Global hypoellipticity for a class of periodic Cauchy operators

Fernando de Ávila Silva

Departamento de Matemática, Universidade Federal do Paraná,
Caixa Postal 19081, Curitiba, PR 81531-990, Brazil

Abstract

This note presents an investigation on the global hypoellipticity problem for Cauchy operators on $\mathbb{T}^{n+1}$ belonging to the class

$L = \prod_{j=1}^{m} (D_t + c_j(t)P_j(D_x))$, where $P_j(D_x)$ is a pseudo-differential operator on $\mathbb{T}^n$ and $c_j = c_j(t)$, a smooth, complex valued function on $\mathbb{T}$. The main goal of this investigation consists in establishing connections between the global hypoellipticity of the operators $L$ and its normal form $L_0 = \prod_{j=1}^{m} (D_t + c_{0,j}P_j(D_x))$. In order to do so, the problem is approached by combining Hörmander’s and Siegel’s conditions on the symbols of the operators $L_j = D_t + c_j(t)P_j(D_x)$.

Keywords: Global hypoellipticity, Pseudo-differential operators, Fourier series, Cauchy operators, Siegel conditions

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1. Introduction

This note discusses the global hypoellipticity problem for operators belonging to the class

$L = \prod_{j=1}^{m} (D_t + c_j(t)P_j(D_x))$, $(t, x) \in \mathbb{T} \times \mathbb{T}^n,$

(1)

Email address: fernando.avila@ufpr.br (Fernando de Ávila Silva)
where $\mathbb{T} \simeq \mathbb{R}/2\pi\mathbb{Z}$ is the Torus and $P_j(D_x)$ is a pseudo-differential operator of order $\nu_j \in \mathbb{R}$ defined by

$$P_j(D_x)u(x) = \sum_{\xi \in \mathbb{Z}^n} e^{ix \cdot \xi} p_j(\xi) \hat{u}(\xi),$$

where its symbol $p_j = p_j(\xi)$ satisfies

$$|p_j(\xi)| \leq C|\xi|^\nu_j, \forall \xi \in \mathbb{Z}^n$$

and $\hat{u}(\xi)$ are the Fourier coefficients

$$\hat{u}(\xi) = (2\pi)^{-n} \int_{\mathbb{T}^n} e^{-ix \cdot \xi} u(x) dx, \xi \in \mathbb{Z}^n.$$ 

In particular, $p_j$ belongs to the class $S^{\nu_j}(\mathbb{Z}^n)$, as shown in Ruzhansky and Turunen [1].

Let us recall that a pseudo-differential operator $\mathcal{P}$ is globally hypoelliptic (GH for short) on the $N$-dimensional torus $\mathbb{T}^N$ if conditions $u \in \mathcal{D}'(\mathbb{T}^N)$ and $\mathcal{P}u \in C^\infty(\mathbb{T}^N)$ imply $u \in C^\infty(\mathbb{T}^N)$.

We emphasize that the study of global hypoellipticity is a highly non-trivial problem and it seems impossible to attack it by a unified approach, even for the classes of vector fields, as we can see in A. Bergamasco, P. Cordaro and G. Petronilho [2], G. Petronilho [3], A. Himonas and G. Petronilho [4] and J. Hounie [5]. Also, the study is considered in different functional spaces, for instance, analytic and Gevrey classes, see A. Bergamasco [6], A. Albanese [7], G. Chinni, P. Cordaro [8] and the references therein.

Furthermore, the problem is analyzed in the class of pseudo-differential operators as proposed by D. Dickson, T. Gramchev and M. Yoshino in [9] where the authors characterizes the global hypoellipticity by means of Siegel type conditions on the symbols of these operators.

