SEMISIMPPLICITY OF ÉTALE COHOMOLOGY OF CERTAIN SHIMURA VARIETIES

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ABSTRACT. Building on work of Fayad and Nekovář, we show that a certain part of the étale cohomology of some abelian-type Shimura varieties is semisimple, assuming the associated automorphic Galois representations exists, and satisfies some good properties. The proof combines an abstract semisimplicity criterion of Fayad-Nekovář with the Eichler-Shimura relations.

1. INTRODUCTION

Let $G$ be a reductive group over $\mathbb{Q}$, and $X$ be a conjugacy class of homomorphisms

$$h : \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$$

such that $(G, X)$ is a Shimura datum. Given a compact open subgroup $K \subset G(\mathbb{A}_f)$, we can form the associated Shimura variety

$$\text{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K,$$

which is a complex manifold for $K$ small enough, and has a canonical model over $E$, for some number field $E$.

Consider a representation valued in complex vector spaces

$$\xi : G_{\mathbb{C}} \to GL(V_\xi)$$

such that $\xi(Z(\mathbb{Q}) \cap K) = 1$, where $Z$ is the center of $G$. This gives rise to a locally constant sheaf of complex vector spaces

$$L_\xi = G(\mathbb{Q}) \backslash V_\xi \times X \times G(\mathbb{A}_f)/K.$$

We now assume that $G^{\text{der}}$ is anisotropic, so that the Shimura variety $\text{Sh}_K(G, X)$ is compact. Similar results should hold with étale cohomology replaced with intersection cohomology of the Baily-Borel compactification $\text{Sh}^{BB}_K(G, X)$ of the Shimura variety.

We now consider the complex analytic cohomology of the tower of the Shimura variety, i.e.

$$H^i(\text{Sh}(G, X)^{\text{an}}, L_\xi) := \lim_{\rightarrow K} H^i(\text{Sh}_K(G, X)^{\text{an}}, L_\xi).$$

Choose $h \in X$, and let $K_\infty$ be the stabilizer of $h$ in $G(\mathbb{R})$. By Matsushima’s formula, we have a decomposition of $H^i(\text{Sh}(G, X)^{\text{an}}, L_\xi)$ in terms of Lie algebra cohomology

$$H^i(\text{Sh}(G, X)^{\text{an}}, L_\xi) = \bigoplus_{\pi = \pi_\infty \otimes \pi_\infty} m(\pi) H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes \xi, \pi_\infty),$$

where $\pi$ runs through unitary automorphic representations of $G(\mathbb{A})$, and $m(\pi)$ is the multiplicity of $\pi$ appearing in $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)$, the space of measurable functions on $G(\mathbb{Q}) \backslash G(\mathbb{A})$ which are square-integrable modulo center, where $\omega$ is the central character of $\pi$. Recall that $\pi_\infty$ is
cohomological in degree $i$ for $\xi$ (i.e. $H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes \xi) \neq 0$) if and only if the central character $\omega_\xi$ of $\xi$ satisfies $\omega_\xi|_{Z(R)} = \omega_{\pi_\infty}^{-1}$.

Fix a prime $l$, and an isomorphism $\iota: \mathbb{Q}_l \sim \to \mathbb{C}$. The representation $\xi$ gives rise to an $l$-adic automorphic sheaf, in the following way. $\xi$ gives rise to a representation valued in $\mathbb{Q}_l$-vector spaces $\xi_{\mathbb{Q}_l} : G_{\mathbb{Q}_l} \to GL(V_{\xi,l})$, and similarly gives rise to a sheaf $L_{\xi,l} = G(\mathbb{Q}) \times X \times G(\mathbb{A}_f)/K$.

If we consider the étale cohomology of the tower

$$H^i_{\text{ét}}(\text{Sh}(G, X)_{\mathbb{Q}}, L_{\xi,l}) = \lim_{\to K} H^i_{\text{ét}}(\text{Sh}_K(G, X)_{\mathbb{Q}}, L_{\xi,l}),$$

then we have an isomorphism

$$H^i_{\text{ét}}(\text{Sh}(G, X)_{\mathbb{Q}}, L_{\xi,l}) \simeq H^i(\text{Sh}(G, X)^{an}, L_\xi) \otimes \mathbb{Q}_l,$$

which is $G(\mathbb{A}_f)$-equivariant, and thus we have a decomposition

$$H^i_{\text{ét}}(\text{Sh}(G, X)_{\mathbb{Q}}, L_{\xi,l}) = \bigoplus_{\pi^\infty} V^i(\pi^\infty) \otimes (\pi^\infty),$$

where $V^i(\pi^\infty) = \text{Hom}_{G(\mathbb{A}_f)}(\pi^\infty, H^i_{\text{ét}}(\text{Sh}(G, X)_{\mathbb{Q}}, L_{\xi,l}))$. We define the Galois representation

$$\rho : \text{Gal}(\overline{E}/E) \to V^i(\pi^\infty).$$

The global Langlands correspondence conjectures that to the automorphic representation $\pi$ we have an associated semisimple Galois representation $\text{Gal}(\overline{E}/E) \to L^G$, and we denote the composition

$$\tilde{\rho} : \text{Gal}(\overline{E}/E) \to L^G \xrightarrow{\gamma} GL(V_{\mu}),$$

where $\mu$ is the minuscule cocharacter associated to the Shimura datum.

When the group $G$ is of the form $\text{Res}_{F/\mathbb{Q}} G'$ for some connected reductive group $G'$, and $F$ is a totally real field of degree $d$, one has a decomposition (perhaps after passing to a finite extension $E'$ of $E$) of $\tilde{\rho}$ as

$$\tilde{\rho} = \bigotimes_{v|\infty} \hat{\rho}_v.$$

Moreover, $\hat{\rho}_v$ should have the following form. Observe that over $\mathbb{C}$, we have a decomposition

$$\text{Res}_{F/\mathbb{Q}} G' \simeq \prod_{v|\infty} G',$$

and the cocharacter $\mu$ also decomposes as $\prod_v \mu_v$. Then we should have

$$\hat{\rho}_v : \text{Gal}(\overline{E'/E'}) \xrightarrow{\phi_v} L^{G'} \xrightarrow{\gamma_{\mu_v}} GL(V_{\mu_v}),$$

where here we view $\pi$ as an automorphic representation of $G'$ over $F$.

The main theorem of this paper is the following:

**Theorem 1.0.1.** Let $(G, X)$ be a Shimura datum of abelian type such that $G = \text{Res}_{F/\mathbb{Q}} G'$ for some connected reductive group $G$, and totally real number field $F$. Let $\pi$ be an automorphic
representation of $G(\mathbb{A}_{Q,f}) = G'(\mathbb{A}_{F,f})$. For all $v$, suppose that the Galois representation associated to $\pi$ exists, and is given by $\tilde{\rho}_v : \text{Gal}(\overline{E}/E) \to L^{G'} \to GL(V_{\mu_v})$. Suppose that moreover we also know that

1. $\tilde{\rho}_v$ is strongly irreducible
2. For all primes $v'$ of $E$ such that $v'|l$, the Hodge-Tate weights of $\tilde{\rho}$, viewed as a $\text{Gal}(\overline{E}_{v'}/E_{v'})$-representation, are distinct.

