HYPERBOLICITY OF THE CYCLIC SPLITTING COMPLEX

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ABSTRACT. We define a new complex on which Out$(F_n)$ acts by simplicial automorphisms, the cyclic splitting complex of $F_n$, and show that it is hyperbolic using a method developed by Kapovich and Rafi in [6].

1. Introduction

Let $S$ be a hyperbolic surface, perhaps with non-empty boundary or punctures. The curve complex $C(S)$ associated to $S$ is a simplicial complex whose 1-skeleton is defined by taking vertices to be homotopy classes of essential simple close curves in $S$, and where two vertices are joined by an edge if they can be represented by disjoint curves in the surface.

A celebrated theorem of Masur-Minsky (see [7]) is that

Theorem (Masur-Minsky). The curve complex $C(S)$ is $\delta$-hyperbolic.

The mapping class group of $S$, denoted by $\text{Mod}(S)$, acts on $C(S)$ in the obvious manner.

In an attempt to mirror the study of surfaces and their mapping class groups, there have been several complexes defined on which $\text{Out}(F_n)$, the group of outer automorphisms of the rank $n$ free group, acts. One such complex is the free factor complex, denoted $\mathbf{FF}_n$, whose vertices are conjugacy classes of proper free factors and which has the structure of a simplicial complex given by the poset structure on the conjugacy classes of free factors of $F_n$. Bestvina and Feighn have shown in [2] that

Theorem (Bestvina-Feighn). The free factor complex $\mathbf{FF}_n$ is $\delta$-hyperbolic.

Another analogue for the curve complex is the free splitting complex, $\mathbf{FS}_n$, whose vertices are 1-edge free splittings of $F_n$, and where two vertices are joined by an edge if the two splittings admit a common refinement. A theorem of Handel and Mosher in [5] is that

Theorem (Handel-Mosher). The free splitting complex $\mathbf{FS}_n$ is $\delta$-hyperbolic.

A recent paper of Kapovich and Rafi [6] shows that the hyperbolicity of the free splitting complex implies the hyperbolicity of the free factor complex. The outline of their argument goes as follows: the authors define an auxiliary complex $\mathbf{FB}_n$ called the complex of free bases, whose vertices are conjugacy classes of free bases of $F_n$, and where two conjugacy classes of bases are adjacent if they have representatives which share an element. They show that $\mathbf{FB}_n$ and $\mathbf{FF}_n$ are quasi-isometric, so it suffices to show that hyperbolicity of $\mathbf{FB}_n$ which they do by applying a consequence of Bowditch’s work (see Theorem 2 below) in [3] to the natural map $\mathbf{FS}_n' \to \mathbf{FB}_n$, where $\mathbf{FS}_n'$ is the barycentric subdivision of $\mathbf{FS}_n$. 
The Cyclic Splitting Complex. In this paper, we define another analogue of the curve complex for surfaces, the cyclic splitting complex, and show that it is hyperbolic using the technology from [6].

The cyclic splitting complex is the simplicial complex whose vertices free splittings of $F_n$, and where two free splittings $X$ and $Y$ are connected by an edge if either (1) they are commonly refined (as in $FS_n$) or (2) if there is an element $w$ in the vertex groups of $X$ and $Y$ and a $Z$-splitting $T$ with edge group $\langle w \rangle$ such that $T$ can be obtained from $X$ and $Y$ by “folding” $\langle w \rangle$ over the trivial edge groups. There is a natural map $FS_n \to FZ_n$.

It is worth noting why one might wish to examine such a complex. By work of Stallings (see [8]) a simple closed curve $c$ in a surface $S$ gives a $Z$-splitting of $\pi_1(S) = A \ast B$ or $\pi_1(S) = A \ast_c B$, and conversely. The definition of $FZ_n$ is constructed so as to mimic this way of thinking of $C(S)$.

We show the following:

Theorem. The cyclic splitting complex $FZ_n$ is $\delta$-hyperbolic.

In further work, the author wishes to describe the elements of $Out(F_n)$ which act hyperbolically on $FZ_n$. In particular, a description of these elements should show that $FZ_n$ is not $Out(F_n)$-equivariantly quasi-isometric to the free factor complex.

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2. A Bowditch Hyperbolicity Condition for Graphs

A graph is a connected 1-dimensional simplicial complex. If $X$ and $Y$ are graphs, a graph map is a continuous map $f : X \to Y$ such that vertices map to vertices. As always, the vertex set of a graph $X$ is denoted by $V(X)$, and the edge set by $E(X)$. From now on, whenever considering a (connected) simplicial complex $Z$ as a metric space, we mean the 1-skeleton of $Z$ with the simplicial metric. We denote a geodesic path from a vertex $x$ to a vertex $y$ by $[x,y]$.