Now, since the main goal of this investigation is to establish connections between the global hypoellipticity of $L$ and its normal form

$$L_0 = \prod_{j=1}^m (D_t + c_{0,j} P_j(D_x)), \ (t, x) \in \mathbb{T}^{n+1},$$
where \( c_{0,j} = (2\pi)^{-1} \int_0^{2\pi} c_j(t) dt \), first we take inspiration from [10], where R. Gonzalez, A. Kirilov, C. Medeira and F. Ávila, present a characterization of the global hypoellipticity for operators in the form
\[
L_j = D_t + c_j(t)P_j(D_x), (t, x) \in \mathbb{T}^{n+1},
\]
by analyzing the functions
\[
t \in \mathbb{T} \mapsto M_j(t, \xi) = c_j(t)p_j(\xi), \xi \in \mathbb{Z}^n,
\]
and its averages
\[
M_{0,j}(\xi) = (2\pi)^{-1} \int_0^{2\pi} M_j(t, \xi) dt, \xi \in \mathbb{Z}^n.
\]
Secondly, by denoting \( Z_{M_j} = \{\xi \in \mathbb{Z}^n; \ M_{0,j}(\xi) \in \mathbb{Z}\} \) and \( L_{0,j} = D_t + c_{0,j}P_j(D_x) \), then the results from [10], that relate to this note, can be summarized by the following:

**Theorem 1.1.** Set \( c_j(t) = a_j(t) + ib_j(t) \) and \( p_j(\xi) = \alpha_j(\xi) + i\beta_j(\xi) \).

(a) if \( L_j \) is globally hypoelliptic, then \( Z_{M_j} \) is finite and \( L_{0,j} \) is globally hypoelliptic;
(b) if \( L_{0,j} \) is globally hypoelliptic and the functions
\[
\mathbb{T} \ni t \mapsto \Im M_j(t, \xi) = a_j(t)\beta_j(\xi) + b_j(t)\alpha_j(\xi)
\]
do not change sign, for a sufficiently large \( |\xi| \), then \( L_j \) is \( (GH) \).
(c) the following statements are equivalent:
   i) \( L_{0,j} \) is globally hypoelliptic;
   ii) there exist positive constants \( C_j, M_j \) and \( R_j \) such that
\[
|\tau + M_{0,j}(\xi)| \geq C_j|\tau, \xi|^{-M_j}, \text{ for all } |\tau| + |\xi| \geq R_j;
\]
   iii) there exist positive constants \( \tilde{C}_j, \tilde{M}_j \) and \( \tilde{R}_j \) such that
\[
|1 - e^{\pm 2\pi i M_{0,j}(\xi)}| \geq \tilde{C}_j|\xi|^{-\tilde{M}_j}, \text{ for all } |\xi| \geq \tilde{R}_j.
\]
In view of the main goal, the first step in our investigations is the study of the constant coefficients operators belonging to the class:

\[ L = \prod_{j=1}^{m} (D_t + \alpha_j P_j(D_x)), \quad (t, x) \in \mathbb{T}^{n+1}, \quad \alpha_j \in \mathbb{C}. \]

In Section 2, we present an analysis for this class that combines Theorem 1.1 and a generalization of Siegel’s condition. This condition was introduced by D. Dickson, T. Gramchev and M. Yoshino in [9]. As it shall be stated in Theorem 2.3, necessary and sufficient conditions for global hypoellipticity are presented by analyzing the behavior of the symbol \( L(\tau, \xi) \).

In Section 3, the case of variable coefficients is studied. A first investigation is performed through Theorem 1.1 and also by assuming the following commutative condition:

\[ [L_j, L_k] = L_j L_k - L_k L_j = 0, \quad j, k \in \{1, \ldots m\}, \]

that will be shown in Theorem 3.2.

In Section 3.1, a second investigation is presented and the approach is motivated by a condition proposed in Hörmander [11]: there exists positive constants \( \eta_j \) such that

\[ \Im M_j(t, \xi) \geq -\eta_j, \quad t \in \mathbb{T}, \quad \forall \xi \in \mathbb{Z}^n. \quad (3) \]

Under this hypothesis, in Theorem 3.3, necessary and sufficient conditions for the global hypoellipticity of the operator \( L_j \) will be exhibited. Moreover, in Theorem 3.4, connections between the hypoellipticity of operators \( L \) and \( L_0 \) will be established. Also, in Theorem 3.9, conditions for the global solvability of the operator \( L_j \) will be presented and, in particular, obstructions for the global hypoellipticity of operator \( L \) will be shown in Theorem 3.10.

