COMPACT EMBEDDED HYPERSURFACES WITH CONSTANT
HIGHER ORDER ANISOTROPIC MEAN CURVATURES

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Abstract. Given a positive function \( F \) on \( S^n \) which satisfies a convexity condition, for \( 1 \leq r \leq n \), we define the \( r \)-th anisotropic mean curvature function \( H^F_r \) for hypersurfaces in \( \mathbb{R}^{n+1} \) which is a generalization of the usual \( r \)-th mean curvature function. We prove that a compact embedded hypersurface without boundary in \( \mathbb{R}^{n+1} \) with \( H^F_r = \text{constant} \) is the Wulff shape, up to translations and homotheties. In case \( r = 1 \), our result is the anisotropic version of Alexandrov Theorem, which gives an affirmative answer to an open problem of F. Morgan.

1. Introduction

Let \( F: S^n \to \mathbb{R}^+ \) be a smooth function which satisfies the following convexity condition:

\[
(D^2 F + FI)_x > 0, \quad \forall x \in S^n,
\]

where \( S^n \) is the standard unit sphere in \( \mathbb{R}^{n+1} \), \( D^2 F \) denotes the intrinsic Hessian of \( F \) on \( S^n \) and \( I \) denotes the identity on \( T_x S^n \), \( > 0 \) means that the matrix is positive definite. We consider the map

\[
\phi: S^n \to \mathbb{R}^{n+1},
\]

\[
x \to F(x)x + (\text{grad}_{S^n} F)_x,
\]

its image \( W_F = \phi(S^n) \) is a smooth, convex hypersurface in \( \mathbb{R}^{n+1} \) called the Wulff shape of \( F \) (see [2], [3], [15], [10], [11], [12], [13], [17], [22], [23]). When \( F \equiv 1 \), the Wulff shape \( W_F \) is just \( S^n \).

Now let \( X: M \to \mathbb{R}^{n+1} \) be a smooth immersion of a compact, orientable hypersurface without boundary. Let \( \nu: M \to S^n \) denote its Gauss map.

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Let \( A_F = D^2F + FI \), \( S_F = -d(\phi \circ \nu) = -A_F \circ d\nu \). \( S_F \) is called the \( F \)-Weingarten operator, and the eigenvalues of \( S_F \) are called anisotropic principal curvatures. Let \( \sigma_r \) be the elementary symmetric functions of the anisotropic principal curvatures \( \lambda_1, \lambda_2, \cdots, \lambda_n \):

\[
\sigma_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} \quad (1 \leq r \leq n).
\]

We set \( \sigma_0 = 1 \). The \( r \)-th anisotropic mean curvature \( H_r^F \) is defined by \( H_r^F = \sigma_r / C_r^n \), also see Reilly [18]. \( H_r^F = H_r^F \) is called the anisotropic mean curvature. If \( F \equiv 1 \), then \( H_r^F = H_r \) is just the \( r \)-th mean curvature of hypersurfaces which has been studied by many authors (see [4], [14], [16], [21]). Thus, the \( r \)-th anisotropic mean curvature \( H_r^F \) generalized the \( r \)-th mean curvature \( H_r \) of hypersurfaces in the \((n + 1)\)-dimensional Euclidean space \( \mathbb{R}^{n+1} \).

For hypersurfaces in \( \mathbb{R}^{n+1} \), we have the following classical Alexandrov Theorem which was proved first by Alexandrov in [1] and later by Reilly in [19], Montiel-Ros in [16] and Hijazi-Montiel-Zhang in [8]:

**Theorem 1.1.** (Alexandrov Theorem) Let \( X : M \to \mathbb{R}^{n+1} \) be a compact hypersurface without boundary embedded in Euclidean space. If \( H = \text{constant} \), then \( X(M) \) is a sphere.

Following from a modification of Reilly’s proof, Ros showed in [20] that the sphere is the only compact embedded hypersurface without boundary with constant scalar curvature in \( \mathbb{R}^{n+1} \), which gave a partial answer to Yau’s conjecture [24]. Thereafter, Ros [21] extended his result to any \( r \)-th mean curvature, and later, Montiel and Ros gave another proof in [16]. Explicitly, they proved:

**Theorem 1.2.** ([16], [21]) Let \( X : M \to \mathbb{R}^{n+1} \) be a compact hypersurface without boundary embedded in Euclidean space. If \( H_r = \text{constant} \) for some \( r = 1, \cdots, n \), then \( X(M) \) is a sphere.

In this paper, we prove the following anisotropic version of Theorem 1.2:

**Theorem 1.3.** Let \( X : M \to \mathbb{R}^{n+1} \) be a compact hypersurface without boundary embedded in Euclidean space. If \( H_r^F = \text{constant} \) for some \( r = 1, \cdots, n \), then up to translations, \( X(M) = \rho W_F \), where \( \rho = -1 / H_1^F \) is a constant.

**Remark 1.1.** For \( n = 1 \), Morgan [15] proved that Theorem 1.3 still holds for a more general condition: \( F \) is only a continuous norm on \( \mathbb{R}^2 \) and \( X : M \to \mathbb{R}^2 \) is a closed curve immersed in \( \mathbb{R}^2 \). In case \( r = 1 \), Theorem 1.3 is actually the anisotropic version of Alexandrov Theorem, which gives an affirmative answer to the following open problem proposed by Morgan in the same paper: Whether an embedded equilibrium, i.e. hypersurfaces with constant anisotropic mean curvature in Euclidean space, must be the Wulff shape? We also note that M. Koiso stated this conjecture in [9].
Remark 1.2. Theorem 1.2 follows by choosing \( F \equiv 1 \) in Theorem 1.3.