Then $\rho$ is a semisimple representation.

To show this result, we first define partial Frobenius isogeny at a positive density of primes $p$, and then show the Eichler-Shimura congruence relations for partial Frobenius for split groups, using results from [Lee20]. More precisely, we show the following:

**Theorem 1.0.2.** Let $(G, X)$ be a Shimura datum of abelian type, such that $G = \text{Res}_{F/Q} G'$ for some connected reductive group $G$, and totally real number field $F$ of degree $d$. Let $p$ be a prime satisfying the conditions in Proposition 2.9.4. Then for all $i = 1, \ldots, d$ we have a partial Frobenius correspondence $\text{Frob}_{p_i}$ such that

$$\text{Frob} = \prod_i \text{Frob}_{p_i},$$

and

$$H_i(\text{Frob}_{p_i}) = 0,$$

where $H_i$ is the renormalized characteristic polynomial of the irreducible representation of $\hat{G}'_i$ with highest weight $\hat{\mu}_i$.

This, combined with a semisimplicity criterion for Lie algebras shown in [PN19], allows us to deduce the main result. In the final section, we apply this result to some abelian-type Shimura varieties attached to similitude groups, following the construction of the automorphic Galois representations in [KS20].

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2. Eichler-Shimura Relations

In this section, we review some key results shown in [Lee20] to prove the Eichler-Shimura relations.

2.1. $p$-divisible groups. For the entirety of this subsection, we fix a prime $p > 2$, and let $G$ be a connected reductive group over $\mathbb{Q}_p$. Let $k$ be a perfect field of characteristic $p$. We denote by $L = W(\mathbb{F}_p)[1/p]$ the maximal unramified extension of $\mathbb{Q}_p$.

**Definition 2.1.1.** A $p$-divisible group with $G$ structure over $k$ consists of a $p$-divisible group $\mathcal{G}/k$ and a collection of $\varphi$-invariant tensors $(s_{\alpha,0})$ which define a reductive subgroup of $GL(\mathbb{D}(\mathcal{G}))$ such that there exists a finite free $\mathbb{Z}_p$-module $U$ and an isomorphism

$$U \otimes_{\mathbb{Z}_p} W(k) \xrightarrow{\sim} \mathbb{D}(\mathcal{G})(W(k))$$

such that under this isomorphism $(s_{\alpha,0})$ correspond to tensors $(s_{\alpha}) \subset U^\otimes$. Moreover, these $s_{\alpha}$ define the reductive subgroup $G_{\mathbb{Z}_p} \subset GL(U)$.
Associated to any $p$-divisible group $\mathcal{G}$ with $G$-structure over $\overline{\mathbb{F}}_p$, we have a $G(W(\overline{\mathbb{F}}_p))-\sigma$-conjugacy class of elements $b \in G(L)$, such that under the isomorphism (2.1.2) the Frobenius on $\mathbb{D}(\mathcal{G})(W(\overline{\mathbb{F}}_p))$ is given by $b\sigma$.

Let $\mu$ be the minuscule cocharacter of $G$ such that $b$ lies in $G(W(\overline{\mathbb{F}}_p))p^\mu G(W(\overline{\mathbb{F}}_p))$ under the Cartan decomposition.

We now define the Rapoport-Zink space for the triple $(G,b,\mu)$ following [Kim18, Def. 4.6].

**Definition 2.1.3.** Let $\text{RZ}(G,b,\mu)$ be the functor which assigns to any $p$-locally nilpotent smooth $W$-algebra $R$ the set of isomorphism classes $((X,\rho,t_\alpha))$ such that

1. $(X,t_\alpha)$ is a $p$-divisible group over $R$ with tensors $t_\alpha \subset \mathbb{D}(X)\otimes$, where $(t_\alpha)$ consists of isomorphisms of crystals $t_\alpha : 1 \rightarrow \mathbb{D}(X)\otimes$ over $\text{Spec}(R)$ such that $t_\alpha : 1[1/p] \rightarrow \mathbb{D}(X)\otimes[1/p]$ is Frobenius equivariant;
2. $\rho : X_{R/p} \rightarrow X_{R/p}$ is a quasi isogeny;
3. For some nilpotent ideal $J \subset R$ containing $(p)$, the pull-back of $t_\alpha$ over $\text{Spec}(R/J)$ is identified with $s_\alpha$ under the isomorphism of isocrystals induced by $\rho$:
$$\mathbb{D}(X_{R/J})[1/p] \overset{s_\alpha}{\rightarrow} \mathbb{D}(X_{0,R/J})[1/p].$$
4. For some (any) formally smooth $p$-adic $W$-algebra $\hat{R}$ of $R$, endowed with the standard PD-structure on $\ker(\hat{R} \rightarrow R) = p^m \hat{R}$ for some $m$, let $(t_\alpha(\hat{R}))$ denote the $\hat{R}$-section of $(t_\alpha)$. Then the $\hat{R}$-scheme
$$P(\hat{R}) := \text{Isom}_R(\mathbb{D}(X)_{\hat{R}}, (t_\alpha(\hat{R})), [\hat{R} \otimes_{\mathbb{Z}_p} \Lambda^*, (1 \otimes s_\alpha)]),$$
classifying isomorphisms matching $(t_\alpha(\hat{R}))$ and $(1 \otimes s_\alpha)$, is a $G_W$-torsor.
5. The Hodge filtration $\text{Fil}^1(X) \subset \mathbb{D}(X)(R)$ is a $\{\mu\}$-filtration with respect to $(t_\alpha(R)) \subset \mathbb{D}(X)(R)\otimes$, where $\{\mu\}$ is the unique $G(W)$-conjugacy class of cocharacters such that $b \in G(W)p^{\mu}G(W)$.

If the group $G$ admits a decomposition over $\mathbb{Q}_p$ as $G = G_1 \times G_2$, then the associated Rapoport-Zink spaces also decompose, as the following proposition [Kim18, Thm 4.9.1] shows:

**Proposition 2.1.4.** Let $b = (b_1,b_2) \in G_1(L) \times G_2(L)$, and $\mu = \mu_1 \times \mu_2$. Then we have an isomorphism
$$\text{RZ}(G_1,b_1,\mu_1) \times_{\text{Spl}W} \text{RZ}(G_2,b_2,\mu_2) \cong \text{RZ}(G,b,\mu)$$
induced by taking product of $p$-divisible groups, and isogenies.

### 2.2. Partial Frobenius for Hodge type

Consider the situation where $G'$ is a connected reductive group over $\mathbb{Q}$, and $G = \text{Res}_{F/\mathbb{Q}}G'$, where $F$ is a totally real field of degree $r$ over $\mathbb{Q}$. Additionally, we suppose that the Shimura datum of interest is of Hodge type.

Suppose that $p$ is a prime which satisfies the following criterion:

1. $p$ splits in $F$.
2. The group $G$ has good reduction at $p$.

Observe that since $G$ is a reductive group, there is some finite extension $K$ of $\mathbb{Q}$ over which the group $G$ splits. Thus, we see that there is a positive density of primes $p$ which satisfies the above two criterion.
We can define the partial Frobenius as follows. We remark here that contrary to previous definitions of the partial Frobenius, we do not define the partial Frobenius over the universal abelian variety \( \mathcal{A} \) over \( \text{Sh}_K(G, X) \). Instead, we will define these elements only over isogeny classes, in a group theoretic way.

