ESTIMATES ON MODULATION SPACES FOR
SCHRÖDINGER EVOLUTION OPERATORS WITH
QUADRATIC AND SUB-QUADRATIC POTENTIALS

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Abstract. In this paper we give new estimates for the solution to
the Schrödinger equation with quadratic and sub-quadratic poten-
tials in the framework of modulation spaces.

1. Introduction

In this paper, we shall give estimates for the solution to the time
dependent Schrödinger equation
\begin{equation}
\begin{cases}
    i\partial_t u(t, x) = -\frac{1}{2}\Delta u(t, x) + V(t, x)u(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\
    u(0, x) = u_0(x), & x \in \mathbb{R}^n
\end{cases}
\end{equation}
in the framework of modulation spaces. Here $i = \sqrt{-1}$, $u(t, x)$ is a
complex valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $V(t, x)$ is a real valued
function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $u_0(x)$ is a complex valued function of
$x \in \mathbb{R}^n$, \(\partial_t = \partial u/\partial t\) and \(\Delta u = \sum_{i=1}^{n} \partial^2 u/\partial x_i^2\).

We shall highlight the case $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ and for all multi-indices
$\alpha$ with $|\alpha| \geq 2$ or $|\alpha| \geq 1$ there exists $C_\alpha > 0$ such that
\begin{equation}
|\partial^\alpha_x V(t, x)| \leq C_\alpha, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n.
\end{equation}

There are a large number of works devoted to study the equation (1).
Particularly, in the context of modulation spaces $M^{p,q}$, these types of
issues were initiated in the works of Bényi-Gröchenig-Okoudjou-Rogers [1],
Wang-Hudzik [16] and Wang-Zhao-Guo [17].

Theorem A. (Bényi-Gröchenig-Okoudjou-Rogers [1]) Let $1 \leq p, q \leq \infty$
and $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. Suppose $V(t, x) = 0$. Then there exists a positive constant $C$
such that
\[\|u(t, \cdot)\|_{M^{p,q}_0} \leq C(1 + |t|)^{n/2}\|u_0\|_{M^{p,q}_0}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n)\]
for all $t \in \mathbb{R}$, where $u(t, x)$ is the solution of (1) with $u(0, x) = u_0(x)$.
Theorem 1.1. In (Wang-Hudzik [10]) let 2 ≤ p ≤ ∞, 1 ≤ q ≤ ∞, 1/p + 1/p' = 1 and ϕ0 ∈ S(R^n) \ {0}. Suppose V(t, x) = 0. Then there exists positive constants C and C' such that
\[ \|u(t, \cdot)\|_{M^{p,q}_{0, \ell}(\cdot)} \leq C(1 + |t|)^{-n(1/2 - 1/p)}\|u_0\|_{M^{p,q}_{0, \ell}}, \quad u_0 \in S(R^n) \]
and
\[ \|u(t, \cdot)\|_{M^{p,q}_{0, \ell}(\cdot)} \leq C'(1 + |t|)^{n(1/2 - 1/p)}\|u_0\|_{M^{p,q}_{0, \ell}}, \quad u_0 \in S(R^n) \]
for all t ∈ R, where u(t, x) is the solution of (1) with u(0, x) = u_0(x).

The studies of this theme have been developed by a number of authors using a large variety of methods (see, for example, Béyi-Okoudjou [2], Cordero-Nicola [3], [4], Kobayashi-Sugimoto [12], Miyachi-Nicola-Rivetti-Tabacco-Tomita [13], Tomita [14], Wang-Huang [15]).

In our previous papers, we have the following estimates.

Theorem C. (Kato-Kobayashi-Ito [9], [10], [11]) Let 1 ≤ p, q ≤ ∞ and ϕ0 ∈ S(R^n) \ {0}.

(i) Suppose V(t, x) = 0. Then
\[ \|u(t, \cdot)\|_{M^{p,q}_{0, \ell}(\cdot)} = \|u_0\|_{M^{p,q}_{0, \ell}}, \quad u_0 \in S(R^n) \]
holds for all t ∈ R.

(ii) Suppose V(t, x) = ±\frac{1}{2}|x|^2. Then
\[ \|u(t, \cdot)\|_{M^{p,q}_{0, \ell}(\cdot)} = \|u_0\|_{M^{p,q}_{0, \ell}}, \quad u_0 \in S(R^n) \]
holds for all t ∈ R.

In (i) and (ii), u(t, x) and ϕ(t, x) denote the solutions of (1) with u(0, x) = u_0(x) and ϕ(0, x) = ϕ_0(x).

We remark that Theorem C covers Theorem A and B (see [9]).

To state our results, we define the Schrödinger operator of a free particle \(e^{-\frac{1}{2}it\Delta}\) by
\[ (e^{-\frac{1}{2}it\Delta}f)(x) = \mathcal{F}^{-1}_{\xi \to x}[e^{-\frac{1}{2}it|\xi|^2}\mathcal{F}f(\xi)](x), \quad f \in S(R^n). \]

Here we use the notation \(\mathcal{F}f(\xi) = \int_{R^n} f(x)e^{-ix\xi}dx\) for the Fourier transform of f and \(\mathcal{F}^{-1}f(x) = \int_{R^n} f(\xi)e^{ix\xi}d\xi\) with \(d\xi = (2\pi)^{-n}d\xi\) for the inverse Fourier transform of f. The following theorems are our main results.

Theorem 1.1. Let 1 ≤ p ≤ ∞, ϕ0 ∈ S(R^n) \ {0} and T > 0. Set
\[ \phi(t, x) = e^{-\frac{1}{2}it\Delta}\phi_0(x). \]
If V ∈ C^∞(R × R^n) satisfies (2) for all multi-indices α with |α| ≥ 2, then there exists CT > 0 such that
\[ \|u(t, \cdot)\|_{M^{p,q}_{0, \ell}(\cdot)} \leq CT\|u_0\|_{M^{p,q}_{0, \ell}}, \quad u_0 \in S(R^n) \]
for all t ∈ [−T, T], where u(t, x) denotes the solution of (1) in C(R; L^2(R^n)) with u(0, x) = u_0(x).
In the above theorem, we cannot expect to replace the $M^{p,p}$ norm with the $M^{p,q}$ norm. In fact, when $V(t,x) = \frac{1}{2}|x|^2$ we have

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(t,\cdot)}} = \|W_{\varphi_0}u_0(x \cos t - \xi \sin t, x \sin t + \xi \cos t)\|_{L^p_{x}} \|_{L^q_{\xi}}$$

and thus

$$\|u(t,\frac{\pi}{2},\cdot)\|_{M^{p,q}_{\varphi(t,\cdot)}} = \|W_{\varphi_0}u_0(\xi, x)\|_{L^p_{\xi}} \|_{L^q_{x}}$$

(refer to \[10\]), but $\|W_{\varphi_0}u_0(\xi, x)\|_{L^p_{\xi}} \|_{L^q_{x}} \leq C \|W_{\varphi_0}u_0(x, \xi)\|_{L^p_{x}} \|_{L^q_{\xi}}$ does not hold generally. However, if we strengthen the assumption of $V$ then we can replace the $M^{p,p}$ norm with $M^{p,q}$ norm.

**Theorem 1.2.** Let $1 \leq p, q \leq \infty$, $\varphi_0 \in S(\mathbb{R}^n) \setminus \{0\}$ and $T > 0$. Set $\varphi(t,x) = e^{\frac{\pi}{2}t} \varphi_0(x)$. If $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfies \[2\] for all multi-indices $\alpha$ with $|\alpha| \geq 1$, then there exists $C_T > 0$ such that

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(t,\cdot)}} \leq C_T \|u_0\|_{M^{p,q}_{\varphi_0}}, \quad u_0 \in S(\mathbb{R}^n)$$

for all $t \in [-T, T]$, where $u(t,x)$ denotes the solution of \[1\] in $C(\mathbb{R}; L^2(\mathbb{R}^n))$ with $u(0,x) = u_0(x)$.

