Independent Sets and Hitting Sets of Bicolored Rectangular Families

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Received: 24 August 2019 / Accepted: 27 January 2021 / Published online: 17 February 2021
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Abstract
A bicolored rectangular family BRF is the collection of all axis-parallel rectangles formed by selecting a bottom-left corner from a finite set of points $A$ and an upper-right corner from a finite set of points $B$. We devise a combinatorial algorithm to compute the maximum independent set and the minimum hitting set of a BRF that runs in $O(n^{2.5}\sqrt{\log n})$-time, where $n = |A| + |B|$. This result significantly reduces the gap between the $\Omega(n^7)$-time algorithm by Benczúr (Discrete Appl Math 129 (2–3):233–262, 2003) for the more general problem of finding directed covers of pairs of sets, and the $O(n^2)$-time algorithms of Franzblau and Kleitman (Inf Control 63(3):164–189, 1984) and Knuth (ACM J Exp Algorithm 1:1, 1996) for BRFs where the points of $A$ lie on an anti-diagonal line. Furthermore, when the bicolored rectangular family is weighted, we show that the problem of finding the maximum weight of an independent set is $\mathsf{NP}$-hard, and provide efficient algorithms to solve it on important subclasses.

Keywords Independent set · Hitting set · Axis-parallel rectangles · Jump number

A short abstract of this work appeared in IPCO 2011 under the name “Jump Number of Two-Directional Orthogonal Ray Graphs” [38]

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1 Introduction

Suppose we are given a collection of axis-parallel closed rectangles in the plane. A subcollection of rectangles that do not pairwise intersect is called an independent set, and a collection of points in the plane intersecting (hitting) every rectangle is called a hitting set. In this paper we study the problems of finding a maximum independent set (MIS) and a minimum hitting set (MHS) of the following collection of rectangles built from bicolored point-sets: given two finite sets \( A, B \subseteq \mathbb{R}^2 \) the bicolor rectangle family (BRF) induced by \( A \) and \( B \) is the set \( \mathcal{R} = \mathcal{R}(A,B) \) of rectangles having bottom-left corner in \( A \) and top-right corner in \( B \). Our results also apply if we are given an arbitrary set \( Z \) in the plane, and we consider the restricted BRF (rBRF) \( \mathcal{R}(A, B, Z) \) consisting of those rectangles in \( \mathcal{R}(A,B) \) that are completely contained in \( Z \).

Our main result is an algorithm that finds a MIS and a MHS of an rBRF \( \mathcal{R}(A, B, Z) \) in \( O(n^{2.5}) \)-time, where \( n = |A| + |B| \). To emphasize the relevance of this result, we need to provide some context. Frank and Jordán [23] gave an intriguing and deep min–max result for covering integer-valued crossing bisupermodular functions over set-pairs that has many consequences for connectivity augmentation problems. As stated in their paper, Frank and Jordán’s theorem (for \( \{0, 1\} \)-valued crossing bisupermodular functions) is a generalization of a beautiful min–max result for point-interval sets shown by Győri [25] eleven years earlier. Győri’s result can be interpreted geometrically in our BRF framework: it states that the MIS and MHS of any BRF \( \mathcal{R}(A,B) \) where points in \( A \) are on the antidiagonal line \( y = -x \) (these are called convex bipartite BRFs) have the same cardinality.

Even though the min–max relation for BRF is, by itself, extremely interesting, we are interested in algorithms to efficiently compute a MIS and a MHS. Frank and Jordán’s difficult result (to grasp the level of difficulty, we note that their result implies Edmonds’ matroid partition theorem) is algorithmic but heavily relies on the ellipsoid method to work in full generality. Benczúr [6] gave a combinatorial pushdown-reduce algorithm for Frank and Jordán’s result in the case of \( \{0, 1\} \) valued functions, which was later generalized to a pseudopolynomial primal-dual algorithm for general valued functions by Benczúr and Végh [40]. The running time of these algorithms, when translated to the BRFs setting are not very practical: if \( n = |A \cup B| \) then these algorithms run in \( \tilde{O}(n^7) \) time.

On the other hand, Győri’s original result was nonconstructive. It was later made algorithmic by Franzblau and Kleitman [24] and a full implementation was given by Knuth [28]. Using BRF ’s notation, their algorithm can compute the MIS and MHS of convex bipartite BRFs in \( O(n^2) \) time. Later, Frank [21] gave a conceptually different algorithm for this result, which, after improvements by Benczúr [5] can be implemented in \( \tilde{O}(n^{2.5}) \). Our work combines Frank’s algorithmic ideas with the geometric framework of rBRFs, greatly reducing the gap between the much simpler algorithms, results and proofs available for point-interval sets (Győri’s result) and the more complex ones available for set-pairs (Frank and Jordán’s result).

A long list of results on seemingly unrelated problems can be unified throughout the geometric prism of BRFs. Most of these connections are non-trivial and
have not been observed before this work. To keep the paper focused on the algorithms, we defer most of this discussion to the appendix, except for a practical application of our results to a scheduling problem known as the jump number problem.

1.1 Main Results

Our main result (Theorem 3) is an algorithm that finds both an independent set \( I \) and a hitting set \( H \) of any rBRF \( \mathcal{R} = \mathcal{R}(A, B, Z) \) with \( |I| = |H| \). By weak linear programming duality, \( I \) and \( H \) are a MIS and a MHS of \( \mathcal{R} \), respectively. In particular, this algorithm provides a new proof for the min–max relation between these two quantities.

Our algorithm is efficient: if \( n \) is the number of points in \( A \cup B \) then both the MIS and the MHS of \( \mathcal{R} \) can be computed deterministically in \( O(n^{2.5} \sqrt{\log n}) \) time. If we allow randomization, we can compute them with high probability in time \( O((n \log n)^\omega) \), where \( \omega < 2.3728639 \) is the exponent for square matrix multiplication (Theorem 5). We implicitly assume that testing if a rectangle is contained in \( Z \) can be done in unit time; otherwise, we need an additional \( O(n^2 T(Z)) \) time, where \( T(Z) \) is the time for testing membership in \( Z \). The bottleneck of our algorithm is the computation of a maximum matching on a bipartite graph needed for an algorithmic version of the classic theorem of Dilworth.

As an additional result, we study a natural linear programming relaxation for the MIS of rBRFs that can be used to compute the optimal size of the MIS. Even though the feasible region is not an integral polytope, we prove, by an uncrossing argument, that the optimal face of that polytope has an integral vertex (Theorem 1), and we provide a way to find that vertex efficiently (Theorem 2). This structural result about this linear program relaxation gives a second (polynomial-time) algorithm to compute the MIS of an rBRF and should be of interest by itself.

We also consider the natural weighted version of MIS, denoted WMIS, where each rectangle in the family has a non-negative weight, and we aim to find a collection of disjoint rectangles with maximum total weight. We show (Theorem 6) that this problem is \( \text{NP} \)-hard even for BRFs and weights in \( \{0, 1\} \). Afterwards, we present algorithmic results for certain subclasses of BRFs defined by the structure of a graph representation \( G(\mathcal{R}) \) which is sufficient for the study of both the MIS and MHS. If \( G \) is a bipartite permutation graph, we provide an \( O(n^2) \) algorithm for arbitrary weights (Theorem 7) and a specialized \( O(n) \) algorithm for binary weights (Theorem 8). If \( G \) is convex bipartite, we note that the algorithm of Lubiw [31] for the maximum weight set of point-interval pairs readily translates into an algorithm for the WMIS that runs in \( O(n^3) \) time (Theorem 10). Results of Correa et al. [16] can be used to extend the \( \text{NP} \)-hardness to interval bigraphs (Theorem 11) and to provide a 2-approximation algorithm for WMIS on this class.

As a practical application, we show a previously unnoticed equivalence between the problems of computing the jump number and the MIS of a BRF graph \( G \). As a consequence, the jump number problem of a convex bipartite graph can be solved in \( O(n^2) \)
This is significantly faster than $O(n^9)$, the fastest asymptotic running time known for this problem. A summary of all the algorithmic results is presented in Table 1.

### 1.2 Paper Organization

Section 2 contains basic definitions about rectangles Sect. 2.1; about BRFs, rBRFs and subclasses Sect. 2.3; and about independent and hitting sets Sect. 2.4. It also contains properties of comparability graphs Sect. 2.5, which is a key ingredient of our algorithms. Section 3 surveys related work on MIS and MHS Sect. 3.1 and establishes the connection between our work and problems on point-interval sets and on set-pairs Sect. 3.2. An application to the jump number problem follows Sect. 3.3. Sections 4 to 6 form the body of the paper. Section 4 covers our simple LP based algorithm for MIS of rBRFs. Section 5 covers the more complex combinatorial algorithm for MIS and MHS of rBRFs, details on its implementation Sect. 5.1 and bounds necessary for the running time guarantee Sect. 5.2. Section 6 provides a list of additional results for the WMIS of BRFs and subclasses (bipartite permutation graphs 6.1, biconvex graphs 6.2, convex bipartite graphs 6.3 and interval bigraphs Sect. 6.4). Section 7 closes the paper with some final remarks. The appendix surveys equivalences between the problems of finding a MIS and MHS of BRFs and seemingly unrelated combinatorial problems. Some of these equivalences have not been noticed before. Appendix 8.1 surveys the minimum biclique cover problem and the maximum cross-free matching problem, while Appendix 8.2 surveys different formulations of the convex bipartite case covered by the theorem of Győri.

The reader interested in the most efficient algorithmic results can safely skip Sections 3 and 4.

| Table 1 Algorithmic results |
|-----------------------------|
|                            | MIS Jump number | MHS | WMIS |
| Bipartite permutation       | $O(n)$ [7, 18]  | $O(n)$ [7, 18] | $O(|\mathcal{R}|) [39] O(n^2)$ |
| Biconvex                    | $O(n^2)$ [7]    | $O(n^2)a$ [7]  | $O(n^3)b$ |
| Convex                      | $O(n^5)$ [17] $O(n^2)a$ | $O(n^2)a$ [3]  | $O(n^3)b$ |
| Interval bigraph/BRF        | $O(n^{2.5} \sqrt{\log n})$, $O((n \log n)^{\omega})c$ | $O(n^{2.5} \sqrt{\log n})$, $O((n \log n)^{\omega})c$ | NP-hard |

The new results, in bold, include the ones obtained via their relation to other known problems, but never considered in the literature.

$^a$Using Franzblau and Kleitman’s result [24] or Knuth’s [28] implementation.

$^b$Using Lubiw’s result [31].

$^c$Deterministic or randomized. $\omega \approx 2.3728596$ is the matrix multiplication exponent.
2 Preliminaries

2.1 Families of Rectangles

We denote the coordinates in the plane $\mathbb{R}^2$ as $x$ and $y$, so that a point $p$ is written as $(p_x, p_y)$. Given two points $a, b \in \mathbb{R}^2$, we write $a \leq_{\mathbb{R}^2} b$ if $a_x \leq b_x$ and $a_y \leq b_y$. For our purposes, a rectangle $R$ is the cartesian product of two closed intervals. For $a \leq_{\mathbb{R}^2} b$, we denote by $\Gamma(a, b) = \{ p \in \mathbb{R}^2 : a_x \leq p_x \leq b_x, a_y \leq p_y \leq b_y \}$ the rectangle with bottom-left corner $a$ and upper-right corner $b$.

We can associate any collection of rectangles $\mathcal{C}$ with a bipartite graph as follows (see Fig. 1). First, we identify each bottom-left (resp. upper-right) corner $a$ (resp. $b$) of rectangles $\Gamma(a, b) \in \mathcal{C}$ with a vertex of a set $A$ (resp. $B$). Then, we identify each rectangle $\Gamma(a, b) \in \mathcal{C}$ with an edge joining $a \in A$ and $b \in B$. The collection $\mathcal{C}$ of rectangles is thus identified with the graph $G = (A \cup B, \mathcal{E})$, which is the graph representation of $\mathcal{C}$. Quite often we work with the subcollection $\mathcal{C}_\downarrow = \{ R \in \mathcal{C} : \forall R' \in \mathcal{C} \setminus \{ R \} : R' \nsubseteq R \}$ of inclusion-wise minimal rectangles in $\mathcal{C}$.

