THE EXPECTED DEPTH OF RANDOM REAL ALGEBRAIC PLANE CURVES

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ABSTRACT. In this note we study asymptotic isotopy of random real algebraic plane curves. More precisely, we obtain a Kac-Rice type formula that gives the expected number of two-sided components (i.e. ovals) of a random real algebraic plane curve winding around a given point. In particular, we show that expected number of such ovals for an even degree Kostlan polynomial is \( \sqrt{d} \) and independent of the given point.

1. INTRODUCTION

The expected number of real zeros of a random real univariate polynomial of a given degree \( d \) was first computed by Kac in [11]. In this work, Kac used a specific probability distribution. The analogous problem for different probability distributions, other questions about the statistics of zeros, the system version of the problem with \( n \) polynomial equations in \( n \) variables and several other variants of the problem were studied by many authors (see [3, 17] and the references therein). Another natural direction for generalization is to study higher dimensional solution sets, for example by looking at the common zero locus of a set of polynomials whose cardinality is less than the dimension of the ambient space. One can then seek results about the “expected topology” of the solution set. The first non-trivial instance of this problem is the zero locus of a single polynomial in the real projective space \( \mathbb{R}P^2 \), in other words, a random real algebraic plane curve. A technique developed by Nazarov and Sodin [14] was then employed by Gayet and Welschinger [7, 8] in order to obtain rather strong estimates for the Betti numbers of such solution sets. In particular, for the Kostlan ensemble they obtained that the number of connected components of a degree \( d \) random real algebraic plane curve is \( O(d) \) (compare to the classical Harnack upper bound for the number of connected components, which is \( O(d^2) \)). The Kostlan ensemble is a probability distribution of special geometric significance since it arises from the Fubini-Study metric on projective space and hence enjoys the unitary invariance property. We refer the reader to the sources [1, 5, 6, 9, 10, 12, 13, 15, 16] for various further results and questions about the expected topology of a positive dimensional solution set.

A significant portion of the classical and current research on real algebraic geometry, in particular of the research on real algebraic plane curves, focuses on the isotopy and rigid isotopy problems of real algebraic varieties. For example, the first part of Hilbert’s 16th problem raises the question of the isotopy classification of sextic curves in the real projective plane with the maximal number of components allowed by Harnack’s bound (M-curves). We refer the readers to [2, 4, 18] and [19] for excellent surveys containing accounts of both the history and the current state of affairs in this area of research.

In this paper, we concentrate on a particular aspect of the isotopy problem of a random real algebraic plane curve. In the deterministic case, Bezout’s theorem puts simple restrictions on the
“deep nests” that a degree \(d\) curve may have; for example no nest can contain more than \(\lfloor d/2 \rfloor\) ovals, as can be seen by intersecting the nest with a line. Defining the depth of a point in \(\mathbb{RP}^2\) to be equal to the number of ovals of the curve that contain the point in its interior, this tells us that no point can have a depth greater than \(\lfloor d/2 \rfloor\). We randomize the question and ask what the expected depth of a given point is, for a degree \(d\) random algebraic curve.

The organization of the paper is as follows: In section 2, we outline the topological preliminaries about real algebraic plane curves and set up the notation. In section 3, we prove Theorem 3.2 which is a double integral formula for winding numbers. This is done by adapting the original argument of Kac in [11] where instead of counting the number of zeros of a univariate polynomial we count the winding number of the components of the curve around a given point. In section 4, we randomize the problem and obtain Theorem 4.2 in which we prove our main result; an integral formula for the expected depth. In section 5, we show two applications of our formula. The first of these is an exact computation of the integral for the Kostlan ensemble; in particular in Theorem 5.1 we compute that the expected depth is equal to \(\sqrt{d/2}\) if \(d\) is even and \(\sqrt{d/2} + a_d\) if \(d\) is odd, where \(|a_d| \leq 1/2\). We also give an estimate of the expected depth for Kac ensemble in §5.2.

2. Topological Preliminaries

Let \(f \in \mathbb{R}[X,Y,Z]\) be a homogeneous polynomial of degree \(d\) and let \(V_f := \{f = 0\}\) be the real algebraic curve in \(\mathbb{RP}^2\) defined by \(f\). Let \(S^2\) denote the unit sphere centered at the origin in \(\mathbb{R}^3\). Recall that \(\mathbb{RP}^2\) can be identified with the set of 1-dimensional subspaces of \(\mathbb{R}^3\), and let \(\pi : S^2 \to \mathbb{RP}^2\) be the 2-to-1 covering map taking the two antipodal intersection points of such a subspace with \(S^2\), to the point in \(\mathbb{RP}^2\) representing this subspace.

