Compact Enumeration for Scheduling One Machine

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Abstract

A Variable Parameter (VP) analysis aims to give precise time complexity expressions of algorithms with exponents appearing solely in terms of variable parameters. A variable parameter is the number of objects with specific properties. Here we describe two VP-algorithms, an implicit enumeration and a polynomial-time approximation scheme for a strongly \(NP\)-hard problem of scheduling \(n\) independent jobs with release and due times on one machine to minimize the maximum job completion time \(C_{\text{max}}\). Our variable parameters are the amounts of some specially defined types of jobs. A partial solution without these jobs is constructed in a low degree polynomial time, and an exponential time procedure (in the number of variable parameters) is carried out to augment it to a complete optimal solution. In the alternative time complexity expressions, the exponential dependence is solely on the some job parameters. Applying the fixed parameter analysis to these estimations, a polynomial-time dependence is achieved. Both, the intuitive probabilistic estimations and our extensive experimental study support our conjecture that the total number of the variable parameters is far less than \(n\) and its ratio to \(n\) converges to 0 asymptotically.

Keywords: algorithm, scheduling, parameterized analysis, polynomial time approximation scheme, time complexity

1 Introduction

A parameterized analysis, that we call the Variable Parameter (VP) analysis, aims to give a precise time complexity expression of algorithms with exponents appearing solely in terms of variable parameters. A variable parameter is the number of objects with specific properties. We can express an exponential-time dependence on the length of the input, instead of terms of \(n\), in the terms of variable parameters and we obtain the accurate time complexity estimations. The main emphasis is on the identification of these “distinguished” objects. The practical efficiency of a given VP-algorithm can be verified by computational experiments revealing the robustness factor of the algorithm, the ratio of the sum of all variable parameters and the length of the input (typically, \(n\)). For variety of optimization problems, VP-analysis can be accomplished using certain structural properties of the problem in question.

In this paper, we describe two VP-algorithms, an implicit enumeration and a polynomial-time approximation scheme for a basic problem of scheduling \(n\) independent jobs with release times and due dates on one machine non-preemptively, with the objective to minimize the maximum job completion time \(C_{\text{max}}\). When expressing an exponential time dependence of an algorithm in terms
of \( n \), we implicitly assume that all \( n \) objects contribute “homogeneously” to the complexity status of the problem. This is not necessarily so for scheduling problems and for numerous other types of optimization problems. As an example, the set cover and the domination problems in graphs are not fixed parameter tractable nor fixed parameter approximable. But they admit VP-analysis. For instance, in the domination problem, one can establish subsets of vertices which must belong to an optimal dominating set, and subsets of vertices which do not belong to that set. Easily constructed the first type of subsets contain all support vertices forming the part of a minimum dominating set and the second type of subsets contain all single degree vertices none of which belong to a minimum dominating set. A deeper analysis could lead to a more refined classification of the set of vertices and the creation of smaller groups of vertices that are to be enumerated for finding a minimum dominating set.

In this research, variable parameters are the numbers of specially defined types of “emerging” jobs that we identify analyzing the problem structure. These parameters do not form a part of the input, but they depend on the input parameters and hence they are not fixed. Nevertheless, the VP-analysis has some similarities with the Fixed Parameter (FP) analysis. For instance, in both cases, an exponential-time dependence is due to the corresponding fixed or variable parameters. Given a variable parameter \( \nu \), the worst-case time complexity of a VP-algorithm can roughly be expressed as \( f(\nu)(n - \nu)^{O(1)} \), for some computable function \( f \) (an exponential dependence on \( n \) is somehow hidden in the variable parameter \( \nu \)). There are other differences. For instance, while the FP-analysis restricts the problem type by fixing some of its parameter(s), the VP-analysis takes the input as it is, and identifies a subset of objects for which an exponential-time enumeration is unlikely to be avoided. Since the so derived parameters depend on the problem instance, they are not fixed. Thus, based on the simplified picture of the problem itself, we obtain refined and exact time complexity expressions in terms of variable parameters.

**Problem formulation.** We describe the scheduling problem, dealt with this paper and then give more details about our methods. The problem is important on its own and also in the solution of more complex multiprocessor and shop scheduling problems. There are \( n \) jobs to be scheduled on one machine. Each job \( j \) becomes available at its release time \( r_j \) (only from time \( r_j \) it can be assigned to the machine). The due date \( d_j \) is the desirable time for the completion of job \( j \). Job \( j \) needs to be processed uninterrupted on the machine during \( p_j \) time units, which is its processing time. We normally assume that these job parameters are non-negative. The machine can handle at most one job at a time. A feasible schedule \( S \) is a mapping that assigns every job \( j \) a starting time \( t_j(S) \) on the machine so that above stated restrictions are satisfied. \( c_j(S) = t_j(S) + p_j \) is the completion time of job \( j \) on the machine. The penalty for late completion of job \( j \) is measured by its lateness \( L_j = c_j(S) - d_j \). The objective is to find an optimal schedule, a feasible one with the minimum maximum job lateness \( L_{\text{max}} = \max_j L_j \). In the standard scheduling notation, the problem is abbreviated as \( 1|r_j|L_{\text{max}} \).

One easily observes that, in a feasible schedule for the above described setting, some jobs may have negative lateness, that makes it difficult to express approximation ratios. To avoid this inconvenience, one may impose that job due dates are negative in problem \( 1|r_j,d_j \leq 0|L_{\text{max}} \). However, the due date negativity constraint can be avoided by considering an alternative and practically intuitive formulation of the problem. In this formulation, besides the processing and release times, every job \( j \) has the delivery time \( q_j \): Job \( j \), once finished on the machine, needs this additional time to be finished. Since the deliveries are accomplished by independent agents, they take no machine time. In particular, the delivery of two or more jobs can be accomplished in parallel without causing any resource conflicts. Job delivery times have immediate practical
sense. For example, they can be machine input/output times or transportation times of finished jobs when they need to be delivered to the customer once they are completely processed by the machine. Since the delivery of two different jobs can be accomplished in parallel, the machine can process some other job during these deliveries. \(C_j(S) = c_j(S) + q_j\) is the full completion time of job \(j\) (including its delivery time) in schedule \(S\). The objective is to minimize the maximum job full completion time \(\max_j C_j(S)\) or the makespan. The problem is abbreviated as \(1|r_j,q_j|C_{\text{max}}\), here \(C_{\text{max}}\) is the maximum job full completion time (where in the scheduling problems without job delivery times, the makespan \(C_{\text{max}}\) normally denotes the maximum job completion time on the machine).

By considering the second setting, we avoid artificial restrictions \(d_j < 0\) for all \(j\). An optimal solution to either of the two settings provides an optimal solution to the other setting, and vice-versa. An instance of \(1|r_j,q_j|C_{\text{max}}\) can easily be transformed to a symmetric instance of \(1|r_j|L_{\text{max}}\) as follows. Take any number \(K \geq \max_j q_j\) and define due date of job \(j\) as \(d_j = K - q_j\). Vice-versa, given an instance of \(1|r_j|L_{\text{max}}\), take a magnitude \(D \geq \max_j d_j\) and define job delivery times as \(q_j = D - d_j\). We refer the reader to Bratley et al. [1] where this kind of transformation is described in detail.

**Outline of this work.** We present two VP-algorithms for the above scheduling problem known to be strongly \(NP\)-hard [3]. We give intuitive rough ideas behind our algorithms without defining the concepts yet. Both, the implicit enumeration algorithm and the polynomial time approximation scheme (PTAS) rely on the observation that only the emerging jobs, ones with some specific properties, contribute to the complexity status of the problem (without these jobs, a polynomial-time algorithm would solve the problem optimally). The whole set of jobs is partitioned into four basic types, the type (1) jobs are emerging jobs, where the remaining (non-emerging) jobs are partitioned into the types (2)-(4). Type (1) jobs, in turn, are partitioned into subsets, the type (1.1) ones and one or more disjoint subsets \(J_{1,2}(K)\) of type (1.2) jobs, where \(K\) is a critical segment, the so-called kernel, in a specially constricted complete feasible schedule. The total number of kernels and hence that of the groups of type (1.2) jobs is trivially bounded by \(n\).

We introduce now some additional notations, yet informally, in order to express time complexity of our algorithms. We denote by \(\nu_{1,1}\) (\(\nu_{1,2}(K)\), respectively) the number of type (1.1) jobs (type (1.2) jobs, associated with kernel \(K\), respectively); \(p_{\text{max}}^{1,1}\) is the maximum processing time of a type (1.1) job, \(p_{\text{min}}(K)\) (\(p_{\text{min}}^{1,2}\), respectively) is the minimum processing time of a type (1.2) job from kernel \(K\) (from a kernel from set \(K\), respectively); \(P_{1,2}(K)\), \(P_{1,2}\) and \(\nu_{1,2}^{\text{ave}}(K)\), respectively, are the total processing time of jobs from set \(J_{1,2}(K)\), the average total processing time and the average number of jobs in a kernel.

We schedule type (2)-(4) jobs optimally in time \(O((n - \nu)^2 \log (n - \nu))\). We carry out an exponential time procedure (in the number of variable parameters) to augment this solution to a complete optimal solution. This results in a worst-case running time \(O(n^2 \log n + n! (n - \nu)n \log n)\) refined to

\[
O((n - \nu)n \log n \nu_{1,1} p_{\text{max}}^{1,1} 2^{\nu_{1,2}^{\text{ave}}(K)}).
\]

Using some “metric” properties of feasible schedules, we obtain an alternative worst-case time complexity expression in terms of job parameters

\[
O((n^2 \log n + n \log n((r_j + p_{\text{max}}^{1,1})/p_{\text{min}}^{1,1})q_{\text{max}}!) + (n - \nu)p_{\text{max}}^{1,1} 2^{(q_{\text{max}}! + P_{1,2})/p_{\text{min}}^{1,2}}).
\]

The PTAS employs the earlier mentioned job partition, and, in addition, partitions the set of jobs into short and long ones, so that the total number of long jobs becomes bounded by a
constant $\kappa$, obtaining an $(1 + 1/\kappa)$-approximation solution. A worst-case running time of the PTAS can straightforwardly be expressed as $O((\kappa_{1,1} + \kappa_{1,2})!\kappa_{1,1}\kappa n \log n)$; applying VP-analysis, it can be rewritten as

$$O(\kappa^4 \log \kappa n \log n \kappa_{1,1}! p_{\text{max}}^{1.1} 2^{\kappa_{1,2} (K)})$$

or alternatively as

$$O(\kappa^4 \log \kappa n \log n ((r_j + p_{\text{max}}^{1.1})/p_{\text{min}}^{1.1})! q_{\text{max}}! + \kappa p_{\text{max}}^{1.1} 2^{(p_{\text{max}}^{1.1} + p_{1.2})/p_{\text{min}}^{1.2}}),$$

where $\kappa_{1,1}$ and $\kappa_{1,2}$ stand for the number of type 1.1 and type 1.2, respectively, long jobs, and $p_{\text{max}}$ and $p_{\text{min}}(K)$ are now taken for long type (1.1) and type (1.2) jobs (in the earlier approximation schemes a complete enumeration of all possible assignments of the starting times of $\kappa$ long jobs yields a factor of $O(\kappa^n)$). Note that $\kappa_{1,1} + \kappa_{1,2} < \kappa$ (in practice, $\kappa_{1,1} + \kappa_{1,2}$ is considerably smaller than $\kappa$, see Section 5). While parameter $\kappa$ is a part of the input for the PTAS, $\kappa_{1,1}$, $\kappa_{1,2}$ and $\nu$ are derived parameters. Our algorithms incorporate permutations and subsets of the type (1) jobs into a partial schedule of the remaining types of jobs, while the PTAS has one additional stage in which a partial schedule of long jobs is completed with short jobs, also in a low degree polynomial time.

**Analysis of our results.** If we take the alternative time complexity expression of VP-algorithm and apply FP-analysis with the input parameters $q_{\text{max}}$, $p_{\text{max}}$ and $p_{\text{min}}(K)$, we obtain a polynomial-time dependence on the length of the input; we obtain a similar kind of an unexpected outcome (for a strongly $NP$-hard problem) for the PTAS, which then can be regarded as a fully polynomial time approximation scheme (since in its alternative time complexity expression there is no exponential-time dependence on $\kappa$).

Considering our exponential in the number of variable parameters time bounds, one may be interested in the probability that an exponential number of permutations will actually be enumerated. Alternatively to our deterministic bounds, we give intuitive probabilistic estimations of the running time of our algorithms (see Sections 3.3 and 4.2.2). For instance, due to a simple probabilistic analysis, the probability that VP-algorithm will verify $\mu$ consecutive dominant permutations is less than $1/4^\mu$.

Since the scheduling problem is $NP$-hard, one naturally expects the existence of “hard” problem instances where $\nu$ is close to $n$. Indeed, there exist such instances. As we will see in Section 5, in a scheduling instance derived from the PARTITION problem, $\nu$ is roughly the number of items in the PARTITION instance. We were keen to know how our VP-analysis works in practice. We carried out an extensive experimental study for over 50 000 problem instances with the number of jobs up to 50 000, generated randomly. While generating our instances, we relied on the standard rules commonly used for this and similar scheduling problems, but we extended the range of possible job processing times (which was set to a constant in earlier studies reported in the literature, see the APPENDIX for details). The variable parameters, in average, turned out to be far less than $n$ and asymptotically the corresponding robustness factors converge to 0 (see again the APPENDIX).

**Related work.** Among the early exact implicit enumeration algorithms for our single-machine scheduling problem, we can mention those given by McMahon & Florian [11] and by Carlier [2], both based on similar ideas. There exist polynomial time approximation schemes for the problem (see later). As to the polynomially solvable special cases, if all job due dates are equal, the problem is easily solvable by a venerable greedy $O(n \log n)$ heuristic by Jackson [9]. This heuristic
can straightforwardly be adopted for the exact solution of an alternative version in which all job release times are equal. Jackson’s heuristic, iteratively, includes the next unscheduled job with the largest delivery time (the smallest due date) at the earliest machine idle time. An extension of this heuristic, described by Schrage [15], gives a 2-approximation solution for problem 1\(|r_j, q_j|C_{\text{max}}\) with job release times (and an exact solution for the preemptive case 1\(|r_j, q_j, pmtn|C_{\text{max}}\)). Iteratively, at each scheduling time \(t\) given by job release or completion time, among the jobs released by that time, the extended heuristic schedules a job with the largest delivery time (smallest due date). We will use this extended heuristic, which has the same time complexity as its predecessor one, as a schedule generator, referring to it as LDT. As it is observed by Horn [8], the heuristic delivers an optimal solution in the case when all the jobs have the unit processing time and job parameters are integers. With real job release times and unit job processing times, the problem can be scaled to an equivalent one in which all job parameters are integers and job processing times are equal (but not unit-length). Then while applying the heuristic for the non-preemptive case, during the execution of a job another more urgent job (i.e., a job with a larger delivery time or a smaller due date) may be released, that basically makes the heuristic non-optimal. If, however, we allow to preempt the former job and schedule the latter one at its release time, the preemptive problem can be solved optimally. The setting with equal (non-unit) length jobs can still be solved in polynomial \(O(n^2 \log n)\) time. Garey et al. [5] used a very sophisticated data structures with a union and find tree with path compression and have achieved the improvement of the time complexity to \(O(n \log n)\).

For the more general setting, where every job processing time divides every larger job processing time, [18] describes an \(O(n^2 \log n \log p_{\text{max}})\) time algorithm, where \(p_{\text{max}}\) is the maximum job processing time. The problem 1\(|r_j|L_{\text{max}}\) becomes strongly \(NP\)-hard if job processing times are from the set \(\{p, 2p, 3p, \ldots\}\), for any integer \(p\). It is fixed parameter tractable for parameters \(p_{\text{max}}\) and \(d_{\text{max}}\), where \(d_{\text{max}}\) is the maximum job due date Vakhania and Werner [17]. For the setting 1\(|r_j, q_j|C_{\text{max}}\), better than 2-approximation polynomial-time algorithms exist. Potts [13] showed that by repeated application of LDT-heuristic \(O(n)\) times, the performance ratio can be improved to 3/2, resulting in an \(O(n^2 \log n)\) time performance. Nowicki and Smutnicki [12] have proposed another 3/2-approximation algorithm with time complexity \(O(n \log n)\). Hall and Shmoys [7] illustrated that the application of the LDT-heuristic to the original and a specially defined reversed problems leads to a further improved approximation of 4/3 in time \(O(n^2 \log n)\).

