CHARACTERIZING TWO-TIMESCALE NONLINEAR DYNAMICS USING FINITE-TIME LYAPUNOV EXPONENTS AND VECTORS*

K. D. MEASE†, U. TOPCU† AND E. AYKUTLUĞ†

Abstract. Finite-time Lyapunov exponents and vectors are used to define and diagnose boundary-layer type, two-timescale behavior and to construct a method for determining an associated slow manifold when one exists. Two-timescale behavior is characterized by a slow-fast splitting of the tangent bundle for the state-space. The slow-fast splitting defined by finite-time Lyapunov exponents and vectors is interpreted in relation to the asymptotic theory of partially hyperbolic sets. The method of determining a slow manifold developed in this paper is potentially more accurate than an existing approach that is based on local eigenvalues and eigenvectors, at the expense of more computation, and is more generally applicable than approaches, such as the singular perturbation method, that require a special coordinate representation. The approach is illustrated via several examples.

Key words. Nonlinear dynamics; Multiple timescales; Slow manifold; Lyapunov exponents and vectors

1. Introduction. When a finite-dimensional nonlinear time-invariant (NTI) dynamical system evolves on multiple timescales, reduced-order analysis may be possible. We consider only two timescales in this paper, referred to as fast and slow, but the discussion and results are relevant to systems with more than two timescales as well. Multiple timescales may induce geometric structure in the flow on the state-space. If the system differential equations can be expressed in terms of coordinates adapted to this structure, the system can be decomposed into lower-order subsystems to simplify the analysis of the dynamical behavior, analogous to modal decomposition for linear time-invariant (LTI) dynamical systems.

As a motivating example, consider the optimal (minimizing a weighted sum of time and fuel consumption) flight of an aircraft between distant locations. The first-order necessary conditions for the optimal solution take the form of a boundary-value problem for a Hamiltonian system. The solution may have a “take-off/cruise/landing structure” [24]. The aircraft spends most of the time in the cruise phase flying on a hyperbolic slow manifold in the state-space for the Hamiltonian system where the most time/fuel efficient flight is possible. On the slow manifold, the aircraft travels from the vicinity of the (longitude, latitude) of the take-off location to the vicinity of the (longitude, latitude) of the landing location. However, this cruise segment does not satisfy all the boundary conditions; for example, the altitude may be 35,000 feet. The take-off and landing phases are transitions to and from the cruise segment on the slow manifold, involving some fast behavior in comparison to the behavior on the slow manifold. For on-board guidance purposes, it would significantly simplify the problem to treat the cruise guidance (guidance on the slow manifold) and the take-off and landing guidance (guidance in the fast boundary-layers) separately, reducing the order of the relevant dynamics in each case and reducing the numerical sensitivity in determining the optimal flight path. This conceptual example has been used be-

---
*This work was supported by the National Science Foundation under Grant CMS-0010085. The work was presented in preliminary form at the AIAA Guidance, Navigation and Control Conference and Exhibit, Paper AIAA 2005-5849, August 2005, San Francisco, California, under the title "Two-Timescale Nonlinear Dynamics and Slow Manifold Determination".
†Department of Mechanical and Aerospace Engineering, University of California, Irvine, CA 92697; (Professor; Graduate Student Researchers); (kmease@uci.edu; utopcu@uci.edu; eaykutlu@uci.edu)
cause it is easy to describe. For an actual, detailed example see [41] in which such a reduced-order approach is proposed for guiding an air-breathing launch vehicle in a near-minimum-fuel manner. Minimum-fuel path planning is carried out on the slow manifold and, because of the order reduction and reduced sensitivity, this is expected to be feasible for near-real-time, onboard computation. A suboptimal feedback control law is then used to track the slow manifold trajectory. Because most of the flight occurs on the slow manifold, the performance is near-optimal. In [41], the order reduction and slow manifold characterization were achieved via the analytical singular perturbation method, with the geometric singular perturbation theory [8], described later, providing insight.

The analytical singular perturbation method [24, 34] starts with the “standard form” of a singularly perturbed dynamical system

\[
\begin{align*}
\dot{y} &= \varepsilon p(y, z, \varepsilon), \\
\dot{z} &= q(y, z, \varepsilon),
\end{align*}
\]

where the system state is \((y, z)\), \(\varepsilon\) is a small parameter, and both \(p\) and \(q\) are of \(O(1)\) with respect to the \(\varepsilon \to 0\) limit. The ‘dot’ over a variable denotes differentiation with respect to time \(t\). With the standard form, the terms slow and fast are due to the properties \(\dot{y} = O(\varepsilon)\) and \(\dot{z} = O(1)\). For \(\varepsilon = 0\), there is a manifold of equilibria given by \(\mathcal{E} = \{(y, z) \in \mathbb{R}^n : q(y, z, 0) = 0\}\). Assuming that the coordinates have been chosen so that \(\partial q / \partial z\) is nonsingular, there exists locally a function \(\psi(y, \varepsilon)\), such that \(q(y, \psi(y, 0), 0) = 0\), with which the slow manifold can be represented as a graph \(S(\varepsilon) = \{(y, z) \in \mathbb{R}^n : z = \psi(y, \varepsilon)\}\). The function \(\psi\) satisfies the partial differential equation (PDE)

\[
q(y, z, \varepsilon) = \varepsilon \frac{\partial \psi}{\partial y}(y, \varepsilon)p(y, z, \varepsilon),
\]

and can be computed as an asymptotic expansion in \(\varepsilon\). Moreover there is a systematic procedure for constructing a solution for particular boundary conditions by matching asymptotic expansions for the segment on the slow manifold and fast boundary layer segments, though determining terms beyond zeroth-order may be difficult, yet needed for sufficient accuracy. A variation on this approach is to pose the PDEs for the slow manifold directly. This approach has been developed and applied in the chemical kinetics context [5, 36]. However, the model required for the successful application of this approach must either be in singularly perturbed form or in coordinates for which it is known how to separate them into dependent and independent variables for the representation of the slow manifold as a graph, which implies knowing the dimension of the slow manifold as well as its orientation. Also the solution of the PDEs is not always feasible [5].

An important step toward greater generality in modeling and characterizing two-timescale behavior was taken by Fenichel [8, 20]. He developed a geometric (coordinate-free) singular perturbation theory for two-timescale behavior starting from a given one-parameter family of vector fields \(x = f_\varepsilon(x)\), \(\varepsilon\) a small constant parameter, avoiding the \(a \text{ priori}\) need for the standard form representation (1.1). He assumed that \(\mathcal{E} = \{x \in \mathbb{R}^n : f_0(x) = 0\}\) is a normally hyperbolic manifold of equilibria (fixed points) and showed that this manifold persists in perturbed form for small nonzero values of \(\varepsilon\). For nonzero \(\varepsilon\), the perturbed manifold, denoted by \(S(\varepsilon)\), such that \(S(0) = \mathcal{E}\), is no longer composed of equilibria; however it is invariant under \(f_\varepsilon(x)\) and for small \(\varepsilon\) the motion on \(S(\varepsilon)\) is slower than the transverse motion, so it is referred to as a slow
invariant manifold. Conveniently eigen-analysis (meaning the use of matrix eigenvalues and eigenvectors) of $Df_0(x) = \frac{\partial f}{\partial x}(x)$ provides the timescales and state-space structure near $S(0) = \mathcal{E}$ in the extended space $\mathbb{R}^n \times \mathbb{R}$, where the extra dimension is for $\varepsilon$. Associated with each $x \in S(0)$ are center, stable and unstable subspaces of the $(n + 1)$-dimensional tangent space. The center manifold theorem [15] connects the center, stable, and unstable subspaces to the corresponding nonlinear invariant manifolds. For a particular value of $\varepsilon$, the slow manifold $S(\varepsilon)$ is a slice of the center manifold. The left side of Fig. (1.1) depicts the geometry for $n = 2$ in the extended space.

Fig. 1.1. Fenichel’s geometric perspective of a two-timescale system.

Starting with a general system $\dot{x} = f(x)$, as depicted for $n = 2$ on the right side of Fig. (1.1), one strategy is thus to suspect two-timescale behavior with a slow invariant manifold $\mathcal{S}$ and seek a coordinate transformation $x = h(y, z, \varepsilon)$, where $y \in \mathbb{R}^{n_s}$ and $z \in \mathbb{R}^{n_f}$ with $n_s + n_f = n$, for which the dynamics take the standard form of (1.1) with the requisite properties for the application of the singular perturbation method. The singular perturbation method has indeed found utility in a number of fields [24, 33, 34]. However, its applicability is limited, for lack of a systematic method of diagnosing two-timescale behavior and generating the required standard form (1.1) representation, whose special coordinates are compatible with representing the slow manifold as a graph. The air-breathing launch vehicle application mentioned above required hypothesizing the two-timescale behavior, choosing an appropriate coordinate representation, and artificially introducing a small parameter into the representation.

The question arises as to whether two-timescale behavior can be defined independently from a singularly perturbed dynamic model, and if so, whether there is a methodology by which it can be diagnosed and exploited for reduced-order modeling and analysis. Timescales, state-space structure, and reduced-order analysis are applicable to nonlinear dynamical systems of the general form $\dot{x} = f(x)$ on $\mathbb{R}^n$, but are significantly more challenging to define and implement. One approach is to analyze the linear variational equation $\dot{v} = Df(x)v$ along orbits of the nonlinear system to identify the timescales and the associated tangent space structure. The tangent space structure is then “transferred” to the local manifold structure of the nonlinear flow in the state-space. When considering asymptotic behavior, the usual tangent space splitting [11, 22] is into stable, center, and unstable subspaces; these subspaces are transferred to the corresponding manifolds. For finite-time behavior, we want to dis-
tistinguish behavior that either dies out quickly in forward or backward time, relative to the characteristic time interval of interest $T_c$, from slower behavior. If the “orbit” under investigation is an equilibrium point $x_e$, the variational system is LTI, so eigen-analysis of $Df(x_e)$ is applicable and can yield an invariant splitting of the tangent space at $x_e$, denoted by $T_{x_e} \mathbb{R}^n = E_f'(x_e) \oplus E_s'(x_e) \oplus E_c'(x_e)$, into fast contracting, slow, and fast expanding linear subspaces. These subspaces can be transferred to fast contracting, slow, and fast expanding manifolds for the nonlinear flow near $x_e$. For a periodic orbit, eigen-analysis can be applied to the corresponding linearized period-one map to determine the tangent space structure, and then this structure transferred to the manifold structure of the nonlinear flow in the neighborhood of the periodic orbit. Away from equilibria and periodic orbits, the characterization of timescales and associated state-space structure is more difficult.

An existing approach to analyzing two (or more) timescale behavior for a general coordinate representation $\dot{x} = f(x)$ away from equilibria and periodic orbits – that does not rely on characterizing it as a singular perturbation – is the intrinsic low-dimensional manifold (ILDM) method \[29\], developed in the context of chemical kinetics. Via eigen-analysis of the system matrix $Df(x)$ for the linear variational dynamics at points $x$ in the state-space region of interest, the tangent space is split into slow and fast subspaces: $T_x \mathbb{R}^n = \hat{E}^s(x) \oplus \hat{E}^f(x)$, where the “hat” denotes that these subspaces are approximations, as clarified later, of the slow and fast subspaces; in particular these subspaces are not invariant with respect to the linear variational dynamics. If a slow invariant manifold $S$ exists, the invariance of the slow manifold with respect to the flow of $\dot{x} = f(x)$, and the fact that the flow on the slow manifold is slow, dictate that at points on $S$, $f(x)$ should lie in the slow subspace, or equivalently that $f(x)$ should be orthogonal to all the vectors in the orthogonal complement to the slow subspace. Using $\hat{E}^s$ to approximate the slow subspace, $n - n^s$ orthogonality conditions are constructed and solved to compute points on the slow manifold, where $n^s$ is the dimension of $\hat{E}^s$ and the corresponding slow manifold. Fig. 1.2 shows an example of a 2D slow manifold in $\mathbb{R}^3$ and the relevant geometric objects. The spectrum shown is consistent with this geometry and would be constructed from the eigenvalues of $Df(x)$ in the ILDM method. Kaper and Kaper \[21\] have analyzed the application of the ILDM method to a two-timescale system in standard singularly perturbed form (1.1) and shown that the error in determining the slow manifold is of order $\varepsilon^2$ and increases proportionally with the curvature of the slow manifold. The eigenvalues and eigenvectors of $Df(x)$ have also been used to express properties of finite-time stable and unstable manifolds, under the assumption that the eigenvectors change sufficiently slowly with $x$ along system trajectories \[16\]. Eigenvectors are indeed simpler to compute, than the Lyapunov vectors we will use, and should be used when they approximate the directions of interest to sufficient accuracy. Our focus however is on what to do when this is not the case (and for that matter, how to know when it is the case).

Because the actual slow and fast subspaces $E^s(x)$ and $E^f(x)$ are invariant under the linear flow, it follows that in tangent vector coordinates adapted to these subspaces, the linear variational system must have a block diagonal structure such that the slow and fast dynamics are uncoupled. In general, the decoupling is not achieved using tangent vector coordinates corresponding to an $x$-dependent eigenvector basis. The computational singular perturbation (CSP) method \[25\] \[20\] includes an iterative procedure that adjusts the eigenvectors of $Df(x)$ to basis vectors that successively reduce the coupling between the slow and fast subsystems. In \[30\], it was noted that
the basis vector refinement procedure in the CSP method is essentially a Lyapunov transformation used previously for subsystem decoupling in linear time-varying (LTV) systems [24]. Zagaris et al. [44] have analyzed the accuracy of the CSP method applied to a two-timescale system in standard singularly perturbed form (1.1) and found that the error in determining the slow manifold is of order $\varepsilon^{q+1}$ after $q$ applications of the CSP basis vector refinement algorithm [44]. The refinement requires accurately computing Lie derivatives of basis vectors [40].

Both the ILDM and CSP methods rely on the eigenvalues of $Df(x)$ to determine the timescales and rely on eigenvectors to either determine (for ILDM) or initialize (for CSP) the tangent space structure. Rather than eigenvalues of $Df(x)$, existing theory [4, 22] for hyperbolic sets and normally hyperbolic invariant manifolds is based on Lyapunov exponents [28] – average rates of tangent vector length contraction and expansion along trajectories. Indeed the general invariant manifold theory developed by Fenichel [5] is based on Lyapunov type numbers, though eigen-analysis was applicable for his geometric singular perturbation theory because it is based on the local
structure for equilibrium points. In the case of a partially hyperbolic compact invariant set $\mathcal{Y} \subset \mathbb{R}^n$, Lyapunov exponents are used to define an invariant splitting of the tangent space $T_x \mathbb{R}^n = E^{fc}(x) \oplus E^s(x) \oplus E^{fe}(x)$ into fast contracting, slow, and fast expanding subspaces at each $x \in \mathcal{Y}$. If the slow distribution $E^s$ is integrable, there is a corresponding foliation of the state-space. If a slow manifold $\mathcal{S}$ exists, it can be characterized as a leaf of the foliation that is invariant with respect to the flow, i.e., at each point on $\mathcal{S}$, $T_x \mathcal{S} = E^s(x)$ and $f(x) \in E^s(x)$. Fig. 1.3 illustrates a 1D hyperbolic slow manifold in $\mathbb{R}^3$ with the tangent space splitting shown at one point $x \in \mathcal{S}$. The associated spectrum of Lyapunov exponents depicted in Fig. 1.3 is consistent with this splitting. Diagnosing and computing such geometric structure, encompassing both the attracting slow manifold (Fig. 1.2) and general hyperbolic slow manifold (Fig. 1.3) cases, is the goal of our work on two-timescale behavior.

