Exact analytical solution to inclined plane motion with constant resistance

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Abstract. This study describes the entire block-sliding movement process from various aspects to reveal the law governing its movement on a slope. An analytical solution of the block motion on an inclined plane is provided based on the variable substitution for various friction coefficient values in this study. The block movement’s initial angle and equivalent friction coefficient are determined by describing its trajectory, kinetic energy change, movement duration, maximum horizontal displacement, and other indicators. The block movement’s initial angle affects its movement duration, whereas the equivalent friction coefficient indirectly represents the degree of influence of friction on the block movement. The influence of singular points on the sliding trajectory is considered under certain initial conditions. Finally, this study provides a calculation method for assessing the block motion state with given initial conditions, which offers a theoretical reference for the application of the block slope-sliding model in the engineering field.

1. Introduction

According to the classic Coulomb friction law, the magnitude of sliding friction is equal to the product of the positive pressure and friction coefficient, and its direction is always opposite to the motion direction [1]. For a one-dimensional system, frictional nonlinear characteristics and motion fractal [2], bifurcation [3], sudden change [4], chaos [5], and other significant nonlinear phenomena have been investigated severally. However, the surface contact phenomenon is more expressed as plane motion, which is a two-dimensional Coulomb friction problem. Inclined plane motion is a classic two-dimensional Coulomb friction problem. It has advantages to solve this problem using a noncontinuous numerical method, however, integral numerical methods always have difficulties with numerical stability and convergence. Alternatively, analytical solutions do exhibit the aforementioned problems, thus they can be used as a benchmark to verify the numerical method [6,7], and they also contain more comprehensive information. However, due to the significant nonlinearity of the two-dimensional Coulomb friction, few studies have conducted in-depth research on the analytical calculation of the two-dimensional Coulomb friction problem, such as the block sliding on an inclined plane.

The analytical calculation difficulty is in obtaining the general solution of the curve motion differential equation under arbitrary initial conditions and friction coefficients. Further, the displacement formula of a block moving in a one-dimensional straight line on an inclined plane can be obtained using Newton's law of motion. However, when the block makes a two-dimensional curve movement on the plane, the equation solving process becomes more challenging due to the friction direction’s transient nature. The usual practice is selecting an auxiliary variable as a parameter to
obtain an analytical solution of the variable, indirectly reflecting the relationship of displacement with time. Zhou [8] used the angle, \( \theta \), between the block velocity and horizontal direction as the independent variable, and presented the parameter equations of the sliding curve trajectory under two limiting conditions of the initial angle, \( \theta_0 = 0 \), and the friction angle equal to the slope angle. Notably, the equation must be integrated again to obtain the relationship between displacement and time, however, no specific integration methods and results are provided. Shunyakov and Lavrik [9] further deduced the relational expressions \( T(\phi), U_s(\phi), U_a(\phi), x(\phi), \) and \( y(\phi) \) of the motion time, velocity and displacement under any initial angle condition by defining a new independent variable, \( \phi \). However, because speed is included in the definition of variable \( \phi \), the speed and displacement formulas are not decoupled from the speed, therefore it is difficult to calculate directly. Aghamohammadi C. and Aghamohammadi A. [10] selected an angle, \( \omega \), between the block’s velocity and the component of gravity along the slope as the independent variable, and presented the relationship between the motion time, velocity, displacement, and the variable, \( \phi \), under any initial angle condition. However, the singularity caused by the friction coefficient was not considered, and the motion laws of the block’s final state, as well as the limit displacement, were not elaborated, thus the formula cannot be applied to more general motion situations. You [11] revealed the influence of the relative relationship between the friction coefficient and slope angle on the sliding characteristics, from the equivalent friction coefficient perspective and demonstrated that a small variation can result in significant changes in movement characteristics when the friction coefficient is close to a certain critical state. Wang [12,13] demonstrated that the block-sliding analytical solution can be used to establish a new numerical algorithm for solving the two-dimensional Coulomb friction system under arbitrary time-varying external load excitation and that it can also be used to solve the three-dimensional Coulomb friction system’s response.

