Gravitational fields as generalized string models

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Abstract

We show that Einstein’s main equations for stationary axisymmetric fields in vacuum are equivalent to the motion equations for bosonic strings moving on a special nonflat background. This new representation is based on the analysis of generalized harmonic maps in which the metric of the target space explicitly depends on the parametrization of the base space. It is shown that this representation is valid for any gravitational field which possesses two commuting Killing vector fields. We introduce the concept of dimensional extension which allows us to consider this type of gravitational fields as strings embedded in D-dimensional nonflat backgrounds, even in the limiting case where the Killing vector fields are hypersurface orthogonal.

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I. INTRODUCTION

In general relativity, stationary axisymmetric solutions of Einstein’s equations play a crucial role for the description of the gravitational field of astrophysical objects. In particular, the black hole solutions and their generalizations that include Maxwell fields are contained within this class.

This type of exact solutions has been the subject of intensive research during the past few decades. In particular, the number of known exact solutions drastically increased after Ernst [1] discovered an elegant representation of the field equations that made it possible to search for their symmetries. These studies led finally to the development of solution generating techniques [2] which allow us to find new solutions, starting from a given seed solution. In particular, solutions with an arbitrary number of multipole moments for the mass and angular momentum were derived in [3] and used to describe the gravitational field of rotating axially symmetric distributions of mass.

The relationship between the Ernst representation and the Lagrangian for nonlinear sigma models [4, 5] was a further result that later allowed to formally develop the isomodromic quantization procedure, based on the total integrability and separation of variables of the underlying sigma model [6]. The analogy between sigma models and vacuum gravitational fields were further studied at the level of the action in [7], where it was shown that the Einstein-Hilbert Lagrangian for gravitational fields with two commuting Killing vector fields can be reduced to the canonical Lagrangian of an $SL(2, R)/SO(2)$ nonlinear sigma model. Since string theories are based upon linear and nonlinear sigma models, some interesting relationships appear between 4-dimensional black hole solutions and string solutions. In particular, in [8] it was shown that any black hole solution can be at the same time the background of $N = 2$ superstrings. Moreover, Tomimatsu-Sato generalizations of black holes can be interpreted as closed string-like circular mass distributions as shown in [9]. The particular case of a Kerr-like solution of axidilaton gravity has been shown to possess a ring singularity in whose vicinity the solution can be interpreted as representing the field around a fundamental heterotic string [10].

In this work we derive an additional representation for stationary axisymmetric vacuum metrics. The fact that the main field equations for this field can be derived from a Lagrangian with only two gravitational variables, depending on only two spacetime coordinates, is what
we use as a starting point in order to develop a formalism which allows us to interpret this special class of gravitational fields as generalized bosonic string models. The formalism is based upon the definition of generalized harmonic maps which are characterized by a new explicit connection between the metrics of the base space and the target space. If the base space is 2–dimensional, generalized harmonic maps can be interpreted as describing the motion of a bosonic string on a nonflat background.

Using the class of stationary axisymmetric vacuum fields as a prime example, we will show that any vacuum gravitational field with two commuting Killing vector fields can be interpreted as a bosonic string “living” on a curved background, whose metric explicitly depends on the parameters that are used to describe the string world-sheet. This interaction between the world-sheet metric and the background metric manifests itself in the appearance of an additional term in the motion equations of the string and in a generalized conservation law for the energy-momentum tensor of the string. These two new constituents of the formalism allow us to identify the motion equations of the string with the main field equations of the gravitational field. This shows that Einstein’s vacuum equations for this class of gravitational fields are equivalent to the motion equations of a generalized bosonic string model. This analogy can be applied to the entire class of spacetimes with two commuting Killing vector fields, and we show it explicitly in the case of Einstein-Rosen gravitational waves and Gowdy cosmological models. Furthermore, particular sets of solutions contained in this class of spacetimes are shown to satisfy boundary conditions for open and closed strings.

This work is organized as follows. In Section II we introduce the notations for the stationary axisymmetric metric, review the $SL(2, R)/SO(2)$ representation of the main field equations, and show the incompatibility with the Polyakov action for bosonic strings. In Section III we present a generalization of harmonic maps which consists in considering metrics on the target space that explicitly depend on the coordinates of the base space. The mathematical properties of this new type of harmonic maps are investigated in Appendix A. This generalization allows us to consider a stationary axisymmetric gravitational field as described by a bosonic string moving on a nonflat background. Analogous results are obtained for Einstein-Rosen waves and Gowdy cosmologies. Section IV is devoted to the discussion of a dimensional extension of the background space which allows us to interpret a gravitational field of this class as a bosonic string moving on a nonflat space of arbitrary
II. STATIONARY AXISYMMETRIC GRAVITATIONAL FIELDS

The first analysis of stationary axially symmetric gravitational fields was carried out by Weyl [11] in 1917, soon after the formulation of general relativity. In particular, Weyl discovered that in the static limit the main part of the vacuum field equations reduces to a single linear differential equation. The corresponding general solution can be written in cylindrical coordinates as an infinite sum with arbitrary constant coefficients. A particular choice of the coefficients leads to the subset of asymptotically flat solutions which is the most interesting from a physical point of view. In this section we review the main properties of stationary axisymmetric gravitational fields. In particular, we show explicitly that the main field equations in vacuum can be represented as the equations of a nonlinear sigma model in which the base space is the 4-dimensional spacetime and the target space is a 2-dimensional conformally Euclidean space.

A. Line element and field equations

Although there exist in the literature many suitable coordinate systems, stationary axisymmetric gravitational fields are usually described in cylindrical coordinates \((t, \rho, z, \varphi)\). Stationarity implies that \(t\) can be chosen as the time coordinate and the metric does not depend on time, i.e. \(\partial g_{ab}/\partial t = 0\). Consequently, the corresponding timelike Killing vector has the components \(\delta^a_t\). A second Killing vector field is associated to the axial symmetry with respect to the axis \(\rho = 0\). Then, choosing \(\varphi\) as the azimuthal angle, the metric satisfies the conditions \(\partial g_{ab}/\partial \varphi = 0\), and the components of the corresponding spacelike Killing vector are \(\delta^a_{\varphi}\).

Using further the properties of stationarity and axial symmetry, together with the vacuum field equations, for a general metric of the form \(g_{ab} = g_{ab}(\rho, z)\), it is possible to show that the most general line element for this type of gravitational fields can be written in the Weyl-Lewis-Papapetrou form as [11, 12, 13] \(ds^2 = f(dt - \omega d\varphi)^2 - f^{-1} \left[ e^{2k} (d\rho^2 + dz^2) + \rho^2 d\varphi^2 \right] \), (1)
where $f$, $\omega$ and $k$ are functions of $\rho$ and $z$, only. After some rearrangements which include the introduction of a new function $\Omega = \Omega(\rho, z)$ by means of

$$
\rho \partial_{\rho} \Omega = f^2 \partial_z \omega, \quad \rho \partial_z \Omega = -f^2 \partial_{\rho} \omega,
$$

(2)

the vacuum field equations $R_{ab} = 0$ can be shown to be equivalent to the following set of partial differential equations

$$
\frac{1}{\rho} \partial_{\rho}(\rho \partial_{\rho} f) + \partial_z^2 f + \frac{1}{f} [(\partial_{\rho} \Omega)^2 + (\partial_z \Omega)^2 - (\partial_{\rho} f)^2 - (\partial_z f)^2] = 0,
$$

