ON THE METRIC DIMENSION OF A CLASS OF DISTANCE TRANSITIVE GRAPHS

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Abstract. Let $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic additive group $\mathbb{Z}_n$, where $S_1 = \{1, n-1\}$, ..., $S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}$, $1 \leq k \leq \lceil \frac{n}{2} \rceil - 1$. In this paper, we consider the problem of determining the cardinality $\psi(\Gamma)$ of minimal doubly resolving sets of $\Gamma$. We prove that if $n$ is an even integer and $k = \frac{n}{2} - 1$, then for $n \geq 8$ every minimal resolving set of $\Gamma$ is also a doubly resolving set, and, consequently, $\psi(\Gamma)$ is equal to the metric dimension of $\beta(\Gamma)$, which is known from the literature. Moreover, we find an explicit expression for the strong metric dimension of $\Gamma$.

Key Words: Metric dimension, resolving set, doubly resolving, strong resolving.

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1. Introduction

We consider graphs $\Gamma = (V, E)$ that are finite, simple and connected, where $V = V(\Gamma)$ is the vertex set and $E = E(\Gamma)$ is the edge set. The size of the largest clique in the graph $\Gamma$ is denoted by $\omega(\Gamma)$ and the size of the largest independent sets of vertices by $\alpha(\Gamma)$. For $u, v \in V(\Gamma)$, the length of a shortest path from $u$ to $v$ is called the distance between $u$ and $v$ and is denoted by $d_\Gamma(u, v)$, or simply $d(u, v)$. The concept of resolving set has been introduced by Slater [15] and also independently by Harary and Melter [7]. This concept has different applications in the areas of network discovery and verification [2], robot navigation [8], chemistry [6], and combinatorial optimization [14]. A vertex $x \in V(\Gamma)$ is said to resolve a pair $u, v \in V(\Gamma)$ if $d_\Gamma(u, x) \neq d_\Gamma(v, x)$. For an ordered subset $W = \{w_1, w_2, ..., w_k\}$ of vertices in a connected graph $\Gamma$ and a vertex $v$ of $\Gamma$, the metric representation of $v$ with respect to $W$ is the $k$-vector $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$. If every pair of distinct vertices of $\Gamma$ have different metric representations then the ordered set $W$ is called a resolving set of $\Gamma$. Indeed, the set $W$ is called a resolving set for $\Gamma$ if $r(u|W) = r(v|W)$ implies that $u = v$ for all pairs $u, v$ of vertices of $\Gamma$. If the set $W$ is as small as possible, then it is called a metric basis of the graph.
We recall that the metric dimension of \( \Gamma \), denoted by \( \beta(\Gamma) \) is defined as the minimum cardinality of a resolving set for \( \Gamma \). If \( \beta(\Gamma) = k \), then \( \Gamma \) is said to be \( k \)-dimensional. Bounds on \( \beta(\Gamma) \) are presented in terms of the order and the diameter of \( \Gamma \). All connected graphs of order \( n \) having metric dimension \( 1, n-1, \) or \( n-2 \) are determined. Notice, for each connected graph \( \Gamma \) and each ordered set \( W = \{ w_1, w_2, ..., w_k \} \) of vertices of \( \Gamma \), that the \( i \)th coordinate of \( r(w_i|W) \) is 0 and that the \( i \)th coordinate of all other vertex representations is positive. Thus, certainly \( r(u|W) = r(v|W) \) implies that \( u = v \) for \( u \in W \).

Therefore, when testing whether an ordered subset \( W \) of \( V(\Gamma) \) is a resolving set for \( \Gamma \), we need only be concerned with the vertices of \( V(\Gamma) - W \).

