Neutral Color Superconductivity Including Inhomogeneous Phases at Finite Temperature

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We investigate neutral quark matter with homogeneous and inhomogeneous color condensates at finite temperature in the frame of an extended NJL model. By calculating the Meissner masses squared and gap susceptibility, the uniform color superconductor is stable only in a temperature window close to the critical temperature and becomes unstable against LOFF phase, mixed phase and gluonic phase at low temperatures. The introduction of the inhomogeneous phases leads to disappearance of the strange intermediate temperature 2SC/g2SC and changes the phase diagram of neutral dense quark matter significantly.

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I. INTRODUCTION

The emergence of gapless color superconductivity[1–4] promoted great interest in the study of dense quark matter[5, 6]. A crucial problem is whether the two flavor gapless (g2SC) or breached pairing[7] color superconductivity is stable. The first instability is the thermodynamical instability when the chemical potential difference between the two species is fixed. This is called Sarma instability[8]. It is now widely accepted that the Sarma instability can be cured under charge neutrality constraint due to the long range electromagnetic gauge interaction[2, 3]. The second instability is the chromomagnetic instability described by negative Meissner mass squared[9–13] or negative superfluid density[14, 15]. In this case, some inhomogeneous condensed phases are energetically favored[16–23]. A specific character of the chromomagnetic instability is the negative Meissner masses squared for the 4-7th gluons in the gapped 2SC phase[24]. The third instability is the instability against the phase separation or mixed phase when the surface energy and long range gauge interaction are excluded.

All these instabilities are confirmed only at zero temperature and in the weak coupling region. While it is recently argued that the breached pairing phase will be free from negative superfluid density or Meissner mass squared in the strong coupling BEC region[25–28], the diquark condensate at moderate baryon density seems not in this region. As for the temperature effect, in the study of isospin asymmetric nuclear matter[29, 30], two flavor color-superconducting quark matter[31] and general breached pairing superfluidity[32], the fermion pairing correlation behaviors strangely: The superfluid/superconductor order parameter is not a monotonously decreasing function of temperature and its maximum is located at finite temperature. Especially, for large enough density asymmetry between the two species at fixed coupling or for weak enough coupling at fixed density asymmetry, the superfluidity appears at intermediate temperature but disappears at low temperature. This strange temperature behavior is quite universal and also discussed in the study of atomic Fermi gas with density imbalance[32–34]. It is found[32, 34] that the superfluid density and number susceptibility of the uniform superfluid phase are positive only in a temperature window near the critical temperature. From the recent instability analysis at finite temperature[32], if LOFF phase is taken into account, this strange temperature behavior will be washed out and the superfluid order parameter remains a monotonous function of temperature. Therefore, for the color superconductivity at moderate density, we may have the following estimations: (1) The superconductor is free from the chromomagnetic instability and also stable against the mixed phase at temperatures near the critical value. (2) The phase diagram of neutral dense quark matter is significantly changed if some non-uniform phase such as LOFF is taken into account. In this paper we will show that these two estimations are true.

The paper is organized as follows. In Section II we study a naive two-species model which possesses the basic mechanism for positive Meissner mass squared and gap susceptibility at finite temperature. We then calculate in an extended NJL model the Meissner masses squared for gluons and the gap susceptibility in two flavor color-superconducting quark matter with charge neutrality in Section III and investigate the LOFF phase and gluonic phase at low temperatures. The introduction of the inhomogeneous phases leads to disappearance of the strange intermediate temperature 2SC/g2SC and changes the phase diagram of neutral dense quark matter significantly.

II. RESULTS IN A TOY MODEL

In this section we study a toy model containing two species of fermions proposed in[11, 13, 26]. While this model only reveals the chromomagnetic instability for the
Nambu-Gorkov spinors defined as \( \Psi = \left( \begin{array}{c} \bar{b} \rho \tau_1 \psi \end{array} \right) \) (1), where \( \psi = (\psi_1, \psi_2)^T \) and \( \bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2) \) are Dirac spinors including two species of fermions, \( \psi^C = C \bar{\psi}^T \) and \( \bar{\psi}^C = \psi^T C \) are charge-conjugate spinors, \( C = \sigma^2 \sigma^0 \) is the charge conjugation matrix, \( m \) the fermion mass, \( \bar{G} \) is the attractive coupling, and \( \tau_1 \) is the first Pauli matrix in the two-species space. In the following we take \( m = 0 \) which corresponds to the quark matter at high baryon density. It is quite convenient to work with the thermodynamic potential can be evaluated as \( \Omega = \frac{\Delta^2}{4G} - \frac{T^2}{2} \sum\limits_n \int \frac{d^3p}{(2\pi)^3} \text{Tr} \ln S^{-1}(i\omega_n, \vec{p}) \) (3) with the mean field fermion propagator in the 4-dimensional Nambu-Gorkov \( \otimes \) two-species space, \[ S^{-1}(i\omega_n, \vec{p}) = \left[ \begin{array}{cc} G_0^+ & i\gamma_5 \Delta \tau_1 \end{array} \right] \left[ \begin{array}{c} G_0^- \end{array} \right]^{-1}, \] \[ \text{ where } \left[ G_0^+ \right]^{-1} = (i\omega_n \pm \mu)\gamma_0 - \vec{\gamma} \cdot \vec{p} \text{ are the inverse of free fermion propagators with frequency } \omega_n = (2n + 1)\pi T, \] \[ n = 1, 2, \ldots \] \[ \text{ We have introduced the chemical potentials } \mu_1 \text{ and } \mu_2 \text{ for the two species via adding the term } L_\mu = \bar{\psi} \mu \gamma^0 \psi \text{ to the original Lagrangian with } \mu = \text{ diag}(\mu_1, \mu_2). \] \[ \text{ After a straightforward calculation, the thermodynamic potential can be evaluated as } \] \[ \Omega = \frac{\Delta^2}{4G} - \sum\limits_{l=1}^4 \int \frac{d^3p}{(2\pi)^3} W(E_l) \] \[ \text{ with } W(E_l) = E_l + 2T \ln \left( 1 + e^{-E_l/T} \right), \] \[ \text{ where the sum runs over all fermionic quasiparticles. The quasiparticle dispersions } E_l(p) \text{ calculated via } \] \[ \text{ det } S^{-1}(p) = 0 \text{ are given by } \] \[ E_1(p) = E^-_\Delta - \delta \mu, \quad E_2(p) = E^+_\Delta + \delta \mu, \] \[ E_3(p) = E^-_\Delta + \delta \mu, \quad E_4(p) = E^+_\Delta + \delta \mu, \] where we have defined the notation \( E^\pm_\Delta = \sqrt{(p \pm \hat{\mu})^2 + \Delta^2} \) and introduced the average chemical potential \( \hat{\mu} = (\mu_1 + \mu_2)/2 \) and the chemical potential mismatch \( \delta \mu = (\mu_2 - \mu_1)/2 \). The gap equation for \( \Delta \) can be derived via minimizing the thermodynamic potential, \[ \frac{\partial \Omega}{\partial \Delta} = 0 \] and the fermion numbers are determined by the definition, \[ n_1 = -\partial \Omega/\partial \mu_1, \quad n_2 = -\partial \Omega/\partial \mu_2. \] The order parameter \( \Delta \) under constraint of fixed numbers \( n_1 \) and \( n_2 \) can be calculated by solving the gap equation (7) together with the number equations (8). Since the gap equation for \( \Delta \) suffers ultraviolet divergence, we introduce a three momentum cutoff \( \Lambda \) to regularize it.

