Reflected Backward Stochastic Difference Equations and Optimal Stopping Problems under $g$-expectation

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Abstract: In this paper, we study reflected backward stochastic difference equations (RBSDEs for short) with finitely many states in discrete time. The general existence and uniqueness result, as well as comparison theorems for the solutions, are established under mild assumptions. The connections between RBSDEs and optimal stopping problems are also given. Then we apply the obtained results to explore optimal stopping problems under $g$-expectation. Finally, we study the pricing of American contingent claims in our context.

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1 Introduction

The theory of nonlinear backward stochastic differential equations was first introduced by Pardoux and Peng [29]. Over the past twenty years, backward stochastic differential equations have been widely used in mathematical finance, stochastic control and other fields. By analogy with the equations in continuous time, Cohen and Elliott [9] consider backward stochastic difference equations (BSDEs) on spaces related to discrete time, finite state processes, establishing fundamental results including the comparison theorem etc. These are studied as entities in their own right, not as approximations to continuous BSDEs, as in [2, 3, 25, 35]. For deeper discussion, the readers may refer to [9–13].

The general theory of reflected backward stochastic differential equations was studied by El Karoui et al. [17]. They considered the case where the

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solution is forced to stay above a given stochastic process (called the obstacle) and introduced an increasing process which pushes the solution to satisfy this condition. This important theory has been applied to optimal stopping problems (see [18]) as well as to problems in financial markets and other related fields (see [1, 13, 20–24, 26]). For this reason, it is interesting to explore reflected backward stochastic difference equations (RBSDEs for short) in the framework of [9], as well as some applications in optimal stopping problems in discrete time.

The RBSDE is formulated in detail in section 2. To associate the RBSDE solution with the classical Skorohod problem, as is done in the continuous time case (see [14]), we first prove that the Skorohod lemma remains valid in our framework. Using the Skorohod lemma, the increasing process which forces the solution is expressed as a supremum. Then we give the main results of this paper including the comparison theorem and the existence and uniqueness theorem. The proof of the comparison theorem is similar to that for nonreflected BSDE’s in [9]. Existence of solutions is established by penalization of the constraints. Moreover we show that the solution of an RBSDE corresponds to the value of an optimal stopping problem. We also show that the solution of the RBSDE in which the coefficient \( f \) is a concave (or convex) function corresponds to the value function of a mixed optimal stopping–optimal stochastic control problem.

With some limitations on the generator \( g \), a BSDE can be used to define a nonlinear expectation \( E^g[\xi] := Y_0 \), which is called \( g \)-expectation (see [30]). A notable property of \( g \)-expectation is its time consistency, namely the property that the conditional expectation \( E^g[\xi|\mathcal{F}_t] \) can be well-defined. Furthermore, it was proved that a dominated and time-consistent nonlinear expectation can be represented as the solution of a BSDE (see [8, 14] in continuous time, [9] in discrete time). So it is interesting to study optimal stopping problems under \( g \)-expectation. In section 3, we first study the \( g \)-expectation theory on spaces related to discrete time, finite state processes. The Doob-Mayer decomposition theorem and optional sampling theorem are obtained. Then we associate \( g \)-martingales with multiple prior martingales which were introduced by Riedel in [34]. We finally show that RBSDE is a convenient tool to solve some optimal stopping problems under \( g \)-expectation.

In section 4, we apply the obtained results to study the pricing of American contingent claims in an incomplete financial market.

## 2 RBSDEs

Following [9], we consider an underlying discrete time, finite state process \( X \) which takes values in the standard basis vectors of \( \mathbb{R}^m \), where \( m \) is the number of states of the process \( X \). In more detail, for each \( t \in \mathcal{N} := \{0, 1, 2, ..., T\} \), \( X_t \in \{e_1, ..., e_m\} \), where \( T > 0 \) is a finite deterministic terminal time, \( e_i = (0, 0, ..., 0, 1, 0, ..., 0)^* \in \mathbb{R}^m \), and \( [\cdot]^* \) denotes vector transposition. We note in passing that \( E[X_t] \) is a vector containing the probabilities of \( X_t \) being in each of its states.

Consider a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)\), where \( \mathcal{F}_t \) is the
completion of the \(\sigma\)-algebra generated by the process \(X\) up to time \(t\) and \(\mathcal{F} = \mathcal{F}_T\).

Define
\[
M_t = X_t - E[X_t \mid \mathcal{F}_{t-1}], \quad t = 1, \ldots, T.
\]

\(M\) is a martingale difference process taking values in \(\mathbb{R}^m\), and we define the following equivalence relation.

**Definition 2.1** We define \(Z^1 \sim_M Z^2\) whenever \(\|Z^1 - Z^2\|_M^2 = 0\) where

\[
\|Z\|_M^2 \triangleq E \text{Tr} \left[ \sum_{0 \leq u < T} Z_u^* \cdot E[M_{u+1} M_u^* \mid \mathcal{F}_u] \cdot Z_u \right]
\]

\[
= \sum_{0 \leq u < T} \text{Tr} E \left[ (Z_u^* M_{u+1})(Z_u^* M_u) + (Z_u M_{u+1})(Z_u^* M_u) \right].
\]

From \cite{9}, Theorem 1, we have the following martingale representation theorem.

**Theorem 2.2** For any \(\{\mathcal{F}_t\}\)-adapted \(\mathbb{R}^K\)-valued martingale \(L\), there exists an adapted \(\mathbb{R}^K \times \mathbb{N}\) valued process \(Z\) such that

\[
L_t = L_0 + \sum_{0 \leq u < t} Z_u M_{u+1},
\]

Moreover, this process is unique up to equivalence \(\sim_M\).

The general form of a backward stochastic difference equation in \cite{9} is for any \(0 \leq t \leq T\),

\[
Y_t = \xi + \sum_{t \leq u < T} f(u, Y_u, Z_u) - \sum_{t \leq u < T} Z_u^* M_{u+1}, \quad P - a.s.
\]

where \(\xi\) is an \(\mathbb{R}\)-valued \(\mathcal{F}_T\)-measurable terminal condition and \(f\) an adapted map \(f : \Omega \times \{0, 1, \ldots, T\} \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}\). The solution \((Y_t, Z_t)\) is adapted to the filtration \(\{\mathcal{F}_t\}\) and takes values in \(\mathbb{R} \times \mathbb{R}^m\). We also assume\footnote{Note that since \(X_t\) only takes finite states, it is clear that \(L^1(\mathcal{F}_T; \mathbb{R}^m) = L^\infty(\mathcal{F}_T; \mathbb{R}^m)\).} that \((Y_t, Z_t) \in L^1(\mathcal{F}_T; \mathbb{R}) \times L^1(\mathcal{F}_T; \mathbb{R}^m)\) for all \(t, \xi \in L^1(\mathcal{F}_T; \mathbb{R})\) and \(f(t, y, z) \in L^1(\mathcal{F}_T; \mathbb{R})\) for all \(t\) and \((y, z) \in \mathbb{R} \times \mathbb{R}^m\).

Theorem 2 in \cite{9} gives the following general existence result.

**Theorem 2.3 (BSDE existence and uniqueness)** The BSDE \cite{2.1} has a unique adapted solution \((Y_t, Z_t)\) if and only if \(f\) satisfies the following two assumptions

(i) For any \(Y\), if \(Z^1 \sim_M Z^2\), then \(f(t, Y_t, Z^1_t) = f(t, Y_t, Z^2_t)\) \(P\)-a.s. for all \(t\);

(ii) For any \(z \in \mathbb{R}^m\), all \(t\) and \(P\)-almost all \(\omega\), the map

\[
y \mapsto y - f(t, y, z)
\]

is a bijection \(\mathbb{R} \rightarrow \mathbb{R}\).
We now consider RBSDEs in this setting.

**Definition 2.4 (Reflected BSDE)** A triple \((\xi, f, S)\) is called ‘standard data’ for an RBSDE if

(i) \(\xi \in L^1(F_T; \mathbb{R})\);

(ii) The map \(f(\cdot, y, z)\) is an adapted process for any \((y, z) \in \mathbb{R} \times \mathbb{R}^m\);

(iii) The obstacle process \(\{S_t, 0 \leq t \leq T\}\) is real-valued, adapted and such that \(S_T \leq \xi\) P-a.s.

**Definition 2.5** A solution of RBSDE with standard data \((\xi, f, S)\) is a triple \(\{(Y_t, Z_t, K_t)\}_{0 \leq t \leq T}\) of adapted processes taking values in \(\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}\) such that,

for all \(0 \leq t \leq T\),

(i) \(Y_t = \xi + \sum_{t \leq u < T} f(u, Y_u, Z_u) + K_T - K_t - \sum_{t \leq u < T} Z_u^* M_{u+1}, P\text{-a.s.}\); (2.2)

(ii) \(Y_t \in L^1(F_t; \mathbb{R}), Z_t \in L^1(F_t; \mathbb{R}^m)\) and \(K_t \in L^1(F_t; \mathbb{R})\);

(iii) \(Y_t \geq S_t\) P-a.s.

(iv) \(\{K_t\}\) is increasing in \(t\), \(K_0 = 0\) and

\[
\sum_{0 \leq t \leq T} (Y_t - S_t)(K_{t+1} - K_t) = 0, \text{ P - a.s.}
\]

### 2.1 Skorohod lemma and a priori estimate

To determine solutions to an RBSDE, it is first useful to consider how it is connected to the Skorohod problem (see §33) in the discrete time case. This will then allow us to obtain a priori estimates on the behaviour of solutions to our RBSDEs.

**Lemma 2.6 (Skorohod problem)** Let \(y\) be a real-valued function on \(\{0, 1, ..., T\}\) such that \(y(0) \geq 0\). There exists a unique pair \((v, g)\) of functions on \(\{0, 1, ..., T\}\) such that, for all \(t \in \{0, 1, ..., T\}\),

(i) \(v(t) = y(t) + g(t)\);

(ii) \(v(t)\) is non-negative;

(iii) \(g(t)\) is increasing, vanishing at zero and

\[\sum_{t \leq i \leq T} v(t)(g(t) - g(t - 1)) = 0.\]
The function \( g \) is moreover given by

\[
g(t) = \sup_{s \leq t} (-y(s) \lor 0).
\]

**Proof.** We first claim that the pair \((g, v)\) defined by

\[
g(t) = \sup_{s \leq t} (-y(s) \lor 0), \quad v(t) = y(t) + g(t)
\]
satisfies properties (i) through (iii).

