Non-autonomous quantum systems with scale-dependent interface conditions.

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Abstract

We consider a class of modified Schrödinger operators where the semiclassical Laplacian is perturbed with \( h \)-dependent interface conditions occurring at the boundaries of the potential’s support. Under positivity assumptions on the potential, we show that this modification produces a small perturbation on the dynamics as \( h \to 0 \), independently from the time scale. In the case of a time dependent potential, this yields uniform-in-\( h \) stability estimates for products of instantaneous propagators. Then, following a standard approach, the non-autonomous dynamical system is defined as a limit of stepwise propagators and its small-\( h \) expansion is provided under suitable regularity assumptions on the potential’s variations.

1 Introduction

Singular perturbations of the 1D Laplacian through non-mixed interface conditions have been used as a technical tool for the study of the adiabatic evolution of shape resonances in the asymptotic regime of quantum wells. A possible approach to this problem consists in using a complex dilation to identify the resonances and write the adiabatic problem as an evolution equation for proper eigenstates of the deformed Hamiltonian with spectral gap condition. This scheme does not allow a uniform-in-time estimate of the resulting dynamical system and prevents a rigorous estimate of the error in the adiabatic limit. An alternative approach developed in \cite{4} consists in modifying the physical Hamiltonian by replacing its kinetic part with a perturbed Laplacian \( \Delta_\theta \), whose domain is the restriction

\[
D(\Delta_\theta) = H^2(\mathbb{R} \setminus \{a,b\}) \cap u : \begin{cases} 
\displaystyle e^{-\frac{\theta}{2}} u(b^+) = u(b^-), & e^{-\frac{\theta}{2}} u'(b^+) = u'(b^-), \\
\displaystyle e^{-\frac{\theta}{2}} u(a^-) = u(a^+), & e^{-\frac{\theta}{2}} u'(a^-) = u'(a^+),
\end{cases}
\]

(here: \( \theta \in \mathbb{C} \) and \( u(x^\pm) \) denote the right and left limits of the function in \( x \)), while the action is given by: \( \Delta_\theta u(x) = u''(x) \) for \( x \in \mathbb{R} \setminus \{a,b\} \). The corresponding modified Hamiltonian: \( H^h_\theta = -h^2 \Delta_\theta + \mathcal{V} \), is defined with a potential \( \mathcal{V} \) compactly supported on \([a,b]\) and possibly depending on \( h \). The potential’s profile can be chosen in order to fix some required spectral condition (as the existence of shape resonances): in this framework, \( h \) is a small parameter fixing the quantum scale of the model.

According to the analytic dilation technique, the resonances of \( H^h_\theta \) identify, in the sector \( \{z \in \mathbb{C} : -2 \Im \varphi < \arg z < 0\} \), with the spectral points of the deformed operator:

\[
H^h_\theta (\varphi) = -h^2 e^{-2\varphi} 1_{\mathbb{R}\setminus\{a,b\}}(x) \Delta_{\varphi+\varphi} + \mathcal{V}
\]
resulting from the sharp exterior dilation: \( x \to e^{\varphi} 1_{\mathbb{R}\setminus\{a,b\}}(x) x \) (see \cite[Proposition 3.6]{4}; see also \cite[Chp. 16]{4} and the references therein for an introduction to the complex deformation method). The interest in these models stands upon the fact that, when \( \varphi = \varphi_0 > 0 \), the perturbed Laplacian \(-i\Delta_\varphi\) transforms into the maximal accretive operator: \(-ie^{-2\varphi} 1_{\mathbb{R}\setminus\{a,b\}}(x) \Delta_{\varphi+\varphi}\) (being \( 1_{\mathbb{R}\setminus\{a,b\}} \) the characteristic function of the exterior domain). Hence, the corresponding complex dilated Hamiltonian: \( H^h_\theta (\varphi) \), although nonselfadjoint, is the generator of a quantum semigroup of contractions (we refer to the Lemma 3.1 in \cite{4}). In the time-dependent case, this allows to rephrase the adiabatic evolution problem for the resonances of \( H^h_\theta (t) = -h^2 \Delta_\theta + \mathcal{V} (t) \) as an adiabatic problem for the corresponding eigenstates of \( H^h_\theta (\varphi, t) \) and, accounting the contractivity property of \( e^{-itH^h_\theta (\varphi, t)} \), a ‘standard’ adiabatic theorem can be developed (e.g. in \cite{11}). This approach led to a version of the adiabatic theorem holding for shape resonances in the regime of quantum wells in a semiclassical island \cite[Theorem 7.1]{4}.

The error introduced using modified Hamiltonians of the type \( H^h_\theta \) is determined by the difference between the modified dynamics and the unitary evolution generated by the corresponding selfadjoint operator \( H^h_\theta \). To justify the use of \( H^h_\theta \) in modelling realistic physical situations, this error needs to be carefully estimated uniformly-in-time when \( \theta \) is assumed to be small. The case of time independent potentials have been considered in \cite{9,10}. It is worthwhile to mention at this concern that, for \( \theta \) small, \( H^h_\theta \) is neither selfadjoint nor symmetric. Hence

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the definition of the quantum dynamics generated by the modified operator does not follows using standard arguments from selfadjoint theory. For \( h = 1 \), an accurate resolvent analysis, and explicit formulas for the generalized eigenfunctions of the modified operator, allow to obtain a small-\( \theta \) expansion of the stationary waves operators for couple \( \{ H^h_0, H^e_0 \} \), provided that \( \mathcal{V} \in L^2(\mathbb{R}) \) is compactly supported on \([a, b]\) and defined positive. This yields an uniform-in-time estimate for the ‘distance’ between the two dynamics according to the expansion: 

\[
e^{-itH^e_0} = e^{-itH^h_0} + \mathcal{O}(\theta),
\]

where \( \mathcal{O}(\cdot) \) is intended in the \( L^2 \)-operator norm sense (see theorem 1.2 in [2]). The case of \( h \)-dependent models is considered in [10] under the asymptotic regime of quantum wells in a semiclassical island; this particular framework, realized with a potential formed by the superposition of a potential barrier and potential wells supported in \((0, b)\) (see theorem 1.1). Using specific spectral assumptions, a small-\( \theta \) expansion of the modified dynamical system \( e^{-itH^h_0} \), similar to the one given in [9], has been obtained in this case (see Theorem 4.4 in [10]); nevertheless, when both \( h \) and \( \theta \) are small, the resulting error term is small uniformly in time only for initial states belonging to an appropriate subspace with prescribed energy conditions. This prevents to extend the result to the most relevant case of time dependent potentials.

In this work we first reconsider the case of autonomous potentials under generic assumptions. In particular, for \( \mathcal{V} \in L^\infty(\mathbb{R}) \) with compact support on \([a, b]\) and \( 1_{[a, b]} \mathcal{V} > 0 \), we provide a small-\( \theta \) expansion of the propagator \( e^{-itH^h_0} \) globally holding on \( L^2(\mathbb{R}) \) uniformly w.r.t. \( t \in \mathbb{R} \) and \( h \in (0, h_0) \). This generalizes the result obtained in [10] and allows us to discuss the case of time dependent potentials by adapting the Kato-Yoshida construction of the modified dynamics in terms of piecewise product of modified propagators. Although not strictly tailored on the case of the quantum well regime, our assumptions also includes the case of \( h \)-dependent potentials, while the positivity constraint on \( \mathcal{V} \) is still coherent with the description of a potential island generating shape resonances; in this connection, the operators concerned with our work can be adapted to the modelling of physical systems involving the interaction with quasi-stationary states corresponding to shape resonances.

## 2 Models and results

We consider the modified Schrödinger operators

\[
D(H^h_0) = D(\Delta_\theta), \quad (H^h_0 u)(x) = -h^2 u''(x) + \mathcal{V}(x) u(x), \quad x \in \mathbb{R} \setminus \{a, b\},
\]

where \( \Delta_\theta \) is defined according to (1.1). The domain \( D(\Delta_\theta) \) is next considered as a Hilbert subspace of \( H^2(\mathbb{R} \setminus \{a, b\}) \) (see the definition (1.1)).

### Theorem 2.1

Let \( h \in (0, h_0), c > 0, |\theta| \leq h^{N_0} \) with \( h_0 \) suitably small and \( N_0 > 2 \). For \( \mathcal{V} \) defined according to

\[
\mathcal{V} \in L^\infty(\mathbb{R}), \quad \text{supp } \mathcal{V} = [a, b], \quad 1_{[a, b]} \mathcal{V} > c,\]

(2.2)

The map \( iH^h_0 \) generates a strongly continuous group of bounded operators both on \( L^2(\mathbb{R}) \) and on the Hilbert space \( D(\Delta_\theta) \) equipped with the \( H^2(\mathbb{R} \setminus \{a, b\}) \)-norm. For \( u \in D(\Delta_\theta) \) the identity: \( i\partial_t \left( e^{-itH^h_0} u \right) = H^h_0 e^{-itH^h_0} u \), holds in the \( L^2(\mathbb{R}) \)-sense.

For a fixed \( t \), \( e^{-itH^h_0} \) is \( \theta \)-analytic w.r.t. the \( L^2(\mathbb{R}) \)-operator norm and allows the expansion

\[
\sup_{t \in \mathbb{R}} \left\| e^{-itH^h_0} - e^{-itH^e_0} \right\|_{L(L^2(\mathbb{R}))} \leq C_{a, b, c} h^{N_0-2},
\]

(2.3)

with \( C_{a, b, c} > 0 \), possibly depending on the data \( a, b, c \).

The case of a time dependent Hamiltonian

\[
H^h_0(t) = -h^2 \Delta_\theta + \mathcal{V}(t)
\]

(2.4)

is analyzed under the assumptions

\[
\mathcal{V}(t) \in C^0([0, T], L^\infty(\mathbb{R}, \mathbb{R})), \quad \text{supp } \mathcal{V}(t) = [a, b], \quad 1_{[a, b]} \mathcal{V}(t) > c,
\]

(2.5)

for some \( c > 0 \), and

\[
\mathcal{V}(t) - \mathcal{V}(s) \in W^{2, \infty}_0([a, b]), \quad \forall t, s \in [T, 0].
\]

(2.6)

where

\[
W^{2, \infty}_0([a, b]) = \{ \psi \in W^{2, \infty}([a, b]) \mid \psi(\alpha) = \psi'(\alpha) = 0, \alpha = a, b \}.
\]

(2.7)

The small-\( \theta \) behaviour of the resulting quantum dynamical system is characterized as follows.
Theorem 2.2 Let $h \in (0, h_0]$, $|\theta| \leq h^{N_0}$ with $h_0$ suitably small and $N_0 > 2$, and assume $H^h_{\theta}(t)$ to be defined according to (2.3). Under the conditions (2.7)-(2.10), there exists a unique family of operators $U^h_{\theta}(t, s)$, bounded and strongly continuous in $t$ and $s$ w.r.t. the $L^2(\mathbb{R})$-operator norm, fulfilling the identities

$$U^h_{\theta}(s, s) = 1_{L^2(\mathbb{R})}, \quad U^h_{\theta}(t, s) = U^h_{\theta}(t, r)U^h_{\theta,n}(r, s), \quad \forall s \leq r \leq t,$$

and such that $U^h_{\theta}(t, s)u$ is the solution of the Cauchy problem

$$i\partial_t u^h_{\theta}(t) = H^h_{\theta}(t)u^h_{\theta}(t), \quad 0 \leq s \leq t \leq T,$$

Moreover, $U^h_{\theta}(t, s)$ is $\theta$-holomorphic in $L(L^2(\mathbb{R}))$ and allows the estimates

$$\sup_{s, t \in [0, T]} \|U^h_{\theta}(t, s) - U^h_{\theta}(t, s)\|_{L(L^2(\mathbb{R}))} \leq M_{a, b, c} \sup_{t \in [0, T]} \|V(t)\|_{L^\infty(\mathbb{R})} h^{N_0-2-\delta},$$

with $M_{a, b, c} > 0$ depending on the data, but independent from $T$, and $\delta > 0$ arbitrarily small.

Our modified model $H^h_{\theta}$ identifies with an extension of the symmetric operator

$$H^h_{\theta,0} = H^h_{\theta} \upharpoonright \{ u \in H^2(\mathbb{R}) \mid u(\alpha) = u'(\alpha) = 0, \alpha = a, b \}.$$

Hence, $H^h_{\theta}$ is explicitly solvable w.r.t. $H^0_{\theta}$ and relevant quantities, as its resolvent or generalized eigenfunctions, can be expressed in terms of corresponding non-modified quantities, related to $H^0_{\theta}$, through non-perturbative formulas. This well-known property of point perturbations (see e.g. in [1]) provides with an useful tool for the perturbative representation of the stationary wave operators related to the couple in the Section 3, we use a ‘Krein-like’ formula for the modified generalized eigenfunctions: this yields a non-perturbative representation of the stationary wave operators. In particular, the positivity assumption for the potential allows to extend the result obtained in [10], providing with the small-$\theta$ expansion (2.3) for the modified quantum propagator holding without restrictions on the initial state.

In the non-autonomous case, considered in the Section 3, the dynamics is approximated by a stepwise product of propagators associated to the ‘instantaneous’ Hamiltonians. The stability of the approximating dynamics w.r.t. the $L(L^2(\mathbb{R}))$ and $L(D(\Delta_\alpha))$ topologies is discussed in the Lemma 2.1 using this result, its uniform convergence is obtained in the Proposition 2.3 by adapting the approach of [5] to our nonselfadjoint framework.

The expansion (2.10), obtained under the conditions (2.7)-(2.10), describes the asymptotic behaviour of the modified dynamical system, as $h, \theta \to 0$, in the case of time-dependent potentials. It is worthwhile to remark that the prescription: $V(t) - V(s) \in W^{2,\infty}_0([a, b])$ in (2.10) is coherent with the modelling of quantum transport systems where the variations in time of the potential, determined by the (possibly nonlinear) time evolution of quasi-stationary states, are expected to be concentrated in small regions of the device (i.e. inside $(a, b)$). Moreover, this result does not depend on the time scale $T$: hence, it can be possibly adapted to the analysis of adiabatic evolution problems, where the potential’s variation rate is fixed by $\varepsilon$ small, and the natural time scale grows according to $1/\varepsilon$.

