Necessary and sufficient conditions for local manipulation of multipartite pure quantum states

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Abstract. Suppose that several parties jointly possess a pure multipartite state, $|\psi\rangle$. Using local operations on their respective systems and classical communication (i.e. LOCC), it may be possible for the parties to transform deterministically $|\psi\rangle$ into another joint state $|\phi\rangle$. In the bipartite case, the Nielsen majorization theorem gives the necessary and sufficient conditions for this process of entanglement transformation to be possible. In the multipartite case, such a deterministic local transformation is possible only if both the states are in the same stochastic LOCC (SLOCC) class. Here, we generalize the Nielsen majorization theorem to the multipartite case, and find necessary and sufficient conditions for the existence of a local separable transformation between two multipartite states in the same SLOCC class. When such a deterministic conversion is not possible, we find an expression for the maximum probability to convert one state to another by local separable operations. In addition, we find necessary and sufficient conditions for the existence of a separable transformation that converts a multipartite pure state into one of a set of possible final states all in the same SLOCC class. Our results are expressed in terms of (i) the stabilizer group of the state representing the SLOCC orbit and (ii) the associate density matrices (ADMs) of the two multipartite states. The ADMs play a similar role to that of the reduced density matrices when considering local transformations that involve pure bipartite states. We show, in particular, that the requirement that one ADM majorizes another is a necessary condition but is, in general, far from also being sufficient as it happens in the bipartite case. In most of the results the twirling operation with respect to
the stabilizer group (of the representative state in the SLOCC orbit) plays an important role that provides a deep link between entanglement theory and the resource theory of reference frames.

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1. Introduction

Every restriction on quantum operations defines a resource theory, determining how quantum states that cannot be prepared under the restriction may be manipulated and used to circumvent the restriction. Much of quantum information theory can be viewed as a theory of interconversions among such resources [1]. When several spatially separated parties are sharing a composite quantum system, they are naturally restricted to act locally on the system. Such a restriction to local operations assisted by classical communication (LOCC) leads to the theory of entanglement, in which multipartite entangled states act as resources for overcoming the LOCC restriction. Hence, exotic multipartite entangled states play an important role in a variety of quantum information processing tasks. These include conventional [2] and measurement-based quantum computation [3], quantum error correction schemes [2], quantum secret sharing [4], quantum simulations [5], and in principle the description of every composite system consisting of more than one subsystem [6, 7].

The amount of information needed in order to describe an $n$-party quantum system grows exponentially with $n$, which makes it very difficult and almost impossible to classify multipartite states by the degree of their resourcefulness (i.e. by their ability to circumvent the limitation of
LOCC). The main reason is that the strength of the LOCC restriction increases with $n$, and in general, the extent to which quantum resources (such as multipartite entangled states) can be interconverted decreases as the strength of the restriction increases [8]. It can be shown [9, 10] that for $n > 3$, the LOCC restriction leads to an uncountable number of inequivalent resources (i.e. multipartite entangled states).

A conventional way to classify multipartite states is to divide the Hilbert space into sets of states that are all related to each other by invertible stochastic local operations and classical communication (SLOCC) [9]. More precisely, we say that $\psi$ and $\phi$ belong to the same (invertible) SLOCC class if both transformations $\psi \rightarrow \phi$ and $\phi \rightarrow \psi$ can be achieved by means of LOCC with non-zero probability. Hence, in general, if $\psi$ and $\phi$ belong to two distinct SLOCC classes, they are incomparable and essentially correspond to two different types of resources. Whereas in three qubits there are only six inequivalent SLOCC classes (see [9]), there are an uncountable number of inequivalent SLOCC classes in four (or more) qubits [9, 10].

Roughly speaking, this fact implies that relative to the dimension of the Hilbert space, only a small number of transformations can be achieved by local means (since the states involved must belong to the same SLOCC class). Nevertheless, the very few transformations that can be achieved locally are extremely useful for QIP tasks, as in the examples of transformations involving the cluster states [3], graph states [11], code states [2] and other novel multipartite entangled states [6].

A fundamental problem in quantum information is therefore to characterize all possible deterministic LOCC transformations among states in the same SLOCC class. That is, if $\psi$ and $\phi$ belong to the same SLOCC class, what are the necessary and sufficient conditions that the transformation $\psi \rightarrow \phi$ can be achieved by LOCC with 100% success (i.e. with probability one)? Despite its importance, this problem was solved completely only for the case of pure bipartite states [6, 7]. In fact, even in the case of three qubits, very little is known about the characterization of local transformations among states in the same GHZ class [12]. For a higher number of qubits or in higher dimensions, much less is known and there are only partial results for specific SLOCC classes of states [13–15].

One of the several difficulties encountered in the effort to solve this problem is the non-elegant mathematical description of (the operationally motivated) LOCC protocols in composite quantum systems consisting of more than two parties. This difficulty has been avoided in the literature by considering a larger set of transformations, such as the positive partial transpose preserving operations [16] or the non-entangling operations in [17]. However, these sets of transformations are much larger than what can be achieved by LOCC and therefore include many transformations that cannot be achieved locally. A much smaller set, which, on the one hand, is local and includes all LOCC transformations and, on the other, is mathematically simple to characterize, is the set of local separable transformations (SEP).

SEP can be characterized as a set of trace-preserving quantum channels such that for each channel in the set, there exists an operator sum representation (or Kraus representation) with the property that each Kraus matrix is a tensor product of $n$ matrices acting on the $n$ subsystems constituting the composite system (see section 2 for the mathematical definition of SEP). In [18] it was shown (somewhat counterintuitively) that SEP is strictly larger than LOCC, whereas in [19] it was shown that for bipartite pure states we have effectively SEP = LOCC; that is, any transformation between two bipartite pure states that can be achieved by SEP can also be achieved by LOCC and vice versa. However, for mixed bipartite states this is no longer true [20].
In this paper, we characterize the set of all transformations taking one multipartite pure state to another by SEP. Clearly, unless the two states belong to the same SLOCC class, such a transformation is impossible. For each SLOCC class, we show that there is a natural state (also known as a normal form [21]) that can be chosen to represent the class, and use the stabilizer group of this state to present our results. Unlike the stabilizer formalism in quantum error correction schemes [22], here the stabilizer group is not a subgroup of the Pauli group, but, in general, it is a subgroup of a much larger group, the group of all invertible matrices.

When it is impossible to convert by SEP one state to another, we find an expression for the maximum probability for the conversion. Our expression is, in general, hard to calculate and we therefore provide simple lower and upper bounds on the maximum probability of conversion. In addition, we generalize our results to include necessary and sufficient conditions for the existence of a separable transformation that converts a multipartite pure state into one of a set of possible final states all in the same SLOCC class.

This paper is organized as follows. In section 2, we discuss some preliminary concepts and theorems about SLOCC classes, separable operations and normal forms. In section 3, we define the stabilizer group and prove several of its properties. We also define the stabilizer twirling operation, and discuss the relation between the stabilizer group and SL-invariant polynomials. In section 4, we define the associate density matrix (ADM) and discuss its relationship to the reduced density matrix. In section 5, we present and prove our main results. Section 6 is devoted to examples demonstrating how to apply our results to specific cases. Finally, in section 7 we end the paper with the conclusions.

2. Preliminaries

The Hilbert space of $n$ qudits is denoted by
\[ \mathcal{H}_n \equiv \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}. \]

The SLOCC groups we will consider are the subgroups
\[ G \equiv \text{SL}(d_1, \mathbb{C}) \otimes \text{SL}(d_2, \mathbb{C}) \otimes \cdots \otimes \text{SL}(d_n, \mathbb{C}), \]
\[ \tilde{G} \equiv \text{GL}(d_1, \mathbb{C}) \otimes \text{GL}(d_2, \mathbb{C}) \otimes \cdots \otimes \text{GL}(d_n, \mathbb{C}) \]
in $\text{GL}(\mathcal{H}_n)$. Here $\text{GL}(d, \mathbb{C})$ is the group consisting of all $d \times d$ complex invertible matrices and $\text{SL}(d, \mathbb{C})$ is a subset of these matrices with determinant 1.

Viewing the SLOCC classes as orbits over the action of $G$, we define the orbit of a state $\psi \in \mathcal{H}_n$ to consist of unit vectors:

**Definition 1.** The orbit $\mathcal{O}_\psi$ of a state $|\psi\rangle \in \mathcal{H}_n$ under the action of $G$ is defined by
\[ \mathcal{O}_\psi \equiv \left\{ \frac{g|\psi\rangle}{\|g|\psi\rangle\|} \mid g \in G \right\}. \]

Note that the orbit above is unchanged if one replaces $G$ by $\tilde{G}$, since we ignore global phase. Note also that any state in the orbit can be chosen to represent the orbit; that is, if $\psi' \in \mathcal{O}_\psi$, then $\mathcal{O}_{\psi'} = \mathcal{O}_\psi$. Sometimes in the literature the term ‘SLOCC class’ is used to refer to a similar set as above, but slightly different, allowing the matrix $g$ in the definition above to be not invertible.
2.1. Separable matrices and separable operations

The set of all states that can be prepared by LOCC is called the set of separable states. Here we define it without the normalization requirement of trace one.

**Definition 2.** A matrix (or operator) \( \rho : \mathcal{H}_n \rightarrow \mathcal{H}_n \) is said to be a separable matrix or a separable operator if it can be written as

\[
\rho = \sum_k \sigma_1^{(k)} \otimes \sigma_2^{(k)} \otimes \cdots \otimes \sigma_n^{(k)},
\]

where all \( \sigma_i^{(k)} : \mathbb{C}^{d_i} \rightarrow \mathbb{C}^{d_i} \) are positive semi-definite matrices (operators).

Later on, we will use the fact that any tensor product of matrices can be completed to the identity matrix by the addition of a separable matrix:

**Lemma 1.** Let \( M = A_1 \otimes A_2 \otimes \cdots \otimes A_n \), where \( A_k \) are \( d_k \times d_k \) positive semi-definite matrices. Denote by \( \lambda_{\text{max}} \) the largest eigenvalue of \( M \). Then, for \( p \in \mathbb{R} \), the matrix \( I - pM \) is separable if and only if \( p \leq 1/\lambda_{\text{max}} \).

**Proof.** Clearly, \( pM \leq I \), so \( I - pM \) is positive semi-definite. To see that it is also separable, denote by \( \lambda_j \) the maximum eigenvalue of \( A_j \), denote by \( I_j \) the identity matrix on subsytem \( j \), and w.l.o.g take \( p = 1/\lambda_{\text{max}} = (\lambda_1 \lambda_2, \ldots, \lambda_n)^{-1} \). Then, the matrix \( I_j - \frac{1}{\lambda_j} A_j \) is positive semi-definite, and therefore \( I_j \) is a sum of two positive semi-definite matrices \( I_j = (I_j - \frac{1}{\lambda_j} A_j) + \frac{1}{\lambda_j} A_j \). Substituting this into \( I = I_1 \otimes I_2 \otimes \cdots \otimes I_n \) completes the proof. \( \square \)

We now give the precise definition of the set of operations that will be used in this paper.

