There are no $C^5$-Regular Pure $y$-Global Landsberg Surfaces

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Abstract

We show that there are not pure $C^5$ regular $y$-global Landsberg surfaced. The proof is based on the averaged connection associated with the linear Chern’s connection and the classification of irreducibles holonomies of torsion-free affine connections. The structure consists on exausting all the possible cases and showing that in dimension 2 Landsberg condition implies Berwald condition.\[2\]

1 Introduction

The existence or not of pure $y$-global Landsberg spaces with enough differentiability conditions on the metric $F$ is an important problem in Finsler Geometry. It has devoted considerable attention and efforts by many Finsler and non-Finsler geometers. Despite the numerous attempts, the solution of the general problem is still elusive.

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However as result of some of these investigations, examples has been found of “relaxed” Landsberg spaces which are not “relaxed” Berwald spaces: if the differentiability conditions are weaker, there are in dimension higher of 2 generalize Landsberg spaces which are not Berwald. These examples are $y$-local (see references on [7] as well as the original paper by Asanov [8]) but not $y$-global.

In this work we are concerned with the original problem, which is the existence of $y$-global pure Landsberg structures. Because the original problem seems quite difficult, we restrict our attention to low dimension and in particular to the two dimensional case. In particular we present the following result:

**Theorem 1.1** Let $M$ be a connected manifold. A $C^5$-regular $y$-global Landsberg surface $(M, F)$ is a Berwald surface.

Because of a theorem from Szabó ([9]), one gets immediately obtains,

**Corollary 1.2** A $C^5$-regular Landsberg surface $(M, F)$ is Riemannian or locally Minkowski.

**Remark.** The requirement of being a $C^5$-regular seems necessary for our proof, because we use the classification of holonomies, which makes use explicitly of the Bianchi identities. However, second Bianchi identities requires $C^5$-regularity of $F$.

Our proof of theorem 1.1 is based on the theory presented in [3], some constraining results on Berwald spaces obtained previously in ref. [2] and on the classification of the holonomy representations of irreducible, torsion-free affine connections ([4]). The main construction used in the proof is the averaged connection ([3]), which is simpler than any ”Finslerian” connection. In particular, the holonomy of the averaged connection is easier to understand than the holonomy of the initial connection, because the averaged connection is affine and torsion-free and is living on $M$. Basically, the holonomy representations of these connections are already classified in ref. [4], for irreducible connections. Also, the holonomy representation of the averaged connection of the Chern’s connection is a Lie sub-group of the holonomy representation of the original Chern’s connection. Therefore, any geometrical property must be invariant under the action of the holonomy group of the averaged connection ([11, paragraph 10.19]). This is the key point of the proof. In particular, Landsberg condition which can be stated as saying the Cartan tensor is covariant constant along the direction of $\ell$ (the distinguish vector field dual to the Hilbert 1-form $w$), must be invariant under the
action of the corresponding holonomy representation of the averaged connection. But this connection is affine, torsion-free. Therefore, the holonomy group of the averaged connection is constrained by the list of Merkulov and Schwachhöfer ([4]). Then, we analyze these possible groups in dimension 2 and conclude that it is not possible to have \( y \)-global Landsberg surfaces which are not Berwald surfaces.

We remark that in higher dimensions our method is far from being conclusive in higher dimensions.

2 Finsler Geometry in dimension 2

In this work all indices run from 1 to 2, being 2 the dimension of the manifold \( M \). However, most of the notions presented in this and next section are applicable to arbitrary dimensions. If it is not stated the contrary, equal indices up and down are summed. The manifold \( M \) is connected.

**Definition 2.1** A Finsler structure \( F \) on the manifold \( M \) is a non-negative, real function \( F : TM \to [0, \infty) \) such that

1. It is smooth in the slit tangent bundle \( N := TM \setminus \{0\} \).
2. Positive homogeneity holds: \( F(x, \lambda y) = \lambda F(x, y) \) for every \( \lambda > 0 \).
3. Strong convexity holds: the Hessian matrix

\[
g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \tag{2.1}
\]

is positive definite on \( N \).

A structure such that both, smoothness of \( F \) and strong convexity hold in the whole \( N \) is called \( y \)-global. Otherwise, if some of the defining properties hold only in a proper open sub-set of \( N \), the structure is called \( y \)-local.