The analytical formulas presented in previous studies lack general applicability, and the analysis of various motion elements in the block-sliding process is not comprehensive enough, thus, it is necessary to conduct a more detailed analysis of the essential factors affecting block sliding. This study presents a calculation method for the block-sliding analytical solution, with the time variable being replaced with the angle variable. The motion law of the highest point, the final state, and the limit displacement of the trajectory are detailed, and the singularity situation caused by the difference in friction coefficient is considered in the motion analysis, making the analytical solution formula suitable for more general motion situations.

2. Analytical solution derivation process

Figure 1 depicts a sliding block along a fixed slope under gravity. The sliding process is assumed to follow the Coulomb friction law (the friction force is constant and is always in the opposite direction of the sliding direction), with negligible air resistance. According to Newton’s second law, the equations of motion are as follows:

\[
\begin{align*}
mx + G_{xy} \cos \omega + R &\quad (1a) \\
mx &\quad R \cdot \sin \omega \quad (1b) \\
m\ddot y &\quad G_{xy} + R \cdot \cos \omega \quad (1c)
\end{align*}
\]

with the following relations,

\[
\begin{align*}
x &\quad \dot s \sin \omega \quad (2a) \\
y &\quad \dot s \cos \omega \quad (2b)
\end{align*}
\]

where \( m \) is the block’s mass, \( G_{xy} \) is the gravity’s component along the slope, \( R \) is the Coulomb friction on the block, \( \dot s \) is the block’s sliding velocity, and \( \omega \) is the angle between \( \dot s \) and the \( y \) axis.
Equation (2a) is organized as follows:

\[
\frac{\ddot{x}}{m} = \frac{G_{xy} \cos \omega + \ddot{s} \cos \omega}{\sin \omega + \dot{s} \omega \cos \omega} \sin \omega
\]

\[
\Rightarrow \dot{s} \omega = -\frac{G_{xy}}{m} \sin \omega #3(3)
\]

Similar organization of Equation (2b), presents the following equation:

\[
\dot{s} = -\frac{G_{xy} \sin \omega}{m} \omega #4(a)
\]

\[
\ddot{x} = \frac{G_{xy} \sin \omega}{m} \sin \omega #4(b)
\]

\[
\ddot{y} = -\frac{G_{xy} \sin \omega}{m} \cos \omega #4(c)
\]

Arranging the acceleration in the horizontal direction presents the following:

\[
\ddot{x} = \frac{R \cdot \sin \omega}{m}
\]

\[
\Rightarrow \frac{d(\dot{x} \sin \omega)}{dt} = \frac{R \cdot \sin \omega}{m}
\]

\[
\Rightarrow \frac{d\omega}{dt} d(\dot{x} \sin \omega) = -\frac{m \ddot{s} R}{G_{xy}} d\omega
\]

\[
\Rightarrow d(\dot{x} \sin \omega) = -\frac{\dot{x} R}{G_{xy}} #5(5)
\]

By analyzing the block’s force, the expressions of friction and sliding force can be obtained as follows:

\[
R = -\mu_k m g \cos \theta #6(6)
\]

\[
G_{xy} = m g \sin \theta #7(7)
\]

where \( \mu_k \) is the friction coefficient. This study assumes a constant \( \mu_k \) value and introduces the equivalent friction coefficient, \( \lambda = \mu_k \cot \theta \). When the slope, \( \theta \), is constant, \( \lambda \) reflects the relationship between the friction and sliding force. \( \lambda > 1 \) means that the friction is greater than the sliding force, and vice versa.

Equation (3) reveals that for any \( \omega \in (0, \pi) \), \( \dot{\omega} \) is always negative, indicating that \( \omega \) decreases with time and eventually approaches 0. When \( \dot{s} \neq 0 \), the expression of \( \dot{\omega} \) can be obtained:

\[
\dot{\omega} = \frac{-g \sin \theta \sin \omega}{\dot{s}} = -\frac{g \sin \theta \sin \omega (\lambda + \cos \omega)}{g \sin \theta (1 - \lambda^2) t + \dot{s}_0 (\lambda + \cos \omega) #8(8)}
\]

In the case of two-dimensional sliding, the value range of \( \omega_0 \) is \((0, \pi/2)\). Thus, the movement of the block will become more complicated due to the friction direction’s transient nature. Notably, the discussions in the following chapters will be based on a two-dimensional situation. Further, the
analytical solutions of the movement velocity and displacement of the block according to the different values of \( \lambda \) will be derived.

