Bakry-Emery meet Villani

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Abstract

We study gradient estimates and convergence to the equilibrium for solutions of
the parabolic equation which is associated to degenerate hypoelliptic diffusion op-
erators. The method relies on a generalized Bakry-Émery type criterion that applies to
this kind of operators. Our approach includes as a special case the kinetic Fokker-
Planck equation and allows, in that case, to recover hypocoercive estimates obtained
by Villani.

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1 Introduction

We study gradient bounds and convergence to equilibrium for the semigroup generated by
a diffusion operator of the form $L = \sum_{i=1}^{n} X_i^2 + Y$, where $X_1, \ldots, X_n, Y$ are vector fields.
Though our results are more general, we are particularly interested in the case where
$L$ is hypoelliptic and $\sum_{i=1}^{n} X_i^2$ is not, that is the hypoellipticity comes from the first
order operator $Y$. The problem of convergence to an equilibrium in this type of situation
has attracted a lot of interest in the literature, because evolution equations involving a
degenerate dissipative operator and a conservative operator naturally arise in many fields
of applied mathematics: We refer to Villani’s memoir [11] and to the references therein for a discussion about this.

There have been different approaches to tackle this problem. A functional analytic approach, based on previous ideas by Kohn and Hörmander, uses pseudo-differential calculus and delicate spectral localization tools to prove exponential convergence to equilibrium with explicit bounds on the rate. For this approach, we refer to Eckmann and Hairer [7], Hérau and Nier [9], and Heffer and Nier [10].

Villani in his memoir [11] introduces the concept of hypocoercivity and derives very general sufficient conditions ensuring the convergence to an equilibrium. The main strategy is to work in a suitable Hilbert space associated to the equation and to find in this Hilbert space a nice norm which is equivalent to the original one, but with respect to which convergence to equilibrium is easy to obtain; We refer to Section 4.1 in [11] for a more precise description.

L. Wu in [13] and Bakry, Cattiaux and Guillin in [5] use the powerful method of Lyapunov functions to prove the exponential convergence to equilibrium.

All these approaches have in common to use global methods to prove the convergence to equilibrium in the sense that the functional inequalities that are used are written in an integrated form with respect to the invariant measure. Our approach parallels the Bakry-Emery approach to hypercontractivity [1] and is only based on local computations: We just compute derivatives. To perform these computations the existence of an invariant measure is not even required. Let us describe this approach and what is the main idea of the present work.

We can associate to $L = \sum_{i=1}^{n} X_i^2 + Y$ its carré du champ operator

$$\Gamma(f, g) = \frac{1}{2} \left( L(fg) - fLg - gLf \right)$$

and its iteration

$$\Gamma_2(f, g) = \frac{1}{2} \left( L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf) \right).$$

If the operator $L$ admits a symmetric measure $\mu$ and if for every $f$, $\Gamma_2(f, f) \geq \rho \Gamma(f, f)$ for some positive constant $\rho$, then it is known the semigroup $P_t$ generated by $L$ will converge exponentially fast to an equilibrium (see [1]). However in a number of interesting situations including the ones described before, it is impossible to bound from below $\Gamma_2$ by $\Gamma$ alone. Actually, the Bakry-Emery criterion $\Gamma_2 \geq \rho \Gamma$ requires some form of ellipticity of the operator $L$ and fails to hold for strictly subelliptic operators. In the recent few years, there have been several works, extending the Bakry-Emery approach to subelliptic operators. We mention in particular Baudoin and Bonnefont [2], Baudoin and Garofalo [3], and Baudoin and Wang [4] where a generalized curvature dimension inequality is shown to
be satisfied for a large class of geometrically relevant subelliptic operators. In particular, under suitable conditions, explicit rates of convergence to equilibrium are obtained for the semigroup. The hypoelliptic situations treated in these works are quite different from the ones we have in mind here, because in [2], [3] or [4] the operator \( \sum_{i=1}^{n} X_i^2 \) is hypoelliptic and \( Y \) is in the linear span of \( X_1, \cdots, X_n \). Here, we really are interested in situations where \( \sum_{i=1}^{n} X_i^2 \) is fully degenerate.

Since it is impossible to bound from below \( \Gamma_2 \) by \( \Gamma \) alone, our main idea will be to introduce a vertical first order bilinear form \( \Gamma^Z \) and to compute the curvature of \( L \) in this new vertical direction:

\[
\Gamma^Z_2(f, g) = \frac{1}{2} \left( L \Gamma^Z(f, g) - \Gamma^Z(f, Lg) - \Gamma^Z(g, Lf) \right).
\]

As it turns out, in the degenerate situations mentioned above, it is possible to find, under some conditions, such a form \( \Gamma^Z \) with the property that

\[
\Gamma_2(f, f) + \Gamma^Z_2(f, f) \geq -K \Gamma(f, f) + \rho \Gamma^Z(f, f), \tag{1.1}
\]

where \( K \in \mathbb{R} \) and where \( \rho > 0 \). The important point here is that \( \rho \) is positive and will hence induce a convergence to equilibrium in the missing vertical direction. The bound (1.1) is local and implies several interesting pointwise bounds for the semigroup \( P_t \). Let us observe that these bounds can not be obtained by Villani’s method where inequalities are always given in an integrated form with respect to an invariant measure which is not even assumed to exist here. In the case when there is an invariant probability measure, that satisfies the Poincaré inequality with respect to the new gradient \( \Gamma + \Gamma^Z \), then the inequality (1.1) implies exponential convergence to equilibrium with an explicit rate, and we recover then an important Villani’s result.

The paper is organized as follows. In Section 2, we study in details the case of the kinetic Fokker-Planck equation. The study of this important equation was the motivation of the present work. In that case, we recover the convergence results by Villani but also prove pointwise gradient bounds that do not seem to have been known before. It is a very special case of equation that we can treat by our methods, however we think that the main ideas are already there so that, for pedagogical reasons, we start with the study of this equation. Section 3 gives more general results. We study in details the structure of the \( \Gamma_2 \) associated to degenerate hypoelliptic operators and give sufficient conditions to obtain lower bounds on \( \Gamma_2 + \Gamma^Z_2 \). As a consequence, we get sufficient conditions for gradient bounds and convergence to the equilibrium.