The manifold \( \pi^*TM \) is a subset of the cartesian product \( TM \times N \). The pull-back bundle \( \pi^*TM \to N \) is defined in such a way that the following diagram commutes:

\[
\begin{array}{ccc}
\pi^*TM & \xrightarrow{\pi_2} & TM \\
\pi_1 \downarrow & & \downarrow \pi \\
N & \xrightarrow{\pi} & M.
\end{array}
\]
Using a trivialization of the bundle $\pi^*TM$, the projection on the first and second factors are

\[ \pi_1 : \pi^*TM \to N, \quad (u, \xi) \mapsto u, \ u \in \pi^{-1}(x) \subset N, \]

\[ \pi_2 : \pi^*TM \to TM, \quad (u, \xi) \mapsto \xi, \ \xi \in \pi_1^{-1}(u). \]

**Definition 2.2** Let $(M, F)$ be a Finsler structure. The fundamental and the Cartan tensors are defined by the equations

1. **Fundamental tensor:**

\[
g(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \pi^* dx^i \otimes \pi^* dx^j. \tag{2.2}
\]

2. **Cartan tensor:**

\[
A(x, y) := \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k} \pi^* dx^i \otimes \pi^* dx^j \otimes \pi^* dx^k = A_{ijk} \pi^* dx^i \otimes \pi^* dx^j \otimes \pi^* dx^k. \tag{2.3}
\]

The “tensor” (2.2) defines on each fiber of $\pi^*TM$ a fiber metric. However, using the coefficients $g_{ij}$ one can define other geometric objects which are also relevant in Finsler Geometry. This is because $g_{ij}$ is tensorial under the local coordinate transformations on $\pi^*TM$ induced from local coordinate transformations on the base manifold $M$. The same applies to the Cartan tensor, which components are also tensorial under induced change of coordinates.

The non-linear connection has the following coefficients:

\[
N^j_k = \gamma^j_{km} y^m F - A^j_{km} \gamma^r_{s} y^r F, \quad j, k, m, s = 1, 2.
\]

The coefficients $\gamma^j_{km}$ are defined in local coordinates by

\[
\gamma^j_{km} = \frac{1}{2} g^{js} \left( \frac{\partial g_{sk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^s} + \frac{\partial g_{sm}}{\partial x^k} \right), \quad j, k, m, s = 1, 2;
\]

\[ A^j_{km} = g^{jl} A_{lk}m \] and $g^{jl} g_{lk} = \delta^j_k$. Using the non-linear connection coefficients one can define a splitting of $TN$, defining a non-linear connection, where the vertical and horizontal subspaces $V_u$ and $H_u$ are such that:
1. Splitting of the tangent space $T\mathbf{N}$ holds,

$$T_u\mathbf{N} = \mathcal{V}_u \oplus \mathcal{H}_u, \forall \ u \in \mathbf{N}$$

2. The right action on $T\mathbf{N}$ induced from the action of the group $GL(2, \mathbb{R})$ on the tangent bundle $T\mathbf{M}$ leaves invariant the above decomposition.

Let us consider the local coordinate system $(x, y, \mathbf{U})$ of the manifold $T\mathbf{M}$. A tangent basis for $T_u\mathbf{N}, \ u \in \mathbf{N}$ is defined by the vectors([1]):

$$\left\{ \left. \frac{\delta}{\delta x^1} \right|_u, ..., \left. \frac{\delta}{\delta x^n} \right|_u, \left. F \frac{\partial}{\partial y^1} \right|_u, ..., \left. F \frac{\partial}{\partial y^n} \right|_u \right\}, \quad \left. \frac{\delta}{\delta x^j} \right|_u = \left. \frac{\partial}{\partial x^j} \right|_u - N_j^i \left. \frac{\partial}{\partial y^i} \right|_u.$$  

The local basis of the dual vector space $T^*_u\mathbf{N}, \ u \in \mathbf{N}$ is

$$\left\{ \left. dx^1 \right|_u, ..., \left. dx^n \right|_u, \left. \frac{\delta y^1}{F} \right|_u, ..., \left. \frac{\delta y^n}{F} \right|_u \right\}, \quad \left. \frac{\delta y^i}{F} \right|_u = \frac{1}{F} \left( \left. dy^i + N_j^i dx^j \right|_u \right).$$

The Chern’s connection is characterized by the following theorem ([1]),

**Theorem 2.3** Let $(\mathbf{M}, F)$ be a Finsler structure. The pull-back vector bundle $\pi^*T\mathbf{M} \rightarrow \mathbf{N}$ admits a unique linear connection determined by the connection 1-forms $\left\{ \omega^i_j, i, j = 1, 2 \right\}$ such that the following structure equations hold:

