Early Exercise Decision in American Options with Dividends, Stochastic Volatility, and Jumps

Antonio Cosma, Stefano Galluccio, Paola Pederzoli, and Olivier Scaillet

Abstract

Using a fast numerical technique, we investigate a large database of investors’ suboptimal nonexercise of short-maturity American call options on dividend-paying stocks listed on the Dow Jones. The correct modeling of the discrete dividend is essential for a correct calculation of the early exercise boundary, as confirmed by theoretical insights. Pricing with stochastic volatility and jumps instead of the Black–Scholes–Merton benchmark cuts the amount lost by investors through suboptimal exercise by one-quarter. The remaining three-quarters are largely unexplained by transaction fees and may be interpreted as an opportunity cost for the investors to monitor optimal exercise.

I. Introduction

Holders of short-maturity American call options on dividend-paying stocks are known to miss exercising their options in an apparently suboptimal way...
(see, e.g., Pool, Stoll, and Whaley (2008)). Our conjecture, which we investigate empirically later on in this paper, is that model choice and transaction costs may be key explanations for that suboptimality. We investigate the suboptimal exercise by allowing alternative models for the underlying stock dynamics. To do this, we compile data on 30 individual dividend-paying stocks listed on the Dow Jones, comprising a total of 101,295 series of short-term options amounting to approximately 9.5 million records. We show that by accounting for stochastic volatility and jumps in the process of the underlying asset, we can explain up to 25% of the gain forgone due to nonoptimal exercise decisions, as computed in Pool et al. (2008). Because financial frictions are a possible explanation of the departure from the expected exercise behavior (e.g., Jensen and Pedersen (2016)), we also show that transaction costs cannot fully explain the nonexercise decisions.

Options written on single stocks are mainly of the American style, which means we can exercise them at any time before maturity. Economic theory gives clear guidance on the early exercise decision: An investor should exercise the options if the proceeds from immediate exercise exceed the value of the contract when kept alive. The value of the underlying stock that makes the investor indifferent between holding onto the option or not is the exercise boundary, or frontier. Such a clear normative dictate is nevertheless accompanied by a strikingly high level of suboptimal nonexercise. The figures reported in Pool et al. (2008) and confirmed in our data show that, in given periods, investors hold on to between one-third and one-half of the contracts they should have exercised. This behavior can potentially be explained in terms of costs, whether explicit, as fees and other forms of financial frictions, or implied, as the opportunity cost of closely monitoring the option. Since the income in case of immediate exercise (i.e., the intrinsic value of the option) is readily known, the main determinant of the decision is the continuation value (i.e., the value of the option when it is not exercised). Our analysis focuses on the correct estimation of the latter. As already noticed in Pool et al. (2008) and in Barraclough and Whaley (2012), there are two ways of addressing this issue. One is the market-based approach. Market prices provide guidance in the exercise decision. If the market time value of the option (i.e., the difference between the market and intrinsic values) vanishes, then it is time for the investor to exercise. In practice, though, due to price discreteness and bid–asks spreads, the time value may not be exactly equal to 0, even when exercise is optimal. We prefer a model-based estimation. With this approach, we obtain an explicit figure for the continuation value; whenever exercise would have been optimal, we can compute how much lower than the intrinsic value it is, thus quantifying the loss due to suboptimally holding on to the contract. This loss gives us an estimate of the opportunity cost or of the implied fees. Market prices are silent on this aspect, since the market time value of the option becoming 0 tells us only that the continuation has reached from above the intrinsic value, not that it has crossed it, or, if so, by how much. The model-based continuation value depends on assumptions on the underlying stock process, which leads us to check how the exercise boundary changes in different modeling environments. The crucial modeling assumption in this context is how to represent the dividend. The most recent theoretical literature on American options (Amin (1993), Adolfsson, Chiarella, Ziogas, and Ziveyi (2013)) models the dividend payment as a continuous dividend yield. We see in
what follows that when assuming a constant dividend yield, the introduction of jumps and stochastic volatility in the stock process may exacerbate, instead of partially explaining, the suboptimality figures.

In our empirical investigation, we show that by correctly modeling the dividend as a discrete payment, the boundary is higher under the Merton (1976) jump-diffusion and Heston (1993) stochastic volatility models than under the Black–Scholes (1973) model. In the subsequent discussion, we provide new theoretical insights into why this is the case. This procedure allows us to revise the share of contracts that are suboptimally unexercised since contracts that should be exercised under Black–Scholes should instead be kept alive under the more sophisticated models. In our calibration, we price by fully considering the discrete nature of the dividend distributed by the underlying stocks and the American style of the call options, and we do so for different specifications of the stock dynamics. In the process, to our knowledge, we are the first to provide comprehensive descriptive statistics of the parameters driving the jumps and the stochastic volatility of the constituents of the Dow Jones Industrial Average Index (DJIA) traded in the period from Jan. 1996 to Dec. 2012. This feature is a peculiarity of our work, given that the standard empirical literature on options focuses mainly on European S&P 500 options with a dividend yield (Bakshi, Cao, and Chen (1997), Eraker, Johannes, and Polson (2003)), and the consensus is that the early exercise premium of American options is negligible. For example, Broadie, Chernov, and Johannes (2007) approximate American option prices on S&P 500 futures with European ones and show that transforming American options to European ones does not matter for calibration purposes when facing a continuous dividend yield since differences in early exercise premia are not so large in that case. This situation is not true with multiple discrete dividend payments, and we provide an example of how neglecting the discrete character of a dividend or its time of payment leads to an incorrect exercise decision. Overlooking this feature biases empirical findings on suboptimality.

To conclude this introduction, we point to the methodological contribution of our work. One of the reasons that researchers have not correctly modeled the dividend as a discrete process is that the repeated option valuations needed to calibrate the different models for the underlying stock to such a large data set (we have approximately 9.5 million records) are too time-consuming given existing numerical methods. To date, no empirical work on options written on dividend-paying stocks exists outside the Black–Scholes world. Broadie et al. ((2007), p. 1484) state, “The computation time required for American options makes calibration to a very large set of options impractical.” As reported previously, what Broadie et al. show in their Appendix A is that transforming American options into European ones does not matter for calibration purposes in their application, but we show in the following that it does make a difference in our study. We develop a new methodology to price American options, which, in the kind of pricing problems we face in our empirical study, is more than one order of magnitude faster than techniques based on the discretization of the pricing partial differential equation and four orders of magnitude faster than simulation-based techniques. Our technique belongs to the quadrature family of numerical routines (see Andricopoulos, Widdicks, Newton, and Duck (2007), Chen, Härkönen, and Newton (2014), and
Fang and Oosterlee (2011), and we refer to it as recursive projections to distinguish it from other variants. We are the first to characterize the convergence properties of a quadrature-based method in the presence of discrete dividends and with the underlying following a dynamic outside the Black–Scholes benchmark.¹

The article is organized as follows: In Section II, we give a numerical example of how introducing jumps and stochastic volatility to the stock process leads to a higher exercise boundary. This example allows us to conjecture that by considering these features of the underlying process, the figures on suboptimal exercise should decrease. This conjecture, together with the discussion that we postpone to Section III.D, is the theoretical background against which we set the empirical analysis of Section III.B. In this section, we verify in the data whether introducing jumps and stochastic volatility goes in the direction of reducing the number of contracts left suboptimally unexercised, as predicted by our theoretical discussion. We can see this procedure as an informal test in which the null assumption is that the proportion of contracts wrongly left nonexercised is unchanged after introducing jumps and stochastic volatility. In Section III.C, we check whether transaction fees can explain the part of suboptimality in the exercise behavior that remains after accounting for jumps and stochastic volatility. Section IV provides some concluding remarks. Appendices A–C provide detailed descriptions of the pricing algorithms used in our empirical strategy and gather further theoretical insights of the behavior of the exercise boundary under the different modeling environments.²

II. Early Exercise Decision: A Numerical Example

In the example that follows, we show how the two most common assumptions made when investigating exercise decisions for American call options written on dividend-paying stocks (i.e., a continuous payment of the dividend, and a geometric Brownian motion for the underlying stock process) reinforce each other in introducing a downward bias in the computation of the exercise frontier. This negative bias increases the number of contracts that appear to be irrationally kept alive. We design this example to provide an intuitive introduction to the impact of modeling choices on the exercise decision of contracts. A more formal treatment is left for Section III.