When \( \lambda = 1 \), the parametric equation for the trajectory of the block is as follows:

\[
\begin{align*}
    s &= -\frac{s^2(1 + \cos \omega_0)^2}{4g\sin \theta} \left( \ln \left( \frac{\omega}{2} - \ln \left( \frac{\omega_0}{2} \right) \right) + \frac{1}{4} \left( \tan^2 \frac{\omega}{2} - \tan^2 \frac{\omega_0}{2} \right) + \frac{1}{4} \left( \tan^2 \frac{\omega}{2} - \tan^2 \frac{\omega_0}{2} \right) \right) \#(9a) \\
    x &= -\frac{s^2(1 + \cos \omega_0)^2}{6g\sin \theta} \left[ 3 \left( \tan \frac{\omega}{2} - \tan \frac{\omega_0}{2} \right) + \left( \tan^3 \frac{\omega}{2} - \tan^3 \frac{\omega_0}{2} \right) \right] \#(9b) \\
    y &= -\frac{s^2(1 + \cos \omega_0)^2}{16g\sin \theta} \left[ 4 \left( \ln \left( \frac{\omega}{2} - \ln \left( \frac{\omega_0}{2} \right) \right) - \left( \tan \frac{\omega}{2} - \tan \frac{\omega_0}{2} \right) \right) \right] \#(9c)
\end{align*}
\]

When \( \lambda \neq 1 \), the equation can be as follows:

\[
\begin{align*}
    s &= -\frac{s^2 \sin^2 \omega_0}{4g\sin \theta \left( \tan \frac{\omega_0}{2} \right)^2} \left[ \left( \tan \frac{\omega}{2} \right)^{2\lambda - 2} - \left( \tan \frac{\omega_0}{2} \right)^{2\lambda - 2} \right] + \left( \tan \frac{\omega}{2} \right)^{2\lambda + 2} - \left( \tan \frac{\omega_0}{2} \right)^{2\lambda + 2} + \left( \tan \frac{\omega}{2} \right)^{2\lambda} - \left( \tan \frac{\omega_0}{2} \right)^{2\lambda} \#(10a) \\
    x &= \left\{ \begin{array}{ll}
        -\frac{s^2 \sin^2 \omega_0}{2g\sin \theta \tan \frac{\omega_0}{2}} \left[ \ln \left( \frac{\omega}{2} - \ln \left( \frac{\omega_0}{2} \right) \right) + \frac{\left( \tan \frac{\omega}{2} \right)^2 - \left( \tan \frac{\omega_0}{2} \right)^2}{2} \right], & \lambda = 0.5 \\
        -\frac{s^2 \sin^2 \omega_0}{2g\sin \theta \left( \tan \frac{\omega_0}{2} \right)^2} \left[ \left( \tan \frac{\omega}{2} \right)^{2\lambda - 1} - \left( \tan \frac{\omega_0}{2} \right)^{2\lambda - 1} \right] + \left( \tan \frac{\omega}{2} \right)^{2\lambda + 1} - \left( \tan \frac{\omega_0}{2} \right)^{2\lambda + 1}, & \lambda \neq 0.5
    \end{array} \right. \#(10b) \\
    y &= -\frac{s^2 \sin^2 \omega_0}{4g\sin \theta \left( \tan \frac{\omega_0}{2} \right)^2} \left[ \left( \tan \frac{\omega}{2} \right)^{2\lambda - 2} - \left( \tan \frac{\omega_0}{2} \right)^{2\lambda - 2} \right] - \left( \tan \frac{\omega}{2} \right)^{2\lambda + 2} + \left( \tan \frac{\omega_0}{2} \right)^{2\lambda + 2} \#(10c)
\end{align*}
\]

The specific derivation process is shown in the Appendix.