2. Preliminaries

Let \( X: M \to \mathbb{R}^{n+1} \) be a compact connected hypersurface immersed in Euclidean space. Let \( \nu: M \to S^n \) denote its Gauss map. Suppose there exists a point where all the principal curvatures with respect to \( \nu \) are positive. By the positiveness of \( A_F \), all the anisotropic principal curvatures are positive at this point. Using the results of Gårding ([5]), we have the following lemma (cf. Montiel-Ros [16]):

Lemma 2.1. Let \( X: M \to \mathbb{R}^{n+1} \) be a compact connected hypersurface without boundary. Suppose that there exists a point where all the principal curvatures are positive. Assume \( H_F \cdot \partial > 0 \) holds on every point of \( M \), then the same holds for \( H_F \cdot k \), \( k = 1, \ldots, r-1 \). Moreover

\[
(H_F \cdot (k-1)/k) \leq H_{k-1}, \quad (H_F \cdot 1/k) \leq H_1, \quad k = 1, \ldots, r.
\]

If \( k \geq 2 \), the equality in the above inequalities happens only at points where all the anisotropic principal curvatures are equal.

Let \( \{e_1, \ldots, e_n\} \) be a local orthogonal frame of \( X: M \to \mathbb{R}^{n+1} \), then we have the structure equations:

\[
\begin{align*}
\text{d}X &= \sum_i \omega_i e_i \\
\text{d}\nu &= -\sum_{ij} h_{ij} \omega_j e_i \\
\text{d}e_i &= \sum_j \omega_{ij} e_j + \sum_j h_{ij} \omega_j \nu \\
\text{d}\omega_i &= \sum_j \omega_{ij} \wedge \omega_j \\
\text{d}\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} &= -\frac{1}{2} \sum_{kl} R_{ijkl} \theta_k \wedge \theta_l
\end{align*}
\]

where \( \omega_{ij} + \omega_{ji} = 0, R_{ijkl} + R_{ijlk} = 0 \), and \( R_{ijkl} \) are the components of the Riemannian curvature tensor of \( M \) with respect to the induced metric \( dX \cdot dX \).

Let \( s_{ij} \) denote the coefficient of \( S_F \) with respect to \( \{e_1, \ldots, e_n\} \), that is

\[
-d(\phi \circ \nu) = -A_F \circ d\nu = \sum_{i,j} s_{ij} \omega_j e_i,
\]

where \( \phi \) is defined in (2).

We call the eigenvalues of \( S_F \) to be anisotropic principal curvatures, and denote them by \( \lambda_1, \ldots, \lambda_n \). From the positive definiteness of \( A_F \), there exists a non-singular matrix \( C \) such that \( A_F = C^T C \), so \( S_F = -A_F \circ d\nu \) is similar to the real symmetric matrix \(-C \circ d\nu \circ C^T\). Thus, the anisotropic principal curvatures are all real. Moreover, if \( \lambda_1 = \cdots = \lambda_n \), we have \( S_F = H_I^F \cdot I \), so \( -d(\phi \circ \nu) = H_I^F dX \) by (4) and (5). Thus, we have the following lemma (cf. [6], [7], [17]):
Lemma 2.2. Let \( X : M \rightarrow \mathbb{R}^{n+1} \) be a compact hypersurface without boundary. If \( \lambda_1 = \lambda_2 = \cdots = \lambda_n = \text{const} \neq 0 \), then up to translations, \( X(M) = \rho W_F \), where \( \rho = -1/H^F \).

We define \( s_{ijk} \) by
\[
d s_{ij} + \sum_k s_{ik} \omega_{kj} + \sum_k s_{kj} \omega_{ki} = \sum_k s_{ijk} \omega_k.
\]

Taking exterior differentiation of (5) and using (4), we get
\[
s_{ijk} = s_{ikj}.
\]

Lemma 2.3. Let \( X : M \rightarrow \mathbb{R}^{n+1} \) be a compact hypersurface without boundary. If \( n \geq 2 \) and \( \lambda_1 = \lambda_2 = \cdots = \lambda_n \neq 0 \), then \( \lambda_1 = \lambda_2 = \cdots = \lambda_n = \text{const} \), so up to translations, \( X(M) = \rho W_F \), where \( \rho = -1/H^F \).

Proof. From (7) and \( s_{ji} = H^F \delta_{ij} \), we have
\[
e_i(H^F) = \sum_j s_{ij} = \sum_j s_{jji} = ne_i(H^F), \quad 1 \leq i \leq n.
\]

Therefore \( \lambda_1 = \lambda_2 = \cdots = \lambda_n = H^F \) is a constant, then the conclusion follows from Lemma 2.2. \( \square \)

We define \( F^* : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) to be (see [2]):
\[
F^*(x) = \sup \left\{ \frac{\langle x, z \rangle}{F(z)} \mid z \in S^n \right\},
\]

Proposition 2.1. Let \( x \in \mathbb{R}^{n+1} \setminus \{0\}, y, z \in S^n \), then we have:

(i) \( \langle \phi(y), z \rangle \leq F(z) \), and the equality holds if and only if \( y = z \);
(ii) \( \langle x, y \rangle \leq F^*(x)F(y) \), and the equality holds if and only if \( x = F^*(x)\phi(y) \).

