PARTIAL AUGMENTATIONS AND BRAUER CHARACTER VALUES OF TORSION UNITS IN GROUP RINGS

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Abstract. For a torsion unit $u$ of the integral group ring $\mathbb{Z}G$ of a finite group $G$, and a prime $p$ which does not divide the order of $u$ (but the order of $G$), a relation between the partial augmentations of $u$ on the $p$-regular classes of $G$ and Brauer character values is noted, analogous to the obvious relation between partial augmentations and ordinary character values. For non-solvable $G$, consequences concerning rational conjugacy of $u$ to a group element are discussed, considering as examples the symmetric group $S_5$ and the groups $\text{PSL}(2,p')$.

1. Introduction

This note is about an observation on torsion units in the integral group ring $\mathbb{Z}G$ of a finite group $G$ which seemingly has been overlooked so far. Sure, we can restrict our attention to units of augmentation (= sum of coefficients) one of $\mathbb{Z}G$, which form a group we denote by $V(\mathbb{Z}G)$. A torsion unit in $V(\mathbb{Z}G)$ is said to be rationally conjugate to a group element if it is conjugate to an element of $G$ by a unit of the rational group ring $\mathbb{Q}G$, and according to a conjecture of Hans Zassenhaus, this should always be the case. For the present state of affairs, we refer the reader to [14, 15, 24, 25].

A close connection with partial augmentations was noted in [23]: a torsion unit $u$ in $V(\mathbb{Z}G)$ is rationally conjugate to a group element if and only if for each power of $u$, all but one of its partial augmentations vanish. Recall that for a group ring element $u = \sum_{g \in G} a_g g$ (all $a_g$ in $\mathbb{Z}$), its partial augmentation with respect to an element $x$ of $G$, or rather its conjugacy class $x^G$ in $G$, is the sum $\sum_{g \in x^G} a_g$; we will denote it by $\varepsilon_x(u)$. Further connections are described in Section 2, but let us mention another known one which is of vital importance in this work. Let $R$ be a Dedekind ring of characteristic zero in which a rational prime $p$ is not invertible, and let $u$ be a $p$-regular torsion unit in $V(RG)$ (the given definitions also make sense for more general coefficient rings). Then $\varepsilon_x(u) = 0$ for every $p$-singular element $x$ of $G$. (A torsion unit is called $p$-regular if $p$ does not divide its order, and $p$-singular otherwise.)

Brauer’s definition of $p$-modular characters (now called Brauer characters) as certain complex-valued functions $\varphi$ on $G_{\text{reg}}$ (= set of $p$-regular elements of $G$) can naturally be extended to make $\varphi(u)$ meaningful for elements $u$ of the set $V(RG)_{\text{tor}}$ of $p$-regular torsion units in $V(RG)$, and then a few elementary facts about Brauer
characters still hold (see Proposition 3.1). From that, we deduce our main observation (Theorem 3.2):

$$\varphi(u) = \sum_{x \in \mathbb{G}; \ x \text{ is } p\text{-regular}} \varepsilon_x(u) \varphi(x) \quad \text{for all } u \in V(RG)_{\text{tor}}^{\text{reg}}.$$ 

This allows us to formulate (in Section 4) a $p$-modular version of a method introduced by Luthar and Passi [21] which imposes constraints on the partial augmentations of $p$-regular torsion units in $\mathbb{Z}G$. This method may prove useful in investigating the Zassenhaus conjecture for non-solvable groups. In Section 5, it is used to verify the Zassenhaus conjecture for the symmetric group $S_5$; based on a computer calculation, this was done before by Luthar and Trama [22]. In Section 6, it is used to explore the Zassenhaus conjecture for the groups $\text{PSL}(2, p^f)$. Some evidence is given that the conjecture might be verified for the groups $\text{PSL}(2, p)$: a complete proof is given for $p = 7, 11, 13$.

We remark that the Luthar–Passi method is of computational nature, and it seems worthwhile to examine more examples on a computer to see how far it will take us. The computer algebra system GAP [13] lends itself to this task. It makes available the tables contained in the Atlas of Brauer Characters [18] and many others, and provides functions for computations with characters and character tables. Since the first version of this paper was written, this line of research has been pursued in [2, 4–7].

2. Partial augmentations

We collect some facts about partial augmentations of torsion units in integral group rings which will be used later on. Only Proposition 2.2 is mathematically new in this section, but has a precursor in [14, Proposition 3.1]. Fix a finite group $G$, and let $R$ be an integral domain of characteristic zero, with quotient field $K$.

Our tool to prove rational conjugacy of torsion units is [23, Theorem 2.5] (formulated in [14, Lemma 2.5] for more general coefficient rings) which, because of its fundamental importance, shall be reproduced here.

**Theorem 2.1.** Let $u$ be a torsion unit in $V(RG)$, with no prime divisor of the order of $u$ being invertible in $R$. Then $u$ is conjugate to an element of $G$ by a unit of $KG$ if and only if for every $m$ dividing the order of $u$, all partial augmentations of $u^m$ but one vanish.

Vanishing of partial augmentations can be caused for arithmetical reasons. A classical result, due to Berman and Higman (see [24, (1.4)]), states that a torsion unit in $V(\mathbb{Z}G)$ is a central element of $G$ provided that its partial augmentation with respect to such an element is nonzero.

It is not known, even for solvable $G$, whether for a given torsion unit $u$ in $V(\mathbb{Z}G)$, there exists an element in $G$ of the same order, possibly such that the associated partial augmentation is nonzero (Research Problem 8 from [24]). The answer is affirmative for units $u$ of prime power order, by [10, Theorem 4.1]. (The used technique is also described in [24, §7], where the result is attributed to Zassenhaus.) Moreover, if the order of $u$ is $r^n$, for a prime $r$, then

$$\sum_{x \in \mathbb{G}; \ x \text{ of order } r^m} \varepsilon_x(u) \equiv 0 \mod r \quad \text{for all } m < n.$$
That these congruences actually might be equalities is known as a conjecture of A. A. Bovdi (rendered in [19]). For $G = \text{PSL}(2, p^f)$, it is confirmed in Proposition 6.5.

The following proposition is an obvious generalization of [14, Proposition 3.1], and can be proved in the exact same manner. Previously, only special cases were known (i.e., [23, Theorem 2.7]).

**Proposition 2.2.** Let $R$ be a Dedekind ring of characteristic zero in which a given rational prime $p$ is not invertible, and let $u$ be a torsion unit in $V(RG)$. Then $\varepsilon_x(u) = 0$ for every element $x$ of $G$ whose $p$-part has strictly larger order than the $p$-part of $u$.

