$. In this case, we only need to extend $X^{m,1}, \ldots, X^{m,k}$. Take an auxiliary nonnegative random element

$$\xi = (\xi^1, \ldots, \xi^k) \in C \left([\tau_j, \infty), \prod_{i=1}^k \mathcal{C}_{rap}(\mathbb{R}) \right)$$

such that the coordinates $(\xi^s; s \in [\tau_j, \infty))$ are independent processes and each of them defines a super-Brownian motion started at $\tau_j$ with initial law

$$\mathcal{L} \left( \xi^s_{\tau_j} \right) = \mathcal{L} \left( X^{m,i}_{\tau_j} \right), \quad \forall i \in \{1, \ldots, k\}. \quad (3.9)$$

Now, our claim is that we can extend $X^{m,1}, \ldots, X^{m,k}$ continuously over $[\tau_j, \tau_{j+1}]$ so that

$$\left( X_{\tau_j}^{m,1}, \ldots, X_{\tau_j}^{m,k} \right) \overset{(d)}{=} \left( \xi_{\tau_j}^1, \ldots, \xi_{\tau_j}^k \right), \quad \left( \xi_{\tau_j}^1 \right)_{r \in [\tau_j, \tau_{j+1}]}, \ldots, \left( \xi_{\tau_j}^k \right)_{r \in [\tau_j, \tau_{j+1}]}. \quad (3.10)$$

In particular, the equality (3.10) in distribution implies that almost surely we have the following equalities:

$$X_{r}^{m,1} + \cdots + X_{r}^{m,k} = X_{r}, \quad \forall r \in [\tau_j, \tau_{j+1}], \quad (3.11)$$

and

$$\mathcal{L} \left( (X^{m,1})_{r \in [\tau_j, \tau_{j+1}]}, \ldots, (X^{m,k})_{r \in [\tau_j, \tau_{j+1}]} \right) \overset{=}{=} \mathcal{L} \left( (X^{m,1})_{r \in [\tau_j, \tau_{j+1}]}, \ldots, (X^{m,k})_{r \in [\tau_j, \tau_{j+1}]} \right), \quad (3.12)$$

where the first equality of (3.12) follows from (3.11), and the second equality follows from the definition of $\xi$. To prove our claim (3.10), first we consider

$$\begin{align*}
\mathbb{P} \left( (X_{r})_{r \in [\tau_j, \tau_{j+1}]} \in \Gamma, X^{m,1}_{\tau_j} \in A_1, \ldots, X^{m,k}_{\tau_j} \in A_k \right) \\
= \sum_{m_1, \ldots, m_k \in \mathbb{Z}} \mathbb{P} \left( (X_{r})_{r \in [\tau_j, \tau_{j+1}]} \in \Gamma \mid X^{m,1}_{\tau_j} \in A_1, \ldots, X^{m,k}_{\tau_j} \in A_k \right) \\
= \mathbb{E} \left[ \text{SBM}_{X^{m,1}_{\tau_j} \mid [\tau_j, \tau_{j+1}]}(\Gamma) \mid X^{m,1}_{\tau_j} \in A_1, \ldots, X^{m,k}_{\tau_j} \in A_k \right] \\
= \mathbb{E} \left[ \text{SBM}_{\sum_{i=1}^k X^{m,i}_{\tau_j} \mid [\tau_j, \tau_{j+1}]}(\Gamma) \mid X^{m,1}_{\tau_j} \in A_1, \ldots, X^{m,k}_{\tau_j} \in A_k \right],
\end{align*}\quad (3.13)$$

where the first and the second equalities use the (time-inhomogeneous) Markov property of $X$ and (3.5), and the last equality follows from the equality (3.7) from induction. Second, by (3.6) from induction and (3.9), we have

$$\left( X_{\tau_j}^{m,1}, \ldots, X_{\tau_j}^{m,k} \right) \overset{(d)}{=} \left( \xi_{\tau_j}^1, \ldots, \xi_{\tau_j}^k \right).$$
Hence, from (3.13), we get
\[
\begin{align*}
P \left( (X_r)_{r \in [\tau_j, \tau_{j+1}]} \in \Gamma, X^{m,1}_{\tau_j} \in A_1, \cdots, X^{m,k}_{\tau_j} \in A_k \right) & = \mathbb{E} \left[ \text{SBM} \sum_{i=1}^k \xi^i_{\tau_j, r \in [\tau_j, \tau_{j+1}]}(\Gamma) ; \xi^1_{\tau_j} \in A_1, \cdots, \xi^k_{\tau_j} \in A_k \right] \\
& = \mathbb{E} \left[ P \left( \sum_{i=1}^k \xi^i_{r \in [\tau_j, \tau_{j+1}]} \in \Gamma \right) ; \xi^1_{\tau_j} \in A_1, \cdots, \xi^k_{\tau_j} \in A_k \right] \\
& = P \left( \sum_{i=1}^k \xi^i_{r \in [\tau_j, \tau_{j+1}]} \in \Gamma, \xi^1_{\tau_j} \in A_1, \cdots, \xi^k_{\tau_j} \in A_k \right). \quad (3.14)
\end{align*}
\]

Here, the second equality follows from the convolution property of the laws of super-Brownian motions:
\[
\text{SBM}_{f_1, [s, t]} \ast \cdots \ast \text{SBM}_{f_k, [s, t]} = \text{SBM}_{\sum_{i=1}^k f_i, [s, t]}.
\]

Then (3.14) implies that
\[
\left( X^{m,1}_{\tau_j}, \cdots, X^{m,k}_{\tau_j}, (X_r)_{r \in [\tau_j, \tau_{j+1}]} \right) \overset{d}{=} \left( \xi^1_{\tau_j}, \cdots, \xi^k_{\tau_j}, \sum_{i=1}^k \xi^i_{r \in [\tau_j, \tau_{j+1}]} \right). \quad (3.15)
\]

Using the boundary condition (3.15) and Theorem 3.1, we can solve the stochastic equation on the left-hand side of (3.10) by a Borel measurable function
\[
F^m_j : \prod_{i=1}^k \mathcal{F}_{\text{rap}}(\mathbb{R}) \times C([\tau_j, \tau_{j+1}]; \mathcal{F}_{\text{rap}}(\mathbb{R})) \times [0, 1] \longrightarrow \prod_{i=1}^k C([\tau_j, \tau_{j+1}], \mathcal{F}_{\text{rap}}(\mathbb{R}))
\]
such that the desired extension of $X^m$ over $[\tau_j, \tau_{j+1}]$ can be defined by
\[
\left( (X^{m,1}_r)_{r \in [\tau_j, \tau_{j+1}]}, \cdots, (X^{m,k}_r)_{r \in [\tau_j, \tau_{j+1}]} \right) = F^m_j \left( X^{m,1}_{\tau_j}, \cdots, X^{m,k}_{\tau_j}, (X_r)_{r \in [\tau_j, \tau_{j+1}]}, U^X_{j+1} \right),
\]
where the independent uniform variable $U^X_{j+1}$ now plays its role to decompose $(X_r)_{r \in [\tau_j, \tau_{j+1}]}$. This proves our claim on the continuous extension of $X^{m,1}, \cdots, X^{m,k}$ over $[\tau_j, \tau_{j+1}]$ satisfying (3.10). By induction and (3.16), the extension of $X^m$ over $[\tau_j, \tau_{j+1}]$ satisfies (3.5) with $j$ replaced by $j+1$; by induction and (3.11), it satisfies (3.7) with $j$ replaced by $j+1$.

Let us verify that (3.6) is satisfied with $j$ replaced by $j+1$. By (3.5), we can write
\[
\begin{align*}
P \left( (X^{m,i}_r)_{r \in [\tau_j, \tau_{j+1}]} \in A_i, (X^{m,i}_s)_{s \in [\tau_j, \tau_j]} \in B_i, \forall i \in \{1, \cdots, k\} \right) & = \mathbb{E} \left[ P \left( (X^{m,i}_r)_{r \in [\tau_j, \tau_{j+1}]} \in A_i, \forall i \in \{1, \cdots, k\} \bigg| \mathcal{F}_{\tau_j} \cup \sigma (U^X_1, \cdots, U^X_j) \right) ; \right. \\
& \left. (X^{m,i}_s)_{s \in [\tau_j, \tau_j]} \in B_i, \forall i \in \{1, \cdots, k\} \right]. \quad (3.17)
\end{align*}
\]

To reduce the conditional probability on the right-hand side to one conditioned on $X^{m,1}_{\tau_j}, \cdots, X^{m,k}_{\tau_j}$,
we review the defining equation (3.16) of $X^m$ over $[\tau_j, \tau_{j+1}]$ and consider the calculation:

$$
\mathbb{E} \left[ g_1 \left( X^m_{\tau_j}, \ldots, X^m_{\tau_{j+k}} \right) g_2 \left( (X_r)_{r \in [\tau_j, \tau_{j+1}]} \right) g_3 \left( U^X_{\tau_{j+1}} \right) \bigg| \mathcal{F}_{\tau_j} \right] \propto \sigma \left( U^X, \ldots, U^X_j \right)
$$

$$
= g_1 \left( X^m_{\tau_j}, \ldots, X^m_{\tau_{j+k}} \right) \mathbb{E} \left[ g_2 \left( (X_r)_{r \in [\tau_j, \tau_{j+1}]} \right) \bigg| \mathcal{F}_{\tau_j} \right] \mathbb{E} \left[ g_3 \left( U^X_{\tau_{j+1}} \right) \bigg| \mathcal{F}_{\tau_j} \right],
$$

$$
= g_1 \left( X^m_{\tau_j}, \ldots, X^m_{\tau_{j+k}} \right) \mathbb{E} \left[ g_2 \left( (X_r)_{r \in [\tau_j, \tau_{j+1}]} \right) \right] \mathbb{E} \left[ g_3 \left( U^X_{\tau_{j+1}} \right) \right],
$$

$$
= \mathbb{E} \left[ g_1 \left( X^m_{\tau_j}, \ldots, X^m_{\tau_{j+k}} \right) g_2 \left( (X_r)_{r \in [\tau_j, \tau_{j+1}]} \right) g_3 \left( U^X_{\tau_{j+1}} \right) \left| \mathcal{F}_{\tau_j} \right] \right],
$$

(3.18)

where the first equality follows again from (3.5) and the second equality follows by using the ($\mathcal{F}_t$)-Markov property of $X$ and considering the “sandwich” of $\sigma$-fields:

$$
\sigma(X_{\tau_j}) \subseteq \sigma \left( X^m_{\tau_j}, \ldots, X^m_{\tau_{j+k}} \right) \cap \mathcal{N} \subseteq \mathcal{F}_{\tau_j} \propto \sigma \left( U^X, \ldots, U^X \right)
$$

with $\mathcal{N}$ being the collection of $\mathbb{P}$-null sets, and the last equality (3.18) follows since $U^X_{\tau_{j+1}}$ is not yet used in the construction of $X^m$ up to time $\tau_j$. Hence, by (3.16) and (3.18), we can continue our calculation in (3.17) as follows:

$$
\mathbb{P} \left( (X^m_{\tau_{r_i}})_{r \in [\tau_j, \tau_{j+1}]} \in A_i, (X^m_{\tau_{s_j}})_{r \in [s_j, \tau_j]} \in B_i, \forall i \in \{1, \ldots, k\} \right)
$$

$$
= \mathbb{E} \left[ \mathbb{P} \left( (X^m_{\tau_{r_i}})_{r \in [\tau_j, \tau_{j+1}]} \in A_i, (X^m_{\tau_{s_j}})_{r \in [s_j, \tau_j]} \in B_i, \forall i \in \{1, \ldots, k\} \right| X^m_{\tau_j}, \ldots, X^m_{\tau_{j+k}} \right],
$$

$$
= \mathbb{E} \left[ \prod_{i=1}^{k} \text{SBM}_{X_{\tau_j}, [\tau_j, \tau_{j+1}]} \left( A_i \right| (X^m_{\tau_{r_i}})_{r \in [s_j, \tau_j]} \in B_i, \forall i \in \{1, \ldots, k\} \right),
$$

where the second equality follows from (3.12). By (3.6) and induction, the foregoing equality implies that (3.6) with $j$ replaced by $j+1$ still holds. This completes our inductive construction for the case (3.8).

We also need to consider the case complementary to (3.8) that $[\tau_{j}, \tau_{j+1}] \subseteq [s_k, s_{k+1}]$ and $\tau_{j+1} = s_{k+1}$ for some $k \geq 2$. In this case, the construction of $X^{m,1}, \ldots, X^{m,k}$ over the time interval $[\tau_j, \tau_{j+1}]$ is the same as before, but the extra coordinate $X^{m,k+1}$ is now defined to be $\psi(1)J_{\tau_{j+1}}^{s_{k+1}}$ at time $\tau_{j+1} = s_{k+1}$. The properties (3.5) and (3.7) with $j$ replaced by $j+1$ remain true, by the argument for the previous case. So does the property (3.6) with $j$ replaced by $j+1$, if in addition we use (iii) of Theorem 2.1 to see that the coordinate $X^{m,k+1}$ is independent of the others by time $\tau_{j+1}$. This completes our inductive construction of $X^m$.

The construction of $Y^m$ is similar to that of $X^m$. We use $\{U^Y_j\}$ to validate decompositions, and the immigration times $\{t_j; j \in \mathbb{N}\}$ are now taken into consideration for the construction. We omit other details.

From the constructions of $X^m$ and $Y^m$, (3.3), and the property (iii) in Theorem 2.1, we see that the following independent landing property is satisfied by $X^m$ and $Y^m$:

$$
\forall i \in \mathbb{N}, \quad \sigma \left( X_{s_i}^{m,j}, Y_{s_i}^{m,j}; s < s_i, j \in \mathbb{N} \right) \perp x_i \text{ and } \sigma \left( X_{t_i}^{m,j}, Y_{t_i}^{m,j}; s < t_i, j \in \mathbb{N} \right) \perp y_i,
$$

(3.19)

(Step 2). We now define a filtration $\mathcal{G}_t^{(m)}$ with respect to which the processes $X^{m,i}$, $Y^{m,i}$, and $W$ perform their defining properties on the grid $D_m$. The filtration $\mathcal{G}_t^{(m)}$ is larger than $\mathcal{F}_t$ and is defined by

$$
\mathcal{G}_t^{(m)} = \begin{cases}
\mathcal{F}_t, & t \in [0, \tau_0], \\
\mathcal{F}_{\tau_{j+1}} \vee \sigma \left( U^X_k, U^Y_k; 1 \leq k \leq j + 1 \right), & t \in (\tau_j, \tau_{j+1}], j \in \mathbb{Z}_+.
\end{cases}
$$
In particular, it follows from (3.5) and the analogue for $Y^m$ that the processes $X_{m,i}$ and $Y_{m,i}$ are all $(\mathcal{G}_{t}^{(m)})$-adapted. Also, it is obvious that $X$, $Y$, and $W(\phi)$ for any $\phi \in L^2(\mathbb{R})$ are $(\mathcal{G}_{t}^{(m)})$-adapted.

We now observe a key feature of $X_m$:

\[ \mathbb{P} \left( X_{m,i}^{s} \in \Gamma \big| \mathcal{G}_{\tau_j}^{(m)} \right) = \text{SBM}_{t-\tau_j} \left( X_{m,i}^{\tau_j}, \Gamma \right), \quad \forall t \in (\tau_j, \tau_{j+1}] \text{ for } s_i \leq \tau_j \text{ and } i \in \mathbb{N} \]

for any Borel measurable subset $\Gamma$ of the space of finite measures on $\mathbb{R}$. To see (3.20), we consider a slight generalization of the proof of (3.18) by adding $\sigma(U_1^Y, \cdots, U_j^Y)$ to the $\sigma$-field $\mathcal{F}_{\tau_j} \vee \sigma(U_1^X, \cdots, U_j^X)$ in the first line therein and then apply (3.6) to obtain

\[ \mathbb{P} \left( X_{m,i}^{s} \in \Gamma \big| \mathcal{G}_{\tau_j}^{(m)} \right) = \mathbb{P} \left( X_{m,i}^{s} \in \Gamma \big| X_{m,i}^{\tau}, X_{m,2}^{\tau}, \cdots \right) 
= \mathbb{P} \left( X_{m,i}^{s} \in \Gamma \big| X_{m,i}^{\tau_j} \right) 
= \text{SBM}_{t-\tau_j} \left( X_{m,i}^{\tau_j}, \Gamma \right), \quad \forall t \in (\tau_j, \tau_{j+1}]. \]

In particular, we deduce from iteration and the semigroup property of $\{\text{SBM}_t\}$ that the following grid Markov property is satisfied:

\[ \mathbb{P} \left( X_{m,i}^{s} \in \Gamma \big| \mathcal{G}_{\tau_j}^{(m)} \right) = \text{SBM}_{t-\tau_j} \left( X_{m,i}^{\tau_j}, \Gamma \right), \quad \forall t \in (\tau_k, \tau_{k+1}] \text{ when } s_i \leq \tau_j \leq \tau_k. \]

We note that the foregoing display does not say that $X_{m,i}$ is a $(\mathcal{G}_{t}^{(m)})_{s \geq s_i}$-super-Brownian motion because the $\sigma$-fields which we can use for verifying the $(\mathcal{G}_{s}^{(m)})_{s \geq s_i}$-Markov property are only $\mathcal{G}_{\tau_j}^{(m)}$, rather than any $\sigma$-field $\mathcal{G}_{s}^{(m)}$. With a similar argument, we also have the grid Markov property of $Y_{m,i}$ stated as

\[ \mathbb{P} \left( Y_{m,i}^{s} \in \Gamma \big| \mathcal{G}_{\tau_j}^{(m)} \right) = \text{SBM}_{t-\tau_j} \left( Y_{m,i}^{\tau_j}, \Gamma \right), \quad \forall t \in (\tau_k, \tau_{k+1}] \text{ when } t_i \leq \tau_j \leq \tau_k. \]

With a much simpler argument, the space-time white noise $W$ has the same grid Markov property:

\[ \mathbb{L} \left( W(\phi) \big| \mathcal{G}_{\tau_j}^{(m)} \right) = \mathcal{N} \left( W_{\tau_j}(\phi), (t-\tau_j), \|\phi\|^2_{L^2(\mathbb{R})} \right), \quad \forall t \in (\tau_k, \tau_{k+1}] \text{ for } \tau_j \leq \tau_k \text{ and } \phi \in L^2(\mathbb{R}), \]

where $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with mean $\mu$ and variance $\sigma^2$.

**Step 3.** To facilitate our argument in the next step, we digress to discuss a general property of space-time white noises.

Let $W^1$ denote a space-time white noise, and suppose that $\{W^2(\phi_n)\}$ is a family of Brownian motions indexed by a countable dense subset $\{\phi_n\}$ of $L^2(\mathbb{R})$ such that $\{W^1(\phi_n)\}$ and $\{W^2(\phi_n)\}$ have the same law as random elements taking values in $\prod_{n=1}^\infty C(\mathbb{R}_+, \mathbb{R})$. Then, whenever $(\phi_n)$ is a subsequence converging to some $\phi$ in $L^2(\mathbb{R})$, the linearity of $W^1$ gives

\[ \mathbb{E} \left[ \sup_{0 \leq s \leq T} \left| W_s^2(\phi_{nk}) - W_s^2(\phi_{nl}) \right|^2 \right] 
= \mathbb{E} \left[ \sup_{0 \leq s \leq T} \left| W_s^1(\phi_{nk} - \phi_{nl}) \right|^2 \right] 
\leq 4T\|\phi_{nk} - \phi_{nl}\|^2_{L^2(\mathbb{R})} \rightarrow 0 \text{ as } k, \ell \rightarrow \infty, \quad \forall T \in (0, \infty), \]
where the inequality follows from Doob’s \( L^2 \)-inequality and the fact that, for any \( \phi \in L^2(\mathbb{R}) \), \( W^1(\phi) \) is a Brownian motion with 
\[
\mathcal{L}(W^1(\phi)) = \mathcal{N}(0, \|\phi\|_2^2). 
\]

The convergence in (3.24) implies that, for some continuous process, say \( W^2(\phi) \), we have
\[
W^2(\phi_n) \to W^2(\phi) \quad \text{uniformly on } [0,T] \text{ a.s., } \forall T \in (0,\infty).
\]

The same holds with \( W^2 \) replaced by \( W^1 \). Hence, making comparisons with the reference space-time white noise \( W^1 \), we obtain an extension of the map \( \phi \mapsto W^2(\phi) \) to the entire space \( L^2(\mathbb{R}) \) such that \( \{W^2(\phi); \phi \in L^2(\mathbb{R})\} \) is a space-time white noise and, in fact, is uniquely defined by \( \{W^2(\phi_n)\} \).

\((\textbf{Step 4})\). In this step, we formalize the infinitesimal description outlined before by shrinking the mesh size of \( D_m \), that is, by passing \( m \to \infty \), and then work with the limiting objects. To use our observation in \((\textbf{Step 3})\), we pick a choice of a countable dense subset \( \{\phi_n\} \) of \( L^2(\mathbb{R}) \).

We have constructed in \((\textbf{Step 1})\) random elements \( X^m \) and \( Y^m \) and hence determined the laws
\[
\mathcal{L}(X,Y,W,(x_i),(y_i),X^m,Y^m), \quad m \in \mathbb{N},
\]
as probability measures on a countably infinite product of Polish spaces. More precisely, our choice of the Polish spaces is through the following identifications. We identify \( X \) as a random element taking values in the closed subset of \( D(\mathbb{R}_+, \mathcal{C}_{\text{rap}}(\mathbb{R})) \) consisting of paths having continuity over each interval \( [s_i, s_{i+1}) \) for \( i \in \mathbb{Z}_+ \), with a similar identification applied to \( Y \) (cf. Proposition 5.3 and Remark 5.4 of [6]). We identify each coordinate \( X^{m,i} \) of \( X^m \) as a random element taking values in \( C([s_i, \infty), \mathcal{C}_{\text{rap}}(\mathbb{R})) \), with a similar identification applied to \( Y^m \). By \((\textbf{Step 3})\), we identify \( W \) as the infinite-dimensional vectors \( (W(\phi_1),W(\phi_2),\ldots,\) whose coordinates are \( C(\mathbb{R}_+,\mathbb{R}) \)-valued random elements. Finally, the identifications of the infinite-dimensional vectors \( (x_i) \) and \( (y_i) \) are plain.

We make an observation for the sequence of laws in (3.25). Note that \( \mathcal{L}(X^m) \) does not depend on \( m \), because, by (3.6), any of its \( i \)-th coordinate \( X^{m,i} \) is a super-Brownian motion with starting measure \( \psi(1)J_x^i \) and started at \( s_i \) and the coordinates are independent. Similarly, \( \mathcal{L}(Y^m) \) does not depend on \( m \). This implies that the sequence of laws in (3.25) is tight in the space of probability measures on the above infinite product of Polish spaces. Hence, by taking a subsequence if necessary, we may assume that this sequence converges in distribution. By Skorokhod’s representation, we may assume the existence of the vectors of random elements in the following display as well as the almost-sure convergence therein:
\[
\left(\tilde{X}^{(m)},\tilde{Y}^{(m)},\{\tilde{x}_i^m\},\{\tilde{y}_i^m\},\tilde{W}^m,\tilde{X}^m,\tilde{Y}^m\right) \xrightarrow{\text{a.s.}} \left(\tilde{X},\tilde{Y},\{\tilde{x}_i\},\{\tilde{y}_i\},\tilde{W},\tilde{X},\tilde{Y}\right). \quad (3.26)
\]
Here,
\[
\mathcal{L}\left(\tilde{X}^{(m)},\tilde{Y}^{(m)},\{\tilde{x}_i^m\},\{\tilde{y}_i^m\},\tilde{W}^m,\tilde{X}^m,\tilde{Y}^m\right)
= \mathcal{L}(X,Y,(x_i),(y_i),W,X^m,Y^m), \quad \forall m \in \mathbb{N}.
\]

\((\textbf{Step 5})\). We take \( (\mathcal{G}_t) \) to be the minimal filtration satisfying the usual conditions to which the limiting objects
\[
\tilde{X},\tilde{Y},\tilde{W},\tilde{X},\tilde{Y}
\]
on the right-hand side of (3.26) are adapted. We will complete the proof in this step by verifying that, with an obvious adaptation of notation, all the limiting objects on the right-hand side of (3.26) along with the filtration \( (\mathcal{G}_t) \) are the required objects satisfying all of the conditions (i)–(v) of Theorem 3.2.

First, let us verify the easier properties (i) and (ii) for \( \{\tilde{X}^i\} \) and the analogues for \( \{\tilde{Y}^i\} \). The statement (i) and its analogue for \( \{\tilde{Y}^i\} \) obviously hold, by the analogous properties of \( \tilde{X}^m \) and \( \tilde{Y}^m \).
(See (b) of (3.6).) To verify the statement (ii), we use the property (3.7) possessed by \((\tilde{X}^m, \tilde{Y}^m)\) and then pass limit, as is legitimate because the infinite series in (3.7) are always finite sums on compact time intervals. Similarly, the analogue of (ii) holds for \((\tilde{Y}, \tilde{Y})\).

The statement (iii) holds by the property (a) of (3.6) satisfied by \(X^m\), except that we still need to verify that each \(\tilde{X}^i\) defines a \((\mathcal{G}_i)_{t \geq s}\)-super Brownian motion, not just a super-Brownian motion in itself. From this point on, our arguments will rely heavily on the continuity of the underlying objects and the fact that \(\bigcup_m D_m\) is dense in \([\tilde{s}, \infty)\). Let \(\frac{s}{2} < t < \infty\) with \(s, t \in \bigcup_m D_m\). Then \(s, t \in D_m\) from some large \(m\) on by the property of the sequence \(\{D_m\}\). For any bounded continuous function \(g\) on the path space of \((\tilde{X}^m, \tilde{Y}^m, \tilde{W}^m, \tilde{X}^m, \tilde{Y}^m)\) restricted to the time interval \([0, s]\), \(\phi \in C^+_c(\mathbb{R})\), and index \(i\) such that \(s_i \leq s\), the grid Markov property (3.21) entails that

\[
\mathbb{E}\left[ g\left( \tilde{X}^m, \tilde{Y}^m, \tilde{W}^m, \tilde{X}^m, \tilde{Y}^m \right) e^{-\langle \tilde{X}_i^m, \phi \rangle} \right] = \mathbb{E}\left[ g\left( \tilde{X}, \tilde{Y}, \tilde{W}, \tilde{X}, \tilde{Y} \right) \int \text{SBM}_{t-s} \left( \tilde{X}^m, \phi \right) e^{-(\nu, \phi)} \right].
\]

The formula of Laplace transforms of super-Brownian motion shows that the map

\[
f \mapsto \int \text{SBM}_{t-s} (f, \nu) e^{-(\nu, \phi)}
\]

has a natural extension to \(C^+_c(\mathbb{R})\) which is continuous. (Cf. Proposition II.5.10 of [18].) Hence, passing \(m \to \infty\) for both sides of (3.27) gives

\[
\mathbb{E}\left[ g\left( \tilde{X}, \tilde{Y}, \tilde{W}, \tilde{X}, \tilde{Y} \right) e^{-\langle \tilde{X}_i, \phi \rangle} \right] = \mathbb{E}\left[ g\left( \tilde{X}, \tilde{Y}, \tilde{W}, \tilde{X}, \tilde{Y} \right) \int \text{SBM}_{t-s} \left( \tilde{X}_s, \phi \right) e^{-(\nu, \phi)} \right].
\]

By the continuity of super-Brownian motion and the denseness of \(\bigcup_m D_m\) in \([\tilde{s}, \infty)\), the foregoing display implies that each coordinate \(\tilde{X}^i\) is truly a \((\mathcal{G}_i)_{t \geq s}\)-super-Brownian motion. A similar argument shows that each \(\tilde{Y}^i\) is a \((\mathcal{G}_i)_{t \geq t_i}\)-super-Brownian motion. We have proved the statement (iii) and its analogue for \(\tilde{Y}^i\) in (iv).

