Turán Density of 2-edge-colored Bipartite Graphs with Application on \( \{2, 3\}\)-Hypergraphs

Shuliang Bai  
Shing-Tung Yau Center  
Southeast University  
Nanjing, 210096, China  
sbai@seu.edu.cn

Linyuan Lu *  
Department of Mathematics  
University of South Carolina  
Columbia, 29208, U.S.A.  
lu@math.sc.edu.

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Abstract

We consider the Turán problems of 2-edge-colored graphs. A 2-edge-colored graph \( H = (V, E_r, E_b) \) is a triple consisting of the vertex set \( V \), the set of red edges \( E_r \) and the set of blue edges \( E_b \) where \( E_r \) and \( E_b \) do not have to be disjoint. The Turán density \( \pi(H) \) of \( H \) is defined to be \( \lim_{n \to \infty} \max_{G_n} h_n(G_n) \), where \( G_n \) is chosen among all possible 2-edge-colored graphs on \( n \) vertices containing no \( H \) as a subgraph and \( h_n(G_n) = \frac{|E_r(G)| + |E_b(G)|}{\binom{n}{2}} \) is the formula to measure the edge density of \( G_n \). We will determine the Turán densities of all 2-edge-colored bipartite graphs. We also give an important application on the Turán problems of \( \{2, 3\}\)-hypergraphs.

Mathematics Subject Classifications: 5D05, 05C65, 05D40

1 Introduction

Given a graph \( H \), the Turán problem asks for the maximum possible number of edges (denoted as \( ex(n, H) \)) in a graph \( G \) on \( n \) vertices without a copy of \( H \) as a subgraph. The Mantel’s theorem [13] states that any graph on \( n \) vertices with no triangle contains at most \( \lfloor n^2/4 \rfloor \) edges. Turán [16] proved that the maximal number of edges in a \( k \)-clique free graph on \( n \) vertices is at most \( (k - 2)n^2/(2k - 2) \). The famed Erdős-Stone-Simonovits Theorem [7, 8] proved that the Turán density of any graph \( H \) is \( \pi(H) = 1 - \frac{1}{\chi(H) - 1} \), where \( \chi(H) \) is the chromatic number of \( H \). For hypergraphs the extremal problems are harder, see Keevash [12] for a complete survey of some results and methods on uniform hypergraphs. Although Turán type problems for graphs and hypergraphs have been actively studied

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for decades, there are only few results on non-uniform hypergraphs, see [14, 15, 10] for related work. Motivated by the study of non-uniform Turán problems [3], in this paper we study a Turán-type problem on edge-colored graphs and show an application on Turán problems of non-uniform hypergraphs of edge size 2 or 3.

A hypergraph \( H = (V, E) \) consists of a vertex set \( V \) and an edge set \( E \subseteq 2^V \). An \( r \)-uniform hypergraph is a hypergraph such that all its hyperedges have size \( r \). Given positive integers \( k \geq r \geq 2 \), and a set of colors \( C \), with \( |C| = k \), a \( k \)-edge-colored \( r \)-uniform hypergraph \( H \) (for short, \( k \)-colored \( r \)-graph) is an \( r \)-uniform hypergraph that allows \( k \) different colors on each hyperedge. We express \( H \) as \( H = (V, E_1, E_2, ..., E_k) \) where \( E_i \) denotes the set of hyperedges colored by \( i \)th color in \( C \), note \( E_1, E_2, ..., E_k \) do not have to be disjoint. We say \( H' \) is a subgraph of \( H \), denoted by \( H' \subseteq H \), if \( V(H') \subseteq V(H), E_i(H') \subseteq E_i(H) \) for every \( i \). Given a family of \( k \)-colored \( r \)-graphs \( H \), we say \( G \) is \( H \)-free if it doesn’t contain any member of \( H \) as a subgraph. To measure the edge density of \( G \) of size \( n \), we use \( h_n(G) \), which is defined by

\[
h_n(G) := \sum_{i=1}^{k} \frac{|E_i(G)|}{\binom{n}{r}},
\]

where \( n = |V(G)| \). Then we define the Turán density of \( H \) as

\[
\pi(H) := \lim_{n \to \infty} \pi_n(H) = \lim_{n \to \infty} \max_{G_n} h_n(G_n),
\]

where the maximum is taken over all \( H \)-free \( k \)-colored \( r \)-graphs \( G_n \) on \( n \) vertices.

By a simple average argument of Katona-Nemetz-Simonovits [11], this limit always exists.

**Theorem 1.** For any fixed family \( H \) of \( k \)-colored \( r \)-graphs, \( \pi(H) \) is well-defined, i.e. \( \lim_{n \to \infty} \pi_n(H) \) exists.

When \( H = \{H\} \), we simply write \( \pi(\{H\}) \) as \( \pi(H) \). Note that \( \pi(H) \) agrees with the definition of

\[
\pi(H) = \frac{ex(n, H)}{\binom{n}{r}},
\]

where \( ex(n, H) \) is the maximum number of hyperedges in an \( n \)-vertex \( H \)-free \( k \)-colored \( r \)-graph.

In this paper, we let \( k = 2 \). A 2-edge-colored graph is a simple graph (without loops) where each edge is colored either red or blue, or both. We call an edge a double-colored edge if it is colored with both colors. For short, we call the 2-edge-colored graphs simply as 2-colored graphs. A 2-colored graph \( H \) can be written as a triple \( H = (V, E_r, E_b) \) where \( V \) is the vertex set, \( E_r \subseteq \binom{V}{2} \) is the set of red edges and \( E_b \subseteq \binom{V}{2} \) is the set of blue edges. Denote \( |E_r| \) and \( |E_b| \) as the size of each set, denote \( H_r, H_b \) as the induced subgraphs of \( H \) generated by all the red edges and all the blue edges respectively. A graph can be considered as a special 2-colored graph with only one color. We say \( H \) is
proper if there exists at least one edge in each class $E_r$ and $E_b$. Throughout the paper, we consider the proper 2-colored graphs. The results in this paper were finished in year 2018 and recently we noticed that our study is similar but different to a Turán problem on edge-colored graphs defined by Diwan and Mubayi [4] in which the authors ask for the minimum $m$, such that the 2-colored graph $G$, if both its red and blue edges are at least $m + 1$, contains a given 2-colored graph $F$? What we do differently in this paper is the study of the Turán density defined above for 2-colored graphs.

It is easy to see that $\pi(H) \geq 1$ for any proper 2-colored graph $H$, since we can take a complete graph with all edges a single color that does not contain a copy of $H$.

**Definition 2.** A 2-colored graph $H$ is called bipartite if $H$ does not contain an odd cycle of length $l \geq 3$ with all edges colored by the same color.

For a 2-colored graph $H$, we say $H$ is degenerate if $\pi(H) = 1$. Note that if $H$ is degenerate, then it must be bipartite. Otherwise, say $H_b = (V, E_b)$ is not a bipartite graph, one may consider the union of the red complete graph and an extremal graph respect to $H_b$, then the resulting graph is a $H$-free 2-colored graph with edge density at least $1 + \pi(H_b) > 1$, a contradiction.

In this paper, we will determine the Turán densities of all 2-colored bipartite graphs and characterize the 2-colored graphs achieving these Turán values. The notation $[n]$ is the set of $\{1, \ldots, n\}$. For convenience, we represent an edge $\{a, b\}$ by $ab$.

