Higher-order Bernoulli and poly-Bernoulli mixed type polynomials

by
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Abstract
In this paper, we consider higher-order Bernoulli and poly-Bernoulli mixed type polynomials and we give some interesting identities of those polynomials arising from umbral calculus.

1 Introduction
The classical polylogarithmic function $Li_k(x)$ is

$$Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \quad k \in \mathbb{Z}, \quad (\text{see [3, 5]}). \quad (1)$$

The poly-Bernoulli polynomials are defined by the generating function to be

$$\frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad (\text{see [3, 5]}), \quad (2)$$

and the Bernoulli polynomials of order $r (r \in \mathbb{Z})$ are given by

$$\left( \frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [2, 4, 7]}). \quad (3)$$

When $x = 0$, $B_n^{(k)} = B_n^{(k)}(0)$ are called the poly-Bernoulli numbers and $B_n^{(r)} = B_n^{(r)}(0)$ are called the Bernoulli numbers of order $r$. In the special case, $r = 1$, $B_n^{(1)}(x) = B_n(x)$ are called the Bernoulli polynomials. When $x = 0$, $B_n = B_n(0)$ are called the ordinary Bernoulli numbers.
The higher-order Bernoulli and poly-Bernoulli mixed type polynomials are defined by the generating function to be

\[
\left( \frac{t}{e^t - 1} \right)^r \frac{Li_k (1 - e^{-t})}{1 - e^{-t}} e^{xt} = \sum_{n=0}^{\infty} s_n^{(r,k)}(x) \frac{t^n}{n!}, \quad \text{(see [5]).} \tag{4}
\]

From (2), (3) and (4), we note that

\[
s_n^{(r,k)}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l}^{(r)} B_l^{(k)}(x). \tag{5}
\]

When \(x = 0\), \(s_n^{(r,k)} = s_n^{(r,k)}(0)\) are called the higher-order Bernoulli and poly-Bernoulli mixed type numbers.

Let \(F\) be the set of all formal power series in variable \(t\) over \(C\) with

\[
F = \left\{ f(t) = \sum_{k=0}^{\infty} a_k t^k \bigg| a_k \in C \right\}. \tag{6}
\]

Let \(P = C[t]\) and let \(P^*\) be the vector space of all linear functional on \(P\). \(\langle L | p(x) \rangle\) denotes the action of linear functional \(L\) on the polynomial \(p(x)\), and it is well known that the vector space operations on \(P^*\) are defined by \(\langle L + M | p(x) \rangle = \langle L | p(x) \rangle + \langle M | p(x) \rangle\), \(\langle cL | p(x) \rangle = c \langle L | p(x) \rangle\), where \(c\) is a complex constant. For \(f(t) \in F\) with \(f(t) = \sum_{k=0}^{\infty} a_k t^k \), let us define the linear functional on \(P\) by setting

\[
\langle f(t) | x^n \rangle = a_n, \quad (n \geq 0), \quad \text{(see [8, 9]).} \tag{7}
\]

From (6) and (7), we note that

\[
\langle t^k | x^n \rangle = n! \delta_{n,k}, \quad (n, k \geq 0), \tag{8}
\]

where \(\delta_{n,k}\) is the Kronecker’s symbol.

