Explicit Bernstein type inequalities for wavelet coefficients in $L_p(\mathbb{R}^n)$

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Abstract
In this paper, we investigate the wavelet coefficients for function spaces $\mathcal{A}_k^p := \{ f : \| (i\omega)^k \hat{f}(\omega) \|_p \leq 1, k \in \mathbb{N}, p \in (1, \infty) \}$ using an important quantity $C_{k,p}(\psi)$. In particular, Bernstein type inequalities associated with wavelets are established. We obtained a sharp inequality of Bernstein type for splines, which induces a lower bound for the quantity $C_{k,p}(\psi)$ with $\psi$ being the semiorthogonal spline wavelets. We also study the asymptotic behavior of wavelet coefficients for both the family of Daubechies orthonormal wavelets and the family of semiorthogonal spline wavelets. Comparison of these two families is done by using the quantity $C_{k,p}(\psi)$.

Keywords: wavelet coefficients, asymptotic estimation, Bernstein type inequality, Daubechies orthonormal wavelets, semiorthogonal spline wavelets

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1. Introduction and Motivations

We say that $\varphi : \mathbb{R} \to \mathbb{C}$ is a 2-refinable function if
\begin{equation}
\varphi = 2 \sum_{\nu \in \mathbb{Z}} a(\nu) \varphi(2 \cdot -\nu),
\end{equation}
where $a : \mathbb{Z} \to \mathbb{C}$ is a finitely supported sequence of complex numbers on $\mathbb{Z}$, called the low-pass filter (or mask) for $\varphi$. In frequency domain, the refinement equation in (1.1) can be rewritten as
\begin{equation}
\hat{\varphi}(2\omega) = \hat{a}(\omega) \hat{\varphi}(\omega), \quad \omega \in \mathbb{R},
\end{equation}
where $\hat{a} is the Fourier series of $a$ given by
\begin{equation}
\hat{a}(\omega) := \sum_{\nu \in \mathbb{Z}} a(\nu)e^{-i\nu\omega}, \quad \omega \in \mathbb{R}.
\end{equation}
The Fourier transform $\hat{f}$ of $f \in L_1(\mathbb{R})$ is defined to be $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-i\omega x}dx$ and can be extended to square integrable functions and tempered distributions.
Usually, a wavelet system is generated by some wavelet function \( \psi \) from a 2-refinable function vector \( \varphi \) as follows:

\[
\psi = 2 \sum_{\nu \in \mathbb{Z}} h(\nu) \varphi(2 \cdot -\nu) \quad \text{or} \quad \hat{\psi}(2\omega) = \hat{h}(\omega) \hat{\varphi}(\omega),
\]

where \( b : \mathbb{Z} \mapsto \mathbb{C} \) is a finitely supported sequence of complex numbers on \( \mathbb{Z} \), called the high-pass filter (or mask) for \( \psi \).

Many wavelet applications, for example, image/signal compression, are based on investigation of the wavelet coefficients \( \langle f, \varphi_{j,v} \rangle \) and \( \langle f, \psi_{j,v} \rangle \) for \( j, v \in \mathbb{Z} \), where \( \langle f, g \rangle := \int_\mathbb{R} f(x) g(x) dx \) and \( \varphi_{j,v} := 2^{j/2} \varphi(2^{j} \cdot -v) \), \( \psi_{j,v} := 2^{j/2} \psi(2^{j} \cdot -v) \). The magnitude of the wavelet coefficients depends on both the smoothness of the function \( f \) and the wavelet \( \psi \). In this paper, we investigate the quantity

\[
C_{k,p}(\psi) = \sup_{f \in \mathcal{A}_p^k} \frac{|(f, \psi)|}{\|f\|_p},
\]

where \( 1 < p, p' < \infty, 1/p + 1/p' = 1, k \in \mathbb{N} \), and \( \mathcal{A}_p^k := \{ f \in L_p(\mathbb{R}) : \|\hat{f}(\omega)\|_{p'} \leq 1 \} \). The classical Bernstein inequality states that for any \( \alpha \in \mathbb{N}_0^n \), one have \( \|\partial^\alpha f\|_p \leq R^{\|\alpha\|_0} \|f\|_p \), where \( f \in L_p(\mathbb{R}^n) \) in an arbitrary function whose Fourier transform \( \hat{f}(\omega) \) is supported in the ball \( |\omega| \leq R \). The quantity \( C_{k,p}(\psi) \) in (1.5) is the best possible constant in the following Bernstein type inequality

\[
|(f, \psi_{j,v})| \leq C_{k,p}(\psi) 2^{-j(k+1/p-1/2)} \|\hat{f}(\omega)\|_p (i\omega)^{-k} \hat{f}(\omega) \|f\|_p
\]

Such type of inequalities plays an important role in wavelet algorithms for the numerical solution of integral equations (cf. [3, 12]) where wavelet coefficients arise by an integral operator to a wavelet and bound of the type (1.6) gives priori information on the size of the wavelet coefficients.

Note that

\[
C_{k,p}(\psi) = \sup_{f \in \mathcal{A}_p^k} \frac{|(f, \psi)|}{\|\hat{f}\|_p} = \sup_{f \in \mathcal{A}_p^k} \frac{|(\hat{f}, \psi)|}{\|\hat{f}\|_p} = \frac{\|\hat{\psi}\|_p}{\|\hat{\psi}\|_p},
\]

where for a function \( f \in L_1(\mathbb{R}) \), \( \hat{f} \) is defined to be the function such that

\[
\hat{f}(\omega) = (i\omega)^{-k} \hat{f}(\omega).
\]

For \( \psi \) that is compactly supported, it is easily shown that the quantity \( C_{k,p}(\psi) < \infty \) is equivalent to

\[
\int_\mathbb{R} \psi(x)x^\nu dx = 0 \quad \text{or} \quad \frac{d^\nu}{dx^\nu} \psi(0) = 0
\]

for \( \nu = 0, \ldots, m - 1 \). That is, \( \psi \) has \( m \) vanishing moments. Consequently, for a wavelet \( \psi \) with \( m \) vanishing moments, we can investigate the magnitude of the wavelet coefficients in the function spaces \( \mathcal{A}_p^0, \ldots, \mathcal{A}_p^m \) for \( 1 < p' < \infty \) using the quantity \( C_{k,p}(\psi) \).

A fundamental question in wavelet application is which wavelet one should choose for a specific purpose. In [3], Keinert used a constant \( G_M \) in the following approximation for comparison of wavelets.

\[
\int_\mathbb{R} f(x)\psi_{j,v}(x) dx \approx 2^{-(j+1)(M+1/2)} \frac{G_M}{M} f(M)(2^{-j}v),
\]
where \( f \) is sufficient smooth, \( \psi \) has exactly \( M \) vanishing moments, and \( G_M \) depends only on \( \psi \). Keinert presented numerical values of \( G_M \) for some commonly used wavelets and provided constructions for wavelets with short support and minimal \( m \), which lead to better compression in practical calculation. By considering the quantity \( C_{k,p}(\psi) \), the "\( \approx \)" in (1.10) can be replaced by precise inequality. In [7], Ehrich investigate the quantity \( C_{k,p}(\psi) \) for \( p = 2 \) and for two important families of wavelets. Precise asymptotic relations of quantities \( C_{k,2}(\psi) \) are established in [7] showing that the quantity for the family of semiorthogonal spline wavelets are generally smaller than that for the family of Daubechies orthonormal wavelets.

In this paper, we shall investigate the quantity \( C_{k,p}(\psi) \) mainly for the family of Daubechies orthonormal wavelets (see [6]) and the family of semiorthogonal spline wavelets (see [5]). Let \( m \) be a positive integer. The Daubechies orthonormal wavelet \( \psi_m^D \) of order \( m \) with mask \( b_m^D \) and its 2-refinable function \( \varphi_m^D \) with mask \( a_m^D \) are determined by

\[
\hat{a}_m^D(\omega) = \cos^{2m}(\omega/2) \sum_{\nu=1}^{m-1} \frac{(m-1+\nu)}{\nu} \sin^{2\nu}(\omega/2),
\]

\[
\hat{b}_m^D(\omega) = e^{-i\omega/2} a_m^D(\omega/2 + \pi),
\]

while the semiorthogonal spline wavelet \( \psi_m^S \) of order \( m \) is given by

\[
\psi_m^S(x) = \sum_{\nu=0}^{2m-2} (-1)^\nu \frac{2m-1}{2m-1} N_{2m}(\nu + 1) N_{2m}(2x - \nu), \quad x \in \mathbb{R},
\]

where \( N_m \) is the B-spline of order \( m \). That is,

\[
N_m(x) = \frac{1}{(m-1)!} \sum_{\nu=0}^{m-1} (-1)^\nu \frac{m!}{\nu!} (x - \nu)_+^{m-1},
\]

or equivalently, \( \tilde{N}_m(\omega) = \frac{1}{\sqrt{2\pi}} (e^{-i\omega/2} \sin(\omega/2))^{m} \). Here for \( k \geq 1 \),

\[
(y)_+^k = \begin{cases} y & y > 0, \\ 0 & y \leq 0 \end{cases} \quad \text{and} \quad (y)_-^k = \begin{cases} 1 & y > 0, \\ \frac{1}{2} & y = 0, \\ 0 & y < 0 \end{cases}
\]

Note that \( \psi_m^S \) is generated from the 2-refinable function \( \varphi_m^S := N_m \) via (1.4) by some mask \( b_m^S \) (cf. [5]).