(3)

$$
\frac{1}{\rho} \partial_{\rho}(\rho \partial_{\rho} \Omega) + \partial_z^2 \Omega - \frac{2}{f} (\partial_{\rho} f \partial_{\rho} \Omega + \partial_z f \partial_z \Omega) = 0,
$$

(4)

$$
\partial_{\rho} k = \frac{\rho}{4f^2} \left[ (\partial_{\rho} f)^2 + (\partial_{\rho} \Omega)^2 - (\partial_z f)^2 - (\partial_z \Omega)^2 \right],
$$

(5)

$$
\partial_z k = \frac{\rho}{2f^2} (\partial_{\rho} f \partial_z f + \partial_{\rho} \Omega \partial_z \Omega).
$$

(6)

It is clear that the field equations for $k$ can be integrated by quadratures, once $f$ and $\Omega$ are known. For this reason, the equations (3) and (4) for $f$ and $\Omega$ are usually considered as the main field equations for stationary axisymmetric vacuum gravitational fields. In the following subsections we will focus on the analysis of the main field equations, only.

Let us consider the special case of static axisymmetric fields. This corresponds to metrics which, apart from being axially symmetric and independent of the time coordinate, are invariant with respect to the transformation $\varphi \rightarrow -\varphi$ (i.e. rotations with respect to the axis of symmetry are not allowed). Consequently, the corresponding line element is given by (1) with $\omega = 0$, and the field equations can be written as

$$
\partial_{\rho}^2 \psi + \frac{1}{\rho} \partial_{\rho} \psi + \partial_z^2 \psi = 0, \quad f = \exp(2\psi),
$$

(7)

$$
\partial_{\rho} k = \rho \left[ (\partial_{\rho} \psi)^2 - (\partial_z \psi)^2 \right], \quad \partial_z k = 2\rho \partial_{\rho} \psi \partial_z \psi.
$$

(8)

We see that the main field equation (7) corresponds to the linear Laplace equation for the metric function $\psi$. The general solution of Laplace’s equation is known and, if we demand additionally asymptotic flatness, we obtain the Weyl solution which can be written as

$$
\psi = \sum_{n=0}^{\infty} \frac{a_n}{(\rho^2 + z^2)^{n+1/2}} P_n(\cos \theta), \quad \cos \theta = \frac{z}{\sqrt{\rho^2 + z^2}},
$$

(9)
where \( a_n (n = 0, 1, \ldots) \) are arbitrary constants, and \( P_n (\cos \theta) \) represents the Legendre polynomials of degree \( n \). The expression for the metric function \( k \) can be calculated by quadratures by using the set of first order differential equations (8). Then

\[
k = - \sum_{n,m=0}^{\infty} \frac{a_n a_m (n + 1)(m + 1)}{(n + m + 2)(\rho^2 + z^2)^{\frac{n+m+2}{2}}} (P_n P_m - P_{n+1} P_{m+1}) .
\]

(10)

Since this is the most general static, axisymmetric, asymptotically flat vacuum solution, it must contain all known solution of this class. In particular, one the most interesting special solutions which is Schwarzschild’s spherically symmetric black hole spacetime must be contained in this class. To see this, we must choose the constants \( a_n \) in such a way that the infinite sum (9) converges to the Schwarzschild solution in cylindric coordinates. But, of course, this representation is not the most appropriate to analyze the interesting physical properties of Schwarzschild’s metric.

B. Representation as a nonlinear sigma model

Consider two (pseudo)-Riemannian manifolds \((M, \gamma)\) and \((N, G)\) of dimension \( m \) and \( n \), respectively. Let \( M \) be coordinatized by \( x^a \), and \( N \) by \( X^\mu \), so that the metrics on \( M \) and \( N \) can be, in general, smooth functions of the corresponding coordinates, i.e., \( \gamma = \gamma(x) \) and \( G = G(X) \). A harmonic map is a smooth map \( X : M \rightarrow N \), or in coordinates \( X : x \mapsto X \) so that \( X \) becomes a function of \( x \), and the \( X \)'s satisfy the motion equations following from the action

\[
S = \int d^m x \sqrt{\gamma} \, \gamma^{ab}(x) \, \partial_a X^\mu \, \partial_b X^\nu \, G_{\mu\nu}(X) ,
\]

(11)

which sometimes is called the “energy” of the harmonic map \( X \). The straightforward variation of \( S \) with respect to \( X^\mu \) leads to the motion equations

\[
\frac{1}{\sqrt{\gamma}} \partial_b \left( \sqrt{\gamma} \gamma^{ab} \partial_a X^\mu \right) + \Gamma^\mu_{\nu\lambda} \, \gamma^{ab} \, \partial_a X^\nu \, \partial_b X^\lambda = 0 ,
\]

(12)

where \( \Gamma^\mu_{\nu\lambda} \) are the Christoffel symbols associated to the metric \( G_{\mu\nu} \) of the target space \( N \). If \( G_{\mu\nu} \) is a flat metric, one can choose Cartesian-like coordinates such that \( G_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(\pm 1, \ldots, \pm 1) \), the motion equations become linear, and the corresponding sigma model is linear. This is exactly the case of a bosonic string on a flat background in which the base space is the 2-dimensional string world-sheet. In this case the action (11) is usually referred to as the Polyakov action [14].
Consider now the case in which the base space $M$ is a stationary axisymmetric spacetime. Then, $\gamma^{ab}$, $a, b = 0, \ldots, 3$, can be chosen as the Weyl-Lewis-Papapetrou metric (11), i.e.

\[
\gamma_{ab} = \begin{pmatrix}
  f & 0 & 0 & -f \omega \\
  0 & -f^{-1} e^{2k} & 0 & 0 \\
  0 & 0 & -f^{-1} e^{2k} & 0 \\
  -f \omega & 0 & 0 & f \omega^2 - \rho^2 f^{-1}
\end{pmatrix}.
\] (13)

Let the target space $N$ be 2-dimensional with metric $G_{\mu\nu} = (1/2) f^{-2} \delta_{\mu\nu}$, $\mu, \nu = 1, 2$, and let the coordinates on $N$ be $X^\mu = (f, \Omega)$. Then, it is straightforward to show that the action (11) becomes

\[
S = \int L \, dt \, d\varphi \, d\rho \, dz , \quad L = \frac{\rho}{2f^2} \left[ (\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2 \right] ,
\] (14)

and the corresponding motion equations (12) are identical to the main field equations (3) and (4).

Notice that the field equations can also be obtained from (11) by a direct variation with respect to $f$ and $\Omega$. This interesting result was obtained originally by Ernst [1], and is the starting point of what today is known as the Ernst representation of the field equations.

The above result shows that stationary axisymmetric gravitational fields can be described as a $(4 \rightarrow 2)$–nonlinear harmonic map, where the base space is the spacetime of the gravitational field and the target space corresponds to a 2-dimensional conformally Euclidean space. A further analysis of the target space shows that it can be interpreted as the quotient space $SL(2, R)/SO(2)$ [6], and the Lagrangian (14) can be written explicitly [7] in terms of the generators of the Lie group $SL(2, R)$. Harmonic maps in which the target space is a quotient space are usually known as nonlinear sigma models [5].

