Matrix generalizations of integrable systems with Lax integro-differential representations

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Abstract.
We present (2+1)-dimensional generalizations of the k-constrained Kadomtsev-Petviashvili (k-cKP) hierarchy and corresponding matrix Lax representations that consist of two integro-differential operators. Additional reductions imposed on the Lax pairs lead to matrix generalizations of Davey-Stewartson systems (DS-I,DS-II,DS-III) and (2+1)-dimensional extensions of the modified Korteweg-de Vries and the Nizhnik equation. We also present an integro-differential Lax pair for a matrix version of a (2+1)-dimensional extension of the Chen-Lee-Liu equation.

1. Introduction
In the modern theory of nonlinear integrable systems, algebraic methods play an important role (see the survey in [1]). Among them are the Zakharov-Shabat dressing method [2–4], Marchenko’s method [5], and the approach based on Darboux-Crum-Matveev transformations [6, 7]. Algebraic methods allow to omit analytical difficulties that arise in the investigation of corresponding direct and inverse scattering problems for nonlinear equations. A significant contribution to such methods has also been made by the Kioto group [8–12]. In particular, they investigated scalar and matrix hierarchies for nonlinear integrable systems of Kadomtsev-Petviashvili type (KP hierarchy).

The KP hierarchy is of fundamental importance in the theory of integrable systems and shows up in various ways in mathematical physics. Several extensions and generalizations of it have been obtained. For example, the multi-component KP hierarchy contains several physically relevant nonlinear integrable systems, including the Davey-Stewartson equation, the two-dimensional Toda lattice and the three-wave resonant interaction system. There are several equivalent formulations of this hierarchy: matrix pseudo-differential operator (Sato) formulation, τ-function approach via matrix Hirota bilinear identities, multi-component free fermion formulation. Another kind of generalization is the so-called “KP equation with self-consistent sources” (KPSCS), discovered by Melnikov [13–17]. In [18–22], k-symmetry constraints of the KP hierarchy were investigated, which have connections with KPSCS. The resulting k-constrained KP (k-cKP) hierarchy contains physically relevant systems like the nonlinear Schrödinger equation, the Yajima-Oikawa system, a generalization of the Boussinesq equation, and the Melnikov system. Multi-component generalizations of the k-cKP hierarchy were considered in [23]. In papers [24–26] the differential type of the gauge transformation
operator was applied to the constrained KP hierarchy at first. A modified k-constrained KP (k-cmKP) hierarchy was proposed in [21,27,28]. It contains, for example, the vector Chen-Lee-Liu and the modified KdV (mKdV) equation. Multi-component versions of the Kundu-Eckhaus and Gerdjikov-Ivanov equations were also obtained in [27], via gauge transformations of the k-cKP, respectively the k-cmKP hierarchy.

Moreover, in [29,30], (2+1)-dimensional extensions of the k-cKP hierarchy were introduced. In [31,32], exact solutions for some representatives of the (2+1)-dimensional k-cKP hierarchy were obtained by dressing binary transformations. (2+1)-dimensional extensions of k-cKP and k-cmKP and their dressings with the help of differential transformations were investigated in [33,34]. The (2+1)-dimensional k-cKP hierarchy in particular contains the DS-III system and a (2+1)-dimensional extension of the mKdV equation. A corresponding Lax representation of the (2+1)-dimensional k-cKP hierarchy consists of one differential and one integro-differential operator. Our aim was to generalize Lax representations of the (2+1)-dimensional k-cKP hierarchy to the case of two integro-differential operators, in order to obtain Lax representations for matrix generalizations of Davey-Stewartson systems DS-I, DS-II, DS-III, and their higher order counterparts. We also present a Lax representation, with two integro-differential operators, for a (2+1)-dimensional generalization of the Chen-Lee-Liu equation, which has been obtained in [28].

This work is organized as follows. In Section 2 we present a short survey of results on constraints for KP hierarchies and their (2+1)-dimensional generalizations. In Sections 3 and 4 we consider integro-differential Lax representations that generalize corresponding representations for the (2+1)-dimensional k-cKP hierarchy. As a result of additional reductions, we obtain matrix generalizations of Davey-Stewartson and (2+1)-dimensional mKdV equations that generalize corresponding systems in the (2+1)-dimensional k-cKP hierarchy. In Section 5, by application of a gauge transformation, we obtain a Lax representation for a (2+1)-dimensional matrix extension of the Chen-Lee-Liu equation. In the final section, we discuss the obtained results and mention problems for further investigations.

