ACTIONS AND COACTIONS OF FINITE QUANTUM GROUPOIDS ON VON NEUMANN ALGEBRAS, EXTENSIONS OF THE MATCH PAIR PROCEDURE

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Abstract. In this work we investigate the notion of action or coaction of a finite quantum groupoid in von Neumann algebras context. In particular we prove a double crossed product theorem and prove the existence of an universal von Neumann algebra on which any finite groupoids acts outerly. In previous works, N. Andruskiewitsch and S. Natale define for any match pair of groupoids two $C^*$-quantum groupoids in duality, we give here an interpretation of them in terms of crossed products of groupoids using a multiplicative partial isometry which gives a complete description of these structures. In a next work we shall give a third description of these structures dealing with inclusions of depth two inclusions of von Neumann algebras associated with outer actions of match pairs of groupoids, and a study, in the same spirit, of an other extension of the match pair procedure.
1. Introduction

Multiplicative partial isometries (mpi) generalize Baaj and Skandalis multiplicative unitaries in finite dimension [BS], [BBS]. They are the finite-dimensional version of so-called pseudo-multiplicative unitaries, which appeared first in a commutative context dealing with locally compact groupoids [Val0], and then in the general case for a very large class of depth two inclusions of von Neumann algebras in a common work with M.Enock [EV], who has developed the theory in the infinite dimensional framework leading to F.Lesieur’s measured quantum groupoids [L].

When it is regular, any mpi generates two involutive subalgebras of the algebra of all bounded linear operators on the corresponding Hilbert space. Using a canonical pairing, these two algebras have structures generalizing involutive Hopf algebras. The first examples of these new structures were discovered by the theoretical physicists Böhm, Szlachanyi and Nill [BoSz] [BoSzNi], they called them weak Hopf $C^*$-algebras.

D.Nikshych and L.Vainerman, using general inclusions of depth two subfactors of type $II_1$ with finite index $M_0 \subset M_1$, and a special pairing between relative commutants $M'_0 \cap M_2$ and $M'_1 \cap M_3$, gave explicit formulas for weak Hopf $C^*$-algebra structures in duality for these two last involutive algebras [NV2] and found a Galois correspondence between intermediate subfactors and involutive coideals for $M'_1 \cap M_3$ [NV4].

In an algebraic construction, N.Andruskiewitsch and S.Natale in [AA], give a construction of weak Hopf $C^*$-algebras dealing with match pairs of groupoids in a sense generalizing directly the group case.

This work is an operator algebra point of view on these match pairs of groupoids, we use a special multiplicative partial isometry to give an interpretation of these examples in terms of groupoids crossed products generalizing in finite dimension previous works in the quantum groups context.

In the second paragraph we give the fundamental definitions and properties of multiplicative partial isometries and their close connection with quantum groupoids.

The third chapter is an approach of the actions of quantum groupoids, the notion of outerness in the groupoid situation, to reach a double crossed product theorem in our context.

The fourth chapter deals with match pairs of groupoids, the quantum groupoids associated with, and their relation with the algebraic point of view in [AA].

So a natural prolongement of this article will be the generalization of these constructions, and other extensions of the match pair procedure, in the direction of Lesieur’s locally compact groupoids. Another will be a characterization of these objects in terms of cleft extensions in the spirit of S.Vaes and L.Vainerman [VV].

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2. Multiplicative partial isometries and quantum groupoids
2.1. Multiplicative partial isometries.

2.1.1. Notations. In this article, $N$ is a finite dimensional von Neumann algebra, so $N$ is isomorphic to a sum of matrix algebras $\oplus \gamma M_{n_\gamma}$, we denote the family of minimal central projections of $N$ by $\{p_\gamma\}$, and we denote a given family of matrix units for $N$ by $\{e_{i,j}\gamma/1 \leq i, j \leq n_\gamma\}$. We shall denote the opposite von Neumann algebra of $N$ by $N^o$, so this is $N$ with the opposite multiplication, hence a matrix unit of $N^o$ is given by the transposed of $N$’s: $\{e_{j,i}\gamma/1 \leq i, j \leq n_\gamma\}$. The element $f = \sum \gamma \sum_{i,j} e_{i,j}\gamma \otimes e_{j,i}\gamma$ is the only projection of $N^o \otimes N$ such that, for any $n$ in $N$: $f(n^o \otimes 1) = f(1 \otimes n)$ and if $f(1 \otimes n) = 0$ then $n = 0$.

Let $M_1, M_2$ be two von Neumann algebras. Let $s$ (resp., $r$) be a faithful non degenerate antirepresentation (resp., a representation) from $N$ to $M_1$ (resp., $M_2$), then $s$ can be viewed as a representation $s^o$ of $N^o$. Let us define:

$$e_{s,r} = (s^o \otimes r)(f) = \sum_{\gamma i,j} \frac{1}{n_\gamma} s(e_{i,j}\gamma) \otimes r(e_{j,i}\gamma).$$

As an obvious generalization of Lemma 2.1.2 in [Val1], $e_{s,r}$ is a projection in $s(N) \otimes r(N)$, $e_{s,r}$ is the only projection $e$ in $M_1 \otimes r(N)$ which satisfies the following two conditions:

a) For every $m_1, m_2$ in $M_1$ and $M_2$ respectively, the relation $e(m_1 \otimes 1) = 0$ implies $m_1 = 0$ and the relation $e(1 \otimes m_2) = 0$ implies $m_2 = 0$,

b) for every $n$ in $N$: $e(s(n) \otimes 1) = e(1 \otimes r(n))$.

If $Tr_{H_2}$ is the canonical trace of $H_2$ and if $H_2$ is finite dimension, $(i \otimes Tr_{H_2})(e_{s,r})$ is positive and invertible in the center of $s(N)$.

Let $H$ be a finite dimensional Hilbert space, and let $\alpha$ (resp., $\beta, \hat{\beta}$) be an injective non degenerate representation (resp., two injective non degenerate antirepresentations) of $N$, which commute two by two pointwise. We also suppose that $tr \circ \alpha = tr \circ \beta = tr \circ \hat{\beta}$, where $tr$ is the canonical trace on $H$ and let us note $\tau = tr \circ \alpha$. One must keep in mind that $\beta$ and $\alpha$ are a representation and an antirepresentation of $N^o$.

2.1.2. Definition. We call a multiplicative partial isometry with the base $(N, \alpha, \beta, \hat{\beta})$ every partial isometry $I$ whose initial (resp., final) support is $e_{\beta,\alpha}$ (resp., $e_{\alpha,\beta}$) and such that:

1) $I$ commutes with $\beta(N) \otimes \hat{\beta}(N)$,

2) For every $n, n'$ in $N$, one has: $I(\alpha(n) \otimes \beta(n')) = (\hat{\beta}(n') \otimes \alpha(n))I$,

3) $I$ satisfies the pentagonal relation: $I_{12}I_{13}I_{23} = I_{23}I_{12}$.

By Lemmas 2.3.1, 2.4.2, 2.4.6 in [Val1], one has:
2.1.3. Notations and lemma. Let $I$ be a multiplicative partial isometry with the base $(N, \alpha, \beta, \hat{\beta})$, let’s denote the set $\{(\omega \otimes i)(I)/\omega \text{ linear form on } \mathcal{L}(H)\}$ by $S$, and let’s denote the set $\{(i \otimes \omega)(I)/\omega \text{ linear form on } \mathcal{L}(H)\}$ by $\hat{S}$. Then $S$ and $\hat{S}$ are non degenerate subalgebras of $\mathcal{L}(H)$.

2.1.4. Lemma. (cf. Lemme 2.6.2 of [Val1]). $S$ and $\hat{S}$ are sub von Neumann algebras of $\mathcal{L}(H)$ if and only if $\{(i \otimes \omega)(\Sigma I)/\omega \text{ is a linear form on } \mathcal{L}(H)\} = \alpha(N)^{\prime}$; in this case one says that $I$ is regular.

2.2. Quantum groupoids. Let us recall the definition of a quantum groupoid (or a weak Hopf $C^*$-algebra):

2.2.1. Definition. (G. Böhm, K. Szlachányi, F. Nill) [BoSzNi]

A weak Hopf $C^*$-algebra is a collection $(A, \Gamma, \kappa, \epsilon)$ where: $A$ is a finite-dimensional $C^*$-algebra (or von Neumann algebra), $\Gamma : A \rightarrow A \otimes A$ is a generalized coproduct, which means that: $(\Gamma \otimes i)\Gamma = (i \otimes \Gamma)\Gamma$, $\kappa$ is an antipode on $A$, i.e., a linear application from $A$ to $A$ such that $(\kappa \circ \ast)^2 = i$ (where $\ast$ is the involution on $A$), $\kappa(xy) = \kappa(y)\kappa(x)$ for every $x, y$ in $A$ with $(\kappa \circ \kappa)\Gamma = \varsigma \Gamma \kappa$ (where $\varsigma$ is the usual flip on $A \otimes A$).

We suppose also that $(m(\kappa \otimes i) \otimes i)(\Gamma \otimes i)\Gamma(x) = (1 \otimes x)\Gamma(1)$ (where $m$ is the multiplication of tensors, i.e., $m(a \otimes b) = ab$), and that $\epsilon$ is a counit, i.e., a positive linear form on $A$ such that $(\epsilon \otimes i)\Gamma = (i \otimes \epsilon)\Gamma = i$, and for every $x, y$ in $A$: $(\epsilon \otimes \epsilon)((x \otimes 1)\Gamma(1)(1 \otimes y)) = \epsilon(xy)$.

2.2.2. Results. (cf. [NV1], [NV3],[BoSzNi]) If $(A, \Gamma, \kappa, \epsilon)$ is a weak Hopf $C^*$-algebra, then the following assertions are true:

1) The sets

$$A_t = \{x \in A/\Gamma(x) = \Gamma(1)(x \otimes 1) = (x \otimes 1)\Gamma(1)\},$$

$$A_s = \{x \in A/\Gamma(x) = \Gamma(1)(1 \otimes x) = (1 \otimes x)\Gamma(1)\}$$

are commuting sub $C^*$-algebras of $A$ and $\kappa(A_t) = A_s$; one calls them respectively target and source Cartan subalgebra of $(A, \Gamma, \kappa, \epsilon)$.

2) There exists a unique faithful positive linear form $\phi$, called the normalized Haar measure of $(A, \Gamma, \kappa, \epsilon)$, satisfying the following three properties:

$\phi \circ \kappa = \phi$, $(i \otimes \phi)(\Gamma(1)) = 1$ and, for every $x, y$ in $A$:

$$(i \otimes \phi)((1 \otimes y)\Gamma(x)) = \kappa((i \otimes \phi)(\Gamma(y)(1 \otimes x))).$$

3) The application $E^s_\phi = (\phi \otimes i)\Gamma$ (resp., $E^t_\phi = (i \otimes \phi)\Gamma$) is the conditional expectation with values in the source (resp., target) Cartan subalgebra, such that: $\phi \circ E^\ast_\phi = \phi$ (resp., $\phi \circ E^t_\phi = \phi$), it is called a source (resp., target) Haar conditional expectation. If $g_s = E^s_\phi(p)^\dagger$ and $g_t = E^t_\phi(p)^\dagger$, then one has $g_t = \kappa(g_s)$. For every $a$ in $A$, $\kappa^2(a) = g_t g_s^{-1} a g_t^{-1} g_s$, and the modular group $\sigma^\phi_{i}$ is given by $\sigma^\phi_{i}(a) = g_t g_s a g_t^{-1} g_s^{-1}$; this leads to a polar decomposition $\kappa = j \circ \text{Ad} \phi_{g}$, where $j$ is the involutive anti-homomorphism of $A$ (coinvolution) defined by $j(y) = g_s \kappa(x)g_s^{-1}$ for any $x$ in $A$.
It is shown in [Val1], that $I$ is regular, if and only if it generates two quantum groupoids in duality:

2.2.3. **Proposition.** If $I$ is regular, then one can define two quantum groupoids in duality $(S, \Gamma, \kappa, \epsilon)$ and $(\hat{S}, \hat{\Gamma}, \hat{\kappa}, \hat{\epsilon})$, by the formulas:

for any $s \in S : \Gamma(s) = I(s \otimes 1)I^*$,

for any $\hat{s} \in \hat{S} : \hat{\Gamma}(\hat{s}) = I^*(1 \otimes \hat{s})I$,

for any $\omega \in \mathcal{L}(H)$, $\kappa(\omega \otimes i)(I) = (\omega \otimes i)(I^*)$, $\hat{\kappa}((i \otimes \omega)(I) = (i \otimes \omega)(I^*)$,

$\epsilon((\omega \otimes i)(I)) = \omega(1)$ and $\hat{\epsilon}((i \otimes \omega)(I)) = \omega(1)$.

These two quantum groupoids are in duality, using the following bracket:

$<(\omega \otimes i)(I), (i \otimes \omega')(I)> = (\omega \otimes \omega')(I)$.

