AVERAGING ALGEBRAS, SCHRÖDER NUMBERS AND ROOTED TREES

LI GUO AND JUN PEI

Abstract. In this paper, we study averaging operators from an algebraic and combinatorial point of view. We first construct free averaging algebras in terms of a class of bracketed words called averaging words. We next apply this construction to obtain one and two variable generating functions for subsets of averaging words when the averaging operator is taken to be idempotent. When the averaging algebra has an idempotent generator, the generating function in one variable is twice the generating function for large Ströder numbers, leading us to give interpretations of large Ströder numbers in terms of bracketed words and rooted trees, as well as a recursive formula for these numbers.

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1. INTRODUCTION

Let $k$ be a unitary commutative ring. An averaging operator on a commutative $k$-algebra $R$ is a linear operator $P$ satisfying the identity

$$P(f P(g)) = P(f)P(g) \text{ for all } f, g \in R.$$  

This operator was already implicitly studied by O. Reynolds [25] in 1895 in turbulence theory under the disguise of a Reynolds operator which is defined by

$$P(f g) = P(f)P(g) + P[(f - P(f))(g - P(g))] \text{ for all } f, g \in R,$$

since an idempotent Reynolds operator is an averaging operator. An important class of such operators used in turbulence theory is the class of averages over one portion of space time of
certain vector fields. For example, the time average of a real valued function $f$ defined on time-
space space
\[
f(x, t) \mapsto \bar{f}(x, t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x, t + \tau) d\tau,
\]
is such an operator.

In the 1930s, averaging operator was explicitly defined by Kolmogoroff and Kampé de Fériet [17].
Then G. Birkhoff [5] continued its study and showed that a positive bounded projection in the
Banach algebra $C(X)$, the algebra of scalar valued continuous functions on a compact Hausdorff
space $X$, onto a fixed range space is an idempotent averaging operator. In 1954, S. T. C. Moy [30]
made the connection between averaging operators and conditional expectation. Furthermore, she
studied the relationship between integration theory and averaging operators in turbulence theory
and probability. Then her results were extended by G. C. Rota [26]. During the same period, the
idempotent averaging operators on $C_\infty(X)$, the algebra of all real valued continuous functions on
a locally compact Hausdorff space $X$ that vanish at the infinity, were characterized by J. L. Kelley
[18].

Later on, more discoveries of averaging operators on various spaces were made. B. Brainerd
[4] considered the conditions under which an averaging operator can be represented as an integra-
tion on the abstract analogue of the ring of real valued measurable functions. In 1964, G. C.
Rota [27] proved that a continuous Reynolds operator on the algebra $L_\infty(S, \Sigma, m)$ of bounded
measurable functions on a measure space $(S, \Sigma, m)$ is an averaging operator if and only if it has
a closed range. J. L. B. Gamlen and J. B. Miller [11, 20] considered averaging operators on noncom-
mutative Banach algebras where the averaging identities are defined by Eq. (3). They
discussed spectrum and resolvent sets of averaging operators on Banach algebras. N. H. Bong [6]
found some connections between the resolvent of a Rota-Baxter operator [3, 13, 28] and that of
an averaging operator on complex Banach algebras. In 1986, Huijsmans generalized the work of
Kelley to the case of $f$-algebras. Triki [31, 32] showed that a positive contractive projection on
an Archimedean $f$-algebra is an idempotent averaging operator.

In the last century, most studies on averaging operators had been done for various special
algebras, such as function spaces, Banach algebras, and the topics and methods had been largely
analytic. In this century, while averaging operators continued to find many applications in analysis
and applied areas [10], they also showed their remarkable algebraic importance.

W. Cao [7] constructed free commutative averaging algebras and studied the naturally induced
Lie algebra structures from averaging operators. J. L. Loday [19] defined the diassociative
algebra as the enveloping algebra of the Leibniz algebra in analogue to the associative algebra as
the enveloping algebra of the Lie algebra. M. Aguiar [1] showed that a diassociative algebra can
be derived from an averaging associative algebra by defining two new operations $x \triangleright y := xP(y)$
and $x \bigtriangledown y := P(x)y$. An analogue process gives a Leibniz algebra from an averaging Lie algebra
by defining a new operation $\{x, y\} := P(x, y)$ and derives a (left) permutative algebra [8] from
averaging commutative associative algebra. In general, an averaging operator could be defined on
any binary operad and this type of process was systematically studied in [22] by relating the aver-
ing actions to a special construction of binary operads called duplicators [12]. Combining
the averaging operators actions with the Rota-Baxter operators [2] actions, we obtained an-
other connection between Rota-Baxter operators and averaging operators: the resulting algebraic
structures given by the actions of the two operators are Koszul dual to each other.

These important developments in theory and applications of the averaging algebra motivate us
to carry out an algebraic and combinatorial study of the averaging algebra in this paper. It is
well known that in the category of any given algebraic structure, the free objects play a central role in studying the other objects. Further the combinatorial nature of the algebraic structure is often revealed by its free objects (see \[24\] for the Lie algebra case). Thus our first step is to construct free averaging algebras, after presenting some preliminary properties and examples of averaging algebras. This is carried out in Section 2, where the free averaging algebra on a set is realized on the free module spanned by a class of bracketed words composed from the set, called averaging words. In Section 3, we begin our combinatorial investigation by enumerating subsets of averaging words for the free averaging algebra on one generator and when the operator is taken to be idempotent. The generating function from the enumeration of averaging words turns out to be twice the generating function of the large Schröder numbers, revealing the combinatorial nature of averaging algebras. Pursuing this numerical connection of averaging algebra with large Schröder numbers further allows us to find two applications of averaging algebras to large Schröder numbers. We obtained two interpretations of large Schröder numbers, one in terms of averaging words, another in terms of decorated rooted trees. We also obtain a recursive formula for large Schröder numbers from such a formula arising from the study of averaging words.

2. Properties and free objects of averaging algebras

Convention. Throughout this paper, all algebras are taken to be nonunitary unless otherwise specified.

In this section, we first give some properties and examples of averaging operators. We then give an explicit construction of the free averaging algebra on a non-empty set $X$.

2.1. Definitions and properties. An averaging operator in the noncommutative context is given as follows.

Definition 2.1. A linear operator $P$ on a $k$-algebra $R$ is called an averaging operator if

\begin{equation}
P(x)P(y) = P(xP(y)) = P(P(x)y) \text{ for all } x, y \in R.
\end{equation}

A $k$-algebra $R$ together with an averaging operator $P$ on $R$ is called an averaging algebra.

As is well-known and easily checked, an idempotent operator is an averaging operator if and only if it is a Reynolds operator defined in Eq. (2). We also give a close relationship between averaging operators and Rota-Baxter operators (of weight zero). The latter operator is defined by the operator equation

\begin{equation}
P(x)P(y) = P(P(x)y) + P(xP(y)) \text{ for all } x, y \in \mathbb{Q}[x]
\end{equation}

and has played an important role in mathematics and physics [8, 13, 28].

Note that an averaging operator is a set operator in the sense that, for any semigroup $(G, \cdot)$, it makes sense to define an averaging operator on $G$ to be a map $P : G \to G$ such that

\begin{equation}
P(x \cdot P(y) = P(x \cdot P(y)) = P(P(x) \cdot y) \text{ for all } x, y \in G.
\end{equation}

Now let $P : \mathbb{Q}[x] \to \mathbb{Q}[x]$ be a linear operator such that, for each $n \geq 0$, we have $P(x^n) = \beta(n)x^{\theta(n)}$ with $\beta(n) \in \mathbb{Q}$ and $\theta(n) > 0$. Then [15] $P$ is a Rota-Baxter operator of weight zero if and only if $\theta$ is an averaging operator on the additive semigroup $\mathbb{Z}_{\geq 1}$:

\begin{equation}
\theta(m) + \theta(n) = \theta(m + \theta(n)) = \theta(\theta(m) + n) \text{ for all } m, n \geq 0.
\end{equation}

In addition to the examples of averaging algebras mentioned in the introduction, we display the following classes of examples. We first give some examples from averaging processes. Note that if $P$ is an averaging operator, then $cP$ is also an averaging operator for any $c \in k$. 
**Proposition 2.2.** Let $R$ be a $k$-algebra.

(a) Let $G$ be a finite group that acts on $R$ (on the right) and preserves the multiplication of $G$: $(xy)^g = x^g y^g$ for all $x, y \in R$ and $g \in G$. Then the linear operator

$$P : R \to R, \quad x \mapsto \sum_{g \in G} x^g$$

is an averaging operator.

(b) Let $a$ be a fixed element in the center of $R$. Define $P_a(x) := ax$ for all $x \in R$. Then $P_a$ is an averaging operator on $R$.

**Proof.** (a) For any $x, y \in R$, we have

$$P(xP(y)) = \sum_{h \in G} \left( \sum_{g \in G} x^g y^h \right)^h = \sum_{h \in G} x^h \sum_{g \in G} y^{gh} = \sum_{h \in G} x^h \sum_{g \in G} y^g = P(x)P(y).$$

We similarly have $P(P(x)y) = P(x)P(y)$.

(b) For any $x, y \in R$, we have

$$P(xP(y)) = a(x(ay)) = P(x)y, \quad P(P(x)y) = a((ax)y) = (ax)(ay) = P(x)y.$$ 

Thus $P$ is an averaging operator. \hfill $\Box$

As an application of Proposition 2.2, consider $F([a, b], k)$, the $k$-algebra of $k$-valued functions on the interval $[a, b), a < b$. For a fixed positive integer $n$, define

$$P : F([a, b], k) \to F([a, b], k), \quad f(x) \mapsto \sum_i f\left(x + \frac{i}{b-a}\right),$$

where the sum is over $i \in \mathbb{Z}$ such that $x + \frac{i}{b-a}$ is in $[a, b)$. Then $P$ is an averaging operator on $F([a, b], \mathbb{R})$. Here we take $G$ to be the cyclic group $\mathbb{Z}/n\mathbb{Z}$ acting on $[a, b)$ by permuting the $n$ subintervals $[a + \frac{i}{b-a}, a + \frac{i+1}{b-a}), 0 \leq i \leq n-1$. This action induces an action of $\mathbb{Z}/n\mathbb{Z}$ on $F([a, b], k)$ and hence Proposition 2.2(b) applies. When $P$ is replaced by $\frac{1}{n}P$ which makes sense whenever $k$ contains $\mathbb{Q}$, then we obtain the usual averaging operator.

As a special case of Proposition 2.2(b), let $G$ be a finite group and let $k[G]$ be the group algebra. Then the $k$-linear operator

$$P : k[G] \to k[G], \quad g \mapsto \sum_{h \in G} hg = \left(\sum_{h \in G} h\right)g, \text{ for all } g \in G,$$

is an averaging operator since $\sum_{h \in G} h$ is in the center of $k[G]$.

There are many averaging operators that do not come from an averaging process. A differential operator on a $k$-algebra $R$ is a linear operator $d : R \to R$ such that

$$d(xy) = d(x)y + xd(y) \quad \text{for all } x, y \in R.$$ 

It is immediately checked that a differential operator $d$ with $d^2 = 0$ is an averaging operator.

Birkhoff showed that an averaging operator on a unital $k$-algebra that preserves the identity $1_R$ must be idempotent: $P^2(x) = P(1_R P(x)) = P(1_R)P(x) = P(x)$ for all $x \in R$. We next determine the conditions for an idempotent linear operator to be an averaging operator. Recall that there is a bijection

$$\{\text{idempotent linear operators on } R\} \leftrightarrow \{\text{linear decompositions } R = R_0 \oplus R_1\}$$
such that \( R_0 = \text{im} P \) and \( R_1 = \ker P \). The linear map \( P \) corresponding to \( R = R_0 \oplus R_1 \) is called the projection onto \( R_0 \) along \( R_1 \).

**Proposition 2.3.** Let \( R \) be a \( k \)-algebra and let \( P : R \to R \) be an idempotent linear map. Let \( R = R_0 \oplus R_1 \) be the corresponding linear decomposition. Then \( P \) is an averaging operator if and only if

\[
R_0R_0 \subseteq R_0, \quad R_0R_1 \subseteq R_1, \quad R_1R_0 \subseteq R_1.
\]

**Proof.** For any \( x, y \in R \), denote \( x = x_0 + x_1 \) and \( y = y_0 + y_1 \) with \( x_i, y_i \in R_i, i = 0, 1 \).

