Spectral triples for higher-rank graph $C^*$-algebras

Carla Farsi, Elizabeth Gillaspy, Antoine Julien, Sooran Kang, and Judith Packer

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Abstract

In this note, we present a new way to associate a spectral triple to the noncommutative $C^*$-algebra $C^*(\Lambda)$ of a strongly connected finite higher-rank graph $\Lambda$. We generalize a spectral triple of Consani and Marcolli from Cuntz-Krieger algebras to higher-rank graph $C^*$-algebras $C^*(\Lambda)$, and we prove that these spectral triples are intimately connected to the wavelet decomposition of the infinite path space of $\Lambda$ which was introduced by Farsi, Gillaspy, Kang, and Packer in 2015. In particular, we prove that the wavelet decomposition of Farsi et al. describes the eigenspaces of the Dirac operator of this spectral triple.

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1 Introduction

Inspired by constructions from Arakelov geometry and Archimedean cohomology, Consani and Marcolli develop in [3] spectral triples associated to certain Cuntz-Krieger algebras. In this note, we expand the applicability of these spectral triples by generalizing the construction of [3] to the setting of higher-rank graphs. We also establish the compatibility of these spectral triples with the representations and wavelets for higher-rank graphs which were developed in [7]. Indeed, both spectral triples and wavelets are algebraic structures which encode geometrical information, so it is natural to ask about the relationship between wavelets and spectral triples.

Our earlier paper [5] was the first to establish a connection between wavelets and spectral triples in the setting of higher-rank graphs $\Lambda$. In that paper, we linked the representations of $C^*(\Lambda)$ from [7], and their associated wavelets, to the eigenspaces of the Laplace–Beltrami operators which arise from the spectral triples of Pearson and Bellissard [18]. The present article establishes that the
wavelets from [7] can also be identified with the eigenspaces of the Dirac operator of a Consani–Marcolli type spectral triple for $C^*(\Lambda)$.

Higher-rank graphs (also called $k$-graphs) were introduced by Kumjian and Pask in [13] to provide a combinatorial model to the higher-dimensional Cuntz-Krieger algebras given by Robertson and Steger in [21]. The $C^*$-algebras $C^*(\Lambda)$ of $k$-graphs $\Lambda$ have been studied by many authors and provided concrete, computable examples of many classifiable $C^*$-algebras. The graphical character of $k$-graphs has also facilitated the analysis of structural properties of $C^*(\Lambda)$, such as simplicity and ideal structure [19, 20, 3, 12, 1], quasidiagonality [2] and KMS states [10, 9, 8].

However, the analysis of the noncommutative geometry of $C^*(\Lambda)$ is in its infancy. Although Pask, Rennie, and Sims establish in [17] that higher-rank graph $C^*$-algebras often provide tractable examples of noncommutative manifolds, the current literature contains only one class of (semifinite) spectral triples for $C^*(\Lambda)$, namely those studied in [16]. In the Pearson–Bellissard spectral triples $(A, H, D)$ which were associated to higher-rank graphs in [5], the algebra $A = C_{Lip}(\Lambda^\infty)$ is commutative. Thus, the spectral triples for the noncommutative $C^*$-algebra $C^*(\Lambda)$, which we construct in Theorem 3.4 below, constitute an important step forward in our understanding of the noncommutative geometry of $C^*(\Lambda)$, in particular because of the link we establish between these spectral triples and wavelet theory for $C^*(\Lambda)$.

Wavelets for higher-rank graphs $\Lambda$ were introduced by four of the authors of the current paper in [7], building on work of Marcoli and Paolucci [15] for Cuntz–Krieger algebras, which in turn was inspired by the wavelets for fractal spaces developed by Jonsson [11] and Strichartz [22]. In all of these settings, the wavelets give an orthogonal decomposition of $L^2(X, \mu)$ for a fractal space $X$, which arises from applying dilation and translation operators to a finite family of “mother wavelets” $f_i \in L^2(X, \mu)$. The dilation and translation operators are determined by the underlying geometry. In Jonsson and Strichartz’ work, the self-similar structure of the fractal space $X$ dictates the dilation and translation operators, while in the higher-rank graph case, the dilation and translation operators arise from the graph structure. (See Section 2.2 for more details.)

To further our understanding of the noncommutative geometry of $C^*(\Lambda)$, we construct in Theorem 3.4 a spectral triple $(A_\Lambda, L^2(\Lambda^\infty, M), D)$, where $A_\Lambda$ is a dense (noncommutative) subalgebra of $C^*(\Lambda)$. This spectral triple was inspired by the spectral triples for Cuntz–Krieger algebras constructed in [3], and offers a very different perspective on the noncommutative geometry of $C^*(\Lambda)$ than the spectral triples of [16]. Theorem 3.5 then establishes our link between spectral triples and wavelets for higher-rank graphs by showing that the eigenspaces of the Dirac operator $D$ of this spectral triple agree with the wavelet decomposition of [7].