This proposition has a reciprocal (see [Val] 3.2.3 and [Val3]):

2.2.4. **Proposition.** Let $(A, \Gamma, \kappa, \epsilon)$ be any quantum groupoid such that $\kappa$ is involutive on Cartan subalgebra, and let $\phi$ be its normalized Haar measure, then with GNS notations, the application $I : \Lambda_{\phi\otimes\phi}(x \otimes y) \mapsto \Lambda_{\phi\otimes\phi}(\Gamma(x)(1 \otimes y))$ is a regular mpi on $H_{\phi}\otimes H_{\phi}$, and the GNS representation $\pi_{\phi}$ is an isomorphism of $(A, \Gamma, \kappa, \epsilon)$ on the quantum groupoid $(S, \Gamma, \kappa, \epsilon)$ given by proposition 2.2.3.

In fact, in [Val2], we proved that any regular mpi $I$ can be modified in an irreducible form for which there exist separating and cyclic vectors $\epsilon$ and $\hat{\epsilon}$ for $S$ and $\hat{S}$ respectively, so that these are in standard form on $H$. With Tomita’s theory notations, the unitary $U = JJ = J\hat{J}$ leads to define a fourth representation $\hat{\alpha} = \text{Ad}U \circ \alpha : N \rightarrow \mathcal{L}(H)$, and two new mpi $\hat{I} = \Sigma(U \otimes 1)I(U \otimes 1)\Sigma$, $\hat{\Gamma} = \Sigma(1 \otimes U)(I(1 \otimes U)\Sigma$ over the base $(\Lambda^0, \beta, \hat{\alpha}, \alpha)$ and $(\Lambda^0, \hat{\beta}, \alpha, \hat{\alpha})$. Applying proposition 2.2.3 to $\hat{I}$ (resp. $\hat{\Gamma}$) leads to define another quantum groups on $\hat{S}'$ (resp. $S'$) the commutant in $\mathcal{L}(H)$ of $\hat{S}$ (resp. $S$), let’s denote them $(S', \Gamma', \kappa', \epsilon')$ and $(\hat{S}', \hat{\Gamma}', \hat{\kappa}', \hat{\epsilon}')$. With these notations one has:

2.2.5. **Proposition (four corners lemma).** One has $S\hat{S} = \hat{\alpha}(N)^0 = \{s\hat{s}/s \in S, \hat{s} \in \hat{S}\}$ (the Weyl algebra), $S \cap \hat{S} = \alpha(N)$, $S \cap \hat{S}' = \beta(N)'$, $S' \cap \hat{S} = \beta(N)$ and $S' \cap \hat{S}' = \hat{\alpha}(N)$.

2.2.6. **Notation.** We shall denote by $\mathfrak{W}$, the Weyl algebra $S\hat{S}$.

2.3. **The commutative example.** Let’s recall that a groupoid $G$ is a small category the morphisms of which are all invertible. In all what follows, $G$ is finite. One can assimilate the set of object, noted $G^0$, to a subset of the morphisms. So a (finite)
groupoid can also be viewed as a set $\mathcal{G}$ together with a not everywhere defined multiplication for which there is a set of unities $\mathcal{G}^0$, two applications $s$, source denoted by $s$ and range by $r$, from $\mathcal{G}$ to $\mathcal{G}^0$ so that the product $xy$ of two elements $x, y \in \mathcal{G}$ exists if and only if $s(x) = r(y)$; every element $x \in \mathcal{G}$ has a unique inverse $x^{-1}$, and one has $x(yz) = (xy)z$ whenever the two members have a sense. We refer to [R] for the fundamental structures and notations for groupoids.

Let’s denote $H = l^2(\mathcal{G})$, with the usual notations. Actually there exists four natural irreducible mpi associated to $\mathcal{G}$ (for an other example see [Val1] 4.1). Let $I_\mathcal{G}$ be the mpi defined for any $x, y \in \mathcal{G}$, $\xi \in l^2(\mathcal{G})$, by:

$$I_\mathcal{G}(\xi)(x, y) = \xi(xy, y)$$

if $s(x) = r(y)$ and $I_\mathcal{G}(\xi)(x, y) = 0$ otherwise.

Here, $\alpha = \hat{\beta}$ and $\hat{\alpha} = \beta$, which are given by the source and target functions $s$ (resp. $r$), so for every $n \in N$: $\alpha(n) = \hat{\beta}(n) = s \circ n$ and $\hat{\alpha}(n) = \beta(n) = r \circ n$. One has $S = C(\mathcal{G})$ and $N = C(\mathcal{G}^0)$ which are the commutative involutive algebras of complex valued functions on $\mathcal{G}$ and $\mathcal{G}^0$ respectively, and $\hat{S} = R(\mathcal{G}) = \{ \sum_{x \in \mathcal{G}} a_x \rho(x) \}$ (the right regular algebra of $\mathcal{G}$) where $\rho(x)$ is the partial isometry given by the formula $(\rho(x)\xi)(t) = \xi(tx)$ if $x \in \mathcal{G}^{s(t)}$ and $= 0$ otherwise, $\hat{S}' = L(\mathcal{G}) = \{ \sum_{x \in \mathcal{G}} a_x \lambda(x) \}$ (the left regular algebra of $\mathcal{G}$), where $\lambda(s)$ is the partial isometry given by the formula $(\lambda(s)\xi)(t) = \xi(s^{-1}t)$ if $t \in \mathcal{G}^{r(s)}$ and $= 0$ otherwise, $S' = S$. The two $C^*$-quantum groupoids structures on $S$ and $\hat{S}$ are given by

- **Coproducts:**
  $$\Gamma_{\mathcal{G}}(f)(x, y) = f(xy)$$
  $$\hat{\Gamma}_{\mathcal{G}}(\rho(s)) = \rho(s) \otimes \rho(s)$$
  otherwise

- **Antipodes:**
  $$\kappa_{\mathcal{G}}(f)(x) = f(x^{-1})$$
  $$\hat{\kappa}_{\mathcal{G}}(\rho(s)) = \rho(s^{-1}) = \rho(s)^*$$

- **Counities:**
  $$\epsilon_{\mathcal{G}}(f) = \sum_{u \in \mathcal{G}^0} f(u), \quad \hat{\epsilon}_{\mathcal{G}}(\rho(s)) = 1$$

3. **Actions of quantum groupoids on von Neumann modules**

The aim of this section is to give a framework for actions of quantum groupoids, for further extrapolation of [N] in the von Neumann algebras context and of [Y] in the quantum groupoids one. Our definitions have direct generalizations to infinite dimension (see [EV] definition 7.1) in the von Neumann algebras context.

In all what follows $I$ will be an irreducible regular mpi over the base $(N, \alpha, \beta, \hat{\beta})$, we shall use the notations of paragraph 2.2 in particular one has: $S_s = \hat{S}_t = \alpha(N)$, $S_t = \beta(N)$ and $\hat{S}_s = \hat{\beta}(N)$, and by [Val2] 3.1, $I$ leads to define two other mpi $\hat{I}$ and $\tilde{I}$, so one has two other quantum groupoids for the commutants $S'$ and $\hat{S}'$....). Let $A$ be a von Neumann acting on an hilbert space $H$.

3.1. **Actions of quantum groupoids.**
3.1.1. Notations. Let $b$ be any unital faithful anti-representation $S_t \to A$, let $i$ be the canonical inclusion $S_t \to S$, we shall denote by $e_{b,i}$ the projection associated to this situation by Definition 2.1.1. Let $b'$ be any unital faithful representation $A_t \to A$ and $\kappa$ viewed as a restriction $S_t \to S_a$, we shall denote by $e'_{\kappa}$ the projection associated with this by Definition 2.1.4 (applied to $S_t'$).

3.1.2. Definition. With notations above, let $b$ be a unital faithful anti-representation $(\text{resp. representation})$ $S_t(=\beta(N)) \to A$, hence $A$ appears to be a right (resp. left) module over $S_t$, one calls a right (resp. left) action of $(S, \Gamma, \kappa)$ on $(A, b)$, any application $\delta$ (resp. $\gamma$) such that:

1) $\delta$ (resp. $\gamma$) is an injective normal homomorphism $A \to A \otimes S$ (not unital in general),

2) $(\delta \otimes i)\delta = (i \otimes \Gamma)\delta$ (resp. $(\gamma \otimes i)\gamma = (i \otimes \Gamma)\gamma$),

3) for any $x \in S_t : \delta(b(x)) = e_{b,i}(1 \otimes \kappa(x))$

(resp. for any $x \in S_t : \gamma(b'(x)) = e'_{\kappa}(1 \otimes x)$)

3.1.3. Definition. We shall call right (resp. left) coaction of $(S, \Gamma, \kappa)$, any right (resp. left) action of $(\hat{S}, \hat{\Gamma}, \hat{\kappa})$.

3.1.4. Remarks. 1) If $S_t \subset Z(S)$ (the center of $S$), then, as for every $a$ in $A$ an $x$ in $S_t$ one has: $\delta(ab(x)) = \delta(b(x)a)$, hence $b(S_t) \subset Z(A)$, in particular, when it is not a quantum group, such a quantum groupoid can not act on a factor.

2) If $\alpha$ is a left action of $(S, \Gamma, \kappa)$ on $(A, b)$ it appears to be a right action of $(S, \gamma, \kappa)$, associated with $\Sigma \hat{\Gamma} \Sigma$, on $(A, b \circ \kappa)$.

3) $\Gamma$ is a right action of $(S, \Gamma, \kappa)$ on $(S, \kappa | \beta(N))$

Hence till the end we shall only deal with right actions.

3.1.5. Lemma and definition. Let $\delta$ be a right action of $(S, \Gamma, \kappa)$ on $(A, b)$, then one has:

$$\{ a \in A/\delta(a) = e_{b,i}(a \otimes 1) \} = \{ a \in A \cap b(S_t)' / \delta(a) = e_{b,i}(a \otimes 1) \},$$

this is a von Neumann subalgebra of $A \cap b(S_t)'$, we shall call the fixed point subalgebra of $\delta$ and note it $A^\delta$.

Proof: As $S_t$ and $\kappa(S_t)$ commute, hence for every $y$ in $S_t$, one has:

$$\delta(b(y)) = e_{b,i}(1 \otimes \kappa(y)) = (1 \otimes \kappa(y))e_{b,i} = e_{b,i}(1 \otimes \kappa(y))e_{b,i},$$

hence for any $a$ in $A$, such that $\delta(a) = e_{b,i}(a \otimes 1)$, one has: $\delta(ab(y)) = e_{b,i}(a \otimes \kappa(y))e_{b,i} = \delta(b(y)a)$, the lemma follows.

3.1.6. Proposition. Let $\delta$ be a right action of $(S, \Gamma, \kappa)$ on $(A, b)$, then the application:

$$T_\delta = (i \otimes \phi)\delta$$

is a faithful conditional expectation $A \to A^\delta$. 


Proof: If \((n_j)\) is a matrix unit for \(N\), then \((\beta(n_j))\) (resp. \(\alpha(n_j)\)) is also a matrix unit for \(S_t\) (resp. \(\hat{S}_t\)), so, for any \(a \in A\), by \(2.1.1\) one has:
\[
T_\delta(a) = (i \otimes \phi)(e_{b,i}(a \otimes 1)) = (i \otimes \phi)(\sum_j b(\beta(n_j)) \otimes \beta(n_j^*))a
\]
\[
= (b \circ \kappa)(i \otimes \phi)(\Gamma(1))a = (b \circ \kappa)(1)a = a
\]
For every \(a \in A\), let’s use the notations \(\delta(a) = a_1 \otimes a_2\), one has:
\[
\delta(T_\delta(a)) = \delta((i \otimes \phi)(\delta(a)) = (i \otimes i \otimes \phi)(\delta \otimes \delta)(\delta(a)) = (i \otimes i \otimes \phi)(i \otimes \Gamma)\delta(a)
\]
\[
= (i \otimes (i \otimes \phi)\Gamma)\delta(a) = (i \otimes E_t)\delta(a) = (i \otimes E_t)(\delta(1)\delta(a)) = e_{b,i}(i \otimes E_t)\delta(a)
\]
\[
e_{b,i}(a_1 \otimes E_t(a_2)) = e_{b,i}(a_1 b(E_t(a_2)) \otimes 1)
\]
So one deduces that:
\[
(i \otimes \phi)\delta(T_\delta(a)) = (i \otimes \phi)(e_{b,i}(a_1 b(E_t(a_2)) \otimes 1)) = (i \otimes \phi)(e_{b,i})a_1 b(E_t(a_2))
\]
\[
= a_1 b(E_t(a_2))
\]
but in an other hand:
\[
(i \otimes \phi)\delta(T_\delta(a)) = (i \otimes \phi)(e_{b,i}(a_1 E_t(a_2))) = (i \otimes \phi)(\delta(1)\delta(a)) = (i \otimes \phi)\delta(a) = T_\delta(a)
\]
Hence one has: \(T_\delta(a) = a_1 b(E_t(a_2))\), replacing this in the expression of \(\delta(T_\delta(a))\), one has:
\[
\delta(T_\delta(a)) = e_{b,i}(a_1 b(E_t(a_2)) \otimes 1) = e_{b,i}(T_\delta(a) \otimes 1)
\]
this implies that \(T_\delta(A) = A^\delta\).

