Geometrical modification of quaternionic quantum mechanics

Vladimir Trifonov
American Mathematical Society
P. O. Box 7621
Cumberland, RI 02864 USA
trifonov@ams.org

Abstract
We present an axiomatic modification of quantum mechanics with a possible worlds semantics capable of predicting essential “nonquantum” features of an observable universe model - the topology and dimensionality of spacetime, the existence, the signature and a specific form of a metric on it, and a set of naturally preferred directions (vistas) in spacetime unrelated to its metric properties.

1 Introduction
The technical purpose of the paper is to provide a formal definition of the notion of an observer and related constructs in order to make the description of quantum systems more compatible with the kinematic structure of general relativity (GR). We consider the following kinematic axioms of GR:

Axiom 1. Spacetime of the universe is a smooth manifold.
Axiom 2. The dimensionality of spacetime is four.
Axiom 3. Spacetime is equipped with a lorentzian metric.

The fact that these still are axioms is somewhat troublesome, at least to the author, who would rather see them as corollaries of a single assertion.

We shall consider changes to the kinematic, dynamic and semantic structure of quaternionic quantum mechanics (QQM, [2]) that deal with notions pertaining to the above axioms and the following assertion:
Assertion 1. The logic of the observer is bivalent boolean.

We shall show that (technically accurate versions of) the statements of the axioms are derivable from (a technically accurate version of) this assertion. We start with three simple observations.

Observation 1. In QQM the quaternionic hilbert space \( \mathbb{V} \) contains a natural principal bundle, \((\mathbb{V}_H^0)\), with the following components:

1. The total space is the set \( \mathbb{V}_H^0 := \mathbb{V} \setminus \{0\} \) of nonzero vectors of \( \mathbb{V} \), with the natural manifold structure canonically generated by the linear structure of \( \mathbb{V} \).

2. The base space is a quaternionic projective space \( \mathbb{P}\mathbb{V}_H^0 \) whose points are quaternionic rays in \( \mathbb{V} \).

3. The standard fiber is the set \( \mathcal{H}^0 = \mathbb{H} \setminus \{0\} \) of nonzero quaternions which is a four dimensional manifold and a lie group \( \mathcal{H}^0 \cong SU(2) \times \mathbb{R}^+ \).

4. The structure group is also \( \mathcal{H}^0 \). It acts on \( \mathbb{V}_H^0 \) from the left, and for each \( \psi \in \mathbb{V}_H^0 \) its orbit (the ray through \( \psi \)) is a copy of \( \mathcal{H}^0 \) via the fiber diffeomorphism.

We shall refer to this bundle as the hyperquantum bundle over \( \mathbb{V} \). The recent discovery \[20\] of natural relativistic structure on \( \mathcal{H}^0 \) turns each fiber of \((\mathbb{V}_H^0)\) into a lorentzian manifold. This principal bundle is a generalization of a principal bundle \((\mathbb{V}_C^0)\) used in geometric quantum mechanics (GQM), the quantum bundle over a complex hilbert space \( \mathbb{V} \) (see \[18\] and references therein). The hyperquantum bundle forms the kinematic foundation of the modification.

Observation 2. The total space \( \mathbb{V}_H^0 \) has the structure of a hyperkähler manifold with the riemannian metric and symplectic forms induced by the decomposition of values of the quaternionic hermitian form on \( \mathbb{V} \) in the canonical basis \((1, i, j, k)\) of quaternions. Hence vector fields and flows on \( \mathbb{V}_H^0 \) are subject to the hyperhamiltonian formalism \[11\], roughly a superposition of three hamiltonian evolutions on \( \mathbb{V}_H^0 \), on which the dynamics of the modification is based.

Observation 3. The observer theory sketched in \[19\] supplies material for a rigorous definition of an observer and perceptible analogues of standard physical constructs such as time, spacetime and a dynamical system, and study
the dependence of their properties on the type of logic of the observer. The perceptible spacetime acquires a group structure, and a perceptible dynamical system becomes a set with a left action of a monoid. The main result of [19] asserts that if the logic of an observer is bivalent boolean then the perceptible spacetime is isomorphic to the lie group of nonzero quaternions, $\mathcal{H}^\circ$, and a perceptible physical system is a set with a left action of $\mathcal{H}^\circ$. A modest category-theoretic generalization of this schema provides a semantic foundation of the modification.

In this paper we restate and combine these observations in a coherent way. It should be stressed that the proposed modification is not a complete theory because the treatment of certain important aspects related to quantum measurement and probability is too sketchy. A complete accurate description of these topics in terms of geodesics of a hyperkähler manifold requires a substantial effort and should be treated in a separate paper. Here our primary objective is complete unambiguity, i.e., axiomatic consistency rather than completeness. The paper is organized as follows.

Section 2 - Basic constructs and notations. The technical material is given very selectively, the purpose being an introduction of notational conventions rather than education. For example, the reader is assumed to have some familiarity with birkhoff categories and hyperkähler geometry. Since this paper utilizes some conventional structures ($\mathbb{F}$-algebras) in an unconventional way, thorough knowledge of [20] is essential for constructive reading. In particular, the notions of a principal inner product on an $\mathbb{F}$-algebra and a principal metric on a unital algebra, introduced in [20], are crucial to our treatment. $\mathbb{N}$ denotes natural numbers and zero, $\mathbb{C}$ an $\mathbb{R}$ are the fields of complex numbers and real numbers, respectively, and $\mathbb{R}$ is assumed taken with its standard linear order and euclidean topology. Small Greek indices, $\alpha, \beta, \gamma$ and small Latin indices $p, q$ always run 0 to 3 and 1 to 3, respectively. The set of $[n]^m$-tensor fields on a smooth manifold $\mathcal{M}$ is denoted $\mathcal{M}[m,n]$.

Section 3 - A slightly nonstandard description of hyperkähler manifolds and basic notions associated with quaternionic hilbert spaces and quaternionic regular maps.

Section 4 - The semantics of the modification. We introduce the notions of an experient and a reality. The reader is warned that we shall neither discuss the philosophical issues involved, nor use the terminology and notation that usually accompany them (see for example [5], [8]). Ours is a simpler and more pragmatic task – to formally redefine certain technical constructs
of physics in terms of elementary experiences of an observer, considered as primitive entities.

Section 5 - We study a particular species of experients called $F$-observers and the associated notions of a temporal reality and a phenomenon.

Section 6 - A specialization of some of the above notions, namely an observer and a robust reality.

Section 7 - We define the notion of a dynamical system, its evolution and its perceptibles. The important notion of propensity is defined.

Section 8 - A special case of a robust reality, a cosmology. This Section contains one of the central results of the paper, namely the essential uniqueness of the cosmology of the observer. We compute several characteristics of the cosmology such as topology and dimensionality of its spacetime, as well as properties of its metric and naturally preferred directions in the spacetime.

Section 9 - We define physical systems, observables and their measurements. It is shown that standard quantum systems of complex quantum mechanics correspond to a degenerate, in a strictly defined sense, kind of physical systems, and the notions of a measurement and propensity seem to reflect certain aspects of quantum measurement and probability, respectively.

Section 10 - We conclude the paper with an informal summary of the results.

2 Technical preliminaries

Definition 2.1. A signature, $\Sigma$, is an ordered pair $(S, s)$, where $S$ is a set of elementary symbols and $s : S \to \mathbb{N}$ is the arity map, assigning to each elementary symbol $s \in S$ a natural number $s(s)$, called the arity of $s$.

