Current in a spin-orbit-coupling system

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The formulae of particle current as well as spin- and angular momentum currents are studied for most spin-orbit coupling (SOC) systems. It is shown that the conventional expression of currents in some literatures are not complete for some SOC systems. The particle current in Dresselhaus system must have extra terms in addition to the conventional one, but no extra term for Rashba, Luttinger model. Furthermore, we also prove that the extra terms of total angular momentum appear in Rashba current in addition to conventional one.

I. INTRODUCTION

In quantum mechanics calculating current is crucial to applications. A widely accepted approach is using the correspondence regulation from classical to quantum mechanics. For example, the formula of charge current density is introduced as

\[
\vec{j}(\vec{r}, t) = e \text{Re}\{\sum_n \rho_n \Psi_n^\dagger(\vec{r}, t) \vec{v} \Psi_n(\vec{r}, t)\},
\]

where \(e\) is an electron (or hole) charge, \(\vec{v}\) the velocity operator defined by \(\vec{v} = \frac{1}{\hbar}[\vec{r}, \hat{H}]\), \(\hat{H}\) is the Hamiltonian, \(\Psi_n(\vec{r}, t)\) is the wave function of an admissible state \(|\Psi_n\rangle\) of the Hamiltonian system and \(\rho_n\) is the probability of \(\Psi_n(\vec{r}, t)\) appeared in the quantum mixed state. For example, if the system is in equilibrium, the distribution of the probability is \(\rho_n = e^{-\beta E_n}\), where \(\beta = 1/(kT)\), \(E_n\) the energy of state \(\Psi_n(\vec{r}, t)\). Summation \(n\) is taken to all admissible states of Hamiltonian. Further, the definition of charge current density is
extended to define some other currents like spin current density

\[ j_y^z = \frac{1}{2} \text{Re} \left\{ \sum_n \rho_n \Psi_n^\dagger(\vec{r}, t) \left( \hat{s} \hat{v}_y + \hat{v}_y \hat{s}^z \right) \Psi_n(\vec{r}, t) \right\} \]

and orbital angular momentum current density

\[ \vec{J}^z = \frac{1}{2} \text{Re} \left\{ \sum_n \rho_n \Psi_n^\dagger(\vec{r}, t) \left[ \hat{l} \hat{v}^\dagger + \hat{v}^\dagger \hat{l}^z \right] \Psi_n(\vec{r}, t) \right\} \]

where \( \hat{l}^z = (\vec{r} \times \vec{p})_z \) is orbital angular momentum along \( z \) direction. And symmetrization in the order of \( \hat{v}^\dagger \) and \( \hat{s}^z \) (or \( \hat{l}^z \)) has been applied. The above conventional formula have been widely accepted as natural definition of currents in literatures, especially in the ones of semiconductor systems including spin-orbit-coupling (SOC).

In this paper, we will prove that the conventional formula of particle current are not generally correct. They can not satisfy the continuity equation for some systems. Based on the continuity equation,[9] we start our calculation from \( \partial w_A(\vec{r}, t)/\partial t \) for observable quantity \( A \), as will be illustrated in Sec.2, and utilize the continuity equation to obtain the definition of \( A \)-current \( \vec{j}_A \). We can find in the case like 2d Rashba, 3d Luttinger and the spin independent Hamiltonian \( \hat{H} = \frac{1}{2m^*} \vec{p}^2 + V(\vec{r}) \), where \( V(\vec{r}) \) can be an arbitrary position dependent potential, the conventional particle current formula are still corrected. However, we prove that the particle current will have a non-trivial additional term except conventional one in some semiconductor, such as the system with Dresselhaus SOC and other model Hamiltonians including the terms \( \vec{p}^n \) of momentum operator with power \( n > 2 \). We will derive the expressions of the extra term for these specific systems.

This paper will focus on some systems with spin-orbit-coupling such as 2d Rashba[1], 3d Luttinger[2] as well as Dresselhaus systems[3] that attracted great interest since the experimental demonstration [4-6] of spin hall effect in some semiconductor materials. Our study in this letter is based on non-relativistic quantum mechanics.
II. PARTICLE CURRENT FOR SOME COMPLEX SYSTEMS

In general, we study an observable quantity $A$. From quantum mechanics, $A$ corresponds to a Hermitian operator $\hat{A}$, then

$$\langle A \rangle = Tr\{\hat{A}\hat{\rho}\} = Tr\{\hat{A}\Psi_n\rho_n\Psi_n\rangle\}
$$

$$= \sum_n \rho_n\langle \Psi_n|\hat{A}|\Psi_n\rangle = \sum_n \rho_n \int d\vec{r}'\langle \Psi_n|\vec{r}'\rangle\langle \vec{r}'|\hat{A}|\Psi_n\rangle
$$

$$= \sum_n \rho_n \int d\vec{r}'\Psi_n^\dagger(\vec{r}',t)\hat{A}\Psi_n(\vec{r}',t),
$$

where we have used $\langle \vec{r}'|\hat{A}|\vec{r}''\rangle = \hat{A}(\vec{r}')\delta(\vec{r}' - \vec{r}'')$ which is diagonal in position representation for the most of cases in quantum mechanics, $\hat{A}(\vec{r}')$ is the operator $\hat{A}$ in position representation and simply expressed by $\hat{A}$. $\hat{\rho}(= \sum_n |\Psi_n\rangle\rho_n\langle \Psi_n|)$ is the density matrix of a mixed state, $\rho_n (\in [0,1])$ the probability of $|\Psi_n\rangle$ appeared in the mixed state and $\sum_n \rho_n = 1$. $|\Psi_n\rangle$ is assumed here to satisfy the time dependent Schrödinger equation $i\hbar\partial(\Psi_n)/\partial t = \hat{H}|\Psi_n\rangle$.

Since $\langle A \rangle$ is real, so the above equation is equivalent to

$$\langle A \rangle \equiv Re\sum_n \rho_n \int d\vec{r}'\langle \Psi_n|\vec{r}'\rangle\langle \vec{r}'|\hat{A}|\Psi_n\rangle = \int w_A(\vec{r}',t) d\vec{r}' \tag{1}
$$

and

$$w_A(\vec{r}',t) = \sum_n \rho_n Re\langle \Psi_n|\vec{r}'\rangle\langle \vec{r}'|\hat{A}|\Psi_n\rangle
$$

$$= \sum_n \rho_n Re\left\{\Psi_n^\dagger(\vec{r}',t)\hat{A}\Psi_n(\vec{r}',t)\right\},
$$

$$w_A(\vec{r}',t) = \sum_n \rho_n w_A^{(n)}(\vec{r}',t),
$$

$$w_A^{(n)}(\vec{r}',t) = Re\left\{\Psi_n^\dagger(\vec{r}',t)\hat{A}\Psi_n(\vec{r}',t)\right\}.
$$

Where $w_A(\vec{r}',t)$ is defined as the density of quantity $\langle A \rangle$ at time $t$ and position $\vec{r}'$. When $[\hat{A},\hat{H}] = 0$,

$$\frac{\partial \langle A \rangle}{\partial t} = \frac{\partial}{\partial t}\sum_n \rho_n Re\langle \Psi_n|\hat{A}|\Psi_n\rangle
$$

$$= \frac{1}{\hbar}\sum_n \rho_n Re\langle \Psi_n|\hat{A}\hat{H} - \hat{H}\hat{A}|\Psi_n\rangle = 0
$$
A is a conserved quantity, therefore it yields a continuity equation:

$$\frac{\partial w_A(\vec{r}, t)}{\partial t} = -\nabla \cdot \vec{j}_A.$$ 

Particle number is one of conserved quantity, here it is relevant to choose $A = NI$, $I$ is an identity operator. In general, one define a conserved current related to mechanical quantity $A$ from above continuity equation. Now the key point is to find out the calculation formula of current $\vec{j}_A$ for each specific Hamiltonian. When $A$ is relevant to the total particle number $N$, $A = NI$, $I$ is identity operator, it yields a definition of particle current $\vec{j}_n$. Total number of charges $eN$ is also a conserved quantity, the relevant operator $A = eNI$ and conserved charge current $\vec{j}_e = e\vec{j}$. In general, $A$ could be others, so one can define other currents.

In this note, we only consider mixed state with time independent probability $\rho_n$. $A$ is also studied within time independent.

For the particle with spin $s = 1/2$, the state $|\Psi\rangle$ can be described by

$$|\Psi\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix},$$

$$\Psi(\vec{r}, t) = \langle \vec{r} | \Psi \rangle = \begin{pmatrix} \langle \vec{r} | \psi_1 \rangle \\ \langle \vec{r} | \psi_2 \rangle \end{pmatrix} = \begin{pmatrix} \psi_1(\vec{r}, t) \\ \psi_2(\vec{r}, t) \end{pmatrix},$$

$$\Psi^\dagger(\vec{r}, t) = \langle \Psi | \vec{r} \rangle = \begin{pmatrix} \psi^*_1(\vec{r}, t) & \psi^*_2(\vec{r}, t) \end{pmatrix},$$

where $\psi^*_i(\vec{r}, t)$ is complex conjugation of $\psi_i(\vec{r}, t)$ ($i = 1, 2$). The Schrödinger (or Pauli) equation is

$$\frac{\partial}{\partial t} |\Psi\rangle = \frac{1}{i\hbar} \hat{H} |\Psi\rangle,$$

where Hamiltonian $\hat{H}$ is a $2 \times 2$ matrix. Since now, for simplifying notation, we will use $\Psi$ to denote $\Psi(\vec{r}, t), \Psi^\dagger$ to $\Psi^\dagger(\vec{r}, t)$.

The time evolution of quantity $w^{(n)}_A(\vec{r}, t)$ is

$$\frac{\partial w^{(n)}_A(\vec{r}, t)}{\partial t} = Re \left\{ \left( \frac{\partial}{\partial t} \Psi^\dagger_n \right) \hat{A} \Psi_n + \Psi^\dagger_n \hat{A} \frac{\partial}{\partial t} \Psi_n \right\}$$

In equation (2), the operator $\hat{A}$ has been represented in position space. Meanwhile, it must bear in mind that $\Psi_n$ and $\Psi^\dagger_n$ are the time and position dependent wave function. When $\hat{A} = \hat{I}_2$ (a $2 \times 2$ identity matrix),

$$w(\vec{r}, t) = n(\vec{r}, t) = Re \sum_n \langle \vec{r} | \Psi_n \rangle \rho_n \langle \Psi_n | \vec{r} \rangle$$

$$\equiv \sum_n \rho_n \Psi^\dagger_n \Psi_n,$$
\( n(\vec{r}, t) \) is the particle density, so the following continuity equation defines the particle current density \( \vec{j} \).

\[
\frac{\partial n(\vec{r}, t)}{\partial t} = -\nabla \cdot \vec{j}(\vec{r}, t).
\]

It describes the conservation of particle number. For a time independent Hamiltonian, the wave function \( \Psi_n(t) = \exp(-iE_n t/\hbar)\Psi_n(0), \) \( \hat{H}\Psi_n(0) = E_n\Psi_n(0) \). Therefore we have \( \partial n(r, t)/\partial t = 0 \). It reduces to \( \nabla \cdot \vec{j}(\vec{r}, t) = 0 \).