Suppose \( P \) is an averaging operator. Then from \( P(R) = R_0 \) and \( P(x)P(y) = P(xP(y)) \) we obtain \( R_0R_0 \subseteq R_0 \). Then we have

\[
P(x)P(y) = x_0y_0, \quad P(P(x)y) = P(x_0y_0 + x_0y_1) = P(x_0y_0) + P(x_0y_1) = x_0y_0 + P(x_0y_1),
\]

\[
P(xP(y)) = P(x_0y_0 + x_1y_0) = P(x_0y_0) + P(x_1y_0) = x_0y_0 + P(x_1y_0).
\]

Thus from Eq. (2) we obtain \( P(x_0y_1) = P(x_1y_0) = 0 \) for all \( x_i, y_i \in R_i, i = 0, 1 \). Therefore Eq. (5) holds since \( R_1 = \ker P \) by the definition of \( P \).

Conversely, suppose Eq. (5) holds. Then we have

\[
P(P(x)y) = P(x_0y_0 + x_0y_1) = P(x_0y_0) + P(x_0y_1) = x_0y_0 + P(x_0y_1),
\]

and similarly \( P(xP(y)) = P(x)P(y) \) for all \( x, y \in R \). Thus \( P \) is an averaging operator. \( \Box \)

Recall that a \( k \)-superalgebra is a \( k \)-algebra \( R \) with a \( k \)-module decomposition \( R = R_0 \oplus R_1 \) such that \( R_iR_j \subseteq R_{i+j} \) where the subscripts are taken modulo 2.

**Corollary 2.4.**

(a) An idempotent algebra endomorphism \( P : R \to R \) is an averaging operator.

In particular, when \( R \) is an augmented \( k \)-algebra with the augmentation map \( \varepsilon : R \to k \), then \( \varepsilon \) regarded as a linear operator on \( R \) is an averaging operator.

(b) Let \( R = R_0 \oplus R_1 \) be a \( k \)-superalgebra. Then the projection \( P \) of \( R \) to \( R_0 \) along \( R_1 \) is an averaging operator on \( A \).

**Proof.** (a) Let \( R_0 := \text{im} P \) and \( R_1 := \ker P \). Then we have \( R = R_0 \oplus R_1 \) and \( P \) is the projection to \( R_0 \) along \( R_1 \). Since \( R_1 \) is an ideal of \( R \), Eq. (2) holds. Hence \( P \) is an averaging operator.

(b) This follows since \( R_0 \) and \( R_1 \) satisfies Eq. (5). \( \Box \)

2.2. The construction of free averaging algebras. Free commutative averaging algebras were constructed in [7].

**Proposition 2.5.** ([7, Theorem 2.6]) Let \( A \) be a commutative \( k \)-algebra and let \( S(A) \) be the symmetric algebra on \( A \). On the tensor product algebra \( \mathcal{A} := \mathcal{A}(A) := A \otimes S(A) \), define the linear operator

\[
P : \mathcal{A} \to \mathcal{A}, \quad P \left( \sum_i a_i \otimes s_i \right) := \sum_i 1 \otimes (a_is_i), \quad \text{for all } a_i \in A, \ s_i \in S(A),
\]

where \( a_is_i \) is the product in \( S(A) \). Then \( P \) is the free commutative averaging algebra on \( A \). When \( A \) is taken to be the polynomial algebra \( k[X] \) on a set \( X \), then \( \mathcal{A}(k[X]) \) is the free commutative averaging algebra on \( X \).

We now construct free (noncommutative) averaging algebras. We carry out the construction in this subsection (Theorem 2.11) and provide the proof of the theorem in the next subsection.
2.2.1. A basis of the free averaging algebra. Let $X$ be a given nonempty set. We will first obtain a linear basis of the free averaging algebra on $X$ from the free operated semigroup $\mathcal{E}(X)$ on $X$ \cite{14, 13}.

For any nonempty set $Y$, let $S(Y)$ be the free semigroup generated by $Y$ and $[Y] := \{[y] \mid y \in Y\}$ be a replica of $Y$. Thus $[Y]$ is a set that is indexed by $Y$ but disjoint with $Y$.

Let $X$ be the nonempty set. Define a direct system as follows. Let

$$
\mathcal{E}_0 := S(X), \quad \mathcal{E}_1 := S(X \cup [\mathcal{E}_0]) = S(X \cup [S(X)]),
$$

with the natural injection

$$i_{0,1} : \mathcal{E}_0 = S(X) \hookrightarrow \mathcal{E}_1 = S(X \cup [\mathcal{E}_0]).$$

Inductively assuming that $\mathcal{E}_{n-1}$ and $i_{n-2,n-1} : \mathcal{E}_{n-2} \hookrightarrow \mathcal{E}_{n-1}$ have been obtained for $n \geq 2$, we define

$$\mathcal{E}_n := S(X \cup [\mathcal{E}_{n-1}])$$

and have the injection

$$[\mathcal{E}_{n-2}] \hookrightarrow [\mathcal{E}_{n-1}].$$

Then by the freeness of $\mathcal{E}_{n-1} = S(X \cup [\mathcal{E}_{n-2}])$, we have

$$\mathcal{E}_{n-1} = S(X \cup [\mathcal{E}_{n-2}]) \hookrightarrow S(X \cup [\mathcal{E}_{n-1}]) = \mathcal{E}_n.$$

Finally, define $\mathcal{E}(X) := \lim \mathcal{E}_n$. Elements in $\mathcal{E}(X)$ are called bracketed words on $X$. Define the depth of $w \in \mathcal{E}(X)$ to be

$$d(w) := \min\{n \mid w \in \mathcal{E}(X)_n\}.$$

Taking the limit in $\mathcal{E}_n = S(X \cup [\mathcal{E}_{n-1}])$, we obtain

$$\mathcal{E}(X) = S(X \cup [\mathcal{E}(X)]).$$

Thus every bracketed word has a unique decomposition, called the standard decomposition,

$$w = w_1 w_2 \cdots w_b,$$

where $w_i$ is in $X$ or $\mathcal{E}(X)$ for $i = 1, 2, \cdots, b$. Then we define $b = b(w)$ to be the breadth of $w$. Elements of $X \cup [\mathcal{E}]$ are called indecomposable. Define the head index $h(w)$ of $w$ to be 0 (resp. 1) if $w_1$ is in $X$ (resp. $[\mathcal{E}]$). Similarly define the tail index $t(w)$ of $w$ to be 0 (resp. 1) if $w_b$ is in $X$ (resp. $[\mathcal{E}]$). If $w$ is indecomposable, then $h(w) = t(w)$, called the index $id(w)$ of $w$.

Further, by combining strings of indecomposable factors in $w$ that are in $X$, we obtain the block decomposition of $w$:

$$w = \omega_1 \cdots \omega_r,$$

where each $\omega_i, 1 \leq i \leq r$, is in either $S(X)$ or $[\mathcal{E}(X)]$.

For example, for $w = x[y[x]]x[y]$, we have $d(w) = 2, b(w) = 5, h(w) = 0, t(w) = 1$. Its block decomposition is $w = x[y[x]](x[y])$. So $r = 4$.

As is known \cite{1, 13}, the free Rota-Baxter algebra on a set is defined on the free $k$-module spanned by the set of Rota-Baxter words $\mathcal{R}(X) \subseteq \mathcal{E}(X)$ consisting of bracketed words that do not contain a subword of the form $[u][v]$ where $u, v \in \mathcal{E}$. It is natural to consider the averaging case in the similar way: Choose the set $\mathcal{B}$ of bracketed words that do not contain a subword of the forms $[u][v]$ and $[[u][v]]$, where $u, v \in \mathcal{E}$. Unfortunately, this restriction is not enough. For example, we have

$$[x[x]^{(2)}] = [x][x]^{(2)} = [[x][x]] = [x[x]]^{(2)}.$$
by the axiom of an averaging operator. Here \( [\cdot]^{(n)} \), \( n \geq 0 \), denotes the \( n \)-th iteration of the operator \( [\cdot] \). Thus only one of the two elements \( [x\{x\}]^{(2)} \) and \( [x\{x\}]^{(2)} \) can be kept in a basis for the free averaging algebra. This motivates us to give the following definition.

**Definition 2.6.** Let \( X \) be a set. A bracketed word \( w \in \mathcal{E}(X) \) is called an **averaging word** if \( w \) does not contain any subword of the form \([u]\{v\}, [u\{v\}] \) or \([u\{v\}]^{(2)} \) for \( u, v \in \mathcal{E}(X) \). The set of averaging words on \( X \) is denoted by \( \mathcal{A} = \mathcal{A}(X) \).

For example, \([x\{x\}], [x\{x\}]^{(2)}, [x\{x\}]^{(2)} \) are averaging words on \( \{x\} \).

We will prove in Theorem 2.11 that the free \( k \)-module \( k\mathcal{A} \) spanned by the set \( \mathcal{A} = \mathcal{A}(X) \), equipped with a suitably defined multiplication and linear operator is the free averaging algebra on \( X \). In order to carry out the construction and proof, we give the following recursive description of \( \mathcal{A} \).

For any nonempty subsets \( G, H \) and \( H' \) of \( \mathcal{E}(X) \), denote

\[
\Lambda(G, H) =: (\bigcup_{r \geq 1} (G[H])^r) \sqcup (\bigcup_{r \geq 1} ([H]G)^r) \sqcup (\bigcup_{r \geq 0} (G[H])^r[H]),
\]
\[
\Lambda^+(G, H, H') =: (\bigcup_{r \geq 1} (G[H])^r[G[H]']) \sqcup (\bigcup_{r \geq 0} (G[H])^r),
\]

where, for a subset \( T \) of \( \mathcal{E}(X) \), \( T^r := \{t_1 \cdots t_r \mid t_i \in T, 1 \leq i \leq r \} \) and \( T^0 := 1 \), the empty word.

We construct direct systems \( \{\mathcal{A}^+_n\}_{n \geq 0}, \{\mathcal{A}'_n\}_{n \geq 0}, \{\mathcal{A}_n\}_{n \geq 0} \) from \( \mathcal{E}(X) \) by the following recursions.

First denote \( \mathcal{A}_0 = \mathcal{A}_0^+ = \mathcal{A}_0^+ = \mathcal{S}(X) \). Then for \( n \geq 0 \), define

\[
\mathcal{A}_{n+1} = \Lambda(\mathcal{A}_n, \mathcal{A}_n^+), \quad \mathcal{A}_{n+1}^+ = \Lambda^+(\mathcal{A}_n, \mathcal{A}_n^+, \mathcal{A}_n^+), \quad \mathcal{A}_{n+1}^+ = \mathcal{A}_{n+1}^+ \cup [\mathcal{A}_n^+].
\]

We have the following properties on \( \mathcal{A}_n, \mathcal{A}_n^+ \) and \( \mathcal{A}_n^+ \).

