TRANSVERSELY AFFINE FOLIATIONS ON PROJECTIVE
MANIFOLDS

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Abstract. We describe the structure of singular transversely affine foliations of codimension one on projective manifolds with zero first Betti number. Our result can be rephrased as a theorem on rank two reducible flat meromorphic connections.

1. Introduction

In this paper we study holomorphic foliations on projective manifolds which are singular transversely affine in the sense of [Sca97]. These are natural generalizations of (smooth) transversely affine foliations as defined in [God91]. The classical definition is weakened at two points: the transverse structure is defined only on the complement of a divisor (but extends meromorphically through this divisor); and its developing map is not necessarily a submersion. A precise definition is given in Section 2.1.

A Theorem due to Singer [Sin92] says that the class of singular transversely affine foliations, roughly speaking, coincides with the class of foliations which admit first integrals that can be obtained by iteration of the following three operations: resolution of algebraic equations, exponentiation, and integration of closed 1-forms. More precisely, there exists a Liouvillian extension (cf. loc. cit. for a definition) of the field of rational functions of the ambient manifold containing a non constant first integral for the foliation.

Our main result describes the structure of singular transversely affine foliations on a projective manifold $X$ rather precisely, at least under the assumption $H^1(X, \mathbb{C}) = 0$.

Theorem A. Let $X$ be a projective manifold with $H^1(X, \mathbb{C}) = 0$ and $\mathcal{F}$ be a singular transversely affine foliation on $X$. Then at least one of following assertions holds true.

1. There exists a generically finite Galois morphism $p : Y \to X$ such that $p^* \mathcal{F}$ is defined by a closed rational 1-form.
2. There exists a transversely affine Ricatti foliation $\mathcal{R}$ on a surface $S$ and a rational map $p : X \dashrightarrow S$ such that $p^* \mathcal{R} = \mathcal{F}$.

Our proof does not use the hypothesis on the topology of $X$ when the transverse affine structure is regular (the connection on $N\mathcal{F}$ has at most logarithmic singularities); or when the monodromy of the transverse affine structure is Zariski dense.

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We do not know if the hypothesis on the topology is necessary in general. The result as stated above probably holds also on compact Kähler manifolds, but at some key points we used results that are only available in the algebraic category. We do know that the result does not hold for compact complex manifolds in general; foliations on Inoue surfaces are perhaps the easiest counterexamples, see [BPT06, Remark 2.1]. Also, the result does not hold for transversely affine germs of foliations, see [Tou03, sec. IV].

There were previous attempts to arrive at a structure theorem for singular transversely affine foliations on projective spaces, see [Sca97, CeSa98, CaSc01] to have a sample of such attempts. All these works approach the problem through the study of the foliation on a neighborhood of the singular divisor of the transverse affine structure based on an analysis of the (generalized) holonomy of this divisor, see also [Pau99]. They use extension results to globalize the semi-local conclusions. The nature of this method leads one to impose restrictions on the type of singularities of the foliation. In contrast, our approach is based on the study of the monodromy representation of the singular transversely affine foliation, and relies on recent results [BCM13, BW12] on the structure of representations of the fundamental groups of quasi-projective manifolds in the affine group Aff(ℂ). We also make use of some classical results on the periods of families of closed rational 1-forms [Del70] combined with basic properties of Picard-Fuchs equations; as well as results on the local/semi-local structure of singular transversely affine foliations.

As a rather concrete application, we provide a classification of Liouvillian integrable 1-forms on ℂⁿ which do not admit invariant algebraic hypersurfaces.

**Corollary B.** Let ω be a polynomial differential 1-form on ℂⁿ. If ω is Liouvillian integrable and has no invariant algebraic hypersurface then there exists a polynomial map P : ℂⁿ → ℂ² and polynomials a, b ∈ ℂ[𝑥] such that

\[ \omega = P^* (dy + (a(x) + b(x)y)dx) \]

The existence of such Liouvillian integrable 1-forms have been recently recognized by [GL12] as a new phenomenon, but as stated above they are nothing but disguised classical Riccati equations.

Our Theorem A can be rephrased as a structure theorem for reducible flat meromorphic sl(2)-connections over projective manifolds. By a sl(2)-connection we mean a connection with zero trace on a rank two vector bundle with trivial determinant.

**Theorem C.** Let X be a projective manifold with h¹(X, ℂ) = 0. Let ∇ be a reducible flat meromorphic sl(2)-connection on a vector bundle V over X. There exists a generically finite Galois morphism p : Y → X such that at least one of the following assertions holds true.

1. The connection matrix of p^* ∇ in a suitable basis of rational sections of p^* V is

\[
\begin{bmatrix}
0 & \omega \\
0 & 0
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
\eta/2 & 0 \\
0 & -\eta/2
\end{bmatrix}
\]

In particular the monodromy of ∇ is virtually abelian.

2. There exists a curve C, a meromorphic flat connection ∇₀ on a rank 2 bundle over C and a rational map π : Y → C such that p^* ∇ is birationally gauge equivalent to π^* ∇₀. Moreover, in this case the degree of p is at most two.
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Contents

1. Introduction \hspace{1cm} 1
2. Transversely affine foliations \hspace{1cm} 3
3. Cohomology jumping loci for local systems \hspace{1cm} 11
4. Factorization of representations \hspace{1cm} 13
5. Proof of Theorem A \hspace{1cm} 14
6. Proof of Corollary B \hspace{1cm} 17
7. Proof of Theorem C \hspace{1cm} 17
References \hspace{1cm} 18

2. Transversely affine foliations

2.1. Definition. Let \(\mathcal{F}\) be a codimension one holomorphic foliation on a complex manifold \(X\) with normal bundle \(N\mathcal{F}\), i.e. \(\mathcal{F}\) is defined by a holomorphic section \(\omega\) of \(N\mathcal{F} \otimes \Omega^1_X\) with zero locus of codimension \(\geq 2\) and satisfying \(\omega \wedge d\omega = 0\). A singular transverse affine structure for \(\mathcal{F}\) is a meromorphic flat connection \(\nabla : N\mathcal{F} \rightarrow N\mathcal{F} \otimes \Omega^1_X(\ast D)\), satisfying \(\nabla(\omega) = 0\);

where \(D\) is a reduced divisor on \(X\) and \(\Omega^1_X(\ast D)\) is the sheaf of meromorphic 1-forms on \(X\) with poles (of arbitrary order) along \(D\). We will always take \(D\) minimal, in the sense that the connection form of \(\nabla\) is not holomorphic in any point of \(D\). The divisor \(D\) is the singular divisor of the transverse affine structure.

A foliation \(\mathcal{F}\) is a singular transversely affine foliation if it admits a singular transverse affine structure. Aiming at simplicity, from now on we will omit the adjective singular when talking about singular transverse affine structures and singular transversely affine foliations.

As will be seen in Example 2.4, the same holomorphic foliation can admit more than one transverse affine structure. When we want to keep track of the transverse affine structure, we write \((\mathcal{F}, \nabla)\) instead of \(\mathcal{F}\).

2.2. Interpretation in terms of rational 1-forms. When \(X\) is an algebraic manifold, the transverse affine structure can be defined by rational 1-forms. If \(\omega_0\) is a rational 1-form defining \(\mathcal{F}\) then the existence of a meromorphic flat connection on \(N\mathcal{F}\) satisfying \(\nabla(\omega) = 0\) is equivalent to the existence of a rational 1-form \(\eta_0\) such that

\[ d\omega_0 = \omega_0 \wedge \eta_0 \quad \text{and} \quad d\eta_0 = 0. \]

Indeed, if \(U\) is an arbitrary open subset of a complex manifold \(X\) where \(N\mathcal{F}\) is trivial then a meromorphic connection on a trivialization of \(N\mathcal{F}\) in \(U\) can be expressed as

\[ \nabla|_U(f) = df + f \otimes \eta_0, \]
where $\eta_0$ is a closed meromorphic 1-form which belongs to $H^0(U, \Omega^1_X(\ast D))$. If $\omega_0$ represents $\omega$ in this very same trivialization then $\nabla_U(\omega_0) = d\omega_0 + \eta_0 \wedge \omega_0$ and $\nabla(\omega) = 0$ is equivalent to $d\omega_0 = \omega_0 \wedge \eta_0$. If $X$ is algebraic we can trivialize $\mathcal{N}F$ in the Zariski topology and get the sought pair of rational 1-forms.

Most of time we will work with $U$ an affine open subset of a projective manifold $X$. At some points we will need to work with open subsets in the analytic topology, as we are going to make use of results on the normal forms of singularities of foliations.

Notice that a change of trivialization does change $\eta_0$, and also changes $\omega_0$. If the pair $(\omega_0, \eta_0)$ represents $(\omega, \nabla)$ in a given trivialization over $U$ then in another trivialization over $\overline{U}$ the representatives will be of the form $(g\omega_0, \eta_0 - d \log g)$ for a suitable $g \in \mathcal{O}^*(U)$.

