The asymptotic properties of Eulerian numbers and refined Eulerian numbers: A Spline perspective

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Abstract

In this paper, the asymptotic formulas for Eulerian numbers, refined Eulerian numbers and the coefficients of descent polynomials are obtained directly from the spline interpretations of these numbers. Having related these numbers directly to B-splines [15], we can take advantage of many powerful spline techniques to derive various results of these numbers. The asymptotic formulas for the Eulerian numbers $A_{d,k}$ agree with the previously known results which were given by L. Carlitz et al. (1972) [2] and S. Tanny (1973) [18], but the convergence order is much better. We also give the asymptotic representations of refined Eulerian numbers which is in terms of the Hermite polynomials.

Keywords: B-splines; Eulerian numbers; Refined Eulerian numbers; Descent polynomials; Asymptotic approximation.

1 Introduction

Eulerian number, denoted here by $A_{d,k}$ is the number of permutations in the symmetric group $S_d$ that have exactly $k - 1$ descents. They play important roles in enumerative combinatorics, geometry [12], statistical applications [2, 18], and spline theory [10, 15]. The asymptotic properties of Eulerian numbers was examined by S. Tanny [18] and L. Carlitz et. al. [2] with the help of central Limit...
they showed that the Eulerian numbers approximate to the Gaussian function with the convergence order \( O(d^{-\frac{3}{4}}) \). Nevertheless, in this paper, we obtain a much better convergence order \( O(d^{-\frac{3}{2}}) \) with the help of the spline theory.

The refined Eulerian number \( A_{d,k,j} \) is the number of permutations in the symmetric group \( S_d \) with \( k \) descents and ending with the element \( j \). Brenti and Welker [9] gave a precise description of the h-vector of the barycentric subdivision of a simplicial complex in terms of the h-vector of the original complex and the refined Eulerian numbers. Later, Martina Kubitzke and Eran Nevo [13] derived new inequalities on \( A_{d,k,j} \) by the algebraic method. A result due to Ehrenborg, Readdy and Steingrímsson [17] gave a geometry interpolation of \( A_{d,k,j} \) as the mixed volumes of two adjacent slices from the unit cube. In [15], Renhong, W., Yan, X. and Zhiqiang, X. gave a spline interpolation of the \( A_{d,k,j} \) by the geometry interpolation and the spline theory. Furthermore, they gave the explicit and recurrence representation of \( A_{d,k,j} \). In [11], the experimental evidence suggested that the refined Eulerian number \( A_{d,k,j} \) is not exactly normal approximation as \( d \) grows large. In this paper, we give the asymptotic representations of refined Eulerian numbers in terms of the Hermite polynomials.

Recently, Steingrímsson [7] generalized the definition of descents and excedances to the elements (called indexed permutations, colored permutations or \( r \)-signed permutations see [17, 7, 5, 4]) of groups \( S^n_d := Z_n \wr S_d \), where \( \wr \) is wreath product with respect to the usual action of \( S_d \) by permutations of \( [d] \). The number of indexed permutations in \( S^n_d \) with \( k \) descents is denoted by \( D(d,n,k) \). In [7], using the work of Brenti [8], Steingrímsson [7] showed that the numbers \( D(d,n,k) \) are unimodal. Moreover, in [13], using the spline theory, the authors gave the explicit expression of \( D(d,n,k) \) and showed the numbers \( D(d,n,k) \) are log-concave. In this paper, we show that \( D(d,n,k) \) is approximately normal in terms of the spline interpretation of \( D(d,n,k) \).
The paper is organized as follows. In Section 2, after recalling some necessary definitions and notations, we show the connection between the B-splines and these combinatorial numbers. In Section 3, we give the main results of this paper. The results show the development of various asymptotic properties of Eulerian numbers, refined Eulerian numbers and the coefficients of descent polynomials, which are obtained directly from the spline interpretations of these numbers. We prove the main results in Section 4.

2 definitions and notations

Eulerian number, denoted here by \( A_{d,k} \) is the number of permutations in the symmetric group \( S_d \) that have exactly \( k-1 \) descents. An recurrence formula for \( A_{d,k} \) is

\[
A_{d,k+1} = (k+1)A_{d-1,k+1} + (d-k)A_{d-1,k}
\]  

(2.1)

with the boundary conditions

\[
A_{0,0} = 1, A_{d,0} = 0, d > 0.
\]

An explicit formula for \( A_{d,k} \) is

\[
A_{d,k} = \sum_{i=0}^{k} \binom{d+1}{i}(-1)^{i}(k-i)^d
\]  

(2.2)

which can be easily verified using the above recurrence.

