KAC-LÉVY PROCESSES

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ABSTRACT. Markov-modulated Lévy processes with two different regimes of restarting are studied. These regimes correspond to the completely renewed process and to the process of Markov modulation, accompanied by jumps. We give explicit expressions for the Lévy-Khintchine exponent in the case of a two-state underlying Markov chain.

For the renewal case, the limit distributions (as \( t \to \infty \)) are obtained. In the case of processes with jumps, we present some results for the exponential functional.

**Keywords:** Markov-modulated Lévy process; Markov-switching model; Goldstein-Kac process; Lévy-Khintchine exponent; Lévy-Laplace exponent; mixture of distributions; exponential functional

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1. Introduction: Kac-Lévy process

New classes of stochastic processes appearing in various fields of applications could be constructed as follows. First, we have the set of underlying processes, the main blocks of the construction, second, the switching mechanism between these blocks should be described, and third, the new starting points of these underlying processes could be determined in various manners.

For instance, let \( \eta_n = \eta_n^\varepsilon(t) \in \mathbb{X}, \ t \geq 0, \ n \geq 0, \) be the sequence of random processes, \( \{T_n\}_{n \geq 0} \) be the sequence of switching times and \( x = x_n \in \mathbb{X}, \ n \geq 0, \) be a sequence of starting points. Here \( \mathbb{X} \) is the phase space (some topological vector space, in general). The composite process \( X \) is defined by

\[
X(t) = \eta_n^\varepsilon(t - T_n) \quad \text{for} \quad t \in [T_n, T_{n+1}), \quad n \geq 0.
\]  

(1.1)

This construction can be accompanied by a simple extension that allows for additional jumps at the times of the state change.

In this paper, the sequence \( \{T_n\}_{n \geq 0} \) is defined by a finite-state Markov process \( \varepsilon = \varepsilon(t) \in E, \ t \geq 0, \) that is \( T_n \) is the switching time of \( \varepsilon \) from one state to another. Assume that \( \{\eta_n\}_{n \geq 0} \) is a sequence of real-valued one-dimensional independent Lévy processes independent of \( \varepsilon. \)

Let us summarise the most important known results. Assume that \( \eta(t), \ t \geq 0, \) is a real-valued one-dimensional Lévy process defined on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), that is the process satisfying the following properties:

- The paths are a. s. right-continuous with left limits, and \( \mathbb{P}(\eta(0) = 0) = 1; \)
- \( \eta(t) - \eta(s) \overset{D}{=} \eta(t - s), \ 0 \leq s \leq t; \)
- \( \eta(t) - \eta(s) \) is independent of the history \( \{\eta_u \mid u \leq s\}. \)

The jump part of the Lévy process \( \eta \) can be characterised by means of the Lévy measure \( \Pi \) (intensity measure),

\[
\Pi(A) := \mathbb{E}[J([0, 1] \times A)], \quad A \in \mathcal{B}(\mathbb{R} \setminus \{0\}),
\]

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satisfying the condition
\[ \int_{\mathbb{R}} (1 \wedge x^2) \Pi(dx) < \infty. \]

Here the jump measure \( J(I, A) \), \( I \in \mathcal{B}([0, \infty)), \ A \in \mathcal{B}(\mathbb{R}) \), is the random measure which is defined by
\[ J(I, A) = \sum_{s \in I} \mathbb{1}_{\Delta \eta(s) \in A \setminus \{0\}}. \]

In general, process \( \eta \) is completely identified by the triplet \( \langle \gamma, \sigma^2, \Pi \rangle \), see e.g. [23, 30]. This means that \( \eta(t) \) may be decomposed into a sum of independent terms (Lévy-Itô decomposition):
\[ \eta(t) = \gamma t + \sigma W(t) + Y(t) + M(t), \quad t \geq 0. \]

Here \( \gamma \) and \( \sigma \) are constants, \( W(t), \ t \geq 0, \) is the standard Brownian motion, \( Y = Y(t) \) is the compound Poisson process,
\[ Y(t) = \int^t_0 \int_{|y| \geq R} y J(ds, dy) = \sum_{n=1}^{N_t} Y_n, \]
where \( \{Y_n\} \) are independent random variables with common distribution supported on \( \{y : |y| \geq R\} \), and \( M = M(t) \) is the square-integrable martingale,
\[ M(t) = \int^t_0 \int_{|y| < R} y J(ds, dy) - t \int_{|y| < R} y \Pi(dy). \]

The distribution of \( \eta(t) \) is determined by the characteristic function \( \mathbb{E} [e^{i \theta \eta(t)}] \) which is given by \( \exp(-t \psi(\theta)) \). According to the Lévy-Khintchine formula, \( \psi(\theta) \) can be written as
\[ \psi(\theta) = \left\{ -i \gamma \theta + \frac{1}{2} \sigma^2 \theta^2 \right\} \\
+ \left\{ \int_{|x| \geq R} (1 - e^{i \theta x}) \Pi(dx) \right\} \\
+ \left\{ \int_{0 < |x| < R} (1 - e^{i \theta x} + i \theta x) \Pi(dx) \right\}, \quad \theta \in \mathbb{R}. \]

Notice that the real part of \( \psi(\theta) \) is nonnegative,
\[ \text{Re}[\psi(\theta)] = \frac{1}{2} \sigma^2 \theta^2 + \int_{|x| \geq R} (1 - \cos \theta x) \Pi(dx) \geq 0, \quad \forall \theta. \quad (1.2) \]

We will also use the Lévy-Laplace exponent
\[ \ell(\xi) = \psi(i \xi) = \mathbb{E} \left[ \exp (-\xi X(t)) \right]; \quad \mathbb{E} \left[ e^{-\xi \eta(t)} \right] = \exp (-t \ell(\xi)). \]

A Markov-modulated Lévy process appears when we try to define a composite process \( X(t) \) that involves switching between the Lévy components once the underlying Markov process \( \varepsilon \) is switched. It is assumed that in the interval \( t \in [T_n, T_{n+1}) \) where \( \varepsilon \) is constant, \( \varepsilon(t) \equiv i \), the process \( X(t) \) is developing as a Lévy process with the triplet \( \langle \gamma_i, \sigma^2_i, \Pi_i \rangle, \ i \in E, \) see (1.1). Aspiring to simplicity, in this paper we study the case of Markov modulation with two states, i.e. \( E = \{0, 1\} \).

In recent decades, the Markov-modulated Lévy processes with various extensions become very popular among researchers from different points of view. Terminology also depends on the scope of application.
This type of process is called the \textit{Markov additive processes}, if it is used in queuing theory, see [1] [22]. Regarding the financial market modeling, this process becomes a \textit{Lévy regime-switching model}, see [6] [9] [10] [12] [16] [17] [29], or a \textit{Markov-switching model}, [13] Section 9.5]. On the other hand, Markov modulated piecewise linear processes have been extensively studied, starting with the seminal paper by Marc Kac, [18].

To unify the terminology and to avoid misunderstandings, I propose the term \textit{Kac-Lévy process} which can be applied to the composite process $X$, (1.1), with Lévy underlying blocks.

In this paper, the two different extensions are studied in detail.

First, in Section 2 we assume that the Kac-Lévy process is completely renewed after each switch, that is, the process is restarted from a new random point. In this case (1.1) becomes

$$X(t) = x_n + \eta_n(t - T_n), \quad T_n \leq t < T_{n+1}, \quad n \geq 0,$$

where the starting points $\{x_n\}$ are independent random variables, independent of the Lévy processes $\{\eta_n(t)\}, \ t \geq 0$. In this case, the composite process $X$ resembles well-studied Markovian growth-collapse processes, which presume a constant trend with additive or multiplicative down jumps. see e.g. [5]. Such models occur in insurance mathematics and related fields, see [1] XIV-5 or [26] Chapters 5 and 11, and in production/inventory models studied by Shanthikumar and Sumita, [31], among others.

Second, in Section 3 we consider the Markov modulation accompanied by jumps. We study a Kac-Lévy process $X$ extended with a jump component,

$$X(t) = X(T_n) + Y_{n-1} + \eta_n(t - T_n), \quad T_n \leq t < T_{n+1}, \quad n \geq 1,$$

$$X(t) = \eta_0(t), \quad 0 \leq t < T_1,$$

where the underlying Lévy processes $\{\eta_n(t), \ t \geq 0\}_{n \geq 0}$ and jump magnitudes $\{Y_n\}_{n \geq 0}$ are independent. These processes are also called \textit{Markov additive processes}, see [22] Proposition 3.1.

In both cases we present the characteristic/moment generating functions of the distribution of $X(t)$.

In particular, if the underlying processes are determined by $\sigma_i = 0, \ \Pi_i = 0$ and $\gamma_i = c_i$, that is $\eta_n(t) = c_i t, \ i \in \{0, 1\}$, then we have the important well-studied example of the composite process $X(t), \ t \geq 0$, which is called (integrated) \textit{telegraph process (Goldstein-Kac process)}, [13] [18], see also [20] [28],

$$X(t) = \int_0^t c_{\varepsilon(s)} ds. \quad (1.4)$$

Process (1.4) is also called \textit{piecewise linear Markov process} or \textit{Markovian fluid}, [1]. This is the persistent random motion switching between two trends at Markovian instants.

The transition densities $p_i(t, y)$ of the telegraph process $X(t)$,

$$p_i(t, y)dy = \mathbb{P}\{X(t) \in dy \mid \varepsilon(0) = i\}, \quad i \in \{0, 1\},$$

follow the coupled integral equations,

$$\begin{cases}
  p_0(t, y) = e^{-\lambda_0 t} \delta(x - c_0 t) + \int_0^t \lambda_0 e^{-\lambda_0 \tau} p_1(t - \tau, y - c_0 \tau) d\tau,
  \\
  p_1(t, y) = e^{-\lambda_1 t} \delta(x - c_1 t) + \int_0^t \lambda_1 e^{-\lambda_1 \tau} p_0(t - \tau, y - c_1 \tau) d\tau,
\end{cases} \quad (1.5)$$

where $\lambda_0$ and $\lambda_1$ are the switching intensities.
Further, the moment generating functions $L_i(t, \xi) := \mathbb{E} [e^{-\xi X(t)} \mid \xi(0) = i]$, $i \in \{0, 1\}$, of $X(t)$ can be expressed by

$$
\begin{align*}
L_0(t, \xi) &= e^{-t(\lambda + c\xi)} \left( \cosh(tD) + (\lambda - c\xi) \frac{\sinh(tD)}{D} \right), \\
L_1(t, \xi) &= e^{-t(\lambda + a\xi)} \left( \cosh(tD) + (\lambda + c\xi) \frac{\sinh(tD)}{D} \right),
\end{align*}
$$

where $a = (c_0 + c_1)/2$, $c = (c_0 - c_1)/2$, $\lambda = (\lambda_0 + \lambda_1)/2$, $\mu = (\lambda_0 - \lambda_1)/2$ and $D = D(\xi) = \sqrt{(\mu + c\xi)^2 + \lambda_0 \lambda_1}$, see [20]. Occupation time distribution for such a process has been analysed by [4]. For the detailed analysis and the properties of the distribution of such a process see [20].

