ON THE $d$-DIMENSIONAL ALGEBRAIC CONNECTIVITY OF GRAPHS

BY

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Dedicated to Nati Linial on the occasion of his 70th birthday

ABSTRACT

The $d$-dimensional algebraic connectivity $a_d(G)$ of a graph $G = (V, E)$, introduced by Jordán and Tanigawa, is a quantitative measure of the $d$-dimensional rigidity of $G$ that is defined in terms of the eigenvalues of stiffness matrices (which are analogues of the graph Laplacian) associated to mappings of the vertex set $V$ into $\mathbb{R}^d$.

Here, we analyze the $d$-dimensional algebraic connectivity of complete graphs. In particular, we show that, for $d \geq 3$, $a_d(K_{d+1}) = 1$, and for $n \geq 2d$,

$$\left\lfloor \frac{n}{2d} \right\rfloor - 2d + 1 \leq a_d(K_n) \leq \frac{2n}{3(d-1)} + \frac{1}{3}.$$
1. Introduction

A \textit{d-dimensional framework} is a pair \((G, p)\) consisting of a graph \(G = (V, E)\) and a mapping of its vertices \(p : V \to \mathbb{R}^d\). A framework \((G, p)\) is called \textbf{rigid} if every continuous motion of the vertices that preserves the lengths of all the edges of \(G\), preserves in fact the distance between every two vertices of \(G\).

The stricter notion of \textbf{infinitesimal rigidity} is defined as follows. For two distinct vertices \(u, v \in V\), let \(d_{uv} \in \mathbb{R}^d\) be defined by

\[
d_{uv} = \begin{cases}
\frac{p(u) - p(v)}{\|p(u) - p(v)\|} & \text{if } p(u) \neq p(v), \\
0 & \text{otherwise},
\end{cases}
\]

and let \(v_{u,v} \in \mathbb{R}^{d|V|}\) be defined by

\[
v_{u,v}^t = \begin{pmatrix} u & v \\
0 & \ldots & 0 & d_{uv} & 0 & \ldots & 0 & d_{vu} & 0 & \ldots & 0 \end{pmatrix}.
\]

Equivalently, \(v_{u,v} = (1_u - 1_v) \otimes d_{uv}\), where \(\{1_u\}_{u \in V}\) is the standard basis of \(\mathbb{R}^{|V|}\) and \(\otimes\) denotes the Kronecker product.

The \textbf{normalized rigidity matrix} of \((G, p)\), denoted by \(R(G, p)\), is the \(d|V| \times |E|\) matrix whose columns are the vectors \(v_{u,v}\) for all \(\{u, v\} \in E\).

Assume that \(|V| > d\). It is known (see [3]) that the rank of \(R(G, p)\) is at most \(|V| - \left(\frac{d+1}{2}\right)\). The framework \((G, p)\) is called \textbf{infinitesimally rigid} if the rank of the \(R(G, p)\) is exactly \(|V| - \left(\frac{d+1}{2}\right)\).\(^1\)

The study of infinitesimal rigidity of graphs has a long history, starting with the study of the rigidity of convex polyhedrons by Dehn [7] (following previous work by Cauchy in the non-infinitesimal setting [6]); see also [1, 10, 3, 4]. Although infinitesimal rigidity is in general stronger than rigidity, it is known that both notions coincide for “generic” embeddings (see [3]). Note that for \(d = 1\), and assuming that \(p\) is injective, both notions of rigidity coincide with the notion of graph connectivity.

Recently Jordán and Tanigawa [12] (building on Zhu and Hu [15, 16] who considered the 2-dimensional case) introduced the following quantitative measure of rigidity:

\(^1\) The usual definition of infinitesimal rigidity is in terms of the unnormalized rigidity matrix, but both definitions are equivalent as scaling of the columns of a matrix does not change its rank.
The stiffness matrix $L(G, p)$ is defined by
\[ L(G, p) = R(G, p) R(G, p)^t \in \mathbb{R}^{|V| \times |V|}. \]
It is easy to check that the rank of $L(G, p)$ equals the rank of $R(G, p)$. Therefore, the kernel of $L(G, p)$ is of dimension at least $(d + 1) \frac{1}{2}$, and equality occurs if and only if the framework $(G, p)$ is infinitesimally rigid.

Let $\lambda_i(L(G, p))$ be the $i$-th smallest eigenvalue of $L(G, p)$. The spectral gap of $L(G, p)$ is its minimal non-trivial eigenvalue $\lambda_{(d + 1)\frac{1}{2} + 1}(L(G, p))$. The $d$-dimensional algebraic connectivity of $G$ is defined by
\[ a_d(G) = \sup \{ \lambda_{(d + 1)\frac{1}{2} + 1}(L(G, p)) \mid p : V \to \mathbb{R}^d \} . \]
For $d = 1$, $a_1(G)$ is the usual algebraic connectivity (a.k.a. spectral gap) of $G$, introduced by Fiedler in [9]. For general $d$, we always have $a_d(G) \geq 0$ (since $L(G, p)$ is positive semi-definite) and $a_d(G) > 0$ if and only if a generic embedding of $G$ in $\mathbb{R}^d$ forms a rigid framework.

Let $K_n$ be the complete graph on $n$ vertices. In [12], a lower bound on $a_d(K_n)$ was used to deduce an improved constant in a threshold result for $d$-rigidity [13]: there exists a constant $C_d$ such that if $pn > C_d \log n$ then a graph $G \in G(n, p)$, the Erdős–Rényi $n$-vertex random graph with edge probability $p$, is asymptotically almost surely $d$-rigid. (For a sharp threshold for $d$-rigidity see our recent [14].) Their estimate on $C_d$ depended on the value of the spectral gap of the stiffness matrix of the regular $d$-simplex graph $K_{d+1}$, denoted $s_d$, which was conjectured to equal 1 for $d \geq 3$.

Motivated by these results, we study in this paper the $d$-dimensional algebraic connectivity of complete graphs. It is well known and easy to check that $a_1(K_n) = n$. For $d = 2$, it was shown by Jordán and Tanigawa [12, Theorem 4.4], based on a result by Zhu [15], that $a_2(K_n) = n/2$ for all $n \geq 3$ (see also Proposition 4.4). For $d \geq 3$ the situation is more complicated, and the only previously known result is the lower bound $a_d(K_n) \geq dn/(2(d + 1)^2) - d$, proved by Jordán and Tanigawa in [12, Thm. 5.2].

Here, we first focus on the case $n = d + 1$. Let $p^\Delta : V(K_{d+1}) \to \mathbb{R}^d$ denote the regular simplex embedding (that is, the vertices of $K_{d+1}$ are mapped bijectively to the vertices of a regular simplex in $\mathbb{R}^d$), and denote
\[ s_d = \lambda_{(d + 1)\frac{1}{2} + 1}(L(K_{d+1}, p^\Delta)) . \]
We prove the following:
Theorem 1.1: The spectrum of $L(K_{d+1}, p^\Delta)$ is
\[
\left\{ 0^{(d(d+1)/2)}, 1^{((d+1)(d-2)/2)}, \frac{d+1}{d+1}^{(d)} , d+1^{(1)} \right\}.
\]
(The superscript $(m)$ indicates multiplicity $m$ of the corresponding eigenvalue, here and throughout the paper.)

This settles a conjecture of Jordán and Tanigawa [12, Conj. 1]. In particular, we obtain $s_d = 1$ for $d \geq 3$. (Note that $s_2 = 3/2$.) Further, we show that this is the largest possible spectral gap for a framework $(K_{d+1}, p)$. That is,

Theorem 1.2: For $d \geq 3$, $a_d(K_{d+1}) = 1$.

However, for $d \geq 3$, $p^\Delta$ is not the only embedding that achieves the maximum value $a_d(K_{d+1}) = 1$, see Proposition 3.1.

Next we consider (balanced) Turán graphs: Let $r, n$ be positive integers such that $r$ divides $n$. Let $V_1, \ldots, V_r$ be pairwise disjoint sets such that $|V_i| = n/r$ for all $i \in [r]$. Let $V = V_1 \cup \cdots \cup V_r$ and $E = \bigcup_{i \neq j \in [r]} \{\{u, v\} : u \in V_i, v \in V_j\}$. The graph $T(n, r) = (V, E)$ is called a Turán graph, or the complete balanced $r$-partite graph on $n$ vertices.

For $r = d+1$ let $q^\Delta : V \to \mathbb{R}^d$ denote the mapping into the vertices of a regular $d$-simplex, such that the preimage of each vertex of the simplex equals $V_i$ for a different $i \in [d+1]$ (note that, for $|V| = d+1$, $q^\Delta$ coincides with the definition of $p^\Delta$ above).

We compute the spectrum of $L(T(n, d+1), q^\Delta)$:

Theorem 1.3: Let $d \geq 2$ and $n \geq d+1$ such that $n$ is divisible by $d+1$. Then, the spectrum of $L(T(n, d+1), q^\Delta)$ is
\[
\left\{ 0^{(d(d+1)/2)}, \frac{n}{2(d+1)}^{((n-d-1)(d-1))}, \frac{n}{d+1}^{((d-2)(d+1)/2)}, \frac{n(n-1)}{2}^{(1)} \right\}.
\]

In particular, its spectral gap for $n \geq 2(d+1)$ is
\[
\lambda_{(d+1)/2}^{(d+1)}(L(T(n, d+1), q^\Delta)) = \frac{n}{2(d+1)}.
\]

This improves upon the lower bound
\[
\lambda_{(d+1)/2}^{(d+1)}(L(T(n, d+1), q^\Delta)) \geq \frac{dn}{2(d+1)^2}
\]
obtained in [12] (after plugging $s_d = 1$).
Similarly, for \( r = 2d \) let \( q^* : V \to \mathbb{R}^d \) denote the mapping into the vertices of a regular \( d \)-crosspolytope, such that the preimage of each vertex of the crosspolytope equals \( V_i \) for a different \( i \in [2d] \) (recall that the regular \( d \)-crosspolytope is defined as the convex hull of the set \( \{ \pm e_1, \ldots, \pm e_d \} \), where \( \{ e_1, \ldots, e_d \} \) is the standard basis of \( \mathbb{R}^d \)). We compute the spectrum of \( L(T(n, 2d), q^*) \):

**Theorem 1.4:** Let \( d \geq 2 \) and \( n \geq 2d \) such that \( n \) is divisible by \( 2d \). Then, the spectrum of \( L(T(n, 2d), q^*) \) is

\[
\left\{ 0^{(d(d+1)/2)} \cdot \frac{n(6d-2^3)}{2d}, \frac{n(6d-2^3)}{d}, \frac{n}{2} \cdot n^{(1)} \right\}.
\]

In particular, its spectral gap is

\[
\lambda_{\frac{d+1}{2}}(L(T(n, 2d), q^*)) = \frac{n}{2d}.
\]

The proofs of Theorems 1.3 and 1.4 follow by computing the eigenbases of the corresponding stiffness matrices. As a corollary of Theorem 1.4, we obtain the following lower bound on \( a_d(K_n) \):

**Theorem 1.5:** Let \( d \geq 3 \) and \( n \geq 2d \). If \( n \) is divisible by \( 2d \), then

\[
a_d(K_n) \geq \frac{n}{2d}.
\]

For general \( n \geq 2d \), we have \( a_d(K_n) \geq \left\lfloor \frac{n}{2d} \right\rfloor - 2d + 1 \).