3 Analysis and discussion

3.1 Movement process analysis

The block movement’s trajectory can be obtained from Equations (9 a–c) and (10 a–c) using \( \omega \) as the intermediate quantity. Figure 2a shows the trajectory when \( \lambda = 0.7 \) and Figure 2b shows the trajectory when \( \lambda = 1.2 \). Figure 2 shows that the initial launch angle and trajectory direction are individually different, but at the end of the trajectory, the \( \omega = 0 \) trend will be the same. The phenomenon of the trajectory’s final \( \omega \) tending to 0 becomes more obvious as the initial angle increases, and an inflection point with a sudden change of direction appears on the trajectory.
Figure 2. Trajectories for different values of the initial angle.

The inflection point is the highest trajectory point and is also the demarcation point for the block motion characteristics’ change. This change is evident in the block’s kinetic energy (Figure 3). Assuming the horizontal plane at the block movement’s starting point is the zero potential energy surface, the position where the inflection point occurs corresponds to the lowest point on the curve in Figure 3a. Simultaneously, the block’s kinetic energy is the smallest, however, the movement does not stop at this point; the kinetic energy increases rapidly after the inflection point. The energy consumed by friction cannot offset the increase in kinetic energy, and the block movement speed will continue to increase. The block can continue to travel toward infinity as long as the slope on which it is placed is long enough. In the case of $\lambda > 1$ (Figure 3b), the kinetic energy continues to decrease from the start of the movement, however, no sudden change in speed is observed at the inflection point. The block’s initial kinetic energy will not be completely consumed until the movement stops. Thus, the degree of influence of friction on the overall block movement is indirectly represented as $\lambda$.

Figure 3. Kinetic energy changes with different values of $\lambda$.

In addition to the equivalent friction coefficient, $\lambda$, the initial launch angle, $\omega_0$, another key factor affecting the block movement. When $\lambda > 1$, the movement’s duration will be affected by $\omega_0$. The larger the initial angle, the shorter the block movement time, and the greater the angle change during the entire movement. A sudden change in the angle occurs when approaching the stop position, which is similar to how the inflection point appears. When the initial angle of the block movement gradually decreases, the block movement time will increase, and the overall angle, $\omega$, change tends to be more stable, even when $\omega_0$ is small enough, exhibiting a phenomenon similar to a linear relationship.

3.2 Limit state analysis

To obtain more accurate block motion information, the limits of Equations (9 a–c) and (10 a–c) can be calculated to get the block’s speed limit and the coordinates of the limit position when $\omega$ approaches 0,
which can be used to assess the trajectory trend. The specific assessment method is summarized as follows.

3.2.1 $\lambda = 1$. When $\lambda = 1$, the block will not stop moving. The block movement speed approaches a certain value as the movement time increases. This value is only related to the initial velocity and angle, and the horizontal component of the velocity approaches zero. When $\omega \to 0_+$, the block movement speed is expressed as follows:

\[
\lim_{\omega \to 0_+} \dot{s} = \frac{\ddot{s}_0 (1 + \cos \omega_0)}{2} \# (11a)
\]

\[
\lim_{\omega \to 0_+} \dot{\theta} = 0 \# (11b)
\]

\[
\lim_{\omega \to 0_+} \dot{\phi} = \frac{\ddot{s}_0 (1 + \cos \omega_0)}{2} \# (11c)
\]

When $\omega \to 0_+$, the block displacement in the horizontal direction approaches a maximum value, and the displacement along the direction of the sliding force continues to increase, thus the block continues to move. The limit position can be expressed by the following formulas:

\[
\lim_{\omega \to 0_+} s = + \infty \# (12a)
\]

\[
\lim_{\omega \to 0_+} x = \frac{s_0^2 (1 + \cos \omega_0)^2}{6 g \sin \theta} \left(3 \tan \frac{\omega_0}{2} + \tan^3 \frac{\omega_0}{2} \right) \# (12b)
\]

\[
\lim_{\omega \to 0_+} y = \infty \# (12c)
\]