The equality $d\omega_0 = \omega_0 \wedge \eta_0$ implies that the (multi-valued) 1-form $\exp(\int \eta_0) \omega_0$ is closed. Its primitives are first integrals for the foliation $\mathcal{F}$. These first integrals belong to Liouvillian extension of the field of rational functions on $X$, and conversely the existence of a non-constant Liouvillian first integral for $\mathcal{F}$ implies that $\mathcal{F}$ is transversely affine, see [Sin92].

Even if $\omega_0$ and $\eta_0$ may have poles in the complement of $D$, the multi-valued function $\int \exp(\int \eta_0) \omega_0$ coincides with the developing map of $\mathcal{F}_{|x-(D, \text{sing, } \mathcal{F})}$ and extends holomorphically to the whole of $X - D$. For any given base point $q \in X - D$, its monodromy is an anti-representation $\varrho$ of the fundamental group of the complement of $D$ in $X$ to the affine group $\text{Aff}(\mathbb{C}) = \mathbb{C}^* \ltimes \mathbb{C}$. The linear part of $\varrho$ will be denoted by $\rho$. It coincides with the monodromy of $\nabla$.

$$
\pi_1(X - D) \xrightarrow{\varrho} \text{Aff}(\mathbb{C}) \xrightarrow{\rho} \mathbb{C}^*
$$

Here and throughout the paper we will deliberately omit the base point of the fundamental groups. Hopefully no confusion will arise.

2.3. Singular divisor and residues. Recall from the previous section that the singular divisor of a transverse affine structure $\nabla$ is nothing but the reduced divisor of poles of $\nabla$.

**Proposition 2.1.** The irreducible components of the singular divisor $D$ of a transverse affine structure $\nabla$ for a foliation $\mathcal{F}$ are invariant by $\mathcal{F}$.

**Proof.** Let $f$ be a local equation for an irreducible component $C$ of $D$ and $(\omega_0, \eta_0)$ be a local pair describing $(\mathcal{F}, \nabla)$ in a small enough trivializing open set for $\mathcal{N}\mathcal{F}$. The equation $d\eta_0 = 0$ imposes $\eta_0 = \alpha + h(f) \frac{df}{f}$ with $\alpha$ holomorphic and $h$ a unit of $\mathbb{C}\{f\}$. The equation $d\omega_0 = \omega_0 \wedge \eta_0$ implies that $f$ divides $\omega_0 \wedge df$, i.e. $C = \{f = 0\}$ is $\mathcal{F}$-invariant. \( \square \)

To each irreducible component $C$ of $D$ we can attach a complex number $\text{Res}_C(\nabla)$, defined as the residue of any local meromorphic 1-form $\eta_0$ defining $\nabla$ at a general point of $C$. The flatness of $\nabla$ implies that this residue is a complex number.

**Proposition 2.2.** Let $X$ be a projective manifold. If $\nabla$ is any flat meromorphic connection on a line-bundle $\mathcal{L}$ then the class of $\sum \text{Res}_C(\nabla)|C|$ in $H^2(X, \mathbb{C})$, with
the summation ranging over the irreducible components of the singular divisor $D$, represents the Chern class of $\mathcal{L}$. Reciprocally, given a $\mathbb{C}$-divisor $R = \sum \lambda_C C$ with the same class in $H^2(X, \mathbb{C})$ as a line bundle $\mathcal{L}$ then there exists a flat meromorphic connection $\nabla_{\mathcal{L}}$ on $\mathcal{L}$ with logarithmic poles and $\text{Res}(\nabla_{\mathcal{L}}) = R$.

**Proof.** When $X$ is a curve it is well-known that the Chern class of a line-bundle $\mathcal{L}$ with a meromorphic connection $\nabla$ can be recovered from $\sum \text{Res}_{C}(\nabla[C])$ in $H^2(X, \mathbb{C})$. The general case can be proved by restriction of $\nabla$ to general curves in $X$.

To realize a $\mathbb{C}$-divisor $R = \sum \lambda_C C$ as the residue divisor of a logarithmic connection with singular divisor $D = \sum C$, look at the exact sequence

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \oplus \mathcal{O}_C \to 0$$

with the first arrow given by the inclusion and the second arrow given by the residue map, cf. [Bru00, Chapter 6]. Notice that the boundary map $\oplus H^0(C, \mathcal{O}_C) \to H^1(X, \Omega^1_X)$ sends a choice of residues to a cocycle of logarithmic 1-forms $\eta_{ij} = \eta_i - \eta_j$ where $\eta_i \in \Omega^1_X(\log D)(U_i)$ is a logarithmic 1-form on an open set $U_i$ with the prescribed residues. If the cocycle $\eta_{ij}$ is cohomologous to a cocycle $d \log g_{ij}$ representing the Chern class of a line-bundle $\mathcal{L}$ in $H^1(X, \Omega^1_{X, \text{closed}})$ then

$$\eta_i - \eta_j + d \log g_{ij} = \beta_i - \beta_j$$

for suitable closed holomorphic 1-forms $\beta_i \in \Omega^1_X(U_i)$. It follows that the 1-forms $\eta_i - \beta_i$ define a flat logarithmic connection on $\mathcal{L}$. \hfill \Box

Actually, the Proposition above holds for arbitrary compact complex manifolds. We use the projectivity of $X$ only to reduce the first part of the proof to the case of curves, but it is not strictly necessary.

Two flat meromorphic connections $\nabla_1$ and $\nabla_2$ on the same line-bundle $\mathcal{L}$ differ by a closed rational 1-form, i.e. $\nabla_1 - \nabla_2 = \beta$ for $\beta$ a rational 1-form satisfying $d\beta = 0$. If the residues of $\nabla_1$ and $\nabla_2$ coincides then $\beta$ has no residues; in particular, when $h^1(X, \mathbb{C}) = 0$, the rational 1-form $\beta$ is the differential of a rational function.

2.4. **Examples and first properties.** We collect below the standard examples of transversely affine foliations and some basic properties concerning the (non) uniqueness of transverse affine structure for a given foliation.

**Example 2.3** (Foliations with rational first integral). If $F : X \to C$ is a dominant rational map to a curve. $\omega_0 = dF$ is a rational form which defines a transversely affine foliation. It has many different transverse affine structures, see example 2.3 below.

**Example 2.4** (Foliations defined by closed 1-forms). If $\mathcal{F}$ is a foliation on a projective manifold defined by a closed rational 1-form $\omega_0$, then $\mathcal{F}$ admits a family of pairwise distinct transverse affine structures parametrized by $\alpha \in \mathbb{C}$. Indeed, for any constant $\alpha \in \mathbb{C}$ we have that $\eta_0 = \alpha \omega_0$ is closed and satisfies $d\omega_0 = \omega_0 \wedge \eta_0$. If $\alpha \neq 0$, since the monodromy is obtained by the analytic continuation of $F = \int \exp(\int \omega_0) = \int \exp(\alpha G)dG = \exp(\alpha G)/\alpha$, where $G = \int \omega_0$, it must be of the form $\gamma \mapsto (\exp \int G) \alpha \in \mathbb{C}^*$. If $\alpha = 0$, $\int \exp(\int \omega_0) = \int \omega_0$, and the monodromy is $\gamma \mapsto \int G \omega_0 \in (\mathbb{C}, +)$. 


Proposition 2.5. Let $\mathcal{F}$ be a foliation on a projective manifold $X$. Suppose $\mathcal{F}$ admits two distinct transverse affine structures. Then $\mathcal{F}$ is defined by a closed rational 1-form. Moreover, if $\mathcal{F}$ does not admit a non-constant rational first integral then every transverse affine structure for $\mathcal{F}$ belongs to the one-parameter affine family presented in Example 2.4.

Proof. If $\mathcal{F}$ admits two distinct transverse affine structures, then there exists a rational 1-form $\omega_0$ defining $\mathcal{F}$ and two distinct closed rational 1-forms $\eta_1$ and $\eta_2$ such that
\[
d\omega_0 = \omega_0 \wedge \eta_i, \quad i = 1, 2.
\]
Therefore $\omega_0 \wedge (\eta_1 - \eta_2) = 0$ and consequently $\eta_1 - \eta_2$ is a closed rational 1-form defining $\mathcal{F}$.

If $\mathcal{F}$ is defined by a closed rational 1-form $\omega_0$ and $(\omega_0, \eta_0)$ represents $(\mathcal{F}, \nabla)$, then $\eta_0$ must satisfy $\omega_0 \wedge \eta_0 = 0$. Therefore $\omega_0 = h\eta_0$ for a suitable rational function. Differentiation shows that $h$ must be constant along the leaves of $\mathcal{F}$, i.e. $h$ is a rational first integral for $\mathcal{F}$. If $\mathcal{F}$ does not admit a non-constant rational first integral then we are in the situation described in Example 2.4.

