Three-Dimensional Gauge Theories and ADE Monopoles

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Abstract
We study three-dimensional $\mathcal{N} = 4$ gauge theories with product gauge groups constructed from ADE Dynkin diagrams. One-loop corrections to the metric on the Coulomb branch are shown to coincide with the metric on the moduli space of well-separated ADE monopoles. We propose that this correspondence is exact.
Introduction

There exists a correspondence between three-dimensional gauge theories and monopole moduli spaces. The Coulomb branch of $N = 4$ supersymmetric $U(n)$ gauge theory with no matter multiplets is conjectured to be the moduli space of $n$ BPS monopoles of $SU(2)$ gauge group. This proposal was first made in [1] for the case $n = 2$ and later generalised in [2]. The case $n = 2$ has subsequently been proven by explicit field theory calculations [3]. For the case of general $n$ the equivalence has been used to predict the leading order exponential corrections to the $n$-monopole $SU(2)$ moduli space [4]. The connection between monopoles and three-dimensional gauge theories appears most naturally in a D-brane setting [5]. Moreover, this technology provides a further generalisation; the Coulomb branches of $N = 4$ three dimensional gauge theories with product gauge groups and certain bi-fundamental matter couplings are conjectured to be the moduli spaces of BPS monopoles of $SU(r + 1)$ gauge groups for any $r \geq 1$.

In the following we propose to extend this correspondence yet further. We construct product gauge groups with matter content determined by ADE Dynkin diagrams. We show that one-loop corrections to the metrics on the Coulomb branches reproduce the asymptotic metrics on the moduli spaces of BPS monopoles of any simply-laced Lie group. We conjecture that this correspondence is exact.

Note added: While writing this paper, reference [6] appeared in which the authors construct the moduli space of $SO$ and $Sp$ monopoles from branes with an orientifold plane. It would be interesting to examine the corresponding three-dimensional gauge theories in these cases.

Three-Dimensional Gauge Theories

Three-dimensional $N = 4$ supersymmetric theories are the dimensional reduction of the minimal supersymmetric models in six dimensions. Representations of the supersymmetry algebra come as either vector or hyper multiplets. The former contains a three-dimensional gauge field, $A_\mu$, three real scalars which we write in vector form, $\vec{\phi}$, and four Majorana fermions. All fields in the vector multiplet transform in the adjoint representation of the gauge group. Matter fields are contained in hypermultiplets which consist of four Majorana fermions, now paired with four real scalars. We will only consider hypermultiplets transforming in the (anti)-fundamental representation of the gauge group. The $N = 4$ action has a $SO(3)_N \times SO(3)_R$ global R-symmetry under which all fields transform. In particular, $\vec{\phi}$ transforms in the 3 of $SO(3)_N$ and is invariant under $SO(3)_R$.

We will denote the three-dimensional gauge group as $G$. It is a product group constructed from the Dynkin diagram of an auxiliary simply laced group, $G$, of rank $r$. Let $\vec{\beta}^A$, $A = 1, .., r$ be the simple roots of $G$ normalised as $\vec{\beta}^A \cdot \vec{\beta}^A = 1$. To
each assign a positive integer, $n_A$. To the node of the Dynkin diagram corresponding to the simple root $\beta_A^A$, we associate the three-dimensional gauge group $U(n_A)$ with coupling constant $e_A$. Thus $G$ is of rank $R = \sum_A n_A$. We also include matter content, $C$. For each link in the Dynkin diagram connecting the $\beta_A^A$ node with the $\beta_B^B$ node, we add a hypermultiplet transforming in the bi-fundamental representation $(n_A, \bar{n}_B)$. Thus the field content is,

$$G = \bigotimes_{A=1}^r U(n_A) ; \quad C = \bigoplus_{A \neq B} a_{AB}(n_A, \bar{n}_B), \quad (1)$$

where $a_{AB} = 1$ whenever $\beta_A^A \cdot \beta_B^B \neq 0$ and is zero otherwise. For $A_n$ Dynkin diagrams, this construction coincides with the brane picture of [5]. The cases of $D_n$ and exceptional Dynkin diagrams are new.

The construction above is similar to Kronheimer’s hyperKähler quotient construction of ALE spaces [6]. These were studied in the context of three-dimensional $N = 4$ theories in [8] (see also [9] for related quiver constructions) where the Coulomb branch was proposed to be the moduli space of instantons of gauge group $G$. The constructions differ in the use of the extended Dynkin diagram in [7] where the integers $n_A$ are also set equal to the Dynkin indices.

