Bound states in Functional Renormalization Group

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A representation of the functional renormalization group (FRG) equations of the QED is proposed with help of auxiliary fields, which span the spectral representation of the 2-to-2 fermion scattering amplitude. The set of auxiliary fields can be interpreted as bound state degrees of freedom. In the nonrelativistic limit the energy (mass) of the ground state can be estimated fairly accurately keeping only a few auxiliary fields.

I. INTRODUCTION

Bound state formation in field theories is a fundamental problem. This is particularly valid in case of strong interactions, where the set of observables is restricted exclusively to bound states of the particles (quarks and gluons) defining the theory.

The nonrelativistic approach to the bound state formation, i.e. the solution of the Schrödinger-equation, works nicely for atomic physics, but for relativistic systems it can not be generalized directly. The main reason is that, because of the retardation of the potential, the Lagrangian becomes nonlocal in time, and this makes the definition of the Hamiltonian cumbersome. Moreover, the propagators of the constituents are not restricted to the mass shell rigidly, and this modifies the naive potential (loop diagrams and “crossed leg” diagrams). As a consequence we are faced with a 2-particle, time-nonlocal problem with an improved potential known as the Bethe-Salpeter equation (BSE) \cite{1,2}. This method was successfully used in the context of many QCD-related questions, c.f. for example \cite{3,4}.

The generalized potential approach of the BSE is still not fully consistent, since the crossing symmetry of the relativistic quantum field theories is not obeyed. Technically speaking, in the diagrammatic expansion only the s-channel diagrams are summed up, the t- and u-channel (if there is allowed such) is treated only perturbatively.

Because of these problems one employs also other frameworks to treat the bound state problem. One such framework is the infinite set of Dyson-Schwinger equations (DSE) \cite{6,7}. The ladder-type structures of the BSE resummation appear in the rainbow-ladder approximation of the DSE \cite{8}. Carrying this approximation consistently for the 4-point function should provide the appropriate bound states of the system. Attempts to step beyond the ladder-summation are based on nPI equations, where also the dynamical evolution of the interaction vertices is included \cite{9,10}.

The Functional Renormalization Group (FRG) equations \cite{11,12}, which are in principle exact (for reviews see \cite{13,14}), must also account for bound state formation. Here we have to choose an Ansatz for the effective quantum action that gives an account for both the fundamental and the bound state degrees of freedom and avoid double counting. Seminal works by Ellwanger and Wetterich \cite{15,16} have shown that a momentum-dependent 4-particle FRG equation is equivalent to the BSE. An efficient algorithm was also proposed and solved numerically for these equations. A disadvantage of this method is that the nonlocal 4-point function is a very complicated object which can be represented with a function of at least 5 variables. But as an alternative, one can characterize the effective action with local terms which include also the interaction with the would-be bound states \cite{17,18}. One can maintain this extended expression of the action during the whole scale evolution, if we apply a Hubbard-Stratonovich transformation after each FRG step to keep only the representatives of the bound states. This method, known also as dynamical hadronization, was used in several QCD studies \cite{19,22}.

In this paper we propose a different approach to avoid the treatment of nonlocal 4-point functions. We will find an exact representation of the 4-point function with the help of auxiliary fields that can be interpreted as the fields describing the bound states of the system. In a given regularization we need a finite, but large number of auxiliary fields for a sufficiently accurate representation, and then our approach is equivalent to the original strategy with a full momentum dependent 4-point function. The point is, however, that concerning the low-lying bound states for a reasonable precision it is enough to consider only a small number of auxiliary fields (to describe the ground state probably it is enough to consider 2-3 auxiliary fields), which makes the problem much more accessible from the point

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of view of explicit calculations. The spirit of this approach is close to the truncated conformal space approach [23], or its massive version [24].

For a pedagogical demonstration of the concept the example of the nonrelativistic Coulomb problem will be discussed in detail. Based on the experience gathered on this example, the complete relativistic formalism will be developed for the QED.

The structure of the paper is as follows. We start with a Prelude (Section II): in the example of the nonrelativistic Coulomb-problem (H-atom) we will demonstrate how can one rewrite the Schrödinger equation to the FRG evolution equation with auxiliary fields. In the course of the reformulation an object is introduced in complete formal analogy with the quantum 4-point function of a field theory. Next we review the relativistic bound state problem, and discuss the BSE for QED in Section III. Then, in Section IV we look for an FRG representation that correctly reproduces the BSE equations. In Section V we consider the representation of the nonlocal four-point function of QED through auxiliary fields. These fields are interpreted as bound state representatives, and a set of flow equations derived for them should be solved for finding the QED bound states. The results of the paper are summarized in a concluding section (Section VI).

II. PRELUDE: THE NONRELATIVISTIC COULOMB PROBLEM WITH LESSONS

The most simple example of bound state formation is the Coulomb-problem (H-atom problem), where we have a quantum particle orbiting around a center, thought to be a very (infinitely) heavy particle. Although it is exactly solvable, it may serve as a testbed where a proposed general method can be confronted with familiar exact results.

In this Prelude the Schrödinger equation of the Coulomb problem will be rewritten in a form that can be interpreted as an FRG equation with auxiliary fields. The results of this Section will be generalized for the full glory of the relativistic QED with two fermion species in the later Sections.

A. From Schrödinger- to FRG-equations: rewriting the Coulomb problem

Let us consider for definiteness an electron in the Coulomb potential. The Schrödinger equation reads

$$\Delta \Psi + \frac{1}{x} \Psi = \bar{E} \Psi,$$

where we introduced $r_0 = \frac{2\hbar^2 \pi \epsilon_0}{me^2}$, $x = \frac{r}{r_0}$, and $\bar{E} = -\frac{2m}{\pi r_0^2} E$. In this paper we will concentrate only on the $s$-states, i.e. rotational invariant states ($\ell = 0$). Then the above equation simplifies to

$$\frac{d^2 u}{dx^2} + \frac{1}{x} u = \bar{E} u.$$  

where $u(x) = x \Psi(x)$. From the analytic solution we know that the energy eigenvalues are

$$\bar{E}_n = \frac{1}{4n^2}, \quad n = 1, 2, \ldots.$$  

This is the standard approach to the quantum mechanical bound states: we look for the eigensystem of the Hamiltonian, that provides the quantized energy values.

The Schrödinger equation approach, however, is hardly generalizable to relativistic systems. To move towards the direction of generalizability, we first perform Fourier-transform on (1), and introduce $v(q) = (\bar{E} + q^2) \Psi(q)$. We obtain

$$\int \frac{d^3 p}{(2\pi)^3} \frac{4\pi}{(q - p)^2 (\bar{E} + p^2)} v(p) = v(q).$$

For the $s$-states, with $\eta(q) = q v(q)$ we find

$$\eta(q) = \frac{1}{\pi} \int_0^\infty dp \frac{1}{\bar{E} + p^2} \log \left| \frac{q + p}{q - p} \right| \eta(p).$$

This is an integral equation, which is seemingly more complicated than the original differential equation (2). Numerically, however, to treat an integral equation is not more difficult than a differential equation. Moreover, this form
is more adequate for relativistic generalizations. In fact, introducing the "matrices" with two spatial momenta as indices:

\[ V_{pq} = \frac{4\pi}{(p-q)^2}, \quad G_{pq}^{(E)} = -\frac{1}{E + p^2} \delta_{pq}, \]

then (4) can be written as

\[ V G^{(E)} v = -v. \]  

(7)

This is formally equivalent to the Bethe-Salpeter equations (32) (in fact it is the nonrelativistic limit of (32)). On the other hand, the eigenvectors of this system clearly correspond to the bound state wave functions.