For many Hamiltonians without spin-orbit-coupling, \( \hat{H} = \hat{H}_0 + V(\vec{r}), \) \( \hat{H}_0 = \hat{p}^2/2m^* = -(\hbar^2/2m^*)\nabla^2 \). In terms of the equality \( \hat{H}_0 = -\frac{1}{2}\nabla \cdot [\vec{r}, \hat{H}_0] = -\frac{1}{2}\nabla \cdot [\vec{r}, \hat{H}] \), equation (1) can be easily calculated as

\[
\frac{\partial n(\vec{r}, t)}{\partial t} = Re \left\{ \left( \frac{1}{i\hbar}\hat{H}_0\Psi_n \right)^\dagger \Psi_n + \Psi_n^\dagger \frac{1}{i\hbar}\hat{H}_0\Psi_n \right\} \\
+ Re \left\{ \left( \frac{1}{i\hbar}V(\vec{r})\Psi_n \right)^\dagger \Psi_n + \Psi_n^\dagger \frac{1}{i\hbar}V(\vec{r})\Psi_n \right\} \\
= -\frac{1}{2}Re \left\{ \left( \nabla \cdot \frac{1}{i\hbar}[\vec{r}, \hat{H}_0]\Psi_n \right)^\dagger \Psi_n + \Psi_n^\dagger (\nabla \cdot \frac{1}{i\hbar}[\vec{r}, \hat{H}_0]\Psi_n) \right\} \\
= -\frac{1}{2}Re \nabla \cdot \left\{ \left( \frac{1}{i\hbar}[\vec{r}, \hat{H}_0]\Psi_n \right)^\dagger \Psi_n + \Psi_n^\dagger (\frac{1}{i\hbar}[\vec{r}, \hat{H}_0]\Psi_n) \right\} \\
= -\nabla \cdot Re \left\{ \Psi_n^\dagger \left( \frac{1}{i\hbar}[\vec{r}, \hat{H}_0]\Psi_n \right) \right\} \equiv -\nabla \cdot Re \left\{ \Psi_n^\dagger \left( \frac{1}{i\hbar}[\vec{r}, \hat{H}]\Psi_n \right) \right\} \quad (3)
\]

In equation (3), we have used the real property of potential \( V(\vec{r}) \) that results \( (\frac{1}{i\hbar}V(\vec{r})\Psi_n)^\dagger \Psi_n + \Psi_n^\dagger \frac{1}{i\hbar}V(\vec{r})\Psi_n = 0 \). Thus we obtain the standard continuity equation and the formula of particle current

\[
\frac{\partial n(\vec{r}, t)}{\partial t} = -\nabla \cdot Re \left\{ \sum_n \rho_n\Psi_n^\dagger \left( \frac{1}{i\hbar}[\vec{r}, \hat{H}]\Psi_n \right) \right\},
\]

\[
\vec{j}(\vec{r}, t) = Re \left\{ \sum_n \rho_n\Psi_n^\dagger (\vec{v}\Psi_n) \right\}. \quad (4)
\]

Equation (4) is called as "conventional formula". The formula is general for any potential \( V(\vec{r}) \) in Hamiltonian \( \hat{H} \) if the potential is position dependent only. However, if potential \( V \) contains momentum operator like the case appeared in some effective Hamiltonians of semiconductors, it will be different and the above current formula may need to be confirmed or to be modified to including some extra terms. In recent literatures, ones usually extend
it to define other kinds of conventional current like spin current, \( \hat{A} = \hat{S} = \frac{\hbar}{2} \hat{\sigma} \),

\[
\vec{j}(\vec{r}, t) = \frac{1}{2} Re \left\{ \sum_n \rho_n \Psi_n^\dagger \left( \hat{\sigma} \hat{S} + \hat{\sigma} \hat{S}^\dagger \right) \Psi_n \right\}.
\]  

(5)

, but not conserved, where symmetric presentation is applied for two non-commuting operators \( \hat{\sigma} \) and \( \hat{S} \).

Now we discuss the particle current in some complex systems with SOC in which the momentum operator appears in "potential" of Hamiltonian. First, we study the current formula for the Dresselhaus Hamiltonian that contains triple power of momentum operator.

The Hamiltonian is

\[
\hat{H} = \hat{H}_0 + \hat{H}_D
\]

\[
\hat{H}_0 = \frac{1}{2m^*} \vec{p}^2 + V(\vec{r})
\]

\[
\hat{H}_D = \eta[p_x (p_y^2 - p_z^2) \sigma_x + p_y (p_z^2 - p_x^2) \sigma_y + p_z (p_x^2 - p_y^2) \sigma_z],
\]

where \( \{\sigma_i : i = x, y, z\} \) are Pauli matrices. The question is whether the particle current can still be calculated by conventional formula (4). We rewrite the Hamiltonian as

\[
\hat{H} = \hat{H}_0 + \hat{H}_D = -\frac{1}{2} \nabla \cdot [\vec{r}, \hat{H}_0] + V(\vec{r}) + \hat{H}_D.
\]

It yields

\[
\frac{\partial n(\vec{r}, t)}{\partial t} = Re \sum_n \rho_n \left[ \Psi_n^\dagger \left( \frac{1}{i\hbar} \hat{H}_0 \Psi_n \right) + \left( \frac{1}{i\hbar} \hat{H}_0 \Psi_n \right)^\dagger \Psi_n \right] \]

\[= Re \sum_n \rho_n \left[ \Psi_n^\dagger \left( \frac{1}{i\hbar} \hat{H}_0 \Psi_n \right) + \left( \frac{1}{i\hbar} \hat{H}_0 \Psi_n \right)^\dagger \Psi_n \right] \]

\[+ Re \sum_n \rho_n \left[ \Psi_n^\dagger \left( \frac{1}{i\hbar} \hat{H}_D \Psi_n \right) + \left( \frac{1}{i\hbar} \hat{H}_D \Psi_n \right)^\dagger \Psi_n \right].
\]

(6)

Similar to the deduction of equation (3) for the terms having \( \hat{H}_0 \) in right side of above equation, we directly have

\[
\frac{\partial n(\vec{r}, t)}{\partial t} = -\nabla \cdot Re \sum_n \rho_n \left[ \Psi_n^\dagger \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_0] \Psi_n \right) \right] \]

\[+ Re \sum_n \rho_n \left[ \Psi_n^\dagger \left( \frac{1}{i\hbar} \hat{H}_D \Psi_n \right) + \left( \frac{1}{i\hbar} \hat{H}_D \Psi_n \right)^\dagger \Psi_n \right].
\]  

(6)
If the second term in right side of above equation equals to $-\nabla \cdot \text{Re} \sum \rho_n \left[ \Psi_n^\dagger \left( \frac{1}{\hbar} [r, \hat{H}_D] \Psi_n \right) \right]$, the final formula of particle current is just the conventional one. But, in deed, they are different. Appendix (A) gives the detail proof to show that an extra term must exist in addition to the conventional current formula. The continuity equation should be

$$\frac{\partial n(\vec{r}, t)}{\partial t} = -\nabla \cdot (\vec{j}_{\text{conv}} + \vec{j}_{\text{extra}}),$$

$$\vec{j}_{\text{conv}} = \text{Re} \left\{ \sum_n \rho_n \Psi_n^\dagger \left( \frac{1}{\hbar} [\vec{r}, \hat{H}] \Psi_n \right) \right\},$$

$$\vec{j}_{\text{extra}} = \text{Re} \sum_n \rho_n \vec{j}_{\text{extra}}^{(n)}$$

(7)

where $\vec{j}_{\text{conv}}$ is the conventional formula of particle current density. In appendix A, we also prove that the term $\vec{j}_{\text{extra}}$ is real and position dependent if the position dependent potential $V(\vec{r})$ is included in Hamiltonian. Therefore, $\nabla \cdot \vec{j}_{\text{extra}} \neq 0$, so the nontrivial extra term of particle current do appear in Dresselhaus system.

We also checked the formulae of particle current for 2d Rashba $(\hat{H} = \hat{H}_0 + \alpha \vec{Z} \cdot (\vec{p} \times \vec{s}))$ and 3d Luttinger $(\hat{H} = \hat{H}_0 \hat{I}_4 + \lambda (\vec{p} \cdot \vec{s})^2)$ model Hamiltonians, where $\vec{s}$ is the $3/2$ angular momentum operator and $\hat{I}_4$ is a $4 \times 4$ identity matrix. It is found that no extra term of particle current appears for these two models. The main source to induce an extra term of particle current for Dresselhaus Hamiltonian seems from the triple power of momentum operator in its Hamiltonian. Indeed, in general, if the Hamiltonian includes momentum operator with order higher than power 2, the extra term of particle current will appear. For Rashba and Luttinger models, the highest power of momentum operator is 2, but Dresselhaus is 3. So this is the reason why the extra term of particle current appears in Dresselhaus, but not in Rashba and Luttinger systems. In section 5, we will show that from an extended Noether’s theorem.
For showing such argument more clearly, we study a simple \( p^4 \) model Hamiltonian without SOC:

\[
\hat{H} = \hat{H}_0 + \hat{H}_I + V(r),
\]
\[
\hat{H}_I = \beta \hat{p}^4, \tag{8}
\]

where \( \beta \) is a coupling constant. In section 4, we will show that from the extended Noether’s theorem

\[
\frac{\partial n(\vec{r}, t)}{\partial t} = -\nabla \cdot (\vec{j}_{\text{conv}} + \vec{j}_{\text{extra}}),
\]
\[
\vec{j}_{\text{extra}} = 2\beta \hbar^3 \sum_n \rho_n \text{Im}\{ \nabla (\Psi_n^\dagger \nabla^2 \Psi_n) \} \tag{9}
\]

For nonhomogeneous system, \( \Psi_n^\dagger \nabla^2 \Psi_n \) is complex including both real and imaginary parts in general. \( \vec{j}_{\text{conv}} \) and \( \vec{j}_{\text{extra}} \) in equation (9) are position dependent, \( \nabla \cdot \vec{j}_{\text{conv}} \) and \( \nabla \cdot \vec{j}_{\text{extra}} \) can be nonzero in general case. Therefore, they do not satisfy the continuity equation separately. But the combination of them gives the following continuity equation for any stead state \( \left( \frac{\partial n(\vec{r}, t)}{\partial t} = 0 \right) \).

\[
\nabla \cdot \vec{j} = \nabla \cdot \vec{j}_{\text{conv}} + \nabla \cdot \vec{j}_{\text{extra}} = 0 \tag{10}
\]

In this example, we clearly show that the extra term of particle current comes from the part of Hamiltonian with momentum \( \vec{p}^n \) \( (n = 4) \).