3.2.2 $\lambda \neq 1$. When $\lambda \neq 1$, the block’s movement should still be considered in different situations. By substituting the condition of $\omega \to 0_+$ in Equations (10 a–c), the movement of the block will eventually stop only when $\lambda > 1$, and the block will stop moving when $\omega = 0$. When $\lambda < 1$, the block’s movement does using expressed by the following formula:

\[
\lim_{\omega \to 0_+} t = \begin{cases} 
+ \infty, \lambda < 1 \\
\frac{\ddot{s}_0 (1 + \cos \omega_0)}{g \sin \theta (\lambda^2 - 1)}, \lambda > 1
\end{cases} \# (13)
\]

When $\omega \to 0_+$, the horizontal component of the block motion’s velocity will infinitely approach 0, but its magnitude will vary depending on the value of $\lambda$; when $\lambda > 1$, the movement will stop, and the movement speed will drop to 0; however, when $\lambda < 1$, the block movement’s speed will continue to increase and eventually approach positive infinity. The expression for speed is as follows:

\[
\lim_{\omega \to 0_+} \dot{s} = \begin{cases} 
+ \infty, \lambda < 1 \\
0, \lambda > 1
\end{cases} \# (14a)
\]

\[
\lim_{\omega \to 0_+} \dot{\theta} = 0 \# (14b)
\]

\[
\lim_{\omega \to 0_+} \dot{\phi} = \begin{cases} 
+ \infty, \lambda < 1 \\
0, \lambda > 1
\end{cases} \# (14c)
\]

By substituting $\omega \to 0_+$ in Equations (10 a–c), the limit position expression can be obtained:

\[
\lim_{\omega \to 0_+} s = \begin{cases} 
\frac{s_0^2 \sin^2 \omega_0}{4 g \sin \theta \left(\tan \frac{\omega_0}{2}\right)^{2k}}, \lambda > 1 \# (15a)
\end{cases}
\]

\[
\lim_{\omega \to 0_+} x = \begin{cases} 
\frac{s_0^2 \sin^2 \omega_0}{2 g \sin \theta \left(\tan \frac{\omega_0}{2}\right)^{2k}} \left[\frac{\tan \frac{\omega_0}{2}^{2k-2}}{2\lambda - 2} + \frac{\tan \frac{\omega_0}{2}^{2k+2}}{2\lambda + 2} + \frac{\tan \frac{\omega_0}{2}^{2k}}{\lambda}\right], \lambda > 0.5 \# (15b)
\end{cases}
\]
\[ \lim_{\omega \to 0^+} y = \begin{cases} \frac{s_0^2 \sin^2 \omega_0}{4g \sin \theta (\tan \frac{\omega_0}{2})^{2\lambda}} \left[ \frac{+\infty, \lambda < 1}{2\lambda - 2} - \frac{\tan \frac{\omega_0}{2}}{2\lambda + 2} \right], & \lambda > 1 \end{cases} \]  

According to Equations (11b), (12b), (14b), and (15b), although the limit velocity in the \( x \)-direction will always be 0, the limit displacement in this direction will asymptotically approach a certain value only when \( \lambda > 0.5 \) and will approach positive infinity when \( \lambda \leq 0.5 \).

### 3.3 Benchmark test of a three-dimensional sliding block problem

The analytical solution may serve as a benchmark test for a numerical algorithm, such as open–close iteration (OCI) and augmented open–close iteration (AOCI). The following parameters are adopted in this test: \( s_0 = 3 \text{ m/s}, \omega_0 = 90^\circ, \theta = 30^\circ, \) and \( \mu_k = \tan 30^\circ \). The numerical solution is given by Wu et al. [6].

The results are presented in Figure 4. Figure 4 shows that the errors in the \( x \)- and \( y \)-displacements are less than 1.4\%, indicating that the analytical solution may be used as the reference value for the numerical algorithm.