In [3] Bowditch develops a criterion for a graph to be $\delta$-hyperbolic. Similar criteria were applied in the Masur-Minsky proof that the curve complex is hyperbolic. Bowditch defines, for constants $B_1, B_2 > 0$ a $(B_1, B_2)$-thin triangles structure in a graph $X$, which is a set of paths $g_{xy}$ between any $x, y \in X$ satisfying some “thinness” conditions (see [6] for details). Bowditch proves the following useful condition for checking hyperbolicity in [3]:

Theorem 1 (Bowditch). Suppose $X$ is a connected graph. If there are $B_1, B_2$ such that $X$ has a $(B_1, B_2)$-thin triangles structure, then there are $\delta > 0$ and $H > 0$ (depending on $B_1, B_2$) such that $X$ is $\delta$-hyperbolic, and every geodesic path from $x$ to $y$ in $X$ is $H$-close to $g_{xy}$.

The following proposition, which is proved as a corollary of the above theorem in [6], will be the main technical tool:

Theorem 2. Suppose $X$ and $Y$ are connected graphs, $X$ is $\delta$-hyperbolic, and $f : X \to Y$ is $L$-Lipschitz for some $L \geq 0$. Suppose there is $S \subseteq V(X)$ such that

1. $f(S) = V(Y)$
2. $S$ is $D$-dense in $V(X)$ for some $D \geq 0$. 

\begin{enumerate}
  \item[(3)] There is an $M > 0$ such that if $x, y \in S$ with $d(f(x), f(y)) \leq 1$ then $\text{diam}(f[x, y]) \leq M$.

Then $Y$ is $\delta'$-hyperbolic for some $\delta'$.
\end{enumerate}

3. The Free Splitting Complex

A tree is a simply connected graph. An action of a group $G$ on a tree $T$, denoted $G \curvearrowright T$, is a homomorphism from $G$ to the group of simplicial automorphisms of $T$. An action $G \curvearrowright T$ is called minimal if there is no proper $G$ invariant subtree of $T$.

Let $F_n$ denote the free group on $n$-generators. Recall from Bass-Serre theory that a minimal action $F_n \curvearrowright T$ with trivial edge stabilizers gives a a graph of groups decomposition of $F_n$ with trivial edge groups (and hence a free splitting), and conversely. We shall often refer to the action $F_n \curvearrowright T$ as a free splitting, as there will be no confusion. A $k$-edge splitting refers to a free splitting whose associated graph of groups decomposition consists of $k$ edges. Two splittings $F_n \curvearrowright T$ and $F_n \curvearrowright T'$ are equivalent if there exists an $F_n$ equivariant homeomorphism $T \to T'$.

An equivariant map $f : T \to T'$ between minimal $F_n$-trees is called a collapse map if the preimage of any point is connected.

Define the free splitting complex of $F_n$, denoted $FS_n$ as follows. For a more complete discussion see [5]. A vertex of $FS_n$ is an equivalence class of 1-edge free splittings. Two vertices $\bar{X}, \bar{Y} \in V(FS_n)$ are connected by an edge if there exists a two edge splitting $T$ and $F_n$-equivariant collapse maps $T \to X$ and $T \to Y$. We say $T$ is a common refinement of $X$ and $Y$. A $k$-simplex in $FS_n$ is a collection of $k + 1$ vertices $X_1, \ldots, X_{k+1}$ such that there exists a $k + 1$ edge splitting $T$ and $F_n$-equivariant collapse maps $T \to X_i$ for each $i = 1, \ldots, k + 1$.

Denote by $FS_n'$ the barycentric subdivision of $FS_n$. This is a simplicial complex whose vertices correspond to free splittings of $F_n$, and where there is an edge between vertices $T$ and $T'$ if there is an equivariant collapse map $T \to T'$ or $T' \to T$. The complexes $FS_n$ and $FS_n'$ are finite dimensional, connected, and have an action of $Out(F_n)$ by simplicial automorphisms such that the quotient is compact (see [5]).

4. Folding Paths in $FS_n$

For a general definition of folding paths in $FS_n'$, see [5]. We need only special types of folding paths between splittings in $CV_n$ which are also discussed in [6], but we will give an explanation here as well following the treatment there.

Let $T$ be a tree. Vertices of valence $\geq 3$ are natural vertices, and connected components of $T \setminus \{\text{natural vertices}\}$ are natural edges.