III. SPIN- AND TOTAL ANGULAR MOMENTUM CURRENT

When the operator \( \hat{A} \) in expression (1) is spin \( \hat{S}^z \) or the total angular momentum (TAM), \( \hat{L}^z = \hat{L}^z + \hat{S}^z \), equation (2) defines the time evolution of spin-density or the TAM density that could relate to the spin-current density or total angular momentum current density (TAMCD) both often appeared in recent literatures [7-10]. For the Rashba Hamiltonian, \( [\hat{S}^z, \hat{H}_R]_w \neq 0 \), the spin \( \hat{S}^z \) is not a conserved quantity. Ref.[8,9] have shown that

\[
\frac{\partial w^{(n)}_{S^z}(\vec{r}, t)}{\partial t} = \Re \frac{\partial}{\partial t} \left[ \Psi_n^\dagger \hat{S}^z \Psi_n \right] = -\nabla \cdot \Re \{ \Psi_n^\dagger \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}] \hat{S}^z \Psi_n \right) \}
\]
\[
+ \Re \{ \Psi_n^\dagger \left( (\alpha \vec{p} \times \vec{\sigma}) \times \vec{\sigma} \right) \Psi_n \}
\]
\[
= -\nabla \cdot \vec{j}^{(n)}_S + \vec{j}^{(n)}_\omega \tag{11}
\]
\[ \vec{j}_S = \sum_n \rho_n \vec{j}_S^{(n)} \] is so called "spin current" widely in recent literatures,

\[ \vec{j}_\omega = \sum_n \rho_n \vec{j}_\omega^{(n)} = Re \left\{ \sum_n \rho_n \Psi_n^\dagger \left( (\alpha(\vec{p} \times \vec{z}) \times \vec{\sigma}) \Psi_n \right) \right\} \]

which is called as spin "torque". The conventional spin current density \( \vec{j}_S \) just partly contribute to the time evolution of spin density \( w_{S_z}(\vec{r},t)(= \sum_n \rho_n w_{S_z}^{(n)}(\vec{r},t)) \). Even \( \partial w_{S_z}(\vec{r},t)/\partial t = 0 \) for a steady state, the conventional spin current density \( \vec{j}_S \) is not conserved ( \( \nabla \cdot \vec{j}_{\text{conv}} \neq 0 \) ) and satisfies the equation \( \nabla \cdot \vec{j}_{\text{conv}} = \vec{j}_\omega \).

Now we study the current of TAM \( J^z = l^z + S^z \) in Rashba system, where \( l^z \) is the \( z \) component of orbital angular momentum. The case is different from particle current. In case of particle current, the number of particles is conserved, but relative operator of particle number is momentum operator independent, \( A = N\hat{I} \). Here \( \hat{J}^z \) is also a conserved quantity since \( [\hat{J}^z, \hat{H}]_\text{-} = 0 \) for Rashba Hamiltonian, but it contains the momentum operator. So we would check whether its related current has extra term even for Rashba Hamiltonian without higher than 2 power of momentum operator. After long careful calculations (see appendix C), we obtain

\[ \partial_t \sum_n \rho_n w_{S_z}^{(n)}(\vec{r},t) = \partial_t \sum_n \rho_n Re[\Psi_n^\dagger \hat{J}^z \Psi_n] = - \nabla \cdot \vec{j}_L, \]

where

\[ \vec{j}_L = Re \sum_n \rho_n \left\{ \Psi_n^\dagger(\vec{r},t) \frac{1}{i\hbar} [\vec{r}, \hat{H}] \hat{J}^z \Psi_n \right\} \]

\[ + \frac{1}{2} \frac{\hbar^2}{2m^*} \sum_n \rho_n \left\{ \Psi_n^\dagger (e_x \nabla_y - e_y \nabla_x) \Psi_n + ((e_x \nabla_y - e_y \nabla_x) \Psi_n)^\dagger \Psi_n \right\} \]

\[ + \frac{1}{2} \frac{\hbar^2}{2m^*} \sum_n \rho_n \{ (\nabla \Psi_n^\dagger) (x \nabla_y - y \nabla_x) \Psi_n + ((x \nabla_y - y \nabla_x) \Psi_n)^\dagger \nabla \Psi_n \}

\[ + \Psi_n^\dagger (x \nabla_y - y \nabla_x) \nabla \Psi_n + ((x \nabla_y - y \nabla_x) \nabla \Psi_n)^\dagger \Psi_n \} \]

The conventional definition for \( \hat{J}^z \) current density in literatures is defined by

\[ \vec{j}_{\text{conv}} = \frac{1}{2} \sum_n \rho_n \left\{ \Psi_n^\dagger \left[ \frac{1}{i\hbar} [\vec{r}, \hat{H}] \hat{J}^z + \hat{J}^z \frac{1}{i\hbar} [\vec{r}, \hat{H}] \right] \Psi_n \right\}. \]

After taking the real part, it becomes:

\[ \vec{j}_{\text{conv}} = \frac{1}{2} Re \sum_n \rho_n \left\{ \Psi_n^\dagger \left[ \frac{1}{i\hbar} [\vec{r}, \hat{H}] \hat{J}^z + \hat{J}^z \frac{1}{i\hbar} [\vec{r}, \hat{H}] \right] \Psi_n \right\}. \]
Now we use the following relation
\[
\left[J_z, \frac{1}{i\hbar} [\vec{r}, \hat{H}] \right] = i\hbar [(p_y/m^* - \alpha \sigma_x) \vec{e}_x - (p_x/m^* + \alpha \sigma_y) \vec{e}_y]
\]
then change the form of \( \overrightarrow{j}_{\text{conv}} \) into
\[
\overrightarrow{j}_{\text{conv}} = \text{Re} \sum_n \rho_n \left\{ \Psi_n^\dagger \frac{1}{i\hbar} [\vec{r}, \hat{H}] \vec{J} \Psi_n \right\} + \frac{1}{2} \text{Re} \sum_n \rho_n \left\{ i\hbar \Psi_n^\dagger [(p_y/m^* - \alpha \sigma_x) \vec{e}_x - (p_x/m^* + \alpha \sigma_y) \vec{e}_y] \Psi_n \right\}
\]
\[= \text{Re} \sum_n \rho_n \left\{ \Psi_n^\dagger \frac{1}{i\hbar} [\vec{r}, \hat{H}] \vec{J} \Psi_n \right\} + \frac{1}{2} \text{Re} \sum_n \rho_n \left\{ i\hbar \Psi_n^\dagger \frac{p_y}{m^*} \vec{e}_x - \frac{p_x}{m^*} \vec{e}_y \right\} \Psi_n \] \hspace{1cm} (13)

In the last step of above equation, we have used that \( \Psi_n^\dagger \sigma \Psi_n \) are real since \( \{\sigma_x, \sigma_y, \sigma_z\} \) are Hermitian and trace has been taken in spin space (but be care for that no trace taken in position space). In appendix C, we obtain
\[
\overrightarrow{j}_{\text{extra}} = \overrightarrow{j}_L - \overrightarrow{j}_{\text{conv}} = -\frac{1}{4m^*} \sum_n \rho_n l_z \vec{p} (\Psi_n^\dagger \Psi_n) \neq 0 \hspace{1cm} (14)
\]
Thus, we conclude that conventional formula of TAMCD, \( \overrightarrow{j}_{\text{conv}} \), is not complete. It must have an extra term even for Rashba system. Therefore, we suggest that one should be careful in using the conventional current formula and do the deduction of formula carefully for every new type Hamiltonian, particularly for some Hamiltonians with SOC in semiconductors. In this note, the TAM, \( \langle J_z^L \rangle \), is not a conserved quantity for Dresselhaus Hamiltonian since \([J_z^L, \hat{H}_{\text{Dresselhaus}}] \neq 0 \). So, we did not attempt to check the existence of its extra part.

IV. NOETHER’S THEOREM AND EXTRA TERM OF CURRENT

Based on time dependent Schrödinger equations and the particle number conservation (see section 2), we have derived new expressions of conserved particle current for some Hamiltonians, which include term that the order of momentum operators in it is higher than 2. In parallel to that, in fact, expressions of particle current can also be derived by Noether’s theorem in field theory from \( U(1) \) gauge invariance of Lagrangian. However, the usual Noether’s theorem only consider systems with term \( p^2/2m \) in the momentum dependent part of Hamiltonian. So the Lagrangian of the system is only a functional of field functions and their first order derivatives, \( \phi(x_\mu), \phi^\dagger(x_\mu), \partial_\mu \phi(x_\mu) \) and \( \partial_\mu \phi^\dagger(x_\mu) \), for a complex
scalar field, where $x_\mu = (t, x, y, z), \mu = 0, 1, 2, 3; (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_t, \partial_x, \partial_y, \partial_z)$. Simply, we denote this Lagrange as $\mathcal{L}(\phi, \phi^\dagger, \partial_\mu \phi, \partial_\mu \phi^\dagger)$. Let us briefly recall the usual Noether theorem. An action is defined by

$$S = \int dx^4 \mathcal{L}(\phi, \phi^\dagger, \partial_\mu \phi, \partial_\mu \phi^\dagger), \quad dx^4 = \prod_\mu dx_\mu.$$  

(15)

Where two variational functions $\delta \phi$ and $\delta \phi^\dagger$ are independent. According to the least action principle, $\delta S = 0$, which gives two conjugate dynamical equations of $\phi$ and $\phi^\dagger$. One is from the term of $\delta \phi$ and another from $\delta \phi^\dagger$. For a complex scalar field, the Lagrangian is usually given by

$$\mathcal{L}(\phi, \phi^\dagger, \partial_\mu \phi, \partial_\mu \phi^\dagger) = \phi^\dagger i \partial_0 \phi - \frac{1}{2m} \sum_{i=1,2,3} (\partial_i \phi)^\dagger (\partial_i \phi) - \phi^\dagger V(\{x_\mu\}) \phi,$$

where $\hbar = 1$. From equation (15), we have

$$\frac{\partial \mathcal{L}}{\partial \phi^\dagger} - \left( \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^\dagger)} \right) = 0 \rightarrow$$

$$i \partial_0 \phi + \frac{1}{2m} \partial_\mu^2 \phi - V(\{x_\mu\}) \phi = 0.$$

It is Schrödinger equation:

$$i \frac{\partial}{\partial t} \phi = H \phi,$$

$$H = -\frac{\nabla^2}{2m} + V(\{x_\mu\}).$$

Equation (15) also yields an equation:

$$\sum_{\mu=0,1,2,3} \partial_\mu F_\mu = 0.$$  

(16)

which defines a 4D current:

$$F_\mu = \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi \right) + \left( \phi \rightarrow \phi^\dagger \right)$$  

