Prandtl–Batchelor Flows on a Disk

Mingwen Fei¹, Chen Gao², Zhiwu Lin³, Tao Tao⁴

¹ School of Mathematics and Statistics, Anhui Normal University, Wuhu 241002, China.
E-mail: mwfei@ahnu.edu.cn
² Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China.
E-mail: gaochen@amss.ac.cn
³ School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA.
E-mail: zlin@math.gatech.edu
⁴ School of Mathematics, Shandong University, Jinan 250100, Shandong, China.
E-mail: taotao@sdu.edu.cn

Received: 21 December 2021 / Accepted: 21 August 2022
Published online: 14 November 2022 – © The Author(s), under exclusive licence to Springer-Verlag GmbH
Germany, part of Springer Nature 2022

Abstract: For steady two-dimensional flows with a single eddy (i.e. nested closed streamlines), Prandtl (Über Flüssigkeitsbewegung bei sehr kleiner Reibung. Verhandlungen des III. Internationalen Mathematiker Kongresses, Heidelberg, 1905) and Batchelor (J Fluid Mech 7(1):177–190, 1956) proposed that in the limit of vanishing viscosity the vorticity is constant in an inner region separated from the boundary layer. In this paper, by constructing higher order approximate solutions of the Navier–Stokes equations and establishing the validity of Prandtl boundary layer expansion, we give a rigorous proof of the existence of Prandtl–Batchelor flows on a disk with the wall velocity slightly different from the rigid-rotation. The leading order term of the flow is the constant vorticity solution (i.e. rigid rotation) satisfying the Batchelor–Wood formula.

1. Introduction

In his famous paper [33] on the birth of boundary layer theory, Prandtl (1904) noted that in the limit of infinite Reynolds number, the vorticity of steady two-dimensional laminar flows becomes constant within a region of nested closed streamlines (i.e. a single eddy). The same property was later rediscovered by Batchelor in [1]. See also the conference abstract [5] of Feynman and Lagerstrom. This class of results is now usually referred to as the Prandtl–Batchelor theory and such laminar flows are called Prandtl–Batchelor flows in the literature. For the Prandtl–Batchelor flows on a circular disk, the formula of the limiting vorticity constant was given in [1, 5, 42] and is usually referred to as the Batchelor–Wood formula. For the Prandtl–Batchelor flows on more general domains, it is more difficult to determine the limiting vorticity constant and partial results were given in [4, 5, 26, 37, 38, 42]. The Prandtl–Batchelor theory plays an important role in many studies involving laminar flows with small viscosity, for example, the nonlinear critical layer theory near shear flows in [27]. It also implies a selection mechanism of the Navier–Stokes equations in the inviscid limit. Although the Euler equations have infinitely many steady solutions, only those Euler solutions satisfying the Prandtl–Batchelor theory can...
appear in the inviscid limit. That is, the vorticity of the compatible Euler solution must be a constant within each eddy.

Moreover, the Prandtl–Batchelor theory can be applied to the general 2D advection-diffusion equation of a passive scalar field \( \theta (x, y) \)

\[
\mathbf{u} \cdot \nabla \theta - R^{-1} \Delta \theta = 0,
\]

where \( \mathbf{u} (x, y) \) is a given steady incompressible flow, and \( R^{-1} \) is the diffusion coefficient. Then the Prandtl–Batchelor theory implies that when \( R^{-1} \) tends to zero, \( \theta \) should tend to be a constant within any single eddy associated with the flow \( \mathbf{u} \). The first example is the homogenization of potential vorticity in the ocean circulation theory [32,35,36], where \( \theta \) is the potential vorticity in the 2D quasi-geostrophic model and \( R \) is the Peclet number. Another example is the flux expulsion in the self-excited dynamo theory [28,29,41], where \( \theta \) is the magnetic potential and \( R \) is the magnetic Reynolds number. Then the Prandtl–Batchelor theory implies that within each eddy of the flow \( \mathbf{u} \), the magnetic field (i.e. \( \nabla \theta \)) should become zero when \( R^{-1} \) tends to zero.

Despite the importance of the Prandtl–Batchelor theory and its wide applications to fluids, there is relatively little mathematical work on this problem. In the existing “proof” of the Prandtl–Batchelor theory stemming from [1,33], the eddy structure of the laminar flow is a priori assumed. In Appendix, we will give such a proof for the case of a single eddy. It is based on the assumptions that: (i) the steady flow of Navier–Stokes equations has a single eddy (i.e. no hyperbolic stagnation point of the stream function); (ii) any interior domain is separated from the boundary layer uniformly for vanishing viscosity; (iii) inside the interior domain, the solutions to the steady Navier–Stokes equations tend to a steady Euler solution in \( C^2 \). However, given a domain and the boundary condition, it is difficult to understand the eddy structures (i.e. streamline structures) of the steady solutions of Navier–Stokes equations and control the boundary layer. The above assumptions were never verified in any case rigorously. In a series of works [20–26], Kim initiated a mathematical study of Prandtl–Batchelor flows on a disk. In particular, when the boundary velocity is slightly different from a constant, the well-posedness of the Prandtl equations under the Batchelor–Wood condition was shown in [21] and some asymptotic study of the boundary layer expansion was given in [20]. However, it remains difficult to show the convergence of the boundary layer expansion to the steady Navier–Stokes solution.

In this paper, we give the first proof of the existence of Prandtl–Batchelor flows on a disk. As in [21], we assume the boundary condition to be slightly different from a constant rotation. We first construct steady solutions of Euler equations (leading order term of the Navier–Stokes equations) to be the constant vorticity solution (i.e. rigid rotation) satisfying the Batchelor–Wood formula. Then, by constructing higher order approximate solutions and establishing the convergence of Prandtl boundary layer expansion, we construct a class of Prandtl–Batchelor flows to the steady Navier–Stokes equations on a disk. More precisely, we consider the steady Navier–Stokes equations in two dimensional disk \( B_1(0) \)

\[
\begin{align*}
\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \nabla p^\varepsilon - \varepsilon^2 \Delta \mathbf{u}^\varepsilon &= 0, \\
\nabla \cdot \mathbf{u}^\varepsilon &= 0,
\end{align*}
\]

(1.1)

with rotating boundary

\[
\mathbf{u}^\varepsilon \big|_{\partial B_1} = (\alpha + \eta f (\theta)) \mathbf{t},
\]

(1.2)
where $\varepsilon > 0$ is reciprocal to Reynolds number, $\mathbf{u}^\varepsilon$ is the velocity, $p^\varepsilon$ is the pressure, $\alpha > 0$, $\eta$ is a small number, $t$ is the unit tangential vector to $\partial B_1$, and $f(\theta)$ is a $2\pi$-periodic smooth function. Let $u^\varepsilon(\theta, r)$ be the tangential component and $v^\varepsilon(\theta, r)$ be the normal component of $\mathbf{u}^\varepsilon$ in polar coordinates, then (1.1) reads

\[
\begin{align*}
\left\{ \begin{array}{l}
u^\varepsilon u^\varepsilon_\theta + ru^\varepsilon v^\varepsilon_\theta + u^\varepsilon + p^\varepsilon_\theta - \varepsilon^2 (ru^\varepsilon_\theta r + u^\varepsilon_\theta r + \frac{u^\varepsilon_\theta}{r} + \frac{2}{r} v^\varepsilon_\theta - \frac{v^\varepsilon_\theta}{r}) = 0, \\
u^\varepsilon v^\varepsilon_\theta + ru^\varepsilon v^\varepsilon_\theta - (u^\varepsilon)^2 + rp^\varepsilon_\theta - \varepsilon^2 (rv^\varepsilon_\theta + v^\varepsilon_\theta r + \frac{v^\varepsilon_\theta}{r} - \frac{2}{r} u^\varepsilon_\theta - \frac{u^\varepsilon_\theta}{r}) = 0,
\end{array} \right.
\end{align*}
\tag{1.3}
\]

where $(\theta, r) \in \Omega := [0, 2\pi] \times (0, 1]$. Formally, as $\varepsilon \to 0$, we obtain the steady Euler equations

\[
\begin{align*}
\left\{ \begin{array}{l}
u e u e_\theta + v e r \partial_r u e + u e v e + \partial_\theta p e = 0, \\
u e v e_\theta + v e r \partial_r v e - (u e)^2 + r \partial_r r p e = 0, \\
\partial_\theta u e + \partial_r (r v e) = 0.
\end{array} \right.
\end{align*}
\tag{1.4}
\]

We will show the existence of solution $(u^\varepsilon, v^\varepsilon)$ to (1.3) which converges to a solution of steady Euler equations (1.4) with constant vorticity as $\varepsilon \to 0$. We first choose a steady Euler flow with constant vorticity (rigid rotation) which satisfies the Batchelor–Wood formula, then construct a solution $(u^\varepsilon, v^\varepsilon)$ to (1.3) by perturbing this steady Euler flow. However, in general, there is a mismatch between the tangential velocities of the Euler flow $u_e$ and the prescribed Navier–Stokes flows $u^\varepsilon$. Due to the mismatch on the boundary, Prandtl in 1904 formally introduced the boundary layer theory to correct this mismatch, but the justification of this formal boundary expansion is a challenging problem. A first step for the justification of this formal boundary expansion is to understand the approximate terms in the formal expansion, that is the steady Prandtl equations and linearized steady Prandtl equations, see [3,12,31,34,39,40]. For the validity of the Prandtl boundary layer expansion, there are some important results in recent years. For the moving boundary, Guo and Nguyen made the first important progress in [9] where they considered the Navier–Stokes equations over a moving plate. Then Iyer in [13] extended Guo–Nguyen’s result to the case of a rotating disk by considering the curvature effect. The leading order of Euler flows in [9] and [13] are both shear flows, and the width of the region in [9] and the angle of sector in [13] are small. Later, Iyer in [14–16] justified the global steady Prandtl expansions over a moving plane under the assumption of the smallness of the mismatch, and considered the situation that Euler flow is a perturbation of shear flow in [17]. For the no-slip boundary, there are also some important works. In [10], Guo and Iyer justified the validity of 2D steady Prandtl layer expansion in a narrow region, and later they extended their result in [11] where they considered the 2D steady Navier–Stokes equations with external force. Gao and Zhang in [6] justified the validity of 2D steady Prandtl layer expansion for shear flow in a wide region by introducing the stream function in the error estimates. Then, Iyer and Masmoudi in [18,19] justified the global-in-x steady Prandtl boundary layer expansion. Very recently, Gao and Zhang in [7] justified the Prandtl expansion under the situation of non-shear Euler flow when the width of the region is small. Moreover, the stability in Sobolev space for some class of shear flows of Prandtl type has been studied by Gerard-Varet and Maekawa in [8], see also [2].

In the current work, we assume that $\int_0^{2\pi} f(\theta)d\theta = 0$ and the leading order of Euler flow $(u_e(\theta, r), v_e(\theta, r))$ is the Couette flow $(u_e, v_e) = (ar, 0)$. 

Prandtl–Batchelor Flows on a Disk 1105
There is an important basis for this choice—Prandtl–Batchelor theory in [1] (see also Appendix C). This theory shows that if the Euler flow \((u_e(\theta, r), v_e(\theta, r))\) in the disk \(B_1(0)\) is the vanishing viscosity limit of Navier–Stokes flow whose streamlines are closed, then it must be the Couette flow 
\[(u_e(\theta, r), v_e(\theta, r)) = (ar, 0),\]
where \(a\) is a constant. Moreover, the Batchelor–Wood formula [42] shows that 
\[a^2 = \alpha^2 + \frac{\eta^2}{2\pi} \int_0^{2\pi} f^2(\theta) d\theta.\]
So we take the leading order of Euler flow \((u_e(\theta, r), v_e(\theta, r))\) as follows 
\[u_e(\theta, r) = u_e(r) := ar, \quad v_e(\theta, r) = 0, \quad (1.5)\]
where 
\[a = \left(\alpha^2 + \frac{\eta^2}{2\pi} \int_0^{2\pi} f^2(\theta) d\theta\right)^{\frac{1}{2}}.\]
Then, we introduce the steady Prandtl equations near \(r = 1\)
\[
\begin{cases}
(u_e(1) + u_p^{(0)}(\theta)) \partial_\theta u_p^{(0)} + (v_p^{(1)}(\theta, r, 0)) \partial_\theta v_p^{(1)} + \partial_\theta u_p^{(0)} + \partial_Y v_p^{(1)} = 0, \\
\left. u_p^{(0)}(\theta, Y) = u_p^{(0)}(\theta + 2\pi, Y), \quad v_p^{(1)}(\theta, Y) = v_p^{(1)}(\theta + 2\pi, Y), \quad u_p^{(0)}(0, Y) = \alpha + \eta f(\theta) - u_e(1), \quad \lim_{Y \to -\infty} (u_p^{(0)}, v_p^{(1)}) = (0, 0) \right. 
\end{cases} \quad (1.6)
\]
The above steady Prandtl equations will be derived by matched asymptotic expansion and the solvability will be studied in the next section.
Our main theorem is stated as follows.

**Theorem 1.1.** Assume that \(f(\theta)\) is a 2\(\pi\)-periodic smooth function which satisfies 
\[\int_0^{2\pi} f(\theta) d\theta = 0,\]
then there exist \(\epsilon_0 > 0, \eta_0 > 0\) such that for any \(\epsilon \in (0, \epsilon_0), \eta \in (0, \eta_0)\), the Navier–Stokes equations (1.3) have a solution \((u^\epsilon(\theta, r), v^\epsilon(\theta, r))\) which satisfies 
\[
\|u^\epsilon(\theta, r) - u_e(r) - u_p^{(0)}(\theta, r - \frac{1}{\epsilon})\|_{L^\infty(\Omega)} \leq C\epsilon, \\
\|v^\epsilon\|_{L^\infty(\Omega)} \leq C\epsilon,
\]
where \((u_e(r), 0)\) is the Couette flow in (1.5), and \(u_p^{(0)}\) is the solution of steady Prandtl equations in (1.6).
Moreover, for any \(r < 1\), there holds 
\[
\lim_{\epsilon \to 0} \|w^\epsilon - 2a\|_{L^\infty(B_r(0))} = 0,
\]
where \(w^\epsilon(\theta, r)\) is the vorticity of \((u^\epsilon(\theta, r), v^\epsilon(\theta, r))\).
Remark 1.2. The condition $\int_0^{2\pi} f(\theta) d\theta = 0$ can be dropped due to the fact
\[
\alpha + \eta f(\theta) = \alpha + \eta \int_0^{2\pi} f(\theta) d\theta + \eta \tilde{f}(\theta),
\]
where $\int_0^{2\pi} \tilde{f}(\theta) d\theta = 0$. Moreover, the smoothness of $f(\theta)$ can be relaxed, but we don’t pursue this issue here.

Now we present a sketch of the proof and some key ideas.

**Step 1: Construction of the approximate solution.** We construct an approximate solution $(u^a, v^a)$ by matched asymptotic expansion. The approximate solution consists of the Euler part $(u^a_e, v^a_e)$ and the Prandtl part $(u^a_p, v^a_p)$, and satisfies the following estimates
\[
|\partial_\theta u^a_e(\theta, r) + v^a_e(\theta, r)| \leq C\varepsilon r, \quad |\partial_\theta v^a_e(\theta, r) - u^a_e(\theta, r) + ar| \leq C\varepsilon r
\]
which will be used frequently in the error estimate. The details of constructing the approximate solution will be given in Sect. 2. After the construction of approximate solution, we derive the Eq. (3.1) for the error $(u, v) := (u^e - u^a, v^e - v^a)$, then establish the well-posedness of (3.1). Noticing that the nonlinear term can be easily handled by higher order approximation, hence we only need to consider the linearized error equations:
\[
\begin{cases}
\begin{aligned}
u u_\theta + \nu u_r u_r + uu_\theta + \nu u^a_\theta + v^a u + uu^a + p^a - \varepsilon^2\left(r u_{rr} + \frac{u_{\theta\theta}}{r} + 2 \frac{\nu}{\varepsilon} + u_r - \frac{u}{r}\right) = F_u,
\nu v_\theta + \nu v_r v_r + uv_\theta + \nu v^a_\theta - 2uu^a + rp^a - \varepsilon^2\left(r v_{rr} + \frac{v_{\theta\theta}}{r} - 2 \frac{u_{\theta}}{r} + v_r - \frac{v}{r}\right) = F_v,
\nu u_\theta + (rv)_r = 0,
\nu (\theta + 2\pi, r) = u(\theta, r), \quad v(\theta + 2\pi, r) = v(\theta, r),
\nu(\theta, 1) = 0, \quad v(\theta, 1) = 0.
\end{aligned}
\end{cases}
\]

**Step 2: Linear stability estimate for (1.7).** Equations (1.7) are the linearized Navier–Stokes equations around the approximate solution $(u^a, v^a)$. The leading order of $(u^a, v^a)$ is $(ar + \chi(r)u_p^0(\theta, r-1, \varepsilon), 0)$, where $\chi(r)$ is a cut-off function, see (2.59). Since $|u^a_p| \leq |u_p^0| \lesssim \eta$, the leading order of the system (1.7) can be simplified as
\[
\begin{cases}
\begin{aligned}
u u_\theta + \nu uu_\theta + uu + p^a - \varepsilon^2(\nu u_{rr} + \frac{u_{\theta\theta}}{r} + 2 \frac{\nu}{\varepsilon} + u_r - \frac{u}{r}) = F_u,
u v_\theta - 2uu^a + rp^a - \varepsilon^2(r v_{rr} + \frac{v_{\theta\theta}}{r} - 2 \frac{u_{\theta}}{r} + v_r - \frac{v}{r}) = F_v,
u u_\theta + (rv)_r = 0,\nu (\theta + 2\pi, r) = u(\theta, r), \quad v(\theta + 2\pi, r) = v(\theta, r),\nu(\theta, 1) = 0, \quad v(\theta, 1) = 0.
\end{aligned}
\end{cases}
\]

where $u^a$ can be regarded as $ar + \chi(r)u_p^0(\theta, r-1, \varepsilon)$. The linear stability estimates consist of a basic energy estimate and a positivity estimate. In fact, it is easy to know that the basic energy estimate is not good enough for obtaining a priori estimate of (1.8) because $\varepsilon$ is small. The key point is the following observation which gives the important positivity estimate: $u^a$ is strictly positive, and we shall make use of the terms $u^a u_\theta$ and $u^a v_\theta$ to obtain a positive quantity from the convective term. To do this, we choose $(u_\theta, v_\theta)$ as the multiplier, and the pressure is
eliminated due to the divergence-free condition and the diffusion terms also vanish. It is easy to obtain
\[
\begin{align*}
&\int_0^{2\pi} \int_0^1 (a^2 u_\theta^2 + r v u_r^a + v u^a) u_\theta d\theta dr + \int_0^{2\pi} (u^a v_\theta - 2 uu^a) v_\theta d\theta dr \\
&= \int_0^{2\pi} \left( \int_0^1 (a u_\theta^2 + 2arv) u_\theta d\theta dr + \int_0^{2\pi} (arv_\theta - 2arv) v_\theta d\theta dr \right) \\
&\quad + \int_0^{2\pi} \left( \chi(r) u_p^{(0)} u_\theta + vr\partial_r (\chi(r) u_p^{(0)}) + v\chi(r) u_p^{(0)} u_\theta d\theta dr + \int_0^{2\pi} (\chi(r) v_\theta - 2u\chi(r) u_p^{(0)}) v_\theta d\theta dr \right).
\end{align*}
\]

It is easy to get
\[
I = a \int_0^1 \int_0^{2\pi} r (u^2_\theta + v^2_\theta) d\theta dr.
\]

Moreover, since \( \chi(r) = 0 \) for \( r < \frac{1}{2} \), by the Hardy inequality, we deduce that
\[
\left| \int_0^{2\pi} \int_0^1 vr\partial_r (\chi(r) u_p^{(0)}) u_\theta d\theta dr \right|
\leq \left| \int_0^{2\pi} \int_0^1 vr\chi'(r) u_p^{(0)} u_\theta d\theta dr \right| + \left| \int_0^{2\pi} \int_0^1 vr \chi(r) \partial_r u_p^{(0)} u_\theta d\theta dr \right|
\leq C\eta \int_0^{2\pi} \int_0^1 r (u^2_\theta + v^2_\theta) d\theta dr
\]
and
\[
\left| \int_0^{2\pi} u\chi(r) u_p^{(0)} v_\theta d\theta dr \right|
\leq \varepsilon \left( \int_0^{2\pi} \int_0^1 \frac{u}{r-1} \chi(r) v_\theta d\theta dr \right) \leq C\varepsilon\eta \left( \int_0^{2\pi} \int_0^1 r u^2_r d\theta dr \right)^{\frac{1}{2}} \left( \int_0^{2\pi} \int_0^1 r v^2_\theta d\theta dr \right)^{\frac{1}{2}}.
\]

The other terms in \( II \) can be handled by the same argument. Thus, we obtain
\[
\begin{align*}
&\int_0^{2\pi} \int_0^1 (a^2 u_\theta^2 + r v u_r^a + v u^a) u_\theta d\theta dr + \int_0^{2\pi} (u^a v_\theta - 2 uu^a) v_\theta d\theta dr \\
&\geq (a - C\eta) \int_0^{2\pi} \int_0^1 r (u^2_\theta + v^2_\theta) d\theta dr - C\varepsilon\eta \int_0^{2\pi} \int_0^1 r u^2_r d\theta dr.
\end{align*}
\]

When \( \eta \) is sufficiently small, \((a - C\eta) \int_0^{2\pi} r (u^2_\theta + v^2_\theta) d\theta dr \) is a positive quantity. The details of the positivity estimate can be found in Lemma 3.3.

However, the positive quantity \( \| \sqrt{r}(u_\theta, v_\theta) \|_2 \) does not contain the zero-frequency term of \((u, v)\), which is needed to obtain the \( L^\infty \) estimate for the error \((u, v)\). Noticing that \( \int_0^{2\pi} v(\theta, r) d\theta = 0 \) because of the divergence-free condition and the boundary condition, we can control \( \|v\|_2 \) by \( \|v_\theta\|_2 \) by using the Poincaré inequality. However,
since \( u_0(r) := \frac{1}{2\pi} \int_0^{2\pi} u(\theta, r) d\theta \neq 0 \), we need to obtain the estimate of \(|u_0(r)|\) from the basic energy estimate.

Now we give a sketch of the basic energy estimate. We choose \( u_0 \) as a multiplier to the first equation in (1.8). The diffusion term is

\[
\int_0^1 \int_0^{2\pi} -\varepsilon^2 (ru_{rr} + \frac{u_{\theta\theta}}{r} + 2\frac{v_{\theta}}{r} + ur - \frac{u}{r})u_0(r) d\theta dr = \varepsilon^2 \int_0^1 \int_0^{2\pi} (|u_0'|^2 + \frac{u_0^2}{r}) d\theta dr.
\]

Recall that \( u^a = ar + \chi(r)u_p^{(0)}(\theta, \frac{r-1}{\varepsilon}) \). It is direct to deduce that

\[
\int_0^1 \int_0^{2\pi} (aru_{\theta} + 2arv + p_{\theta})u_0 d\theta dr = 0,
\]

where we used \( \int_0^{2\pi} v(\theta, r) d\theta = 0 \). Moreover, the Prandtl part can be handled by the Hardy inequality as above

\[
\int_0^1 \int_0^{2\pi} \left( u_p^{(0)}u_{\theta} + vr \psi_r u_p^{(0)} + vu_p^{(0)} \right) u_0 d\theta dr \approx \int_0^1 \int_0^{2\pi} \varepsilon \chi(r) \left( Y u_p^{(0)}u_{\theta} + \frac{vr}{r-1} Y^2 \psi Y u_p^{(0)} + vY u_p^{(0)} \right) \frac{u_0}{r-1} d\theta dr 
\lesssim \eta \varepsilon \| \sqrt{r}(u_{\theta}, v_{\theta}) \|_2 \| \sqrt{r}u_0' \|_2.
\]

Thus, we obtain that \( \varepsilon^2 \| \sqrt{r}u_0' \|_2 \lesssim \| \sqrt{r}(u_{\theta}, v_{\theta}) \|_2 \). The details of the basic energy estimate can be found in Lemma 3.2.

Combining the positivity estimate and basic energy estimate, we obtain the linear stability of Eq. (1.8).

**Step 3: \( H^2 \) estimate for error.** To close the nonlinearity, we need the \( L^\infty \) estimate. However, the quantity \( \| \sqrt{r}(u_{\theta}, v_{\theta}) \|_2 \) cannot be used to control \( \|(u, v)\|_0 \) due to the weight \( r \), and we can only use \( \int_0^1 \int_0^{2\pi} r(u_r^2 + v_r^2) d\theta dr \) which comes from the diffusion term. To do this, we first rewrite the error equations and the associated linear stability estimate in Euler coordinates, then get the \( H^2 \) estimate by Stokes estimates in a smooth bounded domain. Finally \( \|(u, v)\|_0 \) can be obtained by Sobolev embedding.

By combining the linear stability estimate, \( H^2 \) estimate and Sobolev embedding, we can establish the well-posedness of the error equations by contraction mapping theorem.

The paper is organized as follows. In Sect. 2, we construct an approximate solution by matched asymptotic expansion and study the properties of this approximate solution. In Sect. 3, we derive the error equations and establish their linear stability estimates including the basic energy estimate and the positivity estimate. In Sect. 4, we rewrite the error equations and the associated linear stability estimate in Euler coordinates, then obtain \( H^2 \) estimate by the Stokes estimate in a smooth bounded domain. In Sect. 5, we complete the Proof of Theorem 1.1 by combining the linear stability estimate and \( H^2 \) estimate.

**2. Construction of Approximate Solutions**

In this section, we construct an approximate solution of the Navier–Stokes Eq. (1.3) by matched asymptotic expansion.
2.1. Euler expansions. Away from the boundary, we make the following formal expansions

\[
\begin{align*}
    u^\varepsilon(\theta, r) &= u_e^{(0)}(\theta, r) + \varepsilon u_e^{(1)}(\theta, r) + \cdots, \\
    v^\varepsilon(\theta, r) &= v_e^{(0)}(\theta, r) + \varepsilon v_e^{(1)}(\theta, r) + \cdots, \\
    p^\varepsilon(\theta, r) &= p_e^{(0)}(\theta, r) + \varepsilon p_e^{(1)}(\theta, r) + \cdots.
\end{align*}
\]

2.1.1. Equations for \((u_e^{(0)}, v_e^{(0)}, p_e^{(0)})\) By substituting the above expansions into (1.3) and collecting the \(\varepsilon\)-zeroth order terms, we deduce that \((u_e^{(0)}, v_e^{(0)}, p_e^{(0)})\) satisfies the following steady nonlinear Euler equations

\[
\begin{align*}
    u_e^{(0)} \partial_\theta u_e^{(0)} + r v_e^{(0)} \partial_r u_e^{(0)} + u_e^{(0)} v_e^{(0)} + \partial_\theta p_e^{(0)} &= 0, \\
    u_e^{(0)} \partial_\theta v_e^{(0)} + r v_e^{(0)} \partial_r v_e^{(0)} - (u_e^{(0)})^2 + r \partial_r p_e^{(0)} &= 0, \\
    \partial_\theta u_e^{(0)} + \partial_r v_e^{(0)} + v_e^{(0)} &= 0.
\end{align*}
\]

Due to the Prandtl–Batchelor theory in [1], the leading order Euler flows \((u_e^{(0)}, v_e^{(0)})\) with closed streamlines must be the Couette flow in \(\Omega\)

\[
u_e^{(0)}(\theta, r) = u_e(r) =: ar, \quad v_e^{(0)}(\theta, r) = 0,
\]

where the constant \(a\) is determined by the Wood formula (2.11).

