Casimir effect for parallel metallic plates in cosmic string spacetime

E R Bezerra de Mello¹, A A Saharian¹,² and A Kh Grigoryan³

¹ Departamento de Física, Universidade Federal da Paraíba 58.059-970, Caixa Postal 5.008, João Pessoa, PB, Brazil
² Department of Physics, Yerevan State University, 1 Alex Manoogian Street, 0025 Yerevan, Armenia
³ School of Physics and Mathematics, Yerevan State University, Azatutyan Ave, 0037 Yerevan, Armenia

E-mail: emello@fisica.ufpb.br and saharian@ysu.am

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Abstract
We evaluate the renormalized vacuum expectation values (VEVs) of electric and magnetic field squared and the energy–momentum tensor for the electromagnetic field in the geometry of two parallel conducting plates on the background of cosmic string spacetime. On the basis of these results, the Casimir–Polder force acting on a polarizable particle and the Casimir forces acting on the plates are investigated. The VEVs are decomposed into the pure string and plate-induced parts. The VEV of the electric field squared is negative for points with the radial distance to the string smaller than the distance to the plates, and positive for the opposite situation. On the other hand, the VEV for the magnetic field squared is negative everywhere. The boundary-induced part in the VEV of the energy–momentum tensor is different from zero in the region between the plates only. Moreover, this part only depends on the distance from the string. The boundary-induced part in the vacuum energy density is positive for points with a distance to the string smaller than the distance to the plates and negative in the opposite situation. The Casimir stresses on the plates depend non-monotonically on the distance from the string. We show that the Casimir forces acting on the plates are always attractive.

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(Some figures may appear in colour only in the online journal)

1. Introduction
The Casimir effect is a phenomenon common to all systems characterized by fluctuating quantities on which external boundary conditions are imposed (for a review see [1]). It is
among the most striking macroscopic manifestations of the nontrivial properties of the quantum vacuum. The boundary conditions imposed on a quantum field lead to the modification of the spectrum for zero-point fluctuations. As a result of this, the expectation values for physical quantities bilinear in the field are shifted. In particular, the confinement of quantum fluctuations causes forces that act on constraining boundaries. These forces depend on the nature of the quantum field, on the bulk and boundary geometries and on the specific boundary conditions on the field operator.

Among the most interesting topics in the investigations of the Casimir effect is the dependence of physical characteristics on the geometry of the background spacetime. Closed analytic expressions can be obtained for highly symmetric geometries only. In particular, motivated by Randall–Sundrum-type braneworld scenarios, investigations of the Casimir effect in anti-de Sitter spacetime have attracted a great deal of attention. The Casimir energy and corresponding Casimir forces for two parallel branes in anti-de Sitter spacetime have been evaluated in [2] by using either dimensional or zeta function regularization methods. Local Casimir densities were considered in [3]. The Casimir effect in higher dimensional generalizations of the anti-de Sitter spacetime with compact internal spaces has been investigated in [4]. Another popular background in gravitational physics is de Sitter spacetime. The Casimir densities for a massive scalar field with general curvature coupling parameter induced by flat and spherical boundaries on this background have been recently investigated in [5] and [6], respectively (see also [7] for the case of a conformally coupled massless scalar field). In this paper, we consider the Casimir effect for the electromagnetic field in the geometry of two parallel conducting plates in the background of cosmic string spacetime.

Cosmic strings are topologically stable defects which may have been created by phase transitions in the early Universe [8]. They are candidates for the generation of a variety of interesting physical effects, including gravitational lensing, anisotropies in the cosmic microwave background radiation, the generation of gravitational waves, high-energy cosmic rays and gamma ray bursts. Recently, cosmic strings attracted renewed interest related to the appearance of a variant of their formation mechanism within the framework of brane inflation [9]. In the simplest theoretical model describing the infinite straight cosmic string, the spacetime is locally flat except on the string. In quantum field theory, the corresponding nontrivial topology induces non-zero vacuum expectation values (VEVs) for physical observables. In this context, the VEVs of the energy–momentum tensor have been evaluated for scalar, fermionic and electromagnetic fields [10]–[27]. The analysis of the Casimir effect in the cosmic string spacetime has been developed for scalar [28], fermionic [29, 30] and electromagnetic fields [31, 32] for the geometry of a coaxial cylindrical boundary. The Casimir force for massless scalar fields subject to Dirichlet and Neumann boundary conditions in the setting of the conical piston has been recently investigated in [33]. The Casimir densities for a scalar field induced by a flat boundary perpendicular to the string have been considered in [34].