1. "Torsion free" condition,

$$d(dx^i) - dx^j \wedge w^i_j = 0, \quad i, j = 1, 2. \quad (2.4)$$

2. Almost $g$-compatibility condition,

$$dg_{ij} - g_{kj}w^k_i - g_{ik}w^k_j = 2A_{ijk} \frac{\delta y^k}{F}, \quad i, j, k = 1, 2. \quad (2.5)$$

In local coordinates the curvature endomorphisms of a linear connection on $\pi^*T\mathbf{M} \rightarrow \mathbf{N}$ are decomposed in the following way:

$$\Omega^i_j = \frac{1}{2} R^i_{jkl} dx^k \wedge dx^l + P^i_{jkl} dx^k \wedge \frac{\delta y^l}{F} + \frac{1}{2} Q^i_{jkl} \frac{\delta y^j}{F} \wedge \frac{\delta y^l}{F}. \quad (2.6)$$

For the Chern’s connection, the $vv$-curvatures are zero for any arbitrary Finsler structure, $Q^i_{jkl} = 0$.  

Definition 2.4 A Berwald space is a Finsler space if \( \forall x \in M \) there is a local coordinate system containing \( x \) such that in the associated natural coordinates on \( N \) the connection coefficients of the Chern's connection live on the manifold \( M \).

There are several characterizations of Berwald spaces (for instance see chapter 11 of ref. [1]). Between these characterizations, we find quite useful that the Finsler structure \((M, F)\) is a Berwald structure iff the \( hv \)-curvature is zero, \( P_{ijkl}^i = 0, \ i, j, k, l = 1, 2 \).

Definition 2.5 A Finsler structure is a locally Minkowski space if \( \forall x \in M \) there is a local coordinate system containing \( x \) such that the connection coefficients of the Chern's connection does not depend on \( x \).

Definition 2.6 A Finsler structure \((M, F)\) is Landsberg iff the following condition holds:

\[
\dot{A}_{ijk}(x, y) := -\ell^j P_{jkl}(x, y) = 0, \ \ell^j = \frac{y^j}{F(x, y)}.
\]

Examples of Landsberg spaces are Berwald spaces. We call a Landsberg space which is not Berwald.

3 Averaged Connection and a Rigidity condition for Berwald Spaces

The averaged connection was introduced in ref. [3]. In this section we collect without proof the results from [2] and [3] which are necessary in our proof of the theorem 1.1. Let us denote by \( \pi^*_v \Gamma M \) the fiber over \( v \in N \), \( T_x M \) the tensor space over \( x \) and \( T_x^{(p, q)} M \) the fiber over \( x \) of a tensor bundle of order \( (p, q) \); \( F(M) \) is the ring of smooth functions on \( M \). For each \( S_z \in T_z M \) and \( v \in \pi^{-1}(z), z \in U \subset M \) we consider the isomorphisms

\[
\begin{align*}
\pi_{2|v} : \pi^*_v \Gamma M &\rightarrow T_z M, \quad S_v \rightarrow S_z \\
\pi^*_v : T_z M &\rightarrow \pi^*_v \Gamma M, \quad S_z \rightarrow \pi^*_v S_z.
\end{align*}
\]

Definition 3.1 Let \((M, F)\) be a Finsler structure, \( \pi(u) = x \) and \( f \in F(M) \). Then \( \pi^* f \in F(\pi^* \Gamma M) \) is defined by

\[
\pi^*_u f = f(x), \ \forall u \in I_x \subset \pi^{-1}(x) \subset N.
\]
The horizontal lift of vector fields is denoted by
\[ \iota : T\mathcal{M} \rightarrow T\mathcal{N}, \quad X = X^i \frac{\partial}{\partial x^i}|_x \mapsto \iota(X) = X^i \frac{\delta}{\delta x^i}|_u, \quad u \in I_x \subset \pi^{-1}(x) \subset \mathcal{N}, \]
and the horizontal lift is defined by the non-linear connection coefficients \( N^i_j \),
\[ \iota : T\mathcal{M} \rightarrow T\mathcal{N}, \quad \iota(X) = \tilde{X}, \quad |\tilde{X} \in H, \]
such that if \( \pi : T\mathcal{N} \rightarrow \mathcal{N} \) is the canonical projection, \( \pi(\iota(X)) = X \) for \( X \neq 0 \).