In Sections 3.3 and 3.4 the distribution of the exponential functional of the Kac-Lévy process with jumps is studied and some explicit formulae are presented for the Goldstein-Kac process. Some other rare examples with the similar explicit formulae of this type can be found in [21].

2. Kac-Lévy Processes with Renewal Starting Points

Consider a Markov-modulated Lévy process $X = X(t)$ based on the two-state Markov process $\xi = \xi(t) \in \{0, 1\}$, $t \geq 0$, and on the independent of $\xi$ sequence $\{\eta_n(t)\}_{n \geq 0}$, $t \geq 0$, of independent Lévy processes with the alternating marginal distributions $q_0^l(dy)$ and $q_1^l(dy)$. Let $\psi_0$ and $\psi_1$ be the corresponding Lévy-Khintchine exponents.

At every switching time $T_n$, process $X$ starts from a renewed random point $x_n$:

$$
X(t) = x_n + \eta_n(t - T_n), \quad T_n \leq t < T_{n+1}, \quad n \geq 0,
$$

see (1.3). Here $x_n$, $n \geq 0$, are independent random variables with the alternating distributions $g_0 = g_0(dx)$ and $g_1 = g_1(dx)$. Suppose that the variables $\{x_n\}_{n \geq 0}$ are independent of $\{\eta_n\}_{n \geq 0}$ and $\xi$.

Therefore,

$$
P\{X(t) \in dz \mid T_n \leq t < T_{n+1}\} = g_{\varepsilon_n} * q_{\varepsilon_n - T_n}^l(dz). \quad \varepsilon_n = \varepsilon(T_n).
$$

Here $*$ means convolution, such that for any test-function $\varphi$

$$
\int_{\mathbb{R}^2} \varphi(z) [g * q](dz) = \int_{\mathbb{R}^2} \varphi(x + y)g(dx)q(dy).
$$

The distribution of $X(t)$ is determined by the density vector-function

$$
P^X(t, dy) = (p_0^X(t, dy), p_1^X(t, dy))^T,
$$

with the entries $p_i^X(t, dy) = P\{X(t) \in dy \mid \xi(0) = i\}$, $i \in \{0, 1\}$. We have the following representation of $P^X(t, dy)$.

**Theorem 2.1.** Let $\Lambda^{\text{diag}}$ be the diagonal matrix of the switching intensities,

$$
\Lambda^{\text{diag}} = \begin{pmatrix}
\lambda_0 & 0 \\
0 & \lambda_1
\end{pmatrix}
$$

and

$$
B(t) = \frac{1}{2\lambda} \begin{pmatrix}
1 - e^{-2\lambda t} & 1 + \frac{\lambda_0}{\lambda_1} e^{-2\lambda t} \\
1 + \frac{\lambda_1}{\lambda_0} e^{-2\lambda t} & 1 - e^{-2\lambda t}
\end{pmatrix},
$$

(2.2)
where $2\lambda = \lambda_0 + \lambda_1$.

The density function of $X(t)$ is given by

$$p_X(t, dy) = e^{-t\Lambda_{\text{diag}}} g * q'(dy) + \lambda_0 \lambda_1 \int_0^t [B(t-\tau)e^{-\tau\Lambda_{\text{diag}}} g * q''(dy)] d\tau,$$

(2.3)

t > 0, where vector $g * q'(dy)$ is determined by the entries $g_i * q'_i(dy)$, $i \in \{0, 1\}$; see (2.1).

Proof. Due to (2.1) by conditioning on the first switching similarly to (1.5) we have the coupled integral equations

$$\begin{cases}
p_0^X(t, dy) = e^{-\lambda_0 t}g_0 * q'_0(dy) + \int_0^t \lambda_0 e^{-\lambda_0 \tau} p_1^X(t-\tau, dy) d\tau, \\
p_1^X(t, dy) = e^{-\lambda_1 t}g_1 * q'_1(dy) + \int_0^t \lambda_1 e^{-\lambda_1 \tau} p_0^X(t-\tau, dy) d\tau,
\end{cases}$$

for $t \geq 0$.

Therefore, the time-Laplace transforms,

$$p_i(s, \cdot) := \mathcal{L}_{t \to s} [p_i^X(t, \cdot)] = \int_0^\infty e^{-st} p_i^X(t, \cdot) dt,$$

satisfy the algebraic system,

$$\begin{cases}
p_0(s, \cdot) = Q_0(s, \cdot) + \frac{\lambda_0}{\lambda_0 + s} p_1(s, \cdot), \\
p_1(s, \cdot) = Q_1(s, \cdot) + \frac{\lambda_1}{\lambda_1 + s} p_0(s, \cdot),
\end{cases}$$

(2.4)

where $Q_i(s, \cdot)$ is the time-Laplace transform of $e^{-\lambda_i t} g_i * q'_i(\cdot)$:

$$Q_i(s, \cdot) = \mathcal{L}_{t \to s} [e^{-\lambda_i t} g_i * q'_i(\cdot)], \quad i \in \{0, 1\}.$$

The solution of (2.4) can be written in the form

$$\begin{cases}
p_0(s, \cdot) = Q_0(s, \cdot) + \frac{\lambda_0 \lambda_1}{2\lambda} [Q_0(s, \cdot) + Q_1(s, \cdot)] \frac{1}{s} \\
+ \frac{\lambda_0}{2\lambda} [\lambda_1 Q_1(s, \cdot) - \lambda_0 Q_0(s, \cdot)] \frac{1}{2\lambda + s}, \\
p_1(s, \cdot) = Q_1(s, \cdot) + \frac{\lambda_0 \lambda_1}{2\lambda} [Q_0(s, \cdot) + Q_1(s, \cdot)] \frac{1}{s} \\
- \frac{\lambda_1}{2\lambda} [\lambda_0 Q_1(s, \cdot) - \lambda_1 Q_0(s, \cdot)] \frac{1}{2\lambda + s}.
\end{cases}$$

By applying the inverse Laplace transformation $\mathcal{L}_{s \to t}^{-1}$ we obtain (2.3), cf. (2.5).

Theorem 2.1 allows to obtain the Fourier transform $E \left[ e^{i\theta X(t)} \right]$ of $X(t)$.

Let $\psi_0 = \psi_0(\theta)$ and $\psi_1 = \psi_1(\theta)$ be the Lévy-Khintchine exponents of the underlying Lévy blocks $\eta_0$ and $\eta_1$ respectively and

$$\Psi_{\text{diag}}(\theta) = \begin{pmatrix} \psi_0(\theta) & 0 \\ 0 & \psi_1(\theta) \end{pmatrix}.$$ 

Note that the Fourier transform of the convolution $g * q'(dx)$ is given by

$$\mathcal{F}_{x \to \theta}[g * q'(dx)] = \int_{-\infty}^{\infty} e^{i\theta x} g * q'(dx) = e^{-t \Psi_{\text{diag}}(\theta) \underline{G}(\theta)},$$
where \( \hat{g}(\theta) = (\hat{g}_0(\theta), \hat{g}_1(\theta))^\top \),

\[
\hat{g}_0(\theta) = \int_{-\infty}^{\infty} e^{i \theta x} g_0(dx), \quad \hat{g}_1(\theta) = \int_{-\infty}^{\infty} e^{i \theta x} g_1(dx),
\]

are the Fourier transforms of the distributions of starting points.

Therefore, by (2.3) the Fourier transform \( \Phi = (\Phi_0, \Phi_1)^\top \) of \( p_X(t, \cdot) \),

\[
\Phi_i(t, \theta) = E \left( e^{i \theta X(t)} \mid \varepsilon(0) = i \right), \quad i \in \{0, 1\}, \quad t > 0,
\]

is expressed by

\[
\tilde{\Phi}(t, \theta) = \left[ e^{-t(\Lambda^{\text{diag}} + \Psi^{\text{diag}}(\theta))} + \lambda_0 \lambda_1 \int_0^t \mathcal{B}(t - \tau) e^{-\tau(\Lambda^{\text{diag}} + \Psi^{\text{diag}}(\theta))} d\tau \right] \hat{g}(\theta). \tag{2.5}
\]

Integrating in (2.5) by (2.2) we obtain the following formulae for the entries \( \Phi_0(t, \theta) \) and \( \Phi_1(t, \theta) \) of \( \tilde{\Phi}(t, \theta) \).

**Theorem 2.2.** The Fourier transform \( \tilde{\Phi}(t, \theta) \) of \( X(t) \) is given by

\[
\Phi_0(t, \theta) = e^{-t(\lambda_0 \psi_0(\theta))} \hat{g}_0(\theta) + \frac{\lambda_0 \lambda_1}{2 \lambda} \left\{ \Phi_{00}(t, \theta) \hat{g}_0(\theta) + \Phi_{01}(t, \theta) \hat{g}_1(\theta) \right\}, \tag{2.6}
\]

and

\[
\Phi_1(t, \theta) = e^{-t(\lambda_1 \psi_1(\theta))} \hat{g}_1(\theta) + \frac{\lambda_0 \lambda_1}{2 \lambda} \left\{ \Phi_{10}(t, \theta) \hat{g}_0(\theta) + \Phi_{11}(t, \theta) \hat{g}_1(\theta) \right\}, \tag{2.7}
\]

where

\[
\Phi_{00}(t, \theta) = \frac{1 - \exp \left( -\left( \lambda_0 + \psi_0(\theta) \right)t \right)}{\lambda_0 + \psi_0(\theta)} - \frac{\exp(-2\lambda t) - \exp \left( -\left( \lambda_0 + \psi_0(\theta) \right)t \right)}{\psi_0(\theta) - \lambda_1},
\]

\[
\Phi_{01}(t, \theta) = \frac{1 - \exp \left( -\left( \lambda_1 + \psi_1(\theta) \right)t \right)}{\lambda_1 + \psi_1(\theta)} + \frac{\lambda_0 \exp(-2\lambda t) - \exp \left( -\left( \lambda_1 + \psi_1(\theta) \right)t \right)}{\psi_1(\theta) - \lambda_0},
\]

\[
\Phi_{10}(t, \theta) = \frac{1 - \exp \left( -\left( \lambda_0 + \psi_0(\theta) \right)t \right)}{\lambda_0 + \psi_0(\theta)} + \frac{\lambda_1 \exp(-2\lambda t) - \exp \left( -\left( \lambda_0 + \psi_0(\theta) \right)t \right)}{\psi_0(\theta) - \lambda_1},
\]

and

\[
\Phi_{11}(t, \theta) = \frac{1 - \exp \left( -\left( \lambda_1 + \psi_1(\theta) \right)t \right)}{\lambda_1 + \psi_1(\theta)} - \frac{\exp(-2\lambda t) - \exp \left( -\left( \lambda_1 + \psi_1(\theta) \right)t \right)}{\psi_1(\theta) - \lambda_0}.
\]

Formulae (2.6)–(2.7) allows us to prove the limit theorem.

