NOTES ON PROJECTIVE NORMALITY OF REDUCIBLE CURVES

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Abstract. We give some results on quadratic normality of reducible curves canonically embedded and partially extend this study to their projective normality.

Introduction

Let $C$ be a smooth curve of genus $g$ over an algebraically closed field $k$. The canonical bundle $\omega_C$ induces an embedding of $C$ in $\mathbb{P}^{g-1}$ if and only if $C$ is not hyperelliptic; we indicate the power $\omega_C^{\otimes n}$ by $\omega_C^n$ for any $n \in \mathbb{N}$. One says that $C$ is projectively normal if the maps

\begin{equation}
H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(k)) \rightarrow H^0(C, \omega_C^k)
\end{equation}

are surjective for every $k \geq 1$. In other words, $C$ is projectively normal if and only if the hypersurfaces of degree $k$ in $\mathbb{P}^{g-1}$ cut a complete linear series on $C$ for any $k$. If $k = 1$ and the map (1) is surjective, we say that $C$ is linearly normal, which means that the curve is embedded via a complete linear series. If $\omega_C$ is ample, then an equivalent formulation states that $C$ is projectively normal if the maps

\begin{equation}
\text{Sym}^k H^0(C, \omega_C) \rightarrow H^0(C, \omega_C^k)
\end{equation}

are surjective for every $k \geq 1$, because the surjectivity of all these maps when $\omega_C$ is ample implies the very ampleness of $\omega_C$.

If $C$ is a smooth, non-hyperelliptic curve, Castelnuovo and Noether proved that its canonical model is projectively normal (see [ACGH]). When we deal with singular curves, though, the problem becomes harder: for integral curves, in [KM09] the authors generalize Castelnuovo’s approach proving that linear normality is equivalent to projective normality. For reducible curves yet not much is known: properties of the canonical map for Gorenstein curves, i.e. the map induced by the dualising sheaf, are investigated in [CFHR99], whereas in [F04] the author gives a sufficient condition for line bundles on non-reduced curves to be normally generated (see [F91]). The projective normality of reducible curves is studied in [S91]; more in general,
since the problem of studying projective normality reduces to the study of multiplication maps, we refer to [B01] and [F04] for these items.

In this paper we investigate the projective normality of reducible curves restricting the problem to suitable subcurves. The first step is to study the quadratic normality, i.e. the surjectivity of the maps in (1) for $k = 2$. Let $X$ be a connected, reduced and Gorenstein projective curve of genus $g$ with $\omega_X$ very ample. Assume that $X$ has planar singularities at the points lying on at least two irreducible components. Our main result about quadratic normality is the following theorem.

**Theorem 1.** Let $X$ be a curve as above, and set $X = A \cup B$ with $A, B$ connected subcurves being smooth at $D := A \cap B$. If $A \neq \emptyset$ and the map
$$
\mu_{\omega_A, \omega_X|_A}: H^0(A, \omega_A) \otimes H^0(A, \omega_X|_A) \to H^0(X, \omega_A \otimes \omega_X|_A)
$$
denote the multiplication map. Set $\mu_{\omega, X|_A} = \mu_{\omega, M|_A}$. Given the dualizing sheaf $\omega_X$ on $X$, we are interested in studying the surjectivity of the map $\mu_{\omega_X}$. In particular, when we assume that $X$ is canonically embedded this is equivalent to saying that $X$ is quadratically normal. We have

**Proposition 1.1.** Let $X$ be a connected reduced curve of genus $g$ with planar singularities and $\omega_X$ very ample. Assume that $X = A \cup B$, with $A, B$ connected and smooth at $D := A \cap B$. If

(i) $\mu_{\omega_A, \omega_X|_A}$ is surjective,
(ii) $\mu_{\omega_X|_B}$ is surjective.
then \( \mu_{\omega_X} \) is surjective.

In order to prove the proposition, we need some background material. We are going to keep the notation used in the statement of Proposition 1.1. Let \( D := A \cap B \) be the scheme-theoretic intersection. We will view \( D \) also as a subscheme of \( A \) and \( B \). Since both \( A \) and \( B \) are smooth at each point of the support of \( D \), that we denote by \( \text{supp}(D) \), the scheme \( D \) is a Cartier divisor of both \( A \) and \( B \); more in general, this is true if \( X \) has only planar singularities at each point of \( \text{supp}(D) \), because in this case a local equation of \( B \) in an ambient germ of a smooth surface gives a local equation of \( D \) as a subscheme of \( A \).

Remark 1.2. According to the notation above, we have that

(i) It is well known that a curve with planar singularities is Gorenstein.

(ii) Since \( X \) is Gorenstein and locally planar at the points of \( \text{supp}(D) \), then \( A \) and \( B \) are Gorenstein as well, so that \( \omega_A \) and \( \omega_B \) are both line bundles on \( A \) and \( B \).

(iii) Since \( X \) is locally planar at the points of \( \text{supp}(D) \), the adjunction formula gives \( \omega_X|_A = \omega_A(D) \) and \( \omega_X|_B = \omega_B(D) \). Thus \( \deg(\omega_X|_A) = 2g_A - 2 + \delta \) and \( \deg(\omega_X|_B) = 2g_B - 2 + \delta \), where of course \( g_A, g_B \) are the arithmetic genera of \( A \) and \( B \), and \( \delta = \deg(D) \).

Lemma 1.3. Let \( Z \) be a reduced, Gorenstein and connected projective curve. Let \( E \) be an effective Cartier divisor on \( Z \) such that \( E \neq 0 \). Then \( h^0(I_E) = 0 \) and \( h^1(\omega_Z(E)) = 0 \).