(17)
Input usual Lagrangian $\mathcal{L}$ into equation (17), we have

$$
F_0 = i\phi^\dagger \delta \phi,
$$

$$
F_i = -\frac{1}{2m}[ \left( \partial_t \phi \right)^\dagger \delta \phi + \left( \partial_t \phi \right) \delta \phi^\dagger ], \ i = 1, 2, 3.
$$

By $U(1)$ gauge transformation

$$
\phi \rightarrow e^{i\alpha} \phi \approx (1 + i\alpha) \phi = \phi + i\alpha \phi \rightarrow
$$

$$
\delta \phi = i\alpha \phi, \delta \phi^\dagger = -i\alpha \phi^\dagger,
$$

and due to $U(1)$ gauge invariance of the Hamiltonian, we obtain

$$
F_0 = -\alpha \phi^\dagger \phi
$$

$$
F_i = -\frac{i\alpha}{2m} \left\{ \left( \partial_i \phi \right)^\dagger \phi - \phi^\dagger \partial_i \phi \right\} = \frac{\alpha}{m} Re \phi^\dagger i \partial_i \phi
$$

$$
= -\alpha Re \left\{ \phi^\dagger \frac{1}{i} [r_i, H]_- \phi \right\} = -\alpha Re \left\{ \phi^\dagger v_i \phi \right\}, \ i = 1, 2, 3.
$$

We obtain the continuity equation of particle

$$
\frac{\partial \left( \phi^\dagger \phi \right)}{\partial t} + \vec{\nabla} \cdot Re \left\{ \phi^\dagger \frac{1}{i} [\vec{r}, H]_- \phi \right\} = 0.
$$

(18)

This is just the conventional expression of particle current.

$$
\vec{j}_{\text{conv}} = Re \left\{ \phi^\dagger \frac{1}{i} [\vec{r}, H]_- \phi \right\}
$$

(19)

For Rashba and Luttinger systems, the usual Noether’s theorem can be applied since the additional spin-orbit-coupling term in these two Hamiltonians are momentum operator $p$ or $p^2$ dependent. Therefore, it can be understood that why their expressions of the particle currents are also conventional. It is consistent with our conclusion. However, when we study the system like $H = p^2/2m + \beta p^4$ or $H_{\text{dresselhaus}}$, the terms of high order derivatives, inherited from Hamiltonian, should also be included in the Lagrangian: $\mathcal{L}(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi, \cdots; (\phi \rightarrow \phi^\dagger))$, thus the usual Noether’s theorem should be extended. According to variational method,

$$
\delta S = \int_{V_4} dx^4 \delta \mathcal{L}(\left[ \phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi, \cdots; (\phi \rightarrow \phi^\dagger) \right])
$$

$$
= \int_{V_4} dx^4 \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \phi)} \delta (\partial_\mu \partial_\nu \phi) + \cdots + (\phi \rightarrow \phi^\dagger) \right\}
$$

$$
= \int_{V_4} dx^4 \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi + \partial_\nu \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \phi)} \right) \delta \phi + \cdots + (\phi \rightarrow \phi^\dagger) \right\}
$$

$$
+ \int_{V_4} dx^4 \partial_\mu \left\{ \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi - \partial_\nu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \phi)} \right) \delta \phi + \cdots + (\phi \rightarrow \phi^\dagger) \right\}
$$
where $dx^4 = \prod dx_\mu$. Now the 4D current is defined by

$$F_\mu = \left(\frac{\partial L}{\partial (\partial_\mu \phi)} \delta \phi\right) - \partial_\nu \left(\frac{\partial L}{\partial (\partial_\mu \partial_\nu \phi)}\right) \delta \phi + \left(\frac{\partial L}{\partial (\partial_\mu \partial_\nu \phi)} \delta (\partial_\nu \phi)\right) + \cdots + (\phi \rightarrow \phi^\dagger) \quad \text{(20)}$$

and

$$\Delta L = \frac{\partial L}{\partial \phi} - \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \phi)}\right) + \partial_\nu \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \partial_\nu \phi)}\right) + \cdots + (\phi \rightarrow \phi^\dagger)$$

Thus

$$\delta S = \int_{V_4} dx^4 \Delta L \delta \phi + \int_{V_4} dx^4 \partial_\mu F_\mu = 0.$$ 

According to the least action principle, a suitable definition of Lagrangian is necessary for obtaining the correct dynamical equation of $\phi$ from $\Delta L = 0$ and conservation law from $\sum_\mu \partial_\mu F_\mu = 0$. Therefore, the expression of particle current $\vec{j}$ can be obtained from the invariance of $U(1)$ gauge transformation for such extended system:

$$j_i = \left(\frac{\partial L}{\partial (\partial_i \phi)} \delta \phi\right) - \partial_\nu \left(\frac{\partial L}{\partial (\partial_i \partial_\nu \phi)}\right) \delta \phi + \left(\frac{\partial L}{\partial (\partial_i \partial_\nu \phi)} \delta (\partial_\nu \phi)\right) + \cdots + (\phi \rightarrow \phi^\dagger) \quad \text{(21)}$$

Comparing equation (21) with (15), there are some extra terms $F^\text{extra}_\mu$:

$$F^\text{extra}_\mu = -\partial_\nu \left(\frac{\partial L}{\partial (\partial_\mu \partial_\nu \phi)}\right) \delta \phi + \left(\frac{\partial L}{\partial (\partial_\mu \partial_\nu \phi)} \delta (\partial_\nu \phi)\right) + \cdots + (\phi \rightarrow \phi^\dagger)$$

This is the reason why extra terms do exist in particle current for the $p^4$ model and $p^3$-Dresselhaus Hamiltonian. In this paper, as an example, we use the above extended Noether’s theorem to study the $p^4$ model system in detail only. Further study on Noether theorem for other systems must be interesting and it is ongoing.

The central point of Noether’s theorem is to find out a suitable functional expression of Lagrangian, then one can derive the time dependent Schrödinger equation and conservation
law. For $p^4$ mode, we choose the following symmetric Lagrangian

$$\mathcal{L}[\phi(x), \partial_\mu \phi(x), \partial_\mu^2 \phi(x), \partial_\mu \partial_\nu \phi(x), \partial_\mu \partial_\nu \partial_\rho \phi(x); (\phi \to \phi^\dagger)]$$

$$= \phi^\dagger (i\partial_0 \phi) - \frac{1}{2m} \left( \partial_\mu \phi \right)^\dagger \left( \partial_\mu \phi \right) - \frac{1}{5} \beta \cdot \left[ (\partial_\mu^2 \partial_\nu^2 \phi)^\dagger \phi - (\partial_\mu^2 \partial_\nu^\dagger \phi)^\dagger (\partial_\nu \phi) + (\partial_\mu^2 \phi)^\dagger (\partial_\nu^2 \phi) - (\partial_\nu \phi)^\dagger (\partial_\nu \partial_\mu^2 \phi) + \phi^\dagger (\partial_\nu \partial_\mu^2 \phi) \right]$$

After long algebra, but no difficulty, we obtain Schrödinger equation

$$i \frac{\partial}{\partial t} \phi = \left( \frac{p^2}{2m} + \beta p^4 \right) \phi,$$

and the following equation for 4D conserved current

$$F_\mu = \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\nu \phi) - (\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)}) \delta \phi \right.

\left. + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \phi)} \delta (\partial_\nu \partial_\mu \phi) - (\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\nu \partial_\mu \phi)}) \delta (\partial_\nu \phi) \right.

\left. + (\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\nu \partial_\mu \phi)}) \delta (\partial_\nu^2 \phi) + (\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\nu \partial_\mu \phi)}) \delta (\partial_\nu^2 \phi) \right.

\left. - (\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\nu^2 \phi)}) \delta (\partial_\nu \partial_\mu \phi) + (\partial_\mu^2 \frac{\partial \mathcal{L}}{\partial (\partial_\nu \partial_\mu \phi)}) \delta (\partial_\nu \phi) \right.

\left. - (\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\nu \partial_\mu \phi)}) \delta (\partial_\nu \partial_\mu \phi) \right] \left( \phi \to \phi^\dagger \right), \tag{23}$$

and conservation law $\partial_\mu F_\mu = 0$. It is an extension of the conventional expression $F_\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi + h.c.$ for usual Hamiltonian. Now applying $U(1)$ gauge symmetry of the Lagrangian to eq. (22), performing transformation: $\phi \to \phi + i\alpha \phi, \phi^\dagger \to \phi^\dagger - i\alpha \phi^\dagger$, where $\alpha$ is a small constant, we gain the expression of conserved particle current

$$\vec{j}_{\text{cons}} = \frac{-i}{2m} [\phi^\dagger (\nabla \phi) - (\nabla \phi)^\dagger \phi] + i2\beta [\phi^\dagger (\nabla \nabla^2 \phi) - (\nabla \nabla^2 \phi)^\dagger \phi]$$

$$- i \left( \frac{1}{5} \beta \right) [-5 (\nabla \nabla^2 \phi)^\dagger \phi + 5 \phi^\dagger (\nabla \nabla^2 \phi)]$$

$$+ 3 (\nabla \phi)^\dagger (\nabla^2 \phi) + 2 (\nabla \phi)^\dagger \cdot (\nabla \nabla \phi)$$

$$- 3 (\nabla^2 \phi)^\dagger (\nabla \phi) - 2 (\nabla \nabla \phi)^\dagger \cdot (\nabla \phi)] \tag{24}$$

which is essentially the same as eq. (9) in Section 2. We have proved that $\nabla \cdot (\vec{j}_{\text{cons}} - (\vec{j}_{\text{conv}} + \vec{j}_{\text{extra}})) = 0$. 
We also have proved that a suitable symmetric Lagrangian for $p^3$ Dresselhaus model can be found and correct Schrödinger equation and particle current are gained. The expression of particle current is essentially the same as the one in section 2 with a difference $\Delta \vec{j}_{\text{Dresselhaus}} = \nabla \times \vec{A}$, a rot vector, only, so $\nabla \cdot \left( \Delta \vec{j}_{\text{Dresselhaus}} \right) = 0$. Further studies on the extension of Noether’s theorem about the construction of Lagrangian for such unusual system, the multiple correspondence from Hamiltonian to Lagrangian and finding general approach is our next studies.

In summary: we have studied the conventional formula of particle density for some Hamiltonians such as Rashba, Luttinger, Dresselhaus Hamiltonian as well as a $p^4$ model Hamiltonian. It is shown that the conventional formula of particle current density is correct for Rashba and Luttinger Hamiltonians. But it lacks an additional term of current density $\vec{j}_{\text{extra}}$ for $p^3$ Dresselhaus Hamiltonian and $p^4$ one without SOC. The spin and TAM currents are also addressed for Rashba Hamiltonian. Some detailed proofs are presented in appendices to show the existence of extra terms of currents. The results point out that the conventional expression of TAMCD is not complete even for Rashba term. It against the conservation law of the total angular momentum if the extra term is neglected. In deed, the extra term saves the conservation of total angular momentum for Rashba system. In the final, a brief explanation of the extension of Noether’s theorem is presented. It just shows that the deviation of the expression of particle current from conventional formula derived by Noether’s theorem for some system with SOC is reasonable and no doubt. The general extension of Noether’s theorem is still not quite clear and in studying. However, our approach to derive the conserved currents is based on Hamiltonian and time dependent Schrödinger equation, our results are rigorous and unique up to a rot vector (divergence $=0$) without any doubt. A correct formula of particle current is important in studying transport problems in recent SOC systems.