**Remark 1.3.** In Theorem \[1.1\] and Theorem \[1.2\] we assume $V(t,x) \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ but, in fact, it is enough to assume $V(t,x)$ is $C^2$-function in $t$ and $C^2[\frac{n}{4}+4]$-function in $x$.

This paper is organized as follows. In Section 2, we give some notations and recall the definitions and basic properties of wave packet transform and modulation spaces. In Section 3, we give some properties concerning the orbit of the classical mechanics corresponding to the Schrödinger equation \[1\]. In Section 4, we prove Theorem \[1.1\]. Finally, in Section 5, we prove Theorem \[1.2\].

## 2. Preliminaries

### 2.1. Notations.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and a $m \times n$ matrix $A = (a_{ij})$, we denote

$$\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}, \quad \|x\|_\infty = \max_{1 \leq j \leq n} |x_j| \quad \text{and} \quad \|A\|_\infty = \max_{1 \leq j \leq m, 1 \leq k \leq n} |a_{jk}|.$$ 

For a real valued function $V \in C^1(\mathbb{R} \times \mathbb{R}^n)$, we put

$$\nabla_x V(t,x_1,\ldots,x_n) = (\partial_{x_1} V(t,x_1,\ldots,x_n),\ldots,\partial_{x_n} V(t,x_1,\ldots,x_n)).$$

Throughout this paper the letter $C$ denotes a constant, which may be different in each occasion.
2.2. Wave Packet Transform. We recall the definition of the wave packet transform which is defined by Córdoba-Fefferman [5]. Wave packet transform is called short time Fourier transform or windowed Fourier transform in several literatures. Let \( f \in \mathcal{S}'(\mathbb{R}^n) \) and \( \varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\} \). Then the wave packet transform \( W_\varphi f(x, \xi) \) of \( f \) with the wave packet generated by a function \( \varphi \) is defined by

\[
W_\varphi f(x, \xi) = \int_{\mathbb{R}^n} \varphi(y - x)f(y)e^{-iy\cdot\xi}dy.
\]

We call such \( \varphi \) window function. Let \( F \) be a function on \( \mathbb{R}^n \times \mathbb{R}^n \). Then the (informal) adjoint operator \( W_\varphi^* \) of \( W_\varphi \) is defined by

\[
W_\varphi^* F(x) = \int_{\mathbb{R}^n} F(y, \xi)\varphi(x - y)e^{ix\cdot\xi}dyd\xi
\]

with \( d\xi = (2\pi)^{-n}d\xi \). It is known that for \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\} \) satisfying \( \langle \psi, \varphi \rangle \neq 0 \), we have the inversion formula

\[
\frac{1}{\langle \psi, \varphi \rangle}W_\varphi^* W_\psi f = f, \quad f \in \mathcal{S}'(\mathbb{R}^n)
\]

([8, Corollary 11.2.7]).

For the sake of convenience, we use the following notation

\[
W_{\varphi(t, \cdot)} u(t, x, \xi) = W_{\varphi(t, \cdot)}[u(t, \cdot)](x, \xi) = \int_{\mathbb{R}^n} \varphi(t, y - x)u(t, y)e^{-iy\cdot\xi}dy,
\]

where \( \varphi(t, x) \) and \( u(t, x) \) are functions on \( \mathbb{R} \times \mathbb{R}^n \).

2.3. Modulation Spaces. We recall the definition of modulation spaces \( M^{p,q} \). Let \( 1 \leq p, q \leq \infty \) and \( \varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\} \). Then the modulation space \( M^{p,q}_\varphi(\mathbb{R}^n) = M^{p,q} \) consists of all tempered distributions \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that the norm

\[
\|f\|_{M^{p,q}_\varphi} = \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |W_\varphi f(x, \xi)|^pdx \right)^{q/p}d\xi \right)^{1/q} = \||W_\varphi f(x, \xi)||_{L^p_x}||_{L^q_\xi}
\]

is finite (with usual modifications if \( p = \infty \) or \( q = \infty \)).

The space \( M^{p,q}_\varphi(\mathbb{R}^n) \) is a Banach space, whose definition is independent of the choice of the window function \( \varphi \), i.e., \( M^{p,q}_\varphi(\mathbb{R}^n) = M^{p,q}_\psi(\mathbb{R}^n) \) for all \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \) \( \setminus \{0\} \) ([6, Theorem 6.1]). This property is crucial in the sequel, since we choose a suitable window function \( \varphi \) to estimate the modulation space norm. If \( 1 \leq p, q < \infty \) then \( \mathcal{S}(\mathbb{R}^n) \) is dense in \( M^{p,q} \) ([6, Theorem 6.1]). We also note \( L^2 = M^{2,2} \), and \( M^{p_1,q_1} \hookrightarrow M^{p_2,q_2} \) if \( p_1 \leq p_2, q_1 \leq q_2 \) ([6, Proposition 6.5]). Let us define by \( \mathcal{M}^{p,q}(\mathbb{R}^n) \) the completion of \( \mathcal{S}(\mathbb{R}^n) \) under the norm \( \| \cdot \|_{M^{p,q}} \). Then \( \mathcal{M}^{p,q}(\mathbb{R}^n) = M^{p,q}(\mathbb{R}^n) \) for \( 1 \leq p, q < \infty \). Moreover, the complex interpolation theory for these spaces reads as follows: Let \( 0 < \theta < 1 \) and \( \frac{1}{p_1} \leq \frac{1}{p}, \frac{q}{q_1} \leq \infty, i = 1, 2 \). Set \( 1/p = (1 - \theta)/p_1 + \theta/p_2, 1/q = (1 - \theta)/q_1 + \theta/q_2 \), then \( \mathcal{M}^{p_1,q_1}(\mathbb{R}^n) \otimes = \mathcal{M}^{p,q} \) ([6, Theorem 6.1], [15, Theorem 2.3]). We refer to [6] and [8] for more details.
3. Key Lemmas

The orbit of the classical mechanics corresponding to (1) is described by the system of ordinary differential equations

\begin{equation}
\begin{aligned}
\frac{d}{ds}f(s) &= g(s), \\
\frac{d}{ds}g(s) &= -(\nabla_x V)(s, f(s)),
\end{aligned}
\end{equation}

where \( f : \mathbb{R} \to \mathbb{R}^n \) and \( g : \mathbb{R} \to \mathbb{R}^n \) (see also Fujiwara [7]). So, we prepare two lemmas which give some properties of the solutions to the system of ordinary differential equations. Following lemma is used in the proof of Theorem 1.1.

**Lemma 3.1.** Let \( V \in C^\infty(\mathbb{R} \times \mathbb{R}^n) \) satisfy (2) for all multi-indices \( \alpha \) with \( |\alpha| \geq 2 \). Suppose that \( f(s; t, x, \xi) \) and \( g(s; t, x, \xi) \) are solutions to (3) satisfying \( f(t) = x \) and \( g(t) = \xi \) and put

\[
M(s; t, x, \xi) = (w_{i,j})
\]

\[
= \begin{pmatrix}
\frac{\partial f_1(s; t, x, \xi)}{\partial x_1} & \ldots & \frac{\partial f_1(s; t, x, \xi)}{\partial x_n} & \ldots & \frac{\partial f_1(s; t, x, \xi)}{\partial \xi_1} & \ldots & \frac{\partial f_1(s; t, x, \xi)}{\partial \xi_n} \\
\frac{\partial g_1(s; t, x, \xi)}{\partial x_1} & \ldots & \frac{\partial g_1(s; t, x, \xi)}{\partial x_n} & \ldots & \frac{\partial g_1(s; t, x, \xi)}{\partial \xi_1} & \ldots & \frac{\partial g_1(s; t, x, \xi)}{\partial \xi_n} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\partial f_n(s; t, x, \xi)}{\partial x_1} & \ldots & \frac{\partial f_n(s; t, x, \xi)}{\partial x_n} & \ldots & \frac{\partial f_n(s; t, x, \xi)}{\partial \xi_1} & \ldots & \frac{\partial f_n(s; t, x, \xi)}{\partial \xi_n} \\
\frac{\partial g_n(s; t, x, \xi)}{\partial x_1} & \ldots & \frac{\partial g_n(s; t, x, \xi)}{\partial x_n} & \ldots & \frac{\partial g_n(s; t, x, \xi)}{\partial \xi_1} & \ldots & \frac{\partial g_n(s; t, x, \xi)}{\partial \xi_n}
\end{pmatrix}.
\]

Then \( \det M(s; t, x, \xi) = 1 \) for all \( s, t, x \) and \( \xi \).