**Definition 3.** Let \( B(\mathcal{H}_n) \) be the set of density matrices acting on \( \mathcal{H}_n \), and let \( \Lambda : B(\mathcal{H}_n) \rightarrow B(\mathcal{H}_n) \) be a completely positive trace-preserving map. Then, we say that \( \Lambda \) is a separable superoperator if there exists an operator sum representation of \( \Lambda \) such that for any density matrix \( \rho \in B(\mathcal{H}_n) \),

\[
\Lambda(\rho) = \sum_k M_k \rho M_k^\dagger \tag{1}
\]

and

\[
M_k = A_1^{(k)} \otimes A_2^{(k)} \otimes \cdots \otimes A_n^{(k)},
\]

where \( A_i^{(k)} \) are \( d_i \times d_i \) complex matrices.

We denote the set of all separable superoperators by \( \text{SEP} \). In the particular case, when considering only pure bipartite states, LOCC is effectively the same as \( \text{SEP} \), but, in general, LOCC is a strict subset of \( \text{SEP} \).

2.2. The critical set

Let \( \mathfrak{g} \) be the Lie algebra of \( G \) contained in \( \text{End}(\mathcal{H}_n) \). We define the set of critical states as

\[
\text{Crit}(\mathcal{H}_n) \equiv \{ \phi \in \mathcal{H}_n | \langle \phi | X | \phi \rangle = 0, X \in \mathfrak{g} \}.
\]
The GHZ states [9], cluster states [3], graph states [11] and code states [2] (such as the five-qubit code states, the Steane seven-qubit code states and the Shore nine-qubit code states) are all critical. In the bipartite case, with $\mathcal{H}_2 = \mathbb{C}^d \otimes \mathbb{C}^d$, the maximally entangled state is the only critical state up to a local unitary matrix. As we will see shortly, almost all states belong to an orbit of some critical state.

The Kempf–Ness theorem [23] in the context of this paper says (the only hard part of the theorem is the ‘if’ part of 3):

**Theorem 2.** Let $\phi \in \mathcal{H}_n$ and let $K \equiv SU(d_1) \otimes SU(d_2) \otimes \cdots \otimes SU(d_n)$, then
1. $\phi \in \text{Crit}(\mathcal{H}_n)$ if and only if $\|g\phi\| \geq \|\phi\|$ for all $g \in G$.
2. If $\phi \in \text{Crit}(\mathcal{H}_n)$ and $g \in G$, then $\|g\phi\| \geq \|\phi\|$ with equality if and only if $g\phi \in K\phi$. Moreover, if $g$ is positive definite, then the equality condition holds if and only if $g\phi = \phi$.
3. If $\phi \in \mathcal{H}_n$, then $G\phi$ is closed in $\mathcal{H}_n$ if and only if $G\phi \cap \text{Crit}(\mathcal{H}_n) \neq \emptyset$.

The dimension of the set $\text{Crit}(\mathcal{H}_n)$ in the case of $n$ qubits (i.e. $d_1 = d_2 = \cdots = d_n = 2$) is 1 for $n = 3$ (up to a local unitary matrix, only the GHZ state belongs to $\text{Crit}(\mathcal{H}_n)$) and $2^n - 3n$ for $n > 3$.

The following theorem says that critical states are maximally entangled.

**Theorem 3.** [10, 21] Let $\psi \in \mathcal{H}_n$. Then, $\psi \in \text{Crit}(\mathcal{H}_n)$ if and only if the local density matrices are all proportional to the identity (i.e. each qudit is maximally entangled with the rest of the system).

If an SLOCC orbit $O_\psi$ contains a critical state $\psi_c$, then $\psi_c$ can be regarded as the maximally entangled state of the orbit $O_\psi$. In such cases, we will take $\psi_c$ to represent the orbit and denote the orbit by $O_{\psi_c}$. The corollary below implies that almost all SLOCC classes can be represented by a critical state.

**Corollary 4.** [24] The set of all states of the form $g\psi/\|g\psi\|$, where $g \in G$ and $\psi \in \text{Crit}(\mathcal{H}_n)$, is dense in $\mathcal{H}_n$.

Moreover, part 2 of the Kempf–Ness theorem above implies the uniqueness of critical states; that is, up to a local unitary matrix there can be at most one critical state in any SLOCC class $O_\psi$. That is, critical states are the natural representatives of SLOCC orbits, and are the unique maximally entangled states in their SLOCC orbits.

The main results of this paper are expressed in terms of the stabilizer group of the state representing the SLOCC orbit and the multipartite generalization of the reduced density matrices, which we call the ADMs. In sections 3 and 4, we define and prove useful theorems on stabilizer groups and ADMs. These will be useful when we present our main results.

### 3. The stabilizer group

In section 5 we will see that the stabilizer group naturally arises in the analysis of separable operations. However, unlike the stabilizer formalism of quantum error correcting codes [2], in general, the stabilizer group considered in this paper is not a subgroup of the Pauli group. Instead, it is a subgroup of $\tilde{G}$, which is a much bigger group than the Pauli group.

**Definition 4.** The stabilizer group of $\psi \in \mathcal{H}_n$ is a subgroup of $\tilde{G}$ defined by

$$\text{Stab}(\psi) \equiv \{ g \in \tilde{G} | g\psi = \psi \}.$$
Note that for \( g \in \tilde{G} \) we have \( \text{Stab}(g \psi) = g \text{Stab}(\psi)g^{-1} \). That is, the stabilizer groups of two states in the same SLOCC class are related to each other by a conjugation.

**Proposition 5.** If the stabilizer group \( \text{Stab}(\psi) \), for \( \psi \in \mathcal{H}_n \), is finite, then there exists a state \( \phi \in \mathcal{O}_\psi \) such that \( \text{Stab}(\phi) \subset U(d_1) \otimes U(d_2) \otimes \cdots \otimes U(d_n) \).

**Proof.** The proof is by construction. Denote \( \text{Stab}(\psi) \equiv \{ S_k \}_{k=1,2,\ldots,m} \), where \( m \) is the number of elements in \( \text{Stab}(\psi) \). Since \( \text{Stab}(\psi) \) is a subgroup of \( \tilde{G} \), it follows that the elements \( S_k \) can be written as

\[
S_k = A_k^{(1)} \otimes A_k^{(2)} \otimes \cdots \otimes A_k^{(n)},
\]

where \( A_k^{(l)} \in \text{GL}(d_l, \mathbb{C}) \) and \( l = 1, 2, \ldots, n \). Clearly, for a fixed \( l \), the set \( \{ A_k^{(l)} \}_{k=1,2,\ldots,m} \) forms a subgroup of \( \text{GL}(d_l, \mathbb{C}) \). This subgroup is equivalent (up to a conjugation) to a unitary group. To see that, denote

\[
\Delta^{(l)} \equiv \left( \sum_{k=1}^{m} A_k^{(l)} \right)^{1/2}.
\]

Clearly, \( \Delta^{(l)} \in \text{GL}(d_l, \mathbb{C}) \) and the elements \( U_k^{(l)} \equiv \Delta^{(l)} A_k^{(1)} (\Delta^{(l)})^{-1} \) form a unitary representation of the subgroup \( \{ A_k^{(l)} \} \). Now, denote

\[
\Delta \equiv \Delta^{(1)} \otimes \Delta^{(2)} \otimes \cdots \otimes \Delta^{(n)},
\]

\[
U_k \equiv U_k^{(1)} \otimes U_k^{(2)} \otimes \cdots \otimes U_k^{(n)}. \tag{2}
\]

By construction we have

\[
U_k \Delta |\psi\rangle = \Delta |\psi\rangle.
\]

Hence, the stabilizer group of \( \phi \equiv \Delta |\psi\rangle / \| \Delta |\psi\rangle \| \) consists of unitaries.

Note that for generic states of four or more subsystems \cite{24}, the stabilizer group is finite and therefore the proposition above can be applied to most (i.e. a dense subset) of the states in \( \mathcal{H}_n \). We now show that if the orbit of a state with a finite stabilizer contains a critical state, then the stabilizer of the critical state is unitary. Since most orbits contain a critical state, it is most natural to choose the critical states to represent the SLOCC orbit (see figure 1).

**Proposition 6.** If the stabilizer group \( \text{Stab}(\psi) \) for \( \psi \in \mathcal{H}_n \) is finite, and there exists a critical state \( \psi_c \) in the orbit \( \mathcal{O}_\psi \), then \( \text{Stab}(\psi_c) \subset U(d_1) \otimes U(d_2) \otimes \cdots \otimes U(d_n) \).

**Proof.** Let \( g \in \text{Stab}(\psi_c) \subset \tilde{G} \), i.e. \( g |\psi_c\rangle = |\psi_c\rangle \). In its polar decomposition \( g = up \), where \( u \in U(d_1) \otimes U(d_2) \otimes \cdots \otimes U(d_n) \) and \( p \in G \) is a positive definite matrix. Note that \( \| \psi_c \| = \| g |\psi_c\rangle \| = \| p |\psi_c\rangle \| \). From theorem 2 (Kempf–Ness), \( |\psi_c\rangle \) being critical implies that \( \| p |\psi_c\rangle \| \geq \| |\psi_c\rangle \| \) with equality if and only if \( p |\psi_c\rangle = |\psi_c\rangle \). This implies that \( p \in \text{Stab}(\psi_c) \). Moreover, since \( \psi_c \in \mathcal{O}_\psi \), there exists \( h \in G \) such that \( \text{Stab}(\psi_c) = h \text{Stab}(\psi) h^{-1} \). From proposition 5, \( \text{Stab}(\psi) \) is itself conjugate to a unitary subgroup. Thus, \( p \) is similar to a unitary matrix. However, \( p \) is a positive definite matrix and since it is similar to a unitary matrix it must be an identity.

Even though for most states in \( \mathcal{H}_n \) the stabilizer is a finite group, for many important states such as GHZ states, and other graph states, the stabilizer group is not finite. For such non-generic states we show that the stabilizer cannot even be compact.

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**Figure 1.** The state space $\mathcal{H}_n$ can be described heuristically as a horizontal cylinder. The horizontal dimension represents the uncountable number of inequivalent SLOCC classes. Each SLOCC orbit of states is described by a circle perpendicular to the horizontal line. The states $\varphi_1$ and $\varphi_2$ are the representatives of two inequivalent SLOCC orbits. The action of $g \in G$ on $\varphi_1$ and $\varphi_2$ generates $\varphi_1'$ and $\varphi_2'$, respectively. Therefore, the ADMs of both $\varphi_1'$ and $\varphi_2'$ are proportional to $g^\dagger g$. From the results in section 5.1, it follows that if the stabilizers of $\varphi_1$ and $\varphi_2$ are equal, then the transformation $\varphi_1 \rightarrow \varphi_1'$ is possible by SEP iff $\varphi_2 \rightarrow \varphi_2'$ is possible by SEP.