**Proof.** Suppose that $x$ is an element of $G$ whose $p$-part has strictly larger order than the $p$-part of $u$. Let $C$ be an (abstract) cyclic group whose order is the least common multiple of the orders of $u$ and $x$. Enlarging $R$, if necessary, we can assume that $R$ is a complete discrete valuation ring (cf. [11, §4c]). Set $M = RG$, viewed as $RC$-lattice by letting a generator $c$ of $C$ act by $m \cdot c = x^{-1}mu$ for $m \in M$. By [24, (38.12)], we have to show that $\chi(c) = 0$ for the character $\chi$ of $C$ afforded by $KM$. Note that $c$ is $p$-singular, so this will follow once we have shown the stronger statement that $M$ is a projective $RC$-lattice, by Green’s Theorem on Zeros of Characters (see [11, (19.27)]). This in turn follows from the assumption, meaning that the action of the subgroup $P$ of order $p$ in $C$ is given by a multiplication action of the subgroup of order $p$ in $\langle x \rangle$, which shows that $M$ is a free $RP$-lattice. We provide details. Let $\pi$ be a prime element of $R$, and set $k = R/\pi R$ (a residue field of characteristic $p$). It is enough to show that $k \otimes_R M$ as $kC$-module is projective (see [11, (20.10), (30.11)], i.e., that $k \otimes_R M$ is projective relative to a Sylow $p$-subgroup of $C$. It is well-known that this follows from the projectivity of $k \otimes_R M$ as $kP$-module (see [14, Lemma 3.2]).

This applies in particular to torsion units in $ZG$, giving an affirmative answer to Research Problem 9 from [24]:

**Theorem 2.3.** Let $u$ be a torsion unit in $V(ZG)$. Then $\varepsilon_x(u) \neq 0$ is possible only for elements $x$ of $G$ whose order is a divisor of the order of $u$. □

3. Brauer character values

A general reference for the following discussion is Chapter 2 (in particular §17) of [11]. Let $G$ be a finite group, $p$ a prime divisor of its order, and fix a $p$-modular system $(K, R, k)$ such that $\text{char}(K) = 0$, with $K$ sufficiently large relative to $G$. Then also $k = R/\pi R$ is sufficiently large relative to $G$. (Here, $\pi$ is a prime element of the discrete valuation ring $R$.)

Brauer associated with a modular representation $\Theta : G \to \text{GL}(n, k)$ a complex-valued function $\varphi$ on $G_{\text{reg}}$ (= set of $p$-regular elements of $G$), now called a Brauer character of $G$. The way Brauer did this actually shows that we can extend the domain of $\varphi$ to the set $V(RG)_{\text{tor}}$ of $p$-regular torsion units in $V(RG)$. Having the natural map $R \to R/\pi R = k$ at our disposal, we consider the representation $\Theta$ as an algebra homomorphism from $RG$ into the algebra of $n \times n$-matrices over $k$. To define $\varphi(u)$ for a unit $u$ in $V(RG)_{\text{tor}}$, notice that the eigenvalues of $\Theta(u)$ are certain roots of unity in $k$, of order relatively prime to $p$, and their sum is the trace of $\Theta(u)$ (the usual character). We just replace these eigenvalues by the roots of unity in $R$ to which they are in bijection via the map $R \to k$, and call the resulting sum $\varphi(u)$. 
We fix elements $x_1, \ldots, x_r$ of $G$ which are representatives of the $p$-regular conjugacy classes of $G$. If $u$ is a $p$-regular torsion unit in $V(RG)$, then $\varepsilon_y(u) = 0$ for every $p$-singular element $y$ of $G$ (see Proposition 2.2). Hence for an ordinary $K$-character $\chi$ of $G$,

\begin{equation}
\chi(u) = \sum_{l=1}^{r} \varepsilon_{x_l}(u)\chi(x_l) \quad \text{for all } u \in V(RG)_{\text{tor}}. 
\end{equation}

We shall see in a moment that the analogue holds for Brauer characters.

The following elementary facts about these (extended) Brauer characters are proved in the exact same manner as [11, (17.5)].

**Proposition 3.1.** Let $u \in V(RG)_{\text{tor}}$. Then the following holds.

(i) Let $\varphi$ be the Brauer character of a $kG$-module $M$. Let $L$ be a submodule of $M$, let $\phi_L$ be the Brauer character afforded by $L$, and let $\phi_{M/L}$ be the Brauer character afforded by $M/L$. Then $\varphi(u) = \phi_L(u) + \phi_{M/L}(u)$.

(ii) Let $V$ be a $kG$-module with $K$-character $\chi$, and let $\varphi$ be the Brauer character of $M/\pi M$, where $M$ is a full $RG$-lattice in $V$. Then $\chi(u) = \varphi(u)$.

Let $\chi_1, \ldots, \chi_h$ be the ordinary irreducible $K$-characters of $G$, and let $\varphi_1, \ldots, \varphi_r$ be the irreducible Brauer characters of $G$. Let $D = (d_{ij})$ be the decomposition matrix of $G$ (relative to $p$). By its definition, and Proposition 3.1, we have

\begin{equation}
\chi_i(u) = \sum_{j=1}^{r} d_{ij}\varphi_j(u) \quad \text{for all } u \in V(RG)_{\text{tor}}.
\end{equation}

We now prove the main result of this note.

**Theorem 3.2.** Let $\varphi$ be a Brauer character of $G$ (relative to $p$), and let $u$ be a $p$-regular torsion unit in $V(RG)$. Then, with representatives $x_1, \ldots, x_r$ of the $p$-regular conjugacy classes of $G$,

$$
\varphi(u) = \sum_{l=1}^{r} \varepsilon_{x_l}(u)\varphi(x_l).
$$

**Proof.** By Proposition 3.1(i), it is enough to prove the result for an irreducible Brauer character $\varphi_j$. By (3.2) and (3.1), we have

$$
D \begin{pmatrix} \varphi_1(u) \\ \vdots \\ \varphi_r(u) \end{pmatrix} = \begin{pmatrix} \chi_1(u) \\ \vdots \\ \chi_h(u) \end{pmatrix} = \sum_{l=1}^{r} \varepsilon_{x_l}(u) \begin{pmatrix} \chi_1(x_l) \\ \vdots \\ \chi_h(x_l) \end{pmatrix} = \sum_{l=1}^{r} \varepsilon_{x_l}(u) D \begin{pmatrix} \varphi_1(x_l) \\ \vdots \\ \varphi_r(x_l) \end{pmatrix}.
$$

$$
= D \begin{pmatrix} \sum_{l=1}^{r} \varepsilon_{x_l}(u)\varphi_1(x_l) \\ \vdots \\ \sum_{l=1}^{r} \varepsilon_{x_l}(u)\varphi_r(x_l) \end{pmatrix}.
$$

The $r \times r$-matrix $D^t D$ is the Cartan matrix $C$, and $\det(C) \neq 0$. Hence

$$
\begin{pmatrix} \varphi_1(u) \\ \vdots \\ \varphi_r(u) \end{pmatrix} = \begin{pmatrix} \sum_{l=1}^{r} \varepsilon_{x_l}(u)\varphi_1(x_l) \\ \vdots \\ \sum_{l=1}^{r} \varepsilon_{x_l}(u)\varphi_r(x_l) \end{pmatrix},
$$

and we are done. \qed
Remark 3.3. Suppose that \( u \) is a \( p \)-regular torsion unit in \( V(RG) \) which is conjugate to an element \( g \) of \( G \) by a unit of \( KG \). Then \( u \) is even conjugate to \( g \) by a unit of \( RG \) (see [14, Lemma 2.9]), whence the images of \( u \) and \( g \) in \( kG \) are conjugate by a unit of \( kG \). In particular, if \( \Theta \) is a \( p \)-modular representation of \( G \), then \( \Theta(u) \) has the same eigenvalues as \( \Theta(g) \).