In this paper, we evaluate the VEVs of the electric and magnetic field squared and the energy–momentum tensor for two parallel conducting plates perpendicular to the cosmic string axis. These quantities are among the most important local characteristics of the electromagnetic vacuum. Although the corresponding operators are local, due to the global nature of the vacuum state, they contain important information about the topology of the background spacetime. In addition, the VEV of the energy–momentum tensor plays an important role in modelling self-consistent dynamics involving the gravitational field. It is worth calling attention to the fact that the renormalized VEV of the energy–momentum tensor for the electromagnetic field in the geometry of a cosmic string without boundaries is evaluated in [13, 14]. In the problem
under consideration, all calculations can be performed in a closed form and it constitutes an example in which the topological and boundary-induced polarizations of the vacuum can be separated into different contributions. Note that conical defects appear as an effective geometry in a number of condensed matter systems such as crystals, liquid crystals and quantum liquids (see, for example, [35]). In particular, linear defects in crystals named disclination can be dealt with using the same geometric approach as cosmic strings [36]. The results described this paper may shed light on the features of the Casimir forces between conducting parallel plates immersed in this type of medium with a conical defect. These results show that the defect can serve as an additional tool for the control of the forces.

The paper is organized as follows. In the next section, we consider the mode functions for the electric and magnetic fields in the region between two conducting plates. In section 3, these mode functions are used for the evaluation of the VEVs of the electric and magnetic field squared. By making use of the Abel–Plana summation formula, the latter are decomposed as the sum of boundary-free, single-plate- and second-plate-induced parts. The Casimir–Polder force on a polarizable particle is discussed as well. The VEV of the energy–momentum tensor and the Casimir forces acting on the plates are investigated in section 4. Finally, in section 5 the main results are summarized.

2. Mode functions for the electromagnetic field

In this investigation, we consider the effect of two parallel metallic plates on the quantum fluctuations of an electromagnetic field in the background of cosmic string spacetime. For an infinitely long straight cosmic string, the corresponding line element in cylindrical coordinates with the string along the $z$-axis is given by the expression

$$ds^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2,$$

where $0 \leq \phi \leq \phi_0$ and the spatial points $(r, \phi, z)$ and $(r, \phi + \phi_0, z)$ are to be identified. The planar angle deficit is related to the mass per unit length of the string, $\mu_0$, by $2\pi - \phi_0 = \frac{8\pi}{\mu_0}$, where $G$ is the Newton gravitational constant. We assume that the plates are orthogonal to the string and are located at $z = 0$ and $z = a$. In this paper, we are interested in the change of the VEVs of the electric and magnetic field squared and the energy–momentum tensor of the electromagnetic field induced by the plates.

The VEV for a physical quantity $f \{F_i, F_k\}$, bilinear in the electric ($F = E$) or magnetic ($F = B$) fields, can be written as the mode sum

$$\langle 0| f \{F_i, F_k\}|0 \rangle = \sum_\alpha f \{F_{\alpha i}, F^*_{\alpha k}\},$$

where $\{F_{\alpha i}, F^*_{\alpha k}\}$ represents a complete set of normalized mode functions, specified by a set of quantum numbers $\alpha$, and obeying the boundary conditions

$$n \times E_\alpha = 0, \quad n \cdot B_\alpha = 0, \quad z = 0, a,$$

where $n$ is the normal vector to the plates (directed along the $z$-axis). We will consider the VEVs in the region between the plates, $0 \leq z \leq a$. The VEVs for the regions $z \leq 0$ and $z \geq a$ are obtained as limiting cases.

We have two classes of mode functions corresponding to the waves of the transverse magnetic (TM) and transverse electric (TE) types. In the case of TM waves, the corresponding mode functions for the electric field, obeying the boundary conditions (3) on the plate at $z = 0$,
are given by the expressions

\[ E^{(0)}_{a1} = -\beta_0 k \gamma J_{q|m}(yr) \sin(kz) e^{i(qm\phi - \omega t)}, \]
\[ E^{(0)}_{a2} = -i\beta_0 k \frac{qm}{r} J_{q|m}(yr) \sin(kz) e^{i(qm\phi - \omega t)}, \]
\[ E^{(0)}_{a3} = \beta_0 \gamma^2 J_{q|m}(yr) \cos(kz) e^{i(qm\phi - \omega t)}, \]

where \( J_r(x) \) is the Bessel function, \( 0 < \gamma < \infty \), and

\[ q = 2\pi / \phi_0, \quad \omega^2 = \gamma^2 + k^2, \quad m = 0, \pm 1, \pm 2, \ldots \]

In (4), \( E^{(0)}_{a1} \) is the \( l \)th physical component of the electric field vector in cylindrical coordinates and the values \( l = 1, 2, 3 \) correspond to the \( r, \phi, z \) coordinates, respectively. For the mode functions corresponding to the magnetic field, we find

\[ B^{(0)}_{a1} = \beta_0 \omega q \frac{qm}{r} J_{q|m}(yr) \cos(kz) e^{i(qm\phi - \omega t)}, \]
\[ B^{(0)}_{a2} = i\beta_0 \omega \gamma J_{q|m}(yr) \cos(kz) e^{i(qm\phi - \omega t)}, \]
\[ B^{(0)}_{a3} = 0. \]

The eigenvalues for \( k \) are quantized by the boundary conditions (3) on the plate at \( z = a \):

\[ k = k_n = \frac{\pi n}{a}, \quad n = 0, 1, 2, \ldots \]