In our example, the call option has a remaining life of 6 months (i.e., the maturity is \( T = 0.5 \)), and the underlying stock distributes a regular quarterly dividend \( d \). The two dividend payments occur immediately, at \( t_1 = 0 \), and at \( t_2 = 0.25 \), corresponding to a time to maturity of 0.5 and 0.25, respectively. If we correctly model the dividend payment as an event that takes place at a precise point in time, then it is well known that it may be optimal to exercise the call option only immediately before the dividend payment (see, e.g., Hull (2018)). Thus, in this example, we may exercise the option at times \( t = 0 \) and \( t = 0.25 \). It follows that the value of the call option at \( t = 0.25 \) is the larger of the continuation value (i.e., the value of the option just after the dividend is paid if we do not exercise) and the intrinsic value (i.e., the proceeds from exercising immediately before the dividend payment).²

¹Our pricing library (with a ready-to-use Matlab interface) is publicly available on the authors’ Web site (https://paolapederzoli.com/).
²The Supplementary Material presents additional technical details.
payment). Formally, $C(S_t, T, K)$ is the value of the American option as a function of the underlying stock price $S_t$ at time $t$, strike $K$, and maturity $T$. Then, just before the stock goes ex-dividend (i.e., an infinitesimal time interval $\varepsilon$ before $t = 0.25$), it holds that

$$(1) \quad C(S_{0.25}, 0.5, K) = \max\{S_{0.25-\varepsilon} - K, C(S_{0.25} - d, 0.5, K)\}.$$ 

We follow the accepted convention that no jumps other than the drop in value of $S_t$ due to the dividend payment occur at the dividend date so that, by continuity of the stock process, $S_{t-\varepsilon} = S_t$ and $S_t - d$ is the value of the stock immediately after the payment of the dividend.\(^3\) The value $S^*_t$ such that $S^*_t = K = C(S^*_t - d, 0.5, K)$ is the exercise boundary at $t = 0.25$. For values $S_{0.25} \geq S^*_t$, it is optimal to exercise the option, while for $S_{0.25} < S^*_t$, the optimal behavior is to keep the option until maturity. Since no further intermediate payment is foreseen before maturity, $C(S_{0.25} - d, 0.5, K)$ is the value of a European option:

$$(2) \quad C(S_{0.25} - d, 0.5, K) = \mathbb{E}[e^{-r(0.5-0.25)}(S_T - K)|S_t = S_{0.25} - d].$$ 

The expectation in equation (2) is taken with respect to the risk-neutral transition density of the stock value from $t = 0.25$ to $t = 0.5$. For most of the stochastic processes chosen to model the stock price, equation (2) has a closed-form or quasi-closed-form solution. No arbitrage and the intertemporal consistency of asset prices dictate that at $t = 0$, we have to compare the value of immediate exercise with the continuation value after the dividend payment, which now takes the more complex form

$$(3) \quad C(S_0 - d, 0.5, K) = \mathbb{E}[e^{-r(0.25)}C(S_{0.25}, 0.5, K)|S_t = S_0 - d].$$ 

This time, the expectation is taken with respect to the risk-neutral transition density of the stock value from $t = 0$ to $t = 0.25$. The value of the contract at $t = -\varepsilon$ is $\max\{S_{0-\varepsilon} - K, C(S_0 - d, 0.5, K)\}$, and the exercise boundary $S^*_0$ is the value such that $S^*_0 = K = C(S^*_0 - d, 0.5, K)$. Contrary to equation (2), the right side of equation (3) in most cases does not have a simple-to-compute form. The fast recursive projections, the numerical methodology we use in the empirical work of Section III, fully take into account the discrete nature of the dividend payment and give a consistent estimate of $C(S_0 - d, 0.5, K)$. We determine in what follows the exercise boundaries $S^*_t$ at $t = 0$ and $t = 0.25$ by computing the expectations in equations (2) and (3) using the risk-neutral transition densities implied by a pure diffusion model for the underlying stock (Black–Scholes), a stochastic volatility model (Heston), and a process allowing for jumps at discrete moments in time (Merton). A formal definition of the processes is left to Section III.A. For the moment, we are interested only in giving a qualitative overview of how the exercise boundary changes as a consequence of the characteristic features of the three process specifications.

\(^3\)Whenever the dividend $d$ does not appear in the argument of the call value, as in $C(S_t, K, T)$, we are considering the value of the call before the stock goes ex-dividend (i.e., after the maximum in equation (1) is taken). If $d$ appears explicitly, as in $C(S_t - d, T, K)$, we are referring to the continuation value of the call after the dividend payment.
We may be tempted to simplify the problem by approximating the discrete dividend payout process by a continuous dividend payment. In this modeling approach, there are no more a priori known dates when exercise could be optimal. Exercise can be optimal in any moment \( t \) between 0 and 0.5, whenever the stock price is deep enough in the money and higher than the critical value \( S^*_t \), which is the exercise boundary for an American call with a continuous dividend yield. The boundary \( S^*_t \) is defined as the lowest value of \( S_t \) such that \( S_t - K \geq C(S_t, T, K) \). In the continuous dividend approximation, there is therefore a value of \( S^*_t \) for each moment in time until maturity, whereas, in the discrete dividend case, we had only two possible exercise dates. The continuous dividend approach may not appear to be a simplification since the comparison between the intrinsic value and the continuation value as in equation (3) must be done at each point in time. In practice, the comparison takes place on a finite number of points between 0 and \( T \), and the most common numerical methods, such as trees and finite difference methods, adapt much more easily to a continuous dividend yield (which preserves the recombining property of these schemes) than to a process that is discontinuous at each dividend date.

In Graph A of Figure 1, we plot the early exercise boundary for the Heston and Black–Scholes models for an American call option with a continuous dividend yield \( r_d = 0.03 \) (right graph) and with an equivalent quarterly discrete dividend \( d = 1.38 \) (left graph). We choose \( d = 1.38 \) to have an equivalent total annual dividend between the continuous dividend yield \( r_d = 0.03 \) and the discrete dividend case. Indeed, \( 1.38 = 0.03S^*/4 \), where \( S^* = 184 \) is the critical stock price under the Black–Scholes model in the dividend yield case for maturity \( T = 0.5 \). The strike price is \( K = 100 \). We fix the parameters of the underlying processes at values commonly used in studies of options written on single stocks (see Adolfsson et al. (2013)). We use the following set of representative parameters: \( T = 0.5, K = 100, r = 0.05, \sigma_0 = 0.2, \omega = 0.1, \sigma_{tT} = 0.3, \beta = 4, \) and \( \rho = -0.5 \), as in Adolfsson et al. (2013). See Section III.A for definitions of the process parameters. For comparison, we follow Heston (1993) and use the Black–Scholes model with a volatility parameter that matches the (square root of the) variance of the spot return over the life of the option in the Heston model. With a continuous dividend yield, the Heston early exercise boundary is always below the Black–Scholes boundary, whereas we face the opposite situation with the discrete dividend. Indeed, in the only two points in time at which the option may be exercised, just before the dividend payments, the value of the exercise boundary is lower under the Black–Scholes model than under the Heston model. In comparing the exercise boundary under the Black–Scholes and the Merton models, we slightly change the value of the contract parameters to make our discussion comparable with the one in Amin (1993). Maturity is still \( T = 0.5 \) and the dividends are distributed on \( t = 0, 0.25 \). The strike is now \( K = 40 \).

In Graph A of Figure 2, we plot the early exercise boundary for the Merton and Black–Scholes models for an American call option with a continuous dividend yield \( r_d = 0.05 \) (right graph) and, in the case in which the stock pays an
FIGURE 1
Early Exercise Boundary of the Heston and Black–Scholes Models

Graph A of Figure 1 compares the early exercise boundary in the Heston (H) and Black–Scholes (BS) models of an American call with a maturity of 6 months in the case in which the stock pays a dividend yield $r_d = 0.03$ (right) and in the case in which the stock distributes an equivalent quarterly discrete dividend of $d = 1.38$ (left). The remaining parameters are $K = 100$, $r = 0.05$, $\sigma_0 = 0.2$, $\omega = 0.1$, $\sigma_T = 0.3$, $\beta = 4$, and $\rho = -0.5$. We set the volatility parameter in the Black–Scholes model equal to the volatility of the underlying return over the life of the option in the Heston model. Graph B shows the Heston minus Black–Scholes prices of an American call, with $T = 0.25$ for different values of $S_0$ in the case of a discrete dividend (left) and continuous dividend yield (right). The remaining parameters are the same as in Graph A.

Graph A. Heston Early Exercise Boundary

Graph B. Heston Minus Black–Scholes Price

equivalent quarterly discrete dividend, $d = 1.125$ (left graph). For a discrete dividend, we observe a higher boundary in the Merton case than in the Black–Scholes case, which holds true for all maturities. For the continuous dividend case, for short maturities, the boundary is higher under the Merton model than under the Black–Scholes model. For longer maturities, the boundary takes lower values in the Merton case than in the Black–Scholes case, and we observe a crossing of the early exercise boundary in the far-right end of the right graph of Graph A in Figure 2. The exact point where the exercise frontiers under Black–Scholes and Merton cross depends on the configuration of the process parameters. Nevertheless, this example shows how, for a choice of the parameters typically used in the

---

As before, we take $d = 1.125$ because $1.125 = 0.05S^*/4$, where $S^* = 90$ is the critical stock price with the dividend yield $r_d = 0.05$ for maturity $T = 0.5$. The process parameters for the Merton model are $r = 0.08$, $\gamma = 5$, $\sigma^2_y = 0.05$, $\sigma^2_x = 0.05$, and $\mu_x = 0$. See Section III.A for the definition of the process parameters. We set the volatility parameter in the Black–Scholes model equal to the volatility of the underlying return over the life of the option in the Merton model.
FIGURE 2
Early Exercise Boundary of the Merton and Black–Scholes Models

Graph A of Figure 2 compares the early exercise boundary in the Merton (M) and Black–Scholes (BS) models of an American call with a maturity of 6 months in the case in which the stock pays a dividend yield $d = 0.05$ (right) and in the case in which the stock distributes an equivalent quarterly discrete dividend of $d = 1.125$ (left). The other parameters are $K = 40$, $T = 0.5$, $r = 0.08$, $\gamma = 5$, $\sigma_0^2 = 0.05$, $\sigma_z^2 = 0.05$, and $\mu_z = 0$. We set the volatility parameter in the Black–Scholes model equal to the volatility of the underlying return over the life of the option in the Merton model. Graph B shows the Merton minus Black–Scholes price of an American call with the same parameters as those used in Graph A but different values of $S_t$ in the case of a discrete dividend (left) and continuous dividend yield (right).

**Graph A. Merton Early Exercise Boundary**

**Graph B. Merton Minus Black–Scholes Price**



literature on calls on dividend-paying stocks, the crossing takes place in the range of maturities that we use in the empirical investigation.