These two families are widely used in many applications. For example, see [3,10,11,12] for their applications on numerical solution of PDE and signal/image processing. Both of the Daubechies orthonormal wavelet \( \psi_m^D \) and the semiorthogonal spline wavelet \( \psi_m^S \) have vanishing moments of order \( m \) and support length \( 2m-1 \). The Daubechies orthonormal wavelet \( \psi_m^D \) generates an orthonormal basis \( \{2^{j/2} \varphi_m^D(2^{-j} \cdot -\nu) : j, \nu \in \mathbb{Z}\} \) for \( L_2(\mathbb{R}) \) (see [6]). However, the wavelet function \( \psi_m^D \) is implicitly defined and the coefficients in the mask for \( \psi_m^D \) are not rational numbers. In fact,

\[
\hat{\psi}_m^D(\omega) = \frac{1}{\sqrt{2\pi}} \hat{a}_m^D(\omega/2 + \pi) \prod_{\ell=1}^{\infty} \hat{a}_m^D(2^{-\ell} \omega),
\]
and \( a_m^D \) is obtained from (1.11) via Riesz lemma. The semiorthogonal spline wavelets generated by \( \psi^D_m \) are not orthogonal in the same level \( j \). Yet they are orthogonal on different levels. And more importantly, the semiorthogonal spline wavelet \( \psi^D_m \) is explicit defined and the coefficients for its mask are indeed rational numbers, which is a very much desirable property in the implementation of fast wavelet algorithms. We shall see that these two families significantly differ with respect to the magnitude of their wavelet coefficients in terms of \( C_{k,p}(\psi^D_m) \) and \( C_{k,p}(\psi^S_m) \).

The structure of this paper is as follows. In Section 2, for \( k, m \in \mathbb{N} \) fixed and \( p \in (1, \infty) \), we shall investigate the quantity \( C_{k,p}(\psi^S_m) \) in the Bernstein type inequality in (1.6) for the family of semiorthogonal spline wavelets. In Section 3, we shall establish results on the asymptotic behaviors \( (m \to \infty) \) of the quantities \( C_{k,p}(\varphi) \) and \( C_{k,p}(\psi) \) for both the refinable function \( \varphi \) and wavelet function \( \psi \) and for both the two families of wavelets. Finally, we shall generalize our results to high-dimensional wavelets in Section 4.

2. Bernstein Type Inequalities for Splines

In this section, we shall first establish a result of the Bernstein type inequality for splines and then present an upper bound for the quantity \( C_{k,p}(\psi^S_m) \). Throughout this paper, \( p \in \mathbb{R} \) always denotes a constant such that \( p \in (1, \infty) \).

Before we introducing our results, we need some notation and definitions.

A function \( s(x) \) is called a spline of order \( m \) of minimal defect with nodes \( lh, h > 0, l \in \mathbb{Z} \), if

1. \( s(x) \) is a polynomial with real coefficients of the degree \( < m \) at each interval \((h(l - 1), hl), l \in \mathbb{Z} \);
2. \( s(x) \in C^{m-2}(\mathbb{R}) \).

The collection of all such splines is denoted by \( S_{m,h} \). It is well known that any spline \( s \in S_{m,h} \) can be uniquely represented by

\[
s(x) = \sum_{\nu \in \mathbb{Z}} c_{\nu} N_m(x - h\nu) . \tag{2.1}
\]

Here \( N_m \) is the B-spline of order \( m \). It is well known that

\[
N'_m(x) = N_{m-1}(x) - N_{m-1}(x - 1) \quad \text{for} \quad m \geq 2 . \tag{2.2}
\]

and

\[
\sum_{k=-\infty}^{\infty} |\mathcal{N}_m(\omega + 2\pi k)|^2 = \sum_{k=-m+1}^{m-1} N_{2m}(m + k)e^{-ik\omega} . \tag{2.3}
\]

The following result provides an exact upper bound in the Bernstein type inequality for any spline \( s \in S_{m,h} \), which gives estimation of \( k \)th derivative of non-periodic spline in \( L_p(\mathbb{R}) \) by \( L_p(\mathbb{R}) \) norm of the spline \( s \) itself (also cf. [2] for a special case \( p = 2 \)).
Theorem 1. Let $k, m \in \mathbb{N}$, $k < m$, and $h \in \mathbb{Z}$. Let $p \in (1, \infty)$. Then, for any function $s \in S_{m,h}$ such that $\hat{s} \in L_p(\mathbb{R})$, the following sharp inequality holds:

$$||\hat{s}(k)||_p \leq \left(\frac{K_{2(m-k)+1}}{K_{2m+1}}\right)^{1/2}||\hat{s}||_p,$$

where $K_j = \frac{4}{\pi} \sum_{t=0}^{\infty} \frac{(-1)^{j(t+1)}}{(1 + 2t)^{j+1}}$, $j = 0, 1, 2, \ldots$ are the Favard's constants.

Proof. We first show that (2.4) is true for $k = 1$ and $h = 1$.

Since $s'(x) = \sum_{v \in \mathbb{Z}} c_v N^m_m(x - v)$, by (2.2), we have

$$||s'||_p = \int_\mathbb{R} \left| \sum_{v \in \mathbb{Z}} c_v e^{-i\omega v} N_{m-1}(\omega)(1 - e^{-i\omega}) \right|^p d\omega$$

$$= \int_0^{2\pi} \left| \sum_{v \in \mathbb{Z}} c_v e^{-i\omega v} N_{m-1}(\omega)(1 - e^{-i\omega}) \right|^p d\omega$$

$$= \int_0^{2\pi} \left| 1 - e^{-i\omega} \right|^p \sum_{v \in \mathbb{Z}} \left| N_{m-1}(\omega + 2\pi t) \right|^p \sum_{v \in \mathbb{Z}} \left| N_{m}(\omega + 2\pi t) \right|^p d\omega$$

$$\leq \max_{\omega \in [0, 2\pi]} \left| 1 - e^{-i\omega} \right|^p \sum_{v \in \mathbb{Z}} \left| N_{m}(\omega + 2\pi t) \right|^p ||s||_p^p.$$  

Here $\tilde{a}_s(\omega) = \sum_{v \in \mathbb{Z}} c_v e^{-i\omega v}$. Denote $L(\omega) := \frac{|1 - e^{-i\omega} \sum_{v \in \mathbb{Z}} \left| N_{m-1}(\omega + 2\pi t) \right|^2}{\sum_{v \in \mathbb{Z}} \left| N_{m}(\omega + 2\pi t) \right|^2}$. Let us find the maximum of $L(\omega)$ on $[0, 2\pi]$.

A function of complex variables $z$, determined by

$$E_{2m-1}(z) = (2m - 1)! z^{m-1} \sum_{k = -m+1}^{m-1} N_{2m}(m + k) z^k, \forall z \in \mathbb{R}$$

(2.5)
is called Euler-Frobenious’ polynomials of order $2m - 1$ (or degree $2m - 2$). By [4, p.151], we have

$$\sum_{t \in \mathbb{Z}} \left| \overline{N_{m}(\omega + 2\pi t)} \right|^2 = \frac{1}{(2m - 1)!} \prod_{t=1}^{m-1} \frac{1 - 2\alpha_t \cos \omega + \alpha_t^2}{|\alpha_t|},$$

and similarly,

$$\sum_{t \in \mathbb{Z}} \left| \overline{N_{m-1}(\omega + 2\pi t)} \right|^2 = \frac{1}{(2m - 2)!} \prod_{t=1}^{m-2} \frac{1 - 2\beta_t \cos \omega + \beta_t^2}{|\beta_t|},$$

where $\alpha_t$ and $\beta_t$ are the roots of Euler-Frobenious’ polynomials $E_{2m-1}(z)$ and $E_{2m-3}(z)$ respectively. Moreover, $-1 < \lambda_{m-1} < \lambda_{m-2} < \ldots < \lambda_1 < 0$, $-1 < \beta_{m-2} < \beta_{m-3} < \ldots < \beta_1 < 0$. 

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\( \beta_1 < 0, \) and \( \beta_j > \lambda_{j+1} \) for \( j = 1, \ldots, m - 2. \) Then, up to a constant, the following is true

\[
L(\omega) = (1 - \cos \omega) \prod_{j=1}^{m-2} (1 - 2 \beta_j \cos \omega + \beta_j^2) \prod_{j=1}^{m-1} (1 - 2 \lambda_j \cos \omega + \lambda_j^2) =: (1 - \cos \omega) A(\omega).
\]

Note that

\[
A'(\omega) = 2 \sin \omega \cdot A(\omega) \left( \sum_{\ell=1}^{m-2} \frac{\beta_\ell}{1 - 2 \beta_\ell \cos \omega + \beta_\ell^2} - \sum_{\ell=1}^{m-1} \frac{\lambda_\ell}{1 - 2 \lambda_\ell \cos \omega + \lambda_\ell^2} \right).
\]

It is easy to see that \( 1 - 2 \lambda_\ell \cos \omega + \lambda_\ell^2 > 0 \) and \( 1 - 2 \beta_\ell \cos \omega + \beta_\ell^2 > 0. \) Moreover,

\[
\beta_\ell (1 - 2 \lambda_{\ell+1} \cos \omega + \lambda_{\ell+1}^2) - \lambda_\ell (1 - 2 \beta_\ell \cos \omega + \beta_\ell^2) = (\beta_\ell - \lambda_{\ell+1})(1 - \beta_\ell \lambda_{\ell+1}) > 0.
\]

Consequently, \( L'(\omega) = 0 \) has only one root \( \omega = \pi \) on \( (0, 2\pi). \) That is, \( L(\omega) \) attains its maximum at \( \omega = \pi. \) Hence,

\[
\max_{\omega \in [0, 2\pi]} L(\omega) = \frac{\sum_{\ell \in \mathbb{Z}} |\hat{N}_{m-1}(\pi + 2\pi \ell)|^2}{\sum_{\ell \in \mathbb{Z}} |\hat{N}_m(\pi + 2\pi \ell)|^2}.
\]

As in [4],

\[
\sum_{\ell \in \mathbb{Z}} |\hat{N}_m(\pi + 2\pi \ell)|^2 = \frac{2^{2m+2} \sin^{2m+2} \omega}{\sum_{\ell \in \mathbb{Z}} |\pi + 2\pi \ell|^{2m+2}}.
\]

Consequently,

\[
\max_{\omega \in [0, 2\pi]} L(\omega) = \frac{4 2^{2m} \sin^{2m} \frac{\pi}{2} \sum_{\ell \in \mathbb{Z}} |\pi + 2\pi \ell|^{2m+2}}{2^{2m+2} \sin^{2m+2} \frac{\pi}{2}} \left( \frac{2^{2m+1} \sum_{\ell \in \mathbb{Z}} |1 + 2\pi \ell|^{2m+2}}{2^{2m+1} \sum_{\ell \in \mathbb{Z}} |1 + 2\pi \ell|^{2m}} = \pi \frac{K_{2m-1}}{K_{2m+1}} \right),
\]

where \( K_{2m-1}, K_{2m+1} \) are Favard’s constants (cf. [9, p.64-65]). From above calculations, we obtain,

\[
\|\hat{s}\|_p \leq \pi \left( \frac{K_{2m-1}}{K_{2m+1}} \right)^{1/2} \|\hat{s}\|_p.
\]

