Unified bulk semiclassical theory for intrinsic thermal transport and magnetization currents

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We reveal the unexpected role of the material inhomogeneity in unifying and simplifying the formulation of the intrinsic thermal and electric transport as well as magnetization currents. In particular, the inhomogeneity introduces the phase-space Berry curvature, which affects the semiclassical description of carriers in addition to the anomalous velocity, and notably, allows a general and rapid access to thermal transport and magnetization currents displaying the momentum-space Berry curvature physics. As a result, this theory does not need to invoke the edge-current picture, the thermodynamic approach to magnetizations or any mechanical counterpart of statistical forces. By introducing a fictitious inhomogeneity, this theory applies to homogeneous samples as well. Our work thus promotes the material inhomogeneity to be a basic trick in the semiclassical transport theory. We also include more general mechanical driving forces and establish the Mott relation between the resulting transport thermal and electric currents, whereas this relation for these two currents was previously only known when an electric field is the driving force.

I. INTRODUCTION

Momentum-space Berry curvature effects in various nonequilibrium phenomena in crystalline solids driven by statistical forces, namely the gradients of temperature and chemical potential ($\nabla T$ and $\nabla \mu$), have been extensively studied in recent years. Prominent examples include the anomalous and spin Nernst effects [1–4] and the thermal Hall effect [5–13]. A basic issue in these phenomena is that the macroscopic transport current of interest differs from the local one by the so-called magnetization current [1, 14]. An efficient, intuitive and systematic approach to subtracting the magnetization current from the local one is thus vital for understanding the anomalous thermoelectric and thermal transport.

Quantum theories have been formulated to address the aforementioned phenomena. For this purpose, a fictitious gravitational potential [6–10, 17–19] or a thermal vector potential [20] has to be introduced as the mechanical counterpart of the temperature gradient. Alternatively, the gauge theory of gravity (Cartan geometry) combined with the Keldysh perturbation formalism is employed, describing the magnetization of the thermal current thermodynamically as a torsional response [21]. Nevertheless, such approaches are usually technically complicated and less intuitive compared to the semiclassical ones.

Semiclassical theories, based on the Berry-curvature modified semiclassical equations of motion of the carrier wave-packets, are intuitive and thus found many applications in the intrinsic anomalous thermoelectric and thermal Hall effects [1, 2, 6, 7, 11–13]. However, when the thermal (energy) current response is considered, a thorough and systematic understanding of its transport and magnetization parts from the semiclassical theory is still lacking. On one hand, the existing semiclassical theory on the basis of bulk considerations [1] can only deal with the thermal current induced by the electric field $E$, but cannot accommodate that induced by statistical forces. On the other hand, the semiclassical theory based on the edge-current picture [6, 7] cannot include the effect of a uniform $E$ field. And how to identify the local and magnetization currents in this approach is not apparent. Moreover, these semiclassical theories assume no bulk material inhomogeneity, which is too ideal to be true in realistic samples [10].

In fact, dealing with the material inhomogeneity is inconvenient in quantum theories due to the absence of the translational symmetry. The few quantum theories that cover this situation have to avoid the momentum-space formulation [18, 19], thus hindering the manifestation of the momentum-space Berry curvature physics. On the contrary, the material inhomogeneity can be naturally incorporated into the semiclassical theory due to the locality of the wave-packet [22], although this advantage has not been fully exploited in previous considerations on thermal transport.

In this paper, based on pure bulk considerations we formulate a semiclassical theory for the thermal current response in the presence of $(E, \nabla \mu, \nabla T)$ as well as material inhomogeneity. We reveal the unexpected role of the material inhomogeneity in unifying and simplifying the formulation of the intrinsic transport and magnetization currents. It introduces the phase-space Berry curvature, which modifies the semiclassical equations of motion and the phase-space measure [23]. Notably, we show that these ingredients in addition to the anomalous velocity allow a general and rapid access to thermal transport and magnetization currents displaying the momentum-space Berry curvature physics. As a result, this theory does not need to appeal to the edge-current picture, the thermodynamic and electrodynamic approaches to various (orbital, particle, energy and thermal) magnetizations [1, 21, 24] or any mechanical counterpart of statistical forces.

The present theory also applies conveniently to homogeneous samples by introducing an auxiliary material inhomogeneity, thus providing a unified semiclassical treatment for the intrinsic thermal transport in both inhomogeneous and homogeneous systems. Differing from pre-
vious thermal transport theories which tend to avoid the material inhomogeneity in their formulations, our theory promotes the inhomogeneity to be a basic and generic trick in the semiclassical transport theory. Remarkably, unlike the trick of the fictitious gravitational field \([17, 19]\) which mimics the temperature gradient in quantum transport theories, the trick of inhomogeneity is independent of the driving force of transport, thus is not limited to the transport induced by \((E, \nabla \mu, \nabla T)\).