Proof. Proof of (i). It is obvious that \( \langle \phi(y), z \rangle \leq F(z) \) is equivalent to \( \langle \phi(y) - \phi(z), z \rangle \leq 0 \). The function \( \Phi : S^n \times S^n \rightarrow \mathbb{R} \) defined by
\[
\Phi(y, z) = \langle \phi(y) - \phi(z), z \rangle
\]
is smooth, so it attained its maximum at some point \( (y_0, z_0) \) because \( S^n \times S^n \) is compact. By differentiating the function \( \Phi(y, z) \) with respect to \( y \) at the point \( (y_0, z_0) \), we get
\[
\langle A_F \circ dy, z \rangle_{(y_0, z_0)} = 0.
\]
Thus, from the positiveness of \( A_F \), \( z_0 \) is orthogonal to \( S^n \) at the point \( y_0 \), so, we must have \( y_0 = \pm z_0 \). Notice that \( \Phi(z_0, z_0) = 0 \), \( \Phi(-z_0, z_0) = -F(z_0) - F(-z_0) < 0 \), the function \( \Phi \) must attain its maximum 0 at the point \( (z_0, z_0) \), so \( \langle \phi(y), z \rangle \leq F(z) \). If \( \langle \phi(y), z \rangle = F(z) \), then \( \Phi \) obtains its maximum 0 at the point \( (y, z) \), by the same reason we have \( y = z \).
Proof of (ii). It is obvious that \( \langle x, y \rangle \leq F^*(x)F(y) \) by the definition of \( F^* \). Now we suppose that \( \langle x, y \rangle = F^*(x)F(y) \), then the function \( (x - F^*(x)\phi(y), y) \) obtains its maximum 0 at the point \( (x, y) \). So, differentiating it with respect to \( y \), we get
\[
\langle x - F^*(x)\phi(y), dy \rangle = 0.
\]
Thus, it follows that \( x - F^*(x)\phi(y) \) is orthogonal to \( S^n \) at \( y \), that is, \( x - F^*(x)\phi(y) = ky \) for some \( k \). Then from \( \langle x - F^*(x)\phi(y), y \rangle = 0 \), we have \( x - F^*(x)\phi(y) = 0 \).

**Proposition 2.2.** We have:

(i) \( F^*(x) > 0, \forall x \in \mathbb{R}^{n+1} \setminus \{0\} \);
(ii) \( F^*(tx) = tF^*(x), \forall x \in \mathbb{R}^{n+1}, t > 0 \);
(iii) \( F^*(x + y) \leq F^*(x) + F^*(y), \forall x, y \in \mathbb{R}^{n+1} \), and the equality holds if and only if \( x = 0 \), or \( y = 0 \) or \( x = ky \) for some \( k > 0 \).
(iv) \( W_F = \{ x \in \mathbb{R}^{n+1} | F^*(x) = 1 \} \).

**Proof.** (i) and (ii) follow from the definition of \( F^* \). By the definition of \( F^* \) and (ii) of Proposition 2.1, we easily get (iv). We now prove (iii). Suppose \( x, y \neq 0 \). Let \( z \in S^n \) be such that \( F^*(x + y) = \langle x + y, z \rangle / F(z) \), then we have
\[
F^*(x + y) = \langle x + y, z \rangle / F(z) = \langle x, z \rangle / F(z) + \langle y, z \rangle / F(z) \leq F^*(x) + F^*(y),
\]
with the equality holding if and only if \( F^*(x) = \langle x, z \rangle / F(z) \) and \( F^*(y) = \langle y, z \rangle / F(z) \). So, if the equality holds, then from (ii) of Proposition 2.1 we have
\[
x = F^*(x)\phi(z), \quad y = F^*(y)\phi(z).
\]
Thus, \( x = F^*(x) / F^*(y) y \).

From Proposition 2.2, for any \( x \in \mathbb{R}^{n+1} \setminus \{0\} \), we have \( x / F^*(x) \in W_F \), thus there exists a unique \( \psi(x) \in S^n \) such that \( x = F^*(x)\phi(\psi(x)) \). From the implicit function theorem and the convexity of \( F \), the function \( F^*: \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^+ \) and \( \psi: \mathbb{R}^{n+1} \setminus \{0\} \to S^n \) are smooth.

### 3. F-focal point and F-cut point

We define \( d_F: \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R} \) to be \( d_F(x, y) = F^*(y - x) \), then we have \( d_F(x, y) > 0 \) when \( x \neq y \), \( d_F(x, x) = 0 \) and \( d_F(x, z) \leq d_F(x, y) + d_F(y, z) \). Note that in general \( d_F(x, y) \neq d_F(y, x) \); and when \( F \equiv 1 \), \( d_F \) is just the Euclidean distance function \( d \).

For every \( p \in \mathbb{R}^{n+1} \), let \( \exp_p \) be the exponential map in \( \mathbb{R}^{n+1} \) at the point \( p \), then \( \exp_p(u) = p + u \). So, from the definition of \( d_F \), we have
\[
d_F(p, \exp_p(t\phi(Y))) = t, \text{ for every } Y \in S^n \text{ and } t \in \mathbb{R}^+.
\]
Now, let $X: M \to \mathbb{R}^{n+1}$ be a compact embedded hypersurface without boundary, and $\nu$ be the unit inner normal vector field of $M$. For convenience, we identify each point $p \in M$ with its image $X(p) \in \mathbb{R}^{n+1}$.

For each $y \in \mathbb{R}^{n+1}$, define
\begin{equation}
(10) \quad d_F(M, y) = \inf \{d_F(p, y) | p \in M\}.
\end{equation}

We define a function $c: M \to \mathbb{R}^+$ to be such that $c(p)$ is the greatest $t \in (0, \infty)$ satisfying $d_F(M, \exp_p(t \phi \circ (\nu(p)))) = t$. We call $\exp_p(c(p) \phi \circ \nu(p))$ the $F$-cut point of $p \in M$.

For $p \in M$, let $\gamma_p$ be the ray $\gamma_p: [0, \infty) \to \mathbb{R}^{n+1}$ defined by:
\[
\gamma_p(t) = p + t \phi \circ \nu(p), \quad \forall t \in [0, \infty),
\]
and $\Gamma_p = \gamma_p([0, c(p)))$. Then we have

**Lemma 3.1.** For $p, q \in M$, $p \neq q$, we have $\Gamma_p \cap \Gamma_q = \emptyset$.

**Proof.** Suppose $x \in \Gamma_p \cap \Gamma_q$, then there exists $0 < t < \min(c(p), c(q))$ such that
\[
x = \exp_p(t \phi \circ \nu(p)) = \exp_q(t \phi \circ \nu(q)),
\]
by the definition of $c(p), c(q)$ and (9).