For integral-valued shifts \( h \) of splines \( s(x) = \sum_{v \in \mathbb{Z}} c_v N_m(x + hv), \ h \in \mathbb{Z}, \) One can show that

\[
\|\hat{s}\|_p \leq (\pi h) \left( \frac{K_{2m-1}}{K_{2m+1}} \right)^{1/2} \|\hat{s}\|_p.
\]

Now, by induction, it is easy to show that \[ (\omega) \] holds. Finally, we show that the constant in \( (2.4) \) is the best possible one. Let \( \tilde{a}_j(\omega) = \frac{1}{2\pi} \Phi_j(\omega - \omega_0) \) and \( \tilde{s}(\omega) := \tilde{a}_j(\omega) \hat{N}_m(\omega), \) where \( \Phi_j(\omega) \) is a Feyer’s kernel of order \( j \) and \( \omega_0 = \pi \) is the point which realizes the maximum on the right hand
side of inequality (2.4). Note, $\frac{1}{2\pi} \int_0^{2\pi} \Phi_f(\omega) d\omega = 1$. Then,

$$
\|s^*\|_p^p = \int_0^{2\pi} \left| \sum_{\ell \in \mathbb{Z}} c_{s,\ell} e^{-i\ell\omega} \hat{N}_{m-1}(\omega)(1 - e^{-i\omega}) \right|^p d\omega
$$

$$
= \int_0^{2\pi} |\alpha_{s}(\omega)(1 - e^{-i\omega})|^p \sum_{\ell \in \mathbb{Z}} |\hat{N}_{m-1}(\omega + 2\pi\ell)|^p d\omega
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} \frac{|1 - e^{-i\omega}|^p \sum_{\ell \in \mathbb{Z}} |\hat{N}_{m-1}(\omega + 2\pi\ell)|^p}{\sum_{\ell \in \mathbb{Z}} |N_m(\omega + 2\pi\ell)|^p} |\Phi_f(\omega - \omega_0)| \sum_{\ell \in \mathbb{Z}} |N_m(\omega + 2\pi\ell)|^p d\omega
$$

$$
\rightarrow \frac{|1 - e^{-i\omega_0}|^p \sum_{\ell \in \mathbb{Z}} |N_{m-1}(\omega + 2\pi\ell)|^p}{\sum_{\ell \in \mathbb{Z}} |N_m(\omega + 2\pi\ell)|^p} \|s^*\|_p^p, \quad j \to \infty.
$$

Consequently,

$$
\frac{\|s^*\|_p^p}{\|s\|_p^p} \to \max_{\omega \in [0,2\pi]} \frac{|1 - e^{-i\omega}|^p \sum_{\ell \in \mathbb{Z}} |\hat{N}_{m-1}(\omega + 2\pi\ell)|^p}{\sum_{\ell \in \mathbb{Z}} |N_m(\omega + 2\pi\ell)|^p} = \pi^p \left( \frac{K_{2m-1}}{K_{2m+1}} \right)^{p/2},
$$

which completes the proof. \[\Box\]

By Theorem [1] obviously, we have, $C_{k,p}(s) \geq \left( \pi \right)^{-k} \left( \frac{K_{2m+k+1}}{K_{2m-1}} \right)^{1/2}$ for any $s \in S_{m,h}$ such that $s \in L_p$. By the definition of $\psi^S_m$ in (1.12), we have the following corollary.

**Corollary 1.** Let $k \geq 0$ be a fixed integer. Then,

$$
C_{k,p}(\psi^S_m) \geq \left( \frac{1}{2\pi} \right)^k \left( \frac{K_{2m+k+1}}{K_{2m+1}} \right)^{1/2}.
$$

**Proof.** Let $f := \psi^S_m$. Then $\hat{f}(k) = \psi^S_m$. By (1.12),

$$
f(x) = \sum_{\nu=0}^{2m-2} (-1)^\nu N_{2m}(\nu + 1) N_{2m}^{(m-\nu)}(2x - \nu).
$$

Consequently, $f(\cdot / 2) \in S_{m+k,1}$. In view of Theorem [1] we have

$$
\frac{\|\hat{f}(k)\|_p}{\|f\|_p} \leq (2\pi)^k \left( \frac{K_{2m+1}}{K_{2m+k+1}} \right)^{1/2}.
$$

Now, by that $C_{k,p}(\psi^S_m) = \frac{\|\hat{f}\|_p}{\|f\|_p}$, we are done. \[\Box\]

From Corollary [1] when $m$ is large enough, we see that $C_{k,p}(\psi^S_m) \approx (\frac{1}{2\pi})^k$. In next section, we shall study the exact asymptotic behavior of these types of quantities as $m \to \infty$ for both the family of Daubechies orthonormal wavelets and the family of semiorthogonal spline wavelets.
3. The Asymptotic Estimation of Wavelet Coefficients

In this section, we shall study the asymptotic behavior of wavelet coefficients for both Daubechies orthonormal wavelets and semiorthogonal spline wavelets. We first study the asymptotic behavior of the wavelet coefficients for Daubechies orthonormal wavelets the first subsection. In the second subsection, we investigate the asymptotic behavior of the wavelet coefficients for semiorthogonal spline wavelets. In the last subsection, we shall compare the asymptotic behaviors of wavelet coefficients for these two families based on the quantity defined in (1.5).

3.1. The Wavelet Coefficients of Daubechies Orthonormal Wavelets

Let \( H_m(t) \) be a \( 2\pi \)-periodic trigonometric function defined by

\[
H_m(t) = \sum_{\nu=0}^{L} h_\nu e^{-i\nu t}, \quad |H_m(t)|^2 = 1 - c_m \int_0^\ell \sin^{2m-1} \omega d\omega, \tag{3.1}
\]

where \( c_m = \left( \int_\mathbb{R} \sin^{2m-1} \omega d\omega \right)^{-1} = \frac{\Gamma(m+1)}{\sqrt{\pi} \Gamma(m)} \sim \sqrt{\frac{m}{\pi}} \). Then, \( H_m = \alpha_D^m \) is the Daubechies orthonormal mask of order \( m \) (cf. [7]).

To compare with the semiorthogonal spline wavelets, we need the following result for the Daubechies scaling function \( \hat{\varphi}_D^m \).

**Theorem 2.** Let \( \varphi_D^m \) be the Daubechies orthonormal scaling function of order \( m \), i.e., \( \hat{\varphi}_D^m(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{\ell=1}^{\infty} H_m(2^{-\ell}\omega) \). Then

\[
\lim_{m \to \infty} \| \hat{\varphi}_D^m \|_p = \pi^k \left( \frac{2\pi}{1 + pk} \right)^{1/p - 1/2}, \quad k \in \mathbb{N}. \tag{3.2}
\]

**Proof.** Let \( \Phi := \frac{1}{\sqrt{2\pi}} \chi_{[-\pi,\pi]} \). We have

\[
\| \hat{\varphi}_D^m \|_p = \int_{\mathbb{R}} |\omega|^{pk} |\hat{\varphi}_D^m(\omega)|^p d\omega = \int_{\mathbb{R}} |\omega|^{pk} |\hat{\varphi}_D^m(\omega) - \Phi(\omega) + \Phi(\omega)|^p d\omega
\]

Note that

\[
\int_{\mathbb{R}} |\omega|^{pk} |\Phi(\omega)|^p d\omega = \pi^k \left( \frac{2\pi}{1 + pk} \right)^{1/p - 1/2}.
\]

We next prove that

\[
I := \int_{\mathbb{R}} |\omega|^{pk} |\hat{\varphi}_D^m(\omega) - \Phi(\omega)|^p d\omega \to 0, \quad \text{as } m \to \infty.
\]

In fact,

\[
I = \int_{|\omega| > \pi} |\omega|^{pk} |\hat{\varphi}_D^m(\omega)|^p d\omega + \int_{|\omega| \leq \pi} |\omega|^{pk} |\hat{\varphi}_D^m(\omega) - \Phi(\omega)|^p d\omega =: I_1 + I_2.
\]
By the regularity of $\varphi_m^D$, i.e., $|\varphi_m^D(\omega)| \leq C_1|\omega|^{-C_2\log(m)}$, obviously, $I_1 \to 0$ as $m \to \infty$. For $I_2$, let $I := [-\pi, \pi]$, $\delta > 0$ be fixed, and $I_\delta := [-\pi + \delta, \pi - \delta]$. Then

$$I_2 = \int_{I_\delta} |\omega|^p |\varphi_m^D(\omega) - \Phi(\omega)|^p d\omega + \int_{I_\delta} |\omega|^p |\varphi_m^D(\omega) - \Phi(\omega)|^p d\omega := I_{21} + I_{22}.$$  

For $I_{22}$, we have $I_{22} \leq C\delta$ for some $C$ depending only on $p, k$, since $\varphi_m^D$ and $\Phi$ are both bounded. For $I_{21}$, we have

$$I_{21} \leq \int_{I_\delta} |\omega|^p |\varphi_m^D(\omega) - \frac{1}{\sqrt{2\pi}} H_m(\omega/2)|^p d\omega + \int_{I_\delta} |\omega|^p \frac{1}{\sqrt{2\pi}} H_m(\omega/2) - |\Phi(\omega)|^p d\omega \to 0$$

as $m \to \infty$ since $\frac{1}{\sqrt{2\pi}} H_m(\omega/2)$ converges to $\Phi$ uniformly in $I_\delta$ and

$$\begin{align*}
&\int_{I_\delta} |\omega|^p |\varphi_m^D(\omega) - \frac{1}{\sqrt{2\pi}} H_m(\omega/2)|^p d\omega \\
&\leq \int_{I_\delta} |\omega|^p \left( \frac{1}{\sqrt{2\pi}} H_m(\omega/2) \left( \prod_{t=1}^\infty H_m(2^{-t-1} \omega) - 1 \right) \right)^p d\omega \\
&\leq \int_{I_\delta} |\omega|^p \left( \prod_{t=1}^\infty H_m(2^{-t-1} \omega) - 1 \right)^p d\omega \to 0, \quad m \to \infty.
\end{align*}$$

Consequently, we obtain

$$\lim_{m \to \infty} \|\varphi_m^D\|_p = \pi^k \frac{(2\pi)^{1/p-1/2}}{(1 + pk)^{1/p}}, \quad k \in \mathbb{N}.$$ 

More generally, one can also show that for $\alpha \in \mathbb{R}$ such that $1 - p\alpha > 0$,

$$\lim_{m \to \infty} \|\varphi_m^D\|_p = \pi^{-\alpha} \frac{(2\pi)^{1/p-1/2}}{(1 - p\alpha)^{1/p}},$$

where for a real number $\alpha \in \mathbb{R}$, the function $\omega \varphi_m^D$ is similarly defined as in (1.8). However when $1 - p\alpha \leq 0$, i.e., $\alpha \geq 1/p$, the constant $\|\varphi_m^D\|_p \to \infty$ as $m \to \infty$.