To show this generality, we work out the linear electric and thermal transport driven by the first-order spatial gradients of any gauge-invariant static scalar \((\varphi)\) and vector \((F)\) fields that enter the single-particle Hamiltonian in the form of \(\varphi (r) + \vec{\theta} \cdot F (r)\). Here \(\vec{\theta}\) is the observable coupled to the \(F\) field. A particular example covered is the thermoelectric transport driven by the gradient of a Zeeman field. To study transport we need to subtract the orbital and thermal magnetization currents in the presence of the aforementioned perturbations, which have not been addressed in previous theories. We establish for the first time the Mott relation between the transport thermal and electric currents induced by the generic driving forces, whereas, for these two kinds of current, textbooks \([25, 26]\) and previous studies only proved the Mott relation when an electric field is the driving force.

Our paper is organized as follows. Section II is devoted to the preliminaries of the magnetization current and the semiclassical wave-packet dynamics under nonuniform circumstances. In Sec. III we set forth the theory for obtaining the magnetization current at global equilibrium without any statistical inhomogeneity or electric field. The theory is then extended to involve both the magnetization and transport currents in the presence of statistical forces and the electric field in Sec. IV. We show that this approach applies to homogeneous samples as well in Sec. V. Finally, we include more general driving forces into our theory in Sec. VI, and conclude the paper in Sec. VII.

II. PRELIMINARIES

A. Local current and magnetization current

The local thermal current density is given by

\[ j^h (r) \equiv j^E (r) - \mu (r) j^N (r), \tag{1} \]

where \(j^E (r)\) and \(j^N (r)\) are the local energy current and local particle current densities, respectively. Since \(j^E\) and \(j^N\) are conserved currents, there can be a circulating component that is a curl of some bulk quantity and cannot be measured in transport experiments \([10]\). If the quasiparticle number is not conserved, \(j^N\) is not well defined but \(\mu = 0\), thus \(j^h (r) \equiv j^E (r)\). For carriers with a conserved charge, say, electrons with charge \(e\), the particle current implies a charge one \(j^f_e (r) = e j^N (r)\), whose circulating component is the orbital magnetization current \(j^e_{\mu, \text{mag}} = \nabla \times M^e\) according to electromagnetism, with \(M^e\) being the orbital magnetization. Given this convention, the circulating energy and particle currents can be termed as magnetization currents \(j^E_{\mu, \text{mag}} = \nabla \times M^e\) and \(j^N_{\mu, \text{mag}} = \nabla \times M^N\), with \(M^e\) and \(M^N\) being respectively the energy magnetization and particle magnetization, albeit the thermodynamic definitions of these magnetizations are not apparent. The local current density is thus composed of the transport and magnetization parts: \(j^E (r) = j^{E,N} (r) + j^{E,N}_{\mu, \text{mag}} (r)\), and then

\[ j^h (r) = j^{h, tr} (r) + j^{h, \text{mag}} (r). \tag{2} \]

Here the thermal magnetization current density \(j^{h, \text{mag}} = j^{E,N}_{\mu, \text{mag}} - \mu j^{N}_{\text{mag}}\) is given by

\[ j^{h, \text{mag}} = \nabla \times (M^e - \mu M^N) + \nabla \mu \times M^N, \tag{3} \]

which is not simply a curl of some thermal magnetization in the presence of statistical inhomogeneity.

B. Semiclassical description in nonuniform bulk

In the semiclassical theory \([23, 26]\), a Bloch electron is physically identified as a wave-packet \((|\Phi (q_c, r_c, t)\rangle\) that is constructed from the Bloch states in a particular nondegenerate band \((\text{band index } n)\) and is localized around a central position \(r_c\) and a mean crystal momentum \(q_c\). Assuming all the fields are static and vary slowly on the spread of the wave-packet, hence their original position dependence is replaced by the \(r_c\) dependence under the local approximation. The wave-packet dynamics is then described by the equations of motion \([22]\) (we set \(\hbar = 1\))

\[ \dot{r}_c = \partial_{q_c} \varepsilon - \Omega_{q_c, \lambda} \cdot \dot{r}_c - \Omega_{q_c, \lambda} \cdot q_c, \]
\[ \dot{q}_c = - \partial_{r_c} \left[ \varepsilon + e \phi (r_c) \right] + \Omega_{r_c, \lambda} \cdot \dot{r}_c + \Omega_{r_c, q_c} \cdot q_c. \tag{4} \]

Here \((\Omega_{\lambda\lambda})_{ij} = 2 \text{Im} (\partial_{\lambda \lambda} u_i | \partial_{\lambda \lambda} u_j)\) are the Berry curvatures derived from the periodic part \(|u\rangle\) of the Bloch wave, where \(i\) and \(j\) are Cartesian indices, and \(\lambda = r_c, q_c\).

