Sampling for the V-line transform with vertex on a circle

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Abstract
In this paper, we consider a special V-line transform in the two dimensional space. It integrates a given function \( f \) over the V-lines whose vertices are on a circle centered at the origin and whose symmetric axes pass through the origin. We study the sampling problem of this V-line transform. Namely, we consider the problem of recovering the continuous data from its discrete samples. Under suitable conditions, we prove an error estimate of the recovery. The error estimate is explicitly expressed in terms of \( f \). We then elaborate the required conditions for two sampling schemes: standard and interlaced ones. Finally, we analyze the number of sampling points needed for each case.

Keywords: sampling, V-line transform, Compton camera imaging

(Some figures may appear in colour only in the online journal)

1. Introduction

Let us motivate our study of the V-line transform by briefly discussing Compton camera imaging. To this end, we first recall the classical setup of single-photon emission computed tomography (SPECT) (see, e.g. [5]), which is a nuclear medicine tomographic imaging technique using gamma rays. In SPECT, weakly radioactive tracers are injected to the patient’s
blood stream. The tracer distribution shows how the blood flows into the tissues, which is useful information for medical diagnosis. One aims to reconstruct such distribution. The tracer emits gamma ray of photons. A gamma camera is used to record such photons that enter the detector surface perpendicularly (see figure 1). In terms of mathematics, the camera measures the integrals of the tracer distribution over straight lines that are orthogonal to its surface, see figure 1. One then reconstructs the tracer distribution from these data.

The above-described technique removes most photons and only a few photons are recorded. Therefore, a new type of camera for SPECT, which makes use of Compton scattering, was proposed by Everett [12] and Singh [46]. It uses electronic collimation as an alternative to mechanical collimation, which provides both high efficiency and multiple projections of the object. The camera consists of two planar gamma detectors positioned one behind the other. An emitted photon undergoes Compton scattering in the first detector surface $D_1$ and is absorbed by the second detector surface $D_2$. In each detector surface, the position and the energy of the photon are measured. The scattering angle at $D_1$ is determined via the Compton scattering formula $\cos \psi = 1 - \frac{m^2c^2}{E_1E_2}$, where $m$ is the electron mass, $c$ the speed of light, and $E_i$ the photon energy at $D_i$ (for $i = 1, 2$). Therefore, a photon observed at $x_1 \in D_1$ and $x_2 \in D_2$ with the energy $E_1, E_2$ respectively must have been emitted on the surface of the circular cone, whose vertex is at $x_1$, central axis points from $x_2$ to $x_1$, and the half-opening angle is given by $\psi$ (see figure 2).
As a result, a Compton camera gives us the integrals of emission distribution on conical surfaces whose vertices are on $D_1$. The mathematical problem of Compton camera imaging is to reconstruct the emission distribution from such integrals.

There are quite a number of works on the mathematics of Compton camera imaging (e.g. [2, 6, 22, 26, 27, 32–34, 41, 44, 49]). In many cases, Compton camera imaging results in an overdetermined problem. That is the space of collected data is of higher dimension than the space of the object/function to be imaged. Taking advantage of such redundancy was the topic of several works (see, e.g. [31, 48]). Microlocal analysis for conical transform has recently been studied in [50, 52, 53]. In the two dimensional space, the cones become V-lines and the corresponding transforms are called V-line transforms (e.g. [2, 4, 20, 24, 37, 51]). It should be noted that some types of V-line transforms also arise in optical tomography (e.g. [3, 19]).

In this article, we are interested in the two dimensional space setup (i.e. the V-line transform). Namely, let us denote by $D$ the unit disc centered at the origin. Assume that $f$ be an essentially $b$-band-limited function supported in $D$. We consider the V-line transform $Vf$ of $f$ on all the V-lines whose vertex is on the circle of radius $r > 1$ centered at the origin and the symmetric axis passes through the origin, see figure 3.

**Definition 1.1.** Let $f$ be a compactly supported function in $D$. The V-line transform $Vf(\varphi, \psi)$ of $f$ is defined by

$$Vf : [0, 2\pi) \times (0; \pi) \rightarrow \mathbb{R},$$

$$(\varphi, \psi) \mapsto \sum_{\sigma = \pm 1}^{+\infty} \int_{0}^{\infty} f(r\theta(\varphi) - t\theta(\varphi + \sigma\psi)) \, dt.$$ 

Here, $\theta(\varphi)$ is the unit vector that makes an angle $\varphi$ to the $x$-axis.

This transform arises in Compton camera imaging, for example, when taking the Fourier transform of the conical transform along the $z$-axis (see [35]). In [35], an inversion formula in terms of Fourier series was obtained. In this work, we instead, consider the problem of
recovering function $Vf$ from its discrete measurements $Vf(\phi_k, \alpha_m)$. That is, we consider the sampling problem of the V-line transform.

Sampling theory for tomography has quite a notable literature. It was started by Cormack [8] and Rattey and Lindgren [43], and then further developed by Natterer and several other authors [9–11, 13–18, 23, 29, 38–40, 42]. Recent interesting approaches include semi-classical sampling theory for generalized Radon transform [47] (with further investigation by [36]) and resolution analysis for jump singularities [28].

In this paper, we follow the classical approach employed by [38–40]. That is, we make use of the Shannon sampling series for the recovery of the V-line transform from the sampling data. Our main result is an explicit error estimate for the recovery. Furthermore, to realize the conditions needed for the estimate, we consider two sampling schemes. The first one is the standard sampling scheme, which is $(\phi_k, \psi_m) = (\phi_k, \psi_m)$, $0 \leq k \leq N-1, 0 \leq m \leq N-1$ for some given natural numbers $N$ and $N$. Here, $\{\phi_k\}_k$ and $\{\psi_m\}_m$ are evenly spaced in their domains. The second, more efficient, sampling scheme is given by $(\phi_k, \psi_m) = (\phi_k, \psi_{km})$, where $\{\phi_k\}_k$ is evenly spaced and $\{\psi_{km}\}_m$ depends on the parity of $k$. Roughly speaking, the efficient sampling scheme requires about 25% less number of sampling points compared to the standard scheme (see section 4.3).