To model the two flavor dense quark matter with beta equilibrium and charge neutrality, we choose the coupling constant \( G \leq 0.1G_0 \) with \( G_0 = 4\pi^2/\Lambda^2 \) to ensure weak coupling and fix the averaged chemical potential \( \bar{\mu} = 0.615\Lambda \) and the number ratio \( n_1/n_2 = 2 \). In Fig.1 we show the temperature behavior of \( \Delta \) and \( \delta \mu \). For the coupling \( G = 0.095G_0 \), the gapless state satisfying the condition \( \delta \mu > \Delta \) appears at low and high temperature regions but disappear at intermediate temperature. For weaker couplings \( G = 0.08G_0 \) and \( 0.073G_0 \), the superfluid is always gapless. For weak enough coupling such as \( G = 0.073G_0 \), the superfluidity appears only at intermediate temperature.

For a symmetric system with \( n_1 = n_2 \), it is well-known that the temperature effect destroys the pairing, and the pairing gap is a monotonously decreasing function of temperature. However, for an asymmetric system with \( n_1 \neq n_2 \), the two species of fermions have mismatched Fermi surfaces, and the temperature effect not only deforms and reduces the Fermi surfaces which melts the gap, but also makes the overlap region of the two species wider which favors the pairing. The competition of the two opposite effects results in a non-monotonous temperature behavior of the pairing gap. This is the reason why in the low temperature region in Fig.1 the gap increases as the temperature increases and even disappears when the coupling is small enough.

### A. Meissner Mass

Let us assume that the fermions couple to a \((U(1) \otimes U(1))\) gauge field \( A_\mu \) with coupling \( g \), the superfluid becomes a superconductor where the gauge field obtain the so called Meissner mass. The Meissner mass can be calculated from the polarization tensor \[ \Pi^{\nu\nu}(k) = \frac{T}{2} \sum\limits_n \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left[ \hat{\Gamma}^{\nu} S(p) \hat{\Gamma}^{\nu} S(p - k) \right] \] with the vertex \( \hat{\Gamma}^{\nu} = g \gamma^\nu \gamma_5 \otimes \tau_0 \), where \( \gamma_3 \) is the third Pauli matrix in the two-species space, \( \tau_0 \) is the identity.
matrix in the Nambu-Gorkov space, and the propagator \( S \) can be obtained from its inverse \( \Pi \) by taking the technics used in \( [15] \). The Meissner mass is defined as

\[
m_A^2 = \lim_{\bar{p} \to 0} \left( \delta \mu - k_i k_j / |k|^2 \right) \Pi^{ij}(k_0 = 0, \bar{k}).
\]

The calculation of the Meissner mass is quite straightforward but tedious, we quote here only the final result,

\[
m_A^2 = m_d^2 + m_p^2,
\]

the diamagnetic term \( m_d^2 \) and paramagnetic term \( m_p^2 \) are explicitly expressed as

\[
m_d^2 = \frac{4g^2}{3} \int \frac{d^3 \bar{p}}{(2\pi)^3} \left[ \chi^-(p) (f(E_1) + f(E_2) - 1) + \chi^+(p) (f(E_3) + f(E_4) - 1) \right] + m_A^2,
\]

\[
m_p^2 = \frac{g^2}{3} \int \frac{d^3 \bar{p}}{(2\pi)^3} \sum_{l=1}^{4} f'(E_l),
\]

where \( f(x) \) is the Fermi-Dirac distribution function, \( f'(x) \) is its first order derivative, and the functions \( \chi^\pm(p) \) are defined as

\[
\chi^\pm(p) = \pm \frac{(E_\Delta^\pm)^2 - E_0^\pm E_0^\mp + \Delta^2}{(E_\Delta^\pm)^2 - (E_\Delta^\pm)^2} \approx \frac{1}{2p E_\Delta^\pm}
\]

with \( E_0^\pm = p \pm \bar{\mu} \). Since the diamagnetic term suffers ultraviolet divergence, it contains a subtraction term

\[
m_A^2 = \frac{4g^2}{3} \int \frac{d^3 \bar{p}}{(2\pi)^3} \left[ \frac{1}{p} \right] = \frac{g^2 \Lambda^2}{3\pi^2}
\]

to ensure \( m_A^2 (\Delta = 0) = 0 \).

Above the critical temperature \( T_c \) where the pairing gap \( \Delta \) vanishes, we have \( \chi^\pm(p) = 1/(2p) \) and

\[
m_d^2 = \frac{2g^2}{3} \int \frac{d^3 \bar{p}}{(2\pi)^3} \sum_{\mu=\pm} \left[ f(p - \mu) + f(p + \mu) \right],
\]

\[
m_p^2 = \frac{g^2}{3} \int \frac{d^3 \bar{p}}{(2\pi)^3} \sum_{\mu=\pm} \left[ f'(p - \mu) + f'(p + \mu) \right].
\]

Taking partial integration we find

\[
\frac{1}{3} \int \frac{d^3 \bar{p}}{(2\pi)^3} f'(p \pm \mu) = -\frac{2}{3} \int \frac{d^3 \bar{p}}{(2\pi)^3} f(p \pm \mu)
\]

for any \( T \) and \( \mu \). Therefore, the total Meissner mass \( m_A^2 \) is zero in the normal phase, as it is expected.

In weak coupling with \( \Delta < \bar{\mu} \) and \( \delta \mu \ll \bar{\mu} \), the Meissner mass squared at zero temperature can be well approximated as \( [9] \)

\[
m_A^2 \approx \frac{g^2 \bar{\mu}^2}{3\pi^2} \left[ 1 - \frac{\delta \mu \theta(\delta \mu - \Delta)}{\sqrt{\delta \mu^2 - \Delta^2}} \right].
\]

In the gapless phase with \( \Delta < \delta \mu \) the negative Meissner mass squared is quite different from \( m_A^2 = g^2 \bar{\mu}^2 / 3\pi^2 > 0 \) in the symmetric BCS case where there is no paramagnetic contribution.