Observe that since \( p \) splits in \( F \), we see that

\[
G_{\mathbb{Q}_p} \simeq \prod G'_{i, \mathbb{Q}_p},
\]

and since the group \( G_{\mathbb{Q}_p} \) is split, so too are each of the factors \( G'_i \).

The decomposition of the group \( G \) induces a decomposition of the Rapoport-Zink space. We thus have an isomorphism of Rapoport-Zink spaces

\[
\text{RZ}(G, b) = \prod_i \text{RZ}(G'_i, b_i),
\]

and given an isogeny \( f \) over a characteristic \( p \) ring \( R \), we can decompose the isogeny \( f = \prod f_i \) for some \( f_i \in \text{RZ}(G'_i, b_i)(R) \).

In particular, we can apply the above construction to the Frobenius isogeny, to get quasi-isogenies \( \text{Frob}_p \), for \( i = 1, \ldots, r \) and by construction we have the following relationship between the partial Frobenius and the actual Frobenius

\[
\text{Frob}_p = \prod_i \text{Frob}_{p_i}.
\]

**Remark 2.2.1.** Note, moreover, that if we consider \( \text{Frob}_p \) as a \( p \)-power quasi isogeny between \( p \)-divisible groups over \( \overline{\mathbb{F}}_p \), represented by an element \( f \) in \( G(L) \), then we can write \( f = f_1 \cdots f_r \), for elements \( f_i \in G'_i(L) \).

### 2.3. The moduli space \( p - \text{Isog} \)

We again suppose that the Shimura variety is of Hodge type, and recall the constructions in [Lee20] of the associated moduli space \( p - \text{Isog} \).

Let \( T \) be a scheme over \( O_{E,(v)} \), and consider any two points \( x, y \) lying in \( \mathcal{S}_K(G, S)(T) \). For any geometric point \( t \) of \( T \), let \( x_t, y_t \) be the pullback of \( x, y \) to \( t \). From the main construction in [Kis10], we have \( l \)-adic étale and de Rham tensors \( (s_{x_{i, \alpha, l}}) \) for \( l \neq p \), and \( (s_{x_{t, \alpha, dR}}, s_{y_{t, \alpha, dR}}) \) for \( y_t \). Observe that \( k(t) \), the residue field at \( t \), could be of either characteristic 0 or characteristic \( p \). Suppose \( k(t) \) is a field of characteristic 0, i.e. it is an extension of \( E \). Then, we also have \( p \)-adic étale tensors \( (s_{x_{t, \alpha, p}}, s_{y_{t, \alpha, p}}) \). Otherwise, if \( k(t) \) is of characteristic \( p \), it is an extension of \( \kappa \). Similarly, we have crystalline tensors \( (s_{x_{t, \alpha, 0}}, s_{y_{t, \alpha, 0}}) \).

We define a quasi-isogeny between \( x, y \) to be a quasi-isogeny \( f : \mathcal{A}_x \to \mathcal{A}_y \) of abelian schemes over \( T \), such that for any geometric point \( t \), the induced quasi-isogeny \( f_t : \mathcal{A}_{x_t} \to \mathcal{A}_{y_t} \) of abelian varieties over \( k(t) \) preserves all the tensors described above.

We define a \( p \)-quasi-isogeny between \( x, y \) to be a quasi-isogeny as defined above, such that the isomorphism on the rational prime-to-\( p \) Tate modules \( f : \hat{V}^p(\mathcal{A}_x)_{\mathbb{Q}} \xrightarrow{\sim} \hat{V}^p(\mathcal{A}_y)_{\mathbb{Q}} \), induced by the quasi-isogeny \( \mathcal{A}_x \to \mathcal{A}_y \), respects the prime to \( p \)-level structures \( \varepsilon_x^p, \varepsilon_y^p \), i.e. \( \varepsilon_y^p \) is given by the composition

\[
\hat{V}^p_{\mathbb{A}_p} \xrightarrow{\sim} \hat{V}^p(\mathcal{A}_x)_{\mathbb{Q}} \xrightarrow{f} \hat{V}^p(\mathcal{A}_y)_{\mathbb{Q}}.
\]

In particular, we see that the weak polarizations on \( \mathcal{A}_x, \mathcal{A}_y \) differ by some power of \( p \).

Let \( p - \text{Isog} \) be the \( fppf \)-sheaf of groupoids of \( p \)-quasi-isogenies between points on \( \mathcal{S}_K^p(G, X) \). Concretely, for any \( O_{E,(v)} \)-scheme \( T \), points of \( p - \text{Isog} \) are pairs \( (x, f) \), where \( x \in \mathcal{S}_K^p(G, X)(T) \), and \( f \) is a \( p \)-quasi-isogeny \( f : \mathcal{A}_x \to \mathcal{A}_y \), where \( y \in \mathcal{S}_K^p(G, X)(T) \). For \( K_p \subseteq G(k'_p) \), we can
define $p - \text{Isog}_{K_p}$ in a similar way, by setting $K = K^p K_p$ and considering $p$-quasi-isogenies between points on $\mathcal{X}_K(G, X)$ instead. For small enough $K^p$ such that $\mathcal{X}_K(G, X)$ is a scheme, $p - \text{Isog}_{K^p}$ is in fact also a scheme over $O_{E,(v)}$. In the following, we always assume sufficient level structure $K^p$ such that $p - \text{Isog}_{K^p}$ is a scheme, and for notational simplicity we will simply denote this by $p - \text{Isog}$. We have can define projection maps back to $\mathcal{X}_{K_p}(G, X)$ sending a $p$-quasi isogeny $(x, f)$ to $x$ (respectively $y$)

$$s : p - \text{Isog} \to \mathcal{X}_{K_p}(G, X) \quad t : p - \text{Isog} \to \mathcal{X}_{K_p}(G, X).$$

These maps $s, t$ are proper, and surjective.

Consider the closure $\mathcal{J}$ of the generic fiber $p - \text{Isog} \otimes E$ in $p - \text{Isog}$. We abuse notation and still denote the special fiber of $\mathcal{J}$ by $p - \text{Isog} \otimes \kappa$, and the $\mathbb{Q}$-vector space of irreducible components by $\mathbb{Q}[p - \text{Isog} \otimes \kappa]$. Since $\mathcal{J}$ is flat over $O_{E,(v)}$, Irreducible components of $p - \text{Isog} \otimes \kappa$ are hence of dimension $2(p, \mu)$.

