Multidimensional numerical scheme for resistive relativistic magnetohydrodynamics

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ABSTRACT
The paper describes a new upwind conservative numerical scheme for special relativistic resistive magnetohydrodynamics with scalar resistivity. The magnetic field is kept approximately divergence free and the divergence of the electric field is kept consistent with the electric charge distribution via the method of Generalized Lagrange Multiplier. The hyperbolic fluxes are computed using the Harten–Lax–van Leer (HLL) prescription and the source terms are accounted via the time-splitting technique. The results of test simulations show that the scheme can handle equally well both resistive current sheets and shock waves, and thus can be a useful tool for studying phenomena of relativistic astrophysics that involve both colliding supersonic flows and magnetic reconnection.

Key words: magnetic fields – MHD – relativity – waves – methods: numerical.

1 INTRODUCTION
In many phenomena of relativistic astrophysics, such as active galactic nuclei (AGN), gamma ray bursts, quasars, radio galaxies, microquasars, pulsars and magnetars, compact X-ray binaries, etc., the magnetic field is a key dynamic component. On one hand, the magnetic field drives, accelerates and partially collimates relativistic outflows from astrophysical black holes, neutron stars and their accretion discs. On the other hand, magnetic reconnection and dissipation is responsible for bright thermal and non-thermal emissions from these flows. Recent years have seen a remarkable progress in numerical methods for ideal relativistic magnetohydrodynamics (RMHD) (Komissarov 1999; Koide, Shibata & Kudoh 1999; Komissarov 2001; Koldoba, Kuznetsov & Ustyugova 2002; Del Zanna, Bucciantini & Londrillo 2003; Gammie, McKinney & Toth 2003; Duez et al. 2005; Shibata & Sekiguchi 2005; Anderson et al. 2006; Anninos, Fragile & Salmonson 2006; Antón et al. 2006; McKinney 2006; Mizuno et al. 2006; Mignone & Bode 2006; Neilsen et al. 2006; Noble et al. 2006; Del Zanna x al. 2007; Giacomazzo & Rezzolla 2007) and many interesting and important simulations have been carried out already. Quite often the numerical solutions exhibited violent magnetic reconnection. Although it is indeed very likely to occur in the considered astrophysical phenomena as the result of non-vanishing physical resistivity of plasma, both collisional and collisionless, the reconnection observed in the simulations is of purely numerical origin. It is driven by the numerical resistivity due to truncation errors and hence fully depends on the fine details of a numerical scheme and the resolution. A code for resistive RMHD would allow to control magnetic reconnection according to the incorporated physical models of resistivity. Moreover, the relativistic magnetic reconnection by itself is a sufficiently rich and important physical process to warrant the effort of developing such a code. The only numerical study of relativistic magnetic reconnection so far was carried out by Watanabe & Yokoyama (2006). However, their paper gives no details of their numerical scheme and test simulations, and therefore it is not clear as to how accurate their results are and how robust their numerical method is. Since many relevant astrophysical phenomena involve shock waves, including the fast magnetic reconnection of Petchek type (Lyubarsky 2005), a useful code should handle well not only current sheets and filaments but also shock waves. It is well known that codes that do not preserve the magnetic field divergence free can become unstable and crash in the cases with large spatial gradients. Thus, this issue must be addressed too. Moreover, in the relativistic limit the spatial charge density and the advective current can become significant, and thus the electric charge conservation has to be enforced. In this paper, we describe the results of our effort to construct a code that satisfies these criteria. The equations of resistive RMHD are described in Section 2. Section 3 gives brief discussion of the relativistic Ohm law and describes the implemented version of it. Section 4 describes the so-called augmented system of RMHD which is actually integrated by our numerical scheme and its connection to RMHD proper. Section 5 gives the details of numerical integration. The test simulations are described in Section 6 and our conclusions are summarized in Section 7.

2 BASIC EQUATIONS
The covariant Maxwell equations are

\[ \nabla_{\mu} F^{\mu\nu} = 0, \] (1)
\[ \nabla B^\alpha = I^\alpha, \]  
\[ \text{where } F^{\alpha\beta} \text{ is the Maxwell tensor of the electromagnetic field, } \ast F^{\alpha\beta} \text{ is the Faraday tensor and } I^\mu \text{ is the four-vector of electric current (Jackson 1979).} \]

In highly ionized plasma, including pair plasma, the electric and magnetic susceptibilities are essentially zero, and one has

\[ *F^{\alpha\beta} = \frac{1}{2} e^{\alpha\beta\mu\nu} F_{\mu\nu}, \]  
\[ F^{\alpha\beta} = -\frac{1}{2} e^{\alpha\beta\mu\nu} *F_{\mu\nu}, \]  
\[ \text{where } e^{\alpha\beta\mu\nu} = \sqrt{-g} \epsilon^{\alpha\beta\mu\nu} \] is the Levi–Civita alternating tensor of space–time and \( \epsilon^{\alpha\beta\mu\nu} \) is the four-dimensional Levi–Civita symbol.