2. k-constrained KP hierarchy and its extensions

To make this paper somewhat self-contained, we briefly introduce the KP hierarchy [1], its k-symmetry constraints (k-cKP hierarchy), and the extension of the k-cKP hierarchy to the (2+1)-dimensional case [29,30]. A Lax representation of the KP hierarchy is given by

\[ L_{t_n} = [B_n, L], \quad n \geq 1, \]  

(1)

where \( L = D + U_1 D^{-1} + U_2 D^{-2} + \ldots \) is a scalar pseudodifferential operator, \( t_1 := x, \) \( D := \partial_x, \) and \( B_n := (L^n)_+, \) \( (L^n)_{\geq 0} = D^n + \sum_{i=0}^{n-2} u_i D^i \) is the differential operator part of \( L^n. \) The consistency condition (zero-curvature equations), arising from the commutativity of the flows, are

\[ B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0. \]  

(2)

Let \( B_n^r \) denote the formal transpose of \( B_n, \) i.e. \( B_n^r := (-1)^n D^n + \sum_{i=0}^{n-2} (-1)^i D^i u_i^\top, \) where \( ^\top \) denotes the matrix transpose. We use curly brackets to denote the action of an operator on a function whereas, for example, \( B_n q \) means composition of the operator \( B_n \) and the operator of multiplication by the function \( q. \) The k-cKP hierarchy [18–22] is given by

\[ L_{t_n} = [B_n, L], \quad q_{t_n} = B_n \{ q \}, \quad -r_{t_n} = B_n^r \{ r \}, \quad n = 2,3,\ldots, \]  

(3)

with the k-symmetry reduction

\[ L_k := L^k = B_k + q D^{-1} r. \]  

(4)
It admits the Lax representation (here $k \in \mathbb{N}$ is fixed)

$$[B_k + gD^{-1}r, \partial_u - B_n] = 0. \quad (5)$$

Below we will also use the formal adjoint $B^*_n := B^*_n = (-1)^nD^n + \sum_{i=0}^{n-2}(-1)^iD^iu_i^*$ of $B_n$, where $^*$ denotes Hermitian conjugation (complex conjugation and transpose).

In [23], multi-component (vector) generalizations of the $k$-cKP hierarchy were introduced,

$$L^k = B_k + \sum_{i=1}^l \sum_{j=1}^l q_i m_{ij} D^{-1}r_j = B_k + q_i M_0 D^{-1}r^\top, \quad q_i = B_k\{q\}, \quad r_i = -B_k^*\{r\}, \quad \quad (6)$$

where $q = (q_1, \ldots, q_l)$ and $r = (r_1, \ldots, r_l)$ are vector functions, $M_0 = (m_{ij})_{i,j=1}^l$ is a constant $l \times l$ matrix. A corresponding Lax representation is given by

$$[L_k, M_n] = 0, \quad L_k = B_k + q_i M_0 D^{-1}r^\top, \quad M_n = \partial_u - B_n. \quad (7)$$

For $k = 1$, this is a multi-component generalization of the AKNS hierarchy. For $k = 2$ and $k = 3$, one obtains vector generalizations of the Yajima-Oikawa and Melnikov [14,15] hierarchies, respectively.

In [21, 27, 28], a $k$-constrained modified KP (k-cmKP) hierarchy was introduced and investigated. Its Lax representation has the form

$$[\tilde{B}_k + q_i M_0 D^{-1}r^\top D, \partial_u - \tilde{B}_n] = 0, \quad (8)$$

where $\tilde{B}_k = D^k + \sum_{j=1}^{k-1}w_{kj}D^j$. For $k = 1, 2, 3$, this leads to vector generalizations of the Chen-Lee-Liu, the modified multi-component Yajima-Oikawa and Melnikov hierarchies.

An essential extension of the k-cKP hierarchy is its (2+1)-dimensional generalization [29,30], given by

$$[L_k, M_n] = 0, \quad L_k = \beta_k \partial_{\tau_k} - B_k - q_i M_0 D^{-1}r^\top, \quad M_n = \alpha_n \partial_{\tau_n} - A_n, \quad (9)$$

where $u_j$ and $v_i$ are scalar functions, $q$ and $r$ are $l$-component vector-functions. An equivalent system is

$$\alpha_n B_{k,t_n} = \beta_k A_{n,\tau_n} + [A_n, B_k] + ([A_n, q_i M_0 D^{-1}r^\top])_{\geq 0}, \quad \alpha_n q_i = A_n\{q\}, \quad \alpha_n r_i = -A_n^\top\{r\}. \quad (10)$$

We list some members of this (2+1)-dimensional generalization of the k-cKP hierarchy:

(1) $k = 1, n = 2$. Then (9) has the form

$$[L_1, M_2] = 0, \quad L_1 = \beta_1 \partial_{\tau_1} - D - q_i M_0 D^{-1}r^\top, \quad M_2 = \alpha_2 \partial_{\tau_2} - D^2 - v_0, \quad (11)$$

and it is equivalent to the following system,

$$\alpha_2 q_{0z} = q_{zz} + v_0 q, \quad \alpha_2 r_{zz} = -r_{zz} - v_0 r, \quad \beta_1 v_{0,\tau_1} = v_{0,zz} - 2(q_i M_0 r^\top)_{zz}. \quad (12)$$

