FURTHER RESULTS ON STRICTLY LIPSCHITZ SUMMING OPERATORS

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Abstract. We give some new characterizations of strictly Lipschitz $p$-summing operators. These operators have been introduced in order to improve the Lipschitz $p$-summing operators. Therefore, we adapt this definition for constructing other classes of Lipschitz mappings which are called strictly Lipschitz $p$-nuclear and strictly Lipschitz $(p, r, s)$-summing operators. Some interesting properties and factorization results are obtained for these new classes.

1. Introduction and preliminaries

In the last years, many concept of the theory of $p$-summing operators have been developed in several ways, namely the multilinear and Lipschitz settings. The fundamental purpose of nonlinear theory of Lipschitz mappings is to attempt to borrow linear properties in order to make an analog in the nonlinear case. Let $X$ be a pointed metric space and $E$ be a Banach space. It is well known that each Lipschitz operator $T : X \to E$ can be factorized through a Lipschitz map and a linear operator. Let $\mathcal{I}$ be an ideal linear, when we want to form an analogous class of Lipschitz operators, it is natural to think how to preserve the connection between their linearization operators, which appear in the factorization, and the original ideal $\mathcal{I}$. If this property holds for a given Lipschitz class, then it can be represented by [1] as follows

$$\mathcal{I} \circ \text{Lip}_0(X; E) = \mathcal{I}(\mathcal{F}(X); E).$$

(1.1)

Then, the above representation, that we consider interesting, expresses good relation between Lipschitz operators and their linearizations. To make the relation (1.1) attainable, we have improved in [16] the definition of Lipschitz $p$-summing by introducing the strictly Lipschitz $p$-summing operators whose original ideal is $\Pi_p$, the Banach space of $p$-summing operators, and admits a similar representation of (1.1). The goal of this paper is to explore more properties of the

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class of strictly Lipschitz $p$-summing by showing some characterizations of those operators by means of fundamental inequalities and the domination theorem of Pietsch. We give a strong version of this concept, called $M$-strictly Lipschitz $p$-summing, when we consider the class of Lipschitz operators defined on metric spaces. Furthermore, we will introduce the class of strictly Lipschitz $p$-nuclear by generalizing the linear definition introduced by Cohen [6]. Next, we will naturally define the class of strictly Lipschitz $(p, r, s)$-summing. In analogy with the definition of strictly Lipschitz $p$-nuclear, some interesting results are obtained for the class of strictly Lipschitz $(p, r, s)$-summing.

The paper is organized as follows.

First, we recall some standard notations which will be used throughout this paper. In section 2, we study some characterizations of strictly Lipschitz $p$-summing operators which are defined from a metric space into a Banach space. We will give the definition for a general case, called $M$-strictly Lipschitz $p$-summing, where $X$ and $Y$ are metric spaces for which there is a beautiful equivalence between $T$ and its corresponding linear operators for the concept of $p$-summing. In Section 3, the definition of strictly Lipschitz $p$-nuclear operators is given. This class has surprising properties namely their connections with linearization operators. A representation through a Lipschitz tensor product is given for this class. We end this section by investigating certain relations with other classes. Section 4 is devoted to study the class of strictly Lipschitz $(p, r, s)$-summing operators. These operators involve Lipschitz $(p, r, s)$-summing operators which are introduced by [3]. Many considerations in the previous section are analogous to that in section 4.

Now, we recall briefly some basic notations and terminology which we need in the sequel. Throughout this paper, the letters $E, F$ will denote Banach spaces and $X, Y$ will denote metric spaces with a distinguished point (pointed metric spaces) which we denote by $0$. Let $X$ be a pointed metric space, we denote by $X^\#$ the Banach space of all Lipschitz functions $f : X \to \mathbb{R}$ which vanish at 0 under the Lipschitz norm given by

$$Lip (f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\}.$$ 

We denote by $\mathcal{F}(X)$, the Lipschitz-free Banach space over $X$, the closed linear span of the linear forms $\delta_{(x, y)}$ of $Lip_0(X)^*$ such that

$$\delta_{(x, y)} (f) = f(x) - f(y), \text{ for every } f \in Lip_0(X),$$

i.e.,

$$\mathcal{F}(X) = \overline{\text{span}} \left\{ \delta_{(x, y)} : x, y \in X \right\}^{Lip_0(X)^*}.$$ 

We have $X^\# = \mathcal{F}(X)^*$ holds isometrically via the application $Q_X (f)(m) = m (f)$, for every $f \in X^\#$ and $m \in \mathcal{F}(X)$.

For the general theory of free Banach spaces, see [9, 10, 14, 18]. Let $X$ be a pointed metric space and $E$ be a Banach space, we denote by $Lip_0(X; E)$
the Banach space of all Lipschitz functions (Lipschitz operators) $T : X \to E$ such that $T(0) = 0$ with pointwise addition and Lipschitz norm. Note that for any $T \in Lip_0(X; E)$, then there exists a unique linear map (linearization of $T$) $\hat{T} : F(X) \to E$ such that $\hat{T} \circ \delta_X = T$ and $\|\hat{T}\| = Lip(T)$, i.e., the following diagram commutes
\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow \delta_X & & \downarrow \delta_Y \\
F(X) & \xrightarrow{\hat{T}} & F(Y)
\end{array}
\]
where $\delta_X$ is the canonical embedding so that $\langle \delta_X(x), f \rangle = \delta_{(x,0)}(f) = f(x)$ for $f \in X^\#$. If $X$ is a Banach space and $T : X \to E$ is a linear operator, then the corresponding linear operator $\hat{T}$ is given by
\[
\hat{T} = T \circ \beta_X,
\]
where $\beta_X : F(X) \to X$ is linear quotient map which verifies $\beta_X \circ \delta_X = id_X$ and $\|\beta_X\| \leq 1$, see [10, p 124] for more details about the operator $\beta_X$. Let $X, Y$ be two metric spaces. Let $T : X \to Y$ be a Lipschitz operator, then there is a unique linear operator $\tilde{T}$ such that the following diagram commutes
\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow \delta_X & & \downarrow \delta_Y \\
F(X) & \xrightarrow{\tilde{T}} & F(Y)
\end{array}
\]
i.e., $\delta_Y \circ T = \tilde{T} \circ \delta_X$. The Lipschitz adjoint map $T^\# : Y^\# \to X^\#$ of $T$ is defined as follows
\[
T^\#(g)(x) = g(T(x)), \text{ for every } g \in Y^\# \text{ and } x \in X.
\]
The Lipschitz transpose operator $T_{/E^*}^t : E^* \to X^\#$ is the restriction of $T^\#$ on $E^*$. We have,
\[
T^\# = Q^{-1}_X \circ \tilde{T}^* \circ Q_Y \quad \text{and} \quad T^t = Q^{-1}_X \circ \hat{T}^*.
\]
Let $X$ be a metric space and $E$ be a Banach space, by $X \boxtimes E$ we denote the Lipschitz tensor product of $X$ and $E$. This is the vector space spanned by the linear functional $\delta_{(x,y)} \boxtimes e$ on $Lip_0(X; E^*)$ defined by
\[
\delta_{(x,y)} \boxtimes e(f) = \langle f(x) - f(y), e \rangle.
\]
See [4] for more details about the properties of the space $X \boxtimes E$. Now, let $E$ be a Banach space, then $B_E$ denotes its closed unit ball and $E^*$ its (topological) dual. Consider $1 \leq p \leq \infty$ and $n \in \mathbb{N}^*$. We denote by $l^n_p(E)$ the Banach space of all sequences $(e_i)_{i=1}^n$ in $E$ with the norm
\[
\|(e_i)_i\|_{l^n_p(E)} = \left(\sum_{i=1}^n \|e_i\|^p\right)^{\frac{1}{p}},
\]
and by $l^{n,w}_p(E)$ the Banach space of all sequences $(e_i)_{i=1}^n$ in $E$ with the norm
\[
\|(e_i)_i\|_{l^{n,w}_p(E)} = \sup_{e^* \in B_{E^*}} \left(\sum_{i=1}^n |\langle e_i, e^* \rangle|^p\right)^{\frac{1}{p}}.
\]
If $E = \mathbb{K}$, we simply write $l^n_p$ and $l^{n,w}_p$.