In the coordinate basis of global inertial frame of special relativity, these equations split into the familiar set

\[ \nabla \cdot B = 0, \]  
\[ \partial_\nu B + \nabla \times E = 0, \]  
\[ \nabla \cdot E = q, \]  
\[ -\partial_\nu E + \nabla \times B = J, \]  
where

\[ E^\nu = F^{\nu\mu} = \frac{1}{2} e^{\nu\mu\alpha\beta} F_{\alpha\beta}, \]  
\[ B^\nu = * F^{\nu\mu} = \frac{1}{2} e^{\nu\mu\alpha\beta} F_{\alpha\beta}, \]  
\[ q = I^0, \quad J^\alpha = I^\alpha, \] are the electric field, the magnetic field, the electric charge density and the electric current density, respectively, as measured by the inertial observer (\( e^{\nu\mu\alpha\beta} \) is the Levi–Civita tensor of space–time). These equations are consistent with the electric charge conservation

\[ \partial_\nu q + \nabla \cdot J = 0. \]  
\[ \text{In magnetohydrodynamics, Maxwell’s equations are supplemented with the equations of motion of matter and the continuity equation. In the covariant form, the equations of motion can be written as} \]

\[ \nabla_\mu T^{\mu\nu} = 0, \]  
\[ \text{where the total stress-energy momentum tensor,} \]

\[ T^{\mu\nu} = T^{\mu\nu}_{(\text{m})} + T^{\mu\nu}_{(\text{e})}, \]  
\[ \text{is the sum of the stress-energy momentum tensor of electromagnetic field} \]

\[ T^{\mu\nu}_{(\text{e})} = F^{\mu\nu} F_{\nu\gamma} - \frac{1}{4} (F^{\rho\sigma} F_{\rho\sigma}) g^{\mu\nu}, \]  
\[ \text{and the stress-energy momentum tensor of matter} \]

\[ T^{\mu\nu}_{(\text{m})} = w u^\nu u^\mu + p g^{\mu\nu}. \]

Here, \( p \) is the thermodynamic pressure, \( w(p, \rho) \) is the relativistic enthalpy per unit volume as measured in the rest frame of fluid (\( w \) includes the rest energy-density of matter \( \rho \) and \( w' \) is the fluid four-velocity. In the global inertial frame with time-independent coordinate grid equation (14) splits into the energy and momentum conservation laws

\[ \partial_\nu e + \nabla \cdot S = 0, \]  
\[ \partial_\nu P + \nabla \cdot \Pi = 0, \]  
where

\[ e = \frac{1}{2} (E^2 + B^2) + w y^2 - p \] is the energy density,

\[ S = E \times B + w y^2 v, \]  
\[ \text{is the energy flux density,} \]

\[ P = E \times B + w y^2 v \] is the momentum density and

\[ \Pi = -EE - BB + w y^2 vv + \left[ \frac{1}{2} (E^2 + B^2) + p \right] g \] is the stress tensor. Here, \( y \) is the Lorentz factor, \( v \) is the velocity as measured by the inertial observer and \( g \) is the metric tensor of space.

The covariant continuity equation is

\[ \nabla_\nu (\rho u^\nu) = 0, \]  
where \( \rho \) is the rest mass density as measured in the rest frame of fluid. In the inertial frame, this reads

\[ \partial_\nu (\rho y) + \nabla \cdot (\rho y v) = 0. \]  

Equations (6), (7), (8), (13), (18), (19) and (25) constitute the 3+1 partial differential equation system of RMHD in special relativity. Once supplemented with equations of state, which relate various thermodynamic parameters of matter, and with the Ohm law, which couples matter and electromagnetic field, this system closes.

