Hermite-Hadamard Type Inequalities for Preinvex Functions via Right Riemann-Liouville Fractional Integrals

Serap Özcan*

Department of Mathematics, Faculty of Science and Arts, Kırklareli University, Kırklareli, Turkey
*Corresponding author: serapozcann@yahoo.com

Received October 09, 2019; Revised November 11, 2019; Accepted November 20, 2019

Abstract In this paper, with a new approach, some new Hermite-Hadamard type inequalities for preinvex functions are obtained by using only the right Riemann-Liouville fractional integrals. Our results generalize previous studies. Results proved in this paper may stimulate further research in this field.

Keywords: Preinvex functions, Hermite-Hadamard inequalities, right Riemann-Liouville fractional integral

Cite This Article: Serap Özcan, “Hermite-Hadamard Type Inequalities for Preinvex Functions via Right Riemann-Liouville Fractional Integrals.” Turkish Journal of Analysis and Number Theory, vol. 7, no. 6 (2019): 140-144. doi: 10.12691/tjant-7-6-1.

1. Introduction

Let $I$ be a finite interval of real numbers. A function $f:I \to \mathbb{R}$ is said to be convex if the inequality

$$f\left((1-t)\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{a}^{b} f(x) dx \leq \frac{f(a)+f(b)}{2}, a,b \in I.$$ 

Hermite-Hadamard inequality for convex functions has attracted many researchers and as gradually a remarkable of generalizations and extensions in various directions have appeared in the literature, one can see [15-24] and references therein.

2. Preliminaries

Let us recall some definitions and known results concerning invexity and preinvexity.

**Definition 1.** [25] A set $K \subseteq \mathbb{R}$ is said to be invex if there exist a function $\eta: K \times K \to \mathbb{R}$ such that

$$a + \eta(b,a) \in K, \forall a,b \in K, t \in [0,1].$$

The invex set $K$ is also called a $\eta$-connected set.

**Definition 2.** [5] Let $f$ be a function on the invex set $K$. Then, $f$ is said to be preinvex with respect to $\eta$, if

$$f\left((1-t)\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{a}^{b} f(x) dx \leq \frac{f(a)+f(b)}{2}, a,b \in I.$$
inequalities for fractional integrals hold:
\[
\int_a^b \left| f(t) \right| dt \leq \frac{1}{\eta(b,a)} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.
\]

Following definitions of the left and right side Riemann-Liouville fractional integrals are well known in the literature.

**Definition 3.** [27] Let \( f \in L[a, b] \). The left and right Riemann-Liouville fractional integrals \( J_{a^+}^\alpha f \) and \( J_{b^-}^\alpha f \) of order \( \alpha > 0 \) are defined by
\[
J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a
\]
and
\[
J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_b^x (t-x)^{\alpha-1} f(t) dt, \quad x < b
\]
respectively, where \( \Gamma(\alpha) \) is the Gamma function defined by \( \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \).

In [22], Sarıkaya et al. gave the fractional analogue of the inequality (1) as follows:

**Theorem 2.** Let \( f : [a, b] \to \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in L[a, b] \). If \( f \) is a convex function on \( [a, b] \), then the following inequalities for fractional integrals hold:
\[
f\left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2(b-a)\Gamma(\alpha)} J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)
\]
\[
\leq \frac{f(a) + f(b)}{2}
\]
with \( \alpha > 0 \).

In [28], Iscan proved the following Hermite-Hadamard inequalities for preinvex functions via Riemann-Liouville fractional integrals:

**Theorem 3.** Let \( K \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta : K \times K \to \mathbb{R} \) and \( a, b \in K \) with \( a < a + \eta(b, a) \). If \( f : [a, a + \eta(b, a)] \to (0, \infty) \) is a preinvex function, \( f \in L[a, a + \eta(b, a)] \) then the following inequalities for fractional integrals hold:
\[
f\left( \frac{2a + \eta(b, a)}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2\eta^\alpha(b, a)} J_{a^+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b,a))^-}^\alpha f(a)
\]
\[
\leq \frac{f(a) + f(a + \eta(b, a))}{2}
\]
with \( \alpha > 0 \).