But we can go further. Let us modify the integral in (4) in the spirit of FRG, by introducing a regularized momentum as

\[ \int \frac{d^3p}{(2\pi)^3} \frac{4\pi}{(p-q)^2(E + p_k^2)} v_k(p) = v_k(q), \quad p_k^2 = \Theta(p - k)p^2 + \Theta(k - p)k^2. \]

(8)

Note, however, that no regulator is applied in the Fourier transform of the potential. We see that we recover (4) for \( k = 0 \). On the other hand in the formal limit of \( k \to \infty \) we obtain an easily solvable system. There, namely, the \( p \)-integral appears as a convolution, so in real space the above equation reads

\[ k \to \infty : \quad V(x)v_\infty(x) = (\bar{E} + k^2)v_\infty(x), \]  

(9)

which has a solution \( v_\infty(x) \sim \delta(x - r_0) \). Thus, if we can write down a differential equation for the scale dependence of \( v_k(q) \) with \( k \)-derivatives, we can start from a known exact solution at \( k \to \infty \), and arrive at the solution of the actual problem at \( k = 0 \).

For the object obeying a differential equation in \( k \), we introduce the hermitean matrix

\[ \lambda^{(k,E)} = (1 + V G^{(k,E)})^{-1} V, \]

(10)

where \( G^{(k,E)} \) is the same as \( G^{(E)} \) with \( p \to p_k \) substitution. This choice is dictated by the complete formal analogy of (10) with the 4-point function of the relativistic theory (see (30)). We see from (7) that \( \lambda^{(k,E)} \) is singular when \( \bar{E} \) is one of the bound states of the regularized system. For the \( k \)-evolution of \( \lambda^{(k,E)} \) it is easy to determine a differential equation, which is the analogue of the FRG equation. After differentiating (10) with respect to \( k \) we obtain (using the \( k \)-independence of \( V \))

\[ \partial_k \lambda^{-1} = \partial_k G^{(k,E)} = \frac{2k}{(E + k^2)^2} \Theta(k - p) \delta_{pq}, \]

(11)

where the last expression is valid for the Coulomb problem.

This equation has a disadvantage, namely that it is a matrix equation. In the relativistic case it depends on two 4-momenta, not to speak about the spin degrees of freedom. Even when we take into account all symmetries at finite temperature, the solution of a matrix equation is a formidable task (10).

To overcome this difficulty we may apply a trick and represent the \( \lambda \) matrix with its eigenvectors, that is using its spectral decomposition. Taking into account that it is hermitean, we may write

\[ \lambda^{-1} = \sum_n \kappa_n^{(k)} x_n^{(k)} \otimes x_n^{(k)\dagger}, \]

(12)

where \( x_n^{(k)} \) are the (orthonormal) eigenvectors and \( \kappa_n^{(k)} \) are the eigenvalues of \( \lambda^{-1} \) at scale \( k \). Then we have from (11)

\[ \partial_k \sum_n \kappa_n^{(k)} x_n^{(k)} \otimes x_n^{(k)\dagger} = \partial_k G_k. \]

(13)

To simplify the notation, we will omit the superscript \( (k) \) in the sequel.

By performing the differentiation, and multiplying the expression by \( x_m \) we arrive for the index pairs \( m = n \) and \( m \neq n \) the respective equations

\[ \partial_k \kappa_n = x_n^\dagger \partial_k G_k x_n, \]

\[ \partial_k x_n = \sum_m \frac{x_m^\dagger \partial_k G_k x_n}{\kappa_n - \kappa_m} x_m. \]

(14)
We note that these equations are the continuous versions of the quantum mechanical perturbation theory. The evolution equation of the matrix $\lambda^{-1}$ is replaced now by the evolution of the kernel and the eigenfunctions of its spectral decomposition. The latter will correspond to the effective fields representing the bound states of the relativistic field theory. The technical steps for solving (14) are outlined in Appendix A.

Let us analyze the characteristic features of the solution.

The starting $K_n$ eigenvalues (A7) are almost linear functions of $n$. In our numerical example $N = 800$ and $dp = 0.003$ and the initial spectrum can be seen in the left panel of Fig. 1.

Starting the RG evolution from these values we can observe the curves shown in the right panel of Fig. 1. We see that the values are getting smaller as we decrease the scale, and eventually they cross zero. The bound state energies are coming from the condition that $K_n(\bar{E}_n) = 0$ at $k = 0$.

We can also perform an energy scan: in the left panel of Fig. 2 we display $K_n(\bar{E}_n) = \frac{1}{2N_{eff}^2}$ as a function of $N_{eff}$. We see that the kernel is zero at about $N_{eff} \in \mathbb{Z}$, which is the solution of the Coulomb problem. In the right panel we give the resulting section points.

The functions $K_n(\bar{E})$ contain, however, more information than just the bound state energies. This function is the bound state kernel, its inverse is the bound state propagator, c.f. Section VA. Considering the ground state, the pole of the propagator is at $\bar{E}_0 \approx 0.245$. If we represent the propagator as

$$G_0(\bar{E}) = \frac{Z(\bar{E})}{\bar{E} - \bar{E}_0},$$

then the residuum (wave function renormalization, equivalent in non-relativistic quantum mechanics to the absolute square of the ground state wave function) is shown in the left panel of Fig. 3.

These are all interesting features, but the most important message concerns the necessary number of eigenvectors for a reliable estimate of the bound state energies. In the right panel of Fig. 3 we show the estimated energy levels by keeping $N_{eig}$ eigenvectors. It is remarkable that the ground state can be estimated with 8% precision using only three eigenvectors, and for a precision of 28% it is enough to consider just the first two eigenvectors. This means that
we can effectively reduce the dimension of the Hilbert space, concentrating only on the first few eigenvectors (cf. also TCSH approximations [23, 24]). It is also important, that keeping only the ground state is not enough, it leads to a complete misidentification of the ground state energy.

III. BETHE-SALPETER RESUMMATION IN QED

In the previous subsection we demonstrated, that in non-relativistic systems the Schrödinger-equation is equivalent to the Bethe-Salpeter equation, and can also be rephrased as an FRG-like differential equation. In the following we will show that in a relativistic quantum field theory (QFT) it is also possible to write up an FRG equation for the bound states, and these can be represented as a set of scale-dependent eigenvectors of the 4-point function.

Our demonstrative relativistic QFT model studied in this paper is QED with two fermionic degree of freedom: the two oppositely charged fermionic fields are denoted by $\psi$ and $\chi$, the photon is denoted by $A_\mu$. The Lagrangian becomes

$$L = \bar{\psi}(i\partial^\mu - eA^\mu) - m_\psi)\psi + \bar{\chi}(i\partial^\mu + eA^\mu) - m_\chi\chi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{16}$$

We change to the Euclidean description as well as we go over to Fourier transformed quantities. Fourier momenta appear formally as lower "vector-indices". For perturbation theory we choose the Feynman gauge in the present investigation. Then the action reads

$$S_{\text{fund}} = \int_p \left[ \bar{\psi}_p(i\partial^\mu + m_\psi)\psi_p + \bar{\chi}_p(i\partial^\mu + m_\chi)\chi_p + \frac{1}{2}p^2 A_p^\mu A_p^\mu - ie \int_q (\bar{\psi}_p \gamma_5^2 \psi_q - \bar{\chi}_p \gamma_5^2 \chi_q) A_p^\mu q \right], \tag{17}$$

where $a \in \{1, 2, 3, 4\}$, $x_0 = -ix_4$, $\partial_0 = i\partial_4$, $A_0 = iA_4$, $\gamma_j = i\gamma_j^2$ for $j < 4$ and $\gamma_0 = -\gamma_4^2$. Below a simplified notation is used, where the index 'E' is not displayed.