It is easy to prove this lemma by the standard method, but we give the proof for reader’s convenience in Appendix. Next lemma plays an important role in the proof of Theorem 1.2.

**Lemma 3.2.** Let \( f(s; t, x, \xi) \) and \( g(s; t, x, \xi) \) be solutions to (3) with \( f(t) = x \) and \( g(t) = \xi \). If \( V \in C^\infty(\mathbb{R} \times \mathbb{R}^n) \) satisfies (2) for all multi-indices \( \alpha \) with \( |\alpha| \geq 1 \), then there exist \( C_1, C_2 > 0 \) such that

\[
\frac{1}{\langle y - f(s; t, x, \xi) \rangle} \leq \frac{C_1(1 + |t - s|^2)}{\langle y - x + (t - s)\xi \rangle}
\]

and

\[
\frac{1}{\langle \eta - g(s; t, x, \xi) \rangle} \leq \frac{C_2(1 + |t - s|)}{\langle \eta - \xi \rangle}.
\]
Proof. First, we show (1). Since \( f(s; t, x, \xi) \) and \( g(s; t, x, \xi) \) solve (3), we have

\[
\begin{align*}
  f(s; t, x, \xi) &= f(t; t, x, \xi) + \int_t^s g(\tau; t, x, \xi) \, d\tau \\
  &= x + \int_t^s \left( g(t; t, x, \xi) - \int_t^\tau (\nabla_x V)(\sigma, f(\sigma; t, x, \xi)) \, d\sigma \right) \, d\tau \\
  &= x + (s - t)\xi - \int_t^s \int_s^\tau (\nabla_x V)(\sigma, f(\sigma; t, x, \xi)) \, d\tau \, d\sigma \\
  &= x + (s - t)\xi - \int_s^t (\sigma - s)(\nabla_x V)(\sigma, f(\sigma; t, x, \xi)) \, d\sigma.
\end{align*}
\]

(6)

Since \( V \) satisfies (2) for all multi-indices \( \alpha \) with \( |\alpha| \geq 1 \), we have

\[
(7) \quad |(\partial_x^\alpha V)(\sigma, f(\sigma; t, x, \xi))| \leq 2C
\]

for \( j = 1, 2, \ldots, n \). By (6) and (7), we have

\[
|y - x + (t - s)\xi| \leq |y - f(s; t, x, \xi)| + |f(s; t, x, \xi) - x + (t - s)\xi| \\
\leq |y - f(s; t, x, \xi)| + \left| \int_s^t (\sigma - s)(\nabla_x V)(\sigma, f(\sigma; t, x, \xi)) \, d\sigma \right| \\
\leq |y - f(s; t, x, \xi)| + 2\sqrt{nC} \left| \int_s^t |\sigma - s| \, d\sigma \right| \\
= |y - f(s; t, x, \xi)| + \sqrt{nC}|t - s|^2.
\]

So, we have

\[
(9 - x + (t - s)\xi) \leq \{1 + 2(|y - f(s; t, x, \xi)|^2 + nC^2|t - s|^4)\}^{1/2} \\
\leq \sqrt{2(1 + \sqrt{nC}|t - s|^2)} \langle y - f(s; t, x, \xi) \rangle.
\]

Putting \( C_1 = \sqrt{2} \max\{1, \sqrt{nC}\} \), we obtain (11).

Next, we show (5). Since \( f(s; t, x, \xi) \) and \( g(s; t, x, \xi) \) solve (3), we have

\[
\begin{align*}
  g(s; t, x, \xi) &= g(t; t, x, \xi) - \int_t^s \nabla_x V(\sigma, f(\sigma; t, x, \xi)) \, d\sigma \\
  &= \xi + \int_s^t \nabla_x V(\sigma, f(\sigma; t, x, \xi)) \, d\sigma.
\end{align*}
\]

and thus,

\[
|\eta - \xi| \leq |\eta - g(s; t, x, \xi)| + |g(s; t, x, \xi) - \xi| \\
\leq |\eta - g(s; t, x, \xi)| + \left| \int_s^t (\nabla_x V)(\sigma, f(\sigma; t, x, \xi)) \, d\sigma \right| \\
\leq |\eta - g(s; t, x, \xi)| + 2\sqrt{nC}|t - s|.
\]
Hence, we have
\[
\langle \eta - \xi \rangle \leq \{1 + 2(|\eta - g(s; t, x, \xi)|^2 + 4nC^2|t - s|^2)\}^{\frac{1}{2}} \\
\leq \sqrt{2}(1 + 2\sqrt{nC}|t - s|)\langle \eta - g(s; t, x, \xi)\rangle.
\]
Putting \(C_2 = \sqrt{2} \max\{1, 2\sqrt{nC}\}\), we obtain (5). □

4. Proof of Theorem 1.1

We only consider the case \(t \in [0, T]\). We can treat the case \(t \in [-T, 0]\) in the same way. First, by using wave packet transform, we transform (1) into a first order partial differential equation and a lower order term.

By integration by parts, we have
\[
W \phi(t, \cdot)(i\partial_t u)(t, x, \xi) = i\partial_t W \phi(t, \cdot)u(t, x, \xi) + W i\partial_t \phi(t, \cdot)u(t, x, \xi)
\]
and
\[
W \phi(t, \cdot)\left(\frac{1}{2} \Delta u\right)(t, x, \xi)
\]
\[
= W_2 \Delta \phi(t, \cdot)u(t, x, \xi) + i\xi \cdot \nabla_x \phi(t, \cdot)u(t, x, \xi) - \frac{|\xi|^2}{2} W \phi(t, \cdot)u(t, x, \xi),
\]
where \(\phi(t, x) = e^{\frac{i}{2}t \Delta} \phi_0(x)\). Applying Taylor’s theorem to \(V(t, \cdot, \cdot)\), we have, by integration by parts,
\[
W \phi(t, \cdot)(Vu)(t, x, \xi)
\]
\[
= \int_{\mathbb{R}^n} \phi(t, y - x)\left(V(t, x) + \nabla_x V(t, x) \cdot (y - x) + \sum_{j,k=1}^n (y_j - x_j)(y_k - x_k)V_{jk}(t, x, y)\right)u(t, y)e^{-i\xi \cdot y}dy
\]
\[
= \{V(t, x) + i\nabla_x V(t, x) \cdot \nabla_\xi - \nabla_x V(t, x) \cdot \xi\}W \phi(t, \cdot)u(t, x, \xi) + Ru(t, x, \xi),
\]
where
\[
Ru(t, x, \xi)
\]
\[
= \sum_{j,k=1}^n \int_{\mathbb{R}^n} \phi(t, y - x)V_{jk}(t, x, y)(y_j - x_j)(y_k - x_k)u(t, y)e^{-i\xi \cdot y}dy
\]
and
\[
V_{jk}(t, x, y) = \int_0^1 \partial_{x_j} \partial_{x_k} V(t, x + \theta(y - x))(1 - \theta)d\theta.
\]
Since \(i\partial_t \phi(t, x) + \frac{1}{2} \Delta \phi(t, x) = 0\), we have
\[
W i\partial_t \phi(t, \cdot)u(t, x, \xi) + W_2 \Delta \phi(t, \cdot)u(t, x, \xi) = 0.
\]
Combining (8), (9), (10) and (13), the initial value problem (1) is transformed to

\[
\begin{cases}
(i\partial_t + i\xi \cdot \nabla_x - i\nabla_x V(t, x) \cdot \nabla_\xi - \frac{1}{2}||\xi||^2 - V(t, x) \\
+ \nabla_x V(t, x) \cdot x)W_{\varphi(t, \cdot)}u(t, x, \xi) - Ru(t, x, \xi) = 0,
\end{cases}
\]