However, the geometric structure just described has been defined using asymptotic Lyapunov exponents. The Lyapunov exponents indicate average exponential rates over an infinite time interval; i.e., they are defined as limits (limit suprema in the most general case) when $t \to \pm \infty$. The state-space region is assumed to be compact and invariant under the flow. Under these assumptions, the infinite-time Lyapunov exponents converge, at least in the “lim sup” sense, and are metric independent. The tangent space splitting they define is invariant. But for many applications, the state-space region of interest is not invariant and the time span of interest is finite. The particular motivation for our work is to determine if disparate timescales are present in various flight guidance problems, and if they are, to develop accurate reduced-order models to facilitate analysis and control design. The state-space region is the flight envelope for the vehicle under study, and it is not invariant. The behavior outside this region may be different than that inside it, or the mathematical model may not even be valid outside the region, so we do not want the timescale information to be influenced by behavior outside the region. Hence the characteristic exponential rates of interest are averages over a finite-time interval, and we are led to the use of finite-time Lyapunov exponents (FTLEs) and the corresponding finite-time Lyapunov vectors (FTLVs) to characterize the tangent space structure of a two-timescale system. For timescale analysis applicable to the general case of a normally hyperbolic slow manifold, the FTLE and FTLV (FTLE/V for short) approach we develop seems to be most appropriate. If the slow manifold is either attracting or repelling, it can be discovered via simulation in forward or backward time respectively, although even in this case other approaches can be beneficial [5, 29, 39]. There are certainly other special cases, such as determining manifolds associated with equilibria or periodic orbits, where other approaches, for example ones based on eigen-analysis of $Df$, are applicable and may be more efficient. Chemical kinetics researchers have begun to use FTLE/Vs [35], and eigen-analysis of the fundamental solution matrix $\Phi$ for the linear variational equations over a finite time interval was considered in [39]. We note that the FTLE/Vs can be derived from eigen-analysis of $\Phi^T \Phi$, reducing to eigen-analysis of the symmetric part of $Df$ for infinitesimal propagation time, whereas eigen-analysis of $\Phi$ reduces to eigen-analysis of $Df$ for infinitesimal propagation time [7]. Eigen-analyses of $Df$, $\Phi$, and $\Phi^T \Phi$ for characterizing the flow on an attractor were studied and compared in [11]. Researchers in fluid dynamics [42] have used FTLE/Vs to understand local flow features; more recent work [7, 16, 38] has used the maximum FTLE as an indicator of finite-time stable and unstable manifolds. Another approach to stable and unstable manifolds in finite-time vector fields for fluids has been taken in [37].
We refer to the use of Lyapunov exponents and vectors, whether asymptotic or finite-time, to analyze the linear variational equations as Lyapunov Analysis. Pioneering investigations of the properties of FTLE/Vs can be found in [11, 13, 14, 27]. For the specific purpose of characterizing two-timescale behavior, a previous paper [31] focused on the properties of the FTLEs: their relationship to the infinite-time Lyapunov exponents and the kinematic eigenvalues, and their metric and coordinate dependence. Motivation from flight mechanics for decomposing dynamics on the basis of fast and slow behavior, and the relationship of Lyapunov exponents and vectors to the geometry of singularly perturbed systems was described in [32]. In the present paper, we present, in terms of FTLE/Vs, a definition of, and a means of diagnosing, two-timescale behavior of a nonlinear, finite dimensional, time-invariant dynamical system on a non-invariant subset of $\mathbb{R}^n$. This leads to a method of computing a slow manifold, when one exists. The efficacy of the method depends on the convergence rate of certain tangent subspaces defined by FTLVs as the averaging time increases. Previous convergence results [11, 31] are improved by characterizing the convergence in terms of the distance between the critical subspaces rather than in terms of the convergence of individual FTLVs. The scope of the present paper ends with the procedure for identifying points on a slow manifold. A goal of future work is to obtain reduced-order models for the dynamics on the slow manifold and the boundary-layer dynamics. In the context of several application examples, presented at the end of the paper, a first step is taken towards determining effective numerical algorithms for implementing our methodology, directly comparing with other methods, and tackling progressively more complicated and higher-order problems.

The authors recently learned of the work by Adrover et al. [1, 2]. Their approach to timescale analysis is based on Lyapunov exponents and corresponding tangent space directions and subspaces like our approach. There are several distinctions between their work and ours. They view finite-time Lyapunov exponents and vectors as a means of approximating the asymptotic counterparts and draw from asymptotic theory to interpret their numerical results. They consider attracting slow manifolds only, provide numerical evidence for exponential convergence of the tangent space structure, and introduce two numerical methods for performing the computations. We characterize the usefulness of finite-time FTLE/Vs also when it is not appropriate to consider asymptotic limits and propose definitions of two-timescale behavior and a slow manifold suited to the finite-time setting. We derive an explicit exponential bound for the convergence rate of the tangent space structure that depends on the timescale gap. We consider, in addition to attracting slow manifolds, general hyperbolic slow manifolds, necessitating the intersection of forward and backward filtrations. We use existing numerical methods. We have included two application examples from [1] to illustrate our method and facilitate comparison with their work.

The rest of the paper is organized as follows. In section 2, we define the dynamical system to be considered and recall some definitions from geometry. Section 3 covers Lyapunov analysis: first we define FTLE/Vs and describe their use for the identification of the tangent space structure; second the asymptotic theory of partially hyperbolic sets is described briefly; and finally, we address the convergence of the tangent space structure. In section 4 we define a finite two-timescale set and present the conditions satisfied by points on a finite-time slow manifold. The procedure for applying the approach is given in section 5. In section 6 several application examples are presented to demonstrate the use of our method and compare it with other methods. Conclusions are given in section 7.
2. Dynamical System Description and Relevant Geometry. The method we develop will be applied to a particular coordinate representation of a dynamical system. Denoting the vector of coordinates by $x \in \mathbb{R}^n$, $2 \leq n < \infty$, the $x$-representation of the dynamical system is

$$\dot{x} = f(x), \quad (2.1)$$

where the vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth function. The solution of $(2.1)$ for the initial condition $x$ is denoted by $x(t) = \phi(t, x)$, where $\phi(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the $t$-dependent flow associated with the vector field $f$ and $\phi(0, x) = x$. We assume that $\phi$ is complete on $\mathbb{R}^n$ for simplicity in stating some of the results, but the methodology developed will only be applied on a subset of the state space and the properties of the flow outside this subset are irrelevant.

The linearized dynamics associated with $(2.1)$ are

$$\dot{v} = Df(x)v. \quad (2.2)$$

We will analyze the linearized dynamics to characterize the timescales in the nonlinear dynamics. An initial point $(x, v)$ is mapped in time $t$ to the point $(x(t), v(t)) = (\phi(t, x), \Phi(t, x)v)$ where $\Phi$ is the fundamental matrix for the linearized dynamics, defined such that $\Phi(0, x) = I$, the $n \times n$ identity matrix. With this initial condition, we refer to the fundamental matrix as the transition matrix. Geometrically, for a pair $(x, v)$, we view $v$ as taking values in the tangent space at $x$. The coordinates for $v$ correspond to a tangent space frame whose axes are parallel to those of the $x$-frame, but with origin at $x$. The tangent space at a point $x \in \mathbb{R}^n$ is denoted by $T_x \mathbb{R}^n$. The tangent bundle $T\mathbb{R}^n$ is the union of the tangent spaces over $\mathbb{R}^n$ and $(x, v)$ is a point in the tangent bundle, with $v$ the tangent vector and $x$ the base point. We need the interpretation $(x, v) \in T\mathbb{R}^n$, because the analysis of the linearized dynamics will define a subspace decomposition in the tangent space and the orientation of the subspaces will vary with the base point $x$.

We adopt the Euclidean metric for $\mathbb{R}^n$. Consistent with the Euclidean metric, we use the Euclidean norm to define the length of a tangent vector, i.e., for $v \in T_x \mathbb{R}^n$, the length is $||v|| = \langle v, v \rangle^{1/2}$ and $\langle \cdot, \cdot \rangle$ is the standard inner product. We will also use the distance between equidimensional subspaces $S_1$ and $S_2$ of $\mathbb{R}^n$ given by

$$\text{dist}(S_1, S_2) = ||P_1 - P_2||_2, \quad (2.3)$$

where $P_1$ and $P_2$ are orthogonal projections onto $S_1$ and $S_2$, respectively, and $|| \cdot ||_2$ is the induced 2-norm. The distance has a value in the interval $[0, 1]$. The largest principal angle $\theta \in [0, \pi/2]$ between the equidimensional subspaces is defined by

$$\sin \theta = \text{dist}(S_1, S_2).$$

Let $w_1, \ldots, w_k$, $k \leq n$, denote vector fields defined on $\mathbb{R}^n$ that vary smoothly with $x$ and have the property that at each $x \in \mathbb{R}^n$, the vectors $w_1(x), \ldots, w_k(x)$ are linearly independent in $T_x \mathbb{R}^n$. Then at each $x$, $\Lambda(x) = \text{span}\{w_1(x), \ldots, w_k(x)\}$ is a $k$-dimensional subspace. If $k = n$, then $\Lambda(x) = T_x \mathbb{R}^n$ and for each $x$ the set of vectors provides a basis for $T_x \mathbb{R}^n$. If $k < n$, then $\Lambda(x)$ is a linear subspace of $T_x \mathbb{R}^n$ and $\Lambda$ is called a distribution on $\mathbb{R}^n$. A distribution is $\Phi$-invariant, if for any $x \in \mathbb{R}^n$ and $v \in \Lambda(x)$, the property $\Phi(t, x)v \in \Lambda(\phi(t, x))$ holds. Distributions $\Lambda_1, \ldots, \Lambda_m$ allow a splitting of the tangent space, if $T_x \mathbb{R}^n = \Lambda_1(x) \oplus \cdots \oplus \Lambda_m(x)$, where $\oplus$ denotes the direct sum of linear subspaces. If each distribution in the splitting is $\Phi$-invariant, then the splitting is called a $\Phi$-invariant splitting.
A smooth submanifold $\mathcal{M} \subset \mathbb{R}^n$ of dimension $m < n$ is $\phi$-invariant, if for any $x \in \mathcal{M}$, $\phi(t, x) \in \mathcal{M}$ for all $t$. An equivalent requirement for invariance is that $f(x) \in T_x\mathcal{M}$ for all $x \in \mathcal{M}$. We mention three ways of representing such a manifold.

1. **Algebraic constraints:** At least locally, $\mathcal{M} = \{x \in \mathbb{R}^n : h_1(x) = \cdots = h_{n-m}(x) = 0\}$ where $h_i$, $i = 1, \ldots, n - m$ are independent constraints and smooth functions of $x$. Given the invariance of $\mathcal{M}$, for all $x \in \mathcal{M}$ the constraint functions satisfy $L_{tf}h_i(x) = \left(\frac{\partial h_i}{\partial x} (x), f(x)\right) = 0$, $i = 1, \ldots, n - m$ where $L_{tf}h_i$ denotes the Lie derivative of $h_i$ in the direction $f$.

2. **Graph of a function:** At least locally, the coordinates of $x$ can be separated into a vector $x_{\text{indep}}$ of $m$ independent variables and a vector $x_{\text{dep}}$ of $n - m$ dependent variables, and there exists a function $\gamma : \mathbb{R}^m \to \mathbb{R}^{n-m}$ such that $\mathcal{M} = \{x \in \mathbb{R}^n : x_{\text{dep}} = \gamma(x_{\text{indep}})\}$. Given the invariance of $\mathcal{M}$, the function $\gamma$ should satisfy $f_{\text{indep}}(x_{\text{indep}}, \gamma) = \frac{\partial x_{\text{indep}}}{\partial x_{\text{indep}}} f_{\text{indep}}(x_{\text{indep}}, \gamma)$ where $x_{\text{dep}} = f_{\text{dep}}(x_{\text{indep}}, x_{\text{dep}})$ and $x_{\text{indep}} = f_{\text{indep}}(x_{\text{indep}}, x_{\text{dep}})$ are defined consistently with $x = f(x)$.

3. **Tangent space splitting and invariance-based orthogonality conditions:** In this case, we assume that any other smooth invariant manifold containing $\mathcal{M}$ coincides with $\mathcal{M}$. For each $x \in \mathbb{R}^n$, suppose there is a splitting of the tangent space $T_x\mathbb{R}^n = \Gamma(x) \oplus (\Gamma(x))^\perp$ into $m$ and $n - m$ dimensional subspaces, where $(\Gamma(x))^\perp$ is the orthogonal complement to $\Gamma(x)$, the distribution $\Gamma$ is $\Phi$-invariant, and for each $x \in \mathcal{M}$, $T_x\mathcal{M} = \Gamma(x)$. Given a basis $\{w_1(x), \ldots, w_{n-m}(x)\}$ for $(\Gamma(x))^\perp$, the manifold can be defined implicitly by

$$\mathcal{M} = \{x \in \mathbb{R}^n : (w_i(x), f(x)) = 0, \quad i = 1, \ldots, n - m\}, \quad (2.4)$$

where $(\cdot, \cdot)$ denotes the standard inner product.

The orthogonality conditions for $f$ in (2.4) can be viewed as partial-equilibrium conditions, partial in the sense that the vector field $f$ need only be zero in certain directions. If one has constraint functions $h_i$, $i = 1, \ldots, n - m$ for representation 1, then $(\Gamma(x))^\perp = \text{span}(\frac{\partial h_1}{\partial x}(x), \ldots, \frac{\partial h_{n-m}}{\partial x}(x))$. It may be easier however to find a basis for $(\Gamma(x))^\perp$ directly without first finding constraint functions. Not every basis of $(\Gamma(x))^\perp$ can be related to a set of constraint functions. Determining the scalar constraint functions in representation 1 and the vector-valued function $\gamma$ in representation 2 requires the solution of partial differential equations and posing these equations requires a priori knowledge about the manifold, e.g., its dimension and in the case of representation 2, its orientation.

The approach developed in this paper leads to a characterization of points on a slow invariant manifold as in representation 3. The ILDM approach does this also. The distinction is that the ILDM approximates the splitting $\Gamma(x) \oplus (\Gamma(x))^\perp$ based on the eigen-analysis of $Df(x)$, whereas our approach is based on finite-time Lyapunov analysis.

### 3. Lyapunov Analysis

In this section we present the methodology for characterizing the linearized dynamics (2.2), along trajectories of the nonlinear system (2.1), that will enable the definition and diagnosis of two-timescale behavior. We refer to this methodology as *Lyapunov analysis* and think of it as serving the purposes for LTV dynamics that *eigen-analysis* serves for LTI dynamics. In the first subsection, we present a finite-time version of Lyapunov analysis, modeled after the asymptotic version described in Barreira and Pesin [4] and Katok and Hasselblatt [22]. We use
some of the notation and style of those books. In subsection 3.2, a brief account is given of how asymptotic Lyapunov exponents are used to define an (asymptotic) two-timescale set. In the final subsection, the convergence rate of a Lyapunov subspace is characterized, setting the stage for the finite-time approach presented in the remaining sections. Computational methods for Lyapunov analysis are considered briefly in subsection 5.3; see [4] and references therein for the state of the art.

3.1. Finite-Time Lyapunov Exponents/Vectors and Tangent Space Structure. A vector \( \mathbf{v} \in T_x \mathbb{R}^n \), propagated for \( T \) units of time along the trajectory \( \phi(t, \mathbf{x}) \), evolves to the vector \( \Phi(T, \mathbf{x}) \mathbf{v} \) in the tangent space \( T_{\phi(T, \mathbf{x})} \mathbb{R}^n \). The ratio of the Euclidean lengths of an initial non-zero vector and its corresponding final vector, \( \sigma(T, \mathbf{x}, \mathbf{v}) = \| \Phi(T, \mathbf{x}) \mathbf{v} \| / \| \mathbf{v} \| \), is a multiplier that characterizes the net expansion (growth) if \( \sigma(T, \mathbf{x}, \mathbf{v}) > 1 \), or contraction if \( \sigma(T, \mathbf{x}, \mathbf{v}) < 1 \), of the vector over the time interval \([0, T]\). We distinguish variables associated with forward-time propagation and backward-time propagation using the superscripts \( ^{+} \) and \( ^{-} \) respectively.