For later convenience, since we will work in the matrix representation, it seems to be advantageous to work with the conventional (not the Dirac) adjoint. Then we have

$$S_{\text{fund}} = \int_p \left[ \psi_p^\dagger K_p^{(\psi)} \psi_p + \chi_p^\dagger K_p^{(\chi)} \chi_p + \frac{1}{2} A_p^\dagger A_p^{(A)} A_p^{(A)} + \int_q (\psi_p^\dagger \gamma_a^\dagger \psi_q - \chi_p^\dagger \gamma_a^\dagger \chi_q) A_p^{(A)} A_p^{(A)} q \right], \tag{18}$$

where

$$K_p^{(\psi)} = \gamma_0(\gamma^\mu + m_\psi), \quad K_p^{(\chi)} = \gamma_0(\gamma^\mu + m_\chi), \quad K_p^{(A)} = \gamma_a^\dagger, \quad \gamma_a^\dagger = e\gamma_0^\dagger. \tag{19}$$

The inverse matrices of the kernels provide the propagators which will be denoted as $G^{(\psi)}$, $G^{(\chi)}$ and $G^{(A)}$, respectively. In this way the propagators and the couplings (the photon-fermion vertices) all are hermitean matrices

$$G_{\rho\beta}^{(\psi)} = G_{\rho\beta}^{(\psi)}, \quad G_{\rho\beta}^{(\chi)} = G_{\rho\beta}^{(\chi)}, \quad G_{\rho\beta}^{(A)} = G_{\rho\beta}^{(A)}, \quad \gamma_{\rho\beta}^{a} = e\gamma_{\rho\beta}^{a}. \tag{20}$$

As a first step we shall construct the Bethe-Salpeter equations for the $\psi$ -- $\chi$ bound state. To this end we first study the one-loop perturbative expressions.
A. Four point function at one loop level

The central quantity in the study of the bound states is the $\langle \psi \psi \dagger \chi \dagger \rangle$ 4-point function. Since we expect that the bound state consists of a $\psi$ and a $\chi$ particle, we can find it as a pole in the $\psi$-$\chi$ scattering amplitude.

First we will compute the four point function in perturbation theory at one-loop level. The four point function depends on four momenta, but the energy-momentum conservation reduces the number of independent variables to three. For later convenience we will define the amputed 4-point amplitude as

$$M_{\alpha\gamma,\beta\sigma}^{\ell} := -\left\langle \bar{\psi}_\alpha \gamma^\ell \psi_\beta \chi_{\ell-p,\ell-q} \right\rangle_{\text{amputed}}$$

(21)

where the hat means that that line is amputed, and the indices refer to the amputed ends. The quantity $\ell$ denotes the total incoming momentum. The minus sign is here for convenience, it signals that the interaction is attractive.

The indices on the left hand side are arranged in a way that will be convenient to use later on. We will work with matrices with (multi)indices $\alpha\beta$. Relative to these multi-indices the above defined object is hermitean:

$$M_{\alpha\gamma,\beta\sigma}^{\ell\ast} = M_{\beta\sigma,\alpha\gamma}^{\ell}$$

(22)

At tree level we have a single diagram contributing (cf. Fig. 4) which in the non-relativistic limit corresponds to the electrostatic potential between the two fermions. We introduce the notation $V_{\alpha\gamma,\beta\sigma}^{\ell} = (M_{\alpha\gamma,\beta\sigma}^{\ell})^{\text{tree}}$, and its expression reads as

$$V_{\alpha\gamma,\beta\sigma}^{\ell} = \gamma^\alpha p \gamma^\beta q \gamma^\ell \chi_{\ell-p,\ell-q} G_{p-q}^{(A)}.$$  

(23)

At one loop level we show the two contributing 1PI diagrams (cf. Fig. 5) with the analytic expressions

$$(M_{\alpha\gamma,\beta\sigma}^{\ell})_{1\text{-loop}}^{1\text{PI}} = -\int \frac{d^4k}{(2\pi)^4} \left\{ \gamma^\alpha p \gamma^b k \gamma^\beta q \gamma^\ell \chi_{\ell-k,\ell-q} G_{k-q}^{(A)} + \gamma^\alpha p \gamma^b k \gamma^\beta q \gamma^\ell \chi_{\ell-p,\ell-k,\ell-q} G_{k-q}^{(A)} + \gamma^\alpha p \gamma^b k \gamma^\beta q \gamma^\ell \chi_{\ell-p,\ell-k,\ell-q} G_{k-q}^{(A)} \right\},$$

(24)

where we have suppressed the internal spinor indices for readability.

FIG. 4: The tree level Feynman diagram contributing to the connected 4-point function. Thin lines represent $\psi$, thick lines $\chi$, and the curly line stands for the $A$-propagator.

FIG. 5: The 1PI one-loop Feynman diagrams contributing to the connected 4-point function ($k' = \ell - p + k - q$). Thin lines represent $\psi$, thick lines $\chi$, and curly lines stand for $A$ propagators.

In the resummation which leads to BSE (the BS resummation) the crossed leg contribution is included as a perturbative correction into the potential $V^{(0)}$. One writes for the complemented "potential" denoted by $V$

$$V_{\alpha\gamma,\beta\sigma}^{\ell} = V_{\alpha\gamma,\beta\sigma}^{\ell} - \int \frac{d^4k}{(2\pi)^4} \left\{ \gamma^\alpha p \gamma^b k \gamma^\beta q \gamma^\ell \chi_{\ell-p,\ell-k,\ell-q} G_{k-q}^{(A)} \right\}.$$  

(25)

The graphic symbol expressing this definition appears in Fig. 6. To be truly consistent, the vertices and the propagators
here should also be dressed. We do not elaborate on this question, which is just a technical side problem from the point of view of the bound state formation.

In addition to the interaction "potential" we make use also of the 2-fermion propagator

$$G_{p\alpha\gamma,q\beta\sigma} = \delta_{pq} G_{p,\alpha\beta}(\psi) G_{\ell,\gamma\sigma}(\chi).$$  \hfill (26)

In the multi-index notation this is also hermitean

$$G_{\ell}^{\alpha\gamma,q\beta\sigma} = G_{\ell}^{q\beta\sigma,p\alpha\gamma}.$$  \hfill (27)

This expression can be written in a formally simpler way using a scalar product notation. For some quantities $f_{p\alpha\gamma}$ and $g_{p\alpha\gamma}$ the scalar product is defined as

$$fg = \sum_{\alpha\gamma} \int \frac{d^4p}{(2\pi)^4} f_{p\alpha\gamma} g_{p\alpha\gamma}.$$  \hfill (28)

Then we find

$$\mathcal{M}_{p\alpha\gamma,q\beta\sigma} = \bar{V}_{p\alpha\gamma,q\beta\sigma} + \sum_{\alpha'\gamma'\beta'\sigma'} \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \bar{V}_{p\alpha\gamma,k\alpha'\gamma'} G_{k\alpha'\gamma',k'\beta'\sigma'} \bar{V}_{k'\beta'\sigma',q\beta\sigma}.$$  \hfill (29)

We see that with these definitions the index $\ell$ is just a spectator index (it would not be, if we would not include the crossed-leg contribution into the potential). In the following, as far it does not lead to misunderstanding, we will suppress the explicit reference to $\ell$.

**B. The Bethe-Salpeter resummation**

At higher order there are several diagrams that contribute. In the Bethe-Salpeter approximation only the ladder diagrams are taken into account

$$\mathcal{M} = \bar{V} - \bar{V} g \bar{V} + \bar{V} g \bar{V} g \bar{V} + \cdots = (1 + \bar{V} g)^{-1} \bar{V}.$$  \hfill (30)

In graphical representation we see the result in Fig. 7. The 4-point function, as we see, has poles, where the matrix

$$\mathcal{M} = \bar{V} - \bar{V} g \mathcal{M},$$  \hfill (31)

as it is demonstrated in Fig. 8.
All these relations contain matrix inversion which does not exists when the matrix has a zero eigenvalue (then the determinant is also zero). So the condition of having a pole in the four point function is equivalent that there exists a vector $u$ that

$$\tilde{\mathbf{V}} \mathbf{G} u = -u.$$  \hspace{1cm} (32)

This is the *Bethe-Salpeter equation*. Note that in the complete notation $u$ has several indices: $u_{\rho\alpha\gamma}^\ell$.

We remark finally that we can write up the resummed 1PI proper 4-fermion vertex, too:

$$\Gamma^{(4)} = \mathcal{M} - \tilde{\mathbf{V}} = -(1 + \tilde{\mathbf{V}} \mathbf{G})^{-1} \tilde{\mathbf{V}} \mathbf{G} \tilde{\mathbf{V}}.$$  \hspace{1cm} (33)

### IV. FRG STUDY OF EFFECTIVE MODELS OF BOUND STATE FORMATION

The main goal of this Section is to find an adequate Ansatz for the effective action of the QED, for which the renormalization group flow leads to a 4-point function consistent with the results of the Bethe-Salpeter resummation, but also consistent with the characteristics of the exact correlation functions like crossing symmetry.