## 2 The kinetic Fokker-Planck equation

In this section we study the kinetic Fokker-Planck equation which is an important example of equation to which our methods apply and which is the one that motivated our study.
Let $V : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. The kinetic Fokker-Planck equation with confinement potential $V$ is the parabolic partial differential equation:

$$\frac{\partial h}{\partial t} = \Delta_v h - v \cdot \nabla_v h + \nabla V \cdot \nabla_v h - v \cdot \nabla_x h, \quad (x, v) \in \mathbb{R}^{2n}. \quad (2.2)$$

It is the Kolmogorov-Fokker-Planck equation associated to the stochastic differential system

$$\begin{cases}
    dx_t = v_t dt \\
v_t = -v_t dt - \nabla V(x_t) dt + dB_t,
\end{cases}$$

where $(B_t)_{t \geq 0}$ is a Brownian motion on $\mathbb{R}^n$. The operator

$$L = \Delta_v - v \cdot \nabla_v + \nabla V \cdot \nabla_v - v \cdot \nabla_x$$

is not elliptic but it can be written in Hörmander’s form

$$L = \sum_{i=1}^{n} X_i^2 + X_0 + Y,$$

where $X_i = \frac{\partial}{\partial v_i}$, $X_0 = -v \cdot \nabla_v$ and $Y = \nabla V \cdot \nabla_v - v \cdot \nabla_x$. The vectors

$$(X_1, \ldots, X_n, [Y, X_1], \ldots, [Y, X_n])$$

form a basis of $\mathbb{R}^{2n}$ at each point. This implies from Hörmander’s theorem that $L$ is hypoelliptic. The operator $L$ admits for invariant measure the measure

$$d\mu = e^{-V(x)} \frac{||v||^2}{2} dx dv.$$

It is readily checked that $L$ is not symmetric with respect to $\mu$ but that the adjoint $L^*$ in $L^2(\mu)$ is given by

$$L^* = \sum_{i=1}^{n} X_i^2 + X_0 - Y.$$

The operator $L$ is the generator of a strongly continuous sub-Markov semigroup $(P_t)_{t \geq 0}$. If we assume that the Hessian $\nabla^2 V$ is bounded, then $P_t$ is Markovian (that is $P_1 = 1$) and for any bounded Borel function $f : \mathbb{R}^{2n} \to \mathbb{R}$, $(t, x), v \to P_t f(x, v)$ is the unique solution of the Cauchy problem

$$\begin{cases}
    \frac{\partial h}{\partial t} = Lh \\
h(0, x, v) = f(x, v).
\end{cases}$$

One of the main results of Villani (see also Hellfer and Nier [9] for related results) concerning the convergence to equilibrium of $P_t$ is the following theorem:
Theorem 2.1 (Villani [11], Theorem 35) Define $H^1(\mu) = \{ f \in L^2(\mu), \|\nabla f\| \in L^2(\mu) \}$.

Assume that there is a constant $c > 0$ such that $\|\nabla^2 V\| \leq c(1 + \|\nabla V\|)$ and that the normalized invariant measure $d\mu = \frac{1}{Z} e^{-V(x) - \frac{1}{2}\|v\|^2} dx dv$ is a probability measure that satisfies the classical Poincaré inequality

$$\int_{\mathbb{R}^{2n}} \|\nabla f\|^2 d\mu \geq \kappa \left[ \int_{\mathbb{R}^{2n}} f^2 d\mu - \left( \int_{\mathbb{R}^{2n}} f d\mu \right)^2 \right].$$

Then, there exist constants $C > 0$ and $\lambda > 0$ such that for every $f \in H^1(\mu)$, with $\int_{\mathbb{R}^{2n}} f d\mu = 0$,

$$\int_{\mathbb{R}^{2n}} (P_t f)^2 d\mu + \int_{\mathbb{R}^{2n}} \|\nabla P_t f\|^2 d\mu \leq C e^{-\lambda t} \left( \int_{\mathbb{R}^{2n}} f^2 d\mu + \int_{\mathbb{R}^{2n}} \|\nabla f\|^2 d\mu \right)$$

It is worth observing that since $\mu$ is a product, it satisfies the Poincaré inequality as soon as the marginal measure $d\mu_x = e^{-V(x)} dx$ satisfies the Poincaré inequality on $\mathbb{R}^n$.

In this section, we give a new proof of this result under the further assumption that the Hessian $\nabla^2 V$ is bounded. This is a stronger assumption than in Villani’s result. However our method gives pointwise gradient estimates that can not be obtained by Villani’s method. It also provides a better control of the constants $C$ and $\lambda$. Finally, as we shall see in the next section, a very small variation of our method will almost immediately give an entropic convergence of $P_t$, under the assumption that $\mu$ satisfies the log-Sobolev inequality. This latter result is also obtained by Villani under the assumption that $\nabla^2 V$ is bounded (see Theorem 35 in [11]). So, from now on, and in all this section we assume that $\nabla^2 V$ is bounded.

2.1 $\Gamma_2$ calculus for the kinetic Fokker-Planck equation

Following Bakry and Émery [1] we associate to $L = \Delta_v - v \cdot \nabla_v + \nabla V \cdot \nabla_v - v \cdot \nabla_x$ the carré du champ operator,

$$\Gamma(f, g) = \frac{1}{2} (L(fg) - fLg - gLf) = \nabla_v f \cdot \nabla_v g = \sum_{i=1}^n \frac{\partial f}{\partial v_i} \frac{\partial g}{\partial v_i}$$

and its iteration

$$\Gamma_2(f, g) = \frac{1}{2} (L \Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)).$$

For simplicity of notations, we will denote $\Gamma(f) := \Gamma(f, f)$ and $\Gamma_2(f) := \Gamma_2(f, f)$. A straightforward computation shows that:

**Lemma 2.2** For $f \in C^\infty(\mathbb{R}^{2n})$,

$$\Gamma_2(f) = \|\nabla_v^2 f\|^2 + \Gamma(f) + \nabla_x f \cdot \nabla_v f.$$
The term $\nabla_x f \cdot \nabla_v f$ makes impossible to bound from below $\Gamma_2$ by $\Gamma$ alone. As a consequence the Bakry-Émery curvature of $L$ is $-\infty$. The idea is to introduce a carefully chosen vertical gradient and to compute the corresponding curvature of $L$ in this vertical direction. For $i = 1, \ldots, n$, we denote

$$Z_i = 2 \frac{\partial}{\partial x_i} + \frac{\partial}{\partial v_i}.$$ 

We define then

$$\Gamma^Z(f, g) = \sum_{i=1}^{n} Z_i f Z_i g$$

and

$$\Gamma^Z_2(f, g) = \frac{1}{2} (L \Gamma^Z(f, g) - \Gamma^Z(f, Lg) - \Gamma^Z(g, Lf)) .$$

**Lemma 2.3** For $f \in C^\infty(\mathbb{R}^{2n})$,

$$\Gamma^Z_2(f) = \|\nabla_v Z f\|^2 + \frac{1}{2} \Gamma^Z(f) + \frac{1}{2} \nabla_v f \cdot Z f - 2 \nabla^2 V(\nabla_v f, Z f)$$

$$= \sum_{i,j=1}^{n} \frac{\partial}{\partial v_i} Z_j f \right)^2 + \frac{1}{2} \Gamma^Z(f) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial f}{\partial v_i} Z_i f - 2 \sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial x_i \partial x_j} \frac{\partial f}{\partial v_i} Z_j f.$$ 

**Proof.** Let us write

$$L = \sum_{i=1}^{n} X_i^2 + X_0 + Y,$$

where $X_i = \frac{\partial}{\partial v_i}$, $X_0 = -v \cdot \nabla_v$ and $Y = \nabla V \cdot \nabla_v - v \cdot \nabla_x$. We have

$$\Gamma^Z_2(f, g) = \frac{1}{2} (L \Gamma^Z(f, g) - 2 \Gamma^Z(f, Lf))$$

$$= \frac{1}{2} \left( L \left( \sum_{i=1}^{n} (Z_i f)^2 \right) - 2 \sum_{i=1}^{n} Z_i f Z_i L f \right)$$

$$= \sum_{i=1}^{n} \Gamma(Z_i f) + \sum_{i=1}^{n} Z_i f [L, Z_i] f$$

$$= \sum_{i=1}^{n} \Gamma(Z_i f) + \sum_{i=1}^{n} Z_i f [X_0, Z_i] f + \sum_{i=1}^{n} Z_i f [Y, Z_i] f.$$ 

We now compute, 

$$[X_0, Z_i] = X_i$$

and

$$[Y, Z_i] = \frac{1}{2} Z_i - \frac{1}{2} X_i - 2 \sum_{j=1}^{n} \frac{\partial^2 V}{\partial x_i \partial x_j} \frac{\partial}{\partial v_j}.$$ 

The result follows then easily. 