As to the earlier existing polynomial-time approximation schemes for problem 1\(|r_j, q_j|C_{\text{max}}\), the first one was proposed by Hall and Shmoys [7]. They described two PTASs with time complexities \(O(16^k (nk)^{3+4k})\) and \(O(n \log n + n(4k)^{8k^2+8k+2})\). Mastrolilli [10] suggested another approximation scheme for a more general setting with identical processors, and showed that his algorithm has an aggregated time complexity \(O(nk^{O(1)} + k^{O(k)}) = O(n + k^{O(k)})\). These approximation schemes carry out a complete enumeration of all possible combinations yielding a tight exponential-time dependence \(O(k^k)\) (note that the worst-case exponential dependence on \(\nu\) and \(\kappa\) in our algorithms is expressed explicitly without using the big \(O\)-notation). In Section 4.2.1 we give a more detailed analysis of these approximation schemes.

The rest of this paper is organized as follows. Section 2 contains preliminary material, and in Sections 3 and 4 the VP algorithm and the PTAS, respectively, are described. Section 5 contains some final notes, and the APPENDIX describes our experimental study.
2 Preliminaries

We need an existing technical concepts, properties and tools for scheduling problems to build our VP-algorithm and determine the variable parameter \( \nu \). Different notation and terminology have been used in the literature, here we rely on the ones from [16, 18]. Consider an LDT-schedule \( S \) and a longest sequence of the successively scheduled jobs without idle-time intervals in between them in that schedule, such that: (i) for the last job \( o \) of that sequence,

\[
C_o(S) = \max_i \{ C_i(S) \},
\]

and (ii) no job from the sequence has the delivery time less than \( q_o \). We will refer to \( K \) as a kernel in schedule \( S \), and to job \( o = o(K) \) as the corresponding overflow job.

A larger component in an LDT-schedule \( S \) is a block, a consecutive part of that schedule consisting of the successively scheduled jobs without any gap between them, preceded and succeeded by a (possibly a 0-length) gap. Abusing the terminology, we will refer to a kernel and a block (a schedule, respectively) interchangeably as a sequence (a function, respectively) and also as the corresponding job set. We will distinguish the earliest occurred kernel in schedule \( S \) and denote it occasionally by \( K(S) \).

Suppose job \( i \) precedes job \( j \) in LDT-schedule \( S \). Let us consider an auxiliary LDT-schedule \( S(-i) \), in which we prohibit to schedule any job within the execution interval of job \( i \) except job \( j \). We will say that job \( i \) pushes job \( j \) in schedule \( S \) if LDT-heuristic reschedules job \( j \) earlier in schedule \( S(-i) \) compared to schedule \( S \).

Let us consider kernel \( K \in B \) (\( B \in S \)), and suppose that the first job of that kernel starts after its release time being pushed by job \( l \in B \). Then note that \( q_l < q_o \), since otherwise job \( l \) would form part of kernel \( K \). In general, we will refer to a job \( e \) with \( q_e < q_o \) as an emerging job for kernel \( K \) and to the above job \( l \) as the delaying emerging job for kernel \( K \).

Given LDT-schedule \( S \) with the delaying emerging job \( l \) for kernel \( K \), let

\[
\delta(K, S) = c_l(S) - \min_{i \in K} \{ r_i \}
\]

be the delay of kernel \( K \) in that schedule; i.e., \( \delta(K, S) \) is the forced right-shift imposed by the delaying job \( l \) for the jobs of kernel \( K \) (we will omit the argument \( S \) in \( \delta(K, S) \) when it is clear for which schedule this parameter is calculated).

The next property implicitly defines a lower bound \( LB \) on the optimal schedule makespan \( OPT \):

**Property 1** \( \delta(K) < p_l \), hence \(|S| - OPT < p_l \).

Easily seen values for \( LB \) are known, e.g.,

\[
LB \geq \max \{ P, \max_j \{ r_j + p_j + q_j \} \},
\]

where \( P \) stands for the total processing time of all jobs. We may also observe that there are certain values for constant \( k \) for which an \((1 + 1/k)\)-approximation is a priory guaranteed:
Property 2 \( |S|/OPT < (1 + 1/k) \) if \( k \leq LB/p_l \).

Proof. By Property 1 \( |S|/OPT < (OPT + p_l)/OPT = 1 + p_l/OPT \leq 1 + p_l/LB \leq 1 + 1/k \).

Thus the approximation factor of solution \( S \) depends on \( \delta(K) \), which, in turn, depends on the length \( p_l \) of the delaying emerging job \( l \). Based on this, we will distinguish two types of jobs, the short and the long ones. Job \( j \) is short if \( p_j \leq LB/k \); otherwise, it is long.

Lemma 1 An LDT-schedule containing a kernel with a short delaying emerging job \( l \) is an \((1 + 1/k)\)-approximation schedule.

Proof. Similar to the proof of Proposition 2 with the difference that the last inequality immediately follows from \( p_j \leq LB/k \).

Lemma 2 \( \kappa < k \), i.e., there are less than \( k \) long jobs.

Proof. If there were \( k \) or more long jobs then their total length would exceed \( LB \). This is a contradiction since \( LB \) is attainable.

2.1 Conflicts in LDT-schedules

During the construction of an LDT-schedule \( S \), we iteratively update the current scheduling time \( t \) as either the completion time of the job scheduled the last so far or and the release time of the earliest released yet unscheduled job, whichever magnitude is larger. We will use \( S^t \) for the (partial) LDT-schedule constructed by time \( t \), and \( j_t \) the job that is scheduled at time \( t \).

Scheduling time \( t \) is said to be a conflict scheduling time in schedule \( S^t \) if within the execution interval of job \( j_t \) (including its right endpoint) another job \( j \) with \( q_j > q_{j_t} \) is released; i.e., job \( j_t \) is pushing a more urgent job \( j \), that is, jobs \( j \) and \( j_t \) conflict between each other. Let \( \sigma \) be the LDT-schedule obtained by LDT-heuristic applied to the original problem instance.

Lemma 3 If during the construction of LDT-schedule \( \sigma \) no conflict scheduling time occurs, then it is optimal. In particular, at any scheduling time \( t \), no job released within the execution interval of job \( j_t \) can initiate a kernel in schedule \( \sigma \) unless it conflicts with job \( j_t \).

Proof. From the condition, there may exist no kernel in schedule \( \sigma \) possessing the delaying emerging job. A schedule with this property is known to be optimal (e.g., see [17]).

2.2 Creation of alternative LDT-schedules

By modifying the originally given problem instance and applying LDT-heuristic repeatedly, alternative LDT-schedules can be generated. Given an LDT-schedule \( S \) with kernel \( K \) and the delaying emerging job \( l \), we create an alternative LDT-schedule \( S_l \), in which we activate the delaying emerging job \( l \) for kernel \( K \); that is, we force this job to be scheduled after all jobs of kernel
Figure 1: Initial LDT-schedule \( \sigma \)

\( K \), whereas all the emerging jobs, included after kernel \( K \) in schedule \( S \), remain to be included after that kernel. As a result, the earliest job of kernel \( K \) will be scheduled at its release time in a new LDT-schedule \( S_l \).

To construct schedule \( S_l \), we apply LDT-heuristic to a modified problem instance, in which the release time of job \( l \) and that of all emerging jobs included after kernel \( K \) in schedule \( S \) becomes no less than that of any job of kernel \( K \). Then by LDT-heuristic, job \( l \) and any emerging job included after kernel \( K \) in schedule \( S \) will appear after all jobs of kernel \( K \) in schedule \( S_l \). Note that the original schedule \( S \) can easily be restored by the revision of the activated job \( l \), that is, the restoration of job release times and a repeated application of LDT-heuristic to the restored input.

Kernel \( K^1 = K(S_l) \) is determined in LDT-schedule \( S_l \) similarly to schedule \( S \). If kernel \( K^1 \) possesses the delaying emerging job, let \( l_1 \) be that job. Then job \( l_1 \) is activated for kernel \( K^1 \) resulting in another LDT-schedule \( (S_l)_{l_1} \). We proceed similarly creating the next LDT-schedule \( ((S_l)_{l_1})_{l_2} \), where \( l_2 \) is the delaying emerging job for kernel \( K^2 = K(((S_l)_{l_1})_{l_2}) \), and so on. For notational simplicity, we denote LDT-schedule \( \ldots ((S_l)_{l_1})_{l_2} \ldots )_{l_k} \) by \( S_{l_1}l_2 \ldots l_k \).

### 2.3 Example

We give a small problem instance that will be used for the illustrations throughout the paper. Let us consider 13 jobs defined as follows:

\[
\begin{align*}
 r_1 &= 0, \quad p_1 = 12, \quad q_1 = 11, \\
r_2 &= 2, \quad p_2 = 2, \quad q_2 = 50, \\
r_3 &= 5, \quad p_3 = 3, \quad q_3 = 48, \\
r_4 &= 10, \quad p_4 = 5, \quad q_4 = 44, \\
r_5 &= 13, \quad p_5 = 4, \quad q_5 = 43, \\
r_6 &= 1, \quad p_6 = 7, \quad q_6 = 41, \\
r_7 &= 32, \quad p_7 = 10, \quad q_7 = 3, \\
r_8 &= 35, \quad p_8 = 7, \quad q_8 = 15, \\
r_9 &= 37, \quad p_9 = 4, \quad q_9 = 12, \\
r_{10} &= 41, \quad p_{10} = 3, \quad q_{10} = 11, \\
r_{11} &= 45, \quad p_{11} = 2, \quad q_{11} = 11, \\
r_{12} &= 47, \quad p_{12} = 1, \quad q_{12} = 10, \\
r_{13} &= 58, \quad p_{13} = 2, \quad q_{13} = 2.
\end{align*}
\]

The initial LDT-schedule \( \sigma \) is depicted in Fig. 1. Kernel \( K^1 = K(\sigma) \) consists of jobs \((2, 3, 4, 5, 6)\) and possesses the delaying emerging job 1, which is the only emerging job for that kernel. The overflow job in kernel \( K^1 \) is job 6 with the full completion time \( C_6(\sigma) = 33 + 41 = 74 \), the makespan of schedule \( \sigma \). As we can see, the scheduling times \( t = 0 \) and \( t = 34 \) with \( j_0 = 1 \) and \( j_{34} = 7 \) are the conflict scheduling times in schedules \( \sigma^0 \) and \( \sigma^{34} \), respectively. Hence, \( \sigma \) cannot be guaranteed to be optimal.

In Figure 2 an alternative LDT-schedule \( \sigma_1 \) is depicted. We have a newly arisen kernel \( K^2 = \)}
not necessarily a non-split component. Below we briefly describe the decomposition procedure for that kernel. The corresponding overflow job is job 12 with $C_{12}(\sigma_1) = 59 + 10 = 69$.

2.4 Decomposition of kernels

In this subsection we briefly overview the decomposition procedure that exploits a recurrent structure of every kernel and “filters” them. We will consider jobs of a kernel as an independent set of jobs and construct independent LDT-schedules for these jobs. Some relevant terminology is also briefly introduced (the reader is referred to Section 4 of [18] for more details).

The decomposition procedure. A kernel $K \in S$ possessing the delaying emerging job $l$ is not necessarily a non-split component. Below we briefly describe the decomposition procedure for kernel $K$ that proceeds in a number of iterations. Let $K(+l)$ denote the fragment of schedule $S$ containing the delaying emerging job $l$ and the kernel $K$; we will use $K$ also for the fragment of schedule $K(+l)$ without job $l$. Initially at iteration 1, the procedure omits the delaying emerging job $l$ and generates the next partial LDT-schedule $K_1$ by applying LDT-heuristic solely to the set of jobs from schedule $K$. Note that, unlike an alternative LDT-schedule $S_l$, schedule $K_1$ does not contain job $l$ and any job with the delivery time smaller than or equal to $q$. Note also that the first and possibly some following jobs of kernel $K$ are left-shifted in schedule $K_1$, compared to schedule $K(+l)$. Furthermore, a former kernel job, $\lambda_1 \in K$ may become anticipated, i.e., scheduled in a position, earlier than it was scheduled in schedule $K$. As a result, such an anticipated job may become the delaying emerging job in schedule $K_1$. In general, let at iteration $i > 1$, $\lambda_i$ ($o_i$, respectively) be the delaying emerging job (the overflow job, respectively) in the partial schedule $K_{l,\lambda_1,...,\lambda_{i-1}}$ of the previous iteration. Partial schedule $K_{l,\lambda_1,...,\lambda_{i-1}}$ of iteration $i$ is obtained by the application of LDT-heuristic to the set of jobs from schedule $K_{l,\lambda_1,...,\lambda_{i-1}}$ but the delaying emerging job $\lambda_i$ and any other job $e \in K_{l,\lambda_1,...,\lambda_{i-1}}$ with $q_e < q_{o_i}$ (these jobs being omitted at iteration $i$). If there is no delaying emerging job in schedule $K_{l,\lambda_1,...,\lambda_{i-1}}$, then the last created schedule is obtained from the former one by merely omitting any (emerging) job (if any) $e \in K_{l,\lambda_1,...,\lambda_{i-1}}$ with $q_e < q_{o_i}$ (strictly speaking, such a job will not be an emerging job by our earlier definition if it is separated by a gap from the kernel of the former schedule, but we shall still refer to it as an emerging job). Note that in this case $o_i = o_{i-1}$.

Lemma 4 Suppose $m$ is the earliest occurred iteration in the procedure with $o_m = o_{m-1}$. Then there is an optimal solution in which the jobs of kernel $K(K_{l,\lambda_1,...,\lambda_{m-1}})$ are scheduled in the order as they appear in schedule $K_{l,\lambda_1,...,\lambda_{m}}$.

Proof. To keep the notation simple, we illustrate the proof for schedules $K(+l)$ and $K_1$. Let $o$ be the overflow job of kernel $K$ in schedule $K(+l)$. By the condition, $o$ is the overflow job also in schedule $K_1$. Since every job of a kernel has the delivery time, no smaller than that of the overflow job of that kernel, none of the jobs from kernel $K(K_1)$ can be the delaying emerging job in schedule $K_1$. Then schedule $K_1$ contains no delaying emerging job as it consists of only jobs of kernel $K$. The lemma follows since the full completion time of the overflow job of a kernel without delaying emerging job is an easily seen lower bound on the optimum schedule makespan.
Observe that the decomposition procedure results in a partial schedule consisting of all jobs of kernel $K$ except all the occurred (local emerging) jobs omitted during the procedure. This partial schedule, denoted by $S^*[K]$, consists of fragments of sub-schedules separated by the gaps (occurred because of the omission of the latter jobs). The atomic kernel $K^* = K(K_{l,\lambda_1,...,\lambda_{l-1}})$ of the last continuous part in schedule $S^*[K]$ (its atomic component) possesses no delaying emerging job. It cannot be split further and the procedure halts.

**Lemma 5** (i) For any kernel $K$, the decomposition procedure runs in $m$ recursive steps in time $O(m\mu \log \mu)$, where $m$ is the earliest occurred iteration with $o_m = o_{m-1}$ and $\mu (m < \mu < n - \nu_{1,1})$ is the total number of the jobs in kernel $K$.

(ii) The maximum job full completion time in schedule $S^*[K]$ is a lower bound on the optimum schedule makespan.

Proof. The number of the recursive calls in the procedure is bounded by $m$ and Part (i) follows since at every iteration LDT-heuristic with time complexity $O(\mu \log \mu)$ is applied. Part (ii) can be seen similarly as in the proof of Lemma 4.

**Example.** We can observe the decomposition of kernel $K_1$ in Figures 1, 2 and 3. Schedule $K_1(+1)$ consisting of jobs 1 through 6 extending through the time interval $[0,33)$ forms part of the complete LDT-schedule $\sigma$ of Fig. 1. This partial schedule corresponds to the initial iteration 0 in the decomposition procedure. The delaying emerging job is $l = 1$. At iteration 1, schedule $(K_1)_1$ is the fragment of the complete schedule $\sigma_1$ in Figure 2 from time 1 to time 22. Here $\lambda_1 = 6$. Hence, the procedure continues with iteration 2 generating partial schedule $(K_1)_{1,6}$ with the time interval $[2,17)$ that consists of jobs 2,3,4 and 5, see Fig. 3. The overflow job in partial schedules $(K_1)_1$ and $(K_1)_{1,6}$ is the same job 5, and the kernel in schedule $(K_1)_{1,6}$ consisting of jobs 3,4 and 5, possesses no delaying emerging job. The decomposition procedure halts at iteration 2 and outputs partial schedule $S^*[K]_1$ of Fig. 3.