The propagation time \( T \), also referred to as the averaging time, is always taken to be positive whether forward or backward. The forward and backward FTLEs are given by

\[
\begin{align*}
\mu^+(T, \mathbf{x}, \mathbf{v}) &= \frac{1}{T} \ln \sigma^+(T, \mathbf{x}, \mathbf{v}) = \frac{1}{T} \ln \frac{\| \Phi(T, \mathbf{x}) \mathbf{v} \|}{\| \mathbf{v} \|}, \\
\mu^-(T, \mathbf{x}, \mathbf{v}) &= \frac{1}{T} \ln \sigma^-(T, \mathbf{x}, \mathbf{v}) = \frac{1}{T} \ln \frac{\| \Phi(-T, \mathbf{x}) \mathbf{v} \|}{\| \mathbf{v} \|},
\end{align*}
\]

for propagation time \( T \). For \( \mathbf{v} = 0 \), define \( \mu^+(T, \mathbf{x}, 0) = \mu^-(T, \mathbf{x}, 0) = -\infty \). A Lyapunov exponent allows the corresponding multiplier to be interpreted as an average exponential rate, i.e., \( \sigma(T, \mathbf{x}, \mathbf{v}) = \exp[\mu(T, \mathbf{x}, \mathbf{v})T] \); the average is over the time interval \([0, T]\).

Discrete forward and backward Lyapunov spectra, for each \( (T, \mathbf{x}) \), can be defined as follows. Define \( I^+_j(T, \mathbf{x}) \), \( i = 1, \ldots, n \), to be the orthonormal basis of \( T_x \mathbb{R}^n \) with the minimum sum of exponents, i.e., the minimum value of \( \sum_{i=1}^n \mu^+_i(T, \mathbf{x}, I^+_i(T, \mathbf{x})) \) over all orthonormal bases \([6]\). The forward Lyapunov spectrum is the set of exponents corresponding to the minimizing solution, namely, \( \{ \mu^+_i(T, \mathbf{x}), i = 1, \ldots, n \} \). The Lyapunov spectrum is unique, though the minimizing basis is not in general. One way \([6, 31]\) to obtain a minimizing basis and the forward Lyapunov spectrum is to compute the singular value decomposition (SVD) of \( \Phi(T, \mathbf{x}) = N^+(T, \mathbf{x}) \Sigma^+(T, \mathbf{x}) L^+(T, \mathbf{x})^T \), where \( \Sigma^+(T, \mathbf{x}) = \text{diag}(\sigma^+_1(T, \mathbf{x}), \ldots, \sigma^+_n(T, \mathbf{x})) \) contains the singular values, all positive and ordered such that \( \sigma^+_1(T, \mathbf{x}) \leq \sigma^+_2(T, \mathbf{x}) \leq \cdots \leq \sigma^+_n(T, \mathbf{x}) \), and to compute the Lyapunov exponents as \( \mu^+_i(T, \mathbf{x}) = (1/T) \ln \sigma^+_i(T, \mathbf{x}), i = 1, \ldots, n \). The column vectors of the matrix \( L^+(T, \mathbf{x}) \) are the minimizing orthonormal basis vectors \( I^+_i(T, \mathbf{x}) \), \( i = 1, \ldots, n \) for \( T_x \mathbb{R}^n \), and the column vectors of the orthogonal matrix \( N^+(T, \mathbf{x}) \) are denoted \( n^+_i(T, \mathbf{x}), i = 1, \ldots, n \). Rearranging the SVD of \( \Phi(T, \mathbf{x}) \), we can write \( \Phi(T, \mathbf{x}) I^+_i(T, \mathbf{x}) = \exp[\mu^+_i(T, \mathbf{x})T] n^+_i(T, \mathbf{x}) \), which indicates that \( n^+_i(T, \mathbf{x}) \in T_{\phi(T, \mathbf{x})} \mathbb{R}^n \). Geometrically, the unit \( n \)-sphere centered at the origin in \( T_x \mathbb{R}^n \) propagates under the linearized dynamics to an \( n \)-dimensional ellipsoid in \( T_{\phi(T, \mathbf{x})} \mathbb{R}^n \); the principal semi-axes of the ellipsoid are \( \exp[\mu^+_i(T, \mathbf{x})T] n^+_i(T, \mathbf{x}), i = 1, \ldots, n \) and the unit vectors in \( T_x \mathbb{R}^n \) that evolve to these vectors are respectively \( I^+_i(T, \mathbf{x}), i = 1, \ldots, n \).

Similarly, the backward Lyapunov spectrum consists of the exponents for the unit vectors in \( T_x \mathbb{R}^n \) that map to principal axes of an \( n \)-ellipsoid in \( T_{\phi(-T, \mathbf{x})} \mathbb{R}^n \). The backward exponents can be obtained from the singular value decomposition \( \Phi(-T, \mathbf{x}) = N^-(T, \mathbf{x}) \Sigma^-(T, \mathbf{x}) L^-(T, \mathbf{x})^T \), by \( \mu^-_i(T, \mathbf{x}) = (1/T) \ln \sigma^-_i(T, \mathbf{x}), i = 1, \ldots, n \). Assume the ordering on the diagonal of \( \Sigma^-(T, \mathbf{x}) \) is such that \( \sigma^-_1(T, \mathbf{x}) \geq \cdots \geq \sigma^-_n(T, \mathbf{x}) \).
The column vectors of the orthogonal matrix $L^-(T, x)$ are denoted by $l_i^+(T, x)$, $i = 1, \ldots, n$. For the column vectors of $L^+(T, x)$ and the orthogonal matrix $N^-(T, x)$, we have $l_i^-(T, x) \in T_x \mathbb{R}^n$ whereas $n_i^-(T, x) \in T_{\phi(-T,x)} \mathbb{R}^n$.

In summary, a unit $n$-sphere in $T_x \mathbb{R}^n$ is propagated $T$ units of time forward to an $n$-ellipsoid in $T_{\phi(T,x)} \mathbb{R}^n$ and backward to another $n$-ellipsoid in $T_{\phi(-T,x)} \mathbb{R}^n$. In $T_x \mathbb{R}^n$, the $l_i^+(T, x)$ vectors propagate to the principal axes of the forward ellipsoid, whereas the $l_i^-(T, x)$ vectors propagate to the principal axes of the backward ellipsoid. See Figure 3.1 for the case of $n = 2$. The $l_i^+(T, x)$ and the $l_i^-(T, x)$ vectors, for $i = 1, \ldots, n$, referred to as forward and backward FTLVs, respectively, will be used to define subspaces in $T_x \mathbb{R}^n$ associated with different exponential rates. Methods based on QR decomposition provide alternatives to computing FTLE/Vs [6]; see section 5.3.

**Definition 3.1.** [Non-Degenerate Lyapunov Spectra] The forward and backward Lyapunov spectra are non-degenerate for particular arguments $(T, x)$, if there are $n$ distinct forward FTLEs and $n$ distinct backward FTLEs, respectively.

**Assumption 3.1** For all $T$ and $x$ under consideration, the forward and backward FTLE spectra are each non-degenerate.

This assumption simplifies the presentation. It will be modified for the subspace convergence proof presented later. We note however that distinctness is also related to integral separation and the stability of the Lyapunov exponents with respect to perturbations in the linearized system matrix, $Df(x)$ [6]. In some of the application examples presented in section 6, there are degeneracies; however, these degeneracies occur in an initial “transient” phase that is short relative to the time interval under consideration. Modifying the assumption to hold for $T \geq T_o$, for an appropriate value of $T_o$, is sufficient for applying the methodology we develop.

The following subspaces, for $i = 1, \ldots, n$, can be defined by the orthonormal FTLVs

$$L_i^+(T, x) = \text{span}\{l_i^+(T, x), \ldots, l_i^+(T, x)\},$$

$$L_i^-(T, x) = \text{span}\{l_i^-(T, x), \ldots, l_i^-(T, x)\},$$

and will be referred to as finite-time Lyapunov subspaces. For any $i \in \{1, 2, \ldots, n\}$,
for the forward-time case, analogous properties hold for the backward-time exponents. However, for finite $T$, there also exist vectors $v \in T_x\mathbb{R}^n \setminus \mathcal{L}_i^+(T, x)$ for which $\mu^+(T, x, v) \leq \mu_i^+(T, x)$. Although stated only for the forward-time case, analogous properties hold for the backward-time exponents and subspaces.

If a collection of $r \leq n$ linear subspaces of $T_x\mathbb{R}^n$ can be ordered such that $\Lambda_1(x) \subset \Lambda_2(x) \subset \cdots \subset \Lambda_r(x) = T_x\mathbb{R}^n$ with all inclusions strict, then this collection of nested subspaces defines a filtration of $T_x\mathbb{R}^n$. The nested sequences of subspaces

$$
\{0\} = \mathcal{L}_0 \subset \mathcal{L}_1^+(T, x) \subset \mathcal{L}_2^+(T, x) \subset \cdots \subset \mathcal{L}_n^+(T, x) = T_x\mathbb{R}^n,
$$

are forward and backward filtrations \cite{4} \cite{22} of $T_x\mathbb{R}^n$.

We need both forward and backward filtrations, because their intersections are of particular interest, as motivated by the following. Consider a two-dimensional nonlinear system with an equilibrium point $x_e$. Assume the linearized dynamics at $x_e$ are characterized by distinct eigenvalues $\lambda_f$ and $\lambda_s$, with $\lambda_f < \lambda_s < 0$, and corresponding eigenvectors $e_f$ and $e_s$. As $T \to \infty$, the FTLEs approach the eigenvalues, i.e., $\mu_1^+ \to \lambda_f$ and $\mu_2^+ \to \lambda_s$, and the first Lyapunov vector approaches the corresponding eigenvector $I_1^+ \to e_f$. The second Lyapunov vector $I_2^+$ approaches $e_f$, the vector perpendicular to $e_f$. The subspace $\mathcal{L}_1^+(T, x_e)$ thus approaches $E^f(x_e) = \text{span}\{e_f\}$, the eigenspace for $\lambda_f$ as $T \to \infty$, whereas $\mathcal{L}_2^+(T, x_e) = T_{x_e}\mathbb{R}^2$ for any $T$. It would be desirable instead to obtain the invariant splitting $T_{x_e}\mathbb{R}^2 = E^f(x_e) \oplus E^s(x_e)$ where $E^s(x_e) = \text{span}\{e_s\}$. However, asymptotically all the vectors not in $\mathcal{L}_1^+$ have the Lyapunov exponent $\mu_2^+ = \lambda_s$; thus the Lyapunov exponents for forward-time propagation do not distinguish $E^s$. The way to obtain $E^s$ is by repeating the same analysis for backward-time propagation; in this case, the situation is reversed: asymptotically $I_2^- \to e_s$ and $E^s$ can be distinguished, whereas $E^f$ cannot \cite{22} \cite{43}.

Because the Lyapunov exponent and vector information concerns how the lengths of vectors change, this information is in general dependent on how the length of a vector is measured. The FTLEs and corresponding tangent space geometry are invariant with respect to coordinate transformations provided the representation of the metric is transformed consistently. In the asymptotic case the Lyapunov exponents are also metric invariant, but this is not true for the finite-time case. This issue was addressed in \cite{31}. In the present paper, we use the Euclidean metric exclusively, though any Riemannian metric could be used. If two-timescale behavior is not present in the original metric under consideration, there may be another metric for which there is two-timescale behavior, as noted by Greene and Kim \cite{14}. We are not addressing this opportunity directly, although one can apply our method with different metrics.

### 3.2. Asymptotic Lyapunov Exponents and Two-Timescale Set

We draw from \cite{18} to present a brief account of the asymptotic theory, covering only those definitions and results that serve to motivate and support our definitions and results for the finite-time case.

A compact invariant set $\mathcal{Y} \subset \mathbb{R}^n$ is a uniform two-timescale set (called uniform partially hyperbolic in \cite{18}), if there exists an invariant splitting at each $x \in \mathcal{Y}$

$$
T_x\mathbb{R}^n = E^{f_e}(x) \oplus E^s(x) \oplus E^{f_e}(x),
$$

(3.5)
and constants $\mu^s$, $\mu^f$, and $C$, with $0 < \mu^s < \mu^f$ and $C > 0$, such that $\forall t > 0$

$$v \in E^f(x) \Rightarrow \|\Phi(t, x)v\| \leq Ce^{-\mu^f t}\|v\|,$$

$$v \in E^s(x) \Rightarrow C^{-1}e^{-\mu^s t}\|v\| \leq \|\Phi(t, x)v\| \leq Ce^{\mu^s t}\|v\|,$$

$$v \in E^e(x) \Rightarrow \|\Phi(-t, x)v\| \leq Ce^{-\mu^f t}\|v\|.$$ 

Consider a compact, invariant set $\mathcal{Y} \subset \mathbb{R}^n$. When the infinite-time limits ($T \rightarrow \infty$) of the exponents in (3.1) exist at $x \in \mathcal{Y}$ for all $v \in T_x\mathbb{R}^n$, they are denoted by $\mu^+(x, v)$ and $\mu^-(x, v)$ and the system is said to be, respectively, forward regular and backward regular at $x$. There are at most $n$ distinct exponents for the vectors in $T_x\mathbb{R}^n \{0\}$. Consistent with our assumption for the finite-time case, we assume that there are $n$ distinct exponents, denoted $\mu^+_i(x)$, $i = 1, \ldots, n$ for forward-time and $\mu^-_i(x)$, $i = 1, \ldots, n$ for backward-time, with the forward exponents in ascending order and the backward exponents in descending order. Lyapunov subspaces are defined by $L^+_i(x) = \{v \in T_x\mathbb{R}^n : \mu^+_i(x, v) \leq \mu^+_i(x)\}$ and $L^-_i(x) = \{v \in T_x\mathbb{R}^n : \mu^-_i(x, v) \leq \mu^-_i(x)\}$. Forward and backward filtrations are defined as in (3.3) and (3.4). The system is Lyapunov regular [1] at $x$ if (i) it is forward and backward regular at $x$, (ii) $\mu^+_i(x) = -\mu^-_i(x)$, $i = 1, \ldots, n$, (iii) the forward and backward filtrations have the same dimensions, (iv) there exists a splitting $T_x\mathcal{Y} = E_1(x) \oplus \cdots \oplus E_n(x)$ into invariant subbundles such that $L^+_i(x) = E_1(x) \oplus \cdots \oplus E_i(x)$ and $L^-_i(x) = E_1(x) \oplus \cdots \oplus E_n(x)$, $i = 1, \ldots, n$, and (v) for any $v \in E_i(x) \setminus \{0\}$, $\lim_{t \rightarrow \pm \infty} (1/t) \ln \|\Phi(t, x)v\| = \mu^+_i(x)$. The invariant splitting described in parts (iv) and (v) is referred to as Oseledet’s decomposition.

