Inequalities for Wilson loops, cusp singularities, area law and shape of a drum

P.V. Pobylitsa

Institute for Theoretical Physics II, Ruhr University Bochum, D-44780 Bochum, Germany and Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg, 188300, Russia

Abstract
Inequalities are derived for Wilson loops generalizing the well-known Bachas inequality for rectangular contours. The inequalities are compatible with the area law for large contours. The Polyakov cusp anomalous dimension of Wilson lines (playing an important role in QCD applications to hard processes) has a convex angular dependence. This convexity is crucial for the consistency of the inequalities with renormalization. Some parallel properties can be found in the string theory. The Kac-Ray cusp term from the “shape of a drum” problem has the same angular convexity property and plays the role of the cusp anomalous dimension in the effective string model for Wilson loops studied by Lüscher, Symanzik and Weisz (LSW). Using heuristic arguments based on the LSW model, one can find an interesting connection between the inequalities for Wilson loops and inequalities for determinants of two-dimensional Laplacians with Dirichlet boundary conditions on the closed contours associated with Wilson loops.

1 Introduction

1.1 Inequalities for rectangular Wilson loops and properties of static heavy-quark potential

Among various inequalities for Wilson loops [1],

$$W(C) = \frac{1}{N} \left\langle \text{Tr} \ P \left( \exp i \oint_C A \ dx \right) \right\rangle ,$$

(1.1)

an important role is played by Bachas inequality [2] (see also [3, 4]) for rectangular $T \times R$ Wilson loops $W(T, R)$

$$\left[ W \left( T, \frac{R_1 + R_2}{2} \right) \right]^2 \leq W \left( T, R_1 \right) W \left( T, R_2 \right) .$$

(1.2)
This inequality was derived in the lattice gauge theory using the reflection positivity property \([5, 6, 7]\) (which corresponds to the positive metric of the Hilbert space for physical states in the operator formulation of the theory). The large \(T\) behavior of \(\ln W(T, R)\) is described by the static heavy-quark potential

\[
V(R) = -\lim_{T \to \infty} \frac{1}{T} \ln W(T, R).
\]

(1.3)

In Ref. \([2]\) inequality (1.2) was used in order to derive the convexity of the potential

\[
V\left(\frac{R_1 + R_2}{2}\right) \geq \frac{1}{2} [V(R_1) + V(R_2)].
\]

(1.4)

In the continuum limit, this convexity property is equivalent to the inequality

\[
\frac{d^2V(R)}{dR^2} \leq 0.
\]

(1.5)

This inequality has several important consequences. For example, it shows that the potential \(V(R)\) cannot grow faster than linearly, the large distance behavior \(V(R) \sim R^\alpha\) with \(\alpha > 1\) is forbidden. Another consequence is that the \(1/R\) correction to the linear potential \(KR\) can be only negative

\[
V(R) \overset{R \to \infty}{=} KR + 2a_0 + a_{-1} \frac{1}{R} + \ldots,
\]

(1.6)

\[
a_{-1} \leq 0.
\]

(1.7)

### 1.2 Cusp anomalous dimension and its convexity

In Ref. \([8]\) A.M. Polyakov has shown that the renormalization of Wilson loops corresponding to non-smooth contours is accompanied by additional renormalization factors depending on the cusp angles\(\gamma\) (Fig. 1). These renormalization factors can be interpreted in terms of the cusp anomalous dimension \(\Gamma_{\text{cusp}}(\gamma)\). To the leading order in the coupling \(g\), this anomalous dimension for the SU\((N_c)\) gauge group is given by

\[
\Gamma_{\text{cusp}}^{(1)}(\gamma) = -\frac{g^2}{4\pi^2} \frac{N_c^2 - 1}{2N_c} (1 - \gamma \cot \gamma).
\]

(1.8)

A straightforward check shows that \(\Gamma_{\text{cusp}}^{(1)}(\gamma)\) has the convexity property (see Sec. \([8]\) for details)

\[
\Gamma_{\text{cusp}}^{(1)}\left(\frac{\gamma_1 + \gamma_2}{2}\right) \geq \frac{1}{2} \left[\Gamma_{\text{cusp}}^{(1)}(\gamma_1) + \Gamma_{\text{cusp}}^{(1)}(\gamma_2)\right].
\]

(1.9)

---

1 Following the tradition we write cusp anomalous dimension \([10]\) \textit{in the gauge theory} as a function of the deviation angle \(\gamma = \pi - \theta\) where \(\theta\) is the inside-facing angle (see Fig. 1). However, \textit{in the effective string model} discussed in Sec. \([5]\) we use angle \(\theta\).

2 The NLO correction was computed in Ref. \([9]\).
Figure 1: Cusp angles: inside-facing angles $\theta_k$ and deviation angles $\gamma_k = \pi - \theta_k$.

1.3 Polygonal Wilson loops as a common basis for perturbative and nonperturbative bounds

At first sight, the formal coincidence of the convexity properties (1.4), (1.9) for the static potential $V(R)$ and for the cusp anomalous dimension $\Gamma_{\text{cusp}}^{(1)}(\gamma)$ is a pure mathematical curiosity which has nothing to do with physics. The heavy-quark potential $V(R)$ is a nonperturbative quantity describing both large and short distances. The anomalous dimension $\Gamma_{\text{cusp}}^{(1)}(\gamma)$ arises in the context of the perturbative renormalization and plays an important role in the physics of small distances and hard processes [9]. The convexity of the potential $V(R)$ was rigorously established in the lattice gauge theory (with discrete $R$), whereas the cusp anomalous dimension can be discussed only in the continuum limit.

Nevertheless it is possible to find an interesting connection between the convexity of the potential $V(R)$ and the convexity of the anomalous dimension $\Gamma_{\text{cusp}}^{(1)}(\gamma)$. In this paper it will be shown that both convexity properties naturally appear in a common framework provided by inequalities for polygonal Wilson loops. The idea to consider polygonal Wilson loops is quite natural. On the one hand, we already know that the convexity of the static quark potential (1.4) follows from the Bachas inequality for rectangular Wilson loops (1.2). On the other hand, the renormalization of polygonal Wilson loops provides an access to the cusp anomalous dimension.

Certainly, the interpretation of the convexity of the cusp anomalous dimension is not the only motivation for studying inequalities for polygonal Wilson loops. For example, one could be interested in the constraints imposed by the inequalities on the area law behavior in confining theories.

1.4 Example: trapezium-parallelogram inequality

In this article we will derive inequalities for Wilson loops with rather general contours. To begin, let us take a very simple, but still instructive example.

Let us consider a symmetric trapezium with the height $h$ and with two parallel sides $a$ and $b$ as shown in Fig. 2 (a). The corresponding Wilson loop will be denoted $W_{\text{trapezium}}(a, b, h)$. We use notation $W_{\text{parallelogram}}(c, h)$ for the
Figure 2: (a) Trapezium and parallelogram entering in inequality (1.10). (b) Graphic representation of inequality (1.10) for the Wilson loops. (c) Open contours $B_1$ and $B_2$ used for the derivation of inequality (1.10). The end points obey the condition $x_4 = y_4 = 0$.

Wilson loop associated with the parallelogram with side $c$ and height $h$. In Sec. 2.1 we will derive the inequality

$$\left| W_{\text{parallelogram}} \left( \frac{a + b}{2}, h \right) \right| \leq W_{\text{trapezium}}(a, b, h)$$

(1.10)

which is shown graphically in Fig. 2(b).

1.5 Inequalities and renormalization

Trapezium-parallelogram inequality (1.10) is just a simple example of the wide class of inequalities which are discussed in this article. The derivation of inequalities for polygonal Wilson loops follows the same ideas as the derivation of the Bachas inequality for rectangular contours (1.2). However, in the case of polygonal contours one meets several problems:

1) Rectangular loops can be studied in the context of lattice gauge theory. In the case of general polygonal contours we have to work with the continuum version of the theory.
2) The continuum limit is accompanied by renormalization. Although the asymptotic freedom of non-Abelian gauge theories provides a rather reliable perturbative control of the renormalization of nonperturbative quantities, one often has to use heuristic arguments instead of “mathematically rigorous” statements of the lattice gauge theory.

3) Polygonal contours have cusps which give an additional contribution to the renormalization of Wilson loops. In the case of rectangular contours the cusp renormalization factor (determined by the four $90^\circ$ vertices of the rectangle) is the same for all rectangles. On the contrary, the renormalization of cusp singularities in polygonal Wilson loops is not universal. Therefore a special care must be taken about the compatibility of the naive inequalities with renormalization.

Our analysis of the renormalization of inequalities for Wilson loops leads to the following conclusions:

1) One class of inequalities survives under renormalization. In these inequalities, all renormalization constants cancel, so that the correct renormalized form of the inequalities coincides with naive nonrenormalized version.

2) In the second class of inequalities, the balance of the renormalization constants is violated. Therefore in the renormalized version of the inequality some contributions of the original nonrenormalized inequality die out. As a result, one arrives at trivial inequalities.

Although the trivial renormalized inequalities of the second class are not interesting, the fact that these trivial inequalities are correct deserves some attention. This correctness comes as a result of a peculiar property of cusp renormalization factors, which follows from the convexity of the angular dependence of cusp anomalous dimension. Therefore we see that there is a rather interesting connection between the general positivity properties of the theory (underlying the inequality) and properties of the cusp anomalous dimension.

Another interesting issue is the connection between the inequalities discussed in this paper and the area law for Wilson loops expected in gauge theories confining heavy quarks. Here we have the following situation: the area law does not follow from the inequalities but all inequalities are compatible with the area law.

