The Exact Rate Memory Tradeoff for Large Caches with Coded Placement
Vijith Kumar K P, Brijesh Kumar Rai and Tony Jacob

Abstract

The idea of coded caching for content distribution networks was introduced by Maddah-Ali and Niesen, who considered the canonical \((N, K)\) cache network in which a server with \(N\) files satisfy the demands of \(K\) users (equipped with independent caches of size \(M\) each). Among other results, their work provided a characterization of the exact rate memory tradeoff for the problem when \(M \geq \frac{N}{K}(K - 1)\). In this paper, we improve this result for large caches with \(M \geq \frac{N}{K}(K - 2)\). For the case \(\left\lceil \frac{K+1}{2} \right\rceil \leq N \leq K\), we propose a new coded caching scheme, and derive a matching lower bound to show that the proposed scheme is optimal. This extends the characterization of the exact rate memory tradeoff to the case \(M \geq \frac{N}{K}(K - 2 + \frac{(K-2+1/N)}{(K-1)})\). For the case \(1 \leq N \leq \left\lceil \frac{K+1}{2} \right\rceil\), we derive a new lower bound, which demonstrates that the scheme proposed by Yu et al. is optimal and thus extend the characterization of the exact rate memory tradeoff to the case \(M \geq \frac{N}{K}(K - 2)\).

Index Terms

Coded caching, coded placement, exact rate memory tradeoff, content distribution networks.

I. INTRODUCTION

In content distribution networks, some parts of the files in the server are placed in caches distributed across the network during the off-peak traffic time, so as to reduce the load experienced by the network during the peak traffic time. Maddah-Ali and Niesen, in their seminal work [1], noted that traditional caching schemes fail to exploit the multicast coding opportunity available in such networks. They introduced the notion of coded caching in the setting of the canonical \((N, K)\) cache network shown in Fig [1]. The server has \(N\) files \(\{W_1, \ldots, W_N\}\) and is connected through an error free broadcast link to \(K\) users \(\{U_1, \ldots, U_K\}\), each with an isolated cache of size \(M \in [0, N]\). The cache attached to user \(U_k\) is denoted by \(Z_k\). In the first phase of a coded caching scheme, called the placement phase, the server copies some fragments of
the files available to it into the caches, without any knowledge of the files that will be required by each user. Let the demands by the users be represented by a vector \( \mathbf{d} = (W_{d_1}, \ldots, W_{d_K}) \), where \( W_{d_i} \) is the file requested user \( U_i \). In the second phase, called the delivery phase, the server broadcasts a set of packets \( X_d \) of size \( R_d(M) \) in response to the demand \( \mathbf{d} \). Each user recovers its required file from the broadcast packets aided by the contents of its isolated cache. The design of a coded caching scheme involves deciding what to place in the cache attached to each user during the placement phase and what to broadcast for each possible demand such that the shared link experiences the minimum load during the delivery phase. This formulation of the problem of coded caching has been extended in several ways to study decentralised cache networks [2], hierarchical cache networks [3], cache networks with multiple servers [4], coded caching with privacy [5], heterogeneous cache networks [6], networks with shared cache [7], cache aided D2D networks [8] and data shuffling problems with cache aided worker nodes [9].

In the case of the canonical \((N, K)\) cache network, all files and caches are of the same size. Due to the inherent symmetry of the problem, it is natural to group together all demands that are related to each other through a permutation. In [29], Tian showed that corresponding to any caching scheme there exists a symmetric caching scheme which operates with the same or smaller rate. Hence we consider only the class of symmetric caching schemes in this paper. Consider a demand \( \mathbf{d} \), where the user \( U_l \) requires the file \( W_{d_l} \),

\[
\mathbf{d} = (W_{d_1}, \ldots, W_{d_K})
\]

Let \( \pi(.) \) be a permutation operation defined over the set \( \{1, \ldots, K\} \) and \( \pi^{-1}(.) \) be its inverse.
Now consider another demand $\pi d$, which is obtained by permuting the files requested by the users,

$$\pi d = (W_{d_{\pi^{-1}(1)}}, \ldots, W_{d_{\pi^{-1}(K)}}).$$  

(2)

In the demand $\pi d$, the user $U_{\pi(l)}$ requires the file $W_{d_l}$. In response to the demand $\pi d$, the server broadcasts a set of packets $X_{\pi d}$. For a symmetric caching scheme, we have [29]

$$H(W_{d_l}, Z_{\pi(l)}, X_{\pi d}) = H(W_{d_l}, Z_l, X_d)$$  

(3)

$$H(W_n, Z_l) = H(W_p, Z_l)$$  

(4)

where $n, p \in \{1, \ldots, N\}$.

Consider the demands where each of the $N$ files is required by at least one user (and hence $N \leq K$). The set of all such demands is denoted by $D$ and the corresponding rate is denoted by $R(M)$, where

$$R(M) = \max \{R_d(M) \mid d \in D\}. $$  

(5)

For the $(N, K)$ cache network with cache size $M$, the memory rate pair $(M, R)$ is said to be achievable if there is a scheme with $R(M) \leq R$. For a such a scheme, we have

$$H(Z_l) \leq M$$  

(6)

$$H(X_d) \leq R$$  

(7)

$$H(Z_l, X_d) = H(W_{d_l}, Z_l, X_d),$$  

(8)

$$H(W_1, \ldots, W_N, Z_l, X_d) = H(W_1, \ldots, W_N),$$  

(9)

where (6) follows from the fact that size of each cache is $M$, (7) follows from the fact that for any demand in $D$ the size of $X_d$ is at most $R(M) \leq R$, (8) follows from the fact that the file $W_{d_l}$ can be computed from $X_d$ and $Z_l$ by the user $U_l$, and (9) follows from the fact that $Z_l$ and $X_d$ are functions of files $\{W_1, \ldots, W_N\}$. For a given cache size $M$, the smallest $R$ such that $(M, R)$ is achievable is called the exact rate memory tradeoff denoted by

$$R^*(M) = \min \{R : (M, R) \text{ is achievable}\} \tag{10}$$

Maddah-Ali and Niesen in [1] proposed a coding scheme with an uncoded placement phase and a coded delivery phase for the demands in $D$, and demonstrated that the rate achieved by the proposed scheme is within a multiplicative gap of 12 from the optimal rate using cut set arguments. Several improvements to the scheme proposed in [1] were presented in [11]–[14], and
Throughout this paper we use $[L]$ to represent the set $\{1,2,\ldots,L\}$, and $W_{[L]}$ to represent the set $\{W_1,W_2,\ldots,W_L\}$.

## II. Example Networks

As a prelude to the results presented in Section III and IV, we consider two example networks.
A. The (3,4) Cache Network

Here, users \( \{U_1, U_2, U_3, U_4\} \) are connected to a server with files \( \{A, B, C\} \) (each of size \( F \) bits). Each user \( U_k \) has a cache \( Z_k \) of size \( MF \) bits. We now describe a symmetric caching scheme for the case \( M = \frac{25}{12} \). During the placement phase, every file is split into 12 disjoint subfiles, each of size \( \frac{1}{12}F \) bits. The subfiles are:

| File | Subfiles                                      |
|------|----------------------------------------------|
| A    | \( A_{12}, A_{21}, A_{13}, A_{31}, A_{41}, A_{23}, A_{32}, A_{42}, A_{34}, A_{43} \) |
| B    | \( B_{12}, B_{21}, B_{13}, B_{31}, B_{41}, B_{23}, B_{32}, B_{42}, B_{34}, B_{43} \) |
| C    | \( C_{12}, C_{21}, C_{13}, C_{31}, C_{41}, C_{23}, C_{32}, C_{42}, C_{34}, C_{43} \) |

The server places 18 uncoded packets (stage 1) and 7 coded packets (stage 2) in each user’s cache as shown in TABLE II. Each of these packets are of size \( \frac{1}{12}F \) bits and they together occupy \( \frac{25}{12}F \) bits.