This improves the previously known lower bound. Finally, we prove an upper bound on the \( d \)-dimensional algebraic connectivity of the complete graph:

**Theorem 1.6:** Let \( d \geq 3 \) and \( n \geq d + 1 \). Then

\[
a_d(K_n) \leq \frac{2n}{3(d - 1)} + \frac{1}{3}.
\]

This follows by proving a lower bound on the sum of the \( n \) largest eigenvalues of \( L(K_n, p) \) for every embedding \( p \) into \( \mathbb{R}^d \) (see Lemma 5.1).

Most of our results rely on analyzing the lower stiffness matrix of the framework \( (G, p) \), defined by

\[
L^-(G, p) = R(G, p)^t R(G, p) \in \mathbb{R}^{|E| \times |E|}.
\]

It is easy to check that \( \text{rank}(L(G, p)) = \text{rank}(L^-(G, p)) = \text{rank}(R(G, p)) \), and that the non-zero eigenvalues of \( L(G, p) \) are the same as those of \( L^-(G, p) \), namely also with the same multiplicities.
2. The lower stiffness matrix

We start with the following explicit description of $L^-(G, p)$:

**Lemma 2.1:** Let $G = (V, E)$ be a graph and let $p : V \to \mathbb{R}^d$. Let $e_1, e_2 \in E$. Then,

$$L^-(G, p)(e_1, e_2) = \begin{cases} 
2 & \text{if } e_1 = e_2 = \{i, j\} \text{ and } p(i) \neq p(j), \\
\cos(\theta(e_1, e_2)) & \text{if } |e_1 \cap e_2| = 1, \\
0 & \text{otherwise},
\end{cases}$$

where, for $e_1 = \{i, j\}$ and $e_2 = \{i, k\}$, $\theta(e_1, e_2)$ is the angle between $d_{ij} \text{ and } d_{ik}$; that is, $\cos(\theta(e_1, e_2)) = d_{ij} \cdot d_{ik}$ (note that, by convention, if $d_{ij} = 0$ or $d_{ik} = 0$, then $\cos(\theta(e_1, e_2)) = 0$).

**Proof.** For convenience, we identify the vertex set of $G$ with the set $[n]$. Let $e_1 = \{i, j\}$ and $e_2 = \{k, l\}$, where $i < j$ and $k < l$. Let $\{1_i\}_{i \in V}$ be the standard basis for $\mathbb{R}^{|V|}$, and $\{1_e\}_{e \in E}$ be the standard basis for $\mathbb{R}^{|E|}$. Then

$$R(G, p)1_{e_1} = (1_i - 1_j) \otimes d_{ij},$$

and similarly

$$R(G, p)1_{e_2} = (1_k - 1_l) \otimes d_{kl}.$$

Note that

$$L^-(G, p)(e_1, e_2) = (R(G, p)1_{e_1})^t(R(G, p)1_{e_2}) = ((1_i - 1_j) \cdot (1_k - 1_l))(d_{ij} \cdot d_{kl}).$$
If $e_1 = e_2$, we obtain
\[ L^{-}(G, p)(e_1, e_2) = \|1_i - 1_j\|^2 \cdot \|d_{ij}\|^2 = \begin{cases} 2 & \text{if } p(i) \neq p(j), \\ 0 & \text{otherwise.} \end{cases} \]

If $e_1 \cap e_2 = \emptyset$, then
\[ L^{-}(G, p)(e_1, e_2) = (1_i - 1_j) \cdot (1_k - 1_l) (d_{ij} \cdot d_{kl}) = 0 \cdot (d_{ij} \cdot d_{kl}) = 0. \]

Finally, assume $|e_1 \cap e_2| = 1$. Then, either $i = k$, or $i = l$, or $j = k$ or $j = l$.
If $i = k$, then
\[ L^{-}(G, p)(e_1, e_2) = 1 \cdot (d_{ij} \cdot d_{ik}) = \cos(\theta(e_1, e_2)). \]
If $i = l$, then
\[ L^{-}(G, p)(e_1, e_2) = (-1) \cdot (d_{ij} \cdot d_{kl}) = d_{ij} \cdot d_{ik} = \cos(\theta(e_1, e_2)). \]
The other two cases follow similarly.

Remark: In [2], the lower stiffness matrix $L^{-}(G, p)$ was studied in the special case where $G = K_3$, the complete graph on three vertices, and $p$ is an embedding of the vertices in $\mathbb{R}^2$.

2.1. INTERLACING OF SPECTRA. We will use the following special case of Cauchy’s interlacing theorem:

**Theorem 2.2** (See, e.g., [5]): Let $A$ be a real symmetric matrix of size $n \times n$ and $B$ a principal submatrix of $A$ of size $(n-1) \times (n-1)$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the eigenvalues of $A$ and $\mu'_1 \geq \mu'_2 \geq \cdots \geq \mu'_{n-1}$ be the eigenvalues of $B$. Then, for $1 \leq i \leq n - 1$, we have
\[ \mu_i \geq \mu'_i \geq \mu_{i+1}. \]

We obtain the following interlacing result, generalizing a known result for graph Laplacians (see, e.g., [11, Theorem 13.6.2]):

**Theorem 2.3**: Let $G = (V, E)$ be a graph with $|V| = n$, and let $p : V \to \mathbb{R}^d$. Let $e \in E$, and let $G \setminus e = (V, E \setminus \{e\})$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{dn}$ be the eigenvalues of $L(G, p)$ and $\lambda'_1 \leq \cdots \leq \lambda'_{dn}$ be the eigenvalues of $L(G \setminus e, p)$. Let $\lambda_0 = 0$. Then, we have
\[ \lambda_{i-1} \leq \lambda'_i \leq \lambda_i, \]
for all $1 \leq i \leq dn$. 
Proof. Let \( \mu_1' \geq \cdots \geq \mu_{|E|}' \) be the eigenvalues of \( L^- (G, p) \) and \( \mu_1 \geq \cdots \geq \mu_{|E|-1} \) be the eigenvalues of \( L^- (G \setminus e, p) \).

Note that \( L^- (G \setminus \{e\}, p) \) is a principal submatrix of \( L^- (G, p) \), therefore by Theorem 2.2, we have

\[
\mu_i \geq \mu_i' \geq \mu_{i+1}
\]

for \( i = 1, \ldots, |E| - 1 \).

For \( i > |E| - 1 \), let \( \mu_i' = 0 \), and for \( i > |E| \), let \( \mu_i = 0 \). Then, we have

\[
\mu_i \geq \mu_i' \geq \mu_{i+1}
\]

for all \( i \). Since \( L(G, p) \) and \( L^- (G, p) \) have the same positive eigenvalues, we have for all \( i = 1, \ldots, dn \)

\[
\lambda_{dn+1-i} = \mu_i,
\]

and similarly

\[
\lambda'_{dn+1-i} = \mu_i'.
\]

Therefore, we have

\[
\lambda_{dn+1-i} \geq \lambda'_{dn+1-i} \geq \lambda_{dn-i}
\]

for all \( i = 1, \ldots, dn \) (using \( \lambda_0 = 0 \)). So, for \( j = 1, \ldots, dn \), we obtain

\[
\lambda_{j-1} \leq \lambda'_j \leq \lambda_j,
\]

as wanted.  

As an application of Theorem 2.3, we show that restricting attention to maps \( p : V \rightarrow \mathbb{R}^d \) that are embeddings (i.e., injective) does not affect the \( d \)-dimensional algebraic connectivity of a graph \( G = (V, E) \).

**Lemma 2.4:** Let \( G = (V, E) \) and \( d \geq 1 \). Then

\[
a_d(G) = \sup \{ \lambda_{(d+1)/2} (L(G, p)) \mid p : V \rightarrow \mathbb{R}^d, p \text{ is injective} \}.
\]

**Proof.** Let

\[
\tilde{a}_d(G) = \sup \{ \lambda_{(d+1)/2} (L(G, p)) \mid p : V \rightarrow \mathbb{R}^d, p \text{ is injective} \}.
\]

Clearly, \( \tilde{a}_d (G) \leq a_d (G) \). In the other direction, let \( p : V \rightarrow \mathbb{R}^d \). We will show that for any \( \epsilon > 0 \) there exists a \( p' : V \rightarrow \mathbb{R}^d \) such that \( p' \) is injective and

\[
\lambda_{(d+1)/2} (L(G, p')) > \lambda_{(d+1)/2} (L(G, p)) - \epsilon.
\]

Let

\[
E' = \{ \{u, v\} \in E : p(u) \neq p(v) \}
\]
and let \( G' = (V, E') \). Note that \( L(G, p) = L(G', p) \), and in particular
\[
\lambda_{\left\lfloor \frac{d+1}{2} \right\rfloor + 1}(L(G, p)) = \lambda_{\left\lfloor \frac{d+1}{2} \right\rfloor + 1}(L(G', p)).
\]

By Lemma 2.1, for \( p' \) in a neighborhood of \( p \), the entries of the lower stiffness matrix \( L^-(G', p') \) are continuous functions of \( p' \). Therefore, the spectral gap \( \lambda_{\left\lfloor \frac{d+1}{2} \right\rfloor + 1}(L(G', p')) \) is also continuous in \( p' \) (as a root of the characteristic polynomial of \( L^-(G', p') \)). That is, there exists \( \delta > 0 \) such that if \( \|p(u) - p'(u)\| < \delta \) for all \( u \in V \), then
\[
|\lambda_{\left\lfloor \frac{d+1}{2} \right\rfloor + 1}(L(G', p')) - \lambda_{\left\lfloor \frac{d+1}{2} \right\rfloor + 1}(L(G', p))| < \epsilon.
\]

Now, let \( p' : V \to \mathbb{R}^d \) be an embedding satisfying \( \|p(u) - p'(u)\| < \delta \) for all \( u \in V \). Then, we have
\[
\lambda_{\left\lfloor \frac{d+1}{2} \right\rfloor + 1}(L(G', p')) > \lambda_{\left\lfloor \frac{d+1}{2} \right\rfloor + 1}(L(G', p)) - \epsilon = \lambda_{\left\lfloor \frac{d+1}{2} \right\rfloor + 1}(L(G, p)) - \epsilon.
\]

Finally, by Theorem 2.3, we obtain
\[
\lambda_{\left\lfloor \frac{d+1}{2} \right\rfloor + 1}(L(G', p')) \geq \lambda_{\left\lfloor \frac{d+1}{2} \right\rfloor + 1}(L(G', p')) > \lambda_{\left\lfloor \frac{d+1}{2} \right\rfloor + 1}(L(G, p)) - \epsilon.
\]

Thus, \( \tilde{a}_d(G) \geq a_d(G) \), as wanted. \( \blacksquare \)

3. The \( d \)-dimensional algebraic connectivity of the simplex graph

It is known (see [12]) that \( a_1(K_n) = n \) and \( a_2(K_n) = n/2 \). In particular, \( a_2(K_3) = 3/2 \). In this section we prove the following:

