Entanglement-enabled quantum holography

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Holography encodes information using classical light interference with applications ranging from microscopy to data storage. Quantum entanglement enables information processing with capabilities beyond technology based on classical principles. Here we introduce a holographic imaging concept that is conditioned on the coherence, and thus the entanglement, between the qubit terms in a quantum entangled photon state. By harnessing the nonlocal properties of entanglement, we remotely reconstruct an image encoded in the phase of spatial-polarisation hyper-entangled photons. Furthermore, we demonstrate that entanglement-encoded phase images can be retrieved even through dynamic phase disorder and in the presence of strong classical noise, with practical advantages over classical holography.

Holography is an essential tool of modern optics, at the origin of many applications for microscopic imaging, optical security, and data storage. In this respect, holographic interferometry is a widely-used technique that exploits optical interference to retrieve the phase component of a classical optical field through intensity measurements. For example, phase-shifting holography uses four intensity images \( I_\theta (\theta \in \{0, \pi/2, \pi, 3\pi/2\}) \) of a reference optical field \( ae^{i\theta} \) interfering with an unknown field \( be^{i\phi} \) to reconstruct the phase profile

\[
\phi = \arg \left[ I_0 - I_\pi + i(I_{\pi/2} - I_{3\pi/2}) \right].
\]  

Maintaining optical coherence between interfering fields is therefore essential in all holographic protocol. Mechanical instabilities, random phase disorder and the presence of stray light are examples of phenomena that degrade light coherence and hinder the phase reconstruction process.

Whilst holography is based on classical interference of light waves, the quantum properties of light have inspired a range of new imaging modalities, including interaction-free \cite{12} and induced-coherence imaging \cite{13} as well as sensitivity-enhanced \cite{14,15} and super-resolution schemes \cite{16,17}. Non-classical sources of light can also produce holograms \cite{18,19} that were observed with single-photons \cite{20} and photon pairs \cite{21}.

Here, we introduce and experimentally demonstrate a holographic imaging concept that relies on quantum entanglement to carry the image information and cannot be reproduced using classical light. Phase images are encoded in the polarisation-entanglement of hyper-entangled photons and retrieved through spatial intensity correlation measurements (i.e. photon coincidence counting). The nonlocal nature of our measurements removes the need for path overlap, resulting in insensitivity to mechanical instabilities, while polarisation encoding provides robustness against random phase disorder. Furthermore, the measurement of two-photon correlations removes the sensitivity to the presence of stray light.

The conceptual arrangement of our quantum holographic scheme is illustrated in Fig. 1a. Photon pairs entangled in space and polarisation \cite{22} interact with two spatial light modulators (‘Alice SLM’ and ‘Bob SLM’) and are then detected by two single-photon imaging devices, for example two electron multiplied charge coupled device cameras (‘Alice EMCCD’ and ‘Bob EMCCD’). The transverse momentum \( k \) of the photons is mapped onto separated pixels of the SLMs and re-imaged onto the cameras. Alice and Bob shape and detect photons with momentum of negative \( x \)-component \( (k_x < 0) \) and positive \( x \)-component \( (k_x > 0) \), respectively. The quantum state of the photon pair after the SLMs is thus

\[
\sum_k \left[ |V>_k |V>_{-k} \right] + e^{i\psi(k)} |H>_k |H>_{-k}
\]  

in which \( \psi \) is a relative phase, \( |H> \) and \( |V> \) represent horizontal and vertical polarisation states of the photons. For a given momentum \( k (k_x > 0), \psi(k) \) is the sum of three phase terms \( \psi_0(k), \theta_A(-k) \) and \( \theta_B(k) \). \( \psi_0(k) \) is a static phase distortion produced during the photon generation process \cite{23} that is characterised beforehand (see Methods). The phases \( \theta_A(-k) \) and \( \theta_B(k) \) are actively controlled by Alice and Bob by programming pixels at coordinates \(-k\) and \( k\) of their SLMs. This is made possible by the use of parallel aligned nematic liquid-crystal SLMs, that enable the manipulation of the horizontal polarisation of incoming photons but leave the vertical component unchanged.

First, Alice encodes an image \( \theta_A(-k) \) in the phase component of entangled photons by programming her SLM with the corresponding phase pattern. Figure 1b shows the pattern used in our experiments, corresponding to the letters \( UOG \). On the other hand, Bob displays on his SLM a phase mask \( \theta_B(k) = -\psi_0(k) \) to compensate for the phase distortion \( \psi_0 \) (Fig. 1a). This correcting phase remains superimposed to all phase masks that Bob programs throughout the experiment. As a result, the phase of the quantum state after the SLMs equals exactly the encoded image \( \psi(k) = \theta_A(-k) \). In the example shown in Fig. 1b, pixels associated with the letters \( U \) and \( O \) are encoded as the Bell states \( |VV> + |HH> \) \( (\psi = 0) \), while \( f \) and \( G \) are encoded as \( |VV> - |HH> \) \( (\psi = \pi) \). After programming Alice’s phase, we observe that the intensity images measured by both Alice and Bob, shown in Figs. 1c and e, are homogeneous and do
Bob image decoding