| Cache | Subfiles                      | Stage 1                  | Stage 2                  |
|-------|-------------------------------|--------------------------|--------------------------|
| Z1    | \( A_{23}, A_{24}, A_{34}, A_{42}, A_{43} \) | \( A_{12} - A_{13} \) | \( A_{12} - A_{14} \) |
|       | \( B_{23}, B_{24}, B_{34}, B_{42}, B_{43} \) | \( B_{12} - B_{13} \) | \( B_{12} - B_{14} \) |
|       | \( C_{23}, C_{24}, C_{34}, C_{42}, C_{43} \) | \( C_{12} - C_{13} \) | \( C_{12} - C_{14} \) |
|       | \( A_{12} + B_{12} + C_{12} \) |                         |                           |
| Z2    | \( A_{13}, A_{14}, A_{31}, A_{41}, A_{43} \) | \( A_{23} - A_{24} \) | \( A_{23} - A_{21} \) |
|       | \( B_{13}, B_{14}, B_{31}, B_{41}, B_{43} \) | \( B_{23} - B_{24} \) | \( B_{23} - B_{21} \) |
|       | \( C_{13}, C_{14}, C_{31}, C_{41}, C_{43} \) | \( C_{23} - C_{24} \) | \( C_{23} - C_{21} \) |
|       | \( A_{23} + B_{23} + C_{23} \) |                         |                           |
| Z3    | \( A_{21}, A_{24}, A_{14}, A_{42}, A_{41} \) | \( A_{34} - A_{31} \) | \( A_{34} - A_{32} \) |
|       | \( B_{21}, B_{24}, B_{14}, B_{42}, B_{41} \) | \( B_{34} - B_{31} \) | \( B_{34} - B_{32} \) |
|       | \( C_{21}, C_{24}, C_{14}, C_{42}, C_{41} \) | \( C_{34} - C_{31} \) | \( C_{34} - C_{32} \) |
|       | \( A_{34} + B_{34} + C_{34} \) |                         |                           |
| Z4    | \( A_{23}, A_{21}, A_{32}, A_{12}, A_{13} \) | \( A_{41} - A_{42} \) | \( A_{41} - A_{43} \) |
|       | \( B_{23}, B_{21}, B_{32}, B_{12}, B_{13} \) | \( B_{41} - B_{42} \) | \( B_{41} - B_{43} \) |
|       | \( C_{23}, C_{21}, C_{32}, C_{12}, C_{13} \) | \( C_{41} - C_{42} \) | \( C_{41} - C_{43} \) |
|       | \( A_{41} + B_{41} + C_{41} \) |                         |                           |

TABLE II: Cache contents placed in stage 1 and stage 2

To understand how the delivery phase works, consider a demand \( d = (P, P, Q, R) \) where \( P, Q \) and \( R \) are distinct files in \( \{A, B, C\} \). In response to this demand, the server broadcasts a set of
packets

\[ X_d = \begin{cases} 
Q^{13} + R^{14} - P^{12} \\
Q^{23} + R^{24} - P^{21} \\
R^{34} + \frac{1}{2}p^{31} + \frac{1}{2}p^{32} \\
Q^{43} + \frac{1}{2}p^{41} + \frac{1}{2}p^{42} 
\end{cases} \]

As \( X_d \) has four packets, of size \( \frac{1}{12}F \) bits each, the load experienced by the shared link is \( RF = \frac{1}{3}F \) bits and thus the rate is \( R = \frac{1}{3} \).

Let us consider \( U_1 \) to understand how the requested file is obtained from \( X_d \) and \( Z_1 \). Note that the user has subfiles \( P^{23}, P^{32}, P^{24}, P^{34} \) and \( P^{43} \) in \( Z_1 \) and require subfiles \( P^{12}, P^{21}, P^{13}, P^{31}, P^{14} \) and \( P^{41} \) to compute the requested file \( P \). The user can compute subfiles \( P^{21}, P^{31}, P^{41} \) and \( P^{12} \) by combining received packets and cached packets as shown below:

| Received Packet | Cached Packets | Computed Subfile |
|-----------------|----------------|-----------------|
| \( Q^{23} + R^{24} - P^{21} \) | \( Q^{21}, R^{24} \) | \( P^{21} \) |
| \( R^{34} - \frac{1}{2}p^{31} - \frac{1}{2}p^{32} \) | \( R^{34}, p^{32} \) | \( P^{31} \) |
| \( Q^{43} - \frac{1}{2}p^{41} - \frac{1}{2}p^{42} \) | \( Q^{41}, p^{42} \) | \( P^{41} \) |
| \( Q^{13} + R^{14} - P^{12} \) | \( Q^{12} - Q^{13}, R^{12} - R^{14}, P^{12} + Q^{12} + R^{12} \) | \( P^{12} \) |

Combining the subfile \( P^{12} \) with cached packets \( P^{12} - P^{13} \) and \( P^{12} - P^{14} \), \( U_1 \) obtains subfiles \( P^{13} \) and \( P^{14} \). The other users can proceed in similar fashion. We summarise as:

**Lemma 1.** The memory rate pair \( \left( \frac{25}{12}, \frac{1}{3} \right) \) is achievable by symmetric caching schemes for the \( (3,4) \) cache network.

The caching scheme proposed in [1] achieves the memory rate pair \( \left( \frac{9}{4}, \frac{1}{4} \right) \), and by memory sharing between that scheme and the proposed scheme, we can achieve all memory rate pairs \( \left( M, \frac{11}{8} - \frac{1}{2}M \right) \), where \( M \in \left[ \frac{25}{12}, \frac{9}{4} \right] \). We obtain a matching lower bound in the following lemma:

**Lemma 2.** For the \( (3,4) \) cache network, achievable memory rate pairs \( (M, R) \) must satisfy the constraint

\[
4M + 8R \geq 11.
\]
Proof. We have,

\[ 4M + 8R \geq H(Z_1) + H(Z_2) + H(Z_3) + 2H(X_{(A,B,C,A)}) + 2H(X_{(B,C,A,A)}) + 3H(X_{(A,A,B,C)}) \]

\[ + H(X_{(C,A,A,B)}) \]

\[ \geq H(Z_1, X_{(A,B,C,A)}, X_{(B,C,A,A)}) + H(Z_1, X_{(B,C,A,A)}, X_{(A,A,B,C)}) + H(Z_2, X_{(A,B,C,A)}, X_{(A,A,B,C)}) \]

\[ + H(Z_3, X_{(C,A,A,B)}, X_{(A,A,B,C)}) \]

\[ \geq H(A, B, Z_1, X_{(A,B,C,A)}, X_{(B,C,A,A)}) + H(A, B, Z_1, X_{(B,C,A,A)}, X_{(A,A,B,C)}) \]

\[ + H(A, B, Z_2, X_{(A,B,C,A)}, X_{(A,A,B,C)}) + H(A, B, Z_3, X_{(C,A,A,B)}, X_{(A,A,B,C)}) \]

\[ \geq H(A, B, X_{(A,B,C,A)}, X_{(B,C,A,A)}, X_{(A,A,B,C)}) + H(A, B, Z_2, X_{(A,B,C,A)}, X_{(A,A,B,C)}) \]

\[ + H(A, B, Z_3, X_{(C,A,A,B)}, X_{(A,A,B,C)}) + H(A, B, Z_1, X_{(B,C,A,A)}) \]

\[ \geq H(A, B, Z_2, X_{(A,B,C,A)}, X_{(B,C,A,A)}, X_{(A,A,B,C)}) + H(A, B, X_{(A,B,C,A)}, X_{(A,A,B,C)}) \]

\[ + H(A, B, Z_3, X_{(C,A,A,B)}, X_{(A,A,B,C)}) + H(A, B, Z_1, X_{(B,C,A,A)}) \]

\[ \geq H(A, B, C, Z_2, X_{(A,B,C,A)}, X_{(B,C,A,A)}, X_{(A,A,B,C)}) + H(A, B, X_{(A,B,C,A)}, X_{(A,A,B,C)}) \]

\[ + H(A, B, Z_3, X_{(C,A,A,B)}, X_{(A,A,B,C)}) + H(A, B, Z_1, X_{(B,C,A,A)}) \]

\[ \geq H(A, B, C) + H(A, B, X_{(A,B,C,A)}, X_{(A,A,B,C)}) + H(A, B, Z_3, X_{(C,A,A,B)}, X_{(A,A,B,C)}) \]

\[ + H(A, B, Z_1, X_{(B,C,A,A)}) \]

\[ \geq H(A, B, C) + H(A, B, Z_3, X_{(C,A,A,B)}, X_{(A,B,C,A)}, X_{(A,A,B,C)}) + H(A, B, X_{(A,A,B,C)}) \]

\[ + H(A, B, Z_1, X_{(B,C,A,A)}) \]

\[ \geq 2H(A, B, C) + H(A, B, X_{(A,A,B,C)}) + H(A, B, Z_1, X_{(B,C,A,A)}) \]

\[ \geq 2H(A, B, C) + H(A, B, X_{(A,A,B,C)}) + H(A, B, Z_1) \]

\[ \geq 2H(A, B, C) + H(A, B, X_{(A,A,B,C)}) + H(A, B, Z_4) \]

\[ \geq H(A, B, Z_4, X_{(A,A,B,C)}) + H(A, B) + 2H(A, B, C) \]
\[(c) \ H(A, B, C, Z_4, X_{(A,B,C)}) + H(A, B) + 2H(A, B, C) \]
\[(d) \ H(A, B, C) + H(A, B) + 2H(A, B, C) \geq 11 \]

where
(a) follows from (6) and (7),
(b) follows from the submodularity property of entropy,
(c) follows from (8),
(d) follows from (9),
(e) follows from (3).