![Figure 4. Errors of OCI and AOCI: (a) Relative errors in \( x \)-displacement; (b) Relative errors in \( y \)-displacement.](image)

### 4 Conclusion

This study presents an analytical solution of the velocity and displacement of the angle, \( \omega \). The equivalent friction coefficient, \( \lambda \), and initial angle, \( \omega_0 \), affects the block’s trajectory. The degree of influence of friction on the overall sliding movement is indirectly represented as \( \lambda \). When \( \lambda > 1 \), regardless of the angle of the block’s initial velocity, the movement of the block will eventually stop; however, when \( \lambda < 1 \), the movement of the block will not stop, and when \( \lambda < 0.5 \), the movement in the horizontal and sliding force directions is unbounded. \( \omega_0 \) represents the block movement’s initial velocity direction. The larger the initial angle, the shorter the block’s movement time; when the block movement’s initial angle is gradually reduced, the block’s movement time will increase accordingly. In addition, the overall change of the angle tends to be more stable.

Through limit calculation, the speed and position of the block at \( \omega \to 0^+ \) are obtained, and a calculation method for assessing the block movement’s state is obtained. The method is simple to
calculate and can quickly and accurately predict the block movement trend. It realizes the value of verifying the numerical calculation method.

This study conducts derivation and research-based investigations on the premise of the constant friction coefficient. However, in the case of relatively high-speed sliding, test results reveal that the friction coefficient of the sliding surface exhibits a decreasing trend with an increase in normal pressure [14]. In future research, different combinations of the friction coefficient and speed relationship (such as friction coefficient proportional to speed and proportional to the square of the speed) can be introduced to explore the block-sliding motion characteristics under the condition of variable friction coefficient, which can further improve the accuracy of the block slope-sliding simulation.

Appendix

When $\lambda = 1$, Equation (8) becomes

$$\frac{d\omega}{dt} = - \frac{g \sin \theta \sin \omega (\lambda + \cos \omega)}{g \sin \theta (1 - \lambda^2) t + s_0 (\lambda + \cos \omega \omega_0)}$$

$$\Rightarrow (1 - \lambda^2) \sin \omega d\omega = \frac{(1 - \lambda^2) g \sin \theta}{(\lambda + \cos \omega)(1 - \cos \omega)} t + s_0 (\lambda + \cos \omega \omega_0) \frac{g \sin \theta \sin \omega dt}{(1 - \lambda^2) t + s_0 (\lambda + \cos \omega \omega_0)} \quad \#(A.1)$$

The integral of Equation (A.1), which is

$$\int_{\cos \omega_0}^{\cos \omega} \left[ \frac{1}{\lambda + \cos \omega} + \frac{1 - \lambda}{2(1 - \cos \omega)} \right] d(\cos \omega) = \int_0^t (1 - \lambda^2) g \sin \theta \sin \omega \cdot t \cdot (\lambda + \cos \omega \omega_0) \cdot s_0 \quad \#(A.2)$$

$$t = \frac{s_0 (\lambda + \cos \omega_0)}{g \sin \theta (1 - \lambda^2)} \left[ \frac{\sin \omega}{\sin \omega_0} \right] \frac{\sin \omega_0}{\sin \omega} \cdot \sin \omega \cdot \left( \tan \frac{\omega}{2} \right) \frac{\lambda}{\tan \frac{\omega}{2}} - 1 \quad \#(A.3)$$

Equations (4a–c) become

$$\dot{s} = \frac{1 + \cos \omega_0}{\lambda + \cos \omega} \quad \#(A.4a)$$

$$\dot{x} = \frac{1 + \cos \omega_0}{\lambda + \cos \omega} \sin \omega \quad \#(A.4b)$$

$$\dot{y} = \frac{1 + \cos \omega_0}{\lambda + \cos \omega} \cos \omega \quad \#(A.4c)$$

