Topological phase transitions in the 1D multichannel Dirac equation with random mass and a random matrix model

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Abstract – We establish the connection between a multichannel disordered model—the 1D Dirac equation with $N \times N$ matrix random mass—and a random matrix model corresponding to a deformation of the Laguerre ensemble. This allows us to derive exact determinantal representations for the density of states and identify its low-energy ($\varepsilon \to 0$) behaviour $\rho(\varepsilon) \sim |\varepsilon|^{\alpha - 1}$. The vanishing of the exponent $\alpha$ for $N$ specific values of the averaged mass over disorder ratio corresponds to $N$ phase transitions of topological nature characterised by the change of a quantum number (Witten index) which is deduced straightforwardly in the matrix model.

Since the pioneering work of Dorokhov-Mello-Pereyra-Kumar (DMPK) \cite{1,2}, models of multichannel disordered wires have played a prominent role in the theory of disordered systems as they allow to describe a situation intermediate between the strictly one-dimensional (1D) case and higher dimensions. With the dimensionality, another crucial aspect of disordered systems is the presence or not of symmetries, what may strongly affect the localisation properties. This has led to the classification within orthogonal, unitary and symplectic classes, depending on the existence of time reversal and spin rotational symmetries, according to the Wigner-Dyson classification for random matrices, labelled with the Dyson index $\beta \in \{1, 2, 4\}$. Several models of multichannel wires in these symmetry classes were studied by Brouwer and coworkers by extending the DMPK approach \cite{6–9}. The interest for such models has been recently renewed, as they can support topologically protected Majorana zero modes \cite{10,11}, which are stable against perturbations, like a small amount of disorder, what could be used for quantum computation \cite{10,12}.

When sufficiently strong, disorder can however drive topological phase transitions (quantum phase transitions associated with the change of a quantum number of topological nature) \cite{13–15}, which is the matter of interest in this letter.

Although multichannel models can be studied from a symmetry viewpoint \cite{16,17}, detailed analysis of specific models are also useful, in particular for the determination of non-universal properties (microscopic parameter dependences) \cite{18}. In this letter, we propose such a detailed study of 1D models belonging to the chiral classes of disordered systems, and establish the connection with a random matrix model defined by the following matrix distribution:

\begin{equation}
\begin{aligned}
f(Z) &= C_{N,\beta}^{-1} (\det Z)^{-\mu - 1 - \beta(N - 1)/2} \times \exp \left[ -\frac{1}{2} \text{tr} \left\{ G^{-1} (Z + k^2 Z^{-1}) \right\} \right] \\
&= C_{N,\beta}^{-1} (\det Z)^{-\mu - 1 - \beta(N - 1)/2} \times \exp \left[ -\frac{1}{2} \text{tr} \left\{ G^{-1} (Z + k^2 Z^{-1}) \right\} \right]
\end{aligned}
\end{equation}

with matrix argument in the sets of $N \times N$ Hermitian matrices with real ($\beta = 1$), complex ($\beta = 2$) or quaternionic ($\beta = 4$) elements and positive eigenvalues. Integration over these sets will be denoted $\int_{\mathbb{R}^+} DZ f(Z) = 1$, thus $C_{N,\beta}$ ensures normalisation. We will show that several interesting properties of the 1D Dirac equation with random mass (spectral properties and topological index)
can be straightforwardly obtained from this matrix distribution. In the “isotropic case”, when \( G = g1_N \), the distribution is invariant under unitary transformations. Then, eq. (1) interpolates between the Laguerre distribution \( L_{N,\theta}(Z) \propto \exp \left[-\mathrm{tr}(Z)\right] \) with \( \theta > 0 \) (obtained when \( k \to 0 \)) and the “inverse-Laguerre” distribution \( I_{N,\theta}(Z) \propto \exp \left[-\theta\mathrm{tr}(Z^{-1})\right] \) (obtained when \( k \to \infty \)).

The breaking of the invariance of \( f(Z) \) under unitary transformations, when \( G \) is not the identity matrix, is a further deformation of the distribution. Whereas most analytical studies of multichannel disordered models take as a crucial assumption the isotropy among the channels, most of our results will not rely on this hypothesis.