**Proposition 2.7.** For \( n \geq 0 \), we have

\[
\mathcal{A}_n \subseteq \mathcal{A}_{n+1}, \quad \mathcal{A}_n^+ \subseteq \mathcal{A}_{n+1}^+, \quad \mathcal{A}_n^+ \subseteq \mathcal{A}_{n+1}^+.
\]

**Proof.** We prove the inclusions by induction on \( n \). When \( n = 0 \), by definition, we have

\[
\mathcal{A}_0 \subseteq \mathcal{A}_1, \quad \mathcal{A}_0^+ \subseteq \mathcal{A}_1^+, \quad \mathcal{A}_0^+ \subseteq \mathcal{A}_1^+.
\]

Suppose that the inclusions in Eqs. (12)–(14) hold for \( n = k \geq 0 \), that is \( \mathcal{A}_k \subseteq \mathcal{A}_{k+1}, \mathcal{A}_k^+ \subseteq \mathcal{A}_{k+1}^+ \) and \( \mathcal{A}_k^+ \subseteq \mathcal{A}_{k+1}^+ \). Consider the case \( n = k + 1 \). Then we immediately have

\[
\mathcal{A}_{k+1} = \Lambda(\mathcal{A}_0, \mathcal{A}_k^+), \quad \mathcal{A}_{k+1}^+ = \Lambda^+(\mathcal{A}_0, \mathcal{A}_k^+, \mathcal{A}_k^+), \quad \mathcal{A}_{k+1}^+ = \mathcal{A}_{k+1}^+ \cup [\mathcal{A}_k^+].
\]

These complete the induction. \( \square \)

Therefore we can take the direct systems

\[
\mathcal{A}_\infty := \bigcup_{n \geq 0} \mathcal{A}_n = \lim_{n \to \infty} \mathcal{A}_n, \quad \mathcal{A}^+ := \bigcup_{n \geq 0} \mathcal{A}_n^+ = \lim_{n \to \infty} \mathcal{A}_n^+, \quad \mathcal{A}^+ := \bigcup_{n \geq 0} \mathcal{A}_n^+ = \lim_{n \to \infty} \mathcal{A}_n^+.
\]

**Proposition 2.8.** For a given set \( X \), we have \( \mathcal{A} = \mathcal{A}_\infty \).
Because of the proposition, we will omit the notation $A_\infty$ in the rest of the paper.

**Proof.** For now we let $A_{(n)}$, $n \geq 0$, denote the subset of $A$ consisting of its elements of depth less or equal to $n$: $A_{(n)} := A \cap \Xi_n$. Then we have $A = \cup_{n \geq 0} A_{(n)}$. So we just need to verify

(15) 
$$A_{(n)} = A_n \text{ for all } n \geq 0.$$ 

We will prove it by induction on $n \geq 0$.

When $n = 0$, there is nothing to prove since $A_{(0)} = A_0 = S(X)$. Assume that Eq. (15) has been verified for $n \leq k$ for a $k \geq 0$ and consider the subsets $A_{(k+1)}$ and $A_{k+1}$.

Since $A_k := \Lambda^+(A_0, \tilde{A}_{k-1})$ is contained in $A_{(k)} \subseteq A_{(k+1)}$, the subsets $A_k^+$ and $\tilde{A}_k^+$ are contained in $A_{(k+1)}$ by the induction hypothesis. Thus elements in these subsets do not contain elements of the forms excluded in the definition of $A$. Further elements in $\tilde{A}_k^+$ do not contain elements of the forms $[uv][v]$ and $[uv][v]^2$. Thus elements in $[\tilde{A}_k^+]$ do not contain elements of the form $[[u]v]$ and $[[u]v]^2$. Therefore by the definition of $A_{k+1}$ in Eq. (14), elements of $A_{k+1}$ do not contain subwords of the forms excluded in the definition of $A$. Since $A_{k+1}$ is also contained in $\Xi_{k+1}$, we have $A_{k+1} \subseteq A_{(k+1)}$.

Conversely, since elements of $A_{(k+1)}$ have depth less or equal to $k + 1$ and do not contain subwords of the form $[uv][v]$, by the induction hypothesis we have $A_{(k+1)} \subseteq \Lambda(A_0, A_k)$. Since elements in $A_k$ are in brackets in $\Lambda(A_0, A_k)$ and elements of $A_{(k+1)}$ cannot contain elements of the forms $[[u]v]$ and $[[u]v]^2$, we have

$$A_{(k+1)} \subseteq \Lambda^+(A_0, \tilde{A}_k^+) = A_{k+1}.$$ 

This completes the induction. □

**Remark 2.1.** If $v$ is in $[\tilde{A}_k^+]$, then there is unique $s \geq 1$ such that $v = [v']^s$ with $v' \in A^+$. Thus if $v' = v'_1 \cdots v'_n$ is in standard form, then $v'_1 = x$ and $v'_n$ is either $x$ or $[\tilde{v}_n]$ with $\tilde{v}_n \in A^+$ when $n \geq 2$.

Taking the limit on both sides of Eq. (14), we obtain $A = \Lambda(A_0, \tilde{A}^+)$. Thus in the block decomposition $w = \omega_1 \cdots \omega_r$ of $w \in A$, the elements $\omega_1, \cdots, \omega_r$ are alternatively in $A_0 = S(X)$ and $[\tilde{A}^+]$. We show next that these are also sufficient conditions for $w$ to be in $A$.

**Lemma 2.9.** (a) Let $w = \omega_1 \cdots \omega_r$ be the block decomposition of $w \in \Xi(X)$ in Eq. (8). Then $w$ is an averaging word if and only if $w$ is in $\Lambda(X, \Xi(X))$ and each $\omega_i$, $1 \leq i \leq r$, is an averaging word and $id(w_i) \neq id(w_{i+1})$ for $1 \leq i \leq r - 1$.

(b) Let $w = w_1 \cdots w_b$ be the standard decomposition of $w \in \Xi(X)$ in Eq. (8). If $w$ is an averaging word, then each $w_i$, $1 \leq i \leq b$, is an averaging word. Further, if $w$ is an averaging word and $w = uv$ with $u, v \in \Xi(X)$, then $u$ and $v$ are averaging words.

**Proof.** (8) Let $w \in \Xi(X)$ with block decomposition $w = \omega_1 \cdots \omega_r$ is an averaging word. Then by the definition of an averaging word, $w$ does not contain any of the subwords excluded in the definition of averaging words. Thus each $\omega_i$, $1 \leq i \leq r$, does not contain these subwords and hence is an averaging word.

Conversely, let $w = \omega_1 \cdots \omega_r$ be in $\Lambda(X, \Xi(X))$ and $\omega_1, \cdots, \omega_r$ are averaging words. Since $w$ is in $\Xi_k$ for some $k \geq 0$, we just need to prove that $w$ is in $A$ by induction on $k \geq 0$. When $k = 0$, we have $\Xi_0 = S(X)$ which is in $A$. Hence the claim holds. Assume that the claim holds for all $w \in \Xi_k$ where $k \geq 0$ and consider $w \in \Xi_{k+1}$. Thus $w$ is in $\Lambda(X, \Xi)$ and each $\omega_i$, $1 \leq i \leq r$, is in $A_{k+1}$. The second condition means that each $\omega_i$ is either in $S(X)$ or in $[\tilde{A}_k^+]$. Then the first condition means that $w$ is in $\Lambda(S(X), \tilde{A}_k^+) = A_{k+1}$. This completes the induction. □

(8) Both statements follow directly from the definition of averaging words.
2.2.2. Construction of the product and operator. Let \( X \) be a set and let \( kA \) to be the free \( k \)-module generated by \( A := A(X) \). To define a multiplication \( \circ \) on \( kA \), we first define \( u \circ v \) for two words \( u \) and \( v \) in \( A \) by induction on the depth \( d(u) \geq 0 \) of \( u \) as follows.

If \( d(u) = 0 \), then \( u \) is in \( S(X) \) and the product \( \circ \) is the concatenation. Assume that \( u \circ v \) have been defined for all \( u, v \in A \) with \( d(u) \leq k \) where \( k \geq 0 \). Consider \( u, v \in A \) with \( d(u) = k + 1 \). First consider the case when \( u \) and \( v \) are indecomposable, namely are in \( X \sqcup [A] \). For \( u, v \in [A] \), rewrite \( u = [u]^t, v = [v]'^{t} \), where \( s, t \geq 1 \) while \( u', v' \) are in \( A^+ \) as in Remark \([2.3]\) and hence are not in \([A]\). Then define

\[
(16) \quad u \circ v = \begin{cases} uv, & \text{if } u \text{ or } v \text{ is in } X, \\
[u' \circ [v']^{s+t-1}], & \text{if } u = [u']^s \text{ and } v = [v']^t,
\end{cases}
\]

where \( u' \circ [v']^{t} \) is defined by the induction hypothesis since \( d(u') = d(u) - s \) is less than \( k + 1 \).

Next consider the general case when \( u \) and \( v \) are in \( A \). Let \( u = u_1u_2 \cdots u_m, v = v_1v_2 \cdots v_n \) be their standard decompositions in Eq. (8). Then define

\[
(17) \quad u \circ v = u_1u_2 \cdots u_{m-1}(u_m \circ v_1)v_2 \cdots v_n,
\]

where \( u_m \circ v_1 \) is the concatenation or as defined in Eq. (16).

For example, for \( u_1 = x, v_1 = [x], u_2 = [x][x]]^2(2) \) and \( v_2 = [x]^3(3) \), we have

\[
u_1 \circ v_1 = x[x], \quad u_2 \circ v_2 = [x[x] \circ [x]]^4 = [x][x][x]\).
\]

By the concatenation case and Eq. (14), we have

\[
(18) \quad h(u) = h(u \circ v), \quad t(v) = t(u \circ v).
\]

Extending \( \circ \) bilinearly, we obtain a binary operation on \( kA(X) \). The following properties can be derived from the definition of \( \circ \) directly.

**Lemma 2.10.** Let \( w, w' \in A \). Then

(a) \( h(w) = h(w \circ w') \) and \( t(w') = t(w \circ w') \).

(b) If \( t(w) \neq h(w') \) or \( t(w) = h(w') = 0 \), then for any \( w'' \in A \),

\[
(19) \quad (ww') \circ w'' = w(w' \circ w''),
\]

\[
(20) \quad w'' \circ (ww') = (w'' \circ w)w'.
\]

2.2.3. The construction of the operator. We next define a linear operator \( P \) on \( kA \). For \( u \in A \), there is some \( n \) such that \( u \in A_n \). Recall that

\[
(10) \quad A^+ = (\sqcup_{r \geq 1} (A_0[A_{n-1}^+])^{r-1}A_0[A_{n-1}^+]) \cup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r}A_0), \quad \tilde{A}^+ = A^+ \cup [\tilde{A}_{n-1}^+]
\]

and

\[
(21) \quad A_n = (\sqcup_{r \geq 1} (A_0[A_{n-1}^+])^{r-1}A_0[A_{n-1}^+]) \cup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r}A_0) \cup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r}A_0) \sqcup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r}A_0) \sqcup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r}A_0) \sqcup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r}A_0) \sqcup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r}A_0)
\]

Since \( \tilde{A}^+_{n-1} = A^+_{n-1} \cup [\tilde{A}^+_{n-2}] \), the first of the four disjoint union component becomes

\[
\sqcup_{r \geq 1} (A_0[A_{n-1}^+])^{r-1}A_0[A_{n-1}^+] \cup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r-1}A_0[A_{n-1}^+]) \cup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r-1}A_0[A_{n-1}^+]) \sqcup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r-1}A_0[A_{n-1}^+]) \sqcup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r-1}A_0[A_{n-1}^+])
\]

of which the first disjoint union component is exactly the first disjoint union component of \( A^+ \). Also the third disjoint union component of \( A_n \) is the second disjoint union component of \( A^+ \). By collecting the components of \( \tilde{A}^+ \) together in this way, we see that \( A_n \) can be rearranged as

\[
(22) \quad A_n = \tilde{A}^+ \cup (\sqcup_{r \geq 1} (A_0[A_{n-1}^+])^{r-1}A_0[A_{n-1}^+]) \cup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r-1}A_0[A_{n-1}^+]) \cup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r-1}A_0[A_{n-1}^+]) \cup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r-1}A_0[A_{n-1}^+]) \cup (\sqcup_{r \geq 0} (A_0[A_{n-1}^+])^{r-1}A_0[A_{n-1}^+])
\]
So any \( u \in \mathcal{A}_n \) is in one of the above disjoint components. We accordingly define

\[
P_X(u) = \begin{cases} 
[u], & \text{if } u \in \mathcal{A}_n^+, \\
[u_1 \circ [u_2]]^{(s)}, & \text{if } u \in \bigcup_{s \geq 1} (\mathcal{A}_n^+, \mathcal{A}_0) \circ u = [u_1]^{(s)}u_2, \\
[u'_1[u'_2]]^{(s)}, & \text{if } u \in \bigcup_{s \geq 1} (\mathcal{A}_n^+, \mathcal{A}_0) \circ u = [u'_1]^{(s)}u'_2, s \geq 2, \\
[u_1 \circ [u_2][u_3]]^{(s+t-1)}, & \text{if } u \in \bigcup_{s \geq 1} (\mathcal{A}_n^+, \mathcal{A}_0) \circ u = [u_1]^{(s)}u_2[u_3], 
\end{cases}
\]

where \( u_1, u'_2, u_3 \) are in \( \mathcal{A}^+ \) and \( h(u_2) = t(u_2) = 0 \). Thus \( P_X(u) \) is in \( \mathcal{A} \). Then extending \( P_X \) by linearity to a linear operator on \( \mathcal{A}(X) \) that we still denote by \( P_X \).