To study the magnetization current which only manifests itself in the presence of inhomogeneity, we introduce a slowly varying nonuniform mechanical field \(w (r)\) that has reached an equilibrium state with the electron system. Hence the spatial derivative \(\partial_{r_c}\) in the equations of motion acts through \(w (r_c)\). This is a route to manifest the magnetization current in bulk even in the global equilibrium without position dependent temperature or chemical potential, outside of the scope of all previous semiclassical theories \([1, 6, 7, 11]\). The \(w\) field can exist indeed in the system, representing the bulk material inhomogeneity, or just be a auxiliary tool in a homogeneous sample. In the latter case it can be dropped after identifying the magnetization current. The specific form of \(w\) field is not needed, but it should not be a scalar one.

In the equations of motion, \(\varepsilon + e \phi (r_c)\) is the wave-packet energy, with \(\varepsilon\) being its value without the coupling to driving fields that do not equilibrate with the
system, such as the electrostatic potential $e\phi(r_c)$. The dipole moment correction from the gradient of $w(r_c)$ to the wave-packet energy $22$ is not essential for the present topic, as can be easily verified.

Within the validity of the uncertainty principle, the phase-space occupation function $f(q_c, r_c)$ of a grand canonical ensemble of dynamically independent semiclassical Bloch electrons can be defined, and the phase-space measure $D(q_c, r_c)$ has to be introduced. Because of the non-canonical structure of the equations of motion (f phase-space occupation function topic, as can be easily verified.

The number of states within a small phase-space volume is thus only shifts the wave-packet energy and contributes to the anomalous velocity. It does not give rise to a phase-space Berry curvature $\Omega_{q, r_c}$, which is, on the other hand, indispensable in our bulk approach.

The semiclassical wave-packet theory introduced above allows for acquiring the local current densities, which reduce to the magnetization ones in the absence of any statistical inhomogeneity or mechanical driving force. Hence we will first obtain the magnetization current in this case, based on which we can go further to identify the magnetization current in the presence of the statistical inhomogeneity and mechanical driving force. The transport current are thus reached by subtracting the magnetization current from the local one.

III. MAGNETIZATION CURRENT AT GLOBAL EQUILIBRIUM

First we look at the case of global equilibrium without any statistical inhomogeneity or mechanical force driving nonequilibrium states. As is pointed out above, in the presence of a nonuniform mechanical field $w$ that equilibrates with the electron system, $j^E = j^{E, \text{mag}} \neq 0$ and $j^N = j^{N, \text{mag}} \neq 0$ in the bulk. Thus we have

$$j^h = j^{h, \text{mag}} = \nabla \times M^h,$$

where $M^h = M^E - \mu M^N$ is the thermal magnetization in the absence of electric fields, and $\mu$ is a constant.

In the semiclassical theory the local energy current density reads

$$j^E(r) \equiv \int [dq_c] d\mathbf{r}_c D f(\varepsilon) \langle \Phi|\hat{v}\delta(\mathbf{r} - \mathbf{r}_c)|\Phi\rangle,$$

and the local particle current density is

$$j^N(r) \equiv \int [dq_c] d\mathbf{r}_c D f(\varepsilon) \langle \Phi|\hat{v}\delta(\mathbf{r} - \mathbf{r}_c)|\Phi\rangle.$$

Here $[dq_c]$ is shorthand for $\sum_c dq_c/(2\pi)^d$, and $\hat{v}$ is the velocity operator. Expanding the $\delta$ function to first order of $\mathbf{r} - \mathbf{r}_c$ yields

$$j^h(r) = \int D f(\varepsilon) (\varepsilon - \mu) \langle \Phi|\hat{v}\delta(\mathbf{r} - \mathbf{r}_c)|\Phi\rangle,$$

$$-\partial_j \int f(\varepsilon) (\varepsilon - \mu) \langle \Phi|\hat{v}_i(\mathbf{r}_j - \mathbf{r}_j)|\Phi\rangle|_{r_c=r}.$$

Henceforth we omit the center position label $c$, and the notation $\int$ without integral variable is shorthand for $\int [dq_c]$. We are limited to the first order of spatial gradients, thus it is sufficient to set $D = 1$ in the second term in the above expansion.

