Spectral properties of random LC networks with uniformly distributed entries

N A Olekhno¹, Y M Beltukov² and D A Parshin¹
¹Saint Petersburg State Polytechnical University, 195251 Saint Petersburg, Russia
²A. F. Ioffe Physical-Technical Institute, 194021 Saint Petersburg, Russia
E-mail: ¹olekhnon@gmail.com

Abstract. Disordered LC networks are an appropriate model for describing dielectric resonances in random metal-dielectric composites. The existing theoretical methods allow to study only the case when fixed values of \( L \) and \( C \) entries are randomly distributed on a lattice with probabilities \( p \) and \( 1 - p \) correspondingly. In this paper we generalize these approaches to arbitrary values of \( L \) and \( C \) entries and study as an example a 2d random percolative LC square lattice \( (p = 0.5) \) with uniformly distributed values of inverse inductances \( 1/L \) and capacitances \( C \) in the interval \((0,1)\). We show that the spectral density of resonances \( \rho(\omega) \) in this case has a smooth maximum at \( \omega_{\text{max}} \approx 0.25 \). The asymptotic behavior below the maximum is \( \rho(\omega) \propto \omega \) and \( \rho(\omega) \propto 1/\omega^3 \) above the maximum.

1. Introduction
Metal-dielectric composites consisting of metallic nanoparticles incorporated into dielectric matrix (called cermets, such as Au-MgO, Pt-Al\(_2\)O\(_3\), Fe-Al\(_2\)O\(_3\)) are attracting considerable attention last years due to their interesting optical properties both from theoretical and experimental point of view [1, 2, 3].

One can model such systems replacing a continuous medium to a large discrete random network made of capacitors \( C \) and inductances \( L \) (the latter can be in series with a weak resistor \( R \)). Such network description is rather natural and can be derived from discrete version of continuous Maxwell equations for scalar potential of electric field. This approximation is valid if all the inhomogeneity length scales such as size of the nanoparticles \( a \) and distances between them are much smaller compared to the electromagnetic lengths, namely the wavelength \( \lambda \) and the skin depth \( \delta \) (in the case of metal). When this is not the case, the full Maxwell equations must be invoked for a correct description of the physics [4, 5].

Due to well known relation between dielectric permittivity \( \varepsilon \) and conductivity \( \sigma \) of the medium
\[
\varepsilon = i \frac{4\pi \sigma}{\omega}
\]
the capacitors \( C \) model dielectric bridges between metallic nanoparticles the former having conductance \( G_d = i\omega C \) with capacitance
\[
C \simeq \varepsilon_d a.
\]
Here $\varepsilon_d$ is a dielectric permittivity of dielectric. Whereas isolated metallic granules are characterized by almost purely inductive response having conductance $G_m = c^2/i\omega L$ with inductance 

$$L \simeq \frac{c^2}{\omega_p a^2}.$$  (3)

Here $\omega_p$ is the plasma frequency. Indeed the dielectric permittivity of dielectric $\varepsilon_d$ is frequency independent in the investigated range, therefore according to (1) frequency dependent conductivity of dielectric \(\sigma_d \propto i\omega\varepsilon_d\). However dielectric permittivity of metal $\varepsilon_m = 1 - \omega_p^2/\omega^2$. For $\omega \ll \omega_p$, $\varepsilon_m \propto -\omega^2/\omega^2$. The last approximation is valid for frequencies $\omega$ of electromagnetic radiation satisfying to condition $\omega_\tau \ll \omega \ll \omega_p$. Here $\omega_\tau$ is the plasmon relaxation rate [1]. The inductance given by Eq. (3) is the so-called kinetic inductance related with kinetic energy of electrons [6]. It becomes bigger than geometric inductance $L_g \simeq a$ when size of nanoparticles $a < 30$ nm.

Such large random dissipationless $LC$ networks (that is networks made of random mixture of capacitances $C$ and inductances $L$) have a lot of resonance clusters with any size and shape and different resonance frequencies. Due to dipole-dipole interaction between the clusters, eigenmodes in the network are not localized [7] and local electric fields show anomalous fluctuations [8]. These fluctuations are responsible for example for giant surface-enhanced Raman scattering observed in semicontinuous metal films [9].