### 3 Ohm’s Law

Ohm’s law for relativistic plasma was studied by a number of authors (Lichnerowicz 1967; Ardavan 1984; Blackman & Field 1993; Gedalin 1996; Melatos & Melrose 1996; Punsly 2001; Meier 2004) and it can very complex. In general, it has the form of an evolutionary equation that accounts for the finite rise time of the electric current and includes the Lorentz and Hall effects (Meier 2004). Moreover, in strong magnetic field the electric conductivity can be highly anisotropic (Punsly 2001). As the first step, we consider here only the simplest kind of relativistic Ohm’s law that accounts only for the plasma resistivity and which assumes that it is isotropic. In the covariant form, it reads

\[ I_\nu = \sigma F_{\nu\mu} u^\mu + q_0 u_\nu, \]  
\[ \text{where } \sigma = 1/\eta \text{ is the conductivity, } \eta \text{ is the resistivity and } q_0 = -I_\nu u^\nu \text{ is the electric charge density as measured in the fluid frame (Lichnerowicz 1967; Blackman & Field 1993). From this we find that in the inertial frame of special relativity} \]

\[ J = \sigma y (E + v \times B) - (E \cdot v) v + q v. \]  
\[ \text{In particular, in the fluid frame one has the usual Ohm law,} \]

\[ J = \sigma E. \]  
\[ \text{In the limit of infinite conductivity } (\sigma \to \infty) \text{ equation (27) reduces to} \]

\[ E + v \times B - (E \cdot v) v = 0. \]
Splitting this equation into the components that are normal and parallel to the velocity vector, one obtains

\[ E_\perp + v \times B = 0 \]

and

\[ E_\parallel - (E \cdot v)v = 0. \]

Thus, one has the usual result

\[ E = -v \times B, \]

(29)

the electric field is purely inductive.

Now consider the reduced form of Ampere’s law

\[ -\partial_t E = J, \]

(30)

which is of interest for numerical schemes using time-splitting technique. When splitted into components normal and parallel to the velocity vector, this equation reads

\[ \partial_t E_\perp + \sigma \gamma [E_\perp - (E \cdot v)v] = 0, \]

(31)

\[ \partial_t E_\parallel + \sigma \gamma [E_\perp + v \times B] = 0 \]

(32)

(Here, \( v \) and hence \( \gamma \) are assumed to be constant.). The solutions of the initial value problem for these linear equations are

\[ E_\perp = E_\perp^0 \text{exp} \left( -\frac{\sigma}{\gamma} t \right), \]

(33)

and

\[ E_\parallel = E_\parallel^0 + \left( E_\perp^0 - E_\perp^0 \right) \text{exp} (-\sigma \gamma t), \]

(34)

where \( E_\perp^0 = -v \times B \) and suffix 0 denotes the initial components of \( E \). One can see that for relativistic flows the normal component of electric field approaches the inductive value \( E_\perp^0 \) faster than the parallel component approaches zero.

4 AUGMENTED SYSTEM

As well known, the divergence free condition (6) for the magnetic field can be treated as a constraint on the initial solution of Cauchy problem because the Faraday equation (7) will then ensure that the magnetic field remains divergence free at any time. The equation of electric charge conservation is also not independent and follows from the Ampere equation (9) and the Gauss law (8). These properties of the differential equations are not preserved by many numerical schemes. Indeed, the most straightforward way of constructing a self-consistent finite difference counterpart for a differential system like electrodynamics is to ignore all constraints (non-evolution equations) and to leave out all the supplementary laws like the electric charge conservation (otherwise the system of finite difference equations becomes overdetermined). However, it has been discovered that this lack of consistency may lead to strong corruption of numerical solutions in regions with large truncation errors, like strong discontinuities, and may even result in crash. In ideal magnetohydrodynamics (MHD), the divergence free condition has been found particularly important. A number of techniques have been proposed to combat the problem. Here, we adopt the so-called Generalized Lagrange Multiplier method developed by Munz et al. (1999). The main idea is to create a new, augmented system of differential equations, that will include only evolution equations and will have the same solutions of the Cauchy problem as the original system provided the initial solution satisfies the differential constraints of the original system. If, however, the initial solution does not satisfy the constraints then the deviations should decay or at least be carried out of the computational domain by relatively high speed waves. This will ensure that the deviations caused by truncation errors of a numerical method for the augmented system remain small.