Suppose $t < s < c(p)$, then from (iii) of Proposition 2.2, we have
\[
d_F(q, \exp_p(s \phi \circ \nu(p))) < d_F(q, x) + d_F(x, \exp_p(s \phi \circ \nu(p)))
= d_F(p, x) + d_F(x, \exp_p(s \phi \circ \nu(p)))
= d_F(p, \exp_p(s \phi \circ \nu(p)))
= d_F(M, \exp_p(s \phi \circ \nu(p))),
\]
a contradiction. \qed

Consider the map: $\Psi: M \times \mathbb{R} \to \mathbb{R}^{n+1}$, $\Psi(p, t) = \exp_p(t \phi \circ \nu(p))$. If $(p, t)$ is a critical point of the map $\Psi$, then we call $\exp_p(t \phi \circ \nu(p))$ an $F$-focal point of $p \in M$. Because $\exp_p(t \phi \circ \nu(p)) = p + t \phi \circ \nu(p)$, so through direct computation, we have
\begin{equation}
(11) \quad d(\exp_p(t \phi \circ \nu(p))) = (I - t S_F) \circ dp + (\phi \circ \nu(p))dt.
\end{equation}

From (11), $\exp_p(t \phi \circ \nu(p))$ is an $F$-focal point of $p$ if and only if the matrix $I - t S_F$ is degenerate. So, the first $F$-focal point of $p$ along the ray $\gamma_p$ is $\exp_p(1/\lambda_{\max} \phi \circ \nu(p))$, where $\lambda_{\max}$ is the greatest positive anisotropic principal curvature at $p$.

**Remark 3.1.** When $F = 1$, $F$-cut point, $F$-focal point is the cut point and the focal point of hypersurfaces in the Euclidean space respectively.

**Lemma 3.2.** Either $(p, c(p))$ is a critical point of the map $\Psi$, or there exists at least one point $q \in M$, $q \neq p$, such that $d_F(q, \exp_p(c(p) \phi \circ \nu(p))) = c(p)$.
Proof. We choose \( \varepsilon_i > 0 \), such that \( \lim_{i \to \infty} \varepsilon_i = 0 \). Let \( a_i = \exp_p((c(p) + \varepsilon_i) \circ \nu(p)) \), \( a = \exp_p(c(p) \circ \nu(p)) \). The continuity of \( \Psi \) implies that \( \lim_{i \to \infty} a_i = a \). From the definition of \( c(p) \), there exists points \( q_i \in M \), such that \( d_F(q_i, a_i) = d_F(M, a_i) = c(p) + \varepsilon_i' \), where \( \varepsilon_i' < \varepsilon_i \), possibly \( \varepsilon_i' < 0 \). From the compactness of \( M \), there exists a convergent subsequence of \( \{q_i\} \), again denoted by \( \{q_i\} \) such that \( \lim_{i \to \infty} q_i = q \). Then we divided into two cases:

Case 1. \( q \neq p \). In this case we have

\[
\lim_{i \to \infty} d_F(q_i, a_i) = d_F(q, a),
\]

and

\[
\lim_{i \to \infty} d_F(M, a_i) = d_F(M, a) = c(p).
\]

So, we have \( d_F(q, a) = c(p) \), as expected.

Case 2. \( q = p \). Suppose \( (p, c(p)) \) is not a critical point of the map \( \Psi \), then there exists a neighborhood \( U \) of \( (p, c(p)) \in M \times \mathbb{R} \) such that \( \Psi|_U: U \to \Psi(U) \) is a diffeomorphism. And we have \( \lim_{i \to \infty} d_F(q_i, a_i) = d_F(q, a) = c(p) \), so \( \lim_{i \to \infty} \varepsilon_i' = 0 \).

Therefore, for a sufficient large \( i \), we have \( (p, c(p) + \varepsilon_i), (q_i, c(p) + \varepsilon_i') \in U \). But

\[
\exp_p((c(p) + \varepsilon_i) \circ \nu(p)) = \exp_{q_i}((c(p) + \varepsilon_i') \circ \nu(q_i)),
\]

thus we have \( p = q_i \) and \( \varepsilon_i = \varepsilon_i' \), a contradiction. \( \square \)

Lemma 3.3. \( c(p) \leq 1/\lambda_{\text{max}} \), where \( \lambda_{\text{max}} \) is the greatest positive anisotropic principal curvature at \( p \).

Proof. Let \( t > 1/\lambda_{\text{max}} \). We define a function \( h: M \to \mathbb{R} \) by

\[
h(q) = F^*(p + t \circ \nu(p) - q), \forall q \in M.
\]

We prove that \( p \) is not a local minimum point, so \( t > c(p) \), thus the conclusion follows.

Because, \( (p + t \circ \nu(p) - q)/h(q) \in W_F \), we can define a function \( Y: M \to S^n \) by

\[
p + t \circ \nu(p) - q = h(q) \phi \circ Y(q),
\]

and we have \( Y(p) = \nu(p) \).