**Proposition 7.** Let $\phi \in \mathcal{H}_n$ be such that $\text{Stab}(\phi)$ is compact, then $\text{Stab}(\phi)$ is finite.

**Proof.** The stabilizer of an element is a linear algebraic group over $\mathbb{C}$. The identity component of this group is therefore also a linear algebraic group over $\mathbb{C}$. The only connected, linear algebraic group that is compact is the group with one element. \( \square \)

When the stabilizer group of a state $\psi \in \mathcal{H}_n$ is not finite and therefore non-compact, it would be very useful to work with a compact subgroup $T \subset \text{Stab}(\psi)$, defined as the intersection of $\text{Stab}(\psi)$ with $U(d_1) \otimes U(d_2) \otimes \cdots \otimes U(d_n)$. We will discuss it in more detail in sections 5.4 and 6.

### 3.1. The stabilizer twirling operation

In this subsection we define the stabilizer twirling operation that will be used quite often when we discuss our main results in the next section.

**Definition 5.** Let $\psi \in \mathcal{H}_n$ be a state with a finite unitary stabilizer. The $\text{Stab}(\psi)$-twirling operation, $G(\cdot)$, is defined by

$$G(\sigma) \equiv \frac{1}{m} \sum_{k=1}^{m} U_k^\dagger \sigma U_k,$$

for any positive semi-definite operator $\sigma$ acting on $\mathcal{H}_n$.

In [25] it was shown that the twirling operation $G$ can be factorized in terms of the irreps of the group (in our case $\text{Stab}(\psi)$). In the following we make use of techniques introduced in [25];...
for more details see [25]. Under the action of the unitary representation of a compact group $\text{Stab}(\psi)$, a finite-dimensional Hilbert space factorizes as follows:

$$
\mathcal{H} = \sum_q \mathcal{H}_q = \sum_q \mathcal{M}_q \otimes \mathcal{N}_q,
$$

where $q$ labels the irreps of $\text{Stab}(\psi)$, $\mathcal{M}_q$ is the $q$th representation space, and $\mathcal{N}_q$ is the $q$th multiplicity space. The $\text{Stab}(\psi)$-twirling operation has the form

$$
\mathcal{G}(\sigma) = \sum_q \mathcal{D}_{\mathcal{M}_q} \otimes \text{id}_{\mathcal{N}_q} \left( \Pi_{\mathcal{H}_q} \sigma \Pi_{\mathcal{H}_q} \right),
$$

where $\Pi_{\mathcal{H}_q}$ is the projector onto $\mathcal{H}_q$, $\text{id}_{\mathcal{N}_q}$ is the identity map on $\mathcal{N}_q$, and $\mathcal{D}_{\mathcal{M}_q}$ is the completely decohering map on $\mathcal{M}_q$, i.e. it denotes the trace-preserving operation that takes every operator on the Hilbert space $\mathcal{M}_q$ to a constant times the identity. In section 5.1 we will see that this factorization enables the elegant classification of all states $\phi$ to which $\psi$ can be transformed deterministically by separable operations.

As we pointed out above, if the stabilizer group is not finite, then it is also non-compact and therefore there is no $\text{Stab}(\psi)$-twirling. However, for the compact subgroup, $T \subset \text{Stab}(\psi)$ as defined above, there is an analogue for the twirling operation. We extend the definition of the stabilizer twirling operation (see equation (3)) to the compact group $T$.

**Definition 6.** Let $\psi \in \mathcal{H}$ and assume $\text{Stab}(\psi)$ is non-compact. Let $T$ be the intersection of $\text{Stab}(\psi)$ with $U(d_1) \otimes U(d_2) \otimes \cdots \otimes U(d_n)$. Then, for any positive semi-definite operator $\sigma$ acting on $\mathcal{H}_n$, the $T$-twirling operation, $\mathcal{T}(\sigma)$, is defined by

$$
\mathcal{T}(\sigma) \equiv \int dt t^\dagger \sigma t,
$$

where $t \in T$ and $\text{d}t$ is the Haar measure over $T$.

The factorization form in equation (4) holds for the $T$-twirling as well. This $T$-twirling operation will become useful whenever the stabilizer group is non-compact. The simplest example of that is the three-qubit GHZ states (see section 6).

### 3.2. SL-invariant polynomials

In this subsection we discuss the quantification of entanglement in terms of $SL$-invariant polynomials and discuss their relationship to the stabilizer group.

An $SL$-invariant polynomial, $f(\psi)$, is a polynomial in the components of the vector $\psi \in \mathcal{H}_n$, which is invariant under the action of the group $G$. That is, $f(g\psi) = f(\psi)$ for all $g \in G$. In the case of two qubits, there exists only one unique $SL$-invariant polynomial. It is homogeneous of degree 2 and is given by the bilinear form $(\psi, \psi)$:

$$
f_2(\psi) \equiv (\psi, \psi) \equiv \langle \psi^\dagger | \sigma_y \otimes \sigma_y | \psi \rangle, \quad \psi \in \mathbb{C}^2 \otimes \mathbb{C}^2.
$$

Its absolute value is the celebrated concurrence [26].

Also in three qubits there exists a unique $SL$-invariant polynomial. It is homogeneous of degree 4 and is given by

$$
f_4(\psi) = \text{det} \begin{bmatrix} (\varphi_0, \varphi_0) & (\varphi_0, \varphi_1) \\ (\varphi_1, \varphi_0) & (\varphi_1, \varphi_1) \end{bmatrix},
$$

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where the two-qubit states $\varphi_i$ for $i = 0, 1$ are defined by the decomposition $|\psi\rangle = |0\rangle|\varphi_0\rangle + |1\rangle|\varphi_1\rangle$, and the bilinear form $(\varphi_i, \varphi_j)$ is defined above for two qubits. The absolute value of $f_4$ is the celebrated three-tangle $|\bar{\varphi}\rangle$.

In four qubits, however, there are many SL-invariant polynomials and it is possible to show that they are generated by four SL-invariant polynomials (see, e.g., [10] for more details and references).

Any SL-invariant polynomial is generated by a finite number of homogeneous SL-invariant polynomials. Moreover, for some states, all SL-invariant polynomials vanish. For example, the $W$ state

$$|W\rangle = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$$

and (its generalization to $n$-qubits) has the property that $g_i|W\rangle = t|W\rangle$, where $g_i \equiv \text{diag}(t, t^{-1})^\otimes 3$ and $t \in \mathbb{R}$ is non-zero. Note that $g_i \in \text{SL}(2, \mathbb{C})^\otimes 3$. Thus, the value of any homogeneous SL-invariant polynomial, $f$, is zero on $|W\rangle$ (and its generalization to $n$-qubits) since

$$f(|W\rangle) = f(g_i|W\rangle) = f(t|W\rangle) = t^k f(|W\rangle),$$

where $k \in \mathbb{N}$ is the degree of $f$. Nevertheless, the set of states for which all SL-invariant polynomials vanish is of measure zero and is called here the null cone. Since the null cone is of measure zero, for most states there exists at least one homogeneous SL-invariant polynomial that does not vanish on them. The following proposition will be useful for such states.

**Proposition 8.** Let $\psi \in \mathcal{H}_n$ and assume that $\psi$ is not in the null cone. Let $k \in \mathbb{N}$ be the degree of some homogeneous SL-invariant polynomial that is not zero on $\psi$. Then,

$$\text{Stab}(\psi) \subset G_k,$$

where

$$G_k \equiv \left\{ g' \in \tilde{G} | g' = e^{i \frac{2\pi m}{k}} g; \ 1 \leq m \leq k; \ g \in G \right\}$$

is a subgroup of $\tilde{G}$.

**Proof.** Let $f$ be a homogeneous SL-invariant polynomial of degree $k$ such that $f(\psi) \neq 0$. Let $g'$ be an element in $\text{Stab}(\psi)$. Since $\text{Stab}(\psi) \subset \tilde{G}$, there exists $a \in \mathbb{C}$ such that $g' = ag$, where $g \in G$. Thus,

$$f(\psi) = f(g'\psi) = f(a g \psi) = a^k f(\psi) = a^k f(\psi),$$

where the first equality follows from the fact that $g' \in \text{Stab}(\psi)$, whereas the last equality follows from the fact that $g \in G$ and $f$ is an SL-invariant polynomial. Since $f(\psi) \neq 0$ it follows that $a^k = 1$. This completes the proof. \qed

**Remark.** If $k$, in the proposition above, is equal to the dimension, $d_l$, of the subsystem $l$ (here $l = 1, 2, \ldots, n$), then $G_k = G$. This is because $e^{i 2\pi m/d_l} g \in G$ for all $g \in G$, $l = 1, 2, \ldots, n$ and $m = 1, 2, \ldots, d_l$. In particular, for systems of $n$ qubits, $G_2 = G$.

In the following corollary, we show that for most states in $\mathcal{H}_n$ (i.e. a dense subset of $\mathcal{H}_n$), the stabilizer group is a subgroup of $G$ (which is a smaller and somewhat simpler group to work with than $\tilde{G}$).
Corollary 9. Let $f_k$ and $f_k'$ be two homogeneous SL-invariant polynomials of degrees $k$ and $k'$, respectively. Let $\psi \in \mathcal{H}_n$, and assume that $f_k(\psi) \neq 0$ and $f_k'(\psi) \neq 0$. Denote by $l = (k, k')$ the greatest common divisor (gcd) of $k$ and $k'$. Then

$$\text{Stab}(\psi) \subset G_l.$$  

This corollary follows directly from the proposition above. Note also that since most states in $\mathcal{H}_n$ have many non-vanishing SL-invariant homogeneous polynomials, it follows from the corollary above that for these states the stabilizer group is a subgroup of $G$. For example, consider the case of $n$-qubits with even $n$. For such systems their exists a homogeneous SL-invariant polynomial of degree 2. It is possible to show that for a dense set of states in $\mathcal{H}_n$ this polynomial has non-vanishing value. Thus, for this dense set of states the stabilizer is a subgroup of $G_2$, which is equal to $G$ for qubits.

Despite the above corollary, in many important cases (such as the three-qubit GHZ states) the stabilizer is not a subgroup of $G$. For these cases the following corollary will be useful.

Corollary 10. Let $\psi \in \mathcal{H}_n$ and define $\text{St}(\psi)$ to be the intersection of $\text{Stab}(\psi)$ with $G$; that is,

$$\text{St}(\psi) \equiv \{ g \in G | g\psi = \psi \}.$$  

If there exists an homogeneous SL-invariant polynomial of degree $k$ that is not vanishing on $\psi$, then the quotient group $\text{Stab}(\psi)/\text{St}(\psi)$ is a cyclic group of order at most $k$.

This corollary is a direct consequence of proposition 8.

4. The associate density matrix

In this section, we associate density matrices (not necessarily normalized) with states in an orbit $O_\psi$. We call such density matrices ADMs. We will see that the ADM plays a similar role to that of reduced density matrices, when considering local transformations that involve pure bipartite states.