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2 Background material

We begin by detailing some foundational material needed for our results, and in particular reviewing the definition of a higher-rank graph $\Lambda$, the definition of its $C^*$-algebra $C^*(\Lambda)$, and associated
2.1 Higher-rank graphs and their $C^*$-algebras

Throughout this paper, we will view $\mathbb{N} := \{0, 1, 2, \ldots\}$ as a monoid under addition, or as a category. In this interpretation, the natural numbers are the morphisms in $\mathbb{N}$. Thus, for consistency with the standard notation $n \in \mathbb{N}$, we will write

$$\lambda \in \Lambda$$

to indicate that $\lambda$ is a morphism in the category $\Lambda$.

**Definition 2.1.** A higher-rank graph or k-graph by definition is a countable small category $\Lambda$ with a degree functor $d : \Lambda \to \mathbb{N}^k$ satisfying the factorization property: for any morphism $\lambda \in \Lambda$ and any $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n \in \mathbb{N}^k$, there exist unique morphisms $\mu, \nu \in \Lambda$ such that $\lambda = \mu \nu$ and $d(\mu) = m, d(\nu) = n$.

We often think of k-graphs as a generalization of directed graphs, so we call objects $v \in \Lambda^0$ “vertices” and morphisms $\lambda \in \Lambda$ are called “paths.” We write $r, s : \Lambda \to \Lambda^0$ for the range and source maps and $v\Lambda w = \{\lambda \in \Lambda : r(\lambda) = v, s(\lambda) = w\}$. Similarly, for any $n \in \mathbb{N}^k$, we write $v\Lambda^n = \{\lambda \in \Lambda : r(\lambda) = v, d(\lambda) = n\}$.

For $m, n \in \mathbb{N}^k$, we denote by $m \vee n$ the coordinatewise maximum of $m$ and $n$. Given $\lambda, \eta \in \Lambda$, we write

$$\Lambda^\text{min}(\lambda, \eta) := \{ (\alpha, \beta) \in \Lambda \times \Lambda : \lambda \alpha = \eta \beta, d(\lambda \alpha) = d(\lambda) \vee d(\eta) \}.$$  

We say that a k-graph $\Lambda$ is finite if $\Lambda^n$ is a finite set for all $n \in \mathbb{N}^k$ and say that $\Lambda$ has no sources or is source-free if $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. It is well known that this is equivalent to the condition that $v\Lambda e_i \neq \emptyset$ for all $v \in \Lambda$ and all basis vectors $e_i$ of $\mathbb{N}^k$. Also we say that a k-graph is strongly connected if, for all $v, w \in \Lambda^0$, $v\Lambda w \neq \emptyset$.

**Definition 2.2.** If $\Lambda$ is a finite k-graph with no sources, write $C^*(\Lambda)$ for the universal $C^*$-algebra generated by partial isometries $\{s_\lambda\}_{\lambda \in \Lambda}$ satisfying the Cuntz–Krieger conditions:

(CK1) $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;

(CK2) Whenever $s(\lambda) = r(\eta)$ we have $s_\lambda s_\eta = s_{\lambda \eta}$;

(CK3) For any $\lambda \in \Lambda$, $s_\lambda^* s_\lambda = s_{s(\lambda)}$;

(CK4) For all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k$, $\sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^* = s_v$.

Condition (CK4) implies that for any $\lambda, \eta \in \Lambda$ we have

$$s_\lambda^* s_\eta = \sum_{(\alpha, \beta) \in \Lambda^\text{min}(\lambda, \eta)} s_\alpha s_\beta^*,$$

where we interpret empty sums as zero. Consequently, $C^*(\Lambda) = \overline{\text{span}}\{s_\lambda s_\eta^* : \lambda, \eta \in \Lambda\}$.

**Definition 2.3.** Let $\mathcal{A}_\Lambda$ denote the dense $*$-subalgebra of $C^*(\Lambda)$ spanned by $\{s_\lambda s_\eta^*\}_{\lambda, \eta \in \Lambda}$.
An important example of a $k$-graph is the category $\Omega_k$, where
\[
\text{Obj}(\Omega_k) = \mathbb{N}^k, \quad \text{Mor}(\Omega_k) = \{(p, q) \in \mathbb{N}^k : p \leq q\}.
\]
The range and source maps $r, s$ in $\Omega_k$ are given by $r(p, q) = p$, $s(p, q) = q$, and the degree map $d : \Omega_k \to \mathbb{N}^k$ is given by
\[
d(p, q) = q - p.
\]
\textbf{Definition 2.4.} An \textit{infinite path} in a $k$-graph $\Lambda$ is a degree preserving functor $x : \Omega_k \to \Lambda$. We write $\Lambda^\infty$ for the set of infinite paths in $\Lambda$.

Given $\lambda \in \Lambda$, we define the \textit{cylinder set} $[\lambda] \subseteq \Lambda^\infty$ by
\[
[\lambda] := \{x \in \Lambda^\infty : x(0, d(\lambda)) = \lambda\}
\]
to be the infinite paths with initial segment $\lambda$. It is well-known (cf. [13]) that the collection of cylinder sets $\{[\lambda]\}_{\lambda \in \Lambda}$ forms a compact open basis for a locally compact Hausdorff topology on $\Lambda^\infty$. If a $k$-graph $\Lambda$ is finite, then $\Lambda^\infty$ is compact in this topology.

