Long-Range Behavior in Quantum Gravity

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Quantum gravity effects of zeroth order in the Planck constant are investigated in the framework of the low-energy effective theory. A special emphasis is placed on establishing the correspondence between classical and quantum theories, for which purpose transformation properties of the $\hbar^0$-order radiative contributions to the effective gravitational field under deformations of a reference frame are determined. Using the Batalin-Vilkovisky formalism it is shown that the one-loop contributions violate the principle of general covariance, in the sense that the quantities which are classically invariant under such deformations take generally different values in different reference frames. In particular, variation of the scalar curvature under transitions between different reference frames is calculated explicitly. Furthermore, the long-range properties of the two-point correlation function of the gravitational field are examined. Using the Schwinger-Keldysh formalism it is proved that this function is finite in the coincidence limit outside the region of particle localization. In this limit, the leading term in the long-range expansion of the correlation function is calculated explicitly, and the relative value of the root mean square fluctuation of the Newton potential is found to be $1/\sqrt{2}$. It shown also that in the case of a macroscopic gravitating body, the terms violating general covariance, and the field fluctuation are both suppressed by a factor $1/N$, where $N$ is the number of particles in the body. This leads naturally to a macroscopic formulation of the correspondence between classical and quantum theories of gravitation. As an application of the obtained results, the secular precession of a test particle orbit in the field of a black hole is determined.

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I. INTRODUCTION

In this Chapter, quantum properties of the gravitational fields produced by quantized matter will be considered in the framework of the low-energy effective theory. This means that our interest will be in quantum aspects of the gravitational interaction at energy scales much lower than the Planck energy

$$E_P = \sqrt{\frac{\hbar c^5}{G}}.$$

From the practical point of view, this does not present a limitation, since all phenomena observed in the Universe so far fall perfectly into this category. In view of the extreme smallness of the Newton constant $G$, these phenomena are well described by the lowest-order Einstein theory. On the other hand, the results obtained within the lowest-order approximation are model-independent, i.e., they play the role of low-energy theorems. This universality allows one to draw a number of important conclusions concerning the synthesis of quantum theory and gravitation.

An essential part of investigation of this synthesis consists in establishing a correspondence between classical and quantum theories. The formal rules of this correspondence are contained in the Bohr correspondence principle which gives a general recipe for the construction of operators for various physical quantities. Informally, the problem of correspondence is in elucidating actual conditions to be imposed on a system to allow its classical consideration. Identification of these quasi-classical conditions constitutes an integral part of interpretative basis of quantum theory. Right from this point of view the low-energy behavior in quantum gravity will be considered below.

Two characteristics of the gravitational field produced by a quantized system will be of our main concern in this Chapter – the mean value of the gravitational field, and its correlation function. The results of independent investigation of these quantities will lead us to one and the same macroscopic formulation of the quasi-classical conditions in quantum gravity.

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Turning to a more detailed formulation of our approach, let us note, first of all, that the low-energy condition stated above implies a similar condition on the characteristic length scale $L$ of the process under consideration, $L \gg l_P$, where $l_P$ is the Planck length

$$l_P = \sqrt{\frac{G\hbar}{c^3}}. \tag{1}$$

In other words, our interest is in the long-range properties of the gravitational field produced by a given system. As far as this system can be treated classically, $l_P$ is the length scale characterizing quantum properties of its gravitational interactions, because $l_P^2$ is the only parameter entering the theory of quantized gravitational field through the Einstein action

$$S_g = -\frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R. \tag{2}$$

In the presence of quantized matter, however, another parameter with dimension of length comes into play, namely the gravitational radius

$$r_g = \frac{2Gm}{c^2}.$$

This parameter appears, of course, already in classical theory. The question of crucial importance is whether $r_g$ has an independent meaning in quantum domain, representing a scale of specifically quantum phenomena. This question may seem strange at first sight, as $r_g$ does not contain the Planck constant $\hbar$, an inalienable attribute of quantum theory. However, well known is the fact that gravitational radiative corrections do contain pieces independent of $\hbar$. In the framework of the effective theory, they appear as a power series in $r_g/r$, just like post-Newtonian corrections in classical general relativity. This fact was first clearly stated by Iwasaki [2]. The reason for the appearance of $\hbar^0$ terms through the loop contributions is that in the case of gravitational interaction, the mass and “kinetic” terms in a matter Lagrangian determine not only the properties of matter quanta propagation, but also their couplings. Thus the mass term of, e.g., scalar field Lagrangian generates the vertices proportional to