On the Coulomb branch, the hypermultiplet scalars have zero vacuum expectation value (VEV) while the VEVs of the vector multiplet scalars are constrained to live in the $R$-dimensional Cartan subalgebra (CSA), $H$, of $G$.

$$\langle \vec{\phi} \rangle = \vec{v} \cdot H. \quad (2)$$

We assume the VEVs break $G$ to the maximal torus. The adjoint Higgs mechanism gives a mass to all fields except vector multiplet fields in $H$. We denote the massless gauge fields as $A_{\mu} = Tr(A_{\mu} H)$ with similar notation for the other fields. The number of massless bosonic fields is thus $3R$ real scalars and $R$ abelian gauge fields.

We consider the Wilsonian low-energy effective action for the massless degrees of freedom. Introduce $R$ orthonormal vectors, $e_i$, $i = 1, ..., R$, spanning the root space of $G$. In order to later compare with the monopole moduli space it will prove useful to introduce $r$ new numbers $t_B = \sum_{A=1}^n n_A$ where we set $t_1 = 1$. The vectors $e_i$, $t_A \leq i < t_{A+1}$ are associated with the CSA of the $U(n_A)$ factor of the gauge group. We also define the coupling constant $e_i = e_A$ for $t_A \leq i < t_{A+1}$.

At tree level, the bosonic sector of the low-energy effective action is a free abelian theory,

$$S_B = \sum_{i,j=1}^R \int d^3 x \mathcal{M}_{ij} \left( -\frac{1}{4}(F \cdot e_i)(F \cdot e_j) + \frac{1}{2}(\partial \vec{\phi} \cdot e_i)(\partial \vec{\phi} \cdot e_j) \right), \quad (3)$$

where

$$\mathcal{M}_{ij} = \frac{2\pi}{e_i^2} \delta_{ij}. \quad (4)$$

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We have suppressed all space-time indices, and $F \equiv F_{\mu\nu}$ is the abelian field strength.

At weak coupling the low-energy effective action receives contributions from perturbation theory and instantons. We employ the background field method to calculate the former at one-loop. Details of the calculation for $SU(2)$ gauge group can be found in appendix B of [3] and the appendix of [10]. The extension to arbitrary gauge group is simple. Firstly consider the effects of integrating out the vector multiplets. Denote the roots of $U(n_A)$ as $\lambda_m^A, m = 1, \ldots, N_A = n_A(n_A - 1)$. Explicitly, $\{\lambda_m^A; m = 1, \ldots, N_A\} \equiv \{e_i - e_j; i, j = t_A, \ldots, t_A - 1\}$. For each factor $U(n_A)$ of the gauge group, $G$, the low-energy effective action includes the term

$$
\frac{1}{16\pi} \sum_{m=1}^{N_A} \left( \frac{(F \cdot \lambda_m^A)^2 - 2|\partial \phi \cdot \lambda_m^A|^2}{|\vec{v} \cdot \lambda_m^A|} \right).
$$

(5)

For the hypermultiplets, denote the weights of the fundamental representation of $U(n_A)$ as $w_p^A, p = 1, \ldots, n_A$. The weights of the anti-fundamental are $-w_p^A$. We have $\{w_p^A; p = 1, \ldots, n_A\} \equiv \{e_i; i = t_A, \ldots, t_A - 1\}$. Integrating out high momentum modes of the hypermultiplet transforming in the $(n_A, \bar{n}_B)$ representation yields

$$
-\frac{1}{16\pi} \sum_{p=1}^{n_A} \sum_{q=1}^{n_B} \left( \frac{(F \cdot w_p^A - F \cdot w_q^B)^2 - 2|\partial \tilde{\phi} \cdot w_p^A - \partial \tilde{\phi} \cdot w_q^B|^2}{|\vec{v} \cdot w_p^A - \vec{v} \cdot w_q^B|} \right).
$$

(6)

Equations (5) and (6) can both be interpreted as coupling constant renormalisations. There are further one-loop corrections to the low-energy effective action which are not of this form [11]. These will be dictated by supersymmetry.

Combining equations, (3), (5) and (6), the low-energy effective action can be written in the form (3), with

$$
\mathcal{M}_{ii} = \frac{2\pi}{e_i^2} - \frac{1}{2\pi} \sum_{k \neq i} \frac{\alpha_j \cdot \alpha_k}{|\vec{v} \cdot (e_i - e_k)|}
$$

$$
\mathcal{M}_{ij} = \frac{1}{2\pi} \frac{\alpha_j \cdot \alpha_i}{|\vec{v} \cdot (e_i - e_j)|}
$$

(7)

where we have defined $\alpha_j = \beta^A_A$ for $t_A \leq i < t_A + 1$ (recall $\beta^A_A$ are the simple roots of the auxillary simply-laced group, $G$).