Next we describe how the asymptotic Lyapunov exponents can be used to characterize a two-timescale set. For the purpose of motivating the finite-time theory presented in the next section, we assume the system (2.1) is Lyapunov regular at all the points of a compact, invariant set $\mathcal{Y}$, e.g., a periodic orbit. Further, assume that at each $x \in \mathcal{Y}$, there are $n^{fe}$ large negative exponents, $n^s$ small in absolute value exponents, and $n^{fe} + n^s + n^{fe} = n$. That is, there is a splitting of the forward Lyapunov spectrum of the form $Sp^+(x) = Sp^{fe}(x) \cup Sp^s(x) \cup Sp^{fe}(x)$ where $Sp^{fe}(x) = \{\mu^+_1(x), \ldots, \mu^+_r(x)\}$, $Sp^s(x) = \{\mu^+_{n^{fe}+1}(x), \ldots, \mu^+_{n^{fe}+n^s+1}(x)\}$, and $Sp^{fe}(x) = \{\mu^+_{n^{fe}+n^s+1}(x), \ldots, \mu^+_{n^{fe}}(x)\}$ and constants $0 < \mu^s < \mu^f$ defined by

$$\mu^f = \min\{-\max\mu^+_n(x), \min\mu^+_{n^{fe}+n^s+1}(x)\},$$

$$\mu^s = \max\{-\min\mu^+_n(x), \max\mu^+_{n^{fe}+n^s}(x)\},$$

where the maxima and minima are taken over the set $\mathcal{Y}$. If $\mathcal{Y}$ is a periodic orbit, the exponents do not depend on $x$ and there is a zero exponent corresponding to the direction tangent to the orbit, but we will not restrict our discussion to the particular case of a periodic orbit. Using the forward and backward filtrations one can construct the invariant splitting (3.5) by

$$E^{fe}(x) = L^+_{n^{fe}}(x),$$

$$E^s(x) = L^+_{n^{fe}+n^s}(x) \cap L^-_{n^{fe}+1}(x),$$

$$E^e(x) = L^-_{n^{fe}+n^s+1}(x).$$

With $\mu^f$ and $\mu^s$ as defined in (3.6), $C = 1$, and $E^{fe}$, $E^s$, and $E^e$ as defined in (3.7), $\mathcal{Y}$ is an asymptotic uniform two-timescale set.

Although Lyapunov vectors were not used to define the subspaces in the asymptotic case, they can be defined as follows and could be used to define the Lyapunov
subspaces. Let \( \{ l_i^+ \}, i = 1, \ldots, n \) denote an orthonormal basis for \( T_x \mathbb{R}^n \) such that \( \{ l_j^+ \}, j = 1, \ldots, i \) is a basis for \( L_i^+(x) \) for every \( 1 \leq i \leq n \). Let \( \{ l_i^- \}, i = 1, \ldots, n \) denote an orthonormal basis for \( T_x \mathbb{R}^n \) such that \( \{ l_j^- \}, j = i, \ldots, n \) is a basis for \( L_i^-(x) \) for every \( 1 \leq i \leq n \). When there are \( n \) distinct Lyapunov exponents as we are assuming, it follows that these bases are unique up to multiplication of individual vectors by \( \pm 1 \). These are clearly the orthonormal bases that minimize the sum of the asymptotic exponents over the set of orthonormal bases, and hence the basis vectors are the asymptotic counterparts of the FTLVs.

Thus in the asymptotic setting either Lyapunov exponents or vectors (as just defined) can serve to define the Lyapunov subspaces and tangent space splitting, and the results are equivalent. In contrast, the FTLEs and FTLVs define different tangent spaces (defined) can serve to define the Lyapunov subspaces and tangent space splitting, and hence the basis vectors are the Lyapunov vectors \( L^+ \) and \( L^- \). It is this specific convergence rate property on which the methodology described in [11] is addressed explicitly, rather than the convergence of individual Lyapunov vectors. The new element here is that convergence of a particular Lyapunov subspace (as characterized in the next subsection) and \( L_i^+(x) \) converges to \( L_i^+(T, x) \) and thus to \( L_i^+(x) \) as well.

Because the FTLV-defined Lyapunov subspace convergence is exponential in \( T \) (see next subsection), while the Lyapunov exponent convergence is much slower, perhaps proportional to \( 1/T \) [11], in the finite-time setting, we define the Lyapunov subspaces in terms of the FTLVs.

### 3.3. Exponential Lyapunov Subspace Convergence

In this subsection, we relate the finite-time tangent space structure introduced in section 3.1 and the asymptotic tangent space structure described in section 3.2. Theorem 3.5 below gives the exponential rate at which the finite-time Lyapunov subspaces introduced in section 3.1 and expressed in terms of the FTLVs evolve with increasing \( T \) toward their asymptotic limits. Most of the main ideas in Theorem 3.5 and its proof can be found in [11]. The new element here is that convergence of a particular Lyapunov subspace is addressed explicitly, rather than the convergence of individual Lyapunov vectors. It is this specific convergence rate property on which the methodology described in the following section rests. Before presenting the theorem, a couple definitions and a proposition are needed.

The following proposition provides a formula for computing the distance between the subspaces \( L_j^+(T_1, x) \) and \( L_j^+(T_2, x) \) in \( T_x \mathbb{R}^n \) for any value of \( j \) in the index set \( \{1, 2, \ldots, n\} \).

**Proposition 3.2.** Let \( L_j^+(T, x) \) denote the matrix whose columns are the Lyapunov vectors \( l_i^+(T, x), i = 1, \ldots, j \), and \( L_j^+(T, x) \) denote the matrix whose columns are the Lyapunov vectors \( l_i^+(T, x), i = j + 1, \ldots, n \). Then the distance between the subspaces \( L_j^+(T_1, x) \) and \( L_j^+(T_2, x) \) is

\[
\text{dist}(L_j^+(T_1, x), L_j^+(T_2, x)) = \| L_j^+(T_1, x)^T L_j^+(T_2, x) \|_2 \]

(3.8)

**Proof:** Proposition 3.2 is a special case of Theorem 2.6.1 in [12], page 76, and the facts that the columns of \( L_j^+(T, x) \) provide an orthogonal basis for \( L_j^+(T, x) \) and the
columns of $L_j^+(T, x)$ are mutually orthogonal to the columns of $L_j^+(T, x)$.

**Definition 3.3.** \([\text{T}]\) The Lyapunov spectrum is strongly non-degenerate at a point $x$, if there exists positive constants $T_o$ and $\delta$ such that the spectral gap between neighboring pair of forward FTLEs, $\mu_{i+1}(T, x) - \mu_i(T, x)$, is greater than $\delta$ for all $T > T_o$, and likewise for the backward exponents.

To consider the convergence of a Lyapunov subspace $L_j^+(T, x)$ with $T$, we focus on a particular spectral gap and define it more precisely.

**Definition 3.4.** \([\text{Relative Spectral Gap}]\) For a specified $T_o > 0$, the relative spectral gap $\Delta_j(x)$ between neighboring forward FTLEs $\mu_j^+(T, x)$ and $\mu_j^{+1}(T, x)$, for a particular $j \in \{1, 2, \ldots, n - 1\}$, is $\Delta_j(x) = \inf_{T \geq T_o} \mu_{j+1}(T, x) - \mu_j(T, x))$. The relative spectral gap $\Delta_j(x)$ between neighboring backward FTLEs $\mu_{j-1}(T, x)$ and $\mu_j(x)$ is similarly defined.

**Theorem 3.5.** Consider the dynamical system \((2.1)\) on a compact invariant subset $\mathcal{Y}$ of the state-space $\mathbb{R}^n$. At a Lyapunov regular point $x \in \mathcal{Y}$ for which there exists $T_o > 0$ and $\delta > 0$ such that the Lyapunov spectrum is strongly non-degenerate for $T > T_o$, and for which there is a nonzero relative spectral gap $\Delta_j(x)$ for a specific value of $j$, the subspace $L_j^+(T, x)$ approaches the fixed subspace $L_j^+(x)$, defined in section 3.2 in terms of the asymptotic Lyapunov exponent $\mu_j(x)$, at an exponential rate characterized, for every sufficiently small $\Delta T > 0$, by

$$\text{dist}(L_j^+(T, x), L_j^+(T + \Delta T, x)) \leq Ke^{-\Delta_j(x)T}, \quad (3.9)$$

for all $T > T_o$, where $K > 0$ is $\Delta T$ dependent but $T$ independent. Similarly, as $T$ increases, the subspace $L_k^-(T, x)$ approaches the fixed subspace $L_k^-(x)$ at a rate proportional to $\exp(-\Delta_k^-(x)T)$ where $\Delta_k^-(x) = \inf_{T > T_o} (\mu_{k-1}(T, x) - \mu_k(T, x))$.

**Proof:** Using (3.2) we have

$$\text{dist} (L_j^+(T, x), L_j^+(T + \Delta T, x)) = \|L_j^+(T, x)^T L_j^+(T + \Delta T, x)\|_2$$

$$\left\| \begin{bmatrix} 1^+_1(T, x)^T \\ 1^+_2(T, x)^T \\ \vdots \\ 1^+_j(T, x)^T \end{bmatrix} \begin{bmatrix} 1^+_1(T + \Delta T, x) & \cdots & 1^+_n(T + \Delta T, x) \end{bmatrix} \right\|_2$$

$$\left\| \begin{bmatrix} (1^+_1(T, x), 1^+_1(T + \Delta T, x)) & \cdots & (1^+_1(T, x), 1^+_n(T + \Delta T, x)) \\ \vdots \\ (1^+_j(T, x), 1^+_1(T + \Delta T, x)) & \cdots & (1^+_j(T, x), 1^+_n(T + \Delta T, x)) \end{bmatrix} \right\|_2. \quad (3.10)$$

Borrowing a result from \([\text{T}]\), we have for $T > 0$ to $1^{\text{st}}$-order in the time increment $\Delta T$

$$1^+_n(T + \Delta T) = (1 + c\Delta T)1^+_n(T) + \Delta T \sum_{i=1\atop i \neq n}^n \left[ (n^+_i)^T (A^T + A)n^+_i \right] 1^+_i \quad (3.11)$$

where $A = Df(x)$ is the system matrix of the linearized dynamics \((2.2)\), $n^+_i$ is a vector from the SVD of the transition matrix $\Phi(T, x)$ as defined in section 3.1, $c$ is a constant that is inconsequential in the following developments and is thus left unspecified, the
x dependence has been suppressed, and all exponents and vectors in the summation on the right-hand-side are evaluated at $(T, x)$. It follows that the inner products in (3.10) are

$$
\langle I^+_k (T, x), I^+_m (T + \Delta T, x) \rangle = \Delta T \frac{[\langle n^+_k (T^T + A) n^+_m \rangle]}{e(\mu^+_k - \mu^+_m)T - e(\mu^+_k - \mu^+_m)T^T}.
$$

(3.12)

Because $k \in \{1, \ldots, \mu\}$ and $m \in \{j + 1, \ldots, n\}$, we have $\exp[\langle \mu^+_k (T, x) - \mu^+_m (T, x) \rangle T] \leq \exp[-\Delta \mu^+(x) T]$. Let $a = \max_{x \in \mathcal{Y}} \max_{m \in \{1, \ldots, n\}} |\lambda_i (A^T + A)|$, the maximum eigenvalue magnitude of $A^T + A$ over the set $\mathcal{Y}$. And let $\alpha = \exp(-2\Delta \mu^+(x) T)$ for some $T_1 > T_0$. Then for $T \geq T_1 > 0$ we have

$$
|\langle I^+_k (T, x), I^+_m (T + \Delta T, x) \rangle| \leq \frac{\bar{a} \Delta T}{1 - \alpha} e^{-\Delta \mu^+(x) T}.
$$

(3.13)

Upper-bounding the 2-norm by the Frobenius norm and taking $K = \sqrt{j(n - j) + \Delta T}$, the bound in the theorem follows. This bound is conservative, due to the use of the Frobenius norm, but it shows the exponential rate of convergence. Using the bound (3.9), one can show that the sequence of iterates is Cauchy. Moreover this is true for every sufficiently small $\Delta T$. Because the space of $j$-dimensional subspaces in $T_x \mathbb{R}^n$, a Grassmannian, with the distance given in (3.2) as the metric, is complete, we conclude that $\mathcal{L}^+_j (T, x)$ approaches a fixed subspace. This subspace is $\mathcal{L}^+_j (x)$ defined in section 3.2, because all vectors in it have exponents less than or equal to $\mu^+_j (x)$ and one can show that any vector not in the subspace must have a larger exponent. The proof for backward-time is similar. □

The Theorem 3.5 hypothesis that the Lyapunov spectrum is strongly non-degenerate is necessary because the proof is based on the evolution of the individual Lyapunov vectors according to (3.11). We conjecture that the existence of the relative spectral gap is sufficient for the exponential subspace convergence, even if the rest of the spectrum has degeneracies.

4. Finite-Time Two-Timescale Set and Slow Manifold - Theory. We define finite-time two-timescale behavior by first defining a finite-time uniform two-timescale set. A two-timescale set has a special tangent space structure, and allows us to formulate conditions that would be satisfied at points of a slow manifold. If a slow manifold exists, then the nonlinear system has two-timescale behavior of the boundary-layer type and there is an opportunity, though not pursued in this paper, for system decomposition and reduced-order analysis.

We consider the timescale behavior of a system on a set $\mathcal{X} \subset \mathbb{R}^n$ which is in general not $\phi$-invariant. For the purpose of defining and diagnosing two-timescale behavior, $\mathcal{X}$ could be a point, a collection of isolated points, a segment of a trajectory, as examples, but in the search for a slow manifold, we assume no a priori trajectory knowledge and typically need to consider a bounded, connected open set of the state space.

In the finite-time setting, the terms “fast” and “slow” are defined by qualitative properties of the dynamics, relative to a particular time duration $T_c$, namely, “fast” refers to behavior that decays, either in forward or backward time, to a “negligible level” over $T_c$, whereas “slow” refers to behavior that does not. Although in each particular application, one needs to define fast and slow quantitatively, there is no generally appropriate definition; so we do not offer one. The bound $\beta$ in the following definition is the means of quantitatively distinguishing fast from slow. A numerical value of $\beta$ needs to be specified as appropriate for each application. We give some guidelines in section 5.1.
4.1. Finite-Time Two-Timescale Set.

Definition 4.1. A set $\mathcal{X} \subset \mathbb{R}^n$, $n \geq 2$, is a uniform two-timescale set for \([2,1]\), if there exist real numbers $\mu^f$, $\mu^s$, $T_o$ and $T_e$, with $0 < \mu^s < \mu^f$ and $0 < T_o < T_e$, such that, on $[T_o, T_e] \times \mathcal{X}$, there is a uniform splitting of the forward and backward Lyapunov spectra into fast contracting, slow, and fast expanding subsets, separated by gaps of size $\Delta \mu = \mu^f - \mu^s$, where $\Delta \mu (T_c - T_o) \geq \beta$ and the positive constant $\beta$ is a specified lower bound for two-timescale behavior. Specifically, the FTLEs satisfy the following properties for all $(T, x) \in [T_o, T_e] \times \mathcal{X}$:

1. $\mu^f_{n^f}(T, x) \leq -\mu^f$, $-\mu^s \leq \mu^s + n^f + 1(T, x)$, $\mu^f_{n^f+n^s+1} + n^f(T, x) \leq \mu^s$, and $\mu^f \leq \mu^s + n^f + n^s + 1 + 1(T, x)$,

2. $-\mu^s_{n^f}(T, x) \leq -\mu^f$, $-\mu^s \leq -\mu^s + n^f + 1(T, x)$, $-\mu^s_{n^f+n^s+1}(T, x) \leq \mu^s$, and $\mu^f \leq -\mu^s + n^f + n^s + 1 + 1(T, x)$,

where $n^f$, $n^s$ and $n^f$ are constant positive integers, with $n^f + n^s + n^f = n$, that specify the number of exponents associated with fast contracting, slow, and fast expanding behaviors, respectively. Either $n^f$ or $n^f$ is allowed to be zero, but not both. For $n^f = 0$, the conditions on $\mu^f_{n^f+n^s+1}(T, x)$ and $\mu^f_{n^f+n^s+1}(T, x)$ do not apply; similarly, for $n^f = 0$, the conditions on $\mu^f_{n^f+n^s+1}(T, x)$ and $\mu^f_{n^f+n^s+1}(T, x)$ do not apply.

Properties 1 and 2 are illustrated in Figure 4.1, where the bounds and forward and backward exponents are plotted on aligned different copies of the real line for clarity.

