Maximal subbundles and Gromov invariants

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Abstract

In this article we explicitly compute the number of maximal subbundles of rank $k$ of a generically stable bundle of rank $r$ and degree $d$ over a smooth projective curve $C$ of genus $g \geq 2$ over $\mathbb{C}$, when the dimension of the quot scheme of maximal subbundles is zero. Our method is to describe the this number purely in terms of the Gromov invariants of the Grassmannian and then use the formula of Vafa and Intriligator to compute them.

1 Introduction

Let $C$ be a smooth projective curve over $\mathbb{C}$ of genus $g \geq 2$. For a vector bundle of rank $r$ and degree $d$ the $s$-invariant is defined by $s_k(E) = dk - re_{\text{max}}(E)$ where $e_{\text{max}}(E) = \text{Max}\{\deg(F)\}$. Here the maximum is taken over all subbundles $F$ of $E$ of rank $k$. It is known that for any $E$ the $s$-invariant satisfies $s_k(E) \leq k(r-k)g$ (see the results of Mukai-Sakai [13]).

One can get a more specific bound when we put additional structures on the bundle $E$. It is proved by Hirschowitz [7] that for a general stable bundle the invariant $s_k(E)$ is independent of the choice of $E$ and satisfies $s_{\text{min},d} = s_k(E) = k(r-k)(g-1) + \epsilon$ where $\epsilon$ is the unique integer $0 \leq \epsilon < r$ such that $s_{\text{min},d} = kd(\text{mod } r)$. Moreover if $e_{\text{max},d}$ is the degree of a maximal subbundle of a general stable bundle $E$ of degree $d$ then every component of the quot scheme $\text{Quot}^{k,e_{\text{max},d}}(E)$ has dimension $\epsilon$. And if the bundle is sufficiently general then the above quot scheme is itself smooth (see Lemma 2.1, Lange-Newstead [12] when the quot scheme is zero dimensional and in general see Remark 6.6, [8]). Hence when $s_{\text{min}}(d) = k(r-k)(g-1)$ we have a zero dimensional quot scheme which is smooth. This defines the number $m(r,d,k,g)$ as the number of maximal subbundles of a sufficiently stable bundle $E$.

It is known that $m(r,d,1,g) = r^g$ (see Ghione [3], Lange [11], and Segre [17] for $r = 2$ and Okonek-Teleman [14] and Oxbury [15], Theorem 3.1, in general). For the case when $k > 1$ there is a formula to compute $m(r,d,k,g)$ when $k$ and $e_{\text{max},d}$ are relatively prime (see Lange-Newstead [12]).
Our objective in this paper is to compute the number $m(r, d, k, g)$ explicitly for all choices of parameters (see Theorem 1.4). We first define a notion of a twisted Gromov invariants for any vector bundle following the methods of Bertram [2]. The main tool this uses is the fact the quot schemes of our interested are generically smooth of expected dimension. This is the case when $s_e = dk - re$ is sufficiently large. But if the bundle $E$ is sufficiently general then for all values of $e$ for which the quot scheme $\text{Quot}^{k,e}(E)$ is non-empty it does satisfy the above property. This enable us to define the twisted Gromov invariants for all choices of $e \leq e_{\text{max}}(d)$. We then see that the numbers $m(r, d, k, g)$ can be computed purely in terms of these twisted Gromov invariants. These invariants were also defined in [14] and Behrend [1].

We show that these invariants are actually independent of the chosen general stable bundle. Also if $s_e$ is large enough then these invariants do not depend on the choice of the vector bundle of degree $d$. Then using Hecke transformations we compare the twisted Gromov invariants for different choices of $d$. These ideas will now enable us to compute the twisted Gromov invariants purely in terms of the Gromov invariants of the Grassmannian (when $E$ is trivial). And in this case there is a formula of Vafa and Intriligator (see [19], [9], and for proof see Bertram [2] and [3]) which exactly computes these invariants.

One observes that our formula for $m(r, d, k, g)$ and the one obtained in [12] are completely different in appearence. It will be interesting to compare the two expressions directly.

Some of these results can also be generalized to $G$-bundles for a connected reductive algebraic group $G$ and this will be done elsewhere.

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## 2 Twisted Gromov invariants

Let $C$ be a smooth projective curve over $\mathbb{C}$. In this section we define the twisted Gromov invariants for a vector bundle $E$ of rank $r$ and degree $d$, and a positive integer $k \leq r$. Our viewpoint here is that the construction of Gromov invariants given in Bertram [2] holds for any vector bundle.

Let $E$ be a vector bundle of rank $r$ and degree $d$ over $C$. We will be mostly interested in the quot scheme $\text{Quot}^{k,e}(E)$ parameterizing the rank $k$ subsheaves of degree $e$. We have an open subscheme $\text{Quot}^{k,e}_0(E)$ consisting of subbundles. We need some basic facts about these quot schemes which will be used in the sequel. We will denote by $s_e = dk - re$. First we consider the case $k = 1$. Let $e_{\text{max}}(E)$ be the degree of the maximal line subbundle of $E$. We have natural morphisms $i_{e_1} : C_{(e_1-e)} \times \text{Quot}^{1,e_1}(E) \to \text{Quot}^{1,e}(E)$ for each $e < e_1 \leq e_{\text{max}}$ which takes the pair $(D, [L \hookrightarrow E])$ to $[L(-D) \hookrightarrow L \hookrightarrow E]$. Using the universal properties of the quot
schemes we check that this defines a morphism.