In the case of the TE waves, the mode functions have the form

\[ E^{(1)}_{a1} = -\beta_0 \omega q \frac{qm}{r} J_{q|m}(yr) \sin(kz) e^{i(qm\phi - \omega t)}, \]
\[ E^{(1)}_{a2} = -i\beta_0 \omega \gamma J_{q|m}(yr) \sin(kz) e^{i(qm\phi - \omega t)}, \]
\[ E^{(1)}_{a3} = 0, \]

for the electric field and

\[ B^{(1)}_{a1} = \beta_0 k \gamma J_{q|m}(yr) \cos(kz) e^{i(qm\phi - \omega t)}, \]
\[ B^{(1)}_{a2} = i\beta_0 k \frac{qm}{r} J_{q|m}(yr) \cos(kz) e^{i(qm\phi - \omega t)}, \]
\[ B^{(1)}_{a3} = \beta_0 \gamma^2 J_{q|m}(yr) \sin(kz) e^{i(qm\phi - \omega t)}, \]

for the magnetic field with the same notations as in (4). Now for the eigenvalues of \( k \), we have \( k = k_n = \pi n / a, with n = 1, 2, \ldots \). As is seen from the formulas for the mode functions, they are specified by the set \( \alpha = (\lambda, \gamma, m, n) \), where \( \lambda = 0, 1 \) correspond to the TM and TE waves, respectively.

The mode functions are normalized by the condition

\[ \int_0^\infty dr \int_0^{\phi_0} d\phi \int_0^a dz E^{(*)}_{a\lambda} \cdot E^{(*)\ast}_{a\lambda} = 2\pi \omega \delta_{\alpha \alpha'}, \]

where \( \delta_{\alpha \alpha'} \) is understood as the Dirac delta function for continuous components of the collective index \( \alpha \) and as the Kronecker delta for discrete ones. Substituting the expressions for the mode functions for the electric field, it can be seen that the normalization coefficient is given by the expression

\[ \beta^2 = 2q(1 - \delta_{\alpha 0})/\omega ya, \]

for both TM and TE modes.
3. VEVs of the electric and magnetic field squared

In this section, we consider the VEVs of the electric and magnetic field squared for the physical situation specified in the previous section. First, we consider the region between the plates, \(0 \leq z \leq a\). Substituting the mode functions into the corresponding mode-sum formula (2) for these VEVs, we find

\[
\langle 0 | F_{a}^{2} | 0 \rangle = \sum_{a} \sum_{n} \frac{f_{n}^{(a)} \cdot f_{n}^{(a)*}}{a} = \frac{4q}{a} \sum_{m=0}^{\infty} \int_{0}^{\infty} dy \frac{y}{\omega} \times \left[ G_{qm}(y) \sum_{n=1}^{\infty} \left( 2\omega_{n}^{2} + \gamma^{2} \right) f_{1}^{(F)}(k_{a}z) + \gamma^{2} J_{qm}^{2}(y) \sum_{m=0}^{\infty} f_{2}^{(F)}(k_{a}z) \right],
\]

where \(F = E\) and \(F = B\) are for the electric and magnetic fields, respectively. In (12), we have defined the functions

\[ f_{1}^{(E)}(x) = f_{2}^{(B)}(x) = \sin^{2} x, \]

\[ f_{2}^{(E)}(x) = f_{1}^{(B)}(x) = \cos^{2} x, \]

and

\[ G_{qm}(x) = J_{qm}^{2}(x) + (qm/x)^{2} J_{qm}^{2}(x). \]

Of course, expressions (12) are divergent. We assume that a cutoff function is introduced to make them convergent without explicitly writing it. The special form of this function will not be important in the following discussion.

For the evaluation of the sum over \(n\), we apply the Abel–Plana summation formula, written in the form (see, for example, [37])

\[
\frac{\pi}{a} \sum_{n=0}^{\infty} f(\pi n/a) = \int_{0}^{\infty} dx f(x) + i \int_{0}^{\infty} dx \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1},
\]

where the prime on the summation sign means that the term with \(n = 0\) should be taken with the coefficient 1/2. This leads to the following representation of the VEVs:

\[
\langle 0 | F_{a}^{2} | 0 \rangle = \langle F_{a}^{2} \rangle_{1} + \langle F_{a}^{2} \rangle_{2},
\]

where the first and second terms on the right-hand side correspond to the first and second integrals, respectively, in (15). For these separate terms, we obtain the expressions

\[
\langle F_{a}^{2} \rangle_{1} = \frac{4q}{\pi} \sum_{m=0}^{\infty} \int_{0}^{\infty} dk \int_{0}^{\infty} dy \frac{y}{\omega} \times \left[ G_{qm}(y) \omega^{2} f_{1}^{(F)}(kz) + \gamma^{2} J_{qm}^{2}(y) f_{2}^{(F)}(kz) \right],
\]

and

\[
\langle F_{a}^{2} \rangle_{2} = \frac{8q}{\pi} \sum_{m=0}^{\infty} \int_{0}^{\infty} dy \gamma \int_{\gamma}^{\infty} dx \frac{(x^{2} - \gamma^{2})^{1/2}}{e^{2\pi x} - 1} \times \left[ G_{qm}(y) \omega^{2} \gamma^{2} f_{1}^{(F)}(xz) + \gamma^{2} J_{qm}^{2}(y) \gamma^{2} f_{2}^{(F)}(xz) \right],
\]

with the functions

\[ g_{1}^{(E)}(x) = -g_{2}^{(B)}(x) = \sinh^{2} x, \]

\[ g_{2}^{(E)}(x) = -g_{1}^{(B)}(x) = \cosh^{2} x. \]