The example described in this section shows that by not correctly modeling the discrete nature of the dividend payment and by not accounting for important features of the underlying process such as jumps and stochastic volatility, we introduce a bias in the determination of the exercise boundary. When the discrete dividend is modeled correctly, the bias introduced by the simple Black–Scholes model lowers the value of the boundary. A decision of nonexercise that would appear not optimal under the Black–Scholes model may appear much less so under a more realistic Heston or Merton model. For this reason, we expect that the figures of call options left suboptimally nonexercised will decrease by adding jump and stochastic volatility to the process of the stock. In the next section, we empirically test and theoretically justify this expectation.

---

5 In Appendix A, we provide an additional example on how approximating American option prices by European option prices when the stock distributes a dividend introduces a negative bias in the estimation of the exercise boundary.
III. Empirics and Theoretical Insights

In this section, we investigate call holders’ early exercise decision in light of the different values that the exercise boundary can take under distinct modeling assumptions for the underlying asset. Following the procedure suggested by Pool et al. (2008), we first check which contracts should be exercised by comparing the intrinsic value immediately before the dividend payment with the continuation value on the ex-dividend day. We quantify how much is economically lost in the case of a suboptimal nonexercise decision. This amount depends on the continuation value and is model-specific. We compare the results obtained under three modeling environments (i.e., Black–Scholes, Merton jump-diffusion, and Bates), which we formally define in the next paragraph. We observe a lower degree of incorrect exercise decisions if we model the underlying security in the Merton or Bates framework than if we restrict ourselves to the Black–Scholes dynamics, which suggests that investors incorporate features of the more sophisticated models when making the exercise decisions. Finally, whenever we find evidence of a suboptimal nonexercise decision, we show that trading costs alone cannot justify investors’ behavior.

A. Data and Modeling Assumptions

The empirical analyses in this section are carried out using a large data set of call options that have a maturity of less than 6 months and are written on the dividend-paying stocks that enter the Dow Jones Industrial Average Index (DJIA). The sample comprises daily observations between Jan. 1996 and Dec. 2012. The daily data on all option attributes, the stock price, and the dividend distribution details are from Optionmetrics. We obtain the daily data on the interest rates from the U.S. Treasury constant maturities of the H15 report of the Federal Reserve. A total of 101,295 series of short-term options written on 30 stocks enter our database. The total number of records is approximately 9.5 million. Table 1 reports the number of quotes for each stock, with a breakdown by maturity and moneyness. Our article focuses on investors’ early exercise behavior; hence, we focus on the in-the-money options, which are the category of options for which the number of quotes is the highest.

We now give formal definitions of the stochastic processes we use to model the evolution of the underlying stock. Our choice of environments follows the empirical findings of Bakshi et al. (1997), who suggest that jumps and stochastic volatility play a dominant role in pricing short-term options, whereas modeling stochastic interest rates does not seem to significantly improve the pricing performance. We therefore choose a general model that allows for jumps in the mean of the process and for time-varying variance. Bates (1996) was the first to suggest combining these features of the underlying process; we therefore refer to this process specification as the Bates model. Let $X_t = \log(S_t)$, then

---

6We access the Optionmetrics and the H15 databases through the Wharton Research Data Services (WRDS) research platform.
Table 1 reports the number of observations for in-the-money (ITM), at-the-money (ATM), and out-of-the-money (OTM) call option quotes for the stocks that are the constituents of the Dow Jones Industrial Average Index (DJIA). The data are further broken down by maturity. According to the classification of Bollen and Whaley (2004), a call option is considered OTM if its delta is less than 0.375, ATM if its delta ranges between 0.375 and 0.625, and ITM if its delta is above 0.625.

| Number of Option Quotes | Maturity < 60 Days | Maturity > 60 Days |
|-------------------------|---------------------|---------------------|
| Ticker | Stock | OTM | ATM | ITM | OTM | ATM | ITM |
| 1 | AA | ALCOA INC | 42,827 | 6,379 | 4,120 | 6 | 28,788 | 9,621 | 29,371 |
| 2 | AXP | AMERICAN EXPRESS CO | 43,360 | 10,386 | 5,696 | 9,422 | 27,744 | 14,941 | 50,002 |
| 3 | BAC | BANK OF AMERICA CO | 53,785 | 7,767 | 5,779 | 4 | 48,365 | 15,446 | 54,361 |
| 4 | BA | BOEING CO | 40,592 | 8,651 | 55,672 | 28,088 | 11,238 | 51,676 |
| 5 | CAT | CATERPILLAR INC DEL | 78,276 | 8,194 | 63,177 | 44,295 | 14,142 | 55,598 |
| 6 | CHV | CHEVRON CORPORATION | 34,710 | 5,055 | 4,440 | 29,054 | 9,249 | 44,421 |
| 7 | CSCO | CISCO SYS INC | 40,138 | 7,052 | 47,571 | 31,350 | 10,730 | 40,773 |
| 8 | KO | COCA COLA CO | 36,835 | 6,737 | 58,522 | 24,402 | 10,205 | 38,750 |
| 9 | DIS | DISNEY WALT CO | 59,249 | 9,060 | 85,569 | 45,355 | 13,330 | 62,979 |
| 10 | JNJ | JOHNSON & JOHNSON | 34,710 | 5,055 | 4,440 | 29,054 | 9,249 | 44,421 |
| 11 | MCD | MCDONALDS CORP | 33,730 | 5,674 | 47,969 | 27,630 | 10,145 | 48,000 |
| 12 | MRK | MERCK & CO INC | 41,559 | 7,612 | 53,181 | 31,901 | 11,486 | 47,534 |
| 13 | MSFT | MICROSOFT CORP | 68,396 | 8,725 | 78,393 | 50,253 | 14,163 | 65,728 |
| 14 | T | AT&T INC | 42,547 | 5,542 | 52,123 | 32,851 | 9,496 | 46,524 |
| 15 | TRV | TRAVELERS COMPANIES INC | 21,404 | 3,803 | 27,783 | 15,069 | 5,811 | 23,646 |
| 16 | UTX | UNITED TECHNOLOGIES CORP | 34,710 | 5,055 | 4,440 | 29,054 | 9,249 | 44,421 |
| 17 | UNH | UNITEDHEALTH GROUP INC | 39,924 | 8,864 | 58,885 | 32,757 | 16,724 | 54,563 |
| 18 | VZ | VERIZON COMMUNICATIONS INC | 39,642 | 6,461 | 56,527 | 34,306 | 11,100 | 54,563 |
| 19 | WMT | WAL-MART STORES INC | 37,172 | 5,668 | 51,904 | 31,679 | 10,429 | 52,890 |
| 20 | DD | DU PONT E I NEMOURS & CO | 35,262 | 7,404 | 46,302 | 24,077 | 11,097 | 41,173 |

(4) \[ dX_t = \left( r - r_d - \gamma \nu - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dW_{1,t} + \log(\psi) dq_t, \]

\[ d\sigma_t^2 = \beta(\sigma_{LT}^2 - \sigma_t^2) dt + \omega \sqrt{\sigma_t^2} dW_{2,t}, \quad E(dW_{1,t} \cdot dW_{2,t}) = \rho dt. \]

where \( r \) is the risk-free interest rate, \( r_d \) is the continuous dividend yield paid by the asset, \( dW_{1,t} \) and \( dW_{2,t} \) are two correlated Wiener processes, and \( \sigma_t \) is the time-varying volatility of stock returns. In the calibration, we impose \( r_d = 0 \), but we leave it explicitly in model (4) to give the general specification that we used in the numerical exercise of Section II. The Poisson process, \( q(t) \), is independent of \( W_{1,t}, i = 1,2 \), and there is a probability, \( \gamma dt \), that a jump occurs in \( dt \) and a \( 1 - \gamma \) probability that no jump occurs. The parameter \( \gamma \) represents the mean number of jumps per unit of time. The random variable \( \psi \) is such that \( \psi - 1 \) describes the percentage change in the stock price if the Poisson event occurs, and \( \nu = E[\psi - 1] \) is the mean jump size. We further make the standard assumption (for instance, see Amin (1993), Bakshi et al. (1997)) that \( \log(\psi) \sim N(\mu_\psi, \sigma_\psi^2) \). The parameter \( \sigma_{LT}^2 \) denotes the long-term mean of the variance process, and \( \omega \) is the volatility of volatility. If \( \omega = 0 \), the volatility is constant at the level \( \sigma_{LT}^2 \), and equation (4) simplifies to the Merton (1976) jump-diffusion model. When referring to the Merton model, we will denote the (time-constant) volatility as \( \sigma_M \). If we simplify
equation (4) further and set the jump intensity $\gamma = 0$, we recover the standard Black–Scholes model with no jumps. In the following, we will denote the Black–Scholes constant volatility with the symbol $\sigma_{BS}$. If instead $\gamma = 0$ but $\omega \neq 0$, equation (4) is the Heston (1993) model, arguably the most popular process from among the family of stochastic volatility models.