We now consider the $\mu$-ordinary locus $p - \text{Isog}^{\text{ord}} \otimes \kappa$. This is the subspace of $p - \text{Isog} \otimes \kappa$ which maps to the $\mu$-ordinary locus under the map $s$ (equivalently, $t$). The following argument can be extracted from [Lee20]:

**Proposition 2.3.1.** When $G$ is split over $\mathbb{Q}_p$, the $\mu$-ordinary locus is dense in $p - \text{Isog} \otimes \kappa$

**Proof.** The discussion in [Lee20] §6 shows that for any irreducible component in $p - \text{Isog} \otimes \kappa$, it has a dense open subset which corresponds to the Newton strata for some unramified $[b] \in B(G, v)$. If $G$ is split over $\mathbb{Q}_p$, then the only unramified element in $B(G, v)$ is the $\mu$-ordinary $\sigma$-conjugacy class, since we have an isomorphism $B(G, v) \simeq B(G_{\text{ad}}, v_{\text{ad}})$, which maps unramified elements to each other. From [XZ17] 4.2.11, we see that the identity element represents the basic element in $B(M_{\text{ad}}, \mu_{\text{ad}})$ if and only if $[1] \in B(M_{\text{ad}}, \mu_{\text{ad}})$, i.e. $\mu_{\text{ad}} = 0$ in $\pi_1(M_{\text{ad}})^{\Gamma}$. If $M_{\text{ad}}$ is split, then $\mu_{\text{ad}} = 0$ in $\pi_1(M_{\text{ad}})^{\Gamma}$ implies that $\mu_{\text{ad}}$ is the sum of coroots of $M_{\text{ad}}$, and hence $\mu_{\text{ad}}$ is either the identity or it cannot be minuscule. \hfill \qed

### 2.4. Abstract Eichler-Shimura Relations.

We have isomorphisms of Hecke algebras

$$\mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q}) \simeq \bigotimes_i \mathcal{H}(G_i(\mathbb{Q}_p)//K_p, \mathbb{Q}).$$

Similarly, since $G_{\mathbb{Q}_p}$ admits a decomposition, we can write

$$\mu = \prod_i \mu_i$$

where $\mu_i$ is a minuscule cocharacter of $G_i$. If we let $M$ be the centralizer of $\mu$ in $G$, then similarly we also have

$$M_{\mathbb{Q}_p} = \prod_i M_i$$

where $M_i$ is the centralizer of $\mu_i$ in $M$. Thus, we also have an isomorphism of Hecke algebras

$$\mathcal{H}(M(\mathbb{Q}_p)//M_p, \mathbb{Q}) \simeq \bigotimes_i \mathcal{H}(M_i(\mathbb{Q}_p)//M_p, \mathbb{Q}).$$

For a quasi-split reductive group $G$ with standard parabolic subgroup $P$ and Levi subgroup $M$, we can define following algebra homomorphism, known as the twisted Satake homomorphism

$$\mathcal{S}_M^G : \mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q}) \to \mathcal{H}(M(\mathbb{Q}_p)//M_p, \mathbb{Q}),$$
defined as follows. Write \( P = NM \), for \( N \) the unipotent radical of \( P \), and given a function \( f \in \mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q}) \), we have
\[
\hat{S}_M^G(f)(m) = \int_{n \in N} f(nm) dn.
\]
The twisted Satake isomorphism also factors: we have an isomorphism
\[
\hat{S}_M^G = \bigotimes_i \hat{S}_{M_i}^G.
\]
Consider now the representation \( \rho_i : \hat{G} \to GL(V_{\mu_i}) \) of \( \hat{G} \) with highest weight cocharacter \((1, \ldots, \mu_i, \ldots, 1)\), where \( \mu_i \) is in the \( i \)-th position. Observe that \( \rho_{\mu} = \otimes_i \rho_{\mu_i} \). Define the polynomial
\[
H_i(x) \in \mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q})(x)
\]
as the polynomial given by
\[
(2.4.3) \quad H_i(x) = \det(x - p^{nr} \rho_{\mu_i}(\sigma \times \hat{g})).
\]
Note that since \( \mu \) is central in \( M \), \( \mu_i \) is also central in \( M \), hence we can consider the element \( 1_{\mu_i(p)M_c} \in \mathcal{H}(M(\mathbb{Q}_p)//M_p, \mathbb{Q}) \).

**Proposition 2.4.4.** We have the following equality in \( \mathcal{H}(M(\mathbb{Q}_p)//M_p, \mathbb{Q}) \): for all \( i \),
\[
H_i(1_{\mu_i(p)M_c}) = 0.
\]

**Proof.** This follows from the same proof as [B02 Prop 3.4], and the observation that under the decomposition \((2.4.2)\), we see that \( H_i(1_{\mu_i(p)M_c}) \) corresponds to the polynomial with coefficients in \( \mathcal{H}(M(\mathbb{Q}_p)//M_{i,c}, \mathbb{Q}) \) defined similarly as in \((2.4.3)\), where we instead take the determinant of highest weight representation of \( \hat{G}_i \) corresponding to \( \mu_i \). \( \square \)

### 2.5. Newton stratification.

From now on, we will drop the assumption that the Shimura datum is of Hodge type, and consider general abelian type Shimura datum. For any abelian type Shimura datum, we will let \((G_1, X_1)\) denote the Hodge type Shimura datum such that there exists a central isogeny
\[
f : G_1^{der} \to G^{der}
\]
which induces an isomorphism \((G_1^{ad}, X_1^{ad}) \simeq (G^{ad}, X^{ad})\).

We now recall the construction of Newton strata for Shimura varieties of abelian type, as constructed in [SZ17]. Observe that for any connected reductive group \( G \), and minuscule cocharacter \( \nu \), we have an isomorphism \( B(G, \nu) = B(G^{ad}, \nu^{ad}) \). In [SZ17], the Newton strata is first constructed for adjoint groups, and thus we have a stratification on \( \mathcal{S}_K^{ad}(G^{ad}, X^{ad}) \). The Newton strata for \( \mathcal{S}_K(p, G, X) \) is then defined to be the pullback of the Newton strata for \( \mathcal{S}_K^{ad}(G^{ad}, X^{ad}) \) via the natural map
\[
\mathcal{S}_K(p, G, X) \to \mathcal{S}_K^{ad}(G^{ad}, X^{ad}).
\]

Fix a connected component \( X^+ \) of \( X \), and a connected component \( X_1^+ \) of \( X_1 \), such that their images in \( X^{ad} \) are equal to some connected component \( X^{ad, +} \). Let \( \text{Sh}_{K_p}(G, X)^+ \) denote the connected component of \( \text{Sh}_{K_p}(G, X) \) containing \( \{1\} \times X^+ \), and similarly let \( \text{Sh}_{K_1,p}(G_1, X_1)^+ \) denote the connected component of \( \text{Sh}_{K_1,p}(G_1, X_1) \) containing \( \{1\} \times X_1^+ \).
We observe that the Newton strata of $S_{K_p}(G, X)^+$ and $S_{K_{1,p}}(G_1, X_1)^+$ is exactly that pulled back along the maps

$$S_{K_{1,p}}(G_1, X_1)^+ \to S_{K_p}(G, X)^+ \to S_{K_{1,p}^0}(G^{ad}, X^{ad})^+$$

in particular, we see that the $\mu_1$-ordinary locus of $S_{K_{1,p}}(G_1, X_1)^+$ is exactly the preimage of the $\mu$-ordinary locus of $S_{K_p}(G, X)^+$.

2.6. Model for the Hecke correspondences. For general abelian type Shimura varieties, we do not have a moduli interpretation in terms of abelian varieties, and thus we do not have the general formalism of $p$–Isog. However, we can still define models for the Hecke correspondences, defined as follows.