The outline of the letter is as follows: we define our model and introduce the scattering problem. Then we explain how the distribution (1) appears. This will provide the ground from which localisation and spectral properties will be recovered directly (in the isotropic case). The topological phase transitions, identified from the spectral properties, will be characterised through the calculation of a topological quantum number, achieved directly from a detailed analysis of the matrix distribution (1).

**The model.** – The disordered model is the 1D Dirac equation \( H\Psi(x) = \varepsilon\Psi(x) \) for 2N component spinors with

\[
H = i\sigma_2 \otimes 1_N \partial_x + \sigma_1 \otimes M(x) ,
\]

where \( \sigma_i \) being a Pauli matrix. The Hamiltonian exhibits a chiral symmetry \( \sigma_3 H \sigma_3 = -H \), which puts the problem in one of the three chiral classes, depending whether the random \( N \times N \) Hermitian matrix \( M(x) \) has real \( (\beta = 1) \), complex \( (\beta = 2) \) or quaternionic \( (\beta = 4) \) elements. The mass is uncorrelated in space, distributed according to

\[
P[M(x)] \propto e^{-(1/2)\int dx \mathrm{tr}(M(x)-\mu G)^2 G^{-1}(M(x)-\mu G)} .
\]

The correlations and the mean value \( \langle M(x) \rangle = \mu G \) are controlled by the same real symmetric matrix \( G \). The dimensionless parameter \( \mu \) is the averaged mass over disorder ratio and will play a central role as it drives the topological phase transitions.

**Scattering problem, Riccati matrix and random matrix process.** – Our starting point is a scattering formulation: we consider the Dirac equation on the half line with mass \( M(x) \) vanishing for \( x > L \) in order to set a scattering problem. We find gather to the \( N \) independent solutions of the Dirac equation in a \( 2N \times N \) “spinor”, \( \Psi = (\varphi^T, \chi^T)^T \), where the two “components” \( \varphi \) and \( \chi \) are two \( N \times N \) matrices. In the free region \( (x > L) \), we can write the spinor as the superposition of incoming and outgoing plane waves. For \( \varepsilon > 0 \) we have

\[
\Psi(x) = \left( \begin{array}{c} 1_N \\ i1_N \end{array} \right) e^{-i\varepsilon(x-L)} + \mathcal{S}(\varepsilon) \otimes \left( \begin{array}{c} 1_N \\ i1_N \end{array} \right) e^{+i\varepsilon(x-L)} ,
\]

where \( \mathcal{S}(\varepsilon) \) is the \( N \times N \) scattering matrix (here characterising the total reflection). Because the chiral symmetry relates positive and negative energies, we will always choose \( \varepsilon > 0 \) (see ref. [19] for a discussion of the \( \mathcal{S} \)-matrix symmetry). The study of the strictly 1D case \( (N = 1) \) has emphasized the role of a Riccati variable for providing the spectral and localisation informations [20–23]. We extend here this analysis to the multichannel case and emphasize the connection with the scattering problem. We define the Riccati matrix as \( Z_\varepsilon = -\varepsilon \chi \varphi^{-1} \), which obeys the matrix stochastic differential equation (mSDE):

\[
\partial_x Z_\varepsilon(x) = -\varepsilon^2 - Z_\varepsilon(x)^2 - M(x)Z_\varepsilon(x) - Z_\varepsilon(x)M(x) .
\]

The initial condition for \( \Psi(x) \) is chosen such that the chiral symmetry is preserved: we will choose either \( \varphi(0) = 0 \) (which corresponds to \( M(x) \to -\infty \) on \( \mathbb{R}_- \)) or \( \chi(0) = 0 \) \( (M(x) \to +\infty \) on \( \mathbb{R}_- \)) (see footnote 1). In this case, the Riccati matrix is Hermitian. Matching of (4) at the boundary reads \( \chi(L) \varphi(L)^{-1} = i(1_N + S)(I_N - S)^{-1} \) and allows to express the scattering matrix in terms of the Riccati matrix:

\[
\mathcal{S}(\varepsilon) = [\varepsilon - iZ_\varepsilon(L)] [\varepsilon + iZ_\varepsilon(L)]^{-1} .
\]