Let \(f_L(t) = \sum_{k=0}^{\infty} \langle L | x^k \rangle t^k \). Then, by (8), we see that \(\langle f_L(t) | x^n \rangle = \langle L | x^n \rangle\). Additionally, the map \(L \mapsto f_L(t)\) is a vector space isomorphism from \(P^*\) onto \(F\). Henceforth, \(F\) denotes both the algebra of the formal power series in \(t\) and the vector space of all linear functionals on \(P\), and so an element \(f(t)\) of \(F\) will be thought as both a formal power series and a linear functional. We call \(F\)
the umbral algebra. The umbral calculus is the study of umbral algebra. The
order \(O(f(t))\) of the power series \(f(t) \neq 0\) is the smallest integer for which
\(a_k\) does not vanish. If \(O(f(t)) = 0\), then \(f(t)\) is called an invertible series. If
\(O(f(t)) = 1\), then \(f(t)\) is called a delta series. For \(f(t) \in \mathcal{F}\) and \(p(x) \in \mathbb{P}\), we
have
\[
\begin{align*}
f(t) &= \sum_{k=0}^{\infty} \langle f(t) | x^k \rangle \frac{t^k}{k!}, \\
p(x) &= \sum_{k=0}^{\infty} \langle t^k | p(x) \rangle \frac{x^k}{k!}.
\end{align*}
\] (9)
Thus, by (9), we get
\[
\begin{align*}
p^{(k)}(0) &= \langle t^k | p(x) \rangle = \langle 1 | p^{(k)}(x) \rangle, \quad \text{(see [8, 9])},
\end{align*}
\] (10)
where \(p^{(k)}(x) = \frac{d^k p(x)}{dx^k}\).
From (10), we have
\[
t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}.
\] (11)
By (11), we easily see that
\[
e^{yt} p(x) = p(x + y), \quad \langle e^{yt} | p(x) \rangle = p(y).
\] (12)
For \(f(t), g(t) \in \mathcal{F}\) with \(O(f(t)) = 1, O(g(t)) = 0\), there exists a unique
sequence \(s_n(x)\) of polynomials such that \(\langle g(t)f(t)^k | s_n(x) \rangle = n! \delta_{n,k}\), for
\(n, k \geq 0\). The sequence \(s_n(x)\) is called the Sheffer sequence for \((g(t), f(t))\),
which is denoted by \(s_n(x) \sim (g(t), f(t))\).
Let \(p(x) \in \mathbb{P}\) and \(f(t) \in \mathcal{F}\). Then we see that
\[
\begin{align*}
\langle f(t) | xp(x) \rangle &= \langle \partial_t f(t) | p(x) \rangle = \langle f'(t) | p(x) \rangle, \quad \text{(see [8])},
\end{align*}
\] (13)
For \(s_n(x) \sim (g(t), f(t))\), we have
\[
\frac{1}{g(f(t))} e^{yf(t)} = \sum_{k=0}^{\infty} s_k(y) \frac{t^k}{k!}, \quad \text{for all } y \in \mathbb{C},
\] (14)
where \(\bar{f}(t)\) is the compositional inverse for \(f(t)\) with \(\bar{f}(f(t)) = t\), and
\[
f(t)s_n(x) = ns_{n-1}(x), \quad \text{(see [8, 9])},
\] (15)
Let \(s_n(x) \sim (g(t), t)\). Then we see that
\[
\begin{align*}
s_{n+1}(x) &= \left(x - \frac{g'(t)}{g(t)}\right) s_n(x), \quad \text{(see [8])}.
\end{align*}
\] (16)
For \( s_n(x) \sim (g(t), f(t)) \), \( r_n(x) \sim (h(t), l(t)) \), we have
\[
s_n(x) = \sum_{m=0}^{n} c_{n,m} r_m(x),
\]
where
\[
c_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m \right| x^n \right\rangle, \quad (\text{see } [8, 9]).
\]
In this paper, we study higher-order Bernoulli and poly-Bernoulli mixed type polynomials and we give some interesting identities of those polynomials arising from umbral calculus.