We recall some definitions which we need in the sequel. We refer to [6, 7, 13] for more details about the following notions. Let $1 \leq p \leq \infty$ and $p^*$ its conjugate, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$. Let $E, F$ be two Banach spaces and $R : E \to F$ be a linear operator. Then,

- The linear operator $R$ is $p$-summing if there exists a constant $C > 0$ such that, for any $x_1, \ldots, x_n \in X$, we have
  \[ \left( \sum_{i=1}^n \| R(x_i) \|^p \right)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{X^*}} \left( \sum_{i=1}^n |x^*(x_i)| \right)^{\frac{1}{p^*}}. \] \hspace{1cm} (1.4)

The class of $p$-summing linear operators from $E$ into $F$, which is denoted by $\Pi_p(E; F)$, is a Banach space for the norm $\pi_p(R)$, i.e., the smallest constant $C$ such that the inequality (1.4) holds.

- The linear operator $R$ is Cohen strongly $p$-summing if there exists a constant $C > 0$ such that, for any $x_1, \ldots, x_n \in X$, and any $y_1^*, \ldots, y_n^* \in F^*$, we have
  \[ \sum_{i=1}^n |\langle R(x_i), y_i^* \rangle| \leq C \| (x_i) \|_{l^p(E)} \| (y_i^*) \|_{l^{p,w}_p}. \] \hspace{1cm} (1.5)

The class of Cohen strongly $p$-summing operators from $E$ into $F$, which is denoted by $\mathcal{D}_p(E, F)$, is a Banach space for the norm $d_p(R)$, i.e., the smallest constant $C$ such that the inequality (1.5) holds. If $1 \leq p < \infty$, we have by [6, Theorem 2.2.2]
\[ R \in \Pi_p(E; F) \iff R^* \in \mathcal{D}_{p^*}(F^*; E^*). \] \hspace{1cm} (1.6)

- The linear operator $R$ is $p$-nuclear if there exists a constant $C > 0$ such that, for any $x_1, \ldots, x_n \in X$, and any $y_1^*, \ldots, y_n^* \in F^*$, we have
  \[ \sum_{i=1}^n |\langle R(x_i), y_i^* \rangle| \leq C \| (x_i) \|_{l^{p,w}_p} \| (y_i^*) \|_{l^{p^*,w}_p}. \] \hspace{1cm} (1.7)

The class of $p$-nuclear linear operators from $E$ into $F$, which is denoted by $\mathcal{N}_p(E, F)$, is a Banach space for the norm $N_p(R)$, i.e., the smallest constant $C$ such that the inequality (1.7) holds. We have by [6, 12]: $R$ is $p$-nuclear if and only if, $R = R_1 \circ R_2$ where $R_1$ is Cohen strongly $p$-summing and $R_2$ is $p$ summing.

- The linear operator $R$ is $(p, r, s)$-summing if there exists a constant $C > 0$ such that, for any $x_1, \ldots, x_n \in X$, and any $y_1^*, \ldots, y_n^* \in F^*$, we have
  \[ \| \langle (R(x_i), y_i^*) \rangle \|_{l^p} \leq C \| (x_i) \|_{l^{p,w}_p} \| (y_i^*) \|_{l^{p^*,w}_p}. \] \hspace{1cm} (1.8)

The class of $(p, r, s)$-summing linear operators from $E$ into $F$, which is denoted by $\Pi_{p,r,s}(E, F)$, is a Banach space for the norm $\pi_{p,r,s}(R)$, i.e., the smallest constant $C$ such that the inequality (1.8) holds. We have by [13]: $R$ is $(p, r, s)$-summing
if and only if, \( R = R_1 \circ R_2 \) where \( R_1 \) is Cohen strongly \( s^* \)-summing and \( R_2 \) is \( r \)-summing. In a particular case when \( p = 1 \) and \( \frac{1}{r} + \frac{1}{s} = 1 \), we have
\[
N_r(E, F) = \Pi_{1, r, s}(E, F).
\]

2. Characterization of strictly Lipschitz \( p \)-summing operators

Let \( X \) be a pointed metric space and \( E \) be a Banach space. In order to establish a relation between a Lipschitz operator \( T : X \to E \) and its linearization \( \hat{T} : \mathcal{F}(X) \to E \) for the concept of \( p \)-summing, we have introduced in [16] the notion of strictly Lipschitz \( p \)-summing operators. Indeed, both operators are related in the sense that: \( T \) is strictly Lipschitz \( p \)-summing if and only if, \( \hat{T} \) is \( p \)-summing. In fact, that report was not true for the class of Lipschitz \( p \)-summing operators which introduced by Farmer [8], see [15, Remark 3.3]. In this section, we give some characterizations of the class of strictly Lipschitz \( p \)-summing operators and we adapt its definition to the general case where the spaces are metric for which we obtain a good relation between the Lipschitz operator \( T \) and its corresponding linear operators \( \tilde{T} \) and \( T^\# \). Now, we start by the definition of \( d^p \), the corresponding Lipschitz cross-norm of the tensor norm \( d_p \), see [16] for more detail about this norm. Recall the definition of the norms of Chevet-Saphar \( d_p \) and \( g_p \) [5, 17] defined on Banach spaces,
\[
d_p(u) = \inf \left\{ \|(x_i)\|_{p^*,w(E)} \|(y_i)\|_{p^*(F)} \right\},
\]
where the infimum is taking over all representations of \( u \) of the form \( u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F \). The tensor norm \( g_p \) is defined as follows
\[
g_p\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = d^p_p\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = d_p\left(\sum_{i=1}^{n} y_i \otimes x_i\right)
\]
For every \( u = \sum_{k=1}^{l} \delta(x_k,y_k) \otimes s_k \in X \otimes E \), we put
\[
A_u = \left\{ m = \sum_{i=1}^{n} m_i \otimes e_i \in \mathcal{F}(X) \otimes E : m = \sum_{k=1}^{l} \delta(x_k,y_k) \otimes s_k \right\}.
\]
(2.1)
Since the linearization \( \hat{T} \) can be seen as a linear form on \( \mathcal{F}(X) \otimes E^* \), then for every \( m \in A_u \) we have
\[
\sum_{k=1}^{l} (T(x_k) - T(y_k), s_k^*) = \hat{T}(\sum_{k=1}^{l} \delta(x_k,y_k) \otimes s_k^*) = \hat{T}(m) = \sum_{i=1}^{n} \left\langle \hat{T}(m_i), e_i^* \right\rangle = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \langle \lambda_i^j(T(x^i_j) - T(y^i_j)), e_i^* \rangle.
\]
where

\[ m_i = \sum_{j=1}^{k_i} \lambda_i^j \delta(x_i^j, y_i^j) = \sum_{j=1}^{n_2} \lambda_i^j \delta(x_i^j, y_i^j), \]

with \( n_2 = \max_{i=1}^n k_i \) and the terms between \( k_i \) and \( n_2 \) are zero. Now, Let \( \alpha \) be a tensor norm defined on two Banach spaces, by [16, Theorem 3.1], there is a Lipschitz cross-norm \( \alpha^L \) which is defined on Lipschitz tensor product \( X \boxtimes E \). Note that if \( u = \sum_{k=1}^l \delta(x_k, y_k) \boxtimes s_k \in X \boxtimes E \) we have

\[ \alpha^L(\sum_{k=1}^l \delta(x_k, y_k) \boxtimes s_k) = \alpha(\sum_{k=1}^l \delta(x_k, y_k) \boxtimes s_k). \tag{2.3} \]

where \( \sum_{k=1}^l \delta(x_k, y_k) \boxtimes s_k \in \mathcal{F}(X) \boxtimes E \). So, we have

\[ d_p^L(u) = \inf_{m \in A_u} \left\{ \| m_i \|_{\Pi^L_p(\mathcal{F}(X))} \left\| (e_i)_i \right\|_{\pi^L_p(E)} \right\}. \]

**Definition 2.1** [16]. Let \( 1 \leq p \leq \infty \). A Lipschitz operator \( T : X \to E \) is said to be **strictly Lipschitz p-summing** if there exists a positive constant \( C \) such that for every \( x_k, y_k \in X \) and \( s_k^* \in E^* \) \((1 \leq k \leq l)\) we have

\[ \left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| \leq C d_p^L(u), \tag{2.4} \]

where \( u = \sum_{k=1}^l \delta(x_k, y_k) \boxtimes s_k^* \). We denote by \( \Pi_p^{SL}(X, E) \) the Banach space of all strictly Lipschitz p-summing operators from \( X \) into \( E \) which its norm \( \pi_p^{SL}(T) \) is the smallest constant \( C \) verifying (2.4). If we consider linear operators defined on Banach spaces, we have shown in [16, Proposition 3.8] that the three notions: p-summing, Lipschitz p-summing and strictly Lipschitz p-summing are coincide. The following characterization is the main result of this section.