Using the fact that any element of \(A^\delta\) commutes with \(b(S_t)\), for any \(b, c \in A^\delta\) and any \(a \in A\), one has:
\[
T_\delta(cab) = (i \otimes \phi)\delta(cab) = (i \otimes \phi)(\delta(c)\delta(a)\delta(b)) = (i \otimes \phi)(e_{b,i}(c \otimes 1)\delta(a)e_{b,i}(b \otimes 1))
\]
\[
= (i \otimes \phi)((c \otimes 1)e_{b,i}\delta(a)e_{b,i}(b \otimes 1)) = (i \otimes \phi)(\delta(1)\delta(a)\delta(1)b)
\]
\[
e cT_\delta(a)b
\]
The proposition follows. \(\square\)

3.1.7. Definition. Let \(\delta\) (resp. \(\hat{\delta}\)) be a right action (resp. coaction) of \((S, \Gamma, \kappa)\) on a von Neumann module \((A, b)\) (resp. \((A, \hat{b})\), the crossed product \(A \rtimes \delta S\) (resp. \(A \rtimes \hat{\delta} \hat{S}\)) the sub-von Neumann algebra of \(e_{b,i}(A \otimes \mathcal{L}(\mathcal{H}))e_{b,i}\) (resp. \(e_{b,i}(A \otimes \mathcal{L}(\mathcal{H}))e_{b,i}\)) generated by \(\delta(A)\) and \(e_{b,i}(1 \otimes \hat{S})\) (resp. \(\hat{\delta}(A)\) and \(e_{b,i}(1 \otimes S')\)).

3.1.8. Remarks. 1) One must keep in mind that the crossed product is degenerated in \(A \otimes \mathcal{L}(\mathcal{H})\), and it’s unit element is \(e_{b,i}\).

2) If \(\alpha\) is a left action, as it is a right action one can define also a crossed product.

3) As a matter of facts, in this Baaj and Skandalis formalism, \(\hat{S}\) is not equal to the one given by Vaes’ theory, but our crossed product do generalize the quantum groups one.
3.1.9. **Lemma.** The crossed product $A \rtimes S$ is the sub vector space of $e_{b,i}(A \otimes \mathcal{L}(\mathcal{H}))e_{b,i}$ (resp. $e_{b,\kappa}(A \otimes \mathcal{L}(\mathcal{H}))e_{b,\kappa}$) generated by the products $\delta(a)(1 \otimes \hat{b})$, $a \in A$, $\hat{b} \in \hat{S}$.

**Proof:** The equality $(\delta \otimes i)i = (i \otimes \Gamma)\delta$, leads to the fact that for any $a \in A$, one has: $I_{23}(\alpha(a) \otimes 1)I_{23}^* \in \delta(A) \otimes S$, so as $I_{23}^*I_{23} = e_{\hat{\beta},\alpha}$, there exist $a_1, ..., a_k$ in $A$ and $b_1, ..., b_k$ in $S$ such that:

$$I_{23}(\delta(a) \otimes 1) = \sum_i (\delta(a_i) \otimes c_i)I_{23}$$

Hence for any linear form $\omega$ on $\mathcal{L}(\mathcal{H})$, one has:

$$(1 \otimes (i \otimes \omega)(I))\delta(a) = (i \otimes i \otimes \omega)(I_{23}(\delta(a) \otimes 1)) = (i \otimes i \otimes \omega)(\sum_i (\delta(a_i) \otimes c_i)I_{23})$$

$$= \sum_i \delta(a_i)(1 \otimes (i \otimes (\omega.c_i))(I))$$

This leads to the lemma. \hfill $\square$

3.1.10. **Notation.** Let’s denote $I' = \Sigma(\tilde{I}^*\Sigma = \Sigma(U \otimes U)I^*(U \otimes U)\Sigma$, hence $I'$ is a mpi belonging to $S' \otimes \hat{S}$ over the base $(N, \hat{\alpha}, \hat{\beta})$, which gives on $\hat{S}$ the opposite coproduct $\hat{\Gamma}^{opp} = \varsigma \hat{\Gamma}$.

3.1.11. **Proposition.** i) Let $\delta$ be a right action of $(S, \Gamma, \kappa)$ on $(A, b)$, let $\hat{b}$ be the application $\hat{S}_\gamma = \alpha(N) \to A \rtimes S$ defined for every $n \in N$ by:

$$\hat{b}(\alpha(n)) = e_{b,i}(1 \otimes \hat{\beta}(n))$$

let $\hat{\delta}$ be the application defined for every $x \in A \rtimes S$ by:

$$\hat{\delta}(x) = \tilde{I}_{23}(x \otimes 1)\tilde{I}_{23}^*$$

then $(\hat{\delta}, \hat{b})$ is a (right) coaction of $(S, \Gamma, \kappa)$ on $(A \rtimes S, \hat{b})$ and $(A \rtimes S) \rtimes \hat{S}$ is isomorphic to the sub von Neumann algebra of $\delta(1)(A \otimes \mathcal{L}(\mathcal{H}))\delta(1)$ generated by $\delta(A)$ and $\delta(1)(1 \otimes \hat{S}'') = \delta(1)(1 \otimes \beta(N)')$.

**Proof:** By [Val3] 3.1, the initial support of $\tilde{I}$ is $e_{\hat{\alpha},\hat{\beta}}$, but $\hat{\alpha}(N)$ commutes with $S$ and $\hat{S}$, this implies that $\hat{\delta}$ is a normal homomorphism on $A \rtimes S$; as $\tilde{I} \in S' \otimes \hat{S}$ and his final support is $e_{\hat{\beta},\alpha}$, for any $a$ in $A$ one has:

$$\hat{\delta}(\delta(a)) = (1 \otimes e_{\hat{\beta},\alpha})(\delta(a) \otimes 1)$$

and due to [Val2] proposition 3.1.4, for every $\hat{b}$ in $\hat{S}$, one has:

$$\hat{\delta}(e_{b,i}(1 \otimes \hat{b})) = (e_{b,i} \otimes 1)(1 \otimes \Gamma(\hat{b})).$$
So \( \hat{\delta} \) takes values in \( A \rtimes S \otimes \hat{\mathcal{S}} \), and for every \( x \in \hat{\mathcal{S}} \), one has:

\[
\hat{\delta}(\hat{b}(x)) = \hat{\delta}(e_{b,i}(1 \otimes \hat{\kappa}(x))) = (e_{b,i} \otimes 1)(1 \otimes \hat{\Gamma}(\hat{\kappa}(x))) \\
= (e_{b,i} \otimes 1)(1 \otimes e_{\beta,\alpha})(1 \otimes 1 \otimes \hat{\kappa}(x))
\]

But, if \((n_j)\) is a matrix unit for \( N \), then \((\beta(n_j))\) (resp. \(\alpha(n_j)\)) is also a matrix unit for \( S_t \) (resp. \( \hat{\mathcal{S}}_t \)), so by 2.1.1 one has:

\[
e_{b,i} = \sum_j \hat{b}(\alpha(n_j)) \otimes \alpha(n_j^*) = \sum_j e_{b,i}(1 \otimes \hat{\beta}(n_j^*) \otimes \alpha(n_j)) \\
= (e_{b,i} \otimes i)(1 \otimes \sum_j \hat{\beta}(n_j^*) \otimes \alpha(n_j)) \\
= (e_{b,i} \otimes 1)(1 \otimes e_{\beta,\alpha}),
\]

replacing this in the previous calculus:

\[
\hat{\delta}(\hat{b}(x)) = e_{b,i}(1_{A \rtimes S \otimes \hat{\mathcal{S}}} \otimes \hat{\kappa}(x))
\]

Now let’s verify condition 2) of definition 3.1.2

\[
(\hat{\delta} \otimes i)\hat{\delta}(\delta(a)) = (\hat{\delta} \otimes i)((1 \otimes e_{\beta,\alpha})(\delta(a) \otimes 1)) = (\hat{\delta} \otimes i)((1 \otimes e_{\beta,\alpha})(\delta(1) \otimes 1)(\delta(a) \otimes 1)) \\
= \sum_j (\hat{\delta} \otimes i)((1 \otimes \hat{\beta}(n_j) \otimes \alpha(n_j^*)))(e_{b,i} \otimes 1)(\delta(a) \otimes 1)) \\
= \sum_j (\hat{\delta} \otimes i)((e_{b,i}(1 \otimes \hat{\beta}(n_j)) \otimes \alpha(n_j^*)))(\delta(a) \otimes 1)) \\
= \sum_j (\hat{\delta} \otimes i)(e_{b,i}(1 \otimes \hat{\beta}(n_j) \otimes \alpha(n_j^*))(\delta(a) \otimes 1)) \\
= \sum_j \hat{\delta}(e_{b,i}(1 \otimes \hat{\beta}(n_j)) \otimes \alpha(n_j^*))(\delta(a) \otimes 1) \\
= \sum_j ((e_{b,i} \otimes 1)(1 \otimes \hat{\Gamma}(\hat{\beta}(n_j))) \otimes \alpha(n_j^*))((1 \otimes e_{\beta,\alpha})(\delta(a) \otimes 1) \otimes 1) \\
= \sum_j (e_{b,i} \otimes 1)(1 \otimes e_{\beta,\alpha}(1 \otimes \hat{\beta}(n_j)) \otimes \alpha(n_j^*))((1 \otimes e_{\beta,\alpha})(\delta(a) \otimes 1) \otimes 1) \\
= \sum_j (e_{b,i} \otimes 1)(1 \otimes e_{\beta,\alpha}(1 \otimes \hat{\beta}(n_j)) \otimes \alpha(n_j^*))((1 \otimes e_{\beta,\alpha})(\delta(a) \otimes 1) \otimes 1) \\
= (e_{b,i} \otimes 1 \otimes 1)(1 \otimes e_{\beta,\alpha} \otimes 1)(1 \otimes 1 \otimes e_{\beta,\alpha})(\delta(a) \otimes 1) \otimes 1) \\
= (e_{b,i} \otimes 1 \otimes 1)(1 \otimes 1 \otimes e_{\beta,\alpha})(\delta(a) \otimes 1) \otimes 1)
\]

and on the other side:
(i ⊗ \hat{\Gamma})\hat{\delta}(\delta(a)) = (i ⊗ \hat{\Gamma})((1 ⊗ e_{\hat{\beta},\alpha})(\delta(a) ⊗ 1)) = (\hat{\delta} ⊗ i)((1 ⊗ e_{\hat{\beta},\alpha})(\delta(1) ⊗ 1)(\delta(a) ⊗ 1))

= \sum_j (i ⊗ \hat{\Gamma})((1 ⊗ \hat{\beta}(n_j) ⊗ \alpha(n_j^*))((e_{b,i} ⊗ 1)(\delta(a) ⊗ 1))

= \sum_j (i ⊗ \hat{\Gamma})((e_{b,i}(1 ⊗ \hat{\beta}(n_j)) ⊗ \alpha(n_j^*))((e_{b,i} ⊗ 1)(\delta(a) ⊗ 1))

= \sum_j (i ⊗ \hat{\Gamma})(e_{b,i}(1 ⊗ \hat{\beta}(n_j) ⊗ \alpha(n_j^*))((i ⊗ \hat{\Gamma})(\delta(a) ⊗ 1))

= \sum_j (e_{b,i}(1 ⊗ \hat{\beta}(n_j) ⊗ e_{\hat{\beta},\alpha}(\alpha(n_j^*) ⊗ 1)))((\delta(a) ⊗ e_{\hat{\beta},\alpha})

= \sum_j (e_{b,i}(1 ⊗ \hat{\beta}(n_j) ⊗ e_{\hat{\beta},\alpha}(\alpha(n_j^*) ⊗ 1)))((\delta(a) ⊗ e_{\hat{\beta},\alpha})

= (e_{b,i} ⊗ 1 ⊗ 1)(1 ⊗ e_{\hat{\beta},\alpha} ⊗ 1)(1 ⊗ 1 ⊗ e_{\hat{\beta},\alpha})(\delta(a) ⊗ 1 ⊗ 1)

= (\hat{\delta} ⊗ i)\hat{\delta}(\delta(a))

Also for the others generators, and using Sweedler notations, one has:

(\hat{\delta} ⊗ i)\hat{\delta}(e_{b,i}(1 ⊗ \hat{\beta})) = (\hat{\delta} ⊗ i)((e_{b,i} ⊗ 1)(1 ⊗ \hat{\Gamma}(\hat{\beta}))) = (\hat{\delta} ⊗ i)((e_{b,i} ⊗ 1)(1 ⊗ \hat{b}_1 ⊗ \hat{b}_2))

= (\hat{\delta} ⊗ i)(e_{b,i}(1 ⊗ \hat{b}_1 ⊗ \hat{b}_2) = (e_{b,i} ⊗ 1)(1 ⊗ \hat{\Gamma}(\hat{b}_1) ⊗ \hat{b}_2)

= (e_{b,i} ⊗ 1 ⊗ 1)(1 ⊗ (\hat{\Gamma} ⊗ i)\hat{\Gamma}(\hat{b})) = (e_{b,i} ⊗ 1 ⊗ 1)(1 ⊗ (i ⊗ \hat{\Gamma})\hat{\Gamma}(\hat{b}))

= (e_{b,i} ⊗ 1)(1 ⊗ \hat{b}_1 ⊗ \hat{\Gamma}(\hat{b}_2)) = (i ⊗ \hat{\Gamma})(e_{b,i} ⊗ 1)(1 ⊗ \hat{\Gamma}(\hat{b}))

= (i ⊗ \hat{\Gamma})\hat{\delta}(e_{b,i}(1 ⊗ \hat{b}))

One can deduce that: (\hat{\delta} ⊗ i)\hat{\delta} = (i ⊗ \hat{\Gamma})\hat{\delta}.