Cáceres et al. [4] define the notion of a doubly resolving set as follows. Vertices \( x, y \) of the graph \( \Gamma \) of order at least 2, are said to doubly resolve vertices \( u, v \) of \( \Gamma \) if \( d(u,x) - d(u,y) \neq d(v,x) - d(v,y) \). A set \( Z = \{ z_1, z_2, ..., z_l \} \) of vertices of \( \Gamma \) is a doubly resolving set of \( \Gamma \) if every two distinct vertices of \( \Gamma \) are doubly resolved by some two vertices of \( Z \). The minimal doubly resolving set is a doubly resolving set with minimum cardinality. The cardinality of minimum doubly resolving set is denoted by \( \psi(\Gamma) \). The minimal doubly resolving sets for Hamming and Prism graphs has been obtained in [10] and [5], respectively. Since if \( x, y \) doubly resolve \( u, v \), then \( d(u,x) - d(v,x) \neq 0 \) or \( d(u,y) - d(v,y) \neq 0 \), and hence \( x \) or \( y \) resolve \( u, v \). Therefore, a doubly resolving set is also a resolving set and \( \beta(\Gamma) \leq \psi(\Gamma) \).

The strong metric dimension problem was introduced by A. Sebo and E. Tannier [14] and further investigated by O. R. Oellermann and J. Peters-Fransen [13]. Recently, the strong metric dimension of distance hereditary graphs has been studied by T. May and O. R. Oellermann [11]. A vertex \( w \) strongly resolves two vertices \( u \) and \( v \) if \( u \) belongs to a shortest \( v - w \) path or \( v \) belongs to a shortest \( u - w \) path. A set \( N = \{ n_1, n_2, ..., n_m \} \) of vertices of \( \Gamma \) is a strong resolving set of \( \Gamma \) if every two distinct vertices of \( \Gamma \) are strongly resolved by some vertex of \( N \). A strong metric basis of \( \Gamma \) is a strong resolving set of the minimum cardinality. Now, the strong metric dimension of \( \Gamma \), denoted by \( sdim(\Gamma) \) is defined as the cardinality of its strong metric basis. It is easy to see that if a vertex \( w \) strongly resolves vertices \( u \) and \( v \) then \( w \) also resolves these vertices. Hence every strong resolving set is a resolving set and \( \beta(\Gamma) \leq sdim(\Gamma) \).

All three previously defined problems are NP-hard in general case. The proofs of NP-hardness are given for the metric dimension problem in [8], for the minimal doubly resolving set problem in [9] and for the strong metric dimension problem in [13]. The metric dimension of various families of graphs: see [1, 10, 16, 17], for instance.

Let \( G \) be a finite group and \( \Omega \) a subset of \( G \) that is closed under taking inverses and does not contain the identity. A Cayley graph \( \Gamma = Cay(G, \Omega) \) is a graph whose vertex set and edge set are defined as follows:

\[
V(\Gamma) = G; \quad E(\Gamma) = \{ \{ x, y \} \mid x^{-1}y \in \Omega \}.
\]
Let \( \Gamma = \text{Cay}(\mathbb{Z}_n, S_k) \) be the Cayley graph on the cyclic additive group \( \mathbb{Z}_n \), where \( S_1 = \{1, n-1\} \), ..., \( S_k = S_{k-1} \cup \{k, n-k\} \) are the inverse closed subsets of \( \mathbb{Z}_n - \{0\} \) for any \( k \in \mathbb{N}, 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \). Algebraic properties of this class of graphs has been studied in \([12]\). In this paper, we consider the problem of determining the cardinality \( \psi(\Gamma) \) of minimal doubly resolving sets of \( \Gamma \). We prove that if \( n \) is an even integer and \( k = \frac{n}{2} - 1 \), then for \( n \geq 8 \) every minimal resolving set of \( \Gamma \) is also a doubly resolving set, and, consequently, \( \psi(\Gamma) \) is equal to the metric dimension of \( \beta(\Gamma) \), which is known from the literature. Moreover, we find an explicit expression for the strong metric dimension of \( \Gamma \).