Finally, for any set $S \subseteq \mathbb{R}^2$, the projection $\{ s_x : s \in S \}$ of $S$ onto the $x$ axis is denoted by $S_x$. We define $S_y$, the projection of $S$ onto the $y$ axis, analogously.

2.2 Restricted Bicolored Rectangular Families (rBRFs)

Let $A$ and $B$ be finite sets of points on the plane (white and gray colored in the pictures) and $\mathcal{Z} \subseteq \mathbb{R}^2$ be a not necessarily bounded set. The set $\mathcal{R} = \mathcal{R}(A, B, \mathcal{Z}) = \{ \Gamma(a, b) : a \in A, b \in B, a \leq_{\mathbb{R}^2} b \text{ and } \Gamma(a, b) \subseteq \mathcal{Z} \}$ is called the
restricted bicolored rectangular family (rBRF) associated to \( A, B \) and restricted to \( Z \). The graph representation \( G = (A \cup B, R) \) is called an rBRF graph.

We assume that the set \( Z \) is given implicitly, i.e. its size is not part of the input, and that we can test if a rectangle is contained in \( Z \) either in unit time (using an oracle) or in a fixed time \( T(Z) \). A case of interest is that of \( Z \) being the entire plane (see Fig. 2). In this case, we write simply \( R(A, B) \) and we call it a bicolored rectangle family or BRF (i.e., it is not restricted to any set).

We use \( n \) to denote \( |A \cup B| \) and \([k]\) to denote the set \( \{1, 2, \ldots, k\} \). To make the exposition simpler, we will assume without loss of generality that \( A \cap B = \emptyset \), that \( A \cup B \subseteq Z \), that the points in \( A \cup B \) have integral coordinates \( 1 \times [n] \times [n] \), and that no two points of \( A \cup B \) share a common coordinate value.

Below we state a useful property of inclusion-wise minimal rectangles.

**Lemma 1** Let \( R = R(A, B, Z) \) be an rBRF. A rectangle in \( \Gamma(a, b) \in R \) does not contain points in \( A \cup B \) other than its defining corners \( a \) and \( b \).

**Proof** If \( p \in A \cup B \) is contained in \( \Gamma(a, b) \), then either \( \Gamma(a, p) \) or \( \Gamma(p, b) \) is a rectangle in \( R \) that is strictly contained in \( \Gamma(a, b) \) unless \( p = a \) or \( p = b \). \( \square \)

### 2.3 A Hierarchy of Subclasses of BRF Graphs

The class of BRF graphs contain several well-known graph classes. Consider the following graphs on \( A \cup B \):

- A bipartite permutation graph (also known as bipartite two-dimensional graph) is the comparability graph of a two-dimensional partially ordered set of height 2, where \( A \) is the set of minimal elements and \( B \) is the complement of this set. We recall that a two-dimensional partially ordered set is simply a collection of points in \( \mathbb{R}^2 \) with the relation \( \leq \mathbb{R}^2 \).

- A convex bipartite graph is a bipartite graph admitting a labeling \( \{a_1, \ldots, a_k\} \) of \( A \) so that the neighborhood of each \( b \in B \) is a set of consecutive elements of \( A \). A biconvex graph is a convex bipartite graph for which there is also a labeling for \( B = \{b_1, \ldots, b_l\} \) so that the neighborhood of each \( a \in A \) is consecutive in \( B \).

- An interval bigraph is a bipartite graph where each vertex \( v \in A \cup B \) can be associated to a real closed interval \( I_v \) (w.l.o.g. with integral extremes) so that \( a \in A \) and \( b \in B \) are adjacent if and only if \( I_a \cap I_b \neq \emptyset \).

- A two directional orthogonal ray graph (2dorg) is a bipartite graph where each vertex \( v \in A \cup B \) is associated to a point \((v_x, v_y) \in \mathbb{Z}^2 \), so that \( a \in A \) and \( b \in B \) are connected if and only if the rays \([a_x, \infty) \times \{a_y\} \) and \([b_x] \times (-\infty, b_y]\) intersect each other. Since this condition is equivalent to \( \Gamma(a, b) \in R(A, B) \), two directional orthogonal ray graphs are exactly the BRF graphs.

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1 This can easily be done by translating the plane and applying a piecewise linear transformation on the axes.
It is known that the following strict inclusions hold for these classes [8, 37]: bipartite permutation graphs $\subset$ biconvex graphs $\subset$ convex bipartite graphs $\subset$ interval bigraphs $\subset$ 2dorgs (and, by the observations above, 2dorgs = BRF graphs $\subseteq$ rBRF graphs).

We give a simple geometric interpretation of some of the classes presented above as BRF graphs. The equivalence between the definitions below and the ones above are simple so the proof of equivalence is omitted. We refer to Fig. 3. Let $L = \{(x, -x) : x \in \mathbb{Z}\}$ be the integer points of the diagonal line $y = -x$. Similarly, let $L^+ = \{(x, y) \in \mathbb{Z}^2 : y \geq -x\}$ and $L^- = \{(x, y) \in \mathbb{Z}^2 : y \leq -x\}$ be the points weakly above and weakly below $L$. Bipartite permutation graphs are those BRF graphs $G = (A \cup B, R)$ where the points of $A$ and $B$ are integer points in two lines $L_A \subset L^-$ and $L_B \subset L^+$ which are parallel to $L$. Convex bipartite graphs are those BRF graphs where $A \subset L$ and $B \subset L^+$. Interval bigraphs are those BRF graphs where $A \subset L^-$ and $B \subset L^+$. We call the resulting classes of rectangle families bipartite permutation BRFs, convex bipartite BRFs and interval bigraph BRFs.

### 2.4 Independent Sets and Hitting Sets

We say that two rectangles $R$ and $R'$ intersect if they have a non-empty geometric intersection. A collection of pairwise non-intersecting rectangles is called an independent set of rectangles. A point $p$ hits a rectangle $R$ if $p \in R$. A collection of points $H$ is a hitting set of a rectangle family $\mathcal{C}$ if each rectangle in $\mathcal{C}$ is hit by a point in $H$. We denote by $\text{mis}(\mathcal{C})$ and $\text{mhs}(\mathcal{C})$ the sizes of a maximum independent set of rectangles in $\mathcal{C}$ and a minimum hitting set for $\mathcal{C}$ respectively.

Given a collection of rectangles $\mathcal{C}$, the intersection graph $\mathcal{I}(\mathcal{C}) = (\mathcal{C}, E)$ is the graph having edges between intersecting rectangles, that is $E = \{RR' : R \cap R' \neq \emptyset\}$. Independent sets on $\mathcal{C}$ are exactly the stable sets of $\mathcal{I}(\mathcal{C})$. On the other hand, since $\mathcal{C}$ has the Helly property, we can assign to every clique in $\mathcal{I}(\mathcal{C})$ a unique witness point, defined as the leftmost and lowest point contained in all rectangles of the clique. Since different maximal cliques have different witness points, it is easy to prove

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2 To prove the equivalence it is convenient to allow for the defining corners of a BRF to share a coordinate value.
that $\mathcal{C}$ admits a minimum hitting set consisting only of witness points of maximal cliques. In particular, $\text{mhs}(\mathcal{C})$ equals the minimum size of a clique-cover of $\mathcal{I}(\mathcal{C})$.

For both MIS and MHS, we can restrict ourselves to the family $\mathcal{C}_↓$ of inclusion-wise minimal rectangles in $\mathcal{C}$: any maximum independent set in $\mathcal{C}_↓$ is also maximum in $\mathcal{C}$ and any minimum hitting set for $\mathcal{C}_↓$ is also minimum for $\mathcal{C}$. Since the size of every independent set is at most the size of any hitting set, we observe that for any family $\mathcal{C}$,

$$\text{mis}(\mathcal{C}_↓) = \text{mis}(\mathcal{C}) \leq \text{mhs}(\mathcal{C}) = \text{mhs}(\mathcal{C}_↓).$$

### 2.5 Comparability Intersection Graphs

A non-empty intersection between two rectangles $R$ and $S$ in $\mathcal{R}$ is called corner intersection if either rectangle contains a vertex of the other rectangle in its interior. Otherwise, the intersection is called corner-free. A corner-free intersection (CFI) family is a collection of rectangles such that every intersection is corner-free. These intersections are generically shown in Fig. 4.

We now prove that corner-free intersection families have a special structure.

**Proposition 1** If $\mathcal{K}$ is a CFI family then $\mathcal{I}(\mathcal{K})$ is a comparability graph.

**Proof** Consider the relation $\leftrightarrow$ on $\mathcal{K}$ defined by $R \leftrightarrow S$ if $R_x \subseteq S_x$ and $S_y \subseteq R_y$. It is easy to check that $\leftrightarrow$ is a partial order relation. Since $\mathcal{K}$ is a CFI family, $R$ and $S$ intersect if and only if they are comparable under $\leftrightarrow$. Therefore, $\mathcal{I}(\mathcal{K})$ is the comparability graph of $(\mathcal{K}, \leftrightarrow)$. □

Let $\mathcal{K}$ be a family of rectangles such that $\mathcal{I}(\mathcal{K})$ is the comparability graph of a poset $(\mathcal{K}, \leftrightarrow)$. Independent sets in $\mathcal{K}$ correspond then to antichains in $(\mathcal{K}, \leftrightarrow)$; therefore the maximum independent set problem in $\mathcal{K}$ is equivalent to the maximum cardinality antichain problem in $(\mathcal{K}, \leftrightarrow)$.

Rectangles hit by a fixed point are trivially pairwise intersecting; therefore they are chains in $(\mathcal{K}, \leftrightarrow)$. By the Helly property, any family of pairwise intersecting rectangles is hit by a single point. It follows that finding a minimum hitting set of $\mathcal{K}$ is equivalent to finding a minimum size family of chains in $(\mathcal{K}, \leftrightarrow)$ covering all the elements of $\mathcal{K}$. The latter is the description of the minimum chain covering problem.

A classic theorem by Dilworth states that for any partial order, the maximum cardinality of an antichain equals the size of a minimum chain cover. In our context, this translates directly to:
Lemma 2  If $\mathcal{I}(\mathcal{K})$ is a comparability graph then $\text{mis}(\mathcal{K}) = \text{mhs}(\mathcal{K})$. □

From an algorithmic perspective, finding a maximum antichain and a minimum chain covering on a partially ordered set $\mathcal{K}$ can be done by computing a minimum vertex cover and a maximum matching on a bipartite graph (see, e.g. [19]). As a minimum vertex cover can be recovered from a maximum matching in time proportional to the number of edges [36, Section 16.5], determining the matching dominates the computation time. Let matching $(v, e)$ be the time of an algorithm that solves the bipartite matching problem with $v$ nodes and $e$ edges. The following result follows.

Lemma 3 ([19]) The maximum antichain and the minimum chain covering of a partial order $\mathcal{K}$ can be found in $O(\text{matching}(v, e))$, where $v = |\mathcal{K}|$, and $e$ is the number of pairs of comparable elements in $\mathcal{K}$.

□

3  Related Work

3.1 Background on MIS and MHS

Since the MIS problem is $\mathbf{NP}$-hard [20, 27], a significant amount of research has been devoted to heuristics and approximation algorithms. Chalermsook and Chuzhoy [11], and Chalermsook [10] describe two different $O(\log \log m)$-approximation algorithms for the MIS problem on a family of $m$ rectangles, while Chan and Har-Peled [12] provide an $O(\log m / \log \log m)$-approximation factor for WMIS. The approximation factor achieved by these polynomial time algorithms are the best so far for general rectangle families. Adamaszek and Wiese [1], and presented a quasi-polynomial time approximation scheme (QPTAS) for WMIS on general rectangles, and more recently Chuzhoy and Ene [15] presented a faster QPTAS for MIS.

The MHS problem is the dual of the MIS problem, and therefore the value of an optimal solution of MHS is an upper bound for that of MIS. The MHS problem is also $\mathbf{NP}$-hard [20]. The best known approximation is an algorithm of Aronov, Ezra, and Sharir [4] that provides a $O(\log \log \tau)$ approximation algorithm for the MHS on any rectangle family that can be hit by at most $\tau$ points. Achieving a constant factor approximation for MIS or MHS remains a major open problem.