To be able to do a strict comparison we will adapt here a regularization scheme that is best suited for the BS equations. We will regularize only the matter (fermion) propagators, the gauge part remaining untouched. This means that we will use regularized kernel only for $\psi$ and $\chi$:

$$K_p^{(\psi)} \to K_p^{(e)} + R_p^{(\psi)}, \quad K_p^{(\chi)} \to K_p^{(\chi)} + R_p^{(\chi)}.$$  \hspace{1cm} (34)

In QED, where all loops contain at least one matter propagator, this prescription restricts automatically the momentum variation of the photon propagator near the actual momentum scale. For the moment we leave the actual form of these regulators unspecified, actually any choice works equally well.

Having fixed the regularization we have to adapt the form of the effective action to the bound-state problem. Different attempts are discussed in the following subsections.

#### A. A tempting possibility that does not work

The first guess that one tries out comes from the observation that a bound state consists of a $\psi$ and a $\chi$ particle. Therefore in order to account for the bound states we just introduce one field degree of freedom for such an object. This effective field will be called $H$. This bound state field couples to the $\psi$ and a $\chi$ particles to lowest order through a Yukawa-like coupling $\sim H \psi^\dagger \chi^\dagger$. In order to maintain relativistic invariance we have to include its charge conjugated counterpart, too.

Then we can choose an FRG Ansatz that relies on the degrees of freedom $\psi$, $\chi$, $A^a$ and $H$, and contains relevant (renormalizable) operators. In Minkowski space it corresponds to the choice

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \psi^\dagger \gamma_0 (i \nabla - m_\psi) \psi + \chi^\dagger \gamma_0 (i \nabla^\dagger - m_\chi) \chi + H^\dagger K^{(H)}(i \partial) H - H \psi^\dagger \tilde{v} \chi^* - H^\dagger \chi^T \tilde{v}^\dagger \psi,$$  \hspace{1cm} (35)

where the coupling function $\tilde{v}$ is a spin-dependent and eventually nonlocal object, $\nabla$ stands for the covariant derivative. It actually depends also on the momentum of the bosonic bound state, but this is a spectator variable not indicated explicitly. To maintain Lorentz-invariance we may choose $\tilde{v}_{\rho\alpha\beta} = v_\beta \mathcal{C}_E$, where $\mathcal{C}_E = \gamma_0 \gamma_2$ is the charge conjugation operator (c.f. Appendix B). Changing over to the Euclidean theory in Fourier space yields (suppressing spinor indices)

$$\Gamma = \Gamma_{QED} + H_p^\dagger K^{(H)} H_p + H^\dagger \psi^\dagger \tilde{v}_p \chi_{\ell-p} + H^\dagger \chi^T \tilde{v}^\dagger \psi_p,$$  \hspace{1cm} (36)

where the QED part is the same as in [18], $K^{(H)}$ and $v$ are the new scale-dependent parameters.
In matrix notation it can be written as
\[ v \]
where the one-loop correction reads
\[ \Sigma_H(\ell) = - \int \frac{d^4 p}{(2\pi)^4} \tilde{v}_{p\alpha\beta} \tilde{v}_{p'\alpha'\beta'} G^{(v)}(p) G^{(\chi)}(p') = - \text{Tr} v^\dagger \mathcal{G} v, \]
where \( v \) is the vector notation of \( \tilde{v}_{p\alpha\beta} \), and the trace as well as the adjoint is meant in the multi-index notation.

For the one loop correction of \( v_p \) we consider the following expectation value (c.f. Fig. 9):
\[ \langle H'_\ell \psi \psi^\dagger_{p\alpha\beta} \rangle_{\text{amputed}} = \tilde{v}_{p\alpha\beta} - \int \frac{d^4 q}{(2\pi)^4} G^{(v)}(q) G^{(\chi)}(q') \tilde{v}_{q\alpha'} \tilde{v}_{q'\beta'}. \]

In matrix notation it can be written as
\[ \langle H'_\ell \psi \psi^\dagger_{p\alpha\beta} \rangle_{\text{amputed}} = v_{\alpha\beta} - (\mathcal{V} \mathcal{G})_{\alpha\beta,\sigma\sigma'} v_{\sigma\sigma'}. \]

A one-loop correction to the self-energy of \( H \) as well as to the coupling \( v \) can be computed by evaluating appropriate expectation values. For the self-energy of \( H \) we have to compute the amputed 2-point function \(- \langle H H^\dagger \rangle\) (c.f. Fig. 9), the one-loop correction reads
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For the one loop correction of \( v_p \) we consider the following expectation value (c.f. Fig. 9):
\[ \langle H'_\ell \psi \psi^\dagger_{p\alpha\beta} \rangle_{\text{amputed}} = \tilde{v}_{p\alpha\beta} - \int \frac{d^4 q}{(2\pi)^4} G^{(v)}(q) G^{(\chi)}(q') \tilde{v}_{q\alpha'} \tilde{v}_{q'\beta'}. \]

In matrix notation it can be written as
\[ \langle H'_\ell \psi \psi^\dagger_{p\alpha\beta} \rangle_{\text{amputed}} = v_{\alpha\beta} - (\mathcal{V} \mathcal{G})_{\alpha\beta,\sigma\sigma'} v_{\sigma\sigma'}. \]

Now let us determine the FRG equations using this Ansatz. The general Wetterich equation reads
\[ \partial_k \Gamma = \frac{1}{2} \hat{\partial}_k \text{Tr} \ln(\Gamma^{(2)}) + R_k, \]
where \( R_k \) is the regulator and the derivative with respect the scale, \( \hat{\partial}_k \) acts only on the regulator. Using this form one can easily determine the equations for higher derivative \( n \)-point functions, since the corresponding one-loop diagrams just involve larger number of amputed external legs. Therefore we simply present without entering into the derivation steps the RG equations resulting for the \( n \)-point functions.

Since we want to concentrate on the bound states, only the evolution of the bound state kernel \( K^{(H)}_p \) and of the fermion-bound state coupling \( v \) is tracked, the fermion masses and the photon propagator are kept the same at all scales. We find:
\[ \partial_k K^{(H)}_p = - \hat{\partial}_k \text{Tr} v^\dagger \mathcal{G} v \]
\[ \partial_k v = - \hat{\partial}_k (\mathcal{V} \mathcal{G}) v. \]

Now we can use the fact that in our specific regularization scheme the regulator affects only \( \mathcal{G} \). Moreover, since we do not run the QED parameters, in this simple approximation \( \hat{\partial}_k \mathcal{G} = \partial_k \mathcal{G} \) since there is no other \( k \) dependence. So we obtain
\[ \partial_k K^{(H)}_\ell = - \text{Tr} v^\dagger \partial_k \mathcal{G} v \]
\[ \partial_k v = - (\mathcal{V} \mathcal{G}) v. \]

The second equation is very similar to the derivative of the BS-equation \( \mathcal{F}_p \). The difference is that from the derivative of \( \mathcal{F}_p \) we obtain
\[ \partial_k v = - (\mathcal{V} \mathcal{G}) v - \mathcal{G} \partial_k v \quad \Rightarrow \quad \partial_k v = -(1 + \mathcal{V} \mathcal{G})^{-1} \mathcal{V} \partial_k \mathcal{G} v. \]

Thus we do not have the same equation as the one that would come from the BS-equation. And, correspondingly, we also do not have the same solution. The equation for \( v \), namely, can be solved symbolically in form of a "\( k \)-ordered" exponential:
\[ v_k = T_k e^{- \int k^A d^4 x \mathcal{V} \mathcal{G}^{(k)} v_{k=\Lambda}}, \quad \text{vs.} \quad v_{BS,k} = (1 + \mathcal{V} \mathcal{G}^{(k)})^{-1} v_{k=\Lambda}. \]
As we see, the solution of the Ansatz of this section misses the pole which is the signature for the appearance of bound states. So, although tempting is the simplicity of Ansatz (36), it is not adequate to account for the bound states.

To understand what is the conceptual problem with this approach we expand the exponential factor in powers of $V\mathcal{G}(k)$ and recognize the presence of the $1/n!$ suppression factor relative to the expansion of the BS-solution.