1.6 From gauge theory to determinants of two-dimensional Laplacians via effective string model

In the effective string model suggested by Lüscher, Symanzik and Weisz (LSW) [10], [11] for the description of large Wilson loops, contours with cusps also have additional ultraviolet divergences. Similar to gauge theories, these divergences factorize into a product of single cusped contributions. The dependence of the divergence on the inside-facing cusp angle $\theta$ (see Fig. 4) is described by the anomalous dimension

$$\gamma_{\text{cusp}}^{\text{str}}(\theta) = \frac{\theta^2 - \pi^2}{24\pi \theta}. \quad (1.11)$$
The analogy with gauge theories can be continued. Function $\Gamma_{\text{cusp}}(\theta)$ has the same convexity property

$$\Gamma_{\text{cusp}}\left(\frac{\theta_1 + \theta_2}{2}\right) \geq \frac{1}{2} \left[ \Gamma_{\text{cusp}}(\theta_1) + \Gamma_{\text{cusp}}(\theta_2) \right] \quad (0 < \theta < 2\pi) \quad (1.12)$$

as Polyakov’s anomalous dimension (1.8), (1.9).

Expression (1.11) for the cusp anomalous dimension in the string model has an old history. M. Kac has studied the spectrum of the two-dimensional Laplace operator $\Delta_C$ with Dirichlet conditions on the boundary $C$ using the trace of the operator $e^{t\Delta_C}$. In paper “Can one hear the shape of a drum?” [22] he constructed the small-$t$ expansion of $\text{Tr} e^{t\Delta_C}$ for polygonal contours $C$ with cusp angles $\theta_i$:

$$\text{Tr} e^{t\Delta_C} = t^{-1} S(C) - t^{-1/2} L(C) - t^0 \sum\Gamma_{\text{cusp}}(\theta_i) + \ldots \quad (1.13)$$

Here $S(C)$ is the surface area bounded by the contour $C$ and $L(C)$ is the length of $C$. The cusp function $\Gamma_{\text{cusp}}(\theta)$ obtained by M. Kac was represented by a complicated integral in Ref. [22]. The simple expression (1.11) for $\Gamma_{\text{cusp}}(\theta)$ was obtained by D.B. Ray (the derivation is described in Ref. [23]).

Inserting the small-$t$ expansion (1.13) into the proper time representation for determinants

$$\ln \text{Det} \frac{\Delta_1}{\Delta_2} = - \int_0^\infty \frac{dt}{t} \text{Tr} (e^{t\Delta_1} - e^{t\Delta_2}) \quad , \quad (1.14)$$

one can identify the divergence of integral (1.14) at small $t$ with the contribution of cusps into the ultraviolet divergences of the determinant $\text{Det} \Delta_C$ of the Laplace operator.

In the LSW model [10], [11], the determinant of the two-dimensional Laplace operator $\Delta_C$ with Dirichlet boundary conditions on the contour $C$ plays the central role: flat Wilson loops $W(C)$ of the $D$ dimensional gauge theory are approximated in the LSW model by

$$W_{\text{LSW}}(C) = e^{-KS(C)} \left( \text{Det} \Delta_C \right)^{-(D-2)/2} \quad . \quad (1.15)$$

Combining rigorous result (1.13) and model expression (1.15), one obtains the interpretation of the function $\Gamma_{\text{cusp}}(\theta)$ as a cusp anomalous dimension in the effective string model for Wilson loops.

Now one arrives at the conjecture that the convexity of the $\Gamma_{\text{cusp}}(\theta)$ (1.12) is not accidental. It may happen that the inequalities derived in this paper in the framework of the gauge theory also hold in the LSW model. In particular, one can show that Bachas inequality (1.2) (and some of its generalizations) really holds in the model:

$$\left[ W_{\text{LSW}} \left( T, \frac{R_1 + R_2}{2} \right) \right]^2 \leq W_{\text{LSW}} (T, R_1) W_{\text{LSW}} (T, R_2) \quad . \quad (1.16)$$
According to Eq. (1.15) this inequality is equivalent to the following inequality for the determinant of the Laplace operator \( \Delta(T, R) \) with Dirichlet boundary conditions on the rectangle \( T \times R \):

\[
\left[ \text{Det} \Delta \left( T, \frac{R_1 + R_2}{2} \right) \right]^2 \geq [\text{Det} \Delta (T, R_1)] [\text{Det} \Delta (T, R_2)].
\]  

(1.17)

In the above argument, the LSW model played an important heuristic role. However, one has to separate the interesting physical question about the relevance of the LSW model for large Wilson loops in gauge theories [12, 13, 14, 16, 17] from the formal mathematical properties of the determinants of Laplacians. Now we can formulate the mathematical side of the problem in a precise form without appealing to the LSW model: Do inequalities derived in this paper in the framework of the gauge theory also hold if we replace all Wilson loops \( W(C) \) by the determinants of Laplace operators \( \text{Det} \Delta_C \) with \( \nu > 0 \)? A partial positive answer to this interesting question will be given in Sec. 5 for some limited class of inequalities.

1.7 Structure of the paper

In Sec. 2 inequality (1.10) is derived (Sec. 2.1) and its compatibility with the area law is checked (Sec. 2.2).

Sec. 3 is devoted to the renormalization of inequalities. In Sec. 3.1 the basic facts about the renormalization of Wilson loops are briefly described. In Sec. 3.2 we analyze the limit when the trapezium is contracted into a triangle in inequality (1.10). In this singular limit inequalities become trivial but consistent. We show that this consistency is guaranteed by the convexity of the cusp anomalous dimension (Sec. 3.3). Generalizing the example of the singular trapezium-to-triangle limit, we describe in Sec. 3.4 two classes of inequalities quadratic in Wilson loops: one group of inequalities survives in a non-trivial form after the renormalization whereas the second group reduces to trivial but correct inequalities.

In Sec. 4 we show how more general polynomial inequalities can derived. In Sec. 4.1 we derive these inequalities in the form of positivity constraints on determinants of matrices of different Wilson loops. As an example, we consider a generalization of the trapezium-parallelogram inequality in Sec. 4.2. In Sec. 4.3 we check the compatibility of inequalities with the area law. The renormalization properties of these general inequalities are described in Sec. 4.4 and proved in Appendix A. In Sec. 4.5 we comment on specific features of inequalities for

\[\text{Certainly the LSW model has nothing to do with the true small distance physics controlling the cusp behavior in the gauge theory. However, the formal mathematical extrapolation of the LSW model expressions to nonphysical contours (like the above discussion of the connection between the convexity of the cusp anomalous dimension and possible inequalities in the LSW model) can be used as a heuristic way to approach the properties of the determinants of Laplacians. From the rigorous point of view, these conjectures about the determinants should be checked mathematically without any reference to the LSW model.}\]
non-flat Wilson loops. In Sec. 4.6 we discuss the connection between the spectral representations for Wilson loops and inequalities.

Sec. 5 contains several interesting observations concerning the LSW model for Wilson loops [10, 11]. In this model cusp singularities of polygonal contours are described by an anomalous dimension which is different from the gauge theories but still has convexity property (Sec. 5.2). The rectangular loops computed in this string model (Sec. 5.3) obey inequalities derived for Wilson loops in the context of the gauge theory (Sec. 5.4).

2 Trapezium-parallelogram inequality

2.1 Derivation

Our derivation of inequalities is a trivial modification of the original method used by Bachas in Ref. [2]. Let us consider two open paths $B_1$ and $B_2$ with common end points $x, y$ at zero Euclidean time:

$$x_4 = y_4 = 0.$$  \hspace{1cm} (2.1)

We assume that both paths are placed in the semi-space with negative Euclidean times and only the end points reach the zero-time hyperplane as shown in Fig. 2 (c). We use notation $U_{ab}(B_k)$ for the operator Wilson lines taken along the lines $B_k$. Here $a, b$ are gauge indices associated with the open ends of the line $B_k$.

Then the positivity of the squared norm of the linear combination

$$\left\| \sum_{ab} [k_1^{ab} U_{ab}(B_1)|0\rangle + k_2^{ab} U_{ab}(B_2)] |0\rangle \right\|^2 \geq 0$$  \hspace{1cm} (2.2)

with arbitrary coefficients $k_1^{ab}, k_2^{ab}$ leads to the Cauchy inequality

$$|w_{12}|^2 \leq w_{11}w_{22}$$  \hspace{1cm} (2.3)

where

$$w_{kl} = \sum_{ab} \langle 0 | [U_{ab}(B_l)]^+ U_{ab}(B_k)|0 \rangle.$$  \hspace{1cm} (2.4)

Obviously

$$W(C_{kl}) = \frac{1}{N} w_{kl}$$  \hspace{1cm} (2.5)

is nothing else but the Wilson line (1.1) corresponding to the closed contour made of $B_k$ with the time-reflected path $B_l^T$:

$$C_{kl} = B_k \cup B_l^T.$$  \hspace{1cm} (2.6)

Now inequality (2.3) takes the form

$$|W(C_{12})|^2 \leq W(C_{11})W(C_{22}).$$  \hspace{1cm} (2.7)
Choosing the open paths \( B_k \) as shown in Fig. 2 (c), we obtain

\[ W(C_{11}) = W(C_{22}) = W_{\text{trapezium}}(a, b, h), \quad (2.8) \]

\[ W(C_{12}) = W_{\text{parallelogram}} \left( \frac{a + b}{2}, h \right). \quad (2.9) \]

The parallelogram-trapezium inequality (1.10) immediately follows from inequality (2.7).

In the above derivation we could choose arbitrary lines \( B_k \) (with common end points on the zero time hyperplane and all other points having negative time). Therefore inequality (2.7) is valid for arbitrary composite contours \( C_{kl} \) made from open lines \( B_k \) according to (2.6).

Taking inequality (2.7) with open lines \( B_1, B_2 \) shown in Fig. 3 one obtains the inequality for rectangular loops

\[ \left[ W \left( \frac{T_1 + T_2}{2}, R \right) \right]^2 \leq W(T_1, R) W(T_2, R), \quad (2.10) \]

which due to the symmetry

\[ W(T, R) = W(R, T), \quad (2.11) \]

reproduces Bachas inequality (1.2).

**2.2 Compatibility with the area law**

Inequality (1.10) was derived using only the positivity of the norm for physical states. Therefore this inequality is general. It holds both in gauge theories with confinement and in theories without confinement. Still we can check the compatibility of this inequality with the area law. The standard dogma of the area law for the large-size Wilson loops in a confining non-Abelian pure gauge theory reads

\[ W(C) = A(C) \exp \left[ -a_0 L(C) - K S(C) \right]. \quad (2.12) \]
Here \( L(C) \), \( S(C) \) are the perimeter and the surface area of a large flat contour \( C \), whereas the prefactor \( A(C) \) may have only a nonexponential behavior. Coefficients \( a_0, K \) are the same as in the large distance expansion of the static potential \( \text{[1.6]} \).