Definition 7. Let $\psi \in \mathcal{H}_n$ and let $\phi \in O_\psi$ be another state in the SLOCC orbit of $\psi$. Thus, there exists $g \in G$ such that $\phi = g\psi / \|g\psi\|$. An ADM, $\rho_\psi(\phi)$, of $\phi$ with respect to $\psi$, is defined by

$$\rho_\psi(\phi) = \frac{g^\dagger g}{\|g\psi\|^2}, \text{ where } \phi = g\psi / \|g\psi\|.$$  

Note that the ADM is positive definite and that, for $s \in \text{Stab}(\psi)$, both $\rho_\psi(\phi)$ and $s^\dagger \rho_\psi(\phi) s$ are ADMs of $\phi$ with respect to $\psi$. Hence, $\rho_\psi(\phi)$ is defined up to a conjugation by the stabilizer group of $\psi$. Furthermore, note that $\rho_\psi(\phi) = I$ if and only if $\psi$ is related to $\phi$ by a local unitary transformation.

Proposition 11. Let $\psi, \phi \in \mathcal{H}_n$ be two states in the same orbit $O_\psi$ (i.e. $\phi = g\psi / \|g\psi\|$, for some $g \in G$), where $\text{Stab}(\psi) \subset U(d_1) \otimes U(d_2) \otimes \cdots \otimes U(d_n)$. Then, $\text{Stab}(\phi) \subset U(d_1) \otimes U(d_2) \otimes \cdots \otimes U(d_n)$ if and only if

$$[\rho_\psi(\phi), u] = 0 \text{ for all } u \in \text{Stab}(\psi).$$  

Proof. Since $\phi = g\psi / \|g\psi\|$, we have $\text{Stab}(\phi) = g \text{Stab}(\psi) g^{-1}$. That is, if $s \in \text{Stab}(\phi)$, then there exists $u \in \text{Stab}(\psi)$ such that $s = gug^{-1}$. Thus, $s$ is unitary if and only if

$$s^\dagger s = (g^\dagger)^{-1} u^\dagger g^\dagger gug^{-1} = 1 \iff u^\dagger g^\dagger gu = g^\dagger g.$$

This completes the proof. □
4.1. The relationship to the reduced density matrix

Consider the Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$ and the maximally entangled state (it is unnormalized for simplicity)

$$|\psi\rangle = \sum_{i=0}^{d-1} |i\rangle |i\rangle,$$

where {$|i\rangle$} is an orthonormal basis in $\mathbb{C}^d$. Clearly, in this case the orbit $O_\psi$ consists of all states in $\mathbb{C}^d \otimes \mathbb{C}^d$ with the Schmidt number $d$.

The stabilizer group of $\psi$ is given by (see section 6.1 for the proof)

$$\text{Stab}(\psi) = \left\{ S^{-1} \otimes S^T \mid S \in GL(\mathbb{C}^d) \right\},$$

(6)

where $S^T$ is the transpose matrix of $S$.

Any state $\phi \in O_\psi$ can be written as

$$|\phi\rangle = \frac{1}{\|A_1 \otimes A_2 |\psi\rangle\|} A_1 \otimes A_2 |\psi\rangle,$$

where $A_1 \otimes A_2 \in G$ and in this subsection $G = \text{SL}(d, \mathbb{C}) \otimes \text{SL}(d, \mathbb{C})$. However, due to the relatively big stabilizer group, we can always assume without loss of generality that $A_2 = I$.

Thus, the ADM (which is defined up to conjugation by an element from the stabilizer group) can be written as

$$\rho_\psi(\phi) = \frac{A_1^\dagger A_1 \otimes I}{\|A_1 \otimes I |\psi\rangle\|^2} = \rho_r \otimes I,$$

where $\rho_r = \text{Tr}_B |\phi\rangle \langle \phi|$ is the reduced density matrix. Thus, in the bipartite case, the ADM is essentially equivalent to the reduced density matrix. As we will see in the next section, in the multipartite case, the ADM plays the same role as that played by the reduced density matrix in bipartite LOCC transformations.

5. The main results

In this section, we present our main results. As we have seen in the previous sections, for a dense set of states in $\mathcal{H}_n$ the stabilizer group is finite. Hence, in the first three subsections we assume explicitly that the stabilizer is finite, whereas our results for infinite stabilizer groups are presented in the last subsection. More specifically, in section 5.1 we generalize the Nielsen majorization theorem to multi-partite states, and in section 5.2 we introduce necessary and sufficient conditions for non-deterministic SEP transformations. Section 5.3 is devoted to the calculation of the maximum probability to convert one state into another in the case where deterministic SEP is not possible. Finally, in section 5.4 we generalize our theorems to the case where the stabilizer group is non-compact.

5.1. Generalization of the Nielsen majorization theorem to multipartite states

In [28] it was shown that a transformation $|\psi_1\rangle^{AB} \rightarrow |\psi_2\rangle^{AB}$, between two bipartite states in $\mathbb{C}^n \otimes \mathbb{C}^m$, is possible by LOCC if and only if there exists a set of probabilities {$p_k$} and a set of
unitary matrices \{U_k\} such that
\[ \rho_1^A = \sum_k p_k U_k^\dagger \rho_2^A U_k, \]
where \( \rho_i^A \equiv \text{Tr}_p |\psi_i\rangle \langle \psi_i| \) (\( i = 1, 2 \)) are the reduced density matrices. Moreover, it was argued that the relation in the equation above holds true if and only if the eigenvalues of \( \rho_1^A \) are majorized by the eigenvalues of \( \rho_2^A \), i.e. \( \rho_1^A \prec \rho_2^A \). We now generalize this theorem to multipartite states.

**Theorem 12.** Let \( |\psi_1\rangle = g_1 |\psi\rangle / \| g_1 \psi \| \) and \( |\psi_2\rangle = g_2 |\psi\rangle / \| g_2 \psi \| \) be two entangled states in the same orbit \( O_\psi \), where the orbit representative \( |\psi\rangle \in \mathcal{H}_n \) is chosen such that \( \text{Stab}(|\psi\rangle) \) is a finite unitary group (see proposition 5). Let \( m \) be the cardinality of \( \text{Stab}(|\psi\rangle) \). Then, \( |\psi_1\rangle \rightarrow |\psi_2\rangle \) by SEP if and only if there exists a set of probabilities \( \{p_k\} \) such that
\[ \sum_{k=1}^m p_k U_k^\dagger \rho_2 U_k = \rho_1, \tag{7} \]
where \( \rho_i \equiv \rho_i(\psi_i) \) for \( i = 1, 2 \) and \( U_k \in \text{Stab}(|\psi\rangle) \).

**Remark.** The condition on \( \rho_1 \) and \( \rho_2 \) in equation (7) is the multipartite generalization of the condition obtained in [28] for the bipartite case. In particular, equation (7) implies that \( \rho_1 \prec \rho_2 \), i.e. the eigenvalues of \( \rho_1 \) are majorized by the eigenvalues of \( \rho_2 \). In the bipartite case, this majorization condition is both necessary and sufficient. The reason for this is that in the bipartite case the stabilizer group is big and contains all unitary matrices, whereas in the multipartite case the stabilizer usually consists of only a finite number of unitary matrices \( U_k \). Hence, in the multipartite case the majorization condition is far from being sufficient.

**Proof.** Consider the separable operation given in equation (1). If \( \Lambda(|\psi_1\rangle \langle \psi_1|) = |\psi_2\rangle \langle \psi_2| \), then there must exist complex coefficients \( a_k \) such that
\[ M_k |\psi_1\rangle = a_k |\psi_2\rangle \]
for all \( k \). Since the separable operation \( \Lambda \) is invariant under the multiplication of the operators \( M_k \) by phases, we can assume without loss of generality that \( a_k \) are real and non-negative. We therefore denote \( a_k \equiv \sqrt{p_k} \), where \( p_k \) are non-negative real numbers. From the above equation and the completeness relation \( \sum_k M_k^\dagger M_k = I \), it follows that \( \sum_k p_k = 1 \).

Now, since \( |\psi_1\rangle = g_1 |\psi\rangle / \| g_1 \psi \| \) and \( |\psi_2\rangle = g_2 |\psi\rangle / \| g_2 \psi \| \), we obtain
\[ \frac{\| g_2 \psi \|}{\| g_1 \psi \|} \sqrt{p_k} g_2^{-1} M_k g_1 |\psi\rangle = |\psi\rangle. \]
Hence,
\[ \frac{\| g_2 \psi \|}{\| g_1 \psi \|} \sqrt{p_k} g_2^{-1} M_k g_1 = U_k, \]
where \( U_k \in \text{Stab}(|\psi\rangle) \). In the above equation we used the fact that \( \psi \) is \( n \)-way entangled (in the sense that the determinant of each of the \( n \) reduced density matrices obtained by tracing out \( n - 1 \) qudits is non-zero); otherwise, \( M_k \) may have zero determinant. The above equation may be rewritten as
\[ M_k = \frac{\| g_1 \psi \|}{\| g_2 \psi \|} \sqrt{p_k} g_2 U_k g_1^{-1}. \tag{8} \]
The completeness relation \( \sum_k M_k^\dagger M_k = I \) gives
\[
\sum_{k=1}^m p_k U_k^\dagger \rho_2 U_k = \rho_1,
\]
where we substitute \( \rho_i = g_i^\dagger g_i / \| g_i \| \) for \( i = 1, 2 \). This provides the first direction of the proof. The other direction follows the exact same lines backwards. \( \square \)

Taking on both sides of equation (7) the Stab(\( \psi \))-twirling operation gives the following simple necessary condition that we summarize in the following corollary:

**Corollary 13.** Notations as in theorem 12. If the transformation \( |\psi_1\rangle \to |\psi_2\rangle \) can be achieved by SEP, then
\[
G(\rho_2) = G(\rho_1).
\]
Moreover, if \( \rho_1 \) is invariant under the action of the Stab(\( \psi \))-twirling operation, i.e. if
\[
G(\rho_1) = \rho_1,
\]
then the condition in equation (9) is both necessary and sufficient.