Then Eq. (2.1) reduce to

\[
\partial_\theta p_e^{(0)}(\theta, r) = 0, \quad \partial_r p_e^{(0)}(\theta, r) = \frac{1}{r} u_e^2.
\]

Thus, we deduce that

\[
p_e^{(0)}(\theta, r) = p_e(r), \quad p_e'(r) = a^2 r.
\]

2.1.2. Equations for \((u_e^{(1)}, v_e^{(1)}, p_e^{(1)})\)

By collecting the \(\varepsilon\)-order terms, we deduce that \((u_e^{(1)}, v_e^{(1)}, p_e^{(1)})\) satisfies the following linearized Euler equations in \(\Omega\)

\[
\begin{align*}
    ar \partial_\theta u_e^{(1)} + 2ar v_e^{(1)} + \partial_\theta p_e^{(1)} &= 0, \\
    ar \partial_\theta v_e^{(1)} - 2ar u_e^{(1)} + r \partial_r p_e^{(1)} &= 0, \\
    \partial_\theta u_e^{(1)} + \partial_r v_e^{(1)} + v_e^{(1)} &= 0,
\end{align*}
\]

with the boundary conditions

\[
v_e^{(1)}|_{r=1} = -v_p^{(1)}|_{y=0}, \quad v_e^{(1)}(\theta, r) = v_e^{(1)}(\theta + 2\pi, r),
\]

where \(v_p^{(1)}\) is the solution of Prandtl equations which will be derived in the next subsection.

The equations for \((u_e^{(i)}, v_e^{(i)}, p_e^{(i)}), i = 2, 3, 4\) will be derived later.
2.2. Prandtl expansion near $r = 1$.

We introduce the scaled variable $Y = \frac{r-1}{\varepsilon} \in (-\infty, 0]$ and make the following Prandtl expansions near $r = 1$

$$
u^\varepsilon = u_e(r) + u_p^{(0)}(\theta, Y) + \varepsilon[u_e^{(1)}(\theta, r) + u_p^{(1)}(\theta, Y)] + \cdots,$$

$$v^\varepsilon = v_p^{(0)}(\theta, Y) + \varepsilon[v_e^{(1)}(\theta, r) + v_p^{(1)}(\theta, Y)] + \varepsilon^2[v_e^{(2)}(\theta, r) + v_p^{(2)}(\theta, Y)] + \cdots,$$

$$p^\varepsilon = p_e(r) + p_p^{(0)}(\theta, Y) + \varepsilon[p_e^{(1)}(\theta, r) + p_p^{(1)}(\theta, Y)] + \varepsilon[p_e^{(2)}(\theta, r) + p_p^{(2)}(\theta, Y)] + \cdots, \quad (2.6)$$

where as $Y \to -\infty$

$$\partial_\theta^l \partial_Y^m v_p^{(i)}(\theta, Y) \to 0, \quad \partial_\theta^l \partial_Y^m p_p^{(i)}(\theta, Y) \to 0, \quad (2.7)$$

with $l, m \geq 0, i = 0, 1, \ldots$, which satisfy the following boundary conditions

$$u_e^{(0)}(\theta, 1) + u_p^{(0)}(\theta, 0) = \alpha + \eta f(\theta), \quad u_e^{(i)}(\theta, 1) + u_p^{(i)}(\theta, 0) = 0, \quad i \geq 1,$$

$$v_e^{(i)}(\theta, 1) + v_p^{(i)}(\theta, 0) = 0, \quad i \geq 0.$$

The boundary conditions of $u_p^{(i)}(\theta, Y)$ as $Y \to -\infty$ will be given later.

2.2.1. Equations for $(v_p^{(0)}, p_p^{(0)})$

By substituting the above expansions into (1.3) and collecting the $\frac{1}{\varepsilon}$ order terms, we get

$$\partial_Y v_p^{(0)}(\theta, Y) = 0, \quad \partial_Y p_p^{(0)}(\theta, Y) = 0,$$

which together with (2.7) imply

$$v_p^{(0)} = 0, \quad p_p^{(0)} = 0.$$

2.2.2. Equations for $(u_p^{(0)}, v_p^{(1)}, p_p^{(1)})$

By substituting the above expansions into (1.3) and collecting the $\varepsilon$-zeroth order terms, we obtain the following steady Prandtl equations for $(u_p^{(0)}, v_p^{(1)})$

$$
\begin{cases}
(u_e(1) + u_p^{(0)}) \partial_\theta u_p^{(0)} + (v_e^{(1)}(\theta, 1) + v_p^{(1)}(\theta, 0)) \partial_Y u_p^{(0)} - \partial_Y u_p^{(0)} = 0, \\
\partial_\theta u_p^{(0)} + \partial_Y v_p^{(1)} = 0, \\
u_p^{(0)}(\theta, Y) = u_p^{(0)}(\theta + 2\pi, Y), \quad v_p^{(1)}(\theta, Y) = v_p^{(1)}(\theta + 2\pi, Y), \\
u_p^{(0)}|_{Y=0} = \alpha + \eta f(\theta) - u_e(1), \quad \lim_{Y \to -\infty} (u_p^{(0)}, v_p^{(1)}) = (0, 0)
\end{cases} \quad (2.8)
$$

and the pressure $p_p^{(1)}$ satisfies

$$\partial_Y p_p^{(1)}(\theta, Y) = (u_p^{(0)})^2(\theta, Y) + 2u_e(1)u_p^{(0)}(\theta, Y), \quad \lim_{Y \to -\infty} p_p^{(1)}(\theta, Y) = 0. \quad (2.9)$$
2.2.3. Equations for \((u_p^{(1)}, v_p^{(2)})\)

By substituting the above expansions into the first and third equation in (1.3) and collecting the \(\varepsilon\)-order terms, we obtain the following linearized steady Prandtl equations for \((u_p^{(1)}, v_p^{(2)})\)

\[
\begin{aligned}
&\left(\frac{u_e(1) + u_p^{(0)}}{\partial u_p^{(1)}} + (v_e^{(1)}(\theta, 1) + v_p^{(1)})\partial_y u_p^{(1)} + (v_e^{(2)}(\theta, 1) + v_p^{(2)})\partial_y u_p^{(0)}
\right. \\
&+ (u_e^{(1)} + u_e^{(1)}(\theta, 1))\partial_y u_p^{(0)} - \partial_y Y u_p^{(1)} = f_1(\theta, Y), \\
&\partial_y u_p^{(1)} + \partial_y v_p^{(2)} + \partial_y (Y v_p^{(1)}) = 0, \\
&u_p^{(1)}(\theta, Y) = u_p^{(1)}(\theta + 2\pi, Y), \quad v_p^{(2)}(\theta, Y) = v_p^{(2)}(\theta + 2\pi, Y), \\
&u_p^{(1)}|_{Y=0} = -u_e^{(1)}|_{r=1}, \quad \lim_{Y \to -\infty} (\partial_y u_p^{(1)}, v_p^{(2)}) = (0, 0),
\end{aligned}
\]

(2.10)

where

\[
f_1(\theta, Y) = -\partial_\theta p_p^{(1)} + Y \partial_y Y u_p^{(0)} + \partial_y u_p^{(0)} - u_p^{(0)}(\partial_\theta u_e^{(1)}(\theta, 1) + v_e^{(1)}(\theta, 1) + v_p^{(1)}) - u_e^{(1)}(1) Y \partial_\theta u_p^{(0)} - (\partial_r v_e^{(1)}(\theta, 1) + v_e^{(1)}(\theta, 1)) Y \partial_y u_p^{(0)} - (u_e^{(1)}(1) + Y \partial_y u_p^{(0)} + u_e^{(1)}(1)) v_p^{(1)}.
\]

The equations for \((u_p^{(i)}, v_p^{(i+1)}), i = 2, 3, 4\) will be derived later.

2.3. Solvability of Euler equations and Prandtl equations.

The order in which we solve the equations is as follows

\[
(u_e(r), 0) \to (u_p^{(0)}, v_p^{(1)}) \to (u_e^{(1)}, v_e^{(1)}) \to (u_p^{(1)}, v_p^{(2)}) \to \cdots.
\]

2.3.1. Prandtl equations and their solvability

We first derive a necessary condition for the solvability of Prandtl equations.

**Lemma 2.1** (Batchelor–Wood formula \([20, 21]\)). Let \(\int_0^{2\pi} f(\theta) d\theta = 0\). If the nonlinear Prandtl Eq. (2.8) have a solution \((u_p^{(0)}, v_p^{(1)})\) which satisfies

\[
u_e(1) + u_p^{(0)}(\theta, Y) > 0, \quad \forall Y \leq 0, \quad \|u_p^{(0)}\|_0 \leq M,
\]

where \(M > 0\) is a constant, then there holds

\[
u_e^{(1)}(1) = \alpha^2 + \frac{\eta}{2\pi} \int_0^{2\pi} f^2(\theta) d\theta,
\]

(2.11)

**Proof.** We introduce the von Mises variable

\[
\psi = \int_0^Y (u_e(1) + u_p^{(0)}(\theta, z)) dz, \quad \mathcal{U}^{(0)}(\theta, \psi) = u_e(1) + u_p^{(0)}(\theta, Y).
\]

Then from (2.8), we deduce that \(\mathcal{U}^{(0)}\) satisfies

\[
\begin{aligned}
2\mathcal{U}_\theta^{(0)} &= (\mathcal{U}^{(0)})^2 \psi, \\
\mathcal{U}^{(0)}(\theta, \psi) &= \mathcal{U}^{(0)}(\theta + 2\pi, \psi), \\
\mathcal{U}^{(0)}|_{\psi=0} &= \alpha + \eta f(\theta), \quad \lim_{\psi \to -\infty} \mathcal{U}^{(0)} = u_e(1).
\end{aligned}
\]

(2.12)
Here we have used the facts:
\[ \partial_\theta u^{(0)}_p = U^{(0)}_0 + \mathcal{U}^{(0)}(0, z) \partial_\theta z \]
\[ = U^{(0)}_\theta + \mathcal{U}^{(0)}(v^{(1)}_p(0, \theta) - v^{(1)}_p(0, Y)) = U^{(0)}(v^{(1)}_p(0, 1) + v^{(1)}_p(0, Y)), \]
\[ \partial_Y u^{(0)}_p = \mathcal{U}^{(0)}(0, Y). \]

Integrating the first equation in (2.12) from 0 to 2\(\pi\) about \(\theta\) leads to
\[ \frac{\partial^2}{\partial \psi^2} \int_0^{2\pi} (\mathcal{U}^{(0)}(0, \psi) \theta d\theta = 0. \]

Noticing that \(\mathcal{U}^{(0)}\) is bounded at \(\psi \to -\infty\), we deduce that
\[ \frac{\partial}{\partial \psi} \int_0^{2\pi} (\mathcal{U}^{(0)}(0, \psi) \theta d\theta = 0. \]

Therefore combining the boundary condition in (2.12), we deduce that
\[ u^2_e(1) = \frac{1}{2\pi} \int_0^{2\pi} (\alpha + \eta f(\theta))^2 d\theta = \alpha^2 + \frac{\alpha \eta}{\pi} \int_0^{2\pi} f(\theta) d\theta + \frac{\eta^2}{2\pi} \int_0^{2\pi} f^2(\theta) d\theta \]
\[ = \alpha^2 + \frac{\eta^2}{2\pi} \int_0^{2\pi} f^2(\theta) d\theta. \]

Thus, we complete the proof of this lemma. \(\Box\)

The following Corollary is a direct result of Lemma 2.1.

**Corollary 2.2.** If Lemma 2.1 holds, then
\[ a = \left( \alpha^2 + \frac{\eta^2}{2\pi} \int_0^{2\pi} f^2(\theta) d\theta \right)^{\frac{1}{2}}. \]

Next, we aim to solve the steady Prandtl Eq. (2.8) and we will use the formulation of [21].

**Proposition 2.3.** There exists \(\eta_0 > 0\) such that for any \(\eta \in (0, \eta_0)\), the Eq. (2.12) has a unique solution \(\mathcal{U}^{(0)}\) which satisfies the following estimates
\[ \sum_{j+k \leq m} \int_{-\infty}^{0} \int_0^{2\pi} \left| \frac{\partial^j}{\partial \theta^j} \frac{\partial^k}{\partial \psi^k} (\mathcal{U}^{(0)} - u_e(1)) \right|^2 |\psi|^{2l} d\theta d\psi \leq C(m, l) \eta^2, \quad m, l \geq 0 \]

here \(|\psi| = \sqrt{1 + \psi^2} \).

**Proof.** We use the contraction mapping theorem to prove the desired conclusions and divide the proof into five steps.

**Step 1:** Derivation of equivalent equation.
Let $Q(\theta, \psi) := (\mathcal{U}(0)^2(\theta, \psi) - u_e^2(1)$ and we rewrite (2.12) as
\[
\begin{align*}
Q &= \mathcal{U}(0) Q_{\psi}, \\
Q(\theta, \psi) &= Q(\theta + 2\pi, \psi), \\
Q|_{\psi=0} &= a^2 + 2\alpha \eta f(\theta) + \eta^2 f^2(\theta) - u_e^2(1), \\
Q|_{\psi \to -\infty} &= 0.
\end{align*}
\]
(2.13)

Defining
\[
\mathcal{G}(Q) = Q - 2u_e(1)\sqrt{Q + u_e^2(1) + 2u_e^2(1)},
\]
than (2.13) is equivalent to
\[
\begin{align*}
Q_{\theta} - u_e(1) Q_{\psi} &= (\mathcal{G}(Q))_{\theta}, \\
Q(\theta, \psi) &= Q(\theta + 2\pi, \psi), \\
Q|_{\psi=0} &= a^2 + 2\alpha \eta f(\theta) + \eta^2 f^2(\theta) - u_e^2(1), \\
Q|_{\psi \to -\infty} &= 0.
\end{align*}
\]
(2.14)

Let $Q_0$ be the solution to
\[
\begin{align*}
Q_{\theta} - u_e(1) Q_{\psi} &= (\mathcal{H}(Q))_{\theta}, \\
Q(\theta, \psi) &= Q(\theta + 2\pi, \psi), \\
Q|_{\psi=0} &= a^2 + 2\alpha \eta f(\theta) + \eta^2 f^2(\theta) - u_e^2(1), \\
Q|_{\psi \to -\infty} &= 0,
\end{align*}
\]
(2.15)

which will be solved in Appendix A, then (2.14) is equivalent to
\[
\begin{align*}
Q_{\theta} - u_e(1) Q_{\psi} &= (\mathcal{H}(Q))_{\theta}, \\
Q(\theta, \psi) &= Q(\theta + 2\pi, \psi), \\
Q|_{\psi=0} = 0, \\
Q|_{\psi \to -\infty} = 0,
\end{align*}
\]
(2.16)

where
\[
\mathcal{H}(Q) = Q + Q_0 - 2u_e(1)\sqrt{Q + Q_0 + u_e^2(1) + 2u_e^2(1)}.
\]

Defining the linear operator $L : \Lambda \mapsto \Phi$ such that
\[
\begin{align*}
\Phi &= L\Lambda \iff \\
\Phi_{\theta} - u_e(1) \Phi_{\psi} &= \Lambda_{\theta}, \\
\Phi(\theta, \psi) &= \Phi(\theta + 2\pi, \psi), \\
\Phi|_{\psi=0} = 0, \\
\Phi|_{\psi \to -\infty} = 0,
\end{align*}
\]
(2.17)

then (2.16) is equivalent to
\[
\begin{align*}
Q &= L(\mathcal{H}(Q)), \\
Q(\theta, \psi) &= Q(\theta + 2\pi, \psi), \\
Q|_{\psi=0} = 0, \\
Q|_{\psi \to -\infty} = 0.
\end{align*}
\]
(2.18)

Defining the function space $X$ as follows
\[
X = \left\{ Q : Q(\theta, \psi) = Q(\theta + 2\pi, \psi), \; Q|_{\psi=0} = Q|_{\psi \to -\infty} = 0, \right. \\
\|Q\|_X^2 := \sum_{j+k \leq m, l \geq 0} \int_{-\infty}^{0} \int_{0}^{2\pi} |\partial_\theta^j \partial_\psi^k Q|_{\psi=0}^2 d\theta d\psi < +\infty \right\}
\]
and a ball $B_0$ in $X$

$$B_0 = \{ Q \in X : \| Q \|_X \leq r_0 \},$$

where $r_0$ is a small number which will be determined later, $m$ is a positive integer.

In the next three steps, we will verify that $L \circ H$ is a contraction map from $B_0$ to $B_0$ with a suitable small $r_0$.

**Step 2: Boundedness of $L$ in $X$.**

In this step, we prove that for any $m \geq 0$, $l \geq 0$, there holds

$$\sum_{j+k \leq m} \left\| \partial_\theta^j \partial_\psi^k \Phi \psi^l \right\|_2 \leq C(m, l) \sum_{j+k \leq m, q \leq l} \left\| \partial_\theta^j \partial_\psi^k \Lambda \psi^q \right\|_2. (2.19)$$

First, we prove (2.19) for $l = 0$. Multiplying the equation in (2.17) by $\Phi$ and integrating with respect to $(\theta, \psi) \in (0, 2\pi) \times (-\infty, 0)$, we obtain

$$u_\epsilon(1) \| \Phi \psi \|_2 \leq C(\lambda) \| \Lambda \|_2 + \lambda \| \Phi \theta \|_2. (2.20)$$

Then, multiplying the equation in (2.17) by $\Phi_\theta$ and integrating with respect to $(\theta, \psi) \in (0, 2\pi) \times (-\infty, 0)$, one has

$$\| \Phi_\theta \|_2 \leq C(\lambda) \| \Lambda_\theta \|_2 + \lambda \| \Phi_\theta \|_2. (2.21)$$

Combining (2.20)–(2.21) and choosing small $\lambda > 0$, we get

$$\| (\Phi_\psi, \Phi_\theta) \|_2 \leq C(\Lambda, \Lambda_\theta) \|_2. (2.22)$$

Integrating (2.17) with respect to $\theta \in (0, 2\pi)$ gives

$$\frac{d^2}{d\psi^2} \int_0^{2\pi} \Phi(\theta, \psi) d\theta = 0,$$

which combines with $\Phi|_{\psi=0} = \Phi|_{\psi=-\infty} = 0$ implies

$$\int_0^{2\pi} \Phi(\theta, \psi) d\theta = 0. (2.23)$$

Due to (2.23) and the Poincaré inequality, we have

$$\| \Phi \|_2 \leq C \| \Phi_\theta \|_2 \leq C \| (\Lambda, \Lambda_\theta) \|_2. (2.24)$$

For any $j \geq 0, k \geq 2$, from the Eq. (2.17), we deduce that

$$\| \partial_\theta^j \partial_\psi^k \Phi \|_2 \leq C(\| \partial_\theta^j \partial_\psi^{k-2} \Phi_\theta \|_2 + \| \partial_\theta^j \partial_\psi^{k-2} \Lambda_\theta \|_2). (2.25)$$

For any $j \geq 0$, applying $\partial_\theta^j$ to the equation in (2.17), multiplying the resultant equation by $\partial_\theta^j \Phi$, integrating with respect to $(\theta, \psi) \in (0, 2\pi) \times (-\infty, 0)$ and using the Young inequality one has

$$\| \partial_\theta^j \partial_\psi \Phi \|_2 \leq C \| \partial_\theta^{j+1} \Lambda \|_2 + C \| \partial_\theta^j \Phi \|_2. (2.26)$$
For any $j \geq 0$, multiplying the equation in (2.17) by $\partial^{2j-1}_\theta \Phi$, integrating with respect to $(\theta, \psi) \in (0, 2\pi) \times (-\infty, 0)$, one can get

$$\|\partial^j_\theta \Phi\|_2^2 = \int_{-\infty}^{0} \int_0^{2\pi} \partial^{2j-1}_\theta \Phi \Lambda_\theta d\theta d\psi \leq C \|\partial^j_\theta \Phi\|_2 \|\partial^j_\theta \Lambda\|_2. \quad (2.27)$$

Combining (2.25), (2.26) and (2.27), we obtain that for any $m \geq 2$, there holds

$$\sum_{j+k=m} \|\partial^j_\theta \partial^k_\psi \Phi\|_2 \leq C \sum_{j+k \leq m-1} \|\partial^j_\theta \partial^k_\psi \Phi\|_2 + C \sum_{j+k \leq m} \|\partial^j_\theta \partial^k_\psi \Lambda\|_2.$$  

Thus, by induction and (2.22), (2.24), we deduce that for any $m \geq 0$, there holds

$$\sum_{j+k \leq m} \|\partial^j_\theta \partial^k_\psi \Phi\|_2 \leq C(m) \sum_{j+k \leq m} \|\partial^j_\theta \partial^k_\psi \Lambda\|_2. \quad (2.28)$$

Next, we prove (2.19) for $m \leq 1, l \geq 1$. Multiplying the equation in (2.17) by $\Phi^2 l$, integrating with respect to $(\theta, \psi) \in (0, 2\pi) \times (-\infty, 0)$, we have

$$\int_{-\infty}^{0} \int_0^{2\pi} \Phi^2 \psi^{2l} d\theta d\psi \leq C \int_{-\infty}^{0} \int_0^{2\pi} \Phi^2 \psi^{2(l-1)} d\theta d\psi + C(\lambda) \int_{-\infty}^{0} \int_0^{2\pi} \Lambda^2 \psi^{2l} d\theta d\psi + \lambda \int_{-\infty}^{0} \int_0^{2\pi} \Phi^2 \psi^{2l} d\theta d\psi. \quad (2.29)$$

Multiplying the equation in (2.17) by $\Phi^2 \psi^2 l$, integrating with respect to $(\theta, \psi) \in (0, 2\pi) \times (-\infty, 0)$, we obtain

$$\int_{-\infty}^{0} \int_0^{2\pi} \Phi^2 \psi^{2l} d\theta d\psi \leq C \int_{-\infty}^{0} \int_0^{2\pi} \Phi^2 \psi^{2(l-1)} d\theta d\psi + C \int_{-\infty}^{0} \int_0^{2\pi} \Lambda^2 \psi^{2l} d\theta d\psi. \quad (2.30)$$

Combining (2.29)–(2.30), using the Poincaré inequality and choosing small $\lambda > 0$, we get

$$\int_{-\infty}^{0} \int_0^{2\pi} (\Phi^2 + \Phi^2 \psi + \Phi^2_\theta) \psi^{2l} d\theta d\psi$$

$$\leq \int_{-\infty}^{0} \int_0^{2\pi} (\Phi^2 + \Phi^2 \psi + \Phi^2_\theta) \psi^{2(l-1)} d\theta d\psi + C \int_{-\infty}^{0} \int_0^{2\pi} (\Lambda^2 \psi^{2l} + \Lambda^2 \psi^{2l}) d\theta d\psi.$$  

By induction on $l$ and using (2.28), we deduce that for any $l \geq 1$, there holds

$$\| (\Phi \psi, \Phi \theta, \Phi) \psi^l \|_2 \leq C(l) \sum_{q \leq l} \| (\Lambda, \Lambda \theta) \psi^q \|_2. \quad (2.31)$$

Finally, we prove (2.19) for any $m \geq 2, l \geq 1$. When $j + k = m, k \geq 2$, from the Eq. (2.17), we deduce that

$$\|\partial^j_\theta \partial^k_\psi \Phi \psi^l\|_2 \leq C \left( \|\partial^j_\theta \partial^{k-2}_\psi \partial \Phi \psi^l\|_2 + \|\partial^j_\theta \partial^{k-2}_\psi \partial \Lambda \psi^l\|_2 \right). \quad (2.32)$$
Applying $\partial_{\theta}^{m-1}$ to the equation in (2.17), multiplying the resultant equation by $\partial_{\theta}^{2m-1} \Phi \psi^{2l}$, integrating with respect to $(\theta, \psi) \in (0, 2\pi) \times (-\infty, 0)$ and using the Young inequality, one has

$$
\|\partial_{\theta}^{m-1} \partial_{\psi} \Phi \psi^{l}\|_2 \leq C \|\partial_{\theta}^{m} \Lambda \psi^{l}\|_2 + C \|\partial_{\theta}^{m-1} \Phi \psi^{l}\|_2 + C(l) \|\partial_{\theta}^{m-1} \Phi \psi^{l-1}\|_2. \tag{2.33}
$$

Multiplying the equation in (2.17) by $\partial_{\theta}^{2m-1} \Phi \psi^{2l}$, integrating with respect to $(\theta, \psi) \in (0, 2\pi) \times (-\infty, 0)$, one can get

$$
\|\partial_{\theta}^{m} \Phi \psi^{l}\|_2 \leq C \|\partial_{\theta}^{m} \Lambda \psi^{l}\|_2 + C(l) \|\partial_{\theta}^{m-1} \partial_{\psi} \Phi \psi^{l}\|_2. \tag{2.34}
$$

Combining (2.32), (2.33) and (2.34), we obtain that for any $m \geq 2, l \geq 1$, there holds

$$
\sum_{j+k=m} \|\partial_{\theta}^{j} \partial_{\psi}^{k} \Phi \psi^{l}\|_2 \leq C(m, l) \sum_{j+k\leq m-1, q \leq l} \|\partial_{\theta}^{j} \partial_{\psi}^{k} \Phi \psi^{q}\|_2 + C(m, l) \sum_{j+k \leq m} \|\partial_{\theta}^{j} \partial_{\psi}^{k} \Lambda \psi^{l}\|_2. \tag{2.35}
$$

Thus, by induction on $m$ and using (2.31), we deduce that for any $m \geq 2, l \geq 1$, there holds

$$
\sum_{j+k \leq m} \|\partial_{\theta}^{j} \partial_{\psi}^{k} \Phi \psi^{l}\|_2 \leq C(m, l) \sum_{j+k \leq m, q \leq l} \|\partial_{\theta}^{j} \partial_{\psi}^{k} \Lambda \psi^{q}\|_2. \tag{2.36}
$$

This completes the proof of (2.19). Thus, we obtain

$$
\|\mathcal{L} \Lambda\|_X \leq C \|\Lambda\|_X, \quad \forall \Lambda \in X. \tag{2.37}
$$

**Step 3:** $\mathcal{L} \circ \mathcal{H}$ is a continuous map from $B_0$ to $B_0$.

In this section, we first prove that for any $m \geq 2, l \geq 0$, there holds

$$
\sum_{j+k \leq m} \|\partial_{\theta}^{j} \partial_{\psi}^{k} \mathcal{H}(Q)\psi^{l}\|_2 \leq C(m, l) \left( \sum_{j+k \leq m} \|\partial_{\theta}^{j} \partial_{\psi}^{k} Q\psi^{l}\|_2 + \sum_{j+k \leq m} \|\partial_{\theta}^{j} \partial_{\psi}^{k} Q_0\psi^{l}\|_2 \right)^2. \tag{2.38}
$$

Set

$$
\tilde{H}(x) = x - 2u_e(1)\sqrt{x + u_e^2(1) + 2u_e^2(1)}, \quad |x| \ll u_e^2(1),
$$

then $\mathcal{H}(Q) = \tilde{H}(Q + Q_0)$. Direct computation gives

$$
|\tilde{H}'(x)| \leq C|x|, \quad |\tilde{H}^{(k)}(x)| \leq C, \quad k \geq 2. \tag{2.39}
$$

Since

$$
\mathcal{H}(Q) = \left( \frac{Q + Q_0 + u_e^2(1) - u_e(1)}{\sqrt{Q + Q_0 + u_e^2(1) + u_e(1)}} \right)^2 = \left( \frac{Q + Q_0}{\sqrt{Q + Q_0 + u_e^2(1) + u_e(1)}} \right)^2,
$$

it is easy to get

$$
\mathcal{H}^2(Q)\psi^{l}\| \leq C \|Q + Q_0\|_L^2 \left( Q^2\psi^{l} + Q_0^2\psi^{l} \right), \quad l \geq 0.
$$
Using (2.38), we deduce that

\[
(\partial_{\psi} \mathcal{H}(Q))^2|_{\psi = 0} \leq C \| Q + Q_0 \|_{L^2}^2 \left( \| Q_\psi \|_{L^\infty}^2 + (Q_0)_{\psi}^2 \right), \quad l \geq 0.
\]

\[
(\partial_{\theta} \mathcal{H}(Q))^2|_{\psi = 0} \leq C \| Q + Q_0 \|_{L^2}^2 \left( \| Q_{\theta} \|_{L^\infty}^2 + (Q_0)_{\theta}^2 \right), \quad l \geq 0.
\]

Thus, we obtain

\[
\sum_{j+k \leq 1} \left\| \partial_{\theta}^j \partial_{\psi}^k \mathcal{H}(Q) \right\|_2 \leq C \| Q + Q_0 \|_{L^\infty} \left( \sum_{j+k \leq 1} \left\| \partial_{\theta}^j \partial_{\psi}^k Q \right\|_2 + \sum_{j+k \leq 1} \left\| \partial_{\theta}^j \partial_{\psi}^k Q_0 \right\|_2 \right). \tag{2.39}
\]

Since

\[
\partial_{\theta} \mathcal{H}(Q) = \tilde{H}'(Q + Q_0)(\partial_{\theta}Q + \partial_{\theta}Q_0) + \tilde{H}''(Q + Q_0)(\partial_{\theta}Q + \partial_{\theta}Q_0)(\partial_{\theta}Q + \partial_{\theta}Q_0),
\]

by using (2.38), we obtain

\[
\left\| \partial_{\theta} \mathcal{H}(Q) \right\|_{L^2} \leq C \| Q + Q_0 \|_{L^\infty} \left( \left\| Q_{\theta} \right\|_{L^2} + \left\| (Q_0)_{\theta} \right\|_{L^2} \right) \]

\[
+ C \| Q_{\psi} + (Q_0)_{\psi} \|_{L^4} \left( \left\| Q_{\theta} \right\|_{L^4} + \left\| (Q_0)_{\theta} \right\|_{L^4} \right) \]

\[
\leq C \| Q + Q_0 \|_{L^\infty} \left( \left\| Q_{\theta} \right\|_{L^2} + \left\| (Q_0)_{\theta} \right\|_{L^2} \right) \]

\[
+ C \| Q_{\psi} + (Q_0)_{\psi} \|_{H^1} \left( \left\| (Q_0)_{\theta} \right\|_{H^1} \right) .
\]

Hence by the Sobolev embedding, we deduce that

\[
\left\| \partial_{\theta} \mathcal{H}(Q) \right\|_2 \leq C \left( \sum_{j+k \leq 2} \left\| \partial_{\theta}^j \partial_{\psi}^k Q \right\|_2 + \sum_{j+k \leq 2} \left\| \partial_{\theta}^j \partial_{\psi}^k Q_0 \right\|_2 \right)^2 .
\]

Same estimates hold for \( \left\| \partial_{\theta} \mathcal{H}(Q) \right\|_2 \) and \( \left\| \partial_{\psi} \mathcal{H}(Q) \right\|_2 \). Thus, combining the estimate (2.39), we arrive at

\[
\sum_{j+k \leq 2} \left\| \partial_{\theta}^j \partial_{\psi}^k \mathcal{H}(Q) \right\|_2 \leq C \left( \sum_{j+k \leq 2} \left\| \partial_{\theta}^j \partial_{\psi}^k Q \right\|_2 + \sum_{j+k \leq 2} \left\| \partial_{\theta}^j \partial_{\psi}^k Q_0 \right\|_2 \right)^2 .
\]

Using (2.38) and repeating the above argument, we obtain (2.37).