Now we can turn to the question of the compatibility of the area law \( \text{(2.12)} \) with inequality \( \text{(1.10)} \). Note that the parallelogram and the trapezium have the same perimeter and area in Eq. \( \text{(1.10)} \). Therefore inserting \( \text{(2.12)} \) into inequality \( \text{(1.10)} \), we find

\[
A_{\text{parallelogram}} \left( \frac{a + b}{2}, h \right) \leq A_{\text{trapezium}} (a, b, h). \tag{2.13}
\]

Thus the main result is that the exponential area and perimeter growth can be factored out from the inequality.

3 Renormalization and cusp singularities

3.1 Renormalization

Some care must be taken about the ultraviolet divergences and their renormalization. As is well known \([8, 18, 19, 20, 21]\), the renormalization of closed Wilson loops involves two factors:

1) perimeter renormalization constant \( Z_{\text{perimeter}}(\Lambda, L) \) depending on the length of the contour \( L \) and on the ultraviolet cutoff \( \Lambda \) with the exponential dependence on \( L \):

\[
Z_{\text{per}}(\Lambda, L_1 + L_2) = Z_{\text{per}}(\Lambda, L_1)Z_{\text{per}}(\Lambda, L_2), \tag{3.1}
\]

2) cusp renormalization constant factorizable in contributions of separate cusps with angles \( \gamma_k \) (Fig. 1)

\[
\prod_k Z_{\text{cusp}}(\Lambda, \gamma_k). \tag{3.2}
\]

The renormalization of \( W(C) \) is described by the equation \([8, 18, 19, 20, 21]\)

\[
W^{\text{ren}}(C) = \lim_{\Lambda \to \infty} Z(\Lambda, C)W^{\text{nonren}}(C, \Lambda) \tag{3.3}
\]

where

\[
Z(\Lambda, C) = Z_{\text{per}}(\Lambda, L(C)) \prod_{C\text{cusps}} Z_{\text{cusp}}(\Lambda, \gamma). \tag{3.4}
\]

Now we see that

\[
Z_{\text{trapezium}}(\Lambda) = Z_{\text{parallelogram}}(\Lambda) \tag{3.5}
\]

because our parallelogram and trapezium have the same perimeter and the same set of cusp angles.

Therefore renormalization does not change inequality \( \text{(1.10)} \).
3.2 Singularity of the triangle limit

Now let us give an example of an inequality which becomes trivial after renormalization. At first sight, one could take the limit $b \to 0$ in inequality (1.10) as shown in Fig. 4 and obtain the inequality for the Wilson loop corresponding to the triangle with the base $a$ and height $h$:

$$\left|W_{\text{nonren parallelogram}} \left( \frac{a}{2}, h; \Lambda \right) \right| \leq W_{\text{triangle}}(a, h; \Lambda) .$$ (3.6)

This inequality is derived for regularized and nonrenormalized Wilson loops. The renormalization of the Wilson loops is described by equations (3.3), (3.4):

$$W_{\text{ren triangle}}(a, h) = \lim_{\Lambda \to \infty} Z_{\text{per.}}(\Lambda, L_{\text{triangle}}) \times \prod_{k=1}^{3} Z_{\text{cusp}}(\Lambda, \gamma_k^{\text{triangle}}) \times W_{\text{nonren triangle}}(a, h; \Lambda) .$$ (3.7)

$$W_{\text{ren parallelogram}} \left( \frac{a}{2}, h \right) = \lim_{\Lambda \to \infty} Z_{\text{per.}}(\Lambda, L_{\text{parallelogram}}) \times \prod_{k=1}^{4} Z_{\text{cusp}}(\Lambda, \gamma_k^{\text{parallelogram}}) \times W_{\text{nonren parallelogram}} \left( \frac{a}{2}, h; \Lambda \right) .$$ (3.8)

Note that the triangle and the parallelogram have the same perimeter

$$L_{\text{triangle}} = L_{\text{parallelogram}}$$ (3.9)

so that the contribution of the perimeter renormalization cancels in the renormalized version of inequality (3.10). However, the set of cusp angles is different:

$$\frac{\prod_{k=1}^{4} Z_{\text{cusp}}(\Lambda, \gamma_k^{\text{parallelogram}})}{\prod_{k=1}^{3} Z_{\text{cusp}}(\Lambda, \gamma_k^{\text{triangle}})} = \frac{[Z_{\text{cusp}}(\Lambda, \gamma)]^2}{Z_{\text{cusp}}(\Lambda, 2\gamma)} .$$ (3.10)

Here we noticed that the mismatch between cusp angles involves two angles $\gamma$ of the parallelogram and one angle of the triangle $2\gamma$ which is twice larger (see Fig. 4).
Now the renormalized version of inequality (3.6) becomes

\[ \left| W_{\text{ren parallelogram}} \left( \frac{a}{2}, h \right) \right| \leq W_{\text{triangle}}^{\text{ren}}(a, h) \lim_{\Lambda \to \infty} \frac{[Z_{\text{cusp}}(\Lambda, \gamma)]^2}{Z_{\text{cusp}}(\Lambda, 2\gamma)}. \] (3.11)

The ratio of renormalization constants \( Z_{\text{cusp}}(\Lambda, \gamma) \) has a singular behavior in the limit \( \Lambda \to \infty \) and has a nontrivial dependence on \( \gamma \). In principle, two cases could be expected

\[ \lim_{\Lambda \to \infty} \frac{[Z_{\text{cusp}}(\Lambda, \gamma)]^2}{Z_{\text{cusp}}(\Lambda, 2\gamma)} = \begin{cases} \infty, \\ 0. \end{cases} \] (3.12)

In the first case of the infinite limit, inequality (3.11) would be reduced to the triviality

\[ W_{\text{ren parallelogram}}^{\text{ren}} \left( \frac{a}{2}, h \right) \leq \infty. \] (3.13)

In the second case of the zero limit in Eq. (3.12) we would obtain a rather improbable result:

\[ W_{\text{ren parallelogram}}^{\text{ren}} \left( \frac{a}{2}, h \right) = 0. \] (3.14)

From this point of view one would expect that the first case in (3.12)

\[ \lim_{\Lambda \to \infty} \frac{[Z_{\text{cusp}}(\Lambda, \gamma)]^2}{Z_{\text{cusp}}(\Lambda, 2\gamma)} = \infty \] (3.15)

is more natural.

In asymptotically free theories this limit can be studied perturbatively. In the next section we will show that in asymptotically free non-Abelian \( SU(N_c) \) gauge theory we have

\[ \lim_{\Lambda \to \infty} \frac{[Z_{\text{cusp}}(\Lambda, \gamma + \gamma')]^2}{Z_{\text{cusp}}(\Lambda, 2\gamma) Z_{\text{cusp}}(\Lambda, 2\gamma')} = \begin{cases} \infty & \text{if } \gamma \neq \gamma', \\ 1 & \text{if } \gamma = \gamma'. \end{cases} \] (3.16)

in the interval

\[-\frac{\pi}{2} \leq \gamma, \gamma' \leq \frac{\pi}{2}.\] (3.17)

Taking \( \gamma' = 0 \) in this relation (which means no cusp so that \( Z_{\text{cusp}}(\Lambda, 2\gamma') = 1 \), we obtain divergence (3.15).

We see that the limit \( b \to 0 \) in the renormalized version of trapezium-parallelogram inequality (3.6) leads to triviality (3.13). Although our attempt to derive an inequality for renormalized triangular Wilson loops has failed, the fact that one avoids the pathological case (3.14) demonstrates the compatibility of inequality (1.10) with the properties of the cusp anomalous dimension controlling the limit (3.15). These properties will be considered in the next section.
3.3 Convexity of the cusp anomalous dimension

Let us compute the limit (3.16). In the asymptotically free case (e.g. QCD with a limited number of quark flavors \( N_f < 11N_c/2 \)) this limit is determined by the leading order of the cusp anomalous dimension (1.8)

\[
\Gamma^{(1)}_{\text{cusp}}(g, \gamma) = \frac{g^2}{4\pi^2} \Gamma(\gamma), \quad (3.18)
\]

\[
\Gamma(\gamma) = -4C_F \left( 1 - \gamma \cot \gamma \right), \quad (3.19)
\]

\[
C_F = \frac{N_c^2 - 1}{2N_c}, \quad (3.20)
\]

where \( \gamma \) is the cusp angle dual to the inside-facing angle \( \pi - \gamma \) as shown in Fig. 1.

Starting from the trivial inequalities

\[
0 < \gamma < \tan \gamma \quad \left( 0 < \gamma < \frac{\pi}{2} \right), \quad (3.21)
\]

\[
\cot \gamma \leq 0 \quad \left( \frac{\pi}{2} \leq \gamma < \pi \right), \quad (3.22)
\]

one finds

\[
1 - \gamma \cot \gamma > 0. \quad (3.23)
\]

for \( 0 < \gamma < \pi \). Since function \( 1 - \gamma \cot \gamma \) is even, we can extend the validity interval to

\[-\pi < \gamma < \pi \quad (3.24)\]

except for the point \( \gamma = 0 \). Now we can prove the positivity of the second derivative:

\[
\frac{d^2}{d^2\gamma} (1 - \gamma \cot \gamma) = (1 - \gamma \cot \gamma) \frac{2}{\sin^2 \gamma} > 0. \quad (3.25)
\]

This shows that

\[
\Gamma(\gamma) < 0 \quad \left( -\pi < \gamma < \pi, \quad \gamma \neq 0 \right), \quad (3.26)
\]

\[
\frac{d^2}{d^2\gamma} \Gamma(\gamma) < 0 \quad \left( -\pi < \gamma < \pi \right). \quad (3.27)
\]

Hence function \( \Gamma(\gamma) \) is convex

\[
\Gamma(\gamma + \gamma') > \frac{1}{2} \left[ \Gamma(2\gamma) + \Gamma(2\gamma') \right] \quad \left( -\pi/2 < \gamma \neq \gamma' < \pi/2 \right). \quad (3.28)
\]

This is nothing else but convexity property (1.9) which was already mentioned in the introduction.