The form of the Lagrangian (14) with two gravitational field variables, $f$ and $\Omega$, depending on two coordinates, $\rho$ and $z$, suggests a representation as a harmonic map with a 2-dimensional base space. In string theory, this is an important fact that allows one to use the conformal invariance of the base space metric to find an adequate representation for the set of classical solutions. This, in turn, facilitates the application of the canonical quantization procedure. Unfortunately, this is not possible for the Lagrangian (14). Indeed, if we consider $\gamma^{ab}$ as a 2-dimensional metric that depends on the parameters $\rho$ and $z$, the diagonal form of the Lagrangian (14) implies that $\sqrt{\gamma} \gamma^{ab} = \delta^{ab}$. Clearly, this choice is not
compatible with the factor $\rho$ in front of the Lagrangian. Therefore, the reduced gravitational Lagrangian cannot be interpreted as corresponding to a $(2 \to n)$-harmonic map. Nevertheless, we will show in the next section that a modification of the definition of harmonic maps allows us to “absorb” the unpleasant factor $\rho$ in the metric of the target space, and to use all the advantages of a 2-dimensional base space.

Notice that the representation of stationary fields as a nonlinear sigma model becomes degenerate in the limiting case of static fields. Indeed, the underlying geometric structure of the $SL(2, R)/SO(2)$ nonlinear sigma models requires that the target space be 2-dimensional, a condition which is not satisfied by static fields. We will see below that by using a dimensional extension of generalized sigma models, it will be possible to treat the special static case, without affecting the underlying geometric structure.

The analysis performed in this section for stationary axisymmetric fields can be generalized to include any gravitational field containing two commuting Killing vector fields. This is due to the fact that for this class of gravitational fields it is always possible to find the corresponding Ernst representation in which the Lagrangian contains only two gravitational variables which depend on only two spacetime coordinates.

### III. GRAVITATIONAL FIELDS AS GENERALIZED HARMONIC MAPS

A $(m \to n)$–generalized harmonic map is defined as a smooth map $X : M \to N$, satisfying the Euler-Lagrange equations

$$\frac{1}{\sqrt{\gamma}} \partial_b \left( \sqrt{\gamma} \gamma^{ab} \partial_a X^\mu \right) + \Gamma^\mu_{\nu\lambda} \gamma^{ab} \partial_a X^\nu \partial_b X^\lambda + G^\mu\lambda^{ab} \partial_a X^\nu \partial_b G_{\lambda\nu} = 0 , \quad (15)$$

which follow from the variation with respect to the fields $X^\mu$ of the generalized action

$$S = \int L d^m x$$

with the Lagrangian

$$L = \sqrt{\gamma} \gamma^{ab}(x) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X, x) . \quad (16)$$

Here $(M, \gamma)$ and $(N, G)$ are (pseudo-)Riemannian manifolds of dimension $m$ and $n$, and coordinates $x^a$ and $X^\mu$, respectively. Moreover, it is assumed that $\gamma = \gamma(x)$ and $G = G(X, x)$, i.e. the target metric depends explicitly on the coordinates of the base space. This additional dependence is the result of the “interaction” between the base space $M$ and the target space $N$, and leads to an extra term in the motion equations, as can be seen in (15).
In Appendix A we establish the main properties of generalized harmonic maps which will be applied in concrete cases of gravitational fields in this section. First, we will analyze in detail the case of stationary axisymmetric fields and then we will show that these results can be generalized to include other spacetimes with two commuting Killing vector fields, namely, the spacetimes of Einstein–Rosen gravitational waves and Gowdy cosmologies.

### A. Stationary axisymmetric spacetimes

In Section II we described stationary, axially symmetric, gravitational fields as a $(4 \rightarrow 2)$–nonlinear sigma model. There it was pointed out the convenience of having a 2-dimensional base space in analogy with string theory. Now we will show that this can be done by using the generalized harmonic maps defined above.

Consider a $(2 \rightarrow 2)$–generalized harmonic map. Let $x^a = (\rho, z)$ be the coordinates on the base space $M$, and $X^\mu = (f, \Omega)$ the coordinates on the target space $N$. In the base space we choose a flat metric and in the target space a conformally flat metric, i.e.

\[
\gamma_{ab} = \delta_{ab} \quad \text{and} \quad G_{\mu\nu} = \frac{\rho}{2f^2}\delta_{\mu\nu} \quad (a, b = 1, 2; \mu, \nu = 1, 2).
\]  

(17)

A straightforward computation shows that the generalized Lagrangian (16) coincides with the Lagrangian (14) for stationary axisymetric fields, and that the equations of motion (15) generate the main field equations (3) and (4).

For the sake of completeness we calculate the components of the energy-momentum tensor $T_{ab} = \delta\mathcal{L}/\delta\gamma^{ab}$ (cf. Appendix A). Then

\[
T_{\rho\rho} = -T_{zz} = \frac{\rho}{4f^2} \left[ (\partial_\rho f)^2 + (\partial_\rho \Omega)^2 - (\partial_z f)^2 - (\partial_z \Omega)^2 \right],
\]

(18)

\[
T_{\rho z} = \frac{\rho}{2f^2} (\partial_\rho f \partial_z f + \partial_\rho \Omega \partial_z \Omega).
\]

(19)

This tensor is traceless due to the fact that the base space is 2-dimensional. It satisfies the generalized conservation law (A20) on-shell:

\[
\frac{dT_{\rho\rho}}{d\rho} + \frac{dT_{\rho z}}{dz} + \frac{1}{2} \frac{\partial \mathcal{L}}{\partial \rho} = 0,
\]

(20)

\[
\frac{dT_{\rho z}}{d\rho} - \frac{dT_{\rho\rho}}{dz} = 0.
\]

(21)
Incidentally, the last equation coincides with the integrability condition for the metric function $k$, which is identically satisfied by virtue of the main field equations. In fact, as can be seen from Eqs. (56) and (18,19), the components of the energy-momentum tensor satisfy the relationships $T_{\rho\rho} = \partial_\rho k$ and $T_{\rho z} = \partial_z k$, so that the conservation law (21) becomes an identity. Although we have eliminated from the starting Lagrangian (14) the variable $k$ by applying a Legendre transformation on the Einstein-Hilbert Lagrangian (see [7] for details) for this type of gravitational fields, the formalism of generalized harmonic maps seems to retain the information about $k$ at the level of the generalized conservation law.

The above results show that stationary axisymmetric spacetimes can be represented as a $(2 \to 2)$—generalized harmonic map with metrics given as in (17). It is also possible to interpret the generalized harmonic map given above as a generalized string model. Although the metric of the base space $M$ is Euclidean, we can apply a Wick rotation $\tau = i\rho$ to obtain a Minkowski-like structure on $M$. Then, $M$ represents the world-sheet of a bosonic string in which $\tau$ is measures the time and $z$ is the parameter along the string. The string is “embedded” in the target space $N$ whose metric is conformally flat and explicitly depends on the time parameter $\tau$. We will see in the next section that this embedding becomes more plausible when the target space is subject to a dimensional extension. In the present example, it is necessary to apply a Wick rotation in order to interpret the base space as a string world-sheet. This is due to the fact that both coordinates $\rho$ and $z$ are spatial coordinates. However, this can be avoided by considering other classes of gravitational fields with timelike Killing vector fields; examples will be given below.