To proceed, we introduce two functions $g(\varepsilon, \mu, T)$ and $h(\varepsilon, \mu, T)$ which satisfy

$$\frac{\partial g}{\partial \varepsilon} = f(\varepsilon), \quad \frac{\partial h}{\partial \varepsilon} = f(\varepsilon) (\varepsilon - \mu).$$

In fact $g(\varepsilon) = -k_B T \ln[1 + e^{-(\varepsilon - \mu)/k_B T}]$ is the grand potential density contributed by a particular state, whereas $h = -\int_{\varepsilon}^{\infty} df(\eta) (\eta - \mu)$. Then the local thermal (particle) current density reads

$$j^{h(N)} = \int D \frac{\partial F}{\partial \varepsilon} \mathbf{r} + \nabla \times \int \frac{\partial F}{\partial \varepsilon} \mathbf{m}^N$$

for $F = h(g)$. The first and second terms come from the motion of the wave-packet center with velocity $\mathbf{r} = \langle \Phi|\hat{v}\Phi\rangle$ and the wave-packet self-rotation, respectively. $\mathbf{m}^N$ is the particle magnetic moment, which is the vector form of the antisymmetric tensor $\mathbf{m}^N_{ji} = \langle \Phi|\hat{v}_i(\mathbf{r}_j - \mathbf{r}_j)|\Phi\rangle|_{r_c=r}$ (symmetrization between operators is implied) $22$. For $j^h$ the second term reads alternatively $\nabla \times \int f \mathbf{m}^h$, with $\mathbf{m}^h = (\varepsilon - \mu) \mathbf{m}^N$ being the so-called thermal magnetic moment $11$.

Above expressions for the local current are well known. However, now we are in a position to acquire the thermal magnetization current from a pure bulk consideration, outside of the scope of the previous bulk semiclassical theory $1$. In fact, in the latter the inhomogeneity is solely due to the statistical ones ($\nabla T \neq 0, \nabla \mu \neq 0$), thus $j^{h(N)} = 0$ at global equilibrium. Moreover, there the orbital magnetization ($M^O = e M^N$) is acquired separately by the thermodynamic method, namely, as the derivative of the grand potential density with respect to magnetic field $\mu$, and then the magnetization current is obtained. This route poses the basic requirement for a
thermodynamic definition of the thermal magnetization, which is not evident in the familiar context of condensed matter physics. The particle magnetization current of neutral quasi-particles that do not couple to the magnetic field by the Lorentz force suffers from the same situation. In contrast, in the present transport approach, as will be shown, the thermal magnetization and particle magnetization currents emerge naturally.

Plugging the equations of motion in the first order of inhomogeneity and the phase-space measure into the first term of and making use of \( \nabla F = \partial_\varepsilon F \nabla \varepsilon \), we arrive at

\[
\int D \frac{\partial F}{\partial \varepsilon} \hat{r} = \nabla \times \int F(\varepsilon) \Omega_q, \quad (F = g, h), \tag{11}
\]

with \( \Omega_q \) being the vector form of the momentum-space Berry curvature. This is one of the pivotal results of our approach. Since \( \hat{r} \) represents the motion of the wave-packet center, it may not be very apparent to envision that \( \int D \frac{\partial F}{\partial \varepsilon} \hat{r} \) can be totally a part of the magnetization current. In fact, in the previous semiclassical paradigm, \( \int D \frac{\partial F}{\partial \varepsilon} \hat{r} \) was only viewed as a part of the transport current. Therefore, the present approach indicates a new perspective that the motion of the wave-packet center can also give rise to magnetization current in the bulk of nonuniform samples.

According to the above two equations we get

\[
\vec{j}^{(N)} = \nabla \times \int \left( \frac{\partial F}{\partial \varepsilon} m^N + F \Omega_q \right) \quad \text{for } F = h, g. \tag{12}
\]

Since they are just the magnetization currents, we identify the particle magnetization as \( M^N = \int (f m^N + g \Omega_q) \) and the thermal magnetization as

\[
M^h = \int (f m^h + h \Omega_q) \tag{13}
\]

up to a gradient. Our result for \( M^N \) is consistent with the thermodynamic one, and the obtained \( M^h \) coincides with that obtained using the gauge theory of gravity, where the thermal magnetization is thermodynamically defined as the derivative of the grand potential with respect to the torsional magnetic field, further confirming the validity of our theory.