Proof. Since \( Z \) is connected, \( h^0(O_Z) = 1 \). Since \( E \) is effective and non-empty, we get \( h^0(I_E) = 0 \). We apply the duality for locally Cohen-Macaulay schemes, i.e. we apply to the scheme \( X := Z \) and the sheaf \( F := \omega_Z(E) \) the case \( r = p = 1 \) of the theorem at page 1 of [AK70]. We get \( h^1(\omega_Z(E)) = \dim(\text{Ext}^1(\omega_Z(E), \omega_Z)) \), i.e. \( h^1(\omega_Z(E)) = h^0(\text{Hom}(\omega_Z(E), \omega_Z)) \). Since \( \omega_Z \) is assumed to be locally free, we get \( h^1(\omega_Z(E)) = h^0(\text{Hom}(O_Z(E), O_Z)) = 0 \).

Lemma 1.4. Let \( X \) be a connected reduced curve of genus \( g \) with planar singularities and \( \omega_X \) very ample. Assume that \( X = A \cup B \), with \( A, B \) connected and smooth at \( D := A \cap B \). For any subcurve \( Z \) of \( X \) we consider the map

\[ \rho_Z : H^0(X, \omega_X) \to H^0(Z, \omega_X|_Z). \]

Then \( \rho_A \) and \( \rho_B \) are surjective.

Proof. To fix ideas we work on \( Z = A \); let us consider the exact sequence:

\[ 0 \to I_A \otimes \omega_X \to \omega_X \to \omega_X|_A \to 0. \]
We claim that \( I_A \otimes \omega_X = \omega_B \). To prove this, we notice that since \( X \) has only planar singularities, it can be embedded in a smooth surface \( S \), where \( X \), \( A \) and \( B \) are Cartier divisors. Thus \( D \) is a Cartier divisor of \( A \) and of \( B \) (but seldom of \( X \)). By the adjunction formula we have that

\[
\omega_X = \omega_S(A + B)|_X,
\]

then

\[
\omega_B = \omega_S(B)|_B = \omega_S(A + B - A)|_B = (\omega_S(A + B - A)|_X)|_B = (\omega_S(A) \otimes I_A)|_B.
\]

So the claim is proved and the previous sequence becomes

\[
0 \to \omega_B \to \omega_X \to \omega_X|_A \to 0.
\]

The corresponding long exact sequence in cohomology is

\[
0 \to H^0(\omega_B) \to H^0(\omega_X) \to H^0(\omega_X|_A) \to H^1(\omega_B) \to H^1(\omega_X) \to H^1(\omega_X|_A) \to \cdots
\]

Since \( \omega_X|_A = \omega_A(D) \), by lemma \([1.3]\) we have that \( \dim H^1(\omega_X|_A) = 0 \). Moreover, being both \( B \) and \( X \) connected, we have that \( \dim H^1(\omega_B) = 1 \) and \( \dim H^1(\omega_X) = 1 \), so the map \( H^0(\omega_X) \to H^0(\omega_X|_A) \) is surjective. □

We are now able to prove proposition \([1.1]\). Let us consider the composition

\[
H^0(\omega_X) \otimes H^0(\omega_X) \xrightarrow{\mu_{\omega_X}} H^0(\omega_X^2) \xrightarrow{\rho_B^2} H^0(\omega_X^2|_B);
\]

In order to show that \( \mu_{\omega_X} \) is surjective, it suffices, by a basic argument of linear algebra, to prove that

(a) \( \rho_B^2 \circ \mu_{\omega_X} \) is surjective,

(b) \( \ker \rho_B^2 \subseteq \text{Im} \mu_{\omega_X} \).

So let us show (a): we have a commutative diagram

\[
\begin{array}{ccc}
H^0(\omega_X) \otimes H^0(\omega_X) & \xrightarrow{\rho_B \otimes \rho_B \circ \mu_{\omega_X}} & H^0(\omega_X^2|_B) \\
\rho_B \otimes \rho_B & \downarrow & \\
H^0(\omega_X|_B) \otimes H^0(\omega_X|_B) & \xrightarrow{\mu_{\omega_B(D)}} & H^0(\omega_X^2|_B)
\end{array}
\]

where the map \( \rho_B \otimes \rho_B \) is surjective by lemma \([1.4]\) and \( \mu_{\omega_B(D)} \) is surjective by assumption (ii). So, by the commutativity of the diagram we get (a).

In order to prove (b), we notice that

\[
\ker \rho_B^2 = H^0(X, I_B \otimes \omega_X^2),
\]

and take

\[
\mu := \mu_{\omega_X}|_{H^0(X, I_B \otimes \omega_X) \otimes H^0(\omega_X)}.
\]
So we have the following commutative diagram:

\[
\begin{array}{ccc}
H^0(I_B \otimes \omega_X) \otimes H^0(\omega_X) & \xrightarrow{\mu} & H^0(I_B \otimes \omega_X^2) \\
\downarrow \text{id} \otimes \rho_A & & \downarrow \cong \\
H^0(\omega_A) \otimes H^0(\omega_X|_A) & \xrightarrow{\mu_{\omega_A,\omega_X|_A}} & H^0(\omega_A \otimes \omega_X|_A)
\end{array}
\]

The map \( \text{id} \otimes \rho_A \) is surjective by lemma 1.4, while \( \mu_{\omega_A,\omega_X|_A} \) is surjective by assumption (i). Hence \( \mu \) is surjective. Since \( \mu \) is a restriction of \( \mu_{\omega_X} \), we get \( \ker \rho_B^2 \subseteq \text{im} \mu_{\omega_X} \).

\[\square\]

**Definition 1.5.** Fix an integer \( m > 0 \); let \( X \) be a reduced and Gorenstein projective curve. We say that \( X \) is \( m \)-connected (resp. numerically \( m \)-connected) if for any decomposition \( X = U \cup V \) with \( U, V \) subcurves without common irreducible components, the scheme \( U \cap V \) has degree at least \( m \) (resp. \( \deg \omega_X|_U - \deg \omega_U \geq m \) and \( \deg \omega_X|_V - \deg \omega_V \geq m \)).