2. Definitions And Preliminaries

Definition 1. \([3]\) Let \( \Gamma \) be a graph with automorphism group \( \text{Aut}(\Gamma) \). We say that \( \Gamma \) is vertex transitive graph if, for any vertices \( x, y \) of \( \Gamma \) there is some \( \varphi \) in \( \text{Aut}(\Gamma) \), such that \( \varphi(x) = y \). Also, we say that \( \Gamma \) is symmetric if, for all vertices \( u, v, x, y \) of \( \Gamma \) such that \( u \) and \( v \) are adjacent, also, \( x \) and \( y \) are adjacent, there is an automorphism \( \varphi \) such that \( \varphi(u) = x \) and \( \varphi(v) = y \). Finally, we say that \( \Gamma \) is distance transitive if, for all vertices \( u, v, x, y \) of \( \Gamma \) such that \( d(u, v) = d(x, y) \) there is an automorphism \( \varphi \) such that \( \varphi(u) = x \) and \( \varphi(v) = y \).

Remark 1. \([3]\) Let \( \Gamma \) be a graph. It is clear that we have a hierarchy of the conditions is

\[ \text{distance transitive} \Rightarrow \text{symmetric} \Rightarrow \text{vertex transitive} \]

Proposition 1. \([12]\) Let \( \Gamma = \text{Cay}(\mathbb{Z}_n, S_k) \) be the Cayley graph on the cyclic group \( \mathbb{Z}_n \) \((n \geq 4)\), where \( S_1 = \{1, n-1\} \), ..., \( S_k = S_{k-1} \cup \{k, n-k\} \) are the inverse closed subsets of \( \mathbb{Z}_n - \{0\} \) for any \( k \in \mathbb{N}, 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \). Then \( \chi(\Gamma) = \omega(\Gamma) = k + 1 \) if and only if \( k + 1 | n \). (where the chromatic number \( \chi(\Gamma) \) of \( \Gamma \) is the minimum number \( k \) such that \( \Gamma \) is a \( k \)-colorable.)

Lemma 1. For any connected graph \( \Gamma \) on \( n \) vertices which is not a path, \( 2 \leq \beta(\Gamma) \leq n - \text{diam}(\Gamma) \)

Proposition 2. \([12]\) Let \( \Gamma = \text{Cay}(\mathbb{Z}_n, S_k) \) be the Cayley graph on the cyclic group \( \mathbb{Z}_n \) \((n \geq 4)\), where \( S_1 = \{1, n-1\} \), ..., \( S_k = S_{k-1} \cup \{k, n-k\} \) are the inverse closed subsets of \( \mathbb{Z}_n - \{0\} \) for any \( k \in \mathbb{N}, 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \). If \( n \) is an even integer and \( k = \frac{n}{2} - 1 \), then \( \Gamma \) is a distance transitive graph.

3. Main results

Theorem 1. Let \( \Gamma = \text{Cay}(\mathbb{Z}_n, S_k) \) be the Cayley graph on the cyclic group \( \mathbb{Z}_n \) \((n \geq 8)\), where \( S_1 = \{1, n-1\} \), ..., \( S_k = S_{k-1} \cup \{k, n-k\} \) are the inverse closed subsets of \( \mathbb{Z}_n - \{0\} \) for any \( k \in \mathbb{N}, 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \). If \( n \) is an even integer and \( k = \frac{n}{2} - 1 \), then the metric dimension of \( \Gamma \) is \( k + 1 \).

Proof. Let \( V(\Gamma) = \{1, \ldots, n\} \) be the vertex set of \( \Gamma \). By proof of Proposition 3.4 in \([12]\), we know that the diameter of \( \Gamma \) is 2. Hence, for \( x, y \in V(\Gamma) \), the
length of a shortest path from \(x\) to \(y\) is \(d(x, y) = 1\) or \(2\). Also, we know that the size of largest clique in the graph \(\Gamma\) is \(k + 1\). By the following steps we show that the metric dimension of \(\Gamma\) is \(k + 1\).