The term \(\langle F_{a}^{2} \rangle_{1}\) does not depend on the distance between the plates, whereas the term \(\langle F_{a}^{2} \rangle_{2}\) vanishes in the limit \(a \to \infty\). From here it follows that the part \(\langle F_{a}^{2} \rangle_{1}\) corresponds to the VEV
in the geometry of a single plate located at \( z = 0 \) when the second plate is absent. The part \( \langle F^2 \rangle_2 \) is induced by the second plate at \( z = a \).

The single-plate parts can be further decomposed to
\[
\langle F^2 \rangle_1 = \langle F^2 \rangle^{(s)} + \langle F^2 \rangle^{(b)}_1,
\]
where
\[
\langle E^2 \rangle^{(s)} = \langle B^2 \rangle^{(s)} = \frac{2q}{\pi} \sum_{n=0}^{\infty} \int_0^\infty \frac{dk}{k^2} \int_0^\infty d\gamma \times \frac{\gamma}{\omega} \left[ G_{qm}(\gamma r)(2k^2 + \gamma^2) + \gamma^2 J^2_{qm}(\gamma r) \right],
\]
is the corresponding VEV for a boundary-free string geometry. The terms
\[
\langle E^2 \rangle^{(b)}_1 = -\langle B^2 \rangle^{(b)}_1 = -\frac{2q}{\pi} \sum_{n=0}^{\infty} \int_0^\infty \frac{dk}{k^2} \cos(2kz)
\times \int_0^\infty d\gamma \frac{\gamma}{\omega} \left[ G_{qm}(\gamma r)(2k^2 + \gamma^2) - \gamma^2 J^2_{qm}(\gamma r) \right]
\]
are the parts induced by the presence of a single plate at \( z = 0 \). Note that the latter is finite for points away from the plate and the renormalization is necessary for the boundary-free part only. The renormalized boundary-free part is given by the following simple expression [32]:
\[
\langle E^2 \rangle_1^{(b)}_{\text{ren}} = \langle B^2 \rangle_1^{(b)}_{\text{ren}} = -\left( \frac{q^2 - 1}{(q^2 + 1)} \right) \frac{180\pi r^4}{\omega q^2}.
\]
The corresponding VEV is negative.

After the integration over \( k \), the single-plate-induced part is presented in the form
\[
\langle E^2 \rangle^{(b)}_1 = -\langle B^2 \rangle^{(b)}_1 = \frac{2q}{\pi} \sum_{n=0}^{\infty} \int_0^\infty du u^2 \left[ J^2_{qm}(ur/z)K_0(2u) + G_{qm}(ur/z)Q(2u) \right],
\]
with the notation
\[
Q(x) = K_0(x) + 2K_1(x)/x,
\]
where \( K_0(x) \) is the modified Bessel function. As we can see, the boundary-induced part is positive for the electric field and negative for the magnetic field. For a metallic plate in the background of Minkowski spacetime, one has \( q = 1 \) and the summation over \( m \) in (24) can be explicitly done. After the integration over \( u \), we obtain the well-known result \( \langle E^2 \rangle_1^{(b)} \mid_{q=1} = 3/(4\pi z^4) \). For \( r \ll z \), we use the asymptotic expression for the Bessel function for small values of the argument [39]. In this way, to the leading order one finds \( \langle E^2 \rangle_1^{(b)} \approx q/(4\pi z^4) \) for \( q > 1 \). Note that the next-to-leading order term is of the order \( (r/z)^{2q-2} \). In this regime, the total VEV of the electric field squared is dominated by the boundary-free part and it is negative. In the opposite limiting case, \( r \gg z \), to the leading order the boundary-induced VEV coincides with the corresponding quantity in Minkowski bulk. In this limit, the total VEV is dominated by the boundary-induced part and it is positive for the electric field. For the magnetic field, the total VEV is negative everywhere.