Definition 2.2. For a category $\mathcal{E}$ with products and coproducts and a signature $\Sigma = (S, s)$, an endofunctor $\Gamma : \mathcal{E} \to \mathcal{E}$ is called a $\Sigma$-functor (on $\mathcal{E}$) if for each $\mathcal{E}$-object $A$,

$$\Gamma(A) = \coprod_{s \in S} A^{s(s)},$$

(1)

where $\coprod$ denotes coproduct of $\mathcal{E}$-objects, and $A^{s(s)}$ is a product of $s(s)$ copies of $A$.

Definition 2.3. Given an endofunctor $\Gamma : \mathcal{E} \to \mathcal{E}$ on a category $\mathcal{E}$, an algebra, $A$, for $\Gamma$ is an ordered pair $(A, a)$, where $A$ is an $\mathcal{E}$-object, called the carrier, and $a : \Gamma(A) \to A$ is an $\mathcal{E}$-arrow, called the structure map of the algebra. Let
A = (A, a) and B = (B, b) be algebras for Γ. A Γ-morphism, A → B, is a map f : A → B, such that the following diagram commutes:

\[
\begin{array}{ccc}
\Gamma(A) & \xrightarrow{\Gamma(f)} & \Gamma(B) \\
\downarrow^a & & \downarrow^b \\
A & \xrightarrow{f} & B \\
\end{array}
\]  

(2)

Example 2.1. Let \( \mathbb{F} \) be a field, and \( S = \{0, +\} \cup \mathbb{F} \) and \( s(0) = 0, s(+) = 2, s(r) = 1, \forall r \in \mathbb{F} \). A vector space, \( V = (V, v) \), over a field \( \mathbb{F} \) is an algebra for the \( \Sigma \)-functor \( \Gamma : \text{Set} \to \text{Set} \) on the category of sets for the signature \( \Sigma = (S, s) \).

Remark 2.1. Dually, coalgebras for an endofunctor \( \Gamma : \mathcal{E} \to \mathcal{E} \) are defined by reversal of structure maps. Algebras and coalgebras for an endofunctor \( \Gamma \) and their \( \Gamma \)-morphisms form categories denoted \( \mathcal{E}^\Gamma \) and \( \mathcal{E}_\Gamma \), respectively, (see, e. g., [13]).

Example 2.2. A monoid is an example of an algebra, \( M = (M, m) \), for a \( \Sigma \)-functor on \( \text{Set} \) with \( S = \{i, *\}, s(i) = 0, s(*) = 2 \). As with every \( \Sigma \)-functor, the structure map \( m \) can be split into the constituents, giving the more familiar notation \( (M, i, *) \), where \( i \) is understood as a preferred element, the identity of \( M \), and * is a binary operation on \( M \).

Definition 2.4. An endofunctor \( \Gamma \) on \( \mathcal{E} \) is called a monad if there exist two natural transformations, \( \flat : \text{id}(\mathcal{E}) \to \Gamma \) and \( \natural : \Gamma^2 \to \Gamma \) such that the following diagrams commute:

\[
\begin{array}{ccc}
\Gamma^3 & \xrightarrow{\Gamma_\flat} & \Gamma^2 \\
\downarrow^{\natural} & & \downarrow^{\natural} \\
\Gamma^2 & \xrightarrow{\flat} & \Gamma \\
\end{array}
\]  

where \( \Gamma^n \) denotes \( n \) iterations of the functor.

Definition 2.5. The category of algebras, \( \mathcal{E}^\Gamma \), for the monad \( \Gamma : \mathcal{E} \to \mathcal{E} \) is a subcategory of \( \mathcal{E}^\Gamma \) such that the following diagrams commute for each object \( A = (A, a) \) of \( \mathcal{E}^\Gamma \):

\[
\begin{array}{ccc}
\Gamma^2(A) & \xrightarrow{\Gamma(a)} & \Gamma(A) \\
\downarrow^{\natural} & & \downarrow^{\natural} \\
\Gamma(A) & \xrightarrow{id(A)} & A \\
\end{array}
\]  

(4)
For each monoid $M = (M, m)$, an endofunctor $\Gamma$ on $\textbf{Set}$ sending a set $X$ to the set $M \times X$ is a monad. An object, $X = (X, x)$, in the category of algebras, $M\textbf{Set}$, for this monad is a set, $X$, with a left action, $x$, of the monoid $M$, also referred to as an $M$-set. For each $a \in M$, we can define a map $x_a : X \to X$ by $x_a(x) := x(a, x)$, $\forall x \in X$. The arrows $(X, x) \to (X', x')$ are equivariant (i.e., preserving the action) functions $f : X \to X'$, making the following diagram commute:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
x_a \downarrow & & \downarrow x_a' \\
X & \xrightarrow{f} & X'
\end{array}
$$

The category of $M$-sets is of utmost importance in our exposition.

Example 2.4. An $\mathbb{F}$-algebra (a linear algebra over the field $\mathbb{F}$) is an example of an algebra for a $\Sigma$-functor on $\textbf{Set}$ with $\Sigma = \{0, +, \cdot\} \cup \mathbb{F}$ and $s(0) = 0, s(+) = 2, s(\cdot) = 2, s(r) = 1, \forall r \in \mathbb{F}$. For an unconventional description of $\mathbb{F}$-algebras, as vector spaces over $\mathbb{F}$ with a $[\frac{1}{2}]$-tensor on them (structure tensors), crucial to our technical setup, see [20].

Definition 2.6. A complete regularly co-well-powered category with regular epi-mono factorizations is called birkhoff if it has enough regular projectives.

Remark 2.2. For a detailed view of birkhoff categories see [13].

Definition 2.7. A full subcategory of a Birkhoff category is called a birkhoff variety if it is closed under products, subobjects and quotients.

Definition 2.8. An endofunctor $\Gamma : \mathcal{E} \to \mathcal{E}$ is called a varietor if it preserves regular epis, and the forgetful functor $U : \mathcal{E}^{\Gamma} \to \mathcal{E}$ has a left adjoint.

A real vector space $V$ induces a natural manifold structure on its carrier. This manifold which we denote $\mathcal{V}$, is referred to as the linear manifold canonically generated by $V$. Since $V$ and $\mathcal{V}$ have the same carrier, there is a bijection $\mathcal{J}_{V} : \mathcal{V} \to V$. We use the normal $(a, u, \ldots)$ and bold $(\mathbf{a}, \mathbf{u}, \ldots)$ fonts, to denote the elements of $\mathcal{V}$ and $V$, respectively, e.g., $\mathcal{J}_{V}(a) = a$. The tangent space $T_a \mathcal{V}$ is identified with $V$ at each point $a \in \mathcal{V}$ via an isomorphism $\mathcal{J}_{V}^* : T_a \mathcal{V} \to V$ sending a tangent vector to the curve $\mu : \mathbb{R} \to \mathcal{V}, \mu(t) = a + tu$, at the point $\mu(0) = a \in \mathcal{V}$, to the vector $u \in V$, with the “total” map $\mathcal{J}_{V} : TV \to V$. The set of nonzero vectors of $V$ constitutes a submanifold of $\mathcal{V}$, referred to as the punctured manifold (of $V$), denoted $\mathcal{V}^\circ$. 
Definition 2.9. For a real vector space $V$ and a linear map $F : V \rightarrow V$, a vector field $f : \mathcal{V} \rightarrow T\mathcal{V}$ on $\mathcal{V}$, such that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{f} & T\mathcal{V} \\
\downarrow J_{\mathcal{V}} & & \downarrow J_{\mathcal{V}} \\
V & \xrightarrow{F} & V
\end{array}
$$

is called the vector field canonically generated by $F$.