Case 1. Let \(W\) be a clique in the graph \(\Gamma\) such that \(|W| \leq k\). Indeed, for \(x, y \in W\), \(d_\Gamma(x, y) = 1\). It is an easy to prove that if \(|W| < k\) then \(W\) is not a resolving set for \(\Gamma\). In particular if \(|W| = k\), then we show that \(W\) is not a resolving set for \(\Gamma\). Without loss of generality one can assume that an ordered subset \(W = \{1, 2, 3, ..., k\}\) of vertices in the graph \(\Gamma\), and hence \(V(\Gamma) - W = \{k + 1, k + 2, ..., n\}\). On the other hand, by proof of Proposition 3.2 in [12], we know that for any \(x \in V(\Gamma)\), there is exactly one \(y \in V(\Gamma)\) such that \(x^{-1}y = k + 1\), that is \(d(x, y) = 2\). Hence, there exist vertices \(k + 2, k + 3, ..., n - 1\) of \(V(\Gamma) - W\) such that \(d(1, k + 2) = 2, d(2, k + 3) = 2, ..., d(k, n-1) = 2\). So, the metric representations of the vertices \(k + 2, k + 3, ..., n - 1\) \(\in V(\Gamma) - W\) with respect to \(W\) are the \(k\)-vectors \(r(k + 2|W) = (2, 1, 1, ..., 1), r(k + 3|W) = (1, 2, 1, ..., 1), ..., r(n - 1|W) = (1, 1, 1, ..., 2)\). Besides, the metric representations of the vertices \(k + 1, n \in V(\Gamma) - W\) with respect to \(W\) is the \(k\)-vector \(r(k + 1|W) = r(n|W) = (1, 1, 1, ..., 1)\). Therefore, \(W\) is not a resolving set for \(\Gamma\), because the representation of pairs of vertices \(k + 1, n\) are the same.

Case 2. Now, let \(W\) be a clique in the graph \(\Gamma\) such that \(|W| = k + 1\). We show that \(W\) is a resolving set for \(\Gamma\). We may assume without loss that an ordered subset \(W = \{1, 2, 3, ..., k, k + 1\}\) of vertices in the graph \(\Gamma\), and hence \(V(\Gamma) - W = \{k + 2, k + 3, ..., n\}\). Thus, the metric representations of the vertices \(k + 2, k + 3, ..., n \in V(\Gamma) - W\) with respect to \(W\) are the \((k + 1)\)-vectors \(r(k + 2|W) = (2, 1, 1, ..., 1), r(k + 3|W) = (1, 2, 1, ..., 1), ..., r(n|W) = (1, 1, 1, ..., 2)\). Therefore, all the vertices of \(V(\Gamma) - W\) have different representations with respect to \(W\), this implies that \(W\) is a resolving set of \(\Gamma\). Thus the metric dimension of \(\beta(\Gamma) \leq k + 1\).

Case 3. Let, an ordered subset \(W = \{1, 2, ..., k + 1\}\) of vertices in the graph \(\Gamma\) be a resolving set for \(\Gamma\). We show that for each \(x \in V(\Gamma) - W\), \((W \cup x)\) is also a resolving set for \(\Gamma\). Because, without loss of generality one can assume that \(x = k + 2\), and an ordered subset \((W \cup x) = \{1, 2, 3, ..., k + 1, k + 2\}\) of vertices in the graph \(\Gamma\). Then \(V(\Gamma) - (W \cup x) = \{k + 3, k + 4, ..., n\}\). Thus, the metric representations of the vertices \(k + 3, k + 4, ..., n \in V(\Gamma) - (W \cup x)\) with respect to \((W \cup x)\) are the \((k + 2)\)-vectors \(r(k + 3|W) = (1, 2, 1, ..., 1), r(k + 4|W) = (1, 1, 2, ..., 1, 1), ..., r(n|W) = (1, 1, 1, ..., 2, 1)\). Therefore, \((W \cup x)\) is also a resolving set for \(\Gamma\).

