BOUNDARY YAMABE PROBLEM WITH MINIMAL BOUNDARY CASE: A COMPLETE SOLUTION

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Abstract. This article provides a complete solution of the boundary Yamabe problem with minimal boundary scenario, or equivalently, the existence of a real, positive, smooth solution of

$$-\frac{4(n-1)}{n-2} \Delta_g u + S_g u = \lambda u^{\frac{n+2}{n-2}}$$

in $M$, $\frac{\partial u}{\partial \nu} + \frac{n-2}{2} h_g u = 0$ on $\partial M$. In contrast to the classical method of calculus of variation with assumptions on Weyl tensors and classification of types of points on $\partial M$, boundary Yamabe problem is fully solved here in cases classified by the sign of the first eigenvalue $\eta_1$ of the conformal Laplacian with Robin condition. When $\eta_1 < 0$, global sub-solution and super-solution are constructed. When $\eta_1 > 0$, a perturbed boundary Yamabe equation

$$-\frac{4(n-1)}{n-2} \Delta_{g_B} u + u^{\frac{n+2}{n-2}} = 0$$

in $M$, $\frac{\partial u_B}{\partial \nu} + \frac{n-2}{2} h_g u_B = 0$ on $\partial M$ is solved with $\beta < 0$. The boundary Yamabe equation is then solved by taking $\beta \to 0^-$. The signs of scalar curvature $S_g$ and mean curvature $h_g$ play an important role in this existence result.

1. Introduction

In this paper, we will solve the boundary Yamabe problem for minimal boundary case on compact manifolds $(\overline{M}, g)$ with smooth boundary, dim $\overline{M} \geq 3$, by a direct local and global analysis—in contrast to the classical method in calculus of variation—based on earlier works on either a local Riemannian domain $(\Omega, g)$ [24], or closed manifold $(M, g)$ [29]. This work is also inspired by an iterative method in solving Einstein vacuum equation [14, 15] and nonlinear Laplace equation [28], with a long history in PDE theory dating back to [22, 23]. A modification of the monotone iteration method, due to [25], is applied here for the Robin boundary condition on manifolds. One of the advantages of this local analysis is to avoid the use of Weyl tensor in interior points of manifolds, and to avoid the classification of boundary points and vanishing of Weyl tensors at boundary.

In 1992, Escobar [8] proposed the following generalization of the classical Yamabe problem on closed manifolds, which is called boundary Yamabe problem or Escobar problem, and is a far reaching generalization of the uniformization theorem for surfaces:

The Boundary Yamabe Problem. Given a compact Riemannian manifold $(\overline{M}, g)$ of dimension $n \geq 3$ with interior $M$ and smooth boundary $\partial M$, there exists a metric $\tilde{g}$ conformal to $g$ having constant scalar curvature and minimal boundary.

Let $S_g$ be the scalar curvature of $g$ and $h_g$ be the mean curvature on $\partial M$, and let $\tilde{S}$, $\tilde{h}$ be the scalar curvature and mean curvature of the conformal metric $\tilde{g} = e^{2f} g$, respectively. Let $\nu$ be the outward normal vector field along $\partial M$. Set $e^{2f} = u^{\frac{n}{n-2}}$, where $p = \frac{2n}{n-2}$ and $u > 0$. Then

$$\tilde{S} = u^{1-p} \left( -4 \cdot \frac{n-1}{n-2} \Delta_g u + S_g u \right) \text{ in } M;$$

$$\tilde{h} = e^{-f} \left( h_g + \frac{\partial f}{\partial \nu} \right) \text{ on } \partial M.$$

(1)
Setting \( a = 4 \cdot \frac{n - 1}{n - 2} > 0 \), we have that \( \tilde{g} = u^{p - 2}g \) has constant scalar curvature \( \lambda \) and minimal boundary if and only if \( u \) satisfies the boundary Yamabe equation

\[
\Box_g u := -a \Delta_g u + S_g u = \lambda u^{p-1} \text{ in } M;
B_g u := \frac{\partial u}{\partial \nu} + \frac{2}{p - 2} h_g u = 0 \text{ on } \partial M.
\]

(2)

where \( \Delta_g = -d^*d \) is negative definite.

If in addition we drop off the requirement that \( \partial M \) is minimal with respect to \( \tilde{g} \), but instead the nontrivial mean curvature \( \tilde{h} \), as Escobar mentioned in [9], the boundary Yamabe problem on compact Riemannian manifold with smooth boundary \((M, g)\) is equivalent to solve the following PDE

\[
-a \Delta_g u + S_g u = \lambda u^{p-1} \text{ in } M;
\frac{\partial u}{\partial \nu} = \frac{2}{p - 2} \left( -h_g u + \tilde{h} u^{\frac{p}{2}} \right) \text{ on } \partial M.
\]

(3)

Here \( \lambda \) is constant scalar curvature of \( \tilde{g} \) and \( \tilde{h} \) is constant mean curvature on \( \partial M \) with respect to \( \tilde{g} \). Most cases for boundary Yamabe problem with minimal boundary case are resolved by a series of excellent works due to [4], [7], [13], etc. In particular, [20] among others worked on the case when \( S_g = 0 \) and \( \tilde{h} \) is a constant. However, there are several cases with minimal boundary scenario still left open. In [8], the open cases are when \( n \geq 6 \), \( M \) is not locally conformally flat, \( \partial M \) is umbilic, and the Weyl tensor vanishes identically on \( \partial M \). This result was improved in [5] with some extra restrictions on the manifolds. On closed manifolds, [2, 19] provided good survey with classical calculus of variation methods, while a direct analysis can be found in [29]. On non-compact manifolds, results can be found in [3, 12, 16] with certain restrictions.

The main result of this article, which states below, provides a complete solution of boundary Yamabe problem.

**Theorem.** Let \((\tilde{M}, g)\) be a compact manifold with smooth boundary, \( \dim \tilde{M} \geq 3 \). Let \( \eta_1 \) be the first eigenvalue of the boundary value problem \( \Box_g u = \eta_1 u \text{ in } M \), \( B_g u = 0 \text{ on } \partial M \). Then

(i). If \( \eta_1 = 0 \), then (2) has a real, positive solution \( u \in C^\infty(\tilde{M}) \) with \( \lambda = 0 \);
(ii). If \( \eta_1 < 0 \), then (2) has a real, positive solution \( u \in C^\infty(M) \) with \( \lambda < 0 \);
(iii). If \( \eta_1 > 0 \), then (2) has a real, positive solution \( u \in C^\infty(M) \) with \( \lambda > 0 \).

Case (i) is a trivial case, since it’s just an eigenvalue problem. Case (ii) is solved in Theorem 5.2 and [5,14] We first get a solution of (2) when \( h_g > 0 \) everywhere on \( \partial M \); then Theorem 5.3 says that the general case when \( \eta_1 < 0 \) can be converted to the special case just mentioned. Case (iii) is solved in three steps: a perturbed boundary Yamabe equation \(-\frac{4(n-1)}{n-2} \Delta_g u + (S_g + \beta) u \in C^\infty(M) \), \( \frac{\partial u}{\partial \nu} + \frac{n-2}{2} h_g u = 0 \text{ on } \partial M \) is solved in Theorem 5.5 with some \( \beta < 0 \); then the boundary Yamabe problem is solved in Theorem 5.5 for \( \eta_1 > 0 \), \( h_g > 0 \) everywhere on \( \partial M \) and \( S_g < 0 \) somewhere in \( M \); lastly Theorem 5.3 and 5.7 convert every general case to the scenario in Theorem 5.5 which is Theorem 5.8. Inspired by [24] and [29], it turns that a local result with respect to a perturbation of \( \beta \) within a small enough domain \( \Omega \) plays a central role in local analysis the author will use. This smallness of \( \Omega \) is perfectly reflected by Ricci, Riemannian, scalar curvature tensors, etc.

In classical methods of calculus of variation, the goal is looking for the minimizer of the functional

\[
Q(M) = \inf_{u \neq 0} \frac{\int_M |\nabla u|^2 + S_g u^2 \text{Vol}_g + \int_{\partial M} \frac{2}{p-2} h_g u^2 dS}{(\int_M w^p d\text{Vol}_g)^{\frac{2}{p}}}
\]
by showing $Q(M) < Q(S^+_n)$. The existence of the minimization is discussed in several scenarios, classified by whether Weyl tensor vanishes or not, and whether a boundary point is umbilic or not.

There are some critical technical differences between our approach and the classical one. In classical methods, the analysis of Yamabe quotient near the boundary determines the choices of different test functions due to classification of boundary points and the locally conformal flatness of the manifolds. In this article, we apply ideas of Kazdan and Warner [17] in closed manifold case to construct global subsolutions and supersolutions when $\eta_1 < 0$. Historically this is also an easy case.

The hard part is when $\eta_1 > 0$. In classical analysis, subcritical solution are constructed, i.e. solving boundary Yamabe equation with subcritical nonlinear terms $u^{s-1}$, $s \in (2, p)$; then a limiting argument applies by taking $s \to p^-$. This limiting process as well as the argument of nontriviality of the limit require $Q(M) < Q(S^+_n)$. In our method we bypass this by perturbing the coefficient of the zeroth order term of the differential operator, instead of the power of nonlinear term. The advantage is to keep the nonlinear term with the same power $p - 1$, which simplifies the limiting argument significantly in the sense that only $Q(M) \leq Q(S^+_n)$ is required. In particular, we use the local to global analysis developed in closed manifolds [29]: first of all, we construct a local solution of a perturbed Yamabe equation $-a\Delta_g u + (S_g + \beta)u = \lambda u^{p-1}$, $\beta < 0$, in a small domain $\Omega$ with trivial Dirichlet boundary condition; secondly we apply monotone iteration scheme to obtain a global solution of perturbed Yamabe equation with Robin boundary condition $B_g u = 0$; lastly, we pass the limit by taking $\beta \to 0^-$ and obtain the solution. In the last step, weak convergence is enough, and the nontriviality of the limit is easy to obtain.

It is worth mentioning that with the local analysis and perturbation methods applied here and in [29], the problems of conformal deformation on closed manifolds and Escobar problem on compact manifolds with boundary have synchronized methodology: (i) the solvability in both cases are classified by the sign of the first eigenvalue $\eta_1$ of conformal Laplacian only; (ii) when $\eta_1 < 0$, both cases are solved by constructing a global subsolution and supersolution; (iii) when $\eta_1 > 0$, both cases are solved by solving a perturbed PDE followed with a limiting argument which annihilates the perturbed term. We found that it is somehow asynchronous that historically the locally conformally flat cases are hardest cases on closed manifolds with dimensions $n \geq 6$; meanwhile the cases when $M$ is not locally conformally flat were left open on compact manifolds with boundary when $\dim(M) \geq 6$.

This article is organized as follows. In §2, definitions and essential tools are listed and proved if necessary. In §3, we first proved a global $L^p$-regularity theory for second order elliptic PDE with Robin condition in Theorem [3.1] by assuming the existence of the solution of this type of PDE. This work is based on a local result by Agmon, Douglis and Nirenberg [1]. A general elliptic estimate is also given. With injectivity of the second order elliptic operator, a specific elliptic estimate is given in Theorem [3.2] This $L^p$-regularity theory and $L^p$-estimate are then used to prove the existence of the solution of $-a\Delta_g u = F(x, u)$ with Robin condition by monotone iteration method on compact manifolds with boundary in Theorem [3.3] provided the existence of corresponding linear elliptic PDE. In §4, an existence theorem of the elliptic linear PDE $-\Lambda\Delta_g u + \Lambda'u = f$ with Robin boundary condition is given. In §5, the boundary Yamabe problem is fully solved in several steps. In Corollary [5.1] the $\eta_1 = 0$ case is solved; in Theorem [5.2] and [5.3] the $\eta_1 < 0$ case is solved; in Theorem [5.6] and [5.8] the $\eta_1 > 0$ case is solved by a perturbation result in Theorem [5.5] In §5, some discussion of the question “whether functions $f_1, f_2$ can be the prescribed scalar curvature and mean curvature of some metric under conformal change respectively” is given in Theorem [5.3], [5.7] Corollary [5.2] and [5.3].
2. The Preliminaries

In this section, we list necessary definitions and results in order to solve this boundary Yamabe problem. Throughout this section, we consider the spaces with dimensions no less than 3.

Let \( \Omega \) be a connected, bounded, open subset of \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \) equipped with some Riemannian metric \( g \) that can be extended smoothly to \( \bar{\Omega} \). We call \((\Omega, g)\) a Riemannian domain. Furthermore, let \((\bar{\Omega}, g)\) be a compact manifold with boundary.