The main challenge in this exercise is to compute the continuation value at the ex-dividend date. As anticipated in Section II, no closed-form formula is available, and we need to rely on a fast and accurate numerical method. To this end, we develop a new pricing methodology, the recursive projections methodology, which consists of computing at specific moments in time the value of the contract at a fixed (i.e., time-homogeneous) grid of values of the relevant state variables (i.e., the logarithm of the stock price and the stochastic variance). The recursive projections are related to quadrature models in that they allow reconstructing the value of the contract only at the specific points in time that are relevant to our study (i.e., maturity and ex-dividend dates). This characteristic contrasts with finite difference methods and trees, which must compute the value function at close consecutive points in time and allow us to avoid intermediate steps and to obtain fast pricing algorithms. The main feature that permits reconstructing the value function of the call option at a time $t_1$ from the value function at time $t_2$, with $t_2$ and $t_1$ being arbitrarily distant from each other, is that we know the analytical form for the risk-neutral transition density of the stock price and variance level from $t_1$ to $t_2$ (the so-called Green function) either directly, as in the Black–Scholes–Merton case, or its Fourier transform, as in the Heston–Bates case. Finally, the fact that the value of the call is computed at a fixed grid point allows us to exploit some invariance properties of the transition densities that further increase the speed of the method. For the details of the methodology and the convergence properties, see Appendix B.7

B. Estimating the Cost of Suboptimal Nonexercise

The quantification of the suboptimality and the associated costs is computed in two steps: i) At each day before the payment of the dividend, the Black–Scholes, Merton, and Bates models are calibrated separately for each stock, and ii) by looking at the open interest at the end of the day, we calculate the money left on the table by investors who do not optimally exercise their options.

1. Calibration Results

Table 2 reports the results of the calibration for the three modeling frameworks. We obtain the parameters through the minimization of the implied volatility mean squared error, as in Christoffersen and Jacobs (2004).8 The first line of Table 2 displays the average values of the parameters calibrated on our sample of single stocks, and the second line reports the average values that Bakshi et al. (1997) obtain for the same parameters from contracts written on the S&P 500

---

7Section B of the Supplementary Material describes how the invariance properties of the transition densities simplifies the numerical implementation, while Section C makes an extensive comparison between the recursive projections and existing numerical methods.

8Section D of the online Supplementary Material gives a detailed description of the data and the calibration procedure.
Table 2 presents the average values of the parameters of the Black–Scholes (BS), Merton (MRT), and Bates (BTS) models calibrated at each day before the ex-dividend date on the options written on the dividend-paying stocks belonging to the Dow Jones Industrial Average Index (DJIA). In total, we computed 1,701 calibrations, and the reported values are the averages across these calibrations. On average, the in-sample sum of squared error is equal to 0.26 for the Black–Scholes model, 0.20 for the Merton model, and 0.16 for the Bates model with stochastic volatility.

Table 2
Calibrated Parameters

| Underlying | BS | MRT | BTS |
|------------|----|-----|-----|
| All stocks | 0.29 | 1.33 | 0.22 | 0.16 | −0.12 | 0.50 | 0.18 | −0.12 | 0.75 | 0.32 | 1.52 | −0.35 | 0.28 |
| S&P500     | 0.18 | n/a | n/a | n/a | n/a | 0.61 | 0.14 | −0.09 | 0.4 | 0.2 | 3.93 | −0.52 | 0.2 |

index. The parameters that rule the level of the volatility smile (i.e., the Black–Scholes volatility \( \sigma_{BS} \), the long-term volatility \( \sigma_{LT} \), and the spot volatility \( \sigma_0 \)) are much higher in our single-stock calibration than in the index calibration, which reflects the well-known fact that an index is less volatile than its components. Indeed, in our sample, the average Black–Scholes volatility is 29%; \( \sigma_0 \), the instantaneous volatility the day of the calibration, is 28%; and the average long-term implied volatility is 32%; for the index options, the same parameters take the values of 18.15%, 20%, and 20%, respectively. The jump parameters in the Bates model show that jumps are on average less frequent in the single-stock case than in the index case (\( \gamma_{Stocks} = 0.5 \) against \( \gamma_{S&P500} = 0.61 \)), but the amplitude and variability are higher for single stocks (\( \mu_{\psi, Stocks} = −0.12 \) and \( \sigma_{\psi, Stocks} = 0.18 \), respectively) than for the index (\( \mu_{\psi, S&P500} = −0.09 \) and \( \sigma_{\psi, S&P500} = 0.14 \)). The remaining two parameters of the stochastic volatility component of the Bates model, the correlation parameter \( \rho \) and the volatility of volatility \( \omega \), have a specific impact on the shape of the implied volatility smile (Hagan, Kumar, Lesniewski, and Woodward (2002), West (2005)). A negative \( \rho \) implies a negatively sloped smile. The correlation parameter is lower in absolute value in the single-stock case (\( \rho_{Stocks} = −0.35 \)) than for the index (\( \rho_{S&P500} = −0.09 \)). The volatility of volatility \( \omega \) determines the convexity of the implied volatility smile. The difference in the values taken by \( \omega \) is striking. We find \( \omega \) to be 75% for stock options, while Bakshi et al. (1997) find a much smaller value of 40% for short-term index options. This difference is due to the higher convexity of the implied volatility smiles of stock options versus that of index options, another feature documented by Bollen and Whaley (2004). The parameters \( \rho \) and \( \omega \) are related to the smile shape through the higher moments of the distributions of the returns of the underlying. A more negative \( \rho \) generates a more negatively skewed distribution of index returns with respect to stock returns, whereas a higher \( \omega \)
in the single-stock returns leads to a higher kurtosis than in the index return distribution.\footnote{In Section D of the online Supplementary Material, we provide more detailed results, including a breakdown of the calibration by stock, and we show that the values of the calibrated parameters are homogeneous across stocks.}

2. Suboptimality Results

After having calibrated the models, we are able to compute the price $C(S_{t-1} - d, K, T)$ on the day prior to the dividend payment date $t_h$ by using the recursive projections. The price $C(S_{t-1} - d, K, T)$ is the continuation value of the option at date $t_h$, when the dividend $d$ is distributed. By comparing it with the intrinsic value $S_{t-1} - K$, we can assess which options should be exercised on $t_h - 1$. If an option should be exercised (i.e., $C(S_{t-1} - d, K, T) \leq S_{t-1} - K$), then a positive open interest at the end of the day before the ex-dividend date ($OI_{t-1} > 0$) measures investors’ failure to exercise the option. In this case, we calculate the suboptimal nonexercise percentage as the following ratio:

\[
NE_h = \frac{OI_{t-1}}{OI_{t-2}},
\]

that is, $NE$ is the ratio of the number of contracts outstanding at the end of the day $t_h - 1$ to the total number of contracts outstanding at the end of day $t_h - 2$. The quantity defined in equation (5) is actually an approximation of the actual nonexercise ratio, as it neglects a possible issue of new contracts on date $t_h - 1$. This event is unlikely; indeed, Pool et al. (2008) test the approximation on a sub-sample of contracts for which they have real exercise data. They conclude that approximation (5) is a precise description of the actual exercise behavior of option investors. The total amount of money that is left on the table due to suboptimal nonexercise is given by the following formula:

\[
TL = 100 \times OI_{t-1} \times [(S_{t-1} - K) - C(S_{t-1} - d, K, T)].
\]

The continuation value $C(S_{t-1} - d, K, T)$ depends on the model used for pricing; hence, the total loss due to suboptimal nonexercise (TL) is itself model-specific.

Table 3, which presents the results for investors’ suboptimal nonexercise behavior, clearly shows that the optimal early exercise decision depends on the model used for the stock price. Under the Black–Scholes model, approximately 9.5% of the outstanding contracts should be exercised; the percentage decreases (approximately 7.5%) under the alternative models. This result is consistent with the numerical findings of Section II, where we show that, in the case of discrete dividends, the early exercise boundary under the Black–Scholes model is lower than that implied by the Merton and Heston models. As a general rule, contracts that should be exercised under the Merton or Bates models should also be exercised under the Black–Scholes model. In our sample, we find some exceptions to this rule because, in Section II, we choose the model parameters such that the total variance of the returns over the life of the option is the same in all models; in real data, this condition may not hold. To give some examples, 4,680 options should be exercised under the Black–Scholes model but not under the Bates
Table 3 reports the summary results of the total loss due to suboptimal nonexercise for the short-term call option series. The numbers are calculated for each series and each day before the ex-dividend date separately and are then pooled together. The number of contracts outstanding is the total open interest of all contracts 2 days before the ex-dividend date. The contracts that should be exercised under a specific model (i.e., Black–Scholes (BS), Merton (MRT), and Bates (BTS)) are the contracts outstanding for which the continuation value is lower than the exercise proceeds. The number of contracts that are left suboptimally nonexercised is the sum of the open interests one day before the ex-dividend of the contracts that should have been exercised. We compute the other quantities in the table in the following way:

\[
\begin{align*}
\text{Total market value} & = \text{Contracts outstanding} \times \text{Market price} \times 100, \\
\text{Money available} & = \max(0, (S - K - F - \text{Continuation value}) \times \text{Contracts outstanding} \times 100), \\
\text{Total loss} & = \max(0, (S - K - F - \text{Continuation value}) \times \text{Open interest}_{t-1} \times 100),
\end{align*}
\]

where \( F \) is the exercise fee. In the first column, the results are computed considering \( F = 0 \), while in the second column, the results are computed considering the conservative fee of $0.44. The first percentage in parentheses in the total loss due to nonexercise is computed with respect to the money available due to exercise opportunities, while the second is computed with respect to the total market value.

| Model | Without Fee | With Fee |
|-------|-------------|----------|
| Contracts outstanding | 406,414,980 | 99,392,927,000 |
| Total market value | 39,527,586 | 31,551,786 |

**Panel A. Contracts That Should Be Exercised**

| Model | Without Fee | With Fee |
|-------|-------------|----------|
| BS | 31,551,786 | 7.76% |
| MRT | 25,340,009 | 6.23% |
| BTS | 25,050,616 | 6.16% |

**Panel B. Contracts That Are Left Suboptimally Nonexercised**

| Model | Without Fee | With Fee |
|-------|-------------|----------|
| BS | 15,214,908 | 39.49% |
| MRT | 9,404,406 | 30.70% |
| BTS | 8,786,524 | 28.82% |

**Panel C. Money Available Due to Exercise Opportunity**

| Model | Without Fee | With Fee |
|-------|-------------|----------|
| BS | 770,287,766 | 0.77% |
| MRT | 555,922,918 | 0.56% |
| BTS | 617,173,686 | 0.62% |

**Panel D. Total Loss Due to Suboptimal Nonexercise**

| Model | Without Fee | With Fee |
|-------|-------------|----------|
| BS | 209,284,628 | 27.17% |
| MRT | 133,130,786 | (23.9%) |
| BTS | 147,480,996 | 23.9% |
model, whereas the reverse is true for only 249 contracts. Similarly, we find that 2,872 options should be exercised under the Black–Scholes model but not under the Merton model, whereas the opposite occurs with only 53 options. The first important lesson we learn is that, by allowing for more sophisticated models than the Black–Scholes model, the number of contracts that should be optimally exercised decreases by almost 25%. The suboptimality figures are model-dependent, and one may argue that they depend on the implemented calibration procedure. The comparison between our calibration results and the results of Bakshi et al. (1997) is reassuring in terms of the reliability of our calibration method. To justify all the suboptimal behavior found in our sample, we should obtain unreasonably high values for the jump and intensity parameters.