Next, we consider the statement (v). By definition,

\[
\mathcal{L}\left( \tilde{X}^m, \tilde{Y}^m, \{\tilde{x}_i\}, \{\tilde{y}_i\}, \tilde{W}^m \right) = \mathcal{L}\left( X, Y, \{x_i\}, \{y_i\}, W \right), \quad \forall m \in \mathbb{N},
\]

and hence this stationarity gives

\[
\mathcal{L}\left( \tilde{X}, \tilde{Y}, \{\tilde{x}_i\}, \{\tilde{y}_i\}, \tilde{W} \right) = \mathcal{L}\left( X, Y, \{x_i\}, \{y_i\}, W \right).
\]

Now, arguing as in the proof of (3.27) and using the grid Markov property (3.23) of \(\tilde{W}^m\) show that each \(\tilde{W}(\phi_n)\) is a \((\mathcal{G}_i)\)-Brownian motion with

\[
\mathcal{L}\left( \tilde{W}_1(\phi_n) \right) = \mathcal{N}\left( 0, \|\phi_n\|^2_{L^2(\mathbb{R})} \right).
\]

It follows from (3.29) and our discussion in (Step 3) that \(\tilde{W}\) extends uniquely to a \((\mathcal{G}_i)\)-space-time white noise. In addition, one more application of (3.29) shows that the defining equations (2.1) and (2.3) of \(X\) and \(Y\) by \(\{x_i, y_i\}\) and \(W\) carry over to the analogous equations for \(\tilde{X}\) and \(\tilde{Y}\) by \(\{\tilde{x}_i, \tilde{y}_i\}\) and \(\tilde{W}\), respectively. This proves that \((\tilde{X}, \tilde{Y}, \tilde{W})\) satisfies the analogous property described in (i) and (ii) of Theorem 2.1 with \((\mathcal{F}_i)\) replaced by \((\mathcal{G}_i)\). We have obtained the statement (v).
Finally, to obtain the independent landing property (3.2) in the statement (v), we recall that an analogous property is satisfied by \((\tilde{x}^m_i), (\tilde{y}^m_i), \tilde{X}^m, \tilde{Y}^m)\) in (3.19). Hence, arguing in the standard way as in the proof of (3.27) with the use of bounded continuous functions shows that the required independent landing property (3.2) is satisfied by \((\tilde{x}_i, (\tilde{y}_i), \tilde{X}, \tilde{Y})\). This verifies the statement (v) asserted in Theorem 3.2, and the proof is complete. 

\[ \frac{\alpha}{\beta} \]

Remark 3.3. In fact, we have

\[ x_i \in \mathcal{G}_{x_i}, \quad y_i \in \mathcal{G}_{y_i}, \quad \forall \ i \in \mathbb{N}. \tag{3.30} \]

The measurability of \(x_i\) in this statement follows, upon observing that \(\Delta X^i_s = \psi(1)J^i_s\) by (iii) of Theorem 3.2 and \(x_i\) is the center of the topological support of \(J^i_s\) (cf. (1.12)). The measurability of \(y_i\) in (3.30) follows from the similar reasons.

By (iii) of Theorem 3.2, each \(X^i_t\) is a \(\mathcal{G}_{t \geq s}\)-super-Brownian motion and each \(Y^i_t\) is a \(\mathcal{G}_{t \geq i}\)-super-Brownian motion. Hence, by a straightforward generalization of the standard argument of “appending” Brownian motions to solutions of martingale problems (cf. the proof of Theorem VII.2.7 in [20]), we can find, by enlarging the filtered probability space if necessary, two families of \(\mathcal{G}_{t \geq s}\)-white noises \(\{W^X\}\) and \(\{W^Y\}\) such that any of \((X^i_t, W^X)\) and \((Y^i_t, W^Y)\) solves the SPDE of super-Brownian motion with an appropriate translation of time. (See Theorem III.4.2 of [18] for details.) Moreover, by (i) of Theorem 3.2, we can further assume that each of the families \(\{W^X\}\) and \(\{W^Y\}\) consists of independent space-time white noises.

In the remaining of this section, we present a general discussion of covariations of two \(\mathcal{G}_t\)-space-time white noises, say, \(W^1\) and \(W^2\). For such a pair, we can find a random locally bounded signed measure \(\mu_{W^1, W^2}\) on \(\mathcal{F}(\mathbb{R}^2 \times \mathbb{R}_+)\), called the \textit{covariation} of \(W^1\) and \(W^2\), such that

\[
\int_{[0,t]} \int_{\mathbb{R}^2} \phi_1(x)\phi_2(y) d\mu_{W^1, W^2}(x, y, s) = \langle W^1(\phi_1), W^2(\phi_2) \rangle_t \quad \forall \ t \in \mathbb{R}_+ \ \text{a.s.,} \quad \forall \ \phi_1, \phi_2 \in L^2(\mathbb{R}).
\tag{3.31}
\]

Here, a locally bounded signed measure \(\mu\) is one such that \(\mu(\cdot \cap K)\) is a bounded signed measure for any compact subset \(K\). We will rely on the following analogue of the classical Kunita-Watanabe inequality to derive a simple, but important, property of covariations.

\textbf{Proposition 3.4 (Kunita-Watanabe).} Let \(W^1\) and \(W^2\) be two \(\mathcal{G}_t\)-space-time white noises. Then except outside a null set, the inequality

\[
\int_{\mathbb{R}_+} \int_{\mathbb{R}^2} |J(x, y, s)K(x, y, s)| \ |d\mu_{W^1, W^2}(x, y, s)|
\leq \left( \int_{\mathbb{R}_+} \int_{\mathbb{R}} J^2(x, x, s) dx ds \right)^{1/2} \left( \int_{\mathbb{R}_+} \int_{\mathbb{R}} K^2(x, x, s) dx ds \right)^{1/2}
\tag{3.32}
\]

holds for any pair of Borel measurable functions \(J\) and \(K\) on \(\mathbb{R}^2 \times \mathbb{R}_+\).

The proof of Proposition 3.4 is in the same spirit of the proof of the classical Kunita-Watanabe inequality for local martingales, and thus, is omitted here. (Cf. the proof of Proposition IV.1.15 of [20].)

The inequality (3.32) determinates in particular the “worst variation” of covariations, as is made precise in the inequality (3.33) below.
Corollary 3.5. Let $W^1, W^2$ be two $(\mathfrak{G}_t)$-space-time white noises. Then except outside a null event, the inequality

$$\int_{\mathbb{R}^+ \times \mathbb{R}^2} |K(x,y,s)| \, |d\mu_{W^1, W^2}(x,y,s)| \leq \int_{\mathbb{R}^+} \int_{\mathbb{R}} |K(x,x,s)| \, dx \, ds$$

(3.33)

holds for any Borel measurable function $K$ on $\mathbb{R}^2 \times \mathbb{R}^+$. In particular, for any $i,j_1,\cdots,j_n \in \mathbb{N}$ for $n \in \mathbb{N}$ with $j_1 < j_2 < \cdots < j_n$, except outside a null event, the inequality

$$\int_0^t H_s \mathbb{E} \left( X^i(1), \sum_{\ell=1}^n Y^{j_\ell}(1) \right) ds \leq \int_{s \vee t_{j_1}}^t |H_s| \int_\mathbb{R} \left( X^i(x,s) \cdot \sum_{\ell=1}^n Y^{j_\ell}(x,s) \right)^{1/2} dx \, ds, \quad \forall \, t \in [s \vee t_{j_1}, \infty),$$

(3.34)

holds for any locally bounded Borel measurable function $H$ on $\mathbb{R}^+$. Here, $\{X^i\}$ and $\{Y^j\}$ are obtained from the continuous decompositions of $X$ and $Y$, respectively, in Theorem 3.2.

**Proof.** The first assertion follows by writing $|K| = |K|^{1/2} \cdot |K|^{1/2}$ and then using (3.32). To obtain the second assertion, we first write

$$\sum_{\ell=1}^n Y^{j_\ell}(\phi) = \int_0^t \sum_{\ell=1}^n Y^{j_\ell}_x \left( \frac{\Delta}{2} \phi \right) \, ds + \sum_{\ell:1 \leq \ell \leq n \atop t_{j_\ell} \leq t} \psi(\mathbb{I}) J^{j_\ell} \phi$$

$$+ \sum_{\ell=1}^n \int_0^t \int_\mathbb{R} Y^{j_\ell}(x,s)^{1/2} \phi(x) dW^{Y^{j_\ell}}(x,s).$$

Recall that the space-time white noises $W^{Y^{j_1}}, \cdots, W^{Y^{j_n}}$ are independent. Hence, by enlarging the filtered probability space if necessary, we may assume the existence of a $(\mathfrak{G}_t)$-space-time white noise $W^{Y^{j_1}}, \cdots, W^{Y^{j_n}}$ such that

$$\sum_{\ell=1}^n \int_0^t \int_\mathbb{R} Y^{j_\ell}(x,s)^{1/2} \phi(x) dW^{Y^i}(x,s)$$

$$= \int_0^t \int_\mathbb{R} \left( \sum_{\ell=1}^n Y^{j_\ell}(x,s) \right)^{1/2} \phi(x) dW^{Y^{j_1}, \cdots, Y^{j_n}}(x,s).$$

From the last two displays and the analogue of the first one for $X^i$, we obtain

$$\mathbb{E} \left\langle X^i(1), \sum_{\ell=1}^n Y^{j_\ell}(1) \right\rangle_t - \mathbb{E} \left\langle X^i(1), \sum_{\ell=1}^n Y^{j_\ell}(1) \right\rangle_s$$

$$= \int_s^t \int_{\mathbb{R}^2} X^i(x,r)^{1/2} \left( \sum_{\ell=1}^n Y^{j_\ell}(y,r) \right)^{1/2} d\mu_{W^1, W^{Y^{j_1}}, \cdots, W^{Y^{j_n}}}(x,y,r),$$

$$\forall \, s, t \in [s \vee t_{j_1}, \infty) \text{ with } s < t.$$

The second assertion now follows from the foregoing equality and (3.33). The proof is complete. \qed
4. Conditional separation of approximating solutions

4.1. Basic results

We use the processes \( \{X^i; i \in \mathbb{N}\} \) and \( \{Y^i; i \in \mathbb{N}\} \) obtained from the continuous decompositions of \( X \) and \( Y \) in Theorem 3.2 and show conditional separation of the approximating solutions. More precisely, for any \( \varepsilon \in (0, [8\psi(1)]^{-1} \wedge 1] \), we condition on the event that the total mass of a generic cluster \( X^i \) hits 1, and then the conditional separation refers to the separation of the approximating solutions under

\[
\mathbb{Q}_\varepsilon^i(A) \equiv \mathbb{P}_\varepsilon \left( A \mid T_1^{X^i} < \infty \right). \tag{4.1}
\]

Here, the restriction \([8\psi(1)]^{-1}\) for \( \varepsilon \) is just to make sure that \( X^i(1) \) stays in \((0,1)\) initially, and we set

\[
T_x^H \triangleq \inf \{ t \geq 0; H_t(1) = x \} \tag{4.2}
\]

for any nonnegative two-parameter process \( H = (H(x,t); (x,t) \in \mathbb{R} \times \mathbb{R}_+) \). Our specific goal is to study the differences in the local growth rates of masses of \( X \) and \( Y \) over the “initial part” of the space-time support of \( X^i \). In the following, we prove a few basic results concerning \( \mathbb{Q}_\varepsilon^i \).

Let us first represent \( \mathbb{Q}_\varepsilon^i \) via its Radon-Nikodym derivative process with respect to \( \mathbb{P}_\varepsilon \). A standard calculation on scale functions of one-dimensional diffusions gives the following characterization of \( \mathbb{Q}_\varepsilon^i \).

**Lemma 4.1.** For any \( i \in \mathbb{N} \) and \( \varepsilon \in (0, [8\psi(1)]^{-1} \wedge 1] \),

\[
\mathbb{P}_\varepsilon (T_1^{X^i} < T_0^{X^i}) = \psi(1)\varepsilon, \tag{4.3}
\]

and

\[
\mathbb{Q}_\varepsilon^i(A) = \int_A \frac{X_i^i(1)^{T_1^{X^i}}}{\psi(1)\varepsilon} d\mathbb{P}_\varepsilon, \quad \forall A \in \mathcal{G}_t \text{ with } t \in [s_i, \infty). \tag{4.4}
\]

Some basic properties of the total mass processes \( X^i(1) \) and \( Y^j(1) \) for \( t_j > s_i \) under \( \mathbb{Q}_\varepsilon^i \) are stated in the following lemma.

**Lemma 4.2.** Fix \( i \in \mathbb{N} \) and \( \varepsilon \in (0, [8\psi(1)]^{-1} \wedge 1] \). Then we have the following.

1. \( X^i(1)^{T_1^{X^i}} \) under \( \mathbb{Q}_\varepsilon^i \) is a copy of \( \frac{1}{8\psi(1)} \text{BESQ}^4(4\psi(1)\varepsilon) \) started at \( s_i \) and stopped upon hitting 1.
2. For any \( j \in \mathbb{N} \) with \( t_j > s_i \), the process \( (Y^j(1))_{t \geq t_j} \) is a continuous \((\mathcal{G}_t)_{t \geq t_j}\)-semimartingale under \( \mathbb{Q}_\varepsilon^i \) with canonical decomposition

\[
Y^j_t(1) = \psi(1)\varepsilon + I^j_t + M^j_t, \quad t \in [t_j, \infty), \tag{4.5}
\]

where the finite variation process \( I^j \) satisfies

\[
I^j_t = \int_{t_j}^t \frac{1}{X_i^i(1)^{T_1^{X^i}}} d\langle X^i(1)^{T_1^{X^i}}, Y^j(1) \rangle, \tag{4.6}
\]

\[
|I^j_t| \leq \int_{t_j}^t 1_{[0,T_1^{X^i}]}(s) \frac{1}{X_i^i(1)^2} \int_R X^i(x,s)^{1/2} Y^j(x,s)^{1/2} dx ds, \tag{4.7}
\]

for \( t \in [t_j, \infty) \), and \( M^j \) is a true \((\mathcal{G}_t)_{t \geq t_j}\)-martingale under \( \mathbb{Q}_\varepsilon^i \).
(3) For any \( j \in \mathbb{N} \) with \( t_j > s_i \),
\[
X^j(1)[s_i, t_j], \ y_j, \ \text{and } Y^j(1)[t_j, \infty) \text{ are } \mathbb{P}_\varepsilon\text{-independent.} \tag{4.8}
\]

(4) For any \( j \in \mathbb{N} \),
\[
\mathbb{Q}_\varepsilon^j(|y_j - x| \in dx) = \mathbb{P}_\varepsilon(|y_j - x| \in dx), \ x \in \mathbb{R}, \ \mathbb{P}_\varepsilon(y_j \in dx) \leq \frac{\|\psi\|_\infty}{\psi(1)} dx, \ x \in \mathbb{R}. \tag{4.9}
\]

**Proof.** (1). The proof is omitted since it is a straightforward application of Girsanov’s theorem by using Lemma 4.1 and can be found in the proof of Lemma 4.1 of [14].

(2). The total mass process \((Y^j_t(1))_{t \geq t_j}\) for any \( j \in \mathbb{N} \) with \( t_j > s_i \) is a \((\mathcal{G}_t)_{t \geq t_j}\)-Feller process and hence a \((\mathcal{G}_t)_{t \geq t_j}\)-martingale. By Girsanov’s theorem (cf. Theorem VIII.1.4 of [19]), \((Y^j_t(1))_{t \geq t_j}\) for any \( j \in \mathbb{N} \) with \( t_j > s_i \) is a continuous \((\mathcal{G}_t)_{t \geq t_j}\)-semimartingale under \( \mathbb{Q}_\varepsilon^j \) with canonical decomposition, say, given by (4.5). Here, \((M^j_t)_{t \geq t_j}\) is a continuous \((\mathcal{G}_t)_{t \geq t_j}\)-local martingale under \( \mathbb{Q}_\varepsilon^j \) with quadratic variation process
\[
\langle M^j \rangle_t = \int_{t_j}^t Y^j_s(1) ds, \ t \in [t_j, \infty), \tag{4.10}
\]
and by Lemma 4.1 the finite variation process \((I^j_t)_{t \geq t_j}\) is given by (4.6). Applying (3.34) to (4.6), we obtain (4.7) at once.

For the martingale property of \( M^j \) under \( \mathbb{Q}_\varepsilon^j \), we note that the one dimensional marginals of \( Y^j_t(1) \) have \( p \)-th moments which are locally bounded on compacts, for any \( p \in (0, \infty) \). \( Y^j_t(1) \) under \( \mathbb{P}_\varepsilon \) is a Feller diffusion.) Applying this to (4.10) shows that \( \mathbb{E}^{\mathbb{Q}_\varepsilon^j} \langle \langle M^j \rangle_t \rangle < \infty \) for every \( t \in [t_j, \infty) \) and hence \( M^j \) is a true martingale under \( \mathbb{Q}_\varepsilon^j \).

(3). The assertion (4.8) is an immediate consequence of the independent landing property (3.2) and the Markov properties of \( X^i(1) \) and \( Y^j(1) \) (cf. Theorem 3.2 (iii) and (iv)).

(4). We consider (4.9). Recall that \( x_i \in \mathcal{G}_s \) and \( y_j \in \mathcal{G}_j \). If \( t_j > s_i \), then we obtain from (4.4) that
\[
\mathbb{Q}_\varepsilon^j(|y_j - x_i| \in dx) = \frac{1}{\psi(1)} \mathbb{E}^{\mathbb{P}_\varepsilon} \left[ X^j_t(1)^{Y^j_t}; |y_j - x_i| \in dx \right] = \mathbb{P}_\varepsilon(|y_j - x_i| \in dx), \tag{4.11}
\]
where the last equality follows from (4.8). If \( t_j < s_i \), then a similar argument applies (without using (4.4)) since \( X^j_t(1) = \psi(1) \varepsilon \). Hence, the equality in (4.9) holds. The inequality in (4.9) is obvious. The proof is complete. \( \square \)

### 4.2. Setup

In order to state precisely our quantifications of the local growth rates of \( X \) and \( Y \), we need several preliminary results which have similar counterparts in [14]. First, we choose in Proposition 4.3 below a \((\mathcal{G}_t)\)-stopping time \( \tau^i \) satisfying \( \tau^i > s_i \), so that within \([s_i, \tau^i]\) we can explicitly bound from below the growth rate of \( X^i(1) \). Since \( X \geq X^i \), this gives a lower bound for the local growth rate of \( X \) over the initial part of the space-time support of \( X^i \). Our objective is to study the local growth rate of \( Y \) within this part.
Proposition 4.3. For any \( \varepsilon \in (0, [8\psi(1)]^{-1} \wedge 1] \), parameter vector \((\eta, \alpha, L) \in (1, \infty) \times (0, \frac{1}{2}) \times (0, \infty) \), and \( i \in \mathbb{N} \), we define four \( \mathcal{G}_t \)-stopping times by

\[
\tau^{i,(1)} \triangleq \inf \left\{ t \geq s_i ; X^i_t(1)^{T^X_i} < \frac{(t-s_i)^\eta}{4} \right\} \wedge T^Y_i,
\]

\[
\tau^{i,(2)} \triangleq \inf \left\{ t \geq s_i ; \left| X^i_s(1)^{T^X_i} - \psi(1) \varepsilon - (t-s_i) \right| > L \left( \int_{s_i}^t X^i_s(1)^{T^X_i} \, ds \right)^\alpha \right\} \wedge T^Y_i,
\]

\[
\tau^{i,(3)} \triangleq \inf \left\{ t \geq s_i ; \sum_{j: s_i < t_j \leq t} Y^j_i(1) > 1 \right\},
\]

\[
\tau^i \triangleq \tau^{i,(1)} \wedge \tau^{i,(2)} \wedge \tau^{i,(3)} \wedge (s_i + 1).
\]

Then

\[
\forall \rho > 0 \exists \delta > 0 \text{ such that } \sup \left\{ Q^i_\varepsilon(\tau^i \leq s_i + \delta) ; i \in \mathbb{N}, \varepsilon \in \left(0, \frac{1}{8\psi(1)} \wedge 1\right) \right\} \leq \rho. \tag{4.12}
\]

See Section 3.8 of [4] for the proof of Proposition 4.3.

Let us explain the meanings of the parameters \( \eta, \alpha, L \) in this proposition. Since \( X^i(1) \) is a Feller diffusion under \( \mathbb{P}_\varepsilon \), a straightforward application of Girsanov’s theorem (cf. Theorem VIII.1.4 of [20]) shows that \( X^i(1)^{T^X_i} \) under \( Q^i_\varepsilon \) is a \( \frac{1}{4}\)-BESQ\(^4\) stopped upon hitting 1; see Lemma 4.1 of [14] for details. As a result, by the lower escape rate of BESQ\(^4\) (cf. Theorem 5.4.6 of [9]), the time \( \tau^{i,(1)} \) is strictly positive \( Q^i_\varepsilon \)-a.s. for any \( \eta \in (1, \infty) \). In particular, we may take the parameter \( \eta \) close to 1.

The definition of \( \tau^{i,(2)} \) involves the notion of improved modulus of continuity. We will take the parameter \( \alpha \) in the definition of \( \tau^{i,(2)} \) close to \( \frac{1}{4} \) and consider the local Hölder exponent of the martingale part of BESQ\(^4\) in terms of its quadratic variation. The parameter \( L \) bounds the associated local Hölder coefficient. Hence, we have the integral inequality

\[
\left| X^i_t(1)^{T^X_i} - \psi(1) \varepsilon \right| \leq (t-s_i) + L \left( \int_{s_i}^t X^i_s(1)^{T^X_i} \, ds \right)^\alpha \quad \forall t \in [s_i, \tau^i],
\]

\[Q^i_\varepsilon \text{-a.s., } \forall i \in \mathbb{N}, \varepsilon \in \left(0, [8\psi(1)]^{-1} \wedge 1\right),\tag{4.13}\]

by the choice of \( \tau^{i,(2)} \) in Proposition 4.3. The integral inequality (4.13) is reminiscent of the integral inequalities to which Gronwall’s lemma applies, and hence suggests an iteration argument if we aim to bound more explicitly the difference

\[
\left| X^i_t(1)^{T^X_i} - \psi(1) \varepsilon \right|.
\]

A general result for this is given by Lemma 4.4 below, and its proof can be found in Section 3.10 of [4].

Lemma 4.4 (Improved modulus of continuity). Let \( T \in (0, 1], a \in (0, \frac{1}{2}), \) and \( b, c \in \mathbb{R}_+ \). If \( f : [0, T] \to \mathbb{R} \) is a continuous function uniformly bounded by 1 and satisfying

\[
|f(t) - f(0)| \leq bt + c \left( \int_0^t |f(s)| \, ds \right)^a, \quad \forall t \in [0, T],
\]

\[4.14\]
then for any $\xi' \in (0, 1)$ and $N' \in \mathbb{N}$ satisfying
\[
\sum_{j=1}^{N'} a^j \leq \xi' < \sum_{j=1}^{N'+1} a^j,
\] (4.15)
we have
\[
|f(t) - f(0)| \leq \left[ (c^{1/\alpha} + 1) \sum_{j=1}^{N'} |f(0)|^a^j \right] t^a
\]
\[
\left[ b + (c^{1/\alpha} + 1) \sum_{j=1}^{N'} \left( \frac{b^j}{\beta} + c^{1/\alpha} + 1 \right) t^{\xi'}, \forall t \in [0, T].
\] (4.16)

Applying Lemma 4.4 to the random function
\[
t \mapsto X_1^i(1)^{T_1^{X_i}} : [s_i, \tau^i] \rightarrow \mathbb{R}
\]
we obtain from (4.13) that whenever $\xi \in (0, 1)$ and $N_0 \in \mathbb{N}$ satisfies
\[
\sum_{j=1}^{N_0} \alpha^j \leq \xi < \sum_{j=1}^{N_0+1} \alpha^j,
\] (4.17)
the following inequality holds:
\[
\left| X_1^i(1)^{T_1^{X_i}} - \psi(1) \varepsilon \right| \leq K_1^X [\psi(1) \varepsilon]^{\alpha N_0} (t - s_i)^\alpha + K_2^X (t - s_i)^\xi
\]
\[
\forall t \in [s_i, \tau^i], \forall i \in \mathbb{N}, \varepsilon \in \left( 0, [8\psi(1)]^{-1} \wedge 1 \right],
\] (4.18)
where the constants $K_1^X, K_2^X \geq 1$ depend only on $(\alpha, L, \xi, N_0)$. Moreover, since $\alpha$ is close to $\frac{1}{2}$, we can choose $N_0$ large in (4.17) to make $\xi$ close to 1, as is our intention in the sequel. Informally, we can interpret the foregoing inequality as the statement:
\[
t \mapsto X_1^i(1)^{T_1^{X_i}} \text{ is Hölder-1 continuous at } s_i \text{ from the right.}
\]
A similar derivation of the improved modulus of continuity of $Y^j(1)$ will appear in the proof of Lemma 4.13 below.