Firstly we define Sobolev space on compact manifolds \((\bar{\Omega}, g)\) with interior \( \Omega \) and smooth boundary \( \partial \Omega \). The integer ordered Sobolev spaces defined on \((\bar{\Omega}, g)\) are

\[
W^{s,p}(\Omega), \quad s \in \mathbb{N}, 1 \leq p < \infty,
\]

and

\[
W^{s,p}_0(\Omega) = \{ u \in W^{s,p}(\Omega) : u = 0 \text{ on } \partial \Omega \}.
\]

Similarly, \(W^{s,p}(\bar{\Omega}, g)\) is the completion of \(C^\infty(\bar{\Omega})\) with respect to the \(W^{s,p}\)-norm. In particular, \(H^s(\bar{\Omega}, g) : = W^{s,2}(\bar{\Omega}, g)\), \(s \in \mathbb{N}, 1 \leq p < \infty\) are the usual Sobolev spaces, and we similarly define \(H^s_0(\bar{\Omega}, g)\) and \(H^s(\Omega, g)\).

The following result is needed due to Escobar \[8\].
Proposition 2.1. \([8]\) Prop. 1.3.] Let \(\tilde{g} = u^{p-2} g\) be a conformal metric to \(g\). Let \(\eta_1\) and \(\tilde{\eta}_1\) be the first eigenvalue of \(\Box_g\) and \(\Box_{\tilde{g}}\) with boundary conditions \(B_g = 0\) and \(B_{\tilde{g}} = 0\), respectively. Then either the signs of \(\eta_1\) and \(\tilde{\eta}_1\) are the same or \(\eta_1 = \tilde{\eta}_1 = 0\).

A local \(L^p\) regularity is required for some type of Robin boundary condition, due to Agmon, Douglas, and Nirenberg \([1]\).

Proposition 2.2. \([1]\) Thm. 7.3, Thm. 15.2] Let \((\Omega, g)\) be a Riemannian domain where the boundary \(\partial \Omega\) satisfies Lipschitz condition. Let \(\nu\) be the outward unit normal vector along \(\partial \Omega\). Let \(L\) be the second order elliptic operator on \(\Omega\) with smooth coefficients up to \(\partial \Omega\) and \(f \in \mathcal{L}^p(\Omega, g), f' \in W^{1,p}(\Omega, g)\) for some \(p \in (1, \infty)\). Let \(u \in H^1(\Omega, g)\) be the weak solution of the following boundary value problem

\[
Lu = f \text{ in } \Omega, \quad Bu := \frac{\partial u}{\partial \nu} + c(x)u = f' \text{ on } \partial \Omega,
\]

where \(c \in C^\infty(\partial \Omega)\). Then \(u \in W^{2,p}(\Omega, g)\) and the following estimates holds provided \(u \in \mathcal{L}^p(\Omega, g)\):

\[
\|u\|_{W^{2,p}(\Omega, g)} \leq C^* \left(\|Lu\|_{\mathcal{L}^p(\Omega, g)} + \|Bu\|_{W^{1,p}(\Omega, g)} + \|u\|_{\mathcal{L}^p(\Omega, g)}\right).
\]

Here the constant \(C^*\) depends on \(L, p\) and \((\Omega, g)\).

Remark 2.1. It is worth mentioning that the result in Proposition 2.2 holds on a \(n\) dimensional hemisphere denoted by \(\sum_{i=1}^{n-1} x_i^2 + t^2 \leq 1, t \geq 0\) where the boundary condition is only defined on \(t = 0\) and \(u\) in \([8]\) vanishes outside the hemisphere \([1]\) Thm. 15.1]. The Schauder estimates holds in the same manner, see \([1]\) Thm. 7.1, Thm. 7.2]. This is particularly useful since for the global analysis in next section, we will choose a cover of \((\tilde{M}, g)\), and for any boundary chart \((U, \phi)\) of \((\tilde{M}, g)\), the intersection \(\phi(U \cap \tilde{M})\) is a one-to-one correspondence to a hemisphere, provided that \(\partial \tilde{M}\) is smooth enough. It resolves the issue for the boundary charts, as we shall see in later sections.

Another tool required in the future analysis is the \(W^{s,p}\)-type “Peter-Paul” inequality.

Proposition 2.3. \([10]\) Thm. 7.28] Let \((\Omega, g)\) be a Riemannian domain in \(\mathbb{R}^n\) and \(u \in W^{2,p}(\Omega, g)\). Then for any \(\gamma > 0\),

\[
\|\nabla_g u\|_{\mathcal{L}^p(\Omega, g)} \leq \gamma \|u\|_{W^{2,p}(\Omega, g)} + C'_\gamma \gamma^{-1} \|u\|_{\mathcal{L}^p(\Omega, g)}.
\]

Here \(C_\gamma\) only depends on \(\gamma\) and \((\Omega, g)\).

Note that in \([10]\) this inequality is stated in a more general version, we only need the \(W^{2,p}\)-case here. Note also that the result above can be easily obtained by Gagliardo-Nirenberg interpolation inequality when \(u\) is compactly supported in \(\Omega\). We can easily extend this local results to global results.

Proposition 2.4. Let \((\tilde{M}, g)\) be a compact manifold with smooth boundary \(\partial \tilde{M}\). Let \(u \in W^{2,p}(M, g)\). Then for any \(0 < \gamma < 1\),

\[
\|u\|_{W^{1,p}(M, g)} \leq \gamma \|u\|_{W^{2,p}(M, g)} + C_\gamma \gamma^{-1} \|u\|_{\mathcal{L}^p(M, g)}.
\]

Here \(C_\gamma\) only depends on \(\gamma\) and \((M, g)\).

Proof. Taking a finite cover \((U_\xi, \phi_\xi)\) of \((\tilde{M}, g)\) and a smooth partition of unity \(\{\chi_\xi\}\). Applying this, we have

\[
\|u\|_{W^{1,p}(M, g)} = \sum_\xi \left\| \left(\phi_\xi^{-1}\right)^* \chi_\xi u \right\|_{W^{1,p}(\phi_\xi(U_\xi), g)} = \|u\|_{\mathcal{L}^p(M, g)} + \sum_\xi \left\| \nabla_g \left(\left(\phi_\xi^{-1}\right)^* \chi_\xi u\right) \right\|_{\mathcal{L}^p(\phi_\xi(U_\xi), g)}
\]

\[
\leq \|u\|_{\mathcal{L}^p(M, g)} + \sum_\xi \left(\gamma \left\| \left(\phi_\xi^{-1}\right)^* \chi_\xi u \right\|_{W^{2,p}(\phi_\xi(U_\xi), g)} + C'_\gamma \gamma^{-1} \left\| \left(\phi_\xi^{-1}\right)^* \chi_\xi u \right\|_{\mathcal{L}^p(\phi_\xi(U_\xi), g)}\right)
\]

\[
= \gamma \|u\|_{W^{2,p}(M, g)} + C_\gamma \gamma^{-1} \|u\|_{\mathcal{L}^p(M, g)}.
\]
In the last step, we combine the first and the third term.

Sobolev embedding theorem for compact manifolds with boundary plays an important role in regularity arguments.

**Proposition 2.5.** [2, Ch. 2] (Sobolev Embeddings) Let \((\bar{M}, g)\) be a compact manifold with smooth boundary \(\partial M\).

(i) For \(s \in \mathbb{N}\) and \(1 \leq p \leq p' < \infty\) such that
\[
\frac{1}{p} - \frac{s}{n} \leq \frac{1}{p'},
\]
\(W^{s,p}(M,g)\) continuously embeds into \(L^{p'}(M,g)\) with the following estimates:
\[
\|u\|_{L^{p'}(M,g)} \leq K \|u\|_{W^{s,p}(M,g)}.
\]

(ii) For \(s \in \mathbb{N}\), \(1 \leq p < \infty\) and \(0 < \alpha < 1\) such that
\[
\frac{1}{p} - \frac{s}{n} \leq \frac{\alpha}{n},
\]
\(W^{s,p}(M,g)\) continuously embeds in the Hölder space \(C^{0,\alpha}(\bar{M})\) with the following estimates:
\[
\|u\|_{C^{0,\alpha}(\bar{M})} \leq K' \|u\|_{W^{s,p}(M,g)}.
\]

(iii) Both embeddings above are compact embeddings provided that the equalities in (12) and (14) do not hold, respectively.

In order to deal with manifolds with boundaries, a trace theorem is often required. Let \(\iota : \partial M \to \bar{M}\) be the inclusion map and thus \(\partial M\) admits an induced Riemannian metric \(\iota^*g\). The following version is due to Taylor [26].

**Proposition 2.6.** [26, Prop. 4.5] Let \((\bar{M}, g)\) be a compact manifold with smooth boundary \(\partial M\). Let \(u \in H^1(M,g)\). Then there exists a bounded linear operator
\[
T : H^1(M,g) \to L^2(\partial M, \iota^*g)
\]
such that
\[
Tu = u\big|_{\partial M}, \text{ if } u \in C^\infty(\bar{M}) \cap H^1(M,g);
\]
\[
\|Tu\|_{L^2(\partial M, \iota^*g)} \leq K'' \|u\|_{H^1(M,g)}.
\]
Here \(K''\) only depends on \((M, g)\) and is independent of \(u\). Furthermore, the map \(T : H^1(M,g) \to H^{\frac{1}{2}}(\partial M, \iota^*g)\) is surjective.

The following result, which is a local version of perturbed Yamabe equation with trivial Dirichlet boundary condition and a negative constant \(\beta < 0\), plays a central role in boundary Yamabe problem. The author proved this result in [29], and applied this result to proof Yamabe problem on closed manifolds.

**Proposition 2.7.** [29, Prop. 3.3] Let \((\Omega, g)\) be Riemannian domain in \(\mathbb{R}^n\), \(n \geq 3\), with \(C^\infty\) boundary, and with \(Vol_\delta(\Omega)\) and the Euclidean diameter of \(\Omega\) sufficiently small. Let \(\beta < 0\) be any negative constant. Assume \(S_g < 0\) within the small enough closed domain \(\bar{\Omega}\). Then for any \(\lambda > 0\) the following Dirichlet problem
\[
- a \Delta_g u + (S_g + \beta) u = \lambda u^{p-1} \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.
\]
has a real, positive solution \(u \in C_0(\bar{\Omega}) \cap H^1_0(\Omega, g)\) vanishes at \(\partial \Omega\).
3. Monotone Iteration Scheme on Closed Manifolds with Boundary

In this section, an $L^p$ regularity on compact manifolds with smooth boundary will be proven first, then this global $L^p$ regularity will be used to show the existence of solution of second order elliptic PDE on $(\bar{M}, g)$ with appropriate boundary conditions by monotone iteration scheme. Throughout the whole section, the existence of solutions, sub-solutions or super-solutions are assumed. Throughout this section, we assume that $\dim \bar{M} \geq 3$.

The first result is a global $L^p$-regularity with respect to the elliptic operator and oblique boundary conditions. This proof, essentially, is due to Agmon, Douglis and Nirenberg [1], although they only proved a local version.

**Theorem 3.1.** Let $(\bar{M}, g)$ be a compact manifold with smooth boundary $\partial M$. Let $\nu$ be the unit outward normal vector along $\partial M$. Let $L$ be a uniform second order elliptic operator on $M$ with smooth coefficients up to $\partial M$. Let $f \in L^p(M, g)$. Let $u \in H^1(M, g)$ be a weak solution of the following boundary value problem

$$Lu = f \text{ in } M, \frac{\partial u}{\partial \nu} + c(x)u = 0 \text{ on } \partial M. \tag{18}$$

Here $c \in C^\infty(M)$. If, in addition, $u \in L^p(M, g)$, then $u \in W^{2,p}(M, g)$ with the following estimates

$$\|u\|_{W^{2,p}(M, g)} \leq C \left( \|Lu\|_{L^p(M, g)} + \|u\|_{L^p(M, g)} \right). \tag{19}$$

Here $C$ depends on $L, p, c$ and the manifold $(\bar{M}, g)$ and is independent of $u$.