To deal with the divergence free constraint, we modify equation (6) and (7) so that they become

\[ \partial_t \Phi + \nabla \cdot B = -\kappa \Phi, \]

(35)

\[ \partial_t B + \nabla \times E + \nabla \Phi = 0, \]

(36)

where \( \Phi \) is a new dynamic variable (pseudo-potential). From these equations, it follows that \( \Phi \) satisfies the telegraph equation

\[ -\partial_t^2 \Phi - \kappa \partial_t \Phi + \nabla^2 \Phi = 0. \]

Thus, \( \Phi \) is transported by hyperbolic waves propagating with the speed of light and decays if \( \kappa > 0 \). For positive \( \kappa \), the natural evolution of \( \Phi \) is directed towarda \( \Phi(r, t) = 0 \) (unless prevented by the boundary conditions). Equation (35) shows that in such a final state the magnetic field will be divergence free. In fact, it is easy to see that the divergence of magnetic field also satisfies the same telegraph equation

\[ -\partial_t^2 (\nabla \cdot B) - \kappa \partial_t (\nabla \cdot B) + \nabla^2 (\nabla \cdot B) = 0, \]

(38)

and thus evolves in the same fashion.

To deal with the Gauss law, we modify equations (8) and (9) so they read

\[ \partial_t \Psi + \nabla \cdot E = q - \kappa \Psi, \]

(39)

\[ -\partial_t \Psi + \nabla \cdot B - \nabla \Phi = J, \]

(40)

where \( \Psi \) is another new dynamic variable. From these two equations and the electric charge conservation it follows that the evolution of \( \Psi \) is again described by the telegraph equation

\[ -\partial_t^2 \Psi - \kappa \partial_t \Psi + \nabla^2 \Psi = 0. \]

(41)

(Although, in principle, one could use different constants \( \kappa \) for \( \Phi \) and \( \Psi \) this brings no benefit.) Thus, \( \Psi \), naturally evolves in the same fashion as \( \Phi \), ensuring that the electrodynamic solution is kept consistent with the Gauss law. Similarly, one finds that

\[ -\partial_t^2 (\nabla \cdot E - q) - \kappa \partial_t (\nabla \cdot E - q) + \nabla^2 (\nabla \cdot E - q) = 0. \]

(42)

An alternative approach could be to exclude the electric charge conservation law (13) from the set of numerically integrated equations and to compute the electric charge density required in the Ohm law via the Gauss law. This would give a closed system of differential evolution equations but there would be no guarantee that its numerical implementation would keep the electric charge distribution consistent with the evolution of electric current.

Summarizing, the augmented system of relativistic MHD is

\[ \partial_t \Phi + \nabla \cdot B = -\kappa \Phi, \]

(43)

\[ \partial_t B + \nabla \times E + \nabla \Phi = 0, \]

(44)

\[ \partial_t \Psi + \nabla \cdot E = q - \kappa \Psi, \]

(45)

\[ -\partial_t E + \nabla \times B - \nabla \Phi = J, \]

(46)

\[ \partial_t q + \nabla \cdot J = 0, \]

(47)

\[ \partial_t \rho \gamma + \nabla \cdot \rho \gamma v = 0, \]

(48)

\[ \partial_t \rho + \nabla \cdot S = 0, \]

(49)
\[ \partial_t \mathbf{P} + \nabla \cdot \mathbf{P} = 0, \tag{50} \]

where
\[ e = \frac{1}{2} (E^2 + B^2) + w \gamma^2 - p \tag{51} \]
\[ S = E \times B + w \gamma^2 v, \tag{52} \]
\[ P = E \times B + w \gamma^2 v, \tag{53} \]
\[ \Pi = -EE - BB + w \gamma^2 vv + \left[ \frac{1}{2} (E^2 + B^2) + p \right] \mathbf{g} \tag{54} \]

Every differential equation of the system is an evolution equation and a conservation law (with or without a source term), and there is wealth of numerical methods for such systems. For example, one could use Godunov’s upwind scheme (Godunov 1959) or one of its numerous higher order “children”. This simplicity of numerical implementation is the main advantage of the method of Generalized Lagrange Multiplier and it is the reason why we have decided to try it in our numerical method for resistive relativistic MHD.