The above observations improve upon the previous results from [1], [14] and is summarised in TABLE III and Fig. 2.

| Memory | Rate [1], [14] | Lower Bound [1], [14] | New Rate | New Lower Bound |
|--------|----------------|------------------------|----------|----------------|
| \[\frac{25}{12} \leq M \leq \frac{9}{4}\] | \[\frac{3}{2}, \frac{5}{9} M\] | \[R \geq 1 - \frac{1}{3} M\] | \[\frac{11}{8} - \frac{1}{2} M\] | \[R \geq \frac{11}{8} - \frac{1}{2} M\] |

TABLE III: Rate memory tradeoff for the (3, 4) cache network

![Rate Memory Tradeoff](image)

Fig. 2: Exact rate memory tradeoff for the (3, 4) cache network

B. The (2,4) Cache Network

Here, users \(\{U_1, U_2, U_3, U_4\}\) are connected to a server with files \(\{A, B\}\) (each of size \(F\) bits). Each user \(U_k\) has cache \(Z_k\) of size \(MF\) bits. The caching scheme proposed in [14] can achieve
all memory rate pairs \( \left( M, \frac{3}{2} - \frac{5}{6} M \right) \), where \( M \in \left[ 1, \frac{3}{2} \right] \). We obtain a matching lower bound in the following lemma:

**Lemma 3.** For the (2, 4) cache network, achievable memory rate pairs \((M, R)\) must satisfy the constraint

\[
5M + 6R \geq 9.
\]

**Proof.** We have,

\[
5M + 6R \geq 3H(Z_1) + 2H(Z_2) + H(Z_4) + 3H(X_{(A,A,A,B)}) + 2H(X_{(A,A,B,A)}) + H(X_{(A,B,A,A)}) - H(Z_1)
\]

\[
\geq H(Z_1, X_{(A,A,B,A)}) + H(Z_1, X_{(A,A,A,B)}) + H(Z_1, X_{(A,B,A,A)}) + H(Z_2, X_{(A,A,B,A)}) + H(Z_2, X_{(A,A,A,B)}) + H(Z_2, X_{(A,A,B,A)})
\]

\[
+ H(Z_4, X_{(A,A,B,A)}) - H(Z_1)
\]

\[
= H(A, Z_1, X_{(A,A,B,A)}) + H(A, Z_1, X_{(A,A,A,B)}) + H(A, Z_1, X_{(A,B,A,A)}) + H(A, Z_2, X_{(A,A,B,A)}) + H(A, Z_2, X_{(A,A,A,B)})
\]

\[
+ H(A, Z_2, X_{(A,A,A,B)}) + H(A, Z_4, X_{(A,A,B,A)}) - H(Z_1)
\]

\[
\geq H(A, Z_1, X_{(A,A,B,A)}, X_{(A,A,A,B)}, X_{(A,B,A,A)}) + 2H(A, Z_1) + H(A, Z_2, X_{(A,A,B,A)}, X_{(A,A,A,B)}) + H(A, Z_2) + H(A, Z_4, X_{(A,A,B,A)}) - H(Z_1)
\]

\[
= H(A, Z_1, X_{(A,A,B,A)}, X_{(A,A,A,B)}, X_{(A,B,A,A)}) + H(A, Z_2, X_{(A,A,B,A)}, X_{(A,A,A,B)}) + H(A, Z_2, X_{(A,A,A,B)}, X_{(A,A,B,A)})
\]

\[
+ H(A, Z_4, X_{(A,A,B,A)}) + H(A, Z_2) + H(A, Z_1) + H(B, Z_1) - H(Z_1)
\]

\[
\geq H(A, Z_1, X_{(A,A,B,A)}, X_{(A,A,A,B)}, X_{(A,B,A,A)}, Z_4, X_{(A,A,B,A)}) + H(A, Z_2, X_{(A,A,B,A)}, X_{(A,A,A,B)}) + H(A, Z_4, X_{(A,A,B,A)})
\]

\[
+ H(A, Z_3) + H(A, B, Z_1) + H(Z_1) - H(Z_1)
\]

\[
= 2H(A) + H(A, X_{(A,A,B,A)}, X_{(A,A,A,B)}) + H(A, Z_4, X_{(A,A,B,A)}) + H(A, Z_3)
\]
\( (b) \ \geq 2H(A, B) + H(A, Z_4, X_{(A,A,B,A)}, X_{(A,A,A,B)}) + H(A, X_{(A,A,B,A)}) + H(A, Z_3) \)

\( (c) = 2H(A, B) + H(A, B, Z_4, X_{(A,A,B,A)}, X_{(A,A,A,B)}) + H(A, X_{(A,A,B,A)}) + H(A, Z_3) \)

\( (f) = 3H(A, B) + H(A, X_{(A,A,B,A)}) + H(A, Z) \)

\( (b) \ \geq 3H(A, B) + H(A, Z_3, X_{(A,A,B,A)}) + H(A) \)

\( (c) = 3H(A, B) + H(A, B, Z_3, X_{(A,A,B,A)}) + H(A) \)

\( (f) = 4H(A, B) + H(A) \geq 9, \)

where

- (a) follows from (6) and (7),
- (b) follows from the submodularity property of entropy,
- (c) follows from (8),
- (d) follows from (4),
- (e) follows from (3),
- (f) follows from (9)

The above observations improve upon the previous results from [1], [14] and is summarised in TABLE IV and Fig. 3.

| Memory | Rate [1], [14] | Lower Bound [1], [14] | New Lower Bound |
|--------|----------------|------------------------|-----------------|
| \( 1 \leq M \leq \frac{3}{2} \) | \( \frac{3}{2}, \frac{5}{6} M \) | \( R \geq 1 - \frac{1}{2} M \) | \( R \geq \frac{3}{2} - \frac{5}{6} M \) |

TABLE IV: Rate memory tradeoff for the \((2, 4)\) cache network

III. CASE I: \( \lceil \frac{K+1}{2} \rceil \leq N \leq K \)

In this section we propose a new symmetric caching scheme that achieves the memory rate pair

\[ (M_A, R_A) = \left( \frac{N}{K} \left( K - 2 + \frac{(K - 2 + 1/N)}{(K - 1)} \right), \frac{1}{K - 1} \right) \]

(11)

for the \((N, K)\) cache network. This scheme can be seen as a generalization of the scheme presented for \((3, 4)\) cache network in Section II-A and is an extension of the scheme we proposed in [24], [25]. For \( \lceil \frac{K+1}{2} \rceil \leq N \leq K \), we prove a matching lower bound to establish the exact rate memory tradeoff when \( M \geq M_A \). Let \( I \) denote the indicator function and let \( S_k \) denote the set \([K] \setminus \{k\} \).
A. Placement and Delivery Phase

During the placement phase, the server splits every file into \(2^K C_2\) disjoint subfiles of size \(\frac{1}{k(k-1)}F\) bits. Subfiles of the file \(W_n\) are:

\[\{W_n^{ij} : i, j \in [K] \text{ and } i \neq j\}.\]

The placement phase proceeds in two stages. In the first stage, the server copy subfiles \(W_n^{ij}\) in user \(U_k\)’s cache, \(Z_k\), if \(k \notin \{i, j\}\). In the second stage, functions of subfiles are computed and placed into each user’s cache resulting in the cache of the \(k^{th}\) user, \(Z_k\), having the contents:

| Stages   | Packets | Constraints | Number          |
|----------|---------|-------------|-----------------|
| Stage 1  | \(W_n^{ij}\) | \(n \in [N],\) \(i, j \in S_k\) and \(i \neq j\) | \(2N K^{-1}C_2\) |
| Stage 2  | \(W_n^{k(k+1)} - W_n^{kj}\) | \(n \in [N]\) and \(j \in S_k\) | \(N(K-2)\) |
|          | \(\sum_{n=1}^N W_n^{k(k+1)}\) |                           | 1                |

It can be noted that subfiles \(W_n^{kj}\), for \(n \in [N]\) and \(j \in S_k\), are contained in user \(U_k\)’s cache in coded form. The total number of packets, each of size \(\frac{1}{k(k-1)}F\) bits, placed in each user’s cache is,

\[2N K^{-1}C_2 + N(K-2) + 1 = NK(K-2) + 1\]
utilising the entire cache of size $M AF$ bits.