After the reduction $\beta_1 \in \mathbb{R}$, $\alpha_2 \in i\mathbb{R}$, $r = \tilde{q}$, $M_0 = M_0^\top$, $v_0 = \tilde{v}_0$, the operators $L_1$ and $M_2$ in (10) are skew-Hermitian and Hermitian, respectively, and (11) becomes the DS-III system

$$\alpha_2 q_{t_0} = q_{xx} + v_0 q, \quad \beta_1 v_{0,\tau_1} = v_{0,xx} - 2(q_i M_0 q^\top)_{xx}. \quad (13)$$

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(2) \( k = 1, n = 3 \). Now (9) becomes

\[
[L_1, M_3] = 0, \quad L_1 = \beta_1 \partial_{r_1} - D - qM_0D^{-1}r^\top, \quad M_3 = \alpha_3 \partial_{t_3} - D^3 - v_1D - v_0.
\] (13)

After the additional reduction \( \alpha_3, \beta_1 \in \mathbb{R}, M_0 = M_0^R, v_1 = \bar{v}_1, \bar{v}_0 + v_0 = v_{1z} \), the operators \( L_1, M_3 \) in (13) are skew-Hermitian, and the Lax equation (13) is equivalent to the following (2+1)-dimensional generalization of the mKdV system,

\[
\begin{align*}
\alpha_3 q_{t_3} &= q_{xxx} + v_1 q_x + v_0 q, \\
\beta_1 v_{0, r_1} &= v_{0, x} - 3(q_x M_0 q^*)_x, \quad \beta_1 v_{1, r_1} = v_{1, x} - 3(q M_0 q^*)_x.
\end{align*}
\] (14)

This system admits the real version \((M_0^\top = M_0^R, q^* = q^\top)\)

\[
\alpha_3 q_{t_3} = q_{xxx} + v_1 q_x + \frac{1}{2} v_{1, xx} q, \quad \beta_1 v_{1, r_1} = v_{1, x} - 3(q M_0 q^*)_x.
\] (15)

(3) \( k = 2, n = 2 \). (9) takes the form

\[
[L_2, M_2] = 0, \quad L_2 = \beta_2 \partial_{r_2} - D^2 - u_0 - qM_0D^{-1}r^\top, \quad M_2 = \alpha_2 \partial_{t_2} - D^3 - u_0.
\] (16)

Under the reduction \( \alpha_2, \beta_2 \in i\mathbb{R}, r = \bar{q}, M_0 = -M_0^R, u_0 = \bar{u}_0 := u \), the operators \( L_2 \) and \( M_2 \) in (16) become Hermitian and (16) takes the form

\[
\begin{align*}
\alpha_2 q_{t_2} &= q_{xx} + u_q, \\
\alpha_2 u_{t_2} &= \beta_2 u_{r_2} + 2(q M_0 q^*)_x,
\end{align*}
\] (17)

which is a (2+1)-dimensional vector generalization of the Yajima-Oikawa system.

(4) \( k = 2, n = 3 \). Now (9) becomes

\[
\begin{align*}
L_2 &= \beta_2 \partial_{r_2} - D^2 - 2u_0 - qM_0D^{-1}r^\top, \\
M_3 &= \alpha_3 \partial_{t_3} - D^3 - 3u_0D - \frac{3}{2} (u_{0, x} + \beta_2 D^{-1}\{u_{0, r_2}\} + q M_0 r^\top).
\end{align*}
\] (18)

With the additional reduction \( \beta_2 \in i\mathbb{R}, \alpha_3 \in \mathbb{R}, u_0 = \bar{u}_0 := u, M_0 = -M_0^R \) and \( r = \bar{q} \), this is equivalent to the following generalization of the Melnikov system [32],

\[
\begin{align*}
\alpha_3 q_{t_3} &= q_{xxx} + 3q_x + \frac{3}{2} \{u_x + \beta_2 D^{-1}\{u_{r_2}\} + q M_0 q^*\} \bar{q}, \\
\alpha_3 u_{t_3} &= \frac{1}{4} q_{xxx} + 3q_x + \frac{3}{4} (q M_0 q^* - q_x M_0 q^*)_x - \frac{3}{4} \beta_2 (q M_0 q^*)_{r_2} \bigg|_x = \frac{3}{4} \beta_2 u_{r_2 r_2}.
\end{align*}
\] (19)

Remark 2.1 (12) and (17) are related by a linear change of the independent variables.

Thus, for \( k = 1 \) we have the DS-III hierarchy (its first members are DS-III \((k = 1, n = 2)\) and a special \((2+1)\)-dimensional extension of mKdV \((k = 1, n = 3)\), see (12) and (14). For \( k = 2, k = 3 \), we have \((2+1)\)-dimensional generalizations of the Yajima-Oikawa (in particular, it contains (17) and (19)) and the Melnikov hierarchy [15], respectively.