Now let define the application the one to one morphism \gamma defined on A ⊗ \mathcal{L}(\mathcal{H}) by:

\gamma(x) = I_{23}(\delta ⊗ i)(x)I_{23}, \quad \forall x ∈ A ⊗ \mathcal{L}(\mathcal{H})

Obvious calculations give that, for any a in A, \hat{s} in \hat{S}, d in S', one has:

\gamma(\delta(a)) = \hat{\delta}(1)(\delta(a) ⊗ 1), \quad \gamma(\hat{\delta}(1)(1⊗\hat{s})) = \hat{\delta}(1)(1⊗\hat{\Gamma}(\hat{s})), \quad \gamma(\hat{\delta}(1)(1⊗s) = \hat{\delta}(1)(1⊗1⊗s')

Hence, \gamma is an isomorphism between the sub von Neumann algebra of \delta(1)(A ⊗ \mathcal{L}(\mathcal{H}))(1 ⊗ 1) generated by \delta(A) and \delta(1)\hat{\Gamma}(\hat{S})' = \hat{\delta}(1)(1 ⊗ \beta(1 ⊗ s')) and \gamma(A ⊗ \hat{S}) \hat{\Gamma}\hat{\delta}.

\square

3.2. Actions of groupoids. Let’s explain what is an action \alpha of the commutative quantum groupoid \((C(G), \Gamma_G, \kappa_G)\) where \(G\) is any finite groupoid, on a von Neumann module \((A, b)\). In fact, the application \(b\) is clearly equivalent to the given of a decomposition \(A = \bigoplus_{u \in G^0} A_u\), where each \(A_u\) is a von Neumann algebra, the relation is given,
for every \( u \in \mathcal{G}^0 \), by \( b(\delta_u) = 1_u \), where \( \delta_u \) is the Dirac fonction for \( u \) and \( 1_u \) the identity element of \( A_u \) (a projection in \( Z(A) \)). Hence \( A \) appears to be a module over \( C(\mathcal{G}^0) \).

3.2.1. Definition. An action of \( \mathcal{G} \) on \( A \) is any covariant functor from the category \( \mathcal{G} \) to the category whose objects are the element of the set \( \{ A_u, u \in \mathcal{G}^0 \} \) and the morphisms the von Neumann algebras isomorphisms.

Hence, for any \( g \in \mathcal{G} \), it exists a morphism \( \alpha_g : A_{s(g)} \to A_{r(g)} \), in order that for any pair \((g, g')\) of composable elements, one has: \( \alpha_{gg'} = \alpha_g \alpha_{g'} \).

As it can be decomposed in its connected classes, we can suppose that \( \mathcal{G} = \bigsqcup_i X_i \times X_i \times G_i \), where \( X_i \) is a finite set and \( G_i \) is a finite group. In fact one has \( \mathcal{G}^0 = \bigsqcup_i X_i \) and \( G_i \) is isomorphic to the isotropy group \( G_i^u \) for any \( u \in X_i \).

3.2.2. Proposition. Any finite groupoid \( \mathcal{G} \) acts on \( (R^{g^0}, b_{\mathcal{G}^0}) \), where \( R \) is the hyperfinite type \( II_1 \) factor and \( b_{\mathcal{G}^0} : f \mapsto (f(u))_{u \in \mathcal{G}^0} \).

Proof: Due to the previous remark, one can suppose that \( \mathcal{G} = X \times X \times G \) where \( X \) is a finite set and \( G \) is a group. As it is well known, there exists an action \( \beta \) (even outer) of \( G \) on \( R \). Of course \( R^{g^0} \) can be decomposed in it’s cartesian components, each of them is in fact \( R \) itself; up to this identification, one can define for any \((x, y, g) \in \mathcal{G} \)
\( \alpha_{(x, y, g)} = \beta_g \), one easily sees that this is an action. \( \square \)

3.2.3. Lemma. For any action \( \alpha \) of \((C(\mathcal{G}), \Gamma_G, \kappa_G)\) (the image of which can be viewed in \( A \otimes \mathcal{L}(l^2(\mathcal{G})) \)) on \((A, b)\), and for any \( g \in \mathcal{G} \), one has: \((1 \otimes \lambda(t)^*) \alpha(A)(1 \otimes \lambda(t)) \subset \alpha(A)\).

Proof: This is just a generalization of the demonstration of Proposition 1.3 ii) in \([E3]\), replacing the unitary \( W_G \) by the adjoint of the regular mpi defined in \([Val1]\) 4.1. \( \square \)

In fact the two notions of action are equivalent:

3.2.4. Proposition. i) For any action of \( \mathcal{G} \) on \((A, b)\), then the application \( \delta_a \) (resp. \( \gamma_a \)) : \( A \to A \otimes C(G) (= C(G, A)) \) defined for every \( a \in A \) by \( \delta_a(a) : g \mapsto \alpha_g(a_{s(g)}) \) (resp. \( \gamma_a(a) = \alpha_g(a_{r(g)}) \)) is a right (resp. left) action of \((C(\mathcal{G}), \Gamma_G, \kappa_G)\) on \((A, b)\).

ii) For any left (resp. right) action \( \gamma \) (resp. \( \delta \)) of \((C(\mathcal{G}), \Gamma_G, \kappa_G)\) on \((A, b)\) there exists a unique action of \( \mathcal{G} \) on \((A, b)\), such that \( \gamma = \gamma_a \) (resp. \( \delta = \delta_a \)).

Proof: Using lemma 3.2.3, one can exactly use the arguments in \([E3]\) Proposition 1.3 \( \square \)

A third and more synthetic way to define an action on \((A, b)\) is just to consider the groupoid \( Aut(A, b) \) whose base is \( \mathcal{G}^0 \) viewed as \( \{ IdA_u, u \in \mathcal{G}^0 \} \) and morphisms the
isomorphisms $A_u \mapsto A_v$. An action is just a full (i.e. with the same base) subgroupoid of $\text{Aut}(A, b)$.

3.2.5. **Remarks.** 1) Our definition of an action agrees with the algebraic definition due to Vainerman and Nikshych in [NV2]: let $\gamma$ (resp. $\delta$) be a left (resp. right) action of $(C(\mathcal{G}), \Gamma_\mathcal{G}, \kappa_\mathcal{G}, \varepsilon_\mathcal{G})$ on $(A, b)$, then if for any $a$ in $A$ and $h$ in $\mathcal{G}$ one defines $\lambda(h) \triangleright a = \gamma(a_h)$ (resp. $a \triangleleft \rho(h) = \delta(a)_h$), this is a left (resp.right) action and the same formula can be used for the inverse assertion.

2) As $\delta(1) = e_{b,r} \neq 1$, $\delta(A)$ is degenerated in $C(G, A)$, so it’s better convenient to restrict $\delta$ to $e_{b,r}(A \otimes C(\mathcal{G}))e_{b,r}$, which can be identified with $\bigoplus_{g \in \mathcal{G}} A_{r(g)}$, and $\delta(a)$ can be identified with $(\alpha_g(a_{s(g)}))_g$, in that way, $\delta$ appears to be unital.

3.2.6. **Notations.** Let’s consider $H = L^2(A)$, the standard hilbert space of $A$, then $H$ has an orthogonal decomposition $H = \bigoplus_{i \in G} H_i$; for $j \in (s, r)$, $e_{b,j}(A \otimes C(\mathcal{G}))e_{b,j}$ can also be represented as a "diagonal" von Neumann algebra acting on $\bigoplus_{g \in \mathcal{G}} H_{r(g)}$. For any $h \in \mathcal{G}$, one can define the operator $(1_b \otimes_r \rho(h))$ on $\bigoplus_{g \in \mathcal{G}} H_{r(g)}$, by the formula $(1_b \otimes_r \rho(h))(\xi_g)_{g \in \mathcal{G}} = (\eta_g)_{g \in \mathcal{G}}$, where $\eta_g = 0$ if $s(g) \neq r(g)$ and $\eta_g = \xi_{gh}$ otherwise. For any $a \in A$, one also can define the operator $(a_b \otimes_s 1) = \bigoplus a_{s(g)}$ which acts on $\bigoplus H_{s(g)}$. Let’s also denote by $u_g$ the canonical implementation of $\alpha_g$ for any $g \in \mathcal{G}$ (theorem 2.18 of [H]), so the operator $U = \bigoplus u_g$ appears to be a unitary $\bigoplus H_{s(g)} \to \bigoplus H_{r(g)}$. Hence, obviously one has:

3.2.7. **Proposition.** The unitary $U$ implements the action $\delta$, for any $a$ in $A$, one has:

$$\delta(a) = U(a_b \otimes_s 1)U^*.$$  

3.3. **Crossed product by groupoids actions and Jones tower.** In all what follows $\alpha$ is an action of $\mathcal{G}$ on $(A, b)$ and $\delta = \delta_\alpha$, the right action of $(C(\mathcal{G}), \Gamma_\mathcal{G})$ on $(A, b)$ associated with. So one can consider the inclusion $A^d \subset A$. Let’s recall Jones basic construction: if $M_0 \subset M_1$ is an inclusion of von Neumann algebras, and $J$ is the canonical antilinear involutive isometry of $L^2(M_1)$, then one can extend the inclusion by: $M_0 \subset M_1 \subset M_2(=JM_0J)$, that is the basic construction. This paragraph proves simply that $M_2$ is a quotient of the crossed product. First let’s give a simple description of this crossed product.

3.3.1. **Remark.** The crossed product of $(A, b)$ by $\mathcal{G}$ is the sub-von Neumann algebra of $L(\bigoplus_{g \in \mathcal{G}} H_{r(g)})$ generated by $\delta(A)$ and the operators $(1_b \otimes_r \rho(h))$. 

13
3.3.2. **Lemma.** i) For any \(a\) in \(A\) and \(h\) in \(G\), one has:

\[
(1_b \otimes_r \rho(h))\delta(a) = \delta(\alpha_h(a_{s(h)}))(1_b \otimes_r \rho(h)).
\]

ii) The crossed product \(A \rtimes \mathcal{C}(G)\) is the vector space generated by the products \(\delta(a)(1_b \otimes_r \rho(h))\) for any \((a, h)\) in \(A_b \times_r G\)\((= \{(a, h) \in A \times G, a \in A_r(h)\})\).

iii) \(A \rtimes \mathcal{C}(G)\) is the set of elements in \(\mathcal{L}(\oplus_{g \in G} H_{r(g)})\), which can be decomposed in a sum of the form \(\sum_{h \in G} \delta(x^h)(1_b \otimes_r \rho(h))\), where \(x^h \in A_{r(h)}\) for all \(h \in G\), and this decomposition is unique.

**Proof:** For any \(a\) in \(A, h\) in \(G\) and \((\xi_g)_{g \in G}\) in \(\bigoplus_{g \in G} H_{r(g)}\), one has:

\[
(1_b \otimes_r \rho(h))\delta(a)((\xi_g)_{g \in G}) = (1_b \otimes_r \rho(h))((\alpha_g(a_{s(g)})\xi_g)_{g \in G}) = (1_b \otimes_r \rho(h))((\alpha_g(a_{s(g)})\xi_g)_{g \in G}) = (\eta_g)_{g \in G}
\]

where one has:

\[
\eta_g = \begin{cases} 
\alpha_g(\alpha_h(a_{s(h)}))\xi_{gh} & \text{if } s(g) = r(h) \\
0 & \text{otherwise}
\end{cases}
\]

On the other side, for any \(b\) in \(A, h\) in \(G\) and \((\xi_g)_{g \in G}\) in \(\bigoplus_{g \in G} H_{r(g)}\), one has:

\[
\delta(b)(1_b \otimes_r \rho(h))((\xi_g)_{g \in G}) = (\delta(b)(\eta'_g))_{g \in G}
\]

where one has:

\[
\eta'_g = \begin{cases} 
\xi_{gh} & \text{if } s(g) = r(h) \\
0 & \text{otherwise}
\end{cases}
\]

hence

\[
\delta(b)(1_b \otimes_r \rho(h))((\xi_g)_{g \in G}) = (\alpha_g(b_{s(g)})\eta''_g)_{g \in G} = (\eta''_g)_{g \in G},
\]

so one deduces that:

\[
\eta''_g = \begin{cases} 
\alpha_g(b_{r(h)})\xi_{gh} & \text{if } s(g) = r(h) \\
0 & \text{otherwise}
\end{cases}
\]

One deduces that \((1_b \otimes_r \rho(h))\delta(a) = \delta(b)(1_b \otimes_r \rho(h))\) for any \(a, b, h\) such that \(b_{r(h)} = \alpha_h(a_{s(h)}), i\) and ii) follow immediately.