Let \( \gamma: (-\varepsilon, \varepsilon) \to M \) be a smooth curve such that \( \gamma(0) = p \), we denote \( h(s) = h(\gamma(s)), \gamma(s) = X(\gamma(s)), Y(s) = Y(\gamma(s)), \nu(s) = \nu(\gamma(s)) \) for simplicity. Then, we have

\[
p + t \circ \nu(p) - \gamma(s) = h(s) \phi \circ Y(s).
\]

Differentiating (14), we get

\[
h'(s) \cdot \phi(Y'(s)) + h(s) \cdot (\phi(Y))'(s) = -\gamma'(s).
\]

From \( \langle Y, Y \rangle = 1 \), we have \( \langle Y'(s), Y(s) \rangle = 0 \), together with \( (\phi(Y))'(s) = A_F \circ Y'(s) \) we have

\[
\langle (\phi(Y))'(s), Y(s) \rangle = 0.
\]
Thus, by taking inner product with $Y(s)$ in (15), we get
\begin{equation}
 h'(s)F(Y(s)) = -\langle \gamma'(s), Y(s) \rangle.
\end{equation}
From (17) and $Y(0) = \nu(p)$, we have $h'(0) = -\langle \gamma'(0), \nu(p) \rangle/F(\nu(p)) = 0$, so $p$ is an extreme point of the function $h$. And we have
\begin{equation}
 t(\phi(Y))'(0) = tA_F \circ Y'(0) = -\gamma'(0),
\end{equation}
Differentiating (15), we get
\begin{equation}
 h''(s) \cdot \phi(Y(s)) + 2h'(s) \cdot (\phi(Y))'(s) + h(s)(\phi(Y))''(s) = -\gamma''(s).
\end{equation}
Differentiating (16), we get
\begin{equation}
 ((\phi(Y))''(s), Y(s)) = -((\phi(Y))'(s), Y'(s))
\end{equation}
Thus, by taking inner product with $Y(s)$ in (19) and using (16), (20), we get
\begin{equation}
 h''(s) \cdot F(Y(s)) - h(s)\langle A_F \circ Y'(s), Y'(s) \rangle = -\langle \gamma''(s), Y(s) \rangle.
\end{equation}
Evaluating (21) at $s = 0$, using $Y(0) = \nu(p)$ and (18) we have
\begin{equation}
 h''(0) \cdot F(\nu(p)) - \frac{1}{t}(A_F^{-1} \circ \gamma'(0), \gamma'(0)) = -\langle \gamma''(0), \nu(p) \rangle = \langle \gamma'(0), \nu'(0) \rangle.
\end{equation}
Now, let $\gamma$ be such a curve that satisfies $-A_F \circ \nu'(0) = \lambda_{\text{max}} \gamma'(0)$, that is, $\gamma'(0)$ is the eigenvector corresponding to the maximum positive eigenvalue of $S_F = -A_F \circ d\nu$. Then, we have
\begin{equation}
 h''(0) = \frac{\lambda_{\text{max}}}{tF(\nu(p))} \left( \frac{1}{\lambda_{\text{max}}} - t \right) \langle A_F^{-1} \circ \gamma'(0), \gamma'(0) \rangle < 0,
\end{equation}
because $A_F^{-1}$ is positive definite.

So, $p$ is not a local minimum point of the function $h$. \qed

**Lemma 3.4.** The map $c: M \to \mathbb{R}^+$ is continuous.

**Proof.** Let $p_i \in M$ be such that $\lim_{i \to \infty} p_i = p$, we need to prove $\lim_{i \to \infty} c(p_i) = c(p)$. For any $q \in M$, we have
\begin{equation}
 d(q, \exp_q(c(q)\phi(\nu(q)))) = c(q)\sqrt{||\text{grad}_{\nu} F(\nu(q))||^2 + ||F(\nu(q))||^2} < \text{the diameter of } M,
\end{equation}
so the function $c$ is bounded.

Firstly, we prove $\limsup_{i \to \infty} c(p_i) \leq c(p)$. For any $\varepsilon > 0$, there do not exist infinitely many indices $i$ such that $c(p_i) > c(p) + \varepsilon$. Otherwise, by the definition of $c(p_i)$, we have
\begin{equation}
 d_F(p_i, \exp_p((c(p) + \varepsilon)\phi(\nu(p_i)))) = c(p) + \varepsilon,
\end{equation}
and, by the continuity of the function $d_F$, $d_F(p, \exp_p((c(p) + \varepsilon)\phi(\nu(p)))) = c(p) + \varepsilon$, which contradicts the definition of $c(p)$. Therefore $\limsup_{i \to \infty} c(p_i) \leq c(p) + \varepsilon$, for any $\varepsilon > 0$, which proves the claim.
Secondly, we prove lim inf_{i→∞} c(p_i) ≥ c(p). Let \( \bar{t} = \lim inf_{i→∞} c(p_i) \). Consider a subsequence of \{c(p_i)\}, again denoted by c(p_i), which converges to \( \bar{t} \). It is obvious an accumulation point of \( F \)-focal points is \( F \)-focal point, if for any such subsequence, the points \( \exp_{p_i}(c(p_i)φ(ν(p_i))) \) are \( F \)-focal points of \( p_i \), then \( \exp_{p}(c(p)φ(ν(p))) \) is an \( F \)-focal point of \( p \), hence \( \bar{t} \geq c(p) \) by Lemma \ref{lem:3.3}

Suppose, therefore, there exists a subsequence of \( c(p_i) \) (again denoted by \( c(p_i) \)), such that \( \exp_{p_i}(c(p_i)φ(ν(p_i))) \) is not \( F \)-focal point of \( p_i \). By Lemma \ref{lem:3.2} there exists \( q_i \in M \) such that \( d_F(q_i, \exp_{p_i}(c(p_i)φ(ν(p_i)))) = c(p_i) \). Taking, if necessary, a subsequence, we may suppose that \( lim_{i→∞} q_i = q \in M \). If \( p \neq q \), by taking limit we see that,

\[
d_F(q, \exp_{p}(tφ(ν(p)))) = d_F(p, \exp_{p}(tφ(ν(p)))),
\]

hence \( \bar{t} \geq c(p) \). If \( p = q \), then for any neighborhood \( V = U × (\bar{t} - ε, \bar{t} + ε) \) of \( (p, \bar{t}) \), there exists \( i \), such that \( p_i, q_i \in U \) and \( c(p_i) \in (\bar{t} - ε, \bar{t} + ε) \). Choose \( \bar{t} - ε < s < c(p_i) \), then we have \( \exp_{p_i}(sφ(ν(p_i))) \neq \exp_{q_i}(sφ(ν(q_i))) \) by Lemma \ref{lem:3.1} so the map \( Ψ|_V: V → Ψ(V) \) can not be injective. Thus, \( (p, \bar{t}) \) is a critical point of \( Ψ \).