If $\mathcal{F}$ is defined by a closed rational 1-form $\omega_0$, the case $\alpha = 0$ in Example 2.4 says there exists an transversely affine structure for $\mathcal{F}$ which has at worst logarithmic poles at the zeros and poles of $\omega_0$ and has additive monodromy group. The converse of this statement also holds true.

Proposition 2.6. Let $(\mathcal{F}, \nabla)$ be a transversely affine foliation on a projective manifold $X$. If $\nabla$ has at worst logarithmic singularities and the monodromy group of $(\mathcal{F}, \nabla)$ is contained in $(\mathbb{C}, +)$, then $\mathcal{F}$ is defined by a closed rational 1-form.

Proof. We have a locally well-defined closed meromorphic 1-form $\exp(\int \eta_0)\omega_0$ defining $\mathcal{F}$ in $X - D$, the monodromy hypothesis says it is well-defined in $X - D$. We only have to check $\exp(\int \eta_0)\omega_0$ extends meromorphically through $D$. In a neighborhood $U$ of any smooth point of $D$, take a local pair $(\omega, \eta)$ representing $(\mathcal{F}, \nabla)$. Notice that $\eta = \alpha + \lambda \frac{df}{f}$ for a local equation $f$ of $D$, $\alpha$ closed holomorphic 1-form in $U$ and $\lambda \in \mathbb{Z}$. There exists a meromorphic function $g$ on $U$ such that $(\omega_0, \eta_0) = (gf, \eta - \frac{dg}{g})$. Thus $e^{\lambda \eta_0}\omega_0 = g^{-1}e^{\int \eta_0}g\omega = e^{\lambda}f^{\alpha}\omega$ is meromorphic in $U$. Since $U$ is arbitrary, it follows that $e^{\lambda \eta_0}\omega_0$ is a well-defined meromorphic 1-form on $X$.

The hypothesis on the nature of the singularities of $\nabla$ is important in the proposition above. There exist transversely affine foliations $\mathcal{F}$ on projective manifolds which have trivial monodromy group but are not given by a closed rational 1-form.

Example 2.7. A simple example on $\mathbb{P}^1 \times \mathbb{P}^1$ is given in affine coordinates by
\[
\omega_0 = x^3dy + 1/2(x + y)dx \quad \text{and} \quad \eta_0 = \frac{dx}{x} + \frac{dx}{x^3}.\]

The only invariant curves are $\{x = 0\}$ and the $\{y = \infty\}$. Proposition 2.4 implies that this foliation cannot be given by a closed rational 1-form. It is birationally equivalent to the one appearing in GL12 Theorem 3.

If $\mathcal{F}$ is a foliation defined by a closed rational 1-form on a projective manifold $X$ and $\mathcal{F}$ is invariant by a finite group $G \subset \text{Aut}(X)$, then the quotient of $\mathcal{F}$ by $G$, seen in any resolution of $X/G$, is also a transversely affine foliation. Indeed, transversely affine structures behave rather well under rational maps between foliations.
Proposition 2.8. Let $X$ and $Y$ be projective manifolds, $f : X \to Y$ a dominant rational map, and $\mathcal{F}$ a foliation on $Y$. The foliation $f^*\mathcal{F}$ has a transverse affine structure if and only if so does $\mathcal{F}$. If this occurs, the pull-back $(f^*\mathcal{F}, \nabla)$ of any structure $(\mathcal{F}, \nabla)$ has monodromy group a finite index subgroup of the monodromy group of $(\mathcal{F}, \nabla)$.

Proof. The result, phrased in terms of extensions of differential fields, is already implicit in [Sin92]. Except for the finiteness of the index, a geometric proof can be found in [CLL07b, Theorem 2.21].

We can write $f = \pi \circ h$ with $\pi : Z \to Y$ a generically finite rational map from a projective manifold $Z$ to $Y$; and $h : X \to Z$ a rational map with irreducible generic fiber. The monodromy of $h^*(\mathcal{F}, \nabla)$ factors through $h$, and $h$ induces a surjective map of fundamental groups, after restriction to any nonempty Zariski open set. On the other hand, if we restrict $\pi$ over a sufficiently small nonempty Zariski open set, it induces a monomorphism with finite index image between fundamental groups. \qed

Example 2.9 (Quotients). Let $\mathcal{F}$ be a foliation on a projective manifold $X$ defined by a closed rational 1-form $\omega_0$. If $\varphi \in \text{Aut}(X)$ is an automorphism of finite order then $\varphi^*(\omega_0) = \xi\omega_0$ for some root of unity $\xi$, and the quotient of $\mathcal{F}$ by $\varphi$ is a transversely affine foliation with monodromy group equal to an extension of the subgroup of $\mathbb{C}^*$ generated by $\xi$ by a subgroup of $(\mathbb{C}, +)$, determined by the integrals of $\omega_0$ along paths joining points in the same orbit of $\varphi$.

Example 2.10 (Riccati foliations). Let $X$ be a projective manifold and $\pi : X \to Y$ a fibration with generic fiber isomorphic to $\mathbb{P}^1$. If $\mathcal{F}$ is a foliation which has no tangencies with the general fiber of $\pi$ then we say that $\mathcal{F}$ is a Riccati foliation. An arbitrary Riccati foliation does not admit a transverse affine structure. Indeed, it follows from a classical result of Liouville that a Riccati foliation admits a transverse affine structure if and only if there exists a hypersurface $H \subset X$, invariant by $\mathcal{F}$ and which dominates $Y$, i.e. with $\pi(Z) = Y$. For example, if $Z$ intersects the general fiber at only one point then there exists a birational transformation $\varphi : Y \times \mathbb{P}^1 \to X$ such that the strict transform of $Z$ is the section at infinity of $Y \times \mathbb{P}^1 \to Y$. On $Y \times \mathbb{P}^1$ the foliation $\varphi^*\mathcal{F}$ is defined by a rational 1-form $\omega_0 = dy + \alpha + y\beta$, with $\alpha, \beta$ rational 1-forms on $Y$. Since $\omega_0$ is integrable it follows that $\alpha \beta$ is closed. If we take $\eta_0 = \beta$ then $d\omega_0 = \omega_0 \wedge \eta_0$ which shows that $\varphi^*\mathcal{F}$ is a transversely affine foliation.

It follows from the Riemann-Hilbert correspondence that there are no restrictions on the monodromy group of these Riccati foliations, see [LP07]. In particular any finitely generated subgroup of $\text{Aff}(\mathbb{C})$ appears as the monodromy group of a transversely affine Riccati foliation over $Y = \mathbb{P}^1$.

Notice that there exist transversely affine Riccati foliations with trivial monodromy but not given by a closed rational 1-form, e.g. Example 2.7. Similarly, there are Riccati foliations with trivial monodromy which are not transversely affine foliations.

2.5. Holonomy. For a transversely affine foliation $(\mathcal{F}, \nabla)$ with singular divisor $D$, we define the singular leaves of $\mathcal{F}$ as the leaves contained in $D$. Every other leaf of $\mathcal{F}$ will be called a non singular leaf.

Proposition 2.11. The holonomy of any non singular leaf $L$ of $\mathcal{F}$ is linearizable.
Proof. Let $U = X - (D \cup \text{sing } F)$. The holonomy of any leaf of $F|_U$ is in $\text{Aff}(\mathbb{C})$ and fixes a point in $\mathbb{C}$. Any element of $\pi_1(L)$ is represented by a loop in $L \cap U$. □

The determination of the holonomy of the singular leaves of $F$ is more involved.

Proposition 2.12. Let $L$ be a smooth and irreducible component of $D$.

(1) If the singularity of $\nabla$ along $L$ is logarithmic then
   (a) the holonomy of $L$ is linearizable when $1 - \text{Res}_L(\nabla) \notin \mathbb{N}$; and
   (b) the holonomy of $L$ is conjugated to a subgroup of

   \[ \mathcal{H}_k = \left\{ h \in \text{Diff}(\mathbb{C}, 0); h(z) = \frac{\lambda z}{(1 + \mu z^k)^{1/k}} \text{ with } \lambda \in \mathbb{C^*} \text{ and } \mu \in \mathbb{C} \right\} \]

   when $1 - \text{Res}_L(\nabla) = k \in \mathbb{N}$.

(2) If the singularity of $\nabla$ along $L$ is not logarithmic then the holonomy group of $L$ has a finite index subgroup tangent to the identity, and is thus virtually abelian.

Proof. The case of logarithmic singularities is treated in [CeSa98] pages 3076-3077, see also [Sca97] Lemma 3.1. An adaptation of the argument in the former work allows us to treat the case of non logarithmic singularities.