A rose $R_n$ is a graph with one vertex and $n$-edges. Given an identification $F_n = \pi_1(R_n)$, a marking of a graph $\Gamma$ is a homotopy equivalence $f : \Gamma \to R_n$. Two markings $f : \Gamma \to R_n$ and $f' : \Gamma' \to R_n$ are equivalent if there is a homeomorphism $\phi : \Gamma' \to \Gamma$ such that $f' \phi \simeq f'$. In particular, an equivalence class of markings gives an isomorphism $\pi_1(\Gamma) \to F_n$.

Let $\beta$ be a basis for $F_n$. As in [6] define a $\beta$-graph to be a graph $\Gamma$ with a function $\mu : E(\Gamma) \to \beta \cup \beta^{-1}$ such that if $\bar{e}$ is an oriented edge of $\Gamma$ and $\bar{e}$ denotes the edge with the opposite orientation, then $\mu(\bar{e}) = \mu(e)^{-1}$. Let $R_\beta$ be the rose whose (oriented) edges are labelled by elements of $\beta$ and their inverses. This labeling gives an identification of $F_n$ with $\pi_1(R_\beta)$. 

The labeling of a $\beta$-graph $\Gamma$ determines a map $\Gamma \to R_\beta$ by sending each edge of $\Gamma$ to the edge of $R_\beta$ with the same label. In particular, if this map $\Gamma \to R_\beta$ is a homotopy equivalence, then the labeling of $\Gamma$ gives a marking.

**Remark.** A marking of $\Gamma$ corresponds to an action of $F_n$ on the universal cover $\tilde{\Gamma}$, and hence to a point in $FS'_n$. Equivalent markings define the same vertex in $FS'_n$. We will use $\Gamma$ to denote the vertex in $FS'_n$ determined by the marking, hopefully without any confusion.

**Stallings folds.** Let $\Gamma$ be a $\beta$-graph such that there exists two edges $e_1$ and $e_2$ with the same initial vertex and such that $\mu(e_1) = \mu(e_2)$. We obtain another $\beta$-graph $\Gamma'$ by identifying the edges $e_1$ and $e_2$, and labeling the resulting edge by $\mu(e_1) = \mu(e_2)$. This is called a *Stallings fold* (see [10]). There is a quotient map $\Gamma \to \Gamma'$ which is called a *fold map*. Note that if $e_1$ and $e_2$ have distinct terminal vertices, then $\Gamma \to \Gamma'$ is a homotopy equivalence.

Suppose that $\Gamma$ is a $\beta$-graph and that the labeling gives a marking $\Gamma \to R_\beta$. If we have two edges in $\Gamma$ with the same initial vertex and label, we can construct another graph $\Gamma'$ from $\Gamma$ by a Stallings fold, and the marking $\Gamma \to R_\beta$ factors as $\Gamma \to \Gamma' \to R_\beta$. Furthermore, the map $\Gamma' \to R_\beta$ is again a marking.

**Maximal folds.** Suppose $\Gamma$ is a $\beta$-graph and that there exist two edges $e_1$ and $e_2$ in $\Gamma$ with the same initial edge and such that $\mu(e_1) = \mu(e_2)$. Let $\hat{e}_1$ and $\hat{e}_2$ be natural edges containing $e_1$ and $e_2$. Then $\hat{e}_1$ and $\hat{e}_2$ contain maximal initial segments $\hat{e}_1$ and $\hat{e}_2$ which are labeled by the same word in $\beta$. Therefore we can obtain another graph $\Gamma'$ by identifying the segments $\hat{e}_1$ and $\hat{e}_2$. We say $\Gamma'$ is obtained by a maximal *Stallings fold* or just a maximal fold.

**Foldable Maps and Handel-Mosher Folding Paths.** Let $\Gamma$ be a $\beta$-graph, and $f : \Gamma \to R_\beta$ given by the labeling is a marking. We say that the map $f$ is *foldable* if

1. For every vertex $v$ of valence 2, the edges $e_1$ and $e_2$ with initial vertex $v$ have $\mu(e_1) \neq \mu(e_2)$.
2. For every natural vertex $v$, there exist three edges $e_1$, $e_2$, and $e_3$ with the same initial vertex $v$ such that $\mu(e_1)$, $\mu(e_2)$ and $\mu(e_3)$ are pairwise unequal.

There are a few important properties about foldable maps and maximal folds which we need:

- A map $\Gamma \to R_\beta$ is foldable in the sense above if and only if the corresponding map between $F_n$-trees $\tilde{\Gamma} \to \tilde{R}_\beta$ is foldable in the sense of Handel-Mosher in [5].
- If $\Gamma$ is a $\beta$-graph and $\Gamma \to R_\beta$ is foldable, and if $\Gamma'$ is obtained from $\Gamma$ by a maximal fold, the induced map $\Gamma' \to R_\beta$ is foldable.
- If $\Gamma \to \Gamma'$ is a maximal fold, then $d_{FS'_n}(\Gamma, \Gamma') \leq 2$.
- If $\Gamma \to R_\beta$ is a marking then there exists a finite sequence of maximal folds $\Gamma = \Gamma_0 \to \Gamma_1 \to \ldots \to \Gamma_N = R_\beta$.