**Proof.** Choose a finite cover of $(\bar{M}, g)$, say

$$(\bar{M}, g) = \left( \bigcup_{\alpha} (U_{\alpha}, \phi_{\alpha}) \right) \cup \left( \bigcup_{\beta} (U_{\beta}, \phi_{\beta}) \right)$$

where $\{U_{\alpha}, \phi_{\alpha}\}$ are interior charts and $\{U_{\beta}, \phi_{\beta}\}$ are boundary charts. Choose a partition of unity $\{\chi_{\alpha}, \chi_{\beta}\}$ subordinate to this cover, where $\{\chi_{\alpha}\}$ are associated with interior charts and $\{\chi_{\beta}\}$ are associated with boundary charts. The local expression of the differential operator for interior charts is of the form

$L \mapsto (\phi_{\alpha}^{-1})^* L \phi_{\alpha} : C^\infty(\phi_{\alpha}(U_{\alpha})) \to C^\infty(\phi_{\alpha}(U_{\alpha}))$

which can be extended to Sobolev spaces with appropriate orders. The same expression applies for boundary charts. Denote

$$L_{\alpha} = (\phi_{\alpha}^{-1})^* L \phi_{\alpha}, L_{\beta} = (\phi_{\beta}^{-1})^* L \phi_{\beta}, \frac{\partial}{\partial \nu} = (\phi_{\beta}^{-1})^* \left( \frac{\partial}{\partial \nu} \right) \phi_{\beta}$$

$$(\phi_{\alpha}^{-1})^* \chi_{\alpha} = \chi'_{\alpha}, (\phi_{\beta}^{-1})^* \chi_{\beta} = \chi'_{\beta}; (\phi_{\beta}^{-1})^* c = c'_{\beta};$$

$$(\phi_{\alpha}^{-1})^* u = u'_{\alpha}, (\phi_{\beta}^{-1})^* f = f'_{\beta}$$

With these notations, the local expressions of our PDE with respect to $u'_{\alpha}, u'_{\beta}$ associated with (18) in each chart, respectively, are as follows:

$$L_{\alpha} \left( \chi'_{\alpha} u_{\alpha}' \right) - [L_{\alpha}, \chi'_{\alpha}] u_{\alpha}' = \chi'_{\alpha} f_{\alpha}' \text{ in } \phi_{\alpha}(U_{\alpha}), \chi'_{\alpha} u_{\alpha}' = 0 \text{ on } \partial \phi_{\alpha}(U_{\alpha});$$

$$L_{\beta} \left( \chi'_{\beta} u_{\beta}' \right) - [L_{\beta}, \chi'_{\beta}] u_{\beta}' = \chi'_{\beta} f_{\beta}' \text{ in } \phi_{\beta}(U_{\beta}),$$

$$\frac{\partial \chi'_{\beta} u_{\beta}'}{\partial \nu} + c'_{\beta} \chi'_{\beta} u_{\beta}' = \frac{\partial \chi'_{\beta}}{\partial \nu} u_{\beta}' \text{ on } \partial \phi_{\beta}(\bar{U}_{\beta} \cap \bar{M}), \chi'_{\beta} u_{\beta}' = 0 \text{ on } \partial \phi_{\beta}(U_{\beta}) \setminus (\partial \phi_{\beta}(\bar{U}_{\beta} \cap \bar{M})). \tag{20}$$

Here $[L, \chi]$ is a commutator defined as

$$[L, \chi]u = L(\chi u) - \chi(Lu).$$
Since $L$ is a second order differential operator, $[L, \chi]$ is a first order differential operator. Since the existence of solution of \((18)\) is assumed, we conclude that local PDEs in \((20)\) are solvable with $\chi_\alpha' u', \chi_\beta' u \in L^p \cap H^1$ in associated domains, respectively. The boundary conditions on boundary charts are also Robin condition satisfying Proposition 2.2.2. The following analysis is due to Melrose \[21\]. For interior chart, we take $\psi_\alpha \in C^\infty(M)$ such that $\psi_\alpha = 1$ on $\text{supp}(\chi_\alpha)$, denote $\psi_\alpha' = (\phi_\alpha^{-1})^* \psi_\alpha$, thus

\[
\|\chi_\alpha' u_\alpha\|_{W^{2,p}(\phi_\alpha(U_\alpha), g)} \leq C^* \left( \|L_\alpha (\chi_\alpha' u_\alpha)\|_{L^p(\phi_\alpha(U_\alpha), g)} + \|\chi_\alpha' u_\alpha\|_{L^p(\phi_\alpha(U_\alpha), g)} \right)
\]

\[
\leq C^* \left( \|\chi_\alpha' f_\alpha\|_{L^p(\phi_\alpha(U_\alpha), g)} + \|L_\alpha', \chi_\alpha' \psi_\alpha' u_\alpha\|_{L^p(\phi_\alpha(U_\alpha), g)} + \|\chi_\alpha' u_\alpha\|_{L^p(\phi_\alpha(U_\alpha), g)} \right)
\]

\[
\leq C^* \|\chi_\alpha' f_\alpha\|_{L^p(\phi_\alpha(U_\alpha), g)} + C^* \|L_\alpha', \chi_\alpha' \psi_\alpha' u_\alpha\|_{L^p(\phi_\alpha(U_\alpha), g)} + C^* \|\chi_\alpha' u_\alpha\|_{L^p(\phi_\alpha(U_\alpha), g)}
\]

\[
\leq C^* \|\chi_\alpha' f_\alpha\|_{L^p(\phi_\alpha(U_\alpha), g)} + C^* \|\chi_\alpha' u_\alpha\|_{L^p(\phi_\alpha(U_\alpha), g)} + C^* \|\chi_\alpha' u_\alpha\|_{W^{1,p}(\phi_\alpha(U_\alpha), g)} + C^* \|\chi_\alpha' u_\alpha\|_{L^p(\phi_\alpha(U_\alpha), g)}
\]

For boundary chart, we take $\psi_\beta$ correspondingly, and have

\[
\|\chi_\beta' u_\beta\|_{W^{2,p}(\phi_\beta(U_\beta), g)} \leq C^* \left( \|L_\beta (\chi_\beta' u_\beta)\|_{L^p(\phi_\beta(U_\beta), g)} + \left\| \frac{\partial \chi_\beta'}{\partial \nu} u_\beta \right\|_{W^{1,p}(\phi_\beta(U_\beta), g)} \right)
\]

\[
\leq C^* \|\chi_\beta' f_\beta\|_{L^p(\phi_\beta(U_\beta), g)} + C^* \|L_\beta', \chi_\beta' \psi_\beta' u_\beta\|_{L^p(\phi_\beta(U_\beta), g)}
\]

\[
+ C^* \left\| \frac{\partial \chi_\beta'}{\partial \nu} \psi_\beta' u_\beta \right\|_{W^{1,p}(\phi_\beta(U_\beta), g)} + C^* \|\chi_\beta' u_\beta\|_{L^p(\phi_\beta(U_\beta), g)}
\]

\[
\leq C^* \|\chi_\beta' f_\beta\|_{L^p(\phi_\beta(U_\beta), g)} + C^* \|\chi_\beta' \psi_\beta' u_\beta\|_{L^p(\phi_\beta(U_\beta), g)} + C^* \|\chi_\beta' u_\beta\|_{L^p(\phi_\beta(U_\beta), g)} + C^* \|\chi_\beta' u_\beta\|_{W^{1,p}(\phi_\beta(U_\beta), g)} + C^* \|\chi_\beta' u_\beta\|_{L^p(\phi_\beta(U_\beta), g)}
\]

Note that this estimate on boundary chart is legitimate, due to Remark 2.1.1 thanks to [1]. By local estimates, we conclude that $u \in W^{2,p}(M, g)$, since each $\chi_\alpha u, \chi_\beta u$ is in $W^{2,p}$. Sum them up, we have

\[
\|u\|_{W^{2,p}(M, g)} = \sum_\alpha \|\chi_\alpha u_\alpha\|_{W^{2,p}(\phi_\alpha(U_\alpha), g)} + \sum_\beta \|\chi_\beta u_\beta\|_{W^{2,p}(\phi_\beta(U_\beta), g)}
\]

\[
\leq \sum_\alpha \left( C^* \|\chi_\alpha' f_\alpha\|_{L^p(\phi_\alpha(U_\alpha), g)} + C_{\alpha,\alpha} \|\psi_\alpha' u_\alpha\|_{W^{1,p}(\phi_\alpha(U_\alpha), g)} + C^* \|\chi_\alpha' u_\alpha\|_{L^p(\phi_\alpha(U_\alpha), g)} \right)
\]

\[
+ \sum_\beta \left( C^* \|\chi_\beta' f_\beta\|_{L^p(\phi_\beta(U_\beta), g)} + C_{\beta,\beta} \|\psi_\beta' u_\beta\|_{W^{1,p}(\phi_\beta(U_\beta), g)} + C^* \|\chi_\beta' u_\beta\|_{L^p(\phi_\beta(U_\beta), g)} \right)
\]

\[
\leq C^* \|f\|_{L^p(M, g)} + C^* \|u\|_{L^p(M, g)} + C^* \|u\|_{W^{1,p}(M, g)}
\]

The constant $C^*_1$ depends in particular on the choice of finite cover, $\chi_\alpha, \chi_\beta$ and $\psi_\alpha, \psi_\beta$. Applying “Peter-Paul” inequality, we have

\[
\|u\|_{W^{2,p}(M, g)} \leq C^* \|f\|_{L^p(M, g)} + C^* \|u\|_{L^p(M, g)} + C_{1,\gamma} \|u\|_{W^{2,p}(M, g)} + C^*_1 \gamma^{-1} \|u\|_{L^p(M, g)}
\]

Taking $\gamma$ small enough so that we can combine $C^*_1 \gamma \|u\|_{W^{2,p}(M, g)}$ to the left side of the inequality above. With an appropriate choice of $C$, which depends on $\gamma, p, L, c(x)$, $(M, g)$, and the partition of unity, we have

\[
\|u\|_{W^{2,p}(M, g)} \leq C \left( \|Lu\|_{L^p(M, g)} + \|u\|_{L^p(M, g)} \right).
\]
Next we show that the last term \( \|u\|_{L^p(M,g)} \) can be removed when \( L \) is an injective operator on \( W^{2,p}(M,g) \). The following argument is an analogy of [11] §7, Remark 2.

**Theorem 3.2.** Let \((M,g)\) be a compact manifold with smooth boundary \( \partial M \). Let \( \nu \) be the unit outward normal vector along \( \partial M \) and \( p > \dim M \). Let \( L : C^\infty(M) \to C^\infty(M) \) be a uniform second order elliptic operator on \( M \) with smooth coefficients up to \( \partial M \) and can be extended to \( L : W^{2,p}(M,g) \to L^p(M,g) \). Let \( f \in L^p(M,g) \). Let \( u \in H^1(M,g) \) be a weak solution of the following boundary value problem

\[
Lu = f \text{ in } M, \quad \frac{\partial u}{\partial \nu} + c(x)u = 0 \text{ on } \partial M.
\]

Here \( c \in C^\infty(M) \). Assume also that \( \text{Ker}(L) = \{0\} \) associated with this boundary condition. If, in addition, \( u \in L^p(M,g) \), then \( u \in W^{2,p}(M,g) \) with the following estimate

\[
\|u\|_{W^{2,p}(M,g)} \leq C\|Lu\|_{L^p(M,g)}.
\]

Here \( C \) depends on \( L, p, c \) and the manifold \((M,g)\) and is independent of \( u \).

**Proof.** Due to (19), it is suffice to show that there exists some constant \( D \) such that

\[
\|u\|_{L^p(M,g)} \leq D \|Lu\|_{L^p(M,g)}
\]

for all \( u \in W^{2,p}(M,g) \) satisfying the boundary condition in (21). Since \( C^\infty(M) \) is dense in \( W^{2,p}(M,g) \), we show this by assuming, without loss of generality, \( u \in C^\infty(M) \). We show this by contradiction. Suppose that (23) does not hold. Then there exists a sequence \( \{u_n\} \subset C^\infty(M) \), normalized with \( \|u_n\|_{L^p(M,g)} = 1, \forall k \in \mathbb{Z}_{\geq 0} \) such that

\[
\|u_n\|_{L^p(M,g)} \geq n\|Lu_n\|_{L^p(M,g)} \Rightarrow \|Lu_n\|_{L^p(M,g)} \leq \frac{1}{n}, n \in \mathbb{Z}_{\geq 0}.
\]

It follows from estimate in (19) that

\[
\|u_n\|_{W^{2,p}(M,g)} \leq C \left( \|Lu_n\|_{L^p(M,g)} + \|u_n\|_{L^p(M,g)} \right) \leq 2C, \forall n \in \mathbb{Z}_{\geq 0}.
\]

Thus we obtain a sequence \( \{u_n\} \) that is uniformly bounded with \( W^{2,p} \)-norm. Since \( p > \dim M \), Sobolev embedding in Proposition 2.5 implies that a subsequence of \( \{u_n\} \subset C^{1,\alpha}(M) \), say \( \{u_{n_k}\} \subset C^{1,\alpha}(M) \) converges pointwisely to some limit \( u \) with the property that

\[
\lim_{n_k \to \infty} u_{n_k} = Lu = 0, \lim_{n_k \to \infty} u_{n_k} = u, \|u\|_{L^p(M,g)} = 1.
\]

Since \( \text{Ker}(L) \) is trivial, it follows that \( u = 0 \) due to \( Lu = 0 \). Thus \( \|u\|_{L^p(M,g)} = 0 \), which contradicts above. Therefore we conclude that (23) holds. Applying (23) to (19), we conclude that

\[
\|u\|_{W^{2,p}(M,g)} \leq C \left( \|Lu\|_{L^p(M,g)} + \|u\|_{L^p(M,g)} \right) \leq C(1 + D)\|Lu\|_{L^p(M,g)} := C'\|Lu\|_{L^p(M,g)}.
\]

Here \( C' \) is, in particular, independent of \( u \). \( \square \)