Furthermore, also in Section 3.1, an interesting fact shall present itself: in view of condition (3), it becomes possible to obtain a new class of globally hypoelliptic operators, so that the functions \( \Im M_j(\cdot, \xi) \) change sign for infinitely many indexes \( \xi \in \mathbb{Z}^n \). This fact will be discussed at the beginning of the section, where an example will also be proposed.
2. Operators with constant coefficients

In this section we consider the constant coefficient operator

\[ L = \prod_{j=1}^{m} (D_t + \alpha_j P_j(D_x)), \quad (t, x) \in \mathbb{T}^{n+1}, \quad \alpha_j \in \mathbb{C}, \quad (4) \]

and study the global hypoellipticity by analyzing the growth of its symbol

\[ L(\tau, \xi) = \prod_{j=1}^{m} (\tau - \alpha_j p_j(\xi)), \quad (\tau, \xi) \in \mathbb{Z} \times \mathbb{Z}^n. \]

By following the approach introduced by Greenfield and Wallach in [12], we shall characterize the global hypoellipticity of \( L \) by means of a control in \( L(\tau, \xi) \):

**Theorem 2.1.** The operator \( L \) is globally hypoelliptic if and only if there are positive constants \( C, M \) and \( R \) so that

\[ |L(\tau, \xi)| \geq C|\langle \tau, \xi \rangle|^{-M}, \quad |\tau| + |\xi| \geq R. \]

Now, in order to present a complete characterization for the global hypoellipticity, we introduce a Siegel condition, inspired by D. Dickson, T. Gramchev and M. Yoshino in [9].

**Definition 2.2.** We say that a sequence \( \{c(\xi)\}_{\xi \in \mathbb{Z}^n} \) satisfies the generalized Siegel condition if there exists positive constants \( C, M \) and \( R \) such that

\[ |\tau - c(\xi)| \geq C|\langle \tau, \xi \rangle|^{-M}, \quad |\tau| + |\xi| \geq R, \quad (\tau, \xi) \in \mathbb{Z} \times \mathbb{Z}^n. \quad (5) \]

Furthermore, a set \( \Gamma = \{\{c_j(\xi)\}_{\xi \in \mathbb{Z}^n}, \quad j = 1, \ldots, m\} \) satisfies the simultaneous generalized Siegel condition if each \( \{c_j(\xi)\}_{\xi \in \mathbb{Z}^n} \) satisfies (5). In this case we can write \( \Gamma \in (\mathcal{GS})_\infty \).

**Theorem 2.3.** Let \( L \) be the operator defined in (4) and set

\[ \sigma_L = \{\{\alpha_j p_j(\xi)\}_{\xi \in \mathbb{Z}^n}, \quad j = 1, \ldots, m\}. \]

Then, \( L \) is globally hypoelliptic if and only if \( \sigma_L \in (\mathcal{GS})_\infty \).
Proof: Since the coefficients of $L_j$ are constant, we obtain $[L_j, L_k] = 0$, for $j, k \in \{1, \ldots, m\}$. Hence, $L$ is globally hypoelliptic if and only if each $L_j$ is also globally hypoelliptic. Then, if $L$ is (GH), it follows from Theorem 1.1 that all sequences $\{\alpha_j p_j(\xi)\}$ satisfy (5).

Conversely, if $\sigma_L \in (\mathcal{GS})_{\infty}$ it is possible to obtain constants $C_j$, $R_j$ and $M_j$ so that $|\tau - \alpha_j p_j(\xi)| \geq C_j |(\tau, \xi)|^{-M_j}$, for all $|\tau| + |\xi| \geq R_j$, hence

$$|L(\tau, \xi)| = \prod_{j=1}^{m} |\tau - \alpha_j p_j(\xi)| \geq C |(\tau, \xi)|^{-M}, \quad |\tau| + |\xi| \geq R,$$

where $C = \prod C_j$, $M = \sum M_j$ and $R = \max R_j$. The proof is then complete, in view of Theorem 2.1.

\[\square\]

Example 1. Let $L$ be the differential operator

$$L = \prod_{j=1}^{m} (D_t - \alpha_j D_x), \quad (t, x) \in \mathbb{T}^2.$$

In this case, it is possible to write:

$$L(\tau, \xi) = (\tau - \alpha_1 \xi) \ldots (\tau - \alpha_m \xi)$$

and thus $L$ is not globally hypoelliptic if and only if one of the numbers $\alpha_j$ is either a rational or Liouville. It can be noted here that this example recaptures results in [13].