When $k$ fixed and $m \to \infty$, Babenko and Spektor ([1]) show that, for the Daubeches orthonormal wavelet function $\psi_m^D$ with $m$ vanishing moments, one has

$$\lim_{m \to \infty} \|\varphi_m^D\|_p = \frac{(2\pi)^{1/p-1/2}}{\pi^k} \left( \frac{1 - 2^{1-pk}}{pk - 1} \right)^{1/p}, \quad k \in \mathbb{N}.$$ (3.4)

When $k = m$, we can deduce the following estimation, which in turn gives rise to the asymptotic behavior of the constant $[C_{m,p}(\psi_m^D)]^{1/m}$.

**Theorem 3.** Let $\psi_m^D$ be the Daubechies wavelet with $m$ vanishing moments, i.e., $\psi_m^D(\omega) = \frac{1}{\sqrt{2\pi}} H_m(\omega/2 + \pi) \prod_{j=1}^m H_m(2^{j-1} \omega)$. Then

$$\|\psi_m^D\|_p = C \cdot \frac{2^{1/p}}{\sqrt{2\pi}} \cdot \frac{2^{-m} \cdot A(m)}{(\sqrt{mp/2})^{1/p}} \cdot (1 + O(m^{-1/2})).$$ (3.5)
where $C$ is a positive constant independent of $m$ and $\sqrt{\frac{\pi}{2m}} \leq A(m) \leq \sqrt{\frac{\pi}{2}}$.

**Proof.** By definition,

$$
\|m\hat{\varphi}_{m}^{p}\|^p = \int_{\mathbb{R}} |\omega|^{-mp}|\hat{\varphi}_{m}^{p}(\omega)|^{p}d\omega = \left\{ \int_{|\omega| \leq \pi} + \int_{|\omega| > \pi} \right\} |\omega|^{-mp}|\hat{\varphi}_{m}^{p}(\omega)|^{p}d\omega =: I_1 + I_2.
$$

We first estimate $I_2$. Since $|H_m(t)| \leq 1$,

$$
I_2 \leq \frac{2}{\sqrt{2\pi}} \int_{\pi}^{\infty} \omega^{-mp}d\omega \leq \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{mp - 1} \left(\frac{1}{\pi}\right)^{mp-1}, \quad mp > 1.
$$

Next, we show that $I_1 \sim C \cdot c_{m}^{p/2} \cdot (\sqrt{mp/2})^{-1} \cdot 2^{-mp}$. By the definition of $H_m(t)$, for $\omega \in [0, \pi]$, we have

$$
|H_m(\omega/4)|^2 \geq 1 - c_m \frac{\omega}{4} \sin^{2m-1}(\omega/4) \geq 1 - c_m \frac{\pi}{4} \sin^{2m-1}(\pi/4) \geq 1 - \frac{\Gamma(m + \frac{1}{2})}{\sqrt{\pi} \Gamma(m)} \left(\frac{\pi}{4}\right)^{2m},
$$

and

$$
\prod_{\ell=1}^{\infty} |H_m(2^{\ell-3}\omega)|^2 \geq \prod_{\ell=1}^{\infty} |1 - c_m (2^{\ell-3}\omega)^{2m}| \geq \prod_{\ell=1}^{\infty} |1 - c_m \left(\frac{\pi}{4}\right)^{2m} (2^{-2m})| \geq \prod_{\ell=1}^{\infty} |1 - (2^{-2m})| \geq (1 - 2^{-2m})^{1/(1-2^{-2}m)}.
$$

Thus,

$$
I_1 = \frac{1}{(\sqrt{2\pi})^p} \int_{|\omega| \leq \pi} |\omega|^{-mp} \left[|H_m(\omega/2 + \pi)|^2|H_m(\omega/4)|^2|H_m(\omega/8)|^2 \right. 
$$

$$
\times \prod_{\ell=1}^{\infty} \left|H_m(2^{\ell-3}\omega)|^2 \right|^{p/2} d\omega 
$$

$$
\geq (1 - o(1)) \frac{1}{(\sqrt{2\pi})^p} \int_{|\omega| \leq \pi} |\omega|^{-mp}|H_m(\omega/2 + \pi)|^p d\omega.
$$

Obviously,

$$
I_1 \leq \frac{1}{(\sqrt{2\pi})^p} \int_{|\omega| \leq \pi} |\omega|^{-mp}|H_m(\omega/2 + \pi)|^p d\omega.
$$

Now, we use the property of $H_m$ to deduce the asymptotic behavior of

$$
I_{11} := \int_{|\omega| \leq \pi} |\omega|^{-mp}|H_m(\omega/2 + \pi)|^p d\omega.
$$
Let $u = \frac{\sin^2 (\omega/2)}{\sin(\omega/2)}$. We have

$$|H_m(\omega/2 + \pi)|^2 = c_m \int_0^{\omega/2} \sin^{2m-1} t dt = \frac{c_m}{2} \sin^{2m}(\omega/2) \int_0^1 u^{m-1}(1 - u \sin^2(\omega/2))^{-1/2} du$$

Since

$$\frac{1}{m} = \int_0^1 u^{m-1} du \leq \int_0^1 u^{m-1}(1 - u \sin^2(\omega/2))^{-1/2} du \leq \int_0^1 u^{m-1}(1 - u)^{-1/2} du = c_m^{-1}$$

and

$$I_{11} = 2 \int_0^\pi \sin(\omega/2) \left[ \frac{c_m}{2} \sin^{2m}(\omega/2) \int_0^1 u^{m-1}(1 - u \sin^2(\omega/2))^{-1/2} du \right]^{p/2} d\omega,$$

we obtain

$$\left( \frac{c_m}{2m} \right)^{p/2} \cdot 2^{-mp} \cdot \int_0^\pi \left( \frac{\sin(\omega/2)}{\omega/2} \right)^{mp} d\omega \leq \frac{1}{2} I_{11} \leq \left( \frac{1}{2} \right)^{-p/2} \cdot 2^{-mp} \cdot \int_0^\pi \left( \frac{\sin(\omega/2)}{\omega/2} \right)^{mp} d\omega.$$

Now by that $\int_\pi^\pi \left( \frac{\sin(\omega/2)}{\omega/2} \right)^{2mp/2} d\omega = C(\sqrt{mp/2})^{-1}(1 + O(m^{-1/2}))$ and $\frac{1}{2} < \frac{1}{2}$, we conclude that

$$\|\hat{\psi}_m^{0}\|_p = C \cdot \frac{2}{(\sqrt{2\pi})^p} \cdot \frac{2^{-mp} \cdot A(m)^p}{\sqrt{mp/2}} \cdot (1 + O(m^{-1/2})),$$

which completes our proof.

**3.2. The Wavelet Coefficients of Semiorthogonal Spline Wavelets**

In this subsection, we mainly focus on the asymptotic behavior of wavelet coefficients for the semiorthogonal spline wavelets. We shall present the asymptotic estimations of the following quantities: $\|\varphi_m^0\|_p, \|\psi_m^0\|_p$, and $\|\hat{\psi}_m^0\|_p$.

First, for the scaling function $\varphi_m^S$, which is the B-spline $N_m$ of order $m$, we have the following result:

**Theorem 4.** Let $\varphi_m^S := N_m$ be the B-Spline of order $m$. Let $k \geq 0$ be an integer. Then

$$\|\varphi_m^S\|_p = \frac{8^{1/p}}{(\sqrt{2\pi})^{1-1/p}} \cdot \frac{1}{(\Lambda_1 m p)^{1/p}} \cdot (2\xi_1)^{-k} \cdot (\Lambda_1/\xi_1)^{m/2} \cdot (1 + O(m^{-1/2})), \quad (3.6)$$

where

$$\Lambda_1 = \frac{\sin^2(\xi_1)}{\xi_1} = 0.7246..., \quad (3.7)$$

and $\xi_1 = 1.1655...$ is the unique solution of the transcendental equation $\xi_1 - 2 \cot(\xi_1) = 0$ in the interval $(0, \pi)$. 