Now the question is: Is the Meissner mass squared always negative below the critical temperature \( T_c \)? To answer this question, we derive the critical behavior of the Meissner mass squared at \( T_c \). Since the gap equation for \( \Delta \) below \( T_c \) contains only \( \Delta^2 \) and \( \Delta \) continuously approaches to zero at \( T_c \), we have the critical behavior for the order parameter

\[
\Delta(T) \propto (1 - T/T_c)^{3/2}, \quad T \to T_c.
\]
As a consequence, the Meissner mass squared behaviors as
\[ m_A^2(T) \propto 1 - T/T_c > 0, \quad T \to T_c. \] (19)

Combining \( m_A^2 < 0 \) at zero temperature and \( m_A^2 > 0 \)
for \( T \) approaching to \( T_c \), \( m_A^2 \) should be negative at low
temperature but positive in a temperature window close to \( T_c \).

In Fig.2 we show the temperature behavior of the
Meissner mass squared for three coupling values. In any
case, there exists a temperature window close to the criti-
tical temperature where the Meissner mass squared \( m_A^2 \)
is positive which means that the gapless phase is magneti-
cally stable and hence stable against the LOFF phase.

![Graphs showing temperature behavior of Meissner mass squared](image)

**FIG. 2**: The Meissner mass squared \( m_A^2 \) scaled by \( m_0^2 = g^2 \bar{\mu}^2 / 3\pi^2 \) and the gap susceptibility \( \kappa_\Delta \) scaled by \( \kappa_0 = 2 \bar{\mu}^2 / \pi^2 \) as functions of \( T/\Lambda \) for three coupling values.

### B. Density Fluctuations

The general stability condition against changes in num-
ber densities for a two-component system is described by
the total free energy of the system\[46, 47\],
\[ F = \int d^3x \mathcal{F}(n_\sigma(x)) [\sigma = 1, 2]. \]
Combining its fluctuations induced by small number changes \( \delta n_\sigma(x) \), the first-order variation \( \delta F \) vanishes automatically due to the charge
conservation, \( \int d^3x \delta n_\sigma(x) = 0 \), and the second-order variation \( \delta^2 F \) is expressed in the quadratic form
\[ \delta^2 F = \frac{1}{2} \int d^3x \sum_{\sigma, \sigma'} \frac{\partial^2 F}{\partial n_\sigma \partial n_{\sigma'}} \delta n_\sigma \delta n_{\sigma'}. \] (20)

Therefore, to achieve a stable homogeneous phase, the
\( 2 \times 2 \) matrix \( \partial^2 F / \partial n_\sigma \partial n_{\sigma'} \) should be positively definite, namely, it has only positive eigen values. From the relation
between the free energy \( F(\mu_\sigma) \) and thermodynamic potential \( \Omega(\mu_\sigma) \), \( \Omega = F + \mu_1 n_1 + \mu_2 n_2 \), it is easy to check that the stability condition to have positively def-
definite matrix \( \partial^2 F / \partial n_\sigma \partial n_{\sigma'} \) is equivalent to the condition
to have positively definite number susceptibility matrix
\( -\partial^2 \Omega / \partial \mu_\sigma \partial \mu_{\sigma'} \).

For systems without mass difference between the two species, the condition to have positive eigenvalues of \( -\partial^2 \Omega / \partial \mu_\sigma \partial \mu_{\sigma'} \) can be reduced to the condition\[23\] that the imbalance number susceptibility
\( \chi = - (\partial^2 \Omega / \partial \mu^2)_\mu = (\partial \delta n / \partial \delta \mu)_\mu \) should be positive.
For \( \chi < 0 \), the density difference \( \delta n = n_1 - n_2 \) increases
with decreasing chemical potential difference \( \delta \mu \), which
is certainly unphysical and means that the uniform phase
is unstable against density fluctuations. Employing the gap
equation which determines the condensate as a function of
chemical potentials, we can express the imbalance number susceptibility \( \chi \) as a direct and an indirect part,
\[ \chi = \frac{(\partial \delta n)}{(\partial \delta \mu)}_{\bar{\mu}, \Delta} + \left( \frac{(\partial \delta n)}{(\partial \Delta)}_{\bar{\mu}, \delta \mu} \right) \left( \frac{(\partial \Delta)}{(\partial \delta \mu)}_{\bar{\mu}} \right). \] (21)

From the expression
\[ \delta n = \int \frac{d^3 \bar{\mu}}{(2\pi)^3} \left[ f(E_1) - f(E_2) + f(E_3) - f(E_4) \right] \] (22)
and the gap equation \( (\partial \Omega / \partial \Delta)_{\bar{\mu}, \delta \mu} = 0 \), we have\[24\]
\[ \chi = \left( \frac{(\partial \delta n)}{(\partial \delta \mu)}_{\bar{\mu}, \Delta} \right) + \left( \frac{(\partial \delta n)}{(\partial \Delta)}_{\bar{\mu}, \delta \mu} \right)^2 \left( \frac{(\partial^2 \Omega)}{(\partial \Delta^2)}_{\bar{\mu}, \delta \mu} \right)^{-1}. \] (23)

Since \( (\partial \delta n / \partial \delta \mu)_{\bar{\mu}, \Delta} \) is always positive, the stability condi-
tion \( \chi > 0 \) is controlled by the gap susceptibility
\( \kappa_\Delta = (\partial^2 \Omega / \partial \Delta^2)_{\bar{\mu}, \delta \mu} \) which determines if the solution of
the gap equation is the minimum of the thermodynamic
potential.

The gap susceptibility \( \kappa_\Delta \) can be explicitly evaluated
as

\[ \kappa_{\Delta} = 2 \int \frac{d^3 \bar{\rho}}{(2\pi)^3} \left( \frac{\Delta^2}{(E_{\Delta}^2)^2} \left( 1 - f(E_1) - f(E_2) \right) \right. \\
\left. + f'(E_1) + f'(E_2) \right) + \frac{\Delta^2}{(E_{\Delta}^2)^2} \left( 1 - f(E_3) - f(E_4) \right) \\
\left. + f'(E_3) + f'(E_4) \right), \quad (24) \]

where we have considered the gap equation for the condensate \( \Delta \). At weak coupling and at \( T = 0 \), \( \kappa_{\Delta} \) can be evaluated as

\[ \kappa_{\Delta} \approx \frac{2\mu_{\Delta}}{\pi^2} \left[ 1 - \frac{\delta \mu \theta (\delta \mu - \Delta)}{\sqrt{\delta \mu^2 - \Delta^2}} \right] \quad (25) \]

which leads to \( \chi < 0 \) in the gapless phase. Note that while the Sarma instability \( \kappa_{\Delta} < 0 \) can be cured via charge neutrality constraint, the instability \( \kappa_{\Delta} < 0 \) induced by density fluctuations can not be removed by charge neutrality. Similar to the Meissner mass squared, \( \kappa_{\Delta} \) near \( T_c \) takes the form \( \kappa_{\Delta} \propto 1 - T/T_c > 0 \). Therefore, the uniform superfluid phase is stable against density fluctuations at temperature close to \( T_c \).

The temperature behavior of the gap susceptibility \( \kappa_{\Delta} \) is illustrated in Fig. 4. In any case, there exists a temperature window close to the critical temperature where \( \kappa_{\Delta} \) is positive which means that the gapless phase is magnetically stable and hence stable against the phase separation. Note that the stable region against the LOFF phase is larger than the stable region against the phase separation.