A consequence of the previous computations is the following lower bound for $\Gamma_2 + \Gamma^Z_2$. 

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**Proposition 2.4** For every $0 < \eta < \frac{1}{2}$, there exists $K(\eta) \geq -\frac{1}{2}$ such that for every $f \in C^\infty(\mathbb{R}^{2n})$,

$$
\Gamma_2(f) + \Gamma_2^Z(f) \geq -K(\eta)\Gamma(f) + \eta\Gamma^Z(f).
$$

**Proof.** As a consequence of the previous lemmas, we have

$$
\Gamma_2(f) \geq \frac{1}{2}\Gamma(f) + \frac{1}{2}\nabla_v f \cdot Zf
$$

and

$$
\Gamma_2^Z(f) \geq \frac{1}{2}\Gamma^Z(f) + \frac{1}{2}\nabla_v f \cdot Zf - 2\nabla^2 V(\nabla_v f, Zf).
$$

We deduce

$$
\Gamma_2(f) + \Gamma_2^Z(f) \geq \frac{1}{2}\Gamma(f) + \frac{1}{2}\Gamma^Z(f) + \nabla_v f \cdot Zf - 2\nabla^2 V(\nabla_v f, Zf).
$$

We now pick $0 < \eta < \frac{1}{2}$ and $K \in \mathbb{R}$ and write

$$
\frac{1}{2}\Gamma(f) + \frac{1}{2}\Gamma^Z(f) + \nabla_v f \cdot Zf - 2\nabla^2 V(\nabla_v f, Zf)
= \eta\Gamma^Z(f) - K\Gamma(f) + \left(\frac{1}{2} - \eta\right)\Gamma^Z(f) + \left(\frac{1}{2} + K\right)\Gamma(f) + (\text{Id} - 2\nabla^2 V)(\nabla_v f, Zf).
$$

The bilinear form $\left(\frac{1}{2} - \eta\right)\Gamma^Z(f) + \left(\frac{1}{2} + K\right)\Gamma(f) + (\text{Id} - 2\nabla^2 V)(\nabla_v f, Zf)$ can now be made non negative as soon as, in the sense of symmetric matrices,

$$
4 \left(\frac{1}{2} - \eta\right)\left(\frac{1}{2} + K\right) \geq (\text{Id} - 2\nabla^2 V)^2.
$$

The claim follows then from the fact that we assume that $\nabla^2 V$ is bounded on $\mathbb{R}^n$ \qed

**Remark 2.5**

- The keypoint of the previous proposition is the positivity of $\eta$ which will imply the coercivity of $P_t$ in the vertical direction. The sign of $K(\eta)$ is not that relevant in the sense that the coercivity of $P_t$ in the horizontal direction can be obtained as a consequence of a Poincaré inequality for the invariant measure.

- If there are constants $0 < a < b < 1$ such that for every $x \in \mathbb{R}^n$, $a \leq \nabla^2 V \leq b$, then the previous proof shows that we can chose $K(\eta)$ to be negative.
2.2 Gradient bounds

We prove now some global bounds for the gradient of the semigroup \((P_t)_{t \geq 0}\). Related pointwise gradient bounds in this kinetic model were also obtained by Guillin and Wang [8], but our bounds have the advantage to be global. Let us also observe that such bounds can not be obtained by Villani’s method.

The previous computations have shown that for \(f \in C^\infty(\mathbb{R}^{2n})\),
\[
\Gamma_2(f) + \Gamma^Z_Z(f) \geq \lambda(\eta)(\Gamma(f) + \Gamma^Z(f)),
\]
where \(\lambda(\eta) = \min(-K(\eta), \eta)\). With this lower bound in hands we can use the methods introduced by Baudoin-Bonnefont [2] and F.Y. Wang [12] to obtain the following results.

**Lemma 2.6** If \(f\) is a bounded Lipschitz function on \(\mathbb{R}^{2n}\), then for every \(t \geq 0\), \(P_tf\) is a bounded and Lipschitz function. More precisely, with the notations of Lemma 2.4, for every \((v, x) \in \mathbb{R}^{2n}\),
\[
\Gamma(P_tf)(x, v) + \Gamma^Z(P_tf)(x, v) \leq e^{-2\lambda(\eta)t}P_t(\Gamma(f) + \Gamma^Z(f))(x, v),
\]
where \(\lambda(\eta) = \min(-K(\eta), \eta)\).

**Proof.** The heuristic argument is the following: We fix \((v, x) \in \mathbb{R}^{2n}\), \(t > 0\) and consider the functional
\[
\Psi(s) = P_s(\Gamma(P_{t-s}f) + \Gamma^Z(P_{t-s}f))(x, v).
\]
Differentiating \(\Psi\), leads to
\[
\Psi'(s) = 2P_s(\Gamma_2(P_{t-s}f) + \Gamma^Z_2(P_{t-s}f))(x, v) \geq 2\lambda(\eta)P_s(\Gamma(P_{t-s}f) + \Gamma^Z(P_{t-s}f))(x, v) = 2\lambda(\eta)\Psi(s).
\]
Therefore, we obtain \(\Psi(t) \geq e^{2\lambda(\eta)t}\Psi(0)\), which is the claimed inequality. In order to rigorously justify this argument, we observe that since \(\nabla V\) is Lipschitz, the function \(W(x, v) = 1 + \|x\|^2 + \|v\|^2\) is a Lyapunov function such that, for some constant \(C > 0\), \(LW \leq CW\) and \(\|\nabla W\| \leq CW\). We can then use Proposition 2.4 and argue like [2] Proposition 2.2, or [12], Lemma 2.1.

A direction computation shows that, since the vector fields \(X_i\) and \(Z_j\) commute, we do have the following intertwining of the quadratic forms \(\Gamma, \Gamma^Z\): For every \(f \in C^\infty(\mathbb{R}^{2n})\),
\[
\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)).
\]
This intertwining leads to the following entropic type pointwise bound:

**Lemma 2.7** Let \(f \in C^\infty(\mathbb{R}^{2n})\) be a positive function such that \(\sqrt{f}\) is bounded and Lipschitz, then for \(t \geq 0\), \(\sqrt{P_tf}\) is bounded and Lipschitz. More precisely, with the notations of Lemma 2.4, for every \((v, x) \in \mathbb{R}^{2n}\),
\[
P_tf(x, v)\Gamma(\ln P_tf)(x, v) + P_tf(x, v)\Gamma^Z(\ln P_tf)(x, v) \leq e^{-2\lambda(\eta)t}P_t(f\Gamma(\ln f) + f\Gamma^Z(\ln f))(x, v),
\]
where \(\lambda(\eta) = \min(-K(\eta), \eta)\).
Let us observe that the computation of the derivative crucially relies on the fact that by using the Lyapunov function $W$.