From now on, we consider the special case \( L = -a \Delta_g \). Recall that \( a = \frac{4(n-1)}{n-2} \). With the help of Theorem 3.2, we can get a result related to the existence of the solution \( -a \Delta_g u + f(x,u) = 0 \) on \((M,g)\) with oblique boundary condition \( \frac{\partial u}{\partial \nu} + c(x)u = 0 \) on \( \partial M \) provided the existence of some sub-solution and super-solution of the above PDE. This result is an extension of the old result due to Sattinger [25].
Theorem 3.3. Let \((\tilde{M}, g)\) be a compact manifold with smooth boundary \(\partial M\). Let \(\nu\) be the unit outward normal vector along \(\partial M\) and \(p > \dim M\). Let \(-a\Delta_g : C^\infty(\tilde{M}) \to C^\infty(\tilde{M})\) be a uniform second order elliptic operator on \(M\). Let \(F(x, u) : M \times \mathbb{R} \to \mathbb{R}\) be a function smooth in \(x\) and \(C^1\) in \(u\). Furthermore, assume that there exists a positive constant \(A_0\) such that the operator \(-a\Delta_g + A\) is injective on \(C^\infty(\tilde{M})\) for all \(A > A_0\). In addition, assume that the second order linear elliptic PDE

\[
(-a\Delta_g + A)u = f \text{ in } M, \quad \frac{\partial u}{\partial \nu} + c(x)u = 0 \text{ on } \partial M
\]

has a unique solution \(u \in H^1(M, g)\), for all \(A > A_0\) with \(f \in L^p(M, g)\) and \(c \in C^\infty(\tilde{M})\) with \(c(x) > 0\) on \(\partial M\). Suppose that there exist \(u_-, u_+ \in C_0(\tilde{M}) \cap H^1(M, g)\), \(u_- \leq u_+\) such that

\[
-a\Delta_g u_- - F(x, u_-) \leq 0 \text{ in } M, \quad \frac{\partial u_-}{\partial \nu} + c(x)u_- \leq 0 \text{ on } \partial M;
\]

\[
-a\Delta_g u_+ - F(x, u_+) \geq 0 \text{ in } M, \quad \frac{\partial u_+}{\partial \nu} + c(x)u_+ \geq 0 \text{ on } \partial M
\]

holds weakly. Then there exists a solution \(u \in W^{2,p}(M, g)\) of

\[
-a\Delta_g u - F(x, u) = 0 \text{ in } M, \quad \frac{\partial u}{\partial \nu} + c(x)u = 0 \text{ on } \partial M.
\]

Proof. Since \(F(x, u)\) is smooth in both variables, we observe that \(\frac{\partial F}{\partial u}(x)\) is bounded below for all \(x \in \tilde{M}\) and \(u \in [\min_{\tilde{M}} u_-, \max_{\tilde{M}} u_+]\). It follows that we can choose \(A > A_0\) such that

\[
\frac{\partial F}{\partial u}(x, u) + A > 0, \quad \forall x \in \tilde{M}, \forall u \in [\min_{\tilde{M}} u_-, \max_{\tilde{M}} u_+].
\]

Choose \(u_0 = u_+\). Set

\[
(-a\Delta_g + A)u_1 = F(x, u_0) + Au_0 \text{ in } M, \quad \frac{\partial u_1}{\partial \nu} + c(x)u_1 = 0 \text{ on } \partial M.
\]

By hypotheses in the statement, such \(u_1 \in H^1(M, g)\) does exist. By assumption \(-a\Delta_g + A\) is injective, also note that \(u_0 = u_+ \in C_0(\tilde{M})\) and thus \(u_0 \in L^p(M, g)\), thus by Theorem 3.2 we conclude that

\[
\|u_1\|_{W^{2,p}(M, g)} \leq C'\|F(x, u_0) + Au_0\|_{L^p(M, g)} \Rightarrow u_1 \in W^{2,p}(M, g) \Rightarrow u_1 \in C^{1,\alpha}(M, g)
\]

for some \(\alpha < 1 - \frac{2}{p}\) by Sobolev embedding. More importantly, \(u_1 \leq u_0\). To see this, we have

\[
(-a\Delta_g + A)u_0 \geq F(x, u_0) + Au_0
\]

\[
(-a\Delta_g + A)u_1 = F(x, u_0) + Au_0.
\]

Taking subtraction between two formulas above, we have

\[
(-a\Delta_g + A)(u_0 - u_1) \geq 0, \quad B_g(u_0 - u_1) = \frac{\partial(u_0 - u_1)}{\partial \nu} + c(x)(u_0 - u_1) \geq 0.
\]

We claim that

\[
u_0 \geq u_1 \text{ in } \tilde{M}.
\]

To see this, we define \(w = \max\{0, u_1 - u_0\}\). Since \(u_1 - u_0 \in H^1(M, g)\) so is \(w\). Furthermore \(w \geq 0\). In addition, we have \((-a\Delta_g + A)w \leq 0\). Thus we have

\[
0 \geq \int_M w(-a\Delta_g u + A)w \, d\omega = \int_M (a|\nabla w|^2 +Aw^2) \, d\omega + \int_{\partial M} c(x)w^2 \, dS \geq 0.
\]

Thus we must have \(w \equiv 0\), which follows that \(u_1 - u_0 \leq 0\). Hence the claim in (29) holds. By a similar argument, we see that \(u_1 \geq u_-\). Inductively, we take

\[
(-a\Delta_g + A)u_k = F(x, u_{k-1}) + Au_{k-1} \text{ in } M, \quad \frac{\partial u_k}{\partial \nu} + c(x)u_k = 0 \text{ on } \partial M, \quad k \in \mathbb{N}.
\]
By the same argument as above, we conclude that \( u_k \in W^{2,p}(M, g) \) and hence \( u_k \in C^{1,\alpha}(M, g) \) for the same choice of \( \alpha \) as above. We show that \( u_{k+1} \leq u_k \) by assuming that \( u_- \leq u_k \leq u_{k-1} \leq u_+ \) inductively. Observe that
\[
(-a\Delta_g + A)u_{k+1} = F(x, u_k) + Au_k;
\]
\[
(-a\Delta_g + A)u_k = F(x, u_{k-1}) + Au_{k-1}.
\]
Taking subtraction, we have
\[
(-a\Delta_g + A)(u_{k+1} - u_k) = F(x, u_k) + Au_k - F(x, u_{k-1}) - Au_{k-1}.
\]
Based on the choice of \( A \) in \( 26 \), we observe that \( F(x, u_k) + Au_k \leq F(x, u_{k-1}) - Au_{k-1} \) due to mean value theorem as well as the inductive assumption that \( u_- \leq u_k \leq u_{k-1} \leq u_+ \). It follows that
\[
(-a\Delta_g + A)(u_{k+1} - u_k) \leq 0 \Rightarrow u_{k+1} \leq u_k, \forall k \in \mathbb{N}.
\]
By comparing the equations with inductive assumption \( u_{k-1} \geq u_- \),
\[
(-a\Delta_g + A)u_k = F(x, u_{k-1}) + Au_{k-1};
\]
\[
(-a\Delta_g + A)u_- \leq F(x, u_-) + Au_-,
\]
we conclude by the same argument above that
\[
u_k \geq u_- , \forall k \in \mathbb{N}.
\]
Combining \( 30 \), \( 31 \) and \( 32 \), we conclude that
\[
u_- \leq \ldots \leq u_{k+1} \leq u_k \leq u_{k-1} \leq u_{k-2} \leq \ldots \leq u_+ , u_k \in W^{2,p}(M, g), \forall k \in \mathbb{N}.
\]
By Sobolev embedding, we conclude that \( u_k \in C^{1,\alpha}(M, g) \) for all \( k \in \mathbb{N} \). Furthermore, we observe from \( 22 \) that
\[
\|u_k\|_{W^{2,p}(M, g)} \leq C\|(-a\Delta_g + A)u_{k-1}\|_{L^p(M, g)}.
\]
Since \( u_- \leq u_k \leq u_+, \forall k \in \mathbb{N} \), \( \|(-a\Delta_g + A)u_{k-1}\|_{L^p(M, g)} \) has a uniform upper bound, and thus \( \|u_k\| \) is uniformly bounded in \( W^{2,p} \)-norm. When \( \alpha < 1 - \frac{2}{n} \), the embedding \( C^{1,\alpha}(M) \hookrightarrow W^{2,p}(M, g) \) is a compact embedding and thus a subsequence of \( u_k \) converge to some limit \( u \). According to the chain of inequalities in \( 33 \), the whole sequence \( u_k \) converges to \( u \) in \( C^{1,\alpha}(M) \) and hence in \( W^{2,p} \)-sense. Taking the limit, we have
\[
-a\Delta_g u - F(x, u) = 0 \text{ in } M, \frac{\partial u}{\partial \nu} + c(x)u = 0 \text{ on } \partial M.
\]
The boundary condition is achieved in the trace sense. Finally local Schauder estimates [11, Thm. 7.2, Thm. 7.3] indicates that \( u \in C^{2,\alpha}(\bar{M}) \) since regularity is a local property.

4. Solvability of \(-\Lambda\Delta_g u + \Lambda' u = f \) with Oblique Boundary Condition

We expect to apply results in §3 to solve boundary Yamabe problem. In order to apply Theorem \ref{thm:3.3}, we need to show the existence of the weak solution of the following PDE
\[
-\Lambda\Delta_g u + \Lambda' u = f \text{ in } M, Bu := \frac{\partial u}{\partial \nu} + c(x)u = 0 \text{ on } \partial M \tag{34}
\]
with appropriate choices of \( \Lambda, \Lambda' \), provided that \( f \in L^2(M, g) \) and \( c \in C^\infty(\bar{M}) \). A standard Lax-Milgram \cite{18} Ch. 6 will be applied to verify the solvability of \( 34 \). Throughout this section, we denote \( \iota : \partial M \to \bar{M} \) to be the standard inclusion map. The first result below does not require the positivity of \( c \) on \( \partial M \).
Theorem 4.1. Let \((\bar{M}, g)\) be a compact manifold with smooth boundary \(\partial M\). Let \(\nu\) be the unit outward normal vector along \(\partial M\). Let \(f \in L^2(\bar{M}, g)\) and \(c \in C^\infty(\bar{M})\). Then for large enough \(\Lambda, \Lambda'\), \((34)\) has a unique weak solution \(u \in H^1(M, g)\). Furthermore, the operator \(-\Lambda \Delta_g + \Lambda'\) is injective, i.e. \(\text{Ker}(-\Lambda \Delta_g + \Lambda') = \{0\}\) with respect to the oblique boundary condition in \((34)\).

Proof. We consider the Lax-Milgram with respect to \(H^1(M, g)\). Note that the boundary condition is defined in the weak sense by pairing \(H^{-\frac{1}{2}}(\partial M, \nu^*g) \times H^{\frac{1}{2}}(\partial M, \nu^*g)\) with the divergence theorem and \((34)\)

\[
\langle Bu, w \rangle = \int_{\partial M} \left( \frac{\partial u}{\partial \nu} + c(x)u \right) dv + \int_M (\nabla_g v \cdot \nabla_g u + (\Delta_g u) v) d\omega + \int_{\partial M} cu dv.
\]

Due to the trace theorem in Proposition 2.6, we have

\[
\int_{\partial M} c(x) u^2 dS \leq \sup_{\partial M} |c| \int_{\partial M} u^2 dS = \sup_{\partial M} |c| \|u\|^2_{L^2(\partial M, \nu^*g)} \leq \sup_{\partial M} |c| \|(K'')^2\|u\|_{H^1(M, g)}.
\]

Note that the constant \(\sup_{\partial M} |c| \|(K'')^2\) only depends on \(c\) and \((\bar{M}, g)\) and is independent of \(u\). Thus we choose \(\Lambda, \Lambda'\) such that

\[
\Lambda > \sup_{\partial M} |c| \|(K'')^2 + 1, \Lambda' > \sup_{\partial M} |c| \|(K'')^2 + 1.
\]

We observe that the bilinear form of \((34)\) is

\[
B[u, v] = \int_M (\Lambda \nabla_g v \cdot \nabla_g u + \Lambda' uv) d\omega + \int_{\partial M} cu dv, \forall v \in H^1(M, g).
\]

We have

\[
|B[u, v]| \leq \Lambda \|\nabla_g v\|_{L^2(M, g)} \|\nabla_g u\|_{L^2(M, g)} + \Lambda' \|u\|_{L^2(M, g)} \|v\|_{L^2(M, g)}
\]

\[
\leq \sup_{\partial M} |c| \|u\|_{L^2(\partial M, \nu^*g)} \|v\|_{L^2(\partial M, \nu^*g)}
\]

for some constant \(D_1\). On the other hand we apply \((36)\),

\[
B[u, u] = \int_M \Lambda \|\nabla_g u\|^2_{L^2(M, g)} + \Lambda' \|u\|^2_{L^2(M, g)} + \int_{\partial M} c(x) u^2 dS
\]

\[
\geq \Lambda \|\nabla_g u\|^2_{L^2(M, g)} + \Lambda' \|u\|^2_{L^2(M, g)} - \sup_{\partial M} |c| \|u\|^2_{L^2(\partial M, \nu^*g)}
\]

\[
\geq \Lambda \|\nabla_g u\|^2_{L^2(M, g)} + \Lambda' \|u\|^2_{L^2(M, g)} - \sup_{\partial M} |c| \|(K'')^2\|u\|_{H^1(M, g)}^2
\]

\[
= \left( \Lambda - \sup_{\partial M} |c| \|(K'')^2\right) \|\nabla_g u\|_{L^2(M, g)} + \left( \Lambda' - \sup_{\partial M} |c| \|(K'')^2\right) \|u\|_{L^2(M, g)}^2
\]

\[
\geq \|u\|_{H^1(M, g)}^2.
\]

Hence the hypotheses for Lax-Milgram theorem satisfied. Applying Lax-Milgram theorem, we conclude that there exists some \(u \in H^1(M, g)\) that solves \((34)\) weakly.