\section{Higher-order Bernoulli and poly-Bernoulli mixed type polynomials}

From (4) and (14), we note that
\[
s_{n}^{(r,k)}(x) \sim \left( g_{r,k}(t) = \left( \frac{e^t - 1}{t} \right)^r \frac{1 - e^{-t}}{L_{ik} \left( 1 - e^{-t} \right), t} \right).
\]
Thus, by (15), we get
\[
t s_{n}^{(r,k)}(x) = \frac{d}{dx} s_{n}^{(r,k)}(x) = n s_{n-1}^{(r,k)}(x).
\]
From (4), we have
\[
s_{n}^{(r,k)}(x) = \sum_{l=0}^{n} \binom{n}{l} s_{l}^{(r,k)} x^{n-l} = \sum_{l=0}^{n} \binom{n}{l} s_{n-l}^{(r,k)} x^l.
\]
First, we observe that
\[
s_{n}^{(r,k)}(x) = \frac{1}{g_{r,k}(t)} x^n = \left( \frac{t}{e^t - 1} \right)^r \left( \frac{L_{ik} \left( 1 - e^{-t} \right)}{1 - e^{-t}} \right) x^n.
\]
In [3], it is known that
\[
\frac{L_{ik} \left( 1 - e^{-t} \right)}{1 - e^{-t}} x^n = \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) (x-j)^n.
\]
Thus, by (22) and (23), we get

$$s_n^{(r,k)}(x) = \sum_{m=0}^{n} \left( \frac{1}{m+1} \right)^k \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left( \frac{t}{e^t-1} \right)^r (x-j)^n$$

(24)

$$= \sum_{m=0}^{n} \left( \frac{1}{m+1} \right)^k \sum_{j=0}^{m} (-1)^j \binom{m}{j} B_n^{(r)}(x-j).$$

Therefore, by (24), we obtain the following proposition

**Proposition 1.** For $n \in \mathbb{Z}_{\geq 0}$, $r, k \in \mathbb{Z}$, we have

$$s_n^{(r,k)}(x) = \sum_{m=0}^{n} \left( \frac{1}{m+1} \right)^k \sum_{j=0}^{m} (-1)^j \binom{m}{j} B_n^{(r)}(x-j).$$

From (3), we can easily derive the following equation:

$$B_n^{(r)}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l}^{(r)} x^l.$$  

(25)

By (24) and (25), we get

$$s_n^{(r,k)}(x) = \sum_{l=0}^{n} \left\{ \sum_{m=0}^{n} \binom{n}{l} B_{n-l}^{(r)} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{j=0}^{m} \left( \frac{t}{e^t-1} \right)^r (x-j)^n \right\} (x-j)^l.$$  

(26)

In [3], it is known that

$$\frac{L_i e^{-\nu t} (1-e^{-\nu t})^x}{1-e^{-\nu t}} x^n = \sum_{j=0}^{n} \left\{ \sum_{m=0}^{n-j} \left( \frac{1}{m+1} \right)^k \sum_{j=0}^{m} \left( \frac{-1}{n-j-m-j} \right) \binom{n}{j} m! S_2(n-j,m) \right\} x^j,$$  

(27)

where $S_2(n,m)$ is the Stirling number of the second kind.

From (22) and (27), we have

$$s_n^{(r,k)}(x) = \sum_{j=0}^{n} \left\{ \sum_{m=0}^{n-j} \left( \frac{-1}{m+1} \right)^k \binom{n}{j} m! S_2(n-j,m) \right\} \left( \frac{t}{e^t-1} \right)^r x^j$$  

(28)

$$= \sum_{j=0}^{n} \left\{ \sum_{m=0}^{n-j} \left( \frac{1}{m+1} \right)^k \binom{n}{j} m! S_2(n-j,m) \right\} B_j^{(r)}(x)$$

$$= \sum_{j=0}^{n} \left\{ \sum_{l=0}^{n-j} \sum_{m=0}^{n-j} \frac{(-1)^{n-j}}{m+1} \binom{n}{j} \binom{j}{l} \frac{m!}{(m+1)^k} S_2(n-j,m) B_{j-l}^{(r)} \right\} x^l.$$
For any formal power series $f$, it is easy to show that

$$s_{n+1}^{(r,k)}(x) = \left( x - \frac{g_{r,k}(t)}{g_{r,k}(t)} \right) s_n^{(r,k)}(x), \quad (29)$$

where

$$\frac{g_{r,k}(t)}{g_{r,k}(t)} = (\log g_{r,k}(t))' = (r \log (e^t - 1) - r \log t + \log (1 - e^{-t}) - \log \log (1 - e^{-t}))' = \frac{r t e^t - r e^t + r}{t (e^t - 1)} + \frac{t}{e^t - 1} \left( \frac{L_i (1 - e^{-t}) - L_{i-1} (1 - e^{-t})}{t L_i (1 - e^{-t})} \right).$$