Consequently, if we take \( r_0 = \| Q_0 \|_X \) and \( \eta \) small enough, then

\[
\| \mathcal{L} \|_X \leq C \| \mathcal{H}(Q) \|_X \leq 4Cr_0^2 \leq 4C\eta r_0 \leq r_0, \quad \forall Q \in B_0,
\]

where we have used (6.2). Thus, \( \mathcal{L} \mathcal{H} \) is a continuous map from \( B_0 \) to \( B_0 \) for small \( \eta \).
Step 4: $\mathcal{L} \circ \mathcal{H}$ is a contraction map in $B_0$.

First, we note that

$$
\mathcal{H}(Q_1) - \mathcal{H}(Q_2) = (Q_1 - Q_2) \left( 1 - \frac{2u_e(1)}{\sqrt{Q_1 + Q_0 + u_e^2(1)} + \sqrt{Q_2 + Q_0 + u_e^2(1)}} \right)
$$

and

$$
1 - \frac{2u_e(1)}{\sqrt{Q_1 + Q_0 + u_e^2(1)} + \sqrt{Q_2 + Q_0 + u_e^2(1)}} = \frac{Q_1 + Q_0}{\sqrt{Q_1 + Q_0 + u_e^2(1)} + \sqrt{Q_2 + Q_0 + u_e^2(1)}} \left( \frac{Q_2 + Q_0}{\sqrt{Q_1 + Q_0 + u_e^2(1)} + \sqrt{Q_2 + Q_0 + u_e^2(1)}} \right).
$$

By the similar arguments as in Step 3, we can obtain that there exist $\eta_0 > 0$ such that for any $\theta, \psi \in (0, \eta_0)$, there holds

$$\|\mathcal{L}\mathcal{H}(Q_1) - \mathcal{L}\mathcal{H}(Q_2)\|_X \leq C \|\mathcal{H}(Q_1) - \mathcal{H}(Q_2)\|_X \leq C \|Q_1 - Q_2\|_X (\|Q_0\|_X + \|Q_1\|_X + \|Q_2\|_X) \leq C \|Q_0\|_X \|Q_1 - Q_2\|_X \leq \frac{1}{2} \|Q_1 - Q_2\|_X.$$

That is, $\mathcal{L}\mathcal{H}$ is a contraction mapping in $B_0$.

Step 5: Existence and uniqueness of the Eq. (2.12).

By the standard contraction mapping theorem, there exists $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$ and any $m, l \in \mathbb{N} \cup \{0\}$, the equation (2.12) has a unique solution $U(0)$ which satisfies

$$\sum_{j+k \leq m} \int_{-\infty}^{0} \int_{0}^{2\pi} \left| \partial_{\theta}^{j} \partial_{\psi}^{k} (\mathcal{U}(0) - u_e(1)) \right|^2 \langle \psi \rangle^{2l} d\theta d\psi \leq C(m, l) \eta^2.$$

This completes the proof of this proposition. $\square$

Returning to the Eq. (2.8), we have the following result.

Corollary 2.4. There exists $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$, Eqs. (2.8) have a unique solution $(u^{(0)}_p, v^{(1)}_p)$ which satisfies

$$\sum_{j+k \leq m} \int_{-\infty}^{0} \int_{0}^{2\pi} \left| \partial_{\theta}^{j} \partial_{Y}^{k} (u^{(0)}_p, v^{(1)}_p) \right|^2 \langle Y \rangle^{2l} d\theta dY \leq C(m, l) \eta^2, \ m, l \geq 0. \quad (2.40)$$
Noticing that

\[ v_p^{(1)}(\theta, Y) = \int_{-\infty}^{Y} \partial_{Y} v_p^{(1)}(\theta, Y')dY' = -\int_{-\infty}^{Y} \partial_{\theta} u_p^{(0)}(\theta, Y')dY', \]

we have

\[ \int_{0}^{2\pi} v_p^{(1)}(\theta, Y)d\theta = 0, \forall Y \leq 0. \]

Finally, solving (2.9), we obtain \( p_p^{(1)}(\theta, Y) \) which decays very fast as \( Y \to -\infty \).

**Remark 2.5.** Kim established the well-posedness of Prandtl equations (1.6) in [21]. He first rewrote the equations in an equivalent integral form, then constructed a sequence of approximate solutions by Picard iteration and studied the structure of approximate solution sequence by infinite series expansion, finally proved the convergence of approximate solution sequence in some suitable space. In this paper, we use a different approach. Like Kim, we first rewrite the equations in an equivalent integral form, then we establish a priori estimate in a different space by energy method, and the well-posedness of Prandtl Eq. (1.6) can be obtained by the contraction mapping principle directly. The flows constructed above have non-monotone velocity profiles. It was shown by Renardy in [34] that solutions of steady Prandtl equations with monotone profiles do not exist.

Here we give a general strategy for constructing high order approximate solution. We first construct \( (u_e^{(1)}, v_e^{(1)}, p_e^{(1)}) \) by solving the linearized Euler Eq. (2.4) and \( (\tilde{u}_p^{(1)}, \tilde{v}_p^{(2)}) \) by solving the linearized Prandtl Eq. (2.10). Then, because \( A_{1\infty} := \lim_{Y \to -\infty} u_p^{(1)} \) is a nonzero constant, we need to change it to \( \tilde{u}_p^{(1)} = u_p^{(1)} - A_{1\infty} \). Finally, we modify \( u_e^{(1)} \) into \( \tilde{u}_e^{(1)} \) by adding a radial function, see (2.61) for the details. Noticing the structure of the linearized Euler Eq. (2.4) and the fact that only the value of \( u_e^{(1)} \) at \( r = 1 \) appear in the Eq. (2.10), we can easily deduce that the modified \( (\tilde{u}_e^{(1)}, \tilde{v}_e^{(1)}, \tilde{p}_e^{(1)}) \) and \( (\tilde{u}_p^{(1)}, \tilde{v}_p^{(2)}) \) still satisfy the Eqs. (2.4) and (2.10). The higher order approximate solutions \( (\tilde{u}_e^{(i)}, \tilde{v}_e^{(i)}, \tilde{p}_e^{(i)})(i \geq 2) \) and \( (\tilde{u}_p^{(i)}, \tilde{v}_p^{(i+1)})(i \geq 2) \) can be constructed by the same approach.

### 2.3.2. Linearized Euler equations for \( (u_e^{(1)}, v_e^{(1)}, p_e^{(1)}) \) and their solvability

**Proposition 2.6.** The linearized Euler Eq. (2.4) have a solution \( (u_e^{(1)}, v_e^{(1)}, p_e^{(1)}) \) which satisfies

\[ |\partial_{\theta} u_e^{(1)} + v_e^{(1)}|(\theta, r) \leq C \eta r, |\partial_{\theta} v_e^{(1)} - u_e^{(1)}|(\theta, r) \leq C \eta r, \forall (\theta, r) \in \Omega, \]  

\[ \|\partial^j_{\theta} \partial^k_r (u_e^{(1)}, v_e^{(1)})\|_2 \leq C(K, j)\eta, \forall j, k \geq 0, \]  

\[ r^2 \Delta u_e^{(1)} - u_e^{(1)} + 2\partial_{\theta} v_e^{(1)} = 0, \]  

\[ \int_{0}^{2\pi} v_e^{(1)}(\theta, r)d\theta = 0, \]  

where and below, \( \Delta = \partial_{rr} + \frac{\partial}{\partial r} + \frac{\partial_{\theta\theta}}{r^2} \).
Proof. Eliminating the pressure $p_e^{(1)}$ in the Eq. (2.4), we obtain the following equation for $r v_e^{(1)}$ in $\Omega$

$$\begin{cases}
-r \Delta (r v_e^{(1)}) = 0, \\
rv_e^{(1)}|_{r=1} = -v_p^{(1)}|_{r=0}.
\end{cases}$$

(2.44)

Since $\int_0^{2\pi} v_p^{(1)} \, d\theta = 0$, we can assume

$$-v_p^{(1)}(\theta, 0) = \sum_{n=1}^{+\infty} [a_{n1} \cos(n\theta) + b_{n1} \sin(n\theta)].$$

By (2.40), we deduce that

$$|a_{n1}| + |b_{n1}| \leq C \frac{\eta}{n^k}, \quad \forall k \geq 0.$$  

(2.45)

It is easy to justify that

$$v_e^{(1)}(\theta, r) = \sum_{n=1}^{+\infty} [a_{n1} r^{n-1} \cos(n\theta) + b_{n1} r^{n-1} \sin(n\theta)]$$

solves the Eq. (2.44). Set

$$u_e^{(1)}(\theta, r) = \sum_{n=1}^{+\infty} [-a_{n1} r^{n-1} \sin(n\theta) + b_{n1} r^{n-1} \cos(n\theta)],$$

then

$$\partial_\theta u_e^{(1)} + \partial_r (r v_e^{(1)}) = 0, \quad r^2 \Delta u_e^{(1)} - u_e^{(1)} + 2 \partial_\theta v_e^{(1)} = 0,$$

$$\|\partial_\theta^k \partial_r^j (u_e^{(1)}, v_e^{(1)})\|_2 \leq C(k, j) \eta, \quad \forall k, j \geq 0,$$

which give (2.42) and (2.43).

Moreover, there hold

$$\partial_\theta u_e^{(1)}(\theta, r) + v_e^{(1)}(\theta, r) = \sum_{n=2}^{+\infty} [(1 - n)a_{n1} r^{n-1} \cos(n\theta) + (1 - n)b_{n1} r^{n-1} \sin(n\theta)],$$

$$\partial_\theta v_e^{(1)}(\theta, r) - u_e^{(1)}(\theta, r) = \sum_{n=2}^{+\infty} [(1 - n)a_{n1} r^{n-1} \sin(n\theta) + (n - 1)b_{n1} r^{n-1} \cos(n\theta)].$$

Thus, we obtain (2.41) by using (2.45). After obtaining $(u_e^{(1)}, v_e^{(1)})$, we construct $p_e^{(1)}$ as following

$$p_e^{(1)}(\theta, r) := \phi(r) - \int_0^\theta [u_e(r) \partial_\theta u_e^{(1)} + ru_e' v_e^{(1)} + u_e v_e^{(1)}](\theta', r) \, d\theta',$$

where $\phi(r)$ is a function which satisfies

$$r \partial_r \phi(r) + u_e(r) \partial_\theta v_e^{(1)}(0, r) - 2u_e(r) u_e^{(1)}(0, r) = 0.$$

Combining the equation of $(u_e^{(1)}, v_e^{(1)})$, it is direct to obtain

$$u_e \partial_\theta v_e^{(1)} - 2u_e u_e^{(1)} + r \partial_r p_e^{(1)} = 0.$$

Hence, $(u_e^{(1)}, v_e^{(1)}, p_e^{(1)})$ solves the Eq. (2.4).
2.3.3. Linearized Prandtl equations for \((u_p^{(1)}, v_p^{(2)})\) and their solvability

In this subsubsection, we consider the solvability of linearized Prandtl Eq. (2.10).

**Proposition 2.7.** There exists \(\eta_0 > 0\) such that for any \(\eta \in (0, \eta_0)\), Eq. (2.10) have a unique solution \((u_p^{(1)}, v_p^{(2)})\) which satisfies

\[
\sum_{j+k \leq m} \int_{-\infty}^{0} \int_{0}^{2\pi} \left| \partial_{\theta} \partial_{\gamma} (u_p^{(1)} - A_{1\infty}, v_p^{(2)}) \right|^{2} d\theta dY \leq C(m, l)\eta^{2}, \quad m, l \geq 0;
\]

\[
\int_{0}^{2\pi} v_p^{(2)}(\theta, Y) d\theta = 0, \quad \forall Y \leq 0,
\]

where \(A_{1\infty} := \lim_{Y \to -\infty} u_p^{(1)}(\theta, Y)\) is a constant which satisfies \(|A_{1\infty}| \leq C\eta\).

**Proof.** Let \(\eta \in C_{c}^{\infty}((-\infty, 0])\) satisfy

\[
\eta(0) = 1, \quad \int_{-\infty}^{0} \eta(y) dy = 0.
\]

For simplicity, we set

\[
\bar{u} := u_{e}(1) + u_p^{(0)}, \quad \bar{v} := v_{e}^{(1)}(\theta, 1) + v^{(1)},
\]

\[
u := u_p^{(1)} + u_{e}(1, 1)\eta(Y), \quad v := v_p^{(2)} - v_p^{(2)}(\theta, 0) + Y v_{p}^{(1)} - \partial_{\theta} u_{e}(1, 1) \int_{0}^{Y} \eta(z) dz.
\]

Then, the Eq. (2.10) reduce to

\[
\begin{cases}
\bar{u} \partial_{\theta} u + \bar{v} \partial_{Y} u + u \partial_{\theta} \bar{u} + v \partial_{Y} \bar{u} - \partial_{YY} u = \tilde{f}, \\
\bar{u} \partial_{\theta} u + \bar{v} \partial_{Y} v = 0, \\
u(\theta, Y) = u(\theta + 2\pi, Y), \quad v(\theta, Y) = v(\theta + 2\pi, Y), \\
u|_{Y=0} = v|_{Y=0} = 0, \quad \lim_{Y \to -\infty} \partial_{Y} u = 0,
\end{cases}
\]

(2.47)

where \(\tilde{f}(\theta, Y)\) is \(2\pi\)-periodic function which decays fast as \(Y \to -\infty\). We can solve the Eq. (2.47) by consider the following approximate system. Let \(\delta > 0\) be a constant. We consider the following elliptic equation

\[
\begin{cases}
\bar{u} \partial_{\theta} u^{\delta} + \bar{v} \partial_{Y} u^{\delta} + \left[ \int_{0}^{Y} \psi_{\theta} u^{\delta}(\theta, z) dz \right] \partial_{Y} \bar{u} + u^{\delta} \partial_{\theta} \bar{u} - \partial_{YY} u^{\delta} - \partial_{\theta} u^{\delta} = \tilde{f}, \\
u^{\delta}(\theta, Y) = u^{\delta}(\theta + 2\pi, Y), \\
u^{\delta}|_{Y=0} = 0.
\end{cases}
\]

(2.48)

We expect the solution of this equation is in \(H_{0}^{1} = \{u|\psi_{\theta} u \in L^2, \psi_{Y} u \in L^2, u|_{Y=0} = 0\}\) rather than \(H_{0}^{1} = \{u|u \in L^2, \psi_{\theta} u \in L^2, \psi_{Y} u \in L^2, u|_{Y=0} = 0\}\). Now we establish a priori estimates of Eq. (2.48).

Multiplying the first equation in (2.48) by \(u^{\delta}\) and integrating in \((\theta, r) \in (0, 2\pi) \times (-\infty, 0)\), we obtain that
\[
\int_{-\infty}^{0} \int_{0}^{2\pi} \left[ \tilde{u} \partial_\delta u^\delta + \tilde{v} \partial_Y u^\delta + \left( \int_{Y}^{0} \psi_\theta u^\delta(\theta, z) dz \right) \partial_Y \tilde{u} + u^\delta \partial_\theta \tilde{u} - \partial_Y u^\delta - \delta \psi_\theta u^\delta \right] u^\delta d\theta dY \\
= \int_{-\infty}^{0} \int_{0}^{2\pi} \tilde{f} u^\delta d\theta dY.
\]

It is easy to get
\[
\int_{-\infty}^{0} \int_{0}^{2\pi} \left[ - \partial_Y u^\delta - \delta \psi_\theta u^\delta \right] u^\delta d\theta dY = \| \partial_Y u^\delta \|_2^2 + \delta \| \psi_\theta u^\delta \|_2^2.
\]

By the estimates (2.40) and (2.42), we have
\[
|\psi_\theta f \int_{Y}^{0} \tilde{u} - u_e(1)(Y) dY| \leq C(j, k, l) \eta,
\]
\[
|\psi_\theta f \int_{Y}^{0} \tilde{v} - v_e(1)(\theta, 1)(Y)| \leq C(j, k, l) \eta, \quad |\psi_\theta f v_e(1)(\theta, 1)| \leq C(j) \eta.
\]

Thus, we can deduce that
\[
- \int_{-\infty}^{0} \int_{0}^{2\pi} \left[ \tilde{u} \partial_\delta u^\delta + \tilde{v} \partial_Y u^\delta + \left( \int_{Y}^{0} \psi_\theta u^\delta(\theta, z) dz \right) \partial_Y \tilde{u} + u^\delta \partial_\theta \tilde{u} \right] u^\delta d\theta dY \\
+ \int_{-\infty}^{0} \int_{0}^{2\pi} \tilde{f} u^\delta d\theta dY \\
\leq \int_{-\infty}^{0} \int_{0}^{2\pi} \frac{1}{2} \left[ \psi_\theta \tilde{u} + \psi_Y \tilde{v} \right] (u^\delta)^2 d\theta dr + \| Y^2 \tilde{u} \|_2 \| \int_{Y}^{0} \psi_\theta u^\delta dz \|_2 \| u^\delta \|_2 \\
+ \| Y^2 \tilde{u} \|_2 \| \psi_Y u^\delta \|_2^2 + C \| Y \tilde{f} \|_2 \| \psi_Y u^\delta \|_2 \\
\leq C \eta \left[ \| \psi_\theta u^\delta \|_2^2 + \| \psi_Y u^\delta \|_2^2 \right] + C \| Y \tilde{f} \|_2 \| \psi_Y u^\delta \|_2,
\]

where we used \( \tilde{u} + \tilde{v} = 0 \) and the Hardy inequality
\[
\left\| \int_{Y}^{0} \psi_\theta u^\delta dz \right\|_2 \leq C \| \psi_\theta u^\delta \|_2, \quad \left\| \frac{u^\delta}{Y} \right\|_2 \leq C \| \psi_Y u^\delta \|_2.
\]

Combining the above estimates, we obtain
\[
\| \partial_Y u^\delta \|_2^2 + \delta \| \psi_\theta u^\delta \|_2^2 \leq C \eta \left[ \| \psi_\theta u^\delta \|_2^2 + \| \psi_Y u^\delta \|_2^2 \right] + C \| Y \tilde{f} \|_2 \| \psi_Y u^\delta \|_2,
\]

where \( C \) is independent of \( \eta \) and \( \delta \). If \( \eta \) is small enough, there holds
\[
\| \partial_Y u^\delta \|_2^2 + \delta \| \psi_\theta u^\delta \|_2^2 \leq C \eta \| \psi_\theta u^\delta \|_2^2 + C \| Y \tilde{f} \|_2^2. \tag{2.49}
\]

Next, by multiplying the first equation in (2.48) by \( \psi_\theta u^\delta \) and integrating in \( (\theta, r) \in (0, 2\pi) \times (-\infty, 0) \), we arrive at
Thus, we obtain

\[
\int_{-\infty}^{0} \int_{0}^{2\pi} \left[ \tilde{u} \partial_\theta u^\delta + \tilde{v} \partial_Y u^\delta + \left( \int_{Y}^{0} \psi u^\delta(\theta, z) dz \right) \partial_Y \tilde{u} + u^\delta \partial_\theta \tilde{u} - \partial_Y u^\delta - \delta \psi_\theta u^\delta \right] \psi_\theta u^\delta d\theta dY
\]

\[
= \int_{-\infty}^{0} \int_{0}^{2\pi} \tilde{f} \psi_\theta u^\delta d\theta dY.
\]

It is direct to obtain

\[
\int_{-\infty}^{0} \int_{0}^{2\pi} \tilde{u} \partial_\theta u^\delta \psi_\theta u^\delta d\theta dY = \int_{-\infty}^{0} \int_{0}^{2\pi} \tilde{u} \left| \partial_\theta u^\delta \right|^2 d\theta dY \geq (\alpha - C\eta) \| \psi_\theta u^\delta \|^2_2.
\]

The diffusion term can be computed as follows

\[
\int_{-\infty}^{0} \int_{0}^{2\pi} \left[ - \partial_{YY} u^\delta - \delta \psi_\theta u^\delta \right] \psi_\theta u^\delta d\theta dY
\]

\[
= \int_{-\infty}^{0} \int_{0}^{2\pi} \left( \frac{1}{2} \psi_\theta (\partial_Y u^\delta)^2 - \frac{\delta}{2} \psi_\theta (\psi_\theta u^\delta)^2 \right) d\theta dY = 0.
\]

Moreover, there holds

\[
- \int_{-\infty}^{0} \int_{0}^{2\pi} \left[ \tilde{v} \partial_Y u^\delta + \left( \int_{Y}^{0} \psi u^\delta(\theta, z) dz \right) \partial_Y \tilde{u} + u^\delta \partial_\theta \tilde{u} \right] \psi_\theta u^\delta d\theta dY
\]

\[
+ \int_{-\infty}^{0} \int_{0}^{2\pi} \tilde{f} \psi_\theta u^\delta d\theta dY
\]

\[
\leq \| \tilde{v} \| \| \psi \| \| Y \psi \|_2 \| \psi_\theta u^\delta \|_2 + \| Y \tilde{u} \| \| Y \| \| \psi_\theta u^\delta \|_2
\]

\[
+ \| Y \psi_\theta \tilde{u} \| \| u^\delta \|_2 \| \psi_\theta u^\delta \|_2 + \| \tilde{f} \| \| \psi_\theta u^\delta \|_2
\]

\[
\leq C \eta \left[ \| \psi \| \| \psi \|_2 + \| \psi \|_2 \right] + \| \tilde{f} \|_2 \| \psi_\theta u^\delta \|_2.
\]

Thus, we obtain

\[
\alpha \| \psi_\theta u^\delta \|^2_2 \leq C \eta \left[ \| \psi \| \| \psi \|_2 + \| \psi \|_2 \right] + \| \tilde{f} \|_2 \| \psi_\theta u^\delta \|_2.
\]

Since C is independent of \( \eta \) and \( \delta \), and \( \eta \) is small enough, there holds

\[
\alpha \| \psi_\theta u^\delta \|^2_2 \leq C \| \psi \|^2_2 + \| \tilde{f} \|^2_2.
\]  \hspace{1cm} (2.50)

Combining (2.49) and (2.50), we have

\[
\alpha \| \psi_\theta u^\delta \|^2_2 + \| \partial_Y u^\delta \|^2_2 + \delta \| \psi_\theta u^\delta \|^2_2 \leq C \| \psi \|^2_2 + C \| \tilde{f} \|^2_2.
\]  \hspace{1cm} (2.51)

According to the first equation in (2.48), we deduce

\[
\| \psi \| \| \psi \|_2 + 2\delta \| \psi_\theta u^\delta \|^2_2 + \delta^2 \| \psi_\theta u^\delta \|^2_2 = \| \partial_Y u^\delta + \delta \psi_\theta u^\delta \|^2_2
\]

\[
= \| \tilde{u} \partial_\theta u^\delta + \tilde{v} \partial_Y u^\delta + \left( \int_{Y}^{0} \psi u^\delta(\theta, z) dz \right) \partial_Y \tilde{u} + \psi \partial_\theta \tilde{u} - \tilde{f} \|^2_2
\]

\[
\leq C \left[ \| \tilde{f} \|^2_2 + \| \psi \|^2_2 + \| \partial_Y u^\delta \|^2_2 \right] \leq C \| \psi \|^2_2 + C \| \tilde{f} \|^2_2.
\]  \hspace{1cm} (2.52)
Combining the estimates (2.51) and (2.52), we obtain
\[ \alpha \| \psi \theta u \|^2 + \| \partial Y u \|^2 + \| \psi Y u \|^2 + \delta \| \psi \theta u \|^2 + 2 \delta \| \psi Y u \|^2 + \delta^2 \| \psi \theta u \|^2 + 2 \delta \| \psi \theta \|^2 + \delta^2 \| \psi \theta \|^2 \leq C (\langle Y \rangle \tilde{f}^2) \]

The above inequality shows the existence and uniqueness of solutions of the system (2.48) for any \( \delta > 0 \) in space \( \dot{H}^1 \). Moreover, the solution is smooth if \( \tilde{f} \) is smooth enough. Set
\[
u := \lim_{\delta \to 0} \frac{1}{\delta} \int_0^\infty Y \psi \theta u \, \text{d}z,
\]
then
\[ \alpha \| \psi \theta u \|^2 + \| \partial Y u \|^2 + \| \psi Y u \|^2 \leq C (\langle Y \rangle \tilde{f}^2), \quad (2.53) \]

and \([u, v]\) solves the system (2.47).