**2.3 Convergence in $H^1$**

We can now, in particular, recover Villani’s convergence result.

**Theorem 2.8** Assume that the normalized invariant measure $d\mu = \frac{1}{Z}e^{-V(x)-\frac{1}{2}\|x\|^2} dx dv$ is a probability measure that satisfies the Poincaré inequality

$$\int_{\mathbb{R}^{2n}} (\Gamma(f) + \Gamma^Z(f)) d\mu \geq \kappa \left[ \int_{\mathbb{R}^{2n}} f^2 d\mu - \left( \int_{\mathbb{R}^{2n}} f d\mu \right)^2 \right].$$

With the notations of Lemma 2.4:

- If $K(\eta) + \eta > 0$, then for every $f \in H^1(\mu)$, with $\int_{\mathbb{R}^{2n}} f d\mu = 0$,

  $$(\eta + K(\eta)) \int_{\mathbb{R}^{2n}} (P_t f)^2 d\mu + \int_{\mathbb{R}^{2n}} (\Gamma(P_t f) + \Gamma^Z(P_t f)) d\mu \leq e^{-\lambda t} (\eta + K(\eta)) \int_{\mathbb{R}^{2n}} f^2 d\mu + \int_{\mathbb{R}^{2n}} (\Gamma(f) + \Gamma^Z(f)) d\mu,$$

  where $\lambda = \frac{2\eta \kappa}{\kappa + \eta + K(\eta)}$.

- If $K(\eta) + \eta \leq 0$, then for every $f \in H^1(\mu)$, with $\int_{\mathbb{R}^{2n}} f d\mu = 0$,

  $$\begin{cases}
  \int_{\mathbb{R}^{2n}} \Gamma(P_t f) + \Gamma^Z(P_t f) d\mu \leq e^{-2\eta t} \int_{\mathbb{R}^{2n}} (\Gamma(f) + \Gamma^Z(f)) d\mu \\
  \int_{\mathbb{R}^{2n}} (P_t f)^2 d\mu \leq \frac{1}{\kappa} e^{-2\eta t} \int_{\mathbb{R}^{2n}} (\Gamma(f) + \Gamma^Z(f)) d\mu.
  \end{cases}$$

**Remark 2.9** The two norms $(\eta + K(\eta)) \int_{\mathbb{R}^{2n}} f^2 d\mu + \int_{\mathbb{R}^{2n}} (\Gamma(f) + \Gamma^Z(f)) d\mu$ and $\int_{\mathbb{R}^{2n}} \|\nabla f\|^2 d\mu$ are equivalent on $H^1(\mu)$, so the previous result implies Villani’s theorem.

**Proof.** By a density argument we can and will prove these inequalities when $f$ is smooth, bounded and Lipschitz.

Let us first assume that $K(\eta) + \eta > 0$. We fix $t > 0$ and consider the functional

$$\Psi(s) = (K(\eta) + \eta) P_s((P_{t-s} f)^2) + P_s(\Gamma(P_{t-s} f) + \Gamma^Z(P_{t-s} f)).$$
By repeating the arguments of the previous section, we get the differential inequality
\[
\Psi(s) - \Psi(0) \geq 2\eta \int_0^s P_u (\Gamma(P_{t-u}f) + \Gamma^Z(P_{t-u}f)) du.
\]

Denote now \( \varepsilon = \frac{\eta + K(\eta)}{\kappa + \eta + K(\eta)} \). We have
\[
\varepsilon \int_{R^{2n}} \Gamma(P_{t-u}f) + \Gamma^Z(P_{t-u}f) d\mu \geq \varepsilon \kappa \int_{R^{2n}} (P_{t-u}f)^2 d\mu.
\]

Therefore, denoting \( \Theta(s) = \int_{R^{2n}} \Psi(s) d\mu \), we obtain
\[
\Theta(s) - \Theta(0) \geq 2\varepsilon(1 - \varepsilon) \int_0^s \int_{R^{2n}} \Gamma(P_{t-u}f) + \Gamma^Z(P_{t-u}f) d\mu du + 2\varepsilon \kappa \int_0^s (P_{t-u}f)^2 d\mu du
\]
\[
\geq \lambda \int_0^s \Theta(u) du.
\]

We conclude with Gronwall’s differential inequality.

If \( K(\eta) + \eta \leq 0 \), then the argument is identical by considering then the functional
\[
\Psi(s) = P_s (\Gamma(P_{t-s}f) + \Gamma^Z(P_{t-s}f)).
\]

2.4 Entropic convergence to the equilibrium

We now prove the entropic convergence of \( P_t \) to the equilibrium if the invariant measure satisfies log-Sobolev inequality. The result is obtained in a very similar way as the convergence in \( H^1(\mu) \).

Theorem 2.10 Assume that the normalized invariant measure \( d\mu = \frac{1}{Z} e^{-V(x)} dx \) is a probability that satisfies the log-Sobolev inequality
\[
\int_{R^{2n}} (f\Gamma(\ln f) + f\Gamma^Z(\ln f)) d\mu \geq \kappa \left[ \int_{R^{2n}} f \ln f d\mu - \left( \int_{R^{2n}} f d\mu \right) \ln \left( \int_{R^{2n}} f d\mu \right) \right].
\]

With the notations of Lemma 2.4:

- If \( K(\eta) + \eta > 0 \), then for every positive and bounded \( f \in C^\infty(R^{2n}) \), such that \( \sqrt{f} \) is Lipschitz and \( \int_{R^{2n}} f d\mu = 1 \),

\[
2(\eta + K(\eta)) \int_{R^{2n}} P_t f \ln P_t f d\mu + \int_{R^{2n}} (P_t f \Gamma(\ln P_t f) + P_t f \Gamma^Z(\ln P_t f)) d\mu
\]
\[
\leq e^{-\lambda t} \left( 2(\eta + K(\eta)) \int_{R^{2n}} f \ln f d\mu + \int_{R^{2n}} (f \Gamma(\ln f) + f \Gamma^Z(\ln f)) d\mu \right),
\]

where \( \lambda = \frac{2\eta \kappa}{\kappa + 2(\eta + K(\eta))} \).
• If \( K(\eta) + \eta \leq 0 \), then for every positive and bounded \( f \in C^\infty(\mathbb{R}^{2n}) \), such that \( \sqrt{f} \) is Lipschitz and \( \int_{\mathbb{R}^{2n}} f \, d\mu = 1 \),

\[
\begin{align*}
\int_{\mathbb{R}^{2n}} P_t f \Gamma(\ln P_t f) + P_t f \Gamma^Z(\ln P_t f) \, d\mu &\leq e^{-2\eta t} \int_{\mathbb{R}^{2n}} (f \Gamma(\ln f) + f \Gamma^Z(\ln f)) \, d\mu \\
\int_{\mathbb{R}^{2n}} P_t f \ln P_t f \, d\mu &\leq \frac{1}{\kappa} e^{-2\eta t} \int_{\mathbb{R}^{2n}} (f \Gamma(\ln f) + f \Gamma^Z(\ln f)) \, d\mu.
\end{align*}
\]