In the delivery phase, let the server receive a demand $d$. Let $N^d_k$ represent the number of users in the set $S_k$ requesting the file $W_{d_k}$. For each $k \in [K]$, the server constructs a packet,

$$X_d^k = \sum_{s \in S_k} \left( \frac{\alpha^s_k}{N^s_k} \right) W^{ks}_{d_s}$$

(12)

where

$$\alpha^s_k = 1 - 2I\{W_{d_k} = W_{d_s}\}.$$  

(13)

The set of packets broadcast by the server in response to the demand $d$ is,

$$X_d = \{X_d^1, \ldots, X_d^K\}.$$  

(14)

Thus, $K$ packets, each of size $\frac{1}{K(K-1)}F$ bits, are transmitted and the rate corresponding to the demand $d$ is,

$$R = \frac{1}{K-1}.$$  

(15)

B. File Recovery by Users

To understand how the requested files are recovered by the users, let us consider user $U_k$ who needs to recover the file $W_{d_k}$ from its cache contents $Z_k$ and the received packets $X_d$. Subfiles $W_{ij}^d$, for $i, j \in S_k$, are available in $Z_k$. To reconstruct the file $W_{d_k}$, the user needs to compute subfiles $W_{d_k}^{jk}$ and $W_{d_k}^{kj}$, for $j \in S_k$. The user obtains these subfiles in two stages. In the first stage, the user obtains subfiles $W_{d_k}^{jk}$, for $j \in S_k$. One of the packet available in $X_d$ is,

$$X_d^j = \left( \frac{\alpha^j_k}{N^j_k} \right) W_{d_k}^{jk} + \sum_{s \in S_j \setminus \{k\}} \left( \frac{\alpha^s_j}{N^s_j} \right) W_{d_s}^{js}.$$  

(16)

Since subfiles $W_{js}^{n}$, for $s \in S_j \setminus \{k\} = S_k \setminus \{j\}$, are available in $Z_k$, the user can evaluate

$$\sum_{s \in S_j \setminus \{k\}} \left( \frac{\alpha^s_j}{N^s_j} \right) W_{d_s}^{js}$$  

(17)

The subfile $W_{d_k}^{jk}$ can be computed from (16) and (17). In the second stage, the user recovers subfiles $W_{d_k}^{kj}$, for $j \in S_k$. Another packet available in $X_d$ is,

$$X_d^k = \sum_{j \in S_k} \left( \frac{\alpha^j_k}{N^j_k} \right) W_{d_j}^{kj}.$$  

(18)
Since $W_n^{k(k+1)} - W_n^{kj}$, for $j \in S_k$, are available in $Z_k$, the user can evaluate
\[
\sum_{j \in S_k} \left( \frac{\alpha^j_k}{N_k^j} \right) (W_{d_j}^{k(k+1)} - W_{d_j}^{kj})
\] (19)
Combining (18) and (19) the user can compute
\[
\sum_{j \in S_k} \left( \frac{\alpha^j_k}{N_k^j} \right) W_{d_j}^{k(k+1)}
\] (20)
This can be rewritten as
\[
\left( \sum_{n \in [N] \setminus d_k} \sum_{j \in S_k : d_j = n} \left( \frac{\alpha^j_k}{N_k^j} \right) W_{d_j}^{k(k+1)} \right) + \left( \sum_{j \in S_k : d_j = d_k} \left( \frac{\alpha^j_k}{N_k^j} \right) W_{d_j}^{k(k+1)} \right)
\] (21)
Note that when $W_{d_j} \neq W_{d_k}$, $\alpha^j_k = 1$ and when $W_{d_j} = W_{d_k}$, $\alpha^j_k = -1$. Recall that $N^k_k$ denotes the number of users request for file $W_{d_k}$ in set $S_k$. Now the above expression simplifies to
\[
\left( \sum_{n \in [N] \setminus d_k} W_n^{k(k+1)} \right) - I\{N^k_k \neq 0\} W_{d_k}^{k(k+1)}
\] (22)
With the help of the cached function
\[
\sum_{n \in [N]} W_n^{k(k+1)} = \left( \sum_{n \in [N] \setminus \{d_k\}} W_n^{k(k+1)} \right) + W_{d_k}^{k(k+1)}
\] (23)
and (22) the user can compute the subfile $W_{d_k}^{k(k+1)}$. Combining this with $W_{d_k}^{k(k+1)} - W_{d_k}^{kj}$ available in $Z_k$ user can obtain subfiles $W_{d_k}^{kj}$. Using all the recovered subfiles the user can reconstruct the requested file $W_{d_k}$.

The above observations can be summarised as:

**Theorem 1.** The memory rate pair \( \left( \frac{N}{K} (K - 2 + \frac{(K-2+1/N)}{(K-1)}, \frac{1}{K-1} \right) \) is achievable by symmetric caching schemes for the \((N,K)\) cache network.

The caching scheme proposed in [1], [14] achieves the memory rate pair \((\frac{N}{K} (K - 1), \frac{1}{K})\), and by memory sharing between that scheme and the proposed scheme, we can achieve all memory rate pairs \((M, \frac{(KN-1)}{K(N-1)} - \frac{1}{(N-1)} M)\), where \(M \in \left[ \frac{N}{K} (K - 2 + \frac{(K-2+1/N)}{(K-1)}), \frac{N}{K} (K - 1) \right] \). By deriving a matching lower bound, we show that this is the exact rate memory tradeoff when \(\left\lceil \frac{K+1}{2} \right\rceil \leq N \leq K\).
TABLE V: The set of demands \(\{d_l : 1 \leq l \leq K\}\)

\[
\begin{array}{cccccccc}
\text{Demands} & U_1 & \ldots & U_i & \ldots & U_N & U_{N+1} & \ldots & U_{N+i} & \ldots & U_K \\
\hline
d_1 & W_1 & \ldots & W_i & \ldots & W_N & W_1 & \ldots & W_i & \ldots & W_{K-N} \\
d_2 & W_2 & \ldots & W_{i+1} & \ldots & W_1 & W_2 & \ldots & W_{i+1} & \ldots & W_1 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
d_{K-N-i+1} & W_{K-N-i+1} & \ldots & W_{K-N} & \ldots & W_{K-N-i} & W_{K-N-i+1} & \ldots & W_{K-N} & \ldots & W_K \\
d_{K-N-i+2} & W_{K-N-i+2} & \ldots & W_{K-N+1} & \ldots & W_{K-N-i+1} & W_{K-N-i+2} & \ldots & W_1 & \ldots & W_{K-N-i-1} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
d_{K-N-i} & W_N & \ldots & W_{i+1} & \ldots & W_1 & W_{i+1} & \ldots & W_N & \ldots & W_{N-i} \\
d_{K-N-i+1} & W_{N-i+1} & \ldots & W_N & \ldots & W_{2N-K-i} & W_{N-i+1} & \ldots & W_{2N-K} & \ldots & W_{N-i} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
d_K & W_{K-N} & \ldots & W_{i+1} & \ldots & W_{N-i} & W_{K-N} & \ldots & W_{N-i} & \ldots & W_{K-N-i} \\
\end{array}
\]

C. Matching Lower Bound

Consider the demand

\[d_1 = (W_1, W_2, \ldots, W_N, W_1, W_2, \ldots, W_{K-N}).\]  \(\text{(24)}\)

Demands \(\{d_l : 2 \leq l \leq K\}\), are obtained from the demand \(d_1\) by cyclic left shifts as shown in TABLE V. Consider the demand \(b_l\) defined as

\[
b_l = \begin{cases} 
    d_{N-l+1}, & \text{for } 1 \leq l \leq N \\
    d_{K+4-N-l+1}, & \text{for } N+1 \leq l \leq K
\end{cases}
\]

It can be noted that in demand \(b_l\), the user \(U_l\) requests for the file \(W_N\). Let \(X_{d_l}\) denote the set of packets broadcast by the server in response to the demand \(d_l\). For \(S \subseteq \{d_1, \ldots, d_K\}\), let \(X_S\) denote the set of all packets broadcast in response to the demands in the set \(S\). The following lemma are easy to obtain:

**Lemma 4.** For \(S, T \subseteq \{d_1, \ldots, d_K\} \setminus \{b_l\}\), we have the identity,

\[H(W_{[N-1]}, X_{S \cup \{b_l\}}) + H(W_{[N-1]}, Z_l, X_T) \geq H(W_{[N-1]}, X_{S \cup T}) + N\]
Proof. We have,
\[
H(W_{[N-1]}, X_{S \cup \{b_i\}}) + H(W_{[N-1]}, Z_l, X_T) \overset{(a)}{\geq} H(W_{[N-1]}, X_{S \cap T}) + H(W_{[N-1]}, Z_l, X_{S \cup T \cup \{b_i\}})
\]
\[
\overset{(b)}{=} H(W_{[N-1]}, X_{S \cap T}) + H(W_{[N-1]}, W_N, Z_l, X_{S \cup T \cup \{b_i\}})
\]
\[
\overset{(c)}{=} H(W_{[N-1]}, X_{S \cap T}) + H(W_{[N]} = H(W_{[N-1]}, X_{S \cap T}) + N
\]

where

(a) follows from the submodularity property of entropy,

(b) follows from (8),

(c) follow from (9).

