AN INVARIANT FORMULA FOR A STAR PRODUCT WITH SEPARATION OF VARIABLES

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ABSTRACT. We give an invariant formula for a star product with separation of variables on a pseudo-Kähler manifold.

1. THE STANDARD STAR PRODUCT WITH SEPARATION OF VARIABLES

Given a vector space $V$, we say that the elements of the space $V[[\nu]]$ of formal series in a formal parameter $\nu$ with coefficients in $V$ are formal vectors. This way we define formal functions, formal tensors, formal operators, etc.

Let $M$ be a pseudo-Kähler manifold with the metric tensor $g_{ki}$. Its inverse is a Kähler-Poisson tensor, $g^i_k$. The Jacobi identity for $g^i_k$ takes the form

$$g^i_k \frac{\partial g^q_p}{\partial z^k} = g^i_q \frac{\partial g^p_k}{\partial z^k} \quad \text{and} \quad g^i_k \frac{\partial g^q_p}{\partial \bar{z}^l} = g^i_p \frac{\partial g^q_k}{\partial \bar{z}^l},$$

where we assume summation over repeated indices. On a contractible coordinate chart on $M$ there exists a potential $\Phi$ of the pseudo-Kähler metrics such that

$$g_{ki} = \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l}.$$

Throughout the paper we use the conventional notation

$$g_{k_1 \ldots k_r \bar{l}_1 \ldots \bar{l}_s} = \frac{\partial^{r+s} \Phi}{\partial z^{k_1} \ldots \partial z^{k_r} \partial \bar{z}^{\bar{l}_1} \ldots \partial \bar{z}^{\bar{l}_s}}$$

and the formulas

$$\frac{\partial g^i_k}{\partial z^p} = -g^{is} g_{spq} g^q_k \quad \text{and} \quad \frac{\partial g^i_k}{\partial \bar{z}^q} = -g^{is} g_{sqi} g^q_k.$$

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A star product $*$ on $M$ is an associative product on $C^\infty(M)[[\nu]]$ given by the formula

$$f * g = \sum_{r \geq 0} \nu^r C_r(f, g),$$

where $C_r$ are bidifferential operators on $M$, $C_0(f, g) = fg$, and

$$C_1(f, g) - C_1(g, f) = g^l k \left( \frac{\partial f}{\partial \bar{z}^l} \frac{\partial g}{\partial z^k} - \frac{\partial g}{\partial \bar{z}^l} \frac{\partial f}{\partial z^k} \right)$$

(see [1]). We assume that the unit constant is the unity for a star product, so that $f * 1 = 1 * f = f$. It was proved by Kontsevich in [6] that deformation quantizations exist on any Poisson manifold.

Denote by $L_f$ and $R_g$ the left star multiplication operator by $f$ and the right star multiplication operator by $g$, respectively, so that

$$L_f = f * g = R_g f.$$

The associativity of $*$ is equivalent to the condition that $[L_f, R_g] = 0$ for all $f, g$. A star product can be restricted to any open set in $M$.

A star product on $M$ is called a star product with separation of variables if the operators $C_r$ differentiate their first argument only in antiholomorphic directions and the second argument only in holomorphic ones. In particular,

$$C_1(f, g) = g^l k \frac{\partial f}{\partial \bar{z}^l} \frac{\partial g}{\partial z^k}.$$

A star product with separation of variables is characterized by the property that for a local holomorphic function $a$ and a local antiholomorphic function $b$ the operators $L_a$ and $R_b$ are the pointwise multiplication operators by the functions $a$ and $b$, respectively,

$$L_a = a \text{ and } R_b = b.$$

A complete classification of star products with separation of variables on a pseudo-Kähler manifold was given in [4]. In particular, it was proved that on any pseudo-Kähler manifold $M$ there exists a global star-product with separation of variables $*$ uniquely determined on each contractible chart by the condition that

$$L \frac{\partial \Phi}{\partial z^k} = \frac{\partial \Phi}{\partial \bar{z}^k} + \nu \frac{\partial}{\partial z^k} \text{ and } R \frac{\partial \Phi}{\partial \bar{z}^l} = \frac{\partial \Phi}{\partial \bar{z}^l} + \nu \frac{\partial}{\partial \bar{z}^l}.$$

We call it the standard star product with separation of variables. It was independently constructed in [2] using a different method.