By (29) and (30), we get

By (31), (32) and (33), we get

From (16) and (19), we have

$$s_{n+1}^{(r,k)}(x) = x s_n^{(r,k)}(x) - \frac{r}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} (-1)^{n+1-l} B_{n+1-l} s_l^{(r,k)}(x) \quad (31)$$

It is easy to show that

$$\frac{r t e^t - r e^t + r}{e^t - 1} = \frac{r}{2} + \cdots, \quad \frac{L_i (1 - e^{-t}) - L_{i-1} (1 - e^{-t})}{1 - e^{-t}} = \left( \frac{1}{2^k} - \frac{1}{2^{k-1}} \right) t + \cdots \quad (32)$$

For any formal power series $f(t)$ with $O(f(t)) \geq 1$, we have

$$\frac{f(t)}{l} x^n = \frac{f(t)}{l} \frac{1}{n+1} t x^{n+1} = \frac{1}{n+1} f(t) x^{n+1}. \quad (33)$$

By (31), (32) and (33), we get

$$s_{n+1}^{(r,k)}(x) = x s_n^{(r,k)}(x) - \frac{r}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} (-1)^{n+1-l} B_{n+1-l} s_l^{(r,k)}(x) \quad (34)$$

Therefore, by (34), we obtain the following theorem.

**Theorem 2.** For $r, k \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$, we have

$$s_{n+1}^{(r,k)}(x) = x s_n^{(r,k)}(x) - \frac{r}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} (-1)^{n+1-l} B_{n+1-l} s_l^{(r,k)}(x)$$

$$- \frac{1}{n+1} \left\{ s_{n+1}^{(r+1,k)}(x) - s_{n+1}^{(r+1,k-1)}(x) \right\}.\quad (34)$$
From (3), we have

\[ t x s_n^{(r,k)}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l}^{(r)} \left( x B_l^{(k)}(x) \right) \]  
(35)

\[ = \sum_{l=0}^{n} \binom{n}{l} B_{n-l}^{(r)} \left\{ t x B_{l-1}^{(k)}(x) + B_l^{(k)}(x) \right\} \]

\[ = n x \sum_{l=0}^{n-1} \binom{n-1}{l} B_{n-l}^{(r)} B_l^{(k)}(x) + \sum_{l=0}^{n} \binom{n}{l} B_{n-l}^{(r)} B_l^{(k)}(x) \]

\[ = n x s_n^{(r,k)}(x) + s_n^{(r,k)}(x). \]

It is easy to show that

\[ s_n^{(r,k)}(x) = \left( \frac{t}{e^t - 1} \right)^r L_{k-1} \left( \frac{1 - e^{-t}}{1 - e^{-t}} \right) x^n = \left( \frac{t}{e^t - 1} \right)^r B_n^{(k)}(x). \]  
(36)

Applying \( t \) on the both sides of (35) and using (36), we get

\[ (n + 1) s_n^{(r,k)}(x) \]

\[ = n x s_n^{(r,k)}(x) + s_n^{(r,k)}(x) - \frac{r}{n+1} \sum_{l=1}^{n} \binom{n+1}{l} (-1)^{n+1-l} B_{n+1-l}^{(r)} s_n^{(r,k)}(x) \]

\[ - \frac{1}{n+1} \left\{ (n+1) s_n^{(r+1,k)}(x) - (n+1) s_n^{(r+1,k-1)}(x) \right\} \]

\[ = n x s_n^{(r,k)}(x) + s_n^{(r,k)}(x) + n r B_1 s_n^{(r)}(x) - r \sum_{l=0}^{n-2} (-1)^{n-l-1} \binom{n}{l} B_{n-l}^{(r)} s_n^{(r,k)}(x) \]

\[ - s_n^{(r+1,k)}(x) + s_n^{(r+1,k-1)}(x). \]

Thus, by (37), we obtain the following theorem.