If one chooses for any \(u \in G^0\) a base \((a^k_u)\) of \(A_u\), one easily sees that the family \((\delta(a^k_{r(h)})(1_b \otimes_r \rho(h)))\) is free, hence iii) is a consequence of ii).

\[\square\]

3.3.3. **Corollary.** The application \(\sum_{h \in G} \delta(x^h)(1_b \otimes_r \rho(h)) \mapsto x^h \otimes \rho(h)\) leads to an isomorphism of \(A \rtimes \mathcal{C}(G)\) and the corresponding crossed product by L.Vainerman and D.Nikshych.

3.4. **Outer actions of groupoids.**

3.4.1. **Definition.** Let’s call isotropic subgroupoid of \(G\), the subgroupoid of \(G\), denoted \(\text{iso}(G)\), equal to \(\{h \in G, s(h) = r(h)\}\).

3.4.2. **Remark.** Obviously \(\text{iso}(G)\) is the disjoint union of the isotropic groups \(G^0_u\).
3.4.3. Lemma. Let’s suppose that $Z(A)$ is isomorphic to $C(G_0)$ (or equivalently each $A_u$ is a factor). An element $x \in A \rtimes C(\mathcal{G})$ commutes with $\delta(A)$ if and only if for any $h \notin \text{iso}(\mathcal{G})$, one has $x^h = 0$ and for any $h \in \text{iso}(\mathcal{G})$ with $x^h \neq 0$, $\alpha_h$ is inner with $\alpha_h(a) = (x^h)^{-1}ax^h$, for all $a \in A_{r(h)}$ (hence $x^h$ is invertible).

Proof: Due to lemma 3.3.2, for any $x \in A \rtimes C(\mathcal{G})$, if $\delta(a)x = x\delta(a)$ for all $a$ in $A$, it means that, for any $a$ in $A$ and $h$ in $G$, one has: $\delta(a_{r(h)})x^h(1_b \otimes r \rho(h)) = \delta(x^h\alpha_h(a_{s(h)}))(1_b \otimes r \rho(h))$. Hence, one has: $a_{r(h)}x^h = x^h\alpha_h(a_{s(h)})$.

If $h \notin \text{iso}(\mathcal{G})$, let $a$ be the element $\alpha_h^{-1}(x^h)^*$ and so $a_{r(h)} = 0$, then one deduces that: $0 = x^h\alpha_h(a_{s(h)}) = x^h(x^h)^*$, so does $x^h$.

If $h \in \text{iso}(\mathcal{G})$, then for any $a \in A_{r(h)}$, one has: $ax^h = x^ha(a)$, as $A_u$ is a factor, one can suppose that $A$ has no trivial weakly closed two side ideal, but $x^hA$ is a two side weakly closed ideal, so $x^h$ is invertible or equal to zero; the lemma follows. $\square$

3.4.4. Remark. The von Neumann algebra $\delta(A)' \cap A \rtimes C(\mathcal{G})$, the relative commutant of $\delta(A)$ in $A \rtimes C(\mathcal{G})$, contains $Z(A) \rtimes C(G_0)$, whose elements are of the form:

$$\sum_{u \in G_0} \delta(x^u)(1_b \otimes r \rho(u))$$

for $x^u \in Z(A_u)$ which, in the case when $Z(A) = b(C(G_0))$, is just $1_b \otimes r C(G_0)$.

3.4.5. Definition. i) A right action $\delta$ of $C(\mathcal{G})$ on a von Neumann module $(A,b)$ is said to be outer if and only if $\delta(A)' \cap A \rtimes C(\mathcal{G})$ is equal to $Z(A) \rtimes C(G_0)$.

ii) An action $\alpha$ of $\mathcal{G}$ on a von Neumann module $(A,b)$ is said to be outer if and only if for any $h \in \text{iso}(\mathcal{G})$ such that $h \notin \mathcal{G}_0^1$, one has $\alpha_h \in \text{Out}A_{r(h)}$.

3.4.6. Remark. The transitive groupoid $\mathcal{G} = X \times X \times \text{Out}R$ acts outerly on $R^X$.

As a consequence of lemma 3.4.3 one has:

3.4.7. Proposition. In the case when $Z(A) = b(C(G_0))$, any right action $\delta$ of $C(\mathcal{G})$ on a von Neumann module $(A,b)$ is outer if and only if the action of $\mathcal{G}$ on $(A,b)$, canonically associated with $\delta$ is outer.

And finally:
3.4.8. **Proposition.** Any finite groupoid $G$ acts outerly on the von Neumann module $(R^G, b_G)$.

Proof: As in proposition 3.2.2, one can suppose $G = X \times X \times G$ with the same notations, then $iso(G) = \{(x, x, g), x \in X, g \in G\}$, so any $h \in iso(G)$ not in $G^0$, there is $x \in X$ and $g \in G$, $g$ different from the unit element, such that $h = (x, x, g)$, then $\alpha_h = \beta_g$ which can be taken in $Out(R)$ and obviously $Z(A) = b(C(G^0))$, the proposition follows from lemma 3.4.3.

3.5. **Double crossed products.** Now, let’s give a refinement of Proposition 3.1.11 in the commutative case, that is $S = C(G)$ (see also [Y] theorem 6.4 for more general groupoids).

3.5.1. **Lemma.** Let $\delta$ be a right action of a commutative quantum groupoid $(C(G), \Gamma_G, \kappa_G)$ on a von Neumann module $(A, b)$, then $\delta(A)(1 \otimes C(G)) = \delta(1)(A \otimes C(G))$.

Proof: Clearly, one has: $\delta(A)(1 \otimes C(G)) \subset \delta(1)(A \otimes C(G))$. On the other hand, using the identification of $\delta(1)(A \otimes C(G))$ with the set of functions $\phi : G \rightarrow A$ such that for any $g \in G$, one has: $\phi(g) \in A_{\gamma(g)}$, one easily sees that $\phi = \sum_g \delta(\gamma_g^{-1})(\phi(g))(1 \otimes \delta_g)$, the lemma follows.

3.5.2. **Theorem.** Let $\delta$ be a right action of a commutative quantum groupoid $(C(G), \Gamma_G)$ on a von Neumann module $(A, b)$, then the double crossed product $(A \rtimes C(G)) \rtimes R(G)$ is isomorphic to $\delta(1)(A \otimes W(G)) \delta(1)$, where $W(G)$ (the Weyl algebra of $C(G)$) is the commutant in $L(\mathcal{P}(G))$ of $r(\mathbb{I}(G^0)) (= \hat{\beta}(N)'$) which is also the sub von Neumann algebra generated by $C(G)$ and $R(G)$ ($= S\hat{S}$).

Proof: Using Proposition 3.1.11 and the fact that $S = S'$, one deduces that: $(A \rtimes C(G)) \rtimes R(G)$ is isomorphic to $\delta(A)(1 \otimes S\hat{S}) \delta(1)$. Now thanks to lemma 3.5.1 and the fact that $S\hat{S}$ is the vector space generated by $\{s\hat{s} : s \in S, \hat{s} \in \hat{S}\}$ (and also equal to $\beta(N)'$), the double crossed product is also isomorphic to $\delta(1)(A \otimes S\hat{S}) \delta(1)$, the theorem follows.

3.6. **Action of a groupoid on a fibered space over its base.** Let’s suppose now that $A$ is commutative and finite dimensional, hence there is a finite set $X$ such that $A = C(X)$, the von Neumann algebra of functions on $X$, the existence of $b$ leads to a partition $X = \bigsqcup_{g \in G^0} X_u$, and for each $u \in G^0$, one has: $A_u = C(X_u)$.

A left (resp. right) action of the groupoid $G$ on $(A, b)$ is given by a covariant (resp. contravariant) functor between the small category $G$ and the category of sets.
with usual applications \( \{ X^u / u \in \mathcal{G} \} \). So for any \( g \in \mathcal{G} \), there exists an application \( g \triangleright : X^{s(g)} \mapsto X^{r(g)} \) (resp. \( \triangleleft g : X^{r(g)} \mapsto X^{s(g)} \)) such that for any \( g, g' \in \mathcal{G} \) which are composable, one has for any \( x \) in \( X^{s(g')} \) (resp. \( X^{r(g)} \)): \( (g \triangleright (g' \triangleright x)) = gg' \triangleright x \) (resp. \((x \triangleleft g) \triangleleft g' = x \triangleleft g'g\)). The bijection between the two notions is given by the following formulae:

\[
\text{for any } a \text{ in } A \text{ and } g \text{ in } \mathcal{G}: \gamma(a)_g = g^{-1} \triangleright a_{r(g)} \quad \text{(resp. } \delta(a)_g = a_{s(g)} \triangleleft g)\).
\]

The crossed product \( C(\mathcal{G}) \ltimes A \) (resp. \( A \rtimes C(\mathcal{G}) \)) can also be interpreted as the image of a certain \( * \)-algebra representation.

Let’s denote by \( X_b \times_r \mathcal{G} = \{(x, g) \in X \times \mathcal{G} / b(x) = r(g)\} \), the fiber product of \( X \) and \( \mathcal{G} \), and by \( L^1(X_b \times_r \mathcal{G}) \) the vector space of fonctions on this set. One can give to \( L^1(X_b \times_r \mathcal{G}) \) a \( * \)-algebra structure denoted by \( (L^1(X_b \times_r \mathcal{G}), *, ^\#) \) (resp. \( (L^1(X_b \times_r \mathcal{G}), *, ^\delta) \)).

For any fonctions \( F, F' \) in \( L^1(X_b \times_r \mathcal{G}) \) and any \((x, g)\) in \( X_b \times_r \mathcal{G} \), one has:

\[
F \ast F'(x, g) = \sum_{r(h) = r(g)} F(x, h)F'(h^{-1} \triangleright x, h^{-1}g)
\]

\[
F^\#(x, g) = \overline{F(g^{-1} \triangleright x, g^{-1})}
\]

\[
(r \text{resp. } F \ast F'(x, g) = \sum_{r(h) = r(g)} F(x, h)F'(x \triangleleft h, h^{-1}g)
\]

\[
F^\#(x, g) = \overline{F(x \triangleleft g, g^{-1})}
\]

One must keep in mind that these fonctions have the good support. One can define a left (resp.right) regular representation of \( L^1(X_b \times_r \mathcal{G}) \) in \( l^2(X_b \times_s \mathcal{G}) \) (resp. \( l^2(X_b \times_r \mathcal{G}) \)) denoted \( L^\gamma \) (resp. \( R^\delta \)); for any \( \xi \) in \( l^2(X_b \times_s \mathcal{G}) \) (resp \( l^2(X_b \times_r \mathcal{G}) \)) any \( F \) in \( L^1(X_b \times_r \mathcal{G}) \) and any \((x, g)\) in \( X_b \times_s \mathcal{G} \) (resp.\( X_b \times_r \mathcal{G} \)):

\[
L^\gamma(F)\xi(x, g) = \sum_{r(h) = r(g)} F(g \triangleright x, h)\xi(x, h^{-1}g)
\]

\[
(r \text{resp. } R^\delta(F)\xi(x, g) = \sum_{r(h) = s(g)} F(x \triangleleft g, h)\xi(x, gh))
\]

With these definitions one can also formulate an alternative definition of the crossed products: \( L^\gamma(L^1(X_b \times_r \mathcal{G})) = C(\mathcal{G}) \ltimes A \) and \( R^\delta(L^1(X_b \times_r \mathcal{G})) = A \rtimes C(\mathcal{G}) \).
4. Quantum groupoids coming from match pairs of groupoids

4.1. The match pair of groupoids situation. Now let’s explain an extension of the commutative example. Let \( G \) be any groupoid and \( \mathcal{H}, \mathcal{K} \) be two subgroupoids of \( G \) such that \( G = \mathcal{H}\mathcal{K} = \{hk/h \in \mathcal{H}, k \in \mathcal{K}^{s(h)}\} \) and such that \( \mathcal{H} \cap \mathcal{K} \subset \mathcal{C}^0 \), such a pair \( \mathcal{H}, \mathcal{K} \) is called a match pair of groupoids (see [AA] for an abstract point of view). One easily verifies that this implies that \( G^0 = \mathcal{H} \cap \mathcal{K} \) and that for any \( g \in G \) the decomposition \( g = hk, h \in \mathcal{H}, k \in \mathcal{K} \) is unique. Hence one can define two applications, \( p_1 : G \rightarrow \mathcal{H} \) and \( p_2 : G \rightarrow \mathcal{K} \) by the relation \( g = p_1(g)p_2(g) \) for any \( g \in G \). Clearly one has \( s \circ p_2 = s \) and \( r \circ p_1 = r \), but a new application appears, the middle one:

4.1.1. Notations. 1) One has \( s \circ p_1 = r \circ p_2 \), this application will be denoted \( m \).

2) For any \( f \in C(G^0) \), we define \( \alpha, \beta, \hat{\beta} \) by: \( \alpha(f) = f \circ m \) (the middle representation), \( \beta(f) = f \circ r \) (the range representation) and \( \hat{\beta}(f) = f \circ s \) (the source representation).