**Lemma 1.** Let \( K \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta : K \times K \to \mathbb{R} \) and \( a, b \in K \) with \( a < a + \eta(b, a) \). Suppose \( f : K \to \mathbb{R} \) is a differential mapping on \( K \) such that \( f' \in L([a, a + \eta(b, a)]) \). Then the following equality for the right Riemann-Liouville fractional integrals holds:
\[
f(a + \alpha f(a + \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{\eta^\alpha(b, a)} J_{(a+\eta(b,a))^-}^\alpha f(a)
\]
\[
= \frac{\eta(b,a)}{\alpha + 1} \left[ \frac{1}{\alpha + 1} \int_0^{(\alpha+1)a^\alpha - 1} f'(a + \eta(b,a)) dt \right]
\]

Proof. If we apply the partial integration to the right hand side of the above equality, we have
\[
\eta(b,a) \frac{1}{\alpha + 1} \left[ \frac{1}{\alpha + 1} \int_0^{(\alpha+1)a^\alpha - 1} f'(a + \eta(b,a)) dt \right]
\]

This completes the proof.

**Remark 1.** In Lemma 1,
1. If we take \( \eta(b,a) = b - a \), then we get the inequality given in [17, Lemma 3.1].
2. If we take \( \eta(b,a) = b - a \) and \( \alpha = 1 \), then we get the inequality given in [20, Lemma A].

**Theorem 4.** Let \( K \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta : K \times K \to \mathbb{R} \) and \( a, b \in K \) with \( a < a + \eta(b, a) \). Suppose \( f : K \to \mathbb{R} \) is a differential mapping on \( K \) such that \( f' \in L([a, a + \eta(b, a)]) \). If \( f' \) is preinvex on \( [a, a + \eta(b, a)] \), then the following Riemann-Liouville fractional integral inequality holds:
\[
\frac{f(a) + \alpha f(a + \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{\eta^\alpha(b, a)} J_{(a+\eta(b,a))^-}^\alpha f(a)}{\alpha + 1}
\]
\[
\leq \frac{\eta(b,a)}{\alpha + 1} \left[ M_1(\alpha) f'(a) + M_2(\alpha) f'(b) \right]
\]

3. **Main Results**

In this section, we will obtain some generalizations of the right side of the Hermite- Hadamard type inequalities for functions whose first derivatives absolute values are preinvex via right Riemann-Liouville fractional integrals.
Where

\[ M_1(\alpha) = \int_0^1 \left(1 - (\alpha + 1)t^\alpha\right)(1-t)dt = \frac{\alpha\left(2(\alpha + 2)(\alpha + 1)^{-\frac{1}{\alpha}} - 1\right)}{2(\alpha + 2)(\alpha + 1)^{\frac{2}{\alpha}}} \]

\[ M_2(\alpha) = \int_0^1 \left(1 - (\alpha + 1)t^\alpha\right)dt = \frac{\alpha}{2(\alpha + 2)(\alpha + 1)^{\frac{2}{\alpha}}} \]

\[ M_3(\alpha) = \int_0^1 \left(1 - (\alpha + 1)t^\alpha - 1\right)dt = \frac{\alpha\left(2(\alpha + 2)(\alpha + 1)^{-\frac{1}{\alpha}} - 1\right)}{2(\alpha + 2)(\alpha + 1)^{\frac{2}{\alpha}}} \]

\[ M_4(\alpha) = \int_0^1 \left((\alpha + 1)t^\alpha - 1\right)dt = \frac{\alpha\left(2(\alpha + 2)(\alpha + 1)^{-\frac{1}{\alpha}} - 1\right)}{2(\alpha + 2)(\alpha + 1)^{\frac{2}{\alpha}}} \]

with \( \alpha > 0 \).