**Lemma 2.1** The morphism $i_{e_1}$ is a closed embedding. Moreover For $e \leq e_1$, $e_2 \leq e_{\text{max}}$ with $e_1 \neq e_2$, we have

$$i_{e_1}(C_{(e_1-e)} \times \text{Quot}_{0}^{1,e_1}(E)) \cap i_{e_2}(C_{(e_2-e)} \times \text{Quot}_{0}^{1,e_2}(E)) = \emptyset.$$ 

**Proof** If $[L \rightarrow E]$ defines an element of Quot$_{0}^{1,e}(E)$ which is in image of $i_{e_1}$ then we can uniquely recover an effective divisor $D'$ and an element $[L' \hookrightarrow E] \in \text{Quot}_{0}^{1,e'}(E)$ for some $e' \geq e$ using the fact that $L'$ is the saturation of $L$ in $E$ and $D$ is the divisor defined by the inclusion $L \hookrightarrow L'$. This fact already implies that the restriction of the morphism $i_{e_2}$ to $C_{(e_2-e)} \times \text{Quot}_{0}^{1,e_2}(E)$ is injective. Now $i_{e_1}(D_1, [L_1 \hookrightarrow E]) = [L \hookrightarrow E]$ if and only if the inclusion $L \hookrightarrow L'$ factors via $L \hookrightarrow L_1$ up to an isomorphism of $L$. Hence we see that the element defined by $L_1 \hookrightarrow E$ in Quot$_{0}^{1,e_1}(E)$ is in the image of corresponding morphism $C_{(e'-e_1)} \times \text{Quot}_{0}^{1,e'}(E) \rightarrow \text{Quot}_{0}^{1,e_1}(E)$. This proves the second assertions of the lemma.

The above lemma defines a stratification on Quot$_{0}^{1,e}(E)$ by locally closed subschemes $C_{(e_1-e)} \times \text{Quot}_{0}^{1,e_1}(E)$, where $e \leq e_1 \leq e_{\text{max}}$ and for $e_1 = e$ we get an open strata Quot$_{0}^{1,e}(E)$. Note that this stratification is in some sense weak as these quot schemes are not equidimensional and there can be strata with $e_1 > e$ which might contain open subschemes of Quot$_{0}^{1,e}(E)$.

Now we use the above lemma to get a similar understanding of the structure of Quot$_{0}^{k,e}(E)$ for $k > 1$. For any subsheaf $F \hookrightarrow E$ of rank $k$ we can take the $k$-th exterior power to get a line subsheaf $\wedge^{k}(F) \hookrightarrow \wedge^{k}(E)$. This defines the Plücker morphism $P_{e} : \text{Quot}_{0}^{k,e}(E) \rightarrow \text{Quot}_{0}^{1,e}(\wedge^{k}(E))$. Again it is easy to check that this morphism restricted to Quot$_{0}^{k,e}(E)$ is an embedding. We have the following lemma which describes the boundary of Quot$_{0}^{k,e}(E)$ in terms of the stratification defined above.

**Lemma 2.2** The scheme Quot$_{0}^{k,e}(E)$ has a stratification Quot$_{0}^{k,e}(E) \bigsqcup_{e \leq e_1 \leq e_{\text{max}}} B_{e_1}$ where $B_{e_1} = P_{e}^{-1}(C_{(e_1-e)} \times \text{Quot}_{0}^{1,e_1}(\wedge^{k}(E)))$. Moreover the image of the morphism $P_{e}$ when restricted to $B_{e_1}$ is exactly $C_{(e_1-e)} \times P_{e_1}(\text{Quot}_{0}^{k,e_1}(E))$ and the fibers of $\text{pr}_2 \circ P_{e}$ when restricted to $B_{e_1}$ are irreducible and smooth of dimension $k(e-e_1)$.

**Proof** The first assertion in the Lemma follows from the definition. The second follows from the fact that if $F \hookrightarrow E$ is a rank $k$ subsheaf of $E$ then the saturation $F_{\text{sat}}$ of $F$ in $E$ has the property that $\wedge^{k}(F_{\text{sat}})$ is the saturation of $\wedge^{k}(F)$. For the last part one observes that the fiber of the morphism $\text{pr}_2 \circ P_{e}$ at the point $[F \hookrightarrow E]$ is the quot scheme Quot$_{0}^{k,e_1-e}(F)$ of length $e_1 - e$ quotients Quot$_{0}^{k,e_1-e}(F)$ and this space is irreducible and smooth of dimension $k(e_1 - e)$. 

**Remark 2.3** The above lemma also gives us the following dimension estimate

$$\dim(\text{Quot}_{0}^{k,e_1}(E)) \leq \dim(\text{Quot}_{0}^{k,e}(E)) - k(e_1 - e).$$
This estimate is also proved in Popa-Roth \[10\]. It will be used to define the well definedness of the twisted Gromov invariants.