**IV. TRANSPORT AND MAGNETIZATION CURRENTS IN THE PRESENCE OF DRIVING FORCES**

In the presence of statistical inhomogeneity (\( \nabla T \neq 0 \), \( \nabla \mu \neq 0 \)), the local current density consists of both the magnetization and transport parts. Manipulations similar to Eq. lead to the local thermal current density

\[
j^h = \int D \frac{\partial h}{\partial \varepsilon} \hat{r} + \nabla \times \int \frac{\partial h}{\partial \varepsilon} m^N + \nabla \mu \times \int f m^N. \tag{14}
\]

Taking some technical steps similar to those of Eq. and noticing that

\[
\nabla h = \frac{\partial h}{\partial \varepsilon} \nabla \varepsilon + \frac{\partial h}{\partial T} \nabla T + \frac{\partial h}{\partial \mu} \nabla \mu \tag{15}
\]

in the present case, we get

\[
\int D \frac{\partial h}{\partial \varepsilon} \hat{r} = -\left( \nabla \mu \times \frac{\partial J}{\partial \mu} + \nabla T \times \frac{\partial J}{\partial T} \right) \int h \Omega_q + \nabla \times \int h \Omega_q, \tag{16}
\]

which is another pivotal result of our approach. It is of interest because, as long as the material inhomogeneity is absent one would only get \( \int D \frac{\partial h}{\partial \varepsilon} \hat{r} = 0 \), even in the presence of statistical inhomogeneity. However, introducing the material inhomogeneity enables us to show that the motion of the wave-packet center in bulk can contribute to both the transport and magnetization currents.

Then the local thermal current density is given by

\[
j^h = \nabla \mu \times \int \Omega_q s(\varepsilon) T - \nabla T \times \int \frac{\partial h}{\partial T} \Omega_q + \nabla \times M^h + \nabla \mu \times M^N. \tag{17}
\]

Here \( M^h \) and \( M^N \) take the same expressions as those at global equilibrium, with the only difference that \( \mu \) and \( T \) are now position dependent at local equilibrium. The second line of Eq. is thus the zero-electric-field thermal magnetization current. The first line of Eq. is then identified as the transport thermal current.

In the thermal transport current induced by the chemical potential gradient, we recognize the entropy density for a particular state: \( s(\varepsilon) = \frac{[\varepsilon(\varepsilon - g(\varepsilon))]}{T} \). Its appearance is not easy to understand, unless one assumes the Einstein relation, which indicates that the zero-mass Hall entropy current \( j^h/T = -eE \times \int \Omega_q s(\varepsilon) \) arises due to the Berry curvature anomalous velocity transverse to an applied electric field. In the following we will prove this Einstein relation, which has not been shown in a unified semiclassical theory.

In the presence of an electric field as well as statistical inhomogeneity, the local energy current density is still given by Eq. but with the difference that the transported carrier energy changes to be \( \varepsilon + \phi(\varepsilon) \), as shown in the dynamic equations. Note that the electric field does not equilibrate with the electron system thus the equilibrium phase-space distribution function remains as \( f(\varepsilon) \). Manipulations similar to Eq. lead to the local thermal current density

\[
j^h = \int D f(\varepsilon)(\varepsilon - \mu) \hat{r} + \nabla \times \int f m^h + e \phi \int D f \hat{r} + \nabla \times (e \phi \int f m^N) + \nabla \mu \times \int f m^N. \tag{18}
\]

Applying the aforementioned technics again, we get

\[
j^h = (eE - \nabla \mu) \times T \int \frac{\partial q}{\partial T} \Omega_q - \nabla T \times \int \frac{\partial h}{\partial T} \Omega_q + \nabla \times [M^h + e \phi M^N] + \nabla \mu \times M^N. \tag{19}
\]
Here $M^h$ is the zero-electric-field thermal magnetization, and we have used the relation $s = -\partial g / \partial T$.

The presence of the $e\phi M^N$ term just reflects the result in the quantum mechanical linear response theories that the energy magnetization becomes $M^E + e\phi M^N$ in the presence of the electrostatic potential $[17]$. Here $M^E$ stands for the zero-electric-field energy magnetization. The general form of the bulk thermal magnetization current given by the second line of (19) in the presence of not only $(\phi, \nabla \mu, \nabla T)$ but also the material inhomogeneity is consistent with the full quantum theory (Eqs. (76) and (77) in Ref. [13]). This achievement has not been reached in previous semiclassical theories.

The potential field $\phi$ is introduced to produce the electric field $E = -\nabla \phi$, and after achieving this one can always set $\phi (r_e) = 0$. Therefore the thermal magnetization current can be expressed as

$$j^{h,\text{mag}} = \nabla \times M^h -(e E - \nabla \mu) \times M^N. \quad (20)$$

In the absence of the material inhomogeneity ($w$ field) and statistical inhomogeneity, $M^h$ is a constant in bulk. Then $j^{h,\text{mag}} = -E \times M^e$, reducing to the result in the previous bulk semiclassical theory $[1]$. Here this magnetization current is obtained on the basis of a physical argument of the material dependent part of the Poynting vector, while here such an argument is not needed.

