A new inequality measurement tool: The Vinci index

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This paper presents a new inequality measurement tool that gives more weight to inequality at the lower end of the income distribution and offers a new way to synthesize the dispersion of the whole distribution. The proposed measure leverages the mean-dependent differentials that sum to the Gini coefficient and scales them by a type of differential that is exclusively concerned with the inequality between two incomes. The resulting index addresses the limitations of the Gini coefficient and possesses a set of desirable properties, including normalization, scale invariance, population invariance, transfer sensitivity, and weak decomposability.

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**Introduction**

In recent years, there has been increased interest in inequality measures that don’t rely on the mean of the distribution. In general, the focus of classical measures like the Gini coefficient that rely on the mean is on the concentration aspect of inequality, facilitated by increasingly skewed distributions. This makes these mean-based measures insensitive towards inequality at the lower end of the distribution and casts doubt on the ability of these classical measures to describe inequality appropriately. To overcome these limitations different measures have been proposed that take the relative nature of the poor and the rich into account. Among them are inter-decile ratios like the Palma that contrasts the income of the richest ten percent and the poorest 40 percent (Cobham and Sumner, 2013). Recently, Davydov and Greselin (2019) proposed an inequality measure that is based on an exhaustive comparison of the means for equal population shares situated at opposite ends of the distribution, from the comparison of the highest and lowest quantiles to the comparison of the \( n - 1 \) highest quantiles with the \( n - 1 \) lowest. If the relative approach is applied to all possible pairs of income instead of grouped incomes, this results in an even
more exhaustive comparison of incomes. The following has been suggested as an inequality index (Gastwirth, 1974):

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{|y_j - y_i|}{y_j + y_i}
\]

(1)

Differently standardized this can be written as:

\[
\frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{|y_j - y_i|}{y_j + y_i}
\]

(2)

This measure also hinges on the relative nature of inequality, but it does so for each possible pair of income in the distribution. Measures of this type are sensitive to inequality at the lower end of the distribution. They are scale invariant, population invariant, normalized, and weakly decomposable, but they do not satisfy the Pigou-Dalton transfer principle for the high levels of inequality observed in socioeconomics (see also, Cowell, 2000).

The differentials these measures are based on can still be useful if they are combined with differentials that sum to the Gini coefficient. For this synthesis, another differential than that used in equation (2) offers the major advantage that it provides a class of indices that can accommodate non-positive values. These angle-based Arcus differentials are given by:

\[
\left(\frac{|\arctan^2(y_i, y_j) - \arctan^2(y_j, y_i)| \cdot 180}{\pi}\right)/90 = \frac{2}{\pi}
\]

(3)

If income \( y_i \) is plotted at the x-axis and income \( y_j \) at the y-axis, then the function \( \arctan^2 \) returns the angles in rad between the positive x-axis and the radii to points \( P(y_i, y_j) \) and its reflection on the 45-degree line \( P'(y_j, y_i) \). The difference between the two angles \( \alpha \) and \( \beta \) equals angle \( \theta \), see Figure 1. If the differentials are calculated using degrees, the normalized Arcus differential is obtained by dividing \( \theta \) through 90 degrees, the maximum possible angle of the circle quadrant if both incomes are non-negative. The resulting Arcus differential equals the area enclosed by the sector over the total area of the circle quadrant and takes on values between 0 and 1.

The absolute difference between the two angles \( \alpha \) and \( \beta \) is used in the formula because the order of arguments for the function \( \arctan^2 \) is different depending on the computer language. Taking the absolute difference makes sure that the value for angle \( \theta \) is always non-negative.

An alternative approach of measuring inequality with Arcus-type differentials is to use them for the cumulative distribution, which provides inequality measure \( C \). In that case, the differentials are calculated using the points of the Lorenz curve between (0,0) and (1,1), and their reflection on the 45-degree line. If \( p_i \) denotes the cumulative percentage of incomes and \( f_i \) the cumulative percentage of households, then index \( C \) can be written as:
Figure 1: Comparing two incomes, income $y_i$ is twice as large as $y_j$. Points $P(y_i, y_j)$ and its reflection on the 45-degree line $P'(y_j, y_i)$ define a circle sector. The area of the sector over the total area of the circle quadrant equals the Arcus differential.

$$C = \frac{n-1}{n^2} \sum_{i=1}^{n-1} \left( |\tan^{-1}(p_i, f_i) - \tan^{-1}(f_j, p_i)| \right)$$

(4)

Bounded between 0 and 1, index $C$ is scale invariant, satisfies the Pigou-Dalton transfer principle and can accommodate nonpositive values, but it doesn’t satisfy the two standard axioms decomposability and population invariance. Therefore a better option is to combine the Arcus differentials with the differentials that sum to the classical Gini coefficient.