The article is organized as follows. In section 2, we introduce some preliminaries. In section 3 we derive an error estimate for sampling of the V-line transform. Finally, section 4 presents two different sampling schemes that satisfies the sampling condition presented in section 3; they include the standard and interlaced (efficient) schemes described above.

2. Preliminaries

In this section, we introduce some background knowledge needed to our presentation in the next section. We start with the 2D Radon transform (also known as, the x-ray transform), which is arguably one of the most popular transforms in medical imaging [40]. We emphasize the connection between the 2D Radon transform and the V-line transform. We then discuss the sampling theory for periodic functions. Finally, we describe how such theory is applied to the V-line transform.

2.1. The 2D Radon transform

Let us recall that the 2D Radon transform maps a function $f$ on $\mathbb{R}^2$ into the set of its integrals over the lines on $\mathbb{R}^2 \times \mathbb{R}$ and $\theta(\phi) = (\cos \phi, \sin \phi) \in S^1$. The Radon transform $Rf$ of $f$ is defined as

$$Rf(\phi, s) = \int_{x, \theta=x} f(x) d\ell(x)$$

$$= \int_{-\infty}^{+\infty} f(s \cos \phi - t \sin \phi, s \sin \phi + t \cos \phi) dt.$$

That is, $Rf(\phi, s)$ is the integral of $f$ over the line with the normal direction $\theta(\phi)$ and of the distance $s$ from the origin. Let us denote $R_{\phi} f(\cdot) = Rf(\cdot, \phi)$. A useful relation between the 1D Fourier transform of $R_{\phi} f$ and the 2D Fourier transform of $f$ is called the Fourier slice theorem

$$\hat{R_{\phi} f}(\sigma) = (2\pi)^{1/2} \hat{f}(\sigma \theta).$$

This formula has played a central role in sampling theory for the 2D Radon transform [40]. It will be used in section 3 in deriving the sampling theory for the V-line transform. Here, the
Fourier transform of a function $g$ defined on $\mathbb{R}^n$ is given by

$$\hat{g}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(x)e^{-i x \cdot \xi} \, dx.$$ 

Finally, the following relationship between V-line transform and Radon transform

$$V f (\varphi, \psi) = R f \left( \varphi + \frac{\pi}{2}, r \sin \psi \right) + R f \left( \varphi - \frac{\pi}{2}, -r \sin \psi \right)$$

helps to transfer the well established tools from sampling theory of 2D Radon transform to V-line transform.

### 2.2. Sampling of periodic functions

Let $g$ be a function in $\mathbb{R}^n$ that is periodic with respect to $n$ vectors $p_1, p_2, \ldots, p_n$. If the matrix $\mathcal{P} = (p_1, p_2, \ldots, p_n)$ is nonsingular then $g$ is called a $\mathcal{P}$-periodic function. One denotes $L_\mathcal{P} = \mathcal{P} \mathbb{Z}^n$ and its reciprocal lattice $L_{\mathcal{P}}^* = L_{2\pi \mathcal{P}^{-1}}$, where $\mathcal{P}^{-1}$ is the transpose of $\mathcal{P}^{-1}$. We define the discrete Fourier transform of $g$ to be the following function on $L_{\mathcal{P}}^*$:

$$\hat{g}(\xi) = |\det(\mathcal{P})|^{-1} \int_{\mathcal{P}(0,1)^n} g(x)e^{-i x \cdot \xi} \, dx, \xi \in L_{\mathcal{P}}^*.$$ 

Its inverse is defined by

$$\tilde{h}(x) = \sum_{\xi \in L_{\mathcal{P}}^*} h(\xi)e^{i x \cdot \xi}, \quad x \in \mathbb{R}^n.$$ 

Let $W$ be a real nonsingular $n \times n$-matrix, our goal is study the sampling of $g$ on $L_W$. To this end, we assume $L_\mathcal{P} \subset L_W$, which means $\mathcal{P} = W M$ for an integer matrix $M$. That implies, $L_W / L_\mathcal{P} \subset L_{\mathcal{P}}^*$. Let $L_W / L_\mathcal{P}$ be the quotient space whose elements are of the form $y + L_\mathcal{P}$. The following theorem gives us a construction method from sampling on $L_W$ together with an error estimate (see [14, 40]).

**Theorem 2.1.** Suppose $g \in C^\infty(\mathbb{R}^n)$ is a $\mathcal{P}$-periodic function. Let $K \subset L_{\mathcal{P}}^*$ be a finite set such that its translates $K + \eta, \eta \in L_W$, are disjoint, and $\chi_K$ denote the characteristic function of $K$. We define the sampling series

$$S_{W,K} g (x) := \frac{|\det(W)|}{|\det(\mathcal{P})|} \sum_{v \in L_W / L_\mathcal{P}} \chi_K(x - v) g(v).$$

Then,

$$\|S_{W,K} g - g\|_{L^\infty} \leq 2 \int_{L_{\mathcal{P}}^* \setminus K} |\hat{g}(\xi)| \, d\xi.$$ 

\[i.e. \chi_K(\xi) = 1 \text{ if } \xi \in K \text{ and } \chi_K(\xi) = 0 \text{ otherwise.}\]
2.3. Sampling of the V-Line transform