The asymptotic behavior of the renormalization constant at large cutoff \( \Lambda \) is

\[
Z_{\text{cusp}}(\Lambda, \gamma) \xrightarrow{\Lambda \to \infty} [g(\Lambda)]^{-\Gamma(\gamma)/\beta_1} \quad (3.29)
\]
where $\beta_1$ is the first coefficient of the beta function

$$\beta_1 = \frac{11}{3} N_c - \frac{2}{3} N_f.$$  \hfill (3.30)

In the limit $\Lambda \to \infty$, using (3.28) and $g(\Lambda) \to 0$, we find for $-\pi/2 < \gamma \neq \gamma' < \pi/2$

$$\frac{[Z_{\text{cusp}}(\Lambda, \gamma + \gamma')]^2}{Z_{\text{cusp}}(\Lambda, 2\gamma)} \sim \left[ g^2(\Lambda) \right]^{-\left[ 2\Gamma(\gamma + \gamma') - \Gamma(2\gamma) - \Gamma(2\gamma') \right]/\beta_1} \to \infty. \hfill (3.31)$$

Thus the limit (3.16) is computed (the case $\gamma = \gamma'$ trivial). In particular, this completes the derivation of relation (3.15) which was used for the analysis of the singularities accompanying the transformation of a trapezium into a triangle.

### 3.4 Which inequalities remain nontrivial in the continuum limit

The above derivation of inequality (2.7) can be generalized for arbitrary open paths $B_1, B_2$ but one should take care about the renormalization stability of inequalities. The nonrenormalized inequality

$$[W_{\text{nonren}}(C_{12}, \Lambda)]^2 \leq W_{\text{nonren}}(C_{11}, \Lambda) W_{\text{nonren}}(C_{22}, \Lambda) \hfill (3.32)$$

is modified by the renormalization (3.3):

$$[W_{\text{ren}}(C_{12})]^2 \leq W_{\text{ren}}(C_{11}) W_{\text{ren}}(C_{22}) \lim_{\Lambda \to \infty} \frac{[Z(\Lambda, C_{12})]^2}{Z(\Lambda, C_{11}) Z(\Lambda, C_{22})}. \hfill (3.33)$$

In the simplest case when the two open paths $B_1$ and $B_2$ reach the end points with the tangent lines directed along the Euclidean time axis [e.g. the cases shown in Figs. 2(c) and 3] this stability is trivial:

1) The perimeters of contours $C_{kl}$ in (2.7) obey condition

$$2L(C_{12}) = L(C_{11}) + L(C_{22}) = 2L(B_1) + 2L(B_2). \hfill (3.34)$$

Inserting this into (3.11), we obtain

$$Z_{\text{per}}(\Lambda, L(C_{11})) Z_{\text{per}}(\Lambda, L(C_{22})) = [Z_{\text{per}}(\Lambda, L(C_{12}))]^2. \hfill (3.35)$$

2) The combined set of cusp angles of contours $C_{11}$ and $C_{22}$ coincides with the double set of cusp angles of the contour $C_{12}$. Therefore

$$\left[ \prod_{C_{11} \text{ cusps}} Z_{\text{cusp}}(\Lambda, \gamma) \right] \left[ \prod_{C_{22} \text{ cusps}} Z_{\text{cusp}}(\Lambda, \gamma) \right] = \left[ \prod_{C_{12} \text{ cusps}} Z_{\text{cusp}}(\Lambda, \gamma) \right]^2. \hfill (3.36)$$
Figure 5: (a) Open lines $B_k$ and their end point cusp angles $\gamma_k, \gamma'_k$ with respect to the horizontal Euclidean time axis. Note that angles $\gamma_k, \gamma'_k$ can be negative. (b) Corresponding closed contours $C_{kl}$ with the cusp angles $\gamma_k + \gamma_l$ and $\gamma'_k + \gamma'_l$. 

15
Inserting Eqs. (3.35), (3.36) into Eq. (3.4), we find that the renormalization constants for contours $C_{kl}$ obey relation

$$Z(\Lambda, C_{11})Z(\Lambda, C_{22}) = [Z(\Lambda, C_{12})]^2$$

(3.37)

so that renormalization (3.33) does not change inequality (2.7).

The situation becomes more interesting if we allow for lines $B_1$ and $B_2$ which have arbitrary directions of tangent vectors at the end points. In this case

$$\sum_{C_{kl} \text{ cusps}} \Gamma(\gamma) = \sum_{\text{internal } B_1 \text{ cusps}} \Gamma(\gamma) + \sum_{\text{internal } B_2 \text{ cusps}} \Gamma(\gamma) + \Gamma(\gamma_k + \gamma_l) + \Gamma(\gamma_k' + \gamma_l').$$

(3.38)

where $\gamma_k$ and $\gamma_k'$ are “end-point cusp angles” of open lines $B_k$ counted from the Euclidean time axis as shown in Fig. 5 (a). These end-point cusp angles of $B_k$ and $B_l$ generate additional cusp angles $\gamma_k + \gamma_l$ and $\gamma_k' + \gamma_l'$ of the closed contour $C_{kl}$ (see Fig. 5 (b)). Note that angles $\gamma_k, \gamma_k'$ can be positive or negative:

$$-\frac{\pi}{2} < \gamma_k, \gamma_k' < \frac{\pi}{2}$$

(3.39)

The choice of the positive direction for counting angles $\gamma_k$ does not matter ($\Gamma(\gamma)$ is an even) but this choice should be the same for all lines $B_k$.

Using Eq. (3.38), we see that the internal cusp angles of lines $B_k$ cancel in the combination

$$\sum_{C_{11} \text{ cusps}} \Gamma(\gamma) + \sum_{C_{22} \text{ cusps}} \Gamma(\gamma) - 2 \sum_{C_{12} \text{ cusps}} \Gamma(\gamma)$$

$$= [\Gamma(2\gamma_1) + \Gamma(2\gamma_1')] + [\Gamma(2\gamma_2) + \Gamma(2\gamma_2')] - 2 [\Gamma(\gamma_1 + \gamma_2) + \Gamma(\gamma_1' + \gamma_2')]$$

$$= [\Gamma(2\gamma_1) + \Gamma(2\gamma_2) - 2\Gamma(\gamma_1 + \gamma_2)] + [\Gamma(2\gamma_1') + \Gamma(2\gamma_2') - 2\Gamma(\gamma_1' + \gamma_2')].$$

(3.40)

On the RHS we have two terms which are almost strictly negative according to inequality (3.28):

$$\Gamma(2\gamma_1) + \Gamma(2\gamma_2) - 2\Gamma(\gamma_1 + \gamma_2) < 0 \quad \text{if} \quad -\pi/2 < \gamma_1 \neq \gamma_2 < \pi/2,$$

$$\Gamma(2\gamma_1') + \Gamma(2\gamma_2') - 2\Gamma(\gamma_1' + \gamma_2') < 0 \quad \text{if} \quad -\pi/2 < \gamma_1' \neq \gamma_2' < \pi/2.$$ 

(3.41)

(3.42)

Therefore

$$\sum_{C_{11} \text{ cusps}} \Gamma(\gamma) + \sum_{C_{22} \text{ cusps}} \Gamma(\gamma) - 2 \sum_{C_{12} \text{ cusps}} \Gamma(\gamma) \begin{cases} < 0 & \text{if } \gamma_1 \neq \gamma_2 \text{ or } \gamma_1' \neq \gamma_2', \\ = 0 & \text{if } \gamma_1 = \gamma_2 \text{ and } \gamma_1' = \gamma_2'. \end{cases}$$

(3.43)

Combining this with Eq. (3.29), we find

$$\lim_{\Lambda \to \infty} \frac{|Z(\Lambda, C_{12})|^2}{Z(\Lambda, C_{11})Z(\Lambda, C_{22})}$$

$$= \lim_{\Lambda \to \infty} \exp \left\{ \frac{\ln y^2}{\beta_1} \left[ \sum_{C_{11} \text{ cusps}} \Gamma(\gamma) + \sum_{C_{22} \text{ cusps}} \Gamma(\gamma) - 2 \sum_{C_{12} \text{ cusps}} \Gamma(\gamma) \right] \right\}$$

$$= \left\{ \begin{array}{ll} \infty & \text{if } \gamma_1 \neq \gamma_2 \text{ or } \gamma_1' \neq \gamma_2', \\ 1 & \text{if } \gamma_1 = \gamma_2 \text{ and } \gamma_1' = \gamma_2'. \end{array} \right.$$ 

(3.44)
Inserting this into inequality (3.33), we see that a nontrivial result comes after the renormalization only in the case

\[ \gamma_1 = \gamma_2, \quad \gamma'_1 = \gamma'_2. \]  

(3.45)

Otherwise after renormalization we arrive at the triviality

\[ |W(C_{12})| \leq \infty. \]  

(3.46)

We conclude that inequality (2.7) is not changed by the renormalization only in the case when the combined set of cusp angles of contours \( C_{11} \) and \( C_{22} \) coincides with the double set of the cusp angles of contour \( C_{12} \). If this condition is violated then we arrive at the correct but useless result (3.46). Whatever trivial this result is, it is important for us that the ratio of renormalization constants (3.44) diverges to infinity and does not go to zero as it could be in principle. If we had a zero limit in (3.44) then this would correspond to a rather improbable situation with many vanishing Wilson loops. Now one has to remember that the divergence of the ratio (3.44) and the protection from the zero limit originates from the convexity property of the cusp anomalous dimension which was used in Eqs. (3.41), (3.42). Thus we see that the convexity of the cusp anomalous dimension "protects" the consistency of the theory when one tries to take singular limits in inequalities for Wilson loops.

