A note on the pressure of strong solutions to the Stokes system
in bounded and exterior domains

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Abstract

We consider the Stokes problem in an exterior domain \( \Omega \subset \mathbb{R}^n \) with an external force \( f \in L^s(0, T; W^{k, r}(\Omega)) (k \in \mathbb{N}, 1 < r < \infty) \). In the present paper we show that in contrast to \( u \) the boundary regularity of the pressure can be improved according to the differentiability of \( f \) up to order \( k \). In particular, this implies that the pressure is smooth with respect to \( x \in \Omega \) if \( f \) is smooth with respect to \( x \in \Omega \).

Keywords Stokes equations, exterior domain, boundary regularity

Mathematics subject classification 35Q30, 76D03.

1 Introduction

Let \( \Omega \subset \mathbb{R}^n (n \in \mathbb{N}, n \geq 2) \) be an exterior domain, i.e. \( \mathbb{R}^n \setminus \overline{\Omega} \) is a bounded domain in \( \mathbb{R}^n \). Let \( 0 < T < +\infty \). Set \( Q = \Omega \times (0, T) \). In the present paper we consider the Stokes problem

\[
\begin{align*}
(1.1) & \quad \text{div} \ u = 0 \quad \text{in} \ Q, \\
(1.2) & \quad \partial_t u - \Delta u = -\nabla p + f \quad \text{in} \ Q, \\
(1.3) & \quad u = 0 \quad \text{on} \ \partial \Omega \times (0, T), \quad \lim_{|x| \to \infty} u(x, \cdot) = 0, \\
(1.4) & \quad u(0) = 0 \quad \text{in} \ \Omega,
\end{align*}
\]

where \( u = (u^1, \ldots, u^n) \) denotes the unknown velocity of the fluid, \( p \) the unknown pressure and \( f \) the given external force. The Stokes problem has been extensively studied in the past. In particular, for the case \( \Omega \) is the half space or an \( C^2 \) domain with compact boundary the \( L^p \)-theory is well-known. Based on potential theory in [14] Solonnikov proved that for every \( f \in L^q(Q) \) there exists a unique solution \((u, p)\) to (1.1)-(1.4) such that \( \partial_t u, \nabla^2 u \in L^q(Q) \), and \( \nabla p \in L^p(Q) \). By using the semi group approach, similar results have been obtained in [5], [6], [3]. For the corresponding estimates on the pressure we refer to [13]. An optimal result for the anisotropic case when \( f \) belongs to \( L^s(0, T; L^q(\Omega)) \) has been proved in [7] for the cases \( \Omega = \mathbb{R}^n \), \( \Omega = \mathbb{R}^n_+ \), and a \( C^2 \) domain \( \Omega \) with compact boundary.

By standard arguments from the regularity theory of parabolic equations one gets the regularity \( u \) and \( p \) in dependence of the regularity of the right-hand side \( f \) in time and space. However, if \( f \) is only smooth in \( x \in \Omega \) it is not clear whether \( u \) is smooth in \( x \) up to the boundary. In the present paper we will see that such a property at least holds for the pressure \( p \), which...
Theorem 1

Let \( f \) to (1.1)–(1.4) if (1.4)

\[ (1.5) \]

In addition, there holds

\[ \text{For the existence of a strong solution to (1.1)–(1.4) cf. in [7].} \]

Let

\[ \text{Definition 1.1} \]

Now, let us introduce the notion of a strong solution to (1.1) –(1.4).

First we shall introduce the basic notations regarding the function spaces used throughout the paper. By \( W^{k,q}(\Omega) \), \( W^{k,q}_0(\Omega) \) we denote the usual Sobolev spaces. Vector functions and spaces of vector valued functions will be denoted by bold face letters, i.e. we write \( L^q(\Omega), W^{k,q}(\Omega) \), etc. instead of \( L^q(\Omega; \mathbb{R}^n), W^{k,q}(\Omega; \mathbb{R}^n) \), etc. In addition, we use the following spaces of solenoidal functions

\[ L^q_0(\Omega) = \text{closure of } C^\infty_0(\Omega) \text{ w.r.t. the norm } \| \cdot \|_{L^q} \]

\[ W_{0,q}^{k,q}(\Omega) = \text{closure of } C^\infty_0(\Omega) \text{ w.r.t. the norm } \| \cdot \|_{W^{k,q},q}, \]

where \( C^\infty_0(\Omega) \) stands for the space of all smooth solenoidal vector fields with compact support in \( \Omega \). Given a Banach space \( X \) by \( L^q(0, T; X) \) we denote the space of Bochner measurable functions \( f: (0, T) \to X \) such that

\[ \| f \|_{L^q(0,T;X)}^q = \int_0^T \| f(t) \|_X^q \, dt < +\infty \quad \text{if } 1 \leq q < +\infty, \]

\[ \| f \|_{L^\infty(0,T;X)} = \text{ess sup}_{t \in (0,T)} \| f(t) \|_X < +\infty \quad \text{if } q = +\infty. \]

Now, let us introduce the notion of a strong solution to (1.1)–(1.4).

**Definition 1.1** Let \( f \in L^s(0,T;L^q(\Omega)) \) \((1 < s, q < +\infty)\). A pair \((u, p)\) is called a *strong solution* to (1.1)–(1.4) if \( u \in L^s(0,T;W^{1,q}_{0,\sigma}(\Omega)), p \in L^s(0,T;L^{1}_{\text{loc}}(\Omega)) \) and

\[ \partial_i \partial_j u, \partial_i u, \nabla p \in L^s(0,T;L^q(\Omega)), \quad i,j = 1,2,3, \]

such that (1.1), (1.2) holds a.e. in \( Q \), while (1.4) is fulfilled such that \( u = 0 \) a.e. in \( \Omega \times \{0\} \).

For the existence of a strong solution to (1.1)–(1.4) cf. in [7].