**Theorem 2.2.** Let \( 1 \leq p \leq \infty \). Let \( X \) be a pointed metric space and \( E \) be a Banach space. Let \( T : X \to E \) be a Lipschitz operator. The following properties are equivalent.

1) \( T \) is strictly Lipschitz p-summing.

2) \( \tilde{T} \) is p-summing.

3) There exist a constant \( C > 0 \) and a Radon probability \( \mu \) on \( B_{X^*} \) such that for all \( (x^j)_{j=1}^n, (y^j)_{j=1}^n \subset X \) and \( (\lambda^j)_{j=1}^n \subset \mathbb{K}; \) \( n \in \mathbb{N}^* \), we have

\[ \left\| \sum_{j=1}^n \lambda^j (T(x^j) - T(y^j)) \right\| \leq C \left( \int_{B_{X^*}} \left| \sum_{j=1}^n \lambda^j (f(x^j) - f(y^j)) \right|^p d\mu(f) \right)^{\frac{1}{p}}. \tag{2.5} \]
4) There is a constant $C > 0$ such that for every $(x_i^j)_{i=1}^{n_1}, (y_i^j)_{i=1}^{n_1}$ in $X$ and $(\lambda_j^i)_{j=1}^{n_1} \subset \mathbb{K}$; $(1 \leq j \leq n_2)$ and $n_1, n_2 \in \mathbb{N}^*$, we have

$$\sum_{i=1}^{n_1} \left( \sum_{j=1}^{n_2} \lambda_j^i \left( T(x_i^j) - T(y_i^j) \right) \right)^p \leq C \sup_{f \in X^*} \left( \sum_{i=1}^{n_1} \left( \sum_{j=1}^{n_2} \lambda_j^i (f(x_i^j) - f(y_i^j)) \right) \right)^{\frac{p}{p'}}.$$  

(2.6)

**Proof.**

(1) $\implies$ (2) : Theorem 3.5 in [16].

(2) $\implies$ (3) : We apply Pietsch Domination Theorem for $p$-summing linear operators [7, Theorem 2.12], then there is a Radon probability $\mu$ on $B_{X^*}$ such that for any $m \in F(X)$ we have

$$\left\| \hat{T}(m) \right\| \leq C \left( \int_{B_{X^*}} |f(m)|^p \, d\mu(f) \right)^{\frac{1}{p'}}.$$

Now, let $(x_j^i)_{j=1}^{n_1}, (y_j^i)_{j=1}^{n_1} \subset X$ and $(\lambda_j^i)_{j=1}^{n_1} \subset \mathbb{K}$, we put

$$m = \sum_{j=1}^{n_1} \lambda_j^i \delta_{(x_j^i, y_j^i)} \in F(X),$$

Then

$$\left\| \hat{T}\left( \sum_{j=1}^{n_1} \lambda_j^i \delta_{(x_j^i, y_j^i)} \right) \right\| \leq C \left( \int_{B_{X^*}} \left( \sum_{j=1}^{n_1} \lambda_j^i \delta_{(x_j^i, y_j^i)} \right)^p \, d\mu(f) \right)^{\frac{1}{p'}}$$

thus

$$\left\| \sum_{j=1}^{n_1} \lambda_j^i \left( T(x_j^i) - T(y_j^i) \right) \right\| \leq C \left( \int_{B_{X^*}} \left( \sum_{j=1}^{n_1} \lambda_j^i (f(x_j^i) - f(y_j^i)) \right)^p \, d\mu(f) \right)^{\frac{1}{p'}}.$$

(3) $\implies$ (4) : Let $(x_i^j)_{i=1}^{n_1}, (y_i^j)_{i=1}^{n_1}$ in $X$ and $(\lambda_i^j)_{i=1}^{n_1} \subset \mathbb{K}$; $(1 \leq j \leq n_2)$, by (2.5) we have for every $1 \leq i \leq n_1$

$$\left\| \sum_{j=1}^{n_2} \lambda_j^i \left( T(x_i^j) - T(y_i^j) \right) \right\| \leq C \left( \int_{B_{X^*}} \left( \sum_{j=1}^{n_2} \lambda_j^i (f(x_i^j) - f(y_i^j)) \right)^p \, d\mu(f) \right)^{\frac{1}{p'}}.$$

Therefore,

$$\sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \lambda_j^i \left( T(x_i^j) - T(y_i^j) \right) \right\|^p \leq C \int_{B_{X^*}} \left( \sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \lambda_j^i (f(x_i^j) - f(y_i^j)) \right\|^p \right) \, d\mu(f) \leq \sup_{f \in X^*} \sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \lambda_j^i (f(x_i^j) - f(y_i^j)) \right\|^p.$$

Finally, we have

$$\left( \sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \lambda_j^i \left( T(x_i^j) - T(y_i^j) \right) \right\|^p \right)^{\frac{1}{p'}} \leq C \sup_{f \in X^*} \left( \sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \lambda_j^i (f(x_i^j) - f(y_i^j)) \right\|^p \right)^{\frac{1}{p'}}.$$
(4) $\implies$ (1): Let $u = \sum_{k=1}^{l} \delta_{(x_k, y_k)} \otimes s_k^* \in X \boxtimes E^*$. Let $A_u$ the set as defined in (2.1). Let $m = \sum_{i=1}^{n_1} m_i \otimes e_i^* \in A_u (m_i = \sum_{j=1}^{n_2} \lambda_i^j \delta_{(x_i^j, y_i^j)})$, by (2.2)

$$\left| \sum_{k=1}^{l} \langle T (x_k) - T (y_k), s_k^* \rangle \right|
= \left| \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \lambda_i^j \langle T (x_i^j) - T (y_i^j), e_i^* \rangle \right|
\leq \left( \sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j \langle T (x_i^j) - T (y_i^j), e_i^* \rangle \right|^p \right)^{1/p} \left( \sum_{i=1}^{n_1} \| e_i^* \|^p \right)^{1/p}
\leq C \sup_{f \in X^*} \left( \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \lambda_i^j \| f (x_i^j) - f (y_i^j) \| \right)^{1/p} \left( \sum_{i=1}^{n_1} \| e_i^* \|^p \right)^{1/p}
\leq C \|(m_i)\|_{p_1, w(F(X))} \|(e_i^*)\|_{p_1^*, (E)}.$$

By taking the infimum over all representations of $m \in A_u$, we obtain

$$\left| \sum_{k=1}^{l} \langle T (x_k) - T (y_k), s_k^* \rangle \right| \leq C d_{p_1}^L (u).$$

Then, $T$ is strictly Lipschitz $p$-summing. $\blacksquare$

If we put $n_2 = 1$ in the formula (2.6), we obtain exactly the definition of Lipschitz $p$-summing operators. Now, let $X$ and $Y$ be two metric spaces, Farmer [8] has introduced the definition of Lipschitz $p$-summing for the general case of metric spaces. Inspired by the definition of strictly Lipschitz $p$-summing, we give the following general setting.

**Definition 2.3.** Let $1 \leq p \leq \infty$. Let $X$ and $Y$ be two metric spaces. A Lipschitz operator $T : X \to Y$ is said to be $M$-strictly Lipschitz $p$-summing if there exists a positive constant $C$ such that for every $x_k, y_k \in X$ and $g_k \in Y^\# (1 \leq k \leq l)$ we have

$$\left| \sum_{k=1}^{l} g_k (T (x_k)) - g_k (T (y_k)) \right| \leq C d_{p_1}^L (u),$$

where $u = \sum_{k=1}^{l} \delta_{(x_k, y_k)} \otimes g_k \in X \boxtimes Y^\#$. Note that if $Y$ is a Banach space, for every $y^* \in Y^*$ we have $\text{Lip} (y^*) = \| y^* \|$, then

$$\|(y_i^*)\|_{p_1^*, (Y^\#)} = \|(y_i^*)\|_{p_1^*, (Y^*)}.$$

Therefore, the above definition leads to that of strictly Lipschitz $p$-summing. As an interesting characterization of M-strictly Lipschitz $p$-summing operators, we have the following result.