The mSDE (5) may then be related to a Fokker-Planck equation (FPE) describing the matrix random process. Without any further assumption (like the isotropy assumption assumed in [16–19,21]), setting a purely imaginary energy \( \varepsilon = i \kappa \in \mathbb{R}_i \), we have obtained that the stationary distribution for the matrix stochastic process is given by eq. (1) (see the supplementary material)

1A more general boundary condition ensuring the confinement of the particle on \( \mathbb{R}_+ \) is \( e^{2\theta \varepsilon} + \sin \theta \chi(0) = 0 \) [24,25] i.e., \( \cos \theta \varphi(0) + \sin \theta \chi(0) = 0 \). Only \( \theta = 0 \) or \( \pi /2 \) preserves the chiral symmetry.
occurs exactly at $\varepsilon = 0$ in the Dirac equation, as a consequence of the chiral symmetry. The energy $\varepsilon$ thus plays the role of the chiral-symmetry breaking parameter.

Localisation can be studied through the concept of Lyapunov exponents, which measure the exponential growth rate of the wave function envelope in different channels. For $N = 1$, the Lyapunov exponent of the model can be obtained exactly $\forall \varepsilon$ [20,21,23]. For $N > 1$, the determination of the Lyapunov spectrum is extremely complicated in general and is only known explicitly for isotropic disorder at the symmetry point ($\varepsilon = 0$) [6] thanks to an important simplification\(^2\) (see also ref. [16] for a broad perspective on Lyapunov spectra in the various symmetry classes). Instead of dealing with the FPE [6,8], we propose here a more direct derivation of the Lyapunov spectrum from the mSDE.

We consider $G = g1_N$. Taking our inspiration from the $N = 1$ case [26], when $\varepsilon = ik \in \mathbb{R}$ we introduce new variables $z_n = k e^{-2ik\varepsilon}$, which obey the set of coupled SDEs $\partial_k z_n = -k \sin 2\gamma_n + \mu g + 2(\mu + i) \sum_{m \neq n} \text{coth}(\gamma_n - \gamma_m) + m_n(x)$, where $m_n(x)$ are $N$ independent Gaussian white noises of zero mean. The generator of this diffusion coincides with the one involved in the FPE of ref. [8]. For $N = 1$, in the absence of the interaction term, we recover the SDE of [26,27]. For $z = ik = 0$, the vanishing of the confinement allows for a simple analysis of the dynamics as the repulsive forces saturate to $\pm 1$. Choosing $\zeta_1 < \zeta_2 < \cdots < \zeta_N$, one recovers the Lyapunov exponents $\gamma_n = \lim_{x \to \infty} \gamma_n(x)/x$ of ref. [6]\(^3\), directly from the analysis of the SDEs:

$$\gamma_n = \left[ \mu - \frac{\beta}{2}(N - 2n + 1) \right] g$$

for $n \in \{1, \ldots, N\}$. (7)

### Density of states.

The density of states (DoS) of multichannel 1D models in the chiral and BdG classes was studied in several papers when $\mu = 0$ [7,9]; it was shown that the low-energy DoS exhibits a strong dependence in the parity of the channel number $N$. Whereas the case of even $N$ leads to a low-energy DoS which depends on the symmetry index $\beta$, for odd $N$ the properties of the strictly 1D Dirac equation with random mass with zero mean were recovered, irrespectively of $\beta$: Dyson singularity of the DoS $\rho(\varepsilon) \sim 1/|\varepsilon \ln^3 |\varepsilon||$. These features were also obtained in two BdG classes with broken spin rotational symmetry (independently of the parity of $N$) [9]. The existence of common features for distinct symmetry classes was later denoted as “superuniversality” [29], a concept which was used recently for the study of quasi-1D spinless $p$-wave superconducting wires (BdG class D) [15,18].