**Theorem 3.** For \( n \in \mathbb{N} \) with \( n \geq 2 \), we have

\[ n s_n^{(r,k)}(x) + n \left( \frac{1}{2} - 1 \right) s_n^{(r,k)}(x) + r \sum_{l=0}^{n-2} (-1)^{n-l} \binom{n}{l} B_{n-l}^{(r)} s_n^{(r,k)}(x) \]

\[ = - s_n^{(r+1,k)}(x) + s_n^{(r+1,k-1)}(x). \]

For \( r = 0 \), by Theorem 3 we get

\[ n B_n^{(k)}(x) - n x B_{n-1}^{(k)}(x) \]

\[ = - B_n^{(k)}(x) + \frac{1}{2} n B_{n-1}^{(k)}(x) - \sum_{l=0}^{n-2} \binom{n}{l} B_{n-l}^{(l)} B_l^{(k)}(x) + \sum_{l=0}^{n} \binom{n}{l} B_{n-l}^{(k-1)}(x). \]
From (4), we note that

\[ s_n^{(r,k)}(y) = \langle \left( \frac{t}{e^t - 1} \right)^r \frac{L_i(1 - e^{-t})}{1 - e^{-t}} e^{yt} \bigg| x^n \rangle \]  

\[ = \langle \left( \frac{t}{e^t - 1} \right)^r \frac{L_i(1 - e^{-t})}{1 - e^{-t}} e^{yt} \bigg| x^n x^{-1} \rangle \]

\[ = \langle \partial_t \left( \left( \frac{t}{e^t - 1} \right)^r \frac{L_i(1 - e^{-t})}{1 - e^{-t}} e^{yt} \right) \bigg| x^{n-1} \rangle \]

\[ + \langle \left( \frac{t}{e^t - 1} \right)^r \frac{L_i(1 - e^{-t})}{1 - e^{-t}} e^{yt} \bigg| x^{n-1} \rangle \]

\[ + \langle \left( \frac{t}{e^t - 1} \right)^r \frac{L_i(1 - e^{-t})}{1 - e^{-t}} \partial_t e^{yt} \bigg| x^{n-1} \rangle \].

Therefore, by (38), we obtain the following theorem.

**Theorem 4.** For \( n \geq 1, r, k \in \mathbb{Z} \), we have

\[ s_n^{(r,k)}(x) = -r s_n^{(r,k)}(x) + r \sum_{l=0}^{n-1} \frac{(n-1)}{(n+1-l)(n-l)} s_l^{(r+1,k)}(x) \]

\[ + \sum_{l=0}^{n-1} \left\{ (-1)^{n-1-l} \binom{n-1}{l} \sum_{m=0}^{n-1-l} (-1)^m (m+1)! S_2(n-1-l,m) \right\} \]

\[ \times \mathbb{B}_l^{(r)}(x-1) + x s_{n-1}^{(r,k)}(x). \]