To start with, define a simple mixture of distributions. Let

\[
F_{X_1}(dx), \ldots, F_{X_n}(dx)
\]

be the distributions of random variables \( X_1, \ldots, X_n \), and \( \zeta \) be an independent random variable taking values \( 1, \ldots, n \) with probabilities \( p_1, \ldots, p_n \); \( \sum_{k=1}^{n} p_k = 1 \). The mixture of distributions, \( \mathfrak{M}_p(X_1, \ldots, X_n) \), is defined by

\[
\mathfrak{M}_p(X_1, \ldots, X_n) \overset{D}{=} X_\zeta, \quad p = (p_1, \ldots, p_n).
\]

The distribution of \( X_\zeta \) is given by linear combination, \( \sum_{k=1}^{n} p_k F_{X_k}(dx) \).

For example, an asymmetric double exponential distribution (an asymmetric Laplace distribution) with the density

\[
f(x) = p \lambda_0 e^{-\lambda_0 |x|} 1_{\{x > 0\}} + (1 - p) \lambda_1 e^{-\lambda_1 |x|} 1_{\{x < 0\}},
\]

see e. g. [8], is the mixture of \( X_0 \) and \(-X_1\), where \( X_0 \) and \( X_1 \) are independent and exponentially distributed, \( X_i \sim \text{Exp}(\lambda_i) \), \( i \in \{0, 1\} \).
Theorem 2.3. Let $X = X(t), \ t \geq 0$, be defined by \((1.3)\) and $\tau_0, \tau_1$ are independent exponentially distributed random variables with parameters $\lambda_0$ and $\lambda_1$ respectively.

$X(t)$ converges in distribution (when $t \to \infty$) to the $\mathbf{p}$-mixture,

$$X(t) \overset{D}{\to} \mathcal{M}_\mathbf{p}(x_0 + \eta_0(\tau_0), \ x_1 + \eta_1(\tau_1)), \hspace{1cm} (2.8)$$

where $\mathbf{p} = (p_0, \ p_1)$, $p_0 = \frac{\lambda_1}{\lambda_0 + \lambda_1}$, $p_1 = \frac{\lambda_0}{\lambda_0 + \lambda_1}$; $x_0$, $x_1$ are the independent renewal starting points with distributions $g_0$, $g_1$.

The Fourier transform of the limiting distribution $\mathcal{M}_\mathbf{p}(x_0 + \eta_0(\tau_0), \ x_1 + \eta_1(\tau_1))$ is given by

$$\Phi(\theta) := \frac{\lambda_0 \lambda_1}{\lambda_0 + \lambda_1} \left( \frac{\tilde{g}_0(\theta)}{\lambda_0 + \psi_0(\theta)} + \frac{\tilde{g}_1(\theta)}{\lambda_1 + \psi_1(\theta)} \right). \hspace{1cm} (2.9)$$

Proof. Note that $\Re[\psi_0(\theta)] \geq 0$, $\Re[\psi_1(\theta)] \geq 0$, see \((1.2)\). Passing to limit in \((2.6)-(2.7)\) as $t \to \infty$ one can easily obtain $\lim_{t \to \infty} \Phi_0(t, \theta) = \lim_{t \to \infty} \Phi_1(t, \theta) = \Phi(\theta)$, where $\Phi$ is given by \((2.9)\). Since,

$$\mathbb{E} \left[ e^{i \theta \eta_i(\tau)} \right] = \int_0^\infty \lambda_i e^{-\lambda_i \tau} \mathbb{E} \left[ e^{i \theta \eta_i(\tau)} \right] d\tau$$

$$= \int_0^\infty \lambda_i e^{-(\lambda_i + \psi_i(\theta)) \tau} d\tau = \frac{\lambda_i}{\lambda_i + \psi_i(\theta)}, \hspace{1cm} i \in \{0, 1\},$$

the Fourier transform $\Phi$ corresponds to the mixture

$$\mathcal{M}_\mathbf{p}(x_0 + \eta_0(\tau_0), \ x_1 + \eta_1(\tau_1)).$$

Example 2.1. Telegraph process. Let $\eta_0(t) = c_0 t$, $\eta_1(t) = c_1 t$ be deterministic. The limiting distribution of the Goldstein-Kac process $X(t)$ with renewal starting points is determined by \((2.9)\) with $\psi_i(\theta) = -ic_i \theta$, $i \in \{0, 1\}$.

Note that the inverse Fourier transform of $\frac{1}{\alpha - ic \theta}$, $\alpha > 0$, is given by

\[
F_{\theta \to x}^{-1} \left[ \frac{1}{\alpha - ic \theta} \right] = \begin{cases} \frac{1}{|c|} e^{-\alpha x/c} 1_{\{c x > 0\}}, & c \neq 0, \\ \frac{1}{\alpha} \delta(x), & c = 0. \end{cases} \hspace{1cm} (2.10)
\]

In the case of a particle always restarting from the origin, that is $g_0 = g_1 = \delta(dx)$, the limit \((2.8)\) is the mixture of two exponential distributions. In particular, if $c_0, c_1 > 0$, then by \((2.10)\) the limit is characterised by

$$f^*(x) = \frac{\lambda_0 \lambda_1}{\lambda_0 + \lambda_1} \left[ \frac{1}{c_0} e^{-\lambda_0 x/c_0} + \frac{1}{c_1} e^{-\lambda_1 x/c_1} \right] 1_{\{x > 0\}}.$$

If $c_0, c_1 < 0$, then the limiting density function is

$$f^*(x) = \frac{\lambda_0 \lambda_1}{\lambda_0 + \lambda_1} \left[ \frac{1}{|c_0|} e^{-\lambda_0 x/c_0} + \frac{1}{|c_1|} e^{-\lambda_1 x/c_1} \right] 1_{\{x < 0\}}.$$

In the case of the opposite signs, $c_0 > 0 > c_1$, the limit is the Laplace distribution with the density function

$$f^*(x) = \frac{\lambda_0 \lambda_1}{\lambda_0 + \lambda_1} \left[ \frac{1}{c_0} e^{-\lambda_0 x/c_0} 1_{\{x > 0\}} + \frac{1}{|c_1|} e^{-\lambda_1 x/c_1} 1_{\{x < 0\}} \right],$$

cf \[25\]. If $c_0 = 0$ or $c_1 = 0$, then the corresponding term becomes Dirac’s $\delta(x)$. 


Example 2.2. Alternating Brownian motions with alternating drifts. Let \( \eta_0(t) = c_0 t + \sigma_0 W^{(0)}(t) \), \( \eta_1(t) = c_1 t + \sigma_1 W^{(1)}(t) \), where \( W^{(0)} \) and \( W^{(1)} \) are two independent standard Brownian motions, \( c_0, c_1, \sigma_0, \sigma_1 \) are constants. In this case \( \psi_0(\theta) = -ic_0 \theta + \sigma_0^2 \theta^2 / 2 \), \( \psi_1(\theta) = -ic_1 \theta + \sigma_1^2 \theta^2 / 2 \) and (2.9) becomes

\[
\frac{\lambda_0 \lambda_1}{\lambda_0 + \lambda_1} \left[ \frac{\hat{g}_0(\theta)}{\sigma_0^2 \theta^2 / 2 - ic_0 \theta + \lambda_0} + \frac{\hat{g}_1(\theta)}{\sigma_1^2 \theta^2 / 2 - ic_1 \theta + \lambda_1} \right].
\]

Note that

\[
\frac{1}{\sigma^2 \theta^2 / 2 - ic \theta + \lambda} = A \left( \frac{1}{\alpha_2 - i \theta} - \frac{1}{\alpha_1 - i \theta} \right),
\]

where \( A > 0 \) and the numbers \( \alpha_1, \alpha_2 \) are of the opposite signs:

\[
A = (c^2 + 2 \lambda \sigma^2)^{-1/2}, \quad \alpha_1 = \frac{-c - \sqrt{c^2 + 2 \lambda \sigma^2}}{\sigma^2} < 0 < \frac{-c + \sqrt{c^2 + 2 \lambda \sigma^2}}{\sigma^2} = \alpha_2. \tag{2.11}
\]

Hence, by (2.10) the inverse Laplace transform is

\[
F^{-1}_{\theta \rightarrow x} \left[ \frac{1}{\sigma^2 \theta^2 / 2 - ic \theta + \lambda} \right] = A \left[ e^{-\alpha_2 x 1_{\{x > 0\}}} + e^{-\alpha_1 x 1_{\{x < 0\}}} \right],
\]

which gives the Laplace distribution.

Figure 1. Density function \( f^*(x) \), (2.12), of Example 2.2 with \( \lambda_0 = 2 \), \( \lambda_1 = 1 \); \( c_0 = 1 \), \( c_1 = -1 \); \( \sigma_0 = 1 \), \( \sigma_1 = 2 \).
Therefore, the limiting distribution is given by the density function
\[
 f^*(x) = \frac{\lambda_0 \lambda_1}{\lambda_0 + \lambda_1} \left\{ A^{(0)} \left[ \int_{-\infty}^{x} e^{-\alpha_2^{(0)}(x-y)} g_0(dy) + \int_{x}^{\infty} e^{-\alpha_1^{(0)}(x-y)} g_0(dy) \right] \\
+ A^{(1)} \left[ \int_{-\infty}^{x} e^{-\alpha_2^{(1)}(x-y)} g_1(dy) + \int_{x}^{\infty} e^{-\alpha_1^{(1)}(x-y)} g_1(dy) \right] \right\},
\]
where \( A^{(0)}, \alpha_1^{(0)}, \alpha_2^{(0)} \) and \( A^{(1)}, \alpha_1^{(1)}, \alpha_2^{(1)} \) are defined by (2.11) with the sets of parameters \( c_0, \sigma_0, \lambda_0 \) and \( c_1, \sigma_1, \lambda_1 \) respectively.

If \( g_0 = g_1 = \delta_0(dx) \), then the limiting density function is given by the mixture of two Laplace distributions
\[
 f^*(x) = \frac{\lambda_0 \lambda_1}{\lambda_0 + \lambda_1} \left[ \left( A^{(0)} e^{-\alpha_2^{(0)} x} + A^{(1)} e^{-\alpha_2^{(1)} x} \right) 1_{\{x > 0\}} + \left( A^{(0)} e^{-\alpha_1^{(0)} x} + A^{(1)} e^{-\alpha_1^{(1)} x} \right) 1_{\{x < 0\}} \right],
\]
see Fig. 1.