**Proof.** The proof is identical to the proof of Theorem 2.3 by considering now the functional

\[
\Psi(s) = 2 \max(K(\eta) + \eta, 0) P_s (P_{t-s} f \ln P_{t-s} f) + P_s (P_{t-s} f \Gamma(\ln P_{t-s} f) + P_{t-s} f \Gamma^Z(\ln P_{t-s} f)).
\]

\[\square\]

### 3 \( \Gamma_2 \) calculus for degenerate diffusion operators

Our goal is now to considerably extend the scope of the previous section to handle much more general situations. As it appeared, the crucial ingredient is \( \Gamma_2 \) calculus. This section is devoted to the analysis of \( \Gamma_2 \) for a large class of degenerate diffusion operators.

Let \( X_1, \cdots, X_n \) be smooth vector fields on \( \mathbb{R}^d \). We consider on \( \mathbb{R}^d \) a diffusion operator \( L \) that can be written as

\[ L = L_0 + Y, \]

where \( L_0 = \sum_{i=1}^n X_i^2 \) and \( Y \) is a smooth vector field on \( \mathbb{R}^d \). For the range of applications we have in mind (like the kinetic models previously studied), \( L_0 \) can be thought as a diffusion operator which is elliptic on an integral submanifold of \( \mathbb{R}^d \) and \( Y \) can be thought as a vector field such that, at every point, \( \text{span}\{X_1, \cdots, X_n, [Y, X_1], \cdots, [Y, X_n]\} = \mathbb{R}^d \). For instance in the kinetic Fokker-Planck model, \( X_i = \frac{\partial}{\partial v_i} \) and \( Y = -v \cdot \nabla v + \nabla V \cdot \nabla v - v \cdot \nabla x \).

However, our framework covers much more general situations.

#### 3.1 Bochner’s type identities

In order to simplify notations in the subsequent analysis, we introduce the following concept which parallels the notion of relative boundedness of operators introduced by Villani [19]: If \( (T_i)_{i \in I}, (U_j)_{j \in J} \) are two families of vector fields on \( \mathbb{R}^d \), we shall say that the family \( (T_i)_{i \in I} \) is bounded relatively to the family \( (U_j)_{j \in J} \), if there exist smooth and bounded functions \( a_i^j \) on \( \mathbb{R}^d \) such that

\[ T_i = \sum_{j \in J} a_i^j U_j. \]

In that case, we shall denote

\[ (T_i)_{i \in I} \prec (U_j)_{j \in J}. \]

In the coefficients \( a_i^j \) are moreover such that \( \Gamma(a_i^j) \) is always a bounded function, then we shall say that the family \( (T_i)_{i \in I} \) is bounded relatively to the family \( (U_j)_{j \in J} \) with \( \Gamma \)-Lipschitz coefficients.
Our first objective is to prove Bochner’s type identities for the operator $L$. For that purpose, we introduce, as before, the following differential bilinear forms: The carré du champ and its iteration

$$2\Gamma(f, g) = L(fg) - fLg - gLf = \sum_{i=1}^{n} X_i f X_i g,$$

$$2\Gamma_2(f, g) = L\Gamma(f, g) - \Gamma(Lf, g) - \Gamma(f, Lg).$$

We also assume given a family of smooth vector fields $Z_1, \ldots, Z_m$ such that:

$$\{[X_i, X_j]\}_{1 \leq i, j \leq n}, \{[X_i, Z_k]\}_{1 \leq i \leq n, 1 \leq k \leq m}, \{[Y, X_i]\}_{1 \leq i \leq n}, \{[Y, Z_k]\}_{1 \leq k \leq m} \prec \{X_i, Z_k\}_{1 \leq i \leq n, 1 \leq k \leq m},$$

and $\{[X_i, X_j]\}_{1 \leq i, j \leq n}, \{[X_i, Z_k]\}_{1 \leq i \leq n, 1 \leq k \leq m}$ have $\Gamma$-Lipschitz coefficients.

We now denote, as before

$$\Gamma^Z(f, g) = \sum_{i=1}^{n} Z_i f Z_i g$$

and

$$\Gamma^Z_2(f, g) = \frac{1}{2} (L\Gamma^Z(f, g) - \Gamma^Z(f, Lg) - \Gamma^Z(g, Lf)).$$

Finally, we introduce the functions $\omega^k, \gamma^k, \alpha^k, \beta^k, \sigma^k, \lambda^k$ such that

$$[X_i, X_j] = \sum_{k=1}^{n} \omega^k_{ij} X_k + \sum_{k=1}^{m} \gamma^k_{ij} Z_k,$$

$$[X_i, Z_j] = \sum_{k=1}^{n} \alpha^k_{ij} X_k + \sum_{k=1}^{m} \beta^k_{ij} Z_k,$$

$$[Y, Z_i] = \sum_{j=1}^{n} \sigma^j_{i} X_j + \sum_{j=1}^{m} \lambda^j_{i} Z_j.$$

With these preliminaries in hands, we now give the first central result of this section, which can be thought as a horizontal Bochner’s type identity for $L$:

**Theorem 3.1** For $f \in C^\infty(\mathbb{R}^d)$,

$$\Gamma_2(f) = \|\nabla^2_X f\|^2 + \sum_{\ell, j=1}^{n} \left( \sum_{k=1}^{m} \gamma^k_{ij} Z_k f \right)^2 - 2 \sum_{i,j=1}^{n} \sum_{k=1}^{m} \gamma^k_{ij} (X_j Z_k f)(X_i f) + \sum_{i=1}^{n} X_i f T_i f,$$

where

$$\|\nabla^2_X f\|^2 = \sum_{\ell=1}^{n} \left( X_{\ell}^2 f - \sum_{i=1}^{n} \omega^\ell_{i} X_i f \right)^2 + 2 \sum_{1 \leq \ell < j \leq n} \left( \frac{X_j X_{\ell} + X_{\ell} X_j}{2} f - \sum_{i=1}^{n} \frac{\omega^\ell_{ij} + \omega^i_{\ell j}}{2} X_i f \right)^2$$

and the $T_i$’s are vector fields such that $T_i \prec \{X_i, Z_k\}_{1 \leq i \leq n, 1 \leq k \leq m}$.
**Proof.** Let us preliminarily observe that

\[ X_i X_j f = f_{ij} + \frac{1}{2} [X_i, X_j] f, \]

where we have let

\[ f_{ij} = \frac{1}{2} (X_i X_j + X_j X_i) f. \]

Using (3.3), we obtain

\[ X_i X_j f = f_{ij} + \frac{1}{2} \sum_{\ell=1}^{n} \omega_{ij} \ell X_{\ell} f + \frac{1}{2} \sum_{k=1}^{m} \gamma_{ij}^k Z_k f. \]