2.1 Notation

In what follows: $B_D(p)$ is the open disk of radius $\delta$ centered in a point $p \in \mathbb{C}$; $\mathbb{C}^+$ is the upper complex half-plane; $1_{\Omega}(\cdot)$ is the characteristic function of a domain $\Omega$; $\partial^j f$ denotes the derivative of $f$ w.r.t. the $j$-th variable; $C^0(\mathbb{R})$ is the set of $C^0$-continuous functions w.r.t. $\mathbb{R}$, while $H_2(\mathbb{R})$ is the set of holomorphic functions w.r.t. $\mathbb{R}$.

The notation ‘$\leq$’, appearing in some of the proofs, denotes the inequality: ‘$\leq C$’ being $C$ a suitable positive constant. Moreover, the generalization of the Landau notation $O(\cdot)$ is defined according to

Definition 2.3 Let be $X$ a metric space and $f, g : X \to \mathbb{C}$. Then $f = O(g) \iff \forall x \in X$ it holds: $f(x) = p(x)g(x)$, being $p$ a bounded map $X \to \mathbb{C}$.

3 Scattering by interface conditions

Following the analysis developed in [9,10], we next resume the main features of the operators $H^h_{\theta}$, focusing on the scattering couple $\{H^h_{\theta}, H^h_{\theta,0}\}$.
In the case $h = 1$, it has been shown that the interface conditions (11) do not modify the spectrum provided that $\theta$ is small (see [10] Proposition 2.6). In the present case, the dilation: $y = (x - (b + a)/2)/h$ transforms the boundary conditions (11) into
\[
\begin{align*}
  e^{-\theta/2}u(\beta_h^+) &= u(\beta_h^-), \\
  e^{-\theta/3/2}u'(\beta_h^+) &= u'(\beta_h^-), \\
  e^{-\theta/2}u(\alpha_h^-) &= u(\alpha_h^+), \\
  e^{-\theta/3/2}u'(\alpha_h^-) &= u'(\alpha_h^+),
\end{align*}
\]
(3.1)
with: $\alpha_h = -(b - a)/2h$ and $\beta_h = (b - a)/2h$. The corresponding unitary map on $L^2(\mathbb{R})$ transforms $H^h_0$ into the dilated operator
\[
\tilde{H}_h: \left\{ \begin{array}{ll}
  D(\tilde{H}_h) = \{ u \in H^2(\mathbb{R}) \setminus \{ \alpha_h, \beta_h \} \mid (3.4) \text{ holds} \}, \\
  (\tilde{H}_h u)(x) = -u''(x) + \tilde{V}(x) u(x), & x \in \mathbb{R} \setminus \{ \alpha_h, \beta_h \},
\end{array} \right.
\]
(3.2)
where $\tilde{V}(x) = V(hx + (b + a)/2)$ is compactly supported on $[\alpha_h, \beta_h]$.

**Proposition 3.1** Let $h > 0$ fixed and consider the operators $H^h_0$ defined in (2.7) with
\[
\mathcal{V} \subset L^2(\mathbb{R}, \mathbb{R}), \quad \text{supp} \mathcal{V} = [a, b].
\]
(3.3)
For any couple $\theta \in \mathbb{C}$, the essential part of the spectrum is $\sigma_{ess}(H^h_0) = \mathbb{R}_+$. If, in addition, $\mathcal{V}$ is assumed to be defined positive, i.e.
\[
\langle u, \mathcal{V} u \rangle_{L^2((a, b))} > 0 \quad \forall u \in L^2((a, b)) \text{ s.t. } u \neq 0,
\]
(3.4)
it exists $\delta > 0$, possibly depending on $h$, such that: $\sigma(H^h_0) = \mathbb{R}_+$ for all $\theta \in B_\delta(0)$.

**Proof.** From to the Proposition 2.6 in [9], the result holds for $\tilde{H}_h$; then it extends to $H^h_0$ due to the unitarily equivalence of the two operators. ■

**Notice** The essential spectrum of $A$ is here defined according to [12] as $\sigma_{ess}(A) = \mathbb{C} \setminus \mathcal{F}(A)$, being $\mathcal{F}(A)$ the set of complex $\lambda \in \mathbb{C}$ s.t. $(A - \lambda)$ is Fredholm.

The point perturbation model $H^h_0$ can be described as a restriction of the adjoint operator $(H^h_0)^* \quad \text{(see 2.11)}$ through linear relations on an auxiliary Hilbert space. This construction, achieved in [10] using the ‘boundary triples’ technique, allows to express the difference $(H^h_0 - z)^{-1} - (H^h_0 - z)^{-1}$ in terms of a finite rank operator with range: ker $(\tilde{H}_h^h_0)^* - z)$. A basis of this defect space is formed by the Green’s functions associated to the differential operator $(-h^2\partial_x^2 + \mathcal{V} - z)$, and by their first derivatives. Let $z \in res(H^h_0)$ and introduce $G^{z,h}(x, y)$ and $H^{z,h}(x, y)$ as solutions of the boundary value problems
\[
\begin{align*}
  (-h^2\partial_x^2 + \mathcal{V} - z)G^{z,h}(\cdot, y) &= 0, & \text{in } \mathbb{R} \setminus \{ y \}, \\
  G^{z,h}(y^+, y) = G^{z,h}(y^-, y), & h^2(\partial_t G^{z,h}(y^+, y) - \partial_t G^{z,h}(y^-, y)) = -1,
\end{align*}
\]
(3.5)
and
\[
\begin{align*}
  (-h^2\partial_x^2 + \mathcal{V} - z)H^{z,h}(\cdot, y) &= 0, & \text{in } \mathbb{R} \setminus \{ y \}, \\
  h^2(\mathcal{H}^{z,h}(y^+, y) - \mathcal{H}^{z,h}(y^-, y)) &= 1, & \partial_t \mathcal{H}^{z,h}(y^+, y) = \partial_t \mathcal{H}^{z,h}(y^-, y),
\end{align*}
\]
(3.6)
Then ker $(H_{0,0}^h)^* - z) = l.c. \{ \{ \gamma_{z,h,j} \}_{j=1}^4 \}_{z=1}$, with
\[
\begin{align*}
  \gamma_{z,h,1} &= G^{z,h}(x, b), \quad \gamma_{z,h,2} = H^{z,h}(x, b), \quad \gamma_{z,h,3} = G^{z,h}(x, a), \quad \gamma_{z,h,4} = H^{z,h}(x, a).
\end{align*}
\]
(3.7)
Following [10] eq. (2.19) and (2.26), for all $z \in res(H^h_0)$ the identity
\[
(H^h_0 - z)^{-1} - (H^h_0 - z)^{-1} = -\sum_{i,j=1}^4 \left[ (B_\theta q(z, h) - A_\theta)^{-1} B_\theta \right]_{ij} \langle \gamma_{z,h,i}, \gamma_{z,h,j} \rangle_{L^2(\mathbb{R})} \gamma_{z,h,j},
\]
(3.8)
holds with
\[
\begin{align*}
  h^2 A_\theta - 1 &= \begin{pmatrix}
  e^{\theta/2} & e^{\theta/2} \\
  e^{-\theta/2} & e^{-\theta/2}
\end{pmatrix}, \quad B_\theta = \begin{pmatrix}
  0 & 0 & 0 \\
  e^{\theta/2} - 1 & 1 - e^{\theta/2} \\
  0 & e^{-\theta/2} - 1 & 0
\end{pmatrix},
\end{align*}
\]
(3.9)
and \( g(z, h) \) depending on the boundary values of \( G^{z, h}, H^{z, h} \) and \( \partial_t H^{z, h} \) according to

\[
 q(z, h) = \begin{pmatrix}
 G^{z, h}(b, b) & -H^{z, h}(b, b) & -H^{z, h}(b, a) \\
 H^{z, h}(b, b) & G^{z, h}(b, a) & -\frac{1}{2\pi} \partial_t H^{z, h}(b, a) \\
 H^{z, h}(b, a) & -\frac{1}{2\pi} \partial_t H^{z, h}(a, b) & G^{z, h}(a, a) - \frac{1}{2\pi} \partial_t H^{z, h}(a, a)
 \end{pmatrix}
\]  

(3.10)

The Green’s functions \( G^{z, h}, H^{z, h} \) are related to the Jost’s solutions of the equation

\[
 (-h^2 \nabla_y^2 + V) u = \zeta^2 u, \quad \zeta \in \mathbb{C}^+,
\]

next denoted with \( \chi^b_+ (\cdot, \zeta) \), fulfilling the exterior conditions

\[
 \chi^b_+ (\cdot, \zeta)|_{x>b} = e^{i\hat{\xi}x}, \quad \chi^b_+ (\cdot, \zeta)|_{x<a} = e^{-i\hat{\xi}x}.
\]

(3.12)

A detailed analysis of their properties have been given in [13] for generic \( L^1 \)-potentials, while the particular case of a potential barrier is explicitly considered in [9] for \( h = 1 \). The \( h \)-dependent case can be considered as a rescaled problem and the result presented in [9] rephrases as follows

**Lemma 3.2** Let \( V \in L^2(\mathbb{R}, \mathbb{R}) \) s.t.: \( \text{supp} \ V = [a, b] \). For any fixed \( h > 0 \), the solutions \( \chi^b_\pm \) to the problem \( \Delta \chi + V \chi = 0 \) belong to \( C^1_\mathbb{R} (\mathbb{R}, \mathbb{C} (\zeta^+) \) and have continuous extension to the real axis.

**Proof.** With the change of variable: \( y = (x - (b + a)/2) / h \), the problem \( \Delta \chi + V \chi = 0 \) writes as

\[
 \begin{cases}
 -\partial_y^2 \chi^b_+ (\cdot, \zeta) &= \zeta^2 \chi^b_+ (\cdot, \zeta), \quad \zeta \in \mathbb{C}^+, \\
 \chi^b_+ (\cdot, \zeta)|_{y=(b-a)/2h} &= e^{i\zeta y}, \quad \chi^b_- (\cdot, \zeta)|_{y=-(b-a)/2h} = e^{-i\zeta y},
\end{cases}
\]

(3.13)

where \( \tilde{V} \) denotes the dilated potential: \( \tilde{V}(y) = V((b + a) / 2y), \) supported on \([- (b - a) / 2h, (b - a) / 2h]\), while \( \chi^b_\pm \) correspond to the rescaled Jost’s functions

\[
 \chi^b_+ (y) = \chi^b_+ (hy + (b + a) / 2) e^{-i\hat{\xi}(b+a)}.
\]

(3.14)

In this framework the Proposition 2.2 in [9] applies; this yield \( \chi^b_\pm \in C^1_\mathbb{R} (\mathbb{R}, \mathbb{C} (\zeta^+) \) with continuous extensions to the closed complex half-plane \( \mathbb{C}^+ \).

Let \( \zeta \in \mathbb{C}^+ \) be such that: \( \zeta^2 \in \text{res} \ (H_0^b) \); rephrasing the relation [13] Chp. 5, eq. (1.10)] in our framework, we get

\[
 G^{z, h} (\cdot, y) = \frac{1}{h^2 w^b(\zeta)} \begin{pmatrix}
 \chi^b_+ (\cdot, \zeta) \chi^b_+ (y, \zeta), & x \geq y, \\
 \chi^b_- (\cdot, \zeta) \chi^b_- (y, \zeta), & x < y,
\end{pmatrix}
\]

(3.15)

\[
 H^{z, h} (\cdot, y) = \frac{1}{h^2 w^b(\zeta)} \begin{pmatrix}
 \chi^b_+ (\cdot, \zeta) \partial_t \chi^b_+ (y, \zeta), & x \geq y, \\
 \chi^b_- (\cdot, \zeta) \partial_t \chi^b_- (y, \zeta), & x < y,
\end{pmatrix}
\]

(3.16)

where \( w^b(\zeta) \), depending only on \( \zeta \) and \( \mathcal{V} \), denotes the Wronskian associated to the couple \( \{ \chi^b_+ (\cdot, \zeta), \chi^b_- (\cdot, \zeta) \} \) (defined by: \( w(f, g) = fg' - f'g \)). Due to the result of the Lemma 3.2, for each \( h > 0 \), the maps \( z \to G^{z, h}(x, y) \), \( z \to H^{z, h}(x, y) \) are meromorphic in \( \mathbb{C} \setminus \mathbb{R}_+ \) with a branch cut along the positive real axis and poles, corresponding to the points in \( \sigma_p (H_0^b) \) located on the negative real axis. Adapting [13] Chp. 5, eq. (1.9)] to the \( h \)-dependent case, the function \( w^h(k) \) fulfills the identity: \( |w^h(k)|^2 = k^2 / h^2 + |w_0^h(k)|^2 \), where \( w_0^h(k) \) is the Wronskian associated to the couple \( \{ \chi^b_+ (\cdot, -k), \chi^b_- (\cdot, k) \} \). In particular, the inequality

\[
 \frac{1}{|w^h(k)|} \leq \frac{h}{|k|}
\]

(3.17)

implies that the maps \( z \to G^{z, h}(x, y), z \to H^{z, h}(x, y) \) continuously extend up to the branch cut, both in the limits \( z \to k^2 \pm i0 \), with the only possible exception of the point \( z = 0 \).
The above characterization and the definition (3.10) imply that $z \to (B_0 q(z, h) - A_0)$ is meromorphic matrix-valued map in $\mathbb{C} \cup \mathbb{R}_+$ with continuous extension to $z \to k^2 \pm i0$ for $k \neq 0$. Due to the identity (3.8), the conditions: $z \in \text{res} \left( H^0_h \right)$ and $0 \notin \text{res} \left( B_0 q(z, h) - A_0 \right)$ (i.e.: $z$ is a pole for the inverse, matrix-valued, function $z \to (B_0 q(z, h) - A_0)^{-1}$), compel: $z \in \sigma_p \left( H^0_h \right)$. Nevertheless, according to the result of the Proposition 3.1 for defined positive potentials it results: $\text{res} \left( H^0_h \right) = \text{res} \left( H^0_h \right) = \mathbb{C} \cup \mathbb{R}_+$ provided that: $\theta \in B_1 \left( 0 \right)$, for a small $\delta > 0$ possibly depending on $h$. Hence, under these conditions, the inverse $(B_0 q(z, h) - A_0)^{-1}$ exists in $\mathbb{C} \cup \mathbb{R}_+$ and has continuous extensions to the branch cut both in the limits $z = k^2 \pm i0$, with the only possible exception of the origin.

