DISTRIBUTION OF CERTAIN $\ell$-REGULAR PARTITIONS AND TRIANGULAR NUMBERS

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ABSTRACT. Let $pod_\ell(n)$ be the number of $\ell$-regular partitions of $n$ with distinct odd parts. In this article, prove that for any positive integer $k$, the set of non-negative integers $n$ for which $pod_\ell(n) \equiv 0 \pmod{p^k}$ has density one under certain conditions on $\ell$ and $p$. For $p \in \{3, 5, 7\}$, we also exhibit multiplicative identities for $pod_p(n)$ modulo $p$.

1. INTRODUCTION AND STATEMENT OF RESULTS

A partition of a positive integer $n$ is a non-increasing sequence of positive integers $\lambda_1, \lambda_2, \cdots, \lambda_k$ whose sum is $n$. Each $\lambda_i$ is called a part of the partition. If $\ell$ is a positive integer, then a partition is called an $\ell$-regular partition if there is no part divisible by $\ell$. Many mathematicians have studied this partition function and proved several interesting arithmetic and combinatorial properties; see [6, 11, 22]. Furthermore, various other types of partition functions are studied in the literature by imposing certain restrictions on the parts of an $\ell$-regular partitions of $n$. For example, Corteel and Lovejoy [5] introduced the notion of overpartition as a partition of $n$ in which the first occurrence of a number may be overlined. Ray et al. [18, 20] studied arithmetic and density properties of $\ell$-regular overpartitions. In [1, 2, 3, 10], authors considered the partitions with distinct odd parts, and even parts are unrestricted.

If $n$ is a positive integer, then the $n$th triangular number is $T_n = \frac{n(n+1)}{2}$. In this article, we consider $\ell$-regular partitions of $n$ with distinct odd parts and even parts are unrestricted, and denote by $pod_\ell(n)$. The generating function of $pod_\ell(n)$ is as follows (see [9]).

\[
\sum_{n=0}^{\infty} pod_\ell(n) q^n = \frac{\psi(-q^\ell)}{\psi(-q)},
\]

where $\psi(q) = \sum_{n=0}^{\infty} q^{T_n}$.

For a positive integer $M$ and $0 \leq r \leq M$, we define

\[
\delta^p_r(n; M) := \frac{\# \{0 \leq n \leq X : p(n) \equiv r \pmod{M} \}}{X},
\]

where $g : \mathbb{N} \to \mathbb{Z}$ is an arbitrary function. Suppose $p(n)$ counts the number of integer partitions at $n$. Parkin and Shanks [17] made the conjecture that the even and odd values of $p(n)$ are equally distributed, i.e,

\[
\lim_{X \to \infty} \delta^p_0(2; X) = \lim_{X \to \infty} \delta^p_1(2; X) = \frac{1}{2}.
\]

Similar studies are done for some other partition functions, for example see [4, 8, 19, 20]. Recently, Veena et al. [21] proved that the set of all positive integers such that $pod_3(n) \equiv 0 \pmod{3^k}$ have arithmetic density 1, i.e.,

\[
\lim_{X \to \infty} \delta_0^{pod_3(n)}(3^k; X) = 1.
\]

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Here we prove that for any odd positive integer $\ell$, the function $\text{pod}_\ell(n)$ is almost always divisible by $p^a$ with certain conditions, where $a$ is the largest positive integer for which $p^a$ divides $\ell$. In particular, we have the following result.

**Theorem 1.1.** Let $\ell$ be an odd integer greater than one, $p$ a prime divisor of $\ell$, and $a$ the largest positive integer such that $p^a$ divides $\ell$. Then for every positive integer $k$ we have

$$\lim_{X \to \infty} \delta_0^{\text{pod}_\ell(n)}(p^k; X) = 1.$$

The above result is a generalization of (1.2). Furthermore, we get the following corollary as a direct consequence of the above theorem.