To prove the uniqueness of the pair \((g, v)\), we suppose that \((\hat{g}, \hat{v})\) is another pair which satisfies (i) through (iii). Then \( v - \hat{v} = g - \hat{g} \). Note that \( g(0) = \hat{g}(0) = 0 \) and consequently \( v(0) - \hat{v}(0) = 0 \). Thus,

\[
(v(t) - \hat{v}(t))^2
\]

\[
= \sum_{1 \leq s \leq t} [(v(s) - \hat{v}(s))^2 - (v(s - 1) - \hat{v}(s - 1))^2]
\]

\[
= \sum_{1 \leq s \leq t} [(v(s) - \hat{v}(s)) + (v(s - 1) - \hat{v}(s - 1))] [(g(s) - \hat{g}(s)) - (g(s - 1) - \hat{g}(s - 1))]
\]

\[
= \sum_{1 \leq s \leq t} (v(s) - \hat{v}(s)) (g(s) - \hat{g}(s) - (g(s - 1) - \hat{g}(s - 1))]
\]

\[
+ \sum_{1 \leq s \leq t} (g(s - 1) - \hat{g}(s - 1)) (g(s) - \hat{g}(s)) - \sum_{1 \leq s \leq t} (g(s - 1) - \hat{g}(s - 1))^2
\]

\[
= -\sum_{1 \leq s \leq t} v(s) (g(s) - \hat{g}(s - 1)) - \sum_{1 \leq s \leq t} \hat{v}(s) (g(s) - g(s - 1))
\]

\[
- \sum_{1 \leq s \leq t} (g(s - 1) - \hat{g}(s - 1))^2 + \sum_{1 \leq s \leq t} (g(s) - \hat{g}(s)) (g(s - 1) - \hat{g}(s - 1))
\]

\[
\leq -\sum_{1 \leq s \leq t} v(s) (g(s) - \hat{g}(s - 1)) - \sum_{1 \leq s \leq t} \hat{v}(s) (g(s) - g(s - 1))
\]

\[
- \sum_{1 \leq s \leq t} \frac{(g(s - 1) - \hat{g}(s - 1))^2}{2} + \sum_{1 \leq s \leq t} \frac{(g(s) - \hat{g}(s))^2}{2}
\]

As \( v - \hat{v} = g - \hat{g} \), we have that

\[
\frac{(v(t) - \hat{v}(t))^2}{2} \leq -\sum_{1 \leq s \leq t} v(s) (g(s) - \hat{g}(s - 1)) - \sum_{1 \leq s \leq t} \hat{v}(s) (g(s) - g(s - 1)) \leq 0.
\]

Hence \( v(t) = \hat{v}(t) \) and consequently \( g(t) = \hat{g}(t) \). ■

Using Lemma 2.6, we obtain the following estimate for solutions of RBSDEs.

**Proposition 2.7** Let \( \{Y_t, Z_t, K_t\}, 0 \leq t \leq T \) be a solution of the RBSDE [2.2]. Then for each \( t \in \{0, 1, \ldots, T\} \),

\[
K_T - K_t = \sup_{t \leq u \leq T} \left( \xi + \sum_{u \leq s < T} f(s, Y_s, Z_s) - \sum_{u \leq s < T} Z_s^s M_{s+1} - S_u \right).
\]
Proof. Set
\[ y_t = \xi + \sum_{T-t \leq s < T} f(s, Y_s, Z_s) - \sum_{T-t \leq s < T} Z^*_s M_{s+1} - S_{T-t}. \]
Then \( y_0 = \xi - S_T \geq 0 \). Note that
\[ Y_{T-t}(\omega) - S_{T-t}(\omega) = y_t + K_T(\omega) - K_{T-t}(\omega). \]
From the properties of the RBSDE, we can see that
\[ (v(t), g(t)) = (Y_{T-t}(\omega) - S_{T-t}(\omega), K_T(\omega) - K_{T-t}(\omega)) \quad 0 \leq t \leq T, \]
is a solution of the above Skorohod problem. By Lemma 2.6, this solution is unique, and we can write
\[ K_T - K_{T-t} = \sup_{0 \leq u \leq t} \left( \xi + \sum_{T-u \leq s < T} f(s, Y_s, Z_s) - \sum_{T-u \leq s < T} Z^*_s M_{s+1} - S_{T-u} \right). \]
This completes the proof. \( \square \)

2.2 Comparison theorem

We now present a comparison theorem for RBSDEs. Given \( \mathcal{F}_t \), let \( Q_t \) denote the \( \mathcal{F}_t \)-measurable set of indices of possible values of \( X_{t+1} \), i.e.
\[ Q_t \triangleq \{ i : P(X_{t+1} = e_i \mid \mathcal{F}_t) > 0 \}. \]

Theorem 2.8 (Comparison Theorem) Consider two RBSDEs with standard data \((\xi^1, f^1, S^1)\) and \((\xi^2, f^2, S^2)\) respectively. Let \((Y^1, Z^1, K^1)\) and \((Y^2, Z^2, K^2)\) be the associated solutions. Suppose the following conditions hold \( P \)-a.s. for all \( t \):

(i) \( \xi^1 \geq \xi^2 \),
(ii) \( f^1(t, Y^2_t, Z^2_t) \geq f^2(t, Y^2_t, Z^2_t) \)
(iii) \( S^1_t \geq S^2_t \),
(iv) \( f^1(t, Y^2_t, Z^2_t) - f^1(t, Y^1_t, Z^1_t) \geq \min_{i \in Q_t} \{ (Z^2_t - Z^1_t)^+ (e_i - E[X_{t+1} \mid \mathcal{F}_t]) \} \),
(v) if \( Y^1_t - f^1(t, Y^1_t, Z^1_t) \geq Y^2_t - f^1(t, Y^2_t, Z^2_t) \), then \( Y^1_t \geq Y^2_t \).

Then it is true that, for all \( t \),
\[ Y^1_t \geq Y^2_t \quad P \text{-a.s.} \]
Remark 2.9

Proof. It is clear that $Y^1_t - Y^2_t = \xi^1 - \xi^2 \geq 0$ $P$-a.s. For an arbitrary $0 \leq t < T$, suppose that $Y^1_{t+1} - Y^2_{t+1} \geq 0$ $P$-a.s. We then have

$$Y^1_{t+1} - Y^2_{t+1} = Y^1_t - Y^2_t - f^1(t, Y^1_t, Z^1_t) + f^2(t, Y^2_t, Z^2_t) + (Z^1_t - Z^2_t)^*M_{t+1} - (K^1_{t+1} - K^1_t) + (K^2_{t+1} - K^2_t) \geq 0.$$  \hspace{1cm} (2.4)

Since $M_{t+1} = X_{t+1} - E[X_{t+1} | \mathcal{F}_t]$ and $X_{t+1}$ almost surely takes values in $\mathcal{Q}_t$,

$$Y^1_t - Y^2_t - f^1(t, Y^1_t, Z^1_t) + f^2(t, Y^2_t, Z^2_t) - \min_{i \in \mathcal{Q}_t} \{(Z^1_t - Z^2_t)^*(e_i - E[X_{t+1} | \mathcal{F}_t])\} \geq f^1(t, Y^1_t, Z^1_t) - f^2(t, Y^2_t, Z^2_t) - \min_{i \in \mathcal{Q}_t} \{(Z^1_t - Z^2_t)^*(e_i - E[X_{t+1} | \mathcal{F}_t])\} \geq 0.$$  \hspace{1cm} (2.5)

By assumptions (ii) and (iv), we obtain

$$Y^1_t - Y^2_t - f^1(t, Y^1_t, Z^1_t) + f^2(t, Y^2_t, Z^2_t) - \min_{i \in \mathcal{Q}_t} \{(Z^1_t - Z^2_t)^*(e_i - E[X_{t+1} | \mathcal{F}_t])\} \geq \frac{f^1(t, Y^1_t, Z^1_t) - f^2(t, Y^2_t, Z^2_t)}{K^2_{t+1} - K^2_t} - \min_{i \in \mathcal{Q}_t} \{(Z^1_t - Z^2_t)^*(e_i - E[X_{t+1} | \mathcal{F}_t])\} \geq 0.$$

Set

$$A \triangleq \{\omega \mid Y^1_t(\omega) < Y^2_t(\omega)\}.$$

We know that $S^1_t \leq S^1_1 \leq Y^1_1 < Y^2_t$ on $A$, which yields that $K^2_{t+1} - K^2_t$ on $A$. Therefore,

$$Y^1_t - Y^2_t - f^1(t, Y^1_t, Z^1_t) + f^2(t, Y^2_t, Z^2_t) \geq 0$$

on $A$. But, by assumption (v), the above inequality implies $Y^1_t \geq Y^2_t$ on $A$. Thus, we deduce that $P(A) = 0$ and

$$Y^1_t \geq Y^2_t \hspace{0.5cm} P - a.s.$$

This completes the proof.

Remark 2.9 If the map $y \mapsto y - f(\omega, t, y, z)$ is strictly increasing in $y$ for all $t$ and $z$ and $P$-almost all $\omega$, then assumption (v) holds.

Corollary 2.10 Suppose the assumptions of Theorem 2.8 hold. Set $t \in \{0, 1, ..., T\}$. If we also know that $Y^1_s = Y^2_s$ for all $s \in \{0, 1, ..., t\}$, then $K^1_t \leq K^2_t$ $P$-a.s. for all $s \in \{0, 1, ..., (t+1) \wedge T\}$ and $K^1_t - K^2_t$ is decreasing in $s$. Moreover, if $\xi^1 = \xi^2$ and $f^1 = f^2$ $P$-a.s., then $K^1_t = K^2_t$ $P$-a.s. for all $t \in \{0, 1, ..., T\}$.