Let us recall the assumption supp(f) ⊂ \( \mathbb{D} \). Therefore, \( g(\varphi, \psi) := \mathcal{V}f(\varphi, \psi) = 0 \) with \( \psi \in \left[ \frac{\pi}{2}, \pi \right) \). Consequently, \( \mathcal{V}f \) can be extended to an even function in \( \psi \) and 2\( \pi \)-periodic in both variables. We will make use of the two-dimensional form of theorem 2.1 to recover the V-line transform. Since \( g \) is 2\( \pi \)-periodic in each variable, the periodic matrix of \( g \) is \( P = 2\pi I_{2 \times 2} \) and \( L_2^P = \mathbb{Z}^2 \). We chose matrix \( W \) which satisfies the condition \( L_P \subset L^W \) as

\[
W = 2\pi \begin{pmatrix} 1/P & 0 \\ N/(PQ) & 1/Q \end{pmatrix}
\]

(3)

where \( P, Q, N \) are three integers such that \( P, Q > 0 \) and \( 0 \leq N < P \). Therefore,

\[
L_W/L_P = \left\{ (s_j, t_j) : s_j = \frac{2\pi}{P}, t_j = \frac{(l + Nj)2\pi}{Q}, j = 0, \ldots, P - 1, l = 0, \ldots, Q - 1 \right\}.
\]

Theorem 2.1 in this setting becomes

**Theorem 2.2.** Suppose \( f \in C^0_0(\mathbb{D}) \) and \( g(\varphi, \psi) = \mathcal{V}f(\varphi, \psi) \). Let \( K \subset \mathbb{Z}^2 \) be a finite set such that its translates \( K + \eta, \eta \in L^W \), are disjoint. For \( x \in [0, 2\pi) \times [0, 2\pi) \), we define the sampling series

\[
S_{W,K}g(x) := \frac{1}{PQ} \sum_{v \in L_W/L_P} \hat{\chi}_K(x - v) g(v).
\]

Then,

\[
\|S_{W,K}g - g\|_{L^\infty} \leq 2 \sum_{\xi \in K} |\hat{g}(\xi)| d\xi.
\]

3. The main results

Our goal is to use the theorem 2.2 to propose some sampling conditions and derive a corresponding sampling error estimate. For this purpose, we assume \( f(\xi) \) be negligible for \( |\xi| > b \), in the sense that the integral \( \epsilon(f, b) := \int_{|\xi| > b} \int |f(\xi)|^2 d\xi d\xi \) is small for all real number \( d \). Such a function \( f \) is called essentially \( b \)-band-limited. Here is the main result of this article:

**Theorem 3.1.** Let \( f \in C^0_0(\mathbb{D}) \) be essentially \( b \)-band-limited \( (b > 1) \), and \( g(\varphi, \psi) = \mathcal{V}f(\varphi, \psi) \). Let \( \vartheta < 1 \) be such that \( 2 - \vartheta^2 < r \). We define the set (see figure 4)

\[
K = \left\{ (k, m) : |k| < \frac{rb}{\vartheta^2}, |m| - |k| < \frac{rb}{\vartheta} \right\}
\]

in \( \mathbb{R}^2 \). Let \( W \) be a real non-singular \( 2 \times 2 \) matrix of the form (3) such that the sets \( K + \eta, \eta \in L^W \), are mutually disjoint. Define the sampling series \( S_{W,K} \) as in theorem 2.2. Denote

\[
\eta_1(\vartheta, \gamma) = \left( \frac{3}{(1 - \vartheta^2)^3/2} + \frac{9}{(1 - \vartheta^2)^3} \right) \frac{1}{\gamma} \eta(\vartheta, \gamma),
\]

\[
\eta_2(\vartheta, \gamma) = \left( \frac{9}{(1 - \vartheta^2)^3} + \frac{54}{(1 - \vartheta^2)^2} \right) \frac{1}{\gamma} \eta(\vartheta, \gamma),
\]

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where \( \eta(\vartheta, \gamma) = \gamma \vartheta \exp \left( -\frac{\gamma}{2} (1 - \vartheta^2)^{3/2} \right) \). Then for \( b \) being big enough
\[
\| S_{W, K} g - g \|_{L^\infty} \leq \frac{12}{\pi} \eta^*(\vartheta, rb) \| f \|_{L^1(\mathbb{R}^2)} + \frac{16r^2}{\pi \vartheta^3} \epsilon_l(f, b),
\]
where \( \eta^*(\vartheta, rb) = \max \left\{ \frac{2b}{rb} \eta_1(\vartheta, \frac{rb}{\vartheta}); \frac{2}{rb} \eta_2(\vartheta, \frac{rb}{\vartheta}); \frac{1}{25(\vartheta^2 - \bar{r}^2)} \eta_1(\vartheta, \frac{rb}{\vartheta}) \right\} \).

In the rest of this section, we present the proof of this theorem.

3.1. Proof of the main result

We first derive some useful property of Fourier coefficient of V-line transform (see [40]).

**Lemma 3.2.** Suppose \( f \in C^\infty_0(\mathbb{D}) \), \( g(\varphi, \psi) = Vf(\varphi, \psi) \). Let \( \widehat{g}_{k,m} \) be the Fourier coefficient of \( g \). Then
\[
\widehat{g}_{k,m} = \frac{(-1)^m k^2}{4\pi} \int_{\mathbb{R}} \int_0^{2\pi} \tilde{g}(\sigma\theta(\alpha)) \left[ J_{k-m}(\sigma) + J_{k+m}(\sigma) \right] e^{-ik\alpha} \, d\alpha \, d\sigma,
\]
with \( J_k(x) \) is Bessel function of first kind.