The generalized eigenfunctions of our model, next denoted with $\psi^0_h (\cdot, k)$, solve of the boundary value problem

$$
\begin{cases}
(-h^2 \partial_x^2 + \mathcal{V}) u = k^2 u, & x \in \mathbb{R} \setminus \{a, b\}, \ k \in \mathbb{R}, \\
e^{-\theta/2} u(b^+) = u(b^-), & e^{-\theta/2} u'(b^+) = u'(b^-), \quad (3.18)
\end{cases}
$$

and fulfill the exterior conditions

$$
\begin{align*}
\psi^0_h (x, k)|_{x \geq a} &= e^{i \frac{h}{2} x} + R^h (k, \theta) e^{-i \frac{h}{2} x}, \\
\psi^0_h (x, k)|_{x \geq b} &= T^h (k, \theta) e^{i \frac{h}{2} x},
\end{align*}
$$

(3.19)

$$
\begin{align*}
\psi^0_h (x, k)|_{x \geq a} &= T^h (k, \theta) e^{i \frac{h}{2} x}, \\
\psi^0_h (x, k)|_{x \geq b} &= e^{i \frac{h}{2} x} + R^h (k, \theta) e^{-i \frac{h}{2} x},
\end{align*}
$$

(3.20)

describing an incoming wave function of momentum $k$ with reflection and transmission coefficients $R^h$ and $T^h$. For $\theta = 0$, the generalized eigenfunctions of $H^0_h, \psi^0_h (\cdot, k)$, depend on the Jost’s solutions $\chi^\pm$ according to

$$
\psi^0_h (x, k) = \begin{cases}
-\frac{2i k}{h w^H (k)} \chi^\pm (x, k), & \text{for } k \geq 0, \\
\frac{2i k}{h w^H (k)} \chi^\pm (x, -k), & \text{for } k < 0.
\end{cases}
$$

(3.21)

Following an approach similar to the one leading to the Krein-like resolvent formula (3.8), an expansion for the difference: $\psi^0_h (x, k) - \psi^0_h (x, k)$ as $\theta \to 0$ has been provided with In [10, eq. (2.19) and (2.26)]. Let $G^{+|k|, h} (\cdot, y)$ and $H^{+|k|, h} (\cdot, y)$ be defined by

$$
G^{+|k|, h} (\cdot, y) = \lim_{z \to k^2 \pm i0} G^{+h} (\cdot, y), \quad H^{+|k|, h} (\cdot, y) = \lim_{z \to k^2 \pm i0} H^{+h} (\cdot, y),
$$

and denote with $g_{k, h, j}$ and $M^h$ the corresponding limits of $\gamma_{z, h, j}$ and $(B_0 q(z, h) - A_0)$ (see the definitions (3.14), (3.10) and (3.9)); namely, we set

$$
g_{k, h, j} = \lim_{z \to k^2 \pm i0} \gamma_{z, h, j}, \quad M^h (\pm |k|, \theta) = \lim_{z \to k^2 \pm i0} (B_0 q(z, h) - A_0).
$$

(3.23)

Due to (3.15)-(3.10), $G^{+k, h}$ and $H^{+k, h}$ explicitly write as

$$
G^{+k, h} (\cdot, y) = \frac{1}{h^2 w^H (k)} \begin{cases}
\chi^+ (\cdot, k) \chi^h (y, k), & x \geq y, \\
\chi^+ (\cdot, k) \chi^H (y, k), & x < y,
\end{cases}
$$

(3.24)

$$
H^{+k, h} (\cdot, y) = \frac{-1}{h^2 w^H (k)} \begin{cases}
\chi^+ (\cdot, k) \partial_1 \chi^h (y, k), & x \geq y, \\
\chi^+ (\cdot, k) \partial_1 \chi^H (y, k), & x < y,
\end{cases}
$$

(3.25)

while, according to the previous remarks, $g_{k, h, j}$ and $M^h (k, \theta)$ are well defined and continuous w.r.t. $k \in \mathbb{R}$, with the only possible exception of the origin. Denoting with

$$
S^h (\theta) = \{ k \in \mathbb{R} \mid \det M^h (k, \theta) = 0 \},
$$

(3.26)

the set of the singular points of $M^h (k, \theta)$, the representation (see [10, Proposition 2.2])

$$
\psi^0_h (\cdot, k) = \begin{cases}
\psi^0_h (\cdot, k) - \sum_{i,j=1}^{4} \left[ (M^h (k, \theta))^{-1} B_0 \right]_{ij} \Gamma^{k, h}_{ij} g_{k, h, j}, & \text{for } k > 0, \\
\psi^0_h (\cdot, k) - \sum_{i,j=1}^{4} \left[ (M^h (-k, \theta))^{-1} B_0 \right]_{ij} \Gamma^{k, h}_{ij} g_{-k, h, j}, & \text{for } k < 0,
\end{cases}
$$

(3.27)

holds for any fixed $h > 0$, $\theta \in \mathbb{C}$ and $k \in \mathbb{R}^* \backslash S^h (\theta)$, being $\Gamma^{k, h}$ the vector of the boundary values

$$
2 \Gamma^{k, h} = \left( \psi^0_h (b, k), \partial_1 \psi^0_h (b, k), \psi^0_h (a, k), \partial_1 \psi^0_h (a, k) \right).
$$

(3.28)
3.1 Trace estimates

We aim to control the coefficients at the r.h.s. of (3.27) when both \( \theta \) and \( h \) are small. This requires accurate estimates for the boundary values of \( g_{\ell,h,j} \) (occurring in the definition of the matrix \( \mathcal{M}^h (k, \theta) \)) and \( \psi_0^h (\cdot, k) \). In [10] eq. (2.19) and (2.26) these estimates have been provided for a finite energy range when the potential describes quantum wells in a semiclassical island. We next reconsider this problem under a generic condition of positivity for \( 1_{[a,b]} V \). To this aim, we next recall some standard energy estimates; let consider the problem

\[
\begin{cases}
(-h^2 \partial_x^2 + V - \zeta^2) u = 0, & \text{in } (a, b), \\
|h\partial_x + i\zeta| u(a) = \gamma_a, & \text{and } [3.21],
\end{cases}
\]

where: \( V \in L^\infty ((a,b), \mathbb{R}), \gamma_a, \gamma_b \in \mathbb{C}, \) and \( h > 0 \).

**Lemma 3.3** Assume \( \zeta \in \mathbb{C}^+ \) such that: \( V - \Re \zeta^2 > c \) for some \( c > 0 \). The solution of (3.29) fulfills the estimate

\[
h^\frac{1}{2} \sup_{[a,b]} |u| + \|h u'\|_{L^2((a,b), \mathbb{R})} + \|u\|_{L^2((a,b), \mathbb{R})} \leq C_{a,b,c} \frac{1}{h^\frac{1}{2}} (|\gamma_a| + |\gamma_b|),
\]

with \( C_{a,b,c} > 0 \) possibly depending on the data.

**Proof.** From the equation

\[
\langle u, (-h^2 \partial_x^2 + V - \zeta^2) u \rangle = 0,
\]

an integration by parts yields

\[
\|h u'\|_{L^2((a,b), \mathbb{R})}^2 + \int_a^b (V - \zeta^2) |u|^2 \ dx + h^2 (u^* u'(a) - u^* u'(b)) = 0.
\]

Taking into account the boundary conditions in (3.29), our assumptions \((V - \Re \zeta^2 > c \) and \( \Im \zeta \geq 0 \)) imply

\[
\|h u'\|_{L^2((a,b), \mathbb{R})}^2 + c \|u\|_{L^2((a,b), \mathbb{R})}^2 \leq h \Re (|u^* (a)| |\gamma_a| - |u^* (b)| |\gamma_b|).
\]

The estimate (3.30) then follows from (3.33) by taking into account the Gagliardo-Nirenberg inequality: \( \sup_{[a,b]} |\varphi| \leq C_{b-a} \|\varphi\|_{L^2((a,b), \mathbb{R})} \|\varphi\|_{L^2((a,b), \mathbb{R})}^2 \). □

When the differential operator \((-h^2 \partial_x^2 + V - k^2)\) is defined with a potential \( V \in L^\infty (\mathbb{R}, \mathbb{R})\) compactly supported on \([a,b]\), the corresponding Green’s functions solve boundary value problems of the type (3.29) and the Lemma 3.3 applies if: \( V - k^2 > c \). Global-in-k estimates for their boundary values are next considered by combining the explicit representations in terms of the Jost’s solutions, given in (3.24)-(3.25) and (3.21), and energy estimates in the low-energy regime. To this aim, the potential is next assumed to fulfill the stronger condition

\[
V \in L^\infty (\mathbb{R}, \mathbb{R}), \quad \supp V = [a,b], \quad 1_{[a,b]} V > c,
\]

holding for some \( c > 0 \).

**Proposition 3.4** Let \( V \) fulfills the conditions (3.34); the relations

\[
[(1 + k) \psi_0^h (y,k)] + h \ |\partial_1 \psi_0^h (y,k)| \leq |k| C_{a,b,c},
\]

hold for \( y, y' \in [a,b] \) and \( k \in \mathbb{R}, \) being \( C_{a,b,c} > 0 \) possibly depending on the data and \( h \in (0,h_0) \) with \( h_0 > 0 \) small.

**Proof.** From the result of the Lemma 3.2 the Jost’s solutions \( \chi^h_x (x,k) \) are \( C^1_x \)-continuous and the exterior conditions (3.12) can be used for the explicit computation of \( \partial_1^\ell \chi^h_x (y,k) \) when \( y = a, b \) and \( j = 0, 1 \). Then, the relations (3.24), (3.22) and (3.25) allow to obtain (almost) explicit representations of the quantities considered in (3.35)-(3.30). Let start considering the boundary values \( \partial_1^\ell \psi_0^h (y,k); \) we focus on the case \( y = a, \) while similar computations hold for \( y = b. \) If \( k < 0, \) the representation (3.24) and the exterior conditions (3.12) imply

\[
1_{\{k < 0\} (k) \psi_0^h (a,k)} = \frac{2k}{h ne^{-(-k)}} e^{i\frac{1}{h}a}, \quad \text{and} \quad 1_{\{k < 0\} (k) \partial_1 \psi_0^h (a,k)} = -\frac{2k}{h ne^{-(-k)}} e^{i\frac{1}{h}a}.
\]

Recalling that \( w^h (-k) = (w^h (k))^*, \) the inequality (3.17) leads to

\[
1_{\{k < 0\} (k) \psi_0^h (a,k)} \leq 2, \quad \text{and} \quad 1_{\{k < 0\} (k) \partial_1 \psi_0^h (a,k)} \leq 2 |k|.
\]
If \( k \geq 0 \), comparing (3.10) with (3.21), we get the relation
\[
1_{\{k \geq 0\}}(k) T^h(k, 0) = -\frac{2ik}{hw^h(k)}.
\] (3.39)

The identity: \( |T^h(k, 0)|^2 + |R^h(k, 0)|^2 = 1 \), and the representations
\[
1_{\{k \geq 0\}}(k) \psi_0^h(a, k) = e^{i\frac{a}{h}k} + R^h(k, 0)e^{-i\frac{a}{h}k}, \quad 1_{\{k \geq 0\}}(k) \partial_1 \psi_0^h(a, k) = i\frac{k}{h} \left( e^{i\frac{a}{h}k} - R^h(k, 0)e^{-i\frac{a}{h}k} \right),
\] (3.40)
yield
\[
|1_{\{k \geq 0\}}(k) \psi_0^h(a, k)| \leq 2, \quad |1_{\{k \geq 0\}}(k) \partial_1 \psi_0^h(a, k)| \leq 2\frac{k}{h}.
\] (3.41)

From (3.38), (3.41), and from similar computations in the case of \( k = y \), we have
\[
|\psi_0^y(y, k)| + h |k|^{-1} |\partial_1 \psi_0^y(y, k)| \leq C_{a,b,c}, \quad y = a, b.
\] (3.42)

The relations (3.36) are next considered for \( y = a \) (when \( y' = b \) the result follows from similar computations). According to (3.21), we have
\[
|1_{\{k \geq 0\}}(k) (hw^h(k))^{-1} \partial_1^j \chi^h_{\pm}(a, k) + |1_{\{k < 0\}}(k) (hw^h(k))^{-1} \partial_1^j \chi^h_{\pm}(b, -k)| = \left( 2k \right)^{-j} 1_{\{k < 0\}} \partial_1^j \psi_0^h(b, k)
\] (3.43)
then, the estimate (3.42) lead us to
\[
|1_{\{k \geq 0\}}(k) (hw^h(k))^{-1} \partial_1^j \chi^h_{\pm}(a, k) + |1_{\{k < 0\}}(k) (hw^h(k))^{-1} \partial_1^j \chi^h_{\pm}(b, -k)| \leq C_{a,b,c} \left( \frac{|k|}{h} \right)^j,
\] (3.44)
with \( j = 0, 1 \). Both these relations easily extend to all \( k < 0 \) by recalling that: \( w^h(-k) = (w^h(k))^* \) and \( \partial_1^j \chi^h_{\pm}(-, -k) = (\partial_1^j \chi^h_{\pm}(, k))^* \). Finally, the identities: \( \partial_1^j \chi^h_{\pm}(b, k) = \left( i\frac{k}{h} \right)^j e^{i\frac{a}{h}k} \) and \( \partial_1^j \chi^h_{\pm}(a, k) = \left( -i\frac{k}{h} \right)^j e^{-i\frac{a}{h}k} \) (arising from (3.22)) and the inequality (3.17) allow to generalize this result to all \( y = a, b \). This resumes as follows
\[
|\left( hw^h(k) \right)^{-1} \partial_1^j \chi^h_{\pm}(y, k)| \leq C_{a,b,c} \left( \frac{|k|}{h} \right)^j, \quad y = a, b, \quad j = 0, 1.
\] (3.45)

