AN ANALOGUE OF RADFORD’S $S^4$-FORMULA FOR FINITE TENSOR CATEGORIES

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Abstract. We develop the theory of Hopf bimodules for a finite rigid tensor category $C$. Then we use this theory to define a distinguished invertible object $D$ of $C$ and an isomorphism of tensor functors $\delta : V^{**} \to D^{**} V \otimes D^{-1}$. This provides a categorical generalization of Radford’s $S^4$ formula for finite dimensional Hopf algebras [R1], which was proved in [N] for weak Hopf algebras, in [HN] for quasi-Hopf algebras, and conjectured in general in [EO]. When $C$ is braided, we establish a connection between $\delta$ and the Drinfeld isomorphism of $C$, extending the result of [R2]. We also show that a factorizable braided tensor category is unimodular (i.e. $D = 1$). Finally, we apply our theory to prove that the pivotalization of a fusion category is spherical, and give a purely algebraic characterization of exact module categories defined in [EO].

1. Introduction

One of the most important general results about finite dimensional Hopf algebras is Radford’s formula for the forth power of the antipode. This formula reads: for any finite dimensional Hopf algebra $H$ over a field $k$ with antipode $S$,

\begin{equation}
S^4(h) = a^{-1}(\alpha \rightarrow h \leftarrow \alpha^{-1})a,
\end{equation}

where $a$ and $\alpha$ are the distinguished grouplike elements of $H$ and $H^*$, respectively.

Recently, several generalizations of this formula were discovered. Namely, in [N] the formula was extended to weak Hopf algebras, and in [HN] to quasi-Hopf algebras. Finally, in [EO] the authors conjectured a generalization of Radford’s formula to any finite tensor category (Conjecture 2.15). One of the main achievements of the present paper is a proof of this conjecture. 1

More specifically, we show in Theorem 3.3 (which is our main theorem) that for a finite tensor category $C$ there is an invertible object $D$, called the distinguished invertible object of $C$, and an isomorphism of tensor functors $\delta : ?^{**} \cong D^{**} ? \otimes D^{-1}$. We also show that our definition of the distinguished object is the same as in [EO] Definition 2.12], which means that our main theorem implies Conjecture 2.15 from [EO].

Further, we apply the main theorem to braided tensor categories to show that a factorizable braided category is unimodular (in particular, the center $Z(C)$ of a finite tensor category $C$ is unimodular). We also relate the isomorphism $\delta$ with the Drinfeld isomorphism $u : ? \cong ?^{**}$.

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1We note that this result is, essentially, equivalent to the statement that Radford’s formula holds for weak quasi-Hopf algebras. However, weak quasi-Hopf algebras are very cumbersome objects, and we avoid working with them by using a much simpler purely categorical language.
In the case when $C$ is semisimple, we describe the isomorphism $\delta$ in terms of Müger’s squared norms of simple objects, and show that the pivotalization of $C$ is spherical (this was previously known only in characteristic 0 [ENO]).

The paper is organized as follows. In Section 2 we define a category $\mathcal{H}$ of Hopf bimodules over $C$ as the category of right modules over the algebra $A = \text{Hom}(1, 1)$ in $C \boxtimes C^{op}$. We define a tensor category structure on $\mathcal{H}$ and prove in Proposition 2.3 that there is a tensor equivalence between $C$ and $\mathcal{H}$ (which establishes a categorical version of the Fundamental Theorem for Hopf bimodules). We use this result as a major tool in our arguments throughout the paper.

In Section 3 we define the distinguished invertible object of $C$ as the unique up to an isomorphism object $D$ such that $A^* \cong (D \boxtimes 1) \otimes A$, following the idea used in [HN] for quasi-Hopf algebras. We use the category of Hopf bimodules to construct in Theorem 3.3 a natural tensor isomorphism $\delta : \cdot \cong D \otimes \cdot \otimes D^{-1}$.

In Sections 4 and 5 we work with a braided finite tensor category $C$. We introduce a categorical notion of factorizability of $C$ and show that the center of a finite tensor category is factorizable. We prove that a factorizable tensor category is automatically unimodular (i.e., $D \cong 1$), extending the results known for (weak, quasi) Hopf algebras. We also show that in the case of unimodular $C$ one has $\delta = u^{-1} \circ u^*$ where $u : \cdot \cong \cdot \otimes D^{-1}$ is the Drinfeld isomorphism.

In Section 6 we check that our definition of $D$ agrees with the one given in [EO], that is, the projective cover of the unit object $1$ coincides with the injective hull of $D$.

In Section 7 we specialize to the case when $C$ is semisimple and give a convenient numerical characterization of the isomorphism $\delta$ in Theorem 7.3. As a consequence, we obtain that the pivotalization of a semisimple category is spherical.

The appendix contains an algebraic characterization of exact module categories studied in [EO]. We show in Theorem 8.1 that if $\mathcal{M}$ is a module category over $C$, $B$ is a $k$-algebra such that $\mathcal{M} \cong B - \text{mod}$, and $\bar{F} : C \to \text{Vect}_k$ is the fiber functor constructed from the action of $C$ on $\mathcal{M}$, then $\mathcal{M}$ is exact if and only if the algebra $H = \text{End}(\bar{F})$ is a projective $B \otimes B^\circ$-module.

Remark. As was anticipated by G. Kuperberg, the statement and proof of the categorical Radford’s formula do not make an essential use of the additive structure of the category $C$ (and thus can be generalized to the non-additive case). In this respect, they are similar to Theorem 3.10 in [Ku], which gives Radford’s formula for a Hopf object in a rigid monoidal (possibly non-additive) category.

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2. AN ANALOGUE OF THE CATEGORY OF HOPF BIMODULES

We work over an algebraically closed field $k$. All categories considered in this paper are abelian over $k$ with all objects being of finite length and all morphism spaces being finite dimensional.

Such a category $C$ is said to be finite if it has finitely many isomorphism classes of simple objects and every object has a projective cover. That is, $C$ is equivalent to the category of finite dimensional representations of some finite dimensional $k$-algebra.
By a tensor category over $k$ we understand an abelian rigid monoidal category. We refer the reader to \[BK\] \[K\] \[EO\] for the general theory of tensor categories and module categories over them.