To use the support of $X^i$ within which we observe the local growth rate of $Y$, we take a parameter $\beta \in (0, \frac{1}{2})$, which is now close to $\frac{1}{2}$. We use this parameter to get a better control of the supports of $X_i$ and $Y_j$, and this means we use the parabola
\[
P_\beta^X(t) \triangleq \left\{ (x, s) \in \mathbb{R} \times [s_i, t] ; |x - x_i| \leq (\varepsilon^{1/2} + (s - s_i)^{\beta}) \right\}
\] (4.19)
to envelope the space-time support of $X_i^i[s_i, t]$, for $t \in (s_i, \infty)$, with a similar practice applied to other clusters $Y_j$. (See the speed of support propagation of super-Brownian motions in Theorem III.1.3 of [18].) More precisely, we can use the $(\theta_i)$-stopping time
\[
\sigma_\beta^X \triangleq \inf \left\{ s \geq s_i ; \supp(X_i^i) \not\subseteq [x_i - \varepsilon^{1/2} - (s - s_i)^{\beta}, x_i + \varepsilon^{1/2} + (s - s_i)^{\beta}] \right\}
\] (4.20)
as well as the analogous stopping times $\sigma_\beta^Y$ for $Y_j$ to identify the duration of the foregoing enveloping.
below), however, we will consider the super-Brownian motion for the support of \( Y \) is small for small \( t \).

![Figure 4.1](image)

**Fig 4.1.** Parabolas \( \mathcal{P}^{X_i}(t) \), \( \mathcal{P}^{Y_j}(t) \), \( \mathcal{P}^{Y_k}(t) \) and rectangles \( \mathcal{R}^{X_i}(t) \) and \( \mathcal{R}^{Y_j}(t) \), for \( 0 < \beta' < \beta \) and \( t \in [s_i, s_i + 1] \).

We now specify the clusters \( Y^j \) selected for computing the local growth rate of \( Y \). Suppose that at time \( t \) with \( t > s_i \), we can still envelope the support of \( X^i \) by \( \mathcal{P}^{X_i}(t) \) and the analogous enveloping for the support of \( Y^j \) holds for any \( j \in \mathbb{N} \) satisfying \( t_j \in (s_i, t] \). Informally, we can ignore the clusters \( Y^j \) born before \( X^i \), because the probability that they can invade the initial part of the support of \( X^i \) is small for small \( t \) (cf. Lemma 5.2). Under such circumstances, simple geometric arguments show that only the \( Y^j \) clusters born inside the space-time rectangle

\[
\mathcal{R}^{X_i}(t) \triangleq \left[ x_i - 2(\varepsilon^{1/2} + (t - s_i)\beta), x_i + 2(\varepsilon^{1/2} + (t - s_i)\beta) \right] \times [s_i, t]
\]

(4.21)

can invade the initial part of the support of \( X^i \) by time \( t \) (see Lemma 5.2). We remark that this choice of clusters \( Y^j \) for \( (y_j, t_j) \in \mathcal{R}^{X_i}(t) \) is also used in [14].

For technical reasons (cf. Section 4.5 below), however, we will consider the super-Brownian motions \( Y^j \) born inside the slightly larger rectangle \( \mathcal{R}^{X_i}(t) \) for \( t \in (s_i, s_i + 1] \), where \( \beta' \) is another value close to \( \beta \), has the same meaning as \( \beta \), and satisfies \( \beta' < \beta \). See Figure 3.2 for these rectangles as well as an example for three parabolas \( \mathcal{P}^{X_i}(t) \), \( \mathcal{P}^{Y_j}(t) \), and \( \mathcal{P}^{Y_k}(t) \) where \( (y_j, t_j) \in \mathcal{R}^{X_i}(t) \) and \( (y_k, t_k) \notin \mathcal{R}^{X_i}(t) \). The labels \( j \in \mathbb{N} \) of the clusters \( Y^j \) born inside \( \mathcal{R}^{X_i}(t) \) constitute the random index set

\[
\mathcal{J}^{X_i}(t) \triangleq \mathcal{J}^{X_i}_{\beta'}(t, t) \triangleq \mathcal{J}^{X_i}_{\beta'}(t, t') \triangleq \mathcal{J}^{X_i}_{\beta'}(t, t') = \left\{ j \in \mathbb{N} : |y_j - x_i| \leq 2(\varepsilon^{1/2} + (t - s_i)\beta'), s_i < t_j \leq t' \right\}, \quad \forall t, t' \in (s_i, \infty).
\]

(4.23)

**Assumption 4.5 (Choice of auxiliary parameters).** Throughout the remainder of this section and Section 5, we fix a parameter vector

\[
(\eta, \alpha, L, \beta, \beta', \xi, N_0) \in (1, \infty) \times \left( 0, \frac{1}{2} \right) \times (0, \infty) \times \left[ \frac{1}{3}, \frac{1}{2} \right] \times \left[ \frac{1}{3}, \frac{1}{2} \right] \times (0, 1) \times \mathbb{N}
\]

(4.24)
satisfying

\[
\begin{aligned}
(a) \quad & \sum_{j=1}^{N_0} \alpha^j \leq \xi < \sum_{j=1}^{N_0+1} \alpha^j, \\
(b) \quad & \alpha < \frac{\beta'}{\beta} < 1, \\
(c) \quad & \beta' - \frac{\eta}{2} + \frac{3\xi}{2} \alpha > 0, \\
(d) \quad & (\beta' + 1) \wedge \left(\beta' - \frac{\eta}{2} + \frac{3\xi}{2}\right) > \eta.
\end{aligned}
\]  

(Note that we restate (4.17) in (a).) We insist that the parameter vector in (4.24) is chosen to be independent of \( i \in \mathbb{N} \) and \( \varepsilon \in (0, [8\psi(1)]^{-1} \wedge 1] \). For example, we can choose these parameters in the following order: first choose \( \eta, \alpha, \beta', \xi \) according to (c) and (d), choose \( \beta \) according to (b), and finally choose \( N_0 \) according to (a) by enlarging \( \xi \) if necessary; the parameter \( L \), however, can be chosen arbitrarily.

The following theorem gives our quantification of the local growth rates of \( Y \) under \( Q^i_\varepsilon \).

**Theorem 4.6.** Under Assumption 4.5, set three strictly positive constants by

\[
\kappa_1 = (\beta' + 1) \wedge \left(\beta' - \frac{\eta}{2} + \frac{3\xi}{2}\right), \quad \kappa_2 = \frac{\alpha N_0}{4}, \quad \kappa_3 = \beta' - \frac{\eta}{2} + \frac{3\alpha}{2}.
\]

Then there exists a constant \( K^* \in (0, \infty) \), depending only on the parameter vector in (4.24) and the immigration function \( \psi \), such that for any \( \delta \in (0, \kappa_1 \wedge \kappa_3) \), the following uniform bound holds:

\[
\begin{aligned}
Q^i_\varepsilon \left( \exists s \in (s_i, t], \sum_{j \in J \setminus (s \wedge \tau^i \wedge \sigma^Y)} Y^j_\varepsilon(1)^{\tau^i \wedge \sigma^Y \wedge \sigma^Y} > K^* \left[(s - s_i)^{\kappa_1 - \delta} + \varepsilon^{\kappa_2} \cdot (s - s_i)^{\kappa_3 - \delta}\right] \right) \\
\leq \frac{2 \cdot 2^{\kappa_1 \wedge \kappa_3}}{2^{2(N+1)}(1 - 2^{-\delta})}, \quad \forall \ t \in \left[s_i + 2^{-N}, s_i + 2^{-N}, N \in \mathbb{Z}_+, \right. \\
\left. \quad \quad \quad \quad \left(s_i + 2^{-N}, s_i + 2^{-N}, i \in \mathbb{N}, \varepsilon \in \left(0, \frac{1}{8\psi(1)} \wedge 1\right), \right) \right.
\end{aligned}
\]

where the \((\cdot)_i\)-stopping times \( \tau^i \) are defined in Proposition 4.3

**Remark 4.7.** If we follow the aforementioned interpretation of the parameter vector in (4.24) that \((\eta, \beta', \xi)\) is close to \((1, \frac{3}{4}, 1)\), then \( \kappa_1 \) in (4.26) is close to \( \frac{3}{4} \). Informally, if we regard the stopping times \( \tau^i, \sigma^Y, \) and \( \sigma^Y \) as being bounded away from \( s_i \), then by the above reason for choosing the random index sets \( J^i_\varepsilon(\cdot) \) in (4.22), we can regard Theorem 4.6 as a formalization of the statement in (1.15). □

In fact, the proof of Theorem 4.6 is reduced to a study of some nonnegative \((\mathcal{G}_t)_{t \geq s_i}\)-submartingale dominating the process

\[
\sum_{j \in J \setminus (s \wedge \tau^i \wedge \sigma^Y)} Y^j_\varepsilon(1)^{\tau^i \wedge \sigma^Y \wedge \sigma^Y} \quad t \in [s_i, \infty),
\]

(4.28)
in (4.27), and the main task will be to prove Theorem 4.9 below. We explain the reductions as follows.

We observe that by Lemma 4.2 (2), the process in (4.28) is dominated by the nonnegative process

\[
\sum_{j \in \mathcal{J}^i(t, t \wedge \tau^i \wedge \sigma^X_i)} \left( \psi(1) \varepsilon + \int_{t_j}^{t \wedge \tau^i \wedge \sigma^X_i} \frac{1}{X^i(\varepsilon)} \int_{Y^i(x, s)}^{1/2} Y^j(x, s) \frac{1}{2} dx ds \right. \\
+ \left. M^j_{t \wedge \tau^i \wedge \sigma^X_i} \right), \quad t \in [s_i, \infty),
\]

(4.29)

under \( Q^i_\varepsilon \) for any \( i \in \mathbb{N} \) and \( \varepsilon \in (0, [\psi(1)]^{-1} \wedge 1) \). The process in (4.29) is in fact a nonnegative \((\mathcal{G}_t)_{t \geq s_i}\)-submartingale under \( Q^i_\varepsilon \), since for any \( j \in \mathbb{N} \) with \( s_i < t_j \), \( j \in \mathcal{J}^i(t, t \wedge \tau^i \wedge \sigma^X_i) \) if and only if the following \( \mathcal{G}_t \)-event occurs:

\[
\left[ |y_j - x_i| \leq 2(s^{1/2} + (t - s_i)^{\beta^i}) \right] \quad \text{and} \quad t_j \leq t \wedge \tau^i \wedge \sigma^X_i.
\]

(Recall Remark 3.3 for \( y_j \in \mathcal{G}_t \) and \( x_i \in \mathcal{G}_{s_i} \).) It suffices to prove the bound (4.27) of Theorem 4.6 with the involved process in (4.28) replaced by the nonnegative submartingale in (4.29). To further reduce the problem, we resort to the following simple corollary of Doob’s maximal inequality.

**Lemma 4.8.** Let \( F \) be a nonnegative function on \([0, 1]\) such that \( F|([0, 1]) > 0 \) and \( \sup_{s/t, 1 \leq s/t \leq 2} \frac{F(t)}{F(s)} < \infty \). In addition, assume that

\[
\text{for some } \delta > 0, \quad t \mapsto \frac{F(t)}{t^\delta} \text{ is increasing.} \quad (4.30)
\]

Suppose that \( Z \) is a nonnegative submartingale with càdlàg sample paths such that \( \mathbb{E}[Z_t] \leq F(t) \) for any \( t \in [0, 1] \). Then for every \( N \in \mathbb{Z}_+ \),

\[
\sup_{t \in [2^{-(N+1)}, 2^{-N}]} \mathbb{P} \left( \exists s \in (0, t], \ Z_s > \frac{F(s)}{s^\delta} \right) \leq \left( \sup_{s/t, 1 \leq s/t \leq 2} \frac{F(t)}{F(s)} \right) \times \frac{1}{2^{N(1 - 2^{-\delta})}}. \quad (4.31)
\]

**Proof.** For each \( m \in \mathbb{Z}_+ \),

\[
\mathbb{P} \left( \exists s \in \left[ 2^{-(m+1)}, 2^{-m} \right], \ Z_s \geq \frac{F(s)}{s^\delta} \right) \leq \mathbb{P} \left( \sup_{2^{-(m+1)} \leq s \leq 2^{-m}} \frac{Z_s}{s^\delta} \geq F \left( \frac{1}{2(m+1)} \right) / \frac{1}{2(m+1)^\delta} \right) \leq \frac{\mathbb{E}[Z_{1/2m}]}{F \left( \frac{1}{2(m+1)} \right) / \frac{1}{2(m+1)^\delta}} \leq \frac{F \left( \frac{1}{2m} \right)}{F \left( \frac{1}{2(m+1)} \right) / \frac{1}{2(m+1)^\delta}} = \sup_{s/t, 1 \leq s/t \leq 2} \frac{F(t)}{F(s)} \times \frac{1}{2^{(m+1)^\delta}},
\]

where the first inequality follows from (4.30) and the second inequality follows from Doob’s maximal
This completes the proof. \(\square\)

**Theorem 4.9.** Under Assumption 4.5, take the same constants \(\kappa_j\) as in Theorem 4.6. Then we can choose a constant \(K^* \in (0, \infty)\) as stated in Theorem 4.6, such that the following uniform bound holds:

\[
E^\mathbb{Q}_\varepsilon \left[ \sum_{j \in \mathcal{J}_\varepsilon^j(t,t \wedge \tau^i \wedge \sigma^X_j)} \left( \psi(1) \varepsilon + \int_{t_j}^{t \wedge \tau^i \wedge \sigma^X_j} \frac{1}{X^j_s(1)} \int_{\mathbb{R}} \frac{\chi_j}{X^j_s(x,s)^{1/2}} Y^j(x,s)^{1/2} dx ds \right) \right]
\leq K^* \left[ (t-s_i)^{\kappa_1} + \varepsilon^{\kappa_2} \cdot (t-s_i)^{\kappa_3} \right], \quad \forall \ t \in (s_i, s_i + 1], \ i \in \mathbb{N}, \ \varepsilon \in \left( 0, \frac{1}{8\psi(1)} \right] \wedge 1.
\]

Now, we prove the main result of this section, that is Theorem 4.6, assuming Theorem 4.9.

**Proof of Theorem 4.6.** In, and only in, this proof, we denote by \(Z^{(0)}\) the submartingale defined in (4.29).

Since \(j \in \mathcal{J}_\varepsilon^j(t,t \wedge \tau^i \wedge \sigma^X_j)\) \(\in \mathcal{G}_t\), we obtain immediately from Lemma 4.2 (2) that the part

\[
\sum_{j \in \mathcal{J}_\varepsilon^j(t,t \wedge \tau^i \wedge \sigma^X_j)} M^j_{t \wedge \tau^i \wedge \sigma^X_j}, \quad t \in [s_i, \infty),
\]
in the definition of \(Z^{(0)}\) is a true \(\mathbb{Q}_\varepsilon^j\)-martingale with mean zero, for any \(i \in \mathbb{N}\) and \(\varepsilon \in (0, [8\psi(1)]^{-1} \wedge 1)\). Hence, setting

\[
F^{(0)}(s) = K^* \left( s^{\kappa_1} + \varepsilon^{\kappa_2} \cdot s^{\kappa_3} \right), \quad s \in [0, 1],
\]
we see from Theorem 4.9 that

\[
E^\varepsilon[Z^{(0)}_t] \leq F^{(0)}(t-s_i)
\]
for any \(t \in (s_i, s_i + 1], \ i \in \mathbb{N}, \ \varepsilon \in (0, [8\psi(1)]^{-1} \wedge 1)\). Note that

\[
\sup_{s,t:1 \leq t \leq 2} \frac{F^{(0)}(t)}{F^{(0)}(s)} \leq \sup_{s,t:1 \leq t \leq 2} \left( \frac{t^{\kappa_1}}{s^{\kappa_1} + t^{\kappa_3}} \right) \leq 2 \cdot 2^{\kappa_1 / \kappa_3}.
\]

Hence, applying Lemma 4.8 with \((Z,F)\) taken to be \((Z^{(0)}, F^{(0)})\), we see that (4.27) with the involved process in (4.28) replaced by \(Z^{(0)}\) holds. The proof is complete. \(\square\)

The remainder of this section is to prove Theorem 4.9. For this purpose, we need to classify the clusters \(Y^j\) for \(j \in \mathcal{J}_\varepsilon^j(t,t \wedge \tau^i \wedge \sigma^X_j)\). Set

\[
C^i_j(t) \triangleq \{ j \in \mathbb{N} \mid |y_j - x_i| < 2 (\varepsilon^{1/2} + (t_j - s_i)^{\beta_j}), s_i < t_j \leq t \}
\]

\[
L^i_j(t,t') \triangleq \{ j \in \mathbb{N} \mid 2 (\varepsilon^{1/2} + (t_j - s_i)^{\beta_j}) \leq |y_j - x_i| \leq 2 (\varepsilon^{1/2} + (t - s_i)^{\beta_j}), s_i < t_j \leq t' \},
\]

inequality. Hence, whenever \(t \in [2^{-(N+1)}, 2^{-N}]\) for \(N \in \mathbb{Z}_+\), the last inequality gives

\[
P \left( \exists s \in (0,t), \ Z_s > \frac{F(s)}{s^\delta} \right) \leq \sum_{m=N}^{\infty} P \left( \exists s \in \left[2^{-(m+1)}, 2^{-m}\right], \ Z_s > \frac{F(s)}{s^\delta} \right)
\]

\[
\leq \left( \sum_{s,t:1 \leq t \leq 2} \frac{F(t)}{F(s)} \right) \sum_{m=N}^{\infty} \frac{1}{2^{(m+1)\delta}}
\]

\[
= \left( \sum_{s,t:1 \leq t \leq 2} \frac{F(t)}{F(s)} \right) \times \frac{1}{2^{(N+1)\delta}(1 - 2^{-\delta})}.
\]

This completes the proof. \(\square\)
for $t', t \in (s_i, \infty)$ with $t \geq t'$. Hence, as far as the clusters $Y^j$ born inside the rectangle $\mathcal{R}^{X^j}_{\beta'}(t)$ are concerned, the clusters $X^j, j \in C^j_{\beta'}(t)$, are those born inside the double parabola

$$
\left\{(x, s) \in \mathbb{R} \times [s_i, t]; |x - x_i| < 2\left(\varepsilon^{1/2} + (s - s_i)^{\beta'}\right)\right\}
$$

(the light grey area in Figure 3.3), and the clusters $Y^j, j \in C^j_{\beta'}(t, t')$, are those born outside (the dark grey area in Figure 3.3). For any $i \in \mathbb{N}$, we say a cluster $Y^j$ is a **critical cluster** if $j \in C^j_{\beta'}(t)$ and a **lateral cluster** if $j \in C^j_{\beta'}(t, t')$ for some $t, t'$.

Since $\left\{C^j_{\beta'}(t), C^j_{\beta'}(t, t')\right\}$ is a cover of $\mathcal{J}^j_{\beta'}(t, t')$ by disjoint sets, Theorem 4.9 can be obtained by the following two lemmas.

**Lemma 4.10.** Let $\kappa_i$ be as in Theorem 4.6. We can choose a constant $K^* \in (0, \infty)$ as in Theorem 4.6 such that the following uniform bound holds:

$$
\mathbb{E}^{Q_{\varepsilon}} \left[ \sum_{j \in C^i_{\beta'}(t \wedge t', \sigma_{\beta'}^X)} \left( \psi(1) \varepsilon + \int_{t_j}^{t \wedge t' \wedge \sigma_{\beta'}^X} \frac{1}{X^i_s(1)} \int_{\mathbb{R}} X^i(x, s)^{1/2} Y^j(x, s)^{1/2} dx ds \right) \right] 
\leq \frac{K^*}{2} \left[ (t - s_i)^{\kappa_i} + \varepsilon^{\kappa_2} \cdot (t - s_i)^{\kappa_3} \right],
\forall t \in (s_i, s_i + 1], i \in \mathbb{N}, \varepsilon \in \left(0, \frac{1}{8\psi(1)} \wedge 1\right].
$$

(4.32)

**Lemma 4.11.** Let $\kappa_i$ be as in Theorem 4.6. By enlarging the constant $K^*$ in Lemma 4.10 if necessary, the following uniform bound holds:

$$
\mathbb{E}^{Q_{\varepsilon}} \left[ \sum_{j \in C^i_{\beta'}(t \wedge t', \sigma_{\beta'}^X)} \left( \psi(1) \varepsilon + \int_{t_j}^{t \wedge t' \wedge \sigma_{\beta'}^X} \frac{1}{X^i_s(1)} \int_{\mathbb{R}} X^i(x, s)^{1/2} Y^j(x, s)^{1/2} dx ds \right) \right] 
\leq \frac{K^*}{2} \left[ (t - s_i)^{\kappa_i} + \varepsilon^{\kappa_2} \cdot (t - s_i)^{\kappa_3} \right],
\forall t \in (s_i, s_i + 1], i \in \mathbb{N}, \varepsilon \in \left(0, \frac{1}{8\psi(1)} \wedge 1\right].
$$

(4.33)
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\[ \leq \frac{K^*}{2} \left[ (t - s_i)^{\kappa_1} + \varepsilon^{\kappa_2} \cdot (t - s_i)^{\kappa_3} \right], \]
\[ \forall t \in (s_i, s_i + 1], \quad i \in \mathbb{N}, \quad \varepsilon \in \left( 0, \frac{1}{8\psi(1)} \wedge 1 \right). \quad (4.33) \]

Despite some technical details, the methods of proof for Lemma 4.10 and Lemma 4.11 are very similar. For clarity, they are given in Section 4.4 and Section 4.5 separately, with some preliminaries set in Section 4.3 below.

4.3. Auxiliary results and notation

For each \( z, \delta \in \mathbb{R}_+ \), let \((Z, \mathbb{P}_z^\delta)\) denote a copy of \( \frac{1}{4} \text{BESQ}^4(4z) \). We assume that \((Z, \mathbb{P}_z^\delta)\) is defined by a \((\mathcal{H}_t)\)-Brownian motion \( B \), where \((\mathcal{H}_t)\) satisfies the usual conditions. This means that

\[ Z_t = z + \delta t + \int_0^t \sqrt{Z_s} dB_s, \quad \mathbb{P}_z^\delta \text{-a.s.} \]

(Cf. Section XI.1 of [20] for Bessel squared processes.) As we will often investigate \( Z \) before it hits a constant level, we set the following notation similar to (4.2): for any real-valued process \( H = (H_t) \)

\[ T_x^H = \inf \{ t \geq 0; H_t = x \}, \quad x \in \mathbb{R}. \]

For \( \delta = 0 \), \((Z, \mathbb{P}_z^0)\) gives a Feller diffusion and its marginals are characterized by

\[ \mathbb{E}^{P_z^0} \left[ \exp \left( -\lambda Z_t \right) \right] = \exp \left( -\frac{2\lambda z}{2 + \lambda t} \right), \quad \lambda, t \in \mathbb{R}_+. \]

In particular, the survival probability of \((Z, \mathbb{P}_z^0)\) is given by

\[ \mathbb{P}_z^0(Z_t > 0) = \lim_{\lambda \to \infty} \left( 1 - \mathbb{E}^{P_z^0} \left[ \exp \left( -\lambda Z_t \right) \right] \right) = 1 - \exp \left( -\frac{2z}{t} \right), \quad z, t \in (0, \infty). \quad (4.34) \]

Using the elementary inequality \( 1 - e^{-x} \leq x \) for \( x \in \mathbb{R}_+ \), we obtain from the last inequality that

\[ \mathbb{P}_z^0(Z_t > 0) \leq \frac{2z}{t}, \quad z, t \in (0, \infty). \quad (4.35) \]

To save notation in the following Section 4.4 and Section 4.5, we write \( A \leq B \) if \( A \leq CB \) for some constant \( C \in (0, \infty) \) which may vary from line to line but depends only on \( \psi \) and the parameter vector chosen in Assumption 4.5.

4.4. Proof of Lemma 4.10

Fix \( i \in \mathbb{N} \) and \( \varepsilon \in \left( 0, \frac{1}{8\psi(1)} \wedge 1 \right] \), and henceforth we drop the subscripts \( \varepsilon \) of \( \mathbb{P}_z \) and \( \mathbb{Q}_z^\varepsilon \). In addition, we may only consider \( t \in \left[ s_i, s_i + \frac{\varepsilon}{2} \right] \) as there are no immigrants for \( Y \) arriving in \([s_i, s_i + \frac{\varepsilon}{2}]\). We do our analysis according to the following steps.
(Step 1). We start with the simplification:

\[
\sum_{j \in \mathcal{C}_p(t \land \tau^i \land \sigma^j)} \left( \psi(1) \varepsilon + \int_{t_j}^{t \land \tau^i \land \sigma^i} \frac{1}{X^i_s(1)} \int_{\mathbb{R}} X^i(x, s)^{1/2} Y^j(x, s)^{1/2} dx ds \right)
\]

\[
\leq \sum_{j \in \mathcal{C}_p(t \land \tau^i)} \left( \psi(1) \varepsilon + \int_{t_j}^{t \land \tau^i} \frac{1}{X^i_s(1)^{1/2}} [Y^j_s(1)]^{1/2} ds \right)
\]

\[
\leq \sum_{j \in \mathcal{C}_p(t \land \tau^i)} \left( \psi(1) \varepsilon + \int_{t_j}^{t \land \tau^i} \frac{2}{(s - s_i)^{\eta/2}} [Y^j_s(1)]^{1/2} ds \right), \tag{4.36}
\]

where the first inequality follows from the Cauchy-Schwartz inequality and the second one follows by using the component \(\tau^i, (1)\) of \(\tau^i\) in Proposition 4.3.

Now, we claim that

\[
\mathbb{E}^Q \left[ \sum_{j \in \mathcal{C}_p(t \land \tau^i \land \sigma^j)} \left( \psi(1) \varepsilon + \int_{t_j}^{t \land \tau^i \land \sigma^i} \frac{1}{X^i_s(1)} \int_{\mathbb{R}} X^i(x, s)^{1/2} Y^j(x, s)^{1/2} dx ds \right) \right]
\]

\[
\leq \sum_{j : s_i < t_j \leq t} (t_j - s_i)^{\beta'} \varepsilon + \sum_{j : s_i < t_j \leq t} \int_{t_j}^{t} ds \frac{1}{(s - s_i)^{\eta/2}} \mathbb{E}^Q \left[ [Y^j_s(1)]^{1/2} ; s < \tau^i, |y_j - x_i| < 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}) \right]. \tag{4.37}
\]

Note that

\[
\mathbb{E}^Q \left[ \psi(1) \varepsilon \# \mathcal{C}_p(t \land \tau^i) \right] \leq \varepsilon \mathbb{E}^Q \left[ \# \mathcal{C}_p(t) \right]
\]

\[
= \varepsilon \sum_{j : s_i < t_j \leq t} \mathbb{Q}^i \left( |y_j - x_i| < 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}) \right)
\]

\[
\leq \sum_{j : s_i < t_j \leq t} 4(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}) \varepsilon
\]

\[
\leq \sum_{j : s_i < t_j \leq t} (t_j - s_i)^{\beta'} \varepsilon, \tag{4.38}
\]

where the second \(\leq\)-inequality follows from Lemma 4.2 (4), and the last \(\leq\)-inequality follows since

\[
\varepsilon^{1/2} \leq \varepsilon^{\beta'} \leq 2^{\beta'} (t_j - s_i)^{\beta'}, \quad \forall j \in \mathbb{N} \text{ with } t_j > s_i. \tag{4.39}
\]

Our claim (4.37) now follows from (4.36) and (4.38).