Fix a point \(p \in \mathbb{RP}^2\). From now on, we will assume that \(V_f\) is a smooth curve and also that \(p \notin V_f\). We will now recall some facts about topology of real algebraic smooth plane curves; the reader is invited to consult [4], [18] or [19] for further discussion: \(V_f\) has the structure of a smooth 1-manifold, therefore it is diffeomorphic to a disjoint union of finitely many circles. By Harnack’s theorem, the number of connected components of \(V_f\) is bounded above by \(g + 1 = 1 + (d - 1)(d - 2)/2\), where \(g\) denotes the genus of the complexification of \(V_f\). Each connected component of \(V_f\) sits inside \(\mathbb{RP}^2\) in one of the following two essentially different ways: If it has an orientable tubular neighborhood (i.e. diffeomorphic to a cylinder) inside \(\mathbb{RP}^2\), then it is called an oval. If it has a non-orientable tubular neighborhood (i.e. diffeomorphic to a Möbius strip) inside \(\mathbb{RP}^2\) then it is called a pseudoline. It can be checked that a component is an oval if and only if it realizes the trivial homology class in \(H_1(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\) and it is a pseudoline otherwise. If the degree \(d\) of \(V_f\) is even, then \(V_f\) does not have any pseudolines among its components, so all of its components are ovals. If \(d\) is odd, then \(V_f\) has precisely one pseudoline component and all of its other components are ovals.

**Proposition 2.1.** Let \(K\) be a connected component of \(V_f\).

(a) If \(K\) is a pseudoline, then \(\pi^{-1}(K)\) is connected.

(b) If \(K\) is an oval, then \(\pi^{-1}(K)\) has two connected components.

**Proof.** Notice that both \(\pi^{-1}(K)\) and the preimage under \(\pi\) of a tubular neighborhood of \(K\) are symmetric under the antipodal map. After this observation, the proofs of both parts follow from the fact that the image of a single centrally symmetric cylinder contained in \(S^2\) under the antipodal identification is a Möbius strip, whereas the image of two disjoint cylinders under the antipodal identification is a cylinder.

Since for any \(p \in \mathbb{RP}^2\), the set \(\pi^{-1}(p) \subset S^2\) contains two points, we deduce that \(S^2 - \{\pi^{-1}(p)\}\) is homotopy equivalent to a circle. Therefore, \(H_1(S^2 - \{\pi^{-1}(p)\}, \mathbb{Z}) \cong \mathbb{Z}\).
Definition 2.2. Let $V_f$ be as above and $p \notin V_f$. Suppose that $K$ is an oval of $V_f$. We say that $p$ lies in the interior of $K$ if the homology class of one (therefore both) of the components of $\pi^{-1}(K)$ generates $H_1(S^2 - \{\pi^{-1}(p)\}, \mathbb{Z})$. Otherwise, we say that $p$ lies in the exterior of $K$.

Equivalently, $p$ lies in the interior of an oval $K$ if and only if the homology class of a component of $\pi^{-1}(K)$ is not trivial in $H_1(S^2 - \{\pi^{-1}(p)\}, \mathbb{Z})$. Yet another equivalent formulation is as follows: The complement of an oval $K$ in $\mathbb{RP}^2$ has two connected components, one diffeomorphic to a disk and the other diffeomorphic to a Möbius strip. The interior of $K$ is the component which is diffeomorphic to a disk.

Without loss of generality, let us assume that $p = [0 : 0 : 1]$ after a linear change of coordinates. Then the set $\pi^{-1}(p)$ contains the north and south poles of $S^2$. Consider the following 1-form on $S^2 - \{\pi^{-1}(p)\}$:

$$\omega = \frac{-YdX + XdY}{X^2 + Y^2}.$$

The form $\omega$ is a non-trivial element of the first de Rham cohomology group of $S^2 - \{\pi^{-1}(p)\}$ whose integral on the generators of $H_1(S^2 - \{\pi^{-1}(p)\}, \mathbb{Z})$ are equal to $\pm 2\pi$.

Definition 2.3. A generator $C$ of $H_1(S^2 - \{\pi^{-1}(p)\}, \mathbb{Z})$ is called positively oriented if $\int_C \omega = +2\pi$.