Proof. By (2.5), we have

$$K^1_{s+1} - K^1_s \leq K^2_{s+1} - K^2_s.$$
Then
\[ K_{s+1}^1 - K_{s+1}^2 \leq K_s^1 - K_s^2, \]
that is, \( K_s^1 - K_s^2 \) is decreasing in \( s \). Since \( K_0^1 = K_0^2 = 0 \), we obtain that \( K_1^1 \leq K_s^2 \) \( P \)-a.s.

Moreover, if we also have \( \xi^1 = \xi^2 \) and \( f^1 = f^2 \) \( P \)-a.s., then it is easy to see that
\[ K_t^2 \leq K_t^1 \quad P \text{-a.s.} \]
which completes the proof. □

The following example shows that Theorem 2.8 fails when assumption (iv) does not hold.

**Example 2.11** For simplicity, suppose \( T = 1 \). Consider two RBSDEs with standard data \((\xi^1, f^1, S^1)\) and \((\xi^2, f^2, S^2)\) respectively which satisfy the assumptions of Theorem 2.14 in the following section. Let \( \xi^1 = \xi^2 \), \( f^1 = f^2 = f \) and \( S^1 = S^2 \) and the map \( y - f(y, z) \) be strictly increasing in \( y \). By Theorem 2.14, we have \( Y_0^1 = Y_0^2 \), \( K_0^1 = K_0^2 \) and \( K_1^1 = K_1^2 \) \( P \)-a.s.

Suppose that assumption (iv) of Theorem 2.8 does not hold. In particular, we have
\[ f(0, Y_0^2, Z_0^1) - f(0, Y_0^2, Z_0^2) < \min_{i \in \mathbb{Q}} \{(Z_0^1 - Z_0^2)^*(e_i - E[X_1 | F_0])\}. \]

Then we have
\[
0 = Y_1^1 - Y_1^2 \\
= Y_0^1 - Y_0^2 - f(0, Y_0^1, Z_0^1) + f(0, Y_0^2, Z_0^2) + (Z_0^1 - Z_0^2)^*M_1 \\
- (K_1^1 - K_0^1) + (K_1^2 - K_0^2) \\
> Y_0^1 - Y_0^2 - f(0, Y_0^1, Z_0^1) + f(0, Y_0^2, Z_0^2) - (K_1^1 - K_0^1) + (K_1^2 - K_0^2). \\
\]

It follows that
\[
0 = (K_1^1 - K_0^1) - (K_1^2 - K_0^2) \\
> Y_0^1 - Y_0^2 - f(0, Y_0^1, Z_0^1) + f(0, Y_0^2, Z_0^2). \\
\]
As the map \( y \mapsto y - f(y, z) \) is strictly increasing, we deduce \( Y_0^1 < Y_0^2 \), contradicting the conclusion of Theorem 2.8.

### 2.3 Existence and uniqueness

In this subsection, we will explore the existence and uniqueness of solutions of RBSDE basing on approximation via penalization in [17] as well as the comparison theorem obtained in [9].

First, we recall the comparison theorem in [13].

**Theorem 2.12** Consider two BSDEs (2.1) with standard data \((\xi^1, f^1)\) and \((\xi^2, f^2)\) respectively. Suppose \( (Y^1, Z^1) \) and \( (Y^2, Z^2) \) are the associated solutions, and the following conditions also hold \( P \)-a.s. for all \( t \in \{0, 1, \ldots, T\} \),
(i) \( \xi^1 \geq \xi^2 \),
(ii) \( f^1(t, Y^2_t, Z^2_t) \geq f^2(t, Y^2_t, Z^2_t) \),
(iii) \( f^1(t, Y^2_t, Z^1_t) - f^1(t, Y^2_t, Z^2_t) \geq \min_{i \in \Omega_t} \{ [Z^1_t - Z^2_t]^* (e_i - E[X_{t+1} \mid F_t]) \}, \)
(iv) if \( Y^1_t - f^1(t, Y^1_t, Z^1_t) \geq Y^2_t - f^1(t, Y^2_t, Z^2_t) \) then \( Y^1_t \geq Y^2_t \).

Then it is true that, for all \( t \),
\[
Y^1_t \geq Y^2_t \quad \text{P-a.s.}
\]

Corollary 2.13 Suppose Theorem 2.12 holds, and furthermore

- at least one of inequalities (i) and (ii) is strict,
- inequality (iii) is strict unless both sides are zero, and
- the map \( y \mapsto y - f(t, y, Z^1) \) is strictly increasing (guaranteeing inequality (iv)).

Then we have \( Y^1_t > Y^2_t \) P-a.s. for all \( t \).

Proof. By Theorem 2.12, we have \( Y^1_t \geq Y^2_t \) P-a.s. Then, by the same arguments as in Theorem 2.8, we obtain
\[
Y^1_t - Y^2_t - f^1(t, Y^1_t, Z^1_t) + f^1(t, Y^2_t, Z^2_t) > 0.
\]
It follows that \( Y^1_t > Y^2_t \) P-a.s. This completes the proof. \( \blacksquare \)

Theorem 2.14 Consider a RBSDE (2.2) with standard data \( (\xi, f, S) \). The map \( f \) satisfies the following two assumptions P-a.s. for all \( t \)

(i) For any \( Y \), if \( Z^1 \sim_M Z^2 \), then \( f(t, Y_t, Z^1_t) = f(t, Y_t, Z^2_t) \)
(ii) For any \( z \in \mathbb{R}^m \) the map \( y \mapsto y - f(t, y, z) \) is strictly increasing and continuous in \( y \)

Then there exists an adapted solution \( (Y, Z, K) \) for the RBSDE (2.2). Moreover, this solution is unique up to indistinguishability for \( Y \) and equivalence \( \sim_M \) for \( Z \).

Proof. It is clear that the solution \( Y_T = \xi \) at time \( T \). Then we construct the solution for all \( t \) using backward induction. Without loss of generality, we only consider the following one-step RBSDE
\[
Y_t = Y_{t+1} + f(t, Y_t, Z_t) + K_{t+1} - K_t - Z^*_t M_{t+1}.
\]

(1) Existence. We divide the proof into two steps. In the first step, we construct a sequence of BSDEs and prove the convergence of the corresponding
solutions. We prove that the limit obtained in Step 1 is a solution of (2.5) in the second step.

**Step 1.** Consider the following sequence of BSDEs

\[ Y_n^t = Y_{t+1} + f(t, Y_n^t, Z_n^t) + n(Y_n^t - S_t)^- - (Z_n^t)^* M_{t+1}, \quad n \in \mathbb{N} \]  

(2.7)

where \( \mathbb{N} \) is the set of positive integers. Taking a conditional expectation in (2.7), we get

\[ Y_n^t = E[Y_{t+1}|\mathcal{F}_t] + f(t, Y_n^t, Z_n^t) + n(Y_n^t - S_t)^-, \quad n \in N. \]  

(2.8)

Hence,

\[ (Z_n^t)^* M_{t+1} = Y_{t+1} - E[Y_{t+1}|\mathcal{F}_t]. \]

By the Martingale Representation Theorem (Theorem 2.2), there exists a unique process \( Z_t \), up to equivalence \( \sim M \), such that the above equation is satisfied for an arbitrary \( n \). Using this \( Z_t \), (2.8) can be rewritten as

\[ Y_n^t = Y_{t+1} + f(t, Y_n^t, Z_t) + n(Y_n^t - S_t)^- - Z^* M_{t+1}, \quad n \in N. \]  

(2.9)

Let

\[ f_n(t, y, z) = f(t, y, z) + n(y - S_t)^-. \]

Then \( y - f_n(t, y, z) \) is strictly increasing and continuous in \( y \). By theorem 2.12, (2.9) has a unique solution \((Y_n^t, Z_t)\). It is clear that

(i) \( f_{n+1}(t, y, z) \geq f_n(t, y, z), \quad \forall (y, z) \in \mathbb{R} \times \mathbb{R}^m; \)

(ii) \( f(t, Y_t^t, Z_t^t) - f(t, Y_t^t, Z_t^{n+1}) = 0 = \min_{i \in \mathbb{E}_t} \{ [Z_t^n - Z_t^{n+1}]^*(e_i - E[X_{t+1}]) \}, \)

\( Z^n = Z^{n+1} = Z; \)

(iii) since the map \( y - f_n(t, y, z) \) is strictly increasing, we obtain that if

\[ y_1 - f_n(t, y_1, z) \geq y_2 - f_n(t, y_2, z), \]

then \( y_1 \geq y_2 \) \( P \)-a.s.

Therefore, by the Comparison Theorem 2.12 we see that \( Y_{n+1}^t \geq Y_n^t \) \( P \)-a.s.

Thus we have the existence of a limit

\[ Y_t^n \uparrow Y_t \quad P - a.s. \]

From (2.8), on the event \( \{Y_t^n \geq S_t\} \), we have \( Y_t^n = E[Y_{t+1}|\mathcal{F}_t] + f(t, Y_t^n, Z_t) \). Because the map \( y \mapsto y - f(t, y, z) \) is strictly increasing, we deduce that \( Y_t^n \) is essentially bounded on \( \{Y_t^n \geq S_t\} \). On the event \( \{Y_t^n < S_t\} \),

\[ Y_t^n - f(t, Y_t^n, Z_t) = E[Y_{t+1}|\mathcal{F}_t] + n(S_t - Y_t^n). \]
Since \( y - f(t,y,z) \) is strictly increasing and \( n(S_t - Y^n_t) \geq 0 \) on \( \{Y^n_t < S_t\} \), there exists a lower bound for \( Y^n_t \) on this event. Combining these bounds, from Fatou’s Lemma we see that

\[
E \mid Y_t \mid \leq \lim_{n \to \infty} E \mid Y^n_t \mid < \infty.
\]

Define a process \( K^n \) by \( K^n_0 = 0 \) and

\[
K^n_{t+1} - K^n_t = \Delta K^n_t = n(Y^n_t - S_t)^-.
\]

By (2.8), we have

\[
|\Delta K^{n+p}_t - \Delta K^n_t| \leq |f(t,Y^{n+p}_t, Z_t) - f(t,Y^n_t, Z_t)| + |Y^{n+p}_t - Y^n_t|,
\]

for all \( p \in \mathbb{N} \).