4 More general inequalities

4.1 Derivation

We can generalize inequality (2.2) by taking an arbitrary amount of open paths \( B_k \) with common end points:

\[
\left\| \sum_{m=1}^{M} \sum_{ab} \left[ k_m^{ab} U_{ab}(B_m) \right] |0\rangle \right\|^2 \geq 0.
\]  

(4.1)

This means that matrix

\[
W(C_{kl}) = \frac{1}{N} \sum_{ab} (0| U_{ab}(B_l))^\dagger U_{ab}(B_k)|0\rangle
\]  

(4.2)

is positive definite. In particular, for any set of open paths \( B_k \) with common end points we must have the positive determinant

\[
\det_{1 \leq k, l \leq M} W(C_{kl}) \geq 0.
\]  

(4.3)

Note that in the case \( M = 2 \) we reproduce the old inequality (2.7).
Figure 6: (a) Open paths $B_k$ used for the derivation of inequalities. (b) Corresponding contours $C_{kl}$. (c), (d), (e) Graphic representation of inequalities (4.14), (4.15), (4.18).

4.2 Examples

If we take several open paths shown in Fig. 3 then we obtain the following inequality for the rectangular $T \times R$ Wilson loops $W(T, R)$

$$\det_{1 \leq k, l \leq M} W\left(\frac{T_k + T_l}{2}, R\right) \geq 0. \quad (4.4)$$

This inequality should hold for any finite set of points $T_k$.

Concerning applications of inequality (4.3) to nonrectangular contours, let us consider the simple example corresponding to open paths $B_k$ shown in Fig. 6 (a).

Let us introduce a compact notation for the contours $C_{kl}$ shown in Fig. 6.
(b):

\[ W(C_{11}) \equiv r \quad \text{(rectangle)}, \]
\[ W(C_{22}) = W(C_{33}) \equiv t \quad \text{(trapezium)}, \]
\[ W(C_{23}) = W(C_{32}) \equiv p \quad \text{(parallelogram)}, \]
\[ W(C_{12}) = W(C_{13}) = W(C_{21}) = W(C_{31}) \equiv x. \]

Then the positivity of the matrix

\[
W(C_{kl}) = \begin{pmatrix}
    r & x & x \\
    x & t & p \\
    x & p & t
\end{pmatrix}
\]

leads to the positivity of the following minor determinants

\[ \det(r) = r \geq 0, \quad \text{(4.10)} \]
\[ \det(r) = t \geq 0, \quad \text{(4.11)} \]
\[ \det \begin{pmatrix}
    r & x \\
    x & t
\end{pmatrix} = rt - x^2 \geq 0, \quad \text{(4.12)} \]
\[ \det \begin{pmatrix}
    t & p \\
    p & t
\end{pmatrix} = t^2 - p^2 \geq 0, \quad \text{(4.13)} \]

which results in

\[ x^2 \leq rt, \quad \text{(4.14)} \]
\[ |p| \leq t. \quad \text{(4.15)} \]

The positivity of the main determinant

\[ \det \begin{pmatrix}
    r & x & x \\
    x & t & p \\
    x & p & t
\end{pmatrix} = r(t^2 - p^2) - 2x^2(t - p) > 0 \quad \text{(4.16)} \]

results in

\[ r(t - p)(t + p) \geq 2x^2(t - p). \quad \text{(4.17)} \]

If \( t \neq p \) then according to inequality (4.15) we can cancel \( t - p \)

\[ 2x^2 \leq r(t + p). \quad \text{(4.18)} \]

But actually this inequality holds also at \( t = p \) [in this case one has to use
inequality (4.14)].

We can restrict ourselves to the set (4.15), (4.18). Inequality (4.14) is their consequence. Graphically inequalities (4.14), (4.15), (4.18) are shown in Fig. 6
(c,d,e). Certainly the above derivation of these inequalities from the positivity
of the matrix (4.9) is not the shortest way. We simply wanted to illustrate how
the general method based on the positivity of matrix \( W(C_{kl}) \) works.
4.3 Area law

Note that inequalities (4.3) have many basic properties which were established earlier for the simple inequality (2.7). For example, these inequalities are again compatible with the area law. Indeed, keeping in mind that length \( L(C_{kl}) \) and area \( S(C_{kl}) \) of \( C_{kl} \) are additive in \( B_k \) and \( B_l \) for flat contours:

\[
L(C_{kl}) = L(B_k) + L(B_l), \\
S(C_{kl}) = S(B_k) + S(B_l).
\]

we find in the case of the area law (2.12) for large contours:

\[
W(C_{kl}) = \det \begin{array}{c}
A(C_{kl}) \\
\det A(C_{kl}) \geq 0
\end{array}.
\]

Thus inequality (4.3) takes the form

\[
\det_{1 \leq k, l \leq M} \det \begin{array}{c}
A(C_{kl}) \\
\det A(C_{kl}) \geq 0
\end{array}.
\]

We see that the exponentially growing area and perimeter terms are factored out from the inequality.

4.4 Renormalization

Now we want to study the renormalization of the inequality (4.3) which was derived for the nonrenormalized Wilson loops

\[
\det_{1 \leq k, l \leq M} W_{\text{nonren}}(C_{kl}) \geq 0.
\]

The results of our analysis will be quite similar to results obtained in Sec. 3.4 for the case \( M = 2 \):

1) If all open paths \( B_k \) have the same external cusp angles

\[
\gamma_1 = \gamma_2 = \ldots = \gamma_M \equiv \gamma, \\
\gamma'_1 = \gamma'_2 = \ldots = \gamma'_M \equiv \gamma'.
\]

then all \( n! \) terms of the determinant (4.24) have the same renormalization constant so that the renormalized version of the inequality is simply

\[
\det_{1 \leq k, l \leq M} W_{\text{ren}}(C_{kl}) \geq 0.
\]

In the case of non-flat contours, the additivity of the surface area can break down as shown in Sec. 4.5.
2) In the case when conditions (4.25), (4.26) are violated, the renormalization of the inequality (4.24) does not lead to new inequalities. In this case one arrives either at trivial inequalities or at inequalities which can be directly derived from the inequalities obtained in the case (4.25), (4.26).

These properties are proved in Appendix A.

4.5 Further generalizations

So far we used connected open paths $B_k$ with two end points for the derivation of inequalities. In Fig. 7 (a) we show another example with four end points. This choice leads to contours $C_{kl}$ shown in Fig. 7 (b). The general inequality (2.7) for these contours $C_{kl}$ is shown graphically in Fig. 7 (c).

The check of the compatibility of this inequality with the area law is tricky because now we cannot use the additivity of the surface area (4.20). In fact, we must work with the minimal surface. In the case of contours $C_{kl}$ shown in
Fig. 7(b) the minimal surface area is given by
\[ S_{\text{min}}(C_{11}) = \min \{ 4a_1b_1, 2(a_1 + 2b_1)a_2 \}, \]
\[ S_{\text{min}}(C_{22}) = \min \{ 4a_2b_2, 2(a_2 + 2b_2)a_1 \}, \]
\[ S_{\text{min}}(C_{12}) = a_1a_2 + 2(b_1 + b_2) \min(a_1, a_2). \] (4.28)

Using the general inequality
\[ \min(u, v) + \min(x, y) \leq \min(u + y, x + v), \] (4.29)
we find
\[ S_{\text{min}}(C_{11}) + S_{\text{min}}(C_{22}) \leq \min \{ 4a_1b_1 + 2(a_2 + 2b_2)a_1 \}, \]
\[ = 2a_1a_2 + 4 \min \{ (a_1b_1 + a_2b_2), (a_2b_1 + a_2b_2) \}, \]
\[ = 2a_1a_2 + 4(b_1 + b_2) \min(a_1, a_2) = 2S_{\text{min}}(C_{12}). \] (4.30)
Thus
\[ S_{\text{min}}(C_{11}) + S_{\text{min}}(C_{22}) \leq 2S_{\text{min}}(C_{12}), \] (4.31)
which proves the compatibility of inequality (2.7) with the area law.

4.6 Positivity and spectral representation for rectangular Wilson loops

As was already mentioned, inequalities for rectangular Wilson loops [the original Bachas inequality (1.2) and its generalization (4.4)] can be considered as particular cases of the general inequality (4.3). However, in the rectangular case we have much better understanding of the positivity constraints. It comes from the spectral representation for rectangular Wilson loops. Using the gauge \( A_4 = 0 \), we can write
\[ W(T, R) = \frac{1}{N} \sum_{ab} \langle 0 | [U_{ab}(0, R)]^+ e^{-HT} U_{ab}(0, R) | 0 \rangle \]
\[ = \frac{1}{N} \sum_{ab} \sum_n e^{-E_n(R)T} | \langle n | U_{ab}(0, R) | 0 \rangle |^2. \] (4.32)
This can be rewritten in the form
\[ W(T, R) = \int_{E_0(R)}^{\infty} dE e^{-ET} \rho(E, R) \] (4.33)
where
\[ \rho(E, R) = \frac{1}{N} \sum_{ab} \sum_n \delta(E - E_n(R)) e^{-E_n(R)T} | \langle n | U_{ab}(0, R) | 0 \rangle |^2. \] (4.34)
Obviously
\[ \rho(E, R) \geq 0 . \] (4.35)

Starting from the spectral representation (4.33), one can easily reproduce inequality (4.4). Indeed,
\[ \sum_{kl} x_k x_l W \left( \frac{T_k + T_l}{2}, R \right) = \int_{E_0(R)}^\infty dE \left( \sum_k x_k e^{-ET_k/2} \right)^2 \rho(E, R) \geq 0 . \] (4.36)

This shows that the quadratic form \( W \left( \frac{(T_k + T_l)}{2}, R \right) \) is positive definite, which results in inequality (4.4).