The Gini coefficient is usually defined in terms of the Lorenz curve, another popular formula for the Gini coefficient often encountered in the literature is the mean absolute difference $\Delta$ divided by the mean of the distribution.

$$G = \frac{\Delta}{\mu} = \frac{1}{\mu} \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|$$

(5)

However, by algebraic manipulation of formula (5), it is made clear that the
Gini coefficient can also be directly expressed as the sum of pairwise differentials:

\[ G = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{|y_i - y_j|}{n^2 \mu} \]  \hspace{1cm} (6)

Gini differentials are defined by:

\[ \frac{|y_i - y_j|}{n^2 \mu} \]  \hspace{1cm} (7)

Each Gini differential not only equals the contribution of an income pair to the Gini coefficient. It also has a geometrical representation between the Lorenz curve and the 45-degree line either through the area of a triangle for adjacent values or through the difference between the areas of two triangles for non-adjacent values. These triangles are obtained by connecting not only adjacent points of the Lorenz curve between \([0,0]\) to \([1,1]\) with a straight line but also the non-adjacent points. This is best illustrated by an example (see Figures 2-5). It is a simple and intuitive tool that either has not been recognized by researchers until now, or it is critically underreported.

![Figure 2: Gini differentials (GD) for adjacent values of the original distribution 4,8,12,36,40 are geometrically represented by a triangle. GD[4,8] = ∆0AB, GD[8,12] = ∆ABC, GD[12,36] = ∆BCD, GD[36,40] = ∆CDK.](image)

The same absolute difference between two values contributes the same to the Gini coefficient irrespective if it is observed at the top, the middle, or the
Figure 3: Lorenz curve for the original distribution 4, 8, 12, 36, 40. The Gini differential for the comparison of the lowest and the highest income in the distribution is $GD[4, 40]$, it is represented by the area of the triangle $\Delta 0KI$ minus $\Delta IAD$.

bottom of the distribution. Consequently, the triangles representing $GD[4, 8]$, $GD[8, 12]$, and $GD[36, 40]$ have the same area, as indicated in Figure(2). This makes the Gini coefficient insensitive to inequality at the left tail of the distribution, where it arguably matters most. From a normative approach, the Gini does not possess the diminishing transfer property (Kolm, 1976).

A New Synthesis of Inequality Measurement

A new inequality measurement tool called Vinci index that addresses the limitations of the Gini index can be expressed as the sum of products of Arcus and Gini differentials.:

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{2}{\pi} (\text{atan}2(y_i, y_j) - \text{atan}2(y_j, y_i)) \right) \cdot \frac{|y_i - y_j|}{n^2 \mu}$$

(8)
Figure 4: Lorenz curve for the original distribution 4,8,12,36,40. $GD[4,12]$ is represented by the area of the triangle $\Delta OCE$ minus $\Delta EAB$, $GD[8,40]$ by $\Delta AKG$ minus $\Delta GBD$, $GD[12,40]$ by $\Delta BKH$ minus $\Delta HCD$, and $GD[8,36]$ by $\Delta ADF$ minus $\Delta FBC$.

To obtain the class of generalized Vinci-indices the exponent $\delta > 0$ is applied to the Arcus differentials:

$$V^\delta = \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{2}{\pi} (\text{atan}2(y_i, y_j) - \text{atan}2(y_j, y_i)) \right)^\delta \cdot \frac{|y_i - y_j|}{n^2 \mu}$$  \hspace{1cm} (9)

If the differentials from (1) are used instead of Arcus differentials this leads to very similar inequality measures, but they can only accommodate a single zero value:

$$I^\delta = \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{|y_j - y_i|}{y_j + y_i} \right)^\delta \cdot \frac{|y_i - y_j|}{n^2 \mu}$$ \hspace{1cm} (10)

Arcus differentials in equation (9) will approximate unity if the exponent $\delta$ is close to zero. In that case, the Vinci index is practically identical to the Gini coefficient. For non-zero values of the exponent $\delta$ Vinci indices are in general more sensitive towards inequality at the left tail of the distribution than the Gini coefficient, due to the influence of the Arcus differentials. A Vinci index with $\delta > 2.5$ is more sensitive to extreme values at the left tail of the distribution, while a Vinci index with $\delta > 0.5$ indicates higher inequality if there are more incomes at the left tail that are only moderately lower than those of better-off
individuals. The interpretation of the Vinci index with $\delta=1$ is straightforward, it is the sum of the Lorenz-based Gini differentials that have been scaled by the corresponding Arcus differentials. A pairwise comparison will only contribute a high value to the Vinci index if both the Gini and Arcus differential indicate a high level of inequality. This emphasizes differences between the fortunate and the less fortunate in a distribution. Absolute differences between fortunate individuals, often of less social concern, have less influence.