Recall the notion of Deligne’s tensor product $C \boxtimes D$ of abelian categories $C, D,$ see \[DL\]. By definition, this is the universal object for the functor assigning to every abelian category $A$ the category of additive right (equivalently, left) exact bifunctors from $C \times D$ to $A.$ If $C, D$ are tensor categories then the category $C \boxtimes D$ has a natural structure of a tensor category with the tensor product

\[(2) \quad \otimes \times \otimes : (C \boxtimes D) \times (C \boxtimes D) \to C \boxtimes D,
\]

(which we will still denote $\otimes$), and with the unit object $1 \boxtimes 1.$

Deligne’s tensor product can also be applied to functors. Namely, if $F : C \to C'$ and $G : D \to D'$ are additive right (left) exact functors between abelian categories then one can define the functor $F \boxtimes G : C \boxtimes D \to C' \boxtimes D'.$

Let $C$ be a finite tensor category over $k.$ Let $C^{op}$ be the opposite tensor category, that is, $C^{op} = C$ as a category but the tensor product $\otimes^{op}$ in $C^{op}$ is different: $X \otimes^{op} Y = Y \otimes X.$

The category $C$ has a natural structure of an exact module category $\text{[EO]}$ over $(C \boxtimes C^{op}, \otimes)$ coming from

\[(3) \quad \otimes \circ (\otimes \times \text{id}_{C^{op}}) : C \times C \times C^{op} \to C.
\]

We will denote this action by $(X, V) \mapsto X \circ V$ ($X \in C \boxtimes C^{op}, V \in C$).

The internal Hom, $\text{Hom}(V_1, V_2),$ of two objects $V_1, V_2$ in $C$ is an object of $C \boxtimes C^{op}$ representing the functor $\text{Hom}_C(? \circ V_1, V_2)$ (the latter contravariant functor is left exact, so it is representable). This means that there is a natural isomorphism

\[(4) \quad \text{Hom}_C(X \circ V_1, V_2) \cong \text{Hom}_{C \boxtimes C^{op}}(X, \text{Hom}(V_1, V_2)).
\]

We refer to $\text{[EO]}$ for the properties of $\text{Hom}.$ The object $A := \text{Hom}(1, 1)$ has a natural structure of an algebra in the category $C \boxtimes C^{op}.$

**Definition 2.1.** The category of right $A$-modules in $(C \boxtimes C^{op}, \otimes)$ will be called the category of Hopf bimodules over $C.$

Let $\mathcal{H}$ denote the category of Hopf bimodules.

Observe that $(X, Y) \mapsto \text{Hom}_C(\ast Y, X)$ is an additive bifunctor from $C^{op} \times C$ to $\text{Vect}_k.$ Hence it defines an additive functor $H_C : C^{op} \boxtimes C \to \text{Vect}_k.$ Therefore, one can define another tensor product $\otimes$ on $C \boxtimes C^{op}$ by

\[(5) \quad (\text{id}_C \otimes H_C) \boxtimes \text{id}_{C^{op}} : (C \boxtimes C^{op}) \boxtimes (C \boxtimes C^{op}) \cong (C \boxtimes (C^{op} \boxtimes C)) \boxtimes C^{op} \to C \boxtimes C^{op},
\]

where we implicitly used a natural action of $\text{Vect}_k$ on $C.$ One checks directly that $A$ is the unit object for $\otimes.$

**Remark 2.2.** (i) Another way to define $\otimes$ is the following: one identifies $C^{op}$ with the dual category $C^{\vee}$ via the functor $X \mapsto \ast X.$ Then the category $C \boxtimes C^{op}$ is identified with the category of left exact functors from $C$ to itself (for example $X \boxtimes Y \in C \boxtimes C^{op}$ corresponds to the functor $Z \mapsto \text{Hom}(\ast Z, \ast Y)$ and $\otimes$ corresponds to the composition of functors. Under this identification the object $A \in C \boxtimes C^{op}$ representing the functor $X \boxtimes Y \mapsto \text{Hom}(X \otimes Y, 1) = \text{Hom}(X, Y^\ast)$ corresponds to the identity functor (this is why we use $X \mapsto \ast X$ and not $X \mapsto X^\ast$ to identify $C^{op}$ and $C^{\vee}$).
(ii) If $\mathcal{C}$ is the representation category of a Hopf algebra (or quasi-Hopf algebra, or weak Hopf algebra) $H$ then $\mathcal{H}$ is the category of usual $H$-Hopf bimodules $\mathbf{LS}$ and $\odot$ is dual to the usual bimodule tensor product (thus $M \odot N = (N^* \otimes_H M^*)^*$ where the star denotes the dual vector space).

The Fundamental Theorem for Hopf modules over a Hopf algebra $H$ states that the category of $H$-Hopf modules is equivalent to the category of vector spaces. In $\mathbf{HN}$ it was explained that for quasi-Hopf algebras the notion of a Hopf module should be replaced by that of a Hopf bimodule and the Fundamental Theorem for Hopf bimodules over a quasi-Hopf algebra $\mathbf{HN}$ Proposition 3.11] algebra was proved. Proposition 2.3 below provides a categorical version of the Fundamental Theorem for Hopf bimodules and generalizes the results of $\mathbf{LS}$ $\mathbf{HN}$.

**Proposition 2.3.** (a) The category $\mathcal{H}$ is a tensor subcategory of $(\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}, \odot)$ and the functors $(? \boxtimes 1) \otimes A$ and $(1 \boxtimes ?) \otimes A$ from $\mathcal{C}$ to $\mathcal{H}$ are equivalences of tensor categories. (b) There is a natural isomorphism of tensor functors

$$\rho : (? \boxtimes 1) \otimes A \cong (1 \boxtimes ?) \otimes A.$$  

**Proof.** (a) For all objects $V$ in $\mathcal{C}$ we have

$$\text{Hom}(1, V) = \text{Hom}(1, (V \boxtimes 1) \circ 1) \cong (V \boxtimes 1) \otimes \text{Hom}(1, 1) = (V \boxtimes 1) \otimes A,$$

$$\text{Hom}(1, V) = \text{Hom}(1, (1 \boxtimes V) \circ 1) \cong (1 \boxtimes V) \otimes \text{Hom}(1, 1) = (1 \boxtimes V) \otimes A.$$  

It follows from $\mathbf{EO}$, Theorem 3.17 that $(? \boxtimes 1) \otimes A$ is an equivalence between $\mathcal{C}$ and $\mathcal{H}$. To see that it is tensor, observe that under the identification in Remark 2.2 (i) the functor $(? \boxtimes 1) \otimes A$ (and similarly $(1 \boxtimes ?) \otimes A$) sends $V \in \mathcal{C}$ to the functor $V \otimes ?$ from $\mathcal{C}$ to itself. Thus the associativity constraint in the category $\mathcal{C}$ gives rise to a tensor structure on these functors.