Note that the form $\omega$ is invariant under the antipodal map. Therefore, $\omega$ pushes forward to a well-defined 1-form $\pi_* (\omega)$ on $\mathbb{RP}^2 - \{p\}$. Since the map $\pi$ is 2-to-1, passing to the affine coordinates $x = X/Z$ and $y = Y/Z$, we obtain the following formula:

$$\pi_* (\omega) = 2 \frac{-ydx + xdy}{x^2 + y^2}.$$

Lemma 2.4. Let $V_f$ be a smooth real algebraic curve in $\mathbb{RP}^2$ and let $K$ be a connected component of $V_f$, not passing through $p$.

(a) If $K$ is a positively oriented pseudoline, then $\int_K \pi_* (\omega) = 2\pi$.

(b) If $K$ is an oval containing $p$ in its interior, which is positively oriented, then $\int_K \pi_* (\omega) = 4\pi$.

(c) If $K$ is an oval not containing $p$ in its interior, then $\int_K \pi_* (\omega) = 0$.

Proof. The proofs of all three parts directly follow from the equality

$$\int_K \pi_* (\omega) = \int_{\pi^{-1}(K)} \omega$$

and the choice of $\omega$. \qed

3. A Counting Formula for Winding Numbers

Next, we would like to count the number of ovals of $V_f$ that contain $p$ in its interior, by using a surface integral on $\mathbb{RP}^2$. In Theorem 3.2, we prove a Kac type counting lemma for winding numbers \[16\]. For any $\epsilon > 0$, let us define the function $\eta_{\varepsilon} : \mathbb{R} \to \mathbb{R}$ by

$$\eta_{\varepsilon}(x) = \begin{cases} 
\frac{1}{2\varepsilon}, & |x| \leq \varepsilon \\
0, & |x| > \varepsilon.
\end{cases}$$

Let us consider the affine chart $Z \neq 0$ of $\mathbb{RP}^2$ which is isomorphic to $\mathbb{R}^2$. Assume throughout this section that $p \notin \{Z = 0\}$ and $p \notin V_f$. For each connected component $C$ of $V_f$ not totally contained in $\{Z = 0\}$, let $C = C \cap \{Z \neq 0\}$. For each $C$, choose $U_{\tilde{C}}$ to be an open neighborhood of $\tilde{C}$ such that the closures $\overline{U_{\tilde{C}}}$ of these neighborhoods are pairwise disjoint. By some abuse of notation, let $f(x, y)$ denote the defining polynomial of $V_f \cap \{Z \neq 0\}$ in the affine coordinates.
Let $V_f$ be smooth and let $C$ be an oval of $V_f$. The smoothness assumption of $V_f$ implies that the gradient vector $\nabla f$ is never zero on $\tilde{C}$. Therefore, we can find an $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$, the neighborhood $\{ |f| < \epsilon \} \cap U_{\tilde{C}}$ of $\tilde{C}$ can be foliated by gradient curves of $f$. For each point $q \in \tilde{C}$, let $\gamma(q)$ be the gradient curve passing through the point $q$. Let $\vec{n}$ be the unit vector field on this neighborhood tangent to the leaves of the foliation at every point, oriented such that it points towards the exterior of the oval.

**Lemma 3.1.** Let $V_f$ be a smooth real algebraic curve in $\mathbb{RP}^2$ and $C$ be an oval of $V_f$. Then, for sufficiently small $\epsilon > 0$,

$$\int_{\tilde{C}} \pi_*(\omega) = \int_{\int_{\tilde{C}}} (\eta_\epsilon \circ f) \left| \frac{\partial f}{\partial n} \right| d\vec{n} \wedge \pi_*(\omega).$$

**Proof.** For sufficiently small $\epsilon$, the function $f$ is monotone along any gradient curve $\gamma(q)$. Then, clearly

$$\int_{\gamma(q)} (\eta_\epsilon \circ f) \left| \frac{\partial f}{\partial n} \right| d\vec{n} = 1.$$

Hence, by Fubini’s theorem we have

$$\int_{\int_{\tilde{C}}} (\eta_\epsilon \circ f) \left| \frac{\partial f}{\partial n} \right| d\vec{n} \wedge \pi_*(\omega) = \int_{\tilde{C}} \int_{\gamma(q)} (\eta_\epsilon \circ f) \left| \frac{\partial f}{\partial n} \right| d\vec{n} \wedge \pi_*(\omega) = \int_{\tilde{C}} \pi_*(\omega).$$

\[\square\]

Now, we obtain an integral formula for the number of ovals of $V_f$ that contain $p$ in its interior. For this reason, we identify the point $p$ with the point $[0 : 0 : 1]$ in $\mathbb{RP}^2$, hence with the point $(0,0)$ on the affine chart $\{ Z \neq 0 \}$ without loss of generality, in order to simplify the formula.