If we have

\[-\Lambda \Delta_g u + \Lambda' u = 0 \text{ in } M, Bu = 0 \text{ on } \partial M,\]
we pair both sides with \( u \), a very similar argument as above implies that
\[
\int_M \left( -\Lambda \Delta_g u + \Lambda' u \right) u = 0 \Rightarrow \Lambda \| \nabla_g u \|_{L^2(M,g)} + \Lambda' \| u \|_{L^2(M,g)} + \int_{\partial M} cu^2 dS = 0
\]
\[
\Rightarrow 0 \geq \left( \Lambda - \sup_{\partial M} |c(K'')^2 \right) \| \nabla_g u \|_{L^2(M,g)} + \left( \Lambda' - \sup_{\partial M} |c(K'')^2 \right) \| u \|_{L^2(M,g)}^2 \geq \| u \|_{H^1(M,g)}^2
\]
\[
\Rightarrow u \equiv 0.
\]
Hence the operator \(-\Lambda \Delta_g u + \Lambda' u : C^\infty(\tilde{M}) \rightarrow C^\infty(\tilde{M})\) is injective. The extension of this operator to any positive integer order Sobolev spaces is still injective. Note that the solvability of (34) and the injectivity of the operator still hold for any larger \( \Lambda, \Lambda' \).

If we further assume \( c > 0 \) on \( \partial M \), then the above result holds for any positive \( \Lambda \) and any nonnegative \( \Lambda' \). It turns to be a global extension of a local result in Gilbarg and Trudinger [10, Thm. 6.31]. With the condition \( c > 0 \) on \( \partial M \), the following result plays a key role in applying Theorem 3.3.

**Theorem 4.2.** Let \((\tilde{M}, g)\) be a compact manifold with smooth boundary \( \partial M \). Let \( \nu \) be the unit outward normal vector along \( \partial M \). Let \( a, A > 0 \) be any constants. Let \( f \in L^2(M,g) \) and \( c \in C^\infty(\tilde{M}) \) with \( c > 0 \) on \( \partial M \). Then the following PDE
\[
-a \Delta_g u + Au = f \text{ in } M, \quad \frac{\partial u}{\partial \nu} + c(x) u = 0 \text{ on } \partial M \tag{38}
\]
has a unique weak solution \( u \in H^1(M,g) \). Furthermore, \(-a \Delta_g + A\) is injective with this oblique boundary condition.

**Proof.** The bilinear form, similar to (37), is
\[
B[u, v] = \int_M (a \nabla_g u \cdot \nabla_g v + Au v) \, d\omega + \int_{\partial M} cu v dS, \forall v \in H^1(M,g).
\]
The upper bound of \( |B[u, v]| \) is exactly the same as in Theorem 3.3 with different constant. The lower bound of \( B[u, u] \) is
\[
B[u, u] = a \| \nabla_g u \|_{L^2(M,g)}^2 + A \| u \|_{L^2(M,g)}^2 + \int_{\partial M} cu^2 dS \geq \min(a, A) \| u \|_{H^1(M,g)}^2.
\]
Since \( \min(a, A) > 0 \), it follows from Lax-Milgram that (38) has a unique weak solution \( u \in H^1(M,g) \). For injectivity, we observe that if \( B[u, u] = 0 \) then \( \| u \|_{H^1(M,g)} = 0 \) hence \( u \equiv 0 \).

5. **Boundary Yamabe Problem with Minimal Boundary Case**

Recall the boundary value problem associated with boundary Yamabe problem for minimal boundary case.
\[
\square_g u := -a \Delta_g u + S_g u = \lambda u^{p-1} \text{ in } M; \quad B_g u := \frac{\partial u}{\partial \nu} + \frac{2}{p-2} h_g u = 0 \text{ on } \partial M. \tag{40}
\]
In this section, the author would apply the sub-solution and super-solution technique in Theorem 3.3 to solve boundary Yamabe equation for five cases:

(A). \( \eta_1 = 0 \);
(B). \( \eta_1 < 0 \) with \( h_g > 0 \) everywhere on \( \partial M \) and arbitrary \( S_g \);
(C). \( \eta_1 < 0 \) with arbitrary \( h_g \) and \( S_g \);
(D). \( \eta_1 > 0 \) with \( h_g > 0 \) everywhere and \( S_g < 0 \) somewhere;
(E). \( \eta_1 > 0 \) with arbitrary \( h_g \) and \( S_g \).
Throughout this section, we assume dim $\bar{M} \geq 3$. We always assume that $(\bar{M}, g)$ be a compact manifold with smooth boundary $\partial M$ and $\nu$ be the unit outward normal vector along $\partial M$. Let $S_g$ be the scalar curvature and $h_g$ be the mean curvature on $\partial M$. Let $\bar{S}$ and $\bar{h}$ be the scalar and mean curvature with respect to $\bar{g}$ under conformal change. Note that case (B) is a special scenario of (C), and case (D) is a special scenario of case (E). As in closed manifold case [29], we need an extra step when $\eta > 0$: we need to solve the perturbed boundary Yamabe equation with $\beta < 1$

$$- a \Delta_g u_{\beta} + (S_g + \beta) u_{\beta} = \lambda_{\beta} u_{\beta}^{p-1} \text{ in } M, \frac{\partial u_{\beta}}{\partial \nu} + \frac{2}{p-2} h_g u_{\beta} = 0 \text{ on } \partial M$$

(41)

where $\lambda_{\beta}$ is defined to be

$$\lambda_{\beta} = \inf_{u \neq 0} \frac{\int_M |\nabla_g u|^2 d\omega + \int_M (S_g + \beta) u^2 d\omega + \int_{\partial M} \frac{2}{p-2} h_g u^2 dS}{(\int_M u^p d\omega)^\frac{2}{p}}.$$  

(42)

Then we take the limit by $\beta \to 0^-$ to obtain the solution of boundary Yamabe problem.

We need the following results for the eigenvalue problem with respect conformal Laplacian $\Box_g$ and Robin condition.

**Theorem 5.1.** [8 Lemma 1.1] Let $(\bar{M}, g)$ be a compact manifold with boundary. The following eigenvalue problem

$$- a \Delta_g \varphi + S_g \varphi = \eta_1 \varphi \text{ in } M, \frac{\partial \varphi}{\partial \nu} + \frac{2}{p-2} h_g \varphi = 0 \text{ on } \partial M.$$  

(43)

has a real, smooth, positive solution $\varphi \in \mathcal{C}^\infty(\bar{M})$.

In order to apply sub- and super-solutions of the perturbed boundary Yamabe equation, we need a eigenvalue problem result for the perturbed conformal Laplacian

$$\Box_{g, \beta} u = -a \Delta_g u + (S_g + \beta) u.$$  

Next result shows that when $\beta < 0$ with small enough absolute value, the first eigenvalue $\eta_{1, \beta} > 0$ of $\Box_{g, \beta}$ provided $\eta_1 > 0$, with respect to the same Robin condition.

**Lemma 5.1.** Let $\eta_1 > 0$ be the first eigenvalue as in Theorem 5.1. Let $\beta < 0$ be a constant. Then the first eigenvalue $\eta_{1, \beta}$ for the operator

$$\Box_{g, \beta} u = -a \Delta_g u + (S_g + \beta) u$$

with the same Robin condition in (43) is also positive provided that $|\beta|$ is small enough.

**Proof.** The first eigenvalues $\eta_1$ and $\eta_{1, \beta}$ are characterized by

$$\eta_1 = \inf_{\varphi \neq 0} \frac{\int_M |\nabla_g \varphi|^2 d\omega + \int_M S_g \varphi^2 d\omega + \int_{\partial M} \frac{2}{p-2} h_g \varphi^2 dS}{\int_M \varphi^2 d\omega};$$

$$\eta_{1, \beta} = \inf_{\varphi \neq 0} \frac{\int_M |\nabla_g \varphi|^2 d\omega + \int_M (S_g + \beta) \varphi^2 d\omega + \int_{\partial M} \frac{2}{p-2} h_g \varphi^2 dS}{\int_M \varphi^2 d\omega}.$$  

The rest arguments are almost same as in Lemma 4.1 of [29], except a fixed extra term $\int_{\partial M} \frac{2}{p-2} h_g \varphi^2 dS$. We omit the details here. \qed

When $\eta_1 = 0$ in case (A), we can solve Yamabe problem trivially with $\lambda = 0$, this is just an eigenvalue problem. It is worth mentioning that generically zero is not an eigenvalue of conformal Laplacian $\Box$, see [11].

**Corollary 5.1.** Let $(\bar{M}, g)$ be a compact manifold with boundary and $\eta_1 = 0$. Then the boundary Yamabe equation (40) has a real, positive, smooth solution with $\lambda = 0$. 
Proof. It is an immediate consequence of Theorem 5.1.

Now we consider the case when $\eta_1 < 0$, $h_g > 0$ everywhere on $\partial M$. The sign of $S_g$ is not required.

**Theorem 5.2.** Let $(\bar{M},g)$ be a compact manifold with boundary. Let $h_g > 0$ everywhere on $\partial M$. When $\eta_1 < 0$, there exists some $\lambda < 0$ such that the boundary Yamabe equation (40) has a real, positive solution $u \in C^\infty(\bar{M})$.

**Proof.** We apply Theorem 3.3 to construct sub-solution and super-solution here. By Theorem 4.2, any $A_0 > 0$ in hypotheses of Theorem 3.3 works for the solvability of the linear iterations in (30).

We construct the sub-solution first and make choice of $\lambda$. By Theorem 5.1, there exists some $\varphi$ such that

$$-a\Delta_g \varphi + S_g \varphi = \eta_1 \varphi \text{ in } M, \frac{\partial \varphi}{\partial \nu} + \frac{2}{p-2} h_g \varphi = 0 \text{ on } \partial M.$$  

Scaling $\varphi$ by $t\varphi$, we may, without loss of generality, assume that $\sup_{\bar{M}} \varphi < 1$. Since $p-1 > 1$, we have $\varphi^{p-1} \leq \varphi$. Since $\eta_1 < 0$, we can choose $\lambda \in (\eta_1,0)$ and have

$$-a\Delta_g \varphi + S_g \varphi = \eta_1 \varphi \leq \eta_1 \varphi^{p-1} \leq \lambda \varphi^{p-1}. \quad (44)$$

Define

$$F(x,u) := -S_g u + \lambda u^{p-1} : \bar{M} \to \mathbb{R}. \quad (45)$$

It is clear that $F(x,u)$ is smooth in $x$ and $C^1$ in $u$. Set

$$u_- := \varphi > 0, u_- \in C^\infty(\bar{M}).$$

It follows that $u_-$ satisfies

$$-a\Delta_g u_- \leq F(x,u_-) \text{ in } M, B_g u_- \leq 0 \text{ on } \partial M. \quad (46)$$

Next we construct the super-solution. Choose a constant $K_1 > 0$ that is large enough so that for the choice of $\lambda < 0$ in (44),

$$K_1^{p-2} \geq \max \left\{ \frac{\inf_{\bar{M}} S_g}{\lambda}, \sup_{\bar{M}} u_-^{p-2} \right\}. \quad (47)$$

Note that if $S_g \geq 0$ everywhere then the first quantity on the right side above is nonpositive. Set

$$u_+ := K_1, u_+ \in C^\infty(\bar{M}).$$

Since we assume that $h_g \geq 0$, we check that

$$-a\Delta_g u_+ + S_g u_+ - \lambda u_+^{p-1} = K_1(S_g - \lambda K_1^{p-2}) \geq 0 \text{ in } M;$$

$$B_g u_+ = \frac{\partial u_+}{\partial \nu} + \frac{2}{p-2} h_g u_+ = \frac{2}{p-2} h_g K_1 \geq 0 \text{ on } \partial M.$$ 

Hence

$$-a\Delta_g u_+ \geq F(x,u_+) \text{ in } M, B_g u_+ \geq 0 \text{ on } \partial M. \quad (48)$$

In order to apply Theorem 3.3, we need to choose $A$ such that (26) is satisfied. In addition, the choice of $A$ must guarantee the solvability of (27). Here we choose $A$ large enough such that

$$A + \sup_{x \in \bar{M}, u \in [\min_{\bar{M}} u_-, \max_{\bar{M}} u_+]} \frac{\partial F(x,u)}{\partial u} > 0, A > 0. \quad (49)$$

Here $F$ is of the expression in (45). Note that with this choice of $A$, the operator $-a\Delta_g u + A$ is injective. Thus all hypotheses in Theorem 3.3 are satisfied. By (46), (47) and (48), It follows that there exists $u \in W^{2,p}(M,g), p > \dim \bar{M}$ and $0 < u_- \leq u \leq u_+$ such that

$$-a\Delta_g u = F(x,u) \text{ in } M, B_g u = 0 \text{ on } \partial M.$$
It is immediate that
\[-a \Delta_g u = F(x, u) = -S_g u + \lambda u^{p-1} \Rightarrow -a \Delta_g u + S_g u = \lambda u^{p-1}.
\]
Furthermore, a standard bootstrapping argument with Sobolev embedding and Schauder estimates implies that \(u \in C^\infty(\bar{M})\), see e.g. [24, Thm. 2.8] since the regularity is a local argument. □

It is not always possible that \(h_g \geq 0\) everywhere on \(\partial M\). To handle the general case for arbitrary \(h_g, S_g\) with \(\eta_1 < 0\), we show that there exists some conformal metric \(\tilde{g}\) such that \(\tilde{h} \geq 0\) everywhere.

