Local gauge and magnetic translation groups

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The magnetic translation group was introduced as a set of operators \( T(R) = \exp[-iR \cdot (p - eA/c)/\hbar] \). However, these operators commute with the Hamiltonian for an electron in a periodic potential and a uniform magnetic field if the vector potential \( A \) (the gauge) is chosen in a symmetric way. It is shown that a local gauge field \( A_R(r) \) on a crystal lattice leads to operators, which commute with the Hamiltonian for any (global) gauge field \( A = A(r) \). Such choice of the local gauge determines a factor system \( \omega(R, R') = T(R)T(R')T(R + R')^{-1} \), which depends on a global gauge only. Moreover, for any potential \( A \) a commutator \( T(R)T(R')T(R)^{-1}T(R')^{-1} \) depends only on the magnetic field and not on the gauge.

I. INTRODUCTION

The behavior of electrons in crystalline (periodic) potentials in the presence of a constant (external) magnetic field has been studied since the thirties in many papers. In the sixties Brown \\cite{Brown}

\text{and Zak} independently introduced and investigated the so-called magnetic translation groups. Their results have been lately applied to a problem of the quantum Hall effect and relations with the Weyl–Heisenberg group have been also studied. Some interesting results have been presented lately by Geyler and Popov.

The Hamiltonian for an electron in a periodic potential \( V(r) \) and a uniform magnetic field (described by the vector potential \( A \)) is given as

\[
\mathcal{H} = \pi^2/2m + V(r), \quad \text{where} \quad \pi = p + eA/c
\]

is the (vector) operator of the kinetic momentum. Brown introduced a projective representation of the translation group in the following form

\[
T(R) = \exp[-i(p - eA/c) \cdot R/\hbar].
\]

These operators commute with the Hamiltonian \( \mathcal{H} \) if the vector potential \( A \) fulfills the following condition

\[
\partial A_j/\partial x_k + \partial A_k/\partial x_j = 0; \quad \text{for } j, k = 1, 2, 3.
\]

This relation holds, for example, for the gauge \( A(r) = (H \times r)/2 \), which was used by Brown and Zak. On the other hand, this condition is not satisfied by the Landau gauge \( A(r) = [-x_2H_3, 0, 0] \) (for \( H = [0, 0, H_3] \)), which is used in many papers.

The aim of this paper is to find such a gauge \( A' \) that: (i) \( \nabla \times A' = -H \); (ii) operators \( T'(R) = \exp[-i(p + A'c)/c) \cdot R/\hbar] \) commute with the Hamiltonian \( \mathcal{H} \); (iii) a factor system \( \omega(R, R') = T(R)T(R')T(R + R')^{-1} \) depends only on a global gauge \( A \), which defines the magnetic field (and the generalized momentum \( \pi \) in \( \mathcal{H} \)). It should be underlined that only the constant magnetic field \( H \) is considered. It occurs that these conditions are satisfied by a local gauge, i.e. an actual form of \( A'(r) \) depends on a lattice vector \( R \).

II. SOLUTION

For the constant magnetic field \( H = [H_1, H_2, H_3] \) the vector potential (gauge) \( A = [A_1, A_2, A_3] \) can be chosen as a linear function of \( r = [x_1, x_2, x_3] \) and can be written as

\[
A_j = \sum_{k=1}^{3} a_{jk} x_k, \quad \text{with} \quad a_{jk} \in \mathbb{R}, \quad a_{jj} = 0.
\]
Introducing a matrix $A = (a_{jk})$ it can be written as $A = \hbar \mathbf{r}$. Therefore, the magnetic field $\mathbf{H}$ is expressed by the matrix elements $a_{jk}$ as follows

$$H_j = -\sum_{k,l=1}^{3} \varepsilon_{jkl} a_{kl},$$  

(5)

what means that $\mathbf{H}$ is related to antisymmetrized matrix $A$.

The definition (2) of operators $T(R)$ can be rewritten as

$$T(R) = \exp(-i\pi' \cdot R/\hbar),$$  

(6)

where

$$\pi' = \mathbf{p} + eA'/c \quad \text{and} \quad A'(r) = -A(r),$$

so $\nabla \times A' = -\nabla \times A = -\mathbf{H}$. Let us consider $A' = \mathbf{A} - \mathbf{H} \times \mathbf{r} = \kappa^T \mathbf{r}$, i.e.

$$A'_j = A_j - \sum_{k,l=1}^{3} \varepsilon_{jkl} H_k x_l = \sum_{k=1}^{3} a_{kj} x_k.$$  

(7)

It is easy to note that $\nabla \times A' = -\mathbf{H}$. For example, assuming the Landau gauge (for $\mathbf{H} = [0,0,H_3]$) to be given as $\mathbf{A} = [-x_2 H_3,0,0]$ one obtains $A' = [0,-x_1 H_3,0]$, whereas the symmetric gauge $(\mathbf{H} \times \mathbf{r})/2$ yields $A' = - (\mathbf{H} \times \mathbf{r})/2$.