**Definition 3.** Given two $k$-colored $r$-graphs $G$ and $H$, a graph homomorphism is a map $f: V(G) \to V(H)$ which keeps the colored edges, that is, $f(e) \in E_i(H)$ whenever $e \in E_i(G)$ for $i \in [k]$. We say $G$ is $H$-colorable if there is a graph homomorphism from $G$ to $H$.

**Theorem 4.** The Turán densities of all bipartite 2-colored graphs are in the set $\{1, \frac{4}{3}, \frac{3}{2}\}$.

1. A 2-colored graph $H$ is degenerate if and only if it is $T$-colorable, where $T$ is the 2-colored graph with vertices $[4]$ and red edges $\{12, 13, 34\}$, blue edges $\{12, 23, 34\}$.

2. A 2-colored graph $H$ satisfies $\pi(H) = \frac{4}{3}$, then $H$ must be $H_8$-colorable but not $T$-colorable, where $H_8$ is the 2-colored graph with vertices $[8]$, red edges are

   $$E_r(H_8) = \{12, 13, 24, 34, 16, 37, 48, 25, 35, 18, 46, 27\},$$

   blue edges are

   $$E_b(H_8) = \{56, 57, 68, 78, 26, 15, 47, 38, 35, 18, 46, 27\}.$$

3. A 2-colored bipartite graph $H$ satisfies $\pi(H) = \frac{3}{2}$, then $H$ is not $H_8$-colorable.
Our consideration on 2-colored graphs is motivated by the study of Turán density of non-uniform hypergraphs, which was first introduced by Johnston and Lu [10], then studied by us [3]. We refer a non-uniform hypergraph \( H \) as \( R \)-graph, where \( R \) is the set of all the cardinalities of edges in \( H \). For example, \( H \) is a hypergraph on vertices \( \{1, 2, 3, 4\} \) with edges \( \{1\} \), \( \{2, 3\} \) and \( \{1, 2, 4\} \), then the edge type of \( H \) is \( R(H) = \{1, 2, 3\} \) as the cardinalities of all edges are 1, 2, 3. Given a hypergraph \( H \) with edge type \( R(H) \), the Turán density of \( H \) is defined as:

\[
\pi(H) = \lim_{n \to \infty} \max \left\{ \frac{1}{n^{|e|}} \sum_{e \in E(G)} \right\},
\]

where the maximum is taken over all \( H \)-free hypergraphs \( G \) on \( n \) vertices satisfying \( R(G_n) \subseteq R(H) \).

A degenerate \( R \)-graph \( H \) has the smallest Turán density, \( |R| - 1 \), where \( |R| \) is the size of set \( R \). For a history of degenerate extremal graph problems, see [9]. Let \( r \geq 3 \), for \( r \)-uniform hypergraphs the \( r \)-partite hypergraphs are degenerate and they generalize the bipartite graphs. An interesting problem is what the degenerate non-uniform hypergraph look like? In [3], we prove that except for the case \( R \neq \{1, 2\} \), there always exist non-trivial degenerate \( R \)-graphs for any set \( R \) of two distinct positive integers. The degenerate \( \{1, 3\} \)-graphs are characterized in [3], what about the the degenerate \( \{2, 3\} \)-graphs? In the last section of this paper, we will apply the 2-colored graphs to bound the Turán density of some \( \{2, 3\} \)-graphs.

The paper is organized as follows: in Section 2, we show some lemmas on the \( k \)-colored \( r \)-uniform hypergraphs; in Section 3, we classify the Turán densities of all 2-colored bipartite graphs; in Section 4, we give an application of the Turán density of 2-colored graphs on \( \{2, 3\} \)-graphs.

## 2 Lemmas on \( k \)-colored \( r \)-graphs

### 2.1 Supersaturation and Blowing-up

In this section, we give some definitions and lemmas related to the \( k \)-colored \( r \)-graphs for \( k \geq r \geq 2 \). These are natural generalizations from the Turán theory of graphs. We first define the blow-up of a \( k \)-colored \( r \)-graph.

**Definition 1 (Blow-up Families).** For any \( k \)-colored \( r \)-graph \( H \) on \( n \) vertices and positive integers \( s_1, s_2, \ldots, s_n \), the blow-up of \( H \) is a new \( k \)-colored \( r \)-graph, denoted by \( H(s_1, s_2, \ldots, s_n) = (V, E_1, \ldots, E_k) \), satisfying

- \( V := \bigcup_{i=1}^n V_i \), where \( |V_i| = s_i \),
- \( E_j = \bigcup_{F \in E_j(H)} \prod_{i \in F} V_i \), for each \( j \in [k] \).

When \( s_1 = s_2 = \cdots = s_n = s \), we simply write it as \( H(s) \).
Lemma 5 (Supersaturation). For any \( k \)-colored \( r \)-graph \( H \) and \( a > 0 \), then there are \( b, n_0 > 0 \) so that if \( G \) is a \( k \)-colored \( r \)-graph on \( n > n_0 \) vertices with \( h_n(G) > \pi(H) + a \) then \( G \) contains at least \( b \binom{n}{v(H)} \) copies of \( H \).

Proof. Since we have \( \lim_{n \to \infty} \pi_n(H) = \pi(H) \), there exists an \( n_0 > 0 \) so that if \( t > n_0 \) then \( \pi_t(H) < \pi(H) + \frac{a}{2} \). Suppose \( n > t \), and \( G \) is a \( k \)-colored \( r \)-graph on \( n \) vertices with \( h_n(G) > \pi(H) + a \). Let \( T \) represent any \( t \)-set, then \( G \) must contain at least \( \frac{a}{2} \binom{n}{t} \) \( t \)-sets \( T \subseteq V(G) \) satisfying \( h_t(G[T]) > (\pi(H) + \frac{a}{2}) \). Otherwise, we would have

\[
\sum_T h_t(G[T]) \leq \binom{n}{t} (\pi(H) + \frac{a}{2}) + \frac{a}{2} \binom{n}{t} \\
= (\pi(H) + a) \binom{n}{t}.
\]

But we also have

\[
\binom{t}{r} \sum_T h_t(G[T]) = \binom{n-t}{t-r} \binom{n}{r} h_n(G) \\
> \binom{n-t}{t-r} \binom{n}{r} (\pi(H) + a) \\
= (\pi(H) + a) \binom{t}{r} \binom{n}{t}.
\]

A contradiction. Since \( t > n_0 \), it follows that each of the \( \frac{a}{2} \binom{n}{t} \) \( t \)-sets \( T \subseteq V(G) \) satisfying \( h_t(G[T]) > (\pi(H) + \frac{a}{2}) \) contains a copy of \( H \), so the number of copies of \( H \) in \( G \) is at least \( \frac{a}{2} \binom{n}{t} (\pi(H) - v(H)) = \frac{a}{2} \binom{n}{v(H)} / (\pi(H) - v(H)) \). Let \( b = \frac{a}{2} / (\pi(H) - v(H)) \), the result follows.

The ‘blow-up’ does not change the Turán density of \( k \)-colored \( r \)-graphs. The following result and proof are natural generalization of results on uniform hypergraphs, see [12].