**Theorem 3.2.** Consider a real algebraic smooth curve $V_f$ of degree $d$ in $\mathbb{RP}^2$ whose equation in the affine chart $Z \neq 0$ is given by $f(x,y) = 0$. Suppose that $f(0,0) \neq 0$ and the line $\{ Z = 0 \}$ is not a component of $V_f$. Then the number of ovals of $V_f$ containing $p = [0 : 0 : 1]$ in their interior is given by the formula

(a) $\lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_{\mathbb{R}^2} (\eta_\epsilon \circ f)|fx + fy| \frac{dx \wedge dy}{x^2 + y^2}$ if $d$ is even,

(b) $\lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_{\mathbb{R}^2} (\eta_\epsilon \circ f)|fx + fy| \frac{dx \wedge dy}{x^2 + y^2} + a$ if $d$ is odd, where $|a| \leq 1/2$.

**Proof.** First, we choose pairwise disjoint neighborhoods $U_{\tilde{C}}$ of the components on which the vector $\nabla f$ is never zero, by the assumption of smoothness. If $\vec{u}$ denotes the unit vector along $\nabla f$, then $\nabla f \cdot d\vec{u} = fx + fy$. Consequently,

$$\nabla f \cdot d\vec{u} = fxdx + fdydy$$

(3.1)

Let $C$ be an oval of $V_f$. If $\nabla f$ points in the outward normal direction along $C$, then $\vec{u} = \vec{n}$, where $\vec{n}$ is as above. Otherwise $\vec{u} = -\vec{n}$. Also noting that $\frac{dx \wedge dy}{x^2 + y^2}$ is a positive multiple of the area form, we get

$$\left| \frac{\partial f}{\partial n} \right| d\vec{n} \wedge \pi_*(\omega) = 2|fx + fy| \frac{dx \wedge dy}{x^2 + y^2}.$$  

(3.2)

Hence the integrand in the statement of the theorem agrees with $(4\pi)^{-1}$ times the integrand in Lemma [3.1].
Now, let us determine the contribution of each component of $V_f$ to the integral. For any connected component $C$ we have
\[
\int_C \pi_*(\omega) = \int_{\tilde{C}} \pi_*(\omega)
\]
since the form $\pi_*(\omega)$ is uniformly bounded on all of $C$ and $C - \tilde{C}$ has measure 0.

It is clear that we can partition $\mathbb{R}^2$ into neighborhoods $\overline{U}_C$ of the finitely many components of $V_f$, as above, whose interiors do not intersect. We can then apply the lemma 3.1 to each oval. By lemma 2.4, part (b), if $C$ is an oval containing $p$ in its interior, then the contribution of $C$ to the integral in the theorem will be 1. If $C$ is an oval not containing $p$ in its interior, then its contribution will be 0, lemma 2.4, part (c). This proves part (a) of the theorem.

In order to prove part (b), we estimate the contribution of the single pseudoline $C$ of $V_f$ to the integral. In this case, we cannot choose the vector field $\Pi$ consistently in a neighborhood of $C$, however, assuming that $C$ is positively oriented, by triangle inequality we obtain
\[
\left| \frac{1}{2\pi} \int_{U_C} (\nu_2 \circ f) |xf_x + yf_y| \frac{dx \wedge dy}{x^2 + y^2} \right| \leq \frac{1}{4\pi} \int_{C} \pi_*(\omega).
\]
The right hand side equals 1/2 by 2.4 part (a). This proves part (b) of the theorem. \hfill \square

4. Expected depth of a random algebraic curve

We let $W_{2,d}$ denote the real vector space of homogeneous polynomials $f(X, Y, Z)$ of degree $d$. Recall that a centered Gaussian probability measure on the vector space $W_{2,d}$ is given by a real inner product $\langle , \rangle$ and covariance matrix $C$. For simplicity of the notation, in what follows we assume that the random coefficients are uncorrelated, this means that the covariance matrix $C$ is the identity matrix. In particular, the induced Gaussian probability measure $\text{Prob}_d$ is defined by
\[
\text{Prob}_d(A) := \frac{1}{\nu_{2,d}} \int_A e^{-\langle f, f \rangle \over 2} df
\]
where $A \subset W_{2,d}$ is a measurable set, $\nu_{2,d}$ is the normalizing constant to have a probability measure and $f$ is treated as the integration variable that runs over $W_{2,d}$.