The $R$ abelian gauge fields, $A_\mu$, are dual to scalars $\sigma$ which serve as Lagrange multipliers for the Bianchi identity. We add to the action the surface term

$$
S_S = \frac{i}{4\pi} \sum_{i=1}^{R} \int d^3x \epsilon^{\mu\nu\rho} (\partial_\mu F_{\nu\rho} \cdot e_i)(\sigma \cdot e_i)
$$

(8)

where we have restored the space-time indices. With this normalisation, the scalars $\sigma \cdot e_i$ have period $2\pi$ in the background of a single instanton. Performing the Gaussian
integration over $F_{\mu\nu}$ we promote $\sigma$ to a full dynamical field. The low-energy effective action is now a $\sigma$-model with coordinates $\vec{\phi} \cdot e_i$ and $\sigma \cdot e_i$ on the $4R$-dimensional target space,

$$S_{1\text{-loop}} = -\frac{1}{2} \sum_{i,j=1}^{R} \int d^3 x \, M_{ij} (\partial \vec{\phi} \cdot e_i) \cdot (\partial \vec{\phi} \cdot e_j) + \frac{1}{4\pi^2} (M^{-1})_{ij} (\partial \sigma \cdot e_i)(\partial \sigma \cdot e_j).$$

This action inherits the $SO(3)_N$ symmetry of the microsocpic action. It also posseses $R$ global $U(1)$ symmetries,

$$\sigma \rightarrow \sigma + \mathbf{c}$$

for any constant $R$-vector, $\mathbf{c}$. Although broken by instanton effects, these symmetries exist to all orders in perturbation theory as can be seen by integrating (8) by parts in a topologically trivial background.

The global symmetries of the action translate to isometries of the metric on the target space. Moreover, $N = 4$ supersymmetry requires that this metric is hyperKähler \[12\]. $4R$-dimensional hyperKähler metrics with $R$ triholomorphic $U(1)$ isometries have a simple form \[13\]. The hyperKähler condition requires augmenting the bosonic action (9) with extra terms generated at one-loop, corresponding to the replacement,

$$\partial \sigma \cdot e_i \rightarrow \partial \sigma \cdot e_i + \sum_{k=1}^{R} \vec{W}_{ik} \cdot (\partial \vec{\phi} \cdot e_k),$$

where,

$$\vec{\nabla} \times \vec{W}_{ij} = -2\pi \vec{\nabla} M_{ij}.$$ \(12\)

The derivative, $\vec{\nabla}$, is taken with respect to $\vec{v} \cdot e_i$. Equations (9) and (11) define the bosonic one-loop low-energy effective action. It remains to show that this is equivalent to a sigma model on a monopole moduli space.

**Monopole Moduli Spaces**

We consider moduli spaces of BPS monopoles of the simply-laced gauge groups $G$. An adjoint Higgs field with vanishing potential is assumed to break $G$ to the maximal torus, and in doing so defines $r$ simple roots, $\beta_A$, $A = 1, .., r$. The theory contains monopole configurations with magnetic charge defined by an $r$-dimensional vector, $\mathbf{g}$. Topological considerations force $\mathbf{g}_j$ to lie in the root lattice (with suitably normalised roots) of $G$ and we expand $\mathbf{g} = \sum_A n_A \beta_A$.

The moduli space of such a monopole configuration has dimension $4R = 4 \sum_A n_A$ \[14\]. This result has the interpretation that there exist $r$ types of "fundamental" monopoles corresponding to magnetic charges $\mathbf{g}_j = \beta_A$ for $A = 1, .., r$. The only moduli of a fundamental monopole configuration are the position, $\vec{x}$, and the phase,
\( \xi^A \), generated by global gauge transformations within the maximal torus. A general monopole configuration can be thought of as consisting of \( R \) individual fundamental monopoles, \( n_A \) of each type, at least in the asymptotic region of the moduli space. Let the \( i^{th} \) monopole be associated with the simple root, \( \alpha_j = \beta^A \) for some \( A \). For well separated monopoles, we may ascribe positions, \( \vec{x}_i \), and phases \( \xi_i \) to each monopole. We also define \( \vec{r}_{ij} = \vec{x}_i - \vec{x}_j \) and \( r_{ij} = |\vec{r}_{ij}| \).