3) With the exception of the four representations of the base \( N = C(G^0) \), we shall use the same notations than in the commutative case.

4.1.2. Lemma. For any \( h \) in \( \mathcal{H} \), \( \text{Card}(K^{s(h)}) = \text{Card}(K^{r(h)}) \)

Proof: Let’s fix \( h \) in \( \mathcal{H} \); let \( k \) be any element of \( \mathcal{K}^{s(h)} \). As \( G = \mathcal{H}\mathcal{K} \), then also \( G = \mathcal{K}\mathcal{H} \), so there exists a single pair \( (k', h') \) in \( \mathcal{K}\mathcal{H} \) such that \( hk = k'h' \). Let’s prove that the application \( k \mapsto k' \) defines an injection from \( \mathcal{K}^{s(h)} \) into \( \mathcal{K}^{r(h)} \); if \( k_1 \) is any element of \( \mathcal{K}^{r(h)} \) such that \( tk_1' = k' \), then there exists \( h_1' \) for which one has \( h_1k_1 = k'h_1' \), one deduces that \( h^{-1}k' = k_1h_1^{-1} = k'h_1^{-1} \) from which one deduces that \( k = k_1 \) and \( h_1 = h_1' \). So the application is injective and \( \text{Card}(\mathcal{K}^{s(h)}) \leq \text{Card}(\mathcal{K}^{r(h)}) \), applying this to \( h^{-1} \) one also has the inverse inequality. The lemma follows. \( \square \)

4.1.3. Lemma. For any \( u \) in \( G^0 \), one has the following equalities: \( \text{Card}(m^{-1}(u)) = \text{Card}(s^{-1}(u)) = \text{Card}(r^{-1}(u))= \text{Card}(G^u) = \text{Card}(G_u) \). One has: \( tr \circ \alpha = tr \circ \beta = tr \circ \hat{\beta} \).

Proof: The equality \( \text{Card}(s^{-1}(u)) = \text{Card}(r^{-1}(u)) \) is well known and is due to the bijection \( g \mapsto g^{-1} \) which gives \( tr \circ \beta = tr \circ \hat{\beta} \). For every \( f \in C(G^0) \) one has: \( (tr \circ m)(f) = \sum_{u \in G^0} \text{Card}(m^{-1}(u))f(u) \), so the only thing to prove is that for any \( u \in G^0 \), one has: \( \text{Card}(G^u) = \text{Card}(m^{-1}(u)) \).

But the application \( (h, k) \mapsto hk \) is a bijection between \( \mathcal{H}_u \times \mathcal{K}^u \) and \( m^{-1}(u) \), so
\[
\text{Card}(m^{-1}(u)) = \text{Card}(\mathcal{H}_u)\text{Card}(\mathcal{K}^u).
\]
In an other hand, any element \( g \) in \( \mathcal{G}^u \) has a unique decomposition \( g = hk \) where \( h \in \mathcal{H}^u \) and \( k \in \mathcal{K}^{s(h)} \), one easily gets that the image of \( \mathcal{G}^u \) by the bijection \( g \mapsto (h, k) \) is equal to the disjoint union: \( \bigcup_{h \in \mathcal{H}^u} \{h\} \times \mathcal{K}^{s(h)} \), so using lemma 4.1.2 and the last equality, one has:
\[
\text{Card}(\mathcal{G}^u) = \sum_{h \in \mathcal{H}^u} \text{Card}(\mathcal{K}^{s(h)}) = \sum_{h \in \mathcal{H}^u} \text{Card}(\mathcal{K}^{r(h)}) = \text{Card}(\mathcal{H}^u)\text{Card}(\mathcal{K}^u) = \text{Card}(\mathcal{H}_u)\text{Card}(\mathcal{K}^u) = \text{Card}(m^{-1}(u))
\]
\( \square \)
4.1.4. **Lemma.** For any \( x, y \) in \( G \) such that \( m(x) = r(y) \), one has:

1) the elements \( p_2(x)^{-1} \) and \( y \) are composable for the multiplication of \( G \)

2) the same is true for \( x \) and \( p_1(p_2(x)^{-1}y) \),

3) \( m(xp_1(p_2(x)^{-1}y)) = m(y) \).

**Proof:** For any \( x, y \) in \( G \) such that \( m(x) = r(y) \), then \( s(p_2(x)^{-1}) = r(p_2(x)) = m(x) = r(y) \), so \( p_2(x)^{-1} \) and \( y \) are composable. But one has: \( r(p_1(p_2(x)^{-1}y)) = r(p_2(x)^{-1}y) = r(p_2(x)^{-1}) = s(p_2(x)) = m(x) = s(x) \), so \( x \) and \( p_1(p_2(x)^{-1}y) \) are composable too. As for any \((h, k)\) in \( H \times K \) and \( g \) in \( G \), one has: \( m(hgk) = m(g) \), one deduces that: \( m(xp_1(p_2(x)^{-1}y)) = m(p_2(x)p_1(p_2(x)^{-1}y)) \); let \((h_1, k_1)\) be in \( H \times K \) and such that: \( p_2(x)^{-1}y = h_1k_1 \), then: \( m(xp_1(p_2(x)^{-1}y)) = m(p_2(x)h_1) = m(yk_1^{-1}) = m(y) \). \( \square \)

So the following definition is relevant:

4.1.5. **Definition.** We shall denote \( I_{H,K} \) the linear endomorphism of \( l^2(G) \) defined for any \( f \) in \( l^2(G) \) and \( x, y \) in \( G \) by:

\[
I_{H,K}(f)(x, y) = \begin{cases} 
  f(xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y) & \text{if } m(x) = r(y) \\
  0 & \text{otherwise}
\end{cases}
\]

In particular: \( I_{G,G^0} = I_G \) and \( I_{G^0,G} \) is the mpi studied in [Val1] 4.1.

4.1.6. **Proposition.** \( I_{H,K} \) is a mpi over the base \( (C(G^0), \alpha, \beta, \hat{\beta}) \).

**Proof:** An easy computation gives the following formula for \( I_{H,K}^* \), for any \( f \) in \( l^2(G) \) and \( x, y \) in \( G \) one has:

\[
I_{H,K}^*(f)(x, y) = \begin{cases} 
  f(xp_1(y)^{-1}, p_2(xp_1(y)^{-1})y) & \text{if } s(x) = m(y) \\
  0 & \text{otherwise}
\end{cases}
\]

So one has:

\[
I_{H,K}^*I_{H,K}(f)(x, y) = \begin{cases} 
  f(x, y) & \text{if } s(x) = m(y) \\
  0 & \text{otherwise}
\end{cases}
\]

and:

\[
I_{H,K}I_{H,K}^*(f)(x, y) = \begin{cases} 
  f(x, y) & \text{if } m(x) = r(y) \\
  0 & \text{otherwise}
\end{cases}
\]

This means \( I_{H,K} \) is a partial isometry the initial (resp.final) support of which is \( e_{s,m} \) (resp. \( e_{m,r} \)). Let \( f, f' \) be any element in \( C(G^0) \), \( \xi \) be any element in \( l^2(G \times G) \), \( x, y \) be any element in \( G \).

First suppose that \( s(x) = m(y) \) then:

\[
I_{H,K}(\beta(f) \otimes \hat{\beta}(f'))\xi(x, y) = (\beta(f) \otimes \hat{\beta}(f'))\xi(xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y)
= f(r(xp_1(p_2(x)^{-1}y))f'(s(p_2(x)^{-1}y))\xi(xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y)
= f(r(x))f'(s(y))\xi(xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y)
= f(r(x))f'(s(y))I_{H,K}\xi(x, y)
= (\beta(f) \otimes \hat{\beta}(f'))I_{H,K}\xi(x, y)
\]
and, using lemma 4.1.4 3):

\[ I_{\mathcal{H},\mathcal{K}}(\alpha(f) \otimes \beta(f'))\xi(x, y) = (\alpha(f) \otimes \beta(f'))\xi(xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y) \]
\[ = f(m((xp_1(p_2(x)^{-1}y)f'(r(p_2(x)^{-1})\xi((xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y) \]
\[ = f(m(y))f'(r(p_2(x)^{-1})\xi((xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y) \]
\[ = f(m(y))f'(s(x))I_{\mathcal{H},\mathcal{K}}\xi(x, y) \]
\[ = (\hat{\beta}(f') \otimes \alpha(f))I_{\mathcal{H},\mathcal{K}}\xi(x, y) \]

If \( s(x) \neq m(y) \), one has: \( I_{\mathcal{H},\mathcal{K}}(\beta(f) \otimes \hat{\beta}(f'))\xi(x, y) = 0 = (\beta(f) \otimes \hat{\beta}(f'))I_{\mathcal{H},\mathcal{K}}\xi(x, y) \), and also: \( I_{\mathcal{H},\mathcal{K}}(\alpha(f) \otimes \beta(f'))\xi(x, y) = 0 = (\hat{\beta}(f') \otimes \alpha(f))I_{\mathcal{H},\mathcal{K}}\xi(x, y) \). Hence:

\[ I_{\mathcal{H},\mathcal{K}}(\beta(f) \otimes \hat{\beta}(f')) = (\beta(f) \otimes \hat{\beta}(f'))I_{\mathcal{H},\mathcal{K}} \]
\[ I_{\mathcal{H},\mathcal{K}}(\alpha(f) \otimes \beta(f')) = (\hat{\beta}(f') \otimes \alpha(f))I_{\mathcal{H},\mathcal{K}} \]

Now let’s prove the pentagonal relation for \( I_{\mathcal{H},\mathcal{K}} \). Let’s fix some notation: for any \( x, y \) in \( \mathcal{G} \) such that \( m(x) = r(y) \) then one can define: \( V = p_2(x)^{-1}y \) and \( X = xp_1(V) \), if moreover \( z \) is any element of \( \mathcal{G} \) such that \( m(y) = r(z) \), then \( m(p_2(x)^{-1}y) = r(p_2(X)^{-1}z) \) and, by two routine calculations the following relations are true:

\[ (I_{\mathcal{H},\mathcal{K}})_{12}(I_{\mathcal{H},\mathcal{K}})_{13}(I_{\mathcal{H},\mathcal{K}})_{23}\xi(x, y, z) = \]
\[ = \xi(xp_1(V)p_1(p_2(X)^{-1}z), p_2(x)^{-1}yp_1(p_2(V)^{-1}p_2(X)^{-1}z), p_2(V)^{-1}p_2(X)^{-1}z) \]

\[ (I_{\mathcal{H},\mathcal{K}})_{23}(I_{\mathcal{H},\mathcal{K}})_{12}\xi(x, y, z) = \]
\[ = \xi(xp_1(p_2(x)^{-1}yp_1(p_2(y)^{-1}z)), p_2(x)^{-1}yp_1(p_2(y)^{-1}z), p_2(y)^{-1}z) \]

Let \((h, k)\) be in \( \mathcal{H} \times \mathcal{K} \) such that: \( p_2(x)^{-1}p_1(y) = hk \), then \( V = hk p_2(y) \) and \( X = p_1(x)p_2(x)h = p_1(x)p_1(y)k^{-1} \) hence one has:

\[ (1) \quad p_2(V)^{-1}p_2(X)^{-1} = p_2(y)^{-1} \]
\[ (2) \quad p_2(X)^{-1} = k \]
\[ (3) \quad p_1(V) = h \]

So, using (10) and the notation: \( k' = kp_2(y) \), one has:

\[ p_1(p_2(x)^{-1}yp_1(p_2(y)^{-1}z)) = \]
\[ = p_1(hkp_2(y)p_1(p_2(y)^{-1}z)) = hp_1(kp_2(y)p_1(p_2(y)^{-1}z)) = p_1(V)p_1(k'p_1(k'^{-1}kz)) \]

Now let’s define \((h', k'')\) in \( \mathcal{H} \times \mathcal{K} \) such that: \( k'^{-1}kz = h'k'' \), hence using (9), one has:

\[ p_1(p_2(x)^{-1}yp_1(p_2(y)^{-1}z)) = \]
\[ = p_1(V)p_1(k'h') = p_1(V)p_1(kzk'') = p_1(V)p_1(kz) = p_1(V)p_1(p_2(X)^{-1}z) \]
This last equality and (8) gives that for any triple \((x, y, z)\) in \(\mathcal{G}^3\) such that \(m(x) = r(y)\) and \(m(y) = r(z)\):

\[
(I_{H,K})_{12}(I_{H,K})_{13}(I_{H,K})_{23}\xi(x, y, z) = (I_{H,K})_{23}(I_{H,K})_{12}\xi(x, y, z)
\]

but for all the other triples \((x, y, z)\) in \(\mathcal{G}^3\) the two sides of this equality are 0, hence, \(I_{H,K}\) is a mpi.