3.2 From a Specialization (Point-Interval Pairs) to a Generalization (Set-Pairs)

As mentioned in the introduction, our work can be seen as an algorithmic generalization of Győri’s min–max theorem for point-interval pairs [25]. Suppose we are given a fixed ground set $A = \{1, \ldots, n\}$ and a family $\mathcal{F}$ of intervals $I \subseteq A$. A collection of point-interval pairs $(p, I)$ satisfying $p \in I \subseteq A (p_i, I_i)_{i=1,\ldots,m}$ is called independent if, for all $j \neq k$, either $p_j \notin I_k$ or $p_k \notin I_j$. On the other hand, a collection
of intervals is a basis of another family of intervals if every interval of the second family is a union of intervals in the first family. Győri shows that the cardinality of minimum basis for a family \( F \) equals the maximum cardinality of an independent family of point-interval pairs \((p, I)\), where \( I \) is restricted to be in \( F \).

Let us put Győri’s min–max result for point-intervals in our BRF setting. Through the representation of convex bipartite BRFs introduced in Sect. 2.2, we can represent \( A \) as a set of points in the antidiagonal line \( y = -x \), while we can represent \( F \) as points weakly above this line. The rectangles in the convex bipartite graph \((A \cup F, \mathcal{R})\) become the set of all point-interval pairs \((p, I)\) where \( I \in F \). It is easy to see that independent sets of rectangles are in correspondence with independent families of point-interval pairs. Hitting sets of rectangles are also in correspondence with minimum bases of intervals, although not bijectively: we identify a point \( q \) with the interval \( I_q \) of points \( a \in A \) with \( a \leq \mathbb{R}^2 q \); a hitting set \( H \) is transformed into a basis \( \{I_q : q \in H\} \) for \( F \) since \( I = \bigcup_{q \in H} I_q \leq \mathbb{R}^2 I_{I_q} \) for all \( I \in F \). On the other hand, a basis for \( F \) induces a hitting set \( H \) for \( \mathcal{R} \) of the same cardinality, through the inverse identification.

Győri’s min–max result for point-intervals is not constructive [25] and there was significant interest in obtaining simple and efficient algorithmic versions, and possible generalizations of the result, especially due the lack of similarity to previous min–max results in combinatorial optimization. Franzblau and Kleitman [24] (see also, Knuth [28]) present an algorithmic proof using some insights present in Győri’s work. Using their algorithm one can compute the MIS and MHS of convex bipartite BRFs in \( O(n^2) \) time (which is slightly faster than what we can get with our more general algorithm). Later, Lubiw [31] provides a point-weighted generalization of Győri’s theorem, that can be stated in our BRF framework as follows. Given a convex bipartite \( \mathcal{R}(A, B) \) and a non-negative integer valued function \( w : A \to \mathbb{R} \) on the points of \( A \). Define the weight of any rectangle \( \Gamma(a, b) \) in \( \mathcal{R}(A, B) \) equal to the weight of its lower left corner \( a \). Then, the maximum weight independent set of rectangles in \( \mathcal{R}(A, B) \) equals the minimum size of a multi-set of points in the plane \( H \) such that each rectangle in \( \Gamma(a, b) \) is hit by at least \( w(a) \) points of \( H \) (counting multiplicity). The proof is constructive and yields a strongly polynomial algorithm for the point-weighted version. She also shows that computing a maximum weight independent set of point-interval pairs that are arbitrarily weighted (and not just point-weighted) is still possible in polynomial time, using dynamic programming. This implicitly provides an algorithm for the WMIS of convex bipartite BRFs, as described in Theorem 10. In contrast, the min–max relation cannot be extended to the case in which point-intervals are arbitrarily weighted.

A few years later, Frank and Jordán [23] present a much more general min–max theorem about set-pair coverings, whose description we momentaneously defer, that extends even Lubiw’s result. Frank and Jordán’s original proof is non-combinatorial (although, they provide an algorithm that heavily relies on the ellipsoid method). Specializing the proof method of Frank-Jordán to Győri’s point-interval case yields another conceptually different proof of Győri’s theorem. Some years later, Frank [21] presents a simpler algorithmic proof by using more closely the point-interval interpretation of the problem. Frank’s approach is, in fact the starting point of our
Algorithm 1 for BRFs: interpreting his algorithm in our geometric setting, the procedure starts from a convex bipartite BRF $G = (A \cup B, \mathcal{R})$, determines a cross-free intersection family $\mathcal{K} \subseteq \mathcal{R}$, and then uses the theorem of Dilworth to determine a maximum independent set and a hitting set of $\mathcal{K}$, which then manages to transform into a maximum independent set and a hitting set of $\mathcal{R}$. After the additional improvements by Benczúr et al. [5], the complexity of Frank’s algorithm can be reduced to $O(n^{2.5} \log n)$ for both the minimum hitting set and the maximum independent set of convex bipartite BRFs. We note that this is slower than Franzblau and Kleitman’s original algorithm [24]. However, the same algorithm is able to handle cases outside Győri’s setting. In particular, Frank’s algorithm shows that there is also a min–max relation for the natural generalization from point-intervals pairs to point-arcs pairs in cycles. In a second article, Frank [22] also gave an algorithmic proof of the min–max result for the point-weighted version of both point-intervals (Lubiw’s result) and point-arcs in cycles.

All the results discussed so far are covered by Frank and Jordán’s general theorem. In order to describe it, we need some definitions. A collection of pairs $\{(S_i, T_i)\}$ of sets is half-disjoint if for every $j \neq k$, either $S_j \cap S_k$ or $T_j \cap T_k$ are empty. A directed edge $(s, t)$ covers a set-pair $(S, T)$ if $s \in S$ and $t \in T$. A family $\mathcal{S}$ of set-pairs is crossing if whenever $(S, T)$ and $(S', T')$ are in $\mathcal{S}$, so are $(S \cap S', T \cup T')$ and $(S \cup S', T \cap T')$. A function $p$ over a crossing family of set pairs is crossing bisupermodular if the inequality $p(S, T) + p(S', T') \leq p(S \cap S', T \cup T') + p(S \cup S', T \cap T')$ holds for all members $(S, T)$ and $(S', T')$ in the family, such that $S \cap S' \neq \emptyset$, $T \cap T' \neq \emptyset$ and $p(S, T) > 0, p(S', T') > 0$. Frank and Jordán show that for every crossing bisupermodular function $p$ over a crossing family $\mathcal{S}$, the maximum value of $\sum_{(X,Y) \in \mathcal{T}} p(X,Y)$ over every possible half-disjoint subfamily $\mathcal{T}$ of $\mathcal{S}$ is equal to the minimum number of directed edges (counting multiplicity) needed to cover each set pair $(S, T)$ at least $p(S, T)$ times. In particular, by taking $p \equiv 1$, the maximum size of a half-disjoint subfamily of $\mathcal{S}$ equals the minimum size of a collection of directed edges covering $\mathcal{S}$.

Our min–max result also follows from Frank and Jordán’s theorem: the inclusion-wise minimal rectangles $\mathcal{R}_1$ of an rBRF $(A \cup B, \mathcal{R})$, once projected over both axes $\{(R_x, R_y) : R \in \mathcal{R}_1\}$, becomes a crossing family of set-pairs for which half-disjoint subfamilies become independent sets in $\mathcal{R}$, while coverings by directed edges become hitting sets in $\mathcal{R}$. Thus, it may seem that our contribution is merely that of an algorithmic improvement. Yet, the literature that followed [23] shows why this is not the case. The generality of Frank and Jordán also carries significantly more abstract concepts, algorithms and proofs. As mentioned before, Frank and Jordán’s original proof is nonconstructive and already the works of Frank [21] and Benczúr et al. [5] show that a significant effort is required in order to translate the original ideas of Frank and Jordán [23] into an efficient and intuitive algorithmic proof for the theorem of Győri.

Some years later, combinatorial algorithms for Frank and Jordán’s general result appeared. Benczúr [6] gives a polynomial-time algorithm for the case of $\{0,1\}$.

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3 In [5], running times are expressed in terms of $|A|$ and $|B|$, whereas we measure in terms of $n = |A \cup B|$.
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-valued crossing bisupermodular functions, and then Benczúr and Végh devise a pseudo-polynomial time algorithm for general values. Both algorithms could be used for BRFs. However, both the algorithm and the analysis are much more complex and abstract than ours. Furthermore, we estimate the algorithm of Benczúr [6] to run in $O(n^7)$ time.

**Lemma 4** The following pairs of problems are equivalent under polynomial-time reductions:

- Finding the maximum independent set of point-interval pairs and finding the maximum independent set of a convex bipartite BRF.
- Finding the minimum basis for a family of intervals and finding the minimum hitting set of a convex bipartite BRF.
- Finding the maximum weight independent set of point-interval pairs and finding the weighted maximum independent sets of convex bipartite BRFs.

Also under polynomial-time reductions:

- Finding the maximum independent set of a BRF is equivalent to a special case of the problem of finding the maximum half-disjoint collection of a crossing family of set-pairs.
- Finding the minimum hitting set of a BRF is equivalent to a special case of the problem of finding the minimum size of a collection of directed edges covering a crossing family of set-pairs.

Moreover, a reduction exists so that the size of the optima of each problem of the pair coincide.

The algorithmic proof we provide for Theorem 3 has the value of positioning MIS and MHS of BRFs at the same level of complexity as those of convex bipartite BRFs studied by Győri, both conceptually and algorithmically. A concrete contribution of our work is to adapt Frank’s algorithm to the much broader set of rBRFs, with no additional overhead in the running time. We remark that although we use some ideas from [5] to tweak the algorithm, our problem is significantly more complex and our analysis does not follow from [5] nor the original paper of Frank [21].

### 3.3 An Application to the Jump Number Problem

The *jump number* of a partial order (poset) $P$ with respect to a linear extension $L$ is the number of pairs of consecutive elements in $L$ that are incomparable in $P$. The jump number problem is, given a poset $P$, to find its minimum jump number over all linear extensions. In the scheduling interpretation of this problem we want to schedule a set of tasks on a single machine/processor while minimizing the number of changeovers. The partial order $P$ serves a dual purpose, inducing precedence
constraints on the tasks and defining the changeovers that occurs whenever a task is scheduled immediately after an incomparable task.

Chaty and Chein [13] show that computing the jump number of a poset is equivalent to finding a maximum alternating-cycle-free matching in the underlying comparability graph. Müller [33] has shown that this problem is \text{NP}-hard on chordal bipartite graphs, a class of graphs where alternating-cycle-free matchings and cross-free matchings coincide. In Appendix 8, we show that on BRF graphs, which are chordal bipartite, the maximum cross-free matching problem is equivalent to the MIS problem. Therefore,

**Lemma 5** On unrestricted BRFs graphs \( G \) the following problems are equivalent: the jump number of \( G \) (as a comparability graph) and the maximum independent set of the geometric representation of \( G \).

Let \( n \) and \( m \) denote the number of vertices and edges of the comparability graph. In order of inclusion, there are \( O(m) \) time algorithms for the jump number problem on bipartite permutation graphs [7, 18, 39], an \( O(n^2) \) time algorithm on biconvex bipartite graphs [7] and an \( O(n^9) \) time algorithm for convex bipartite graphs [17].

### 4 An LP Based Algorithm for MIS on rBRFs

In this section we show an integrality result for a natural linear relaxation of the MIS problem on rBRFs. Consider an arbitrary family of rectangles \( C \) (not necessarily an rBRF) whose members have vertices in \([n]^2\). We associate a variable \( x_R \) to each \( R \in C \). The *fractional clique constrained independent set polytope* associated to \( C \) is the polyhedron

\[
\text{QSTAB}(C) \equiv \{x \in \mathbb{R}^C : \sum_{R \ni \ell} x_R \leq 1 \text{ for all } \ell \in [n]^2; x \geq 0\}.
\]

Integral solutions to QSTAB(\( C \)) satisfy \( x_R \in \{0, 1\} \) for all \( R \in C \); they are indicators of independent sets in \( C \).