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Proof. By \( \varphi_m^p(t) = \frac{1}{\sqrt{2\pi}}(e^{-i\omega/2})^{\frac{m}{p}} \),

\[
\|k\varphi_m^p\|_p = \int_{\mathbb{R}} |\omega|^{-kp} \cdot \left| \frac{\sin(\omega/2)}{\omega/2} \right|^{mp} d\omega = \frac{2^{1-kp}}{\sqrt{2\pi}} \int_{\mathbb{R}} |\omega|^{-kp} \cdot \left| \frac{\sin(\omega)}{\omega} \right|^{mp} d\omega
\]

\[
= \frac{2^{2-kp}}{\sqrt{2\pi}} \int_0^\infty \omega^{-p(m/2+k)} \cdot \left( \frac{\sin^2(\omega)}{\omega} \right)^{mp/2} d\omega
\]

\[
= \frac{2^{2-kp}}{\sqrt{2\pi}} \left\{ \int_0^\pi + \int_{\pi}^\infty \right\} \omega^{-p(m/2+k)} \cdot \left( \frac{\sin^2(\omega)}{\omega} \right)^{mp/2} d\omega
\]

\[
= \frac{2^{2-kp}}{\sqrt{2\pi}} (I_1 + I_2).
\]

For \( I_2 \) with \( mp > 1 \), we have

\[
I_2 \leq \int_\pi^{\infty} \omega^{-mp} d\omega = \frac{1}{mp - 1} \left( \frac{1}{\pi} \right)^{mp-1}.
\]

To estimate \( I_1 \), we use the same technique as in the proof of [2, Lemma 4]. Let \( \xi_1 \) be the point where \( \sin^2(\omega)/\omega \) takes its maximum value \( \lambda_1 \) in \( (0, \pi) \), i.e., \( \xi_1 = 1.1655... \) is the root of the transcendental equation \( \xi_1^{-1} - 2 \cot(\xi_1) = 0 \) and \( \lambda_1 = \frac{\sin(\xi_1)}{\xi_1} = 0.72461... \). Separate \( I_1 \) to two parts as follows

\[
I_1 = \left\{ \int_0^\xi_1 + \int_{\xi_1}^\pi \right\} \left( \frac{\sin^2(\omega)}{\omega} \right)^{mp/2} \cdot \omega^{-p(m/2+k)} d\omega = : I_{11} + I_{12}.
\]

We first estimate \( I_{11} \). Let

\[
t = t(\omega) = \ln \frac{\xi_1 - \omega}{\sin^2(\xi_1 - \omega)} - \ln \frac{\xi_1}{\sin^2(\xi_1)} = \ln \frac{\lambda_1(\xi_1 - \omega)}{\sin^2(\xi_1 - \omega)}, \quad \omega \in (0, \xi_1).
\]

Then,

\[
t(\omega) \sim a_2 \omega^2 + a_3 \omega^3 + \cdots \sim a_2 \omega^2 \left( 1 + \frac{d_3}{d_2} \omega + \cdots \right), \quad \omega \to 0,
\]

where

\[
a_2 = \Lambda_1 = -\frac{1}{2} \frac{d^2}{d\omega^2} \ln \frac{\sin^2(\xi_1 - \omega)}{\xi_1 - \omega} \bigg|_{\omega=0} = 0.81597... .
\]

Then, similar to the proof of [2, Lemma 4], we can obtain

\[
\omega = \omega(t) \sim (\Lambda_1)^{-1/2} \sqrt{t}(1 + c_1 t^{1/2} + c_2 t + \cdots),
\]

\[
\frac{d\omega}{dt} \sim \frac{1}{2 \sqrt{\Lambda_1 t}}(1 + d_1 t^{1/2} + d_2 t + \cdots),
\]

\[
\xi_1 - \omega(t) \sim \xi_1(1 - e_1 t^{1/2} - e_2 t + \cdots),
\]

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for \( t \to 0 \). Changing the variable of \( I_{11} \), we have
\[
I_{11} = \int_0^\xi \left( \frac{\sin^2(\omega)}{\omega} \right)^{mp/2} \cdot (\xi - \omega)^{-p(m/2+k)} d\omega
\]
\[
= \int_0^\xi \left( \frac{\sin^2(\xi_1 - \omega)}{\xi_1 - \omega} \right)^{mp/2} \cdot (\xi_1 - \omega)^{-p(m/2+k)} d\omega
\]
\[
= \lambda_{m/2}^p \int_0^\infty e^{-\xi_1 t} q(t) dt,
\]
where
\[
q(t) = \left( \xi_1^{p(m/2+k)} \sqrt{\Lambda_1 t} \right)^{-1} (1 + f_1 t + f_2 t + \cdots).
\]
Now by Watson’s lemma, we have
\[
I_{11} = \lambda_{m/2}^p \cdot \left( \xi_1^{p(m/2+k)} \sqrt{\Lambda_1} \right)^{-1} \cdot \frac{\sqrt{\pi}_{\Lambda_1^{mp/2}}}{\Lambda} \cdot (1 + O(m^{-1/2}))
\]
\[
= -\frac{\sqrt{2\pi}}{\Lambda_1^{mp/2}} \cdot (\xi_1)^{-k} \cdot \left( \frac{\lambda_1}{\xi_1} \right)^{mp/2} \cdot (1 + O(m^{-1/2})).
\]
For \( I_{12} \), we use
\[
t = t(\omega) = \ln \frac{\xi_1 + \omega}{\sin^2(\xi_1 + \omega)} \quad \text{and} \quad \xi_1 = \ln \frac{\lambda_1(\xi_1 + \omega)}{\sin^2(\xi_1 + \omega)}, \quad \omega \in (0, \pi - \xi_1).
\]
Similarly, we have
\[
I_{12} = \frac{2\sqrt{\pi}_{\Lambda_1^{mp/2}}}{\Lambda_1^{mp/2}} \cdot (\xi_1)^{-k} \cdot \left( \frac{\lambda_1}{\xi_1} \right)^{mp/2} \cdot (1 + O(m^{-1/2})).
\]
Consequently,
\[
I_1 = \frac{2\sqrt{\pi}_{\Lambda_1^{mp/2}}}{\Lambda_1^{mp/2}} \cdot (\xi_1)^{-k} \cdot \left( \frac{\lambda_1}{\xi_1} \right)^{mp/2} \cdot (1 + O(m^{-1/2})).
\]
Noting that \( \frac{1}{\pi} = 0.31830... < \left( \frac{4}{\pi} \right)^{1/2} = 0.78846... \), we conclude
\[
\| \psi_m^S \|_p^p = \frac{2^{2-k}}{\sqrt{2\pi}} \cdot \frac{2\sqrt{\pi}_{\Lambda_1^{mp/2}}}{\Lambda_1^{mp/2}} \cdot (\xi_1)^{-k} \cdot \left( \frac{\lambda_1}{\xi_1} \right)^{mp/2} \cdot (1 + O(m^{-1/2}))
\]
\[
= \frac{8}{(\sqrt{2\pi})^{p-1}} \cdot \frac{1}{\Lambda_1^{mp/2}} \cdot (2\xi_1)^{-k} \cdot \left( \frac{\lambda_1}{\xi_1} \right)^{mp/2} \cdot (1 + O(m^{-1/2})),
\]
which completes our proof.

Next, for the spline wavelet function \( \psi_m^S \), we have the following estimation.

**Theorem 5.** Let \( k \in \mathbb{N} \cup \{0\} \) be a fixed nonnegative integer. Let \( \psi_m^S \) be the semiorthogonal spline wavelet of order \( m \), i.e.,
\[
\psi_m^S(x) := \sum_{v=0}^{2m-2} (-1)^v N_{2m}(v + 1) N_{2m}^m(2x - v), \quad x \in \mathbb{R},
\]
(3.8)

Then
\[
\| \psi_m^S \|_p = \frac{2^{3/p}}{(\sqrt{2\pi})^{1/p}} \cdot \frac{(2\pi - 4\xi_2)^{-k}}{(\sqrt{2\Lambda_2^{mp/2}})^{1/p}} \cdot \lambda_2^m \cdot (1 + O(m^{-1/2})).
\]
(3.9)
where

\[ \lambda_2 = \frac{\sin^2(\xi_2 - \pi/2) \sin^2(\xi_2)}{(\pi/2 - \xi_2)\xi_2^2} = 0.69706... \]

\[ \Lambda_2 = -\frac{1}{2} \frac{d^2}{du^2} \ln \frac{\sin^2(u - \pi/2) \sin^2(u)}{(\pi/2 - u)u^2} \bigg|_{u=\xi_2} = 1.2292..., \tag{3.10} \]

and \( \xi_2 = 0.2853... \) is the unique solution of the transcendental equation

\[ (2\pi \xi_2 - 4\xi_2^2) \cos(2\xi_2) + (3\xi_2 - \pi) \sin(2\xi_2) = 0, \quad \xi \in (0, \pi/2). \]

**Proof.** Using the Fourier transform of the B-spline and the definition of Euler-Frobenius polynomial \( E_{2m-1}(z) \) for \( z = e^{i\omega} \):

\[ \frac{E_{2m-1}(z)}{(2m-1)!} = \sum_{\nu=0}^{2m-2} N_{2m}(\nu + 1) z^\nu = e^{-i(m-1)\omega}(2 \sin(\omega/2))^{2m} \sum_{l=-\infty}^{\infty} \frac{1}{(\omega + 2\pi l)^{2m}}, \]

we can derive (c.f. [7, Lemma 4])

\[ |\nu\hat{\psi}_{m}(\omega)| = \frac{2^{-2k}}{\sqrt{2\pi}} \left| \frac{\sin^2(\omega/4)}{\omega/4} \right| \frac{\omega}{4} \left| \frac{E_{2m-1}(\tilde{z})}{(2m-1)!} \right| \]

\[ = \frac{2^{-2k}}{\sqrt{2\pi}} \left| \frac{\sin^2(\omega/4)}{\omega/4} \right| \frac{\omega}{4} \left| 2 \sin(\omega/2) \right|^{2m} \sum_{l=-\infty}^{\infty} \frac{1}{(\omega + 2\pi l)^{2m}}, \]

where \( \tilde{z} = e^{i\omega} \) and \( \tilde{\omega} = \pi - \omega/2 \). Then,

\[ \|\nu\hat{\psi}_{m}(\omega)\|_p = \frac{2^{-2kp}}{(2\pi)^p} \int_{\mathbb{R}} \left| \frac{\sin^2(\omega/4)}{\omega/4} \right|^p \frac{\omega}{4} \left| 2 \sin(\omega/2) \right|^{2mp} \sum_{l=-\infty}^{\infty} \frac{1}{(\omega + 2\pi l)^{2mp}} \left| \omega \right|^p d\omega \]

\[ = \frac{2^{-2kp}}{(2\pi)^p} \int_{\mathbb{R}} \left( \frac{\sin^2(u - \pi/2)}{u - \pi/2} \right)^{2m} (u - \pi/2)^{-2k} (2 \sin(u))^{4m} \]

\[ \times \left( \sum_{l=-\infty}^{\infty} \frac{1}{(2u + 2\pi l)^{2m}} \right)^{2p/2} 4du \]

\[ = \frac{4 \cdot 2^{-2kp}}{(2\pi)^p} \left\{ \int_{-\infty}^{-\pi/2} + \int_{-\pi/2}^{-\xi_2} + \int_{-\xi_2}^{\pi/2} + \int_{\pi/2}^{3\pi/2} + \int_{3\pi/2}^{\infty} \right\} \left( \frac{\sin^2(u - \pi/2)}{u - \pi/2} \right)^{2m} \]

\[ \times (u - \pi/2)^{-2k} (\sin(u))^{4m} \left( \sum_{l=-\infty}^{\infty} \frac{1}{(u + \pi l)^{2m}} \right)^{2p/2} du \]

\[ = \frac{4 \cdot 2^{-2kp}}{(2\pi)^p} (I_1 + I_2 + I_3 + I_4 + I_5), \]

Here, \( \xi_2 \) is the point where the function

\[ g(u) := \frac{\sin^2(u - \pi/2) \sin^2(u)}{(\pi/2 - u)u^2} \]
takes its maximum value in $(0, \pi/2)$, i.e., $\xi_2 = 0.28532\ldots$ is the root of the transcendental equation

$$h(u) := (2\pi u - 4u^2) \cos(2u) + (3u - \pi) \sin(2u).$$

Note that $g'(u) = \frac{\sin(2u)}{4(2\pi - 4u^2 - u)}$, $h(u)$ and $\lambda_2 = g(\xi_2) = 0.69706\ldots$.