4.2. Crossed products and match pairs of groupoids. The situation of a match pair of groupoids \(\mathcal{G} = \mathcal{H}\mathcal{K}\), leads to a natural right action of the groupoid \(\mathcal{H}\) on the fibered space \(\mathcal{K}\) and a left action of the groupoid \(\mathcal{K}\) on the fibered space \(\mathcal{H}\). Using the inverse map, one has \(\mathcal{G} = \mathcal{H}\mathcal{K} = \mathcal{K}\mathcal{H}\). Hence, for any \(k \in \mathcal{K}\) and \(h \in \mathcal{H}\), there exist a unique \(h' \in H\) and a unique \(k' \in K^{s(h')}\) such that \(kh = h'k'\).

4.2.1. Lemma and definition. Let \(\mathcal{G} = \mathcal{H}\mathcal{K}\) be a match pair of groupoids, and for any \(k \in \mathcal{K}\) and \(h \in \mathcal{H}^{s(k)}\), let’s denote by \(k \triangleright h\) (resp. \(k \triangleleft h\)) the unique element in \(H\) (resp. \(K^{s(k)}\)) such that:

\[
kh = (k \triangleright h)(k \triangleleft h),
\]

then \(\triangleright\) (resp. \(\triangleleft\)) is a left action of the groupoid \(\mathcal{K}\) on the fibered space \(\mathcal{H}\) (resp. right action of the groupoid \(\mathcal{H}\) on the fibered space \(\mathcal{K}\)).

Proof: Left to the reader.

Let us denote by \(\mathcal{G}/\mathcal{K}\) (resp. \(\mathcal{H}\backslash \mathcal{G}\)) the set of right (resp.left) classes in \(\mathcal{G}\) modulo \(\mathcal{K}\) (resp.\(\mathcal{H}\)), that is \(\{gK^{s(g)} / g \in \mathcal{G}\}\) (resp. \(\{H_{r(g)}g / g \in \mathcal{G}\}\)). In that case, the application \(h \rightarrow hK^{s(h)}\) (resp. \(k \rightarrow H_{r(k)}k\)) is a natural bijection between \(\mathcal{H}\) and \(\mathcal{G}/\mathcal{K}\) (resp. \(\mathcal{K}\) and \(\mathcal{H}\backslash \mathcal{G}\)). Using these applications, \(\mathcal{G}/\mathcal{K}\) and \(\mathcal{H}\backslash \mathcal{G}\) are fibered by \(\mathcal{G}^0\): for any \(u \in \mathcal{G}^0\), one can define \((\mathcal{G}/\mathcal{K})^u = \{gK^{s(g)} / r(g) = u\}\) and \((\mathcal{H}\backslash \mathcal{G})^u = \{H_{r(g)}g / s(g) = u\}\). Also \(\mathcal{K}\) (resp. \(\mathcal{H}\)) has a left action on \(\mathcal{G}/\mathcal{K}\) (resp. \(\mathcal{H}\backslash \mathcal{G}\)) by multiplication: for any \(h \in \mathcal{K}\), \(g \in \mathcal{G}_{r(h)}\) and \(g' \in \mathcal{G}^{s(h)}\), one can define \(\delta_h(H_{r(g)}g) = H_{r(g)}gh\) and \(\gamma_k(gK^{s(g')}) = kg'K^{s(g')}\). Using the natural bijections below, and slightly abusing notations, one easily sees that the right action of \(\mathcal{H}\) on \(\mathcal{K}\) (resp. left action of \(\mathcal{K}\) on \(\mathcal{H}\)) is exactly the one coming from lemma [4.2.1].

Now, in the following, let’s remember that \(\chi_p\) (resp. \(\chi_Z\)) denotes the characteristic function of the singleton \(\{p\}\) (resp. set \(Z\)).

4.2.2. Proposition and notations. The mpi \(I_{H,K}\) is regular and the \(C^*\)-algebra \(S\) (resp.\(\hat{S}\)) associated to \(I_{H,K}\) is isomorphic to the crossed product \(C(\mathcal{K}) \rtimes_{\delta^K} C(\mathcal{H})\) (resp. \(C(\mathcal{K}) \rtimes_{\delta^K} C(\mathcal{H})\)), where \(\gamma^K\) is the left action of \(C(\mathcal{K})\) on \(C(\mathcal{H})\) associated with \(\triangleright\) (resp. \(\delta^K\) is the right action of \(C(\mathcal{H})\) on \(C(\mathcal{K})\) associated with \(\triangleleft\)). Hence \(C(\mathcal{K}) \rtimes_{\delta^K} C(\mathcal{H})\) and \(C(\mathcal{K}) \rtimes_{\gamma^K} C(\mathcal{H})\) have weak Hopf \(C^*\)-algebras structures in duality, we shall note them \((C(\mathcal{K}) \rtimes_{\gamma^K} C(\mathcal{H}), \Gamma^\gamma, \kappa^\gamma, e^\gamma)\) and \((C(\mathcal{K}) \rtimes_{\delta^K} C(\mathcal{H}), \Gamma^\delta, \kappa^\delta, e^\delta)\).
Proof: For any \( g, g', p, q \) in \( G \) and \( \xi \) in \( l^2(G) \), one has:

\[
(i \otimes \omega_{\chi_p, \chi_q})(I_{H,K})(\xi)(g) = ((i \otimes \omega_{\chi_p, \chi_q})(I_{H,K})(\xi))(g) = (\omega_{\xi, \chi_g} \otimes \omega_{\chi_p, \chi_q})(I_{H,K}) = (I_{H,K}(\xi \otimes \chi_p), \chi_g \otimes \chi_q) = I_{H,K}(\xi \otimes \chi_p)(g, q)
\]

Hence, \((i \otimes \omega_{\chi_p, \chi_q})(I_{H,K})(\xi)(g) = 0\) if \( m(g) \neq r(q) \); otherwise:

\[
(i \otimes \omega_{\chi_p, \chi_q})(I_{H,K})(\xi)(g) = I_{H,K}(\xi \otimes \chi_p)(g, q) = \chi_p(p_2(g)^{-1}q)(\chi_p(p_2(g)^{-1}q))
\]

This also implies that \((i \otimes \omega_{\chi_p, \chi_q})(I_{H,K})(\xi)(g) \neq 0\) only if there exists \( k \in K \) such that \( q = kp \) and \( p_2(g) = k \); one can see that these two conditions imply that \( m(g) = r(q) \).

To resume, for any \( k \) in \( K \), \( p \) in \( G^{\times(k)} \) and \( g \) in \( G \), one has: \((i \otimes \omega_{\chi_p, \chi_q})(I_{H,K})(\xi)(g) = 1_{p_2^{-1}(k)}(g)(\chi_p(1_{p_2}(p)))\), and if \( q \) is not in \( Kp \), one has: \((i \otimes \omega_{\chi_p, \chi_q})(I_{H,K}) = 0\). So \( S \) is generated by the operators : \( \xi \mapsto (g \mapsto \chi_{hk}(g)(\xi(gh))) \), for any \((k, h)\) in \( K \times_r H \), up to the natural identification of \( G \) with \( K \times_r H \), this is \( R^\delta(\chi(k,h))\) so \( S \) is isomorphic to the crossed product \( C(K) \rtimes C(H) \), hence \( I_{H,K} \) is regular. In a very similar way, \( S \) is generated by the operators \((\omega_{\chi_{hk}, \chi_k} \otimes i)(I_{H,K})\) for \((h, k)\) in \( K \times_r H \) which appear, up to the identification of \( H \times_r K \) with \( G \), to be equal to \( L^\gamma(\chi_{k\circ h,k}) \) (observe that \( r(k) = r(k \triangleright h) \)); hence \( S \) is isomorphic to \( C(K) \rtimes C(H) \).

Let’s compare these structures to ones defined in by N. Andruskiewitsch and S.Natale [AN]. Let’s use the notations of [AN] theorem 3.1, and let’s identify \( K \times_r H \) and \( T \), the double groupoid associated with the match pair \( K \mathcal{H} \) by [AN] proposition 2.9, using the bijection:

\[
(k, h) \mapsto \begin{array}{c}
\hline
\hline
k \\
\hline
\hline
h
\end{array}
\]

As vector spaces \( C(K) \rtimes C(H) = \mathbb{C}T \). As one can define on \( T \) an horizontal and a vertical product (see lemma 1.5 of [AN]), the above identification gives rise to compositions laws on \( K \times_r H \).

4.2.3. Definition and notations. We shall denote by \( \rightdownarrow \) the horizontal product defined for every \((k, h), (k', h')\) in \( K \times_r H \) such that \( h' = k \triangleright h \) by:

\[
(k, h) \rightdownarrow (k', h') = (kk', h').
\]

We shall denote by \( \leftdownarrow \) the vertical product defined for every \((k, h), (k', h')\) in \( K \times_r H \) such that \( k' = k \triangleleft h \) by:

\[
(k, h) \leftdownarrow (k', h') = (k, hh').
\]

Now we shall give a complete description of the \( C^* \)-quantum groupoid structure given by proposition 4.2.2 to \( C(K) \rightdownarrow C(H) \), which proves that it is isomorphic to \( \mathbb{C}T \).
4.2.4. **Lemma.** One has: $\Gamma^\delta \delta^H = (\delta^H \otimes \delta^H) \Gamma_K$, so for any $k$ in $K$ and any $g, g'$ in $G$, one gets:

$$\Gamma^\delta(\delta^H(\chi_k)) \xi(g, g') = \chi_k(p_2(g)p_2(g'))\xi(g, g')$$

**Proof:** Left to the courageous reader. □

4.2.5. **Theorem.** 1) For every $(k, h), (k', h')$ in $K_s \times_r H$, one has:

$$\chi((k, h))_{\delta} \star \chi((k', h')) = \begin{cases} \chi((k, h))_{\delta} \downarrow (k', h') & \text{if } k' = k < h \\ 0 & \text{otherwise} \end{cases}$$

and $\chi((k, h))_{\delta}^\# = \chi((k, h), h^{-1})$.

The identification of $K_s \times_r H$ with $T$ given by the application: $(k, h) \mapsto \begin{array}{c} k \\ h \end{array}$, leads to a $C^*$-isomorphism between and $C(K) \rtimes C(H)$ and $\mathbb{C}T$.

2) For every $(k, h)$ in $K_s \times_r H$, one has:

$$\Gamma^\delta(R^\delta(\chi_{(k, h)})) = \sum_{(k_1, h_1) \sqcup (k_2, h_2) = (k, h)} R^\delta(\chi_{(k_1, h_1)}) \otimes R^\delta(\chi_{(k_2, h_2)}).$$

$$\kappa^\delta(R^\delta(\chi_{(k, h)})) = R^\delta(\chi_{[k^{-1}, (k^{-1})^{-1}]}))$$

$$\epsilon(R^\delta(\chi_{(k, h)})) = \begin{cases} 1 & \text{if } k = r(h) \\ 0 & \text{otherwise} \end{cases}$$

$R^\delta$ is an isomorphism of $C^*$-quantum groupoids between $\mathbb{C}T$ associated with the match pair $KH$ by proposition 3.4 of [AN] and $(C(K) \rtimes C(H), \Gamma^\delta, \kappa^\delta, \epsilon^\delta)$.

**Proof:** An easy computation gives that, for all $(k, h), (k', h')$ in $K_s \times_r H$, one has:

$$\chi((k, h))_{\delta} \star \chi((k', h')) = \begin{cases} \chi((k, h))_{\delta} \downarrow (k', h') & \text{if } k' = k < h \\ 0 & \text{otherwise} \end{cases}$$

and $\chi((k, h))_{\delta}^\# = \chi((k, h), h^{-1})$, but $(k < h, h^{-1})$ appears to be the inverse of $(h, k)$ for the vertical product.

Hence, if one denotes by $\text{char}(\begin{array}{c} k \\ h \end{array})$ the characteristic function of $\begin{array}{c} k \\ h \end{array}$ in $\mathbb{C}T$, the application:

$$\text{char}(\begin{array}{c} k \\ h \end{array}) \mapsto R^\delta(\chi_{(k, h)})$$

gives an explicit $*$-algebras isomorphism between $\mathbb{C}T$ and $C(K) \rtimes C(H)$.