Now, we compute

\[ \langle \left( \frac{t}{e^t - 1} \right)^r \frac{L_i(1 - e^{-t})}{1 - e^{-t}} \bigg| x^{n+1} \rangle \]

in two different ways.
On the one hand,

\[
\left\langle \left( \frac{t}{e^{t} - 1} \right)^{r} \text{Li}_{k} \left( 1 - e^{-t} \right) \right|_{x^{n+1}} \rightangle = \left\langle \left( \frac{t}{e^{t} - 1} \right)^{r} \frac{\text{Li}_{k} \left( 1 - e^{-t} \right)}{1 - e^{-t}} \right|_{x^{n+1}} \rightangle = \left\langle \left( \frac{t}{e^{t} - 1} \right)^{r} \frac{\text{Li}_{k} \left( 1 - e^{-t} \right)}{1 - e^{-t}} \right|_{x^{n+1} - (x - 1)^{n+1}} \rightangle = \sum_{m=0}^{n} \binom{n+1}{m} (-1)^{n-m} \left\langle 1 \right| s_{m}^{(r,k)}(x) \right\rangle = \sum_{m=0}^{n} \binom{n+1}{m} (-1)^{n-m} s_{n}^{(r,k)}.
\]

On the other hand, we have

\[
\left\langle \left( \frac{t}{e^{t} - 1} \right)^{r} \text{Li}_{k} \left( 1 - e^{-t} \right) \right|_{x^{n+1}} \rightangle = \left\langle \text{Li}_{k} \left( 1 - e^{-t} \right) \left| \frac{t}{e^{t} - 1} \right|^{r} x^{n+1} \right\rangle = \left\langle \text{Li}_{k} \left( 1 - e^{-t} \right) \left| B_{n+1}^{(r)}(x) \right\rangle = \left\langle \int_{0}^{t} \left( \text{Li}_{k} \left( 1 - e^{-s} \right) \right)^{r} ds \left| B_{n+1}^{(r)}(x) \right\rangle = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \binom{l}{m} (-1)^{l-m} B_{n}^{(k-1)} \frac{1}{l!} \left\langle \int_{0}^{t} s^{l} ds \left| B_{n+1}^{(r)}(x) \right\rangle = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{l}{m} (-1)^{l-m} B_{m}^{(k-1)} \frac{1}{(l+1)!} \left\langle 1 \right| s^{l+1} B_{n+1}^{(r)}(x) \right\rangle = \sum_{l=0}^{n} \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} \binom{n+1}{l+1} B_{m}^{(k-1)} B_{n-l}^{(r)}.
\]

Therefore, by (39) and (40), we obtain the following theorem.

**Theorem 5.** For \( n \in \mathbb{Z}_{\geq 0}, \ r, k \in \mathbb{Z}, \) we have

\[
\sum_{m=0}^{n} \binom{n+1}{m} (-1)^{n-m} s_{n}^{(r,k)} \]

\[
= \sum_{l=0}^{n} \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} \binom{n+1}{l+1} B_{m}^{(k-1)} B_{n-l}^{(r)}.
\]
Lemma 6 ([5]). For \( k \in \mathbb{Z} \) and \( m \geq 1 \), we have
\[
\left( \sum_{l=0}^{m} \left[ \begin{array}{c} m \\ l \end{array} \right] \partial_{t}^{l} \right) \frac{L_{i}k (1 - e^{-t})}{1 - e^{-t}} \quad (41)
\]
\[
= \frac{1}{(e^{t} - 1)^{m}} \sum_{l=0}^{m} (-1)^{m-l} \left[ \begin{array}{c} m + 1 \\ l + 1 \end{array} \right] \frac{L_{i}k-l (1 - e^{-t})}{1 - e^{-t}},
\]
where \( \left[ \begin{array}{c} m \\ l \end{array} \right] = |S_{1}(m, l)| \) and \( S_{1}(m, l) \) is the stirling number of the first kind.

Now, we compute \( \left( \frac{t}{e^{t} - 1} \right)^{m} \sum_{l=0}^{m} (-1)^{m-l} \left[ \begin{array}{c} m + 1 \\ l + 1 \end{array} \right] \frac{L_{i}k-l (1 - e^{-t})}{1 - e^{-t}} \bigg|_{x^{n}} \) in two different ways.