Refined Eulerian number \( A_{d,k,j} \) is the number of permutations in the symmetric group \( S_d \) with \( k \) descents and ending with the element \( j \). The recurrence formulas for \( A_{d,k,j} \) are \[15\]

\[
A_{d+1,k,d-j+1} = (k+1)A_{d,k,d-j} + (d-k)A_{d,k-1,d-j},
\]

\[
A_{d+1,k,d-j+1} = kA_{d,k,d-j+1} + (d-k+1)A_{d,k-1,d-j+1}.
\]

(2.3)

An explicit formula for \( A_{d,k,j} \) is \[15\]

\[
A_{d,k,j} = \sum_{i=0}^{k} \binom{d}{i}(-1)^{i}(k-i)^{d-j}(k-i+1)^{j-1}.
\]
Descent polynomials, denoted by $D_n^d(t)$, are defined as

$$D_n^d(t) = \sum_{k=0}^{d} D(d, n, k)t^k,$$

where $D(d, n, k)$ is the number of indexed permutations in $S^n_d$ with $k$ descents. The indexed permutation of length $d$ and with indices in $\{0, 1, \ldots, n-1\}$ is an ordinary permutation in the symmetric group $S_d$ where each letter has been assigned an integer between 0 and $n-1$. Indexed permutations, or $r$-signed permutations, are a generalization of permutations (see [17, 7]). We will follow the notation in [7]. The set of all such indexed permutations is denoted by $S^n_d$. The numbers $D(d, n, k)$ satisfy a simple three-term recurrence [7],

$$D(d, n, k) = (nk + 1)D(d-1, n, k) + (n(d-k) + (n-1))D(d-1, n, k-1). \quad (2.4)$$

And also has an explicit formula [15]:

$$D(d, n, k) = \sum_{i=0}^{k} \binom{d+1}{i} (-1)^i (n(k-i) + 1)^d.$$

We turn to the definitions of B-splines and Hermite polynomials, then show the relations among Eulerian numbers, the refined Eulerian numbers, the descent polynomials and B-splines.

Let $p \in \mathbb{R}^{\geq 1}$ and $L^p(\mathbb{R})$ as usual denote the set

$$L^p(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} | f \text{ measurable, } \int_{-\infty}^{+\infty} |f(t)|^p \, dt < \infty \}.$$ 

For $p = 2$ and $f, g \in L^2(\mathbb{R})$ define the inner product

$$\langle f, g \rangle := \int_{-\infty}^{+\infty} f(t)\overline{g(t)} \, dt$$

and the norm

$$\|f\| := \sqrt{\langle f, f \rangle}$$

making $L^2(\mathbb{R})$ to a Hilbert space.
For $f \in L^1(\mathbb{R})$ define the Fourier transform $\hat{f}$ and the inverse Fourier transform $\check{f}$ as

$$\hat{f}(\omega) := \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt,$$

$$\check{f}(\omega) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt.$$

This definition can be extended to functions $f \in L^2(\mathbb{R})$, see for example [6].

**B-splines with order $d$,** which is denoted as $B_d(\cdot)$, is defined by the induction as

$$B_1(x) = \begin{cases} 1 & \text{if } x \in [0,1), \\ 0 & \text{otherwise,} \end{cases}$$

and for $d \geq 1$

$$B_d = B_1 \ast B_{d-1},$$

where $\ast$ denotes the operation of convolution which is defined:

$$(f \ast g)(t) := \int_{-\infty}^{+\infty} f(t-y)g(y)dy$$

for $f$ and $g$ in $L^2(\mathbb{R})$.

Then $B_d$ has the compact support $[0,d]$ and is in $C^{d-1}(\mathbb{R})$. A well known explicit formula for $B_d(\cdot)$ is

$$B_d(x) = \frac{1}{(d-1)!} \sum_{i=0}^{d} (-1)^i \binom{d}{i} (x-i)^{d-1}$$

(2.5)

where the one-sided power function is defined by:

$$x^d_+ = \begin{cases} x^d & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

And the recurrence relation is

$$B_d(x) = \frac{x}{d-1} B_{d-1}(x) + \frac{d-x}{d-1} B_{d-1}(x-1).$$

(2.6)

For all $B_d$, $d \geq 1$, it holds $B_d \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$B_d^\wedge(\omega) = sinc^d\left(\frac{\omega}{2}\right)$$

(2.7)
where $sinc(t)$ denotes:

$$sinc(t) := \begin{cases} \frac{\sin t}{t} & t \neq 0, \\ 1 & t = 0. \end{cases}$$

(2.8)

B-splines play important roles in approximation, computer-aid design (CAD), signal processing and discrete geometry. They have been well developed during the past few decades. For extensive monographs see [3, 19].