The corollary above implies in particular that the transformation \( \psi \to \psi_2 \) can be achieved by SEP if and only if
\[
G(\rho_2) = I.
\]

5.2. Non-deterministic transformations

Transformations involving quantum measurements are usually not deterministic. In this subsection, we provide necessary and sufficient conditions for the existence of a local procedure that converts a finite multipartite pure state into one of a set of possible final states. For the bipartite case, this problem was solved in [29]. In particular, a minimal set of necessary and sufficient conditions were found for the local conversion of a bipartite state \( |\psi\rangle_{AB} \) into the ensemble of bipartite states \( \{ p_j, |\phi_j\rangle_{AB} \} \), i.e. \( |\psi\rangle_{AB} \) is converted into \( |\phi_j\rangle_{AB} \) with probability \( p_j \). The conditions are
\[
E_k(|\psi\rangle_{AB}) \geq \sum_j p_j E_k(|\phi_j\rangle_{AB}),
\]
where \( \{ E_k \} \) are Vidal’s entanglement monotones [30]:
\[
E_k(|\psi\rangle_{AB}) \equiv \sum_{m=k}^n \lambda_m, \quad \text{for } k = 1, 2, \ldots, n,
\]
where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) are the ordered eigenvalues of the reduced density matrix \( \rho^A \equiv \text{Tr}_B |\psi\rangle_{AB} \langle \psi| \). The following theorem generalizes these conditions to the multipartite case.
Theorem 15. Let $\phi \in \mathcal{H}_n$ be a normalized state in the orbit $\mathcal{O}_\psi$, where the orbit representative $|\psi\rangle$ is chosen such that $\text{Stab}(\psi)$ consists of $m$ unitary matrices (see proposition 5). Let $\{p_j, \phi_j\}$ be an ensemble of states also in the orbit $\mathcal{O}_\psi$. Then, the transformation $\phi \rightarrow \{p_j, \phi_j\}$ can be achieved by SEP (i.e. there exists a separable operation taking $\psi$ to $\phi_j$ with probability $p_j$) if and only if there exist probabilities $\{p_{jk}\}$ such that

$$
\sum_{k=1}^{m} p_{jk} = p_j, \quad \sum_{j} \sum_{k=1}^{m} p_{jk} U_k^\dagger \rho_j U_k = \rho, \quad (12)
$$

where $\rho \equiv \rho_\phi(\phi_j), \rho_j \equiv \rho_\psi(\phi_j)$ and $U_k \in \text{Stab}(\psi)$.

Remark. The relationship between $\rho$ and $\rho_j$ in equation (12) is the multipartite generalization of the condition obtained in [29] for the bipartite case. In particular, equation (12) implies that

$$
E_k (|\phi\rangle) \geq \sum_j p_j E_k (|\phi_j\rangle), \quad (13)
$$

for all $k = 1, 2, \ldots, \dim \mathcal{H}_n$. On multipartite states the functions $\{E_k\}$ are defined by

$$
E_k (|\phi\rangle) \equiv \sum_{m=k}^{\dim \mathcal{H}_n} \lambda_m, \quad \text{for} \quad k = 1, 2, \ldots, \dim \mathcal{H}_n, \quad (14)
$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\dim \mathcal{H}_n}$ are the ordered eigenvalues of the ADM of $|\phi\rangle$. In the bipartite case, these conditions are both necessary and sufficient since the stabilizer group contains all unitary matrices, whereas in the multipartite case the stabilizer usually consists of only a finite number of unitary matrices $U_k$. Hence, in the multipartite case the conditions in equation (13) are far from being sufficient.

Proof. The proof follows similar lines as the proof of theorem 12. Consider the SEP $\Lambda = \sum_j \Lambda_j$, where

$$
\Lambda_j (\rho) \equiv \sum_{k=1}^{m} M_{jk} \rho M_{jk}^\dagger
$$

for any density matrix $\rho$ acting on $\mathcal{H}_n$. If the transformation $\phi \rightarrow \{p_j, \phi_j\}$ can be achieved by such an SEP, then

$$
\Lambda_j (|\psi\rangle \langle \psi|) = p_j |\phi_j\rangle \langle \phi_j|.
$$

Thus, there exist complex coefficients $a_{jk}$ such that

$$
M_{jk} |\phi\rangle = a_{jk} |\phi_j\rangle
$$

for all $j$ and $k$. Since the separable operation $\Lambda$ is invariant under the multiplication of the operators $M_{jk}$ by phases, we can assume without loss of generality that $a_{jk}$ are real and non-negative. We therefore denote $a_{jk} \equiv \sqrt{p_{jk}}$, where $p_{jk}$ are non-negative real numbers. From the above equation it follows that $\sum_k p_{jk} = p_j$.

Now, since $|\phi\rangle = g|\psi\rangle / \|g|\psi\|$ and $|\phi_j\rangle = g_j|\psi\rangle / \|g_j|\psi\|$ for some $g, g_j \in G$, we obtain

$$
\frac{\|g_j|\psi\|}{\|g|\psi\| \sqrt{p_{jk}}} g_j^{-1} M_{jk} g |\psi\rangle = |\psi\rangle.
$$
Hence,
\[
\frac{\|g_j\psi\|}{\|g\psi\|} \sqrt{p_{jk} g_j^{-1} M_{jk}} g = S_k,
\]
where \( S_k \in \text{Stab}(\psi) \). In the above equation, we used the fact that \( \psi \) is \( n \)-way entangled (in the sense that the determinant of each of the \( n \) reduced density matrices, obtained by tracing out \( n - 1 \) qudits, is non-zero); otherwise \( M_{jk} \) may have zero determinant. The above equation may be rewritten as
\[
M_{jk} = \frac{\|g\psi\|}{\|g_j\psi\|} \sqrt{p_{jk} g_j S_k} g^{-1}.
\]

Thus, the completeness relation \( \sum_{jk} M_{jk}^* M_{jk} = I \) gives
\[
\sum_j \sum_{k=1}^m p_{jk} U_k^* \rho_j U_k = \rho,
\]
where we substitute \( \rho \equiv \rho_\psi(\phi_j) \) and \( \rho_j \equiv g_j^* g_j / \|g_j\psi\|^2 \). This provides the first direction of the proof. The other direction follows the exact same lines backwards. \( \Box \)

Taking on both sides of equation (12) the Stab(\( \psi \))-twirling operation gives the following simple necessary condition which we summarize in the following corollary:

**Corollary 16.** Notations as in theorem 15. If the transformation \( |\phi\rangle \rightarrow \{p_j, |\phi_j\rangle\} \) can be achieved by SEP, then
\[
\sum_j p_j G(\rho_j) = G(\rho). \tag{15}
\]
Moreover, if \( \rho \) is invariant under the action of the Stab(\( \psi \))-twirling operation, i.e. if 
\[
G(\rho) = \rho,
\]
then the condition in equation (15) is both necessary and sufficient.

The corollary above implies in particular that the transformation \( \psi \rightarrow \{p_j, \phi_j\} \) can be achieved by SEP if and only if
\[
\sum_j p_j G(\rho_j) = I.
\]

### 5.3. Probabilities

From the theorems above, we see that when the conversion of one state into another is possible with some probability, i.e. the states in the same SLOCC class, this probability is usually less than one. A natural question to ask is: What is the maximum possible probability, \( P_{\text{max}} \), to convert one state into another if we know that the two states are in the same SLOCC class? Equation (13) provides an immediate upper bound on \( P_{\text{max}} \), which we summarize in the following corollary.

**Corollary 17.** Let \( \psi_1 \) and \( \psi_2 \) be as in theorem 12. Then,
\[
P_{\text{max}}(\psi_1 \rightarrow \psi_2) \leq \min_k \left\{ \frac{E_k(\psi_1)}{E_k(\psi_2)} \right\}, \tag{16}
\]
where \( k = 1, 2, \ldots, \dim \mathcal{H}_n \), and \( E_k \) are the functions defined in equation (14).
Proof. The procedure converting \( |\psi_1\rangle \) into \( |\psi_2\rangle \) with probability \( P_{\text{max}} \) also converts \( |\psi_1\rangle \) into some other states with the remaining probability \( 1 - P_{\text{max}} \). The value of \( E_k \) for these states is non-zero and therefore from equation (13) it follows that \( P_{\text{max}} \leq E_k(\psi_1)/E_k(\psi_2) \). \( \square \)

In [30] it was shown that for the bipartite case the upper bound in equation (16) is always achievable. In the multipartite case, unless \( \psi_2 \) is critical, this is usually not the case, and as we will see below, in some cases it is even possible to find better upper bounds. In particular, it is possible that for all \( k \), \( E_k(\psi_1) \geq E_k(\psi_2) \) and yet the probability \( P_{\text{max}} \) is strictly less than 1 (or even arbitrarily close to zero).

In the following we first derive a mathematical expression for the maximum probability. Since in general the expression can be difficult to calculate (at least as hard as determining whether a multipartite density matrix is separable or not), we simplify it dramatically for the case in which the states \( \psi_1 \) and \( \psi_2 \) are critical. Then, for the more general case, we find lower and upper bounds for \( P_{\text{max}} \). These upper and lower bounds remain the same if we replace SEP by LOCC.

5.3.1. General formula for \( P_{\text{max}} \).

**Theorem 18.** Let \( \psi \), \( \psi_1 \), \( \psi_2 \), \( \rho_1 \), \( \rho_2 \) and \( m \) be as in theorem 12. For any set of probabilities \( \{p_k\}_{k=1}^m \) with \( \sum_{k=1}^m p_k \leq 1 \), denote

\[
\Delta_{\psi_1 \rightarrow \psi_2}(\{p_k\}) \equiv \rho_1 - \sum_{k=1}^m p_k U_k^\dagger \rho_2 U_k,
\]

where \( U_k \in \text{Stab}(\psi) \). Then, the maximum probability to convert \( \psi_1 \) into \( \psi_2 \) is given by the following expression:

\[
P_{\text{max}}(\psi_1 \rightarrow \psi_2) = \max \left\{ \sum_{k=1}^m p_k \left| \Delta_{\psi_1 \rightarrow \psi_2}(\{p_k\}) \text{ is separable} \right. \right\}.
\]

That is, the maximum is taken over all sets of probabilities \( \{p_k\} \) such that \( \sum_{k=1}^m p_k \leq 1 \) and \( \Delta_{\psi_1 \rightarrow \psi_2}(\{p_k\}) \) is a positive semi-definite separable matrix.