The transport thermal current is given by the first line of Eq. (19) and can be rewritten as

$$j^{h,\text{tr}} = (E - \frac{1}{e} \nabla \mu) \times \nabla T \times \nabla \phi - \nabla \times \nabla \phi T, \quad (21)$$

where $M^{e,B} = e \int \Omega_q$ and $M^{h,B} = \int h \Omega_q$ are the Berry-curvature parts of the orbital magnetization and thermal magnetization, respectively. Similarly, the transport electric current is obtained as

$$j^{e,\text{tr}} = (e E - \nabla \mu) \times \nabla T \times \frac{\partial M^{e,B}}{\partial \mu}, \quad (22)$$

thus one immediately verifies the Einstein relation, the Onsager relation and the Wiedemann-Franz law, in the presence of material inhomogeneity ($M^{e,B}$ and $M^{h,B}$ are implicitly dependent on local material parameters).

Equations (21) and (22) embody the Streda-type formulas $[11]$, which link the response coefficients for the transport currents to the derivatives of magnetizations with respect to $\mu$ or $T$. In insulators $M^{e,B}$ and $M^{h,B}$ can be replaced by $M^e$ and $M^h$, respectively, recovering the standard Streda formulas $[28, 29]$.

V. APPLICATION TO HOMOGENEOUS SAMPLES

The obtained transport and magnetization currents [Eqs. (20) to (22)] are also valid in the absence of material inhomogeneity. In fact, in homogeneous samples, which all previous semiclassical thermoelectric and thermal transport theories $[1, 4, 6, 7, 11, 13]$ are designed for, one can introduce an auxiliary nonuniform $w$ field that is removed at last to reach the above results. Our approach thus unifies the treatment and understanding of the intrinsic thermal transport in both homogeneous and inhomogeneous samples.

With the identified thermal magnetization current $[20]$, we can also understand the intrinsic thermal transport current in a relatively familiar way in homogeneous samples, where $\nabla \phi = \nabla \mu = \nabla T = 0$. Only when the content of the magnetization current is known is this approach feasible, whereas our theory appealing to an auxiliary material inhomogeneity can provide the needed magnetization current.

The result of the edge-current approach $[6, 7]$, which only applies to case with merely statistical inhomogeneities, can be derived from the present bulk theory as well. To see this we turn off the electric field in Eq. (23), which then reduces to $j^h = \nabla \times \int f m^h + \nabla \mu \times \int f m^N$. Subtracting the magnetization current $[20]$ from this equation just recovers the transport thermal current $[21]$. An approach of the same spirit was employed in the previous bulk semiclassical theory $[1]$ in the simpler case where $\nabla \mu = \nabla T = 0$. Furthermore, observe that the intrinsic thermal transport current thus unifies the treatment and understanding of the intrinsic linear electric and thermal transport.

VI. GENERALIZATION OF DRIVING FORCES

Above our theory has promoted the material inhomogeneity to be a basic trick in the unified semiclassical theory of the intrinsic linear electric and thermal transport...
induced by the driving forces \(\mathbf{E}, \nabla \mu, \nabla T\). Recall that in the quantum thermal and thermochemical transport theories it is the fictitious gravitational field \([6, 17-19]\) which serves as a vital trick to mimic the temperature gradient. However, unlike the gravitational field, the trick of inhomogeneity has nothing to do with the driving force of transport, thus is not limited to the transport induced by \(\mathbf{E}, \nabla \mu, \nabla T\).

In this section we generalize our theory to the linear electric and thermal transport driven by the spatial gradients of any gauge-invariant static scalar (\(\varphi\)) and vector (\(\mathbf{F}\)) fields that enter the single-particle Hamiltonian in the form of \(\varphi(\mathbf{r}) + \hat{\theta} \cdot \mathbf{F}(\mathbf{r})\). The genuine or auxiliary material inhomogeneity is still represented by the nonuniform field. Here \(\hat{\theta}\) is the physical observable operator coupled to the \(\mathbf{F}\) field. A specific example of \(\hat{\theta} \cdot \mathbf{F}\) is the Zeeman coupling \(\hat{s} \cdot \mathbf{Z}(\mathbf{r})\), where \(\hat{s}\) is the carrier spin operator and \(\mathbf{Z}(\mathbf{r})\) is the Zeeman field. In fact, both \(\hat{\theta}\) and \(\mathbf{F}\) can be tensors in general, and the product denotes the contraction between them. Thus \(\hat{\theta} \cdot \mathbf{F}\) also includes the specific case where \(\hat{\theta} = e\) is the carrier charge and \(\mathbf{F} = \phi\) is the electrostatic potential and hence the transport is driven by the electric field. Only the spatial gradients of \(\varphi(\mathbf{r})\) and \(\mathbf{F}(\mathbf{r})\) matter and are assumed to be uniform, and one can always choose \(\varphi(\mathbf{r}_n) = 0\) and \(\mathbf{F}(\mathbf{r}_n) = 0\) at the last of the linear response calculation.