Our main result is the following

**Theorem 1** Let \( \Omega \subset \mathbb{R}^3 \) be an exterior domain or a bounded domain with \( \partial \Omega \in C^{2+k}(k \in \mathbb{N}). \) For \( f \in L^s(0,T;W^{k,q}(\Omega)) \) \((1 < s, q < +\infty; k \in \mathbb{N})\), let \((u, p)\) be the strong solution to (1.1)–(1.4). Then,

\[ \nabla p \in L^s(0,T;W^{k,q}(\Omega)). \]

In addition, there holds

\[ \| \nabla^{k+1} p \|_{L^s(0,T;L^s(\Omega))} \leq c \| f \|_{L^s(0,T;W^{k,q}(\Omega))}, \]

where \( c = \text{const} > 0 \) depending only on \( s,q,k \) and the geometric properties of \( \partial \Omega \).
2 Remarks on the equation \( \text{div } v = f \)

Let \( G \subset \mathbb{R}^n \) be a bounded domain, star-shaped with respect to a ball \( B_R \). It is well known that for all \( f \in L^q(G) \) with \((f)_G = 0\) the equation \( \text{div } v = f \) has a solution \( v \in W^{1,q}_0(G) \) such that

\[
\| \nabla v \|_{L^q(G)} \leq c \| f \|_{L^q(G)}
\]

with \( c = \text{const} > 0 \), depending on \( n, q \) and \( G \) (cf. \cite{2}, \cite{8}). In fact, the constant \( c \) depends on the geometric property of \( G \), namely the ratio of \( G \) which is defined by

\[
\text{ratio}(G) := \frac{R_a(G)}{R_i(G)},
\]

where

\[
R_a(G) = \inf \{ R > 0 \mid \exists B_R(x_0) : G \subset B_R(x_0) \},
\]

\[
R_i(G) = \sup \{ r > 0 \mid \exists B_r(x_0) : G \text{ is star-shaped w.r.t } B_r(x_0) \}.
\]

For instance \( \text{ratio}(G) = 1 \) if \( G \) is a ball, and \( \text{ratio}(G) = \sqrt{n} \) if \( G \) is a cube. Moreover, the ratio is invariant under translation and scaling, i.e.

\[
\text{ratio}(\lambda G) = \text{ratio}(G) \quad \forall \lambda > 0.
\]

Now, let \( G \) such that \( 2 < R_i(G) < 3 \). In particular, \( G \) is star shaped with respect to a ball \( B_2 = B_2(x_0) \). Without loss of generality we may assume that \( x_0 = 0 \). Let \( \phi \in C_0^\infty(B_2) \). We define

\[
\mathcal{B}_\phi f(x) = \int_{\mathbb{R}^n} f(x - y) K_\phi(x, y) dy, \quad x \in \mathbb{R}^n, \quad f \in C_0^\infty(G),
\]

where

\[
K_\phi(x, y) = \frac{y}{|y|^n} \int_0^\infty \phi \left( x + r \frac{y}{|y|} \right) (|y| + r)^{n-1} dr, \quad (x, y) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}).
\]

As in \cite{2}, \cite{8} it has been proved that \( \mathcal{B}_\phi f \in C_0^\infty(G) \) for all \( f \in C_0^\infty(G) \). In addition, there holds

\[
\| \nabla^k \mathcal{B}_\phi f \|_{L^q(G)} \leq c \| \nabla^{k-1} f \|_{L^q(G)} \quad \forall f \in C_0^\infty(G)
\]

with a constant depending on \( n, k, q, \phi \) and \( \text{ratio}(G) \) only. Furthermore, there holds

\[
\text{div } \mathcal{B}_\phi f = f \int_{B_1} \phi(y) dy - \phi \int_{G} f(y) dy \quad \text{in } \quad G.
\]

In particular, if \( \int_{B_1} \phi(y) dy = 1 \) and \( \int_{G} f(y) dy = 0 \) then \( v = \mathcal{B}_\phi f \) solves the equation \( \text{div } v = f \).

Finally, by (2.1) we may extend \( \mathcal{B}_\phi \) to an operator \( \mathcal{L}(W^{k-1,q}(G), W^{k,q}(G)) \) denoted again by \( \mathcal{B}_\phi \).

---

\(^1\) Let \( A \subset \mathbb{R}^n \) be a measurable set with \( \text{mes}(A) \). Given \( v \in L^1(A) \) by \( (v)_A \) we denote the mean value

\[
\frac{1}{\text{mes}(A)} \int_A v(x) dx.
\]
Let $i, j \in \{1, \ldots, n\}$. Observing, that
\[
\begin{align*}
\partial_j \mathcal{B}_\phi(\partial_i f) &= \partial_i \partial_j \mathcal{B}_\phi(f) - \partial_j \mathcal{B}_\phi(\partial_i f) \quad \text{in} \quad G, \\
\partial_i \partial_j \mathcal{B}_\phi(f) &= \partial_i \mathcal{B}_\phi(\partial_j f) + \partial_j \mathcal{B}_\phi(\partial_i f) \quad \text{in} \quad G,
\end{align*}
\]
we see that
\[
\partial_j \mathcal{B}_\phi(\partial_i f) = \partial_i \mathcal{B}_\phi(\partial_j f) + \partial_i \partial_j \mathcal{B}_\phi(f) - \partial_j \mathcal{B}_\phi(\partial_i f) \quad \text{in} \quad G.
\]
By the aid of (2.1), and Poincaré’s inequality, using the above identity, we get
\[
\begin{align*}
\|\partial_j \mathcal{B}_\phi(\partial_i f)\|_{L^q(G)} &\leq c(\|\partial_j f\|_{L^q(G)} + \|f\|_{L^q(G)}) \quad \forall f \in W_0^{1,q}(G), \\
\|\nabla^2 \partial_j \mathcal{B}_\phi(\partial_i f)\|_{L^q(G)} &\leq c(\|\partial_j \nabla \phi\|_{L^q(G)} + \|\partial_j \partial_n f\|_{L^q(G)} + \|\nabla^2 f\|_{L^q(G)})^{2) \\
\forall f &\in W_0^{3,q}(G),
\end{align*}
\]
where $c = \text{const} > 0$, depending on $n, q$ and ratio($G$).