Since \( f \) is continuous in \( y \) and \( Y^n_t \uparrow Y_t \) \( P \)-a.s., we obtain

\[
|\Delta K^{n+p}_t - \Delta K^n_t| \to 0, \quad \text{as } n \to \infty.
\]

Consequently, there exist random variables \( \Delta K_t \) such that \( \Delta K^n_t \to \Delta K_t = K_{t+1} - K_t \) as \( n \to \infty \). Define the limiting process \( K \) by

\[
K_0 = 0 \quad \text{and} \quad K_t = \sum_{0 \leq u < t} \Delta K_u.
\]

Then as \( n \to \infty \), (2.9) becomes

\[
Y_t = Y_{t+1} + f(t,Y_t, Z_t) + K_{t+1} - K_t - Z_t^* M_{t+1}.
\]

**Step 2.** It is clear that the triple \((Y_t, Z_t, K_t)\) obtained above satisfies (i) and (ii) of Definition 2.5. It remains to check (iii) and (iv).

First, note that \( K_t \) is increasing, as \( \Delta K_t \) is non-negative. As

\[
(Y^n_t - S_t)\Delta K^n_t = n(Y^n_t - S_t)(Y^n_t - S_t)^- = -n[(Y^n_t - S_t)^-]^2 \leq 0,
\]

we have that

\[
(Y_t - S_t)(K_{t+1} - K_t) \leq 0.
\]

On the other hand, as \( Y^n \) is increasing in \( n \),

\[
(Y^{n+1}_t - S_t)^- \leq (Y^n_t - S_t)^-.
\]

By (2.8), we have

\[
(Y^n_t - S_t)^- = \frac{Y^n_t - E[Y_{t+1} | \mathcal{F}_t] - f(t,Y^n_t, Z_t)}{n}.
\]

Then, as \( n \to \infty \),

\[
(Y^n_t - S_t)^- \downarrow 0 \quad \text{and} \quad (Y_t - S_t)^- = \lim_{n \to +\infty} (Y^n_t - S_t)^- = 0.
\]
It follows that $Y_t \geq S_t$. Hence

$$(Y_t - S_t)(K_{t+1} - K_t) \geq 0 \quad \text{P-a.s.}$$

Thus, we obtain $(Y_t - S_t)(K_{t+1} - K_t) = 0 \quad \text{P-a.s.}$

\[ \text{(2) Uniqueness.} \] Suppose that there exist two solutions $(Y_t, Z_t, K_t)$ and $(Y'_t, Z'_t, K'_t)$ of the RBSDE \eqref{eq:2.2}. Without loss of generality, suppose $Y_t > Y'_t$ and $Y_s = Y'_s$ for all $s \in \{0, 1, \ldots, (t-1)\}$. Then $Y_t > Y'_t \geq S_t$. It follows that $K_{t+1} - K_t = 0$ and \eqref{eq:2.6} can be simplified to

$$Y_t = Y_{t+1} + f(t, Y_t, Z_t) - Z_t^* M_{t+1}.$$ 

On the other hand,

$$Y'_t = Y_{t+1} + f(t, Y'_t, Z'_t) + K'_{t+1} - K'_t - Z'_t^* M_{t+1}.$$ 

By Theorem 2.12, we have $Y_t \leq Y'_t \quad \text{P-a.s.}$ This leads to contradiction. Thus, we have $Y_t = Y'_t \quad \text{P-a.s.}$

By Corollary 2.10, we have $K_t = K'_t$, $K_{t+1} = K'_{t+1}$, and consequently

$$Z^*_t M_{t+1} = Y_{t+1} - E[Y_{t+1} | \mathcal{F}_t] + K'_{t+1} - K'_t - E[K'_{t+1} - K'_t | \mathcal{F}_t]$$

$$= Y_{t+1} - E[Y_{t+1} | \mathcal{F}_t] + K_{t+1} - K_t - E[K_{t+1} - K_t | \mathcal{F}_t]$$

$$= Z_t^* M_{t+1}$$

By Definition 2.1, we have $Z \sim_{\mathcal{M}} z'$. \[ \square \]

### 2.4 Relation to optimal stopping problems

We now show that the solution $(Y_t)$ of the RBSDE \eqref{eq:2.2} corresponds to the value of an optimal stopping problem.

**Proposition 2.15** Let $\{Y_t, Z_t, K_t\}_{0 \leq t \leq T}$ be a solution of the RBSDE \eqref{eq:2.2}. Then for each $t \in \{0, 1, \ldots, T\}$,

$$Y_t = \sup_{\theta \in \mathcal{J}_t} E \left[ \sum_{t \leq u < \theta} f(u, Y_u, Z_u) + S_\theta 1_{\{\theta < T\}} + \xi 1_{\{\theta = T\}} | \mathcal{F}_t \right],$$

where $\mathcal{J}$ is the set of all stopping times dominated by $T$ and $\mathcal{J}_t \triangleq \{ \theta \in \mathcal{J}; t \leq \theta \leq T \}$.

**Proof.** For a given stopping time $\theta \in \mathcal{J}_t$, we have

$$Y_t = Y_\theta + \sum_{t \leq u < \theta} f(u, Y_u, Z_u) + K_\theta - K_t - \sum_{t \leq u < \theta} Z_u^* M_{u+1}, \quad 0 \leq t \leq T.$$ 

Taking the conditional expectation,

$$Y_t = E[Y_\theta + \sum_{t \leq u < \theta} f(u, Y_u, Z_u) + K_\theta - K_t | \mathcal{F}_t]$$

$$\geq E[ \sum_{t \leq u < \theta} f(u, Y_u, Z_u) + S_\theta 1_{\{\theta < T\}} + \xi 1_{\{\theta = T\}} | \mathcal{F}_t].$$
In order to obtain the reversed inequality, we define

\[ D_t = \begin{cases} T, & \text{if } Y_u > S_u \text{ for all } t \leq u \leq T; \\ \inf \{ u : t \leq u \leq T, Y_u = S_u \}, & \text{otherwise.} \end{cases} \]

Note that \( \sum_{0 \leq t \leq T} (Y_t - S_t)(K_{t+1} - K_t) = 0 \) implies that \( K_s = K_{s-1} \) for any \( t + 1 \leq s \leq D_t \), and so

\[ K_{D_t} - K_t = \sum_{t+1 \leq s \leq D_t} (K_s - K_{s-1}) = 0, \quad 0 \leq t \leq T. \]

From this, we see that

\[ Y_t = E[Y_{D_t} + \sum_{t \leq u < D_t} f(u, Y_u, Z_u) + K_{D_t} - K_t | \mathcal{F}_t] \]

\[ = E[Y_{D_t} + \sum_{t \leq u < D_t} f(u, Y_u, Z_u) | \mathcal{F}_t] \]

\[ \leq \sup_{\theta \in \mathcal{J}_t} E\left[ \sum_{t \leq u < \theta} f(u, Y_u, Z_u) + S_\theta 1_{\{ \theta < T \}} + \xi_1 1_{\{ \theta = T \}} | \mathcal{F}_t \right] \]

which completes the proof. □

**Example 2.16** Set \( L_t = \sum_{t \leq u < T} Z^*_u M_{u+1} \). Consider the special case \( f = C \), \( S_T = \xi \geq 0 \) where \( C \) is a constant. If \( (Y, Z, K) \) is a solution, then

\[ Y_0 = E[\xi + CT + K_T] = E[\xi + \sup_{0 \leq t \leq T} (S_t + L_t - C(T - t) - \xi)^+] \].

Since \( S_T = \xi \), it is easy to check that

\[ Y_0 = \sup_{\theta \in \mathcal{J}_0} E[S_\theta + C\theta] = E\left[ \sup_{0 \leq t \leq T} (S_t + L_t + Ct) \right]. \]

When \( C = 0 \), we have

\[ Y_0 = \sup_{\theta \in \mathcal{J}_0} E[S_\theta] = E\left[ \sup_{0 \leq t \leq T} (S_t + L_t) \right]. \]

By Proposition 2.15, the solution of the RBSDE in which \( f \) is a given stochastic process is the value function of an optimal stopping problem. In the following, we shall particularly investigate the cases where \( f(t, y, z) \) is a linear function or concave (convex) function. In the latter case, the solution \( \{Y_t\}_{0 \leq t \leq T} \) is shown to be the value function of a mixed optimal stopping–optimal stochastic control problem. Note that El. Karoui et al. [17] studied similar problems for reflected backward stochastic differential equations in continuous time.

Without loss of generality, we consider the one-step RBSDE in our framework, i.e.

\[ Y_t = Y_{t+1} + f(t, Y_t, Z_t) - Z^*_t M_{t+1} + K_{t+1} - K_t, \quad 0 \leq t < T. \quad (2.10) \]

By Proposition 2.15, we have the following results.
Proposition 2.17 Consider the RBSDE (2.2) with coefficient \( f = \alpha_t \), where \( \{\alpha_t\}_{0 \leq t \leq T} \) is a given adapted process and takes values in \( R \), and \( S \) is a given adapted boundary process. Then the unique solution \((Y, Z, K)\) satisfies

\[
Y_t = \sup_{\theta \in \mathcal{J}_t} E \left[ \sum_{t \leq s < \theta} \alpha_s + S\ell_{\theta < T} + \xi_{\theta = T} \bigg| \mathcal{F}_t \right].
\]

Moreover, if we consider equation (2.10), then we have

\[
Y_t = S_t \lor (\alpha_t + E[Y_{t+1}|\mathcal{F}_t]). \tag{2.11}
\]

Proof. In fact, we can directly obtain (2.11) from the definition of an RBSDE solution (Definition 2.5). Denote \( \rho_t = \alpha_t + E[Y_{t+1}|\mathcal{F}_t] \). Taking the conditional expectation for (2.10), we have

\[
Y_t = \rho_t + E[K_{t+1} - K_t|\mathcal{F}_t].
\]

There are then two cases:

(i) \( \rho_t \geq S_t \). Then \( Y_t - S_t \geq E[K_{t+1} - K_t|\mathcal{F}_t] \geq 0 \), since \( K \) is an increasing process. However, by condition (iv) of Definition 2.5, it follows that \( K_{t+1} - K_t = 0 \), so \( Y_t = \rho_t \).