Lemma 5. For a sequence of sets \(S_i \subset \{d_1, \ldots, d_K\} \setminus \{b_i\}\), such that \(S_i = S_{i+1} \cup \{b_{i+1}\}\), we have the identity

\[
H(W_{[N-1]}, X_{S_i}) + \sum_{i=l+1}^{j} H(W_{[N-1]}, Z_l, X_{S_i}) \geq H(W_{[N-1]}, X_{S_l}) + (j-l)N
\]

Proof. We have,
\[
H(W_{[N-1]}, X_{S_l}) + \sum_{i=l+1}^{j} H(W_{[N-1]}, Z_l, X_{S_i}) = H(W_{[N-1]}, X_{S_l}) + H(W_{[N-1]}, Z_{l+1}, X_{S_{l+1}})
\]
\[
+ \sum_{i=l+2}^{j} H(W_{[N-1]}, Z_l, X_{S_i})
\]
\[
= \left( H(W_{[N-1]}, X_{S_{l+1}}, X_{b_{l+1}}) + H(W_{[N-1]}, Z_{l+1}, X_{S_{l+1}}) \right) + \sum_{i=l+2}^{j} H(W_{[N-1]}, Z_l, X_{S_i})
\]
\[
\overset{(a)}{\geq} H(W_{[N]}) + \left( H(W_{[N-1]}, X_{S_{l+1}}) + H(W_{[N-1]}, Z_{l+2}, X_{S_{l+2}}) \right) + \sum_{i=l+3}^{j} H(W_{[N-1]}, Z_l, X_{S_i})
\]
\[
\overset{(a)}{\geq} 2H(W_{[N]}) + H(W_{[N-1]}, X_{S_{l+2}}) + H(W_{[N-1]}, Z_{l+3}, X_{S_{l+3}}) + \sum_{i=l+4}^{j} H(W_{[N-1]}, Z_l, X_{S_i})
\]
\[
\overset{(b)}{\geq} (j-l)H(W_{[N]}) + H(W_{[N-1]}, X_{S_j}) = H(W_{[N-1]}, X_{S_j}) + (j-l)N
\]

where

(a) follows from Lemma 4 with \(S = T = S_{l+1}\),

(b) follows from repeated use of Lemma 4 with \(S \cup \{b_i\} = S_i\) and \(T = S_{l+1}\) for \(l + 3 \leq i \leq j\).

□
In a similar fashion, for a sequence of sets $T_i \subset \{d_1, \ldots, d_K\} \setminus \{b_i\}$, such that $T_i = T_{i-1} \cup \{b_{i-1}\}$, we can obtain

$$H(W_{[N-1]}, X_{T_{i+l}}) + \sum_{j=1}^{j} H(W_{[N-1]}, Z_j, X_{T_j}) \geq H(W_{[N-1]}, X_{T_j}) + (j - l + 1)N \quad (25)$$

For $1 \leq i \leq N$, let us consider the sets of demands as shown below:

| Set | Demands | Number | Files Requested by $U_i$ |
|-----|---------|--------|--------------------------|
| $A_i$ | $d_1, \ldots, d_{N-i}$ | $N-i$ | $W_1, \ldots, W_{(N-1)}$ |
| $B_i$ | $d_{K-i+2}, \ldots, d_K$ | $i-1$ | $W_1, \ldots, W_{i-1}$ |
| $C_i$ | $d_{K-N-i+2}, \ldots, d_{N-i}$ | $2N-K-1$ | $W_{(K-N+1)}, \ldots, W_{N-1}$ |
| $J$ | $d_{N+1}, \ldots, d_K$ | $K-N$ | $W_1, \ldots, W_{K-N}$ |

These set are also indicated in TABLE V. Note that

$$A_N = B_1 = C_N = \phi \quad (26)$$

$$A_{i+1} \cup \{b_{i+1}\} = A_i \quad (27)$$

$$B_i \cup \{b_{N+i}\} = B_{i+1} \quad (28)$$

$$A_i \cap C_i = C_i \quad (29)$$

$$B_K-N \cup \{b_K\} = B_K-N+1 = J \quad (30)$$

$$B_i \cap J = \begin{cases} B_i \text{ when } 1 \leq i \leq K - N \\ J \text{ when } K - N + 1 \leq i \leq N \end{cases} \quad (31)$$

It can be noted that in the demand set $B_i$, both users $U_i$ and $U_{N+i}$ are requesting for the same set of files (for $1 \leq i \leq K - N$). Note that $|A_i \cup B_i| = |J \cup C_i| = N - 1$. Thus, we have

$$M + (N - 1)R \geq H(Z_i) + H(X_{A_i \cup B_i}) \geq H(Z_i, X_{A_i \cup B_i}) \quad (32)$$

Similarly,

$$M + (N - 1)R \geq H(Z_i) + H(X_{J \cup C_i}) \geq H(Z_i, X_{J \cup C_i}) \quad (33)$$

Now we can obtain the following result:

**Theorem 2.** For the $(N, K)$ cache network, when $\left\lceil \frac{K+1}{2} \right\rceil \leq N \leq K$, achievable memory rate pairs $(M, R)$ must satisfy the constraint

$$KM + K(N - 1)R \geq KN - 1.$$
Proof. We have,

\[ KM + K(N - 1)R = N(M + (N - 1)R) + (K - N)(M + (N - 1)R) \]

\[ \geq \sum_{i=1}^{N} H(Z_i, X_{A_i \cup B_i}) + \sum_{i=1}^{K-N} H(Z_i, X_{J_i \cup C_i}) \]

\[ \geq \sum_{i=1}^{N} H(W_{[N-1]}, Z_i, X_{A_i \cup B_i}) + \sum_{i=1}^{K-N} H(W_{[N-1]}, Z_i, X_{J_i \cup C_i}) \]

\[ = \sum_{i=1}^{K-N} \left( H(W_{[N-1]}, Z_i, X_{A_i \cup B_i}) + H(W_{[N-1]}, Z_i, X_{J_i \cup C_i}) \right) + \sum_{j=K-N+1}^{N} H(W_{[N-1]}, Z_j, X_{A_j \cup B_j}) \]

\[ \geq \sum_{i=1}^{K-N} \left( H(W_{[N-1]}, Z_i, X_{A_i \cup B_i}) + H(W_{[N-1]}, Z_i, X_{J_i \cup C_i}) \right) + \sum_{j=K-N+1}^{N} H(W_{[N-1]}, Z_j, X_{J_j \cup A_j}) \]

\[ \geq \sum_{i=1}^{K-N} H(W_{[N-1]}, Z_i, X_{A_i \cup B_i}) \]

\[ \geq (K - N - 1)N + \left( H(W_{[N-1]}, X_{A_{K-N} \cup J}) + \sum_{j=K-N+1}^{N} H(W_{[N-1]}, Z_j, X_{J_j \cup A_j}) \right) \]

\[ + \sum_{i=1}^{K-N} H(W_{[N-1]}, Z_i, X_{J_i \cup C_i}) \]

\[ \geq (N - 1)N + H(W_{[N-1]}, X_{A_{N} \cup J}) + \sum_{i=1}^{K-N} H(W_{[N-1]}, Z_i, X_{B_i \cup C_i}) \]

\[ \geq (N - 1)N + H(W_{[N-1]}, X_{A_{N} \cup J}) + \sum_{i=1}^{K-N} H(W_{[N-1]}, Z_i, X_{B_i}) \]

\[ \geq (N - 1)N + H(W_{[N-1]}, X_{J_i}) + \sum_{i=1}^{K-N} H(W_{[N-1]}, Z_i, X_{B_i}) \]

\[ \geq (N - 1)N + H(W_{[N-1]}, X_{J_i}) + \sum_{i=1}^{K-N} H(W_{[N-1]}, Z_{i+N}, X_{B_i}) \]

\[ \geq (N - 1)N + H(W_{[N-1]}, X_{B_{K-N+1}}) + \sum_{i=1}^{K-N} H(W_{[N-1]}, Z_{i+N}, X_{B_i}) \]

\[ \geq (K - 1)N + H(W_{[N-1]}) \geq KN - 1 \]
where

(a) follows from (32) and (33),
(b) follows from (8) and the definition of sets \( A_i, B_i, C_i \) and \( J \),
(c) follows from the facts that \( A_i \cap C_i = C_i, J \cap B_i = B_i \) for \( 1 \leq i \leq K - N \) (refer (29) and (31)) and the submodularity property of entropy,
(d) follows from Lemma 5 with \( S_i = A_i \cup J, l = 1, j = K - N \) and (27),
(e) follows from Lemma 5 with \( S_i = A_i \cup J, l = K - N, j = N \), the fact that \( J \cap B_i = J \)
for \( K - N + 1 \leq i \leq K \) (refer (31)) and (27),
(f) follows from (26),
(g) follows from (3),
(h) follows from (30),
(i) follows from (25) with \( T_i = B_i, l = 1, j = K - N \) and (28).