Figure 1. Schematic of the quantum holographic reconstruction. a, Space-polarisation hyperentangled photon pairs propagate through two spatial light modulators (Alice SLM and Bob SLM) and are detected by two electron multiplied charge couple device cameras (Alice EMCCD and Bob EMCCD). Transverse momentums $k$ of photons with negative $x$-component ($k_x < 0$) are mapped to pixels on Alice’s SLM and camera, while those with positive $x$-component ($k_x > 0$) are mapped to pixels on Bob’s SLM and camera. Parallel aligned nematic liquid-crystal SLMs allow Alice and Bob to modulate at any pixel the horizontal polarisation of incoming photons with spatial phases $\theta_A$ and $\theta_B$. Two polarisers oriented at 45 degrees are inserted between SLMs and cameras. b, Phase image $\theta_A(-k)$ displayed on Alice SLM. c and d, Intensity images measured by Alice and Bob on their cameras, respectively. e, SLM pattern displayed on Bob SLM to compensate for the static phase distortion $\psi_0$. f-i, Intensity correlation images measured by Bob for different constant phase shift programmed on his SLM: $+0$ (f), $+\pi/2$ (g), $+\pi$ (h) and $+3\pi/2$ (i). Each image is obtained by measuring intensity correlations between Bob camera pixels $k$ and their symmetric on Alice camera $-k$. j, Phase image reconstructed by Bob, with a signal-to-noise ratio (SNR) over 19 and a normalised mean square error (NMSE) of 5%. A total of $2.5 \times 10^6$ frames was acquired to retrieve the phase. Intensity and intensity correlation values are in arbitrary units and the same scales are used in all the figures of the manuscript.

not reveal the phase-encoded image. This observation remains valid when including polarisers in front of the cameras, at any orientation.

In the holographic reconstruction step of the process, Bob decodes the image by performing intensity correlation measurements between pixels at $k$ of his camera and symmetric pixels at $-k$ on Alice’s camera with two polarisers oriented at 45 degrees (see Methods). This measurement is repeated four times for four different constant phase shifts $\theta$ applied on Bob’s SLM, resulting in intensity correlation image $R_\theta(k) \propto 1 + \cos(\theta_A(k) + \theta)$ (see Methods). Figures 1f-i show four intensity correlation images measured for $\theta \in \{0, \pi/2, \pi, 3\pi/2\}$ that partially reveal the hidden phase. Following a similar approach to classical holography, Bob then reconstructs the encoded image by using equation 1 after replacing $I_\theta$ by $R_\theta$. As shown in Fig. 1j, the retrieved image is 180 degrees rotated and is of high quality, with a signal-to-noise ratio (SNR) over 19 and a normalised mean square error (NMSE) of 5%. While the SNR measures the intrinsic quality of the image retrieved by Bob in term of noise level, the NMSE quantifies its resemblance to the original image encoded by Alice (see Methods).

In our protocol, the photon pair spatial correlations provide the high-dimensional image space while polarisation entanglement carries the grey-scale information at each pixel. The presence of polarisation entanglement is therefore essential to this scheme. For example, Fig. 2 shows results of quantum holography performed with the same encoded image as in Fig. 1 but using a source of photon pairs that are entangled in space but not in polarisation (see Methods). As in the previous case, intensity images measured by Alice and Bob in Fig 2b and c do not reveal information about the encoded phase. However, Figs. 2f-i show that the intensity correlation images acquired during the phase-shifting process do not reveal any image information either, and the phase image cannot be retrieved (NMSE=95%), as shown in Fig. 2j. Non-zero
values in intensity correlation images also confirm that (classical) correlations between photon polarisations are present without entanglement; the existence of a phase $\Psi$ is conditioned on the coherence, and thus the entanglement, between the two-qubit terms $|VV\rangle$ and $|HH\rangle$ of the state.\(^{22}\)

Beyond the intrinsic interest in imaging schemes that rely exclusively on nonlocal entanglement, our quantum holographic protocol is also robust against dynamic phase disorder. Figure 3 describes an experimental apparatus in which space-polarisation entangled photons propagate through a thin diffuser (figure inset) positioned on a motorised translation stage that is placed in the image plane of both SLMs and cameras. In this configuration, the diffuser introduces a time varying random phase $\Phi_k(t)$ in all spatial modes $k$ (see Methods). At time $t$, the state detected by Alice and Bob thus becomes

$$\sum_k e^{i[\Phi_k(k)+\Phi_k(-k)]} \left[ |V\rangle_k |V\rangle_{-k} + e^{i\Psi(k)} |H\rangle_k |H\rangle_{-k} \right].$$

The phase disorder terms factorise in equation 3 leaving the term $\Psi$ undisturbed. Figures 4a-b show the experimental reconstruction of a phase image through a dynamic diffuser. The diffuser induces a small blur of the edges in the intensity images measured by Alice and Bob (Figs. 4b and c), yet the image encoded by Alice as shown in Fig. 4a is very accurately reconstructed by Bob in Fig. 4h (SNR = 21 and NMSE = 2%). It is important to note that each photon of a pair experiences a phase disorder independent of that experienced by its twin (i.e. they propagate through spatially separated parts of the diffuser). Therefore, our holographic protocol can operate with Alice and Bob placed at any distance from each other, without being sensitive either to mechanical instability and vibrations.