From the representation (3.24), the boundary values \( G^{\pm|k,h}(y, a) \) write as
\[
G^{k,h}(y, a) = \frac{1}{h^2w^h(k)} \chi^h_{\pm}(y, k) e^{-i\frac{a}{h}k}, \quad \text{for } y \in \{a, b\}.
\] (3.46)

Let \( c > 0 \) be such that (3.34) holds at consider at first the case \( k^2 \geq c \); using (3.13) with \( j = 0 \), we get
\[
|1_{\{k^2 \geq c\}}(k) G^{k,h}(y, a)| \leq 1_{\{k^2 \geq c\}}(k) \frac{C_{a,b,c}}{h|k|} \leq \frac{C_{a,b,c}}{h(1 + |k|)}, \quad y = a, b.
\] (3.47)
for a suitable \( C_{a,b,c} > 0 \) (depending on \( a, b, c \)). The function \( G^{\pm|k,h}(\cdot, a) \) solves an equation of the type-(3.29) with: \( \gamma_0 = -\frac{1}{h} \) and \( \gamma_0 = 0 \); when \( k^2 \leq c \), the lemma (3.33) applies, allowing to control the boundary values \( G^{\pm|k,h}(y, a) \) according to
\[
|1_{\{k^2 \leq c\}}(k) G^{k,h}(y, a)| \leq C_{a,b,c} h^{-2}, \quad y = a, b.
\] (3.48)

Hence, (3.37) - (3.48) lead us to
\[
|(1 + |k|) G^{k,h}(y, a)| \leq K_{a,b,c} h^{-2}, \quad y = a, b.
\] (3.49)

Using the representation (3.25), the exterior condition: \( 1_{\{x < a\}}(x) \chi^h_{\pm}(x, k) = e^{-i\frac{a}{h}x} \) and the \( C^1 \)-regularity of \( \chi^h_{\pm}(\cdot, k) \), it follows
\[
1_{\{a,b\}} H^{k,h}(y, a) = -\frac{ik}{h^2w^h(k)} \chi^h_{\pm}(y, k) e^{-i\frac{a}{h}k}, \quad y = a, b,
\] (3.50)
and the inequality (3.17), \( j = 0 \), yields
\[
|H^{k,h}(y, a)| \leq \frac{C_{a,b,c}}{h^2}, \quad y = a, b.
\] (3.51)
Using once more (3.24), we have
\[
\partial_1 H^{k,h}(y,a) = \frac{-ik}{\hbar^3 w^h(k)} \partial_1 \chi^{h}_+(y,k) e^{-i\theta a}, \quad y = a, b. \tag{3.52}
\]

Then, (3.45), \( j = 1 \), implies
\[
|\partial_1 H^{k,h}(a,b)| \leq C_{a,b} \frac{|k|}{\hbar^3}, \quad y = a, b. \tag{3.53}
\]

For \( y' = a \), the inequality (3.36) follows from (3.49), (3.51) and (3.52) with a suitable \( C_{a,b,c} \).

Finally, we reconsider the bound (3.42); according to (3.24), the relation (3.36) yields
\[
C^{k,h}(b,a) = \left( \hbar^2 w^h(k) \right)^{-1} e^{\beta - \alpha - a} = \mathcal{O}(\hbar^{-2}), \tag{3.54}
\]

uniformly w.r.t. \( k \in \mathbb{R} \). It follows: \( \min \| w^h(k) \| > c_0 \) for a suitable \( c_0 > 0 \) possibly depending on the data, while, taking into account (3.17), we get: \( w^h(k)^{-1} = \mathcal{O}\left( \frac{1}{1 + k} \right) \). Hence, the relations (i)-(3.37), (3.39) and (i)-(3.40), yield: \( |\psi^h_0(a,k)| = \mathcal{O}(k \left( 1 + k \right)^{-1}) \); this improves the previous estimates according to (3.35).

**Remark 3.5** The result presented in (3.4) stands upon the regularity of the Jost’s solution at the boundaries \([a,b]\). This property, considered in the Lemma 3.3, generically holds for positive defined and compactly supported potentials, while the trace estimates (3.37), (3.39) do not depend on the particular shape of \( V \), provided that it fulfills the conditions (3.34).

### 3.2 Generalized eigenfunctions expansion

The result of the Proposition 3.3 can be implemented to obtain an expansion of the modified generalized eigenfunctions \( \psi^h_{\theta}(\cdot, k) \) when both \( \theta \) and \( h \) are small. Using the notation introduced in (3.22) and (3.23), a direct computation leads to
\[
\mathcal{M}^h(k,\theta) = \begin{pmatrix}
\beta(\theta) H^{k,h}(b^+,b) & -\beta(\theta) \partial_1 H^{k,h}(b,b) & \beta(\theta) H^{k,h}(a,b) & -\beta(\theta) \partial_1 H^{k,h}(b,a) \\
-\beta(\theta) H^{k,h}(b,b) & \beta(\theta) H^{k,h}(b',b) & -\beta(\theta) H^{k,h}(b,a) & \beta(\theta) H^{k,h}(b,a) \\
\beta(-\theta) H^{k,h}(b,a) & -\beta(-\theta) \partial_1 H^{k,h}(a,b) & \beta(-\theta) H^{k,h}(a',a) & -\beta(-\theta) \partial_1 H^{k,h}(a,a) \\
\beta(-\theta) H^{k,h}(a,b) & -\beta(-\theta) H^{k,h}(a,b) & \beta(-\theta) H^{k,h}(a,a) & -\beta(-\theta) H^{k,h}(a',a)
\end{pmatrix}
- \frac{1}{\hbar^2} \begin{pmatrix}
\alpha(\theta) + \frac{h^2}{2} \beta(\theta) \\
\alpha(\theta) - \frac{h^2}{2} \beta(\theta) \\
\alpha(-\theta) - \frac{h^2}{2} \beta(-\theta) \\
\alpha(-\theta) + \frac{h^2}{2} \beta(-\theta)
\end{pmatrix},
\tag{3.55}
\]

where \( \alpha(\theta) \) and \( \beta(\theta) \) are defined by
\[
\alpha(\theta) = 1 + e^{\frac{\theta}{2}}, \quad \beta(\theta) = 1 - e^{\frac{\theta}{2}}. \tag{3.56}
\]

As consequence of the estimates (3.35)-(3.39), for defined positive potentials the above relation rephrases as
\[
\mathcal{M}^h(k,\theta) = \begin{pmatrix}
\alpha(\theta) & \alpha(\theta) & \alpha(-\theta) & \alpha(-\theta) \\
\beta(\theta) \mathcal{O}(h^{-2}) & \beta(\theta) \mathcal{O}(|k| h^{-3}) & \beta(\theta) \mathcal{O}(h^{-2}) & \beta(\theta) \mathcal{O}(|k| h^{-3}) \\
\beta(-\theta) \mathcal{O}(h^{-2}) & \beta(-\theta) \mathcal{O}(|k| h^{-3}) & \beta(-\theta) \mathcal{O}(h^{-2}) & \beta(-\theta) \mathcal{O}(|k| h^{-3}) \\
\beta(-\theta) \mathcal{O}(h^{-2}) & \beta(-\theta) \mathcal{O}(|k| h^{-3}) & \beta(-\theta) \mathcal{O}(h^{-2}) & \beta(-\theta) \mathcal{O}(|k| h^{-3})
\end{pmatrix},
\tag{3.57}
\]

being the symbols \( \mathcal{O}(\cdot) \) referred to the metric space \( \mathbb{R} \times (0, \hbar_0] \) and defining continuous functions of \( k \in \mathbb{R} \).

From the definition of \( \alpha(\theta), \beta(\theta) \), the coefficients of \( \mathcal{M}^h(k,\theta) \) result \( \theta \)-holomorphic and continuous w.r.t. \( (k,\theta) \in \mathbb{C} \times \mathbb{R} \) while, using the expansions: \( \alpha(\theta) = 2 + \mathcal{O}(\theta) \) and \( \beta(\theta) = \mathcal{O}(\theta) \), follows
\[
\mathcal{M}^h(k,\theta) = -\frac{2}{\hbar^2} 1_{c_+} + \theta m^h(k,\theta), \tag{3.58}
\]

\[9\]
where the remainder term is
\[
m^h(k, \theta) = \begin{pmatrix}
O(h^{-2}) & O(|k|h^{-3}) & O(h^{-2}) & O(|k|h^{-3}) \\
O(h^{-2} (1 + |k|)^{-1}) & O(h^{-2}) & O(h^{-2}) & O(h^{-2}) \\
O(h^{-2}) & O(|k|h^{-3}) & O(h^{-2}) & O(|k|h^{-3}) \\
O(h^{-2} (1 + |k|)^{-1}) & O(h^{-2}) & O(h^{-2} (1 + |k|)^{-1}) & O(h^{-2})
\end{pmatrix},
\] (3.59)

Hence, \( \mathcal{M}^h(k, \theta) \) is invertible for all \( k \) provided that \( \theta \) is small depending on \( h \), while, from the representation (3.27), an expansion of \( \psi^h_\theta(k, h) \) for small values of \( \theta \) follows.

**Proposition 3.6** Assume \( h \in (0, h_0], |\theta| \leq h^2 \) and let \( \mathcal{V} \) be defined according to (3.34) for some \( c > 0 \). For a suitably small \( h_0 \), the solutions \( \psi^h_\theta(\cdot, k) \) of the generalized eigenfunctions problem (3.18), (3.19) - (3.20) allow the expansion
\[
\psi^h_\theta(\cdot, k) - \psi^0_\theta(\cdot, k) = \mathcal{O}\left( \frac{\theta k}{1 + k} \right) \mathcal{G}^{[1], h}(\cdot, b) + \mathcal{O}\left( \frac{\theta k}{1 + k} \right) H^{[1], h}(\cdot, a) + \mathcal{O}\left( \frac{\theta k}{1 + k} \right) H^{[1], h}(\cdot, a),
\] (3.60)
where the symbols \( \mathcal{O}(\cdot) \) denote functions of the variables \( \{\theta, k, h\} \in B_{h^2}(0) \times \mathbb{R} \times (0, h_0] \) holomorphic w.r.t. \( \theta \) and continuous in \( k \).

**Proof.** The coefficients of the remainder \( m^h(k, \theta) \) in (3.58) - (3.59), depending on the variables \( \{\theta, k, h\} \), are \( \mathcal{O}(h^{-3}) \); hence, for \( |\theta| \leq h^2 \) and \( h \in (0, h_0] \) with \( h_0 \) suitably small, the expansion (3.58), rephrasing as: \( \mathcal{M}^h(k, \theta) = -2h^{-2}1_{\mathbb{C}^+} + \mathcal{O}(h^{-1}) \), defines an invertible matrix for all \( k \in \mathbb{R} \) and the representation (3.27) globally holds. In particular, from (3.58) - (3.59), a direct computation yields
\[
\det \mathcal{M}^h(k, \theta) = h^{-8} (16 + \mathcal{O}(h)),
\] (3.61)
and
\[
(\mathcal{M}^h(k, \theta))^{-1} = h^2 \begin{pmatrix}
-1/2 + \mathcal{O}(h) & \mathcal{O}(hk) & \mathcal{O}(h) & \mathcal{O}(hk) \\
-1/2 + \mathcal{O}(h) & \mathcal{O}(h) & \mathcal{O}(h) & \mathcal{O}(hk) \\
\mathcal{O}(h) & \mathcal{O}(h) & -1/2 + \mathcal{O}(h) & \mathcal{O}(hk) \\
\mathcal{O}(h) & \mathcal{O}(h) & \mathcal{O}(h) & -1/2 + \mathcal{O}(h)
\end{pmatrix}.
\] (3.62)

From the relations (3.55) and the definitions (3.9), (3.28), follows
\[
\mathcal{B}_h \Gamma^{k, h} = \left\{ \mathcal{O}\left( \frac{\theta k}{h} \right), \mathcal{O}\left( \frac{\theta k}{1 + k} \right), \mathcal{O}\left( \frac{\theta k}{1 + k} \right), \mathcal{O}\left( \frac{\theta k}{h} \right) \right\}.
\] (3.63)
where the symbols \( \mathcal{O}(\cdot) \) are referred to the metric space \( B_{h^2}(0) \times \mathbb{R} \times (0, h_0] \). Making use of the above relations, we get
\[
(\mathcal{M}^h(k, \theta))^{-1} \mathcal{B}_h \Gamma^{k, h} = \left\{ \mathcal{O}\left( \frac{\theta k}{h} \right), \mathcal{O}\left( \frac{\theta k}{1 + k} \right), \mathcal{O}\left( \frac{\theta k}{1 + k} \right), \mathcal{O}\left( \frac{\theta k}{h} \right) \right\}.
\] (3.64)

Then, the expansion (3.60) follows from the formula (3.27) - (3.28) by taking into account (3.64) and the definition of \( g_{k, h, j} \).