Remark 2.4. It is possible to consider operators belonging to the class

$$\mathcal{P} = D_t^m + \sum_{j=1}^{m} P_j(D_x) D_t^{m-j}, \quad (t, x) \in \mathbb{T}^{n+1}, \quad c_j \in \mathbb{C},$$

rewriting their symbols as

$$\mathcal{P}(\tau, \xi) = \prod_{j=1}^{m} (\tau - c_j(\xi)), \quad (\tau, \xi) \in \mathbb{Z} \times \mathbb{Z}^n.$$
where $\sigma_j(\xi)$ satisfies $\mathcal{P}(\sigma_j(\xi), \xi) = 0$. It follows from Theorem 2.3 that $\mathcal{P}$ is globally hypoelliptic provided that

$$
\sigma_{\mathcal{P}} = \{ \{ \sigma_j(\xi) \}_{\xi \in \mathbb{Z}^n}, \; j = 1, \ldots, m \} \in (\mathcal{G}\mathcal{S})_\infty.
$$

**Example 2.** Consider $\Delta_x = -\sum_{j=1}^n \partial_x^2$ defined on $T^n$ and set

$$
\mathcal{P} = D_t^2 - 2\alpha D_t(-\Delta_x)^{1/2} + \beta^2 \Delta_x,
$$

with $\alpha, \beta \in \mathbb{R}$. In this case, $\mathcal{P}(\tau, \xi) = (\tau - |\xi| \rho_1)(\tau - |\xi| \rho_2)$, where

$$
\rho_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \beta^2}.
$$

If $\alpha^2 - \beta^2 < 0$, then $\rho_{1,2}$ is not real and $\mathcal{P}$ is globally hypoelliptic. However, the case of real roots could be much more complicated since, in general, the usual approximations by rational numbers can't be applied, once $\tau/|\xi|$ may be irrational. For instance, when $\alpha = \beta$ we have $\mathcal{P} = (D_t - \alpha(-\Delta_x)^{1/2})^2$ and

$$
\mathcal{P}(\tau, \xi) = |\xi|^2 \left( \frac{\tau}{|\xi|} - \alpha \right)^2.
$$

The reader can find a complete discussion and examples for this type of approximations in [10].

**3. Operators with variable coefficients**

In this section we study the global hypoellipticity of the operator (1), which is recalled by

$$
L = \prod_{j=1}^m (D_t + c_j(t)P_j(D_x)), \; (t, x) \in T \times T^n.
$$

Given that the main goal is to exhibit connections between the hypoellipticity of operators $L$ and

$$
L_0 = \prod_{j=1}^m (D_t + c_{0,j}P_j(D_x)),
$$

we start by observing that:
(i) when each $L_j$ is globally hypoelliptic, then $L$ and $L_0$ are globally hypoelliptic and the sets $Z_{M_j}$ are finite;
(ii) $L_0$ is globally hypoelliptic if and only if each $L_{0,j}$ is globally hypoelliptic;
(iii) let $\rho$ be a permutation of the set $\{1, \ldots, m\}$ and define
\[
L_\rho = \prod_{j=1}^{m} L_{\rho(j)} = L_{\rho(1)} \circ \ldots \circ L_{\rho(m)}. \tag{6}
\]

If some $L_k$ is not globally hypoelliptic, then for any permutation $\beta$ of $\{1, \ldots, m\} \setminus \{k\}$ we obtain $L_{\beta} \circ L_k$ not (GH). This is the case for $L$ when $L_m$ is not globally hypoelliptic.