**Proof.** We have
\[
\widehat{g}_{k,m} = \frac{1}{4\pi} \int_0^{2\pi} \int_{-\pi}^\pi \tilde{g}(\varphi, \psi) e^{-ik(\varphi + m\psi)} \, d\varphi \, d\psi
\]
\[
= \frac{1}{4\pi} \int_0^{2\pi} \int_{-\pi}^\pi \tilde{Rf} \left( \varphi + \psi - \frac{\pi}{2}, r\theta \left( \varphi + \psi - \frac{\pi}{2} \right) \theta(\varphi) \right) \, d\varphi \, d\psi.
\]
\[
+ Rf \left( \varphi - \psi - \frac{\pi}{2}, r\theta \left( \varphi - \psi - \frac{\pi}{2} \right) \vartheta (\varphi) \right) e^{-i(k\varphi + m\psi)} \, d\varphi \, d\psi.
\]

That is, we can write
\[
\hat{g}_{k,m} = I_1 + I_2,
\]
where
\[
I_{1,2} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Rf \left( \varphi \pm \psi - \frac{\pi}{2}, r\theta \left( \varphi \pm \psi - \frac{\pi}{2} \right) \vartheta (\varphi) \right) e^{-im\varphi} e^{-ik\psi} \, d\varphi \, d\psi.
\]

Denoting \( \alpha = \varphi + \psi - \frac{\pi}{2} \), the integral inside of \( I_1 \) becomes
\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Rf \left( \varphi + \psi - \frac{\pi}{2}, r\theta \left( \varphi + \psi - \frac{\pi}{2} \right) \vartheta (\varphi) \right) e^{-im\varphi} \, d\varphi
\]
\[
= (-i)^m \int_{\varphi - \pi}^{\varphi} Rf \left[ \alpha, r\theta (\alpha) \vartheta (\varphi) \right] e^{-im\varphi} \, d\alpha
\]
\[
= \frac{(-i)^m}{2} e^{im\varphi} \int_{0}^{2\pi} Rf \left[ \alpha, r\theta (\alpha) \vartheta (\varphi) \right] e^{-im\varphi} \, d\alpha
\]
\[
= \frac{(-i)^m}{2\sqrt{2\pi}} e^{im\varphi} \int_{0}^{2\pi} \left( \frac{\int_{0}^{2\pi} Rf(\alpha, \sigma) e^{ir\theta(\alpha) \sigma} \, d\sigma}{R} \right) e^{-im\varphi} \, d\alpha
\]
\[
= \frac{1}{2} \frac{(-i)^m}{2} e^{im\varphi} \int_{0}^{2\pi} \int_{0}^{R} \tilde{f}(\sigma \theta (\alpha)) e^{-im\varphi + ir\sigma \cos(\varphi - \alpha)} \, d\sigma \, d\alpha.
\]

Hence,
\[
I_1 = \frac{(-i)^m}{8\pi^2} \int_{0}^{2\pi} \int_{0}^{2\pi} \tilde{f}(\sigma \theta (\alpha)) \int_{0}^{R} e^{-i(k\varphi - m\psi)} e^{-im\varphi + ir\sigma \cos(\varphi - \alpha)} \, d\varphi \, d\sigma \, d\alpha
\]
\[
= \frac{(-i)^m}{8\pi^2} \int_{0}^{2\pi} \int_{0}^{2\pi} \tilde{f}(\sigma \theta (\alpha)) \left( \int_{0}^{2\pi} e^{-i(k\varphi - m\psi) + ir\sigma \cos(\varphi - \alpha)} \, d\varphi \right) \, d\alpha \, d\sigma.
\]

Recalling the formula [40, formula 7.3.16] (see also [1, formula 9.1.21])
\[
\int_{0}^{2\pi} e^{-ik\varphi + ix \cos \varphi} \, d\varphi = 2\pi i \mathcal{J}_k(x).
\]

we obtain
\[
\int_{0}^{2\pi} e^{-i(k\varphi - m\psi) + ir\sigma \cos(\varphi - \alpha)} \, d\varphi = 2\pi \times i^{k-m} \times \mathcal{J}_{k-m}(r\sigma).
\]
Therefore,
\[
I_1 = \frac{(-1)^{m+k}}{4\pi} \int_0^{2\pi} \int_{\mathbb{R}} f(\sigma \theta(\alpha)) e^{-ik\alpha} J_{k-m}(r\sigma) \, d\alpha \, d\sigma. \tag{4}
\]

Similarly,
\[
I_2 = \frac{(-1)^{m+k}}{4\pi} \int_0^{2\pi} \int_{\mathbb{R}} f(\sigma \theta(\alpha)) e^{-ik\alpha} J_{k+m}(r\sigma) \, d\alpha \, d\sigma. \tag{5}
\]

Combining (4) and (5), we conclude
\[
\hat{g}_{k,m} = \frac{(-1)^{m+k}}{4\pi} \int_0^{2\pi} \int_{\mathbb{R}} f(\sigma \theta(\alpha)) \left[ J_{k-m}(r\sigma) + J_{k+m}(r\sigma) \right] e^{-ik\alpha} \, d\alpha \, d\sigma.
\]

In order to estimate $\hat{g}_{m,k}$ we will need the following result

**Lemma 3.3.** Suppose $f \in C^\infty_0(D)$. Then, for any $\tau > 0$,
\[
\left| \int_0^{2\pi} \int_{\mathbb{R}} f(\sigma \theta(\alpha)) J_{l}(r\sigma) e^{-ik\alpha} \, d\alpha \, d\sigma \right| \leq \int_D \left| f(x) \right| \left| \int_{\tau}^{\gamma} J_l(r\sigma) |\sigma| \, d\sigma \right| \, dx + 2\epsilon^{-1}(f, \tau).
\]

**Proof.** Since the Bessel function $J_l(s)$ is bounded by 1,
\[
\left| \int_0^{2\pi} \int_{\mathbb{R}} f(\sigma \theta(\alpha)) J_{l}(r\sigma) e^{-ik\alpha} \, d\alpha \, d\sigma \right| \leq \frac{1}{4\pi} \int_{\tau}^{\gamma} \int_0^{2\pi} f(\sigma \theta) e^{-ik\alpha} \, d\alpha \, d\sigma \, dx + \frac{1}{2\pi} \epsilon^{-1}(f, \tau).
\]