The proofs of these are elementary and found in [5]. We will also need to following result:

**Theorem 3** (Handel-Mosher [5]). The path in $FS'_n$ given by connecting each $\Gamma_i$ and $\Gamma_{i+1}$ by an edge path of length $\leq 2$ is an unparametrized quasi-geodesic in $FS'_n$. 

5. The Cyclic Splitting Complex \( FZ_n \)

First, let \( F_n \rtimes T \) be a free splitting. Let \( v \) be a vertex of \( T \), and let \( G_v \) be the stabilizer of \( v \) in \( F_n = G \). Suppose \( w \in G_v \) and let \( \langle w \rangle \) denote the cyclic subgroup generated by \( w \). Construct a new \( F_n \)-tree \( T' \) as follows: choose an edge \( e \) with initial vertex \( v \). Then for every \( \gamma \in G_v \), identify \( \gamma e \) with its orbit under the conjugate \( \langle \gamma w \gamma^{-1} \rangle \subseteq G_v \).

The resulting tree \( T' \) corresponds to a graph of groups decomposition with an edge group \( \langle w \rangle \). In particular, if \( T \) is a one edge free splitting, then \( T' \) is a one edge splitting with cyclic edge group. See figures 1 and 2 below.

We say \( T' \) is obtained from \( T \) by an equivariant edge fold. The natural map \( T \to T' \) is called an edge folding map.

**Example 1.** Suppose \( F_4 = \langle a, b, c, d \rangle \). Consider the one-edge free splitting \( A \ast B \) given by \( A = \langle a, b \rangle \) and \( B = \langle c, d \rangle \). Then the one edge \( \mathbb{Z} \)-splitting \( A \ast \langle [a, b] \rangle \langle B, [a, b] \rangle \) is obtained from \( A \ast B \) by an edge fold.

It is a theorem of Bestvina and Feighn (see Lemma 4.1 in \([1]\)) that any \( \mathbb{Z} \)-splitting can be “unfolded.”

**Theorem 4** (Bestvina-Feighn \([1]\)). Let \( \Gamma \) be a graph of groups decomposition of the free group \( F_n \) with cyclic edge groups. Then either all of the edge groups of \( \Gamma \) are trivial or there exists an edge \( e \) with stabilizer \( G_e \cong \mathbb{Z} \) and a vertex \( v \) which is an endpoint of \( e \) such that the inclusion \( i : G_e \to G_v \) has image a free factor of \( G_v \) and furthermore for any edge \( e' \) incident at \( v \) the image of \( G_{e'} \to G_v \) lies in a complementary free factor.

In particular, Theorem 4 generalizes an earlier theorem of Shenitzer, Stallings, and Swarup (see \([9], [11], [12]\)) that any \( \mathbb{Z} \)-splitting \( A \ast_2 B \) is obtained by edge folding from a free splitting as in the above example.

Define a complex \( FZ_n \), the cyclic splitting complex of \( F_n \) as follows. The vertices of \( FZ_n \) are 1-edge free splittings of \( F_n \). Free splittings \( X \) and \( Y \) are connected by an edge if

- there exists a 2-edge splitting and \( F_n \)-equivariant collapse maps \( T \to X \) and \( T \to Y \).
- there exists a \( \mathbb{Z} \)-splitting \( T \) and equivariant edge folds \( X \to T, Y \to T \).

Distinct vertices \( X_1, \ldots, X_{k+1} \) define a \( k \)-simplex if each \( X_i \) and \( X_j \) with \( i \neq j \) are pairwise adjacent.

Note that there is a natural inclusion \( i : FS_n \to FZ_n \). If two free splittings are connected by an edge in \( FS_n \), then their images are also connected by an edge of the first type in \( FZ_n \). \( Out(F_n) \) acts on \( FZ_n \) by simplicial automorphisms in the obvious way.

We can now extend this map \( i \) to a map \( f \) from the barycentric subdivision of \( FS_n \) to \( FZ_n \) as follows: A vertex \( V \) of \( FS_n \) is a \( k \)-edge splitting of \( F_n \). Define \( f(V) \) to be the splitting obtained by collapsing all edges but one to a point. The map is only coarsely well-defined, but for any choice of edge in \( V \), the 1-edge splittings obtained will be at most distance 1 apart. Then extend to a graph map from \( FS_n \to FZ_n \).
Furthermore, \( f \) restricts to \( i \) on the vertices of \( FS_n \) (these are already 1-edge splittings), and \( f \) is clearly 1-Lipschitz as well. We will need the following useful lemma.