For the special case with integer values of the parameter \( q \), the summation over \( m \) in (24) can be done by using the formulas [15, 32, 38]
\[
\sum_{m=0}^{\infty} J^2_{qm}(y) = \frac{1}{2q} \sum_{l=0}^{q-1} J_0(2ys_l),
\]
\[
\sum_{m=0}^{\infty} G_{qm}(y) = \frac{1}{2q} \sum_{l=0}^{q-1} \cos(2\pi l/q)J_0(2ys_l).
\]
with

\[ s_l = \sin(\pi l/q). \]  

(27)

In this case, for the single-plate-induced parts, we find

\[ \langle E^2 \rangle_{1}^{(b)} = \frac{1}{4\pi} \sum_{l=0}^{q-1} \frac{(3 - 4s_l^2) z^2 - r^2 s_l^2}{(z^2 + r^2 s_l^2)^{3/2}}. \]

(28)

The \( l = 0 \) term in this expression coincides with the corresponding VEV for a plate in the background of Minkowski spacetime: \( \langle E^2 \rangle_{1}^{(b)} \big|_{q=1} = 3/(4\pi z^4) \). In figure 1, we plot the VEVs for the electric (full curves) and magnetic (dashed curves) field squared in the geometry of a single conducting plate located at \( z = 0 \) as a function of the ratio \( r/z \). The numbers near the curves are the corresponding values of the parameter \( q \). The dot-dashed curves correspond to the boundary-induced part in the VEV of the field squared, \( z^4 \langle E^2 \rangle_{1}^{(b)} \). Note that, in general, the boundary-induced part \( \langle E^2 \rangle_{1}^{(b)} \) is a non-monotonic function of \( r \). The corresponding VEVs for a plate in Minkowski spacetime would be presented by the horizontal lines \( 3/(4\pi) \) and \( -3/(4\pi) \) for the electric and magnetic fields, respectively.

Now, we turn to the second-plate-induced parts given by (18). By using the expansion \( (e^y - 1)^{-1} = \sum_{n=1}^{\infty} e^{-ny} \), they can be presented in the form

\[ \langle E^2 \rangle_2 = \frac{1}{a^4} C(r/a) + \frac{1}{a^4} \sum_{j=\pm 1} D(r/a, jz/a), \]

\[ \langle B^2 \rangle_2 = \frac{1}{a^4} C(r/a) - \frac{1}{a^4} \sum_{j=\pm 1} D(r/a, jz/a), \]

(29)
with the functions
\[
C(x) = -\frac{4q}{\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_0^\infty dy \gamma^3 \left[ G_{qm}(\gamma x)Q(2n\gamma) - J_{qm}^2(\gamma x)K_0(2n\gamma) \right],
\]
\[
D(x, y) = \frac{2q}{\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_0^\infty dy \gamma^3 \left[ G_{qm}(\gamma x)Q(2(n - y)\gamma) + J_{qm}^2(\gamma x)K_0(2(n - y)\gamma) \right].
\]

Note that the \(z\)-dependent parts for the VEVs of the electric and magnetic fields have opposite signs. In particular, they are cancelled in the corresponding energy density (see the next section). The quantity \(\langle E^2 \rangle_z\) is finite on the plate at \(z = 0\) and diverges on the plate at \(z = a\). The divergence comes from the \(n = 1\) term in the expression for the function \(D(x, y)\). Note that the contribution of this term to the VEV of the electric field squared coincides with (24) with the replacement \(z \to a - z\). Hence, it presents the VEV induced by the plate at \(z = a\) when the plate at \(z = 0\) is absent. Combining the results for the single-plate- and second-plate-induced parts, the total VEV is presented in the form
\[
\langle F^2 \rangle = \langle F^2 \rangle_{\text{ren}}^{(n)} = \frac{4q}{\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_0^\infty dy \gamma^3 \left[ G_{qm}(\gamma r)Q(2na\gamma) - J_{qm}^2(\gamma r)K_0(2na\gamma) \right]
\]
\[+ \delta_1 \frac{2q}{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \int_0^\infty dy \gamma^3 \times \left[ G_{qm}(\gamma r)Q(2(n - a)\gamma) + J_{qm}^2(\gamma r)K_0(2(n - a)\gamma) \right],
\]
where \(\delta_1 = 1\) and \(\delta_2 = -1\). The single-plate parts in this expression are presented by \(n = 0\) and \(n = 1\) terms in the last summation on the right-hand side. The VEV is not changed under the replacement \(z \to a - z\), which is a direct consequence of the problem symmetry with respect to the plane \(z = a/2\).

For integer values of the parameter \(q\), by using formulas (26), one finds
\[
C(x) = -\frac{1}{2\pi} \sum_{l=0}^{q-1} \sum_{n=1}^{\infty} \frac{(1 - 4s_l^2)n^2 + x^2s_l^2}{(n^2 + x^2s_l^2)^3},
\]
\[
D(x, y) = \frac{1}{4\pi} \sum_{l=0}^{q-1} \sum_{n=1}^{\infty} \frac{(3 - 4s_l^2)(n - y)^2 - x^2s_l^2}{[(n - y)^2 + x^2s_l^2]^3}.
\]