5 Inequalities and cusp divergences in the effective string model for Wilson loops

5.1 Effective string model

In Refs. [10], [11] it was suggested to consider Nambu-Goto string with the boundary fixed on a closed contour \( C \) as an effective model for Wilson loops \( W(C) \) with large smooth contours \( C \). In this model the Wilson loop \( W(C) \) is approximated
\[ W_{\text{LSW}}(C) \sim Z(C) \] (5.1)
by the string “partition function” \( Z(C) \) in \( D \)-dimensional space-time computed for open strings fixed with Dirichlet boundary conditions (up to a reparametrization) on contour \( C \). For flat contours \( C \) this partition function (computed in the quadratic approximation in deviations from the minimal surface) reduces to the determinant of Laplace operator \( \Delta_C \) with Dirichlet boundary conditions on contour \( C \)
\[ Z(C) \sim e^{-KS(C)} [\text{Det} (-\Delta_C)]^{-(D-2)/2} . \] (5.2)

Applying this model to (smoothed) rectangular contours, one can extract [10], [11] the \( 1/R \) correction (1.6) to the potential (1.6) at large distances:
\[ V(R) \underset{R \to \infty}{=} KR + 2a_0 - \frac{\pi(D - 2)}{24} \frac{1}{R} . \] (5.3)

Without touching the interesting question about the theoretical status of the LSW model, we would like to concentrate on the consistency of the model with the inequalities derived in this paper for Wilson loops.

5.2 Cusp singularities

The small-\( t \) expansion (1.11) leads to the factorized structure of the ultraviolet divergences for the polygonal contours
\[ [\text{Det} (-\Delta_C)]_{\text{ren}} = \lim_{\Lambda \to \infty} \left[ \exp \left[ \text{const} S(C)\Lambda^2 + \text{const} L(C)\Lambda \right] \prod_i \Lambda^{-2\Gamma_{\text{cusp}}(\theta_i)} \right] \times [\text{Det} (-\Delta_C)]_{\text{nonren,}\Lambda} . \] (5.4)
The renormalization constant appearing in Eq. (5.4) should be taken to the power \((- (D - 2)/2\) according to Eq. (5.2). Comparing the string (5.4) and gauge (3.20) cusp factors

\begin{align*}
\text{Gauge theories : } & \quad Z_{\text{cusp}}(\Lambda, \gamma) \xrightarrow{\Lambda \to \infty} [g(\Lambda)]^{-\Gamma(\gamma)/\beta_1} \sim [\ln \Lambda]^{\Gamma(\gamma)/(2\beta_1)}, \quad (5.5) \\
\text{String model : } & \quad Z_{\text{cusp}}(\Lambda, \gamma) \xrightarrow{\Lambda \to \infty} \Lambda^{(D-2)\Gamma_{\text{str}}(\theta_i)}, \quad (5.6)
\end{align*}

we see that the dependence on \(\Lambda\) is different. Therefore the direct comparison of Wilson loops \(W(C)\) and string functionals \(Z(C)\) for contours with cusps makes no sense. In fact, the physical assumptions standing behind the effective string model also suggest that this model may be relevant only for large smooth contours. On the other hand, one can construct ratios of Wilson loops where cusp singularities cancel. For these ratios the comparison between gauge theories and the string model is still possible. In order to avoid uncertainties related to the area and perimeter dependent divergences, the ratios of string functionals \(Z(C)\) should be organized so that the exponential perimeter and area dependences cancel.

Thus we want to concentrate on ratios of Wilson lines \(W(C)\) (and ratios of corresponding string functionals \(Z(C)\)) for which

1) the perimeter and area exponential factors cancel,

2) the full set of cusp angels associated with the numerator of the ratio coincides with the full set of the denominator angles.

But these are exactly the properties of the ratios of Wilson loops which appear in the inequalities discussed in this article. Now one can ask the question whether these inequalities hold in the string model. A serious analysis of this problem is beyond the scope of this paper. But several arguments in favor of this possibility can be suggested now.

First, the string cusp anomalous dimension (1.11) obeys convexity inequality (1.12), which is a trivial consequence of the negative second derivative \(d^2\Gamma_{\text{str}}(\theta)/d\theta^2 < 0\). This is an analog of the convexity property of the cusp anomalous dimension in gauge theories (1.12). If polygon inequalities hold in the string model then the property (1.12) would keep the consistency of inequalities with respect to the renormalization of cusp singularities. Although the \(\Lambda\) dependence of cusp renormalization factors (5.5), (5.6) is different (logarithmic in gauge theories and power in the string model) the qualitative features are the same: both \(\Lambda\) and \(\ln \Lambda\) grow to infinity when \(\Lambda \to \infty\). Therefore the stability of inequalities under the renormalization is completely controlled by the convexity properties which are the same for the gauge and string anomalous dimensions. Strictly speaking, there is one small difference. The gauge anomalous dimension is invariant under \(\gamma \to -\gamma\) whereas the corresponding transformation \(\theta \to 2\pi - \theta\) changes the string anomalous dimension (the inside polygon angle \(\theta\) is implied in (1.11)). This is the reason why we prefer to write \(\Gamma_{\text{str}}\) as a function of the inside angle \(\theta\) \((0 < \theta < 2\pi)\) and not the deviation angle \(\gamma = \pi - \theta\) as in the case of the gauge theory. But one can show that the absence of the \(\gamma \to -\gamma\) symmetry does not break the arguments of Secs. 3.2, 3.4 and the same logic can be applied to the string model.
5.3 Rectangular contour

Another evidence for the validity of inequalities comes from the analysis of the simple rectangular case. The results for rectangular loops discussed here are well known \[25\]. We use them in order to illustrate how inequalities (5.1), (5.2) work in the effective string model.

In the case of the rectangular \( T \times R \) contour, the cusp renormalization factor generated by four angles \( \theta_i = \pi/2 \) and appearing in Eq. (5.4) equals according to Eq. (1.11)

\[
\prod_{i=1}^{4} \Lambda^{-2\text{cusp}(\theta_i)} = \Lambda^{-8\text{cusp}(\pi/2)} = \Lambda^{1/2}.
\]  

(5.7)

Therefore

\[
Z_{\text{ren}}(T, R) = \lim_{\Lambda \to \infty} \left\{ \frac{\Lambda^{1/2}}{\Lambda} \exp \left[ \text{const} S(C)\Lambda^2 + \text{const} L(C)\Lambda \right] \right\}^{-(D-2)/2} 
\times Z_{\text{nonren}}(\Lambda, T, R).
\]

(5.8)

Combining this with the dimensional counting, we see that

\[
Z_{\text{ren}}(T, R) = e^{-KR-2a_0(T+R)} \left[ (RT)^{1/4} f(T/R) \right]^{(D-2)/2}
\]

(5.9)

where function \( f \) depends only on the ratio \( T/R \). Function \( f(T/R) \) was computed in Ref. \[25\] and the result is\[^5\] (we omit constant factors depending on the renormalization scheme)

\[
Z_{\text{ren}}(T, R) = e^{-KRT-2a_0(T+R)} \left[ R^{-1/2} \eta(iT/R) \right]^{-(D-2)/2}.
\]

(5.10)

Here \( \eta(\tau) \) is Dedekind eta function \([B.1]\) whose properties are briefly described in Appendix \[B\]. Expression (5.10) also agrees with the determinant of the lattice regularized Laplacian computed in Ref. \[27\].

Using the property (B.3) of the \( \eta \) function, one can see that \( Z_{\text{ren}}(T, R) \) is symmetric

\[
Z_{\text{ren}}(T, R) = Z_{\text{ren}}(R, T)
\]

(5.11)

as it should be.

Taking the limit of large \( T \to \infty \) in the rectangular contour of Eq. (5.10), one can easily reproduce Lüscher term \([5.3]\). Indeed, using Eq. (B.2), we find from Eq. (5.10)

\[
Z_{\text{ren}}(T, R) \overset{T \to \infty}{=} e^{-KRT-2a_0(T+R)} \left[ R^{-1/2} e^{-(\pi/12)T/R} \right]^{-(D-2)/2}.
\]

(5.12)

Combining this with Eqs. (1.3) and (5.1), one arrives at Eq. (5.3).

\[^5\] In the case of general polygonal contours \( C \) the determinant of Laplace operator \( \Delta_C \) was studied in Ref. \[26\].
5.4 Spectral decomposition and inequalities for rectangular contours

Inserting the power series for $[\eta(i\tau)]^{-\alpha}$ into Eq. (5.10), one finds

\[
Z_{\text{ren}}(T, R) = e^{-KRT-2a_0(T+R)} \left[ R^{-1/2} \eta(iT/R) \right]^{-\frac{(D-2)}{2}}
\]

\[
= e^{-KRT-2a_0(T+R)} \left[ R e^{\pi/6} T/R \right]^{(D-2)/4} \sum_{n=0}^{\infty} A_n \left( \frac{D-2}{2} \right) e^{-2\pi n T/R}.
\]

where according to (B.6)

\[
A_n \left( \frac{D-2}{2} \right) > 0.
\]

Eq. (5.13) can be rewritten in the form

\[
Z_{\text{ren}}(T, R) = \sum_{n=0}^{\infty} C_n(R) e^{-E_n(R) T}
\]

where

\[
E_n(R) = KR + 2a_0 + \left( -\frac{D-2}{24} + n \right) \frac{\pi}{R},
\]

\[
C_{2n}(R) = e^{-2a_0 R (D-2)/4} A_n \left( \frac{D-2}{2} \right) > 0,
\]

\[
C_{2n+1}(R) = 0.
\]

The full spectrum is twice larger than can be seen in expansion (5.13) because the odd coefficients $C_{2n+1}(R)$ vanish. The lost part of the spectrum $E_n(R)$ can be found if one considers the partition function of the string model

\[
Z_\beta(R) = \sum_{n=0}^{\infty} w_n e^{-\beta E_n(R)}
\]

where $\beta$ is inverse temperature and $w_n$ are integer degeneracy weights of levels $E_n(R)$. Since partition function $Z_\beta(R)$ corresponds to periodic boundary conditions in the functional integral, one has to modify the boundary conditions in Eq. (5.2)

\[
Z_\beta(R) \sim [\text{Det} (-\Delta_{\text{mixed}})]^{-(D-2)/2},
\]

\[
\Delta_{\text{mixed}} : \begin{cases}
T & \text{periodic} \\
R & \text{Dirichlet}
\end{cases}
\]

The result of Ref. [25] for these boundary conditions is

\[
Z_\beta(R) = [\eta \left( \frac{i \beta}{2R} \right)]^{-(D-2)}.
\]
Using expansion (B.7) for the $\eta$ function, one finds

$$Z_\beta(R) = e^{-KRT-2a_0(T+R)}e^{\beta(D-2)\pi/(24R)} \sum_{n=0}^{\infty} A_n(D-2)e^{-\pi n \beta/R}. \quad (5.23)$$

Comparing this with (5.19), one obtains spectrum (5.16) and the degeneracies

$$w_n = A_n(D-2). \quad (5.24)$$

Note that numbers $A_n(D-2)$ are integer (see Eq. (B.12) in Appendix B).