4. The Luthar–Passi method

The method introduced by Luthar and Passi [21] imposes constraints—once ordinary characters of the finite group \( G \) are known—on the partial augmentations of a torsion unit in \( ZG \). The idea behind is to exploit the simple fact that the character values of a torsion unit, which are determined by its partial augmentations and the character table, are sums of roots of unity. It is quite obvious that Theorem 3.2 allows us to formulate, for \( p \)-regular torsion units in \( ZG \), a \( p \)-modular version of this method, which can be applied once Brauer characters of \( G \) are known.

Let \( (K,R,k) \) be a \( p \)-modular system as in Section 3, and let the field \( F \) be either \( K \) or \( k \). Let \( U \) be an invertible matrix with entries in \( F \) such that \( U^n \) is the identity matrix for some natural number \( n \). (In a moment, \( U \) will be taken to be the image of a torsion unit in \( ZG \) under an ordinary or modular representation of \( G \).) If \( F = k \) we assume that \( n \) is not divisible by the characteristic \( p \) of \( k \). We also assume that \( K \) contains a primitive \( n \)-th root of unity \( \zeta \). The canonical map \( R \to R/\pi R = k \) will be denoted by \( r \mapsto \bar{r} \) \( (r \in R) \); if \( F = k \), it induces an isomorphism \( \langle \zeta \rangle \cong \langle \bar{\zeta} \rangle \).

The method to be described rests on the observation that the eigenvalues of \( U \) will be denoted by \( r \), we assume that \( \bar{r} \) is a \( \zeta \)-th root of unity in \( K \). Let \( \lambda \) be an \( n \)-th root of unity in \( K \). Let \( \mu_\xi \) be the multiplicity of \( \xi \) as an eigenvalue of \( U \) or \( \mu_\bar{\xi} \) be the multiplicity of \( \bar{\xi} \) as an eigenvalue of \( U \), as the case may be. Having the possibility to diagonalize \( U \) shows that

\[
\mu_\xi = \frac{1}{n} \sum_{i=0}^{n} \text{trace}(U^i)\xi^{-i} \quad \text{or} \quad \mu_\bar{\xi} = \frac{1}{n} \sum_{i=0}^{n} \lambda(U^i)\xi^{-i}
\]

where in the first case the ordinary trace of the matrices is taken while in the second \( \lambda(U^i) \) is, in the sense of Brauer, the sum of the roots of unity in \( \langle \xi \rangle \) corresponding to all the eigenvalues of \( U^i \). Collecting summands for which the matrices \( U^i \) have the same order, i.e., taking the Galois action into account yields

\[
\mu_\xi = \frac{1}{n} \sum_{d|n} \sum_{\sigma \in \text{Gal}(Q(\xi^d)/Q)} \text{trace}(U^d)^\sigma (\xi^{-d})^\sigma
\]

\[
= \frac{1}{n} \sum_{d|n} \text{Tr}_{Q(\xi^d)/Q}(\text{trace}(U^d)\xi^{-d})
\]

if \( U \) is written over \( K \), and in the modular case

\[
\mu_\bar{\xi} = \frac{1}{n} \sum_{d|n} \text{Tr}_{Q(\bar{\xi}^d)/Q}(\lambda(U^d)\xi^{-d}).
\]

Now let \( u \) be a torsion unit of order \( n \) in \( V(ZG) \). Let \( \chi \) be an ordinary character of \( G \) afforded by a representation \( D \) (written over \( K \)), and let \( \varphi \) be a Brauer character of \( G \) afforded by a \( p \)-modular representation \( \Theta \) (written over \( k \)). We write \( \mu(\xi,u,\chi) \) for the multiplicity of \( \xi \) as an eigenvalue of \( D(u) \), and \( \mu(\xi,u,\varphi) \) for the multiplicity of \( \bar{\xi} \) as an eigenvalue of \( \Theta(u) \). Let \( \psi \) stand for \( \chi \) or \( \varphi \). If \( \varphi \) is considered we assume
that \( p \) does not divide the order \( n \) of \( u \), but the order of \( G \) (the latter to avoid triviality). We have seen that

\[
\mu(\xi, u, \psi) = \frac{1}{n} \sum_{d|n} \text{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\psi(u^d)\xi^{-d}).
\]

We split the sum into the two parts

\[
a(\xi, u, \psi) = \frac{1}{n} \sum_{d|n, d\neq 1} \text{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\psi(u^d)\xi^{-d}),
\]

\[
b(\xi, u, \psi) = \frac{1}{n} \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\psi(u)\xi^{-1}).
\]

Note that the multiples \( n \cdot a(\xi, u, \psi) \) and \( n \cdot b(\xi, u, \psi) \) are rational integers.

We immediately note the following useful relation for units of prime power order:

\[
(4.1) \quad a(\xi, u, \psi) = \frac{1}{r} \mu(\xi^r, u^r, \psi) \quad \text{if} \quad u \text{ has order a power of a prime } r.
\]

If one tries to verify the Zassenhaus conjecture for \( G \), i.e., if one tries to show that \( u \) is rationally conjugate to a group element, then one can assume—by induction on the order of \( u \)—that the values \( a(\xi, u, \psi) \) are “known.” Moreover, since the partial augmentations of \( u \) are rational numbers, we obviously have

\[
b(\xi, u, \chi) = \sum_{x \in G} \varepsilon_x(u) b(\xi, x, \chi)
\]

where the sum runs over the conjugacy classes of \( G \). Further, it follows from Theorem 3.2 that

\[
b(\xi, u, \varphi) = \sum_{x \in G: x \text{ is } p\text{-regular}} \varepsilon_x(u) b(\xi, x, \varphi).
\]

Altogether, we obtain the equation given by Luthar and Passi:

\[
(4.2) \quad \mu(\xi, u, \chi) = a(\xi, u, \chi) + \sum_{x \in G} \varepsilon_x(u) b(\xi, x, \chi)
\]

and its modular counterpart

\[
(4.3) \quad \mu(\xi, u, \varphi) = a(\xi, u, \varphi) + \sum_{x \in G: x \text{ is } p\text{-regular}} \varepsilon_x(u) b(\xi, x, \varphi).
\]

The values \( b(\xi, x, \chi) \) and \( b(\xi, x, \varphi) \) can be computed from the (ordinary) character table of \( G \) and the Brauer character table of \( G \) (modulo \( p \)), respectively. Both (4.2) and (4.3) should be seen as a linear system of equations, indexed by the irreducible characters, in the unknown partial augmentations \( \varepsilon_x(u) \). Note that the natural integers \( \mu(\xi, u, \chi) \) are bounded above by the degree of \( \chi \).

If \( G \) is \( p \)-solvable, (4.3) provides nothing new since then each irreducible Brauer character is the “reduction modulo \( p \)” of an ordinary irreducible character, by the Fong–Swan–Rukolaine Theorem (see [11, (22.1)]). For other groups, (4.3) may impose stronger conditions on the partial augmentations than (4.2). This is illustrated by means of a few examples in the next sections.
5. The symmetric group $S_5$

Luthar and Trama [22] verified the Zassenhaus conjecture for the symmetric group $S_5$. Their proof is based on a computer calculation which yields certain congruences between matrix entries with reference to a particular Wedderburn embedding of $\mathbb{Z}S_5$, and uses a few ad hoc arguments. Here, we show that Theorem 3.2 provides enough information to yield another proof.