\textbf{Remark 2.4} If \( Q \) is an irreducible component of the quot scheme \( \text{Quot}^{k,e}(E) \) such that every element in \( Q \) corresponds to a quotient of \( E \) which is not locally free then there exists an \( e_1 \geq e \) such that \( B_{e_1} \) is actually dense open subset of \( Q \). This has the effect that we obtain a dimension bound \( \dim(Q) \leq \text{Quot}^{k,e_1}_0(E) + k(e_1 - e) \). This estimate will also be used later.

Let \( x \in C \) be a point. We will denote by \( E(x) \) the fiber of \( E \) at \( x \). Let \( Gr_k(E(x)) \) denote the Grassmannian of \( k \) dimensional subspaces of \( E(x) \). Hence we have the universal rank \( k \) subbundle \( S(x) \to E(x) \otimes \mathcal{O}_{Gr_k(E(x))} \). There is a natural action of the general linear group \( GL(E(x)) \) of automorphisms of \( E(x) \) on \( Gr_k(E(x)) \). Let \( H \subset E(x)^* \) be a subspace of dimension \( n \leq k \). We then have a special Schubert variety \( Y_H \) defined by the degeneracy locus of the canonical map \( H \otimes \mathcal{O}_{Gr_k(E(x))} \to S(x)^* \). The variety \( Y_H \) is irreducible and reduced of codimension \( k + 1 - n \) and it represents the \((k + 1 - n)\)-th Chern class of \( S(x)^* \). For any element \( g \in GL(E(x)) \) we have a \( g \) translate of \( Y_H \) defined by the degeneracy locus associated to the translate \( gH \).

For a Schubert variety \( Y_H \) associated to \( H \) we define the generalized Schubert scheme \( W_e(x, H) = ev_x^{-1}(Y_H) \), where \( ev_x \) is the evaluation map \( \text{Quot}^{k,e}_0(E) \to Gr_k(E(x)) \) at \( x \) which takes an element \([F \leftrightarrow E]\) to the subspace \( F(x) \leftrightarrow E(x) \).

We need the following lemma.

\textbf{Lemma 2.5} Given a vector bundle \( E \) of rank \( r \) over \( C \) there is an integer \( \overline{s}(E) \) such that for each \( e \) with \( s_e \geq \overline{s}(E) \), every component of the the quot scheme \( \text{Quot}^{k,e}(E) \) is generically smooth of expected dimension and a general element in every component corresponds to a subbundle of \( E \).

\textbf{Proof} This is proved in Popa-Roth \[10\], Theorem 5.14. In fact they prove a stronger version namely that there is an integer \( \overline{s}_1(E) \) such that for each \( e \) with \( s_e \geq \overline{s}_1(E) \) the quot scheme \( \text{Quot}^{k,e}(E) \) is also irreducible.\( \square \)

The above lemma allows us to define the invariant \( \overline{s}(E) \) (namely minimal such integer satisfying the above lemma) for any vector bundle.

Now we define the twisted Gromov invariants as the intersection numbers in the quot scheme. Let \( e \) be such that \( s_e \geq \overline{s}(E) \). Let \((x, H)\) be a pair with \( x \in C \) and \( H \) a subspace of \( E(x)^* \) of rank \( n \leq k \). We define the subscheme \( \text{V}_e(x, H) \subset \text{Quot}^{k,e}(E) \) as the the degeneracy locus of the composite morphism \( H \otimes \mathcal{O}_{\text{Quot}^{k,e}(E)} \to (\text{pr}_1^*E)(x) \to F^*(x) \), where \((F^*)(x)\) is the restriction of the dual of the universal subsheaf \( F \subset \text{pr}_1^*E \) of rank \( k \) and degree \( e \) to \( \{x\} \times \text{Quot}^{k,e}(E) \) and \( \text{pr}_1 : C \times \text{Quot}^{k,e}(E) \to C \) is the first projection. By definitions we can check that \( W_e(x, H) \) is the scheme theoretic intersection of \( \text{V}_e(x, H) \) and \( \text{Quot}^{k,e}_0(E) \).
Lemma 2.6 With the notations above if \( g \in GL_x(E) \) general then every component of \( V_e(x, gH) \) has codimension exactly \( k - n + 1 \) in \( \text{Quot}^{k,e}(E) \) and \( V_e(x, gH) \) represents the \( (k - n + 1) \)-th Chern class of \( \mathcal{F}^*(x) \).

Proof  Firstly the conditions \( s_e \geq \overline{\pi}(E) \) ensures that the quot scheme \( \text{Quot}^{k,e}(E) \) has the properties mentioned in the Lemma 2.5. This has the effect that the above quot scheme is locally of complete intersection hence Cohen-Macaulay (see for example Kollár, I, Theorem 5.17, [10]). Hence the subscheme \( V_e(x, H) \) represents the \( k - n + 1 \)-th Chern class of \( \mathcal{F}^*(x) \) if we show that the codimension of every irreducible component of \( V_e(x, H) \) is \( k - n + 1 \) (see Fulton, Theorem 14.3, Fulton [4]). Now we use the Lemma 2.2 Lemma 2.5 we can continue as in the proof of the first part of the Theorem 1.4 of [2] to check the codimensionality condition. □

The above lemma allows us to define the twisted Gromov invariants as follows. Let \( E \) be a rank \( r \) and degree \( d \) vector bundle over \( C \). Let \( \overline{\pi}(E) \) be as defined in Lemma 2.5. Let \( X_1, \ldots, X_k \) be weighted variables such that weight of \( X_i \) is \( i \). Let \( P(X_1, \ldots, X_k) \) be a homogeneous polynomial of weighted degree \( s_e + k(r - k)(1 - g) \) with \( s_e = d - r e \) and \( s_e \geq \overline{\pi}(E) \).