Proof. Using Lemma 1 and the preinvexity of \( |f'| \), we have

\[ \frac{f(a) + \alpha f(a + \eta(b,a))}{\alpha + 1} \leq \frac{\eta(b,a)}{\alpha + 1} \int_0^1 - (\alpha + 1)t^\alpha \left| f'(a + \eta(b,a)) \right| dt \]

\[ \leq \frac{\eta(b,a)}{\alpha + 1} \int_0^1 (1 - (\alpha + 1)t^\alpha)(1-t)\left| f'(a) + t f'(b) \right| dt \]

\[ + \int_0^1 \left((\alpha + 1)t^\alpha - 1\right)[(1-t)\left| f'(a) + t f'(b) \right|] dt \]

\[ \leq \frac{\eta(b,a)}{\alpha + 1} \left[ \frac{\alpha\left(2(\alpha + 2)(\alpha + 1)^{-\frac{1}{\alpha}} - 1\right)}{2(\alpha + 2)(\alpha + 1)^{\frac{2}{\alpha}}} \right] \left| f'(a) \right| \]

\[ + \frac{\alpha}{2(\alpha + 2)(\alpha + 1)^{\frac{2}{\alpha}}} \left| f'(b) \right| \]

So, the proof is completed.

**Remark 2.** In Theorem 4,

1. If we take \( \eta(b,a) = b - a \), then we get the inequality given in [[17], Theorem 4.1].
2. If we take \( \eta(b,a) = b - a \) and \( \alpha = 1 \), then we get the inequality given in [[29], Theorem 2.2].

**Theorem 5.** Let \( K \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta: K \times K \rightarrow \mathbb{R} \) and \( a, b \in K \) with \( a < a + \eta(b,a) \). Suppose \( f: K \rightarrow \mathbb{R} \) is a differentiable mapping on \( K \) such that \( f' \in L([a,a + \eta(b,a)]) \). If \( |f'|^q \) is preinvex on \([a,a + \eta(b,a)]\) for \( q \geq 1 \), then the following Riemann-Liouville fractional integral inequality holds:

\[ \left| \frac{f(a) + \alpha f(a + \eta(b,a))}{\alpha + 1} \right| \leq \frac{\eta(b,a)}{\alpha + 1} \left( \frac{2\alpha}{(\alpha + 1)^{\frac{2}{\alpha}}} \right)^{\frac{1}{q}} \]

\[ \times \left[ \frac{M_1(\alpha)}{|f'(a)|^q} + \frac{M_2(\alpha)}{|f'(b)|^q} \right] \]

\[ + \frac{M_3(\alpha)}{|f'(a)|^q} + \frac{M_4(\alpha)}{|f'(b)|^q} \]

where \( M_1(\alpha), M_2(\alpha), M_3(\alpha) \) and \( M_4(\alpha) \) are given as Theorem 4 and \( \alpha > 0 \).

Proof. Using Lemma 1, power mean inequality and the preinvexity of \( |f'| \), we have

\[ \left| \frac{f(a) + \alpha f(a + \eta(b,a))}{\alpha + 1} \right| \leq \frac{\eta(b,a)}{\alpha + 1} \int_0^1 - (\alpha + 1)t^\alpha \left| f'(a + \eta(b,a)) \right| dt \]

\[ \leq \frac{\eta(b,a)}{\alpha + 1} \int_0^1 (1 - (\alpha + 1)t^\alpha)(1-t)\left| f'(a) + t f'(b) \right| dt \]

\[ + \int_0^1 \left((\alpha + 1)t^\alpha - 1\right)[(1-t)\left| f'(a) + t f'(b) \right|] dt \]

\[ \leq \frac{\eta(b,a)}{\alpha + 1} \left[ \frac{\alpha\left(2(\alpha + 2)(\alpha + 1)^{-\frac{1}{\alpha}} - 1\right)}{2(\alpha + 2)(\alpha + 1)^{\frac{2}{\alpha}}} \right] \left| f'(a) \right| \]

\[ + \frac{\alpha}{2(\alpha + 2)(\alpha + 1)^{\frac{2}{\alpha}}} \left| f'(b) \right| \]
Riemann-Liouville fractional integral inequality holds:

\[
\frac{1}{\sqrt[q]{\alpha+1}} \left( \frac{\eta(b,a)}{\alpha+1} \right)^{1-\frac{1}{q}} \times \left[ M_1(\alpha) \left\| f'(a) \right\|^q + M_2(\alpha) \left\| f'(b) \right\|^q \right] \\
+ \frac{1}{\sqrt[q]{\alpha+1}} \left( \frac{\eta(b,a)}{\alpha+1} \right)^{1-\frac{1}{q}} \times \left[ M_3(\alpha) \left\| f'(a) \right\|^q + M_4(\alpha) \left\| f'(b) \right\|^q \right]
\]

This completes the proof.