Hermite polynomials $H_n(x)$ with degree $d$, are set of orthogonal polynomials over the domain $(-\infty, +\infty)$ with weighting function $e^{-x^2}$. The Hermite polynomials are defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

(2.9)

and satisfy

$$H_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k n!(2x)^n}{k!(n-2k)!}.$$ 

Being the limiting case of several families of classical orthogonal polynomials, they are of fundamental importance in asymptotic analysis.

In this paper, we shall be concerned primarily with the development of various asymptotic properties of Eulerian numbers, refined Eulerian numbers and the coefficients of descent polynomials, which are obtained directly from the spline interpretations of these numbers [15]. The theory of B-splines is a well developed area of applied numerical analysis and interpolation theory. Having related these numbers directly to B-splines [15], we can take advantage of many powerful spline techniques to derive various results of these numbers. It is precisely this fact which motivates the need for more purely spline interpretations of these numbers to which we now turn. To state conveniently, we use $[\lambda^j]f(\lambda)$ to denote the coefficient of $\lambda^j$ in $f(\lambda)$ for any given power series $f(\lambda)$.

**Lemma 2.1.** [15]

(i) \[ A_{d,k} = d! \cdot B_{d+1}(k); \]
(ii) \[ D(d, n, k) = d! \cdot n^d \cdot B_{d+1} \left( k + \frac{1}{n} \right); \]

(iii) \[ A_{d+1,k,d-j+1} = d! \cdot [\lambda^j] \left( (\lambda + 1)^d B_{d+1} \left( k + \frac{1}{\lambda + 1} \right) \right) \cdot \binom{d}{j}, \lambda \geq 0. \]

3 Main results

After the necessary definitions have been provided, we can come to the main results of this article.

**Theorem 3.1.** Let be \( k \in \mathbb{N} \) then for \( d > k + 2 \) the sequence of the \( k \)-th derivatives \( B_d^{(k)} \) of the B-spline converges to the \( k \)-th derivative of the Gaussian function,

\[
\left( \frac{d}{12} \right)^{\frac{k+1}{2}} B_d^{(k)} \left( \sqrt{\frac{d}{12}} x + \frac{d}{2} \right) = \frac{1}{\sqrt{2\pi}} D^k \exp \left( -\frac{x^2}{2} \right) + O \left( \frac{1}{d} \right), \tag{3.1}
\]

and

\[
\lim_{d \to \infty} \left\{ \left( \frac{d}{12} \right)^{\frac{k+1}{2}} B_d^{(k)} \left( \sqrt{\frac{d}{12}} x + \frac{d}{2} \right) \right\} = \frac{1}{\sqrt{2\pi}} D^k \exp \left( -\frac{x^2}{2} \right), \tag{3.2}
\]

where the limit may be taken point-wise or in \( L^p(\mathbb{R}), p \in [2, \infty) \).

Sommerfeld in 1904 [1] show that the Gaussian function can be approximated by B splines point-wise. In 1992, Unser and colleagues [14] proved that the sequence of normalized and scaled B-splines \( B_d \) tends to the Gaussian function as the order \( d \) increases in \( L^p(\mathbb{R}) \). A result due to Ralph Brinks [16] generalize Unser’s result to the derivatives of the B-splines. In this paper, we reprove the theorem and give the order of the convergence.

**Theorem 3.2.** For \( x_d = \sqrt{\frac{d+1}{12}} x + \frac{d+1}{2} \), we have

\[
\frac{1}{d!} A_{d,\left[x_d\right]} = \sqrt{\frac{6}{\pi(d+1)}} \exp \left( -\frac{x^2}{2} \right) + O \left( d^{-\frac{3}{2}} \right). \tag{3.3}
\]

L. Carlitz, D. C. Kurtz, R. Scoville and O. P. Stackelberg in [2] showed [3.3] with the help of the central limit theorem of probability theory that the expression on the right side of (3.3) has the order \( O(d^{-\frac{3}{2}}) \). Nevertheless, using
the spline interpretation of Eulerian numbers, we obtain the same asymptotic forms of $A_{n,k}$ with a much better convergence order $O(d^{-\frac{3}{2}})$.