From the display (4.37), we see the necessity to obtain the order of

\[
\mathbb{E}^Q \left[ [Y^j_s(1)]^{1/2} ; s < \tau^i, |y_j - x_i| < 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}) \right],
\]

\[
s \in (t_j, t], \ s_i < t_j < t, \tag{4.40}
\]

in \(s_i, t_j, s, t\).

We subdivide our analysis of a generic term in (4.40) into the following (Step 2-1)–(Step 2-4), with a summary given in (Step 2-5).
We convert the $\mathbb{Q}^i$-expectations in (4.40) to $\mathbb{P}$-expectations. Recalling that $x_i, y_j \in \mathcal{G}_i$, (cf. Remark 3.3), we can use Lemma 4.1 to get
\[
\mathbb{E}^{\mathbb{Q}^i} \left[ \left[ Y^i_j \right]^{1/2} ; s < \tau^i, |y_j - x_i| < 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}) \right] < \frac{1}{\psi(1) \varepsilon} \mathbb{E}^{\mathbb{P}} \left[ X^i_j (1)^{T^i X^i} ; s < \tau^i, |y_j - x_i| < 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}) \right].
\]

We break the $\mathbb{P}$-expectation in (4.41) into finer pieces by considering the following. For $s > t_j$, $X^i(1)^{T^i X^i}$ is nonzero on the union of the two disjoint events:
\[
\left[ X^i_j (1)^{T^i X^i} > 0, T^i_0 \leq t_j \right] = \left[ T^i_1 < T^i_0 \leq t_j \right]
\]
and
\[
\left[ X^i_j (1)^{T^i X^i} > 0, t_j < T^i_0 \right].
\]

Here, the equality in (4.42) holds $\mathbb{P}$-a.s. since $0$ is an absorbing state of $X^i(1)$ under $\mathbb{P}$. In fact, $X^i(1)^{T^i X^i} = 1$ on the event in (4.42). To invoke the additional order provided by the improved modulus of continuity of $X^i(1)$ at its starting point $s_i$, we use the trivial inequality
\[
X^i_j (1)^{T^i X^i} \leq \left| X^i_j (1)^{T^i X^i} - \psi(1) \varepsilon \right| + \psi(1) \varepsilon
\]
on the event (4.43).

Putting things together, we see from (4.41) that
\[
\mathbb{E}^{\mathbb{Q}^i} \left[ \left[ Y^i_j \right]^{1/2} ; s < \tau^i, |y_j - x_i| < 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}) \right] \leq \frac{1}{\psi(1) \varepsilon} \mathbb{E}^{\mathbb{P}} \left[ \left[ Y^i_j \right]^{1/2} ; s \leq T^i_1 \right] + \frac{1}{\psi(1) \varepsilon} \mathbb{E}^{\mathbb{P}} \left[ \left| X^i_j (1)^{T^i X^i} - \psi(1) \varepsilon \right| \left[ Y^i_j \right]^{1/2} ; s < \tau^i, |y_j - x_i| < 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}) \right] + \frac{1}{\psi(1) \varepsilon} \mathbb{E}^{\mathbb{P}} \left[ \left[ Y^i_j \right]^{1/2} ; s \leq T^i_1 \right] + \frac{1}{\psi(1) \varepsilon} \mathbb{E}^{\mathbb{P}} [ |y_j - x_i| < 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}) ; t_j < T^i_0 \right],
\]
where for the first and the third terms on the right-hand side, it is legitimate to replace the event $[s < \tau^i]$ by the larger one $[s \leq T^i_1 \varepsilon]$ since, in Proposition 4.3, $\tau^i(3)$ is a component of $\tau^i$, and for the third term we replace the event in (4.43) by the larger one $[t_j < T^i_0 \varepsilon]$.

In (Step 2-2)–(Step 2-4) below, we derive a bound for each of the three terms in (4.44) which involves only Feller’s diffusion. We use the notation in Section 4.3.

(Step 2-1). Consider the first term on the right-hand side of (4.44), and recall the notation in
Section 4.3. It follows from (4.8) and (4.9) that
\[
\frac{1}{\psi(\varepsilon)} \mathbb{E}^p \left[ \left| Y_s^i(\mathbb{1}) \right|^{1/2} : s \leq T^Y_i, \right.
\]
\[
|y_j - x_i| < 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta^*}), T^X_i < T^0 \leq t_j
\]
\[
\leq \frac{1}{\varepsilon} \mathbb{P} \left( T^X_i < T^0 \leq t_j \right) \left( \varepsilon^{1/2} + (t_j - s_i)^{\beta^*} \right) \mathbb{E}^p_{\psi(\varepsilon)} \left[ (Z_{s-t_j})^{1/2} : s - t_j \leq T^Z_i \right]
\]
\[
\leq \frac{1}{\varepsilon} \mathbb{P} \left( T^X_i < T^0 \leq t_j \right) \left( \varepsilon^{1/2} + (t_j - s_i)^{\beta^*} \right) \mathbb{E}^p_{\psi(\varepsilon)} \left[ (Z_{s-t_j})^{1/2} : s - t_j \leq T^Z_i \right]
\]
\[
\leq (t_j - s_i)^{\beta^*} \mathbb{E}^p_{\psi(\varepsilon)} \left[ (Z_{s-t_j})^{1/2} : s - t_j \leq T^Z_i \right], \quad \forall \ s \in (t_j, t], \ s_i < t_j < t, \quad (4.45)
\]
where the last inequality follows from (4.39) and Lemma 4.1.

(Step 2-3). Let us deal with the second term in (4.44). We claim that
\[
\frac{1}{\psi(\varepsilon)} \mathbb{E}^p \left[ X^i(\mathbb{1})^{T^X_i} - \psi(\varepsilon) \right] \left| Y_s^i(\mathbb{1}) \right|^{1/2} : s \leq \tau^i,
\]
\[
|y_j - x_i| < 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta^*}), X^i(\mathbb{1})^{T^X_i} > 0, t_j < T^X_i
\]
\[
\leq \mathbb{E}^p_{\psi(\varepsilon)} \left[ (Z_{s-t_j})^{1/2} : s - t_j \leq T^Z_i \right], \quad \forall \ s \in (t_j, t], \ s_i < t_j < t. \quad (4.46)
\]
Fix such s throughout (Step 2-3).

First, let us transfer the improved modulus of \( X^i(\mathbb{1}) \) under \( Q^i \) to one under \( \mathbb{P} \). It follows from (4.18) that on \( [s < \tau^i, X^i(\mathbb{1})^{T^X_i} > 0] \) \( \mathcal{G}^i \), we have
\[
\left| X^i(\mathbb{1})^{T^X_i} - \psi(\varepsilon) \right| \leq K_1^X |\psi(\varepsilon)|^{\alpha N_0} (s - s_i)^{\alpha} + K_2^X (s - s_i)^{\xi}, \quad \mathbb{Q}^i\text{-a.s.}
\]
and hence
\[
0 = \mathbb{Q}^i \left( X^i(\mathbb{1})^{T^X_i} - \psi(\varepsilon) \right) > K_1^X |\psi(\varepsilon)|^{\alpha N_0} (s - s_i)^{\alpha} + K_2^X (s - s_i)^{\xi}, \quad s < \tau^i, X^i(\mathbb{1})^{T^X_i} > 0
\]
\[
\frac{1}{\psi(\varepsilon)} \mathbb{E}^p \left[ X^i(\mathbb{1})^{T^X_i} : |X^i(\mathbb{1})^{T^X_i} - \psi(\varepsilon)| < K_1^X |\psi(\varepsilon)|^{\alpha N_0} (s - s_i)^{\alpha} + K_2^X (s - s_i)^{\xi}, s < \tau^i, X^i(\mathbb{1})^{T^X_i} > 0 \right], \quad (4.47)
\]
where the last equality follows from Lemma 4.1 since the event evaluated under \( Q^i \) is a \( \mathcal{G}^i \)-event. Using the restriction \( X^i(\mathbb{1})^{T^X_i} > 0 \), we see that the equality (4.47) implies
\[
\left| X^i(\mathbb{1})^{T^X_i} - \psi(\varepsilon) \right| \leq K_1^X |\psi(\varepsilon)|^{\alpha N_0} (s - s_i)^{\alpha} + K_2^X (s - s_i)^{\xi}, \quad \mathbb{P}\text{-a.s. on } [s < \tau^i, X^i(\mathbb{1})^{T^X_i} > 0]. \quad (4.48)
\]
Now, using (4.48) gives
\[
\frac{1}{\psi(1)\varepsilon} E^\varepsilon \left[ X^i_s(1)^{T_s^X} - \psi(1)\varepsilon \right] \left[ Y^i_s'(1) \right]^{1/2}; s < r^i, \\
|y_j - x_i| < 2(\varepsilon^{1/2} + (t_j - s_i)\beta'), X^i_s(1)^{T_s^X} > 0, t_j < T_0^X, \\
\leq \frac{\varepsilon^{1/2}(s - s_i)^{\alpha} + (s - s_i)\varepsilon^{1/2}}{\varepsilon} E^\varepsilon \left[ Y^i_s'(1) \right]^{1/2}; s < T_1, \\
|y_j - x_i| < 2(\varepsilon^{1/2} + (t_j - s_i)\beta'), t_j < T_0^X, \\
(4.49)
\]
where in the last inequality we use the component \(\tau^i(3)\) of \(\tau^i\) in Proposition 4.3 and discard the event \([X^i_s(1)^{T_s^X}] > 0\). Applying (4.8) and (4.9) to (4.49) gives
\[
\frac{1}{\psi(1)\varepsilon} E^\varepsilon \left[ X^i_s(1)^{T_s^X} - \psi(1)\varepsilon \right] \left[ Y^i_s'(1) \right]^{1/2}; s < r^i, \\
|y_j - x_i| < 2(\varepsilon^{1/2} + (t_j - s_i)\beta'), X^i_s(1)^{T_s^X} > 0, t_j < T_0^X, \\
\leq \frac{\varepsilon^{1/2}(s - s_i)^{\alpha} + (s - s_i)\varepsilon^{1/2}}{\varepsilon} \left( \varepsilon^{1/2} + (t_j - s_i)\beta' \right) P \left( t_j < T_0^X \right), \\
E^\varepsilon \left[ (Z_{s-t})^{1/2}; s - t_j \leq T_1^2 \right], \\
(4.50)
\]
We have
\[
P \left( t_j < T_0^X \right) \leq \frac{2\psi(1)\varepsilon}{t_j - s_i} \\
(4.51)
\]
by (4.35). Applying the last display and (4.39) to the right-hand side of (4.50) then gives the desired inequality (4.46).

**Step 2-4.** For the third term in (4.44), the arguments (Step 2-3) (cf. (4.49) and (4.50)) readily give
\[
\frac{1}{\psi(1)\varepsilon} \psi(1)\varepsilon E^\varepsilon \left[ Y^i_s'(1) \right]^{1/2}; s \leq T_1, |y_j - x_i| \leq 2(\varepsilon^{1/2} + (t_j - s_i)\beta'), t_j < T_0^X, \\
\leq (t_j - s_i)\beta' - 1 \varepsilon E^\varepsilon \left[ (Z_{s-t})^{1/2}; s - t_j \leq T_1^2 \right], \\
\forall s \in (t_j, t], s_i < t_j < t. \\
(4.52)
\]

**Step 2-5.** We note that in (4.45), (4.46), and (4.52), there is a common fractional moment, or more precisely
\[
E^\varepsilon \left[ (Z_{s-t})^{1/2}; s - t_j \leq T_1^2 \right], \\
(4.53)
\]
left to be estimated, as will be done in this step.

Recall the filtration \(\mathcal{H}_t\) defined in Section 4.3.

**Lemma 4.12.** Fix \(z, T \in (0, \infty)\). Under the conditional probability measure \(P_z(T)\) defined by
\[
P_z(T)(A) \triangleq P^0_z(A|Z_T > 0), \quad A \in \mathcal{H}_t, \\
(4.54)
\]
the process \((Z_t)_{0 \leq t \leq T}\) is a continuous \((\mathcal{H}_t)\)-semimartingale with canonical decomposition
\[ Z_t = z + \int_0^t F \left( \frac{2Z_s}{T-s} \right) ds + M_t, \quad 0 \leq t \leq T. \] (4.55)
Here, \(F : \mathbb{R}_+ \to \mathbb{R}_+\) defined by
\[ F(x) \triangleq \begin{cases} \frac{e^{-x}}{1-e^{-x}}, & x > 0, \\ 1, & x = 0, \end{cases} \] (4.56)
is continuous and decreasing, and \(M\) is a continuous \((\mathcal{H}_t)\)-martingale under \(\mathbf{P}_z^{(T)}\) with quadratic variation \((M)_t \equiv \int_0^t Z_s ds\).

**Proof.** The proof of this lemma is a standard application of Girsanov’s theorem (cf. Theorem VIII.1.4 of [20]), and we proceed as follows.

First, let \((D_t)_{0 \leq t \leq T}\) denote the \((\mathcal{H}_t, \mathbf{P}_z^0)\)-martingale associated with the Radon-Nikodym derivative of \(\mathbf{P}_z^{(T)}\) with respect to \(\mathbf{P}_z^0\), that is,
\[ D_t \equiv \frac{\mathbf{P}_z^0(Z_T > 0 | \mathcal{H}_t)}{\mathbf{P}_z^0(Z_T > 0)}, \quad 0 \leq t \leq T. \] (4.57)
To obtain the explicit form of \(D\) under \(\mathbf{P}_z^0\), we first note that the \((\mathcal{H}_t, \mathbf{P}_z^0)\)-Markov property of \(Z\) and (4.34) imply
\[ \mathbf{P}_z^0(Z_T > 0 | \mathcal{H}_t) = \mathbf{P}_z^0(Z_{T-t} > 0) = 1 - \exp \left( -\frac{2Z_t}{T-t} \right), \quad 0 \leq t < T. \] (4.58)
Hence, it follows from Itô’s formula and the foregoing display that, under \(\mathbf{P}_z^0\),
\[ D_t = \frac{1}{\mathbf{P}_z^0(Z_T > 0)} \left[ 1 - \exp \left( -\frac{2Z_t}{T} \right) \right] \]
\[ + \frac{1}{\mathbf{P}_z^0(Z_T > 0)} \int_0^t \exp \left( -\frac{2Z_s}{T-s} \right) \cdot \left( \frac{2}{T-s} \right) \sqrt{Z_s} dB_s, \quad 0 \leq t < T. \] (4.59)
We now apply Girsanov’s theorem and verify that the components of the canonical decomposition of \((Z_t)_{0 \leq t \leq T}\) under \(\mathbf{P}_z^{(T)}\) satisfy the asserted properties. Under \(\mathbf{P}_z^{(T)}\), we have
\[ Z_t = z + \int_0^t D_s^{-1} d\langle D, Z\rangle_s + M_t, \quad 0 \leq t \leq T, \]
Here,
\[ M_t = \int_0^t \sqrt{Z_s} dB_s - \int_0^t D_s^{-1} d\langle D, Z\rangle_s, \quad 0 \leq t \leq T \]
is a continuous \((\mathcal{H}_t, \mathbf{P}_z^{(T)})\)-local martingale with the asserted quadratic variation \((M)_t \equiv \int_0^t Z_s ds\), which implies that \(M\) is a true martingale under \(\mathbf{P}_z^{(T)}\). In addition, it follows from (4.58) and (4.59) that the finite variation process of \(Z\) under \(\mathbf{P}_z^{(T)}\) is given by
\[ \int_0^t D_s^{-1} d\langle D, Z\rangle_s = \int_0^t \frac{1}{\mathbf{P}_z^0(Z_T > 0 | \mathcal{H}_s)} d\mathbf{P}_z^0(Z_T > 0)D, Z \]
\[ = \int_0^t \exp \left( -\frac{2Z_s}{T-s} \right) \frac{2Z_s}{T-s} ds \]
\[ = \int_0^t F \left( \frac{2Z_s}{T-s} \right) ds, \quad 0 \leq t \leq T, \]
where \(F\) is given by (4.56). The proof is complete. \(\square\)
Lemma 4.13. For any $p \in (0, \infty)$, there exists a constant $K_p \in (0, \infty)$ depending only on $p$ and $(\alpha, \xi, N_0)$ such that
\[
\mathbb{P}^\alpha_z \left[ (Z_T)^p; T \leq T_1^Z \right] \leq K_p \left[ (z^{p\alpha})^{N_0} T^{p\alpha} + z^p \right] \mathbb{P}_z^0 (Z_T > 0) + z^p T^{p\xi - 1},
\]
\[\forall z, T \in (0, 1].\]  

Proof. Recall the conditional probability measure $\mathbb{P}^{(T)}_z$ defined in (4.54) and write
\[
\mathbb{P}^\alpha_z \left[ (Z_T)^p; T \leq T_1^Z \right] \leq \mathbb{P}_z^0 (Z_T > 0) \mathbb{E}^{\mathbb{P}_z^0} \left[ \left( \frac{Z_{T \wedge T_1^Z}}{Z_T} \right)^p \right] \mathbb{P}_z^0 (Z_T > 0) \mathbb{P}^{(T)}_z \left[ \left( \frac{Z_{T \wedge T_1^Z}}{Z_T} \right)^p \right].
\]

Henceforth, we work under the conditional probability measure $\mathbb{P}^{(T)}_z$.

We turn to the improved modulus of continuity of $Z$ at its starting time 0 under $\mathbb{P}^{(T)}_z$ in order to bound the right-hand side of (4.61). We first claim that, by enlarging the underlying probability space if necessary,
\[
|Z_t - z| \leq t + C^Z_\alpha \left( \int_0^t Z_s ds \right)^\alpha \quad \forall t \in [0, T \wedge T_1^Z] \text{ under } \mathbb{P}^{(T)}_z,
\]
where the random variable $C^Z_\alpha$ under $\mathbb{P}^{(T)}_z$ has distribution depending only on $\alpha$ and finite $\mathbb{P}^{(T)}_z$-moment of any finite order. We show how to obtain (4.62) by using the canonical decomposition of the continuous $(\mathcal{H}_t, \mathbb{P}^{(T)}_z)$-semimartingale $(Z_t)_{0 \leq t \leq T}$ in (4.55). First, since its martingale part $M$ has quadratic variation $\int_0^t Z_s ds$, the Dambis-Dubins-Schwarz theorem (cf. Theorem V.1.6 of [20]) implies that, by enlarging of the underlying probability space if necessary,
\[
M_t = \tilde{B} \left( \int_0^t Z_s ds \right), \quad t \in [0, T \wedge T_1^Z],
\]
for some standard Brownian motion $\tilde{B}$ under $\mathbb{P}^{(T)}_z$. Here, the random clock $\int_0^t Z_s ds$, $t \in [0, T \wedge T_1^Z]$, for $\tilde{B}$ is bounded by 1 by the assumption that $z, T \leq 1$. On the other hand, recall that the chosen parameter $\alpha$ lies in $(0, \frac{1}{2})$ and the uniform Hölder-$\alpha$ modulus of continuity of standard Brownian motion on compacts has moments of any finite order. (See, e.g., the discussion preceding Theorem 1.2.2 of [20] and its proof.) Hence,
\[
\left| \tilde{B} \left( \int_0^t Z_s ds \right) \right| \leq C^Z_\alpha \left( \int_0^t Z_s ds \right)^\alpha, \quad t \in [0, T \wedge T_1^Z],
\]
where the random variable $C^Z_\alpha$ is as in (4.62). Second, Lemma 4.12 also states that the finite variation process of $Z$ under $\mathbb{P}^{(T)}_z$ given by (4.55) is a time integral with integrand uniformly bounded by 1. This and the last two displays are now enough to obtain our claim (4.62).

With the integral inequality (4.62) and the distributional properties of $C^Z_\alpha$, we obtain the following improved modulus of continuity of $Z$ (cf. Lemma 4.4):
\[
|Z_{T \wedge T_1^Z} - z| \leq K_1^Z z^{\alpha N_0} T^{\alpha} + K_2^Z T^{\xi}
\]
(4.63)
for some random variables $K_1^Z, K_2^Z \in \bigcap_{q \in (0, \infty)} L^q (\mathbb{P}^{(T)}_z)$ obeying a joint law under $\mathbb{P}^{(T)}_z$ depending only on $(\alpha, \xi, N_0)$ by the analogous property of $C^Z_\alpha$ and Lemma 4.4.
We now return to the calculation in (4.61). Applying (4.63), we get
\[
\mathbb{E}^{\mathbf{P}_{\psi}^{0}}[(Z_{T})^{p}; T \leq T_{Z}^{\beta}] \leq \mathbb{P}_{\psi}^{0}(Z_{T} > 0) \mathbb{E}^{\mathbf{P}_{\psi}^{0}}[(Z_{T} \wedge T_{z}^{\beta})^{p}]
\]
\[
\leq \mathbb{P}_{\psi}^{0}(Z_{T} > 0) (2^{p-1} \vee 1) \mathbb{E}^{\mathbf{P}_{\psi}^{0}}[\|Z_{T} \wedge T_{z}^{\beta} - z\|^{p} + z^{p}]
\]
\[
\leq \mathbb{P}_{\psi}^{0}(Z_{T} > 0) K'_{p}\left(z^{p\alpha_{N_{0}}}T^{p\alpha} + T^{p\xi} + z^{p}\right)
\]  
(4.64)
for some constant $K'_{p}$ depending only on $p$ and $(\alpha, \xi, N_{0})$ by (4.63) and the distributional properties of $K_{Z}^{\beta}$, where the second inequality follows from the elementary inequality
\[
(x + y)^{p} \leq (2^{p-1} \vee 1) \cdot (x^{p} + y^{p}), \quad \forall \ x, y \in \mathbb{R}_{+}.
\]
The desired result follows by applying (4.35) to (4.64). The proof is complete.  

(Step 2-6). At this step, we summarize our results in (Step 2-1)–(Step 2-4), using Lemma 13. We apply (4.45), (4.46), and (4.52) to (4.44). This gives
\[
\mathbb{E}^{\mathbf{Q}^{0}}[(Y_{s}(1))^{1/2}; s < t, |y_{j} - x_{i}| < 2(e^{1/2} + (t - s))^{\beta}]
\]
\[
\lesssim \left((t_{j} - s_{i})^{\beta} + (t_{j} - s_{i})^{\beta - 1} \left(e^{\alpha N_{0}}(s - s_{i})^{\alpha} + (s - s_{i})^{\xi} + \varepsilon\right)\right)
\times \mathbb{E}^{\mathbf{P}_{\psi}^{0}(1)}[(Z_{s - t_{j}}) \wedge T_{Z}^{\beta}]^{1/2}, s - t_{j} \leq T_{Z}^{\beta}
\]
\[
\lesssim \left((t_{j} - s_{i})^{\beta} + (t_{j} - s_{i})^{\beta - 1} \left(e^{\alpha N_{0}}(s - s_{i})^{\alpha} + (s - s_{i})^{\xi} + \varepsilon\right)\right)
\times \left\{\left(e^{\alpha N_{0}}(s - t_{j})^{\alpha} + \varepsilon\right) + (s - t_{j})^{\frac{\beta - 1}{2}}\right\}
\]
\[
\times \mathbb{P}_{\psi(1)}^{0}(Z_{s - t_{j}} > 0) + \varepsilon(s - t_{j})^{\frac{\beta - 1}{2}}
\]
\[
\lesssim (t_{j} - s_{i})^{\beta - 1} \times \left((t_{j} - s_{i}) + e^{\alpha N_{0}}(s - s_{i})^{\alpha} + (s - s_{i})^{\xi} + \varepsilon\right)
\times \varepsilon^{\alpha N_{0}}(s - t_{j})^{\frac{\alpha}{2}}(s - t_{j})^{\frac{\beta}{2} - 1}
\]
\[
+ (t_{j} - s_{i})^{\beta - 1} \times \varepsilon(s - t_{j})^{\frac{\beta}{2} - 1}
\]
\[
+ (t_{j} - s_{i})^{\beta - 1} \times (e^{\alpha N_{0}}(s - s_{i})^{\alpha} + \varepsilon) \times \varepsilon(s - t_{j})^{\frac{\beta}{2} - 1}
\]
\[
+ (t_{j} - s_{i})^{\beta - 1} \times (s - s_{i})^{\xi} \times \varepsilon(s - t_{j})^{\frac{\beta}{2} - 1}, \quad \forall \ s \in (t_{j}, t], \ s_{i} < t_{j} < t,
\]
where the last $\lesssim$-inequality follows by some algebra.