5 NUMERICAL METHOD

The evolution equations (43) and (50) can be written as conservation laws in Cartesian coordinates, and this is the only type of coordinates we use in the paper, the system can be written as a single phase vector equation
\[ \frac{\partial \mathcal{Q}(\mathcal{P})}{\partial t} + \frac{\partial \mathcal{F}^n(\mathcal{P})}{\partial x^m} = \mathcal{S}(\mathcal{P}), \tag{55} \]

where
\[ \mathcal{Q} = \begin{pmatrix} \Phi \\ B^i \\ \psi \\ E^i \\ q \\ \rho \gamma \\ e \\ p^i \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} \Phi \\ B^i \\ \psi \\ E^i \\ q \\ \rho \gamma \\ e \\ p^i \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} -\kappa \psi \\ 0' \\ q - \kappa \psi \\ -J^i \\ 0 \\ 0 \\ 0' \end{pmatrix} \]

are the vectors of conserved quantities, primitive quantities and sources, respectively, and
\[ \mathcal{F}^n = \begin{pmatrix} B^m \\ e^{im} E_k + \Phi \delta^m_{k} \\ E^i \\ -e^{im} B_k + \psi g^m \\ J^m \\ \rho u^m \\ S^m \\ \Pi^m \end{pmatrix} \]

is the vector of corresponding hyperbolic fluxes. Here, \( i = \gamma v \) are the spatial components of four-velocity, \( g^i \) are the components of the metric tensor of space (given by Kronecker’s delta \( \delta^i_j \)) and \( e^{ijk} \) is the Levi–Civita alternating tensor of space.

We have found it useful to split the source term into two parts
\[ \mathcal{S}_b(\mathcal{P}) = \begin{pmatrix} 0 \\ 0' \\ q \\ -q v' \\ 0 \\ 0 \\ 0 \\ 0' \end{pmatrix}, \quad \mathcal{S}_a(\mathcal{P}) = \begin{pmatrix} -\kappa \Phi \\ 0' \\ -\kappa \psi \\ -J^i \\ 0 \\ 0 \\ 0' \end{pmatrix}, \]

where
\[ J^i = \sigma \gamma [E + v \times B - (E \cdot v) v] \]
is the conductivity current. The source term \( \mathcal{S}_b \) is potentially stiff (in the case of low resistivity) and is treated via the time-step splitting technique by Strang (1968). That is first the solution is advanced via integration of equation
\[ \frac{\partial \mathcal{Q}(\mathcal{P})}{\partial t} = \mathcal{S}_b(\mathcal{P}), \tag{56} \]

over the half time-step, \( \Delta t/2 \). Then, the solution is advanced via second-order accurate numerical integration of equation
\[ \frac{\partial \mathcal{Q}(\mathcal{P})}{\partial t} + \frac{\partial \mathcal{F}^n(\mathcal{P})}{\partial x^m} = \mathcal{S}_a(\mathcal{P}) \tag{57} \]

over the full time-step. Finally, the solution is advanced via integration of equation (56) over the half time-step once more. Thanks to the fact that all equations in (56) are linear they can be integrated exactly, in particular, we utilize at this step solutions (33, 34). This removes the stability constraints of the time-step otherwise imposed by \( \mathcal{S}_a \). In principle, all source terms could be passed to equation (56) but this somehow results in reduction of accuracy.

Equation (57) is integrated explicitly
\[ \mathcal{Q}_{n+1} = \mathcal{Q}_n + \Delta t \sum_{i=1}^{N_2} \mathcal{F}_{n+\frac{1}{2},m+\frac{1}{2}} - \mathcal{F}_{n+\frac{1}{2},m+\frac{1}{2}}, \tag{58} \]

The interface fluxes \( \mathcal{F}_{m+\frac{1}{2},a+\frac{1}{2}} \) are computed using the Harten–Lax–van Leer (HLL) prescription (Harten, Lax & van Leer 1983):
\[ \mathcal{F}_{m+\frac{1}{2},a+\frac{1}{2}} = \frac{\mathcal{F}_{m+\frac{1}{2},a} + \mathcal{F}_{m+\frac{1}{2},a} - Q_{m+\frac{1}{2},a} - Q_{m+\frac{1}{2},a}}{2}, \tag{59} \]

where indexes \( L \) and \( R \) refer to the states, respectively, to the left-hand side and to the right-hand side of the interface (which can be considered as the location of discontinuity in the solution at \( t = t_n \)).
due to the fact that the maximum characteristic speed of the system in each direction equals exactly to the speed of light (unity in our dimensionless equations). Obviously, the HLL solver is very easy to implement. It has already been tested in several codes for ideal RMHD and found to be very robust. It is also more diffuse compared to the linear Riemann solver based on full spectral decomposition, which is particularly notable in problems involving stationary shocks (Komissarov 2006). In the case of resistive RMHD, this advantage of linear Riemann solver becomes much more important as the physical resistivity introduces smooth shock structure, and thus reduces the effects of numerical diffusion. Moreover, the characteristic modes of resistive equations describefully decoupled electromagnetic and hydrodynamic waves. As a result, they do not provide with accurate representation of the flow unless the conductivity is indeed low (This is also the most likely reason for the relatively high numerical diffusion revealed in the test simulations with high conductivity. See below.). Thus, we do not expect the method of linear Riemann solver to be any superior to the HLL flux prescription in resistive MHD.