Finally, we demonstrate that our approach also works in the presence of stray, classical light falling on the detectors. As illustrated in Fig. 3 images of two different cat-shaped objects illuminated by a laser are superimposed onto both Alice’s and Bob’s sensors. These classical images are clearly visible in the intensity images as shown in Figs. 4i and g. However, because photons emitted by the classical source are spatially uncorrelated, the cat-shaped images do not appear in the intensity correlation images\(^{23}\) used for phase reconstruction, as shown in Figs. 4i and j. Therefore, a high quality (SNR = 20 and NMSE = 4%) phase image, encoded by Alice (Fig. 4i), is retrieved by Bob (Fig. 4j). This experiment highlights another practical advantage of the protocol, namely that it remains operational in natural environments containing sources of stray light.

In summary, we have presented a quantum imaging technique enabled by nonlocal entanglement. Grey scale information about the image is encoded in the relative phases between the entangled polarisation qubits and distributed over many spatial positions through the high-dimensional structure of spatial entanglement. Bob cannot retrieve the phase information encoded by Alice by classical holography. Furthermore, because the holography happens in a degree of freedom that is not affected by phase disorder in the image basis (i.e. polarisation) and is based on correlation measurements, it is robust against the presence of dynamic diffusers and stray light, two factors that would respectively disrupt and inhibit a phase reconstruction process using other techniques.

Beyond imaging, the use of quantum entanglement as an information carrier holds potential for developing quantum-secured imaging applications.\(^{23}\) In particular, the possibility to arbitrarily enlarge the distance between Alice and Bob makes our scheme promising for quantum communication. The measurements performed by Bob in the holographic process correspond exactly to projections in the diagonal ($\theta_B \in \{0, \pi\}$) and circular ($\theta_B \in \{\pi/2, 3\pi/2\}$) polarisation basis. Similarly, Alice can use her SLM to perform measurements in the corresponding rotated basis $\theta_A \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$, instead of encoding an image. As shown in Fig. 5, one may use these measurement settings to show spatially-resolved violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality between Alice and Bob (see Methods). This implies that in principle Ekert’s quantum key distribution protocol\(^{25}\) could be implemented to share secret keys across many spatial channels (i.e. pixel pairs). The ability to exploit many channels in parallel could considerably increase the secret key rates of QKD protocols\(^{26}\) compared to the conventional single channel.
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Figure 3. **Detailed experimental setup with dynamic diffusers and stray light.** Light emitted by a laser diode at 405nm and polarised at 45 degrees illuminates a pair of β-Barium Borate (BBO) crystals (0.5mm thickness each) whose optical axes are perpendicular to each other to produce pairs of photons entangled in space and polarisation by type-I spontaneous parametric down-conversion (SPDC). After the crystals, pump photons are filtered out by a combination of long-pass and band-pass filters. The momentum of red photons is mapped onto an SLM divided in two parts (Alice SLM and Bob SLM) by Fourier imaging with lens $f_1$. Lenses $f_2 - f_3$ image the SLM plane onto a thin diffuser (inset) positioned on a motorised translation stage and lenses $f_4 - f_5$ image it on an EMCCD camera split in two parts (Alice EMCCD and Bob EMCCD). A polariser oriented at 45 degrees is positioned between lenses $f_4$ and $f_5$. Two cat-shaped objects illuminated by a laser (810nm) are imaged on the camera and superimposed on top of quantum light using two lenses $f_6 - f_5$ and an unbalanced beam splitter (BS 90T/10R). For clarity, only two propagation paths of entangled photons at k and $-k$ are represented, while they have a higher dimensional spatial structure (> 500 modes) ; SLMs, EMCCD cameras and diffusers are represented by pairs, while they are single devices spatially divided into two independent parts; the SLM is represented in transmission, while it operates in reflection. See Methods for further information.

Figure 4. **Quantum holography through dynamic phase disorder and in the presence of stray light.** a, Phase image encoded by Alice. b and c, Intensity images measured respectively by Alice and Bob through the dynamic diffuser with no stray light. d, Phase image reconstructed by Bob with SNR=21 and NMSE=2%. e, Phase image by Alice. f and g, Intensity images measured by Alice and Bob through the dynamic diffuser and in the presence of stray light taking the form of two cat-shaped images. h, Phase image reconstructed by Bob with SNR=20 and NMSE=4%. i-j, Intensity correlation images measured by Bob for phase shifts: +0 (i) and $+\pi$ (j). $5 \times 10^6$ frames were acquired in total for each case.
Figure 5. Spatially-resolved Clauser-Horne-Shimony-Holt (CHSH) inequality violation. **a**, Poincar spheres representing the projections performed by Alice and Bob with their SLMs. $|D\rangle$ and $|A\rangle$ are two eigenstates of the diagonal polarisation basis and $|R\rangle$ and $|L\rangle$ are the two eigenstates of the circular polarisation basis. Bob performs measurements directly in the diagonal and circular basis, while Alice operates in a similar basis rotated by $\pi/4$. **b**, Values of $S$ measured at different pairs of pixel of Alice’s and Bob’s sensors. The image is calculated from intensity correlation measured for 16 combinations of angles $\theta_A$ and $\theta_B$. Spatial averaged value is $S = 2.23 \pm 0.02 > 2$. See Methods for further details.
Experimental layout. A paired set of BBO crystals have dimensions of 0.5 × 5 × 5 mm each and are cut for type I SPDC at 405 nm. They are optically contacted with one crystal rotated by 90 degrees about the axis normal to the incidence face. Both crystals are slightly rotated around horizontal and vertical axis to ensure near-collinear phase matching of photons at the output (i.e. rings collapsed into disks). The pump is a continuous-wave laser at 405 nm (Coherent OBIS-LX) with an output power of approximately 200 mW and a beam diameter of 0.8 ± 0.1 mm. A 650 nm-cut-off long-pass filter is used to block pump photons after the crystals, together with a band-pass filter centred at 810 ± 5 nm. The SLM is a phase only modulator (Holoeye Pluto-2-NIR-015) with 1920 × 1080 pixels and a 8 µm pixel pitch. The camera is an EMCCD (Andor Ixon Ultra 897) that operates at −60°C, with a horizontal pixel shift readout rate of 17 MHz, a vertical pixel shift every 0.3 µs, a vertical clock amplitude voltage of +4V above the factory setting and an amplification gain set to 1000. It has a 16 µm pixel pitch. Exposure time is set to 3 ms. The classical source is a superluminescent diode laser (Qphotonics) with a spectrum of 810 ± 5 nm. The SLM is a phase only modulator (Holoeye Pluto-2-NIR-015) with 1920 × 1080 pixels and a 8 µm pixel pitch. The camera is an EMCCD (Andor Ixon Ultra 897) that operates at −60°C. During the holographic process, Alice encodes a phase \(\theta_A(-k)\) and Bob applies a phase shift \(\theta\) superimposed over the phase compensation pattern \(-\Psi_0(k)\).