**Remark 1.6.** If every point of \( X \) lying on at least two irreducible components of \( X \) is a planar singularity of \( X \), then \( X \) is \( m \)-connected if and only if it is numerically \( m \)-connected (see [CFHR99], Remark 3.2).

**Notation 1.7.** Given a reduced curve \( X \), we will denote by \( X_{\text{mult}} \subset X \) the set of points of \( X \) lying on at least two irreducible components of \( X \) and by \( X_{\text{sm}} \) the open set of smooth points of \( X \).

**Lemma 1.8.** Let \( X \) be a connected, reduced and Gorenstein curve of genus \( g \) with \( \omega_X \) very ample. Assume that \( X \) has planar singularities at the points of \( X_{\text{mult}} \). Then \( X \) is \( 3 \)-connected.

**Proof.** Let us fix any decomposition \( X = U \cup V \) of \( X \), with \( U, V \) subcurves and \( \dim(U \cap V) = 0 \). Set \( D := U \cap V \). Since \( X \) has planar singularities at the points of \( \text{supp}(D) \), \( D \) is a Cartier divisor of \( U \). To prove the lemma it is sufficient to show the inequality \( \deg(D) \geq 3 \). Assume \( \deg(D) \leq 2 \). Since \( \omega_X \) is globally generated, \( X \) is \( 2 \)-connected (see [Ca81], Theorem D). Assume, then, \( \deg(D) = 2 \). Remark 1.2 gives \( \omega_X|_U \cong \omega_U(D) \). Since \( X \) is \( 2 \)-connected and \( \deg(D) = 2 \), we easily see that \( U \) is connected. By lemma 1.3 we get that \( \dim H^1(\omega_U(D)) = 0 \). Thus Riemann-Roch gives

\[\dim H^0(\omega_U(D)) = \dim H^0(\omega_U) + 1.\]

Since \( D \) is a Cartier divisor of \( U \), we get \( I_D \otimes \omega_U(D) \cong \omega_U \). Thus

\[\dim H^0(I_D \otimes \omega_X|_U)|_U = \dim H^0(\omega_X|_U) - 1,\]

hence the restriction to \( D \) of the morphism induced by \( |\omega_X| \) is not very ample, contradiction. \[\square\]
**Definition 1.9.** One says that a line bundle $L$ on a curve $X$ is *normally generated* if the maps

$$H^0(X, L)^k \to H^0(X, L^k)$$

are surjective for any $k \geq 1$.

Now we need to recall Theorem B in [F04].

**Theorem 1.10 (Franciosi).** Let $C$ be a connected reduced curve and let $\mathcal{H}$ be an invertible sheaf on $C$ such that

$$\deg \mathcal{H}|_Z \geq 2p_a(Z) + 1$$

for all subcurves $Z \subseteq C$. Then $\mathcal{H}$ is normally generated on $C$.

We are now able to prove the following lemma.

**Lemma 1.11.** Let $X = A \cup B$, with $A, B \neq \emptyset$ and assume that $X$ is Gorenstein, with planar singularities at the points of $X_{\text{mult}}$. Let $\omega_X$ be very ample. Then $\omega_X|_A$ and $\omega_X|_B$ are normally generated.

**Proof.** Let us prove the conclusions for $B$. By Theorem 1.10 it is sufficient to prove that $\deg \omega_X|_Z \geq 2p_a(Z) + 1$ for every subcurve $Z \subseteq B$. Since $A \neq \emptyset$, we have that $Z \subseteq X$. But since $\omega_X$ is very ample, by lemma 1.8 we have that $X$ is 3-connected, hence the conclusions.

We are now ready to prove Theorem 1.

**Proof of theorem 1.** We recall that $X$ is a connected, reduced and Gorenstein projective curve of genus $g$ with $\omega_X$ very ample. By hypothesis we assume that $X$ has planar singularities at the points of $X_{\text{mult}}$, and that $X = A \cup B$ with $A, B$ connected subcurves being smooth at $D := A \cap B$. Since $\mu_{\omega_X|_A}$ is surjective, by proposition 1.1 it suffices to show that (ii) holds. But this is true by lemma 1.11.

In what follows we will investigate when condition (i) of proposition 1.1 holds. If $X$ is any curve, we denote by $X_{\text{sm}}$ its smooth locus. We recall a result from [B01]; before doing this, let us introduce some notation: if $L$ is a line bundle on a curve $C$ globally generated and such that $\dim H^0(C, L) = r$, it induces a morphism

$$h_L : C \to \mathbb{P}^{r-1}.$$  

**Lemma 1.12 (Ballico).** Let $C$ be an integral projective curve with $C \neq \mathbb{P}^1$ and $R \in \text{Pic} C$, $R$ globally generated and such that $h_R$ is birational onto its image. Then the multiplication map

$$\mu_{\omega_C, R} : H^0(C, \omega_C) \otimes H^0(C, R) \to H^0(C, \omega_C \otimes R)$$

is surjective.