**Theorem 1.2:** For \( d \geq 3 \), \( a_d(K_{d+1}) = 1 \).

3.1. **Lower bound:** \( a_d(K_{d+1}) \geq 1 \). Recall that \( p^\Delta : V \to \mathbb{R}^d \) is the embedding that maps each vertex of \( K_{d+1} \) to one of the vertices of the regular \( d \)-dimensional simplex. The lower bound follows from the following result, conjectured in [12, Conj. 1]:

**Theorem 1.1:** The spectrum of \( L(K_{d+1}, p^\Delta) \) is
\[
\left\{ 0^{\left\lfloor \frac{d(d+1)}{2} \right\rfloor}, 1^{\left\lfloor \frac{(d+1)(d-2)}{2} \right\rfloor}, \frac{d + 1^{(d)}}{2}, d + 1^{(1)} \right\}.
\]

**Proof.** Let \( K_{d+1} = (V, E) \), where \( V = [d + 1] \) and \( E = \binom{[d+1]}{2} \). Since the angle between every two intersecting edges of the regular simplex is 60°, we have,
by Lemma 2.1,

\[ L^-(K_{d+1}, p^\Delta)(e_1, e_2) = \begin{cases} 2 & \text{if } e_1 = e_2, \\ \frac{1}{2} & \text{if } |e_1 \cap e_2| = 1, \\ 0 & \text{otherwise}, \end{cases} \]

for every \( e_1, e_2 \in E \). We can write

\[ L^-(K_{d+1}, p^\Delta) = I + \frac{1}{2} Q, \]

where \( Q \in \mathbb{R}^{|E| \times |E|} \) is defined by

\[ Q(e_1, e_2) = \begin{cases} 2 & \text{if } e_1 = e_2, \\ 1 & \text{if } |e_1 \cap e_2| = 1, \\ 0 & \text{otherwise}, \end{cases} \]

for every \( e_1, e_2 \in E \).

Let \( M \in \mathbb{R}^{(d+1) \times |E|} \) be the signless incidence matrix of \( K_{d+1} \), defined by

\[ M(i, e) = \begin{cases} 1 & \text{if } i \in e, \\ 0 & \text{otherwise}, \end{cases} \]

for \( i \in V = [d+1] \) and \( e \in E \). Then, we have

\[ Q = M^t M. \]

Let

\[ \tilde{Q} = MM^t \in \mathbb{R}^{(d+1) \times (d+1)}. \]

The matrix \( \tilde{Q} \) is the signless Laplacian of \( K_{d+1} \), namely

\[ \tilde{Q}(i, j) = \begin{cases} d & \text{if } i = j, \\ 1 & \text{otherwise}. \end{cases} \]

Therefore, \( \tilde{Q} = (d - 1)I + J \), where \( J \) is the all-ones matrix. The spectrum of \( J \) is \( \{0^{(d)}, d + 1^{(1)}\} \); therefore, the spectrum of \( \tilde{Q} \) is \( \{d - 1^{(d)}, 2d^{(1)}\} \). Since the non-zero eigenvalues of \( Q \) and \( \tilde{Q} \) are the same, the spectrum of \( Q \) is

\[ \{0^{((d-2)(d+1)/2)}, d - 1^{(d)}, 2d^{(1)}\}. \]

Thus, the spectrum of \( L^-(K_{d+1}, p^\Delta) \) is

\[ \{1^{((d-2)(d+1)/2)}, \frac{d + 1^{(d)}}{2}, d + 1^{(1)}\}. \]
Finally, since the non-zero eigenvalues of \(L^-(K_{d+1}, p^\Delta)\) and \(L(K_{d+1}, p^\Delta)\) are the same, the spectrum of \(L(K_{d+1}, p^\Delta)\) is
\[
\left\{ 0^{(d/2)(d+1)/2}, 1, \left((d-2)(d+1)/2\right), \frac{d+1}{2}, d+1 \right\},
\]
as wanted.

As a consequence of Theorem 1.2, we obtain \(a_d(K_{d+1}) \geq 1\) for all \(d \geq 3\). We are left to show that \(a_d(K_{d+1}) \leq 1\). Before doing that, let us remark that for \(d \geq 3\) there are embeddings \(p \neq p^\Delta\) such that \(\lambda_{(d+1)+1}^d(L(K_{d+1}, p)) = 1\):

**Proposition 3.1:** Let \(p_1, p_2, p_3 \in \mathbb{R}^2\) be the vertices of an equilateral triangle with sides of length 1 centered at the origin. For \(h \geq 0\), let \(p_h : [4] \to \mathbb{R}^3\) be defined by
\[
p_h(i) = \begin{cases} (p_i, 0) & \text{if } i \in [3], \\ (0, h) & \text{if } i = 4. \end{cases}
\]
Then, the spectrum of \(L(K_4, p_h)\) is
\[
\left\{ 0^{(6)}, 1^{(2)}, \left(3 - \left(h^2 + \frac{1}{3}\right)^{-1}\right)^{(1)}, \left(\frac{3}{2} + \frac{1}{2} \left(h^2 + \frac{1}{3}\right)^{-1}\right)^{(2)}, 4^{(1)} \right\}.
\]
In particular, for \(h \geq \frac{1}{\sqrt{6}}\), we have \(\lambda_7(L(K_4, p_h)) = 1\).

**Proof.** Denote the edges of the tetrahedron formed by the image of \(p_h\) by \(e_{ij}\), \(1 \leq i \neq j \leq 4\). Let \(\ell\) be the length of the edges \(e_{ij}\) (for \(i \in [3]\)). Note that \(\ell = \sqrt{h^2 + \frac{1}{3}}\). It is easy to check that for three distinct indices \(i, j, k\)
\[
\cos(\theta(e_{ij}, e_{jk})) = \begin{cases} \frac{1}{2} & \text{if } i, j, k \in [3], \\ \frac{1}{2\ell} & \text{if } i, j \in [3], k = 4, \\ \frac{1}{2\ell^2} & \text{if } i, k \in [3], j = 4. \end{cases}
\]
Therefore, by Lemma 2.1, we have
\[
L^-(K_4, p_h) = \begin{pmatrix}
2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2\ell} & \frac{1}{2\ell} & 0 \\
\frac{1}{2} & 2 & \frac{1}{2} & \frac{1}{2\ell} & 0 & \frac{1}{2\ell} \\
\frac{1}{2} & \frac{1}{2} & 2 & 0 & \frac{1}{2\ell} & \frac{1}{2\ell} \\
\frac{1}{2\ell} & \frac{1}{2\ell} & 0 & 2 & 1 - \frac{1}{2\ell^2} & 1 - \frac{1}{2\ell^2} \\
\frac{1}{2\ell} & 0 & \frac{1}{2\ell} & 1 - \frac{1}{2\ell^2} & 2 & 1 - \frac{1}{2\ell^2} \\
0 & \frac{1}{2\ell} & \frac{1}{2\ell} & 1 - \frac{1}{2\ell^2} & 1 - \frac{1}{2\ell^2} & 2
\end{pmatrix}.
\]
The spectrum of $L^{-}(K_4,p_h)$ can now be computed with the help of a computer algebra system. We obtain the following eigenvalues:

$$1^{(2)}, (3 - ℓ^{-2})^{(1)}, \left(\frac{3}{2} + \frac{1}{2} ℓ^{-2}\right)^{(2)}, 4^{(1)}.$$  

Since the non-zero eigenvalues of $L^{-}(K_4,p_h)$ are the same as those of $L(K_4,p_h)$, and $ℓ^2 = h^2 + \frac{1}{3}$, the spectrum of $L(K_4,p_h)$ is

$$\{0^{(6)}, 1^{(2)}, (3 - (h^2 + \frac{1}{3})^{-1})^{(1)}, \left(\frac{3}{2} + \frac{1}{2} (h^2 + \frac{1}{3})^{-1}\right)^{(2)}, 4^{(1)}\},$$

as wanted. Finally, note that for $h \geq \frac{1}{\sqrt{6}}$ we obtain

$$3 - (h^2 + \frac{1}{3})^{-1} \geq 3 - 2 = 1,$$

and therefore $\lambda_7(K_4,p_h) = 1$. □

### 3.2. Upper Bound: $a_d(K_{d+1}) \leq 1$

First, we prove the 3-dimensional case:

**Proposition 3.2:**

$$a_3(K_4) = 1.$$  

**Proof.** Let $p : V \rightarrow \mathbb{R}^3$. Note that, since $|E| = 6 = dn - \binom{d+1}{2}$ (for $d = 3$ and $n = 4$),

$$\lambda_7(L(K_4,p)) = \lambda_{\binom{d+1}{2}+1}(L(K_4,p))$$

is equal to the minimal eigenvalue of $L^{-}(K_4,p)$. Thus, for every $0 \neq x \in \mathbb{R}^6$

$$\lambda_7(L(K_4,p)) \leq \frac{x^tL^{-}(K_4,p)x}{\|x\|^2}.$$  

Let $e_1, e_1', e_2, e_2', e_3, e_3'$ be the edges of $K_4$, such that $e_i \cap e_i' = \emptyset$ for all $i$. Let $\ell_1, \ell_1', \ell_2, \ell_2', \ell_3, \ell_3'$ be the lengths of the images of $e_1, e_1', e_2, e_2', e_3, e_3'$ (that is, if $e_i = \{u,v\}$, $\ell_i = \|p(u) - p(v)\|$).