A second piece of evidence that stands out from Table 3 is that the percentage of investors who leave the options suboptimally nonexercised is higher under the Black–Scholes model than under the other models (39% versus approximately 30%, respectively). We compute these percentages in accordance with definition (5). If we restrict our attention to the 1,965 contracts in our sample that should be exercised under the Black–Scholes model but not under the Merton or Bates models, we find a striking 81% of no exercise. These results may suggest that investors do not limit themselves to a Black–Scholes world when evaluating their options but rely instead on more sophisticated models that include jumps or stochastic volatility. Even if this evidence is a considerable step toward understanding investors’ decision-making process, it does not fully solve the puzzle. Indeed, even in the Merton and Bates models, we still find a high percentage of suboptimal nonexercises, which leads to a global loss of approximately $130–$140 million, down approximately 30% from the loss of $206 million in the Black–Scholes model. The possible interpretation of this residual amount in terms of hidden fees or of the opportunity cost of closely monitoring the options leads us to the discussion in Section III.C.

As a further reliability check on our calibration procedure, we compare our results regarding the exercise decision with those obtained by Pool et al. (2008) for the Black–Scholes case. In that work, the authors apply the early exercise decision rule to all options series by using the Black–Scholes model with historical volatility and find that 53.1% of investors leave their options unexercised when they should have been exercised instead. Their data span over 10 years (from 1996 to 2006). To compare our results with theirs, we divide our sample into two subsamples, the first spanning 1996–2006 and the second spanning 2006–2012. We calculate the average percentage of suboptimal nonexercise in the two subsamples and find that the percentage of suboptimal nonexercise under the Black–Scholes model is approximately 47% in the first subsample and 37% in the second. The decrease in the nonexercise behavior with time intimates that investors became more attentive in monitoring their investments. There is a small difference between our results (47%) and the 53.1% found in Pool et al. (2008). The explanation is most likely our focus on the constituents of the Dow Jones Industrial Average, whereas Pool et al. (2008) consider all option series. It is likely that, stock and option prices are monitored more closely for large-cap companies than they are on average.

Throughout our empirical investigation, we choose a model-based approach to calculate the continuation value of the option $C(S_{t_{n-1}} - d, K, T)$. We could
also have used a market-based approach in which the continuation value is the market price of the option. The market-based approach checks whether the quantity $C_{MKT}(S_{t_{b-1}}, K, T) - (S_{t_{b-1}} - K)_+$ equals 0, where $C_{MKT}(S_{t_{b-1}}, K, T)$ is the observed market price at $t=t_{b-1}$. As discussed by Pool et al. (2008) and Barraclough and Whaley (2012), the market-based approach has shortcomings, the most important of which is that it does not make it possible to calculate the total loss due to suboptimal nonexercise, which we do in equation (6). In addition, the bid–ask spread and the discreteness of the prices make it difficult to decide which $C_{MKT}$ should be used. For all these reasons, we follow Pool et al. (2008) and Barraclough and Whaley (2012) and use a model-based approach with equation (5) to account for options investors’ actual exercise behavior. Barraclough and Whaley (2012) use only the market-based approach as a useful model-free test to confirm the presence of suboptimal nonexercise behavior. They find that the market-based approach gives a magnitude of suboptimality comparable to that implied by the model-based approach. This last piece of evidence is an additional argument against the possible objection that an incorrect calibration of the model parameters may be the source of the suboptimal exercise figures.

C. The Role of Fees

According to the recent literature on option prices (Jensen and Pedersen (2016), Christoffersen, Goyenko, Jacobs, and Karoui (2017)), trading costs and financial frictions in general strongly affect both the option prices and the early exercise decision of American options. In this section, we investigate whether investors’ suboptimal nonexercise behavior is due to the trading costs that investors face when exercising their options.

Following Pool et al. (2008), we model the costs of exiting a long call position as a per share lump sum $\mathcal{F}$ that the investor must pay at the moment he decides to exercise. The specific value of $\mathcal{F}$ depends on how the exit is accomplished according to the investor’s different possible objectives. The most expensive value of the fee $\mathcal{F}$ is attained when the investor wants to exercise the option and reenter the same call position. Pool et al. (2008) estimate an average value for the rollover costs $\mathcal{F}$ by using the commissions of the high-cost brokers, and they obtain a very conservative amount of $\mathcal{F} = $0.4446/share. A detailed description of the components of the fee $\mathcal{F}$ can be found in Pool et al. (2008).

To understand the role of fees in the early exercise decision, we perform two empirical exercises. As a first check, we reperform the exercise in Section III.B.2 and compute the loss due to a suboptimal nonexercise decision, this time using $C(S_{t_{b-1}} - d, K + \mathcal{F}, T)$ as the continuation value and $(S_{t_{b-1}} - K - \mathcal{F})$ as the intrinsic value. The fee value is $\mathcal{F} = 0.4446$. The fee $\mathcal{F}$ enters both in the exercise proceeds and in the continuation value. Indeed, at the moment of the exercise decision, the investor should decide whether to exercise and hence pay the exercise fee immediately or to not exercise and postpone the payment of the exercise fee.

Accordingly, the calculation of the total amount of money lost due to suboptimal nonexercise is given by

$$
(7) \quad TL_{\mathcal{F}} = 100 \times OI_{t_{b-1}} \times [(S_{t_{b-1}} - K - \mathcal{F}) - C(S_{t_{b-1}} - d, K + \mathcal{F}, T)].
$$
The second column in Table 3 shows the summary results including the fee. They are not very different from those obtained without considering the fee (first column of Table 3). We can conclude that the inclusion of trading costs does not change the big picture regarding investors’ suboptimal nonexercise, as outlined in the previous paragraph.

As a second empirical exercise, and in order to detect possible additional costs that are not considered in the fee $F$, we calculate the value of the fee that would justify investors’ nonexercise decision. To do so, for each option for which $C(S_{t_0} - d, K, T) < (S_{t_0} - K)$ but that is not optimally exercised by some investors, we compute the value of the implied fee $F$ that would justify the nonexercise decision. It amounts to numerically finding the 0 of the following function:

$$f(F) = C(S_{t_0} - d, K + F, T) - (S_{t_0} - K - F).$$

The results are reported in Table 4. The average implied fee is between $7 and $8 per share, incredibly high compared to the already conservative fee of $0.4446/share estimated by Pool et al. (2008). No realistic hidden fees can sum up to $7/share, and the trading costs of exiting a long call option position cannot fully justify investors’ suboptimal nonexercise behavior. We can interpret the difference between the implied fee of $8 and the conservative fee of 0.4446 as an implied opportunity cost for the holder of the option to monitor the optimal exercise of the American option. Investors may not respond immediately to favorable stock price movements and may take some time before reacting and optimally exercising their option, which would be in line with the behavior on the early prepayment of mortgages documented by Stanton (1995).

| Underlying | BS | MRT | BTS | BS | MRT | BTS |
|------------|----|-----|-----|----|-----|-----|
| All stocks | 7.54 | 7.27 | 7.23 | 94% | 93% | 93% |

**D. Discussion of the Findings on the Early Exercise Decision**

In Section III.B.2, we documented a decrease in the suboptimal exercise behavior when jumps and stochastic volatility are added to the stock process. The decrease is explained by an increase of the value of the early exercise boundary under the alternative modeling environments. We now extend the discussion on the early exercise boundary of Section II and provide theoretical insights that justify our empirical findings.

In the discussion of Section II, we found that the early exercise boundary reacted differently when moving from Black–Scholes to Merton and Heston...
depending on the modeling assumption of the dividend payment. The findings in the case of a continuous dividend do not provide an unambiguous prediction on the effect of the underlying process on the exercise boundary and are in line with those of Adolfsson et al. (2013) for the stochastic volatility case and those of Amin (1993) for the jump-diffusion case. The former work documents a lower exercise boundary in the Heston case (for all maturities) with respect to the Black–Scholes case, while the latter shows evidence of a lower boundary in the Merton case than under Black–Scholes case for longer maturities, while the opposite is true for shorter maturities. Adolfsson et al. (2013) and Amin (1993) back up these findings using simulations but do not explain why it is so. In Appendix C, we provide theoretical insights on these results.