Proof. Any separable transformation that takes \( \psi_1 \) to \( \psi_2 \) with maximum probability \( P_{\text{max}} \) takes \( \psi_1 \) to some other states that are not \( \psi_2 \) with the remaining probability \( 1 - P_{\text{max}} \). However, since any state can be transformed to the product state \( |00\ldots0\rangle \) with 100% success, we can assume, without loss of generality, that the separable map that takes \( \psi_1 \) to \( \psi_2 \) with maximum probability \( P_{\text{max}} \) takes \( \psi_1 \) to the product state \( |00\ldots0\rangle \) with probability \( 1 - P_{\text{max}} \). A separable transformation with such a property can be written as \( \Lambda = \Lambda_1 + \Lambda_2 \), where

\[
\Lambda_1(\cdot) = \sum_{k=1}^m M_k(\cdot) M_k^\dagger
\]
is a separable map that takes \( \psi_1 \) to \( \psi_2 \) with probability \( \text{Tr}[\Lambda_1(\psi_1)] \) and

\[
\Lambda_2(\cdot) = \sum_{k>m} M_k(\cdot) M_k^\dagger
\]
is a separable map that takes \( \psi_1 \) to \( |00\ldots0\rangle \) with probability \( \text{Tr}[\Lambda_2(\psi_1)] = 1 - \text{Tr}[\Lambda_1(\psi_1)] \). The form of \( M_k \) for \( k \leq m \) is given in equation (8). Note that equation (8) also implies that
there are at most \( m \) (i.e. the cardinality of the stabilizer group) distinct Kraus operators in the (separable) operator sum representation of \( \Lambda_1 \). If there are strictly less than \( m \) Kraus operators in the representation of \( \Lambda_1 \), then we can complete it to \( m \) Kraus operators by adding zeros (i.e. taking some of the \( p_k \)s in equation (8) to be zero). With these notations, the map \( \Lambda_1 \) converts \( \psi_1 \) into \( \psi_2 \) with probability

\[
\text{Tr}[\Lambda_1(\psi_1)] = \sum_{k=1}^{m} p_k.
\]

Now, the completeness relation for the tace-preserving CP map \( \Lambda \) can be written as

\[
I = \sum_{k} M_k^\dagger M_k = \sum_{k=1}^{m} M_k^\dagger M_k + \sum_{k>m} M_k^\dagger M_k
\]

\[
= \|g_1\psi\|^2 \sum_{k=1}^{m} p_k \left( g_1^\dagger \right)^{-1} U_k \rho_2(\phi) U_k^{-1} g_1^\dagger + \sum_{k>m} M_k^\dagger M_k,
\]

where we have used equation (8). Hence, the probabilities \( \{p_k\} \) for \( \Lambda_1 \) are chosen such that

\[
\Delta_{\psi_1 \rightarrow \psi_2}(\{p_k\}) = \frac{1}{\|g_1\psi\|^2} g_1^\dagger \left( \sum_{k>m} M_k^\dagger M_k \right) g_1.
\]

That is, \( \Delta_{\psi_1 \rightarrow \psi_2}(\{p_k\}) \) is a positive semi-definite separable matrix. This completes the proof.

The requirement in theorem 18 that \( \Delta_{\psi_1 \rightarrow \psi_2}(\{p_k\}) \) is a separable matrix implies that \( G(\Delta_{\psi_1 \rightarrow \psi_2}(\{p_k\})) \) is also a separable matrix. This leads to the following necessary condition:

**Corollary 19.** Notations as in theorem 18. If \( P_m \equiv P_{\text{max}}(\psi \rightarrow \phi) \), then the matrix

\[
G(\rho_1) - P_m G(\rho_2)
\]

is separable.

As we will see shortly, this leads to an upper bound on \( P_m \).

5.3.2. A simple formula for states with unitary stabilizer. If the state \( \psi_2 \) in theorem 18 is the critical state \( \psi \), then \( \rho_2 = I \) and the expression \( \Delta_{\psi_1 \rightarrow \psi_2}(\{p_k\}) \) in theorem 18 is given simply by

\[
\Delta_{\psi_1 \rightarrow \psi_2}(\{p_k\}) = \rho_1 - \sum_{k=1}^{m} p_k I.
\]

Now, from similar arguments as in lemma 1, it follows that \( \Delta_{\psi_1 \rightarrow \psi_2}(\{p_k\}) \) is separable if and only if \( \sum_{k=1}^{m} p_k \) is not greater than the smallest eigenvalue of \( \rho_1 \). We therefore have the following corollary:

**Corollary 20.** Let \( \psi \in \mathcal{H}_n \) be a critical state and let \( \psi_1 \in \mathcal{O}_\psi \). Then, the maximum possible probability to convert \( \psi_1 \) into \( \psi \) is given by

\[
P_{\text{max}}(\psi_1 \rightarrow \psi) = \lambda_{\text{min}}[\rho_1],
\]

where \( \lambda_{\text{min}}[\rho_1] \) is the minimum eigenvalue of \( \rho_1 \).
Note that in this case the upper bound given in corollary 16 is saturated by this value. That is, the probability to convert a state to the maximally entangled state of the SLOCC orbit (i.e. the critical state) is given by the same formula that is given in the bipartite case.

The expression in theorem 18 can also be simplified if $\psi_1$ is critical or if the stabilizer group of $\psi_1$ is unitary, i.e. if $\psi_1$ is also a natural representative of the orbit $O_\psi$. Since most states are generic, their stabilizer group is finite, and if they are also critical, then from proposition 6 it follows that their stabilizer group is unitary. For such states the following corollary applies.

**Corollary 21.** Add to the assumptions in theorem 18 the assumption that $\text{Stab}(\psi_1)$ is unitary. Denote $\sigma_p \equiv \rho_1 - pG(\rho_2)$, where $G(\cdot)$ is the $\text{Stab}(\psi)$-twirling operation (see equation (3)). Then,

$$P_{\text{max}}(\psi_1 \rightarrow \psi_2) = \max\{p \mid \sigma_p \text{ separable}\}.$$  

**Remark.** Note that $\sigma_p = G(\rho_1 - p\rho_2)$ since we assume that $\text{Stab}(\psi_1)$ is unitary. Moreover, if $p$ is small enough (see lemma 1), then $\rho_1 - p\rho_2$ is a positive semi-definite separable matrix, and therefore $G(\rho_1 - p\rho_2)$ is also a positive semi-definite separable matrix. This observation provides a lower bound on $P_{\text{max}}$, which will be discussed after the proof.

**Proof.** Based on theorem 18, we require that $\Delta_{\psi_1 \rightarrow \psi_2}(\{p_k\})$ is a separable operator. However, since $G(\rho_1) = \rho_1$ (see proposition 11) we obtain that

$$G(\Delta_{\psi_1 \rightarrow \psi_2}(\{p_k\})) = G(\rho_1) - pG(\rho_2) = \rho_1 - pG(\rho_2) = \Delta_{\psi_1 \rightarrow \psi_2}(\{p/m\}) $$

is also a separable matrix. Here, $p \equiv \sum_{k=1}^{m} p_k$ and $\{p/m\}$ is a set of $m$ probabilities all equal to $p/m$. This completes the proof.  

\[ \square \]

**5.3.3. Lower and upper bounds.** The formula in theorem 18 for the maximum probability to convert one state to another is very hard to calculate, especially if the stabilizer of $\psi_1$ is not unitary. One of the reasons for this is that the optimization in equation (17) needs to be taken over all sets $\{p_k\}$ such that $\Delta_{\psi \rightarrow \phi}(\{p_k\})$ is separable, but to check whether a multipartite mixed state is separable or not is an NP hard problem [31]. Therefore, it would be very useful to find lower and upper bounds to $P_{\text{max}}$ in terms of much simpler expressions. Fortunately, we were able to find such expressions.

**5.3.4. An upper bound.** Notations as in theorem 18. A simple upper bound to $P_m$ follows immediately from corollary 19. The requirement that the matrix in equation (18) is separable implies that the matrix is also positive semi-definite. We therefore have the following upper-bound:

**Corollary 22.** The maximum probability to convert $\psi_1$ into $\psi_2$ is bound from above by

$$P_{\text{max}}(\psi_1 \rightarrow \psi_2) \leq \frac{\lambda_{\text{max}}[G(\rho_1)]}{\lambda_{\text{max}}[G(\rho_2)]},$$

where $\lambda_{\text{max}}[\cdot]$ is the maximum eigenvalue of the matrix in square brackets.

Note that another upper bound can be found by replacing the requirement in theorem 18 that $\Delta_{\psi \rightarrow \phi}(\{p_k\})$ is separable with the weaker requirement that it is only a positive semi-definite matrix. However, this will still lead to a non-trivial optimization problem.
5.3.5. A lower bound. The maximum probability to convert one state to another is always greater than zero if both states belong to the same orbit. In the following corollary, we find a positive lower bound for this probability.

**Corollary 23.** Notations as in theorem 18. The maximum probability to convert $\psi_1$ into $\psi_2$ is bound from below by

$$P_{\text{max}}(\psi_1 \rightarrow \psi_2) \geq \frac{1}{\lambda_{\text{max}}[\rho_1^{-1}\rho_2]},$$

where $\lambda_{\text{max}}[\cdot]$ is the maximum eigenvalue of the matrix in square brackets.

**Remark.** Since the ADM is defined up to a conjugation by an element in the stabilizer group, we can replace $\rho_2$ in the equation above by $U_k^\dagger \rho_1 U_k^\dagger$ for any $U_k \in \text{Stab}(\psi)$. Since $\lambda_{\text{max}}[\rho_1^{-1}\rho_2]$ is in general not invariant under such a conjugation, we can improve the bound by taking an optimization over all $m$ elements of the stabilizer; that is,

$$P_{\text{max}}(\psi_1 \rightarrow \psi_2) \geq \max_{k=1,...,m} \frac{1}{\lambda_{\text{max}}[\rho_1^{-1}U_k^\dagger \rho_2 U_k]}.$$ 

Note also that

$$\frac{1}{\lambda_{\text{max}}[\rho_1^{-1}\rho_2]} \geq \frac{\lambda_{\text{max}}[\rho_1]}{\lambda_{\text{max}}[\rho_2]}$$

with equality if $[\rho_1, \rho_2] = 0$. Hence, the maximum eigenvalue of the ADM can only increase under SEP. This implies, for example, that $-\log \lambda_{\text{max}}[\rho_1]$ is an entanglement monotone. In the bipartite case, it is called entanglement of teleportation [32].

5.4. Infinite stabilizer group

In this section, we generalize some of the results obtained in previous sections for the case where the stabilizer group is not finite. Although for most states the stabilizer group is finite, for many important states, such as the GHZ-type states, W-type states or cluster states, the stabilizer group has non-zero dimension. Furthermore, as we will see in the examples of the next section, the larger the stabilizer group, the greater the number of conversions between multipartite states that can be achieved by local separable operations. In this context, the fact, for example, that the maximally bipartite entangled state can be converted by LOCC into any other bipartite state is a consequence of a very large stabilizer group (see equation (6)).

When the stabilizer group of a state $\psi$ is infinite, it is impossible, in general, to find a state in the orbit of $\psi$ for which the stabilizer group is a unitary group. A simple example of that can be found in the stabilizer of the three-qubit GHZ states discussed in the next section. Nevertheless, theorem 12 in section 5.1 can be easily generalized to infinite groups, but the conditions are more complicated.

**Theorem 24.** Let $|\psi_1\rangle = g_1|\psi\rangle / \|g_1\psi\|$ and $|\psi_2\rangle = g_2|\psi\rangle / \|g_2\psi\|$ be two entangled states in the same orbit $O_\psi$. Then, $|\psi_1\rangle \rightarrow |\psi_2\rangle$ by SEP if and only if there exists $m \in \mathbb{N}$ and a set of probabilities $p_0, p_1, \ldots, p_m$ (i.e. $p_k \geq 0$ and $\sum_{k=1}^{m} p_k = 1$) such that

$$\sum_{k=1}^{m} p_k S_k^\dagger \rho_1 S_k = \rho_1,$$

for some $S_k \in \text{Stab}(\psi)$, where $\rho_i \equiv \rho_\psi(\psi_i)$ for $i = 1, 2$. 