More in general we have the following result.
Theorem 1.13. Let $A$ be a reduced, connected and Gorenstein projective curve such that $\omega_A$ is very ample and the map $\mu_{\omega_A}$ is surjective. Let $E \subset A_{sm}$ be an effective divisor on $A$ such that $\deg E \geq 2$. Then $\mu_{\omega_A,\omega_A(E)}$ is surjective.

Proof. Since $A$ is connected, lemma 1.3 gives $H^1(\omega_A(D)) = 0$ for every effective and nonzero Cartier divisor $D$ on $A$. Thus

$$\dim H^0(\omega_A(D)) = g_A + \deg D - 1$$

for every such $D$. We use induction on $e := \deg E$.

(a) Let us first assume $e = 2$. We check that $\omega_A(E)$ is globally generated. Set $E = p_1 + p_2$, where $p_1, p_2$ are smooth points for $A$. Since $\omega_A$ is globally generated, then $\omega_A(E)$ is globally generated outside $\{p_1, p_2\}$. We just proved that

$$\dim H^0(\omega_A(p_i)) = \dim H^0(\omega_A(p_1 + p_2)) - 1.$$

Thus there is at least one section of $\omega_A(E)$ that doesn’t vanish at $p_i$, with $i = 1, 2$. Hence $\omega_A(E)$ is globally generated. The divisor $E$ induces two inclusions $j : \omega_A \hookrightarrow \omega_A(E)$ and $j' : \omega_A^2 \hookrightarrow \omega_A^2(E)$, which in turn induce the linear maps $j_\ast : H^0(\omega_A) \rightarrow H^0(\omega_A(E))$ and $j'_\ast : H^0(\omega_A^2) \rightarrow H^0(\omega_A^2(E))$ which have respectively corank 1 and 2. Consider the following diagram:

$$
\begin{array}{ccc}
H^0(\omega_A) \otimes H^0(\omega_A) & \xrightarrow{id \otimes j'_\ast} & H^0(\omega_A) \otimes H^0(\omega_A(E)) \\
\mu_{\omega_A,\omega_A} & & \mu_{\omega_A,\omega_A(E)} \\
H^0(\omega_A^2) & \xrightarrow{j'_\ast} & H^0(\omega_A^2(E))
\end{array}
$$

Since by hypothesis $\mu_{\omega_A,\omega_A}$ is surjective and

$$\dim H^0(\omega_A^2(E)) = \dim H^0(\omega_A^2) + 2,$$

then $j'_\ast(\text{Im}(\mu_{\omega_A,\omega_A}))$ is the codimension 2 linear subspace $\Gamma := H^0(\mathcal{I}_E \otimes \omega_A(E))$ of $H^0(\omega_A^2(E))$. Since the subspace $j'_\ast(\text{Im}(\mu_{\omega_A,\omega_A}))$ is contained in $\text{Im}(\mu_{\omega_A,\omega_A(E)})$, in order to get the conclusions for $e = 2$ it suffices to prove the existence of two elements of $\text{Im}(\mu_{\omega_A,\omega_A(E)})$ which together with a basis of $j'_\ast(\text{Im}(\mu_{\omega_A,\omega_A}))$, i.e. of $\Gamma$, are linearly independent. Since $\omega_A(E)$ is globally generated, there exists $\alpha \in H^0(\omega_A(E))$ not vanishing at $p_1$ and $p_2$. Since $\omega_A$ is globally generated, there is $\beta \in H^0(\omega_A)$ not vanishing at $p_1$ and $p_2$ as well. Since $\omega_A$ is very ample, there is $\gamma \in H^0(\omega_A)$ vanishing at $p_1$ but not at $p_2$, or, in the case when $p_1 = p_2$, vanishing at $p_1$ with order exactly 1. Now the section $\sigma := \mu_{\omega_A,\omega_A(E)}(\gamma \otimes \alpha)$ doesn’t belong to $\Gamma$; indeed, if $p_1 \neq p_2$, $\sigma$ doesn’t vanish at $p_2$, and if $p_1 = p_2$, it vanishes at $p_1$ with order exactly 1. Since the section $\mu_{\omega_A,\omega_A(E)}(\beta \otimes \alpha)$ does not vanish at $p_1$, it is not contained in the linear span of $\Gamma$ and $\sigma$. Thus

$$\dim \text{Im}(\mu_{\omega_A,\omega_A(E)}) \geq \dim \Gamma + 2.$$
Thus \( \mu_{\omega_A, \omega_A(E)} \) is surjective in the case \( e = 2 \).

(b) Let now \( e \geq 3 \). We use induction on \( e \). We fix a point \( p \) contained in the support of the divisor \( E \), and set \( F := E - p \). We check that \( \omega_A(E) \) is globally generated. By inductive hypothesis the line bundle \( \omega_A(F) \) is globally generated, hence so is \( \omega_A(E) \) outside \( p \). Since \( \dim H^1(\omega_A(F)) = 0 \), Riemann-Roch gives \( \dim H^0(\omega_A(E)) > \dim H^0(\omega_A(F)) \). Thus \( \omega_A(F) \) has a section not vanishing at \( p \). Hence \( \omega_A(E) \) is globally generated. We define two inclusions: \( \iota : \omega_A(F) \hookrightarrow \omega_A(E) \) and \( \iota' : \omega_A^2(F) \hookrightarrow \omega_A^2(E) \), which induce the linear maps \( \iota_* : H^0(\omega_A(F)) \rightarrow H^0(\omega_A(E)) \) and \( \iota'_* : H^0(\omega_A^2(F)) \rightarrow H^0(\omega_A^2(E)) \), both having corank 1. We consider the diagram

\[
\begin{array}{ccc}
H^0(\omega_A) \otimes H^0(\omega_A(F)) & \xrightarrow{id \otimes \iota_*} & H^0(\omega_A) \otimes H^0(\omega_A(E)) \\
\mu_{\omega_A, \omega_A(F)} & & \mu_{\omega_A, \omega_A(E)} \\
H^0(\omega_A^2(F)) & \xrightarrow{\iota'_*} & H^0(\omega_A^2(E))
\end{array}
\]