The results of Section II that are more relevant to our empirical application are the ones in the case of a discrete dividend. These are, to our knowledge, entirely new. They consistently predict an increase of the early exercise boundary. In the following, we explain why this is the case. We first consider the stochastic volatility case. Assume that there is only one discrete dividend to be paid. The continuation value of the call option immediately after the ex-dividend date is that of a European call with the remaining time to maturity. When the correlation \( \rho \leq 0 \), the price of European options for a deep in-the-money call, for which early exercise could be optimal, is higher in the Heston case than in the Black–Scholes case (see Heston (1993), Hull and White (1987)). For instance, in the left graph of Graph B of Figure 1, for a time to maturity of 0.25, this would be the case in the range of stock prices of approximately 150. Even by accounting for the dividend drop in computing the continuation value, the ex-dividend stock price should remain in the region where the Heston price is higher. We can repeat the same argument for a number of discrete dividends that are sufficiently small (typically of the order of a couple percentage points) to prevent the stock price from falling into the price range in which the call has more value under the Black–Scholes model. A higher continuation value under the Heston model in turn implies a higher early exercise boundary.

We move on to consider the behavior of the early exercise boundary under the Merton model (i.e., model (4) with the volatility of volatility \( \omega = 0 \) and \( \sigma_t \) set constant to \( \sigma_M \)). To interpret the graphs in Figure 2, we must make an important distinction. For short-maturity options, the jump component in the first line of equation (4) dominates the diffusion component. As explained in Amin (1993) and Merton (1976), the result is higher prices for short-maturity, in-the-money call options under the Merton model than under the Black–Scholes model. We call this effect the jump effect. For longer maturities, the jump effect no longer dominates the diffusion component but instead creates an interplay that makes the jump-diffusion process observationally similar to a stochastic volatility process. For a discrete dividend, both the jump effect and the stochastic volatility effect, as previously discussed in the Heston case, predict a higher boundary in the Merton case than in the Black–Scholes case, which holds true for all maturities. The full Bates specification (4) must imply a higher exercise boundary than the Black–Scholes model in the discrete divided case, as it displays both jumps and stochastic volatility, and both features push the boundary upward.
IV. Concluding Remarks

We investigate investors’ exercise behavior in a large database of 101,295 series of short-term American call options (9.5 million prices). Pool et al. (2008) find that more than 40% of investors fail to optimally exercise their contracts. We extend these authors’ analysis by including stochastic volatility and jumps to the process of the underlying stock. In order to deal with the large option database and the repeated calculations required for the calibration and pricing, we develop an option-pricing technique that is simultaneously fast, precise, and able to handle both multidimensional dynamics and cash dividend distributions. By applying our technique to the data set, we can explain up to 25% of the gain forgone due to suboptimal exercise decisions, as computed in Pool et al. (2008). This result confirms the insights we obtain from the theoretical developments of the article. Indeed, we show that the exercise boundary is higher under the Merton and Heston models than under the Black–Scholes model if the dividend is discrete. This result underlines the importance of correctly modeling the dividend distribution. We show that by modeling the dividend as a continuous yield instead of as a discrete cash flow, the exercise frontier in the Bates model could have been lower instead of higher, and the suboptimal exercise behavior of Pool et al. (2008) could have been reinforced instead of mitigated. We further try to check whether we can explain the remaining part of the suboptimal behavior in terms of transaction costs (Jensen and Pedersen (2016)). We show that hidden transaction costs would need to be unrealistically large to explain the entire amount forgone by investors. This observation leads us to interpret the implied transaction fee as a monitoring cost.

As a final remark, the new pricing methodology (recursive projections) we develop is a powerful tool that allows new empirical research investigations. On one hand, American option pricing is only a particular case of stochastic optimal control problems. We can think of applying the recursive projection method to other problems, such as the optimal portfolio allocation involving complex and path-dependent financial assets or incentives contracts (Hodder and Jackwerth (2007)). Currently, these types of complex problems are solved by using Monte Carlo simulations (Detemple, Garcia, and Rindisbacher (2003)), and our method could offer a more efficient computational alternative. On the other hand, our method facilitates the analysis based on individual stocks paying discrete dividends and sophisticated pricing models, similar to recent work by Kelly, Lustig, and Van Nieuwerburgh (2016b) and Christoffersen, Fournier, and Jacobs (2017). We can also think, for example, of extending the recent studies of tail risk (Andersen, Fusari, and Todorov (2015), (2017)) and political uncertainty (Kelly, Pástor, and Veronesi (2016a)) to single-name stock options. We can also use our method to efficiently compute the Greeks of the options, which are the risk metrics most frequently used by option traders. All of these analyses are outside the scope of this article and are left for future research.
Appendix A. Approximating American Option Prices by European Prices

In Appendix A, we provide further evidence on the importance of correctly modeling the dividend as a discrete cash flow when computing the early exercise boundary.

The setup is the same as that in Section II of the main text. There are two dividends to be paid over the remaining life of the option, which is 6 months. One is to be paid immediately (\(t = 0\)), while the second is to be paid in 3 months (\(t = 0.25\)). We compute the early exercise boundary i) by correctly taking into account that the dividends are discrete and ii) by using the “escrow dividend” approximation. That is, we price the contract at \(t = 0\) as if it were a European option, with the continuation value after the first dividend payment computed not at \(S_0 - d\) but at \(S_0 - d - d \cdot e^{-r0.25}\). We compute the early exercise boundary under the three models (Black–Scholes, Merton, and Bates), with the following representative parameters: \(\gamma = 5, \mu_d = 0, \sigma_d = 0.2, \sigma_{MRT} = 0.2, \sigma_{BS} = 0.2, \omega = 0.1, \sigma_{LT} = 0.3, \beta = 4, \rho = -0.5,\) and \(r = 0.05\). Figure A1 displays the results of our numerical study. We call \(S_{A,t}\) the early exercise boundary computed under assumption i) and \(S_{E,t}\) the one under assumption ii). The two boundaries \(S_{A,t}\) and \(S_{E,t}\) coincide after the dividend at \(t = 0.25\) is paid out, since there are no more intermediate cash flows before maturity, and the continuation value is in both cases that of a European call with a time to maturity of 3 months. At \(t = 0\), the two boundaries are different, with \(S_{E,t} < S_{A,t}\), which means that an investor who bases his exercise decision on the approximation \(S_{E,t}\) may suboptimally decide to exercise the option at \(t = 0\), if \(S_{E,t} < S_0 < S_{A,t}\). He would then incur a loss given by \(C(S_0 - d, 0) - (S_0 - K)\), where \(C(S_0 - d, 0)\) is the correct price of an American option computed at the stock value \(S_0 - d\). The upper bound for the loss coming from the escrowed dividend approximation is reached exactly when \(S_0 = S_{E,t}\). In this case, the following formulae give the maximum percentage loss at \(t = 0\) under the three modeling environments considered and with the model parameters given previously:

\[
\begin{align*}
DL\%_{BS} & = \frac{C_{BS}(S_{E,BS}) - (S_{E,BS} - K)}{C_{BS}(S_{E,BS})} = 0.63\%, \\
DL\%_{MRT} & = \frac{C_{MRT}(S_{E,MRT}) - (S_{E,MRT} - K)}{C_{MRT}(S_{E,MRT})} = 0.48\%, \\
DL\%_{BTS} & = \frac{C_{BTS}(S_{E,BTS}) - (S_{E,BTS} - K)}{C_{BTS}(S_{E,BTS})} = 1.14\%,
\end{align*}
\]

where we have suppressed for readability the dependence from time and where the indexes BS, MRT, and BTS mean that we have computed the relevant quantities under the Black–Scholes, Merton and Bates models, respectively. The investors can lose up to 1% of the market value of their option if they exercise according to the wrong boundary.

We finally check our database to determine whether there are situations in which the stock reaches the value \(S_{E,t}\) but not \(S_{A,t}\) on the days before the ex-dividend dates. We find that this occurrence does happen, and not sporadically. For example, on May 10, 2006, the Dupont stock closed at $45.71. The call option with \(K = 30\) and \(T = 0.45\) should not be exercised if the continuation value is computed correctly, but the option should be exercised if the continuation value is approximated with a European price. In this case, if an investor exercises his option incorrectly, he will suffer a loss given by \(DL\%_{BS} = 0.07\%, \ DL\%_{MRT} = 2\%, \) and \(DL\%_{BTS} = 0.06\%.\) These figures are not small relative to the bid–ask spreads.
FIGURE A1
Early Exercise Boundary with the Escrow Dividend Approximation

In Figure A1, for each model, $S^*_E$ is the true early exercise boundary, while $S^*_A$ is calculated by approximating the continuation value of the option with the price of a European option, where the starting value of the stock is set equal to $S_0$ minus the present value of all future dividends.

Appendix B. Valuation by Fast Recursive Projections

In Appendix B, we detail the recursive projection method that we developed to compute the option prices and, consequently, the exercise boundaries in our study. The main innovation of the methodology is that it can handle in a simple and fast way both multidimensional processes and discrete dividend distributions for the underlying asset.