### 3.3 Stationary wave operators and uniform-in-time estimates for the dynamical system

Following [3], we next construct a similarity between \( H^h_0 \) and \( H^h_0 \) by making use of the stationary waves operators related to the scattering system \( \{H^h_0, H^h_0\} \). Let us recall that, for potentials defined as in (3.3), the generalized Fourier transform associated to \( H^h_0 \),
\[
(\mathcal{F}^h_0 \varphi)(k) = \int_{\mathbb{R}} \frac{dx}{(2\pi h)^{1/2}} (\psi^h_0(x, k))^* \varphi(x), \quad \varphi \in L^2(\mathbb{R}),
\] (3.65)
is a bounded operator on \( L^2(\mathbb{R}) \) with a right inverse coinciding with the adjoint \( (\mathcal{F}^h_0)^* \)
\[
(\mathcal{F}^h_0)^* f(x) = \int_{\mathbb{R}} \frac{dk}{(2\pi h)^{1/2}} \psi^h_0(x, k) f(k),
\] (3.66)
and it results: \( F^h_a (F^h_b) = 1 \) in \( L^2 (\mathbb{R}) \) in \( L^2 (\mathbb{R}) \), while the product \((F^h_a)^* F^h_b\) defines the projector on the absolutely continuous subspace of \( H^h_a (\text{cf. } [3]) \). In addition, when \( V \) is positive defined, \( H^h_a \) has a purely absolutely continuous spectrum coinciding with \( \mathbb{R}_+ \); in this case \( F_V \) is an unitary map with range \( L^2 (\mathbb{R}) \) and the representation: \( 1_{L^2 (\mathbb{R})} = (F^h_a)^* (F^h_b) \) holds. According to the above notation, the standard Fourier transform operator, corresponding to the case \( V = 0 \), is next denoted with \( F^h_0 \). We consider the maps \( \phi^h_\alpha \) and \( \psi^h_\alpha \), acting on \( L^2 (\mathbb{R}) \) as

\[
\phi^h_\alpha (\varphi, f) = \int_{\mathbb{R}} \frac{dk}{(2\pi h)^{1/2}} f(k) G^{(k), h}_\alpha (\cdot, \varphi) (F^h_\varphi) (k), \quad \alpha \in \{a, b\},
\]

\[
\psi^h_\alpha (\varphi, f) = \int_{\mathbb{R}} \frac{dk}{(2\pi h)^{1/2}} g(k) H^{(k), h}_\alpha (\cdot, \varphi) (F^h_\varphi) (k), \quad \alpha \in \{a, b\}.
\]

Here \( G^{k,h} \) and \( H^{k,h} \) are the limits of the Green’s functions on the branch cut (see the definition in (3.74)-(3.75)), while \( f \) is an auxiliary function, possibly depending on \( h \) and \( \theta \) aside from \( k \).

**Lemma 3.7** Let \( h \in (0, h_0) \) and \( V \) be defined according to (3.37) with \( h_0 \) suitably small. Assume \( f_{j=1,2} \in L^2 (\mathbb{R}) \) such that: \( f = \mathcal{O} (k) \) and \( g = \mathcal{O} \left( \frac{k}{1+k} \right) \). Then it results

\[
h \left\| \phi^h_\alpha (\cdot, f) \right\|_{L^2 (\mathbb{R})} + \left\| \psi^h_\alpha (\cdot, g) \right\|_{L^2 (\mathbb{R})} \leq C_{a,b,c} h^{-2},
\]

and

\[
h \left\| \phi^h_\alpha (\cdot, f) \right\|_{L^2 (\mathbb{R})} + \left\| \psi^h_\alpha (\cdot, g) \right\|_{L^2 (\mathbb{R})} \leq C_{a,b,c} h^{-2},
\]

where \( C_{a,b,c} \) is a positive constant depending on the data.

**Proof.** We show that each of the maps \( \phi^h_\alpha (\cdot, f) \) and \( \psi^h_\alpha (\cdot, g) \), \( \alpha = a, b \), can be expressed as a superpositions of terms having the following form

\[
1_{\{x \geq 0\}} T^h_\alpha (\mu_1 + \mathcal{P} \mu_2) F^h_a + 1_{\{x < 0\}} T^h_\alpha (\mu_3 + \mathcal{P} \mu_4) F^h_b,
\]

where \( \mu_i \in L^\infty (\mathbb{R}) \), \( T^h_\alpha = (F^h_0)^* \) or \( T^h_\alpha = (F^h_b)^* \) depending on \( \alpha = a, b \), while \( \mathcal{P} \) denotes the parity operator: \( \mathcal{P} u(t) = u(-t) \). The estimate (3.69) is a direct consequence of this representation. Let us focus on the case \( \alpha = b \) and explicitly consider \( \phi^h_\alpha (\cdot, f) \). As it follows from (3.24)-(3.21), the functions \( G^{(k), h}_\alpha (\cdot, b) \) and \( H^{(k), h}_\alpha (\cdot, b) \) allow the representations

\[
G^{(k), h}_\alpha (x, b) = 1_{\{x \geq 0\}} (x) \frac{1}{h^2 w^h (|k|)} e^{i\frac{h}{2} \frac{x}{|k|}} \varphi^b_0 (b, |k|) \right] + 1_{\{x < 0\}} (x) \left( -\frac{1}{2} \frac{x}{|k|} \right) \psi^b_0 (b, |k|) e^{i\frac{h}{2} \varphi^b_0} \right),
\]

\[
H^{(k), h}_\alpha (x, b) = 1_{\{x \geq 0\}} (x) \frac{1}{h^2 w^h (|k|)} e^{i\frac{h}{2} \frac{x}{|k|}} \partial_1 \varphi^b_0 (b, |k|) \right] + 1_{\{x < 0\}} (x) \left( -\frac{|k|}{2h^2} \right) \psi^b_0 (b, |k|) e^{i\frac{h}{2} \varphi^b_0} \right).
\]

The condition \( f(k) = \mathcal{O} (k) \) and the estimates (3.45) implies: \( (h^2 w^h (k))^{-1} f(k) \varphi^b_0 (b, |k|) = \mathcal{O} (h^{-1}) \) and \( \frac{1}{2h^2} f(k) e^{i\frac{h}{2} \varphi^b_0} = \mathcal{O} (h^{-1}) \); thus, using (3.72) for \( x \geq b \) we get

\[
1_{\{x \geq b\}} \phi^h_\alpha (\varphi, f) =
\]

\[
1_{\{x \geq b\}} \left( \int_0^{+\infty} \frac{dk}{(2\pi h)^{1/2}} \mathcal{O} (h^{-1}) e^{i\frac{h}{2} \varphi^b_0} (F^h_\varphi) (k) + \int_{-\infty}^{0} \frac{dk}{(2\pi h)^{1/2}} \mathcal{O} (h^{-1}) e^{-i\frac{h}{2} \varphi^b_0} (F^h_\varphi) (k) \right).
\]

The previous identity rephrases as

\[
1_{\{x \geq b\}} \phi^h_\alpha (\varphi, f) = 1_{\{x \geq b\}} (F^h_0)^* \left[ 1_{\{k \geq 0\}} (k) \left( \mathcal{O} (h^{-1}) + \mathcal{P} \mathcal{O} (h^{-1}) \right) F^h_\varphi \right],
\]

where the symbols \( \mathcal{O} (\cdot) \), denoting functions of the variables \( k \) and \( h \), are defined in the sense of the metric space \( \mathbb{R} \times (0, h_0] \). Using (3.72) for \( x < b \) leads to

\[
1_{\{x \leq b\}} \phi^h_\alpha (\varphi, f) =
\]

\[
1_{\{x < b\}} \left( \int_0^{+\infty} \frac{dk}{(2\pi h)^{1/2}} \mathcal{O} (h^{-1}) \psi^b_0 (\cdot, -k) (F^h_\varphi) (k) + \int_{-\infty}^{0} \frac{dk}{(2\pi h)^{1/2}} \mathcal{O} (h^{-1}) \psi^b_0 (\cdot, k) (F^h_\varphi) (k) \right),
\]

(3.76)
and, proceeding as before, we get
\[
1_{\{x < b\}} h(\varphi, f) = 1_{\{x < b\}} (\mathcal{F}_h) \ast \left[ 1_{\{k < 0\}} (k) (\mathcal{P} \circ \mathcal{O} (h^{-1}) + \mathcal{O} (h^{-1})) \right] (\mathcal{F}_h) \varphi .
\] (3.77)

From (3.75) and (3.77) we get a representation of the type given in (3.71)
\[
\phi_h^b (\varphi, f) = 1_{\{x > b\}} (\mathcal{F}_h) \ast \left[ 1_{\{k > 0\}} (k) (\mathcal{P} \circ \mathcal{O} (h^{-1}) + \mathcal{O} (h^{-1})) \right] (\mathcal{F}_h) \varphi
\]
+ \left[ 1_{\{x < b\}} (\mathcal{F}_h) \ast \left[ 1_{\{k < 0\}} (k) (\mathcal{P} \circ \mathcal{O} (h^{-1}) + \mathcal{O} (h^{-1})) \right] (\mathcal{F}_h) \varphi .
\] (3.78)

From which it follows: \(|\phi_h^b (\varphi, f)|_{L^2(\mathbb{R})} \leq \frac{1}{h} |\varphi|_{L^2(\mathbb{R})} .\) Recall that the generalized Fourier transform \(\mathcal{F}_h^b\) is a bounded map from \(H^2 (\mathbb{R})\) to the weighted space \(L^{2,2} (\mathbb{R})\), defined by
\[
L^{2,2} (\mathbb{R}) = \left\{ u \in L^2 (\mathbb{R}) : (1 + k^2)^{\alpha/2} u \in L^2 (\mathbb{R}) \right\} ,
\] (3.79)

namely, we have
\[
(\mathcal{F}_h^b)^* \in \mathcal{B} (H^2 (\mathbb{R}), L^{2,2} (\mathbb{R})) , \quad \text{and} \quad (\mathcal{F}_h^b)^* \in \mathcal{B} (L^{2,2} (\mathbb{R}), H^2 (\mathbb{R})) .
\] (3.80)

Since \(h (\mathcal{P} \circ \mathcal{O} (h^{-1}))\) is bounded on \(L^{2,2} (\mathbb{R})\) uniformly w.r.t. \(h \in (0, h_0]\), from (3.78) we also have: \(|\phi_h^b (\varphi, f)|_{H^2 (\mathbb{R})} \leq \frac{1}{h} |\varphi|_{L^2 (\mathbb{R})} .\) In the case of \(\psi_h^b (\varphi, g)\), the representation (3.71) allows similar computations leading to: \(|\psi_h^b (\varphi, g)|_{L^2(\mathbb{R})} \leq \frac{1}{h^2} |\varphi|_{L^2 (\mathbb{R})} \text{ and } |\psi_h^b (\varphi, g)|_{H^2 (\mathbb{R})} \leq \frac{1}{h^2} |\varphi|_{H^2 (\mathbb{R})} .\) For \(\alpha = \alpha\), a representation of the type (3.71) for the maps (3.67)-(3.68) is obtained by a suitably adaptation of the previous arguments. \(\blacksquare\)

The stationary waves operators \(\mathcal{W}_0^h\) are defined by the integral kernel
\[
\mathcal{W}_0^h (x, y) = \int_{\mathbb{R}} \frac{dk}{2\pi h} \psi_0^h (x, k) \psi_h^0 (x, k) * .
\] (3.81)

These have been considered in a slightly different framework in [10], where, using an energy cutoff (corresponding to a cutoff in \(k\) in (3.81) and suitable spectral assumptions, estimates of the type (3.66) are obtained and a small-\(\theta\) expansion of \(\mathcal{W}_0^h\) is provided with for \(h \in (0, h_0]\) (see [10] Lemma 4.2 and Proposition 4.3)). In our setting, the positivity condition (3.34) allows generalizing this result as follows.

**Proposition 3.8** Let \(h \in (0, h_0]\), with \(h_0\) suitably small, \(V\) be defined according to (3.74) and \(|\theta| \leq h^{N_0}\), with \(N_0 > 2\). Then \(\{\mathcal{W}_0^h, \theta \in B_{h^{N_0}} (0)\}\) form an analytic family of bounded and invertible operators on \(L^2 (\mathbb{R})\) fulfilling the expansion
\[
||\mathcal{W}_0^h - 1_{L^2 (\mathbb{R})}\|_{\mathcal{L}(L^2 (\mathbb{R}))} + \| (\mathcal{W}_0^h)^{-1} - 1_{L^2 (\mathbb{R})}\|_{\mathcal{L}(L^2 (\mathbb{R}))} \leq C_{a,b,c} h^{N_0-2} ,
\] (3.82)

where \(C_{a,b,c}\) is a positive constant depending on the data.

For each \(\theta \in B_{h^{N_0}} (0), \mathcal{W}_0^h \in \mathcal{B} (H^2 (\mathbb{R}), H^2 (\mathbb{R})\setminus \{a, b\})\) with ran \(\mathcal{W}_0^h \cap H^2 (\mathbb{R})\) = \(D (\Delta_0)\) and it results
\[
H_0^h \mathcal{W}_0^h = \mathcal{W}_0^h H_0^h .
\] (3.83)

In particular, \(\mathcal{W}_0^h\) is an isomorphism: \(H^2 (\mathbb{R}) \to D (\Delta_0)\) (considered as an Hilbert subspace of \(H^2 (\mathbb{R})\setminus \{a, b\}\)).