Explicit formulas for star products with separation of variables expressed in terms of graphs encoding bidifferential operators were given
In this paper we define an invariant total symbol $T$ of the formal bidifferential operator
\begin{equation}
\sum_{r=0}^{\infty} \nu^r C_r
\end{equation}
which determines the standard star product with separation of variables $\ast$ on a pseudo-Kähler manifold $M$. The symbol $T$ is a formal covariant tensor on $M$ separately symmetric in the holomorphic and antiholomorphic indices. The star product $\ast$ can be immediately recovered from the total symbol $T$. The main result of this paper is a closed invariant formula for the symbol $T$.

2. A closed formula for a symbol of a left star multiplication operator

In this section we introduce a locally defined total symbol $\sigma(A)$ of a differential operator $A$ on a pseudo-Kähler manifold $M$. Then we give a closed formula for the total symbol $\sigma(L_f)$ of the left star multiplication operator by a function $f$ with respect to the standard deformation quantization with separation of variables on $M$.

We will work on a contractible coordinate chart on $M$ with local holomorphic coordinates $\{z^k\}$. Introduce the following locally defined operators,
\begin{equation}
\bar{D}_l = g_{lk} \frac{\partial}{\partial z^k}, \quad \bar{D}_l = \frac{\partial}{\partial \bar{z}^l} - \frac{\partial^2 \Phi}{\partial \bar{z}^l \partial z^q} g^{qp} \frac{\partial}{\partial z^p} - \frac{\partial^2 \Phi}{\partial \bar{z}^l \partial z^q} \bar{D}^q.
\end{equation}

Lemma 1. The operators $\bar{z}^l, \frac{\partial \Phi}{\partial \bar{z}^l}, \bar{D}_l, \bar{D}^l$ satisfy canonical relations. In particular, for all $l, q$,
\[ [\bar{D}_l, \frac{\partial \Phi}{\partial \bar{z}^q}] = 0, \quad [\bar{D}_l, \bar{D}^q] = 0, \quad [\bar{D}_l, \bar{z}^q] = \delta^q_l, \quad \text{and} \quad [\bar{D}^l, \frac{\partial \Phi}{\partial \bar{z}^q}] = \delta^l_q. \]

Lemma 1 can be checked by direct calculations using the Jacobi identity (1) and formulas (2).

A differential operator $A$ can be uniquely represented as a finite sum of the form
\begin{equation}
A = \sum_{s, t} \bar{D}_{l_1} \ldots \bar{D}_{l_s} f_{q_1, \ldots, q_t} (z, \bar{z}) \bar{D}^{q_1} \ldots \bar{D}^{q_t}.
\end{equation}

The middle factor in each monomial in (6) is the pointwise multiplication operator by a function of $z, \bar{z}$. We will call this representation normal. Using auxiliary variables $\bar{\eta}^l$ and $\bar{\zeta}_l$, we introduce symbols of
differential operators as functions of $z, \bar{z}, \zeta, \eta$ polynomial in $\zeta$ and $\eta$ such that the symbol of the operator $A$ is

$$\sigma(A) = \sum_{s,t} \sum_{l_1, \ldots, l_s} f^{l_1, \ldots, l_s}_{q_1, \ldots, q_t}(z, \bar{z}) \zeta^{l_1} \cdots \zeta^{l_s} \eta^{q_1} \cdots \eta^{q_t}.$$  

The composition of operators induces a composition of symbols which will be denoted by $\circ$. It is easy to check using expressions (6), (7) and Lemma 1 that

$$\frac{\partial \Phi}{\partial \bar{z}^l} \circ F = \frac{\partial \Phi}{\partial \bar{z}^l} F, \quad F \circ \frac{\partial \Phi}{\partial \bar{z}^l} = F \frac{\partial \Phi}{\partial \bar{z}^l} + \frac{\partial F}{\partial \bar{\eta}^l},$$  

$$F \circ \bar{\eta}^l = F \bar{\eta}^l, \quad \bar{\eta}^l \circ F = F \bar{\eta}^l + \bar{D}^l F,$$

$$\bar{\zeta}^l \circ F = F \bar{\zeta}^l, \quad F \circ \zeta^l = F \zeta^l - \bar{D}^l F.$$  

We have the following commutators of symbols from Eqn. 8,

$$\left[ F, \frac{\partial \Phi}{\partial \bar{z}^l} \right]_o = \frac{\partial F}{\partial \bar{\eta}^l}, \quad [\bar{\eta}^l, F]_o = \bar{D}^l F, \quad [\bar{\zeta}^l, F]_o = \bar{D}^l F.$$  

The following lemma can be obtained from (8) and (9) by standard methods.