For example, if \( u = [x|y]|z, v = x|y|^{(2)} \) and \( w = [x|y]|z|x|^{(2)} \) then we have

\[
P_X(u) = [x|y] \circ [z] = [x|y|z], \quad P_X(v) = [x|y]^{(2)},
\]

\[
P_X(w) = [x|y] \circ [z|x|]^{(2)} = [x|y|z|x|]^{(2)}.
\]

Let \( j_X : X \rightarrow S(X) \rightarrow \mathcal{A} \rightarrow \mathbb{k}\mathcal{A} \) denote the natural injection from \( X \) to \( \mathcal{A}(X) \).

**Theorem 2.11.** Let \( X \) be a non-empty set. Then

(a) The pair \( (\mathbb{k}\mathcal{A}, \circ) \) is an algebra;
(b) the triple \( (\mathbb{k}\mathcal{A}, \circ, P_X) \) is an averaging algebra;
(c) the quadruple \( (\mathbb{k}\mathcal{A}, \circ, P_X, j_X) \) is the free averaging algebra on set \( X \). More precisely, for any averaging algebra \( B \) and a map \( f : X \rightarrow B \), there is a unique averaging \( \mathbb{k}\)-algebra homomorphism \( \bar{f} : \mathbb{k}\mathcal{A} \rightarrow B \) such that \( f = \bar{f} \circ j_X \).

The proof of the theorem is given in the next subsection.

2.3. **The proof of Theorem 2.11.** We now prove Theorem 2.11.

2.3.1. *Proof of Theorem 2.11.* We only need to verify the associativity of \( \circ \):

\[
(W \circ W') \circ W'' = W \circ (W' \circ W'') \quad \text{for all } W, W', W'' \in \mathcal{A}.
\]

For this we prove by induction on the depth \( d(W) \geq 0 \) of \( W \). If \( d(W) = 0 \), then \( W \) is in \( S(X) \). Then \( W \circ W' = WW' \) and \( W \circ (W' \circ W'') = W(W' \circ W'') \) and the associativity follows from the definition of \( \circ \) in Eqs. (16) and (17). Suppose Eq. (23) has been verified for all \( W, W', W'' \in \mathcal{A} \) with \( d(W) \leq k \) for \( k \geq 0 \) and consider \( W, W', W'' \in \mathcal{A} \) with \( d(W) = k + 1 \). We consider three cases.

**Case I. Suppose** \( t(W) \neq h(W') \) or \( t(W) = h(W') = 0 \): Then by Lemma 2.10, we have

\[
(W \circ W') \circ W'' = (WW') \circ W'' = W(W' \circ W'') = W \circ (W' \circ W'').
\]

**Case II. Suppose** \( t(W') \neq h(W'') \) or \( t(W') = h(W'') = 0 \): This case is proved similarly.

**Case III: Suppose** \( t(W) = h(W') = 1 \) and \( t(W') = h(W'') = 1 \): We divide this case into the following four subcases.

(i) **Suppose** \( b(W') \geq 2 \): Then \( W' = w'_1w'_2 \) with \( w'_1, w'_2 \in \mathcal{A} \) and either \( t(w'_1) \neq h(w'_2) \) or \( t(w'_1) = h(w'_2) = 0 \). Then we have

\[
(W \circ W') \circ W'' = (W \circ (w'_1w'_2)) \circ W'' \\
= ((W \circ w'_1)w'_2) \circ W'' \quad \text{(by Eq. (20))} \\
= (W \circ w'_1)(w'_2 \circ W'') \quad \text{(by Eq. (19))} \\
= W \circ (w'_1 \circ (w'_2 \circ W'')) \quad \text{(by Case I)} \\
= W \circ ((w'_1w'_2) \circ W'') \quad \text{(by Eq. (24))} \\
= W \circ (W' \circ W'').
\]
(ii) **Suppose** \(b(W) \geq 2\): Then \(W = w_1w_2\) with \(w_1 \in A\), \(b(w_2) = 1\) and either \(t(w_1) \neq h(w_2)\) or \(t(w_1) = h(w_2) = 0\). By Eq. (24), we have

\[
(W \circ W' \circ W'' = ((w_1w_2) \circ W') \circ W'' = (w_1(w_2 \circ W')) \circ W'' = w_1((w_2 \circ W') \circ W'')
\]

and

\[
W \circ (W' \circ W'') = (w_1w_2) \circ (W' \circ W'') = w_1(w_2 \circ (W' \circ W'')).
\]

Thus \((W \circ W') \circ W'' = W \circ (W' \circ W'')\) holds if and only if \((w_2 \circ W') \circ W'' = w_2 \circ (W' \circ W'')\) holds. Therefore this case is reduced to the case when \(b(W) = 1\) in Subcase (iv).

(iii) **Suppose** \(b(W'') \geq 2\): Then \(W'' = w_1'w_2'\) with \(w_2' \in A\), \(b(w_2') = 1\) and either \(t(w_1') \neq h(w_2')\) or \(t(w_1') = h(w_2') = 0\). Similarly to the case when \(b(W) \geq 2\), we obtain \((W \circ W') \circ W'' = W \circ (W' \circ W'')\) holds if and only if \((W \circ W') \circ w_1'' = W \circ (W' \circ w_1'')\) holds. Thus this case is also reduced to the case when \(b(W'') = 1\) in Subcase (iv).

In summary we have reduced the proof of Case III to the proof of the following special case:

(iv) **Suppose** \(b(W) = b(W') = b(W'') = 1\): Then all the three words are in \([\overline{A}^+\]). Then \(W = [w]^{(r)}, W' = [w']^{(s)}, W'' = [w'']^{(t)}\), where \(r, s, t \geq 1\), \(w, w', w'' \in A^+\) and \(d(W) = k + 1\). We have

\[
(W \circ W') \circ W'' = ([w \circ [w']]^{r+s-1}) \circ [w'']^{(t)} = ([w \circ [w']] \circ [w''])^{r+s+t-2},
\]

\[
W \circ (W' \circ W'') = [w]^{(r)} \circ ([w' \circ [w'']]^{s+t-1}) = [w \circ ([w' \circ [w'']])^{r+s+t-2}.
\]

Since \(d(w) = k + 1 - s\), by the induction hypothesis we have

\[
(w \circ [w']) \circ [w''] = w \circ ([w'] \circ [w'']) = w \circ ([w' \circ [w'']])
\]

and then \((W \circ W') \circ W'' = W \circ (W' \circ W'')\).

This completes the inductive proof of Eq. (23).

2.3.2. **The proof of Theorem** 2.71. We only need to verify the equations

(25) \(P_X(u) \circ P_X(v) = P_X(P_X(u) \circ v), \quad P_X(u) \circ P_X(v) = P_X(u \circ P_X(v))\) for all \(u, v \in A\).

We first recall by Remark 2.7 that for any \(u, v \in A\), there exist unique \(u', v' \in A^+\) and \(s, t \geq 1\) such that

(26) \(P_X(u) = [u']^{(s)}, P_X(v) = [v']^{(t)}\)

Then we have

(27) \(P_X(u) \circ P_X(v) = [u']^{(s)} \circ [v']^{(t)} = [u' \circ [v']]^{(s+t-1)}\).

We verify the first equation in Eq. (25) by considering the two cases when \(h(v) = 0, 1\).

**Case 1.** Suppose \(h(v) = 0\): When \(t(v) = 0\), we have \(v \in A^+\). Then \(P_X(v) = [v] = [v']^{(t)}\) in Eq. (26) with \(v' = v\) and \(t = 1\). When \(t(v) = 1\), we rewrite \(v\) as \([v_1][v_2]^{(t)}\), \(v_2 \in A^+\). Then \(P_X(v) = [v_1][v_2]^{(t)} = [v']^{(t)}\) with \(v' = v_1[v_2]\) and \(t = \ell\). Then by Eq. (27), we have

\[
P_X(P_X(u) \circ v) = P_X([u']^{(s)}v) = \begin{cases} [u' \circ [v']^{(t)}], & t(v) = 0 \\ [u' \circ [v_1][v_2]^{(t)}]^{(s+t-1)}, & t(v) = 1 \end{cases} = [u' \circ [v']^{(s+t-1)}] = P_X(u) \circ P_X(v).
\]

**Case 2.** Suppose \(h(v) = 1\): When \(t(v) = 0\), we rewrite \(v\) as \([v_1][v_2]^{(t)}\), \(v_1 \in A^+\). Then \(P_X(v) = [v_1][v_2]^{(t)} = [v']^{(t)}\) with \(v' = v_1[v_2]\) and \(t = \ell\). When \(t(v) = 1\), we rewrite \(v\) as
Then $P_X(v) = [v_1 \circ [v_2 v_3]]^{(ℓ+q-1)} = [v']^{(s)}$ with $v' = v_1 \circ [v_2 v_3]$ and $t = ℓ + q - 1$. Then by Eq. (27), we have
\[
\begin{align*}
P_X(P_X(u) \circ v) &= P_X([u']^{(s)} \circ v) \\
&= \begin{cases}
P_X([u' \circ [v_1]]^{(s+ℓ-1)} v_2), & t(v) = 0 \\
P_X([u' \circ [v_1]]^{(s+ℓ-1)} v_2 [v_3]^{(s)}), & t(v) = 1 \end{cases} \\
&= \begin{cases}
([u' \circ [v_1]] \circ [v_2])^{(ℓ+q-1)}, & t(v) = 0 \\
([u' \circ [v_1]] \circ [v_2] [v_3]^{(s)}), & t(v) = 1 \end{cases} \\
&= \begin{cases}
[u' \circ [v_2]]^{(ℓ+q-1)}, & t(v) = 0 \\
[u' \circ [v_2] [v_3]^{(s)}], & t(v) = 1 \end{cases} \\
&= [u' \circ [v']^{(s+ℓ-1)}] \\
&= P_X(u) \circ P_X(v).
\end{align*}
\]

Thus the first equation in Eq. (25) is verified.

To verify the second equation in Eq. (25), we also consider the two cases when $h(u)$ is 0 and 1.