Remark 2.3. In particular, for a unital algebra $A$, the linear manifold canonically generated by its underlying vector space $A$ is denoted by $A$ and the punctured manifold by $A^\circ$.

Definition 2.10. For real vector spaces $U, V$, and a map $F : U \rightarrow V$, the map $F : \mathcal{U} \rightarrow \mathcal{V}$, such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{F} & \mathcal{V} \\
\downarrow J_{\mathcal{U}} & & \downarrow J_{\mathcal{V}} \\
U & \xrightarrow{F} & V
\end{array}
$$

is called the linear induction of $F$.

3 Quaternionic maps and hyperhamiltonian vector fields

Definition 3.1. For the quaternion algebra $\mathbb{H} = (H, H)$ a basis on the vector space $H$ is called canonical if the components of the structure tensor $H$ are given by the entries of the following matrices

$$
H^0_{\alpha\beta} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
H^1_{\alpha\beta} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix},
H^2_{\alpha\beta} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
H^3_{\alpha\beta} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
$$

(8)
Remark 3.1. The set \( \{i\} \) of canonical bases \((i_\beta)\) is parametrized by the elements of \(SO(3)\). We shall refer to a bijection \(\Xi : SO(3) \rightarrow \{i\} \) as an array.

Remark 3.2. For a set \(X\), the values of a map \(f : X \rightarrow \mathbb{H}\) can be decomposed in a canonical basis \((i_\beta)\), producing an ordered quadruple \((f_\beta)\) of real valued maps on \(X\) which we call the constituents of \(f\) in the basis \((i_\beta)\). We shall be careful to distinguish these from components of tensorial objects.

Let \(M = (M, g, (\tilde{\omega}^p))\) be hyperkähler manifold. Then, for an array \(\Xi : SO(3) \rightarrow \{i\}\), we can consider the ordered quadruple \((g, \tilde{\omega}^1, \tilde{\omega}^2, \tilde{\omega}^3)\) as constituents \((\tilde{\omega}^\beta)\), where \(\tilde{\omega}^0 := g\), of a quaternion valued map \(\tilde{\omega} : TM \times TM \rightarrow \mathbb{H}\) in the standard basis \(\Xi(1)\), assigning to each ordered pair \((u, v)\) of tangent vectors at each point \(\phi \in M\) a quaternion \(\tilde{\omega}^\beta(u, v)i_\beta\), which allows us to define the constituents \(\tilde{\omega}^p\) in every canonical basis \((i_\beta)\). Similarly, we can assign the ordered quadruple \((J_\beta)\), where \(J_0\) is the identity map on \(TM\), and \(J_p\) is the complex structure corresponding to \(\tilde{\omega}^p\), to each canonical basis \((i_\beta)\), as constituents of the hypercomplex structure \(J\).

Definition 3.2. For an array \(\Xi\) and a hyperkähler manifold \(M\), the ordered pair \(\Omega = (\tilde{\omega}, J)\) is called a hyperkähler structure on \(M\) generated by \(\Xi\). For a canonical basis \((i_\beta)\) the maps \(\tilde{\omega}^p\) and \(J_p\) are called the symplectic and complex constituents of \(\Omega\), respectively, in the basis \((i_\beta)\).

Definition 3.3. A left module \(V\) over \(\mathbb{H}\) is called a (left) quaternionic vector space.

Remark 3.3. A right quaternionic vector space is defined similarly, as well as the right versions of constructs based on it.

Definition 3.4. A map \(\hat{F} : V \rightarrow V\) is called a (left) quaternion linear operator, if
\[
\hat{F}(a\phi) = a\hat{F}(\phi), \quad \forall \phi \in V, \quad \forall a \in \mathbb{H}.
\]

Definition 3.5. A (left) quaternionic hilbert space, \(V\), is a quaternionic vector space, \(V\), together with a map \(\langle \cdot | \cdot \rangle : V \times V \rightarrow \mathbb{H}\), called a quaternionic hermitian inner product, such that
\[
\langle \phi | \psi + \xi \rangle = \langle \phi | \psi \rangle + \langle \phi | \xi \rangle,
\]
\[
\langle \phi | \psi \rangle = \overline{\langle \psi | \phi \rangle}, \quad \langle a\phi | \psi \rangle = a\langle \phi | \psi \rangle,
\]
\[
\|\phi\|^2 := \langle \phi | \phi \rangle \in \mathbb{R}, \quad \|\phi\|^2 > 0, \quad \forall \phi \neq 0,
\]
\[
\forall \phi, \psi, \xi \in V, \forall a \in \mathbb{H},
\]

Remark 3.4. For a set \(X\), the values of a map \(f : X \rightarrow \mathbb{H}\) can be decomposed in a canonical basis \((i_\beta)\), producing an ordered quadruple \((f_\beta)\) of real valued maps on \(X\) which we call the constituents of \(f\) in the basis \((i_\beta)\). We shall be careful to distinguish these from components of tensorial objects.
and the diagonal $\| \cdot \|$ induces a topology on $V$, relative to which $V$ is separable and complete.

**Definition 3.6.** For a quaternionic hilbert space $V$ and a quaternion linear operator $\hat{F}$ on $V$, its *quaternionic adjoint (with respect to $\langle \cdot | \cdot \rangle$)* is a quaternion linear operator $\hat{F}^\dagger$ on $V$, such that

$$\langle \phi | \hat{F}(\psi) \rangle = \langle \hat{F}^\dagger(\phi) | \psi \rangle, \quad \forall \phi, \psi \in V.$$

**Definition 3.7.** We refer to a quaternion linear operator $\hat{F}$ as *quaternionic (anti)hermitian* if it coincides with (the negative of) its quaternionic adjoint.

**Remark 3.4.** Given a basis, $(e_j)$, on an $n$ dimensional quaternionic hilbert space $V$, and a canonical basis, $(i_\beta)$, on $H$, $V$ induces a real $4n$ dimensional vector space, $V_H$, and the latter canonically generates a real linear $4n$ dimensional manifold $V_{\mathbb{H}}$ and the punctured manifold $V_{\mathbb{H}}^\circ$.

**Definition 3.8.** For an operator $\hat{F} : V \to V$ on a quaternionic hilbert space $V$ its *expectation operator* is a map $\hat{F} : V \to \mathbb{H}$ assigning to each $\phi \in V$ a quaternion $\langle \phi | \hat{F}(\phi) \rangle$.

**Lemma 3.1.** For each quaternionic antihermitian operator, $\hat{F}$, its expectation operator $\hat{F}$ has the following property:

$$\hat{F}(\phi) = \text{Im}(\hat{F}(\phi)), \quad \forall \phi \in V. \quad (9)$$

**Proof.**

$$\langle \phi | \hat{F}(\phi) \rangle = \langle -\hat{F}(\phi) | \phi \rangle = -\langle \hat{F}(\phi) | \phi \rangle = -\langle \phi | -\hat{F}(\phi) \rangle. \quad (10)$$

Thus, for each $\phi \in V$ the quaternion $\langle \phi | \hat{F}(\phi) \rangle$ coincide with the negative of its adjoint, which means that its real part is zero.

**Definition 3.9.** For a quaternionic antihermitian operator $\hat{F} : V \to V$ the linear induction $F : V_{\mathbb{H}} \to \mathcal{H}$ of its expectation operator $\hat{F}$ is called the *expectation of $\hat{F}$*.