### III. NEUTRAL 2SC/G2SC PHASE

We investigate in this section the two flavor color superconductivity in an extended NJL model. The Lagrangian density of the model including quark-quark interaction sector is defined as

\[ \mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu - m_0) \psi + G_S \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \tau \psi)^2 \right] + G_D \left[ (\bar{\psi} C i \gamma^5 \epsilon \epsilon^\dagger \psi) \right], \quad (26) \]

where \( G_S \) and \( G_D \) are, respectively, coupling constants in color singlet and anti-triplet channels, the quark field \( \psi_i \), with flavor index \( i \) and color index \( \alpha \) is a flavor doublet and a color triplet as well as a four-component Dirac spinor, \( \tau = (\tau_1, \tau_2, \tau_3) \) are Pauli matrices in flavor space, and \( (\epsilon)_{ij} \equiv \epsilon^{ij} \) and \( (\epsilon^\dagger)_{\alpha\beta} \equiv \epsilon^{\alpha\beta} \) are, respectively, totally antisymmetric tensors in flavor and color spaces. We focus in the following on the color symmetry breaking phase with nonzero diquark condensates defined as

\[ \phi_1 = -2G_D \langle \bar{\psi} C i \gamma_5 \epsilon \epsilon^\dagger \psi \rangle, \]
\[ \phi_2 = -2G_D \langle \bar{\psi} i \gamma_5 \epsilon \epsilon^\dagger \psi \rangle, \quad \gamma = 1, 2, 3. \quad (27) \]

To ensure color and electric neutrality, one should introduce a set of color chemical potentials \( \mu_a \) with respect to color charges \( Q_1, Q_2, ..., Q_8 \) and an electric chemical potential \( \mu_e \) with respect to the electric charge \( Q_e \). The ground state of the system is determined by minimizing the thermodynamical potential, \( \partial \Omega / \partial \delta \mu \epsilon = 0 \) under the charge neutrality constraint \( Q_e = 0 \) and \( Q_a = 0 \) (for \( a = 1, 2, ..., 8 \)).

Since the model Lagrangian is invariant under the color \( SU(3) \) transformation, we can choose a specific color symmetry breaking direction. The most convenient choice is

\[ \phi_1 = \phi_2 = 0, \quad \phi_3 \equiv \Delta \neq 0. \quad (28) \]

To simplify the calculation, we consider the chiral limit with \( m_0 = 0 \) and assume chiral symmetry restoration in the color superconducting phase. This assumption is confirmed when the coupling constant \( G_D \) in the diquark channel is not too large. In the specific case \( Q_2, Q_4, ..., Q_7 \) vanish automatically and only \( Q_8 \) can be nonzero. Therefore, one can introduce the color chemical potential \( \mu_8 \) only, and the quark chemical potential matrix elements can be expressed as

\[ \mu_{ij}^{\alpha\beta} = (\mu \delta_{ij} - \mu_e Q_{ij}) \delta_{\alpha\beta} + \frac{2}{\sqrt{3}} \mu_8 \delta_{ij} (T_8)_{\alpha\beta}. \quad (29) \]

In mean field approximation the thermodynamical potential \( \Omega \) of the system can be expressed as

\[ \Omega = \frac{\Delta^2}{4G_D} - \frac{T}{2} \sum_i \int \frac{d^3 \bar{\rho}}{(2\pi)^3} \text{Tr} \ln S^{-1} + \Omega_e, \quad (30) \]

where \( \Omega_e \) is the contribution from the free electron gas

\[ \Omega_e = -\frac{1}{12\pi^2} \left( \mu_e^4 + 2\pi^2 \mu_e^2 T^2 + \frac{7\pi^4}{15} T^4 \right). \quad (31) \]

The inverse of the quark propagator \( S^{-1} \) in the Nambu-Gorkov space can be written as

\[ S^{-1} = \left( \begin{array}{c} [G_0^+]^{-1} \\
-i\gamma_5 \epsilon^3 \Delta \\
-G_0^{-1} \end{array} \right) \quad (32) \]

with \( [G_0^+]^{-1} = (i\omega_n \pm \epsilon_{ij} \gamma_0 - \gamma_5 \bar{p} \gamma_\tau) \). After a straightforward algebra \( \Omega \) can be evaluated as

\[ \Omega = \frac{\Delta^2}{4G_D} - \sum_i N_i \int \frac{d^3 \bar{\rho}}{(2\pi)^3} W(E_i) + \Omega_e, \quad (33) \]

where the sum runs over all quasiparticles. The quasiparticle dispersions \( E_i(p) \) calculated by \( \text{det} S^{-1}(p) = 0 \) are given by \( (3) \) and

\[ E_3(p) = p + \mu_a, \quad E_6(p) = p - \mu_a, \]
\[ E_7(p) = p + \mu_3, \quad E_8(p) = p - \mu_3, \quad (34) \]

with the chemical potential mismatch \( \delta \mu = \mu_e/2 \) and averaged chemical potential \( \bar{\mu} = \mu - \mu_e/6 + \mu_8/3 \) for
paired quarks, where $\mu = \mu_B/3$ is related to the baryon chemical potential $\mu_B$. The degenerate factor $N_I$ is 2 for $I = 1, 2, 3, 4$ and 1 for $I = 5, 6, 7, 8$. Minimizing the thermodynamic potential $\Omega$,
\[
\frac{\partial \Omega}{\partial \Delta} = 0,
\]
and considering the charge neutrality conditions
\[
\frac{\partial \Omega}{\partial \mu_8} = 0, \quad \frac{\partial \Omega}{\partial \mu_e} = 0,
\]
we can determine simultaneously the order parameter $\Delta$ and chemical potentials $\mu_e$ and $\mu_8$ in the neutral uniform color superconductor. At weak coupling, the color chemical potential $\mu_8$ is only a few MeV, and the electric charge neutrality plays the role of the condition $n_1/n_2 = 2$ in the toy two-species model.

The numerical solutions of the condensate $\Delta$ and chemical potential mismatch $\delta \mu$ are demonstrated in Fig. 3 at fixed quark chemical potential $\mu = 400$ MeV for several values of coupling $G_D$. There are three parameters in the model. The momentum cutoff $\Lambda$ and coupling $G_S$ can be fixed by fitting the pion decay constant and chiral condensate in the vacuum\[9\], and we denote the coupling $G_D$ by the ratio $\eta = G_D/G_S$. For weak coupling such as $\eta = 0.75, 0.70, 0.66$, the temperature behavior of $\Delta$ and $\delta \mu$ is similar to that in the toy two-species model.

The uniform color superconducting phase disappears at the critical coupling $\eta_c \simeq 0.63$. For strong coupling such as $\eta = 0.85$, the quark matter is in gapped phase in a wide temperature region and in gapless phase only in a small window close to $T_c$. Again, the strange behavior of the gap in the low temperature region for small $\eta$ is due to the competition of the two opposite temperature effects for pairings with mismatched number densities.