\[\Box\]

4. An integral inequality of compact hypersurfaces

In this section we derive an integral inequality of compact hypersurfaces without boundary embedded in Euclidean space (Theorem \ref{thm:4.2}) which plays an important role in the proof of our main theorem. First, we recall the following integral formulas of Minkowski type for compact hypersurfaces in \( \mathbb{R}^{n+1} \).

**Theorem 4.1.** (\cite{6}, \cite{7}) Let \( X: M → \mathbb{R}^{n+1} \) be an \( n \)-dimensional compact hypersurface without boundary, \( F: S^n → \mathbb{R}^{+} \) be a smooth function which satisfies (1), then we have the following integral formulas of Minkowski type:

\[
\int_M (H_r^F \circ ν + H_{r+1}^F(X, ν))dA = 0, \quad r = 0, 1, \cdots, n - 1.
\]

Now, we let \( X: M → \mathbb{R}^{n+1} \) be a compact embedded hypersurface without boundary, then \( M \) is a boundary of some compact domain \( D ⊂ \mathbb{R}^{n+1} \), let \( ν \) be the unit inner normal vector field of \( M \).

**Lemma 4.1.** For any fixed point \( y \in D \setminus X(M) \), there exists at least a point \( p ∈ M \) such that

\[
y - p = d_F(M, y)φ \circ ν(p).
\]

**Proof.** From the compactness of \( M \) and the continuity of the function \( d_F \), there exists \( p ∈ M \) such that \( d_F(p, y) = inf\{d_F(q, y) | q ∈ M\} \).

Let \( Z: M → S^n \) be defined by

\[
y - q = F^*(y - q)φ \circ Z(q).
\]
Then we have
\begin{equation}
\label{eq:26}
d_F(q, y) = \frac{\langle y - q, Z(q) \rangle}{F(Z(q))}.
\end{equation}

Differentiating \eqref{eq:26}, we get
\begin{equation}
\label{eq:27}
dd_F(q, y) = -\frac{\langle dq, Z(q) \rangle}{F(Z(q))}.
\end{equation}

So, from the minimum of \( p \), we get \( \langle dp, Z(p) \rangle = 0 \), thus \( Z(p) = \pm \nu(p) \). If \( Z(p) = -\nu(p) \), then \( \langle \phi \circ Z(p), \nu(p) \rangle = -F(Z(p)) < 0 \), so the line segment connecting \( p \) and \( y \) must intersect \( X(M) \) at another point \( \tilde{p} \), therefore \( F^*(y - q) \) can not attain its minimum at \( p \). Thus, \( Z(p) = \nu(p) \) is the unit inner normal vector. \( \square \)

**Lemma 4.2.** Let \( X : M \to \mathbb{R}^2 \) be a simple closed curve and denote its arc parameter by \( s \). Suppose \( c(p) = 1/\lambda(p) \) for some point \( p \in M \), then we must have \( \lambda'(p) = 0 \), where \( \lambda \) is the anisotropic curvature and \( ' \) denote derivative with respect to the arc parameter.

**Proof.** Let \( x_0 = p + c(p)\phi \circ \nu(p) \), define \( \tau : M \to \mathbb{R} \) by:
\[ \tau(q) = F^*(x_0 - q), \]
then there exists a function \( W : M \to S^1 \) such that
\begin{equation}
\label{eq:28}
x_0 - q = \tau(q)\phi \circ W(q), \quad W(p) = \nu(p).
\end{equation}

From the definition of \( c(p) \), we have
\begin{equation}
\label{eq:29}
\tau(q) \geq c(p), \forall q \in M, \quad \tau(p) = c(p).
\end{equation}

Differentiating \eqref{eq:28}, we get
\begin{equation}
\label{eq:30}
-T(q) = \tau'(q)\phi \circ W(q) + \tau(q)a(W(q))W'(q),
\end{equation}
where \( T \) denotes the tangent vector of \( M \), \( a = D^2F + F \).

By taking inner product with \( W(q) \) in \eqref{eq:30}, we obtain
\begin{equation}
\label{eq:31}
F(W(q))\tau'(q) = -\langle T(q), W(q) \rangle.
\end{equation}

Thus, we have \( \tau'(p) = 0 \) by \( W(p) = \nu(p) \). Then, from \eqref{eq:30},
\begin{equation}
\label{eq:32}
W'(p) = -T(p)/(ac(p)) = -\lambda(p)T(p)/a = -k(p)T(p) = \nu'(p),
\end{equation}
where \( k \) is the curvature of \( M \).