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V. APPENDIX

A. Extra term of particle current for Dresselhaus system

In this part, we would show some details leading to the extra term in the current density for Dresselhaus Hamiltonian \( \hat{H} = \hat{H}_0 + \hat{H}_D \). Let’s begin from the equation of (6) in the body of the text

\[
\frac{\partial n}{\partial t} = \left( -\frac{1}{2\hbar} \nabla \cdot \left[ \vec{r}, \hat{H}_0 \right] \Psi_n \right)^\dagger \Psi_n - \Psi_n^\dagger \left( -\frac{1}{2\hbar} \nabla \cdot \left[ \vec{r}, \hat{H}_0 \right] \right) \Psi_n \\
+ \left( \frac{1}{i\hbar} \hat{H}_D \Psi_n(\vec{r}, t) \right)^\dagger \Psi_n(\vec{r}, t) + \Psi_n^\dagger \left( \frac{1}{i\hbar} \hat{H}_D \Psi_n(\vec{r}, t) \right).
\]

(A.1)

Now we focus on the deduction of the last two terms of Eq. (A.1). Considering SOC part of Hamiltonian

\[
\hat{H}_D = \eta \left[ p_x \left( p_y^2 - p_z^2 \right) \sigma_x + p_y \left( p_z^2 - p_x^2 \right) \sigma_y + p_z \left( p_x^2 - p_y^2 \right) \sigma_z \right],
\]

we obtain

\[
\left( \frac{1}{i\hbar} \left[ x, \hat{H}_D \right] + 2\eta \left( p_y p_x \sigma_y - p_x p_y \sigma_z \right) \right) = \eta \left( p_y^2 - p_z^2 \right) \sigma_x,
\]

(A.2)

\[
\left( \frac{1}{i\hbar} \left[ y, \hat{H}_D \right] + 2\eta \left( p_z p_y \sigma_z - p_y p_z \sigma_x \right) \right) = \eta \left( p_z^2 - p_x^2 \right) \sigma_y,
\]

(A.3)

\[
\left( \frac{1}{i\hbar} \left[ z, \hat{H}_D \right] + 2\eta \left( p_x p_z \sigma_x - p_y p_z \sigma_y \right) \right) = \eta \left( p_x^2 - p_y^2 \right) \sigma_z.
\]

(A.4)

\[
\hat{H}_D = \vec{p} \cdot \frac{1}{i\hbar} \left[ \vec{r}, \hat{H}_D \right] + 2\eta \left( p_y \sigma_y - p_z \sigma_z \right) p_x^2 \\
+ 2\eta \left( p_z \sigma_z - p_x \sigma_x \right) p_y^2 + 2\eta \left( p_x \sigma_x - p_y \sigma_y \right) p_z^2 \\
= \vec{p} \cdot \frac{1}{i\hbar} \left[ \vec{r}, \hat{H}_D \right] - 2\hat{H}_D.
\]

\[
\hat{H}_D = -\frac{1}{3} \nabla \cdot \left[ \vec{r}, \hat{H}_D \right]
\]
Then we have

\[
\left( \frac{1}{i\hbar} \hat{H}_D \Psi_n \right)^\dagger \Psi_n + \Psi_n^\dagger \left( \frac{1}{i\hbar} \hat{H}_D \Psi_n \right)
= -\frac{1}{3} \left\{ \left( \frac{1}{i\hbar} \nabla \cdot [\vec{r}, \hat{H}_D] \Psi_n \right)^\dagger \Psi_n + \Psi_n^\dagger \left( \frac{1}{i\hbar} \nabla \cdot [\vec{r}, \hat{H}_D] \Psi_n \right) \right\}
= -\frac{1}{3} \nabla \cdot \left\{ \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_D] \Psi_n \right)^\dagger \Psi_n + \Psi_n^\dagger \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_D] \Psi_n \right) \right\}
+ \frac{1}{3} \left\{ \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_D] \Psi_n \right)^\dagger \cdot (\nabla \Psi_n) + (\nabla \Psi_n)^\dagger \cdot \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_D] \Psi_n \right) \right\}
\]

Inserting above result into the equation (A.1), we obtain

\[
\frac{\partial n}{\partial t} = -\nabla \cdot Re \left\{ \Psi_n^\dagger \frac{1}{i\hbar} [\vec{r}, \hat{H}] \Psi_n \right\} + \frac{1}{3} \nabla \cdot Re \left\{ \Psi_n^\dagger \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_D] \Psi_n \right) \right\}
+ \frac{2}{3} Re \left\{ (\nabla \Psi_n)^\dagger \cdot \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_D] \Psi_n \right) \right\}
\]

Now we try to express \( \frac{2}{3} Re \left\{ (\nabla \Psi_n)^\dagger \cdot \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_D] \Psi_n \right) \right\} \) by \( \nabla \cdot j_{\text{add}} \), then the analytic formula of extra contribution to particle current can be obtained. By using (A.2), (A.2) and (A.4), \( \frac{1}{i\hbar} [\vec{r}, \hat{H}_D] \) can be substituted as following

\[
\frac{1}{3} \left\{ \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_D] \Psi_n \right)^\dagger \cdot \nabla \Psi_n + \Psi_n^\dagger \cdot \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_D] \Psi_n \right) \right\}
= -\frac{1}{3} \hbar^2 \eta \left\{ \left[ (\nabla_y^2 - \nabla_z^2) \sigma_x - 2\nabla_y \nabla_x \sigma_y + 2\nabla_z \nabla_x \sigma_z \right] \Psi_n \right\} \cdot \nabla \Psi_n
+ \frac{1}{3} \hbar^2 \eta \left\{ \left[ (\nabla_y^2 - \nabla_z^2) \sigma_x - 2\nabla_y \nabla_x \sigma_y + 2\nabla_z \nabla_x \sigma_z \right] \Psi_n \right\}
+ \frac{1}{3} \hbar^2 \eta \left\{ \left[ (\nabla_x^2 - \nabla_z^2) \sigma_y - 2\nabla_x \nabla_z \sigma_x + 2\nabla_y \nabla_z \sigma_x \right] \Psi_n \right\} \cdot \nabla \Psi_n
+ \frac{1}{3} \hbar^2 \eta \left\{ \left[ (\nabla_x^2 - \nabla_y^2) \sigma_z - 2\nabla_x \nabla_y \sigma_x + 2\nabla_z \nabla_y \sigma_x \right] \Psi_n \right\} \cdot \nabla \Psi_n
= -\frac{1}{3} \hbar^2 \eta \left\{ \left[ (\nabla_y^2 - \nabla_z^2) \sigma_x \Psi_n \right] \cdot \nabla \Psi_n + \nabla \Psi_n \cdot \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_D] \Psi_n \right) \right\}
\]

It can be changed to

\[
\frac{1}{3} \left\{ \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_D] \Psi_n \right)^\dagger \cdot \nabla \Psi_n + \nabla \Psi_n \cdot \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_D] \Psi_n \right) \right\}
= -\frac{1}{3} \hbar^2 \eta \left\{ \left( \nabla_y^2 \sigma_x \Psi_n \right) \cdot \nabla \Psi_n + \nabla \Psi_n \cdot \left( \nabla_x \sigma_x \Psi_n \right)
+ \left( 2\nabla_x \nabla_y \sigma_x \Psi_n \right) \cdot \nabla \Psi_n + \left( \nabla_x \sigma_x \Psi_n \right) \cdot \nabla \Psi_n
- \left( 2\nabla_x \nabla_z \sigma_x \Psi_n \right) \cdot \nabla \Psi_n - \left( \nabla_x \sigma_x \Psi_n \right) \cdot \nabla \Psi_n \right\} + C.P.
Where C.P. is cyclic permutation of index \{x, y, z\}. Above equation can be changed to

\[
\frac{1}{3} \left\{ \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_D] \Psi_n \right) \right\}^\dagger \cdot \nabla \Psi_n + \nabla \Psi_n \cdot \left( \frac{1}{i\hbar} [\vec{r}, \hat{D}_D] \Psi_n \right) \right\} \\
= -\frac{1}{3} \hbar^2 \eta \{ \nabla_y (\nabla_y \sigma_x \Psi_n)^\dagger \nabla_x \Psi_n + (\nabla_x \Psi_n)^\dagger (\nabla_y \sigma_x \Psi_n) \} \\
+ \nabla_x [(\nabla_y \sigma_x \Psi_n)^\dagger \nabla_y \Psi_n] \\
- \nabla_z [(\nabla_z \sigma_x \Psi_n)^\dagger \nabla_z \Psi_n + (\nabla_z \Psi_n)^\dagger (\nabla_z \sigma_x \Psi_n)] \\
- \nabla_x [(\nabla_z \sigma_x \Psi_n)^\dagger \nabla_z \Psi_n] + C.P.
\]

where

\[
K_x^{(n)} = \frac{1}{3} \hbar^2 \eta \{ (\nabla_y \sigma_x \Psi_n)^\dagger \nabla_y \Psi_n + (\nabla_y \sigma_x \Psi_n)^\dagger \nabla_z \Psi_n + (\nabla_z \Psi_n)^\dagger (\nabla_y \sigma_x \Psi_n) \\
- (\nabla_z \sigma_x \Psi_n)^\dagger \nabla_z \Psi_n - (\nabla_z \sigma_x \Psi_n)^\dagger \nabla_y \Psi_n - (\nabla_y \Psi_n)^\dagger (\nabla_y \sigma_x \Psi_n) \}
\]

\[
K_y^{(n)} = \frac{1}{3} \hbar^2 \eta \{ (\nabla_z \sigma_y \Psi_n)^\dagger \nabla_z \Psi_n + (\nabla_y \sigma_y \Psi_n)^\dagger \nabla_x \Psi_n + (\nabla_x \Psi_n)^\dagger (\nabla_y \sigma_y \Psi_n) \\
- (\nabla_x \sigma_y \Psi_n)^\dagger \nabla_x \Psi_n - (\nabla_y \sigma_y \Psi_n)^\dagger \nabla_z \Psi_n - (\nabla_x \Psi_n)^\dagger (\nabla_y \sigma_y \Psi_n) \}
\]

\[
K_z^{(n)} = \frac{1}{3} \hbar^2 \eta \{ (\nabla_x \sigma_z \Psi_n)^\dagger \nabla_x \Psi_n + (\nabla_z \sigma_y \Psi_n)^\dagger \nabla_y \Psi_n + (\nabla_y \Psi_n)^\dagger (\nabla_x \sigma_z \Psi_n) \\
- (\nabla_y \sigma_z \Psi_n)^\dagger \nabla_y \Psi_n - (\nabla_z \sigma_x \Psi_n)^\dagger \nabla_z \Psi_n - (\nabla_x \Psi_n)^\dagger (\nabla_x \sigma_z \Psi_n) \}
\]