Let $q \in L$ be a point where the foliation is smooth. In suitable local analytic coordinates at a neighborhood of $q$ the foliation $F$ is defined by a 1-form $\omega = dy$ and the connection form is $\eta = \lambda \frac{dz}{y} + d(1/a(y)) + a(0) = 0$. Let $\Sigma$ be a transversal to $F$ at $q$ with coordinate $y$. The restriction of the (multi-valued) first integral $\int \exp(\int \eta) \omega$ to $\Sigma$ is (one of the determinations of) $f(y) = \int_q^y s^\lambda \exp(1/a(s))ds$. If $h : (\Sigma, q) \rightarrow (\Sigma, q)$ is a holonomy map then

\[ f(y) = \alpha f(h(y)) + \beta \]

for suitable $\alpha \in \mathbb{C^*}$ and $\beta \in \mathbb{C}$. After differentiating we can express $\alpha$ as

\[ \alpha = \frac{f'(y)}{f'(h(y))h'(y)} = \frac{1}{h'(y)} \left( \frac{y}{h(y)} \right)^{\lambda} \exp \left( \frac{1}{a(y)} - \frac{1}{a(h(y))} \right) \cdot \exp(\int \eta) \omega \]

Since $\alpha$ is constant, the parameter in the exponential must be holomorphic as all the other factors in the product have moderate growth. This implies $h'(0)^k = 1$, where $k$ is the vanishing order of $a(y)$ at $y = 0$.

We know that the holonomy group $H$ of $L$ is a solvable subgroup of $\text{Diff}(\mathbb{C}, 0)$, see for instance [Pau99] or [BLL01]. From the paragraph above we know that the linear part of the members of $H$ are all roots of unity of order $\leq k$, they form a finite subgroup of $\mathbb{C^*}$. Therefore $H$ admits a finite index subgroup consisting of germs of diffeomorphisms tangent to the identity. But solvable subgroups of $\text{Diff}_1(\mathbb{C}, 0)$ are abelian, [Lor99] pages 3-4], and we conclude that the holonomy of $L$ is virtually abelian. □

2.6. Transversely affine foliations as transversely projective foliations. A codimension one foliation $F$ on a projective manifold $X$ is a singular transversely projective foliation if there exists

(1) $\pi : P \rightarrow X$ a $\mathbb{P}^1$-bundle over $X$;
(2) $\mathcal{H}$ a codimension one singular holomorphic foliation of $P$ transverse to the generic fiber of $\pi$;
(3) $\sigma : X \rightarrow P$ a rational section generically transverse to $\mathcal{H}$;
such that \( \mathcal{F} = \sigma^*\mathcal{H} \). The triple \( \mathcal{P} = (P, \mathcal{H}, \sigma) \) is, by definition, a transverse projective structure for \( \mathcal{F} \). This definition of singular transversely projective foliation is essentially equivalent to the one given in \([\text{Sca}97]\), for a comparison between the two definitions and thorough discussion see \([\text{LP}07]\). As in the case of singular transverse affine structures/foliations we will deliberately omit the adjective singular, and refer to this class of foliations from now on as transversely projective foliations.

Any two such triples \( \mathcal{P} = (P, \mathcal{H}, \sigma) \) and \( \mathcal{P}' = (P', \mathcal{H}', \sigma') \) are said birationally equivalent when they are conjugate by a birational bundle transformation \( \phi : P \to P' \) satisfying \( \phi^*\mathcal{H}' = \mathcal{H} \), and \( \phi \circ \sigma = \sigma' \).

The polar divisor of the transverse structure, denoted by \((\mathcal{P})_\infty\), is the divisor on \( X \) defined by the direct image under \( \pi \) of the tangency divisor between \( \mathcal{H} \) and the one-dimensional foliation induced by the fibers of \( \pi \).

The monodromy representation of a projective structure \( \mathcal{P} = (P \to X, \mathcal{H}, \sigma) \) is the (anti-)representation of \( \pi_1(X \setminus (\mathcal{P})_\infty) \) into \( \text{PSL}(2, \mathbb{C}) \) obtained by lifting paths on \( X \setminus (\mathcal{P})_\infty \) to the leaves of \( \mathcal{H} \). Notice that the monodromy representation does not depend on \( \sigma \).

Over a sufficiently small open subset \( U \) of \( X \), the foliation \( \mathcal{H} \) is the projectivization of a foliation on a rank two vector bundle over \( U \) given by the flat sections of a meromorphic flat \( \mathfrak{sl}(2) \)-connection. We say that a transverse projective structure \( (P, \mathcal{H}, \sigma) \) has regular singularities when the corresponding flat meromorphic connection is regular in the sense of \([\text{Del}70], \text{Chapter II}]\).

**Lemma 2.13.** If we have two transverse projective structures \( (P, \mathcal{H}, \sigma) \) and \( (P', \mathcal{H}', \sigma') \) on \( X \), both having regular singularities then they have the same monodromy if and only if there exists a birational bundle map \( \phi : P \to P' \) such that \( \phi^*\mathcal{H}' = \mathcal{H} \).

**Proof.** To compare the monodromies, we have to consider structures with the same polar locus \( D \); to be in that situation we may take \( D \) the union of both original polar loci. Once this is done, suppose these monodromies are the same (or rather conjugated).

Over \( X \setminus D \), there exists a \( \mathbb{P}^1 \)-bundle isomorphism \( \psi \) such that \( \psi^*\mathcal{H}'|_{X \setminus D} = \mathcal{H}|_{X \setminus D} \), since both foliations are defined over \( X \setminus D \) by the suspension of the corresponding representations. We want to show \( \psi \) extends to \( \phi \), a bimeromorphic \( \mathbb{P}^1 \)-bundle map, defined on the whole of \( X \). We first show that \( \psi \) extends meromorphically in the neighborhood of any smooth point \( q \) of \( D \). Let \( U \) be a sufficiently small neighborhood of \( q \) and \( f \) a local equation in \( U \) for the irreducible component \( C \) of \( D \) through \( q \). We take \( \mathfrak{sl}(2) \)-connections \( (E, \nabla), (E', \nabla') \) which are local lifts for \( (P, \mathcal{H}), (P', \mathcal{H}') \) as in \([\text{LP}07], \text{Section 2.1}]\).

By regularity of the connections, and up to bimeromorphic transformations of \( E|_U, E'|_U \) which are biholomorphic outside of \( E|_C \), we can suppose both vector bundles are trivial and that the connections have at worst logarithmic poles. Moreover, by coincidence of monodromy, we can suppose the spectra \( \{\theta/2, -\theta/2\}, \{\theta'/2, -\theta'/2\} \) of the residues at \( C \) are the same. Thus, we can suppose both connections have the same connection matrix, in normal form.

This implies \( \psi|_{U \setminus D} \) is a symmetry \( \tilde{z} = A(f) \cdot z \) of \( \mathcal{H}_0 : dz = \theta z \frac{df}{f} \) or \( \mathcal{H}_0 : dz = (n z + f^n) \frac{df}{f} \), \( n = \theta \in \mathbb{N} \); where \( A : (\mathbb{C}, 0)^* \to \text{GL}_2(\mathbb{C}) \) is holomorphic. In any case, plugging the coordinate functions of \( A \) in the equations for the symmetry of \( \mathcal{H}_0 \)
shows that $A$ is meromorphic at $f = 0$. Since $q \in D$ is an arbitrary smooth point, it follows that we can meromorphically extend $\psi$ to the complement of a codimension two subset of $P$. Levi extension theorem allow us to meromorphically extend $\psi$ to the whole $X$, and obtain the sought birational bundle map $\phi$. \qed

Every transversely affine foliation $(\mathcal{F}, \nabla)$ on a projective manifold $X$ carries a natural transverse projective structure $\mathcal{P}_\mathcal{F}$. It is given by $(\mathcal{P}, \mathcal{H}, \sigma)$ as follows.

- $\mathcal{P} = \mathcal{P}(E)$ where $E = \mathcal{N}\mathcal{F} \oplus \mathcal{O}_X$;
- $\sigma : X \to \mathcal{P}$ the section corresponding to the inclusion $\mathcal{O}_X \to \mathcal{N}\mathcal{F} \oplus \mathcal{O}_X$;
- $\mathcal{H}$ the foliation on $P$ defined by projectivization of a flat connection $\mathcal{D}$ on $E$; its leaves are the projections of horizontal sections of $\mathcal{D}$. We now describe $\mathcal{D}$. We have a flat connection $\hat{\nabla}$ on $E$ given by $\hat{\nabla} = \nabla + d$, we also have a map $i : \mathcal{N}\mathcal{F} \otimes \Omega^1_X \to \text{End}(E) \otimes \Omega^1_X$ induced by the composition of natural maps
  
  \[ \mathcal{N}\mathcal{F} \simeq \text{Hom}(\mathcal{O}, \mathcal{N}\mathcal{F}) \hookrightarrow \text{End}(E); \]
- let $\omega \in H^0(X, \mathcal{N}\mathcal{F} \otimes \Omega^1_X)$ be a section defining $\mathcal{F}$, we define $\mathcal{D}$ to be the translated of $\hat{\nabla}$ by $i(\omega)$: $\mathcal{D} = \hat{\nabla} + i(\omega)$. It is easily checked that $\mathcal{D}$ is flat.