By the method of characteristics, we obtain

\[
W_{\varphi(0, \cdot)}u(0, x, \xi) = W_{\varphi_0}u_0(x, \xi).
\]

By taking $L$-transformed to

\[
\text{So it follows that}
\]

\[
W_{\varphi(t, \cdot)}u(t, x, \xi) = e^{-i\int_0^t h(s; t, x, \xi)ds} \left( W_{\varphi_0}u_0(f(0; t, x, \xi), g(0; t, x, \xi)) \right)
\]

\[
- i \int_0^t e^{i\int_0^\tau h(s; t, x, \xi)ds} Ru(\tau, f(\tau; t, x, \xi), g(\tau; t, x, \xi))d\tau,
\]

where $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ are solutions to (3) with $f(t) = x$ and $g(t) = \xi$, and

\[
h(s; t, x, \xi)
\]

\[
= \frac{1}{2}[g(s; t, x, \xi)]^2 + V(s, f(s; t, x, \xi)) - \nabla_x V(s, f(s; t, x, \xi)) \cdot f(s; t, x, \xi).
\]

By taking $L^p$-norm with respect to $x$ and $\xi$ on both sides of (14), we have

\[
\|u(t, \cdot)\|_{L^p_{\varphi(t, \cdot)}} = \|W_{\varphi(t, \cdot)}u(t, x, \xi)\|_{L^p_{x, \xi}} \leq \|I_1\|_{L^p_{x, \xi}} + \int_0^t \|I_2\|_{L^p_{x, \xi}} d\tau,
\]

where

\[
I_1 = W_{\varphi_0}u_0(f(0; t, x, \xi), g(0; t, x, \xi)), \quad I_2 = Ru(\tau, f(\tau; t, x, \xi), g(\tau; t, x, \xi)).
\]

Now we consider the change of variables $X = f(0; t, x, \xi)$ and $\Xi = g(0; t, x, \xi)$. From Lemma 3.1 and the implicit function theorem, we have

\[
\frac{\partial(x, \xi)}{\partial(X, \Xi)} = 1.
\]

So it follows that

\[
\|I_1\|_{L^p_{x, \xi}} = \left( \int_{\mathbb{R}^{2n}} |W_{\varphi_0}u_0(X, \Xi)|^p \left| \frac{\partial(x, \xi)}{\partial(X, \Xi)} \right| dX d\Xi \right)^{\frac{1}{p}} = \|u_0\|_{M_{\varphi_0}^p}.
\]

On the other hand, from (11) and the inversion formula of wave packet transform for $u$, we have

\[
Ru(t, x, \xi) = \frac{1}{\|\varphi(t, \cdot)\|_{L^2}^2} \sum_{j,k=1}^n \int_{\mathbb{R}^{2n}} \varphi_{jk}(t, y - x) V_{jk}(t, x, y) \varphi(t, y - z) 
\]

\[
\times W_{\varphi(t, \cdot)}u(t, z, \eta) e^{iy(\eta - \xi)} dz d\eta dy,
\]
where \( \varphi_{jk}(t, y) = y_j y_k \varphi(t, y) \). Take \( N \in \mathbb{N} \) satisfying \( 2N > n \). From
\[
(1 - \Delta_y)^N e^{iy(\eta - g(t, x, \xi))} = (\eta - g(t, x, \xi))^N e^{iy(\eta - g(t, x, \xi))},
\]
we have
\[
\|I_2\|_{L^p_{t, \xi}} = \|Ru(t, f(\tau; t, x, \xi), g(\tau; t, x, \xi))\|_{L^p_{t, \xi}}
\leq \frac{1}{\|\varphi(\tau, \cdot)\|_{L^2_x}} \sum_{j,k=1}^N \left\| \iint_{\mathbb{R}^3n} \left(1 - \Delta_y\right)^N \left\{ \varphi_{jk}(\tau, y - f(\tau; t, x, \xi)) \times V_{jk}(\tau, f(\tau; t, x, \xi), y) \varphi(\tau, y - z) \right\} \left| \frac{|W_{\varphi(\tau, \cdot)} u(\tau, z, \eta)}{(\eta - g(\tau; t, x, \xi))^2N} d\eta dy \right\|_{L^p_{t, \xi}} \right\|
\leq \frac{1}{\|\varphi(\tau, \cdot)\|_{L^2_x}} \sum_{j,k=1}^N \sum_{|\beta_1| + |\beta_2| + |\beta_3| \leq 2N} \left\| \iint_{\mathbb{R}^3n} \left| \partial^{\beta_1}_{\eta} \varphi_{jk}(\tau, y - f(\tau; t, x, \xi)) \times \partial^{\beta_2}_{\tau} V_{jk}(\tau, f(\tau; t, x, \xi), y) \partial^{\beta_3}_{\tau} \varphi(\tau, y - z) \right| \left| \frac{|W_{\varphi(\tau, \cdot)} u(\tau, z, \eta)}{(\eta - g(\tau; t, x, \xi))^2N} d\eta dy \right\|_{L^p_{t, \xi}} \right\|
\}
\]
Since \( |\partial^{\beta_2}_{\tau} V_{jk}(\tau, f(\tau; t, x, \xi), y)| \leq C_{\beta_2} \) for \( C_{\beta_2} > 0 \), we have, by the change of variables \( X = f(\tau; t, x, \xi) \) and \( \Xi = g(\tau; t, x, \xi) \), Young’s inequality and Lemma 3.1
\[
\|I_2\|_{L^p_{t, \xi}} \leq \frac{1}{\|\varphi(\tau, \cdot)\|_{L^2_x}} \sum_{j,k=1}^N \sum_{|\beta_1| + |\beta_2| + |\beta_3| \leq 2N} C_{\beta_2} \left\{ \iint_{\mathbb{R}^2n} \left( \iint_{\mathbb{R}^2n} \left| \partial^{\beta_1}_{\eta} \varphi_{jk}(\tau, y - X) \right| \left| \frac{|W_{\varphi(\tau, \cdot)} u(\tau, z, \eta)}{(\eta - \Xi)^2N} \right| d\eta d\Xi \right)^{\frac{p}{2}} \right\}^{\frac{1}{p}}
\leq \frac{1}{\|\varphi(\tau, \cdot)\|_{L^2_x}} \sum_{j,k=1}^N \sum_{|\beta_1| + |\beta_2| + |\beta_3| \leq 2N} C_{\beta_2} \left\| \partial^{\beta_1}_{\eta} \varphi_{jk}(\tau, y) \right\|_{L^1_{t, \eta}}^p
\]
(18)
\[
\leq C_T \|u(\tau, \cdot)\|_{M^{p}_{\varphi(\tau, \cdot)}}
\]
for \( t \in [0, T] \) and \( \tau \in [0, t] \). From (15), (17) and (18), we have
\[
\|u(t, \cdot)\|_{M^{p}_{\varphi(\cdot, \cdot)}} \leq \|u_0\|_{M^{p}_{\varphi(\cdot, \cdot)}} + C_T \int_0^t \|u(\tau, \cdot)\|_{M^{p}_{\varphi(\tau, \cdot)}} d\tau
\]
for \( t \in [0, T] \). Then Gronwall’s inequality yields \( \|u(t, \cdot)\|_{M^{p}_{\varphi(\cdot, \cdot)}} \leq C_T \|u_0\|_{M^{p}_{\varphi(\cdot, \cdot)}} \) for \( t \in [0, T] \). \( \square \)
5. Proof of Theorem 1.2

We only consider the case \( t \in [0, T] \), since we can treat the case \( t \in [-T, 0] \) in the same way. First, we consider the case \((p, q) = (\infty, 1)\), next \((p, q) = (1, \infty)\) and finally general \((p, q)\).