So

\[
\vec{j}^{(n)} = \vec{j}_{\text{conv}}^{(n)} + \vec{j}_{\text{conv}}^{(n)} + \vec{K}^{(n)} = \vec{j}_{\text{conv}}^{(n)} + \vec{j}_{\text{extra}}^{(n)} \quad (A.5)
\]

where

\[
\vec{j}_{\text{conv}}^{(n)} = Re \left\{ \Psi_n^\dagger \frac{1}{i\hbar} [\vec{r}, \hat{H}] \Psi_n \right\} \\
\vec{j}_{\text{conv}}^{(n)} = -\frac{1}{3} Re \left\{ \Psi_n^\dagger (\frac{1}{i\hbar} [\vec{r}, \hat{D}_D] \Psi_n) \right\} \\
\vec{j}_{\text{extra}}^{(n)} = \vec{j}_{\text{conv}}^{(n)} + \vec{K}^{(n)}.
\]
And
\[
\vec{j}^{(n)}_x = -\frac{1}{3} \text{Re} \left\{ \Psi_n^\dagger \frac{1}{\sqrt{h}} [x, \hat{H}_D] \Psi_n \right\}
\]

\[
= -\frac{1}{6} \Psi_n^\dagger \left\{ \eta \left( (p_x^2 - p_y^2) \sigma_x - 2p_y p_x \sigma_y + 2p_z p_x \sigma_z \right) \Psi_n \right. \\
+ \left. \eta \left( (p_y^2 - p_z^2) \sigma_x - 2p_y p_x \sigma_y + 2p_z p_x \sigma_z \right) \Psi_n \right\}^\dagger \Psi_n
\]

So the \(x\) component of the extra particle current density is

\[
(\vec{j}^{(n)}_{\text{extra}})_x = K_x + \vec{j}^{(n)}_x
\]

\[
= \frac{1}{3} \hbar^2 \eta \left\{ (\nabla_y \sigma_x \Psi_n)^\dagger \nabla_y \Psi_n + (\nabla_x \sigma_z \Psi_n)^\dagger \nabla_z \Psi_n + (\nabla_z \sigma_y \Psi_n)^\dagger \nabla_z \Psi_n \right. \\
- \left. (\nabla_y \Psi_n)^\dagger (\nabla_x \sigma_y \Psi_n) - (\nabla_z \sigma_y \Psi_n)^\dagger \nabla_z \Psi_n - (\nabla_x \sigma_y \Psi_n)^\dagger \nabla_y \Psi_n \right\} \\
+ \frac{1}{6} \hbar^2 \eta \left\{ \Psi_n^\dagger \left[ \left( \nabla_y^2 - \nabla_z^2 \right) \sigma_x - 2 \nabla_y \nabla_z \sigma_y + 2 \nabla_z \nabla_x \sigma_y \right] \Psi_n \right. \\
+ \left. \left( \left( \nabla_y^2 - \nabla_z^2 \right) \sigma_x - 2 \nabla_y \nabla_x \sigma_y + 2 \nabla_z \nabla_x \sigma_y \right) \Psi_n \right\} \\
= \frac{1}{6} \hbar^2 \eta \left\{ (\nabla_y^2 \sigma_x \Psi_n)^\dagger \nabla_y \Psi_n + (\nabla_z^2 \sigma_x \Psi_n)^\dagger \nabla_z \Psi_n + 2 (\nabla_y \sigma_x \Psi_n)^\dagger \nabla_y \Psi_n \\
- \Psi_n^\dagger (\nabla_z^2 \sigma_x \Psi_n) - (\nabla_z^2 \sigma_x \Psi_n)^\dagger \Psi_n - 2 (\nabla_z \sigma_x \Psi_n)^\dagger \nabla_z \Psi_n \\
- 2 \nabla_x \nabla_y [\Psi_n^\dagger (\sigma_y \Psi_n)] + 2 \nabla_x \nabla_z [\Psi_n^\dagger (\sigma_z \Psi_n)] \right\}
\]

\[
+ 2 \nabla_x \nabla_z [\Psi_n^\dagger (\sigma_z \Psi_n)] \}
\]

(A.6)

And we can obtain other components of \(\vec{j}^{(n)}_{\text{extra}}\) by cyclic permutation of indices.

\[
(\vec{j}^{(n)}_{\text{extra}})_y = \frac{1}{6} \hbar^2 \eta \left\{ (\nabla_z^2 - \nabla_x^2) [\Psi_n^\dagger (\sigma_y \Psi_n)] \\
- 2 \nabla_y \nabla_z [\Psi_n^\dagger (\sigma_z \Psi_n)] + 2 \nabla_y \nabla_x [\Psi_n^\dagger (\sigma_x \Psi_n)] \right\}
\]

(A.7)

\[
(\vec{j}^{(n)}_{\text{extra}})_z = \frac{1}{6} \hbar^2 \eta \left\{ (\nabla_x^2 - \nabla_y^2) [\Psi_n^\dagger (\sigma_z \Psi_n)] \\
- 2 \nabla_z \nabla_x [\Psi_n^\dagger (\sigma_x \Psi_n)] + 2 \nabla_z \nabla_y [\Psi_n^\dagger (\sigma_y \Psi_n)] \right\}
\]

(A.8)

which is the extra term of the current density \(\vec{j}_{\text{extra}}\) in Dresselhaus Hamiltonian as stated in the body of the text. It is not zero in general since \(\Psi_n^\dagger (\vec{c} \Psi_n)\) is real and position dependent for general Hamiltonian. Therefore the terms in above equations are real and not zero after the derivation of position in general.
B. Extra term of particle current for \( \hat{H}_I = \beta \hat{p}^4 \) system

\[
\hat{H} = \hat{H}_0 + \hat{H}_I + V(r), \\
\hat{H}_I = \beta \hat{p}^4.
\]

\[
\frac{1}{i\hbar}[\hat{r}, \hat{H}_I] = 4\beta \hat{p}^3,
\]

\[
\frac{\partial}{\partial t} [\Psi_n^\dagger(\vec{r}) \Psi_n(\vec{r})] = -Re \nabla \cdot \{ \Psi_n^\dagger \frac{1}{i\hbar} [\vec{r}, \hat{H}_0] \Psi_n \} \\
- \frac{1}{4} \Psi_n^\dagger \left( \nabla \cdot \frac{1}{i\hbar} [\vec{r}, \hat{H}_I] \Psi_n \right) - \frac{1}{4} \nabla \cdot \left( \Psi_n^\dagger \frac{1}{i\hbar} [\vec{r}, \hat{H}_I] \Psi_n \right)^\dagger \\
- \frac{1}{4} \nabla \cdot \left\{ \left( \frac{1}{i\hbar} [\vec{r}, \hat{H}_I] \Psi_n \right)^\dagger \Psi_n \right\} + \frac{1}{4} \left( \nabla \Psi_n^\dagger \right) \cdot \left\{ \frac{1}{i\hbar} [\vec{r}, \hat{H}_I] \Psi_n(\vec{r}) \right\} \\
+ \frac{1}{4} \left\{ \frac{1}{i\hbar} [\vec{r}, \hat{H}_I] \Psi_n(\vec{r}) \right\}^\dagger \cdot (\nabla \Psi_n)
\]

And since

\[
\frac{1}{4} \left( \nabla \Psi_n^\dagger \right) \cdot \frac{1}{i\hbar} [\vec{r}, \hat{H}_I] \Psi_n + \frac{1}{4} \frac{1}{i\hbar} [\vec{r}, \hat{H}_I] \Psi_n^\dagger \cdot (\nabla \Psi_n) \\
= \beta \left\{ \left( \nabla \Psi_n^\dagger \right) \cdot \hat{p}^3 \Psi_n + (\hat{p}^3 \Psi_n(\vec{r}))^\dagger \cdot (\nabla \Psi_n) \right\} \\
= \beta (-i\hbar)^3 \left\{ \left( \nabla \Psi_n^\dagger \right) \cdot (\nabla^3 \Psi_n) - (\nabla^3 \Psi_n)^\dagger \cdot (\nabla \Psi_n) \right\} \\
= \beta (-i\hbar)^3 \nabla \cdot \left\{ \left( \nabla \Psi_n^\dagger \right) (\nabla^2 \Psi_n) - (\nabla^2 \Psi_n)^\dagger (\nabla \Psi_n) \right\} \\
- \beta (-i\hbar)^3 \left\{ \left( \nabla^2 \Psi_n^\dagger \right) (\nabla^2 \Psi_n) - (\nabla^2 \Psi_n)^\dagger (\nabla^2 \Psi_n) \right\} \\
= \beta (-i\hbar)^3 \nabla \cdot \left\{ \left( \nabla \Psi_n^\dagger \right) (\nabla^2 \Psi_n) - (\nabla^2 \Psi_n)^\dagger (\nabla \Psi_n) \right\},
\]

we have

\[
\frac{\partial}{\partial t} [\Psi_n^\dagger(\vec{r}) \Psi_n(\vec{r})] \\
= -\nabla \cdot \left\{ Re \left\{ \Psi_n^\dagger \frac{1}{i\hbar} [\vec{r}, \hat{H}_0] \Psi_n \right\} \right\} \\
- \nabla \cdot \left\{ \frac{1}{2} Re \left\{ \Psi_n^\dagger \frac{1}{i\hbar} [\vec{r}, \hat{H}_I] \Psi_n \right\} \right\} \\
+ \nabla \cdot \beta (-i\hbar)^3 \left\{ \left( \nabla \Psi_n^\dagger \right) (\nabla^2 \Psi_n) - (\nabla^2 \Psi_n)^\dagger (\nabla \Psi_n) \right\}
\]
So we get the conserved current

\[ \vec{j} = Re\{\psi_n^\dagger \frac{1}{i\hbar} [\vec{r}, \hat{H}_0] \psi_n\} \]

\[ + \frac{1}{2} Re\{\psi_n^\dagger \frac{1}{i\hbar} [\vec{r}, \hat{H}_I] \psi_n\} \]

\[ - \beta (-ih)^3 \left\{ (\nabla \psi_n^\dagger) (\nabla^2 \psi_n) - (\nabla^2 \psi_n)^\dagger (\nabla \psi_n) \right\} \]

Compare with the conventional definition,

\[ \vec{j}_{\text{conv}} = Re\{\psi_n^\dagger \frac{1}{i\hbar} [\vec{r}, \hat{H}_0] \psi_n\} \]

\[ + Re\{\psi_n^\dagger \frac{1}{i\hbar} [\vec{r}, \hat{H}_I] \psi_n\} \]

we find the extra term of particle current appears.