In the papers [2, 10] considering as an example a 2d square lattice it was shown that the problem of resonances of a random $LC$ network (with random positions of capacitors $C$ and inductances $L$) can be reduced to some generalized eigenvalue problem. However the reduction was done only for the very particular case when all values of inductances $L$ (and capacitances $C$) are equal. Then in the system there is a typical frequency $\omega_0 = 1/\sqrt{LC}$.

In the present paper we have extended these results to the general case when values of inductances $L$ and capacitances $C$ are also random and can fluctuate from bond to bond. We have shown that in this case frequencies of the resonances can be found from the generalized eigenvalue problem

$$\sum_j K_{ij} \varphi_j - \omega^2 \sum_j C_{ij} \varphi_j = 0.$$  (4)

Here $K_{ij} = 1/L_{ij}$ is the matrix of inverse inductances, $C_{ij}$ is the matrix of capacitances between sites $i$ and $j$ and $\varphi_j$ is a potential on site $j$. The diagonal elements of these matrices are equal to the minus sum of non diagonal ones

$$C_{ii} \equiv -\sum_{j \neq i} C_{ij}, \quad K_{ii} \equiv -\sum_{j \neq i} K_{ij}.$$  (5)

2. Kirchhoff equations for arbitrary values of $L$ and $C$ entries

To derive Kirchhoff equations in the general case for arbitrary values of $L$ and $C$ entries, we will numerate sites of our $LC$ network using integer index $i$ taking values from 1 to $N$, here $N$ is the total number of cites (see Fig. 1). Let $\varphi_i$ being the potential of site $i$. Also we attribute to each site $i$ a charge $q_i$. It is equal to the sum of charges of capacitor plates adjoined site $i$ (see shaded regions shown on Fig. 1 around sites 6 and 7). If no capacitor adjoins site $i$, then the corresponding charge $q_i = 0$. The charges $q_i$ and the potentials $\varphi_i$ are connected with each other by the system of equations

$$q_i = \sum_{j \neq i} C_{ij} (\varphi_i - \varphi_j),$$  (6)
Figure 1. A piece of random LC network. Shaded are regions with charge accumulated on sites 6 and 7 respectively.

where \( C_{ij} \) is a capacity of the capacitor between sites \( i \) and \( j \). The matrix \( C \) is by definition symmetric \( C_{ij} = C_{ji} \). If there is no capacitor between sites \( i \) and \( j \) then \( C_{ij} = 0 \). Let us define the diagonal matrix element \( C_{ii} \) as a minus sum of the non-diagonal elements

\[
C_{ii} \equiv - \sum_{j \neq i} C_{ij}, \tag{7}
\]

then from Eq. (6) we get

\[
q_i = - \sum_j C_{ij} \varphi_j. \tag{8}
\]

The time derivative \( \dot{q}_i \) by definition is equal to the sum of incoming electric currents \( I_{ji} \) to the site \( i \) from all neighboring sites \( j \)

\[
\dot{q}_i = \sum_{j \neq i} I_{ji}. \tag{9}
\]

Here \( I_{ji} \) is the electric current from site \( j \) to site \( i \). The matrix \( I_{ji} \) by definition is antisymmetric \( I_{ij} = -I_{ji} \). In its turn the time derivative of electric current \( \dot{I}_{ij} \) is related to voltage \( \varphi_i - \varphi_j \) in the corresponding inductance coil

\[
L_{ij} \dot{I}_{ij} = \varphi_i - \varphi_j. \tag{10}
\]

Here \( L_{ij} \) is an inductance of the coil between sites \( i \) and \( j \). The matrix \( L \) is a symmetric matrix, \( L_{ij} = L_{ji} \). If there is no coil between sites \( i \) and \( j \), then the corresponding value of \( L_{ij} = \infty \).