(b) The tensoriality of the natural isomorphism $\rho$ is obvious from description in (a). It is also equivalent to commutativity of the following diagram

$$\begin{array}{ccc}
(V \boxtimes 1) \otimes (W \boxtimes 1) \otimes A & \cong & ((V \boxtimes W) \boxtimes 1) \otimes A \\
\rho_V \otimes \rho_W & \cong & \rho_V \otimes \rho_W \\
((1 \boxtimes V) \otimes A) \odot (1 \boxtimes (V \boxtimes W) \otimes A) & \rightarrow & (1 \boxtimes (V \otimes W)) \otimes A.
\end{array}$$

□

**Remark 2.4.** Another way to state Proposition 2.3 (b) is to say that the following diagram commutes:

$$\begin{array}{ccc}
(V \boxtimes 1) \otimes (W \boxtimes 1) \otimes A & \rightarrow & ((V \boxtimes W) \boxtimes 1) \otimes A \\
\rho_V \otimes \rho_W & \cong & \rho_V \otimes \rho_W \\
(V \boxtimes 1) \otimes (1 \boxtimes W) \otimes A & \cong & (1 \boxtimes (V \boxtimes W)) \otimes A \\
\text{id} \otimes \rho_W & \cong & \text{id} \otimes \rho_W \\
(1 \boxtimes W) \otimes (V \boxtimes 1) \otimes A & \cong & (1 \boxtimes W) \otimes (1 \boxtimes V) \otimes A.
\end{array}$$
3. Construction of an isomorphism between duality functors

Let \( A = \text{Hom}(1, 1) \) be the algebra in \( \mathcal{C} \otimes \mathcal{C}^{op} \) defined in the previous Section and let \( M \) be a left \( A \)-module. Recall [OT] that \( M^* \) has a natural structure of a right \( A \)-module with the action given by

\[
M^* \otimes A \xrightarrow{m_A \otimes \text{id}_A} M^* \otimes A^* \otimes A \xrightarrow{\text{id}_{M^*} \otimes \text{coev}_A} M^*,
\]

where \( m_A : A \otimes M \to M \) is the left action of \( A \) on \( M \) and \( \text{coev}_A \) is the coevaluation morphism of \( A \). In particular \( A^* \) has a canonical \(^2\) structure of a Hopf bimodule. Thus according to Proposition \( \mathbb{A}3 \)(a) there exists a unique up to an isomorphism \( \rho \) defined by (10)

\[
(D \boxtimes 1) \otimes A \cong A^*
\]
as Hopf bimodules. Moreover isomorphism (10) is unique up to scaling. It follows immediately from the definition that the Frobenius-Perron dimension (see [EO]) of \( D \) equals to 1 and thus \( D \) is invertible.

**Definition 3.1.** The object \( D \) defined by (10) is called a *distinguished invertible object* of \( \mathcal{C} \).

**Remark 3.2.** Our definition of the distinguished invertible object of \( \mathcal{C} \) categorically extends definitions of distinguished group-like element, or modulus, of a Hopf algebra [R1], quasi-Hopf algebra [HN], or weak Hopf algebra [N]. In Section 6 we show that Definition 3.1 is equivalent to [EO] Definition 2.12.

The classical formula of D. Radford [R1] expresses the fourth power of the antipode of a finite-dimensional Hopf algebra \( H \) in terms of distinguished group-like elements of \( H \) and \( H^* \). The categorical version of this formula below is a main result of this paper.

**Theorem 3.3.** Let \( \mathcal{C} \) be a finite tensor category. There is a natural isomorphism of tensor functors,

\[
\delta : ?^{**} \to D \otimes ?^{**} \otimes D^{-1}.
\]

**Proof.** Isomorphism (10) produces a canonical isomorphism of algebras

\[
A^{**} = \text{Hom}(A^*, A^*) \cong \text{Hom}((D \boxtimes 1) \otimes A, (D \boxtimes 1) \otimes A) = (D \boxtimes 1) \otimes A \otimes (D \boxtimes 1)^*.
\]

We will identify these algebras using this isomorphism.

Recall that we have a tensor isomorphism \( \Phi \rho_V : (V \boxtimes 1) \otimes A \cong (1 \boxtimes V) \otimes A \). Its double dual \( \rho_V^* : (V^{**} \boxtimes 1) \otimes A^{**} \cong (1 \boxtimes V^{**}) \otimes A^{**} \) is also tensor (i.e. the diagram analogous to \( \Phi \) commutes).

Thus we have a tensor isomorphism of right \( A \)-modules

\[
\hat{\rho}_V : (V^{**} \boxtimes 1) \otimes (D \boxtimes 1) \otimes A \cong (1 \boxtimes V^{**}) \otimes (D \boxtimes 1) \otimes A
\]
defined by \( \hat{\rho}_V \otimes \text{id}_{(D \boxtimes 1)^*} = \rho_V^* \).

Now define \( \hat{\delta}_V \) as the following composition:

\[
(V^{**} \boxtimes 1) \otimes A = ((V^{**} \otimes D^*) \boxtimes 1) \otimes (D \boxtimes 1) \otimes A \xrightarrow{\hat{\rho}_V \otimes \text{id}_{D^*}} (1 \boxtimes (V^{**} \otimes D^*)) \otimes (D \boxtimes 1) \otimes A =
\]

\[
= (D \boxtimes 1) \otimes (1 \boxtimes ((V \otimes D^*) \boxtimes 1)) \otimes A \xrightarrow{\text{id} \otimes \rho^{**} \otimes D^*} (D \boxtimes 1) \otimes ((V \otimes D^*) \boxtimes 1) \otimes A
\]

\(^2\)In this paper, “canonical” means that there is a distinguished choice which should be obvious to the reader.
$$= ((D \otimes **V \otimes D^*) \boxtimes 1) \otimes A.$$  

Obviously, the isomorphism $\tilde{\delta}_V$ is tensor (again, the diagram analogous to (8) commutes).

Finally, define the isomorphism $\delta_V : V^{**} \to D \otimes **V \otimes D^*$ by the condition $\tilde{\delta}_V \otimes \text{id}_A = \tilde{\varphi}_V$. Since $\tilde{\delta}_V$ is a morphism of right $A$–modules Proposition 2.3 (a) implies that $\delta_V$ is well defined. The fact that $\tilde{\delta}_V$ is tensor translates into the fact that $\delta_V$ is an isomorphism of tensor functors. The Theorem is proved.  

\[ \Box \]

**Corollary 3.4.** There is a positive integer $N$ such that the $N$th powers of tensor functors $?^{**}$ and $??$ are naturally isomorphic.