Now, let $G$ be a bounded domain, star-shaped with respect to a ball $B$. Let $R := \frac{1}{2}R_i(G)$. Thus, there exist $B_R(x_0)$ such that $G$ is star shaped to the ball $B_R(x_0)$. Without loss of generality we may assume that $x_0 = 0$. Let $\phi \in C_0^\infty(B_1)$ with $\int_{B_1} \phi(y)dy = 1$. We define
\[
\mathcal{B} : W_0^{k-1,q}(G) \rightarrow W_0^{k,q}(G)
\]
by setting
\[
\mathcal{B}(f)(x) = R \mathcal{B}_\phi(\tilde{f})\left(\frac{x}{R}\right), \quad x \in G, \quad f \in W_0^{k-1,q}(G),
\]
where $\tilde{f}(y) = f(Ry)$ ($y \in R^{-1}G$). Using the transformation formula of the Lebesgue integral, in view of (2.1), we see that
\[
\|\nabla^k \mathcal{B}(f)\|_{L^q(G)} = R^{n/q-k+1} \|\nabla^k \mathcal{B}_\phi(\tilde{f})\|_{L^q(R^{-1}G)} \leq cR^{n/q-k+1} \|\nabla^{k-1} \tilde{f}\|_{L^q(R^{-1}G)}
\]
\[
= c \|\nabla^{k-1} f\|_{L^q(G)},
\]
where $c = \text{const} > 0$ depends on $n, q$ and ratio($R^{-1}G$) = ratio($G$). In addition, from (2.3), and (2.4) we deduce
\[
\begin{align*}
\|\partial_j \mathcal{B}(\partial_i f)\|_{L^q(G)} &\leq c(\|\partial_j f\|_{L^q(G)} + R^{-1} \|f\|_{L^q(G)}) \quad \forall f \in W_0^{1,q}(G), \\
\|\nabla^2 \partial_j \mathcal{B}(\partial_i f)\|_{L^q(G)} &\leq c(\|\partial_j \nabla \phi\|_{L^q(G)} + \|\partial_j \partial_n f\|_{L^q(G)} + R^{-1} \|\nabla^2 f\|_{L^q(G)})
\forall f &\in W_0^{3,q}(G),
\end{align*}
\]
(i, j = 1, \ldots, n) with a constant $c$, depending on $n, q$ and ratio($G$) only. Furthermore, from (2.2) we get
\[
\text{div} \mathcal{B}(f)(x) = f(x) - \phi\left(\frac{x}{R}\right)R^{-n} \int_G f(y)dy \quad \text{for a.e.} \ x \in G.
\]
\footnote{Here $\nabla_*$ denotes the reduced gradient $(\partial_1, \ldots, \partial_{n-1})$.}
3 Proof of Theorem 1

Proof 1° By decomposing the right-hand side into a solenoidal field, and a gradient field, we are able to reduce the problem to the case $\text{div } f = 0$. Let $E : W^{k,q}(\Omega) \to W^{k,\hat{q}}(\mathbb{R}^n)$ denote an extension operator such that

$$\|Ev\|_{W^{k,q}(\mathbb{R}^n)} \leq c\|v\|_{W^{k,\hat{q}}(\Omega)} \quad \forall v \in W^{k,\hat{q}}(\Omega).$$

Let $P : W^{k,\hat{q}}(\mathbb{R}^n) \to W^{k,\hat{q}}(\mathbb{R}^n)$ denote the Helmholtz-Leray projection. Given $v \in W^{k,\hat{q}}(\Omega)$ we have

$$v = PEv + (I - P)Ev \quad \text{a.e. in } \Omega.$$ 

In addition, there exists a constant $c > 0$ depending only on $n, q, k$ and $\Omega$ such that

$$(3.1) \quad \|PEv\|_{W^{k,\hat{q}}(\Omega)} + \|(I - P)Ev\|_{W^{k,\hat{q}}(\Omega)} \leq c\|v\|_{W^{k,\hat{q}}(\Omega)} \quad \forall v \in W^{k,\hat{q}}(\Omega).$$

Now, for $f \in L^s(0, T; W^{k,r}(\Omega))$ let $(u, p)$ be a strong solution to (1.1)–(1.4). Observing $I - P = \nabla(\Delta^{-1} \text{div})$ recalling the definition of $E$ we get

$$Ef = PEf + (I - P)Ef = PEf + \nabla(\Delta^{-1} \text{div } Ef) \quad \text{a.e. in } Q.$$ 

Since, $\nabla(\Delta^{-1} \text{div } Ef) = (\Delta^{-1} \nabla \text{div } Ef) \in L^s(0, T; W^{k,\hat{q}}(\mathbb{R}^n))$ we see that $PEf \in L^s(0, T; W^{k,\hat{q}}(\mathbb{R}^n))$. Thus, we can replace $f$ by the restriction of $PEf$ on $Q$, and $p$ by the restriction of $-\Delta^{-1} \text{div } Ef + p$ on $Q$. Hence, in what follows without loss of generality we may assume that

$$\text{div } f = 0, \quad \text{and } \Delta p = 0 \quad \text{a.e. in } Q.$$ 

2° Secondly, we recall a well-known result by Giga and Sohr [7] which is the following