**Corollary 1.2.** Let $p$ be an odd prime. Then for any positive integer $k$, $\text{pod}_p(n)$ is almost always divisible by $p^k$, i.e.,

$$\lim_{X \to \infty} \delta_0^{\text{pod}_p(n)}(p^k; X) \equiv 1.$$

Next, we consider 3-regular partitions of $n$ with distinct odd parts $\text{pod}_3(n)$. In [7], Gireesh et al. proved the following congruences:

$$\text{pod}_3(n) \equiv \text{pod}_3(9n + 2) \pmod{3^2},$$

$$\text{pod}_3(3n + 2) \equiv \text{pod}_3(27n + 20) \pmod{3^3},$$

$$\text{pod}_3(27n + 20) \equiv \text{pod}_3(243n + 182) \pmod{3^4}.$$ 

For any positive integers $k$ and $n$, Veena et al. [21] proved the following result

$$\text{pod}_3\left(3^{2k}n + \frac{3^{2k} - 1}{4}\right) \equiv \text{pod}_3(n) \pmod{3}. \quad (1.3)$$

We have generalized (1.3) and deduced the following infinite families of multiplicative formulas for $\text{pod}_3(n)$ modulo 3 using the theory of Hecke eigenforms.

**Theorem 1.3.** Let $k$ be a positive integer, $p$ a prime with $p \equiv 3 \pmod{4}$, and $\delta$ a non-negative integer such that $p$ divides $4\delta + 3$. Then for all $j \geq 0$ we have

$$\text{pod}_3\left(p^{k+1}n + p^\delta + \frac{3p - 1}{4}\right) \equiv \text{pod}_3\left(p^{k-1}n + \frac{4\delta + 3 - p}{4p}\right) \pmod{3}.$$ 

**Corollary 1.4.** Let $k$ be a positive integer and $p$ a prime with $p \equiv 3 \pmod{4}$. Then for all $n \geq 0$ we have

$$\text{pod}_3\left(p^{2k}n + \frac{p^{2k} - 1}{4}\right) \equiv \text{pod}_3(n) \pmod{3}.$$ 

**Corollary 1.5.** Let $k$ be a positive integer and $p$ a prime with $p \equiv 3 \pmod{4}$. Then for all $n \geq 0$ we have

$$\text{pod}_3\left(p^{2k+1}n + \frac{p^{2k} - 1}{4}\right) \equiv \text{pod}_3(np) \pmod{3}.$$ 

For example, if we consider $p = 7$ and $k = 1$, then from Corollary 1.5 we have the following congruences:

$$\text{pod}_3(343n + 24) \equiv \text{pod}_3(7n) \pmod{3}.$$ 

In addition, we prove the following results for $\text{pod}_5$ and $\text{pod}_7$. Let $\sigma_k(n) = \sum_{d|n} d^k$ be the standard divisor function and $\sigma_2,\chi(n) = \sum_{d|n} \chi(d)d^2$ be the generalized divisor function, where $\chi$ is the Dirichlet character modulo 4 with $\chi(1) = 1$ and $\chi(3) = -1.$
Theorem 1.6. For any positive integer \( n \), we have
\[
\text{pod}_5(n) \equiv (-1)^n\sigma_1(2n + 1) \pmod{5}, \text{ and }
\]
\[
\text{pod}_7(n) \equiv (-1)^{n+1}\frac{1}{8}\sigma_2\chi(4n + 3) \pmod{7}.
\]

We use Mathematica [12] for our computations.

2. Preliminaries

In this section, we recall some necessary facts and notation coming from modular forms (for further details we refer the reader to [13] and [15]). For a positive integer \( k \) denote by \( \mathcal{M}_k(\Gamma) \) the complex vector space of modular forms of weight \( k \) for a congruence subgroup \( \Gamma \), and let \( \mathbb{H} \) be the complex upper half plane.

Definition 2.1. [15, Definition 1.15] Let \( \chi \) be a Dirichlet character modulo \( N \). Then a modular form \( f \in \mathcal{M}_k(\Gamma_1(N)) \) has Nebentypus character \( \chi \) if
\[
f \left( \frac{az + b}{cz + d} \right) = \chi(d)(cz + d)^\ell f(z)
\]
for all \( z \in \mathbb{H} \) and all \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) \). We denote the space of such modular forms by \( \mathcal{M}_k(\Gamma_0(N), \chi) \).