Lemma 2. The composition $\circ$ of symbols $F, G$ is given by the formula

$$F \circ G = \mu \left( \exp \left\{ \frac{\partial}{\partial \bar{\eta}^l} \otimes \bar{D}^l - \bar{D}^l \otimes \frac{\partial}{\partial \zeta^l} \right\} (F \otimes G) \right),$$  

where $\mu(A \otimes B) = AB$ is the pointwise product of functions.

The exponential formal series in (10) terminates. If symbols $F, G$ do not depend on the variables $\zeta$, their composition can be given by the formula

$$F \circ G = \sum_{r=0}^{\infty} \frac{1}{r!} \frac{\partial^r F}{\partial \bar{\eta}^{l_1} \cdots \partial \bar{\eta}^{l_r}} \bar{D}^{l_1} \cdots \bar{D}^{l_r} G.$$  

Given a formal function $f = f_0 + \nu f_1 + \ldots$, denote by $A = L_f$ the left star multiplication operator by $f$ with respect to the standard deformation quantization with separation of variables. Then $A$ is determined by the conditions that it commutes with the local operators $R_{z^l} = z^l$ and

$$R_{\frac{\partial \Phi}{\partial z^l}} = \frac{\partial \Phi}{\partial z^l} + \nu \frac{\partial}{\partial z^l},$$  

for all $l$ and $A1 = f$. Set $F := \sigma(A)$. Since $A$ commutes with the multiplication by the variables $z^l$, its symbol does not depend on the variables $\zeta$, i.e., $F = F(\nu, z, \bar{z}, \eta)$. The condition $A1 = f$ means that

$$F \bigg|_{\eta = 0} = f.$$  

The operator $R_{\frac{\partial}{\partial z^i}}$ can be rewritten with the use of (5) in the normal form as follows,

$$R_{\frac{\partial}{\partial z^i}} = \frac{\partial \Phi}{\partial z^i} + \nu \bar{D}_l + \nu \frac{\partial^2 \Phi}{\partial z^i \partial z^q} \bar{D}^q.$$  

(12)

Therefore, the symbol of the operator $R_{\frac{\partial}{\partial z^i}}$ is

$$\sigma \left( R_{\frac{\partial}{\partial z^i}} \right) = \frac{\partial \Phi}{\partial \bar{z}^l} + \nu \bar{\zeta}_l + \nu \frac{\partial^2 \Phi}{\partial \bar{z}^l \partial \bar{z}^q} \bar{\eta}^q.$$  

(13)

The condition that the symbols of $A$ and $R_{\frac{\partial}{\partial z^i}}$ commute,

$$\left[ \frac{\partial F}{\partial \bar{\eta}^l} + \nu \bar{D}_l F + \nu \sum_{r=0}^{\infty} \frac{1}{r!} \left( \bar{D}^{l_1} \ldots \bar{D}^{l_r} \frac{\partial^2 \Phi}{\partial z^i \partial \bar{z}^q} \right) \bar{\eta}^q - \nu \frac{\partial^2 \Phi}{\partial z^i \partial z^q} (F \bar{\eta}^q + \bar{D}^q F) = 0, \right.$$  

using the definition (5) of the operator $D_l$ we can further simplify (14):

$$\frac{\partial F}{\partial \bar{\eta}^l} = \nu \left( \frac{\partial F}{\partial \bar{z}^l} - \sum_{r=1}^{\infty} \frac{1}{r!} \left( \bar{D}^{l_1} \ldots \bar{D}^{l_r} \frac{\partial^2 \Phi}{\partial z^i \partial \bar{z}^q} \right) \bar{\eta}^q - \frac{\partial^r F}{\partial \bar{\eta}^{l_1} \ldots \partial \bar{\eta}^{l_r}} \right).$$  