3. Lax representations for matrix generalizations of Davey-Stewartson systems

In this section we introduce an essential extension of the Lax pair (10) for the \((2+1)\)-dimensional extended KP hierarchy to the case of two integro-differential operators. It leads to DS-I and DS-II systems. It is known that the Davey-Stewartson (DS-I) system is connected with the non-stationary Dirac operator (see [35]). By using a matrix differential representation for the DS system, involving the non-stationary Dirac operator, and the representation for DS-III in the \((2+1)\)-dimensional k-cKP hierarchy, we obtain the following Lax integro-differential Lax pair,

\[
L_1 = \partial_y - qM_0D^{-1}r^\top,
\]
where \( \alpha, c_1, c_2 \in \mathbb{C} \). \( \mathbf{q} = \mathbf{q}(x, y, t_2) \), \( \mathbf{r} = \mathbf{r}(x, y, t_2) \) and \( S_1 = S_1(x, y, t_2) \) are matrix functions with dimensions \( N \times M \) and \( N \times N \), respectively. \( \mathbf{M}_0 \) is a constant \( M \times M \) matrix. The Lax equation \([L_1, M_2] = 0\) is equivalent to the following system,

\[
\begin{align*}
\alpha_2 \mathbf{q}_{t_2} &= \mathbf{c}_1 \mathbf{q}_{xx} + \mathbf{c}_2 \mathbf{q}_{yy} - 2\mathbf{c}_1 \mathbf{S}_1 \mathbf{q} - 2\mathbf{c}_2 \mathbf{q} \mathbf{M}_0 \mathbf{S}_2, \\
-\alpha_2 \mathbf{r}_{t_2} &= \mathbf{c}_1 \mathbf{r}_{xx} + \mathbf{c}_2 \mathbf{r}_{yy} - 2\mathbf{c}_1 \mathbf{r}_1 \mathbf{S}_1 - 2\mathbf{c}_2 \mathbf{S}_2 \mathbf{M}_0 \mathbf{r}, \\
S_{1y} &= (\mathbf{q} \mathbf{M}_0 \mathbf{r})_x, \quad S_{2x} = (\mathbf{r}^\top \mathbf{q})_y.
\end{align*}
\]

After the reduction \( c_1, c_2 \in \mathbb{R} \), \( \alpha_2 \in i\mathbb{R} \); \( \mathbf{r}^\top = \mathbf{q}^* \), \( \mathbf{M}_0 = \mathbf{M}_0^\top \), the operators \( L_1 \) and \( M_2 \) are skew-Hermitian and Hermitian, respectively, and (21) takes the form

\[
\begin{align*}
\alpha_2 \mathbf{q}_{t_2} &= \mathbf{c}_1 \mathbf{q}_{xx} + \mathbf{c}_2 \mathbf{q}_{yy} - 2\mathbf{c}_1 \mathbf{S}_1 \mathbf{q} - 2\mathbf{c}_2 \mathbf{q} \mathbf{M}_0 \mathbf{S}_2, \\
S_{1y} &= (\mathbf{q} \mathbf{M}_0 \mathbf{q}^*)_x, \quad S_{2x} = (\mathbf{q}^* \mathbf{q})_y.
\end{align*}
\]

This has the following two interesting subcases:

1. \( c_2 = 0 \). Then we have

\[
\alpha_2 \mathbf{q}_{t_2} = \mathbf{c}_1 \mathbf{q}_{xx} - 2\mathbf{c}_1 \mathbf{S}_1 \mathbf{q}, \quad S_{1y} = (\mathbf{q} \mathbf{M}_0 \mathbf{q}^*)_x.
\]

2. \( c_1 = 0 \). Then (22) takes the form

\[
\alpha_2 \mathbf{q}_{t_2} = \mathbf{c}_2 \mathbf{q}_{yy} - 2\mathbf{c}_2 \mathbf{q} \mathbf{M}_0 \mathbf{S}_2, \quad S_{2x} = (\mathbf{q}^* \mathbf{q})_y.
\]

The systems (23) and (24) are two different matrix generalizations of the Davey-Stewartson equation (DS-III) [2, 30, 37]. In the vector case \((N = 1)\), (23) has been obtained in [28] as a member of the DS-III hierarchy.

**Remark 3.1** If \( N = 1 \), the change of variables \( \tilde{x} = x + y \), \( \tau_1 = \beta_1 y \), \( \mathbf{q}(\tilde{x}, \tau_1) = \mathbf{q}(x, y) \), \( v_0(\tilde{x}, \tau_1) = -2S_1(x, y) \) maps (23) to the DS-III equation (12) (which we obtained from the \((2+1)\)-dimensional k-cKP hierarchy).