Recall that Dedekind’s eta-function is defined by
\[
\eta(z) := q^{1/24} \psi(q), \quad q = e^{2\pi iz} \quad z \in \mathbb{H}.
\]
A function \( f(z) \) is called an eta-quotient if it is expressible as a finite product of the form
\[
f(z) = \prod_{\delta | N} \eta(\delta z)^{r_{\delta}},
\]
where \( N \) is a positive integer and each \( r_{\delta} \) is an integer. The following two theorems allow one to determine whether a given eta-quotient is a modular form.

Theorem 2.2. [15, Theorem 1.64] Suppose that \( f(z) = \prod_{\delta | N} \eta(\delta z)^{r_{\delta}} \) is an eta-quotient such that
\[
\ell = \frac{1}{2} \sum_{\delta | N} r_{\delta} \in \mathbb{Z},
\]
\[
\sum_{\delta | N} \delta r_{\delta} \equiv 0 \pmod{24} \quad \text{and} \quad \sum_{\delta | N} \frac{N}{\delta} r_{\delta} \equiv 0 \pmod{24}.
\]

Then
\[
f \left( \frac{az + b}{cz + d} \right) = \chi(d)(cz + d)^\ell f(z)
\]
for every \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) \). Here
\[
\chi(d) := \left( \frac{(-1)^\ell \prod_{\delta | N} \delta^{r_{\delta}}}{d} \right).
\]

If the eta-quotient \( f(z) \) satisfy the conditions of Theorem 2.2 and holomorphic at all of the cusps of \( \Gamma_0(N) \), then \( f \in \mathcal{M}_k(\Gamma_0(N), \chi) \). To determine the orders of an eta-quotient at each cusp is the following.
**Theorem 2.3.** [15, Theorem 1.65] Let $c, d,$ and $N$ be positive integers with $d \mid N$ and $\gcd(c, d) = 1$. If $f(z)$ is an eta-quotient satisfying the conditions of Theorem 2.2 for $N$, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is

$$\frac{N}{24} \sum_{d \mid N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, \frac{N}{d})} \delta.$$ 

Let $m$ be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N), \chi)$. Then the action of Hecke operator $T_m$ on $f(z)$ is defined by

$$f(z)|T_m := \sum_{n=0}^{\infty} \left( \sum_{d \mid \gcd(n, m)} \chi(d) d^{\ell-1} a \left( \frac{nm}{d^2} \right) \right) q^n. \quad (2.1)$$

We note that $a(n) = 0$ unless $n$ is a non-negative integer. The modular form $f(z)$ is called a Hecke eigenform if for every $m \geq 2$ there exists a complex number $\lambda(m)$ for which

$$f(z)|T_m = \lambda(m) f(z). \quad (2.2)$$

**3. PROOFS OF THEOREM 1.1**

To prove Theorem 1.1, we need the following lemmas.

**Lemma 3.1.** Let $\ell$ be an odd integer greater than one, $p$ a prime divisor of $\ell$, and $a$ the largest positive integer such that $p^a$ divides $\ell$. Then for any positive integer $k$ we have

$$\frac{\eta(24z)^{p^a+k-1} \eta(48z) \eta(24\ell z) \eta(96\ell z)}{\eta(96z) \eta(48\ell z) \eta(24p^a z)^{p^k}} \equiv \sum_{n=0}^{\infty} p_{\ell, n}(n) q^{24n+3(\ell-1)} \pmod{p^k}. \quad (3.1)$$

**Proof.** Consider

$$A(z) = \prod_{n=1}^{\infty} \frac{(1 - q^{24n})^{p^n}}{(1 - q^{24p^a n})} = \frac{\eta(24z)^{p^n}}{\eta(24p^a z)}. \quad (3.2)$$

By the binomial theorem, for any positive integers $r, k,$ and prime $p$ we have

$$(q^r; q^r)_\infty^{p^k} \equiv (q^{pr}; q^{pr})_\infty^{p^{k-1}} \pmod{p^k}. \quad (3.3)$$