We first estimate $I_2$. By \cite[Lemma 3]{}, we have

$$I_2 = \int_{-\pi/2}^{\xi_2} [g(u)]^{2m}(u - \pi/2)^{-2k}(1 + R_1 + r(u))^{2\rho/2} du = I_{21} + \tilde{R},$$

where $|R_1| \leq (2m - 1)^{-1}$,

$$r(u) = \begin{cases} \left(\frac{u}{\pi - u}\right)^{2m}, & -\pi/2 < u \leq 0, \\ \left(\frac{u}{\pi + u}\right)^{2m}, & 0 \leq u < \xi_2. \end{cases}$$

$$I_{21} := \int_{-\pi/2}^{\xi_2} [g(u)]^{2m}(u - \pi/2)^{-2k}(1 + R_2(u))^{2\rho/2} du,$$

where

$$R_2(u) = \begin{cases} R_1 + r(u), & -\pi/2 + \delta < u < \xi_2, \\ R_1, & -\pi/2 < u < -\pi/2 + \delta. \end{cases}$$

$0 < \delta < \pi/2 - \xi_2$ is fixed. Hence

$$|R_2(u)| \leq \frac{1}{2m - 1} + \left(\frac{\pi/2 - \delta}{\xi_2 + \delta}\right)^{2m},$$

and

$$\tilde{R} = \int_{-\pi/2}^{-\pi/2 + \delta} [g(u)]^{2m}(u - \pi/2)^{-2k}\rho/2 \cdot [(1 + R_1 + r(u))^\rho - (1 + R_1)^\rho] du.$$

$$\leq (p2^p + o(1)) \int_{-\pi/2}^{-\pi/2 + \delta} [g(u)]^{2m}(u - \pi/2)^{-2k}\rho/2 du$$

$$\leq (p2^p + o(1))\delta \cdot \frac{\sin^{2m} \delta}{(\pi - \delta)^{2m+2k}} \leq (p2^p + o(1))\delta \cdot \frac{\sin^{2m} \delta}{(\pi - \delta)^{2m+2k}}.$$

For the estimation of $I_{21}$, we shall employ the Watson’s lemma. We introduce

$$t = t(v) := \ln g(\xi_2) - \ln g(\xi_2 - v) = \ln \frac{\lambda_2}{g(\xi_2 - v)}, \quad \frac{dt}{dv} = \frac{g'(\xi - v)}{g(\xi - v)},$$

for $v \in [0, \pi/2 + \xi_2]$. We have $t \to 0$ as $v \to 0$ and $t$ goes from 0 to $\infty$ monotonically as $v$ increases from 0 to $\pi/2 + \xi_2$. We can state the asymptotic expansion of $t(v)$ near $v = 0$ as follows:

$$t(v) \sim a_2v^2 + a_3v^3 + \cdots \sim a_2v^2(1 + a_3/a_2v + \cdots),$$

where

$$a_2 = \Lambda_2 = -\frac{1}{2} \frac{d^2}{dv^2} \ln g(\xi_2) \bigg|_{v=0} = -\frac{h'(\xi_2)}{2\xi_2(\pi/2 - \xi_2)\sin(2\xi_2)} = 1.2229\ldots.$$
Let $s = \sqrt{t}$. Then
\[ s(v) \sim \sqrt{\Lambda_2}v(1 + b_1v + \cdots), \quad v \to 0. \]
Now $s'(v) \neq 0$, we can reverse this expansion,
\[ v = v(t) \sim \Lambda_2^{-1/2} s(1 + c_1 s + c_2 s^2 + \cdots) - \Lambda_2^{-1/2} t^{1/2}(1 + c_1 t^{-1/2} + c_2 t + \cdots). \]
Also,
\[ \frac{dv}{dt} = \frac{(\pi/2 + v - \xi_2)(\xi_2 - v) \sin 2(\xi_2 - v)}{h(\xi_2 - v)} \]
Asymptotic expansion of numerator and denominator at $v = 0$ and division yields
\[ \frac{dv}{dt} \sim \frac{(\pi/2 - \xi_2)\xi \sin(2\xi)}{-h'(\xi_2)v(t)}(1 + d_1v(t)^2 + \cdots) \]
\[ \sim \frac{1}{2\Lambda_2v(t)}(1 + d_1v(t)^2 + \cdots) \]
\[ \sim \frac{1}{2 \sqrt{\Lambda_2}}(1 + e_1 t^{1/2} + e_2 t + \cdots). \]
Now changing the variable in $I_{21}$ and noting $g(\xi_2 - v) = \lambda_2 e^{-t}$, we have
\[ I_{21} \sim \int_{-\pi/2}^{\xi_2} [g(u)^{2m}(u - \pi/2)^{2k}]^{1/2} du \]
\[ = \Lambda_2^{mp} \int_{0}^{\xi_2+\pi/2} [(g(\xi_2 - v) / \lambda_2)^{2m}(\xi_2 - v - \pi/2)^{2k}]^{1/2} dv \]
\[ = \Lambda_2^{mp} \int_{0}^{\infty} e^{-mp} q(t) dt, \]
where
\[ q(t) = (\pi/2 + v(t) - \xi_2)^{-kp} \cdot \frac{dv}{dt} \]
\[ \sim \frac{(\pi/2 - \xi_2)^{-kp}}{2 \sqrt{\Lambda_2}}(1 + f_1 t^{1/2} + f_2 t + \cdots)^{-kp}(1 + e_1 t^{1/2} + e_2 t + \cdots) \]
\[ \sim \frac{(\pi/2 - \xi_2)^{-kp}}{2 \sqrt{\Lambda_2}}(1 + g_1 t^{1/2} + g_2 t + \cdots). \]
By Watson’s lemma and choosing $\delta$ such that $\sin^2 \delta/(\pi - \delta) < \lambda_2$, we conclude that
\[ I_2 \sim I_{21} \sim \Lambda_2^{mp} \cdot \frac{(\pi/2 - \xi_2)^{-kp}}{2 \sqrt{\Lambda_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} (1 + O(m^{-1/2})). \]
Similarly, we can estimate the asymptotic behavior of $I_3$. We use
\[ t = t(v) = \ln g(\xi_2) - \ln g(\xi + v) = \ln \frac{\lambda_2}{g(\xi + v)}, \quad v \in (0, \pi/2 - \xi_2). \]
Same technique implies

\[ I_3 \sim A_2^{mp} \cdot \frac{\left(\frac{\pi}{2} - \xi_2\right)^{-kp}}{2 \sqrt{\Lambda_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} \cdot (1 + O(m^{-1/2})). \]

Next, for \( I_4 \), observing the period of \( \sum_{l=-\infty}^{\infty} \frac{1}{(u + \pi \ell)^{2m}} \) is \( \pi \), we have

\[
I_4 = \int_{\pi/2}^{3\pi/2} \left( \frac{\sin^2(u - \pi/2) \sin^2(u)}{u - \pi/2} \right)^{2m} (u - \pi/2)^{-2k} \left( \sum_{\ell=-\infty}^{\infty} \frac{1}{(u + \pi \ell)^{2m}} \right)^{p/2} \, du
\]

Consequently,

\[ I_4 \sim 2A_2^{mp} \cdot \frac{\left(\frac{\pi}{2} - \xi_2\right)^{-kp}}{2 \sqrt{\Lambda_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} \cdot (1 + O(m^{-1/2})). \]

Next, we estimate \( I_5 \). By \( E_{2m-1}(z) = (2m - 1)! \sum_{v=0}^{2m-2} N_{2m}(v + 1)z^v \), we derive that \( |E_{2m-1}(z)| \leq (2m - 1)! N_{2m} \), for \( |z| = 1 \) and

\[
I_5 = \int_{\pi/2}^{\pi} \left( \frac{\sin^2(u - \pi/2)}{u - \pi/2} \right)^{2m} (u - \pi/2)^{-2k} \left| E_{2m-1}(e^{2iu}) \right|^{p/2} \, du
\]

Similarly,

\[
I_1 = \int_{-\pi/2}^{-\pi} \left( \frac{\sin^2(u - \pi/2)}{u - \pi/2} \right)^{2m} (u - \pi/2)^{-2k} \left| E_{2m-1}(e^{2iu}) \right|^{p/2} \, du
\]

In summary, we have

\[ I_1 \sim I_5 \leq \frac{1}{(mp - 1)\pi^{2k}} \left( \frac{1}{\pi} \right)^{mp-1} \]

and

\[ I_2 \sim I_3 \sim \frac{1}{2} I_4 \sim A_2^{mp} \cdot \frac{\left(\frac{\pi}{2} - \xi_2\right)^{-kp}}{2 \sqrt{\Lambda_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} \cdot (1 + O(m^{-1/2})). \]
Due to $\frac{1}{\pi} = 0.31830... < \lambda_2 = 0.69706...$, we conclude that

\[
\|\tilde{\psi}_m\|_p = \frac{4 \cdot 2^{-2kp}}{\sqrt{2\pi}p} \cdot 4\lambda_2^{mp} \cdot \frac{(\pi/2 - \xi_2)^{-kp}}{2\sqrt{N_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} \cdot (1 + O(m^{-1/2}))
\]

\[
= \frac{8}{\sqrt{2\pi}p-1} \cdot \frac{(2\pi - 4\xi_2)^{-kp}}{\sqrt{2N_2mp}} \cdot \lambda_2^{mp} \cdot (1 + O(m^{-1/2}))
\]

which completes our proof.