The most studied solutions belonging to the class of stationary axisymmetric fields are the asymptotically flat solutions. Asymptotic flatness imposes conditions on the metric functions which in the cylindrical coordinates used here can be formulated in the form

$$\lim_{x^a \to \infty} f = 1 + O \left( \frac{1}{x^a} \right), \quad \lim_{x^a \to \infty} \omega = c_1 + O \left( \frac{1}{x^a} \right), \quad \lim_{x^a \to \infty} \Omega = O \left( \frac{1}{x^a} \right) \quad (22)$$

where $c_1$ is an arbitrary real constant which can be set to zero by appropriately choosing the angular coordinate $\varphi$. If we choose the domain of the spatial coordinates as $\rho \in [0, \infty)$ and $z \in (-\infty, +\infty)$, from the asymptotic flatness conditions it follows that the coordinates of the target space $N$ satisfy the boundary conditions

$$\dot{X}^\mu (\rho, -\infty) = 0 = \dot{X}^\mu (\rho, \infty), \quad X^{\prime \mu} (\rho, -\infty) = 0 = X^{\prime \mu} (\rho, \infty) \quad (23)$$
where the dot stands for a derivative with respect to $\rho$ and the prime represents derivation with respect to $z$. These relationships are known in string theory [14] as the Dirichlet and Neumann boundary conditions for open strings, respectively, with the extreme points situated at infinity. We thus conclude that if we assume $\rho$ as a “time” parameter for stationary axisymmetric gravitational fields, an asymptotically flat solution corresponds to an open string with endpoints attached to $D$–branes situated at plus and minus infinity in the $z$–direction.

B. Einstein–Rosen gravitational waves

Consider the line element for Einstein-Rosen gravitational waves [2]

$$ds^2 = e^{2(\gamma-\psi)}dt^2 - \frac{e^{-2\psi}}{2}\left[ (\rho^2 d\rho^2 + \rho^2 d\varphi^2) - e^{2\psi}(dz + \omega d\varphi)^2 \right]$$

(24)

where $\psi$, $\omega$ and $\gamma$ are functions of $t$ and $\rho$. These spacetimes are characterized by the existence of two spacelike, commuting Killing vector fields $\xi^a_I = \delta^a_\varphi$ and $\xi^a_{II} = \delta^a_z$. They describe the field of gravitational waves that propagate inward in vacuum, implode on the axis of symmetry situated at $\rho = 0$, and finally propagate outward to spatial infinity. The special case in which the Killing vectors are hypersurface orthogonal corresponds to linearly polarized gravitational waves with $\omega = 0$.

The reduced Einstein-Hilbert Lagrangian is obtained neglecting all the terms which can be represented as surface terms. The final result can be written as [7]

$$\mathcal{L}_{ER} = 2\rho\left( (\partial_t \psi)^2 - (\partial_\rho \psi)^2 \right) + \frac{1}{2}\rho e^{-4\psi}\left[ (\partial_t \Omega)^2 - (\partial_\rho \Omega)^2 \right],$$

(25)

where the function $\Omega$ is defined by $\rho\Omega_t = e^{4\psi}\omega_\rho$ and $\rho\Omega_\rho = e^{4\psi}\omega_t$. The function $\gamma$ has been eliminated by means of a Legendre transformation. Comparing the particular Lagrangian (25) with the general Lagrangian (16), it is easy to establish that it corresponds to a $(2 \rightarrow 2)$–generalized harmonic map with a Minkowski-like base space $(M, \gamma(x))$, i. e.,

$$x^1 = t, \quad x^2 = \rho, \quad \gamma_{ab} = \text{diag}(1, -1),$$

(26)

and a curved target space $(N, G(X, x))$ with

$$X^1 = \psi, \quad X^2 = \Omega, \quad G_{\mu\nu} = \text{diag}\left[ 2\rho, (\rho/2)e^{-4\psi} \right].$$

(27)
The field equations \([15]\) for this particular generalized harmonic map can be written as

\[
\begin{align*}
\partial^2_\rho \psi + \frac{1}{\rho} \partial_\rho \psi - \partial^2_t \psi + \frac{1}{2} e^{-4\psi} [(\partial_\rho \Omega)^2 - (\partial_t \Omega)^2] &= 0 , \\
\partial^2_\rho \Omega + \frac{1}{\rho} \partial_\rho \Omega - \partial^2_t \Omega + 4 \partial_t \partial_\rho \psi - \partial_\rho \Omega \partial_\rho \psi &= 0 ,
\end{align*}
\]

and coincide with the main Einstein’s field equations in empty space for this kind of gravitational waves.

As for the energy-momentum tensor associated with the string metric \(\gamma_{ab}\), the components read

\[
\begin{align*}
T_{tt} &= T_{\rho \rho} = \rho [(\partial_t \psi)^2 + (\partial_\rho \psi)^2] + \frac{1}{4} \rho e^{-4\psi} [(\partial_t \Omega)^2 + (\partial_\rho \Omega)^2] , \\
T_{t \rho} &= 2 \rho \partial_t \psi \partial_\rho \psi + \frac{1}{2} \rho e^{-4\psi} \partial_t \Omega \partial_\rho \Omega .
\end{align*}
\]

Finally, it can be shown that \(\partial_\rho \gamma = T_{tt}\) and \(\partial_t \gamma = T_{t \rho}\) so that the integrability condition for the function \(\gamma\) corresponds to the generalized conservation law \([A20]\).

The above results show that Einstein-Rosen gravitational waves can be interpreted as a particular generalized harmonic map and that particular solutions of the field equations correspond to a string spatially situated along the coordinate \(\rho\) and moving along the time coordinate \(t\). The string propagates on a 2-dimensional nonflat background with metric \(G\).

As for the boundary conditions of this type of strings, if we choose a particular wave solution with a regular curvature behavior everywhere in spacetime, except at the wave front, the metric functions \(\psi, \omega\) and \(\Omega\) must satisfy certain relationships (see, for instance, \([15]\)) which can be expressed as the Dirichlet and Neumann conditions for and open string in the form

\[
\dot{X}^\mu(t, 0) = 0 = \dot{X}^\mu(t, \infty) , \quad X'^\mu(t, 0) = 0 = X'^\mu(t, \infty) .
\]

Here the dot stands for a derivative with respect to the time coordinate \(t\) and the prime represents derivation with respect to the spatial coordinate \(\rho\). The endpoints are situated on the axis of symmetry, \(\rho = 0\), and at infinity. We see that an Einstein-Rosen gravitational wave can be interpreted as an open string attached to \(D\)−branes located on the axis and at infinity in the \(\rho\)−direction. Since the wave propagates inwards and outwards in empty space, its singular front reaches the endpoints at some moment, say at \(t_0\) and at \(t_\infty\), where the metric and its curvature diverge so that the analogy with \(D\)−branes breaks down.
C. Gowdy cosmological models

Consider the Gowdy cosmological models whose line element in the unpolarized $T^3$ case can be written as

\[ ds^2 = e^{-\frac{(\lambda+3\tau)}{2}}d\tau^2 - e^{-\frac{(\lambda-\tau)}{2}}d\theta^2 - e^{-\frac{\tau}{2}}[e^P(d\sigma + Qd\delta)^2 + e^{-P}d\delta^2] , \]  

where $P$, $Q$, and $\lambda$ are functions of $\tau$ and $\theta$ only. The spacelike Killing vector fields are associated to the coordinates $\sigma$ and $\delta$, i.e., $\xi^a_I = \delta^a_\sigma$ and $\xi^a_{II} = \delta^a_\delta$. These spacetimes are the simplest, inhomogeneous, spatially closed cosmological models in vacuum. They are expected to describe the geometric behavior of cosmological inhomogeneities and are useful in the study of the geometric properties of initial cosmological singularities. The special case in which $Q = 0$ is usually known as the polarized model and corresponds to the limiting case of hypersurface orthogonal Killing vectors.