(14)

Introduce the following operator on symbols,

$$Q := \bar{\eta}^l \frac{\partial}{\partial \bar{z}^l} - \sum_{r=1}^{\infty} \frac{1}{r!} \left( \bar{D}^{l_1} \ldots \bar{D}^{l_r} \frac{\partial^2 \Phi}{\partial z^i \partial \bar{z}^q} \right) \bar{\eta}^q - \frac{\partial^r F}{\partial \bar{\eta}^{l_1} \ldots \partial \bar{\eta}^{l_r}}.$$  

(15)

We see from (15) that the symbol $F$ of the operator $L_f$ satisfies the equation

$$E(F) = \nu Q(F),$$  

(16)

where

$$E = \bar{\eta}^l \frac{\partial}{\partial \bar{\eta}^l}$$

is the Euler operator with respect to the variables $\bar{\eta}$. Denote by $\mathcal{F}$ the space of formal symbols which do not depend on the variables $\bar{\zeta}$. The elements of $\mathcal{F}$ are the formal symbols $S = S_0 + \nu S_1 + \nu^2 S_2 + \ldots$, where each component $S_r = S_r(z, \bar{z}, \bar{\eta})$ is polynomial in the variables $\bar{\eta}$. So, in particular, $F \in \mathcal{F}$. The operator $E$ acts on $\mathcal{F}$. We have that

$$\mathcal{F} = \ker E \oplus \text{im } E.$$
The kernel \( \ker E \) of \( E \) in \( \mathcal{F} \) consists of the formal functions in the variables \( z, \bar{z} \) and the image \( \text{im} E \) consists of the symbols \( S = S(\nu, z, \bar{z}, \bar{\eta}) \) such that

\[
S \big|_{\bar{\eta} = 0} = 0.
\]

The Euler operator \( E \) is invertible on \( \text{im} E \). Denote its inverse on \( \text{im} E \) by \( E^{-1} \). Observe that the operator \( Q \) maps \( \mathcal{F} \) to \( \text{im} E \). Therefore, the operator \( E^{-1}Q \) is well defined on \( \mathcal{F} \). Represent \( F \) as the sum

\[
F = f + H
\]

for some formal symbol \( H \). Then \( H \in \text{im} E \) and equation (16) can be rewritten as follows:

\[
(E - \nu Q)(H) = \nu Q(f).
\]

We see that

\[
(1 - \nu E^{-1}Q)(H) = \nu E^{-1}Q(f).
\]

The operator \( (1 - \nu E^{-1}Q) \) is invertible on \( \mathcal{F} \). We have

\[
F = f + H = f + \nu(1 - \nu E^{-1}Q)^{-1}E^{-1}Q(f) = (1 - \nu E^{-1}Q)^{-1}(f).
\]

We have proved the following theorem:

**Theorem 1.** Given a formal function \( f \), the symbol \( F \) of the left multiplication operator \( L_f \) with respect to the standard star product with separation of variables is given by the following formula,

\[
F = (1 - \nu E^{-1}Q)^{-1}(f) = \sum_{r=0}^{\infty} \nu^r (E^{-1}Q)^r (f).
\]

Observe that the component \( F_r \) of \( F = F_0 + \nu F_1 + \nu^2 F_2 + \ldots \) is given by the formula

\[
F_r = (E^{-1}Q)^r (f).
\]

It is the symbol of the operator \( C_r(f, \cdot) \).