**Proof.** Since $C$ has finitely many non-isomorphic invertible objects, there exists $N$ such that $D \otimes N \cong 1$.  

\[ \Box \]

### 4. Unimodularity of factorizable categories

**Definition 4.1.** We will say that $C$ is unimodular if its distinguished invertible object is isomorphic to $1$.

Equivalently, $C$ is unimodular if $A$ is a self-dual object of $C \boxtimes C^{op}$.

Let $(C, \sigma)$ be a braided finite tensor category, where $\sigma$ is a natural isomorphism of bifunctors $\sigma : \otimes \cong \otimes^{op}$ satisfying hexagon axioms [BN, K]. Let $Z(C)$ be the center of $C$. Recall (see e.g. [K]) that the objects of $Z(C)$ are pairs $(X, e_X(\cdot))$ where $e_X(\cdot)$ is a functorial isomorphism $e_X(Y) : X \otimes Y \cong Y \otimes X$ defined for all $Y \in C$ and satisfying certain axioms. Now we define a tensor functor $G : C \boxtimes C^{op} \to Z(C)$ in the following way:

\[(13) \quad G(X \boxtimes Y) = (X \otimes Y, e_{X \otimes Y})\]

where

$$e_{X \otimes Y}(Z) : X \otimes Y \otimes Z \to X \otimes Z \otimes Y \to Z \otimes X \otimes Y.$$

The functor $G$ has a natural structure of a braided tensor functor.

**Definition 4.2.** We will say that a finite braided tensor category $C$ is factorizable if $G$ is an equivalence of tensor categories.

**Remark 4.3.** Factorizable Hopf algebras were introduced and studied by N. Reshetikhin and M. Semenov-Tian-Shansky in [RS]. This notion was extended to weak Hopf algebras in [NVY] and to quasi-Hopf algebras in [BT]. One can directly check that our Definition 4.2 extends the previous definitions. E.g., using a computation analogous to the one given by H.-J. Schneider for Hopf algebras in [S, Theorem 4.3] one shows that a weak Hopf algebra $H$ is factorizable if and only if its representation category $\text{Rep}(H)$ is factorizable in the sense of Definition 4.2 so the two definitions agree in this case.

The notion of a factorizable braided tensor category also extends that of a modular category to the case when $C$ is not necessarily ribbon or semisimple. Indeed, every semisimple finite tensor category is equivalent to the representation category of some semisimple weak Hopf algebra [OT], and it was shown in [NTV] that a semisimple ribbon weak Hopf algebra is modular if and only if it is factorizable.

An example of a factorizable category is given by the center of a tensor category.
Proposition 4.4. Let \( \mathcal{C} \) be a finite tensor category. Then its center \( \mathcal{Z}(\mathcal{C}) \) is factorizable.

Proof. Let \( \mathcal{M} \) be an exact module category over a finite tensor category \( \mathcal{D} \) and let \( \mathcal{D}_\mathcal{M}^\ast \) be the dual tensor category, whose objects are \( \mathcal{D} \)-module endofunctors of \( \mathcal{M} \), see [EO, \text{O1}] for definitions. By [EO, Corollary 3.35] there is a canonical tensor equivalence of tensor categories \( Q : \mathcal{Z}(\mathcal{D}) \cong \mathcal{Z}(\mathcal{D}_\mathcal{M}) \) that assigns to every object in \( \mathcal{Z}(\mathcal{D}) \) its module action on \( \mathcal{M} \). We will apply this result to the case when \( \mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}^\text{op} \) and \( \mathcal{M} = \mathcal{C} \).

Recall that there is a tensor equivalence \( \mathcal{Z}(\mathcal{C}) \cong (\mathcal{C} \boxtimes \mathcal{C}^\text{op})^\ast \mathcal{C} \) \([EO, \text{O1}]\). It is straightforward to check that the tensor functor \( G : \mathcal{Z}(\mathcal{C}) \boxtimes \mathcal{Z}(\mathcal{C})^\text{op} \to \mathcal{Z}(\mathcal{Z}(\mathcal{C})) \) defined as in equation (13) is the composition of the obvious equivalence \( \mathcal{Z}(\mathcal{C}) \boxtimes \mathcal{Z}(\mathcal{C})^\text{op} \cong \mathcal{Z}(\mathcal{C} \boxtimes \mathcal{C}^\text{op}) \) and \( Q : \mathcal{Z}(\mathcal{C} \boxtimes \mathcal{C}^\text{op}) \cong \mathcal{Z}(\mathcal{Z}(\mathcal{C})) \) defined in the previous paragraph. Therefore, \( G \) is an equivalence, i.e., \( \mathcal{Z}(\mathcal{C}) \) is factorizable. \( \Box \)

Next, we establish a categorical generalization of another Radford’s result [R3] stating that a factorizable Hopf algebra is unimodular (see also [BT] for quasi-Hopf algebras).

Proposition 4.5. If \( \mathcal{C} \) is a factorizable tensor category then \( \mathcal{C} \) is unimodular.

Proof. Observe that \( \mathcal{C} \) is a \( \mathcal{Z}(\mathcal{C}) \)-module category via the forgetful functor \( F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C} \),

\[
Z \bullet V := F(Z) \otimes V,
\]

for all objects \( Z \in \mathcal{Z}(\mathcal{C}) \) and \( V \in \mathcal{C} \). By the factorizability of \( \mathcal{C} \) there is a natural isomorphism of functors \( \bullet 1 \cong G^{-1}(?) \circ 1 \). Let \( I : \mathcal{C} \to \mathcal{Z}(\mathcal{C}) \) be the induction functor [EO, Lemma 3.38], i.e., the right adjoint functor of \( F \). There is a sequence of natural isomorphisms:

\[
\text{Hom}_{\mathcal{Z}(\mathcal{C})}(Z, I(V)) \cong \text{Hom}_{\mathcal{C}}(F(Z), V) = \text{Hom}_{\mathcal{C}}(Z \otimes 1, V) \cong \text{Hom}_{\mathcal{C}}(G^{-1}(Z) \circ 1, V) \cong \text{Hom}_{\mathcal{C}}(G^{-1}(Z), \text{Hom}(1, V)) \cong \text{Hom}_{\mathcal{Z}(\mathcal{C})}(Z, G(\text{Hom}(1, V))).
\]