The next result follows directly from Lovasz’s Perfect Graph Theorem [30].

Fig. 5 A BRF with non-integral independent set polytope. The point \( x \in \mathbb{R}^R \) with \( x_R = 1/2 \) for the given rectangles is a vertex with non-integral coordinates.
Proposition 2 Given a family of rectangles $C$ with vertices in $[n]^2$, the polytope $QSTAB(C)$ is integral if and only if $I(C)$ is a perfect graph. 

For example, this is the case (by Proposition 1) when $C$ is a CFI family, as comparability graphs are known to be perfect. Unfortunately, the same does not hold for rBRFs or BRFs. Figure 5 shows a BRF $R$ for which $QSTAB(R)$ has a non-integral vertex. Because this polytope is not a full description of the convex hull of characteristic vectors of independent sets, linear optimization over $QSTAB(R)$ may lead to non-integral solutions.

Nevertheless, we will prove that the LP obtained when we optimize on the all-ones $(1, 1, \ldots, 1)$ direction:

$$\text{mis}_{LP}(R) = \max \left\{ \sum_{R \in R} x_R : x \in QSTAB(R) \right\},$$

has an optimal integral solution. Recalling that $\text{mis}(R)$ is the integral version of $\text{mis}_{LP}(R)$, this result can be stated as follows.

Theorem 1 Let $R$ be a non-empty rBRF. There is an optimal integral solution for $\text{mis}_{LP}(R)$. In particular, $\text{mis}_{LP}(R) = \text{mis}(R)$.

To prove Theorem 1, we look at optimal solutions for $\text{mis}_{LP}(R)$ minimizing the geometric area. More precisely, we consider the modified linear program $\text{mis}_{LP}(R)$, where $area(R)$ denotes the geometric area of a rectangle $R \in R$ and $z^*$ denotes the optimal cost of $\text{mis}_{LP}(R)$:

$$\text{mis}_{LP}(R) = \min \left\{ \sum_{R \in R} \text{area}(R) x_R : \sum_{R \in R} x_R = z^*; x \in QSTAB(R) \right\}.$$

Here we are using the function $area : R \rightarrow \mathbb{R}$, given by $area([a_x, b_x] \times [a_y, b_y]) = (b_x - a_x) \cdot (b_y - a_y)$, but the argument that follows works for any function satisfying (1) Nonnegativity: $area(R) \geq 0$, (2) Strict monotonicity: If $R \subset S$, then $area(R) < area(S)$, and (3) Strict crossing bisubmodularity: If $R = I \times J$, $S = I' \times J'$ are two corner-intersecting rectangles in $R_\perp$ then $area((I \cup I') \times (J \cap J')) + area((I \cap I') \times (J \cup J')) < area(R) + area(S)$.

Let $x^*$ be an arbitrary optimal extreme point of $\text{mis}_{LP}(R)$ and let $R_0 = \{R \in R : x^*_R > 0\}$ be the support of $x^*$. Since $x^*$ minimizes weighted area, $R_0$ is a subset of the collection of inclusion-wise minimal rectangles $R_\perp$. 

[Diagram and Figure 6]
By Lemma 1, the only type of corner-intersection that could occur in $\mathcal{R}_I$ is shown in Fig. 6a. We show that those intersections do not arise in $\mathcal{R}_0$.

**Proposition 3** The family $\mathcal{R}_0$ is CFI.

**Proof** Suppose that $R = \Gamma(a, b)$ and $R' = \Gamma(a', b')$ are rectangles in $\mathcal{R}_0$ with corner-intersection as in Fig. 6a. We modify $x^*$, first reducing the values of $x_R^*$ and $x_{R'}^*$ by $\lambda \equiv \min\{x_R^*, x_{R'}^*\}$ and then increasing the values of $x_{R''}^*$ and $x_{R'''}^*$ by $\lambda$. Let $\tilde{x}^*$ be this modified solution.

Since the weight of the rectangles covering any point in the plane can only decrease (that is, $\sum_{S \ni q} x_S^* \geq \sum_{S \ni q} \tilde{x}_S^*, \forall q \in [n]^2$), all the constraints $\sum_{S \ni q} \tilde{x}_S^* \leq 1$ must hold. We also have $\sum_{S \in \mathcal{R}} \tilde{x}_S^* = \sum_{S \in \mathcal{R}} x_S^* = z^*$, and therefore $\tilde{x}^*$ is a feasible solution for $\text{mis}_{LP}(\mathcal{R})$. But the total weighted area of $\tilde{x}^*$ is smaller than the total weighted area of the original solution $x^*$ by a difference of $\lambda((\text{area}(R) + \text{area}(R')) - (\text{area}(R'') + \text{area}(R'''))) > 0$, which contradicts its optimality. \hfill $\square$

The fact that the support of $x^*$ forms a corner-free intersection family is all we need to prove the integrality of $x^*$:

**Proposition 4** The point $x^*$ is integral. In particular, $\mathcal{R}_0$ is a maximum independent set of $\mathcal{R}$.

**Proof** Since $\mathcal{R}_0$ is the support of $x^*$, the linear program $\text{mis}_{LP}(\mathcal{R}_0)$ also has $x^*$ as optimal solution. But $I(\mathcal{R}_0)$ is a comparability graph by Proposition 1; therefore, $\text{QSTAB}(\mathcal{R}_0)$ is integral. To conclude, we show that $x^*$ is an extreme point of $\text{QSTAB}(\mathcal{R}_0)$. Indeed, if this does not hold, then $x^*$ can be written as a convex combination of two points in $\text{QSTAB}(\mathcal{R}_0) \subseteq \text{QSTAB}(\mathcal{R})$, contradicting the fact that $x^*$ is extreme in $\text{QSTAB}(\mathcal{R})$. \hfill $\square$

The previous proposition concludes the proof of Theorem 1. Furthermore, since $\text{QSTAB}(\mathcal{R})$ has polynomially many variables and constraints, we obtain the following algorithmic result.

**Theorem 2** We can compute a MIS of an rBRF in polynomial time. \hfill $\square$

**Proof** Consider any polynomial time algorithm $\mathcal{A}$ to find an optimal vertex to a linear programming problem (e.g., the ellipsoid method). Use $\mathcal{A}$ to find the optimal cost $z^*$ of $\text{mis}_{LP}(\mathcal{R})$. Then use $\mathcal{A}$ to solve $\text{mis}_{LP}(\mathcal{R})$. The rectangles in the support of this solution give a MIS of $\mathcal{R}$.
In this section we devise an algorithm that constructs a maximum independent set $I^*$ and a minimum hitting set $H^*$ of a given rBRF $\mathcal{R}(A, B, Z)$. The overall description of our algorithm is as follows. We first construct the family $\mathcal{R}_i$ of inclusion-wise minimal rectangles of $\mathcal{R}$. Using a greedy procedure, we then find a maximal CFI family of rectangles $\mathcal{K} \subseteq \mathcal{R}_i$. Next, we use Lemma 3 to construct an independent set $I^*$ and a hitting set $H_0$ of the same size for the family $\mathcal{K}$. Afterwards, we use a flipping procedure to transform $H_0$ into a set $H^*$ of the same size in such a way that $H^*$ is also a hitting set for $\mathcal{R}_i$, and therefore for $\mathcal{R}$. Since $I^*$ and $H^*$ have the same size, they are both optimal.

The following notation will be useful to describe our algorithm. Given a rectangle $R$, let $u_1(R)$, $u_2(R)$, $u_3(R)$ and $u_4(R)$ be the bottom-left corner, top-right corner, top-left corner and bottom-right corners of $R$, respectively. The notation is chosen so that if $R$ is a rectangle of the rBRF $\mathcal{R}(A, B, Z)$, then $u_1(R) \in A$ and $u_2(R) \in B$ are the two defining corners of $R$.

Let $\mathcal{R}_i$ be the set of inclusion-wise minimal rectangles of $\mathcal{R}$. The construction of the CFI family $\mathcal{K}$ uses a specific order in $\mathcal{R}_i$: We say that a rectangle $R$ precedes a rectangle $S$ in right-top order if the top-left corner of $R$ is lexicographically smaller than the top-left corner of $S$, that is, $u_1(R) < u_1(S)$. In Fig. 7 there is an example of rectangles sorted in this way.

To construct $\mathcal{K}$, we process the rectangles in the sequence $(R_i)_{i=1}^t$ one by one. The rectangle $R_i$ being processed is added to $\mathcal{K}$ if its addition keeps $\mathcal{K}$ corner-free. The set $\mathcal{K}$ obtained by this simple greedy procedure is a maximal CFI subfamily of $\mathcal{R}_i$. As stated before, we then use Lemma 3 to compute a maximum independent set $I^*$ and a minimum hitting set $H_0$ for $\mathcal{K}$. As we will prove later, $I^*$ is actually a maximum independent set of the entire family $\mathcal{R}$. We can further assume that the points in $H_0$ have integral coordinates in $[n]^2$, because every rectangle hit by $(p_x, p_y)$ is also hit by $([p_x], [p_y])$.

The last step of the algorithm consists of a flipping procedure that essentially moves the points of $H_0$ around so that, in their new position, they not only hit $\mathcal{K}$ but the entire family $\mathcal{R}$. As we discuss in Sect. 3.2, the underlying ideas for this flipping algorithm are...
are already present in Frank’s algorithmic proof of Győri’s min–max result on intervals [21].

Consider a set $H \subseteq \mathbb{Z}^2$ and two points $p, q \in H$ with $p_x < q_x$ and $p_y < q_y$. By flipping $p$ and $q$ in $H$ we mean to move these two points to the new coordinates $r = (p_x, q_y)$ and $s = (q_x, p_y)$. More precisely, this means replacing $H$ by $H' = H \setminus \{p, q\} \cup \{r, s\}$.

The following lemma states that flipping points of $H$ that are inside a rectangle of $R_\downarrow$ can only increase the family of rectangles of $R_\downarrow$ that are hit by $H$.

**Lemma 6** Let $R$ be a rectangle in $R_\downarrow$ that contains $p$ and $q$. If $S \in R_\downarrow$ is hit by $H$ then it is also hit by $H' = H \setminus \{p, q\} \cup \{r, s\}$.

**Proof** Assume by contradiction that $S$ is hit by $H$ but not by $H'$. Assume also that $S$ is hit by $p$ (the case where $S$ is hit by $q$ is analogous). Since $S$ does not contain $r$ nor $s$, the point $b = \mathbf{B}(S)$ must be in the region $[p_x, q_x - 1] \times [p_y, q_y - 1]$. In particular, $b \in R \setminus \{A(R), \mathbf{B}(R)\}$. By Lemma 1, this contradicts the inclusion-wise minimality of $R$. 

We can now describe the flipping procedure (see Fig. 8). Let $K_1, \ldots, K_k$ be the sequence of rectangles in $K$ in right-top order, and $H$ be a set initially equal to $H_0$. In the $j$-th iteration of this procedure, we find the point $p$ in $H \cap K_j$ of minimum $y$-coordinate and the point $q$ in $H \cap K_j$ of maximum $x$-coordinate, breaking ties arbitrarily. Since we maintain $H$ as a hitting set for $K$, both points $p$ and $q$ exist (they may be the same). We then check if the points can be flipped (i.e. if $p_x < q_x$ and $p_y < q_y$) and in that case, we update $H$ by flipping $p$ and $q$. Our entire algorithm is depicted as Algorithm 1.
To analyze the correctness of Algorithm 1 we need an auxiliary definition. Recall that if a rectangle $R_i$ in the sorted sequence $R \downarrow$ is not included in $K$, then $R_i$ must have corner-intersection with a rectangle $R_i \in K$ with $i < i'$. The rectangle $R_i \in K$ with largest index that has corner-intersection with $R_i$ is denoted as the witness of $R_i$, and written as $wit(R_i) = \text{wit}(R_i)$. By Lemma 1, the fact that $\text{wit}(R_i)$ precedes $R_i$ in right-top order implies that $D(\text{wit}(R_i)) \in \text{int}(R_i)$ and $C(R_i) \in \text{inf}(\text{wit}(R_i))$.  