Assume without loss of generality that $\ell_3^2 + \ell_3'^2 \leq \min\{\ell_1^2 + \ell_1'^2, \ell_2^2 + \ell_2'^2\}$. Let

$$x = \ell_1 1_{e_1} + \ell_1' 1_{e_1'} - \ell_2 1_{e_2} - \ell_2' 1_{e_2'}.$$  

Then, $\|x\|^2 = \ell_1^2 + \ell_1'^2 + \ell_2^2 + \ell_2'^2$, and

$$x^tL^{-}(K_4,p)x = 2\|x\|^2 - 2(\ell_1\ell_2 \cos(\theta(e_1,e_2)) + \ell_1\ell_2' \cos(\theta(e_1,e_2'))) + \ell_1'\ell_2 \cos(\theta(e_1',e_2)) + \ell_1'\ell_2' \cos(\theta(e_1',e_2'))).$$

By the law of cosines, we obtain

$$x^tL^{-}(K_4,p)x = 2\|x\|^2 + 2(\ell_3^2 + \ell_3'^2 - \ell_1^2 - \ell_1'^2 - \ell_2^2 - \ell_2'^2) = 2(\ell_3^2 + \ell_3'^2).$$
So
\[ \frac{x^t L^-(K_4, p)x}{\|x\|^2} = \frac{2(\ell_3^2 + \ell_1^2)}{\ell_1^2 + \ell_1^2 + \ell_2^2 + \ell_2^2} \leq 1. \]

Therefore, we obtain
\[ \lambda_7(L(K_4, p)) \leq 1. \]

Finally, we show that the bound for general \( d \) follows from the \( d = 3 \) case:

**Proposition 3.3:** For all \( d \geq 4 \),
\[ a_d(K_{d+1}) \leq a_3(K_4) = 1. \]

**Proof.** Let \( K_{d+1} = (V, E) \), where \( V = [d + 1] \) and \( E = \binom{[d+1]}{2} \). We will show that for every \( d \),
\[ a_d(K_{d+1}) \leq a_{d-1}(K_d). \]

Let \( G \) be the graph obtained by adding an isolated vertex \( v \) to \( K_d \). Then, \( G \) is obtained from \( K_{d+1} \) by removing the \( d \) edges containing \( v \). Therefore, for every \( p : V \to \mathbb{R}^d \), we have by Theorem 2.3
\[ \lambda_1, (d+1) + 1 (L(K_{d+1}, p)) \leq \lambda_1, (d+1) + d+1 (L(G, p)). \]

Let \( H \) be an affine hyperplane containing \( p(V \setminus \{v\}) \). Identify \( H \) with \( \mathbb{R}^{d-1} \), and let \( p' = p|_{V \setminus \{v\}} : V \setminus \{v\} \to \mathbb{R}^{d-1} \). Note that \( L^-(G, p) = L^-(K_d, p') \); therefore, the non-zero eigenvalues of \( L(G, p) \) and \( L(K_d, p') \) are the same. Since \( L(K_d, p') \in \mathbb{R}^{(d-1)d \times (d-1)d} \) and \( L(G, p) \in \mathbb{R}^{d(d+1) \times d(d+1)} \), this means in particular that
\[ \lambda_1, (d+1) + 1 (L(K_d, p')) = \lambda_1, (d+1) + 2d+1 (L(G, p)) = \lambda_1, (d+1) + d+1 (L(G, p)). \]

So, we obtain
\[ \lambda_1, (d+1) + 1 (L(K_{d+1}, p)) \leq \lambda_1, (d+1) + 1 (L(K_d, p')) \leq a_{d-1}(K_d). \]

Since this holds for every \( p : V \to \mathbb{R}^d \), we obtain
\[ a_d(K_{d+1}) \leq a_{d-1}(K_d), \]
as wanted. Therefore, the bound
\[ a_d(K_{d+1}) \leq a_3(K_4) = 1 \]
follows by induction on \( d \).
4. Spectra of Turán graphs $T(n, d+1)$ and $T(n, 2d)$

Recall that $T(n, r)$ is the complete balanced $r$-partite graph with $n$ vertices. For $r = d + 1$, we denote by $q^\Delta : [n] \to \mathbb{R}^d$ the mapping that maps the vertices of each of the $d + 1$ parts of $T(n, d+1)$ to one of the vertices of the regular $d$-dimensional simplex. Similarly, we denote by $q^\diamond : [n] \to \mathbb{R}^d$ the mapping that maps the vertices of each of the $2d$ parts of $T(n, 2d)$ to one of the vertices of the regular $d$-dimensional crosspolytope. In this section we determine the spectra of $L(T(n, d+1), q^\Delta)$ and $L(T(n, 2d), q^\diamond)$. In fact, in each case we provide a basis of $\mathbb{R}^E$ consisting of eigenvectors of the corresponding lower stiffness matrix.

Note that $L(K_n, q^\Delta) = L(T(n, d+1), q^\Delta)$ and $L(K_n, q^\diamond) = L(T(n, 2d), q^\diamond)$. As a consequence, we obtain a lower bound on the $d$-dimensional algebraic connectivity of $K_n$ (Theorem 1.5).

We begin with the following result of Jordán and Tanigawa:

**Lemma 4.1 ([12, Lemma 4.3]):** Let $p : [n] \to \mathbb{R}^d$. If $p$ is not constant, then the largest eigenvalue of $L(K_n, p)$ is $n$.

In [12] it was shown that $p$ itself (when considered as a vector in $\mathbb{R}^{dn}$) is an eigenvector of $L(K_n, p)$ with eigenvalue $n$. The corresponding eigenvector for $L^{-}(K_n, p)$ is $\phi = R(G, p)^t p$, which satisfies

$$\phi(i, j) = \|p(i) - p(j)\|$$

for each $i \neq j \in [n]$.

The following result shows that for mappings $p$ satisfying certain “spherical symmetry”, $n/2$ is an eigenvalue of $L(K_n, p)$ of high multiplicity:

**Proposition 4.2:** Let $p : [n] \to \mathbb{R}^d$ such that $\|p(i)\| = 1$ for all $i \in [n]$ and $\sum_{i=1}^{n} p(i) = 0$. Assume that the image of $p$ is of size at least 3. Then, $n/2$ is an eigenvalue of $L(K_n, p)$ with multiplicity at least $n - 1$.

**Proof.** Since the non-zero eigenvalues of $L(K_n, p)$ and $L^{-}(K_n, p)$ are the same, it is enough to look at $L^{-}(K_n, p)$.

Let $f \in \mathbb{R}^n$ such that $\sum_{i=1}^{n} f_i = 0$. For every $i \neq j \in [n]$, let $\ell_{ij} = \|p(i) - p(j)\|$. Let $E = E(K_n) = \{\{i, j\} : 1 \leq i < j \leq n\}$, and let $\phi_f \in \mathbb{R}^E$ be defined by

$$\phi_f(i, j) = (f_i + f_j)\ell_{ij}.$$ 

We will show that $\phi_f$ is an eigenvector of $L^{-}(K_n, p)$ with eigenvalue $n/2$. 

For $i,j,k \in [n]$, let $\theta_{ijk}$ be the angle between $p(i) - p(j)$ and $p(k) - p(j)$. By the law of cosines, we have
\[
\ell_{ij}\ell_{jk}\cos(\theta_{ijk}) = \frac{1}{2}(\ell_{ij}^2 + \ell_{jk}^2 - \ell_{ik}^2).
\]
Let $I' \in \mathbb{R}^{E \times E}$ be a diagonal matrix with elements $I'_{e,e} = 1$ if $e = \{i, j\}$ where $p(i) \neq p(j)$ and $I'_{e,e} = 0$ otherwise. Let $A = L^-(K_n,p) - 2I'$. Let $e = \{i, j\} \in E$.

First, assume that $p(i) \neq p(j)$. Then, by Lemma 2.1, we have
\[
A\phi_f(e) = \sum_{k \neq i,j} (\cos(\theta_{ijk})\phi_f(\{j,k\}) + \cos(\theta_{jik})\phi_f(\{i,k\}))
\]
\[
= \sum_{k \neq i,j} (\ell_{jk}\cos(\theta_{ijk})(f_j + f_k) + \ell_{ik}\cos(\theta_{jik})(f_i + f_k))
\]
\[
= \sum_{k \neq i,j} \left( \frac{\ell_{ij}^2 + \ell_{jk}^2 - \ell_{ik}^2}{2\ell_{ij}}(f_j + f_k) + \frac{\ell_{ij}^2 + \ell_{ik}^2 - \ell_{jk}^2}{2\ell_{ij}}(f_i + f_k) \right)
\]
\[
= \sum_{k \neq i,j} \ell_{ij}f_k + \sum_{k \neq i,j} (f_i + f_j) \frac{\ell_{ij}}{2} + \frac{f_i - f_j}{2\ell_{ij}} \sum_{k \neq i,j} (\ell_{jk}^2 - \ell_{ik}^2).
\]

Note that, since $\sum_{k=1}^n f_k = 0$, we have
\[
\sum_{k \neq i,j} f_k = -(f_i + f_j).
\]

Also, since $\|p(x)\| = 1$ for all $x \in [n]$, for all $x, y \in [n]$ we have
\[
\ell_{xy}^2 = \|p(x)\| + \|p(y)\|^2 - 2p(x) \cdot p(y) = 2 - 2p(x) \cdot p(y).
\]

So, since $\sum_{k=1}^n p(k) = 0$, we obtain
\[
\sum_{k \neq i,j} (\ell_{jk}^2 - \ell_{ik}^2) = 2(p(i) - p(j)) \cdot \sum_{k \neq i,j} p(k)
\]
\[
= 2(p(j) - p(i)) \cdot (p(i) + p(j)) = 2(\|p(j)\| - \|p(i)\|^2) = 0.
\]

Therefore, we obtain
\[
A\phi_f(e) = -(f_i + f_j)\ell_{ij} + \frac{n-2}{2}(f_i + f_j)\ell_{ij} = \frac{n-4}{2}\phi_f(e).
\]

So, $L^-(K_n,p)\phi_f(e) = (n/2)\phi_f(e)$. Now, assume $p(i) = p(j)$. Note that $\phi_f(e) = (f_i + f_j)\ell_{ij} = 0$. Then, since $\cos(\theta_{ijk}) = 0$ and $\cos(\theta_{jik}) = 0$ for all $k \neq i, j$, we obtain
\[
A\phi_f(e) = \sum_{k \neq i,j} (\cos(\theta_{ijk})\phi_f(\{j,k\}) + \cos(\theta_{jik})\phi_f(\{i,k\})) = 0,
\]
and therefore $L^-(K_n, p) \phi_f(e) = 0 = (n/2) \phi_f(e)$. Thus, $\phi_f$ is an eigenvector of $L^-(K_n, p)$ with eigenvalue $n/2$.

Finally, we will show that the dimension of the subspace 

$$U = \left\{ \phi_f : f \in \mathbb{R}^n, \sum_{i=1}^n f_i = 0 \right\}$$

is $n - 1$. Indeed, let $W = \{ f \in \mathbb{R}^n : \sum_{i=1}^n f_i = 0 \}$. Clearly $\dim(W) = n - 1$. We have $U = \Phi(W)$, where $\Phi \in \mathbb{R}^{|E| \times n}$ is defined by

$$\Phi(e, i) = \begin{cases} 
\ell_{ij} & \text{if } e = \{i, j\} \text{ for some } j \in [n], \\
0 & \text{otherwise.}
\end{cases}$$

Let $g \in \mathbb{R}^n$ such that $\Phi g = 0$. Let $i \in [n]$. Since the image of $p$ is of size at least 3, there exist $j, k \in [n]$ such that $p(i), p(j), p(k)$ are pairwise distinct. Then, we have

$$0 = (\Phi g)_{i,j} = (g_i + g_j) \ell_{ij},$$
$$0 = (\Phi g)_{i,k} = (g_i + g_k) \ell_{ik},$$
$$0 = (\Phi g)_{j,k} = (g_j + g_k) \ell_{jk}.$$ 

We obtain $g_i = -g_j = g_k = -g_i$. That is, $g_i = 0$. Therefore, $g = 0$. Thus, $\Phi$ has linearly independent columns, and so

$$\dim(U) = \dim(\Phi(W)) = \dim(W) = n - 1,$$

as wanted.

Hence, $n/2$ has multiplicity at least $n - 1$ as an eigenvalue of $L^-(K_n, p)$ (and thus also as an eigenvalue of $L(K_n, p)$). 