(3.4)

Starting from the definition of \( \Gamma_2(f) \), we have

\[ \Gamma_2(f) = \sum_{i,j=1}^{n} (X_j X_i f)^2 - 2 \sum_{i,j=1}^{n} X_i f [X_i, X_j] X_j f + \sum_{i,j=1}^{n} X_i f [[X_i, X_j], X_j] f + \sum_{i=1}^{n} X_i f [Y, X_i] f. \]

From (3.4) we have

\[ \sum_{i,j=1}^{n} (X_j X_i f)^2 = \sum_{i,j=1}^{n} f_{ij}^2 + \frac{1}{2} \sum_{1 \leq i < j \leq n} \left( \sum_{\ell=1}^{n} \omega_{ij} \ell X_{\ell} f \right)^2 + \frac{1}{2} \sum_{1 \leq i < j \leq n} \left( \sum_{k=1}^{m} \gamma_{ij}^k Z_k f \right)^2 \]

\[ + \sum_{1 \leq i < j \leq n} \sum_{\ell=1}^{n} \sum_{k=1}^{m} \omega_{ij} \ell \gamma_{ij}^k Z_k f X_{\ell} f \]

and therefore,

\[ \Gamma_2(f) = \sum_{i,j=1}^{n} f_{ij}^2 - 2 \sum_{i,j=1}^{n} X_i f [X_i, X_j] X_j f + \sum_{i,j=1}^{n} X_i f [[X_i, X_j], X_j] f \]

\[ + \sum_{i=1}^{n} X_i f [Y, X_i] f + \frac{1}{2} \sum_{1 \leq i < j \leq n} \left( \sum_{\ell=1}^{n} \omega_{ij} \ell X_{\ell} f \right)^2 + \frac{1}{2} \sum_{1 \leq i < j \leq n} \left( \sum_{k=1}^{m} \gamma_{ij}^k Z_k f \right)^2 \]

\[ + \sum_{1 \leq i < j \leq n} \sum_{\ell=1}^{n} \sum_{k=1}^{m} \omega_{ij} \ell \gamma_{ij}^k Z_k f X_{\ell} f. \]
We now compute
\[
\sum_{i,j=1}^{n} f_{ij}^2 - 2 \sum_{i,j=1}^{n} X_i f [X_i, X_j] X_j f
= \sum_{\ell=1}^{n} \left( f_{\ell\ell}^2 - 2 \left( \sum_{i=1}^{n} \omega_{i\ell}^2 X_i f \right) f_{\ell\ell} \right)
\]
\[+ 2 \sum_{1 \leq \ell < j \leq n} \left( f_{\ell j}^2 - 2 \sum_{1 \leq \ell < j \leq n} \left( \sum_{i=1}^{n} \frac{\omega_{ij}^2}{2} X_i f \right) f_{\ell j} \right)
\]
\[\quad - \sum_{i,j,k,\ell=1}^{n} \sum_{i=1}^{n} \omega_{ij} \omega_{\ell j} X_k f X_i f
- \sum_{i,j=1}^{n} \sum_{\ell=1}^{m} \sum_{k=1}^{n} \omega_{ij} \gamma_{\ell j} Z_k f X_i f
\]
\[\quad - 2 \sum_{i,j=1}^{n} \sum_{k=1}^{m} \gamma_{ij} Z_k f X_i f \cdot X_j f.
\]
Completing the squares in the latter expression we find
\[
\sum_{i,j=1}^{n} f_{ij}^2 - 2 \sum_{i,j=1}^{n} X_i f [X_i, X_j] X_j f
\]
\[= \sum_{\ell=1}^{n} \left( f_{\ell\ell}^2 - \sum_{i=1}^{n} \omega_{i\ell}^2 X_i f \right)^2 + 2 \sum_{1 \leq \ell < j \leq n} \left( f_{\ell j}^2 - \sum_{i=1}^{n} \frac{\omega_{ij}^2 + \omega_{i\ell}^2}{2} X_i f \right)^2
\]
\[\quad - \sum_{\ell=1}^{n} \left( \sum_{i=1}^{n} \omega_{i\ell}^2 X_i f \right)^2 - 2 \sum_{1 \leq \ell < j \leq n} \left( \sum_{i=1}^{n} \frac{\omega_{ij}^2 + \omega_{i\ell}^2}{2} X_i f \right)^2
\]
\[\quad - \sum_{i,j,k,\ell=1}^{n} \sum_{i=1}^{n} \omega_{ij} \omega_{\ell j} X_k f X_i f - \sum_{i,j=1}^{n} \sum_{\ell=1}^{m} \sum_{k=1}^{n} \omega_{ij} \gamma_{\ell j} Z_k f X_i f
\]
\[\quad - 2 \sum_{i,j=1}^{n} \sum_{k=1}^{m} \gamma_{ij} Z_k f X_i f - 2 \sum_{i,j=1}^{n} \sum_{k=1}^{m} \gamma_{ij} [Z_k, X_j] f X_i f.
\]
The conclusion then easily follows by putting the \(\Gamma_2\) pieces together. \(\square\)

The second central result of the section is the following vertical Bochner’s formula:

**Theorem 3.2** For \(f \in C^\infty(\mathbb{R}^d)\),

\[
\Gamma_2^Z(f) = \|\nabla^2 f\|^2 + \sum_{j=1}^{m} \left( \lambda_j^k + \sum_{i,\ell=1}^{n} \alpha_{i\ell}^j \gamma_{i\ell}^j + \sum_{i=1}^{n} X_i \beta_{i\ell}^j + \sum_{i,\ell=1}^{m} \beta_{i\ell}^k (\beta_{i\ell}^j - \beta_{i\ell}^j) \right) Z_j f Z_k f
\]
\[+ 2 \sum_{k=1}^{m} \sum_{i,j=1}^{n} \alpha_{i\ell}^j Z_k f X_i X_j f + \sum_{i=1}^{n} X_i f U_i f.
\]
where
\[ \|\nabla_{X,Z}^2 f\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} \left( X_i Z_j f + \sum_{k=1}^{m} \beta_{ik}^j Z_k f \right)^2. \]
and the \( U_i \)'s are vector fields such that \( U_i \prec \{ X_i, Z_k \}_{1 \leq i \leq n, 1 \leq k \leq m} \).