For torsion units in $V(\mathbb{Z}S_5)$ which are not of order 2, 4 or 6, the Luthar–Passi method (4.2) yields rational conjugacy to group elements (as noted in [22]), so we only deal with the remaining cases, for which we will make use only of the Brauer character table for the prime 5, shown in Table 1 together with the decomposition matrix, in the form obtained by requiring CharacterTable("SS") mod 5 in GAP [13] (dots indicate zeros). Let $u$ be a torsion unit in $V(\mathbb{Z}S_5)$, of order 2, 4 or 6.

| $\phi_{1a}$ | $\phi_{1b}$ | $\phi_{3a}$ | $\phi_{3b}$ | $\phi_{5a}$ | $\phi_{5b}$ |
|------------|------------|------------|------------|------------|------------|
| $\chi_{1a}$ | 1          | .          | .          | .          | .          | $\varphi_{1a}$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_{1b}$ | .          | 1          | .          | .          | .          | $\varphi_{1b}$ | 1 | 1 | $-1$ | $-1$ | $-1$ |
| $\chi_6$   | .          | .          | 1          | 1          | .          | $\varphi_{3a}$ | 3 | $-1$ | . | 1 | $-1$ | $-2$ |
| $\chi_{4a}$ | 1          | .          | 1          | .          | .          | $\varphi_{3b}$ | 3 | $-1$ | . | $-1$ | 1 | 2 |
| $\chi_{4b}$ | .          | 1          | .          | 1          | .          | $\varphi_{5a}$ | 5 | 1 | $-1$ | 1 | $-1$ | 1 |
| $\chi_{5a}$ | .          | .          | .          | 1          | .          | $\varphi_{5b}$ | 5 | 1 | $-1$ | $-1$ | 1 | $-1$ |
| $\chi_{5b}$ | .          | .          | .          | .          | 1          |             |     |     |     |     |     |     |

Table 1. Decomposition matrix and Brauer character table of $S_5$ (mod 5)

Let $\varepsilon_{1a}, \varepsilon_{2a}, \varepsilon_{2b}, \varepsilon_{3a}, \varepsilon_{5a}$ and $\varepsilon_{6a}$ be its partial augmentations, so that $\varepsilon_{2a}$, for example, denotes the partial augmentation of $u$ with respect to one of the two conjugacy classes of elements of order 2. We have $\varepsilon_{1a} = 0$ by the Berman–Higman result. By [23, Theorem 2.7] (cf. Theorem 2.3), we have $\varepsilon_{3a} = \varepsilon_{5a} = \varepsilon_{6a} = 0$ if $u$ is of order 2 or 4, and $\varepsilon_{5a} = 0$ if $u$ is of order 6.

We denote by $\Theta_\ast$ a representation of $G$, over a sufficiently large field, which affords the Brauer character $\varphi_\ast$, and write $\Theta_\ast(u) \sim \text{diag}(\lambda_1, \ldots, \lambda_{\varphi_\ast(1)})$ to indicate that $\Theta_\ast(u)$ has eigenvalues $\lambda_1, \ldots, \lambda_{\varphi_\ast(1)}$ (multiplicities taken into account), which will, as happened before, be identified with complex roots of unity.

**When $u$ has order 2.** To simplify things, we shall use that $\varepsilon_{4a} = 0$ for theoretical reason, viz. Theorem 2.3. Then taking augmentation of $u$ gives $\varepsilon_{2a} + \varepsilon_{2b} = 1$, and $\varphi_{1b}(u) = \varepsilon_{2a} - \varepsilon_{2b} = 2\varepsilon_{2a} - 1$. Note that $\varphi_{1b}(u) = \pm 1$ since $\varphi_{1b}$ is of degree one. It follows that $\varepsilon_{2a} = 1$ or $\varepsilon_{2a} = 0$, and we are done.

**When $u$ has order 4.** Taking augmentation of $u$ gives

$$
\varepsilon_{2a} + \varepsilon_{2b} + \varepsilon_{4a} = 1.
$$

Since $u^2$ is rationally conjugate to $2a$ or $2b$, the matrix $\Theta_{3a}(u^2)$ is conjugate to $\Theta_{3a}(2a)$ or $\Theta_{3a}(2b)$ by Remark 3.3, and as both $\varphi_{3a}(2a)$ and $\varphi_{3a}(2b)$ are not equal to the degree of $\varphi_{3a}$, it follows that $\Theta_{3a}(u)$ is of order 4. Together with $\Theta_{3a}(u) \in \mathbb{Z}$,
this implies $\Theta_{3a}(u) \sim \mathrm{diag}(i, -i, \pm 1)$, so $\varphi_{3a}(u) = \pm 1$. Similarly, $\varphi_{3b}(u) = \pm 1$.

Thus
\begin{align*}
(5.2) & \quad \varphi_{3a}(u) = -\varepsilon_{2a} + \varepsilon_{2b} - \varepsilon_{4a} = \pm 1, \\
(5.3) & \quad \varphi_{3b}(u) = -\varepsilon_{2a} - \varepsilon_{2b} + \varepsilon_{4a} = \pm 1.
\end{align*}

Adding (5.1) and (5.2) shows that
\begin{equation}
\varepsilon_{2b} = 1 \quad \text{or} \quad \varepsilon_{2b} = 0.
\end{equation}

If $\varphi_{3b}(u) = -1$, then adding (5.1) and (5.3) shows that $\varepsilon_{4a} = 0$, which contradicts [24, (7.3)]. Thus $\varphi_{3b}(u) = 1$, and adding (5.1) and (5.3) shows that $\varepsilon_{4a} = 1$. Consequently $\varepsilon_{2a} + \varepsilon_{2b} = 0$ by (5.1). So $\varepsilon_{2a} = -1$ if $\varepsilon_{2b} = 1$. But then $\varphi_{1b}(u) = \varepsilon_{2a} - \varepsilon_{2b} - \varepsilon_{4a} = 1 - 1 - 1 = -3$, contradicting $\varphi_{1b}(u) = \pm 1$. Hence $\varepsilon_{2b} = 0$ by (5.4), and $\varepsilon_{2a} = 0$, so $\varepsilon_{4a}$ is the only partial augmentation which does not vanish.

**When $u$ has order 6.** Taking augmentation of $u$ gives
\begin{equation}
\varepsilon_{2a} + \varepsilon_{2b} + \varepsilon_{4a} + \varepsilon_{3a} + \varepsilon_{6a} = 1.
\end{equation}

Also,
\begin{equation}
\varphi_{1b}(u) = \varepsilon_{2a} - \varepsilon_{2b} - \varepsilon_{4a} + \varepsilon_{3a} - \varepsilon_{6a} = \pm 1.
\end{equation}

The unit $u^3$ of order 2 is rationally conjugate to either $2a$ or $2b$. If $u^3$ is conjugate to $2a$, then $u^3$, and hence also $u$, maps to 1 under the natural map $\mathbb{Z}S_5 \rightarrow \mathbb{Z}S_5/A_5 = \mathbb{Z}C_2$, so $\varphi_{1b}(u) = 1$. Adding (5.5) and (5.6) then shows:
\begin{equation}
\varepsilon_{2a} + \varepsilon_{3a} = 1.
\end{equation}

Also, if $u^3$ is conjugate to $2b$, then $u^3$, and hence also $u$, maps to $-1$ under the natural map $\mathbb{Z}S_5 \rightarrow \mathbb{Z}C_2$, so $\varphi_{1b}(u) = -1$, and adding (5.5) and (5.6) shows:
\begin{equation}
\varepsilon_{2a} + \varepsilon_{3a} = 0.
\end{equation}