Therefore,

$$A^{p^k}(z) = \frac{\eta(24z)^{p^a+k}}{\eta(24p^a z)^{p^k}} \equiv 1 \pmod{p^{k+1}}. \quad (3.4)$$

Define $B_{\ell,p,k}(z)$ by

$$B_{\ell,p,k}(z) = \frac{\eta(48z) \eta(24\ell z) \eta(96\ell z)}{\eta(24z) \eta(96z) \eta(48\ell z)} A^{p^k}(z). \quad (3.5)$$
Now, modulo $p^{k+1}$, we have
\begin{equation}
\mathcal{B}_{\ell,p,k}(z) = \frac{\eta(48z)\eta(24\ell z)\eta(96\ell z)}{\eta(24z)\eta(96z)\eta(48\ell z)} \eta(24p^a z)^{p^k} = \frac{\eta(48z)\eta(24\ell z)\eta(96\ell z)}{\eta(24z)\eta(96z)\eta(48\ell z)} = q^{3(\ell-1)} \psi(-q^{24\ell}) \psi(-q^{24}).
\end{equation}
(3.1)
Since
\begin{equation}
\mathcal{B}_{\ell,p,k}(z) = \frac{\eta(24z)p^{a+k-1}\eta(48z)\eta(24\ell z)\eta(96\ell z)}{\eta(96z)\eta(48\ell z)\eta(24p^a z)^{p^k}},
\end{equation}
combining (1.1) and (3.1), we obtain the required result.

\textbf{Lemma 3.2.} Let $\ell$ be an odd integer greater than one, $p$ a prime divisor of $\ell$, and $a$ the largest positive integer such that $p^a$ divides $\ell$. Then, for a positive integer $k > a$, we have
\begin{equation}
\mathcal{B}_{\ell,p,k}(z) \in M_{\frac{k(a-1)}{2}}(\Gamma_0(384 \cdot \ell), \chi(\bullet)),
\end{equation}
where the Nebentypus character
\begin{equation}
\chi(\bullet) = \left(\frac{-1}{2}\frac{(24)^{p^a+k-1} \cdot 48 \cdot 24\ell \cdot 96 \ell \cdot (96)^{-1} \cdot (48)^{-1} \cdot (24p^a)^{p^k}}{_{24}}\right).
\end{equation}

\textbf{Proof.} First we verify the first, second and third hypotheses of Theorem 2.2. The weight of the \textit{eta}-quotient $\mathcal{B}_{\ell,p,k}(z)$ is $\frac{1}{2}(p^{a+k}-p^k) = \frac{p^k}{2}(p^a - 1)$.

Suppose the level of the \textit{eta}-quotient $\mathcal{B}_{\ell,p,k}(z)$ is $96\ell u$, where $u$ is the smallest positive integer satisfying the following identity.
\begin{equation}
(p^{a+k} - 1) \frac{96\ell u}{24} + \frac{96\ell u}{48} + \frac{96\ell u}{24\ell} + \frac{96\ell u}{96} - \frac{96\ell u}{96} - 96\ell u - p^k \frac{96\ell u}{24p^a} \equiv 0 \pmod{24}.
\end{equation}
(3.2)
Equivalently, we have
\begin{equation}
u \left(4\ell p^{k-a} (p^{2a} - 1) - 3(\ell - 1)\right) \equiv 0 \pmod{24}.
\end{equation}
For all prime $p \neq 3$, note that $p^{2a} - 1$ is multiple of $3$. Hence, from (3.2), we conclude that the level of the \textit{eta}-quotient $\mathcal{B}_{\ell,p,k}(z)$ is $384\ell$ for $k > a$.