The reduced Einstein-Hilbert Lagrangian can be expressed as

\[ \mathcal{L}_{Gow} = \frac{1}{2} \left\{ (\partial_\tau P)^2 - e^{-2\tau}(\partial_\theta P)^2 + e^{2P}[\partial_\tau Q)^2 - e^{-2\tau}(\partial_\theta Q)^2] \right\} , \]  

where a Legendre transformation has been used to eliminate the cyclic function $\lambda$. The corresponding field equations can be obtained by varying this Lagrangian density with respect to $P$ and $Q$ independently. As in the previous examples, to establish the relationship with generalized harmonic maps, we compare the particular Lagrangian (34) with the general Lagrangian (16). It is then easy to identify the coordinates and metric of the base space $M$ as

\[ x^1 = \tau , \ x^2 = \theta , \ \gamma^{ab} = \text{diag}(1, -e^{-2\tau}) , \]  

and of the target space $N$ as

\[ X^1 = P , \ X^2 = Q , \ G_{\mu\nu} = \frac{1}{2}e^{-\tau}\text{diag}(1, e^{2P}) . \]  

Moreover, the motion equations motion equations (15) lead to the set

\[ \partial_\tau^2 P - e^{-2\tau}\partial_\theta^2 P - e^{2P}[\partial_\tau Q)^2 - e^{-2\tau}(\partial_\theta Q)^2] = 0 , \]  

\[ \partial_\tau^2 Q - e^{-2\tau}\partial_\theta^2 Q + 2[\partial_\tau P \partial_\tau Q - e^{-2\tau}\partial_\theta P \partial_\theta Q] = 0 , \]  

which are equivalent to the main Einstein field equations in empty space.
Using Eq. (35), the base space \((M, \gamma(x))\) in this case can be shown to correspond to a 2-dimensional pseudo-Riemannian manifold of negative constant curvature, whereas the target manifold \((N, G(X, x))\) is in general characterized by a non-constant curvature. This means that any Gowdy cosmological model is at the same time a string with constant local curvature which propagates on a 2-dimensional curved background space.

Finally, the components of the energy-momentum tensor \(T_{ab}\) (cf. Eq. (A18)) are

\[
T_{\tau\tau} = e^{-2\tau}T_{\theta\theta} = \frac{1}{4} \left\{ (\partial_\tau P)^2 + e^{-2\tau}(\partial_\theta P)^2 + e^{2P}[(\partial_\tau Q)^2 + e^{-2\tau}(\partial_\theta Q)^2] \right\} ,
\]

\[
T_{\tau\theta} = \frac{1}{2} \left( \partial_\tau P \partial_\theta P + e^{2P} \partial_\tau Q \partial_\theta Q \right) .
\]

The generalized conservation law (A20) is equivalent to the integrability condition for the function \(\lambda\), since \(\partial_\tau \lambda = 4T_{\tau\tau}\) and \(\partial_\theta \lambda = 4T_{\tau\theta}\). Moreover, the condition \(T^a_a = 0\) is identically satisfied due to the conformal invariance of the string metric.

We will now establish a relationship between a class of Gowdy cosmologies and the boundary conditions of the string. The most important class of Gowdy spacetimes are the so called asymptotically velocity term dominated (AVTD) cosmologies which are expected to describe the initial cosmological singularity \((\tau \to \infty)\) from a geometrical point of view \cite{16}. It can be shown that AVTD cosmologies behave at \(\tau \to \infty\) as \cite{17}

\[
P = \ln[a(e^{-cr} + b^2e^{cr})] , \quad Q = \frac{b}{a(e^{-2cr} + b^2)} + d ,
\]

where \(a, b, c\) and \(d\) are arbitrary real functions of \(\theta\). Since the angular coordinate \(\theta\) is defined in the interval \([0, 2\pi]\), from the functional dependence of the string metric \(\gamma_{ab}\) and from the arbitrariness of the functions entering the AVTD expressions for \(P\) and \(Q\) in (41), it can be shown that in this case the boundary conditions for a closed string \cite{14}

\[
\gamma_{ab}(\tau, 0) = \gamma_{ab}(\tau, 2\pi) , \quad X^\mu(\tau, 0) = X^\mu(\tau, 2\pi) , \quad X'^\mu(\tau, 0) = X'^\mu(\tau, 2\pi) ,
\]

are satisfied. Here the prime denotes differentiation with respect to \(\theta\). This results establishes that an AVTD Gowdy cosmology can be interpreted as a closed string with a constant curvature geometry propagating on a nonflat background.

**IV. DIMENSIONAL EXTENSION**

In order to further analyze the analogy between gravitational fields and bosonic string models, we perform an arbitrary dimensional extension of the target space \(N\), and study
the conditions under which this dimensional extension does not affect the field equations of the gravitational field. Consider an \((m \to D)\)--generalized harmonic map. As before we denote by \(\{x^a\}\) the coordinates on \(M\). Let \(\{X^\mu, X^\alpha\}\) with \(\mu = 1, 2\) and \(\alpha = 3, 4, ..., D\) be the coordinates on \(N\). The metric structure on \(M\) is again \(\gamma = \gamma(x)\), whereas the metric on \(N\) can in general depend on all coordinates of \(M\) and \(N\), i.e. \(G = G(X^\mu, X^\alpha, x^a)\). The general structure of the corresponding field equations is as given in \([15]\). They can be divided into one set of equations for \(X^\mu\) and one set of equations for \(X^\alpha\). According to the results of the last section, the class of gravitational fields under consideration can be represented as a \((2 \to 2)\)--generalized harmonic map so that we can assume that the main gravitational variables are contained in the coordinates \(X^\mu\) of the target space. Then, the gravitational sector of the target space will be contained in the components \(G_{\mu\alpha}\) \((\mu, \nu = 1, 2)\) of the metric, whereas the components \(G_{\alpha\beta}\) \((\alpha, \beta = 3, 4, ..., D)\) represent the sector of the dimensional extension.