Let $\Theta$ be one of $\Theta_{3a}$ and $\Theta_{3b}$, with Brauer character $\varphi$. Then we can write
\begin{equation}
\Theta(u) \sim \mathrm{diag}(\nu_1 \zeta^{\alpha_1}, \nu_2 \zeta^{\alpha_2}, \nu_3 \zeta^{\alpha_3})
\end{equation}
for some $\nu_i \in \{\pm 1\}$, a primitive third root of unity $\zeta$, and $\alpha_i \in \mathbb{Z}$. Since $\varphi(u^2) = \varphi(3a) = 0$ we have $\zeta^{\alpha_1 + \alpha_2 + \alpha_3} = 0$. Together with $\Theta(u) \in \mathbb{Z}$, it follows that
\begin{equation}
\Theta(u) \sim \pm \mathrm{diag}(1, -\zeta, -\zeta^2).
\end{equation}

If $\Theta(u) \sim \mathrm{diag}(-1, \zeta, \zeta^2)$, then $\varphi(u^4) = 1$, so $\varphi = \varphi_{3a}$ and $u^3$ is conjugate to $2a$. Suppose that $u^3$ is conjugate to $2a$. Then the preceding observations show that $\varphi_{3a}(u) = \varphi_{3b}(u) = 2$. It follows that $-2\varepsilon_{2a} = \varphi_{3a}(u) + \varphi_{3b}(u) = 4$, and $\varepsilon_{2a} = -2$. Now $\varepsilon_{3a} = 3$ from (5.7). But then $\varphi_{5a}(u) + \varphi_{5b}(u) = 2\varepsilon_{2a} - 2\varepsilon_{3a} = -4 - 6 = -10$, implying $\varphi_{5a}(u^2) = \varphi_{5a}(1)$ which is impossible.

Hence $u^3$ is conjugate to $2b$. So $\varphi_{3a}(u^3) = 1$ and $\varphi_{3b}(u^3) = -1$, and (5.9) gives $\varphi_{3a}(u) = -2$ and $\varphi_{3b}(u) = 2$. It follows that $-2\varepsilon_{2a} = \varphi_{3a}(u) + \varphi_{3b}(u) = 0$, i.e., $\varepsilon_{2a} = 0$. Now $\varepsilon_{3a} = 0$ from (5.8). Using (5.5), which now reads $\varepsilon_{2b} + \varepsilon_{4a} + \varepsilon_{6a} = 1$, we get $\varphi_{5a}(u) = \varepsilon_{2b} - \varepsilon_{4a} + \varepsilon_{6a} = -1 - 2\varepsilon_{4a}$. Subtracting (5.5) from $\varepsilon_{2b} - \varepsilon_{4a} - 2\varepsilon_{6a} = \varphi_{3a}(u) = -2$ gives $2\varepsilon_{4a} + 3\varepsilon_{6a} = 3$, and we further obtain $\varphi_{5a}(u) = -2 + 3\varepsilon_{6a}$. Since $| -2 + 3\varepsilon_{6a} | < \varphi_{5a}(1) = 5$, it follows that $\varepsilon_{6a} \in \{0, 1, 2\}$. If $\varepsilon_{6a} = 0$ or $\varepsilon_{6a} = 2$, then $\varepsilon_{4a} = 3/2$ or $\varepsilon_{4a} = -3/2$, which is impossible. Hence $\varepsilon_{6a} = 1$, and $\varepsilon_{4a} = 0$, $\varepsilon_{2b} = 0$ easily follows. Thus $\varepsilon_{6a}$ is the only partial augmentation which does not vanish.
Remark 5.1. Suppose that the Zassenhaus conjecture is verified for a symmetric group $S_n$. Then the Zassenhaus conjecture also holds for the symmetric groups $S_m$, $m < n$, since there is no fusion of conjugacy classes of $S_m$ in $S_n$.

6. The groups $\text{PSL}(2, p^f)$

The projective special linear group $\text{PSL}(2, p^f)$ may be seen as a prototype for intended applications. Most of its conjugacy classes are $p$-regular, and a $p$-singular group element is of order $p$, so it seems particular promising to work with Brauer characters in the defining characteristic $p$. These characters are known and it is also advantageous that there exist some of small degree.

We remark that $\text{PSL}(2,3)$ and $\text{PSL}(2,5)$ are the alternating groups of order 12 and 60, respectively, for which the Zassenhaus conjecture has been verified [21].

Group-theoretical properties of $\text{PSL}(2, p^f)$ are described in [17, Kapitel II, §8]. Its order is $\frac{1}{2}(p^f - 1)p^f(p^f + 1)$ where $k = \gcd(p^f - 1, 2)$. Of importance to us will be that $\text{PSL}(2, p^f)$ has cyclic subgroups of order $\frac{1}{2}(p^f - 1)$ and $\frac{1}{2}(p^f + 1)$, and that each $p$-regular element of $\text{PSL}(2, p^f)$ lies in a conjugate of one of these subgroups. Moreover, if $g$ is a $p$-regular element of $\text{PSL}(2, p^f)$ of order greater than 2, then $g^{-1}$ is its only distinct conjugate in $(g)$.

The representation theory of $\text{PSL}(2, p^f)$ is well understood, and some references are given below (we will only use characters). The $p$-modular representation theory of $\text{SL}(2, p)$ is described in Alperin’s beautiful book [1], and there is also a nice introductory article [16] by Humphreys on the representations of $\text{SL}(2, p)$. The character table of $\text{SL}(2, p)$ was first obtained by Frobenius; shortly afterwards, Schur and (independently) H. Jordan found the characters of $\text{SL}(2, p^f)$. Dornhoff’s version [12, §38] is a readable account of Schur’s work. The modular irreducible representations of $\text{SL}(2, p^f)$ in describing characteristic were given by Brauer and Nesbitt [8, §30]; they also gave the decomposition matrix for $\text{PSL}(2, p)$, $p$ odd. The decomposition matrices for all groups $\text{PSL}(2, p^f)$ were given by Burkhardt [9] (see also [27]). In [26], the Brauer character table (relative to $p$) of $\text{SL}(2, p^f)$ (of which the Brauer character table of $\text{PSL}(2, p^f)$ is a part) is given explicitly.

First, we shall consider $p$-singular torsion units when $p$ is odd. For $\text{PSL}(2, p)$ one can give definite results. The ordinary character table of $\text{PSL}(2, p^f)$ for odd $p$ is shown in Table 2.

The following two results were proved by Wagner [28] who did the formal calculations needed for application of the Luthar–Passi method (4.2). The calculation of (4.2) for the following result in the case $f = 1$ is reported in [3]. We shall give more direct arguments.

Proposition 6.1. Let $G = \text{PSL}(2, p^f)$ for an odd prime $p$, and $f \leq 2$. Then units of order $p$ in $V(\mathbb{Z}G)$ are rationally conjugate to elements of $G$.

Proof. Let $u$ be a torsion unit in $V(\mathbb{Z}G)$ of order $p$, and let $\alpha$ be its partial augmentation with respect to the class of $c$, an element of order $p$. Then $1 - \alpha$ is its partial augmentation with respect to the other class of elements of order $p$.