During the holographic process, Alice encodes a phase \(\theta_A(-k)\) and Bob applies a phase shift \(\theta\) superimposed over the phase compensation pattern \(-\Psi_0(k)\). As a result, intensity correlation measurements performed by Bob for a given \(\theta\) are given by \(R_0(k) = \frac{1}{2} [1 + \cos(\theta_A(k) + \theta)]\). As in classical holography (equation 1), Bob then reconstructs the phase image \(\theta_A(k)\) image using four successive measurements: \(\theta_A(k) = \arg \left[ R_0(k) - R_1(k) + i (R_2(k) - R_3(k)) \right]\).

Spatial and polarisation entanglement. Spatial entanglement in the photon source is characterized by performing intensity correlation measurements between positions and momentum of photons. Correlation width measurements return values of \(\sigma_x = 10.85 \pm 0.06 \mu m\) for position and \(\sigma_k = [2.033 \pm 0.001] \times 10^3 \text{rad.m}^{-1}\) for momentum. These values show violation of EPR criteria \(\sigma_x \sigma_k = [2.21 \pm 0.01] \times 10^{-2} \lessgtr \frac{\sqrt{2}}{2}\) and allows estimation of the number of entangled modes \(K = 514 \pm 5\).

Polarisation entanglement is certified by demonstrating violation of Bell inequalities between all spatially correlated pairs of pixels at the output, with a mean value of \(S = 2.23 \pm 0.02 > 2\) (Fig. 5).
Signal-to-noise, normalised mean square error and spatial resolution. Signal-to-noise ratio (SNR) is obtained by calculating an averaged value of the phase in a region of the retrieved image where it is constant, and then dividing it by the standard deviation of the noise in the same region. To have a common reference, SNR values are calculated using areas where the phase is constant and equals $\pi$. For a fixed exposure time and pump power, the SNR varies as $\sqrt{N}$, where $N$ is the number of images used to reconstruct the intensity correlation images\cite{54}. The normalised mean square error (NMSE)\cite{45} quantifies the resemblance between an image reconstructed by Bob and the ground truth image encoded by Alice. The NMSE is calculated using the formula:

$$NMSE = \frac{|M_0 - M_x|}{M_0}$$  \hfill (8)

where $M_0$ is the mean square error (MSE) measured between the ground truth and the retrieved image and $M_x$ is an average value of MSE measured between the ground truth and a set of images composed of phase values randomly distributed between 0 and $2\pi$. The MSE between two images composed of $P$ pixels with values denoted respectively $\{x_i\}_{i\in\{1,P\}}$ and $\{y_i\}_{i\in\{1,P\}}$ is defined as $M = 1/P \sum_{i=1}^{P} |x_i - y_i|^2$. Values of NMSE range between 1 (retrieved image is a random phase image) and 0 (retrieved image is exactly the ground truth). Spatial resolution in the retrieved image is determined by the spatial correlation width of entangled photons. In our experiment, its value is estimated to $d = 45 \pm 3 \mu\text{m}$, which corresponds to approximately 3 camera pixels. See the Supplementary Information for further details on the signal-to-noise ratio and spatial resolution characterisation.

Photons without polarisation entanglement. Results shown in Fig. 3 are obtained using a quantum state defined by the following density operator:

$$\frac{1}{2} \sum_k [|H\rangle_k |H\rangle_{-k} \langle H|_k \langle H|_{-k} + |V\rangle_k |V\rangle_{-k} \langle V|_k \langle V|_{-k}]$$ \hfill (9)

Experimentally, it is produced by switching the polarisation of the pump laser between vertical and horizontal polarisations, which is equivalent of using a unpolarised pump. Because entanglement originates fundamentally from a transfer of coherence properties between the pump and the down-converted fields in SPDC\cite{31,32} the lack of coherence in the pump polarisation induces the absence of polarisation entanglement in the produced two-photon state, while spatial and temporal entanglement are maintained. See the Supplementary Information for further details on state entangled in space but not in polarisation.

