Properties of the renormalized quark mass in the Schrödinger functional with a non–vanishing background field

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We compute the current quark mass in the Schrödinger functional with a non–vanishing background field at one loop order of perturbation theory. The results are used to obtain the critical mass at which the renormalized quark mass vanishes, and some lattice artefacts at one loop order.

1. THE SCHÖDINGER FUNCTIONAL

The Schrödinger Functional method treats QCD on a space-time cylinder \( L^3 \times T \).

\[
\exp(aC_k) \quad \exp(aC'_k)
\]

\( x_0 = 0 \)

\( x_0 = T \)

The gauge fields have Dirichlet boundary conditions in time direction:

\[
U(x,k)|_{x_0=0} = \exp\{aC_k\}, \quad U(x,k)|_{x_0=T} = \exp\{aC'_k\}
\]

in time direction. Here, \( C_k \) and \( C'_k \) are constant and diagonal, imposing a constant colour electric background field \( V(x,\mu) \).

The quark fields have Dirichlet boundary conditions in time direction:

\[
P_+ \psi(x)|_{x_0=0} = \rho(x), \quad P_- \psi(x)|_{x_0=T} = \rho'(x)
\]

\[
\bar{\psi}(x)P_-|_{x_0=0} = \bar{\rho}(x), \quad \bar{\psi}(x)P_+|_{x_0=T} = \bar{\rho}'(x)
\]

with the projectors

\[
P_\pm = \frac{1}{2}(1 \pm \gamma_0),
\]

and are periodic in space: \( \psi(x + \hat{k}L) = e^{i\theta} \psi(x) \), \( \bar{\psi}(x + \hat{k}L) = e^{-i\theta} \bar{\psi}(x) \).

The observables are gauge invariant combinations of the fields inside the cylinder, \( \psi(x), \bar{\psi}(x), U(x,\mu) \), and of the “boundary quark fields”

\[
\zeta(x) = \frac{\delta}{\delta \rho(x)}, \quad \zeta'(x) = \frac{\delta}{\delta \rho'(x)}.
\]

\[
\bar{\zeta}(x) = -\frac{\delta}{\delta \rho(x)}, \quad \bar{\zeta}'(x) = -\frac{\delta}{\delta \rho'(x)}.
\]

2. THE CURRENT QUARK MASS

2.1. Definition of the current quark mass

For the quark mass, we adopt the definition of \cite{1} based on the PCAC relation. For this purpose, we define the bare correlation functions

\[
f_A(x_0) = -\frac{a^3}{L^3} \sum_{x,y,z} \frac{1}{3} \left< A^a_0(x) \bar{\zeta}(y) \gamma_5 \frac{1}{2} \tau^a \zeta(z) \right>,
\]

\[
f_P(x_0) = -\frac{a^3}{L^3} \sum_{x,y,z} \frac{1}{3} \left< P^a(x) \bar{\zeta}(y) \gamma_5 \frac{1}{2} \tau^a \zeta(z) \right>,
\]

where \( A^a \) and \( P^a \) denote the axial current and density.
Now we can define the quark mass

\[
m(x_0) = \frac{1}{2} \left( \partial_0^2 + \partial_0 \right) f_A(x_0) + c_A a \partial_0 f_P(x_0)
\]

which depends on \(x_0\) due to cutoff effects. Here, \(c_A\) is the improvement coefficient which is needed to cancel the \(O(a)\) discretization error of the axial current.

As an unrenormalized quark mass, we take \(m\) in the centre of the lattice

\[
m_1 = \begin{cases} 
    m \left( \frac{T}{2} \right) & \text{for even } T/a, \\
    \frac{1}{2} \left( m \left( \frac{T-a}{2} \right) + m \left( \frac{T+a}{2} \right) \right) & \text{for odd } T/a.
\end{cases}
\]

(13)

Analogously, a mass \(m'\) can be computed using the boundary quark fields \(\zeta'\) and \(\zeta'\) at \(x_0 = T\) instead of \(\zeta\) and \(\zeta\). An alternative mass \(m_2\) may be defined as the average of \(m\) and \(m'\) in the centre of the lattice. \(m_1\) and \(m_2\) differ only because of lattice effects. Hence, in the improved theory, the difference between \(m_1\) and \(m_2\) is of order \(a^2\).

2.2. Aims

We want to compute the critical quark mass at which the renormalized mass vanishes at 1–loop order of perturbation theory, which amounts to calculating \(m_c^{(1)}\) in the series

\[
m_c = m_c^{(0)} + m_c^{(1)} g_0^2 + O(g_0^4).
\]

(14)

Since \(m_1\) is only renormalized multiplicatively, it is sufficient to require \(m_1 = 0\).

Furthermore, we want to estimate the size of the lattice artefacts in the mass calculation. In order to do this, we compute two different discretization errors at 1–loop order of perturbation theory. One is the difference

\[
d(L/a) = m_2(L/a) - m_1(L/a),
\]

(15)

the other one is the difference

\[
e(L/a) = m_1(2L/a) - m_1(L/a).
\]

(16)

For these purposes, \(f_A\) and \(f_P\) have to be expanded up to 1–loop order.

3. \(f_A\) AND \(f_P\) AT 1–LOOP ORDER

\(f_A\) and \(f_P\) may be written as

\[
f_{A,P}(x_0) = c_t^2 a^9 \sum_{x,y,z} \frac{1}{2} \langle \text{tr} \{ P_i \Gamma P_- \}

\]

\[
U(z - a \hat{0}, 0) S(z, x) \Gamma S(x, y)

\]

\[
U(y - a \hat{0}, 0)^{-1} \rangle \big|_{y_0 = z_0 = a}
\]

(17)

where \(\Gamma = \gamma_0 \gamma_5\) for \(f_A\) and \(\Gamma = \gamma_5\) for \(f_P\), and \(S(x,y)\) is the quark propagator. \(c_t\) is a coefficient needed for \(O(a)\) improvement.

\(S(x,y)\) and \(U(x, \mu)\) are expanded up to order \(g_0^2\), where \(U(x, \mu)\) is expanded around the background field \(V(x, \mu)\)

\[
U(x, \mu) = V(x, \mu) \left( 1 + g a \partial_\mu \right)

\]

\[
+ \frac{1}{2} g_3 a^2 q_\mu(x)^2 + O(g_0^4) \bigg). \quad (18)
\]

Collecting all contributions of order \(g_0^2\) amounts to summing the following diagrams.
The dotted lines are the links between $x_0 = 0$ and $x_0 = a$, and the cross denotes the insertion of the axial current or density.

4. RESULTS

4.1. The critical quark mass

The results for the critical mass at 1-loop order are shown below for the case of two light flavours, the quenched case and the bermion case (i.e. $N_f = -2$). They seem to converge quickly to the $N_f$ independent continuum limit $a m_c^{(1)} = -0.2700753495(2)$.

4.2. Lattice artefacts

The results for the lattice artefacts are shown below.

As one expects for discretization errors, the continuum limit is zero. The lattice artefacts are small at 1-loop level, but the perturbative results for $m_2(L/a) - m_1(L/a)$ appear to be bigger than the results obtained by Monte Carlo simulations.

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