### 3. An Invariant Formula for a Star Product with Separation of Variables

Given an affine connection \( \nabla \) on a manifold \( M \), there exists a global vector field on the total space of the tangent bundle \( \pi : TM \to M \) defined as follows. A point \( y \in TM \) represents a tangent vector \( v_y \in T_{\pi(y)}M \). There exists a unique vector \( w_y \in T_y(TM) \) horizontal with respect to the connection \( \nabla \) such that \( \pi_*(w_y) = v_y \). Then \( y \mapsto w_y \) is a global vector field on \( TM \). We will denote it \( w_\nabla \). In local coordinates \( \{x^i\} \) on \( M \) the connection \( \nabla \) is determined by the Christoffel symbols
\( \Gamma_{ij} \). If \{y^i\} are the fibre coordinates on \( TM \) corresponding to \{x^i\}, then
\[
w_\nabla = y^i \frac{\partial}{\partial x^i} - \Gamma_{ij}^k y^j y^k \frac{\partial}{\partial y^k}.
\]
A symmetric covariant tensor \( S_{i_1...i_r} \) of degree \( r \) on \( M \) can be equivalently described as the fibrewise polynomial function
\[S := S_{i_1...i_r} y^{i_1} \ldots y^{i_r}\]
of degree \( r \) on \( TM \). Then the function \( w_\nabla S \) corresponds to the covariant tensor of degree \( r + 1 \) which is the symmetrization of the covariant derivative of the tensor \( S_{i_1...i_r} \),
\[
(17) \quad \nabla (i_1 S_{i_2...i_{r+1}}) = \frac{1}{r+1} \sum_{\sigma \in C_{r+1}} \nabla_{i_{\sigma(1)}} S_{i_{\sigma(2)}...i_{\sigma(r+1)}},
\]
where \( C_{r+1} \) is the group of cyclic permutations of the set \{1, \ldots, r + 1\}.

Introduce the following operators,
\[D^k = g^{lk} \frac{\partial}{\partial \bar{z}^l}.
\]
It can be shown using \[\|\] that they pairwise commute. Consider a differential operator \( A \) that commutes with the multiplication operators by the variables \( \bar{z} \). It can be written in the normal form as a finite sum of operators of the form
\[A = \sum_{r} f_{i_1...i_r} (z, \bar{z}) \bar{D}^{i_1} \ldots \bar{D}^{i_r}.
\]
Its symbol is the following polynomial in \( \eta \),
\[\sigma(A) = \sum_{r} f_{i_1...i_r} (z, \bar{z}) \eta^{i_1} \ldots \eta^{i_r}.
\]
Using Lemma \[\|\] one can recover the symbol of \( A \) by the formula
\[
(18) \quad \left( e^{-\eta^j \frac{\partial}{\partial z^j}} A e^{\eta^j \frac{\partial}{\partial z^j}} \right) 1 = \sum_{r} f_{i_1...i_r} (z, \bar{z}) \eta^{i_1} \ldots \eta^{i_r}.
\]

Similarly, if an operator \( \tilde{A} \) commutes with the multiplication operators by the variables \( z \), it can be written as a finite sum
\[\tilde{A} = \sum_{r} \tilde{f}_{k_1...k_r} (z, \bar{z}) D^{k_1} \ldots D^{k_r}.
\]
Using the fact that
\[
\left[ D^k, \frac{\partial \Phi}{\partial z^p} \right] = \delta^k_p
\]
one can show that
\[(19) \quad \left( e^{-\eta^k \frac{\partial \Phi}{\partial z^k}} \tilde{A} e^{\eta^k \frac{\partial \Phi}{\partial z^k}} \right) 1 = \sum_r \tilde{f}_k \ldots k_r (z, \bar{z}) \eta^{k_1} \ldots \eta^{k_r}. \]

Recall that we obtain the contravariant derivative of a tensor by lifting the index of the covariant derivative,
\[\nabla^k = g^{ik} \nabla_i \quad \text{and} \quad \nabla^{\bar{i}} = g^{\bar{i}k} \nabla_k.\]

**Lemma 3.** Given a function \( f(z, \bar{z}) \), the following expressions,
\[D^{k_1} \ldots D^{k_r} f \quad \text{and} \quad \bar{D}^{\bar{l}_1} \ldots \bar{D}^{\bar{l}_r} f,\]
are symmetric contravariant tensors.

**Proof.** The fact that the Christoffel coefficients of the Kähler connection with mixed indices are all equal to zero implies that consecutively contravariantly differentiating a function \( f \) in holomorphic directions we obtain the following equality,
\[\nabla^{k_1} \ldots \nabla^{k_r} f = D^{k_1} \ldots D^{k_r} f.\]
This tensor is symmetric because the operators \( D^k \) pairwise commute.