The exponents for particular values of $T$ and $x$ are pictured, but Properties 1 and 2 require this structure for all $(T, x) \in [T_o, T_e] \times \mathcal{X}$. The use of times up to $T_e$ means that the averaging in computing the Lyapunov exponents and vectors involves trajectories, though they begin in $\mathcal{X}$, extend into the larger (unless $\mathcal{X}$ is $\phi$-invariant) set

$$\mathcal{X}_{ext} = \{ y \in \mathbb{R}^n : y = \phi(t, x) \text{ for some } (t, x) \in [-T_e, T_e] \times \mathcal{X} \}. \quad (4.1)$$

**Fig. 4.1.** Spectra of forward and backward FTLEs illustrating the gaps (at the point $x \in \mathcal{X}$ for the averaging time $T$).

In Definition 4.1, the fast and slow behaviors are characterized by the exponent bounds $\mu^f$ and $\mu^s$, respectively. To simplify the definition we have used the same magnitudes $\mu^s$ and $\mu^f$ to define both gaps; however the symmetry of the two gaps with respect to zero is not necessary. Properties 1 and 2 ensure that, uniformly in $T$ and $x$, the forward Lyapunov spectrum can be divided into fast contracting, slow, and fast expanding subsets (Property 1) and the backward Lyapunov spectrum can be divided into fast contracting, slow, and fast expanding subsets (Property 2). Properties 1 and 2 also ensure that common gaps in the forward and backward Lyapunov spectra not only exist, but also separate the spectrum in a dimensionally consistent manner. Although Properties 1 and 2 only apply to the exponents for points in $\mathcal{X}$, they imply uniform timescale structure over $\mathcal{X}_{ext}$, because the exponents represent averages over this larger set; in particular, the “kinematic eigenvalues” \([3, 31]\), whose averages produce the exponents, must be similar over $\mathcal{X}_{ext}$. $T_o$ provides a grace period over which
the bounds on the exponents do not have to be satisfied, in order to accommodate initial transients which could otherwise violate the bounds. \( T \) is the characteristic maximum time over which the uniformity in the exponents holds and also is the time duration relative to which the terms fast and slow are relevant. One could use two different times; we have used \( T_e \) for the dual role to simplify the definition. The bound \( \beta \) also has a dual role: (i) it ensures that the fast motion decays to a negligible level, in either forward or backward time, over the duration \( T_e \), because the decay factor is \( \exp(-\mu^f(T_e - T_o)) \) and is small if \( \Delta \mu(T_e - T_o) = (\mu^f - \mu^s)(T_e - T_o) > \beta \), and (ii) it ensures that the gap is large enough (once its value is specified) relative to \( T_e - T_o \) that the critical subspaces can be resolved, as clarified next. We note that if there are additional gaps in the spectrum, a multiple-timescale set could be defined similarly.

**Proposition 4.2.** When \( \mathcal{X} \) is a uniform two-timescale set, at each \( x \in \mathcal{X} \), the subspaces

\[
\begin{align*}
E^{fc}(T, x) &= L^+_{n_f}(T, x), \\
E^s(T, x) &= L^+_{n_f + n_e}(T, x) \cap L^-_{n_f + 1}(T, x), \\
E^{fc}(T, x) &= L^-_{n_f + n_e + 1}(T, x),
\end{align*}
\]

have, for all \( T \in [T_o, T_e) \), the properties:

\[
\begin{align*}
v \in E^{fc}(T, x) \setminus \{0\} &\implies \|\Phi(T, x)v\| \leq e^{-\mu^f T}\|v\|, \\
v \in E^s(T, x) \setminus \{0\} &\implies \|\Phi(T, x)v\| \leq e^{\mu^s T}\|v\|, \\
v \in E^s(T, x) \setminus \{0\} &\implies \|\Phi(-T, x)v\| \leq e^{\mu^r T}\|v\|, \\
v \in E^{fc}(T, x) \setminus \{0\} &\implies \|\Phi(-T, x)v\| \leq e^{-\mu^f T}\|v\|,
\end{align*}
\]

and in addition, the subspaces \( E^{fc}(T, x) \), \( E^s(T, x) \) and \( E^{fc}(T, x) \) approach, in the sense of \( (3.9) \), fixed subspaces, with increasing \( T \), at least at a rate proportional to \( \exp(-\Delta \mu \cdot T) \), where \( \Delta \mu = \mu^f - \mu^s \).

**Proof:** The four properties in (4.3) follow from the definitions of the subspaces \( L^+_{n_f}(T, x) \), \( L^-_{n_f + 1}(T, x) \), \( L^+_{n_f + n_e}(T, x) \), and \( L^-_{n_f + n_e + 1}(T, x) \) (see (3.2)) and of a uniform two-timescale set (see Definition 4.1). Given the properties of a uniform two-timescale set in Definition 4.1 an argument similar to that used in the proof of Theorem 3.5 can be used to show that \( L^+_{n_f}(T, x) \), \( L^-_{n_f + 1}(T, x) \), \( L^+_{n_f + n_e}(T, x) \), and \( L^-_{n_f + n_e + 1}(T, x) \) approach with increasing \( T \) fixed subspaces at least at a rate proportional to \( e^{-\Delta \mu T} \). Because of the relationships in (4.2), it follows that \( E^{fc}(T, x) \), \( E^s(T, x) \) and \( E^{fc}(T, x) \) approach fixed subspaces at the same rate. Due to the finite-time constraint, this is not to say that convergence is achieved.

Proposition 4.2 says that there is tangent space structure associated with the slow and fast exponential rates. However the decay/growth bounds (4.3) hold for the subspaces \( E^{fc}(T, x) \), \( E^s(T, x) \) and \( E^{fc}(T, x) \) for all \( T \in [T_o, T_e] \). Which value of \( T \) should we use, i.e., which subspace structure is most appropriate? Proposition 4.2 states also that the subspaces are approaching fixed subspaces as \( T \) increases. If the hypotheses of Theorem 3.5 were applicable (if \( \mathcal{X} \) is a subset of a compact invariant set, etc.) and the \( T \to \infty \) limits could be computed, then we could compute \( E^{fc}(T, x) \), \( E^s(T, x) \) and \( E^{fc}(T, x) \) at each point of \( \mathcal{X} \) for arbitrarily large averaging times \( T \) and these subspaces would converge to \( \Phi \)-invariant subspaces that depend only on \( x \) [4]. Limited to \( T \in [T_o, T_e] \) we should use \( T = T_e \) to obtain subspaces that not only have the decay/growth bounds (4.3) but also approximate invariant subspaces as closely as possible within the available averaging times. If, however, \( \Delta \mu(T_e - T_o) \) is larger than the prescribed \( \beta \), it is sufficient to use the value of \( T \) satisfying \( \Delta \mu(T - T_o) = \beta \).
4.2. Finite-Time Slow Manifold. \( \mathcal{X} \) being a finite-time uniform two-timescale set establishes the potential for the existence of a slow manifold. To define a finite-time slow manifold, we now assume \( \mathcal{X} \) is an open set of \( \mathbb{R}^n \).

**Definition 4.3.** A finite-time slow manifold is a submanifold of \( \mathcal{X} \) denoted \( \mathcal{S}(T) \) such that \( f(x) \in E^s(T, x) \) for all \( x \in \mathcal{S}(T) \).

The set
\[
\{ x \in \mathcal{X} : \langle f(x), w \rangle = 0, \forall w \in [E^s(T, x)]^\perp \},
\]
thus satisfies a necessary condition and constitutes a candidate finite-time slow manifold. The motion at each point in the set is slow, because \( f(x) \in E^s(x) \), and thus there is no fast motion, the components of \( f(x) \) in \( E^{fc}(x) \) and \( E^{fc}(x) \) being zero. That this set is a manifold has to be verified. If a finite-time slow manifold exists, it will in general not be relatively invariant with respect to \( \mathcal{X} \), because \( T_x\mathcal{S}(T) \) and \( E^s(T, x) \) at points of the set do not coincide, and thus the invariance condition \( f(x) \in T_x\mathcal{S}(T) \) is not satisfied. However, if \( E^s(T, x) \) is close to the (hypothetical) \( \Phi \)-invariant asymptotic limit \( E^s(x) \), we conjecture that \( \mathcal{S}(T) \) will be close to the corresponding (hypothetical) \( \phi \)-invariant slow manifold. The examples in section 6, in which the asymptotic limits are relevant and can be determined, support this conjecture.

5. Finite-Time Two-Timescale Set and Slow Manifold - Procedure. If the goal is only to diagnose two-timescale behavior and determine the tangent space structure, then \( \mathcal{X} \) can be any subset of \( \mathbb{R}^n \). For example one could take \( \mathcal{X} \) to consist of a single equilibrium point (a fixed point of the vector field), although eigen-analysis would be applicable and more efficient for this particular case. If one also wants to search for a slow manifold, then usually \( \mathcal{X} \) is an open set, because it will be necessary to iteratively search for points that satisfy slow manifold conditions in a state space region of full dimension. As an example, consider a normally hyperbolic periodic orbit in \( \mathbb{R}^3 \), for which the transverse motion is faster than the motion along the periodic orbit. Let \( \mathcal{X} \) be the neighborhood of a segment of the periodic orbit. The invariant slow manifold in \( \mathcal{X} \) is the segment of the periodic orbit; the finite-time slow manifold, for a given \( T_c \), would approximate the invariant slow manifold. In the application of the methodology we do not require that \( \mathcal{X} \) is embedded in an invariant set; this hypothesis was only used to relate our approach to the asymptotic theory and characterize the Lyapunov subspace convergence.

5.1. Diagnosing a Finite-Time Two-Timescale Set. FTLEs are computed for a grid of points on \( \mathcal{X} \) to determine if \( \mathcal{X} \) is a two-timescale set according to Definition 4.1. One needs to see a pattern as illustrated in Figure 4.1 uniformly in \( T \) and \( x \) and to verify that the spectral gap is sufficiently large relative to \( T_c \). Regarding uniformity, the individual exponents can vary with \( T \) and \( x \) as long as there is a sufficiently large uniform gap. Thinking of the reciprocals of \( \mu^f \) and \( \Delta \mu^f \) as time constants, a guideline is that the slow behavior should decay/grow over several time constants and the finite-time subspaces \( E^{fc}(T_c, x) \), \( E^s(T_c, x) \), and \( E^{fc}(T_c, x) \) should converge over several time constants toward their hypothetical infinite-time limits. If decay/growth and convergence over \( m \) time constants is desired, then we require

---

*Because the FTLEs are only examined on a grid and at a finite set of values of \( T \), some experimentation with the grid (in \( T \) and \( x \)) is required to ensure that it is sufficiently fine.

*For an exponential function of time, \( e^{-t/\tau} \), the time constant \( \tau > 0 \) is the time \( t \) at which the function equals \( e^{-1} \).
\[ \mu(T_e - T_o) > m \text{ and } \Delta \mu(T_e - T_o) > m; \] the latter requirement is the most demanding, and in applying Def. 4.1 we should set \( \beta = m \). For a given value of \( \beta \), the smaller the gap is, the larger \( T_e \) must be. However, unless \( \mathcal{X} \) is \( \phi \)-invariant, the set \( \mathcal{X}_{\text{ext}} \) (see (4.1)) grows with \( T_e \) and at some point the timescale behavior may not be uniform on this extended set.

We note that the convergence of the subspaces can be checked directly by monitoring the distance between the subspaces with increasing averaging time (illustrated in section 6). To fill in the timescale structure around the grid points, one can either compute the FTLE/Vs at additional points or use the equations for propagating the Lyapunov subspaces (actually the orthonormal basis given by the Lyapunov vectors) along a trajectory given in [13].

5.2. Computing the Finite-Time Slow Manifold. Provided that \( \mathcal{X} \) satisfies Definition 4.1, we can take the next step which is to look for a slow manifold in \( \mathcal{X} \), where \( \mathcal{X} \) is now assumed to be an open set in \( \mathbb{R}^n \). Within \( \mathcal{X} \), the points on a candidate finite-time slow manifold \( \mathcal{S}(T) \) are defined implicitly by the orthogonality conditions in (4.1). Rather than use local eigenvectors to form an approximate basis for the orthogonal complement to \( E^s \) as in the ILDM method, we propose using the appropriate Lyapunov vectors to form the basis for \( (E^s)^\perp \) as given by the following proposition.

**Proposition 5.1.** The orthogonal complement of \( E^s(T, x) \) can be represented as

\[
(E^s(T, x))^\perp = \text{span}\{I_1^-(T, x), \ldots, I_{n+}\text{fc}+1(T, x), \ldots, I_n(T, x)\}. \tag{5.1}
\]

**Proof:** From Proposition 4.2, we have defined the slow subspace \( E^s(T, x) = \mathcal{L}_{n+\text{fc}+n}^+(T, x) \cap \mathcal{L}_{n+\text{fc}+1}^-(T, x) \). Using an identity [10], we have \( (E^s(T, x))^\perp = (\mathcal{L}_{n+\text{fc}+1}^-(T, x))^\perp \oplus (\mathcal{L}_{n+\text{fc}+n}^+(T, x))^\perp \). The proposition then follows from the facts: \( (\mathcal{L}_{n+\text{fc}+1}^-(T, x))^\perp = \text{span}\{I_1^-(T, x), \ldots, I_{n+\text{fc}}(T, x)\} \) and \( (\mathcal{L}_{n+\text{fc}+n}^+(T, x))^\perp = \text{span}\{I_{n+\text{fc}+n+1}^+(T, x), \ldots, I_n^+(T, x)\} \). \( \square \)

In order to obtain solutions of the algebraic equations, we designate \( n^s \) components of \( x \) as independent variables, fix their values, and determine the values of the remaining \( n - n^s \) components, the dependent variables, that minimize

\[
J = \sum_{i=1}^{n+\text{fc}} |I_i^-(T, x), f(x)|^2 + \sum_{n+\text{fc}+1}^{n} |I_{n+\text{fc}+n+1}^+(T, x), f(x)|^2, \tag{5.2}
\]

for the particular value of \( T \) that has been chosen. Ideally at points on a finite-time slow manifold, the minimum value of \( J \) would be zero, but we use a numerical iterative solution procedure that is stopped once \( J \) is below a specified tolerance. This is repeated for a grid on the independent variable space. The directions of the Lyapunov vectors indicate how to separate the coordinates of \( x \) into independent and dependent variables. The independent variables must be chosen such that their coordinate axes are not parallel to any directions in \( (E^s)^\perp \). Different independent variables might be required in different regions of \( \mathcal{X} \). Within \( \mathcal{X} \), there could be zero, one, or more than one slow manifold. Hence, for fixed values of the independent variables, there could be zero, one or several minima. The set of points satisfying the orthogonality conditions must be further examined to identify whether there is a finite-time slow manifold.

This procedure for determining a slow manifold is much the same as the ILDM method, except for the important difference that FTLE/Vs are used instead of eigenvalues and eigenvectors (in practice ILDM is often implemented using Schur vectors,
an orthogonal basis generated from the eigenvectors). Determining the Lyapunov vector basis for \((E')\) requires more computation than determining the eigenvectors, but the potentially greater accuracy may be needed, if the timescale separation is modest or if the slow manifold has significant curvature, since these are the conditions \([21]\) for which the accuracy of the eigenvector-based ILDM is reduced. The improved accuracy is demonstrated in the following section. Whether or not improved accuracy can indeed be achieved in a numerical implementation of our method on more complicated and/or higher-dimensional applications than those considered in the next section remains to be determined.

5.3. Numerical Methods. The numerical methods used for the computations presented in the next section are described in this subsection. All the computations are done in the Matlab® environment. The numerical integration of the nonlinear state equations and the corresponding linear variational equations is performed with the ‘ode45’ integrator.

The FTLEs and FTLVs associated with an initial state \(\mathbf{x}\) are computed for an averaging time \(T\) either by SVD or QR factorization. Only the computation of the forward-time FTLE/Vs is described, since the computation of the backward-time FTLE/Vs is analogous. The first step of both methods is to integrate the nonlinear state equations from \(t = 0\) to \(t = T\) and save the values of \(\phi(t, \mathbf{x})\) at the \(N\) equally spaced times \(\Delta t, 2\Delta t, \ldots, N\Delta t\), where \(N\Delta t = T\).