The above observations improve upon the previous results from [1], [14] as shown in TABLE VI.

| Memory | Rate [1], [14] | Lower Bound [1], [14] | New Rate | New Lower Bound |
|--------|----------------|------------------------|----------|-----------------|
| \( N \) \( K - 2 + \frac{(K-2+1/N)}{(K-1)} \) \( \leq M \) \( \leq \frac{N(K-1)}{K} \) | \( \frac{(K^2+K-2)}{K(K-1)} - \frac{(K+1)M}{N(K-1)} \) | \( R \geq 1 - \frac{M}{N} \) | \( \frac{(KN-1)}{K(N-1)} - \frac{M}{N(N-1)} \) |

TABLE VI: Rate memory tradeoff when \( \lceil \frac{K+1}{2} \rceil \leq N \leq K \)

We summarise as:

**Theorem 3.** For the \((N, K)\) cache network, when \( \lceil \frac{K+1}{2} \rceil \leq N \leq K \), the exact rate memory tradeoff is given by

\[
R^\ast(M) = \frac{(KN - 1)}{K(N - 1)} - \frac{1}{(N - 1)} M \tag{34}
\]

where \( M \geq \frac{N}{K} \( K - 2 + \frac{(K-2+1/N)}{(K-1)} \) \).

**IV. CASE II: \( 1 \leq N \leq \lceil \frac{K+1}{2} \rceil \)**

The caching scheme proposed by Yu et al. in [14] can achieve all memory rate pairs

\[
\left( M, \frac{K^2 + K - 2}{K(K - 1)} - \frac{(K+1)}{N(K-1)} M \right). \tag{35}
\]
TABLE VII: Demand set \( \{d_l : 1 \leq l \leq K \} \)

where \( M \in \left[ \frac{N(K-2)}{K}, \frac{N(K-1)}{K} \right] \), for the \((N, K)\) cache network. By deriving a matching lower bound, we show that this is the exact rate memory tradeoff when \( 1 \leq N \leq \left[ \frac{K+1}{2} \right] \). Consider the demand

\[d_1 = (W_1, W_2, \ldots, W_N, W_1, W_2, \ldots, W_{N-1}, W_1, W_1, \ldots, W_1)\]  

(36)

Demands \( \{d_l : 2 \leq l \leq K \} \), are obtained from the demand \( d_1 \) by cyclic left shifts as shown in TABLE VII. Consider the demand \( b_1 \) defined as,

\[ b_1 = \begin{cases} 
  d_{N-l+1}, & \text{for } 1 \leq l \leq N \\
  d_{K+N-l+1}, & \text{for } N+1 \leq l \leq K 
\end{cases} \]

It can be noted that in demand \( b_1 \), the user \( U_l \) requests for the file \( W_N \). The following lemma is easy to obtain:
Lemma 6. Let \( S, T \subset \{d_1, \ldots, d_K\} \setminus \{b_i\} \) be such that for every demand in \( T \), user \( U_1 \) requests the file \( W_1 \). We have

\[
H(W_{[N-1]}, Z_i, X_S) + \sum_{j \in T} H(X_j) + \frac{|T|}{N} H(Z_i) \geq H(W_{[N-1]}, Z_i, X_{S \cup T}) + |T|
\]

Proof. We have,

\[
H(W_{[N-1]}, Z_i, X_S) + \sum_{j \in T} H(X_j) + \frac{|T|}{N} H(Z_i)
\]

\[
= H(W_{[N-1]}, Z_i, X_S) + \sum_{j \in T} H(X_j) + \frac{|T|}{N} H(Z_i)
\]

\( \geq H(W_{[N-1]}, Z_i, X_S) + \sum_{j \in T} H(Z_i, X_j) - \frac{|T|}{N} (N-1) H(Z_i) \) \((a)\)

\( = H(W_{[N-1]}, Z_i, X_S) + \sum_{j \in T} H(W_1, Z_i, X_j) - \frac{|T|}{N} (N-1) H(Z_i) \) \((b)\)

\( \geq H(W_{[N-1]}, Z_i, X_{S \cup T}) + \frac{|T|}{N} (N-1) H(Z_i) - \frac{|T|}{N} (N-1) H(Z_i) \) \((c)\)

\( \geq H(W_{[N-1]}, Z_i, X_{S \cup T}) + \frac{|T|}{N} (N-1) H(Z_i) - \frac{|T|}{N} (N-1) H(Z_i) \) \((d)\)

where

\( (a) \) follows from submodularity property of entropy,

\( (b) \) follows from (8),

\( (c) \) follows from (4),

\( (d) \) follows from (9).

For \( 1 \leq i \leq N \), let us consider the set of demands as shown below:

| Set | Demands | Number | Files Requested by \( U_i \) |
|-----|---------|--------|-----------------------------|
| \( A_i \) | \( d_1, \ldots, d_{N-i} \) | \( N-i \) | \( W_1, \ldots, W_{N-1} \) |
| \( B_i \) | \( d_{K-i+2}, \ldots, d_K \) | \( i-1 \) | \( W_1, \ldots, W_{i-1} \) |
| \( E_i \) | \( d_{N+i+1}, \ldots, d_{2N-i} \) | \( N-i \) | \( W_1, \ldots, W_{N-i} \) |
| \( G_i \) | \( d_{2N-i+1}, \ldots, d_{K-i+1} \) | \( K-2N+1 \) | \( W_1 \) |
These set are also indicated in TABLE VII. We also have a set of demands $L_i$ defined as

$$L_i = A_i \cup B_i \cup E_i \cup G_i.$$  \hspace{1cm} (37)

Note that

$$A_N = B_1 = E_N = \phi$$  \hspace{1cm} (38)

$$L_{i+1} \cup \{b_{i+1}\} = L_i$$  \hspace{1cm} (39)

$$B_i \cup \{b_{N+i}\} = B_{i+1}$$  \hspace{1cm} (40)

It can be noted that in the demand set $B_i$, both users $U_i$ and $U_{N+i}$ are requesting for the same set of files (for $1 \leq i \leq N - 1$). Note that $|A_i \cup B_i| = |B_i \cup E_i| = N - 1$. Thus, we have

$$M + (N - 1)R \geq H(Z_i) + H(X_{A_i \cup B_i}) \geq H(Z_i, X_{A_i \cup B_i})$$  \hspace{1cm} (41)

Similarly,

$$M + (N - 1)R \geq H(Z_i) + H(X_{B_i \cup E_i}) \geq H(Z_i, X_{B_i \cup E_i})$$  \hspace{1cm} (42)

The following lemma is easy to obtain:

**Lemma 7.** The demand sets $B_i$ and $L_i$, defined as above, satisfy

$$KM + (KN - 2N + 1)R \geq H(W_{[N-1]}, X_{[N-1]}) + \sum_{i=1}^{N-1} \left( H(W_{[N-1]}, Z_{N+i}, X_{B_i}) \right) + N(K - N)$$

**Proof.** We have,

$$KM + (KN - 2N + 1)R = N \left( M + (N - 1)R + \frac{(K - 2N + 1)}{N} M + (K - 2N + 1)R \right)$$
$$+ (N - 1)(M + (N - 1)R)$$

\(a\) \hspace{1cm} $$\geq \sum_{i=1}^{N} \left( H(Z_i, X_{A_i \cup B_i}) + \frac{(K - 2N + 1)}{N} H(Z_i) + \sum_{l \in G_i} H(X_l) \right) + \sum_{i=1}^{N-1} \left( H(Z_i, X_{B_i \cup E_i}) \right)$$