**Lemma 1.** Suppose \( X \) and \( Y \) are one-edge, two-vertex splittings connected by an edge of the second type in \( FZ_n \). Then there exist vertices \( X' \) and \( Y' \) with \( d(X, X') \leq 1 \) and \( d(Y, Y') \leq 1 \) such that \( d(X', Y') \leq 1 \) and which share a vertex group.

**Proof.** Let \( \langle w \rangle \) be the edge group of the \( Z \)-splitting to which \( X \) and \( Y \) fold, and let \( A \) be the smallest free factor containing \( \langle w \rangle \) (in which case, we say that the element \( w \) fills \( A \)). Then there exist \( X' = A \ast B \) and \( Y' = A \ast B' \) such that \( X \) and \( X' \), and \( Y \) and \( Y' \) share a common refinement. \( \square \)

## 6. Hyperbolicity of \( FZ_n \)

We use the map \( f : FS'_n \to FZ_n \) and the method pioneered in [6] to prove the following theorem:

**Theorem 5.** The cyclic splitting complex \( FZ_n \) is \( \delta \)-hyperbolic.

**Proof.** Let \( S \) be the set of 1-edge splittings in \( FS_n \). Conditions (1) and (2) of Theorem 2 are clearly satisfied.

Since \( FS_n \) is \( \delta \)-hyperbolic by [5], by Theorem 2 it suffices to show condition (3) is true: that there exists an \( M > 0 \) such that for any 1-edge free splittings \( X \) and \( Y \), if \( d_{FZ_n}(f(X), f(Y)) \leq 1 \), then \( \text{diam}(f[X, Y]) \leq M \).

Suppose \( X \) and \( Y \) are 1-edge free splittings of \( F_n \) which are joined by an edge of the second type in \( FZ_n \) (note that it suffices to cover this case because an edge of type 1 corresponds to being distance 1 in \( FS'_n \)). Suppose \( T \) is the \( Z \)-splitting such that there exist edge folds \( X \to T \) and \( Y \to T \).

There are two cases to cover: (1) the graph of groups of the \( Z \)-splitting \( T \) is a segment and (2) is a loop:
Case 1. By lemma 1, choosing splittings at most distance 1 away we may assume $X = A \ast B$ and $Y = A \ast B'$, and that the element $w$ fills $A$. Then the condition that $X$ and $Y$ fold to $T$ is exactly the condition that $B \ast \langle w \rangle = B' \ast \langle w \rangle$. Let $R$ and $R'$ be free splittings defined as follows: both have underlying graphs which are roses, and the loops of $R$ represent elements of bases of $A$ and $B$; in particular choose a basis $\{a_1, \ldots, a_k\}$ of $A$ and a basis $\{b_1, \ldots, b_l\}$ of $B$ and label the edges of $R$ by the collection of $a_i$’s and $b_j$’s. Denote the basis of $F_n$ formed by this collection by $\beta$.

Choose a basis $\{b'_1, \ldots, b'_l\}$ for $B'$ and define $R'$ as the rose whose edges represent the elements in the basis $\beta' = \{a_1, \ldots, a_k, b'_1, \ldots, b'_l\}$. Label the edges of $R'$ by the elements of $\beta'$ written in the basis $\beta$ (subdividing the edges of $R'$ as necessary) so that both are $\beta$-graphs. Note that we see a subgraph labelled by the $a_i$’s in both. This gives a homotopy equivalence $R' \rightarrow R$. By perhaps conjugating, we may assume that the map $R' \rightarrow R$ is foldable; indeed $R' \rightarrow R$ fails to be foldable exactly when the word labeling each edge starts and ends with some $a_i$, so by conjugating every label we can be sure this does not happen without changing the splitting. Note that $R$ (resp. $R'$) has $d(f(R), X) \leq 1$ (resp. $d(f(R'), Y) \leq 1$).

Then choose a Handel-Mosher folding path $R' = \Gamma_0 \rightarrow \Gamma_1 \rightarrow \cdots \rightarrow \Gamma_N = R$ as follows: recall that $B \ast \langle w \rangle = B' \ast \langle w \rangle$ so that the basis elements $b'_1, \ldots, b'_l$ written in terms of $\beta$ are just words in $w$ and $b_1, \ldots, b_l$. Each maximal fold $\Gamma_i \rightarrow \Gamma_{i+1}$ occurs as one of the following two types, either (1) fold a loop labelled by some $b'_j$ over a letter $a_i$ in the word $w$ or (2) fold maximal initial segments of two distinct loops labelled by some $b'_j$ and $b'_l$. We require to fold an entire $w$ before moving on to a fold of the second type: if we do a maximal fold of type 1 and fold only a proper subword of $w$, then after this fold we still see natural edges with the same initial label. By [5], regardless of the order in which the edges are folded, we still end up at $R$, so we continue doing maximal folds until we have folded out the entire word $w$.