For the auxiliary half time-step, these left- and right-hand states are found via the piecewise constant reconstruction of numerical solution in each spatial direction

\[
P_i = \mathcal{P}_n \quad \text{for} \quad x_n^m = \frac{\Delta x^m}{2} < x^m < x_n^m + \frac{\Delta x^m}{2},
\]

where \(x_n^m\) is the coordinate of the cell centre and \(\mathcal{P}_n\) is the phase state vector of the cell.

The auxiliary solution is then used for the second now quadratic reconstruction of numerical solution within each cell

\[
\mathcal{P}_{i+\frac{1}{2}} = \mathcal{P}_{i-\frac{1}{2}} + a_1 (x^m - x_i^m) + \frac{a_2}{2} (x^m - x_i^m)^2
\]

for \(x_i^m - \frac{\Delta x^m}{2} < x^m < x_i^m + \frac{\Delta x^m}{2}\).\n
Obviously, \(a_1\) and \(a_2\) are the first- and the second-order derivatives of the reconstructed solution and these should be derived using the auxiliary solution by means of one of many existing non-linear limiters (needed to avoid spurious oscillations). In this particular paper, \(a_1\) is found using the same limiter as in our ideal MHD code (Komissarov 1999)

\[
a_1 = \text{av} (\mathcal{P}_i^r, \mathcal{P}_i^e) - \text{av} (\mathcal{P}_{i-1}^r, \mathcal{P}_{i-1}^e),
\]

where

\[
\mathcal{P}_i^r = \frac{\mathcal{P}_i - \mathcal{P}_{i-1}}{\Delta x^m}, \quad \mathcal{P}_i^e = \frac{\mathcal{P}_{i+1} - \mathcal{P}_i}{\Delta x^m}.
\]

are the left- and the right-hand side numerical approximations of the first derivative (\(i\) is the cell index along the direction of \(x^m\)) and

\[
\text{av}(a, b) = \begin{cases} 
0 & \text{if } ab < 0 \text{ or } a^2 + b^2 = 0, \\
\frac{a^2 b + ab^2}{a^2 + b^2} & \text{if } ab \geq 0 \text{ and } a^2 + b^2 \neq 0.
\end{cases}
\]

To find \(a_2\), we use a similar procedure. First, we compute the left-hand side, centre and right-hand side numerical approximations for the second derivative, \(\mathcal{P}_{i+\frac{1}{2}}^r, \mathcal{P}_{i+\frac{1}{2}}^c\) and \(\mathcal{P}_{i+\frac{1}{2}}^e\) and then we feed them to the minmod function with three arguments

\[
a_2 = \text{minmod}(\mathcal{P}_{i+\frac{1}{2}}^r, \mathcal{P}_{i+\frac{1}{2}}^c, \mathcal{P}_{i+\frac{1}{2}}^e).
\]

where

\[
\text{minmod}(a, b, c) = \begin{cases} 
0 & \text{if } ab \leq 0 \text{ or } bc \leq 0, \\
\min(a, b, c) & \text{if } a, b, c > 0 \\
\max(a, b, c) & \text{if } a, b, c < 0.
\end{cases}
\]

The left- and the right-hand states of each cell interface that are found via this second reconstruction are then used to compute HLL fluxes \(\mathcal{F}_{i+\frac{1}{2}+\frac{1}{2}}\) of equation (58). The resulting scheme is second-order accuracy in time and third-order accuracy in space.

6 TEST SIMULATIONS

In these test simulations, we use the polytropic equation of state

\[
w = \rho + \frac{\Gamma}{\Gamma - 1} p
\]

with the ratio of specific heats \(\Gamma = 4/3\).