Let us assume that we need the value of an option at some specific points in time, $(t_i)_{i=1,...,L}$. We use the notation $V(x, t_i)$ for the value of the option at time $t_i$ and for a value of the underlying $S_0 = x$. The value function $V(x, t_i)$ does not need to be the value of a call option. The recursive projections are a flexible tool that can value a wide range of path-dependent contracts (see Andrícopoulos et al. (2007) for treatment of the kind of path dependency that we can address). For readability, we do not include the contract parameters $T$ and $K$ in the arguments of the value function. At each time $t_i$, the holder of the option decides whether to exercise. Thus, the value of the option at time $t_i$ is given by the maximum between the intrinsic value (i.e., the proceeds from exercising the option) and the continuation value of the following option:

$$V(x, t_i) = \max \{ H(x, t_i), \mathbb{E}[e^{-r(t_{i+1}-t_i)}V(S_{t_{i+1}}, t_{i+1})|S_0 = x] \}.$$  \hspace{1cm} (B-1)

$H(x, t_i)$ is a payoff function, for instance $H(x, t_i) = \max \{ x - K, 0 \}$ in the case of a call option or $H(x, t_i) = \mathbb{I}(x < K)$, $\mathbb{I}(\cdot)$ being an indicator function for a digital put. At maturity, $t_L = T$, $V(x, T) = H(x, T)$. The main requirement to be able to apply the recursive projections is that the discounted probability distribution (the so-called Green function), with respect to which the conditional expectation in equation (B-1) is taken, be known analytically, whether in direct form or through its Fourier or Laplace transform. In the simple case, when we have a direct analytical form for the Green function, such as in the Black–Sholes case, we can write the expectation in equation (B-1) as

$$\mathbb{E}[e^{-r(t_{i+1}-t_i)}V(S_{t_{i+1}}, t_{i+1})|S_0 = x] = \int G(x, t_i; y, t_{i+1})V(y, t_{i+1})dy.$$  \hspace{1cm} (B-2)

The integral in equation (B-2) is taken over the values $S_{t_{i+1}}$ that the underlying asset can take, and $G(x, t_i; y, t_{i+1})$ is the discounted transition density of the underlying asset form value $x$ at $t_i$ to the value $y$ at time $t_{i+1}$. We now restrict our attention to the value of the derivative at a fixed grid of values of $S_t$: $(y_1, \ldots, y_N)$ with a constant step $\Delta y = y_i - y_{i-1}$. 

---

Cosma, Galluccio, Pederzoli, and Scaillet 351
The grid is the same for each value \( t \in (t_{i}), i = 1, \ldots, T \). Then, we can write the following equality in matrix form:

\[
\phi(t_{i}) \overset{\text{def}}{=} \max \{ H(t_{i}), G(t_{i}; t_{i+1})V(t_{i+1}) \},
\]

where \( V(t_{i+1}) \) is an \( N \times 1 \) vector with entries \( V(t_{i+1}) = V(y, t_{i+1}). \) \( G(t_{i}; t_{i+1}) \) is an \( N \times N \) matrix with entries \( G_{ij} = G(y, t_{i}; y_{j}, t_{i+1}) \Delta y \), and \( H(t_{i}) \) an \( N \times 1 \) vector with entries \( H(t_{i}) = H(y, t_{i}) \). It is easy to check that the multiplication of \( V(t_{i+1}) \) by the \( i \)th row of \( G(t_{i}; t_{i+1}) \) is a discretization of the expectation on the right side of equation (B-1) for \( x = y \). The vector \( \phi(t_{i}) \) is an approximation of the value of the contract \( V(t_{i}) \) at \( t_{i} \) and can be fed again into equation (B-3) to obtain \( \phi(t_{i-1}) \):

\[
\phi(t_{i-1}) \overset{\text{def}}{=} \max \{ H(t_{i}), G(t_{i-1}; t_{i})V(t_{i}) \}.
\]

The main difference between equations (B-3) and (B-4) is that in the former, we assumed \( V(t_{i+1}) \) is known, for instance, at maturity, where \( V(T) = H(T) \), whereas in the latter, \( \phi(t_{i}) \) is an approximation of the real value. Equation (B-4) is at the heart of the recursive projections. It provides a recursive formula that relates the value of a contract at different points on a grid of values for the underlying and at successive points in time. Equation (B-3) shows how the pricing of options that can be exercised at specific moments, such as Bermudan contracts, can be expressed as a series of matrix multiplied by vector operations. Furthermore, if the time interval \( \tau = t_{i+1} - t_{i}, l = 1, \ldots, L - 1 \) is constant and if the pricing operator enjoys a stationarity property (time translation invariance), then the matrix \( G(t_{i}; t_{i+1}) = G(\tau) \) has constant entries and the algorithm involves only a single computation of the matrix.

The methodology easily extends in the presence of discrete dividends paid on potential exercise dates. We need only to add the dividend \( \delta \) to the continuation value in equation (B-4). Hence, in order to price an American option on a dividend-paying stock, equation (B-4) must be modified by computing the state price density \( G(x, t_{i}; y, t_{i+1}) \) at the grid \( (y_{j} - \delta(y_{j})), j = 1, \ldots, N \) for the conditioning value \( x \) whenever \( t_{i} \) is the dividend distribution date. The entries of the matrix \( G(t_{i}; t_{i+1}) \) then become \( G_{ij} = G(y_{j} - \delta; t_{i}; y_{j}, t_{i+1}) \Delta y \). Given the freedom in choosing where to sample \( G, \delta(x) \) could be any function of \( x \). If \( \delta(x) = r_{x} x \), then we can accommodate for a proportional dividend. If \( \delta(x) = d \), then we can accommodate for a discrete dividend amount \( d \). If \( \delta(x) = 0 \), then we return to the Bermudan option case. The value function \( \phi(t_{i}) \) still gives the value of the contract at the grid points \( (y_{j}, \ldots, y_{j}) \); thus, we can use its approximation \( \phi(t_{i}) \) as the input for the following step of the algorithm, and the recursive property of the algorithm is maintained. Figure B1 shows how the recursive projections easily accommodate dividend payments.

To carry out the analysis of Section III.B.2, we must be able to incorporate stochastic volatility and jumps in the description of the process. In the class of stochastic volatility models, there are two state variables: the underlying asset \( S_{0} \) and the variance \( \sigma_{v}^2 \). One of the most popular models belonging to this class is the Heston model. In the Heston model, we know the analytical form of the Fourier transform of the bivariate state price density \( G_{2}(S_{0}, \sigma_{v}^2; t_{i}; y, w, t_{i+1}) \), which describes the discounted transition probability density from the asset level \( S_{0} \) and variance level \( \sigma_{v}^2 \) at time \( t_{i} \) to the asset level \( y = S_{0} + \) and variance level \( w = \sigma_{v}^2 \) at time \( t_{i+1} \). Let its Fourier transform be \( \hat{G}_{2}(S_{0}, \sigma_{v}^2; t_{i}; \lambda, \kappa, t_{i+1}), \) so that \( G_{2}(S_{0}, \sigma_{v}^2; t_{i}; y, w, t_{i+1}) = \frac{1}{4\pi^{2}} \int d\lambda d\kappa e^{-i(\lambda y + \kappa w)} \hat{G}_{2}(S_{0}, \sigma_{v}^2; t_{i}; \lambda, \kappa, t_{i+1}), \) where \( i \) is the

---

\[10\] When we consider discrete dividends, we rule out arbitrage situations in which the dividend is too large with respect to the stock price (see Haug, Haug, and Lewis (2003) for a discussion).

\[11\] For an explicit formula, see Griebsch (2013).
FIGURE B1
Discrete Dividends in the Recursive Projection Methodology

Figure B1 presents a recursive scheme without dividends (Graph A) and with discrete dividends (Graph B). In Graph A, at date $t = t_l + 1$, the intrinsic value $H(y_i, t_l + 1) = \max\{y_i - K, 0\}$ is compared with the continuation value $C(y_i, T, K)$ computed at the same grid point $y_i$ (black ball). In Graph B, at the ex-dividend date $t = t_l + 1$, the intrinsic value $H(y_i, t_l + 1)$ at grid point $y_i$ (black ball) is compared with the continuation value $C(y_i - \delta, T, K)$ at $y_i - \delta$ (red ball).

Imaginary unit. There are different ways to obtain a numerical approximation of $G_2$. One straightforward possibility is to consider the function $\Gamma_2(y_i, w_p, t_l; y_j, w_q, t_l+1)$, obtained by numerically inverting $\hat{G}_2$ by using a Fast Fourier Transform (FFT), which gives the approximated discounted transition probability density from the asset level $S_{t_l} = y_i$ and variance level $\sigma_{t_l}^2 = w_p$ at time $t_l$ to the asset level $y_j = S_{t_l+1}$ and variance level $w_q = \sigma_{t_l+1}^2$ at time $t_{l+1}$, where $(w_1, \ldots, w_W)$ is fixed, equispaced of values taken by the stochastic variance $\sigma_t^2$, with step $\Delta w$.

In the stochastic volatility framework, the recursive pricing relation for an option that can be exercised on a discrete set of dates consists of moving backward in time as in equation (B-4), with

\[
V(x, \xi, t_l) = \max\{H(x, t_l), E\left[e^{-r(t_{l+1}-t_l)}V(S_{t_{l+1}}, \sigma_{t_{l+1}}^2, t_{l+1})\right|S_{t_l} = x, \sigma_{t_l}^2 = \xi]\}
\]

\[\text{In Cosma, Galluccio, Pederzoli, and Scaillet (2016), we describe a more general approach to obtain an approximation of the } G_2 \text{ function in terms of functional projections. The function } \Gamma_2(y_i, w_p, t_l; y_j, w_q, t_{l+1}) \text{ is an approximation of } G_2(S_{t_l}, \sigma_{t_l}^2, t_l; y, w, t_{l+1}) \text{ since it is obtained by a discrete transform; in place of the continuous transform we would need to recover the true } G_2 \text{ transition density.}\]

\[\]
Thus, the recursive projections in the Heston model are the discrete counterpart of (B-5). In the following proposition, we give the convergence properties of the algorithm:

**Proposition 1.** Let $H(y, w, T)$ be such that $|H(y, w, T) - H(y', w', T)| < C\Delta$ for a positive constant $C$, and for $|y - y'| < \Delta y$, $|w - w'| < \Delta w$ and $\Delta = \sqrt{(\Delta y)^2 + (\Delta w)^2}$. Let $v_p(t)$ be defined for a set of dates $(t_i)_{i=1,...,L}$, with $t_L = T$, as follows:

\[
\begin{align*}
(B-6) \quad v_p(t_l) &= \max \left\{ \sum_{q=1}^{\infty} \Gamma_2(y_i, w_p, t_l; y_j, w_q, t_{l+1}) H(y_j, w_q, t_{l+1}) \sqrt{(\Delta y)(\Delta w)} \right\}, \\
& \quad \text{for } l = L - 1, \\
(B-7) \quad v_p(t_l) &= \max \left\{ \sum_{q=1}^{\infty} \Gamma_2(y_i, w_p, t_l; y_j, w_q, t_{l+1}) v_p(t_{l+1}) \sqrt{(\Delta y)(\Delta w)} \right\}, \\
& \quad \text{for } l = 1, \ldots, L - 2.
\end{align*}
\]

Then, for each $t_l$ in $(t_1, \ldots, t_{L-1})$, the approximated values $v_p(t_l)$ defined in equations (B-6) and (B-7) converge to the true value $V(y, w_p, t)$ with an approximation error of the order $O(\Delta^2)$.