We conjecture that for the mappings considered in Proposition 4.2, $n/2$ is the second largest eigenvalue:

**Conjecture 4.3:** Let $d \geq 3$, and let $p : [n] \to \mathbb{R}^d$ such that $\|p(i)\| = 1$ for all $i \in [n]$ and $\sum_{i=1}^n p(i) = 0$. Assume that the image of $p$ is of size at least 3. Then, the second largest eigenvalue of $L(K_n, p)$ is $n/2$, and its multiplicity is exactly $n - 1$.

In the case $d = 2$ we can say more:
**Proposition 4.4:** Let \( p : [n] \to \mathbb{R}^2 \) such that \( \|p(i)\| = 1 \) for all \( i \in [n] \) and \( \sum_{i=1}^{n} p(i) = 0 \). Assume that \( p \) is injective. Then, the spectrum of \( L(K_n, p) \) is
\[
\{0(3), \frac{n}{2}(2n-4), n(1)\}.
\]

Proposition 4.4 extends Zhu’s result [15, Remark 3.4.1], which states that the same conclusion holds under the more restrictive assumption that \( p \) maps \([n]\) to the roots of unity of order \( n \). In particular, this shows that there are infinitely many embeddings \( p : [n] \to \mathbb{R}^2 \) attaining the supremum \( a_2(K_n) = n/2 \). For completeness, we include a proof, following the proof of the special case sketched by Zhu in [15].

**Proof.** Let \( x, y \in \mathbb{R}^{2n} \) be defined by
\[
x = (1, 0, 1, 0, \ldots, 1, 0)^t, \quad y = (0, 1, 0, 1, \ldots, 0, 1)^t.
\]

For \( i \in [n] \), let \( p_x(i), p_y(i) \) be the two coordinates of \( p(i) \), and let
\[
p^\perp(i) = (-p_y(i), p_x(i))^t.
\]

Define \( r \in \mathbb{R}^{2n} \) by
\[
r = (p_x^\perp(1), p_y^\perp(1), p_x^\perp(2), p_y^\perp(2), \ldots, p_x^\perp(n), p_y^\perp(n))^t
= (-p_y(1), p_x(1), -p_y(2), p_x(2), \ldots, -p_y(n), p_x(n))^t.
\]

It is easy to check (using the fact that \( \sum_{i=1}^{n} p(i) = 0 \)) that \( x, y, r \) belong to the kernel of \( L(K_n, p) \), and moreover, form an orthogonal set.

We identify the mapping \( p \) with the vector
\[
p = (p_x(1), p_y(1), \ldots, p_x(n), p_y(n))^t \in \mathbb{R}^{2n}.
\]

By Lemma 4.1, \( p \) is an eigenvector of \( L(K_n, p) \) with eigenvalue \( n \).

We will show that
\[
L(K_n, p) = \frac{n}{2} I + \frac{1}{2} (pp^t - xx^t - yy^t - rr^t).
\]

Then, it immediately follows that every vector in \( \mathbb{R}^{2d} \) orthogonal to \( p, x, y \) and \( r \) is an eigenvector of \( L(K_n, p) \) with eigenvalue \( n/2 \), as wanted.

We can write \( L(K_n, p) \) as an \( n \times n \) block matrix (see [12, Section 4.4]), formed by \( 2 \times 2 \) blocks
\[
[L(K_n, p)]_{i,j} = -d_{ij} d_{ij}^t
\]
for $i \neq j \in [n]$, and

$$[L(K_n, p)]_{i,i} = \sum_{j \in [n] \setminus \{i\}} d_{ij} d_{ij}^t$$

for $i \in [n]$. It is then easy to check that proving (1) is equivalent to showing that, for all $i \in [n]$

$$\sum_{j \in [n] \setminus \{i\}} d_{ij} d_{ij}^t = \frac{1}{2}p(i)p(i)^t - \frac{1}{2}p^\perp(i)(p^\perp(i))^t + \frac{n-1}{2}I,$$

and for all $i \neq j \in [n]$

$$-d_{ij} d_{ij}^t = \frac{1}{2}p(i)p(j)^t - \frac{1}{2}p^\perp(i)(p^\perp(j))^t - \frac{1}{2}I.$$

First, note that (2) follows from (3). Indeed, let $i \in [n]$. By (3), and using the fact that $\sum_{j \in [n] \setminus \{i\}} p(j) = -p(i)$ (and similarly, $\sum_{j \in [n] \setminus \{i\}} p^\perp(j) = -p^\perp(i)$), we obtain

$$\sum_{j \in [n] \setminus \{i\}} d_{ij} d_{ij}^t = \sum_{j \in [n] \setminus \{i\}} \left( \frac{1}{2}I - \frac{1}{2}p(i)p(j)^t + \frac{1}{2}p^\perp(i)(p^\perp(j))^t \right)$$

$$= \frac{n-1}{2}I - \frac{1}{2}p(i) \left( \sum_{j \in [n] \setminus \{i\}} p(j)^t \right) + \frac{1}{2}p^\perp(i) \left( \sum_{j \in [n] \setminus \{i\}} p^\perp(j)^t \right)$$

$$= \frac{n-1}{2}I + \frac{1}{2}p(i)p(i)^t - \frac{1}{2}p^\perp(i)p^\perp(i)^t.$$

So, we are left to show that (3) holds for all $i \neq j \in [n]$. Let $i \neq j \in [n]$, and let

$$M = d_{ij} d_{ij}^t + \frac{1}{2}p(i)p(j)^t - \frac{1}{2}p^\perp(i)(p^\perp(j))^t - \frac{1}{2}I.$$

We will show that $M = 0$. We denote the four coordinates of $M$ by $M_{xx}, M_{xy}, M_{yx}$ and $M_{yy}$. Since $\|p(i)\| = \|p(j)\| = 1$, we have

$$\|p(i) - p(j)\|^2 = 2(1 - p(i) \cdot p(j)),$$

and therefore

$$d_{ij} d_{ij}^t = \frac{(p(i) - p(j))(p(i) - p(j))^t}{2(1 - p(i) \cdot p(j))}. $$
Using this and the fact that $p_x(i)^2 + p_y(i)^2 = p_x(j)^2 + p_y(j)^2 = 1$, we obtain

\[
M_{xx} = \frac{(p_x(i) - p_x(j))(p_y(i) - p_y(j))}{2(1 - p_x(i)p_x(j) - p_y(i)p_y(j))} + \frac{1}{2}p_x(i)p_x(j) - \frac{1}{2}p_y(i)p_y(j) - \frac{1}{2}.
\]

Similarly, $M_{yy} = 0$. Finally, we have

\[
M_{xy} = \frac{(p_x(i) - p_x(j))(p_y(i) - p_y(j))}{2(1 - p_x(i)p_x(j) - p_y(i)p_y(j))} + \frac{1}{2}p_x(i)p_y(j) + \frac{1}{2}p_y(i)p_x(j) = 0.
\]

and similarly $M_{yx} = 0$. So $M = 0$ as wanted. \(\square\)

The following very simple Lemma will be useful for finding the additional eigenvectors of $L^-(T(n, d + 1), q^\infty)$ and $L^-(T(n, 2d), q^\infty)$.

**Lemma 4.5:** Let $G = (V, E)$ and $p : V \to \mathbb{R}^d$ such that $p(i) \neq p(j)$ for all $\{i, j\} \in E$. Let $L^- = L^-(G, p)$ and $\phi \in \mathbb{R}^E$. Let $\text{supp}(\phi) \subset E$ be the support of $\phi$. Assume that the following conditions hold:

1. For all $e \in E \setminus \text{supp}(\phi)$, $\cos(\theta(e, e')) = \cos(\theta(e, e''))$ for every $e', e'' \in \text{supp}(\phi)$ such that $|e \cap e'| = |e \cap e''| = 1$.
2. For all $v \in V$, $\sum_{e \in E : e \ni v} \phi(e) = 0$.
3. There is $\lambda \in \mathbb{R}$ such that for all $e \in \text{supp}(\phi)$

\[
\sum_{e' \in E : |e \cap e'| = 1} \cos(\theta(e, e')) \phi(e') = (\lambda - 2)\phi(e).
\]

Then, $\phi$ is an eigenvector of $L^-$ with eigenvalue $\lambda$. 

Proof. By Lemma 2.1, conditions (1) and (2) imply that \( L^\phi(e) = 0 = \phi(e) \) for every \( e \in E \setminus \text{supp}(\phi) \), and condition (3) says exactly that \( L^\phi(e) = \lambda \phi(e) \) for all \( e \in \text{supp}(\phi) \). Therefore, we obtain \( L^\phi = \lambda \phi \), as wanted. \( \blacksquare \)

4.1. The spectrum of \( L(T(n, d + 1), q^\triangle) \).

Theorem 1.3: Let \( d \geq 2 \) and \( n \geq d + 1 \) such that \( n \) is divisible by \( d + 1 \). Then, the spectrum of \( L(T(n, d + 1), q^\triangle) \) is

\[
\{ 0^{\lfloor d(d+1)/2 \rfloor}, \frac{n}{2(d+1)}^{((n-d-1)(d-1))}, \frac{n}{d+1}^{((d-2)(d+1)/2)}, \frac{n(n-1)}{2}, n^{(1)} \}.
\]

Note that for \( n = d + 1 \) the proof below gives a second proof of Theorem 1.1.

Proof. Denote \( T(n, d + 1) = (V, E) \). First note that, as for all frameworks, 0 is an eigenvalue of \( L(T(n, d + 1), q^\triangle) \) with multiplicity at least \( \lfloor d(d+1)/2 \rfloor = d(d+1)/2 \).

Moreover, note that \( L(T(n, d + 1), q^\triangle) = L(K_n, q^\triangle) \). Thus, by Lemma 4.1, \( n \) is an eigenvalue of \( L(T(n, d + 1), q^\triangle) \). Also, since \( \|q^\triangle(v)\| = 1 \) for all \( v \in V \) and

\[
\sum_{v \in V} q^\triangle(v) = 0,
\]

then by Proposition 4.2, \( n/2 \) is an eigenvalue of \( L(T(n, d + 1), q^\triangle) \) with multiplicity at least \( n - 1 \).

Therefore, we are left to show that \( n/(d+1) \) is an eigenvalue with multiplicity at least \( (d-2)(d+1)/2 \) and that \( n/(2(d+1)) \) is an eigenvalue with multiplicity at least \( (n-d-1)(d-1) \).

Since the non-zero eigenvalues of

\[
L(T(n, d + 1), q^\triangle) \quad \text{and} \quad L^- = L^-(T(n, d + 1), q^\triangle)
\]

are the same, we will find the corresponding eigenvectors for \( L^- \).