Expressing $\hat{f}(\sigma\theta)$ by its definition, we obtain
\[
\int_0^{2\pi} f(\sigma \theta(\alpha)) e^{-ik\alpha} \, d\alpha = \frac{1}{2\pi} \int_{\mathbb{R}} \int_D f(x)e^{-ix\theta(\psi)} x^{-ik\alpha} \, dx \, d\alpha
\]
\[
= \frac{1}{2\pi} \int_D f(x) \int_0^{2\pi} e^{-i|\sigma| \cos(\alpha - \psi) + i\alpha(1 - \psi)} \, d\alpha \, dx
\]
\[
= i^{k} \int_D f(x) J_k(-|\sigma| \cos(\alpha - \psi) + i\alpha(1 - \psi)) \, dx.
\]

Above, we have decomposed $x$ into the polar form $x = |x| \theta(\psi)$. We, therefore, obtain the desired inequality. $\square$

The following inequality in [45]
\[
J_n(v) \leq \frac{s^v \exp \left( v(1 - s^2)^{1/2} \right)}{(1 + (1 - s^2)^{1/2})^{1/2}}, \quad v \geq 0, \quad 0 < s \leq 1,
\]
yields
\[
J_{\sigma}(\nu s) \leq e^{-\frac{\nu}{2}(1 - \rho^2)^{1/2}}, \quad \nu \geq 0, \quad 0 < s \leq 1.
\]

We obtain for any \( k \geq 0, \)
\[
\sup_{|x| \leq 1 - \delta_k} \int |J_{\sigma}(|x|)| \, d\sigma \leq 2k \sup_{|x| \leq 1} \int J_{\sigma}(x|\rho|) \, d\sigma \\
\leq 2k \int_0^1 e^{-\frac{1}{2}(1 - \rho^2)^{1/2}} \, d\sigma = 2\eta(\rho, k),
\]
where
\[
\eta(\rho, \gamma) = \gamma \rho \exp \left( -\frac{\gamma}{3}(1 - \rho^2)^{1/2} \right).
\]

Lemma 3.4. For \( \tau = \rho |l| / r, \) we have
\[
\int |f(x)| \int_{-\tau}^\tau J_{\sigma}(\rho \sigma |l|) \, d\sigma \, dx \leq \frac{2}{\tau} \eta(\rho, |l|) \| f \|_{L^1(\mathbb{R}^2)}.
\]

Proof. Indeed, using the fact that \( |J_{\sigma}(s)| \leq 1, \) we obtain
\[
\int |f(x)| \int_{-\tau}^\tau J_{\sigma}(\rho \sigma |l|) \, d\sigma \, dx \leq \left( \int_{|\sigma| \leq \rho |l| / r} |J_{\sigma}(\rho \sigma |l|) | \, d\sigma \right) \| f \|_{L^1(\mathbb{R}^2)} \\
\leq \left( \frac{1}{r} \int_{|\sigma| \leq \rho |l|} |J_{\sigma}(\rho \sigma |l|) | \, d\sigma \right) \| f \|_{L^1(\mathbb{R}^2)} \\
\overset{(6)}{\leq} \frac{2}{\tau} \eta(\rho, |l|) \| f \|_{L^1(\mathbb{R}^2)}.
\]

We will also need the following result

Lemma 3.5. Suppose \( f \in C_0^\infty(\mathbb{D}) \) and \( \tau := 2 - \rho^2 < r, \) then for \( |l| \geq \rho^{-1} |l|, \)
\[
\left| \int_0^{2\pi} \int_0^\rho \tilde{f}(\sigma \theta(\alpha)) J_{\sigma}(\sigma) e^{-ik\alpha} \, d\alpha \, d\sigma \right| \leq \frac{1}{2\pi(r^2 - \rho^2)} \eta(\rho, k) \| f \|_{L^1(\mathbb{R}^2)}.
\]

Proof. From [42], for any positive real number \( \epsilon \) such that \( \rho := \sqrt{\cosh 2\epsilon / \rho} \leq r, \) we have
\[
\left| \int_0^{2\pi} \int_0^\rho \tilde{f}(\sigma \theta(\alpha)) J_{\sigma}(\sigma) e^{-ik\alpha} \, d\alpha \, d\sigma \right| \leq \frac{I(f)}{2\pi} \exp \left( -\epsilon |l| + \frac{\delta}{|l|} \right)
\]
where \( \delta := \int_0^\epsilon \sqrt{\cosh 2t} \, dt \) and
\[
I(f) := \int_{\mathbb{D}} \frac{|f(x)|}{r^2 - \rho^2 |x|^2} \, dx.
\]
Simple calculations give, for all $\epsilon < 1$,

$$\delta = \int_0^\epsilon \sqrt{\cosh 2t} \, dt \leq \epsilon \sqrt{\cosh 2\epsilon} < \epsilon (1 + \epsilon).$$

Choosing $\epsilon = 1 - \vartheta^2$, we obtain

$$\epsilon - \frac{\delta \vartheta}{r} \geq \epsilon \left(1 + \frac{1 - \epsilon}{r} \vartheta \right) \geq \epsilon (1 - \vartheta) \geq \epsilon (1 - \sqrt{1 - \epsilon}) \geq \frac{1}{3} \epsilon^{3/2} = \frac{1}{3} (1 - \vartheta^2)^{3/2}.$$

Therefore, for $|k| \geq \vartheta^{-1} ||l||$,

$$\exp \left(-\epsilon |k| + \frac{\delta}{r} ||l|| \right) \leq \exp \left(-\frac{1}{3} (1 - \vartheta^2)^{3/2} \right).$$

Moreover, since $\rho = \cosh(2\epsilon) < 1 + \epsilon = \bar{r}$,

$$I(f) := \int_D |f(x)| \sqrt{r^2 - \rho^2 |x|^2} \, dx \leq \frac{1}{r^2 - \bar{r}^2} \|f\|_{L^1(\mathbb{R}^2)}.$$

The above two inequalities finish our proof. \hfill \square

We are now ready to prove theorem 3.1.