Let’s denote by $\Theta$ the image by this isomorphism of the coproduct $\Delta$ given by theorem 3.1 of [AN], so for any $(k, h)$ in $K_s \times_r H$:
\[ \Theta(R^\delta(\chi(k,h))) = \sum_{(k_1,h_1) \in \Gamma(k_2,h_2) = (k,h)} R^\delta(\chi(k_1,h_1)) \otimes R^\delta(\chi(k_2,h_2)) \]

Let's first prove the first formulae of the theorem when \( h = s(k) \), (i.e. \( \Gamma^\delta(\delta^\mathcal{H}(\chi_k)) = \Theta(\delta^\mathcal{H}(\chi_k)) \)). One can easily observe that for any \( (k_1,h_1),(k_2,h_2) \) in \( K \times_r \mathcal{H} \), one has \( (k_1,k_1) \Gamma(k_2,h_2) = (k,s(k)) \) if and only if \( k_1k_2 = k \) and \( h_1 = s(k_1), h_2 = s(k_2) \), hence for any \( g, g' \) in \( \mathcal{G} \) and \( \xi \) in \( l^2(\mathcal{G} \times \mathcal{G}) \), by lemma 4.2.4 one has:

\[
\sum_{(k_1,h_1) \in \Gamma(k_2,h_2) = (k,h)} (R^\delta(\chi(k_1,h_1)) \otimes R^\delta(\chi(k_2,h_2)))\xi(g,g') = \sum_{k_1,k_2=k} (R^\delta(\chi(k_1,s(k_1))) \otimes R^\delta(\chi(k_2,s(k_2))))\xi(g,g') = \sum_{k_1,k_2=k} \sum_{r(h')=s(g')} \sum_{r(h'')=s(g')} \chi(k_1,s(k_1))(p_2(g),h')\chi(k_2,s(k_2))(p_2(g'),h'')\xi(gh',g'h'')
\]

\[
= \sum_{k_1,k_2=k} \chi_k(p_2(g))\chi_{k_2}(p_2(g'))\xi(g,g') = \chi_k(p_2(g)p_2(g'))\xi(g,g') = \Gamma^\delta(\delta^\mathcal{H}(\chi_k))\xi(g,g') = \Gamma^\delta(R^\delta(\chi(k,h)))\xi(g,g').
\]

Let's denote that the fourth equality is due to the fact that: \( p_2(g) = k_1 \) implies that \( s(k_1) = s(g) \) and that \( p_2(g') = k_2 \) implies that \( s(k_2) = s(g') \). So one has:

\[ \Gamma^\delta(\delta^\mathcal{H}(\chi_k)) = \Theta(\delta^\mathcal{H}(\chi_k)) \text{ for any } k \text{ in } \mathcal{K} \]

Now let's denote by \( \rho \) the right regular representation of \( \mathcal{H} \), so for any \( h, h' \) in \( \mathcal{H} \), \( \xi \) in \( l^2(\mathcal{H}) \): \( \rho(h)\xi(h') = \xi(h'h) \). Then in \( C(\mathcal{K}) \rtimes C(\mathcal{H}) \), one has: \( 1 \otimes \rho(h) = \sum_{s(k)=r(h)} R^\delta(\chi(k,h)) \). As by definition, for any \( h \) in \( \mathcal{H} \), \( \xi \) in \( l^2(\mathcal{G} \times \mathcal{G}) \), \( g, g' \) in \( \mathcal{G} \), one has:

\[ \Gamma^\delta(1_\mathcal{K} \otimes \rho(h))\xi(g,g') = I_{\mathcal{H},\mathcal{K}}(1_\mathcal{G} \otimes (1_\mathcal{K} \otimes \rho(h)))I_{\mathcal{H},\mathcal{K}}\xi(g,g'), \]

an easy computation gives that:

\[ \Gamma^\delta(1_\mathcal{K} \otimes \rho(h))\xi(g,g') = \begin{cases} 
\xi(gp_1(p_2(g')h),g'h) & \text{if } s(g) = m(g') \text{ and } s(g') = r(h) \\
0 & \text{otherwise}
\end{cases} \]

In an other hand, for any \( k_1, k_2, K \) in \( \mathcal{K} \), \( h_1, h_2, H \) in \( \mathcal{H} \), one has \( (h_1,k_1) \Gamma(k_2,h_2) = (k,h) \) if and only if \( r(k_1) = r(k), h_1 = k_1^{-1}k \triangleright h, k_2 = k_1^{-1}k, h_2 = h \). Hence, we can write that:
\[
\Theta(1_k \otimes \rho(h))\xi(g, g') = \sum_{s(k)=r(h)} \sum_{r(k_1)=r(k)} R^\delta(\chi_{(k_1, k^{-1}k\circ h)}) \otimes R^\delta(\chi_{(k^{-1}k, h)})\xi(g, g')
\]
\[
= \sum_{s(k)=r(h) \cap k^{-1}k=r(k) \cap s(g)} \sum_{r(k_1)=r(k)} \chi_{k_1}(p_2(g))\chi_{k^{-1}k\circ h}(h')\chi_{k^{-1}k}(p_2(g'))\chi_{h}(h'')\xi(gh', g'')
\]

Due to the characteristic functions, all terms of the sum are zero except when:

\[
\begin{align*}
  k_1 &= p_2(g) \\
  k_2^{-1}k \triangleright h &= h' \\
  k_2^{-1}k &= p_2(g') \\
  h'' &= h
\end{align*}
\]

In order to have non-zero terms, one needs that \( k = p_2(g)p_2(g') \), hence \( s(g) = m(g') \), and also \( s(g') = s(k) \), which implies \( s(g') = r(h) \). In these conditions there is a single term in the sum, it is obtained for: \( k = p_2(g)p_2(g'), k_1 = p_2(g), h' = p_2(g) \triangleright h = p_1(p_2(g)h) \), hence: \( \Theta(1_k \otimes \rho(h))\xi(g, g') = \xi(gp_1(p_2(g')h), g'h) \). One deduces that:

\[
(5) \quad \Gamma^\delta(1_k \otimes \rho(h)) = \Theta(1_k \otimes \rho(h))
\]

As both \( \Gamma^\delta \) and \( \Theta \) are multiplicative, using (4) and (5), one deduces that for any \((k, h)\) in \( K_s \times_r H \):

\[
\Theta(R^\delta(\chi_{(k, h)})) = \sum_{(k_1, h_1) \triangleright (k_2, h_2) = (k, h)} R^\delta(\chi_{(k_1, h_1)}) \otimes R^\delta(\chi_{(k_2, h_2)})
\]

Now for any \( g \) in \( G \), any \((k, h)\) in \( K_s \times_r H \), and any \( \xi \) in \( l^2(G) \), one has:

\[
\begin{align*}
  \kappa^\delta(R^\delta(\chi_{(k, h)}))\xi(g) &= (i \otimes \omega_{\chi_h, \chi_{kh}})(I^*_{H,K})\xi(g) = I^*_{H,K} \otimes \chi_h)(g, kh) \\
  &= \xi(gp_1(kh)^{-1})\chi_h(p_2(gp_1(kh)^{-1})kh)
\end{align*}
\]

One easily sees that \( p_2(gp_1(kh)^{-1})kh = h \) if and only if \( p_2(g) = (k \triangleright h)^{-1} \)
So: \( \kappa^\delta(R^\delta(\chi_{(k, h)}))\xi(g) = \xi(g(k \triangleright h)^{-1})\chi_{H(k \triangleright h)}(g) = R^\delta(\chi_{((k \triangleright h)^{-1}, (k \triangleright h)^{-1})})\xi(g) \)

Hence:

\[
\kappa^\delta(R^\delta(\chi_{(k, h)})) = R^\delta(\chi_{((k \triangleright h)^{-1}, (k \triangleright h)^{-1})})
\]

For any \((k, h)\) in \( K_s \times_r H \), one has:

\[
\epsilon(R^\delta(\chi_{(k, h)})) = \epsilon((i \otimes \omega_{\chi_h, \chi_{kh}})(I_{H,K}) = \omega_{\chi_h, \chi_{kh}}(1) = \sum_{g \in G}\chi_h(g)\bar{\chi}_{kh}(g)
\]

Hence:

\[
\epsilon(R^\delta(\chi_{(k, h)})) = \begin{cases} 1 & \text{if } k = r(h) \\ 0 & \text{otherwise} \end{cases}
\]

The theorem follows immediately. \qed
4.2.6. **Remark.** Using the natural identification of \( \mathcal{G} \) and \( \mathcal{K} \times_r \mathcal{H} \), one can also express the \( C^* \)-quantum groupoid structure \((C(\mathcal{K}) \rtimes C(\mathcal{H}), \Gamma_\delta, \kappa_\delta, \epsilon_\delta)\) by the following formulae:

- \( \Gamma_\delta(R^\delta(x_g)) = \sum_{(k_1,h_1)\Box (k_2,h_2) = \langle p_2(g),p_1(g) \rangle} R^\delta(\chi_{k_1}) \otimes R^\delta(\chi_{k_2}) \)
- \( \kappa_\delta(R^\delta(x_g)) = R^\delta(x_{g^{-1}}) \)
- \( \epsilon(R^\delta(x_g)) = \begin{cases} 1 & \text{if } g \in \mathcal{H} \\ 0 & \text{otherwise} \end{cases} \)

Naturally, using the identification of \( \mathcal{G} \) and \( \mathcal{K} \times_r \mathcal{H} \), one also has a characterization of the \( C^* \)-quantum groupoid structure for \((C(\mathcal{K}) \rtimes_r C(\mathcal{H}), \Gamma_\gamma, \kappa_\gamma, \epsilon_\gamma)\) dual to \((C(\mathcal{K}) \rtimes_r C(\mathcal{H}), \Gamma_\delta, \kappa_\delta, \epsilon_\delta)\). Let’s recall that \( \mathcal{T}^t \) (the transpose of \( \mathcal{T} \)) is by definition equal to \( \mathcal{T} \) as a set but the horizontal and vertical laws are exchanged and due to Proposition 3.11 [AN] \( \mathcal{C}\mathcal{T}^t \) is the dual of \( \mathcal{C}\mathcal{T} \).

The application \( (h,k) \mapsto h \begin{array}{c} k \end{array} \) gives a bijection between \( \mathcal{H} \times_r \mathcal{K} \) and \( \mathcal{T}^t \) (equal to \( \mathcal{T} \) as a set).

The same calculations than above give the following result:

4.2.7. **Proposition.** There exists an isomorphism of \( C^* \)-quantum groupoids between \((C(\mathcal{K}) \rtimes_r C(\mathcal{H}), \Gamma_\gamma, \kappa_\gamma, \epsilon_\gamma)\) and \( \mathcal{C}\mathcal{T}^t \).

One can give explicitly this isomorphism. If \( \text{char}( \begin{array}{c} h \end{array} k) \) is the characteristic function of \( \begin{array}{c} h \\ k \end{array} \) in \( \mathcal{C}\mathcal{T}^t \), then:

\[
L^\gamma(\chi_{(h,k)}) \mapsto \text{char}( \begin{array}{c} h \\ k \end{array} )
\]

In [BH] is given a very deep study of inclusions of the form \( R^H \subset R \rtimes K \) where \( H \) and \( K \) are subgroups of a group \( G \) acting properly and outerly on the hyperfinite type \( II_1 \) factor \( R \), in such a way we can identify \( G \) with a subgroup of \( \text{Out} R \), in particular there is here no ambiguity for \( H \cap K \subset \text{Out} R \): let’s call \( \alpha \) the action of \( K \) and \( \beta \) the one of \( H \) here these actions coincide on \( H \cap K \). In [BH], it’s proved that this inclusion is finite depth if and only if the group generated by \( H \) and \( K \) in \( \text{Out} R \) is finite, and, in that situation, it is irreducible and depth two when \( H, K \) is a match pair. Relaxing the match pair property, considering the case \( \text{card}(H \cap K) \neq 1 \), let’s prove now that the inclusion \( R^H \subset R \rtimes K \) does not come from the match pair of groupoids procedure.

4.2.8. **Lemma(HB).** In the preceeding conditions, the algebra \( (R^H)' \cap R \rtimes K \) is isomorphic to the group algebra \( C[H \cap K] \).

**Proof:** If \( u_h \) and \( v_h \) are canonical implementations of \( \alpha \) and \( \beta \) on \( L^2(R) \), one can suppose \( u_x = v_x \) for any \( x \) in \( H \cap K \), these \( u_x \) generate a *-algebra isomorphic to
C[H ∩ K] and are clearly in \((R^H)' \cap R \rtimes K\), in an other hand, using the calculation in the proof of 4.1 in [HB], one has: \(\dim((R^H)' \cap R \rtimes K) = \text{card}(H \cap K)\). The lemma follows. □

4.2.9. **Corollary.** The algebra \((R^H)' \cap R \rtimes K\) is commutative if and only if \(H \cap K\) is abelian. Hence, when \(H \cap K\) is non abelian, the inclusion \(R^H \subset R \rtimes K\) does not come from the match pair of groupoids procedure..

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