### A. Meissner Mass

The Meissner masses for gluons and photon can be evaluated via the polarization tensor\[9\]
\[
\Pi^{\mu\nu}_{ab}(k) = \frac{T}{2} \sum_n \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left[ \tilde{\Gamma}_a S(p) \tilde{\Gamma}_b S(p-k) \right],
\]
where the vertex in the color-flavor space is defined as $\tilde{\Gamma}_a = \text{diag} (g_\gamma \gamma^\mu T_a, -g_\gamma \gamma^\mu T_a^T)$ for $a = 1, \ldots, 8$ and $\tilde{\Gamma}_8 = \text{diag} (e\gamma^\mu Q, -e\gamma^\mu Q)$ for $a = \gamma$. In the 2SC/g2SC phase with the conventional choice \[28\], the diquark condensate breaks the gauge symmetry group $SU(3)_c \otimes U(1)_{em}$ down to the $SU(2)_c \otimes \tilde{U}(1)_{em}$ subgroup. Therefore, we need to calculate the Meissner masses for the 4-8th gluons and the photon only. Most of the analytic calculation is presented in \[9\]. With the function $A, A', B, C, D, H, J$.
listed in Appendix A, we obtain

\[ m_{\gamma}^2 = \frac{4 \alpha_s}{9 \pi} \int dp \, p^2 [A + A' - 2B + 4(C - D)], \]  
\[ m_{\gamma\gamma}^2 = \frac{4 \alpha_s}{9 \pi} \int dp \, p^2 [4A + A' + 4B + 2(5C + 4D)], \]  
\[ m_{S8}^2 = \frac{8 \sqrt{\alpha_s \alpha_s}}{9 \sqrt{3} \pi} \int dp \, p^2 [2A - A' - B + 2(C - D)] \]

with \( \alpha_s = g_s^2 / (4\pi) \) and \( \alpha_c = e^2 / (4\pi) \) for the 8th gluon and photon and

\[ m_4^2 = m_{44}^2 = m_{55}^2 = m_{66}^2 = m_{77}^2 = \frac{4 \alpha_s}{3\pi} \int dp \, p^2 (H + 2J) \]

for the 4-7th gluons. Due to the nonzero \( m_{\gamma\gamma}^2 \), the 8th gluon and the photon mix with each other, and the physical Meissner masses squared are given by the eigen values

\[ m_{S8}^2 = \frac{1}{2} \left[ m_{S8}^2 + m_{\gamma\gamma}^2 + \sqrt{(m_{S8}^2 - m_{\gamma\gamma}^2)^2 + 4m_s^4} \right], \]
\[ m_{S4}^2 = \frac{1}{2} \left[ m_{S8}^2 + m_{\gamma\gamma}^2 - \sqrt{(m_{S8}^2 - m_{\gamma\gamma}^2)^2 + 4m_s^4} \right]. \]

When the coupling is not very large, we have approximately \( m_{S8}^2 \simeq 0 \), which is consistent with the analysis of symmetry breaking in weak coupling limit. Since \( \alpha_s \sim 1 >> \alpha_c \simeq 1/137 \), we have

\[ m_8^2 \equiv m_{S8}^2 \simeq m_{S8}^2. \]

With the explicit form of the functions \( A, A', B, C, D \) shown in Appendix A, \( m_8^2 \) takes the same expression as \( m_A^2 \) in the toy two-species model,

\[ m_8^2 = \frac{2}{3} m_A^2, \]

and therefore, \( m_4^2 \) and \( m_A^2 \) have exactly the same temperature behavior. Now the most important task is to investigate whether \( m_4^2 \) can be positive at finite temperature. For \( T \rightarrow T_c \), we have \( \mu_s \rightarrow 0 \) and the functions \( H \) and \( J \) are reduced to

\[ H = \sum_{i=0}^{d} [f(p - \mu_i) + f(p + \mu_i)], \quad J = H/p \]

with \( \mu_i = \mu - 2\mu_c / 3 \) and \( \mu_d = \mu + \mu_c / 3 \). Employing the same trick used in Section III we find \( m_4^2 = 0 \) for \( T \geq T_c \) and

\[ m_4^2(T) \propto 1 - T/T_c, \quad T \rightarrow T_c \]

for \( T \) below but close to \( T_c \). Therefore, there must exist a temperature window near the critical temperature where the two flavor color superconductor is free from chromomagnetic instability.

In Fig 4, we illustrate the temperature behavior of the Meissner masses squared \( m_4^2 \) and \( m_8^2 \) at fixed chemical potential \( \mu = 400 \) MeV for several coupling values. For weak coupling \( 0.62 < \eta < 0.8 \), there exist two intermediate temperatures \( T_4 \) and \( T_8 \), \( m_4^2 \) is negative at \( 0 < T < T_4 \) and positive at \( T_4 < T < T_c \), and \( m_8^2 \) is negative at \( 0 < T < T_8 \) and positive at \( T_8 < T < T_c \). In a wide range of coupling, \( T_4 \) is larger than \( T_8 \). Only for sufficiently small coupling \( \eta < 0.66 \), \( T_4 \) coincides with \( T_8 \) or even becomes less than \( T_8 \). For strong coupling \( \eta > 0.8 \),
the 8th gluon is free from chromomagnetic instability at any temperature, but the 4-7th gluons suffer negative Meissner mass squared in the low temperature region $0 < T < T_4$ which indicates the instability against the gluonic phase\cite{21}.

B. Density Fluctuation

Like the toy two-species model, we can study the stability of color superconductivity against the number fluctuations of the paired quarks\cite{49}. Similarly, the stability condition can be reduced to $\chi = (\partial \delta n / \partial \delta \mu)_\mu > 0$ where $\delta n = n_{d1} - n_{u2} = n_{d2} - n_{u1}$ is the density imbalance between the paired quarks. Using the techniques in Section II, the gap susceptibility $\kappa_\Delta$ which controls the stability takes almost the same form (24), the only difference is the replacement of the factor of 2 in front of the momentum integration in (24) by the factor of 4.

In Fig.5 we show the temperature behavior of the gap susceptibility $\kappa_\Delta$ for four coupling values in the diquark channel. In any case, there indeed exists a temperature window where $\kappa_\Delta$ is positive and therefore the gapless phase is magnetically stable and hence stable against the phase separation. Recently, it is argued that the Higgs instability\cite{39} indicates the instability of 2SC/g2SC against the mixed phase. The gap susceptibility is the long wavelength limit of the Higgs instability. For a complete study we may need to check the Higgs instability at finite temperature.

IV. INHOMOGENEOUS PHASES

We have calculated the Meissner masses squared and the gap susceptibility in the neutral 2SC/g2SC phase, and found that there exists a temperature window close to the critical temperature $T_c$ where they are both positive. Therefore, the uniform phase is free from the chromomagnetic instability and stable against the mixed phase in this temperature region. In general case, the stable region for the 4-7th gluons is smaller than the stable region for the 8th gluon, namely $T_4 > T_8$. Only for sufficiently small coupling $\eta < 0.66$, the 4-7th gluons can have larger stable region than the 8-th gluon, namely $T_4 \leq T_8$. For large enough coupling, the 8th gluon is free from the chromomagnetic instability at any temperature and there exists only a turning temperature $T_4$ where $m_4^2$ changes sign.