**Algorithm 1** for MIS and MHS of an rBRF $\mathcal{R}(A, B, Z)$. 

1. Construct $\mathcal{R}_1$ and sort them as $(R_i)_{i=1}^t$ in right-top order. 
2. Greedily construct a maximum CFI family $\mathcal{K}$ from the sequence $(R_i)_{i=1}^t$. Let $(K_j)_{j=1}^b$ be the family $\mathcal{K}$ sorted in right-top order. 
3. Find a maximum independent set $I^*$ and a minimum hitting set $H$ for $\mathcal{K}$. 
4. for $j = 1$ to $k$ do  
   - Flipping procedure starts.  
5. Let $p$ be a point of minimum $y$-coordinate in $H \cap K_j$ and $q$ be a point of maximum $x$-coordinate in $H \cap K_j$.  
6. if $p_x < q_x$ and $p_y < q_y$ then  
   - $H \leftarrow H \setminus \{p, q\} \cup \{(p_x, q_y), (q_x, p_y)\}$  
   - Flip $p$ and $q$ in $H$.  
7. end if  
8. end for  
9. Return $I^*$ and $H^* \leftarrow H$.  

To analyze the correctness of Algorithm 1 we need an auxiliary definition. Recall that if a rectangle $R_i$ in the sorted sequence $\mathcal{R}_1$ is not included in $\mathcal{K}$, then $R_i$ must have corner-intersection with a rectangle $R_i' \in K$ with $i' < i$. The rectangle $R_i' \in K$ with largest index that has corner-intersection with $R_i$ is denoted as the witness of $R_i$, and written as $R_i = \text{wit}(R_i)$. By Lemma 1, the fact that $\text{wit}(R_i)$ precedes $R_i$ in right-top order implies that $D(\text{wit}(R_i)) \in \text{int}(R_i)$ and $C(R_i) \in \text{inf}(\text{wit}(R_i))$. 

**Lemma 7** After iteration $j$ of the flipping procedure, the set $H$ hits every rectangle in $\mathcal{K}$ and every rectangle in $\mathcal{R}_1 \setminus \mathcal{K}$ with witness in $\{K_l : l \leq j\}$. 

**Proof** The algorithm only flips pairs of points that are inside a given rectangle $K_j$ of $\mathcal{R}_1$. By Lemma 6, the collection of rectangles hit by $H$ cannot decrease. So we only need to prove that after iteration $j$ all the rectangles witnessed by $\{K_1, \ldots, K_j\}$ are hit. We do this by induction on $j$. Before entering the loop (we call this the end of the $j = 0$-th iteration), we do not need to hit any rectangle, so let us assume that $j \geq 1$. If every rectangle witnessed by $K_j$ is hit at the end of iteration $j - 1$, we are done. Suppose then, that at the end of iteration $j - 1$ there is at least one rectangle $R$ in $\mathcal{R}_1 \setminus \mathcal{K}$ witnessed by $K_j$ that has not been hit yet. Let $a = A(K_j), b = B(K_j), a' = A(R)$ and $b' = B(R)$ so that $K_j = \Gamma(a, b)$ and $R = \Gamma(a', b')$. Since $K_j$ precedes $R$ in right-top order,...
order and they have a corner-intersection, the relative position of both rectangles must be as depicted in Fig. 9.

The rectangles $S = \Gamma(a', b')$ and $T = \Gamma(a, b')$ have no points in $A \cup B$ other than their defining corners, therefore they are in $\mathcal{R}_1$. We claim that $S \in \mathcal{K}$ and that $T$ is already hit by $H$.

Let us prove the first claim. Assume this is not the case; then $S \in \mathcal{R}_1 \setminus \mathcal{K}$ must have a witness $U = \text{wit}(S)$ in $\mathcal{K}$. Let $d = \mathbf{D}(U)$ be the bottom-right corner of $U$. Since $S$ and $U$ have corner-intersection and $U$ precedes $S$, the point $d$ is in the interior of $S$. Furthermore, since $U$ and $K_j$ are both in $\mathcal{K}$, $d$ cannot be in the interior of $K_j$. We conclude that $d \in Z_1 := \text{int}(S) \setminus \text{int}(K_j) = (a_x', b_x') \times (a_y', a_y]$. See Fig. 9 for reference.

But since $U$ contains the top-left corner of $S$ in its interior (because $U$ and $S$ have corner-intersection) and $U$ does not contain $a$ in its interior (as otherwise $U$ would not be inclusion-wise minimal), we conclude that $K_j$ precedes $U$ in right-top order. But then, $U$ is a rectangle in $\mathcal{K}$ having corner-intersection with $R$ and appearing after $K_j$ in right-top order. This contradicts the definition of $K_j$ as witness of $R$, and concludes the proof of the first claim.

Let us prove the second claim. If $T = \Gamma(a, b')$ is in $\mathcal{K}$, then we are done as $H$ is a hitting set for $\mathcal{K}$. So assume that $T \in \mathcal{R}_1 \setminus \mathcal{K}$. Let $W = \text{wit}(T)$ and $d' = \mathbf{D}(W)$. Since $W$ and $T$ have corner-intersection and $W$ precedes $T$, $d'$ is in the interior of $T$. Furthermore, since both $W$ and $K_j$ are in $\mathcal{K}$, $d'$ is not in the interior of $K_j$. We conclude that $d' \in Z_2 := \text{int}(T) \setminus \text{int}(K_j) = (b_x', b_y') \times (a_y', b_y')$. Since $W$ contains the top-left corner of $T$ in its interior (because $W$ and $T$ have corner-intersection), we deduce that $W$ precedes $K_j$ in right-top order. But then $W = K_j$ for $j' < j$ and so, by the induction hypothesis, rectangle $T$ was hit at the end of iteration $j'$. This ends the proof of the second claim.

Recall that $R$ is not hit by $H$ at the end of iteration $j - 1$. Since $T$ is hit by $H$, the set $H \cap T \setminus R$ must be nonempty. In particular, the point $p$ chosen by the algorithm of minimum $y$-coordinate in $K_j \cap H$ must be in the zone $Z_3 := [a_x, a_x') \times [a_y, b_y']$. Similarly, since $S \in \mathcal{K}$, $S$ must be hit by $H$ and so the set $H \cap S \setminus R$ is nonempty. Therefore, the point $q$ chosen by the algorithm of maximum $x$-coordinate in $K_j \cap H$ is in the zone $Z_4 := [a_x', b_x] \times (b_y', b_y]$. In particular, $p_x < q_x$, $p_y < q_y$ and the point $s = (q_x, p_y)$ is in $R$. We conclude that after flipping $p$ and $q$ in $H$, rectangle $R$ is hit.

Thanks to the previous lemma we obtain our main combinatorial result.

**Theorem 3** Algorithm 1 returns an independent set $I^*$ and a hitting set $H^*$ of $\mathcal{R}(A, B, \mathcal{Z})$ of the same cardinality. In particular, they are both optimal and $\text{mis}(\mathcal{R}) = \text{mhs}(\mathcal{R})$.

**Proof** The fact that $I^*$ is an independent set of $\mathcal{R}$ follows by construction and the fact that $\mathcal{K} \subseteq \mathcal{R}$. By Lemma 7, the set $H^*$ returned by the algorithm hits all the elements in $\mathcal{R}_1$. Since every rectangle in $\mathcal{R}$ contains a rectangle in $\mathcal{R}_1$, $H^*$ is also a hitting set of $\mathcal{R}$. Finally, since $H^*$ was constructed from $H_0$ via a sequence of flips which preserve cardinality and since $|H_0| = |I^*|$ by Lemma 2, we obtain that $|I^*| = |H^*|$. As
every hitting set has cardinality at least as large as every independent set, we conclude that both are optimal.

It is quite simple to give a polynomial-time implementation of Algorithm 1. In Sect. 5.1, we discuss an efficient implementation that runs in time \(O(\text{matching}(n \log n, n^2) + n^2 \log^2 n)\), where \(n = |A \cup B|\). Here we assume that testing containment in \(\mathcal{Z}\) can be done in unit time. Otherwise, we need additional \(O(n^2 T(\mathcal{Z}))\) time, where \(T(\mathcal{Z})\) is the time for testing containment in \(\mathcal{Z}\). The \(O(\text{matching}(n \log n, n^2))\) time is what it takes to find a maximum independent set and a minimum hitting set of \(\mathcal{K}\) using Lemma 3; the specific matching bound follows from tighter bounds on \(|\mathcal{K}|\) and \(E(\mathcal{K})\) that are relevant for this lemma.

### 5.1 Implementation of the Combinatorial Algorithm

To implement our algorithm it will be useful to have access to a data structure for dynamic orthogonal range queries. That is, a structure to store a dynamic collection \(P\) of points in the plane, supporting insertions, deletions and queries of the following type: given an axis-parallel rectangle \(Q\), is there a point \(p\) in \(Q \cap P\)? And if there is one, report any.

There are many data structures we can use. For our purposes, it would be enough to use any structure in which each operation takes polylogarithmic time. For concreteness, we use the following result of Willard and Lueker [41], specialized to two-dimensional Euclidean space.

**Theorem 4** (Willard and Lueker) There is a point data structure for orthogonal range queries in the plane on \(n\) points supporting insertion, deletion and queries in time \(O(\log^2 n)\), and using space \(O(n \log n)\).

Using this data structure, we can easily implement Algorithm 1.

- **Construction of \(\mathcal{R}_\downarrow\)**: Start by inserting the points in \(A\) and \(B\) to the point data structure and creating an empty list \(\mathcal{R}_\downarrow\). For each point \(a\) in \(A\) sorted from left to right and each point \(b\) in \(B\) from bottom to top check if \(\Gamma(a, b)\) is a rectangle in \(\mathcal{R}\) (i.e., if \(a \leq_{\mathbb{R}^2} b\) and if \(\Gamma(a, b) \subseteq \mathcal{Z}\)) and if it has no points of \(A \cup B\) in its interior using the data structure.\(^4\) If both conditions hold, add \(\Gamma(a, b)\) to the end of \(\mathcal{R}_\downarrow\). Note that by going through \(A \times B\) in this order, the list \(\mathcal{R}_\downarrow\) is sorted in right-top order. It is easy to see that the entire procedure takes time \(O(n^2 \log^2 n + n^2 T(\mathcal{Z}))\).

- **Construction of \(\mathcal{K}\)**: Start by creating an empty list \(\mathcal{K}\) and a point data structure \(\mathcal{D}\) containing the bottom-right corners of all rectangles in \(\mathcal{R}_\downarrow\). Then, go through the ordered list \(\mathcal{R}_\downarrow\) once again and add the current rectangle \(R \in \mathcal{R}_\downarrow\) to the end of \(\mathcal{K}\) if \(R\) does not contain a point of \(\mathcal{D}\) in its interior. It is easy to see that we

---

\(^4\) This requires \(a\) and \(b\) to be removed from \(A \cup B\) while querying if \(\Gamma(a, b)\) has no point from such set.
obtain the sorted CFI family \( \mathcal{K} \) described in our algorithm in this way and that the entire procedure takes time \( \mathcal{O}(|\mathcal{R}_1| \log^2(|\mathcal{K}|)) = \mathcal{O}(n^2 \log^2(|\mathcal{K}|)) \).

- Initial independent set and hitting set: Construct the intersection graph \( \mathcal{I}(\mathcal{K}) \) in \( \mathcal{O}(|\mathcal{K}|^2) \) time and use Lemma 3 to get a maximum independent set \( I^\ast \) and a minimum hitting set \( H_0 \) of \( \mathcal{K} \) in time \( \mathcal{O}(\text{matching}(|\mathcal{K}|, |E(\mathcal{I}(\mathcal{K}))|)) \).

- Flipping procedure: we initialize a point data structure containing \( H \) at every moment. Note that \( |H| \leq n \) as \( A \cup B \) is itself a hitting set of \( \mathcal{R} \). In each of the \( |\mathcal{K}| \) iterations of the flipping procedure we need to find the lowest point and the rightmost point of a range query. This can be done using binary search and the query operation of the data structure losing an extra logarithmic factor, i.e., in time \( \mathcal{O}(\log^2 n) \). To flip two points of \( H \), we perform two deletions and two insertions in time \( \mathcal{O}(\log^2 n) \). The entire flipping procedure takes time \( \mathcal{O}(|\mathcal{K}| \log^3 n) \).