After the summation over \(n\), the function \(C(x)\) can also be presented in the form
\[
C(x) = -\frac{1}{2\pi} \sum_{l=0}^{q-1} \left[ (1 - 4s_l^2)h_2(xs_l) + 4s_l^2h_3(xs_l) \right],
\]
with the notations
\[
h_2(b) = \sum_{n=1}^{\infty} \frac{1}{(n^2 + b^2)^3} = -\frac{1}{2b^2} + \frac{\pi}{4b^3} \left[ \coth(\pi b) + \frac{\pi b}{\sinh^2(\pi b)} \right],
\]
\[
h_3(b) = \sum_{n=1}^{\infty} \frac{b^2}{(n^2 + b^2)^3} = -\frac{1}{2b^2} + \frac{\pi}{16b^3} \left[ 3\coth(\pi b) + \frac{3\pi b}{\sinh^2(\pi b)} + \frac{2\pi b^2}{\sinh^2(\pi b)} \coth(\pi b) \right].
\]
Note that \(h_2(0) = \pi^4/90\). For integer \(q\), the expression of the total VEV takes the form
In figure 2, we display the dependence of the electric (left plot) and magnetic (right plot) field squared on \( r/a \) and \( z/a \) for the VEVs in the region between the plates for a cosmic string with \( q = 2 \). For the VEV of the electric field squared, the boundary-induced part dominates near the plates and it is positive in this region. Near the string, the VEV is dominated by the boundary-free part and it is negative. The VEV of the magnetic field squared is negative everywhere.

In the region \( z < 0 \), the VEVs for the field squared are given by (20), where the plate-induced part is still given by expression (22). The same decomposition (20) is valid for the region \( z > a \), with the difference that now the single-plate part is given by expression (22) with the replacement \( z \rightarrow z - a \).

The VEV of the electric field squared determines the Casimir–Polder force acting on a polarizable particle. In the static limit, when the dispersion of polarizability can be neglected, the corresponding potential is given by the expression

\[
U(r, z) = -\frac{1}{2} \alpha_0 \langle E^2 \rangle,
\]

where \( \alpha_0 \) is the static polarizability. The Casimir–Polder force in the boundary-free cosmic string geometry has been recently investigated in [40] for the general case of anisotropic polarizability with dispersion. In the isotropic case, the corresponding force is repulsive with respect to the cosmic string. From (24) it can be easily seen that \( \partial_z \langle E^2 \rangle^{(b)} < 0 \), and hence, the \( z \)-component of the Casimir–Polder force in the geometry of a single plate is attractive with respect to the plate. In the region between two plates, this component vanishes at \( z = a/2 \) and it is attractive to the nearest plate for \( z \neq a/2 \). In the presence of the plates, the radial component of the total Casimir–Polder force remains repulsive with respect to the string, though the boundary-induced part separately can be either repulsive or attractive.
4. Energy–momentum tensor and the Casimir force

In this section, we consider the VEV of the energy–momentum tensor for the geometry of two parallel metallic plates. This VEV is obtained from the general formula (2) considering the standard expression for the energy–momentum tensor of the electromagnetic field. Substituting the mode functions for the electric and magnetic fields, given in section 2, in a way similar to that for the field squared, it can be seen that the single-plate parts in the VEV of the energy–momentum tensor vanish, \((T_{\mu\nu})^{(1)} = 0\). In the region between the plates, the total VEV is presented in the decomposed form

\[
\langle T_{\mu\nu} \rangle = \langle T_{\mu\nu} \rangle^{(s)} + \langle T_{\mu\nu} \rangle^{(b)},
\]

where the renormalized energy–momentum tensor for the pure string part is given by the expression [13, 14]

\[
\langle T_{\mu\nu}^{(s)} \rangle = -\frac{(q_1^2 - 1)(q_1^2 + 11)}{720 \pi^2 r^4} \text{diag}(1.1, -3.1).
\]

The energy density in (38) can be directly obtained from the expressions of the electric and magnetic field squared. Providing the energy density, the spatial components are obtained by using the zero trace condition, \(\langle T_{\mu\nu}^{(s)} \rangle = 0\), and the covariant continuity equation, \(\nabla_{\mu}\langle T_{\mu\nu}^{(s)} \rangle = 0\). For the boundary-induced parts in the separate components, we obtain the following expressions:

\[
\langle T_0^{(b)} \rangle = -\frac{q}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_0^\infty d\gamma \gamma^3 \left[ G_{qm}(\gamma r) Q(2any) - J_{qm}(\gamma r) K_0(2any) \right],
\]

\[
\langle T_1^{(b)} \rangle = -\frac{q}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_0^\infty d\gamma \gamma^3 \left\{ J_{qm}(\gamma r) + \left[ 1 - \frac{qm}{\gamma r} \right]^2 \right\} J_{qm}(\gamma r) K_0(2any),
\]

\[
\langle T_2^{(b)} \rangle = \frac{q}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_0^\infty d\gamma \gamma^3 \left\{ J_{qm}(\gamma r) - \left[ 1 + \frac{qm}{\gamma r} \right]^2 \right\} J_{qm}(\gamma r) K_0(2any),
\]

\[
\langle T_3^{(b)} \rangle = \frac{q}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_0^\infty d\gamma \gamma^3 \left[ G_{qm}(\gamma r) Q(2any) + J_{qm}^2(\gamma r) K_0(2any) \right],
\]

and the off-diagonal components vanish. It is easily checked that the boundary-induced part is traceless. Moreover, it does not depend on the \(z\)-coordinate.