**Theorem 3.3.** For $x_d = \sqrt{\frac{d+1}{12}} x + \frac{d+1}{2}$, we have

$$
\frac{1}{d! \cdot n^d} D(d, n, [x_d]) = \frac{6}{\pi (d+1)} \exp \left( -\frac{(x + \frac{1}{2})^2}{2} \right) + O \left( d^{-\frac{3}{2}} \right) \quad (3.4)
$$

Using the spline theory, we can also get the asymptotic representations of refined Eulerian numbers in terms of the Hermite polynomials.

**Theorem 3.4.** Let $x_d = \sqrt{\frac{d+1}{12}} (x - 1) + \frac{d+1}{2}$ then

$$
A_{d+1, [x_d], d-j+1} = d! \left( \frac{6}{\pi (d+1)} \exp \left( -\frac{x^2}{2} \right) \sum_{i=0}^{j} \frac{1}{(d-j+i)} \left( \frac{d+1}{12} \right)^{-\frac{i}{2}} H_i(x) + O \left( d^{-\frac{3}{2}} \right) \right),
$$

where $H_n(x)$ are the Hermite polynomials are defined by

$$
H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}. \quad (3.5)
$$

### 4 Proofs of the main results

The proof of Theorem 3.1 is based on the following lemmata given in [16] which shows an upper bound for standardized Sinc functions.

**Lemma 4.1.** [16] For $k, d \in \mathbb{N}$ and $d \geq k+2$, there is a constant $c_k \in \mathbb{R}^+$ such that for

$$
G_k(x) := \chi_{\mathbb{R}\setminus[-1,1]}(x) \frac{c_k}{\pi^2 x^2} + \pi^k |x|^k \exp(-x^2)
$$

where $\chi_A(x)$ is characteristic function on set $A$, it holds

$$
G_k \in L^p(\mathbb{R}), \forall p \in [1, \infty),
$$

and

$$
\pi^k |x|^k \left| \text{sinc} \left( \frac{\pi x}{\sqrt{d}} \right) \right|^d \leq G_k(x).
$$

**Proof.** We start with the inequality

$$
\forall x \in [0, 1], \text{sinc}(x) \leq 1 - x^2,
$$
where the right-hand side term is the parabola that goes through the two extreme values of \( \text{sinc}(x) \) within this interval. We will first show that

\[
\forall x \in [0, \sqrt{d}], \text{sinc}^n \left( \frac{x}{\sqrt{d}} \right) \leq \left( 1 - \frac{x^2}{d} \right)^d \leq \exp \left( -x^2 \right). \tag{4.1}
\]

For this purpose, we define the positive function

\[
p(x) = \exp \left( x^2 \right) \left( 1 - \frac{x^2}{d} \right)^d.
\]

The derivative of \( p(x) \) is given by

\[
\frac{\partial p(x)}{\partial x} = -2 \exp(x^2)x^3 \left( 1 - \frac{x^2}{d} \right)^d/(1 - x^2/d)
\]

and is always negative for \( x \in [0, \sqrt{d}] \). Therefore, the maximum of \( p(x) \) within the interval occurs at \( x = 0 \)

\[
\sup_{x \in (0, \sqrt{d})} p(x) = p(0) = 1,
\]

which proves (4.1).

Second, for \( x \geq \sqrt{d} \), set

\[
c_k := \max_{n \geq k+2} \left( \frac{d^{k+2}}{\pi^{d-k-2}} \right).
\]

We note that \( \pi^k |x|^k \cdot \left| \text{sinc}(\pi x/\sqrt{d}) \right|^d \) is also bounded by

\[
\pi^k |x|^k \cdot \left| \text{sinc} \left( \frac{\pi x}{\sqrt{d}} \right) \right|^d \leq \frac{c_k}{x^2 \pi^2}. \tag{4.2}
\]

The existence of \( c_k \) follows from the convergence of the sequence

\[
\left( \frac{d^{k+2}}{\pi^{d-k+2}} \right)_{d \geq k+2}.
\]

Then one follows

\[
\left( \frac{\sqrt{d}}{x \pi} \right)^d \leq \left( \frac{\sqrt{d}}{x} \right)^{k+2} \cdot \frac{1}{\pi^d} \leq \frac{c_k}{x^{k+2} \pi^{k+2}}.
\]

We get a bound \( \frac{\sqrt{d}}{x^2 \pi^2} \) that is independent of \( d \) by noticing that

\[
\left| \text{sinc}^d \left( \frac{\pi x}{\sqrt{d}} \right) \right| \leq \left( \frac{\sqrt{d}}{\pi x} \right)^d.
\]
Finally, we define the function $G_k(x)$, which is independent of $d$, by suitably combining the right-hand sides of (4.1) and (4.2). The function $\pi^k |x|^k \cdot \left| \text{sinc} \left( \frac{\pi x}{\sqrt{n}} \right) \right|^n$ is uniformly bounded from above by an $L_p(-\infty, +\infty)$ function $G_k(x)$ where $p \in [1, +\infty)$.