Lemma 6. For any \( s > 1 \) and any \( k \)-colored \( r \)-graph \( H \), \( \pi(H(s)) = \pi(H) \).

Proof. First, since any \( H \)-free \( r \)-graph \( G \) is also \( H(s) \)-free, we have \( \pi(H) \leq \pi(H(s)) \). We will show that for any \( a > 0 \), \( \pi(H(s)) < \pi(H) + a \).

By the supersaturation lemma, for any \( a > 0 \), there are \( b, n_0 > 0 \) so that if \( G \) is a \( k \)-colored \( r \)-graph on \( n > n_0 \) vertices with \( h_n(G) > \pi(H) + a \) then \( G \) contains at least \( b \binom{n}{v(H)} \) copies of \( H \). Consider an auxiliary \( v(H) \)-graph \( U \) on the same vertex set as \( G \) such that the edges of \( U \) correspond to copies of \( H \) in \( G \). Note that \( U \) contains at least \( b \binom{n}{v(H)} \) edges. For any \( S > 0 \), if \( n \) is large enough we can find a copy \( K \) of \( K_{v(H)}(s) \) in \( U \). Note that \( K \) is the complete \( v(H) \)-partite \( v(H) \)-graph with \( S \) vertices in each part, then \( \pi(K) = 0 \). Fix one such \( K \) in \( U \). Color each edge of \( K \) with one of the \( v(H)! \) colors corresponding to the possible orderings with which the vertices of \( H \) are mapped into the parts of \( K \). By Ramsey theory, one of the color classes contains at least \( S^v / v! \) edges. For large enough \( S \) (such that \( S^v / v! \geq s \)) it follows that \( U \) contains a monochromatic copy of \( K_{v(H)}(s) \), which gives a copy of \( H(s) \) in \( G \). Thus \( \pi(H(s)) < \pi(H) + a \).
Note when we say \( G \) is \( H \)-colorable, it is equivalent to say \( G \) is a subgraph of a blow-up of \( H \). It is easy to prove the following lemmas.

**Lemma 7.** Let \( \mathcal{H} \) be a family of \( k \)-colored \( r \)-graphs. If \( G \) is \( H \)-colorable for any \( H \in \mathcal{H} \), then \( \pi(G) \leq \pi(\mathcal{H}) \).

**Definition 2.** Given two \( k \)-colored \( r \)-graphs \( G_1 \) and \( G_2 \) with vertices set \( V_1 \) and \( V_2 \), we define the product of \( G_1 \) and \( G_2 \), denoted by \( G_1 \times G_2 = (V_1 \times V_2, E_1, \ldots, E_k) \), where for any \( i \in [k] \),

\[
E_i = E_i(G_1) \times E_i(G_2) = \{ e \times f \mid e \in E_i(G_1), f \in E_i(G_2) \},
\]

where \( e \times f \) is defined through the following way: denote \( e = \{v_1, \ldots, v_r\} \in E_i(G_1) \), \( f = \{u_1, \ldots, u_r\} \in E_i(G_2) \), then \( e \times f = \bigcup_{\sigma \in S_r} \{(v_1, u_{\sigma(1)}), \ldots, (v_r, u_{\sigma(r)})\} \), where \( \sigma = (\sigma(1), \ldots, \sigma(r)) \) takes over all permutations of \([r]\).

**Lemma 8.** A \( k \)-colored \( r \)-graph \( G \) is \( G_1 \) and \( G_2 \) colorable, then it's \((G_1 \times G_2)\)-colorable.

**Proof.** There exist two graph homomorphisms \( f_1 : V(G) \rightarrow V(G_1) \) and \( f_2 : V(G) \rightarrow V(G_2) \) such that for any edge \( e \in \{v_1, \ldots, v_r\} \in E(G) \), without loss of generality, let \( e \in E_1(G) \), we have

\[
f_1(e) = \{f_1(v_1), \ldots, f_1(v_r)\} \in E_1(G_1),
\]

and

\[
f_2(e) = \{f_2(v_1), \ldots, f_2(v_r)\} \in E_1(G_2).
\]

Define a map \( f := f_1 \times f_2 \) from \( V(G) \) to \( V(G_1) \times V(G_2) \), such that \( f(v) = (f_1(v), f_2(v)) \) for any \( v \in V(G) \). Then we have

\[
f(e) = \{(f_1(v_1), f_2(v_1)), \ldots, (f_1(v_r), f_2(v_r))\} \in f_1(e) \times f_2(e) \subset E_1(G_1 \times G_2).
\]

Thus the map \( f \) is a graph homomorphism. Hence \( G \) is \((G_1 \times G_2)\)-colorable. \( \square \)

### 2.2 Construction of 2-colored graphs

To compute the lower bound of \( \pi(H) \), we need to construct a family of \( H \)-free 2-colored graphs \( G_n \) with \( h_n(G_n) \) as large as possible. Here are three useful constructions.

**\( G_A \):** A 2-colored graph \( G_A \) on \( n \) vertices is generated by partitioning the vertex set into two parts such that \( V(G_A) = X \cup Y \) and the red edges either meet two vertices in \( X \) or meet one vertex in \( X \) plus the other in \( Y \), the blue edges meet one vertex in \( X \) plus the other in \( Y \). In other words, the red edges \( E_r(G_A) = \{ \binom{X}{2} \} \cup \{ \binom{X}{1} \times \binom{Y}{1} \} \) and blue edges \( E_b(G_A) = \{ \binom{X}{1} \times \binom{Y}{1} \} \). Let \( |V(G_A)| = n, |X| = xn \) and \( |Y| = (1 - x)n \) for some real number \( x \in (0, 1) \). We have

\[
h_n(G_A) = \frac{\binom{|X|}{2}}{2} + 2 \frac{\binom{|X|}{1} \binom{|Y|}{1}}{\binom{n}{2}}
\]

\[
= 4x - 3x^2 + o_n(1),
\]

which reaches the maximum \( \frac{4}{3} \) at \( x = \frac{2}{3} \).
\[ G_A: h_n(G_A) = \frac{4}{3} + o_n(1) \text{ at } |X| = \frac{2}{3}n. \]

**G_B:** It is obtained from \( G_A \) by simply exchanging red edges with blue edges. In other words, the red edges \( E_r(G_B) = \{ (X_1)^1 \times (Y_1)^1 \} \) and blue edges \( E_b(G_B) = \{ (X_1)^1 \} \cup \{ (X_1)^3 \} \).

\[ G_B: h_n(G_B) = \frac{4}{3} + o_n(1) \text{ at } |X| = \frac{2}{3}n. \]

**G_C:** A 2-colored graph \( G_C \) on \( n \) vertices is generated by partitioning the vertex set into two parts such that \( V(G_C) = A \cup B \) and the red edges either meet two vertices in \( A \) or meet one vertex in \( A \) plus the other in \( B \), the blue edges either meet two vertices in \( B \) or meet one vertex in \( A \) plus the other in \( B \). In other words, the red edges \( E_r(G_C) = \{ (A)^2 \} \cup \{ (A)^1 \times (B)^1 \} \) and blue edges \( E_b(G_C) = \{ (A)^1 \times (B)^1 \} \cup \{ (B)^2 \} \).

\[ G_C: h_n(G_C) = \frac{3}{2} + o_n(1) \text{ at } |A| = \frac{1}{2}n. \]

**G_D and G_E:** Two variations of \( G_C \) are the following constructions:

\[ G_D: h_n(G_D) = \frac{3}{2} + o_n(1). \]
\[ G_E: h_n(G_E) = \frac{3}{2} + o_n(1). \]

Following a similar description of above constructions, the red/blue edges of \( G_D \) are in the sets \( E_r(G_D) = \{ (X)^2 \} \times (Y)^1 \} \) and \( E_b(G_D) = \{ (V(G_D))^2 \} \setminus E_r(G_D) \) respectively; the blue/red edges of \( G_E \) are in the sets \( E_b(G_E) = \{ (C)^1 \times (D)^1 \} \) and \( E_r(G_E) = \{ (V(G_E))^2 \} \setminus E_b(G_E) \) respectively.