Similarly,
\[\nabla^{\bar{l}_1} \ldots \nabla^{\bar{l}_r} f = \bar{D}^{\bar{l}_1} \ldots \bar{D}^{\bar{l}_r} f.\]
is a symmetric contravariant tensor. \(\Box\)

The standard star product with separation of variables can be uniquely written in the form
\[u * v = \sum_{r,s} T_{k_1 \ldots k_r} I_{l_1 \ldots l_s} (D^{k_1} \ldots D^{k_r} u) (\bar{D}^{l_1} \ldots \bar{D}^{l_s} v),\]
where \( T_{k_1 \ldots k_r l_1 \ldots l_s} \) is a formal covariant tensor separately symmetric in the indices \( k_i \) and \( \bar{l}_j \). Set
\[T := \sum_{r,s} T_{k_1 \ldots k_r} I_{l_1 \ldots l_s} \eta^{k_1} \ldots \eta^{k_r} \bar{\eta}^{l_1} \ldots \bar{\eta}^{l_s}.\]
We have from (18) and (19) that
\[(20) \quad T = e^{-\eta^k \frac{\partial \Phi}{\partial z^k} + \bar{\eta}^{\bar{l}} \frac{\partial \Phi}{\partial \bar{z}^\bar{l}}} \left( e^{\eta^k \frac{\partial \Phi}{\partial z^k} * e^{\bar{\eta}^{\bar{l}} \frac{\partial \Phi}{\partial \bar{z}^\bar{l}}} \right).\]
The tensor \( T \) completely determines the star product \( * \). It can be thought of as an invariant total symbol of the formal bidifferential operator (4) that determines the star product \( * \).

The operator \( Q \) is a global operator on \( TM \). To see it, consider the \((0,1)\)-component of the vector field \( w_\nabla \). Denote it by \( \nabla \). We have
\[\nabla = \bar{\eta}^{\bar{l}} \frac{\partial}{\partial z^{\bar{l}}} - \Gamma_{\bar{l} q}^{\bar{k}} \eta^{\bar{k}} \frac{\partial}{\partial \eta^q},\]
where

$$\Gamma^\bar{t}_{l\bar{q}} = g^\bar{t}{}_s g_{s\bar{l}}\bar{q} = \bar{D}^t \frac{\partial^2 \Phi}{\partial \bar{z}^l \partial \bar{z}^q}$$

is the Christoffel symbol of the Kähler connection with antiholomorphic indices. Lowering the upper index in the curvature tensor of the Kähler connection, we obtain the tensor

$$R_{kp\bar{l}\bar{q}} := g_{kp}^{\bar{n}} g_{\bar{m}\bar{l}\bar{q}} = \bar{D}^t \frac{\partial^2 \Phi}{\partial \bar{z}^l \partial \bar{z}^q}.$$  

(21)

It is separately symmetric in the holomorphic and antiholomorphic indices. Lifting the holomorphic indices in (21) we obtain a tensor with antiholomorphic indices only,

$$R^{\bar{l}_1\bar{l}_2}_{l\bar{q}} := g^{\bar{l}_1 k_1} g^{\bar{l}_2 k_2} R_{k_1 k_2 l\bar{q}}.$$  

It is separately symmetric in the lower and upper indices. It can be shown that

$$R^{\bar{l}_1\bar{l}_2}_{l\bar{q}} = -\bar{D}^{\bar{l}_1} \bar{D}^{\bar{l}_2} \frac{\partial^2 \Phi}{\partial \bar{z}^l \partial \bar{z}^q}.$$  

Given $r \geq 2$, contravariantly differentiating the tensor (22) $r - 2$ times in antiholomorphic directions we obtain the tensor

$$R^{\bar{l}_1...\bar{l}_r}_{l\bar{q}} := -\bar{D}^{\bar{l}_1} \ldots \bar{D}^{\bar{l}_r} \frac{\partial^2 \Phi}{\partial \bar{z}^l \partial \bar{z}^q}.$$  

The operator $Q$ can be written in an invariant form as follows:

$$Q = \nabla + \sum_{r=2}^{\infty} \frac{1}{r!} R^{\bar{l}_1...\bar{l}_r}_{l\bar{q}} \frac{\partial^r}{\partial \bar{\eta}^{\bar{l}_1} \ldots \partial \bar{\eta}^{\bar{l}_r}}.$$  