Finally, we show that the derivatives of \( u, v \) decay fast as \( Y \to -\infty \). Let
\[
\tilde{u}(\theta, \psi) = u(\theta, Y(\theta, \psi)), \quad \tilde{v}(\theta, \psi) = v(\theta, Y(\theta, \psi)), \quad F(\theta, \psi) = \frac{\tilde{f}(\theta, Y(\theta, \psi))}{\tilde{u}(\theta, Y(\theta, \psi))},
\]
then there hold
\[
\begin{cases}
\partial_\theta \tilde{u} - \partial_\psi (a(\theta, \psi) \partial_\psi \tilde{u} + b(\theta, \psi) \tilde{u} + c(\theta, \psi) \tilde{v}) = F(\theta, \psi), \\
\tilde{u}(\theta, 0) = 0, \quad \lim_{\psi \to -\infty} \partial_\psi \tilde{u}(\theta, \psi) = 0,
\end{cases}
\quad (2.54)
\]

where
\[
a(\theta, \psi) = \tilde{u}(\theta, Y(\theta, \psi)), \quad b(\theta, \psi) = \frac{\partial_\theta \tilde{u}(\theta, Y(\theta, \psi))}{\tilde{u}(\theta, Y(\theta, \psi))}, \quad c(\theta, \psi) = \frac{\partial_\psi \tilde{u}(\theta, Y(\theta, \psi))}{\tilde{u}(\theta, Y(\theta, \psi))}.
\]

Noticing that there exists \( \eta_0 > 0 \) such that for any \( \eta \in (0, \eta_0) \), there holds
\[
\alpha \frac{1}{2} \leq \tilde{u}(\theta, Y) \leq \alpha, \quad \forall (\theta, Y) \in [0, 2\pi] \times (-\infty, 0].
\]

Thus, we deduce that
\[ \frac{\alpha}{2} \leq \frac{\| \psi \|}{|Y|} \leq \alpha. \]

We claim that for any \( l \in \mathbb{N} \), there holds
\[
\| \partial_\theta \tilde{u}^l \|^2 + \| \partial_\psi \tilde{u} \tilde{u}^l \|^2 + \| \partial_\psi \tilde{u}^l \|^2 \leq C(\delta, l) \| F[\psi]^{l+1} \|_2, \quad (2.55)
\]

where
\[
\tilde{u} = \tilde{u} - u_0(\psi), \quad u_0(\psi) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}(\theta, \psi) \, \text{d}\theta.
\]

From (2.53), we deduce that
\[
\| \partial_\theta \tilde{u} \|^2 + \| \partial_\psi \tilde{u} \|^2 + \| \partial_\psi \tilde{u} \|^2 \leq C(\delta) \| F[\psi] \|_2,
\]
thus (2.55) holds for \( l = 0 \).
For any \( l \geq 1 \), multiplying \( \tilde{u} \neq \psi^{2l} \) in (2.54) and integrating in \([0, 2\pi] \times (-\infty, 0]\), we obtain

\[
\int_{0}^{2\pi} \int_{-\infty}^{0} \partial_{\psi} \tilde{u} \neq \psi^{2l} d\psi d\theta - \int_{0}^{2\pi} \int_{-\infty}^{0} \partial_{\psi} (a(\theta, \psi) \partial_{\psi} \tilde{u}) \neq \psi^{2l} d\psi d\theta
\]

\[
= \int_{0}^{2\pi} \int_{-\infty}^{0} \left\{ F(\theta, \psi) - b(\theta, \psi) \tilde{u} - c(\theta, \psi) \tilde{v} \right\} \tilde{u} \neq \psi^{2l} d\psi d\theta.
\]

Obviously, \( I_1 = 0 \). Due to the fast decay of \( b(\theta, \psi), c(\theta, \psi) \) as \( \psi \to -\infty \), we deduce that

\[
|I_3| \leq C \| F(\psi)^{l+1} \|_2 \| \partial_{\psi} \tilde{u} \neq \psi^{l-1} \|_2 + C(\delta) (\| \partial_{\psi} \tilde{u} \|^2 + \| \partial_{\psi} \tilde{u} \|^2_2)
\]

\[
\leq C(\delta) \| F(\psi)^{l+1} \|_2^2 + C(\delta) \| \partial_{\psi} \tilde{u} \|_2^{l-1} \|_2^2.
\]

Moreover, there holds

\[
I_2 = \int_{0}^{2\pi} \int_{-\infty}^{0} a(\theta, \psi) \partial_{\psi} \tilde{u} \partial_{\psi} \tilde{u} \neq \psi^{2l} d\psi d\theta + 2l \int_{0}^{2\pi} \int_{-\infty}^{0} a(\theta, \psi) \partial_{\psi} \tilde{u} \neq \psi^{2l-1} d\psi d\theta.
\]

Noticing that \( a(\theta, \psi) = u_e(1) + u^{(0)}_p(\theta, Y(\theta, \psi)) \), we deduce that

\[
I_{21} = u_e(1) \int_{0}^{2\pi} \int_{-\infty}^{0} \partial_{\psi} \tilde{u} \partial_{\psi} \tilde{u} \neq \psi^{2l} d\psi d\theta
\]

\[
+ \int_{0}^{2\pi} \int_{-\infty}^{0} u^{(0)}_p(\theta, Y(\theta, \psi)) \partial_{\psi} \tilde{u} \partial_{\psi} \tilde{u} \neq \psi^{2l} d\psi d\theta
\]

\[
= u_e(1) \int_{0}^{2\pi} \int_{-\infty}^{0} \partial_{\psi} \tilde{u} \partial_{\psi} \tilde{u} \neq \psi^{2l} d\psi d\theta
\]

\[
+ \int_{0}^{2\pi} \int_{-\infty}^{0} u^{(0)}_p(\theta, Y(\theta, \psi)) \partial_{\psi} \tilde{u} \partial_{\psi} \tilde{u} \neq \psi^{2l} d\psi d\theta
\]

\[
\geq \frac{\alpha}{2} \| \partial_{\psi} \tilde{u} \|^{l} \|_2 - C(\delta) \| \partial_{\psi} \tilde{u} \|^2_2.
\]

Moreover, by the Hölder inequality and Poincaré inequality, there holds

\[
|I_{22}| \leq C(\delta, l) \| \partial_{\psi} \tilde{u} \neq \psi^{l} \|_2 \| \partial_{\psi} \tilde{u} \|^{l-1} \|_2.
\]

Thus, we obtain

\[
\| \partial_{\psi} \tilde{u} \neq \psi^{l} \|_2 \leq C(\delta, l) \| \partial_{\psi} \tilde{u} \|^{l-1} \|_2 + C(\delta) \| F(\psi)^{l+1} \|_2.
\]

(2.56)

Moreover, we can easily get

\[
\partial_{\psi} \tilde{u} - a(\theta, \psi) \partial_{\psi} \tilde{u} = F(\theta, \psi) + \partial_{\psi} a(\theta, \psi) \partial_{\psi} \tilde{u} - b(\theta, \psi) \tilde{u} - c(\theta, \psi) \tilde{v},
\]
Prandtl–Batchelor Flows on a Disk 1127

thus there holds

$$\|\partial_\theta \tilde{u} - a(\theta, \psi) \partial_\psi \tilde{u}\|_2^2 = \|F(\theta, \psi) + \partial_\psi a(\theta, \psi) \partial_\psi \tilde{u} - b(\theta, \psi) \tilde{u} - c(\theta, \psi) \tilde{v}\|_2^2.$$ 

The righthand side above can be controlled by

$$C(\delta)\|F(\psi)\|_2^2 + C(\delta)(\|\partial_\theta \tilde{u}\|_2^2 + \|\partial_\psi \tilde{u}\|_2^2).$$

Moreover, there holds

$$\|\partial_\theta \tilde{u}\|_2^2 + \|a(\theta, \psi) \partial_\psi \tilde{u}\|_2^2 - 2\int_0^{2\pi} \int_{-\infty}^{0} a(\theta, \psi) \partial_\theta \tilde{u} \partial_\psi \tilde{u} \psi^2 d\psi d\theta$$

$$\geq \|\partial_\theta \tilde{u}\|_2^2 + \|a(\theta, \psi) \partial_\psi \tilde{u}\|_2^2 - 2\int_0^{2\pi} \int_{-\infty}^{0} a(\theta, \psi) \partial_\theta \tilde{u} \partial_\psi \tilde{u} \psi^2 d\psi d\theta.$$

Integrating by parts, we deduce that

$$I = 2 \int_0^{2\pi} \int_{-\infty}^{0} \partial_\psi a(\theta, \psi) \partial_\theta \tilde{u} \partial_\psi \tilde{u} \psi^2 d\psi d\theta$$

$$+ 4l \int_0^{2\pi} \int_{-\infty}^{0} a(\theta, \psi) \partial_\theta \tilde{u} \partial_\psi \tilde{u} \psi^{2l-1} d\psi d\theta$$

$$+ 2 \int_0^{2\pi} \int_{-\infty}^{0} a(\theta, \psi) \partial_\theta \psi \tilde{u} \partial_\psi \tilde{u} \psi^2 d\psi d\theta.$$

Obviously, there holds

$$|I_1| + |I_3| \leq C(\delta)(\|\partial_\theta \tilde{u}\|_2^2 + \|\partial_\psi \tilde{u}\|_2^2).$$

Moreover,

$$|I_2| = 4lu_{e}(1) \int_0^{2\pi} \int_{-\infty}^{0} \partial_\theta \tilde{u} \psi^{2l-1} \partial_\psi \tilde{u} \psi d\psi d\theta$$

$$+ 4l \int_0^{2\pi} \int_{-\infty}^{0} u_p^{(0)}(\theta, Y(\theta, \psi)) \partial_\theta \tilde{u} \psi^{2l-1} \partial_\psi \tilde{u} \psi d\psi d\theta$$

$$\leq C(l)\|\partial_\theta \tilde{u}\|_2^2 \|\partial_\psi \tilde{u}\|_2^2 + C(\delta, l)(\|\partial_\theta \tilde{u}\|_2^2 + \|\partial_\psi \tilde{u}\|_2^2).$$
Thus, we obtain
\[
\| \partial_\theta \tilde{u} \psi^l \|_2 + \| \partial_\psi \tilde{u} \psi^l \|_2 \leq C(\delta, l) \| \partial_\psi \tilde{u} \psi^{l-1} \|_2 + C(\delta, l) \| F(\psi)^{l+1} \|_2. \tag{2.57}
\]
Combining the estimate (2.56) and (2.57), we obtain that for any \( l \geq 1 \), there holds
\[
\| \partial_\theta \tilde{u} \psi^l \|_2 + \| \partial_\psi \tilde{u} \psi^{l-1} \|_2 + \| \partial_\psi \tilde{u} \psi^l \|_2 \leq C(\delta, l)(\| \partial_\psi \tilde{u} \psi^{l-1} \|_2 + \| \partial_\theta \tilde{u} \psi^l \|_2) + C(\delta, l) \| F(\psi)^{l+1} \|_2.
\]
Thus, by induction we obtain (2.55).

Furthermore, by the Hardy inequality, we have
\[
\| \partial_\psi \tilde{u} \psi^{l-1} \|_2 \leq C \| \partial_\psi \tilde{u} \psi^l \|_2.
\]
Hence there holds
\[
\| \partial_\theta \tilde{u} \psi^l \|_2 + \| \partial_\psi \tilde{u} \psi^{l-1} \|_2 + \| \partial_\psi \tilde{u} \psi^l \|_2 \leq C(\delta) \| F(\psi)^{l+1} \|_2.
\]

Returning to the original variable, we obtain that for any \( l \geq 1 \), there holds
\[
\| \langle Y \rangle^l \psi_Y u \|_2^2 + \| \langle Y \rangle^{l-1} \psi_{\theta} u \|_2^2 + \| \langle Y \rangle^{l-1} \psi_Y u \|_2^2 \leq C(\delta, l) \| \langle Y \rangle^{l+1} \tilde{f} \|_2^2.
\]

Furthermore, by induction we obtain that for any \( m \in \mathbb{N}_+ \), \( l \in \mathbb{N} \), there holds
\[
\sum_{j+k\leq m} \left( \| \langle Y \rangle^l \psi_{\theta} \psi^k \psi_Y u \|_2^2 + \| \langle Y \rangle^{l-1} \psi_{\theta} \psi^k \psi_{\theta} u \|_2^2 + \| \langle Y \rangle^{l-1} \psi_{\theta} \psi^k \psi_Y u \|_2^2 \right) \leq C(\delta, m, l) \sum_{j+k\leq m} \| \langle Y \rangle^{l+1} \psi_{\theta} \psi^k \tilde{f} \|_2^2 \leq C(\delta, m, l) \eta^2.
\]

Noting that \( \lim_{Y \to -\infty} (u_{\theta}, u_Y) = 0 \) and \( A_{1\infty} := \lim_{Y \to -\infty} u(\theta, Y) \) is a constant independent of \( \theta \), then by the Hardy inequality we have for any \( l \geq 2 \)
\[
\| Y^{l-2} (u - A_{1\infty}) \|_2 \leq C(\delta, l) \| Y^{l-1} \partial_Y u \|_2 \leq C(\delta, l) \eta^2.
\]
This completes the proof of this proposition. \( \square \)

Remark 2.8. Unlike the usual case, we can not expect \( \lim_{Y \to -\infty} u^{(1)}_p = 0 \) due to the periodic condition on \( \theta \). However, the behaviour of boundary layer profile \( u^{(1)}_p \) as \( Y \to -\infty \) does not affect the out flow \([u^{(1)}_e, v^{(1)}_e]\) because \([u^{(1)}_e + A_{1\infty}, v^{(1)}_e]\) also solves the Eq. (2.4). Motivated by this observation, we modify the Euler flow \([u^{(1)}_e, v^{(1)}_e]\) in (2.61).

Next, we construct the pressure \( p^{(2)}_p(\theta, Y) \). Consider the equation
\[
\partial_Y p^{(2)}_p(\theta, Y) = g_1(\theta, Y), \quad \lim_{Y \to -\infty} p^{(2)}_p(\theta, Y) = 0, \tag{2.58}
\]
where
\[
g_1(\theta, Y) = -Y\partial_Y p_p^{(1)} + \partial_Y v_p^{(1)} - u_\epsilon(1)\partial_\theta v_p^{(1)} - u_0(1)\partial_\theta w_\epsilon^{(1)}(\theta, 1) + \partial_\theta v_p^{(1)}
- \partial_Y v_p^{(1)}(v_\epsilon^{(1)}(\theta, 1) + v_p^{(1)}) - 2(Yu'_\epsilon(1)u_p^{(0)} + u_\epsilon(1)\tilde{u}_p^{(1)} + [u_\epsilon^{(1)}(\theta, 1) + A_1]
\]
\[
u_p^{(0)} + u_p^{(0)}\tilde{u}_p^{(1)},
\]
here and below,
\[
\tilde{u}_p^{(1)} = u_p^{(1)} - A_1\infty.
\]
The term \(g_1(\theta, Y)\) can be obtained by replacing \(u_p^{(1)}\) by \(\tilde{u}_p^{(1)}\) in the expansion (2.6) and putting the new expansion into the second equation of (1.3), then collecting the \(e^1\)-order terms together. Noticing that \(g_1(\theta, Y)\) decays fast as \(Y \to -\infty\), we can get \(p_p^{(2)}(\theta, Y)\) by solving (2.58) and deduce that \(p_p^{(2)}(\theta, Y)\) decays fast as \(Y \to -\infty\).

2.3.4. Linearized Euler equations for \((u_\epsilon^{(2)}, v_\epsilon^{(2)}, p_\epsilon^{(2)})\) and their solvability

Let \(\chi(r) \in C^\infty([0, 1])\) be an increasing smooth function such that
\[
\chi(r) = \begin{cases} 
0, & r \in [0, \frac{1}{2}], \\
1, & r \in [\frac{3}{4}, 1]. 
\end{cases}
\]
(2.59)

Then, let \(\phi_1(r) = -A_1\infty(r\chi''(r) + \chi'(r) - \frac{\chi(r)}{r})\) and
\[
A_1(r) := a_1r + r \int_0^r \frac{\phi_1(s)}{2s} - \frac{1}{r} \int_0^r \frac{s\phi_1(s)}{2} ds,
\]
where \(a_1\) is a constant such that \(A_1(1) = 0\). Obviously, \(|a_1| \leq C\eta\) and
\[
\begin{align*}
rA_1''(r) + A_1'(r) - \frac{A_1(r)}{r} &= -\phi_1(r), \\ A_1(1) &= 0.
\end{align*}
\]
(2.60)

Direct computation gives \(\|\partial_\theta^k A_1(r)\|_\infty \leq C(k)\eta\). Moreover, noticing that \(\chi(r) = 0\) for \(r \leq \frac{1}{2}\), we deduce that \(A_1(r) = a_1r\) for \(r \leq \frac{1}{2}\).

Set
\[
\tilde{u}_\epsilon^{(1)}(\theta, r) := u_\epsilon^{(1)}(\theta, r) + \chi(r)A_1\infty + A_1(r), \\
\tilde{v}_\epsilon^{(1)}(\theta, r) := v_\epsilon^{(1)}(\theta, r), \\
\tilde{p}_\epsilon^{(1)}(\theta, r) := p_\epsilon^{(1)}(\theta, r) + 2a_1 \int_0^r [\chi(s)A_1\infty + A_1(s)] ds,
\]
(2.61)
then \((\tilde{u}_\epsilon^{(1)}, \tilde{v}_\epsilon^{(1)}, \tilde{p}_\epsilon^{(1)})\) also satisfies the linearized Euler Eq. (2.4) with the boundary conditions (2.5). Moreover, there holds
\[
\|\partial_\theta\tilde{u}_\epsilon^{(1)} + \tilde{v}_\epsilon^{(1)}\|/(\theta, r) \leq C\eta r, \|\partial_\theta\tilde{v}_\epsilon^{(1)} - \tilde{u}_\epsilon^{(1)}\|/(\theta, r) \leq C\eta r, \forall (\theta, r) \in \Omega,
\]
\[
\|\partial_\theta^j\partial_\theta^k(\tilde{u}_\epsilon^{(1)}, \tilde{v}_\epsilon^{(1)})\|_2 \leq C(k, j)\eta, \forall j, k \geq 0;
\]
\[
r^2\triangle\tilde{u}_\epsilon^{(1)} - \tilde{u}_\epsilon^{(1)} + 2\partial_\theta\tilde{v}_\epsilon^{(1)} = 0, \int_0^{2\pi} \tilde{v}_\epsilon^{(1)} d\theta = 0.
\]
(2.62)
Putting
\[
\begin{align*}
u^e(\theta, r) &= u_e(r) + \varepsilon \tilde{u}_e(1)(\theta, r) + \varepsilon^2 u_e^{(2)}(\theta, r) + \cdots, \\
v^e(\theta, r) &= \varepsilon \tilde{v}_e(1)(\theta, r) + \varepsilon^2 v_e^{(2)}(\theta, r) + \cdots, \\
p^e(\theta, r) &= p_e(r) + \varepsilon \tilde{p}_e(1)(\theta, r) + \varepsilon^2 p_e^{(2)}(\theta, r) + \cdots.
\end{align*}
\]

into the Navier–Stokes Eq. (1.3), we obtain the following linearized Euler equations for

\[
\begin{align*}
\begin{cases}
ar \partial_\theta u_e^{(2)} + 2ar v_e^{(2)} + \partial_\theta p_e^{(2)} + \tilde{u}_e^{(1)} \partial_\theta \tilde{u}_e^{(1)} + \tilde{v}_e^{(1)} r \partial_r \tilde{v}_e^{(1)} + \tilde{u}_e^{(1)} v_e^{(1)} &= 0, \\
ar \partial_\theta v_e^{(2)} - 2ar u_e^{(2)} + r \partial_r p_e^{(2)} + \tilde{u}_e^{(1)} \partial_\theta \tilde{v}_e^{(1)} + \tilde{v}_e^{(1)} r \partial_r \tilde{v}_e^{(1)} - (\tilde{u}_e^{(1)})^2 &= 0,
\end{cases}
\end{align*}
\]

(2.63)

with the boundary conditions
\[
v_e^{(2)}|_{r=1} = -v_p^{(2)}|_{\gamma=0}, \quad v_e^{(2)}(\theta, r) = v_e^{(2)}(\theta + 2\pi, r).
\]

Proposition 2.9. The linearized Euler Eq. (2.63) have a solution \((u_e^{(2)}, v_e^{(2)}, p_e^{(2)})\) which satisfies

\[
\begin{align*}
|\partial_\theta u_e^{(2)} + v_e^{(2)}| \leq C \eta r, \quad |\partial_\theta v_e^{(2)} - u_e^{(2)}| \leq C \eta r, \quad \forall (\theta, r) \in \Omega, \\
\|\partial_\theta^k \partial_r^j (u_e^{(2)}, v_e^{(2)})\|_2 \leq C(k, j) \eta, \quad \forall j, k \geq 0, \\
r^2 \Delta u_e^{(2)} - u_e^{(2)} + 2\partial_\theta v_e^{(2)} &= 0, \\
\int_0^{2\pi} v_e^{(2)} d\theta &= 0.
\end{align*}
\]

Proof. Eliminating the pressure \(p_e^{(2)}\) in the Eq. (2.63), we obtain

\[-ar^2 \Delta (rv_e^{(2)}) - \tilde{u}_e^{(1)} r \Delta (v_e^{(1)}) + \tilde{v}_e^{(1)} (r^2 \Delta \tilde{u}_e^{(1)} - \tilde{u}_e^{(1)} + 2\partial_\theta \tilde{v}_e^{(1)}) = 0.
\]

Recalling that \(\Delta (r v_e^{(1)}) = 0\) and using (2.62), we obtain the following equation for \(rv_e^{(2)}\) in \(\Omega\)

\[
\begin{align*}
\begin{cases}
-ar^2 \Delta (rv_e^{(2)}) &= 0, \\
rv_e^{(2)}|_{r=1} &= -v_p^{(2)}(\theta, 0).
\end{cases}
\end{align*}
\]

Then, we can complete the proof of this proposition by following the argument of Proposition 2.6 line by line, and we omit the details. \(\square\)

2.3.5. Linearized Prandtl equations for \((u_p^{(2)}, v_p^{(3)})\) and their solvability

Putting the expansion

\[
\begin{align*}
u^e(\theta, r) &= u_p(r) + u_p^{(0)}(\theta, Y) + \varepsilon \tilde{u}_p^{(1)}(\theta, r) + \tilde{u}_p^{(1)}(\theta, Y) + \varepsilon^2 \left[ u_e^{(2)}(\theta, r) + v_e^{(2)}(\theta, Y) \right] + \cdots, \\
v^e(\theta, r) &= \varepsilon \tilde{v}_p^{(1)}(\theta, r) + v_p^{(1)}(\theta, Y) + \varepsilon^2 \left[ v_e^{(2)}(\theta, r) + v_p^{(2)}(\theta, Y) \right] + \cdots, \\
p^e(\theta, r) &= p_p(r) + \varepsilon \tilde{p}_p^{(1)}(\theta, r) + p_p^{(1)}(\theta, Y) + \varepsilon^2 \left[ p_e^{(2)}(\theta, r) + p_p^{(2)}(\theta, Y) \right] + \cdots.
\end{align*}
\]
with the boundary conditions
\[ u_ε^2(θ, 1) + u_p^2(θ, 0) = 0, \ v_ε^3(θ, 1) + v_p^3(θ, 0) = 0, \ \lim_{Y \to -∞} (∂Y u_p^2, v_p^3) = (0, 0), \]
into the first and third equation of (1.3), collecting ε²-order terms together, we obtain the following linearized steady Prandtl equations for \( (u_p^2, v_p^3) \)
\[
\begin{align*}
\left( u_ε^1 + u_p^0 \right) \partial_θ u_p^2 + \left( v_ε^1(θ, 1) + v_p^0 \right) \partial_Y u_p^2 + \left( v_ε^3(θ, 1) + v_p^3 \right) \partial_Y u_p^0 \\
+ (u_p^2 + u_ε^2(θ, 1)) \partial_θ u_p^0 - \partial_Y Y u_p^2 = f_2(θ, Y),
\end{align*}
\]
where
\[
f_2(θ, Y) = -\partial_θ p_p^2 + Y \partial_Y Y u_p^1 + \partial_Y u_p^1 + \partial_θ u_p^0 - u_p^0 - u_ε^1 \partial_θ u_p^1 - v_p^2 \partial_Y u_p^1
\]

Proposition 2.10. There exists \( η_0 > 0 \) such that for any \( η \in (0, η_0) \), Eq. (2.65) have a unique solution \((u_p^2, v_p^3)\) which satisfies
\[
\sum_{j+k \leq m} \int_{-∞}^{0} \int_0^{2π} \left| \partial_θ^j \partial_Y^k (u_p^2 - A_{2∞}, v_p^3) \right|^2 \left| Y \right|^{2l} dθ dY \leq C(m, l) η^2, \ m, l \geq 0,
\]
\[
\int_0^{2π} v_p^3(θ, Y) dθ = 0, \ ∀ Y \leq 0, \quad (2.66)
\]
where \( A_{2∞} := \lim_{Y \to -∞} u_p^2(θ, Y) \) is a constant which satisfies \( |A_{2∞}| \leq Cη. \)

The proof is the same as Proposition 2.7 by noticing that \( f_2(θ, Y) \) decays fast as \( Y \to -∞ \), and we omit the details.