**Example 1.** The product of \( G_A \) and \( G_B \) is a blow-up of \( T \), where \( V(T) = [4] \), the red edges \{12, 13, 34\} and the blue edges \{12, 23, 34\}:
We define a map \( f : V(H) \to \{1, 2, 3, 4\} \) as follows:

1. If \( v \) appears in \( X \) of \( G_A \) and in \( Y \) of \( G_B \), set \( f(v) = 1 \).
2. If \( v \) appears in \( Y \) of \( G_A \) and in \( X \) of \( G_B \), set \( f(v) = 2 \).
3. If \( v \) appears in \( X \) of \( G_A \) and in \( X \) of \( G_B \), set \( f(v) = 3 \).
4. If \( v \) appears in \( Y \) of \( G_A \) and in \( Y \) of \( G_B \), set \( f(v) = 4 \).

One can check \( f \) is a graph homomorphism from the product \( G_A \times G_B \) to \( T \).

3 Turán density of bipartite 2-colored graphs

In this section, we will prove results in Theorem 4. We first give a boundary to divide the Turán densities of 2-colored non-bipartite graphs and 2-colored bipartite graphs.

Lemma 9.

1. For any 2-colored non-bipartite graph \( H \), \( \pi(H) \geq \frac{3}{2} \).
2. For any 2-colored bipartite graph \( H \), \( \pi(H) \leq \frac{3}{2} \).

Before proceeding to the proof, we see several important 2-colored graphs whose Turán density achieves value \( \frac{3}{2} \), and we will use these results to prove Lemma 9. The following lemma will be used in the proof of Lemma 12 which is useful to prove item 2 of Lemma 9.

Lemma 10. Let \( K_3 \) be a triangle with three double-colored edges, i.e.

\[
K_3 = ([3], \{12, 13, 23\}, \{12, 13, 23\}).
\]

Then

\[
ex(n, K_3) = \left( \frac{n}{2} \right) + \left\lceil \frac{n^2}{4} \right\rceil.
\]

In particular, \( \pi(K_3) = \frac{3}{2} \).

Proof. Observe that \( K_3 \) is not contained in \( G_C \), thus \( \pi(K_3) \geq \frac{3}{2} \). Now we prove the other direction. Let \( n \) be a positive integer and \( G \) be any \( K_3 \)-free 2-colored graph on \( n \) vertices. Construct an auxiliary graph \( F \) on the same vertex set \( V(G) \) and with the edge sets consisting of all double-colored edges in \( G \). Let \( H = E_r(F) \) consisting of all red colored edges of \( F \). Notice that \( H \) is triangle-free. By Mantel’s theorem, we have

\[
|E(H)| \leq \left\lceil \frac{n^2}{4} \right\rceil.
\]
Note that $H$ is a subgraph of $G$ and the number of the rest of edges in $G$ is at most $\binom{n}{2}$. Therefore, we have
\[ |E(G)| \leq \left(\frac{n}{2}\right) + |E(H)| \leq \left(\frac{n}{2}\right) + \left\lfloor \frac{n^2}{4} \right\rfloor = \left(\frac{3}{2} + o(1)\right) \left(\frac{n}{2}\right).\]
This implies that $\pi(K_3) = \frac{3}{2}$.

**Corollary 11.** Let $K_3^- = ([3], \{12, 13, 23\}, \{12, 13\})$, then $\pi(K_3^-) = \frac{3}{2}$.

**Proof.** Since $K_3^-$ is a subgraph of $K_3$, then $\pi(K_3^-) \leq \frac{3}{2}$. By Lemma 9, $\pi(K_3^-) \geq \frac{3}{2}$. The result follows.

Except the 2-colored non-bipartite graph, some bipartite graphs also achieves $\pi(H) = \frac{3}{2}$. See the following 2-colored graph on four vertices $\{1, 2, 3, 4\}$:

![Graph](image)

**Lemma 12.** $T_1 = ([4], \{12, 34, 13, 24\}, \{12, 34, 14, 23\})$. Then
\[ ex(n, T_1) = \left(\frac{n}{2}\right) + \left\lfloor \frac{n^2}{4} \right\rfloor \text{ for any } n \neq 3\]
and $ex(3, T_1) = 6$. In particular, we have $\pi(T_1) = \frac{3}{2}$.

**Proof.** When $n \leq 3$, the complete 2-colored graph does not contain $T_1$. Thus $ex(n, T_1) = 0, 0, 2, 6$ when $n = 0, 1, 2, 3$, respectively. The assertion holds for $n \leq 3$. It is sufficient to prove for $n \geq 4$. Since $T_1$ is not contained in $G_C$, we have
\[ ex(n, T_1) \geq \left(\frac{n}{2}\right) + \left\lfloor \frac{n^2}{4} \right\rfloor.\]
Now we prove the other direction by induction. We may assume $n \geq 4$. Let $n$ be a positive integer and $G$ be any $T_1$-free 2-colored graph on $n$ vertices.

Note $K_3$ is referring to a triangle with 3 double colored edges.

**Case 1:** $G$ doesn’t contain $K_3$ as a subgraph, by Lemma 10, we have
\[ |E(G)| \leq ex(n, K_3) = \left(\frac{n}{2}\right) + \left\lfloor \frac{n^2}{4} \right\rfloor.\]

**Case 2:** $G$ contains a copy of $K_3$, let $V_1 = \{a, b, c\}$ be the vertices of this triangle and $V_2 = V(G) \setminus V_1$. Then there are at most 4 edges from any vertex in $V_2$ to $V_1$. To see this, suppose there are 5 edges from the vertex $w \in V_2$ to $V_1$, then there are only two possible graphs on $V_1 \cup \{w\}$ and each of them contains a copy of $T_1$. A contradiction.
Applying the inductive hypothesis to $G[V_2]$, we have

$$|E(G[V_2])| \leq \left( \frac{n-3}{2} \right) + \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + \epsilon.$$ 

Here $\epsilon = 1$ if $n = 6$ and 0 otherwise.