**Definition 3.10.** For a quaternion linear operator $\hat{F} : V \to V$ a vector field $f : V_{\mathbb{H}} \to TV_{\mathbb{H}}$ canonically generated by $-\hat{F}$ is called the *hyperfield of $\hat{F}$*. 
Remark 3.5. The set of the integral curves of the hyperfield of $\hat{F}$ can be formally represented by the following differential equation

$$\dot{\psi} = -\hat{F}(\psi).$$

Hyperfields are a generalization of *schrödinger vector fields* of geometric quantum mechanics [18]. Indeed, due to the simple relationship between eigenvalues of hermitian and antihermitian operators in the complex case we can use either kind to represent observables. If we take antihermitian operators, the Schrödinger equation has the form (11), with $\hat{F} = \hat{H}$, which defines the set of integral curves of the schrödinger vector field of the hamiltonian operator $\hat{H}$.

Given an array $\Xi$, the quaternionic hermitian product $\langle \cdot | \cdot \rangle$ induces a map $TV_H \times TV_H \rightarrow \mathbb{H}$ whose constituents in a canonical basis $(i_\beta)$ can be identified with the symplectic constituents in $(i_\beta)$ of a hyperkähler structure $\Omega$ on $V_H$ generated by $\Xi$. Similarly, the maps $(i'_\beta)$ defined by

$$i'_\beta(\phi) := i_\beta \phi, \quad \forall \phi \in \mathbb{V}$$

produce the complex constituents of $\Omega$.

Example 3.1. The quaternion algebra $\mathbb{H}$ is a quaternionic vector space. Together with a natural quaternionic hermitian inner product defined by

$$\langle a | b \rangle := a\overline{b}, \quad \forall a, b \in \mathbb{H},$$

it is a quaternionic hilbert space. Therefore $\mathcal{H}$ and $\mathcal{H}^\circ$ possess natural hyperkähler structures.

**Definition 3.11.** Let $M$ and $N$ be hyperkähler manifolds with hyperkähler structures $\Omega^M$ and $\Omega^N$, respectively, generated by an array $\Xi$. A smooth map $f : M \rightarrow N$ is called *quaternionic (regular)* if there exists an $SO(3)$ matrix $B$ such that

$$\sum_{p,q=1}^3 B_{pq} \eta^N_q \circ df \circ \eta^M_p = df,$$

where $\eta^N_p$ and $\eta^N_q$ are the complex constituents, in a canonical basis $(i_\beta) = \Xi(B)$, of the hyperkähler structures of $M$ and $N$, respectively, and $df$ is the differential of $f$. 
Remark 3.6. Quaternionic maps generalize holomorphic functions of complex analysis (see e. g., [9]).

Definition 3.12. For a quaternionic map \( f : \mathcal{M} \to \mathcal{H} \), its \textit{hyperhamiltonian vector field} is a vector field \( f \) on \( \mathcal{M} \), such that

\[
(df_0)(u) = g(f, u), \quad \forall u \in \mathcal{M}[\mathbb{I}],
\]

(14)

where \( g \) is the riemannian metric on \( \mathcal{M} \), and \( df_0 \) is the differential of \( f_0 \). The maps \( f \) and \( f_0 \) are referred to as the \textit{generating map} and the \textit{main generator} of \( f \), respectively.

Remark 3.7. Since \( f_0 \) is invariant under a canonical basis change, so is the definition of a hyperhamiltonian vector field.

Theorem 3.1. For an array \( \Xi \) and a quaternionic hilbert space \( \mathcal{V} \) let \( f \) be the hyperhamiltonian vector field of a quaternionic map \( f : \mathcal{V}_\mathbb{H}^0 \to \mathcal{H}^0 \), such that \( \text{Im}(f) \) is the expectation of a quaternion antihermitian operator \( \hat{F} \) on \( \mathcal{V} \). Then \( f \) is the hyperfield of \( \hat{F} \), and for each canonical basis \( (i_\beta) \) there exists an ordered triple \( (f_p) \) of vector fields on \( \mathcal{V} \) such that

\[
f = f_1 + f_2 + f_3, \quad (df_p)(u) = \tilde{\omega}^p(f_p, u), \quad \forall u \in \mathcal{X}[\mathbb{I}],
\]

(15)

(no summation on \( p \)) where \( df_p \) is the differential of \( f_p \), and \( \tilde{\omega}^p \) is a symplectic constituent of the hyperkähler structure on \( \mathcal{V}_\mathbb{H}^0 \) generated by \( \Xi \), in the basis \( (i_\beta) \).

Proof. The hyperfield of \( \hat{F} \) is clearly a hyperhamiltonian vector field whose generating map is \( f \) [11]. It was shown in [11] and [17] that given a quaternionic map \( f \), the conditions (15) are satisfied for a unique vector field \( f \), and hence the hyperfield of \( \hat{F} \) coincides with \( f \). □

4 Semantics

The observer theory outlined in [19] describes an observer as an \textit{experient}, as opposed to an \textit{occupant}, of the environment. This can be presented within a framework similar to what is known as the \textit{initial algebra} and \textit{final coalgebra} approach to syntax and semantics of formal languages ([1], [21]). The underlying idea is that perception and comprehension are somehow dual to
each other, and each experience can be considered as both a percept and a concept. This idea has its origin in logic and theoretical computer science where similar dualities are considered (a symbol and its meaning, syntax of a formal language and its semantics). We start with a category $E$ representing the totality of experiences, and an endofunctor $\Gamma$ on $E$ interpreted as the language of thought of the experient (see [10] for a philosophical discussion). The categories of algebras, $E^{\Gamma}$, and coalgebras, $E_{\Gamma}$, for this endofunctor represent perception and comprehension of the experient, respectively. If the perception, $E^{\Gamma}$, has an initial object, $I$, the latter serves as the syntax of the experient’s language of thought. Then an algebra, $A$, for $\Gamma$ is considered a paradigm or a model for the language of thought, and the unique arrow $I \to A$ as a meaning function or an interpretation.

Definition 4.1. An experient, $E$, is an ordered pair $(E, \Gamma)$, where $E$ is a birkhoff category, called the metauniverse, and $\Gamma : E \to E$ is a varietor called the construer. The objects and arrows of $E$ are called metaphenomena and metalinks, respectively. $E$-elements of metaphenomena are called reflexors.

The category of algebras $E^{\Gamma}$ is called the perception category. Objects and arrows of $E^{\Gamma}$ are referred to as paradigms and shifts, respectively. An experient is coherent if the metauniverse is a topos. An experient is called boolean if the metauniverse is a boolean topos.

Remark 4.1. It follows from the Definition 4.1 that $E^{\Gamma}$ is also a birkhoff category (see [13], p. 125).

Remark 4.2. Intuitively, metaphenomena are complex percepts composed out of elementary ones (reflexors). For a coherent experient $E = (E, \Gamma)$, we are to think of the internal logic of $E$ as the metalogic of the experient. In particular, the metalogic of a boolean experient is boolean.

Definition 4.2. For a paradigm $A$ of an experient $E$, and a natural number $n$, a reality of rank $n$ is an ordered pair $R = (A, R)$, where $R$ is a subobject of $A^n$. We refer to $A$ as the underlying paradigm, and $R$ is called the ontology of $R$.

Remark 4.3. Intuitively, a reality of rank $n$ is an $n$-ary relation on a collection of reflexors. From this point on we are interested exclusively in realities of rank 2, henceforth referred to simply as realities.