If we perform the construction of $\mathcal{D}$ starting with another section $\lambda \omega$, $\lambda \in C^*$, then we obtain $\mathcal{D}'$, the transform of $\mathcal{D}$ by the automorphism $\lambda \oplus 1$ of $E$. So that the isomorphism class of $\mathcal{D}$ is canonically defined by the transverse affine structure of $\mathcal{F}$.

If we use a trivialization coordinate $z : \mathcal{N}\mathcal{F}|_U \to \mathbb{C}$ on $\mathcal{N}\mathcal{F}|_U$ and $z \oplus \text{id}$ on $E|_U$, we see that $\mathcal{D} = d + \Omega$ has connection matrix $\Omega = \begin{bmatrix} \eta & \hat{\omega} \\ 0 & 0 \end{bmatrix}$, where $\hat{\omega}$ and $\eta$ represent respectively $\omega$ and the connection matrix of $\nabla$ in the trivializations.

In such a trivialization, $\mathcal{H}$ coincides with the foliation defined by the meromorphic $1$-form

\[ dz + \hat{\omega} + z\eta \]

on the open subset $U \times \mathbb{P}(\mathcal{N}\mathcal{F}|_U \oplus 1)$. Over $U$, the section $\sigma$ is $z = 0$.

**Proposition 2.14.** Let $\mathcal{F}$ be a transversely projective foliation on a projective manifold $X$ with transverse projective structure $\mathcal{P} = (P, \mathcal{H}, \sigma)$. Suppose there exists a fibration $f : X \to C$ with connected fibers such that the polar divisor $(\mathcal{P})_\infty$ of $\mathcal{P}$ intersects the general fiber of $f$ at most on logarithmic poles and that the monodromy representation $\rho$ of $\mathcal{P}$ factors through $f$, i.e. there exists a divisor $F$ supported on finitely many fibers of $f$ and a representation $\rho_0$ from the fundamental group of $C_0 = f(X - ((\mathcal{P})_\infty + F))$ to $\text{PSL}(2, \mathbb{C})$ fitting in the diagram below.

\[ \begin{array}{ccc}
\pi_1(X - ((\mathcal{P})_\infty + F)) & \xrightarrow{p} & \text{PSL}(2, \mathbb{C}) \\
\downarrow f & & \\
\pi_1(C_0) & \xleftarrow{\rho_0} & \\
\end{array} \]

Then there exists a $\mathbb{P}^1$-bundle $S$ over $C$; a Riccati foliation $\mathcal{R}$ on $S$; and a rational map $p : X \dashrightarrow S$ such that $p^*\mathcal{R} = \mathcal{F}$.

**Proof.** Let $\pi$ denote the projection of $P$, and $\mathcal{G}$ be the codimension 2 foliation of $P$ obtained as the intersection of $\mathcal{H}$ with the foliation determined by $f$. The leaves of $\mathcal{G}$ are the leaves of the restrictions $\mathcal{H}_{(f^{-1}(y))}$ over the fibers of $f$. For generic $y \in C$, $\mathcal{H}_{(f^{-1}(y))}$ has logarithmic poles and trivial monodromy, hence it is
birationally equivalent to the trivial horizontal foliation on the trivial \( P^1 \)-bundle over \( f^{-1}(y) \). The main result of [CG89] (see also [Per01] Section 8) implies that \( G \) is defined by the levels of a rational dominant map with connected fibers \( F: P \to S \), with \( S \) a smooth projective surface. By construction, \( f \circ \pi \) factors through \( F: f \circ \pi = q \circ F \), for some rational map \( q: S \to C \). Up to birational transformation of \( S \), we can suppose \( q \) is holomorphic. It follows from [CLLPT07] Lemma 3.1 that \( \mathcal{H} \) projects to a foliation \( R \) on \( S \) such that \( F^* R = \mathcal{H} \).

For general \( x \in X \), the restriction of \( F \) to \( \pi^{-1}(x) \simeq P^1 \) is not constant, and it separates the points of \( \pi^{-1}(x) \) as they correspond to different leaves of \( G \). Therefore \( F \) takes a general fiber of \( \pi \) biholomorphically into a fiber of \( q \). We conclude that \( R \) is a Riccati foliation on \( S \), with adapted fibration \( q \). Defining \( p := F \circ \sigma \) yields the conclusion.

Proposition 2.15. Let \( X \) and \( Y \) be projective manifolds, \( f: X \to Y \) a dominant rational map, and \( F \) a foliation on \( Y \). If \( f^* F \) is a pull-back of a transversely affine Riccati equation on a surface then \( F \) is a pull-back of a transversely affine Riccati equation on a surface, or there exists a dominant rational map \( g: Z \to Y \) such that \( g^* F \) is given by a closed rational 1-form.

Proof. It suffices to consider the case where \( \dim X = \dim Y \) since we can replace \( X \) by a general submanifold with the same dimension as \( Y \). Let \( r: Z \to X \) be a dominant rational map between manifolds of the same dimension such that the composition \( g = f \circ r \) defines a Galoisian field extension \( g^*: \mathbb{C}(Y) \to \mathbb{C}(Z) \), i.e., the group of birational transformations \( \varphi: Z \to Z \) which satisfy \( g \circ \varphi = g \) acts transitively on the general fiber of \( g \). Notice that \( g^* F \) admits a transverse affine structure and is the pull-back of a Riccati foliation on a surface.

If the transverse affine structure for \( g^* F \) is not unique then \( g^* F \) is defined by a closed rational 1-form according to Proposition 2.13.

If the transverse affine structure for \( g^* F \) is unique then it must be invariant under birational maps \( \varphi: Z \to Z \) such that \( g \circ \varphi = g \). In other words, if we consider the projective structure \((P, \mathcal{H}, \sigma)\) naturally associated to \( g^* F \) then every birational transformation \( \varphi \) of \( g \) lifts to a birational map \( \Phi: P \to P \), which preserves \( \mathcal{H} \) and \( \sigma \), i.e., \( \Phi^* \mathcal{H} \to \mathcal{H} \) and \( \Phi \circ \sigma = \sigma \). Since \( g^* F \) is a pull-back of a Riccati foliation \( \mathcal{H}_0 \) on a surface \( S \), the same holds true for \( \mathcal{H} \). Notice that the fibers of the pull-back map \((P, \mathcal{H}) \to (S, \mathcal{H}_0)\) define a codimension two foliation \( G \) by algebraic subvarieties tangent to \( \mathcal{H} \). To prove that \( F \) is also a pull-back of a Riccati foliation on a surface it suffices to verify that \( G \) is invariant by \( \Phi \). Since \( \mathcal{H} \) is invariant by \( \Phi \), if \( \Phi^* G \neq G \) then \( G \) would be another foliation by codimension two algebraic subvarieties tangent to \( \mathcal{H} \). But this would imply that the leaves of \( \mathcal{H} \) are algebraic and the same would hold true for \( g^* F \) and \( F \). This contradiction shows that \( \Phi^* G = G \), and therefore \( F \) is also a pull-back of a Riccati foliation on a surface under a rational map. \( \square \)

3. Cohomology jumping loci for local systems

Let \( X \) be a projective manifold and \( U \subset X \) be the complement of a divisor \( D \) of \( X \). In this section we are going to review results on the structure of representations

\[
\rho: \pi_1(U) \to \text{Aff}(\mathbb{C})
\]

which will be essential in what follows.
3.1. Group cohomology. Let $\Gamma$ be a group, $V$ a finite dimension vector space, and $\rho: \Gamma \to \text{GL}(V)$ a morphism of groups. The homomorphism $\rho$ endows $V$ with the structure of a $\Gamma$-module which we will denote by $V_{\rho}$. The first cohomology group of $\Gamma$ with values in $V_{\rho}$ can be defined as the quotient of 1-cocycles and 1-coboundaries
\[
H^1(\Gamma, V_{\rho}) = \frac{\{ \varphi: \Gamma \to V; \varphi(\gamma_1 \cdot \gamma_2) = \varphi(\gamma_1) + \rho(\gamma_1)\varphi(\gamma_2) \}}{\{ \varphi: \Gamma \to V; \exists v \in V \text{ such that } \varphi(\gamma) = \rho(\gamma)v - v, \forall \gamma \in \Gamma \}}.
\]

Let $\rho: \Gamma \to \text{Aff}(V)$ a representation in the affine group $\text{Aff}(V) = \text{GL}(V) \rtimes V$. If $\gamma$ belongs to $\Gamma$ then we can write $\rho(\gamma)(z) = \rho(\gamma)z + \tau(\gamma)$, where $\rho: \Gamma \to \text{GL}(V)$ is a homomorphism; and $\tau: \Gamma \to V$ is a 1-cocycle with values in $V_{\rho}$. The class of $\tau$ in $H^1(\Gamma, V_{\rho})$ is trivial if and only if the action of $\rho(\Gamma)$ fixes a point, so that we can write $\rho(\gamma)(v) = \rho(\gamma) \cdot (v - v_0) + v_0$. In particular, if $\rho$ is the trivial homomorphism then $H^1(\Gamma, V_{\rho}) = \text{Hom}(\Gamma, V)$.