Lemma 3.1 Let $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+, \Omega$ bounded or $\Omega$ an exterior domain with $\partial \Omega \subset C^2$. For every $g \in L^{s}(0, T; L^{q}_{\text{loc}}(\Omega))$ $(1 < s, q, < +\infty)$ there exists a unique solution $(\psi, \pi) \in L^s(0, T; W^{2,\hat{q}}_{\text{loc}}(\Omega)) \times L^s(0, T; W^{1,\hat{q}}_{\text{loc}}(\Omega))$ to the Stokes problem

$$\partial_t \psi - \Delta \psi = -\nabla \pi + g \quad \text{and} \quad \text{div } \psi = 0 \quad \text{in} \quad \Omega \times (0, T),$$

$$\psi = 0 \quad \text{on} \quad \partial \Omega \times (0, T),$$

$$\psi(0) = 0 \quad \text{on} \quad \Omega \times \{0\},$$

such that $\partial_t \psi, \partial_{ij} \psi, \nabla \pi \in L^s(0, T; L^{q}(\Omega))$ $(i, j = 1, \ldots, n)$, and there holds,

$$(3.3) \quad \|\partial_t \psi\|_{L^s(0, T; L^s(\Omega))} + \|\nabla^2 \psi\|_{L^s(0, T; L^s(\Omega))} + \|\nabla \pi\|_{L^s(0, T; L^q(\Omega))} \leq c\|g\|_{L^s(0, T; L^q(\Omega))},$$

where the constant $c$ depends only on $n, s, q$ and $\Omega$.

As a consequence of Lemma 3.1 we get the existence of a unique solution $(u, p) \in L^s(0, T; W^{2,\hat{q}}_{\text{loc}}(\Omega)) \times L^s(0, T; W^{1,\hat{q}}_{\text{loc}}(\Omega))$ to the Stokes system (1.1)–(1.4), such that

$$(3.4) \quad \|\partial_t u\|_{L^s(0, T; L^s(\Omega))} + \|\nabla^2 u\|_{L^s(0, T; L^s(\Omega))} + \|\nabla p\|_{L^s(0, T; L^q(\Omega))} \leq c\|f\|_{L^s(0, T; L^s(\Omega))}.$$
bounded open sets such that \( \overline{G} \subset G' \) and \( \overline{G'} \subset G'' \). Set \( \Omega'' = \mathbb{R}^3 \setminus \overline{G''} \) and \( \Omega' = \mathbb{R}^3 \setminus \overline{G'} \). Then, let \( \zeta \in C^\infty(\mathbb{R}^3) \) denote a cut-off function such that \( \zeta \equiv 1 \) on \( \Omega'' \), and \( \zeta \equiv 0 \) in \( G' \). In particular, \( \text{supp}(\nabla \zeta) \subset G'' \setminus G' \). Observing \( \text{div}(u(t)\zeta) = u(t) \cdot \nabla \zeta \), it follows that \( \text{supp}(u(t) \cdot \nabla \zeta) \subset G'' \setminus G' \) for a.e. \( t \in (0, T) \).

Next, let \( 1 < R < +\infty \) such that \( G'' \subset B_R \). By \( \mathcal{B} : W_0^{k-1,q}(B_R) \to W_0^{k,q}(B_R) \) we denote the Bogowski operator defined in Section 2. We now define

\[
z(t) = \mathcal{B}(u(t) \cdot \nabla \zeta), \quad t \in [0, T).
\]

Let \( t \in (0, t) \). Since \( \int_{B_R} u(t) \cdot \nabla \zeta \, dx = 0 \), in view of (2.8) we have

\[
\text{div} \, z(t) = u(t) \cdot \nabla \zeta \quad \text{a.e. in } B_R.
\]

Thanks to (2.6), recalling that \( \text{ratio}(B_R) = 1 \), there exists a constant \( c \) depending only on \( q \) and \( n \) such that

\[
\|z(t)\|_{W^{3,q}(B_R)} \leq c \|u(t) \cdot \nabla \zeta\|_{W^{2,q}(B_R)} \quad \text{for a.e. } t \in (0, T).
\]

Making use of the embedding \( W_0^{3,q}(B_R) \hookrightarrow W^{3,q}(\mathbb{R}^n) \) the above inequality implies that \( z \in L^q(0, T; W^{3,q}(\mathbb{R}^n)) \). Together with (3.4), and the Sobolev-Poincaré inequality we obtain

\[
\|z\|_{L^q(0,T;W^{3,q}(\mathbb{R}^n))} \leq c \|u\|_{L^q(0,T;W^{2,q}(\Omega \setminus B_R))} \leq c \|f\|_{L^q(0,T;L^q(\Omega))}.
\]

By an analogous reasoning taking into account \( \partial_t z = \mathcal{B}(\partial_t u \cdot \nabla \zeta) \) a.e. in \( \mathbb{R}^n \times (0, T) \) we see that \( \partial_t z \in L^q(0, T; W^{1,q}(\mathbb{R}^n)) \). In addition, by virtue of (3.4) we obtain

\[
\|
abla \partial_t z\|_{L^q(0,T;W^{1,q}(\mathbb{R}^n))} \leq c \|
abla \partial_t u\|_{L^q(0,T;L^q(\Omega))} \leq c \|f\|_{L^q(0,T;L^q(\Omega))}.
\]

Next, let \( k \in \{1, \ldots, n\} \) be fixed. We define

\[
\begin{aligned}
v(x,t) &= \partial_k(u(x,t)\zeta(x) - z(x,t)), \quad (x,t) \in (G'' \setminus G') \times (0, T), \\
v(x,t) &= -\partial_k z(x,t), \quad (x,t) \in (\mathbb{R}^n \setminus (G'' \setminus G')) \times (0, T),
\end{aligned}
\]

and

\[
\begin{aligned}
\pi(x,t) &= \partial_k(p(x,t)\zeta(x)), \quad (x,t) \in (G'' \setminus G') \times (0, T), \\
\pi(x,t) &= 0, \quad (x,t) \in (\mathbb{R}^n \setminus (G'' \setminus G')) \times (0, T).
\end{aligned}
\]