In the proof of Theorem 1.1, we have already obtained

\[
|W_{\varphi(t, \cdot)}u(t, x, \xi)| \leq |I_1| + \int_0^t |I_2| \, d\tau,
\]

where \(I_1\) and \(I_2\) are defined by (16). Take \(N \in \mathbb{N}\) satisfying \(2N > n\). From

\[
(1 - \Delta_y)^N e^{iy(\eta - g(0; t, x, \xi))} = \langle \eta - g(0; t, x, \xi) \rangle^{2N} e^{iy(\eta - g(0; t, x, \xi))},
\]

we have, by the inversion formula of wave packet transform for \(u_0\) and (5) in Lemma 3.2,

\[
|I_1| = \frac{1}{\|\varphi_0\|_{L^2}^2} \left| \iiint_{\mathbb{R}^3} \varphi_0(y - f(0; t, x, \xi))\varphi_0(y - z) \right.
\]
\[
\times W_{\varphi_0}u_0(z, \eta) e^{iy(\eta - g(0; t, x, \xi))} \, dz \, \partial \sigma \, dy
\]
\[
\leq \frac{1}{\|\varphi_0\|_{L^2}^2} \iiint_{\mathbb{R}^{3n}} |(1 - \Delta_y)^N \{ \varphi_0(y - f(0; t, x, \xi))\varphi_0(y - z) \}|
\]
\[
\times \frac{|W_{\varphi_0}u_0(z, \eta)|}{\langle \eta - g(0; t, x, \xi) \rangle^{2N} } \, dz \, \partial \sigma \, dy
\]
\[
\leq C(1 + |t|)^{2N} \sum_{|\beta_1| + |\beta_2| \leq 2N} \iiint_{\mathbb{R}^{3n}} \left| \partial_{\beta_2}^2 \varphi_0(y - z) \right| \frac{|W_{\varphi_0}u_0(z, \eta)|}{\langle \eta - \xi \rangle^{2N} } \, dz \, \partial \sigma \, dy.
\]  

(19)

By Fubini’s theorem, we have

\[
\|I_1\|_{L^\infty} \leq C(1 + |t|)^{2N} \sum_{|\beta_1| + |\beta_2| \leq 2N} \|\partial_{\beta_2}^2 \varphi_0\|_{L^1}
\]
\[
\times \left\| \iiint_{\mathbb{R}^{2n}} \left| \partial_{\beta_2}^2 \varphi_0(y - f(0; t, x, \xi)) \right| \frac{|W_{\varphi_0}u_0(z, \eta)|}{\langle \eta - \xi \rangle^{2N} } \, dz \, \partial \sigma \, dy \right\|_{L^\infty} \|L^1\|
\]
\[
\leq C(1 + |t|)^{2N} \sum_{|\beta_1| + |\beta_2| \leq 2N} \|\partial_{\beta_2}^2 \varphi_0\|_{L^1} \|\partial_{\beta_1}^2 \varphi_0\|_{L^1} \|W_{\varphi_0}u_0(z, \eta)\|_{L^\infty} \|L^1\|
\]
\[
\leq C_T \|u_0\|_{M_{p_0}^{\infty, 1}}
\]
for \( t \in [0, T] \). Similarly, we have

\[
|I_2| \leq \frac{1}{\| \varphi(\cdot, \cdot) \|^2_{L^2}} \sum_{j,k=1}^{n} \iint_{\mathbb{R}^n} \left| (1 - \Delta_y)^N \{ \varphi_{jk}(\tau, y - f(\tau; t, x, \xi)) \} \right| \left| W_{\varphi(\cdot, \cdot)} u(\tau, z, \eta) \right| \left| \frac{1}{(\eta - g(\tau; t, x, \xi))^2} \right| dz \eta dy \\
\leq C(1 + |t - \tau|)^{2N} \| \varphi(\cdot, \cdot) \|^2_{L^2}
\]

\[
\times \sum_{j,k=1}^{n} \sum_{|\beta_1| + |\beta_2| + |\beta_3| \leq 2N} \iint_{\mathbb{R}^n} |\partial_y^{\beta_1} \varphi_{jk}(\tau, y - f(\tau; t, x, \xi))| \\
\times |\partial_y^{\beta_2} V_{jk}(\tau, f(\tau; t, x, \xi), y)\partial_y^{\beta_3} \varphi(\tau, y - z)| \left| W_{\varphi(\cdot, \cdot)} u(\tau, z, \eta) \right| \left| \frac{1}{(\eta - \xi)^2} \right| dz \eta dy,
\]

where \( \varphi_{jk}(t, y) = y_j y_k \varphi(t, y) \) and \( V_{jk} \) is defined by (12). Since

\[
|\partial_y^{\beta_2} V_{jk}(\tau, f(\tau; t, x, \xi), y)| \leq C_{\beta_2}
\]

for \( C_{\beta_2} > 0 \), we have

\[
\| I_2 \|_{L^\infty} \leq C(1 + |t - \tau|)^{2N} \| \varphi(\cdot, \cdot) \|^2_{L^2}
\]

\[
\times \sum_{j,k=1}^{n} \sum_{|\beta_1| + |\beta_2| + |\beta_3| \leq 2N} C_{\beta_2} \left\| \iint_{\mathbb{R}^n} |\partial_y^{\beta_1} \varphi_{jk}(\tau, y - f(\tau; t, x, \xi))| \\
\times |\partial_y^{\beta_2} \varphi(\tau, y - z)| \left| W_{\varphi(\cdot, \cdot)} u(\tau, z, \eta) \right| \left| \frac{1}{(\eta - \xi)^2} \right| dz \eta dy \right\|_{L^\infty} \| L^2 \|
\]

\[
\leq C'(1 + T)^{2N} \| \varphi(\cdot, \cdot) \|^2_{L^2} \sum_{j,k=1}^{n} \sum_{|\beta_1| + |\beta_2| + |\beta_3| \leq 2N} C_{\beta_2} |\partial_y^{\beta_3} \varphi(\tau, z)| \| L^2 \|
\times |\partial_y^{\beta_2} \varphi_{jk}(\tau, y)| \| L^2 \| |\varphi(\cdot, \cdot)| - 2N | W_{\varphi(\cdot, \cdot)} u(\tau, z, \eta) | \| L^\infty \| \| L^2 \|
\]

\[
\leq C_T \| u(\cdot, \cdot) \|_{M^{\infty, 1}_\varphi}
\]

for \( t \in [0, T] \) and \( \tau \in [0, t] \). Hence, we have

\[
\| u(t, \cdot) \|_{M^{\infty, 1}_\varphi} \leq \| \| I_1 \|_{L^\infty} \|_{L^2} + \int_0^t \| \| I_2 \|_{L^\infty} \|_{L^2} d\tau
\]