### 3 The VP algorithm

The VP-algorithm has stages 0 and 1. Stage 0 at first creates an auxiliary partial schedule without emerging jobs, based on which it forms the (initial) job partition. Repeatedly, it forms the next permutation (subset) of the emerging jobs and calls stage 1. Stage 1 incorporates this permutation into the above auxiliary schedule. In this way, it generates a complete schedule with the minimum makespan respecting that permutation, in which the type (1.1) jobs appear in the order of that permutation. At stage 0, each next permutation of emerging jobs will be formed according to the dominance relations which are determined at stage 1.

We keep track of a current state of computations in a configuration, consisting of the current set of kernels and the corresponding job partition. We denote by $K$ the set of all kernels in the current configuration.
3.1 Partitioning the set of jobs at stage 0

The initial job partition is created once the following iterative procedure forms an auxiliary partial schedule of non-emerging jobs. It decomposes every kernel occurred during the construction of that schedule and omits every emerging job occurred during the decomposition of that kernel.

The procedure starts by setting $J' := J$ as the set of all jobs, $\sigma^0 = \sigma$ as the schedule of iteration 0 and $\mathcal{K}^0$ as the set of the kernels in schedule $\sigma^0$. We denote by $\mathcal{K}^h$ and $\sigma^h$ the set of kernels and the schedule, respectively, formed by iteration $h$.

At iteration $h + 1 \geq 1$, the earliest kernel $K^{h+1} := K(\sigma^h)$ in schedule $\sigma^h$ and the set of emerging jobs $E^{h+1}$ from the current set $J'$ for that kernel are determined, $\mathcal{K}^{h+1} := \mathcal{K}^h \cup K^{h+1}$ and $J' := J' \setminus E^{h+1}$. Schedule $\sigma^{h+1}$ is created by firstly replacing the segment of schedule $\sigma^h$ containing kernel $K^{h+1}$ with partial schedule $S^*[K^{h+1}]$. Let $E[K^{h+1}]$ be the set of the emerging jobs omitted during the decomposition of kernel $K^{h+1}$ (note that all jobs from this set get omitted). The remaining parts of schedule $\sigma^{h+1}$ are obtained by applying LDT-heuristic to the jobs from the set $J' \setminus S^*[K^{h+1}] \setminus E[K^{h+1}]$. If $E^{h+1} = \emptyset$, partial schedule $S^*[K^{h+1}]$ coincides with kernel $K^{h+1}$ and $\sigma^{h+1} = \sigma^h$. The procedure halts at iteration $\xi = h + 1$ if there is an overflow job in schedule $\sigma^{h+1}$ from partial schedule $S^*(K)$, for some $K \in \mathcal{K}^h$. Let $\mathcal{K}^{\xi}$ be the set of all kernels in schedule $\sigma^\xi$. The schedule $\sigma^{\xi}$ is updated by replacing similarly every segment corresponding to a kernel $K \in \mathcal{K}^{\xi}$, $K \notin \mathcal{K}^{\xi}$, with the partial schedule $S^*[K]$. We let the resultant set of kernels be $\mathcal{K} := \mathcal{K}^{\xi} \cup \mathcal{K}^{\xi}$ (note that every kernel in schedule $\sigma^{\xi}$ is an atomic kernel $K^*$, for some $K \in \mathcal{K}$).

**Observation 1** The partial schedule $\sigma^{\xi}$ is a well-defined feasible schedule obtained in less than $n$ iterations.

Proof. Since schedule $\sigma^{\xi}$ contains no emerging job for any of the kernels from set $\mathcal{K}$, there occurs a gap before any $S^*[K'$, $K \in \mathcal{K}$ and hence no overlapping of any included job from the current set $J'$ with any of these parts may occur. Furthermore, $\xi < n$, i.e., stage 0 runs in less than $n$ new overflow jobs may occur.

The initial job partition is determined as follows (we use $K^-$ ($K^+$, respectively) for the kernel from set $\mathcal{K}$ immediately preceding (immediately succeeding, respectively) kernel $K$). Type (1) jobs are the emerging jobs, divided into two sub-types.

1. A type (1.1) job is an emerging job for a kernel $K \in \mathcal{K}$.
2. The set of the type (1.2) jobs associated with kernel $K \in \mathcal{K}$ is formed by the emerging jobs from set $E^{h+1}$ (omitted in schedule $S^*[K]$ during the decomposition of kernel $K$).
3. The type (2) jobs associated with a kernel $K \in \mathcal{K}$, are the jobs of the atomic kernel $K^*$ (the last kernel $K(K_{i_{l, \lambda_{l, \ldots, \lambda_{m-1}}})$ occurred in the decomposition of kernel $K$, see Lemma 4).
4. The type (3) jobs associated with a kernel $K \in \mathcal{K}$, are formed by the remaining jobs of partial schedule $S^*[K]$.
5. All the remaining (non-emerging, non-kernel) jobs are the type (4) jobs. Note that a type (4) is scheduled either before the earliest kernel from set $\mathcal{K}$ or between two partial schedules $S^*[K']$ and $S^*[K'']$, for two neighboring kernels $K'$ and $K''$ from set $\mathcal{K}$, or after the last of these partial schedules in schedule $\sigma^{\xi}$. 

Thus with every kernel $K \in \mathcal{K}$ its own type (1.1), type (1.2), type (2) and type (3) jobs are associated. At the same time, a type (1.1) job is associated with one or more (successive) kernels (with the ones for which it is an emerging job). Note that schedule $\sigma^e$ contains no type (1.1) or (1.2) job. A type (4) job can be associated with a single kernel or with a pair of two neighboring kernels, depending on its position. If a type (4) job $j$ is scheduled between two adjacent partial schedules $S^*[K']$ and $S^*[K'']$ (before the first or after the last such partial schedule), then it cannot be an emerging job for none of the corresponding kernels; hence, it must be scheduled in between these two partial schedules (before the first and after the last such partial schedule, respectively). Note that any type (3) job associated with kernel $K$ is included before the type (2) jobs in schedule $S^*[K]$. The corresponding type (1.2) jobs (which do not belong to schedule $S^*[K]$) are to be included immediately before, within or immediately after the jobs of schedule $S^*[K]$ and before the next such partial schedule (see also Lemma 7). So the partial schedule $\sigma^e$, that we further denote by $\sigma(2, 3, 4)$, contains all the type (2)-(4) jobs and does not include the type (1.1) and type (1.2) jobs.

Example. Fig. 4 represents schedule $\sigma(2, 3, 4)$ delivered by the procedure of stage 0. The procedure outputs the set of kernels $\{K_1, K_2\}$. We easily observe that type (1.1) jobs are 1 and 7, the type (1.2) job associated with kernel $K_1$ is job 6. Note that there is no type (1.2) job associated with kernel $K_2$. Type (2) jobs associated with kernel $K_1$ are jobs 3, 4 and 5. Type (2) jobs associated with kernel $K_2$ are jobs 8, 9, 10, 11 and 12. There is a single type (3) job 2 associated with kernel $K_1$, and there is no type (3) job associated with kernel $K_2$. There is only one type (4) job 13.

Lemma 6 The sets of type (1)-(4) jobs form a partition of set $J$.

Proof. It is straightforward to see that any job is one of the types (1)-(4). At the same time, these sets may have no (non-empty) intersection. Now we show that the set of type (1.1) jobs may have no intersection with the sets of the other types of jobs. Suppose $e$ is a type (1.1) job for a kernel $K$ and it is also part of a succeeding kernel $K'$. So, job $e$ gets included behind the jobs of kernel $K$. Since job $e$ is released before any job of kernel $K$, it is no more urgent than any job included after kernel $K$ preceding job $e$. Since job $e$ belongs to kernel $K'$, there may exist no emerging job for kernel $K'$ scheduled after kernel $K$. But it implies that kernel $K$ cannot exist, i.e., the set of jobs in $K$ and $K'$ must form part of one (larger) kernel (to which job $e$ belongs). Thus either $e$ is not an emerging job or it does not belong to any kernel, i.e., there may exist no intersection of the set of type (1.1) jobs with the sets of type (1.2) and type (2)-(3) jobs. The remaining implications are straightforward to see. 

Lemma 7 There is an optimal schedule $S_{OPT}$ in which:

(a) Any type (4) job is included between the intervals of adjacent partial schedules $S^*[K']$ and $S^*[K'']$, before the interval of the first of these partial schedules or after that of the last of them. In particular, there is no intersection of the execution interval of a type (4) job in schedule $\sigma(2, 3, 4)$ with the time interval of any type (2) and type (3) job from that schedule.
(b) If a type (1.1) job is an emerging job for two or more kernels $K^i, \ldots, K^{i+l}$, then it is scheduled either before or after partial schedules $S^*[K^i], \ldots, S^*[K^{i+l}]$ in schedule $S_{OPT}$; a type (1.1) job which is not an emerging job for kernel $K^\lambda$ does not appear after the jobs of partial schedule $S^*[K^\lambda]$ in that schedule.

(c) A type (1.2) job associated with a kernel $K$ is scheduled within the interval of partial schedule $S^*[K]$ or after that interval before the interval of the following partial schedule, whereas a job not from kernel $K$ is not to be included in between the jobs of that kernel.

(d) A type (2) or type (3) job associated with a kernel $K$ can be scheduled after the initial time interval of partial schedule $S^*[K]$ only if it gets pushed by a type (1.1) job or/and by a type (1.2) job associated with kernel $K$.

Proof. Part (a) holds as no type (4) job can be an emerging job for the corresponding kernel: a type (4) job scheduled between partial schedules $S^*[K']$ and $S^*[K'']$ is less urgent than any job from schedule $S^*[K']$ and it is more urgent than any job from schedule $S^*[K'']$ (otherwise it would be a type (1.1) job for kernel $K''$). Hence, it cannot be included after partial schedule $S^*[K'']$ in schedule $S_{OPT}$. Likewise, there can be no benefit in including such jobs in between the jobs of these two partial schedules. The type (4) jobs included after the last partial schedule $S^*[\hat{K}]$ in schedule $\sigma(2,3,4)$ can be included after all jobs of that sequence (since they are less urgent than all jobs from schedule $S^*[\hat{K}]$ and hence there will be no benefit in rescheduling them earlier). This proves part (a). Part (b), stating that the type (1.1) jobs can be dispelled in between the corresponding kernel sequences, easily follows. As to part (c), note that no type (1.2) job associated with kernel $K^i$ is released before the interval of partial schedule $S^*[K']$ and it cannot be scheduled after the initial execution interval of that kernel without causing the increase in the makespan. At the same time, only a type (1.1) job may potentially be scheduled in between the jobs of kernel $K$. But this would yield a forced left-shift of a more urgent job from the same kernel and cannot yield a smaller makespan. Part (d) follows since only a type (1.1) job can potentially be included before the initial time interval of partial schedule $S^*[K]$ and only a type (1.2) job can potentially be included within this time interval.

**Lemma 8** The makespan of partial schedule $\sigma(2,3,4)$ is a lower bound on the optimal schedule makespan.

Proof. By Lemma 5 and the fact that two kernel segments do not overlap in schedule $\sigma(2,3), \sigma(2,3)$ is a feasible partial schedule such that the maximum full job completion time in it is a lower bound on the optimal schedule makespan. We show that this magnitude cannot be surpassed by any other job from schedule $\sigma(2,3,4)$. Indeed, any job $j \in \sigma(2,3,4)$ is either from partial schedule $S^*[K]$ for some $K \in K$, or it is a type (4) job. In the latter case, our claim follows since job $j$ does not belong to any kernel. If now $j \in S^*[K]$, then our claim follows from the fact that no type (4) job may push job $j$ in schedule $\sigma(2,3,4)$. Indeed, job $j$ may potentially be pushed only by a type (4) job in schedule $\sigma(2,3,4)$. But any type (4) job originally scheduled before the delaying emerging job of kernel $K$, completes before the starting time of that job. But the latter job is omitted in schedule $\sigma(2,3,4)$ and hence no job can push job $j$. It follows that the full completion time of job $j$ is a lower bound on the optimum schedule makespan.
3.2 Stage 1: Generating complete schedules respecting permutations of the type (1) jobs

Recall that each partial schedule \(S^*[K] \ (K \in \mathcal{K})\) from schedule \(\sigma(2, 3, 4)\) consists of the jobs of kernel \(K\) except the type (1.2) jobs omitted during the decomposition of that kernel. At stage 1, the type (1.2) jobs associated with every kernel \(K \in \mathcal{K}\) are incorporated locally into partial schedule \(S^*[K]\), while type (1.1) jobs are incorporated globally into the partial schedule \(\sigma(2, 3, 4)\), as we describe below. For the sake of conciseness, we denote these sets by \(J_{1,2}(K)\) and \(J_{1,1}\), and we let \(\nu_{1,2}(K) = \lvert J_{1,2}(K) \rvert\) and \(\nu_{1,1} = \lvert J_{1,1} \rvert\).

**Steady permutations of type (1) jobs.** Stage 1 generates the first complete schedule by expanding partial schedule \(\sigma(2, 3, 4)\) with the type (1) jobs using a variation of LDT-heuristic. The scheduling time \(t\), at which the next type (1) job is included, is the earliest idle-time moment in the current partial augmented schedule such that there is yet unscheduled type (1) job released by that time. Among all such jobs, ties are broken by selecting one with the maximum delivery time (further ties can be broken by selecting any shortest job); if the so selected job overlaps with an earlier scheduled job from schedule \(\sigma(2, 3, 4)\), the latter and the following jobs are right-shifted accordingly. In this way, the jobs from set \(J_{1,2}(K)\) will be included within the interval of schedule \(S^*[K]\), for every \(K \in \mathcal{K}\), since they become released within that time interval and they are more urgent than any already released yet unscheduled type (1.1) job (see Lemma 6). Observe that the so formed partial schedules of the jobs of each kernel \(K\), are non-idle (i.e., they contain no gap) and hence they have the minimum length among all partial schedules consisting of the jobs of kernel \(K\). As a consequence, they impose smallest possible delay for the succeeding jobs (although the makespan of such partial schedule is not necessarily the minimum possible).

We denote the complete feasible schedule created by this procedure by \(\sigma(\pi^*)\). The order of the type (1.1) jobs (type (1.2) jobs from set \(J_{1,2}(K)\), respectively) in that schedule defines the steady permutation \(\pi^*\) (permutations \(\psi^*(K)\), for every \(K \in \mathcal{K}\), respectively).

**Generating alternative complete schedules.** We may generate additional complete schedule(s) respecting the steady permutation \(\pi^*\) and other permutations of the type (1) jobs. We deal with permutations of type (1.1) and type (1.2) jobs separately. Recall that by point (c) in Lemma 7, the type (1.2) jobs associated with kernel \(K\) cannot be scheduled before any type (1.2) job associated with a kernel preceding kernel \(K\) and after any type (1.2) job associated with a kernel succeeding kernel \(K\), and no other type (1) job is to be included in between these type (1.2) jobs. As a consequence, jobs of each group \(J_{1,2}(K), \ K \in \mathcal{K}\), can be scheduled “locally” within the schedule segment associated with kernel \(K\). This already reduces a worst-case term \((\nu_{1,1} + \sum_{K \in \mathcal{K}} \nu_{1,2}(K))!\) to \(\nu_{1,1}! \prod_{K \in \mathcal{K}} \nu_{1,2}(K)\)!. We will be dealing with subsets of jobs from each group \(J_{1,2}(K)\) instead of permutations of these jobs replacing the latter bound by

\[
\nu_{1,1}! \ n \log n \ (n - \nu)p_{1,1}^\nu \max (n - \nu)2^{\psi^*(K)}.
\]

We will use letters \(\pi\) (\(\psi(K)\), respectively) for a permutation of type (1.1) (type (1.2) jobs from set \(J_{1,2}(K)\), respectively). We expand the partial schedule \(\sigma(2, 3, 4)\) with type (1) jobs according to the order of the jobs in these permutations while keeping the processing order of the jobs from schedule \(\sigma(2, 3, 4)\) unchanged. Let \(\pi = (i_1, \ldots, i_{\nu_{1,1}})\). Initially at iteration 0, \(\sigma(\pi, 0) := \sigma(2, 3, 4)\); at iteration \(i \geq 1\), schedule \(\sigma(\pi, i)\) is obtained from schedule \(\sigma(\pi, i - 1)\) by including job \(i_\nu\) at the earliest idle-time interval at or after time \(r_{i_\nu}\) that occurs in schedule \(\sigma(\pi, i - 1)\) behind the jobs \(i_1, \ldots, i_{\nu_i - 1}\); if the overlapping with an earlier included job from schedule \(\sigma(\pi, i - 1)\) occurs, the latter job and following ones are right-shifted by the required amount of time units.
(the processing order of these jobs in schedule $\sigma(\pi, i - 1)$ being kept unchanged, similarly to the earlier described construction for the steady permutation). The jobs from a permutation $\psi(K)$ are similarly incorporated into the schedule $S^+[\psi(K)]$ resulting in the partial schedule $S^+ [\psi(K), \pi]$ of the jobs of kernel $K$ respecting permutation $\psi(K)$, for each $K \in \mathcal{K}$. We denote by $\sigma(\pi, \psi)$ the so constructed complete schedule respecting permutations $\pi$ and $\psi(K)$ (we may omit argument $\psi$ when this will case no ambiguity).