The treatment regarding \(\varphi\) is completely analogous to that of the electrostatic potential, so here we focus on the transport induced by the gradient of the \(\mathbf{F}\) field. The local electric current density is given by \(\mathbf{j}^e = e\mathbf{F}\) and Eq. \([10]\). The perturbed wave-packet energy needed in the calculation reads \(\varepsilon + \theta \cdot \mathbf{F}(\mathbf{r}_n) + \delta \varepsilon\), where \(\theta = \langle \hat{\theta} | \hat{\theta} \rangle\), and

\[
\delta \varepsilon = d_{ij}^0 \partial_i \mathbf{F}_j
\]

is the energy correction induced by the dipole moment of operator \(\hat{\theta}\) on a finite-size wave-packet \([22]\). \((\partial_i \equiv \partial \varepsilon)\). This dipole moment takes the explicit form of

\[
d_{ij}^0 = \text{Im} \sum_{n, j, n'} \frac{u_n^* u_{n'}^* \langle u_n | \hat{\theta} | u_{n'} \rangle}{\varepsilon_n - \varepsilon_{n'}} d^0_{ij}, \tag{26}
\]

where \(n\) is the index of the band we are considering and \(n'\) denotes other bands in the system. Therefore, up to the first order of inhomogeneity we obtain

\[
\int Df(\varepsilon) \dot{\mathbf{r}}_i = \int f(\varepsilon) \left[ \partial_{q_i} (\varepsilon + \delta \varepsilon) + \Omega_{q_i q_j} \partial_{q_j} \varepsilon \right]
+ \Omega_{q_i q_j} (\partial_{q_j} \varepsilon + \theta_{ij}^0 \partial_j \mathbf{F}_i - \Omega_{q_i q_j} \partial_{q_j} \varepsilon),
\]

where \(\theta_{ij}^0 = \langle u_n | \hat{\theta} | u_{n'} \rangle\). Since the driving \(\mathbf{F}\) field does not equilibrate with the electron system, the argument of the equilibrium distribution function remains as \(\varepsilon\). Some manipulations similar to those leading to Eq. \([11]\) give rise to \(J^e = J^{e, \text{tr}} + \nabla \times M^N\), with the transport electric current reading

\[
J^{e, \text{tr}} = e \int f(\varepsilon) \left[ \partial_{q_i} d_{ij}^0 + \Omega_{q_i q_j} \theta_{ij}^0 \right] \partial_j \mathbf{F}_i. \tag{27}
\]

In the case where \(\hat{\theta} = e\) is the carrier charge and \(\mathbf{F} = \phi\) is the electrostatic potential, we have \(\delta \varepsilon = 0\) since the electric dipole moment of a wave-packet is zero \([22]\) (as can be directly verified using Eq. \([26]\)). Hence the transport electric current \([27]\) reduces to the familiar one induced by an electric field: \(J^{e, \text{tr}} = -e^2 \int f(\varepsilon) \Omega_{q_i q_j} E_j\). When \(\hat{\theta}\) is not a conserved quantity, its dipole moment contributes to the transport current driven by the \(\mathbf{F}\)-field gradient.

Similarly, the local thermal current density is given by

\[
J^h = \int Df(\varepsilon) (\varepsilon + \delta \varepsilon - \mu) \dot{\mathbf{r}} + \nabla \times \int f(\varepsilon) m^h
+ \int Df(\varepsilon) \theta_i F_i \dot{\mathbf{r}} + \nabla \times \int f(\varepsilon) \theta_i^0 F_i m^N.
\]

Applying the aforementioned techniques again, up to the first order of inhomogeneity we find

\[
J^h = \int s(\varepsilon) T \left[ \partial_{q_i} \delta \varepsilon + \theta_{ij}^0 \nabla F_i \times \Omega_q \right]
+ \nabla \times \left[ M^h + \int \theta_i^0 F_i (f m^N + g \Omega_q) \right]. \tag{28}
\]

The second line is the thermal magnetization current density. As is anticipated, the energy magnetization in the presence of the perturbation \(\hat{\theta} \cdot \mathbf{F}\) changes by \(\int \theta_i^0 F_i (f m^N + g \Omega_q)\). Accordingly, this form of the thermal magnetization current density can be envisioned. Nevertheless, one can easily check that, it cannot be obtained by all the previous semiclassical transport theories \([1, 6, 7, 11]\). In the case that \(\hat{\theta} = e\) and \(\mathbf{F} = \phi\), the above equation indeed reduces to Eq. \([19]\) in the absence of \(\nabla \mu, \nabla T\).