**Remark 3.** In Theorem 5,

1. If we take \( \eta(b,a) = b - a \), then we get the inequality given in [17, Theorem 4.2].

2. If we take \( \eta(b,a) = b - a \) and \( \alpha = 1 \), then we get the inequality given in [20, Theorem 1].

**Theorem 6.** Let \( K \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta: K \times K \rightarrow \mathbb{R} \) and \( a, b \in K \) with \( a < b + \eta(b,a) \). Suppose \( f: K \rightarrow \mathbb{R} \) is a differentiable mapping on \( K \) such that \( f' \in L([a, a + \eta(b,a)]) \). If \( \left\| f' \right\|^q \) is preinvex on \( [a, a + \eta(b,a)] \) for \( q > 1 \), then the following Riemann-Liouville fractional integral inequality holds:

\[
\left( \frac{\eta(b,a)}{\alpha+1} \right)^{1-\frac{1}{q}} \times \left[ M_5(\alpha, p) \left\| f'(a) \right\|^q + M_6(\alpha, p) \left\| f'(b) \right\|^q \right] \left( \frac{1}{2} \right)
\]

Where \( M_5(\alpha, p) = \int_0^1 \left( 1-(\alpha+1)t^\alpha \right)^p dt, \)
\( M_6(\alpha, p) = \int_0^1 ((\alpha+1)t^\alpha - 1)^p dt, \)

with \( p^{-1} + q^{-1} = 1 \) and \( \alpha > 0 \).

This completes the proof.

**Remark 4.** In Theorem 6,

1. If we take \( \eta(b,a) = b - a \), then we get the inequality given in [17, Theorem 4.3].

2. If we take \( \eta(b,a) = b - a \) and \( \alpha = 1 \), then we get the inequality given in [29, Theorem 2.3].

### 4. Conclusion

We have derived new fractional Hermite-Hadamard type integral inequalities via preinvex functions involving only the right Riemann-Liouville fractional integral. We have obtained new generalizations of the right side of the Hermite-Hadamard type inequalities for functions whose first derivatives absolute values are preinvex via right Riemann-Liouville fractional integrals. It has shown that previously known results can be obtained as special cases from our results. It is expected that idea of this article may attract interested readers.

### Acknowledgements

The author is thankful to the Editor and anonymous referees for their constructive comments and valuable suggestions. This research article is supported by Kırklareli University Scientific Research Projects Coordination Unit. Project Number: KLUBAP-191.
References

[1] Pecaric, J. E., Proschan, F. and Tong, Y. L., Convex Functions, Partial Orderings and Statistical Applications, Academic Press, Boston, 1992.

[2] Hanson, M. A., On Sufficiency of the Kuhn-Tucker Conditions, J. Math. Anal. Appl., 1 (1981) 545-550.

[3] Ben-Israel, A. and Mond, B., What is Invexity, J. Aust. Math. Soc., Ser. B, 28(1) (1986) 1-9.

[4] Pini, R., Invexity and Generalized Convexity, Optimization, 22 (1991), 513-523.

[5] Weir, T. and Mond, B., Preinvex Functions in Multiple Objective Optimization, J. Math. Anal. Appl., 136 (1998), 29-38.

[6] Noor, M. A., Variational Like Inequalities, Optimization, 30 (1994) 323-330.

[7] Barani, A., Ghazanfari, A. G. and Dragomir, S. S., Hermite-Hadamard Inequality Through Prequasivex Functions, RGMIA Research Report Collection, 14 (2011) Article 48, 7 pp.

[8] Barani, A., Ghazanfari, A. G. and Dragomir, S. S., Hermite-Hadamard Inequality for Functions Whose Derivatives Absolute Values are Preinvex, RGMIA Research Report Collection, 14 (2011), Article 64, 11 pp.

[9] Iscan, I., Kadakal, M. and Kadakal, H., On Two Times Differentiable Preinvex and Prequasivex Functions, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 68(1) (2019) 950-963.