(ii) \( \rho_t < S_t \). It follows that \( K_{t+1} - K_t > 0 \). By condition (iv) of Definition 2.5 we have \( Y_t = S_t \).

To sum up, \( Y_t = S_t \lor \rho_t \), i.e. \( Y_t = S_t \lor (\alpha_t + E[Y_{t+1}|\mathcal{F}_t]) \).

To neatly consider linear RBSDEs, we need the following definition.

Definition 2.18 Recall from (2.6) that \( Q_t \) defines the set of possible jumps of \( X \) at time \( t \). We shall say that \( \gamma \) is a \( Q \)-vector process if it is an adapted process in \( L^1(\mathbb{R}^m) \) which satisfies \( \sum \langle \gamma_t, e_i \rangle = 0 \) and \( \langle \gamma_t, e_i \rangle = 0 \) for all \( e_i \not\in Q_t \). We write \( \mathbb{R}^m_{Q_t} \) for the space of \( Q \)-vectors at time \( t \), and note this is a subspace of \( \mathbb{R}^m \).

Lemma 2.19 For \( \gamma \) a \( Q \)-vector process, the function \( f(\omega,t,z) = \langle \gamma_t, z \rangle \) satisfies Theorem 2.3 condition (i). Furthermore, the solution process \( Z \) can be taken to lie in \( \mathbb{R}^m_{Q_t} \) without loss of generality, and is unique in this space.

Proof. Simply note that \( Z \sim_M Z' \) if and only if, for every \( t \), \( Z_t \) and \( Z'_t \) differ at most by a constant and by the values of \( \langle Z_t, e_i \rangle \) for \( e_i \not\in Q_t \), both of which are in the kernel of the linear map \( \langle \gamma_t, \cdot \rangle \).

Proposition 2.20 Let \( \{\alpha_t, \beta_t, \gamma_t\}_{0 \leq t \leq T} \) be adapted processes taking values in \( \mathbb{R} \times [0,1] \times \mathbb{R}^m \), and let \( \gamma \) be a \( Q \)-vector process. Let \( S \) be a given adapted boundary process. Consider the RBSDE (2.10) with

\[ f(t, y, z) = \alpha_t + \beta_t y + \langle \gamma_t, z \rangle. \]

Then the solution \((Y, Z, K)\) satisfies

\[
Y_t = S_t \lor (\alpha_t + \beta_t Y_t + \langle \gamma_t, z \rangle + E[Y_{t+1}|\mathcal{F}_t]). \tag{2.12}
\]
Proof. It is easy to check that Theorem 2.14 applies, so the solution \((Y, Z, K)\) exists and is unique. By Proposition 2.17 applied with \(f(t, Y_t, Z_t)\) as the fixed term, we obtain (2.12).

Note that this also yields a simple method of calculating solutions. We have that \(Z_t\) satisfies \(Z_t^* M_{t+1} = Y_{t+1} - E[Y_{t+1} | \mathcal{F}_t]\) from the proof of Theorem 2.14 and so we obtain
\[ \hat{Y}_t = \frac{1}{1 - \beta_t} (\alpha_t + \langle \gamma_t, Z_t \rangle + E[Y_{t+1} | \mathcal{F}_t]) \]
For each \((\omega, t)\), if \(\hat{Y}_t > S_t\), then \(Y_t = \hat{Y}_t\) is as desired; otherwise, \(Y_t = S_t\).

Remark 2.21 If \(\beta_t = 1\), then
\[ y - f(t, y, z) = -\alpha_t - \langle \gamma_t, z \rangle \]
which violates assumption (ii) of Theorem 2.14 as the right hand side is independent of \(y\). Thus, we can not guarantee that there exists a unique solution of the linear RBSDE (and typically, no solution will exist).

2.4.1 Concave coefficients

We now suppose that for each fixed \((\omega, t)\), the driver \(f(t, y, z)\) is a concave function of \((y, z)\). For each \((\omega, t, \beta, \gamma) \in \Omega \times \{0, 1, \ldots, T\} \times \mathbb{R} \times \mathbb{R}^m\), define the conjugate function \(F(\omega, t, \beta, \gamma)\) as follows:
\[ F(\omega, t, \beta, \gamma) = \sup_{(y, z) \in \mathbb{R} \times \mathbb{R}^m} (f(t, y, z) - \beta y - \langle \gamma, z \rangle) \]
\[ D_F^\omega(t) = \{(\beta, \gamma) \in \mathbb{R} \times \mathbb{R}^m; F(\omega, t, \beta, \gamma) < \infty\} \]
It follows that
\[ f(t, y, z) = \inf_{(\beta, \gamma) \in D_F^\omega(t)} \{F(t, \beta, \gamma) + \beta y + \langle \gamma, z \rangle\}, \]
the infimum is achieved at \((\beta', \gamma') \in D_F^\omega(t)\) and the set \(D_F^\omega(t)\) is a.s. bounded (refer to [19]). If the function \(f\) also satisfies conditions (iii) and (iv) of the Comparison Theorem (Theorem 2.12), we see that the infimum is attained in the smaller set
\[ C_F^\omega(t) = \{(\beta, \gamma) \in D_F^\omega(t); |\beta_t| < 1, (\gamma_t, z) \geq (\epsilon_t - E[X_{t+1} | \mathcal{F}_t], z) \text{ for all } z \in \mathbb{R}^m, \epsilon_t \in \mathcal{Q}_t \} \]
(2.13)

Denote the solution of RBSDE with coefficient \(f^\beta(\omega, t, y, z) = f(t, \beta_t, \gamma_t) + \beta_t y + \langle \gamma_t, z \rangle\) by \(\{(Y^\beta_t, Z^\beta_t, K^\beta_t)\}_{0 \leq t \leq T}\) (resp. \(\{(Y_t, Z_t, K_t)\}_{0 \leq t \leq T}\) for the RBSDE with coefficient \(f(t, y, z)\)). Then, P-a.s. for all \(t\), we have
\[ f(t, Y_t, Z_t) = f(t, \beta_t, \gamma_t) + \beta_t Y_t + \langle \gamma_t, Z_t \rangle \]
\[ (Y_t, Z_t, K_t) = (Y^\beta_t, Z^\beta_t, K^\beta_t) \]
and so $Y_t = Y_t^{\beta', \gamma'}$ can be interpreted as the value functions of an optimization problem.

**Theorem 2.22** For each $(\beta_t, \gamma_t) \in C^F_t$ with $|\beta_t| < 1$ P-a.s., we have

$$Y_t^{\beta, \gamma} = S_t \vee (F(t, \beta_t, \gamma_t) + \beta_t Y_t^{\beta, \gamma} + \langle \gamma_t, Z_t \rangle + E[Y_{t+1}|\mathcal{F}_t]);$$

$$Y_t = S_t \vee (F(t, \beta'_t, \gamma'_t) + \beta'_t Y_t + \langle \gamma'_t, Z_t \rangle + E[Y_{t+1}|\mathcal{F}_t]).$$

Moreover,

$$Y_t = \inf_{(\beta, \gamma) \in C^F_t} Y_t^{\beta, \gamma}$$

$$= \inf_{(\beta, \gamma) \in C^F_t} (S_t \vee (F(t, \beta, \gamma) + \beta Y_t^{\beta, \gamma} + \langle \gamma, Z_t \rangle + E[Y_{t+1}|\mathcal{F}_t]));$$

$$= S_t \vee \inf_{(\beta, \gamma) \in C^F_t} (F(t, \beta, \gamma) + \beta Y_t^{\beta, \gamma} + \langle \gamma, Z_t \rangle + E[Y_{t+1}|\mathcal{F}_t]).$$

In other words, $Y_t$ is the value function of a minimax control problem, and the triple $(\beta', \gamma', D_t)$, where $D_t = \inf\{s : t \leq s \leq T, Y_s = S_s\}$ is optimal.

**Proof.** The first statement is apparent from Proposition 2.20. By the Comparison Theorem 2.8, we have

$$Y_t \leq Y_t^{\beta, \gamma}, \quad \text{for all } (\beta, \gamma) \in C^F_t.$$  

On the other hand,

$$Y_t = Y_t^{\beta', \gamma'} \geq \inf_{(\beta, \gamma) \in D^F_t} Y_t^{\beta, \gamma},$$

which immediately leads to

$$Y_t = \inf_{(\beta, \gamma) \in C^F_t} Y_t^{\beta, \gamma}.$$

Finally, it is easy to see that the operators inf and ∨ can be exchanged. ■

**Remark 2.23** If $f$ is a convex function of $(y, z)$, by essentially the same argument, we have a similar representation of the form

$$Y_t = S_t \vee \sup_{(\beta, \gamma) \in C^F_t} (F(t, \beta, \gamma) + \beta Y_t^{\beta, \gamma} + \langle \gamma, Z_t \rangle + E[Y_{t+1}|\mathcal{F}_t]);$$

where $F(t, \beta, \gamma) = \inf_{(y, z) \in \mathbb{R} \times \mathbb{R}_+^m} (f(t, y, z) - \beta y - \langle \gamma, z \rangle)$, and $C^F_t$ is as defined in 2.13.