**Definition 3.2** Consider the family of operators
\[ A := \{ A_w : \pi^*_w T^{(p,q)}M \rightarrow \pi^*_w T^{(p,q)}M \} \]
with \( w \in \pi^{-1}(x) \). The average of this family of operators is defined to be the operator
\[ A_x : T^{(p,q)}_x M \rightarrow T^{(p,q)}_x M \]
with \( x \in \mathcal{M} \) given by the action:
\[ < A_u > := \left< \pi^2_{\mid u} A_u \pi^*_u \right>_u S_x = \frac{1}{\text{vol}(I_x)} \left( \int_{I_x} \pi^2_{\mid u} A_u \pi^*_u \text{dvol} \right) S_x, \]
\[ u \in \pi^{-1}(x), \; S_x \in T^{(p,q)}_x M; \]
\[ \text{dvol} \text{ is the standard volume form induced on the indicatrix } I_x \text{ from the Riemannian volume of the Riemannian structure } (T_x M \setminus \{0\}, g_x), \]
where the fiber metric is \( g_x := g_{ij}(x, y) dy^i \otimes dy^j \), with fixed \( x \in \mathcal{M} \) and \( y \in T_x M \setminus \{0\} \).

**Remark 1.** In the formula (3.3) the integration must be understood as an integration on the manifold \( I_x \) of an operator.

**Remark 2.** \( I_x \) can be understood as a sub-manifold on \( \mathcal{N} \) or, due to a canonical embedding of \( \mathcal{N} \) on \( \mathcal{M} \), as a sub-manifold on the tangent bundle. The second interpretation is geometrical and will be given elsewhere. The first interpretation was adopted in my previous work (for instance in [3]). However, there is not dramatic difference between both interpretations.

**Theorem 3.3** (Averaged connection [2]) Let \( (\mathcal{M}, F) \) be a Finsler structure and \( \nabla \) a linear connection on \( \pi^*\mathcal{T}\mathcal{M} \). Then, there is defined on \( \mathcal{M} \) a linear covariant derivative along \( X \), \( \tilde{\nabla}_X \) characterized by the following conditions:
1. \( \forall X \in T_xM \) and \( Y \in \Gamma TM \), the covariant derivative of \( Y \) in the direction \( X \) is given by the following average operation:
\[
\hat{\nabla}_X Y := \langle \nabla > X Y := \langle \pi_2 |_u \nabla_{\tau_u(X)} \pi_u^* Y >_u, \ u \in I_x \subset \pi^{-1}(x) \subset N, \tag{3.4}
\]

2. For every smooth function \( f \in \mathcal{F}(M) \) the covariant derivative is given by the following average:
\[
\hat{\nabla}_X f := \langle \nabla > X f := \langle \pi_2 |_u \nabla_{\tau_u(X)} \pi_u^* f >_u, \ u \in I_x \subset \pi^{-1}(x) \subset N. \tag{3.5}
\]

**Remark.** Equation (3.4) implies \( \hat{\nabla}_X f = X(f) \).

The following properties are important for our application. The proof can be find in ref. [3, section 4], Proposition 3.4

The following properties hold:

1. **Torsion of the averaged connection.** The torsion tensor \( T_{\hat{\nabla}} \) of the averaged connection \( \hat{\nabla} \) is the averaged of the torsion tensor \( T_{\nabla} \) of the original linear connection \( \nabla \),
\[
T_{\hat{\nabla}}(X, Y) = \pi_2 |_u (\nabla_{\tau_u(X)} \pi_u^* Y - \nabla_{\tau_u(Y)} \pi_u^* X - \pi_u^* [X, Y] >_u = \langle \pi_2 T_{\nabla}(\pi_u^* X, \pi_u^* Y) >. \tag{3.6}
\]

2. **Parallel transport of the averaged connection.** The parallel transport associated with \( \hat{\nabla} \) is given by the average of the parallel transport operation of the original connection \( \nabla \),
\[
(\tau^t_{t+\delta})_{x_t} S := \langle \pi_2 |_u \tau^t_{t+\delta} \pi^* u_{(t+\delta)} S(x_t+\delta) >_{u(t)}, \ S_2 \in \Gamma x_tM. \tag{3.7}
\]