For each $m \in \mathbb{N}^k$, we have a shift map $\sigma^m$ on $\Lambda^\infty$ given by
\[
\sigma^m(x)(p, q) = x(p + m, q + m).
\]
(1)

for $x \in \Lambda^\infty$ and $(p, q) \in \Omega_k$. In duality to the shift map $\sigma^m$, for each $\lambda \in \Lambda$ we also have a prefixing map $\sigma_\lambda : [s(\lambda)] \to [\lambda]$ given by
\[
\sigma_\lambda(x) = \lambda x = \begin{cases} 
(\lambda, q) & q \leq d(\lambda) \\
(x(p - d(\lambda), q - d(\lambda)), p \geq d(\lambda)) & p \geq d(\lambda) \\
(x(p, d(\lambda)), x(0, q - d(\lambda)), p < d(\lambda) < q) & p < d(\lambda) < q
\end{cases}
\]
(2)

According to [10] Proposition 8.1, for any finite and strongly connected $k$-graph $\Lambda$, there is a unique self-similar Borel probability measure $M$ on $\Lambda^\infty$. To describe $M$, we require more definitions.

\textbf{Definition 2.5.} For a finite $k$-graph $\Lambda$ and $1 \leq i \leq k$, the \textit{vertex matrix} $A_i \in M_{\Lambda^0}(\mathbb{N})$ is
\[
A_i(v, w) = \#(v\Lambda^iw).
\]

Lemma 3.1 of [10] establishes that if $\Lambda$ is finite and strongly connected, then there exists a unique vector $\kappa^\Lambda \in (0, \infty)^{\Lambda^0}$, called the Perron–Frobenius eigenvector of $\Lambda$, such that
\[
\sum_{v \in \Lambda^0} \kappa^\Lambda_v = 1 \quad \text{and} \quad A_i\kappa^\Lambda = \rho_i\kappa^\Lambda \quad \forall 1 \leq i \leq k.
\]

The unique self-similar Borel probability measure $M$ of [10] is given on cylinder sets by
\[
M([\lambda]) = (\rho(\Lambda))^{-d(\lambda)}\kappa^\Lambda_{s(\lambda)} \quad \text{for} \quad \lambda \in \Lambda.
\]

Here $\rho(\Lambda) = (\rho_1, \ldots, \rho_k)$, where $\rho_i$ denotes the spectral radius of the vertex matrix $A_i \in M_{\Lambda^0}(\mathbb{N})$, and $(\rho(\Lambda))^n := \rho_1^{n_1} \cdots \rho_k^{n_k}$ for $n = (n_1, \ldots, n_k) \in \mathbb{R}^k$. We call the measure $M$ the Perron–Frobenius measure on $\Lambda^\infty$. 

4
2.2 Wavelets on higher-rank graphs

According to Proposition 3.4 and Theorem 3.5 of [7], there is a separable representation \( \pi \) of \( C^*(\Lambda) \) on \( L^2(\Lambda^\infty, M) \) when \( \Lambda \) is a finite, strongly connected \( k \)-graph. Theorem 3.3 below identifies a Dirac operator \( D \) for which this representation gives a spectral triple \((A_\Lambda, L^2(\Lambda^\infty, M), D)\).

Before stating Theorem 3.3 we review the definition of the representation \( \pi \) and the associated wavelet decomposition of \( L^2(\Lambda^\infty, M) \). For \( p \in \mathbb{N}^k \) and \( \lambda \in \Lambda \), let \( \sigma^p \) and \( \sigma_\lambda \) be the shift and prefixing maps on \( \Lambda^\infty \) given in (1) and (2). If we let \( S_\lambda := \pi(s_\lambda) \), the image of the standard generator \( s_\lambda \) of \( C^*(\Lambda) \) under the representation \( \pi \), then [7, Theorem 3.5] tells us that \( S_\lambda \) is given on characteristic functions of cylinder sets by

\[
S_\lambda \chi_{[\eta]}(x) = \chi_{[\lambda]}(x) \rho(\Lambda)^{d(\lambda)/2} \chi_{[\eta]}(\sigma^{d(\lambda)}(x)) = \begin{cases} 
\rho(\Lambda)^{d(\lambda)/2} & \text{if } x = \lambda \eta y \text{ for some } y \in \Lambda^\infty \\
0 & \text{otherwise}
\end{cases}
\]

(3)

Moreover, the adjoint \( S_\lambda^* \) of \( S_\lambda \) is given on characteristic functions of cylinder sets by

\[
S_\lambda^* \chi_{[\eta]}(x) = \chi_{[\sigma(\eta)]}(x) \rho(\Lambda)^{-d(\lambda)/2} \chi_{[\eta]}(\sigma(x)) = \begin{cases} 
\rho(\Lambda)^{-d(\lambda)/2} & \text{if } \lambda x = \eta y \text{ for some } y \in \Lambda^\infty \\
0 & \text{otherwise}
\end{cases}
\]

(4)

We can think of the operators \( S_\lambda \) as combined “scaling and translation” operators, since they change both the size and the range of a cylinder set \([\eta]\), and are intimately tied to the geometry of the \( k \)-graph \( \Lambda \).