We now make some simplifications for the right-hand side of (4.65) before going further. We remark that some orders in $\varepsilon$ and other variables will be discarded here. We bound the survival probability in (4.65) by
\[
\mathbb{P}_{\psi(1)}^{0}(Z_{s - t_{j}} > 0) \leq \left(\frac{2\psi(1)\varepsilon}{s - t_{j}}\right)^{1 - \alpha N_{0}},
\]  
(4.66)
as follows from the elementary inequalities $x \leq x^{\gamma}$ for any $x \in [0, 1]$ and $\gamma \in (0, 1]$, and then (4.35). Assuming $s \in (t_{j}, t]$ for $s_{i} < t_{j} < t$, we have the inequalities
\[
1 \geq s - s_{i} \geq t_{j} - s_{i} \geq \frac{\varepsilon}{2},
\]
\[
s - t_{j} \leq 1,
\]
\[
0 < \alpha + \alpha N_{0} < \xi < 1
\]
(cf. (4.25)-(a) for the third inequality). These and (4.66) imply that the first term of (4.65) satisfies
\[
(t_j - s_i)^{\beta'-1} \times \left( (t_j - s_i) + \varepsilon^{\alpha_{N_0}}(s - s_i)^{\alpha} + (s - s_i)^{\xi} + \varepsilon \right) \\
\times \varepsilon^{a N_0} (s - t_j)^{\frac{\alpha}{2}} P_{\Psi^{(l)}}(Z_{s-t_j} > 0) \\
\lesssim (t_j - s_i)^{\beta'-1}(s - s_i)^{\alpha} (s - t_j)^{\frac{\alpha}{2} + \frac{a N_0}{4} - 1} \varepsilon^{1 + \frac{a N_0}{4}} ,
\]
the second term of (4.65) satisfies
\[
(t_j - s_i)^{\beta'-1} \times \left( (t_j - s_i) + \varepsilon^{\alpha_{N_0}}(s - s_i)^{\alpha} + (s - s_i)^{\xi} + \varepsilon \right) \\
\times \varepsilon^{a N_0} (s - t_j)^{\frac{\alpha}{2}} P_{\Psi^{(l)}}(Z_{s-t_j} > 0) \\
\lesssim (t_j - s_i)^{\beta'+\frac{1}{2}}(s - s_i)^{\alpha} (s - t_j)^{\frac{\alpha}{4} + \frac{a N_0}{4} - \frac{1}{2} \varepsilon^{1 + \frac{a N_0}{4}}}
\]
and, finally, the fourth term of (4.65) satisfies
\[
(t_j - s_i)^{\beta'-1} \times \left( \varepsilon^{\alpha_{N_0}}(s - s_i)^{\alpha} + \varepsilon \right) \times \varepsilon(s - t_j)^{\frac{\alpha}{2} - 1} \\
\lesssim (t_j - s_i)^{\beta'-1}(s - s_i)^{\alpha} (s - t_j)^{\frac{\alpha}{4} + \frac{a N_0}{4} - 1} \varepsilon^{1 + \frac{a N_0}{4}} \\
\lesssim (t_j - s_i)^{\beta'-1}(s - s_i)^{\alpha} (s - t_j)^{\frac{\alpha}{4} + \frac{a N_0}{4} - 1} \varepsilon^{1 + \frac{a N_0}{4}}.
\]

Note that the bounds in (4.67) and (4.69) coincide. Using (4.67)-(4.69) in (4.65), we obtain
\[
\mathbb{E}^{Q_t} \left[ \left| Y_s(x) \right|^{1/2} ; s < \tau^1, |y_j - x_i| \leq 2 (1/2 + (t_j - s_i)^{\beta'}) \right] \\
\lesssim (t_j - s_i)^{\beta'-1}(s - s_i)^{\alpha} (s - t_j)^{\frac{\alpha}{2} + \frac{a N_0}{4} - 1} \varepsilon^{1 + \frac{a N_0}{4}} \\
+ (t_j - s_i)^{\beta'+\frac{1}{2}}(s - s_i)^{\alpha} (s - t_j)^{\frac{\alpha}{4} + \frac{a N_0}{4} - \frac{1}{2} \varepsilon^{1 + \frac{a N_0}{4}}} \\
+ (t_j - s_i)^{\beta'-1}(s - s_i)^{\xi} (s - t_j)^{\frac{\alpha}{2} - 1} \varepsilon, \quad \forall s \in (t_j, t], s_i < t_j < t.
\]

**Step 3.** We digress to a conceptual discussion for some elementary integrals which will play an important role in the forthcoming calculations in (Step 4). First, for \(a, b, c \in \mathbb{R}\) and \(T \in (0, \infty)\), a straightforward application of Fubini’s theorem and changes of variables shows that
\[
I(a, b, c) \triangleq \int_0^{T} drr^a \int_r^{T} dss^b(s - r)^c < \infty \quad \iff a, c \in (-1, \infty) \quad \text{and} \quad a + b + c > -2.
\]
Furthermore, when \(I(a, b, c)_T\) is finite, it can be expressed as
\[
I(a, b, c)_T = \left( \int_0^{1} drr^a(1 - r)^c \right) \cdot \frac{T^{a+b+c+2}}{a + b + c + 2}.
\]
Given \(a + b + c > -2\) with \(a, c \in (-1, \infty)\), we consider alternative ways to show that the integral \(I(a, b, c)_T\) is finite while preserving the same order \(T^{a+b+c+2}\) in \(T\), according to \(b \geq 0\) and \(b < 0\). If \(b \geq 0\), then

\[
I(a, b, c)_T \leq \int_0^T dr r^a \times T^b \times \int_0^T dss = \frac{1}{a+1} \frac{1}{c+1} T^{a+b+c+2},
\]

(4.72)

where the first inequality follows since \(s^b \leq T^b\) for any \(s \in [r, T]\). For the case that \(b < 0\), we decompose the function \(s \mapsto s^b\) in the following way. For \(b_1, b_2 < 0\) such that \(b_1 + b_2 = b\), we have

\[
I(a, b, c)_T \leq \int_0^T dr r^{a+b_1} \int_r^T ds (s-r)^{b_2+c} \leq \int_0^T dr r^{a+b_1} \times \int_0^T dss^{b_2+c}
\]

(4.73)

where the first inequality follows since for \(s > r\), \(s^{b_1} \leq r^{b_1}\) and \(s^{b_2} \leq (s-r)^{b_2}\). Using the following elementary lemma, we obtain from (4.73) that

\[
I(a, b, c)_T \leq \frac{1}{a+b_1+1} \frac{1}{b_2+c+1} T^{a+b+c+2}.
\]

**Lemma 4.14.** For any reals \(a, c > -1\) and \(b < 0\) such that \(a + b + c > -2\), there exists a pair \((b_1, b_2) \in (-\infty, 0) \times (-\infty, 0)\) such that \(b = b_1 + b_2\) and \(a + b_1 > -1\) and \(b_2 + c > -1\).

The two simple concepts for the inequalities (4.72) and (4.73) will be applied later on in (Step 4) to bound Riemann sums by integrals of the type \(I(a, b, c)_T\).

**(Step 4).** We complete the proof of Lemma 4.10 in this step. Applying the bound (4.70) to the right-hand side of the inequality (4.37). We have

\[
\mathbb{E}^q \left[ \sum_{j \in C_{\mu}^t(t \wedge t') \wedge \sigma^*_{x_i}} \left( \psi(I) + \int_{t_j}^{t \wedge t' \wedge \sigma^*_{s_j} \wedge \sigma^*_{x_i}} \frac{1}{X^t_s(\iota, s)} \int_R X^t(x, s)^{1/2} Y^j(t, s)^{1/2} dx ds \right) \right] \lesssim \sum_{j:s_i < t_j \leq t} (t_j - s_i)^{\beta'} \varepsilon
\]

\[
+ \sum_{j:s_i < t_j \leq t} (t_j - s_i)^{\beta' - \frac{1}{2}} \int_{t_j}^{t} (s - s_i)^{-\frac{1}{2}} (s - t_j)^{-\frac{1}{2} + \frac{\beta_0}{2}} ds \cdot \varepsilon^{1 + \frac{\beta_0}{2}}
\]

\[
+ \sum_{j:s_i < t_j \leq t} (t_j - s_i)^{\beta' + \frac{1}{2} - 1} \int_{t_j}^{t} (s - s_i)^{-\frac{1}{2} + \alpha} (s - t_j)^{\frac{\beta_0}{2}} ds \cdot \varepsilon^{1 + \frac{\beta_0}{2}}
\]

\[
+ \sum_{j:s_i < t_j \leq t} (t_j - s_i)^{\beta'} \int_{t_j}^{t} (s - s_i)^{-\frac{1}{2}} (s - t_j)^{\frac{1}{2} - 1} ds \cdot \varepsilon
\]

\[
+ \sum_{j:s_i < t_j \leq t} (t_j - s_i)^{\beta' - 1} \int_{t_j}^{t} (s - s_i)^{-\frac{1}{2} + \xi} (s - t_j)^{\frac{1}{2} - 1} ds \cdot \varepsilon.
\]

(4.74)

Recall the notation \(I(a, b, c)\) in (4.71). It should be clear that, up to a translation of time by \(s_i\), the first, the fourth, and the fifth sums are Riemann sums of

\[
I(\beta', 0, 0)_{t-s_i},
\]

\[
I \left( \beta', -\frac{\eta}{2}, \frac{\xi}{2} - 1 \right)_{t-s_i},
\]

\[
I \left( \beta' - 1, -\frac{\eta}{2} + \xi, \frac{\xi}{2} - 1 \right)_{t-s_i}.
\]
respectively, and so are the second and the third sums after a division by $\varepsilon^{-\frac{N_0}{4}}$ with the corresponding integrals equal to

$$I \left( \beta' - 1, -\frac{\eta}{2} + \frac{\alpha}{2} + \frac{N_0}{4} - 1 \right)_{t-s_i},$$

$$I \left( \beta' + \frac{\alpha}{2} - 1, -\frac{\eta}{2} + \frac{\alpha}{2} + \frac{N_0}{4} - 1 \right)_{t-s_i},$$

respectively. It follows from (4.25)-(c) and (d) and (4.71) that all of the integrals in the last two displays are finite.

We now aim to bound each of the five sums in (4.74) by suitable powers of $\varepsilon$ and $t$, using integral comparisons. Observe that, whenever $\gamma \in (-1, \infty)$, the monotonicity of $r \mapsto (r-s_i)^\gamma$ over $(s_i, \infty)$ implies

$$\sum_{j:s_i < t_j \leq t} (t_j - s_i)^\gamma \cdot \varepsilon \leq 2 \int_{s_i}^{t+\varepsilon} (r-s_i)^\gamma dr = \frac{2}{\gamma+1} (t+\varepsilon-s_i)^{\gamma+1} = \frac{2 \cdot 3^{\gamma+1}}{\gamma+1} (t-s_i)^{\gamma+1}$$

since $t \geq s_i + \frac{\varepsilon}{2}$. (The constant 2 is used to accommodate the case that $\gamma < 0$.) Hence, the first sum in (4.74) can be bounded as:

$$(4.75)$$

$$\sum_{j:s_i < t_j \leq t} (t_j - s_i)^\beta' \varepsilon \leq (t-s_i)^{\beta'+1}. \quad (4.76)$$

Consider the other sums in (4.74). Recall our discussion of some alternative ways to bound $I(a,b,c)$ for given $a+b+c > -2$ and $a,c \in (-1, \infty)$ according to $b \geq 0$ or $b < 0$; see (4.72) and (4.73). We use Lemma 4.14 in the following whenever necessary. Now, the second sum in (4.74) can be bounded as

$$\sum_{j:s_i < t_j \leq t} (t_j - s_i)^{\beta'-1} \int_{t_j}^{t} (s-s_i)^{-\frac{\eta}{2}+\alpha} (s-t_j)^{\frac{\alpha}{2}+\frac{N_0}{4}-1} ds \cdot \varepsilon^{1+\frac{N_0}{2}}$$

$$= \sum_{j:s_i < t_j \leq t} (t_j - s_i)^{\beta'-1} \int_{t_j-s_i}^{t-s_i} s^{-\frac{\eta}{2}+\alpha} [s-(t_j-s_i)]^{\frac{\alpha}{2}+\frac{N_0}{4}-1} ds \cdot \varepsilon^{1+\frac{N_0}{2}}$$

$$\leq (t-s_i)^{\beta'-\frac{\eta}{2}+\frac{\alpha}{2}+\frac{N_0}{2}} \cdot \varepsilon^{\frac{N_0}{2}}$$

$$\leq (t-s_i)^{\beta'-\frac{\eta}{2}+\frac{\alpha}{2}+\frac{N_0}{2}} \cdot \varepsilon^{\frac{N_0}{2}}. \quad (4.77)$$

Here, in the foregoing $\leq$-inequality, we use the integral comparison discussed in (Step 3) (with Lemma 4.14 to algebraically allocate the exponent $-\frac{\eta}{2} + \frac{\alpha}{2}$ if necessary) and the Riemann-sum bound (4.75). The other sums on the right-hand side of (4.74) can be bounded similarly as follows. The third sum satisfies

$$\sum_{j:s_i < t_j \leq t} (t_j - s_i)^{\beta'+\frac{\eta}{2}-1} \int_{t_j}^{t} (s-s_i)^{-\frac{\eta}{2}+\alpha} (s-t_j)^{\frac{N_0}{4}+1} ds \cdot \varepsilon^{1+\frac{N_0}{2}}$$

$$\leq (t-s_i)^{\beta'-\frac{\eta}{2}+\frac{2\alpha}{2}+\frac{N_0}{2}} \cdot \varepsilon^{\frac{N_0}{2}}. \quad (4.78)$$
The fourth sum satisfies
\[
\sum_{j:s_i < t_j \leq t} (t_j - s_i)^{\beta'} \int_{t_j}^t (s - s_i)^{-\frac{\beta}{2}} (s - t_j)^{\frac{\beta}{2} - 1} ds \cdot \varepsilon \lesssim (t - s_i)^{\beta' - \frac{\beta}{2} + \frac{\beta}{2} + 1}
\]
where the last inequality applies since \( \xi \in (0, 1) \). The last sum satisfies
\[
\sum_{j:s_i < t_j \leq t} (t_j - s_i)^{\beta' - 1} \int_{t_j}^t (s - s_i)^{-\frac{\beta}{2} + \xi} (s - t_j)^{\frac{\beta}{2} - 1} ds \cdot \varepsilon \lesssim (t - s_i)^{\beta' - \frac{\beta}{2} + \frac{\beta}{2}}.
\]
(4.79)

The proof of Lemma 4.10 is complete upon applying (4.76)–(4.80) to the right-hand side of (4.74).

4.5. Proof of Lemma 4.11

As in Section 4.4, we fix \( t \in [s_i + \frac{\beta}{2}, s_i + 1] \), \( i \in \mathbb{N} \), and \( \varepsilon \in (0, [8\psi(1)]^{-1} \wedge 1) \) and drop the subscripts of \( P_e \) and \( Q_e \). For the proof of Lemma 4.11, the arguments in Section 4.4 work essentially. Now, we begin to use the condition (4.25)-(b) in Assumption 4.5 and the upper limit \( \sigma^X_\beta \wedge \sigma^Y_\beta \) in the time integral in (4.33), which are neglected when we prove Lemma 4.10.

To motivate our adaptation of the arguments for critical clusters in Section 4.4, we discuss some parts of Section 4.4. First, it is straightforward to modify the proof of (4.38) and obtain
\[
Q^i \left( 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}) \leq |y_j - x_i| \leq 2(\varepsilon^{1/2} + (t - s_i)^{\beta'}) \right) \lesssim (t - s_i)^{\beta'}.
\]
(4.81)

If we proceed as in (4.37) and use (4.81) in the obvious way, then this leads to
\[
\mathbb{E}^Q^i \left[ \sum_{j \in \mathcal{L}_e^i(t, t \wedge t' \wedge \sigma^X_\beta)} \left( \psi(1) \varepsilon + \int_{t_j}^{t \wedge t' \wedge \sigma^X_\beta} \frac{1}{X^i(1)} \int_{\mathbb{R}} X^i(x, s)^{1/2} Y^j(x, s)^{1/2} dx ds \right) \right]
\lesssim \sum_{j:s_i < t_j \leq t} (t - s_i)^{\beta'} \varepsilon + \sum_{j:s_i < t_j \leq t} \int_{t_j}^t ds \frac{1}{(s - s_i)^{\eta/2}} \mathbb{E}^Q^i \left[ \left[ Y^j(1) \right]^{1/2} ; s < x^i, 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}) \leq |y_j - x_i| \leq 2(\varepsilon^{1/2} + (t - s_i)^{\beta'}) \right] .
\]

(Compare this with (4.37) for critical clusters.) If we argue by using (4.81) repeatedly in the steps analogous to (Step 2-2)–(Step 2-4) of Section 4.4, then we obtain the following \( \lesssim \)-inequality similar to (4.74):
\[
\mathbb{E}^Q^i \left[ \sum_{j \in \mathcal{L}_e^i(t, t \wedge t' \wedge \sigma^X_\beta)} \left( \psi(1) \varepsilon + \int_{t_j}^{t \wedge t' \wedge \sigma^X_\beta} \frac{1}{X^i(1)} \int_{\mathbb{R}} X^i(x, s)^{1/2} Y^j(x, s)^{1/2} dx ds \right) \right]
\]
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\[ \lesssim \sum_{j: s_i < t_j \leq t} (t - s_i)^{\beta'} \varepsilon \\
+ \sum_{j: s_i < t_j \leq t} (t - s_i)^{\beta'} (t_j - s_i)^{-1} \int_{t_j}^t (s - s_i)^{-\frac{\alpha}{2} + \alpha} (s - t_j)^{\frac{\alpha}{2} + \frac{4N_0}{\nu} - 1} ds \cdot \varepsilon^{1 + \frac{2N_0}{\nu}} \\
+ \sum_{j: s_i < t_j \leq t} (t - s_i)^{\beta'} (t_j - s_i)^{-1} \int_{t_j}^t (s - s_i)^{-\frac{\alpha}{2} + \alpha} (s - t_j)^{\frac{\alpha}{2} + \frac{4N_0}{\nu} - 1} ds \cdot \varepsilon^{1 + \frac{2N_0}{\nu}} \\
+ \sum_{j: s_i < t_j \leq t} (t - s_i)^{\beta'} \int_{t_j}^t (s - s_i)^{-\frac{\alpha}{2} + \alpha} (s - t_j)^{\frac{\alpha}{2} - 1} ds \cdot \varepsilon \\
+ \sum_{j: s_i < t_j \leq t} (t - s_i)^{\beta'} (t_j - s_i)^{-1} \int_{t_j}^t (s - s_i)^{-\frac{\alpha}{2} + \alpha} (s - t_j)^{\frac{\alpha}{2} - 1} ds \cdot \varepsilon, \]

(4.82)

taking into account some simplifications similar to (4.67)–(4.69) where some orders are discarded. (We omit the derivation of the foregoing display, as it will not be used for the proof of Lemma 4.11.) In other words, replacing the factor \((t_j - s_i)^{\beta'} \) for each of the sums in (4.74) by \((t - s_i)^{\beta'} \) gives the bound in the foregoing display. Applying integral domination to the second and the last sums of the foregoing display as in (Step 4) of Section 4.5 results in bounds which are divergent integrals.

Examining the arguments in (Step 2–2)–(Step 2–4) of Section 4.4 shows that the problematic factor

\[ (t_j - s_i)^{-1} \]

in (4.82) results from using the bound (4.51) for the survival probability \(\mathbb{P}(t_j < T_0^X)\). The exponent \(-1\) in the foregoing display, however, is critical, and any decrease in this value will lead to convergent integrals. Also, we recall that (4.9) is used repeatedly in (Step 2–2)–(Step 2–4) of Section 4.4, while (4.9) is a consequence of (4.8) and the proof of (4.8) uses in particular the Markov property of \(Y_j(1)\) at \(t_j\). These observations lead us to consider modifying the arguments in Section 4.4 by replacing \(t_j\) with a “larger” value, subject to the condition that certain \(\mathbb{P}\)-independence, similar to (4.8) with \(t_j\) replaced by the resulting value, still holds.

Let us start with identifying the value to replace \(t_j\). The idea comes from the following observation.

**Observation** The support process of a lateral cluster \(Y_j\) takes a positive amount of time after its birth to meet the support of \(X^i\), thereby leading to a time \(t_j^c\) larger than \(t_j\). Moreover, prior to \(t_j^c\), the supports of \(X^i\) and \(Y_j\) separate. (Cf. Figure 3.2.)

We first formalize the definition of this time \(t_j^c\). Let \(j \in \mathbb{N}\) with \(t_j \in (s_i, s_i + 1]\). Recall that the range for the possible values \(y\) of \(y_j\) associated with a lateral cluster is

\[ 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}) \leq |y - x_i| \leq 2(\varepsilon^{1/2} + (t - s_i)^{\beta'}) \]

(4.84)

and we use \(\mathcal{P}_\beta^{X^i}(\cdot)\) and \(\mathcal{P}_\beta^{Y^j}(\cdot)\) to envelope the support processes of \(X^i\) and \(Y^j\), respectively. Let the processes of parabolas \(\{\mathcal{P}_\beta^{X^i}(t); t \in [s_i, \infty)\}\) and \(\{\mathcal{P}_\beta^{Y^j}(t); t \in [t_j, \infty)\}\) evolve in the deterministic way, and consider the **support contact time** \(t_j^c(y_j)\), that is, the first time \(t\) when \(\mathcal{P}_\beta^{X^i}(t)\) and \(\mathcal{P}_\beta^{Y^j}(t)\) intersect. Here, for any \(y\) satisfying (4.84), \(t_j^c(y) \in (t_j, \infty)\) solves

\[
\begin{align*}
&x_i + \varepsilon^{1/2} + (t_j^c(y) - s_i)^{\beta} = y - \varepsilon^{1/2} - (t_j^c(y) - t_j)^{\beta}, & \text{if } y > x_i, \\
&x_i - \varepsilon^{1/2} - (t_j^c(y) - s_i)^{\beta} = y + \varepsilon^{1/2} + (t_j^c(y) - t_j)^{\beta}, & \text{if } y < x_i.
\end{align*}
\]

(4.85)
By simple arithmetic, we see that the minimum of \( t_j^\varepsilon(y) \) for \( y \) satisfying (4.84) is attained at the boundary cases where \( y \) satisfies \( 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}) = |y - x_i| \). Let us consider the worst case of the support contact time as

\[
t_j^* \triangleq \min \{ t_j^\varepsilon(y); y \text{ satisfies } (4.84) \}.
\]  

(4.86)

Recall that \( \beta' < \beta \) by (4.25)-(b).

**Lemma 4.15.** Let \( j \in \mathbb{N} \) with \( t_j \in (s_i, s_i + 1] \).

1. The number \( t_j^* \) defined by (4.86) satisfies

\[
t_j^* = s_i + A(t_j - s_i) \cdot (t_j - s_i)^{\beta'},
\]

where \( A(r) \) is the unique number in \( (r^{1-\frac{\beta'}{\beta}}, \infty) \) solving

\[
A(r)^\beta + \left[ A(r) - r^{1-\frac{\beta'}{\beta}} \right]^\beta = 2, \quad r \in (0, 1].
\]

(4.88)

2. The function \( A(\cdot) \) defined by (4.88) satisfies

\[
1 \leq A(r) \leq 1 + r^{1-\frac{\beta'}{\beta}}, \quad \forall r \in (0, 1].
\]

(4.89)

**Proof.** Without loss of generality, we may assume that \( t_j^* = t_j^\varepsilon(y) \) for \( y \) satisfying

\[
x_i - y = 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}). \]

Using this particular value \( y \) of \( y_j \) in (4.85), we see that \( t_j^* \) solves the equation

\[
x_i - \varepsilon^{1/2} - (t_j^* - s_i)^\beta = y + \varepsilon^{1/2} + (t_j^* - t_j)^\beta
\]

\[
= x_i - \varepsilon^{1/2} - 2(t_j - s_i)^{\beta'} + (t_j^* - t_j)^\beta.
\]

Taking \( t_j^* = s_i + A \cdot (t_j - s_i)^{\frac{\beta'}{\beta'}} \) for some constant \( A \in (0, \infty) \) left to be determined, we obtain from the foregoing equality that

\[
2(t_j - s_i)^{\beta'} = A \cdot (t_j - s_i)^{\beta'} + \left[ A \cdot (t_j - s_i)^{\frac{\beta'}{\beta'}} - (t_j - s_i) \right]^\beta
\]

\[
= A^\beta \cdot (t_j - s_i)^{\beta'} + \left[ A - (t_j - s_i)^{1-\frac{\beta'}{\beta'}} \right]^\beta \cdot (t_j - s_i)^{\beta'},
\]

which shows that \( A = A(t_j - s_i) \) for \( A(\cdot) \) defined by (4.88) upon cancelling \( (t_j - s_i)^{\beta'} \) on both sides. We have obtained (1).

From the definition (4.88) of \( A(\cdot) \), we obtain

\[
2A(r)^\beta \geq A(r)^\beta + \left[ A(r) - r^{1-\frac{\beta'}{\beta}} \right]^\beta = 2,
\]

\[
2\left[ A(r) - r^{1-\frac{\beta'}{\beta}} \right]^\beta \leq A(r)^\beta + \left[ A(r) - r^{1-\frac{\beta'}{\beta}} \right]^\beta = 2,
\]

and both inequalities in (4.89) follow. The proof is complete.

As a result of Lemma 4.15, we have

\[
\mathbb{P}(t_j^* < T_0^{X^*}) \leq \varepsilon(t_j - s_i)^{\frac{\beta'}{\beta}},
\]

(4.90)

where the exponent \( -\frac{\beta'}{\beta} \) is now an improvement in terms of our preceding discussion about the factor (4.83). The value \( t_j^* \) will serve as the desired replacement of \( t_j \).

We then turn to show how \( t_j^* \) still allows some independence similar to (4.8).
Lemma 4.16 (Orthogonal continuation). Let \((\mathcal{F}_t)\) be a filtration satisfying the usual conditions, and \(U\) and \(V\) be two \((\mathcal{F}_t)\)-Feller diffusions such that \(U_0 \equiv V_0\) and, for some \((\mathcal{F}_t)\)-stopping \(\sigma^+\), \((U,V)^{\sigma^+} \equiv 0\). Then by enlarging the underlying filtered probability space if necessary and writing again \((\mathcal{F}_t)\) for the resulting filtration with a slight abuse of notation in this case, we can find a \((\mathcal{F}_t)\)-Feller diffusion \(\hat{U}\) such that \(\hat{U} \equiv V\) and \(\hat{U} = V\) over \([0,\sigma^+]\).

Proof. We only give a sketch of the proof here, and leave the details, calling for standard arguments, to the readers. Using Lévy’s theorem, we can define a Brownian motion \(\hat{B}\) by

\[
\hat{B}_t = \int_0^{t_0^U \wedge \sigma^+ \wedge t} \frac{1}{\sqrt{U_s}} dU_s + \int_0^t 1_{\{r \leq \sigma^+\}} dB_s,
\]

for some independent Brownian motion \(B\). We can use \(\hat{B}\) to solve for a Feller diffusion \(\hat{U}\) with initial value \(U_0\). Then the proof of pathwise uniqueness for Feller diffusions (cf. [24]) gives \(\hat{U} = U\) on \([0,\sigma^+]\). Note that \((\hat{U},V) \equiv 0\), and consider the martingale problem associated with a two-dimensional independent Feller diffusions with initial values \(U_0\) and \(V_0\). By its uniqueness, \(\hat{U} \equiv V\). Hence, \(\hat{U}\) is the desired continuation of \(U\) beyond \(\sigma^+\). \(\square\)

We now apply Lemma 4.16 to the total mass processes \(X^i(1)\) and \(Y^j(1)\) under \(\mathbb{P}\) to give the following analogue of (4.8).