Then the number of edges in $G$ is:

if $n \neq 6$,

$$|E(G)| = |E(G[V_1])| + |E(G[V_2])| + |E(V_1, V_2)|$$

$$\leq 6 + \left( \frac{n-3}{2} \right) + \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + 4(n-3)$$

$$= \left( \frac{n}{2} \right) + n + \left\lfloor \frac{n^2 - 6n + 9}{4} \right\rfloor$$

$$= \left( \frac{n}{2} \right) + \left\lfloor \frac{n^2 - 2n + 9}{4} \right\rfloor$$

$$= \left( \frac{n}{2} \right) + \left\lfloor \frac{n^2}{4} \right\rfloor.$$ 

if $n = 6$,

$$|E(G)| = |E(G[V_1])| + |E(G[V_2])| + |E(V_1, V_2)|$$

$$\leq 6 + \left( \frac{n-3}{2} \right) + \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + \epsilon + 4(n-3)$$

$$= 24$$

$$= \left( \frac{6}{2} \right) + \left\lfloor \frac{6^2}{4} \right\rfloor.$$ 

The induction step is finished. It follows that $h_n(G) \leq \frac{3}{2}$. Therefore, $\pi(T_1) = \frac{3}{2}$. 

Proof of Lemma 9. For Item 1, let $H$ be a 2-colored non-bipartite graph, without loss of generality, assume $H$ contains an odd cycle with red edges. For any $n$, let $G$ be a 2-colored graph generated by construction $G_D$, then $H$ can not be contained in $G$. Similarly, if $H$ contains an odd cycle with blue edges, then it is not contained in any 2-colored graph generated by construction $G_E$. Thus $\pi(H) \geq \frac{3}{2}$. 

For Item 2, it is sufficient to prove that any 2-colored bipartite graph $H$ is $T_1$-colorable. For any 2-colored bipartite graph $H$, the subgraph $H_r$ can be partitioned into two disjoint parts $V_1(H_r)$ and $V_2(H_r)$ such that the red edges form a bipartite graph between $V_1(H_r)$ and $V_2(H_r)$. Similarly for the subgraph $H_b$, the blue edges form a bipartite graph between
Let $S$ be the set of vertices incidents to double colored edges, then $S$ can be divided into four classes: $V_1(H_r) \cap V_1(H_b)$, $V_1(H_r) \cap V_2(H_b)$, $V_2(H_r) \cap V_1(H_b)$, and $V_2(H_r) \cap V_2(H_b)$. We define a map $f : V(H) \to \{1, 2, 3, 4\}$ as follows:

1. If $v \in V_1(H_r) \cap V_1(H_b)$, set $f(v) = 1$.
2. If $v \in V_1(H_r) \cap V_2(H_b)$, set $f(v) = 4$.
3. If $v \in V_2(H_r) \cap V_1(H_b)$, set $f(v) = 3$.
4. If $v \in V_2(H_r) \cap V_2(H_b)$, set $f(v) = 2$.
5. If $uv \in E_r(H) \setminus E_b(H)$, set $f(u) = 1, f(v) = 2$.
6. If $uv \in E_b(H) \setminus E_r(H)$, set $f(u) = 3, f(v) = 4$.

One can verify that this map $f$ is a graph homomorphism from $H$ to $T_1$. By Lemma 12, we have $\pi(H) \leq \frac{3}{2}$.

### 3.1 The degenerate 2-colored graphs

In this part, we will determine the degenerate 2-colored graphs. We will see that the 2-colored bipartite graph $T = ([4], \{12, 13, 34\}, \{12, 23, 34\})$ shown in Example 1 plays an important role.

**Lemma 13.** Let $n$ be a positive integer, for any $T$-free 2-colored graph $G$ on $n$ vertices, $G$ has at most $\binom{n+1}{2}$ edges. Thus $T$ is degenerate.

**Proof.** We will prove this lemma by induction on $n$. It is trivial for $n = 1, 2, 3, 4$. Assume $n \geq 5$. We assume that the statement holds for any $T$-free 2-colored graphs on less than $n$ vertices.

Let $G = (V, E_r, E_b)$ be a $T$-free 2-colored graph on $n$ vertices. We also assume $G$ contains at least one double-colored edge $uv$, or else $|E_r(G)| + |E_b(G)| \leq \binom{n}{2} < \binom{n+1}{2}$. Then $G$ is one of the following cases.

**Case 1:** There exists a vertex $w$ so that both $uw$ and $vw$ are double-colored edges. Since $G$ is $T$-free, there is no double-colored edges from $u, v, w$ to the rest of the vertices. By inductive hypothesis, when $G$ is restricted to the complement set of $\{u, v, w\}$, the number of edges of $G[V \setminus \{u, v, w\}]$ is at most $\binom{n-2}{2}$. Thus, $G$ has at most

$$6 + 3(n-3) + \binom{n-2}{2} = \binom{n+1}{2}.$$  

**Case 2:** Now we assume no such $w$ exists. Let $X = \{x \in V : |E(\{x\}, \{u, v\})| \geq 3\}$. That is, for each vertex $x \in X$, $x$ has exactly 3 edges connecting to $u$ and $v$. Since $G$ is $T$-free, for each $x \in X$, $x$ has no double-colored edges to any vertex not in $\{u, v, x\}$. In particular, the induced subgraph $G[X]$ of $G$ has no double-colored edge. Let
$V_1 = \{u, v\} \cup X$ and $V_2$ be the complement set. Then the induced subgraph $G[V_1]$ has at most
\[
2 + 3|X| + \left(\frac{|X|}{2}\right) < \left(\frac{|X| + 3}{2}\right) = \left(\frac{|V_1| + 1}{2}\right)
\]
edges. Applying the inductive hypothesis to $G[V_2]$, then $G[V_2]$ has at most $\left(\frac{|V_2| + 1}{2}\right)$ edges. Note that all edges from $X$ to $V_2$ are single colored and the number of edges from $\{u, v\}$ to each vertex in $V_2$ is at most 2. Thus the total number of edges from $V_1$ to $V_2$ is at most $|V_1||V_2|$ edges. Combining these facts together, we have $G$ has at most $N$ edges, where
\[
N = \left(\frac{|V_1| + 1}{2}\right) + |V_1||V_2| + \left(\frac{|V_2| + 1}{2}\right) = \left(\frac{|V| + 1}{2}\right).
\]

We finish the inductive step. Then we have
\[
\pi(T) = \lim_{n \to \infty} \max_{G_n} h_n(G_n) \leq \lim_{n \to \infty} \left(\frac{n+1}{2}\right) = 1,
\]
implying $\pi(T) = 1$. $T$ is degenerate.

**Proof of Item 1 of Theorem 4.** Assume $H$ is a degenerate 2-colored graph, then it must be $G_A$ and $G_B$-colorable. By Lemma 8, it must be $G_A \times G_B$-colorable. Note that the product of these two graphs is $T$-colorable. Thus $H$ is $T$-colorable, see Example 1. By Lemma 13, the result follows.

**Remark** 14. Note both $h_n(G_A)$ and $h_n(G_B)$ are equal to $\frac{4}{3} + o_n(1)$, then any 2-colored graph $H$ with $\pi(H) < \frac{4}{3}$ is $G_A$ and $G_B$-colorable, from above proof, $H$ is then $T$-colorable, thus further implies $\pi(H) = 1$.