Thus, the Vinci-indices are novel inequality measurement tools that allow more weight to be given to inequality at the lower end of the distribution. This makes these measures a counterpart of the class of indices known as $\alpha$-Ginis (Chameni Nembua, 2006). Depending on the exponent $\alpha$ that is applied to the absolute differences between incomes this class of indices measures inequality with a focus on the concentration aspect of inequality that is at least as high as that of the classical Gini coefficient.

The proposed new class of Vinci inequality measures satisfies the normative axioms that have been proposed to aid in identifying appropriate inequality indices:

1. $V$-indices are normalized. If all incomes are the same $V=0$, The upper bound is reached if one person receives all the income. For finite $n$ the upper bound is given by $\frac{(n-1)}{n}$, such that the index value lies always between 0 and 1 for distributions without negative values.

2. $V$-indices satisfy the axiom of scale invariance, the index value does not change if all incomes are changed proportionally.

3. $V$-indices satisfy the axiom of population invariance, the index value does not change if the original population is replicated.

4. $V$-indices satisfy the well-known Pigou-Dalton transfer principle. If a progressive transfer does not change the rank of the individuals in the distribution the index value decreases.

5. $V$-indices with exponent $\delta \geq 1 \leq 2$ satisfy Kolm’s principle of strong diminishing transfer (Kolm, 1976), for distributions without negative values. This principle requires that if there are two income pairs with the same absolute income difference in the distribution, that the inequality reducing effect of a progressive transfer is stronger if it occurs between the income pair that has a lower share of total income.

6. $V$-indices are decomposable, total inequality can be decomposed into the inequality of subgroups and the inequality between subgroups.
The Vinci indices possess the property of weak decomposability (Ebert, 2010). If a population is partitioned into subgroups and the Vinci index is calculated for the subgroups as if they were separate populations, then the Vinci index for the total population $V_T$ can be expressed as the weighted sum of within-group terms that are based on the Vinci indices of the subgroups ($V_{G1}, V_{G2}$), and a between-group term that is based on all comparisons of incomes between members of different subgroups. The within-group terms are obtained by multiplying the Vinci index value for a subgroup of size $n_1$ with $\frac{n_1^2}{2}$ to adjust for subgroup size and the number of comparisons within each subgroup. In addition, the index value of the subgroups is weighted by the subgroup mean over the population mean $\frac{\mu_{G1}}{\mu}$. The resulting within-group terms and the aggregate of the between-group comparisons then are summed up. To get the value of the Vinci index for the population this value has to be adjusted for the number of all possible comparisons in the population and population size through division by $\frac{(n-1)^2}{2}$.

Vinci indices also can accommodate non-positive values. For distributions without negative values, there is no limit to the number of zero values that can be accommodated, as long as the mean of the distribution is positive. The Arcus differential can reach 3 for the comparison of a negative income with a zero or very small positive income, in that case, angle $\theta$ equals $270^\circ$. The Pigou-Dalton transfer principle is not violated for Vinci-indices with exponent $\delta \leq 1$, if half of the values are negative, zero and near-zero incomes.

**Summary**

The most used inequality measure is probably the Gini coefficient, a measure that focuses mostly on the concentration aspect of inequality. In this paper the Vinci index is introduced, a new tool that gives more weight to inequality at the lower end of the income distribution. The new measure is based on two types of differentials that are calculated for all possible income pairs in the distribution: angle-based Arcus differentials and the differentials that sum to the Gini coefficient. It emerges that the novel Vinci measure can be expressed as an aggregate of the products of Arcus and Gini differentials. A class of generalized Vinci indices is provided and the properties of the novel measures are discussed. The novel inequality measures satisfy the axioms of scale and population invariance, normalization, weak decomposability, and transfer sensitivity. They also can accommodate non-positive values. Furthermore, the differentials it is based on have simple geometrical representations. For concreteness, the novel measures are discussed in the context of income inequality, but they can also be used to study the dispersion of other transferable quantities.
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