Then the pair \((v, \pi)\) solves the Stokes system

\[
\begin{aligned}
\text{div} \, v &= 0 \quad \text{in } \mathbb{R}^n \times (0, T), \\
\partial_t v - \Delta v &= -\nabla \pi + g \quad \text{in } \mathbb{R}^n \times (0, T), \\
v &= 0 \quad \text{on } \mathbb{R}^n \times \{0\},
\end{aligned}
\]

where

\[
g = (p - p_{BR}) \nabla \zeta - 2\partial_k(\nabla u \cdot \nabla \zeta) - \partial_k(u \Delta \zeta) - \partial_k \partial_t z + \partial_k \Delta z + \partial_k(f \zeta) \quad \text{a.e. in } \mathbb{R}^n \times (0, T).
\]
In view of (3.3), (3.5), and (3.6) we see that \( g \in L^s(0,t;L^q(\mathbb{R}^n)) \). In addition, there holds
\[
\|g\|_{L^s(0,t;L^q(\mathbb{R}^n))} \leq c\|f\|_{L^s(0,t;W^{1,q}(\Omega))}.
\]
Thus, applying Lemma 3.1 with \( \Omega = \mathbb{R}^n \), and using the last inequality we see that
\[
\|
\partial_t v \|_{L^s(0,t;L^q(\mathbb{R}^n))} + \|\nabla^2 v\|_{L^s(0,t;L^q(\mathbb{R}^n))} + \|\nabla\pi\|_{L^s(0,t;L^q(\mathbb{R}^n))}
\]
\[
\leq c\|g\|_{L^s(0,t;L^q(\mathbb{R}^n))} + c\|f\|_{L^s(0,t;W^{1,q}(\Omega))}.
\]
Recalling the definition of \( \mathbf{v} \), making use of (3.5), (3.6), and (3.4), we infer from above
\[
\|\zeta \partial_k \partial_k \mathbf{u}\|_{L^s(0,t;L^q(\Omega))} + \|\zeta \nabla^2 \partial_k \mathbf{u}\|_{L^s(0,t;L^q(\Omega))} + \|\zeta \nabla \partial_k p\|_{L^s(0,t;L^q(\Omega))}
\]
\[
\leq c\|f\|_{L^s(0,t;W^{1,q}(\Omega))}.
\]
Iterating the above argument \( k \) times, we get
\[
\|\partial_k \mathbf{u}\|_{L^s(0,t;W^{k,q}(\Omega))} + \|\mathbf{u}\|_{L^s(0,t;W^{k+2,q}(\Omega))} + \|\nabla p\|_{L^s(0,t;W^{k,q}(\Omega))}
\]
\[
(3.7) \quad \leq c\|f\|_{L^s(0,t;W^{k,q}(\Omega))}
\]
\((k \in \mathbb{N})\), where \( c = \text{const} > 0 \), depending on \( s, q, k \), and \( \Omega \) only.

4° Boundary regularity Let \( x_0 \in \partial \Omega \). Up to translation and rotation we may assume that \( x_0 = 0 \) and \( \mathbf{n}(0) = -e_n \), where \( \mathbf{n}(0) \) denotes the outward unite normal on \( \Omega \) at \( x_0 \). According to our assumption on the boundary of \( \Omega \) there exists \( 0 < R < +\infty \), and \( h \in C^{2+k}(B'_R) \) such that
\[
(\text{i}) \quad \partial \Omega \cap (B'_R \times (-R, R)) = \{(y', h(y')); y' \in B'_R\};
\]
\[
(\text{ii}) \quad \{(y', y_n); y' \in B'_R, h(y') < y_n < h(y') + R\} \subset \Omega;
\]
\[
(\text{iii}) \quad \{(y', y_n); y' \in B'_R, -R + h(y') < y_n < h(y')\} \subset \Omega^c \quad [3].
\]
Set \( U_R = B'_R \times (-R, R), U_R^+ = B'_R \times (0, R)\), and define \( \Phi : U_R \rightarrow \Phi(U_R) \) by
\[
\Phi(y) = (y', h(y') + y_n)^T, \quad y \in U_R.
\]
Elementary,
\[
D\Phi(y) = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ \partial_1 h(y) & \partial_2 h(y) & \ldots & \partial_n h(y) & 1 \end{pmatrix},
\]
\[
(D\Phi(y))^{-1} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ -\partial_1 h(y) & -\partial_2 h(y) & \ldots & -\partial_n h(y) & 1 \end{pmatrix}.
\]

\(^3\) Here \( y' = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1} \), and \( B'_R \) denotes the two dimensional ball \( \{(y_1, \ldots, y_{n-1}) : y_1^2 + \ldots + y_{n-1}^2 < R^2\} \).
For the outward unit normal at \( x = \Phi(y) \) we have
\[
n(x) = N(y) = \frac{(\partial_1 h(y), \ldots, \partial_{n-1} h(y), -1)}{\sqrt{1 + |\nabla h(y)|^2}}, \quad y \in B'_R \times \{0\}.
\]

In addition, one calculates
\[
(3.8) \quad \partial_{x_i} \circ \Phi = \partial_{y_i} - (\partial_{x_i} h) \partial_{y_n}, \quad \text{in} \quad U_R, \quad i = 1, \ldots, n. \tag{3.9}
\]

We set \( U = u \circ \Phi, P = p \circ \Phi \) and \( F = f \circ \Phi \) a.e. in \( U_R^+ \times (0, T) \). By the aid of (3.8) we easily get
\[
(3.10) \quad (\text{div}_x u) \circ \Phi = \text{div}_x U - \nabla h \cdot \partial_{y_n} U = 0,
\]
\[
(3.11) \quad (\Delta_x u) \circ \Phi = \Delta_y U - 2\nabla h \cdot \nabla y \cdot \partial_{y_n} U + |\nabla h|^2 \partial_{y_n} \partial_{y_n} U - (\Delta h) \partial_{y_n} U,
\]
\[
(3.12) \quad \text{div}_x U = \nabla h \cdot \partial_{y_n} U \quad \text{a.e. in} \quad U_R^+ \times (0, T),
\]
and with help of (3.10) and (3.11) the equation (1.2) turns into
\[
\partial_t U - \Delta U = -\nabla P + (\partial_{y_n} P) \nabla h - 2\nabla h \cdot \nabla y \cdot \partial_{y_n} U + |\nabla h|^2 \partial_{y_n} \partial_{y_n} U - (\Delta h) \partial_{y_n} U + F \tag{3.13}
\]
a.e. in \( U_R^+ \times (0, T) \).