Differentiating \eqref{eq:31}, we get
\begin{equation}
\label{eq:33}
(F \circ W)'(q)\tau'(q) + F(W(q))\tau''(q) = -\langle T'(q), W(q) \rangle - \langle T(q), W'(q) \rangle.
\end{equation}

Evaluating \eqref{eq:33} at \( q = p \), we get
\[ F(\nu(p))\tau''(p) = -\langle k(p)\nu(p), \nu(p) \rangle - \langle T(p), -k(p)T(p) \rangle = 0, \]
so $\tau''(p) = 0$. Thus, differentiating (33) and evaluating it at $p$, we obtain

\begin{equation}
F'(\nu(p))\tau'''(p) = -(T''(p), \nu(p)) - 2\langle T'(p), W'(p) \rangle - \langle T(p), W''(p) \rangle.
\end{equation}

Differentiating (30) and evaluating at $p$, we get

\[-T'(p) = c(p)((a \circ W)'(p)W'(p) + a \circ \nu(p)W''(p)).\]

As $W'(p) = \nu'(p)$, so $(a \circ W)'(p) = (a \circ \nu)'(p)$, through a direct calculation, we have

\begin{equation}
W''(p) = \frac{k(p)(a \circ \nu)'(p)}{a \circ \nu(p)}T(p) - k^2(p)\nu(p).
\end{equation}

From (32), (35) and $T'' = -k^2T + k'\nu$, we obtain

\begin{equation}
F'(\nu(p))\tau'''(p) = -k'(p) - \frac{k(p)(a \circ \nu)'(p)}{a \circ \nu(p)} = -\frac{\chi'(p)}{a \circ \nu(p)}.
\end{equation}

As $\tau(q) \geq \tau(p)$ holds for all $q \in M$, and $\tau'(p) = \tau''(p) = 0$, so we must have $\tau'''(p) = 0$, thus $\chi'(p) = 0$ as expected. \hfill \square

Let $f : D \to \mathbb{R}$ be an integrable function. From Lemma 3.1, Lemma 3.4 and Lemma 4.1, we have the following formula of integration

\begin{equation}
\int_D f dV = \int_M \int_0^{\exp_p(t\phi(\nu(p)))} f(\exp_p(t\phi(\nu(p)))) E(p, t) dt dA,
\end{equation}

where $E(p, t)$ is given by

\begin{equation}
dV(\exp_p(t\phi(\nu(p)))) = E(p, t) dt dA.
\end{equation}

If $x$ denotes the position vector in $\mathbb{R}^{n+1}$, we have $\bar{\Delta}|x|^2 = 2(n+1)$, where $\bar{\Delta}$ is the Euclidean Laplacian. From the Stokes Theorem, we have

\begin{equation}
-\int_M \langle X, \nu \rangle dA = (n+1)V,
\end{equation}

where $V$ the volume of $D$. From (11), we have

\[dV(p + t\phi(\nu(p))) = \det(I - tSF)F \circ \nu dt dA = (1 - t\lambda_1) \cdots (1 - t\lambda_n) F \circ \nu dt dA.\]

Letting $f \equiv 1$ in (37) and taking into account that $E(p, t) = (1 - t\lambda_1) \cdots (1 - t\lambda_n) F \circ \nu$, we have

\begin{equation}
V = \int_M \int_0^{\exp_p(t\phi(\nu(p)))} (1 - t\lambda_1) \cdots (1 - t\lambda_n) F \circ \nu dt dA.
\end{equation}

**Theorem 4.2.** Let $X : M \to \mathbb{R}^{n+1}$ be a compact hypersurface without boundary embedded in Euclidean space. If the anisotropic mean curvature $H_1^F$ of $X$ with respect to the unit inner normal $\nu$ is everywhere positive on $M$, then we have

\begin{equation}
\int_M \frac{F \circ \nu}{H_1^F} dA \geq (n+1)V,
\end{equation}

where $V$ is the volume of the compact domain determined by $M$. Moreover, the equality holds in (41) if and only if up to translations, $X(M) = \rho W_F$, where $\rho = -1/H_1^F$ is a constant.

**Proof.** Firstly, if $X(M) = \rho W_F$, then $H_1^F = -1/\rho = \text{constant}$. So, by the integral equalities of Minkowski type (24) and (39), the equality in (41) holds.

For $p \in M$, by Lemma 3.3, we have

$$c(p) \leq 1/\lambda_{\text{max}} \leq 1/H_1^F(p).$$

Moreover, if $t \in [0, c(p))$, we have

$$(1 - t\lambda_1) \cdots (1 - t\lambda_n) \leq (1 - tH_1^F)^n,$$

the equality holds only at points where $\lambda_1 = \cdots = \lambda_n$. Thus, by putting (42), (43) into (40), we get

$$V \leq \int_M \int_0^{1/H_1^F} (1 - tH_1^F)^n F \circ \nu \, dt \, dA = \frac{1}{n + 1} \int_M F \circ \nu \, dA,$$

and the equality holds if and only if $\lambda_1 = \cdots = \lambda_n = 1/c(p)$. Therefore, by Lemma 2.3 if $n \geq 2$ and the equality holds, then up to translations, $X(M) = \rho W_F$, where $\rho = -1/H_1^F$. If $n = 1$, then from Lemma 4.2 $\lambda = H_1^F$ is a constant, so from Lemma 2.2 up to translations, $X(M) = \rho W_F$, where $\rho = -1/H_1^F$. $\square$

**Remark 4.1.** By Lemma 2.2 Theorem 1.3 is true for $n = 1$ even without the assumption of embedding. So, in order to prove Theorem 1.3 we actually don’t need to prove the case $n = 1$ of Theorem 4.2. We prove it here only for completeness.

If $F \equiv 1$ in Theorem 4.2 then we obtain

**Corollary 4.1.** ([16], [21]) Let $X: M \to \mathbb{R}^{n+1}$ be a compact hypersurface without boundary embedded in Euclidean space. If the mean curvature $H$ of $X$ with respect to the unit inner normal $\nu$ is everywhere positive on $M$, then we have

$$\int_M \frac{1}{H} \, dA \geq (n + 1)V,$$

where $V$ is the volume of the compact domain determined by $M$. Moreover, the equality holds if and only if $X(M)$ is a round sphere.