Hence, \( I(?) \) is naturally isomorphic to \( G(\text{Hom}(1, ?)) \). For all objects \( V \) in \( \mathcal{C} \) we have \( F(\ast V) = \ast F(V) \) by the definition of duality in \( \mathcal{Z}(\mathcal{C}) \), i.e., \( F \) commutes with the left dual functor. Therefore the adjoint functor \( I \) commutes with the right dual functor. In particular, \( I(1)^\ast = I(1) \) and \( \text{Hom}(1, 1) = \text{Hom}(1, 1)^\ast \) (note that the tensor functor \( G \) commutes with duality), i.e., \( \mathcal{C} \) is unimodular. \( \Box \)

5. Relation with the Drinfeld isomorphism in the braided case

Let \( \mathcal{C} \) be a braided tensor category with braiding \( \sigma : \otimes \cong \otimes^\text{op} \). It is well known that in this case there is a natural isomorphism \( u : ? \to ?^\ast \ast \), called the Drinfeld isomorphism, given by

\[
(15) \quad u_V : V \xrightarrow{\text{coev}_V} V \otimes V^\ast \otimes V^{\ast \ast} \xrightarrow{\sigma V \otimes V^\ast} V^\ast \otimes V \otimes V^{\ast \ast} \xrightarrow{\text{ev}_V} V^{\ast \ast},
\]

where \( \text{coev}_V : 1 \to V \otimes V^\ast \) and \( \text{ev}_V : V^\ast \otimes V \to 1 \) are the coevaluation and evaluation morphisms attached to an object \( V \).
Let $\mathcal{C} = \text{Rep}(H)$ be the category of representations of a finite-dimensional Hopf algebra $H$. Then a result of Radford \cite{Radford2003} relates the Drinfeld isomorphism $u_V : V \cong V^{**}$ and the tensor isomorphism $\delta_V : V^{**} \rightarrow D \otimes V \otimes D^{-1}$ constructed in Theorem 5.1. An extension of this result to weak Hopf algebras was obtained in \cite{ENO} Lemma 5.12]. The proofs use Hopf algebra language and techniques. Below we give a categorical generalization of these results (we restrict ourselves to the unimodular case).

**Theorem 5.1.** Let $\mathcal{C}$ be a unimodular braided finite tensor category. Let $\delta_V : V^{**} \rightarrow V^{**}$ be the tensor isomorphism constructed in Theorem 5.1 and let $u_V : V \rightarrow V^{**}$ be the Drinfeld isomorphism. Then there is an equality of natural isomorphisms

$$\delta_V = u_{**V} \circ u_V^*.$$

**Proof.** Let $\rho_V : (V \boxtimes 1) \otimes A \cong (1 \boxtimes V) \otimes A$ be as in Theorem 5.1. Recall that by definition,

$$\rho_V \boxtimes \text{id}_A) \otimes \text{id}_A = \rho_{**V} \circ \rho_V^*.$$

Let $\Sigma = \sigma \otimes \sigma^{-1}$ be the braiding on $\mathcal{C} \boxtimes \mathcal{C}^{op}$. Define a natural isomorphism

$$\iota : (? \boxtimes 1) \otimes A \rightarrow (?^{**} \boxtimes 1) \otimes A$$

as the following composition:

$$(V \boxtimes 1) \otimes A \xrightarrow{\rho_V} (1 \boxtimes V) \otimes A \xrightarrow{\Sigma \boxtimes V \Sigma_{G,V,A}} A \otimes (1 \boxtimes V) \xrightarrow{\rho_V^*} A \otimes (V^{**} \boxtimes 1) \xrightarrow{\Sigma_{A,V,\Sigma_{G,1}}} (V^{**} \boxtimes 1) \otimes A.$$

Observe that in terms of Hom spaces the isomorphism $\iota_V$ is given as the following sequence of natural isomorphisms:

$$\text{Hom}(X_1 \boxtimes X_2, (V \boxtimes 1) \otimes A) \cong \text{Hom}(V^* \otimes X_1 \otimes X_2, 1)$$

$$\cong \text{Hom}(X_1 \otimes V^* \otimes X_2, 1)$$

$$\cong \text{Hom}(X_1 \otimes X_2 \otimes V^*, 1)$$

$$\cong \text{Hom}(X_1 \boxtimes X_2, (V^{**} \boxtimes 1) \otimes A)$$

for all objects $X_1, X_2 \in \mathcal{C}$, where the two isomorphisms in the middle come from the braiding in $\mathcal{C}$. On the other hand the isomorphism $(u_V \boxtimes \text{id}_1) \otimes \text{id}_A$ is given by

$$\text{Hom}(X_1 \boxtimes X_2, (V \boxtimes 1) \otimes A) \cong \text{Hom}(V^* \otimes X_1 \otimes X_2, 1)$$

$$\cong \text{Hom}(X_1 \otimes X_2 \otimes V^*, 1)$$

$$\cong \text{Hom}(X_1 \boxtimes X_2, (V^{**} \boxtimes 1) \otimes A),$$

therefore, the hexagon identity and Proposition 2.3(a) imply that

$$\iota_V = (u_V \boxtimes \text{id}_1) \otimes \text{id}_A.$$

Next, using the definition of $\iota_V$ in (19) and the naturality of braiding, we compute:

$$\iota_V^{-1} \circ \iota_V^* =$$

$$= (\rho_{**V} \circ \Sigma_{G,1} \circ \rho_{**V} \circ \Sigma_{G,1}^{-1}) \circ (\rho_{**V} \circ \Sigma_{G,1} \circ \rho_{**V} \circ \Sigma_{G,1}^{-1})$$

$$= \rho_{**V} \circ \Sigma_{G,1} \circ \rho_{**V} \circ \Sigma_{G,1}^{-1}$$

$$= \rho_{**V} \circ \rho_{**V}^*.$$
On the other hand,
\[
\iota_{bV}^{-1} \circ \iota_{aV} = (u_{bV}^{-1} \circ u_{aV} \boxtimes \text{id}_1) \otimes \text{id}_A,
\]
therefore Proposition 2.3(a) and equation (17) imply the result. \(\square\)

6. Comparison with [EO]

In this Section we show that our Definition 3.1 of a distinguished invertible object \(D\) of \(\mathcal{C}\) agrees with [EO, Definition 2.12].