**Proposition 3.1.** Consider $L_\rho$ an operator as in (6).

(a) If $L_\rho$ is globally hypoelliptic for any permutation $\rho$, then $L_0$ is also globally hypoelliptic. In particular, all sets $Z_{M_k}$ are finite.
(b) If some $L_\rho$ is globally hypoelliptic, then at least one of the operators $L_{0,k}$ is globally hypoelliptic.

**Proof:** Assume that every $L_\rho$ is (GH) and consider the equation $L_k u = f \in C^\infty(T^{n+1})$, for some $k$. If $\beta$ is a permutation of $\{1, \ldots, m\} \setminus \{k\}$, then the operator $L_\beta \circ L_k$ is (GH), which implies $u \in C^\infty(T^{n+1})$, since $L_\beta(C^\infty(T^{n+1})) \subset C^\infty(T^{n+1})$. It follows from Theorem 1.1 that $L_{0,k}$ is globally hypoelliptic and a) is proved.

To verify b) it is enough to observe that if $L_\rho$ is (GH), then $L_{\rho(m)}$ is necessarily globally hypoelliptic and consequently $L_{0,\rho(m)}$.

□

We point out that, by Proposition 3.1, when $L$ is globally hypoelliptic, then at least $L_{0,m}$ is (GH). The next result improves this conclusion if commutative conditions are admitted.

**Theorem 3.2.** Assume that $L$ satisfies the commutative hypothesis
\[
[L_j, L_k] = L_j L_k - L_k L_j = 0,
\]
for every $j, k \in \{1, \ldots, m\}$. Then:
(a) $L$ is (GH) if and only if every $L_k$ is (GH);
(b) if $L$ is (GH), then $L_0$ is (GH).

**Proof:** From the commutative hypothesis we have $L = L_\rho$, for any permutation $\rho$ of $\{1, \ldots, m\}$. If $L$ is globally hypoelliptic, then each $L_k$ is also globally hypoelliptic by the preceding result. In particular, every $L_{0,k}$ is (GH), and consequently so is $L_0$.

\[\square\]

3.1. Hörmander conditions

In this section we study the global hypoellipticity problem by considering operators $L_j$ satisfying the following condition: there exists positive constants $\eta_j$ such that
\[
\Im M_j(t, \xi) \geq -\eta_j, \ t \in \mathbb{T}, \ \forall \xi \in \mathbb{Z}^n.
\] (7)

Notice that there are no novelties if $\eta_j = 0$ is admitted, since in this case $\Im M_j(\cdot, \xi)$ does not change sign for every $\xi$, and we may apply Theorem 1.1. Otherwise, by assuming $\eta_j > 0$, the existence of a globally hypoelliptic operator $L_j$, with $\Im M_j(t, \cdot)$ changing sign, becomes possible, as follows from the next result.

**Theorem 3.3.** An operator $L_j$ satisfying (7) is globally hypoelliptic if and only if $L_{0,j}$ is globally hypoelliptic.

Before the proof of this theorem, we present a direct consequence.

**Theorem 3.4.** Admit that each operator $L_j$ satisfies (7). If $L_0$ is globally hypoelliptic, then $L$ is globally hypoelliptic.

**Proof:** If $L_0$ is globally hypoelliptic, then each $L_{0,j}$ is also (GH) and, by the previous theorem, each $L_j$ is globally hypoelliptic.

\[\square\]
Remark 3.5. We emphasize that the phenomena of hypoellipticity with changes of sign in the imaginary part $\Im M_j(t, \xi)$, for operators with symbols satisfying the logarithm growth

$$|p_j(\xi)| = O(\log(|\xi|)), \quad \text{when } |\xi| \to \infty,$$

is discussed in [10], as the reader can see in Sections 4 and 5 of that paper.

In the next example, we exhibit an operator satisfying (7) which cannot be captured by the approach presented in [10].