Note that the assumption \( n(0) = -e_n \) implies \( \nabla h(0) = 0 \). We now choose \( 0 < \delta < +\infty \) sufficiently small, which will be specified later. Since \( \nabla h \in C^0(U_R) \), there exists \( 0 < \rho < \frac{R}{2} \) such that
\[
(3.14) \quad |\nabla h(y)| \leq \delta \quad \forall \, y \in U_{2\rho}.
\]

Let \( \zeta \in C_0^\infty(U_{2\rho}) \) denote a cut-off function such that \( 0 \leq \zeta \leq 1 \) in \( U_{2\rho} \), and \( \zeta \equiv 1 \) on \( U_\rho \). We define \( \tilde{U} : \mathbb{R}_+^n \times (0, T) \rightarrow \mathbb{R}^n \) by
\[
\tilde{U}(y, t) = \zeta(y) U(y, t), \quad y \in U_{2\rho}^+ \times (0, T), \quad \tilde{U}(y, t) = 0 \quad \text{if} \quad y \in \mathbb{R}_+^n \setminus U_{2\rho}^+ \times (0, T).
\]

Let \( \mathcal{B} : W_0^{k-1,q}(U_{2\rho}^+) \rightarrow W^{k,q}(\mathbb{R}_+^n) \) denote the Bogowski\(^4\) operator defined in Section 2. We set
\[
\begin{align*}
\zeta_1(y, t) &= \mathcal{B}(\zeta \nabla h \cdot \partial_{y_n} U)(y, t), \\
\zeta_2(y, t) &= \mathcal{B}(\nabla \zeta \cdot U)(y, t), \quad (y, t) \in \mathbb{R}_+^n \times (0, T).
\end{align*}
\]

Let \( k \in \{1, \ldots, n-1\} \) be fixed. We define
\[
\begin{align*}
V(y, t) &= \partial_k(\tilde{U}(y, t) - \zeta_1(y, t) - \zeta_2(y, t)), \\
\Pi(y, t) &= \partial_k(\zeta(y) P(y, t)).
\end{align*}
\]

\(^4\) Since \( h \) is independent on \( y_n \) there holds \( \partial_{x_n} \circ \Phi = \partial_{y_n} \).
\((y, t) \in \mathbb{R}^n_+ \times (0, T)\). Observing that

\[
\int_{U_{2p}^+} \zeta \nabla h \cdot \partial_n U(t) + \nabla \zeta \cdot U(t) dy = \int_{U_{2p}^+} \text{div}_y \tilde{U}(t) dy = 0 \quad \text{for a.e. } t \in (0, T),
\]

by the aid of (2.8) we calculate

\[
(3.15) \quad \text{div}_y V = \partial_k \left( \zeta \nabla h \cdot \partial_n U + \nabla \zeta \cdot U - \zeta \nabla h \cdot \partial_n U - \nabla \zeta \cdot U \right) = 0
\]
a.e. in \(\mathbb{R}^n_+ \times (0, T)\). In addition, taking into account (3.13), we find

\[
\partial_t V - \Delta V = \partial_k \left( \zeta \partial_t U - \zeta \Delta U - 2 \nabla \zeta \cdot \nabla U - (\Delta \zeta) U \right)
\]

\[
- \partial_k (\partial_t z_1 - \Delta z_1) - \partial_k (\partial_t z_2 - \Delta z_2)
\]

\[
= - \nabla \Pi + \partial_k \left( (P - P_{U_{2p}^+}) \nabla \zeta \right) - \partial_k \left( 2 \nabla \zeta \cdot \nabla U + (\Delta \zeta) U \right)
\]

\[
- \partial_k (\partial_t z_1 - \Delta z_1) - \partial_k (\partial_t z_2 - \Delta z_2)
\]

\[
+ \partial_k \left( \zeta (\partial_t P) \nabla h - 2 \zeta \nabla h \cdot \nabla \partial_n U + \zeta |\nabla h|^2 \partial_n \partial_n U \right)
\]

\[
- \zeta (\Delta h) \partial_n U + \zeta F
\].

Thus, \((V, \Pi)\) solves the following Stokes system

\[
\begin{align*}
\text{div} V &= 0 & \text{in } \mathbb{R}^n_+ \times (0, T), \\
\partial_t V - \Delta V &= - \nabla \Pi + G & \text{in } \mathbb{R}^n_+ \times (0, T), \\
V &= 0 & \text{on } \partial \mathbb{R}^n_+ \times (0, T),
\end{align*}
\]

where \(G = G_1 + \ldots + G_6\) with

\[
G_1 = \partial_{y_k} ((P - P_{U_{2p}^+}) \nabla \zeta),
\]

\[
G_2 = - \partial_t (2 \nabla \zeta \cdot \nabla U + (\Delta \zeta) U),
\]

\[
G_3 = - \partial_t (\partial_t z_1 - \Delta z_1),
\]

\[
G_4 = - \partial_t (\partial_t z_2 - \Delta z_2),
\]

\[
G_5 = \partial_k \left( \zeta (\partial_t P) \nabla h - 2 \zeta \nabla h \cdot \nabla \partial_n U + \zeta |\nabla h|^2 \partial_n \partial_n U \right),
\]

\[
G_6 = \partial_k (\Delta h) \partial_n U + \zeta F.
\]