By Theorem 2.3, the cusps of $\Gamma_0(384\ell)$ are given by $\frac{d}{a}$ where $d \mid 384\ell$ and $\gcd(c, d) = 1$. Now $\mathcal{B}_{\ell,p,k}(z)$ is holomorphic at a cusp $\frac{d}{a}$ if and only if
\begin{equation}
(p^{a+k} - 1) \frac{\gcd(d, 24)^2}{24\ell} + \frac{\gcd(d, 48)^2}{48\ell} + \frac{\gcd(d, 24\ell)^2}{24\ell} + \frac{\gcd(d, 96\ell)^2}{96\ell} - \frac{\gcd(d, 96\ell)^2}{48\ell} - p^k \frac{\gcd(d, 24p^a)^2}{24p^a} \geq 0.
\end{equation}
Equivalently, if and only if
\begin{equation}
4\ell(p^{a+k} - 1) \frac{\gcd(d, 24)^2}{\gcd(d, 96\ell)^2} + 2\ell \frac{\gcd(d, 48)^2}{\gcd(d, 96\ell)^2} + 4\ell \frac{\gcd(d, 24\ell)^2}{\gcd(d, 96\ell)^2} - \ell \frac{\gcd(d, 96\ell)^2}{\gcd(d, 96\ell)^2} - 2\frac{\gcd(d, 48\ell)^2}{\gcd(d, 96\ell)^2} - 4\ell p^{k-a} \frac{\gcd(d, 24p^a)^2}{\gcd(d, 96\ell)^2} + 1 \geq 0.
\end{equation}
(3.3)
To check the positivity of (3.3), we have to find all the possible divisors of $384\ell$. We define three sets as follows.

$$
\mathcal{H}_1 = \left\{ 2^r 3^s tp^s : 0 \leq r_1 \leq 3, 0 \leq r_2 \leq 1, t \mid \ell \text{ but } p \nmid t, \text{ and } 0 \leq s \leq a \right\},$
$$
\mathcal{H}_2 = \left\{ 2^r 3^s tp^s : r_1 = 4, 0 \leq r_2 \leq 1, t \mid \ell \text{ but } p \nmid t, \text{ and } 0 \leq s \leq a \right\},$
$$
\mathcal{H}_3 = \left\{ 2^r 3^s tp^s : 5 \leq r_1 \leq 7, 0 \leq r_2 \leq 1, t \mid \ell \text{ but } p \nmid t, \text{ and } 0 \leq s \leq a \right\}.
$$

Note that $\mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$ contains all positive divisors of $384\ell$. In the following table, we compute all necessary data to prove the positivity of (3.3).

| Values of $d$ such that $d \mid 384\ell$ | $\gcd(d, 24)^2$ | $\gcd(d, 48)^2$ | $\gcd(d, 96)^2$ | $\gcd(d, 24p^a)^2$ | $\gcd(d, 24\ell)^2$ | $\gcd(d, 48\ell)^2$ |
|---------------------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $d \in \mathcal{H}_1$                     | $1/t^2 p^{2s}$  | $1/t^2 p^{2s}$  | $1/t^2 p^{2s}$  | $1/t^2$         | $1$             | $1$             |
| $d \in \mathcal{H}_2$                     | $1/4 t^2 p^{2s}$| $1/t^2 p^{2s}$  | $1/t^2 p^{2s}$  | $1/4 t^2$       | $1/4$           | $1$             |
| $d \in \mathcal{H}_3$                     | $1/16 t^2 p^{2s}$| $1/4 t^2 p^{2s}$| $1/t^2 p^{2s}$  | $1/16 t^2$      | $1/16$          | $1/4$           |

**Case (i).** If $d \in \mathcal{H}_1$, then left side of (3.3) can be written as:

$$
\frac{\ell}{t^2} \left[ 4p^k \left( \frac{p^a}{p^{2s}} - \frac{1}{p^a} \right) - 3 \frac{1}{p^{2s}} \right] + 3.
$$

When $s = a$, the above quantities, $\left( 3 - \frac{3\ell}{r^2 p^a} \right) \geq 0$, as $p^{2a} \geq \ell$. For $0 \leq s < a$ it is clear that $\frac{p^a}{p^{2s}} - \frac{1}{p^a} > 0$. Therefore,

$$
\frac{p^a}{p^{2s}} - \frac{1}{p^a} - \frac{1}{p^{2s}} \geq \frac{p^{2a} - p^{2(a-1)} - p^a}{p^{a+2s}} = \frac{p^a(1 - \frac{1}{p}) - 1}{p^{a+2s}} > 0.
$$

The last inequality holds because $p^a(1 - \frac{1}{p^a}) > 1$ for all prime $p$. Hence the quantities in (3.4) are greater than or equal to 0 when $p^{2a} \geq \ell$.