Recall that it was proved in [EO] that in a finite tensor category any projective object is injective and vice versa. In particular the projective cover \(P_0\) of the unit object \(1 \in \mathcal{C}\) coincides with the injective hull of some object \(\tilde{D} \in \mathcal{C}\). It was shown in [EO] that \(\tilde{D}\) is an invertible object.

Theorem 6.1. The object \(\tilde{D}\) is isomorphic to \(D\).

Proof. Let \(I\) be a set indexing the isomorphism classes of simple objects in \(\mathcal{C}\); for \(\alpha \in I\) let \(L_\alpha, P_\alpha, I_\alpha\) denote a simple object corresponding to \(\alpha\), its projective cover, and its injective hull. We will assume that \(0 \in I\) and \(L_0 = 1\). Let \(i\) runs through \(I\). We are going to compute \(\dim \text{Hom}(P_0 \boxtimes L_i, A^*)\) in two ways.

First calculation:
\[
\dim \text{Hom}(P_0 \boxtimes L_i, A^*) = \dim \text{Hom}(P_0 \boxtimes L_i, (D \boxtimes 1) \otimes A) = \dim \text{Hom}(P_0 \otimes L_i, D) = \begin{cases} 1 & \text{if } L_i = D, \\ 0 & \text{otherwise.} \end{cases}
\]

Second calculation:
\[
\dim \text{Hom}(P_0 \boxtimes L_i, A^*) = \dim \text{Hom}(A^* (P_0 \boxtimes L_i)) = \dim \text{Hom}((P_0 \boxtimes 1) \otimes A, 1 \boxtimes L_i^*).
\]

Let us look closely at the object \((P_0 \boxtimes 1) \otimes A\) in \(\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}\).

Lemma 6.2. The object \((P_0 \boxtimes 1) \otimes A\) is injective.

Proof. Observe that the functor
\[
\text{Hom}(X \boxtimes Y, (P_0 \boxtimes 1) \otimes A) = \text{Hom}((P_0^* \boxtimes X) \boxtimes Y, A) = \text{Hom}(P_0^* \boxtimes X \boxtimes Y, 1)
\]
is exact in both variables \(X, Y\) since \(P_0^* \boxtimes X \boxtimes Y\) is injective, see [EO]. Thus the functor \(\text{Hom}(?, (P_0 \boxtimes 1) \otimes A)\) is exact, see [D]. The Lemma is proved. \(\square\)

We continue the proof of the Theorem. By Lemma 6.2
\[
(P_0 \boxtimes 1) \otimes A = \sum_{\alpha, \beta \in I} M_{\alpha\beta} L_\alpha \boxtimes I_\beta
\]
for some non-negative integer multiplicities \(M_{\alpha\beta}\). We have
\[
M_{\alpha\beta} = \dim \text{Hom}(L_\alpha \boxtimes L_\beta, (P_0 \boxtimes 1) \otimes A) = \dim \text{Hom}(P_0^* \boxtimes L_\alpha \otimes L_\beta, 1) = \dim \text{Hom}(L_\alpha \otimes L_\beta, P_0)
\]
\[
= [L_\alpha \otimes L_\beta : \tilde{D}],
\]
where \([X : L_i]\) denotes the multiplicity of a simple object \(L_i\) in the Jordan-Hölder series of \(X\). To calculate \(\dim \text{Hom}((P_0 \boxtimes 1) \otimes A, 1 \boxtimes L_i^*)\) it is enough to consider
the summands with $I_\alpha = P_0$. In this case $L_\alpha = \hat{D}$ and $[L_\alpha \otimes L_\beta : \hat{D}] = [L_\beta : 1]$.

Thus

$$\dim \text{Hom}((P_0 \boxtimes 1) \otimes A, 1 \boxtimes L_i^*) = \begin{cases} 1 & \text{if } I_0 \text{ covers } L_i^*, \\ 0 & \text{otherwise.} \end{cases}$$

(20)

Since $I_0 = P_0^*$ covers $\hat{D}^{-1}$ the Theorem follows. \hfill \Box

**Remark 6.3.** Note that Theorems 3.3 and 6.1 together are equivalent to [EO, Conjecture 2.15].

**Corollary 6.4.** A semisimple finite tensor category is unimodular.

Here is another application of Lemma 6.2 (which in the case of Hopf algebras is a well-known statement).

**Proposition 6.5.** Let $f : A \to A^{**}$ be a morphism in $C \boxtimes C^{op}$. Assume that $\text{Tr}(f) \neq 0$. Then the category $C$ is semisimple.

**Proof.** By definition $\text{Tr}(f)$ is the following morphism:

$$\text{Tr}(f) : 1 \boxtimes 1 \xrightarrow{\text{coev}_A} A \otimes A^{*} \xrightarrow{f \otimes \text{id}_A} A^{**} \otimes A^{* \otimes A^*} \boxtimes 1 \boxtimes 1.$$  

(21)

In particular, if $\text{Tr}(f) \neq 0$ then 1 is a direct summand of $A \otimes A^{*}$. Hence $P_0 \boxtimes 1$ is a direct summand of $(P_0 \boxtimes 1) \otimes A \otimes A^{*}$. By Lemma 6.2 $(P_0 \boxtimes 1) \otimes A$ is projective and therefore $(P_0 \boxtimes 1) \otimes A \otimes A^{*}$ is projective. Thus $P_0 \boxtimes 1$ is projective and consequently 1 is projective. Hence $C$ is semisimple. \hfill \Box

7. Fusion categories

In this section we specialize the previous considerations to the case when the category $C$ is semisimple (and thus $C$ is a fusion category, see [ENO]). Let $\{L_i\}_{i \in I}$ be a set of representatives of isomorphism classes of simple objects in $C$.

Let us describe the structure of the algebra $A$. It follows immediately from definitions that we have a canonical isomorphism

$$A = \bigoplus_{i \in I} L_i \boxtimes L_i^*.$$  

(22)

Using the definitions one describes the multiplication in the algebra $A$ in the following way (cf. [Mu]): we have $A \otimes A = \bigoplus_{i,j \in I} (L_i \otimes L_j) \boxtimes (L_i \otimes L_j)$; for any $m \in I$ the vector spaces $\text{Hom}(L_i \otimes L_j, L_m)$ and $\text{Hom}((L_i \otimes L_j), L_m^*) = \text{Hom}(L_m, L_i \otimes L_j)$ are canonically dual to each other via the pairing

$$\text{Hom}(L_i \otimes L_j, L_m) \otimes \text{Hom}(L_m, L_i \otimes L_j) \to \text{Hom}(L_m, L_m) = k$$

and hence there is a canonical morphism $(L_i \otimes L_j) \boxtimes (L_i \otimes L_j) \to L_m \boxtimes L_m^*$. Then the multiplication $A \otimes A \to A$ is just a direct sum (over $m \in I$) of all such morphisms.