5. **Proof of Theorem 1.3**

We divide into two cases:

**Case 1.** $\nu$ is the unit inner normal vector field. Since $M$ is compact without boundary, one can find a point where all the principal curvatures with respect to $\nu$ are positive. It follows from the positive definiteness of $A_F$ that all the anisotropic...
principal curvatures at this point with respect to \( \nu \) are positive too. Thus \( H^F_r \) is a positive constant. From Lemma 2.1 we have that \( H^F_1, \cdots, H^F_{r-1} > 0, (H^F_r)^{1/r} \leq H^F_1 \) and \( H^F_{r-1} \geq (H^F_r)^{(r-1)/r} \). Using Theorem 4.2 we have

\[
(n + 1)(H^F_r)^{1/r} V \leq \int_M F \circ \nu dA,
\]

and the equality holds if and only if up to translations, \( X(M) = -\rho W_F \), where \( \rho = 1/H^F_1 \) is a constant.

Since \( H^F_r \) is a positive constant and \( (H^F_r)^{1/r} \leq H^F_1 \), by Theorem 4.1 we have

\[
0 = \int_M (H^F_{r-1} F \circ \nu + H^F_r \langle X, \nu \rangle) dA \geq \int_M (H^F_1)^{(r-1)/r} F \circ \nu + H^F_r \langle X, \nu \rangle) dA = (H^F_r)^{(r-1)/r} \int_M (F \circ \nu + (H^F_r)^{1/r} \langle X, \nu \rangle) dA.
\]

As \( H^F_r \) is a positive constant, using (39) we have

\[
\int_M F \circ \nu dA - (n + 1)(H^F_r)^{1/r} V = \int_M (F \circ \nu + (H^F_r)^{1/r} \langle X, \nu \rangle) dA \leq 0.
\]

Hence, the equality in (44) holds, so up to translations, \( X(M) = \rho W_F \), where \( \rho = -1/H^F_1 \) is a constant.

**Case 2.** \( \nu \) is the unit outer normal vector field. The conclusion follows as in Case 1 by considering the function \( \tilde{F}: S^n \to \mathbb{R}^+ \) defined by \( \tilde{F}(x) = F(-x) \) instead of \( F \). This completes the proof of Theorem 1.3.

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**References**

1. A. D. Alexandrov, *Uniqueness theorems for surfaces in the large I*, Vestnik Leningrad Univ., 11(1956), 5-17.
2. J. E. Brothers and F. Morgan, *The isoperimetric theorem for general integrands*, Michigan Math. J., 41(1994), 419-431.
3. U. Clarenz, *The Wulff-shape minimizes an anisotropic Willmore functional*, Interfaces and Free Boundaries, 6(2004), 351-359.
4. L. F. Cao and H. Li, *r-minimal submanifolds in space forms*, Ann. Global Anal. Geom., 32(2007), 311-341.
5. L. Gårding, *An inequality for hyperbolic polynomials*, J. Math. Mech., 8(1959), 957-965.
6. Y. J. He and H. Li, *Integral formula of Minkowski type and new characterization of the Wulff shape*, arXiv:math.DG/0703187, 2007, to appear in Acta Math. Sinica.
7. Y. J. He and H. Li, *A new variational characterization of the Wulff shape*, Diff. Geom. App., (2007), doi:10.1016/j.difgeo.2007.11.030.
8. O. Hijazi, S. Montiel and Xiao Zhang, *Dirac operator on embedded hypersurfaces*, Math. Res. Lett., 8(2001), 195-208.
9. M. Koiso, *Geometry of surfaces with constant anisotropic mean curvature*, Symposium on the differential geometry of submanifolds, Valenciennes, July, 2007, 113-125.
10. M. Koiso and B. Palmer, *Geometry and stability of surfaces with constant anisotropic mean curvature*, Indiana Univ. Math. J., 54(2005), No.6, 1817-1852.
11. M. Koiso and B. Palmer, *Stability of anisotropic capillary surfaces between two parallel planes*, Calc. Var. Partial Differential Equations, 25(2006), 275-298.
12. M. Koiso and B. Palmer, *Anisotropic capillary surfaces with wetting energy*, Calc. Var. Partial Differential Equations, 29(2007), 295-345.
13. M. Koiso and B. Palmer, *Uniqueness theorems for stable anisotropic capillary surfaces*, Siam J. Math. Anal., 39(2007), 721-741.
14. H. Li, *Hypersurfaces with constant scalar curvature in space forms*, Math. Ann., 305(1996), 665-672.
15. F. Morgan, *Planar Wulff shape is unique equilibrium*, Proc. Amer. Math. Soc, 133(2004), 809-813.
16. S. Montiel and A. Ros, *Compact hypersurfaces: The Alexandrov theorem for higher order mean curvatures*, in Lawson, B. and Tenenblat, K. (eds), Differential Geometry, Pitman Monographs, Vol. 52, Longman, Essex, 1991, 279-296.
17. B. Palmer, *Stability of the Wulff shape*, Proc. Amer. Math. Soc., 126(1998), 3661-3667.
18. R. Reilly, *The relative differential geometry of nonparametric hypersurfaces*, Duke Math. J., 43(1976), 705-721.
19. R. Reilly, *Applications of the Hessian operator in a Riemannian manifold*, Indiana Univ. Math. J., 26(1977), 459-472.
20. A. Ros, *Compact hypersurfaces with constant scalar curvature and a congruence theorem*, J. Diff. Geom., 27(1988), 215-220.
21. A. Ros, *Compact hypersurfaces with constant higher order mean curvatures*, Revista Mathematica Iberoamericana, 3(1987), 447-453.
22. J. Taylor, *Crystalline variational problems*, Bull. Amer. Math. Soc., 84(1978), 568-588.
23. S. Winklmann, *A note on the stability of the Wulff shape*, Arch. Math., 87(2006), 272-279.
24. S.-T. Yau, *Problem section*, Seminar on Differential Geometry, Annals Math. Studies, No. 102, Princeton University Press, Princeton, NJ, 1982.

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