The origin of this suppression factor is the following. When we solve an equation $\partial_k x = A(k)x$ recursively, then we obtain a series of ladder diagrams. But the momentum of the subsequent ladder diagrams is strictly ordered, since we can insert a rung only at the left end of the series containing rungs with higher $k$. Therefore the exponential function corresponds to ladder diagrams with rungs of ordered momenta. If we release the momentum ordering, then the rungs can appear in all possible sequence, yielding a factor of $n!$ growth at $n$th order.

### B. The pure fermion representation of the QED effective action

To overcome the difficulty pointed out in the preceding subsection we turn to a more complete description of the problem which requires the inclusion of the (non-renormalizable, irrelevant) four fermion function. This requirement is very similar to the scalar electrodynamics case [18], where the authors have taken into account the generated 4-point functions, and in each RG step they also performed a Hubbard-Stratonovich transformation, projecting the theory back to the fundamental particles-bound-state sector.

In this work we follow a similar strategy, with the difference that here we project the system on a pure fermion theory with a momentum dependent 4-fermion function that plays here a crucial role. Formally we take the effective action $\Gamma[\psi, \chi, A^a]$, and associate an effective action to it $\Gamma_{eff}[\psi, \chi]$, where all the fermionic observables are the same but the photon field is missing. The physics comes from the connected $n$-point functions: so what we have to do it to produce the connected $n$-point functions from the proper vertices in both theories, and match them equal.

We will use the Ansatz

$$\Gamma_{eff} = \int_p \left[ \bar{\psi}(p)p\gamma^\mu \psi(p) + \bar{\chi}(p)p\gamma^\mu \chi(p) \right] + \int d^4 p \int d^4 q \int d^4 \ell \left\{ \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{d^4 \ell}{(2\pi)^4} \lambda^\ell_{\rho\gamma,q\beta\sigma} \bar{\psi}(p)\gamma^\lambda \bar{\psi}(q)\gamma^\ell \bar{\chi}(q)\gamma^\rho \chi(p) \right\}. \tag{45}$$

On scale $k = \Lambda$ the original $\Gamma[\psi, \chi, A^a]$ is the QED action [18]. The matching condition for the connected $\psi-\chi$ 4-point function in the two models reads (cf. [23])

$$\mathcal{M}^\ell_{\rho\gamma,q\beta\sigma} = V^\ell_{\rho\gamma,q\beta\sigma} = \lambda^\ell_{\rho\gamma,q\beta\sigma}|_{k=\Lambda} \tag{46}$$

Of course other correlation functions of the two theories differ from each other, but we can hope that, since this specific 4-point function is the most important from the point of view of producing bound states, the above Ansatz will catch the essence of bound state formation. The scale-evolution certainly induces deviations between the effective 4-fermion coupling and the coupling arising from a complete (dressed) photon exchange. If the cut-off is not much higher than the bound-state scale, this deviation will not introduce large systematic error in the characterization of the bound state.

We first compute radiative corrections to the four-point function. Technically $\lambda$ plays the same role as the earlier introduced $V$. Therefore we find for the 4-point 1PI function (c.f. Fig. 10)

\[ \Gamma_{\rho\gamma,q\beta\sigma}^{(4)\ell} = \lambda_{\rho\gamma,q\beta\sigma}^\ell - \int \frac{d^4 r}{(2\pi)^4} \left[ \lambda_{\rho\gamma,r\beta'\sigma'}^\ell G_{r,\beta'\sigma'}^{\ell} \lambda_{r\gamma',q\beta\sigma}^\ell + \lambda_{\rho\gamma',r\beta\sigma'}^\ell G_{r,\beta\sigma'}^{\ell} \lambda_{r\gamma,q\beta\sigma'}^\ell \right], \tag{47} \]

where the correction has a very similar structure as $(\mathcal{M}_{\rho\gamma,q\beta\sigma}^\ell)^{1-loop}$ in [24]. The first term corresponds to the ladder-type diagram, the second one corresponds to the crossed leg diagram.
For the FRG equation of the coupling $\lambda$ one finds:

$$
\partial_k \lambda_{\rho\gamma,q\beta\sigma} = -\partial_k \int \frac{d^4r}{(2\pi)^4} \left[ \lambda_{\rho\gamma,r\beta'\sigma'} G_{r\beta'\sigma'}^{\ell} \lambda_{\rho\gamma',q\beta\sigma} + \lambda_{\rho\gamma,r\beta'\sigma'} G_{r\beta'\sigma'}^{\ell+2k-p-q} \lambda_{\rho\gamma',q\beta\sigma'}^{\ell+r-p} \right].
$$

(48)

Using our special regularization, where the $k$-dependence comes solely from the regularization of $G$, we can also write

$$
\partial_k \lambda_{\rho\gamma,q\beta\sigma} = -\int \frac{d^4r}{(2\pi)^4} \left[ \lambda_{\rho\gamma,r\beta'\sigma'} \partial_k G_{r\beta'\sigma'}^{\ell} \lambda_{\rho\gamma',q\beta\sigma} + \lambda_{\rho\gamma,r\beta'\sigma'} \partial_k G_{r\beta'\sigma'}^{\ell+2r-p-q} \lambda_{\rho\gamma',q\beta\sigma'}^{\ell+r-p} \right].
$$

(49)

The initial condition for the flow, as mentioned above, is $\lambda(k = \Lambda) = V$. This is the complete equation which, as it comes from a consistent Ansatz, respects all the symmetries of the quantum field theory like the crossing symmetry.

BS resummation is known to violate crossing symmetry, and so we can not expect that the above result fully agrees with the BS results. Therefore to have a connection to the BS result, we have to simplify our FRG equations, taking into account only the first (ladder) term in the differential equation. Then the FRG equation can be written in matrix form as before:

$$
\partial_k \lambda = -\lambda (\partial_k G), \quad \lambda(k = \Lambda) = V.
$$

(50)

This equation can be solved explicitly

$$
\partial_k \lambda^{-1} = \partial_k G \Rightarrow \lambda^{-1} = G_k - G_{\Lambda}.
$$

(51)

Assuming that $G(\Lambda) = 0$ due to the regulator, we find

$$
\lambda_k = (1 + V G_k)^{-1} V.
$$

(52)

We see that at $k = 0$ we obtain the same result as from the BS resummation, cf. (30).

This means that this version of FRG contains enough information to correctly reproduce the bound state spectrum, but in addition it is a fully consistent resummation method (cf. also [15, 16]). The bad news is that we have to keep the complete momentum dependence in order to obtain the correct result. Now we may wonder if we could find an easier representation of this 4-fermion theory with help of auxiliary (bound) states.

V. REPRESENTATION WITH MANY BOUND STATES

Once we have an equation for the scale dependence of the 4-fermion function of QED, like (49), we might attempt to interpret it as emerging from bound state exchanges. It is worth to emphasize that the notion “bound state” has no pre-defined meaning, we only introduce them as auxiliary quantities. The only requirement is that the effective bound state theory should match the results of the FRG flow equations (49) or the BS equations (32).

We first study which type of effective theories are appropriate to fully reproduce the nonlocal 4-point functions, then we discuss various representations, each corresponding to a different approximation.

A. Introducing bound state effective theory

Let us try to introduce new degrees of freedom into the system so that the physics it describes is the same as using the original degrees of freedom of quantum electrodynamics. “Physics” here means identical $n$-fermion connected correlation functions.

In the effective model we exploit the freedom to introduce (infinitely) many auxiliary fields $H_{pn}$, where $p$ is its momentum, $n$ counts the different fields. The auxiliary fields are coupled to two fermions through a Yukawa-type interaction like in (36). The Ansatz reads

$$
\Gamma_{\text{eff}} = \int p \left[ \psi_p \Gamma_{p}(\psi_p) \chi_p + \lambda_p \Gamma_{p}(\chi_p) + \sum_n H_{pn}^{\dagger} K_{pn}(H) H_{pn} + \sum_n \int p_e \left[ H_{en} \psi_{pn}^{\dagger} \Gamma_{en}(H) \chi_{en}^{\dagger} + H_{en}^{\dagger} \chi_{pn}^{\dagger} \Gamma_{en}(H) \psi_{pn} \right] + \sum_n \int p_{e} \left[ H_{en} \psi_{pn}^{\dagger} \Gamma_{en}(H) \chi_{en}^{\dagger} + H_{en}^{\dagger} \chi_{pn}^{\dagger} \Gamma_{en}(H) \psi_{pn} \right] \right].
$$

(53)

This is similar to (45), as far there are no photons in this action, but this Ansatz contains neither 4-fermion vertices. The interactions are replaced by the exchange of (infinitely many) auxiliary fields.