Note that, by using the relation

\[
J_{qm}^2(\gamma r) + \left[ 1 - \frac{qm}{\gamma r} \right]^2 \right\} J_{qm}(\gamma r) = \frac{2}{r} \int_0^r dx x J_{qm}^2(\gamma x),
\]

the expression for the radial stress may also be written in the form

\[
\langle T_1^{(b)} \rangle = -\frac{2q}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_0^\infty d\gamma \gamma^3 K_0(2any) \int_0^r dx x J_{qm}^2(\gamma x),
\]

from which it follows that the boundary-induced part in the vacuum pressure along the radial direction, \(P_{1}^{(b)} = -\langle T_1^{(b)} \rangle\), is always positive. The same is the case for the boundary-free part. Note that, from (39) we also have \(\langle T_1^{(b)} \rangle + \langle T_2^{(b)} \rangle \leq 0\). It can be explicitly checked that the boundary-induced parts obey the covariant continuity equation, \(\nabla_{\mu}\langle T_{\mu\nu}^{(b)} \rangle = 0\), for which the geometry under consideration is reduced to the single equation \(\partial_r (r \langle T_1^{(b)} \rangle) = \langle T_2^{(b)} \rangle\).
Let us consider the behaviour of the boundary-induced part in the VEV of the energy-momentum tensor near the string assuming that \( r \ll a \). Assuming that \( q > 1 \) and by using the formula for the Bessel function for small arguments to the leading order, we find

\[
\langle T_{00}^{(b)} \rangle (b) \approx \langle T_{33}^{(b)} \rangle (b) \approx - \langle T_{11}^{(b)} \rangle (b) \approx - \langle T_{22}^{(b)} \rangle (b) \approx \frac{q\pi^2}{720a^4}.
\]

In particular, we see that near the string, the boundary-induced part in the energy density is positive. Note that in this region the total VEV is dominated by the boundary-free part. At large distances from the string, \( r \gg a \), the effects induced by the nontrivial topology of the string are small and, to the leading order, the VEV coincides with the corresponding expression for metallic plates in Minkowski spacetime [41]:

\[
\langle T_\mu^\nu \rangle \approx \langle T_\mu^\nu \rangle (b) \approx \langle T_\mu^\nu \rangle_M = - \frac{\pi^2}{720a^4} \text{ diag}(1, 1, 1, -3).
\]

In this region, both the boundary-free and the boundary-induced parts in the vacuum energy density are negative.

For integer values of \( q \), by using formulas (26), one finds the following expressions (no summation over \( \mu \)):

\[
\langle T_{\mu\mu}^{(b)} \rangle = - \frac{a^{-4}}{8\pi^2} \sum_{l=0}^{q-1} [f_{\mu,2}(s_l)h_2(rs_l/a) + f_{\mu,3}(s_l)h_3(rs_l/a)],
\]

where

\[
f_{0,2}(x) = 1 - 4x^2, \quad f_{0,3}(x) = 4x^2, \quad f_{1,2}(x) = 1, \quad f_{1,3}(x) = 0, \quad f_{2,2}(x) = 1, \quad f_{2,3}(x) = -4, \quad f_{3,2}(x) = 4x^2 - 3, \quad f_{3,3}(x) = 4 - 4x^2.
\]

The \( l = 0 \) terms in (44) coincide with the corresponding quantities for parallel plates in Minkowski spacetime (see (43)). In figure 3, we present the ratio of the boundary-induced part of the energy density to the corresponding quantity for parallel plates in Minkowski spacetime as a function of the distance from the string. The numbers near the curves are the values of the parameter \( q \).
Figure 4. The ratio of the Casimir stress on the plate to the corresponding quantity in Minkowski spacetime as a function of the distance $r/a$ from the string for separate values of the parameter $q$.

In the regions $z < 0$ and $z > a$, the boundary-induced part in the VEV of the energy–momentum tensor vanishes and in these regions $\langle T_{\mu\nu} \rangle = \langle T_{\mu\nu}^{(s)} \rangle$.

In the discussion above, we have considered an idealized model for a cosmic string with the core of zero thickness. The corresponding classical energy–momentum tensor has the form

$$T_{\nu}^{\mu} = \delta_{\nu}^{\mu} \delta(x) \delta(y) \text{diag}(1, 0, 0, 1),$$

with $x$ and $y$ being the Cartesian coordinates in the plane perpendicular to the string. This tensor is located on the string, whereas the vacuum energy–momentum tensor $\langle T_{\mu\nu} \rangle$ is distributed over all the space. In a realistic point of view, the cosmic strings have a characteristic core radius, $r_0$, determined by the energy scale where the gauge symmetry of the system is spontaneously broken. At distances $r \gg r_0$, the geometry of a realistic cosmic string is well approximated by the line element (1) and the VEV of the energy–momentum tensor for the electromagnetic field in the presence of conducting plates is given by expression (37). For the energy density inside the string core, one has $\langle T_{0}^{0} \rangle \sim \mu_0/(\pi r_0^2)$. Assuming that $a \gg r_0$, for points near the string core, $r \gtrsim r_0$, the vacuum energy density is dominated by the pure string part given by (38). In this case for the ratio of the energy densities, we obtain (in standard units)

$$\frac{|\langle T_{0}^{0} \rangle|}{\langle T_{0}^{0} \rangle^{(s)}} \sim 10^{-2} (q - 1) \frac{\hbar}{(c \mu_0 r_0^2)}.$$  