**Example 2.3. Alternating compound Poisson processes with exponentially distributed jumps.** Let \( \eta_0 = c_0 t + \sum_{m=1}^{M_0(t)} Y_m^{(0)} \) and \( \eta_1 = c_1 t + \sum_{m=1}^{M_1(t)} Y_m^{(1)} \), where \( \{Y_m^{(0)}\}, \{Y_m^{(1)}\} \) are two independent sets of iid random variables, independent of two independent counting Poisson processes \( M_0(t), M_1(t) \) with intensities \( \nu_0 \) and \( \nu_1 \) respectively. Let \( Y_m^{(0)} \sim \text{Exp}(a_0) \) and \( Y_m^{(1)} \sim \text{Exp}(a_1) \), \( a_0, a_1 > 0 \).

The Lévy-Khintchine exponents are
\[
 \psi_i(\theta) = -i \theta \left( c_i + \frac{\nu_i}{a_i - i \theta} \right), \quad i \in \{0, 1\},
\]
and (2.9) becomes
\[
 \frac{\lambda_0 \lambda_1}{\lambda_0 + \lambda_1} \left[ \frac{a_0 - i \theta}{\nu_i} \frac{a_0 - i \theta}{a_0 - i \theta} \right] \frac{\theta}{a_0 - i \theta(a_0 c_0 + \nu_0 + \lambda_0)} - \frac{\theta}{a_1 - i \theta(a_1 c_1 + \nu_1 + \lambda_1)} - \frac{\theta}{a_1 - i \theta(a_1 c_1 + \nu_1 + \lambda_1)} - \frac{\theta}{a_1 - i \theta(a_1 c_1 + \nu_1 + \lambda_1)} - \frac{\theta}{a_1 - i \theta(a_1 c_1 + \nu_1 + \lambda_1)} - \frac{\theta}{a_1 - i \theta(a_1 c_1 + \nu_1 + \lambda_1)} - \frac{\theta}{a_1 - i \theta(a_1 c_1 + \nu_1 + \lambda_1)} - \frac{\theta}{a_1 - i \theta(a_1 c_1 + \nu_1 + \lambda_1)}.
\]
In this case, we manage like in Example 2.2. Note that
\[
 \frac{a - i \theta}{a \lambda - i \theta(ac + \nu + \lambda) - c \theta^2} = \begin{cases} \frac{A_1}{\alpha_1 - i \theta} + \frac{A_2}{\alpha_2 - i \theta}, & \text{if } c < 0, \\
\frac{1}{\nu + \lambda} + \frac{A_3}{\alpha_3 - i \theta}, & \text{if } c = 0. \end{cases}
\]
Here \( \alpha_1 \) and \( \alpha_2 \) are the roots of the equation \( \phi(\alpha) := a^2 - (ac + \nu + \lambda) \alpha + a \lambda = 0 \), \( c \neq 0 \).
Since \( \phi(0) = a \lambda > 0 \) and \( \phi(a) = -\nu a < 0 \) we have two positive roots \( \alpha_1, \alpha_2 > 0 \), if \( c > 0 \); in the case \( c < 0 \) the roots are of the opposite signs, \( \alpha_2 < 0 < \alpha_1 < a \). Precisely,
\[
 \alpha_1 = \frac{ac + \nu + \lambda - \sqrt{D}}{2c}, \quad \alpha_2 = \frac{ac + \nu + \lambda + \sqrt{D}}{2c},
\]
and
\[
 A_1 = \frac{a - \alpha_1}{\sqrt{D}}, \quad A_2 = \frac{\alpha_2 - a}{\sqrt{D}}, \quad A_3 = (a - \alpha_3)/(\nu + \lambda) = av/(\nu + \lambda)^2 > 0,
\]
where \( D = (ac + \nu + \lambda)^2 - 4ac\lambda, \) \( D > 0. \) Note that if \( c > 0, \) then \( A_1, A_2 > 0; \) if \( c < 0, \) then \( A_1 > 0 > A_2. \)

Due to (2.10) the inverse Fourier transform

\[
F(x; a, c, \nu, \lambda) = \mathcal{F}^{-1}_{\theta \to x} \left[ \frac{a - i\theta}{a\lambda - i\theta(ac + \nu + \lambda) - c\theta^2} \right]
\]

can be expressed explicitly,

\[
F(x; a, c, \nu, \lambda) = \begin{cases} 
A_1e^{-\alpha_1x} + A_2e^{-\alpha_2x}1_{\{x>0\}}, & \text{if } c > 0, \\
A_1e^{-\alpha_1x}1_{\{x>0\}} - A_2e^{-\alpha_2x}1_{\{x<0\}}, & \text{if } c < 0, \\
\frac{\delta(x)}{\nu + \lambda} + A_3e^{-\alpha_3x}1_{\{x>0\}}, & \text{if } c = 0,
\end{cases}
\]

where \( \alpha_k \) and \( A_k, k = 1, 2, 3, \) are defined by (2.13) and (2.14).

So the limiting density function is given by the convolutions

\[
f^*(x) = \frac{\lambda_0\lambda_1}{\lambda_0 + \lambda_1} \left[ F(\cdot; a_0, c_0, \nu_0, \lambda_0) * g_0(x) + F(\cdot; a_1, c_1, \nu_1, \lambda_1) * g_1(x) \right].
\]

The case of \( g_0 = g_1 = \delta_0(dx) \) can be described as in the previous examples.

3. KAC-LÉVY PROCESSES WITH JUMPS

Unlike the case of Section 2 suppose that the Markov-modulated process \( X = X(t) \) (with alternating intensities \( \lambda_0, \lambda_1 \)) is jumping after each switching:

\[
\begin{align*}
X(t) &= X(T_n -) + Y_{n-1} + \eta_n(t - T_n), & T_n \leq t < T_{n+1}, & n \geq 1, \\
X(t) &= \eta_0(t), & 0 \leq t < T_1,
\end{align*}
\]

(3.1)

where the underlying Lévy processes \( \{\eta_n(t), t \geq 0\}_{n \geq 0} \) and jump magnitudes \( \{Y_n\}_{n \geq 0} \) are independent with alternating distributions. Such Markov-modulated Lévy process \( X \) can be considered as an example of Itô-Lévy process, see e.g. [11].

3.1. Lévy-Laplace exponent of \( X(t) \). Let \( \ell_0 \) and \( \ell_1 \) be alternating Lévy-Laplace exponents of the underlying processes \( \eta_n(t) \) and \( h_0(dy) \) and \( h_1(dy) \) be alternating distributions of jump magnitudes \( Y_n \) with the Laplace transforms

\[
\tilde{h}_0(\xi) = \int_{-\infty}^{\infty} e^{-\xi y}h_0(dy), \quad \tilde{h}_1(\xi) = \int_{-\infty}^{\infty} e^{-\xi y}h_1(dy).
\]

We denote

\[
\begin{align*}
\lambda := (\lambda_0 + \lambda_1)/2, \quad \mu := (\lambda_0 - \lambda_1)/2, \quad (3.2) \\
\ell(\xi) := (\ell_0 + \ell_1)/2, \quad m(\xi) := (\ell_0 - \ell_1)/2. \quad (3.3)
\end{align*}
\]

Let

\[
\mathcal{L}(\xi) = \begin{pmatrix}
\lambda_0 + \ell_0(\xi) & -\lambda_0\tilde{h}_0(\xi) \\
-\lambda_1\tilde{h}_1(\xi) & \lambda_1 + \ell_1(\xi)
\end{pmatrix}.
\]

(3.4)

Similar to (1.5), the Laplace transform \( \mathcal{L}(t, \xi) = (L_0(t, \xi), L_1(t, \xi))^T \) of \( X(t), \)

\[
L_i(t, \xi) = \mathbb{E}\left[e^{-\xi X(t)} \mid \varepsilon(0) = i\right], \quad i \in \{0, 1\},
\]

can be obtained explicitly.
Theorem 3.1. For $t \geq 0$

\[
L_0(t, \xi) = \frac{1}{2} e^{-\lambda t + \ell t} \left[ e^{tD(\xi)} + e^{-tD(\xi)} + \frac{\lambda_0 \tilde{h}_0(\xi) - \mu - m(\xi)}{D(\xi)} \left( e^{tD(\xi)} - e^{-tD(\xi)} \right) \right],
\]

\[
L_1(t, \xi) = \frac{1}{2} e^{-\lambda t + \ell t} \left[ e^{tD(\xi)} + e^{-tD(\xi)} + \frac{\lambda_1 \tilde{h}_1(\xi) + \mu + m(\xi)}{D(\xi)} \left( e^{tD(\xi)} - e^{-tD(\xi)} \right) \right],
\]

where $D(\xi) = (m(\xi) + \mu)^2 + \lambda_0 \lambda_1 \tilde{h}_0(\xi) \tilde{h}_1(\xi) \right)^{1/2} > 0$.

Here $\lambda, \mu$ and $\ell(\xi), m(\xi)$ are defined by (3.2) and (3.3).

Proof. A commonplace is that

\[
\tilde{L}(t, \xi) = \exp \left( -tL(\xi) \right) 1, \quad 1 = (1,1)^T,
\]

where $L(\xi)$ is defined by (3.4), cf. [1 Chap. XI, Proposition 2.2]. Therefore,

\[
\tilde{L}(t, \xi) = \exp \left( -tL(\xi) \right) 1 = e^{-\alpha_1 t} e_1 + e^{-\alpha_2 t} e_2, \quad t \geq 0,
\]

where $\alpha_1$ and $\alpha_2$ are the eigenvalues of the matrix $L(\xi)$ and $e_1, e_2$ are corresponding eigenvectors with $e_1 + e_2 = 1$.