Definition 2.7 We define the twisted Gromov invariant \( N_{d,e}(P(X_1, \ldots, X_k), E) \) for a vector bundle \( E \), and an integer \( e \) such that \( s_e > \overline{\pi}(E) \), as the intersection number

\[
P(c_1(\mathcal{F}^*(x)), \ldots, c_k(\mathcal{F}^*(x)))[\text{Quot}^{k,e}(E)]
\]

Here \([\text{Quot}^{k,e}(E)]\) denotes the fundamental cycle.

Remark 2.8 The above number is independent of the single point \( x \) chosen. More generally we could choose points \( x_i \) for \( i = 1, \ldots, n \) and subspace \( H_i \subset E(x_i)^* \) of dimension \( a_i \leq k \) such that \( \sum (k - a_i + 1) = s_e + k(r - k)(1 - g) \) then the number \( N_{d,e}(\prod_{i=1}^n X_k-a_i+1, E) \) can be exactly computed as the intersection number \( \cap V_e(x_i, g_i H_i) \) for general elements \( g_i \in GL(E(x_i)) \).

The following proposition shows how to compute the above intersection as an intersection in the open subscheme \( \text{Quot}^{k,e}_0(E) \).

Proposition 2.9 Let \( E \) be a rank \( r \) vector bundle of degree \( d \) and let \( e \) be such that \( s_e \geq \overline{\pi}(E) \). For \( i = 1, \ldots, N \), let \( x_i \in C \) be distinct points and \( H_i \subset E^*(x_i) \) be subspaces of dimension \( a_i \leq k \) such that \( \sum_{i=1}^N (k - a_i + 1) = s_e + k(r - k)(1 - g) \). Then for general choices of \( g_i \in GL(E(x_i)) \), the twisted Gromov invariants \( N_{d,e}(\prod_{i=1}^N X_k-a_i+1, E) \) can be exactly computed as the number of intersections \( \cap_{i=1}^N W_e(x_i, g_i H_i) \) (counted with multiplicities).

Proof The proof of the proposition exactly follows second part of the proof of Theorem 1.4 of [2]. Here we can use the dimension bound obtained in the Remark
to ensure that for general translates the intersections in the boundary are trivial for any choice of $e$ such that $s_e \geq \overline{s}(E)$.

Now we compare the intersection numbers obtained in the above theorem for different rank $r$ vector bundles of a fixed degree $d$.

Let $E_1$ and $E_2$ be two rank $r$ vector bundles over $C$ of degree $d$. It can be proved using the smoothness and irreducibility of the moduli stack bundles of rank $r$ and degree $d$ that there is a smooth irreducible variety $B$, a family of bundles $\mathcal{E}$ over $C \times B$ and two points $x_1$ and $x_2$ of $B$ such that $\mathcal{E}|_{C \times x_i} \cong E_i$ for $i = 1, 2$. We will denote by $\mathcal{E}_x$ the bundle $\mathcal{E}|_{C \times x}$ for $x \in B$. We need the following generalization of the Lemma 2.3.

**Lemma 2.10** There exists an integer $s_{\mathcal{E}}$ independent of $x \in B$ such that for all $s \geq s_{\mathcal{E}}$ the quot scheme $\text{Quot}^{k,e}(\mathcal{E}_x)$ is generically smooth of expected dimension and satisfies the property that general elements in every irreducible component of $\text{Quot}^{k,e}(\mathcal{E}_x)$ lies in $\text{Quot}^{k,e}(\mathcal{E}_x)$.

**Proof** The lemma follows from the arguments similar to the proof of Theorem 6.4 of [1] once we establish the existence of an integer $s_{\mathcal{E}}$ independent of $x \in B$ such that $\text{Quot}^{k,e}(\mathcal{E}_x)$ is generically smooth of expected dimension for all $e$ with $s_e \geq s_{\mathcal{E}}$. The last statement can be again proved by going through the arguments of Proposition 6.1 and Theorem 6.2 of [1] (Also see Proposition 5.11, [2] where the integer $s_{\mathcal{E}}$ is described in terms of instability degree of a principal bundle).

We can use the above Lemma to define the invariant $s_{\mathcal{E}}$ (namely minimal such integer satisfying the above lemma) for a family of vector bundles $\mathcal{E}$ over $C \times B$.

**Proposition 2.11** If $\mathcal{E}$ is a family of rank $r$ vector bundles of degree $d$ on $C \times B$ with $B_1$ being a smooth curve and $e$ is chosen such that $s_e \geq s_{\mathcal{E}}$ then the Gromov invariants $N_{d,e}(P(X_1, \ldots, X_k), \mathcal{E}_x)$ are independent of the choice of points $x \in B$.