The actual set of independent fermion–bound-state Yukawa-couplings $v$ is found by associating a different coupling with each irreducible Lorentz-representation built from the fermion bilinears. We can observe that the combination
$\psi^\dagger \bar{R}^s C E \chi^*$ is a vector operator in the representation of $R = \{S, P, V, A, T\}$ ($\Gamma^{RS} = \{1, \gamma_5, \gamma_\mu, \gamma_5 \gamma_\mu, \sigma^{\mu\nu}\}$ and $\bar{\Gamma}^{RS} = \gamma_0 \Gamma^{RS} \gamma_0$, cf. Appendix B). The Yukawa coupling also depends on the momentum in a generic way, so the p-integral in fact represents a form like $\gamma_{\mu}(45)$ representing QED, the connected, amputed 4-point function is just the earlier introduced state and in the pure fermion theory. Since $\Gamma$ is an effective action, it contains the proper vertices, thus to find the Gies and Wetterich [18], Alkofer et al. projected back to the auxiliary field (bound state) representation. This technique is very similar to the procedure of and in the result we keep only the 4-fermion terms. We perform the FRG step in this model, then the result is the bound states of the system. Therefore we follow the technique that we eliminate the auxiliary degrees of freedom, 4-fermion interactions, and these, as it was demonstrated in the previous section, are crucial for correctly representing proven to be inaccurate. Instead we take into account that in an FRG step there appear also other operators like technique of using solely those relevant operators that appear in the Ansatz for setting up an FRG equation was

The condition that we describe the same physics with both models is simply that [27]

$$\lambda^\ell_{\mu\nu,\gamma\beta} = - \sum_n \psi^\dagger_{\mu\nu} G^{(H)}_{\ell n} \psi_{\gamma\beta}.$$  \hspace{1cm} (57)

or nicely written in matrix notation:

$$\lambda = - \sum_n G^{(H)}_{n} \psi_n \otimes \psi^\dagger_n.$$  \hspace{1cm} (58)

The most simple solution to fulfill this requirement is based on the spectral representation of $\lambda$ through its eigensystem:

$$\lambda = \sum_n c_n x_n \otimes x^\dagger_n, \quad \lambda x_n = c_n x_n, \quad |x_n|^2 = 1.$$  \hspace{1cm} (59)

Therefore, we may choose

$$\psi_n = x_n, \quad G^{(H)}_n = -c_n.$$  \hspace{1cm} (60)

In this case, using the initial condition for $\lambda$ [16] implies that $\{c_n, \psi_n\}_{k=1}^K$ is the eigensystem of $V$. At this point it becomes clear that for an exact representation of the original system as many auxiliary fields are needed as many eigenvectors span the spectral decomposition of the operator $\lambda$. This is typically a large (mostly infinite) number.

Although it is the most straightforward representation, we may generalize this setup. We can find a representation based not on the original $\lambda$, but some of its transformed form:

$$A \lambda A^\dagger x_n = c'_n x_n, \quad |x_n|^2 = 1.$$  \hspace{1cm} (61)
where $\mathbf{A}$ is any (invertible) matrix. This provides a representation

$$A\lambda A^\dagger = \sum_n c_n x_n \otimes x_n^\dagger, \quad \lambda = \sum_n c_n'(A^{-1}x_n) \otimes (A^{-1}x_n)^\dagger. \tag{62}$$

This suggests that we can also choose a set of non-orthogonal $\mathbf{v}_n$ Yukawa-couplings:

$$\mathbf{v}_n = A^{-1}x_n, \quad G_n^{(H)} = -c_n'. \tag{63}$$

Usually the eigenvalues also depend on the choice of $\mathbf{A}$ (unless it is unitary). Exceptions are the zero or infinite eigenvalues: for a zero eigenvalue that had originally an eigenvector $x_0$, then

$$\lambda x_0 = 0 \Rightarrow A\lambda A^\dagger(A^{-1})_0 = 0, \tag{64}$$

so $\lambda A^\dagger$ has also a zero eigenvalue. Infinite eigenvalue strictly speaking means that the matrix does not exists, or that $Q = \lambda^{-1}$ is not invertible, since it has a zero eigenvalue. That also means that $A^{-1}QA^{-1}$ has also a zero eigenvalue, it is not invertible, so the corresponding inverse matrix $A\lambda A^\dagger$ must have an “infinite” eigenvalue.

This means that although the auxiliary field propagators are not unique, their poles are representation independent, and so they can be considered as physical quantities. Therefore the bound state mass is well defined, but for example the bound state scattering amplitudes depend on the accurate definition of the off-shell parts of the bound states.

Summarizing the correspondence found in this subsection, we can faithfully represent the 4-fermion interaction through a Yukawa-type theory with infinite number of auxiliary fields. The choice of these fields is not unique, but the poles of their propagators represent real physical singularities of the 4-point function. They provide unique characterization for what one should call the physical bound states. The last question is, how the FRG equations for the bound state propagators and couplings look like. This is discussed in the next subsection.

### B. FRG equations of the bound state system: the BS approximation

Before we turn to the full expression, let us discuss the FRG equations that faithfully represent the BS approximation. We have seen earlier that eq. [50] can reproduce the BS resummation of the subsection IV B. Therefore we will use this equation with the representation [59]. Sandwiching the evolution equation with the eigenvectors of $\mathbf{A}$ we find

$$x_m^\dagger(\partial_\lambda) x_n = \delta_{nm}\partial c_n + c_n x_m^\dagger \partial x_n + c_m \partial x_m^\dagger x_n = -c_n c_m x_m^\dagger(\partial_\mathbf{G}) x_n. \tag{65}$$

Using the fact that $x_n$ are orthonormal, we have

$$\delta_{nm}\partial c_n + (c_n - c_m) x_m^\dagger \partial x_n = -c_n c_m x_m^\dagger(\partial_\mathbf{G}) x_n. \tag{66}$$

This yields

$$\partial c_n = -c_n^2 x_n^\dagger(\partial_\mathbf{G}) x_n, \quad \partial x_n = \sum_{\ell \neq n} \frac{c_n c_m}{c_\ell - c_n} x_n(\partial_\mathbf{G}) x_n. \tag{67}$$

In physical terms $x \rightarrow \mathbf{v}$ and $c \rightarrow -G^{(H)}$ (c.f. [60]). Then it is worth to work with the $H$-field kernels using $G^{(H)} = 1/K^{(H)}$. We find

$$\partial c_n = -v_n^\dagger(\partial_\mathbf{G}) v_n, \quad \partial v_n = \sum_{m \neq n} \frac{1}{K_m - K_n} v_m(\partial_\mathbf{G}) v_n. \tag{68}$$

Writing out the indices, the matrix elements of $\partial_\mathbf{G}$ read

$$v_m^\dagger(\partial_\mathbf{G}) v_n = v_{\alpha\beta\gamma\delta}^{m\ell}(\partial_\mathbf{G})_{\alpha\beta\gamma\delta}^{\ell} v_{\alpha\beta\gamma\delta}^{n\ell}. \tag{69}$$

To compute this expression we just need to perform a $p$ integral and spinor index summations.