Now using equations (9) and (10) we can relate the second time derivative of charge \( \ddot{q}_i \) with voltages \( \varphi_j - \varphi_i \)

\[
\ddot{q}_i = \sum_{j \neq i} \frac{1}{L_{ij}} (\varphi_j - \varphi_i). \tag{11}
\]

To proceed further let us introduce matrix \( K \) of inverse inductances (between sites \( i \) and \( j \))

\[
K_{ij} = \frac{1}{L_{ij}}. \tag{12}
\]

The matrix \( K \) is symmetric, i.e. \( K_{ij} = K_{ji} \). If sites \( i \) and \( j \) are not connected by inductance coil, then by definition \( K_{ij} = 0 \). Similar with the case of matrix \( C \) let us define the diagonal matrix
elements of the matrix of inverse inductances $K$ as minus sum of the non-diagonal elements (compare with Eq. (7))

$$K_{ii} = - \sum_{j \neq i} K_{ij}. \quad (13)$$

Then from (11) we get

$$\ddot{q}_i = \sum_j K_{ij} \varphi_j. \quad (14)$$

Comparing Eq. (8) and Eq. (14), we obtain finally a linear system of ordinary differential equations for site potentials

$$\sum_j C_{ij} \ddot{\varphi}_j = - \sum_j K_{ij} \varphi_j. \quad (15)$$

Looking for solution of these equations in the exponential form $\varphi_i \propto \exp(i\omega t)$ we get

$$\omega^2 \sum_j C_{ij} \varphi_j = \sum_j K_{ij} \varphi_j. \quad (16)$$

It is identical to Eq. (4). Non-zero solution of this linear system of equations exist when

$$\det (K - \omega^2 C) = 0. \quad (17)$$

As a result we have reduced calculation of eigenfrequencies of our random impedance network to a generalized eigenvalue problem.

3. **Resonances in the network with uniformly distributed entries**

![Figure 2](image-url)  

**Figure 2.** The spectral density of resonances $\rho(\omega)$ in the case of uniform distribution of $C$ and $1/L$ values in the interval $(0,1)$ with concentrations $p_L = p_C = 0.5$.

Using Eqs. (4) we as an example have considered a particular case when capacitances $C$ and inverse inductances $1/L$ are uniformly distributed in the interval $(0,1)$ (in appropriate units) and placed randomly with equal concentrations on a 2d square lattice. The later corresponds to
percolation threshold for both types of $L$ and $C$ bonds. The uniform distribution of $C \propto a$ and $1/L \propto a$ values according to (2) and (3) models the realistic case when size of the metallic grains and distance between them $a$ are distributed uniformly from zero to some maximum value. The spectral density of resonances $\rho(\omega)$ is shown on Fig. 2. It has a smooth maximum at frequency $\omega_{\text{max}} \approx 0.25$. Asymptotically at small frequencies below the maximum $\rho(\omega) \propto \omega$ and at large frequencies above the maximum the spectral density decays as $\rho(\omega) \propto 1/\omega^3$. One can show that such behavior follows directly from the chosen distributions.

Due to dual symmetry of Eqs (4) in our case the $\omega^2$ and $1/\omega^2$ values have the same distribution function (compare with [2]). It follows from the fact that matrices $K$ and $C$ have the same distribution function for matrix elements (uniform in our case). To show that let us introduce the new variable

$$\lambda = \frac{\omega^2}{1 + \omega^2}. \quad (18)$$

The value of $\omega = 0$ corresponds to $\lambda = 0$, whereas $\omega = \infty$ corresponds to $\lambda = 1$. The replacement of $\omega \to 1/\omega$ corresponds to replacement $\lambda \to 1 - \lambda$. The distribution of $\lambda$ is shown on Fig. 3. It is a symmetric function of $\lambda$ relative to the point $\lambda = 1/2$. In other words the distribution $\rho(\lambda) = \rho(1 - \lambda)$

![Figure 3. The spectral density of resonances $\rho(\lambda)$.](image)

4. Conclusion

We generalized the problem of random LC networks to the case when values of capacitances $C$ and inductances $L$ in a lattice can fluctuate from bond to bond. As an example we have calculated the spectral density of resonances for uniform distribution of capacitances $C$ and inverse inductances $1/L$ in the interval $(0,1)$. Such distribution corresponds to the uniform distribution of metallic grain sizes $a$. One can use the developed approach for many other actual distributions.

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