For a random polynomial $f \in W_{2,d}$ we let $\ell_p(f)$ denote the number of ovals of $V_f \subset \mathbb{R}P^2$ that contain the point $p$ in their interior. It is easy to see that for almost every $f \in W_{2,d}$ its zero locus $V_f$ is non-singular, $f(p) \neq 0$ and $V_f$ does not contain $\{Z = 0\}$ as a connected component. Hence,
\[
\ell_p : (W_{2,d}, \text{Prob}_d) \to \mathbb{N}
\]
defines a random variable and we wish to compute its expected value $\mathbb{E}[\ell_p]$.

Using the above notation, we may identify the point $p \in \mathbb{R}P^2$ with the point $[0 : 0 : 1]$, hence with the origin $(0, 0)$ in the affine chart $\{Z \neq 0\}$ and assume that the equation of $V_f$ in the affine chart is given by $f(x, y) = 0$ where
\[
f(x, y) = \sum_{|J| \leq d} a_J c_J x^{j_1} y^{j_2}
\]
$J = (j_1, j_2)$, $|J| = j_1 + j_2$, the $a_J$ are i.i.d. standard Gaussian $N(0, 1)$ and $c_J$ are non-zero deterministic constants determined by the scaler inner product $\langle , \rangle$.

For $q = (x, y) \in \mathbb{R}^2$, we consider the Gaussian variables
\[
W_q = f(x, y) \text{ and } T_q = xf_x(x, y) + yf_y(x, y).
\]
We also let
\[
\Sigma_q := \begin{bmatrix} \alpha_q & \beta_q \\ \beta_q & \gamma_q \end{bmatrix}
\]
be the covariance matrix of the Gaussian vector \((W_q, T_q)\) where
\[
\alpha_q = \text{Cov}(W_q, W_q), \quad \beta_q = \text{Cov}(W_q, T_q) \quad \text{and} \quad \gamma_q = \text{Cov}(T_q, T_q).
\]
We observe that
\[
\alpha_q = \mathbb{E}[(f(x, y))^2], \quad \beta_q = \mathbb{E}[f(x, y)(xf_x(x, y) + yf_y(x, y))] \quad \text{and} \quad \gamma_q = \mathbb{E}[(xf_x(x, y) + yf_y(x, y))^2].
\]

We will need the following lemma

**Lemma 4.1.** Let \(f(x, y) = 0\) be a Gaussian random real algebraic curve of degree \(d\). Then

\[
\int \int_{\mathbb{R}^2} \frac{\sqrt{\det \Sigma}}{2\pi} \frac{\alpha}{\alpha^2} \frac{x^2 + y^2}{dx dy} < \infty.
\]

**Proof.** Since \(a_j\) are standard Gaussian we have \(\alpha_q = \sum_{|j| \geq 0} c_j^2 x^{2j_1} y^{2j_2}, \beta_q = \sum_{|j| \geq 1} c_j^2 (j_1 + j_2) x^{2j_1} y^{2j_2}\) and \(\gamma_q = \sum_{|j| \geq 1} c_j^2 (j_1 + j_2)^2 x^{2j_1} y^{2j_2}\). The assertion follows by expressing the integral in the polar coordinates. \(\square\)

Now, we obtain the following Kac-Rice type formula for the expected depth of a random real algebraic curve:

**Theorem 4.2.** Let \(f(x, y) = 0\) be a Gaussian random real algebraic curve of degree \(d\). Then the expected depth of \(f(x, y)\) is given by

\[
\mathbb{E}[\ell_p] = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} \mathbb{E}[(xf_x + yf_y)|f = 0] \frac{dx dy}{x^2 + y^2} + a_d
\]

\[
= \frac{1}{2\pi} \int \int_{\mathbb{R}^2} \frac{\sqrt{\det \Sigma}}{2\pi \alpha} \frac{dx dy}{x^2 + y^2} + a_d.
\]

where the constant \(a_d = 0\) if \(d\) is even and satisfies \(|a_d| \leq \frac{1}{2}\) if \(d\) is odd.