First, suppose that $G = \text{PSL}(2, p)$, and set $\zeta = e^{2\pi i/p}$. The character table of $G$ involves Gaussian sums:

$$\frac{1}{p}(\varepsilon + \sqrt{p}) = \delta_{\varepsilon,1} + \sum_{i \text{ square in } (\mathbb{Z}/p\mathbb{Z})^\times} \zeta^i, \quad \frac{1}{p}(\varepsilon - \sqrt{p}) = \delta_{\varepsilon,1} + \sum_{j \text{ non-square in } (\mathbb{Z}/p\mathbb{Z})^\times} \zeta^j$$
\begin{table}[h]
\centering
\begin{tabular}{|c|ccc|cc|}
\hline
\text{class of} & 1 & c & d & a^l & b^m \\
\hline
\text{order} & 1 & p & p & o(a) = \frac{q-1}{2} & o(b) = \frac{q+1}{2} \\
\hline
\psi & q & 0 & 1 & 1 & 1 \\
\chi_i & q+1 & 1 & 1 & \rho^{il} + \rho^{-il} & 0 \\
\theta_j & q-1 & -1 & -1 & 0 & -(\sigma^{jm} + \sigma^{-jm}) \\
\eta_1 & \frac{1}{2}(q + \varepsilon) & \frac{1}{2}(\varepsilon + \sqrt{\varepsilon q}) & \frac{1}{2}(\varepsilon - \sqrt{\varepsilon q}) & (-1)^i \delta_{\varepsilon,1} & (-1)^m \delta_{\varepsilon,-1} \\
\eta_2 & \frac{1}{2}(q + \varepsilon) & \frac{1}{2}(\varepsilon - \sqrt{\varepsilon q}) & \frac{1}{2}(\varepsilon + \sqrt{\varepsilon q}) & (-1)^i \delta_{\varepsilon,1} & (-1)^m \delta_{\varepsilon,-1} \\
\hline
\end{tabular}
\caption{Character table of PSL(2,\(q\)), \(q = p^f\geq 5\), odd prime \(p\)}
\end{table}

Entries: sign \(\varepsilon\) such that \(q \equiv \varepsilon \mod 4\),
\(\delta_{\varepsilon,\pm 1}\) Kronecker symbol,
\(\rho = e^{4\pi i/(q-1)}\), \(\sigma = e^{4\pi i/(q+1)}\),
\(\varepsilon = 1: 1 \leq i \leq \frac{1}{2}(q - 5)\), \(1 \leq j, l, m \leq \frac{1}{2}(q - 1)\),
\(\varepsilon = -1: 1 \leq i, j, l \leq \frac{1}{2}(q - 3)\), \(1 \leq m \leq \frac{1}{2}(q + 1)\).

(see, for example, [20, Chapter VI, Theorem 3.3]). Thus
\[\eta_1(u) = \alpha \frac{1}{2}(\varepsilon + \sqrt{\varepsilon p}) + (1 - \alpha) \frac{1}{2}(\varepsilon - \sqrt{\varepsilon p}) = \delta_{\varepsilon,1} + (\alpha - 1) + (2\alpha - 1) \sum_{i \text{ square in } (\mathbb{Z}/p\mathbb{Z})^\times} \zeta^i.\]

If \(\alpha > 0\), then this character value is a sum of \(\frac{1}{2}(p + \varepsilon)\) roots of unity if and only if \(\alpha = 1\). By symmetry (we may also use \(\eta_2\)), it follows that one of the partial augmentations on elements of order \(p\) vanish.

Now suppose that \(G = \text{PSL}(2, p^2)\). Then \(\varepsilon = 1\) as \(p^2 \equiv 1 \mod 4\), and we have
\[\eta_1(u) = \alpha \frac{1}{2}(1 + p) + (1 - \alpha) \frac{1}{2}(1 - p) = \frac{1}{2}((2\alpha - 1)p + 1).\]

Note that if \(n\) is a natural number, and \(-n\) is a sum of \(m\) \(p\)-th roots of unity, then \(m \geq n(p - 1)\). Hence if \(\alpha < 0\), then
\[\frac{1}{2}(p^2 + 1) = \eta_1(1) \geq |\eta_1(u)|(p - 1) = \frac{1}{2}(-(2\alpha - 1)p - 1)(p - 1) \geq \frac{1}{2}(3p - 1)(p - 1) > \frac{1}{2}(p^2 + 1),\]
which is impossible. Thus \(\alpha \geq 0\). By symmetry (use \(\eta_2\)), also \(1 - \alpha \geq 0\). So \(\alpha = 0\) or \(\alpha = 1\), and we are done. \(\square\)

\textit{Remark 6.2.} Let \(G = \text{PSL}(2, p^f)\) for an odd prime \(p\), and set \(H = \text{PSL}(2, p^{2f})\). Considering \(G\) as a subgroup of \(H\), the two conjugacy classes of elements of order \(p\) in \(G\) fuse in \(H\). Thus a unit of order \(p\) in \(V(\mathbb{Z}G)\) is conjugate to an element of \(G\) by a unit of \(\mathbb{Q}H\).
Apparently, we exploited the fact that some character degrees are small when compared with the orders of the involved elements. This is also what underlies the proof of the next proposition.

**Proposition 6.3.** Let $G = \text{PSL}(2, p)$ for an odd prime $p$. If the order of a torsion unit $u$ in $V(ZG)$ is divisible by $p$, then $u$ is of order $p$.

**Proof.** We can assume that $p > 5$ (see [21]).

Since $G$ has no elements of order $p^3$, there are no torsion units of order $p^2$ in $V(ZG)$ (see [24, (7.3)]). Suppose that a torsion unit $u$ in $V(ZG)$ is of order $pr$, for some prime $r$ different from $p$. Let $\varepsilon_c$ and $\varepsilon_d$ be the partial augmentations of $u$ with respect to the two conjugacy classes of $G$ of elements of order $p$. Write $u = \sum_{g \in G} a_g g$ (all $a_g$ in $\mathbb{Z}$). Elementary calculus shows (see [24, §7]):

$$u^p \in \sum_{g \in G} a_g g^p + [ZG, ZG] + pZG.$$  

Thus the 1-coefficient of $u^p$ lies in $\varepsilon_c + \varepsilon_d + p\mathbb{Z}$. By the Berman–Higman result, it follows that $\varepsilon_c + \varepsilon_d = mp$ for some integer $m$. Similarly, we obtain that the sum of all partial augmentations of $u$ with respect to conjugacy classes of elements of order $r$ is divisible by $r$. By Theorem 2.3, all other partial augmentations of $u$ vanish. Thus $m \neq 0$ as $u$ has augmentation one. Suppose that $r \mid \frac{1}{2}(p - 1)$. Then $\theta_1(x) = 0$ for all $r$-elements $x$ of $G \setminus \{1\}$, so $\theta_1(u) = -mp$, which is impossible since $\theta_1$ has degree $p - 1$. Hence $r \mid \frac{1}{2}(p + 1)$. So $\chi_1(x) = 0$ for all $r$-elements $x$ of $G \setminus \{1\}$, and $\chi_1(u) = \varepsilon_c + \varepsilon_d = mp$. As $\chi_1$ has degree $p + 1$, it follows that $m = \pm 1$, and $\pm p$ is the sum of $p + 1$ (pr)-th roots of unity. This can only happen if $p = 3$ (and $r = 2$), a case which we excluded at the beginning. The proposition is proved. \[\square\]

We now consider $p$-regular torsion units. We only prove partial results, but perhaps more can be expected when the $p$-modular version of the Luther–Passi method is used more rigorously.