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The proof of the theorem above follows the exact same lines as in the proof of theorem 12. Similarly, theorem 15 can also be generalized to non-compact groups in this way. However, since the set of \( m \) elements, \( \{S_i\} \), is only a finite subset of the infinite group \( \text{Stab}(\psi) \), it can be very difficult to find such a finite subset of \( \text{Stab}(\psi) \) that satisfies equation (20). Moreover, note that according to proposition 7 the \( \text{Stab}(\psi) \) is also non-compact, which makes it impossible to generalize the stabilizer twirling operation (see equation (3)) to this case. Nevertheless, we will see, in the three-qubit examples of the next section, that the twirling operation in equation (5) can be a very useful tool for determining local conversions of multipartite states with non-compact stabilizers. The idea to define the group \( T \) as the intersection of \( \text{Stab}(\psi) \) with the group of local unitary matrices is in general useful (not only for the three-qubit examples below), but we leave the general analysis for a future work.

6. Examples

In this section, we give examples demonstrating the application of the main results to specific cases. We start by showing how our techniques produce all the known results about the necessary and sufficient conditions for local interconversion among pure bipartite states. Then, we focus on Hilbert spaces of three and four qubits. We start with the four-qubit cases because the stabilizer of generic states in four qubits is finite, while the stabilizer of the three-qubit GHZ states turns out to be a two-dimensional (2D) non-compact group, and therefore the analysis is slightly more complicated.

6.1. The bipartite case

In this subsection we consider the bipartite example with a Hilbert space \( \mathcal{H}_2 = \mathbb{C}^d \otimes \mathbb{C}^d \). Up to local unitaries, the only critical state is the maximally entangled state

\[
|\psi\rangle = \sum_{i=0}^{d-1} |i\rangle |i\rangle,
\]

where \( \{|i\rangle\} \) is an orthonormal basis in \( \mathbb{C}^d \). Clearly, in this case the orbit \( \mathcal{O}_\psi \) is dense and consists of all states in \( \mathbb{C}^d \otimes \mathbb{C}^d \) with the Schmidt number \( d \). In order to apply the theorems from previous sections, we need to calculate first the stabilizer group.

**Lemma 25.** The stabilizer group of \( \psi \) is given by

\[
\text{Stab}(\psi) = \{S^{-1} \otimes S^T | S \in GL(\mathbb{C}^d)\},
\]

where \( S^T \) is the transpose matrix of \( S \).

**Proof.** It is straightforward to check that \( S^{-1} \otimes S^T |\psi\rangle = |\psi\rangle \). Now, if \( A \otimes B \in \text{Stab}(\psi) \), then

\[
|\psi\rangle = A \otimes B |\psi\rangle = (A \otimes B) \left( A^{-1} \otimes A^T \right) |\psi\rangle = I \otimes BA^T |\psi\rangle.
\]

Thus, \( BA^T = I \), which implies that \( A \otimes B \) is of the form given in the lemma. \( \square \)

Clearly, the lemma above implies that \( \text{Stab}(\psi) \) is not compact. However, in [33] it was shown that when considering only pure bipartite states, any LOCC transformation can be achieved by the following protocol: Alice carries out a quantum measurement and sends the result to Bob.
Based on this result, Bob performs a unitary matrix. Hence, all LOCC protocols can be achieved by the following completely positive map. For all $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$, 

$$\Lambda (|\phi\rangle \langle \phi|) = \sum_k M_k |\phi\rangle \langle \phi| M_k^\dagger, \quad M_k = A_k \otimes B_k,$$

where $B_k$ is a unitary matrix. Thus, without loss of generality, for the bipartite case we can replace $\text{Stab}(\psi)$ in theorem 24 with the subgroup $T \subset \text{Stab}(\psi)$ given by 

$$T = \{ V^\dagger \otimes V^T | V \in U(d) \}.$$

Note that $T$ is a compact unitary group.

Now, let $|\psi_1\rangle$ and $|\psi_2\rangle$ be, as in theorem 12, applied to the bipartite case, and denote by $\rho_1$ and $\rho_2$ their ADMs, respectively. Towards the end of section 4 we have seen that the ADMs $\rho_1$ and $\rho_2$ can be written as 

$$\rho_i = \rho_i^{(1)} \otimes I, \quad \text{for } i = 1, 2,$$

where $\rho_i^{(1)}$ and $\rho_i^{(2)}$ are the reduced density matrices of $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively. Thus, theorem 24 implies that the map $|\psi_1\rangle \rightarrow |\psi_2\rangle$ can be implemented by LOCC if and only if their exist probabilities $\{p_k\}$ and elements $S_k \in T$ such that 

$$\sum_{k=1}^m p_k S_k^\dagger \rho_2 S_k = \rho_1.$$

But this condition is equivalent to 

$$\sum_{k=1}^m p_k V_k^\dagger \rho_i^{(2)} V_k = \rho_i^{(1)},$$

where $\{V_k\}$ is a set of $m$ unitary matrices. This is the exactly same condition as that obtained in [28] for pure bipartite LOCC transformations and is equivalent to the majorization condition. In the same way, we can show that when applying theorem 15 to the bipartite case, we can obtain the same results as those found in [29] for non-deterministic LOCC transformations.

### 6.2. Four qubits

In this subsection, we consider the Hilbert space of four qubits $\mathcal{H}_4 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The SLOCC group is given by $G = \text{SL}(2, \mathbb{C}) \otimes \text{SL}(2, \mathbb{C}) \otimes \text{SL}(2, \mathbb{C}) \otimes \text{SL}(2, \mathbb{C})$. The analysis of four-qubit SLOCC classes can be found in [10, 21, 24, 34] (and references therein); for more details of some of the statements we make here see these references.

In four-qubits the critical set can be characterized elegantly by the following 4D subspace:

$$\text{Crit}(\mathcal{H}_4) = K.A,$$

where $K \equiv \text{SU}(2) \otimes \text{SU}(2) \otimes \text{SU}(2) \otimes \text{SU}(2)$ is the set of special local unitary matrices, 

$$A \equiv \{ z_0 u_0 + z_1 u_1 + z_2 u_2 + z_3 u_3 | z_0, z_1, z_2, z_3 \in \mathbb{C} \},$$

with 

$$u_0 = |\phi^+\rangle |\phi^+\rangle, \quad u_1 = |\phi^-\rangle |\phi^-\rangle, \quad u_2 \equiv |\psi^+\rangle |\psi^+\rangle, \quad u_3 \equiv |\psi^-\rangle |\psi^-\rangle$$

and $|\phi^{\pm}\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\psi^{\pm}\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. In [34] the class $A$ was denoted by $G_{abcd}$. 

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Note that not all the states in $A$ are generic. For example, the state $u_0$ is not generic. It is possible to show [24] that a four-qubit state $\phi$ is generic if and only if (a) the dimension of the (non-normalized) orbit $G\phi$ is maximal (i.e. 12) and (b) the orbit $G\phi$ is closed in $H_4$.

The set of all generic states in Crit($H_4$) is given by $KD$, where

$$D \equiv \{z_0u_0 + z_1u_1 + z_2u_2 + z_3u_3 | z_i^2 \neq z_j^2 \text{ for } i \neq j \}.$$

In the following theorem, we prove that for all these states the intersection of the stabilizer group with $G$ is the Klein group, consisting of four elements.

**Proposition 26.** Let $\psi \in D$ and denote by $St(\psi)$ the intersection of $Stab(\psi)$ with $G$ (see corollary 10). Then,

$$St(\psi) = \{I, \tilde{X}, \tilde{Y}, \tilde{Z}\},$$

where $X$, $Y$ and $Z$ are the three $2 \times 2$ Pauli matrices and $\tilde{C} \equiv C \otimes C \otimes C \otimes C$ for $C = X, Y, Z$.

**Proof.** If $\phi \in A$ and $\phi = \sum_{i=0}^{4} z_i u_i$ and $z_i^2 \neq z_j^2$ for $i \neq j$, then we say that $\phi$ is generic. If $g \in G$ and $gA = A$, then if $g\beta = \sum x_i u_i$ then $g\beta = \sum \epsilon_i x_{\sigma(i)}u_i$ with $\sigma$ being a permutation, $\epsilon_i = \pm 1$ and $\sum \epsilon_i \epsilon_j = 1$. We will show that the stabilizer in $G$ of a generic element in $A$ has the property that it leaves $A$ invariant. We therefore see that if $g\phi = \phi$ with $\phi$ being generic, then $g|_A = I$.

We now present more details (see [24]).

To complete the discussion, we reformulate the action of $G$ on $H_4$. We first consider $SL(2, C) \otimes SL(2, C)$ acting on $H_4$. There is a non-degenerate symmetric form $\cdots$ on $H_4$ that is invariant under $SL(2, C) \otimes SL(2, C)$ (take the tensor product of the symplectic form on $C^2$). Relative to an orthonormal basis of $H_4$ the image of $SL(2, C) \otimes SL(2, C)$ is $SO(4, C)$. Thus the action of $G$ on $H_4 = H_2 \otimes H_2$ is equivalent to the action of $SO(4, C) \otimes SO(4, C)$ on $C^4 \otimes C^4$. We can use the form $\cdots$ to identify $C^4 \otimes C^4$ with $M_4(C)$ with $SO(4, C) \times SO(4, C)$ acting by $(g, h) X = gXh^T = gXh^{-1}$. One can check that with these identifications we can take $A$ to be the diagonal elements. If we consider the group $SO(8, C)$, then we can consider $G$ to be the subgroup

$$\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} | g, h \in SO(4, C).$$

The space $H_4$ can be imbedded in $\text{Lie}(SO(4, C))$ as

$$\begin{bmatrix} 0 & X \\ -X^T & 0 \end{bmatrix} | X \in M_4(C).$$

The space $A$ now corresponds to a Cartan subalgebra of $\text{Lie}(SO(8, C))$ and the assertion above is implied by the fact that the Weyl group of $SO(8, C)$ is given as above. This implies that the stabilizer of a regular element in $A$ fixes every element in $A$. In particular, if $(g, h)$ is in the stabilizer, then

$$gIh^T = I,$$

so $g = h$. Also if for all $X$ we have $gXg^{-1} = X$, then $g$ is diagonal. Since $g$ is orthogonal, $gg^T = I$, so $g^2 = I$. Hence $g$ is diagonal with entries $\pm 1$. Now unwind the identifications. □

In four-qubits, there are four generating $SL$-invariant polynomials [10]. One of these generating polynomials is of order 2 and is given by

$$f(\psi) \equiv \langle \psi^* | \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y | \psi \rangle.$$

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The absolute value of this SL-invariant polynomial is called the four-tangle. If \( \psi \in \text{Crit}(\mathcal{H}_4) \), then
\[
f(\psi) = z_0^2 + z_1^2 + z_2^2 + z_3^2.
\]

Now, from proposition 8 (see also the remark after the proof of proposition 8), it follows that if \( f(\psi) \neq 0 \), then
\[
\text{St}(\psi) = \text{Stab}(\psi).
\]

That is, for all the generic states in \( \mathcal{D} \) with \( f(\psi) \neq 0 \), the stabilizer group is the four-element Klein group \( \text{Stab}(\psi) = \{ I, \bar{X}, \bar{Y}, \bar{Z} \} \). Hence, the results in 5.1 imply that a state \( \psi \in \mathcal{D} \) with \( f(\psi) \neq 0 \) can be converted into \( |\phi\rangle \equiv g|\psi\rangle/\|g|\psi\rangle\| \) by SEP if and only if
\[
\mathcal{G}(\sigma) = I, \quad \sigma \equiv \frac{g^\dagger g}{\|g|\psi\|},
\]

where \( \mathcal{G} \) is the \( \text{Stab}(\psi) \)-twirling operation defined in definition 3. Moreover, it is possible to characterize completely all the ADMs \( \sigma \) that satisfy the equation above. This can be done by applying the factorization in equation (4) to the Klein group.