We construct the pressure \( p_p^3(θ, Y) \) by considering the equation
\[
\partial_Y p_p^3(θ, Y) = g_2(θ, Y), \quad \lim_{Y \to -∞} p_p^3(θ, Y) = 0, \quad (2.67)
\]
where
\[ g_2(\theta, Y) = \partial_Y Y v_p(2) + Y \partial_Y Y v_p(1) + \partial_Y v_p(1) - 2 \partial_\theta u_p(0) - Y \partial_\theta p_p(2) - \sum_{i+j=3} v_p(i) \partial_Y v_p(j) \]
\[ - \sum_{i+j=2} \left( \tilde{u}_p(i) \partial_\theta v_p(j) + v_p(i) Y \partial_Y v_p(j) - \tilde{u}_p(i) \tilde{u}_p(j) + \tilde{v}_e(i, \theta, 1) Y \partial_Y v_p(j) + v_p(i) r \tilde{v}_e(j, \theta, 1) \right) \]
\[ - \frac{1}{k} \sum_{k=0}^2 \sum_{i+j=2-k} \left( \frac{\partial^k \tilde{u}_e(i)}{k!} Y^k \partial_\theta v_p(j) + \frac{\partial^k \partial_\theta \tilde{v}_e(j)}{k!} Y^k \partial_\theta \tilde{u}_p(i) \right) \]
\[ - \frac{2}{k} \sum_{k=0}^2 \sum_{i+j=2-k} \left( \frac{\partial^k \tilde{u}_e(i)}{k!} Y^k \tilde{u}_p(j) + \frac{\partial^k \tilde{v}_e(j)}{k!} Y^k \tilde{u}_p(i) \right), \]
in which \( \tilde{u}_p(0) = u_p(0), \tilde{u}_e(0) = u_e(r), \tilde{u}_e(2) = u_e^2 + A_{2\infty}, \tilde{v}_e(2) = v_e^2, \) and here and below \( \tilde{u}_p(2) = u_p(2) - A_{2\infty}. \) The term \( g_2(\theta, Y) \) can be derived by the same argument as \( g_1(\theta, Y). \) Moreover, noticing that \( g_2(\theta, Y) \) decays fast as \( Y \to -\infty, \) we can obtain \( p_p(2) \) by solving (2.67) and \( p_p(2) \) also decays fast as \( Y \to -\infty. \)

### 2.3.6 Linearized Euler equations for \((u_e^3, v_e^3, p_e^3)\) and their solvability

Let \( \phi_2(s) = -A_{2\infty}(r \chi''(r) + \chi'(r) - \frac{2}{r}) \) and
\[ A_2(r) := a_2 r + r \int_0^r \frac{\phi_2(s)}{2s} - \frac{1}{r} \int_0^r \frac{s \phi_2(s)}{2} ds, \]
where \( a_2 \) is a constant such that \( A_2(1) = 0. \) Obviously, \( |a_2| \leq C \eta, \)
\[ \left\{ \begin{array}{l}
 r A''_2(r) + A'_2(r) - \frac{A_2(r)}{r} = -\phi_2(r), \quad 0 < r < 1 \\
 A_2(1) = 0,
\end{array} \right. \]
\[ (2.68) \]
and \( \| \partial_r A_2(r) \|_\infty \leq C(k) \eta. \) Moreover, noticing that \( \chi(r) = 0 \) for \( r \leq \frac{1}{2}, \) we deduce that \( A_2(r) = a_2 r \) for \( r \leq \frac{1}{2}. \) Set
\[ \tilde{u}_e(2)(\theta, r) := u_e(2)(\theta, r) + \chi(r) A_{2\infty} + A_2(r), \]
\[ \tilde{v}_e(2)(\theta, r) := v_e(2)(\theta, r), \]
\[ \tilde{p}_e(2)(\theta, r) := p_e(2)(\theta, r) + 2a \int_0^r [\chi(s) A_{2\infty} + A_2(s)] ds, \]
then \((\tilde{u}_e^2, \tilde{v}_e^2, \tilde{p}_e^2)\) also satisfies the linearized Euler Eq. (2.63) with the boundary conditions (2.64). Moreover, there holds
\[ |\partial_\theta \tilde{u}_e^2 + \tilde{v}_e^2| / (\theta, r) \leq C \eta r, \quad |\partial_\theta \tilde{v}_e^2 - \tilde{u}_e^2| / (\theta, r) \leq C \eta r, \quad \forall (\theta, r) \in \Omega, \]
\[ \| \partial_\theta^k \partial_r^j (\tilde{u}_e^2, \tilde{v}_e^2) \|_2 \leq C(k, j) \eta, \quad \forall j, k \geq 0; \]
\[ r^2 \Delta \tilde{u}_e^2 - \tilde{u}_e^2 + 2 \partial_\theta \tilde{v}_e^2 = 0, \quad r \int_0^{2\pi} \tilde{v}_e^2 d\theta = 0. \]
\[ (2.69) \]
Putting
\[ u^e(\theta, r) = u_e(r) + \sum_{i=1}^2 \varepsilon^i \tilde{u}_e^{(i)}(\theta, r) + \varepsilon^3 u_e^{(3)} + \cdots , \]
\[ v^e(\theta, r) = \sum_{i=1}^2 \varepsilon^i \tilde{v}_e^{(i)}(\theta, r) + \varepsilon^3 v_e^{(3)} + \cdots , \]
\[ p^e(\theta, r) = p_e(r) + \sum_{i=1}^2 \varepsilon^i \tilde{p}_e^{(i)}(\theta, r) + \varepsilon^3 p_e^{(3)} + \cdots \]
into the Navier–Stokes Eq. (1.3), we find that \((u_e^{(3)}, v_e^{(3)}, p_e^{(3)})\) satisfies the following linearized Euler equations in \(\Omega\)

\[
\begin{aligned}
\begin{cases}
  & ar^2 \partial_\theta u_e^{(3)} + 2ar v_e^{(3)} + \partial_\theta p_e^{(3)} + f_e(\theta, r) = 0, \\
  & ar^2 \partial_\theta v_e^{(3)} - 2ar u_e^{(3)} + r \partial_r p_e^{(3)} + g_e(\theta, r) = 0, \\
  & \partial_\theta u_e^{(3)} + r \partial_r v_e^{(3)} + v_e^{(3)} = 0,
\end{cases}
\end{aligned}
\]

(2.70)

with the boundary conditions
\[ r v_e^{(3)}|_{r=1} = -v_p^{(3)}|_{\gamma=0}, \quad v_e^{(3)}(\theta, r) = v_e^{(3)}(\theta + 2\pi, r), \]

(2.71)

where
\[
\begin{aligned}
f_e(\theta, r) &= \tilde{u}_e^{(1)} \partial_\theta \tilde{u}_e^{(2)} + \tilde{u}_e^{(2)} \partial_\theta \tilde{u}_e^{(1)} + \tilde{v}_e^{(1)} r \partial_r \tilde{u}_e^{(2)} + \tilde{v}_e^{(2)} r \partial_r \tilde{u}_e^{(1)} + \tilde{u}_e^{(1)} \tilde{v}_e^{(2)} + \tilde{u}_e^{(2)} \tilde{v}_e^{(1)} \\
&\quad - \left( \frac{\partial_\theta \tilde{u}_e^{(1)} r}{r} + r \partial_r \tilde{u}_e^{(1)} + \partial_r \tilde{u}_e^{(1)} + \frac{2}{r} \partial_\theta \tilde{v}_e^{(1)} - \frac{\tilde{u}_e^{(1)}}{r} \right), \\
&\quad = I_1 \\
g_e(\theta, r) &= \tilde{u}_e^{(1)} \partial_\theta \tilde{v}_e^{(2)} + \tilde{u}_e^{(2)} \partial_\theta \tilde{v}_e^{(1)} + \tilde{v}_e^{(1)} r \partial_r \tilde{v}_e^{(2)} + \tilde{v}_e^{(2)} r \partial_r \tilde{v}_e^{(1)} - 2\tilde{u}_e^{(1)} \tilde{u}_e^{(2)} \\
&\quad - \left( \frac{\partial_\theta \tilde{v}_e^{(1)} r}{r} + r \partial_r \tilde{v}_e^{(1)} + \partial_r \tilde{v}_e^{(1)} - \frac{2}{r} \partial_\theta \tilde{u}_e^{(1)} - \frac{\tilde{v}_e^{(1)}}{r} \right), \\
&\quad = I_2
\end{aligned}
\]

We claim that \(I_1 = I_2 = 0\). In fact,
\[ I_1 = \frac{1}{r} [r^2 \Delta \tilde{u}_e^{(1)} - \tilde{u}_e^{(1)}] = 0, \quad I_2 = \Delta (r v_e^{(1)}) - \frac{2(\partial_\theta u_e^{(1)} + \partial_r (r v_e^{(1)}))}{r} = 0, \]

where we have used (2.62).

**Proposition 2.11.** The linearized Euler Eq. (2.70) have a solution \((u_e^{(3)}, v_e^{(3)}, p_e^{(3)})\) which satisfies

\[
\begin{aligned}
& |\partial_\theta u_e^{(3)} + v_e^{(3)}(\theta, r)| \leq C \eta, \quad |\partial_\theta v_e^{(3)} - u_e^{(3)}(\theta, r)| \leq C \eta, \quad \forall (\theta, r) \in \Omega, \\
& \|\partial_\theta ^k \partial_r ^j (u_e^{(3)}, v_e^{(3)})\|_2 \leq C(j, k) \eta, \quad j, k \geq 0; \\
& r^2 \Delta u_e^{(3)} - u_e^{(3)} + 2\partial_\theta v_e^{(3)} = 0, \quad \int_0^{2\pi} v_e^{(3)} d\theta = 0.
\end{aligned}
\]

(2.72)
Proof. Using the same argument as Proposition 2.9, we deduce that

\[
\begin{cases}
-ar^2 \Delta (r v_e^{(3)}) + v_e^{(2)} (r^2 \Delta \tilde{u}_e^{(1)} - \tilde{u}_e^{(1)}) + v_e^{(1)} (r^2 \Delta \tilde{u}_e^{(2)} - \tilde{u}_e^{(2)}) = 0, \\
r v_e^{(3)} |_{r=1} = -v_p^{(3)} (\theta, 0),
\end{cases}
\]

where we used \( \Delta (r v_e^{(1)}) = \Delta (r v_e^{(2)}) = 0 \). Using (2.62) and (2.69), we obtain the following equation for \( r v_e^{(3)} \) in \( \Omega \)

\[
\begin{cases}
-ar^2 \Delta (r v_e^{(3)}) = 0, \\
r v_e^{(3)} |_{r=1} = -v_p^{(3)} (\theta, 0).
\end{cases}
\]

Thus, we can complete the proof of this proposition by the same argument as Proposition 2.6. \( \square \)

2.3.7. Linearize Prandtl equations for \((u_p^{(3)}, v_p^{(4)})\) and their solvability

Let

\[
\begin{align*}
u_e^{(3)}(\theta, r) &= u_e(r) + u_p^{(0)}(\theta, Y) + \sum_{i=1}^{2} \varepsilon^i \left[ \tilde{u}_e^{(i)}(\theta, r) + \tilde{u}_p^{(i)}(\theta, Y) \right] \\
&\quad + \varepsilon^3 [u_e^{(3)}(\theta, r) + u_p^{(3)}(\theta, Y)] + \cdots, \\
v_e^{(4)}(\theta, r) &= \sum_{i=1}^{2} \varepsilon^i \left[ \tilde{v}_e^{(i)}(\theta, r) + v_p^{(i)}(\theta, Y) \right] + \varepsilon^3 [v_e^{(3)}(\theta, r) + v_p^{(3)}(\theta, Y)] \\
&\quad + \varepsilon^4 p_p^{(4)}(\theta, Y) + \cdots,
\end{align*}
\]

with the following boundary conditions

\[
u_e^{(3)}(\theta, 1) + u_p^{(3)}(\theta, 0) = 0, \quad v_e^{(4)}(\theta, 1) + v_p^{(4)}(\theta, 0) = 0, \quad \lim_{Y \to \infty} (\partial_Y u_p^{(3)}, v_p^{(4)}) = (0, 0).
\]

As the derivation of equations for \((u_p^{(2)}, v_p^{(3)})\), we obtain the following linearized Prandtl problem for \((u_p^{(3)}, v_p^{(4)})\)

\[
\begin{cases}
(u_e(1) + u_p^{(0)}) \partial_\theta u_p^{(3)} + (v_e^{(1)}(\theta, 1) + v_p^{(1)}) \partial_Y u_p^{(3)} + (u_p^{(3)} + u_e^{(3)}(\theta, 1)) \partial_\theta u_p^{(0)} \\
(\partial_\theta u_p^{(3)} + \partial_Y v_p^{(4)} + \partial_Y (Y v_p^{(3)}) = 0, \\
u_p^{(3)}(\theta, Y) = u_p^{(3)}(\theta + 2\pi, Y), \quad v_p^{(4)}(\theta, Y) = v_p^{(4)}(\theta + 2\pi, Y), \\
u_p^{(3)} |_{Y=0} = -u_e^{(3)} |_{r=1}, \quad \lim_{Y \to \infty} (\partial_Y u_p^{(3)}, v_p^{(4)}) = (0, 0),
\end{cases}
\]

(2.73)
where

\[
f_3(\theta, Y) = -\partial_\theta p_p^{(3)} + Y \partial_YY u_p^{(3)} + \partial_Y u_p^{(3)} + \sum_{k=0}^{1} \frac{(-1)^k Y^k \partial_\theta \bar{u}_p^{(1-k)}}{k!} + 2\partial_\theta v_p^{(1)} - 2Y \partial_\theta v_p^{(1)}
\]

\[
- \frac{1}{k!} \sum_{k=0}^{1} (-1)^k Y^k u_p^{(1-k)} - \sum_{i+j=3, i+j \leq 2} \tilde{u}_p^{(i)} \partial_\theta \bar{u}_p^{(j)} - \sum_{i+j=3} [v_p^{(i)} Y \partial_Y \bar{u}_p^{(j)} + \tilde{u}_p^{(i)} v_p^{(j)}]
\]

\[
- \frac{3}{k!} \sum_{k=0}^{1} \sum_{i+j=3-k, (k,j) \neq (0,3)} \left( \frac{\partial_{(k,j)} \tilde{v}_e^{(i)}}{k!} Y^{k+1} \partial_Y \bar{u}_p^{(j)} + v_p^{(i)} \partial_{(k,j)} \bar{u}_e^{(j)} (\theta, 1) \frac{1}{k!} \right)
\]

\[
- \frac{3}{k!} \sum_{k=0}^{1} \sum_{i+j=4-k, (k,j) \neq (0,3), (0,0), i \leq 3} \left( \frac{\partial_{(k,j)} \tilde{v}_e^{(i)}}{k!} Y^{k} \partial_Y \bar{u}_p^{(j)} + v_p^{(i)} \partial_{(k,j)} \bar{u}_e^{(j)} (\theta, 1) \frac{1}{k!} \right)
\]

\[
- \frac{3}{k!} \sum_{k=0}^{1} \sum_{i+j=3-k} \left( \frac{\partial_{(k,j)} \tilde{v}_e^{(i)}}{k!} Y^{k} \partial_Y \bar{u}_p^{(j)} + v_p^{(i)} \partial_{(k,j)} \bar{u}_e^{(j)} (\theta, 1) \frac{1}{k!} \right)
\]

with \( \tilde{u}_p^{(0)} = u_p^{(0)}, \tilde{v}_e^{(0)} = v_e(r), (\tilde{u}_e^{(3)}, \tilde{v}_e^{(3)}) = (u_e^{(3)}, v_e^{(3)}).

**Proposition 2.12.** There exists \( \eta_0 > 0 \) such that for any \( \eta \in (0, \eta_0) \), Eq. (2.73) has a unique solution \((u_p^{(3)}, v_p^{(4)})\) which satisfies

\[
\sum_{j+k \leq m} \int_{0}^{2\pi} \left| \partial_\theta p_p^{(3)} - A_3^{\infty}, v_p^{(4)} \right|^2 |Y|^{2l} d\theta dY \leq C(m, l) \eta^2, \quad m, l \geq 0,
\]

\[
\int_{0}^{2\pi} v_p^{(3)}(\theta, Y) d\theta = 0, \quad \forall Y \leq 0,
\]

where \( A_3^{\infty} := \lim_{Y \to -\infty} u_p^{(3)} \) is a constant which satisfies \( |A_3^{\infty}| \leq C \eta. \)

The proof is the same as Proposition 2.7, and we omit the details. Moreover, we can construct \( p_p^{(4)}(\theta, Y) \) by solving the equation

\[
\partial_Y p_p^{(4)}(\theta, Y) = g_3(\theta, Y), \quad \lim_{Y \to -\infty} p_p^{(4)}(\theta, Y) = 0,
\]

where

\[
g_3(\theta, Y) = \partial_YY v_p^{(3)} + Y \partial_YY v_p^{(2)} + \partial_Y v_p^{(2)} + \partial_\theta v_p^{(1)}
\]

\[
- \frac{1}{k!} \sum_{k=0}^{1} (-1)^k \frac{Y^k}{Y^{k+1}} \partial_\theta \bar{u}_p^{(2-k)} - v_p^{(1)} - Y \partial_YY p_p^{(3)}
\]

\[
- \sum_{i+j=3} \left[ \tilde{u}_p^{(i)} \partial_\theta \bar{u}_p^{(j)} + v_p^{(i)} Y \partial_Y \bar{u}_p^{(j)} - \tilde{u}_p^{(i)} \bar{u}_p^{(j)} \right] - \sum_{i+j=4} v_p^{(i)} Y \partial_Y v_p^{(j)}
\]

\[
- \frac{2}{k!} \sum_{k=0}^{1} \sum_{i+j=4-k} \left( \frac{\partial_{(k,j)} \tilde{v}_e^{(i)}}{k!} Y^{k} \partial_Y \bar{u}_p^{(j)} \right)
\]
Moreover, there holds
\[
\sum_{k=0}^{3} \sum_{i+j=3-k} \left( \frac{\partial^k \tilde{u}_e(i) (\theta, 1)}{k!} Y^k \partial v_p(j) + \tilde{u}_p(i) \frac{\partial^k \partial^i \tilde{v}_e(j) (\theta, 1)}{k!} Y^k \right)
\]
\[
+ \sum_{k=0}^{3} \sum_{i+j=3-k} \left( \frac{\partial^k \tilde{u}_e(i) (\theta, 1)}{k!} Y^k \tilde{u}_p(j) + \tilde{u}_p(i) \frac{\partial^k \partial^i \tilde{v}_e(j) (\theta, 1)}{k!} Y^k \right)
\]
\[
- \sum_{k=0}^{3} \sum_{i+j=3-k} \left( \frac{\partial^k \tilde{v}_e(i) (\theta, 1)}{k!} Y^{k+1} \partial y v_p(j) + v_p(i) \frac{\partial^k \partial^i \partial r \tilde{v}_e(j) (\theta, 1)}{k!} Y^k \right)
\]

with \(\tilde{u}_p(0) = u_p(0), \tilde{u}_e(0) = u_e(r), (\tilde{u}_e(3), \tilde{v}_e(3)) = (u_e(3) + A_3\infty, v_e(3)),\) and here and below \(\tilde{u}_p(3) = u_p(3) - A_3\infty.\) The term \(g_3(\theta, Y)\) is obtained by the same argument as \(g_1(\theta, Y).\) Moreover, since \(g_3(\theta, Y)\) decays fast as \(Y \to -\infty,\) it follows that \(p_p^{(4)}\) also decays fast as \(Y \to -\infty.\)

### 2.3.8. Linearized Euler equations for \((u_e^{(4)}, v_e^{(4)}, p_e^{(4)})\) and their solvability

Let \(\phi_3(s) = -A_3\infty(r \chi''(r) + r \chi'(r) - \frac{\chi(r)}{r})\) and

\[
A_3(r) := a_3 r + r \int_0^r \frac{\phi_3(s)}{2s} - \frac{1}{r} \int_0^r \frac{s \phi_3(s)}{2} ds,
\]

where \(a_3\) is a constant such that \(A_3(1) = 0.\) Obviously, \(|a_3| \leq C \eta,\)

\[
\begin{cases}
ra_3''(r) + A_3'(r) - \frac{A_3(r)}{r} = -\phi_3(r), & 0 < r \leq 1 \\
A_3(1) = 0,
\end{cases}
\]

(2.75)

and \(\|\partial^k A_3(r)\|_\infty \leq C(k) \eta.\) Moreover, noticing that \(\chi(r) = 0\) for \(r \leq \frac{1}{2},\) we deduce that \(A_3(r) = a_3 r\) for \(r \leq \frac{1}{2}.\)

Set

\[
\begin{align*}
\tilde{u}_e^{(3)}(\theta, r) & := u_e^{(3)}(\theta, r) + \chi(r) A_3\infty + A_3(r), \\
\tilde{v}_e^{(3)}(\theta, r) & := v_e^{(3)}(\theta, r), \\
\tilde{p}_e^{(3)}(\theta, r) & := p_e^{(3)}(\theta, r) + 2a \int_0^r [\chi(s) A_3\infty + A_3(s)] ds,
\end{align*}
\]

then \((\tilde{u}_e^{(3)}, \tilde{v}_e^{(3)}, \tilde{p}_e^{(3)})\) also satisfies the linearized Euler Eq. (2.70) with the boundary condition (2.71). Moreover, there holds

\[
\begin{align*}
|\partial_\theta \tilde{u}_e^{(3)}(\theta, r)| & \leq C \eta r, |\partial_\theta \tilde{v}_e^{(3)}(\theta, r)| \leq C \eta r, \forall (\theta, r) \in \Omega, \\
\|\partial^k \partial^j_r (\tilde{u}_e^{(3)}, \tilde{v}_e^{(3)})\|_2 & \leq C(k, j) \eta, \quad \forall j, k \geq 0; \\
r^2 \Delta \tilde{u}_e^{(3)} - \tilde{u}_e^{(3)} + 2 \partial_\theta \tilde{v}_e^{(3)} & = 0, \quad \int_0^{2\pi} \tilde{v}_e^{(3)} d\theta = 0. \quad (2.76)
\end{align*}
\]
Putting

\[ u^e(\theta, r) = u_e(r) + \sum_{i=1}^{4} \varepsilon^i \tilde{u}_e^{(i)}(\theta, r) + \cdots, \]

\[ v^e(\theta, r) = \sum_{i=1}^{4} \varepsilon^i \tilde{v}_e^{(i)}(\theta, r) + \cdots, \]

\[ p^e(\theta, r) = p_e(r) + \sum_{i=1}^{4} \varepsilon^i \tilde{p}_e^{(i)}(\theta, r) + \cdots \]

into the Navier–Stokes Eq. (1.3), we find that \((\tilde{u}_e^{(4)}, \tilde{v}_e^{(4)}, \tilde{p}_e^{(4)})\) satisfies the following linearized Euler equations in \(\Omega\)

\[
\begin{cases}
   ar\partial_\theta \tilde{u}_e^{(4)} + 2ar\tilde{v}_e^{(4)} + \partial_\theta \tilde{p}_e^{(4)} + f_{4e}(\theta, r) = 0, \\
   ar\partial_\theta \tilde{v}_e^{(4)} - 2ar\tilde{u}_e^{(4)} + r\partial_r \tilde{p}_e^{(4)} + g_{4e}(\theta, r) = 0, \\
   \partial_\theta \tilde{u}_e^{(4)} + r\partial_r \tilde{v}_e^{(4)} + \tilde{v}_e^{(4)} = 0,
\end{cases}
\]

(2.77)

with the boundary conditions

\[ r\tilde{v}_e^{(4)}|_{r=1} = -v_p^{(4)}|_{\gamma=0}, \quad \tilde{v}_e^{(4)}(\theta, r) = \tilde{v}_e^{(4)}(\theta + 2\pi, r), \]

where

\[
f_{4e}(\theta, r) = \tilde{u}_e^{(1)}\partial_\theta \tilde{u}_e^{(3)} + \tilde{u}_e^{(3)}\partial_\theta \tilde{v}_e^{(1)} + \tilde{v}_e^{(1)}r\partial_r \tilde{u}_e^{(3)} + \tilde{v}_e^{(3)}r\partial_r \tilde{v}_e^{(1)} + \tilde{u}_e^{(3)}\tilde{v}_e^{(1)} + \tilde{u}_e^{(1)}\tilde{v}_e^{(1)}
\]

\[
+ \tilde{v}_e^{(2)}\partial_\theta \tilde{u}_e^{(2)} + \tilde{u}_e^{(2)}r\partial_r \tilde{u}_e^{(2)} + \tilde{u}_e^{(2)}\tilde{v}_e^{(2)}
\]

\[
- \left( \frac{\partial_\theta \tilde{u}_e^{(2)}}{r} + r\partial_r \tilde{u}_e^{(2)} + \partial_r \tilde{v}_e^{(2)} + \frac{2}{r} \partial_\theta \tilde{v}_e^{(2)} - \frac{\tilde{u}_e^{(2)}}{r} \right),
\]

\[
g_{4e}(\theta, r) = \tilde{u}_e^{(1)}\partial_\theta \tilde{v}_e^{(3)} + \tilde{v}_e^{(3)}\partial_\theta \tilde{v}_e^{(1)} + \tilde{v}_e^{(1)}r\partial_r \tilde{v}_e^{(3)} + \tilde{v}_e^{(3)}r\partial_r \tilde{v}_e^{(1)} - 2\tilde{u}_e^{(1)}\tilde{v}_e^{(3)}
\]

\[
+ \tilde{v}_e^{(2)}\partial_\theta \tilde{v}_e^{(2)} + \tilde{v}_e^{(2)}r\partial_r \tilde{v}_e^{(2)} - (\tilde{u}_e^{(2)})^2
\]

\[
- \left( \frac{\partial_\theta \tilde{v}_e^{(2)}}{r} + r\partial_r \tilde{v}_e^{(2)} + \partial_r \tilde{v}_e^{(2)} - \frac{2}{r} \partial_\theta \tilde{v}_e^{(2)} - \frac{\tilde{v}_e^{(2)}}{r} \right).
\]

Proposition 2.13. The linearized Euler Eq. (2.77) have a solution \((\tilde{u}_e^{(4)}, \tilde{v}_e^{(4)}, \tilde{p}_e^{(4)})\) which satisfies

\[
|\partial_\theta \tilde{u}_e^{(4)} + \tilde{v}_e^{(4)}|(\theta, r) \leq C\varepsilon r, \quad |\partial_\theta \tilde{v}_e^{(4)} - \tilde{u}_e^{(4)}|(\theta, r) \leq C\varepsilon r, \quad \forall (\theta, r) \in \Omega,
\]

\[
\|\partial_\theta^j \partial_r^k (\tilde{u}_e^{(4)}, \tilde{v}_e^{(4)})\|_2 \leq C(k, j)\varepsilon, \quad \forall j, k \geq 0;
\]

\[
r^2 \Delta \tilde{u}_e^{(4)} - \tilde{u}_e^{(4)} + 2\partial_\theta \tilde{v}_e^{(4)} = 0, \quad \int_0^{2\pi} \tilde{v}_e^{(4)} d\theta = 0.
\]

(2.78)

Proof. The proof is the same as Proposition 2.11, and we omit the details. \(\square\)
2.3.9. Linearize Prandtl equations for \((u^{(4)}_p, v^{(5)}_p)\) and their solvability

Let

\[
u^{(4)}(\theta, r) = u_e(\theta) + u^{(0)}_p(\theta, Y) + \sum_{i=1}^{3} \varepsilon^i [\tilde{u}^{(i)}_e(\theta, r) + \tilde{u}^{(i)}_p(\theta, Y)] \\
+ \varepsilon^4 [\tilde{u}^{(4)}_e(\theta, r) + u^{(4)}_p(\theta, Y)] + \cdots,
\]

\[
v^{(5)}(\theta, r) = \sum_{i=1}^{4} \varepsilon^i [\tilde{v}^{(i)}_e(\theta, r) + v^{(i)}_p(\theta, Y)] + \varepsilon^5 v^{(5)}_p(\theta, Y) + \cdots,
\]

\[
p^{(5)}(\theta, r) = p_e(\theta) + \sum_{i=1}^{4} \varepsilon^i [\tilde{p}^{(i)}_e(\theta, r) + p^{(i)}_p(\theta, Y)] + \varepsilon^5 p^{(5)}_p(\theta, Y) + \cdots
\]

with the boundary conditions

\[
\tilde{u}^{(4)}_e(\theta, 1) + u^{(4)}_p(\theta, 0) = 0, \quad \lim_{Y \to \infty} \partial_Y u^{(4)}_p(\theta, Y) = v^{(5)}_p(\theta, 0) = 0, \quad p^{(5)}_p(\theta, 0) = 0.
\]

Similar to the derivation of equations for \((u^{(2)}_p, v^{(3)}_p)\), we obtain the following linearized Prandtl problem for \((u^{(4)}_p, v^{(5)}_p)\)

\[
\left\{
\begin{aligned}
\left( u^{(1)} + u^{(0)}_p \right) \partial_Y u^{(4)}_p + (v^{(1)}_e(\theta, 1) + v^{(1)}_p) \partial_Y u^{(4)}_p + u^{(4)}_p \partial_{\theta} u^{(0)}_p \\
v^{(5)}_p \partial_Y u^{(0)}_p - \partial_{Y Y} u^{(4)}_p = f_4(\theta, Y) \\
\partial_{\theta} u^{(4)}_p + \partial_Y v^{(5)}_p + \partial_Y (Y v^{(4)}_p) = 0, \\
u^{(4)}_p(\theta, Y) = u^{(4)}_p(\theta + 2\pi, Y), \quad v^{(5)}_p(\theta, Y) = v^{(5)}_p(\theta + 2\pi, Y), \\
u^{(4)}_p|_{Y=0} = -\tilde{u}^{(4)}_e|_{r=1}, \quad \lim_{Y \to \infty} \partial_Y u^{(4)}_p(\theta, Y) = v^{(5)}_p(\theta, 0) = 0
\end{aligned}
\right.
\]

(2.79)

and the pressure \(p^{(5)}_p\) satisfies

\[
\partial_Y p^{(5)}_p(\theta, Y) = g_4(\theta, Y), \quad p^{(5)}_p(\theta, 0) = 0,
\]