Let us consider (22) in the case where \( u := \mathbf{q} \) and \( \mathbf{M}_0 := \mu \) are scalars. Then (22) becomes

\[
\alpha_2 u_{t_2} = \mathbf{c}_1 u_{xx} + \mathbf{c}_2 u_{yy} - 2\mathbf{c}_1 \mathbf{S}_1 u - 2\mathbf{c}_2 \mathbf{S}_2 u, \quad S_{1y} = \mu(|u|^2)_x, \quad S_{2x} = (|u|^2)_y.
\]

Setting \( c_1 = c_2 = 1 \) and \( \mu = 1 \), as a consequence of (25) we obtain

\[
\alpha_2 u_{t_2} = u_{xx} + u_{yy} - 2Su, \quad S_{xy} = (|u|^2)_{xx} + (|u|^2)_{yy}.
\]

where \( S = S_1 + S_2 \). This is the well-known Davey-Stewartson system (DS-I) and (22) is therefore a matrix generalization.

Now we present integro-differential Lax pairs for generalizations of the second Davey-Stewartson equation (DS-II). By replacing the real variables \( x \) and \( y \) in the operators \( L_1, M_2 \) in (20) by a complex variable \( z = x + iy \) and its conjugate \( \bar{z} \), respectively, we obtain the following pair,

\[
\begin{align*}
L_1 &= \partial_z - \mathbf{q} \mathbf{M}_0 \mathbf{D}_z^{-1} \mathbf{r}^\top, \\
M_2 &= \alpha_2 \partial_{z_2} - \mathbf{c}_1 D^2_{zz} - \mathbf{c}_2 \partial_{z_2}^2 + 2\mathbf{c}_1 \mathbf{S}_1 + 2\mathbf{c}_2 \mathbf{q} \mathbf{M}_0 \mathbf{D}_z^{-1} \mathbf{r}_z^\top + 2\mathbf{c}_2 \mathbf{q} \mathbf{M}_0 \mathbf{D}_z^{-1} \mathbf{r}_z^\top \partial_{z_2},
\end{align*}
\]

(27)
where \( q = q(z, \bar{z}, t_2) \), \( r = r(z, \bar{z}, t_2) \) and \( S_1 = S_1(z, \bar{z}, t_2) \) are matrix functions with dimensions \( N \times M \) and \( N \times N \), respectively. \( M_0 \) is a constant \( M \times M \) matrix. The Lax equation \([L_1, M_2] = 0\) is equivalent to the system

\[
\begin{align*}
\alpha_2 q_{x_2} = c_1 q_{x_2} &+ c_2 q_{y_2} - 2c_1 S_1 q - 2c_2 q M_0 S_2, \\
-\alpha_2 r_{t_2} = c_1 r_{x_2} &+ c_2 r_{y_2} - 2c_1 r S_1 - 2c_2 r M_0 r^\top, \\
S_{1z} &= (q M_0 r_2^\top)_z, ~ S_{2z} = (r_2^\top q)_z.
\end{align*}
\]

In terms of the real variables \( t_2, x \), and \( y \), after setting \( c_1 = c_2 = 1 \), the system (28) becomes

\[
\begin{align*}
2\alpha_2 q_{x_2} &= q_{xx} - q_{yy} - 4S_1 q - 4q M_0 S_2, \\
-2\alpha_2 r_{t_2} &= r_{xx} - r_{yy} - 4r S_1 - 4S_2 M_0 r^\top, \\
S_{1x} + iS_{1y} &= (q M_0 r^\top)_x - i(q M_0 r^\top)_y, ~ S_{2x} - iS_{2y} = (r^\top q)_x + i(r^\top q)_y.
\end{align*}
\]

If \( N = M \) (so that \( q \) and \( r \) are square matrices), and \( \alpha_2 \in \mathbb{R} \), (29) admits the reduction \( M_0 r^\top = q, ~ S_1 = S_2 \), and then takes the following form,

\[
\begin{align*}
2\alpha_2 q_{x_2} &= q_{xx} - q_{yy} - 4S_1 q - 4S_1, \\
S_{1x} + iS_{1y} &= (q q)^{\top} - i(q q)^{\top}.
\end{align*}
\]

In the scalar case \((N = M = 1)\), writing \( u = q \), we obtain the following consequence of (30),

\[
2\alpha_2 u_{x_2} = u_{xx} - u_{yy} - 8S u, ~ S_{xx} + S_{yy} = |u|_{xx}^2 - |u|_{yy}^2,
\]

where \( \hat{S} = \text{Re}(S_1) \). This is the second Davey-Stewartson system (DS-II).