**Proof of Theorem 3.1** Due to theorem 2.2, it now suffices to prove that

$$2 \sum_{Z \subseteq K} |\hat{g}_{k,m}| \leq \frac{12}{\pi} \eta^* (\vartheta, rb) \|f\|_{L^1(\mathbb{R}^2)} + \frac{4r^2}{\pi \vartheta^2} (2b + 1) \epsilon_1(f, b).$$

Indeed, from lemma 3.2, we obtain

$$|\hat{g}_{k,m}| \leq \frac{1}{2\pi} \left| \int_\mathbb{R} \int_0^{2\pi} \hat{f} (\sigma \theta (\alpha)) J_{k-m} (\sigma) e^{-ik\alpha} \, d\sigma \, d\alpha \right| + \frac{1}{2\pi} \left| \int_\mathbb{R} \int_0^{2\pi} \hat{f} (\sigma \theta (\alpha)) J_{k+m} (\sigma) e^{-ik\alpha} \, d\sigma \, d\alpha \right|.$$

Therefore,

$$\sum_{Z \subseteq K} |\hat{g}_{k,m}| \leq S_1 + S_2,$$

where

$$S_{1,2} = \sum_{Z \subseteq K} \frac{1}{2\pi} \left| \int_\mathbb{R} \int_0^{2\pi} \hat{f} (\sigma \theta (\alpha)) J_{k+m} (\sigma) e^{-ik\alpha} \, d\sigma \, d\alpha \right|.$$

Let us denote

$$\eta_1 (\vartheta, k) = \sum_{m < k} \eta (\vartheta, m), \quad \eta_2 (\vartheta, k) = \sum_{m > k} \eta_1 (\vartheta, m).$$
We note that $\eta(\vartheta, \gamma)$ exponentially decays as $\gamma \to \infty$ and
\[
\sum_{m=0}^{\infty} m^d \eta(\vartheta, m) \leq \int_{k}^{\infty} s^d \eta(\vartheta, s) \, ds = \int_{k}^{\infty} \vartheta s^{d+1} \exp\left(-\frac{s}{3}(1-\vartheta^2)^{3/2}\right) \, ds,
\]
for $d \geq -1$ and $k$ big enough. Direct calculations then show
\[
\bar{\eta}_1(\vartheta, \gamma) \leq \left(\frac{3}{(1-\vartheta^2)^{3/2}} + \frac{9}{(1-\vartheta^2)^3} \right) \eta(\vartheta, \gamma) = \eta_1(\vartheta, \gamma),
\]
(7)
\[
\bar{\eta}_2(\vartheta, \gamma) \leq \left(\frac{3}{(1-\vartheta^2)^{3/2}} + \frac{9}{(1-\vartheta^2)^3} \right) \eta(\vartheta, \gamma) = \eta_2(\vartheta, \gamma).
\]
(8)

3.1.1 Part 1: estimate $S_1$. We decompose $S_1 = S_{11} + S_{12} + S_{13}$ where each $S_{ij}$ is the sum ranging over the region $\Sigma_{ij}$, where (see figure 5)
\[
\Sigma_{11} = \{ (k, m) \in \mathbb{Z}^2 : |k| < \frac{r b}{\vartheta^2}; |m| - |k| \geq \frac{r b}{\vartheta}\},
\]
\[
\Sigma_{12} = \{ (k, m) \in \mathbb{Z}^2 : |k| \geq \frac{r b}{\vartheta^2}; |k| > |k - m| \},
\]
\[
\Sigma_{13} = \{ (k, m) \in \mathbb{Z}^2 : |k| \geq \frac{r b}{\vartheta^2} ; |k| \leq \frac{r b}{\vartheta^2} \}.
\]

- For the sum $S_{11}$, we note that $|k - m| \geq \vartheta |k|$ in $\Sigma_{11}$. Choosing $\sigma = \vartheta |k - m| / r$ and using lemmas 3.3 and 3.4, we obtain
\[
S_{11} \leq \sum_{|k| \leq \frac{r b}{\vartheta^2}} \sum_{|k - m| > \frac{r b}{\vartheta}} \left( \frac{1}{2 \pi} \eta(\vartheta, |k - m|) \|f\|_{L^1(\mathbb{R}^2)} + \frac{1}{2 \pi} \epsilon_{-1} \left(f, \frac{\vartheta |k - m|}{r}\right) \right)
\]
\[
\leq \sum_{|k| \leq \frac{r b}{\vartheta^2}} \sum_{|k - m| > \frac{r b}{\vartheta}} \left( \frac{1}{\pi r} \eta(\vartheta, b) \|f\|_{L^1(\mathbb{R}^2)} + \frac{1}{\pi r} \epsilon_{-1} \left(f, \frac{\vartheta b}{r}\right) \right)
\]
\[
\overset{(7)}{\leq} \frac{2 r b}{\pi \vartheta^2} \eta_1(\vartheta, \frac{r b}{\vartheta}) \|f\|_{L^1(\mathbb{R}^2)} + \frac{2 r b}{\pi \vartheta^2} \epsilon_{1}(f, b)
\]
\[
\leq \frac{2 b}{\pi \vartheta^2} \eta_1(\vartheta, \frac{r b}{\vartheta}) \|f\|_{L^1(\mathbb{R}^2)} + \frac{2 r b}{\pi \vartheta^2} \epsilon_{1}(f, b),
\]
where we have used, $0 < \mu$ and $b > 1$ (see, e.g. [40])
\[
\sum_{l \geq b / \mu} \epsilon_d (f, \mu l) \leq \frac{1}{\mu} \epsilon_{d+1} (f, b),
\]
for $\mu = \vartheta / r$. 