**Case 1.** Suppose $h(u) = 0$: When $t(u) = 0$, we have $u \in \mathcal{A}^\dagger$. So $P_X(u) = [u] = [u']^{(s)}$ as in Eq. (26) with $u' = u$ and $s = 1$. When $t(u) = 1$, we rewrite $u$ as $u_1 [u_2]^{(ℓ)}$, where $u_2 \in \mathcal{A}^\dagger$. Then $P_X(u) = [u_1 [u_2]]^{(ℓ)} = [u']^{(s)}$ with $u' = u_1 [u_2]$, $s = ℓ$. Then by Eq. (27) we have
\[
\begin{align*}
P_X(u \circ P_X(v)) &= P_X([v']^{(s)}) \\
&= \begin{cases}
[u \circ [v']^{(s)}], & t(u) = 0 \\
P_X(u_1 [u_2] \circ [v']^{(s+ℓ-1)}), & t(u) = 1 \end{cases} \\
&= \begin{cases}
[u \circ [v']^{(s)}], & t(u) = 0 \\
[u_1 [u_2] \circ [v']^{(s+ℓ-1)}], & t(u) = 1 \end{cases} \\
&= [u' \circ [v']^{(s+ℓ-1)}] \\
&= P_X(u) \circ P_X(v).
\end{align*}
\]

**Case 2.** Suppose $h(u) = 1$: When $t(u) = 0$, we rewrite $u$ as $[u_1]^{(ℓ)} u_2$, $u_1 \in \mathcal{A}^\dagger$. Then $P_X(u) = [u_1 \circ [u_2]]^{(ℓ)} = [u']^{(s)}$ as in Eq. (26) with $u' = u_1 \circ [u_2]$ and $s = ℓ$. When $t(u) = 1$, we rewrite $u$ as $[u_1]^{(ℓ)} u_2 [u_3]^{(q)}$. Then $P_X(u) = [u_1 \circ [u_2] [u_3]^{(q+ℓ)}] = [u']^{(s)}$ with $u' = u_1 \circ [u_2 [u_3]]$ and $s = ℓ + q - 1$. Then we have
\[
\begin{align*}
P_X(u \circ P_X(v)) &= P_X([v']^{(s)}) \\
&= \begin{cases}
P_X([u_1]^{(ℓ)} u_2 [v']^{(s)}), & t(u) = 0 \\
P_X([u_1]^{(ℓ)} u_2 [u_3] \circ [v']^{(q+ℓ-1)}), & t(u) = 1 \end{cases} \\
&= \begin{cases}
[u_1 \circ [u_2] [v']^{(s+ℓ-1)}], & t(u) = 0 \\
[u_1 \circ [u_2 [u_3] \circ [v']^{(s+ℓ+q-2)}], & t(u) = 1 \end{cases} \\
&= \begin{cases}
([u_1 \circ [u_2]] [v']^{(s+ℓ-1)}), & t(u) = 0 \\
([u_1 \circ [u_2 [u_3]]] [v']^{(s+ℓ+q-1)}], & t(u) = 1 \end{cases} \\
&= [u' \circ [v']^{(s+ℓ-1)}].
\end{align*}
\]
2.3.3. The proof of Theorem [27,14]. Let \((B, Q)\) be an averaging algebra and \(*\) be the product in \(B\). Let \(f : X \rightarrow B\) be a map. We will construct a \(k\)-linear map \(\tilde{f} : \mathbb{K}A \rightarrow B\) by defining \(\tilde{f}(w)\) for \(w \in \mathbb{A}\). We achieve this by using induction on \(n\) for \(w \in \mathbb{A}_n\). For \(w = x_1x_2 \cdots x_m \in \mathbb{A}_0 = S(X)\), where \(x_i \in X, 1 \leq i \leq m\), define \(\tilde{f}(w) = f(x_1) * f(x_2) \cdots * f(x_m)\). Suppose \(\tilde{f}(w)\) has been defined for \(w \in \mathbb{A}_n\) and consider \(w \in \mathbb{A}_{n+1}\) which is defined by
\[
\mathbb{A}_{n+1} = (\bigcup_{i \geq 1} (\mathbb{A}_0 \mathbb{A}_n^+)') \cup (\bigcup_{i \geq 1} (\mathbb{A}_n^+ \mathbb{A}_0)' \cup (\bigcup_{i \geq 0} (\mathbb{A}_i \mathbb{A}_0)') \cup (\bigcup_{i \geq 0} (\mathbb{A}_i + \mathbb{A}_0)^')\).
\]

If \(w\) is in \(\bigcup_{i \geq 1} (\mathbb{A}_0 \mathbb{A}_n^+)')\), then \(w = \bigcup_{i=1}^r \mathbb{A}_0 \mathbb{A}_n^+\), where \(w_{2i-1} \in \mathbb{A}_0\) and \(w_{2i} \in \mathbb{A}_n^+\). By the construction of the multiplication \(\odot\) and the averaging operator \(P_X\), we also can express it by
\[
w = \odot_{i=1}^r (w_{2i-1} \circ P_X(w_{2i})).
\]
Thus there is only one possible way to define \(\tilde{f}(w)\) in order for \(\tilde{f}\) to be an averaging homomorphism:
\[
\tilde{f}(w) = \odot_{i=1}^r (\tilde{f}(w_{2i-1}) * Q(\tilde{f}(w_{2i}))).
\]
\(\tilde{f}(w)\) can be similarly defined if \(w\) is in the other unions. This proves the existence of \(f\) as a map and its uniqueness.

We next prove that the map \(\tilde{f}\) defined in Eq. \((28)\) is indeed an averaging algebra homomorphism. We will first show that \(\tilde{f}\) is an algebra homomorphism, that is, for any \(W, W' \in \mathbb{A}\)
\[
\tilde{f}(W * W') = \tilde{f}(W) * \tilde{f}(W').
\]
We will prove Eq. \((29)\) by induction on \(b := b(W) + b(W') \geq 2\).

If \(b(W) + b(W') = 2\), then \(b(W) = b(W') = 1\). We then prove Eq. \((29)\) by induction on \(d(W)\). When \(d(W) = 0\), that is \(W \in S(X)\). Then \(W \odot W'\) is the concatenation and, by definition,
\[
\tilde{f}(W \odot W') = \tilde{f}(W) * \tilde{f}(W').
\]
Suppose Eq. \((29)\) holds for \(W, W'\) with \(0 \leq d(W) \leq k\) and consider \(W, W' \in \mathbb{A}\) with \(d(W) = k + 1\). If \(d(W') = 0\), a similar argument as in the case of \(d(W) = 0\) proves Eq. \((29)\). If \(d(W') > 0\), then \(W = [w]^s\) and \(W' = [w']^t\) with \(w, w' \in \mathbb{A}^+\). Hence
\[
\tilde{f}(W \odot W') &= \tilde{f}([w] \odot [w']^s) \quad (\text{by definition})
\]
\[
= \tilde{Q}^s \left( \tilde{f}(w) \circ \tilde{f}(w') \right) \quad (\text{by induction hypothesis})
\]
\[
= \tilde{Q}^s \left( \tilde{f}(w) * Q(\tilde{f}(w')) \right) \quad (\text{by definition})
\]
\[
= \tilde{Q}^s(\tilde{f}(w)) * Q(\tilde{f}(w')) \quad (Q \text{ is an averaging operator})
\]
\[
= \tilde{f}([w]^s) * \tilde{f}([w']^t) \quad (\text{by definition})
\]
\[
= \tilde{f}(W) * \tilde{f}(W').
\]
Now assume that Eq. \((29)\) holds for all \(W, W' \in \mathbb{A}\) with \(2 \leq b \leq k\). Consider \(W, W' \in \mathbb{A}\) with \(b = k + 1\). If \(t(W) \neq h(W')\) or \(t(W) = h(W') = 0\), then \(W \odot W'\) is the concatenation. So we have
\[
\tilde{f}(W \odot W') = \tilde{f}(WW') = \tilde{f}(W) * \tilde{f}(W')
\]
by Eq. (28). Now let $t(W) = h(W') = 1$. Since $k \geq 2$, we have $k + 1 \geq 3$, so at least one of $W$ and $W'$ have breadth $\geq 2$. Without loss of generality, we can assume $b(W) \geq 2$. Then $W$ can be written as $w_1w_2$ with $b(w_2) = 1$. Thus we have $t(w_2) = t(W) = h(W') = 1$, $t(w_1) = 0, h(w_2) = 1$, and $b(w_2) + b(W') < b(W) + b(W') = k + 1$. Applying the associativity of the product and the induction hypothesis, we have

$$
\tilde{f}(W \circ W') = \tilde{f}((w_1_2) \circ W') = \tilde{f}(w_1_2 \circ W')
= \tilde{f}(w_1 \circ (w_2 \circ W'))
= \tilde{f}(w_1) \ast \tilde{f}(w_2 \circ W') \quad (t(w_1) \neq h(w_2))
= \tilde{f}(w_1) \ast (\tilde{f}(w_2) \ast \tilde{f}(W')) \quad \text{(by induction hypothesis)}
= (\tilde{f}(w_1) \ast \tilde{f}(w_2)) \ast \tilde{f}(W')
= \tilde{f}(w_1 \circ w_2) \ast \tilde{f}(W') \quad (d(w_1) + d(w_2) \leq k)
= \tilde{f}(W) \ast \tilde{f}(W').
$$

Now we are left to show that, for any word $w \in \mathcal{A}$, we have

$$
\tilde{f}(P_X(w)) = Q(\tilde{f}(w)).
$$

Corresponding to the four components of the disjoint union decomposition in Eq. (22), we have the following four cases to consider.

If $w$ is in $\mathcal{A}^+$, then by definition $P_X(w) = [w] \in \mathcal{A}$. By definition (Eq. (28)) we have

$$
\tilde{f}(P_X(w)) = \tilde{f}([w]) = Q(\tilde{f}(w)).
$$

If $w$ is in $\sqcup_{r \geq 1}(\mathcal{A}_0 \circ \mathcal{A}_n) \ast \mathcal{A}_0 \ast \mathcal{A}_n \ast \mathcal{A}_{n-2}^{(2)}$, then write $w = w_1[w_2]^{(s)}$, $s \geq 2$ with $w_2 \in \mathcal{A}^+$. Then $P_X(w) = [w_1][w_2]^{(s)}$. We have

$$
\tilde{f}(P_X(w)) = \tilde{f}([w_1][w_2]^{(s)}) = Q^s(\tilde{f}(w_1[w_2])) = Q^s(\tilde{f}(w_1) \ast Q(\tilde{f}(w_2))).
$$

Since $Q$ is an averaging operator, we further have

$$
\tilde{f}(P_X(w)) = Q(\tilde{f}(w_1)) \ast \tilde{f}(w_2) = Q(\tilde{f}(w_1) \ast Q(\tilde{f}(w_2))).
$$

If $w$ is in $\sqcup_{r \geq 1}(\mathcal{A}_0 \ast \mathcal{A}_n)^r \ast \mathcal{A}_0$, then write $w = [w_1]^{(s)} w_2$ with $w_1 \in \mathcal{A}^+$. Then by definition $P_X(w) = [w_1 \circ w_2]^{(s)}$. Thus

$$
\tilde{f}(P_X(w)) = \tilde{f}([w_1] \circ [w_2])^{(s)} = Q^s(\tilde{f}(w_1) \ast \tilde{f}(w_2)) = Q^s(\tilde{f}(w_1) \ast \tilde{f}(w_2)) = Q^s(\tilde{f}(w_1) \ast Q(\tilde{f}(w_2))).
$$

Since $Q$ is an averaging operator, we further have

$$
\tilde{f}(P_X(w)) = Q(\tilde{f}(w_1)) \ast \tilde{f}(w_2) = Q(\tilde{f}(w_1) \ast Q(\tilde{f}(w_2))).
$$

If $w$ is in $\sqcup_{r \geq 1}(\mathcal{A}_0 \ast \mathcal{A}_n)^r \ast \mathcal{A}_0$, then write $w = [w_1]^{(s)} w_2 [w_3]^{(t)}$ with $w_1, w_2, w_3 \in \mathcal{A}^+$. Then $P_X(w) = [w_1 \circ w_2]^{(s)} [w_3]^{(t)}$. We have

$$
\tilde{f}(P_X(w)) = \tilde{f}([w_1 \circ w_2]^{(s)} [w_3]^{(t)})
= Q^{s+t-1}(\tilde{f}(w_1) \ast [w_2][w_3])
= Q^{s+t-1}(\tilde{f}(w_1) \ast \tilde{f}([w_2][w_3]))
= Q^{s+t-1}(\tilde{f}(w_1)) \ast Q(\tilde{f}(w_2) \ast Q(\tilde{f}(w_3)))
= Q^s(\tilde{f}(w_1)) \ast Q^t(\tilde{f}(w_2) \ast Q(\tilde{f}(w_3)))
= Q^t(\tilde{f}(w_1)) \ast Q^s(\tilde{f}(w_2) \ast Q(\tilde{f}(w_3))).
$$

(Q is an averaging operator)
\[
\begin{align*}
&= Q(Q'(\tilde{f}(w_1)) \ast (\tilde{f}(w_2) \ast Q'(\tilde{f}(w_3)))) \\
&= Q(\tilde{f}(\lfloor w_1 \rfloor^{(3)}) \ast (\tilde{f}(w_2) \ast \tilde{f}(\lfloor w_3 \rfloor^{(3)}))) \\
&= Q(\tilde{f}(w)).
\end{align*}
\]

Therefore \( \tilde{f} \) commutes with the averaging operator. This completes the proof of Theorem 2.11. The proof of Theorem 2.11 is now completed.