$$\left(\frac{mc}{\hbar}\right)^2$$

containing inverse powers of $\hbar$. Naively, one expects these be cancelled by $\hbar$’s coming from the propagators when combining an amplitude. One should remember, however, that such a counting of powers of $\hbar$ in Feynman diagrams is a bad helmsmate in the presence of massless particles. Virtual propagation of gravitons interacting with matter field quanta near their mass shells results in a root non-analyticity of the massive particle form factors at zero momentum transfer ($p$). For instance, the low-energy expansion of the one-loop diagram in Fig. 2(a) begins with terms proportional to $(-p^2)^{-1/2}$, rather than integer powers (or logarithms) of $p^2$. It is this singularity that is responsible for the appearance of $r_g/r$ in the long-range expansion of the loop contributions. The question we ask is of what nature, classical or quantum, these pieces are. This is precisely the question of correspondence in quantum gravity.

Suppose that the matter producing gravitational field satisfies the usual quantum mechanical quasi-classical conditions, e.g., consider sufficiently heavy particles. Then the quasi-classical conditions for the gravitational field can be inferred from the requirement that the mean value of the spacetime metric, $\langle g_{\mu\nu} \rangle$, coincides with the classical solution of Einstein equations, corresponding to the same distribution of gravitating matter. Practically, the most convenient way of looking for these conditions is to compare transformation properties of the quantities involved under deformations of the reference frame, thus avoiding explicit calculation of the expectation values. The latter point of view takes advantage of the fact that the transformation law of classical solutions is known in advance. Hence, we have to check whether $\langle g_{\mu\nu} \rangle$ transforms covariantly with respect to transitions between different reference frames. In other words, we have to consider the question of general covariance in quantum gravity. This is the way we follow in Sec. III. Alternatively, one can infer the quasi-classical conditions from the requirement of vanishing of field fluctuations. We take this rout in Sec. IV. Some results of the Schwinger-Keldysh and Batalin-Vilkovisky formalisms, used in our investigation, are summarized in Sec. III. Section V contains concluding remarks.

Condensed notations of DeWitt [1] are in force throughout this chapter. Also, right and left derivatives with respect to the fields and the sources, respectively, are used. The dimensional regularization of all divergent quantities is assumed.
II. PRELIMINARIES

Before going into detailed discussion of the question of general covariance in quantum gravity, let us describe the general setting we will be working in.

A. Frame of reference and interacting fields.

First of all, we should set a frame of reference, i.e., a system of idealized reference bodies with respect to which the 4-position in spacetime can be fixed. Let us assume, for definiteness, that the frame of reference is realized by means of an appropriate distribution of electrically charged matter. For simplicity, the energy-momentum of matter, as well as of the electromagnetic field it produces, will be assumed sufficiently small so as not to alter the gravitational field under consideration. The 4-position in spacetime can be determined by exchanging electromagnetic signals with a number of charged matter species. The electric charge distributions \( \sigma_a \) of the latter are thus supposed to be in a one-to-one correspondence with the spacetime coordinates \( x_\mu \),

\[ \sigma_a \leftrightarrow x_\mu, \]

where index \( a \) enumerates the species. The \( \sigma_a(x) \) will be assumed smooth scalar functions. Physical properties of the reference frame are determined by the action \( S_\phi \) which specific form is of no importance for us.

Next, let us consider a system of interacting gravitational and matter fields. The latter are arbitrary species, bosons or fermions, self-interacting or not, denoted collectively by \( \Phi \), or fermions, self-interacting or not, denoted collectively by \( \phi \). Dynamics of the system is described by the action \( S = S_g + S_\phi + S_\sigma \) where \( S_\phi \) is the matter action, and \( S_g \) is given by Eq. (2).