Case 4. Let \(W\) be a clique in the graph \(\Gamma\) such that \(x \in W\), and \(|W| = k\). Let \(y \in V(\Gamma) - W\) such that \(d_\Gamma(x, y) = 2\), we show that an ordered subset \((W \cup y)\) of vertices in the graph \(\Gamma\) is not a resolving set for \(\Gamma\). In this case, without loss of generality one can assume that an ordered subset \(W = \{1, 2, ..., k\}\) of vertices in the graph \(\Gamma\), and \(x = 1, y = k + 2\). Thus, the metric representations of the vertices \(k + 1, n \in V(\Gamma) - (W \cup y)\) with respect to \((W \cup y)\) is the \((k + 1)\)-vector \(r(k + 1|W) = r(n|W) = (1, 1, 1, ..., 1)\). Therefore, \((W \cup y)\) is not a resolving set for \(\Gamma\).
Case 5. Finally, let $W$ be an ordered set contains the vertices $x, y \in V(\Gamma)$ such that $d(x, y) = 2$ and the size of $|W| = k + 1$. It is an easy to show that $W$ is not a resolving set for $\Gamma$.

From above cases, we conclude that the minimum cardinality of a resolving set for $\Gamma$ is $k + 1$. Also, it is well known that every Cayley graph is vertex transitive, hence every minimal resolving set for $\Gamma$ has the same size. \hfill $\Box$

**Theorem 2.** Let $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ ($n \geq 8$), where $S_1 = \{1, n-1\}$, ..., $S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}$, $1 \leq k \leq \lceil \frac{n}{2} \rceil - 1$. If $n$ is an even integer and $k = \frac{n}{2} - 1$, then the cardinality of minimum doubly resolving set of $\Gamma$ is $k + 1$.

**Proof.** First consider the natural ordering $V(\Gamma) = \{1, ..., n\}$, relabelling if necessary by $\{v_1, v_2, ..., v_n\}$, where $v_i = i$ for $1 \leq i \leq n$. We know that an ordered subset $Z = \{1, 2, 3, ..., k, k+1\}$ of vertices in the graph $\Gamma$ is a resolving set for $\Gamma$ of size $k+1$, also by Theorem [1], the metric dimension of $\Gamma$ is $\beta(\Gamma) = k+1$. Besides, $B(\Gamma) \leq \psi(\Gamma)$.

We show that an ordered subset $Z = \{1, 2, 3, ..., k, k+1\}$ of vertices in the graph $\Gamma$ is a doubly resolving set of $\Gamma$. It is sufficient to show that for two vertices $v_i$ and $v_j$ of $\Gamma$ there are vertices $x, y \in Z$ such that $d(v_i, x) - d(v_j, x) \neq d(v_i, y) - d(v_j, y)$. Consider two vertices $v_i$ and $v_j$ of $\Gamma$. Without loss of generality $i < j$. By the following steps we show that the cardinality of minimum doubly resolving set of $\Gamma$ is $k + 1$.

Case 1. If $1 \leq i < j \leq k + 1$ then $v_i, v_j \in Z$, thus $d(v_i, v_j) = 1$. So, we can assume that $x = v_i \in Z$ and $y = v_j \in Z$. Hence, we have $-1 = 0 - 1 = d(v_i, x) - d(v_j, x) \neq d(v_i, y) - d(v_j, y) = 1 - 0 = 1$. Thus, vertices $x$ and $y$ of $Z$ doubly resolve $v_i, v_j$.