We have all the ingredients to prove our results now.

**Proof of Theorem 3.1.** Set

$$L_n(x) := d \ln \left( \frac{\text{sinc} \left( \frac{x}{2} \sqrt{\frac{12}{d}} \right)}{1 - \frac{x^2}{2} d + O \left( \frac{1}{d^2} \right)} \right).$$

(4.3)

Due to the symmetry, we may assume $x \geq 0$. By Taylor’s theorem, for any $x \in [0, 1]$ and $d \in \mathbb{N}$, it holds

$$\text{sinc} \left( \frac{x}{2} \sqrt{\frac{12}{d}} \right) = 1 - \frac{x^2}{2} d + O \left( \frac{1}{d^2} \right)$$

and

$$\ln(1 + x) = x + O(x^2).$$

Then for any $x \in [0, 1]$ and $d \in \mathbb{N}$, it holds

$$L_n(x) = d \ln \left( 1 - \frac{x^2}{2} d + O \left( \frac{1}{d^2} \right) \right) = -\frac{x^2}{2} d + O \left( \frac{1}{d} \right).$$

(4.4)

Combining (4.3) and (4.4), we have

$$\text{sinc}^d \left( \frac{x}{2} \sqrt{\frac{12}{d}} \right) = \exp \left( -\frac{x^2}{2} d \right) \left( 1 + O \left( \frac{1}{d} \right) \right).$$

(4.5)

Furthermore, it holds $B_d^\wedge (\omega) = \text{sinc}^d \left( \frac{\omega}{2} \right)$, and $B_d \in C^{d-1}(\mathbb{R})$. The later yields for $k \leq d - 1$:

$$\left[ B_d^{(k)} \right]^\wedge (\omega) = i^k \omega^k \text{sinc}^d \left( \frac{\omega}{2} \right).$$

Consequently, one obtains

$$\left( \frac{d}{12} \right)^\wedge^{\frac{d+1}{2}} \left[ B_d^{(k)} \left( \sqrt{\frac{d}{12}} x + \frac{d}{2} \right) \right]^\wedge (\omega) = i^k \omega^k \text{sinc}^d \left( \frac{\omega}{2} \sqrt{\frac{12}{d}} \right)$$

$$= i^k \omega^k \exp \left( -\frac{\omega^2}{2} d \right) \left( 1 + O \left( \frac{1}{d} \right) \right)$$

$$= \left[ D^k \left\{ \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) \cdot \left( 1 + O \left( \frac{1}{d} \right) \right) \right\} \right]^\wedge (\omega)$$
\[
(d/12)^{k+1} B^{(k)}_d \left( \sqrt{d/12} x + d/2 \right) = D^k \left( \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) \right) + O \left( \frac{1}{d} \right)
\]
for \( d \to \infty \)

\[
\lim_{d \to \infty} \left\{ (d/12)^{k+1} B^{(k)}_d \left( \sqrt{d/12} x + d/2 \right) \right\} = \frac{1}{\sqrt{2\pi}} D^k \exp \left( -\frac{x^2}{2} \right)
\]

(4.6)

where the limit in equation (4.6) follows from Lemma 4.1 and is meant to be taken point-wise. Clearly, \( B^{(k)}_d(\omega) \) is bounded by the \( L_p \) function \( G_k(x) \) defined in Lemma 4.1 and this bound is independent of \( d \). The use of Lebesgue’s dominated convergence Theorem provides the \( L_p \) convergence for \( P \in [1, +\infty) \). The \( L_q \) convergence with \( q \in [2, +\infty] \) in the time domain follows as a consequence of Titchmarsh inequality, which states that for \( 1 \leq p \leq 2 \) and \( p^{-1} + q^{-1} = 1 \), the Fourier transform is a bounded linear operator from \( L_p(-\infty, +\infty) \) into \( L_q(-\infty, +\infty) \).