**Proof** We consider the relative Quot scheme $f : \text{Quot}^{k,e}(\mathcal{E}) \rightarrow B_1$ which has the property that for each $x \in B_1$ the fiber is exactly $\text{Quot}^{k,e}(\mathcal{E}_x)$. If $s_e \geq s_{\mathcal{E}}$ then by Lemma 2.10 we see that each of $\text{Quot}^{k,e}(\mathcal{E}_x)$ are generically smooth of expected dimension. Hence $f$ is a locally complete intersection morphism (see I, Theorem 5.17, [1]), in particular flat. Now the proposition follows from Lemma 1.6 of [2].

The above proof can also be used to show that the intersection numbers are actually depend on only the genus $g$ of the smooth projective curve $C$ (for the case of trivial bundles, this is the Proposition 1.5 [2]).

We also have a simple relation between the twisted Gromov invariants when the vector bundle is tensored with a line bundle.

**Lemma 2.12** Let $E$ be a rank $r$ vector bundle of degree $d$ and $L$, a line bundle of degree $d_1$. Then we have $\overline{s}(E) = \overline{s}(E \otimes L)$ and the twisted Gromov invariants of $E$
and $E \otimes L$ are related by the following

$$Nd_{d+rd_1,e+kd_1}(P(X_1, \ldots, X_k), E \otimes L) = Nd_{d,e}(P(X_1, \ldots, X_k), E).$$

**Proof** This follows from the fact that the quot schemes $\text{Quot}^{k,e+kd_1}(E \otimes L)$ and $\text{Quot}^{k,e}(E)$ are naturally isomorphic and the isomorphism preserves the all the degeneracy loci. \hfill \square

## 3 Generically stable bundles

In this section we will assume that $C$ is a smooth projective curve over $\mathbb{C}$ of genus $g \geq 2$.

We first prove some basic facts about the structure of the quot schemes for generically stable bundles and these fact will be used later.

We first begin by the following result which will enable us to define twisted Gromov invariants when the degrees of the subbundles is not very small. Let $M^s(r, d)$ denote the coarse moduli space of stable bundles of rank $r$ and degree $d$. Let $0 < k < r$. For a vector bundle $E$ over $C$ of rank $r$ and degree $d$, recall the notion of the $s$ invariant by $s_k(E) = dk - r e_{\text{max}}(E)$ where $e_{\text{max}} = \text{Max}\{\deg(F)\}$. Here the maximum is taken over all rank $k$ subbundles $F$ of $E$. It is known that if the vector bundle $E$ is generically stable then $s_{\text{min},d} = s_k(E) = k(r-k)(g-1) + \epsilon$ where $\epsilon$ is the unique integer with $0 \leq \epsilon \leq r - 1$ such that $s = kd (\text{mod } r)$. Let $e_{\text{max},d}$ be the degree of the maximal subbundle of a generically stable bundle of degree $d$.

**Proposition 3.1** There exists a non empty open subset $U(r, d) \subset M(r, d)$ with the property that for each $E \in U(r, d)$ and for every $e \leq e_{\text{max},d}$, every component of quot scheme $\text{Quot}^{k,e}(E)$ is generically smooth of expected dimension ($= dk - re + k(r-1)(g-1)$) and satisfies the property that general elements in every irreducible component corresponds to subbundles of $E$.

**Proof** It is already known that there is a non empty open subset $U(r, d) \subset M(r, d)$ such that each $E$ in $U(r, d)$ satisfies $s_{\text{min},d} = s_k(E) = dk - r e_{\text{max},d}$ and the dimension estimate dim$(\text{Quot}^{k,e}(E)) \leq s_e + k(r-k)(1-g)$ holds for all $e \leq e_{\text{max}}$ (see Example 5.4 [10]). This along with the deformation theoretic lower bounds implies that every component of the quot scheme has the above dimension. Moreover it is also known that every irreducible component of $\text{Quot}^{k,e}(E)$ is generically smooth (see for example Proposition 6.8, [8]). So we only have to only make sure that these quot schemes have no pathological components. Let $Q$ be an irreducible component of $\text{Quot}^{k,e}(E)$ such that every element in $Q$ corresponds to quotient which is not torsion free. Then by Remark 2.4 there is an $e_1 > e$ such that dim$(Q) \leq \text{dim}(\text{Quot}^{k,e_1}(E)) + k(e_1 - e)$. Now this leads to a contradiction once we put the values of the dimensions of these spaces. \hfill \square
The above result shows that if \( E \in U(r, d) \) then \( \overline{\sigma}(E) = s_{\min,d} \). This enables us to define the twisted Gromov invariant \( N_{d,e}(P(X_1, \ldots, X_k), E) \) for all possible values of \( e \leq e_{\max,d} \). Moreover if we stay in the open subset \( U(r, d) \) then we see by Proposition \[2.11\] that these numbers are actually independent of the choice of \( E \in U(r, d) \). This enable us to define \( N_{d,e}(P(X_1, \ldots, X_n)) \) to be \( N_{d,e}(P(X_1, \ldots, X_n), E) \) for some \( E \in U(r, d) \) and \( e \) with \( e \leq e_{\max,d} \).

Also the Proposition \[2.11\] shows that if \( E_1 \) is any vector bundle of rank \( r \) and degree \( d \) then for any \( e \) with \( s_e \geq \overline{\sigma}(E_1) \) we have

\[
N_{d,e}(P(X_1, \ldots, X_n)) = N_{d,e}(P(X_1, \ldots, X_n), E_1).
\]

Now we will compare the above invariants when the degree of the vector bundle changes. The main result is the following.