Case 2. Know, let $1 \leq i \leq k + 1 < j \leq n$, hence $v_i \in Z$ and $v_j \notin Z$. Thus $d(v_i, v_j) = 1$ or 2. Let $d(v_i, v_j) = 1$. Therefore, we can assume without loss of generality that $v_i = 1$ and $v_j = n$. Hence by taking $x = v_i \in Z$ and $y = k+1 \in Z$, we have $-1 = 0 - 1 = d(v_i, x) - d(v_j, x) \neq d(v_i, y) - d(v_j, y) = 0 - 2 = -2$. Thus vertices $x$ and $y$ of $Z$ doubly resolve $v_i, v_j$. Now, let $d(v_i, v_j) = 2$, without loss of generality $v_i = 1$ and $v_j = k + 2$. Hence by taking $x = 1 \in Z$ and $y = 2 \in Z$ we have $-2 = 0 - 2 = d(v_i, x) - d(v_j, x) \neq d(v_i, y) - d(v_j, y) = 1 - 1 = 0$. Thus, vertices $x$ and $y$ of $Z$ doubly resolve $v_i, v_j$.

Case 3. Finally, let $1 < k + 1 < i < j \leq n$. Hence $v_i \notin Z$ and $v_j \notin Z$, thus $d(v_i, v_j) = 1$. We know that for $v_i \in V(\Gamma) - Z$, there is exactly one $x \in Z$ such that $v_i^{-1}x = k + 1$, indeed $d(v_i, x) = 2$. Also, for $v_j \in V(\Gamma) - Z$, there is exactly one $y \in Z$ such that $v_j^{-1}y = k + 1$, indeed $d(v_j, y) = 2$, hence $1 = 2 - 1 = d(v_i, x) - d(v_j, x) \neq d(v_j, y) - d(v_i, y) = 1 - 2 = -1$. Thus, vertices $x$ and $y$ of $Z$ doubly resolve $v_i, v_j$.

From above cases, we conclude that the minimum cardinality of a doubly resolving set for $\Gamma$ is $k + 1$. \hfill $\Box$
Theorem 3. Let $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ ($n \geq 8$), where $S_1 = \{1, n-1\}$, ..., $S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}$, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$. If $n$ is an even integer and $k = \frac{n}{2} - 1$, then the strong metric dimension of $\Gamma$ is $k + 1$.

Proof. First consider the natural ordering $V(\Gamma) = \{1, \ldots, n\}$, relabelling if necessary by $\{v_1, v_2, \ldots, v_n\}$, where $v_i = i$ for $1 \leq i \leq n$. We know that an ordered subset $N = \{1, 2, 3, \ldots, k, k+1\}$ of vertices in the graph $\Gamma$ is a resolving set for $\Gamma$ of size $k + 1$, also by Theorem [1], the metric dimension of $\Gamma$ is $\beta(\Gamma) = k + 1$.

Besides, $B(\Gamma) \leq \text{sdim}(\Gamma)$. We show that subset $N = \{1, 2, 3, \ldots, k, k+1\}$ of vertices in the graph $\Gamma$ is a strong resolving set of $\Gamma$. Consider two vertices $v_i$ and $v_j$ of $\Gamma$. Without loss of generality $i < j$. So, it is sufficient to prove that there exists a vertex $w \in N$ such that $v_i$ belongs to a shortest $v_j - w$ path or $v_j$ belongs to a shortest $v_i - w$ path. Let $1 < k + 1 < i < j \leq n$. Hence $v_i \not\in N$ and $v_j \not\in N$, thus $d(v_i, v_j) = 1$. We know that for $v_i \in V(\Gamma) - N$, there is exactly one $w \in N$ such that $v_i^{-1}w = k + 1$, indeed $d(v_i, w) = 2$. Hence, $d(v_j, w) = 1$. So, $d(v_i, w) = d(v_i, v_j) + d(v_j, w)$, that is, vertex $v_j$ belongs to a shortest $v_i - w$ path, and, therefore, $w$ strongly resolves vertices $v_i$ and $v_j$. Also, for $v_i \in N$ or $v_j \in N$, vertex $v_i$ or vertex $v_j$ obviously strongly resolves pair $v_i, v_j$. Therefore, $N$ is a strong resolving set. Thus, the minimum cardinality of a strong resolving set for $\Gamma$ is $k + 1$.

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