Representation (3.7) can be derived as a consequence of the coupled integral equations arising by conditioning on the first pattern’s switch,

\[
\begin{cases}
q_0^X(t, \cdot) = e^{-\lambda_0 t} q_0^X(\cdot) + \int_0^t \lambda_0 e^{-\lambda_0 \tau} \left[ p_1^X(t - \tau, \cdot) * q_0^X \ast h_0 \right](\cdot) d\tau, \\
p_1^X(t, \cdot) = e^{-\lambda_1 t} q_1^X(\cdot) + \int_0^t \lambda_1 e^{-\lambda_1 \tau} \left[ p_0^X(t - \tau, \cdot) * q_1^X \ast h_1 \right](\cdot) d\tau,
\end{cases}
\]

where the convolution $*$ is defined by

\[
\int_{\mathbb{R}^3} \varphi(z) [p(t - \tau, \cdot) * q^\tau \ast h](dz) = \int_{\mathbb{R}^3} \varphi(x + y + z) p(t - \tau, dx) q^\tau(dy) h(dz)
\]

for any test-function $\varphi$. From (3.8) one can obtain the following coupled integral equations for the Laplace transform of $X(t)$,

\[
\begin{cases}
L_0(t, \xi) = \exp \left\{ - (\lambda_0 + \ell_0(\xi)) t \right\} + \lambda_0 \tilde{h}_0(\xi) \int_0^t e^{-\lambda_0 \tau} e^{-\ell_0(\xi) \tau} L_1(t - \tau, \xi) d\tau, \\
L_1(t, \xi) = \exp \left\{ - (\lambda_1 + \ell_1(\xi)) t \right\} + \lambda_1 \tilde{h}_1(\xi) \int_0^t e^{-\lambda_1 \tau} e^{-\ell_1(\xi) \tau} L_0(t - \tau, \xi) d\tau,
\end{cases}
\]

which is equivalent to the initial value problem:

\[
\frac{d \tilde{L}}{dt}(t, \xi) = - L(\xi) \tilde{L}(t, \xi), \quad t > 0,
\]

\[
\tilde{L}(0, \xi) = 1,
\]

where $L(\xi)$ is defined by (3.4). Formula (3.7) is the solution of (3.10).
The eigenvalues $\alpha_1$, $\alpha_2$ and the corresponding eigenvectors $e_1$, $e_2$ of the matrix $L(\xi)$ can be found explicitly. We have

$$\alpha_1 = \lambda + \ell(\xi) - D(\xi), \quad \alpha_2 = \lambda + \ell(\xi) + D(\xi)$$  \hspace{1cm} (3.11)

and

$$e_1 = \frac{1}{2} \left( 1 - \frac{\mu + m(\xi) - \lambda \overline{h}_0(\xi)}{D(\xi)}, \ 1 + \frac{\mu + m(\xi) + \lambda \overline{h}_1(\xi)}{D(\xi)} \right)^T, \hspace{1cm} (3.12)$$

$$e_2 = \frac{1}{2} \left( 1 + \frac{\mu + m(\xi) - \lambda \overline{h}_0(\xi)}{D(\xi)}, \ 1 - \frac{\mu + m(\xi) + \lambda \overline{h}_1(\xi)}{D(\xi)} \right)^T. \hspace{1cm} (3.13)$$

Formulae (3.5)-(3.6) follow from (3.7) and (3.11)-(3.13).

**Remark 3.1.** Notice that in the case of Kac process which corresponds to the Lévy-Laplace exponents $\ell_0 = c_0 \xi$, $\ell_1 = c_1 \xi$ formulae (3.5)-(3.6) of Theorem 3.1 are equivalent to (3.3), Theorem 3.1, where the characteristic functions of $X(t)$ are presented explicitly.

Assume that the Markov modulation $\varepsilon = \varepsilon(t)$ is determined by the underlying Lévy processes. For example, let $\varepsilon(t)$ be switching at times of “big” jumps. Let $\ell_0$ switches to $\ell_1$ just after a “big” negative jump of the current Lévy pattern $\eta$, that is if $\Delta \eta_0(t) < -R_0$; vice versa, $\ell_1$ switches to $\ell_0$ just after a “big” positive jump, that is if $\Delta \eta_1(t) > R_1$.

**Corollary 3.1.** The Laplace transform $\hat{L} = (L_0, L_1)^T$ of this process can be obtained in the following form:

$$L_0(t, \xi) = \frac{1}{2} e^{-\alpha(\xi) + \ell(\xi)t} \left[ e^{tD(\xi)} + e^{-tD(\xi)} \right] \left[ \lambda_0 a_0(\xi) - b(\xi) - m(\xi) \right] \left( e^{tD(\xi)} - e^{-tD(\xi)} \right),$$

$$L_1(t, \xi) = \frac{1}{2} e^{-\alpha(\xi) + \ell(\xi)t} \left[ e^{tD(\xi)} + e^{-tD(\xi)} \right] \left[ \lambda_1 a_1(\xi) + b(\xi) + m(\xi) \right] \left( e^{tD(\xi)} - e^{-tD(\xi)} \right).$$

Here $\ell(\xi)$ and $m(\xi)$ are defined by (3.3) and $D(\xi)^2 = (m(\xi) + b(\xi))^2 + \lambda_0 \lambda_1 a_0(\xi) a_1(\xi)$, where

$$\lambda_0 = \Pi_0 \{ x : x < -R_0 \}, \quad \lambda_1 = \Pi_1 \{ x : x > R_1 \},$$

$$a_0(\xi) = \int_{x < -R_0} e^{-\xi x} \Pi_0(dx), \quad a_1(\xi) = \int_{x > R_1} e^{-\xi x} \Pi_1(dx);$$

$$a(\xi) := \frac{1}{2} \left( a_0(\xi) + a_1(\xi) \right), \quad b(\xi) := \frac{1}{2} \left( a_0(\xi) - a_1(\xi) \right).$$

**Proof.** To prove it, note that in this case, instead of (3.3), we have equations

$$L_0(t, \xi) = \exp \left\{ - \left( \lambda_0 + \overline{t}_0(\xi) \right) t \right\} + \lambda_0 a_0(\xi) \int_0^t e^{-\lambda_0 \tau} e^{-\overline{t}_0(\xi) \tau} L_1(t - \tau, \xi) d\tau,$$

$$L_1(t, \xi) = \exp \left\{ - \left( \lambda_1 + \overline{t}_1(\xi) \right) t \right\} + \lambda_1 a_1(\xi) \int_0^t e^{-\lambda_1 \tau} e^{-\overline{t}_1(\xi) \tau} L_0(t - \tau, \xi) d\tau.$$

Here $\lambda_0 = \Pi_0 \{ x : x < -R_0 \}$, $\lambda_1 = \Pi_1 \{ x : x > R_1 \}$ and $\overline{t}_0(\xi)$, $\overline{t}_1(\xi)$ are defined by

$$\overline{t}_0(\xi) = c_0 \xi - \frac{1}{2} \sigma_0^2 \xi^2 + \int_{-\infty}^{-R_0} \left( 1 - e^{-\xi x} - \xi 1_{\{ x < -R_0 \}} \right) \Pi_0(dx),$$

$$\overline{t}_1(\xi) = c_1 \xi - \frac{1}{2} \sigma_1^2 \xi^2 + \int_{-\infty}^{R_1} \left( 1 - e^{\xi x} - \xi 1_{\{ x > R_1 \}} \right) \Pi_1(dx).$$
Further,
\[ \lambda_0 + \ell_0(\xi) = a_0(\xi) + \ell_0(\xi), \quad \lambda_1 + \ell_1(\xi) = a_1(\xi) + \ell_1(\xi). \]

Representation \((3.14)-(3.15)\) follows from \((3.5)-(3.6)\).

3.2. Kac-Lévy subordinated Kac-Lévy process with jumps. Let \(S_0 = S_0(t)\) and \(S_1 = S_1(t)\) be two subordinators (increasing Lévy processes) with the Lévy-Laplace exponents \(\ell^S_0(\xi)\) and \(\ell^S_1(\xi)\) respectively. Let \(\epsilon^S = \epsilon^S(t)\) be a two-state Markov process with switching intensities \(\lambda^S_0\) and \(\lambda^S_1\). We define a Markov-modulated subordinator as a Markov-modulated Lévy process \(Z\) based on \(S_0, S_1\) and \(\epsilon^S\).

Let \(X = X(t)\) be a Kac-Lévy process with jumps based on the two-state Markov process \(\epsilon^X\) with switching intensities \(\lambda^X_0\) and \(\lambda^X_1\) and on the independent Lévy processes \(\eta_0\) and \(\eta_1\) with the Lévy-Laplace exponents \(\ell^X_0, \ell^X_1\). Assume that \(X\) and \(Z\) are independent.

Consider the Kac-Lévy subordinated process \(X \circ Z(t)\). The Laplace transform of \(X \circ Z(t)\),
\[ L_{ij}(t, \xi) = E\{e^{-\xi X \circ Z(t)} | \epsilon^X(0) = i, \epsilon^S(0) = j\}, \quad i, j \in \{0, 1\}, \]
can be expressed explicitly.

**Theorem 3.2.** Let \(\tilde{L}_j(t, \xi) = (L_0(t, \xi), L_{1j}(t, \xi))\), \(j \in \{0, 1\}\). Then
\begin{align*}
\tilde{L}_0(t, \xi) &= e^{-\left(\lambda^S + \ell^S(\alpha_1(\xi))\right)t} \left[ \cosh(tD^S(\alpha_1(\xi))) + \frac{\lambda^S - m^S(\alpha_1(\xi))}{D^S(\alpha_1(\xi))} \sinh(tD^S(\alpha_1(\xi))) \right] e_1 \\
&\quad + e^{-\left(\lambda^S + \ell^S(\alpha_2(\xi))\right)t} \left[ \cosh(tD^S(\alpha_2(\xi))) + \frac{\lambda^S - m^S(\alpha_2(\xi))}{D^S(\alpha_2(\xi))} \sinh(tD^S(\alpha_2(\xi))) \right] e_2, \\
&\quad (3.16) \\
\tilde{L}_1(t, \xi) &= e^{-\left(\lambda^S + \ell^S(\alpha_1(\xi))\right)t} \left[ \cosh(tD^S(\alpha_1(\xi))) + \frac{\lambda^S + m^S(\alpha_1(\xi))}{D^S(\alpha_1(\xi))} \sinh(tD^S(\alpha_1(\xi))) \right] e_1 \\
&\quad + e^{-\left(\lambda^S + \ell^S(\alpha_2(\xi))\right)t} \left[ \cosh(tD^S(\alpha_2(\xi))) + \frac{\lambda^S + m^S(\alpha_2(\xi))}{D^S(\alpha_2(\xi))} \sinh(tD^S(\alpha_2(\xi))) \right] e_2. \\
&\quad (3.17)
\end{align*}

Here \(D^S(\cdot) = \left[(m^S(\cdot) + \mu^S)^2 + \lambda^S_0 \lambda^S_1\right]^{1/2}\); \(\alpha_i(\xi), \ e_i, \ i \in \{0, 1\}\) are identified by \((3.11)-(3.13)\).

**Proof.** By \((3.7)\)
\[ E\left(e^{-\xi X \circ Z(t)} | Z(t)\right) = e^{-\alpha_1(\xi)Z(t)} e_1(\xi) + e^{-\alpha_2(\xi)Z(t)} e_2(\xi), \]
where \(\alpha_1, \alpha_2, e_1, e_2\) are defined by \((3.11)-(3.13)\). Formulæ \((3.16)-(3.17)\) follow from \((3.5)-(3.6)\).

For the Lévy subordination of Lévy processes see [30, Theorem 30.1].