3. Enumeration in averaging algebras and large Schröder numbers

In this section, we study the enumeration and generating functions of free averaging algebras. We first give the generating function of averaging words in two variables parameterizing the number of appearances of the variable and the operator respectively. We then observe that the generating function in one variable for averaging words (resp. indecomposable averaging words) with one idempotent operator and one idempotent generator is twice (resp. \( z \) times) the generating function of large Schröder numbers. This motivates us to give two interpretations of large Schröder numbers in terms of averaging words and a class of decorated rooted trees. As a result, we obtain a recursive formula for large Schröder numbers.

3.1. Enumeration of averaging words. In this section, we restrict ourselves to the set of averaging words with one generator and one idempotent operator and then give some results on enumerations of this set. For an idempotent operator, \( P \) is an averaging operator if and only if it is a Reynolds operator. Interestingly, in most applications of averaging algebras in physics (invariant theory and fluid dynamics), function spaces, Banach algebras, the operators are idempotent.

Under the condition that the operator \( \lfloor \rfloor \) is idempotent, no two pairs of brackets can be immediately adjacent or nested in an averaging word. Enumerations of Rota-Baxter words are given in [17]. We will follow the similar notations and apply the similar method to solve the enumeration problem of averaging words.

For an averaging word \( w \), an \( x \)-run is any occurrence in \( w \) of consecutive products of \( x \) of maximal length. Let \( v \) be either a positive integer or \( \infty \) and let \( \mathcal{A}_v \) be the subset of \( \mathcal{A} \) (including \( 1 \)) where the length of \( x \)-runs is \( \leq v \) with the convention that there is no restriction on \( x \)-runs when \( v = \infty \). The number of balanced pairs of brackets (resp. of \( x \)) in an averaging word is called its degree (resp. arity). For \( n \geq 1 \), let \( \mathcal{A}_v(n) \) denote the subset of \( \mathcal{A}_v \) consisting of all averaging words of degree \( n \). For \( m \geq 1 \), let \( \mathcal{A}_v(n, m) \) denote the subset of \( \mathcal{A}_v \) consisting of averaging words with degree \( n \) and arity \( m \). Moreover, for \( 1 \leq k \leq m \), we let \( \mathcal{A}_v(n, m; k) \) be the subset of \( \mathcal{A}_v(n, m) \) consisting of averaging words where the \( m \) \( x \)'s are distributed into exactly \( k \) \( x \)-runs.

Let \( G(m, k, v) \) be the set of compositions of the integer \( m \) into \( k \) positive integer parts, with each part at most \( v \) and let \( g(m, k, v) \) be the size of this set (\( v = \infty \) means there is no restrictions on the size of each part). Then we have \([17, 21]\)

\[
G_{k,v}(t) := \sum_{m=1}^{\infty} g(m, k, v) t^m = t^k \left( \frac{1 - t^v}{1 - t} \right)^k,
\]

and

\[
G_{k,\infty}(t) := \sum_{m=1}^{\infty} g(m, k, \infty) t^m = \left( \frac{t}{1 - t} \right)^k.
\]

In particular, by Eq. (31), we have \( v = 1 \) implies \( G_{k,1}(t) = t^k \).

\(^1\)In order to consider \( x \) to be an associate in Section 2.2, we add the trivial averaging word \( 1 \) to \( \mathcal{A} \).
By the definition of \( \mathcal{A}_v(n, m) \), we have the disjoint union
\[
\mathcal{A}_v(n, m) = \bigsqcup_{k=1}^m \mathcal{A}_v(n, m; k).
\]

By the definition of \( \mathcal{A}_1(n, m; k) \), we have that \( \mathcal{A}_1(n, m; k) \neq \emptyset \) implies \( m = k \). Then we have \( \mathcal{A}_1(n, m) = \mathcal{A}_1(n, k) \). We define a map
\[
\Phi_{v,n,m} : \mathcal{A}_v(n, m) \rightarrow \mathcal{A}_1(n, k)
\]
by sending \( w \in \mathcal{A}_v(n, m) \) to the averaging word \( \Phi_{v,n,m}(w) \) in \( \mathcal{A}_1(n, k) \) obtained by replacing each of the \( x \)-runs appearing in \( w \) by a single \( x \). This map is clearly surjective for each pair \((n, k)\). Further, each of the fiber (inverse image) of \( \Phi_{v,n,m} \) has \( g(m, k, n) \) elements, giving rise to a bijection
\[
\Psi : \mathcal{A}_v(n, m) \leftrightarrow \mathcal{A}_1(n, k) \times G(m, k, v).
\]

Therefore by Eq. (33), the numbers \( a_v(n, m) \) of averaging words of degree \( n \) and arity \( m \) in the set \( \mathcal{A}_v \) are given by
\[
a_v(n, m) = \sum_{k=1}^m g(m, k, v) a_1(n, k).
\]

We next determine the expressions of the generating functions \( \mathcal{A}_v(z, t) \) of the number sequences \( a_v(n, m), n, m \geq 0 \), for \( 1 \leq v \leq \infty \).

**Theorem 3.1.** Let \( 1 \leq v \leq \infty \). The generating function \( \mathcal{A}_v(z, t) \) for the number \( a_v(n, m), n, m \geq 0 \), of averaging words is given by \( \mathcal{A}_v(z, t) = \mathcal{A}_1(z, G_{1,v}(t)) \). where \( G_{1,v} \) is given by Eq. (37) for finite \( v \) and by Eq. (32) for infinite \( v \).

**Proof.** We have
\[
\mathcal{A}_v(z, t) = \sum_{n,m \geq 1} a_v(n, m) z^n t^m
\]
\[
= \sum_{n,m \geq 1} \sum_{k=1}^m g(m, k, v) a_1(n, k) z^n t^m
\]
\[
= \sum_{m \geq 1} \left( \sum_{k \geq 1} g(m, k, v) \left( \sum_{n \geq 1} a_1(n, k) z^n \right) \right) t^m
\]
\[
= \sum_{k \geq 1} \left( \sum_{n \geq 1} a_1(n, k) z^n \right) (G_{1,v}(t))^k
\]
\[
= \mathcal{A}_1(z, G_{1,v}(t)).
\]

By Eq. (32), we also have

**Corollary 3.2.** We have the generating function
\[
\mathcal{A}_\infty(z, t) = \mathcal{A}_1 \left( z, \frac{t}{1-t} \right).
\]

Now we have reduced the problem of finding the explicit expression of the generating function for \( a_\infty(n, m) \) to the problem of finding \( \mathcal{A}_1(z, t), \) to be considered in the next subsection.
3.2. The generating function in the case of idempotent generator and operator. In this section, we will focus on the generating function \( \mathcal{U}_1(z, t) \) for \( a_1(n, m) \). First, we give some descriptions of the word structures of averaging words. Note that \( \mathcal{U}_1 \) is the subset consisting of 1 and averaging words \( w \) composed of \( x \)'s and pairs of balanced brackets such that no two \( x \)'s are adjacent, and no two pairs of brackets can be immediately adjacent or nested. This special case can be considered as the case that we have \( X = \{x\} \) and \( x^2 = x, [ ]^2 = [ ] \).

In the rest of this section, all averaging words are assumed to be in \( \mathcal{U}_1 \).

For \( n > 0 \), let \( B(n) \) be the subset of \( \mathcal{U}_1(n) \) consisting of averaging words that begin with a left bracket and end with a right bracket and words in \( B(n) \) are said to be bracketed. By pre- or post-concatenating a bracketed averaging word \( w \) with \( x \), we get three new averaging words: \( xw, wx, \) and \( xwx \), which are called respectively the left, right, and bilateral associate of \( w \). We also consider \( x \) to be an associate of the trivial word \( 1 \). Any nontrivial averaging word is either bracketed or an associate. Thus for \( n > 0 \), the set \( C(n) \) of all associates is the disjoint union

\[
C(n) = xB(n) \sqcup B(n)x \sqcup xB(n)x, \quad n > 0.
\]

and forms the complement of \( B(n) \) in \( \mathcal{U}_1(n) \). Define \( C^-(n) := B(n)x \). We call the words in the subset \( xB(n) \sqcup B(n)x \) admissible. The set of bracketed averaging words is further divided into two disjoint subsets. The first subset \( I(n) \) consists of all indecomposable bracketed averaging words whose beginning left bracket and ending right bracket are paired, like \( [x]x[j] \). The second subset \( D(n) \) consists of all decomposable bracketed averaging words whose beginning left bracket and ending right bracket are not paired, like \( [x]x[j] \). For the convenience in counting, we define \( B(0), I(0), D(0) \) to be the empty set and note that \( C(0) = \{x\} \). With our convention, denote \( \mathcal{U}(0) := \mathcal{U}_1(0) = \{1, x\} \).

The following table lists these various types of averaging words in lower degrees

| deg | I(n) | D(n) | C(n) | C^-(n) | B(n) |
|-----|-----|-----|-----|--------|-----|
| 0   | \( \{x\} \) | \( x \) | \( \{x\} \) | \( \{x\} \) | \( \{x\} \) |
| 1   | \( [x] \) | \( x[x], [x][x]x \) | \( x[x]x \) | \( [x] \) |
| 2   | \( [x][x], [x][x]x \) | \( 9 \) associates | \( B(2)x \) | \( I(2) \sqcup D(2) \) |

The production rules will be

\[
\begin{align*}
(36) \quad & \langle AW \rangle \rightarrow 1 \mid \langle \text{bracketed} \rangle \mid \langle \text{associate} \rangle \\
(37) \quad & \langle \text{associate} \rangle \rightarrow x \mid x(\langle \text{bracketed} \rangle \mid \langle \text{bracketed} \rangle x \mid x(\langle \text{bracketed} \rangle x \\
(38) \quad & \langle \text{bracketed} \rangle \rightarrow \langle \text{indecomposable} \rangle \mid \langle \text{decomposable} \rangle \\
(39) \quad & \langle \text{indecomposable} \rangle \rightarrow \langle \text{admissible} \rangle \\
(40) \quad & \langle \text{decomposable} \rangle \rightarrow \langle \text{bracketed} \rangle x(\langle \text{bracketed} \rangle)
\end{align*}
\]

An averaging word \( w \) has arity \( m \) means the number of occurrences of \( x \) in \( w \) is \( m \). For any \( m \geq 0 \), let \( \mathcal{U}(n, m) \) be the subset of \( \mathcal{U} \) consisting of words with degree \( n \) and arity \( m \), and define similarly the notations \( C(n, m), C^-(n, m), B(n, m), I(n, m), \) and \( D(n, m) \). These are all finite sets. Let their sizes be \( a_n, c_n, c_n^-, b_n, i_n, d_n \) respectively. For initial values, we have

\[
\begin{align*}
& a_{0,0} = 1; \quad c_{0,0} = b_{0,0} = i_{0,0} = d_{0,0} = 0; \\
& a_{0,1} = c_{0,1} = 1; \quad b_{0,1} = i_{0,1} = d_{0,1} = 0; \\
& a_{1,1} = b_{1,1} = i_{1,1} = 1; \quad c_{1,1} = d_{1,1} = 0; \\
& a_{1,2} = c_{1,2} = 2; \quad b_{1,2} = i_{1,2} = d_{1,2} = 0;
\end{align*}
\]
\[ a_{1,3} = c_{1,3} = 1; \quad b_{1,3} = i_{1,3} = d_{1,3} = 0; \]
\[ a_{0,m} = c_{0,m} = b_{0,m} = i_{0,m} = d_{0,m} = 0 \quad \text{for } m \geq 2; \]
\[ a_{1,m} = c_{1,m} = b_{1,m} = i_{1,m} = d_{1,m} = 0 \quad \text{for } m \geq 4; \]
\[ a_{n,0} = c_{n,0} = b_{n,0} = i_{n,0} = d_{n,0} = 0 \quad \text{for } n \geq 1; \]
\[ a_{n,1} = c_{n,1} = b_{n,1} = i_{n,1} = d_{n,1} = 0 \quad \text{for } n \geq 2. \]