**Proof.** Due to the assumptions: \(|\theta| < h^2\), the formula (3.60) applies and the action of \(\mathcal{W}_0^h\) on \(\varphi \in L^2 (\mathbb{R})\) writes as
\[
\mathcal{W}_0^h \varphi = (\mathcal{F}_h) \ast (\mathcal{F}_h^b) \varphi (k) + \sum_{\alpha=a,b} \int_{\mathbb{R}} \frac{dk}{2\pi h} \left[ \mathcal{O} (\theta k/h) G_{\{k, \theta, h\}} (\alpha, \alpha) + \mathcal{O} (\theta k/(1 + k)) H_{\{k, \theta, h\}} (\alpha, \alpha) \right] (\mathcal{F}_h^b) \varphi (k) .
\] (3.84)

where \(\mathcal{O} (\cdot)\) here denote bounded functions of the variables \(\{k, \theta, h\}\), holomorphic w.r.t. \(\theta\). With the notation introduced in (3.67)-(3.68), the identities: \(\mathcal{O} (g) = g \mathcal{O} (1)\) (see the definition 2.3) and \(1_{L^2 (\mathbb{R})} = (\mathcal{F}_h^b) \ast (\mathcal{F}_h^b)\) yield
\[
(\mathcal{W}_0^h - 1_{L^2 (\mathbb{R})}) \varphi = \sum_{\alpha=a,b} \left[ \frac{\theta}{h} \phi_0^b (\varphi, O (k)) + \theta \phi_0^b (\varphi, O (k (1 + k)^{-1})) \right] ,
\] (3.85)

Then, (3.69) applies to the r.h.s. of (3.85) and using \(|\theta| < h^{N_0}\), we conclude that
\[
||\mathcal{W}_0^h - 1_{L^2 (\mathbb{R})}\|_{\mathcal{L}(L^2 (\mathbb{R}))} \leq C_{a,b,c} h^{N_0-2} .
\] (3.86)
Then, for \( h_0 \) suitably small, \( W^h_\theta \in \mathcal{B}(L^2(\mathbb{R})) \) is invertible and \( (W^h_\theta)^{-1} \) fulfills an analogous estimate; this yields (3.82). The action of \( W^h_\theta \) over \( L^2(\mathbb{R}) \) is defined using the expansion (3.85). Each of the maps \( \phi^h_\alpha(\cdot, \mathcal{O}(k)) \) and \( \psi^h_\alpha(\cdot, \mathcal{O}(k(1 + k)^{-1})) \), appearing in this formula expresses as a superposition of the form (cf. (3.71))

\[
1_{\{x > \alpha\}} T^h_\alpha (\mu_{1,\alpha} + \mathcal{P} \phi_{1,\alpha}) + 1_{\{x < \alpha\}} T^h_\alpha (\mu_{3,\alpha} + \mathcal{P} \mu_{4,\alpha})
\]

(3.87)

\( \mu_{1,\alpha} \) being, in our case, bounded functions of \( \{k, \theta, h\} \), holomorphic w.r.t. \( \theta \). Thus, \( \phi^h_\alpha(\cdot, \mathcal{O}(k)) \) and \( \psi^h_\alpha(\cdot, \mathcal{O}(k(1 + k)^{-1})) \) define holomorphic families of bounded maps on \( L^2(\mathbb{R}) \) and, due to (3.82), this still holds for \( W^h_\theta \).

Let us consider the action of \( W^h_\theta \) on \( H^2(\mathbb{R}) \); using (3.86), (3.70), we get

\[
\|W^h_\theta - 1_{H^2(\mathbb{R})}\|_{L(H^2(\mathbb{R}), H^2(\mathbb{R} \setminus \{a, b\}))} \leq C_{\alpha, h} e^{h N_0^{-2}}.
\]

(3.88)

This implies: \( W^h_\theta \in \mathcal{B}(H^2(\mathbb{R}), H^2(\mathbb{R} \setminus \{a, b\})) \), while, from the definitions (3.18) and (3.81), \( W^h_\theta \varphi \) fulfills the interface conditions (1.1); it follows

\[
\text{ran } (W^h_\theta \mid H^2(\mathbb{R})) \subseteq D(\Delta_\theta).
\]

(3.89)

Let \( \varphi \in H^2(\mathbb{R}) \); using the functional calculus of \( H^h_\theta \), we have: \( (F^h_\theta(H^h_\theta \varphi)) (k) = k^2 (F^h_\theta \varphi) (k) \), and, from the definition (3.81), the r.h.s. of (3.83) writes as

\[
W^h_\theta H^h_\theta \varphi = \int_{\mathbb{R}} \frac{dk}{2\pi h} \psi^h_\theta(\cdot, k) k^2 (F^h_\theta \varphi) (k).
\]

(3.90)

Using once more the definition (3.81) and the relation: \( (H^h_\theta - k^2) \psi^h_\theta(\cdot, k) = 0 \), the l.h.s. of (3.83) identifies with

\[
H^h_\theta W^h_\theta \varphi = \int_{\mathbb{R}} \frac{dk}{2\pi h} \psi^h_\theta(\cdot, k) k^2 (F^h_\theta \varphi) (k).
\]

(3.91)

From (3.90)–(3.91) we get the intertwining relation (3.83). Since \( (W^h_\theta)^{-1} \) exists, we also have: \( (W^h_\theta)^{-1} H^h_\theta = H^h_\theta (W^h_\theta)^{-1} \), which implies

\[
\text{ran } (W^h_\theta)^{-1} \mid D(\Delta_\theta) \subseteq H^2(\mathbb{R}).
\]

(3.92)

From (3.89) and (3.92), follows: \( \text{ran } (W^h_\theta \mid H^2(\mathbb{R})) = D(\Delta_\theta) \). Thus, the restriction \( W^h_\theta \mid H^2(\mathbb{R}) \) is injective (since \( (W^h_\theta)^{-1} \) exists in \( L(L^2(\mathbb{R})) \)), and surjective onto \( \text{ran } (W^h_\theta \mid H^2(\mathbb{R})) = D(\Delta_\theta) \).

**Remark 3.9** The explicit bounds for the factors \( \mathcal{O}(\cdot) \) appearing in (3.69) depend on the trace estimates provided by the Proposition 3.4. According to the Remark 3.5, these are independent from the potential’s profile, provided that the assumptions are fulfilled. As a consequence, the expansion (3.69), as well as the relations (3.69)–(3.70) and the expansion (3.86) hold uniformly w.r.t. any family of potentials for which the conditions (3.37) hold for a fixed \( c > 0 \).

Due to the result of the Proposition (3.8) \( W^h_\theta \) is an invertible map as far as \( h \in (0, h_0) \) and \( |\theta| \leq h^{N_0} \) (with \( N_0 > 2 \) and \( h_0 \) small); under these conditions, the intertwining property (3.83) yields a similarity between \( H^h_\theta \) and \( H^h_0 \); this allows to define the quantum dynamics generated by \( H^h_\theta \) by conjugation.

**Proof of the Theorem**. The first part of the statement follows from the Proposition 3.8.

For the second part, let us introduce

\[
e^{-it H^h_\theta} = W^h_\theta e^{-it H^h_0} (W^h_\theta)^{-1}.
\]

(3.93)

Under our assumptions on the potential, \( i H^h_\theta \) generates a strongly continuous group of unitary maps both on \( L^2(\mathbb{R}) \) and on \( H^2(\mathbb{R}) \). According to the results of the Proposition 3.8 \( W^h_\theta \) is bounded and invertible on \( L^2(\mathbb{R}) \), while its restriction \( \text{ran } (W^h_\theta \mid H^2(\mathbb{R})) \in \mathcal{B}(H^2(\mathbb{R}), H^2(\mathbb{R} \setminus \{a, b\})) \) has \( \text{ran } (W^h_\theta \mid H^2(\mathbb{R})) = D(\Delta_\theta) \). Hence, \( W^h_\theta \) is a bijection: \( H^2(\mathbb{R}) \to D(\Delta_\theta) \) and the modified propagator \( e^{-it H^h_\theta} \) is strongly continuous both on \( L^2(\mathbb{R}) \) and \( D(\Delta_\theta) \) (w.r.t. the corresponding topologies). Moreover, from the identity: \( i \partial_t e^{-it H^h_\theta} \psi = H^h_\theta e^{-it H^h_\theta} \psi \), holding in \( L^2(\mathbb{R}) \) for any \( \psi \in H^2(\mathbb{R}) \), it follows

\[
i \partial_t (e^{-it H^h_\theta} u) = H^h_\theta e^{-it H^h_\theta} u, \quad u \in D(\Delta_\theta).
\]

(3.94)

Then \( e^{-it H^h_\theta} \) identifies with the quantum dynamical system generated by \( i H^h_\theta \).

Finally, since \( W^h_\theta \) and \( (W^h_\theta)^{-1} \) are analytic w.r.t. \( \theta \), \( e^{-it H^h_\theta} \) has the same regularity and the expansion (3.93) follows from (3.82).
4 The time dependent case

We consider the time dependent family of modified operators $H^b_\theta(t)$ defined according to (2.4) when the potential is a continuous function of the time fulfilling the conditions

$$V(t) \in C^0([0,T], L^\infty(\mathbb{R}, \mathbb{R})),$$

$$\text{supp } V(t) = [a,b], \quad 1_{[a,b]} V(t) > c,$$  \hspace{1cm} (4.1)

for a suitable $c > 0$. The $\theta$-dependent time propagator $U^{b}_\theta(t, s)$ associated to $H^b_\theta(t)$ solves the evolution problem

$$i\partial_t U^{b}_\theta(t, s) u = H^b_\theta(t) U^{b}_\theta(t, s) u,$$

$$U^{b}_\theta(s, s) u = u, \quad u \in D(\Delta_\theta), \quad 0 \leq s \leq t \leq T.$$  \hspace{1cm} (4.2)

A standard strategy in the definition of the quantum dynamical system generated by a non-autonomous Hamiltonian, consists in using an approximating sequence whose terms are stepwise products of propagators associated to the 'instantaneous' Hamiltonians (cf. [14]). This approach requires stability estimates for the product of the instantaneous propagators in suitable spaces.

4.1 Stability estimates

In the following, $D(\Delta_\theta)$ is considered as an Hilbert subspace of $H^2(\mathbb{R} \setminus \{a,b\})$ (see the definition (1.1)) and the notation $\mathcal{L}(D(\Delta_\theta))$ refers to the linear operators on $D(\Delta_\theta)$ w.r.t. its topology.

In the time dependent case, the instantaneous propagators $e^{-itH^b_\theta(s)}$ verify the relations

$$e^{-itH^b_\theta(s)} = \mathcal{W}^b_\theta(s) e^{-itH^b_\theta(s)} (\mathcal{W}^b_\theta(s))^{-1},$$  \hspace{1cm} (4.3)

where the maps $\mathcal{W}^b_\theta(t)$ now depend on time according to $V(t)$. Let us recall from results of the Proposition 3.8 that, for any $t \in [0, T]$, $\mathcal{W}^b_\theta(t)$ is bounded and invertible on $L^2(\mathbb{R})$, while its restriction to $H^2(\mathbb{R})$ is a bijection: $H^2(\mathbb{R}) \rightarrow D(\Delta_\theta)$. In particular, following the Remark 3.9, from the estimate (3.86) we get

$$\sup_{t \in [0,T], \theta \in (0,h_0]} \left\{ \left\| \mathcal{W}^b_\theta(t) \right\|_{\mathcal{L}(L^2(\mathbb{R}))} + \left\| (\mathcal{W}^b_\theta(t))^{-1} \right\|_{\mathcal{L}(L^2(\mathbb{R}))} \right\} \leq A_{a,b,c},$$  \hspace{1cm} (4.4)

for some $A_{a,b,c} > 0$ depending on the data, provided that $h_0$ is suitably small and $|\theta| \leq h^{N_0}$, with $N_0 > 2$. The selfadjointness of $H^b_\theta(s)$ then yields

$$\sup_{t \in [0,T], s \in [0,T], \theta \in (0,h_0]} \left\| e^{-itH^b_\theta(s)} \right\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq K_{a,b,c}, \quad \forall \; |\theta| \leq h^{N_0}, \; N_0 > 2,$$  \hspace{1cm} (4.5)

for a suitable $K_{a,b,c} > 0$. Under the same assumptions, the estimate (3.88) and the Remark 3.9 suggest

$$\sup_{t \in [0,T], \theta \in (0,h_0]} \left\| \mathcal{W}^b_\theta(t) \right\|_{\mathcal{L}(H^2(\mathbb{R}), H^2(\mathbb{R} \setminus \{a,b\}))} \leq B_{a,b,c},$$  \hspace{1cm} (4.6)

According to the results of the Proposition 3.8 ran $(\mathcal{W}^b_\theta(t) \upharpoonright H^2(\mathbb{R})) = D(\Delta_\theta)$ for any $t$ and $(\mathcal{W}^b_\theta(t))^{-1}$ exists in $\mathcal{B}(D(\Delta_\theta), H^2(\mathbb{R}))$. Then (4.6) yields

$$\sup_{t \in [0,T], \theta \in (0,h_0]} \left\| (\mathcal{W}^b_\theta(t))^{-1} \right\|_{\mathcal{L}(D(\Delta_\theta), H^2(\mathbb{R}))} \leq C_{a,b,c}.$$  \hspace{1cm} (4.7)

Since $e^{-itH^b_\theta(s)}$ defines a unitary (strongly continuous) flow on $H^2(\mathbb{R})$, from the definition (1.3) we obtain

$$\sup_{t \in [0,T], \theta \in (0,h_0]} \left\| e^{-itH^b_\theta(s)} \right\|_{\mathcal{L}(D(\Delta_\theta))} \leq \tilde{K}_{a,b,c}, \quad \forall \; |\theta| \leq h^{N_0}, \; N_0 > 2.$$  \hspace{1cm} (4.8)

For $T > 0$ we next introduce the partition $[0,T] = \bigcup_{j=1}^n [t_{j-1}, t_j]$, where $t_j = jT/n$ and $t_0 = 0$; the step-propagators $U^{b}_{\theta,n}(t, s)$ are defined by

$$U^{b}_{\theta,n}(t, s) = \begin{cases} e^{-i(t-s)H^b_\theta(t_j)}, & \text{if } s \in [t_{j-1}, t_j], \\ U^{b}_{\theta,n}(t, k+j-1) U^{b}_{\theta,n}(k+j-1, k+j-2) \cdots U^{b}_{\theta,n}(t_j, s), & \text{if } s \in [t_{k+1}, t_{k+j}], \; t \in [t_{k-1}, t_{k+j}]. \end{cases}$$  \hspace{1cm} (4.9)
Under the assumption of the Theorem 2.1 each factor in \( U^h_{\theta,n}(t,s) \) defines a \( \theta \)-holomorphic family of bounded operators on \( L^2(\mathbb{R}) \), strongly continuous w.r.t. the time variables. Then for each \( n \), \( U^h_{\theta,n}(t,s) \) is \( \theta \)-holomorphic and strongly continuous in \( t \) and \( s \) on \( L^2(\mathbb{R}) \), while, according to its definition, we have

\[
U^h_{\theta,n}(s,s) = 1_{L^2(\mathbb{R})}, \quad U^h_{\theta,n}(t,s) = U^h_{\theta,n}(t,r) U^h_{\theta,n}(r,s), \quad \forall \, s \leq r \leq t.
\] (4.10)

The result in Theorem 2.1 also imply that each factor in \( U^h_{\theta,n}(t,s) \) is bounded on \( D(\Delta_\theta) \) and strongly continuous in the time variables (w.r.t. the \( \mathcal{L}(D(\Delta_\theta)) \) topology). Thus, \( U^h_{\theta,n}(t,s) \) is bounded strongly continuous in \( t \) and \( s \) on \( D(\Delta_\theta) \) and introducing: \( H^h_{\theta,n}(t) = H^h_{\theta}(\frac{T}{n} \, \lfloor \frac{t}{T} \rfloor) \) (\( \lfloor \cdot \rfloor \) denotes the floor function), from (3.94) the identity

\[
i\partial_t U^h_{\theta,n}(t,s) u = H^h_{\theta,n}(t) U^h_{\theta,n}(t,s) u, \quad \forall \, 0 \leq s \leq t \leq T, \quad u \in D(\Delta_\theta),
\] (4.11)

holds in \( L^2(\mathbb{R}) \). The additional condition (see the definition (2.7))

\[
V(t) - V(s) \in W^{2,\infty}_0([a,b]), \quad \forall \, t, s \in [T,0).
\] (4.12)

is next used to obtain stability estimates for the sequence \( U^h_{\theta,n}(t,s) \).