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For the sum $S_{12}$, we notice that $|k - m| < \vartheta |k|$. Using lemma 3.5,

$$S_{12} \leq \sum_{|k| \geq \frac{r b}{\vartheta^2}} \frac{1}{4 \pi^2 (r^2 - \overline{r}^2)} \eta (\vartheta, |k|) \|f\|_{L^1(\mathbb{R}^2)}$$

$$\leq \frac{1}{2 \pi^2 (r^2 - \overline{r}^2)} \sum_{l \geq \frac{r b}{\vartheta^2}} \eta (\vartheta, l) \|f\|_{L^1(\mathbb{R}^2)}$$

$$\leq \left(7\right) \frac{1}{2 \pi^2 (r^2 - \overline{r}^2)} \eta_1 \left(\vartheta, \frac{r b}{\vartheta^2}\right) \|f\|_{L^1(\mathbb{R}^2)}.$$

In $\Sigma_{13}$, we get the sum for the $m$ first. Hence, from lemma 3.4

$$S_{13} \leq \sum_{|k| \geq \frac{r b}{\vartheta^2}} \sum_{|k-m| \geq \vartheta |k|} \left(\frac{1}{2 \pi^2} \eta (\vartheta, |k-m|) \|f\|_{L^1(\mathbb{R}^2)} + \frac{1}{2 \pi} \epsilon_{-1} \left(f, \frac{\vartheta |k-m|}{r}\right)\right)$$

$$\leq \sum_{|k| \geq \frac{r b}{\vartheta^2}} \left(\frac{1}{2 \pi} \eta_1 (\vartheta, \vartheta |k|) \|f\|_{L^1(\mathbb{R}^2)} + \frac{r}{\pi \vartheta} \epsilon_0 \left(f, \frac{\vartheta^2 |k|}{r}\right)\right)$$

$$\leq \sum_{l \geq \frac{r b}{\vartheta^2}} \left(\frac{2}{\pi} \eta_1 (\vartheta, \vartheta |l|) \|f\|_{L^1(\mathbb{R}^2)} + \frac{2 r}{\pi \vartheta} \epsilon_0 \left(f, \frac{\vartheta^2 l}{r}\right)\right)$$

$$\leq \left(8\right) \frac{2}{\pi} \eta_2 \left(\vartheta, \frac{r b}{\vartheta^2}\right) \|f\|_{L^1(\mathbb{R}^2)} + \frac{2 r^2}{\pi \vartheta^2} \epsilon_1 (f, b).$$
3.1.2. **Part 2: estimate** $S_2$. Similarly, we consider the sums $S_{21}, S_{22}, S_{23}$ over the regions

\[
\Sigma_{21} := \left\{ (k, m) \in \mathbb{Z}^2 : |k| < \frac{rb}{\vartheta}; |m| - |k| \geq \frac{rb}{\vartheta} \right\}
\]

\[
\Sigma_{22} := \left\{ (k, m) \in \mathbb{Z}^2 : |k| \geq \frac{rb}{\vartheta}; |k| > \frac{|k + m|}{\vartheta} \right\}
\]

\[
\Sigma_{23} := \left\{ (k, m) \in \mathbb{Z}^2 : |k| \geq \frac{rb}{\vartheta}; |k| \leq \frac{|k + m|}{\vartheta} \right\}.
\]

Then (see figure 6),

\[
S_{21} \leq \frac{2b}{\pi \vartheta^2} \eta_1 \left( \vartheta, \frac{rb}{\vartheta} \right) \| f \|_{L^1(\mathbb{R}^2)} + \frac{2r^2}{\pi \vartheta^3} \epsilon_1(f, b),
\]

\[
S_{22} \leq \frac{1}{2\pi r^2 (r^2 - \bar{r}^2)} \eta_2 \left( \vartheta, \frac{rb}{\vartheta} \right) \| f \|_{L^1(\mathbb{R}^2)},
\]

\[
S_{23} \leq \frac{2}{\pi r} \eta_2 \left( \vartheta, \frac{rb}{\vartheta} \right) \| f \|_{L^1(\mathbb{R}^2)} + \frac{2r^2}{\pi \vartheta^3} \epsilon_1(f, b).
\]

3.1.3. **Finishing the proof.** Combining the estimates in parts 1 and 2, we obtain

\[
2 \sum_{(k,m) \in \mathbb{Z}^2} |\hat{g}(k,m)| \leq \frac{12}{\pi} \eta^* (\vartheta, rb) \| f \|_{L^1(\mathbb{R}^2)} + \frac{16r^2}{\pi \vartheta^3} \epsilon_1(f, b),
\]

where $\eta^* (\vartheta, rb) = \max \left\{ \frac{2b}{\vartheta^2} \eta_1 \left( \vartheta, \frac{rb}{\vartheta} \right) ; \frac{1}{2\pi r^2 (r^2 - \bar{r}^2)} \eta_2 \left( \vartheta, \frac{rb}{\vartheta} \right) ; \frac{1}{2\pi (r^2 - \bar{r}^2)} \eta_1 \left( \vartheta, \frac{rb}{\vartheta} \right) \right\}.$
According to theorem 2.2, we conclude

$$\|S_{W,K}g - g\|_{L^\infty} \leq \frac{12}{\pi} \eta^2 (\vartheta, rb) \|f\|_{L^1(\mathbb{R}^2)} + \frac{16r^2}{\pi \vartheta^3} \mathcal{C}(f, b).$$

□

4. Sampling schemes for $g(\varphi, \psi)$

In this section, we consider two schemes that satisfy the conditions in theorem 3.1. The first one is the standard scheme, where the sampling locations is the Cartesian product. The second one, more efficient, is an interlaced scheme. The central idea is to find proper matrix $W$ such that the following condition holds:

**Condition 4.1.** The translates $K + \eta$, $\eta := 2\pi W^{-T}m \in L_{W}^{\perp}$, are mutually disjoint for all $m \in \mathbb{Z}^2$.