If $T_4 > T_8$, the single-plane wave LOFF phase can not completely cure the chromomagnetic instability, since there exists a region $T_4 < T < T_8$ where only $m_4^2$ is negative. In this section we will focus on possible inhomogeneous phases in two limits, very weak coupling and very strong coupling. For very weak coupling, we have $T_4 < T_8$, it has been shown in \cite{22} that the neutral LOFF phase is free from the chromomagnetic instability at least at $T = 0$. For very strong coupling, only 4-7th gluons suffer instability, the introduction of gluonic phase may be sufficient to cure the instability.
A. LOFF Phase

We firstly discuss the LOFF state. The numerical calculation will be performed at very weak coupling $\eta \leq 0.06$. The order parameters $\phi_+$ and $\phi_-$ are complex conjugate to each other and can be set to be real only in a uniform color superconductor. In the LOFF phase the phase factor of the order parameter is non-uniform. For the sake of simplicity we take the following single wave LOFF ansatz

$$\phi_+ = \Delta e^{2i\vec{q} \cdot \vec{x}}, \quad \phi_-^* = \Delta e^{-2i\vec{q} \cdot \vec{x}}. \quad (45)$$

It has been shown that in the two flavor color superconductor the LOFF momentum $\vec{q}$ can be regarded as the 8th gluon condensate\cite{22} or a spontaneously generated Nambu-Goldstone current\cite{19}, namely

$$\vec{q} = \frac{1}{2\sqrt{3}} g \langle \hat{A}_8 \rangle = \langle \nabla \varphi_8 \rangle. \quad (46)$$

With a transformation for the quark fields,

$$\chi(\vec{x}) = \psi(\vec{x}) e^{-iq \cdot x}, \quad \chi^C(\vec{x}) = \psi^C(\vec{x}) e^{iq \cdot x}, \quad (47)$$

we can express the thermodynamic potential as

$$\Omega = \frac{\Delta^2}{4G_D} - \frac{T}{2} \sum_n \int \frac{d^3p}{(2\pi)^3} \text{Tr} \ln S^{-1}_q(i\omega_n, \vec{p}) \quad (48)$$

with the inverse of the mean field propagator in LOFF state

$$S^{-1}_q(i\omega_n, \vec{p}) = \begin{pmatrix} G^+_q \mu + \bar{\gamma}_0 - \epsilon \gamma^3 \Delta & -i\gamma^5 \epsilon \gamma^3 \Delta \\ -i\gamma^5 \epsilon \gamma^3 \Delta & [G_q^-]^{-1} \end{pmatrix}, \quad (49)$$

where the $q$ dependent free propagators are defined as

$$[G^\pm_q]^{-1} = (i\omega_n \pm \mu) \gamma_0 - \gamma^5 (\vec{p} \pm \vec{q}). \quad (50)$$

The neutral LOFF state should be determined self-consistently by the gap equations for the condensate and pair momentum and the charge neutrality constraints,

$$\frac{\partial \Omega}{\partial \Delta} = 0, \quad \frac{\partial \Omega}{\partial q} = 0, \quad \frac{\partial \Omega}{\partial \mu_c} = 0, \quad \frac{\partial \Omega}{\partial \mu_8} = 0, \quad (51)$$

where we have chosen a suitable frame with $\vec{q} = (0, 0, q)$. Since $\mu_8$ is very small in weak coupling we will simply set $\mu_8 = 0$ in numerical calculations.

Following the treatment in \cite{10}, we expand the thermodynamic potential in powers of the pair momentum $q$ in the vicinity of $q = 0$,

$$\Omega(q) - \Omega(0) = \frac{\partial \Omega}{\partial q} \bigg|_{q=0} q + \frac{1}{2} \frac{\partial^2 \Omega}{\partial q^2} \bigg|_{q=0} q^2 + \cdots. \quad (52)$$

The linear term vanishes automatically because of the gap equation for $q$. Using the relation

$$S^{-1}_q = S^{-1}_{\vec{q} = 0} - \tau_3 \gamma^5 \cdot \vec{q}, \quad (53)$$

and taking the derivative expansion of $\Omega$ in powers of the gauge field, we can easily obtain the relation between the momentum susceptibility $\kappa_q$ and the Meissner mass squared for the 8th gluon

$$\kappa_q = \partial^2 \Omega / \partial q^2 \bigg|_{q=0} = 12m_8^2 / g^2. \quad (54)$$

On the other hand, since $q = 0$ must be a solution of the gap equation $\partial \Omega / \partial q = 0$ which corresponds to the homogeneous 2SC/g2SC phase, the first order derivative of $\Omega$ with respect to $q$ must take the form\cite{23}

$$\partial \Omega / \partial q = qQ(q). \quad (55)$$

The momentum solution for the LOFF state is given by $Q(q) = 0$. The case here is similar to the gap equation for the pairing gap $\Delta$ which contains a trivial solution $\Delta = 0$ corresponding to the normal phase and a finite solution $\Delta \neq 0$ corresponding to the superfluid phase. For the formal proof here we do not need the explicit function $Q(q)$. From the identity

$$\partial^2 \Omega / \partial q^2 = Q(q) + Q'(q) \quad (56)$$

and (54), we obtain

$$m_8^2 = g^2 Q(0) / 12, \quad (57)$$

which means $Q(0) = 0$ at $T = T_8$ where $m_8^2$ changes sign. Therefore, the LOFF momentum $q$ must vanish at $T = T_8$, providing that the neutral LOFF solution is unique for a given $T$ and $\mu$. This indicates that there is no neutral LOFF solution at $T > T_8$. Below but near the temperature $T_8$, the LOFF momentum $q$ is very small, the small $q$ expansion is valid

$$\Omega(q) - \Omega(0) = 6m_8^2 q^2 / g^2. \quad (58)$$

For negative $m_8^2$ below $T_8$, the neutral LOFF state has lower free energy than the uniform state at least at $T \lesssim T_8$. Since the small $q$ expansion is like a Ginzburg-Landau expansion, we conclude that the LOFF momentum near the critical point behaviors as

$$q(T) \sim (1 - T/T_8)^{1/2}, \quad T \to T_8. \quad (59)$$

Now we turn to the numerical calculation of neutral LOFF state. The explicit form of $\Omega$ can be obtained if we neglect the mixing between the particles and antiparticles\cite{22} since we have $\Delta, q \ll \mu$ at weak coupling. It reads

$$\Omega = \frac{\Delta^2}{4G_D} - \int \frac{d^3\vec{p}}{(2\pi)^3} \left[ \sum_{l=1}^{8} W(E_l) + \frac{1}{2} \sum_{l=9}^{16} W(E_l) \right] + \Omega_c, \quad (60)$$