Note that in schedule $\sigma(\pi, \psi)$, partial schedule $S^+ [\psi(K), \pi]$ can start no earlier than at the completion time of the last type (1.1) job included immediately before the interval of the partial schedule $S^+[K]$, i.e., permutation $\pi$ imposes this earliest possible starting time for the first scheduled job of kernel $K$ in any complete schedule respecting permutation $\pi$. If this time is less than the starting time of the partial schedule $S^+[K]$, the first type (1.2) job from set $J_{1, 2}(K)$ can be included at its release time within the gap immediately before schedule $S^+[K]$; otherwise, the first type (1.2) job from schedule $S^+ [\psi(K), \pi]$ can only be started behind its release time.

For each kernel $K \in \mathcal{K}$, we will deal with two kinds of complete schedules respecting permutation $\pi$ at stage 1. In a tight (loose, respectively) complete schedule for kernel $K$, the first included job of that kernel starts after (at, respectively) time $r(K)$. Note that the above described procedure of stage 1 creates a tight (for all kernels, unless no type (1) job is available for the inclusion before some kernel) complete schedule respecting permutation $\pi$. We also may create a loose complete schedule for kernel $K$ respecting permutation $\pi$ rescheduling the type (1.1) delaying emerging job of kernel $K$ right after the last included job of kernel $K$ (leaving in this way a gap before the first scheduled job of kernel $K$).

The steady permutation $\psi^*(K, \pi)$ of the jobs from set $J_{1, 2}(K)$ respecting permutation $\pi$ is formed during the construction of a tight complete schedule for kernel $K$, similarly to the initial steady permutation $\psi^*(K)$. In the latter schedule, the first job of set $J_{1, 2}(K)$ starts right at the completion time of the corresponding type (1.1) delaying emerging job. Note that the so obtained partial schedule $S^+[\psi^*(K), \pi]$ of jobs from set $J_{1, 2}(K)$ remains non-idle and has the minimum possible length among all partial schedules for the jobs of kernel $K$ respecting permutation $\pi$.

**Halting condition for atomic kernels.** Next, we introduce the optimality condition and the dominance relation related to atomic kernels.

**Lemma 9** For any permutation $\pi$, schedule $\sigma(\pi)$ is optimal if it contains a kernel, which is an atomic kernel $K^*$ possessing no delaying emerging job, for some $K \in \mathcal{K}$.

Proof. Since $K^*$ is the kernel from the atomic component of a kernel from set $\mathcal{K}$ and it possesses no delaying emerging job, the full completion time of the overflow job of this kernel is a lower bound on the optimum schedule makespan by Lemma 8 and the lemma follows.

If atomic kernel $K^*$ possesses the delaying emerging job, then we provide a non-delay starting of the first job of that kernel creating a loose complete schedule for that kernel respecting permutation $\pi$ (see the formal description at the end of this section).

We can immediately observe that any tight for kernel $K^*$ complete schedule in which the first job of that kernel is right-shifted by a magnitude at least $\beta(K) = \delta(K^*, \sigma(\pi))$ will have a makespan no less than that of schedule $\sigma(\pi)$. We will further refer to a permutation of type (1.1) jobs that yields such forced delay for kernel $K^*$ as one dominated due to Observation 2 by permutation $\pi$:
Observation 2 Suppose schedule $\sigma(\pi)$ contains an atomic kernel $K^*$ ($K \in \mathcal{K}$) possessing delaying emerging job. Then no tight for kernel $K^*$ schedule respecting a permutation of type (1.1) jobs dominated due to Observation 2 by permutation $\pi$ can have the makespan, less than that of an already created complete schedule respecting permutation $\pi$ and hence it can be discarded.

We keep the magnitude $\delta(K^*, \sigma(\pi))$ in global variable $\beta(K^*)$, which is to be updated each time a tight complete schedule respecting the next dominant due to Observation 2 permutation (one, yielding less than the current $\beta(K)$ delay for the atomic kernel $K^*$) is generated.

New kernels in alternative complete schedules. If the condition in Lemma 9 for schedule $\sigma(\pi, \psi^*)$ is not satisfied, additional schedule(s) respecting permutation $\pi$ might be generated. At stage 1, a kernel in schedule $\sigma(\pi, \psi)$, which is not a kernel from schedule $\sigma(2, 3, 4)$, will be referred to as a new kernel. A new kernel may consist of only type (4) jobs; otherwise it contains jobs of some kernel(s) from set $\mathcal{K}$ (note that if it contains a single job of such kernel, it will contain all the remaining jobs from that kernel). Let $K \in \mathcal{K}$ be the last kernel contained in a new kernel. Then the latter new kernel will be referred to as a secondary kernel of kernel $K$ and will commonly be denoted by $\bar{K}$. A new kernel consisting of only type (4) jobs will be referred to as a non-secondary kernel.

Observation 3 (i) A new kernel in schedule $\sigma(\pi, \psi)$ possesses the delaying emerging job.
(ii) The delaying emerging job of a secondary kernel is a type (1.1) job, whereas that of a non-secondary kernel is either a type (1.1) or a type (4) job.
(iii) The overflow job of a non-secondary kernel is either a type (1.1) or a type (4) job, whereas that of a secondary kernel $\bar{K}$ is either the overflow job of the atomic kernel $K^*$ or a type (1.1) or a type (1.2) job.

Proof. Let $K$ be a new kernel in schedule $\sigma(\pi, \psi)$ and $o$ be the overflow job from that kernel. Without loss of generality, assume $o \in \sigma(2, 3, 4)$. The completion time of job $o$ in schedule $\sigma(\pi, \psi)$ is less than that of any overflow job from this schedule. Obviously, this time could not have been increased in schedule $\sigma(\pi, \psi)$ without a forced right-shift, which could have been caused only by the corresponding (type (1.1) delaying emerging job, and part (i) follows. To see the first claim of part (ii), note that any type (4) job is no-less urgent than a job of any kernel following this job and hence it cannot be an emerging job for that kernel, whereas neither a type (2) nor a type (3) job can be an emerging job for any new kernel. The second claim also follows since obviously, the delaying emerging job of a non-secondary kernel cannot be any type (1.2) job. The first claim of part (iii) follows since no job of a kernel from set $\mathcal{K}$ can belong to a non-secondary kernel. To see the second claim, note that no type (4) job scheduled after the secondary kernel $\bar{K}$ may be right-shifted by more time units than the last scheduled job of kernel $K$ and hence it cannot become the overflow job of the former kernel.

Lemma 10 For any permutation $\pi$, if schedule $\sigma(\pi, \psi)$ contains a (secondary) kernel containing a type (1.1) job $e$, then a schedule with the minimum makespan respecting permutation $\pi$ is among the already generated complete schedules.

Proof. Let $K'$, $K' = \bar{K}$ ($K \in \mathcal{K}$) be a secondary kernel in schedule $\sigma(\pi, \psi)$ with a type (1.1) job $e$. By definition, kernel $K'$ contains all jobs of the atomic kernel $K^*$. Either (1) job $e$ was
activated in one of the (earlier) generated LDT-schedules respecting permutation $\pi$ or (2) not. In case (2), for any job $j$ preceding job $e$ in kernel $K'$ (including any type (1.2) job from set $J_{1,2}(K)$), $q_j \geq q_e$ and the lemma follows if $e$ is the first job in permutation $\pi$ since kernel $K'$ may possess no delaying emerging job. If $e$ is not the first job in permutation $\pi$, then it should have been included in the first available gap after the type (1.1) job immediately preceding it in permutation $\pi$. Again, for any job $j$ preceding job $e$ in kernel $K'$, $q_j \leq q_e$ and the lemma similarly follows if kernel $K'$ possesses no delaying emerging job. Otherwise, the delaying emerging job $d$ of kernel $K'$ must be a type (1.1) job preceding job $j$ in permutation $\pi$. Hence the full completion time of the overflow job of kernel $K'$ may only potentially be reduced by rescheduling job $d$ behind job $j$. But such a schedule does not respect permutation $\pi$.

For case (1), let $\sigma'(\pi)$ be the first generated LDT-schedule respecting permutation $\pi$ in which job $e$ was activated, say, for kernel $K''$. We show that there can exist no delaying emerging job for kernel $K'$. Indeed, suppose $\epsilon$ is such a job. Since there is a gap immediately before kernel $K''$ in schedule $\sigma'(\pi)$ and in any further generated LDT-schedule respecting permutation $\pi$, the delaying emerging job for kernel $K'$ must be included between kernel $K''$ and job $e$ in schedule $\sigma'(\pi)$. But no job included between kernel $K''$ and job $e$ can be a type (1.1) job from $\pi$ since otherwise schedule $\sigma'(\pi)$ would not respect permutation $\pi$ (since job $e$ was included before such job in an earlier created schedule respecting permutation $\pi$). Hence, $\epsilon$ does not exist. It follows that there exists no LDT-schedule respecting permutation $\pi$ with a makespan, less than the makespan of an already generated schedule respecting permutation $\pi$, and the lemma follows.

Suppose that stage 1 halts by Lemma 10 and let $\rho$ be a permutation of type (1.1) jobs. Consider a sub-schedule of a partial schedule respecting a permutation $\rho$, that consists of the jobs of kernel $K$ (one, that contains a type (1.1) overflow job). Suppose now the full completion time of that type (1.1) job is no-less than that of the overflow job of the secondary kernel $K'$ from schedule $\sigma(\pi, \psi)$. Then we will say that permutation $\rho$ is dominated by Lemma 10 by permutation $\pi$. By Lemma 10, the makespan of any complete schedule respecting permutation $\rho$ will be at least as large as that of the schedule $\sigma(\pi, \psi)$. The next result follows:

**Observation 4** Suppose stage 1 invoked for permutation $\pi$ halts by Lemma 10 for that permutation. Then any permutation dominated by Lemma 10 by permutation $\pi$ can be discarded.

We examine the earliest occurred new kernel in schedule $\sigma(\pi, \psi)$ if Lemmas 9 and 10 do not apply. If that kernel is non-secondary, the current configuration is updated respectively and stage 1 is repeatedly invoked for this new configuration (with this new kernel). Otherwise, the earliest kernel in schedule $\sigma(\pi, \psi)$ is secondary This case is dealt with in what follows.

**Instances of alternatives (a) and (b) for secondary kernels.** Suppose the earliest occurred kernel in schedule $\sigma(\pi, \psi)$ is a secondary kernel $\bar{K}, \bar{K} \in \mathcal{K}$. We will say that an instance of alternative (a) (IA(a), for short) in that schedule with kernel $\bar{K}$ occurs if the overflow job in that kernel is the same as that of schedule $S^*[K]$, i.e., it is the last scheduled job of the atomic kernel $K^*$.

If now no IA(a) in schedule $\sigma(\pi, \psi)$ occurs and the overflow job of kernel $\bar{K}$ is not a type (1.1) job (see Lemma 10), then, by Observation 3, the overflow job must be from set $J_{1,2}(K)$. In the letter case, we will say that an instance of alternative (b) (IA(b), for short) with that job (or kernel $\bar{K}$) occurs.
In what follows, we complete the description of our framework by specifying how IA(a) and IA(b) are dealt with in our algorithm. Part (i) of the next lemma states the optimality condition for an IA(a). We will define dominated permutations of type (1.1) and type (1.2) jobs for instances of alternative (a) using part (ii).

**Lemma 11** Let \( \overline{K} \) be a secondary kernel of kernel \( K \in \mathbb{K} \) in schedule \( \sigma(\pi, \psi) \) in which an IA(a) with that kernel occurs, and let \( j \) be the first scheduled job from the atomic kernel \( K^* \).

(i) If schedule \( \sigma(\pi, \psi) \) is loose for kernel \( K^* \), i.e., job \( j \) starts at its release time in that schedule, then it is optimal.

(ii) If job \( j \) starts after its release time in schedule \( \sigma(\pi, \psi) \), then the makespan of a complete schedule respecting a permutation of type (1.1) jobs (that of the jobs in set \( J_{1,2}(K) \)), in which the starting time of job \( j \) is no smaller than that in schedule \( \sigma(\pi, \psi) \) cannot be smaller than that of schedule \( \sigma(\pi, \psi) \) and hence it can be discarded.

**Proof.** Recall that since \( K^* \) is an atomic kernel, the sequence of jobs of that kernel in any complete LDT-schedule is the same and is optimal. Part (i) follows from the condition, as the last scheduled job of kernel \( K^* \) is an overflow job in schedule \( \sigma(\pi, \psi) \). Part (ii) similarly follows, as in the claimed schedules (in which job \( j \) is right-shifted) the former overflow job will correspondingly be right-shifted (by the same amount of time units as job \( j \)) and hence the resultant makespan will correspondingly be increased.

VP-algorithm halts if condition (i) in Lemma 11 is satisfied. If condition (ii) holds, in any further generated complete schedule, job \( j \) is to be delayed by the magnitude less than \( \beta(K^*) = \delta(K^*, \sigma(\pi, \psi)) \). We will refer to a permutation \( \pi \) of type (1.1) jobs as *dominated due to an IA(a)* for kernel \( K \) if it imposes the delay at least \( \beta(K^*) \) for the first scheduled job of the atomic kernel \( K^* \) in a tight for kernel \( K \) schedule \( \sigma(\pi, \psi) \) (note that this delay also depends on permutations \( \psi(K') \), \( K' \in \mathbb{K} \)).

Suppose now an IA(b) in schedule \( \sigma(\pi, \psi) \) occurs. Let \( \alpha(K) \) be the the starting time of the earliest included job from set \( J_{1,2}(K) \) in schedule \( \sigma(\pi, \psi) \). Note that any (tight complete) schedule in which the first job from permutation \( \psi(K) \) starts no earlier than in schedule \( \sigma(\pi, \psi) \) will have a makespan, no smaller than that of schedule \( \sigma(\pi, \psi) \) (and analogously, such a schedule can be discarded). The corresponding dominance relation is as follows. We will refer to a permutation \( \pi \) of type (1.1) jobs as *dominated due to an IA(b)* for kernel \( K \) and permutation \( \psi(K) \) if it imposes the starting time at least \( \alpha(K) \) for the first scheduled job of the permutation \( \psi(K') \) in (a tight for kernel \( K \)) schedule \( \sigma(\pi, \psi) \) (note that \( \alpha(K) \) also depends on permutations \( \psi(K') \), \( K' \in \mathbb{K} \)). The next observation follows.