Proposition 4.17. Let \(i, j \in \mathbb{N}\) be given so that \(s_i < t_j\). Suppose that \(\sigma^+\) is a \((\mathcal{G}_t)\)-stopping time such that \(\sigma^+ \geq t_j\) and \((X^i(1), Y^j(1))^{\sigma^+} \equiv 0\). Then for \(r_2 > r_1 \geq t_j\) and nonnegative Borel measurable functions \(H_{11}, H_{12}, H_{21}\), and \(h\),

\[
\begin{align*}
\mathbb{E}^{{\mathcal{G}_t}}[H_1 (Y^j_2(1); r \in [t_j, r_2]) & \cdot H_2 (X^i_1(1); r \in [s_i, r_1]) h(y_j, x_i); r_1 \leq \sigma^+] \\
\leq & \mathbb{E}^{{\mathcal{G}_t}}[H_1 (Y^j_2(1); r \in [t_j, r_2])] \cdot \mathbb{E}^{{\mathcal{G}_t}}[H_2 (X^i_1(1); r \in [s_i, r_1]) h(y_j, x_i); r_1 \leq \sigma^+] \quad (4.91)
\end{align*}
\]

Proof. By the monotone class theorem, we may only consider the case that

\[
\begin{align*}
H_1 (Y^j_2(1); r \in [t_j, r_2]) = & H_{11} (Y^j_2(1); r \in [t_j, r_1]) H_{12} (Y^j_2(1); r \in [r_1, r_2]), \\
H_2 (X^i_1(1); r \in [s_i, r_1]) = & H_{21} (X^i_1(1); r \in [s_i, t_j]) H_{22} (X^i_1(1); r \in [t_j, r_1]),
\end{align*}
\]

for nonnegative Borel measurable functions \(H_{k,t}\) of \(\mathcal{G}_t\).

As the first step, we condition on \(\mathcal{G}_{r_1}\) and obtain

\[
\begin{align*}
\mathbb{E}^{{\mathcal{G}_t}}[H_1 (Y^j_2(1); r \in [t_j, r_2]) & \cdot H_2 (X^i_1(1); r \in [s_i, r_1]) h(y_j, x_i); r_1 \leq \sigma^+] \\
= & \mathbb{E}^{{\mathcal{G}_{r_1}}} \left[ H_{11} (Y^j_2(1); r \in [t_j, r_1]) \mathbb{E}^{{\mathcal{G}_{r_1}}} \left[ H_{12} (Y^j_2(1); r \in [r_1, r_2]) \cdot h(y_j, x_i); r_1 \leq \sigma^+ \right] \right] \quad (4.92)
\end{align*}
\]

Since \(Y^j(1)\) is a \((\mathcal{G}_t)\)-Feller process, we know that

\[
\begin{align*}
\mathbb{E}^{{\mathcal{G}_{r_1}}} \left[ H_{12} (Y^j_2(1); r \in [r_1, r_2]) \cdot h(y_j, x_i); r_1 \leq \sigma^+ \right] = & \mathbb{E}^{{\mathcal{G}_{r_1}}} \left[ H_{12} (Y^j_2(1)) \right] \quad (4.93)
\end{align*}
\]

for some nonnegative Borel measurable function \(\hat{H}_{12}\). Hence, from (4.92), we get

\[
\begin{align*}
\mathbb{E}^{{\mathcal{G}_t}}[H_1 (Y^j_2(1); r \in [t_j, r_2]) & \cdot H_2 (X^i_1(1); r \in [s_i, r_1]) h(y_j, x_i); r_1 \leq \sigma^+] \\
= & \mathbb{E}^{{\mathcal{G}_{r_1}}} \left[ H_{11} (Y^j_2(1); r \in [t_j, r_1]) \hat{H}_{12} (Y^j_2(1)) \right] \cdot \mathbb{E}^{{\mathcal{G}_{r_1}}} \left[ H_{12} (Y^j_2(1); r \in [r_1, r_2]) \cdot h(y_j, x_i); r_1 \leq \sigma^+ \right] \quad (4.94)
\end{align*}
\]
Next, since \( Y^i_t(1) \equiv \psi(1) \varepsilon \) is obviously \( \mathbb{P} \)-independent of \( X^i_t(1) \) and \( \sigma^+ \geq t_j \) by assumption, we can do an orthogonal continuation of \( X^i(1) \) over \([\sigma^+, \infty)\) by Lemma 4.16. This gives a Feller diffusion \( \hat{X}^i \) such that \( \hat{X}^i \perp Y^j(1) \) under \( \mathbb{P} \) and \( \hat{X}^i_{\sigma^+} = X^i(1)_{\sigma^+} \). Hence,

\[
X^i(1) = \hat{X}^i \quad \text{over } [s_i, r_1] \quad \text{on } [r_1 \leq \sigma^+]
\]

and from (4.94) we get

\[
\begin{align*}
\mathbb{E}^\mathbb{P} \left[ H_1 \left( Y^i_t(1); r \in [t_j, r_2] \right) H_2 \left( X^i_t(1); r \in [s_i, r_1] \right) h(y_j, x_i); r_1 \leq \sigma^+ \right] \\
= \mathbb{E}^\mathbb{P} \left[ H_{1,1} \left( Y^i_t(1); r \in [t_j, r_1] \right) \hat{H}_{1,2} \left( Y^i_{r_1}(1) \right) \right] \\
\leq \mathbb{E}^\mathbb{P} \left[ H_{1,1} \left( Y^i_t(1); r \in [t_j, r_1] \right) \hat{H}_{1,2} \left( Y^i_{r_1}(1) \right) \hat{H}_{2,2} \left( \hat{X}^i_{r_1}; r \in [t_j, r_1] \right) h(y_j, x_i) \right], \\
\end{align*}
\]

(4.95)

where the last inequality follows from the non-negativity of \( \hat{H}_{1,2}, H_{k,\ell}, \) and \( h \).

Next, we condition on \( \mathcal{G}_{t_j} \). From (4.95), we get

\[
\begin{align*}
\mathbb{E}^\mathbb{P} \left[ H_1 \left( Y^i_t(1); r \in [t_j, r_2] \right) H_2 \left( X^i_t(1); r \in [s_i, r_1] \right) h(y_j, x_i); r_1 \leq \sigma^+ \right] \\
\leq \mathbb{E}^\mathbb{P} \left[ \mathbb{E}^\mathbb{P} \left[ H_{1,1} \left( Y^i_t(1); r \in [t_j, r_1] \right) \hat{H}_{1,2} \left( Y^i_{r_1}(1) \right) \hat{H}_{2,2} \left( \hat{X}^i_{r_1}; r \in [t_j, r_1] \right) \mathcal{G}_{t_j} \right] \right] \\
\mathbb{E}^\mathbb{P} \left[ H_{1,1} \left( Y^i_t(1); r \in [t_j, r_1] \right) \hat{H}_{1,2} \left( Y^i_{r_1}(1) \right) \hat{H}_{2,2} \left( \hat{X}^i_{r_1}; r \in [t_j, r_1] \right) h(y_j, x_i) \right], \\
\end{align*}
\]

(4.96)

To evaluate the conditional expectation in the last term, we use the independence between \( \hat{X}^i \) and \( Y^j(1) \) and deduce from the martingale problem formulation and Theorem 4.4.2 of [6] that the two-dimensional process \((\hat{X}^i, Y^j(1)) | [t_j, \infty)\) is \((\mathcal{G}_{t_j})_{t_j \geq t_j}\)-Markov with joint law

\[
\mathcal{L}(\hat{X}^i | [t_j, \infty)) \otimes \mathcal{L}(Y^j(1) | [t_j, \infty)).
\]

Hence,

\[
\begin{align*}
\mathbb{E}^\mathbb{P} \left[ H_{1,1} \left( Y^i_t(1); r \in [t_j, r_1] \right) \hat{H}_{1,2} \left( Y^i_{r_1}(1) \right) \hat{H}_{2,2} \left( \hat{X}^i_{r_1}; r \in [t_j, r_1] \right) \mathcal{G}_{t_j} \right] \\
= \mathbb{E}^\mathbb{P} \left[ H_{1,1} \left( Y^i_t(1); r \in [t_j, r_1] \right) \hat{H}_{1,2} \left( Y^i_{r_1}(1) \right) \right] \\
\times \mathbb{E}^\mathbb{P} \left[ \mathcal{P}^{\hat{X}^i_{t_j}} [H_{2,2}(Z_r; r \in [0, r_1 - t_j])], \right],
\end{align*}
\]

where we recall that \((Z, \mathcal{P}^0_t)\) denotes a copy of \( \frac{1}{2}\text{BESQ}^0(4\varepsilon) \). (The value of \( Y^j(1) \) at \( t_j \) is \( \psi(1) \varepsilon \).) Applying the foregoing equality to (4.96) and using (4.93), we obtain

\[
\begin{align*}
\mathbb{E}^\mathbb{P} \left[ H_1 \left( Y^i_t(1); r \in [t_j, r_2] \right) H_2 \left( X^i_t(1); r \in [s_i, r_1] \right) h(y_j, x_i); r_1 \leq \sigma^+ \right] \\
\leq \mathbb{E}^\mathbb{P} \left[ H_1 \left( Y^i_t(1); r \in [t_j, r_2] \right) \right] \\
\times \mathbb{E}^\mathbb{P} \left[ \mathcal{P}^{\hat{X}^i_{t_j}} [H_{2,2}(Z_r; r \in [0, r_1 - t_j])], \right] \\
\mathbb{E}^\mathbb{P} \left[ H_{2,1} \left( \hat{X}^i_{r_1}; r \in [s_i, t_j] \right) h(y_j, x_i) \right], \\
\end{align*}
\]

(4.97)
where the last equality follows since we only redefine $X^t(\mathbb{I})$, for $t \geq \sigma^j$ to obtain $\tilde{X}^t$, whereas $\sigma^j \geq t_j$. The rest is easy to obtain. Using (4.8), we see that (4.97) gives

\[
\mathbb{E}^P \left[ H_1 \left( Y_r^j(\mathbb{I}); r \in [t_j, r_2] \right) H_2 \left( X^t(\mathbb{I}); r \in [s_i, r_1] \right) h(y_j, x_i); r_1 \leq \sigma^j \right] \\
\leq \mathbb{E}^P \left[ H_1 \left( Y_r^j(\mathbb{I}); r \in [t_j, r_2] \right) \right] \\
\times \mathbb{E}^P \left[ \mathbf{P}^{X(\cdot), t_j} \left[ H_{2,2} \left( Z_r; r \in [0, r_1 - t_j] \right) \right] H_{2,1} \left( X^t(\mathbb{I}); r \in [s_i, t_j] \right) \right] \\
= \mathbb{E}^P \left[ H_1 \left( Y_r^j(\mathbb{I}); r \in [t_j, r_2] \right) \right] \mathbb{E}^P \left[ H_2 \left( X^t(\mathbb{I}); r \in [s_i, r_1] \right) \right] \mathbb{E}(h(y_j, x_i)).
\]

We have obtained the desired inequality, and the proof is complete. \(\square\)

We are now ready to prove Lemma 4.11 with arguments similar to those in Section 4.4. The following steps are labelled in the same way as their counterparts in Section 4.4, except that (Step 2-5) and (Step 3) below correspond to (Step 2-6) and (Step 4) in Section 4.4, respectively. Due to the similarity, we will only point out the key changes, leaving other details to readers.

Recall that we fix $t \in [s_i + \frac{t}{2}, s_i + 1]$, $i \in \mathbb{N}$, and $\varepsilon \in (0, [8\psi(\mathbb{I})]^{-1} \land 1]$.

**Step 1.** We begin with a simple observation for the integral term

\[
\int_{t_j}^{t \land \tau^i \land \sigma^X \land \sigma^Y} \frac{1}{X^t(\mathbb{I})} \int_{\mathbb{R}} X^t(x, s)^{1/2} Y^j(x, s)^{1/2} dx ds
\]

in (4.33), for $y_j = y$ satisfying (4.84) and $j \in \mathbb{N}$ with $t_j \in (s_i, s_i + 1]$. For $s \in [t_j, t \land \tau^i \land \sigma^X \land \sigma^Y]$, with $s < t_j$, the support processes of $X^t$ and $Y^j$ can be enveloped by $\mathcal{P}^X_{\sigma^X}(\cdot)$ and $\mathcal{P}^Y_{\sigma^Y}(\cdot)$ up to time $s$, respectively, and $\mathcal{P}^X_{\sigma^X}(s) \cap \mathcal{P}^Y_{\sigma^Y}(s) = \varnothing$ by the definition of $t_j$ in (4.86). Hence, for such $s$,

\[
\int_{\mathbb{R}} X^t(x, s)^{1/2} Y^j(x, s)^{1/2} dx = 0.
\]

Using the bound (4.81), we obtain as for (4.37) that

\[
\mathbb{E}^Q \left[ \sum_{j \in L_{\sigma^X}(t, t \land \tau^i \land \sigma^X)} \left( \psi(\mathbb{I}) \varepsilon + \int_{t_j}^{t \land \tau^i \land \sigma^X \land \sigma^Y} \frac{1}{X^t(\mathbb{I})} \int_{\mathbb{R}} X^t(x, s)^{1/2} Y^j(x, s)^{1/2} dx ds \right) \right]
\]

\[
\leq \sum_{j: s_i < t_j \leq t} (t - s_i)^{\beta^j} \varepsilon + \sum_{j: s_i < t_j \leq t} \int_{t_j}^{t} ds \int_{t_i \land s_i \leq s} \frac{1}{(s - s_i)^{\eta/2}} \mathbb{E}^Q \left[ \left| \mathbf{Y}_s^j(\mathbb{I}) \right|^{1/2} \right];
\]

\[
2(\varepsilon^{1/2} + (t_j - s_i)^{\beta^j}) \leq |y_j - x_i| \leq 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta^j} \varepsilon).
\]

Hence, for lateral clusters, we consider

\[
\mathbb{E}^Q \left[ \left| \mathbf{Y}_s^j(\mathbb{I}) \right|^{1/2}; s < t \land \sigma^X \land \sigma^Y \right],
\]

\[
2(\varepsilon^{1/2} + (t_j - s_i)^{\beta^j}) \leq |y_j - x_i| \leq 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta^j}),
\]

\[
s \in (t_j, t], s_i < t_j < t, t_j^* < t.
\]

\(\square\)
(Step 2-1). We partition the event \([X_s^j(1)^{T_s^X} > 0]\) into the two events in (4.42) and (4.43) with \(t_j\) replaced by \(t_j^*\). Then as in (4.44), we write
\[
\mathbb{E}^P \left[ \left( Y_s^j(1) \right)^{1/2} \right] ; s < \tau^\epsilon \wedge \sigma^X_{\beta} \wedge \sigma^Y_{\beta}, 2(\epsilon^{1/2} + (t_j - s_i)^{\beta'}) \leq |y_j - x_i| \\
\leq 2(\epsilon^{1/2} + (t - s_i)^{\beta'}) \leq 2(\epsilon^{1/2} + (t - s_i)^{\beta'}), \\
T_s^X < T_0^{X^j} \leq t_j^*
\]
\[
\leq \frac{1}{\psi(1)} \mathbb{E}^P \left[ \left( Y_s^j(1) \right)^{1/2} \right] ; s < \tau^\epsilon \wedge \sigma^X_{\beta} \wedge \sigma^Y_{\beta}, 2(\epsilon^{1/2} + (t_j - s_i)^{\beta'}) \leq |y_j - x_i| \\
\leq 2(\epsilon^{1/2} + (t - s_i)^{\beta'}) \leq 2(\epsilon^{1/2} + (t - s_i)^{\beta'}), \\
X_s^j(1)^{T_s^X} > 0, t_j^* < T_0^{X^j}(4.99)
\]
\[
\leq \frac{1}{\psi(1)} \mathbb{E}^P \left[ \left( Y_s^j(1) \right)^{1/2} \right] ; s < \tau^\epsilon \wedge \sigma^X_{\beta} \wedge \sigma^Y_{\beta}, 2(\epsilon^{1/2} + (t_j - s_i)^{\beta'}) \leq |y_j - x_i| \\
\leq 2(\epsilon^{1/2} + (t - s_i)^{\beta'}) \leq 2(\epsilon^{1/2} + (t - s_i)^{\beta'}), t_j^* < T_0^{X^j}, \\
\forall s \in (t_j^*, t], s_i < t_j < t, t_j^* < t,
\]

where we replace the event \([X_s^j(1)^{T_s^X} > 0, t_j^* < T_0^{X^j}]\) by the larger one \([t_j^* < T_0^{X^j}]\) for the third term.

(Step 2-2). Consider the first term on the right-hand side of (4.99). We have
\[
\frac{1}{\psi(1)} \mathbb{E}^P \left[ \left( Y_s^j(1) \right)^{1/2} \right] ; s < \tau^\epsilon \wedge \sigma^X_{\beta} \wedge \sigma^Y_{\beta}, 2(\epsilon^{1/2} + (t_j - s_i)^{\beta'}) \leq |y_j - x_i| \\
\leq 2(\epsilon^{1/2} + (t - s_i)^{\beta'}) \leq 2(\epsilon^{1/2} + (t - s_i)^{\beta'}), \\
T_s^X < T_0^{X^j} \leq t_j^*(4.100)
\]

We then apply Proposition 4.17, taking
\[
\sigma^j = \left( \sigma^X_{\beta} \wedge \sigma^Y_{\beta} \wedge t_j^* \right) \vee t_j, r_1 = t_j^*, r_2 = s. (4.101)
\]
Hence, from (4.81) and (4.100), we obtain
\[
\frac{1}{\psi(1)} \mathbb{E}^P \left[ \left( Y_s^j(1) \right)^{1/2} \right] ; s < \tau^\epsilon \wedge \sigma^X_{\beta} \wedge \sigma^Y_{\beta}, 2(\epsilon^{1/2} + (t_j - s_i)^{\beta'}) \leq |y_j - x_i| \\
\leq 2(\epsilon^{1/2} + (t - s_i)^{\beta'}) \leq 2(\epsilon^{1/2} + (t - s_i)^{\beta'}), T_s^X < T_0^{X^j} \leq t_j^* \\
\leq \frac{1}{\epsilon} \mathbb{E} \left( T_s^X < T_0^{X^j} \leq t_j^* \right) (t - s_i)^{\beta'} \mathbb{E}^P_{\psi(t)} \left[ (Z_{t-s_j}) \right]^{1/2} ; s - t_j \leq T_1^2
\]
\[
\leq (t - s_i)^{\beta'} \cdot \mathbb{E}^{P_0^{(1)} \epsilon} \left[ (Z_{s-t_j})^{1/2}; s - t_j \leq T_1^Z \right], \\
\forall s \in (t_j^*, t], s_i < t_j < t, t_j^* < t. 
\]

(Step 2-3). Let us consider the second term in (4.99). As before, using (4.48) gives
\[
\frac{1}{\psi(1)\varepsilon} \mathbb{E}^\mathcal{F} \left[ X_i^j(1)^{T_i^X}; - \psi(1)\varepsilon \left| Y_j^j(1) \right|^{1/2}; s < \tau^i \land \sigma^X_{\beta} \land \sigma^Y_{\beta} \right], \\
2(e^{1/2} + (t_j - s_i)^{\beta'}) \leq |y_j - x_i| \leq 2(e^{1/2} + (t_j - s_i)^{\beta'}), X_i^j(1)^{T_i^X} > 0, t_j^* < T_0^{X_i} \\
\leq \varepsilon^{\alpha_N}(s - s_i)^{\alpha} + (s - s_i)^{\xi} \mathbb{E}^\mathcal{F} \left[ Y_j^j(1) \right]^{1/2}; s \leq T_1^{Y_i}, t_j^* \leq \sigma^X_{\beta} \land \sigma^Y_{\beta} \\
2(e^{1/2} + (t_j - s_i)^{\beta'}) \leq |y_j - x_i| \leq 2(e^{1/2} + (t_j - s_i)^{\beta'}), t_j^* < T_0^{X_i}, \\
\forall s \in (t_j^*, t], s_i < t_j < t, t_j^* < t. 
\]

(4.103)

Taking the choice (4.101) again, we obtain from Proposition 4.17, (4.81), and the last display that
\[
\frac{1}{\psi(1)\varepsilon} \mathbb{E}^\mathcal{F} \left[ X_i^j(1)^{T_i^X}; - \psi(1)\varepsilon \left| Y_j^j(1) \right|^{1/2}; s < \tau^i \land \sigma^X_{\beta} \land \sigma^Y_{\beta} \right], \\
2(e^{1/2} + (t_j - s_i)^{\beta'}) \leq |y_j - x_i| \leq 2(e^{1/2} + (t_j - s_i)^{\beta'}), X_i^j(1)^{T_i^X} > 0, t_j^* < T_0^{X_i} \\
\leq \varepsilon^{\alpha_N}(s - s_i)^{\alpha} + (s - s_i)^{\xi} \mathbb{E}^\mathcal{F} \left[ t_j^* < T_0^{X_i} \right] \mathbb{E}^{P_0^{(1)} \epsilon} \left[ (Z_{s-t_j})^{1/2}; s - t_j \leq T_1^Z \right], \\
\forall s \in (t_j^*, t], s_i < t_j < t, t_j^* < t. 
\]

(4.104)

Hence, by a computation similar to (4.50) and Lemma 4.15, the foregoing display gives
\[
\frac{1}{\psi(1)\varepsilon} \mathbb{E}^\mathcal{F} \left[ X_i^j(1)^{T_i^X}; - \psi(1)\varepsilon \left| Y_j^j(1) \right|^{1/2}; s < \tau^i \land \sigma^X_{\beta} \land \sigma^Y_{\beta} \right], \\
2(e^{1/2} + (t_j - s_i)^{\beta'}) \leq |y_j - x_i| \leq 2(e^{1/2} + (t_j - s_i)^{\beta'}), X_i^j(1)^{T_i^X} > 0, t_j^* < T_0^{X_i} \\
\leq (\varepsilon^{\alpha_N}(s - s_i)^{\alpha} + (s - s_i)^{\xi}) \mathbb{E}^\mathcal{F} \left[ t_j^* - s_i \right] \mathbb{E}^{P_0^{(1)} \epsilon} \left[ (Z_{s-t_j})^{1/2}; s - t_j \leq T_1^Z \right], \\
\forall s \in (t_j^*, t], s_i < t_j < t, t_j^* < t. 
\]

(4.105)

(Step 2-4). For the third term in (4.99), the calculation in the foregoing (Step 2-3) readily shows
\[
\frac{1}{\psi(1)\varepsilon} \cdot \psi(1)\varepsilon \mathbb{E}^\mathcal{F} \left[ Y_j^j(1) \right]^{1/2}; s < \tau^i \land \sigma^X_{\beta} \land \sigma^Y_{\beta} \right], \\
2(e^{1/2} + (t_j - s_i)^{\beta'}) \leq |y_j - x_i| \leq 2(e^{1/2} + (t_j - s_i)^{\beta'}), t_j^* < T_0^{X_i} \\
\leq (t - s_i)^{\beta'}(t_j - s_i)^{-\beta'} \cdot \varepsilon \cdot \mathbb{E}^{P_0^{(1)} \epsilon} \left[ (Z_{s-t_j})^{1/2}; s - t_j \leq T_1^Z \right], \\
\forall s \in (t_j^*, t], s_i < t_j < t, t_j^* < t. 
\]

(4.106)

(Step 2-5). At this step, we apply (4.102), (4.104), and (4.105) to (4.99) and give a summary as
We complete the proof of Lemma 4.11 in this step. Applying the bound (4.107) to the

\[ E^{\mathcal{Q}'} \left[ \left[ Y^2 \left( 1 \right) \right]^{1/2} : s < \tau^i \wedge \sigma^{X^i} \wedge \sigma^{Y^j} \right], \]

\[ 2 \left( \varepsilon^{1/2} + (t_j - s_i)^{\beta'} \right) \leq |y_j - x_i| \leq 2 \left( \varepsilon^{1/2} + (t - s_i)^{\beta'} \right) \]

\[ \leq (t - s_i)^{\beta'} (t_j - s_i) - \frac{\varepsilon^\prime}{\pi} \left( (t_j - s_i) \frac{\sigma}{\pi} + \varepsilon^{N_0} (s - s_i)^\alpha + (s - s_i)^\xi + \varepsilon \right) \]

\times \varepsilon^{\frac{\beta}{2}} \left( \frac{\varepsilon}{s - t_j} \right)^{1 - \frac{\alpha N_0}{2}}

+ (t - s_i)^{\beta'} (t_j - s_i) - \frac{\varepsilon^\prime}{\pi} \left( (t_j - s_i) \frac{\sigma}{\pi} + \varepsilon^{N_0} (s - s_i)^\alpha + (s - s_i)^\xi + \varepsilon \right)

+ (t - s_i)^{\beta'} (t_j - s_i) - \frac{\varepsilon^\prime}{\pi} (s - s_i)^\xi (s - t_j)^{\frac{\xi - 1}{2}},

\forall s \in \left( t^*_j, t_j \right], \quad s_i < t_j < t, \quad t^*_j < t,

where as in (Step 2-6) of Section 4.4, the last "\( \triangleq \)"-inequality follows again from Lemma 4.13, some arithmetic, and an application of (4.66).