In particular, each type of fold (1) or (2) either leaves the splitting $f(\Gamma_{i+1})$ (coarsely) equal to $f(\Gamma_i)$ or it gives another splitting $A \ast B_i$ within distance 1 of $f(\Gamma_i)$ so that $B_i \ast \langle w \rangle = B' \ast \langle w \rangle$. In either case, at each step of the folding path $\Gamma_i$, we have $d(f(\Gamma_i), X) \leq 3$.

Case 2. Suppose the vertex group of $X$ is $A \ast B$ and the vertex group of $Y$ is $A \ast B'$, where $A$ is the smallest free factor containing $\langle w \rangle$. $X$ and $Y$ are adjacent to a common $\mathbb{Z}$-splitting $T$ exactly when $A \ast B \ast \langle w^t \rangle = A \ast B' \ast \langle w^t \rangle$, where $t$ is the element of $F_n$ corresponding to the non-trivial loop in the graph of groups, and $\langle w \rangle$ is the edge group of $T$.

We follow the same basic outline as in the segment case: choose splittings $R$ and $R'$ as follows: Let $\{a_1, \ldots, a_k\}$ be a basis of $A$, and $\{b_1, \ldots, b_l\}$ a basis of $B$. Let $R$ be the splitting with underlying graph a rose and whose edges are labelled by the elements in the basis $\beta = \{a_1, \ldots, a_k, t, b_1, \ldots, b_l\}$. Choose a basis $\{b'_1, \ldots, b'_l\}$ for $B'$ and let $\beta' = \{a_1, \ldots, a_k, t, b'_1, \ldots, b'_l\}$. We choose $R'$ to be the rose whose edges are represent the elements in the basis $\beta'$. Label the edges of $R'$ by this elements of the basis $\beta'$ written in the letters of $\beta$ so that both $R$ and $R'$ become $\beta$-graphs.

By conjugating, we may assume that the homotopy equivalence $R' \rightarrow R$ given by the markings is foldable. Also recall that $A \ast B \ast \langle w^t \rangle = A \ast B' \ast \langle w^t \rangle$ so that every basis element $b'_j$ is written as a word in $A \ast B \ast \langle w^t \rangle$. Thus, there is a Handel-Mosher folding path $R' \rightarrow \Gamma_0 \rightarrow \Gamma_1 \rightarrow \cdots \rightarrow \Gamma_N = R$ all of whose maximal folds $\Gamma_i \rightarrow \Gamma_{i+1}$ either (1) fold an edge labelled by some $b'_j$ over a edge whose label is a letter in the
word \( w^t \) or (2) folds an initial segment of an edge labelled by \( b_i \) with the initial segment of a different edge labelled by \( a_j \) or \( b_j \). As above, with the first type of fold we make sure to fold an entire word \( w^t \) before moving on. In both cases, either the splitting \( f(\Gamma_{i+1}) \) is within distance 1 of \( f(\Gamma_i) \) or \( f(\Gamma_i) \) is distance 1 from a 1-edge splitting whose vertex group is of the form \( A \ast B_i \ast \langle w^t \rangle \).

In either case, for each \( i \) we have \( d(f(\Gamma_i), X) \leq 3 \).

By [5], since each of the maps \( \Gamma_i \to \Gamma_{i+1} \) are maximal folds, \( d(\Gamma_i, \Gamma_{i+1}) \leq 2 \). Since the map \( f \) is Lipschitz, this implies that the image of this folding path is contained in a bounded neighborhood of the splitting \( T \). Furthermore, since folding paths are unparametrized quasi-geodesics by Theorem 3, this path is uniformly close to \([R',R]\), and because \( FS'_n \) is hyperbolic, the geodesic \([X,Y]\) is uniformly close to \([R',R]\). Putting all this together, we see that there exists a constant \( M \geq 0 \) such that \( \text{diam}(f[X,Y]) \leq M \).

\[ \square \]

7. Other definitions

There are a few other candidate complexes that we might have called the cyclic splitting complex. We will show that all these complexes are \( \text{Out}(F_n) \)-equivariantly quasi-isometric.