Example 3. Let $P(D_x)$ be a pseudo-differential operator on $T$ defined by the symbol $p(\xi) = \alpha(\xi) + i\beta(\xi)$, where

$$\alpha(\xi) = \begin{cases} 
\xi^{-1}, & \text{if } \xi < 0 \text{ is odd,} \\
|\xi|, & \text{if } \xi < 0 \text{ is even,}
\end{cases} \quad \text{and} \quad \beta(\xi) = \begin{cases} 
1, & \text{if } \xi < 0, \\
\xi, & \text{if } \xi \geq 0.
\end{cases}$$

Clearly, both real and imaginary parts of $p(\xi)$ do not satisfy the logarithm condition.

Now, let $a(t)$ and $b(t)$ be two real, smooth and positive functions on $T$ with disjoint supports and define $Q = D_t + (a(t) + ib(t))P(D_x)$. In this case, we have $\Im M(t, \xi) = 0$ when $t \notin \text{supp}(a) \cup \text{supp}(b)$, for any $\xi \in \mathbb{Z}$. On the other hand,

$$t \in \text{supp}(a) \Rightarrow \Im M(t, \xi) = a(t)\beta(\xi),$$

and

$$t \in \text{supp}(b) \Rightarrow \Im M(t, \xi) = b(t)\alpha(\xi).$$

Note that $\Im M(\cdot, \xi)$ changes sign if $\xi < 0$ is odd. Indeed, in this case $t \in \text{supp}(a)$ implies $\Im M(t, \xi) = a(t) > 0$, while $\Im M(t, \xi) = b(t)\xi^{-1} < 0$ when $t \in \text{supp}(b)$.

Also, condition (7) is fulfilled by choosing $\eta = \max_{t \in T} b(t)$. It follows that $Q$ is globally hypoelliptic if and only if $Q_0$ is globally hypoelliptic.

To prove Theorem 3.3 we make use of a standard result in literature:
Lemma 3.6. Let \( u \in D'(\mathbb{T}^{n+1}) \) be a distribution and its \( x \)-Fourier coefficients be

\[
\hat{u}(t, \xi) = (2\pi)^{-n} < u(t, \cdot), e^{-ix\xi} >, \quad \xi \in \mathbb{Z}^n.
\]

Given a sequence \( \{c_\xi(t)\}_{\xi \in \mathbb{Z}^n} \) of smooth functions on \( \mathbb{T} \), the formal series

\[
u(t) = \sum_{\xi \in \mathbb{Z}^n} c_\xi(t)e^{ix\cdot\xi}
\]

converges in \( D'(\mathbb{T}^{n+1}) \) if and only if for any \( \alpha \in \mathbb{Z}_+ \) there exist positive constants \( N, C \) and \( R \) such that

\[
|\partial_\alpha c_\xi(t)| \leq C|\xi|^{-N}, \quad |\xi| \geq R.
\]

(8)

Moreover, \( u \in C^\infty(\mathbb{T}^{n+1}) \) if and only if estimate (8) holds true for every \( N > 0 \). In both cases, \( c_\xi(\cdot) = \hat{u}(\cdot, \xi) \).

Proof of Theorem 3.3

The necessary part is a consequence of Theorem 1.1. Conversely, let \( u \in D'(\mathbb{T}^{n+1}) \) be a solution of equation \( L_ju = f \in C^\infty(\mathbb{T}^{n+1}) \). By taking the \( x \)-Fourier coefficients of \( u \) and \( f \) we get the O.D.E’s

\[
\partial_t\hat{u}(t, \xi) + iM_j(t, \xi)\hat{u}(t, \xi) = \hat{f}(t, \xi), \quad t \in \mathbb{T}, \quad \xi \in \mathbb{Z}^n.
\]

(9)

Since \( L_{0,j} \) is globally hypoelliptic, then \( M_{0,j}(\xi) \) satisfies the Siegel condition (5) and the set \( Z_{M_j} \) is finite. When \( \xi \notin Z_{M_j} \), the solutions (9) can be written in the form

\[
\hat{u}(t, \xi) = \frac{1}{e^{2\pi i M_{0,j}(\xi)}} - \int_0^{2\pi} \exp \left( i \int_{t}^{t+s} M_j(r, \xi) dr \right) \hat{f}(t+s, \xi)ds.
\]