In almost all applications, $H(y, w, t)$ depends only on the value $y$ taken by the underlying asset at $t_l$ and the computed price $v_p(t_{l-1})$ depends on the stochastic variance only through the conditioning value $\sigma^2_{t_{l-1}} = w_p$.

**Proof.** See Section A of the Supplementary Material. \qed

**Appendix C. Exercise Boundary with a Continuous Dividend**

In this section, we provide theoretical insights into why the early exercise boundary in the stochastic volatility models is lower than the boundary implied by the Black–Scholes model when the underlying stock distributes a continuous dividend yield. The discussion also applies to jump-diffusion processes for longer maturities because, in this case (as explained in Section D), they are observationally similar to stochastic volatility processes.

Following Kim (1990) and Jamshidian (1992), we can decompose the value $V(S, t)$ of an American option into two components, (i.e., the European value $V^E(S_t, t)$ and the early exercise premium $V^A(S_t, t)$), such that

\[
V(S, t) = V^E(S_t, t) + V^A(S_t, t) = e^{-r(t-s)} \mathbb{E}\left[ (S_T - K)^+ | S_s, \sigma^2_s \right] + \int_s^T e^{-r(t-s)} \mathbb{E}\left[ (r_d S_t - r_K) I_{\{S_T > S_t\}} | S_t, \sigma^2_t \right] ds,
\]

where $S^*_T$ is the early exercise boundary at time $s$ and $\mathbb{I}_{\{S_T > S_t\}}$ equals 1 if the stock is in the exercise region at time $s$, and 0 otherwise. We can interpret $V^A(S_t, t)$ as a continuum of European call options with maturity $T - s$, strike price $S^*_T$, and payoff $r_d S_t - r K$. For each of these European options, we can apply the results of Table II in Hull and White (1987), who compare the values of European options under general stochastic volatility dynamics with the Black–Scholes price. Call values under the stochastic volatility assumption are lower when the contracts are at-the-money and $\rho \leq 0$. The continuum of contracts composing the $V^A(S_t, t)$ are at-the-money when $S_t = S^*_T$. As confirmed from our numerical simulations, the $S^*_T$ values are distributed in the region immediately above $S = 150$ (i.e., exactly where the price of the American option under the Heston model is lower than that under the Black–Scholes model, such that the early exercise premium is lower under the former of the two modeling environments). This situation, in turn, explains the negative bump in the right graph of Graph B in Figure 1.
Supplementary Material

Supplementary Material for this article is available at https://doi.org/10.1017/S0022109018001229.

References

Adolfsson, T.; C. Chiarella; A. Ziogas; and J. Ziveyi. “Representation and Numerical Approximation of American Option Prices under Heston Stochastic Volatility Dynamics.” Research Paper 327, University of Technology Sydney Quantitative Finance Research Centre (2013).

Amin, K. I. “Jump Diffusion Option Valuation in Discrete Time.” Journal of Finance, 48 (1993), 1833–1863.

Andersen, T. G.; N. Fusari; and V. Todorov. “The Risk Premia Embedded in Index Options.” Journal of Financial Economics, 117 (2015), 558–584.

Andersen, T. G.; N. Fusari; and V. Todorov. “Short-Term Market Risks Implied by Weekly Options.” Journal of Finance, 72 (2017), 1335–1386.

Andricopoulos, A. D.; M. Widdicks; D. P. Newton; and P. W. Duck. “Extending Quadrature Methods to Value Multi-Asset and Complex Path Dependent Options.” Journal of Financial Economics, 83 (2007), 471–499.

Bakshi, G.; C. Cao; and Z. Chen. “Empirical Performance of Alternative Option Pricing Models.” Journal of Finance, 52 (1997), 2003–2049.

Bakshi, G.; N. Kapadia; and D. Madan. “Stock Return Characteristics, Skew Laws, and the Differential Pricing of Individual Equity Options.” Review of Financial Studies, 16 (2003), 101–143.

Barronough, K., and R. E. Whaley. “Early Exercise of Put Options on Stocks.” Journal of Finance, 67 (2012), 1423–1456.

Bates, D. S. “Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options.” Review of Financial Studies, 9 (1996), 69–107.

Black, F., and M. Scholes. “The Pricing of Options and Corporate Liabilities.” Journal of Political Economy, 81 (1973), 637–654.

Bollen, N. P. B., and R. E. Whaley. “Does Net Buying Pressure Affect the Shape of Implied Volatility Functions?” Journal of Finance, 59 (2004), 711–753.

Broadie, M.; M. Chernov; and M. Johannes. “Model Specification and Risk Premia: Evidence from Futures Options.” Journal of Finance, 62 (2007), 1453–1490.

Chen, D.; H. J. Härkönen; and D. P. Newton. “Advancing the Universality of Quadrature Methods to Any Underlying Process for Option Pricing.” Journal of Financial Economics, 114 (2014), 600–612.

Christoffersen, P.; M. Fournier; and K. Jacobs. “The Factor Structure in Equity Options.” Review of Financial Studies, 31 (2017), 595–637.

Christoffersen, P.; R. Goyenko; K. Jacobs; and M. Karoui. “Illiquidity Premia in the Equity Options Market.” Review of Financial Studies, 31 (2017), 811–851.

Christoffersen, P., and K. Jacobs. “The Importance of the Loss Function in Option Valuation.” Journal of Financial Economics, 72 (2004), 291–318.

Cosma, A.; S. Galluccio; P. Pederzoli; and O. Scaillet. “Early Exercise Decision in American Options with Dividends, Stochastic Volatility and Jumps.” Research Paper No. 16-73, Swiss Finance Institute (2016).

Detemple, J. B.; R. Garcia; and M. Rindisbacher. “A Monte Carlo Method for Optimal Portfolios.” Journal of Finance, 58 (2003), 401–446.

Eraker, B.; M. Johannes; and N. Polson. “The Impact of Jumps in Volatility and Returns.” Journal of Finance, 58 (2003), 1269–1300.

Fang, F., and C. W. Oosterlee. “A Fourier-Based Valuation Method for Bermudan and Barrier Options under Heston’s Model.” SIAM Journal on Financial Mathematics, 2 (2011), 439–463.

Griebisch, S. A. “The Evaluation of European Compound Option Prices under Stochastic Volatility Using Fourier Transform Techniques.” Review of Derivatives Research, 16 (2013), 135–165.

Hagan, P. S.; D. Kumar; A. S. Lesniewski; and D. E. Woodward. “Managing Smile Risk.” Wilmott Magazine, 1 (2002), 84–108.

Haug, E. G.; J. Haug; and A. Lewis. “Back to Basics: A New Approach to the Discrete Dividend Problem.” Wilmott Magazine, 9 (2003), 37–47.

Heston, S. L. “A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options.” Review of Financial Studies, 6 (1993), 327–343.

Hodder, J. E., and J. C. Jackwerth. “Incentive Contracts and Hedge Fund Management.” Journal of Financial and Quantitative Analysis, 42 (2007), 811–826.
Hull, J. *Options, Futures and Other Derivatives*, 9th global ed. Edinburgh, UK: Pearson Education (2018).
Hull, J., and A. White. “The Pricing of Options on Assets with Stochastic Volatilities.” *Journal of Finance*, 42 (1987), 281–300.
Jamshidian, F. “An Analysis of American Options.” *Review of Futures Markets*, 11 (1992), 72–80.
Jensen, M. V., and L. H. Pedersen. “Early Option Exercise: Never Say Never.” *Journal of Financial Economics*, 121 (2016), 278–299.
Kelly, B.; H. Lustig; and S. Van Nieuwerburgh. “Too-Systemic-to-Fail: What Option Markets ImPLY about Sector-Wide Government Guarantees.” *American Economic Review*, 106 (2016b), 1278–1319.
Kelly, B.; L. Pástor; and P. Veronesi. “The Price of Political Uncertainty: Theory and Evidence from the Option Market.” *Journal of Finance*, 71 (2016a), 2417–2480.
Kim, I. “The Analytic Valuation of American Options.” *Review of Financial Studies*, 3 (1990), 547–572.
Merton, R. C. “Option Pricing When Underlying Stock Returns Are Discontinuous.” *Journal of Financial Economics*, 3 (1976), 125–144.
Pool, V. K.; H. R. Stoll; and R. E. Whaley. “Failure to Exercise Call Options: An Anomaly and a Trading Game.” *Journal of Financial Markets*, 11 (2008), 1–35.
Stanton, R. “Rational Prepayment and the Valuation of Mortgage-Backed Securities.” *Review of Financial Studies*, 8 (1995), 677–708.
West, G. “Calibration of the SABR Model in Illiquid Markets.” *Applied Mathematical Finance*, 12 (2005), 371–385.