**Case (ii).** If $d \in \mathcal{H}_2$ or $d \in \mathcal{H}_3$, then left side of (3.3) can be written respectively as:

$$
\frac{p^k \ell}{t^2} \left( \frac{p^a}{p^{2s}} - \frac{1}{p^a} \right), \quad \text{and}
$$

$$
\frac{\ell}{4t^2} \left[ p^k \left( \frac{p^a}{p^{2s}} - \frac{1}{p^a} \right) - 3 \frac{1}{p^{2s}} \right] + \frac{3}{4}.
$$

By the similar argument as case (i), the quantities in (3.6) and (3.5) are greater than or equal to 0.

Therefore, by **Case (i)** and **Case (ii)**, the orders of vanishing of $B_{\ell,p,k}(z)$ at the at the cusp $\zeta_q$ is nonnegative. So $B_{\ell,p,k}(z)$ is holomorphic at every cusp $\zeta_q$. We have also verified the Nebentypus character by Theorem 2.2. Hence $B_{\ell,p,k}(z)$ is a modular form of weight $\frac{k(p^2-1)}{2}$ on $\Gamma_0(384 \cdot \ell)$ with Nebentypus character $\chi(\bullet)$. □

We state the following result of Serre, which is useful to prove Theorem 1.1.
1.1

Theorem 3.3. [15, Theorem 2.65] Let \( k, m \) be positive integers. If \( f(z) \in M_k(\Gamma_0(N), \chi(\bullet)) \) has the Fourier expansion
\[
f(z) = \sum_{n=0}^{\infty} c(n)q^n \in \mathbb{Z}[q],
\]
then there is a constant \( \alpha > 0 \) such that
\[
\# \{ n \leq X : c(n) \not\equiv 0 \pmod{m} \} = O\left( \frac{X}{\log^\alpha X} \right).
\]

Proof of Theorem 1.1. Suppose \( k > 1 \) is a positive integer. From Lemma 3.2, we have
\[
B_{\ell,p,k}(z) = \frac{\eta(24z)p^{\ell+k-1} \eta(48\ell z)\eta(24\ell^2 z)}{\eta(96\ell^2 z)\eta(48\ell^2 z)\eta(24p^\ell z)p^k} \in M_{\frac{k^2}{2}}(\Gamma_0(384 \cdot \ell), \chi(\bullet)).
\]

Also the Fourier coefficients of the \( \eta \)-quotient \( B_{\ell,p,k}(z) \) are integers. So, by Theorem 3.3 and Lemma 3.1, we can find a constant \( \alpha > 0 \) such that
\[
\# \{ n \leq X : pod_\ell(n) \not\equiv 0 \pmod{p^k} \} = O\left( \frac{X}{\log^\alpha X} \right).
\]

Hence
\[
\lim_{X \to \infty} \frac{\# \{ n \leq X : pod_c(n) \equiv 0 \pmod{p^k} \}}{X} = 1.
\]

This allows us to prove the required divisibility by \( p^k \) for all \( k > a \) which trivially gives divisibility by \( p^k \) for all positive integer \( k \leq a \). This completes the proof of Theorem 1.1. \( \square \)

4. PROOFS OF THEOREM 1.3 - 1.6

In this section, we discuss about the multiplicative nature of \( pod_3(n) \) and we get Corollary 1.4 - 1.5 as a special case of Theorem 1.3. Then we obtain some congruence formulas for \( pod_5(n) \) and \( pod_7(n) \) in the Theorem 1.6-1.6.