The analysis of the stochastic process $Z_n(x)$ allows for a direct determination of the spectral properties, what was used in the $N = 1$ case [20–23]. Extending this idea when $N > 1$, we introduce the characteristic function $\Omega = \text{tr}\{\langle Z_n(x) + M(x) \rangle\}$, from which the DoS can be deduced. The vanishing or the divergence of $\text{det} Z_n(L)$, i.e., of one of its eigenvalues, corresponds to satisfy a second boundary condition for the spinor $\text{det} \chi(L) = 0$ or $\text{det} \varphi(L) = 0$ at $x = L$ for one of the eigenmodes. As a consequence the integrated DoS per unit length is given by $N(\varepsilon) = -(1/\pi) \text{Im} \Omega$, as for $N = 1$ [21–23]. The mean value $\langle Z_n(x) \rangle$ can be computed from the stationary distribution of the matrix process. Using the analyticity of $\Omega$, we can consider a purely imaginary energy $\varepsilon = ik \in \mathbb{R}$: the stationary distribution is then an equilibrium distribution, eq. (1), which highly simplifies the calculation of $\Omega$. Considering, without loss of generality, $G = \text{diag}\{g_1, \cdots, g_N\}$, we deduce the mean value from the normalisation constant $\langle Z_n(x) + M(x) \rangle_i = g_i^2 \text{tr} \text{C}_n,\beta/g_i$. For the isotropic case, $G = g1_N$, we write equivalently

$$\Omega = -g \left( N \mu + k \frac{\partial \text{tr} \text{C}_n,\beta}{\partial k} \right).$$

(8)

$\text{C}_n,\beta$ can then be written as an integral over the eigenvalues of the Riccati matrix, $\text{C}_n,\beta = \int_0^\infty dz_1 \cdots dz_N \prod_{i < j} |z_i - z_j|^\beta \prod_i \phi(z_i) = z^{\beta - 1 - \beta(N - 1)/2} \exp[-(z + k^2/2z)/(2g)]$, which is computed using standard technique of random matrix theory [30].

The unitary case ($\beta = 2$) is the simplest one: we obtain the form of a Hankel determinant

$$_{2}^{3}$$

\(C_{n,2} = N! 2^N k^{-N}\mu \text{ det} \{K_{m + 1 + N - i - j}(k/g)\},\) (9)

where $1 \leq i, j \leq N$. $k_i(z)$ is the MacDonald function [31]. We can deduce exact expressions for the DoS: we show in fig. 1 the DoS for $N = 2$ and $N = 3$ channels for various values of $\mu$. The case $\beta = 1$ is discussed in the SM. Assuming that (1) is also the stationary distribution for $\beta = 4$, we get the Pfaffian

$$_{2}^{3}$$

\(C_{n,4} = N! 2^N k^{-N}\mu \text{ pf} \{j - i K_{m + 1 + 2N - i - j}(k/g)\},\) (10)

where $1 \leq i, j \leq 2N$. For $N = 1$, we check that (9,10) give the known result [20,21]. Similar results were obtained for $\mu = 0$ by a slightly different approach in ref. [9].

We can analyse in detail the low-energy behaviour of the DoS, which involves the two leading order terms of a $k \to 0$ expansion of $\text{C}_{n,\beta}$. The DoS presents a power law behaviour $N(\varepsilon) \sim e^{\alpha}$. For $\beta = 2$, eq. (9) shows that the next leading order term of the $k \to 0$ expansion is controlled by an exponent which is a non-monotonic function of $\mu$, what originates from the expansion of $K_0(z)$. For $\beta = 1$ and 4, see the SM. The exponent is $\alpha = 2\mu - \beta(N - 1)$ for $\mu > \beta(N - 1)/2$ and vanishes $N$ times below this threshold, presenting a saw behaviour (fig. 2) (see the SM). Each
The dots correspond to the critical points where the IDoS presents a sign of superuniversality \([29]\)). When the ex-

For a continuous spectrum, \(\Delta(\tilde{\beta})\) presents a non-trivial \(\tilde{\beta}\)-dependence, however \(\Delta(\infty) = (\delta_+(0) - \delta_-(0))/(2\pi)\) provides the information on the number of zero modes.