Clearly, the set of differential equations for \(X^\mu\) also contains the variables \(X^\alpha\) and its derivatives \(\partial_a X^\alpha\). For the gravitational field equations to remain unaffected by this dimensional extension we demand the vanishing of all the terms containing \(X^\alpha\) and its derivatives in the equations for \(X^\mu\). It is easy to show that this can be achieved by imposing the conditions

\[
G_{\mu\alpha} = 0, \quad \frac{\partial G_{\mu\nu}}{\partial X^\alpha} = 0, \quad \frac{\partial G_{\alpha\beta}}{\partial X^\mu} = 0. \tag{43}
\]

That is to say that the gravitational sector must remain completely invariant under a dimensional extension, and the additional sector cannot depend on the gravitational variables, i.e., \(G_{\alpha\beta} = G_{\alpha\beta}(X^\gamma, x^a), \gamma = 3, 4, ..., D\). Furthermore, the variables \(X^\alpha\) must satisfy the differential equations

\[
\frac{1}{\sqrt{|\gamma|}} \partial_b \left( \sqrt{|\gamma|} \gamma^{ab} \partial_a X^\alpha \right) + \Gamma^\alpha_{\beta\gamma} \gamma^{ab} \partial_a X^\beta \partial_b X^\gamma + G^\alpha_{\beta\gamma} \gamma^{ab} \partial_a X^\gamma \partial_b G_{\beta\gamma} = 0. \tag{44}
\]

This shows that any given \((2 \to 2)\)--generalized map can be extended, without affecting the field equations, to a \((2 \to D)\)--generalized harmonic map.

It is worth mentioning that the fact that the target space \(N\) becomes split in two separate parts implies that the energy-momentum tensor \(T_{ab} = \delta \mathcal{L}/\delta \gamma^{ab}\) separates into one part belonging to the gravitational sector and a second one following from the dimensional extension, i.e. \(T_{ab} = T_{ab}(X^\mu, x) + T_{ab}(X^\alpha, x)\). The generalized conservation law as given in \([A20]\) is satisfied by the sum of both parts.
Consider the example of stationary axisymmetric fields given the metrics (17). Taking into account the conditions (43), after a dimensional extension the metric of the target space becomes

\[
G = \begin{pmatrix}
\frac{\rho}{2f^2} & 0 & 0 & \cdots & 0 \\
0 & \frac{\rho}{2f^2} & 0 & \cdots & 0 \\
0 & 0 & G_{33}(X^\alpha, x) & \cdots & G_{3D}(X^\alpha, x) \\
& \cdots & \cdots & \cdots & \cdots \\
0 & 0 & G_{D3}(X^\alpha, x) & \cdots & G_{DD}(X^\alpha, x)
\end{pmatrix} .
\]

(45)

Clearly, to avoid that this metric becomes degenerate we must demand that \( \det(G_{\alpha\beta}) \neq 0 \), a condition that can be satisfied in view of the arbitrariness of the components of the metric.

With the extended metric, the Lagrangian density gets an additional term

\[
\mathcal{L} = \frac{\rho}{2f^2} \left[ (\partial_\rho f)^2 + (\partial_z f)^2 + (\partial_\rho \Omega)^2 + (\partial_z \Omega)^2 \right] + \left( \partial_\rho X^\alpha \partial_\rho X^\beta + \partial_z X^\alpha \partial_z X^\beta \right) G_{\alpha\beta} ,
\]

which nevertheless does not affect the field equations for the gravitational variables \( f \) and \( \Omega \). On the other hand, the new fields must be solutions of the extra field equations

\[
\left( \partial_\rho^2 + \partial_z^2 \right) X^\alpha + \Gamma^\alpha_{\beta\gamma} \left( \partial_\rho X^\beta \partial_\rho X^\gamma + \partial_z X^\beta \partial_z X^\gamma \right) + G^{\alpha\gamma} \left( \partial_\rho X^\beta \partial_\rho G_{\beta\gamma} + \partial_z X^\beta \partial_z G_{\beta\gamma} \right) = 0 .
\]

(47)

An interesting special case of the dimensional extension is the one in which the extended sector is Minkowskian, i.e. for the choice \( G_{\alpha\beta} = \eta_{\alpha\beta} \) with additional fields \( X^\alpha \) given as arbitrary harmonic functions. This choice opens the possibility of introducing a “time” coordinate as one of the additional dimensions, an issue that could be helpful when dealing with the interpretation of gravitational fields in this new representation.

The dimensional extension finds an interesting application in the case of static axisymmetric gravitational fields. As mentioned in Section II A, these fields are obtained from the general stationary fields in the limiting case \( \Omega = 0 \) (or equivalently, \( \omega = 0 \)). If we consider the representation as an \( SL(2, R)/SO(2) \) nonlinear sigma model or as a \( (2 \rightarrow 2) \)-generalized harmonic map, we see immediately that the limit \( \Omega = 0 \) is not allowed because the target space becomes 1-dimensional and the underlying metric is undefined. To avoid this degeneracy, we first apply a dimensional extension and only then calculate the limiting case \( \Omega = 0 \).

In the most simple case of an extension with \( G_{\alpha\beta} = \delta_{\alpha\beta} \), the resulting \( (2 \rightarrow 2) \)-generalized
map is described by the metrics $\gamma_{ab} = \delta_{ab}$ and
\[
G = \begin{pmatrix}
\rho^2 & 0 \\
\frac{1}{\rho^2} & 0 \\
0 & 1
\end{pmatrix}
\] (48)
where the additional dimension is coordinatized by an arbitrary harmonic function which does not affect the field equations of the only remaining gravitational variable $f$. This scheme represents an alternative method for exploring static fields on nondegenerate target spaces. Clearly, this scheme can be applied to the gravitational fields mentioned in the Appendix and, in general, to the case of gravitational fields possessing two hypersurface orthogonal Killing vector fields.

Our results show that a stationary axisymmetric field can be represented as a string “living” in a $D$-dimensional target space $N$. The string world-sheet is parametrized by the coordinates $\rho$ and $z$. The gravitational sector of the target space depends explicitly on the metric functions $f$ and $\Omega$ and on the parameter $\rho$ of the string world-sheet. The sector corresponding to the dimensional extension can be chosen as a $(D-2)$-dimensional Minkowski spacetime with time parameter $\tau$. Then, the string world-sheet is a 2-dimensional flat hypersurface which is “frozen” along the time $\tau$.

V. CONCLUSIONS

In this work, we introduced the concept of generalized harmonic maps which are characterized by a new explicit interaction between the metric of the base space and the metric of the target space. This interaction is realized by means of an explicit dependence of the target space metric in terms of the coordinates of the base space. The action of the generalized harmonic map becomes directly influenced by the existence of the additional interaction. As a result of this new dependence, an additional term appears in the differential equations that determine the harmonic map. Furthermore, a generalized conservation law is satisfied by the energy-momentum tensor obtained by varying the action with respect to the metric of the base space. In the case of a 2-dimensional base space we interpret a generalized map as describing the behavior of a string embedded in the target space.

We showed that any vacuum gravitational field with two commuting Killing vector fields accepts a representation as a $(2 \rightarrow 2)$–generalized harmonic map and, consequently, can be interpreted as a bosonic string “living” on a curved background, whose metric explicitly
depends on the parameters that are used to describe the string world-sheet. This result indicates that Einstein’s vacuum equations for this class of gravitational fields are equivalent to the motion equations of a generalized bosonic string model. The case of stationary axisymmetric vacuum fields was used throughout the work to illustrate the details of this new representation. In this particular example we saw that the base space is flat and the target space defines a conformally flat background. Moreover, in the case of Einstein-Rosen gravitational waves and a class of Gowdy cosmological models the reinterpretation in terms of generalized string models holds, with more general metrics for the base space and the target space. It was shown that physical conditions imposed on the behavior of the spacetime metrics correspond to boundary conditions on the string models. For instance, asymptotic flatness in stationary axisymmetric spacetimes corresponds to Dirichlet or Neumann boundary conditions for an open string with endpoints situated at infinity. A regular Einstein-Rosen gravitational wave can be interpreted as an open string with endpoints localized at the symmetry axis and at infinity. Finally, the so called asymptotically velocity term dominated (AVTD) Gowdy cosmologies are at the same time closed strings with a constant curvature geometry, propagating on a nonflat background. We expect that this analogy between physical conditions of the spacetime metrics and boundary conditions of the string models holds in more general cases.