(20)

\[
\leq C_T \| u_0 \|_{M^{\infty, 1}_\varphi} + C_T' \int_0^t \| u(\cdot, \cdot) \|_{M^{\infty, 1}_\varphi} d\tau
\]

for \( t \in [0, T] \). Applying Gronwall’s inequality to (20), we obtain

\[
\| u(t, \cdot) \|_{M^{\infty, 1}_\varphi} \leq C_T'' \| u_0 \|_{M^{\infty, 1}_\varphi}
\]

for \( t \in [0, T] \).
Next, we consider \((p, q) = (1, \infty)\). Take \(N \in \mathbb{N}\) satisfying \(2N > n\). For all multi-indices \(\beta_1\), we have
\[
\|\partial_y^{\beta_1} \varphi(\tau, y - f(\tau; t, x, \xi))\|_{L^1_t}
\]
\[
\leq C(1 + |t - \tau|)^{2N} \int_{\mathbb{R}^n} \frac{|y - f(\tau; t, x, \xi)|^{2N}}{|y - x + (t - \tau)\xi|^{2N}} \|\partial_y^{\beta_1} \varphi(\tau, y - f(\tau; t, x, \xi))\| \, dx
\]
\[
\leq C(1 + T^2)^{2N} \sup_{\tau \in [0,T], y \in \mathbb{R}^n} (y)^{2N} \|\partial_y^{\beta_1} \varphi(\tau, y)\| \int_{\mathbb{R}^n} \frac{1}{|y - x + (t - \tau)\xi|^{2N}} \, dx
\]
(22)
\[
\leq C_T
\]
for \(t \in [0,T]\) and \(\tau \in [0,t]\). Here, we have used \((14)\) in Lemma 3.2. Thus, we have, by \((11)\), \((22)\) and Fubini’s theorem,
\[
\|\|I_1\|_{L^1_t}\|_{L^1_\xi} \leq C(1 + |t|)^{2N} \sum_{|\beta_1| + |\beta_2| \leq 2N} \left\|\int_{\mathbb{R}^n} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \partial_y^{\beta_1} \varphi_0(y - f(0; t, x, \xi)) \, dx \, dy \, dz \right\|
\]
\[
\leq C_T \|u_0\|_{M^{1,\infty}_{\nu_0}}
\]
for \(t \in [0,T]\). In the similar way as above, it follows that
\[
\|\|I_2\|_{L^1_t}\|_{L^1_\xi} \leq \frac{C(1 + |t - \tau|)^{2N}}{\|\varphi(\tau, \cdot)\|_{L^2_\xi}}
\]
\[
\times \sum_{j,k=1}^n \sum_{|\beta_1| + |\beta_2| + |\beta_3| \leq 2N} \left\|\int_{\mathbb{R}^n} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \partial_y^{\beta_1} \varphi_{j,k}(\tau, y - f(\tau; t, x, \xi)) \, dx \, dy \, dz \right\|
\]
\[
\leq C_T \|u(\tau, \cdot)\|_{M^{1,\infty}_{\nu_{\varphi(\tau, \cdot)}}}
\]
for \(\tau \in [0,t]\) and \(t \in [0,T]\). Thus, we have
\[
\|u(t, \cdot)\|_{M^{1,\infty}_{\nu_{\varphi(\tau, \cdot)}}} \leq \|\|I_1\|_{L^1_t}\|_{L^1_\xi} \leq C_T \|\|I_1\|_{L^1_t}\|_{L^1_\xi} + \int_0^t \|\|I_2\|_{L^1_t}\|_{L^1_\xi} \, d\tau
\]
(23)
\[
\leq C_T \|u_0\|_{M^{1,\infty}_{\nu_0}} + C_T' \int_0^t \|u(\sigma, \cdot)\|_{M^{1,\infty}_{\nu_{\varphi(\sigma, \cdot)}}} \, d\tau
\]
for \(t \in [0,T]\). Applying Gronwall’s inequality to (23), we obtain
\[
\|u(t, \cdot)\|_{M^{1,\infty}_{\nu_{\varphi(\cdot)}}} \leq C_T'' \|u_0\|_{M^{1,\infty}_{\nu_0}}
\]
for \(t \in [0,T]\).
Finally, we consider the general case. From Theorem 1.1 we have
\begin{equation}
\|u(t, \cdot)\|_{M^{p, r}_{\sigma(t, \cdot)}} \leq C_T \|u_0\|_{M^{p, r}_0}.
\end{equation}
Combing (24), (24) and (25), we have, by the complex interpolation theorem for modulation space,
\[\|u(t, \cdot)\|_{M^{p, q}_{\sigma(t, \cdot)}} \leq \overline{C}_T \|u_0\|_{M^{p, q}_0}\]
for \(t \in [0, T]\). Therefore we obtain the desired result. \(\square\)

**Appendix A.**

First, We remark that if \(V(t, x) \in C^\infty(\mathbb{R} \times \mathbb{R}^n)\) satisfies (2) for all multi-indices with \(|\alpha| \geq 2\) then the ordinary differential equation (3) with the initial condition \(f(t) = x\) and \(g(t) = \xi\) has a unique solution on \(\mathbb{R}\). In fact, the existence of the solution on \([t - \sigma, t + \sigma]\) for some \(\sigma > 0\) is proved by Picard’s iteration scheme. Here is the outline. Let \(f^{(0)}(s) \equiv x\) and \(g^{(0)}(s) \equiv \xi\) and set
\[f^{(k+1)}(s) = x + \int_t^s g^{(k)}(\tau)d\tau\]
and \(g^{(k+1)}(s) = \xi - \int_t^s (\nabla_x V)(\tau, f^{(k)}(\tau))d\tau\).

Then \(\{f^{(k)}\}\) and \(\{g^{(k)}\}\) converge uniformly to some functions \(f(s)\) and \(g(s)\) on \([t - \sigma, t + \sigma]\) and the functions \(f(s)\) and \(g(s)\) satisfy the initial value problem and belong to \(C^\infty([t - \sigma, t + \sigma])\). Moreover, by using following Lemma A.1, we can show that above fact holds not only on \([t - \sigma, t + \sigma]\) but also on \(\mathbb{R}\), easily. Lemma A.1 is also used in the proof of Lemmas A.2 and A.3.

**Lemma A.1.** Let \(V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)\) satisfy (2) for all multi-indices with \(|\alpha| \geq 2\). Then, for all multi-indices \(\beta\) with \(|\beta| \geq 1\), \(\partial^\beta_x V(t, x)\) is Lipschitz continuous with respect to \(x\), more precisely, there exists \(C_\beta > 0\) such that
\[|(\partial^\beta_x V)(t, y) - (\partial^\beta_x V)(t, z)| \leq C_\beta n \|y - z\|_{\infty}\]
for all \(t \in \mathbb{R}\), \(y, z \in \mathbb{R}^n\).

**Proof.** Let \(t \in \mathbb{R}\), \(y = (y_1, \ldots, y_n) \in \mathbb{R}^n\) and \(z = (z_1, \ldots, z_n) \in \mathbb{R}^n\). Since \(|y_k - z_k| \leq \|y - z\|_{\infty}\) for \(k = 1, \ldots, n\), it is enough to show that
\[|(\partial^\beta_x V)(t, y) - (\partial^\beta_x V)(t, z)| \leq C_\beta \sum_{k=1}^n |y_k - z_k|\]
Set \(F(\theta) = (\partial^\beta_x V)(t, \theta(y - z))\). We note that \(F(\theta) \in C^\infty([0, 1])\), \(F(0) = (\partial^\beta_x V)(t, z)\) and \(F(1) = (\partial^\beta_x V)(t, y)\). By the fundamental theorem
of calculus, we have
\[
(\partial_x^\alpha V)(t, y) - (\partial_x^\beta V)(t, z) = \int_0^1 \frac{d}{d\theta} F(\theta)d\theta
\]
(26)
\[
= \int_0^1 \sum_{k=1}^n (y_k - z_k)(\partial_{x_k} \partial_x^\beta V)(t, z + \theta(y - z))d\theta.
\]
Since \( V \) satisfies (2) for \( |\alpha| \geq 2 \), we obtain
(27)
\[
|\langle \partial_{x_k} \partial_x^\beta V \rangle(t, z + \theta(y - z))| \leq C_\beta
\]
for \( k = 1, 2, \ldots, n \). Combining (26) and (27), we obtain the desired result. \( \square \)

Next, we establish one more lemma relating to the Lemma A.3.