3.3. Exponential functional. Let \(X\) be an alternating Markov-modulated Lévy process with jumps based on a Markov process \(\epsilon = \epsilon(t) \in \{0, 1\}\), with alternating switching intensities \(\lambda_0, \lambda_1\), the sequence of jumps \(\{Y_n\}_{n \geq 1}\) with alternating distributions and the independent alternating Lévy processes \(\{\eta_n\}_{n \geq 0}\).
We study the exponential functional
\[
I_\infty(\gamma) = \int_0^\infty e^{-\gamma X(t)} dt = \sum_{n=0}^{\infty} \int_{T_n}^{T_{n+1}} e^{-\gamma X(t)} dt
\]
\[
= \int_0^{T_1} e^{-\gamma \eta_0(t)} dt + \sum_{n=1}^{\infty} \int_{T_n}^{T_{n+1}} e^{-\gamma (X(T_n) - Y_{n-1}) - \gamma Y_{n-1}} dt
\]
\[
= \mathcal{I}_0 + \sum_{n=1}^{\infty} (Z_n)^\gamma \cdot \mathcal{I}_{n, \gamma}, \quad \gamma > 0,
\]
where
\[
\mathcal{I}_{n, \gamma} = \int_{T_n}^{T_{n+1}} \exp(-\gamma \eta_n(t - T_n)) dt = \int_0^{\Delta T_n} \exp(-\gamma \eta_n(t)) dt,
\]
\[
\Delta T_n = T_{n+1} - T_n, \text{ and } Z_n = \exp(-X(T_n) - Y_{n-1}), \ n \geq 0.
\]
By definition (3.1)
\[
Z_n = \exp(-X(T_n) - Y_{n-1}) = \exp(-X(T_n))
\]
\[
= \exp\left(-\sum_{k=1}^{n} [X(T_k) - X(T_{k-1})]\right) = \exp\left(-\sum_{k=1}^{n} [\eta_{k-1}(\Delta T_k) + Y_{k-1}]\right), \ n \geq 1.
\]
Note that \(\mathcal{I}_{n, \gamma}\) are mutually independent and independent of \(Z_n\).

Similar to (3.18) the exponential functional
\[
I_\infty = I_\infty(1) = \int_0^\infty e^{-X(t)} dt.
\]
can be represented by
\[
I_\infty = \mathcal{I}_0 + \sum_{n=1}^{\infty} Z_n \cdot \mathcal{I}_n,
\]
where \(\mathcal{I}_n = \int_0^{\Delta T_n} e^{-\gamma \eta_n(t)} dt, \ n \geq 0.

First, we study the convergence of the exponential functional \(I_\infty\). To begin, we need the following two important statements.

**Lemma 3.1.** Let \(I_\infty(\gamma)\) be defined by (3.18) with alternating Lévy-Laplace exponents \(\ell_0(\xi), \ell_1(\xi)\). If \(E[I_\infty(\gamma)] < \infty\) for some \(\gamma, \gamma \in (0, 1]\), such that
\[
\ell_0(\gamma) + \lambda_0 > 0, \quad \ell_1(\gamma) + \lambda_1 > 0,
\]
then \(I_\infty < \infty\) a.s.

**Proof.** Let \(\varepsilon_n = \varepsilon(T_n), \ n \geq 0\). If \(\gamma\) satisfies (3.20), then the expectations
\[
E[\mathcal{I}_{n, \gamma}] = E \left[ \int_0^{\Delta T_n} e^{-\gamma \eta_n(t)} dt \right] = \int_0^{\infty} \lambda_{\varepsilon_n} e^{-\lambda_{\varepsilon_n} s} \left[ \int_0^{s} e^{-t\ell_{\varepsilon_n}(\gamma)} dt \right]
\]
\[
= \frac{1}{\lambda_{\varepsilon_n} + \ell_{\varepsilon_n}(\gamma)}
\]
exist, positive and alternating. Let \(E[I_\infty(\gamma)] < \infty\). By (3.18)
\[
E[I_\infty(\gamma)] = E[\mathcal{I}_{0, \gamma}] + \sum_{n=1}^{\infty} E[(Z_n)^\gamma] \cdot E[\mathcal{I}_{n, \gamma}] < \infty.
\]
Therefore the series of \(E[(Z_n)^\gamma]\) also converges,
\[
\sum_{n=1}^{\infty} E[(Z_n)^\gamma] < \infty.
\]
(3.21)
Further, for $\gamma \leq 1$ we have
\[
\sum_{n \geq 1} E \left[ (Z_n \cdot \mathcal{T}_{n,1}) \wedge 1 \right] \leq \sum_{n \geq 1} E \left[ (Z_n \cdot \mathcal{T}_{n,1})^{\gamma} \wedge 1 \right]
\]
\[
\leq \sum_{n \geq 1} E \left[ (Z_n \cdot \mathcal{T}_{n,1})^{\gamma} \right] = \sum_{n \geq 1} E \left[ (Z_n)^{\gamma} \right] \cdot E \left[ (\mathcal{T}_{n,1})^{\gamma} \right].
\]

Moreover,
\[
0 < E \left[ (\mathcal{T}_{n,1})^{\gamma} \right] = E \left[ \left( \int_0^\infty \lambda_{\epsilon_n} e^{-\lambda_{\epsilon_n} t} dt \int_0^t e^{-\gamma_{\epsilon_n}(s)} ds \right)^\gamma \right]
\]
\[
= E \left[ \left( \int_0^\infty e^{-\gamma_{\epsilon_n}(s)} ds \right)^\gamma \right] = \left( \frac{1}{\lambda_{\epsilon_n} + \ell_{\epsilon_n}(1)} \right)^\gamma.
\]

By Hölder’s inequality, for $\gamma < 1$
\[
E \left[ (\mathcal{T}_{n,1})^{\gamma} \right] = E \left[ \left( \int_0^\infty e^{-\gamma_{\epsilon_n}(s)} ds \right)^\gamma \right]
\]
\[
\leq \left( E \int_0^\infty e^{-\gamma_{\epsilon_n}(s)} ds \right)^\gamma = \left( \frac{1}{\lambda_{\epsilon_n} + \ell_{\epsilon_n}(1)} \right)^\gamma.
\]

By (3.21) the series
\[
\sum_{n \geq 1} E \left[ (Z_n \cdot \mathcal{T}_{n,1}) \wedge 1 \right] \leq \sum_{n \geq 1} E \left[ (Z_n)^{\gamma} \right] \cdot E \left[ (\mathcal{T}_{n,1})^{\gamma} \right]
\]

converges, since $E \left[ (\mathcal{T}_{n,1})^{\gamma} \right] < \infty$ are positive and alternating.

Therefore, by [19, Proposition 3.14]
\[
I_\infty = \mathcal{T}_{0,1} + \sum_{n \geq 1} Z_n \cdot \mathcal{T}_{n,1} < \infty, \quad a.s.
\]

Lemma 3.2. $E \left[ I_\infty(\gamma) \right] < \infty$ if and only if
\[
\lambda_0 + \lambda_1 + \ell_0(\gamma) + \ell_1(\gamma) > 0, \quad (3.22)
\]
\[
\ell_0(\gamma)\ell_1(\gamma) + \lambda_0\ell_1(\gamma) + \lambda_1\ell_0(\gamma) + \lambda_0\lambda_1 \left[ 1 - \tilde{h}_0(\gamma)\tilde{h}_1(\gamma) \right] > 0. \quad (3.23)
\]

Proof. By (3.7) $E \left[ \exp \left( -\gamma X(t) \right) \right] \to 0$ as $t \to \infty$, and
\[
E \left[ \int_0^\infty e^{-\gamma X(t)} dt \right] < \infty,
\]

if and only if both eigenvalues $\alpha_1, \alpha_2$ of the matrix $\mathcal{L}(\gamma)$ are positive, i.e. in the case $\text{Tr} \mathcal{L}(\gamma) > 0$ and $\text{Det} \mathcal{L}(\gamma) > 0$. This is equivalent to (3.22) - (3.23).

Conversely, if conditions (3.22) - (3.23) are met, then by (3.7) the expectation
\[
E \left[ I_\infty(\gamma) \mid \varepsilon(0) = i \right] = \int_0^\infty E \left[ e^{-\gamma X(t)} \mid \varepsilon(0) = i \right] dt = \alpha_1^{-1} e_1^{(i)} + \alpha_2^{-1} e_2^{(i)},
\]
is finite, $i \in \{0, 1\}$.

Note that if $\gamma$ satisfies condition (3.20), then condition (3.22) holds. We have the following result.

Theorem 3.3. Let (3.24) be true for some $\gamma, \gamma \in (0, 1)$, satisfying (3.20).
Therefore the exponential functional $I_\infty = I_\infty(1)$ is a.s. finite.

We consider the following tractable examples. Let the distributions of jumps $\{Y_n\}$ satisfy
\[
E \left[ e^{-\xi(Y_n + 1)} \right] \sim 1 - b_\xi^\beta, \quad \xi \to 0, \quad \beta > 0. \quad (3.24)
\]
Example 3.1. Let $X$ is based on independent Poisson processes $\eta_n(t) = N_n(t)$ with alternating parameters $\mu_0$, $\mu_1 > 0$, that is their Lévy-Laplace exponents are given by $\ell(1) = \mu_i (1-e^{-\gamma}) \sim \mu_i \gamma$, $\gamma \to 0$. Condition (3.20) holds for $\gamma > 0$.

Let jump magnitudes follow (3.24) and $\beta > 1$. In this case for sufficiently small $\gamma > 0$ condition (3.23) becomes

$$\mu_0 \mu_1 \gamma^2 + (\lambda_0 \mu_1 + \lambda_1 \mu_0) \gamma + \lambda_0 \lambda_1 b \gamma^\beta > 0$$

Note that inequality (3.25) has a solution $\gamma$, $\gamma \in (0,1]$, since $\lambda_0 \mu_1 + \lambda_1 \mu_0 > 0$. Thus $I_{\infty} < \infty$ a.s.

If (3.24) holds with $\beta = 1$, then for (3.25) it is sufficient to assume that

$$b \geq - \left( \frac{\mu_0}{\lambda_0} + \frac{\mu_1}{\lambda_1} \right),$$

If $\beta < 1$ and $b > 0$, then the solution $\gamma$, $0 < \gamma < 1$ of (3.25) also exists.

Finally, notice that if $Y_0 + Y_1 = 0$ a.s., that is $b = 0$, then (3.25) holds, so $I_{\infty} < \infty$ a.s.