In the SVD method, the transition matrix is computed and then the SVD is applied. The transition matrix is computed by integrating, simultaneously, the nonlinear equations and the linear variational equations over each segment of the base space trajectory, with the state initialized with the saved value at the beginning of the segment and the transition matrix initialized with the identity matrix. Using the notation \(\Phi^\Delta_{k\Delta t} = \Phi(\Delta t, \phi((k-1)\Delta t, \mathbf{x}))\) for \(k = 1, 2, \ldots, N\), the transition matrix is constructed from the transition matrices for the segments as \(\Phi(T, \mathbf{x}) = \Phi^\Delta_{N\Delta t} \cdots \Phi^\Delta_{2\Delta t} \Phi^\Delta_{\Delta t}\).

The resulting transition matrix is then factored as \(\Phi(T, \mathbf{x}) = N^+ \Sigma^+ (L^+)^T\) using the ‘svd’ command in Matlab®. Each FTLE, \(\mu_i(T, \mathbf{x})\), is obtained by \(\mu_i^+(T, \mathbf{x}) = \frac{1}{2} \ln \sigma_i^+\), where \(\sigma_i\) is the \(i\)th singular value of \(\Phi\), the positive square root of the \(i\)th diagonal element of \(\Sigma^+\). If this procedure does not produce FTLEs in the ascending order we have assumed in our notation, the procedure is modified to achieve ascending order. The FTLVs \(I_i^+(T, \mathbf{x}), i = 1, \ldots, n\) are the column vectors of \(L^+\).

For a given trajectory from \(\mathbf{x}\) to \(\phi(T, \mathbf{x})\), for a particular \(T\), we have the option of computing certain Lyapunov vectors at \(\mathbf{x}\) and at \(\phi(T, \mathbf{x})\) by forward or backward integration. Because \(\Phi(-T, \phi(T, \mathbf{x})) = \Phi(T, \mathbf{x})^{-1}\), it follows that \(L^+(T, \mathbf{x}) = N^-(T, \phi(T, \mathbf{x}))\) and \(N^+(T, \mathbf{x}) = L^-(T, \phi(T, \mathbf{x}))\).

In the QR method, a segment by segment approach is also used \([6]\). For the \(k\)th-segment, after the transition matrix is computed as described in the previous paragraph, the \(Q_{k-1}\) matrix associated with the state at the end of the previous segment is propagated by the transition matrix to the end of the \(k\)th-segment and the \(Q_k R_k\) factorization of the resulting matrix is obtained, as summarized by

\[
\Phi^\Delta_{k\Delta t} Q_{k-1} = Q_k R_k \tag{5.3}
\]

This sequence of operations for \(k = 1, \ldots, N\) must be initialized by prescribing \(Q_0\); typically the identity matrix is used \([6][10]\). It then follows that

\[
\Phi(T, \mathbf{x}) Q_0 = Q(T, \mathbf{x}) R \tag{5.4}
\]
where \( Q(T, x) = Q_N \) and \( R = R_N R_{N-1} \ldots R_2 R_1 \). Note that if we choose \( Q_o = L^+(T, x) \), then \( Q(T, x) = N^+(T, x) \), or equivalently \( Q(T, x) = L^-(T, \phi(T, x)) \), and \( R = \Sigma^+ \). For almost every \( Q_o \), \( Q(T, \phi(T, x)) \) will approach \( N^+(T, \phi(T, x)) \) and \( R \) will approach \( \Sigma^+ \) in the absence of numerical errors. We have found the QR method to be more numerically reliable. In the following section, both methods produced essentially identical results for the 2D and 3D examples. For the 5D example, we found that the QR method allowed longer averaging times without exhibiting numerical problems.

Adrover et al. [1, 2] have considered two alternative computational approaches. One is based on exterior algebra, that calculates, in addition to the FTLEs, the Lyapunov subspaces directly, rather than calculating vectors that span these subspaces as in the SVD and QR methods. The other propagates bases for the tangent and cotangent spaces along trajectories, with periodic adjustments to keep the basis vectors from all aligning with the direction associated with the largest Lyapunov exponent. We do not have any computational experience with either method.

The approximation to the slow invariant manifold is computed from the orthogonality conditions. We fix the values of \( n_s \) components of the state vector, \( x \), as the independent variables, and solve the unconstrained minimization problem (5.2) to find the values of the remaining \( n - n_s \) components using the ‘fminunc’ function in the Matlab® Optimization Toolbox.

6. Application Examples. Several application examples are presented to demonstrate the use of the finite-time Lyapunov analysis (FTLA) method and compare it to the ILDM method.

6.1. Two-Dimensional Linear Time-Varying Example. The purpose of this example is to distinguish clearly between the timescale information provided by Lyapunov analysis and the timescale information provided by eigen-analysis. Consider the LTI system

\[
\dot{w} = \Lambda w = \begin{bmatrix} \lambda_f & 0 \\ 0 & \lambda_s \end{bmatrix} w,
\]

where \( w = (w_1, w_2)^T \) and the eigenvalues of \( \Lambda \) are real with \( \lambda_f < \lambda_s < 0 \). We introduce the coordinate transformation

\[
x = R(t)w = \begin{bmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{bmatrix} w,
\]

where \( \theta = \omega t \) and \( \omega \) is a constant. In terms of \( x \), the system is

\[
\dot{x} = A(t)x = (\dot{R}R^T + R\Lambda R^T)x = R(R^T \dot{R} + \Lambda)R^T x.
\]

The solution for the \( w \)-system for the initial condition \( w(t_1) = w_1 \) is \( w(t) = \exp(\Lambda(t - t_1))w_1 \) where \( \exp(\Lambda(t - t_1)) \) is the transition matrix. The corresponding transition matrix for the LTV \( x \)-system is

\[
\Phi(t, t_1) = R(t)e^{\Lambda(t-t_1)}R^T(t_1).
\]

In the \( w \)-representation the behavior is composed of fast and slow exponentially contracting modes with fixed directions given by the eigenvectors of \( \Lambda \). In the \( x \)-representation there are also fast and slow exponentially contracting modes, but the fast and slow directions are rotating. We will show that the Lyapunov exponents and
vectors identify the slow and fast exponential modes, whereas in general the eigenvalues and eigenvectors for $A(t)$ and $\Phi$ do not. This example serves as an idealization of the linearized dynamics of a nonlinear system whose slow and fast directions rotate along a trajectory, as would be the case along a trajectory on a slow manifold with curvature. To be more consistent with the notation of the previous section, we use $T$ to denote the propagation time and let $t$ denote the time at which the Lyapunov vectors are computed.

The transition matrix $\Phi(t + T, t)$ for the $x$-system has the SVD

$$\Phi(t + T, t) = N^+(t + T)\Sigma^+(t + T, t)(L^+(t))^T = R(t + T)\exp(\Lambda T)R^T(t).$$

The Lyapunov exponents are $\mu_1^+ = \lambda_f$ and $\mu_2^+ = \lambda_s$ and are independent of $t$ and $T$ for this example. The Lyapunov vectors, independent of $T$, are the columns of $L^+(t) = L^-(t) = R(t)$. The SVD of $\Phi$ thus identifies the exponential rates of the two modes in $\Sigma^+$ and the rotating directions of these modes in $L^+$ and $L^-$. Specifically we have the fast subspace $E^f(t) = L^+_1(t) = \text{span}\{1_1^+(t)\}$ and the slow subspace $E^s(t) = L^+_2(t) = \text{span}\{1_2^+(t)\}$, where the fast direction is $1_1^+(t) = [\cos \theta(t) \sin \theta(t)]^T$ and the slow direction is $1_2^+(t) = [-\sin \theta(t) \cos \theta(t)]^T$. In this example the rotational motion is periodic and Floquet theory is applicable; but, the SVD would also characterize different and irregular rotations of the fast and slow directions.

The eigenvalues of $A(t) = (\dot{R}RR^T + R\Lambda R^T)$ are \[^3\] the same as the eigenvalues of the matrix

$$(R^T \dot{R} + \Lambda) = \begin{bmatrix} \lambda_f & -\omega \\ \omega & \lambda_s \end{bmatrix},$$

because the two matrices are related by a similarity transformation. The two eigenvalues of $A(t)$ are

$$1/2 \left( \lambda_s + \lambda_f \pm ( (\lambda_s - \lambda_f)^2 - 4\omega^2 )^{1/2} \right),$$

and are denoted by $\lambda_+$ and $\lambda_-$ based on which sign is taken. The corresponding eigenvectors of $A(t)$ are

$$v_+ = R(t) \begin{bmatrix} -\omega \\ \lambda_+ - \lambda_f \end{bmatrix},$$

$$v_- = R(t) \begin{bmatrix} \lambda_- - \lambda_s \\ \omega \end{bmatrix}.$$
cos ωT = 0, the eigenvalues are pure imaginary; only for T such that cos ωT = 1 are the eigenvalues e^{λ_1 T} and e^{λ_2 T}. Hence the eigenvalues of A and Φ are not always reliable indicators of the fast and slow exponential modes. The eigenvectors of A provide the directions associated with distinct exponential rates only if these directions change slowly enough.

6.2. Davis-Skodje 2D System: Attracting Slow Manifold. Davis and Skodje (DS) [5] introduced a two-dimensional nonlinear system for which the slow manifold can be determined analytically. The DS system is

\[
\begin{align*}
\dot{x}_1 &= -x_1 \\
\dot{x}_2 &= -γx_2 + \frac{(γ-1)x_1 + γx_1^2}{1+x_1}
\end{align*}
\]  

(6.10)

defined on the state space \( \{(x_1, x_2) ∈ \mathbb{R}^2 : x_1 ≥ 0 \text{ and } x_2 ≥ 0\} \) with constant \( γ > 1 \). The origin is a globally attracting equilibrium point, but more importantly in the present context, for sufficiently large \( γ \), trajectories are first attracted on a faster timescale to the 1D slow manifold

\[
S = \{(x_1, x_2) ∈ \mathbb{R}^2 : x_2 = x_1/(1 + x_1)\},
\]  

(6.11)

and then follow \( S \) to the origin on a slower timescale. The two timescales are evident in the analytic solution for the flow associated with the vector field in (6.10)

\[
\phi(t; x_1, x_2) = \begin{bmatrix}
x_1 e^{-t} \\
\left(x_2 - \frac{x_1}{1+x_1}\right) e^{-γt} + \frac{x_1}{1+x_1} e^{-t}
\end{bmatrix}.
\]  

(6.12)

Note that if the initial state is on the slow manifold, there is no fast timescale behavior because the coefficient of \( e^{-γt} \) in (6.12) is zero. The slow manifold \( S \) and several other trajectories are shown in Fig. 6.1 for \( γ = 10 \). The time interval between asterisks on the trajectories is 0.1, illustrating faster motion off \( S \) than on \( S \). From the analytical representation (6.11) for the slow manifold, we know that for any \( x ∈ S \),

\[
T_xS = \text{span}\{(1 + x_1)^2 1^T\}.
\]  

(6.13)

The linearized dynamics of the DS system are given by \( \dot{v} = Df(x)v \), where

\[
Df = \begin{bmatrix}
-1 & 0 \\
(γ-1+γx_1)/(1+x_1)x_1 & -γ
\end{bmatrix}.
\]  

(6.14)

Given the presence of the equilibrium point, other approaches based on eigenanalysis at the equilibrium point are applicable: for example, integrating (6.10) backward from an initial state perturbed slightly from the origin in the direction of the eigenvector associated with the largest eigenvalue to compute \( S \). However our purpose here is to demonstrate the methodology developed in this paper, methodology that does not require the presence of an equilibrium point.

6.2.1. Finite-Time Lyapunov Analysis Method. We now demonstrate the numerical application of our approach over a finite-time interval for the case \( γ = 3 \), the case also investigated in [5]. We consider the set \( X = \{(x_1, x_2) ∈ \mathbb{R}^2 : 0 ≤ x_1 ≤ 2.0 \)
and $0 \leq x_2 \leq 1.0$} and check if the system (6.10) with $\gamma = 3.0$, satisfies the conditions in Definition 4.1 for a finite-time uniform two-timescale set. Figure 6.2 shows the superposition of the forward and backward FTLEs, as functions of $T$, for a uniform grid of points in $\mathcal{X}$; there is one fast-contracting exponent and one slow exponent uniformly on $\mathcal{X}$. Taking $\beta = 4.0$, $\mu^s = 1.0$, $\mu^f = 3.0$, $n^{fc} = 1$, $n^s = 1$, $n^f = 0$, $T_o = 0.02$, and $T_c \geq 2.0$, we find that the Definition 4.1 conditions are satisfied and we conclude that $\mathcal{X}$ is a uniform two-timescale set. We note that with $\beta = 4.0$, $T_c = 2.0$ is sufficient. For the DS system, it can be verified that the timescale behavior is globally uniform, so that there is no upper limit on $T_c$. The FTLVs that approximate the fast and slow directions are $\mathbf{l}^{+1}(T, x)$ and $\mathbf{l}^{-2}(T, x)$. The approximations $E^{fc}(x) \cong \text{span}\{\mathbf{l}^{+1}(T, x)\}$ and $E^s(x) \cong \text{span}\{\mathbf{l}^{-2}(T, x)\}$ improve with increasing $T$ sufficiently rapidly that, using $T = 2.0$, accurate approximations of the invariant subspaces $E^{fc}(x)$ and $E^s(x)$ can be achieved, and the invariant fast-slow splitting can be accurately approximated by

$$
T_x \mathbb{R}^2 = E^{fc}(x) \oplus E^s(x) \cong \text{span}\{\mathbf{l}^{+1}(2.0, x)\} \oplus \text{span}\{\mathbf{l}^{-2}(2.0, x)\}. 
$$

Figure 6.3 shows points that are solutions to the orthogonality condition $\langle f(x), \mathbf{l}^{-1}(T, x) \rangle = 0$ for averaging time $T = 2.0$ illustrating that the FTLA method provides an accurate approximation of $\mathcal{S}$. Both QR and SVD methods produced essentially identical results.

In Fig. 6.4 the slow manifold tangent direction is compared to the approximation offered by the slow Lyapunov vector $\mathbf{l}^{-2}$ at points along the slow manifold $\mathcal{S}$, indexed by $x_1$. The Lyapunov vector $\mathbf{l}^{-2}$ provides a uniformly accurate approximation when a sufficiently large averaging time is used. $T = 2.0$ is large enough here, whereas $T = 0.2$ is not.

6.2.2. Asymptotic Lyapunov Analysis. For the DS system, because the timescale structure is uniform on the entire state space (positive quadrant), the progress toward convergence in the first 2 units of time continues and it is possible to compute the asymptotic Lyapunov exponents and vectors. The infinite-time
limits of the FTLEs can be determined analytically to be $\mu_1^+ = -\gamma$, and $\mu_2^+ = -1$. The backward time limits are $(\mu_1^-, \mu_2^-) = (\gamma, 1) = (-\mu_1^+, -\mu_2^+)$. The fast and slow stable directions on the tangent plane at a point $x = (x_1, x_2)$, are given by $l_1^+(T, x)$ and $l_2^-(T, x)$ respectively. We can analytically compute $l_2^-(T, x)$ as the eigenvector of $\Phi(-T, x)^T\Phi(-T, x)$ corresponding to the slow exponent in backward time, $\mu_2^-(T, x)$. As time goes to negative infinity, $l_2^-(T, x)$ can be shown to converge to

$$l_2^-(x) = \alpha(x_1, x_2) \begin{bmatrix} (1 + x_1)^2 \\ 1 \end{bmatrix}$$

(6.16)

where $\alpha(x_1, x_2)$ is a scalar function. For $l_2^-$ to be a unit vector, $\alpha(x_1, x_2)$ should be chosen appropriately. Similarly, as $T$ goes to infinity, $l_1^+(T, x)$ can be shown to converge to

$$l_1^+(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(6.17)

independent of $x$.