4.1. Standard sampling scheme

We consider the arrangement of the sets $K + \eta$, $\eta \in L_{W}^{\perp}$, as in figure 7. There, $L_{W}^{\perp}$ is a rectangular lattice with the primitive (column) vectors: $v_1 = \left[\frac{2rb}{\vartheta^2}, 0\right]^T$, $v_2 = \left[0, \frac{2rb}{\vartheta} (1 + \vartheta)\right]^T$.

Therefore, we can choose

$$2\pi W^{-T} = \begin{bmatrix} 2rb & 0 \\ \vartheta^2 & 2rb(1 + \vartheta) \end{bmatrix}.$$  

That is,

$$W = \begin{bmatrix} \vartheta^2 / rb & 0 \\ 0 & \vartheta^2 / rb(1 + \vartheta) \end{bmatrix}.$$  

This matrix, however, may not be of the form (3). We instead choose

$$W = \begin{bmatrix} 2\pi / N_\varphi & 0 \\ 0 & 2\pi / N_\psi \end{bmatrix},$$

where $N_\varphi, N_\psi$ are two integers satisfying

$$N_\varphi \geq \frac{2rb}{\vartheta^2}, \quad N_\psi \geq \frac{2rb(1 + \vartheta)}{\vartheta^2}.$$  

(10)
Then condition 4.1 is satisfied, and consequently, theorem 3.1 holds. In this situation, the sampling points have the form

\[(\varphi_k, \psi_m) = \left( \frac{k 2\pi}{N_{\varphi}}, \frac{m 2\pi}{N_{\psi}} \right), \quad 0 \leq k \leq N_{\varphi} - 1, \quad 0 \leq m \leq N_{\psi} - 1.\]

Since \(f\) is supported in \(D\), \(m\) can be restricted to \(|m| < \frac{N_{\psi} \arcsin(1/r)}{2\pi}\). Because the function \(g(\varphi, \psi)\) is even in \(\psi\), we only consider \(\psi \geq 0\). This yields the standard detector system \(g_{k,m} = g(\varphi_k, \psi_m)\), where

\[
\varphi_k = \frac{k 2\pi}{N_{\varphi}}, \quad \text{for} \quad 0 \leq k \leq N_{\varphi} - 1
\]

\[
\psi_m = \frac{m 2\pi}{N_{\psi}}, \quad \text{for} \quad 0 \leq m < \frac{N_{\psi} \arcsin(1/r)}{2\pi}.
\]

### 4.2. Efficient sampling scheme

We consider the arrangement of the sets \(K + \eta, \eta \in L_{\frac{1}{2}}\), as in figure 8. The lattice is generated by the two primitive vectors \(v_1 = \left[\frac{2r\pi}{\vartheta^2}, 0\right]^T, \quad v_2 = \left[-\frac{rb}{\vartheta^2}, \frac{r}{\vartheta} (2 + \frac{1}{\vartheta})\right]^T\). It corresponds to

\[
2\pi W^{-T} = \begin{bmatrix}
\frac{2rb}{\vartheta^2} & \frac{rb}{\vartheta} \\
0 & \frac{r}{\vartheta} (2 + \frac{1}{\vartheta})
\end{bmatrix},
\]

for which

\[
W = \begin{bmatrix}
\frac{\pi \vartheta^2}{rb} & 0 \\
\frac{rb}{\vartheta^2} & \frac{2\pi \vartheta^2}{rb (1 + 2\vartheta)}
\end{bmatrix},
\]
In order to make sure that $\mathcal{W}$ is of the form (3), we (re-)choose
\[
\mathcal{W} = \begin{bmatrix}
\frac{2\pi}{N_\varphi} & 0 \\
\pi/N_\psi & 2\pi/N_\psi
\end{bmatrix},
\]
with
\[
N_\varphi \geq \frac{2rb}{\varrho^2}, \quad N_\psi \geq \frac{rb(1 + 2\vartheta)}{\varrho^2}.
\]  \hfill (11)

We arrive at the interlaced sampling scheme $(\varphi_k, \psi_{km})$, defined by
\[
\varphi_k = \frac{k2\pi}{N_\varphi}, \quad \psi_{km} = \frac{\pi(k + 2m)}{N_\psi}.
\]

Let $l = k + 2m$ and $\alpha_{kl} = \psi_{lm}$, the sampling scheme takes the data points $(\varphi_k, \alpha_{kl})$ where
\[
\varphi_k = \frac{k2\pi}{N_\varphi} \quad \text{for} \quad 0 \leq k \leq N_\varphi - 1
\]
\[
\alpha_{kl} = \frac{\pi l}{N_\psi} \quad \text{for} \quad k, l \text{ having the same parity and } 0 \leq l < \frac{N_\psi \arcsin(1/r)}{\pi}.
\]

4.3. Further discussion

Let us elaborate the numbers of sampling points need for each sampling scheme. Taking the limit $\vartheta \to 1$, (10) becomes
\[
N_\varphi \geq 2rb, \quad N_\psi \geq 4rb,
\]
and (11) becomes
\[
N_\varphi \geq 2rb, \quad N_\psi \geq 3rb.
\]

Therefore, the number sampling points $M_0$ and $M_1$ needed for the standard and efficient schemes must satisfy $M_0 > \frac{4rb^2}{\varrho^2} \arcsin(1/r)$ and $M_1 > \frac{3rb^2}{\varrho^2} \arcsin(1/r)$, respectively. Roughly speaking, the efficient sampling scheme requires about 25% less number of sampling points compared to the standard scheme.
One can minimize the number of sampling points, for both schemes, by choosing the proper value for $r$. Namely, one can choose $r$ to be the minimum point of $r^2 \arcsin(1/r)$, which is approximately 1.088. However, one may not choose such a value of $r$ for technical reasons (for example, the object is too close to the gamma camera).

To illustrate the two sampling schemes, we draw them for the case $r = 1.5$ and $b = 8$ (figures 9 and 10).

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Data availability statement

No new data were created or analyzed in this study.

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