Consider the Hecke correspondence $C \subset Sh_K(G, X) \times Sh_K(G, X)$ given by $1_{K_pB_K}$ on the generic fiber, and let $C'$ be the closure of $C$ in $\mathcal{J}_K(G, X) \times \mathcal{J}_K(G, X)$. For any correspondence $C'$ over $\mathcal{J}_K(G, X)$, we will let $C_0$ denote the special fiber, which is a correspondence over $\mathcal{J}_K(G, X)_{\kappa}$. We have a similar construction for Hecke correspondences for the groups $G_1, G^{ad}$. Observe that the Hecke operators for $G, G^{ad}$ are related as follows. Let $g^{ad}$ denote the image of $g \in G(\mathbb{Q}_p)$ in $G^{ad}$. $C^{ad} \subset Sh_{K^{ad}}(G^{ad}, X^{ad}) \times Sh_{K^{ad}}(G^{ad}, X^{ad})$ given by $1_{K_p^{ad}g^{ad}K_p}$ on the generic fiber. Let $C$ be the closure of $C^{ad}$ in $\mathcal{J}_{K^{ad}}(G^{ad}, X^{ad}) \times \mathcal{J}_{K^{ad}}(G^{ad}, X^{ad})$, and let $C_0^{ad}$ denote the special fiber.

Our key observation is the following: $C^{ad}$ is the image of $C$ under the (finite) projection maps

$$Sh_K(G, X) \times Sh_K(G, X) \to Sh_{K^{ad}}(G^{ad}, X^{ad}) \times Sh_{K^{ad}}(G^{ad}, X^{ad}).$$

Thus, we see that $C_0^{ad}$ is the image of the projection of $C_0$ to a correspondence on $\mathcal{J}_{K^{ad}}(G^{ad}, X^{ad})_{\kappa}$.

We can also define the $\mu$-ordinary locus $C_0^{ad}$ of $C_0$ to be the subspace of $C_0$ which maps to the $\mu$-ordinary locus under the natural projection maps to $\mathcal{J}_K(G, X)$. The discussion of Newton strata above shows that $C_0^{ad, ord}$ is the image of the projection to $\mathcal{J}_{K^{ad}}(G^{ad}, X^{ad})_{\kappa}^{ord} \times \mathcal{J}_{K^{ad}}(G^{ad}, X^{ad})_{\kappa}^{ord}$ of $C_0^{ord}$, and also $C_0^{ord}$. 

**Proposition 2.6.1.** Let $G$ be split over $\mathbb{Q}_p$, and suppose $g \in G(\mathbb{Q}_p)$ is such that there exists $g_1 \in G_1(\mathbb{Q}_p)$ such that $g^{ad} = g_1^{ad} \in G^{ad}(\mathbb{Q}_p)$. Consider the correspondence $C$ associated with $1_{K_pB_K}$, as above. Then we have $C_0$ has a dense $\mu$-ordinary locus.

**Proof.** Note that since we have an isomorphism $B(G, \mu) \simeq B(G^{ad}, \mu^{ad})$, the condition that $G$ is split implies that we also have exactly one unramified $\sigma$-conjugacy class in $B(G_1, \mu_1)$, and thus $p – Isog(G_1, X_1) \otimes \kappa$ has a dense $\mu$-ordinary locus, by Proposition 2.3.1.

By the above arguments, we know that the Hecke correspondence $C_0^{ad}$ has a dense $\mu$-ordinary locus, since it is the image under a finite map of the Hecke correspondence $C_{1,0}$ on $Sh_{K_1}(G_1, X_1)$, which will have a dense $\mu$-ordinary locus since $p – Isog(G_1, X_1) \otimes \kappa$ has a dense $\mu$-ordinary locus.

Thus, since the image of $C_0$ under a finite map to $\mathcal{J}_{K^{ad}}(G^{ad}, X^{ad})_{\kappa} \times \mathcal{J}_{K^{ad}}(G^{ad}, X^{ad})_{\kappa}$ has a dense $\mu$-ordinary locus, the original Hecke correspondence must $C_0$ must have a dense $\mu$-ordinary locus as well.

Observe that from the definition of the $H_{G,X}(t)$ that the Hecke correspondences which appear as coefficients in $H_{G,X}(t)$ are closed subschemes of Hecke correspondences lying in the subring $R$ of $H(G(\mathbb{Q}_p)//K_p)$ generated by $1_{K_p\mu(p)K_p}$. Let $\mu_1$ be the cocharacter of $G_1, \mathbb{Q}_p$ associated to $X_1$. Observe that we have $\mu^{ad} = \mu_1^{ad}$ when after projecting to cocharacters of $G^{ad}$. Thus, if the group $G$ is split, then the proposition holds for the coefficients of the Hecke polynomial.
2.7. Canonical liftings of $\mu$-ordinary points. Consider now any $\mu$-ordinary point $x \in \mathcal{S}_{K_p}(G, X)(\mathbb{F}_p)$. We suppose now that the point $x$ lies in $S_{K_p}(G, X)^+$. Note that this is always possible up to the action of some $g^p \in G(\mathbb{A}_p^p)$, since by [Kis10, 2.2.5] $G(\mathbb{A}_p^p)$ acts transitively on $S_{K_p}(G, X)$.

We suppose that $x$ is the image of some $\mu$-ordinary point $x_1 \in \mathcal{S}_{K_1,p}(G_1, X_1)^+(\mathbb{F}_p)$. We know from [SZ16] that for every ordinary point in $\mathcal{S}_{K_1,p}(G_1, X_1)^+(\mathbb{F}_p)$, there exists a special point lifting $\tilde{x}_1$, with associated cocharacter $\mu_{\tilde{x}_1}$ satisfying $\mu_{\tilde{x}_1, Q_p} = \mu_1$.

Note that since the map $X \to X^{ad}$ is injective, and takes a special point $x \in X$ to a special point in $X^{ad}$, if we consider the image $\tilde{x}$ of $\tilde{x}_1$ in $\mathcal{S}_{K_p}(G, X)^+$, then $\tilde{x}$ is a special point whose reduction mod $p$ is the point $x$. Moreover, note that the cocharacter $\mu_{\tilde{x}}$ is determined by the map to the adjoint group $\mu_{\tilde{x}}^{ad}$, since for any map $\mathbb{G}_m \to G$, it is determined by the induced maps to $G/G^{der}$ and $G^{ad}$, and since $G/G^{der}$ is commutative, the map $\mathbb{G}_m \to G/G^{der}$ is constant for all elements $x \in X$. Thus, we see that the associated cocharacter $\mu_{\tilde{x}}$ satisfies $\mu_{\tilde{x}, Q_p} = \mu$, since $\mu^{ad} = \mu_1^{ad}$. Thus, we have the following corollary:

**Corollary 2.7.1.** Let $x$ be a $\mu$-ordinary point in $\mathcal{S}_{K_p}(G, X)(\mathbb{F}_p)$. Then $x$ admits a lifting to a special point $\tilde{x}$, and the $p^n$-Frobenius map on $\tilde{x}$ is given by

$$j = \mu(p)\sigma(\mu(p)) \cdots \sigma^{n-1}(\mu(p)) \in G(\mathbb{Q}_p).$$

We now want to show the following proposition, which is a generalization of [B02, Lemma 4.5].