Spatially-resolved Clauser-Horne-Shimony-Holt (CHSH) measurement. A set of 16 intensity correlations images $R_{\theta_A, \theta_B}$ is first measured using all combi-
Supplementary information

I. DETAILS ON INTENSITY CORRELATION MEASUREMENT

This section provides more details about the intensity correlation measurement performed by the EMCCD camera. Further theoretical details can be found in Ref.20.

An EMCCD camera can be used to reconstruct the spatial (a) intensity distribution $I(k)$ and (b) intensity correlation distribution $\Gamma(k_1, k_2)$ of photon pairs, where $k, k_1$ and $k_2$ correspond to positions of camera pixels. To do that, the camera first acquires a set of $N$ frames $\{I_l\}_{l=1}^N$ using a fixed exposure time. Then:

(a) The intensity distribution is reconstructed by averaging over all the frames:

$$I(k) = \frac{1}{N} \sum_{l=1}^N I_l(k)$$  \hspace{1cm} (12)

(b) The intensity correlation distribution is reconstructed by performing the following substation:

$$\Gamma(k_1, k_2) = \frac{1}{N} \sum_{l=1}^N I_l(k_1)I_l(k_2) - \frac{1}{N-1} \sum_{l=1}^{N-1} I_l(k_1)I_{l+1}(k_2)$$  \hspace{1cm} (13)

Under illumination by the photon pairs, intensity correlations in the left term of the subtraction originate from detections of both real coincidences (two photons from the same entangled pair) and accidental coincidences (two photons from two different entangled pairs), while intensity correlations in the second term originate only from photons from different entangled pairs (accidental coincidence) because there is zero probability for two photons from the same entangled pair to be detected in two successive images. A subtraction between these two terms leaves only genuine coincidences, that is proportional to the spatial joint probability distribution of the pairs.

In our work, we use this technique for measuring correlation between pairs of symmetric pixels $k$ and $-k$ to reconstruct intensity correlation images $R(k)$. These images correspond exactly to the anti-diagonal component of the complete intensity correlation distribution $R(k) = \Gamma(k, -k)$. Fig. 6 illustrates these different types of measurements in the case of the experiment described in Fig. 1 of the manuscript when Bob displays a phase shift $+\pi$ on his SLM. Fig. 6a and b show respectively intensity images measured by Alice (pixels $k = (k_x < 0, k_y)$) and Bob (pixels $k = (k_x > 0, k_y)$). These images do not provide any information about the correlation between photon pairs. Fig. 6c is the conditional image $\Gamma(k, k_1)$ that represents the probability of measuring a photon from a pair at pixel $k$ in Alice side conditioned by the detection of its twin photon at pixel $k_1$ in Bob side. We observe a strong peak of correlation centred around the symmetric pixel $-k_1$ due to the strong anti-correlation between momentum of the pairs (zoom in inset). Similarly, Fig. 6d is the conditional image $\Gamma(k, k_2)$ relative to position $k_2$. In this last case, the peak of correlation is very weak (zoom in inset). Finally, Fig. 6e shows the intensity correlation image $R(k) = \Gamma(k, -k)$. In this last image, the value at pixel $k_1$ corresponds to the value of the peak of correlation at $-k_1$ shown in Fig. 6c and the value at pixel $k_2$ corresponds to the value of the peak of correlation at $-k_2$ in Fig. 6d.

II. DETAILS ON SPATIAL AND POLARISATION ENTANGLEMENT OF THE SOURCE

As described in Fig. 3 of the manuscript, entangled photon pairs are generated by type I SPDC using a pair of BBO crystals. These pairs are entangled in both their polarisation and spatial degree of freedom21,22,23.

A. Spatial entanglement

Spatial entanglement between photons is characterised by performing intensity correlation measurement between (a) positions and (b) momentum of photons21,22,23. Fig. 7 describes the corresponding experimental apparatus:

(a) Positions $r$ of photons are mapped onto pixels of the camera using a two-lens imaging configuration $f_1 - f_2$. After measuring the intensity correlation distribution $\Gamma(r_1, r_2)$, its projection along the minus-coordinate axis $r_1 - r_2$ is shown in Fig. 7b. The peak of correlation at its center is a signature of strong correlations between positions of photons. Its width $\sigma_r = 10.85 \pm 0.06 \mu m$ is measured by fitting with a Gaussian model23.

(b) Momentum $k$ of photons are mapped onto pixels of the camera by replacing the lens $f_2$ by a lens with twice its focal length. After measuring the intensity correlation distribution $\Gamma(k_1, k_2)$, its projection along the sum-coordinate axis $k_1 + k_2$ is shown in Fig. 7c. The peak of correlation at its center is a signature of strong anti-correlations between momentum of photons. Its width $\sigma_k = [2.033 \pm 0.001] \times 10^3 \text{ rad.m}^{-1}$ is measured by fitting with a Gaussian model23.