### 3.2 Non-degenerate 2-colored bipartite graphs

In this part, we will further classify the non-degenerate 2-colored bipartite graphs. By Lemma 9, the largest possible Turán density of a 2-colored bipartite graph $H$ is $\frac{3}{2}$, so if $\pi(H) < \frac{3}{2}$, it must be contained in the construction $G_c$ and its variations $G_D, G_E$, thus it must be colored by the product of these constructions. While the product of graphs generated by the three constructions is a blow-up of following graph $H_8$. Let $ACX$ stand for the vertex in $A \times C \times X$, similar for other labels:

![Diagram](https://example.com/diagram.png)

$H_8$
To compute the Turán density of $H_8$, we need the following 2-colored graph $T_2 = ([4], \{12, 14, 23, 24, 34\}, \{12, 13, 14, 23, 34\})$. $T_2$ is not contained in a variation of $G_C$, thus $\pi(T_2) \geq \frac{3}{2}$.

![Graph T2](image)

**Lemma 15.** For any positive integer $n$, let $G$ be a $\{T_1, T_2\}$-free 2-colored graph on $n$ vertices. Then $|E(G)| \leq \binom{n}{2} + \left\lfloor \frac{n^2 + 3n}{6} \right\rfloor$. Thus $\pi(\{T_1, T_2\}) \leq \frac{4}{3}$.

**Proof.** It is not hard to check the cases for $n \leq 3$. Let $n \geq 4$, by induction on $n$ we assume the statement holds for any $\{T_1, T_2\}$-free graph on less than $n$ vertices. Note if $G$ contains no double-colored edge, the result is trivial. Thus we assume $G$ contains at least one double-colored edge. Then $G$ is one of the following cases.

**Case 1:** $G$ contains a triangle consisting of three double-colored edges, let $V_1 = \{a, b, c\}$ be the vertices of this triangle and $V_2 = V(G) \setminus V_1$. By Lemma 12 “Case 2”, for any vertex $w \in V_2$, there are at most 4 edges from $w$ to $V_1$.

**Case 2:** $G$ contains $V_1 = \{a, b, c\}$ such that $|E(G[V_1])| = 5$, without loss of generality, let $ab, bc$ be double colored edges, and $ac$ is blue colored edge. Let $V_2 = V(G) \setminus V_1$. For any vertex $w \in V_2$, there are at most 4 edges to $V_1$. If there are 5 edges from $w$ to $V_1$, then the following graphs include all of the possibilities and they contain $T_1, T_2$ as subgraph respectively.

![Graph Case 2](image)

**Case 3:** $G$ contains two incident double-colored edges $ab$ and $bc$, but no edge connecting $a$ and $c$. Let $V_1 = \{a, b, c\}$, $V_2 = V(G) \setminus V_1$. Then there cannot be 5 edges from any vertex $w \in V_2$ to $V_1$, otherwise, $G$ is a graph either in Case 1 or in Case 2. Thus there are at most 4 edges from any vertex in $V_2$ to $V_1$.

**Case 4:** If $G$ is not the above three cases, then for any double-colored edge connecting $a$ and $b$, there are at most 2 edges from any other vertex to $\{a, b\}$.

Applying the inductive hypothesis to $G[V_2]$, we have

$$|E(G[V_2])| \leq \left( \frac{|V_2|}{2} \right) + \left\lfloor \frac{|V_2|^2 + 3|V_2|}{6} \right\rfloor.$$

Then the number of edges in $G$ is: for the first three cases,
We first prove Lemma 16.

|E(G)| = |E(G[V_1])| + |E(G[V_2])| + |E(V_1, V_2)|
\leq 6 + \left(\frac{n - 3}{2}\right) + \left[\frac{(n - 3)^2 + 3(n - 3)}{6}\right] + 4(n - 3)
= \left(\frac{n + 1}{2}\right) + \left[\frac{(n - 3)^2 + 3(n - 3)}{6}\right]
= \left(\frac{n}{2}\right) + \left[\frac{n^2 - 6n + 9 + 3(n - 3) + 6n}{6}\right]
= \left(\frac{n}{2}\right) + \left[\frac{n^2 + 3n}{6}\right].

for Case 4,

|E(G)| = |E(G[V_1])| + |E(G[V_2])| + |E(V_1, V_2)|
\leq 2 + \left(\frac{n - 2}{2}\right) + \left[\frac{(n - 2)^2 + 3(n - 2)}{6}\right] + 2(n - 3)
= \left(\frac{n}{2}\right) - 1 + \left[\frac{(n - 2)^2 + 3(n - 2)}{6}\right]
= \left(\frac{n}{2}\right) + \left[\frac{(n - 2)^2 + 3(n - 2) - 6n}{6}\right]
= \left(\frac{n}{2}\right) + \left[\frac{n^2 - 7n - 2}{6}\right]
< \left(\frac{n}{2}\right) + \left[\frac{n^2 + 3n}{6}\right].

The induction step is finished. It follows that \(\pi(\{T_1, T_2\}) \leq \frac{4}{5}\).

Lemma 16. \(\pi(H_8) = \frac{4}{3}\).

Proof. We first prove \(\pi(H_8) \leq \frac{4}{3}\). To show this, we prove that \(H_8\) is \(T_1\) and \(T_2\)-colorable, i.e. there are graph homomorphisms from \(H_8\) to \(T_1\) and from \(H_8\) to \(T_2\).

For \(T_1\): We define a map \(f\) by \(f(ACX) = f(BCX) = 4, f(ADY) = f(BDY) = 3, f(ACY) = f(BCY) = 2, f(ADX) = f(BDX) = 1\). One can check that \(f\) is a graph homomorphism from \(H_8\) to \(T_1\).

For \(T_2\): We define a map \(g\) by \(g(ACX) = g(ADX) = 1, g(ADY) = g(ACY) = 3, g(BDX) = g(BDY) = 2, g(BCY) = g(BCX) = 4\). It is easy to check that \(g\) is a graph homomorphism from \(H_8\) to \(T_2\).

For any positive integer \(n\), let \(G_n\) be a 2-colored graph on \(n\) vertices such that \(h_n(G_n) \geq \pi(T_1, T_2) + \epsilon = \pi(T_1(s), T_2(s)) + \epsilon\) for any \(s \geq 2\) and \(\epsilon > 0\). Then \(G_n\) contains \(T_1(s)\) or \(T_2(s)\) as subgraph, further \(G_n\) contains \(H_8\) as subgraph. Then \(\pi(H_8) \leq \pi(\{T_1, T_2\})\). By Lemma 15, \(\pi(H_8) \leq \frac{4}{3}\). By Remark 14, if \(\pi(H_8) < \frac{4}{3}\), then \(\pi(H_8) = 1\), while \(H_8\) is not \(T\)-colorable, a contradiction. Thus it must be the case \(\pi(H_8) = \frac{4}{5}\). \(\square\)
Remark 17. As we know, if $\pi(H) < \frac{3}{2}$, it must be colorable by $G_c$ and its variations, then it must be be colorable by $H_8$ according to Lemma 8. Thus $\pi(H) \in \{1, \frac{4}{3}\}$.

For convenience, we use numbers to represent vertices: $ACX = 1, ADY = 2, ACY = 3, ADX = 4, BDX = 5, BCY = 6, BCX = 7, BDX = 8$. Then $H_8$ has edges:

$$E_r(H_8) = \{12, 13, 24, 34, 16, 37, 48, 25, 35, 18, 46, 27\};$$

$$E_b(H_8) = \{56, 57, 68, 78, 26, 15, 47, 38, 35, 18, 46, 27\}.$$

Now we are ready to finish the proof of Theorem 4.