On the one hand,
\[
\left( \frac{t}{e^{t} - 1} \right)^{m} \sum_{l=0}^{m} (-1)^{m-l} \left[ \begin{array}{c} m + 1 \\ l + 1 \end{array} \right] \frac{L_{i}k-l (1 - e^{-t})}{1 - e^{-t}} \bigg|_{x^{n}} \quad (42)
\]
\[
= \left[ 1 \right] \sum_{l=0}^{m} (-1)^{m-l} \left[ \begin{array}{c} m + 1 \\ l + 1 \end{array} \right] \frac{t^{m} L_{i}k-l (1 - e^{-t})}{1 - e^{-t}} x^{n}
\]
\[
= \sum_{l=0}^{m} (-1)^{m-l} \left[ \begin{array}{c} m + 1 \\ l + 1 \end{array} \right] s_{n}^{(m-k-l)}(x)
\]

On the other hand, by Lemma 6 it is equal to
\[
\left( \frac{t}{e^{t} - 1} \right)^{m} \sum_{l=0}^{m} (-1)^{m-l} \left[ \begin{array}{c} m + 1 \\ l + 1 \end{array} \right] \frac{L_{i}k-l (1 - e^{-t})}{1 - e^{-t}} \bigg|_{x^{n}} \quad (43)
\]
\[
= \left( \frac{t^{m} L_{i}k (1 - e^{-t})}{1 - e^{-t}} \right) \bigg|_{x^{n}}
\]
\[
= \sum_{l=0}^{m} \left[ \begin{array}{c} m \\ l \end{array} \right] \partial_{t}^{l} \frac{L_{i}k (1 - e^{-t})}{1 - e^{-t}} \bigg|_{x^{n}}
\]
\[
= \left\{ \begin{array}{ll}
(n)_{m} \sum_{l=0}^{m} \left[ \begin{array}{c} m \\ l \end{array} \right] \partial_{t}^{l} \frac{L_{i}k (1 - e^{-t})}{1 - e^{-t}} \bigg|_{x^{n-m}} , & \text{if } n \geq m, \\
0, & \text{if } 0 \leq n \leq m - 1.
\end{array} \right.
\]
For $n \geq m$, we have

$$
(n)_m \left\langle \sum_{l=0}^{m} \left[ \begin{array}{c} m \\ l \end{array} \right] \frac{L_i k (1 - e^{-t})}{1 - e^{-t}} x^{n-m-l} \right\rangle = (n)_m \left\langle \frac{L_i k (1 - e^{-t})}{1 - e^{-t}} \left( \sum_{l=0}^{m} \left[ \begin{array}{c} m \\ l \end{array} \right] x^l \right) x^{n-m-l} \right\rangle
$$

$$
= (n)_m \sum_{l=0}^{m} \left[ \begin{array}{c} m \\ l \end{array} \right] \left( \frac{L_i k (1 - e^{-t})}{1 - e^{-t}} \right) x^{n-m-l}
$$

$$
= (n)_m \sum_{l=0}^{m} \left[ \begin{array}{c} m \\ l \end{array} \right] B_{n-m+l}^{(k)}.
$$

Therefore, by (42), (43) and (44), we obtain the following theorem.

**Theorem 7** ([5]). For $k \in \mathbb{Z}$, $m \geq 1$, we have

$$
\sum_{l=0}^{m} (-1)^{m-l} \left[ \begin{array}{c} m + 1 \\ l + 1 \end{array} \right] s_{n}^{(m,k-l)}
$$

$$
= \begin{cases} 
(n)_m \sum_{l=0}^{m} \left[ \begin{array}{c} m \\ l \end{array} \right] B_{n-m+l}^{(k)}, & \text{if } n \geq m, \\
0, & \text{if } 0 \leq n \leq m - 1.
\end{cases}
$$

Now, we consider the following two Sheffer sequences:

$$
s_{n}^{(r,k)}(x) \sim \left( \left( e^{t} - \frac{1}{t} \right)^{r} \frac{1 - e^{-t}}{L_i k (1 - e^{-t})}, t \right)
$$

and

$$
E_{n}^{(s)}(x) \sim \left( \left( e^{t} + \frac{1}{2} \right)^{s}, t \right).
$$