3.2. Cohomology jumping loci for quasi-projective manifolds. Let $U$ be a quasi-projective manifold. The characteristic varieties of $U$ are defined as
\[
\Sigma^k_*(U) = \{ \rho \in \text{Hom}(\pi_1(U), \mathbb{C}^*); \dim H^1(X, \mathbb{C}_\rho) \geq k \}.
\]
When $U$ is compact they have been studied by Green-Lazarsfeld, Beauville, Catanese, Simpson, Campana, Delzant and others, and when $U$ is not proper they have been studied by Arapura, Dimca, Bartolo-Cogolludo-Matei, Budur-Wang and others. See [BCM13], [BW12] and references therein.

Of particular interest for us, is the first characteristic variety which is described by the following theorem which combines results by Arapura and Bartolo-Cogolludo-Matei, and is stated in [BCM13] in a slightly different form which we present afterwards.

**Theorem 3.1.** If $U$ is a quasi-projective manifold then $\Sigma^1_*(U)$ is a finite union of torsion translates of subtori of $\text{Hom}(\pi_1(U), \mathbb{C}^*)$. Moreover, each irreducible component of $\Sigma^1_*(U)$ of positive dimension is a translate of a subtori of the form
\[
f^* \text{Hom}(\pi_1(C), \mathbb{C}^*)
\]
where $C$ is a quasi-projective curve and $f: U \to C$ is a morphism.

In particular, if $\rho: \pi_1(U) \to \mathbb{C}^*$ belongs to a positive dimensional component of $\Sigma^1_*(U)$ then there exists an étale covering $p: V \to U$ and $\rho': \pi_1(V) \to \mathbb{C}^*$ such that the following diagram commutes.
\[
\begin{array}{ccc}
\pi_1(V) & \xrightarrow{(f \circ p)_*} & \pi_1(C) \\
p_* \downarrow & & \downarrow \rho' \\
\pi_1(U) & \xrightarrow{\rho} & \mathbb{C}^*
\end{array}
\]

The covering $p$ is determined by the torsion character used to translate the subtori and is related to the presence of multiple fibers of the fibration $f: U \to C$. This factorization is more succinctly stated in the language of orbifolds: there exist $C$ an orbifold of dimension one, $f: U \to C$ a morphism of orbifolds, and a representation $\rho': \pi_1^{orb}(C) \to \mathbb{C}^*$ such that
\[
\rho = \rho' \circ f_*.
\]
This is the statement of [BCM13, Theorem 1].
4. Factorization of representations

The result below is an easy consequence of Theorem 3.1, and is well-known to the specialists. Indeed, in [BCM13, Theorem 5.1] it appears as an important intermediate step toward the proof of Theorem 3.1. Nevertheless, we present a proof using Theorem 3.1 as a black-box, since the argument is short and clarifies what sort of obstructions one may find when trying to factorize a representation in \( \text{Aff}(\mathbb{C}) \) through a fibration.

**Theorem 4.1.** Let \( U \) be a quasi-projective manifold and \( \varrho : \pi_1(U) \to \text{Aff}(\mathbb{C}) \) be a representation in the affine group. If the image of \( \varrho \) is Zariski dense in \( \text{Aff}(\mathbb{C}) \) then there exists an orbifold \( C \) of dimension one, a morphism of orbifolds \( f : U \to C \), and a representation \( \tilde{\varrho} : \pi_1^{orb}(C) \to \text{Aff}(\mathbb{C}) \) factoring \( \varrho \) as in the diagram below.

\[
\begin{array}{ccc}
\pi_1(U) & \xrightarrow{\varrho} & \text{Aff}(\mathbb{C}) \\
\downarrow f_* & & \downarrow \tilde{\varrho} \\
\pi_1^{orb}(C) & & \\
\end{array}
\]

**Proof.** Let \( \rho : \pi_1(U) \to \mathbb{C}^* \) be the linear part of \( \varrho \), i.e. \( \rho \) is the composition of \( \varrho \) with the natural projection \( \text{Aff}(\mathbb{C}) \to \mathbb{C}^* \). Since \( \varrho \) has Zariski dense image, it must be non abelian and therefore \( k = h^1(U, \mathbb{C}_\rho) > 0 \). Since \( \rho \) is not torsion, Theorem 3.1 implies that the germ \( \Sigma_\rho \) of \( \Sigma_1 \) at \( \rho \) is smooth of positive dimension, and every \( \rho' \in \Sigma_\rho \) satisfies \( h^1(U, \mathbb{C}_{\rho'}) = h^1(U, \mathbb{C}_\rho) \). Consequently we have a morphism \( \Psi : \Sigma_\rho \to \text{Hom}(\pi_1(U), \text{Aff}(\mathbb{C})) \) such that \( \Psi(\rho) = \varrho \) and \( \Psi(\rho') \) is a representation with linear part \( \rho' \), see [Sim93, Lemma 2.1].

Let \( f : U \to C \) be the morphism of orbifolds given by Theorem 3.1. Let \( U_0 \subset U \) be a Zariski open subset such that the restriction of \( f \) to \( U_0 \) is smooth fibration, locally trivial in the \( C^\infty \) category, over \( C_0 = f(U_0) \). Let us compare the long exact sequence for the homotopy groups of a fibration with the factorization of the linear part of \( \varrho \).

\[
\begin{array}{cccccc}
0 & \to & (\mathbb{C}, +) & \to & \text{Aff}(\mathbb{C}) & \to & \mathbb{C}^* & \to & 1 \\
\varrho \downarrow & & \varrho \downarrow & & \rho \downarrow & & \\
0 & \to & \pi_1(F) & \to & \pi_1(U_0) & \to & \pi_1(C_0) & \to & 1 \\
\end{array}
\]

From this diagram we deduce that \( \varrho(\pi_1(F)) \) is a finitely generated normal subgroup of \( \varrho(\pi_1(U_0)) = \varrho(\pi_1(U)) \) with trivial linear part. Since \( \rho(\pi_1(U_0)) \) is an infinite group, this implies that, \( \varrho(\pi_1(F)) = 0 \) or that \( \rho(\pi_1(U_0)) \) is contained in the ring of algebraic integers of some number field \( K \subset \mathbb{C} \).

Notice that a general \( \rho' \in \Sigma_\rho \) is not defined over a number field and therefore the representation \( \Psi(\rho') \) factors. Since the factorization is equivalent to the triviality of \( \Psi(\rho') \) over a general fiber of \( f \) and this is a closed property, it follows that the representation \( \varrho = \Psi(\rho) \) also factors as wanted. \( \square \)
5. Proof of Theorem A

Let \((\mathcal{F}, \nabla)\) be a transversely affine foliation on a projective manifold \(X\) with singular divisor \(D\) and complement \(U = X - D\). Let \(\varrho: \pi_1(U) \to \text{Aff}(\mathbb{C})\) be its monodromy representation and \(\rho: \pi_1(U) \to \mathbb{C}^*\) be its multiplicative part.

We will divide the proof of Theorem A according to the properties of \(\varrho\).

5.1. Zariski dense monodromy. Under the assumption that \(\varrho\) has Zariski dense image we are able to prove Theorem A on arbitrary projective manifolds as already mentioned in the Introduction.

**Theorem 5.1.** Let \(X\) be a projective manifold and \((\mathcal{F}, \nabla)\) be a transversely affine foliation on \(X\). If the monodromy of \((\mathcal{F}, \nabla)\) is Zariski dense in \(\text{Aff}(\mathbb{C})\) then there exists a transversely affine Ricatti foliation \(\mathcal{R}\) on a projective surface \(S\) and a rational map \(p: X \dasharrow S\) such that \(p^*\mathcal{R} = \mathcal{F}\).

**Proof.** Suppose that \(\varrho\) has Zariski dense image in \(\text{Aff}(\mathbb{C})\). Theorem 4.1 implies that the representation \(\varrho\) factors through a morphism of orbifolds \(f: U \to C_0\) where \(C_0\) is a quasi-projective orbicurve. Include in \(D\) the fibers of \(f\) over the multiple fibers of \(f\). Over the new \(U = X - D\), \(f\) is just a regular morphism to a quasi-projective curve \(C_0\), restriction of a rational map \(f: X \dasharrow C\) between projective manifolds. Modulo resolving the indeterminacies of this map, we can assume that \(f: X \to C\) is regular, and its restriction to a Zariski open subset \(U\) factors the monodromy of \((\mathcal{F}, \nabla)\) through a quasi-projective curve \(C_U = f(U)\), i.e. there exists \(\varrho_f: \pi_1(C_U) \to \text{Aff}(\mathbb{C})\) such that \(\varrho_f \circ f_* = \varrho\).