Now, for any \( s \in \left( t^*_j, t \right] \) with \( s_i < t_j < t \) and \( t^*_j < t \), we have

\[ (t_j - s_i) \frac{\sigma}{\pi} + \varepsilon^{N_0} (s - s_i)^\alpha + (s - s_i)^\xi + \varepsilon \triangleq (s - s_i)^\alpha, \]

which results from (4.25)-(a), (4.25)-(b), Lemma 4.15, and \( t_j - s_i \geq \frac{\sigma}{\pi} \). Hence, with some simplifications similar to (4.67)-(4.69), we obtain

\[ E^{\mathcal{Q}'} \left[ \left[ Y^2 \left( 1 \right) \right]^{1/2} : s < \tau^i \wedge \sigma^{X^i} \wedge \sigma^{Y^j} \right], \]

\[ 2 \left( \varepsilon^{1/2} + (t_j - s_i)^{\beta'} \right) \leq |y_j - x_i| \leq 2 \left( \varepsilon^{1/2} + (t - s_i)^{\beta'} \right) \]

\[ \leq (t - s_i)^{\beta'} (t_j - s_i) - \frac{\varepsilon^\prime}{\pi} (s - s_i)^\alpha (s - t_j)^{\frac{\alpha N_0}{2} - 1} \varepsilon^1 + \frac{\alpha N_0}{2}\]

\[ (t - s_i)^{\beta'} (t_j - s_i) - \frac{\varepsilon^\prime}{\pi} (s - s_i)^\alpha (s - t_j)^{\frac{\alpha N_0}{2} - 1} \varepsilon^1 + \frac{\alpha N_0}{2}\]

\[ + (t - s_i)^{\beta'} (s - t_j)^{\frac{\xi - 1}{2}} \]

\[ + (t - s_i)^{\beta'} (s - t_j)^{\frac{\xi - 1}{2}} \varepsilon, \]

\[ \forall s \in \left( t^*_j, t \right], \quad s_i < t_j < t, \quad t^*_j < t. \]
right-hand side of the inequality (4.98). We have

\[
\mathbb{E}^\mathbb{Q} \left[ \sum_{j \in \mathcal{L}_\beta^\prime(\cdot, t) \cap \mathcal{L}_\beta(\cdot, s)} \left( \psi(\mathbb{1}) \varepsilon + \int_{t_j}^{t \wedge \tau \wedge \sigma^X_\beta \wedge \sigma^X_\beta \wedge \sigma^Y_\beta} \frac{1}{\mathbb{X}^\varepsilon_\beta(\mathbb{1})} \int_\mathbb{R} X^1(x, s)^{1/2} Y^j(x, s)^{1/2} dx ds \right) \right] \\
\leq (t-s_1)^{\beta'} + (t-s_1)^{\beta'} \sum_{j : s_i < t_j \leq t} (t_j - s_i)^{-\frac{\varepsilon}{\alpha}} \int_{t_j}^{t} (s - s_i)^{-\frac{\varepsilon}{\alpha} + \eta} (s - t_j)^{-\frac{\varepsilon}{\alpha} + \eta} ds \cdot \varepsilon^{-\frac{\varepsilon}{\alpha} + \eta} \\
+ (t-s_1)^{\beta'} \sum_{j : s_i < t_j \leq t} (t_j - s_i)^{-\frac{\varepsilon}{\alpha} + \eta} \int_{t_j}^{t} (s - s_i)^{-\frac{\varepsilon}{\alpha} + \eta} ds \cdot \varepsilon^{-\frac{\varepsilon}{\alpha} + \eta} \varepsilon^{-\frac{\varepsilon}{\alpha} + \eta} \\
+ (t-s_1)^{\beta'} \sum_{j : s_i < t_j \leq t} (t_j - s_i)^{-\frac{\varepsilon}{\alpha} + \eta} \int_{t_j}^{t} (s - s_i)^{-\frac{\varepsilon}{\alpha} + \eta} (s - t_j)^{-\frac{\varepsilon}{\alpha} + \eta} ds \cdot \varepsilon^{-\frac{\varepsilon}{\alpha} + \eta}.
\]

Thanks to the second inequality in (4.25)-(b), the integral domination outlined in (Step 3) of Section 4.4 can be applied to each term on the right-hand side of the foregoing \(\leq\)-inequality, giving bounds which are convergent integrals. As in (Step 4) of Section 4.4, we obtain

\[
\mathbb{E}^\mathbb{Q} \left[ \sum_{j \in \mathcal{L}_\beta^\prime(\cdot, t) \cap \mathcal{L}_\beta(\cdot, s)} \left( \psi(\mathbb{1}) \varepsilon + \int_{t_j}^{t \wedge \tau \wedge \sigma^X_\beta \wedge \sigma^X_\beta \wedge \sigma^Y_\beta} \frac{1}{\mathbb{X}^\varepsilon_\beta(\mathbb{1})} \int_\mathbb{R} X^1(x, s)^{1/2} Y^j(x, s)^{1/2} dx ds \right) \right] \\
\leq (t-s_1)^{\beta'} + (t-s_1)^{\beta'} + (t-s_1)^{\beta'} \cdot \varepsilon^{-\frac{\varepsilon}{\alpha} + \eta} + (t-s_1)^{\beta'} \cdot \varepsilon^{-\frac{\varepsilon}{\alpha} + \eta},
\]

which proves Lemma 4.11.

5. Uniform separation of approximating solutions

In this section, we prove the main theorem on the pathwise non-uniqueness of nonnegative solutions of the SPDE (1.2), and the result is summarized in Theorem 5.7. We will need the uniform separation of approximating solutions, and our tasks will be to show how it can be obtained from the conditional separation implied by Theorem 4.6 (cf. Remark 4.7) and obtain appropriate probability bounds. We continue to suppress the dependence on \(\varepsilon\) of the approximation solutions and use only \(\mathbb{P}_\varepsilon\) for emphasis, unless otherwise mentioned.

Our program is sketched as follows. For small \(r \in (0, 1]\), we choose a number \(\Delta(r) \in (0, \infty)\) and an event \(S(r)\) satisfying the following properties. First, \(\Delta(r)\) depends only on the parameter vector in Assumption 4.5 and \(r\). Second, for every small \(\varepsilon \in (0, 1]\), \(S(r) = S_\varepsilon(r)\) is defined by the approximating solutions \(X\) and \(Y\) such that

\[
S(r) \subseteq \left[ \sup_{0 \leq s \leq 2r} \|X_s - Y_s\|_{\text{sup}} \geq \Delta(r) \right] \\
(5.1)
\]

and

\[
\liminf_{\varepsilon \downarrow 0} \mathbb{P}_\varepsilon(S(r)) > 0. \\
(5.2)
\]
(Recall the definition of $\| \cdot \|_{\text{rap}}$ in (1.5).)
Let us define the events $S_{\varepsilon}(r)$. First, recall the parameter vector chosen in Assumption 4.5 as well as the constants $\kappa_j$ defined in Theorem 4.6. In the following, we need to use small portions of the constants $\kappa_1$ and $\kappa_3$, and by (4.25)-(d) we can take
$$\varepsilon \in (0, \kappa_1 \wedge \kappa_3)$$
such that $\kappa_1 - \varepsilon > \eta$.
We insist that $\varepsilon$ depends only on the parameter vector in (4.24). For any $i \in \mathbb{N}$, $\varepsilon \in (0, [8\psi(1)]^{-1} \wedge 1]$, and random time $T \geq s_i$, let $G^i(T) = G^i(T)$ be the event defined by
$$G^i(T) = \left\{ \begin{array}{ll}
X_i^i([x_i - \varepsilon^{1/2} - (s - s_i)\beta, x_i + \varepsilon^{1/2} + (s - s_i)\beta]) \\
\geq \frac{(s - s_i)^{\eta}}{4} \quad \text{and} \\
Y_s([x_i - \varepsilon^{1/2} - (s - s_i)\beta, x_i + \varepsilon^{1/2} + (s - s_i)\beta]) \\
\leq K^* \left[ (s - s_i)^{\kappa_1 - \varepsilon} + \varepsilon^{\kappa_2} (s - s_i)^{\kappa_3 - \varepsilon} \right], \forall s \in [s_i, T]
\end{array} \right\}, \quad (5.3)$$
where the constant $K^* \in (0, \infty)$ is as in Theorem 4.6. Note that $G^i(\cdot)$ is decreasing in the sense that, for any random times $T_1, T_2$ with $T_1 \leq T_2$, $G^i(T_1) \supseteq G^i(T_2)$. We then choose
$$S(r) = S_{\varepsilon}(r) = \bigcup_{i=1}^{\left\lfloor \frac{r}{r^{\varepsilon^{-1}}} \right\rfloor} G^i(s_i + r), \quad r \in (0, \infty), \varepsilon \in \left(0, \frac{1}{8\psi(1)} \wedge 1\right]. \quad (5.4)$$
We will explain later on that the events $G^i(\cdot)$ roughly capture the main feature of the events considered in Theorem 4.6 as well as the initial behaviour of $X^i(\cdot)$ under $\mathbb{Q}^i_t$ discussed in Section 4.2. The inclusion (5.1) is now a simple consequence of our choice of the events $G^i(\cdot)$.

**Lemma 5.1.** For some $r_0 \in (0, 1]$, we can find $\varepsilon_0(r) \in (0, r \wedge [8\psi(1)]^{-1} \wedge 1]$ and $\Delta(r) \in (0, \infty)$ for any $r \in (0, r_0]$ so that the inclusion (5.1) holds almost surely for any $\varepsilon \in (0, \varepsilon_0(r)]$. The constant $\Delta(r)$ depends only on $r$ and the parameter vector chosen in Assumption 4.5.

**Proof.** We first specify the strictly positive numbers $r_0, \varepsilon_0(r)$, and $\Delta(r)$. Since the small portion $\varepsilon$ taken away from $\kappa_1$ and $\kappa_3$ satisfies $\kappa_1 - \varepsilon > \eta$, we can choose $r_0 \in (0, 1]$ such that
$$\frac{r_0^{-\eta}}{4} - 2K^* r_0^{-\kappa_1 - \varepsilon} > 0, \quad \forall r \in (0, r_0]. \quad (5.5)$$
Then we choose, for every $r \in (0, r_0]$, a number $\varepsilon_0(r) \in (0, r \wedge [8\psi(1)]^{-1} \wedge 1]$ such that
$$0 < \varepsilon_0(r) \leq r^{\kappa_2} \leq r^{\kappa_1 - \kappa_3}. \quad (5.6)$$
Finally, we set
$$\Delta(r) \triangleq \frac{1}{2} \left[ \left( \frac{r_0^{-\eta}}{2} - 2K^* r_0^{-\kappa_1 - \varepsilon} \right) \wedge 1 \right] > 0, \quad r \in (0, r_0]. \quad (5.7)$$
We now check that the foregoing choices give (5.1). Fix $r \in (0, r_0], \varepsilon \in (0, \varepsilon_0(r)]$, and $1 \leq i \leq \left\lfloor r\varepsilon^{-1} \right\rfloor$. Note that $r\varepsilon^{-1} \geq 1$ since $\varepsilon \leq r$. The arguments in this paragraph are understood to be valid on $G^i(s_i + r)$. By definition,
$$Y_s([x_i - \varepsilon^{1/2} - (s - s_i)\beta, x_i + \varepsilon^{1/2} + (s - s_i)\beta]) \leq K^* \left[ (s - s_i)^{\kappa_1 - \varepsilon} + \varepsilon^{\kappa_2} (s - s_i)^{\kappa_3 - \varepsilon} \right], \quad \forall s \in [s_i, s_i + r]. \quad (5.8)$$
In particular, (5.6) and (5.8) imply that

\[ Y_{s_i + r}(|x_i - \varepsilon^{1/2} - r^\beta, x_i + \varepsilon^{1/2} + r^\beta|) \leq 2K^*r^{\kappa_1 - \varphi}. \]

Since \( X \geq X^1 \), the last inequality and the definition of \( G^i(s_i + r) \) imply

\[
X_{s_i + r}(|x_i - \varepsilon^{1/2} - r^\beta, x_i + \varepsilon^{1/2} + r^\beta|)
- Y_{s_i + r}(|x_i - \varepsilon^{1/2} - r^\beta, x_i + \varepsilon^{1/2} + r^\beta|)
\geq \frac{r^n}{4} - 2K^*r^{\kappa_1 - \varphi},
\]

where the lower bound is strictly positive by (5.5). To carry this to the \( C_{\text{rap}}(\mathbb{R}) \)-norm of \( X_{s_i + r} - Y_{s_i + r} \), we make an elementary observation: if \( f \) is Borel measurable, integrable on a finite interval \( I \), and satisfies \( f > A \), then there must exist some \( x \in I \) such that \( f(x) > A/\ell(I) \), where \( \ell(I) \) is the length of \( I \). Using this, we obtain from the last inequality that, for some \( x \in [x_i - \varepsilon^{1/2} - r^\beta, x_i + \varepsilon^{1/2} + r^\beta] \),

\[
X(x, s_i + r) - Y(x, s_i + r) \geq \frac{r^n}{2\varepsilon^{1/2} + 2r^\beta} - 2K^*r^{\kappa_1 - \varphi},
\]

so the definition of \( \| \cdot \|_{\text{rap}} \) (in (1.5)) and the definition (5.7) of \( \Delta(r) \) entail

\[
\Delta(r) \leq \| X_{s_i + r} - Y_{s_i + r} \|_{\text{rap}} \leq \sup_{0 \leq s \leq 2r} \| X_s - Y_s \|_{\text{rap}},
\]

where the second inequality follows since \( s_i = \frac{(2i + 1)\varepsilon}{2} \) and \( 1 \leq i \leq |r\varepsilon^{-1}| \).

In summary, we have shown that (5.1) holds because each component \( G^i(s_i + r) \) of \( S(r) \) satisfies the analogous inclusion. The proof is complete.

We move on to show that (5.2) holds whenever \( r > 0 \) is small enough. To use Theorem 4.6, we need to bring the involved stopping times into the events \( G^i(\cdot) \) and change the statements about \( Y \).

First, we define \( \Gamma_i(r) = \Gamma_{\varepsilon}^i(r) \) by

\[
\Gamma_i(r) \triangleq \begin{cases} 
\mathcal{P}_\beta X^i(s_i + r) \cap \left( \bigcup_{j : t_j \leq s_i} \text{supp}(Y^j) \right) = \emptyset \cap \bigcap_{j : t_j \leq s_i + r} \left[ \sigma^X_j > t_j + 3r \right] \\
\cap \left[ \sigma^X_i > s_i + 2r \right], & r \in (0, 1], \ i \in \mathbb{N}, \ \varepsilon \in \left( 0, \frac{1}{8\psi(1)} \right] \end{cases}
\]

(5.9)

where \( \text{supp}(Y^j) \) denotes the topological support of the two-parameter (random) function \( (x, s) \mapsto Y^j(x, s) \). Hence, through \( \Gamma_i(r) \), we confine the ranges of the supports of \( Y^j \), for \( j \in \mathbb{N} \) satisfying \( t_j \leq s_i + r \), and \( X^i \). As will become clear in passing, one of the reasons for considering this event is to make precise the informal argument of choosing \( J^i_{\beta}(\cdot) \), as discussed in Section 4.2.

**Lemma 5.2.** Let \( r \in (0, \infty) \). Then for any \( j \in \mathbb{N} \) with \( t_j \in (s_i, s_i + r] \) and \( |y_j - x_i| > 2(\varepsilon^{1/2} + r^\beta) \),

\[
\mathcal{P}_\beta X^i(s_i + r) \cap \mathcal{P}_\beta Y^j(s_i + r) = \emptyset.
\]

The proof of Lemma 5.2 follows from elementary algebra and can be found in Lemma 3.53 in [4]. With this result, we now identify the potential \( Y^j \)-invaders of the support of \( X^i \).
Lemma 5.3. Fix $r \in (0,1]$, $i \in \mathbb{N}$, and $\varepsilon \in (0,[8\psi(1)]^{-1} \wedge 1]$. Then on the event $\Gamma^i(r)$ defined by (5.9), we have

$$Y_s([x_i - \varepsilon^{1/2} - (s - s_i)^{\beta}, x_i + \varepsilon^{1/2} + (s - s_i)^{\beta}]) = \sum_{j \in J^i_\beta(s)} Y^j_s([x_i - \varepsilon^{1/2} - (s - s_i)^{\beta}, x_i + \varepsilon^{1/2} + (s - s_i)^{\beta}]),$$

$$(5.10)$$

$\forall s \in [s_i, s_i + r].$

In particular, on $\Gamma^i(r),$

$$Y_s([x_i - \varepsilon^{1/2} - (s - s_i)^{\beta}, x_i + \varepsilon^{1/2} + (s - s_i)^{\beta}]) \leq \sum_{j \in J^i_\beta(s)} Y^j_s(1), \ \forall s \in [s_i, s_i + r].$$

$$(5.11)$$

Proof. In this proof, we argue on the event $\Gamma^i(r)$ and call $\Theta_s \triangleq \{x; (x,s) \in \Theta\}$ the $s$-section of a subset $\Theta$ of $\mathbb{R} \times \mathbb{R}_+$ for any $s \in \mathbb{R}_+$. Consider (5.10). Since the $s$-section $\text{supp}(Y^j)_s$ contains the support of $Y^j_s(\cdot)$, it suffices to show that, for any $s \in [s_i, s_i + r]$ and $j \in \mathbb{N}$ with $t_j \leq s$ and $j \notin J^i_\beta(s)$,

$$[x_i - \varepsilon^{1/2} - (s - s_i)^{\beta}, x_i + \varepsilon^{1/2} + (s - s_i)^{\beta}] \cap \text{supp}(Y^j)_s = \emptyset. \quad (5.12)$$

If $j \in \mathbb{N}$ satisfies $t_j \leq s_i$, then using the first item in the definition (5.9) of $\Gamma^i(r)$ gives

$$\mathcal{P}^X(s_i + r) \cap \text{supp}(Y^j) = \emptyset.$$ 

Hence, taking the $s$-sections of both $\mathcal{P}^X(s_i + r)$ and $\text{supp}(Y^j)$ shows that $Y^j$ satisfies (5.12).

Next, suppose that $j \in \mathbb{N}$ satisfies $s_i < t_j \leq s$ but $j \notin J^i_\beta(s)$. On one hand, this choice of $j$ implies

$$|y_j - x_i| > 2 (\varepsilon^{1/2} + (s - s_i)^{\beta}) \geq 2 (\varepsilon^{1/2} + (s - s_i)^{\beta}),$$

where the second inequality follows from the assumption $r \in (0,1]$ and the choice $\beta' < \beta$ by (4.25)-(b), so Lemma 5.2 entails

$$\mathcal{P}^X(s) \cap \mathcal{P}^Y(s) = \emptyset.$$ 

$$(5.13)$$

On the other hand, using the second item in the definition of $\Gamma^i(r)$, we deduce that

$$\text{supp}(Y^j) \cap (\mathbb{R} \times [t_j, t_j + 3r]) \subseteq \mathcal{P}^Y(t_j + 3r).$$

Using $t_j + r > s_i + r \geq s$ and taking $s$-sections of $\text{supp}(Y^j)$ and $\mathcal{P}^Y(t_j + 3r)$, we obtain from the foregoing inclusion that

$$\text{supp}(Y^j)_s \subseteq [y_j - \varepsilon^{1/2} - (s - t_j)^{\beta}, y_j + \varepsilon^{1/2} + (s - t_j)^{\beta}]
= \mathcal{P}^Y(t_j + 3r)_s
= \mathcal{P}^Y(s)_s.$$ 

$$(5.14)$$

Now, since

$$\mathcal{P}^X(s)_s = [x_i - \varepsilon^{1/2} - (s - s_i)^{\beta}, x_i + \varepsilon^{1/2} + (s - s_i)^{\beta}],$$

(5.13) and (5.14) give our assertion (5.12) for $j \in \mathbb{N}$ satisfying $t_j \leq s$ and $j \notin J^i_\beta(s)$. We have considered all cases for which $j \in \mathbb{N}$, $t_j \leq s$, and $j \notin J^i_\beta(s)$. The proof is complete. $\square$
Recall \(r_0 \in (0,1]\) and \(\varepsilon_0(r) \in (0,r \wedge [8\psi(1)]^{-1} \wedge 1]\) chosen in Lemma 5.1 and the events \(S(r)\) in (5.4).

**Lemma 5.4.** For some \(r_1 \in (0,r_0]\), we can find \(\varepsilon_1(r) \in (0,\varepsilon_0(r)]\) for any \(r \in (0,r_1]\) such that

\[
\inf_{\varepsilon \in (0,\varepsilon_1(r)]} \mathbb{P}_\varepsilon (S(r)) > 0. \tag{5.15}
\]

The proof of Lemma 5.4 relies heavily on Proposition 5.5, and the proofs for the statements of the latter can be adapted from their counterparts in \([\psi]\). See Section 3.12 of \([4]\) for details.

**Proposition 5.5.** (1). There is a constant \(C^0_{\text{supp}} \in (0,\infty)\) depending only on the immigration function \(\psi\) and the parameter \(\beta \in [\frac{1}{3},\frac{1}{2}]\) such that

\[
\mathbb{P}_\varepsilon \left( \sigma^X_i - s_i \leq r \right) + \mathbb{P}_\varepsilon \left( \sigma^{Y_i} - t_i \leq r \right) \leq C^0_{\text{supp}} \varepsilon \vee \varepsilon,
\]

\[
\forall \varepsilon, r \in (0,1], i \in \mathbb{N}. \tag{5.16}
\]

(2). There exists a constant \(C^1_{\text{supp}} \in (0,\infty)\) depending only on the immigration function \(\psi\) such that whenever \(\beta \in [\frac{1}{3},\frac{1}{2}]\),

\[
Q^T_\varepsilon \left( \mathcal{P}^X_i(s_i + r) \cap \left( \bigcup_{j: t_j \leq s_i} \text{supp}(Y_j^i) \right) \right) \neq \emptyset, \quad \min_{j: t_j \leq s_i} \left( \sigma^Y_j - t_j \right) > 3r, \sigma^X_i - s_i > 2r,
\]

\[
\leq C^1_{\text{supp}} r^{1/2}, \quad \forall i \in \mathbb{N} \text{ with } s_i \leq 1, r \in [s_i,1], \varepsilon \in (0,r].
\]

**Proof of Lemma 5.4.** For any \(i \in \mathbb{N}, \varepsilon \in (0,[8\psi(1)]^{-1} \wedge 1]\), and random time \(T \geq s_i\), we define \(\hat{G}^i(\cdot) = \hat{G}^i(\cdot)\) by

\[
\hat{G}^i(T) = \left\{ X^i_s(1) \geq \frac{(s-s_i)^n}{4} \text{ and } \sum_{j \in J^i_s(s)} Y^i_j(1) \leq K^*(s-s_i)^{\kappa_1-\nu} + \varepsilon \kappa_2 (s-s_i)^{\kappa_3-\nu}, \quad \forall s \in [s_i,T] \right\}. \tag{5.17}
\]

Note that \(\hat{G}^i(\cdot)\) is decreasing, and its definition about the masses of \(Y\) is the same as the event considered in Theorem 4.6 except for the restrictions from stopping times \(\tau^i, \sigma^X_i, \text{ and } \sigma^{Y_i}\).

The connection between \(\hat{G}^i(\cdot)\) and \(G^i(\cdot)\) is as follows. First note that by (5.11), the statement about the masses of \(Y\) in \(\hat{G}^i(r) \cap \Gamma^i(r)\) implies that in \(G^i(r) \cap \Gamma^i(r)\). Also, the statements in \(G^i(\cdot)\) and \(\hat{G}^i(\cdot)\) concerning the masses of \(X^i\) are linked by the obvious equality:

\[
X^i_s(1) = X^i_s(\left[x_i - \varepsilon^{1/2} - (s-s_i)^{\beta}, x_i + \varepsilon^{1/2} + (s-s_i)^{\beta}\right]), \quad \forall s \in [s_i,\sigma^X_i].
\]

Since \(\sigma^X_i > s_i + 2r\) on \(\Gamma^i(r)\), we are led to the inclusion

\[
\hat{G}^i \left( \tau^i \wedge (s_i + r) \right) \cap \Gamma^i(r) \subseteq G^i \left( \tau^i \wedge (s_i + r) \right) \cap \Gamma^i(r) \tag{5.18}
\]

for any \(r \in (0,1], i \in \mathbb{N}, \varepsilon \in (0,[8\psi(1)]^{-1} \wedge 1]\) \((\tau^i\) is defined in Proposition 4.3). We can also write (5.18) as

\[
\hat{G}^i \left( \tau^i(s_i + r) \wedge (s_i + r) \right) \cap \Gamma^i(r) \subseteq G^i \left( \tau^i(s_i + r) \wedge (s_i + r) \right) \cap \Gamma^i(r), \tag{5.19}
\]
where
\[ \hat{\tau}^i(s_i + r) \triangleq \tau^i \land \sigma^Y_{\beta} \land \bigwedge_{j:s_j < t_j \leq s_i + r} \sigma^Y_j. \tag{5.20} \]

Here, although the restriction \( \sigma^Y_{\beta} \land \bigwedge_{j:s_j < t_j \leq s_i + r} \sigma^Y_j \) is redundant in (5.19) (because \( \sigma^Y_{\beta} > t_j + 3r > s_i + r \) for each \( j \in \mathbb{N} \) with \( s_i < t_j \leq s_i + r \) by the definition of \( \Gamma^i(r) \)), we emphasize its role by writing it out.

We now start bounding \( \mathbb{P}_\varepsilon(S(r)) \). For any \( r \in (0, r_0] \) and \( \varepsilon \in (0, \varepsilon_0(r)] \), we have
\[
\begin{align*}
\mathbb{P}_\varepsilon(S(r)) & \geq \mathbb{P}_\varepsilon\left( \bigcup_{i=1}^{\lfloor r\varepsilon^{-1} \rfloor} G^i_i \left( \hat{\tau}^i(s_i + r) \land (s_i + r) \right) \cap \Gamma^i(r) \cap \left[ T_{1}^{X^i} < T_{0}^{X^i} \right] \cap \left[ \hat{\tau}^i(s_i + r) \geq s_i + r \right] \right) \\
& \geq \mathbb{P}_\varepsilon\left( \bigcup_{i=1}^{\lfloor r\varepsilon^{-1} \rfloor} \hat{G}^i_i \left( \hat{\tau}^i(s_i + r) \land (s_i + r) \right) \cap \Gamma^i(r) \cap \left[ T_{1}^{X^i} < T_{0}^{X^i} \right] \cap \left[ \hat{\tau}^i(s_i + r) \geq s_i + r \right] \right),
\end{align*}
\]

where the last inequality follows from the inclusion (5.19). We make the restrictions \( \left[ T_{1}^{X^i} < T_{0}^{X^i} \right] \) in order to invoke \( \mathbb{Q}^\varepsilon \)-probabilities later on. By considering separately \( \hat{\tau}^i(s_i + r) \geq s_i + r \) and \( \hat{\tau}^i(s_i + r) < s_i + r \), we obtain from the last inequality that
\[
\begin{align*}
\mathbb{P}_\varepsilon(S(r)) & \geq \mathbb{P}_\varepsilon\left( \bigcup_{i=1}^{\lfloor r\varepsilon^{-1} \rfloor} \hat{G}^i_i \left( \hat{\tau}^i(s_i + r) \land (s_i + r) \right) \cap \Gamma^i(r) \cap \left[ T_{1}^{X^i} < T_{0}^{X^i} \right] \right) \\
& \quad - \mathbb{P}_\varepsilon\left( \bigcup_{i=1}^{\lfloor r\varepsilon^{-1} \rfloor} \left[ \hat{\tau}^i(s_i + r) < s_i + r \right] \cap \left[ T_{1}^{X^i} < T_{0}^{X^i} \right] \right),
\end{align*}
\]

Applying another inclusion-exclusion to the first term on the right-hand side of (5.21) now gives the main inequality of this proof:
\[
\begin{align*}
\mathbb{P}_\varepsilon(S(r)) & \geq \mathbb{P}_\varepsilon\left( \bigcup_{i=1}^{\lfloor r\varepsilon^{-1} \rfloor} \hat{G}^i_i \left( \hat{\tau}^i(s_i + r) \land (s_i + r) \right) \cap \left[ T_{1}^{X^i} < T_{0}^{X^i} \right] \right) \\
& \quad - \mathbb{P}_\varepsilon\left( \bigcup_{i=1}^{\lfloor r\varepsilon^{-1} \rfloor} \left[ \Gamma^i(r) \cap \left[ T_{1}^{X^i} < T_{0}^{X^i} \right] \right] \right) \\
& \quad - \mathbb{P}_\varepsilon\left( \bigcup_{i=1}^{\lfloor r\varepsilon^{-1} \rfloor} \left[ \hat{\tau}^i(s_i + r) < s_i + r \right] \cap \left[ T_{1}^{X^i} < T_{0}^{X^i} \right] \right),
\end{align*}
\]

(5.22)

\[ \forall r \in (0, r_0], \varepsilon \in (0, \varepsilon_0(r)]. \]

In the rest of this proof, we bound each of the three terms on the right-hand side of (5.22) and then choose according to these bounds the desired \( r_1 \) and \( \varepsilon_1(r) \) for (5.15).