**Proposition 2.4.** Let $1 \leq p \leq \infty$. Let $X$ and $Y$ be two metric spaces. The following properties are equivalent.
1) $T : X \to Y$ is $M$-strictly Lipschitz $p$-summing.
2) The linearization operator $\tilde{T} : F (X) \to F (Y)$ is $p$-summing.
3) The Lipschitz adjoint $T^\# : Y^\# \to X^\#$ is Cohen strongly $p$-summing.

For the proof, we need the following Lemma.
Lemma 2.5. Let $X$ be a pointed metric space and $E, F$ be two Banach spaces. Suppose that $E$ and $F$ are isometrically isomorphic via the application $Q$. Let $u = \sum_{k=1}^{l} \delta_{(x_k,y_k)} \otimes s_k \in X \otimes E$, then
\[ d_p^L (u) = d_p^L (w), \]
where $w = \sum_{k=1}^{l} \delta_{(x_k,y_k)} \otimes Q (s_k) \in X \otimes F$.

Proof. The identification $\mathcal{F} (X) \otimes E = \mathcal{F} (X) \otimes F$ holds via the transformation
\[ \sum_{i=1}^{n} m_i \otimes e_i \mapsto \sum_{i=1}^{n} m_i \otimes Q (e_i). \]
Let $m = \sum_{i=1}^{n} m_i \otimes e_i \in A_u$, then $\sum_{i=1}^{n} m_i \otimes Q (e_i) \in A_w$. We have
\[ d_p^L (w) \leq \| m_i \|_{p,w,\mathcal{F}(X)} \| (Q (e_i))_i \|_{p^*,F} \]
\[ \leq \| m_i \|_{p,w,\mathcal{F}(X)} \| (e_i)_i \|_{p^*,E}, \]
by taking the infimum on $A_u$, we find $d_p^L (w) \leq d_p^L (u)$. With the same argument, the inverse inequality is immediate. 

Proof of Proposition 2.4. (1) $\Rightarrow$ (2) : Let $(x_k)_{k=1}^{l} \subset X$ and $(f_k)_{k=1}^{l} \subset \mathcal{F} (Y)^*$. We will show that $\delta_Y \circ T$ is strictly Lipschitz $p$-summing. So, there is $(g_k)_{k=1}^{l} \subset Y^*$ such that $f_k^* = Q_Y (g_k)$ for every $1 \leq k \leq l$. We have
\[ \left| \sum_{k=1}^{l} \langle \delta_Y \circ T (x_k) - \delta_Y \circ T (y_k), f_k^* \rangle \right| = \left| \sum_{k=1}^{l} \langle \delta (T(x_k), T(y_k)), Q_Y (g_k) \rangle \right| \]
\[ = \left| \sum_{k=1}^{l} g_k (T(x_k)) - g_k (T(y_k)) \right| \leq Cd_p^L (u) = Cd_p^L (w), \]
where $u = \sum_{k=1}^{l} \delta_{(x_k,y_k)} \otimes g_k \in X \otimes Y^*$ and $w = \sum_{k=1}^{l} \delta_{(x_k,y_k)} \otimes f_k^* \in X \otimes \mathcal{F} (Y)^*$. Then, $\delta_Y \circ T$ is strictly Lipschitz $p$-summing. Now, by the factorization (1.2) we have
\[ \delta_Y \circ T = \tilde{T} \circ \delta_X, \]
this shows that the linearization of $\delta_Y \circ T$ is $\tilde{T}$. Then, by Theorem 2.2, $\tilde{T}$ is $p$-summing.

(2) $\Rightarrow$ (3) : We have by (1.3)
\[ T^# = Q_X^{-1} \circ \tilde{T}^* \circ Q_Y, \]
hence, by (1.6) and the ideal property, $T^#$ is Cohen strongly $p^*$-summing.

(3) $\Rightarrow$ (1) : Suppose that $T^#$ is Cohen strongly $p^*$-summing. Let $u = \sum_{k=1}^{l} \delta_{(x_k,y_k)} \otimes g_k \in X \otimes Y^*$ then
\[ \left| \sum_{k=1}^{l} g_k (T(x_k)) - g_k (T(y_k)) \right| \]
\[ = \left| \sum_{k=1}^{l} \langle T^# (g_k), \delta_{(x_k,y_k)} \rangle \right|. \]
\[ \leq C g_p (\sum_{k=1}^{l} g_k \otimes \delta_{x_k,y_k}) \text{ (we know that } g_p = d_p^t) \]
\[ \leq C d_p (\sum_{k=1}^{l} \delta_{x_k,y_k} \otimes g_k) \text{ (by (2.3))} \]
\[ = C d_{p_L} (u). \]
Therefore, \( T \) is \( M \)-strictly Lipschitz \( p \)-summing. ■

The definition of Lipschitz dual of a given operator ideal \( \mathcal{I} \) is given in [1] as follows
\[ \mathcal{I}^{Lip_{0-dual}} (X,E) = \{ T \in Lip_0 (X,E) : T^t \in \mathcal{I} (E^*,X^#) \} . \]

**Corollary 2.6.** Let \( \mathcal{D}_p \) be the linear ideal of Cohen strongly \( p \)-summing operators. Then, we have
\[ \mathcal{D}_p^{Lip_{0-dual}} (X,E) = \Pi_{p_L}^{SL} (X,E) . \]

**Proof.** Let \( T \in \Pi_{p}^{SL} (X,E) \), by Theorem 2.2 its linearization \( \hat{T} \) is \( p^* \)-summing. By (1.3) and the ideal property, \( T^t \) is Cohen strongly \( p \)-summing, then \( T \in \mathcal{D}_p^{Lip_{0-dual}} (X,E) \). Conversely, let \( T \in \mathcal{D}_p^{Lip_{0-dual}} (X,E) \), then \( T^t : E^* \rightarrow X^# \) is Cohen strongly \( p \)-summing. We have
\[ \hat{T}^* = Q_X \circ T^t , \]
then \( \hat{T} \) is \( p^* \)-summing by (1.6) and the result follows by Theorem 2.2. ■

3. **Strictly Lipschitz \( p \)-nuclear operators**

In this section, we adopt the same procedure of the previous section for defining the strictly Lipschitz \( p \)-nuclear operators. Then, we obtain a nice class that has many interesting properties with good relations with other classes of Lipschitz mappings where the linearization operators play the key of all obtained results. Among the results of this section, we obtain an analog of Pietsch domination theorem and then the same factorization result to Cohen-Kwapień which is given for \( p \)-nuclear linear operators. Let \( p \in [1, \infty] \) and \( E, F \) be two Banach spaces. The definition of tensor norm \( w_p \) on \( E \otimes F \) is given in [6] by
\[ w_p (u) = \inf \left\{ \| (x_i) \|_{l_p (E)} \| (y_i) \|_{l_p (F)} \right\} , \]
where the infimum is taken over all representations of \( u \) of the form \( u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F \). Then by [6, Lemma 2.5.1], we have the following identification
\[ \mathcal{N}_p (E,F) = (E \otimes_{w_p} F^*)^* . \]

Next, we will give an analog approach for the class of strictly Lipschitz \( p \)-nuclear operators. Let \( X \) be a pointed metric space and \( E \) be a Banach space. Let \( u = \sum_{k=1}^{l} \delta_{(x_k,y_k)} \otimes s_k^* \in X \otimes E^* \) and \( A_u \) be the set as in (2.1). We consider
\[ w_p^{SL} (u) = \inf_{m \in A_u} \left\{ \| (m_i) \|_{l_p (F(X))} \| (e_i^*) \|_{l_p (E^*)} \right\} . \]

The following Proposition shows that \( w_p^{SL} \) is nothing else than the corresponding Lipschitz cross-norm to the tensor norm \( w_p \).
**Proposition 3.1.** Let $X$ be a pointed metric space and $E$ be a Banach space. For every $u \in X \boxtimes E$ we have

$$w_p^{SL}(u) = w_p^L(u),$$

where $w_p^L$ is the Lipschitz cross-norm corresponding to the tensor norm $w_p$.

**Proof.** Let $u = \sum_{k=1}^{l} \delta_{(x_k, y_k)} \boxtimes s_k \in X \boxtimes E$, we have

$$w_p^{SL}(u) = \inf_{m \in A_u} \left\{ \| (m) \|_{l_p^n(\mathcal{F}(X))} \| (e_i) \|_{l_p^m(E)} \right\}$$

$$= w_p(\sum_{k=1}^{l} \delta_{(x_k, y_k)} \otimes s_k) \text{ (by (2.3))}$$

$$= w_p^L(u). \quad \blacksquare$$

It is not difficult to prove the following results.