Proof of Items 2 and 3 in Theorem 4. By Remark 14, Remark 17 and Lemma 9, the Turán densities of all bipartite 2-colored graphs are in the set $\{1, \frac{4}{3}, \frac{3}{2}\}$. To show Item 2, let $H$ be a 2-colored graph with $\pi(H) = \frac{4}{3}$, then $H$ must be $H_8$-colorable. One can check if $H$ does not contain $T$ as a subgraph, then $H$ must be $T$-colorable, implying $\pi(H) = 1$, a contradiction. By excluding the bipartite 2-colored graphs in Item 2, we obtain the result in Item 3.

$\square$

Example 2. Let $T_3$ be the following 2-colored graph, $T_3$ is non-degenerate and $\pi(T_3) = \frac{4}{3}$.

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {1};
\node (2) at (1,0) {2};
\node (3) at (0.5,1) {3};
\node (4) at (-0.5,1) {4};
\draw (1) -- (2);
\draw (1) -- (3);
\draw (1) -- (4);
\draw (2) -- (3);
\draw (2) -- (4);
\draw (3) -- (4);
\end{tikzpicture}
\end{center}

4 The degenerate $\{2, 3\}$-graphs

In this section, we study degenerate $\{2, 3\}$-graphs and show an application of the study of 2-edge-colored graphs on the Turán density of $\{2, 3\}$-graphs. A $\{2, 3\}$-graph is a non-uniform hypergraph where each edge consists of 2 or 3 vertices. Given a $\{2, 3\}$-graph $G$, we call an edge of cardinality $i$ as an $i$-edge, and use $E_i(G)$ to represent the set of $i$-edges. Thus $G$ can be represented by $G = (V(G), E_2(G), E_3(G))$. A 2-edge $e$ is called a double edge if $e \subset f$, for some 3-edge $f \in E_3(G)$. For convenience, we use the form of $ac$ to denote the edge $\{a, b\}$ and use $abc$ to denote the edge $\{a, b, c\}$. The notation $H_n^{\{2,3\}}$ represents a $\{2, 3\}$-graph on $n$ vertices, $K_n^{\{2,3\}}$ represents the complete hypergraph on $n$ vertices with edge set $\binom{[n]}{2} \cup \binom{[n]}{3}$.

Given a family of $\{2, 3\}$-graphs $\mathcal{H}$, the Turán density of $\mathcal{H}$ is defined to be:

$$\pi(\mathcal{H}) = \lim_{n \to \infty} \pi_n(\mathcal{H}) = \lim_{n \to \infty} \max \left\{ \frac{|E_2(G)|}{\binom{n}{2}} + \frac{|E_3(G)|}{\binom{n}{3}} \right\},$$

where the maximum is taken over all $H$-free hypergraphs $G$ on $n$ vertices satisfying $G \subseteq K_n^{\{2,3\}}$, and $G$ is $\mathcal{H}$-free $\{2, 3\}$-graph. Please refer to [3] for details on the Turán density of non-uniform hypergraphs.

Next let us see some definitions and results for $\{2, 3\}$-graphs.
Definition 3. [10] Let $H$ be a hypergraph containing some 2-edges. The 2-subdivision of $H$ is a new hypergraph $H'$ obtained from $H$ by subdividing each 2-edge simultaneously. Namely, if $H$ contains $t$ 2-edges, add $t$ new vertices $x_1, \ldots, x_t$ to $H$ and for $i = 1, 2, \ldots, t$ and replace the 2-edge $\{u_i, v_i\}$ with $\{u_i, x_i, v_i\}$.

Theorem 18. [10] Let $H'$ be the 2-subdivision of $H$. If $H$ is degenerate, then so is $H'$.

Definition 4. [10] The suspension of a hypergraph $H$, denoted by $S(H)$, is the hypergraph with $V = V(H) \cup \{v\}$ where $\{v\}$ is a new vertex not in $V(H)$, and the edge set $E = \{e \cup \{v\} : e \in E(H)\}$. We write $S^t(H)$ to denote the hypergraph obtained by iterating the suspension operation $t$-times, i.e. $S^2(H) = S(S(H))$ and $S^3(H) = S(S(S(H)))$, etc.

Proposition 1. [10] For any family of hypergraphs $\mathcal{H}$ we have that $\pi(S(\mathcal{H})) \leq \pi(\mathcal{H})$.

Theorem 19. [3] Let $R$ be a set of distinct positive integers with $|R| \geq 2$ and $R \neq \{1, 2\}$. Then a non-trivial degenerate $R$-graph always exists.

A chain $C^R$ is a special $R$-graph containing exactly one edge of each size such that any pair of these edges are comparable under inclusion relation. In [3], we say a degenerate $R$-graph is trivial if it is a subgraph of a blow-up of the chain $C^R$. By Theorem 19, there exist non-trivial degenerate $\{2, 3\}$-graphs. The $\{2, 3\}$-graph $H = \{12, 123\}$ is a chain, thus it is degenerate. By Theorem 18, the subdivision $H' = \{14, 24, 123\}$ is also degenerate, but it is non-trivial. As showed in [10], $H^0 = S(K^1_{2,3}) = \{13, 12, 123\}$ is not degenerate, and $\pi(H^0) = \frac{5}{4}$.

So what does the degenerate $\{2, 3\}$-graph look like? To answer this question, we may need to construct a family of $\{2, 3\}$-graphs $G_n$ with $h_n(G_n) > (1 + \epsilon)$ for some $\epsilon > 0$. Here are three $\{2, 3\}$-graphs with edge density greater than 1.

Note that for any $R$-graph $H$ (with possible loops), one can construct the family of $H$-colorable $R$-graph by blowing up $H$ in certain way. The langrangian of $H$ is the maximum edge density of the $H$-colorable $R$-graph that one can get this way. For more details of $R$-graphs with loops, blow-up, and Lagrangian, please refer to [3]. In this part, we will use an easy-understood way to calculate the edge densities.

Example 3. A $\{2, 3\}$-graph $G^{(2,3)}_1$ is a blowing-up of the general hypergraph $H_1$ with vertex set $\{a, b, c\}$ and edge set $\{aa, ab, ac, abc\}$, if there exists a partition of vertex set such that $V(G^{(2,3)}_1) = A \cup B \cup C$ and every 2-edge meets two vertices in $A$ (or $B$, or $C$), every 3-edge meets $A, B, C$ one vertex respectively. In other words, $E(G^{(2,3)}_1) = \binom{A}{2} \cup \binom{A}{1} \binom{B}{1} \cup \binom{A}{1} \binom{C}{1} \cup \binom{A}{1} \binom{B}{1} \binom{C}{1}$.