The total action \( S = S_g + S_\phi + S_\sigma \) is invariant under the gauge transformations

\[
\begin{align*}
\delta h_{\mu\nu} &= \xi^\alpha \partial_{\alpha} h_{\mu\nu} + (\eta_{\mu\alpha} + h_{\mu\alpha}) \partial_{\alpha} \xi^\alpha + (\eta_{\nu\alpha} + h_{\nu\alpha}) \partial_{\mu} \xi^\alpha \equiv G_{\mu\nu}^\alpha \xi^\alpha, \\
\delta \phi_i &= G_i^\alpha \xi^\alpha, \\
\delta \sigma_a &= \sigma_{a,\alpha} \xi^\alpha
\end{align*}
\]

(3) (4) (5)

The generators \( G_{\mu\nu}, G_i \) span the closed algebra

\[
G_{\mu\nu}^{\alpha\sigma\lambda} G_{\sigma\lambda}^{\beta} - G_{\mu\nu}^{\beta\sigma\lambda} G_{\sigma\lambda}^{\alpha} = f^{\alpha\beta\gamma} G_{\mu\nu}^{\gamma},
\]

\[
G_i^{\alpha,k} G_k^{\beta} - G_i^{\beta,k} G_k^{\alpha} = f^{\alpha\beta\gamma} G_i^{\gamma},
\]

(6)

where the ”structure constants” \( f^{\alpha\beta\gamma} \) are defined by

\[ f^{\alpha\beta\gamma} \xi^\alpha \eta_\beta = \xi^\alpha \partial^\sigma \eta_\gamma - \eta_\alpha \partial^\alpha \xi_\gamma. \]

(7)

Let the gauge-fixing action be written in the form

\[
S_{gf} = \left( F_\alpha - \frac{1}{2} \pi^\beta \dot{\zeta}_\beta \alpha \right) \pi^\alpha,
\]

(8)

where \( F_\alpha \) is a set of functions of the fields \( h_{\mu\nu} \), fixing general invariance, \( \pi^\alpha \) auxiliary fields introducing the gauge, and \( \zeta_\alpha \) a non-degenerate symmetric matrix weighting the functions \( F_\alpha \); the particular choice \( \zeta_\alpha = \eta_\alpha \) corresponds to the well-known Feynman weighting of the gauge conditions. Introducing the ghost fields \( c_\alpha, \bar{c}^\alpha \), and denoting all the fields collectively by \( \Phi \), we write the Faddeev-Popov action \( S_{FP} \)

\[
S_{FP}[\Phi] = S + S_{gf} + \bar{c}^\beta F_\beta^{\mu\nu} G_{\mu\nu}^\alpha c_\alpha.
\]

(9)

\footnote{Our notation is \( R_{\mu\nu} \equiv R_{\mu\nu}^\alpha = \partial_\alpha \Gamma_{\mu\nu}^\alpha - \cdots \), \( R \equiv R_{\mu\nu} g^{\mu\nu}, g \equiv \det g_{\mu\nu}, g_{\mu\nu} = \text{sgn}(+,-,-,-), \eta_{\mu\nu} = \text{diag}\{+1,-1,-1,-1\}. \) The Minkowski tensor \( \eta \) is used to raise and lower tensor indices. The units in which \( \hbar = c = 16\pi G = 1 \) are chosen in what follows.}
\( S_{FP}[\Phi] \) is invariant under the following Becchi-Rouet-Stora-Tyutin (BRST) transformations [4]

\[
\begin{align*}
\delta h_{\mu\nu} &= G^\alpha_{\mu\nu} c_\alpha \lambda, \\
\delta \phi_i &= G^\alpha_i c_\alpha \lambda, \\
\delta \sigma_a &= \sigma_{a\alpha} c^\alpha \lambda, \\
\delta c_\gamma &= - \frac{1}{2} f^{\alpha\beta\gamma} c_\alpha c_\beta \lambda, \\
\delta \pi^\alpha &= \pi^\alpha \lambda, \\
\delta \pi^0 &= 0, \\
\end{align*}
\]