In order to be able to apply Proposition 2.14 we have to exclude the existence of irreducible components \(H\) of the singular set of \(\nabla\) which are not logarithmic and have image under \(f\) dominating \(C\). Since the monodromy of \((\mathcal{F}, \nabla)\) factors through \(C_U\), it induces a representation \(g_H: \pi_1(U_H) \to \text{Aff}(\mathbb{C})\), where \(U_H = f^{-1}(f(U)) \cap H\), such that \(\varrho_f \circ (f_{|H})_* = g_H\). Moreover, according to Proposition 2.12 there exists a finite index subgroup \(G\) of \(\pi_1(U_H)\) whose image under holonomy is abelian.

Since elements in \([G,G]\) have trivial holonomy, representatives of them lift to leaves of \(\mathcal{F}\) nearby \(H\), and consequently \(g_H([G,G]) = \{\text{id}\}\). To arrive at a contradiction with the density of the monodromy group notice that \(f, \pi_1(U_H)\) has finite index in \(\pi_1(C_U)\), provided \(f_{|U_H}\) is étale, which is the case if \(U\) is initially well chosen. This proves that such \(H\) cannot exist, and the Theorem follows from Proposition 2.14.

5.2. Virtually additive monodromy.

**Theorem 5.2.** Let \(X\) be a projective manifold with \(h^1(X, \mathcal{O}_X) = 0\) and \((\mathcal{F}, \nabla)\) be a transversely affine foliation on \(X\). If the monodromy of \((\mathcal{F}, \nabla)\) is contained in a finite extension of \((\mathcal{O}, +) \subset \text{Aff}(\mathbb{C})\) then there exists a generically finite Galois morphism \(p: Y \to X\) such that \(p^*\mathcal{F}\) is defined by a closed rational 1-form; or there exist a transversely affine Ricatti foliation \(\mathcal{R}\) on a surface \(S\) and a rational map \(p: X \dasharrow S\) such that \(p^*\mathcal{R} = \mathcal{F}\).

**Proof.** If \(\nabla\) is logarithmic, with connection form \(\eta_0\) in a Zariski open set, then \(\exp(\int \eta_0)\) is a multi-valued algebraic function. Its branches determine a generically finite Galois morphism \(p: Y \to X\) such that \(p^*\mathcal{F}\) is defined by a closed rational 1-form.
Suppose that \( \nabla \) is not logarithmic. The connection \( \nabla \) can be written as the sum of a logarithmic connection on \( NF \) with a closed rational 1-form \( \beta \) without residues. Since \( h^1(X, \mathbb{C}) = 0 \), the 1-form \( \beta \) is exact in the sense that there exists a rational function \( g \in \mathbb{C}(X) \) such that \( \beta = dg \). We can assume that \( g \) defines a regular morphism (no indeterminacies) from \( X \) to \( \mathbb{P}^1 \). Let \( h : X \to \mathbb{P}^1 \) be the Stein factorization of \( g \). Of course the target is still \( \mathbb{P}^1 \) as we are assuming \( h^1(X, \mathbb{C}) = 0 \).

We want to show that the monodromy representation factors through \( h \). To that end suppose first that all the residues of \( \nabla \) are integers. In this case we can choose a pair of rational 1-forms \( \omega_0, \eta_0 \) in a Zariski open subset \( U \) such that \( \eta_0 = h^* \beta_0 \) for some rational 1-form \( \beta_0 \) on \( \mathbb{P}^1 \). The equation \( d\omega_0 = \omega_0 \wedge h^* \beta_0 \) implies that \( \omega_0 \) is closed when restricted to the general fiber of \( h \). To prove the factorization of the monodromy it suffices the restriction of \( \omega_0 \) to a general fiber of \( h \) is not only closed, but exact.

Let \( p \) be a non-logarithmic pole of \( \beta_0 \) and let us suppose that \( U \) intersects the fiber over \( p \), otherwise we can start with a different \( U \). Replacing \( U \) by a smaller open subset we can suppose that \( h \) restricted to \( U^* = U \setminus h^{-1}(p) \) is a locally trivial \( C^\infty \)-fibration over \( T^* = h(U^*) \). The 1-form \( \omega_0 \) can be interpreted as a family \( \omega_{0,t} \) of closed rational 1-forms on the quasi-projective manifolds \( U_t = U \cap h^{-1}(t) \) parametrized by \( t \in T^* \). We want to prove that for a general \( t \in T^* \) and any \( \delta_t \in H_1(U_t, \mathbb{Z}) \) the integral \( \int_{\delta_t} \omega_{0,t} \) is equal to zero.

The multi-valued function

\[
F : T^* \to \mathbb{C}
\]

\[
t \mapsto \int_{\delta_t} \omega_{0,t}
\]

obtained by continuous deformation of \( \delta_t \in H_1(U_t, \mathbb{Z}) \) satisfies the so called Picard-Fuchs equation, see [AGZV12, Chapters 10 and 12] specially §10.2.4 and §12.2.1. In our setting, it is nothing but

\[
d\frac{F}{F} = \beta_0.
\]

Therefore, if one of the periods of \( \omega_{0,t} \) is not zero then the function \( F(t) \) does not have moderate growth when we approach \( p \). But this contradicts [Del70, Théorème 1.8, page 125] (see also [AGZV12, Theorem 12.3]) which roughly says that the periods of holomorphic families of rational 1-forms are functions with moderate growth at infinity. We conclude that periods of \( \omega_{0,t} \) are zero for any \( t \in T^* \), i.e. \( \omega_0 \) is exact on the general fiber of \( h \) and therefore the monodromy factors through \( h \).

We apply Proposition 2.14 to conclude that \( \mathcal{F} \) is a pull-back of a Riccati foliation over a rational surface.

If the residues of \( \nabla \) are not integers in general, they must be rational and we can apply the above arguments to the pull-back of \( \mathcal{F} \) under a generically finite rational map \( p : Y \to X \) to conclude that \( p^* \mathcal{F} \) is the pull-back of a Riccati foliation on a surface. Proposition 2.15 allows us to conclude.

5.3. Multiplicative monodromy.

Theorem 5.3. Let \( X \) be a projective manifold with \( h^1(X, \mathbb{C}) = 0 \) and \( (\mathcal{F}, \nabla) \) be a transversely affine foliation on \( X \). If the monodromy of \( (\mathcal{F}, \nabla) \) is contained in \( \left( \mathbb{C}^*, \times \right) \subset \text{Aff}(\mathbb{C}) \) then there exists a generically finite Galois morphism \( p : Y \to X \) such that \( p^* \mathcal{F} \) is defined by a closed rational 1-form; or there exists a transversely
affine Ricatti foliation $\mathcal{R}$ on a surface $S$ and a rational map $p : X \dashrightarrow S$ such that $p^* \mathcal{R} = \mathcal{F}$.

Proof. Let $(P, \mathcal{H}, \sigma)$ be the transverse projective structure naturally associated to $(\mathcal{F}, \nabla)$. Since the monodromy is multiplicative, on the complement of the singular divisor of $\nabla$ we have two sections $\Sigma_1, \Sigma_2$ of $P$ invariant by $\mathcal{H}$. If $\nabla$ is logarithmic these sections are indeed meromorphic over all $X$ and it follows that $\mathcal{H}$ is defined by a closed rational 1-form with polar divisor containing these two sections. The pull-back of this closed rational 1-form under the section $\sigma$ gives a closed rational 1-form on $X$ defining $\mathcal{F}$.

Suppose that $\nabla$ is not logarithmic, and as in the proof of Theorem 5.2 consider the decomposition of $\nabla$ in a logarithmic connection $\nabla_{\log}$ and a closed rational 1-form $h^* \beta_0$, where $h : X \to \mathbb{P}^1$ is a rational map with irreducible fibers. Such a decomposition exists since $h^1(X, \mathbb{C}) = 0$.

If there is no irreducible component $H$ in the singular divisor of $\nabla$ with $\text{Res}_H(\nabla) \not\in \mathbb{Z}$ and $b(H) = \mathbb{P}^1$, then we claim that the representation factors through $h$. To verify that consider the residue divisor of $\nabla$. It can be written in the form $\text{Res}(\nabla) = V + T$ where $V$ is a $\mathbb{C}$-divisor supported on fibers of $h$ and $T$ is a $\mathbb{Z}$-divisor with irreducible components transverse to $h$, i.e. not contained in fibers of $h$. Since $c(N\mathcal{F})$ lies in $H^2(X, \mathbb{Z})$ it follows that the class of $V$ in the quotient $H^2(X, \mathbb{C})/H^2(X, \mathbb{Z})$ is zero. Thus we can write $\text{Res}(\nabla) = V' + T'$ where now $V'$ is a $\mathbb{C}$-divisor with zero Chern class supported on fibers of $h$ and $T'$ is a $\mathbb{Z}$-divisor. Hodge index Theorem implies that $V'$ is a $\mathbb{C}$-linear combination of fibers of $h$. At this point it is clear that the logarithmic connection $\nabla_{\log}$ can be written as a sum $\nabla_{\log, \mathbb{Z}} + \eta_{\log}$ with $\nabla_{\log, \mathbb{Z}}$ a logarithmic connection with integral residues and $\eta_{\log}$ is the pull-back under $h$ of a logarithmic 1-form on $\mathbb{P}^1$ having at least one residue not belonging to $\mathbb{Z}$. The monodromy representation of the connection $\nabla_{\log, \mathbb{Z}}$ is trivial around its poles, and since $H^1(X, \mathbb{C}) = 0$, this suffices to conclude that it is finite. After passing to a ramified covering, we can argue as in the proof of Theorem 5.2 using Picard-Fuchs equation, and use Proposition 2.15 to conclude that $\mathcal{F}$ is the pull-back of a transversely affine Ricatti foliation on a surface.