\(b\) \hspace{1cm} $$\geq \sum_{i=1}^{N} \left( H(W_{[N-1]}, Z_i, X_{A_i \cup B_i}) + \frac{(K - 2N + 1)}{N} H(Z_i) + \sum_{l \in G_i} H(X_l) \right)$$
$$+ \sum_{i=1}^{N-1} H(W_{[N-1]}, Z_i, X_{B_i \cup E_i})$$

\(c\) \hspace{1cm} $$\geq \sum_{i=1}^{N} \left( H(W_{[N-1]}, Z_i, X_{A_i \cup B_i \cup G_i}) + (K - 2N + 1) \right) + \sum_{i=1}^{N-1} H(W_{[N-1]}, Z_i, X_{B_i \cup E_i})$$
\begin{align*}
&= H(W_{[N-1]}, Z_N, X_{A_N \cup B_N \cup G_N}) + \sum_{i=1}^{N-1} \left( H(W_{[N-1]}, Z_i, X_{A_i \cup B_i \cup G_i}) + H(W_{[N-1]}, Z_i, X_{B_i}) \right) \\
&+ N(K - 2N + 1) \\
&\geq H(W_{[N-1]}, Z_N, X_{A_N \cup B_N \cup G_N \cup E_N}) + \sum_{i=1}^{N-1} \left( H(W_{[N-1]}, Z_i, X_{A_i \cup B_i \cup E_i \cup G_i}) + H(W_{[N-1]}, Z_i, X_{B_i}) \right) \\
&+ N(K - 2N + 1) \\
&\overset{(d)}{=} \sum_{i=1}^{N} H(W_{[N-1]}, Z_i, X_{L_i}) + \sum_{i=1}^{N-1} H(W_{[N-1]}, Z_i, X_{B_i}) + N(K - 2N + 1) \\
&\geq \left( H(W_{[N-1]}, X_{L_1}) + \sum_{i=2}^{N} H(W_{[N-1]}, Z_i, X_{L_i}) \right) + \sum_{i=1}^{N-1} H(W_{[N-1]}, Z_i, X_{B_i}) + N(K - 2N + 1) \\
&\overset{(f)}{=} (N - 1)N + H(W_{[N-1]}, X_{L_N}) + \sum_{i=1}^{N-1} H(W_{[N-1]}, Z_i, X_{B_i}) + N(K - 2N + 1) \\
&\overset{(g)}{=} H(W_{[N-1]}, X_{L_N}) + \sum_{i=1}^{N-1} H(W_{[N-1]}, Z_{N+i}, X_{B_i}) + N(K - N)
\end{align*}

where

(a) follows from (41) and (42),

(b) follows from (8) and definition of sets \( A_i, B_i \) and \( E_i \),

(c) follows from Lemma 6 with \( S = A_i \cup B_i \) and \( T = G_i \),

(d) follows from the submodularity property of entropy and the fact that \( E_N = \phi \),

(e) follows from (37),

(f) follows from Lemma 5 with \( S_l = L_i, l = 1, j = N \) and (39),

(g) follows from (3).

Now, for \( 2N \leq j \leq K \), consider another set of demands as shown below:

| Set | Demands | Number | Files Requested by \( U_j \) |
|-----|---------|--------|------------------|
| \( P_j \) | \( d_{K+N-i+2}, \ldots, d_{K+2N-j} \) | \( N-1 \) | \( W_1, \ldots, W_{N-1} \) |
| \( Q_j \) | \( d_{K+2N-j+1}, \ldots, d_K \) | \( j-2N \) | \( W_1 \) |

These set are also indicated in TABLE VIII. We also have a set of demands \( T_j \) defined as

\[ T_j = P_j \cup Q_j \] (43)
Note that
\[ Q_{2N} = \phi \]  
(44)
\[ T_{j+1} \cup \{b_{j+1}\} = T_j \]  
(45)
\[ T_K \cup \{b_K\} = L_N \]  
(46)
\[ B_{N-1} \cup \{b_{2N-1}\} = T_{2N} \]  
(47)

Note that \(|P_j| = N - 1\). Thus, we have
\[ M + (N - 1)R \geq H(Z_j) + H(X_{P_j}) \geq H(Z_j, X_{P_j}) \]  
(48)

The following lemma is easy to obtain:

**Lemma 8.** The demand set \(T_j\), defined as above, satisfy
\[
\frac{K(K - 2N + 1)}{2N}M + \frac{(K - 2)(K - 2N + 1)}{2}R \geq \sum_{j=2N}^{K} H(W_{N-1}, Z_j, X_{T_j}) + \frac{(K - 2N + 1)(K - 2N)}{2}
\]

**Proof.** We have,
\[
\frac{K(K - 2N + 1)}{2N}M + \frac{(K - 2)(K - 2N + 1)}{2}R = (K - 2N + 1)(M + (N - 1)R) + \sum_{j=2N}^{K} \left( \frac{(j - 2N)}{N}M + (j - 2N)R \right)
\]

\[
\geq \sum_{j=2N}^{K} \left( H(Z_j, X_{P_j}) + \frac{(j - 2N)}{N}H(Z_j) + \sum_{l \in Q_j} H(X_l) \right)
\]

\[
\geq \sum_{j=2N}^{K} \left( H(W_{N-1}, Z_j, X_{P_j}) + \frac{(j - 2N)}{N}H(Z_j) + \sum_{l \in Q_j} H(X_l) \right)
\]

\[
\geq \sum_{j=2N}^{K} \left( H(W_{N-1}, Z_j, X_{P_j \cup Q_j}) + \frac{(j - 2N)}{N}H(W_{N-1}) \right)
\]

\[
\geq \sum_{j=2N}^{K} \left( H(W_{N-1}, Z_j, X_{T_j}) + \frac{(K - 2N + 1)(K - 2N)}{2} \right)
\]

where

(a) follows from (48) and the fact that \(Q_{2N} = \phi\),
(b) follows from (48) and definition of set \(P_j\),
(c) follows from Lemma 6 with \(S = P_j\) and \(T = Q_j\),
(d) follows from (43).
Using the above lemma, we can obtain the following result:

**Theorem 4.** For the \((N, K)\) cache network, when \(1 \leq N \leq \left\lceil \frac{K+1}{2} \right\rceil\), achievable memory rate pairs \((M, R)\) must satisfy the constraint

\[
\frac{K(K + 1)}{2N}M + \frac{K(K - 1)}{2}R \geq \frac{K^2 + K - 2}{2}
\]

**Proof.** We have,

\[
\frac{K(K + 1)}{2N}M + \frac{K(K - 1)}{2}R = KM + (KN - 2N + 1)R + \frac{K(K - 2N + 1)}{2N}M + \frac{(K - 2)(K - 2N + 1)}{2}R
\]

\[
\geq H(W_{N-1}, X_{L_N}) + \sum_{i=1}^{N-1} H(W_{N-1}, Z_{N+i}, X_{B_i}) + N(K - N) + \sum_{j=2N}^{K} H(W_{N-1}, Z_j, X_{T_j})
\]

\[
+ \frac{(K - 2N + 1)(K - 2N)}{2}
\]

\[
= H(W_{N-1}, X_{T_K}, X_{B_K}) + \sum_{j=2N}^{K} H(W_{N-1}, Z_j, X_{T_j}) + \sum_{i=1}^{N-1} H(W_{N-1}, Z_{N+i}, X_{B_i})
\]

\[
+ \frac{(K + 1)(K - 2N) + 2N^2}{2}
\]

\[
\geq (K - 2N + 1)N + \left( H(W_{N-1}, X_{T_{2N}}) + \sum_{i=1}^{N-1} H(W_{N-1}, Z_{N+i}, X_{B_i}) \right)
\]

\[
+ \frac{(K + 1)(K - 2N) + 2N^2}{2}
\]

\[
= H(W_{N-1}, X_{B_{N-1}}, X_{B_{2N-1}}) + \sum_{i=1}^{N-1} H(W_{N-1}, Z_{N+i}, X_{B_i})
\]

\[
+ \frac{(K + 1 + 2N)(K - 2N) + 2N(N + 1)}{2}
\]

\[
\geq (N - 1)N + H(W_{N-1}, X_{B_1}) + \frac{(K + 1 + 2N)(K - 2N) + 2N(N + 1)}{2}
\]

\[
= H(W_{N-1}) + \frac{K^2 + K - 2N}{2} \geq \frac{K^2 + K - 2}{2}
\]

where

(a) follows from Lemma 7 and Lemma 8,

(b) follows from (46),

(c) follows from (25) with \(T_i = T_j, l = 2N, j = K\) and (45),

(d) follows from (47),

(e) follows from (25) with \(T_i = B_i, l = 1, j = N - 1\) and (40),

(f) follows from (38).
The above observations improve upon the previous results from [1], [14] as shown in TABLE VIII.