Definition 4.3. An existence mode, $E(V)$, of an experient $E = (E, \Gamma)$ is an ordered pair $(E, V)$, where $V$ is a birkhoff variety of $E^{\Gamma}$. A paradigm $A$ such
that it is (not) a $\mathcal{V}$-object is called (non)existent with respect to $\mathcal{E}(\mathcal{V})$. A shift $A \to B$ such that it is (not) a $\mathcal{V}$-arrow is called (im)possible with respect to $\mathcal{E}(\mathcal{V})$.

Remark 4.4. It should be noted that we thus consider every paradigm $A$ of an experient also a paradigm of the experient in a certain existence mode, although $A$ may be nonexistent with respect to the latter. Given an existence mode, $\mathcal{E}(\mathcal{V})$, of an experient $E$, we say that the experient is in the mode $\mathcal{V}$, or that the experient is the $\mathcal{V}$-observer.

5 $\mathbb{F}$-observers

Different existence modes possess different amount of structure to allow for a definition of notions that we normally associate with observers. We shall focus on a class of boolean experients whose metauniverse is $\text{Set}$, and the construer is a $\Sigma$-functor for the following signature:

$$\Sigma = (S, s); \quad S = \{+, \cdot, 0\} \cup \mathbb{F}, \quad s(+) = 2, s(\cdot) = 2, s(0) = 0, s(r) = 1, \forall r \in \mathbb{F}, \quad (16)$$

where $\mathbb{F}$ is a field. They are introduced implicitly in [19], where a particular existence mode, an $\text{Alg}\{\mathbb{F}\}$-observer, where $\text{Alg}\{\mathbb{F}\}$ is the category of $\mathbb{F}$-algebras, is studied. $\text{Alg}\{\mathbb{F}\}$-observers are sufficiently fine structured to define the fundamental notion of a temporal reality, and at the same time they are relatively simple: their metaphenomena and reflexors are sets and set elements, respectively. For the rest of the paper we shall deal exclusively with $\text{Alg}\{\mathbb{F}\}$-observers, to whom we henceforth refer simply as $\mathbb{F}$-observers.

Definition 5.1. For an existent paradigm $A$ of an $\mathbb{F}$-observer the underlying vector space $A$ is called the sensory domain of $A$, with the principal inner products of $A$ referred to as sensory forms of the paradigm. A basis on $A$ is called a sensory basis of the paradigm. The dual vector space $A^*$ is called the ether domain of $A$, with the elements called ether forms. The motor domain of $A$ is the multiplicative subgroupoid, $M_A = (M_A, \cdot, *)$, of $A$, generated by the set of nonzero reflexors of the paradigm $A$. We refer to elements of $M_A$ as effectors. An existent finite dimensional paradigm, $A$, is called rational if its motor domain is a monoid; otherwise $A$ is called irrational. A paradigm is called transient if it is neither rational nor irrational. The set $A^{\bullet}$ of invertible reflexors of a rational paradigm $A$ is called the perception domain of $A$. 
Remark 5.1. For a rational paradigm $A$ of an $F$-observer, the perception domain $A^\bullet$ is a group, and the ontology, $R$, of a reality $\mathcal{R} = (A, R)$ is a binary relation on the carrier of $A$.

**Definition 5.2.** A reality of an $F$-observer is called immanent if its underlying paradigm is existent and its ontology is a partial order. Otherwise the reality is called transcendent.

**Definition 5.3.** For an $F$-observer, an immanent reality $(F, \preceq)$ is called a temporal template if $F$ is an ordered field with a complete linear order $\preceq$.

Remark 5.2. Not every existence mode has a temporal template. For instance, complex numbers, $\mathbb{C}$, do not admit linear orders ([15], p. 304). Therefore the $\mathbb{C}$-observer has no temporal templates.

**Definition 5.4.** For a rational paradigm $A$ of an $F$-observer the $A$-universe is the category of $M_A$-sets, $M_A Set$ (see Example 2.3), with objects and arrows called $A$-phenomena and $A$-links, respectively. For an $A$-phenomenon $X = (X, x)$, the metaphenomenon $X$ is called the propensity realm, with elements called propensity modes of $X$. Each metalink $X \rightarrow A$, where $A$ is the carrier of $A$, is referred to as an attribute of $X$.

Remark 5.3. We refer to the internal logic of the topos $M_A Set$ as the operational logic of the $F$-observer with respect to $A$, or as the logic of the $A$-universe.

**Definition 5.5.** For each $A$-phenomenon $X = (X, x)$, the $A^\bullet$-set $(X, \bar{x})$, where $\bar{x} : A^\bullet \times X \rightarrow X$ is the restriction of $x$ to $A^\bullet$, is called the perceptible part of $X$. For each $\phi \in X$ the orbits, $W_\phi$ and $\overline{W}_\phi$, of $\phi$, with respect to the actions of $M_A$ and $A^\bullet$, respectively, are called an existence mode and a presence mode of $X$, respectively.

Remark 5.4. In other words, $W_\phi$ and $\overline{W}_\phi$ are the sets

$$W_\phi = \{ \psi \in X \mid \psi = x(\xi, \phi), \ \forall \xi \in M_A \},$$

$$\overline{W}_\phi = \{ \psi \in X \mid \psi = \bar{x}(\xi, \phi), \ \forall \xi \in A^\bullet \}. \quad (17)$$

Intuitively, each presence mode of an $A$-phenomenon is the perceptible part of one of its existence modes.
**Definition 5.6.** An \( \mathcal{A} \)-phenomenon \( \mathcal{X} \), together with a map \( \sigma : \overline{W} \to \mathcal{A}^\bullet \) for each presence mode \( \overline{W} \), is called a *stable phenomenon* if each \( \sigma \) is a bijection. We refer to \( \sigma \) as a *proper view* of \( \overline{W} \). For a stable phenomenon, the propensity modes \( \phi \in (W_\phi \setminus \overline{W}_\phi) \) are called *(im)perceptible.*

**Definition 5.7.** For an \( \mathcal{F} \)-observer, a rational paradigm \( \mathcal{A} \) is *consistent* if the \( \mathcal{A} \)-universe is a boolean topos; a consistent paradigm of maximal dimensionality is called a *home paradigm* of the \( \mathcal{F} \)-observer.

*Remark 5.5.* Intuitively, the consistency condition requires the logic of the \( \mathcal{A} \)-universe to match the logic of the metauniverse (the metalogic of the \( \mathcal{F} \)-observer). \( \mathcal{F} \)-observers without home paradigms, referred to as *Wanderers,* may or may not be of interest, but our main concern will be precisely with home paradigms, and more specifically with home paradigms of the \( \mathbb{R} \)-observer and their \( \mathcal{A} \)-universes, due to the following result.

**Theorem 5.1** (Trifonov, 1995). *Every home paradigm of the \( \mathbb{R} \)-observer is isomorphic to the quaternion algebra \( \mathbb{H} \) with a family of minkowski sensory forms.*

**Definition 5.8.** For an \( \mathcal{F} \)-observer, a reality \( \mathcal{R} = (\mathcal{A}, \mathcal{R}) \) is called *stable* if it is immanent and \( \mathcal{A} \) is a rational paradigm; otherwise the reality is called *unstable* (or *virtual*).