6.1 One-dimensional test problems

6.1.1 Stationary Fast Shock

To set up this test, we solved the ideal relativistic MHD shock equations describing stationary shocks. The selected particular solution is

(i) Left-hand state.

\[
\mathbf{B} = (5.0, 15.08, 0.0), \quad \gamma = (4.925, 0.0, 0.0), \quad \rho = 1.0, \quad p = 10.0, \quad q = 0, \quad \Phi = 0, \quad \Psi = 0.
\]

(ii) Right-hand state.

\[
\mathbf{B} = (5.0, 28.92, 0.0), \quad \gamma = (0.6209, 0.1009, 0.0), \quad \rho = 7.930, \quad p = 274.1; \quad q = 0, \quad \Phi = 0, \quad \Psi = 0.
\]
The electric field is found via the ideal equation

\[ E = -v \times B. \]  

The computational grid is uniform and has 100 cells in \([-1, +1]\) and the initial solution is set as a discontinuity at \(x = 0\). The resistivity is \(\eta = 0.01\). Fig. 1 shows the numerical solution at \(t = 3.0\) by when the secondary waves created during the development of the dissipative shock structure have left the grid. One can see that the shock jump is captured very well. The fact that there are only three grid points in the shock structure tells that the shock is unresolved and suggests that the shock structure might be dominated by numerical dissipation. This is confirmed by the simulations with higher resistivity (see Fig. 3).

6.1.2 Stationary Slow Shock

To set up this test, we also solved the ideal relativistic MHD shock equations describing stationary shocks. Now the selected particular solution is

(i) Left-hand state.

(ii) Right-hand state.

6.1.3 Alfvén wave

To set up this test, we utilized the analytical solution for ideal MHD Alfvén waves obtained in Komissarov (1997). In this test \(\rho = 1.0\),
Figure 4. Alfvén wave. The solid lines show the exact ideal MHD solution at times $t = 0$ and 1.5. The circles show the numerical solution at $t = 1.5$. $p = 1.0, B^\prime = 1.0$ and the Alfvén speed $c_A = 0.4079$. Initially, the wave occupies the zone $x_0 < x < x_1$, with $x_0 = -0.8, x_1 = 0.0$. To the left-hand side of the wave $B = (1.0, 0.1, 0.0), \gamma v = 0$. In the wave, the angle $\theta$ between the tangential component of magnetic field and the $y$-axis varies as

$$\theta = 2\pi (3\xi^2 - 2\xi^3), \quad \xi = (x - x_0)/(x_1 - x_0)$$

that gives vanishing first derivatives at $x_0, 1$. The initial electric field is computed via equation (68) and the electric charge density via equation (8). The resistivity is set to a relatively small value, $\eta = 0.003$, in order to get closer to the ideal case. The simulations are continued up to $t = 1.5$ and then compared with the exact solution of ideal MHD at the same time (Fig. 4). One can see that the agreement is pretty good. The ideal solution keeps the wave profile totally invariant, however, the numerical solution is a little distorted, mainly due to numerical dissipation (this is confirmed by studying the dependence on $\eta$).

When the zero gradient boundary conditions (free-flow) are utilized in the simulations then both the fast and the slow waves do not get reflected of boundaries and cleanly pass through. However, the Alfvén waves exhibit notable reflection (in contrast to the results obtained with our ideal MHD code). We have not figured out yet as to how to avoid such a reflection.

6.1.4 Self-similar current sheet

Assume that $B = [0, B(x, t), 0]$, the magnetic pressure is much smaller than the gas pressure and $B(x, 0)$ changes sign within a thin current layer of width $\Delta l$. Provided the initial solution is in equilibrium, $p = \text{constant}$, the evolution of this layer is a slow diffusive expansion.

$$\partial_t B - \eta \partial^2_x B = 0,$$

As the width of the layer becomes much larger than $\Delta l$ the expansion becomes self-similar

$$B(x, t) = B_0 \text{erf} \left( \frac{1}{2\sqrt{\eta t}} \right), \quad \xi = t/x^2,$$

where erf is the error function, and this analytic results can be used to test the resistive part of the code. In the test problem that is presented here the initial solution has uniform distribution of $P = 50.0, \rho = 1.0, E = 0$ and $\gamma v = 0$ and the initial magnetic field is given by equation (69) for $B_0 = 1.0, t = 1$ and $\eta = 0.01$. The computational grid is uniform and has 200 cells in $[-1.5, +1.5]$. The numerical simulations are continued up to $t = 8$ and then the numerical solution is compared with the solution (69) for $t = 9$. The results are shown in Fig. 5 – one cannot see the difference between the solutions.

6.2 Multidimensional tests

All the one-dimensional problems, that are described above, have been used to test both the two-dimensional and three-dimensional versions of the code via application in all two/three directions of the Cartesian grid. The results are almost identical to that of one-dimensional tests. In addition, we considered several generically multidimensional problems.