\[ \vec{j}_{\text{extra}} = \vec{j} - \vec{j}_{\text{conv}} = -\frac{1}{2} Re\{\psi_n^\dagger \frac{1}{i\hbar} [\vec{r}, \hat{H}_I] \psi_n\} \]

\[ - \beta (-ih)^3 \left\{ (\nabla \psi_n^\dagger) (\nabla^2 \psi_n) - (\nabla^2 \psi_n)^\dagger (\nabla \psi_n) \right\} \]

\[ = - \beta (-ih)^3 \left\{ \psi_n^\dagger \nabla \nabla^2 \psi_n + (\nabla \psi_n^\dagger) (\nabla^2 \psi_n) - (\nabla^2 \psi_n)^\dagger (\nabla \psi_n) \right\} \]

\[ = 2\beta \hbar^3 Im\{\nabla (\psi_n^\dagger \nabla^2 \psi_n)\} \]

And in general case, the state \( \psi_n \) can be expressed as

\[ \psi_n(\vec{r}, t) = \sum_{\vec{k}_n} A(\vec{k}_n, t)e^{i\vec{k}_n \cdot \vec{r}} \]

\[ \vec{j}_{\text{extra}} = -\beta i\hbar^3 \left\{ \psi_n^\dagger \nabla \nabla^2 \psi_n + (\nabla \psi_n^\dagger) (\nabla^2 \psi_n) - (\nabla^2 \psi_n)^\dagger (\nabla \psi_n) - (\nabla \nabla^2 \psi_n)^\dagger \psi_n \right\} \]

\[ = -\beta i\hbar^3 \sum_{\vec{k}'_n} \sum_{\vec{k}_n} A^*(\vec{k}'_n, t)A(\vec{k}_n, t)[ik'_n k_n - ik_n k'_n - ik'_n 2 k_n + ik_n 2 k'_n]e^{-i(\vec{k}_n - \vec{k}'_n) \cdot \vec{r}} \]

\[ = \beta \hbar^3 \sum_{\vec{k}'_n} \sum_{\vec{k}_n} A^*(\vec{k}'_n, t)A(\vec{k}_n, t)[(k'_2 - k'_2) (k_n - k'_n)]e^{-i(\vec{k}_n - \vec{k}'_n) \cdot \vec{r}} \]

\[ \neq 0 \]

We reached our conclusion that, in general case, \( \psi_n(\vec{r}, t) \) includes many components \( \{A(\vec{k}_n, t)e^{i\vec{k}_n \cdot \vec{r}}\} \), therefore the extra charge current density \( \vec{j}_{\text{extra}} \) is position dependent and not zero.
C. Extra term of total angular momentum current for Rashba system

First, we have Rashba Hamiltonian: $\hat{H} = \hat{H}_0 + \hat{H}_R$, and it is easy to check following relations:

$$\frac{1}{\hbar} [\hat{\tau}, \hat{H}_0] = \frac{1}{m^*} \hat{J}$$
$$\frac{1}{\hbar} [\hat{\tau}, \hat{H}_R] = \alpha \sigma_y \hat{e}_x - \alpha \sigma_x \hat{e}_y$$
$$\frac{1}{\hbar} \hat{H}_0 = -\frac{1}{2} \nabla \cdot \left( \frac{1}{\hbar} [\hat{r}, \hat{H}_0] \right)$$
$$\frac{1}{\hbar} \hat{H}_R = -\nabla \cdot \left( \frac{1}{\hbar} [\hat{r}, \hat{H}_R] \right)$$

The time evolution of the density of TAM $\hat{J}^z$ satisfies following equation

$$\frac{\partial}{\partial t} (\Psi_n^\dagger \hat{J}^z \Psi_n) = \Psi_n^\dagger \hat{J} \frac{1}{\hbar} \hat{H} \Psi_n - \frac{1}{\hbar} (\hat{H} \Psi_n)^\dagger \hat{J}^z \Psi_n$$

$$= \frac{1}{\hbar} \left( \Psi_n^\dagger (\hat{H}_0 + \hat{H}_R) \hat{J}^z \Psi_n \right) - \frac{1}{\hbar} \left\{ (\hat{H}_0 + \hat{H}_R) \Psi_n \right\}^\dagger \hat{J}^z \Psi_n$$

$$= -\frac{1}{2} \Psi_n^\dagger \nabla \cdot \left( \frac{1}{\hbar} [\hat{\tau}, \hat{H}_0] \hat{J}^z \Psi_n \right) - \Psi_n^\dagger \nabla \cdot \left( \frac{1}{\hbar} [\hat{\tau}, \hat{H}_R] \hat{J}^z \Psi_n \right)$$

$$= -\frac{1}{2} \left( \nabla \cdot \left[ \frac{1}{\hbar} [\hat{r}, \hat{H}_0] \Psi_n \right] \right) \hat{J}^z \Psi_n - \left( \nabla \cdot \left[ \frac{1}{\hbar} [\hat{r}, \hat{H}_R] \Psi_n \right] \right) \hat{J}^z \Psi_n$$

where, in the second step, we used the fact that $\hat{J}^z$ commutes with the Hamiltonian $\hat{H}$.

$$\frac{\partial (\Psi_n^\dagger \hat{J}^z \Psi_n)}{\partial t} = -\frac{1}{2} \nabla \cdot \left( \Psi_n^\dagger \frac{1}{\hbar} [\hat{\tau}, \hat{H}_0] \hat{J}^z \Psi_n \right) - \frac{1}{2} \nabla \cdot \left( \left[ \frac{1}{\hbar} [\hat{\tau}, \hat{H}_0] \Psi_n \right]^\dagger \hat{J}^z \Psi_n \right)$$

$$- \nabla \cdot \left[ \Psi_n^\dagger \left( \frac{1}{\hbar} [\hat{\tau}, \hat{H}_R] \hat{J}^z \Psi_n \right) \right] - \nabla \cdot \left( \left[ \frac{1}{\hbar} [\hat{\tau}, \hat{H}_R] \Psi_n \right]^\dagger \hat{J}^z \Psi_n \right)$$

$$+ \frac{1}{2} \left( \nabla \Psi_n \right)^\dagger \cdot \left( \frac{1}{\hbar} [\hat{\tau}, \hat{H}_0] \hat{J}^z \Psi_n \right) + \frac{1}{2} \left[ \frac{1}{\hbar} [\hat{\tau}, \hat{H}_0] \Psi_n \right]^\dagger \cdot \nabla \left( \hat{J}^z \Psi_n \right)$$

$$+ \left( \nabla \Psi_n \right)^\dagger \cdot \left( \frac{1}{\hbar} [\hat{\tau}, \hat{H}_R] \hat{J}^z \Psi_n \right) + \left[ \frac{1}{\hbar} [\hat{\tau}, \hat{H}_R] \Psi_n \right]^\dagger \cdot \nabla \left( \hat{J}^z \Psi_n \right)$$

Considering

$$\frac{1}{2} \left( \nabla \Psi_n \right)^\dagger \cdot \left( \frac{\hbar}{im^*} \nabla \hat{J}^z \Psi_n \right) + \frac{1}{2} \left( \frac{\hbar}{im^*} \nabla \Psi_n \right)^\dagger \cdot \nabla \left( \hat{J}^z \Psi_n \right) = 0$$
(\nabla \Psi_n)^\dagger \cdot \left( \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_R] \hat{J}^z \Psi_n \right) + \left[ \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_R] \Psi_n \right]^\dagger \cdot \nabla \left( \hat{J}^z \Psi_n \right) \\
= (\nabla \Psi_n)^\dagger \cdot \left( (\alpha_{\sigma_y} \hat{e}_x - \alpha_{\sigma_x} \hat{e}_y) \hat{J}^z \Psi_n \right) + \left[ (\alpha_{\sigma_y} \hat{e}_x - \alpha_{\sigma_x} \hat{e}_y) \Psi_n \right]^\dagger \cdot \nabla \left( \hat{J}^z \Psi_n \right) \\
= \nabla \cdot \left\{ \Psi_n^\dagger \left( (\alpha_{\sigma_y} \hat{e}_x - \alpha_{\sigma_x} \hat{e}_y) \hat{J}^z \Psi_n \right) \right\} = \nabla \cdot \left\{ \Psi_n^\dagger \left( \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_R] \hat{J}^z \Psi_n \right) \right\}

Where we have used the following relation
\[(\alpha_{\sigma_y} \hat{e}_x - \alpha_{\sigma_x} \hat{e}_y) \cdot \nabla \Psi_n]^\dagger \left( \hat{J}^z \Psi_n \right) = (\nabla \Psi_n)^\dagger \cdot \left( (\alpha_{\sigma_y} \hat{e}_x - \alpha_{\sigma_x} \hat{e}_y) \hat{J}^z \Psi_n \right)\]

Therefore, we obtain
\[
\partial \Psi_n^\dagger (\hat{\mathbf{r}}, t) \hat{J}^z \Psi_n / \partial t = -\frac{1}{2} \nabla \cdot \left( \Psi_n^\dagger \left[ \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_0] \hat{J}^z \Psi_n \right] \right) - \frac{1}{2} \nabla \cdot \left( \left[ \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_0] \Psi_n \right]^\dagger \hat{J}^z \Psi_n \right) \\
- \nabla \cdot \left[ \Psi_n^\dagger \left[ \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_R] \hat{J}^z \Psi_n \right] \right] - \nabla \cdot \left[ \left( \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_R] \Psi_n \right)^\dagger \hat{J}^z \Psi_n \right] \\
+ \nabla \cdot \left\{ \Psi_n^\dagger \left( \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_R] \hat{J}^z \Psi_n \right) \right\}
\]

Since \( \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_R] = \alpha_{\sigma_y} \hat{e}_x - \alpha_{\sigma_x} \hat{e}_y \), it yields
\[
\nabla \cdot \left\{ \Psi_n^\dagger \left( \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_R] \hat{J}^z \Psi_n \right) \right\} = \nabla \cdot \left\{ \left( \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_R] \Psi_n \right)^\dagger \hat{J}^z \Psi_n \right\}
\]

Thus, we have
\[
\partial \Psi_n^\dagger \hat{J}^z \Psi_n / \partial t = -\frac{1}{2} \nabla \cdot \left( \Psi_n^\dagger \left[ \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_0] \hat{J}^z \Psi_n \right] \right) - \frac{1}{2} \nabla \cdot \left( \left[ \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_0] \Psi_n \right]^\dagger \hat{J}^z \Psi_n \right) \\
- \frac{1}{2} \nabla \cdot \left[ \Psi_n^\dagger \left[ \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_R] \hat{J}^z \Psi_n \right] \right] - \frac{1}{2} \nabla \cdot \left[ \left( \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}_R] \Psi_n \right)^\dagger \hat{J}^z \Psi_n \right] \\
= -\frac{1}{2} \nabla \cdot \left( \Psi_n^\dagger \left( \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}] \hat{J}^z \Psi_n \right) \right) - \frac{1}{2} \nabla \cdot \left( \left( \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}] \Psi_n \right)^\dagger \hat{J}^z \Psi_n \right)
\]