Under certain assumptions that are discussed in Ref.21, the presence of spatial entanglement between photons is certified by violating a EPR-type inequality: $\sigma_r \sigma_k = [2.21 \pm 0.01] \times 10^{-2} < \frac{1}{2}$. Moreover, the dimension of
entanglement $K = 514 \pm 5$ is calculated using the formula \cite{8}:

$$K = \frac{1}{4} \left[ \sigma_r \sigma_k + \frac{1}{\sigma_r \sigma_k} \right]^2$$ \hspace{1cm} (14)

**B. Polarisation entanglement**

The presence of polarisation entanglement between photons can be demonstrated by violating Clauser-Horne-Shimony-Holt (CHSH) inequality \cite{9}. A simplified version of the experimental apparatus used to perform this measurement is shown in Fig. 7d. In this case, the combination of Alice and Bob SLMs with a 45 degrees polariser positioned in front of the cameras play the role of the rotating polarisers used in a conventional CHSH violation experiment \cite{9}. When a constant phase $\theta_B (\theta_A)$ is programmed onto Bob SLM (Alice SLM), each pixel of Bob camera (Alice) performs a measurement that corresponds to an operator of the form:

$$\hat{B}_{\theta_B} = \frac{1}{2} [ |H\rangle \langle H| + |V\rangle \langle V| + e^{i\theta_B} |H\rangle \langle V| + e^{-i\theta_B} |V\rangle \langle H| ]$$ \hspace{1cm} (15)

In the first step of this experiment, Alice and Bob measure intensity correlation images $R_{\theta_A, \theta_B}$ for 16 combinations of phase values $\theta_A$ and $\theta_B$ programmed on their SLMs. On the one hand, Bob uses the same set than the one used in the holographic process $\theta_B \in [0, \pi/2, \pi, 3\pi/2]$, where $\theta_B = 0$ and $\theta_B = \pi$ correspond to measurements performed in the diagonal polarisation basis $B_{\theta_B = 0/\pi} = (|V\rangle \pm |H\rangle)/\sqrt{2}$ and values $\theta_B = \pi/2$ and $\theta_B = 3\pi/2$ correspond to measurements performed in the circular polarisation basis $B_{\theta_B = \pi/2} = (|V\rangle \pm i|H\rangle)/\sqrt{2}$. On the other hand, Alice programs phase values $\theta_B = \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$ of her SLM to perform measurements in diagonal and circular basis rotated by $\pi/4$: $\hat{A}_{\theta_B = \pi/4} = (|V\rangle \pm i|H\rangle)/(|V\rangle \pm i|H\rangle)$ and $\hat{A}_{\theta_B = 3\pi/4} = (|V\rangle \pm i|H\rangle)/(|V\rangle \pm i|H\rangle)$. In the second step, Alice and Bob combine these 16 intensity correlation images to compute $E_{\theta_A, \theta_B}$ using the following formula \cite{9}:

$$E_{\theta_A, \theta_B} = \frac{R_{\theta_A, \theta_B} - R_{\theta_A, \theta_B + \pi} - R_{\theta_A + \pi, \theta_B} + R_{\theta_A + \pi, \theta_B + \pi}}{R_{\theta_A, \theta_B} + R_{\theta_A, \theta_B + \pi} + R_{\theta_A + \pi, \theta_B} + R_{\theta_A + \pi, \theta_B + \pi}}$$ \hspace{1cm} (16)

Finally, an image of $S$ values is obtained using the following equation:

$$S = |E_{\pi/2, \pi/4} - E_{\pi/2, 5\pi/4}| + |E_{\pi/4} + E_{5\pi/4}|$$ \hspace{1cm} (17)

This image is shown in Fig.5.b of the manuscript and in Fig. 7e. We observe that almost all values of $S$ measured between in Alice and Bob correlated pixels show violation of CHSH inequality $S > 2$. A spatial averaged value of $S = 2.23 \pm 0.02 > 2$ is estimated by calculating the mean and variance of $S$ values over a region of a smaller region of the sensor where $S$ is homogeneous ($k_x \in [10, 50]$ and $k_y \in [-40, 40]$). Violation of CHSH inequality demonstrates the presence of polarisation entanglement between photon pairs.

**III. DETAILS ON SIGNAL-TO-NOISE AND SPATIAL RESOLUTION**

This section provides more details about the signal-to-noise ratio (SNR) and spatial resolution of the phase image retrieved by quantum holography.
A. Signal-to-noise ratio.

In our work, we define the SNR as the ratio between π and the standard deviation of the noise measured in a region of the image in which phase values equal π. It is for example the case for the area within the letters U and o in the phase image shown in Fig.1.b. For a constant source intensity and a fixed exposure time, the factor that most influences the SNR is the total number of images N acquired to measure the four intensity correlation images used to reconstruct the phase image. Each phase image shown in the manuscript has been retrieved using \( N = 2.5 \times 10^6 \) images and show SNR values ranging between 19 and 21. Fig.8.a shows SNR values measured for different values of \( N \) (black crosses). As predicted by the theory, \( \sqrt{N} \) and demonstrated by fitting the data (blue dashed curve), the SNR evolves as \( \sqrt{N} \). Fig.8.b-c show three images of the retrieved phase for respectively \( N = 2.5 \times 10^4 \) images, \( N = 2.5 \times 10^5 \) and \( N = 2.5 \times 10^7 \) images.

B. Spatial resolution.

To measure the spatial resolution in the retrieved phase image, a radial resolution target (Siemens star with 16 branches) is programmed by Alice. Fig.8.e shows the retrieved image. The area of the resolution target that is not spatially resolved is a disk of diameter 29 pixels, which corresponds to a spatial resolution of \( d = 45 \pm 3 \mu m \) or approximately 3 pixels.