### 3 Optimal stopping under g-expectation

In order to study optimal stopping problems under $g$-expectation, we first study the $g$-expectation theory on spaces related to discrete time, finite state processes. Then we give the connection between multiple prior martingales and $g$-martingales. Last, we show that some optimal stopping problems with multiple priors can be solved by computing the corresponding RBSDEs.
3.1 $g$-expectation

Peng [30] and [31] introduced the notions of $g$-expectation and conditional $g$-expectation as well as $g$-martingale via backward stochastic differential equations. The aim of this section is to study the $g$-expectation theory in our framework.

**Definition 3.1** We say a driver $f$ satisfying the conditions of Theorem 2.14 is normalised if

$$f(\omega, t, y, 0) = 0 \quad \text{P-a.s. for all } t \in \{0, 1, \ldots, T\}, y \in \mathbb{R}.$$

**Definition 3.2** A system of operators

$$G(\cdot | \mathcal{F}_t) : L^1(\mathcal{F}_T; \mathbb{R}) \to L^1(\mathcal{F}_t; \mathbb{R})$$

is called a filtration consistent nonlinear expectation if it satisfies, for all $\xi \in L^1(\mathcal{F}_T; \mathbb{R})$,

(i) If $\xi \geq \xi'$ P-a.s. then $G(\xi | \mathcal{F}_t) \geq G(\xi' | \mathcal{F}_t)$.

(ii) For any $\mathcal{F}_t$-measurable $\xi$, we have $G(\xi | \mathcal{F}_t) = \xi$.

(iii) We have the tower property $G(G(\xi | \mathcal{F}_s) | \mathcal{F}_t) = G(\xi | \mathcal{F}_s)$ for all $s < t$.

(iv) For any $A \in \mathcal{F}_t$, we have $I_A G(\xi | \mathcal{F}_t) = G(I_A \xi | \mathcal{F}_t)$.

It is said to be translation invariant if

(v) For any $q \in L^1(\mathcal{F}_t, \mathbb{R})$ we have $G(\xi + q | \mathcal{F}_t) = G(\xi | \mathcal{F}_t) + q$.

We have the following representation theorem from [9], Theorem 7.

**Theorem 3.3** The following are equivalent.

(i) $G$ is a filtration consistent, translation invariant nonlinear expectation

(ii) $Y_t = G(\xi | \mathcal{F}_t)$ is the solution to a BSDE with coefficient $f : \Omega \times \{0, 1, \ldots, T\} \times \mathbb{R}^m \to \mathbb{R}$, and $f$ satisfies the conditions of Theorem 2.14 (so BSDE solutions exist) and conditions (iii) and (iv) of Theorem 2.12 (so the comparison theorem holds) and is normalised.

Furthermore, the function $f$ is unique, and can be obtained from the relation $f(\omega, t, z) = G(z \ast M_{t+1} | \mathcal{F}_t)$.

From [30] and [31], an operator $G$ defined by a BSDE solution of this type is called a $g$-expectation. This result therefore shows that $g$-expectations and filtration consistent, translation invariant nonlinear expectations coincide. For the sake of brevity, we will therefore use the term $g$-expectation.

Now we study the Doob–Meyer decomposition and optional sampling theorem under $g$-expectation.
Definition 3.4 A process \( \{X_t\} \) will be called a \( g \)-supermartingale if \( X_t \in L^1(\mathcal{F}_t; \mathbb{R}) \) for all \( t \) and \( X_s \geq G(X_t | \mathcal{F}_t) \) for all \( s \leq t \). In a similar way, we define submartingales and martingales.

Recall that, in this setting, a process \( K \) is predictable if \( K_{t+1} \) is \( \mathcal{F}_t \)-measurable for all \( t \).

Theorem 3.5 (Doob–Meyer Decomposition) Let \( X \) be a \( g \)-supermartingale (resp. \( g \)-submartingale). Then there exists a unique predictable increasing (resp. decreasing) process \( K \) such that \( X + K \) is a \( g \)-martingale and \( K_0 = 0 \).

Proof. Let \( K_0 = 0 \) and define \( K \) recursively by

\[
K_{t+1} = K_t + G(X_{t+1} | \mathcal{F}_t) - X_t.
\]

Then simple calculation verifies the result. ■

In the following, we study the optional sampling theorem for \( g \)-super and \( g \)-sub-martingales in our framework.

Theorem 3.6 Let \( X \) be a \( g \)-supermartingale. Then for any stopping times \( \sigma, \tau \) with \( 0 \leq \sigma \leq \tau \leq T \), we have

\[
X_\sigma \geq G(X_\tau | \mathcal{F}_\sigma).
\]

Similarly for \( g \)-submartingales.

Proof. From the Doob–Meyer decomposition, we know that there is an increasing process \( K \) such that \( X + K \) is a \( g \)-martingale. Hence, by the recursive nature of BSDEs and normalisation

\[
X_\sigma + K_\sigma = X_\tau + K_\tau + \sum_{\sigma \leq t < \tau} f(t, Z_t) - \sum_{\sigma \leq t < \tau} Z_t^* M_{t+1}.
\]

Rearranging the terms, we have

\[
X_\sigma = X_\tau + \sum_{\sigma \leq t < \tau} f^K(t, Z_t) - \sum_{\sigma \leq t < \tau} Z_t^* M_{t+1},
\]

where

\[
f^K(t, Z_t) = f(t, Z_t) + K_{t+1} - K_t.
\]

As \( K \) is an increasing predictable process, \( f^K \) satisfies the requirements of Theorem 2.8 and \( f^K(t, Z_t) \geq f(t, Z_t) \). Therefore, \( X_\sigma \geq G(X_\tau | \mathcal{F}_\sigma) = Y_\sigma \), where \( Y_\sigma \) solves the BSDE

\[
Y_\sigma = Y_\tau + \sum_{\sigma \leq t < \tau} f(t, Z_t) - \sum_{\sigma \leq t < \tau} Z_t^* M_{t+1}.
\]

The corresponding argument when \( X \) is a submartingale, and so \( K \) is a decreasing process, also holds. ■
3.2 Multiple prior martingales and \( g \)-Martingales

Riedel [34] developed a theory of optimal stopping under multiple priors. He defined the process \( \{U_t\} \) to be a ‘multiple prior martingale’ if it satisfies

\[
U_t = \inf_{p \in \Lambda} E^p[U_{t+1}|\mathcal{F}_t],
\]

where \( \Lambda \) is a set of time-consistent measures, as defined in the following.

**Definition 3.7** A family \( \Lambda \) of probability measures will be called ‘time-consistent’ if, for any \( Q, Q' \in \Lambda \), any \( A \in \mathcal{F}_t \), we have \( Q'' \in \Lambda \), where

\[
Q''(B) = E_Q[I_A E_{Q'}[I_B|\mathcal{F}_t] + I_{A^c} I_{B^c}],
\]

See the \( m \)-stability of [16], Proposition 3.6 in [27], Theorem 2.2 in [5] and Definition 13 in [7] for discussion of this and related concepts. In particular, we have the following result, which is proven in each of these references in varying degrees of generality (any of which is sufficient for our setting here).

**Lemma 3.8** The family of operators \( G(\cdot|\mathcal{F}_t) = \inf_{p \in \Lambda} E^p[U_{t+1}|\mathcal{F}_t] \) is filtration consistent (in particular satisfies (ii-iv) of Definition 3.2) if and only if \( \Lambda \) is a time-consistent family of measures.

In this subsection, we will study special cases of multiple prior martingales using the theory of (R)BSDEs.

We denote by \( Q \) the set of all probability measures \( Q \sim P \). For any \( Q \in \mathcal{Q} \), set

\[
W_t := E^P\left[ \frac{dQ}{dP} | \mathcal{F}_t \right].
\]

Then \( W_t \) is a martingale and \( W_0 = 1 \). By Theorem 2.2 there exists an adapted process \( \bar{z} \) such that

\[
W_t = 1 + \sum_{0 \leq s < t} \bar{z}_s^* M_{s+1}.
\]

Let

\[
\psi_t = \text{var}(X_{t+1} | \mathcal{F}_t) = E[X_{t+1} X_{t+1}^* | \mathcal{F}_t] - E[X_{t+1} | \mathcal{F}_t] E[X_{t+1}^* | \mathcal{F}_t] = E[M_{t+1} M_{t+1}^* | \mathcal{F}_t],
\]

so \( \psi \) is a symmetric positive semidefinite matrix with null space orthogonal to the space of \( Q \)-vectors. We note that \( \psi \) also appeared in Definition 2.1. As the \( \bar{z} \) process in the martingale representation theorem is only defined up to equivalence \( \sim_M \), writing

\[
\theta_t = \frac{1}{W_t} \psi_t \bar{z}_t,
\]

and \( \psi^+ \) for the Moore–Penrose pseudoinverse of \( \psi \), we have \( \{W_t \psi_t^+ \theta_t\}_{0 \leq t \leq T} \sim_M z \). Without loss of generality, we can take \( \theta_t \) to be a \( Q \)-vector.
Therefore, we can write

\[ W_t = \prod_{0 \leq s < t} (1 + \theta^*_s \psi_t^+ M_{s+1}) \]

and

\[ \frac{dQ}{dP} = W_T = \prod_{0 \leq s < T} (1 + \theta^*_s \psi_t^+ M_{s+1}). \] (3.1)

Thus, for any \( Q \in \mathcal{Q} \), \( \frac{dQ}{dP} \) can be generated by \( (\theta_t) \) through (3.1). The probability measure generated by \( (\theta_t) \) is denoted by \( Q^\theta \). It is classical that this is a probability measure if and only if \( \theta^*_s \psi_s^+ M_{s+1} > -1 \) a.s.