Proof of Theorem 1.3. By (1.1) we have
\[
\sum_{n=0}^{\infty} pod_3(n)q^n = \frac{\psi(-q^3)}{\psi(-q)} \equiv \psi(-q)^2 \pmod{3}.
\]

Using Theorem 2.2 and Theorem 2.3 we have
\[
\frac{\eta(4z)^2\eta(16z)^2}{\eta(8z)^2} \in M_1(\Gamma_0(64), \chi_{-1}(\bullet)), \text{ where } \chi_{-1} \text{ is defined by } \chi_{-1}(\bullet) = (\frac{-1}{\ell}).
\]

Therefore the above eta-quotient has a Fourier expansion and consider
\[
\frac{\eta(4z)^2\eta(16z)^2}{\eta(8z)^2} = q - 2q^5 + 3q^9 - 6q^{13} + 7q^{17} - 10q^{21} + \cdots = \sum_{n=1}^{\infty} a(n)q^n.
\]

From (1.1) and (4.2), for any positive integer \( n \), we have
\[
pod_3(n) \equiv a(4n+1) \pmod{3}.
\]

From [14], we know that \( \frac{\eta(4z)^2\eta(16z)^2}{\eta(8z)^2} \) is a Hecke eigenform. Using (2.1) and (2.2) we obtain
\[
\left| T_p \right| = \sum_{n=1}^{\infty} \left( a(pn) + \left( \frac{-1}{p} \right) a\left( \frac{n}{p} \right) \right) q^n = \lambda(p) \sum_{n=1}^{\infty} a(n)q^n.
\]

Since \( a(p) = 0 \) for all \( p \equiv 3 \pmod{4} \), equating the coefficients on the both sides, we have \( \lambda(p) = 0 \), and
\[
a(pn) + \left( \frac{-1}{p} \right) a\left( \frac{n}{p} \right) = 0.
\]
Replacing $n$ by $4p^k n + 4\delta + 3$ in (4.4), we get the following

$$a \left( 4 \left( p^{k+1} n + p\delta + \frac{3p - 1}{4} \right) + 1 \right) = (-1) \left( \frac{-1}{p} \right) a \left( 4 \left( p^{k-1} n + \frac{4\delta + 3 - p}{4p} \right) + 1 \right). \quad (4.5)$$

Note that $\frac{3p - 1}{4}$ and $\frac{4\delta + 3 - p}{4p}$ are integers. Using (4.3) and (4.5) we obtain

$$\text{pod}_3 \left( p^{k+1} n + p\delta + \frac{3p - 1}{4} \right) \equiv (-1) \left( \frac{-1}{p} \right) \text{pod}_3 \left( p^{k-1} n + \frac{4\delta + 3 - p}{4p} \right) \pmod{3} \quad (4.6)$$

Since $\left( \frac{-1}{p} \right) = -1$, for all prime $p \equiv 3 \pmod{4}$, our result now follows from (4.6).

**Proof of Corollary 1.4.** Let $p \equiv 3 \pmod{4}$ be a prime. Now replacing $n$ with $np^{k-1}$ in (4.6) and then considering $4\delta + 3 = p^{2k-1}$ we have

$$\text{pod}_3 \left( p^{2k}(4n + 1) - 1 \right) \equiv \text{pod}_3 \left( p^{2(k-1)}(4n + 1) - 1 \right) \pmod{3}.$$ 

By repeating the above relation for $(k - 1)$-times, we obtain

$$\text{pod}_3 \left( p^{2k}(4n + 1) - 1 \right) \equiv \text{pod}_3(n) \pmod{3}.$$ 

Then the result directly follows from the above congruence.

**Proof of Corollary 1.5.** We can prove Corollary 1.5 in a similar fashion as Corollary 1.4.

For a positive integer $k$, let $t_k(n)$ denote the number of representations of $n$ as a sum of $k$ triangular numbers. It is easy to see that if $k > 1$ then

$$\psi(q)^k = \sum_{n=0}^{\infty} t_k(n)q^n.$$ 

If $p$ an odd prime number, then applying the binomial theorem on (1.1) we have

$$\sum_{n=0}^{\infty} \text{pod}_p(n)q^n \equiv \psi(-q)^{p-1} \pmod{p}.$$ 

Therefore for a positive integer $n$ and a odd prime $p$ we obtain

$$\text{pod}_p(n) \equiv (-1)^n t_{p-1}(n) \pmod{p} \quad (4.7)$$

**Proof of Theorem 1.6.** Suppose $t_4(n)$ is the number of representations of $n$ as sum of 4 triangular numbers. Then Ono et al. [16, Theorem 3] showed that

$$t_4(n) = \sigma_1(2n + 1).$$

Now (1.4) directly follows from the above equation and (4.7). Similarly, the result (1.5) follows from [16, Theorem 4] and (4.7).