We have to check whether the operators $T(R)$ determined by the gauge (2) commute with the Hamiltonian. It suffices to calculate commutators $[\pi_j,\pi'_k]$ for $j,k = 1,2,3$, for which one obtains

$$[\pi_j,\pi'_k] = [-i\hbar \partial_j + eA_j/c,-i\hbar \partial_k + eA'_k/c] = -ie\hbar (\partial_j, A'_k) + [A_j,\partial_k]/c = 0.$$

To find a factor system of the above determined (projective) representations one has to calculate commutators $[X_j\pi'_j, X'_k\pi'_k]$ ($\mathbf{R} = [X_1,X_2,X_3]$):

$$[X_j\pi_j, X'_k\pi'_k] = X_j X'_k [-i\hbar \partial_j + eA'_j/c,-i\hbar \partial_k + eA'_k/c]$$

$$= -iX_j X'_k e\hbar/c (\partial_j A'_k - \partial_k A'_j) = -iX_j X'_k e\hbar/c (a_{jk} - a_{kj}).$$

On the other hand we have

$$(\mathbf{R} \times \mathbf{R}') \cdot \mathbf{H} = \sum_{l=1}^{3} \left( \sum_{j,k=1}^{3} \varepsilon_{ljk} X_j X'_k \right) \left( \sum_{p,q=1}^{3} \varepsilon_{lpq} a_{pq} \right) = \sum_{j,k=1}^{3} X_j X'_k (a_{jk} - a_{kj})$$

and, therefore,

$$T(R) T(R') = T(R + R') \exp[-i(e/\hbar c)(\mathbf{R} \times \mathbf{R}') \cdot \mathbf{H}/2].$$

It is interesting that this result does not depend on the chosen gauge $A$ (and $A'$).

Let us consider now a local gauge determined as $A'_R(r) = A(r + R/2)$, i.e.

$$(A'_R)_j(r) = \sum_{k=1}^{3} a_{kj} (x_k + X_k/2).$$  

(8)

Similar, but a bit more tedious, calculations lead to the following results:

1. $\nabla \times A'_R = -\mathbf{H}$;
2. Operators $T(R)$ determined by this gauge commute with the Hamiltonian (2);
3. The projective representation $T(R)$ is characterized by a factor system

$$\omega(R,R') = T(R) T(R') T(R + R')^{-1} = \exp[(-i\hbar c)(\mathbf{R} \cdot A(R'))].$$  

(9)

Note that a scalar product in the last equation can be also written as a bilinear form.
\[
R \cdot A(R') = \sum_{j,k=1}^{3} a_{jk} X_j X'_k = R \cdot A = A^T R \cdot R',
\]

(10)
i.e. it is fully determined by the matrix \(A\). It should be stressed that calculating \(\omega(R, R')\) one has to take into account that

\[
T(R + R') = \exp[-i(p + eA_{R+R'}/c) \cdot (R + R')/\hbar].
\]
The obtained factors (9) allow to find the commutator \(T(R)T(R')T(R)^{-1}T(R')^{-1}\) as

\[
\omega(R, R')\omega(R', R)^{-1} = \exp((-ie/\hbar)c R \cdot (A - A^T) R').
\]

Applying (10) this result can be also written as

\[
\exp \left[ (-ie/\hbar) \sum_{j,k=1}^{3} X_j X'_k (a_{jk} - a_{kj}) \right].
\]

Hence, we have showed that

\[
T(R)T(R')T(R)^{-1}T(R')^{-1} = \exp[(ie/\hbar)c \Phi],
\]
where \(\Phi = (R \times R') \cdot H\) is a magnetic flux through the cell spanned by lattice vectors \(R\) and \(R'\).

### III. CONCLUSION

It was showed that a projective representation (6) of the translation group determined by the local gauge (8) has the following properties:

1. Operators \(T(R)\) commute with the Hamiltonian (1).
2. The factor system \(\omega(R, R')\) depends on the global gauge \(A\), i.e. on the matrix \(A\).
3. The commutator of (magnetic) translations \(T(R)\) and \(T(R')\) depends only on the magnetic field \(H\).

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1. L. Landau, Z. Physik 64, 629 (1930).
2. R. Peierls, Z. Physik 80, 763 (1933).
3. E. Brown, Bull. Am. Phys. Soc. 8, 256 (1963); Phys. Rev. 133, A1038 (1964).
4. J. Zak, Phys. Rev. 134, A1602, A1607 (1964).
5. J. Zak, Phys. Rev. 136, A776 (1964).
6. I. Dana, J. Zak, Phys. Rev. B28, 811 (1983); Phys. Rev. B32, 3612 (1985).
7. I. Dana, Y. Avron, J. Zak, J. Phys. C: Solid State Phys. 18, L679 (1985).
8. J. Zak, Phys. Lett. A116, 195 (1986); Phys. Rev. B39, 694 (1989).
9. V. A. Geyler, I. Yu. Popov, Z. Physik B93, 437 (1994); Phys. Lett. A201, 359 (1995).
10. P. Wiegmann, A. Zabrodin, Phys. Rev. Lett. 72, 1890 (1994).