At this stage, we use Proposition 4.3 and Theorem 4.6 in the following way. For any \( \rho \in (0, \frac{1}{2}) \),
we choose $\delta_1 \in (0, 1]$, independent of $i \in \mathbb{N}$ and $\varepsilon \in (0, \left[\frac{8\psi(1)}{1} \right]^{-1} \wedge 1)$, such that

$$\sup \left\{ Q^i_\varepsilon (r^i \leq s_i + \delta_1); i \in \mathbb{N}, \varepsilon \in \left(0, \frac{1}{8\psi(1)} \wedge 1\right) \right\} \leq \rho, \quad (5.23)$$

$$\sup \left\{ Q^i_\varepsilon \left( \exists s \in (s_i, s_i + \delta_1], \sum_{j \in J^i_\varepsilon (s \wedge r^i \wedge \sigma^p_i)} Y^j_s (I) \tau^j \wedge \sigma^q_i \wedge \sigma^p_i > \right) K^*[(s - s_i)^{\kappa_1 - \rho} + \varepsilon^{\kappa_2} \cdot (s - s_i)^{\kappa_3 - \rho}] \right\}; i \in \mathbb{N}, \varepsilon \in \left(0, \frac{1}{8\psi(1)} \wedge 1\right) \leq \rho. \quad (5.24)$$

Consider the first probability on the right-hand side of (5.22). We use the elementary inequality: for any events $A_1, \cdots, A_n$ for $n \in \mathbb{N},$

$$\mathbb{P} \left( \bigcup_{j=1}^n A_j \right) \geq \sum_{j=1}^n \mathbb{P}(A_j) - \sum_{i=1}^n \sum_{j=1}^n \mathbb{P}(A_i \cap A_j).$$

Then

$$\mathbb{P}_\varepsilon \left( \bigcup_{i=1}^{[r \varepsilon^{-1}]} \tilde{G}^i \left( \tilde{S}^i (s_i + r) \wedge (s_i + r) \right) \cap \left[ T^X_i < T^0_0 \right] \right) \geq \sum_{i=1}^{[r \varepsilon^{-1}]} \mathbb{P}_\varepsilon \left( \tilde{G}^i \left( \tilde{S}^i (s_i + r) \wedge (s_i + r) \right) \cap \left[ T^X_i < T^0_0 \right] \right) - \sum_{i=1}^{[r \varepsilon^{-1}]} \sum_{j=1}^{[r \varepsilon^{-1}]} \mathbb{P}_\varepsilon \left( T^X_i < T^0_0, T^X_j < T^0_0 \right), \quad (5.25)$$

$$\forall r \in (0, r_0], \varepsilon \in (0, \varepsilon_0(r)).$$

The first term on the right-hand side of (5.25) can be written as

$$\sum_{i=1}^{[r \varepsilon^{-1}]} \mathbb{P}_\varepsilon \left( \tilde{G}^i \left( \tilde{S}^i (s_i + r) \wedge (s_i + r) \right) \cap \left[ T^X_i < T^0_0 \right] \right) = \sum_{i=1}^{[r \varepsilon^{-1}]} \psi(I) \varepsilon \cdot Q^i_\varepsilon \left( \tilde{G}^i \left( \tilde{S}^i (s_i + r) \wedge (s_i + r) \right) \right) \quad (5.26)$$

by the definition of $Q^i_\varepsilon$ in (4.1). By inclusion-exclusion, we have

$$Q^i_\varepsilon \left( \tilde{G}^i \left( \tilde{S}^i (s_i + r) \wedge (s_i + r) \right) \right) \geq Q^i_\varepsilon \left( Y^i_s (I) \geq \frac{(s - s_i)^{\eta}}{4}, \forall s \in \left( s_i, \tilde{S}^i (s_i + r) \wedge (s_i + r) \right) \right) - Q^i_\varepsilon \left( \exists s \in \left( s_i, \tilde{S}^i (s_i + r) \wedge (s_i + r) \right), \right.$$  \begin{equation} \sum_{j \in J^i_\varepsilon (s)} Y^j_s (I) > K^*[(s - s_i)^{\kappa_1 - \rho} + \varepsilon^{\kappa_2} \cdot (s - s_i)^{\kappa_3 - \rho}] \right), \quad (5.27)$$

$$\forall i \in \mathbb{N}, \varepsilon \in \left(0, \frac{1}{8\psi(1)} \wedge 1\right).$$
Recall that $\tau^{i,1} \leq \tau^i$ and $X^i_s(1) = \psi(1)\varepsilon > 0$. Hence, by the definition of $\hat{\tau}^i(s_i + r)$,

$$
Q^i_\varepsilon \left( X^i_s(1) \geq \frac{(s - s_i)^\eta}{4}, \forall s \in [s_i, \hat{\tau}^i(s_i + r) \wedge (s_i + r)] \right) = 1, \quad \forall i \in \mathbb{N}, \varepsilon \in \left(0, \frac{1}{8\psi(1)} \wedge 1\right).
$$

(5.28)

For

$$
r \in (0, \delta_1], \ i \in \mathbb{N}, \text{ and } \varepsilon \in \left(0, \frac{1}{8\psi(1)} \wedge 1\right),
$$

the second probability in (5.27) can be bounded as

$$
Q^i_\varepsilon \left( \exists s \in (s_i, \tau^i(s_i + r) \wedge (s_i + r)], \sum_{j \in \mathcal{J}_0}(s) \right) Y^j_s(1) > K^*[(s - s_i)^{\kappa_1 - \rho} + \varepsilon^\kappa_2 \cdot (s - s_i)^{\kappa_3 - \rho}] \leq Q^i_\varepsilon \left( \exists s \in (s_i, \tau^i \wedge (s_i + r)], \sum_{j \in \mathcal{J}_0}(s \wedge \tau^i \wedge \sigma^\beta_0) \right) Y^j_s(1)^{\tau^i \wedge \sigma^\beta_0} \wedge \sigma^\beta \leq \tau^i \wedge \sigma^\beta \wedge \sigma^\beta \leq 2\rho.
$$

(5.29)

Here, the first inequality follows since for $s \in (s_i, \tau^i(s_i + r) \wedge (s_i + r)], s \leq \tau^i \wedge \sigma^\beta_0$ and and

$$
j \in \mathcal{J}_0(s) \implies j \in \mathcal{J}_0(s_i + r)
$$

$$
\quad \implies t_j \in (s_i, s_i + r]
$$

$$
\quad \implies \bar{\tau}^i(s_i + r) \leq \sigma^\beta Y_j
$$

$$
\quad \implies s \leq \sigma^\beta Y_j
$$

with the third implication following from the definition of $\bar{\tau}^i(s_i + r)$ in (5.20). The first term in the second inequality follows by considering the scenario $\tau^i > s_i + \delta_1$ and using $r \in (0, \delta_1]$ and the last inequality follows from (5.23) and (5.24). Applying (5.28) and (5.29) to (5.27), we get

$$
Q^i_\varepsilon \left( \bar{G}^i(\tau^i(s_i + r) \wedge (s_i + r)) \right) \geq 1 - 2\rho, \quad \forall \ r \in (0, \delta_1 \wedge r_0], \ i \in \mathbb{N}, \varepsilon \in (0, \varepsilon_0(r)].
$$

(5.30)

From (5.26) and the last inequality, we have shown that

$$
\sum_{i=1}^{[\tau^{r-1}_i]} \mathbb{P}_\varepsilon \left( \bar{G}_i(\tau^i(s_i + r) \wedge (s_i + r)) \cap \left[ T^{X_i}_1 < T^{X_i}_0 \right] \right) \geq \psi(1) (r - \varepsilon) (1 - 2\rho), \quad \forall \ r \in (0, \delta_1 \wedge r_0], \varepsilon \in (0, \varepsilon_0(r)].
$$
Recalling \( \varepsilon_0(r) \leq r \). The second term on the right-hand side of (5.25) is relatively easy to bound. Indeed, by using the independence between the clusters \( X^i \) and Lemma 4.1,

\[
\sum_{i=1}^{[r^{-1}]} \sum_{1 \leq j \leq [r^{-1}]} P_\varepsilon \left( T_1^{X^i} < T_0^{X^i}, T_1^{X^j} < T_0^{X^j} \right) = \sum_{i=1}^{[r^{-1}]} \sum_{1 \leq j \leq [r^{-1}]} P_\varepsilon \left( T_1^{X^i} < T_0^{X^i} \right) P_\varepsilon \left( T_1^{X^j} < T_0^{X^j} \right) \leq \psi(1)^2 r^2,
\]

\( \forall r \in (0, 1], \varepsilon \in \left( 0, \frac{1}{8\psi(1)} \right) \).

Recalling (5.25) and using the last two displays, we have the following bound for the first term on the right-hand side of (5.22):

\[
P_\varepsilon \left( \bigcup_{i=1}^{[r^{-1}]} \widehat{G}_i \left( \widehat{G}^i (s_i + t) \land (s_i + r) \right) \cap \left[ T_1^{X^i} < T_0^{X^i} \right] \right) \geq \psi(1)(r - \varepsilon)(1 - 2\rho) - \psi(1)^2 r^2, \quad \forall r \in (0, \delta_1 \land \tau_0], \varepsilon \in (0, \varepsilon_0(r)].
\]

Next, we consider the second probability on the right-hand side of (5.22). By the definition of \( \Gamma^i(r) \) in (5.9) and the general inclusion

\[
(A_1 \cap A_2 \cap A_3)^C \subseteq \left( A_1^C \cap A_2 \cap A_3 \right) \cup A_2^C \cup A_3^C,
\]

we have

\[
\Gamma^i(r)^C \subseteq \left[ \left[ P_{\beta}^{X^i}(s_i + r) \cap \left( \bigcup_{j: t_j \leq s_i} \text{supp}(Y^j) \right) \neq \emptyset \right] \cap \bigcap_{j: t_j \leq s_i} \left[ \sigma_{\beta}^{Y^j} > t_j + 3r \right] \cap \left[ \sigma_{\beta}^{X^i} > s_i + 2r \right] \right. \cup \left. \bigcup_{j: t_j \leq s_i + r} \left[ \sigma_{\beta}^{Y^j} \leq t_j + 3r \right] \cup \left[ \sigma_{\beta}^{X^i} \leq s_i + 2r \right] \right],
\]

where we note that the indices \( j \) in

\[
\bigcap_{j: t_j \leq s_i} \left[ \sigma_{\beta}^{Y^j} > t_j + 3r \right]
\]

now range only over \( j \in \mathbb{N} \) with \( t_j \leq s_i \). Hence,

\[
P_\varepsilon \left( \bigcup_{i=1}^{[r^{-1}]} \Gamma^i(r)^C \cap \left[ T_1^{X^i} < T_0^{X^i} \right] \right) \leq P_\varepsilon \left( \bigcup_{i=1}^{[r^{-1}]} \left[ \left[ P_{\beta}^{X^i}(s_i + r) \cap \left( \bigcup_{j: t_j \leq s_i} \text{supp}(Y^j) \right) \neq \emptyset \right] \cap \bigcap_{j: t_j \leq s_i} \left[ \sigma_{\beta}^{Y^j} > t_j + 3r \right] \cap \left[ \sigma_{\beta}^{X^i} > s_i + 2r \right] \right) \right)
\]

\[
+ P_\varepsilon \left( \bigcup_{j=1}^{[2r^{-1}]+1} \left[ \sigma_{\beta}^{Y^j} \leq t_j + 3r \right] \right) + P_\varepsilon \left( \bigcup_{i=1}^{[r^{-1}]} \left[ \sigma_{\beta}^{X^i} \leq s_i + 2r \right] \right),
\]

(5.33)
where we have the second probability in the foregoing inequality since
\[ t_j \leq s_{r\varepsilon^{-1}} + r \implies t_j \leq 2r \implies j \leq [2r\varepsilon^{-1}] + 1. \]
Resorting to the conditional probability measures \( Q^i \), we see that the first probability in (5.33) can be bounded as
\[
P_\varepsilon \left( \bigcup_{i=1}^{[r\varepsilon^{-1}]} \left\{ \mathcal{P}_0^{X^i}(s_i + r) \cap \left( \bigcup_{j : t_j \leq s_i} \text{supp}(Y^j) \right) \right\} \right)
\leq \sum_{i=1}^{[r\varepsilon^{-1}]} \psi(\mathbf{1})\varepsilon C^1 \supp \leq \psi(\mathbf{1})C^1 \supp^2, \quad \forall r \in (0, r_0), \varepsilon \in (0, \varepsilon_0(r)],
\]
where the next to the last inequality follows from Proposition 5.5 (2) and the constant \( C^1 \supp \in (0, \infty) \) is independent of \( r \in (0, r_0] \) and \( \varepsilon \in (0, \varepsilon_0) \). (Here, we use the choice \( \beta \in \left[ \frac{1}{3}, \frac{1}{2} \right) \) to apply this proposition.) By Proposition 5.5 (1), the second probability in (5.33) can be bounded as (recall \( \varepsilon_0(r) \leq r \))
\[
P_\varepsilon \left( \bigcup_{j=1}^{[2r\varepsilon^{-1}]+1} \left[ \sigma^X_j \leq t_j + 3r \right] \right)
\leq C^0 \supp (2r\varepsilon^{-1} + 1) \cdot 3\varepsilon r
\leq 9C^0 \supp^2, \quad \forall r \in (0, r_0), \varepsilon \in (0, \varepsilon_0(r)],
\]
where \( C^0 \supp \) is a constant independent of \( r \in (0, r_0] \) and \( \varepsilon \in (0, \varepsilon_0(r)] \). Similarly,
\[
P_\varepsilon \left( \bigcup_{i=1}^{[r\varepsilon^{-1}]} \left[ \sigma^X_j \leq s_i + 2r \right] \right) \leq 2C^0 \supp^2, \quad \forall r \in (0, r_0], \varepsilon \in (0, \varepsilon_0(r)].
\]
From (5.33) and the last three displays, we have shown that the second probability in (5.22) satisfies the bound
\[
P_\varepsilon \left( \bigcup_{i=1}^{[r\varepsilon^{-1}]} \Gamma^i(r)^c \cap \left[ T^X_i < T^X_0 \right] \right) \leq 11C^0 \supp^2 + \psi(\mathbf{1})C^1 \supp^2, \quad \forall r \in (0, r_0], \varepsilon \in (0, \varepsilon_0(r)].
\]
It remains to consider the last probability on the right-hand side of (5.22). Recall the number \( \delta_1 \)
chosen for (5.23). Similar to the derivation of (5.33), we have

\[
P_\varepsilon \left( \bigcup_{i=1}^{\lfloor r \varepsilon^{-1} \rfloor} \left[ \bar{\tau}^i(s_i + r) < s_i + r \right] \cap \left[ T_1^{s_i} < T_0^{s_i} \right] \right) 
\leq P_\varepsilon \left( \bigcup_{i=1}^{\lfloor r \varepsilon^{-1} \rfloor} \left[ \sigma_{\delta}^{s_i} \leq s_i + r \right] \right) + P_\varepsilon \left( \bigcup_{i=1}^{2\lfloor r \varepsilon^{-1} \rfloor + 1} \left[ \sigma_{\delta}^{s_i} \leq t_i + r \right] \right) 
+ \sum_{i=1}^{\lfloor r \varepsilon^{-1} \rfloor} \psi(1)\varepsilon Q_{\varepsilon}^{s_i} (\tau^i < s_i + r) 
\leq 11C_0^0 \varepsilon^2 + \psi(1)r \rho, \quad \forall \varepsilon \in (0, \varepsilon_0(r)], \quad (5.37)
\]

where in the last inequality we also use (5.34) and (5.35).

We apply the three bounds (5.31), (5.36), and (5.37) to (5.22). This shows that for any $\rho \in (0, \frac{1}{2})$, there exist $\delta_1 > 0$ such that for any $r \in (0, \delta_1 \wedge r_0]$ and $\varepsilon \in (0, \varepsilon_0(r)]$ (note that $\varepsilon_0(r) \leq r \wedge 1$),

\[
P_\varepsilon(S(r)) \geq \left[ \psi(1)(r - \varepsilon)(1 - 2\rho) - \psi(1)^2 r^2 \right] - \left( 11C_0^0 \varepsilon^2 + \psi(1)C_1^1 r^{7/6} \right) 
- (11C_0^0 \varepsilon^2 + \psi(1)r \rho) 
= r \left[ \psi(1)(1 - 3\rho) - \psi(1)^2 + 22C_0^0 \right] - r \psi(1)C_1^1 r^{1/6} 
- \psi(1)\varepsilon (1 - 2\rho).
\]

Finally, to attain the uniform lower bound (5.15), we choose $\rho \in (0, \frac{1}{2})$ and $r_1 \in (0, \delta_1 \wedge r_0]$ such that

\[
\psi(1)(1 - 3\rho) - \psi(1)^2 + 22C_0^0 r - \psi(1)C_1^1 r^{1/6} \geq \frac{\psi(1)}{2}, \quad \forall r \in (0, r_1],
\]

and then $\varepsilon_1(r) \in (0, \varepsilon_0(r)]$ such that

\[
\psi(1)\varepsilon_1(r)(1 - 2\rho) \leq \frac{\psi(1)r}{4}.
\]

Putting things together, we obtain

\[
P_\varepsilon(S(r)) \geq \frac{\psi(1)r}{4}, \quad \forall \varepsilon \in (0, \varepsilon_1(r)], \quad r \in (0, r_1],
\]

and hence (5.15) follows. The proof is complete. \(\square\)

We use Lemma 5.4 to give the proof for a more precise version of our main theorem, namely Theorem 1, in Theorem 5.7 below. Before this, we state the following proposition whose proof can be found in Section 3.9 of [4].

**Proposition 5.6.** For any $(\varepsilon_n) \subseteq (0, [8\psi(1)]^{-1} \wedge 1]$ such that $\varepsilon_n \rightarrow 0$, the sequence of laws of $(\langle X, Y \rangle, P_{\varepsilon_n})$ is relatively compact in the space of probability measures on the product space $D(\mathbb{R}_+, \mathcal{C}_{\text{rap}}(\mathbb{R})) \times D(\mathbb{R}_+, \mathcal{C}_{\text{rap}}(\mathbb{R}))$ and any of its limits is the law of a pair of nonnegative solutions of the SPDE (1.2) subject to the same space-time white noise.

**Theorem 5.7 (Separation of solutions).** For any $(\varepsilon_n) \subseteq (0, [8\psi(1)]^{-1} \wedge 1]$ with $\varepsilon_n \searrow 0$ such that the sequence of laws of $(\langle X, Y \rangle, P_{\varepsilon_n})$ converges to the law of $(\langle X, Y \rangle, P_0)$ of a pair of nonnegative solutions of the SPDE (1.2) in the space of probability measures on the product space $D(\mathbb{R}_+, \mathcal{C}_{\text{rap}}(\mathbb{R})) \times D(\mathbb{R}_+, \mathcal{C}_{\text{rap}}(\mathbb{R}))$, we have

\[
P_0 \left( \sup_{0 \leq s \leq 2r_1} \| X - Y \|_{\text{rap}} \geq \frac{\Delta(r_1)}{2} \right) \geq \inf_{\varepsilon \in (0, \varepsilon_1(r_1)]} P_\varepsilon(S(r_1)) > 0.
\]
where $\Delta(r_1) > 0$ is chosen in Lemma 5.1 and $r_1, \varepsilon_1(r_1) \in (0, 1]$ are chosen in Lemma 5.4.

**Proof.** By Skorokhod’s representation theorem, we may take $(X^{(\varepsilon_n)}, Y^{(\varepsilon_n)})$ to be copies of the $\varepsilon_n$-approximating solutions which live on the same probability space, and assume that $(X^{(\varepsilon_n)}, Y^{(\varepsilon_n)})$ converges almost surely to $(X^{(0)}, Y^{(0)})$ in the product (metric) space $D(\mathbb{R}^+, \mathcal{C}_{rap}(\mathbb{R}))$. By Skorokhod’s representation theorem, we may take $X^{(\varepsilon_n)}$ and $Y^{(\varepsilon_n)}$ to be copies of the $\varepsilon_n$-approximating solutions which live on the same probability space, and assume that $(X^{(\varepsilon_n)}, Y^{(\varepsilon_n)})$ converges almost surely to $(X^{(0)}, Y^{(0)})$ in the product (metric) space $D(\mathbb{R}^+, \mathcal{C}_{rap}(\mathbb{R}))$.

Now, it follows from Lemma 5.1 and Lemma 5.4 that

$$\inf_{n: \varepsilon_n \leq \varepsilon_1(r_1)} \mathbb{P} \left( \sup_{0 \leq s \leq 2r_1} \left| X^{(\varepsilon_n)}_s - Y^{(\varepsilon_n)}_s \right|_{rap} \geq \Delta(r_1) \right) \geq \inf_{\varepsilon \in (0, \varepsilon_1(r_1)]} \mathbb{P}_\varepsilon \left( S(r_1) > 0 \right).$$

Hence, by Fatou’s lemma, we get

$$0 < \inf_{\varepsilon \in (0, \varepsilon_1(r_1)]} \mathbb{P}_\varepsilon \left( S(r_1) \right) \leq \limsup_{n \to \infty} \mathbb{P} \left( \sup_{0 \leq s \leq 2r_1} \left| X^{(\varepsilon_n)}_s - Y^{(\varepsilon_n)}_s \right|_{rap} \geq \Delta(r_1) \right)$$

$$\leq \mathbb{P} \left( \limsup_{n \to \infty} \sup_{0 \leq s \leq 2r_1} \left| X^{(\varepsilon_n)}_s - Y^{(\varepsilon_n)}_s \right|_{rap} \geq \Delta(r_1) \right)$$

$$\leq \mathbb{P} \left( \sup_{0 \leq s \leq 2r_1} \left| X^{(0)}_s - Y^{(0)}_s \right|_{rap} \geq \frac{\Delta(r_1)}{2} \right),$$

where the last inequality follows from the convergence

$$X^{(\varepsilon_n)} \xrightarrow{a.s., n \to \infty} X^{(0)} \quad \text{and} \quad Y^{(\varepsilon_n)} \xrightarrow{a.s., n \to \infty} Y^{(0)}$$

in the Skorokhod space $D(\mathbb{R}^+, \mathcal{C}_{rap}(\mathbb{R}))$, the continuity of $X^{(0)}$ and $Y^{(0)}$, and Proposition 3.6.5 (a) of [6]. The proof is complete.

**References**

[1] Bass, R. F., Burdzy, K., and Chen, Z.-Q. (2007). Pathwise uniqueness for a degenerate stochastic differential equation. *Ann. Probab.* 35 2385–2418.

[2] Burdzy, C., Mueller, C. and Perkins, E. A. (2011). Nonuniqueness for nonnegative solutions of parabolic stochastic partial differential equations. *Illinois J. Math.* 54 1481–1507.

[3] Chen, Y.-T. and Delmas, J.-F. (2012). Smaller population size at the MRCA time for stationary branching processes. *Ann. Probab.* 40 2034–2068.

[4] Chen, Y.-T. (2013). Stochastic models for spatial populations. Ph.D. thesis, U. British Columbia, Vancouver, BC.

[5] Etheridge, A. M. (2000). *An Introduction to Superprocesses*. University Lecture Series 20. American Mathematical Society, Providence, RI.

[6] Ethier, S. N. and Kurtz, T. G. (2005). *Markov Processes: Characterization and Convergence*, reprint edition. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New Jersey.

[7] Jacod, J. and Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 288. Springer Verlag, Berlin.

[8] Kallenberg, O. (2002). *Foundations of Modern Probability*, 2nd edition. Probability and its Applications, Springer Verlag, Berlin.

[9] Knight, F. B. (1981). *Essentials of Brownian Motion and Diffusion*. Mathematical Surveys 18. American Mathematical Society, Providence, RI.

[10] Konno, N. and Shiga, T. (1988). Stochastic partial differential equations for some measure-valued diffusions. *Probab. Theory Related Fields* 79 201–225.
[11] Kurtz, T. G. (2007). The Yamada-Watanabe-Engelbert theorem for general stochastic equations and inequalities. *Electronic J. Probab.* **12** 951–965.

[12] Le Gall, J.-F. (1999). *Spatial Branching Processes, Random Snakes and Partial Differential Equations. Lectures in Mathematics ETH Zürich*. Birkhäuser Verlag, Basel.

[13] Mueller, C. and Perkins, E. A. (1992). The compact support property for solutions to the heat equation with noise. *Probab. Theory Related Fields* **93** 325–358.

[14] Mueller, C., Mytnik, L. and Perkins, E. A. (2013). Nonuniqueness for a parabolic SPDE with $3/4 - \epsilon$ diffusion coefficients. To appear in *Ann. Probab.*

[15] Mytnik, L. (1998). Weak uniqueness for the heat equation with noise. *Ann. Probab.* **26** 968–984.

[16] Mytnik, L. and Perkins, E. A. (2011). Pathwise uniqueness for stochastic heat equations with Hölder continuous coefficients; the white noise case. *Probab. Theory Related Fields* **149** 1–96.

[17] Mytnik, L., Perkins, E. A. and Sturm, A. (2006). On pathwise uniqueness for stochastic heat equations with non-Lipschitz coefficients. *Ann. Probab.* **34** 1910–1959.

[18] Perkins, E. A. (2002). Dawson-Watanabe superprocesses and measure-valued diffusions. In *Lectures on probability theory and statistics (Saint-Flour, 1999)*, *Lecture Notes in Math.* **1781** 125–324.

[19] Reimers, M. (1989). One dimensional stochastic partial differential equations and the branching measure diffusion. *Probab. Theory Related Fields* **81** 319–340.

[20] Revuz, D. and Yor, M. (2005). *Continuous Martingales and Brownian Motion*, corrected 3rd printing of the 3rd ed. Springer Verlag, Berlin.

[21] Rogers, L. C. G. and Williams D. (1987). *Diffusions, Markov Processes, and Martingales. Volume 2: Itô Calculus*. Cambridge Mathematical Library. Cambridge University Press, Cambridge.

[22] Shiga, T. (1994). Two contrasting properties of solutions for one-dimensional stochastic partial differential equations. *Canad. J. Math.* **46** 415–437.

[23] Walsh, J. B. (1986). An introduction to stochastic partial differential equations. In *École d’été de probabilités de Saint-Flour, XIV—1984, Lecture Notes in Math.* **1180** 265–439. Springer-Verlag, Berlin.

[24] Yamada, T. and Watanabe, S. (1971). On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.* **11** 155–167.