*Remark 5.6.* Given a stable reality \( \mathcal{R} = (\mathcal{A}, \mathcal{R}) \) we refer to the logic of the \( \mathcal{A} \)-universe also as the *logic of \( \mathcal{R} \).*

**Definition 5.9.** For an \( \mathcal{F} \)-observer with a temporal template \( (\mathcal{F}, \preceq) \), a reality \( \mathcal{R} = (\mathcal{A}, \mathcal{R}) \), together with an \( \mathcal{F} \)-valued map \( T : \mathcal{A} \to \mathcal{F} \), is called a *temporal reality* if it is stable and

\[
a \mathcal{R} b \iff (T(a) \preceq T(b) \land a \neq b) \lor (a = b), \forall a, b \in \mathcal{A}.
\]  

Otherwise the reality is called *atemporal.* The ontology of a temporal reality is called its *temporal order,* and the map \( T \) is referred to as the *global time* of the reality. The *perceptible time* of \( \mathcal{R} \) is the restriction, \( \mathcal{T} : \mathcal{A}^\bullet \to \mathcal{F} \), of \( T \) to the perception domain.

*Remark 5.7.* It should be emphasized that a temporal reality is defined only with respect to a certain temporal template. Some \( \mathcal{F} \)-observers may have several temporal templates, and some may have none. For example, since
the \( \mathbb{C} \)-observer has no temporal templates, all realities of such an observer are atemporal. It is easy to see that a temporal template and a global time uniquely determine the temporal order of \( \mathcal{R} \).

**Definition 5.10.** For an \( \mathbb{F} \)-observer and a presence mode, \( \overline{W} \), of his stable \( A \)-phenomenon, the pullback, \( T_{\overline{W}} := T \circ \sigma \), of the perceptible time \( T \) under the proper view \( \sigma \) is referred to as the perceptible time of \( \overline{W} \).

**Definition 5.11.** For an \( \mathbb{F} \)-observer, let \( X \) be a stable \( A \)-phenomenon, \( \mathcal{R} = (A, R) \) a temporal reality, and \( T \) a map \( X \to \mathbb{F} \). An ordered triple \( X = (X, \mathcal{R}, T) \) is called a realization (of \( X \) in \( \mathcal{R} \)) if the following diagram commutes for each \( \overline{W} \),

\[
\begin{array}{ccc}
\overline{W} & \xrightarrow{\sigma} & A^* \\
j & \downarrow & \downarrow \\
X & \xrightarrow{T} & \mathbb{F}
\end{array}
\]

where \( j \) is the inclusion map. We refer to \( X, \mathcal{R} \) and \( T \) as the underlying phenomenon, the background reality and the ambient time of the realization, respectively.

## 6 Observers

Due to the results of [19] and [20], in the remainder of the paper we shall deal exclusively with the \( \mathbb{R} \)-observer, henceforth referred to simply as the observer. If not mentioned explicitly, it is assumed in the following that the constructs under consideration always refer to the observer.

**Lemma 6.1.** Any rational paradigm of the observer is a unital algebra.

**Proof.** Since nonzero elements of \( A \) obey associative multiplication, nonassociativity can occur only in the permutations of \( (ab)0 \), which is impossible since \( b0 = 0, \forall b \in A \). Thus, \( A \) is associative, finite dimensional, and the identity of the motor domain is the identity of \( A \). Therefore it is unital. \( \blacksquare \)

**Corollary 6.1.** The perception domain \( A^* \) of a rational paradigm \( A \) is a lie group with respect to the multiplication of \( A \).

**Definition 6.1.** For a rational paradigm \( A \), the linear manifold \( A \), canonically generated by the sensory domain \( A \) is called the sensory manifold of \( A \), and each reflexor \( a \in A \) is called a viewpoint. Viewpoints \( a \in (A \setminus A^*) \) are called (im)proper.
Definition 6.2. For a rational paradigm $A$, a reflexor $u \in A$, and a proper viewpoint $a \in A^\bullet$, the integral curve, through $a$, of the left invariant vector field generated by $u$ is called a $(u,a)$-vista.

Remark 6.1. Intuitively, $(u,a)$-vistas indicate naturally distinguished directions within the perception domain.

Lemma 6.2. The observer has a unique temporal template.

Proof. Indeed, there is a unique complete linear order on $\mathbb{R}$, namely the standard order $\leq$ (see, e.g., [4], p. 245). Thus, $(\mathbb{R}, \leq)$ is unique. □

Remark 6.2. The previous result makes it unnecessary to mention the temporal order explicitly, and we use the simplified notation $\mathcal{R} = (A, T)$ for a temporal reality of the observer.

Remark 6.3. It is shown in [19] that besides the home paradigm, the observer has exactly two (up to an $\mathbb{R}$-algebra isomorphism) consistent paradigms, namely the one dimensional $\mathbb{R}$-algebra of reals, $\mathbb{R}$, and the two dimensional $\mathbb{R}$-algebra of complex numbers, $\mathbb{C}$, both subalgebras of $\mathbb{H}$.

Definition 6.3. A reality is called robust if it is temporal and there exists a principal metric, $\mathcal{T}$, on the perception domain $A^\bullet$ of the underlying paradigm with the perceptible time $T$ as its generating function. Given a robust reality $\mathcal{R} = (A, T)$, the structure field of $A^\bullet$ is called the structure (field) of the reality, and we refer to $dT$ and $\mathcal{T}$ as the ether (field), and the metric of $\mathcal{R}$, respectively. The ordered pair $S = (A^\bullet, \mathcal{T})$ is called the spacetime of $\mathcal{R}$.

Definition 6.4. A realization $X = (X, \mathcal{R}, T)$ is called robust if the background reality is robust, and the perceptible part of the underlying phenomenon $X$ is a principal $A^\bullet$-bundle $(X)$, such that the propensity realm $X$ is its total space and each proper view $\sigma : W \to A^\bullet$ is a fiber diffeomorphism. The dimensionality of $X$ is referred to as the rank of $X$. For a presence mode, $W$, of the underlying phenomenon the pullback $\mathcal{T}_W$, of the metric of the background reality under the proper view $\sigma$ is called the metric of $W$. The ordered pair $W = (W, \mathcal{T}_W)$ is referred to as a (possible) world of $X$. The bundle $(X)$ is called the monocosm of the realization.
7 Dynamical systems

Definition 7.1. A realization $X = (X, R, T)$ is called a dynamical system if it is robust and its propensity realm is a riemannian manifold $X = (X, g)$. The riemannian metric $g$ is referred to as the propensity metric of $X$.

Remark 7.1. There is a natural connection on the monocosm $(X)$ of a physical system: the horizontal space at any point $ψ$ is defined as the set of tangent vectors orthogonal to the world $W_ψ$ with respect to the propensity metric. We refer to this connection as the fundamental connection of the dynamical system.

Definition 7.2. For a dynamical system $X = (X, R, T)$ a perceptible is a smooth map $f : X \to A^\ast$. A temporal evolution of a dynamical system $X$ is an integral curve of a vector field $f_T$ on $X$, called the temporal evolution vector field of $X$, such that

$$(dT)(u) = g(f_T, u), \quad \forall u \in X[1], \tag{19}$$

The propensity realm of a dynamical system is referred to as its state space, and propensity modes are called states. A state $ψ$ such that the vector $f_T(ψ)$ is vertical is called the proper state of $X$, and the possible world $W_ψ$ is called an accessible world.

Remark 7.2. A perceptible is a smooth restriction of an attribute of the underlying phenomenon to the perception domain. Intuitively, a temporal evolution of a dynamical system is the motion of the observer’s proper viewpoint across possible worlds of the system along its temporal evolution vector field. At each point of an evolution the observer encounters a possible world, a diffeomorphically perturbed copy of the spacetime of the background reality, which contains perceptible information about the system.