This perspective enabled four of the authors of the current paper to use the representation \( \pi \) to construct a wavelet decomposition of \( L^2(\Lambda^\infty, M) \); we recall the details from [7, Section 4]. For each vertex \( v \) in \( \Lambda \), let

\( D_v = v\Lambda^{(1,\ldots,1)} \).

One can show (cf. [10, Lemma 2.1(a)]) that \( D_v \) is always nonempty when \( \Lambda \) is strongly connected.

Enumerate the elements of \( D_v \) as \( D_v = \{\lambda_0, \ldots, \lambda_{\#(D_v) - 1}\} \). Observe that if \( \lambda \in \Lambda \) is a 1-element set, then \([v] = [\lambda]\). If \( \#(D_v) > 1 \), then for each \( 1 \leq i \leq \#(D_v) - 1 \), we define

\[
\left( f^{i,v} \right)_{i,v} = \frac{1}{M[\lambda_i]} \chi_{[\lambda_i]} \quad \text{and} \quad \chi_{[\lambda_0]} = \frac{1}{M[\lambda_0]} \chi_{[\lambda_0]}.
\]

(5)

One easily checks that in \( L^2(\Lambda^\infty, M) \), \( \left( f^{i,v}, \chi_{[w]} \right) = 0 \) for all \( i \) and all vertices \( v, w \), and that

\[
\{ f^{i,v} : v \in \Lambda^0, 1 \leq i \leq \#(D_v) - 1 \}
\]

is an orthogonal set. Therefore, the functions \( \{ f^{i,v} \}_{i,v} \) span the subspace \( \mathcal{W}_0,\Lambda \subseteq L^2(\Lambda^\infty, M) \) from [7, Theorem 4.2], which we will henceforth call \( \mathcal{W}_0 \).

The following Theorem, which was proved in [7], justifies our labeling of the orthogonal decomposition (6) as a wavelet decomposition: the subspaces \( \mathcal{W}_n \) are given by applying “scaling and translation” operators \( S_\lambda \) to the finite family of “mother functions” \( \{ f^{i,v} \}_{i,v} \).
Theorem 2.6. [7, Theorem 4.2] Let $\Lambda$ be a finite, strongly connected $k$-graph and define $\mathcal{V}_0 := \text{span}\{x_v : v \in \Lambda^0\}$. Let $\mathcal{V}_0 := \text{span}\{x_v : v \in \Lambda^0\}$, and set

$$\mathcal{W}_n = \text{span}\{S_\lambda f^{i,s(\lambda)} : d(\lambda) = (n, \ldots, n), 1 \leq i \leq \#(D_{s(\lambda)}) - 1\}$$

for each $n \in \mathbb{N}$. Then $\{S_\lambda f^{i,s(\lambda)} : d(\lambda) = (n, \ldots, n), 1 \leq i \leq \#(D_{s(\lambda)}) - 1\}$ is a basis for $\mathcal{W}_n$ and

$$L^2(\Lambda^\infty, M) \cong \mathcal{V}_0 \oplus \bigoplus_{n=0}^{\infty} \mathcal{W}_n. \quad (6)$$

3 Spectral triples of Consani-Marcolli type for strongly connected finite higher-rank graphs

In Section 6 of [3], Consani and Marcolli construct a spectral triple for the Cuntz-Krieger algebra $\mathcal{O}_A$ associated to a matrix $A \in M_n(\mathbb{N})$. Recall from [14] that if $E$ is the 1-graph with adjacency matrix $A$, then $\mathcal{O}_A \cong C^*(E)$.

In this section, we generalize the construction of Consani and Marcolli to build spectral triples for higher-rank graph $C^*$-algebras $C^*(\Lambda)$. For these spectral triples (described in Theorem 3.4 below), it is shown in Theorem 3.5 that the eigenspaces of the Dirac operator agree with the wavelet decomposition from [7]. We also discuss in Remark 3.6 at the end of the section how to modify the construction of the spectral triple to make the eigenspaces of the Dirac operator compatible with the $J$-shape wavelets of [6].

Definition 3.1. Let $\Lambda$ be a finite, strongly connected $k$-graph. Define $\mathcal{R}_{-1} \subset L^2(\Lambda^\infty, M)$ to be the linear subspace of constant functions on $\Lambda^\infty$. For $s \in \mathbb{N}$, define $\mathcal{R}_s \subset L^2(\Lambda^\infty, M)$ by

$$\mathcal{R}_s = \text{span}\{x_\eta : \eta \in \Lambda, \sup\{d(\eta)_i : 1 \leq i \leq k\} \leq s\},$$

where $d(\eta) = (d(\eta)_1, \ldots, d(\eta)_k) \in \mathbb{N}^k$.

Let $\Xi_s$ be the orthogonal projection in $L^2(\Lambda^\infty, M)$ onto the subspace $\mathcal{R}_s$. For a pair $(s, r) \in \mathbb{N} \times (\mathbb{N} \cup \{-1\})$ with $s > r$, let

$$\hat{\Xi}_{s,r} = \Xi_s - \Xi_r.$$  

Since $\mathcal{R}_r \subset \mathcal{R}_s$, $\hat{\Xi}_{s,r}$ is the orthogonal projection onto the subspace $\mathcal{R}_s \cap (\mathcal{R}_r)^\perp$.