Finally, when $k = m$, we obtain the following estimation.

**Theorem 6.** Let $\psi^S_m$ be the spline wavelet defined in (3.8). Then

\[
\|\tilde{\psi}^S_m\|_p = \frac{2^{1/p}}{(\sqrt{2\pi}1^{1/p})} \cdot \left(\frac{\pi}{\sqrt{\pi} - 8}\right) \cdot \frac{1}{(\sqrt{2mp})^{1/p}} \cdot \frac{16}{\sqrt{\pi^2}} \cdot (1 + O(m^{-1/2})). \quad (3.11)
\]

**Proof.** By definition, $n\psi^S_m(x) = \sum_{n=0}^{2m-2} (-1)^n N_{2m}(v) N_{2m}(2x - v)$. Hence,

\[
\|\tilde{\psi}^S_m(\omega)\| = \frac{2^{-2m}}{\sqrt{2\pi}} \cdot \left(\frac{\sin(\omega/4)}{\omega/4}\right)^{2m} \cdot \left|\frac{E_{2m-1}(\bar{\omega})}{(2m - 1)!}\right|
\]

\[
= \frac{2^{-2m}}{\sqrt{2\pi}} \cdot \left(\frac{\sin(\omega/4)}{\omega/4}\right)^{2m} \cdot \left(2 \sin(\bar{\omega}/2)\right)^{2m} \cdot \left|\sum_{n=0}^{\infty} \frac{1}{(\bar{\omega} + 2\pi)^{2m}}\right|
\]

where $E_{2m-1}$ is the Euler-Frobenius polynomial, $\bar{\omega} = \pi - \omega/2$. Setting $u = \bar{\omega}/2 = \pi/2 - \omega/4$, we obtain

\[
\|\tilde{\psi}^S_m\|_p = \frac{4 \cdot 2^{-2mp}}{\sqrt{2\pi}p} \cdot \left(\frac{\sin(u - \pi/2)}{u - \pi/2}\right)^{4m} \cdot \left(\frac{\sin(u)}{u}\right)^{4m} \cdot \left(\sum_{n=0}^{\infty} \frac{1}{(u + 2\pi)^{2m}}\right)^{2/p/2} \cdot du
\]

\[
= \frac{4 \cdot 2^{-2mp}}{\sqrt{2\pi}p} \cdot \left(\frac{\sin(u - \pi/2)}{u - \pi/2}\right)^{4m} \cdot \left(\frac{\sin(u)}{u}\right)^{4m} \cdot \left(\sum_{n=0}^{\infty} \frac{1}{(u + 2\pi)^{2m}}\right)^{2/p/2} \cdot du
\]

\[
= \frac{4 \cdot 2^{-2mp}}{\sqrt{2\pi}p} \cdot \left(1 + I_1 + I_2 + I_3 + I_4\right)
\]

Let

\[
g(u) := \frac{(\sin(u - \pi/2) \sin(u))}{(u - \pi/2)u}
\]

Then $g$ is symmetric about $u = \pi/4$ and $g(u) \leq g(\pi/4) = 64/\pi^4$. Similarly, using Lemma 3], we have

\[
I_2 \sim \int_{-\pi/2}^{\pi/4} (g(u))^{mp} du
\]
Introducing
\[ t = t(v) = \ln \frac{g(\pi/4)}{g(\pi/4 - v)}, \quad v \in [0, \frac{3}{4} \pi], \]
we can derive
\[ q(t) := \frac{dv}{dt} \sim \frac{\pi}{4} (\pi^2 - 8)^{-1/2} t^{-1/2} (1 + e_1 t^{1/2} + e_2 t + \cdots). \]

Changing the variable \( u \rightarrow \pi/4 - v \) in \( I_2 \) and using Watson’s lemma, we deduce
\[
\frac{4 \cdot 2^{-2mp}}{(\sqrt{2\pi})^p} I_2 \sim \frac{4 \cdot 2^{-2mp}}{(\sqrt{2\pi})^p} \left[ g(\pi/4) \right]^{mp} \int_0^{\infty} e^{-mp q(t)} dt \\
\sim \frac{1}{\sqrt{2\pi}^{p-1}} \cdot \frac{\pi}{\sqrt{\pi^2 - 8}} \left( \frac{16}{\pi^4} \right)^{mp} \cdot \frac{1}{\sqrt{2}^{2mp}} \cdot (1 + O(m^{-1/2}))
\]

It is easily seen that \( I_3 = I_2 \) due to the symmetry of \( g(u) \). Also, by the symmetry, we have \( I_1 = I_4 \). Using the fact that \( |E_{2m-1}(\pi)| \leq (2m-1)! \) for \( |z| = 1 \), we have
\[
\frac{4 \cdot 2^{-2mp}}{(\sqrt{2\pi})^p} I_4 = \frac{2^{-2mp}}{(\sqrt{2\pi})^p} \int_0^{\infty} \left( \frac{\sin(u - \pi/2)}{u - \pi/2} \right)^{2mp} \left[ |E_{2m-1}(\pi)| \right]^{mp} (2m-1)! \, d\omega \\
\leq \frac{2^{-2mp}}{(\sqrt{2\pi})^p} \int_0^{\infty} \left( \frac{\sin(u - \pi/2)}{u - \pi/2} \right)^{2mp} \, d\omega \\
\leq \frac{2^{-2mp}}{(\sqrt{2\pi})^p} \frac{1}{2mp - 1} \left( \frac{2}{\pi} \right)^{2mp} = \frac{1}{2mp - 1} \left( \frac{1}{\sqrt{2\pi}} \right)^{mp}.
\]

Noting that \( 1/\pi^2 \leq 16/\pi^4 \), we conclude that
\[
\|m\vec{\phi}_m^D\|_p^p = \frac{2}{\sqrt{2\pi}^{p-1}} \cdot \frac{\pi}{\sqrt{\pi^2 - 8}} \cdot \frac{1}{\sqrt{2}^{2mp}} \cdot \left( \frac{16}{\pi^4} \right)^{mp} \cdot (1 + O(m^{-1/2})),
\]
which completes our proof. \( \square \)

3.3. \\Comparison of Daubechies orthonormal Wavelets and Semiorthogonal Wavelets

Now, by the results we obtained in the above two subsections, we can compare the Daubechies orthonormal wavelets and the semiorthogonal spline wavelets using the constants \( C_{k,p}(f) \). Note that both Daubechies orthonormal wavelets and the semiorthogonal spline wavelets have the same support length and number of vanishing moments, thereby a comparison is possible in this respect.

We first consider the situation when \( k \) is fixed and let \( m \rightarrow \infty \). For Daubechies family, by Theorem 2 and (3.4), we can deduces the following result.

**Corollary 2.** Let \( \psi_m^D \) and \( \psi_m^D \) be the Daubechies orthonormal scaling function and wavelet function of order \( m \), respectively. Let \( k \geq 0 \) be a nonnegative integer. Then
\[
\lim_{m \rightarrow \infty} C_{k,p}(\psi_m^D) = \lim_{m \rightarrow \infty} \frac{\| -k \psi_m^D \|_p}{\| \psi_m^D \|_p} = \frac{\pi^k}{(1 + pk)^{1/p}} \tag{3.12}
\]
\[
\lim_{m \to \infty} C_{k,p}(\psi^S_m) = \lim_{m \to \infty} \frac{\|\hat{k}\psi^S_m\|_p}{\|\hat{k}\psi^S_m\|_p} = \pi^{-k} \left(1 - \frac{2^{1-pk}}{pk - 1}\right)^{1/p}.
\] (3.13)

For the semiorthogonal spline wavelet family, by Theorems 4 and 5, we can deduce the following result.

**Corollary 3.** Let \(\varphi^S_m\) and \(\psi^S_m\) be the semiorthogonal spline wavelet of order \(m\), respectively. Let \(k \geq 0\) be an integer. Then

\[
\lim_{m \to \infty} C_{k,p}(\varphi^S_m) = \lim_{m \to \infty} \frac{\|\hat{k}\varphi^S_m\|_p}{\|\hat{k}\varphi^S_m\|_p} = (2\pi - 4\xi_1)^{-k} = (2.331...)^{-k}
\] (3.14)

and

\[
\lim_{m \to \infty} C_{k,p}(\psi^S_m) = \lim_{m \to \infty} \frac{\|\hat{k}\psi^S_m\|_p}{\|\hat{k}\psi^S_m\|_p} = (2\pi - 4\xi_2)^{-k} = (5.1419...)^{-k},
\] (3.15)

where \(\xi_1 = 1.1655..., \xi_2 = 0.2853...\) are constants given in Theorems 4 and 5.

Comparing Corollaries 2 and 3, we obtain that for every \(k \in \mathbb{N} \cup \{0\}\), the semiorthogonal spline wavelets are better than the Daubechies orthonormal wavelets in the sense of asymptotically smaller constants. More precisely, we have

**Corollary 4.** Let \(\varphi^D_m\) and \(\psi^D_m\) be Daubechies orthonormal wavelet the semiorthogonal spline wavelet of order \(m\), respectively. Then

\[
\lim_{k \to \infty} \lim_{m \to \infty} \frac{C_{k,p}(\varphi^S_m)}{C_{k,p}(\varphi^D_m)}^{1/k} = \frac{\pi}{2\pi - 4\xi_2} = 0.61098...\] (3.16)

That is, the semiorthogonal spline wavelet constant \(C_k(\varphi^S_m)\) is exponentially better than Daubechies orthonormal wavelet constant \(C_k(\varphi^D_m)\) for increasing \(k\).

Since the number of vanishing moments increases with \(m\), it is natural to consider the behavior of the constants \(C_k(\psi^S_m)\) with \(k = k(m) = m\). In this situation, from Theorems 3 and 6, we have the following result, which shows that for smooth functions, the ratios in (3.16) when \(k = m\) is even more in favor of the spline wavelets.