Series (5.15) is a spectral decomposition of type (4.33) with the spectral density made of discrete delta-functions. As was shown in section 4.6, representation (4.33) guarantees that inequality (4.4) holds. Thus the string model for Wilson loops (5.1), (5.2) satisfies inequalities (4.4) for rectangular contours $C$.

In terms of two-dimensional Laplacians $\Delta(T,R)$ with Dirichlet boundary conditions on $T \times R$ rectangular contours these inequalities can be rewritten in the form

$$\det_{1 \leq k,l \leq M} \left\{ \left[ \text{Det} \Delta \left( \frac{T_k + T_l}{2}, R \right) \right]^{-\nu} \right\} \geq 0 \quad (5.25)$$

where $\nu > 0$.

6 Conclusions

We have considered two ways of the generalization of the original Bachas inequality (1.2):

1) extension to nonrectangular contours (2.7),

2) determinant inequalities (4.3) containing several composite contours.

The transition from the rectangular contours to arbitrary polygons does not allow us to use the lattice regularization (at least its standard cubic version). Therefore one should take a special care about the renormalization of inequalities. We have found that naive polygon inequalities survive in the renormalized theory only if the balance of cusp angles (3.45) holds. In the absence of this balance the inequalities become trivial in the continuum limit but still correct. This correctness follows from the convexity property of cusp anomalous dimensions (3.28).

In this paper we used polygonal contours in order to concentrate on the properties of cusp singularities. The general inequality (1.27) also holds for curved contours with cusps or without them.

Since the inequalities are too general and hold in any gauge theory, the area law cannot be derived from these inequalities without using an additional dynamic input. Nevertheless it is interesting that the inequalities are compatible with the area law and the area and perimeter exponential terms can be factored out from the inequalities.

Our analysis of the inequalities for the determinants of two-dimensional Laplacians is incomplete. Only rectangular inequalities (5.25) were checked.
Although the discussion was presented in the context of the LSW model but actually one deals with a rather interesting pure mathematical problem.

Acknowledgements. I am grateful to E. Antonov, I. Cherednikov, D.I. Diakonov, M. Eides, N. Kivel, V. Kudryavtsev, A. Losev, V.Yu. Petrov, M.V. Polyakov and N.G. Stefanis for useful discussions. This work was supported by DFG and BMBF.

A Combinatorics of the renormalization

In this appendix we prove the renormalization properties of inequality (4.24) formulated in Sec. 4.4.

Using general renormalization equations (3.3), (3.4), we reduce inequality (4.24) to

$\det_{1 \leq k,l \leq M} \left[ \prod_{C_{kl} \text{ cusps}} Z_{\text{per}}^{-1}(\Lambda, L(C_{kl})) \prod_{C_{kl} \text{ cusps}} Z_{\text{cusp}}^{-1}(\Lambda, \gamma) W_{\text{ren}}(C_{kl}) \right] \geq 0$.  \hspace{1cm} (A.1)

Combining Eqs. 3.1 and 4.19, we find

$Z_{\text{per}}(\Lambda, L(C_{kl})) = Z_{\text{per}}(\Lambda, L(B_k)) Z_{\text{per}}(\Lambda, L(B_l))$.  \hspace{1cm} (A.2)

Hence

$\det_{1 \leq k,l \leq M} \left\{ \prod_{C_{kl} \text{ cusps}} Z_{\text{cusp}}^{-1}(\Lambda, \gamma) W_{\text{ren}}(C_{kl}) \right\} = \left[ \prod_{k=1}^{M} Z_{\text{per}}(\Lambda, L(B_k)) \right]^{-2} \det_{1 \leq k,l \leq M} \left\{ \prod_{C_{kl} \text{ cusps}} Z_{\text{cusp}}^{-1}(\Lambda, \gamma) W_{\text{ren}}(C_{kl}) \right\}$

and inequality (4.24) reduces to

$\det_{1 \leq k,l \leq M} \left\{ \prod_{C_{kl} \text{ cusps}} Z_{\text{cusp}}^{-1}(\Lambda, \gamma) W_{\text{ren}}(C_{kl}) \right\} \geq 0$.  \hspace{1cm} (A.3)

If path $B_k$ has external cusp angles $\gamma_k, \gamma'_k$ (see Fig. 5) then

$\prod_{C_{kl} \text{ cusps}} Z_{\text{cusp}}^{-1}(\Lambda, \gamma) = \left[ \prod_{B_k \text{ cusps}} Z_{\text{cusp}}^{-1}(\Lambda, \gamma) \right] \left[ \prod_{B_l \text{ cusps}} Z_{\text{cusp}}^{-1}(\Lambda, \gamma) \right] \times Z_{\text{cusp}}^{-1}(\Lambda, \gamma_k + \gamma_l) Z_{\text{cusp}}^{-1}(\Lambda, \gamma'_k + \gamma'_l)$.  \hspace{1cm} (A.5)
Therefore

\[
\det_{1 \leq k,l \leq M} \left\{ \prod_{C_{kl} \text{ cusps}} Z_{\text{cusp}}^{-1}(A, \gamma) \right\} W^\text{ren}(C_{kl}) = \left[ \prod_{B_k \text{ cusps}} Z_{\text{cusp}}^{-1}(A, \gamma) \right]^2
\]

\times \det_{1 \leq k,l \leq M} \left\{ Z_{\text{cusp}}^{-1}(A, \gamma_k + \gamma_l) Z_{\text{cusp}}^{-1}(A, \gamma_k' + \gamma_l') W^\text{ren}(C_{kl}) \right\}

(A.6)

and inequality (A.4) simplifies to

\[
\det_{1 \leq k,l \leq M} \left\{ Z_{\text{cusp}}^{-1}(A, \gamma_k + \gamma_l) Z_{\text{cusp}}^{-1}(A, \gamma_k' + \gamma_l') W^\text{ren}(C_{kl}) \right\} \geq 0.
\]

(A.7)

Now we have to consider different cases separately. In the simplest case when all open paths \( B_k \) have equal external cusp angles (4.25), (4.26) we arrive at

\[
\det_{1 \leq k,l \leq M} \left\{ Z_{\text{cusp}}^{-1}(A, \gamma_k + \gamma_l) Z_{\text{cusp}}^{-1}(A, \gamma_k' + \gamma_l') W^\text{ren}(C_{kl}) \right\} = \left[ Z_{\text{cusp}}^{-1}(A, 2\gamma) Z_{\text{cusp}}^{-1}(A, 2\gamma') \right]^M \det_{1 \leq k,l \leq M} \left\{ W^\text{ren}(C_{kl}) \right\}
\]

(A.8)

and inequality (A.7) becomes

\[
\det_{1 \leq k,l \leq M} W^\text{ren}(C_{kl}) \geq 0.
\]

(A.9)

Thus in the case (4.25), (4.26), the renormalized inequality (A.9) has the same form as the nonrenormalized inequality (4.24). This completes the derivation of inequality (4.27) announced in Sec. 4.4.

As was already announced in Sec. 4.4, inequalities (A.9) derived under assumptions (4.25), (4.26) provide all information: in the case when paths \( B_k \) do not obey cusp balance conditions (4.25), (4.26) the renormalization of inequality (4.24) will not lead to new inequalities. Let us prove this statement.

In this case we must return to inequality (A.7). We can represent the determinant as a sum over all permutations \( P \) with signum factor \( \varepsilon(P) \):

\[
\det_{1 \leq k,l \leq M} \left[ Z_{\text{cusp}}^{-1}(A, \gamma_k + \gamma_l) Z_{\text{cusp}}^{-1}(A, \gamma_k' + \gamma_l') W^\text{ren}(C_{kl}) \right] = \sum_P \varepsilon(P) \prod_{k=1}^{M} \left[ Z_{\text{cusp}}^{-1}(A, \gamma_k + \gamma_{P(k)}) Z_{\text{cusp}}^{-1}(A, \gamma_k' + \gamma'_{P(k)}) W^\text{ren}(C_{k,P(k)}) \right] \geq 0.
\]

(A.10)

Now we multiply this inequality by

\[
\prod_{k=1}^{M} \left[ Z_{\text{cusp}}(A, 2\gamma_k) Z_{\text{cusp}}(A, 2\gamma_k') \right].
\]

(A.11)
Then
\[
\sum_P \varepsilon(P) \prod_{k=1}^M \left[ \frac{Z_{cusp}^{-1}(\Lambda, \gamma_k + \gamma_{P(k)})}{Z_{cusp}^{-1}(\Lambda, 2\gamma_k)} \frac{Z_{cusp}^{-1}(\Lambda, \gamma'_k + \gamma'_{P(k)})}{Z_{cusp}^{-1}(\Lambda, 2\gamma'_k)} W^{ren}(C_{k,P(k)}) \right] \geq 0.
\] (A.12)

Next we have to identify the terms giving the dominant contribution at large \( \Lambda \). We will need the following property
\[
\lim_{\Lambda \to \infty} \prod_{k=1}^M \frac{Z_{cusp}^{-1}(\Lambda, \gamma_k + \gamma_{P(k)})}{Z_{cusp}(\Lambda, 2\gamma_k)} = \begin{cases} 
1 & \text{if } \gamma_k = \gamma_{P(k)} \text{ for all } k \\
0 & \text{otherwise}
\end{cases}.
\] (A.13)

The first case, when \( \gamma_k = \gamma_{P(k)} \) for all \( k \), is trivial. Therefore we must concentrate on the case when \( \gamma_k \neq \gamma_{P(k)} \) at least one value of \( k \). According to Eq. (3.29)
\[
\prod_{k=1}^M \frac{Z_{cusp}^{-1}(\Lambda, \gamma_k + \gamma_{P(k)})}{Z_{cusp}(\Lambda, 2\gamma_k)} \xrightarrow{\Lambda \to \infty} \exp \left\{ [\ln g^2(\Lambda)] \sum_{k=1}^M [\Gamma(\gamma_k + \gamma_{P(k)}) - \Gamma(2\gamma_k)] \right\}.
\] (A.14)

Taking into account that
\[
\sum_{k=1}^M \Gamma(2\gamma_{P(k)}) = \sum_{k=1}^M \Gamma(2\gamma_k),
\] (A.15)

we can write
\[
\sum_{k=1}^M [\Gamma(\gamma_k + \gamma_{P(k)}) - \Gamma(2\gamma_k)] = \frac{1}{2} \sum_{k=1}^M [2\Gamma(\gamma_k + \gamma_{P(k)}) - \Gamma(2\gamma_k) - \Gamma(2\gamma_{P(k)})]\] (A.16)

According to inequality (3.28)
\[
\sum_{k=1}^M [2\Gamma(\gamma_k + \gamma_{P(k)}) - \Gamma(2\gamma_k) - \Gamma(2\gamma_{P(k)})] \geq 0
\] (A.17)

and the equality occurs only if \( \gamma_k = \gamma_{P(k)} \) for all \( k \). Combining Eqs. (A.14), (A.16) and (A.17), we prove (A.13).