**Lemma A.2.** Let \( h \in \mathbb{R}\setminus\{0\} \), \( T > 0 \) and \( V \in C^\infty(\mathbb{R} \times \mathbb{R}^n) \) satisfy (2) for all multi-indices \( \alpha \) with \( |\alpha| \geq 2 \). Suppose that \( f(s, t, x, \xi) \) and \( g(s, t, x, \xi) \) are solutions to (3) with \( f(t) = x \) and \( g(t) = \xi \). For \( k = 1, \ldots, n \), we set
\[
\phi_{h,k}(s; t, x, \xi) = (f(s; t, x + he_k, \xi) - f(s; t, x, \xi), g(s; t, x + he_k, \xi) - g(s; t, x, \xi))
\]
and
\[
\psi_{h,k}(s; t, x, \xi) = (f(s; t, x, \xi + he_k) - f(s; t, x, \xi), g(s; t, x, \xi + he_k) - g(s; t, x, \xi)),
\]
where \( e_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n \). Then
\[
\lim_{h \to 0} \sup_{|s-t| \leq T} \|\phi_{h,k}(s; t, x, \xi)\|_{\infty} = 0 \text{ and } \lim_{h \to 0} \sup_{|s-t| \leq T} \|\psi_{h,k}(s; t, x, \xi)\|_{\infty} = 0.
\]

**Proof.** Here, we only show that \( \lim_{h \to 0} \sup_{|s-t| \leq T} \|\phi_{h,1}(s; t, x, \xi)\|_{\infty} = 0 \). We can treat the other cases in the same way. Put
\[
F(s; t, x, \xi) = f(s; t, x + he_1, \xi) - f(s; t, x, \xi)
\]
and
\[
G(s; t, x, \xi) = g(s; t, x + he_1, \xi) - g(s; t, x, \xi).
\]
Since \( F(t; t, x, \xi) = f(t; t, x + he_1, \xi) - f(t; t, x, \xi) = he_1 \) and
\[
\frac{d}{ds} F(s; t, x, \xi) = g(s; t, x + he_1, \xi) - g(s; t, x, \xi) = G(s; t, x, \xi),
\]
we have
\[
F(s; t, x, \xi) = he_1 + \int_t^s G(\tau; t, x, \xi) d\tau.
\]
Thus, we have
(28)
\[
\|F(s; t, x, \xi)\|_{\infty} \leq |h| + \left| \int_t^s \|G(\tau; t, x, \xi)\|_{\infty} d\tau \right|.
\]
On the other hand, since $G(t; t, x, \xi) = 0$ and
\[
\frac{d}{ds} G(s; t, x, \xi) = -\nabla_x V(s, f(s; t, x + he_1, \xi)) + \nabla_x V(s, f(s; t, x, \xi)),
\]
we have
\[
G(s; t, x, \xi) = -\int_s^t \left\{ \nabla_x V(\tau, f(\tau; t, x + he_1, \xi)) - \nabla_x V(\tau, f(\tau; t, x, \xi)) \right\} d\tau.
\]
By Lemma A.1, there exists $C > 0$ such that
\[
\|G(s; t, x, \xi)\|_\infty \leq |h| + C\left| \int_t^s \|\nabla_x V(\tau, f(\tau; t, x + he_1, \xi)) - \nabla_x V(\tau, f(\tau; t, x, \xi))\|_\infty d\tau \right|.
\]
From (28) and (29), we have
\[
\|\phi_{h,1}(s; t, x, \xi)\|_\infty \leq |h| + C'\left| \int_t^s \|\phi_{h,1}(\tau, t, x, \xi)\|_\infty d\tau \right|,
\]
where $C' = \max\{1, Cn\}$. Since $|s - t| \leq T$, applying Gronwall’s inequality to (30) gives
\[
\|\phi_{h,1}(s; t, x, \xi)\|_\infty \leq |h| e^{C'|s-t|} \leq |h| e^{C'T}.
\]
Hence, we obtain the desired result.

Next, we show the differentiability of the solution to (3) in initial datum.

**Lemma A.3.** Let $T > 0$, $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfy (2) for all multi-indices $\alpha$ with $|\alpha| \geq 2$. Suppose that $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ are solutions to (3) satisfying $f(t) = x$ and $g(t) = \xi$. Then $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ are $C^\infty$-function with respect to $x$ and $\xi$ in $|s - t| \leq T$.

**Proof.** Let $A(s, y)$ be the $2n \times 2n$ matrix defined by
\[
A(s, y) = \begin{pmatrix}
O_n & -\frac{\partial^2}{\partial x_1 \partial x_1} V(s, y) & \cdots & -\frac{\partial^2}{\partial x_1 \partial x_n} V(s, y) \\
\vdots & \ddots & \vdots \\
-\frac{\partial^2}{\partial x_n \partial x_1} V(s, y) & \cdots & -\frac{\partial^2}{\partial x_n \partial x_n} V(s, y) \\
E_n & O_n
\end{pmatrix},
\]
where $O_n$ is the $n \times n$ zero matrix and $E_n$ is the $n \times n$ identity matrix. Let $h \in \mathbb{R} \backslash \{0\}$ and put

$$
\phi_{h,j}(s; t, x, \xi)
= (f(s; t, x + he_j, \xi) - f(s; t, x, \xi), g(s; t, x + he_j, \xi) - g(s; t, x, \xi))
$$

and

$$
\psi_{h,j}(s; t, x, \xi)
= (f(s; t, x, \xi + he_j) - f(s; t, x, \xi), g(s; t, x, \xi + he_j) - g(s; t, x, \xi)),
$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$ and $j = 1, 2, \ldots, n$. Suppose that

$$
w^{(k)}(s; t, x, \xi) = (w_{1,k}(s; t, x, \xi), \ldots, w_{2kn}(s; t, x, \xi))
$$

is the solution of

$$
\begin{cases}
\frac{dw(s)}{ds} = w(s)A(s, f(s; t, x, \xi)) \\
w(t) = (0, \ldots, 0, 1, 0, \ldots, 0),
\end{cases}
$$

where $k = 1, 2, \ldots, 2n$.