where \( \lambda \) is a constant anticommuting parameter. Finally, the generating functional of Green functions has the form

\[
Z[J, K] = \int \mathcal{D}\Phi \exp \{i[\Sigma[\Phi, K] + \bar{\beta}^\alpha c_\alpha + \beta_\alpha c^\alpha + t^\mu h_{\mu\nu} + j^i \phi_i + s^a \sigma_a]\},
\]

where

\[
\Sigma[\Phi, K] = S_{FP}[\Phi] + k^{\mu\nu} G^\alpha_{\mu\nu} c_\alpha + q^i G^\alpha_i c_\alpha + r^a \sigma_{a\alpha} c^\alpha - \frac{l^0}{2} f^{\alpha\beta\gamma} c_\alpha c_\beta + n_\alpha \pi^\alpha,
\]

\{t, j, s, \bar{\beta}, \beta, 0\} \equiv J \text{ ordinary sources, and } \{k, q, r, l, n, 0\} \equiv K \text{ the BRST-transformation sources [5] for the fields } \\
\{h, \phi, \sigma, c, \bar{c}, \pi\} \equiv \Phi, \text{ respectively, and integration is carried over all field configurations satisfying} \]

\[
\Phi^\pm \to 0, \quad \text{for } t \to \mp \infty,
\]

where the superscripts + and − denote the positive- and negative-frequency parts of the fields, respectively. The \( K \)-sources are introduced into \( Z \) for the purpose of studying gauge dependence of observable quantities.

### B. Effective fields and correlation functions

The central objects of our investigation in the sections below will be the expectation value of the gravitational field produced by an elementary particle, and its correlation function. They can be obtained from the one- and two-point Green functions of the gravitational field, respectively,

\[
h_{\mu\nu}^{\text{eff}}(j) = \frac{\delta W[J, K]}{\delta h_{\mu\nu}} \bigg|_{K=0}^{J=0},
\]

\[
C_{\mu\nu\alpha\beta}(j) = \frac{\delta^2 W[J, K]}{\delta h_{\mu\nu} \delta h_{\alpha\beta}} \bigg|_{K=0}^{J=0},
\]

with the help of the well-known reduction formula applied to the external matter lines to go over onto the mass shell of the particle. In Eqs. (12), (13), \( W \) is the generating functional of connected Green functions,

\[
W[J, K] = -i \ln Z[J, K],
\]

and the symbol \( J \setminus j \) means that the source \( j \) is excluded from the set \( J \).

However, under the Feynman asymptotic conditions specified above, Eqs. (12), (13) give the in-out matrix elements of operators, rather than the in-in expectation values we are interested in. As is well known, in order to find the latter, the usual Feynman rules for constructing the matrix elements must be modified. According to the so-called closed time path formalism of Schwinger and Keldysh [6, 7] (modern reviews of this formalism can be found in Refs. [8, 9]), this amounts to using in Eqs. (12), (13) the generating functional

\[
Z_{SK}[J_\pm, K_\pm] = \int \mathcal{D}\Phi - \int \mathcal{D}\Phi_+ \exp \{i[\Sigma[\Phi_\pm, K_\pm] + \bar{\beta}^\alpha c_\alpha + \beta_\alpha c^\alpha + t^\mu h_{\mu\nu} + j^i \phi_i + s^a \sigma_a]\},
\]

where

\[
\Sigma[\Phi_\pm, K_\pm] = S_{FP}[\Phi_+] - S_{FP}[\Phi_-] + k^{\mu\nu} G^\alpha_{\mu\nu} c_\alpha + q^i G^\alpha_i c_\alpha + r^a \sigma_{a\alpha} c^\alpha - \frac{l^0}{2} f^{\alpha\beta\gamma} c_\alpha c_\beta + n_\alpha \pi^\alpha,
\]
instead of (11). Here the subscript + (−) shows that the time argument of the integration variable runs from \(-\infty\) to \(+\infty\) (from \(+\infty\) to \(-\infty\)). Integration is over all fields satisfying

\[
\Phi^+ \to 0, \quad \text{for} \quad t \to -\infty, \\
\Phi_- = \Phi_+ \quad \text{for} \quad t \to +\infty.
\]