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REFERENCES

[1] George E. Andrews. A generalization of the Göllnitz-Gordon partition theorems. *Proc. Amer. Math. Soc.*, 18:945–952, 1967.
[2] George E. Andrews. Partitions and Durfee dissection. *Amer. J. Math.*, 101(3):735–742, 1979.
[3] Alexander Berkovich and Frank G. Garvan. Some observations on Dyson’s new symmetries of partitions. *J. Combin. Theory Ser. A*, 100(1):61–93, 2002.
[4] Kathrin Bringmann and Jeremy Lovejoy. Rank and congruences for overpartition pairs. *Int. J. Number Theory*, 4(2):303–322, 2008.
[5] Sylvie Corteel and Jeremy Lovejoy. Overpartitions. *Trans. Amer. Math. Soc.*, 356(4):1623–1635, 2004.
[6] Su-Ping Cui and Nancy S. S. Gu. Arithmetic properties of ℓ-regular partitions. *Adv. in Appl. Math.*, 51(4):507–523, 2013.
[7] D. S. Gireesh, M. D. Hirschhorn, and M. S. Mahadeva Naika. On 3-regular partitions with odd parts distinct. *Ramanujan J.*, 44(1):227–236, 2017.
[8] Basil Gordon and Ken Ono. Divisibility of certain partition functions by powers of primes. *Ramanujan J.*, 1(1):25–34, 1997.
[9] B. Hemanthkumar, H. S. Sumanth Bharadwaj, and M. S. Mahadeva Naika. Arithmetic properties of 9-regular partitions with distinct odd parts. *Acta Math. Vietnam.*, 44(3):797–811, 2019.
[10] Michael D. Hirschhorn and James A. Sellers. Arithmetic properties of partitions with odd parts distinct. *Ramanujan J.*, 22(3):273–284, 2010.
[11] Qing-Hu Hou, Lisa H. Sun, and Li Zhang. Quadratic forms and congruences for ℓ-regular partitions modulo 3, 5 and 7. *Adv. in Appl. Math.*, 70:32–44, 2015.
[12] Wolfram Research, Inc. Mathematica, Version 10.0. Champaign, IL, 2014.
[13] Neal Koblitz. *Introduction to elliptic curves and modular forms*, volume 97 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.
[14] Yves Martin. Multiplicative η-quotients. *Trans. Amer. Math. Soc.*, 348(12):4825–4856, 1996.
[15] Ken Ono. The web of modularity: arithmetic of the coefficients of modular forms and q-series, volume 102 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2004.
[16] Ken Ono, Sinai Robins, and Patrick T. Wahl. On the representation of integers as sums of triangular numbers. *Aequationes Math.*, 50(1-2):73–94, 1995.
[17] Thomas R. Parkin and Daniel Shanks. On the distribution of parity in the partition function. *Math. Comp.*, 21:466–480, 1967.
[18] Chiranjit Ray and Rupam Barman. Infinite families of congruences for k-regular overpartitions. *Int. J. Number Theory*, 14(1):19–29, 2018.
[19] Chiranjit Ray and Rupam Barman. On Andrews’ integer partitions with even parts below odd parts. *J. Number Theory*, 215:321–338, 2020.
[20] Chiranjit Ray and Kalyan Chakraborty. Certain eta-quotients and ℓ-regular overpartitions. *Ramanujan J.*, 2020.
[21] V. S. Veena and S. N. Fathima. Arithmetic properties of 3-regular partitions with distinct odd parts. *Abh. Math. Semin. Univ. Hambg.*, 2021.
[22] Ernest X. W. Xia. Congruences for some ℓ-regular partitions modulo ℓ. *J. Number Theory*, 152:105–117, 2015.

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