**Observation 5** Suppose at stage 1 invoked for permutation \( \pi \) an IA(a) (an IA(b), respectively) with a secondary kernel \( \overline{K} \) occurs in schedule \( \sigma(\pi, \psi) \). Then the makespan of a complete (tight for kernel \( K \)) schedule respecting a permutation of type (1.1) jobs dominated due to an IA(a) (IA(b), respectively) by permutation \( \pi \) for kernel \( K \) and permutation \( \psi(K) \) cannot be smaller than that of schedule \( \sigma(\pi, \psi) \) and hence it can be discarded.
3.2.1 Enumeration of subsets of type (1.2) jobs for repeated IA(a)/IA(b)

We will treat instances of alternatives (a) and (b), roughly, by trying possible permutations/subsets of jobs from $J_{1,2}(K)$. Suppose we wish to decrease the starting time of job $j$ (as defined in Lemma 11) and the full completion time of the overflow job from kernel $K^*$ for an IA(a). This can be done only by either (a) reducing the total processing time of the jobs from set $J_{1,2}(K)$ scheduled before the atomic kernel $K^*$ or (b) creating a loose complete schedule for kernel $K$ (with a gap before the first scheduled job of that kernel) or (c) replacing the current permutation $\pi$ with another permutation of type (1.1) jobs, that yields a right-shift, smaller than $\delta(K, \sigma(\pi, \psi))$ for the first scheduled job of kernel $K$. Options (a) and (b) are dealt with in the current call of stage 1 for permutation $\pi$. Option (a) can be realized either by (a1) altering the set of jobs from $J_{1,2}(K)$ scheduled before the atomic kernel $K^*$ or by (a2) rescheduling the job $d_{1,2} \in J_{1,2}(K)$ included in schedule $\sigma(\pi, \psi)$ immediately before job $j$, immediately after the last job of the atomic kernel $K^*$, creating a loose complete schedule $\sigma_{d_{1,2}}(\pi, \psi(K))$ for the atomic kernel $K^*$ respecting permutation $\pi$ and the current permutation $\psi(K)$ of jobs in set $J_{1,2}(K)$. With option (b), we create a loose complete schedule $\sigma_{d_{1,1}}(\pi, \psi(K))$ for kernel $K$ respecting permutations $\pi$ and $\psi(K)$ (by rescheduling the type (1.1) delaying emerging job $d_{1,1}$ of kernel $K$ after that kernel). Likewise, we may wish to decrease the full completion time of the last scheduled job of set $J_{1,2}(K)$ for an IA(b) (note that, while option (a2) provides a non-delay starting of job $j$, the forced right-shift for the type (1.2) jobs following the atomic kernel $K^*$ that it would yield, can be larger than that with option (a1)).

We distinguish two subsets of set $J_{1,2}(K)$ to treat repeated occurrences of IA(a) and IA(b). The first one contains the jobs that will be scheduled before, and the second one contains the jobs that will be scheduled after the atomic kernel $K^*$. Observe that no type (1.2) job scheduled before the atomic kernel $K^*$ may have the full completion time, larger than that of a job of that kernel. Hence, the jobs from the first above subset, say $J'$, can be scheduled in any order that leaves no avoidable gap (see the details below). This will not affect a current makespan. At the same time, since all jobs from the second above subset are released by the completion time of the last job of the atomic kernel $K^*$, they will be sequenced optimally by LDT-heuristic. We arrived at the following observation.

**Observation 6** For repeated instances of alternatives (a) and (b), the search for a complete schedule with the minimum makespan respecting a permutation of type (1.1) jobs can be restricted to subsets of type (1.2) jobs, the ones to be included before the jobs of the atomic kernel $K^*$.

Scheduling jobs from set $J' \subset J_{1,2}(K)$. Recall that initially at stage 1, the steady permutations $\psi^*(K)$ for each kernel $K \in \mathcal{K}$ (together with the steady permutation $\pi^*$) are formed by scheduling type (1) jobs within schedule $\sigma(2, 3, 4)$ using a variation of LDT-heuristic. We employ another variation of LDT-heuristic, procedure $\text{LDT}_{1,2}(K)$, that schedules type (1.2) jobs from set $J_{1,2}(K)$ enumerating the subsets of jobs from set $J_{1,2}(K)$. Every call of procedure $\text{LDT}_{1,2}(K)$ creates the next (yet not enumerated) subset $J' \subset J_{1,2}(K)$ and schedules the jobs from that subset, quite similarly as the earlier variation of LDT-heuristic. Instead of schedule $\sigma(2, 3, 4)$, $\text{LDT}_{1,2}(K)$ includes the jobs of a formed subset within another partial schedule that contains all jobs as in schedule $\sigma(\pi, \psi)$ except the jobs from the set $J_{1,2}(K)$.

While scheduling the jobs from set $J'$, procedure $\text{LDT}_{1,2}(K)$ respects the current order of the type (3) and type (2) jobs in kernel $K$. I.e., all type (3) jobs remain to be scheduled before all type (2) jobs (recall that by definition, any such type (3) job is more urgent than any type (2)
job from kernel $K$). The first job from set $J'$ is included at the earliest possible starting time, the largest magnitude between the completion time of the last type (1.1) job included before the atomic kernel $K^*$ in schedule $\sigma(\pi, \psi)$ and the earliest release time of a job from set $J'$. If a type (1.2) job, included before a type (3) job $i$, overlaps with the current execution interval of job $i$, this latter job and the following jobs, if any, are right-shifted correspondingly.

Let $k$ be the first type (1.2) job from subset $J'$ pushing a type (2) job from the atomic kernel $K^*$. We consider two subsets derived from set $J'$, the first one yielding a loose and the second one a tight for kernel $K^*$ schedule (both respecting the same permutation of jobs from set $J_{1.2}(K)$). The loose subset does not contain job $k$, and the tight one contains job $k$. Thus in the loose schedule, job $k$ is included immediately after kernel $K^*$, where in the tight one this job is scheduled immediately before that atomic kernel and it is pushing the first job of that kernel (again, the type (2) jobs from the atomic kernel $K^*$ are right-shifted correspondingly).

The above two alternative subsets are not considered only if the type (1.2) job scheduled immediately before job $k$ completes exactly at time $r(K^*)$ (the minimum release time of a job in kernel $K^*$). Then only one loose subset (the one not containing job $k$) is enumerated (the corresponding complete schedule being similarly generated).

Suppose, once job $k$ gets included (before or after the atomic kernel $K^*$), there remains a yet unscheduled job from subset $J'$. Since any such job is less urgent than any type (2) job of the atomic kernel $K^*$, it can be included after the jobs of that kernel. In this way, subset $J'$ gets discarded being dominated by $J''$, the subset of set $J'$ not containing the latter (non-fitted) jobs from set $J'$. Note that any other subset containing the jobs from subset $J''$ and any other job from set $J_{1.2}(K)$ is also dominated by subset $J''$, and is also to be discarded. Note also that the jobs from set $J' \setminus J''$ and the remaining jobs from set $J_{1.2}(K)$ are released by the completion time of the last job of the atomic kernel $K^*$ and hence they are sequenced optimally by LDT-heuristic.

We will complete the description of the procedure once we prove the following lemma.

**Lemma 12** At each call of stage 1 for given permutations $\pi$ and $\psi(K)$, $K \in \mathcal{K}$, and for any kernel $K' \in \mathcal{K}$, the total number of dominant subsets of set $J_{1.2}(K')$ (for repeated instances of alternatives (a)/(b)) is upper bounded by $2^{p_{\max}^{1.1} + P_{1.2}(K')}/p_{\min}^{1.1}(K')$. Furthermore, the total number of complete schedules generated for different subsets of type (1.2) jobs (for repeated instances of alternatives (a)/(b)) in VP algorithm is upper bounded by $p_{\max}^{1.1}(n - \nu)2^{p_{\max}^{1.1}(K')}$ or alternatively by $p_{\max}^{1.1}(n - \nu)2^{(p_{\max} + P_{1.2})/p_{\min}^{1.1}}$.

Proof. Consider the interval where dominant subsets of set $J_{1.2}(K)$, to be included before the atomic kernel $K^*$, are scheduled. There are $p_{\max}^{1.1}$ possible starting times for this interval. Indeed, either the first scheduled job from this interval starts at its release time or otherwise it is pushed by the corresponding delaying emerging job. A forced delay for the first scheduled job of set $J_{1.2}(K)$ in this case will vary from 1 to $p_{\max}^{1.1} - 1$, see Property 1. Then the length of the above interval is clearly less than $p_{\max} + P_{1.2}(K)$. We obtain an upper bound $(p_{\max} + P_{1.2}(K))/p_{\min}^{1.1}(K)$ on the total number of jobs in a subset of jobs from set $J_{1.2}(K)$ which are to be included before the atomic kernel $K^*$, and the first claimed bound follows. The second bound follows since the total number of kernels in set $\mathcal{K}$ is upper bounded by $n - \nu$ and there are $p_{\max}^{1.1}$ possible starting times (configurations) for the first scheduled job of each set $J_{1.2}(K)$, and the third bound similarly follows.


As we just saw, there are $p_{\text{max}}$ possible starting times for the first scheduled job of set $J_{1.2}(K)$, each of them defining a particular configuration for Procedure $\text{LDT}_{1.2}(K)$. Each time at stage 1 invoked for permutation $\pi$, the next $\text{IA}(a)/\text{IA}(b)$ with kernel $K$ occurs, the next dominant subset of set $J_{1.2}(K)$ is created for a particular configuration, determined by the early starting time of the first scheduled job of set $J_{1.2}(K)$ in the current schedule $\sigma(\pi, \psi)$. If for another call of stage 1 (for a different permutation) a repeated $\text{IA}(a)/\text{IA}(b)$ with kernel $K$ occurs (here the corresponding kernel can be a secondary kernel of kernel $K$ or an atomic kernel $K^*$), the procedure first determines the current configuration and resumes with the next yet unconsidered dominant subset of set $J_{1.2}(K)$ with that configuration. In this way, it is invoked for kernel $K$ once or more times from the same or different calls of stage 1 for the same or different configurations.

### 3.2.2 Bounds on the number of dominant permutations of type (1.1) jobs

At stage 1, we restricted the search to dominant subsets of type (1.2) jobs, and defined dominance relations on the set of permutations of type (1.1) jobs. In this subsection we introduce our last dominant relation on the permutations of type (1.1) jobs and we give bounds on the total number of dominated and dominant permutations that we consider. First, we give a bound on the total number of permutations dominated due to $\text{IA}(a)$ or $\text{IA}(b)$ that we may need to consider.

**Observation 7** It suffices to consider $n - \nu$ permutations of type (1.1) jobs dominated due to $\text{IA}(a)$ or $\text{IA}(b)$.

Proof. Let $\rho$ be a permutation dominated due to $\text{IA}(a)$ and $\text{IA}(b)$, and let it be the $l$th such permutation considered by the algorithm. There are $l$ different kernels in set $\mathcal{K}$ for which an $\text{IA}(a)$ or $\text{IA}(b)$ during earlier calls of stage 1 has occurred. Let $\mathcal{K}'$ ($\mathcal{K}' \subset \mathcal{K}$) be the set of all these kernels. By Observation 3 for any kernel from set $\mathcal{K}'$, only a loose complete schedule respecting a permutation dominated due to $\text{IA}(a)$ or $\text{IA}(b)$ is created. In particular, permutation $\rho$ cannot be dominated for any of the kernels from set $\mathcal{K}'$ and a complete schedule $\sigma(\rho)$ respecting this permutation will be loose for all kernels from set $\mathcal{K}'$. Suppose $K'' \in \mathcal{K}'$ is a kernel in schedule $\sigma(\rho)$ and it contains no type (1.1) job. If $\text{LDT}_{1.1}(K'') = \text{NIL}$, then a schedule with the current makespan cannot be improved by any further rearrangement of the jobs of kernel $K''$ and stage 1 can halt for permutation $\rho$; otherwise, procedure $\text{LDT}_{1.1}(K'')$ resumes. If now kernel $K''$ contains a type (1.1) job, then only dominant by Lemma 10 permutations need to be considered Observation 4 (which number is bounded due to Corollary 1, see a bit later).

It follows that permutation $\rho$ can be dominated only for a kernel from set $\mathcal{K} \setminus \mathcal{K}'$ and claimed bound follows since set $\mathcal{K}$ may contain no more than $n - \nu$ kernels (i.e., $l \leq n - \nu$). 

Our last dominance relation is urged by implicit restrictions imposed by job release times. Given a permutation $\pi = \{i_1, \ldots, i_{\nu_1}\}$, while constructing a complete schedule respecting that permutation, a potentially avoidable gap may occur at iteration $\iota$ if job $i_\iota$ is released earlier than its predecessor job $i_{\iota-1}$. Job $i_\iota$ may potentially be included before job $i_{\iota-1}$ without causing any non-permissible delay. So a permutation in which job $i_\iota$ comes after job $i_{\iota-1}$ may yield a non-active schedule and it might be neglected.

The just described scenario is dealt with by our last dominance relation. Suppose at iteration $\iota$ (i) there is a gap $g$ in schedule $\sigma(\pi, \iota - 1)$ such that $r_{i_\iota} \in g$; if there is no such gap, then (ii) let $g$ be the earliest gap preceding job $i_{\iota-1}$. Let, further $\sigma(\pi, \iota, \iota - 1)$ be an extension of
schedule \(\sigma(\pi, \iota - 1)\) in which job \(i_e\) is included within the gap \(g\) at time \(r_{i_e}\) (in case (i)) or at the beginning of the gap \(g\) (in case (ii)), and the following jobs from schedule \(\sigma(\pi, \iota - 1)\) are right-shifted correspondingly. If the makespan of schedule \(\sigma(\pi, \iota - 1)\) is no larger than that of schedule \(\sigma(\pi, \iota)\), then permutation \(\pi\) is said to be dominated by release time by permutation \(\{i_1, \ldots, i_\iota, i_{\iota - 1}, \ldots, i_\nu\}\). The next observation easily follows.

**Observation 8** If permutation \(\{i_1, \ldots, i_{\iota - 1}, i_\iota, \ldots, i_\nu\}\) is dominated by by release time by an enumerated permutation \(\{i_1, \ldots, i_\iota, i_{\iota - 1}, \ldots, i_\nu\}\), then the former permutation can be discarded.

Now we switch to the dominance relations related to the halting conditions of stage 1. Recall that the first complete schedule that we generate defines the steady permutation \(\pi^*\) of type (1.1) jobs. Another permutation of type (1.1) jobs will be considered only if either (for repeated instances of alternative (a)/(b)) stage 1 halts for a given permutation of type (1.1) jobs (after verifying a required amount of dominant subsets of type (1.2) jobs) or it halts due to the condition in Lemma 10 Consider first the latter case (see step (3.1) in the formal description of stage 1), where \(e\) is a type (1.1) job from kernel \(\bar{K}\). Note that job \(e\) is less urgent than any job from kernel \(K\), it is scheduled after all these jobs and it is the overflow job of kernel \(K'\) (since it belongs to kernel \(\bar{K}\)). Let \(\pi(i, e)\) be the permutation, obtained from \(\pi\) by interchanging the positions of jobs \(i\) and \(e\), where job \(i\) with \(q_i < q_e\) precedes job \(e\) in permutation \(\pi\). Consider now a chain of permutations, each of them obtained from permutation \(\pi\) by a series of such interchanges, and let \(\pi((i_1, e_1)(i_2, e_2)\ldots(i_k, e_k))\) be the permutation obtained from permutation \(\pi\) by the \(k\) interchanges, where \(e_x\) is a type (1.1) job from the kernel of schedule \(\sigma(\pi((i_1, e_1)(i_2, e_2)\ldots(i_{k - 1}, e_{x - 1})))\), for \(x = 1, \ldots, k\), where \(e_1 = e\).

**Lemma 13** Suppose the condition in Lemma 10 is met. Then permutation \(\pi\) dominates by Lemma 10 all yet unconsidered permutations of type (1.1) jobs if among the jobs preceding job \(e\) in that permutation, there is no job \(i\) with \(q_i < q_e\). Otherwise, any permutation, not dominated by Lemma 10 by permutation \(\pi\) has the form \(\pi((i_1, e_1)(i_2, e_2)\ldots(i_x, e_x))\), \(x \in \{1, \ldots, k\}\).

Proof. Note that any permutation of type (1.1) jobs that imposes a type (1.1) job \(i\) with \(q_i \geq q_e\) to be included after jobs of kernel \(K\) is dominated by Lemma 10 by permutation \(\pi\). At the same time, if job \(e\), together with all type (1.1) jobs preceding it in permutation \(\pi\), are included before the jobs of kernel \(K\), then the completion time of the last scheduled job of kernel \(K\) would become more than the makespan of schedule \(\sigma(\pi)\). By the condition in the first claim, for any type (1.1) job \(i\) included before kernel \(K\) in schedule \(\sigma(\pi)\), \(q_i \geq q_e\). Hence, if alternatively, job \(e\) (job \(i\), respectively) gets rescheduled before (after, respectively) the jobs of kernel \(K\), the resultant makespan cannot be decreased. It follows that in a complete schedule respecting a permutation \(\rho\) not dominated by Lemma 10 by permutation \(\pi\), a type (1.1) job \(i\) with \(q_i < q_e\) is to be included after jobs of kernel \(K\). Then job \(i\) with \(q_i < q_e\) must appear before job \(e\) in permutation \(\pi\) (and \(\{i_i, \ldots, i_k\} \neq \emptyset\)), and in a complete schedule respecting permutation \(\rho\), at least one of the jobs \(i_1, \ldots, i_k\) is to appear after the jobs of kernel \(K\). It is now easy to see that both claims in the lemma follow.