**Proof.** We start from the definition
\[ \Gamma_2^Z(f) = \frac{1}{2} \left( L \Gamma^Z(f) - 2 \Gamma^Z(f, Lf) \right), \]
which by similar computations as before easily leads to
\[
\Gamma_2^Z(f) = \sum_{i=1}^{n} \sum_{k=1}^{m} (X_i Z_k)^2 + 2 \sum_{i=1}^{n} \sum_{k=1}^{m} Z_k f X_i [X_i, Z_k] f + \sum_{i=1}^{n} \sum_{k=1}^{m} (Z_k f) [\{X_i, Z_k\}, X_i] f + \sum_{k=1}^{m} Z_k f [Y, Z_k] f \\
= \sum_{i=1}^{n} \sum_{k=1}^{m} (X_i Z_k)^2 + 2 \sum_{i=1}^{n} \sum_{k=1}^{m} Z_k f X_i \left( \sum_{j=1}^{n} \alpha_{ik}^j X_j f + \sum_{\ell=1}^{m} \beta_{ik}^\ell Z_\ell f \right) \\
+ \sum_{i=1}^{n} \sum_{k=1}^{m} (Z_k f) [\{X_i, Z_k\}, X_i] f + \sum_{k=1}^{m} Z_k f [Y, Z_k] f \\
= \sum_{i=1}^{n} \sum_{k=1}^{m} (X_i Z_k)^2 + 2 \sum_{i=1}^{n} \sum_{k,\ell=1}^{m} \beta_{ik}^\ell Z_k f X_i Z_\ell f + 2 \sum_{k=1}^{m} \sum_{i,j=1}^{n} \alpha_{ik}^j Z_k f X_i X_j f + 2 \sum_{i,j,k=1}^{n} X_i \beta_{ik}^j Z_k f Z_j f + 2 \sum_{i,j,k=1}^{n} X_i \alpha_{ik}^j X_j f Z_k f \\
+ \sum_{i=1}^{n} \sum_{k=1}^{m} (Z_k f) [\{X_i, Z_k\}, X_i] f + \sum_{k=1}^{m} Z_k f [Y, Z_k] f
\]

We then complete the squares
\[
\sum_{i=1}^{n} \sum_{k=1}^{m} (X_i Z_k f)^2 + 2 \sum_{i=1}^{n} \sum_{j,k=1}^{m} \beta_{ik}^j Z_k f X_i Z_j f \\
= \sum_{i=1}^{n} \sum_{j=1}^{m} (X_i Z_j f + \sum_{k=1}^{m} \beta_{ik}^j Z_k f)^2 - \sum_{i=1}^{n} \sum_{j=1}^{m} \left( \sum_{k=1}^{m} \beta_{ik}^j Z_k f \right)^2
\]
and compute that
\[
\sum_{k=1}^{m} \sum_{i=1}^{n} ([X_i, Z_k], X_i] f Z_k f = \sum_{k=1}^{m} \sum_{i=1}^{n} \left[ \sum_{j=1}^{n} \alpha_{ik}^j X_j + \sum_{\ell=1}^{m} \beta_{ik}^\ell Z_\ell, X_i \right] f Z_k f.
\]
We then have

\[
\sum_{j=1}^{n} \alpha^j_{ik} X_j + \sum_{\ell=1}^{m} \beta^\ell_{ik} Z_\ell, X_i
\]

\[
= \sum_{j=1}^{n} \alpha^j_{ik} [X_j, X_i] + \sum_{\ell=1}^{m} \beta^\ell_{ik} [Z_\ell, X_i] - \sum_{j=1}^{n} (X_j \alpha^j_{ik}) X_j - \sum_{\ell=1}^{m} (X_\ell \beta^\ell_{ik}) Z_\ell,
\]

and the result follows by putting the $\Gamma^Z_2$ pieces together. \hfill \Box

Combining the two Bochner’s identities leads to the following result:

**Theorem 3.3** Assume that there exists a constant $\rho \in \mathbb{R}$ such that for every $f \in C^\infty(\mathbb{R}^d)$,

\[
\sum_{j,k=1}^{n} \left( \lambda^j_k + \sum_{i=1}^{n} (\alpha^j_{ik} + \gamma^j_{ii}) \gamma^j_{li} - \alpha^j_{ik} \alpha^j_{ij} + \sum_{i=1}^{n} X_i \beta^j_{ik} + \sum_{i=1}^{n} \beta^j_{ik} (\beta^j_{ii} - \beta^j_{ij}) \right) Z_j f Z_k f \geq \rho \Gamma^Z(f).
\]

Then, for every $\eta < \rho$, there exists $K(\eta) \in \mathbb{R}$ such that for every $f \in C^\infty(\mathbb{R}^d)$,

\[
\Gamma_2(f) + \Gamma^Z_2(f) \geq -K(\eta) \Gamma(f) + \eta \Gamma^Z(f).
\]

**Proof.** Adding the formulas for $\Gamma_2$ and $\Gamma^Z_2$ and completing the squares in the sums

\[
\|\nabla^2_{X,Z} f\|^2 - 2 \sum_{i,j=1}^{n} \sum_{k=1}^{m} \gamma^j_{ij} (X_j Z_k f)(X_i f)
\]

and

\[
\|\nabla^2_{X} f\|^2 + 2 \sum_{k=1}^{m} \sum_{i,j=1}^{n} \alpha^j_{ik} Z_k f X_i X_j f
\]

leads to the inequality

\[
\Gamma_2(f) + \Gamma^Z_2(f) \geq \rho \Gamma^Z(f) + \sum_{i=1}^{n} X_i f V_i f,
\]

where the $V_i$’s are vector fields such that $V_i \prec \{X_i, Z_k\}_{1 \leq i \leq n, 1 \leq k \leq m}$. If we now pick $\eta < \rho$ it is clear that for $K(\eta)$ big enough the bilinear form

\[
(\rho - \eta) \Gamma^Z(f) + \sum_{i=1}^{n} X_i f V_i f + K(\eta) \Gamma(f)
\]

can be made positive. As a consequence, we have

\[
\Gamma_2(f) + \Gamma^Z_2(f) \geq -K(\eta) \Gamma(f) + \eta \Gamma^Z(f).
\]
In order to use the previous results in concrete situations, we of course need to identify the vector fields $Z_i$'s and try to chose them in such a way that $\rho > 0$.

The following result is an explicit example of choice for the $Z_i$'s under boundedness conditions on the vector fields $[X_i, X_j]$, $[X_i, [Y, X_j]]$, and $[Y,[Y, X_i]]$. It is comparable to Theorem 24 in Villani [11] and applies to kinetic type Fokker-Planck operators.

**Proposition 3.4** Assume that the family of vector fields $\{[X_i, X_j]\}_{1 \leq i,j \leq n}$ is bounded relatively with respect to the family $\{X_i\}_{1 \leq i \leq n}$ with $\Gamma$-Lipschitz coefficients and that the two families $\{[X_i, [Y, X_j]]\}_{1 \leq i \leq n, 1 \leq j \leq n}$ and $\{[Y,[Y, X_i]]\}_{1 \leq i \leq n}$ are bounded relatively to $\{X_i, [Y, X_k]\}_{1 \leq i \leq n, 1 \leq k \leq n}$ with $\Gamma$-Lipschitz coefficients, then there exist vector fields $Z_1, \ldots, Z_m$ and a constant $\rho > 0$ such that for every $\eta < \rho$, there exists $K(\eta) \in \mathbb{R}$ such that for every $f \in C^\infty(\mathbb{R}^d)$

$$\Gamma_Z^2(f) \geq -K(\eta)\Gamma(f) + \eta \Gamma_Z(f).$$

**Proof.** We can write

$$[X_i, X_j] = \sum_{k=1}^n \omega_{ij}^k X_k$$

$$[X_i, [Y, X_j]] = \sum_{k=1}^n \alpha_{ij}^k X_k + \sum_{k=1}^n \beta_{ij}^k [Y, X_k],$$