By the inductive hypothesis the map \( \mu_{\omega_A, \omega_A(F)} \) is surjective. Thus the linear subspace \( \iota'_*(\text{Im}(\mu_{\omega_A, \omega_A(F)})) \) has codimension 1 in \( H^0(\omega_A^2(E)) \). Fix \( \eta \in H^0(\omega_A) \) not vanishing at \( p \) and \( \tau \in H^0(\omega_A(E)) \) not vanishing at \( p \). Since \( \mu_{\omega_A, \omega_A(E)}(\eta \otimes \tau) \) does not vanish at \( p \), it doesn’t belong to \( \iota'_*(\text{Im}(\mu_{\omega_A, \omega_A(F)})) \). Thus \( \mu_{\omega_A, \omega_A(E)} \) is surjective.

\[\square\]

2. \( k \)-normality in higher degree

We are now interested in studying the surjectivity of higher order maps, i.e. of

\[\text{Sym}^k(H^0(\omega_X)) \rightarrow H^0(\omega_X^k)\]

when \( k \geq 3 \), but since \( \text{Sym}^k(H^0(\omega_X)) \) is a quotient of \( H^0(\omega_X)^{\otimes k} \), we can equivalently study the surjectivity of

\[H^0(\omega_X)^{\otimes k} \rightarrow H^0(\omega_X^k)\]

We observe that by applying part (b) in the proof of theorem 1.13 we get the following:

**Proposition 2.1.** Let \( A \) be a reduced, connected and Gorenstein curve such that \( \omega_A \) is globally generated. Fix a globally generated \( R \in \text{Pic}A \) such that \( H^1(R) = 0 \) and \( \mu_{\omega_A, R} \) is surjective. Let \( D \subset A_{\text{sm}} \) be any effective divisor. Then \( \mu_{\omega_A, R(D)} \) is surjective.

As a corollary of theorem 1.13 we get the following result.

**Corollary 2.2.** Let \( A \) be a reduced, connected and Gorenstein projective curve such that \( \omega_A \) is very ample and \( \mu_{\omega_A} \) is surjective. Let \( E \subset A_{\text{sm}} \) be an effective divisor such that \( \deg E \geq 2 \). Then the maps \( \mu_{\omega_A, \omega_A^k(kE)} \) are surjective for all \( k \geq 2 \).
We are now going to give some definitions in order to state a result;

**Definition 2.3.** A simple $(r-1)$-secant is a configuration of $r-1$ smooth points $p_1, \ldots, p_{r-1}$ on a curve $X \subset \mathbb{P}^N$, spanning a $\mathbb{P}^{r-2}$ and such that $X \cap \mathbb{P}^{r-2} = \{p_1, \ldots, p_{r-1}\}$ as schemes.

**Definition 2.4.** Let $R$ be a globally generated line bundle on a curve $X$, inducing a map $h_R : X \rightarrow \mathbb{P}^r$, $r := \dim H^0(R) - 1$, which is birational onto the image. A good $(r-1)$-secant of $R$ is a set $S := \{p_1, \ldots, p_{r-1}\}$ such that $\dim H^0(R(-\sum_{i=1}^{r-1} p_i)) = 2$, $R(-\sum_{i=1}^{r-1} p_i)$ is still globally generated, and $h_R$ is an embedding at each $p_i$.

We recall the following result from [B01]

**Lemma 2.5** (Ballico). Let $X$ be a one-dimensional projective locally Cohen-Macaulay scheme with $\dim H^0(O_X) = 1$ and $R \in \text{Pic} X$ globally generated and such that $\dim H^0(R) = 2$. Then the multiplication map

$$\mu_{\omega_X,R} : H^0(\omega_X) \otimes H^0(R) \rightarrow H^0(\omega_X \otimes R)$$

is surjective.

**Lemma 2.6.** Let $A$ be a connected, projective curve, $L, M \in \text{Pic} A$, $M$ globally generated, and such that $\dim H^0(M) = 2$ and $\dim H^1(L \otimes M^\vee) = 0$. Then $\mu_{L,M}$ is surjective.

**Proof.** Obvious by the base point free pencil trick. \qed

**Proposition 2.7.** Let $A$ be a connected, Gorenstein curve with $\omega_A$ globally generated, $R \in \text{Pic} A$ with $R$ globally generated, with $h_R$ birational onto its image and with a good $(r-1)$-secant, where $r := h^0(R) - 1$. Then the maps $\mu_{\omega_A,R^k}$ are surjective for all $k \geq 1$.