As an elementary illustration, we consider first the free case \((g = 0)\), when the channels are uncoupled and the spectrum gapped. Setting \(M(x) = -\infty\) \((i.e., \varphi(0) = 0)\) and \(M(x) = m_0 l_N\) on \([0, L]\), we get the reflection phase \(\delta(\varepsilon)/2 = \arg(\kappa \cosh \kappa L - m_0 \sinh \kappa L + \varepsilon \sinh \kappa L e^{i\varepsilon/\sqrt{\kappa}})\), where \(\kappa = \sqrt{m_0^2 - \varepsilon^2}\). Letting first \(L \to \infty\) we deduce that \(\lim_{L \to 0} \delta(\varepsilon) = \pi \varepsilon_0(m_0) + \pi/2\), where \(\varepsilon_0(x)\) is the Heaviside function, leading to \(\Delta(\infty) = (N/2) \varepsilon_0(m_0)\) (see footnote \(5\)). Each channel is characterised by a Witten index equal to \(1/2\), which is interpreted as a fermion fractionization phenomenon characterising a topologically non-trivial state \([34]\). A crucial question is to understand how these states are affected by the disorder. Quite remarkably the zero modes, which are midgap states in the absence of disordered, exist in strongly disordered wires \([15, 18]\) with gapless spectrum, as we show below.

**Topological phase transitions.** – The spectral density (bulk information) was related to the mean value \((Z_{\varepsilon})\); the distribution \(f(Z)\) however contains a lot more of information.

We now determine the topological index \(\Delta(\infty)\) by a direct analysis of \((1)\). The relation between \(\Delta(\infty)\) and \(f(Z)\) is made more clear by using the supersymmetry of the Dirac equation: denoting \(Z_\varepsilon^\pm(x)\) the solution of \((5)\) for \((M(x)) = \pm \mu G\), we get \(Z_\varepsilon^-(x) \varepsilon_0^{-1} e^{2\varepsilon Z_\varepsilon^+(x)^{-1}} \) \([21]\) \((equality\ in\ law\ means\ the\ same\ statistical\ properties)\). For \(\varepsilon \in \mathbb{R}\) we deduce \((\delta_+(\varepsilon) - \delta_-(\varepsilon))/(2\pi) = (1/2) \sum_{n=1}^N (\varepsilon(z_n))\). For \(\varepsilon \in \mathbb{R}\), we get

\[
\frac{1}{2\pi} (\delta_+(ik) - \delta_-(ik)) = \frac{1}{2}\sum_{n=1}^N (\varepsilon(z_n - k)).
\]

Below, we will obtain \(\Delta(\infty)\) from \((1), (12)\).

For \(G = g l_N\), the distribution of the Riccati eigenvalues for \((M(x)) = \pm mg l_N\) is \(P_{\pm}(z_1, \cdots, z_N) \propto \prod_{j<k} |z_i - z_j|^\beta \prod_i \phi(z_i)\) where \(\phi(z) = z^{-\mu - 1/2} e^{-1/\beta(N-1)} \exp[-(z + k^2/2)]\) \((setting\ \mu = 1\ for\ simplicity)\). In the Coulomb gas approach, \(P_{\pm}\) is interpreted as the Gibbs measure for a 1D gas of \(N\) “charges” trapped in a confining potential \(-\ln \phi(z)\) and with logarithmic interactions. We show below that the limit \(k \to 0\) involves the distribution of eigenvalues of the Laguerre ensemble \(L_{N_0, \theta}(\lambda_1, \cdots, \lambda_N) = A_{N_0,\theta}^{1/2} \prod_{j} |\lambda_i - \lambda_j|^{\beta} \prod_{i<j} \phi(z_i - z_j)^{\beta} \prod_i \tau_i^{-\beta(N-1)/2} e^{-1/\beta(N-1)}\), normalisable for \(\theta > 0\), and the one of the “inverse-Laguerre” ensemble \(L_{N_0, \theta}(\tau_1, \cdots, \tau_N) = A_{N_0,\theta}^{1/2} \prod_{i<j} |\tau_i - \tau_j|^{\beta} \prod_i \tau_i^{-\beta(\theta(N-1)-1/2)} e^{1/(2\pi \tau_i)}\), i.e., the distribution of \(\tau_i = 1/\lambda_i^{\theta}\). We first remark that, for