We let \(\gamma(K) := q_e\) each time the condition in Lemma 10 is fulfilled. A dominant permutation for kernel \(K\) will be formed by accomplishing the corresponding interchange of a (yet untried) candidate job \(e'\) with \(q_{e'} < q_e\) and job \(e\) if the condition in Lemma 10 is met; among all candidate jobs, ties can be broken by selecting the latest scheduled one (that yields the smallest gap after the interchange).
Corollary 1 For each permutation $\pi$ of type (1.1) jobs and every kernel $K \in \mathcal{K}$, it suffices to consider no more than $\gamma(K)$ permutations dominant by Lemma 10.

Proof. By Lemma 13 the delivery time of the overflow job in the complete schedule respecting the next dominant by Lemma 10 permutation is at least one less than that of the overflow job from the complete schedule respecting the previous dominant permutation and the corollary follows.

Now we proceed with the other case when stage 1 also halts for permutation $\pi$; i.e., suppose that for repeated instances of alternatives (a) and (b), it enumerated dominant subsets of type (1.2) jobs. We derive an upper bound in terms of job parameters on the total number of jobs in a dominant permutation $\rho$ of type (1.1) jobs. Suppose, at stage 1 invoked for permutation $\pi$, after repeated occurrences of instance(s) of alternatives (a)/(b) with kernel $K$, schedule $\sigma(\pi, \psi)$ is a generated schedule the minimum makespan. Let $j$ be the earliest included job of the atomic kernel $K^*$ (if the last occurrence of IA(a)/IA(b) was an IA(a)) or that of the permutation $\psi(K)$ (if the last occurrence of IA(a)/IA(b) was an IA(b)), and let $o$ be the overflow job in schedule $\sigma(\pi, \psi)$ (which is the last scheduled job of the atomic kernel $K^*$ in the case of an IA(a) and a type (1.2) job in the case of an IA(b)).

Observation 9 Suppose stage 1 halts for permutation $\pi$ after considering dominant subsets of type (1.2) jobs of kernel $K$. Then for a dominant permutation $\rho$, the total number of the type (1.1) jobs from that permutation included in the segment of schedule $\sigma(\rho)$ before (after, respectively) kernel $K$ is no more than $(r_j + p_{1\max}^{11})/p_{1\min}^{11}$ $(q_o/p_{1\min}^{11}$, respectively). Furthermore, $q_{\rho_i} < q_o$, for any $i = l + 1, \ldots, \nu_{1,1}$.

Proof. We immediately observe that in a complete schedule $\sigma(\rho)$ with a makespan no larger than that of schedule $\sigma(\pi)$, job $j$ is to be started no later than it starts in schedule $\sigma(\pi, \psi)$. At the same time, recall that any type (1.1) job is included either before or after the jobs of kernel $K$, whereas the time interval before the jobs of kernel $K$ in schedule $\sigma(\pi, \psi)$ is no longer than $r_j + p_{1\max}^{11}$. Hence, the time interval in schedule $\sigma(\rho)$ before jobs of kernel $K$ is no longer than $r_j + p_{1\max}^{11}$ and the total number of the type (1.1) jobs included in that segment is no more than $(r_j + p_{1\max}^{11})/p_{1\min}^{11}$.

Likewise, the total number of the type (1.1) jobs included in schedule $\sigma(\pi, \psi)$ after the jobs of kernel $K$ is bounded by $q_o/p_{1\min}^{11}$. Indeed, the processing time of every type (1.1) job is at least $p_{1\min}^{11}$ and the makespan of any feasible schedule in which more than $q_o/p_{1\min}^{11}$ jobs are included after job $o$ will be more than $C_o(\sigma(\pi, \psi))$.

Now we show the last claim. Let $l$ be the maximum job index in permutation $\rho$ such that

$$\max\{r(K'), \sum_{i=1}^{l} p_{\rho_i} + P_{2,3,4}(K_-)\} \leq \alpha(K),$$

where $P_{2,3,4}(K_-)$ is the total processing time of the type (2)-(4) jobs included before the jobs of kernel $K$ in schedule $\sigma(\rho)$. Note that in the above inequality, the expression before the inequality sign defines the time moment when a job of kernel $K'$ will start in schedule $\sigma(\rho)$. Then

$$c_k(\sigma(\rho)) = \max\{r(K), \sum_{i=1}^{l} p_{\rho_i} + P_{2,3,4}(-K)\} + P(K),$$

and hence $C_k(\sigma(\rho)) = c_k(\sigma(\rho)) + q_k$, where $k$ is the last scheduled job of kernel $K$ in schedule $\sigma(\rho)$ and $P(K)$ is the total processing time of the jobs of kernel $K$. Then the makespan of any complete
3.3 Formal description of stage 1

In the formal description of stage 1, LDT_{1,2}(K) stands for the complete schedule generated by the procedure for the next enumerated subset in the current configuration (recall from Section 3.2.1 that this subset is of the type J'' \{k\} (J'', respectively) for the loose (tight, respectively) cases, see Section 3.2.1); we let LDT_{1,2}(K) := NIL if all dominating subsets for the corresponding configuration are already enumerated. It straightforwardly follows from this section’s earlier results and the construction of stage 1, that for every permutation of type (1.1) jobs, it creates a complete schedule with the minimum makespan respecting that permutation.

(0) Generate schedule \(\sigma(\pi, \psi^*)\) \{respecting the steady permutation \(\pi^*\) and steady permutations \(\psi^*(K)\), for each kernel \(K \in \mathcal{K}\}\}; \(\sigma(\pi) := \sigma(\pi, \psi^*)\)

(1) if there is an atomic kernel in schedule \(\sigma(\pi)\) with no delaying emerging job, output \(\sigma(\pi)\) and halt VP algorithm

(2) let \(K'\) be the first kernel in schedule \(\sigma(\pi)\) \{either \(K' = \bar{K}\), i.e., \(K'\) is a secondary kernel of kernel \(K \in \mathcal{K}\) or \(K' = K^*\), i.e., \(K'\) is the atomic kernel of kernel \(K \in \mathcal{K}\}\)

(2.1) if \(\sigma(\pi)\) is loose for kernel \(K\) schedule or such schedule was already generated (at this or a previous call of stage 1), proceed with step (3.1)

(2.2) if no loose for kernel \(K\) schedule was earlier generated, let \(e\) be the type (1.1) delaying emerging job for kernel \(K'\) in schedule \(\sigma(\pi)\); proceed with two alternative complete schedules: (i) the tight for kernel \(K\) schedule \(\sigma(\pi) := \sigma_e(\pi, \psi(K'))\) \{in which job \(e\) pushes the first job of kernel \(K'\}\}, and (ii) the loose for kernel \(K\) schedule \(\sigma(\pi) := \sigma_e(\pi, \psi(K'))\) \{in which job \(e\) is rescheduled immediately after kernel \(K'\)\}; proceed with step (3.1) for case (i) and step (1.1) for case (ii)

(3.1) if \(K'\) contains a type (1.1) job, output a created complete schedule with the minimum makespan and halt for permutation \(\pi\)

(3.2) if an IA(a) or an IA(b) with kernel \(K'\) occurs

(3.2.1) if LDT_{1,2}(K) = NIL, output a created complete schedule with the minimum makespan and halt for permutation \(\pi\)

(3.2.2) if LDT_{1,2}(K) \neq NIL, \(\sigma(\pi) := \text{LDT}_{1,2}(K)\) and proceed with step (1.1)

(4) if kernel \(K'\) is non-secondary: \(\mathcal{K} := \mathcal{K} \cup K'\); \{invoke the decomposition procedure for kernel \(K'\)\} update schedule \(\sigma(\pi)\) by replacing in it the segment containing kernel \(K'\) with partial schedule \(S^+[\psi^*(K'), \pi]\) and repeat from step (1.1) with the updated \(\sigma(\pi)\).

3.4 Overall VP-algorithm

We first summarize the iterative step of stage 0. Recall that, initially, stage 0 forms the steady permutation \(\pi^*\) and calls stage 1 for that permutation that finds a complete schedule with the minimum makespan respecting this permutation. Iteratively, each time stage 1 completes for a current permutation \(\pi\), stage 0 resumes if stage 1 did not halt the whole algorithm. This happens if either (1) step (3.1) (Lemma 10) or (2) step (3.2.1) is entered. Stage 0 then generates the
next dominant permutation of type (1.1) jobs, which is created at stage 0 depending on whether
the last call of stage 1 has halted at step (3.1) or at step (3.2.1) and calls stage 1 for that new
permutation, which is determined as follows:

1) Stage 1 halts for permutation \( \pi \) at step (3.1): if there exists a yet untried candidate job
e with \( q_\sigma < \gamma(K) \), call stage 1 for a dominant permutation \( \rho \) (obtained from permutation \( \pi \) by
interchanging jobs \( e \) and \( e' \), see Corollary 1).

2) Stage 1 halts for permutation \( \pi \) at step (3.2.1): create the next dominant (by IA(a)/IA(b))
permutation \( \rho = (\rho_1, \ldots, \rho_1, \ldots, \rho_{\nu_1,1}) \) (such that in schedule \( \sigma(\rho) \) no more than \( (r_j + p_{\max})/p_{\min} \)
jobs from permutation \( \pi \) are included before the jobs of kernel \( K \) and \( q_{\rho_i} < q_\sigma \), for all \( i =
(1 + \ldots, \nu_{1,1} \) see the proof of Observation 9).

**Theorem 1** VP-algorithm generates an optimal solution to problem 1 \( |r_j, q_j| C_{\max} \) in time
\[
O(n^2 \log n + n \log n \ \nu_{1,1}! \ p_{\max}^{1.1} \ (n - \nu) 2^{\nu_{1,1}^{\text{new}}(K)})
\]
or alternatively in time
\[
O((n^2 \log n + n \log n ((r_j + p_{\max}^{1.1})/p_{\min}^{1.1})!q_{\max}! + \ (n - \nu)p_{\max}^{1.1} \ 2^{\nu_{1,2}^{\text{new}}(}\max(r_j+p_{\max}^{1.1}+p_{1,2})/p_{\min}^{1.2})).
\]

Proof. The correctness straightforwardly follows from the construction and earlier established
results. As to the time complexity, at the initial call of stage 0, the initial set of kernels is formed
in at most \( n - \nu \) iterations, since each next iteration is invoked only if a new kernel in the LDT-
schedule of the previous iteration arises, whereas there may arise at most \( n - \nu \) different kernels.
Since at each iteration LDT-heuristic with cost \( O(n \log n) \) is applied, the total cost of this procedure is
\( O((n - \nu)n \log n) \). For a kernel with \( i \) jobs, the decomposition procedure with cost \( O(i^2 \log i) \)
is invoked to create schedule \( \sigma(2, 3, 4) \) (Lemma 5). For the purpose of this estimation, suppose
that each of the \( n - \nu \) kernels have the same number of jobs (which yields the maximum overall
cost). Then the total cost of all calls of the decomposition procedure is bounded by \( O(n^2 \log n) \)
(maintaining the jobs of each type in separate binary search tree, the total cost of all job partition
updates can be reduced to \( O(n \log n) \), but this will not change our overall estimation). Thus
the overall cost of stage 0 for creating the initial configuration and schedule \( \sigma(2, 3, 4) \) is \( O(n^2 \log n) \).

There are \( \nu_{1,1}! \) possible permutations of type (1.1) jobs, while the cost of the generation of
a complete schedule respecting each permutation at stage 1 is \( O(n \log n) \), that of LDT-heuristic
(note that the cost of declaring each new kernel and replacing the corresponding schedule part at
step (4) is absorbed in the cost of stage 0). There are \( O(p_{\max}(n - \nu)2^{\nu_{1,2}^{\text{new}}(K)}) \) possible complete
schedules generated for subsets of type (1.2) jobs and the first time complexity expression follows
(see Observation 6 and Lemma 12). In the second time complexity expression, we replace \( \nu_{1,1}! \)
and \( 2^{\nu_{1,2}^{\text{new}}(K)} \) from the first time complexity expression by alternative expressions, basically, in
terms of job parameters. Recall that each call of stage 1 is carried out for a unique permutation
of type (1.1) jobs. This will happen only if the previous call of that stage ended either at step
(3.1) or at step (3.2.1). We have a bound \( q_{\max} \) from Corollary 1 on the total number of dominant
permutations in the first case. For the second case, from Observation 9 we obtain a bound
\( (r_j + p_{\max}^{1.1})/p_{\min}^{1.1}!q_{\max}! \) on the total number of dominant permutations of type (1.1) jobs. The
total number of dominated permutations of type (1.1) jobs that the algorithm may generate is no
more than \( (n - \nu) \) (Observation 7); this term is absorbed by the following term in the expression.
The total number of complete schedules generated for different subsets of type (1.2) jobs is upper
bounded by $p_{\text{max}}^{11}(n - \nu)2(\nu_{\text{max}}^{11} + P_{1,2})/\nu_{\text{min}}^{12}$ due to Lemma 12, hence the corresponding cost is multiplied by that of LDT-heuristic, and the second time complexity expression follows.

A probabilistic estimation. Now we give an intuitive probabilistic estimation of the running time of our algorithm. Assume that job delivery times are uniformly distributed within the interval $[0, q_{\text{max}}]$. Suppose, once stage 1 completes with the steady permutation $\pi^*$, stage 0 creates another (dominant) permutation $\rho$ of type (1.1) jobs and calls stage 1 for that permutation. Such an event may occur when either step (3.1) or step (3.2.2) was entered during the previous call of stage 1 so that stage 0 generated a dominant permutation $\rho$. Suppose the previous call ended by the condition at step (3.2.2) with secondary kernel $K$ and the type (1.2) overflow job $o$. By Observation 9 for any type (1.1) job $e'$ scheduled after the jobs of kernel $K$ in schedule $\sigma(\rho)$, $q_{e'} < q_o$. Assuming that there are just 2 such type (1.1) jobs (in average, one may expect to have about $\nu_{1,1}/2$ of them), the probability that $\rho$ is a dominant permutation is

$$\frac{q_{\text{max}}}{2(q_{\text{max}} + 1)} < 1/4.$$  

So the probability that $\mu$ such dominant permutations will be considered is less than $1/4^\mu$ (which is close 0 for $\mu = \nu_{1,1}$). If the previous call of stage 1 ended by the condition at step (3.1) with secondary kernel $K$ and the type (1.1) overflow job $e$, for any type (1.1) job $e'$ included in schedule $\sigma(\rho)$ after the jobs of kernel $K$, $q_{e'} < q_e$ (see Lemma 13). Similarly to the above case, the probability that two consecutive dominant permutations will be generated is less than 1/4.

**Theorem 2** The probability that stage 0 will create $2\mu$ consecutive dominant permutations of type (1.1) jobs is less than $\frac{1}{4^\mu}$. 

Example. We illustrate the algorithm using our problem instance. There are six possible permutations of the 3 type (1.1) jobs 1, 6 and 7 from our example. Permutations (1, 7, 6), (7, 1, 6) and (7, 6, 1) are dominated by permutation (1, 6, 7), and permutation (6, 7, 1) is dominated by permutation (6, 1, 7). So there are left only two permutations (1, 6, 7) and (6, 1, 7) to consider. Permutation (1, 6, 7) yields a complete schedule $\sigma(\pi, \nu) = \sigma((1, 6, 7), 3)$, which coincides with the one of Fig. 1. The secondary kernel consisting of jobs of kernel $K^1$ possesses the delaying emerging job 1 and schedule $\sigma_1((1, 6, 7), 3)$ of Fig. 5 is generated. There arises a secondary kernel in schedule $\sigma_1((1, 6, 7), 3)$ with type (1.1) overflow job 6, and no further schedule respecting permutation (1, 6, 7) is created.