**Proposition 3.2.** Let $X$ be a pointed metric space and $E$ be a Banach space.

1. For every $u \in X \boxtimes E$ we have $w_p^{SL}(u) \leq d_p^L(u)$.
2. If $p = 1$, we have $w_1^{SL}(u) = d_1^L(u)$.

Now, we give the following definition of strictly Lipschitz $p$-nuclear operators for which we use the Lipschitz cross-norms.

**Definition 3.3.** Let $1 \leq p \leq \infty$. The Lipschitz operator $T : X \to E$ is strictly Lipschitz $p$-nuclear if for every $x_k, y_k \in X$ and $s_k^* \in E^*$ $(1 \leq k \leq l)$ we have

$$\left| \sum_{k=1}^{l} \langle T(x_k) - T(y_k), s_k^* \rangle \right| \leq C w_p^{SL}(u), \quad (3.2)$$

where $u = \sum_{k=1}^{l} \delta_{(x_k, y_k)} \boxtimes s_k^*$. The class of all strictly Lipschitz $p$-nuclear operators from $X$ into $E$ is denoted by $\mathcal{N}_p^{SL}(X, E)$, which is a Banach space with the norm $\mathcal{N}_p^{SL}(T)$ which is the smallest constant $C$ such that the inequality (3.2) holds.

As an immediate consequence of the Proposition 3.2, the following assertions can be checked easily.

**Proposition 3.4.** Let $X$ be a pointed metric space and $E$ be a Banach space.

1. For every $1 < p \leq \infty$ we have

$$\mathcal{N}_p^{SL}(X, E) \subset \Pi_p^{SL}(X, E).$$

2. If $p = 1$, then

$$\mathcal{N}_1^{SL}(X, E) = \Pi_1^{SL}(X, E).$$

In the following result, we connect between a Lipschitz operator and its linearization for the concept of $p$-nuclear. So, this connection helps in considering the space of these operators as a Lipschitz ideal generated by the composition method from the linear ideal $\mathcal{N}_p$. 
Theorem 3.5. Let $1 \leq p \leq \infty$. Let $X$ be a pointed metric space and $E$ be a Banach space. Let $T : X \to E$ be a Lipschitz operator. The following properties are equivalent.

1) $T$ is strictly Lipschitz $p$-nuclear.
2) The linearization operator $\hat{T}$ is $p$-nuclear.

On the other hand

$$N^S_L(X, E) = N_p(\mathcal{F}(X), E)$$

holds isometrically.

Proof. Suppose that $\hat{T}$ is $p$-nuclear. Let $x_k, y_k$ in $X$ and $e_k^* \in E^*(1 \leq k \leq l)$, we put $u = \sum_{k=1}^l \delta_{(x_k,y_k)} \otimes s_k^*$. Let $A_u$ the set as defined in (1.2). For every $m \in A_u$, by (2.2) we have

$$\sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle = \sum_{i=1}^{n_1} \langle \hat{T}(m_i), e_i^* \rangle .$$

Now, let us prove that $T$ is strictly Lipschitz $p$-nuclear. Let $m = \sum_{i=1}^{n_1} m_i \otimes e_i^* \in A_u$, by (1.7) we have

$$\left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| = \left| \sum_{i=1}^{n_1} \langle \hat{T}(m_i), e_i^* \rangle \right| \leq N_p(\hat{T}) \|m_i\|_{p^1,w(\mathcal{F}(X))} \|e_i^*\|_{l_{p^1,w}(E)} .$$

By taking the infimum over all representations of $m \in A_u$, we obtain

$$\left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| \leq N_p(\hat{T}) w_p(\sum_{k=1}^l \delta_{(x_k,y_k)} \otimes s_k^*) \leq N_p(\hat{T}) w^S_p(u) .$$

Then, $T$ is strictly Lipschitz $p$-nuclear and

$$N^S_L(T) \leq N_p(\hat{T}) .$$

Conversely, let $m_i \in \mathcal{F}(X)$ and $e_i^* \in E^*(1 \leq i \leq n_1)$. Then

$$\sum_{i=1}^{n_1} \left| \langle \hat{T}(m_i), e_i^* \rangle \right|$$

$$= \sup_{(\xi_i)_{i \in B_{\infty}}} \left| \sum_{i=1}^{n_1} \xi_i \langle \hat{T}(m_i), e_i^* \rangle \right|$$

$$= \sup_{(\xi_i)_{i \in B_{\infty}}} \left| \sum_{i=1}^{n_1} \xi_i \left( \sum_{j=1}^{n_2} \lambda_i^j \langle T(x_i^j) - T(y_i^j), e_i^* \rangle \right) \right|$$

$$= \sup_{(\xi_i)_{i \in B_{\infty}}} \left| \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} T(x_i^j) - T(y_i^j), \lambda_i^j \xi_i e_i^* \right|$$

$$\leq \sup_{(\xi_i)_{i \in B_{\infty}}} N^S_p(T) w^S_p(u) ,$$
where

\[ u = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \delta(x_i^j, y_i^j) \otimes \lambda_i^j \xi_i \epsilon_i^* \]

\[ = \sum_{i=1}^{n_1} \xi_i \sum_{j=1}^{n_2} \lambda_i^j \delta(x_i^j, y_i^j) \otimes \epsilon_i^* \]

\[ = \sum_{i=1}^{n_1} \xi_i m_i \otimes \epsilon_i^* \]

Therefore,

\[ \sum_{i=1}^{n_1} \left| \left\langle \hat{T}(m_i), \epsilon_i^* \right\rangle \right| \leq \sup_{(\xi_i) \in B_{X^*}} N_p^{SL}(T) \left\| (\xi_i m_i) \right\|_{p^1,1}(F(X)) \left\| (\epsilon_i^*) \right\|_{p^1,1}(E^*) \]

\[ \leq N_p^{SL}(T) \left\| (m_i) \right\|_{p^1,1}(F(X)) \left\| (\epsilon_i^*) \right\|_{p^1,1}(E^*) . \]

Then, by (1.7) \( \hat{T} \) is \( p \)-nuclear and

\[ N_p(\hat{T}) \leq N_p^{SL}(T). \] ■

The Pietsch domination theorem is one of the interesting characterizations which is verified by the class of strictly Lipschitz \( p \)-nuclear. It can be proved by means of the linearization operators. Consequently, there is an equivalent definition in which we use only fundamental inequalities.

**Theorem 3.6.** Let \( X \) be a pointed metric space and \( E \) be a Banach space. Let \( T : X \rightarrow E \) be a Lipschitz operator. The following properties are equivalent.

1) \( T \) is strictly Lipschitz \( p \)-nuclear.
2) There exist a constant \( C > 0 \), a Radon probability \( \mu \) on \( B_{X^*} \) and \( \eta \in B_{E^{**}} \) such that for every \( (x_i^j)_{j=1}^n, (y_i^j)_{j=1}^n \subseteq X \), \( (\lambda_i^j)_{j=1}^n \subseteq \mathbb{K} \) and \( e_i^* \in E^* \), we have

\[ \left| \left\langle \sum_{j=1}^n \lambda_i^j (T(x_i^j) - T(y_i^j)), e_i^* \right\rangle \right| \leq \left( \sum_{j=1}^n \lambda_i^j \left( f(x_i^j) - f(y_i^j) \right) \right)^p d\mu(f) \| e_i^* \|_{L_p,\eta} . \] (3.3)

3) For any \( (x_i^j)_{i=1}^{n_1}, (y_i^j)_{i=1}^{n_1} \subseteq X_j \), \( (\lambda_i^j)_{i=1}^{n_1} \subseteq \mathbb{K} \) and \( (e_i^*)_{i=1}^{n_1} \subseteq E^* \), we have

\[ \sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)), e_i^* \right| \leq C \sup_{f \in B_{X^*}} \left( \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \lambda_i^j \left( f(x_i^j) - f(y_i^j) \right) \right)^{\frac{p}{2}} \left\| (e_i^*) \right\|_{p^1,1} . \] (3.4)