Our approach allows a dimensional extension in which the class of gravitational fields with two commuting Killing vectors can be represented as \((2 \rightarrow D)\)–generalized harmonic maps. In particular, we used this extension to show that it is possible to investigate the limiting case of static gravitational fields as a generalized map, avoiding the problem of the degeneracy of the target space.

It would be interesting to investigate the possibility of using the present representation in the context of canonical quantization. In fact, one important result of string theory is that when one quantizes a string on a flat background, one obtains an infinite tower of massive states which are partially identified with elementary particles. In our case, however, gravitational fields are represented by strings moving on nonflat backgrounds. Furthermore, one of the main reasons why the canonical quantization of the bosonic string on curved backgrounds presents serious difficulties is because exact solutions of the corresponding field equations are very difficult to be found. Nevertheless, for the gravitational fields under consideration, this problem has already been solved. In fact, the special case of static
solutions can be solved in general, as we mentioned in Section II A. Solution generating techniques [2] can be used to find the general stationary solution, for instance, in terms of multipole moments [3]. We believe that this advantage can be used to formulate quantization schemes for this special class of gravitational fields.

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APPENDIX A: GENERALIZED HARMONIC MAPS

Consider two (pseudo-)Riemannian manifolds \((M, \gamma)\) and \((N, G)\) of dimension \(m\) and \(n\), respectively. Let \(x^a\) and \(X^\mu\) be coordinates on \(M\) and \(N\), respectively. This coordinatization implies that in general the metrics \(\gamma\) and \(G\) become functions of the corresponding coordinates. Let us assume that not only \(\gamma\) but also \(G\) can explicitly depend on the coordinates \(x^a\), i.e. let \(\gamma = \gamma(x)\) and \(G = G(X, x)\). This simple assumption is the main aspect of our generalization which, as we will see, lead to new and nontrivial results.

A smooth map \(X : M \to N\) will be called an \((m \to n)\)-generalized harmonic map if it satisfies the Euler-Lagrange equations

\[
\frac{1}{\sqrt{|\gamma|}} \partial_b \left( \sqrt{|\gamma|} \gamma^{ab} \partial_a X^\mu \right) + \Gamma^\mu_{\nu\lambda} \gamma^{ab} \partial_a X^\nu \partial_b X^\lambda + G^{\mu\nu} \gamma^{ab} \partial_a X^\nu \partial_b G_{\lambda\nu} = 0 ,
\]

which follow from the variation of the generalized action

\[
S = \int d^m x \sqrt{|\gamma|} \gamma^{ab}(x) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X, x) ,
\]

with respect to the fields \(X^\mu\). Here the Christoffel symbols, determined by the metric \(G_{\mu\nu}\), are calculated in the standard manner, without considering the explicit dependence on \(x\). Notice that the new ingredient in this generalized definition of harmonic maps, i.e., the term \(G_{\mu\nu}(X, x)\) in the Lagrangian density implies that we are taking into account the “interaction” between the base space \(M\) and the target space \(N\). This interaction leads to an extra term in the motion equations, as can be seen in (A1). It turns out that this interaction is the result of the effective presence of the gravitational field.
Notice that the limiting case of generalized linear harmonic maps is much more complicated than in the standard case. Indeed, for the motion equations (A1) to become linear it is necessary that the conditions

$$\gamma^{ab}(\Gamma^{\mu\nu\lambda}_{\lambda} \partial_b X^\lambda + G^{\mu\lambda} \partial_b G_{\lambda\nu})\partial_a X^\nu = 0 ,$$  \hspace{1cm} (A3)

be satisfied. One could search for a solution in which each term vanishes separately. The choice of a (pseudo-)Euclidean target metric $G_{\mu\nu} = \eta_{\mu\nu}$, which would imply $\Gamma_{\nu\lambda}^\mu = 0$, is not allowed, because it would contradict the assumption $\partial_b G_{\mu\nu} \neq 0$. Nevertheless, a flat background metric in curvilinear coordinates could be chosen such that the assumption $G^{\mu\nu} \partial_b G_{\mu\nu} = 0$ is fulfilled, but in this case $\Gamma_{\nu\lambda}^\mu \neq 0$ and (A3) cannot be satisfied. In the general case of a curved target metric, conditions (A3) represent a system of $m$ first order nonlinear partial differential equations for $G_{\mu\nu}$. Solutions to this system would represent linear generalized harmonic maps. The complexity of this system suggests that this special type of maps is not common.

1. **Symmetries of the action**

Let us consider the symmetries of the generalized action (A2), i.e. transformations involving the “variables” $x$, $X$, $\gamma$ and $G$ such that $S$ remains invariant.

The first obvious symmetry follows from the application of diffeomorphisms of the target space,

$$X^\mu \rightarrow X'^\mu = X'^\mu(X),$$ \hspace{1cm} (A4)

which leave invariant the metric structure of the base space $\gamma = \gamma(x)$, but they affect the metric $G$ of the target space and the partial derivatives of the fields $X^\mu$,

$$G'_{\mu\nu} = \frac{\partial X^\alpha}{\partial X'^\mu} \frac{\partial X^\beta}{\partial X'^\nu} G_{\alpha\beta},$$ \hspace{1cm} and \hspace{1cm} $\partial_a X'^\mu = \frac{\partial X^\mu}{\partial X^\beta} \partial_a X^\beta.$ \hspace{1cm} (A5)

Then, it follows that the form of the Lagrangian density of the action (A2) is left unchanged.

The diffeomorphism invariance or reparametrization of the base space $x^a \rightarrow x'^a = x'^a(x)$ requires more attention because of the explicit dependence of the metric of the target space on the coordinates of the base space, $G_{\mu\nu} = G_{\mu\nu}(X,x)$. The volume element $\sqrt{|\gamma|}d^m x$ in (A2) is by definition an invariant. Let us introduce the notation

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X,x) ,$$ \hspace{1cm} (A6)
so that the integrand of (A2) can be written as $\gamma^{ab}(x)h_{ab}(X, x)$. By construction, we know that the contravariant form of the metric of the base space $\gamma^{ab}$ transforms as a $(2, 0)$ rank tensor. Then, the expression $\gamma^{ab}h_{ab}$ will transform as a scalar only if $h_{ab}$ transforms as a $(0, 2)$ tensor. In other words, the invariance of the generalized action (A2) is fulfilled if $h_{ab}(X, x)$ is a $(0, 2)$ tensor which corresponds to the metric induced on the target space $N$ by means of the map $X : M \rightarrow N$. This is equivalent to say that $h = X^*(G)$, where $X^*$ is the pullback associated to the map $X$. Now we will show that in fact $h_{ab}$ transforms as the components of an induced metric. Let us recall that the transformation law

$$\gamma'_{ab} = \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} \gamma_{cd}, \quad (A7)$$

implies for an infinitesimal diffeomorphism

$$x^a \rightarrow x'^a = x^a + \epsilon \xi^a(x), \quad (A8)$$

where $\epsilon$ is an infinitesimal parameter, that

$$\gamma_{ab} \rightarrow \gamma'_{ab} = \gamma_{ab} + \epsilon (\partial_c \gamma_{ab} \xi^c - \gamma_{cb} \partial_c \xi^a - \gamma_{ac} \partial_c \xi^b) = \gamma_{ab} + \epsilon \mathcal{L}_\xi \gamma_{ab}. \quad (A9)$$