The other permutation (6, 1, 7) yields a complete schedule $\sigma(\pi, \nu) = \sigma((6, 1, 7), 3)$ that coincides with the schedule of Fig. 2. The secondary kernel consisting of jobs of kernel $K^2$ possesses the delaying emerging job 7, and schedule $\sigma_2((6, 1, 7), 3)$ of Fig. 6 is generated. The next secondary kernel consists of jobs of kernel $K^1$ and the overflow job 5 with $c_5(\sigma_7((6, 1, 7), 3)) = 22 + 43 = 65$. A type (1.2) job 6 is the delaying emerging job. Hence, the next LDT-schedule is $\sigma_7(6, 1, 7), 3)$, see Fig. 7. There arises a secondary kernel in that schedule with the overflow job job 13, consisting of all jobs except job 2. Stage 1 halts again by Lemma 10 and the VP algorithm also halts as all dominant permutations were considered.

Thus from $13! = 6227020800$ possible permutations of the 13 jobs, the algorithm verified only two permutations of the 3 type (1.1) jobs (from $3!$ possible permutations with $\nu_{1,1} = 3$). Among the five created complete solutions, the ones of Figures 6 and 7 turned out to be optimal.
4 Polynomial-time approximation scheme

In the PTAS, previous section’s algorithm is used for the construction of partial schedules with only long jobs. All the dealt from here on permutations/subsets are formed solely from type (1) long jobs. We basically keep the notation from the previous section, that is now applied to solely long jobs.

PTAS incorporates yet unscheduled short jobs into each enumerated partial schedule of long jobs, as we describe in the following subsection. Initial partial schedule of long type (2)-(4) jobs is constructed by applying the polynomial-time procedure of stage 0 from Section 3. Long type (1.1) and type (1.2) jobs are incorporating while forming initial steady permutations $\pi^*$ and $\psi^*(K)$, for $K \in \mathcal{K}$, similarly as in Section 3, resulting in the initial partial schedule $\sigma(\pi^*, \psi^*, \kappa)$ containing all $\kappa$ long jobs. In general, a partial schedule $\sigma(\pi, \psi, \kappa)$ of with all long jobs is formed similarly as before, by incorporating permutations $\pi$ and $\psi(K)$, $K \in \mathcal{K}$ into the above partial schedule of long type (2)-(4) jobs.

4.1 Incorporating short jobs

The main procedure incorporates unscheduled $n - \kappa$ short jobs into schedule $\sigma(\pi, \kappa) = \sigma(\pi, \psi, \kappa)$ to form a complete feasible schedule respecting permutation $\pi$. The flowchart of the procedure can be represented by a solution tree $T(\pi)$ in which breadth-first search is carried out. With every node (iteration) $h$, (partial) schedule $\pi_h$ (including all long and some shorts jobs) is associated. Initially, a single extension of schedule $\sigma(\pi, \kappa)$ with all short jobs, represented by the leftmost branch of tree $T(\pi)$, is created. Iteratively, backtracking to a node from the already created partial tree is accomplished and additional complete extensions of schedule $\sigma(\pi, \kappa)$ are generated. In each generated complete schedule, some optimality conditions are verified, depending on which, PTAS may halt or the procedure may halt for permutation $\pi$.

In the detailed description of the main procedure below, at iteration $h > 0$, $r_h$ stands for the release time of an earliest released currently unscheduled job (one not in $\pi^*_{h-1}$), and $r^*_h$ is the release time of an earliest released yet unscheduled short job (as we will see, $r_h$ can be smaller than $r^*_h$). Among all short jobs released at time $r^*_h$, the short incoming job $j_h$ is one with the largest delivery time (further ties are broken by selecting any longest job).

Suppose the incoming job $j_h$ is such that time moment $r^*_h = r_{j_h}$ falls within the execution...
interval of a long job $i \in \pi_{h-1}$ (i.e., the latter job starts before time $r^*_h$ in schedule $\pi^*_h$) and the portion of job $i$ from time $r^*_h$, $\hat{p}_i$ is long. Then job $i$ is said to cover job $j_h$ if $q_i < q_{j_h}$. Thus (a) if we leave the execution interval of long job $i$ unchanged, it will impose a “significant” delay $\hat{p}_i$ for a more urgent short job $j_h$. Alternatively, (b) if we (temporarily) remove long job $i$ from its current execution interval, short job $j_h$ can be started without any delay, but this would yield the rise of a new gap (these two possibilities are realized in the main and backtracking procedures, respectively, by creating two alternative branches from node $h-1$). In case (b), the omitted long job $i$ is declared to be a pending long job $p$ in that branch. $t_h$ is used for the time moment when the job at iteration $h$ gets scheduled. If that job overlaps with a job from schedule $\pi_{h-1}$, the latter job and the succeeding long jobs from that schedule are right-shifted correspondingly. In the formal description, we assume that such right-shifts are accomplished at every iteration when the current schedule is extended by a new job; $\pi_h := \pi_{h-1} \setminus \{i\} \cup j$ means that long job $i$ is omitted in schedule $\pi_h$ and it becomes a pending long job, whereas the short incoming job $j$ gets included in that schedule; $p = \text{nil}$ means that there is no pending job at the current iteration.

(0) $h := 0; \pi_0 := \sigma(\pi^*, \psi^*, \kappa); p := \text{nil};$ call Procedure MAIN($h$)

Procedure MAIN($h$):

If at iteration $h$ a complete schedule is generated

then call procedure BACKTRACK($h$) else $h := h + 1$ {and proceed as follows}

(1) If time moment $r_h$ falls in a gap in schedule $\pi_{h-1}$ then

(1.1) If $p = \text{nil}$ or $q_{j_h} > q_p$ then {schedule short job $j_h$ at time $t_h = r_h = r^*_h$}

$t_h := r^*_h; \pi_h := \pi_{h-1} \cup j_h$

(1.2) If $p \neq \text{nil}$ and $q_{j_h} \leq q_p$ then {schedule long job $p$ at time $t_h = r_h$}

$t_h := r_h; \pi_h := \pi_{h-1} \cup p.$

(2) Else {time moment $r_h$ falls within the execution interval of (2.1) a short job or (2.2) a long job from schedule $\pi_{h-1}$}

(2.1) {Time moment $r_h$ falls within the execution interval of a short job}

Let $t'_h$ be the completion time of the latest scheduled short job in schedule $\pi_{h-1}$ {among all the short jobs released by time $t'_h$; $j_h$ is one with the largest delivery time}

(2.1.1) If the machine is idle immediately after time $t'_h$ {there is no right-shifted (long) job scheduled after time $t'_h$ in $\pi_{h-1}$} then {schedule either job $j_h$ or job $p$ at time $t'_h$}

(2.1.1.1) If $p = \text{nil}$ or $q_{j_h} > q_p$ then {schedule short job $j_h$ at time $t_h = t'_h$}

$t_h := t'_h; \pi_h := \pi_{h-1} \cup j_h$

(2.1.1.2) If $p \neq \text{nil}$ and $q_{j_h} \leq q_p$ then {schedule long job $p$ at time $t_h = t'_h$}

$t_h := t'_h; \pi_h := \pi_{h-1} \cup p.$

(2.1.2) If the machine is not idle immediately after time $t'_h$, let $i$ be the long job scheduled at time $t'_h$ in schedule $\pi_{h-1}$

(2.1.2.1) If $p = \text{nil}$ and $q_{j_h} > q_i$ then {schedule short job $j_h$ at time $t_h = t'_h$}

$t_h := t'_h; \pi_h := \pi_{h-1} \cup j_h$

(2.1.2.2) If $p = \text{nil}$ and $q_{j_h} \leq q_i$ then {retain long job $i$ scheduled at time $t_h = t'_h$}

$t_h := t'_h; \pi_h := \pi_{h-1} \cup i.$

(2.1.2.3) If $p \neq \text{nil}$ and $q_{j_h} > q_p$ then {schedule short job $j_h$ at time $t_h = t'_h$}

$t_h := t'_h; \pi_h := \pi_{h-1} \cup j_h$

(2.1.2.4) If $p \neq \text{nil}$ and $q_{j_h} \leq q_p$ then {schedule long job $p$ at time $t_h = t'_h$}

$t_h := t'_h; \pi_h := \pi_{h-1} \cup p.$
(2.2) {Time moment $r_h$ falls within the execution interval of a long job $i \in \pi_{h-1}$}

(2.2.1) If $p \neq \text{nil}$ then \{schedule either job $j_h$ or job $p$ at time $r_h$\}

(2.2.1.1) If $q_{j_h} > q_p$ then \{schedule short job $j_h$ at time $t_h = r_h$\}

\[ t_h := r_h; \pi_h := \pi_{h-1} \cup j_h \]

(2.2.1.2) If $q_{j_h} \leq q_p$ then \{schedule long job $p$ at time $t_h = r_h$\}

\[ t_h := r_h; \pi_h := \pi_{h-1} \cup p \]

(2.2.2) If $p = \text{nil}$ then \{retain long job $i$ scheduled as in schedule $\pi_{h-1}$\}

\[ t_h := \text{the starting time of job $i$ in schedule $\pi_{h-1}$} \]

\[ \text{call Procedure MAIN} (h). \]

4.1.1 Procedure BACKTRACK

Procedure BACKTRACK creates alternative branches in tree $T(\pi)$ verifying the following conditions. If last created schedule $\pi_h$ contains a kernel with either a short delaying emerging job or with a long delaying emerging job $i$ such that $\hat{p}_i$ is short, it outputs that schedule and stops. Otherwise, either (i) every kernel in schedule $\pi_h$ possesses a long delaying emerging job $i$ such that $\hat{p}_i$ is long or (ii) there is a kernel $K \in \pi_h$ with no delaying emerging job. In case (i) the procedure backtracks to the predecessor $h - 1$ of node $h$ and considers the earliest kernel $K(\pi_h)$ (with a long delaying emerging job $i$) in schedule $\pi_{h-1}$. It creates the second son $g$ of node $h - 1$ and an alternative partial schedule $\pi_g$, in which job $i$ is activated for kernel $K(\pi_h)$ and short incoming job $j_h$ is included at its release time $r_h = r_{j_h}$; job $i$ becomes the pending long job $p$ in this branch and Procedure MAIN($g$) is invoked to continue the newly created branch.

In case (ii) above, there are two possibilities. Kernel $K$ is either (ii,1) the kernel of the atomic component from the partial schedule $\sigma(2,3,4)$ of type (2)-(4) long jobs, or otherwise (ii,2) it is a secondary kernel containing a type (1.1) job. In case (ii,1) PTAS halts with solution $\pi_h$ (Lemma 9), and in case (ii,2) the construction of tree $T(\pi)$ is complete (Lemma 10).

Procedure BACKTRACK($h'$):

(0) if the complete schedule $\pi_{h'}$ contains a kernel with either a short delaying emerging job or with a long delaying emerging job $i$ such that $\hat{p}_i$ is short, then finish the construction of tree $T(\pi)$ and stop \{$\pi_{h'}$ is an $(1 + 1/\kappa)$-approximation schedule\}

(i) if every kernel in schedule $\pi_{h'}$ possesses the long delaying emerging job $i$ such that $\hat{p}_i$ is long, then \{continue the construction of tree $T(\pi)$\} backtracks to the node $h - 1$ of the current branch such that $i$ is a long delaying emerging job for the kernel $K(\pi_{h'})$ and $\hat{p}_i$ is long \{create the second alternative son $g$ of node $h - 1$ in which short job $j_h$ is scheduled at time $t_h = r_h$ \[ t_g := r_h; \pi_g := \pi_{h-1} \setminus \{i\} \cup \{j_h\}; p := i; \text{call Procedure MAIN}(g) \]

(ii) else \{schedule $\pi_{h'}$ contains kernel $K$ with no delaying emerging job\}

(ii,1) if $K$ is an atomic kernel, return schedule $\pi_{h'}$ and halt

\{schedule $\pi_{h'}$ is optimal\}

(ii,2) else \{$K$ is a secondary kernel with a type (1.1) job\},

\{return a schedule from tree $T(\pi)$ with the smallest makespan.\}
4.2 Time complexity

The PTAS initially calls procedure MAIN for the partial schedule of long jobs respecting the steady permutation π∗; repeatedly, it invokes procedure MAIN for the partial schedule of long jobs respecting following permutations from a priority list until one of the halting conditions of procedure BACKTRACK is satisfied.

Theorem 3 Procedure MAIN invoked for permutation π of the type (1.1) jobs delivers an \((1 + 1/\kappa)\)-approximation solution respecting that permutation in time \(O(\kappa_1 n^2 \log n)\). Hence, time complexity of PTAS can directly be expressed as

\[ O((\kappa_1 + \kappa_1!)\kappa_1 n^2 \log n), \]

or applying VP-analysis, it is

\[ O(\kappa_1 n^2 \log n(\kappa^2 \log \kappa + \kappa^2 \log \kappa \kappa_1 ! P_{\text{max}}^{1.1} 2^{h_1} (K))) = O(\kappa^3 \log \kappa n^2 \log n \kappa_1 ! P_{\text{max}}^{1.1} 2^{h_1} (K)) \]

or alternatively it is

\[ O(\kappa^3 \log \kappa n^2 \log n (((r_j + P_{\text{max}}^{1.1})/P_{\text{min}}^{1.1})!q_{\text{max}}! + \kappa P_{\text{max}}^{1.1} 2(r_{\text{max}}^{1.1} + P_{\text{max}}^{1.1} + P_{\text{min}}^{1.1}))/P_{\text{min}}^{1.1})). \]

Proof. For schedule \(\pi_h\) created by Procedure MAIN, one of the cases \((0), (i), (ii,1)\) and \((ii,2)\) occur. In case \((0)\) \(\pi_h\) contains a kernel with either a short delaying emerging job or with a long delaying emerging job \(i\) such that \(\hat{p}_i\) is short, schedule \(\pi_h\) is an \((1 + 1/\kappa)\)-approximation solution by Lemma 1. In case \((ii,1)\) \(\pi_h\) contains kernel \(K\) of some atomic component and with no delaying emerging job, schedule \(\pi_h\) is optimal, see Lemma 9. In case \((ii,2)\) (schedule \(\pi_h\) contains a secondary kernel with a type \((1,1)\) delaying emerging job), a schedule with the minimum makespan respecting permutation \(\pi\) should already belong to solution tree \(T(\pi)\), see Lemma 10.

Let us now consider case \((i)\) assuming that every kernel in schedule \(\pi_h\) possesses a long delaying emerging job. In a schedule with a better makespan, the full completion time of the overflow job \(i\) must be pushed by a single short job and schedule \(\pi_{ih_1}\) already gives the desired approximation if \(K(\pi_{ih_1}) = K(\pi_{hi})\), similarly as in case \((0)\).

Suppose \(K(\pi_{ih_1}) \neq K(\pi_{hi})\). The two kernels \(K(\pi_{ih_1})\) and \(K(\pi_{hi})\) may have a job in common or not. As it is easy to see, in the former case, all jobs of kernel \(K(\pi_{ih_1})\) belong to kernel \(K(\pi_{hi})\) and initiate the latter kernel, so that job \(i_{h_1}\) belongs to kernel \(K(\pi_{ih_1})\). Furthermore, by the construction of schedule \(\pi_{ih_1}\), the delaying emerging job of kernel \(K(\pi_{ih_1})\) in schedule \(\pi_{ih_1}\) (if any) is short. Again, the difference between the makespan of schedule \(\pi_{ih_1}\) and that of any complete schedule in which job \(i_{h_1}\) is in the state of activation for kernel \(K(\pi_{hi})\) and which respects permutation \(\pi\) is less than the length of the latter short job (Lemma 1). Hence no other schedule in which job \(i_{h_1}\) is in the state of activation for kernel \(K(\pi)\) and which respects permutation \(\pi\) needs to be generated. It follows that all the potentially optimal schedules respecting permutation
\(\pi\) in which job \(i_{h_1}\) is and is not in the state of activation for kernel \(K(\pi)\) are already created in solution tree \(T(\pi)\) and Procedure MAIN can halt for permutation \(\pi\).