In what follows we shall establish some important estimates of \(z_1\) and \(z_2\), where we will make essential use of the properties of \(B\) (cf. Section 2). Starting with \(z_1\), we write \(z_1 = z_{1,1} + z_{1,2}\), where

\[
z_{1,1} = \mathcal{B}(\partial_n (\zeta \nabla h \cdot U)), \quad z_{1,2} = - \mathcal{B}(\partial_n \zeta \nabla h \cdot U).
\]

Let \(t \in (0, T)\) be fixed. Using (2.5), (2.6) with \(j = k, i = n\) and \(f = \zeta \nabla h \cdot U\), and observing \(\partial_t \mathcal{B} = \mathcal{B} \partial_t\), we see that

\[
\|\partial_t \partial_k z_1(t)\|_{L^q(\mathbb{R}^n_+)} \leq \|\partial_k \partial_t z_{1,1}(t)\|_{L^q(\mathbb{R}^n_+)} + \|\partial_t \partial_k z_{1,2}(t)\|_{L^q(\mathbb{R}^n_+)}
\]

\[
\leq c \|\partial_t \partial_k (\zeta \nabla h \cdot U)(t)\|_{L^q(\mathbb{R}^n_+)} + c \rho^{-1} \|\partial_t \partial_k U(t)\|_{L^q(\mathbb{R}^n_+)}
\]

\[
\leq c \delta \|\partial_t \partial_k \tilde{U}(t)\|_{L^q(\mathbb{R}^n_+)} + c (\|h\|_{C^2} + \rho^{-1}) \|\partial_t U(t)\|_{L^q(U_{2p}^+)}.
\]

9
Taking the above inequality to the $s$-th power, and integrating the resulting equation in time over $(0, T)$, we get
\begin{equation}
\| \partial_t \partial_k z_1 \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} \leq c\delta \| \partial_t \partial_k \tilde{U} \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} + c(\| h \|_{C^2} + \rho^{-1}) \| \partial_t U \|_{L^s(0,T; L^q(\mathbb{R}^n_+))}.
\end{equation}
(3.16)

On the other hand, using $[2,5]$, [2,7] with $j = k$, $i = n$, and $f = \zeta \nabla h \cdot U(t)$, we see that
\begin{align*}
\| \nabla^2 \partial_k z_1(t) \|_{L^q(\mathbb{R}^n_+)} &\leq c \| \partial_n \nabla \zeta \nabla h \cdot U(t) \|_{L^q(\mathbb{R}^n_+)} + c \| \partial_n \partial_n \partial_k (\zeta \nabla h \cdot U) \|_{L^q(\mathbb{R}^n_+)} \\
&\quad + c \rho^{-1} \| \nabla^2 (\zeta \nabla h \cdot U)(t) \|_{L^q(\mathbb{R}^n_+)} + c \| \nabla^2 (\partial_n \zeta \nabla h \cdot U)(t) \|_{L^q(\mathbb{R}^n_+)}.
\end{align*}

By means of product rule and Poincaré’s inequality we find
\begin{align*}
\| \nabla^2 \partial_k z_1(t) \|_{L^q(\mathbb{R}^n_+)} \leq c \delta \| \nabla^2 \nabla \zeta \tilde{U}(t) \|_{L^q(\mathbb{R}^n_+)} + c(\| h \|_{C^3} + \rho^{-1}) \| \nabla^2 U(t) \|_{L^q(\mathbb{R}^n_+)}.
\end{align*}

We now take the above inequality to the $s$-th power, integrating the result in time over $(0, T)$, we obtain
\begin{equation}
\| \nabla^2 \partial_k z_1 \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} \leq c \delta \| \nabla^2 \nabla \zeta \tilde{U} \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} + c(\| h \|_{C^3} + \rho^{-1}) \| \nabla^2 U \|_{L^s(0,T; L^q(\mathbb{R}^n_+))}.
\end{equation}
(3.17)

By an analogous reasoning, making use of $[2,5]$, and Poincaré’s inequality, we infer
\begin{align*}
\| \partial_t z_2 \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} + \| \nabla^2 z_2 \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} &\leq c \rho^{-1} \left( \| \partial_t U \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} + \| \nabla^2 U \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} \right).
\end{align*}
(3.18)

We are now in a position to estimate $G_1, \ldots, G_6$. First by virtue of Poincaré’s inequality we easily estimate
\begin{align*}
\| G_1 \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} &\leq c \rho^{-1} \| \nabla P \|_{L^s(0,T; L^q(\mathbb{R}^n_+))}.
\end{align*}

Analogously,
\begin{align*}
\| G_2 \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} &\leq c \rho^{-1} \| \nabla^2 U \|_{L^s(0,T; L^q(\mathbb{R}^n_+))}.
\end{align*}

Next, with the help of (3.16), (3.17), and (3.18) we see that
\begin{align*}
\| G_3 \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} + \| G_4 \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} &\leq c \delta \left( \| \partial_t \partial_k \tilde{U} \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} + \| \nabla^2 \nabla \tilde{U} \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} \right) \\
&\quad + c(\| h \|_{C^3} + \rho^{-1}) \left( \| \partial_t \partial_k \tilde{U} \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} + \| \nabla^2 \nabla \tilde{U} \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} \right).
\end{align*}

Then applying the product rule, and using Poincaré’s inequality, we get
\begin{align*}
\| G_5 \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} &\leq c \delta \left( \| \nabla \Pi \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} + \| \nabla \nabla^2 \tilde{U} \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} \right) \\
&\quad + c(\| h \|_{C^2} + \rho^{-1}) \left( \| \nabla P \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} + \| \nabla^2 U \|_{L^s(0,T; L^q(\mathbb{R}^n_+))} \right).
\end{align*}
Finally, we estimate
\[
\|G_6\|_{L^*(0,T;L^q(\mathbb{R}^3_+))} \leq c(\|h\|_{C^3} + \rho^{-1}) \left( \left\| \nabla P \right\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} + \left\| \nabla^2 U \right\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} \right) 
+ c\rho^{-1}\|F\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} + c\|\partial_y P\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))}.
\]