Next, we describe the canonical isomorphism $A \cong A^{**}$. We have canonically

$$A^{**} \cong \bigoplus_{i \in I} L_i^{**} \boxtimes \cdots \boxtimes L_i \cong \bigoplus_{i \in I} L_i \boxtimes \cdots \boxtimes L_i.$$  

Thus to specify a morphism $A \to A^{**}$ in $C \boxtimes C^{op}$ is the same as to specify a collection of morphisms $\psi_i : L_i \to \cdots \boxtimes L_i$ in $C$. Recall [Mu] [ENO] that for any simple object $L$ in $C$ one defines its squared norm $|L|^2$ as follows: choose an isomorphism $\phi : L \to L^{**}$ (such an isomorphism always exists and is unique up to a scaling) and
Lemma 7.1. The canonical isomorphism $A \to A^{**}$ corresponds to the collection of morphisms $\psi_i : *L_i \to *****L_i$ characterized by the following property: for any isomorphism $\phi_i : *L_i \to ***L_i$ one has $\text{Tr}(\phi_i^{-1}) \text{Tr}(\phi_i \circ \psi_i^{-1}) = |L_i|^2$.

Proof. The statement is immediate from definitions since the isomorphism is the composition of the isomorphism $A$ isomorphism set $|\cdot|$ does not change after rescaling the evaluation and coevaluation morphisms in $C$.

Corollary 7.2. The trace of the canonical isomorphism $A \cong A^{**}$ is equal to $\dim(C) := \sum_{i \in I} |L_i|^2$.

Let us relate the canonical isomorphism $\delta : *\cong ***$ from Theorem 3.3 with squared norms of simple objects of $C$.

Theorem 7.3. Let $L \in C$ be a simple object. The canonical isomorphism $\delta_L : L^{**} \cong ***L$ can be characterized in the following way: for any isomorphism $\phi : L^{**} \to L$ one has $\text{Tr}(\phi^{-1}) \text{Tr}(\phi \circ \delta_L^{-1}) = |L|^2$.

Proof. Recall that $A$ represents the functor $X \boxtimes Y \mapsto \text{Hom}(X \otimes Y, 1)$ and $A^{**}$ represents the functor $X \boxtimes Y \mapsto \text{Hom}(**X \otimes Y^{**}, 1) = \text{Hom}(X \otimes Y^{***}, 1)$. It follows immediately from definitions that the canonical isomorphism $A \to A^{**}$ corresponds to the natural transformation $X \boxtimes Y \xrightarrow{id \boxtimes \delta^{-1}} X \boxtimes Y^{***}$. Now the Theorem is an immediate consequence of Lemma 7.1.

Corollary 7.4. Let $V \in C$ be an object and let $\phi : V \to **V$ be a morphism. Then $\text{Tr}(\phi^{**}) = \text{Tr}(\phi \circ \delta_{V^{**}}^{-1})$.

Proof. It is enough to prove the statement for $V = L_i$ and any isomorphism $\phi : L_i \to ***L_i$. But this is an immediate consequence of Theorem 7.3.

Remark 7.5. The following example shows that the statement of Corollary 7.4 is not true if $C$ is only assumed to be unimodular. Let $q$ be a primitive $p$th root of unity and let $U_q(sl_2)$ be the corresponding finite dimensional quantum $sl_2$ Hopf algebra. Let $H = \text{gr}(U_q(sl_2))$ be the associated graded Hopf algebra of $U_q(sl_2)$. This is a Hopf algebra of dimension $p^3$ defined like $U_q(sl_2)$ except that $EF - FE = 0$. Since $U_q(sl_2)$ is unimodular, so is $\text{gr}(U_q(sl_2))$. Then the statement of Corollary 7.4 would say that $\text{Tr}(K) = \text{Tr}(K^{-1})$ in any finite dimensional representation, which is false in 1-dimensional representations.

Recall (see [ENO]) that for any fusion category $C$ one constructs a twice bigger category $\hat{C}$ called its pivotalization. By definition, the objects of $\hat{C}$ are pairs $(X, f)$ where $X$ is an object of $C$ and $f : X \to X^{**}$ is an isomorphism satisfying $f^{**}f = d_X$. It is easy to see that the category $\hat{C}$ has a canonical pivotal structure.

Corollary 7.6. The category $\hat{C}$ is spherical, that is $\dim(X) = \dim(X^*)$ for any $X \in \hat{C}$.

Proof. Follows from Corollary 7.4.

Remark 7.7. The statement of corollary was proved in [ENO] under assumption that $\dim(C) \neq 0$, which is automatically satisfied in characteristic zero. Thus our result is new only in positive characteristic.
In the case when $C$ is the representation category of a semisimple Hopf algebra $H$, Corollary 7.4 becomes the following statement, which appears to be new in the case of positive characteristic (however, see [LR2, Theorem 2.2] and [R4, Propositions 9 and 10] for related statements).

**Proposition 7.8.** If $H$ is a semisimple Hopf algebra over an algebraically closed field $k$ and $g$ the distinguished grouplike element in $H$ (so that $g^{-1} x g = S^4(x)$). Then for any element $a \in H$ such that $axa^{-1} = S^2(x)$ and any irreducible $H$-module $V$, one has

$$\text{Tr}_V(a) = \text{Tr}_V(ga).$$

**Remark 7.9.** Let us give another, purely Hopf-algebraic proof of this statement. Consider a new Hopf algebra $K$ obtained by extending $H$ by a grouplike element $a$ such that $a^2 = g - 1$ and $axa^{-1} = S^2(x)$ [So]. It is well known that $S^2$ is an inner automorphism of $H$, so every irreducible representation $V$ of $H$ extends to a representation of $K$ (in two ways). We must show that $\text{Tr}_V(a) = \text{Tr}_V(ga)$ for either of these extensions.