**Proposition 3.2** For a polynomial \( P(X_1, \ldots, X_k) \) of weighted degree \( s_e + k(r-k)(g-1) - k \) with \( e \) satisfying \( e \leq e_{\max,d-1} \) We have

\[
N_{d-1,e}(P(X_1, \ldots, X_k)) = N_{d,e}(X_kP(X_1, \ldots, X_k))
\]

We first recall a basic lemma which will be a step in the proof of the above result. Let \( x \in C \) be a point fixed. For a bundle \( E \) of rank \( r \) and degree \( d \) and a quotient space \( l : E(x) \to C \) (or equivalently a line \( l \subset E^*(x) \)) of dimension \( 1 \) at a point \( x \) we have the Hecke transform \( F_l = \ker(\tilde{l}) \), where \( \tilde{l} \) is the map \( E \to C \) defined by \( l \). Then the degree of \( F_l \) is equal to \( d - 1 \). Hence for any point \( g \in GL(E(x)) \) we have a vector bundle \( F_{gl} \) defined by the above procedure.

**Lemma 3.3** For any two non-empty open sets \( U \subset M^s(r, d) \) and \( U_1 \subset M^s(r, d-1) \) there exists a vector bundle \( E \in U \) and a non empty open set \( V \subset GL(E(x)) \) such that for each \( g \in V \) the vector bundle \( F_{gl} \) got by Hecke transform with respect to \( gl \) lies in \( U_1 \).

**Proof** Let \( \text{Vect}(r, d) \) denote the moduli stack of vector bundles of rank \( r \) and degree \( d \). We have the Hecke stack \( \mathcal{H} \) defined by triples \( (F, E, i) \) where \( F \) and \( E \) are rank \( r \) bundles of degree \( d-1 \) and \( d \) respectively and an inclusion \( i : F \leq E \) whose cokernel is supported at a single point \( x \). We also have two morphisms \( h_1 : \mathcal{H} \to \text{Vect}(r, d-1) \) and \( h_2 : \mathcal{H} \to \text{Vect}(r, d) \) which takes such a triple to \( F \) and \( E \) respectively. It is easy to check that these two morphisms are surjective. Now the stack \( \mathcal{H} \) is irreducible as is a projective bundle over \( \text{Vect}(r, d) \), which is irreducible, whose fiber at \( E \) is given by \( Gr_m(E(x)) \). Since \( M^s(r, d) \) and \( M^s(r, d-1) \) are the coarse moduli spaces of open substack of \( \text{Vect}(r, d) \) and \( \text{Vect}(r, d-1) \) respectively, hence we can check that the inverse images of the open sets \( U \) and \( U_1 \) under the morphism \( h_1 \) and \( h_2 \) respectively intersect in a non empty open substack of \( \mathcal{H} \). This proves the lemma.

\[\square\]
Now we return to the proof of the Proposition 3.2. Let \( x \in C \) be a fixed point. Choose a vector bundle \( E \in U(r, d) \) prescribed by the Lemma 5.3 for the open subsets \( U_1 = U(r, d - 1) \) and \( U = U(r, d) \). We then obtain a non empty open subset \( V \subset GL(E(x)) \) such that for each \( g \in V \) the bundle \( F_{gl} \) lies in \( U_1 \). Let \( x_1, \ldots, x_N \) be distinct points of the curve \( C \) which are also distinct from \( x \) and \( H_i \subset E^t(X_i) \) be subspaces of rank \( a_i \leq k \) such that \( \sum_{i=1}^N (k - a_i + 1) = s_e + k(r - k)(1 - g) - k \). Then by Proposition 2.11 the twisted Gromov invariant \( N_{d,e}(X_k \Pi_{i=1}^N X_{k-a_i+1}, E) \) can be exactly computed as the number of intersections \( V(x, g) \cap_{i=1}^N V_e(x, g_i H_i) \) for general elements of \( g_i \)'s and \( g \), where \( V_e(x, g_i H_i) \) (and \( V(x, g) \)) are the degeneracy loci in \( Quot^{k,e}(E) \) associated to \( H_i \) (and \( l \)).

We have a natural morphism \( f : Quot^{k,e}(F_{gl}) \to Quot^{k,e}(E) \) which takes \([H \leftrightarrow F_{gl}]\) to \([H \leftrightarrow F_{gl} \leftrightarrow E]\). This morphism can be checked to be an embedding.

Now we check by definitions that the degeneracy locus \( V(x, gl) \) is exactly the isomorphic image of \( f \) and for each \( i \) the scheme \( f^{-1} V_e(x, g_i H_i) \) is exactly the degeneracy locus associated to \( F_{gl} \) for the subspace \( g_i H_i \subset F_{gl}^t(x) \). Hence for general \( g \) and \( g \) there is a natural isomorphism between the the zero dimensional schemes

\[
V(x, gl) \cap_{i=1}^N V_e(x, g_i H_i) \cong \cap_{i=1}^N f^{-1}(V_e(x, g_i H_i)).
\]

Hence by Proposition 2.9 we see that the left hand side of the above isomorphism computes the twisted Gromov invariants for \( E \) and the right hand side of the of the above isomorphism computes the twisted Gromov invariants for \( F_{gl} \). This completes the proof of the Proposition 3.2. \( \square \)

Remark 3.4 the last part of the proof of the above proposition also follows from the excess intersection formula (see Fulton, Theorem 6.3 and Proposition 6.6, [4]) since the quotient schemes in question satisfies the local complete intersection property.