**Proof.** We begin with estimating the expression

\[
\mathbb{E}[(\eta_c \circ f)|xf_x + yf_y|].
\]

Denoting the Gaussian vector \(U = \begin{bmatrix} W \\ T \end{bmatrix}\) we have

\[
\mathbb{E}[(\eta_c \circ f)|xf_x + yf_y] = \frac{1}{2\pi} \int_{|W| < \epsilon} \int_{\mathbb{R}} |T| \frac{1}{2\pi \sqrt{\det \Sigma}} e^{-\frac{1}{2}(\Sigma^{-1}U,U)} dTdW
\]

\[
= \frac{1}{2\pi} \int_{|W| < \epsilon} \int_{\mathbb{R}} |T| \frac{1}{2\pi \sqrt{\det \Sigma}} e^{-\frac{1}{2\alpha \det \Sigma}(\gamma W^2 - 2\beta WT + \alpha T^2)} dTdW
\]

\[
= \frac{1}{2\pi} \int_{|W| < \epsilon} e^{-\frac{1}{2\alpha} W^2} \left( \int_{\mathbb{R}} \frac{1}{2\pi \sqrt{\det \Sigma}} e^{-\frac{\alpha}{2\det \Sigma}(T - \frac{\beta W}{\alpha})^2} dT \right) dW
\]

\[
= \frac{1}{2\pi} \int_{|W| < \epsilon} e^{-\frac{1}{2\alpha} W^2} \left( \int_{\mathbb{R}} |T| d\Gamma_{\frac{\alpha W}{\alpha}, \frac{\det \Sigma}{\alpha}} \right) dW
\]

where \(d\Gamma_{m, \sigma^2} := \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(T-m)^2}{2\sigma^2}} dT\) is the law of normal distribution \(N(m, \sigma)\) on \(\mathbb{R}\). Note that \(\phi(W) := \int_{\mathbb{R}} |T| d\Gamma_{\frac{\beta W}{\alpha}, \frac{\det \Sigma}{\alpha}}\) is a continuous function of \(W\) and this implies that

\[
\lim_{\epsilon \to 0^+} \mathbb{E}[(\eta_c \circ f)|xf_x + yf_y] = \phi(0)
\]

\[
= \frac{1}{\sqrt{2\pi\alpha}} \int_{\mathbb{R}} |T| d\Gamma_{0, \frac{\det \Sigma}{\alpha}}
\]
Moreover, by Jensen’s inequality we have
\[ \mathbb{E}[|T|] \leq \sqrt{\mathbb{E}[T^2]} \leq \sqrt{\text{Var}(T) + \mathbb{E}[T]^2} \]
where \(\text{Var}(T)\) denotes the variance of \(T\). This in turn implies that
\[
\int_\mathbb{R} |T| d\Gamma_{(0, \frac{\beta W}{\alpha}, \frac{\det \Sigma}{\alpha})} \leq \sqrt{\frac{\det \Sigma}{\alpha} + \frac{\beta^2 W^2}{\alpha^2}} \\
\leq \sqrt{\frac{\det \Sigma}{\alpha} + |\beta W|}. 
\]

Hence, for \(\epsilon \leq 1\) we have
\[
(4.2) \quad \mathbb{E}[(\eta_k \circ f)|xf_x + yf_y|] \leq \frac{1}{\sqrt{2\pi}} \left( \frac{\sqrt{\det \Sigma}}{\alpha} + \frac{|\beta|}{\alpha^2} \right). 
\]

Then it follows from Lemma 4.1, Theorem 3.2, Harnack’s bound, Lebesgue dominated convergence theorem and Tonelli’s theorem that
\[
(4.3) \quad \mathbb{E}[\ell_p] = \mathbb{E} \left[ \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_{\mathbb{R}^2} (\eta_k \circ f)|xf_x + yf_y| \frac{dx dy}{x^2 + y^2} \right] + a_d \\
= \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathbb{E}[\eta_k \circ f |xf_x + yf_y|] \frac{dx dy}{x^2 + y^2} + a_d \\
= \frac{1}{2\pi} \int_{\mathbb{R}^2} \left[ \frac{1}{\sqrt{2\pi \alpha}} \int_\mathbb{R} |T| d\Gamma_{(0, \frac{\beta W}{\alpha}, \frac{\det \Sigma}{\alpha})} \right] \frac{dx dy}{x^2 + y^2} + a_d \\
= \frac{1}{\sqrt{\det \Sigma}} \int_\mathbb{R} |T| d\Gamma_{(0, \frac{\beta W}{\alpha}, \frac{\det \Sigma}{\alpha})}. 
\]

where the constant \(a_d = 0\) if \(d\) is even and satisfies \(|a_d| \leq \frac{1}{2}\) if \(d\) is odd. Then we compute
\[
(4.4) \quad \frac{1}{\sqrt{2\pi \alpha}} \int_\mathbb{R} |T| d\Gamma_{(0, \frac{\beta W}{\alpha}, \frac{\det \Sigma}{\alpha})} = \frac{1}{\pi \sqrt{\det \Sigma}} \int_0^\infty T e^{-\frac{\alpha^2 T^2}{2}} dT \\
= \frac{\sqrt{\det \Sigma}}{\pi \alpha}. 
\]