where the sum runs over all quasiparticles. The quasiparticle dispersions $E_l(\vec{p})$ calculated by $\det S^{-1} = 0$ are
given by
\begin{align*}
E_{1,2}(\vec{p}) &= E_{\Delta q}^- + \delta E_q \pm \delta \mu, \\
E_{3,4}(\vec{p}) &= E_{\Delta q}^- - \delta E_q \pm \delta \mu, \\
E_{5,6}(\vec{p}) &= E_{\Delta q}^+ + \delta E_q \pm \delta \mu, \\
E_{7,8}(\vec{p}) &= E_{\Delta q}^+ - \delta E_q \pm \delta \mu, \\
E_{9,10}(\vec{p}) &= |\vec{p} + \vec{q}| - \mu + \mu_s \pm \delta \mu, \\
E_{11,12}(\vec{p}) &= |\vec{p} - \vec{q}| - \mu + \mu_s \pm \delta \mu, \\
E_{13,14}(\vec{p}) &= |\vec{p} + \vec{q}| + \mu - \mu_s \pm \delta \mu, \\
E_{15,16}(\vec{p}) &= |\vec{p} - \vec{q}| + \mu - \mu_s \pm \delta \mu.
\end{align*}

where \( E_{\Delta q}^\pm \) and \( \delta E_q \) are defined as
\begin{align*}
E_{\Delta q}^\pm &= \sqrt{(|\vec{p} + \vec{q}| + |\vec{p} - \vec{q}| - 2\mu)^2 / 4 + \Delta^2}, \\
\delta E_q &= \frac{1}{2} (|\vec{p} + \vec{q}| + |\vec{p} - \vec{q}|).
\end{align*}

Notice that in our ultra-relativistic system, the thermodynamic potential of the LOFF state suffers from a unphysical term \( \sim q^2 \Lambda^2 \). In fact, \( \Omega(\Delta, q) \) should recover the result for the free quark gas when \( \Delta \to 0 \). Thus to study the neutral LOFF state, we define the following subtracted thermodynamic potential
\[ \Omega_{\text{sub}}(\Delta, q) = \Omega(\Delta, q) - \Omega(0, q) + \Omega(0, 0). \]

Thus we can set the momentum \( \vec{q} \) to be zero in the dispersion of quasiparticles 9 – 16 and treat the UV divergence for the quasiparticles 1 – 8. It is obvious that we recover the thermodynamic potential of the 2SC phase when \( q = 0 \) and the thermodynamic potential of the free quark gas when \( \Delta = 0 \). We can further make approximation on \( E_{\Delta q}^\pm \) and \( \delta E_q \) since \( \Delta, q \ll \mu \) at weak coupling and all integrals are dominated over the Fermi surface \( |\vec{p}| = \bar{\mu} \). In this approximation we have
\begin{align*}
E_{\Delta q}^\pm &\simeq E_{\Delta}^\pm + \frac{q^2}{2 \mu} E_{\Delta q}^\pm (1 - \cos^2 \theta), \\
\delta E_q &\simeq q \cos \theta,
\end{align*}
with \( \theta \) being the angle between \( \vec{p} \) and \( \vec{q} \). In this case the subtraction term at \( T = 0 \) can be analytically evaluated as
\[ \Omega_\Lambda = 8 \int \frac{d^3\vec{p}}{(2\pi)^3} (1 - \cos^2 \theta) \frac{q^2}{2 \mu} \frac{2\Lambda^2 q^2}{3 \pi^2} \]
which is from the kinetic term of the quasiparticle dispersion and is indeed diverges as \( \sim \Lambda^2 q^2 \). At finite temperature, however, we do not find an analytical expression for \( \Omega_\Lambda \).

In the numerical calculation of the neutral LOFF state at finite temperature, we then use the subtracted thermodynamic potential \( \Omega_{\text{sub}}(\Delta, q) \). The gap parameter \( \Delta \) and pair momentum \( q \) as well as the electron chemical potential \( \mu_e \) in the neutral LOFF state are determined by self-consistently solving the coupled set of equations \[ \text{[51]} \] but with the subtracted thermodynamic potential \( \Omega_{\text{sub}} \). For simplicity, we will neglect the color neutrality condition and set \( \mu_s = 0 \).

In Fig. 6 we display the numerical result of neutral LOFF state for \( \eta = 0.66 \) and 0.70. The neutral LOFF solution exists only at low temperature, namely at \( T < T_\Lambda \), and the LOFF momentum approaches to zero continuously at \( T = T_\Lambda \). Since the LOFF phase has lower thermodynamic potential than the uniform phase (we checked this numerically), the LOFF phase is energetically more favored than the uniform superconductivity. Considering both the stable LOFF state at low temperature and the stable uniform superconductivity at high temperature, the strange intermediate temperature superconductivity disappears, and the order parameter becomes a monotonously decreasing function of temperature, like in the conventional BCS case. Especially, for \( \eta = 0.66 \), the uniform superconductivity does not appear at \( T = 0 \), but LOFF phase starts at \( T = 0 \). For sufficiently weak coupling, the uniform superconductivity disappears at any temperature, but the LOFF phase can survive at low temperature. In Fig. 7 we show the neutral LOFF state for \( \eta = 0.63 \) where the uniform superconductivity starts to disappear. The temperature behavior of the order parameter is similar to that in the conventional BCS case.

We conclude that when LOFF phase is taken into account, the phase diagram [14–15] of neutral quark matter is significantly changed.

FIG. 6: The pairing gap \( \Delta \) in 2SC/g2SC phase (dashed line) and LOFF phase (solid line) and the LOFF momentum \( q \) as functions of \( T \) for two coupling values \( \eta = 0.70 \) and 0.66.
the 2SC/g2SC phase starts to disappear. Since the 4-7th gluons form a complex doublet, we can introduce only the 4-th gluon condensate and may appear at intermediate coupling too\[22, 40\]. In this phase, the off-diagonal gluon condensate or the spontaneous generated off-diagonal Nambu-Goldstone current is nonzero. Since the 4-7th gluons form a complex doublet, we can introduce only the 4-th gluon condensate

\[ \bar{\rho} = g \langle \bar{A}_4 \rangle = \langle \nabla \varphi_4 \rangle. \]  

(66)

Including this condensate, the thermodynamic potential of the system becomes \( \Omega(\Delta, \bar{\rho}) \). Following the same procedure used above, we can prove:

(1) The neutral gluonic phase with \( \bar{\rho} \neq 0 \) exists only in the temperature region \( T < T_4 \) where the Meissner mass squared \( m_4^2 \) is negative.

(2) The gluonic phase has lower free energy than the uniform phase in the region \( T < T_4 \).

(3) At \( T = T_4 \), the value of \( \bar{\rho} \) approaches to zero continuously.