When \( k = 0 \), the Theorem 3.1 turns to

**Corollary 4.1.** For \( d \in \mathbb{N} \), the B-spline converges to the Gaussian function,

\[
\sqrt{d/12} B_d \left( \sqrt{d/12} x + d/2 \right) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) + O \left( \frac{1}{d} \right),
\]

(4.7)

and

\[
\lim_{d \to \infty} \left\{ \sqrt{d/12} B_d \left( \sqrt{d/12} x + d/2 \right) \right\} = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right),
\]

(4.8)

where the limit may be taken point-wise or in \( L^p(\mathbb{R}) \), \( p \in [2, +\infty) \).

**Proof of Theorem 3.4** We firstly prove Theorem 3.4.

Set

\[
p(\lambda) = \sum_{j=0}^{d} \binom{d}{j} A_{d+1,k,d-j+1} \lambda^j
\]

for \( \lambda > -1 \). Then \( p(\lambda) \) is the polynomial of \( \lambda \) with \( d \) degree. The lemma 2.1 states that

\[
A_{d+1,k,d-j+1} = d! \cdot \lambda^j \left( \frac{1}{\lambda+1} \right)^{d+1} \frac{(d+1)_j}{(d+1)_j} , \lambda \geq 0.
\]
which implies
\[
p(\lambda) = \sum_{j=0}^{d} \binom{d}{j} A_{d+1,k,d-j+1} \lambda^j = d!(\lambda + 1)^d B_{d+1} \left( k + \frac{1}{\lambda + 1} \right) .
\] (4.9)

Taking the \(j\)-th derivative of both sides of equation (4.9), it holds:
\[
A_{d+1,k,d-j+1} = \sum_{i=0}^{j} (-1)^i \frac{i! (d-j)!}{(d-j+i)!} B_{d+1}^{(i)}(k+1)
\]
\[
= d! \sum_{i=0}^{j} (-1)^i \frac{1}{(d-j+i)!} B_{d+1}^{(i)}(k+1).
\]

Combining Theorem 3.1 with the definition of Hermit polynomials (2.9), we obtain that
\[
A_{d+1,[x_d],d-j+1} = d! \sqrt{\frac{6}{\pi(d+1)}} \sum_{i=0}^{j} (-1)^i \frac{1}{(d-j+i)!} \left( \frac{d+1}{12} \right)^{-\frac{i}{2}} D^i \exp \left( -\frac{x^2}{2} \right) + O \left( d^{-2} \right)
\]
\[
= d! \sqrt{\frac{6}{\pi(d+1)}} \exp \left( -\frac{x^2}{2} \right) \sum_{i=0}^{j} (-1)^i \frac{1}{(d-j+i)!} \left( \frac{d+1}{12} \right)^{-\frac{i}{2}} H_i(x) + O \left( d^{-2} \right).
\]

\[
\]

Proofs of Theorem 3.2 and Theorem 3.3. By using Lemma 2.1

(i) \( A_{d,k} = d! \cdot B_{d+1}(k) \),

(ii) \( D(d,n,k) = d! \cdot n^d \cdot B_{d+1} \left( k + \frac{1}{n} \right) \),

and Corollary 4.1
\[
\sqrt{\frac{d}{12}} B_d \left( \sqrt{\frac{d}{12}} x + \frac{d}{2} \right) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) + O \left( \frac{1}{d} \right),
\]
consequently, for \( x_d = \sqrt{\frac{d+1}{12}} x + \frac{d+1}{2} \), we have
\[
\frac{1}{d!} A_{d,[x_d]} = \sqrt{\frac{6}{\pi(d+1)}} \exp \left( -\frac{x^2}{2} \right) + O \left( d^{-\frac{3}{2}} \right)
\]
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and
\[ \frac{1}{d! \cdot n^d} D(d, n, [x_d]) = \sqrt{\frac{6}{\pi(d + 1)}} \exp \left( -\frac{(x + \frac{1}{n})^2}{2} \right) + O \left( d^{-\frac{3}{2}} \right). \]

**Remark 4.1.** We can also prove Theorem 3.2 by using Theorem 3.4. By the combinatorial interpretations of \( A_{d,k} \) and \( A_{d,k,j} \), we have \( A_{d+1,k} = A_{d+1,k+1} \). Combining with Theorem 3.4 one obtains
\[ \frac{1}{d!} A_{d,[x_d]} = \sqrt{\frac{6}{\pi(d + 1)}} \exp \left( -\frac{x^2}{2} \right) + O \left( d^{-\frac{3}{2}} \right). \] (4.10)

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