Basic calculation yields that

\[ E_{Q^\theta}[X_{t+1} | \mathcal{F}_t] = E[(1 + \theta^*_t \psi_t^+ M_{t+1}) X_{t+1}] + \psi_t^+ \psi_t \cdot \theta_t. \]

In particular, for \( \theta_t \) a \( Q \)-vector (and so orthogonal to the null space of \( \psi \)),

\[ E_{Q^\theta}[M_{t+1} | \mathcal{F}_t] = \theta_t. \]

**Proposition 3.9** If \( \Lambda \) is a time consistent family of measures, by Lemma 3.8 and Theorem 3.3, the process \( Y_t = G(\xi | \mathcal{F}_t) \) solves a BSDE with terminal value \( Y_T = \xi \) and driver

\[ f(\omega, t, z) = \inf_{(\theta, Q^\theta) \in \Lambda} \{ z^\star \theta_t \}. \]

Conversely, we can verify that \( Q^\theta \) defined in this way is a probability measure (absolutely continuous with respect to \( P \)) provided that

- \( \theta_t \) is a.s. a \( Q \)-vector for all \( t \) and
- \( 0 \leq E[X_{t+1} | \mathcal{F}_t] + \theta_t \leq 1 \) a.s., the inequality being taken componentwise.

The measure \( Q^\theta \) is equivalent to \( P \) if and only if the inequality is strict in all components where \( \theta_t \neq 0 \).

### 3.2.1 \( \kappa \)-ignorance model

We now consider a concrete example, inspired by the \( \kappa \)-ignorance model in [5]. Suppose there exists a nonnegative process \( \kappa_t \) such that

\[ 0 \leq E[X_{t+1} | \mathcal{F}_t] + \theta_t \leq 1 \]

for all \( Q \)-vectors \( \theta_t \) with \( \| \theta_t \|_M \leq \kappa_t \). Consider the associated set of probability measures

\[ \mathcal{B} = \{ Q^\theta : \| \theta_t \|_M \leq \kappa_t \text{ P-a.s. for all } t \}. \] (3.2)

**Lemma 3.10** \( \mathcal{B} \) is a time consistent family of measures.
Proof. If $Q^0, Q^{0'} \in \mathcal{B}$, then for any time $t$, any $A \in \mathcal{F}_t$, the measure defined by
\[ Q''(B) = E_{Q^0}[I_A E_{Q^{0'}}[I_B|\mathcal{F}_t] + I_A^c I_B] \]
will have representation $Q'' = Q^{0''}$, where
\[ \theta''_s = \theta_{s\wedge t} + ((\theta_s - \theta_t) I_{A^c} + (\theta'_s - \theta'_t) I_{A^c}) I_{s>t}. \]
Therefore, as $\theta''$ will also satisfy $\|\theta''_s\|_M \leq \kappa_s$, the measure $Q''$ is also in $\mathcal{B}$. ■

**Definition 3.11** Suppose $\xi \in L^1(\mathcal{F}_T; \mathbb{R})$. Let
\[ G(\xi|\mathcal{F}_t) = \inf_{Q \in \mathcal{B}} \{ E_Q[\xi|\mathcal{F}_t] \}, \quad 0 \leq t \leq T. \]
Then we call $G(\xi|\mathcal{F}_t)$ the minimal conditional expectation of $\xi$ about $\mathcal{B}$. Similarly, we can define the corresponding maximal conditional expectation.

As $\mathcal{B}$ is a time consistent family, by Lemma 3.8 and Theorem 3.3 we see the following relation.

**Theorem 3.12** For any $\xi \in L^1(\mathcal{F}_T; \mathbb{R})$, $Y_s = G(\xi|\mathcal{F}_s)$ is the solution to the BSDE
\[ Y_t = \xi - \sum_{t \leq s < T} \kappa_s Z_s \|_M - \sum_{t \leq s < T} Z_s^* M_{s+1} \]

Proof. We know from Theorem 3.3 that $Y_s = G(\xi|\mathcal{F}_s)$ solves a BSDE with driver $f(\omega, t, z) = G(z^* M_{t+1}|\mathcal{F}_t)$. Hence
\[ f(\omega, t, z) = \inf_{Q \in \mathcal{B}} \{ E_Q[z^* M_{t+1}|\mathcal{F}_t] \} = \inf_{\theta: \|\theta\|_M \leq \kappa} \{ z^* \theta \}. \]
By the Cauchy–Schwarz inequality, this infimum is realised at $\theta \sim_M -\kappa \psi z$, where we have $f(\omega, t, z) = -\kappa \|z\|_M$. ■

### 3.2.2 Scenario perturbation model

An alternative similar model is where a collection of perturbation vector processes $\{\pi^i\}_{i=1}^n$ are given, each of which takes values in the probability vectors in $\mathbb{R}^m$. We assume these are absolutely continuous with respect to $\pi^0_i := E[X_{t+1}|\mathcal{F}_t]$, in the sense that if a component of $\pi^0_i$ is zero, then so is the corresponding component of $\pi^i$. These vectors can be thought of as ‘scenarios’, or mixtures of scenarios, which with $t$ we will stress-test our outcome, and correspond to measures where $X_{t+1}$ has $\mathcal{F}_t$ conditional expectation $\pi^1_i$.

For a given parameter $\kappa \leq 1$, we define a scenario perturbation measure to be a measure $Q^\theta$ where
\[ \theta_t = \lambda_t (\pi^i_{t+1} - \pi^0_i) \]
for some adapted processes $\lambda$ and $i$ with $\lambda_t \leq \kappa$ and $i(t) \in \{0, \ldots, n\}$. Again, one can verify that the associated family of measures is time consistent, and the corresponding minimal conditional expectations are given by the BSDE solutions
\[ G(\xi|\mathcal{F}_t) = Y_t = \xi - \sum_{t \leq s < T} \kappa \min_i \{ Z_s^* (\pi^i_s - \pi^0_s) \} - \sum_{t \leq s < T} Z_s^* M_{s+1} \]
3.3 Nonlinear expectations and optimal stopping

Riedel [34] considered the optimal stopping problem under ambiguity as follows:

\[
\maximize \inf_{Q \in \Lambda} E_Q[U_\tau] \quad \text{over all stopping times} \quad \tau \leq T
\]

for a finite horizon \(T < \infty\), where \(\Lambda\) is a time-consistent set of priors and \((U_t)_{t \in \mathcal{N}}\) is an essentially bounded and adapted process.

To solve the above problem, Riedel [34] introduced the multiple prior Snell envelope \(\bar{U}\) defined by \(\bar{U}_T = U_T\) and

\[
\bar{U}_t = \max\{U_t, \inf_{Q \in \mathcal{B}} E_Q[\bar{U}_{t+1} + K_{t+1} - K_t]\}, \quad t \in \{0, 1, \ldots, T - 1\}.
\]

We now study the relation between the multiple prior Snell envelope and RBSDEs. For a given time consistent family of measures \(\Lambda\), let

\[
\Theta = \{\theta : Q^\theta \in \Lambda\} \cap \{Q\text{-vectors}\}.
\]

Consider the following RBSDE:

\[
\begin{cases}
Y_t = Y_{t+1} + \inf_{\theta \in \Theta} \{Z_t^* \theta_t\} - Z_t^* M_{t+1} + K_{t+1} - K_t \\
Y_T = U_T, \\
Y_t \geq U_t, \\
(Y_t - U_t)(K_{t+1} - K_t) = 0
\end{cases}
\] (3.3)

By Theorem 2.13, provided the infimum is almost surely finite (which is guaranteed by the fact that \(\theta\) generates a measure), (3.3) has a unique solution \((Y_t, Z_t, K_t)\).

**Theorem 3.13** Suppose \(U_T \in L^1(\mathcal{F}_T; R)\). Then the solution \(Y_t\) of (3.3) is the multiple prior Snell envelope of \(U\) with multiple prior set \(\Lambda\), that is, \(Y = \bar{U}\).

**Proof.** By Theorem 2.13 we know that for any \(\xi \in L^1(\mathcal{F}_{t+1}; \mathbb{R})\)

\[
\inf_{Q \in \Lambda} E^Q[\xi | \mathcal{F}_t] = \inf_{\theta \in \Theta} \{z^* \theta_t\} + E[\xi | \mathcal{F}_t]
\]

where \(z^* M_{t+1} = \xi - E[\xi | \mathcal{F}_t]\). For \((Y, Z, K)\) the solution of (3.3), as \(f(t, z) = \inf_{\theta \in \Theta} \{z^* \theta_t\}\) is concave, we know from Theorem 2.22 that

\[
Y_t = U_t \vee \inf_{\theta \in \Theta} \{z^* \theta_t + E[\xi | \mathcal{F}_t]\} = \bar{U}_t
\]

as desired. ■

By the above theorem and Proposition 2.15 as well as other properties of RBSDEs, we can deduce the following useful results:

(i) \(\bar{U}\) is the smallest \(g\)-supermartingale (for the driver \(f(t, z) = \inf_{\theta \in \Theta} \{z^* \theta_t\}\)) which dominates \(U\);
(ii) $\bar{U}$ is the value process of the following optimal stopping problem under ambiguity, i.e.

$$\bar{U}_t = \sup_{\tau \in \mathcal{J}_t} \inf_{P \in \mathcal{B}} E^P[U_{\tau} | \mathcal{F}_t];$$

(iii) an optimal stopping rule can be given by

$$\tau^* = \inf\{t \geq 0 : \bar{U}_t = U_t\}.$$

Now we reconsider a simple example which was discussed in [28] and [6].

**Example 3.14** Suppose the value process of an asset is governed by

$$\begin{cases}
S_{t+1} - S_t = \mu S_t + \sigma S_t (M_{t+1} - M_t) \\
S_0 = s > 0
\end{cases} \tag{3.4}$$

where $s, \mu \in \mathbb{R}^+, \sigma \in \mathbb{R}^n \setminus \{0\}$ are given constants. We want to find the optimal time $\tau^* \in \{0, 1, \ldots, T\}$ to sell this asset.

We first suppose that there does not exist ambiguity and the risk only comes from the martingale difference process. This problem can be formulated as follows

$$\sup_{0 \leq \tau \leq T} E[S_\tau].$$

From (3.4), we know

$$E[S_{t+1} - S_t] = \mu E[S_t].$$

Since $S_0 = s > 0$, we have $E[S_1] > S_0 > 0$. It is easy to see that $E[S_{t+1}] \geq E[S_t] > 0$. Thus, the optimal time is $\tau^* = T$, which implies that the owner is better to hold this asset until the maturity time $T$.