If there exists an irreducible component $H$ of the singular divisor of $\nabla$ with non-integral residue generically transverse to the fibers of $h$, then we are going to prove that $\mathcal{F}$ is defined by a closed rational 1-form. The proof of CLNS92 Lemma 9 shows that a codimension 1 foliation $\mathcal{F}$ on a projective manifold is defined by a rational closed 1-form if and only if the same holds true for the restriction of $\mathcal{F}$ to a sufficiently general hyperplane section. Therefore it suffices to consider the case where $X$ is a surface. We can further assume that $\mathcal{F}$ is a reduced foliation in the sense of Seidenberg.

Let us consider a general point $q$ in the intersection of $H$ (which is now a curve) with a non-logarithmic pole of $\nabla$. This point of intersection is a saddle node singularity for the foliation $\mathcal{F}$ which has formal normal form

$$\theta_{k, \mu} = x^{k+1}dy - y(1 + \mu x^k)dx.$$ 

If it is not analytically conjugated to its formal normal form then BT99 Proposition 5.5 implies that every transverse affine structure for $\mathcal{F}$ has integral residues. Therefore, our assumptions implies that the saddle node is analytically conjugated to its formal normal form. Hence every transverse affine structure for the germ of $\mathcal{F}$ at $q$ is defined by a pair $(\theta_{k, \mu}/yx^{k+1}, \lambda \theta_{k, \mu}/yx^{k+1})$, with $\lambda \in \mathbb{C}$. It follows that
the sections $\Sigma_1, \Sigma_2$ defined on the first paragraph of the proof extend to meromorphic sections over a neighborhood of any $H$ generically transverse to the fibers of $h$. Adding to $H$ fibers of $h$ not contained in the singular divisor of $\nabla$ we obtain a neighborhood of an ample divisor where the sections $\Sigma_1$ and $\Sigma_2$ extend as meromorphic sections. We can apply [And63 Théorème 5] or [Har68 Corollary 6.8] to extend these sections to the whole of $X$. We conclude that $F$ in this case is defined by a closed rational 1-form.

6. Proof of Corollary B

If $\omega$ is a polynomial Liouvillian integrable 1-form on $\mathbb{C}^n$ without invariant algebraic hypersurfaces then $d\omega = dQ \wedge \omega$ for some polynomial $Q \in \mathbb{C}[x_1, \ldots, x_n]$. Let $F$ be the extension to $\mathbb{P}^n = \mathbb{C}^n \cup H_\infty$ of the foliation of $\mathbb{C}^n$ defined by $\omega$.

According to Theorem A there exists a rational map $F : \mathbb{P}^n \dashrightarrow S$, where $S$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, and a transversely affine Riccati foliation $R$ on $S$ such that $F = F^* R$. We will denote by $\pi : S \to \mathbb{P}^1$ the reference fibration of the Riccati foliation $R$, i.e. $R$ is everywhere transverse to the general fiber of $\pi$.

With a suitable choice of coordinates we can identify the restriction to $\mathbb{C}^n$ of the composition $\pi \circ F$ with a polynomial $P$ such that $P - c$ is irreducible for a general $c \in \mathbb{C}$ and $Q = A \circ P$ for some polynomial $A$ in one variable.

Up to birational transformations on $S$, we can also assume that $S$ is a compactification of $\mathbb{C}^2$ such that the restriction of $R$ to $\mathbb{C}^2$ has no invariant algebraic curves and such that the divisor at infinity is $R$ invariant and has no dicritical singularities.

The pre-image under $F$ of the divisor at infinity must be therefore invariant by $F$. Since $F$ has no algebraic invariant hypersurfaces on $\mathbb{C}^n$, it follows that this pre-image must be contained in the hyperplane at infinity. Hence $F$, in suitable coordinates, is nothing but a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^2$.

7. Proof of Theorem C

The statement of Theorem C is about the birational equivalence class of $\nabla$. Therefore we can assume that $\nabla$ is a meromorphic flat connection on the trivial rank two vector bundle. Since it takes values in $\mathfrak{sl}(2)$, the connection matrix $\Omega$ has zero trace; and the reducibility hypothesis allows us to assume that $\Omega$ is an upper triangular matrix. Therefore we can write

$$
\Omega = \begin{bmatrix} \eta/2 & \omega \\ 0 & -\eta/2 \end{bmatrix}
$$

for suitable rational 1-forms $\omega$ and $\eta$. Since $\nabla$ is flat, the integrability equation $d\Omega + \Omega \wedge \Omega = 0$ holds true. This equation is equivalent to the pair of equations $d\omega = \omega \wedge \eta$ and $d\eta = 0$.

If $\omega$ is zero then there is nothing else to prove. Otherwise $\omega$ defines a transversely affine foliation $F$ on $X$, with transversely affine structure given by the pair $(\omega, \eta)$.

Assume first that the foliation $F$ has a non constant rational first integral $f \in \mathbb{C}(X)$. Then $g\omega = df$ for some rational functions $f, g : X \to \mathbb{P}^1$. We can replace $f$ by its Stein factorization, which still takes values in $\mathbb{P}^1$ since $h^1(X, \mathbb{C}) = 0$. If we apply the birational transformation $\sqrt{g} \oplus 1/\sqrt{g}$ on $Y$, the resolution of a double covering of $X$ determined by $\sqrt{g}$, the connection matrix becomes

$$
\begin{bmatrix}
\eta/2 - 1/2d\log g \\ 0
\end{bmatrix}
\begin{bmatrix}
df \\ -\eta/2 + 1/2d\log g
\end{bmatrix}
= \begin{bmatrix}
hdf/2 \\ 0
\end{bmatrix}
\begin{bmatrix}
df \\ -hdf/2
\end{bmatrix}.
$$
with \( h \in \mathbb{C}(X) \subset \mathbb{C}(Y) \) satisfying \( dh \wedge df = 0 \). Since the general fiber of \( f \) is irreducible, there exists \( H \in \mathbb{C}(\mathbb{P}^1) \) such that \( h = H \circ f \), and it becomes clear that the induced connection on \( Y \) is birationally equivalent to the pull-back of a meromorphic connection on \( \mathbb{P}^1 \).

From now on we will assume that \( F \) does not admit a non constant rational first integral. In particular, if \( F \) is defined by a closed rational 1-form \( \tilde{\omega} \) then every other closed rational 1-form defining \( F \) is a constant multiple of \( \tilde{\omega} \).

If there exists a generically finite Galois morphism \( p : Y \to X \) such that \( p^*F \) is defined by a closed rational 1-form (case (1) of Theorem A) then after applying a gauge transformation of the form \( \sqrt{g} \oplus 1/\sqrt{g} \), with \( g \in \mathbb{C}(Y) \), we can assume that \( \omega \) is closed, and consequently \( \eta = \lambda \omega \), for some \( \lambda \in \mathbb{C} \). If \( \lambda = 0 \) there is nothing else to prove, otherwise we apply the gauge transformation with matrix

\[
\begin{pmatrix}
1 & 1/\lambda \\
0 & 1
\end{pmatrix}
\]

to obtain a diagonal connection matrix and we have the result.

Now, we suppose we are not in case (1) of Theorem A, in particular \( F \) is a rational pull-back of a Riccati foliation \( \mathcal{H}_0 \) on a surface \( S \) (case (2) of Theorem A). Then there exists \( f, g, h \in \mathbb{C}(X) \) and \( \alpha, \beta \) rational 1-forms on \( \mathbb{P}^1 \) such that

\[
g \omega = dh + f^*\alpha + (f^*\beta)h.
\]

After applying the gauge transformation \( \sqrt{g} \oplus 1/\sqrt{g} \) we can assume that \( g = 1 \), thus by proposition 2.3, \( \eta = f^*\beta \). If we apply the gauge transformation given by the matrix

\[
\begin{pmatrix}
1 & h \\
0 & 1
\end{pmatrix}
\]

we obtain the connection form

\[
\begin{pmatrix}
f^*\beta/2 & f^*\alpha \\
0 & -f^*\beta/2
\end{pmatrix}
\]

which is clearly a pull-back from a curve. \( \square \)

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