Consider the other possibility that the two kernels \(K(\pi_{i_{h_1}})\) and \(K(\pi_h)\) have no job in common. If there is a kernel in schedule \(\pi_{i_{h_1}}\) with a short delaying emerging job then this schedule again gives the desired approximation and the procedure can halt. Otherwise, the delaying emerging job of kernel \(K(\pi_{i_{h_1}})\) is the incoming job \(i_{h_2}\) of some iteration \(h_2\). Procedure BACKTRACK generates an alternative LDT-schedule \(\pi_{i_{h_1}}, i_{h_2}\) in which job \(i_{h_2}\) is in the state of activation for kernel \(K(\pi_{i_{h_1}})\) (where job \(i_{h_1}\) remains in the state of activation for kernel \(K(\pi_h)\)). The reasoning that we have just applied to schedule \(\pi_{i_{h_1}}\) can now be repeatedly applied to schedule \(\pi_{i_{h_1}}; i_{h_2}\), and to the following generated schedules until one of the above two considered halting conditions is satisfied.

We show that in no more than \(\kappa_{1,1} n\) steps, one of the above halting conditions will be satisfied; in other words, the total number of times a long delaying emerging job gets activated is bounded by \(\kappa n\). First, it is not difficult to see that long delaying emerging job \(i_{h_1}\), once activated for kernel \(K(\pi_h)\), is to remain in the state of activation for that kernel at any following iteration in any branch of tree \(T(\pi)\). Indeed, no type (1.1) job preceding job \(i_{h}\) in permutation \(\pi\), can be rescheduled behind kernel \(K(\pi_h)\) in any complete feasible schedule respecting permutation \(\pi\) with job \(i_{h}\) being scheduled before that kernel. But then no such schedule can have a makespan, less than that of schedule \(\pi_h\). So, a long delaying emerging job can be activated at most once for the same kernel in tree \(T(\pi)\), whereas there are less than \(n - \kappa_{1,1} - \kappa_{1,2}\) kernels in set \(\mathcal{K}\). Then the total number of the complete schedules that can be created in solution tree \(T(\pi)\) is bounded by \(\kappa_{1,1} n\) since there may occur no more than \(\kappa_{1,1}\) long delaying emerging jobs. Procedure MAIN uses a variation of LDT-heuristics with the cost \(O(n \log n)\) for the generation of each complete schedule. Hence, the overall cost of the procedure invoked for permutation \(\pi\) is \(O(\kappa_{1,1} n^2 \log n)\).

Then the first expression for the overall cost of the PTAS is immediately obtained by verifying all \((\kappa_{1,1} + \kappa_{1,2})!\) permutations of type (1) long jobs. By considering dominant subsets of type (1.2) jobs instead of permutations (see Observation 6), the second expression is obtained, and we get the third alternative bound due to the second time complexity expression in Theorem 1.

\[\square\]

### 4.2.1 Alternative time complexities

The running time of PTAS from Theorem 3 can slightly be improved by restricting the number of possible job release times, as it was done in the earlier approximation schemes. First, we give a brief analysis of these schemes, which rely on the fact that the set of job release times can be reduced to a subset with a constant number of release times, i.e., a number, dependent only on \(\kappa\) and not dependent on \(n\). As it is shown in [4], this can be done without affecting essentially the approximation ratio. In the first approximation scheme from [7], job processing times are also rounded to the nearest multiples of the total job processing time \(P\) divided by \(nk\). Job release and delivery times are scaled respectively and an instance in which the total number of job processing times is restricted by \(nk\) (instead of \(P\)) is obtained. This instance is solved by a dynamic programming algorithm in which the whole scheduling horizon in divided into sub-intervals, and all possible distributions of the total processing time of jobs assigned to each of these intervals is considered (given a particular such distribution, jobs in each interval are scheduled by LDT-heuristics that includes all the released jobs optimally in non-increasing order of their delivery times). Since every interval is at most \(P\) long and there are \(O(\kappa)\) intervals, there are at most \(P^{O(\kappa)}\) possible distributions to be considered by the dynamic programming algorithm. But because of the accomplished rounding of job processing times, this magnitude converts to
In the second approximation scheme described in [7], as in one from [10], two types of jobs, short and long (or small and large) ones, are treated differently. The essential difference between these two types of jobs makes the fact that the total number of long jobs is bounded by $O(\kappa)$ (it is not dependent of $n$). The whole scheduling horizon is again divided into $O(\kappa)$ (disjoint) subintervals according to the reduced set of job release times. All possible ways in which long jobs can be accommodated into these subintervals are considered. The set of short jobs to be included in every subinterval is determined by “guessing” again their total processing time. LDT-rule is applied to schedule every set of short jobs together with the corresponding long jobs in each subinterval (again, since all these jobs are released within this subinterval, LDT-heuristic will include them in an optimal order).

In the approximation scheme from [10] short and long jobs are formed in different ways yielding a constant number of also small jobs. The original problem instance is again converted into one in which the number of possible job release, processing and delivery times is bounded by $O(\kappa)$, and the whole set of jobs is partitioned into (disjoint) subsets in such a way that the jobs in each subset have the same release and delivery times. The number of such subsets becomes bounded by $O(\kappa^2)$ in the modified problem instance. Then jobs from every subset are merged resulting in longer composed jobs with a common release and delivery time as the component (merged) jobs. In every subset at most one small (non-composed) job remains. It is shown that the approximation ratio will be kept within the allowable range by considering so formed set of jobs. Since the number of the above subsets is constant, the total number of small jobs is also constant, and the number of large (composed) jobs is also constant. Then the complete enumeration already yields a non-exponential time dependence on $n$. The time complexity is adjusted by considering only a constant number of possible starting times solely for large jobs, similarly as in the second approximation scheme from [7] (and this still guarantees the desired approximation ratio). Again, for every such potentially useful configuration, the corresponding feasible schedule is created by LDT-heuristic.

Thus for the purpose of polynomial-time approximation, the restriction on the total number of job release times can be imposed without almost no loss in the approximation ratio. The approximation schemes from [7] and [10] essentially rely on this observation. Let us define a constant $\eta = 1/(2\kappa) \cdot LB$, and round down all job release times to the nearest multiple of $\eta$. Without loss of generality, it can be assumed that $LB = 1$ as all job parameters can be divided by $LB$ and an equivalent scaled instance can be obtained, as described in [10]. In the modified instance, the number of different release times is bounded by $2\kappa$. Then, a solution obtained by an $(1 + 1/(2\kappa))$-approximation algorithm for the modified instance can be transformed to an $(1 + 1/\kappa)$-approximation algorithm for the original instance, as described in the proof of Lemma 2 of [7].

Because of the above observations, the number of different kernels and delaying emerging jobs becomes bounded by $2\kappa$. Furthermore, the number of iterations in procedure MAIN where a long job may cover a short one is now $\kappa$. Indeed, a long job released simultaneously with a short job cannot cover that short job. Hence, $\kappa$ distinct release times are left vacant and the latter claim follows from the fact that from all simultaneously released short jobs, a long job may cover at most one of them. From these observations, the time complexity of procedure MAIN invoked for a permutation of the type (1) jobs can be expressed as $O(\kappa n \log n)$ and hence we have the following alternative bound:
Corollary 2: For a given integer $\kappa > 0$ the PTAS delivers an $(1 + 1/\kappa)$-approximation solution in time $O((\kappa_{1.1} + \kappa_{1.2})!\kappa_{1.1} n \log n)$.

An alternative probabilistic estimation for the running time. Using an intuitive probabilistic model, similar to that of Section 3.3, we can give an alternative probabilistic bound for the running time of PTAS. In particular, we wish to establish the probability that, for the next considered permutation $\pi$, procedure BACKTRACK halts at step (0); in other words, that schedule $\pi_h$ contains a kernel with either a short delaying emerging job or a long delaying emerging job $i$ such that $\hat{p}_i$ is short. Given that a kernel $K$ possesses the delaying emerging job $e$, $r_e < r_o$ and $q_e < g_o$ hold, where $o \in K$ is the overflow job. Similarly as for the probabilistic bound of Section 3.3, the probability that such an $e$ exists is less than $1/4$. There are $\nu$ type (1) jobs, $\kappa$ type (1) long jobs, and hence $\nu - \kappa$ type (1) short jobs. Assuming that job processing times are uniformly distributed, the probability that the delaying emerging job of kernel $K$ is long is $\kappa/\nu$, and the probability that it is short is $(\nu - \kappa)/\nu$. Just ignoring a possible event that $e$ is a long delaying emerging job with short $\hat{p}_e$, the probability that PTAS does not halt after enumerating permutation $\pi$ is less than $1/4 + \kappa/\nu$. Since $\kappa << \nu$, we have that $\kappa/\nu << 3/4$, and the probability that PTAS enumerates $\kappa!$ permutations is about $(1/4 + \kappa/\nu)^{\kappa!}$.

5 Concluding notes

Depending on a particular relationship of object parameters with the objective function, the VP-analysis can be applied to non-homogeneously structured optimization problems. Following our scheduling problem we showed that the classification of jobs into different types may lead to a better understanding of inherent problem properties. This is true for a number of scheduling problems and also for other optimization problems. The VP-analysis led to a polynomial time approximation scheme which “starts” with a reduced set of feasible solutions. Our framework can similarly be used for the construction of other types of approximation algorithms. We can incorporate lower bounds in the constructed solution tree and obtain a branch-and-bound algorithm.

There exist artificial “hard” scheduling instances, for which our variable parameters will approach the total number of objects. Such instances can be generated using any PARTITION instance. We associate with every element in a PARTITION instance a partition job with the same weight (duration). We also create a separator job $j$ that forces all jobs from a subset of partition jobs to complete at a given time moment in any optimal solution (due to the release time of the separator job). In such a scheduling instance, there occurs a single kernel containing (type (2)) job $j$ and the remaining jobs are type (1.1) jobs. In an optimal solution, the type (1.1) jobs need to fill in intervals before and after the separator job in an optimal fashion, so that either to create a schedule without a delaying emerging job or to create an appropriate forced delay for the separator job. This is where a solution to the PARTITION instance can be useful. So, consider a complete schedule $\sigma(\pi)$ created for permutation $\pi$ of the type (1.1) partition jobs and the corresponding two fragments in that schedule, the first one consisting of the jobs included before job $j$ and the second one consisting of the jobs included after job $j$. If in schedule $\sigma(\pi)$ job $j$ starts at its release time and no type (1.1) job realizes the maximum full job completion time in that schedule, then it is optimal. Otherwise, in an optimal schedule, the first above non-critical fragment is filled in by some partition jobs of permutation $\pi$ so that job $j$ is pushed by an appropriate amount of time units. If all type (1.1) jobs are released at time 0, the time interval of the first fragment in an optimal
solution is one of the following time intervals \([0, r_j], [0, r_j + 1], [0, r_j + 2], \ldots, [0, r_j + \delta(K)]\), i.e., job \(j\) is pushed by an integer magnitude \(\Delta \in [0, \delta(K)]\). So, a solution to PARTITION gives a desired packing of the first fragment and hence also of the second one. The application of the standard dynamic programming algorithm yields the time complexity \(O((r + \Delta)\nu)\) for a given \(\Delta\). Using binary search, we restrict the possible values for \(\Delta\) and obtain an overall cost \(O((r + \Delta)\nu \log(\delta(K)))\) for a procedure that creates the corresponding schedules and selects a schedule with the minimum makespan.

Finally, there are real-life problems where variable parameters are reasonably bounded from the above. As an example, consider an airline agent (a machine) serving transit passengers. Each passenger (a job) has a well predictable release time and due date, according to the corresponding flight arrival and departure times. For the airline, it is non-profitable to have passengers that wait too long in the airport (extra expenses, limited space in the airport, etc.). So the release and due times of most of the passengers are tight enough and these passengers form the set of kernel jobs. The remaining \(\nu_{1,1}\) passengers, those ones with a considerable difference between their arrival and departure times, form the set of type (1.1) jobs.

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7 APPENDIX

In this section we briefly overview the results of our experimental study. It aimed to establish the values of our variable parameters for real problem instances. We tested over 50000 problem instances of 26 sizes from \( n = 10 \) to \( n = 50000 \) jobs. We report our results for the 3900 ”hardest” of them and also for 900 additional benchmark instances (the reader can access a more detailed description and the complete experimental data at [14]). Although we relied on the standard rules from literature for the generation of our instances, we observed an interesting dependence of the variable parameter \( \nu_{1,1} \) on the upper limit for the time interval from which job processing times were derived. Therefore, we extended the standard derivation rules by trying different intervals for job processing times. For each size \( n \), we verified up to 500 upper limits (about 1800 problem instances) to find the upper limit resulting the maximum robustness factor of the parameter \( \nu_{1,1} \). Once this limit was found, we created 50 additional problem instances with this upper limit, and also 50 instances for each of the limits 50 and 100, 150 additional instances of each size \( n \).

Even for the upper limits maximizing \( \nu_{1,1} \), the parameter converges to 0 (see Table 1 and Figure 8). Figures 9 and 10 show the dependence of the average and minimum robustness factors of parameter \( \nu_{1,1} \), respectively, on the size of the instances. As to the parameters \( \nu_{1,2}(K) \), surprisingly,
Robustness factor of type 1.1 jobs

\[ p_{\text{max}} = 50 \]
\[ p_{\text{max}} = 100 \]

that maximizes \( \rho_{\text{avg}} \)

\[ \rho_{\text{avg}} \]

\[ \rho_{\text{max}} \]

\[ \rho_{\min} \]

\[ n \]

Table 1: The robustness factor of type 1.1 jobs

they turned out to be neglectable, attaining the maximum total value of 4 (for \( n = 20 \)), 0 for all \( n \) except 5 small \( n \)-s with the average over all the tested instances equal to practically 0, see Table 2.
Figure 8: Graphics for the maximum number of type 1.1 jobs

Figure 9: Graphics for the average number of type 1.1 jobs
The number of type 1.2 jobs

| n         | $\nu_{avg}^{1.2}$ | $\nu_{max}^{1.2}$ | $\nu_{min}^{1.2}$ | $\rho_{avg}^{1.2}$ | $\rho_{max}^{1.2}$ | $\rho_{min}^{1.2}$ |
|------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 50        | 0.02              | 1.02              | 0.01              | 116.0             | 0.00              | 1.00              |
| 100       | 0.01              | 1.01              | 0.00              | 128.0             | 0.00              | 1.00              |
| 200       | 0.00              | 1.00              | 0.00              | 130.0             | 0.00              | 1.00              |
| 400       | 0.00              | 1.00              | 0.00              | 132.0             | 0.00              | 1.00              |
| 600       | 0.00              | 1.00              | 0.00              | 137.0             | 0.00              | 1.00              |
| 800       | 0.00              | 1.00              | 0.00              | 138.0             | 0.00              | 1.00              |
| 1000      | 0.00              | 1.00              | 0.00              | 140.0             | 0.00              | 1.00              |
| 2000      | 0.00              | 1.00              | 0.00              | 144.0             | 0.00              | 1.00              |
| 3000      | 0.00              | 1.00              | 0.00              | 152.0             | 0.00              | 1.00              |
| 4000      | 0.00              | 1.00              | 0.00              | 159.0             | 0.00              | 1.00              |
| 5000      | 0.00              | 1.00              | 0.00              | 161.0             | 0.00              | 1.00              |
| 6000      | 0.00              | 1.00              | 0.00              | 167.0             | 0.00              | 1.00              |
| 7000      | 0.00              | 1.00              | 0.00              | 171.0             | 0.00              | 1.00              |
| 8000      | 0.00              | 1.00              | 0.00              | 174.0             | 0.00              | 1.00              |
| 9000      | 0.00              | 1.00              | 0.00              | 177.0             | 0.00              | 1.00              |
| 10000     | 0.00              | 1.00              | 0.00              | 180.0             | 0.00              | 1.00              |
| 20000     | 0.00              | 1.00              | 0.00              | 320.0             | 0.00              | 1.00              |
| 30000     | 0.00              | 1.00              | 0.00              | 430.0             | 0.00              | 1.00              |
| 40000     | 0.00              | 1.00              | 0.00              | 520.0             | 0.00              | 1.00              |
| 50000     | 0.00              | 1.00              | 0.00              | 620.0             | 0.00              | 1.00              |
| 60000     | 0.00              | 1.00              | 0.00              | 720.0             | 0.00              | 1.00              |
| 70000     | 0.00              | 1.00              | 0.00              | 820.0             | 0.00              | 1.00              |
| 80000     | 0.00              | 1.00              | 0.00              | 920.0             | 0.00              | 1.00              |
| 90000     | 0.00              | 1.00              | 0.00              | 1020.0            | 0.00              | 1.00              |
| 100000    | 0.00              | 1.00              | 0.00              | 1120.0            | 0.00              | 1.00              |

Table 2: The number of type 1.2 jobs