Dividing Equations (A.4a–c) by Equation (A.1) yields

$$\frac{ds}{d\omega} = \frac{s}{\dot{s}} = \frac{s_0^2 (1 + \cos \omega_0)^2}{g \sin \theta (1 + \cos \omega_0) \sin \omega} \cdot \frac{1}{(1 + \cos \omega_0)^2} \sin \omega \quad \#(A.5a)$$

$$\frac{dx}{d\omega} = \frac{x}{\dot{x}} = \frac{s_0^2 (1 + \cos \omega_0)^2}{g \sin \theta (1 + \cos \omega_0) \sin \omega} \cdot \frac{1}{(1 + \cos \omega_0)^2} \quad \#(A.5b)$$

$$\frac{dy}{d\omega} = \frac{y}{\dot{y}} = \frac{s_0^2 (1 + \cos \omega_0)^2}{g \sin \theta (1 + \cos \omega_0) \sin \omega} \cdot \cos \omega \quad \#(A.5c)$$

which can be integrated and yields a parametric equation for the trajectory of the block as follows:

$$s = - \frac{s_0^2 (1 + \cos \omega_0)^2}{4g \sin \theta} \left[ \frac{(\ln \tan \frac{\omega}{2} - \ln \tan \frac{\omega_0}{2}) + 4}{4} (\tan^2 \frac{\omega}{2} - \tan^2 \frac{\omega_0}{2}) + \frac{1}{4} \tan^2 \frac{\omega}{2} - \tan^4 \frac{\omega_0}{2} \right] \quad \#(A.6a)$$

$$x = - \frac{s_0^2 (1 + \cos \omega_0)^2}{6g \sin \theta} \left[ 3 \left( \tan \frac{\omega}{2} - \tan \frac{\omega_0}{2} \right) + 3 \tan^2 \frac{\omega}{2} - \tan^4 \frac{\omega_0}{2} \right] \quad \#(A.6b)$$

$$y = - \frac{s_0^2 (1 + \cos \omega_0)^2}{16g \sin \theta} \left[ 4 \left( \ln \tan \frac{\omega}{2} - \ln \tan \frac{\omega_0}{2} \right) - \tan^4 \frac{\omega}{2} - \tan^4 \frac{\omega_0}{2} \right] \quad \#(A.6c)$$
When $\lambda \neq 1$, Equation (8) can be written as

$$
\dot{\omega} = \frac{d\omega}{dt} = -\frac{g \sin \theta \sin \omega (\lambda + \cos \omega)}{g \sin \theta (1 - \lambda^2) t + s_0 (\lambda + \cos \omega_0)}
$$

$$
\Rightarrow -\frac{(1 - \lambda^2) \sin \omega d\omega}{(\lambda + \cos \omega)(1 - \cos^2 \omega)} = \frac{g \sin \theta (1 - \lambda^2) t + s_0 (\lambda + \cos \omega_0)}
$$

#(A.7)

The integral of Equation (A.7), which is

$$
\int_{\cos \omega_0}^{\cos \omega} \left[ \frac{1}{\lambda + \cos \omega} + \frac{1 - \lambda}{2(1 - \cos \omega)} - \frac{1 + \lambda}{2(1 + \cos \omega)} \right] d(\cos \omega) = \int_0^t \frac{(1 - \lambda^2) g \sin \theta}{(1 - \lambda^2) g \sin \theta \cdot t + (\lambda + \cos \omega_0) s_0} dt
$$

$$
t = \frac{s_0 (\lambda + \cos \omega_0)}{g \sin \theta (1 - \lambda^2)} \left[ \frac{(\lambda + \cos \omega_0) \sin \omega_0}{(\lambda + \cos \omega_0) \sin \omega_0} \left(\tan \frac{\omega}{\omega_0}\right)^\lambda - 1 \right] #(A.9)
$$

Equations (4a–c) become

$$
\dot{s} = \dot{s}_0 \sin \omega_0 \left(\tan \frac{\omega}{\omega_0}\right)^\lambda \frac{1}{\sin \omega} #(A.10a)
$$

$$
\dot{x} = \dot{s}_0 \sin \omega_0 \left(\tan \frac{\omega}{\omega_0}\right)^\lambda #(A.10b)
$$

$$
\dot{y} = \dot{s}_0 \sin \omega_0 \left(\tan \frac{\omega}{\omega_0}\right)^\lambda \cot \omega #(A.10c)
$$