IV. DETAILS ON PHASE DISTORTION CHARACTERISATION

This section provides more details about the characterisation of the phase distortion \( \Psi_0 \).

Phase distortion \( \Psi(k) \) originates from the SPDC process used to produce pairs of photons. To characterise it, Alice and Bob perform a quantum holographic experiment using the scheme described in Fig.1.a. In this case, Alice programs a flat phase pattern \( \theta_A = 0 \) (Fig.9.a) and Bob programs successive phase shifts patterns with values \( +0 \) (Fig.9.b), \( +\pi/2 \) (Fig.9.b), \( +\pi \) (Fig.9.b) and \( +3\pi/2 \) (Fig.9.b), without any correction phase mask superimposed on them. Fig.9.f-i show intensity correlations images measured for each of the four SLM patterns displayed by Bob. Therefore, the resulting reconstructed phase image corresponds exactly to the phase distortion \( \Psi_0(k) \) (Fig.9.j). The phase image is then fitted by a quadratic function of the form \( \Psi_0(k_x,k_y) = 4.09k_x^2 + 5.04k_y^2 + 0.02 \) (Fig.9.k). Result of the fit is then used to construct and program the phase
compensation pattern on Bob SLM (Fig. 9).}

V. DETAILS ON QUANTUM HOLOGRAPHY WITHOUT POLARISATION ENTANGLEMENT

This section provides more details about quantum holography performed with photons entangled in space but not in polarisation.

Fig. 2 of the manuscript shows results of quantum holographic experiment performed with photons that are entangled in space but not in polarisation. In this experiment, the state generated at the output of the crystals is defined by the density operators:

$$\frac{1}{2} \sum_k \left[ |H_k\rangle\langle H_k| - 1 + |V_k\rangle\langle V_k| - 1 \right]$$

This state is composed of a balanced statistical mixture of two pure states: $$\sum_k |H_k\rangle\langle H_k|$$ and $$\sum_k |V_k\rangle\langle V_k|$$.

VI. DETAILS ON THE CHARACTERISATION OF THE DYNAMIC PHASE DISORDER

This section provides more details about the dynamic phase disorder and its characterisation.

Fig. 11a shows the experimental setup used to characterise properties of the phase disorder introduce by the presence of the diffuser, that is a plastic sleeve layer of thickness < 100µm. Without diffuser, a sine-shaped phase image programmed on the SLM (Fig. 11b) generates a very specific diffraction pattern on the camera (Fig. 11c). After introducing the static diffuser, the diffraction pattern becomes a speckle pattern (Fig. 11c). The presence of the diffuser erases all information about the phase image, that cannot be retrieved by classical holographic techniques such as phase retrieval and phase-stepping holography. When the diffuser is moving, the speckle pattern takes the form of a diffuse halo if the exposure time of the camera is larger (0.5s) than the typical decorrelation time of the speckle. The halo width is estimated to 1.3mm by Gaussian fitting which corresponds approximately to a diffusing angle of 1 degree and a surface roughness of 46µm. Moreover, the dynamic properties of the disorder are estimated by measuring the speckle decorrelation time. Speckle correlation coefficients are calculated by acquiring a series of speckle patterns using short exposure times (3ms) and correlating them with a reference speckle image. Fig. 11 shows the decrease of speckle correlation with time (black crosses). A typical decorrelation time of 183ms is measured by fitting data with an exponential model (dashed blue curve).
Figure 9. **Phase distortion characterisation.** a. Flat phase pattern programmed on Alice SLM. b–e. Phase-shifted patterns programmed on Bob SLM: +0 (b), +π/2 (c), +π (d) and +3π/2 (e). f–i. Intensity correlation images measured for each phase mask displayed on Bob SLM. j. Phase image retrieved by Bob. k. Fit of the phase image by a quadratic function of the form: \[ \Psi_0(k_x, k_y) = 4.69k_x^2 + 5.04k_y^2 + 0.02. \] l. Correction phase pattern resulting from the fitting process.
Figure 10. **Generation of a mixed state without polarisation entanglement.**

a. Experimental configurations used to generate the mixed state: $\sum_k [\langle H|k\rangle|H\rangle\langle H|k\rangle|H\rangle + \langle V|k\rangle|V\rangle\langle V|k\rangle|V\rangle]$. Polarisation of pump laser is alternated between vertical and horizontal.

b. Frames acquired in each configuration are summed. Intensity images shown in Fig.2.b,c of the manuscript and intensity correlation images shown in Fig.2.e-f of the manuscript are reconstructed from the set of summed frames.
Figure 11. **Characterisation of the dynamic phase disorder.**

a, A classical laser (810nm; beam diameter 0.8mm) illuminates an SLM that is imaged onto the diffuser by two lenses $f_1 = 150\text{mm}$ and $f_2 = 75\text{mm}$. Lens $f_3 = 75\text{mm}$ Fourier-images the diffuser onto the camera.

b, Sine-shaped phase pattern programmed on the SLM.

c, Intensity image measured on the camera without diffuser.

d, Intensity image measured on the camera with the moving diffuser using an exposure time of 0.5s. Width of the diffuse halo is estimated to 1.3mm by fitting with a Gaussian function.

