On the Complexity of the Circular Chromatic Number

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Abstract

Circular chromatic number, $\chi_c$ is a natural generalization of chromatic number. It is known that it is $\text{NP}$-hard to determine whether or not an arbitrary graph $G$ satisfies $\chi(G) = \chi_c(G)$. In this paper we prove that this problem is $\text{NP}$-hard even if the chromatic number of the graph is known. This answers a question of Xuding Zhu. Also we prove that for all positive integers $k \geq 2$ and $n \geq 3$, for a given graph $G$ with $\chi(G) = n$, it is $\text{NP}$-complete to verify if $\chi_c(G) \leq n - \frac{1}{k}$.

1 Introduction

We follow [3] for terminology and notation not defined here, and we consider finite undirected simple graphs. Given a graph $G$, an edge $e = xy$ of $G$ and a triple $(H;a,b)$ where $a$ and $b$ are distinct vertices of the graph $H$, by replacing the edge $e$ by $(H;a,b)$, we mean taking the disjoint union of $G - e$ and $H$, and identifying $x$ with $a$ and $y$ with $b$. For our purposes, it does not matter whether $x$ is identified with $a$ or with $b$.

For two positive integers $p$ and $q$, a $(p,q)$-coloring of a graph $G$ is a vertex coloring $c$ of $G$ with colors $\{0, 1, 2, \ldots, p-1\}$ such that

$$(x,y) \in E(G) \implies q \leq |c(x) - c(y)| \leq p - q.$$ 

The circular chromatic number is defined as

$$\chi_c(G) = \inf \{p/q : G \text{ is } (p,q)\text{-circular colorable}\}.$$ 

So for a positive integer $k$, a $(k,1)$-coloring of a graph $G$ is just an ordinary $k$-coloring of $G$. The circular chromatic number of a graph was introduced by Vince [3]
as “the star-chromatic number” in 1988. He proved that for every finite graph $G$, the infimum in the definition of the circular chromatic number is attained, so the circular chromatic number $\chi_c(G)$ is always rational. He also proved, among other things, that $\chi - 1 < \chi_c \leq \chi$, and $\chi_c(K_n) = n$.

For a $(p, q)$-coloring $\phi$ of a graph $G$, let $D_\phi(G)$ be the digraph with vertex set $V(G)$ and for every edge $xy$ in $G$ there is a directed edge $(x, y)$ in $D_\phi(G)$, if $\phi(y) - \phi(x) = q \pmod{p}$.

**Lemma A.** [1] For a graph $G$, $\chi_c(G) < p/q$ if and only if $D_\phi(G)$ is acyclic for some $(p, q)$-coloring $\phi$ of $G$.

The question determining which graphs have $\chi_c = \chi$ was raised by Vince [3]. It was shown by Guichard [1] that it is NP-hard to determine whether or not an arbitrary graph $G$ satisfies $\chi_c(G) = \chi(G)$. In [3] X. Zhu surveyed many results on circular chromatic number and posed some open problems on this topic, among them the following problem ([3], Question 8.23).

**Problem 1** What is the complexity of determining whether or not $\chi_c(G) = \chi(G)$, if the chromatic number $\chi(G)$ is known?

We answer this question, using the following theorem.

**Theorem A.** [2] It is NP-hard to determine whether a graph is 3-colorable or any coloring of it requires at least 5 colors.

## 2 Complexity

Consider the graph $K^-$ which is obtained from a copy of $K_4$ with vertices $v_1, v_2, v_3$, and $v_4$, by removing the edge $\{v_1, v_2\}$. In the following trivial lemma all equalities are in $\mathbb{Z}_4$.

**Lemma 1** In every $(4, 1)$-coloring $c$ of $K^-$,

(a) if $c(v_1) = c(v_2)$, then $D_c(K^-)$ is acyclic and has no directed path between $v_1$ and $v_2$.

(b) if $c(v_1) - c(v_2) = 1$, then $D_c(K^-)$ is acyclic and has a directed path from $v_1$ to $v_2$.

(c) if $c(v_1) - c(v_2) = 2$, then $D_c(K^-)$ has a cycle.

Consider the graph $H$ shown in Figure [1]. One can easily check that $\chi(H) = 4$ and we have the following Lemma.
Lemma 2 Consider the graph $H$ shown in Figure 1.

(a) For every $(4,1)$-coloring $c$ of $H$, if $c(a) = c(b)$, then $D_c(H)$ has a cycle.

(b) For every $0 \leq x < y \leq 2$, there is a coloring $c$ for $H$ such that $c(a) = x$, $c(b) = y$, and $D_c(H)$ is acyclic and has no directed path from $b$ to $a$.

Proof. (a) Without loss of generality assume that $c(a) = c(b) = 0$. For all cases except when $c(c) = c(d) = 1$ and $c(c) = c(d) = 3$, one can easily check by Lemma 1(c) that $D_c(H)$ has a cycle. Without loss of generality assume that $c(c) = c(d) = 1$. Now by Lemma 1(b) there are directed paths from $d$ to $b$, $b$ to $c$, $c$ to $a$ and $a$ to $d$. Thus $D_c(H)$ has a cycle.

(b) Such colorings are given in Figures 1(a), 1(b), 1(c).