**Proposition 2.7.2.** Let $x$ be a $\mu$-ordinary point in $\mathcal{S}_{K_p}(G, X)(\mathbb{F}_p)$, and let $\tilde{x} = [gK_p \times h]$ be the lifting constructed in Corollary [2.7.1]. Let $U$ denote the unipotent radical of the parabolic subgroup of $G$ associated to $\mu$. Then for any $u \in U(\mathbb{Q}_p)$, we have that the mod $p$ reductions of the points

$$[gK_p \times h] = [guK_p \times h]$$

are equal.

**Proof.** Up to the action of some $g^p \in G(\mathbb{A}_p^p)$, we may assume that $x \in \mathcal{S}_{K_p}(G, X)^+(\mathbb{F}_p)$. Observe that we have an isomorphism of root systems $\Phi(G, T) = \Phi(G^{ad}, T^{ad}) = \Phi(G_1, T_1)$. Hence if we consider $U_1$ the unipotent radical of the standard parabolic subgroup of $G_1$ corresponding to $\mu_1$, then we can identify $U_1$ with $U$. In particular, since this result is true for the lift $\tilde{x}_1$, for any $u \in U_1(\mathbb{Q}_p)$, the same is true for the action of $n \in U(\mathbb{Q}_p)$ on $\tilde{x}$. \qed

2.8. We now let $\tilde{G}$ be the simply connected cover of $G^{der}_1$. Let $Z_G$ denote the center of the group $G$. Recall that we have a central isogeny

$$Z \times \tilde{G} \to G,$$

and thus, for a maximal torus $T$ of $G$ defined over $\mathbb{Q}$, we have a injective map with finite cokernel

$$X_*(Z_G) \oplus X_*(T^{der}) \hookrightarrow X_*(T)$$

where $\tilde{T}$ in $\tilde{G}$ is a maximal torus of $G^{der}$.

In particular, observe that for any cocharacter $\lambda \in X_*(T)$, there exists some positive integer $m$ such that $\lambda^m$ lifts (up to some cocharacter in $X_*(Z_G)$) to a cocharacter of $\tilde{G}$. 


By [MS82, 3.4], there exists a Shimura variety \( \text{Sh}(G', X') \) and a map with central kernel \( G' \rightarrow G_1 \) such that \( G^\text{der} = \tilde{G} \) and via the composition map

\[
\tilde{G} \rightarrow G^\text{der}_1 \rightarrow G^\text{der}
\]

there is an isomorphism of Shimura data \( (G'^\text{ad}, X'^\text{ad}) \simeq (G^\text{ad}_1, X^\text{ad}_1) \simeq (G^\text{ad}, X^\text{ad}) \).

Now, we consider the Shimura variety \( \text{Sh}(\tilde{G}, \tilde{X}) \). Since \( \tilde{G} \) is simply connected, observe that the action of any \( g' \in \tilde{G}(\mathbb{Q}_p) \) preserves connected components. We now choose a connected component \( X'^+ \) of \( X' \) which maps to \( X^\text{ad,}^+ \), and let \( \text{Sh}(G, X)^+ \) be the connected component which contains \( X'^+ \times \{1\} \). In particular, we see that \( g' \) maps \( \text{Sh}(G', X')^+ \) back to itself.

Let \( g \in G^\text{der}(\mathbb{Q}_p) \) be the image of \( g' \) under the central isogeny \( \tilde{G} \rightarrow G^\text{der} \). Moreover, note that on geometrically connected components we have

\[
\text{Sh}_K(G, X)^+ = \text{Sh}_{K'}(G', X')^+ / \Delta
\]

where \( \Delta = \ker(\mathcal{A}(G')^0 \rightarrow \mathcal{A}(G)) \), where the groups \( \mathcal{A}(G')^0 \) and \( \mathcal{A}(G) \) are as defined in [Kis10, §3.3]. Thus, we see that since the action of \( g' \) preserves \( \text{Sh}(G', X')^+ \), so too does the action of \( g \) preserve connected components of \( \text{Sh}_K(G, X) \), and moreover the action of \( g \) on \( \text{Sh}_K(G, X)^+ \) is exactly the quotient by \( \Delta \) of the action of \( g' \) on \( \text{Sh}(G', X')^+ \).

If we let \( g_1 \in G^\text{der}_1(\mathbb{Q}_p) \) be the image of \( g' \) under the central isogeny \( \tilde{G} \rightarrow G^\text{der}_1 \), a similar result holds for the action of \( g_1 \).

Moreover, we see that from [Kis17 3.7.10] we have a surjective map

\[
Z_G(\mathbb{Q}) \rightarrow Z_G(\mathbb{Q}_p)/Z_G(\mathbb{Z}_p).
\]

2.9. **Partial Frobenius for abelian type.** Similar to the situation for the partial Frobenius for Shimura varieties of Hodge type, we would like to define the partial Frobenius to be the \( p \)-power quasi-isogeny represented by \( \mu_i(p) \). Since we are in the abelian type case, we cannot work directly with \( p \)-divisible groups. Instead, here we will define the partial Frobenius correspondence, at least over the ordinary locus.

Suppose that \( p \) is a prime which satisfies the following criterion:

1. \( p \) splits in \( F \)
2. The group \( G \) has good reduction at \( p \)

2.9.1. Firstly we will assume the group \( G \) is adjoint. By [Kis17 4.6.6], there is a Hodge type Shimura datum \((G_1, X_1)\) such that

1. \((G^\text{ad}_1, X^\text{ad}_1) \simeq (G, X)\) and \( Z_{G_1} \) is a torus;
2. if \((G, X)\) has good reduction at \( p \), then \((G_1, X_1)\) in (1) can be chosen to have good reduction at \( p \), and such that \( E(G, X)_p = E(G_1, X_1)_p \).

Since \( Z_{G_1} \) is an unramified torus, by [Ama69, Corollary 2], we know that \( H^1(\mathbb{Q}_p, Z_{G_1}) \) is trivial, and thus we have a surjective map

\[
G_1(\mathbb{Q}_p) \rightarrow G(\mathbb{Q}_p).
\]

More precisely, it tells us that the element \( \mu_i(p) \in G(\mathbb{Q}_p) \) lifts to an element \( \tilde{\mu}_i(p) \in G_1(\mathbb{Q}_p) \) for some cocharacter \( \tilde{\mu}_i \) of \( G_1 \). Now, observe that \( \tilde{\mu}_i(p) \) lies in the center of \( M_1 \), the centralizer of \( \mu_1 \). Thus, we may consider the section of \( p-\text{Isog}_{G_1} \) given by the image of \( 1 \cdot \tilde{\mu}_i(p)_{M_1}(\mathbb{Z}_p) \). This gives me a correspondence on \( \mathcal{A}(G_1, X_1)_K \), which we project to get a correspondence on \( \mathcal{A}(G, X)_K \).
This is the partial Frobenius correspondence $\text{Frob}_{p_i}$. Observe that by construction we have

$$\text{Frob} = \prod_i \text{Frob}_{p_i},$$

since the product of the images of $1_{\tilde{\mu}_i}(p) M_1(\mathbb{Z}_p)$ is the image of $1_{\tilde{\mu}(p)} M_1(\mathbb{Z}_p)$, where $\tilde{\mu}$ is a cocharacter whose image in $G(\mathbb{Q}_p)$ is $\mu(p)$, which corresponds to the Frobenius over the ordinary locus.