The next result works for both signs of \(\eta_1\). Note that by Proposition 2.1, \(\text{sgn}\tilde{\eta}_1 = \text{sgn}(\eta_1)\) after conformal change.

**Theorem 5.3.** Let \((\bar{M}, g)\) be a compact manifold with boundary. There exists a conformal metric \(\tilde{g}\) associated with mean curvature \(\tilde{h} > 0\) everywhere on \(\partial M\).

**Proof.** By scaling the metric \(g\), we can, without loss of generality, assume that \(\frac{2}{p-2} \sup_{\partial M} |h_g| \leq 1\).

We start with this metric. Pick up a positive function \(H \in C^\infty(\partial M)\) such that \(\inf_{\partial M} H \geq 2\). We can always find a smooth function \(W \in C^\infty(\bar{M})\) such that \(\frac{\partial W}{\partial \nu} = H\) on \(\partial M\).

This \(W\) can be obtained by the unique smooth solution of the PDE
\[-\Delta_g W + W = 0 \text{ in } M, \frac{\partial W}{\partial \nu} = H \text{ on } \partial M.\]

Let \(\tilde{W}(x) = e^x\) as a map between \(\mathbb{R}\). From the following composition
\[\bar{M} \xrightarrow{W} \mathbb{R} \xrightarrow{e^x} \mathbb{R}\]

We define
\[u = \tilde{W} \circ W = e^W : \bar{M} \to \mathbb{R}.\] (49)

Clearly \(u > 0\) and is smooth on \(\bar{M}\); in addition \(u\) has the same expression by Proposition 2.1.

By (49), we define
\[\tilde{h} = \frac{p-2}{2} u^\frac{2}{p} \left( \frac{\partial u}{\partial \nu} + \frac{2}{p-2} h_g u \right) \text{ on } \partial M;\]
\[\tilde{S} = u^{-p} (-a \Delta_g u + S_g u) \text{ in } M.\] (50)

Definitions in (50) implies that the following boundary value problem
\[-a \Delta_g u + S_g u = \tilde{S} u^{p-1} \text{ in } M; \frac{\partial u}{\partial \nu} = \frac{2}{p-2} \left( -h_g u + \tilde{h} u^\frac{2}{p} \right) \text{ on } \partial M\]

has a real, smooth, positive solution, i.e. there exists a conformal change \(\tilde{g} = u^{p-2} g\) associated with scalar curvature \(\tilde{S}\) and mean curvature \(\tilde{h}\).

We check the sign of \(\tilde{h}\). By (50),
\[\tilde{h} = \frac{p-2}{2} u^\frac{2}{p} \left( \frac{\partial u}{\partial \nu} + \frac{2}{p-2} h_g u \right) = \frac{p-2}{2} u^\frac{2}{p} \left( e^W \frac{\partial W}{\partial \nu} |_{\partial M} + \frac{2}{p-2} h_g e^W \right)
\[= \frac{p-2}{2} u^\frac{2}{p} e^W \left( H + \frac{2}{p-2} h_g \right) \geq \frac{p-2}{2} u^\frac{2}{p} e^W \left( \inf_{\partial M} H - \frac{2}{p-2} \sup_{\partial M} |h_g| \right) > 0.\] □
Similarly, we can find a conformal change with \( \tilde{h} < 0 \) everywhere on \( \partial M \), by exactly the same argument as above.

**Corollary 5.2.** Let \((\tilde{M}, \tilde{g})\) be a compact manifold with boundary. There exists a conformal metric \( \tilde{g} \) associated with mean curvature \( \tilde{h} < 0 \) everywhere on \( \partial M \).

**Proof.** After possible scaling we start with a metric with \( \frac{2}{p-2} \sup_{\partial M} |h_g| \leq 1 \). Now we choose a negative smooth function \( H \in C^\infty(\partial M) \) such that \( \sup_{\partial M} H \leq -2 \). The rest follows exactly the same as in Theorem 5.3.

With the help of Theorem 5.3 we can deal with the general case when \( \eta_1 \) without restriction of \( h_g \) and \( S_g \).

**Theorem 5.4.** Let \((M, g)\) be a compact manifold with boundary. When \( \eta_1 < 0 \), there exists some \( \lambda < 0 \) such that the boundary Yamabe equation (47) has a real, positive solution \( u \in C^\infty(M) \).

**Proof.** If \( h_g > 0 \) everywhere on \( \partial M \), it reduces to the case in Theorem 5.2. If not, then applying Theorem 5.3, we have a conformal metric \( \tilde{g}_1 = u^{p-2}g \) associate with scalar curvature \( S_1 \) and mean curvature \( \tilde{h}_1 > 0 \) everywhere on \( \partial M \). Here \( u \in C^\infty(M) \) is real and positive. By Theorem 5.2, it follows that there exists a real and positive function \( v \in C^\infty(M) \) such that the metric \( \tilde{g} = v^{p-2}\tilde{g}_1 \) admits a constant scalar curvature \( \lambda < 0 \) and the boundary is minimal with respect to \( \tilde{g} \). Hence

\[
\tilde{g} = v^{p-2}\tilde{g}_1 = v^{p-2}u^{p-2}g = (uv)^{p-2}g.
\]

When the first eigenvalue \( \eta_1 > 0 \), we need to control not only the sign of \( h_g \) on \( \partial M \), but also the sign of \( S_g \). We show the special case when \( h_g > 0 \) everywhere on \( \partial M \) and \( S_g < 0 \) somewhere, provided that \( \eta_1 > 0 \). The following argument is very close to Theorem 4.3 in [29]. We start with the perturbed boundary Yamabe equation (11) first. It is clear that when \( \eta_1 > 0 \), the Yamabe invariant \( Q(M) \) introduced at very beginning is also positive. Hence by the same argument as in Lemma 5.1, we conclude that \( \lambda_\beta > 0 \) when \( \beta < 0 \) with small enough \( |\beta| \).

**Theorem 5.5.** Let \((M, g)\) be a compact manifold with boundary. Let \( \beta < 0 \) be a fixed constant with small enough constant. Let \( \lambda_\beta \) be given in (42) for the fixed \( \beta \). Assume \( S_g < 0 \) somewhere in \( M \) and \( h_g > 0 \) everywhere on \( \partial M \). When \( \eta_1 > 0 \), the perturbed boundary Yamabe equation (11) has a real, positive solution \( u \in C^\infty(M) \).

**Proof.** Again we apply Theorem 5.3 to construct sub-solution and super-solution here. By Theorem 4.2, any \( A_0 > 0 \) in hypotheses of Theorem 5.3 works for the solvability of the linear iterations in (30). When \( \eta_1 > 0 \) and \( |\beta| \) small enough, we have \( \lambda_\beta > 0 \).

Let \( \beta < 0 \) such that \( |\beta| \) is small enough, thus by Lemma 5.1, the eigenvalue problem

\[
-a\Delta_g \varphi + (S_g + \beta) \varphi = \eta_1 \beta \varphi \quad \text{in} \quad M, \quad \frac{\partial \varphi}{\partial \nu} + \frac{2}{p-2} h_g \varphi = 0 \quad \text{on} \quad \partial M
\]

has a real, positive solution \( \varphi \in C^\infty(M) \). Note that any scaling \( \delta \varphi \) is also an eigenvalue with respect to \( \eta_1 \beta \). For the given \( \lambda_\beta \), we want

\[
\eta_1 \beta \inf_M (\delta \varphi) > 2^{p-2} \lambda_\beta \sup_M (\delta \varphi) \Leftrightarrow \frac{\eta_1 \beta}{2^{p-2} \lambda_\beta} > \frac{\sup_M \varphi}{\inf_M \varphi}.
\]

For fixed \( \eta_1 \beta \), \( \lambda_\beta \), \( \varphi \), this can be done by letting \( \delta \) small enough. We denote \( \phi = \delta \varphi \). It follows that

\[
-a\Delta_g \phi + (S_g + \beta) \phi = \eta_1 \beta \phi \quad \text{in} \quad M;
\]

\[
\eta_1 \inf_M \phi > 2^{p-2} \lambda_\beta \sup_M \phi > 2^{p-2} \lambda_\beta \phi > \lambda_\beta \phi \quad \text{in} \quad M.
\]

(51)
Set
\[ \beta' = \eta_{1,\beta} \sup_M \phi - 2p-2 \lambda_\beta \inf_M \phi^{p-1} > \eta_{1,\beta} \phi - 2p-2 \lambda_\beta \phi^{p-1} \text{ pointwise.} \] (52)

Thus we have
\[-a\Delta_g \phi + (S_g + \beta) \phi = \eta_{1,\beta} \phi > 2p-2 \lambda_\beta \phi^{p-1} > \lambda_\beta \phi^{p-1} \text{ in } M \text{ pointwise,} \]
\[ \frac{\partial \phi}{\partial \nu} + \frac{2}{p-2} h_g \phi = 0 \text{ on } \partial M. \] (53)

Now we construct the sub-solution by applying Proposition 2.7. Pick up a small enough interior Riemannian domain \((\Omega, g)\) in which \(S_g < 0\) such that the Dirichlet boundary value problem (17) with the given \(\lambda_\beta\) above has a positive solution \(u_1 \in C_0(\Omega) \cap H^1(\Omega, g)\), i.e.
\[-a\Delta_g u_1 + (S_g + \beta) u_1 = \lambda_\beta u_1^{p-1} \text{ in } \Omega, u_1 = 0 \text{ on } \partial M. \] (54)

Extend \(u_1\) by zero on the rest of \(\bar{M}\), we define
\[ u_- := \begin{cases} u_1(x), & x \in \bar{\Omega} \\ 0, & \bar{M} \setminus \bar{\Omega}. \end{cases} \] (55)

Clearly \(u_- \in C_0(\bar{M})\). Since \(u_1 \in H^1(\Omega, g)\), \(u_1\) can be approximated by \(\{v_k\} \subset C_c^\infty(\Omega)\) in \(H^1\)-sense. We extend \(v_k\) by zero to the rest of \(M\) and the extensions are still \(C_c^\infty(\bar{M})\). The extensions converge to the limit \(u_-\) in \(H^1\)-sense, hence \(u_- \in H^1(M, g)\). Therefore \(u_- \in C_0(\bar{M}) \cap H^1(M, g)\). Define \(F(x, u) = -(S_g + \beta) u + \lambda u^{p-1}\) as in (45), we observe that
\[-a\Delta_g u_- \leq F(x, u_-) \text{ in } M, B_g u_- \leq 0 \text{ on } \partial M. \] (56)

We construct the super-solution here. Pick up \(\gamma \ll 1\) such that
\[ 0 < 20\lambda \gamma + 2\gamma \sup_M |S_g| < \frac{\beta'}{2}, 31\lambda (\varphi + \gamma)^{p-2} < \frac{\beta'}{2}. \] (57)

The choice of \(\gamma\) is dimensional specific. Set
\[ V = \{x \in \Omega : u_1(x) > \varphi(x)\}, V' = \{x \in \Omega : u_1(x) < \varphi(x)\}, D = \{x \in \Omega : u_1(x) = \varphi(x)\}, \]
\[ D' = \{x \in \Omega : |u_1(x) - \varphi(x)| < \gamma\}, D'' = \{x \in \Omega : |u_1(x) - \varphi(x)| > \frac{\gamma}{2}\}. \]