Denote by \( V_1, \ldots, V_{d+1} \) the sides of \( T(n, d + 1) \). For every \( i \neq j \in [d+1] \), let \( E(V_i, V_j) \) be the set of edges with one endpoint in \( V_i \) and the other in \( V_j \). Let \( i, j, k \in [d + 1] \) such that \( i \neq j, k \). Let \( v \in V_i, u \in V_j \) and \( w \in V_k \). Denote by \( \theta_{uvw} \) the angle between \( q^\triangle(v) - q^\triangle(u) \) and \( q^\triangle(v) - q^\triangle(w) \). Then, we have

\[
\cos(\theta_{uvw}) = \begin{cases} 
\frac{1}{2} & \text{if } j \neq k, \\
1 & \text{if } j = k.
\end{cases}
\]
EIGENVALUE $n/(d + 1)$. Assume $d \geq 3$. Let $i_1, i_2, i_3, i_4 \in [d+1]$ be four distinct integers. Define $\Phi_{i_1, i_2, i_3, i_4} \in \mathbb{R}^E$ by

$$
\Phi_{i_1, i_2, i_3, i_4}(e) = \begin{cases} 
1 & \text{if } e \in E(V_{i_1}, V_{i_2}) \cup E(V_{i_3}, V_{i_4}), \\
-1 & \text{if } e \in E(V_{i_2}, V_{i_3}) \cup E(V_{i_1}, V_{i_4}), \\
0 & \text{otherwise.}
\end{cases}
$$

Note that for every $e \notin \text{supp}(\Phi_{i_1, i_2, i_3, i_4})$ and every $e' \in \text{supp}(\Phi_{i_1, i_2, i_3, i_4})$ such that $|e \cap e'| = 1$, $\cos(\theta(e, e')) = 1/2$, and that for all $v \in V$, we have

$$
\sum_{e \in E : v \in e} \Phi_{i_1, i_2, i_3, i_4}(e) = 0.
$$

Moreover, let $e = \{u, v\} \in \text{supp}(\Phi_{i_1, i_2, i_3, i_4})$. Assume $u \in V_{i_1}$ and $v \in V_{i_2}$ (the other cases are analyzed similarly). Then, using (4), we obtain

$$
\sum_{e' \in E : |e \cap e'| = 1} \cos(\theta(e, e')) \Phi_{i_1, i_2, i_3, i_4}(e') = \sum_{w \in V_{i_1} \setminus \{u\}} 1 + \sum_{w \in V_{i_2} \setminus \{v\}} 1 - \sum_{w \in V_{i_3}} 1/2 - \sum_{w \in V_{i_4}} 1/2
$$

$$
= \frac{n}{d + 1} - 2 = \left(\frac{n}{d + 1} - 2\right) \Phi_{i_1, i_2, i_3, i_4}(e).
$$

Therefore, by Lemma 4.5, $\Phi_{i_1, i_2, i_3, i_4}$ is an eigenvector of $L^-$ with eigenvalue $n/(d + 1)$.

Let

$$
I = \{(1, 2, 3, k), (1, 3, 2, k) : k \in [d + 1] \setminus \{1, 2, 3\}\}
$$

$$
\cup \{(2, 3, j, k) : j, k \in [d + 1] \setminus \{1, 2, 3\}, j < k\},
$$

and let

$$
B = \{\Phi_{i_1, i_2, i_3, i_4} \, | \, (i_1, i_2, i_3, i_4) \in I\}.
$$

Note that

$$
|B| = |I| = 2(d - 2) + \binom{d - 2}{2} = \frac{(d - 2)(d + 1)}{2}.
$$

We will show that $B$ is a linearly independent set:

For each $(i_1, i_2, i_3, i_4) \in I$, let $\alpha_{i_1, i_2, i_3, i_4} \in \mathbb{R}$. Assume that

$$
\sum_{(i_1, i_2, i_3, i_4) \in I} \alpha_{i_1, i_2, i_3, i_4} \Phi_{i_1, i_2, i_3, i_4} = 0.
$$
Let \( j, k \in [d + 1] \setminus \{1, 2, 3\} \) such that \( j < k \). Note that for every \( u \in V_j \) and \( w \in V_k \), \( \Phi_{2,3,j,k} \) is the only vector in \( B \) containing \( \{u, w\} \) in its support. Therefore, we must have \( \alpha_{2,3,j,k} = 0 \).

Now, let \( k \in [d + 1] \setminus \{1, 2, 3\} \). For every \( u \in V_1 \), \( w \in V_k \), the only vectors in \( B \) containing \( \{u, w\} \) in their support are \( \Phi_{1,2,3,k} \) and \( \Phi_{1,3,2,k} \). Therefore, we must have \( \alpha_{1,2,3,k} = - \alpha_{1,3,2,k} \). Hence, we obtain a linear relation

\[
\sum_{k \in [d + 1] \setminus \{1,2,3\}} \alpha_{1,2,3,k} (\Phi_{1,2,3,k} - \Phi_{1,3,2,k}) = 0.
\]

Note that \( \Phi_{1,2,3,k} - \Phi_{1,3,2,k} = \Phi_{1,2,k,3} \). So, we obtain

\[
\sum_{k \in [d + 1] \setminus \{1,2,3\}} \alpha_{1,2,3,k} \Phi_{1,2,k,3} = 0.
\]

But note that each vector \( \Phi_{1,2,k,3} \) contains a unique edge in its support (for example, any edge \( \{u, w\} \) where \( u \in V_2 \) and \( w \in V_k \)). Therefore,

\[
\{ \Phi_{1,2,k,3} \}_{k \in [d + 1] \setminus \{1,2,3\}}
\]

are independent, and hence \( \alpha_{1,2,3,k} = \alpha_{1,3,2,k} = 0 \) for all \( k \in [d + 1] \setminus \{1, 2, 3\} \). Thus, \( B \) is linearly independent. So, \( n/(d + 1) \) is an eigenvalue of \( L^- \) with multiplicity at least \( (d - 2)(d + 1)/2 \).

**Eigenvector** \( n/(2(d + 1)) \). Let \( i, j, k \in [d + 1] \) be three distinct indices, and let \( u, v \in V_i \). Define

\[
\Psi_{u,v,j,k} = \sum_{w \in V_j} (1_{\{u,w\}} - 1_{\{v,w\}}) + \sum_{w \in V_k} (1_{\{v,w\}} - 1_{\{u,w\}}).
\]

Let \( e = \{x, y\} \notin \text{supp}(\Psi_{u,v,j,k}) \). Assume \( x \in V_r \) and \( y \in V_t \). If \( \{r, t\} = \{i, j\} \) or \( \{r, t\} = \{i, k\} \), then \( \cos(\theta(e,e')) = 1 \) for all \( e' \in \text{supp}(\Psi_{u,v,j,k}) \) such that \( |e \cap e'| = 1 \). Otherwise, \( \cos(\theta(e,e')) = 1/2 \) for all \( e' \in \text{supp}(\Psi_{u,v,j,k}) \) such that \( |e \cap e'| = 1 \).

Also, it is easy to check that for every \( w \in V \), \( \sum_{e \in E, w \in e} \Psi_{u,v,j,k}(e) = 0 \).
Finally, let $e = \{x, y\} \in \text{supp}(\Psi_{u, v, j, k})$. Assume $x = u$ and $y \in V_j$ (the other cases are similar). Then, by (4), we have

$$\sum_{e' \in E : |e \cap e'| = 1} \cos(\theta(e, e')) \Psi_{u, v, j, k}(e') = \sum_{w \in V_j \setminus \{y\}} \cos(\theta_{yuw}) - \sum_{w \in V_k} \cos(\theta_{uwv}) = \sum_{w \in V_j \setminus \{y\}} 1 - \sum_{w \in V_k} \frac{1}{2} - 1 = \frac{n}{2(d + 1)} - 2 = \left(\frac{n}{2(d + 1)} - 2\right) \Psi_{u, v, j, k}(e).$$

Therefore, by Lemma 4.5, $\Psi_{u, v, j, k}$ is an eigenvector of $L^-$ with eigenvalue $n/(2(d + 1))$.

For every $i \in [d + 1]$, fix some $j(i) \in [d + 1] \setminus \{i\}$ and $u_i \in V_i$, and let

$$J_i = \{(u_i, v, j(i), k) : v \in V_i \setminus \{u_i\}, k \in [d + 1] \setminus \{i, j(i)\}\}$$

and

$$B_i = \{\Psi_{u, v, j, k} : (u, v, j, k) \in J_i\}.$$

Let $B = \bigcup_{i=1}^{d+1} B_i$. Note that

$$|B| = \sum_{i=1}^{d+1} |J_i| = (d + 1)\left(\frac{n}{d + 1} - 1\right)(d - 1) = (n - d - 1)(d - 1).$$

We will show that $B$ is a linearly independent set. Let $i \neq i' \in [d + 1]$ and let $(u_i, v, j, k) \in J_i$ and $(u_{i'}, v', j', k') \in J_{i'}$. We will show that $\Psi_{u_i, v, j, k}$ and $\Psi_{u_{i'}, v', j', k'}$ are orthogonal. Indeed, the supports of the two vectors are disjoint unless $i' \in \{j, k\}$ and $i \in \{j', k'\}$. In this case, the intersection of the two supports consists of the four edges $\{u_i, u_{i'}\}$, $\{u_i, v'\}$, $\{v, u_{i'}\}$, $\{v, v'\}$, and it is easy to check that we have $\Psi_{u_i, v, j, k} \cdot \Psi_{u_{i'}, v', j', k'} = 0$.

Therefore, it is enough to check that for each $i \in [d + 1]$, $B_i$ is linearly independent. But this follows from the fact that for every $(u_i, v, j(i), k) \in J_i$, there is a unique edge in the support of $\Psi_{u_i, v, j(i), k}$ (any edge of the form $\{v, w\}$ where $w \in V_k$). So $B$ is linearly independent, and therefore $n/(2(d + 1))$ is an eigenvector of $L^-$ with multiplicity at least $(n - d - 1)(d - 1)$. ■
4.2. The spectrum of $L(T(n, 2d), q^\circ)$. 

**Theorem 1.4:** Let $d \geq 2$ and $n \geq 2d$ such that $n$ is divisible by $2d$. Then, the spectrum of $L(T(n, 2d), q^\circ)$ is

$$\left\{0 \left(d(d+1)/2\right), \frac{n}{2d} \left(n(d-1)-d^2\right), \frac{n}{d} \left(n(d-1)/2\right), \frac{n}{2}, n^{(1)}\right\}.$$

**Proof.** Denote $T(n, 2d) = (V, E)$. First, note that, as for all frameworks, 0 is an eigenvalue of $L(T(n, 2d), q^\circ)$ with multiplicity at least $d(d+1)/2$.

Since the non-zero eigenvalues of $L(T(n, 2d), q^\circ)$ and $L^-(T(n, 2d), q^\circ)$ are the same, we will find the corresponding eigenvectors for $L^-$.

Let $u \in V_{i,x}$ and $v \in V_{j,y}$, where $i \neq j \in [d]$ and $x, y \in \{1, -1\}$. Let $w \neq u, v$, and assume $w \in V_{k,z}$ for $k \in [d]$ and $z \in \{1, -1\}$.