For states \( \psi \in \mathcal{D} \) with \( f(\psi) = 0 \), one can use corollary 10 to find the stabilizer group. For example, the state (see \[10\] for important properties of this state)
\[
|L\rangle \equiv \frac{1}{\sqrt{3}}(u_0 + \omega u_1 + \bar{\omega} u_2)
\]
is in \( \mathcal{D} \) and satisfies \( f(|L\rangle) = 0 \). In fact it is possible to show that it vanishes on all SL-invariant polynomials of degree less than 6 and that there exists an SL-invariant polynomial of degree 6 on which it does not vanish [10]. Hence, from corollary 10 it follows that \( \text{Stab}(|L\rangle)/\text{St}(|L\rangle) \) is a cyclic group with three elements. The generator of the group is given by \( g = \omega p_3 \), where \( p_3 \in K \) is the local unitary permutation that takes \( u_0 \rightarrow u_2, u_1 \rightarrow u_0, u_2 \rightarrow u_1 \) and \( u_3 \rightarrow u_3 \). Hence, now that we have found \( \text{Stab}(|L\rangle) \), we can apply all the theorems in the previous section to determine the states into which \( |L\rangle \) can be converted by SEP.

6.3. Three-qubits

In this subsection, we apply the results from the previous section to the case of three-qubits. In particular, we will consider local transformations between two states in the GHZ SLOCC class [9]. Since the stabilizer group of the GHZ state is non-compact, we will only be able to solve the problem partially.

We denote the GHZ state by \( \phi = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \). A straightforward calculation shows that the stabilizer of \( \phi \) in \( G \) is the group
\[
\text{St}(\phi) = \left\{ t_1 \otimes t_2 \otimes t_3 \mid t_j = \begin{bmatrix} s_j & 0 \\ 0 & s_j^{-1} \end{bmatrix}, \quad s_1 s_2 s_3 = 1 \right\}.
\]

Note that if \( t = t_1 \otimes t_2 \otimes t_3 \) as above, then we have
\[
\begin{align*}
|000\rangle & = |000\rangle, \quad |t|111\rangle = |111\rangle, \\
|001\rangle & = s_3^{-2} |001\rangle, \quad |t|010\rangle = s_2^{-2} |010\rangle, \quad |t|100\rangle = s_1^{-2} |100\rangle, \\
|011\rangle & = s_1^2 |011\rangle, \quad |t|101\rangle = s_2^2 |101\rangle, \quad |t|110\rangle = s_3^2 |110\rangle.
\end{align*}
\]
Now, the three-tangle of the GHZ state is non-zero. Since the tangle is the absolute value of a homogeneous SL-invariant polynomial of degree 4, from corollary 10 it follows that Stab(φ)/St(φ) is a cyclic group with two elements. The two elements are
\[ \text{Stab}(\phi)/\text{St}(\phi) = \{ I, X \otimes X \otimes X \}, \]
where \( X \) is the \( x \)-Pauli matrix. Stab(\( \phi \)) is therefore the union of St(\( \phi \)) and \( h\text{St}(\phi) \), where \( h \equiv X \otimes X \otimes X \).

Set \( T \) equal to the intersection of Stab(\( \phi \)) with the group of local unitary transformations, and \( T_0 \) the intersection of St(\( \phi \)) with the group of local unitary transformations. The \( T \)-twirling operation is therefore
\[ T(\sigma) = \frac{1}{2} T_0(\sigma) + \frac{1}{2} h T_0(\sigma) h, \]
where
\[ T_0(\sigma) \equiv \int dt \ i^\sigma t \]
with \( dt = \frac{1}{4\pi^2} d\theta_1 d\theta_2 \) for \( s_j = e^{i\theta_j}, \theta_1 + \theta_2 + \theta_3 = 0 \).

Now, from theorem 24 it follows that if \( \psi_1 \) and \( \psi_2 \) are two states in the GHZ class, then it is possible to convert \( \psi_1 \) into \( \psi_2 \) by SEP if and only if there exist probabilities \( \{ p_k \} \) and elements \( g_k \in \text{Stab}(\phi) \) such that
\[ \sum_k p_k g_k^\dagger \rho_2 g_k = \rho_1, \]
where \( \rho_1 \) and \( \rho_2 \) are the ADMs of \( \psi_1 \) and \( \psi_2 \), respectively. In order to get a simpler condition, we apply the \( T_0 \)-twirling operation on both sides of this equation. This gives
\[ \sum_k p_k T_0(g_k^\dagger \rho_2 g_k) = T_0(\rho_1). \]
Since St(\( \phi \)) is commutative and \( h\text{St}(\phi) = \text{St}(\phi) h \), we obtain
\[ \sum_k p_k g_k^\dagger T_0(\rho_2) g_k = T_0(\rho_1). \]
Now, without loss of generality, assume that \( g_k \in \text{St}(\phi) \) for \( k \leq a \), and \( g_k = h f_k \) with \( f_k \in \text{St}(\phi) \) for \( k > a \). Then,
\[ \sum_k p_k g_k^\dagger T_0(\rho_2) g_k = \sum_{k \leq a} p_k g_k^\dagger g_k T_0(\rho_2) + \sum_{k > a} p_k f_k^\dagger f_k T_0(h \rho_2 h), \]
where we have used the fact that \( T_0(\rho_2) \) commutes with every element of \( T_0 \), and since St(\( \phi \)) is the complexification of \( T_0 \), we see that \( T_0(\rho_2) \) commutes with every element of St(\( \psi \)). This implies that
\[ \sum_{k \leq a} p_k g_k^\dagger g_k T_0(\rho_2) + \sum_{k > a} p_k f_k^\dagger f_k T_0(h \rho_2 h) = T_0(\rho_1). \]
In the case where \( T_0(\rho_1) = \rho_1 \), the above condition is both necessary and sufficient. In particular, if we take \( \psi_1 \) to be the GHZ state \( \phi \), then \( \rho_1 = I \) and we obtain
\[ \sum_{k \leq a} p_k g_k^\dagger g_k T_0(\rho_2) + \sum_{k > a} p_k f_k^\dagger f_k T_0(h \rho_2 h) = I. \]
The above condition provides the necessary and sufficient condition for local separable conversion of the GHZ state $\psi_1 = \phi$ into some other state $\psi_2$. In particular, the condition implies that the GHZ state can be converted by SEP into any state $\psi_2$ with $T(\rho_2) = I$, since we can take, in this case, $g_k$ and $f_k$ to be unitaries. However, it is left open whether there are other states that can be obtained by SEP from the GHZ state.

7. Conclusions

Quantum information can be viewed as a theory of interconversions among different resources. Multipartite entangled states, such as cluster states, are resources for quantum information and quantum computing. In this paper, we considered transformations that can be implemented on pure multipartite entangled states by local separable operations. Quite remarkably, we were able to generalize both the Nielsen majorization theorem [28] and the Jonathan and Plenio theorem [29] for transformations involving multipartite pure states. This generalization was expressed in terms of the stabilizer group and the ADMs of the states involved.

For the bipartite case with a Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$, both the majorization theorem in [28] and the conditions in [29] can be expressed in terms of Vidal’s entanglement monotones [30] defined by

$$E_k(\psi) = \sum_{j=k}^{d} \lambda_j, \quad k = 1, 2, \ldots, d,$$

where $\lambda_j$ are the ordered (from the biggest to the smallest) eigenvalues of the reduced density matrix, and $\psi$ is a bipartite state. Here we were able to extend this definition to the multipartite case by taking the $\lambda_j$'s to be the ordered eigenvalues of the ADM of $\psi$. However, unlike the bipartite case, for local transformations involving multipartite states, the conditions given in terms of these functions are only necessary but not sufficient. It is therefore impossible to express the necessary and sufficient conditions only in terms of functions of the eigenvalues of the ADMs involved in the transformation. Instead, we express the conditions for local transformations in terms of matrix equalities given in theorems 12 and 15.

The necessary and sufficient conditions get extremely elegant for transformations that involve the maximally entangled state of the SLOCC orbit. That is, every SLOCC orbit with a finite stabilizer group has a maximally entangled state (this is usually a critical state, i.e. a normal form). Its stabilizer group consisting of unitary matrices, and the necessary and sufficient conditions to convert this state to another state in its SLOCC orbit, are given in equation (10) and expressed elegantly in terms of the stabilizer twirling operation (see definition 3). This simple expression made it possible to characterize all the states to which a maximally entangled state can be converted by SEP.

Both theorems 12 and 15 assume that the stabilizer group of the state representing the SLOCC orbit is finite. We also argued that for a dense set of states in $\mathcal{H}_n$ with $n > 3$, the stabilizer is finite. Therefore, these theorems hold true for most states in $\mathcal{H}_n$. Moreover, from continuity arguments, it follows that the necessary conditions in equation (13) given in terms of the continuous functions $E_k(\psi)$ hold true for all states in $\mathcal{H}_n$. That is, the necessary conditions that generalize the majorization theorems in [28] and [29] hold true for all the states in the Hilbert space. In particular, this implies that $E_k$ are entanglement monotones.

When local deterministic transformations are not possible, we found an expression for the maximum probability to convert one multipartite state to another by SEP. This expression
can be simplified dramatically if one of the two states involved is critical. However, since in general the expression is relatively complicated, we found simple lower and upper bounds for the probability of local conversion.

There are many important SLOCC orbits, such as the GHZ class, that have a non-finite (in fact, non-compact) stabilizer group. For such orbits we also found necessary and sufficient conditions for a local conversion among states in the same orbit, but these conditions are more complicated since the stabilizer group is not unitary. We are willing to conjecture that, for such orbits, theorems [28] and [29] hold true if one replaces the non-compact stabilizer group with the subgroup obtained by the intersection of the stabilizer with the local unitary group.

There are many other interesting related problems that were not discussed here. One such problem concerns the asymptotic rates to interconvert one multipartite state into another. We hope the work presented here will be useful for such a research direction.

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