Let $|A| = xn$ and $|B| = |C| = \frac{1-x}{2}n$ for some value $x \in (0, 1)$. We have

$$h_n(G^{(2,3)}_1) = \frac{\binom{xn}{2} + \binom{xn}{1}(\frac{1-x}{1}) \binom{xn}{1} + \binom{xn}{1}(\frac{1-x}{2}) \binom{xn}{2}}{\binom{n}{3}}$$

$$= x^2 + 2x(1-x) + \frac{3}{2}x(1-x)^2 + o_n(1)$$

$$= \frac{7}{2}x - 4x^2 + \frac{3}{2}x^3 + o_n(1).$$
The above value reaches the maximum value $\frac{245}{243} + o_n(1)$ at $x = \frac{7}{9}$.

$$G^{(2,3)}_1: \quad h_n(G^{(2,3)}_1) = \frac{245}{243} \text{ at } |A| = \frac{7}{9}n.$$  

**Example 4.** A $\{2,3\}$-graph $G^{(2,3)}_2$ is a blowing-up of the general hypergraph $H_2$ with vertex set $\{x, y\}$ and edge set $\{xy, xxx, xxy\}$, if there exists a partition of vertex set such that $V(G^{(2,3)}_2) = X \cup Y$ and every 2-edge meets one vertex in $X$ and one vertex in $Y$, every 3-edge either meet three vertices in $X$ or two vertices in $X$ plus one vertex in $Y$. Actually $G^{(2,3)}_2$ is $H_2$-colorable. In other words,

$$E(G^{(2,3)}_2) = \binom{X}{3} \cup \binom{X}{2} \binom{Y}{1} \cup \binom{X}{1} \binom{Y}{1}.$$  

Let $|X| = xn$ and $|Y| = (1 - x)n$ for some value $x \in (0, 1)$, we have

$$h_n(G^{(2,3)}_2) = \frac{x^n}{3} + \binom{x^n}{2} \frac{(1-x)^n}{1} + \frac{xn(1-x)n}{2}$$  

$$= x^3 + 3x^2(1-x) + 2x(1-x) + o_n(1)$$  

$$= 2x + x^2 - 2x^3 + o_n(1).$$  

The above value reaches the maximum value $\frac{19 + 13\sqrt{13}}{54} + o_n(1) \approx 1.21985\ldots + o_n(1)$ at $x = \frac{1 + \sqrt{13}}{6}$.  

$$G^{(2,3)}_2: \quad h_n(G^{(2,3)}_2) \approx 1.21985 \text{ at } |X| = \left(\frac{1 + \sqrt{13}}{6}\right)n.$$  

**Example 5.** A $\{2,3\}$-graph $G^{(2,3)}_3$ is a blowing-up of the general hypergraph $H_3$ with vertex set $\{e, f\}$ and edge set $\{ee, eef\}$, if there exists a partition of vertex set such that $V(G^{(2,3)}_2) = E \cup F$ and every 2-edge meets two vertices in $E$, every 3-edge meets two vertices in $E$ plus one vertex in $F$. Actually $G^{(2,3)}_3$ is $H_3$-colorable. In other words,

$$E(G^{(2,3)}_3) = \binom{E}{2} \cup \binom{E}{2} \binom{Y}{1}.$$
Let $|E| = xn$ and $|F| = (1 - x)n$ for some value $x \in (0, 1)$, we have

$$h_n(G_3^{(2,3)}) = \binom{xn}{2} \binom{(1-x)n}{1} \binom{n}{3}$$

$$= x^2 + 3x^2(1 - x) + o_n(1)$$

$$= 4x^2 - 3x^3 + o_n(1).$$

The above value reaches the maximum value $\frac{256}{243} + o_n(1)$ at $x = \frac{8}{9}$.

A degenerate \{2, 3\}-graph must appear as subgraphs in all above \{2, 3\}-graphs $G_1^{(2,3)}$, $G_2^{(2,3)}$ and $G_3^{(2,3)}$, thus it must appear as subgraph in the product of these hypergraphs. By taking this product, we get a 12-vertex \{2, 3\}-graph which is $H_9^{(2,3)}$-colorable. Thus we have

**Lemma 20.** The degenerate \{2, 3\}-graphs must be $H_9^{(2,3)}$-colorable.

The following theorem shows a relation between such \{2, 3\}-graphs and the 2-colored graphs and can help us determine the upper bound for the Turán density of some \{2, 3\}-graphs.

**Theorem 21.** Let $H = (V, E_r, E_b)$ be a 2-colored graph, and $H' = (V', E_2, E_3)$ be a \{2, 3\}-graph obtained from $H$ by adding a new vertex $v \not\in V$ such that $V' = V \cup \{v\}$ and $E_2 = E_r$, and $E_3 = \{e'|e' = e \cup v, e \in E_b\}$. Then $\pi(H') \leq \pi(H)$. 

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Proof. Let \( n \) be positive integer, let \( G = (V, E_2(G), E_3(G)) \) be an arbitrary \( H' \)-free \( \{2, 3\} \)-graph on \( n \) vertices. For any vertex \( v \in V(G) \), let \( G_v = (V(G) \setminus \{v\}, E_{v,2}, E_{v,3}) \) be a 2-colored graph obtained form \( G \), such that the red edges are \( E_{v,2} = E_2(G) \), the blue edges are \( E_{v,3} = \{u, w| \{vuw \} \in E_3 \} \). Observe that \( G_v \) is \( H \)-free since \( G \) is \( H' \)-free. Thus \( h_{n-1}(G_v) \leq \pi_n(H) \). Since

\[
\left| E_2(G) \right| = \frac{1}{n-2} \sum_{v \in V(G)} |E_{v,2}| \quad \text{and} \quad \left| E_3(G) \right| = \frac{1}{3} \sum_{v \in V(G)} |E_{v,3}|
\]

Then

\[
h_n(G) = \frac{|E_2(G)|}{\binom{n}{2}} + \frac{|E_3(G)|}{\binom{n}{3}} = \sum_{v \in V(G)} \frac{|E_{v,2}|}{(n-2)\binom{n}{2}} + \sum_{v \in V(G)} \frac{|E_{v,3}|}{3\binom{n}{3}} = \frac{1}{n} \sum_{v \in V(G)} \frac{|E_{v,2}|}{\binom{n-1}{2}} + \frac{1}{n} \sum_{v \in V(G)} \frac{|E_{v,3}|}{\binom{n-1}{2}} \leq \frac{1}{n} \sum_{v \in V(G)} h_{n-1}(G_v) \leq \pi(H).
\]

Therefore \( \pi(H') \leq \pi(H) \). \qed

So far we couldn’t give an upper bound of \( \pi(H_9^{\{2,3\}}) \), but we can show a subgraph of \( \pi(H_9^{\{2,3\}}) \) are degenerate using above theorem. Let us observe that if we remove a single vertex \( AXF \) and edges connecting to it, the resulting sub-hypergraph is \( H_5^{\{2,3\}} \)-colorable, where \( H_5^{\{2,3\}} = ([5], \{12, 13, 34, 125, 135, 345\}) \).

![Diagram](image1.png)

Observe that we can also obtain \( H_5^{\{2,3\}} \) from \( T \) by adding vertex 5, and connect it with blue edges. Thus we have \( \pi(H_5^{\{2,3\}}) = 1 \).
In $H_9^{(2,3)}$, removing a single 2-edge connecting vertices $AXE$ and $AYE$, the resulting subgraph is $H_6^{(2,3)}$ colorable, where $H_6^{(2,3)} = ([6], \{34, 35, 134, 235, 456\})$. However, we don’t know the Turán density of $H_6^{(2,3)}$. We remark that determining the degenerate $\{2,3\}$-hypergraph is still unknown.

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