If \(\varphi \geq u_1\) pointwisely, then \(\phi\) is a super-solution. If not, a good candidate of super-solution will be \(\max\{u_1, \phi\}\) in \(\Omega\) and \(\varphi\) outside \(\Omega\), this is an \(H^1\cap C_0\)-function. Let \(\nu\) be the outward normal derivative of \(\partial V\) along \(D\). If \(\frac{\partial u_1}{\partial \nu} = -\frac{\partial \varphi}{\partial \nu}\) on \(D\) then the super-solution has been constructed. However, this is in general not the case. If not, then \(\frac{\partial u_1}{\partial \nu} - \frac{\partial \varphi}{\partial \nu} \neq 0\), which follows that \(0\) is a regular point of the function \(u_1 - \varphi\) and hence \(D\) is a smooth submanifold of \(\Omega\). Define
\[ \Omega_1 = V \cap D'', \Omega_2 = V' \cap D'', \Omega_3 = D'. \] (58)

Construct a specific smooth partition of unity \(\{\chi_i\}\) subordinate to \(\{\Omega_i\}\) as in Theorem 4.3 of [29], we define
\[ \bar{u} = \chi_1 u_1 + \chi_2 \varphi + \chi_3 (\varphi + \gamma). \] (59)

Without loss of generality, we may assume that all \(\Omega_i, i = 1, 2, 3\) are connected. Due to the same argument in Theorem 4.3 of [29], we conclude that \(\bar{u} \in C^\infty(\Omega)\) is a super-solution of the same Yamabe equation in \(\Omega\) pointwise. By the definition of \(\bar{u}\), it is immediate that \(\bar{u} \geq u_1\). Then define
\[ u_+ := \begin{cases} \bar{u}, & \text{in } \Omega; \\ \varphi, & \text{in } M \setminus \Omega. \end{cases} \] (60)
It follows that $u_+ \in C^\infty(M)$ since $\tilde{u} = \varphi$ near $\partial\Omega$. By (53) we conclude that
\[ -a\Delta_g u_+ \geq F(x, u_+) \text{ in } M, B_g u_+ \geq 0. \] (61)
Critically, $0 \leq u_- \leq u_+$ and $u_- \not\equiv 0$ on $M$. As discussed in Theorem 3.3 we can then apply
Theorem 3.3. With the aids of (56) and (61), we conclude that there exists a real, nonnegative solution $u \in W^{2,p}(M, g)$ such that
\[ -a\Delta_g u + (S_g + \beta) u = \lambda u^{p-1} \text{ in } M, B_g u = 0 \text{ on } \partial M. \]
By bootstrapping method mentioned as above, we conclude that there exists a real, nonnegative solution $u \in W^{2,p}(M, g)$ such that
\[ -a\Delta_g u + (S_g + \beta) u = \lambda u^{p-1} \text{ in } M, B_g u = 0 \text{ on } \partial M. \]
Finally, we show first that $u \in C^\infty(M)$ is small enough, i.e.
\[ -a\Delta_g u + (S_g + \beta) u = \lambda u^{p-1} \text{ in } M, \frac{\partial u}{\partial \nu} + \frac{2}{p-2} h_g u = 0 \text{ on } \partial M. \] (62)
We show first that $\{\lambda_\beta\}$ is bounded above, and is increasing and continuous when $\beta \to 0^-$. We may assume $\int_M d\omega = 1$ for this continuity verification, since otherwise only an extra term with respect to $\text{Vol}_g$ will appear. Recall that
\[ \lambda_\beta = \inf_{u \neq 0, u \in H^1(M)} \left\{ \frac{\int_M a|\nabla_g u|^2 d\omega + \int_M (S_g + \beta) u^2 d\omega + \int_{\partial M} \frac{2}{p-2} h_g u^2 dS}{(\int_M u^p d\omega)^{\frac{2}{p}}} \right\} \]
It is immediate that if $\beta_1 < \beta_2 < 0$ then $\lambda_{\beta_1} \leq \lambda_{\beta_2}$. For continuity we assume $0 < \beta_2 - \beta_1 < \gamma$. For each $\epsilon > 0$, there exists a function $u_0$ such that
\[ \frac{\int_M a|\nabla_g u|^2 d\omega + \int_M (S_g + \beta_1) u_0^2 d\omega + \int_{\partial M} \frac{2}{p-2} h_g u_0^2 dS}{(\int_M u_0^p d\omega)^{\frac{2}{p}}} < \lambda_{\beta_1} + \epsilon. \]
It follows that
\[ \lambda_{\beta_2} \leq \frac{\int_M a|\nabla_g u_0|^2 d\omega + \int_M (S_g + \beta_2) u_0^2 d\omega + \int_{\partial M} \frac{2}{p-2} h_g u_0^2 dS}{(\int_M u_0^p d\omega)^{\frac{2}{p}}} \]
\[ \leq \frac{\int_M a|\nabla_g u|^2 d\omega + \int_M (S_g + \beta_1) u_0^2 d\omega + \int_{\partial M} \frac{2}{p-2} h_g u_0^2 dS}{(\int_M u_0^p d\omega)^{\frac{2}{p}}} + (\beta_2 - \beta_1) \frac{\int_M u_0^2 d\omega}{(\int_M u_0^p d\omega)^{\frac{2}{p}}} \]
\[ \leq \lambda_{\beta_1} + \epsilon + \beta_2 - \beta_1 < \lambda_{\beta_1} + \epsilon + \beta_2 - \beta_1. \]
Since $\epsilon$ is arbitrarily small, we conclude that
\[ 0 < \beta_2 - \beta_1 < \gamma \Rightarrow |\lambda_{\beta_2} - \lambda_{\beta_1}| \leq 2\gamma. \]
By equation (4) in [8 §1], we conclude that
\[ \lambda_\beta \leq \lambda(S_+^n) \]
Fix some $\beta_0 < 0$ with $\eta_{1,\beta_0} > 0$, we have
\[
\lambda_{\beta_0} \leq \lambda_0 \leq \lambda(S_+^n), \ \forall \beta \in [\beta_0, 0], \ \lim_{\beta \to 0^-} \lambda_\beta = \lambda.
\] (63)

Next we show $\|u_\beta\|_{H^2(M)} \leq C$ uniformly when $\beta \in [\beta_0, 0]$. Pairing the PDE (62) with $u_\beta$, it follows that
\[
\begin{aligned}
& a \|\nabla_g u_\beta\|^2_{L^2(\Omega, g)} + \frac{2}{p-2} \int_{\partial M} h_g u_\beta^2 dS = \lambda_s \|u_\beta\|^p_{L^p(M, g)} - \int_M (S_g + \beta) u_\beta^2 d\omega
\leq \lambda_s \|u_\beta\|^p_{L^p(M, g)} + \left( \sup_M |S_g| + |\beta_0| \right) \|u_\beta\|^2_{L^2(\Omega, g)};
\end{aligned}
\] (64)
\[
\Rightarrow a \|\nabla_g u_\beta\|^2_{L^2(\Omega, g)} \leq \lambda_s \|u_\beta\|^p_{L^p(M, g)} + \left( \sup_M |S_g| + |\beta_0| \right) \|u_\beta\|^2_{L^2(\Omega, g)}.
\]

Thus it suffices to show
\[
\|u_\beta\|_{L^p(M, g)} \leq C', \ \forall \beta \in [\beta_0, 0]
\]
due to Hölder’s inequality. Denote the local solutions of (17) by $\{\tilde{u}_\beta\}$, i.e.
\[
-a \Delta_g \tilde{u}_\beta + (S_g + \beta) \tilde{u}_\beta = \lambda_\beta \tilde{u}_\beta^{p-1} \text{ in } \Omega, \ \tilde{u}_\beta = 0 \text{ on } \partial \Omega
\]
with fixed domain $\Omega$. Again we pair $\tilde{u}_\beta$ on both sides,
\[
\begin{aligned}
a \|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega, g)} = \lambda_s \|\tilde{u}_\beta\|^p_{L^p(\Omega, g)} - \int_M (S_g + \beta) u_\beta^2 d\Vol_g
\Rightarrow \lambda_s \|\tilde{u}_\beta\|^p_{L^p(\Omega, g)} \leq a \|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega, g)} + \left( \sup_M |S_g| + |\beta| \right) \|u_\beta\|^2_{L^2(\Omega, g)}.
\end{aligned}
\] (65)

Recall the functional
\[
J(u) = \int_\Omega \left\{ \frac{1}{2} \sum_{i,j} a_{ij}(x) \partial_i u \partial_j u - \lambda_\beta \sqrt{\det(g)} \frac{1}{p} u^p - \frac{1}{2} (S_g + \beta) u^2 \sqrt{\det(g)} \right\} dx
\]
and the fixed constant $K_0$ in [29] §3, due to Theorem 1.1 of [27], each solution $\tilde{u}_\beta$ satisfies
\[
J(\tilde{u}_\beta) \leq K_0 \Rightarrow \frac{a}{2} \|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega, g)} - \lambda_\beta \|\tilde{u}_\beta\|^p_{L^p(\Omega, g)} - \frac{1}{2} \int_M (S_g + \beta) \tilde{u}_\beta^2 d\Vol_g \leq K_0.
\] (66)

Apply the estimate (65) into (66), we have
\[
\frac{a}{2} \|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega, g)} \leq K_0 + \frac{1}{p} \left( \sup_M |S_g| + |\beta| \right) \|u_\beta\|^2_{L^2(\Omega, g)}
\]
\[
\leq K_0 + \frac{a(n-2)}{2n} \|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega, g)}
\]
\[
+ \left( \frac{n-2}{2n} + 1 \right) \left( \sup_M |S_g| + |\beta| \right) \cdot \lambda_1^{-1} \|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega, g)};
\]
\[
\Rightarrow \left( \frac{a}{n} - \left( \frac{n-2}{2n} + 1 \right) \left( \sup_M |S_g| + |\beta| \right) \cdot \lambda_1^{-1} \right) \|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega, g)} \leq K_0.
\]

We can choose $\Omega$ small enough so that
\[
\frac{a}{n} - \left( \frac{n-2}{2n} + 1 \right) \left( \sup_M |S_g| + |\beta| \right) \cdot \lambda_1^{-1}
\]
holds for $\beta_0$, thus for all $\beta \in [\beta_0, 0]$, the above inequality holds due to the smallness of $\lambda^{-1}_1$. It follows from above that there exists a constant $C'_0$ such that
\[
\|\nabla_g \bar{u}_\beta\|^2_{L^2(\Omega, g)} \leq C'_0, \forall \beta \in [\beta_0, 0].
\]
Apply (65) with the other way around, we conclude that
\[
\lambda_s \|\bar{u}_\beta\|^p_{L^p(\Omega, g)} \leq a \|\nabla_g \bar{u}_\beta\|^2_{L^2(\Omega, g)} + \left(\sup_M |S_g| + |\beta|\right) \|\bar{u}_\beta\|^2_{L^2(\Omega, g)}
\] \[
\leq \left( a + \left(\sup_M |S_g| + |\beta|\right) \lambda^{-1}_1\right) \|\nabla_g u_\beta\|^2_{L^2(\Omega, g)}.
\]
We conclude that
\[
\|u_\beta\|^p_{L^p(\Omega, g)} \leq C'_1, \forall \beta \in [\beta_0, 0].
\]
(67)
Recall the construction of super-solution of each $u_\beta$ in Theorem 5.5, we have
\[
0 < u_\beta \leq u_{\beta^+} = \begin{cases} \bar{u}_\beta, & \text{in } \Omega \\ \phi_\beta, & \text{in } M \setminus \Omega. \end{cases}
\]
where $\bar{u}_\beta$ is of the form
\[
\bar{u}_\beta = \chi_1 \bar{u}_\beta + \chi_2 \phi_\beta + \chi_3 (\phi_\beta + \gamma).
\]
It follows from the uniform upper bound of (67) that
\[
\|u_\beta\|^p_{L^p(M, g)} \leq \|u_{\beta^+}\|^p_{L^p(M, g)} \leq E_0 \left(\|\bar{u}_\beta\|^p_{L^p(\Omega, g)} + \|\phi_\beta\|^p_{L^p(M, g)}\right).
\]
Recall in determining $\phi_\beta$ we require
\[
\frac{\eta_{1, \beta}}{2p - 2\lambda_\beta} > \delta^{p - 2} \sup_M \varphi^{p - 1} \inf_M \varphi.
\]
Since when $\beta \in [\beta_0, 0]$, we have $\lambda_\beta \in [\lambda_\beta_0, \lambda(S^+)]$, we can choose a fixed scaling $\delta$ for all $\beta$, thus $\|\phi_\beta\|^p_{L^p(M, g)}$ is uniformly bounded. It follows that there exists a constant $C'$ such that
\[
\|u_\beta\|^p_{L^p(M, g)} \leq C', \forall \beta \in [\beta_0, 0].
\]
Due to Hölder’s inequality, we conclude that
\[
\|u_\beta\|^p_{L^2(M, g)} \leq \|u_\beta\|^p_{L^p(M, g)} \cdot \text{Vol}_g(M)^{\frac{p - 2}{p}}.
\]
Thus by (64) we conclude that
\[
\|u_\beta\|_{H^1(M, g)} \leq C, \forall \beta \in [\beta_0, 0].
\]
(68)
We can thus extract a subsequence of $\{u_\beta\}$, denoted by $\{u_{\tilde{\beta}}\}$, such that when $\tilde{\beta} \to 0^-$, we have
\[
\begin{align*}
&u_{\tilde{\beta}} \to u \text{ in } H^1(M) \text{ weakly, } u_{\tilde{\beta}} \rightharpoonup u \text{ in } \mathcal{L}^q(M) \text{ strongly } \forall q \in (1, p); \\
u_{\tilde{\beta}} &\to u \text{ in } \mathcal{L}^{p-1}(M) \text{ weakly in } (\mathcal{L}^p(M, g))^*; \\
u_{\tilde{\beta}} &\to u \text{ on } \partial M \text{ strongly;} \\
u &\in H^1(M).
\end{align*}
\]
The strong convergence on $\partial M$ is due to the facts [26, Prop. 4.4, 4.5] that the inclusion map $H^{s+\sigma}(M, g) \hookrightarrow H^s(M, g)$ for all $s \geq 0, \sigma > 0$ is compact and the trace operator $\tau$ is a bounded
linear operator $\tau : H^s(M, g) \to H^{s-\frac{1}{2}}(\partial M, \tau^*g)$ for $s > \frac{1}{2}$ implies that $T : H^1(M, g) \to L^2(\partial M, g)$ is compact. Pairing any test function $v \in C^\infty(M)$ with (62), we have
\[
\int_M \nabla_g \bar{v}_\beta \cdot \nabla_g v \, d\omega + \int_{\partial M} \frac{2}{p-2} h_g \bar{v}_\beta u_\beta^2 \, dS + \int_M \left( S_g + \beta \right) u_\beta^2 \, d\omega = \lambda_\beta \int_M u_\beta^{p-1} v \, d\omega;
\]
\[
\Rightarrow \lim_{\beta \to 0^-} \int_M \nabla_g \bar{v}_\beta \cdot \nabla_g v \, d\omega + \lim_{\beta \to 0^-} \int_{\partial M} \frac{2}{p-2} h_g \bar{v}_\beta u_\beta^2 \, dS + \lim_{\beta \to 0^-} \int_M \left( S_g + \beta \right) u_\beta^2 \, d\omega
\]
\[
= \lim_{\beta \to 0^-} \left( \lambda_\beta \int_M u_\beta^{p-1} v \, d\omega \right);
\]
\[
\Rightarrow \int_M \nabla_g u \cdot \nabla_g v \, d\omega + \int_{\partial M} \frac{2}{p-2} h_g u_\beta^2 \, dS + \int_M S_g u_\beta^2 \, d\omega = \lambda \int_M u_\beta^{p-1} v \, d\omega.
\]
It follows that the Yamabe equation (2) holds in the $H^1$ sense. Due to [6], we conclude that $u \in C^\infty(M)$.