2.9.2. More generally, if $G$ is not adjoint, then we will consider the Hecke correspondence

$$h(1_{K_p \mu_i(p) K_p}) \rightarrow h(1_{K_p \mu_i(p) K_p})$$

in the ring $\text{Corr}(\mathcal{S}, \mathcal{S})$. Then, we define $\text{Frob}_{p_i}$ to be the preimage of $\text{Frob}^{ad}_{p_i}$ under the projection map

$$h(1_{K_p \mu_i(p) K_p}) \rightarrow h(1_{K_p \mu_i(p) K_p}).$$

It remains to check that

$$\text{Frob} = \prod_i \text{Frob}_{p_i},$$

which we can check over the ordinary locus. This follows from the corresponding observation for $\text{Frob}^{\circ, ad}$, and noting that over the ordinary locus closed points in $\text{Frob}_{p_i}$ consist of the reduction mod $p$ of pairs $(\tilde{x}, \mu_i(p) \cdot \tilde{x})$, where $\tilde{x}$ is the special point lift of $x$ constructed previously. Thus, we have the following proposition:

**Proposition 2.9.4.** Let $(G, X)$ be a Shimura datum of abelian type, and let $(G_1, X_1)$ be a Hodge type Shimura datum constructed in (2.9.1) for $(G^{ad}, X^{ad})$. Let $p$ be a prime such that

1. $p$ splits in $F$
2. The group $G$ has good reduction at $p$

Then we have a partial Frobenius correspondence $\text{Frob}_{p_i}$ such that

$$\text{Frob} = \prod_i \text{Frob}_{p_i}.$$  

Remark 2.9.6. Since we do not have a moduli interpretation for general abelian type Shimura varieties, outside of the $\mu$-ordinary locus, it is not clear what $\text{Frob}_{p_i}$ has to do with partial Frobenii. However, from the discussion in Section 2.2, if the Shimura variety is of Hodge type, then we have a well-defined notion of a partial Frobenius isogeny (at least at closed points), and indeed the correspondence $\text{Frob}_{p_i}$ is given as the section defined by the partial Frobenius at $i$ for all closed points.

2.10. **Proof of Eichler-Shimura relations.**

**Proposition 2.10.1.** Let $(G, X)$ be a Shimura datum of abelian type, such that $G = \text{Res}_{F/\mathbb{Q}} G'$, and $p$ a prime satisfying the conditions in Proposition 2.9.4. Let $H_i(t)$ be the polynomial defined in (2.4.3), viewed as a polynomial with coefficients in $\text{Corr}(\mathcal{S}, \mathcal{S})$ via the map $h$. Then we have the equality

$$H_i(\text{Frob}_{p_i}) = 0.$$  

in the ring $\text{Corr}(\mathcal{S}, \mathcal{S})$. 
Proof. Firstly, observe that from Proposition 2.6.1, all the terms appearing in $H_i(\text{Frob}_{p_i})$ have a dense $\mu$-ordinary locus, hence it suffices to show the result where we restrict all the terms to the $\mu$-ordinary locus. Applying [102 A.6], to show that $H_i(\text{Frob}_{p_i}) = 0$, it suffices to show that $x \cdot H_i(\text{Frob}_{p_i}) = 0$

for all $x \in \mathcal{S}_{K_p}(G, X)(\overline{\mathbb{F}}_p)^{\text{ord}}$.

Let the Hecke polynomial be

$$H_i(t) = \sum_j A_j t^j,$$

for elements $A_j \in \mathcal{H}(G(\mathbb{Q}_p)//K_p)$. Let $h(A_j)$ denote the mod $p$ algebraic cycle in $\text{Corr}(\mathcal{S}_k, \mathcal{S}_k)$ corresponding to $A_j$. Thus, we want to show that

$$\sum_j \text{Frob}_{p_i} \cdot h(A_j) = 0$$

where $\text{Frob}_{p_i}$ is the correspondence defined above.

The proof then follows as in [102 Thm 4.7]. As constructed above, we let $\tilde{x}$ be the special point lift of $x$. Write $\tilde{x} = [gK \times h]$, and we write the coefficients of the Hecke polynomial $A_j$ in terms of left $K_p$-cosets of $G(\mathbb{Q}_p)$

$$A_j = \sum_k n^{(j)}_k g^{(j)}_k K_p$$

It remains for us to show that

$$\sum_{j, k} n^{(j)}_k g^{(j)}_k (p^{-k}) g^{(j)}_k K \times h = 0,$$

since (2.10.2) is the mod $p$ reduction of (2.10.3), and we observe that the partial Frobenius acting on $\tilde{x}$ is given by $\mu_i(p)$. The equality in (2.10.3) follows from the abstract Eichler-Shimura relation and the Proposition 2.7.2.

3. Semisimplicity Criterion

3.1. An abstract semisimplicity criterion. We first recall the following theorem of Fayad and Nekovár [FN19 Theorem 1.7].

**Definition 3.1.1.** A representation $\rho : \Gamma \to GL(V)$ is strongly irreducible if the restriction to any open finite index subgroup $U \rho_U$ is still irreducible.

**Theorem 3.1.2.** Let $\Gamma$ be a profinite group, $V, W_1, \ldots, W_r$ non-zero vector spaces of finite dimension over $\mathbb{Q}$. Let $\rho : \Gamma \to \text{Aut}_\mathbb{Q}(V)$ and $\rho_i : \Gamma \to \text{Aut}_\mathbb{Q}(W)$ be representations of $\Gamma$ with Lie algebras

$$\mathfrak{g}_i = \text{Lie}(\rho_i(\Gamma)), \quad \mathfrak{g} = \text{Lie}(\rho(\Gamma)).$$

We denote $\overline{\mathfrak{g}}_i = \mathfrak{g}_i \otimes \overline{\mathbb{Q}}, \overline{\mathfrak{g}} = \mathfrak{g} \otimes \overline{\mathbb{Q}}$. If the following three conditions hold, then the representation $\rho = \rho^* \otimes \overline{\mathbb{Q}}$ is semisimple.

1. Each $\rho_i$ is strongly irreducible (which implies that each $\mathfrak{g}_i$ is a reductive $\overline{\mathbb{Q}}$-Lie algebra and each element of its centre acts on $W_i$ by a scalar).

2. For each $i = 1, \ldots, r$, every (equivalently, some) Cartan subalgebra $h_i$ of $\mathfrak{g}_i$ acts on $W_i$ without multiplicities (i.e., all weight spaces of $h_i$ on $W_i$ are one-dimensional).
(3) There exists an open subgroup $\Gamma' \subset \Gamma$ and a dense subset $\Sigma \subset \Gamma'$ such that for each $g \in \Sigma$ there exists a finite dimensional vector space over $\mathbb{Q}$ (depending on $g$) $V(g) \supset V$ and elements $u_1, \ldots, u_r \in \text{Aut}_{\mathbb{Q}}(V(g))$ such that $u_i u_j = u_j u_i$, $P_{\rho(g)}(u_i) = 0$ for all $i$, $j = 1, \ldots, r$, and $V$ is stable under $u_1 \ldots u_r$ and $u_1 \ldots u_r|_V = \rho(g)$.

We state here a theorem of Sen [Sen73, Theorem 1] which we will use to find representations which satisfy condition (2) of the theorem above.