The full metric on the monopole moduli space is not known. In the asymptotic region the monopoles are well-separated and the metric may be calculated using the techniques of \([15, 16]\). The calculation for arbitrary gauge group was performed in \([17]\) and the metric has the form

\[
\text{d}s^2 = \text{M}_{ij} \text{d}\vec{x}_i \cdot \text{d}\vec{x}_j + \frac{1}{4\pi^2} (M^{-1})_{ij} (\text{d}\xi^i + \vec{W}_{ik} \cdot \text{d}\vec{x}^k)(\text{d}\xi^j + \vec{W}_{jl} \cdot \text{d}\vec{x}^l) \tag{13}
\]

where,

\[
M_{ii} = m_i - \sum_{k \neq i} \frac{\alpha_j \cdot \alpha_k}{2\pi r_{ik}}
\]

\[
M_{ij} = \frac{\alpha_j \cdot \alpha_i}{2\pi r_{ij}} \tag{14}
\]

\( m_i \) is the mass of the \( i^{th} \) fundamental monopole associated with the simple root \( \alpha_j \) and \( \vec{W}_{ij} \) is defined by \( \vec{\nabla} \times \vec{W}_{ij} = -2\pi \vec{\nabla} M_{ij} \).

Comparing (13) to the sigma model metric defined by (9) and (11) we find the metrics do indeed coincide as advertised with

\[
m_i = 2\pi/e_i^2 \\
\vec{x}_i = \vec{\phi} \cdot e_i \\
\xi_i = \sigma \cdot e_i \tag{15}
\]

The metric (13) is singular at \( r_{ij} = 0 \) for \( \alpha_j \cdot \alpha_i > 0 \). These singularities are resolved by exponential corrections to the metric corresponding to instanton corrections in the three-dimensional gauge theory. Assuming the correspondence between monopole moduli spaces and three-dimensional gauge theories is exact, the leading order exponential corrections could be calculated along the lines of [1].

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\footnote{In the notation of [17] we set \( g = 2 \) where \( g \) is the coupling constant for monopole interactions. This parameter has no counterpart in the three-dimensional theory.}
References

[1] N. Seiberg and E. Witten, in “The Mathematical Beauty of Physics”, p.333, Eds. J. M. Drouffe and J.-B. Zuber (World Scient., 1997), hep-th/9607163.

[2] G. Chalmers and A. Hanany, Nucl. Phys. B489 (1997) 223, hep-th/9608105.

[3] N. Dorey, V. V. Khoze, M. P. Mattis, D. Tong and S. Vandoren, Nucl. Phys. B502 (1997) 59, hep-th/9703228.

[4] C. Fraser and D. Tong, Instantons, Three-Dimensional Gauge Theories and Monopole Moduli Spaces, hep-th/9710098.

[5] A. Hanany and E. Witten, Nucl. Phys. B492 (1997) 152, hep-th/9611230.

[6] C. Ahn and B. Lee, \textit{SO/Sp Monopoles and Branes with Orientifold 3 Plane}, hep-th/9803069.

[7] P. B. Kronheimer, Jour. Diff. Geom. 29 (1989) 665.

[8] K. Intriligator and N. Seiberg, Phys.Lett. B387 (1996) 513, hep-th/9607207.

[9] J. de Boer, K. Hori, H. Ooguri and Y. Oz Nucl.Phys. B493 (1997) 101, hep-th/9611063.

[10] N. Dorey, D. Tong and S. Vandoren, Instanton Effects in Three-Dimensional Supersymmetric Gauge Theories with Matter, hep-th/9803065.

[11] N. Seiberg, Phys. Lett B384 (1996) 81, hep-th/9606017.

[12] L. Alvarez-Gaumé and D. Z. Freedman, Commun. Math. Phys. 80 (1981) 443.

[13] N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, Commun. Math. Phys. 108 (1987) 535.
H. Pederson and Y.S. Poon, Commun. Math. Phys. 80 (1988) 569.

[14] E. Weinberg, Nucl. Phys. B197 (1980) 500.

[15] N. Manton, Phys. Lett. B154, 397 (1985); (E) B157B, 475 (1985)

[16] G. Gibbons and N. Manton, Phys. Lett. B356 32 (1995), hep-th/9506052.

[17] K. Lee, E.J. Weinberg, P. Yi, Phys. Rev. D54 (1996) 1633, hep-th/9602167.