4. Integro-differential Lax representations for a \((2+1)\)-dimensional matrix generalization of the mKdV equation

In this section we generalize the Lax representation (13) for the \((2+1)\)-dimensional mKdV equation to the case of two integro-differential operators. More precisely, we consider an extension of the operator \( M_3 \) in (13). As in the case of the operator \( M_2 \) in (20), by using a differential representation for the \((2+1)\)-dimensional mKdV equation that involves the non-stationary Dirac operator \([35]\), we obtain the following integro-differential Lax pair,

\[
\begin{align*}
L_1 &= \partial_y - q M_0 D^{-1} r^\top, \\
M_3 &= \alpha_3 \partial_t + c_1 D^3 + c_2 D^3 - c_1 v_1 D - 3c_1 v_3 + 3c_2 q y M_0 D^{-1} r^y \\
&- 3c_2 q M_0 \partial_y D^{-1} r^x q M_0 D^{-1} r^y + 3c_2 q M_0 D^{-1} \{r^x q\}_y M_0 D^{-1} r^y \\
&+ 3c_2 \partial_y q M_0 D^{-1} r^\top q, \partial_y
\end{align*}
\]

where \( q = q(x, y, t_3), \) \( r = r(x, y, t_3) \) and \( v_1 = v_1(x, y, t_3), \) \( v_3 = v_3(x, y, t_3) \) are \( N \times M \), respectively \( N \times N \), matrix functions. As in the case of the pair \( L_1, M_2 \) in (20), a change of variables shows that in the case \( c_2 = 0 \) and \( N = 1 \) the Lax pairs (32) and (13) are equivalent. The Lax equation \([L_1, M_3] = 0\) results in the system

\[
\begin{align*}
\alpha_3 q_{t_3} + c_1 q r_{x_3} - c_2 r_{y_3} - 3c_1 v_1 q_x + 3c_2 q y M_0 v_2 + 3c_2 q M_0 v_2 y - 3c_1 v_3 - 3c_2 q M_0 v_4 \\
- 3c_2 q M_0 r_2^y q M_0 v_2 - M_0 v_2 M_0 r^\top q \\
\end{align*}
\]

\[
\begin{align*}
\alpha_3 r_{t_3} + c_1 r r_{x_3} - c_2 r_{y_3} - 3c_1 r^x v_1 + 3c_2 v_2 M_0 r^y + 3c_1 r^y v_3 + 3c_2 v_4 M_0 r^\top \\
- 3c_2 D^{-1} \{v_2 M_0 r^\top q - r^\top q M_0 v_2\} M_0 r^\top \\
\end{align*}
\]

\[
\begin{align*}
v_{1y} = (q M_0 r^\top)_x, ~ v_{2x} = (r^\top q)_y, ~ v_{3y} = (q x M_0 r^\top)_x + [q M_0 r^\top, v_1], ~ v_{4x} = (r^\top q)_y.
\end{align*}
\]

Let us consider some reductions of this system:
In this section we apply a gauge transformation to the Lax pair (20) in order to obtain an generalization of the Chen-Lee-Liu equation.

In the scalar case \((N = M = 1)\), setting \(\mathcal{M}_0 = \mu\) and \(q = q(x, y, t_3) = q(x, y, t_3)\), (34) reads

\[
\alpha_3 q_{t_3} + c_1 q_{xxx} - c_2 q_{yyy} - 3 c_1 v_1 q_x + 3 c_2 v_y \mathcal{M}_0 v_2 + 3 c_2 q \mathcal{M}_0 v_2 y - 3 c_1 v_3 q - 3 c_2 q \mathcal{M}_0 v_4 - 3 c_2 q D^{-1} \left\{ \mathcal{M}_0 q q \mathcal{M}_0 v_2 - \mathcal{M}_0 v_2 \mathcal{M}_0 q q \right\} = 0,
\]

\[
v_{1y} = (q_{\mathcal{M}_0 q}^\top)_x, v_{2x} = (q^\top q)_y, v_{3y} = (q_x \mathcal{M}_0 q^\top)_x + [q \mathcal{M}_0 q^\top, v_1], v_{4x} = (q_y^\top q)_y. \tag{34}
\]

In the real case \(q = \tilde{q}\), (34) becomes

\[
\alpha_3 q_{t_3} + c_1 q_{xxx} - c_2 q_{yyy} - 3 c_1 v_1 q_x + 3 c_2 v_y \mathcal{M}_0 v_2 + 3 c_2 q \mathcal{M}_0 v_2 y - 3 c_1 v_3 q - 3 c_2 q \mathcal{M}_0 v_4 - 3 c_2 q D^{-1} \left\{ \mathcal{M}_0 (q^\top q) \mathcal{M}_0 v_2 - \mathcal{M}_0 v_2 (q^\top q) \right\} = 0,
\]

\[
v_{1y} = (q_{\mathcal{M}_0 q^\top})_x, v_{2x} = (q^\top q)_y, v_{3y} = (q_x \mathcal{M}_0 q^\top)_x + [q \mathcal{M}_0 q^\top, v_1], v_{4x} = (q_y^\top q)_y. \tag{36}
\]

In the scalar case \((N = M = 1)\), writing \(\mathcal{M}_0 = \mu\) and \(q = q(x, y, t_3) = q(x, y, t_3)\), after setting \(y = x\) and \(c_1 - c_2 = 1\), (36) has the form

\[
\alpha_3 q_{t_3} + q_{xxx} - 6 \mu q^2 q_x = 0, \tag{37}
\]

which is the mKdV equation. The systems (34) and (36) are therefore, respectively, complex and real, spatially two-dimensional matrix generalizations of it.