It is thus globally defined on $TM$. It follows from Theorem 1 and formula (19) that

$$T = e^{-\eta^k \frac{\partial \Phi}{\partial x^k}} (1 - \nu E^{-1} Q)^{-1} \left( e^{\eta^k \frac{\partial \Phi}{\partial x^k}} \right).$$

Equivalently, $T$ can be obtained by applying the operator

$$e^{-\eta^k \frac{\partial \Phi}{\partial x^k}} (1 - \nu E^{-1} Q)^{-1} e^{\eta^k \frac{\partial \Phi}{\partial x^k}}$$

to the unit constant. Observing that

$$e^{-\eta^k \frac{\partial \Phi}{\partial x^k}} Q e^{\eta^k \frac{\partial \Phi}{\partial x^k}} = Q + \gamma,$$

where $\gamma := g_{pq} \bar{\eta}^p \bar{\eta}^q$, we arrive at the following theorem.

Theorem 2. The tensor $T$ is given by the following invariant formula:

$$T = (1 - \nu E^{-1} (Q + \gamma))^{-1}.$$
According to [5], around every point $x$ of a pseudo-Kähler manifold for any $N$ one can choose normal holomorphic coordinates such that at $x$

$$g_{k_1,...,k_r,ar{l}_j} = 0 \text{ and } g_{k_1,...,k_r,ar{l}_s} = 0$$

for all $r, s \leq N$. For every $r, s \geq 2$ there exists a canonical tensor

$$R_{k_1...k_r,\bar{l}_1...\bar{l}_s}$$

separately symmetric in the indices $k_i$ and $\bar{l}_j$ such that it coincides with $-g_{k_1...k_r,\bar{l}_1...\bar{l}_s}$ at $x$ in normal coordinates around $x$ for a sufficiently large $N$ (see [7]). It is expressed through the tensor $g^{\bar{l}k}$ and covariant derivatives of the tensor (21). In particular,

$$R_{k_1...k_r,\bar{l}_1...\bar{l}_s} = \nabla_{k_1} \cdots \nabla_{k_{r-2}} R_{k_{r-1}k_1\bar{l}_1\bar{l}_2}$$

and

$$R_{k_1k_2...\bar{l}_s} = \nabla_{\bar{l}_1} \cdots \nabla_{\bar{l}_{s-2}} R_{k_1k_2\bar{l}_{s-1}\bar{l}_s}.$$

For $r \geq 2$, the tensor $R_{k_1...k_r,\bar{l}_1...\bar{l}_2}$ can be obtained from the tensor (23) by lowering the indices $\bar{l}_1,...,\bar{l}_r$. Similarly, for $s \geq 2$, the tensor $R_{kj\bar{l}_1...\bar{l}_s}$ can be obtained from the tensor

$$-D^{k_1} \cdots D^{k_r} \frac{\partial^2 \Phi}{\partial z^k \partial z^p}$$

by lowering the indices $k_1,...,k_s$. Set

$$\rho_{r,s} := R_{k_1...k_r,\bar{l}_1...\bar{l}_s} \eta^{k_1} \cdots \eta^{k_s} \bar{\eta}^{l_1} \cdots \bar{\eta}^{l_s}.$$

Using Theorem 2 and formula (17) we can easily calculate the tensor $T$ up to $\nu^4$, which allows to recover the operators $C_r$ for $r \leq 4$:

$$T = 1 + \nu \gamma + \frac{1}{2} \nu^2 \gamma^2 + \nu^3 \left( \frac{1}{6} \gamma^3 + \frac{1}{4} \rho_{2,2} \right) +$$

$$\nu^4 \left( \frac{1}{24} \gamma^4 + \frac{1}{4} \gamma \rho_{2,2} + \frac{1}{12} \rho_{2,3} + \frac{1}{12} \rho_{3,2} + \frac{1}{8} \bar{\rho} \right) + \ldots,$$

where

$$\bar{\rho} = R_{k_1k_2\bar{l}_1\bar{l}_2} g^{\bar{l}_1p_1} g^{\bar{l}_2p_2} R_{p_1p_2\bar{l}_1\bar{l}_2} \eta^{k_1} \eta^{k_2} \bar{\eta}^{l_1} \bar{\eta}^{l_2}.$$

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