(10)

Recall that by Theorem 1.1, there are positive constants \( C_1, N_1 \) and \( R_1 \) such that

\[
|1 - e^{2\pi i M_{0,j}(\xi)}|^{-1} \leq C_1|\xi|^{N_1}, \text{ for all } |\xi| \geq R_1.
\]

Now, given \( \alpha \in \mathbb{Z}_+ \) and \( \tilde{N} > 0 \) there are positive constants \( C_2, C_3 \) and \( R_2 \) such that: \( |p_j(\xi)| \leq C_2|\xi|^\nu \) (see (2)) and

\[
\sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} |\partial_\xi^{\alpha-\beta}\hat{f}(t, \xi)| \leq C_3|\xi|^{-\tilde{N}}, \quad \forall |\xi| \geq R_2.
\]
Hence, we get

\[
|\partial^\alpha t \hat{u}(t, \xi)| \leq C_1 C_2 C_3 C_4 |\xi|^{N_1 - N_2 + \nu_j} \int_0^{2\pi} \exp \left( - \int_t^{t+s} \Im M_j(r, \xi) \, dr \right) \, ds
\]

\[
\leq 2\pi C_1 C_2 C_3 C_4 |\xi|^{N_1 - \tilde{N} + \nu_j + 2\pi \nu_j},
\]

with \( C_4 \) depending on \( \alpha \) and \( c_j(t) \).

Finally, for a fixed \( \alpha \in \mathbb{Z}_+ \) it is possible to obtain for any \( N > 0 \), positive constants \( C \) and \( R \) such that

\[
|\partial^\alpha t \hat{u}(t, \xi)| \leq C |\xi|^{-N}, \quad t \in \mathbb{T}, \ |\xi| \geq R.
\]

It follows from Lemma 3.6 that \( u \in C^\infty(\mathbb{T}^{n+1}) \) and, finally, the proof is finished. \( \square \)

**Remark 3.7.** We emphasize that condition (7) may be replaced by

\[
\Im M_j(t, \xi) \leq \eta_j, \quad t \in \mathbb{T}, \ \forall \xi \in \mathbb{Z}^n,
\]

for \( j \in \{1, \ldots, m\} \), since

\[
\hat{u}(t, \xi) = \frac{1}{1 - e^{-2\pi i M_{0,j}(\xi)}} \int_0^{2\pi} \exp \left( -i \int_{t-s}^t M_j(r, \xi) \, dr \right) \hat{f}(t - s, \xi) ds
\]

is an equivalent expression for (10). In this case,

\[
\sup_{s \in [0, 2\pi]} \left| \exp \left( -i \int_{t-s}^t M_j(r, \xi) \, dr \right) \right| = \sup_{s \in [0, 2\pi]} \left| \exp \left( \int_{t-s}^t \Im M_j(r, \xi) \, dr \right) \right| \leq e^{2\pi \eta_k}.
\]

**3.2. A link with the solvability problem**

To introduce this section, let us consider the operator \( L = L_1 \circ L_2 \). It is already possible to state that \( L \) is not globally hypoelliptic when \( L_2 \) is not globally hypoelliptic. Hence, all that is needed is a search for conditions so that the same conclusion holds true when \( L_1 \) is not globally hypoelliptic.
This inquiry is equivalent to the following problem: if \( u \) is a singular solution of \( L_1 \), namely, \( u \in D'((\mathbb{T}^{n+1})^c) \) and \( L_1 u \in \mathcal{C}_c(\mathbb{T}^{n+1}) \), under which conditions is there \( u \in L_2(D'((\mathbb{T}^{n+1})))? \)

Clearly, a solution for this question is connected with solvability properties and we discuss this subject in the next few paragraphs. The first step consists in introducing the following:

**Definition 3.8.** Let \( E_j \) be the space of distributions \( v \in \mathcal{D}'(\mathbb{T}^{n+1}) \) such that
\[
\hat{v}(\cdot, \xi) \in \mathcal{C}_c(\mathbb{T}), \text{ for each } \xi \in \mathbb{Z}^n, \text{ and }
\int_0^{2\pi} \exp \left( i \int_0^t \mathcal{M}_j(r, \xi) dr \right) \hat{v}(t, \xi) dt = 0, \text{ if } \xi \in Z_{M_j}.
\]

**Theorem 3.9.** Let \( L_j \) be a globally hypoelliptic operator satisfying (7). Thus, for each \( v \in E_j \), there exist \( u \in \mathcal{D}'(\mathbb{T}^{n+1}) \) such that \( L_j u = v \).