Let $a$ and $b$ be elements of $\text{PSL}(2, p^f)$ of order $\frac{1}{k}(p^f - 1)$ and $\frac{1}{k}(p^f + 1)$, respectively, where $k = \gcd(p^f - 1, 2)$. Let $\rho$ and $\sigma$ be roots of unity of the same order as $a$ and $b$, respectively. Relative to the prime $p$, the group $\text{PSL}(2, p^f)$ has Brauer characters $\varphi_1, \varphi_3, \varphi_5, \ldots$ of degree $1, 3, 5, \ldots$ arising from the action of $\text{SL}(2, p^f)$ on homogeneous polynomials in two variables of degree 0, 2, 4, …, such that if $\Theta_{2l+1}$ is a representation affording $\varphi_{2l+1}$, then (in the notation from Section 5):

- $\Theta_{2l+1}(a) \sim \text{diag}(\rho^l, \rho^{-1}, \ldots, \rho, 1, \rho^{-1}, \ldots, \rho^{-(l-1)}, \rho^{-l})$,
- $\Theta_{2l+1}(b) \sim \text{diag}(\sigma^l, \sigma^{-1}, \ldots, \sigma, 1, \sigma^{-1}, \ldots, \sigma^{-(l-1)}, \sigma^{-l})$.

The proof of our first result is straightforward.

**Proposition 6.4.** Let $G = \text{PSL}(2, p^f)$, and let $r$ be a prime different from $p$. Then any torsion unit $u$ in $V(ZG)$ of order $r$ is rationally conjugate to an element of $G$.

**Proof.** Only for elements $x$ of $G$ which are of order $r$ we possibly have $\varepsilon_x(u) \neq 0$, by Theorem 2.3. So we can assume that $r \neq 2$ as there is only one conjugacy class of elements of order 2 in $G$. Let $x$ be an element of order $r$ in $G$. Then $\varphi_3(x) = 1 + \zeta + \zeta^{-1}$ for a primitive $r$-th root of unity $\zeta$. By Theorem 3.2,

$$\varphi_3(u) = \sum_{i=1}^{(r-1)/2} \varepsilon_{x^i}(u)(1 + \zeta^i + \zeta^{-i}).$$
\( G \) elements of 

Inductively, we may assume that 

\( \mu(\varphi) = 0 \) (otherwise all \( \varepsilon_x(u) \) would be equal to \(-2\), which is impossible), and further \( \varepsilon_x(u) = 1 \), with all other partial augmentations of \( u \) vanishing. \( \Box \)

The next proposition confirms a conjecture of Bovdi (see Section 2) for the groups \( \text{PSL}(2, p^l) \). Its proof is already a little bit more sophisticated.

**Proposition 6.5.** Let \( G = \text{PSL}(2, p^l) \), and let \( r \) be a prime different from \( p \). Suppose that \( u \) is an \( r \)-torsion unit in \( V(\mathbb{Z}G) \), of order \( r^n \) (say). For any integer \( m \), let \( T_{r^m} \subset G \) be a set of representatives of the conjugacy classes of elements of order \( r^m \) in \( G \). Then

\[
\sum_{x \in T_{r^m}} \varepsilon_x(u) = 0 \quad \text{for all } 0 \leq m < n.
\]

Moreover, there exist elements of \( G \) of the same order as \( u \), and for such an element \( g \), we have \( \mu(1, u, \varphi) = \mu(1, g, \varphi) \) for each \( p \)-modular character \( \varphi \) of \( G \) (notation as in Section 4).

**Proof.** There exists an element \( g \) in \( G \) of the same order as \( u \) (see [24, (7.3)]). We shall use induction, firstly on \( n \) and secondly on \( m \), to prove \( \mu(1, u, \varphi) = \mu(1, g, \varphi) \) for each \( p \)-modular character \( \varphi \) of \( G \), and \( \sum_{x \in T_{r^m}} \varepsilon_x(u) = 0 \) for all \( 0 \leq m < n \). By Proposition 6.4, and the Berman–Higman result, this holds for \( n = 1 \), and we can assume that \( n > 1 \), \( m \geq 1 \). Suppose that the assertion holds for smaller values of \( n \) or \( m \).

We also write \( T_{r^m} \) for a set of representatives of the conjugacy classes of \( r \)-elements of \( G \) of order equal or less than \( r^m \) in \( G \), and \( T_{r^m} \) for such a set of \( r \)-elements whose order is greater than \( r^m \), but equal or less than \( r^n \). (The latter condition is just a technical one, which makes sense in view of Theorem 2.3). Let \( \zeta \) be a primitive \( r^n \)-th root of unity, and set \( T_{r^m} = T_{r^m}/\langle \zeta \rangle \). Note that for any integer \( l \),

\[
\text{Tr}(\zeta^l) = \begin{cases} 
  r^{n-1}(r - 1) & \text{if } \zeta^l = 1, \\
  -r^{n-1} & \text{if } \zeta^l \neq 1 \text{ and } \zeta^{lr} = 1, \\
  0 & \text{otherwise}.
\end{cases}
\]

Set \( k = 2r^{n-1} + 1 \). Then

\[
(6.1) \quad \frac{1}{r^{m}} \text{Tr}(\varphi_k(x)) = \begin{cases} 
  (r - 1)/r & \text{if } x \in T_{r^m}, \\
  (r - 3)/r & \text{if } x \in T_{r^m}.
\end{cases}
\]

By (4.1) and (4.3),

\[
\mu(1, u, \varphi_k) = \frac{1}{r^m} \mu(1, u^r, \varphi_k) + \frac{1}{r^{m+1}} \sum_{x \in T_{r^m}} \varepsilon_x(u) \text{Tr}(\varphi_k(x)).
\]

Inductively, we may assume that \( \mu(1, u^r, \varphi_k) = \mu(1, g^r, \varphi_k) = 1 \). Note that for elements \( x \) and \( y \) of \( \langle g \rangle \) of the same order, \( \varphi_k(x) \) and \( \varphi_k(y) \) are algebraically conjugate,
so \( \text{Tr}(\varphi_k(x)) = \text{Tr}(\varphi_k(y)) \). Hence for \( 0 \leq l < m \), we obtain inductively\( \sum_{x \in T_{r,l}} \varepsilon_x(u) \text{Tr}(\varphi_k(x)) = \text{Tr}(\varphi_k(g^{r^n l})) \sum_{x \in T_{r,l}} \varepsilon_x(u) = 0. \)

Together with (6.1), it follows that \( \mu(1, u, \varphi_k) = 1 \frac{r}{r + r^3 r} \sum_{x \leq T_{r,m}} \varepsilon_x(u) \text{Tr}(\varphi_k(u)) + 1 \frac{r - 1}{r - 1} \sum_{x > r,m} \varepsilon_x(u) \).

Hence \( \mu(1, u, \varphi_k) \equiv 1 \mod 2 \) since \( \sum_{x \in T_{r,m}} \varepsilon_x(u) \) is divisible by \( r \), see (2.1).

We may expect that results about \( p \)-regular torsion units of mixed order can be achieved as well. The following proposition was proved in [14, Example 3.6] for \( \text{PSL}(2,7) \) by an ad hoc argument. The proof turns out to be pretty simple.