Given an increasing sequence $\alpha = \{\alpha_q\}_{q \in \mathbb{N}}$ of positive real numbers with $\lim_{q \to \infty} \alpha_q = \infty$, we define an operator $D$ on $L^2(\Lambda^\infty, M)$ by

$$D := \sum_{q \in \mathbb{N}} \alpha_q \hat{\Xi}_{q,q-1}. \quad (7)$$

Note first that the operator $D$ has eigenvalues $\alpha_q$ with eigenspaces $\mathcal{R}_q \cap \mathcal{R}_{(q-1)}^\perp$ by construction. Also note that when $\Lambda$ has one vertex, $\mathcal{R}_{-1} = \mathcal{R}_0$ and the orthogonal projection $\hat{\Xi}_{0,-1}$ is the zero projection.

Proposition 3.2. The operator $D$ on $L^2(\Lambda^\infty, M)$ of Equation (7) is unbounded and self-adjoint.
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for some
n \in \mathbb{N}
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generates the topology on \Lambda^\infty, and hence span\{\chi_{[\eta]} : d(\eta) = (n,n,\ldots,n), n \in \mathbb{N}\} is dense in \textit{L}^2(\Lambda^\infty,M). Given such a “square” cylinder set \eta with d(\eta) = (s,\ldots,s), since \chi_{[\eta]} \in \mathcal{R}_s, we can write \chi_{[\eta]} = \sum_{r \leq s} \hat{\Xi}_{r,r-1}(\chi_{[\eta]}). Then,

\[ D(\chi_{[\eta]}) = \sum_{r \leq s} \alpha_r \hat{\Xi}_{r,r-1}(\chi_{[\eta]}), \]

which is a finite linear combination of vectors with finite \textit{L}^2-norm, and hence is in \textit{L}^2(\Lambda^\infty,M).

In other words, for any finite linear combination \xi of characteristic functions of square cylinder sets, \textit{D}\xi is in \textit{L}^2(\Lambda^\infty,M). Thus \textit{D} is defined on (at least) the finite linear combinations of square cylinder sets, which form a dense subspace of \textit{L}^2(\Lambda^\infty,M).

Moreover, our definition of \textit{D} as a diagonal operator on \textit{L}^2(\Lambda^\infty,M) with real eigenvalues implies that \textit{D} = \textit{D}^* formally; since the operators \textit{D} and \textit{D}^* are given by the same diagonal formula, their domains also agree, and hence we do indeed have \textit{D} = \textit{D}^* as unbounded operators.

**Proposition 3.3.** Let \textit{D} be the operator on \textit{L}^2(\Lambda^\infty,M) given in (7). For all complex numbers \lambda \notin \{\alpha_n\}_{n \in \mathbb{N}}, the resolvent \textit{R}_\lambda(\textit{D}) := (\textit{D} - \lambda)^{-1} is a compact operator on \textit{L}^2(\Lambda^\infty,M).

**Proof.** By definition, \textit{D} is given by multiplication by \alpha_q on \textit{R}_q \cap \mathcal{R}_q^\perp. Consequently, for all \lambda \in \mathbb{R}, (\textit{D} - \lambda)^{-1} is given by multiplication by \frac{1}{\alpha_q - \lambda} on \textit{R}_q \cap \mathcal{R}_q^\perp.

Since \lambda \notin \{\alpha_n\}_{n \in \mathbb{N}} and lim_{n \to \infty} \alpha_n = \infty, given \epsilon > 0, we can choose \textit{N} so that for all \textit{n} \geq \textit{N}, \frac{1}{|\alpha_n - \lambda|} < \epsilon. Fix \textit{s} \in \mathbb{N}; then for any \textit{f} \in \mathcal{R}_s \cap \mathcal{R}_s^\perp of norm 1,

\[ \| \left( \sum_{q=1}^{\textit{N}} \frac{1}{\alpha_q - \lambda} \hat{\Xi}_{q,q-1}(\textit{f}) \right) - (\textit{D} - \lambda)^{-1}(\textit{f}) \| = \| \sum_{q>N} \frac{1}{\alpha_q - \lambda} \hat{\Xi}_{q,q-1}(\textit{f}) \| \]

\[ = \begin{cases} \left| \frac{1}{\alpha_s - \lambda} \right| \| \textit{f} \| & \text{if } \textit{s} > \textit{N} \\
0 & \text{if } \textit{s} \leq \textit{N} \end{cases} < \epsilon, \]

since \|\textit{f}\| = 1 by hypothesis. Since the subspaces \{\mathcal{R}_s \cap \mathcal{R}_s^\perp : s \in \mathbb{N}_0\} span \textit{L}^2(\Lambda^\infty,M), it follows that \( (\textit{D} - \lambda)^{-1} \) is the norm limit of finite rank operators and hence is compact.