Note that
\[
(4.5) \quad \mathbb{E}[|xf_x + yf_y| |f = 0] = \frac{1}{\sqrt{2\pi \alpha}} \int_\mathbb{R} |T| d\Gamma_{(0, \frac{\beta W}{\alpha}, \frac{\det \Sigma}{\alpha})}. 
\]

Hence, combining (4.3), (4.4) and (4.5) the assertion follows. \(\square\)

Recall that
\[ \alpha_q = \mathbb{E}[f(x, y)^2] =: K_d(x, y). \]

Denoting the differential operators by
\[ D_1 := \frac{x}{2} \frac{\partial}{\partial x} \quad \text{and} \quad D_2 := \frac{y}{2} \frac{\partial}{\partial y} \]
we obtain
\[ \beta_q = (D_1 + D_2)K_d(x, y) \quad \text{and} \quad \gamma_q = (D_1 + D_2)^2K_d(x, y) \]
which yields
\[
(4.6) \quad \frac{\sqrt{\det \Sigma}}{\alpha} = \sqrt{(D_1 + D_2)^2 \log K_d(x, y)}. 
\]
Thus, we obtain the following formula (cf. [3, Theorem 2.2]):

\[
E[\ell_p] = \frac{1}{2\pi^2} \int_{\mathbb{R}^2} \sqrt{(D_1 + D_2)^2 \log K_d(x, y)} \frac{dx dy}{x^2 + y^2} + a_d.
\]

where \(a_d\) as in Theorem 4.2

5. Applications

In this section we will apply Theorem 4.2 to various ensembles of random polynomials.

5.1. Kostlan Ensemble. In this part, we endow the vector space \(W_{2,d}\) with the real inner product

\[
\langle \sum_{|J|=d} a_J x^{j_1} y^{j_2} z^{j_3}, \sum_{|J|=d} b_J x^{j_1} y^{j_2} z^{j_3} \rangle = \sum_{|J|=d} \left( \begin{array}{c} d \\ j_1, j_2, j_3 \end{array} \right) a_J b_J.
\]

Thus, in the affine coordinates \(Z \neq 0\), we obtain Gaussian random polynomials of the form

\[
f(x, y) = \sum_{|J|\leq d} a_J \sqrt{\binom{d}{j_1, j_2}} x^{j_1} y^{j_2}
\]

where \(J = (j_1, j_2), \left( \begin{array}{c} d \\ j_1, j_2 \end{array} \right) = \frac{d!}{j_1! j_2! (d - j_1 - j_2)!}\) and the \(a_J \sim N(0, 1)\) are independent standard normal random variables. The corresponding ensemble of random polynomials is known as Kostlan ensemble (cf. [3]). We remark that the probability distribution of this family is invariant under orthogonal transformations. In particular, the expected value \(E[\ell_p]\) is independent of the point \(p \in \mathbb{RP}^2\).

**Theorem 5.1.** For any point \(p \in \mathbb{RP}^2\) the expected number of ovals of a degree \(d\) random Kostlan polynomial containing \(p\) in its interior satisfies

\[
E[\ell_p] = \frac{\sqrt{d}}{2} + a_d
\]

where \(a_d\) as in Theorem 4.2

**Proof.** The proof is a direct application of Theorem 4.2. Namely,

\[
\alpha_q = K_d(x, y) = (1 + x^2 + y^2)^d \text{ for } (x, y) \in \mathbb{R}^2
\]

and

\[
\frac{\sqrt{\det \Sigma}}{\alpha} = \sqrt{(D_1 + D_2)^2 \log K_d(x, y)} = \frac{\sqrt{d(x^2 + y^2)}}{1 + x^2 + y^2}.
\]