Lastly we show $u > 0$. According to the definition of $\lambda_\beta$, we apply the solution of (62) with the form (64), it follows that for all $\beta \in [\beta_0, \beta]$.
\[
\lambda_\beta \leq \frac{\int_M a |\nabla_g \bar{v}_\beta|^2 \, d\omega + \int_M \left( S_g + \beta \right) u_\beta^2 \, d\omega + \int_{\partial M} \frac{2}{p-2} h_g \bar{v}_\beta u_\beta^2 \, dS}{\left( \int_M u_\beta^{p} \, d\omega \right)^{\frac{2}{p}}}
\]
\[
\Rightarrow \lambda_\beta \leq \lambda_\beta \|u_\beta\|^p_{L^p(M, g)} \Rightarrow \|u_\beta\|_{L^p(M, g)} \geq 1.
\]
The last cancellation is due to the fact that $\lambda_\beta \geq \lambda_{\beta_0} > 0$. For any test function $v \in C^\infty(M)$, we apply the weak convergence of the subsequence, we see that for an arbitrarily small $\epsilon$, there exists some $\bar{\beta}$ very close to zero such that
\[
|\int_M u_\beta^{p-1} \, d\omega - \int_M u_\beta^{p-1} \, d\omega| < \epsilon \Rightarrow \int_M u_\beta^{p-1} \, d\omega \leq \int_M u_\beta^{p-1} \, d\omega + \epsilon.
\]
Taking $v = u_{\bar{\beta}}$, Hölder’s inequality implies that
\[
1 \leq \|u_{\bar{\beta}}\|^p_{L^p(M, g)} \leq \left( \int_M u^{p} \, d\omega \right)^{\frac{p-1}{p}} \left( \int_M u_{\bar{\beta}}^{p} \, d\omega \right)^{\frac{1}{p}} + \epsilon \leq \|u\|_{L^p(M, g)}^p \cdot C' + \epsilon.
\]
Hence $u \neq 0$. Since $u$ is limit of a positive sequence almost everywhere, $u \geq 0$, thus by maximum principle, $u > 0$ in the interior $M$. By the same argument in [8] §1, we conclude that $u > 0$ on $M$. \qed

As discussed before, in general $h_g > 0$ everywhere on $\partial M$ is not the case. Similarly when $\eta_1 > 0$, it is not always possible that $S_g < 0$ somewhere. The next two result, analogous to Theorem 4.4 in [29], shows the existence of metric $\bar{g}$ under conformal change such that $\bar{S}$ is negative somewhere and the sign of $\bar{h}$ will be the same as the sign of $h_g$ pointwisely, provided that $S_g \geq 0$ everywhere. Note again that by Proposition 2.4, the signs of first eigenvalues keep same under conformal change.

**Theorem 5.7.** Let $(\bar{M}, g)$ be a compact manifold with smooth boundary. Let $S_g \geq 0$ everywhere. There exists a conformal metric $\bar{g}$ associated with scalar curvature $\bar{S}$ and mean curvature $\bar{h}$ such that $\bar{S} < 0$ somewhere, and $\text{sgn}( h_g ) = \text{sgn} ( \bar{h} )$ pointwisely on $\partial M$.

**Proof.** By scaling we can, without loss of generality, assume that $|S_g| \leq 1$ on $\bar{M}$. Based on exactly the same construction in Theorem 4.4 of [29], there exists a smooth function $F \in C^\infty(M)$ such that (i) $\int_M F \, d\omega = 0$; (ii) $F$ is very negative at some interior point $p \in \bar{M}$; (iii) $\|F\|_{H^{s-2}(M, g)}$ is small.
enough, here $s = \frac{n}{2} + 1$ if $n$ is even and $s = \frac{n+1}{2}$ if $n$ is odd. The largeness and smallness will be determined later. Consider the following linear PDE with Neumann boundary condition

$$-a \Delta_g u' = F \text{ in } M, \frac{\partial u'}{\partial \nu} = 0 \text{ on } \partial M. \quad (69)$$

By standard elliptic theory, we conclude that there exists $u' \in C^\infty(\bar{M})$ solves (69) uniquely up to constants. By a standard elliptic regularity [26, Prop. 7.4], we conclude that

$$\|u'\|_{H^s(M,g)} \leq C^* \left( \|F\|_{H^{s-2}(M,g)} + \|u'\|_{L^2(M,g)} \right).$$

Pairing both side of (69) by $u'$, we conclude that

$$\|u'\|_{L^2(M,g)} \leq C_0 \|\nabla_g u'\|_{L^2(M,g)} \leq C_0 \|F\|_{L^2(M,g)}.$$  

This can be done by taking $u' \mapsto u' + \epsilon_1$ so that $\int_M (u' + \epsilon_1) d\omega = 0$. Since $u' + \epsilon_1$ also solves (69) we assume without loss of generality that $\int_M u' d\omega = 0$ thus the Poincaré inequality holds. Therefore there exists some $C_1$ such that

$$\sup_{\bar{M}} |u| \leq C_1 \|F\|_{H^{s-2}(M,g)}$$

with $s = \frac{n}{2} + 1$ when $n$ is even or $s = \frac{n+1}{2}$ when $n$ is odd. Then, same as in Theorem 4.4 of [29], pick up some $C > 1$ we choose can choose $F$ such that

$$F(q) < -\frac{C}{2}, |u'| < \frac{C}{8} \text{ in } \bar{M}.$$  

Finally we choose

$$u := u' + \frac{C}{4}. \quad (70)$$

It follows that this positive function $u \in [\frac{C}{8}, \frac{3C}{8}]$, $u \in C^\infty(\bar{M})$ and $u$ solves (69) since constant functions are in the kernel of $-a \Delta_g$ with Neumann boundary condition. Using the choice of $u$ in (70), we define

$$h = \frac{p - 2}{2} u^2 \left( \frac{\partial u}{\partial \nu} + \frac{2}{p - 2} h_g u \right) \text{ on } \partial M;$$

$$S = u^{1-p} (-a \Delta_g u + S_g u) \text{ in } M. \quad (71)$$

The first line in (71) says

$$h = \frac{p - 2}{2} u^2 \left( 0 + \frac{2}{p - 2} h_g u \right) = u^p h_g.$$  

Since $u > 0$, hence at each point of $\partial M$, the sign of $h$ is the same as the sign of $h_g$. From the second line of (71), we see that at the point $q$,

$$S(q) = u(q)^{1-p} (F(q) + S_g(q) u_q) \leq u(q)^{1-p} \left( F(q) + \sup_{\bar{M}} S_g |\sup_{\bar{M}} u| \right) \leq u(q)^{-p} \left( -\frac{C}{2} + \frac{3C}{8} \right) < 0.$$  

Lastly, we notice that since a real, positive $u \in C^\infty(\bar{M})$ solves the boundary value problem (71), there exists a conformal metric $\tilde{g} = u^{p-2} g$ associated with $\tilde{S}$ and $\tilde{h}$, where $\tilde{S}$ and $\tilde{h}$ has desired properties.

There is an immediate consequence with respect to some metric $g$ associate with $S_g \leq 0$ everywhere.

**Corollary 5.3.** Let $(\bar{M}, g)$ be a compact manifold with smooth boundary. Let $S_g \leq 0$ everywhere. There exists a conformal metric $\tilde{g}$ associated with scalar curvature $\tilde{S}$ and mean curvature $\tilde{h}$ such that $\tilde{S} > 0$ somewhere, and $\text{sgn}(h_g) = \text{sgn}(\tilde{h})$ pointwisely on $\partial M$.

**Proof.** Choosing $F' = -F$ as above and everything follows exactly the same as in Theorem 5.5. \(\square\)
With the help of Theorem 5.3 and 5.5, we can handle the general case when \( \eta_1 > 0 \) with arbitrary \( h_g \) and \( S_g \).

**Theorem 5.8.** Let \((\bar{M},g)\) be a compact manifold with smooth boundary. When \( \eta_1 > 0 \), there exists some \( \lambda > 0 \) such that \((40)\) has a real, positive solution \( u \in \mathcal{C}^{\infty}(\bar{M}) \).

**Proof.**

Start with the metric \( g \), we discuss the solvability in three cases:

(i) When \( S_g < 0 \) somewhere and \( h_g > 0 \) everywhere on \( \partial M \), this is exactly Theorem 5.3.

(ii) When \( S_g \geq 0 \) everywhere and \( h_g > 0 \) everywhere on \( \partial M \). In this case, we apply Theorem 5.7 and find out some \( \tilde{g}_1 = w^{p-2}g \) such that \( \bar{S}_1 < 0 \) somewhere and \( \bar{h} > 0 \) everywhere; then by Theorem 5.5, there exists some real, positive \( v \in \mathcal{C}^{\infty}(\bar{M}) \) such that \( \tilde{g} = v^{p-2}\tilde{g}_1 \) associates with constant scalar curvature and the boundary is minimal. Therefore,

\[
\tilde{g} = v^{p-2}\tilde{g}_1 = v^{p-2}w^{p-2}g = (uw)^{p-2}g
\]

is the desired conformal change.

(iii) When either \( S_g < 0 \) somewhere or \( S_g \geq 0 \) everywhere, and \( h_g \) changes sign on \( \partial M \). In this case, we first apply Theorem 5.3 and obtain some \( \tilde{g}_1 = w^{p-2}g \) with \( \bar{h}_1 > 0 \) everywhere. If \( \tilde{S}_1 < 0 \) somewhere, then we apply Theorem 5.5 directly to get some metric with constant scalar curvature, two steps. If not, i.e. \( \tilde{S}_1 \geq 0 \) everywhere, then we apply Theorem 5.7 to get some \( \tilde{g}_2 = v^{p-2}\tilde{g}_1 \) such that \( \tilde{S}_2 < 0 \) somewhere and we still have \( \bar{h}_2 > 0 \) everywhere on \( \partial M \). Lastly, applying Theorem 5.5 and thus we get \( g_3 = w^{p-2}\tilde{g}_2 \) which admits a constant scalar curvature with minimal boundary. Therefore

\[
\tilde{g}_3 = w^{p-2}\tilde{g}_2 = w^{p-2}v^{p-2}\tilde{g}_1 = w^{p-2}v^{p-2}w^{p-2}g = (uvw)^{p-2}g
\]

has the desired property. \( \square \)

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