Recall the definition of the standard Gromov invariants \( N_e(P(X_1, \ldots, X_k), g) \) for the Grassmannian from [2] for a polynomial \( P(X_1, \ldots, X_k) \) of weighted degree \( e r + k(r - k)(g - 1) \) with \(-e \) large enough. This invariant in our notations is equal to \( N_{0,-e}(P(X_1, \ldots, X_k)) \) for the choices of \( e \). It is also proved in Lemma 5.3, [4] (Also see [2], Remark before the Proposition 1.7) that the above invariant can be consistently defined for larger values of \( e \) by the following is defined for larger \( e \) by the recurrence \( N_{0,e}(P(X_1, \ldots, X_k)) = N_{0,e-k}(X^e_k P(X_1, \ldots, X_k)) \). This equality also follows from Lemma 2.12.

We have the following explicit formula for the twisted Gromov invariants in terms of the standard Gromov invariants of the Grassmannian.

**Theorem 3.5** Let \( r \) and \( k \) be fixed. Let \( d = ar - b \) with \( 0 \leq b < r \) and \( e \leq e_{\text{max}}(d) \).

Let \( P(X_1, \ldots, X_k) \) be a polynomial of weighted degree \( d k - re + k(r - k)(1 - g) \). Then all the twisted Gromov invariants are computable in terms of the actual Gromov invariants of the Grassmannian by the following

\[
N_{d,e}(P(X_1, \ldots, X_k)) = N_{0,e-ak}(X^b_k P(X_1, \ldots, X_k))
\]
Proof  This just follows from the Proposition 3.2 and Lemma 2.12. □

Remark 3.6  The above result shows that the twisted Gromov invariants are independent of the choice of the genus $g$ curve $C$.

Now we record here the formula of Vafa and Intriligator, proved by A. Bertram (see [2], [3]) and Siebert-Tian (see [18]) about explicit computation of Gromov invariants $N_{0,e}(P(X_1, \ldots, X_k))$.

Let $P(X_1, \ldots, X_k) = \prod_{i=1}^{m} X_{a(i)}$ be a polynomial with $0 < a_i \leq k$ such that the weighted degree of $P$ is $\sum_i (k - a_i + 1) = -er + k(r - k)(1 - g)$. Then we have the following.

Proposition 3.7  For the polynomial $P = \prod_{i=1}^{m} X_{a(i)}$ as above, the Gromov invariant $N_{0,e}(P(X_1, \ldots, X_k))$ is given by the following.

$$
\frac{r^k(g-1)(-1)^{e(k-1)+(g-1)k(k-1)/2}}{k!} \sum_{\{(\rho_1, \ldots, \rho_k)\mid \rho_i = 1; \rho_i \neq \rho_j\}} \left(\prod_{i=1}^{m} \sigma_{k-a(i)+1}(\rho)\right) \left(\prod_{i=1}^{m} \rho_i \prod_{i \neq j} (\rho_i - \rho_j)^{(g-1)}\right)
$$

where $\sigma_j(\rho)$ is the $j$-th symmetric polynomial in $\rho_i$'s.

The above proposition along with the Theorem 3.5 we can explicitly compute the twisted Gromov invariants.

4 Maximal Subbundles

In this section we will relate the twisted Gromov invariants to the number of Maximal subbundles of a generically stable bundles. We will assume that the genus of the curve is at least 2. Recall that for a generically stable bundle $E$ of rank $r$ and degree $d$ we have $s_{\min,d} = s_k(E) = k(r - k)(g - 1) + \epsilon$ where $\epsilon$ is a unique integer $0 \leq \epsilon \leq r - 1$ such that $s_{\min,d} = kd(\mod r)$. Also recall the definition of $e_{\max,d}$ from the previous section.

We need the following proposition in the sequel.

Proposition 4.1  Let $r$, $k$, $d$ and $e_{\max,d}$ be as before. Then for a sufficiently general stable bundle $E$ the quot scheme $\text{Quot}^{k,e_{\max,d}}(E)$ is a smooth scheme of dimension $s_{\min,d} + k(r - k)(1 - g)$ consisting only of vector bundle quotients.

Proof  We have already seen that for generically stable bundle $E$ every component of the quot scheme $\text{Quot}^{k,e_{\max}}(E)$ is of expected dimension. Now the result of Mukai and Sakai [13], as $e = e_{\max}$, ensures there is no boundary. Hence we have to only show that that for a sufficiently general $E$ the scheme $\text{Quot}^{k,e_{\max}}(E) = \text{Quot}^{k,e_{\max}}(E)$ is smooth. This follows from Remark 5.4, of Holla [8] where this has been worked out for a principal $G$-bundle with $G$ a connected reductive algebraic group. □
Remark 4.2 The above result follows from Lemma 2.1 of \cite{12} for the case when \( s_{\text{min},d} = k(r - k)(g - 1) \). This is the only case when we will use the Proposition \[4.1\] Also the above result has been worked out in \cite{13}, Proposition 1.4 when \( k = 1 \).