Let $\theta_{uvw}$ be the angle between $q^\circ(u) - q^\circ(v)$ and $q^\circ(u) - q^\circ(w)$. Then,

$$\cos(\theta_{uvw}) = \begin{cases} 
\frac{1}{2} & \text{if } k \neq i, j, \\
\frac{1}{\sqrt{2}} & \text{if } k = i, z = -x, \\
1 & \text{if } k = j, z = y, \\
0 & \text{otherwise.}
\end{cases}$$
Let \( i < j \in [d] \). Let \( \phi_{ij} \in \mathbb{R}^E \) be defined by

\[
\phi_{ij}(e) = \begin{cases} 
1 & \text{if } e \in E(V_{i,1}, V_{j,1}) \cup E(V_{i,-1}, V_{j,-1}), \\
-1 & \text{if } e \in E(V_{i,-1}, V_{j,1}) \cup E(V_{i,1}, V_{j,-1}), \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( e \notin \text{supp}(\phi_{ij}) \). If \( e \in E(V_{i,1}, V_{i,-1}) \cup E(V_{j,1}, V_{j,-1}) \), then \( \cos(\theta(e, e')) = 1/\sqrt{2} \) for every \( e' \in \text{supp}(\phi_{ij}) \) such that \( |e \cap e'| = 1 \). If \( e \in E(V_{k,1}, V_{k,-1}) \) for some \( k \in [d] \setminus \{i, j\} \), then there are no edges \( e' \in \text{supp}(\phi_{ij}) \) such that \( |e \cap e'| = 1 \). If \( e \notin \bigcup_{k=1}^{d} E(V_{k,1}, V_{k,-1}) \), then \( \cos(\theta(e, e')) = 1/2 \) for every \( e' \in \text{supp}(\phi_{ij}) \) such that \( |e \cap e'| = 1 \).

Also, note that for all \( v \in V \), we have \( \sum_{e \in E : v \in e} \phi_{ij}(e) = 0 \).

Moreover, let \( e = \{u, v\} \in \text{supp}(\phi_{ij}) \). Assume \( u \in V_{i,1} \) and \( v \in V_{j,1} \) (the other cases are analyzed similarly). Then, using (5), we obtain

\[
\sum_{e' \in E : |e \cap e'| = 1} \cos(\theta(e, e')) \phi_{ij}(e') = \sum_{w \in V_{i,1} \setminus \{u\}} 1 + \sum_{w \in V_{j,1} \setminus \{v\}} 1 = \frac{n}{d} - 2 = \left(\frac{n}{d} - 2\right) \phi_{ij}(e).
\]

So, by Lemma 4.5, \( \phi_{ij} \) is an eigenvector of \( L^- \) with eigenvalue \( n/d \). Since the supports of the vectors \( \{\phi_{ij}\}_{i < j \in [d]} \) are pairwise disjoint, they form a linearly independent set. Thus, \( n/d \) is an eigenvalue of \( L^- \) with multiplicity at least \( d(d - 1)/2 \).

**Eigenvalue** \( n/(2d) \). Let \( i \neq j \in [d] \). Define \( \psi_{ij} \in \mathbb{R}^E \) by

\[
\psi_{ij}(e) = \begin{cases} 
1 & \text{if } e \in E(V_{i,1}, V_{j,1}) \cup E(V_{i,-1}, V_{j,1}), \\
-1 & \text{if } e \in E(V_{i,1}, V_{j,-1}) \cup E(V_{i,-1}, V_{j,-1}), \\
0 & \text{otherwise}.
\end{cases}
\]

For \( k \neq i, j \), let \( \Psi_{ijk} = \psi_{ij} - \psi_{kj} \).

Let \( e \notin \text{supp}(\Psi_{ijk}) \). If \( e \in \bigcup_{t=1}^{d} E(V_{t,1}, V_{t,-1}) \), then \( \cos(\theta(e, e')) = 1/\sqrt{2} \) for every \( e' \in \text{supp}(\Psi_{ijk}) \) such that \( |e \cap e'| = 1 \) (note that if \( t \neq i, j, k \) then there are no such edges \( e' \)). If \( e \notin \bigcup_{t=1}^{d} E(V_{t,1}, V_{t,-1}) \), then \( \cos(\theta(e, e')) = 1/2 \) for every \( e' \in \text{supp}(\Psi_{ijk}) \) such that \( |e \cap e'| = 1 \).

Also, note that for all \( v \in V \), we have \( \sum_{e \in E : v \notin e} \Psi_{ijk}(e) = 0 \).
Moreover, let \( e = \{u, v\} \in \text{supp}(\Psi_{ijk}) \). Assume \( u \in V_{i,1} \) and \( v \in V_{j,1} \) (the other cases are analyzed similarly). Then, using (5), we obtain
\[
\sum_{e' \in E: |e \cap e'| = 1} \cos(\theta(e, e'))\Psi_{ijk}(e') = \sum_{w \in V_{i,1}\{u\}} 1 + \sum_{w \in V_{j,1}\{v\}} 1 - \sum_{w \in V_{k,1} \cup V_{k,-1}} \frac{1}{2} = \frac{n}{2d} - 2 = \left(\frac{n}{2d} - 2\right)\Psi_{ijk}(e).
\]

Therefore, by Lemma 4.5, \( \Psi_{ijk} \) is an eigenvector of \( L^- \) with eigenvalue \( n/(2d) \).

For each \( j \in [d] \), fix \( i(j) \in [d] \setminus \{j\} \). Let
\[
I_j = \{(i(j), j, k): k \in [d] \setminus \{j, i(j)\}\}
\]
and \( I = \bigcup_{j \in [d]} I_j \). Note that \( |I| = d(d - 2) \). We will show that \( \{\Psi_{ijk}(i,j,k)\}_{(i,j,k) \in I} \) is a linearly independent set.

First, note that the vectors \( \{\psi_{ij}\}_{i \neq j \in [d]} \) form an orthogonal set. Indeed, let \( i, j, k, s \in [d] \) such that \( i \neq j \) and \( k \neq s \). If \( \{i, j\} \neq \{k, s\} \) then \( \psi_{ij} \) and \( \psi_{ks} \) have disjoint supports, and therefore \( \psi_{ij} \cdot \psi_{ks} = 0 \). Otherwise, if \( i = s \) and \( j = k \) we obtain
\[
\psi_{ij} \cdot \psi_{jk} = \sum_{e \in E(V_{i,1}, V_{j,1})} 1 \cdot 1 + \sum_{e \in E(V_{i,1}, V_{j,-1})} (-1) \cdot 1 + \sum_{e \in E(V_{i,-1}, V_{j,1})} 1 \cdot (-1) + \sum_{e \in E(V_{i,-1}, V_{j,-1})} (-1) \cdot (-1) = 0.
\]

Hence, for \( \{i, j, k\} \in \binom{[d]}{3} \) and \( \{i', j', k'\} \in \binom{[d]}{3} \) such that \( j \neq j' \), we have
\[
\Psi_{ijk} \cdot \Psi_{i'j'k'} = 0.
\]

Therefore, we are left to show that for all \( j \in [d] \), \( \{\Psi_{ijk}(i,j,k)\}_{(i,j,k) \in I_j} \) is linearly independent. But this follows from the fact that for every \( (i,j,k) \in I_j \) there is an edge in the support of \( \Psi_{ijk} \) that is unique to \( \Psi_{ijk} \) (for example, any edge \( \{u,v\} \) where \( u \in V_{j,1} \) and \( v \in V_{k,1} \)).

We now complete the set \( \{\Psi_{ijk}(i,j,k)\}_{(i,j,k) \in I} \) to an eigenbasis of \( n/2d \). Let \( i \neq j \in [d] \) and \( x \in \{1, -1\} \). Let \( u \neq v \in V_{i,x} \).

Define \( f_{ij}^{u,v} \in \mathbb{R}^E \) by
\[
f_{ij}^{u,v} = \sum_{w \in V_{j,1}} (1_{\{u,w\}} - 1_{\{v,w\}}) + \sum_{w \in V_{j,-1}} (-1_{\{u,w\}} + 1_{\{v,w\}}).
\]
Let \( e \notin \text{supp}(f_{ij}^{u,v}) \). If \( e \in E(V_{i,x}, V_{j,1}) \cup E(V_{i,x}, V_{j,-1}) \), then \( \cos(\theta(e, e')) = 1 \) for all \( e' \in \text{supp}(f_{ij}^{u,v}) \) such that \( |e \cap e'| = 1 \). If \( e \in E(V_{i,1}, V_{i,-1}) \cup E(V_{j,1}, V_{j,-1}) \), then \( \cos(\theta(e, e')) = 1/\sqrt{2} \) for all \( e' \in \text{supp}(f_{ij}^{u,v}) \) such that \( |e \cap e'| = 1 \).
If \( e \in E(V_{i,-x}, V_{j,1}) \cup E(V_{i,-x}, V_{j,-1}) \), then \( \cos(\theta(e, e')) = 0 \) for all \( e' \in \text{supp}(f^u,v) \) such that \( |e \cap e'| = 1 \). Otherwise, \( \cos(\theta(e, e')) = 1/2 \) for all \( e' \in \text{supp}(f^u,v) \) such that \( |e \cap e'| = 1 \).

In addition, it is easy to check that for every \( w \in V \),

\[
\sum_{e \in E, w \in e} f^u,v(e) = 0.
\]

Finally, let \( e = \{a, b\} \in \text{supp}(f^u,v) \). Assume \( a = u \) and \( b \in V_{j,1} \) (the other cases are similar). Then, by (5), we have

\[
\sum_{e' \in E : |e \cap e'| = 1} \cos(\theta(e, e')) f^u,v(e')
\]

\[
= \sum_{w \in V_{j,1} \setminus \{b\}} \cos(\theta_{buw}) - \sum_{w \in V_{j,-1}} \cos(\theta_{buw}) - \cos(\theta_{ubv})
\]

\[
= \sum_{w \in V_j \setminus \{b\}} 1 - \sum_{w \in V_k} 0 - 1 = \frac{n}{2d} - 2 = \left(\frac{n}{2d} - 2\right) f^u,v(e).
\]

Therefore, by Lemma 4.5, \( f^u,v \) is an eigenvector of \( L^- \) with eigenvalue \( n/(2d) \).

For each \( i \in [d] \) and \( x \in \{1, -1\} \), fix some \( u(i, x) \in V_{i,x} \). For \( j \in [d] \setminus \{i\} \), let

\[
J_{i,x,j} = \{ (u(i, x), v, j) : u(i, x) \neq v \in V_{i,x} \},
\]

and let

\[
J = \bigcup_{i \in [d]} \bigcup_{x \in \{1, -1\}} \bigcup_{j \in [d] \setminus \{i\}} J_{i,x,j}.
\]

Note that \( |J| = 2d(d - 1)(n/(2d) - 1) \).

We will show that \( \{f^u,v\}_{(u,v,j) \in J} \) are linearly independent.