Meanwhile
\[
\left( \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}] \Psi_n \right)^\dagger \hat{J}^z \Psi_n = \left[ \left( \frac{1}{im^*} \hat{J}^z + \alpha_{\sigma_y} \hat{e}_x - \alpha_{\sigma_x} \hat{e}_y \right) \Psi_n \right]^\dagger \hat{J}^z \Psi_n \\
= \left[ \left( \frac{\hbar}{im^*} \nabla + \alpha_{\sigma_y} \hat{e}_x - \alpha_{\sigma_x} \hat{e}_y \right) \Psi_n \right]^\dagger \hat{J}^z \Psi_n \\
= -\frac{\hbar}{im^*} \nabla \{ \Psi_n^\dagger \hat{J}^z \Psi_n \} + \frac{\hbar}{im^*} \Psi_n^\dagger \nabla \Hat{\mathbf{r}} \nabla \hat{J}^z \Psi_n \\
+ \Psi_n^\dagger (\alpha_{\sigma_y} \hat{e}_x - \alpha_{\sigma_x} \hat{e}_y) \hat{J}^z \Psi_n \\
= -\frac{\hbar}{im^*} \nabla \{ \Psi_n^\dagger \hat{J}^z \Psi_n \} + \Psi_n^\dagger (\Hat{\mathbf{r}}, t) \frac{1}{i\hbar} [\hat{\mathbf{r}}, \hat{H}] \hat{J}^z \Psi_n
It results
\[
\frac{\partial (\Psi_n^\dagger \hat{J}^z \Psi_n)}{\partial t} = -\frac{1}{2} \nabla \cdot \left\{ \left( \Psi_n^\dagger \frac{1}{\imath \hbar} [\hat{r}, \hat{H}] \hat{J}^z \Psi_n \right) + \left( \frac{1}{\imath \hbar} [\hat{r}, \hat{H}] \Psi_n \right)^\dagger \hat{J}^z \Psi_n \right\}
\]
\[
= -\nabla \cdot \left\{ \Psi_n^\dagger \frac{1}{\imath \hbar} [\hat{r}, \hat{H}] \hat{J}^z \Psi_n \right\} + \nabla \cdot \left\{ \frac{\hbar}{2i \imath m^*} \nabla \{ \Psi_n^\dagger \hat{J}^z \Psi_n \} \right\}. \tag{C.1}
\]

And use the fact that $\Psi_n^\dagger \sigma_j \Psi_n^\dagger$ is real, we have
\[
-\text{Re} \left\{ \frac{\hbar}{2 \imath m^*} \nabla (\Psi_n^\dagger \hat{J}^z \Psi_n) \right\}
\]
\[
= -\text{Re} \left\{ \frac{\hbar}{2 \imath m^*} \nabla (\Psi_n^\dagger \hat{J}^z \Psi_n) \right\} - \text{Re} \left\{ \frac{\hbar}{2 \imath m^*} \nabla (\Psi_n^\dagger \hat{J}^z \Psi_n) \right\}
\]
\[
= -\frac{1}{2} \left\{ \frac{\hbar}{2 \imath m^*} \nabla (\Psi_n^\dagger (x \nabla_y - y \nabla_x) \Psi_n) - \frac{\hbar}{2 \imath m^*} \nabla ((x \nabla_y - y \nabla_x) \Psi_n^\dagger \Psi_n) \right\}
\]
\[
= \frac{\hbar^2}{2 \imath m^*} \left\{ (\nabla \Psi_n^\dagger) (x \nabla_y - y \nabla_x) \Psi_n + ((x \nabla_y - y \nabla_x) \Psi_n)^\dagger \nabla \Psi_n
+ \Psi_n^\dagger (x \nabla_y - y \nabla_x) \nabla \Psi_n + ((x \nabla_y - y \nabla_x) \Psi_n)^\dagger \Psi_n
+ \frac{1}{2 \imath m^*} \left\{ \Psi_n^\dagger (e_x \nabla_y - e_y \nabla_x) \Psi_n + ((e_x \nabla_y - e_y \nabla_x) \Psi_n)^\dagger \Psi_n \right\} \right\} \].
\]

Therefore,
\[
\vec{j}_L = \text{Re} \sum_n \rho_n \left\{ \Psi_n^\dagger (\hat{r}, t) \frac{1}{\imath \hbar} [\hat{r}, \hat{H}] \hat{J}^z \Psi_n \right\}
\]
\[
+ \frac{\hbar^2}{2 \imath m^*} \sum_n \rho_n \left\{ \Psi_n^\dagger (e_x \nabla_y - e_y \nabla_x) \Psi_n + ((e_x \nabla_y - e_y \nabla_x) \Psi_n)^\dagger \Psi_n \right\}
\]
\[
+ \frac{\hbar^2}{2 \imath m^*} \sum_n \rho_n \left\{ (\nabla \Psi_n^\dagger) (x \nabla_y - y \nabla_x) \Psi_n + ((x \nabla_y - y \nabla_x) \Psi_n)^\dagger \nabla \Psi_n
+ \Psi_n^\dagger (x \nabla_y - y \nabla_x) \nabla \Psi_n + ((x \nabla_y - y \nabla_x) \Psi_n)^\dagger \Psi_n \right\}. \tag{C.3}
\]

The conventional current of TAM $J^z_L$:
\[
\vec{j}_{\text{conv}} = \frac{1}{2} \text{Re} \sum_n \rho_n \Psi_n^\dagger \left\{ \frac{1}{\imath \hbar} [\hat{r}, \hat{H}] \hat{J}^z + \hat{J}^z \frac{1}{\imath \hbar} [\hat{r}, \hat{H}] \right\} \Psi_n \]
Due to the following relation for Rashba Hamiltonian:

\[
\left[ J_z, \frac{1}{i\hbar} \langle \vec{r}, \hat{H} \rangle \right] = \hbar i [(p_y/m^* - \alpha \sigma_x) \vec{e}_x - (p_x/m^* + \alpha \sigma_y) \vec{e}_y]
\]  

we have

\[
\vec{J}_{\text{conv}} \equiv \frac{1}{2} \sum_n \rho_n \Psi_n^\dagger \left\{ \frac{1}{i\hbar} \langle \vec{r}, \hat{H} \rangle \hat{J}^2 + \hat{J} \frac{1}{i\hbar} \langle \vec{r}, \hat{H} \rangle \right\} \Psi_n
\]

\[
= \text{Re} \left\{ \sum_n \rho_n \Psi_n^\dagger \frac{1}{i\hbar} \langle \vec{r}, \hat{H} \rangle \hat{J}^2 \Psi_n \right\}
\]

\[
+ \frac{1}{2} \sum_n \rho_n \left\{ \hbar i \Psi_n^\dagger [(p_y/m^*) \vec{e}_x - (p_x/m^*) \vec{e}_y] \Psi_n \right\}
\]

\[
= \text{Re} \left\{ \sum_n \rho_n \Psi_n^\dagger \frac{1}{i\hbar} \langle \vec{r}, \hat{H} \rangle \hat{J}^2 \Psi_n \right\}
\]

\[
+ \frac{\hbar^2}{4m^*} \left\{ \Psi_n^\dagger (\nabla_y \vec{e}_x - \nabla_x \vec{e}_y) \Psi_n + \left( (\nabla_y \vec{e}_x - \nabla_x \vec{e}_y) \Psi_n \right)^\dagger \Psi_n \right\}
\]  

where, in the second step, the property that \( \Psi_n^\dagger \sigma_z \Psi_n \) is real is again used. Compare (C.6) with (C.3), we have

\[
\vec{J}_{\text{extra}}^{(n)} = \vec{J}_{\text{extra}}^{(n)} - \vec{J}_{\text{extra}}^{(n)}
\]

\[
= \frac{\hbar^2}{2m^*} \left\{ (\nabla \Psi_n^\dagger) (x \nabla_y - y \nabla_x) \Psi_n + (x \nabla_y - y \nabla_x) \Psi_n^\dagger \nabla \Psi_n
\]

\[
+ \Psi_n^\dagger (x \nabla_y - y \nabla_x) \nabla \Psi_n + (x \nabla_y - y \nabla_x) \nabla \Psi_n^\dagger \Psi_n \right\}
\]

So the components of \( \vec{J}_{\text{extra}}^{(n)} \) are

\[
J_{\text{extra}}^{(n)x} = \frac{\hbar^2}{2m^*} \left\{ (\nabla_x \Psi_n^\dagger) (x \nabla_y - y \nabla_x) \Psi_n + (x \nabla_y - y \nabla_x) \Psi_n^\dagger \nabla_x \Psi_n
\]

\[
+ \Psi_n^\dagger (x \nabla_y - y \nabla_x) \nabla_x \Psi_n + (x \nabla_y - y \nabla_x) \nabla_x \Psi_n^\dagger \Psi_n \right\}
\]

\[
= \frac{\hbar^2}{4m^*} \left[ x \nabla_x \nabla_y \Psi_n^\dagger \Psi_n + x(\nabla_y \Psi_n^\dagger) \nabla_x \Psi_n + x \Psi_n^\dagger \nabla_y \nabla_x \Psi_n
\]

\[
+ x(\nabla_x \nabla_y \Psi_n^\dagger) \Psi_n - y \nabla^2_x (\Psi_n^\dagger \Psi_n) \right]
\]

\[
= \frac{\hbar^2}{4m^*} \left[ x \nabla_x \nabla_y (\Psi_n^\dagger \Psi_n) - y \nabla^2_x (\Psi_n^\dagger \Psi_n) \right]
\]

\[
= \frac{\hbar^2}{4m^*} \left[ x \nabla_y - y \nabla_x \right] \nabla_x (\Psi_n^\dagger \Psi_n)
\]

\[
= -\frac{1}{4m^*} \hbar^2 p_x (\Psi_n^\dagger \Psi_n)
\]

Similarly, we have

\[
J_{\text{extra}}^{(n)y} = \frac{\hbar^2}{4m^*} \left[ x \nabla_y - y \nabla_x \right] \nabla_y (\Psi_n^\dagger \Psi_n) = -\frac{1}{4m^*} \hbar^2 p_y (\Psi_n^\dagger \Psi_n)
\]  

(C.7)
So finally we arrive at

\[ \vec{j}_{\text{extra}} = -\frac{1}{4m^*} \sum_n \rho_n \vec{l} \cdot \vec{p} (\Psi_n^\dagger \Psi_n) \]  
(C.8)

Where \( \Psi_n^\dagger \Psi_n \) is the density of particles and real. It must be position dependent in general, thus \( \vec{j}_{\text{extra}} = \frac{\hbar^2}{4m^*} [x \nabla_y - y \nabla_x] \nabla (\Psi_n^\dagger \Psi_n) \neq 0 \).

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