Define the complex \( \mathcal{FZ}_n \) as follows: vertices of \( \mathcal{FZ}_n \) are one edge free or \( \mathbb{Z} \)-splittings of \( F_n \). Two such vertices \( X, Y \) are connected if (1) \( X \) and \( Y \) are free splittings which admit a common refinement or (2) \( X \) can be obtained from \( Y \) by an edge fold.

**Proposition 1.** \( \mathcal{FZ}_n \) and \( \mathcal{FZ}_n \) are \( \text{Out}(F_n) \)-equivariantly quasi-isometric.

**Proof.** Define a map \( \phi : \mathcal{FZ}_n \to \mathcal{FZ}_n \) in the obvious way: send a vertex of \( \mathcal{FZ}_n \) to the corresponding free splitting in \( \mathcal{FZ}_n \). Extend to a map of the entire complex. \( \phi \) is clearly equivariant.

Let \( X, Y \in V(\mathcal{FZ}_n) \) with \( d(X, Y) \leq 1 \). Then by definition of \( \mathcal{FZ}_n \), at worst \( d(\phi(X), \phi(Y)) \leq 2 \). Hence \( d(X, Y) \leq 2d(\phi(X), \phi(Y)) \). Furthermore, if \( d(f(X), f(Y)) \leq 2 \), then \( X \) and \( Y \) are joined by a path of length at most 2. In particular,

\[
\frac{1}{2} d(X, Y) \leq d(\phi(X), \phi(Y)) \leq 2d(X, Y)
\]

so \( \phi \) is quasi-isometry as desired. \[ \square \]

There is third complex whose definition more closely resembles the definition of \( FS_n \). Define a complex \( C_n \) whose vertices are 1-edge free or \( \mathbb{Z} \)-splittings and where two vertices \( X \) and \( Y \) are connected by an edge if the corresponding splittings have a 2-edge common refinement.

**Proposition 2.** \( \mathcal{FZ}_n \) and \( C_n \) are \( \text{Out}(F_n) \)-equivariantly quasi-isometric.

**Proof.** We will actually show that there is an \( \text{Out}(F_n) \)-equivariant quasi-isometry \( \phi : \mathcal{FZ}_n \to C_n \). The vertex sets of \( \mathcal{FZ}_n \) and \( C_n \) are the same, so set \( \phi \) to be the identity on vertices. Then extend \( \phi \) to a map of graphs.

Let \( X \) and \( Y \) be vertices of \( \mathcal{FZ}_n \) such that \( d(X, Y) \leq 1 \). Then by folding the edge group \( \langle w \rangle \) “half-way” over the edge of the splitting, we get a two-edge splitting which commonly refines both \( X \) and \( Y \). More precisely, if \( X \) and \( Y \) are one-vertex splittings, consider the 2-edge splitting in which the edges are adjacent at both
endpoints, the vertex groups are the vertex group of $X$ and the edge group $\langle w \rangle$ of the $Z$-splitting, and the edge groups are trivial and $\langle w \rangle$.

Similarly, if $X$ and $Y$ are two-vertex splittings, say $X = A * B$ and $Y = A * (w) \langle B, w \rangle$, then we can consider the two edge splitting $A * (w) \langle w \rangle * B$ which refines $X$ and $Y$. Hence $d(\phi(X), \phi(Y)) \leq d(X, Y)$.

Now suppose $X$ and $Y$ are cyclic splittings which are commonly refined by a two-edge splitting (so $d(\phi(X), \phi(Y)) \leq 1$). We need to find a uniform $L > 0$ such that $d(X, Y) \leq L$.

Suppose that $X$ is a $Z$-splitting and $Y$ is a free splitting. Then we can unfold the edge group from $X$ to get a free splitting $X'$. Since $X$ and $Y$ are commonly refined, writing down the vertex groups of $X'$ we see that $X'$ is commonly refined with $Y$, and hence $d(X, Y) \leq 2$ in $FZ_n$. We will do one case carefully - the others are similar and left to the reader. Suppose the graphs of groups corresponding to both $X$ and $Y$ are segments. Then the common refinement is $A * (w) (B * C)$, with $X = A * (w) (B * C)$ and $Y = (A * (w) B) * C$.

Suppose furthermore that $A * (w) (B * C)$ unfolds to $X' = A * (B' * C)$. Then $X'$ and $Y$ are commonly refined by the two-edge splitting $A * B' * C$ since the factors $A * (w) B$ and $A * B'$ are equal. The other cases are similar, so we have the above inequality for $L = 2$.