Example 3.2. Consider the integrated telegraph process $X$ with the states $(\lambda_i, c_i)$, $i \in \{0,1\}$, accompanying with jumps $\{Y_n\}$ which satisfy (3.24). The Lévy-Laplace exponents are $\ell(1) = c_0 \gamma$, $\ell(2) = c_1 \gamma$ and for sufficiently small $\gamma$ condition (3.23) becomes

$$c_0 c_1 \gamma^2 + (c_0 \lambda_1 + c_1 \lambda_0) \gamma + \lambda_0 \lambda_1 b \gamma^\beta > 0.$$}

The exponential functional $I_{\infty}$ is a.s. finite in the following three cases:

1. if $\beta > 1$ and $c_0 \lambda_1 + c_1 \lambda_0 > 0$;
2. if $\beta = 1$ and $b \geq - \left( \frac{c_0}{\lambda_0} + \frac{c_1}{\lambda_1} \right)$;
3. if $\beta < 1$ and $b > 0$.

If $Y_0 + Y_1 = 0$ a.s. (that is $b = 0$), then condition (3.25) holds if

$$c_0 \lambda_1 + c_1 \lambda_0 > 0.$$ (3.26)

The next example is a bit more sophisticated.

Let $\alpha_0, \alpha_1 > 0$, $\alpha_0, \alpha_1 \in (0,1)$, $\alpha_0 \geq \alpha_1$. Let $\{\eta_n\}$ be independent stable subordinators with the alternating Lévy-Laplace exponents $\ell(2) = a_0 \xi^{\alpha_0}$ and $\ell(1) = a_1 \xi^{\alpha_1}$, $\xi \geq 0$.

Let $X^+$ be the Markov-modulated Lévy process based on such Lévy processes $\{\eta_n\}$, (3.1), and $X_0^+$ and $X_1^+$ be the Markov-modulated Lévy processes based on $\pm \eta_n$ with alternating signs, such that the Lévy-Laplace exponents of the underlying Lévy blocks are $-\ell(2)$ and $\ell(1)$ and $\ell(2)$ and $-\ell(1)$ respectively.

Since (3.20) holds for $\pm \ell$ (for sufficiently small $\gamma$, $\gamma > 0$), we have the following result.

Theorem 3.4. Let $I_{\infty}^+$ and $I_{0,\infty}^+$, $I_{1,\infty}^-$ be the exponential functionals (3.19) of $X^+$, $X_0^-$ and $X_1^-$ respectively.

1. $I_{\infty}^+ < \infty$ a.s. in the following cases:
   a. $\beta > \alpha_1$;
   b. $\beta = \alpha_1$ and $a_1 + b \geq 0$,
   c. $\beta < \alpha_1$ and $b \geq 0$.

2. $I_{0,\infty}^+ < \infty$ a.s. in the following cases:
   a. $\beta > \alpha_1$;
   b. $\beta = \alpha_1$ and $a_1 + b > 0$;
   c. $\beta < \alpha_1$ and $b \geq 0$.

3. $I_{1,\infty}^- < \infty$ a.s. in the following cases:
with some $\gamma$, $\gamma \in (0, 1]$. This is equivalent to

$$a_0 a_1 + \frac{\lambda_0 a_1}{\gamma a_0} + \frac{\lambda_1 a_0}{\gamma a_1} + \frac{\lambda_0 \lambda_1 b}{\gamma a_0 + a_1 - \beta} + o(1) > 0,$$

as $\gamma \to 0$. It is easy to see that in the cases (1a), (1b) and (1c) inequality (3.28) holds for sufficiently small $\gamma$, $\gamma \to 0$.

Second, $I_{0,\infty}^- < \infty$ a. s., if for sufficiently small $\gamma$

$$-a_0 a_1 - \frac{\lambda_1 a_0}{\gamma a_1} + \frac{\lambda_0 a_1}{\gamma a_0} + \frac{\lambda_0 \lambda_1 b}{\gamma a_0 + a_1 - \beta} > 0,$$

which is true in the cases (2a), (2b) and (2c).

In the case of $I_{\infty}^- (3.27)$ becomes

$$- a_0 a_1 - \frac{\lambda_0 a_1}{\gamma a_0} + \frac{\lambda_1 a_0}{\gamma a_1} + \frac{\lambda_0 \lambda_1 b}{\gamma a_0 + a_1 - \beta} > 0,$$

which holds in the cases of (3a) and (3b). Note that if $\beta > \alpha_1$ inequality (3.29) does not asymptotically hold (as $\gamma \to 0$).

We will denote by $I_{\infty}^0$ and $I_{\infty}^1$ the functionals (3.19), when the alternating Lévy process $X$ starts from the state 0 and 1 respectively. Let $T$ be the first switching time. By (3.1) we have the following identities in law:

$$I_{\infty}^0 \overset{D}{=} \mathcal{T}^0(T) + \exp(-Y_0 - \eta_0(T)) \tilde{I}_{\infty}^1,$$

$$I_{\infty}^1 \overset{D}{=} \mathcal{T}^1(T) + \exp(-Y_1 - \eta_1(T)) \tilde{I}_{\infty}^0,$$

where

$$\mathcal{T}^0(T) = \int_0^T e^{-\eta_0(t)} dt, \quad \mathcal{T}^1(T) = \int_0^T e^{-\eta_1(t)} dt,$$

and $\tilde{I}_{\infty}^0$ and $\tilde{I}_{\infty}^1$ are independent copies of $I_{\infty}^0$ and $I_{\infty}^1$. By identities (3.30) for any $t$, $t > 0$,

$$\begin{cases} \mathbb{P}\{I_{\infty}^0 > t\} = \mathbb{P}\{I_{\infty}^1 > \exp(Y_0 + \eta_0(T))(t - \mathcal{T}^0(T))\}, \\
\mathbb{P}\{I_{\infty}^1 > t\} = \mathbb{P}\{I_{\infty}^0 > \exp(Y_1 + \eta_1(T))(t - \mathcal{T}^1(T))\}. \end{cases}$$

Since $T$ is exponentially distributed, the density functions $f_0(t)$ and $f_1(t)$, $t \geq 0$,

$$f_0(t)dt = \mathbb{P}\{I_{\infty} \in dt \mid \varepsilon(0) = 0\}, \quad f_1(t)dt = \mathbb{P}\{I_{\infty} \in dt \mid \varepsilon(0) = 1\},$$

of the distributions of $I_{\infty}^0$ and $I_{\infty}^1$ follow the coupled integral equations:

$$\begin{cases} f_0(t) = \int_0^\infty \lambda_0 e^{-\lambda_0 \tau} \mathbb{E} \left[e^{Y_0 + \eta_0(\tau)} f_1 \left(e^{Y_0 + \eta_0(\tau)} (t - \mathcal{T}^0(\tau))\right)\right] d\tau, \\
f_1(t) = \int_0^\infty \lambda_1 e^{-\lambda_1 \tau} \mathbb{E} \left[e^{Y_1 + \eta_1(\tau)} f_0 \left(e^{Y_1 + \eta_1(\tau)} (t - \mathcal{T}^1(\tau))\right)\right] d\tau. \end{cases}$$

In some particular cases the distributions of $I_{\infty}^0$ and $I_{\infty}^1$ can be written explicitly by solving system (3.31). For instance, system (3.31) can be solved in the important particular
case of the integrated telegraph process $X$ with the states $(\lambda_i, c_i)$, $i \in \{0, 1\}$, $c_0 > c_1$, accompanying with the deterministic jumps $y_0, y_1$.

3.4. **Exponential functional for the jump-telegraph process.** Consider an example of the Kac-Lévy process with jumps and with the Lévy-Laplace exponents $\ell_0(\xi) = c_0 \xi$, $\ell_1(\xi) = c_1 \xi$. Note that if $c_0, c_1 \leq 0$, then $I_\infty^{(0)} = \infty$, $I_\infty^{(1)} = \infty$, a.s. On the contrary, if both $c_0, c_1$ are positive and $y_0 + y_1 \geq 0$, then the variables $I_\infty^{(0)}$ and $I_\infty^{(1)}$ are a.s. bounded.

In what follows we assume $c_0 > c_1$, $c_0 > 0$ and

$$y_0 + y_1 = 0, \quad y_0 \leq 0.$$  

(3.32)

Note that by (3.32) the following inequalities hold $\forall t, t > 0$, a.s.

$$y_0 + c_1 t < X^{(0)}(t) < c_0 t, \quad c_1 t < X^{(1)}(t) < y_1 + c_0 t,$$

and system (3.31) becomes

$$
\begin{cases}
  f_0(t) = \int_0^\infty \lambda_0 e^{-\lambda_0 \tau + y_0 + c_0 \tau} f_1 \left( e^{y_0 + c_0 \tau} (t - T^{(0)}(\tau)) \right) d\tau, \\
  f_1(t) = \int_0^\infty \lambda_1 e^{-\lambda_1 \tau + y_1 + c_1 \tau} f_0 \left( e^{y_1 + c_1 \tau} (t - T^{(1)}(\tau)) \right) d\tau.
\end{cases}
$$

(3.35)

Here

$$T^{(0)}(\tau) = \frac{1 - e^{-\alpha_0 \tau}}{c_0}, \quad T^{(1)}(\tau) = \frac{1 - e^{-\alpha_1 \tau}}{c_1}$$

(if $c_1 = 0$, then $T^{(1)}(\tau) = \tau$).

First, we present the formulae for the density functions $f_0$ and $f_1$ in the case of nonnegative trends, $c_0 > c_1 \geq 0$.

**Theorem 3.5. Nonnegative $c_0$ and $c_1$.** Assume that $c_0 > c_1 \geq 0$.

Therefore, $I_\infty^{(0)} < \infty$, $I_\infty^{(1)} < \infty$, a.s.

- If $0 < c_1 < c_0$, then the density functions $f_0$ and $f_1$ have a compact support, and they are given by

$$
\begin{align*}
  f_0(t) &= A_0 (t - a)^{\alpha - 1} (b - y_0 - t) \beta 1_{\{t < b - y_0\}}, \\
  f_1(t) &= A_1 (t - a e^{-y_1}) \beta 1_{\{t < a e^{-y_1} \}}.
\end{align*}
$$

(3.36)

where $a = 1/c_0$, $\alpha = \lambda_0/c_0$, $b = 1/c_1$, $\beta = \lambda_1/c_1$ and

$$
A_0 = \frac{(b - y_0 - a)^{-\alpha - \beta}}{B(\alpha, \beta + 1)}, \quad A_1 = \frac{(b - a e^{-y_1})^{-\alpha - \beta}}{B(\alpha + 1, \beta)}.
$$

$B(\cdot, \cdot)$ is beta-function.