Let \( u \neq v \in V_{i,x} \) and \( u' \neq v' \in V_{i',x'} \). Let \( j \neq i \) and \( j' \neq i' \). Assume \( (i, x, j) \neq (i', x', j') \). We will show that \( f^u,v \) and \( f^{u',v'} \) are orthogonal. Indeed, the supports of the two vectors are disjoint unless \( i' = j \) and \( j' = i \). But it is easy to check that also in this case we have

\[
f^u,v \cdot f^{u',v'} = 0.
\]

Therefore, it is enough to show that for every \( i \neq j \in [d] \) and \( x \in \{1, -1\} \),

\[
\{f^u,v\}_{(u,v,j) \in J_{i,x,j}} \text{ is linearly independent.}
\]

But again, this follows from the fact that for every \((u, v, j) \in J_{i,x,j}\) there is an edge in the support of \( f^u,v \) that is unique to it (for example, the edge \( \{v, w\} \) for any \( w \in V_{j,1} \)).
Finally, note that $\psi_{ij} \cdot f_{u,v}^k = 0$ for every $i \neq j \in [d]$ and $(u,v,k) \in J$. Therefore, $\Psi_{ijk} \cdot f_{u,v}^m = 0$ for all $(i,j,k) \in I$ and $(u,v,m) \in J$. Thus, the eigenvectors $\{\Psi_{ijk}\}_{(i,j,k) \in I} \cup \{f_{u,v}^j\}_{(u,v,j) \in J}$ form a linearly independent set.

So, $n/(2d)$ is an eigenvalue of $L^-$ with multiplicity at least

$$d(d-2) + 2d(d-1)(n/(2d) - 1) = n(d - 1) - d^2,$$

as wanted.

It was shown in [12, Lemma 4.5] that the removal of a vertex reduces the $d$-dimensional algebraic connectivity of a graph by at most 1. Hence, as an immediate consequence of Theorem 1.4, we obtain:

**Theorem 1.5:** Let $d \geq 3$ and $n \geq 2d$. If $n$ is divisible by $2d$, then

$$a_d(K_n) \geq \frac{n}{2d}.$$

For general $n \geq 2d$, we have $a_d(K_n) \geq \left\lceil \frac{n}{2d} \right\rceil - 2d + 1$.

5. The $n$ largest eigenvalues of the complete graph $K_n$

We proceed to establish a lower bound for the sum of the largest $n$ eigenvalues of $L(K_n, p)$ for all embeddings $p : [n] \to \mathbb{R}^d$. As a corollary, we derive an upper bound for $a_d(K_n)$.

**Lemma 5.1:** Let $p : [n] \to \mathbb{R}^d$ be injective. Then

$$\sum_{j=(d-1)n+1}^{dn} \lambda_j(L(K_n, p)) \geq \frac{n^2}{3} + n.$$

**Theorem 1.6:** Let $d \geq 3$ and $n \geq d + 1$. Then

$$a_d(K_n) \leq \frac{2n}{3(d-1)} + \frac{1}{3}.$$

**Proof of Theorem 1.6.** Let $p : V \to \mathbb{R}^d$ be injective. We compute the trace of $L(K_n, p)$ in two ways. By Lemma 2.1, all the diagonal entries of $L^-(K_n, p)$ equal 2, since $p$ is injective. Thus we have

$$\text{Tr}(L(K_n, p)) = \text{Tr}(L^-(K_n, p)) = 2 \binom{n}{2} = n^2 - n.$$
On the other hand, let $\lambda = \lambda_{\binom{d+1}{2}+1}(L(K_n, p))$. We deduce from Lemma 5.1 that

$$\text{Tr}(L(K_n, p)) = \sum_{j=1}^{dn} \lambda_j(L(K_n, p))$$

(7)

$$= \sum_{j=1}^{(d-1)n} \lambda_j(L(K_n, p)) + \sum_{j=(d+1)n+1}^{dn} \lambda_j(L(K_n, p))$$

$$\geq \left((d - 1)n - \binom{d + 1}{2}\right) \lambda + \frac{n^2}{3} + n.$$

By combining (6) and (7) we derive that

$$\lambda \leq \frac{2n^2/3 - 2n}{(d - 1)n - \binom{d + 1}{2}}.$$

Finally, we have

$$\frac{2n^2/3 - 2n}{(d - 1)n - \binom{d + 1}{2}} - \frac{2n}{3(d - 1)} = \frac{n(d(d + 1) - 6(d - 1))}{3(d - 1)((d - 1)n - \binom{d + 1}{2})}$$

$$\leq \frac{n(d + 1) - 6(d - 1)}{3n(d - 1)^2}$$

$$= \frac{d(d + 1) - 6(d - 1)}{3(d - 1)^2} \leq \frac{1}{3}.$$

Therefore, we obtain

$$\lambda \leq \frac{2n}{3(d - 1)} + \frac{1}{3}.$$

Thus, by Lemma 2.4, we obtain

$$a_d(K_n) \leq \frac{2n}{3(d - 1)} + \frac{1}{3},$$

as claimed.

To prove Lemma 5.1 we will need the following theorem due to Ky Fan:

**Theorem 5.2** (Fan [8, Thm. 1]): Let $A \in \mathbb{R}^{m \times m}$ be a symmetric matrix, and let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ be its eigenvalues. Then, for every $k \leq m$,

$$\sum_{i=1}^{k} \mu_i = \max \{\text{Tr}(AP) : P \in \mathcal{P}_k\}$$

where $\mathcal{P}_k$ consists of all orthogonal projection matrices into $k$-dimensional subspaces of $\mathbb{R}^m$. 
Proof of Lemma 5.1. Let $E = \left(\begin{bmatrix} n \\ 2 \end{bmatrix}\right)$. For $i \in [n]$, let $v^{(i)} \in \mathbb{R}^E$ be defined by

$$v^{(i)}_e = \begin{cases} 1 & \text{if } i \in e, \\ 0 & \text{otherwise}. \end{cases}$$

We claim that the $E \times E$ symmetric matrix $P$ defined by

$$P_{e,e'} = \begin{cases} \frac{2}{n-1}, & e = e' \\ \frac{n-3}{(n-1)(n-2)}, & |e \cap e'| = 1 \\ \frac{1}{(n-1)^2}, & e \cap e' = \emptyset \end{cases}$$

is the orthogonal projection on the subspace spanned by \{v^{(i)} : i \in [n]\}. Indeed, if $e = \{i, j\}$, then the $e$-th row of $P$ is equal to

$$P_{e,} = \frac{1}{n-1} \left(v^{(i)} + v^{(j)} - \frac{1}{n-2} \sum_{k \neq i, j} v^{(k)}\right).$$

Therefore, $Pw = 0$ for every $w$ that is orthogonal to all the vectors $v^{(i)}, i \in [n]$. In addition, a straightforward computation shows that $Pv^{(i)} = v^{(i)}$ for every $i \in [n]$ since $(v^{(i)})^Tv^{(j)} = 1$ if $i \neq j$ and $\|v^{(i)}\|^2 = n - 1$.

We apply Theorem 5.2 for $L^- := L^-(K_n, p)$ and $P$ to find that

$$\sum_{i=(d-1)n+1}^{dn} \lambda_i(L(K_n, p)) = \sum_{i=\binom{n}{2}}^{\binom{n}{2}+1} \lambda_i(L^-) \geq \text{Tr}(L^-P) = \sum_{e,e'} P_{e,e'}L^-_{e,e'}.$$

Recall the precise description of $L^-$ in Lemma 2.1. Since $p$ is injective, the contribution of the diagonal terms $e = e'$ to the summation is

$$\sum_{e \in E} P_{e,e}L^-_{e,e} = \binom{n}{2} \cdot \frac{2}{n-1} \cdot 2 = 2n.$$

In addition, since $L^-_{e,e'} = 0$ if $e \cap e' = \emptyset$, the contribution of the non-diagonal terms is

$$\sum_{e \neq e'} P_{e,e'}L^-_{e,e'} = \sum_{i=1}^{n} \sum_{j \neq i} \sum_{j' \neq i, j} \frac{n-3}{(n-1)(n-2)} L^-_{\{i,j\}, \{i,j'\}}.$$

Note that all the $\binom{n}{3}$ triples $\{i, j, j'\}$ of vertices satisfy

$$L^-_{\{i,j\}, \{i,j'\}} + L^-_{\{i,j\}, \{j,j'\}} + L^-_{\{j,i\}, \{j',j\}} \geq 1.$$

Indeed, since $p$ is injective, this is the sum of the cosines of the angles in the (possibly flat) triangle $p(i), p(j), p(j')$—which is bounded from below by 1. In
addition, note that each term $L^{-}_{\{i,j\}, \{i,j'\}}$ appears twice in (9) since $j,j'$ are ordered. Consequently,

\[
\sum_{e \neq e'} P_{e,e'} L^{-}_{e,e'} \geq \binom{n}{3} \cdot \frac{2(n-3)}{(n-1)(n-2)} = \frac{n^2}{3} - n.
\]

Joining (8) and (10), we obtain

\[
\sum_{i=(d-1)n+1}^{dn} \lambda_i(L(K_n,p)) \geq \frac{n^2}{3} + n.
\]

We believe that the bound in Lemma 5.1 can be improved:

**Conjecture 5.3:** Let $p : [n] \to \mathbb{R}^d$ be injective. Then the sum of the $n$ largest eigenvalues of $L(K_n,p)$ is at least $\frac{n(n+1)}{2}$.

Recall that, by Lemma 4.1, for every non-constant $p : [n] \to \mathbb{R}^d$, the largest eigenvalue of $L(K_n,p)$ is $n$. Thus, Conjecture 5.3 is equivalent to saying that the average of the next $n-1$ largest eigenvalues is at least $n/2$.

6. Concluding remarks

6.1. COMPLETE GRAPHS. We conjecture that the lower bound of Theorem 1.5 is essentially tight:

**Conjecture 6.1:** Let $n \geq 2d$. Then

\[
\left| \frac{n}{2d} \right| \leq a_d(K_n) \leq \frac{n}{2d}.
\]

6.2. REGULAR GRAPHS. From some computer calculations, it seems possible that the following generalization of [12, Conj. 2] holds:

**Conjecture 6.2:** For $d \geq 1$

\[
\lim_{n \to \infty} \max \{a_d(G) : G \text{ is } 2d\text{-regular on } n \text{ vertices}\} = 0.
\]

6.3. REPEATED POINTS. In line with our analysis of the minimal nontrivial eigenvalue of $L(G,p)$, where $G$ is the Turán graph $T(n,d+1)$ (resp. $T(n,2d)$) and $p = q^\Delta$ (resp. $p = q^\diamond$) in Theorem 1.3 and Theorem 1.4, we conjecture the following general phenomena regarding the effect of repeated points on the spectral gap.
For an injective \( p : [n] \to \mathbb{R}^d \) and graph \( G \) with vertex set \([n]\), denote \( \lambda(G, p) = \lambda_{d+1}^{(d+1)}(L(G, p)) \). For \( k \geq 1 \) let \( p^k : [kn] \to \mathbb{R}^d \) be a mapping such that \( |(p^k)^{-1}(p(v))] = k \) for every \( v \in [n] \). In words, we put \( k \) vertices on each point of the image of \( p \).

**Conjecture 6.3:** For every injective mapping \( p : [n] \to \mathbb{R}^d \) and every \( k \geq 2 \)

\[
\lambda(K_{kn}, p^k) = \frac{k}{2} \lambda(K_{2n}, p^2).
\]

We remark that for \( k = 1 \) the assertion fails, as demonstrated by the regular simplex embedding \( p^\triangle \).

**Acknowledgments.** Part of this research was done while A.L. was a postdoctoral researcher at the Einstein Institute of Mathematics at the Hebrew University. We thank the anonymous referee for their helpful remarks and suggestions.

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