If a point $x$ is on $S$, then, using the asymptotic Lyapunov vector $l_1^-(x)$, the orthogonality condition characterizing points on $S$ is in agreement with (6.11). These asymptotic results lend credence to the finite-time results, but the most important message is that in 2 units of time, the two-timescale behavior can be diagnosed and an accurate approximation of the slow manifold can be obtained.
6.2.3. Invariant Slow Manifold Approximation Using Eigenvectors (ILDM Method). The eigenvalues of $Df$ in (6.14) are $-\gamma$ and $-1$; in this case they indicate the two-timescale behavior correctly. Assuming that the eigenvector, denoted $e_s$, associated with the slow eigenvalue $-1$, spans the slow subspace of the tangent plane, the ILDM method [29] estimates points on $S$ by computing solutions to the orthogonality condition $\langle f(x), (e_s)^\perp \rangle = 0$, where $(e_s)^\perp$ is orthogonal to $e_s$. The slow eigenvector $e_s$ can be obtained analytically and is

$$e_s = \begin{bmatrix} (1 + x_1)^3 \\ 1 + \frac{(\gamma+1)}{\gamma-1} x_1 \end{bmatrix}$$

(6.18)

The ILDM approximation to the slow manifold is

$$x_2 = \frac{x_1}{1 + x_1} + \frac{2x_1^2}{\gamma^2} \left[ \frac{1}{(1 - \frac{1}{\gamma})(1 + x_1)^3} \right]$$

(6.19)

and is also shown in Fig. 6.3. The ILDM approximation is accurate around the equilibrium point (small $x_1$) but gets worse away from the origin. The error is proportional to $\varepsilon^2$, where $\varepsilon = 1/\gamma$, consistent with the analysis of the ILDM method in [21]. In Fig. 6.4 the slow manifold tangent direction, specifically the angle of the tangent relative to the direction of the $x_1$-axis, is compared to the approximations offered by the slow eigenvector $e^s$ and the slow Lyapunov vector $L^-_2$ at points along the slow manifold.
$S$, indexed by $x_1$. As expected, the slow eigenvector accurately approximates the slow manifold tangent direction near the equilibrium point (not shown in Fig. 6.4), but is in error farther away from the equilibrium point.

6.3. 3D Nonlinear System: Hyperbolic Slow Manifold. Consider a nonlinear time invariant (NTI) system

$$\begin{align*}
\dot{x}_1 &= \lambda_s x_1 \\
\dot{x}_2 &= \lambda_{fc} x_2 + \alpha_1 (\lambda_{fc} - 2\lambda_s) x_1^2 \\
\dot{x}_3 &= \lambda_{fe} x_3 + \alpha_2 (\lambda_{fe} - 2\lambda_s) x_1^2 
\end{align*}$$

(6.20)

For the numerical results, the constants are assigned the values $\lambda_s = -0.2$, $\lambda_{fc} = -3$, $\lambda_{fe} = 3$, and $\alpha_1 = \alpha_2 = 2$.

6.3.1. Finite-Time Lyapunov Analysis Method. First the FTLEs are computed on a grid on the cubic region $X = [-10, 10]^3$ in the state space $\mathbb{R}^3$. Figure 6.5 shows a superposition of all the forward and backward FTLEs for different averaging times for the values of $x$ on the $X$ grid, indicating that $X$ has uniform timescale structure. We see from Figure 6.5 that there is one fast stable exponent, one slow exponent, and one fast unstable exponent. Taking $\beta = 6$, $\mu^s = 0.8$, $\mu^f = 3.5$, $n^s = n^f = n^u = 1$, $T_o = 0.002$, $T_c = 3.0$, the system satisfies the conditions given in Definition 4.1 to be a uniform two-timescale set. $T = 3.0$ will provide accurate FTLVs based on the bound given in Theorem 3.5.

Having diagnosed two timescales and both fast contracting and fast expanding behavior, there may be a 1D slow manifold and, if so, it is hyperbolic. Because there is sufficient averaging time for $(E^\delta(T, x))^\perp = \text{span}\{l_1^-(T, x), l_1^+(T, x)\}$, the application of the general result (5.1), to be a good approximation of the orthogonal complement to the corresponding invariant slow subspace, good approximations to the orthogonality conditions for points on slow manifold can be formulated.

For the point $x = [5 \ 10 \ 10]^T$, Fig. 6.6 shows, as $T$ increases, the distance, as defined in Proposition 3.2, between $l_1^-(T, x)$ and $l_1^-(T + \Delta T, x)$ and $l_1^+(T, x)$ and
Fig. 6.5. Superposition of finite-time Lyapunov exponents in forward and backward time for grid on \( X \). For forward time, the curves are red: \( \mu^+_1(T,x) \), green: \( \mu^+_2(T,x) \), blue: \( \mu^+_3(T,x) \). For backward time, the curves are red: \( \mu^-_3(T,x) \), green: \( \mu^-_2(T,x) \), blue: \( \mu^-_1(T,x) \).

\[
\begin{align*}
\mu^+_1(T+\Delta T, x) \quad \text{respectively, for } \Delta T = 0.006, \text{ confirming that the exponential bound given in Theorem 3.5 is satisfied. To compute the bound, we calculated the maximum eigenvalue of the symmetric part of the Jacobian matrix on the grid, and took } T_1 = 0.006. \text{ The bound given in the theorem is conservative as stated earlier. In the plot, the bound has been reduced by a factor of four, to make it tighter. Although the FTLVs vary with } x, \text{ the convergence behavior as } T \text{ increases is represented uniformly over } X \text{ by the behavior at } [5 10 10]^T. \text{ The Lyapunov vectors for } x = [5 10 10]^T \text{ and } T = 3.0 \text{ are}
\end{align*}
\]

\[
\begin{align*}
I^+_1 &= [0.00 \ -1.00 \ 0.00]^T, \quad I^+_2 = [0.05 \ 0.00 \ -1.00]^T, \quad I^+_3 = [1.00 \ 0.00 \ 0.05]^T \\
I^-_1 &= [1.00 \ 0.05 \ 0.00]^T, \quad I^-_2 = [0.05 \ -1.00 \ 0.00]^T, \quad I^-_3 = [0.00 \ 0.00 \ 1.00]^T
\end{align*}
\]

Based on the results in (6.21) and the FTLVs at the other grid points of \( X \), we chose \( x_1 \) as the independent variable, because its coordinate axis is not parallel to any of the directions in \( (E^s(T,x))^\perp \). For each of the values on the \( x_1 \) grid, we compute the values of \( x_2 \) and \( x_3 \) that satisfy the orthogonality conditions. The resulting finite-time approximation of the slow manifold for values of \( x_1 \) from -10 to 10 is plotted in Figure 6.7.

6.3.2. Exact Slow Manifold. For this problem, there is an independent means of determining the slow manifold, which allows the accuracy of the FTLA approach to be assessed. Over a time interval short relative to the fast timescale, yet long relative to the slow timescale, trajectories approach the 2D manifolds \( \mathcal{M}^+ \) and \( \mathcal{M}^- \), in forward (+) and backward (-) time respectively, given by
The intersection of these sets is the slow manifold: that is \( S = \mathcal{M}^+ \cap \mathcal{M}^- \). These manifolds and their intersection are shown in Fig. 6.8. This intersecting of manifolds in the state-space (i.e., in the base space) is the counterpart of the subspace intersections in the tangent space employed in the FTLA method.

At a point \( x \in S \), the vectors normal to \( \mathcal{M}^+ \) and \( \mathcal{M}^- \) are given by \( \eta_1 = [2\alpha_1 x_1 1 0]^T \) and \( \eta_2 = [2\alpha_2 x_1 0 1]^T \) respectively. Points on \( S \), due to its invari-
ance with respect to the flow, satisfy the orthogonality conditions

\begin{align}
0 &= < \eta_1, f(x) > \\
&= < [2\alpha_1 x_1 \ 1 \ 0]^T, f(x) > \\
&= 2\alpha_1 \lambda_s x_1^2 + \lambda_{fe} x_2 + \alpha_1 (\lambda_{fe} - 2\lambda_s) x_1^2 \\
&= \lambda_{fe} (x_2 + \alpha_1 x_1^2) \\
\end{align}

\begin{align}
0 &= < \eta_2, f(x) > \\
&= < [2\alpha_2 x_1 \ 0 \ 1]^T, f(x) > \\
&= 2\alpha_2 \lambda_s x_1^2 + \lambda_{fe} x_3 + \alpha_2 (\lambda_{fe} - 2\lambda_s) x_1^2 \\
&= \lambda_{fe} (x_3 + \alpha_2 x_1^2)
\end{align}

(6.23)

where \( f(x) \) is the vector field given in (6.20). We now show that the FTLVs conform to this geometry. As \( T \rightarrow \infty \), \( \mathbf{T}_s^T (T, x) \) and \( \mathbf{I}_s^T (T, x) \) converge to tangent vectors of \( T_x M^+ \) and \( I_s^{-1} (T, x) \) converge to tangent vectors of \( T_x M^+ \). It follows that as \( T \) increases, \( \mathbf{I}_s^T \) should approach \( \eta_2 \) and \( \mathbf{I}_s^{-1} \) should approach \( \eta_1 \), as the results in (6.21) indicate. For a given \( x_1 \), if we let \( (x_1, \hat{x}_2, \hat{x}_3) \) denote the approximation of the slow manifold point \( (x_1, 2x_1^2, 2x_1^2) \), then approximation error is \( [(x_2 + 2x_1^2)^2 + (x_3 + 2x_1^2)^2]^{1/2} \).

We see that the error for FTLA approximation with \( T = 3.0 \), plotted in Fig. 6.9 is significantly smaller than that for the ILDM approximation. Figure 6.10 shows that the FTLVs closely approximate the normal vectors to the exact slow manifold for \( T > 2 \).

6.3.3. Invariant Slow Manifold Approximation Using Eigenvectors (ILDM Method). Applying the ILDM method, it is assumed that the eigenvector corresponding to the slow eigenvalue spans the slow subspace. The matrix for the linear variational equations corresponding to the system (6.20) is given by

\[
Df = \begin{bmatrix}
\lambda_s & 0 & 0 \\
2\alpha_1 (\lambda_{fe} - 2\lambda_s) x_1 & \lambda_{fe} & 0 \\
2\alpha_2 (\lambda_{fe} - 2\lambda_s) x_1 & 0 & \lambda_{fe}
\end{bmatrix}
\]

(6.24)

the eigenvector corresponding to the slow eigenvalue, \( \lambda_s \), can be written as

\[
v_s = \begin{bmatrix} 1 & -2\alpha_1 (1 - 2\frac{\lambda_s}{\lambda_{fe}}) x_1 & -2\alpha_2 (1 - 2\frac{\lambda_s}{\lambda_{fe}}) x_1 \end{bmatrix}^T
\]

(6.25)

Two linearly independent vectors orthogonal to \( v_s \) are

\[
(v_s^1)_1 = \begin{bmatrix} 2\alpha_1 x_1 - 4\frac{\lambda_s}{\lambda_{fe}} \alpha_1 x_1 & 1 \end{bmatrix}^T, (v_s^1)_2 = \begin{bmatrix} 2\alpha_1 x_1 - 4\frac{\lambda_s}{\lambda_{fe}} \alpha_1 x_1 & 0 & 1 \end{bmatrix}^T
\]

(6.26)
Points on the slow manifold are characterized as solutions to the orthogonality conditions

\[ \langle (v_s^+)_1, f(x) \rangle = 0, \quad \langle (v_s^+)_2, f(x) \rangle = 0 \quad (6.27) \]

For a given \( x_1 \), the magnitudes of the errors in \( x_2 \) and \( x_3 \) relative to the correct values for \( S \) are \( 4\alpha_1(\lambda_s/\lambda_{fc})^2x_1^2 \) and \( 4\alpha_2(\lambda_s/\lambda_{fe})^2x_1^2 \) respectively. The slow manifold approximation error, as defined in section 6.3.3, for the ILDM method is plotted in Fig. 6.9. The ILDM error is similar to that for the FTLA method when the averaging time is short, but the FTLA method gives greater accuracy as the averaging time is increased.

6.3.4. Roussel-Fraser Method. The Roussel-Fraser method is based on parameterizing the slow manifold as a graph and solving PDEs to obtain the graph. Parameterizing the slow manifold requires separating the state coordinates appropriately into independent and dependent variables for the graph. The FTLE/Vs provide a general means; however in this problem, the eigenvalues and eigenvectors suggest the appropriate choices. Assuming we have determined that \( x_1 \) is the independent variable and \( x_2 \) and \( x_3 \) are the dependent variables, we get the following PDEs

\[ \epsilon_1 \frac{\partial x_2}{\partial x_1} = \frac{x_2}{x_1} + (1 - 2\epsilon_1)\alpha x_1^2 \quad (6.28) \]
\[ \epsilon_2 \frac{\partial x_3}{\partial x_1} = \frac{x_3}{x_1} + (1 - 2\epsilon_2)\alpha x_1^2 \quad (6.29) \]

where \( \epsilon_1 := \lambda_s/\lambda_{fc} \), and \( \epsilon_2 := \lambda_s/\lambda_{fe} \). In general, the solution of the PDEs that arise in this method must be approached with numerical methods, and may be problematic.
even then \[5\], but in this case the solution is straightforward. Assuming \( x_2(x_1) = \sum_{i=0}^{\infty} \epsilon_i^2 x_i \) and \( x_3(x_1) = \sum_{i=0}^{\infty} \epsilon_i^3 x_i \), we obtain \( x_2 = -\alpha x_1^2 \) and \( x_3 = -\alpha x_1^2 \). This is the correct graph for the slow invariant manifold.

6.4. 5D System: Chaotic Attractor. The dynamics of a five-mode truncation of the Navier-Stokes equations for a two-dimensional incompressible fluid on a torus were studied by Fracceschi and Tebaldi \[9\] and later by Adrover et al. \[1\]. The five-dimensional dynamics depending on the constant parameter \( r \), the Reynolds number, are

\[
\begin{align*}
\dot{x}_1 &= -2x_1 + 4x_2x_3 + 4x_4x_5, \\
\dot{x}_2 &= -9x_2 + 3x_1x_3, \\
\dot{x}_3 &= -5x_3 - 7x_1x_2 + r, \\
\dot{x}_4 &= -5x_4 - x_1x_5, \\
\dot{x}_5 &= -x_5 - 3x_1x_4.
\end{align*}
\]

For \( r = 33 \), the case investigated here and in \[1\], there is a chaotic attractor.

We demonstrate the diagnostics derivable from FTLA, as a particular trajectory is followed in the state space. The trajectory begins at \( x = (1, 10, 1, 10, 10) \). The exponents corresponding to this trajectory, for averaging times \( T \) up to 15, are shown in Fig. 6.11 and were computed using the QR method. The FTLEs for \( T = 2.0 \) were determined to be \( \mu_1^+ = -11.61, \mu_2^+ = -7.04, \mu_3^+ = -2.87, \mu_4^+ = -1.66, \) and \( \mu_5^+ = 1.18 \); and the FTLEs for \( T = 15.0 \) were determined to be \( \mu_1^+ = -12.63, \mu_2^+ = -8.27, \mu_3^+ = -1.35, \mu_4^+ = -0.17, \) and \( \mu_5^+ = 0.42 \). For comparison, Adrover et al. \[1\], using an exterior-algebra based method \[2\], determined the Lyapunov exponents to be \( \mu_1^+ = -11.97, \mu_2^+ = -9.1, \mu_3^+ = -1.27, \mu_4^+ = 0, \) and \( \mu_5^+ = 0.34 \), the sum of which is -22.0, for motion on the chaotic attractor. Their exponents are intended
to approximate the asymptotic Lyapunov exponents; for example, using their other computational method, they \cite{1} use an averaging time of T=60. Because of our short averaging times, there should be no expectation that the FTLEs should match the asymptotic values. Also properties associated with the asymptotic exponents such as independence of the initial state \( x \) and a zero exponent associated with the direction of the vector field \( f(x) \) (if the attractor is a compact, invariant set without fixed points, e.g., a periodic orbit) will not in general apply to the FTLEs.