(2) \(\mathcal{M}_0 r^\top = \nu\) with a constant matrix \(\nu\). In terms of \(u := q \nu\), (33) takes the form

\[
\alpha_3 u_{t_3} + c_1 u_{xxx} - c_2 u_{yyy} - 3 c_1 D \left\{ \left( \int u_x dy \right) u \right\} + 3 c_2 \partial_y \left\{ u \left( \int u_y dx \right) \right\} - 3 c_1 \left( \int [u, v_1] dy \right) u - 3 c_2 u \left( \int [u, v_2] dx \right) = 0,
\]

\[
\nu \left( c_1 \int [u, v_1] dy - c_2 \int [v_2, u] dx \right) = 0, \quad v_{1y} = u_x, \quad v_{2x} = u_y. \tag{38}
\]

In the scalar case \((N = 1, M = 1)\), this reduces to

\[
\alpha_3 u_{t_3} + c_1 u_{xxx} - c_2 u_{yyy} - 3 c_1 D \left\{ \left( \int u_x dy \right) u \right\} + 3 c_2 \partial_y \left\{ u \left( \int u_y dx \right) \right\} = 0, \tag{39}
\]

which is the Nizhnik equation [35]. The system (38) thus generalizes the latter to the matrix case.

5. Integro-differential Lax representation for a spatially two-dimensional matrix generalization of the Chen-Lee-Liu equation

In this section we apply a gauge transformation to the Lax pair (20) in order to obtain an integro-differential Lax pair for the (2+1)-dimensional Chen-Lee-Liu equation. The resulting Lax pair generalizes a corresponding Lax pair obtained from the (2+1)-dimensional k-cmKP
Let us consider some of its reductions:

Let \( q(x, y, t_2) \) be an \( M \times M \) matrix solution of \( L_1 \{ f \} = 0 \), where the operator \( L_1 \) has the form given in (20). Consider the following gauge transformation,

\[
\tilde{q} := f^{-1}(q) \quad \tilde{r} := -D^{-1}\{r^\top f\}. \]

where \( \tilde{q} := f^{-1}(q) \). \( \tilde{r} \) is also possible to apply a similar transformation to the operator \( M_2 \) in (20). In order to simplify notation, in the following we write \( q, r \) instead of \( \tilde{q}, \tilde{r} \). As a result of the gauge transformation, we obtain a Lax pair of the form

\[
L_1 = \partial_y - qM_0D^{-1}r^\top D, \tag{40}
\]

\[
M_2 = \alpha_2\partial_{t_2} - c_1D^2 - c_2\partial_y^2 + 2c_1S_1D + 2c_2qM_0D^{-1}\partial_yr^\top D, \tag{41}
\]

where \( q = q(x, y, t_2) \), \( r = r(x, y, t_2) \) and \( S_1 = S_1(x, y, t_2) \) are \( N \times M \), respectively \( N \times N \), matrix functions. \( M_0 \) is a constant \( M \times M \) matrix. The condition \([L_1, M_2] = 0\) is then equivalent to the following system,

\[
\begin{align*}
\alpha_2q_{t_2} - c_1q_{xx} - c_2q_{yy} + 2c_1S_1q_x - 2c_2qM_0S_2 + 2c_2qM_0\{r^\top q\}_y = 0, \\
\left(\alpha_2r_{t_2}^\top + c_1r_{xx}^\top + c_2r_{yy}^\top + 2c_1r_{x}^\top S_1 + 2c_2S_2M_0r^\top\right)_x = 0, \\
S_{1y} = (qM_0r^\top)_x + [qM_0r^\top, S_1], \\
S_{2x} = (r^\top q)_y.
\end{align*}
\tag{42}
\]

Let us consider some of its reductions:

1. The system (42) contains three different \((2+1)\)-dimensional matrix generalizations of the Chen-Lee-Liu system: a) \( c_1 = 0 \), \( \alpha_2 \in \mathbb{R} \); b) \( c_2 = 0 \), \( \alpha_2 \in \mathbb{R} \); c) \( c_1 \neq 0 \) and \( c_2 \neq 0 \).

In case b), \( M_2 \) becomes a differential operator. With the restriction \( N = 1 \), this case appeared in [28].