**Lemma 4.1** Let \( V(t) \) fulfills the conditions (4.7), \( h \in (0,h_0] \), with \( h_0 \) suitably small, and \( |\theta| \leq h^{N_0} \), with \( N_0 > 2 \). There exist \( C_{a,b,c} \) and \( \bar{C}_{a,b,c} \) positive and possibly depending on the data, such that

\[
\sup_{t,s \in [0,T]} \| U^h_{\theta,n}(t,s) \|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \exp \left( C_{a,b,c} \sup_{t \in [0,T]} \| V(t) \|_{L^\infty(\mathbb{R})} \right), \quad t \geq s.
\] (4.13)

If in addition (4.12) holds, then

\[
\sup_{t,s \in [0,T]} \| U^h_{\theta,n}(t,s) \|_{\mathcal{L}(D(\Delta_\theta))} \leq \exp \left( \bar{C}_{a,b,c} \sup_{t \in [0,T]} \| V(t) - V(s) \|_{W^{2,\infty}(a,b)} \right), \quad t \geq s.
\] (4.14)

**Proof.** Let us fix \( j \in \{1, \ldots, n\} \), \( s \in [t_j-1, t_j) \) and consider \( U^h_{\theta,n}(t,s) \); if \( t \in [t_j-1, t_j) \), then (4.13) is a consequence of (4.9). If \( t \in [t_{k+1}, t_{k+2}) \) for some \( k \in \{1, \ldots, n-j\} \), then \( U^h_{\theta,n}(t,s) \) is a product of \( k+1 \) terms and writes as

\[
U^h_{\theta,n}(t,s) = e^{-i(t-t_{k+1})H^h_{\theta}(t_{k+1})} \left( \prod_{\ell=1}^{k-1} U^h_{\theta,n}(t_{k+\ell-1}, t_{k+\ell-1}) \right) e^{-i(t-s)H^h_{\theta}(t_j-1)}
\] (4.15)

Let \( m \in \{1, \ldots, n\} \) and

\[
I^h_{\theta,n}(\tau, r, m) = -i \int_r^\tau e^{-i(x-r)H^h_{\theta}(m-1)} \left( V(t_{m-1}) - V(t_{j-1}) \right) e^{-i(x-s)H^h_{\theta}(m-1)} \, dx.
\] (4.16)

Each factor in \( U^h_{\theta,n}(t,s) \) allows the representation

\[
U^h_{\theta,n}(\tau, r, m) = e^{-i(t-r)H^h_{\theta}(m-1)} = e^{-i(t-r)H^h_{\theta}(t_j-1)} + I^h_{\theta}(\tau, r, m), \quad \tau, r \in [t_{m-1}, t_m],
\] (4.17)

and the identity (4.13) rephrases as

\[
U^h_{\theta,n}(t,s) = \left( e^{-i(t-t_{k+1})H^h_{\theta}(t_{k+1})} + I^h_{\theta}(t, t_{k+1}, k+j) \right) \circ \left( \prod_{\ell=1}^{k-1} \left( e^{-i(t_{k+\ell-1}-t_{k+\ell-1})H^h_{\theta}(t_{k+\ell-1})} + I^h_{\theta}(t_{k+\ell-1}, t_{k+\ell-1}, k+j) \right) \right)
\] (4.18)

To simplify the notation let us fix \( n_0 = k + j \leq n \) and assume, without loss of generality, assume that \( j = 1 \) (which implies \( t_{j-1} = t_0 = 0 \)); it follows

\[
U^h_{\theta,n}(t,s) = \left( e^{-i(t-t_{n-1})H^h_{\theta}(0)} + I^h_{\theta}(t, t_{n-1}, n_0) \right) \circ \left( \prod_{\ell=1}^{n_0-2} \left( e^{-i(t_{n-\ell-1}-t_{n-\ell-1})H^h_{\theta}(0)} + I^h_{\theta}(t_{n-\ell-1}, t_{n-\ell-1}, n_0 - \ell) \right) \right) \left( e^{-i(t_{1-1})H^h_{\theta}(0)} + I^h_{\theta}(t_1, s, 1) \right)
\] (4.19)
Each contribution to the sum obtained by expanding this product of $n_0$ binomials is a product of $n_0$ factors corresponding either to the quantum propagators associated to $H^h_\theta (0)$ either to the operators $I^h_\theta$. Recalling that in $\left( \frac{n_0}{m} \right)$ terms of this sum the factors $I^h_\theta$ appear $m$ times, we get

$$U^h_{\theta,n}(t,s) = \sum_{m=0}^{n_0} \sum_{\ell=1}^{b_m} F^h_{\theta,m}(t,s,\ell), \quad b_m = \left( \frac{n_0}{m} \right), \quad (4.20)$$

where $F^h_{\theta,m}(t,s,\ell)$, possibly depending on $t$ and $s$, denote the contributions to $U^h_{\theta,n}(t,s)$ where $m$ terms of the type $I^h_\theta$ appear; using the group properties of $e^{-i\tau H^h_\theta(0)}$ and the definition (4.10), these are factorized according to

$$F^h_{\theta,m}(t,s,\ell) = \left[ \prod_{p=1}^{m} \left( -i \int_{t_{p-1}}^{t_p} e^{-i(\tau_p-x)H^h_\theta(0)} \left( \mathcal{V}(t_{p-1}) - \mathcal{V}(0) \right) e^{-i(x-r_p)H^h_\theta(t_{p-1})} dx \right) \right] e^{-i(z_p-s)H^h_\theta(0)}, \quad (4.21)$$

being $j_p$ is a strictly decreasing subsequence of $\{1,2,...,n_0\}$ and $\tau_p \geq \tau_{p+1}$, $z_p \geq s$ suitable values in $[0,T]$ depending on $t$ and $s$.

The estimate (4.3) and

$$\left\| \int_{t_{p-1}}^{t_p} e^{-i(\tau_p-x)H^h_\theta(0)} \left( \mathcal{V}(t_{p-1}) - \mathcal{V}(0) \right) e^{-i(x-r_p)H^h_\theta(t_{p-1})} dx \right\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \frac{2K^2_{a,b,c}}{n} \sup_{t\in[0,T]} \|\mathcal{V}(t)\|_{L^\infty(\mathbb{R})}, \quad (4.22)$$

imply

$$\|F^h_{\theta,m}(t,s,\ell)\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \left( \frac{C_{a,b,c}}{n} \sup_{t\in[0,T]} \|\mathcal{V}(t)\|_{L^\infty(\mathbb{R})} \right)^m, \quad (4.23)$$

for some $C_{a,b,c} > 0$ depending on the data; setting: $C_{a,b,c,\mathcal{V}} = C_{a,b,c} \sup_{t\in[0,T]} \|\mathcal{V}(t)\|_{L^\infty(\mathbb{R})}$, from (4.20) follows

$$\|U^h_{\theta,n}(t,s)\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \sum_{n=0}^{n_0} \left( \frac{n_0}{m} \right) \left( \frac{C_{a,b,c,\mathcal{V}}}{n} \right)^m \leq \sum_{n=0}^{n_0} \left( \frac{n}{m} \right) \left( \frac{C_{a,b,c,\mathcal{V}}}{n} \right)^m = \left( 1 + \frac{C_{a,b,c,\mathcal{V}}}{n} \right)^n. \quad (4.24)$$

Since this bound holds independently from $t,s \in [0,T]$ and $h \in (0,h_0)$ provided that $|\theta| \leq h^{N_0}$, with $N_0 > 2$, we get

$$\sup_{t,s \in [0,T]} \|U^h_{\theta,n}(t,s)\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \left( 1 + \frac{C_{a,b,c,\mathcal{V}}}{n} \right)^n, \quad t \geq s. \quad (4.25)$$

Then, the uniform estimate (4.13) follows from the limit: $\lim_{n \to \infty} \left( 1 + \frac{C_{a,b,c,\mathcal{V}}}{n} \right)^n = e^{C_{a,b,c,\mathcal{V}}}$.

Let us remark that a function $\psi \in L^\infty(\mathbb{R}) \cap W^{2,\infty}_0([a,b])$ such that: $\text{supp}\ \psi = [a,b]$ is a multiplier of $D(\Delta_\theta)$; in particular, for any $u \in D(\Delta_\theta)$ it results: $\psi u \in H^1_0(\mathbb{R} \setminus \{a,b\}) \subset D(\Delta_\theta)$ and $\|\psi u\|_{H^2(\mathbb{R} \setminus \{a,b\})} \leq \|\psi\|_{W^{2,\infty}_0([a,b])} \|u\|_{H^2(\mathbb{R} \setminus \{a,b\})}$.

Then, the estimate (4.3) and the assumption (4.12) yield

$$\left\| \int_{t_{p-1}}^{t_p} e^{-i(\tau_p-x)H^h_\theta(0)} \left( \mathcal{V}(t_{p-1}) - \mathcal{V}(0) \right) e^{-i(x-r_p)H^h_\theta(t_{p-1})} dx \right\|_{\mathcal{L}(D(\Delta_\theta))} \leq \frac{K^2_{a,b,c}}{n} \sup_{t,s \in [0,T]} \|\mathcal{V}(t) - \mathcal{V}(s)\|_{W^{2,\infty}_0([a,b])}. \quad (4.26)$$

Proceeding as before we have

$$\|F^h_{\theta}(t,s,\ell)\|_{\mathcal{L}(D(\Delta_\theta))} \leq \left( \frac{C_{a,b,c}}{n} \sup_{t,s \in [0,T]} \|\mathcal{V}(t) - \mathcal{V}(s)\|_{W^{2,\infty}_0([a,b])} \right)^m. \quad (4.27)$$

for some $\tilde{C}_{a,b,c} > 0$ depending on the data, and the representation (4.20) - (4.24) lead us to (4.14) \.

**Remark 4.2** The constants $C_{a,b,c}$ and $\tilde{C}_{a,b,c}$ in (4.13), (4.14) do not depend on $T$ once the assumptions (4.1), (4.12) are fulfilled.
4.2 The existence of the dynamics

We next show that $U^h_{\theta,n}(t,s)$ approximates the dynamical system $U^h_{\theta}(t,s)$ introduced in [12]; the proof adapts the strategy used in [8, Theorems 4.1 and 5.1] to our framework.

**Proposition 4.3** Under the assumptions of the Lemma [11, 12] the sequence $U^h_{\theta,n}(t,s)$ uniformly converges in the $\mathcal{L}(L^2(\mathbb{R}))$ topology to a limit operator $U^h_{\theta}(t,s)$ such that

$$
\sup_{t,s \in [0,T], \quad h \in (0,h_0]} \|U^h_{\theta}(t,s)\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \exp \left( C_{a,b,c} \sup_{t \in [0,T]} \|V(t)\|_{L^\infty(\mathbb{R})} \right), \quad t \geq s, \quad (4.28)
$$

Moreover $U^h_{\theta}(t,s) \in \mathcal{L}(D(\Delta_0))$ with

$$
\sup_{t,s \in [0,T], \quad h \in (0,h_0]} \|U^h_{\theta}(t,s)\|_{\mathcal{L}(D(\Delta_0))} \leq \exp \left( \tilde{C}_{a,b,c} \sup_{t,s \in [0,T]} \|V(t) - V(s)\|_{W^{2,\infty}([a,b])} \right), \quad t \geq s. \quad (4.29)
$$

The positive constants $C_{a,b,c}, \tilde{C}_{a,b,c}$ possibly depending on the data, are independent from $T$.