Now if there exists ambiguity, which can be described by a family of time-consistent probability measures $\Lambda$ with associated set $\Theta = \{\theta : Q^\theta \in \Lambda\}$. Then an ambiguity averse decision maker wishes to solve

$$\sup_{\tau} \inf_{\theta \in \Theta} E^{Q^\theta}[S_\tau].$$

This problem solved by considering the following RBSDE,

$$\begin{cases}
\bar{U}_t = \bar{U}_{t+1} - \mu \bar{U}_t + \inf_{\theta \in \Theta} \{Z_t^\theta \theta_t\} - Z_t^\theta M_{t+1} + K_{t+1} - K_t \\
\bar{U}_T = S_T, \quad \bar{U}_t \geq S_t \\
(\bar{U}_t - S_t)(K_{t+1} - K_t) = 0.
\end{cases}$$

By Proposition 2.15, we know

$$\tau^* = \inf\{u \leq T : \bar{U}_u = S_u\} \wedge T.$$

Hence, ambiguity aversion can encourage earlier stopping.
4 Applications to pricing of American contingent claims

It is well-known that the price of an American option corresponds to the solution of a reflected BSDE, where the information flow is generated by the Brownian motion \[4, 18\]. In this section, we explore this pricing problem in the discrete time and finite state cases, where the martingale difference process replace the Brownian motion.

We begin with the classical set-up for discrete time asset pricing: the basic securities consist of \( m + 1 \) assets \( \{S^i_t\}_{0 \leq t \leq T, i \in \{0, 1, \ldots, m\}} \), one of which is a non-risky asset with price process as follows:

\[
S^0_{t+1} - S^0_t = r_t S^0_t,
\]

where \( r_t \) is the interest rate. The other \( k \) risky asset (the stocks) are traded discretely, of which the price process \( S^i_t \) for one share of \( i \)th stock is governed by the linear difference equation

\[
S^{i}_{t+1} - S^{i}_t = S^{i}_t \left( b^{i}_t + \sum_{j=1}^{m} \sigma^{i,j}_t M^j_{t+1} \right)
\]

where \( M_t = (M^1_t, M^2_t, \ldots, M^m_t)^* \) is our martingale difference sequence on \( \mathbb{R}^m \).

We assume that

(i) The short interest rate \( r \) is a predictable process which is generally non-negative.

(ii) The stock appreciation rates \( b = (b^1, b^2, \ldots, b^m)^* \) is a predictable process.

(iii) The volatility matrix \( \sigma = (\sigma^{i,j}) \) is a predictable process in \( \mathbb{R}^{k \times m} \).

(iv) There exists a predictable \( Q \)-vector process \( \theta \), called the risk premium, such that

\[
b_t - r_t 1 = \sigma_t \theta_t, \quad dt \times dP - a.e..
\]

where \( 1 \) is the vector whose every component is 1. Denote by \( \Theta \) the family of all such processes.

We note that each \( \theta \in \Theta \) corresponds to a measure where

\[
E^Q[S^i_{t+1} - S^i_t | \mathcal{F}_t] = S^i_t \left( b^{i}_t + \sum_{j=1}^{m} \sigma^{i,j}_t \theta^j_t \right) = S^i_t r_t
\]

and so \( \Theta \) is a representation of the equivalent martingale measures of the discounted processes \( S^i_t \prod_{s \leq t} (1 + r_s)^{-1} \).
Definition 4.1 A predictable process $H = (H^0, H^1, ..., H^k)$ is called self-financing if $(H_t, S_t) = (H_t, S_t)$, where $S = (S^0, S^1, ..., S^k)$. The value $V$ of the corresponding self-financing portfolio can be formulated as follows (refer to [32]):

$$V_t = H^0_t S^0_t + \sum_{i=1}^{k} H^i_t S^i_t = H^0_t S^0_t + \sum_{i=1}^{k} H^i_{t+1} S^i_t.$$ 

So

$$V_{t+1} - V_t = H^0_{t+1} (S^0_{t+1} - S^0_t) + \sum_{i=1}^{k} H^i_{t+1} (S^i_{t+1} - S^i_t)$$

$$= r_t V_t + \sum_{i=1}^{k} H^i_{t+1} S^i_t (b^i_t - r_t + \sum_{j=1}^{m} \sigma^{i,j}_t M^j_{t+1})$$

$$= r_t V_t - z_t^* (\theta_t + M_{t+1}),$$

where $z_t^* = -\sum_{i=1}^{k} H^i_{t+1} S^i_t \sigma^{i,j}_t$ and $\theta \in \Theta$.

We then have the following subreplication result.

Theorem 4.2 Let $f(t, y, z) = -r y + \inf_{\theta \in \Theta} \{z^* \theta_t\}$. Then the solution $(Y, Z)$ to the BSDE with driver $f$ and terminal value $Y_T = \xi$ is equal to the largest subreplication price of the European contingent claim $\xi$.

Proof. Let $R_t = \prod_{s \leq t} (1 + r_s)^{-1}$. By standard duality results (see [32]), we know that the largest subreplication value of $\xi$ is given by $\inf_{Q \in \Lambda} E^Q[\xi R_T / R_t | F_t]$, where $\Lambda$ is the family of equivalent martingale measures for the discounted processes $S^i_t R_t$. By construction $\Lambda = \{Q^\theta : \theta \in \Theta\}$, and so by Proposition 3.9, this price is given by the solution to the stated BSDE.

Remark 4.3 In a similar way, we can obtain the minimal superreplication price as the solution of the BSDE with driver $f(t, y, z) = -r y + \sup_{\theta \in \Theta} \{z^* \theta_t\}$, by considering subreplication of $-\xi$. Note that (4.1) then shows that, as expected, the self-financing portfolios have the same subreplication and superreplication prices, as (4.1) holds for all $\theta \in \Theta$.

Let us consider the valuation problem of an American contingent claim with possible payoffs $\{\xi_t\}_{0 \leq t \leq T}$. The holder can exercise only once, at a stopping time $\tau \in \{0, 1, ..., T\}$. For notational convenience, we define

$$\tilde{f}(t, y, z) = -r y + \sup_{\theta \in \Theta} \{z^* \theta_t\},$$

$$\tilde{f}(t, y, z) = -r y + \inf_{\theta \in \Theta} \{z^* \theta_t\},$$

and we assume that this sup and inf are attained.

It is well known that this kind of claim cannot be replicated by a self-financing portfolio, and that it is necessary to introduce self-financing superstrategies with a cumulative consumption process.
**Definition 4.4** A self-financing super-strategy is a vector process \((V, H, K)\), where \(V\) is the value process, \(H\) is the portfolio process and \(K\) is the cumulative consumption process, such that

\[ V_{t+1} - V_t = (H_t, S_{t+1} - S_t) - (K_{t+1} - K_t), \]

where \(K\) is an increasing adapted process with \(K_0 = 0\). Equivalently, it is a process such that \(V_t \geq E^Q[V_{t+1}|F_t]\) for all \(Q \in \Lambda\). If \(-V\) is a super-strategy, then we say that \(V\) is a sub-strategy.

**Definition 4.5** Given a payoff process \(\{\xi_t\}_{t \in \{0,1,...,T\}}\), a super-strategy is called a superreplication strategy if

\[ V_t \geq \xi_t \quad \text{for all} \quad t \in \{0,1,...,T\}, \quad P-a.s. \]

in which case \(V\) is called a superreplication price. The value \(\inf V_t\), where the infimum is taken over all superreplication prices, is called the minimal superreplication price.

In a similar way, with the inequality reversed, we define sub-replication strategies (which are sub-strategies with \(V_t \geq \xi_t\) for all \(t\)) and the maximal subreplication price.

**Theorem 4.6** Let \((Y, Z, K)\) be the solution to the RBSDE with driver \(\bar{f}\), terminal value \(\xi_T\) and lower barrier \(\xi_t\). Then \(Y_t\) is equal to the smallest superreplication price of the American contingent claim with payoff \(\{\xi_t\}\). Similarly, the RBSDE with driver \(f\) yields the largest subreplication price for the claim.

**Proof.** We consider the superreplication price only, the subreplication price is similar. Recall from Proposition 2.15 that if \((Y, Z, K)\) is the solution of the RBSDE with driver \(f\) and lower barrier \(\xi_\tau\), then

\[ Y_t = \sup_{\tau} E \left[ \sum_{t \leq s < \tau} \bar{f}(s, Y_s, Z_s) + \xi_\tau | F_t \right]. \tag{4.2} \]

The one-step dynamics for \(Y\) are

\[ Y_t - (K_{t+1} - K_t) = Y_{t+1} + f(t, Y_t, Z_t) - Z_t^* M_{t+1} \]
\[ = Y_{t+1} - r_t Y_t + \sup_{\theta \in \Theta} \{Z_t^* \theta_t\} - Z_t^* M_{t+1} \]
\[ = \sup_{Q \in \Lambda} E^Q [Y_{t+1} R_{t+1} / R_t | F_t]. \]

As \(K\) is an increasing process we see that \(Y\) corresponds to a super-strategy, and so \(Y_t\) is a superreplication price for \(\xi\). Conversely, if \(Y'\) is a superreplication price for \(\{\xi_t\}\), then from (4.1) there would exist a process \(z'\) and an increasing consumption process \(K'\) such that

\[ Y'_{t+1} - Y'_t = r_t Y'_t - (z'_t)^*(\theta_t + M_{t+1}) + K'_{t+1} - K'_t \]

and \(Y'_t \geq \xi_t\). By the comparison theorem for RBSDEs, this implies \(Y'_t \geq Y_t\), so \(Y\) is the minimal superreplication price for \(\{\xi_t\}\). 

26
Remark 4.7 By replacing $\xi_t$ with $-\xi_t$, we can consider the perspective of the seller of a claim, who will have to pay out when it is exercised. After changing sign again, this results in the supremum over $\tau$ in (4.2) being replaced with an infimum, as the seller cannot control the exercise time of the option.

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