We may compare the above equations with [62]. We see that the kernel equation is the same as earlier (with the difference that now we have a lot of auxiliary fields instead of one). The evolution equation of the Yukawa-couplings, however, is different [28].
To provide an initial condition, we use the fact that the potential is $V$ defined in (23), where the gamma matrices are momentum independent. Therefore for $V$ the eigenvalue equation is a convolution in the momentum space, and so a product in the real space:

$$
\lambda(k = \Lambda) = G_{\rho q}^{(A)\gamma_{\alpha\beta}^a}_{\gamma_{\alpha\beta}^a} \Rightarrow \lambda(k = \Lambda) = G_{\rho q}^{(A)\gamma_{\alpha\beta}^a}_{\gamma_{\alpha\beta}^a}.
$$

(70)

The eigenvalue equation can be solved by the Ansatz

$$
v^\rho_{\alpha\beta} = v^\rho_{\alpha\beta},
$$

(71)
then we have

$$
4G^{(A)}v^\rho_{\alpha\beta} = c_n v^\rho_{\alpha\beta} \Rightarrow v^\rho_{\alpha\beta} = \delta(x - n), \quad c_n = 4G^{(A)}.
$$

(72)

The advantage of the representation with the bound states lies in the fact that it may allow to keep the information about the system much more economically. The 4-point coupling has a momentum structure $\lambda_{\rho q}^{\ell}$. Taking into account Lorentz invariance this function may depend on $\ell^2$, $p^2$, $q^2$, $\ell p$, and $pq$, i.e. six invariants. In the BS-approximation $\ell$ is a spectator index, so $\ell^2$ should not be taken into account, but even then we have a function with five arguments. The 3-point function, on the other hand, has a momentum and species structure $v^\rho_{\alpha\beta}^{pq}$, in the relativistic invariant case it may depend on $n$, $\ell^2$, $p^2$, and $\ell p$. These are four parameters, and in the BS approximation we have a function with only three arguments. Even better, one of these parameters is a species index, and if we trust the argument that IR physics is dominated by the eigenvectors with the smallest eigenvalues, then we may use a reduced number of species.

C. The general FRG equations

We can repeat the results of the previous subsection with a general evolution equation. With the representation we find, just like before:

$$
\delta_{nm}\partial_k c_n + (c_n - c_m)x_m^\dagger \partial_k x_n = x_m^\dagger (\partial_k \lambda) x_n
$$

(73)
which yields

$$
\partial_k c_n = x_m^\dagger (\partial_k \lambda) x_n, \quad \partial_k x_n = \sum_{\ell \neq n} \frac{1}{c_n - c_\ell} x_i (x_\ell^\dagger (\partial_k \lambda) x_n).
$$

(74)

Here $\partial_k \lambda$ is given by (49) where we have to use the representation (59). Diagrammatically we have the contributions for $\partial_k \lambda$ as shown in Fig. (11). Its matrix elements in the $x_n^\dagger$ basis read as

$$
x_m^\dagger (\partial_k \lambda) x_n^\ell = -c_n c_m x_m^\dagger (\partial_k G) x_n^\ell - L_m^\ell,
$$

(75)
where

$$
L_m^\ell = x_{\rho \sigma \gamma \delta}^{m^n_{\rho \sigma \gamma \delta}^{\ell + r - p - q}} \frac{G^{\ell + 2r - p - q}_{\rho \sigma \gamma \delta} x_{m^n_{\rho \sigma \gamma \delta}^{\ell + r - p - q}} x_{n^{\ell + r - p - q}}}{x_{n^{\ell + r - p - q}}}
$$

(76)

This is a very complicated object containing integration over $p$, $q$ and $r$, therefore it is practically unapproachable if we keep all auxiliary fields. However, if we keep only a few of them, this term also simplifies significantly. Thus it may be included in the full analysis in this approximation.
D. Ground state approximation

As a last point we will argue that like it was the case of the non-relativistic Coulomb problem, also in the relativistic case it is not enough to keep only the ground state in a specific channel to reliably estimate its ground state energy. Keeping just the smallest energy level means that we try to represent the four-point function with a single particle exchanged in the s-channel:

\[ \lambda = c_0 P_0, \quad P_0 = v_0 \otimes v_0^\dagger. \] (77)

Having a look on (68) we see that \( v_0 \) is constant, while for \( K_0 = -1/c_0 \) we obtain

\[ \partial_k K_0 = -v_0^\dagger \partial_k G v_0 \] (78)

that has the solution, assuming \( G_\Lambda = 0 \)

\[ K_0 k = K_0 \Lambda - v_0^\dagger G k v_0. \] (79)

The initial condition \( \lambda_\Lambda = V \) is hardly can be satisfied, since \( \lambda \) is now a pure projector. The best approximation is that we start from \( \lambda_\Lambda = PVP \), which means \( c_0 = v_0^\dagger V v_0 = -1/\Lambda \). Thus we have

\[ K_{0k} = \frac{1 + v_0^\dagger V v_0 v_0^\dagger G k v_0}{v_0^\dagger V v_0}. \] (80)

The condition to find a bound state thus reads

\[ v_0^\dagger V v_0 v_0^\dagger G v_0 = -1 \] (81)

(where \( G = G_{k=0} \)). This equation should determine the bound state energy. We note, however, that the exact equation would be (c.f. (32)) \( V G v_0 = -v_0 \), or \( v_0 V G v_0 = -1 \). It is numerically conceivable that the extra projector between \( V \) and \( G \) does not count, but in general it is not true. For example, one can construct examples with \( 2 \times 2 \) matrices where \( V \) and \( G \) are hermitean, \( V G v_0 = -v_0 \), but \( v_0^\dagger V v_0 = 0 \) while \( v_0^\dagger G v_0 = 1 \).

VI. CONCLUSIONS

In this paper we studied methods for the treatment of bound states in the framework of the Functional Renormalization Group. We have investigated the construction of the FRG flow equations of QED in various approximations for finding relativistic bound states with two oppositely charged fermionic constituents. The central issue of our study was the consistency of these techniques with the Bethe-Salpeter equation. Therefore we have not discussed the structure of the renormalisation group flow arising from the various RG-equations. This subject will be covered in a separate publication [26].

The relation of the different approaches is schematically sketched in Fig. 12.

For the practical use of the FRG we need an Ansatz, and in fact it is the quality of the Ansatz that determines the precision of the FRG results. As it turned out, the most simple representation of the bound states through a kinetic term and a Yukawa potential for the field representing the ground state, does not lead to a pole in the propagator (see Section IV A). In a further step, with an effective model, that contains only fermionic kinetic terms and a 4-fermion interaction, it was possible to represent the bound states of the QED (see Section IV B) in a consistent QFT framework.

The effective model with 4-fermion interaction, however, has disadvantages. Physically it represents the bound states as poles in the 4-fermion theory, the bound states do not appear explicitly in the Lagrangian. Technically its drawback is that to run the FRG equations we have to consider tensor quantities. To cure these disadvantages we considered an effective model where we introduced infinitely many auxiliary scalar fields coupled to the fermions with generalized (momentum dependent) Yukawa couplings to represent the 4-fermion interaction (c.f. Section V). In the treatment of this model, however, we had to take into account the appearance of the four-fermion interaction, but after each scale change we could redefine the auxiliary fields to fully represent this term. The auxiliary fields can be interpreted as the bound states, although the representation is not unique, only the poles can be considered representation independent, physical quantities.

If we keep all the auxiliary fields, the bound state effective action is as complicated as the four-fermion interaction. However, as the experience with the non-relativistic case could demonstrate, for understanding only the ground state
properties, it is enough to keep only few (2-3) auxiliary bound state fields. If this property is inherited to the
relativistic case, then the bound state effective theory can very efficiently solve relativistic bound state models. In the
present paper, however, we only studied the non-relativistic case, the relativistic calculation is the task of a future
publication.

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may introduce
satisfactory accuracy for the lower energy eigenvalues. We use discrete momentum values
K
We can treat the collection of eigenvalues
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addition, we still need an antihermitean matrix ¯η to avoid adding a lot of zeros, it is simpler to introduce a non-square matrix ¯η
from (A1) we can introduce a projector nm
η
Then we pursue the following algorithm.