(2.80)

where

\[
f_4(\theta, Y) = - \partial_{\theta} p^{(4)}_p + Y \partial_{Y Y} u^{(3)}_p + \partial_{Y} u^{(3)}_p + \sum_{k=0}^{2} (-1)^k \frac{Y^k}{k!} \partial_{\theta} \tilde{u}^{(2-k)}_p + 2\partial_{\theta} v^{(2)}_p - 2Y \partial_{\theta} v^{(1)}_p
\]

\[
- \sum_{k=0}^{2} (-1)^k \frac{Y^k u^{(2-k)}_p}{k!} - \sum_{i+j=4, i \leq j} \tilde{u}^{(i)}_p \partial_{\theta} \tilde{u}^{(j)}_p - \sum_{i+j=4} [v^{(i)}_p Y \partial_Y \tilde{u}^{(j)}_p + \tilde{u}^{(i)}_p \partial_{Y} v^{(j)}_p]
\]

\[
- \sum_{k=0}^{4} \sum_{i+j=4-k, (i,j) \neq (0,4)} \left( \frac{\partial_{\theta} \tilde{u}^{(i)}_e(\theta, 1) Y^{k} \partial_{\theta} \tilde{u}^{(j)}_p}{k!} + \tilde{u}^{(i)}_p \frac{\partial_{\theta} \tilde{u}^{(j)}_e(\theta, 1)}{k!} Y^{k} \right)
\]

\[
- \sum_{k=0}^{4} \sum_{i+j=4-k} \left( \frac{\partial_{\theta} \tilde{u}^{(i)}_e(\theta, 1) Y^{k+1} \partial_{Y} \tilde{u}^{(j)}_p}{k!} + v^{(i)}_p \frac{\partial_{\theta} (r \partial_{Y} \tilde{u}^{(j)}_e(\theta, 1))}{k!} Y^{k} \right)
\]
Proposition 2.14. There exists \( \eta_0 > 0 \) such that for any \( \eta \in (0, \eta_0) \), Eq. (2.79) have a unique solution \( (u_p^{(4)}, v_p^{(5)}) \) which satisfies

\[
- \sum_{k=0}^{4} \sum_{i+j=5-k, (k,j) \neq (0,4), i \leq 4} \frac{\partial^k \tilde{v}_e^{(i)}(\theta, 1)}{k!} Y^k \partial_Y \tilde{u}_p^{(j)}
\]

\[
- \sum_{k=0}^{4} \sum_{i+j=4-k} \left( \frac{\partial^k \tilde{u}_e^{(i)}(\theta, 1)}{k!} Y^k v_p^{(j)} + \tilde{u}_p^{(i)} \frac{\partial^k \tilde{v}_e^{(j)}(\theta, 1)}{k!} Y^k \right)
\]

and

\[
g_4(\theta, Y) = \partial_Y v_p^{(4)} + \partial_Y Y v_p^{(3)} + \partial_Y u_p^{(3)} + [\partial_{\theta \theta} v_p^{(2)} - Y \partial_{\theta \theta} v_p^{(1)}]
\]

\[
- \sum_{k=0}^{2} \frac{(-1)^k Y^k}{k!} \partial_{\theta} \tilde{u}_p^{(2-k)} - [v_p^{(2)} - Y v_p^{(1)}]
\]

\[- Y \partial_Y p_p^{(4)} - \sum_{i+j=4} [\tilde{u}_p^{(i)} \partial_{\theta} v_p^{(j)} + v_p^{(i)} Y \partial_Y v_p^{(j)} - \tilde{u}_p^{(i)} \tilde{v}_p^{(j)}] - \sum_{i+j=5} v_p^{(i)} Y \partial_Y v_p^{(j)}
\]

\[
- \sum_{k=0}^{4} \sum_{i+j=4-k} \left( \frac{\partial^k \tilde{u}_e^{(i)}(\theta, 1)}{k!} Y^k \partial_Y v_p^{(j)} + \tilde{u}_p^{(i)} \frac{\partial^k \tilde{v}_e^{(j)}(\theta, 1)}{k!} Y^k \right)
\]

\[+ \sum_{k=0}^{4} \sum_{i+j=4-k} \left( \frac{\partial^k \tilde{v}_e^{(i)}(\theta, 1)}{k!} Y^k \tilde{u}_p^{(j)} + \tilde{u}_p^{(i)} \frac{\partial^k \tilde{u}_e^{(j)}(\theta, 1)}{k!} Y^k \right)
\]

\[- \sum_{k=0}^{3} \sum_{i+j=5-k} \frac{\partial^k \tilde{v}_e^{(i)}(\theta, 1)}{k!} Y^k \partial_Y v_p^{(j)}
\]

\[- \sum_{k=0}^{2} \sum_{i+j=4-k} \left( \frac{\partial^k \tilde{v}_e^{(i)}(\theta, 1)}{k!} Y^k v_p^{(j)} + v_p^{(i)} \frac{\partial^k (r \partial_r \tilde{v}_e^{(j)})(\theta, 1)}{k!} Y^k \right)
\]

with \( \tilde{u}_p^{(0)} = u_p^{(0)}, \tilde{u}_p^{(4)} = u_p^{(4)}, \tilde{v}_e^{(0)} = u_e(r) \).

Proposition 2.14. There exists \( \eta_0 > 0 \) such that for any \( \eta \in (0, \eta_0) \), Eq. (2.79) have a unique solution \( (u_p^{(4)}, v_p^{(5)}) \) which satisfies

\[
\sum_{0 < j + k \leq m} \int_{0}^{2\pi} \int_{0}^{2\pi} \left| \partial_\theta^j \partial_Y^l (u_p^{(4)}, v_p^{(5)}) \right|^2 \left| Y \right|^{2l} d\theta dY \leq C(m, l) \eta^2, \quad m \geq 1, l \geq 0,
\]

\[
\int_{0}^{2\pi} v_p^{(5)}(\theta, Y) d\theta = 0, \quad \forall \ Y \leq 0,
\]

and

\[
\|(u_p^{(4)}, v_p^{(5)})\|_\infty \leq C \eta.
\]

The proof is the same as Proposition 2.7, and we omit the details. Moreover, we can construct \( p_p^{(5)} \) by solving the Eq. (2.80).
2.4. Approximate solutions.

In this subsection, we construct an approximate solution of Navier–Stokes Eq. (1.3). Set

\[
\tilde{u}_p^a(\theta, r) := \chi(r) \left( u_p^{(0)}(\theta, Y) + \sum_{i=1}^{3} \varepsilon^i \tilde{u}_p^{(i)}(\theta, Y) + \varepsilon^4 u_p^{(4)}(\theta, Y) \right) := \chi(r) u_p^a,
\]

\[
\tilde{v}_p^a(\theta, r) := \chi(r) \left( \sum_{i=1}^{5} \varepsilon^i \tilde{v}_p^{(i)}(\theta, Y) \right) := \chi(r) v_p^a,
\]

\[
\tilde{p}_p^a(\theta, r) := \chi^2(r) \left( \sum_{i=1}^{5} \varepsilon^i \tilde{p}_p^{(i)}(\theta, Y) \right) := \chi^2(r) p_p^a,
\]

and

\[
\tilde{u}_e^a(\theta, r) := u_e(r) + \sum_{i=1}^{4} \varepsilon^i \tilde{u}_e^{(i)}(\theta, r),
\]

\[
\tilde{v}_e^a(\theta, r) := \sum_{i=1}^{4} \varepsilon^i \tilde{v}_e^{(i)}(\theta, r),
\]

\[
\tilde{p}_e^a(\theta, r) := p_e(r) + \sum_{i=1}^{4} \varepsilon^i \tilde{p}_e^{(i)}(\theta, r).
\]

We construct an approximate solution

\[
u^a(\theta, r) := \tilde{u}_e^a(\theta, r) + \tilde{u}_p^a(\theta, r) + \varepsilon^5 h(\theta, r),
\]

\[
v^a(\theta, r) := \tilde{v}_e^a(\theta, r) + \tilde{v}_p^a(\theta, r),
\]

\[
p^a(\theta, r) := \tilde{p}_e^a(\theta, r) + \tilde{p}_p^a(\theta, r),
\]

(2.82)

where the corrector \(h(\theta, r)\) will be given in Appendix B which satisfies

\[
h(\theta, 1) = 0, \quad \|\partial^j_{\theta} \partial^k_r h\|_2 \leq C(j, k)\varepsilon^{-k}
\]

and \((u^a, v^a)\) is divergence-free

\[
u_\theta^a + ru_r^a + v^a = 0.
\]

Moreover, \((u^a, v^a)\) satisfies the following boundary conditions

\[
u^a(\theta + 2\pi, r) = u^a(\theta, r), \quad \nu^a(\theta + 2\pi, r) = v^a(\theta, r),
\]

\[
u^a(\theta, 1) = \alpha + \eta f(\theta), \quad \nu^a(\theta, 1) = 0,
\]

and \(v^a\) satisfies

\[
\int_0^{2\pi} v^a d\theta = 0.
\]

(2.83)
Using (2.62), (2.69), (2.76), (2.78) and (2.40), (2.46), (2.66), (2.74), (2.81), we deduce that
\[
\|u_e^a - ar\|_0 + \|\partial_r(u_e^a - ar)\|_0 \leq C\varepsilon\eta, \quad \|v_e^a\|_0 + \|\partial_r v_e^a\|_0 \leq C\varepsilon\eta.
\]

and
\[
|\partial_\theta u_e^a(\theta, r) + v_e^a(\theta, r)| \leq C\varepsilon\eta, \quad |\partial_\theta v_e^a(\theta, r) - u_e^a(\theta, r) + ar| \leq C\varepsilon\eta, \quad \forall (\theta, r) \in \Omega.
\]

Finally, set
\[
R_u^a := u^a u_\theta^a + v^a r u_r^a + u^a v_r^a + p^a_\theta - \varepsilon^2 \left( r u_{rr}^a + \frac{u^a}{r} + 2 \frac{v^a}{r} + u_r^a - \frac{u^a}{r} \right),
\]
\[
R_v^a := u^a v_\theta^a + v^a r v_r^a - (u^a)^2 + r p^a_\theta - \varepsilon^2 \left( r v_{rr}^a + \frac{v^a}{r} - 2 \frac{u^a}{r} + v_r^a - \frac{v^a}{r} \right),
\]
then there hold
\[
\left( \int_0^1 \int_0^{2\pi} \frac{(R_u^a)^2(\theta, r)}{r} d\theta dr \right)^{\frac{1}{2}} \leq C\varepsilon^5, \quad \left( \int_0^1 \int_0^{2\pi} \frac{(R_v^a)^2(\theta, r)}{r} d\theta dr \right)^{\frac{1}{2}} \leq C\varepsilon^5
\]

and
\[
\left\{ \begin{array}{l}
\|u^a u_\theta^a + v^a r u_r^a + u^a v_r^a + p^a_\theta - \varepsilon^2 \left( r u_{rr}^a + \frac{u^a}{r} + 2 \frac{v^a}{r} + u_r^a - \frac{u^a}{r} \right) = R_u^a, \quad (\theta, r) \in \Omega, \\
\|u^a v_\theta^a + v^a r v_r^a - (u^a)^2 + r p^a_\theta - \varepsilon^2 \left( r v_{rr}^a + \frac{v^a}{r} - 2 \frac{u^a}{r} + v_r^a - \frac{v^a}{r} \right) = R_v^a, \quad (\theta, r) \in \Omega, \\
\|u^a(\theta + 2\pi, r) = u^a(\theta, r), \quad \|v^a(\theta + 2\pi, r) = v^a(\theta, r), \quad (\theta, r) \in \Omega, \\
u^a(\theta, 1) = \alpha + f(\theta)\eta, \quad v^a(\theta, 1) = 0, \quad \theta \in [0, 2\pi].
\end{array} \right.
\]

In fact, when \( r \leq \frac{1}{2} \), there hold
\[
R_u^a = u^a \partial_\theta u_e^a + v^a r \partial_r u_e^a + u^a v_e^a + \partial_\theta p_e^a - \varepsilon^2 \left( r \partial_{rr} u_e^a + \frac{\partial_\theta u_e^a}{r} + 2 \frac{\partial_r v_e^a}{r} + \partial_r u_e^a - \frac{u^a}{r} \right)
\]
\[
= \sum_{m=5}^8 \varepsilon^m \left( \sum_{i+j=m, i \leq 4} \tilde{u}_e^{(i)} \partial_{\theta} u_e^{(j)} + \tilde{v}_e^{(i)} \partial_r u_e^{(j)} \right) + \sum_{i+j=m, i \leq 4} \tilde{v}_e^{(i)} \partial_r v_e^{(j)},
\]
\[
R_v^a = u^a \partial_\theta v_e^a + v^a r \partial_r v_e^a - (u_e^a)^2 + r \partial_r p_e^a - \varepsilon^2 \left( r \partial_{rr} v_e^a + \frac{\partial_\theta v_e^a}{r} + 2 \frac{\partial_r u_e^a}{r} + \partial_r v_e^a - \frac{v^a}{r} \right)
\]
\[
= \sum_{m=5}^8 \varepsilon^m \left( \sum_{i+j=m, i \leq 4} \tilde{u}_e^{(i)} \partial_{\theta} v_e^{(j)} - \tilde{u}_e^{(i)} \partial_r v_e^{(j)} \right) + \sum_{i+j=m, i \leq 4} \tilde{v}_e^{(i)} \partial_r v_e^{(j)}.
\]

Using (2.62), (2.69), (2.76) and (2.78), we deduce that
\[
|R_u^a(\theta, r)| + |R_v^a(\theta, r)| \leq C\varepsilon^5 r, \quad r < \frac{1}{2}.
\]
Moreover, when \( \frac{1}{2} \leq r < 1 \), there holds \( \|R_u^a\|_2 + \|R_v^a\|_2 \leq C\varepsilon^5 \). Thus, we obtain (2.86).
3. Linear Stability Estimates of Error Equations

In this section, we derive the error equations and establish linear stability estimates.

3.1. Error equations.

Set the error
\[ u := u^e - u^a, \quad v := v^e - v^a, \quad p := p^e - p^a, \]
then there hold
\[
\begin{align*}
& u^a u_\theta + uu_\theta + uu_r + v^a u_r + vr u_r + vr u_r + uu + uu + p_\theta - \varepsilon^2 (ru_{rr} + \frac{uu}{r} + 2\frac{uu}{r} + u_r - \frac{u}{r}) = R_u^a, \\
& u^a v_\theta + uu_\theta + v^a v_r + v^a v_r + vu^2 - (u^2 + 2uu^2) + rp_r - \varepsilon^2 (rv_{rr} + \frac{v^2}{r} - 2\frac{v^2}{r} + v_r - \frac{v}{r}) = R_v^a, \\
& u_\theta + (rv)_r = 0, \\
& u(\theta + 2\pi, r) = u(\theta, r), \quad v(\theta + 2\pi, r) = v(\theta, r), \\
& u(\theta, 1) = 0, \quad v(\theta, 1) = 0.
\end{align*}
\]

Let
\[
S_u := u^a u_\theta + v^a r u_r + uu_\theta + vr u_r + vu + uu, \\
S_v := u^a v_\theta + v^a r v_r + uv_\theta + v^a v_r - 2uu^a,
\]
then the error equations become
\[
\begin{align*}
-\varepsilon^2 (ru_{rr} + \frac{uu}{r} + 2\frac{uu}{r} + u_r - \frac{u}{r}) + p_\theta + S_u &= R_u, \\
-\varepsilon^2 (rv_{rr} + \frac{v^2}{r} - 2\frac{v^2}{r} + v_r - \frac{v}{r}) + rp_r + S_v &= R_v, \\
(3.1)
\end{align*}
\]
where
\[
R_u := R_u^a - uu_\theta - vr u_r - uu, \quad R_v := R_v^a - uv_\theta - vr v_r + u^2.
\]

3.2. Linear stability estimate.

In this subsection, we consider the linear equations
\[
\begin{align*}
-\varepsilon^2 (ru_{rr} + \frac{uu}{r} + 2\frac{uu}{r} + u_r - \frac{u}{r}) + p_\theta + S_u &= F_u, \\
-\varepsilon^2 (rv_{rr} + \frac{v^2}{r} - 2\frac{v^2}{r} + v_r - \frac{v}{r}) + rp_r + S_v &= F_v, \\
(3.2)
\end{align*}
\]
and establish the stability estimates. Due to the divergence-free condition and the boundary condition of \(v\), we deduce that
\[
\int_0^{2\pi} v(\theta, r)d\theta = 0, \quad r \in (0, 1].
\]
So Poincaré inequality implies that
\[
\int_0^{2\pi} v^2 d\theta \leq \int_0^{2\pi} b^2 d\theta. \quad (3.3)
\]
The following Hardy-type inequality will be used later frequently.
Lemma 3.1. If $u(r)$ is a bounded function in $[0, 1]$ and $u(1) = 0$, then there hold
\[
\int_0^1 r^\alpha u^2 dr \leq C(\alpha) \int_0^1 r^{\alpha+2} u^2_r dr, \quad \forall \alpha > -1, \tag{3.4}
\]
\[
\int_0^1 \frac{ru^2}{(1-r)^2} dr \leq C \int_0^1 ru^2_r dr. \tag{3.5}
\]

Proof. By integrating by parts, we deduce that
\[
\int_0^1 r^\alpha u^2 dr = -\frac{2}{\alpha + 1} \int_0^1 r^{\alpha+2} uu_r dr \leq C(\alpha) \left( \int_0^1 r^\alpha u^2 dr \right)^{\frac{1}{2}} \left( \int_0^1 r^{\alpha+2} u^2_r dr \right)^{\frac{1}{2}}.
\]
Thus, there holds
\[
\int_0^1 r^\alpha u^2 dr \leq C(\alpha) \int_0^1 r^{\alpha+2} u^2_r dr, \quad \forall \alpha > -1.
\]
By integrating by parts,
\[
\int_0^1 \frac{ru^2}{(1-r)^2} dr = \int_0^1 ru^2 d\frac{1}{1-r} = -\int_0^1 \frac{1}{1-r} d(ru^2) = -\int_0^1 \left( \frac{u^2}{1-r} + \frac{ruu_r}{1-r} \right) dr.
\]
So we deduce that
\[
\int_0^1 \frac{ru^2}{(1-r)^2} dr + \int_0^1 \frac{u^2}{1-r} dr = -\int_0^1 2ruu_r \frac{1}{1-r} dr \leq C \left( \int_0^1 \frac{ru^2}{(1-r)^2} dr \right)^{\frac{1}{2}} \left( \int_0^1 ru^2_r dr \right)^{\frac{1}{2}}.
\]
Thus, there holds
\[
\int_0^1 \frac{ru^2}{(1-r)^2} dr \leq C \int_0^1 ru^2_r dr.
\]

\[\square\]

Since $\chi(r)$ is a smooth cut-off function satisfying $\chi|_{[0, \frac{1}{2}]} = 0$, by inequality (3.5), one has
\[
\int_0^1 \left( \frac{\chi u^2}{(1-r)^2} + \frac{\left| \chi' \right| u^2}{(1-r)^2} \right) dr \leq C \int_0^1 \frac{ru^2}{(1-r)^2} dr \leq C \int_0^1 ru^2_r dr. \tag{3.6}
\]

3.2.1. Basic energy estimate
In this subsection, we establish the following basic energy estimate (3.2).

Lemma 3.2. Let $(u, v)$ be a bounded solution of (3.2), then there holds
\[
\varepsilon^2 \int_0^1 \int_0^{2\pi} \frac{(u_\theta + \nu)^2 + (v_\theta - \nu_u)^2}{r} d\theta dr + \varepsilon^2 \int_0^1 \int_0^{2\pi} \frac{r(u_r^2 + v_r^2)}{r} d\theta dr \leq C \int_0^1 \int_0^{2\pi} [uF_u + vF_v] d\theta dr. \tag{3.7}
\]
Proof. Multiplying the first equation in (3.1) by \( u \), the second equation in (3.1) by \( v \), adding them together and integrating in \( \Omega \), we obtain that

\[
-\varepsilon^2 \int_{0}^{1} \int_{0}^{2\pi} \left[ r u_{rr} + \frac{u_{\theta\theta}}{r} + 2 \frac{v_{\theta}}{r} + u_r - \frac{u}{r} \right] u + \left( r v_{rr} + \frac{v_{\theta\theta}}{r} - 2 \frac{u_{\theta}}{r} + v_r - \frac{v}{r} \right) v \, d\theta dr
\]

\[
= \int_{0}^{1} \int_{0}^{2\pi} \left[ \left( \phi u + \rho + p r v \right) \right] d\theta dr
\]

\[
= \int_{0}^{1} \int_{0}^{2\pi} \left[ u F_u + v F_v \right] d\theta dr.
\]

We deal with them term by term. **The diffusion term:** Integrating by parts, we obtain

\[
-\varepsilon^2 \int_{0}^{1} \int_{0}^{2\pi} \left[ r u_{rr} + \frac{u_{\theta\theta}}{r} + 2 \frac{v_{\theta}}{r} + u_r - \frac{u}{r} \right] u + \left( r v_{rr} + \frac{v_{\theta\theta}}{r} - 2 \frac{u_{\theta}}{r} + v_r - \frac{v}{r} \right) v \, d\theta dr
\]

\[
= \varepsilon^2 \int_{0}^{1} \int_{0}^{2\pi} \left( \frac{u^2}{r} + v^2 + 2 \frac{u v}{r} \right) d\theta dr + \varepsilon^2 \int_{0}^{1} \int_{0}^{2\pi} r \left( u_r^2 + v_r^2 \right) d\theta dr
\]

\[
= \varepsilon^2 \int_{0}^{1} \int_{0}^{2\pi} \left( \frac{u_{\theta}^2}{r} + (v - u)^2 \right) d\theta dr + \varepsilon^2 \int_{0}^{1} \int_{0}^{2\pi} r \left( u_r^2 + v_r^2 \right) d\theta dr.
\]

**Pressure term:** Integrating by parts and using the divergence-free condition, we deduce that

\[
\int_{0}^{1} \int_{0}^{2\pi} \left( \rho u + \rho r (r v) \right) d\theta dr = 0.
\]

**Convective term:** Integrating by part and using the divergence-free condition, we obtain

\[
\int_{0}^{1} \int_{0}^{2\pi} \left( u F_u + v F_v \right) d\theta dr
\]

\[
= \int_{0}^{1} \int_{0}^{2\pi} \left[ u^a u_{\theta} u + r v^a u_{r} u + u u^a u + v v^a u + (v^a u + u u^a) u \right] d\theta dr
\]

\[
+ \int_{0}^{1} \int_{0}^{2\pi} \left[ u^a v_{\theta} v + r v^a v_{r} v + u v^a v + v v^a v + 2 u v^a v - 2 u u^a v \right] d\theta dr
\]

\[
= \int_{0}^{1} \int_{0}^{2\pi} \left[ u^2 u^a_{\theta} + v^2 u^a_{r} + v^a u^2 + 2 u v^a u + u v^a u + u v (r u^a_r - u^a) \right] d\theta dr,
\]

where we used

\[
\int_{0}^{1} \int_{0}^{2\pi} (u^a u_{\theta} u + r v^a u_{r} u) d\theta dr = \int_{0}^{1} \int_{0}^{2\pi} (u^a v_{\theta} v + r v^a v_{r} v) d\theta dr = 0.
\]
Thus,
\[
\int_0^1 \int_0^{2\pi} (uS_u + vS_v) d\theta dr \\
= \int_0^1 \int_0^{2\pi} \left[ u^2(u^a_\theta + v^a) + uv(v^a_\theta - u^a + ar) + v^2r v^a_\theta + uv(r u^a_\theta - ar) \right] d\theta dr \\
= \int_0^1 \int_0^{2\pi} (v^2r v^a_\theta + uv(r u^a_\theta - ar)) d\theta dr + \int_0^1 \int_0^{2\pi} u^2(u^a_\theta + v^a) d\theta dr \\
+ \int_0^1 \int_0^{2\pi} uv(v^a_\theta - u^a + ar) d\theta dr.
\]

1) Estimate of \( I_1 \): We divide it into the Euler part and the Prandtl part. By (2.84), the Poincaré inequality (3.3) and the Hardy-type inequality (3.4), we obtain

\[
\left| \int_0^1 \int_0^{2\pi} (v^2r \partial_r v^a_e + uv(r \partial_r u^a_e - ar)) d\theta dr \right| \\
\leq C \varepsilon \eta \int_0^1 \int_0^{2\pi} (rv^2 + r|uv|) d\theta dr \\
\leq C \varepsilon \eta \int_0^1 \int_0^{2\pi} rv^2 d\theta dr + C \varepsilon \eta \left( \int_0^1 \int_0^{2\pi} rv^2 d\theta dr \right)^{\frac{1}{2}} \left( \int_0^1 \int_0^{2\pi} ru^2 d\theta dr \right)^{\frac{1}{2}} \\
\leq C \varepsilon \eta \int_0^1 \int_0^{2\pi} rv^2 d\theta dr + C \varepsilon \eta \left( \int_0^1 \int_0^{2\pi} rv^2 d\theta dr \right)^{\frac{1}{2}} \left( \int_0^1 \int_0^{2\pi} ru^2 d\theta dr \right)^{\frac{1}{2}} \\
\leq C \int_0^1 \int_0^{2\pi} rv^2 d\theta dr + \frac{\varepsilon^2}{20} \int_0^1 \int_0^{2\pi} ru^2 d\theta dr.
\]

Moreover, noticing that \( \chi(r) = 0 \) for \( r \leq \frac{1}{2} \), using (2.85), the Poincaré inequality (3.3), and the Hardy inequality (3.6), we have

\[
\left| \int_0^1 \int_0^{2\pi} uvr \partial_r (\chi u^a_p) d\theta dr \right| \\
\leq \left| \int_0^1 \int_0^{2\pi} uvr \chi'(r) u^a_p d\theta dr \right| + \left| \int_0^1 \int_0^{2\pi} \frac{ruv}{r - 1} \chi(r) Y \partial_y u^a_p d\theta dr \right| \\
\leq \varepsilon \left( \int_0^1 \int_0^{2\pi} rv^2 d\theta dr \right)^{\frac{1}{2}} \left( \int_0^1 \int_0^{2\pi} ru^2 \left( \frac{r - 1}{r - 1} \right)^2 d\theta dr \right)^{\frac{1}{2}} \\
+ C \varepsilon \left( \int_0^1 \int_0^{2\pi} rv^2 d\theta dr \right)^{\frac{1}{2}} \left( \int_0^1 \int_0^{2\pi} ru^2 \left( \frac{r - 1}{r - 1} \right)^2 d\theta dr \right)^{\frac{1}{2}} \\
\leq C \varepsilon \left( \int_0^1 \int_0^{2\pi} rv^2 d\theta dr \right)^{\frac{1}{2}} \left( \int_0^1 \int_0^{2\pi} ru^2 d\theta dr \right)^{\frac{1}{2}}.
Moreover, noticing that 

\[ \chi(r) = -\frac{1}{r^2} \int_0^r \theta_{\ast} d\theta \]

and the Poincaré inequality (3.3), we deduce that

\[ u \in L^4(\Omega) \]

In the last inequality above, we used the divergence-free condition \( u_\theta = -(rv)_\theta \). By (2.85) and the Poincaré inequality (3.3), we deduce that

\[ \left| \int_0^1 \int_0^{2\pi} r^2 \partial_r (\chi(r) v^a_p) d\theta dr \right| \leq C \int_0^1 \int_0^{2\pi} r v^2_{\partial} d\theta dr. \]

Thus, we obtain

\[ |I_1| \leq C \int_0^1 \int_0^{2\pi} r (u^2_{\partial} + v^2_{\partial}) d\theta dr + \frac{\varepsilon^2}{20} \int_0^1 \int_0^{2\pi} r u^2_{\partial} d\theta dr. \]