From the production rules (36)–(40), we see that for \( n \geq 1, m \geq 2, \)
\[ a_{n,m} = b_{n,m} + c_{n,m}, \]
\[ c_{n,m} = 2b_{n,m-1} + b_{n,m-2}, \]
\[ b_{n,m} = i_{n,m} + d_{n,m}, \]
\[ i_{n,m} = c_{n-1,m} - c_{n-1,m} = c_{n-1,m} - b_{n-1,m-1}, \]
where Eq. (44) follows from Eq. (35) and \( C^{-}(n) = B(n)x. \)

Now for \( n \geq 2, m \geq 2 \) and any \( w \in D(n, m) \), we can write \( w \) uniquely as \( w_{n_1}xw_{n_2}\cdots xw_{n_p} \) where \( w_{n_j} \in I(n_j) \) and \( n_1 + \cdots + n_p \) is a composition of \( n \) using \( p \) positive integers. Let \( m_j \) be the arity of \( w_{n_j}. \) Then clearly, \( m_1 + \cdots + m_p = m - p + 1. \) So we have
\[ d_{n,m} = \sum_{p=2}^{\min(n,m)} \sum_{(m_1,\ldots,m_p,m-p+1)} \sum_{(n_1,\ldots,n_p,m)} (i_{n_1,m_1})\cdots(i_{n_p,m_p}), \]
The case when \( p = 1 \) corresponds to a single summand \( i_{n,m}, \) and then
\[ b_{n,m} = i_{n,m} + d_{n,m} = \sum_{p=1}^{\min(n,m)} \sum_{(m_1,\ldots,m_p,m-p+1)} \sum_{(n_1,\ldots,n_p,m)} (i_{n_1,m_1})\cdots(i_{n_p,m_p}). \]
Now from Eqs. (42) and (44), we have
\[ i_{n,m} = c_{n-1,m} - b_{n-1,m-1} = 2b_{n-1,m-1} + b_{n-1,m-2} - b_{n-1,m-1} = b_{n-1,m-1} + b_{n-1,m-2}. \]

Define the bivariate generating series
\[ \mathcal{A}(z, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n,m}z^nm^t \]
and similarly define \( B(z, t), I(z, t), D(z, t), \) and \( C(z, t). \) Note that for \( B(z, t), I(z, t), \) and \( D(z, t), \) it does not matter whether the series indices \( n, m \) start at 0 or 1. We will multiply both sides of Eq. (47) by \( z^n t^m \) and sum up for \( n \geq 2, m \geq 2. \) Since \( i_{0,m} = i_{n,0} = 0 \) for all \( m, n \geq 0 \) and \( i_{1,m} = i_{n,1} = 0 \) for all \( m, n \geq 2, \) then the left hand side gives
\[ \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} i_{n,m}z^nm^t = I(z, t) - zt. \]

Now, we sum the right hand side of Eq. (47) one term at a time.
\[ \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} b_{n-1,m-1}z^nm^t = zt \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{n,m}z^nm^t = ztB(z, t), \]
\[ \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} b_{n-1,m-2}z^nm^t = zt^2 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} b_{n,m}z^nm^t = zt^2(B(z, t) + \sum_{n=1}^{\infty} b_{n,0}z^n) = zt^2B(z, t). \]
Hence, we have the identity
\[(48)\quad I(z, t) - zt = zt(1 + t)B(z, t).\]

Using Eq. \((46)\), we have
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{n,m} z^n t^m = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \min(n,m) \sum_{p=1}^{m} \sum_{(m_1, \ldots, m_p, m-p+1)} \sum_{(n_1, \ldots, n_p, n-p)} (i_{n_1,m_1}) \cdots (i_{n_p,m_p}) z^n t^m
\]
\[
= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \min(n,m) \sum_{p=1}^{m} \sum_{(m_1, \ldots, m_p, m-p+1)} \sum_{(n_1, \ldots, n_p, n-p)} (i_{n_1,m_1} z^{n_1} t^{m_1}) \cdots (i_{n_p,m_p} z^{n_p} t^{m_p}) z^n t^m
\]
\[
= \sum_{p=1}^{\infty} \left(\sum_{k=1}^{\infty} i_{k,\ell} z^{k} t^{\ell}\right)^p t^{p-1}
\]
and hence
\[(49)\quad B(z, t) = \frac{I(z, t)}{1 - tI(z, t)}.\]

Thus we obtained the identity defining \(I(z, t)\) as
\[(50)\quad I(z, t) - zt = zt(1 + t)\frac{I(z, t)}{1 - tI(z, t)}.
\]

Solving this quadratic equation in \(I(z, t)\) and using the initial conditions, we find
\[(51)\quad I(z, t) = \frac{1 - zt - \sqrt{z^2 t^2 - (2t + 4t^2)z + 1}}{2t}.
\]

Then by Eq. \((49)\), we have
\[(52)\quad B(z, t) = \frac{1 - zt - 2zt^2 - \sqrt{z^2 t^2 - (2t + 4t^2)z + 1}}{2zt^2(1 + t)}.
\]

Furthermore, from Eq. \((46)\), we obtain
\[(53)\quad D(z, t) = B(z, t) - I(z, t)
\]
\[
= \frac{1 - 2zt - 3zt^2 + z^2 t^2 + z^2 t^3 + (zt + zt^2 - 1) \sqrt{z^2 t^2 - (2t + 4t^2)z + 1}}{2zt^2(1 + t)}.
\]

We can also obtain the bivariate generating series for \(c_{n,m}\):
\[
C(z, t) = \sum_{n=0}^{\infty} c_{n,0} z^n t^m + \sum_{m=2}^{\infty} c_{n,m} z^n t^m
\]
\[
= t + \sum_{m=2}^{\infty} c_{0,m} t^m + \sum_{n=1}^{\infty} \left(\sum_{m=2}^{\infty} c_{n,m} z^n t^m\right)
\]
\[
= t + \sum_{n=1}^{\infty} \sum_{m=2}^{\infty} (2b_{n,m-1} + b_{n,m-2}) z^n t^m.
\]
\[ A(z, t) = \frac{2 + t - 2zt - 3zt^2 - (2 + t) \sqrt{z^2t^2 - (2t + 4t^2)z + 1}}{2zt(1 + t)} \]

giving

by Eq. (52).

Finally, utilizing \( \mathcal{A}(z, t) = 1 + B(z, t) + C(z, t) \), Eqs. (52) and (54) we derive the following generating function \( \mathcal{A}_1(z, t) = \mathcal{A}(z, t) \) for the sequences \( a_{n,m} = a_1(n, m) \).

Recall that the averaging words in \( \mathcal{A}_1 \) can be considered as the case that we have \( X = \{x\} \) and \( x^2 = x, \lceil \rceil^2 = \lceil \rceil \). We conclude our discussion on generating functions by ignoring the arity and only focus on the degree. For \( n > 0 \), let \( a_n \) (resp. \( c_n \), resp. \( b_n \), resp. \( i_n \), resp. \( d_n \)) be the number of all (resp. associate, resp. bracketed, resp. indecomposable, resp. decomposable) averaging words with \( n \) pairs of (balanced) brackets. For example, we have \( a_0 = 2 \) since such averaging words with no operators are \( \{1, x\} \); while \( a_1 = 4 \) since those with one operators are \( \{1, x, [x]x, x[x, x]\} \).

Putting \( t = 1 \) in the generating functions of \( \mathcal{A}(z, t), B(z, t), I(z, t), D(z, t) \) and \( C(z, t) \), we obtain

**Theorem 3.3.** The generating series for \( a_n, b_n, i_n, d_n \) and \( c_n \) are given by

\[
\begin{align*}
\mathcal{A}(z) & = \sum_{n=0}^{\infty} a_n z^n = \frac{1 - z - \sqrt{z^2 - 6z + 1}}{z}, \\
B(z) & = \sum_{n=0}^{\infty} b_n z^n = \frac{1 - 3z - \sqrt{z^2 - 6z + 1}}{4z}, \\
I(z) & = \sum_{n=0}^{\infty} i_n z^n = \frac{1 - z - \sqrt{z^2 - 6z + 1}}{2}, \\
D(z) & = \sum_{n=0}^{\infty} d_n z^n = \frac{1 - 5z + 2z^2 + (2z - 1) \sqrt{z^2 - 6z + 1}}{4z}, \\
C(z) & = \sum_{n=0}^{\infty} c_n z^n = \frac{3 - 5z - 3 \sqrt{z^2 - 6z + 1}}{4z}.
\end{align*}
\]

From the theorem we obtain

**Corollary 3.4.** (a) The sequence \( a_n, n \geq 0 \), of averaging words of degree \( n \) with one idempotent generator and one idempotent operator, is twice the sequence \( s_n \) of large Schröder numbers: \( a_n/2 = s_n, n \geq 0 \). The first few terms of \( a_n, n \geq 0 \), are

\[ 2, 4, 12, 44, 180, 788, 3612, 17116, \ldots. \]

(b) The sequence \( i_n, n \geq 1 \), of bracketed indecomposable averaging words of degree \( n \) is the sequence \( s_n, n \geq 0 \), of large Schröder numbers: \( i_{n+1} = s_n, n \geq 0 \). The first few terms of \( i_n, n \geq 0 \), are

\[ 0, 1, 2, 6, 22, 90, 394, 1806, \ldots. \]
Proof. Both results are proved by comparing the corresponding generating functions with the generating function
\[ S(z) := \sum_{n=0}^{\infty} s_n z^n = \frac{1 - z - \sqrt{z^2 - 6z + 1}}{2z} \]
of large Schröder sequence (A006318 in the On-line Encyclopedia of Integer Sequences [29]).

On the other hand, the number \( d_n \) of bracketed decomposable averaging words of degree \( n \) \((n \geq 0)\) is a sequence which starts with
\[ 0, 0, 1, 5, 23, 107, 509, 2473, \ldots. \]
This sequences is a new integers sequence which can not be found in [23].

3.3. Averaging words and large Schröder numbers. The sequence \( s_n, n \geq 0 \), of large Schröder numbers (A006318 in [29]) is an important sequence of integers with numerous interesting properties and interpretations. For example, \( s_n \) counts the number of Schröder paths of semilength \( n \) is a lattice path on the plane from \((0, 0)\) to \((2n, 0)\) that does not go below the \( x \)-axis and consists of up steps \( U = (1, 1) \), down steps \( D = (1, -1) \) and horizontal steps \( H = (2, 0) \). See [29] for more details.

Corollary 3.4 gives two more interpretations of large Schröder numbers in terms of averaging words with a single idempotent generators and idempotent operator. Motivated by this, we next give another interpretation of the sequence of large Schröder numbers in terms of decorated trees.

Definition 3.5. (a) Let \( \mathcal{X} \) denote the set of planar reduced rooted trees together with the trivial tree \( \mathcal{X} = \{x\} \). By a \((\omega, \iota)\)-decorated tree we mean a tree \( t \) in \( \mathcal{X} \) together with a decoration on the vertices of \( t \) by the symbol \( \omega \) and a decoration on the leaves of \( t \) by \( \omega \) or another symbol \( \iota \). Let \( D(t) \) denote the set of decorated trees from \( t \) and denote
\[ D(\mathcal{X}) := \bigsqcup_{t \in \mathcal{X}} D(t). \]

(b) Let \( t \) be in \( \mathcal{X} \) whose number of leafs is greater than 1. Then there exists an integer \( m \) such that \( t \) can be written uniquely as the grafting \( \bigvee_{i=1}^{m} t_i := t_1 \lor \cdots \lor t_m \) of trees \( t_1, t_2, \cdots t_m \). The trees \( t_1, \cdots, t_m \) are called the branches of \( t \).

(c) Let \( \tau \in D(t) \) where \( t \) is \( \mathcal{X} \) with grafting \( t = t_1 \lor \cdots \lor t_m \). Then there are \( \tau_i \in D(t_i), 1 \leq i \leq m \), such that \( \tau \) is the grafting \( \omega(\bigvee_{i=1}^{m} \tau_i = \omega(\tau_1 \lor \cdots \lor \tau_m) \) of \( \tau_1, \cdots, \tau_m \) with the new root decorated by \( \omega \).