**Proof:** By the hypothesis, \( L_{0,j} \) is globally hypoelliptic, \( \mathcal{M}_{0,j}(\xi) \) satisfies condition (5) and the set \( Z_{M_j} \) is finite. For \( \xi \not\in Z_{M_j} \) it is possible to define \( \hat{u}(t, \xi) \) by expression (10), while in the case of \( \xi \in Z_{M_j} \) we set

\[
\hat{u}(t, \xi) = \exp \left( -i \int_0^t \mathcal{M}_j(r, \xi) dr \right) \int_0^t \exp \left( i \int_0^s \mathcal{M}_j(r, \xi) dr \right) \hat{v}(s, \xi) ds.
\]

(11)

Since \( v \in E_j \), we obtain \( \hat{u}(\cdot, \xi) \in \mathcal{C}_c(\mathbb{T}) \) and

\[
\partial_t \hat{u}(t, \xi) + i \mathcal{M}_j(t, \xi) \hat{u}(t, \xi) = \hat{v}(t, \xi), \ t \in \mathbb{T}, \ \xi \in \mathbb{Z}^n.
\]

Now, we shall prove that \( \{ \hat{u}(t, \xi) \}_{\xi \in \mathbb{Z}^n} \) defines an element \( u \in \mathcal{D}'(\mathbb{T}^{n+1}) \). Firstly, since the set \( Z_{M_j} \) is finite, no estimates for (11) are needed. For the general case, an argument similar to the one in the proof of Theorem 3.3 shows that

\[
|\partial_t^\alpha \hat{u}(t, \xi)| \leq 2\pi C_1 C_2 C_3 C_4 |\xi|^{N_1 - N_2 + \nu_j + 2\pi \eta_j},
\]

where \( C_3 \) and \( R_2 \) now satisfy

\[
\sum_{\beta=0}^\alpha \binom{\alpha}{\beta} |\partial_t^{\alpha-\beta} \hat{v}(t, \xi)| \leq C_3 |\xi|^{-N_2}, \ \forall |\xi| \geq R_2,
\]

13
for some $N_2$. Then, it is possible to choose positive constants $C$, $N$ and $R$ such that
\[ |\partial_t^\alpha \hat{u}(t, \xi)| \leq C|\xi|^{-N}, \quad t \in \mathbb{T}, \quad |\xi| \geq R, \]
which implies
\[ u = \sum_{\xi \in \mathbb{Z}^n} \hat{u}(t, \xi)e^{ix \cdot \xi} \in \mathcal{D}'(\mathbb{T}^{n+1}). \]
Clearly, $L_j u = v$ and the proof is done.

**Theorem 3.10.** Admit that each $L_j$ satisfies (7). If some $L_k$ has a singular solution $u_k$ satisfying
\[ u_{k+j} = L_{k+j+1}u_{k+j+1}, \quad j \in \{0, \ldots, m-k-1\}, \]
then $L$ is not globally hypoelliptic.

**Proof:** It is enough to consider the case $m = 2$ and $k = 1$. By hypothesis, we have $L_1 u_1 = f \in C^\infty(\mathbb{T}^{n+1})$ with $u_1 \notin C^\infty(\mathbb{T}^{n+1})$. Since $u_1 \in E_2$ we obtain $u_2 \in \mathcal{D}'(\mathbb{T}^{n+1})$ such that $L_2 u_2 = u_1$. Clearly $u_2 \notin C^\infty(\mathbb{T}^{n+1})$ and $f = L_1 \circ L_2(u_2)$, hence the proposition is proved.

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