Now we can apply Eq. (A.13) to the analysis of the LHS of inequality (A.12)
\[
\prod_{k=1}^M \left[ \frac{Z_{cusp}^{-1}(\Lambda, \gamma_k + \gamma_{P(k)})}{Z_{cusp}(\Lambda, 2\gamma_k)} \right] \xrightarrow{\Lambda \to \infty} \begin{cases} 
1 & \text{if } \gamma_k = \gamma_{P(k)} \text{, } \gamma'_k = \gamma'_{P(k)} \text{ for all } k \\
0 & \text{otherwise}
\end{cases}.
\] (A.18)

Therefore inequality (A.10) takes the form
\[
\sum_{P \in \Pi_0} (-1)^{\varepsilon(P)} \prod_{k=1}^M W^{ren}(C_{k,P(k)}) \geq 0.
\] (A.19)
Here $\Pi_0$ is the subgroup of permutations obeying the condition

$$
\Pi_0 = \left\{ P : \gamma_k = \gamma_{P(k)} , \gamma'_k = \gamma'_{P(k)} \text{ for all } k \right\} . \tag{A.20}
$$

Let us say that two indices $k$ and $l$ belong to the same class if $\gamma_k = \gamma_l$ and $\gamma'_k = \gamma'_l$. Then the full set of indices can be divided into classes of equivalence $J_\alpha$:

$$
\{1, 2, \ldots, M\} = \bigcup_\alpha J_\alpha , \tag{A.21}
$$

$$
J_\alpha \cap J_\beta = \emptyset \quad (\alpha \neq \beta) . \tag{A.22}
$$

It is easy to see that subgroup $\Pi_0$ is made of permutations which leave indices within their equivalence classes. Therefore the sum over $P$ in inequality (A.19) reduces to the product of determinants restricted to different classes:

$$
\sum_{P \in \Pi_0} \varepsilon(P) \prod_{k=1}^M W_{\text{ren}}(C_{k,P(k)}) = \prod_\alpha \det_{k,l \in J_\alpha} W_{\text{ren}}(C_{kl}) . \tag{A.23}
$$

Thus our inequality (A.19) reduces to

$$
\prod_\alpha \det_{k,l \in J_\alpha} W_{\text{ren}}(C_{kl}) \geq 0 . \tag{A.24}
$$

Now it remains to note that instead of disentangling the problems caused by the violation of the cusp balance conditions (4.25), (4.26) we could concentrate from the very beginning on a separate class of equivalent indices $k \in J_\alpha$ and consider the corresponding set of open lines $B_k$ generating inequality

$$
\det_{k,l \in J_\alpha} W_{\text{ren}}(C_{kl}) \geq 0 . \tag{A.25}
$$

Inequality (A.24) is simply the product of these elementary inequalities which we have already derived in (A.9).

A small comment is needed about the case when the equivalence class $J_\alpha$ contains only one index $m$. In this case inequality (A.25) becomes trivial:

$$
\det_{k,l \in J_\alpha} W_{\text{ren}}(C_{kl}) = W_{\text{ren}}(C_{mm}) \geq 0 . \tag{A.26}
$$

## B Dedekind $\eta$ function

Dedekind $\eta$ function [28] is defined by the infinite product

$$
\eta(\tau) = e^{i\pi/12} \prod_{n=1}^\infty (1 - e^{2\pi i n \tau}) . \tag{B.1}
$$

for $\Im \tau > 0$. 

31
At large \( \text{Im} \tau \to +\infty \)

\[
\eta(\tau) \xrightarrow{\text{Im} \tau \to +\infty} e^{i(\pi/12)\tau}.
\]  

(B.2)

Dedekind \( \eta \) function has the property

\[
\sqrt{z} \eta(iz) = \eta(iz^{-1})
\]

(B.3)

where the branch of \( \sqrt{z} \) is chosen so that \( \sqrt{z} \) is real for \( z > 0 \).

Note that in the power series

\[
(1 - q)^{-\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + k)}{k!\Gamma(\alpha)} q^k
\]

(B.4)

all coefficients are positive at \( \alpha > 0 \). Expanding the infinite product

\[
\prod_{n=1}^{\infty} (1 - q^n)^{-\alpha} = \prod_{n=1}^{\infty} \left[ \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + k)}{k!\Gamma(\alpha)} q^{nk} \right] = \sum_{n=0}^{\infty} A_n(\alpha) q^n,
\]

(B.5)

one finds that here all coefficients are also positive

\[
A_n(\alpha) > 0 \ (\alpha > 0).
\]

(B.6)

Applying this to Eq. (B.1), we see that

\[
[\eta(i\tau)]^{-\alpha} = \left[ e^{-(\pi/12)\tau} \prod_{n=1}^{\infty} (1 - e^{-2\pi n\tau}) \right]^{-\alpha} = e^{(\pi/12)\alpha\tau} \sum_{n=0}^{\infty} A_n(\alpha)e^{-2\pi n\tau}.
\]

(B.7)

In the case \( \alpha = 1 \) expansion (B.5) simplifies to

\[
\prod_{n=1}^{\infty} (1 - q^n)^{-1} = \prod_{n=1}^{\infty} \sum_{k=0}^{\infty} q^{kn} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \cdots \delta_{n,n_1+2n_2+3n_3+\ldots} q^n
\]

\[
= \sum_{n=0}^{\infty} P(n)q^n
\]

(B.8)

where \( P(n) \)

\[
P(n) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \cdots \delta_{n,n_1+2n_2+3n_3+\ldots}
\]

(B.9)

is the number of partitions of the integer \( n \), i.e. the number of representations of \( n \) in the form

\[
n = n_1 + 2n_2 + 3n_3 + \ldots
\]

(B.10)

Taking an integer \( m \) power of Eq. (B.8)

\[
\left[ \prod_{n=1}^{\infty} (1 - q^n)^{-1} \right]^m = \left[ \sum_{n=0}^{\infty} P(n)q^n \right]^m = \sum_{n=0}^{\infty} A_n(m)q^n,
\]

(B.11)
one can see that
\[ A_n(m) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} \delta_{n,n_1+n_2+\ldots+n_m} P(n_1)P(n_2)\ldots P(n_m). \] (B.12)

This shows that all \( A_n(m) \) are integer positive numbers if \( m, n \) are integer.

References

[1] K. Wilson, Phys. Rev. D10 (1974) 2445.
[2] C. Bachas, Phys. Rev. D33 (1986) 2723.
[3] E. Seiler, Phys. Rev. D18 (1978) 482.
[4] C. Borgs and E. Seiler, Commun. Math. Phys. 91 (1983) 329.
[5] K. Osterwalder and R. Schrader, Commun. Math Phys. 42 (1975) 281.
[6] K. Osterwalder and E. Seiler, Ann. Phys. (N.Y.) 110 (1978) 440.
[7] D. Brydges, J. Fröhlich and E. Seiler, Ann. Phys. (N.Y.) 121 (1979) 227.
[8] A.M. Polyakov, Nucl. Phys. B164 (1979) 171.
[9] G.P. Korchemsky and A.V. Radyushkin, Nucl. Phys. B283 (1987) 342.
[10] M. Lüscher, K. Symanzik and P. Weisz, Nucl. Phys. B173 (1980) 365.
[11] M. Lüscher, Nucl. Phys. B180 [FS2] (1981), 317.
[12] O. Alvarez, Phys. Rev. D24 (1981) 440.
[13] M. Lüscher and P. Weisz, JHEP 0207 (2002) 049.
[14] J. Greensite, Prog. Part. Nucl. Phys. 51 (2003) 1.
[15] M. Lüscher and P. Weisz, JHEP 0407 (2004) 014.
[16] M. Caselle, M. Hasenbusch, M. Panero, JHEP 0503 (2005) 026.
[17] H.B. Meyer, Nucl. Phys. B758 (2006) 204.
[18] V.S. Dotsenko and S.N. Vergeles, Nucl. Phys. B169 (1980) 527.
[19] I.Ya. Aref’eva, Phys. Lett. 93B (1980) 347.
[20] R.A. Brandt, F. Neri and M.-A. Sato, Phys. Rev. D24 (1981) 879.
[21] R.A. Brandt, A. Gocksch, M.-A. Sato and F. Neri, Phys. Rev. D26 (1982) 3611.
[22] M. Kac, Amer. Math. Monthly 73 (1966) 1.
[23] H.P. McKean and I.M. Singer, J. Diff. Geometry 1 (1967) 43.

[24] T. Filk, preprint BONN-HE-81-16 (unpublished).

[25] K. Dietz and T. Filk, Phys. Rev. D27 (1983) 2944.

[26] E. Aurell and P. Salomonson, Commun. Math. Phys. 165 (1994) 233.

[27] B. Duplantier and F. David, J. Stat. Phys. 51 (1988) 32.

[28] T. M. Apostol, Modular Functions and Dirichlet Series in Number Theory, 2nd ed. New York, Springer-Verlag, 1997.