Here $\mathcal{L}_\xi$ is the Lie derivative with respect to the vector field $\xi^a$ tangent to the integral curves of the diffeomorphism. Vice versa, if the expression $\gamma_{ab}$ transforms under an infinitesimal diffeomorphism as in (A9), it can be shown that the corresponding finite diffeomorphism satisfies the transformation law (A7). For the components of $h_{ab}$ consider the geometric object

$$h = h_{ab}dx^a dx^b = G_{\mu\nu}(X, x)\partial_\mu X^\mu \partial_\nu X^\nu dx^a dx^b. \quad (A10)$$

It is straightforward to show that under an infinitesimal diffeomorphism of the form (A4), the expressions entering this object transform as

$$X^\mu(x) \rightarrow X^\mu(x') = X^\mu(x) + \epsilon \partial_c X^\mu(x) \xi^c, \quad (A11)$$

$$G_{\mu\nu}(X(x), x) \rightarrow G_{\mu\nu}(X(x'), x') = G_{\mu\nu}(X(x), x) + \epsilon \left( \partial_\lambda G_{\mu\nu} \partial_\lambda X^\lambda \xi^c + \partial_c G_{\mu\nu} \xi^c \right). \quad (A12)$$

Applying now the infinitesimal diffeomorphism to $h$ as given in (A10), and considering only terms up to the first order in $\epsilon$, after some algebraic manipulations we obtain

$$h' = \left\{ G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \epsilon [\partial_c (G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu) \xi^c + G_{\mu\nu} \partial_c X^\mu \partial_\alpha X^\nu \partial_b \xi^c + G_{\mu\nu} \partial_\alpha X^\mu \partial_c X^\nu \partial_b \xi^c] \right\} dx^a dx^b, \quad (A13)$$
where we dropped the arguments for the sake of simplicity. Now it is easy to prove that the latter expression can be written as

\[ h' = (h_{ab} + \epsilon \mathcal{L}_\xi h_{ab}) dx^a dx^b, \quad (A14) \]

showing that under an infinitesimal diffeomorphism the components of \( h_{ab} \) transform as \( h_{ab} \to h'_{ab} = h_{ab} + \mathcal{L}_\xi h_{ab} \). The finite version of this infinitesimal diffeomorphism leads to the standard transformation law of a \((0,2)\) rank tensor. Consequently, \( h \) is a well-defined metric structure, induced by the map \( X \) on the base space \( M \). This proves the invariance of the action \([A2]\) under reparametrizations of the base space. It is worth noting that this invariance is a consequence of the diffeomorphism invariance at the level of the Einstein-Hilbert action and the corresponding field equations. Indeed, for stationary axisymmetric fields in the Weyl-Lewis-Papapetrou representation, diffeomorphism invariance of spacetime reduces to invariance with respect to arbitrary transformations relating the coordinates \( \rho \) and \( z \), and this is exactly the reparametrization invariance of the base space as discussed above.

Finally, an important symmetry exists if the base space is 2-dimensional. In fact, in this case the change of the metric \( \gamma^{ab} \) under an infinitesimal transformation \([A9]\) can be used in order to bring it into the conformally flat form \( \gamma^{ab} = e^{2\phi(x)} \eta^{ab} \), where \( \eta^{ab} (a,b = 1,2) \) is the \((pseudo-)Euclidean metric, and \( \phi(x) \) is a smooth function. This property allows us to introduce the Weyl transformation

\[ \gamma_{ab} \to \gamma'_{ab} = e^{\sigma(x)} \gamma_{ab}, \quad (A15) \]

which preserves the form of the generalized action \([A2]\). In fact, the expression \( \sqrt{|\gamma|} \gamma^{ab} \) is invariant under a Weyl transformation since \( \gamma'^{ab} = e^{-\sigma(x)} \gamma^{ab} \) and \( \sqrt{|\gamma'|} = e^{\sigma(x)} \sqrt{|\gamma|} \). This symmetry is associated with a local rescaling of the metric \( \gamma_{ab} \).

2. Conservation laws

As we mentioned before, the generalized action \([A2]\) includes an interaction between the base space \( N \) and the target space \( M \), reflected on the fact that \( G_{\mu\nu} \) depends explicitly on the coordinates of the base space. Clearly, this interaction must affect the conservation laws of the physical systems we attempt to describe by means of generalized harmonic maps. To
see this explicitly we calculate the covariant derivative of the generalized Lagrangian density
\[
\mathcal{L} = \sqrt{|\gamma|} \gamma^{ab}(x) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X, x),
\]
and replace in the result the corresponding motion equations (A1). Then, the final result can be written as
\[
\nabla_b \tilde{T}^b_a = -\frac{\partial \mathcal{L}}{\partial x^a},
\]
where \(\tilde{T}^b_a\) represents the canonical energy-momentum tensor
\[
\tilde{T}^b_a = \frac{\partial \mathcal{L}}{\partial (\partial_b X^\mu)} (\partial_a X^\mu) - \delta^b_a \mathcal{L} = 2\sqrt{\gamma} G_{\mu\nu} \left( \gamma^{bc} \partial_a X^\mu \partial_c X^\nu - \frac{1}{2} \delta^b_a \gamma^{cd} \partial_c X^\mu \partial_d X^\nu \right).
\]
The standard conservation law is recovered only when the Lagrangian does not depend explicitly on the coordinates of the base space. Even if we choose a flat base space \(\gamma_{ab} = \eta_{ab}\), the explicit dependence of the metric of the target space \(G_{\mu\nu}(X, x)\) on \(x\) generates a term that violates the standard conservation law. This term is due to the interaction between the base space and the target space which, consequently, is one of the main characteristics of the generalized harmonic maps introduced in this work.

An alternative and more general definition of the energy-momentum tensor is by means of the variation of the Lagrangian density with respect to the metric of the base space, i.e.
\[
T_{ab} = \frac{\delta \mathcal{L}}{\delta \gamma_{ab}}.
\]
A straightforward computation shows that for the action under consideration here we have that \(\tilde{T}_{ab} = 2T_{ab}\) so that the generalized conservation law (A17) can be written as
\[
\nabla_b T^b_a + \frac{1}{2} \frac{\partial \mathcal{L}}{\partial x^a} = 0.
\]
For a given metric on the base space, this represents in general a system of \(m\) differential equations for the “fields” \(X^\mu\) which must be satisfied “on-shell”.

If the base space is 2-dimensional, we can use a reparametrization of \(x\) to choose a conformally flat metric, and the invariance of the Lagrangian density under arbitrary Weyl transformations to show that the energy-momentum tensor is traceless, \(T^a_a = 0\).

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