In this case, we have

\[ N_p^{SL}(T) = \inf \{ C : \text{verifying (3.3)} \} \]

\[ = \inf \{ C : \text{verifying (3.4)} \} . \]
Proof. (1) $\Rightarrow$ (2): Let $(x^j)_{j=1}^n, (y^j)_{j=1}^n \subset X$, $(\lambda^j)_{j=1}^n \subset \mathbb{K}$ and $e^* \in E^*$. We put

$$m = \sum_{j=1}^n \lambda^j \delta_{(x^j, y^j)}.$$ 

By Theroem 3.5, the linearization operator $\hat{T}$ is $p$-nuclear. Then, by [12, Proposition 2], there exist a Radon probability $\mu$ on $B_{X^\#}$ and $\eta \in B_{E^{**}}$ such that

$$\left| \left\langle \sum_{j=1}^n \lambda^j (T(x^j) - T(y^j)), e^* \right\rangle \right| \leq \left( \int_{B_{X^\#}} |f(m)|^p \, d\mu(f) \right)^{\frac{1}{p}} \|e^*\|_{L_p(\eta)}.$$ 

By Hölder

$$\left| \left\langle \sum_{j=1}^n \lambda^j (f(x^j) - f(y^j)), e^*_i \right\rangle \right| \leq C \left( \int_{B_{X^\#}} \left| \sum_{j=1}^{n_2} \lambda^j_i (f(x^j) - f(y^j)) \right|^p \, d\mu(f) \right)^{\frac{1}{p}} \|e^*_i\|_{L_p(\eta)}.$$ 

By (3.3) we have for every $1 \leq i \leq n_1$

$$\left| \left\langle \sum_{j=1}^{n_2} \lambda^j_i (T(x^j) - T(y^j)), e^*_i \right\rangle \right| \leq C \left( \int_{B_{X^\#}} \left| \sum_{j=1}^{n_2} \lambda^j_i (f(x^j) - f(y^j)) \right|^p \, d\mu(f) \right)^{\frac{1}{p}} \|e^*_i\|_{L_p(\eta)}.$$ 

So, we have

$$\sum_{i=1}^{n_1} \left| \left\langle \sum_{j=1}^{n_2} \lambda^j_i (T(x^j) - T(y^j)), e^*_i \right\rangle \right| \leq C \int_{B_{X^\#}} \left| \sum_{j=1}^{n_2} \lambda^j_i (f(x^j) - f(y^j)) \right|^p \, d\mu(f) \right)^{\frac{1}{p}} \|e^*_i\|_{L_p(\eta)}.$$ 

By Hölder

$$\left| \left\langle \sum_{j=1}^{n_2} \lambda^j_i (f(x^j) - f(y^j)), e^*_i \right\rangle \right| \leq C \left( \int_{B_{X^\#}} \left| \sum_{j=1}^{n_2} \lambda^j_i (f(x^j) - f(y^j)) \right|^p \, d\mu(f) \right)^{\frac{1}{p}} \left( \int_{B_{E^{**}}} \left| \sum_{i=1}^{n_1} |e^* (e^*_i)|^p \, d\eta(e^{**}) \right|^{\frac{1}{p}} \right)^{\frac{1}{p}}.$$ 

(2) $\Rightarrow$ (3): Let $(x^j)_{j=1}^{n_1}, (y^j)_{j=1}^{n_1} \subset X$, $(\lambda^j)_{j=1}^{n_1} \subset \mathbb{K}$ $(1 \leq j \leq n_2)$ and $e_1^*, ..., e_{n_1}^* \in E^*$. Then, by (3.3) we have for every $1 \leq i \leq n_1$

$$\left| \left\langle \sum_{j=1}^{n_2} \lambda^j_i (T(x^j) - T(y^j)), e^*_i \right\rangle \right| \leq C \left( \int_{B_{X^\#}} \left| \sum_{j=1}^{n_2} \lambda^j_i (f(x^j) - f(y^j)) \right|^p \, d\mu(f) \right)^{\frac{1}{p}} \|e^*_i\|_{L_p(\eta)}.$$ 

By (2.6)

$$\sum_{k=1}^{l} (T(x_k) - T(y_k), s^*_k) = \sum_{i=1}^{n_1} \left| \left\langle \sum_{j=1}^{n_2} \lambda^j_i (T(x^j) - T(y^j)), e^*_i \right\rangle \right| \leq C \sup_{f \in B_{X^\#}} \left( \int_{B_{X^\#}} \left| \sum_{j=1}^{n_2} \lambda^j_i (f(x^j) - f(y^j)) \right|^p \, d\mu(f) \right)^{\frac{1}{p}} \|e^*_i\|_{L_p(\eta)}.$$ 

By taking the infimum over all representations of $m \in A_u$, we obtain

$$\sum_{k=1}^{l} (T(x_k) - T(y_k), s^*_k) \leq C w_p^{SL}(u).$$ 

Then, $T$ is strictly Lipschitz $p$-nuclear. ■
In the following proposition, we give new examples of strictly Lipschitz $p$-nuclear operators.

**Proposition 3.7.** Let $1 \leq p \leq \infty$. Let $X$ be a pointed metric space, $E$ and $F$ be Banach spaces. Let $v \in \mathcal{B}(E, F)$ and $T \in \text{Lip}_0(X, E)$.

1. If $T$ is strictly Lipschitz $p$-nuclear, then $v \circ T$ strictly Lipschitz $p$-nuclear. We have
   $$N_p^{SL}(v \circ T) \leq \|v\|N_p^{SL}(T).$$
2. If $L \in \Pi_p^{SL}(X, E)$ and $v \in \mathcal{D}_p(E, F)$, then $v \circ L$ is strictly Lipschitz $p$-nuclear and
   $$N_p^{SL}(v \circ L) \leq d_p(v)\pi_p^{SL}(L).$$

**Theorem 3.8.** Let $X$ be a pointed metric space and $E$ be a Banach space. Let $T : X \to E$ be a Lipschitz operator. The following properties are equivalent.

1. $T$ is strictly Lipschitz $p$-nuclear.
2. There exist a Banach space $G$, a strictly Lipschitz $p$-summing operator $R : X \to G$ and a Cohen strongly $p$-summing linear operator $S : G \to E$ such that $T = S \circ R$.

**Proof.** The second implication is immediate by Proposition 3.7. For the first, let $T$ be a strictly Lipschitz $p$-nuclear operator. Since $\hat{T}$ is $p$-nuclear, it factors as follows

$$\hat{T} : \mathcal{F}(X) \rightarrow E$$

$$\downarrow L \quad G \quad \uparrow S$$

where $L$ is $p$-summing and $S$ is Cohen strongly $p$-summing linear operators. Consequently, $T$ admits the following factorization

$$T : X \rightarrow E$$

$$\delta_X \downarrow \quad \uparrow S$$

$$\mathcal{F}(X) \downarrow L \quad G$$

On the other hand, $T = S \circ R$ where $R = L \circ \delta_X$. So, as $\hat{R} = L$ which is $p$-summing, $R$ is strictly Lipschitz $p$-summing by Theorem 2.2. ■

According to the last results and by applying [16, Corollary 3.2], we can identify the space of strictly Lipschitz $p$-nuclear operators with the dual of the space $X \boxtimes E$ endowing with the norm $w_p^{SL}$. This identification views as a Lipschitz analog of (3.1).

**Corollary 3.9.** Let $X$ be a pointed metric space and $E$ be a Banach space. Then

$$\mathcal{N}_p^{SL}(X, E) = (X \boxtimes_{w_p^{SL}} E^*)^* = (\mathcal{F}(X) \boxtimes_{w_p} E^*)^*.$$
linear case we know that a \( p \)-nuclear operator is compact whenever its domain or range is reflexive. In [11], there is an equivalence between a Lipschitz map and its linearization, so we have the following fact.

**Corollary 3.10.** Let \( 1 < p \leq \infty \). Let \( X \) be a pointed metric space and \( E \) be a reflexive Banach space. Then, every strictly Lipschitz \( p \)-nuclear operator is compact.

**Corollary 3.11.** Let \( 1 < p \leq \infty \). Let \( X \) be a pointed metric space and \( E \) be a reflexive Banach space. Then,

1. \( \mathcal{N}_p^{SL}(X, E) \subset \mathcal{D}_p^L(X, E) \).
2. \( \mathcal{N}_p^{SL}(X, E) \subset \mathcal{P}_p^{SL}(X, E) \subset \mathcal{P}_p^L(X, E) \).

Next, we have the same linear inclusion and a good relation with Lipschitz G-integral operators which introduced in [2]. By [2, Proposition 2.4], [6, Theorem 2.5.2] and Theorem 3.5, we have the following inclusion.