**Proposition 3.1.3.** Let $\mathfrak{g} = \mathbb{Q}_l \cdot \text{Lie}(\tilde{\rho}(\text{Gal}E)) \subset \mathfrak{g}(n, \mathbb{Q}_l)$ be the $\mathbb{Q}_l$-Lie algebra generated by the image of $\tilde{\rho}$. Then any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ acts on $\mathbb{Q}_l^n$ by the $n$ Hodge-Tate weights of $\tilde{\rho}|_{\text{Lie}G_K}$, for any $\tau : F \to \mathbb{Q}_l$.

**3.2. Proof of Main Theorem.** We prove in this subsection Theorem [1.0.1]

**Proof.** Observe that to show that the representation $\rho$ is semisimple, it suffices to show that $\rho|_U$ is semisimple for any open finite index subgroup of $\text{Gal}(\bar{E}/E)$. Thus, it suffices to find a set of primes $p$ of positive density such that $\rho(\text{Frob}_p)$ is a semisimple endomorphism of $V^i(\pi^\infty)$.

From the Eichler-Shimura relation for split primes, observe that if we let $H$ be the Hecke polynomial, then we have

$$H(\text{Frob}_p|H^i(\text{Sh}_K)) = 0$$

as endomorphisms of $H^i(\text{Sh}_K)$. Applying the decomposition of $H^i(\text{Sh}_K)$, observe that since each summand $V^i(\pi^\infty) \otimes (\pi^\infty)^K$ is Frobenius stable, we can consider $\text{Frob}_p$ as an endomorphism of $V^i(\pi^\infty)$.

Replacing each element of the Hecke algebra with its eigenvalue on $\pi_p^K$, we obtain

$$H|_{\pi_p^K} (\text{Frob}_p|_{V^i(\pi^\infty) \otimes (\pi^\infty)^K}) = 0.$$

Observe that the polynomial on the right hand side is, by the definition of the Hecke polynomial, $\det(t - \tilde{\rho}(\text{Frob}_p)) =: P_p$.

In order to show semisimplicity of the endomorphism given by $\text{Frob}_p$, it suffices to show that the $P_p$ has distinct roots. This is an open condition on $\text{Gal}(\bar{E}/E)$, hence it suffices to exhibit an element $u \in \text{Gal}(\bar{E}/E)$ such that the characteristic polynomial of $\rho(u)$ has distinct roots. To see this, observe that Proposition [3.1.3] applies, hence the Lie algebra

$$\tilde{\mathfrak{g}} = \mathbb{Q}_l \cdot \text{Lie}(\tilde{\rho}(\text{Gal}E))$$

contains a semisimple element whose eigenvalues on $V_i(\pi^\infty)$ act by the Hodge-Tate weights of $\rho_i$. By assumption (2), the Hodge-Tate weights of $\rho_i$ are all distinct. This implies that there is an open subset of $\text{Gal}(\bar{E}/E)$ where the characteristic polynomial of $\rho(u)$ has distinct roots.

Finally, we conclude using the criterion in Theorem [3.1.2] since we have constructed a Zariski dense set of elements which are semisimple, that $\rho$ is semisimple. □

**4. Applications to some Shimura varieties**

We discuss here some examples of Shimura varieties where Theorem [1.0.1] applies. Consider the Shimura variety associated to the group $\text{Res}_{F/Q} G$, where $G$ is some inner form of $GSp_{2n,F}$ which is compact modulo center for at least one place $v|\infty$ of $F$. Let $\pi$ be a cuspidal $L$-algebraic automorphic representation of $G(\mathbb{A}_F)$, and we assume that $\pi$ satisfies

(1) There is a finite $F$-place $v_{SL}$ such that $\pi_{v_{SL}}$ is the Steinberg representation of $GSp_{2n}(F_{v_{SL}})$ twisted by a character.
(2) $\pi^{\infty}\mid\text{sim}^{n(n+1)/4}$ is $\xi$-cohomological for an irreducible algebraic representation $\xi = \otimes_{y:F} \mathbb{C} \xi_{y}$ of the group $(\text{Res}_{F/Q} \text{GSp}_{2n})_{\mathbb{C}}$, where $\text{sim}$ is the similitude factor map $\text{sim} : \text{GSp}_{2n} \to \mathbb{G}_{m}$.

Under these conditions on $\pi$, Kret and Shin [KS20, Theorem A] construct the Galois representation

$$\rho_{\pi} : \text{Gal}(\bar{F}/F) \to \text{GSpin}_{2n+1}(\mathbb{Q}_{l})$$

associated to $\pi$. If we moreover assume that the Zariski closure of the image of $\rho_{\pi}$ maps onto $\text{SO}_{2n+1}$ (which should hold generically), then at all places $v|\infty$ where the group is not compact modulo center, the associated representation

$$\tilde{\rho}_{\pi,v} : \text{Gal}(\bar{F}/F) \to \text{GSpin}_{2n+1}(\mathbb{Q}_{l}) \xrightarrow{\text{spin}} \text{GL}(V),$$

$\tilde{\rho}_{\pi,v}$ will be strongly irreducible, since it is irreducible and has connected image, hence the Zariski closure of the image of any finite index open subgroup is also $\text{SO}_{2n+1}$, and hence irreducible.

If we moreover assume that

(1) The representation $\pi_{v}$ is spin-regular at every infinite place $v$ of $F$,

then [KS20, Theorem C] implies $\pi$ is potentially automorphic. We will further assume that $\pi$ is automorphic. Moreover, spin-regularity implies that the Hodge-Tate weights of $\tilde{\rho}_{\pi,v}$ are distinct. Under all these conditions, we have may apply the Main Theorem, and we can deduce the following:

**Theorem 4.0.1.** Let $\pi$ be a cuspidal $L$-algebraic automorphic representation of $G(\mathbb{A}_{F})$, satisfying

(1) There is a finite $F$-place $v_{St}$ such that $\pi_{v_{St}}$ is the Steinberg representation of $\text{GSp}_{2n}(F_{v_{St}})$ twisted by a character.

(2) $\pi^{\infty}\mid\text{sim}^{n(n+1)/4}$ is $\xi$-cohomological for an irreducible algebraic representation $\xi = \otimes_{y:F} \mathbb{C} \xi_{y}$ of the group $(\text{Res}_{F/Q} \text{GSp}_{2n})_{\mathbb{C}}$, where $\text{sim}$ is the similitude factor map $\text{sim} : \text{GSp}_{2n} \to \mathbb{G}_{m}$.

(3) The representation $\pi_{v}$ is spin-regular at every infinite place $v$ of $F$.

If moreover the l-adic Galois representation $\rho_{\pi} : \text{Gal}(\bar{E}/E) \to \text{GSpin}_{2n+1}$ satisfies

(1) $\text{spin} \circ \rho_{\pi}$ is automorphic

(2) The Zariski closure of the image of $\rho_{\pi}$ maps onto $\text{SO}_{2n+1}$,

Then the Galois module

$$\text{Hom}_{G(\mathbb{A}_{f})}(\pi^{\infty}, H^{*}_{\text{et}}(\text{Sh}(G,X), \mathbb{Q}_{l}))$$

(which is finite-dimensional over $\mathbb{Q}_{l}$) is semisimple.

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