Appendix A: Solution of the non-relativistic flow equation
Here the technical steps for solving the equations in (14) are described. In case of the s-state Coulomb problem we may introduce
ηn(p) = pxn(p), \quad \text{and} \quad C_{nm} = \frac{1}{2\pi^2} \int_0^k dq \eta_n^*(q)\eta_m(q), \quad (A1)
then we have
\partial_k\mathcal{K}_n = \frac{2kC_{nn}}{(E+k^2)^2},
\partial_k\eta_n(p) = \frac{2k}{(E+k^2)^2} \sum_{m\neq n} \eta_m(p)C_{mn}\mathcal{K}_n - C_{nn}, \quad (A2)
The numerical solution of this system allows us to assess the necessary number of bound state fields to achieve satisfactory accuracy for the lower energy eigenvalues. We use discrete momentum values
p_\ell = \left(\ell + \frac{1}{2}\right) dp, \quad \ell \in \{0, 1, 2, \ldots N - 1\}. \quad (A3)
From the function \eta_n(p_\ell) we create a matrix
\eta_\ell n = \sqrt{\frac{dp}{2\pi^2}} \eta_n(p_\ell) \quad (A4)
This matrix is normalized to be unitary, since
\delta_{nm} = \frac{1}{2\pi^2} \int_0^\infty dp \eta_n^*(p)\eta_n(p) \to \frac{dp}{2\pi^2} \sum_{\ell=0}^{N-1} \sqrt{\frac{2\pi^2}{dp}} \eta_n^* \sqrt{\frac{2\pi^2}{dp}} \eta_n = \sum_{\ell=0}^{N-1} \eta_{n\ell}^* \eta_n. \quad (A5)
We can treat the collection of eigenvalues \mathcal{K}_n \rightarrow \mathcal{K} as a vector.
In order to compute C_{nm} from (A1) we can introduce a projector P_{q\ell} = \Theta(k - q)\delta_{q\ell}, then C = \eta^T P \eta. Technically, to avoid adding a lot of zeros, it is simpler to introduce a non-square matrix \eta_\ell n = \eta_n, for \ell < \kappa, then C = \eta^T \eta. In addition, we still need an antihermitean matrix D_{m\neq n} = \frac{C_{mn}}{\mathcal{K}_n - \mathcal{K}_m}, with D_{nn} = 0.
Then we pursue the following algorithm.
1. Parameter setting: the external parameters are the value of the energy \bar{E}, the number of the points N and the resolution of the integrals dp. We need \Lambda = N dp > \bar{E}.
2. Initialization: we diagonalize the potential: in matrix notation V is a symmetric (in general hermitean) matrix, and we are looking for \eta orthogonal (in general unitary) matrix that satisfies
\eta^T V \eta = \text{diag}(V_n). \quad (A6)
The starting value for the scale \kappa = \Lambda = N dp, and the initial value for \mathcal{K} reads
\mathcal{K}_n = \frac{1}{V_n} - \frac{1}{E + \Lambda^2}. \quad (A7)
3. **Recursion:** for a given $k = \kappa dp$ (where $\kappa = N, \ldots, 1$) we determine the $\kappa \times N$ matrix $\eta$, and from that we determine the symmetric $C$ and antisymmetric $D$ matrices as

$$ C = \eta^T \eta, \quad D_{nm} = \frac{C_{nm}}{K_m - K_n}. \quad (A8) $$

Then we update

$$ K_n \to K_n - \alpha dp C_{nn}, \quad \eta \to \eta(1 - \alpha dp D), \quad \alpha = \frac{2k}{(E + k^2)^2}. \quad (A9) $$

4. **End:** do the recursion until $\kappa = 1$. Performing the above algorithm for different $\bar{E}$ values, we obtain the function $K_n(\bar{E})$ which is just the bound state kernel. The location of the crossing $K_n(\bar{E}_n) = 0$ provides the bound state energies $\bar{E}_n$.

**Appendix B: Charge conjugation**

Although it is a standard piece of knowledge, for completeness we describe charge conjugation operations. Let us start from a Dirac-like Lagrangian

$$ \mathcal{L} = \bar{\zeta}(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\zeta, \quad (B1) $$

where $\zeta$ is an anticommuting field. Choose new degrees of freedom as $\zeta = C\chi^*$

$$ \mathcal{L} = \chi^T C^\dagger \gamma_0(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)C\chi^* = -\bar{\chi}\gamma_0 C^T (-i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\gamma_0 C^* \chi = \chi\gamma_0 C^T (i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu + m)\gamma_0 C^* \chi, \quad (B2) $$

where we have performed a partial integration, used the anticommuting nature of $\chi$, and used the fact that $\gamma_0^T = \gamma_0$. Now we require

$$ \gamma_0 C^T \gamma^\mu \gamma_0 C^* = \gamma^\mu, \quad \gamma_0 C^T \gamma_0 C^* = -1, \quad (B3) $$

then we obtain

$$ \mathcal{L} = \bar{\chi}(i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu - m)\chi, \quad (B4) $$

the same form with opposite charge. We use that $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, that in the Dirac or Weyl representation $\gamma_2\gamma_0^T \gamma_2 = \gamma^\mu$, and $\gamma_0^T \gamma_0 = \gamma^0$. Both in Dirac and Weyl representation $\gamma_2^T = \gamma_2$ and $\gamma_2^T = -\gamma_2$. Thus a good choice is $C = \alpha\gamma_2$ where $|\alpha|^2 = 1$. In this case, namely

$$ \gamma_0 \alpha\gamma_2^T \gamma^\mu \gamma_0 \alpha^* \gamma_2^* = -\gamma_0 \gamma_2 \gamma^\mu \gamma_0 \gamma_2 = \gamma_0^T \gamma_2 \gamma^\mu \gamma_2 \gamma_0 = \gamma_0^T \gamma_2 \gamma_0 \gamma_2 \gamma_0 = \gamma_0 \gamma_2 \gamma_0 \gamma_2 \gamma_0 = \gamma^\mu, \quad (B5) $$

and

$$ \gamma_0 \gamma_2 \gamma_0 \alpha^* \gamma_2^* = -\gamma_0 \gamma_2 \gamma_0 \gamma_2 = \gamma_0 \gamma_2 \gamma_0 \gamma_2 = -1. \quad (B6) $$

If we take two fields $\zeta$ and $\psi$ with the same charge, then a mass term of a form $\bar{\psi}\zeta + \bar{\zeta}\psi$ is possible. In the language of the $\chi$ fields we have

$$ \bar{\zeta}\psi = \chi^T C^\dagger \gamma_0 \psi, \quad \bar{\psi}\zeta = \bar{\psi}C\gamma_0(\bar{\chi})^T. \quad (B7) $$

Choosing $C^\dagger = C$ means that a relativistic invariant term of the form

$$ \chi^T C \psi + \bar{\psi}C(\bar{\chi})^T \quad (B8) $$

is allowed. The hermiticity of $C$ can be ensured by $\alpha = -i$, then

$$ C = i\gamma_0 \gamma_2. \quad (B9) $$

If we want to work with the normal adjoint fields instead, then the Lorentz invariant forms are

$$ \psi^\dagger \gamma_0 \zeta + \zeta^\dagger \gamma_0 \psi = \alpha \psi^\dagger \gamma_0 \gamma_2 \chi^* + \alpha^* \chi^\dagger \gamma_2 \gamma_0 \psi. \quad (B10) $$
Then it is advantageous to choose $\alpha = 1$ and $C_E = \gamma_0 \gamma_2$, then

$$\psi^\dagger \gamma_0 \zeta + \zeta^\dagger \gamma_0 \psi = \psi^\dagger C_E \chi^* + \chi^T C_E \psi. \quad (B11)$$

In this case $C_E^\dagger = C_E$.

Similarly, we can consider the vector operators $\bar{\psi} \Gamma^{Rs} \zeta$ which transforms under the Lorentz group as the $R$ irreducible representation with a definite parity. Here $R = \{S, P, V, A, T\}$ are the scalar, pseudoscalar, vector, axialvector and tensor representations, the corresponding $\Gamma^{Rs}$ matrices are $\Gamma^{Rs} = \{1, \gamma_5, \gamma_\mu \gamma_5, \sigma_{\mu \nu}\}$, respectively. We can rewrite them with the $\chi$ field as

$$\psi^\dagger \gamma_0 \Gamma^{Rs} \zeta = \psi^\dagger \bar{\Gamma}^{Rs} \gamma_0 C \gamma_0 \chi^* = \psi^\dagger \bar{\Gamma}^{Rs} C_E \chi^*, \quad (B12)$$

where $\bar{\Gamma}^{Rs} = \gamma_0 \Gamma^{Rs} \gamma_0$. 