- If $0 = c_1 < c_0$, then the density functions $f_0$ and $f_1$ are given by

$$
\begin{align*}
  f_0(t) &= \frac{(\lambda_1 e^{y_0})^\alpha}{\Gamma(\alpha)} (t - a)^{\alpha - 1} \exp(-\lambda_1 e^{y_0} (t - a)) 1_{\{t > a\}}, \\
  f_1(t) &= \frac{\lambda_1^{\alpha - 1}}{\Gamma(\alpha + 1)} (t - a e^{-y_1})^\alpha \exp(-\lambda_1 (t - a e^{-y_1})) 1_{\{t > a e^{-y_1}\}}.
\end{align*}
$$

(3.37)

where $a = 1/c_0$, $\alpha = \lambda_0/c_0$ and $\Gamma(\cdot)$ is gamma-function.
Proof. Let $0 < c_1 < c_0$. By (3.33)-(3.34) we have
\[ a < I^{(0)}_\infty b^{-y_0}, \quad ae^{-y_1} < I^{(1)}_\infty b, \quad a.s. \]
where $a = 1/c_0$, $b = 1/c_1$.

By applying the change of variables
\[ u = e^{y_0+c_0\tau}(t - \tau^{(0)}(\tau)) = e^{y_0}[a + e^{c_0\tau}(t-a)], \quad te^{y_0} < u < b \]
\[ \iff \tau = a\log \frac{ue^{-y_0} - a}{t-a}, \]
in the first equation of (3.35) with $a < t < be^{-y_0}$; and
\[ u = e^{y_1+c_1\tau}(t - \tau^{(1)}(\tau)) = e^{y_1}[b - e^{c_1\tau}(b-t)], \quad a < u < te^{y_1} \]
\[ \iff \tau = b\log \frac{b - ue^{-y_1}}{b-t} \]
in the second one with $ae^{-y_1} < t < b$, we found that system (3.35) is equivalent to
\begin{align*}
f_0(t) &= \alpha(t-a)^{-\alpha-1} \int_{te^{y_0}}^{b} (ue^{-y_0} - a)^{-\alpha} f_1(u) du, \quad a < t < be^{-y_0}, \quad (3.38) \\
f_1(t) &= \beta(b-t)^{-\beta-1} \int_{a}^{te^{y_1}} (b - ue^{-y_1})^{-\beta} f_0(u) du, \quad ae^{-y_1} < t < b. \quad (3.39)\end{align*}

Setting
\[ f_0(t) = A_0(t-a)^{-\alpha-1}(b - te^{-y_1})^\beta 1_{\{a < t < be^{-y_0}\}}, \]
and after the corresponding change of variables, we easily get

\[ f_1(t) = A_1(t - ae^{-y_0})^\alpha (b - t)^{\beta - 1} 1_{(ae^{-y_1} < t < b)}, \]

where by (3.38) \( A_1 = \frac{\beta A_0}{\alpha}. \) The expressions for \( A_0 \) and \( A_1 \) follow from [15, 3.196.3].

Let \( c_1 = 0 \) and \( c_0 > 0 \). Hence, condition (3.22)-(3.23) (with \( \gamma = 1 \)) hold and

\[ E\{I_0^{(0)}\} < \infty, \quad E\{I_1^{(1)}\} < \infty. \]

and \( I_0^{(0)} < \infty, I_1^{(1)} < \infty, \ a.s. \)

System (3.35) becomes

\[
\begin{align*}
& f_0(t) = \int_0^\infty \lambda_0 e^{-\lambda_0 \tau + y_0 + a\tau} f_1(e^{y_0 + a\tau}(t - T^{(0)}(\tau)))d\tau, \quad a < t, \\
& f_1(t) = \int_0^\infty \lambda_1 e^{-\lambda_1 \tau + y_1} f_0(e^{y_1}(t - \tau))d\tau, \quad ae^{-y_1} < t,
\end{align*}
\]

and, after the corresponding change of variables, \( \tau = a \log \frac{ue^{-y_0} - a}{t - a} \) in the first equation and \( \tau = t - ae^{-y_1} \) in the second one, we get

\[
\begin{align*}
& f_0(t) = \alpha(t - a)^{\alpha - 1} \int_0^\infty (ue^{-y_0} - a)^{-\alpha} f_1(u)du, \quad a < t < \infty, \\
& f_1(t) = \lambda_1 e^{-\lambda_1 t} \int_a^{te^{y_1}} \exp(\lambda_1 ue^{-y_1}) f_0(u)du, \quad ae^{-y_1} < t.
\end{align*}
\]

Taking into the second equation

\[ f_0(t) = A_0(t - a)^{\alpha - 1} \exp(-\lambda_1 te^{y_0}) 1_{t > a} \]

we easily get

\[ f_1(t) = A_1(te^{y_1} - a)^{\alpha} e^{-\lambda_1 t} 1_{t > ae^{-y_1}}, \]

with \( A_1 = \lambda_1 A_0/\alpha. \) The expressions for \( A_0 \) and \( A_1 \) follow by the definition of gamma-function:

\[ 1 = A_0 \int_\alpha^\infty (t - a)^{\alpha - 1} e^{-\lambda_1 te^{y_0}} dt = A_0 \frac{\Gamma(\alpha) \exp(-\lambda_1 ae^{y_0})}{(\lambda_1 e^{y_0})^\alpha}. \]

The theorem is proved.

The case of \( c_0 > 0 > c_1 \) with directions of jumps to be opposite to the sign of current trend is most important for financial applications, see [20].

**Theorem 3.6.** **Trends** \( c_0 \) and \( c_1 \) **are of opposite signs.** Let \( c_1 < 0 < c_0 \) and \( \alpha = \lambda_0/c_0 > 0, \beta = \lambda_1/c_1 < 0. \)

- **If** \( \alpha + \beta \geq 0, \) **then**
  \[ \mathbb{P}\{I_0^{(0)} = \infty\} = \mathbb{P}\{I_1^{(1)} = \infty\} = 1. \]  \[ (3.40) \]

- **If** \( \alpha + \beta < 0, \) **that is,**
  \[ \frac{\lambda_0}{c_0} + \frac{\lambda_1}{c_1} < 0, \]  \[ (3.41) \]
  **then**
  \[ \mathbb{P}\{I_0^{(0)} < \infty\} = \mathbb{P}\{I_1^{(1)} < \infty\} = 1. \]
The density functions of the distribution in this case are given by
\[
\begin{align*}
    f_0(t) &= \frac{(a - be^{-y_0})^{-\alpha - \beta}}{B(-\alpha - \beta, \alpha)} (t - a)^{\alpha - 1} (t - be^{-y_0})^\beta 1_{\{t > a\}}, \\
    f_1(t) &= \frac{(ae^{-y_1} - b)^{-\alpha - \beta}}{B(-\alpha - \beta, \alpha + 1)} (t - ae^{-y_1})^\alpha (t - b)^{\beta - 1} 1_{\{t > ae^{-y_1}\}}. 
\end{align*}
\] (3.42)

**Figure 3.** Density functions \(f_0\) and \(f_1\), (3.42), with \(\lambda_0 = \lambda_1 = 1; c_0 = 2, c_1 = -0.1; y_0 = -0.5, y_1 = 0.5\).

**Proof.** System (3.35) for the density functions \(f_0\) and \(f_1\) after the corresponding change of variables becomes
\[
\begin{align*}
    f_0(t) &= \alpha(t - a)^{\alpha - 1} \int_{te^{y_0}}^\infty (ue^{-y_0} - a)^{-\alpha} f_1(u)du, \quad a < t, \\
    f_1(t) &= -\beta(t - b)^{\beta - 1} \int_{ae^{y_1}}^te^{y_1} (ue^{-y_1} - b)^{-\beta} f_0(u)du, \quad ae^{-y_1} < t. 
\end{align*}
\] (3.43)

Here \(\alpha, a > 0, \beta, b < 0\).

Similarly to the proof of Theorem 3.5 one can obtain the solution of (3.43) in the form
\[
\begin{align*}
    f_0(t) &= A(t - a)^{\alpha - 1} (te^{-y_1} - b)^\beta 1_{\{t > a\}}, \\
    f_1(t) &= -\frac{\beta}{\alpha} A(te^{-y_0} - a)^\alpha (t - b)^{\beta - 1} 1_{\{t > ae^{-y_1}\}}. 
\end{align*}
\] (3.44)

with an indefinite coefficient \(A\).

Let \(\alpha + \beta \geq 0\). Note that in this case the integrals \(\int_a^\infty f_0(t)dt\) and \(\int_{ae^{-y_1}}^\infty f_1(t)dt\) of functions \(f_0\) and \(f_1\) defined by (3.42), diverge, if \(A \neq 0\). Hence \(A = 0\) and we have (3.40).
Let \( \alpha + \beta < 0 \). This condition coincides with (3.26), see Example 3.2. Thus \( I_\infty < \infty \) a.s. and formulae (3.42) follow from (3.44) and \( \int_0^\infty f_0(t)dt = 1 \), see [15, 3.196.2].

**Remark 3.2.** The Laplace transforms of these distributions which are obtained in Theorem 3.6 and Theorem 3.7

\[
\tilde{f}_i(s) = \int_0^\infty e^{-st} f_i(t) dt, \quad s > 0,
\]
can be expressed in terms of the hypergeometric functions:

- if \( c_0 > c_1 > 0 \), then
  \[
  \tilde{f}_0(s) = \,_{1}F_{1}(\alpha, \alpha + \beta + 1; -(be^{-y_0} - a)s) \exp(-as)
  \]
  and
  \[
  \tilde{f}_1(s) = \,_{1}F_{1}(\alpha + 1, \alpha + \beta + 1; -(b - ae^{-y_1})s) \exp(-ae^{-y_1}s);
  \]

- if \( c_0 > 0 = c_1 \), then
  \[
  \tilde{f}_0(s) = \left( \frac{\lambda_1 e^{y_0}}{\lambda_1 e^{y_0} + s} \right)^\alpha \exp(-as)
  \]
  and
  \[
  \tilde{f}_1(s) = \left( \frac{\lambda_1}{\lambda_1 + s} \right)^{\alpha+1} \exp(-ae^{-y_1}s);
  \]

- if \( c_0 > 0 > c_1 \), then
  \[
  \tilde{f}_0(s) = \frac{\Gamma(-\beta)}{\Gamma(-\alpha - \beta)} \Psi(\alpha, \alpha + \beta + 1; (a - be^{-y_0})s) \exp(-as)
  \]
  and
  \[
  \tilde{f}_1(s) = \frac{\Gamma(-\beta + 1)}{\Gamma(-\alpha - \beta)} \Psi(\alpha + 1, \alpha + \beta + 1; (ae^{-y_1} - b)s) \exp(-ase^{-y_1}),
  \]
  where \( \beta = \lambda_1/c_1 < 0 \) and by (3.31) \( \alpha + \beta = \lambda_0/c_0 + \lambda_1/c_1 < 0 \).

Here \( _1F_1 \) is the Kummer and \( \Psi \) the Tricomi confluent hypergeometric functions. For these formulae see [24, 2.1.3], (3.36)-(3.37) and (3.42).

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