**Corollary 5.** Let \(\varphi^D_m\) and \(\psi^S_m\) be Daubechies orthonormal wavelet the semiorthogonal spline wavelet of order \(m\), respectively. Then

\[
\lim_{m \to \infty} \frac{C_{m,p}(\varphi^D_m)}{C_{m,p}(\psi^S_m)}^{1/m} = \frac{1}{2}, \quad \lim_{m \to \infty} \frac{C_{m,p}(\psi^D_m)}{C_{m,p}(\varphi^S_m)}^{1/m} = \frac{16}{\lambda_2\pi^4},
\] (3.17)

and

\[
\lim_{m \to \infty} \frac{C_{m,p}(\varphi^S_m)}{C_{m,p}(\psi^D_m)}^{1/m} = \frac{32}{\lambda_2\pi^4} = 0.47128...\] (3.18)

In order to study the high dimensional tensor product wavelets, we need to compare the asymptotic behaviors between the scaling function \(\varphi\) and the wavelet function \(\psi\) for both the Daubechies orthonormal wavelets and semi-orthonormal spline wavelets.

For the Daubechies orthonormal wavelets, again, by Theorems 2 and 3, we have the following result.
Corollary 6. Let $\varphi_m^D$ and $\psi_m^D$ be the Daubechies orthonormal scaling function and wavelet function of order $m$, respectively. Let $k_1, k_2 \geq 0$ be nonnegative integers. Then

$$\lim_{m \to \infty} \frac{\| \hat{\varphi}_{k_1}^D \|_p}{\| \hat{\psi}_{k_2}^D \|_p} = \frac{\pi^{k_1+k_2}}{(1 - 2^{-1-pk_2})^{1/p}} \left( \frac{pk_1 + 1}{pk_2 - 1} \right)^{1/p}. \quad (3.19)$$

For the semiorthogonal wavelets, similarly, using the results of Theorems 4 and 5, we have

Corollary 7. Let $\varphi_m^S$ and $\psi_m^S$ be the semiorthogonal spline scaling function and wavelet function of order $m$, respectively. Let $k_1, k_2 \geq 0$ be nonnegative integers. Then

$$\lim_{m \to \infty} \left( \frac{\| \hat{\varphi}_{k_1}^S \|_p}{\| \hat{\psi}_{k_2}^S \|_p} \right)^{1/m} = \sqrt[2m]{\frac{\xi_1 \xi_2}{A_1}} = 1.1311\ldots. \quad (3.20)$$

4. High-dimensional Wavelet Coefficients

One of the simplest way to construct high-dimensional wavelets is using tensor product. In this section, we discuss the wavelet coefficients for high-dimensional tensor product wavelets. We shall mainly focus on dimension two while results of high dimensions can be similarly obtained due to the properties of tensor product.

Let $\varphi, \psi$ be the refinable function and wavelet function that generates a wavelet basis in $L_2(\mathbb{R})$. Then, in two-dimensional case, the refinable function $\Phi(x_1, x_2) = \varphi(x_1)\varphi(x_2)$ and we have three wavelets instead of one,

$$\Psi^1(x_1, x_2) := \psi(x_1)\varphi(x_2),$$
$$\Psi^2(x_1, x_2) := \varphi(x_1)\psi(x_2),$$
$$\Psi^3(x_1, x_2) := \psi(x_1)\psi(x_2). \quad (4.1)$$

Let $k = (k_1, k_2) \in \mathbb{Z}^2$ be a two-dimensional index. Then, for a two-dimensional wavelet function $\Psi$, we can define $C_{k,p}(\Psi)$ similar to (1.5) by

$$C_{k,p}(\Psi) = \sup_{f \in \mathcal{H}^p} \frac{|\langle f, \Psi \rangle|}{\| \Psi \|_p} \| \hat{\Psi} \|_p, \quad (4.2)$$

where $1 < p, p' < \infty$, $1/p + 1/p' = 1$ and $\mathcal{H}^p := \{ f \in L_{p'}(\mathbb{R}^2) : \| (i\omega)^{\xi} f(\omega) \|_{p'} \leq 1 \}$. Here, for $x = (x_1, x_2) \in \mathbb{R}^2$, $k = (k_1, k_2) \in \mathbb{Z}^2$, $x^k := x_1^{k_1}x_2^{k_2}$. And for a function $f \in L_1(\mathbb{R}^2)$, $\hat{f}$ is defined to be a function such that $\hat{f}(\omega) = (i\omega)^{\xi} \hat{f}$, where $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$. In particular, when $\Psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$, one can easily show that $C_{k,p}(\Psi) = C_{k,p}(\psi_1)C_{k,p}(\psi_2)$.

In two-dimensional case, the semiorthogonal wavelets can be represented by

$$\Psi_m^{S,1}(x_1, x_2) := \varphi_m^S(x_1)\varphi_m^S(x_2),$$
$$\Psi_m^{S,2}(x_1, x_2) := \varphi_m^S(x_1)\psi_m^S(x_2),$$
$$\Psi_m^{S,3}(x_1, x_2) := \psi_m^S(x_1)\psi_m^S(x_2), \quad (4.3)$$

We can obtain that following corollaries using results in previous sections and the properties of tensor product.
Corollary 8. Let $\Psi_m^{s,1}$, $\Psi_m^{s,2}$, and $\Psi_m^{s,3}$ be defined in (4.3). Let $k = (k_1, k_2) \in \mathbb{N}^2$. Then,

\[
C_{k,p}(\Psi_m^{s,1}) \geq \frac{1}{2^{k_2}} \left( \frac{1}{\pi} \right)^{k_1+k_2} \frac{\sqrt{K_{2(m+k_2)+1}}}{K_{2m+1}},
\]

\[
C_{k,p}(\Psi_m^{s,2}) \geq \frac{1}{2^{k_2}} \left( \frac{1}{\pi} \right)^{k_1+k_2} \frac{\sqrt{K_{2(m+k_2)+1}}}{K_{2m+1}},
\]

\[
C_{k,p}(\Psi_m^{s,3}) \geq \left( \frac{1}{2\pi} \right)^{k_1+k_2} \frac{\sqrt{K_{2(m+k_2)+1}}}{K_{2m+1}},
\]

where $K_j$’s are the Favard’s constants. And

\[
\lim_{m \to \infty} C_{k,p}(\Psi_m^{s,1}) = (2\pi - 4\xi_2)^{-k_2},
\]

\[
\lim_{m \to \infty} C_{k,p}(\Psi_m^{s,2}) = (2\pi - 4\xi_1)^{-k_1},
\]

\[
\lim_{m \to \infty} C_{k,p}(\Psi_m^{s,3}) = (2\pi - 4\xi_2)^{-k_1-k_2},
\]

where $\xi_1, \xi_2$ are constants in Corollary 3.

In two-dimensional case, Daubechies wavelets can be represented by

\[
\Psi_m^{D,1}(x_1, x_2) := \psi_m^{D,1}(x_1)\psi_m^{D,2}(x_2),
\]

\[
\Psi_m^{D,2}(x_1, x_2) := \phi_m^{D,1}(x_1)\psi_m^{D,2}(x_2),
\]

\[
\Psi_m^{D,3}(x_1, x_2) := \psi_m^{D,1}(x_1)\phi_m^{D,2}(x_2).
\]

Similarly, we have the following result.

Corollary 9. Let $\Psi_m^{D,1}$, $\Psi_m^{D,2}$, and $\Psi_m^{D,3}$ be defined in (4.5). Then,

\[
\lim_{m \to \infty} C_{k,p}(\Psi_m^{D,1}) = \pi^{-k_2} \left( \frac{1 - 2^{-1-pk_1}}{pk_1 - 1} \right)^{1/p} \frac{\pi^{-k_2}}{(1-pk_2)^{1/p}}, \quad k_2 \leq 1/p, k_1 \in \mathbb{N},
\]

\[
\lim_{m \to \infty} C_{k,p}(\Psi_m^{D,2}) = \pi^{-k_2} \left( \frac{1 - 2^{-1-pk_1}}{pk_2 - 1} \right)^{1/p} \frac{\pi^{-k_1}}{(1-pk_1)^{1/p}}, \quad k_1 \leq 1/p, k_2 \in \mathbb{N},
\]

\[
\lim_{m \to \infty} C_{k,p}(\Psi_m^{D,3}) = \pi^{-k_1-k_2} \left( \frac{1 - 2^{-1-pk_1}}{pk_1 - 1} \right)^{1/p} \frac{1 - 2^{-1-pk_2}}{pk_2 - 1}, \quad (k_1, k_2) \in \mathbb{N}^2.
\]

References

[1] V.F. Babenko, S.A. Spektor, *Estimates for wavelet coefficients on some classes of functions*, Ukrainian Mathematical Journal. Vol. 59, 12 (2007), 1791–1799.

[2] V.F. Babenko and S.A. Spektor, *Inequalities similar to those of Bernstein for non-periodic splines in L2 space*, Vestnik DNU. Vol. 1, 6 (2008), 21–29.

[3] G. Beylkin, R. Coifman and V. Rokhlin, *Fast wavelet transforms and numerical algorithms I*, Comm. Pure Appl. Math., 44 (1991), 141–183.
[4] C.K. Chui, An Introduction to Wavelets, Wavelet Analysis and Its Applications, Vol. 1 (Academic Press 1992).

[5] C.K. Chui and J. Wang, On compactly supported spline wavelets and duality principle, Trans. Amer. Math. Soc. 330 (1992), 903–915.

[6] I. Daubechies, Ten Lectures on Wavelets, SIAM, CBMS Series, 1992.

[7] S. Ehrich, On the estimate of wavelet coefficients, Adv. Comput. Math., 13 (2000), 105–129.

[8] F. Keinert, Biorthogonal wavelets for fast matrix computations, Appl. Comput. Harm. Anal., 1 (1994), 147–156.

[9] N.P. Kornejchuk, Splines in the Approximation Theory, Moscow, (1984).

[10] T. Petersdorff and C. Schwab, Wavelet approximations for first kind boundary integral equations on polygons, Numer. Math., 74 (1996), 479–519.

[11] A. Rathsfeld, A wavelet algorithm for the solution of the double layer potential equation over polygonal boundaries, J. Integral Equation Appl. 7 (1995), 47–97.

[12] M. Unser, Ten good reasons for using spline wavelets, in: Wavelet Applications in Signal and Image Processing V, Proc. SPIE, 3169 (1997), 422–432.