3For finite \(L \gg 1/m_0\), the midgap state is split into two low-energy states, \(\varepsilon_{\pm} \approx \pm 2m_0 e^{-\pi L/4}\) \((for\ \varphi(0) = \chi(L) = 0)\), which is also reflected in the phase shift. Distribution of the ground-state energy has been obtained in \([25, 27]\).
One of the spectra (7) plotted in fig. 3: this suggests that the creation of a pair of zero modes can be possible by the delocalisation in one of the \( N \) channels.

Conclusion. In this article we have shown a connection between multichannel 1D disordered models with\( \theta = \mu - \beta(N-1)/2 > 0 \), the limit \( k \to 0 \) of the joint distribution \( P_{\cdot}(z_1, \ldots, z_N) \) corresponds to the Laguerre ensemble. Equation (12) leads to \( \Delta(\infty) = N/2 \). For \( \tilde{\mu} = \mu - \beta(N-2n-1)/2 \in [0, \beta] \) with \( n \in \{1, \ldots, N\} \), the limit \( k \to 0 \) is more subtle: we obtain that the N charges split into two independent sets, as the distribution behaves as

\[
P_{\cdot}(z_1, \ldots, z_N) \propto \frac{1}{k^{2n}} T_{n, \beta - \tilde{\mu}} \left( \frac{z_1}{k^2}, \ldots, \frac{z_n}{k^2} \right) \times \mathcal{L}_{N-n, \beta} (z_{n+1}, \ldots, z_N).
\]

This distribution describes the “condensation” of \( n \) charges towards the origin, while the \( N-n \) remaining charges are described by a distribution which “freezes” as \( k \to 0 \). Interestingly, the parameter \( k \) drives a phase transition in the Coulomb gas (the splitting of the charges into two groups), which occurs for a finite \( N \), whereas several phase transitions were observed up to now in the thermodynamic limit \( N \to \infty \) in various contexts [35–37]. As the non-isotropic case, \( \gamma \) can be generalised. Using these generalisations and the distribution (1), we were able to show that both the exponent \( \alpha \) and the Witten index present the same dependence with \( \mu \) as in the isotropic case (figs. 2 and 3). The Lyapunov analysis is much more difficult to extend. Using a formalism based on the QR decomposition, we have obtained analytical expressions for the Lyapunov exponents (at \( \gamma = 0 \) for \( N = 2 \) channels, in the unitary case \( \beta = 2 \)) with non-isotropic disorder

\[
\gamma_1 = 2 - \frac{g_1 g_2}{g_1 + g_2} + \frac{\mu^2}{g_1 g_2} - \frac{g_1 + g_2}{g_1 g_2} \left( 1 + \mu - \mu \right) - \frac{g_2}{g_1 + g_2} \left( 1 + \mu - \mu \right)
\]

where \( B(z; a, b) \) is the incomplete Beta function, and \( F_1 \) is the Appell hypergeometric function with two arguments:

\[
F_1(a, b, b'; c, x, y) = \sum_{m \geq 0, n \geq 0} \frac{(a)_m (b)_n (b')_n}{(c)_{m+n} m! n!} x^m y^n,
\]

with \( (a)_n = \Gamma(a + n)/\Gamma(a) \). The second Lyapunov exponent follows from the sum rule \( \sum_n \gamma_n = \mu \text{ tr } \{G\} \) (valid at \( \gamma = 0 \)), \( \gamma = \mu (g_1 + g_2) \gamma_1 \). For \( N > 2 \), we have performed a numerical analysis. Our results also exhibit the vanishing of one Lyapunov at each phase transition, as expected (fig. 5). Details will be published elsewhere.
Although we are providing here the first exact results for a non-isotropic multichannel disorder, to the best of our knowledge, we stress that we have not considered the most general situation, which would involve the distribution of disorder of the form \[ \int dx \text{Tr} \{ A^{-1}M(x)B^{-1}M(x) \} \]. This remains a challenging problem.

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