In Sect. 5.2 we prove that \( |\mathcal{K}| = \mathcal{O}(n \log n) \) and \( |E(\mathcal{I}(\mathcal{K}))| = \mathcal{O}(n^2) \). By using these bounds and the previous discussion, we conclude that we can implement Algorithm 1 in time \( \mathcal{O}(\text{matching}(n \log n, n^2) + n^2 \log^2 n + n^2 T(\mathcal{Z})) \).

Hopcroft and Karp’s [26] algorithm for maximum matching on a bipartite graph with \( v \) vertices and \( e \) edges runs in time \( \mathcal{O}(e \sqrt{v}) \). Specializing this to \( v = \mathcal{O}(n \log n) \) and \( e = \mathcal{O}(n^2) \), matching \( (n \log n, n^2) \) is time \( \mathcal{O}(n^{2.5} \sqrt{\log n}) \). On the other hand, Mucha and Sankowski [32] have devised a randomized algorithm that returns with high probability a maximum matching of a bipartite graph in time \( \mathcal{O}(v^\omega) \), where \( \omega \) is the exponent for square matrix multiplication. The current best upper bound for \( \omega \) is approximately 2.3728596 by Alman and Williams [2]. From this discussion, we obtain the following result.

**Theorem 5** We can implement Algorithm 1 to run in time

\[
\mathcal{O}\left( \text{matching}(n \log n, n^2) + n^2 \log^2 n + n^2 T(\mathcal{Z}) \right),
\]

where \( n = |A \cup B| \) and \( T(\mathcal{Z}) \) is the time needed to test if a rectangle is contained in \( \mathcal{Z} \). Using Hopcraft and Karp’s implementation, this is

\[
\mathcal{O}(n^{2.5} \sqrt{\log n} + n^2 T(\mathcal{Z})).
\]

Using Mucha and Sankowski’s randomized algorithm, time can be reduced to

\[
\mathcal{O}(\sqrt{n \log n}^\omega + n^2 T(\mathcal{Z})),
\]

where \( \omega < 2.3728596 \) is the exponent for square matrix multiplication. \( \square \)

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5 More precisely, the algorithm gives a chain partition of \( \mathcal{I}(\mathcal{K}) \). This can be transformed into a hitting set of \( \mathcal{K} \) by selecting on each chain returned the bottom-left point of the mutual intersection of all rectangles in the chain. The extra processing time needed is dominated by \( \mathcal{O}(|\mathcal{K}|) \).

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Fig. 10 The broken lines, thin lines and thick lines correspond to $L_0$, $L_1$ and $L_2$, respectively. White, light-gray and dark-gray rectangles are assigned to $K_0$, $K_1$ and $K_2$, respectively.

5.2 Bounds for CFI Families

The bounds we prove in this section are valid for every corner-free intersection subfamily of $R_1$. Given one such family $K$, define its lower-left corner set $A(K) = \{A(K) : K \in K\}$ and its upper-right corner set $B(K) = \{B(K) : K \in K\}$. We start with a simple result.

**Lemma 8** If all the rectangles of $K$ intersect a fixed vertical (or horizontal) line $\lambda$, then $|K| \leq |A(K)| + |B(K)| - 1$.

**Proof** Project all the rectangles in $K$ onto the line $\lambda$ to obtain a collection of intervals in the line. Since $K$ is a CFI family, the collection of intervals forms a laminar family: if two intervals intersect, then one is contained in the other. Let $X$ be the collection of extreme points of the intervals. Since by assumption no two points in $A \cup B$ share coordinates, $|X| = |A(K)| + |B(K)|$ and furthermore, every interval is a non-singleton interval. To conclude the proof of the lemma we use the following known fact (see, e.g., the proof of [29, Prop. 4.1.7]): every laminar family of non-singleton intervals with extreme points in $X$ has cardinality at most $|X| - 1$.

Now we consider the situation where the family $K$ is not necessarily stabbed by a single line.

**Lemma 9** Let $\mathcal{L} = \{\lambda_1, \ldots, \lambda_r\}$ be a collection of vertical lines that intersect all the rectangles in $K$, then $|K| \leq n(1 + \lfloor \log_2(r) \rfloor) - r$.

**Proof** Assume that $\lambda_1, \ldots, \lambda_r$ are sorted from left to right. Consider the following collections of vertical lines:
Let \( \mathcal{L}_0 = \{ \lambda_{2k+1} : k \geq 0 \} = \{ \lambda_1, \lambda_3, \lambda_5, \lambda_7, \lambda_9 \ldots \} \).
\( \mathcal{L}_1 = \{ \lambda_{4k+2} : k \geq 0 \} = \{ \lambda_2, \lambda_6, \lambda_{10}, \lambda_{14}, \ldots \} \). 
\( \mathcal{L}_2 = \{ \lambda_{8k+4} : k \geq 0 \} = \{ \lambda_4, \lambda_{12}, \lambda_{20}, \lambda_{28}, \ldots \} \).
\[
\vdots
\]
\( \mathcal{L}_t = \{ \lambda_{2^{(2k+1)}} : k \geq 0 \} \).

The collections \( \mathcal{L}_0, \ldots, \mathcal{L}_{\lceil \log_2(r) \rceil} \) form a partition of \( \mathcal{L} \). Include each rectangle \( R \) in \( \mathcal{K} \) into the set \( \mathcal{K}_t, 0 \leq t \leq \lfloor \log_2(r) \rfloor \), of largest index such that \( \mathcal{L}_t \) contains a vertical line that intersects \( R \) (see Fig. 10).

Fix an index \( t \). Every rectangle in \( \mathcal{K}_t \) intersects a unique line in \( \mathcal{L}_t \) (if it intersects two or more, then it would also intersect a line in \( \mathcal{L}_{t+1} \)). For a given line \( \lambda \in \mathcal{L}_t \), let \( \mathcal{K}_\lambda \) be the family of rectangles in \( \mathcal{K}_t \) intersecting \( \lambda \). By Lemma 8, the number of rectangles in \( \mathcal{K}_\lambda \) is at most \( |\mathcal{A}(\mathcal{K}_\lambda)| + |\mathcal{B}(\mathcal{K}_\lambda)| - 1 \). Every point \( a \) in \( \mathcal{A}(\mathcal{K}) \) belongs to exactly one set in \( \{ \mathcal{A}(\mathcal{K}_\lambda) : \lambda \in \mathcal{L}_t \} \). It belongs to the one corresponding to the first line \( \lambda \) on or to the right of \( a \). Therefore, \( \sum_{\lambda \in \mathcal{L}_t} |\mathcal{A}(\mathcal{K}_\lambda)| \leq |A| \), and similarly, \( \sum_{\lambda \in \mathcal{L}_t} |\mathcal{B}(\mathcal{K}_\lambda)| \leq |B| \). Altogether we get

\[
|\mathcal{K}_t| = \sum_{\lambda \in \mathcal{L}_t} |\mathcal{K}_\lambda| \leq |\mathcal{A}(\mathcal{K})| + |\mathcal{B}(\mathcal{K})| - |\mathcal{L}_t|.
\]

Summing over \( t \) we obtain

\[
|\mathcal{K}| = \sum_{t=0}^{\lfloor \log_2(r) \rfloor} |\mathcal{K}_t| \leq (1 + \lfloor \log_2(r) \rfloor)(|\mathcal{A}(\mathcal{K})| + |\mathcal{B}(\mathcal{K})|) - |\mathcal{L}|.
\]

\[\square\]

If \( \mathcal{K} \) is the CFI family obtained with Algorithm 1, then \( n \) vertical lines are enough to intersect all rectangles in \( \mathcal{K} \). Therefore, \( |\mathcal{K}| = O(n \log n) \). Let us now bound the number of edges of the intersection graph \( \mathcal{I}(\mathcal{K}) \).

Lemma 10 \( |E(\mathcal{I}(\mathcal{K}))| = O(n^2) \).

Proof Let \( \ell = |E(\mathcal{I}(\mathcal{K}))| \). Let also \( \Lambda(n) \) be the maximum possible value of \( \ell \) as a function of \( n \). Recall that the vertices of all rectangles in \( \mathcal{K} \) are in the grid \( [n]^2 \). Consider the vertical line \( \lambda = \{([n/2], x) : x \in \mathbb{R} \} \) that divides the grid in two roughly equal parts. Count the edges in \( \mathcal{I}(\mathcal{K}) \) as follows. Let \( E_1 \) be the edges connecting pairs of rectangles that are totally to the left of \( \lambda \), \( E_2 \) be the edges connecting pairs of rectangles that are totally to the right of \( \lambda \), and \( E_3 \) be the remaining edges. We trivially get that \( |E_1| \leq \Lambda([n/2]) \) and \( |E_2| \leq \Lambda([n/2]) \). We bound the value of \( |E_3| \) in a different way.

Let \( \mathcal{K}_\lambda \) be the rectangles intersecting the vertical line \( \lambda \). Then, \( E_3 \) is exactly the collection of edges in \( \mathcal{I}(\mathcal{K}) \) having an endpoint in \( \mathcal{K}_\lambda \). By Lemma 8, \( |\mathcal{K}_\lambda| \leq n \). Now we bound the degree of each element of \( \mathcal{K}_\lambda \) in \( \mathcal{I}(\mathcal{K}) \). Consider one rectangle \( R = \Gamma(a, b) \in \mathcal{K}_\lambda \). Every rectangle intersecting \( R \) must intersect one of the four lines defined by its sides. By using again Lemma 8, we conclude that the degree of \( R \) in \( \mathcal{I}(\mathcal{K}) \) is at most \( 4n \). Therefore, the number of edges having an endpoint in \( \mathcal{K}_\lambda \) is at most \( 4n^2 \).

We conclude that \( \Lambda(n) \) satisfies the recurrence \( \Lambda(n) \leq \Lambda([n/2]) + \Lambda([n/2]) + 4n^2 \), from which, \( \Lambda(n) = O(n^2) \). \[\square\]
We remark that even though the expressions in the bounds used here look similar to those achieved by Benczúr et al. [5] to achieve a fast implementation of Frank’s algorithm [21], they are in fact different since in their case all the rectangles have lower left corner on the same diagonal whereas in ours rectangles can be in general position.

It is also worth noting that there are examples showing that our bounds on $|\mathcal{K}|$ and $|E(\mathcal{I}(\mathcal{K}))|$ are tight.

6 WMIS of BRFs

We now consider the problem of finding a maximum weight independent set of the rectangles in an rBRF $\mathcal{R}(A, B, Z)$ with weights $\{w_R\}_{R \in \mathcal{R}}$. We will focus mostly on special subclasses of BRFs since this case is already significantly harder than the unweighted counterpart.

**Theorem 6** The maximum weight independent set problem is NP-hard even for BRFs with weights in $\{0, 1\}$.

**Proof** We reduce from the maximum independent set of rectangles problem, which is NP-hard even if the vertices of the rectangles are all distinct [20]. Given an instance $\mathcal{I}$ with the previous property, let $A$ (resp. $B$) be the set of lower-left (resp. upper-right) corners of rectangles in $\mathcal{I}$. Note that $\mathcal{I} \subseteq \mathcal{R}(A, B)$ so we can find a maximum independent set of $\mathcal{I}$ by finding a maximum weight independent set in $\mathcal{R}$, where we give unit weight to each rectangle $R \in \mathcal{I}$, and zero weight to every other rectangle. \qed

We now show efficient algorithms for the WMIS on some of the subclasses of BRFs graphs introduced in Sect. 2.2, using the simple geometric representation of these subclasses based on the line $L = \{(x, -x) : x \in \mathbb{Z}\}$. 
6.1 Bipartite Permutation BRFs

Let \( R(A,B) \) be a bipartite permutation BRF such that the points of \( A \) and \( B \) lie on two lines \( L_A \) and \( L_B \) which are parallel to \( L \). Let \( \setminus \) be the following partial order on its rectangles. We say that \( R \setminus S \) if \( R \) and \( S \) are disjoint and \( R \) is positioned to the left or above \( S \) (this is, \( R(S)_x < A(S)_x \) or \( A(R)_y > B(S)_y \)). It is not hard to verify (see, e.g., Brandstädt [7]) that \((R, \setminus)\) is a partial order whose comparability graph is the complement of \( \mathcal{I}(R) \). In what follows, we use this fact to devise polynomial-time algorithms for the maximum weight independent set of bipartite permutation BRFs.