**Corollary 3.12.** Let \( 1 < p \leq \infty \). Let \( X \) be a pointed metric space and \( E \) be a Banach space. Then

\[ Lip_{0GI}(X, E) \subset \mathcal{N}_p^{SL}(X, E). \]

If \( E \) is an \( \mathcal{L}_{p^*} \)-space, we obtain by [6, Theorem 3.3.3] the following coincidence.

**Corollary 3.13.** Let \( 1 < p \leq \infty \). Let \( X \) be a pointed metric space and \( E \) is an \( \mathcal{L}_{p^*} \)-space. Then, the Lipschitz G-integral operators and strictly Lipschitz \( p \)-nuclear operators are coincide.

As a particular case, every Hilbert space \( H \) is an \( \mathcal{L}_2 \)-space, we have

\[ \mathcal{N}_2^{SL}(X, H) = Lip_{0GI}(X, H). \]

**Corollary 3.14.** Let \( X, E \) be two Banach spaces and \( \mathcal{N}_p \) the linear ideal of \( p \)-nuclear operators. Then we have,

\[ \mathcal{N}_p^{Lip_{0-dual}}(X, E) = \mathcal{N}_p^{rSL}(X, E). \]

**Proof.** Let \( T \in \mathcal{N}_p^{SL}(X, E) \), then by Theorem 3.5, its linearization \( \hat{T} \) is \( p^* \)-nuclear. By the ideal property and (1.3), \( T^t \) is \( p \)-nuclear, then

\[ \mathcal{N}_p^{SL}(X, E) \subset \mathcal{N}_p^{Lip_{0-dual}}(X, E). \]

Let \( T \in \mathcal{N}_p^{Lip_{0-dual}}(X, E) \), then \( T^t : E^* \to X^# \) is \( p \)-nuclear, then

\[ T^t = v_1 \circ v_2 \]

where \( v_1 \) is Cohen strongly \( p \)-summing and \( v_2 \) is \( p \)-summing. Therefore, \( Q_X^{-1} \circ \hat{T}^* = v_1 \circ v_2 \), and then

\[ \hat{T}^* = Q_X \circ v_1 \circ v_2, \]
which is $p$-nuclear by the ideal property. So, $\hat{T}$ is $p^*$-nuclear by [6, Theorem 2.2.4]. Finally, $T$ is strictly Lipschitz $p^*$-nuclear. ■

4. Strictly Lipschitz $(p, r, s)$-summing operators

In the same circle of ideas, we study the strong version of Lipschitz $(p, r, s)$-summing linear operators. The linear class has been stated by Lapresté in [13] and generalized to Lipschitz case by Chávez-Domínguez [3]. The results of this section are analogous of the setting of strictly Lipschitz $p$-nuclear operators. Now, we recall the following definition as stated in [3].

**Definition 4.1.** Let $X$ be a pointed metric space and $E$ be a Banach space. Let $T : X \to E$ be a Lipschitz map. $T$ is Lipschitz $(p, r, s)$-summing if there is a constant $C > 0$ such that for any $n \in \mathbb{N}^*$, $(x_i)_i, (y_i)_i$ in $X; (e_i)_i$ in $Y^*$ and $(\lambda_i)_i, (k_i)_i$ in $\mathbb{R}_+^*$ ($1 \leq i \leq n$), we have

$$\| (\lambda_i (T(x_i) - T(\lambda_i y_i))_{i=1}^n \|_{p, r} \leq C \| w_r^{Lip} ((\lambda_i k_i^{-1}, x_i, y_i)_i) \|_{p, w} \| (k_i e_i)_i \|_{p, w} \|_{E^*} ; \quad (4.1)$$

where $w_r^{Lip} ((\lambda_i k_i^{-1}, x_i, e_i)_{i=1}^n)$ is the weak Lipschitz $p$-norm defined by

$$w_r^{Lip} ((\lambda_i, x_i, y_i)_{i=1}^n) = \sup_{f \in B_{X^*}} \left( \sum_{i=1}^n |\lambda_i (f(x_i) - f(y_i))|^r \right)^{\frac{1}{r}} \quad = \| (\lambda_i \delta_{(x_i,y_i)}) \|_{p, w} (\mathcal{F}(X)) .$$

We denote by $\Pi_{p, r, s}^L (X, E)$ the Banach space of all Lipschitz $(p, r, s)$-summing operators with the norm $\pi_{p, r, s}^L (T)$ which is the smallest constant $C$ such that the inequality (4.1) holds.

**Definition 4.2.** Let $0 < p, r, s < \infty$ such that $\frac{1}{p} \leq \frac{1}{r} + \frac{1}{s}$. Let $X$ be a pointed metric space and $E$ be a Banach space. The Lipschitz operator $T : X \to E$ is strictly Lipschitz $(p, r, s)$-summing if there is a constant $C > 0$ such that for any $n \in \mathbb{N}^*$, $(x_i)_i_{i=1}^{n_1}, (y_i)_i_{i=1}^{n_2} \subset X$, $(\lambda_i)_{i=1}^{n_1} \subset \mathbb{K}$ ($j = 1, ..., n_2$) and any $e_i^*, ..., e_i^{n_1} \in E^*$, we have

$$\left( \sum_{j=1}^{n_2} \lambda_j^r (T(x_j^i) - T(y_j^i)), e_i^* \right)_{l^p} \right)^{\frac{1}{p}} \leq C \sup_{f \in B_{X^*}} \left( \sum_{j=1}^{n_2} \left| \sum_{j=1}^{n_2} \lambda_j^r (f(x_j^i) - f(y_j^i)) \right| \right)^{\frac{1}{r}} \| (e_i^*) \|_{l^{p, w}} . \quad (4.2)$$

The class of all strictly Lipschitz $(p, r, s)$-summing operators from $X$ into $E$ is denoted by $\Pi_{p, r, s}^{SL} (X, E)$, which is a Banach space with the norm $\pi_{p, r, s}^{SL} (T)$ which is the smallest constant $C$ such that the inequality (4.2) holds.

**Remark 4.3.** (1) If we put $n_2 = 1$, we obtain

$$\left( \sum_{i=1}^{n_1} |\lambda_i (T(x_i) - T(y_i)) e_i^*| |^p \right)^{\frac{1}{p}} \leq C \sup_{f \in B_{X^*}} \left( \sum_{i=1}^{n_1} | \sum_{j=1}^{n_1} \lambda_j^{-1} \lambda_i (f(x_i) - f(y_i)) |^r \right)^{\frac{1}{r}} \| (k_i e_i^*) \|_{l^{p, w}} .$$
\[ \Pi_{p,r,s}^{SL}(X, E) \subset \Pi_{p,r,s}^L(X, E). \]

(2) If \( p = 1, r \) and \( s \) verify \( \frac{1}{p} + \frac{1}{r} + \frac{1}{s} = 1 \), the definition coincides with the definition of strictly Lipschitz \( r \)-nuclear.

**Theorem 4.4.** Suppose that \( \frac{1}{p} = \frac{1}{r} + \frac{1}{s} \). Let \( X \) be a pointed metric space and \( E \) be a Banach space. Let \( T : X \to E \) be a Lipschitz operator. The following properties are equivalent.

1) \( T \) is strictly Lipschitz \((p, r, s)\)-summing.

2) \( \hat{T} \) is \((p, r, s)\)-summing.