**Theorem 3.4.** Let \Lambda be a finite, strongly connected \textit{k}-graph, and denote by \pi the representation of \textit{C}^*(\Lambda) on \textit{L}^2(\Lambda^\infty,M) given in Equations (3) and (1). Let \mathcal{A}_\Lambda be the dense \ast\text{-subalgebra of} \textit{C}^*(\Lambda) given in Definition 2.3 and let \textit{D} be the operator given in (7). If there exists a constant \textit{C} \geq 0 such that the sequence \alpha = \{\alpha_q\}_{q \in \mathbb{N}} satisfies

\[ |\alpha_{q+1} - \alpha_q| \leq \textit{C}, \forall q \in \mathbb{N}, \]

then the commutator [\textit{D}, \pi(\textit{a})] is a bounded operator on \textit{L}^2(\Lambda^\infty,M) for any \textit{a} \in \mathcal{A}_\Lambda.

Combined with the above results, this implies that the data \( (\mathcal{A}_\Lambda, \textit{L}^2(\Lambda^\infty,M), \textit{D}) \) gives a spectral triple for \textit{C}^*(\Lambda).
Proof. To prove that \((A_\Lambda, L^2(\Lambda^\infty, M), D)\) is a spectral triple we need to show that \(D\) is self-adjoint, \((D^2+1)^{-1}\) is compact and \([D, \pi(a)]\) is bounded for all \(a \in A_\Lambda\). The first statement is the content of Proposition 3.2, and the second follows from Proposition 3.3 thanks to the fact that \(\pm i \not\in \{\alpha_n\}_{n \in \mathbb{N}}\) and hence \((D \pm i)^{-1}\) is compact. Thus, to complete the proof of the Theorem, we will now show that \([D, \pi(a)]\) is bounded for all finite linear combinations \(a = \sum_{i \in F} c_i s_{\lambda_i} s^{*}_{n_i} \in A_\Lambda\), where \(c_i \in \mathbb{C}\).

Given \(\lambda \in \Lambda\), write \(\max_\lambda = \max_j \{d(\lambda)_j\}\) and \(\min_\lambda = \min_j \{d(\lambda)_j\}\). Then the formula (3) implies immediately that, for any fixed \(s \in \mathbb{N}\), the operator \(S_\lambda\) on \(L^2(\Lambda^\infty, M)\) takes \(\mathcal{R}_s\) to \(\mathcal{R}_{s+\max_\lambda}\).

Moreover, Equation (4) implies that the operator \(S_\lambda^*\) on \(L^2(\Lambda^\infty, M)\) takes \(\mathcal{R}_s\) to \(\mathcal{R}_{s-\min_\lambda}\) if \(\min_\lambda \leq s\), and to \(\mathcal{R}_0\) otherwise. To see this, suppose \(\chi_{[\eta]} \in \mathcal{R}_s\) and \(d(\eta) = (n_1, \ldots, n_k)\). Then \(S_\lambda^* \chi_{[\eta]}\) is a linear combination of cylinder sets \(\chi_{[\xi]}\) with

\[
d(\xi)_i = \begin{cases} 0, & d(\lambda)_i \geq d(\eta)_i \\ d(\eta)_i - d(\lambda)_i, & d(\lambda)_i < d(\eta)_i \end{cases}
\]

Consequently, we see that (as desired)

\[
\max \{d(\xi)_i\} = \max \{0, n_i - d(\lambda)_i : 1 \leq i \leq k\} \leq s - \min_\lambda.
\]

If \(s < \min_\lambda\), then \(n_i - d(\lambda)_i \leq 0\) for all \(i\), so \(S_\lambda^* \chi_{[\eta]} \in \mathcal{R}_0\) for all \(\chi_{[\eta]} \in \mathcal{R}_s\).

Similarly, if \(f \in \mathcal{R}_s^\perp\), then \(S_\lambda f \in \mathcal{R}_{s+\min_\lambda}\). Namely, if \(\langle f, h \rangle = 0\) for all \(h \in \mathcal{R}_s\), then our description of \(S_\lambda^*\) above yields

\[
\langle f, S_\lambda^* g \rangle = 0 \quad \forall \ g \in \mathcal{R}_{s+\min_\lambda}.
\]

An analogous argument shows that \(S_\lambda^*\) takes \(\mathcal{R}_s^\perp\) to \(\mathcal{R}_{s-\max_\lambda}^\perp\) if \(s \geq \max_\lambda\).

Now fix \(q \in \mathbb{N}\), \(f \in \mathcal{R}_q \cap \mathcal{R}_q^\perp\), and fix \(\lambda, \mu \in \Lambda\) with \(s(\lambda) = s(\mu)\). We use the reasoning of the previous paragraphs to identify the subspaces \(\mathcal{R}_s, \mathcal{R}_s^\perp\) which contain \(S_\lambda S_\mu^* f\).

If \(\max_\mu \geq q\), then we cannot guarantee that \(S_\mu^* f\) is orthogonal to any \(\mathcal{R}_t\) with \(t \geq 0\); in order to do so, we must have \(\langle S_\mu^* f, \xi \rangle = \langle f, S_\mu^* \xi \rangle = 0\) for all \(\xi \in \mathcal{R}_t\). In other words, we must have \(S_\mu \xi \in \mathcal{R}_{q-1}\) for all \(\xi \in \mathcal{R}_t\). However, \(S_\mu\) takes \(\mathcal{R}_t\) into \(\mathcal{R}_{t+\max_\mu} \supsetneq \mathcal{R}_{q-1}\) if \(\max_\mu \geq q\) and \(t \geq 0\).