2) Estimate of \( I_2 \): We decompose \( u(\theta, r) = u_0(r) + \tilde{u}(\theta, r) \), where \( u_0(r) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta, r) d\theta \). Noticing that \( \int_0^{2\pi} v^a(r, \theta) d\theta = 0 \), \( \forall r \in (0, 1] \), we obtain

\[ \int_0^1 \int_0^{2\pi} u^2 (u^2_{\partial} + v^2_p) d\theta dr = \int_0^1 \int_0^{2\pi} (u^2_{\partial} + v^2_p) (2u_0 \tilde{u} + \tilde{u}^2) d\theta dr \\
\quad = \int_0^1 \int_0^{2\pi} (\partial_{\theta} u^2_{p} + v^2_p) (2u_0 \tilde{u} + \tilde{u}^2) d\theta dr \\
\quad + \int_0^1 \int_0^{2\pi} (\partial_{\theta} \tilde{u}^2_{p} + v^2_p) (2u_0 \tilde{u} + \tilde{u}^2) d\theta dr. \]

Moreover, using (2.84), the Poincaré inequality and Hardy inequality (3.4), we deduce that

\[ |I_{21}| \leq C \varepsilon \int_0^1 \int_0^{2\pi} r^2 |2u_0 \tilde{u} + \tilde{u}^2| d\theta dr \\
\leq C \varepsilon \int_0^1 \int_0^{2\pi} r u^2_{\partial} d\theta dr + C \varepsilon \int_0^1 \int_0^{2\pi} r \tilde{u}^2 d\theta dr \\
\leq C \varepsilon \int_0^1 \int_0^{2\pi} r u^2_{\partial} d\theta dr + C \varepsilon \int_0^1 \int_0^{2\pi} r u^2_{\partial} d\theta dr \\
\leq C \int_0^1 \int_0^{2\pi} r^2 u^2_{\partial} d\theta dr + \frac{\varepsilon^2}{20} \int_0^1 \int_0^{2\pi} r u^2_{\partial} d\theta dr. \]

Moreover, noticing that \( \chi(r) = 0 \) for \( r \leq \frac{1}{2} \), by (2.85) and the Hardy inequality (3.6), we deduce that

\[ |I_{22}| \leq 2\varepsilon \left( \int_0^1 \int_0^{2\pi} r \tilde{u}^2 d\theta dr \right)^{\frac{1}{2}} \left( \int_0^1 \int_0^{2\pi} r^2 u^2_{\partial} (r - 1)^2 d\theta dr \right)^{\frac{1}{2}} \]

\[ \leq C \varepsilon \left( \int_0^1 \int_0^{2\pi} r \tilde{u}^2 d\theta dr \right)^{\frac{1}{2}} \left( \int_0^1 \int_0^{2\pi} r^2 u^2_{\partial} (r - 1)^2 d\theta dr \right)^{\frac{1}{2}} \]
\[ + C \varepsilon \left( \int_0^1 \int_0^{2\pi} r u^2 \, d\theta \, dr \right)^{\frac{1}{2}} \left( \int_0^1 \int_0^{2\pi} r u_0^2 (r) \, d\theta \, dr \right)^{\frac{1}{2}} + C \int_0^1 \int_0^{2\pi} r u_0^2 \, d\theta \, dr \]

\[ \leq C \varepsilon \eta \left( \int_0^1 \int_0^{2\pi} r u^2 \, d\theta \, dr \right)^{\frac{1}{2}} \left( \int_0^1 \int_0^{2\pi} r (u_0)'(r)^2 \, d\theta \, dr \right)^{\frac{1}{2}} + C \int_0^1 \int_0^{2\pi} r u_0^2 \, d\theta \, dr \]

\[ \leq C \int_0^1 \int_0^{2\pi} r u_0^2 \, d\theta \, dr + \frac{\varepsilon^2}{20} \int_0^1 \int_0^{2\pi} r u_0^2 \, d\theta \, dr. \] (3.10)

Thus, we obtain

\[ |I_2| \leq C \int_0^1 \int_0^{2\pi} r u_0^2 \, d\theta \, dr + \frac{\varepsilon^2}{10} \int_0^1 \int_0^{2\pi} r u_0^2 \, d\theta \, dr. \]

3) Estimate of \( I_3 \): First, there holds

\[ I_3 = \int_0^1 \int_0^{2\pi} v (\partial_\theta v e^a - u e^a + ar) \, d\theta \, dr + \int_0^1 \int_0^{2\pi} v \left( \partial_\theta \tilde{v}_p^a - \tilde{u}_p^a \right) \, d\theta \, dr. \]

Using (2.84) and (3.4), we deduce that

\[ |I_{31}| \leq C \varepsilon \eta \int_0^1 \int_0^{2\pi} r |uv| \, d\theta \, dr \]

\[ \leq C \int_0^1 \int_0^{2\pi} r u_0^2 \, d\theta \, dr + \frac{\varepsilon^2}{20} \int_0^1 \int_0^{2\pi} r u_0^2 \, d\theta \, dr. \]

The similar estimate as (3.10) gives

\[ |I_{32}| \leq C \int_0^1 \int_0^{2\pi} r u_0^2 \, d\theta \, dr + \frac{\varepsilon^2}{20} \int_0^1 \int_0^{2\pi} r u_0^2 \, d\theta \, dr. \]

Thus, there holds

\[ |I_3| \leq C \int_0^1 \int_0^{2\pi} r u_0^2 \, d\theta \, dr + \frac{\varepsilon^2}{10} \int_0^1 \int_0^{2\pi} r u_0^2 \, d\theta \, dr. \]

Finally, summing the estimates of \( I_1, I_2 \) and \( I_3 \), we obtain

\[ \left| \int_0^1 \int_0^{2\pi} (u S_u + v S_v) \, d\theta \, dr \right| \leq \frac{\varepsilon^2}{3} \int_0^1 \int_0^{2\pi} r (u_r^2 + v_r^2) \, d\theta \, dr \]

\[ + C \int_0^1 \int_0^{2\pi} r (v_0^2 + u_0^2) \, d\theta \, dr. \] (3.11)

Combining the estimates (3.8), (3.9) and (3.11), we obtain (3.7). \( \square \)
3.2.2. Positivity estimate

In this subsection, we establish the following positivity estimate.

**Lemma 3.3.** Let \((u, v)\) be a bounded solution of (3.2), then there exists \(\eta_0 > 0\) such that for any \(\eta \in (0, \eta_0)\), there holds

\[
\int_0^1 \int_0^{2\pi} r(u_\theta^2 + v_\theta^2) d\theta dr \leq C \varepsilon^2 \eta \left( \int_0^1 \int_0^{2\pi} \frac{(u_\theta + v)^2 + (v_\theta - u)^2}{r} d\theta dr \\
+ \int_0^1 \int_0^{2\pi} r(u_\theta^2 + v_\theta^2) d\theta dr \right) \\
+ C \int_0^1 \int_0^{2\pi} [u_\theta F_u + v_\theta F_v] d\theta dr.
\] (3.12)

**Proof.** Multiplying the first equation by \(u_\theta\) and the second equation by \(v_\theta\), integrating in \(\Omega\) and summing two terms together, we arrive at

\[
\int_0^1 \int_0^{2\pi} \left[ -\varepsilon^2 (ru_{rr} + \frac{u_\theta \theta}{r} + \frac{2v_\theta}{r} + u_r - \frac{u}{r}) + p_\theta + S_u \right] u_\theta d\theta dr \\
+ \int_0^1 \int_0^{2\pi} \left[ -\varepsilon^2 (rv_{rr} + \frac{v_\theta \theta}{r} - \frac{2u_\theta}{r} + v_r - \frac{v}{r}) + r p_r + S_v \right] v_\theta d\theta dr \\
= \int_0^1 \int_0^{2\pi} [u_\theta F_u + v_\theta F_v] d\theta dr.
\] (3.13)

**Positivity term:** We first deal with these terms which are related to \(S_u, S_v\). To simplify notations, we set \(\bar{u} = u^a - ar, \bar{v} = v^a\). Then we deduce that

\[
\int_0^1 \int_0^{2\pi} [S_u u_\theta + S_v v_\theta] d\theta dr \\
= \int_0^1 \int_0^{2\pi} (aru_\theta + v r a + v a r) u_\theta d\theta dr + \int_0^1 \int_0^{2\pi} (a rv_\theta - 2u a r) v_\theta d\theta dr \\
+ \int_0^1 \int_0^{2\pi} (\bar{u} u_\theta + \bar{v} r u_r + u u_\theta + v r \bar{u}_r + \bar{v} u + \bar{v} \bar{u}) u_\theta d\theta dr \\
+ \int_0^1 \int_0^{2\pi} (\bar{v} v_\theta + \bar{v} r v_r + u \bar{v}_\theta + v r \bar{v}_r - 2u \bar{u}) v_\theta d\theta dr \\
= \underbrace{\int_0^1 \int_0^{2\pi} (aru_\theta + 2ar v) u_\theta d\theta dr}_{I_1} + \int_0^1 \int_0^{2\pi} (a rv_\theta - 2ar u) v_\theta d\theta dr \\
+ \underbrace{\int_0^1 \int_0^{2\pi} (\bar{v} r u_\theta + \bar{u} r v_r v_\theta) d\theta dr}_{I_2} + \int_0^1 \int_0^{2\pi} (r u_\theta u_\theta + r v_\theta v_\theta) d\theta dr}_{I_3} \\
+ \int_0^1 \int_0^{2\pi} (ru_r u_\theta + rv_r v_\theta) d\theta dr + \int_0^1 \int_0^{2\pi} (rv_r u_\theta + rv_r v_\theta) d\theta dr.
\]
Prandtl–Batchelor Flows on a Disk

\[ + \int_0^1 \int_0^{2\pi} (\bar{u}_\theta + \bar{v})uu_\theta d\theta dr + \int_0^1 \int_0^{2\pi} \bar{u}_\theta (v + u_\theta) d\theta dr \]

\[ + \int_0^1 \int_0^{2\pi} (v_\theta - u)\bar{u}v_\theta d\theta dr + \int_0^1 \int_0^{2\pi} u\bar{u}_\theta (\bar{v}_\theta - \bar{u}) d\theta dr . \]

Direct computation gives that

\[ I_1 = \int_0^1 \int_0^{2\pi} ar(u_\theta^2 + v_\theta^2) d\theta dr. \]

Next, we estimate \( \{I_i\}_{i=2}^7 \) term by term.

(1) **Estimate of \( I_2 \).** Using (2.84) and (2.85), we obtain

\[
\left| \int_0^1 \int_0^{2\pi} \bar{v}ru_ru_\theta d\theta dr \right| \\
\leq C \varepsilon \eta \left( \int_0^1 \int_0^{2\pi} ru_r^2 d\theta dr \right)^{1/2} \left( \int_0^1 \int_0^{2\pi} ru_\theta^2 d\theta dr \right)^{1/2} \\
\leq \eta \int_0^1 \int_0^{2\pi} ru_\theta^2 d\theta dr + C \varepsilon^2 \eta \int_0^1 \int_0^{2\pi} ru_r^2 d\theta dr.
\]

Similarly, there holds

\[
\left| \int_0^1 \int_0^{2\pi} \bar{v}r_v r_v v_\theta d\theta dr \right| \leq \eta \int_0^1 \int_0^{2\pi} r v_\theta^2 d\theta dr + C \varepsilon^2 \eta \int_0^1 \int_0^{2\pi} r v_r^2 d\theta dr.
\]

Thus, we obtain

\[ I_2 \leq \eta \int_0^1 \int_0^{2\pi} r (u_\theta^2 + v_\theta^2) d\theta dr + C \varepsilon^2 \eta \int_0^1 \int_0^{2\pi} r (u_r^2 + v_r^2) d\theta dr. \]

(2) **Estimate of \( I_3 \).** Firstly, by (2.84) and the Poincaré inequality (3.3), we obtain

\[
\left| \int_0^1 \int_0^{2\pi} r v \partial_r (u_e^a - ar) u_\theta d\theta dr \right| \leq C \varepsilon \eta \left( \int_0^1 \int_0^{2\pi} ru_e^2 d\theta dr \right)^{1/2} \left( \int_0^1 \int_0^{2\pi} ru_\theta^2 d\theta dr \right)^{1/2},
\]

\[
\left| \int_0^1 \int_0^{2\pi} r v \partial_r v_e^a v_\theta d\theta dr \right| \leq C \varepsilon \eta \int_0^1 \int_0^{2\pi} r v_\theta^2 d\theta dr.
\]

Then, noticing that \( \chi(r) = 0 \) for \( r \leq \frac{1}{2} \), by (2.85) and the Hardy inequality (3.6), we have

\[
\left| \int_0^1 \int_0^{2\pi} r v \partial_r (\chi u_p^a) u_\theta d\theta dr \right|
\]

\[ = \left| \int_0^1 \int_0^{2\pi} r v \chi'_p u_p^a u_\theta d\theta dr + \int_0^1 \int_0^{2\pi} r v \chi \partial_r u_p^a u_\theta d\theta dr \right| \]
\[
\begin{align*}
&= \int_0^1 \int_0^{2\pi} rv \chi' u^a_p u \theta d\theta dr + \int_0^1 \int_0^{2\pi} \frac{rv}{r-1} \chi Y \partial_Y u^a_p u \theta d\theta dr \\
&\leq C \eta \int_0^1 \int_0^{2\pi} r(u^2_\theta + v^2_\theta) d\theta dr.
\end{align*}
\]

By (2.85) and the Poincaré inequality (3.3), we deduce that
\[
\left| \int_0^1 \int_0^{2\pi} rv \partial_r (\chi v^a_p) v \theta d\theta dr \right| = \left| \int_0^1 \int_0^{2\pi} rv \chi' v^a_p v \theta d\theta dr + \int_0^1 \int_0^{2\pi} rv \chi \partial_r v^a_p v \theta d\theta dr \right| \leq C \eta \int_0^1 \int_0^{2\pi} rv^2_\theta d\theta dr.
\]

Thus, there holds
\[
|I_3| \leq C \eta \int_0^1 \int_0^{2\pi} r(u^2_\theta + v^2_\theta) d\theta dr.
\]

(3) Estimate of \(I_4\) and \(I_7\). We first decompose \(I_4\) into two parts
\[
I_4 = \int_0^1 \int_0^{2\pi} (\partial_\theta u^a_p + v^a_p) uu_\theta d\theta dr + \int_0^1 \int_0^{2\pi} (\partial_\theta \tilde{u}^a_p + \tilde{v}^a_p) uu_\theta d\theta dr.
\]

By (2.84) and (3.4), we deduce that
\[
\begin{align*}
|I_{41}| &\leq C \varepsilon \eta \int_0^1 \int_0^{2\pi} ruu_\theta dr d\theta \\
&\leq C \varepsilon \eta \left( \int_0^1 \int_0^{2\pi} ru^2_\theta dr d\theta \right)^{\frac{1}{2}} \left( \int_0^1 \int_0^{2\pi} ru^2_\theta dr d\theta \right)^{\frac{1}{2}} \\
&\leq \eta \int_0^1 \int_0^{2\pi} ru^2_\theta dr d\theta + C \varepsilon^2 \eta \int_0^1 \int_0^{2\pi} ru^2_\theta dr d\theta.
\end{align*}
\]

Moreover, noticing that \(\chi(r) = 0\) for \(r \leq \frac{1}{2}\), by (2.85) and the Hardy inequality (3.6), we deduce that
\[
\begin{align*}
|I_{42}| &= \varepsilon \left| \int_0^1 \int_0^{2\pi} Y(\partial_\theta \tilde{u}^a_p + \tilde{v}^a_p) \frac{uu_\theta}{r-1} d\theta dr \right| \\
&\leq \eta \int_0^1 \int_0^{2\pi} ru^2_\theta dr d\theta + C \varepsilon^2 \eta \int_0^1 \int_0^{2\pi} ru^2_\theta dr d\theta.
\end{align*}
\]

Thus, there holds
\[
|I_4| \leq \eta \int_0^1 \int_0^{2\pi} ru^2_\theta dr d\theta + C \varepsilon^2 \eta \int_0^1 \int_0^{2\pi} ru^2_\theta dr d\theta.
\]

Same argument gives
\[
|I_7| \leq \eta \int_0^1 \int_0^{2\pi} rv^2_\theta dr d\theta + C \varepsilon^2 \eta \int_0^1 \int_0^{2\pi} rv^2_\theta dr d\theta.
\]
Thus, there holds

\[ |I_4| + |I_7| \leq \eta \int_0^1 \int_0^{2\pi} r(u_\theta^2 + v_\theta^2) d\theta dr + C \varepsilon^2 \eta \int_0^1 \int_0^{2\pi} ru^2_\gamma d\theta dr. \]

(4) Estimate of \(I_5\) and \(I_6\). By (2.84) and (2.85), we deduce that

\[ |I_5| \leq \left| \int_0^1 \int_0^{2\pi} \chi(r)u_p^{(0)}u_\theta (v + u_\theta) d\theta dr \right| \]

\[ + C \varepsilon \eta \left( \int_0^1 \int_0^{2\pi} ru^2_\gamma d\theta dr \right)^{\frac{1}{2}} \left( \int_0^1 \int_0^{2\pi} \frac{(u_\theta + v)^2}{r} d\theta dr \right)^{\frac{1}{2}} \]

\[ \leq I_{51} + \eta \int_0^1 \int_0^{2\pi} ru^2_\gamma d\theta dr + C \varepsilon^2 \eta \int_0^1 \int_0^{2\pi} ru^2_\gamma d\theta dr. \]

Moreover, noticing that \(\chi(r) = 0\) for \(r < \frac{1}{2}\), by (2.85) and the Hardy inequality (3.6), we deduce

\[ I_{51} \leq C \eta \int_0^1 \int_0^{2\pi} ru^2_\gamma d\theta dr + C \varepsilon \eta \left( \int_0^1 \int_0^{2\pi} ru^2_\gamma d\theta dr \right)^{\frac{1}{2}} \left( \int_0^1 \int_0^{2\pi} ru^2_\gamma d\theta dr \right)^{\frac{1}{2}}. \]

Thus, there holds

\[ |I_5| \leq C \eta \int_0^1 \int_0^{2\pi} ru^2_\gamma d\theta dr + C \varepsilon^2 \eta \left( \int_0^1 \int_0^{2\pi} \frac{(u_\theta + v)^2}{r} d\theta dr + \int_0^1 \int_0^{2\pi} ru^2_\gamma d\theta dr \right). \]

Similar argument gives that

\[ |I_6| \leq C \eta \int_0^1 \int_0^{2\pi} rv^2_\gamma d\theta dr + C \varepsilon^2 \eta \left( \int_0^1 \int_0^{2\pi} \frac{(v_\theta - u)^2}{r} d\theta dr + \int_0^1 \int_0^{2\pi} ru^2_\gamma d\theta dr \right). \]

Thus, we deduce that

\[ |I_5| + |I_6| \]

\[ \leq C \eta \int_0^1 \int_0^{2\pi} r(u_\theta^2 + v_\theta^2) d\theta dr \]

\[ + C \varepsilon^2 \eta \left( \int_0^1 \int_0^{2\pi} \frac{(u_\theta + v)^2}{r} + \frac{(v_\theta - u)^2}{r} d\theta dr + \int_0^1 \int_0^{2\pi} ru^2_\gamma + \frac{v^2_\gamma}{r} d\theta dr \right). \]

Finally, collecting the estimates for \(I_1, \ldots, I_7\), we can choose \(\eta_0 > 0\) such that for any \(\eta \in (0, \eta_0)\), there holds
\[
\int_0^1 \int_0^{2\pi} [S_u u_\theta + S_v v_\theta] d\theta dr \\
\geq (a - C\eta) \int_0^1 \int_0^{2\pi} r(u_\theta^2 + v_\theta^2) d\theta dr \\
-C\varepsilon^2 \eta \left( \int_0^1 \int_0^{2\pi} r(u_\theta^2 + v_\theta^2) d\theta dr + \int_0^1 \int_0^{2\pi} \frac{(u_\theta + v)^2 + (v_\theta - u)^2}{r} d\theta dr \right).
\]

\text{(3.14)}

**Pressure estimate:** Integrating by parts, we deduce that
\[
\int_0^1 \int_0^{2\pi} p_\theta u_\theta d\theta dr + \int_0^1 \int_0^{2\pi} r p_r v_\theta d\theta dr = 0.
\]

\text{(3.15)}

**Diffusion term:** Finally, we deal with the diffusion term:
\[
\int_0^1 \int_0^{2\pi} \left[ -\varepsilon^2 \left(r u_{\theta\theta} + \frac{u_\theta}{r} + 2 \frac{v_\theta}{r} + u_r - \frac{u}{r} \right) \right] u_\theta d\theta dr \\
+ \int_0^1 \int_0^{2\pi} \left[ -\varepsilon^2 \left(r v_{\theta\theta} + \frac{v_\theta}{r} - 2 \frac{u_\theta}{r} + v_r - \frac{v}{r} \right) \right] v_\theta d\theta dr \\
= -\varepsilon^2 \int_0^1 \int_0^{2\pi} \left(r u_{\theta r} + u_r\right) u_\theta d\theta dr - \varepsilon^2 \int_0^1 \int_0^{2\pi} \left(r v_{\theta r} + v_r\right) v_\theta d\theta dr = 0.
\]

\text{(3.16)}

Finally, collecting the estimates (3.14), (3.15) and (3.16) together, we obtain (3.12) which completes the proof of this lemma.

\text{\hfill \Box}

3.2.3. Linear stability estimate.

**Proposition 3.4.** Let \((u, v)\) be a bounded solution of (3.2), then there exist \(\varepsilon_0 > 0, \eta_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0)\), there holds
\[
\int_0^1 \int_0^{2\pi} \left(r u_\theta^2 + v_\theta^2\right) d\theta dr + \varepsilon^2 \left( \int_0^1 \int_0^{2\pi} \frac{(u_\theta + v)^2 + (v_\theta - u)^2}{r} d\theta dr \\
+ \int_0^1 \int_0^{2\pi} r \left(u_r^2 + v_r^2\right) d\theta dr \right) \\
\leq C \varepsilon^{-2} \int_0^1 \int_0^{2\pi} \left( \frac{F_u^2}{r} + \frac{F_v^2}{r} \right) d\theta dr.
\]

\text{(3.17)}

**Proof.** By combining the estimates (3.7) and (3.13), we can choose \(\varepsilon_0 > 0, \eta_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0)\), there holds
\[
\int_0^1 \int_0^{2\pi} \left(r u_\theta^2 + v_\theta^2\right) d\theta dr + \varepsilon^2 \left( \int_0^1 \int_0^{2\pi} \frac{(u_\theta + v)^2 + (v_\theta - u)^2}{r} d\theta dr \\
+ \int_0^1 \int_0^{2\pi} r \left(u_r^2 + v_r^2\right) d\theta dr \right) \\
\leq C \int_0^1 \int_0^{2\pi} \left(u F_u + v F_v\right) d\theta dr + C \int_0^1 \int_0^{2\pi} \left[u_\theta F_u + v_\theta F_v\right] d\theta dr.
\]

\text{(3.18)}
By the Hölder inequality, we deduce that
\[
\int_0^1 \int_0^{2\pi} (u_\theta F_u + v_\theta F_v) d\theta dr \leq \left( \int_0^1 \int_0^{2\pi} r(u_{\theta\theta}^2 + v_{\theta\theta}^2) d\theta dr \right)^{\frac{1}{2}}
\left( \int_0^1 \int_0^{2\pi} \left( \frac{F_u^2}{r} + \frac{F_v^2}{r} \right) d\theta dr \right)^{\frac{1}{2}},
\]
\[
\int_0^1 \int_0^{2\pi} (u F_u + v F_v) d\theta dr \leq \left( \int_0^1 \int_0^{2\pi} r(u_{\theta r}^2 + v_{\theta r}^2) d\theta dr \right)^{\frac{1}{2}}
\left( \int_0^1 \int_0^{2\pi} (F_u^2 r + F_v^2 r) d\theta dr \right)^{\frac{1}{2}},
\]
where we used (3.4) in the second inequality. Putting this into (3.18), we obtain (3.17).

4. Existence of Error Equations

In this section, we rewrite the error equations and the associated linear stability estimate in Euler coordinates, then obtain the \( H^2 \) estimate by the Stokes estimate in a smooth bounded domain.

4.1. Error equations in Euler coordinates.

Let
\[
\vec{e}_\theta = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}, \quad \vec{e}_r = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}
\]
and
\[
\begin{pmatrix} \tilde{u}(x, y) \\ \tilde{v}(x, y) \end{pmatrix} = u(\theta, r) \vec{e}_\theta + v(\theta, r) \vec{e}_r, \quad \begin{pmatrix} \tilde{u}^a(x, y) \\ \tilde{v}^a(x, y) \end{pmatrix} = u^a(\theta, r) \vec{e}_\theta + v^a(\theta, r) \vec{e}_r, \quad \begin{pmatrix} \tilde{p}(x, y) \\ \tilde{R}_u^a(x, y) \\ \tilde{R}_v^a(x, y) \end{pmatrix} = \begin{pmatrix} \tilde{p}(\theta, r) \\ \frac{R_u^a(\theta, r)}{r} \vec{e}_\theta + \frac{R_v^a(\theta, r)}{r} \vec{e}_r \end{pmatrix}.
\]

Then the error Eq. (3.1) can be written in Euler coordinates as follows
\[
\begin{align}
\tilde{u}^a & \partial_x \tilde{u} + \tilde{v}^a \partial_y \tilde{u} + \tilde{u} \partial_x \tilde{u}^a + \tilde{v} \partial_y \tilde{u}^a + \partial_x \tilde{p} + \tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_y \tilde{u} - \varepsilon^2 \Delta \tilde{u} = \tilde{R}_u^a, \\
\tilde{u}^a & \partial_x \tilde{v} + \tilde{v}^a \partial_y \tilde{v} + \tilde{u} \partial_x \tilde{v}^a + \tilde{v} \partial_y \tilde{v}^a + \partial_y \tilde{p} + \tilde{u} \partial_x \tilde{v} + \tilde{v} \partial_y \tilde{v} - \varepsilon^2 \Delta \tilde{v} = \tilde{R}_v^a, \\
\partial_x \tilde{u} + \partial_y \tilde{v} & = 0, \\
(\tilde{u}, \tilde{v})|_{\partial B_1} & = (0, 0). \quad (4.1)
\end{align}
\]

4.2. Linear stability estimate in Euler coordinates.

Let
\[
\begin{pmatrix} \tilde{F}_u(x, y) \\ \tilde{F}_v(x, y) \end{pmatrix} = \begin{pmatrix} F_u(\theta, r) \vec{e}_\theta + \frac{F_v(\theta, r)}{r} \vec{e}_r \end{pmatrix}.
\]
Then, the Eq. (3.2) become
\[
\begin{align*}
\bar{\mathbf{a}}^a \partial_x \bar{u} + \bar{\mathbf{a}}^a \partial_y \bar{u} + \bar{\mathbf{b}}^a \partial_x \bar{a}^a + \bar{\mathbf{b}}^a \partial_y \bar{a}^a + \partial_x \bar{p} - \varepsilon^2 \Delta \bar{u} &= \bar{F}_u, \\
\bar{\mathbf{a}}^a \partial_x \bar{v} + \bar{\mathbf{a}}^a \partial_y \bar{v} + \bar{\mathbf{b}}^a \partial_x \bar{a}^a + \bar{\mathbf{b}}^a \partial_y \bar{a}^a + \partial_y \bar{p} - \varepsilon^2 \Delta \bar{v} &= \bar{F}_v, \\
\partial_x \bar{u} + \partial_y \bar{v} &= 0, \\
(\bar{u}, \bar{v})|_{\partial B_1} &= (0, 0).
\end{align*}
\] (4.2)

Moreover, from the linear stability estimate (3.17), we can deduce that there exist \(\varepsilon_0 > 0, \eta_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0)\) and any solution \((\bar{u}, \bar{v})\) of (4.2), there holds
\[
\varepsilon \|\nabla(\bar{u}, \bar{v})\|_2 \leq C \varepsilon^{-1} \|\bar{F}_u, \bar{F}_v\|_2.
\] (4.3)

4.3. Existence of error equations.

We apply the contraction mapping theorem to prove the existence of the error Eq. (3.1).