For GUT scale cosmic strings $\mu_0 \sim 10^{22}$ g cm$^{-1}$, $r_0 \sim 10^{-29}$ cm and $|\langle T_{0}^{0} \rangle|/\langle T_{0}^{0} \rangle^{(s)} \ll 1$. At distances $r \gtrsim a$, the vacuum energy density is dominated by the boundary-induced part.

The Casimir force acting per unit surface of the plate is determined by the component $\langle T_{3}^{3} \rangle$. For the corresponding effective pressure, one has $p_3 = -\langle T_{3}^{3} \rangle$. The boundary free part of the pressure is the same for both sides of the plate, and it does not contribute to the net force. Hence, the force per unit surface of the plate is determined by the boundary-induced part of the pressure along the $z$-direction:

$$p_3 = \frac{q}{\pi^2 a^2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \int_0^{\infty} d\gamma \gamma^3 \left[ G_{\gamma}(\gamma r/a)Q(2n\gamma) + J_{\gamma}(\gamma r/a)K_0(2n\gamma) \right].$$  

Unlike the case of Minkowski bulk, the Casimir stress on the plates is not uniform. The effective pressure (46) is always negative, and hence, the corresponding Casimir force is always attractive. For integer values of the parameter $q$, we have

$$p_3 = -\frac{a - q}{8\pi^2} \sum_{l=0}^{q-1} \left[ (3 - 4s_l^2)h_2(rs_l/a) - 4(1 - s_l^2)h_3(rs_l/a) \right].$$  

(47)
with \( s_l \) being defined by (27). The \( l = 0 \) term in this expression coincides with the corresponding quantity for plates in Minkowski spacetime: \( p_{M,3} = -\pi^2 a^{-4}/240 \). In figure 4, we plot the ratio \( p_M^{(b)}/p_{M,3} \) as a function of the distance from the string (in units of the separation between the plates) for separate values of the parameter \( q \) (numbers near the curves).

5. Conclusion

In this paper, we have investigated the combined effects from nontrivial topology and boundaries on properties of the electromagnetic vacuum. The nontrivial topology of the spacetime is induced by the presence of the cosmic string, and for the boundaries we have considered the classical Casimir geometry with two parallel conducting plates. Among the most important characteristics of the vacuum state are the VEVs of the electric and magnetic field squared and the energy–momentum tensor. In order to evaluate these VEVs, we have employed the direct mode summation technique. By applying the Abel–Plana summation formula to the mode sum over the eigenvalues of the wave vector component along the cosmic string, we have explicitly decomposed the VEVs into the pure string and boundary-induced parts. The pure string parts in the VEVs of the electric and magnetic field squared coincide and are negative everywhere. The presence of the cosmic string increases the boundary-induced parts in the VEVs of the field squared when compared with the Minkowski spacetime results.

The boundary part in the VEVs of the field squared is further split into the single-plate- and second-plate-induced parts. Single-plate parts are given by expression (24) and they have opposite signs for the electric and magnetic fields. For points near the string, \( r \ll z \), the VEV of the electric field squared is dominated by the pure string part and it is negative. Near the plate, \( z \ll r \), the plate-induced part dominates and the VEV is positive. The VEV of the magnetic field squared is negative everywhere. The second-plate-induced parts are presented in the form (29) with the functions defined in (30). The \( z \)-dependent parts in the expressions for the electric and magnetic field squared have opposite signs and they are cancelled in the expression for the vacuum energy density. The general formulas are simplified in a special case of integer values of the parameter \( q \). The corresponding expressions take the form (28) and (32). The VEV of the electric field squared determines the Casimir–Polder force on a polarizable particle. In the geometry of a single plate, the force is attractive with respect to the plate and repulsive with respect to the string. In the region between two plates, the Casimir–Polder force is attractive to the nearest plate and remains repulsive with respect to the string.

The boundary-induced part in the VEV of the energy–momentum tensor for the electromagnetic field is different from zero in the region between the plates only. The expressions for the separate components are given by formulas (39). The VEVs are uniform with respect to the coordinate along the cosmic string and depend on the distance from the string only. The boundary-induced part in the vacuum energy density is positive near the string and negative at large distances from the string. The radial effective pressure is always positive. The general formulas for the VEV of the energy–momentum tensor are simplified for integer values of the parameter \( q \). The corresponding expressions are given by (44). Unlike the geometry of two conducting plates in Minkowski spacetime, in the geometry with cosmic string the Casimir stresses on the plates are non-uniform. They depend on the distance from the string, and this dependence is not monotonic. The Casimir forces acting on the plates are always attractive. They are given by (46) for the general case and by (47) for integer values of the parameter \( q \).
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