[10] Kadakal, H., Kadakal, M. and Iscan, I., New Type Integral Inequalities for Three Times Differentiable Preinvex and Prequasivex Functions, Open J. Math. Anal., 2(1) (2018), 33-46.

[11] Latif, M. A. and Shoaib, M., Hermite-Hadamard Type Integral Inequalities for Differentiable m-Preinvex and (a,m)-Preinvex Functions, J. Egyptian Math. Soc., 23 (2015) 236-241.

[12] Noor, M. A., Noor, K. I., Awan, M. U. and Li, J., On Hermite-Hadamard Inequalities for h-Preinvex Functions, Filomat, 28(7) (2014), 1463-1474.

[13] Noor, M. A., Noor, K. I., Awan, M. U. and Qi, F., Integral Inequalities of Hermite-Hadamard Type for Logarithmically h-Preinvex Functions, Cogent Mathematics, 2(1) (2015), Article Number: 1035856, 10 pages.

[14] Ozcan, S., On Refinements of Some Integral Inequalities for Differentiable Prequasivex Functions, Filomat, 33(14) (2019), 4377-4385.

[15] Kadakal, M., Hermite-Hadamard and Simpson Type Inequalities for Multiplicatively Harmonically P-Functions, Sigma: Journal of Engineering & Natural Sciences, 37(4) (2019), 1311-1320.

[16] Kirmaci, U. S., Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Midpoint Formula, Appl. Math. Comp., 147 (2004) 137-146.

[17] Kurt, M., Karapinar, D., Turhan, S. and Iscan, I., The Right Riemann-Liouville Fractional Hermite-Hadamard Type Inequalities for Convex Functions, J. Ineq. Spec. Func., 9(1) (2018), 45-57.

[18] Ozcan, S., Some Integral Inequalities for Harmonically (α,s)-Convex Functions, J. Func. Spaces, Vol. 2019 (2019) Article ID 2394021, 8 pages.

[19] Ozcan, S. and Iscan, I., Some New Hermite-Hadamard Type Inequalities for s-Convex Functions and Their Applications, J. Ineq. Appl., Article number: 2019:201 (2019), 11 pages.

[20] Pearce, C. E. M. and Pecaric, J., Inequalities for Differentiable Mappings with Application to Special Means and Quadrature Formulae, Appl. Math. Lett. 13 (2000), 51-55.

[21] Sarikaya, M. Z. and Budak, H., Generalized Hermite-Hadamard Type Integral Inequalities for Fractional Integrals, Filomat, 30(5) (2016), 1315-1326.

[22] Sarikaya, M. Z., Set, E., Yaldiz, H. and Basak, N., Hermite-Hadamard's Inequalities for Fractional Integrals and Related Fractional Inequalities, Math. Comput. Mod. 57(9) (2013), 2403-2407.

[23] Set, E., Iscan, I., Sarikaya, M. Z. and Ozdemir, M. E., On New Inequalities of Hermite-Hadamard-Fejer Type for Convex Functions via Fractional Integrals, Appl. Math. Comp., 259 (2015), 875-881.

[24] Zhang, Y. and Wang, J. R., On some new Hermite-Hadamard Inequalities Involving Riemann-Liouville Fractional Integrals, J. Ineq. Appl., Article number: 2013:220 (2013), 27 pages.

[25] Yang, X. M. and Li. D., On Properties of Preinvex Functions, J. Math. Anal. Appl., 256 (2001), 229-241.

[26] Noor, M. A., Hermite-Hadamard Integral Inequalities for Log-Preinvex Functions, J. Math. Anal. Approx. Theory, 2 (2007) 126-131.

[27] Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J., Theory and Applications of Fractional Differential Equations , Elsevier, Amsterdam, 2006.

[28] Iscan, I., Hermite-Hadamard's Inequalities for Preinvex Functions via Fractional Integrals and Related Fractional Inequalities, American Journal of Mathematical Analysis, 1(3) (2013), 33-38.

[29] Dragomir, S. S. and Agarwal, R. P., Two Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Trapezoidal Formula, Appl. Math. Lett. 11(5) (1998), 91-95.