**Proposition 4.1.** There exist \(\varepsilon_0 > 0, \eta_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0)\), the error Eq. (3.1) have a unique solution \((u, v)\) which satisfies
\[
\|u, v\|_\infty \leq C \varepsilon.
\]

**Proof.** For each smooth function \((\bar{u}, \bar{v})\) which satisfies
\[
\begin{align*}
\partial_x \bar{u} + \partial_y \bar{v} &= 0, \\
(\bar{u}, \bar{v})|_{\partial B_1} &= (0, 0),
\end{align*}
\] (4.4)

we consider the following linear system
\[
\begin{align*}
\bar{\mathbf{a}}^a \partial_x \bar{u} + \bar{\mathbf{a}}^a \partial_y \bar{u} + \bar{\mathbf{b}}^a \partial_x \bar{a}^a + \bar{\mathbf{b}}^a \partial_y \bar{a}^a + \partial_x \bar{p} - \varepsilon^2 \Delta \bar{u} &= \bar{\mathbf{R}}^a_u - \bar{u} \partial_x \bar{u} - \bar{v} \partial_y \bar{u}, \\
\bar{\mathbf{a}}^a \partial_x \bar{v} + \bar{\mathbf{a}}^a \partial_y \bar{v} + \bar{\mathbf{b}}^a \partial_x \bar{a}^a + \bar{\mathbf{b}}^a \partial_y \bar{a}^a + \partial_y \bar{p} - \varepsilon^2 \Delta \bar{v} &= \bar{\mathbf{R}}^a_v - \bar{u} \partial_x \bar{v} - \bar{v} \partial_y \bar{v}, \\
\partial_x \bar{u} + \partial_y \bar{v} &= 0, \\
(\bar{u}, \bar{v})|_{\partial B_1} &= (0, 0).
\end{align*}
\] (4.5)

By the linear stability estimate (4.3), we deduce that there exist \(\varepsilon_0 > 0, \eta_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0)\), there holds
\[
\varepsilon \|\nabla(\bar{u}, \bar{v})\|_2 \leq C \varepsilon^{-1} \|\bar{\mathbf{R}}^a_u - \bar{u} \partial_x \bar{u} - \bar{v} \partial_y \bar{u}, \bar{\mathbf{R}}^a_v - \bar{u} \partial_x \bar{v} - \bar{v} \partial_y \bar{v}\|_2
\leq C \varepsilon^{-1} \|\bar{\mathbf{R}}^a_u, \bar{\mathbf{R}}^a_v\|_2 + C \varepsilon^{-1} \|\bar{u}, \bar{v}\|_\infty \|\nabla(\bar{u}, \bar{v})\|_2.
\] (4.6)

Then, by Stokes estimates in smooth domain, we deduce that
\[
\varepsilon^2 \|\nabla^2(\bar{u}, \bar{v})\|_2 \leq C \|\bar{\mathbf{R}}^a_u - \bar{u} \partial_x \bar{u} - \bar{v} \partial_y \bar{u} - (\bar{\mathbf{a}}^a \partial_x \bar{u} + \bar{\mathbf{a}}^a \partial_y \bar{u} + \bar{\mathbf{b}}^a \partial_x \bar{a}^a + \bar{\mathbf{b}}^a \partial_y \bar{a}^a)\|_2
+ C \|\bar{\mathbf{R}}^a_v - \bar{u} \partial_x \bar{v} - \bar{v} \partial_y \bar{v} - (\bar{\mathbf{a}}^a \partial_x \bar{v} + \bar{\mathbf{a}}^a \partial_y \bar{v} + \bar{\mathbf{b}}^a \partial_x \bar{a}^a + \bar{\mathbf{b}}^a \partial_y \bar{a}^a)\|_2
\leq C \|\bar{\mathbf{R}}^a_u, \bar{\mathbf{R}}^a_v\|_2 + C \|\bar{u}, \bar{v}\|_\infty \|\nabla(\bar{u}, \bar{v})\|_2
+ C \|\bar{u}^a \partial_x \bar{u} + \bar{v}^a \partial_y \bar{u} + \bar{u}^a \partial_x \bar{a}^a + \bar{v}^a \partial_y \bar{a}^a\|_2
+ C \|\bar{u}^a \partial_x \bar{v} + \bar{v}^a \partial_y \bar{v} + \bar{u}^a \partial_x \bar{a}^a + \bar{v}^a \partial_y \bar{a}^a\|_2.
\]

By using \(\|(\bar{\mathbf{a}}^a, \bar{\mathbf{b}}^a)\|_\infty \leq C\), it is easy to obtain
\[
\|\bar{\mathbf{a}}^a \partial_x \bar{u} + \bar{\mathbf{b}}^a \partial_y \bar{u}\|_2 + \|\bar{\mathbf{a}}^a \partial_x \bar{v} + \bar{\mathbf{b}}^a \partial_y \bar{v}\|_2 \leq C \|\nabla(\bar{u}, \bar{v})\|_2.
\]
Moreover, let
\[
\left( \begin{array}{c}
\tilde{u}(x, y) \\
\tilde{v}(x, y)
\end{array} \right) = \hat{u}(\theta, r)\tilde{e}_\theta + \hat{v}(\theta, r)\tilde{e}_r,
\]
then there holds
\[
\|\tilde{u}\partial_x\tilde{u}^a + \tilde{v}\partial_y\tilde{u}^a\|^2_2 + \|\tilde{u}\partial_x\tilde{v}^a + \tilde{v}\partial_y\tilde{v}^a\|^2_2
\leq C \int_0^1 \int_0^{2\pi} r[\hat{u}^2(\theta, r) + \hat{v}^2(\theta, r)]d\theta dr + C \int_0^1 \int_0^{2\pi} \frac{r\hat{v}^2(\theta, r)}{(r - 1)^2} d\theta dr.
\]
Obviously, there holds
\[
\int_0^1 \int_0^{2\pi} r[\hat{u}^2(\theta, r) + \hat{v}^2(\theta, r)]d\theta dr = \|\tilde{u}, \tilde{v}\|^2_2.
\]
By the Hardy inequality (3.5), there holds
\[
\int_0^1 \int_0^{2\pi} \frac{r\hat{v}^2(\theta, r)}{(r - 1)^2} d\theta dr \leq C \int_0^1 \int_0^{2\pi} r(\partial_r\hat{v})^2 d\theta dr \leq C \|\nabla(\tilde{u}, \tilde{v})\|^2_2.
\]
Hence, we have
\[
\varepsilon^2 \|\nabla^2(\tilde{u}, \tilde{v})\|_2 \leq C\|((\tilde{R}^a_u, \tilde{R}^a_v))\|_2 + C\|(\tilde{u}, \tilde{v})\|_\infty\|\nabla(\tilde{u}, \tilde{v})\|_2 + C\|\nabla(\tilde{u}, \tilde{v})\|_2. \tag{4.7}
\]
Set
\[
\|\!(u, v)\!\|_Y := \varepsilon\|\nabla(u, v)\|_2 + \varepsilon^3 \|\nabla^2(u, v)\|_2.
\]
By (4.6) and (4.7), we deduce that for any \(\varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0),\) there holds
\[
\|((\tilde{u}, \tilde{v}))\|_Y \leq C\varepsilon^{-1}\|((\tilde{R}^a_u, \tilde{R}^a_v))\|_2 + C\varepsilon^{-1}\|((\tilde{u}, \tilde{v}))\|_\infty\|\nabla(\tilde{u}, \tilde{v})\|_2. \tag{4.8}
\]
Moreover, by the Sobolev embedding, we have
\[
\|((\tilde{u}, \tilde{v}))\|_\infty \leq C\|\nabla(\tilde{u}, \tilde{v})\|^\frac{3}{2} \|\nabla^2(\tilde{u}, \tilde{v})\|^\frac{1}{2}.
\]
Thus, we obtain
\[
\|((\tilde{u}, \tilde{v}))\|_Y \leq C\varepsilon^{-1}\|((\tilde{R}^a_u, \tilde{R}^a_v))\|_2 + C\varepsilon^{-\frac{2}{3}}\|((\tilde{u}, \tilde{v}))\|_Y^\frac{2}{3}. \tag{4.9}
\]
Let \(Y = \{(u, v) \in C^\infty : (u, v) \text{ satisfies (4.4) and } \|((u, v))\|_Y < +\infty\}.\) Thus, due to
\[
\|((\tilde{R}^a_u, \tilde{R}^a_v))\|_2 \leq C\varepsilon^5,
\]
there exist \(\varepsilon_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0),\) the mapping
\[
((\tilde{u}, \tilde{v})) \mapsto ((\tilde{u}, \tilde{v})
\]
maps the ball \(\{(u, v) : \|((u, v))\|_Y \leq 2C^2\varepsilon^4\}\) in \(Y\) into itself.
Moreover, for every two pairs \((\tilde{u}_1, \tilde{v}_1)\) and \((\tilde{u}_2, \tilde{v}_2)\) in this ball, we have
\[
\| (\tilde{u}_1 - \tilde{u}_2, \tilde{v}_1 - \tilde{v}_2) \|_Y \leq C \varepsilon^{-2} \left( \| (\tilde{u}_1, \tilde{v}_1) \|_Y + \| (\tilde{u}_2, \tilde{v}_2) \|_Y \right) \| (\tilde{u}_1 - \tilde{u}_2, \tilde{v}_1 - \tilde{v}_2) \|_Y.
\]
(4.10)

In fact, set
\[
\tilde{U} := \tilde{u}_1 - \tilde{u}_2, \quad \tilde{V} := \tilde{v}_1 - \tilde{v}_2, \quad \tilde{P} = \tilde{p}_1 - \tilde{p}_2,
\]
then there holds
\[
\begin{cases}
-\varepsilon^2 \Delta \tilde{U} + \partial_x \tilde{P} + \tilde{v}^a \partial_x \tilde{U} + \tilde{v} \partial_y \tilde{U} + \tilde{U} \partial_x \tilde{u}^a + \tilde{V} \partial_y \tilde{u}^a = \tilde{R}_U, \\
-\varepsilon^2 \Delta \tilde{V} + \partial_y \tilde{P} + \tilde{u}^a \partial_y \tilde{V} + \tilde{V} \partial_y \tilde{v}^a + \tilde{U} \partial_x \tilde{v}^a = \tilde{R}_V, \\
\partial_x \tilde{U} + \partial_y \tilde{V} = 0,
\end{cases}
\]
where
\[
\begin{align*}
\tilde{R}_U &= -\tilde{u}_1 \partial_x (\tilde{u}_1 - \tilde{u}_2) - \tilde{v}_1 \partial_y (\tilde{u}_1 - \tilde{u}_2) - (\tilde{u}_1 - \tilde{u}_2) \partial_x \tilde{u}_2 - (\tilde{v}_1 - \tilde{v}_2) \partial_y \tilde{u}_2, \\
\tilde{R}_V &= -\tilde{u}_1 \partial_x (\tilde{v}_1 - \tilde{v}_2) - \tilde{v}_1 \partial_y (\tilde{v}_1 - \tilde{v}_2) - (\tilde{u}_1 - \tilde{u}_2) \partial_x \tilde{v}_2 - (\tilde{v}_1 - \tilde{v}_2) \partial_y \tilde{v}_2.
\end{align*}
\]

Thus, following the estimate (4.9) line by line, we obtain (4.10).

Hence, there exist \(\varepsilon_0 > 0, \eta_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0)\), the mapping
\[(u, v) \mapsto (\tilde{u}, \tilde{v})\]
maps the ball \(\{(u, v) : \| (u, v) \|_Y \leq 2C^2 \varepsilon^4\}\) in \(Y\) into itself and is a contraction mapping. Thus, for any \(\varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0)\), the error Eq. (4.1) have a unique solution \((\tilde{u}, \tilde{v})\) which satisfies
\[
\| (\tilde{u}, \tilde{v}) \|_Y \leq C \varepsilon^4.
\]

Hence, there holds \(\| (\tilde{u}, \tilde{v}) \|_\infty \leq C \varepsilon\). Noticing that
\[
u(\theta, r) = \left( \frac{\tilde{u}(x, y)}{\tilde{v}(x, y)} \right) \cdot \tilde{e}_\theta, \quad v(\theta, r) = \left( \frac{\tilde{u}(x, y)}{\tilde{v}(x, y)} \right) \cdot \tilde{e}_r,
\]
we deduce \(\| (u, v) \|_\infty \leq C \varepsilon\). This completes the proof of this proposition. \(\square\)

5. Proof of Theorem 1.1

Finally, we give the Proof of Theorem 1.1.

Proof. Combining Proposition 4.1 and the approximate solution (2.82) of the Navier–Stokes Eq. (1.3), we easily obtain Theorem 1.1. \(\square\)
Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

6. Appendix

Appendix A: Constant coefficient periodic PDE. In this appendix, we give a brief argument to solve the following problem

\[
\begin{align*}
(Q_0)_\theta &= u_e(1)(Q_0)_\psi, \\
Q_0(\theta, \psi) &= Q_0(\theta + 2\pi, \psi), \\
Q_0|_{\psi=0} &= g(\theta), \\
Q_0|_{\psi\to-\infty} &= 0,
\end{align*}
\]

where

\[g(\theta) = \alpha^2 + 2\alpha\eta f(\theta) + \eta^2 f^2(\theta) - u_e^2(1) = 2\alpha\eta f(\theta) + \eta^2 f^2(\theta) - \frac{\eta^2}{2\pi} \int_0^{2\pi} f^2(\theta)d\theta.\]

Let \(Q_0(\theta, \psi) = \sum_{k \in \mathbb{Z}} e^{ik\theta} Q_{0k}(\psi)\) and substitute it into (6.1), we obtain

\[
\begin{align*}
\begin{cases}
  ik Q_{0k} = u_e(1) Q''_{0k}, \\
  Q_{0k}|_{\psi=0} = \hat{g}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} g(\theta)d\theta, \\
  Q_{0k}|_{\psi\to-\infty} = 0.
\end{cases}
\end{align*}
\]

It is easy to get

\[Q_{0k}(\psi) = \hat{g}(k)e^{\alpha_k \psi}\]

with \(\alpha_k = \sqrt{\frac{|k|}{2\pi u_e(1)} (1 + \text{sgn} k \cdot i)}\). Then

\[Q_0(\theta, \psi) = \sum_{k \in \mathbb{Z}} e^{ik\theta} \hat{g}(k)e^{\alpha_k \psi} \in X\]

and

\[\|Q_0\|_X \leq C\eta.\]
Appendix B: Construction of corrector $h(\theta, r)$. In this section, we give a construction of the corrector $h(\theta, r)$. Firstly, we give a simple lemma.

**Lemma 6.1.** Assume that $K(\theta, r)$ is a $2\pi$-periodic smooth function which satisfies

$$\int_0^{2\pi} K(\theta, r)d\theta = 0, \ \forall r \in (0, 1]; \ K(\theta, 1) = 0,$$

then there exists a $2\pi$-periodic function $h(\theta, r)$ such that

$$\partial_\theta h(\theta, r) = K(\theta, r); \ h(\theta, 1) = 0;$$

$$\int_0^{2\pi} h(\theta, r)d\theta = 0, \ \|\partial_\theta \partial_r^k h\|_2 \leq C \|\partial_\theta \partial_r^k K\|_2. \quad (6.3)$$

**Proof.** Let

$$K(\theta, r) = \sum_{n \neq 0} K_n(r)e^{in\theta}, \ K_n(1) = 0.$$  

Set

$$h(\theta, r) = \sum_{n \neq 0} \frac{K_n(r)}{in} e^{in\theta}.$$  

It is easy to justify that $h(\theta, r)$ satisfies (6.3) which completes the proof.  

Next, we construct the corrector $h(\theta, r)$ by the above lemma. Direct computation gives

$$u^a_\theta + rv^a_r + v^a = \varepsilon^5 \partial_\theta h(\theta, r) + K(\theta, r),$$

where

$$K(\theta, r) = \varepsilon^5 \chi(r)[Y \partial_Y v_p^{(5)}(\theta, Y) + v_p^{(5)}(\theta, Y)] + r \chi'(r)\left(\sum_{i=1}^{5} \varepsilon^i v^{(i)}_p(\theta, Y)\right).$$

Noticing that $\chi'(r) = 0, \ r \in [0, \frac{1}{2}] \cup [\frac{3}{4}, 1]$ and the property of $v^{(i)}_p$, we deduce that $K(\theta, r) = O(\varepsilon^5)$ and $K(\theta, 1) = 0$.

Moreover, noticing that

$$\int_0^{2\pi} v^{(i)}_p(\theta, Y)d\theta = 0, \ \forall Y \leq 0, \ i = 1, \cdots, 5,$$

we deduce that

$$\int_0^{2\pi} K(\theta, r)d\theta = 0, \ \forall r \in (0, 1].$$

Thus, we can choose $h(\theta, r)$ by Lemma 6.1 such that

$$\varepsilon^5 \partial_\theta h(\theta, r) + K(\theta, r) = 0, \ h(\theta, 1) = 0, \ \|\partial_\theta \partial_r^k h\|_2 \leq C \varepsilon^{-k}.$$
Appendix C: Prandtl–Batchelor theory on disk. For the convenience of readers, we give an introduction to the Prandtl–Batchelor theory. One can see [1,20,21,30,42] for more details and physical backgrounds.

**Theorem 6.2.** We consider the steady Navier–Stokes equations in two dimensional simply-connected domain $\Omega$

$$
\begin{align*}
\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \nabla p^\varepsilon - \varepsilon^2 \Delta \mathbf{u}^\varepsilon &= 0, \\
\nabla \cdot \mathbf{u}^\varepsilon &= 0, \\
\mathbf{u}^\varepsilon \cdot \mathbf{n}|_{\partial \Omega} &= 0, \quad \mathbf{u}^\varepsilon \cdot \mathbf{t}|_{\partial \Omega} = g,
\end{align*}
$$

(6.4)

where $\mathbf{n}$ is the unit normal vector to $\partial \Omega$, $\mathbf{t}$ is the unit tangential vector to $\partial \Omega$, and $g$ is a smooth function. We assume that (i) the stream function $\psi^\varepsilon$ of Eq. (6.4) has no hyperbolic critical point (i.e. nested closed streamlines and single eddy); (ii) for any $\Omega_1 \subset \subset \Omega$, there exist $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, $\Omega_1$ is away from the boundary layer of Eq. (6.4) and $\mathbf{u}^\varepsilon \to \mathbf{u}^e$ in $C^2(\Omega_1)$, where $\mathbf{u}^e$ is a solution of steady Euler equations in $\Omega$. Then the vorticity $\omega^\varepsilon = \nabla \times \mathbf{u}^\varepsilon$ is a constant in $\Omega$.

**Proof.** Let $\mathbf{u}^\varepsilon = (u^\varepsilon, v^\varepsilon)$ and $\omega^\varepsilon = \partial_\theta u^\varepsilon - \partial_r v^\varepsilon$ be the vorticity, then it is easy to obtain that

$$
\mathbf{u}^\varepsilon \cdot \nabla \omega^\varepsilon - \varepsilon \Delta \omega^\varepsilon = 0.
$$

The boundary is taken to be defined by $\psi^\varepsilon = 0$ and $0 < \psi^\varepsilon < c_1$ throughout the interior of the eddy. For any $0 < c < c_1$, integrating the Navier–Stokes equations over the domain which is surrounded by the closed streamline $\{(x, y)|\psi^\varepsilon(x, y) = c\}$ and using the divergence theorem, we obtain

$$
\int_{\{|\psi^\varepsilon = c\}} \frac{\partial \omega^\varepsilon}{\partial n} \, dl = 0, \quad \forall c \in (0, c_1).
$$

(6.5)

Moreover, due to $\mathbf{u}^\varepsilon \to \mathbf{u}^e$ in $C^2$, then there holds $\omega^\varepsilon \to \omega^e$ in $C^1$.

Let $\psi^e$ be the associated stream function of Euler equations, then $\omega^e = F(\psi^e)$. In fact, we introduce the action-angle transform $(x, y) \to (r^e, \theta^e)$, where

$$
r^e = \psi^e(x, y), \quad \frac{2\pi}{v^e(r)} = \oint_{\{|\psi^\varepsilon = r\}} \frac{1}{|\nabla \psi^\varepsilon|}, \quad \theta^e = v^e(r) \int_0^l \frac{dl'}{|\nabla \psi^\varepsilon|},
$$

and $l$ is the arc-length variable on the curve $\{|\psi^\varepsilon = r^e\}$. Then the transformation $(x, y) \to (r^e, \theta^e)$ has Jacobian 1 and the operator $\mathbf{u}^e \cdot \nabla$ becomes $u^e(r)\partial_\theta$. In $\Omega_1$, $(r^0, \theta^0)$ in $C^1$ and $r^0 = \psi^e$. Then $(\psi^e, \theta^0)$ is a coordinate system and the steady vorticity $\omega^e$ is a single-valued function $F(\psi^e)$.

Taking $\varepsilon \to 0$ in (6.5), we obtain

$$
F'(c) = 0, \quad \forall c \in (0, c_1).
$$

Thus, $\omega_e$ is a constant in $\Omega$. \(\Box\)

**Remark 6.3.** On the disk $B_1(0)$: due to $\tilde{u}_e = u_e(\theta, r)\tilde{e}_\theta + v_e(\theta, r)\tilde{e}_r$, we deduce

$$
\omega_e = \frac{1}{r}(\partial_r(ru_e) - \partial_\theta v_e) = a, \quad v_e|_{\partial \Omega} = 0.
$$

Solving this equation, we obtain

$$
(u_e, v_e) = \left(\frac{a}{2} r, 0\right).
$$

where $a$ is any constant.
References

1. Batchelor, G.K.: On steady laminar flow with closed streamlines at large Reynolds number. J. Fluid Mech. 7(1), 177–190 (1956)
2. Chen, Q., Wu, D., Zhang, Z.: On the stability of shear flows of Prandtl type for the steady Navier–Stokes equations, arXiv:2106.04173
3. Dalibard, A., Masmoudi, N.: Separation for the stationary Prandtl equation. Publ. Math. Inst. Hautes Études Sci. 130, 187–297 (2019)
4. Edwards, D.A.: Viscous boundary-layer effects in nearly inviscid cylindrical flows. Nonlinearity 10, 277–290 (1997)
5. Feynman, R.P., Lagerstrom, P.A.: Remarks on high Reynolds number flows in finite domains. Proc. IX Int. Cong. Appl. Mech. Brussels 3, 342–343 (1956)
6. Gao, C., Zhang, L.: On the steady Prandtl boundary layer expansion, arXiv:2001.10700
7. Gao, C., Zhang, L.: Remarks on the steady Prandtl boundary layer expansion, arXiv:2107.08372
8. Gerard-Varet, D., Maekawa, Y.: Sobolev stability of Prandtl expansions for the steady Navier–Stokes equations. Arch. Ration. Mech. Anal. 233(3), 1319–1382 (2019)
9. Guo, Y., Nguyen, T.: Prandtl boundary layer expansions of steady Navier-Stokes flows over a moving plate, Ann. PDE (2017). https://doi.org/10.1007/s40818-016-0020-6
10. Guo, Y., Iyer, S.: Validity of steady Prandtl layer expansions, arXiv:1805.05891v5
11. Guo, Y., Iyer, S.: Steady Prandtl layer expansions with external forcing, arXiv:1810.06662
12. Iyer, S.: Steady Prandtl boundary layer expansions over a rotating disk. Arch. Ration. Mech. Anal. 224(2), 421–469 (2017)
13. Iyer, S.: Global steady Prandtl boundary layer over a moving boundary. Peking Math. J., I: 2(2), 155–238 (2019)
14. Iyer, S.: Global steady Prandtl boundary layer over a moving boundary. Peking Math. J., II: 2(3–4), 353–437 (2019)
15. Iyer, S.: Global steady Prandtl boundary layer over a moving boundary. Peking Math. J., III: 3(1), 47–102 (2019)
16. Iyer, S.: Steady Prandtl boundary layer over a moving boundary: nonshear Euler flow. SIAM J. Math. Anal. 51(3), 1657–1685 (2019)
17. Iyer, S., Masmoudi, N.: Boundary layer expansions of steady Navier–Stokes equation, arXiv:2103.09170
18. Iyer, S., Masmoudi, N.: Global-in-x stability of steady Prandtl expansions for 2D Navier–Stokes flows, arXiv:2008.12347
19. Kim, S.-C.: On Prandtl-Batchelor theory of a cylindrical eddy: asymptotic study. SIAM J. Appl. Math. 58, 1394–1413 (1998)
20. Kim, S.-C.: On Prandtl-Batchelor theory of a cylindrical eddy: existence and uniqueness. Z. Angew. Math. Phys. 51, 674–686 (2000)
21. Kim, S.-C.: Asymptotic study of Navier–Stokes flows. Trends Math. Inf. Center Math. Sci. 6, 29–33 (2003)
22. Kim, S.C., Childress, S.: Vorticity selection with multiple eddies in two-dimensional steady flow at high Reynolds number. SIAM J. Appl. Math. 61(5), 1605–1617 (2001)
23. Kim, S.C.: A free-boundary problem for Euler flows with constant vorticity. Appl. Math. Lett. 12, 101–104 (1999)
24. Kim, S. C.: On Prandtl-Batchelor theory of steady flow at large Reynolds number, Ph.D Thesis, New York University, (1996)
25. Kim, S.C.: Batchelor-Wood formula for negative wall velocity. Phys. Fluids 11, 1685–1687 (1999)
26. Maslowe, S.A.: Critical layers in shear flows. Ann. Rev. Fluid Mech. 18(1), 405–432 (1986)
27. Moffatt, H.K., Dormy, E.: Self-exciting Fluid Dynamos. Cambridge University Press, Cambridge (2019)
28. Okamoto, H.: A variational problem arising in the two-dimensional Navier-Stokes equation with vanishing viscosity. Appl. Math. Lett. 7(1), 29–33 (1994)
29. Oleinik, O.A., Samokhin, V.N.: Mathematical Models in Boundary Layer Theory, Applied Mathematics and Mathematical Computation, 15. Champan and Hall/CRC, Boca Raton, FL (1999)
30. Pedlosky, J.: Ocean Circulation Theory. Springer-Verlag, Berlin (1996)
31. Prandtl, L.: Über Flüssigkeitsbewegung bei sehr kleiner Reibung. Verhandlungen des III. Internationalen Mathematiker Kongresses, Heidelberg, 1904, pp. 484–491, Teubner, Leizig. See Gesammelte Abhandlungen II, pp. 575–584 (1905)
32. Renardy, M.: On non-existence of steady periodic solutions of the Prandtl equations. J. Fluid Mech. 717(R7), 1–5 (2013)
35. Rhines, P.B., Young, W.R.: How rapidly is a passive scalar mixed within closed streamlines. J. Fluid Mech. 133, 133–145 (1983)
36. Rhines, P.B., Young, W.R.: Homogenization of potential vorticity in planetary gyres. J. Fluid Mech. 122, 347–367 (1982)
37. Riley, N.: High Reynolds number flows with closed streamlines. J. Eng. Math. 15, 15–27 (1981)
38. Van Wijngaarden, L.: Prandtl–Batchelor flows revisited. Fluid Dyn. Res. 39, 267–278 (2007)
39. Shen, W., Wang, Y., Zhang, Z.: Boundary layer separation and local behavior for the steady Prandtl equation. Adv. Math., 389,107896, 25pp, (2021)
40. Wang, Y., Zhang, Z.: Global $C^\infty$ regularity of the steady Prandtl equation with favorable pressure gradient. Ann. Inst. H. Poincaré Anal. Non Linéaire 38(6), 1989–2004 (2021)
41. Weiss, N.O.: The expulsion of magnetic flux by eddies. Proc. R. Soc. London, Ser. A 293, 310–328 (1966)
42. Wood, W.W.: Boundary layers whose streamlines are closed. J. Fluid Mech. 1(2), 77–87 (1957)

Communicated by A. Ionescu