Definition 7.3. For a dynamical system $X$, an ordered triple $(f, φ, ψ)$, where $f$ is a perceptible and $φ, ψ$ are states, is called an $f$-observation. The states $φ$ and $ψ$ are called the initial and the final states, and the worlds $W_φ$ and $W_ψ$ are called the source and the target worlds of the $f$-observation, respectively. The propensity $ρ(φ, ψ) \in R \cup \{∞\}$ of an $f$-observation $(f, φ, ψ)$ is defined by

$$ρ(φ, ψ) := \begin{cases} \infty & : W_φ = W_ψ \\ d^{-1} & : W_φ \neq W_ψ \end{cases}, \tag{20}$$

where $d$ is the length of the shortest geodesics between $φ$ and $ψ$. 
Remark 7.3. Intuitively, propensity roughly quantifies accessibility of the target world - the farther away, the less accessible it is.

8 Cosmologies

Remark 8.1. A left invariant vector field $\hat{u}$ on $\mathcal{H}^{\circ}$, generated by a vector $u \in H$ with the components $(u^\beta)$ in a canonical basis $(i_\beta)$, assigns to each point $a \in \mathcal{H}^{\circ}$ with canonical coordinates $(w, x, y, z)$ a vector $\hat{u}(a) \in T_a\mathcal{H}^{\circ}$ with the components $\hat{u}^\beta(a) = (au)^\beta$ in the basis $(\partial_w, \partial_x, \partial_y, \partial_z)(a)$ on $T_a\mathcal{H}^{\circ}$:

$$\begin{align*}
\hat{u}^0(a) &= wu^0 - xu^1 - yu^2 - zu^3, \\
\hat{u}^1(a) &= wu^1 + xu^0 + yu^3 - zu^2, \\
\hat{u}^2(a) &= wu^2 - xu^3 + yu^0 + zu^1, \\
\hat{u}^3(a) &= wu^3 + xu^2 - yu^1 + zu^0. 
\end{align*} \tag{21}$$

Definition 8.1. A robust reality is called a cosmology if its underlying paradigm is isomorphic to the home paradigm of the observer.

Remark 8.2. As follows from Theorem 4.1 of [20], the choice of cosmologies of the observer is extremely limited. In fact, there is a unique, up to the functional variable $T$, cosmology $(H^{\circ}, T)$. Let us review its basic properties.

1. The perception domain $\mathcal{H}^{\circ}$ of the underlying paradigm $H$ is the lie group of nonzero quaternions with the $\mathbb{R} \times S^3$ topology, the product of the real line and a three-sphere.

2. The spacetime $S = (\mathcal{H}^{\circ}, T)$ of the cosmology is a smooth four dimensional manifold with a closed FLRW metric.

3. Given a canonical sensory basis $(i_\beta)$ and, for a differentiable function $R : \mathbb{R} \to \mathbb{R} \setminus \{0\}$, a system of natural spherical coordinates $(\eta, \chi, \theta, \varphi)$ on $\mathcal{H}^{\circ}$, related to the canonical coordinates by

$$\begin{align*}
w &= R(\eta) \cos(\chi), \\
x &= R(\eta) \sin(\chi) \sin(\theta) \cos(\varphi), \\
y &= R(\eta) \sin(\chi) \sin(\theta) \sin(\varphi), \\
z &= R(\eta) \sin(\chi) \cos(\theta),
\end{align*}$$

the metric has the following components in the spherical frame $(\partial_\eta^R, \partial_\chi^R, \partial_\theta^R, \partial_\varphi^R)$

$$\mathcal{J}_{\alpha\beta} = \text{diag}(1, -a^2, -a^2 \sin^2 \chi, -a^2 \sin^2 \chi \sin^2 \theta), \quad a := \sqrt{|\dot{T}|},$$

with 

$$R = \exp \int \frac{d\eta}{\sqrt{|\dot{T}|}}.$$

\tag{22}$$
4. The perceptible time $\mathcal{T}$ is a monotonous function of $\eta$.

5. The structure field of the cosmology has the following components in the frame $(\partial^R_\eta, \partial^R_\chi, \partial^R_\theta, \partial^R_\phi)$:

$$
\mathcal{H}^0_{\alpha\beta} = \begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & -\lambda^{-1} & 0 & 0 \\
0 & 0 & -\lambda^{-1} \sin^2 \chi & 0 \\
0 & 0 & 0 & -\lambda^{-1} \sin^2 \chi \sin^2 \theta
\end{pmatrix},
$$

$$
\mathcal{H}^1_{\alpha\beta} = \begin{pmatrix}
0 & \lambda & 0 & 0 \\
\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & \sin^2 \chi \sin \theta \\
0 & 0 & -\sin^2 \chi \sin \theta & 0
\end{pmatrix},
$$

$$
\mathcal{H}^2_{\alpha\beta} = \begin{pmatrix}
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & -\sin \theta \\
\lambda & 0 & 0 & 0 \\
0 & \sin \theta & 0 & 0
\end{pmatrix},
$$

$$
\mathcal{H}^3_{\alpha\beta} = \begin{pmatrix}
0 & 0 & 0 & \lambda \\
0 & 0 & 1/\sin \theta & 0 \\
0 & -1/\sin \theta & 0 & 0 \\
\lambda & 0 & 0 & 0
\end{pmatrix},
$$

where $\lambda := \dot{R}/R$.

6. The ether field of the cosmology has the following components in the frame $(\partial^R_\eta, \partial^R_\chi, \partial^R_\theta, \partial^R_\phi)$:

$$
d\mathcal{T} = (\dot{\mathcal{T}}, 0, 0, 0).
$$

7. For a canonical sensory basis $(i_\beta)$ and the corresponding canonical coordinate system $(w, x, y, z)$, let $u$ be a reflexor with the components $(u^\beta)$ in $(i_\beta)$, and $\alpha$ - a proper viewpoint with coordinates $(\bar{w}, \bar{x}, \bar{y}, \bar{z})$. Then the $(u, \alpha)$-vista can be computed by solving the system of differential equations $[21]$ with a parameter $t$:

$$
w(t) = \exp(u^0 t)(\frac{-u^1 \bar{x} - u^2 \bar{y} - u^3 \bar{z}}{\omega} \sin \omega t + \bar{w} \cos \omega t)
$$

$$
x(t) = \exp(u^0 t)(\frac{u^1 \bar{w} - u^2 \bar{z} + u^3 \bar{y}}{\omega} \sin \omega t + \bar{x} \cos \omega t)
$$

$$
y(t) = \exp(u^0 t)(\frac{u^1 \bar{z} + u^2 \bar{w} - u^3 \bar{x}}{\omega} \sin \omega t + \bar{y} \cos \omega t)
$$

$$
z(t) = \exp(u^0 t)(\frac{-u^1 \bar{y} + u^2 \bar{x} + u^3 \bar{w}}{\omega} \sin \omega t + \bar{z} \cos \omega t), \quad (23)
$$
where \( \omega := \sqrt{u_1^2 + u_2^2 + u_3^2} \).

**Remark 8.3.** The theory of the \( \mathbb{R} \)-observer we have developed so far is left invariant (utilizing left invariant vector fields on perception domains). It is easy to show that the metric of the right invariant cosmology coincide with (22), but the \((u,a)\)-vistas are different. This can be used, in principle, by the observer to determine the “chirality” of his contemporary reality.

## 9 Physical systems

**Definition 9.1.** A dynamical system \( X = (X, \mathcal{R}, T) \) is called a physical system if \( \mathcal{R} \) is a cosmology and the state space is a hyperkähler manifold.