**Proof.** Fix a good $(r-1)$-secant set $S = \{q_1, \ldots, q_{r-1}\}$. Thus the linear span $\langle h_R(q_1), \ldots, h_R(q_{r-1}) \rangle$ has dimension $r-2$, $h_R(A) \cap \langle h_R(q_1), \ldots, h_R(q_{r-1}) \rangle = \{h_R(q_1), \ldots, h_R(q_{r-1})\}$ as schemes and

$$h_R^{-1}(\{h_R(q_1), \ldots, h_R(q_{r-1})\}) = \{q_1, \ldots, q_{r-1}\}.$$

Set $M := R(-S)$. We start by examining the case $k = 1$. Since $\omega_A$ is globally generated, we have $A \neq \mathbb{P}^1$. Since the map $h_R$ induced by $R$ is birational onto its image, we have $r \geq 2$. The first condition on the good $(r-1)$-secant points gives $\dim H^0(M) = 2$. The last two conditions give that $M$ is globally generated. Since $h^0(R) = h^0(M) + r - 1$, we also get $h^0(M(q_1)) = h^0(M) + 1$. Thus there is $\eta \in H^0(M(q_1))$ such that $\eta(q_1) \neq 0$. The factorization shown
in the following diagram

\[
\begin{array}{ccc}
H^0(\omega_A) \otimes H^0(M) & \rightarrow & H^0(\omega_A \otimes M) \\
\downarrow & & \downarrow j \\
H^0(\omega_A) \otimes H^0(M) & \rightarrow & H^0(\omega_A \otimes M(q_1))
\end{array}
\]

shows that the image of \( \varphi \) contains a copy of \( H^0(\omega_A \otimes M) \) as a hyperplane.

Since \( q_1 \) is not a base point for \( M \) and \( \omega_A \) is globally generated, there is \( \sigma \in H^0(\omega_A) \otimes H^0(M(q_1)) \) that doesn’t vanish on \( q_1 \). Hence the image of \( \sigma \) via \( \varphi \) doesn’t vanish on \( q_1 \), and we get the surjectivity of \( \varphi \). Repeating this argument for all the points \( q_1, \ldots, q_{r-1} \) adding them one by one we get that \( \mu_{\omega_A,M} \) is surjective.

Now we assume \( k \geq 2 \) and use induction on \( k \). The inductive assumption gives the surjectivity of the map \( H^0(\omega_A) \otimes H^0(R^{k-1}) \rightarrow H^0(\omega_A \otimes R^{k-1}) \).

We use the following commutative diagram:

\[
\begin{array}{ccc}
H^0(\omega_A) \otimes H^0(R^{k-1}) & \otimes & H^0(R^k) \\
\downarrow & \otimes & \downarrow \phi \\
H^0(\omega_A) \otimes H^0(R) & \rightarrow & H^0(\omega_A \otimes R^k)
\end{array}
\]

It suffices to prove that \( \phi \) is surjective, indeed, if it is, then \( \phi \circ \psi \) is surjective, hence \( \mu \) must be surjective. We proved that \( M \) is globally generated and \( \dim H^0(M) = 2 \). Moreover we notice that

\[
\omega_A \otimes R^{k-1} \otimes M^\vee = \omega_A \otimes R^{k-2}(S).
\]

Since \( k \geq 2 \) and \( S \neq \emptyset \), we have that \( \dim H^1(\omega_A \otimes R^{k-2}(S)) = 0 \). The base point free pencil trick applied to \( \omega_A \otimes R^{k-1} \) and \( M \) gives the surjectivity of \( \mu_{\omega_A\otimes R^{k-1},M} \). By Riemann-Roch theorem we get that

\[
\dim H^0(\omega_A \otimes R^k) = \dim H^0(\omega_A \otimes R^{k-1} \otimes M) + \sharp S.
\]

Arguing as in case \( k = 1 \) we get that the map \( \mu_{\omega_A,R} \) is surjective. \( \square \)

**Definition 2.8.** We say that a line bundle \( L \) on a curve \( X \) is \( k \)-normally generated if the map

\[
H^0(\omega_X)^{\otimes k} \rightarrow H^0(\omega_X^k)
\]

is surjective.

For instance “quadratically normal” means “linearly normal” plus “2-normally generated”.

**Proposition 2.9.** Let \( X \) be a connected, reduced, Gorenstein projective curve with planar singularities and \( \omega_X \) very ample. Assume that \( X = A \cup B \), with \( A, B \) connected and smooth at \( D := A \cap B \). Fix \( k \geq 3 \); if
(i) \( \omega_X \) is \((k-1)\)-normally generated,
(ii) \( \mu_{\omega_A \omega_{k}^X} \) is surjective for \( 1 \leq j \leq k \),
(iii) \( \omega_X | B \) is \( j \)-normally generated for \( 1 \leq j \leq k \),
then \( \omega_X \) is \( k \)-normally generated.

Proof. The proof is similar to the one of proposition [1.1] we just change notation slightly, denoting the multiplication maps in an easier way. We notice that in order to prove that the map

\[
H^0(\omega_X)^{\otimes k} \xrightarrow{\mu_k} H^0(\omega_X^k)
\]

is surjective, by factorizing we get

\[
H^0(\omega_X) \otimes H^0(\omega_X)^{\otimes k-1} \xrightarrow{\mu \otimes \mu_{k-1}} H^0(\omega_X) \otimes H^0(\omega_X^{k-1}) \xrightarrow{\tilde{\mu}} H^0(\omega_X^k),
\]

so it suffices to see that the map \( \tilde{\mu} \) is surjective. We consider the diagram

\[
\begin{array}{ccc}
H^0(\omega_X) \otimes H^0(\omega_X^{k-1}) & \xrightarrow{\tilde{\mu}} & H^0(\omega_X^k) \\
\downarrow{\eta} & & \downarrow{\psi} \\
H^0(\omega_X | B) \otimes H^0(\omega_X^{k-1} | B) & \xrightarrow{\phi} & H^0(\omega_X^k | B)
\end{array}
\]

where the map \( \tilde{\mu} = \mu_{\omega_X \omega_{X}^{k-1}} \). We know that \( \phi \) is surjective by (iii), and if