First, we show that

$$
\lim_{h \to 0} \sup_{|s-t| \leq T} \left\| \frac{\phi_{h,j}(s; t, x, \xi)}{h} - w^{(j)}(s; t, x, \xi) \right\|_{\infty} = 0.
$$

From (33) and (26), it is easy to see that

$$
\begin{aligned}
\frac{d}{ds} \left( \frac{\phi_{h,j}(s; t, x, \xi)}{h} \right)
&= \frac{1}{h} \int_0^1 \phi_{h,j}(s, t, x, \xi) \\
&\quad \times A(s, f(s; t, x + he_j, \xi) + \theta(f(s; t, x, \xi) - f(s; t, x + he_j, \xi))) d\theta \\
&= \frac{1}{h} \phi_{h,j}(s, t, x, \xi)A(s, f(s; t, x, \xi)) + \gamma_{h,j}(s, t, x, \xi),
\end{aligned}
$$

where

$$
\gamma_{h,j}(s; t, x, \xi) = \frac{1}{h} \int_0^1 \phi_{h,j}(s, t, x, \xi) \left\{ A(s, f(s; t, x + he_j, \xi) \\
+ \theta(f(s; t, x, \xi) - f(s; t, x + he_j, \xi)) - A(s, f(s; t, x, \xi)) \right\} d\theta.
$$
By the definition of $A(s, y)$ and Lemma A.1, there exists $C > 0$ such that

$$\left\| A\left(s, f(s; t, x + h e_j, \xi) + \theta \left( f(s; t, x, \xi) - f(s; t, x + h e_j, \xi) \right) \right) - A(s, f(s; t, x, \xi)) \right\|_{\infty}$$

$$\leq C n (1 - \theta) \| f(s; t, x + h e_j, \xi) - f(s, t, x, \xi) \|_{\infty}$$

(37)

$$\leq C n \| \phi_{h,j}(s; t, x, \xi) \|_{\infty}$$

for $\theta \in [0, 1]$. Thus (31), (36) and (37) yield

$$\| \gamma_{h,j}(s; t, x, \xi) \|_{\infty} \leq \frac{2 C n^2}{|h|} \| \phi_{h,j}(s; t, x, \xi) \|_{\infty}^2$$

(38)

$$\leq 2 C n^2 \| \phi_{h,j}(s; t, x, \xi) \|_{\infty} e^{C'|s-t|}$$

for $C' > 0$. As $V$ satisfies the estimate (2) for all multi-indices $\alpha$ with $|\alpha| \geq 2$, there exists $M > 0$ such that

$$\| A(s, f(s; t, x, \xi)) \|_{\infty} \leq M.$$  

(39)

Since $\phi_{h,j}(t; t, x, \xi)/h - w^{(j)}(t; t, x, \xi) = 0$, we have, by (38) and (39),

$$\frac{\phi_{h,j}(s; t, x, \xi)}{h} - w^{(j)}(s; t, x, \xi)$$

$$= \int_t^s \left( \frac{1}{h} \frac{d}{d \tau} \phi_{h,j}(\tau; t, x, \xi) - \frac{d}{d \tau} w^{(j)}(\tau; t, x, \xi) \right) d \tau$$

$$= \int_t^s \gamma_{h,j}(\tau; t, x, \xi) d \tau$$

$$+ \int_t^s \left( \frac{\phi_{h,j}(\tau; t, x, \xi)}{h} - w^{(j)}(\tau; t, x, \xi) \right) A(\tau, f(\tau; t, x, \xi)) d \tau.$$  

So we have, by (38) and (39),

$$\left\| \frac{\phi_{h,j}(s; t, x, \xi)}{h} - w^{(j)}(s; t, x, \xi) \right\|_{\infty}$$

$$\leq 2 C n^2 e^{C'T} \int_{t-T}^{t+T} \| \phi_{h,j}(\tau; t, x, \xi) \|_{\infty} d \tau$$

(40)

$$+ 2 n M \left| \int_t^s \left\| \frac{\phi_{h,j}(\tau; t, x, \xi)}{h} - w^{(j)}(\tau; t, x, \xi) \right\|_{\infty} d \tau \right|$$

for $|s - t| \leq T$. Applying Gronwall’s inequality to (40), we obtain

$$\left\| \frac{\phi_{h,j}(s; t, x, \xi)}{h} - w^{(j)}(s; t, x, \xi) \right\|_{\infty}$$

$$\leq 2 C n^2 e^{(2 n M + C')T} \int_{t-T}^{t+T} \| \phi_{h,j}(\tau; t, x, \xi) \|_{\infty} d \tau$$
for $|s - t| \leq T$. By Lemma A.2, we obtain (34). Thus, we have
\[
\frac{\partial f_l(s; t, x, \xi)}{\partial x_k} = w_{l,k}(s; t, x, \xi) \in C(\mathbb{R})
\]
and
\[
\frac{\partial g_l(s; t, x, \xi)}{\partial x_k} = w_{n+l,k}(s; t, x, \xi) \in C(\mathbb{R})
\]
for $k, l = 1, \ldots, n$.

On the other hand, in the similar calculation as above, we have
\[
\lim_{h \to 0} \sup_{|s - t| \leq T} \left\| \frac{\psi_{h,j}(s; t, x, \xi) - w^{(n+j)}(s; t, x, \xi)}{h} \right\|_{\infty} = 0.
\]
Thus,
\[
\frac{\partial f_l(s; t, x, \xi)}{\partial \xi_k} = w_{l,n+k}(s; t, x, \xi) \in C(\mathbb{R})
\]
and
\[
\frac{\partial g_l(s; t, x, \xi)}{\partial \xi_k} = w_{n+l,n+k}(s; t, x, \xi) \in C(\mathbb{R})
\]
for $k, l = 1, \ldots, n$. Hence, $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ are $C^1$-function with respect to $x$ and $\xi$.

Using above fact, we can easily show, by induction, that if $V(t, x)$ is $C^{r+1}$-function in $x$ then $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ are $C^r$-function with respect to $x$ and $\xi$. Therefore we obtain the desired result. \(\square\)

**Proof of Lemma 3.1.** Put $w^{(k)}(s; t, x, \xi) = (w_{1,k}, w_{2,k}, \ldots, w_{2n,k})$ for $1 \leq k \leq 2n$. By Lemma A.3, $w^{(k)}(s)$ are the solutions of
\[
\begin{cases}
\frac{dw(s)}{ds} = w(s)A(s, f(s; t, x, \xi)), \\
w(t) = (0, \ldots, 0, 1, 0, \ldots, 0),
\end{cases}
\]
where $A(s, f(s; t, x, \xi)) = (a_{ij})$ is defined by (32). Then
\[
\frac{d(\det M(s; t, x, \xi))}{ds} = \left| \begin{array}{ccc}
\frac{dw_{1,1}}{ds} & \cdots & \frac{dw_{1,2n}}{ds} \\
\frac{dw_{2,1}}{ds} & \cdots & \frac{dw_{2,2n}}{ds} \\
\vdots & \cdots & \vdots \\
\frac{dw_{2n,1}}{ds} & \cdots & \frac{dw_{2n,2n}}{ds}
\end{array} \right| + \cdots + \left| \begin{array}{ccc}
w_{1,1} & \cdots & w_{1,2n} \\
w_{2,1} & \cdots & w_{2,2n} \\
\vdots & \cdots & \vdots \\
\frac{dw_{2n,1}}{ds} & \cdots & \frac{dw_{2n,2n}}{ds}
\end{array} \right|.
\]
Since \( \frac{dw^{(k)}(s)}{ds} = w^{(k)}(s)A(s, f(s; t, x, \xi)) \), we have \( \frac{dw^{(k)}(s)}{ds} = \sum_{j=1}^{2n} a_{j}w_{j,k} \) and then

\[
\begin{vmatrix}
\frac{dw_{1,1}}{ds} & \cdots & \frac{dw_{1,2n}}{ds} \\
\frac{dw_{2,1}}{ds} & \cdots & \frac{dw_{2,2n}}{ds} \\
\vdots & \ddots & \vdots \\
\frac{dw_{2n,1}}{ds} & \cdots & \frac{dw_{2n,2n}}{ds}
\end{vmatrix} = 2n \sum_{j=1}^{2n} a_{j}.
\]

From (41) and (42), we have

\[
\begin{aligned}
\frac{d}{ds}(\det M(s)) &= (a_{11} + \cdots + a_{2n2n})\det M(s; t, x, \xi).
\end{aligned}
\]

Since \( \text{tr} A(s, f(s; t, x, \xi)) = 0 \), we have \( \frac{d}{ds}(\det M(s; t, x, \xi)) = 0 \). Therefore

\[
\det M(s; t, x, \xi) = \det M(t; t, x, \xi) = \det E_{2n} = 1. \quad \square
\]

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