Let us assume that

$$
s_{n}^{(r,k)}(x) = \sum_{m=0}^{n} C_{n,m} E_{m}^{(s)}(x).
$$
Then, from (18), we have
\[
C_{n,m} = \frac{1}{m!} \left\langle \left( \frac{e^t - 1}{t} \right)^s \frac{t}{e^t - 1} \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} \right| x^n \right\rangle \tag{47}
\]
\[
= \frac{1}{m!} \left\langle \left( \frac{e^t + 1}{2} \right)^s \left( \frac{t}{e^t - 1} \right)^r \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} \right| x^m x^n \right\rangle \tag{48}
\]
\[
= \left( \frac{n}{m} \right)^s \sum_{j=0}^{s} \left( \frac{1}{j} \right) \left\langle \left( \frac{e^t - 1}{t} \right)^s \frac{t}{e^t - 1} \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} x^{n-m} \right| x^n \right\rangle \tag{49}
\]
Therefore, by (46) and (47), we obtain the following theorem.

**Theorem 8.** For \( r, k \in \mathbb{Z}, \) \( n, m \in \mathbb{Z}_{\geq 0}, \) we have
\[
s_{n}^{(r,k)}(x) = \frac{1}{2^s} \sum_{m=0}^{n} \left\{ \left( \frac{n}{m} \right) \sum_{j=0}^{s} \left( \frac{s}{j} \right) s_{n-m}^{(r,k)}(j) \right\} E_{m}^{(s)}(x). \tag{48}
\]
Let us consider the following two Sheffer sequences:
\[
s_{n}^{(r,k)}(x) \sim \left( \frac{e^t - 1}{t} \right)^r \frac{1 - e^{-t}}{Li_k(1 - e^{-t})}, \quad \mathbb{B}_{n}^{(s)}(x) \sim \left( \frac{e^t - 1}{t} \right)^s \tag{49}
\]
Let
\[
s_{n}^{(r,k)}(x) = \sum_{m=0}^{n} C_{n,m} \mathbb{B}_{m}^{(s)}(x). \tag{50}
\]
From (18), we note that
\[
C_{n,m} = \frac{1}{m!} \left\langle \left( \frac{t}{e^t - 1} \right)^{r-s} Li_k(1 - e^{-t}) \right| x^m \right\rangle \tag{51}
\]
\[
= \left( \frac{n}{m} \right) \left\langle \left( \frac{t}{e^t - 1} \right)^{r-s} Li_k(1 - e^{-t}) x^{n-m} \right| x^n \right\rangle \tag{52}
\]
Therefore, by (49) and (50), we obtain the following theorem.
**Theorem 9.** For \( r, s \in \mathbb{Z} \), \( n \in \mathbb{Z}_{\geq 0} \), we have

\[
s_n^{(r,k)}(x) = \sum_{m=0}^{n} \binom{n}{m} s_{n-m}^{(r-s,k)} B_m^{(s)}(x).
\]

We note that

\[
s_n^{(r,k)}(x) \sim \left( \left( \frac{e^t - 1}{t} \right)^r \frac{1 - e^{-t}}{L_i(k \cdot (1 - e^{-t}))}, t \right),
\]

and

\[
H_n^{(s)}(x|\lambda) \sim \left( \left( \frac{e^t - \lambda}{1 - \lambda} \right)^s, t \right), \quad \text{(see [1, 6])}.
\]

By the same method, we get

\[
s_n^{(r,k)}(x) = \frac{1}{(1 - \lambda)^s} \sum_{m=0}^{n} \left\{ \binom{n}{m} \sum_{j=0}^{s} \binom{s}{j} (-\lambda)^{s-j} s_{n-m}^{(r,k)}(j) \right\} H_m^{(s)}(x|\lambda). \quad (51)
\]

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