6.2.1 Strong cylindrical explosion

Strong symmetric explosions are useful standard tests for MHD codes even if there are no exact analytic solutions to work with.
Figure 6. Two-dimensional strong cylindrical explosion. Top left-hand panel: $B^x$ and magnetic field lines; Top right-hand panel: $B^y$ and magnetic field lines; Bottom left-hand panel: $\log_{10} p$, gas pressure; Bottom right-hand panel: Lorentz factor.

Figure 7. Two-dimensional strong cylindrical explosion. This plot shows slices along the $y$-axis (dashed line and stars) and along the $x$-axis (solid lines and circles) for gas pressure (left-hand panel), Lorentz factor (middle panel) and $B^x$ (right-hand panel).
Figure 8. Three-dimensional strong spherical explosion. Top left-hand panel: $B^i$; top right-hand panel: $q$, electric charge density; Bottom left-hand panel: $p$, gas pressure; bottom right-hand panel: Lorentz factor.

Figure 9. Three-dimensional strong spherical explosion. In both the panels, stars and solid line show the solution along the $y$-axis and circles show the solution along the $z$-axis (the initial solution is axisymmetric.) Left-hand panel: log$_{10} p$, gas pressure. Right-hand panel: Lorentz factor.

This is because the generated shocks make all possible angles both to the grid and to the magnetic field, and such diversity of conditions allows to detect well-hidden bugs and potential weaknesses. In this problem, the Cartesian computational domain is $(-6.0, +6.0) \times (-6.0, +6.0)$ with 200 equidistant grid points in each direction. The initial explosion zone is a cylinder of radius $r = 1$ centred on the origin. Its pressure and density are set to $p = 1$ and $\rho = 0.01$ for $r < 0.8$ and exponentially decrease for $0.8 < r < 1.0$. The ambient gas has $p = \rho = 0.001$. The initial magnetic field is uniform, $B = (0.1, 0.0, 0.0)$. Fig. 6 shows the two-dimensional solution at $t = 4$ for
\( \eta = 0.018 \) and \( \eta_0 = 1/\kappa = 0.18 \). It exhibits the same features as the ideal MHD solution of a similar test problem (Komissarov 1999), which is expected given the low value of \( \eta \). One can see no features that could be suspected as artifacts. For more detailed future comparisons with other codes Fig. 7 shows slices of the solution along \( x = 0 \) and \( y = 0 \).

The same problem has been used to test the three-dimensional code with identical results.

6.2.2 Strong spherical explosion

Finally, we tested our three-dimensional code on the problem of spherical explosion. All parameters of the explosion are the same as in the cylindrical case with the exception of explosion zone – now this is a sphere of unit radius. The computational domain is \( \eta \times \eta \times \eta \). The slice in each direction. Figs 8 and 9 show the numerical solution for \( \eta = 0.0257 \) and \( \eta_0 = 1/\kappa = 0.257 \) at \( t = 4.0 \). The general structure of the solution is similar to that of the cylindrical case but with much stronger central rarefaction. One qualitatively new feature is the non-vanishing electric charge density (top right-hand panel of Fig. 8). Given the axial symmetry of the problem one expects the solutions to be the same in the planes \( z = 0 \) and \( y = 0 \). Fig. 9 shows that this is indeed the case.

7 CONCLUSIONS

We have constructed a multidimensional upwind scheme for resistive RMHD. At the moment only the simplest form of relativistic Ohm’s law with scalar resistivity has been implemented and more work has to be done to incorporate more realistic versions of generalized Ohm’s law. The results of test simulations show that the scheme is robust in the regime of small to moderate magnetization, which can be described by the ratio of the electromagnetic energy density to the total mass-energy density of matter. The regime of high magnetization is still problematic as the truncation errors for the energy-momentum of matter become large, often making impossible conversion of the conserved quantities into the primitive ones. This is a well-known problem of all conservative schemes for relativistic MHD. Compared to our scheme for ideal RMHD (Komissarov 1999), which utilizes the method of constrained transport in order to preserve the magnetic field divergence free, this scheme performs a little bit worse in this regime which is probably related to the additional errors caused by the non-vanishing divergence of magnetic field. The numerical shock structure in the limit of high physical conductivity is also not as sharp, indicating higher numerical diffusion. Apart from these disadvantages, the scheme can handle quite well both resistive current sheets and shock waves, and thus can serve as a useful tool for studying phenomena of relativistic astrophysics that involve both colliding supersonic flows and magnetic reconnection. It is very simple, does not use staggered grid, can be easily combined with adaptive grid codes and parallelized.

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