Dividing Equations (A.10a–c) by Equation (A.7) yields

$$
\frac{ds}{d\omega} = \frac{\dot{s}}{\dot{\omega}} = -\frac{\dot{s}_0^2 \sin^2 \omega_0}{g \sin \theta \left(\tan \frac{\omega}{\omega_0}\right)^{2\lambda}} \left(\tan \frac{\omega}{2}\right)^{2\lambda} \left(\tan \frac{\omega}{2}\right)^{2\lambda} \left(\tan \frac{\omega}{2}\right)^{2\lambda} \sin^2 \omega #(A.11a)
$$

$$
\frac{dx}{d\omega} = \frac{\dot{x}}{\dot{\omega}} = -\frac{\dot{s}_0^2 \sin^2 \omega_0}{g \sin \theta \left(\tan \frac{\omega}{\omega_0}\right)^{2\lambda}} \left(\tan \frac{\omega}{2}\right)^{2\lambda} \left(\tan \frac{\omega}{2}\right)^{2\lambda} \left(\tan \frac{\omega}{2}\right)^{2\lambda} \sin^2 \omega #(A.11b)
$$

$$
\frac{dy}{d\omega} = \frac{\dot{y}}{\dot{\omega}} = -\frac{\dot{s}_0^2 \sin^2 \omega_0}{g \sin \theta \left(\tan \frac{\omega}{\omega_0}\right)^{2\lambda}} \left(\tan \frac{\omega}{2}\right)^{2\lambda} \left(\tan \frac{\omega}{2}\right)^{2\lambda} \cos \omega #(A.11c)
$$

which can be integrated and leads to the following parametric equation for the trajectory of the block:

$$
s = -\frac{\dot{s}_0^2 \sin^2 \omega_0}{4 g \sin \theta \left(\tan \frac{\omega_0}{2}\right)^{2\lambda}} \left[ \left(\tan \frac{\omega}{2}\right)^{2\lambda - 2} - \left(\tan \frac{\omega}{2}\right)^{2\lambda - 2} + \left(\tan \frac{\omega}{2}\right)^{2\lambda + 2} - \left(\tan \frac{\omega_0}{2}\right)^{2\lambda + 2} + \left(\tan \frac{\omega}{2}\right)^{2\lambda} - \left(\tan \frac{\omega_0}{2}\right)^{2\lambda} \right] \#(A.12)
$$
\[
x = \begin{cases} 
- \frac{s_0^2 \sin^2 \omega_0}{2 g \sin \theta \tan \omega_0} \left[ \left( \ln \frac{\omega}{2} - \ln \tan \frac{\omega_0}{2} \right) + \left( \tan \frac{\omega}{2} - \left( \tan \frac{\omega_0}{2} \right)^2 \right) \right], \lambda = 0.5 \\
- \frac{s_0^2 \sin^2 \omega_0}{2 g \sin \theta \left( \tan \frac{\omega_0}{2} \right)^{2\lambda}} \left[ \left( \tan \frac{\omega}{2} \right)^{2\lambda-1} - \left( \tan \frac{\omega_0}{2} \right)^{2\lambda-1} \right] + \left( \tan \frac{\omega}{2} \right)^{2\lambda+1} - \left( \tan \frac{\omega_0}{2} \right)^{2\lambda+1}, \lambda \neq 0.5 
\end{cases}
\]

\[
y = -\frac{s_0^2 \sin^2 \omega_0}{4 g \sin \theta \left( \tan \frac{\omega_0}{2} \right)^{2\lambda}} \left[ \left( \tan \frac{\omega}{2} \right)^{2\lambda-2} - \left( \tan \frac{\omega_0}{2} \right)^{2\lambda-2} \right] - \left( \tan \frac{\omega}{2} \right)^{2\lambda+2} - \left( \tan \frac{\omega_0}{2} \right)^{2\lambda+2} 
\]

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