e, Intensity image measured on the camera with the moving diffuser with an exposure time of 0.5s. Width of the diffuse halo is estimated to 1.3mm by fitting with a Gaussian function.

f, Measurement of speckle correlation values with time (black crosses) together with a theoretical fit of the form $1 - 0.77(1 - e^{-t/0.183})$ (blue dashed line). Typical decorrelation time equals to 183ms.
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1. D. Gabor, Nature 161, 777 (1948).
2. P. Marquet, B. Rappaz, P. J. Magistretti, E. Cuche, Y. Emery, T. Colomb, and C. Depeursinge, Optics Letters 30, 468 (2005).
3. P. Refregier and B. Javidi, Optics Letters 20, 767 (1995).
4. J. F. Heanue, M. C. Bashaw, and L. Hesselink, Science 265, 749 (1994).
5. I. Yamaguchi and T. Zhang, Optics Letters 22, 1268 (1997).
6. P.-A. Moreau, E. Toninelli, T. Gregory, and M. J. Padgett, Nature Reviews Physics 1, 367 (2019) arXiv: 1908.03034.
7. A. G. White, J. R. Mitchell, O. Nairz, and P. G. Kwiat, Physical Review A 58, 605 (1998).
8. T. B. Pittman, Y. H. Shih, D. V. Strekalov, and A. V. Sergienko, Physical Review A 52, R3429 (1995).
9. G. B. Lemos, V. Borish, G. D. Cole, S. Ramelow, R. Lapkiewicz, and A. Zeilinger, Nature 512, 409 (2014).
10. M. B. Nasr, B. E. A. Saleh, A. V. Sergienko, and M. C. Teich, Physical Review Letters 91, 083601 (2003).
11. G. Brida, M. Genovese, and I. R. Berchera, Nature Photonics 4, 227 (2010).
12. T. Ono, R. Okamoto, and S. Takeuchi, Nature Communications 4, 2426 (2013).
13. R. Tenne, U. Rossman, B. Rephael, Y. Israel, A. Krupinski-Ptaszek, R. Lapkiewicz, Y. Silberberg, and D. Oron, Nature Photonics 1, 498 (2001).
14. A. F. Abouraddy, B. E. A. Saleh, A. V. Sergienko, and M. C. Teich, Optics Express 9, 498 (2001).
15. S. Asban, K. E. Dorfman, and S. Mukamel, Proceedings of the National Academy of Sciences 116, 11673 (2019).
16. R. Chrapkiewicz, M. Jachura, K. Banaszek, and W. Wasilewski, Nature Photonics 10, 576 (2016).
17. F. Devaux, A. Mosset, F. Bassignot, and E. Lantz, Physical Review A 99, 033854 (2019).
18. J. T. Barreiro, N. K. Langford, N. A. Peters, and P. G. Kwiat, Physical Review Letters 95, 260501 (2005).
19. S. F. Hegazy and S. S. A. Obayya, JOSA B 32, 445 (2015).
20. H. Defienne, M. Reichert, and J. W. Fleischer, Physical review letters 120, 203604 (2018).
21. J. C. Howell, R. S. Bennink, S. J. Bentley, and R. W. Boyd, Physical Review Letters 92, 210403 (2004).
22. P. G. Kwiat, S. Barraza-Lopez, A. Stefanov, and N. Gisin, Nature 409, 1014 (2001).
23. H. Defienne, M. Reichert, J. W. Fleischer, and D. Faccio, Science Advances 5, eaa0307 (2019).
24. M. Malik, O. S. Magaa-Loaiza, and R. W. Boyd, Applied Physics Letters 101, 241103 (2012).
25. A. K. Ekert, Physical Review Letters 67, 661 (1991).
26. J. Fang, P. Huang, and G. Zeng, Physical Review A 89, 022315 (2014).
27. P.-A. Moreau, J. Mougin-Sisini, F. Devaux, and E. Lantz, Physical Review A 86, 010101 (2012).
28. D. S. Tasca, F. Izdebski, G. S. Buller, J. Leach, M. Agnew, M. J. Padgett, M. P. Edgar, R. E. Warburton, and R. W. Boyd, Nature Communications 3, 984 (2012).
29. M. Reichert, H. Defienne, and J. W. Fleischer, Physical Review A 98, 013841 (2018).
30. R. C. Gonzalez and P. Wintz, Reading, Mass., Addison-Wesley Publishing Co., Inc.(Applied Mathematics and Computation , 451 (1977).
31. A. K. Jha and R. W. Boyd, Physical Review A 81, 013828 (2010).
32. G. Kulkarni, V. Subrahmanyam, and A. K. Jha, Physical Review A 93, 063842 (2016).
33. G. Kulkarni, P. Kumar, and A. K. Jha, JOSA B 34, 1637 (2017).
34. J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Physical review letters 23, 880 (1969).
35. M. V. Fedorov, Y. M. Mikhailova, and P. A. Volkov, Journal of Physics B: Atomic, Molecular and Optical Physics 42, 175503 (2009).
36. M. V. Fedorov, Physica Scripta 90, 074048 (2015).
37. A. Aspect, P. Grangier, and G. Roger, Physical review letters 49, 91 (1982).
38. S. F. Hegazy, Y. A. Badr, and S. S. A. Obayya, Optical Engineering 56, 026114 (2017).
39. J. R. Fienup, Applied Optics 21, 2758 (1982).