Remark 4.3 The Theorem \[3.5\] allows us to compute explicitly all the Chern numbers of the vector bundle \( \mathcal{F}^*(x) \), where \( \mathcal{F} \) is the universal subbundle of \( \text{pr}_1^*E \) on \( C \times \text{Quot}^{k,e_{\text{max},d}}(E) \) in terms of the Gromov invariants. From here one should be able to compute all the Chern numbers of the quot scheme itself. Since the Proposition \[4.1\] ensures that these quot schemes are smooth projective schemes hence it would be interesting to know what kind of objects these are. For the case \( k = 1 \) it is a conjecture of Oxbury (see \cite{15}, Conjecture 2.8) that these quot schemes are irreducible when they are of dimension greater than zero. We ask whether this conjecture is true for other \( k \).

Now we will handle the case when the choice of \( k, r \) and \( d \) are made such that \( k(r - k)(g - 1) = k d \pmod{r} \). In this case we can write down \( e_{\text{max},d} = dk + k(r - k)(1 - g) \). This has the effect that for a bundle \( E \) which is generically stable in the sense defined in Proposition \[4.1\] then the quot scheme \( \text{Quot}^{k,e_{\text{max},d}}(E) \) is of dimension 0 and smooth.

Hence we can count the number points in the quot scheme. We denote this number by \( m(r, d, k, g) \).

We have the following explicit formula for the number \( m(r, d, k, g) \).

**Theorem 4.4** Let \( d = ar - b \) with \( 0 \leq b < r \). The number \( m(r, d, k, g) \) is calculated by the following formula.

\[
m(r, d, k, g) = \frac{r^k(g-1)(-1)^{(k-1)(bk-(g-1)k^2)/r}}{k!} \sum_{(\rho_1, \ldots, \rho_k) | \rho'_i = 1; \rho_i \neq \rho_j} \frac{(\Pi_{i=1}^m \rho_i)^{b-g+1}}{(\Pi_{i \neq j} (\rho_i - \rho_j))^{(g-1)}}.
\]

**Proof** This follows from the fact that the number \( m(r, d, k, g) \) we are interested in is exactly \( N_{d,e_{\text{max},d}}(1) \) for our choice of vector bundle \( E \) in Proposition \[1.1\] as the expected dimension of \( \text{Quot}^{k,e_{\text{max}}}(E) \) is 0. Now using the Theorem \[3.5\] we obtain \( m(r, d, k, g) = N_{0,e_{\text{max}}-ak}(X^k_b) \). Now the Proposition \[3.7\] implies the theorem. \( \square \)

**Corollary 4.5** \( m(r, d, 1, g) = r^g \)

The above result recovers a part of the Theorem 3.1 of Oxbury \[15\].

The next case is when \( k = 2 \). We write \( d = ar - b \) and in this case the zero dimensionality of the quot scheme gives us the condition \( 2b - 4(g - 1) = 0 \pmod{r} \). Now using the fact that \( \sum_{\{\nu' = 1, \nu \neq 1\}} \nu^i \) is equal to \( -1 \), if \( 0 < i < r \) and is equal to \( r - 1 \) if \( i = 0, r \), the Theorem \[4.4\] gives

\[
m(r, d, 2, g) = \frac{r^{2(g-1)+1}(-1)^{g-1+(2b-4(g-1))/r}}{2} \sum_{\{z | z^r = 1, z \neq 1\}} \frac{z^{b-g+1}}{(1 - z)^{2g-1}}.
\]
The last summation can be computed using some combinatorial identities which we will now briefly describe. We define \( B(k, m) = \sum_{\{z \mid z^r = 1, z \neq 0\}} z^m (1 - z)^{-k} \). Then we can compute \( B(k, m) \) using the recursive relation \( B(k, m) = B(k, m - 1) + B(k - 1, m - 1) \) and the condition \( \sum_{i=0}^{r-1} B(k, i) = 0 \). These relations can easily be verified and one can hence obtain the following recursive formula for \( m > 0 \)

\[
B(k, m) = B(k, 0) - \sum_{i=0}^{m-1} B(k - 1, i)
\]

and \( B(k, 0) \) can be computed by the following formula.

\[
B(k, 0) = \frac{1}{r} \sum_{i=0}^{r-2} (r - i - 1)B(k - 1, i)
\]

For example using the above formulas we can calculate for \( 0 < m < r - 1 \)

\[
B(2, m) = -(r^2 + r(6 - 6m) + 6m^2 - 12m + 5)/12.
\]

Let \( r \geq 3 \). If we assume that the genus of the curve is 2, then the condition for the quot scheme to be zero dimensional is \( 2b - 4 = 0(\mod r) \). Since \( 0 \leq b < r \), hence either \( b = 2 \) or \( 2b - 4 = r \). Hence we have the following corollary

**Corollary 4.6** Let \( g = 2 \) and \( r \geq 3 \). If \( b = 2 \) then we have \( m(r, d, 2, 2) = r^3(r^2 - 1)/24 \) and if \( r = 2b - 4 \) then we have \( m(r, d, 2, 2) = r^3(r^2 + 2)/48 \).

The formula for \( r = 2b - 4 \) was also obtained by Lange and Newstead (Theorem 4.1 [12]).

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