The transport thermal current is then identified as

\[
J_{i}^{h, \text{tr}} = \int s(\varepsilon) T \left[ \partial_{q_i} d_{ij}^0 + \Omega_{q_i q_j} \theta_{ij}^0 \right] \partial_j F_i. \tag{29}
\]

As the entropy density \(s(\varepsilon)\) goes to zero at the zero-temperature limit, the above transport thermal current behaves well in this limit. One could then ask if there is the Mott relation linking the transport thermal and electric currents, when they are driven by the first-order spatial gradient of the \(\mathbf{F}\) field. For these two kinds of current, textbooks and previous studies only proved the Mott relation in the case that an electric field is the driving force. Here the present theory enables us to extend the regime of validity of the Mott relation to the case where the \(\mathbf{F}\)-field gradient serves as the driving force. In fact, when the temperature is much less than the distances between the chemical potential and band edges, the Sommerfeld expansion can be used \([30]\), yielding the entropy density \(s(\varepsilon) = \frac{\pi^2 k_B^2 T}{3} \delta (\mu - \varepsilon)\). Then we arrive at the standard form of the Mott relation

\[
J_{i}^{h, \text{tr}} / T = \frac{\pi^2 k_B^2 T}{3 \varepsilon} \frac{\partial J_{i}^{e, \text{tr}}(\varepsilon)}{\partial \varepsilon} \big|_{\varepsilon = \mu}, \tag{30}
\]

where \(J_{i}^{e, \text{tr}}(\varepsilon)\) is the zero-temperature transport electric current with Fermi energy \(\varepsilon\). 

VII. CONCLUSION

In conclusion, we have provided a semiclassical description for the linear thermal transport induced by the electric field and the gradients of chemical potential and temperature in the presence of material inhomogeneity, based on pure bulk considerations. In our method, applying simply the semiclassical equations of motion in the presence of material inhomogeneity leads to a systematic and efficient approach to both the magnetization and transport currents. The results that previously can only be obtained in different theories even in the absence of material inhomogeneity now emerge in a unified theory.

As long as the Berry-curvature modified semiclassical equations of motion hold for the considered quasiparticles, the intrinsic thermal (energy) Hall response falls into the present framework, irrespective of the specific content of Berry-curvatures in different physical contexts \[1, 2, 4, 6, 11, 13, 14\]. A particular example of current interest is the thermal Hall effect mediated by the Bogoliubov quasiparticles in superconductors \[31\] with time-reversal broken pairing like \(d + id\) \[2\]. However, to derive the equations of motion \[1\] for bogolons is nontrivial and has not been done up to now. This issue deserves an elaborate investigation in the near future.

By introducing a fictitious inhomogeneity, this theory also applies conveniently to homogeneous samples. Therefore, our theory promotes the material inhomogeneity to be a basic and generic trick in the unified semiclassical theory of the intrinsic linear thermoelectric and thermal transport. Compared to quantum theories, our theory is physically intuitive and technically simple. Moreover, the trick of inhomogeneity is independent of the driving force of transport, thus is not limited to transport induced by temperature gradient, in contrast to the trick of the fictitious gravitational field \[17, 19\] in quantum transport theories. We have included more generic driving forces in our theory and established the Mott relation between the resulting transport thermal and electric currents, whereas in textbooks \[23, 24\] and previous studies the Mott relation for these two currents was only known when they are driven by an electric field. In addition, to generalize our theory to cover the nonlinear thermal transport and linear thermoelectric responses of various magnetizations as well as the equilibrium thermal (energy) quadrupole density seems easier than quantum theories. This potential may inspire further studies in the future.

ACKNOWLEDGMENTS

We thank Liang Dong, Bangguo Xiong, Yinhan Zhang and Yang Gao for insightful discussions. Q.N. is supported by DOE (DE-FG03-02ER45958, Division of Materials Science and Engineering) on the geometric formulation in this work. C.X. is supported by Welch Foundation (F-1255).

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