3. **Curvatures of the averaged connection.** The curvature endomorphisms \( \hat{\Omega}(X, Y) \) of the averaged connection \( \hat{\nabla} \) is obtained as the averaged of the curvature endomorphisms \( \Omega(X, Y) \) of the original linear connection \( \nabla \),
\[
\hat{\Omega}(X, Y)Z := \pi_2 |_u \Omega(X^{v,h}, Y^{v,h}) |_u \pi_u^* Z >_u, \tag{3.8}
\]

where \( X^{v,h} \) and \( Y^{v,h} \) denotes the corresponding vertical and horizontal lifting of \( X, Y \in \Gamma TM \), using the non-linear connection on \( TN \).
4. Holonomy of the averaged connection. Since the holonomy is essentially parallel transports along loops, one has the following homomorphism,

\[ < \cdot >_H : Hol(\nabla) \longrightarrow Hol(\bar{\nabla}) \]

\[ \tau_\gamma u \pi^* S \longrightarrow < \pi_2 |_{u} \tau_\gamma u \pi^* |_{u} S >, \]

\[ \tau_\gamma u \in Hol(\nabla), \ S \in \Gamma TM \]

where \( \gamma_u \) is the horizontal lift of the path \( \gamma \) on the bundle \( TN \).

Remark. From the above properties one can argue that the most powerful averaged is on the connection, instead of the metric averaging. They are also an argument suggesting the notion of convex invariance ([3, section 8]).

Let us consider the injection of \( I_x M \) on \( T_x M \). Then, we can pull-back this manifold to \( \pi^* TM \). In ref. [2] it was proved in a straightforward way the following

**Theorem 3.5** ([2], [10]) Let \( (M, F) \) be a Finsler structure. Then there is a Riemannian metric \( h \) such that its Levi-Civita connection leaves invariant the indicatrix \( I_x \) of \( F \) iff the structure is Berwald.

Remark. The metric \( h \) appearing in theorem 3.5 is not necessarily the averaged metric \( h(x) = < g_{ij} > dx^i \otimes dx^j \).

4 Proof of the Theorem 1.1

Let us fix the Finslerian connection on \( \pi^* TM \) to be the Chern’s connection. In this case, the \( \psi \)-curvature is zero, \( Q = 0 \). The average procedure induces a group homomorphism between the holonomy groups of the Chern connection,

\[ < \cdot >_H : Hol(\nabla^{ch}) \longrightarrow Hol(< \nabla^{ch} >). \] (4.1)

This homomorphism is surjective. To proof this fact, note that because theorem 3.4.4 and because the Ambrose-Singer theorem on holonomies([5]), the Lie algebra of the holonomy representation of the averaged connection associated with the linear Chern’s connection is generated by the curvature endomorphism \( \{ < R_i^j >, < P_i^j > \} \) and we know from theorem 3.4.3 that the curvature endomorphisms of the averaged connection are the average of the curvature endomorphisms of the original linear connection. Therefore the homomorphism (4.1) is surjective. This fact implies that the induced homomorphism of Lie algebras is also surjective. Therefore, let us consider the action of the pull-back group of the averaged holonomy group. Therefore, look for two-dimensional holonomy representation which are not:
1. Reducible: if the holonomy representation is reducible, must be a product of one dimensional representations. Since the holonomy representation of a 1-dimensional manifold is trivial, Therefore the product is trivial as well and the surface must be locally Minkowski.

2. Locally symmetric affine connections: in this case, one has to be aware that they are not locally symmetric spaces of dimension two.

3. Riemann metrizable connections: it is known ([6], [10], [2], [3]) that in this case \((M, F)\) is a Berwald space.

The holonomy group \(Hol(<\nabla^{ch}>)\) is the holonomy group of a torsion free affine connection ([3]). Therefore, from the list of groups that can be holonomy groups of irreducible, non locally symmetric torsion-free connection which are non-metric, the possibilities are restricted by the classification of Merkulov-Schwachhöfer ([4]). There is one possibility in dimension 4:

1. \(T_{\mathbb{R}} \cdot SL(2, \mathbb{R})\).

\(T_{\mathbb{R}}\) is any connected Lie sub-group of \(\mathbb{R}\), which is trivial.