In the basis provided by the backward Lyapunov vectors, we can write

\[
    f(x) = \sum_{i=1}^{5} w_i(x) l_i^{-}(T, \phi(T, x))
\]

where \( w_1, w_2, w_3, w_4, \) and \( w_5 \) are the coordinates of \( f(x) \) in the backward FTLV basis. The Euclidean distances from \( f(x) \) to the backward Lyapunov subspaces defined in \eqref{3.2} are

\[
    
    
    \begin{align*}
    d_1 &= \text{dist}(f(\phi(T, x)), \mathcal{L}^{-}_2(T, \phi(T, x))) = |w_1|, \\
    d_2 &= \text{dist}(f(\phi(T, x)), \mathcal{L}^{-}_3(T, \phi(T, x))) = (w_1^2 + w_2^2)^{1/2}, \\
    d_3 &= \text{dist}(f(\phi(T, x)), \mathcal{L}^{-}_4(T, \phi(T, x))) = (w_1^2 + w_2^2 + w_3^2)^{1/2}, \\
    d_4 &= \text{dist}(f(\phi(T, x)), \mathcal{L}^{-}_5(T, \phi(T, x))) = (w_1^2 + w_2^2 + w_3^2 + w_4^2)^{1/2}.
    \end{align*}
\]

We consider the system behavior along this trajectory over 2 units of time. The results in Fig. 6.12 show how these distances evolve with time. The distances \( d_1, d_2, \) and \( d_3, \) are analogous, but not equivalent, to the “mode amplitudes” in the CSP method \cite{25, 26} and in the work of Adrover et al. \cite{1}; we will refer to them as mode amplitudes anyway. The interpretation of Fig. 6.12 is not straightforward because (i) as time increases we are looking at what is going on in the tangent space at different points along the trajectory and (ii) as we move along the trajectory, the subspaces are converging to fixed subspaces due to the benefit of having averaged over a longer time interval. The mode amplitudes \( d_1, d_2, \) and \( d_3 \) of \( f(x) \) decay by 5 time constants respectively at the times 0.43, 0.71, and 3.00. The results shown in Fig. 6.12 indicate that the two fastest contracting “modes” have decayed significantly. However, for the decay of a particular mode to be relative to a fixed subspace, the subspace the mode amplitude is measured relative to must have converged (in a numerical sense, 5 time constants being the criterion we are using). The convergence bound in Theorem \ref{3.5} allows us to estimate that \( \mathcal{L}^{-}_2(T, \phi(T, x)) \), \( \mathcal{L}^{-}_3(T, \phi(T, x)) \) and \( \mathcal{L}^{-}_4(T, \phi(T, x)) \) will converge in times of 1.09, 1.20, and 4.13, respectively, using 5 time constants for the appropriate \( \Delta \mu \) in each case. \( \mathcal{L}^{-}_2(T, \phi(T, x)) \) and \( \mathcal{L}^{-}_3(T, \phi(T, x)) \) converge sufficiently fast, as do the associated mode amplitudes, \( d_1 \) and \( d_2 \), that the decay of those modes is apparent in Fig. 6.12 and there is a clear indication that the trajectory has reached a 3D manifold by \( T = 2 \) with tangent space given by \( \mathcal{L}^{-}_2(2, \phi(2, x)) \) at the base point \( \phi(2, x) \). On the other hand, for \( d_3 \), though the decay rate is large enough to be observed within 2 units of time, the subspace \( \mathcal{L}^{-}_4(T, \phi(T, x)) \) that it is measured relative to takes 4.13 units of time to converge, so it is not resolved by \( T = 2 \).

Due to their convergence rates, for the second half of the time interval in Fig. 6.12 \( \mathcal{L}^{-}_2(T, \phi(T, x)) \) and \( \mathcal{L}^{-}_3(T, \phi(T, x)) \) are very close to the fixed subspaces they would converge to if \( T \) were increased even farther and the timescale behavior were uniform. The FTLVs used to represent these subspaces were calculated by forward integration using the QR method with the initial state always \( x = (1, 10, 1, 10, 10) \). If we wanted better accuracy for the subspaces at trajectory points corresponding to the times between 0 and 1, we would back up the initial condition along the trajectory under
consideration. For example to double $T$, one would start at $\phi(-T, x)$ and obtain $L_2(2T, \phi(T, x))$ by computing $N^+_2(2T, \phi(-T, x))$ with forward integration.

This analysis for a single trajectory has allowed us to demonstrate the use of FTLA to produce the type of results sought by CSP and the method of Adrover et al., and to illustrate the utility of the additional subspace convergence information. The computation of the FTLE/Vs for this example was more challenging. Whereas both the SVD and QR methods worked well and gave essentially identical results in the previous two examples, for this one the QR method permitted sufficiently large averaging time, whereas the SVD method did not. Further analysis of the system could involve computing FTLE/Vs more extensively and using the information to determine points on the lower dimensional manifolds to which trajectories are attracted, including the chaotic attractor.

![Finite-time Lyapunov exponents for the FT system with $r = 33.0$, for the initial condition $x = (1, 10, 1, 10, 10)$.](image)

**Fig. 6.11.** *Finite-time Lyapunov exponents for the FT system with $r = 33.0$, for the initial condition $x = (1, 10, 1, 10, 10)$.*

# 7. Conclusions

Two-timescale behavior of a finite dimensional, nonlinear time-invariant dynamical system on a not necessarily invariant subset of the state-space has been defined in terms of finite-time Lyapunov exponents and vectors, in a manner guided by the asymptotic theory of partially hyperbolic sets. Two-timescale behavior is characterized by a gap in the spectrum of finite-time Lyapunov exponents. There is a corresponding splitting of the tangent bundle into slow and fast subbundles defined by the finite-time Lyapunov vectors. The other desired property of a slow-fast splitting is that it is invariant under the linearized flow. In principle, determining an invariant slow-fast splitting requires computing asymptotic limits of the finite-time Lyapunov exponents. However, building on previous results, we have shown that under certain conditions the finite-time slow-fast splitting approaches an invariant slow-fast splitting exponentially fast as the time interval, over which the finite-time
Lyapunov exponents and vectors are calculated, increases. The larger the spectral gap is, the faster the convergence. This is an important step toward establishing the feasibility of using finite-time Lyapunov exponents and vectors for timescale analysis. We have also provided evidence that the finite-time Lyapunov exponents and vectors more accurately characterize the timescales and associated geometric structure of the state-space than do the eigenvalues and eigenvectors associated with the “frozen-time” linear flow.

When the tangent bundle has a slow-fast splitting, a slow manifold may exist. One approach for computing a slow manifold is to identify state-space points where the vector field is orthogonal to the directions normal to the slow subspace. In the intrinsic low-dimensional manifold method, the normal directions are calculated approximately from the eigenvectors of the Jacobian matrix associated with the vector field, using the eigenvalues of this matrix to identify slow and fast directions. The alternative of determining the normal directions from finite-time Lyapunov exponents and vectors offers the potential for greater accuracy in determining a slow manifold. This advantage has been demonstrated in several application examples of increasing dimension and complexity. The examples illustrated that, consistent with existing theory, the accuracy of the eigenvector-based approach decreases as the curvature of the slow manifold increases and as the spectral gap decreases. The finite-time Lyapunov analysis method can yield more accurate normal directions even when there is significant curvature in the slow manifold. It can also yield accurate normal directions for a small spectral gap, if the gap is large enough relative to the available averaging time.

Finite-time Lyapunov analysis of the linear variational equations provides an alternative diagnostic approach to eigen-analysis of the associated system matrix (the Jacobian matrix associated with the vector field). Though we have used this finite-time information to improve the performance of the intrinsic low-dimensional manifold type approach for determining points on a slow manifold, the finite-time information
could potentially be used (a) to suggest a transformation of coordinates leading to the standard form required for the analytical singular perturbation approach, (b) to initialize the basis vectors in the computational singular perturbation method, and (c) to guide the selection of independent and dependent variables in the application of the Roussel-Fraser partial differential equation approach.

Further attention to numerical algorithms and additional application experience, along with direct comparison with other slow manifold determination methods, are needed to complete the development and assessment of the methodology. Although in this paper we have stopped with the computation of points on the slow manifold, our ultimate objective is to translate the geometric structure into reduced-order models.

Acknowledgments. Stimulating discussions with S.-H. Lam started the first author on this research. Helpful discussions with L.-S. Young and Y. B. Pesin are gratefully acknowledged. Discussions with B. Villac during the course of revising the paper led to significant improvements in the paper.

REFERENCES

[1] A. Adrover, F. Creta, M. Giona, M. Valorani and V. Vitacolonna, Natural tangent dynamics with recurrent biorthonormalizations: A geometric computational approach to dynamical systems exhibiting slow manifolds and periodic/chaotic limit sets, Physica D, (2006), pp. 121 – 146.

[2] A. Adrover, S. Cerbelli and M. Giona Exterior algebra-based algorithms to estimate Lyapunov spectra and stretching statistics in high dimensional and distributed systems, International Journal of Bifurcation and Chaos, 12(2) (2002), pp. 353–368.

[3] U. M. Ascher, R. M. M. Mattheij, and R. D. Russell, Numerical Solution of Boundary Value Problems for Ordinary Differential Equations, SIAM, Philadelphia, 1995.

[4] L. Barreira and Y. B. Pesin, Lyapunov Exponents and Smooth Ergodic Theory, University Lecture Series, Vol. 23, American Mathematical Society, Providence, 2002.

[5] M. J. Davis and R. T. Skodje, Geometric investigation of low-dimensional manifolds in systems approaching equilibrium, J. Chemical Physics, 111 (1999), pp. 859–874.

[6] L. Dieci and E. S. Van Vleck, Lyapunov spectral intervals: theory and computation, SIAM J. Numerical Analysis, 40(2) (2002), pp. 516–542.

[7] R. Doerner, B. Hübinger, W. Martienssen, S. Grossmann, and S. Thomae, Stable manifolds and predictability of dynamical systems, Chaos, Solitons, and Fractals, 10(11) (1999), pp. 1759–1782.

[8] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations, J. Differential Equations, 31 (1979), pp. 53–98.

[9] V. Frenceshini and C. Tebaldi, Sequences of infinite bifurcations and turbulence in a five mode truncation of the Navier-Stokes equations, J. Statistical Physics, 27 (1987), pp. 311-337.

[10] G. H. Golub and C. F. Van Loan, Matrix Computations, 3rd Edition, The Johns Hopkins University Press, Baltimore, 1996.

[11] I. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, New York, 1983, p. 127.

[12] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, New York, 1983, p. 127.

[13] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, New York, 1983, p. 127.

[14] J. Haller, Finding finite-time invariant manifolds in two-dimensional velocity fields, Chaos, 10(1) (2000), pp. 99–108.

[15] J. Haller, Distinguishing material surfaces and coherent structures in three-dimensional fluid flows, Physica D, 149 (2001), pp. 248–277.
[18] B. Hasselblatt and Y. B. Pesin, *Partially Hyperbolic Dynamical Systems*, in B. Hasselblatt and A. Katok (Eds.), Handbook of Dynamical Systems, Vol. 1B, Elsevier, New York, 2005.
[19] A. Isidori, *Nonlinear Control Systems*, 3rd Edition, Springer-Verlag, London, 1995, pp. 21.
[20] C. K. R. T. Jones, *Geometric Singular Perturbation Theory*, in R. Johnson (Ed.), Dynamical Systems, Lecture Notes in Math 1609, Springer, Berlin, 1995.
[21] H. G. Kaper and T. J. Kaper, *Asymptotic analysis of two reduction methods for systems of chemical reactions*, Physica D, 165 (2002), pp. 66–93.
[22] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, New York, 1995.
[23] H. J. Kelley, *Aircraft maneuver optimization by reduced-order approximations*, Control and Dynamic Systems, ed. C. T. Leondes, Academic Press, New York, 1973, pp. 131–178.
[24] P. V. Kokotovic, H. K. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design*, Academic Press, New York, 1986.
[25] S. H. Lam, *Singular perturbation for stiff equations using numerical computations*, Lectures in Applied Mathematics, 24 (1986), pp. 3–19.
[26] S. H. Lam and D. A. Goussis, *The CSP method for simplifying kinetics*, Int. J. Chemical Kinetics, 26 (1994), pp. 461–486.
[27] E. N. Lorenz, *The local structure of a chaotic attractor in 4-dimension*, Physica D, 13 (1984), pp. 90–104.
[28] A. M. Lyapunov, *The general problem of stability of motion*, Intern. J. of Control, 55(3) (1992), pp. 531–773. (reprint of Lyapunov's 1892 Thesis)
[29] U. Maas and S. B. Pope, *Simplifying chemical kinetics: intrinsic low-dimensional manifolds in composition space*, Combustion and Flame, 88 (1992), pp. 239–264.
[30] K. D. Mease, *Geometry of computational singular perturbations*, in Nonlinear Control System Design, Vol. 2, A. J. Krener and D. Q. Mayne, Oxford, UK, Pergamon, 1996, pp. 855–861.
[31] K. D. Mease, S. Bharadwaj, and S. Iravanchy, *Timescale analysis for nonlinear dynamical systems*, J. Guidance, Control and Dynamics, 26 (2003), pp. 318–330.
[32] K. D. Mease, *Multiple timescales in nonlinear flight mechanics: diagnosis and modeling*, Applied Mathematics and Computation, 164 (2005), pp. 627–648.
[33] D. S. Naidu and A. J. Calise, *Singular perturbations and timescales in guidance and control of aerospace systems: a survey*, J. Guidance, Control and Dynamics, 24(6) (2001), pp. 1057–1078.
[34] R. E. O'Malley, *Singular Perturbation Methods for Ordinary Differential Equations*, Springer-Verlag, New York, 1991.
[35] Z. Ren and S. B. Pope, *The geometry of reaction trajectories and attracting manifolds in composition space*, Combustion Theory and Modeling, 10(3) (2006), pp. 361–388.
[36] M. R. Roussel and S. J. Fraser, *Geometry of the steady-state approximation: perturbation and accelerated convergence methods*, J. Chem. Phys., 93(3) (1990), pp. 1072–1081.
[37] B. Sandstede, S. Balasuriya, C. K. R. T. Jones, and P. Miller, *Melnikov theory for finite-time vector fields*, Nonlinearity, 13 (2000), pp. 1357–1377.
[38] S. C. Shadden, F. Lekien, and J. E. Marsden, *Definition and properties of Lagrangian coherent structures from finite-time Lyapunov exponents in two-dimensional aperiodic flows*, Physica D, 212 (2005), pp. 271–304.
[39] R. T. Skodje and M. J. Davis, *Geometrical simplification of complex kinetic systems*, J. Phys. Chem. A, 105 (2001), pp. 10556–10365.
[40] M. Valorani, D.A. Goussis, F. Creta, and H.N. Naim, *Higher order corrections in the approximation of low-dimensional manifolds and the construction of simplified problems with the CSP method*, J. Comp. Phys., 209 (2005), pp. 754–786.
[41] M. A. Van Buren and K. D. Mease, *Geometric synthesis of aerospace plane ascent guidance*, Automatica, 30(12) (1994), pp. 1839–1849.
[42] J. A. Vastano and R. D. Moser, *Short-time Lyapunov exponent analysis and the transition to chaos in Taylor-Couette flow*, J. Fluid Mechanics, 233 (1991), pp. 83–118.
[43] Young, L.-S., *Ergodic Theory of Differentiable Dynamical Systems*, in: Real and Complex Dynamics, eds. Branner and Hiorth, NATO ASI Series, Kluwer Academic Publishers, 1995, pp. 201–226.
[44] A. Zagaris, H. G. Kaper and T. J. Kaper, *Analysis of the computational singular perturbation reduction method for chemical kinetics*, J. Nonlinear Sci., 14 (2004), pp. 59–91.