(a) \( \psi \circ \tilde{\mu} \) is surjective,
(b) \( \text{Ker} \psi \subseteq \text{Im} \tilde{\mu} \),

then by linear algebra we get that \( \tilde{\mu} \) is surjective. In order to prove (a), by [9] we equivalently show that the map \( \phi \circ \eta \) is surjective. We claim that \( \eta \) is surjective. Indeed, since \( \omega_X \) is locally free we have the exact sequence

\[
0 \to \mathcal{I}_B \otimes \omega_X \to \omega_X \to \omega_X | B \to 0.
\]

If we tensor by \( \omega_X^{k-2} \), we get

\[
0 \to \mathcal{I}_B \otimes \omega_X^{k-1} \to \omega_X^{k-1} \to \omega_X | B \otimes \omega_X^{k-2} \to 0,
\]

which is equivalent to

\[
0 \to \omega_A \otimes \omega_X^{k-2} \to \omega_X^{k-1} \to \omega_X^{k-1} | B \to 0.
\]

The corresponding long exact sequence in cohomology is

\[
0 \to H^0(\omega_A \otimes \omega_X^{k-2}) \to H^0(\omega_X^{k-1}) \to H^0(\omega_X^{k-1} | B) \to H^1(\omega_A \otimes \omega_X^{k-2}) \to \to H^1(\omega_X^{k-1} | B) \to \cdots
\]

Now we consider \( H^1(\omega_A \otimes \omega_X^{k-2}) \); we have that \( \omega_A \otimes \omega_X^{k-2} = \omega_A \otimes \omega_X^{k-2} | A = \omega_A \otimes \omega_A^{k-2}((k-2)D) \), hence by lemma [1.3] we obtain that \( H^1(\omega_A \otimes \omega_X^{k-2}((k-2)D)) = 0 \), therefore the map

\[
H^0(\omega_X^{k-1}) \to H^0(\omega_X^{k-1} | B)
\]
is surjective, and we get (a).

Now we want to prove (b). We notice that
\[ \text{Ker} \psi = H^0(\mathcal{I}_B \otimes \omega_X^k) \]
and set
\[ \mu := \tilde{\mu} |_{H^0(X, \mathcal{I}_B \otimes \omega_X) \otimes H^0(\omega_X^{k-1})}. \]

We have the following commutative diagram:

\[
\begin{array}{ccc}
H^0(\mathcal{I}_B \otimes \omega_X) \otimes H^0(\omega_X^{k-1}) & \xrightarrow{\mu} & H^0(\mathcal{I}_B \otimes \omega_X^k) \\
\downarrow{\gamma} & & \downarrow{\cong} \\
H^0(\omega_A) \otimes H^0(\omega_X^{k-1} | A) & \xrightarrow{\mu_{\omega_A \omega_X^{k-1} | A}} & H^0(\omega_A \otimes \omega_X^{k-1} | A)
\end{array}
\]

Now we have that \( \mathcal{I}_B \otimes \omega_X \cong \omega_A \) and applying the previous argument to \( A \) rather than to \( B \), we obtain that
\[ H^0(\omega_X^{k-1}) \rightarrow H^0(\omega_X^{k-1} | A) \]
is surjective, hence so is \( \gamma \) in (10). Applying hypothesis (ii) we have that \( \mu \) is surjective, hence as in the proof of 1.1 we get that \( \text{Ker} \psi = \text{Im} \mu \subseteq \text{Im} \tilde{\mu} \). \( \square \)

We notice that when \( k \) grows, the hypothesis in proposition 2.9 can be simplified:

**Proposition 2.10.** Let \( X \) be a connected, reduced, Gorenstein projective curve of genus \( g \), with \( \omega_X \) globally generated. Fix \( k \geq 4 \) and assume that \( \omega_X \) is \( (k-1) \)-normally generated. Then \( \omega_X \) is \( k \)-normally generated.

**Proof.** As in the proof of 2.9, looking at the factorization

\[ H^0(\omega_X) \otimes H^0(\omega_X^{k-1}) \xrightarrow{\mu_{\omega_X \omega_X^{k-1}}} H^0(\omega_X^k) \]

by hypothesis it suffices to prove that \( \mu_{\omega_X \omega_X^{k-1}} \) is surjective. We use Proposition 8 in [F07] in the following way: we take \( \mathcal{F} := \omega_X \) and \( \mathcal{H} := \omega_X^{k-1} \), so we have that \( H^0(\mathcal{F}) \) is globally generated. Moreover we have that

\[ H^1(\mathcal{H} \otimes \mathcal{F}^{-1}) = H^1(\omega_X^{k-2}) = 0 \]
if \( k \geq 4 \), so we get that the \( \mu_{\omega_X \omega_X^{k-1}} \) is surjective. \( \square \)

3. Applications

In the sequel we are going to study some cases where we can apply our results.

**Lemma 3.1.** Let \( Z \) be a connected and Gorenstein curve such that \( \omega_Z \) is globally generated. Let \( D \subset Z_{\text{sm}} \) be an effective Cartier divisor such that \( \deg(D) \geq 2 \). Then \( \omega_Z(D) \) is globally generated.
Proof. Since $\omega_Z(D)$ is a line bundle, it is globally generated if and only if for every $q \in Z$ there is $s \in H^0(\omega_Z(D))$ such that $s(q) \neq 0$. Since $\omega_Z$ is assumed to be globally generated and $D$ is effective, the sheaf $\omega_Z(D)$ is globally generated outside the finitely many points appearing in $\text{supp}(D)$. Fix $p \in \text{supp}(D)$ and set $D' := \mathcal{I}_p \otimes D$. Since $p \in X_{\text{sm}}$, $D'$ is a Cartier divisor of degree $\deg(D) - 1$. Moreover, since $p \in \text{supp}(D)$, $D'$ is effective. Thus Lemma 3.1 gives $h^1(\omega_Z(D')) = 0$. Riemann-Roch gives $h^0(\omega_Z(D')) = h^0(\omega_Z(D)) + 1$. Thus there is $s \in H^0(\omega_Z(D))$ such that $s(p) \neq 0$. □

Corollary 3.2. Let $X$ be a connected reduced curve with two irreducible non-rational components $C_1, C_2$ meeting at planar singularities for $X$ and both smooth at $C_1 \cap C_2$; assume that $\omega_X$ is very ample. Then $X$ is canonically embedded is projectively normal.