**Proposition 6.6.** Let \( G = \text{PSL}(2, p^f) \) with \( p \neq 2,3 \). Then any torsion unit in \( V(\mathbb{Z}G) \) of order 6 is rationally conjugate to an element of \( G \).

**Proof.** Suppose that there is a torsion unit \( u \) of order 6 in \( V(\mathbb{Z}G) \). Then \( G \) has elements of order 3 which form a single conjugacy class. Of course, the elements of order 2 are all conjugate in \( G \). Further, if \( G \) has elements of order 6, they also form a single conjugacy class. We denote the corresponding partial augmentations of \( u \) by \( \varepsilon_2, \varepsilon_3 \) and \( \varepsilon_6 \), agreeing that \( \varepsilon_6 = 0 \) if \( G \) contains no element of order 6.

By Theorem 2.3, these are the only partial augmentations of \( u \) which are possibly nonzero. The Brauer characters \( \varphi_3 \) and \( \varphi_5 \) have the following values on the relevant classes (the last column may be left out):

|        | 2a | 3a | 6a |
|--------|----|----|----|
| \( \varphi_3 \) | -1 | 0  | 2  |
| \( \varphi_5 \) | 1  | -1 | 1  |
By Proposition 6.4 and Remark 3.3, we have

$$\Theta_5(u^3) \sim \text{diag}(1, -1, 1, -1, 1), \quad \Theta_5(u^2) \sim \text{diag}(\zeta^2, \zeta, 1, \zeta^{-1}, \zeta^{-2})$$

for a primitive third root of unity $\zeta$. By Theorem 3.2, $\varphi_5(u) = \varepsilon_2 - \varepsilon_3 + \varepsilon_6$. In particular $\varphi_5(u) \in \mathbb{Z}$, forcing $\Theta_5(u) \sim \text{diag}(\zeta^2, 1, -\zeta, -\zeta^{-2})$. Thus $\varphi_5(u) = 1$. Taking augmentation of $u$ gives $\varepsilon_2 + \varepsilon_3 + \varepsilon_6 = 1$, so that $\varepsilon_3 = 0$ follows. For the same reasons as before, we further have

$$\Theta_3(u^3) \sim \text{diag}(-1, 1, -1), \quad \Theta_3(u^2) \sim \text{diag}(\zeta, 1, \zeta^{-1})$$

and $\varphi_3(u) = -\varepsilon_2 + 2\varepsilon_6$. This forces $\Theta_3(u) \sim \text{diag}(-\zeta^2, 1, -\zeta)$ as $\varphi_3(u) \in \mathbb{Z}$, and $\varphi_3(u) = 2$. Using $-\varepsilon_2 = -1 + \varepsilon_6$ which results from taking augmentation, it follows that $\varepsilon_6 = 1$, and $\varepsilon_2 = 0$. Thus $u$ is rationally conjugate to a group element, by [23, Theorem 2.5] and Proposition 6.4.

The last result is as follows.

**Proposition 6.7.** Let $G = \text{PSL}(2, p^f)$. Then for any $p$-regular torsion unit $u$ in $V(\mathbb{Z}G)$, there is an element of $G$ of the same order as $u$.

**Proof.** Let $u$ be a $p$-regular torsion unit in $V(\mathbb{Z}G)$. If $u$ is of prime power order the claim follows from [24, (7.3)]. Thus we can assume that $u$ has order $rs$ for different primes $r$ and $s$ such that $r \mid p^f - 1$ and $s \mid p^f + 1$, and have to reach a contradiction.

By Proposition 6.4 and Remark 3.3, we have

$$\Theta_3(u^r) \sim \text{diag}(\zeta_r, 1, \zeta_s^{-1}), \quad \Theta_3(u^s) \sim \text{diag}(\zeta_r, 1, \zeta_r^{-1})$$

for a primitive $s$-th root of unity $\zeta_s$ and a primitive $r$-th root of unity $\zeta_r$. This shows that $\Theta_3(u)$ has a primitive $rs$-th root of unity as an eigenvalue, and replacing the roots of unity by suitable powers, if necessary, we either have $\Theta_3(u) \sim \text{diag}(\zeta_r \zeta_s, \zeta_r^{-1}, \zeta_s^{-1})$ or $\Theta_3(u) \sim \text{diag}(\zeta_r, \zeta_s, 1, \zeta_r^{-1}, \zeta_s^{-1})$. By [24, (7.3)], $G$ has elements $x$ and $y$ of order $r$ and $s$, respectively. By Theorem 3.2 and Theorem 2.3, these elements can be chosen such that

$$\varphi_3(u) = 1 + \sum_{i \in I} \varepsilon_{x^i}(u)(\zeta_r^i + \zeta_r^{-i}) + \sum_{j \in J} \varepsilon_{y^j}(u)(\zeta_s^j + \zeta_s^{-j}) \quad (6.2)$$

where the $x^i$ ($i \in I$) are representatives of the conjugacy classes of elements of order $r$, and the $y^j$ ($j \in J$) are representatives of the conjugacy classes of elements of order $s$. In particular, $\varphi_3(u)$ is a real number, which shows that only $\Theta_3(u) \sim \text{diag}(\zeta_r \zeta_s, 1, \zeta_r^{-1}, \zeta_s^{-1})$ is possible, and

$$\varphi_3(u) = 1 + \zeta_r \zeta_s + \zeta_r^{-1} \zeta_s^{-1} \quad (6.3)$$

Suppose that $r = 2$. Then $\varphi_3(u) = 1 - \zeta_s - \zeta_s^{-1}$, and by (6.2)

$$\varphi_3(u) = 1 - 2\varepsilon_x(u) + \sum_{j=1}^{(s-1)/2} \varepsilon_{y^j}(u)(\zeta_s^j + \zeta_s^{-j})$$

$$= 1 + \sum_{j=1}^{(s-1)/2} (2\varepsilon_x(u) + \varepsilon_{y^j}(u))(\zeta_s^j + \zeta_s^{-j}),$$
so $2\varepsilon_x(u) + \varepsilon_y(u) = -1$, and $\varepsilon_{y'}(u) = -2\varepsilon_x(u)$ for $1 < j \leq (s - 1)/2$. It follows that

$$1 = \varepsilon_x(u) + \sum_{j=1}^{(s-1)/2} \varepsilon_{yj}(u) = -1 + (2 - s)\varepsilon_x(u),$$

so $\varepsilon_x(u) = 2/(2-s)$. By Proposition 6.6, we have $s > 3$. But then $0 \neq \varepsilon_x(u) < 1$, a contradiction.

Thus we can assume that both $r$ and $s$ are odd. Let $\tau$ be a Galois automorphism satisfying $\zeta^r = \zeta^{-1}$ and $\zeta^s = \zeta^1$. By (6.2), $\varphi_3(u)^\tau = \varphi_3(u)$, so $\zeta^{-1}_{r}\zeta_{s} + \zeta_{r}\zeta^{-1}_{s} = \zeta_{r}\zeta_{s} + \zeta_{r}^{-1}\zeta_{s}^{-1}$ by (6.3), which is impossible. The proposition is proved.

In concluding, we record that the Zassenhaus conjecture is verified for the groups $PSL(2, \mathbb{F}_p)$, $p = 7, 11, 13$, by Propositions 6.1, 6.3, 6.4, 6.6 and 6.7.

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