Observe that \( \mathcal{I}(R) \) is a perfect graph (because so is its complement [30]); therefore, using Proposition 2 we can compute a maximum weight independent set of \( R \) in polynomial time by finding an optimal vertex of

\[
\text{mis}_{\text{LP}}(R, w) \equiv \max \left\{ \sum_{R \in \mathcal{R}} w_R x_R : \sum_{R \in q} x_R \leq 1, \text{ for all } q \in [n]^2 : x \geq 0 \right\}.
\]

Steiner and Stewart [39] provide an \( O(|\mathcal{R}|^2) = O(n^4) \) time dynamic programming algorithm for WMIS on this graph class, which they further specialize to an \( O(|\mathcal{R}|) = O(n^2) \) time algorithm for MIS. By noting that the maximum weight independent sets in \( R \) are exactly the maximum weight chains on the weighted partially ordered set \((R, \setminus)\), we obtain a faster dynamic program for WMIS on this class of graphs.

**Theorem 7** There is an \( O(n^2) \) algorithm for the WMIS of bipartite permutation BRFs.

**Proof** For simplicity, let us assume that all the weights are different. Our algorithm exploits the geometric structure of the independent sets in \( R \).

Let \( R^+ \subset R \) be any maximum weight independent set and let \( R \setminus S \setminus T \) be three consecutive rectangles in \( R^+ \). Figure 11 shows the four ways in which the relative position (above-below, left-right) of these rectangle may look like.

For \( a \in A \), let \( V_1(a) \) be the maximum weight of a chain using only rectangles strictly below \( a \). Similarly, for \( b \in B \), let \( V_\rightarrow(b) \) be the maximum weight of a chain using only rectangles strictly to the right of \( b \).

The first rectangle \( S \) in an optimal chain defining \( V_1(a) \) must be below \( a \). The rectangle \( T \) that succeeds \( S \) is positioned either

- below \( S \) (Down-Down scenario). Then \( S \) is the heaviest rectangle \( S(a', a) \) below \( a \) with corner \( a' = A(S) \). From this, \( V_1(a) \) must be at least

\[
\max_{a' \in A, a > a'} \left\{ V_1(a') + w_{S(a', a)} \right\}
\]

- to the right of \( S \) (Down-Right scenario). Then \( S \) is the heaviest rectangle \( S(b') \) with corner \( b' = B(S) \). From this, \( V_1(a) \) must be at least

\[
\max_{b' \in B} \left\{ V_\rightarrow(b') + w_{S(b', a)} \right\}
\]
and therefore $V_\downarrow(a)$ is the maximum of these two quantities. Whenever $S(a', a)$ or $S(b')$ do not exist, the corresponding weight is set to zero. The base case occurs when no rectangle $S$ exists ($V_\downarrow(a) = 0$). A similar recursion can be developed for $V_\rightarrow(b)$. The recursions for $V_\downarrow(a)$ and $V_\rightarrow(b)$ can be solved simultaneously, going from right to left on both $A$ and $B$.

In principle, these recursions can be solved in $O(n^2)$: there are $O(n)$ terms $V_\downarrow(a)$, $V_\rightarrow(b)$ to compute, each requiring to find the maximum of $O(n)$ values. In order to query these values in time $O(1)$, we want constant-time access to the rectangles $S(a', a)$ and $S(b)$. We precompute them in time $O(n^2)$: for each $a' \in A$, it is easy to traverse all the points $a \in A$ backward, identifying $S(a', a)$ along the way. Precomputing $S(b)$, for each $b \in B$, is trivial.

In order to find the WMIS, we add a point $a_0$ to $A$ so that all rectangles are below $a_0$, and then compute for $V_\downarrow(a_0)$ recursively. \qed

We can further improve the running time of the previous algorithm when the weights are in $\{0, 1\}$ and a suitable description of the input is given.

**Theorem 8** We can compute a WMIS of a bipartite permutation BRF in $O(n)$ time when the input graph and its weights are given by matrices $M$ and $M'$ satisfying the following conditions:

1. $M \in \{0, 1\}^{A \times B}$ is the biadjacency matrix. $A$ and $B$ are sorted according to the x-coordinate, and we can access the first and last 1 of every row and column in $O(1)$ time.
2. In the weight matrix $M' \in \{0, 1\}^{A \times B}$, the points of $A$ and $B$ are sorted according to the x-coordinate, and we can access the first and last 1 of every row and column in $O(1)$ time.

**Proof** Let us ignore the rectangles of weight 0 from this discussion, as they do not contribute to the weight of any independent set. We simplify the algorithm for arbitrary weights (Theorem 7) when we restrict it to this unweighted case.

Note that we can modify any maximum weighted independent set so that, in the four scenarios described in Fig. 11, the corner defining $S$ (namely, $A(S)$ for Down-Down and Right-Down, $B(S)$ for Down-Right and Right-Right) is located as much to the left as possible. This allows us to simplify the recursion for $V_\downarrow(\cdot)$ and $V_\rightarrow(\cdot)$. Recall that the computation of $V_\downarrow(a)$ assumes a first rectangle $S$ in an optimal chain which is below $a$. If the (potentially) next rectangle $T$ in the chain is below $S$, we can now build a chain of length $1 + V_\downarrow(a')$, for the $a' \in A$ maximizing $a'$, such that $S(a', a)$ exists. If $T$ is to the right of $S$, then we can build a chain of length at least $V_\downarrow(a) = 1 + V_\rightarrow(b')$ for the $b' \in B$ maximizing $b'$ for which $S(b')$ exist.

Therefore, when $S$ exists we have the recursion
for the two points \( a' \in A \) and \( b' \in B \) just defined. When \( S \) does not exist, we define \( V \downarrow (a) = 0 \) as the base case. A similar recursion can be developed for \( V \rightarrow (b) \).

To solve the recursion in \( O(n) \) we use the matrices \( M \) and \( M' \) to access \( a \) and \( b \) in \( O(1) \), for any given \( a \in A \). For simplicity, let us assume that both \( M \) and \( M' \) have no zero rows or columns (otherwise, they can simply be ignored).

Finding \( b' \) is straightforward, we locate the last element \( \overline{b} \) with \( M(a, \overline{b}) = 1 \), which corresponds to the minimum height rectangle with corner \( a \), regardless of the weight. We define \( b' \) as the corner that follows \( \overline{b} \) in \( B \).

Finding \( a' \) is a bit more complicated, and it must be precomputed in \( O(n) \) for all \( a \in A \). First, for each \( b \in B \), we determine the minimum height rectangle \( S \) with \( B(S) = b \), which is given by the first 1-entry in the column of \( M' \) associated to \( b \). With one additional traversal of these rectangles, we can keep track of the rectangle \( S(b) \) weakly below \( b \) with highest \( A(S) \). Finally, for each \( a \in A \) we can find the corresponding \( a' \) in \( O(1) \) by first finding the last element \( b \) with \( M(a, b) = 1 \) and then returning \( S(b) \), where \( b \) is the element of \( B \) immediately after \( b \).

Note that the assumptions made about the input in this theorem are not completely unrealistic. They hold, for example, when the ones of each row and column of \( M \) and \( M' \) are connected by a doubly linked list.

### 6.2 Biconvex Graphs

Unlike the bipartite permutation case, the intersection graph of a biconvex graph is not always perfect, so we cannot directly compute the WMIS of biconvex graphs using linear programming. At least that is the case for arbitrary weight functions. However, if we could show that the intersection graph obtained by considering only inclusion-wise minimal rectangles of a biconvex graphs is perfect, then we could still solve the WMIS problem using linear programming, for interesting classes of weight functions (not just the cardinality case). Indeed, if for a given BRF \( \mathcal{R} \), one consider weight functions which are decreasing for inclusion (\( R \subseteq R' \) implies \( w(R') \leq w(R) \)), or that are point-weighted (i.e. there is a weight function on the points in \( A \), and the weight of each rectangle is defined as the weight of its lower-left
corner) then the maximum weight independent sets of $R_i$ are also of maximum weight in $R$.

It turns out that the intersection graph of inclusion-wise minimal rectangles of a biconvex graph $G$ is not always perfect (see Fig. 12). However, as we will see, perfection depends on the geometric representation chosen for $G$. The following result addresses one particular representation in which there is perfection (note that in this representation different vertices may be mapped to points sharing coordinates).

**Theorem 9** Let $G' = (A' \cup B', R')$ be a biconvex graph. Let $A' = \{a'_1, \ldots, a'_s\}$ and $B' = \{b'_1, \ldots, b'_t\}$ be labelings of $A'$ and $B'$ so that the neighborhood of each $b'_i \in B'$ is a set of consecutive elements of $A'$ and vice versa. Map each $a'_i \in A'$ to the point $a_i = (i, -i)$ and each $b'_i \in B'$ to the point $b_i = (r(i), -l(i))$, where $l(i)$ (resp. $r(i)$) are the minimum (resp. maximum) index $j$ such that $\Gamma(a'_j, b'_i) \in R'$. Then $G = (A \cup B, R)$ is a representation of $G'$ for which $I(R'_\downarrow)$ is perfect.

**Proof** We use the strong perfect graph theorem [14], proving by contradiction that $I(R'_i)$ has neither odd-holes nor odd-antiholes. First, suppose there is an odd-hole $H = \{R_1, R_2, \ldots, R_k\} \subseteq R'_i$ formed by rectangles $R_i = \Gamma(a_i, b_i)$ that (only) intersect $R_{i-1}$ and $R_{i+1}$ (mod $k$). Assume that $a_{i_1}$ is the leftmost defining corner. The three values $i_1, i_2$ and $i_k$ must be different: $i_2$ and $i_k$ are different because the corresponding rectangles do not intersect, while $i_1$ is different from $i_2$ (or $i_k$) since otherwise any rectangle intersecting the thinnest of these two rectangles would intersect the other one. The rest of the argument, sketched below, refers to the lines $\lambda_1$ and $\lambda_2$ and the zones $Z_1, Z_2, Z_3$ and $Z_4$ defined in Fig. 13(left), where $i_1 < i_2 < i_k$ is assumed without lost of generality. Zones $Z_1$ and $Z_4$ are closed regions, while $Z_2$ and $Z_3$ are open. The next claims are easily verified.

- The point $\lambda_1 \cap \lambda_2$ is in both $R_1$ and $R_k$.
- For all $1 < l < k$, $i_l < i_k$. Indeed, the region $\bigcup_{j=2}^l R_j$ is connected, so it contains a continuous path from $a_{i_2}$ to $a_{i_l}$. Having $i_l > i_k$ would imply that such path crosses $\lambda_1$ or $\lambda_2$, and therefore some rectangle in $\{R_{j} \}_{j=2,\ldots,l}$ intersects $R_1$ or $R_k$ (contra-
dicting that \( \mathcal{H} \) is a hole). The equality \( i_l = i_k \) could only hold if \( l = k - 1 \), but \( i_{k-1} < i_k \) by the same reason that \( i_1 < i_2 \).

- \( i_2 < i_{k-1} \), otherwise \( R_2 \) and \( R_{k-1} \) would intersect.
- Corners \( b_{i_1} \) and \( b_{i_k} \) lie in \( Z_4 \).
- Corner \( b_{i_2} \) lies in \( Z_2 \): being in any other zone would contradict the intersecting structure among the rectangles in the hole; being in \( \lambda_2 \) would contradict the inclusion-wise minimality of \( R_1 \). Analogously, \( b_{i_{k-1}} \) lies in \( Z_3 \).
- Corners \( b_{i_1}, \ldots, b_{i_{k-2}} \) lie in \( Z_1 \), otherwise the corresponding rectangles would intersect \( R_1 \) or \( R_k \).
- Either \( b_{i_3} \) lies above \( b_{i_{k-1}} \) or \( b_{i_{k-2}} \) lies to the right of \( b_{i_{k-1}} \): if not, \( R_2 \) and \( R_{k-1} \) would intersect. In what follows, set \( j = i_3 \) in the first case and \( j = i_{k-2} \) in the second one.