Since the rooted trees we are considering are reduced, we have \( m \geq 2 \) in any grafting of trees. Now, we define a special subset of \( D(\mathcal{X}) \).

Definition 3.6. (a) A \((\omega, \iota)\)-decorated tree \( \tau \in D(\mathcal{X}) \) is called a Schröder tree if either \( \tau \) is the trivial tree decorated by \( \omega \) or \( \tau \) satisfies the following conditions: For each vertex \( v \) of \( \tau \), let \( \tau_v \) be the subtree of \( \tau \) with root \( v \). Then
(i) the leftmost branch of \( \tau_v \) is a leaf decorated by \( \iota \);
(ii) the branches of \( \tau_v \) are alternatively a leaf decorated by \( \iota \) and a subtree that is not a leaf decorated by \( \iota \) (the latter means that the subtree is either not a leaf or a leaf decorated by \( \omega \)).
To put it in another way, let \( \tau_{v,i} \) be of the form \( \omega(\tau_{v,i+1} \lor \cdots \tau_{v,i+k}) \), then each \( \tau_{v,i} \) for \( i \) odd is a leaf decorated by \( \iota \) and each \( \tau_{v,i} \) for \( i \) even is either not a leaf or is a leaf decorated by \( \omega \).
(b) Let $\text{ST}$ denote the set of Schröder trees and let $\text{ST}_n$ denote the set of Schröder trees with $n$ vertices or leaves decorated by $\omega$.

**Theorem 3.7.**  
(a) The sequence $|\text{ST}_n|_{n \geq 1}$ counting Schröder trees is the sequence $\{i_n\}_{n \geq 1}$ counting indecomposable averaging words: $|\text{ST}_n| = i_n, n \geq 1$.
(b) The sequence $|\text{ST}_n|_{n \geq 1}$ coincides the sequence $\{s_n\}_{n \geq 0}$ of large Schröder numbers. In other words,

$$|\text{ST}_{n+1}| = i_{n+1} = s_n, n \geq 0.$$  

(c) The sequence $\{s_n\}_{n \geq 0}$ of large Schröder numbers satisfies the following recursion:

$$s_0 = 1,$$

$$s_n = 2 \sum_{j=1}^{n} \sum_{(p_1, \cdots, p_j) \in G(n, j)} s_{p_1-1} \cdots s_{p_j-1}.  \tag{57}$$

We illustrate the theorem by listing the first three elements of $I(n)$ and $\text{ST}_n, n \geq 1$.

| I(1)    | [x] | 1     |
|---------|-----|-------|
| I(2)    | [x][x], [x][x][x] | 2     |
| I(3)    | [x][x][x], [x][x][x][x], [x][x][x][x][x], [x][x][x][x][x][x], [x][x][x][x][x][x][x] | 6     |

2. $\text{ST}_1 : \omega$;

3. $\text{ST}_2 : \omega \omega, \omega \omega \omega, \omega \omega \omega \omega$

4. $\text{ST}_3 : \omega \omega \omega, \omega \omega \omega \omega, \omega \omega \omega \omega \omega, \omega \omega \omega \omega \omega \omega$

**Proof.** We just need to prove that the sequence $|\text{ST}_n|_{n \geq 1}$ and $\{i_n\}_{n \geq 1}$ satisfy the same recursion relation and the same initial condition.

Let $G(n, k)$ be the set of compositions of the integer $n$ into $k$ positive integer parts. Let $I(n)$ denote the set of bracketed indecomposable averaging words of degree $n$ and $I$ by the set of all bracketed indecomposable averaging words.

First we have $I(1) = \{|x|\}$ and $\text{ST}_1 = \{|\}$, Hence $|\text{ST}_1| = 1 = i_1$.

Next, let $n \geq 2$. For any word $W = [w_1 w_2 \cdots w_m] \in I(n)$, we have $w_i = x$ for $i$ odd and $w_i \in I$ for $i$ even. Then there exist $p_i, 1 \leq i \leq k := \lfloor m/2 \rfloor$ such that $w_{2i}$ is in $I(p_i)$. Thus we have $p_1 + \cdots + p_k = n - 1$ and so $(p_1, \cdots, p_k) \in G(n-1, k)$. Note that there are two $W$’s that give the same $(p_1, \cdots, p_k)$: one is $[w_1 \cdots w_{2k}]$, the another is $[w_1 \cdots w_{2k+1}]$.

Conversely, let $(p_1, \cdots, p_k) \in G(n-1, k)$ and take $w_2 \in I(p_1), \cdots, w_{2k} \in I(p_k)$ and $w_1 = \cdots = w_{2k-1} = w_{2k+1} = x$. Then $[w_1 \cdots w_{2k}]$ and $[w_1 \cdots w_{2k+1}]$ are in $I(n)$.

Therefore, we have the following recursive formula for $i_n, n \geq 2$:

$$i_n = 2 \left( i_{n-1} + \cdots + \sum_{(p_1, \cdots, p_j) \in G(n-1, j)} i_{p_1} \cdots i_{p_j} \right) = 2 \sum_{j=0}^{n-1} \sum_{(p_1, \cdots, p_j) \in G(n-1, j)} i_{p_1} \cdots i_{p_j}. \tag{59}$$

On the other hand, if $\tau \in \text{ST}_n$, then there exists $k \geq 0$ such that either $\tau = \omega(\sqrt{2} \tau_i)$ or $\omega(\sqrt{2} \tau_i)$. Furthermore, each $\tau_i$ with $i$ odd is a leaf decorated by $\omega$ and each $\tau_{2i}, 1 \leq i \leq k$, is in
ST_{p_i} for some integer \( p_i \geq 1 \). Since \( \omega \) does not appear in \( \tau_i \) for \( i \) odd, we have \( p_1 + p_2 + \cdots + p_k = n - 1 \). That is, \((p_1, \cdots, p_k)\) is in \( G(n - 1, k) \).

Conversely, let \((p_1, \cdots, p_k)\) be in \( G(n - 1, k) \). Take \( \tau_{2i} \in ST_n, 1 \leq i \leq k \) and take \( \tau_i \) for odd \( i \) to be a leaf decorated by \( i \). Then the \((\omega, i)\)-decorated trees \( \omega(\sqrt{2^k} \tau_i) \) and \( \omega(\sqrt{2^{k+1}} \tau_i) \) are in \( ST_n \).

By the above argument, we obtain the following recursive formula for \(|ST_n|, n \geq 2\).

\[
|ST_n| = 2 \left( |ST_{n-1}| + \cdots + \sum_{(p_1, \cdots, p_j) \in G(n-1, j)} (|ST_{p_1}| \cdots |ST_{p_j}|) \right) + \cdots + |ST_1|^{n-1}
\]

\[
= \sum_{j=0}^{n-1} \sum_{(p_1, \cdots, p_j) \in G(n-1, j)} (|ST_{p_1}| \cdots |ST_{p_j}|).
\]

In summary, \( i_n, n \geq 1 \) and \(|ST_n|, n \geq 1\) have the same initial value and the same recursive relation. Therefore \( i_n = |ST_n|, n \geq 1 \).

\( \blacksquare \) This follows from Item \( \blacksquare \) and Corollary \( \blacksquare \).

\( \blacksquare \) This follows from Item \( \blacksquare \) and Eq. \( \blacksquare \).

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References

[1] M. Aguiar, Pre-Poisson algebras, Lett. Math. Phys. 54 (2000) 263-277.
[2] C. Bai, O. Bellier, L. Guo and X. Ni, Splitting of operations, Manin products and Rota-Baxter operators, Int. Math. Res. Not. (2012), no. 3, 485-524.
[3] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, Pacific J. Math. 10 (1960), 731-742.
[4] B. Brainerd, On the structure of averaging operators, J. Math. Anal. 5 (1962) 347-377.
[5] G. Birkhoff, Moyennes de fonctions bornées, Coll. Internat. Centre Nat. Recherthe Sci. (Paris), Algèbre Théorie Nombres 24 (1949), 149-153.
[6] N. H. Bong, Some apparent connection between Baxter and averaging operators, J. Math. Anal. Appl. 56 (1976), 330-345.
[7] W. Cao, An algebraic study of averaging operators, Ph.D. Thesis, Rutgers University at Newark, 2000.
[8] F. Chapoton, Un endofoncteur de la catégorie des opérades, in: Dialgebras and related operads, Lect. Notes Math., Springer-Verlag, 2001.
[9] K. Ebrahimi-Fard and L. Guo, Rota-Baxter algebras and dendriform algebras, Jour. Pure Appl. Algebra 212 (2008) 320-339.
[10] W. Fechner, Inequalities connected with averaging operators, Indag. Math. 24 (2013), 305-312.
[11] J.L.B. Gamlen, J.B. Miller, Averaging operators and Reynolds operators on Banach algebras II. spectral properties of averaging operators, J. Math. Anal. Appl 23 (1968), 183-197.
[12] V. Yu. Gubarev and P. S. Kolesnikov, On embedding of dendriform algebras into Rota-Baxter algebras, Cent. Eur. Jour. Math 11 (2013), 226-245.
[13] L. Guo, An Introduction to Rota-Baxter Algebra, International Press, 2012.
[14] L. Guo, Operated semigroups, Motzkin paths and rooted trees, J. Algebraic Combinatorics 29 (2009), 35-62.
[15] L. Guo, M. Rosenkranz and S. Zheng, Rota-Baxter operators on the polynomial algebra, preprint.
[16] L. Guo, W. Y. Sit, Enumeration and generating functions of Rota-Baxter words (with W. Sit), Math. Comput. Sci. 4 (2010), 313-337.
[17] J. Kampde Féret, L’état actuel du problème de la turbuance (I and II), La Sci. Aérienne 3 (1934) 9-34, 4 (1935), 12-52.
[18] J.L. Kelley, Averaging operators on $C_\infty(X)$, Illinois J. Math. 2 (1958), 214-223.

[19] J.-L. Loday, Dialgebras, in Dialgebras and related operads, Lecture Notes in Math. 1763 (2002) 7-66.

[20] J.B. Miller, Averaging and Reynolds operators on Banach algebra I, Representation by derivation and antiderivations, J. Math. ANAL. APPL. 14 (1966) 527-548.

[21] P.A. MacMahon, Combinatory Analysis. Chelsea Pub. Co., NewYork, third edition, 1984.

[22] J. Pei, C. Bai, L. Guo and X. Ni, Replicating of binary operads, Koszul duality, Manin products and averaging operators, arXiv:1212.0177v2.

[23] J. Pei, C. Bai, L. Guo and X. Ni, Dissuccessors and duplicators of operads, Manin products and operators, In “Symmetries and Groups in Contemporary Physics”, Nankai Series in Pure, Applied Mathematics and Theoretical Physics 11 (2013) 191-196.

[24] C. Reutenauer, Free Lie Algebras. London Mathematical Society Monographs New Series 7, The Clarendon Press Oxford University Press, 1993.

[25] O. Reynolds, On the dynamic theory of incompressible viscous fluids, Phil. Trans. Roy. Soc. A 136 (1895), 123-164.

[26] G.C. Rota, On the representation of averaging operator, Rendiconti del Seminario Matematico della Universita di Padova 30 (1960), 52-64.

[27] G.C. Rota, Reynolds operators, Proceedings of Symposia in Applied Mathematics, Vol. XVI (1964), Amer. Math. Soc., Providence, R.I., 70-83.

[28] G.-C. Rota, Baxter algebras and combinatorial identities I, II, Bull. Amer. Math. Soc. 75 (1969), 325-329, 330-334.

[29] Sloane, N., et al. On-Line Encyclopedia of Integer Sequences, URL: http://www.research.att.com/jas/sequences/index.html

[30] S.-T. C. Moy, Characterizations of conditional expectation as a transformation on function spaces, Pacific J. Math. 4 (1954), 47-63.

[31] A. Triki, Extensions of positive projections and averaging operators, J. Math. Anal. 153 (1990), 486-496.

[32] A. Triki, A note on averaging operators, function spaces (Ed-wardsville, IL, 1998), (1999), 345-348.

Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102
E-mail address: liguo@rutgers.edu

Department of Mathematics, Lanzhou University, Lanzhou, Gansu 730000, China
E-mail address: peitsun@163.com