Suppose both $X$ and $Y$ are $Z$-splittings which are commonly refined by a two-edge splitting, say $A * (s) B * (t) C$, so in particular $X = (A * (s) B) * (t) C$ and $Y = A * (s) (B * (t) C)$. By Theorem 4 above, one of the edge groups $\langle s \rangle$ or $\langle t \rangle$ can be unfolded to get a splitting with one trivial edge stabilizer, and one $Z$-stabilizer. We can then unfold the remaining $Z$-edge to get a free splitting, say $A' * B' * C'$ which is a common refinement of the 1-edge free splittings $(A' * B') * C'$ and $A' * (B' * C')$. Furthermore, we can fold the element $t$ over the edge of the splitting $(A' * B') * C'$ to get a $Z$-splitting equal to $X$ and folding $s$ over the edge of $A' * (B' * C')$ we obtain the splitting $Y$.

Hence, $d(X, Y) \leq 3$. Therefore, we have

$$\frac{1}{3} d(X, Y) \leq d(\phi(X), \phi(Y)) \leq 3d(X, Y)$$

so $\phi : FZ_n \to C_n$ is a quasi-isometry.

Remark

There are natural $Out(F_n)$-equivariant maps: $FS_n \to FZ_n$ discussed above and $FZ_n \to FF_n$, which is given by sending a vertex of $FZ_n$ to one of the edge groups of the corresponding free splittings. There is also a natural map $FS_n \to FF_n$ defined in the same way, which clearly factors through the above maps. It is a priori unclear that these maps are not quasi-isometries.

**Proposition 3.** For $n \geq 3$, the natural $Out(F_n)$-equivariant map $FS_n \to FZ_n$ is not a quasi-isometry.

**Proof.** We assume the following result claimed in [3], and which is to appear in second part of their work on the free splitting complex: An element $\phi \in Out(F_n)$ acts hyperbolically on $FS_n$ there exists an attracting lamination $\Lambda$ of $\phi$ whose support is all of $F_n$, i.e. $\Lambda$ is not carried by a proper free factor.
Let $S$ be a surface so that $\pi_1(S)$ is free of rank $n \geq 3$, and suppose there exists a (non-separating) simple closed curve $C$ in $S$ such that $S \setminus C = \Sigma$ admits a pseudo-Anasov mapping class. $C$ gives a $\mathbb{Z}$-splitting $\pi_1(S) = A \ast C$. Such a surface exists for all $n \geq 3$; indeed, for $n \geq 4$, we can take a surface of genus $\geq 2$ with either one or two boundary components. To get $n = 3$, take a twice-punctured torus. We can cut along a curve $C$ to get a 4-times punctures sphere, which admits a pseudo-Anasov homeomorphism.

Let $\phi$ be a pseudo-Anasov mapping class of $\Sigma$. Hence $\phi$ can also be thought of as an outer automorphism of $\pi_1(S)$ which fixes the splitting $A \ast C$. It remains to show that the expanding lamination $\Lambda$ of $\phi$ is not carried by a proper free factor. Since $\phi$ is pseudo-Anasov, it follows that $\Lambda$ restricted to $\Sigma$ is minimal and filling.

Suppose not, and let $H$ be a factor which carries a leaf $L$ of $\Lambda$ (and hence carries all of $\Lambda$ by minimality). Let $\tilde{S}$ be the cover of $S$ corresponding to $H$. Since $H$ is finitely generated, there exists a compact subsurface $S_H$ of $\tilde{S}$ which has fundamental group $H$. Let $\bar{L}$ be a lift of $L$ to $\tilde{S}$ which is contained in $S_H$. Let $\Sigma'$ be the smallest subsurface of $S_H$ which $\bar{L}$ fills. Suppose $\bar{L}_0$ is any other lift of $L$ which meets $\Sigma'$. If $\bar{L}_0$ is not contained in $\Sigma'$, it enters through some boundary component. By Theorem 5.2 in [4] (this is for foliations, but the result for laminations is similar), $\bar{L}_0$ exits through a boundary component of $\Sigma'$. In particular, by cutting $\Sigma'$ along $\bar{L}_0$ we would obtain a smaller surface which $\bar{L}$ fills, a contradiction. Hence, any lift of $\bar{L}$ meeting $\Sigma'$ must be contained in $\Sigma'$.

In particular, $\Sigma'$ is (homotopic to) a finite cover of $\Sigma$, so $H$ contains a finite index subgroup of $\pi_1(\Sigma) = A$, which is impossible unless $H = F_n$. Indeed, $\pi_1(\Sigma)$ cannot be contained in a proper free factor; looking at Euler characteristics we see $\chi(S) = \chi(\Sigma)$ so $\chi(\tilde{\Sigma}) \leq \chi(S)$, so the rank of $\pi_1(\tilde{\Sigma})$ is at least $n$.

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