Appealing to Lemma 3.1 (cf. [7]) for the case \( \Omega = \mathbb{R}^n_+ \) using the above estimates for \( G_1, \ldots, G_6 \), we obtain
\[
\|\partial_t V\|_{L^*(0,T;L^1(\mathbb{R}^3_+)))} + \|\nabla^2 V\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} + \|\nabla\partial_t U\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} 
\leq c\|G_1 + \ldots + G_6\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} 
\leq c\delta \left( \|\partial_t \nabla \tilde{U}\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} + \|\nabla^2 \nabla \tilde{U}\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} + \|\nabla\partial_t U\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} \right) 
+ c(\|h\|_{C^3} + \rho^{-1}) \left( \left\| \partial_t \nabla U \right\|_{L^*(0,T;L^q(U^+_R)))} + \left\| \nabla^2 U \right\|_{L^*(0,T;L^q(U^+_R)))} \right) 
+ \|\nabla P\|_{L^*(0,T;L^q(U^+_R)))} + \|F\|_{L^*(0,T;W^{1,q}(U^+_R)))},
\]

(3.19)

Recalling \( V = \partial_t(\tilde{U} - z_1 - z_2) \), making use of (3.16), (3.17) and (3.18), from the last inequality we infer
\[
\|\partial_t \nabla \tilde{U}\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} + \|\nabla^2 \nabla \tilde{U}\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} + \|\nabla\partial_t U\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} 
\leq c_0\delta \left( \|\partial_t \nabla \tilde{U}\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} + \|\nabla^2 \nabla \tilde{U}\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} + \|\nabla\partial_t U\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} \right) 
+ c_1 \left( \left\| \partial_t \nabla U \right\|_{L^*(0,T;L^q(U^+_R)))} + \left\| \nabla^2 U \right\|_{L^*(0,T;L^q(U^+_R)))} \right) 
+ \|\nabla P\|_{L^*(0,T;L^q(U^+_R)))} + \|F\|_{L^*(0,T;W^{1,q}(U^+_R)))}.
\]

(3.20)

where \( c_0 = c_0(n,q,s) \) and \( c_1 = c_1(n,q,s,\|h\|_{C^3},\rho) \). On the other hand, recalling the definition of \( U, P, \) and \( F, \) with the help of (3.10), (3.11), and (3.7) we find
\[
\|\partial_t U\|_{L^*(0,T;L^q(U^+_R)))} + \|\nabla^2 U\|_{L^*(0,T;L^q(U^+_R)))} 
+ c\|\nabla P\|_{L^*(0,T;L^q(U^+_R)))} + \|F\|_{L^*(0,T;W^{1,q}(U^+_R)))} \leq c\|f\|_{L^*(0,T;W^{1,q}(\Omega)))}
\]
with a constant \( c \) depending on \( n, q, s \) and \( h \). Now, in (3.19) we take \( \delta = \frac{1}{2c_0} \) and estimate the right-hand side of (3.19) by the aid of (3.20). This leads to
\[
\|\partial_t \nabla \tilde{U}\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} + \|\nabla^2 \nabla \tilde{U}\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} + \|\nabla\partial_t U\|_{L^*(0,T;L^q(\mathbb{R}^3_+)))} 
\leq c_2\|f\|_{L^*(0,T;W^{1,q}(\Omega)))},
\]
where \( c_2 = c_2(n,q,s,\|h\|_{C^3},\rho) \).

By a standard iteration argument we obtain
\[
\|\partial_t \nabla^k U\|_{L^*(0,T;L^q(U^+_R)))} + \|\nabla^2 \nabla^k U\|_{L^*(0,T;L^q(U^+_R)))} + \|\nabla\nabla^k P\|_{L^*(0,T;L^q(U^+_R)))} 
\leq c\|f\|_{L^*(0,T;W^{1,q}(\Omega)))},
\]
where \( c = \text{const depending only on } n, q, s, k, \|h\|_{C^{k+2}} \) and \( \rho \).

5° Estimation of the full pressure gradient Recalling that \( \Delta x P = 0 \), with the help of (3.10) we calculate
\[
0 = \Delta x P \circ \Phi = \Delta x P - 2\nabla h \cdot \nabla \partial_y n P + |\nabla h|^2 \partial_y n \partial_y n P - (\Delta h) \partial_y n P 
= (1 + |\nabla h|^2) \partial_y n \partial_y n P + \Delta' x P - 2\nabla h \cdot \nabla \partial_y n P - (\Delta h) \partial_y n P
\]
a.e. in $U_R^+$. Thus,

$$(1 + |\nabla h|^2)\partial_{y_n} \partial_{y_n} P = -\Delta'_y P + 2\nabla h \cdot \nabla \partial_{y_n} P + (\Delta h)\partial_{y_n} P$$

a.e. in $U_R^+$. From this identity along with (3.21) with $k = 1$ it follows that

$$\|\nabla^2_y P\|_{L^q(0,T;L^q(U_R^+))} \leq c \left( \|\nabla \partial_{y_n} P\|_{L^q(0,T;L^q(U_R^+))} + \|\nabla_y P\|_{L^q(0,T;L^q(U_R^+))} \right)$$

$$\leq c \|f\|_{L^q(0,T;W^{1,q}(\Omega))}.$$ 

Choosing $\rho \in \left(0, \frac{R}{2}\right)$ sufficiently small, and applying the above argument $k$-times, we get

$$(3.22) \quad \|\nabla^{k+1}_y P\|_{L^q(0,T;L^q(U_R^+))} \leq c \|f\|_{L^q(0,T;W^{k,q}(\Omega))}$$

with a constant $c$ depending on $n, q, s, k, \|h\|_{C^{k+2}}$, and $\rho$.

Finally a standard covering argument, together with (3.22), and (3.7) gives the estimate (1.5), which completes the proof of the Theorem 1. 

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