Moreover, if \(q < \min_\mu\), then \(S_\mu^* f \in \mathcal{R}_0\). Thus,

\[
q < \min_\mu \Rightarrow S_\lambda S_\mu^* f \in \mathcal{R}_{\max_\lambda}; \quad \min_\mu \leq q \leq \max_\mu \Rightarrow S_\lambda S_\mu^* f \in \mathcal{R}_{q+\max_\lambda+\min_\mu};
\]

\[
q > \max_\mu \Rightarrow S_\lambda S_\mu^* f \in \mathcal{R}_{q+\max_\lambda-\min_\mu} \cap \mathcal{R}_{(q-1)+\min_\lambda-\max_\mu}^\perp.
\]

For now, assume \(q > \max_\mu\). Writing \(g = S_\lambda S_\mu^* f\), we have

\[
g = \left(\Xi_{q+\max_\lambda - \min_\mu} - \Xi_{(q-1)+\min_\lambda - \max_\mu}\right) g = \sum_{w=q+\min_\lambda - \max_\mu}^{q+\max_\lambda - \min_\mu} \left(\Xi_w - \Xi_{w-1}\right) g
\]

and consequently

\[
D(S_\lambda S_\mu^* f) =: Dg = \sum_{w=q+\min_\lambda - \max_\mu}^{q+\max_\lambda - \min_\mu} D\left(\left(\Xi_w - \Xi_{w-1}\right) g\right) = \sum_{w=q+\min_\lambda - \max_\mu}^{q+\max_\lambda - \min_\mu} \alpha_w \left(\left(\Xi_w - \Xi_{w-1}\right) g\right).
\]

It now follows that, if \(f \in \mathcal{R}_q \cap \mathcal{R}_q^\perp\) for \(q > \max_\mu\),

\[
[D, S_\lambda S_\mu^* f] = D S_\lambda S_\mu^* f - S_\lambda S_\mu^* D f = \sum_{w=q+\min_\lambda - \max_\mu}^{q+\max_\lambda - \min_\mu} \left(\alpha_w - \alpha_q\right) \left(\left(\Xi_w - \Xi_{w-1}\right) S_\lambda S_\mu^* f\right).
\]
Consequently, since \(|\alpha_w - \alpha_{w-1}| \leq C\) for all \(w\),
\[
\| [D, S_\lambda S_\mu^*]f \| \leq \sum_{w=q+\min_\lambda - \max_\mu}^{\max_\lambda - \min_\mu} |\alpha_w - \alpha_q| \| S_\lambda S_\mu^* f \|
\leq \| S_\lambda S_\mu^* f \| \sum_{w=q+\min_\lambda - \max_\mu}^{\max_\lambda - \min_\mu} C|w - q| = \| S_\lambda S_\mu^* f \| C \sum_{t=\min_\lambda - \max_\mu}^{\max_\lambda - \min_\mu} |t|.
\]
Since \(S_\lambda S_\mu^*\) is a partial isometry and hence norm-preserving, whenever \(f \in \mathcal{R}_q \cap \mathcal{R}_{q-1}^\perp\) for \(q > \max_\mu\), \(\| [D, S_\lambda S_\mu^*]f \|\) is bounded above by a constant which depends only on \(\lambda\) and \(\mu\).

If we have \(\min_\mu \leq q \leq \max_\mu\), since we no longer know that \(S_\lambda S_\mu^* f \in \mathcal{R}_t^\perp\) for any \(t\), in calculating \(\| [D, S_\lambda S_\mu^*]f \|\) we have to begin our summation over \(w\) at zero, rather than at \(q + \min_\lambda - \max_\mu\). In this case, the final inequality above becomes
\[
\| [D, S_\lambda S_\mu^*]f \| \leq \sum_{t=1}^{\max_\lambda - \min_\mu} Ct\|S_\lambda S_\mu^* f \| + \sum_{t=1}^{q} Ct\|S_\lambda S_\mu^* f \|.
\]
In this case, \(q \leq \max_\mu\), so we obtain the norm bound
\[
\| [D, S_\lambda S_\mu^*]f \| \leq \| S_\lambda S_\mu^* f \| C \left( \frac{(\max_\lambda - \min_\mu)(\max_\lambda - \min_\mu + 1)}{2} + \max_\mu (\max_\mu + 1) \right).
\]
In other words, \(\| [D, S_\lambda S_\mu^*]f \|\) is again bounded by a constant which only depends on \(\lambda\) and \(\mu\).