for some bounded and $\Gamma$-Lipschitz functions $\omega_{ij}^k, \alpha_{ij}^k, \beta_{ij}^k$. Consider then

$$Z_i = X_i + \varepsilon [Y, X_i],$$

where $\varepsilon > 0$ is to be chosen later. We have

$$[X_i, Z_j] = [X_i, X_j] + \varepsilon [X_i, [Y, X_j]]$$

$$= \sum_{k=1}^n \omega_{ij}^k X_k + \varepsilon \sum_{k=1}^n \alpha_{ij}^k X_k + \varepsilon \sum_{k=1}^n \beta_{ij}^k [Y, X_k]$$

$$= \sum_{k=1}^n (\omega_{ij}^k + \varepsilon \alpha_{ij}^k) X_k + \varepsilon \sum_{k=1}^n \beta_{ij}^k (Z_k - X_k)$$

$$= \sum_{k=1}^n (\omega_{ij}^k + \varepsilon \alpha_{ij}^k - \varepsilon \beta_{ij}^k) X_k + \varepsilon \sum_{k=1}^n \beta_{ij}^k Z_k$$

and

$$[Y, Z_i] = [Y, X_i] + \varepsilon [Y, [Y, X_i]]$$

$$= \frac{1}{\varepsilon} (Z_i - X_i) + \varepsilon [Y, [Y, X_i]].$$

We can then chose $\varepsilon$ small enough in such a way that the assumptions of Proposition 3.3 are satisfied with $\rho > 0$. \qed
3.2 Gradient estimates and convergence to the equilibrium

Collecting the previous $\Gamma_2$ estimates and reasoning as for the kinetic Fokker-Planck model we can now generalize the results of Section 2. As in the previous section, we consider

$$L = L_0 + Y,$$

and we now assume that there exist smooth vector fields $Z_1, \ldots, Z_m$ and constants $\rho_1 \geq 0, \rho_2 > 0$ such that for every $f \in C^\infty(\mathbb{R}^d)$,

$$\Gamma_2(f) + \Gamma_Z^2(f) \geq -\rho_1 \Gamma(f) + \rho_2 \Gamma_Z(f).$$

This is for instance the case if the assumptions of Proposition 3.4 are fulfilled.

Borrowing an hypothesis introduced by F.Y. Wang [12], we also assume that there exists a function $W$ with compact level sets such that $W \geq 1$, $\Gamma(f) + \Gamma_Z(f) \leq CW^2$, $LW \leq CW$ for some constant $C > 0$. In that case, $L$ is the generator of a Markov semigroup $(P_t)_{t \geq 0}$ and the following result is a consequence of [2] Proposition 2.2, or [12], Lemma 2.1.

**Lemma 3.5** If $f$ is a bounded Lipschitz function on $\mathbb{R}^d$, then for every $t \geq 0$, $P_t f$ is a bounded and Lipschitz function. More precisely, for every $x \in \mathbb{R}^d$,

$$\Gamma(P_t f)(x) + \Gamma_Z(P_t f)(x) \leq e^{-2\lambda t} P_t (\Gamma(f) + \Gamma_Z(f))(x),$$

where $\lambda = \min(-\rho_1, \rho_2)$.

**Proof.** The proof is identical to the proof of Lemma 2.6. $\square$

We also have the following result:

**Lemma 3.6** Assume that for every $f \in C^\infty(\mathbb{R}^{2n})$,

$$\Gamma(f, \Gamma_Z(f)) = \Gamma_Z(f, \Gamma(f)).$$

Let $f \in C^\infty(\mathbb{R}^d)$ be a positive function such that $\sqrt{f}$ is bounded and Lipschitz, then for $t \geq 0$, $\sqrt{P_t f}$ is bounded and Lipschitz. More precisely, for every $x \in \mathbb{R}^d$,

$$P_t f(x) \Gamma(\ln P_t f)(x) + P_t f(x) \Gamma_Z(\ln P_t f)(x) \leq e^{-2\lambda t} P_t (f \Gamma(\ln f) + f \Gamma_Z(\ln f))(x),$$

where $\lambda(\eta) = \min(-\rho_1, \rho_2)$.

**Proof.** The proof is identical to the proof of Lemma 2.7. $\square$

The following theorems are obtained in the very same way as Theorems 2.8 and 2.10.

**Theorem 3.7** Assume that the operator $L$ admits an invariant probability measure $\mu$ that satisfies the Poincaré inequality

$$\int_{\mathbb{R}^d} (\Gamma(f) + \Gamma_Z(f)) d\mu \geq \kappa \left[ \int_{\mathbb{R}^{2n}} f^2 d\mu - \left( \int_{\mathbb{R}^{2n}} f d\mu \right)^2 \right].$$
Then, for every bounded function \( f \) such that \( \Gamma(f) + \Gamma^Z(f) \) is bounded and \( \int_{\mathbb{R}^d} f d\mu = 0, \)

\[
(\rho_1 + \rho_2) \int_{\mathbb{R}^d} (P_t f)^2 d\mu + \int_{\mathbb{R}^d} (\Gamma(P_t f) + \Gamma^Z(P_t f)) d\mu \\
\leq e^{-\lambda t} \left( (\rho_1 + \rho_2) \int_{\mathbb{R}^d} f^2 d\mu + \int_{\mathbb{R}^d} (\Gamma(f) + \Gamma^Z(f)) d\mu \right),
\]

where \( \lambda = \frac{2\rho_2 \kappa}{\kappa + \rho_1 + \rho_2} \).

**Theorem 3.8** Assume that for every \( f \in C^\infty(\mathbb{R}^d) \),

\[ \Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)). \]

Assume also that the operator \( L \) admits an invariant probability measure \( \mu \) that satisfies the log-Sobolev inequality

\[
\int_{\mathbb{R}^d} (f \Gamma(\ln f) + f \Gamma^Z(\ln f)) d\mu \geq \kappa \left[ \int_{\mathbb{R}^d} f \ln f d\mu - \left( \int_{\mathbb{R}^d} f d\mu \right) \ln \left( \int_{\mathbb{R}^d} f d\mu \right) \right].
\]

Then for every positive and bounded \( f \in C^\infty(\mathbb{R}^d) \), such that \( \Gamma(\sqrt{f}) + \Gamma^Z(\sqrt{f}) \) is bounded and \( \int_{\mathbb{R}^d} f d\mu = 1, \)

\[
2(\rho_1 + \rho_2) \int_{\mathbb{R}^d} P_t f \ln P_t f d\mu + \int_{\mathbb{R}^d} (P_t f \Gamma(\ln P_t f) + P_t f \Gamma^Z(\ln P_t f)) d\mu \\
\leq e^{-\lambda t} \left( 2(\rho_1 + \rho_2) \int_{\mathbb{R}^d} f \ln f d\mu + \int_{\mathbb{R}^d} (f \Gamma(\ln f) + f \Gamma^Z(\ln f)) d\mu \right),
\]

where \( \lambda = \frac{2\rho_2 \kappa}{\kappa + 2(\rho_1 + \rho_2)} \).

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