To do this, we recall that the left integral in $K^*$ is given by the formula [LR1, Proposition 2.4(a)]

$$\lambda(x) = \sum_{W \in \text{Irrep} K} \text{Tr}_W(xa) \text{Tr}_W(a^{-1}).$$

Also, recall that under right multiplication in $K^*$, $\lambda$ changes according to the character $g$. So for any $f \in K^*$ we get

$$\sum_W \text{Tr}_W(xa)f(x)\text{Tr}_W(a^{-1}) = \sum_W \text{Tr}_W(xa)\text{Tr}_W(a^{-1})f(g).$$

(we use the Sweedler notation $\Delta(z) = z(1) \otimes z(2)$, implying the summation as usual).

Set $f(z) = \text{Tr}_V(za)$, and $x = I$ (the integral of $K$ acting by 1 in the trivial module; it exists since $K$ is semisimple). Then we get

$$\sum_W \text{Tr}_W(Ia)\text{Tr}_V(Ia)\text{Tr}_W(a^{-1}) = \sum_W \text{Tr}_W(Ia)\text{Tr}_W(a^{-1})\text{Tr}_V(ga).$$

The right hand side of this equation is obviously equal to $\text{Tr}_V(ga)$ (only the term with $W = V$ survives). As to the left hand side, it can be written as

$$\sum_W \text{Tr}_{W \otimes V}(Ia)\text{Tr}_W(a^{-1}) = \sum_W \text{Tr}_{W \otimes V}(I)\text{Tr}_W(a^{-1}).$$

Here the only nonzero summand comes from $W = V^*$, and it yields $\text{Tr}_{V^*}(a^{-1}) = \text{Tr}_V(S(a)) = \text{Tr}_V(a)$. Thus, $\text{Tr}_V(ga) = \text{Tr}_V(a)$, and we are done.

**Corollary 7.10.** If $H$ is a semisimple Hopf algebra over $k$ and $S^4 = 1$ in $H$ then $H^*$ is unimodular.

**Proof.** In this case $g$ is central, so we get $\text{Tr}_V(a) = g_V \text{Tr}_V(a)$, where $g_V$ is the eigenvalue of $g$ on $V$. But $\text{Tr}_V(a)$ is nonzero, so $g_V = 1$ and hence $g = 1$. Thus $H^*$ is unimodular. \qed

**Remark 7.11.** In the case $S^2 = 1$, this result is contained in [L].
8. Appendix

We will use the notation of [EO]. Let $\mathcal{C}$ be a finite tensor category, and $\mathcal{M}$ be an abelian category which carries the structure of a module category over $\mathcal{C}$. Let $B$ be a finite dimensional algebra over $k$, and suppose that an equivalence of categories $\mathcal{M} \to B - \text{mod}$ is fixed. In this case, any object $X \in \mathcal{C}$ defines an exact functor $X \otimes : B - \text{mod} \to B - \text{mod}$. This means that we have a tensor functor $F: \mathcal{C} \to B - \text{bimod}$, such that $X \otimes M = F(X) \otimes_B M$ for $M \in B - \text{Mod}$.

Let $\bar{F} := F \circ \text{Forget} : \mathcal{C} \to \text{Vect}_k$ be the fiber functor, and $H := \text{End} \bar{F}$. Thus $\mathcal{C} = H - \text{mod}$ as an abelian category. (In fact, $H$ has the structure of a Hopf algebroid, reflecting the fact that $\mathcal{C}$ is a finite tensor category). In particular, we have a homomorphism $\mu : B \otimes B^\circ \to H$, coming from the functor $F$. Thus $H$ is a module over $B \otimes B^\circ$, where $B^\circ$ is the algebra opposite to $B$, via $a \circ h := \mu(a)h$ for all $h \in H$ and $a \in B \otimes B^\circ$.

Recall [EO] that $\mathcal{M}$ is called an exact module category if for any projective object $P \in \mathcal{C}$ and any $M \in \mathcal{M}$, the object $P \otimes M$ is projective. Equivalently, $\mathcal{M}$ is exact if any $\mathcal{C}$-module additive functor from $\mathcal{M}$ is exact.

The main result of this appendix is the following algebraic characterization of the property of exactness.

**Theorem 8.1.** $\mathcal{M}$ is exact if and only if $H$ is a projective $B \otimes B^\circ$-module.

**Proof.** Recall that a module $V$ over a finite dimensional algebra $A$ is projective if and only if it is flat. Indeed, $V \otimes_A W = \text{Hom}_A(V,W^*)$ for any two finite dimensional left $A$-modules $V,W$, and hence the functor $V \otimes_A ?$ is exact iff so is $\text{Hom}_A(V,?)$.

Suppose $\mathcal{M}$ is an exact module category over $\mathcal{C}$. We see that to prove the required statement, it suffices to show that the module $H$ over $B \otimes B^\circ$ is flat.

Since $H$ is a left $H$-module, we can regard $H$ as an object of $\mathcal{C}$. Thus, the functor $H \otimes_B ?$ is exact, and for any finite dimensional left $B$-module $M$, the module $H \otimes_B M$ is projective.

To show that $H$ is flat, let $M$ be a left $B$-module, $N$ a right $B$-module, and let us compute the derived tensor product $H \otimes^L_{B \otimes B^\circ} (M \otimes N)$. By the Künneth formula we have

$$H \otimes^L_{B \otimes B^\circ} (M \otimes N) \cong (H \otimes^L_B M) \otimes^L_{B^\circ} N.$$  

Now, since the functor $H \otimes_B$ is exact, we have

$$(H \otimes^L_B M) \otimes^L_{B^\circ} N \cong (H \otimes_B M) \otimes^L_{B^\circ} N.$$  

Further, since the module $H \otimes_B M$ is projective, it is also flat. Thus,

$$(H \otimes_B M) \otimes^L_{B^\circ} N \cong (H \otimes_B M) \otimes_{B^\circ} N.$$  

We conclude that

$$H \otimes^L_{B \otimes B^\circ} (M \otimes N) \cong H \otimes^L_{B \otimes B^\circ} (M \otimes N),$$  

which implies that $H$ is flat, as desired.

Conversely, assume that $H$ is a projective module. Then $H \cong \oplus_{i,j} V_{ij} \otimes P_i \otimes P_j^\circ$, where $V_{ij}$ are vector spaces, and $P_i, P_j^\circ$ are the projective covers of irreducible modules over $B, B^\circ$, respectively. Thus, for any left $B$-module $M$ we have

$$H \otimes_B M = \oplus_{i,j} \left[ \oplus_{j} V_{ij} \otimes (P_j^\circ \otimes_B M) \right] \otimes P_i.$$  

This $B$-module is obviously projective, so $\mathcal{M}$ is exact and we are done. \qed
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