Proof. First of all we have to prove that $X$ is quadratically normal, so let us use the set-up of proposition 1.1 and set $A = C_1$, $B = C_2$. We look at hypothesis (i) and (ii) of the theorem; hypothesis (i) is verified by applying 1.12 to $C_1$. Indeed in our situation $R = \omega_X|_{C_1}$, i.e. $R = \omega_{C_1}(D)$ where $D$ is the divisor on $C_1$ and $C_2$ corresponding to $C_1 \cap C_2$. Hence by lemma 3.1 we have that $R$ is globally generated and birational onto the image, and we get (i). Concerning (ii), it suffices to apply 1.11 and then by 1.1 we obtain that $X$ is quadratically generated. Now we want to study the 3-normal generation of $X$. So we look at the hypothesis of 2.9 we know that $\omega_X$ is quadratically normal, and of course (iii) holds by lemma 1.11. So it remains to prove (ii): but this is a consequence of corollary 2.2, indeed we have that $\mu_{\omega_A}$ is surjective since $A$ is irreducible and hence projectively normal, moreover, being $\omega_X$ very ample, $A \cdot B \geq 3$. Now when $k \geq 4$ we just apply 2.10 and get the conclusions. □

Remark 3.3. We observe that in the case of nodal connected curves with two non-rational irreducible components, the corollary above says that if the two components $C_1$ and $C_2$ meet at least at 3 points, then $X = C_1 \cup C_2$ canonically embedded is projectively normal. The corollary leaves out the curves having at least one $\mathbb{P}^1$ as a component, and in particular binary curves (i.e. a curve $X$ is binary if it is composed of two $\mathbb{P}^1$'s meeting at $g + 1$ points where $g$ is the genus of $X$), but for the latter special class of curves we can use [S91] (see 3.6) and easily get projective normality. Concerning the class of curves $X = C_1 \cup C_2$ with $C_1 \neq \mathbb{P}^1$ and $C_2 = \mathbb{P}^1$, we get the projective normality by applying the same proof as in corollary 3.2 once we denote by $A$ the component $C_1$. Indeed the hypothesis $C_1 \neq \mathbb{P}^1$ is used only when we apply 1.12 to $A$.

We can generalize the previous result:
Corollary 3.4. Let $X$ be a connected reduced Gorenstein curve with $\omega_X$ very ample and with planar singularities. Assume that $X = A \cup B$ with $A \neq \mathbb{P}^1$ irreducible and let $B$ be a connected curve. Let $A$ and $B$ be smooth at $A \cap B$. Then $\omega_X$ is $k$-normally generated for any $k \geq 2$.

Proof. The proof is straightforward once we notice that we can apply 1.12 to $A$ and by Theorem 1 we get quadratic normality of $X$; for $k = 3$ we apply 2.9 since both 1.11 for $B$ and 1.12 for $A$ hold, and when $k \geq 4$ we apply 2.10. □

Corollary 3.5. Let $X$ be a connected reduced Gorenstein curve with $\omega_X$ very ample and with planar singularities. Assume that $X = A \cup B$ as in theorem 1.13 and let $B$ be a connected curve. Let $A$ and $B$ be smooth at $A \cap B$. Then $X$ canonically embedded is projectively normal.

Proof. The proof is as in corollary 3.4, we just apply theorem 1.13 to $A$. □

We give now an example; before doing this, we recall an important result from [S91]:

Theorem 3.6 (Schreyer). Let $X \subset \mathbb{P}^{g-1}$ be a canonical curve of genus $g$. If $X$ has a simple $(g - 2)$-secant, then $X$ is projectively normal.

Schreyer’s theorem can be used in the most general setting once one is able to verify the existence of a simple $(g - 2)$-secant. In [S91] pp.86 gave an example of a reducible canonically embedded curve admitting no simple $(g - 2)$-secant. In the following example we show that our theorem applies to that case.

Example 3.7. Let $X = X_1 \cup X_2 \cup X_3 \cup X_4$, with $X_i$ smooth of genus $g_i$ and such that the components intersect in 6 distinct points $p_{ij} = X_i \cap X_j$ that are ordinary nodes for $X$. Then $X$ has genus $g = g_1 + g_2 + g_3 + g_4 + 3$. We have that $\omega_X$ is a very ample line bundle; if $g_i = 0$ for every $i$ we have a graph curve, and it is projectively normal, as we see in [BE91]. Hence we can assume $g_i \neq 0$ for some $i$, say $g_1 > 0$. Set $A := X_1$, $B := X_2 \cup X_3 \cup X_4$. Since $A \neq \mathbb{P}^1$ we can apply 1.12 and get that the multiplication map $\mu_{\omega_A, \omega_X|A}$ is surjective. Since the conditions on the degree of $\omega_X|B$ in 1.10 are satisfied, the map $\mu_{\omega_X|B}$ is surjective and we can apply proposition 1.1 and get that $X$ is quadratically normal.

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