The proof is quite similar to that for the LOFF phase. We need only the potential curvature definition of the Meissner masses squared \[12\]

\[ \partial^2 \Omega/\partial \rho^2 |_{\rho=0} = m_4^2/g^2 \]  

(67)

and consider the fact that the gap equation for \( \bar{\rho} = (0, 0, \rho) \) can be generally expressed as

\[ \partial \Omega/\partial \rho = \rho K(\rho) = 0. \]  

(68)

Recently, the numerical calculation on neutral gluonic phase is presented at \( T = 0 \) in \[40\].

\[ \text{V. SUMMARY} \]

We have investigated the stability of neutral two flavor color superconducting quark matter at finite temperature. The main conclusions are:

(1) There exists a temperature window below and close to the critical temperature of the superconductivity where the uniform 2SC/g2SC phase is stable, namely the Meissner masses squared and the gap susceptibility are both positive. The Meissner mass squared is positive at temperatures \( T_4 < T < T_c \) for the 4-7th gluons and at \( T_8 < T < T_c \) for the 8th gluon.

(2) In a wide range of coupling in the diquark channel, we have \( T_4 > T_8 \), the introduction of LOFF phase can not completely solve the problem of chromomagnetic instability at finite temperature, since in the region \( T_8 < T < T_4 \) only the 4-7th gluons suffer instability.

(3) The LOFF phase can exist only below the turning temperature \( T_8 \) where the Meissner mass squared for the 8th gluon changes sign and the LOFF momentum approaches to zero. Similarly, the gluonic phase can exist only below the turning temperature \( T_4 \) where the Meissner mass squared for the 4-7th gluons changes sign and the off-diagonal gluon condensate vanishes.

(4) When the LOFF phase is taken into account, the strange temperature behavior of the uniform color superconducting order parameter \[3\] disappears and the corresponding phase diagram of neutral quark matter \[11, 45\] is significantly changed. This situation is quite like the recent studies for non-relativistic Fermi gas with population imbalance \[32, 34\].

For further investigation, one needs to make a detailed calculation for the phase with gluonic condensation or Nambu-Goldstone current and to check the chromomagnetic stability of the LOFF phase and gluonic phase. Since the LOFF momentum \( \vec{q} \) and the gluonic condensate \( \bar{\rho} \) approach to zero continuously at the turning temperatures \( T_8 \) and \( T_4 \), one can at least expand the thermodynamic potential in terms of \( \vec{q} \) and \( \bar{\rho} \) in the neighborhood of \( T_8 \) and \( T_4 \). We defer the research in this direction to be a future work.

\[ \text{Acknowledgement:} \] After having completed this work, we knew that O.Kiriyama did a work \[48\] where some results are similar to ours. We thank him for useful discussions. This work is supported by the grants NSFC10428510, 10435080 10575058 and SRFDP20040003103.
Appendix A: Functions $A, A', B, C, D, H$ and $J$

We list in this appendix the functions $A, A', B, C, D, H$ and $J$ used to express the Meissner masses squared (38) and (39) for the 4-8th gluons and photon. Using the coefficients $C_{ij}^{ij}(p)$ and $C_{ij}^{ij}(p)$ defined in (9) and taking the trick of replacement $p_0 \rightarrow -p_0$ in calculating the Matsubara frequency summation, we can prove that only 7 of the coefficients are independent,

\[ C_{++}^{11} = C_{-+}^{11} = A, \quad C_{++}^{11} = C_{-+}^{22} = A', \quad C_{++}^{11} = C_{-+}^{21} = C_{+}^{12} = C_{-+}^{21} = B, \]

\[ C_{++}^{11} = C_{-+}^{11} = C_{++}^{22} = C_{-+}^{22} = C_{++}^{12} = C_{-+}^{12} = C_{++}^{21} = C_{-+}^{21} = C_{++}^{21} = C_{-+}^{12} = D, \]

\[ C_{++}^{44} = C_{-+}^{44} = H, \quad C_{++}^{44} = C_{-+}^{44} = J \]

(A1)

with the explicit expressions of $A, A', B, C, D, H, J$ as functions of $p \equiv |p|$, 

\[
A(p) = u_1^2 v_2^2 \left( \frac{f(E_1) + f(E_2)}{E_\Delta} + u_1^2 v_2^2 \frac{f(E_3) + f(E_4)}{E_\Delta} \right) + u_1^4 f'(E_2) + v_1^4 f'(E_4) + u_1^4 f'(E_3) + v_1^4 f'(E_4),
\]

\[
A'(p) = u_1^2 v_2^2 \left( \frac{f(E_1) + f(E_2)}{E_\Delta} + u_1^2 v_2^2 \frac{f(E_3) + f(E_4)}{E_\Delta} \right) + u_1^4 f'(E_2) + v_1^4 f'(E_4) + u_1^4 f'(E_3) + v_1^4 f'(E_4),
\]

\[
B(p) = u_1^2 v_2^2 \left( \frac{f(E_1) + f(E_2)}{E_\Delta} + u_1^2 v_2^2 \frac{f(E_3) + f(E_4)}{E_\Delta} \right) - u_1^2 v_2^2 \left( f'(E_1) + f'(E_2) \right) - u_1^2 v_2^2 \left( f'(E_3) + f'(E_4) \right),
\]

\[
C(p) = \left( \frac{E_\Delta^2 - E_0^2}{E_\Delta^2 - (E_0^2)^2} \right) (f(E_1) + f(E_2) - 1) - \left( \frac{E_\Delta^2 - E_0^2}{E_\Delta^2 - (E_0^2)^2} \right) (f(E_3) + f(E_4) - 1) + \frac{1}{p},
\]

\[
D(p) = -\frac{\Delta^2}{(E_\Delta^2 - (E_0^2)^2)^2} \left( f(E_1) + f(E_2) - 1 - \frac{f(E_3) + f(E_4) - 1}{E_\Delta} \right),
\]

\[
H(p) = u_1^2 \left( \frac{f(E_0) - f(E_2)}{E_0 - E_2} + \frac{f(E_0) - f(E_1)}{E_0 - E_1} \right) + v_2^2 \left( \frac{f(E_0) - f(E_1)}{E_0 - E_1} + \frac{f(E_0) - f(E_2)}{E_0 - E_2} \right),
\]

\[
J(p) = u_1^2 \left( \frac{f(E_0) - f(-E_2)}{E_0 + E_2} + \frac{f(E_0) - f(-E_1)}{E_0 + E_1} \right) + v_2^2 \left( \frac{f(E_0) - f(-E_1)}{E_0 + E_1} + \frac{f(E_0) - f(-E_2)}{E_0 + E_2} \right),
\]

where the coherent coefficients $u_1^2$ and $v_2^2$ are defined as $u_1^2 = (1 + E_0^2 / E_\Delta^2) / 2$ and $v_2^2 = (1 - E_0^2 / E_\Delta^2) / 2$. Note that we have added the terms $1/p$ to $C$ and $2/p$ to $J$ to cancel the vacuum contribution. In this way the Meissner masses squared are guaranteed to be zero in the normal phase with $\Delta = 0$.

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