A similar argument shows that if \(\lambda, \mu\) given in Theorem 3.4, the eigenspaces of the Dirac operator \(D\) given in (1) agree with the wavelet decomposition
\[
L^2(\Lambda^\omega, M) = \mathcal{V}_0 \oplus \bigoplus_{q=0}^\infty \mathcal{W}_q
\]
of Theorem 2.6 above (also see [7] Theorem 4.2). In particular,
\[
\mathcal{V}_0 = \mathcal{R}_0 \supseteq \mathcal{R}_{-1}\quad \text{and}\quad \mathcal{W}_q = \mathcal{R}_{q+1} \cap \mathcal{R}_{q-1}^\perp, \quad q \geq 0.
\]

\textbf{Proof}. By definition, \(\mathcal{R}_{-1} \subseteq \mathcal{R}_0 = \mathcal{V}_0 = \text{span}\{\chi_v : v \in \Lambda^0\}\). For the second assertion, recall that \(\mathcal{W}_n = \text{span}\{S_\lambda f : f \in \mathcal{W}_0, \quad d(\lambda) = (q, q, \ldots, q)\}\). Since \(\max_\lambda = \min_\lambda = q\) for all such \(\lambda\), each such \(S_\lambda\) takes \(\mathcal{R}_s \cap \mathcal{R}_{s-1}^\perp\) to \(\mathcal{R}_{s+q} \cap \mathcal{R}_{s+q-1}^\perp\). Thus, it suffices to see that \(\mathcal{W}_0 \subseteq \mathcal{R}_1 \cap \mathcal{R}_0^\perp\), and that \(\mathcal{W}_q\) and \(\mathcal{R}_q \cap \mathcal{R}_{q-1}^\perp\) have the same dimension for all \(q \in \mathbb{N}\).

For the first statement, recall that \(\mathcal{W}_q\) was constructed precisely to be the span of a family \(\{f^{i,v}\}\) of functions (see Equation (5)) which were orthogonal to \(\mathcal{V}_0 = \mathcal{R}_0\). Moreover, every function
$f^{i,v}$ is a linear combination of characteristic functions $\chi_\eta$ with $d(\eta) = (1, \ldots , 1)$, and therefore lies in $\mathcal{R}_1 \cap \mathcal{R}_0^\perp$.

From the fact that $\{S_\lambda f^{i,s(\lambda)} : d(\lambda) = (q,q, \ldots , q), \ 1 \leq i \leq \#(D_{s(\lambda)}) - 1\}$ is a basis for $\mathcal{W}_q$, and the factorization rule in $\Lambda$, it follows that $\mathcal{W}_q$ has dimension

$$\sum_{v \in \Lambda^0} \#(\Lambda^{(q,\ldots, q)}v) \cdot (\#(v\Lambda^{(1,\ldots, 1)}) - 1) = \#(\Lambda^{(q+1,\ldots , q+1)}) - \#(\Lambda^{(q,\ldots, q)})$$

Moreover, we know from [7, Lemma 4.1] that “square” cylinder sets generate the topology on $\Lambda^\infty$; it follows that $\mathcal{R}_s$ is spanned by $\{\chi_{[\lambda]} : d(\lambda) = (s, \ldots , s)\}$. Indeed, this set forms a basis for $\mathcal{R}_s$: if $d(\lambda) = d(\mu) = (s, \ldots , s)$, then the factorization rule implies that

$$\langle \chi_{[\lambda]}, \chi_{[\mu]} \rangle = \int_{\Lambda^\infty} \chi_{[\lambda]}(\lambda) \chi_{[\mu]}(\lambda) \, dM = \delta_{\lambda,\mu} M([\lambda]).$$

Consequently, $\mathcal{R}_{q+1} \cap \mathcal{R}_q^\perp$ also has dimension $\#(\Lambda^{(q+1,\ldots , q+1)}) - \#(\Lambda^{(q,\ldots, q)})$. Hence, $\mathcal{W}_q = \mathcal{R}_{q+1} \cap \mathcal{R}_q^\perp$ for all $q \in \mathbb{N}$, as desired.

Remark 3.6. Fix $J \in \mathbb{N}^k$ with $J_i > 0$ for all $i$. We described in Section 5 of [6] how to construct wavelets with “fundamental domain” $J$ – the original construction in Section 4 of [7] used $J = (1, \ldots , 1)$. By defining

$$\tilde{\mathcal{R}}_s = \text{span}\{\chi_{[\eta]} : d(\eta) \leq sJ\}$$

we can construct a Dirac operator $\tilde{D}$ on $L^2(\Lambda^\infty, M)$ which gives rise to a spectral triple $(\mathcal{A}_\Lambda, L^2(\Lambda^\infty, M), \tilde{D})$ whose eigenspaces agree with the wavelet decomposition given in Theorem 5.2 of [6]. We omit the details here as they are completely analogous to the proofs of Theorems 3.4 and 3.5 above.

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Carla Farsi, Judith Packer: Department of Mathematics, University of Colorado at Boulder, Boulder, Colorado, 80309-0395, USA.
E-mail address: carla.farsi@colorado.edu, packer@euclid.colorado.edu

Elizabeth Gillaspy: Department of Mathematical Sciences, University of Montana, 32 Campus Drive #0864, Missoula, MT 59812-0864.
E-mail address: elizabeth.gillaspy@msu.umt.edu

Antoine Julien: Nord University Levanger, Høgskoleveien 27, 7600 Levanger, Norway.
E-mail address: antoine.julien@nord.no
Sooran Kang: College of General Education, Chung-Ang University, 84 Heukseok-ro, Dongjak-gu, Seoul, Republic of Korea.

E-mail address, sooran09@cau.ac.kr