2. \( \alpha_2 \in \mathbb{R} \), \( c_1, c_2 \in \mathbb{R} \), \( M_0 = -M_0^* \), \( r^\top = q^\top \), \( S_1 = S_1^* \). Then \( L_1 \) and \( M_2 \), given by (40) and (41), are \( D \)-skew-Hermitian \((L_1^* = -DL_1D^{-1})\) and \( D \)-Hermitian \((M_2^* = DM_2D^{-1})\), respectively. (42) has the following form,

\[
\begin{align*}
\alpha_2q_{t_2} - c_1q_{xx} - c_2q_{yy} + 2c_1S_1q_x - 2c_2qM_0S_2 + 2c_2qM_0\{q^\top q\}_y = 0, \\
S_{1y} = (qM_0q^\top)_x + [qM_0q^\top, S_1], \\
S_{2x} = (q^\top q)_y.
\end{align*}
\tag{43}
\]

In the scalar case \((N = M = 1)\), setting \( c_1 = 1 \), \( c_2 = 0 \), \( y = x \), and writing \( \mu = M_0, q = q(x, y, t_2) = q(x, y, t_2) \), (43) reduces to the Chen-Lee-Liu equation [38]

\[
\alpha_2q_{t_2} - q_{xx} + 2\mu|q|^2q_x = 0.
\]

3. \( M_0r^\top = \nu \) with a constant matrix \( \nu \). Then the operators (40) and (41) are purely differential and, in terms of \( u := \nu \), (42) reads

\[
\begin{align*}
\alpha_2u_{t_2} - c_1u_{xx} - c_2u_{yy} + 2c_1S_1u_x + 2c_2uu_y = 0, \\
S_{1y} = u_x + [u, S_1].
\end{align*}
\tag{44}
\]

This is a spatially two-dimensional matrix generalization of the Burgers equation. A generalization of (44) to an arbitrary number of spatial dimensions has been considered in [39].

It is possible to apply a similar gauge transformation to the pair of operators (32). Corresponding Lax representations that lead to matrix versions of spatially two-dimensional higher Chen-Lee-Liu equations were investigated in [39].
6. Conclusions

In this paper we considered several members of a (2+1)-dimensional generalization of the k-cKP hierarchy (9). Originally, this substantial generalization of the k-cKP hierarchy had been proposed in [29, 30]. For some members of this hierarchy (e.g. (17) and (19)), solutions were obtained via the binary transformation dressing method [31, 32]. The (2+1)-dimensional extension of the k-cKP hierarchy has been rediscovered more recently in [33], also see [34, 40] for further investigations. The authors of [33] and [34] considered the (2+1)-dimensional k-cKP and (2+1)-dimensional k-cmKP hierarchies with two operators of the Lax pair having different orders of differentiation with respect to $x$. This excludes, for example, (2+1)-dimensional vector generalizations of the Yajima-Oikawa (17) and Drinfeld-Sokolov-Wilson equation [41–43]:

$$q_t = q_{xxx} + 3uq_x + \frac{3}{2}u_xq, \quad u_t = u_{r3} + 3(qM_0q^\top)_x.$$  \hspace{1cm} (45)

It is obtained from $[L_3, M_3] = 0$ with

$$L_3 = \partial_{r3} - D^3 - 3uD - \frac{3}{2}u_x - qM_0D^{-1}q^\top, \quad M_3 = \partial_{r3} - D^3 - 3uD - \frac{3}{2}u_x.$$ \hspace{1cm} (46)

**Remark 6.1** Analogously to Remark 2.1, it can be shown that the (2+1)-dimensional vector generalization of the Drinfeld-Sokolov-Wilson equation (45) is equivalent to the real version of the (2+1)-dimensional multi-component mKdV equation (15) via a linear change of independent variables.

The aim of our work was also to generalize Lax representations for members of the (2+1)-dimensional k-cKP hierarchy [29, 30, 33, 34], in order to obtain integrable equations that do not belong to the (2+1)-dimensional k-cKP hierarchy. In particular, we constructed Lax integro-differential representations for matrix generalizations of the Davey-Stewartson systems (DS-I, DS-II, DS-III), matrix generalizations of (2+1)-dimensional extensions of the mKdV, the Nizhnik [35] and the Chen-Lee-Liu [38] equations. Representations for some of those systems in the algebra of purely differential operators with matrix coefficients can be found in [44].

One of the problems left for further investigation are the dressing methods for the corresponding Lax representations. It was shown that for the Lax representations for the (2+1)-dimensional k-cKP hierarchy [29, 30, 34], one can use differential operators for the dressing. The most interesting systems obtained from the (2+1)-dimensional k-cKP hierarchy and in Sections 3-5 arise after a Hermitian conjugation reduction. This imposes nontrivial constraints on the dressing differential operator. It was shown for the k-cKP hierarchy [45, 46] that it is more suitable to use a binary transformation operator in this situation. The dressing method by binary transformations for evolution integro-differential operators that arise in the (2+1)-dimensional k-cKP case has been considered in [36]. We plan to apply this method to the systems in Sections 3-5 in a forthcoming paper. Another interesting question is the possibility of generalizations of other representations from the (2+1)-dimensional k-cKP hierarchy to the case of two integro-differential operators, e.g. (2+1)-dimensional extensions of the Yajima-Oikawa and the Melnikov systems.

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