**Proof.** Let $s,t \in [0,T]$ and $t \geq s$; from (4.11), the relation

$$(U^h_{\theta,n}(t,s) - U^h_{\theta,m}(t,s)) u = -i \int_{s}^{t} U^h_{\theta,n}(t,t') \left(H^h_{\theta,n}(t') - H^h_{\theta,m}(t')\right) U^h_{\theta,m}(t',s) u \, dt', \quad (4.30)$$

holds for any $u \in D(\Delta_0)$. The difference at the r.h.s. writes as

$$H^h_{\theta,n}(t') - H^h_{\theta,m}(t') = V \left( \frac{T}{n} \left\lfloor \frac{nt'}{T} \right\rfloor \right) - V \left( \frac{T}{m} \left\lfloor \frac{mt'}{T} \right\rfloor \right), \quad (4.31)$$

and (4.30) rephrases as

$$(U^h_{\theta,n}(t,s) - U^h_{\theta,m}(t,s)) u = -i \int_{s}^{t} U^h_{\theta,n}(t,t') \left(V \left( \frac{T}{n} \left\lfloor \frac{nt'}{T} \right\rfloor \right) - V \left( \frac{T}{m} \left\lfloor \frac{mt'}{T} \right\rfloor \right)\right) U^h_{\theta,m}(t',s) u \, dt'. \quad (4.32)$$

Since both the l.h.s and at the r.h.s. of (4.32) define bounded operators on $L^2(\mathbb{R})$, the density of the inclusion $D(\Delta_0) \subset L^2(\mathbb{R})$ allows to extend this identity to the whole space. Using the result of the Lemma 4.1 we get the estimate

$$\| (U^h_{\theta,n}(t,s) - U^h_{\theta,m}(t,s)) u \|_{L^2(\mathbb{R})} \leq M^2_{a,b,c} \int_{0}^{T} \left\| V \left( \frac{T}{n} \left\lfloor \frac{nt'}{T} \right\rfloor \right) - V \left( \frac{T}{m} \left\lfloor \frac{mt'}{T} \right\rfloor \right) \right\|_{L^\infty(\mathbb{R})} dt', \quad (4.33)$$

while the regularity of $V(t)$ yields

$$\lim_{n,m \to \infty} \left\| V \left( \frac{T}{n} \left\lfloor \frac{nt'}{T} \right\rfloor \right) - V \left( \frac{T}{m} \left\lfloor \frac{mt'}{T} \right\rfloor \right) \right\|_{L^\infty(\mathbb{R})} = 0. \quad (4.34)$$

Hence, for any $u \in L^2(\mathbb{R})$, $U^h_{\theta,n}(t,s) u$ forms a Cauchy sequence in $L^2(\mathbb{R})$, uniformly w.r.t. $t,s \in [0,T]$ and $h \in (0,h_0]$. As a consequence, $U^h_{\theta,n}(t,s) u$ uniformly converges to a limit $U^h_{\theta}(t,s) u$ allowing the bound (see 4.13)

$$\sup_{s,t \in [0,T], \quad h \in (0,h_0]} \| U^h_{\theta}(t,s) u \|_{L^2(\mathbb{R})} \leq \sup_{s,t \in [0,T], \quad n \in \mathbb{N}^*, \quad h \in (0,h_0]} \| U^h_{\theta,n}(t,s) u \|_{\mathcal{L}(L^2(\mathbb{R}))} \| u \|_{L^2(\mathbb{R})}$$

$$\leq \exp \left( C_{a,b,c} \sup_{t \in [0,T]} \| V(t) \|_{L^\infty(\mathbb{R})} \right) \| u \|_{L^2(\mathbb{R})}, \quad (4.35)$$

where, according to the Remark 4.12, $C_{a,b,c} > 0$ is independent from $T$. Then, $U^h_{\theta,n}(t,s) u$ uniformly converges to $U^h_{\theta}(t,s)$ in the $\mathcal{L}(L^2(\mathbb{R}))$ topology with

$$\sup_{s,t \in [0,T], \quad h \in (0,h_0]} \| U^h_{\theta}(t,s) u \|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \exp \left( C_{a,b,c} \sup_{t \in [0,T]} \| V(t) \|_{L^\infty(\mathbb{R})} \right) \| u \|_{L^2(\mathbb{R})}, \quad (4.36)$$
When \( u \in D(\Delta) \), the sequence \( U_{\theta,n}^h(t, s)u \) is uniformly bounded in \( D(\Delta) \) (see (4.14)). Being \( D(\Delta) \) reflexive (as a subspace of \( H^2(\mathbb{R} \setminus \{a, b\}) \)), there exists a subsequence of \( U_{\theta,n}^h(t, s)u \) weakly convergent in \( D(\Delta) \). Hence: \( U_{\theta}^h(t, s)u \in D(\Delta) \), and the bound

\[
\sup_{s, t \in [0, T]} \|U_{\theta}^h(t, s)u\|_{L^2(D(\Delta))} \leq \exp \left( \tilde{C}_{n,h} \sup_{t, s \in [0, T]} \|\mathcal{V}(t) - \mathcal{V}(s)\|_{W^{2, \infty}([a, b])} \right) \|u\|_{D(\Delta)},
\]

follows from (4.14) with \( \tilde{C}_{n,h} \) are independent from \( T \). \( \blacksquare \)

For \( t \in [0, T] \) fixed, let us denote with \( \tilde{U}_{\theta,n}^h \) the semigroup sequence obtained following the construction in the trivial case of the time independent Hamiltonian \( \tilde{H}_\theta^h = H_\theta^h(t) \). It results

\[
\tilde{U}_{\theta,n}^h(t, s) = e^{-i(t-s)H_{\theta}^h} \gamma H_{\theta}^h(t), \quad \forall n \in \mathbb{N}^*, \text{ and } t \geq s,
\]

while using (4.11) for \( \tilde{U}_{\theta,n}^h(t, s) \) and \( U_{\theta,n}^h(t, s) \), we obtain the identity

\[
\left( \tilde{U}_{\theta,n}^h(t, s) - U_{\theta,n}^h(t, s) \right) u = -i \int_s^t \tilde{U}_{\theta,n}^h(t, s) \left( \mathcal{V}(t) - \mathcal{V} \left( \frac{T}{n} \left[ \frac{m}{T} \right] \right) \right) U_{\theta,n}^h(t', s) u dt',
\]

holding for all \( u \in L^2(\mathbb{R}) \) (due the density of the inclusion \( D(\Delta) \subset L^2(\mathbb{R}) \)). Taking the limit as \( n \to \infty \) and using (4.38), the result of the Proposition 4.3 yields

\[
e^{-i(t-s)H_{\theta}^h} \gamma H_{\theta}^h(t) \gamma H_{\theta}^h(t, s) u = -i \int_s^t e^{-i(t-t')H_{\theta}^h(t)} (\mathcal{V}(t) - \mathcal{V}(t')) U_{\theta}^h(t', s) u dt'.
\]

**Theorem 4.4** Under the assumptions of the Lemma 4.1, there exists an unique family of operators \( U_{\theta}^h(t, s) \), strongly continuous in \( t \) and \( s \) w.r.t. the \( L^2(\mathbb{R}) \) topology, fulfilling the identities

\[
U_{\theta}^h(s, s) = 1_{L^2(\mathbb{R})}, \quad U_{\theta}^h(t, s) = U_{\theta}^h(t, r) U_{\theta,n}^h(r, s), \quad \forall s \leq r \leq t,
\]

and such that \( U_{\theta}^h(t, s)u \) is the solution of the problem (4.4) for all \( u \in D(\Delta) \).

**Proof.** For each \( n, U_{\theta,n}^h(t, s) \) is strongly continuous in \( t \) and \( s \) w.r.t. the \( L^2(\mathbb{R}) \) topology, and fulfills the identities (4.10). The uniform convergence of the sequence in \( L^2(\mathbb{R}) \) allows to extend this characterization to its limit \( U_{\theta}^h(t, s) \). Let \( u \in D(\Delta), \tilde{t} \in [0, T] \) and \( \delta > 0 \); the relation (4.40) yields

\[
U_{\theta}^h(t, s)u = e^{-i(t-s)H_{\theta}^h(t)} + \int_s^t e^{-i(t-t')H_{\theta}^h(t')}(\mathcal{V}(t') - \mathcal{V}(t))U_{\theta}^h(t', s) u dt'.
\]

It follows

\[
\left( U_{\theta}^h(t + \delta, s) - U_{\theta}^h(t, s) \right) u = \left( e^{-i(t+\delta-s)H_{\theta}^h(t)} - e^{-i(t-s)H_{\theta}^h(t)} \right) u
\]

\[
+ i \left( e^{-i\delta H_{\theta}^h(t)} - 1 \right) \int_s^t e^{-i(t-t')H_{\theta}^h(t')}(\mathcal{V}(t') - \mathcal{V}(t))U_{\theta}^h(t', s) u dt'
\]

\[
+ \frac{i}{\delta} e^{-i\delta H_{\theta}^h(t)} \int_s^{t+\delta} e^{-i(t-t')H_{\theta}^h(t')}(\mathcal{V}(t') - \mathcal{V}(t))U_{\theta}^h(t', s) u dt'.
\]

with

\[
\left( e^{-i(t-s)H_{\theta}^h(t)} - U_{\theta}^h(t, s) \right) u = -i \int_s^t e^{-i(t-t')H_{\theta}^h(t')}(\mathcal{V}(t') - \mathcal{V}(t))U_{\theta}^h(t', s) u dt'
\]

Since \( d/dt e^{-itH_{\theta}^h(t)}u = -iH_{\theta}^h(t) e^{-itH_{\theta}^h(t)}u \) for all \( u \in D(\Delta) \), we get

\[
\lim_{\delta \to 0^+} \frac{1}{\delta} \left( U_{\theta}^h(t + \delta, s) - U_{\theta}^h(t, s) \right) u = -iH_{\theta}^h(t) e^{-i(t-s)H_{\theta}^h(t)}u
\]

\[
+ H_{\theta}^h(t) \int_s^t e^{-i(t-t')H_{\theta}^h(t')}(\mathcal{V}(t') - \mathcal{V}(t))U_{\theta}^h(t', s) u dt'
\]

\[
+ \lim_{\delta \to 0^+} \frac{i}{\delta} e^{-i\delta H_{\theta}^h(t)} \int_s^{t+\delta} e^{-i(t-t')H_{\theta}^h(t')}(\mathcal{V}(t') - \mathcal{V}(t))U_{\theta}^h(t', s) u dt'.
\]

\[
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\]
which leads to

\[
\lim_{\delta \to 0^+} 1/\delta \left( U^h_0(t + \delta, s) - U^h_0(t, s) \right) u = -i H^h_0(t) U^h_0(t, s) u
\]

\[
+ \lim_{\delta \to 0^+} i/\delta e^{-it\Delta}(t) \int_t^{t+\delta} e^{-i(t-t')\Delta}(t') \left( \mathcal{V}(t') - \mathcal{V}(t) \right) U^h_0(t', s) u \, dt'.
\]  

(4.46)

In particular, choosing \( \ell = t \), we have

\[
\left\| \int_t^{t+\delta} e^{-i(t-t')\Delta}(t') \left( \mathcal{V}(t') - \mathcal{V}(t) \right) U^h_0(t', s) u \, dt' \right\|_{L^2(\mathbb{R})} \leq \delta C_{\alpha,\beta,\gamma} \sup_{t' \in [t+\delta, t]} \| \mathcal{V}(t) - \mathcal{V}(t') \| = o(\delta),
\]

and the previous limit reduces to

\[
\lim_{\delta \to 0^+} 1/\delta \left( U^h_0(t + \delta, s) - U^h_0(t, s) \right) u = -i H^h_0(t) U^h_0(t, s) u.
\]  

(4.47)

This shows that: \( D^+_{\Delta} U^h_0(t, s) u = -i H^h_0(t) U^h_0(t, s) u \) for any \( u \in D(\Delta_0) \). Following the same line for the left derivative, we obtain

\[
\frac{d}{dt} U^h_0(t, s) u = -i H^h_0(t) U^h_0(t, s) u.
\]  

(4.48)

Assume that \( \mathcal{V}^h_\theta(t, s) \) is a solution of (4.22); then it expresses as

\[
(\mathcal{V}^h_\theta(t, s) - U^h_{\theta,n}(t, s)) u = -i \int_s^t U^h_{\theta,n}(t', s) \left( H^h_\theta(t') - H^h_{\theta,n}(t') \right) \mathcal{V}^h_\theta(t', s) u \, dt'.
\]  

(4.49)

Estimating the difference at the r.h.s. as in (4.33)-(4.34), we get the identity

\[
\mathcal{V}^h_\theta(t, s) = \lim_{n \to \infty} U^h_{\theta,n}(t, s) = U^h_\theta(t, s),
\]

(4.50)

both in the \( \mathcal{L}(L^2(\mathbb{R})) \) and in the \( \mathcal{L}(D(\Delta_0)) \)-norm sense. This yields the uniqueness of the solution.

We are now in the position to conclude the proof of the Theorem 2.2.

**Proof of the Theorem 2.2.** The first part of the statement follows from the results in the Theorem 4.4. Let us discuss the estimate (2.10). For each \( n, U^h_{\theta,n}(t, s) \) is \( \theta \)-holomorphic in \( \mathcal{L}(L^2(\mathbb{R})) \) and the uniform convergence of the sequence implies that \( U^h_\theta(t, s) \) is \( \theta \)-holomorphic w.r.t. the \( L^2(\mathbb{R}) \)-operator norm in the ball: \( |\theta| \leq h^{N_0}, N_0 > 2 \). Let us fix \( t, s \) and \( h \); it results

\[
U^h_\theta(t, s) - U^h_0(t, s) = \theta D^h_\theta(t, s),
\]

(4.51)

where the operator \( D^h_\theta(t, s) \) fulfills the estimate

\[
\sup_{|\theta| \leq h^{N_0}} \| D^h_\theta(t, s) \|_{\mathcal{L}(L^2(\mathbb{R}))} \leq m^h(t, s),
\]

(4.52)

with \( m^h(t, s) > 0 \) possibly depending on \( h \) and on the couple \( t, s \). Fixing \( \theta = h^{2+\delta} \) with \( \delta > 0 \) arbitrarily small, it follows

\[
\| U^h_\theta(t, s) - U^h_0(t, s) \|_{\mathcal{L}(L^2(\mathbb{R}))} \leq h^{2+\delta} m^h(t, s),
\]

(4.53)

The uniform bound (4.28) implies

\[
\sup_{t, s \in [0,T]} \| U^h_\theta(t, s) - U^h_0(t, s) \|_{\mathcal{L}(L^2(\mathbb{R}))} \leq 2 \exp \left( C_{\alpha,\beta,\gamma} \sup_{t \in [0,T]} \| \mathcal{V}(t) \|_{L^\infty(\mathbb{R})} \right), \quad t \geq s,
\]

(4.54)

Hence, (4.53) yields

\[
\sup_{t, s \in [0,T]} m^h(t, s) \leq \frac{M_{\alpha,\beta,\gamma}}{h^{2+\delta}} \sup_{t \in [0,T]} \| \mathcal{V}(t) \|_{L^\infty(\mathbb{R})},
\]

(4.55)

with \( M_{\alpha,\beta,\gamma} > 0 \) possibly depending on the data, but independent from \( T \). The estimate (2.10) finally follows from (4.51)-(4.52) and (4.55).
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