**Definition 9.2.** For a physical system \( X = (X, \mathcal{R}, T) \), a quaternionic regular perceptible \( f \) is called an observable. For an observable \( f \), we refer to its hyperhamiltonian field \( f \) as the \( f \)-field. A state \( \psi \) such that the vector \( f(\psi) \) is vertical is called an \( f \)-proper state of \( X \). A possible world \( \mathcal{W} \) is called \( f \)-(in)accessible iff there is (not) an \( f \)-proper state \( \psi \), such that \( \mathcal{W} = \mathcal{W}_\psi \). For each \( f \)-proper state \( \psi \) the value \( f(\psi) \) is called a relative perceptible property of \( X \).

**Remark 9.1.** The above definition generalizes the notions of GQM, where points at which the schrödinger vector field of an observable becomes vertical, and the corresponding values of \( f \) parametrize the eigenvectors and eigenvalues of \( \hat{F} \), respectively \([18]\).

**Definition 9.3.** For a physical system \( X = (X, \mathcal{R}, T) \), its hamiltonian is an observable \( h \) such that the temporal evolution vector field of \( X \) coincides with the \( h \)-field.

**Remark 9.2.** A physical system can be thought of as a sufficiently smooth fine-graining of a cosmology. It is a natural generalization of the notion of a quantum system of GQM.

**Definition 9.4.** For a physical system, an \( f \)-observation \((f, \phi, \psi)\) is called successful if \( f \) is an observable, the propensity \( \rho(\phi, \psi) \) exists, and the target world \( \mathcal{W}_\psi \) is \( f \)-accessible, in which case we refer to \( \mathcal{W}_\psi \) as the actual world of the \( f \)-observation. Otherwise the \( f \)-observation is called failed (or unsuccessful), and the world \( \mathcal{W}_\psi \) is called virtual.
Remark 9.3. A successful $f$-observation is also referred to as a measurement.

**Definition 9.5.** For a measurement $(f, \phi, \psi)$ its result is an ordered pair $(W_\psi, T[\psi])$, where $W_\psi$ is the actual world and $T[\psi]$ is the level set of the perceptible time of $W_\psi$ containing $\psi$. We refer to $T[\psi]$ as the hypersurface of the present. The value $f(\psi)$ is called a relative perceptible property of the physical system.

Remark 9.4. Intuitively, the final state of an $f$-measurement marks the “landing spot” of the observer (or, more correctly, of his proper viewpoint) in a (target) world whose properties may differ, in a strictly defined sense, from the respective properties of the source world. Propensity carries information about the “likelihood” of the “collapse” of $\phi$ to $\psi$: the farther away $\psi$ is from $\phi$, the less likely it is that the final state is $\psi$. We are to think of a relative perceptible property as a generalization of a non-normalized eigenvalue of an observable in CQM. The main difference between the two notions is this: instead of a single real number it is an element of the quaternion algebra, which, given a canonical sensory basis, is an ordered quadruple of real numbers, the first of which can be used to represent the “landing” moment in the ambient time of the system, and the other three - to quantify various features of the system. Indeed, for a measurement to be a meaningful procedure, the time of its occurrence must be known to the observer. Even then, if a measurement of, say, a position of a particle in CQM results in a particular value of one of the coordinates, the position of the particle is still unknown unless the other two coordinates are known.

**Definition 9.6.** A physical system is called a hyperquantum system if its monocosm is a hyperquantum bundle $(V_\omega \circ H)$ over a quaternionic hilbert space $V$, and the imaginary part $\text{Im}(\hat{h})$ of its hamiltonian coincides with the restriction to $V_\omega$ of the expectation of a quaternionic antihermitian operator $\hat{H}$ on $\mathbb{V}$.

Remark 9.5. It seems tempting to obtain quantum systems of GQM by demanding the existence of a canonical sensory basis in which the hamiltonian has a unique nonzero constituent $h_\nu$. However, this would not be quite correct because within our framework the underlying paradigm of such systems would correspond to the two dimensional consistent (boolean) paradigm $\mathbb{C}$, and their background reality is not a cosmology. In this sense quantum systems are a degenerate case of hyperquantum systems: two out of four dimensions are collapsed in each possible world.
Definition 9.7. A dynamical system is called a quantum system if its background reality is $(\mathbb{C}, T)$, its monocosm is a quantum bundle $(\mathcal{V}_C^\circ)$ over a complex hilbert space $\mathcal{V}$, and the complex imaginary part of its hamiltonian coincides with the restriction to $\mathcal{V}_C^\circ$ of the expectation of a complex antihermitian operator $\hat{H}$ on $\mathcal{V}$.

Remark 9.6. It is a standard result in GQM that a schrödinger evolution of a quantum system is also a hamiltonian evolution with the expectation of $\hat{H}$ as its generating function [18], so the above definition is equivalent to the description of a quantum system in GQM. For an $f$-measurement $(f, \phi, \psi)$ of a quantum system the propensity $\rho(\phi, \psi)$ can be expressed in terms of probability of obtaining a particular result ([6] and references therein).

Example 9.1. The Old World $\Upsilon$. The monocosm of $\Upsilon$ is the degenerate hyperquantum bundle with a single possible world, hence the source, target and actual worlds coincide for each measurement, and the ambient time of $\Upsilon$ is the perceptible time of the background cosmology. A temporal evolution of the system is orthogonal to the level sets of the perceptible time with respect to the propensity metric of $\Upsilon$. Since the evolution vector field is vertical everywhere, every state of the system is proper. For each measurement of $\Upsilon$ the metric of its actual world is the metric of the background cosmology. In conventional terms this hyperquantum system seems to correspond to the classical coarse-grained view of the universe - an observer at rest relative to CMB in the spacetime of a spatially closed FLWR cosmology.

10 Summary

As we mention in the introduction, the technical purpose of the paper is to provide formal definitions of observer related notions which are normally considered too ambiguous for constructive discussion within mainstream physics. Below is an informal overview of some of them.

The observer is represented by an existence mode of a boolean experient, and is capable of perceiving various realities, each based on a paradigm. In some realities the observer tends to distinguish dynamical systems, collections of experiences stable in a strictly defined sense. Dynamical systems spend most of their ambient time roaming their possible worlds according to the equation [19] which is the main dynamical equation of the modification. An observation of a dynamical system perturbs its evolution resulting occa-
sionally in a creation of a *perceptible property* of the system with respect to its *actual world*. For reasons that are beyond the scope of this theory, the contemporary *operational logic* of the observer seems to be bivalent boolean (*Assertion 1*), and hence the largest immediate environment conforming to this requirement corresponds to a *robust reality* of his *home paradigm* (a *cosmology*). Then the kinematic axioms of GR follow: the spacetime of the cosmology is a smooth manifold, because it is a lie group, its dimensionality is indeed four, and it has a lorentzian metric of a very special type (closed FLRW). It is curious that the requirement of booleanity alone is sufficient, and bivalence follows (see [12], p. 121). The internal mathematics of the H-universe, although boolean, is nonclassical: the axiom of choice fails in the topos $\mathcal{H} Set$ (see [12], p. 301), making the basic tool for obtaining, say, countable additivity of the Lebesgue measure unavailable to the observer. Since spacetime acquires a locally compact lie group structure, the observer can use the Haar measure whose properties are less dependent on the axiom of choice and can be described constructively ([7], [3]). The nonstandard integration over spacetime may have some application to the problem of apparently non-Newtonian behavior of large structures in contemporary observational astrophysics (see [16] and references therein).

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