Finally, observe that the four indices \( j_1 = i_2, j_2 = j, j_3 = i_{k-1} \) and \( j_4 = i_1 \) are such that the associated first three intervals satisfy \( l(j_1) < l(j_2) < l(j_3) \) and \( r(j_1) < r(j_2) < r(j_3) \) while \( l(j_4) < l(j_2) \) and \( r(j_2) < r(j_4) \). It can be checked that no biconvex labeling of \( B \) can comply with these inequalities, which gives the contradiction.

Now suppose that \( \mathcal{I}(R_j) \) has an odd-antihole \( A = \{ R_1, R_2, \ldots, R_k \} \) of length at least 7. We keep the notation consistent from the odd-hole case; in particular, each \( R_i \) intersects all others but \( R_{i-1} \) and \( R_{i+1} (\mod k) \). Assume that \( a_{i_1} \) and \( a_{i_m} \) are the leftmost and rightmost defining-corners in \( A \), respectively. Using Fig. 13(right) as reference, the following is easy to see.

- Rectangles \( R_1 \) and \( R_m \) are the only ones with corners \( a_1 \) and \( a_m \), respectively: this is by the same argument as with odd-holes.
- We must have \( m \neq k \): if \( m = k \), \( R_4 \) and \( R_5 \) would intersect both \( R_1 \) and \( R_m \). Therefore, \( \lambda_1 \cap \lambda_2 \) lies in \( R_4 \cap R_5 \), which contradicts the intersecting structure of the antihole (since \( R_4 \) and \( R_5 \) do not intersect).
- We also have \( m \neq 2 \): this is proved in the same way as \( m \neq k \). Therefore, \( m \in \{ 3, \ldots, k - 1 \} \) and so \( R_1 \) and \( R_m \) intersect.
- There is a rectangle \( R_j \) intersecting both \( R_1 \) and \( R_m \) (because \( k \geq 7 \)).

Rectangles \( R_{j-1} \) and \( R_{j+1} \) intersect each other, but do not intersect \( R_j \), so they both lie either in zones \( Z_1 \) or in \( Z_2 \). This contradicts the fact that \( R_{j-1} \) and \( R_{j+1} \) intersect \( R_1 \) and \( R_m \). We conclude that \( \mathcal{I}(\mathcal{R}_1) \) does not contain odd-holes or odd-antiholes, and hence, it is a perfect graph. \( \square \)

### 6.3 Convex Bipartite BRFs

Recall that convex bipartite BRFs are those BRFs with \( A \subset L \) and with the points of \( B \) weakly above the line \( L \). As we discussed in Sect. 3.2, the maximum weight independent set of convex bipartite BRFs is equivalent to find the maximum weight point-interval set of a collection of intervals. For the latter problem, Lubiw [31] provides a polynomial-time algorithm that directly translates into an \( O(n^3) \)-algorithm [38].
Theorem 10  (Based on [31]) We can compute a WMIS of a convex bipartite BRF in $O(n^3)$ using Lubiw’s algorithm for maximum weight point-interval set.

6.4 Interval Bigraphs

In the natural geometric representation of an interval bigraph (see Sect. 2.3) all the rectangles intersect the diagonal line $L$. We use this property and a recent result of Correa et al. [16] to strengthen Theorem 6 as follows.

Theorem 11 Computing a WMIS of an interval bigraph is \textbf{NP}-hard even for weights in \{0, 1\}.

Proof The problem of computing a MIS of a family $I$ of rectangles intersecting $L$ is \textbf{NP}-hard [16]. Our hardness proof reduces this problem, by transforming a collection of rectangles intersecting $L$ into a subset of rectangles of an interval bigraph (by translating and piecewise scaling the plane), and then using weights in \{0, 1\} to distinguish the rectangles inside the collection from those outside when solving the WMIS. The proof is similar to that of Theorem 6.

Currently, convex bipartite BRFs is the largest natural class of BRFs for which the WMIS problem is solvable in polynomial time. Nevertheless, Correa et al. [16] gave a dynamic programming algorithm to compute WMIS of families of rectangles intersecting a diagonal line having the following property: if two rectangles intersect then they share a point below the diagonal. Based on this, they devise a 2-approximation for WMIS of rectangle families intersecting the diagonal whose running time is quadratic in the number of rectangles. Using their result we directly conclude that there is an $O(n^4)$ time algorithm to compute a 2-approximation for the WMIS of an interval bigraph.

7 Conclusions

It is worth noting that the min–max result also applies to rBRFs that are drawn in a cylinder $\mathbb{S}^1 \times \mathbb{R}$ or a torus $\mathbb{S}^1 \times \mathbb{S}^1$. In both surfaces axis-aligned rectangles are well-defined as cartesian products of closed intervals or, more precisely, arc segments. We only consider rectangles that are contractible to a point (i.e., they don’t wrap around the torus or the cylinder). Given two finite sets of points $A$ and $B$ and an arbitrary set $Z$ in a surface $S$ that can be either a cylinder or a torus, we can still define the collection $R_\downarrow$ of inclusion-wise minimal axis-aligned rectangles contained in $Z$ with bottom-left corner in $A$ and top-right corner in $B$. It is easy to see that $\{(R_x, R_y) : R_x \times R_y \in R_\downarrow\}$ is a crossing family of set-pairs. Applying Frank and Jordán’s theorem, the size of a maximum independent set in $R_\downarrow$ equals the size of a minimum hitting set. We believe it is not hard to modify our combinatorial algorithm to work in this case too. It is also not hard to see that if we consider only
rectangles in the torus that touch the anti-diagonal line \( y = 1 - x \) of the fundamental square \([0, 1] \times [0, 1]\) on their bottom-left corner (and don’t touch it again), then we also get as special case the one for point-arc pairs in cycles considered by Frank [21]. We left open the problem of generalizing our algorithm to handle the point-weighted version considered in [22] and its natural generalization to cylinder or torus drawn rBRFs.

**Appendix**

**Biclique Covers and Cross-Free Matchings**

Our results have consequences for two graph theoretical problems: minimum biclique cover and maximum cross-free matching. Before stating the relation we give some background on these problems.

A **biclique** of a bipartite graph is the edge set of a complete bipartite subgraph. A **biclique cover** is a collection of bicliques whose union is the entire edge set. Two edges \( e \) and \( f \) cross if there is a biclique containing both. A **cross-free matching** is a collection of pairwise non-crossing edges.

Orlin [35] has shown that finding a minimum biclique cover of a bipartite graph is \( \text{NP} \)-hard. Müller [34] extended this result to chordal bipartite graphs. To our knowledge, the only classes of graphs for which this problem has been explicitly shown to be polynomially solvable before our work are \( C_4 \)-free bipartite graphs [34], distance hereditary bipartite graphs [34], bipartite permutation graphs [34], domino-free bipartite graphs [3] and convex bipartite graphs [24, 25]. The proof of polynomiality for this last class can be deduced from the statement that the minimum biclique cover problem on convex bipartite graphs is equivalent to finding the minimum basis of a family of intervals on this class. The latter was originally studied and solved by Győri [25], but its connection with the minimum biclique problem was only noted several years later [3] (see Sect. 8.2 for more details).

It is an easy exercise to check that for any rBRF graph \( G = (A \cup B, \mathcal{R}) \), two edges \( R \) and \( R' \) cross if and only if \( R \) and \( R' \) intersect as rectangles. In particular, the cross-free matchings of \( G \) are in correspondence with the independent sets of \( \mathcal{R} \). Similarly, the maximal bicliques of \( G \) (in the sense of inclusion) corresponds to cliques in the intersection graph \( I(\mathcal{R}) \) and thus, the minimum biclique cover problem of \( G \) is equivalent to both the minimum clique-cover of \( I(\mathcal{R}) \) and the minimum hitting set problem on \( \mathcal{R} \).

Our results imply then that for BRF graphs, the maximum size of a cross-free matching equals the minimum size of a biclique cover and both optima can be computed in polynomial time. We summarize these equivalences in the following lemma.
Lemma 11 On unrestricted BRFs:

- The following problems are equivalent: the maximum independent set of its geometric representation, the maximum cross-free matching of its graph representation and the optimal jump number of $G$ of its poset representation.
- The following problems are equivalent: the minimum hitting set of its geometric representation and the minimum biclique cover of its graph representation.

In the case of the MIS, the equivalence also extends to the weighted case.

The Min–Max Theorem of Győri and Its Many Facets

Győri’s min–max theorem for point-interval pairs has many facets. To see this, it is helpful to describe some previous works.

A natural problem in discrete geometry is that of covering an orthogonal polygon (i.e. a simply connected polygon whose sides are parallel to the axes) with the minimum number of axis-aligned rectangles. It can be assumed that the polygon is formed by finite subsets of unit squares in a grid, so it can be represented as a 0-1 matrix whose entries of value 1 correspond to the unit grid squares forming the polygon. Chvátal (see [24] for more history on the problem) asked whether minimum size of such a rectangle cover equals the maximum number of unit grid squares, no two in a common rectangle inside in $P$ (a set of squares satisfying this property is called an anti-rectangle). For general orthogonal polygons, the answer turned out to be negative. However, Chaiken et al. [9] proved that the desired min–max relation holds for biconvex polygons, i.e. for polygons that are convex with respect to horizontal and vertical segments.

Győri [25] extends this result to polygons that are only vertically convex by proving a min–max result on a seemingly unrelated pair of problems. A sequence $(I_1, I_2, \ldots, I_k)$ of intervals of a total ordered set is called irredundant if every interval contains a point not contained in the union of preceding intervals. Győri shows the following remarkable result: the cardinality of a minimal basis (see Sect. 3.2 for a definition) of a collection $F$ of intervals (where the intervals in the base are not constrained to be elements of $F$) equals the maximum cardinality of an irredundant sequence of intervals in $F$. To obtain the related min–max result for vertically convex polygons $P$, represent them as the biadjacency matrices $M(G)$ of a convex bipartite graph whose rows are sorted according to its corresponding convex labeling (as defined in Sect. 2.3). Every maximal rectangle in $M(G)$ is defined uniquely by an interval of columns. Denote by $F$ the collection of intervals of columns induced by the maximal rectangles in $M(G)$. Győri shows that from any base of $F$ one can directly construct a rectangle cover of $M(G)$ of the same cardinality and that from any irredundant sequence of intervals in $F$ one can select a collection of squares (1-entries) in $M(G)$ that forms an anti-rectangle of the same size.

One can use Győri’s result to compute minimum biclique covers and maximum cross-free matchings in $G$ by using a different construction. Győri observes that
from any irredundant family of intervals one can obtain an independent family of point-interval pairs (see Sect. 3.2 for a definition) of the same size and vice-versa. Using this observation and Győri’s theorem one obtains that the cardinality of minimum basis for a family \( \mathcal{F} \) equals the maximum cardinality of an independent family of point-interval pairs \((p,I)\), where \(I\) is restricted to be in \( \mathcal{F} \).

Note that the containment relation on \((A \times \mathcal{F})\) defines a convex bipartite graph \(G\). Amilhastre et al. [3] noted many years later that finding the minimum biclique cover problem on \(G\) is equivalent to finding the minimum basis of the family \(\mathcal{F}\). It is not hard to see also, that independent families of point-intervals \((p,I)\) with \(I \in \mathcal{F}\) correspond to cross-free matchings of \(G\). Surprisingly, this last connection has not been noticed in the literature of cross-free matchings or the jump number until our work (see [38]).

Acknowledgements

The first author was partially supported by ANID-CHILE via FONDECYT 1181180, and PIA AFB170001. The second author was partially supported by ANID-CHILE via FONDECYT 11200616 and by the FSR Incoming Post-doctoral Fellowship of the Catholic University of Louvain (UCL), funded by the French Community of Belgium. The authors gratefully acknowledge Prof. Andreas S. Schulz for many stimulating discussions during the early stages of this paper.

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