In this case we have \( \pi_{p,r,s}(\hat{T}) = \pi_{p,r,s}^{SL}(T) \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( T \) be a strictly Lipschitz \((p, r, s)\)-summing operator. Let \( m_i \in \mathcal{F}(X) \) of the form \( m_i = \sum_{j=1}^{n_2} \lambda_i \delta_{(x_j, y_j)} \) and \( e_1^*, \ldots, e_{n_1}^* \in E^* \). Then,

\[
\left( \sum_{i=1}^{n_1} \left( \sum_{j=1}^{n_2} \lambda_i^* \left( T(x_j^i) - T(y_j^i) \right), e_i^* \right)^p \right)^{\frac{1}{p}} \leq \pi_{p,r,s}^{SL}(T) \sup_{f \in B_{X^\#}} \left( \sum_{i=1}^{n_1} \| f(m_i) \|^r \right)^{\frac{1}{r}} \| (e_i^*) \|_{\ell_1^{n_1 w}} \leq \pi_{p,r,s}^{SL}(T) \| (m_i) \|_{\ell_1^{p, w}(\mathcal{F}(X))} \| (e_i^*) \|_{\ell_1^{n_1 w}}.
\]

Then, \( \hat{T} \) is \((p, r, s)\)-summing and we have

\[ \pi_{p,r,s}(\hat{T}) \leq \pi_{p,r,s}^{SL}(T). \]

(2) \( \Rightarrow \) (1) Let \( (x_j^i)_{i=1}^{n_1}, (y_j^i)_{i=1}^{n_1} \in X, (\lambda_i^j)_{i=1}^{n_1} \subset \mathbb{K}(j = 1, \ldots, n_2) \) and any \( e_1^*, \ldots, e_{n_1}^* \in E^* \),

\[
\left( \sum_{i=1}^{n_1} \left( \sum_{j=1}^{n_2} \lambda_i^j \left( T(x_j^i) - T(y_j^i) \right), e_i^* \right)^p \right)^{\frac{1}{p}} \leq \pi_{p,r,s}(\hat{T}) \| (m_i) \|_{\ell_1^{p, w}(\mathcal{F}(X))} \| (e_i^*) \|_{\ell_1^{n_1 w}} \leq \pi_{p,r,s}(\hat{T}) \sup_{f \in B_{X^\#}} \left( \sum_{i=1}^{n_1} \left( \sum_{j=1}^{n_2} \lambda_i^j \left( f(x_j^i) - f(y_j^i) \right) \right)^r \right)^{\frac{1}{r}} \| (e_i^*) \|_{\ell_1^{n_1 w}}.
\]

Then, \( T \) is Lipschitz \((p, r, s)\)-summing and we have

\[ \pi_{p,r,s}^{SL}(T) \leq \pi_{p,r,s}(\hat{T}). \]

The following theorem gives an integral characterization of the strictly Lipschitz \((p, r, s)\)-summing operators. The proof is an adaptation of the one in the last section, for this we will omit it.
Theorem 4.5. Suppose that $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. Let $X$ be a pointed metric space and $E$ be a Banach space. Let $T : X \rightarrow E$ be a Lipschitz operator. The following properties are equivalent.

1) $T$ is strictly Lipschitz $(p, r, s)$-summing.

2) There exists a constant $C > 0$ and a Radon probability $\mu$ on $B_X^{\#}$ and $\eta \in B_{E^{**}}$ such that for all $x^j, y^j \in X$, $\lambda^j \in \mathbb{R} (1 \leq j \leq n)$ and $e^* \in E^*$, we have

$$\left| \sum_{j=1}^{n} \lambda^j \langle T(x^j) - T(y^j) \rangle, e^* \right| \leq C \left( \int_{B_X^\#} \sum_{j=1}^{n} \lambda^j \left( f(x^j) - f(y^j) \right)^{r} \, d\mu(f) \right)^{\frac{1}{r}} \|e^*\|_{L_s(\eta)}.$$

Corollary 4.6. Let $p \in [0, \infty]$ and $r, s \geq 1$. Suppose that $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. Let $X$ be a pointed metric space and $E$ be a Banach space. Let $T : X \rightarrow E$ be a Lipschitz operator. The following properties are equivalent.

1) $T$ is strictly Lipschitz $(p, r, s)$-summing.

2) There exists a Banach space $G$, a strictly Lipschitz $s^*$-summing operator $R : X \rightarrow G$ and a Cohen strongly $r$-summing linear operator $S : G \rightarrow E$ such that

$$T = S \circ R.$$

Let $1 \leq p, r, s < \infty$ such that $\frac{1}{p} + \frac{1}{r} + \frac{1}{s} = 1$. Let $E, F$ be two Banach spaces, Lapreste [13] has defined the norm $\mu_{p, r, s}$ as follows

$$\mu_{p, r, s}(u) = \inf \left\{ \| (\lambda_i) \|_{L_p^T} \| (e_i) \|_{L_r^E} \| (y_i) \|_{L_s^w(F)} \right\},$$

where the infimum is taken over all representations of $u = \sum_{i=1}^{n} \lambda_i e_i \otimes y_i \in E \otimes F$. For the Lipschitz case, Chávez-Domínguez in [3], has defined the corresponding Lipschitz norm $\mu_{p, r, s}$ (with the same notation of Lapreste) on the space of molecules $\mathcal{F}(X; E)$ and shown that the class $\Pi_{p, r, s}^L(X, E^*)$ coincides with the dual of $\mathcal{F}(X; E^*)$, $\mu_{p, r, s}$. Note that the space $\mathcal{F}(X; E)$ plays the same role of Lipschitz tensor product $X \otimes E$. Let $u = \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \in X \otimes E^*$, the norm $\mu_{p, r, s}$ is defined as follows

$$\mu_{p, r, s}(u) = \inf \left\{ \| (\lambda_i) \|_{L_p^T} \| (\lambda_i^{-1} \kappa_i^{-1} \delta_{(x_i, y_i)}) \|_{L_r^E(\mathcal{F}(X))} \| (\kappa_i e_i^*) \|_{L_s^w(E^*)} \right\},$$

where the infimum is taken over all representations of $u$ in $X \otimes E^*$ and $\lambda_i, \kappa_i > 0$.

Let $X$ be a pointed metric space and $E$ be a Banach space. Let $u = \sum_{k=1}^{l} \delta_{(x_k, y_k)} \otimes s_k^* \in X \otimes E^*$ and

$$A_u = \left\{ m = \sum_{i=1}^{n} \lambda_i m_i \otimes e_i^* \in \mathcal{F}(X) \otimes E^* : m = \sum_{k=1}^{l} \delta_{(x_k, y_k)} \otimes s_k \right\}.$$

We consider

$$\mu_{p, r, s}^{SL}(u) = \inf_{m \in A_u} \left\{ \| (\lambda_i) \|_{L_p^T} \| (m_i) \|_{L_r^E(\mathcal{F}(X))} \| (e_i^*) \|_{L_s^w(E^*)} \right\}.$$

In fact, in the Lipschitz definition of the norm $\mu_{p, r, s}$ we have just used elements in $X \otimes E^*$ of the form $\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i^*$ which equals to $u$ and $\lambda_i, \kappa_i > 0$, but in
the definition of $\mu_{p,r,s}^{SL}$, we have to consider all elements of the set $A_u$. Therefore, the infimum in $\mu_{p,r,s}^{SL}$ will in general be smaller.

**Proposition 4.7.** Let $X$ be a pointed metric space and $E$ be a Banach space. For every $u \in X \otimes E$ we have

$$\mu_{p,r,s}^{SL}(u) = \mu_{p,r,s}^{L}(u),$$

where $\mu_{p,r,s}^{L}$ is the Lipschitz cross-norms corresponding to the Lapreste tensor norm $\mu_{p,r,s}$.

**Proof.** Let $u = \sum_{k=1}^{l} \delta(x_k, y_k) \otimes s_k^* \in X \otimes E^*$. Let $m = \sum_{i} \lambda_i m_i \otimes e_i^* \in A_u$, we have

$$\mu_{p,r,s}^{SL}(u) = \inf_{m \in A_u} \left\{ \|\lambda_i\|_{p,r,s} \|m_i\|_{p,r,s}(\|e_i^*\|_{p,r,s}(F(X)) \|\gamma^*(X)\|_{p,r,s}(Y^*)\} \right\}$$

$$= \mu_{p,r,s}^{L}(\sum_{k=1}^{l} \delta(x_k, y_k) \otimes s_k^*) \text{ (by (2.3))}$$

$$= \mu_{p,r,s}^{L}(u). \quad \blacksquare$$

**Corollary 4.8.** Let $1 \leq p \leq \infty$. The following properties are equivalent.

1) The Lipschitz operator $T : X \to E$ strictly Lipschitz $(p, r, s)$-summing.

2) There exists a positive constant $C$ such that for every $x_k, y_k \in X$ and $s_k^* \in E^* (1 \leq k \leq l)$ we have

$$\left| \sum_{k=1}^{l} \langle T(x_k) - T(y_k), s_k^* \rangle \right| \leq C \mu_{p,r,s}^{SL}(u),$$

where $u = \sum_{k=1}^{l} \delta(x_k, y_k) \otimes s_k^*$.

Then, by [16, Corollary 3.2] we get the following coincidence result.

**Corollary 4.9.** Let $X$ be a pointed metric space and $E$ be a Banach space. Then

$$\Pi_{p,r,s}^{SL}(X, E^*) = \left( X \otimes_{\mu_{p,r,s}^{SL}} E^* \right)^* = \left( F(X) \otimes_{\mu_{p,r,s}^{L}} E^* \right)^*.$$

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