Hence, by (4.7)

\[
E[\ell_p] = \frac{1}{2\pi^2} \int_{\mathbb{R}^2} \frac{\sqrt{d(x^2 + y^2)}}{(1 + x^2 + y^2)(x^2 + y^2)} dx dy + a_d
\]

\[
= \sqrt{d} \int_{0}^{2\pi} \int_{0}^{\infty} \frac{1}{1 + r^2} dr d\theta + a_d
\]

\[
= \frac{\sqrt{d}}{2} + a_d.
\]

where \(a_d\) as in Theorem 4.2.
5.2. **Kac Ensemble with Square Newton Polygon.** Next, we consider random polynomials of the form

$$f(x, y) = \sum_{0 \leq i, j \leq d} a_{ij}x^i y^j$$

where $a_{ij} \sim N(0, 1)$ are independent unit normal random variables. In this case,

$$K_d(x, y) = \sum_{0 \leq i, j \leq d} x^{2i} y^{2j} = \left( \frac{1 - x^{2d+2}}{1 - x^2} \right) \left( \frac{1 - y^{2d+2}}{1 - y^2} \right)$$

which yields

$$\left( D_1 + D_2 \right) \log K_d(x, y) = D_1 \log \left( \frac{1 - x^{2d+2}}{1 - x^2} \right) + D_2 \log \left( \frac{1 - y^{2d+2}}{1 - y^2} \right)$$

$$= \left[ \frac{x^2}{(x^2 - 1)^2} - \frac{(d + 1)^2 x^{2d}}{(x^{2d+2} - 1)^2} \right] + \left[ \frac{y^2}{(y^2 - 1)^2} - \frac{(d + 1)^2 y^{2d}}{(y^{2d+2} - 1)^2} \right]$$

Thus, by (4.7) we obtain

**Theorem 5.2.** The expected number of ovals of a degree $d$ random Kac polynomial containing $(0, 0)$ in its interior satisfies

$$\mathbb{E}[\ell_{(0,0)}] = \frac{1}{2\pi^2} \int_{\mathbb{R}^2} \sqrt{x^2 \left[ \frac{1}{(x^2 - 1)^2} - \frac{(d + 1)^2 x^{2d}}{(x^{2d+2} - 1)^2} \right] + y^2 \left[ \frac{1}{(y^2 - 1)^2} - \frac{(d + 1)^2 y^{2d}}{(y^{2d+2} - 1)^2} \right]} \ dx \ dy + a_d$$

where $a_d$ as in Theorem 4.2.

Next, we estimate the growth order of the expected number of ovals $\mathbb{E}[\ell_{(0,0)}]$. To this end, we write the right hand side of (5.3) in polar coordinates and by symmetry we obtain

$$\mathbb{E}[\ell_{(0,0)}] = \frac{1}{2\pi^2} \int_0^\infty \int_0^{2\pi} \sqrt{\cos^2(\theta) \phi_d(r \cos \theta) + \sin^2(\theta) \phi_d(r \sin \theta)} \, d\theta \, dr + a_d$$

$$= \frac{4}{\pi^2} \int_0^\infty \int_0^{\frac{\pi}{2}} \sqrt{\cos^2(\theta) \phi_d(r \cos \theta) + \sin^2(\theta) \phi_d(r \sin \theta)} \, d\theta \, dr + a_d$$

where

$$\phi_d(s) = \frac{1}{(s^2 - 1)^2} - \frac{(d + 1)^2 s^{2d}}{(s^{2d+2} - 1)^2} \text{ for } s \neq 1$$

and $\phi_d$ has a removable singularity near 1. We remark that the function $\frac{1}{\pi} \sqrt{\phi_d(s)}$ is the density of expected number of real roots of Kac polynomials [11] in dimension one. Moreover, by [3], Theorem 2.2] we have

$$\frac{1}{\pi} \int_0^\infty \sqrt{\phi_d(s)} \, ds = \frac{2}{\pi} \log d + C_1 + \frac{2}{d\pi} + O\left(\frac{1}{d^2}\right) \text{ as } d \to \infty$$

where $C_1 > 0$ does not depend on $d$. Hence, the expected number of ovals of a degree $d$ random Kac polynomial containing $(0, 0)$ in its interior satisfies

$$c_1 \log d + o(1) \leq \mathbb{E}[\ell_{(0,0)}] \leq c_2 \log d + o(1)$$

for some constants $c_1, c_2 > 0$ independent of $d$. In particular, we observe that
Corollary 5.3. The expected number of components of a degree $d$ random Kac polynomial in dimension two satisfies

$$\mathbb{E}[b_0(V_f)] \geq c \log d$$

for some $c > 0$ independent of $d$.

Finally, to our knowledge finding the precise growth order of expected number of components for Kac ensemble in dimension two is an open question.

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