Theorem 1 Given a graph $G$ and its chromatic number, the problem of determining whether or not $\chi_c(G) = \chi(G)$ is NP-hard.

Proof. For every graph $G'$, we construct a graph $G$ such that $\chi(G) = 4$, and if $G'$ is 3-colorable, then $\chi_c(G') < 4$, and if $G'$ is not 4-colorable, then $\chi_c(G') = 4$. Thus by Theorem A the result is proven.

Construct the graph $G$ by replacing every edge of $G'$ by $(H; a, b)$. Obviously, for every nontrivial graph $G'$, $\chi_c(G) = 4$.

First suppose that $G'$ is 3-colorable. So we can properly color the vertices of $G'$ with 0, 1, and 2. Now by Lemma 2(b), this coloring can be expanded to a $(4,1)$-coloring $c$ of $G$ such that in $D_c(G)$ the copies of $H$ are acyclic, and also for every two vertices $u$ and $v$ of $G'$, there is no path from $u$ to $v$ in $D_c(G)$ if $c(u) > c(v)$. This implies that $D_c(G)$ is acyclic. So $\chi_c(G) < 4$.

Next suppose that $G'$ is not 4-colorable. So in any $(4,1)$-coloring $c$ of $G$ there are two adjacent vertices $u$ and $v$ of $G$ such that $c(u) = c(v)$. So by Lemma 2(a) for the copy of $H$ which is between $u$ and $v$ there exists a cycle in $D_c(H)$. Hence $\chi_c(G) = 4$.

Now we prove that it is NP-complete to verify that the difference between chromatic number and circular chromatic number of a given graph is greater than or
equal to $\frac{1}{k}$, when $k \geq 2$ is an arbitrary positive integer is \textbf{NP}-complete. Let $K$ be a graph with vertex set \{a, b, v_1, \ldots, v_{n-1}\} in which each $v_i$ is adjacent to every other $v_j, a$ is adjacent to $v_1, \ldots, v_{n-2}$, and $b$ is adjacent to $v_{n-1}$.

\textbf{Lemma 3} For all integers $0 \leq x, y \leq kn - 1$, $K$ has a $(kn - 1, k)$-coloring $c$ with $c(a) = x$ and $c(b) = y$ if and only if $x \neq y$.

\textbf{Proof.} If $x = y$, then a $(kn - 1, k)$-coloring of $K$ can be transformed to a $(kn - 1, k)$-coloring of $K_n$ by identifying $a$ and $b$. And this is impossible because $\chi_c(K_n) = n$. If $x \neq y$ without loss of generality we can assume that $x = 0$ and $0 < y \leq \frac{kn-1}{2}$.

First suppose that $y \geq k$. In this case define a desired $(kn - 1, k)$-coloring $c$ by $c(a) = 0$, $c(b) = y$, $c(v_i) = ik$ for $1 \leq i \leq n - 2$ and $c(v_{n-1}) = 0$.

Next suppose that $y < k$. In this case define a desired $(kn - 1, k)$-coloring $c$ by $c(a) = 0$, $c(b) = y$, $c(v_i) = ik$ for $1 \leq i \leq n - 2$, and $c(v_{n-1}) = y - k$.

\textbf{Theorem 2} For all positive integers $k \geq 2$ and $n \geq 3$, the following problem is \textbf{NP}-complete. A graph $G$ is given where $\chi(G) = n$, and it is asked whether $\chi_c(G) \leq n - \frac{1}{k}$?

\textbf{Proof.} Clearly, the problem is in \textbf{NP}. We reduce \textsc{Vertex Coloring} to this problem. Consider a graph $G'$ as an instance of \textsc{Vertex Coloring}. It is asked whether the vertices of $G'$ can be colored with $kn - 1$ colors. We construct a new graph $G$ with the property that $\chi_c(G) \leq n - \frac{1}{k}$ if and only if the vertices of $G'$ can be colored with $kn - 1$ colors.

Construct a graph $G$ by replacing every edge $uv$ of $G' \sqcup K_n$, the disjoint union of $G'$ and a copy of $K_n$, by $(K; a, b)$. Obviously, $\chi(G) \leq n$. Since in every $(n - 1)$-coloring of $K$ the vertices $a$ and $b$ must have different colors, thus $\chi(G) = n$. We know that $\chi_c(G) \leq n - \frac{1}{k}$ if and only if there exists a $(kn - 1, k)$-coloring $c$ for $G$.

First suppose that $\chi(G') \leq kn - 1$, and $c$ is a $(kn - 1)$-coloring of $G' \sqcup K_n$. For all copies of $K$ in $G$, we have $c(a) \neq c(b)$. By Lemma 3, $c$ can be extended to a $(kn - 1, k)$-coloring of $G$. Thus $\chi_c(G) \leq n - \frac{1}{k}$.

Next suppose that $\chi(G') > kn - 1$ and $c$ is a $(kn - 1, k)$-coloring of $G$. There exist two adjacent vertices $u$ and $v$ in $G'$ such that $c(u) = c(v)$. But by Lemma 3 the copy of $K$ between $u$ and $v$ has no $(kn - 1, k)$-coloring. This is a contradiction. Thus $\chi_c(G) > n - \frac{1}{k}$.

\textbf{Acknowledgements}

The authors wish to thank Hossein Hajiabolhassan who drew their attention to this subject, and for suggesting the problem.
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