Until know we didn’t use the fact that \((M, F)\) is a Landsberg space. For \(C^4\) regular structures, the notion of Landsberg spaces is given by a tensor condition, write in tensor coordinates as:

\[
\dot{A}_{ijk} = 0.
\]

(4.2)

**Key part of the proof:** This tensor condition must be invariant by the holonomy of the averaged connection because:

1. Assume that eq. (4.2) holds at one given point \(x \in M\). We want that this condition holds everywhere in \(M\). Therefore one has to extend this condition by parallel transport everywhere on \(M\) (since \(M\) is connected and by the Holonomy Fundamental Principle ([11, paragraph 10.19]). Therefore, the condition must be invariant under the holonomy of the linear Chern’s connection (which is the connection that we are using).

2. The homomorphism \(< \cdot >\) is surjective. Therefore, we have the following relation

\[
Hol(\nabla^{ch}) \supset Hol(\pi^* < \nabla^{ch}>) \simeq Hol(< \nabla^{ch}>).
\]

(4.3)

From here one has the requirement that

\[
\pi^* < \nabla^{ch} > \dot{A} = \pi^* < \nabla^{ch} > 0 = 0.
\]
This is also because (both for Berwald and Chern connections) $\nabla^{ch}$ connection has an associated zero covariant derivative in vertical directions.

Therefore, we have to check the invariance of the Landsberg condition for the groups $GL(2, \mathbb{R})$ and $SL(2, \mathbb{R})$.

**Proposition 4.1** In a convenient basis, the Landsberg condition is

$$P^2_{ijk} = \dot{A}_{ijk} = 0. \quad (4.4)$$

**Proof:** This relation is obtained from a known relation of the $P$ curvature of the Chern’s connection ([1, pg 57]),

$$\ell^l P_{l ijk} = -\dot{A}_{ijk}. \quad (4.5)$$

We can write this condition as follows,

$$\frac{1}{F} \ell^l P_{l ijk} = \frac{1}{F} \ell^l P^l_{ijk} = -\dot{A}_{ijk}$$

Let us consider a basis of the dual of the fiber space with respect to the fundamental tensor $g$

$$\{e_1, e_2 = \frac{y_j}{F} \pi^* e^j \} \leftrightarrow \{\tilde{e}_1, \tilde{e}_2 = \frac{y_j}{F} \pi^* e^j \}, \quad y_j = g_{jk} y^k.$$

Then equation (4.5) becomes (4.4).

The relation $\dot{A}_{ijk} = 0$ is a tensorial relation which in addition must be invariant under the holonomy group of the pull-back of the averaged connection, as we have discussed. Therefore, let us consider $\dot{A}_{ijk} = \bar{P}^2_{ijk} = 0$ write in the following way:

$$0 = \bar{P}^2_{ijk} = M^2_{ijk} M^m_{ip} M^p_m M^q_j M^q_k,$$

where $M$ is an arbitrary element of $Hol(\pi^* < \nabla >)$. The matrix $M$ is invertible and therefore it is equivalent to the following expression,

$$0 = M^2_{ijk} P^l_{ijk}.$$

The elements $M^2_{ijk}$ can be chosen zero independently, because the matrix $M$ is on the averaged holonomy group. Let us consider the action of an arbitrary holonomy element $M^a_b$,

$$0 = M^2_{ijk} P^l_{ijk} = M^2_{1} P^1_{ijk} + M^2_{2} P^2_{ijk} \Rightarrow M^2_{1} P^1_{ijk} = M^2_{2} P^2_{ijk} = 0.$$
Since the group $SL(2, \mathbb{R})$ contain elements such tat $M_1^2$ are independently zero, one gets that $P_{ijk}^1 = P_{ijk}^2 = 0$.

Note that the groups $O(2)$ and $SO(2)$ are not eligible, because they correspond to Riemannian holonomies.

The condition of $C^5$ regular structure is required because of the use of the classification of irreducible representations of affine connections, where it is used. If the Landsberg structure $(M, F)$ is of less degree of smoothness, we cannot use Ambrose-Singer theorem on algebra of holonomies.

5 Discussion

It has been proved that pure regular $y$-global Landsberg surfaces doesn’t exist. However, the situation in higher dimensions seems more involved. Indeed, there are difficulties to generalize the result to higher dimensions:

1. The proliferation of holonomy groups of torsion-free affine connections.
2. The absence of the De Rham decomposition theory for affine connections, in contraposition to the (pseudo)-Riemannian category,
3. The existence of non-trivial local symmetric spaces, where the isotropy group corresponds to the holonomy group.

Nevertheless, we hope that the averaged method can be useful in the point 2 before. To treat points 1 and 3, a combination of other methods with exterior systems seems a possible avenue to try.

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