| Memory | Rate [1], [14] | Lower Bound [1], [14] | New Lower Bound |
|--------|---------------|-----------------------|-----------------|
| $\frac{N(K-2)}{K} \leq M \leq \frac{N(K-1)}{K}$ | $\frac{K^2+K-2}{K(K-1)} - \frac{(K+1)M}{N(K-1)}$ | $R \geq 1 - \frac{1}{N}M$ | $R \geq \frac{K^2+K-2}{K(K-1)} - \frac{(K+1)M}{N(K-1)}$ |

TABLE VIII: Rate memory tradeoff when $1 \leq N \leq \left\lceil \frac{K+1}{2} \right\rceil$

We summarise as:

**Theorem 5.** For the $(N, K)$ cache network, when $1 \leq N \leq \left\lceil \frac{K+1}{2} \right\rceil$, the exact rate memory tradeoff is given by

$$R^*(M) = \frac{K^2 + K - 2}{K(K-1)} - \frac{(K+1)M}{N(K-1)}$$

(49)

where $M \geq \frac{N}{K}(K-2)$.

V. CONCLUSIONS

In this paper, we considered the problem of characterizing the exact rate memory tradeoff for the canonical $(N, K)$ cache network, where we focused on the case of large caches. For $\left\lfloor \frac{K+1}{2} \right\rfloor \leq N \leq K$, in Section III, we proposed a new coded caching scheme and derived a matching lower bound leading to the characterization of the exact rate memory tradeoff when

$$M \geq \frac{N}{K} \left( K - 2 + \frac{(K-2+1/N)}{(K-1)} \right).$$

For $1 \leq N \leq \left\lceil \frac{K+1}{2} \right\rceil$, in Section IV, we derived a new lower bound matching the scheme proposed in [14], thereby providing a characterization of the exact rate memory tradeoff when $M \geq \frac{N}{K}(K-2)$.

REFERENCES

[1] M. A. Maddah-Ali and U. Niesen, “Fundamental limits of caching,” IEEE Transactions on Information Theory, vol. 60, no. 5, pp. 2856–2867, 2014.
[2] M. A. Maddah-Ali and U. Niesen, “Decentralized coded caching attains order-optimal memory-rate tradeoff,” IEEE/ACM Transactions on Networking, vol. 23, no. 4, pp. 1029–1040, 2015.
[3] N. Karamchandani, U. Niesen, M. A. Maddah-Ali, and S. N. Diggavi, “Hierarchical coded caching,” IEEE Transactions on Information Theory, vol. 62, no. 6, pp. 3212–3229, 2016.
[4] S. P. Shariatpanahi, S. A. Motahari, and B. H. Khalaj, “Multi-server coded caching,” IEEE Transactions on Information Theory, vol. 62, no. 12, pp. 7253–7271, 2016.
Fig. 4: Exact rate memory tradeoff for the \((N, K)\) cache network

[5] V. Ravindrakumar, P. Panda, N. Karamchandani, and V. M. Prabhakaran, “Private coded caching,” *IEEE Transactions on Information Forensics and Security*, vol. 13, no. 3, pp. 685–694, 2017.

[6] A. M. Daniel and W. Yu, “Optimization of heterogeneous coded caching,” *IEEE Transactions on Information Theory*, vol. 66, no. 3, pp. 1893–1919, 2019.

[7] E. Parrinello and P. Elia, “Coded caching with optimized shared-cache sizes,” in *Information Theory Workshop*, IEEE, 2019, pp. 1–5.

[8] Ç. Yapar, K. Wan, R. F. Schaefer, and G. Caire, “On the optimality of D2D coded caching with uncoded cache placement and one-shot delivery,” *IEEE Transactions on Communications*, vol. 67, no. 12, pp. 8179–8192, 2019.

[9] K. Wan, D. Tuninetti, M. Ji, and P. Piantanida, “Fundamental limits of distributed data shuffling,” in *Annual Allerton Conference on Communication, Control, and Computing*, IEEE, 2018, pp. 662–669.

[10] K. Wan, D. Tuninetti, and P. Piantanida, “On the optimality of uncoded cache placement,” in *Information Theory Workshop*, IEEE, 2016, pp. 161–165.

[11] M. M. Amiri, Q. Yang, and D. Gündüz, “Coded caching for a large number of users,” in *Information Theory Workshop*, IEEE, 2016, pp. 171–175.

[12] K. Wan, D. Tuninetti, and P. Piantanida, “On caching with more users than files,” in *International Symposium on Information Theory*, IEEE, 2016, pp. 135–139.

[13] K. Wan, D. Tuninetti, and P. Piantanida, “An index coding approach to caching with uncoded cache placement,” *IEEE Transactions on Information Theory*, vol. 66, no. 3, pp. 1318–1332, 2020.

[14] Q. Yu, M. A. Maddah-Ali, and A. S. Avestimehr, “The exact rate-memory tradeoff for caching with uncoded prefetching,” *IEEE Transactions on Information Theory*, vol. 64, no. 2, pp. 1281–1296, 2017.

[15] H. Ghasemi and A. Ramamoorthy, “Improved lower bounds for coded caching,” *IEEE Transactions on Information Theory*, vol. 63, no. 7, pp. 4388–4413, 2017.

[16] N. Ajaykrishnan, N. S. Prem, V. M. Prabhakaran, and R. Vaze, “Critical database size for effective caching,” in *National Conference on Communications*, IEEE, 2015, pp. 1–6.
[17] A. Sengupta, R. Tandon, and T. C. Clancy, “Improved approximation of storage-rate tradeoff for caching via new outer bounds,” in *International Symposium on Information Theory*, IEEE, 2015, pp. 1691–1695.

[18] C. Y. Wang, S. S. Bidokhti, and M. Wigger, “Improved converses and gap results for coded caching,” *Transactions on Information Theory*, vol. 64, no. 11, pp. 7051–7062, 2018.

[19] Q. Yu, M. A. Maddah-Ali, and A. S. Avestimehr, “Characterizing the rate-memory tradeoff in cache networks within a factor of 2,” *IEEE Transactions on Information Theory*, vol. 65, no. 1, pp. 647–663, 2018.

[20] Z. Chen, P. Fan, and K. B. Letaief, “Fundamental limits of caching: Improved bounds for users with small buffers,” *IET Communications*, vol. 10, no. 17, pp. 2315–2318, 2016.

[21] M. M. Amiri and D. Gündüz, “Fundamental limits of coded caching: Improved delivery rate-cache capacity tradeoff,” *IEEE Transactions on Communications*, vol. 65, no. 2, pp. 806–815, 2017.

[22] J. Gómez-Vilardebo, “Fundamental limits of caching: Improved rate-memory trade-off with coded prefetching,” *IEEE Transactions on Communications*, vol. 66, no. 10, pp. 4488–4497, 2018.

[23] C. Tian and J. Chen, “Caching and delivery via interference elimination,” in *International Symposium on Information Theory*, IEEE, 2016, pp. 830–834.

[24] K. P. Vijith Kumar, B. K. Rai, and T. Jacob, “Towards the exact rate memory tradeoff in coded caching,” in *National Conference on Communications*, IEEE, 2019, pp. 1–6.

[25] K. P. Vijith Kumar, B. K. Rai, and T. Jacob, “Fundamental limits of coded caching: The memory rate pair (K-1-1/K, 1/(K-1)),” in *International Symposium on Information Theory*, IEEE, 2019, pp. 2624–2628.

[26] S. Shao, J. Gómez-Vilardebo, K. Zhang, and C. Tian, “On the fundamental limit of coded caching systems with a single demand type,” in *Information Theory Workshop*, IEEE, 2019, pp. 1–5.

[27] S. Shao, J. Gómez-Vilardebo, K. Zhang, and C. Tian, “On the fundamental limits of coded caching systems with restricted demand types,” *arXiv preprint [arXiv:2006.16557]* 2020.

[28] S. Sahraei and M. Gastpar, “K users caching two files: An improved achievable rate,” in *Conference on Information Science and Systems*, IEEE, 2016, pp. 620–624.

[29] C. Tian, “Symmetry, outer bounds, and code constructions: A computer-aided investigation on the fundamental limits of caching,” *MDPI Entropy*, vol. 20, no. 8, p. 603, 2018.