We do not distinguish the plus and minus field components in the source terms explicitly, implying that summation over repeated Greek indices includes summation over ± as well as spacetime integration. For instance,

\[
l^\mu f^{\alpha\beta\gamma\delta} c_\alpha c_\beta \equiv \int d^4 x \left( t^\mu f^{\alpha\beta\gamma\delta} c_\alpha c_\beta + l^\gamma f^{\alpha\beta\gamma\delta} c_\alpha c_\beta \right), \quad \text{etc.}
\]

With the help of the new generating functional, the in-in expectation value of, e.g., the product \( \hat{h}_{\mu\nu}(x)\hat{h}_{\alpha\beta}(x') \) can be written as

\[
\langle \text{in} | \hat{h}_{\mu\nu}(x)\hat{h}_{\alpha\beta}(x') \text{in} \rangle = -Z_{SK}^{-1} \left. \delta^2 Z_{SK} | J_x, K_\pm \rangle_{K=\pm} \right|_{J_x=0}. \tag{16}
\]

It is seen that \( \langle \text{in} | \hat{h}_{\mu\nu}(x)\hat{h}_{\alpha\beta}(x') \text{in} \rangle \) is given by the ordinary functional integral but with the number of fields doubled, and unusual boundary conditions specified above. Accordingly, diagrammatics generated upon expanding this integral in perturbation theory consists of the following elements. There are four types of pairings for each component of the set \( \Phi \), corresponding to the four different ways of placing two field operators on the two branches of the time path. They are conveniently combined into 2 \( \times \) 2 matrices according to\(^2\)

\[
\mathcal{D} = \begin{pmatrix} D_{++} & D_{+-} \\ D_{-+} & D_{--} \end{pmatrix}.
\]

For instance, pairings matrices for the gravitational and scalar matter fields read

\[
\mathcal{D}_{ik}(x,y) = \begin{pmatrix} i\langle T \hat{\phi}_i(x)\hat{\phi}_k(y) \rangle_0 & i\langle \hat{\phi}_k(y)\hat{\phi}_i(x) \rangle_0 \\ i\langle \hat{\phi}_i(x)\hat{\phi}_k(y) \rangle_0 & i\langle T \hat{\phi}_i(x)\hat{\phi}_k(y) \rangle_0 \end{pmatrix},
\]

\[
\mathcal{D}_{\mu\nu\alpha\beta}(x,y) = \begin{pmatrix} i\langle T \hat{h}_{\mu\nu}(x)\hat{h}_{\alpha\beta}(y) \rangle_0 & i\langle \hat{h}_{\alpha\beta}(y)\hat{h}_{\mu\nu}(x) \rangle_0 \\ i\langle \hat{h}_{\mu\nu}(x)\hat{h}_{\alpha\beta}(y) \rangle_0 & i\langle T \hat{h}_{\mu\nu}(x)\hat{h}_{\alpha\beta}(y) \rangle_0 \end{pmatrix},
\]

where the operation of time ordering \( T \) \((\hat{T})\) arranges the factors so that the time arguments decrease (increase) from left to right, and \( \langle \cdot \rangle_0 \) denotes vacuum averaging. The “propagators” \( \mathcal{D}_{\mu\nu\alpha\beta}, \mathcal{D}_{ik} \) satisfy the following matrix equations

\[
\mathcal{G}^{\mu\nu\alpha\beta} \mathcal{D}_{\alpha\beta\sigma\lambda} = -\varepsilon^{\mu\nu}, \quad \mathcal{G}^{\mu\nu\alpha\beta} = i \frac{\delta^2 S^{(2)}}{\delta h_{\mu\nu}\delta h_{\alpha\beta}}, \quad \delta_{\mu\nu}^{\alpha\beta} = \frac{1}{2} \left( \delta_{\alpha\beta}^{\mu\nu} + \delta_{\mu\nu}^{\alpha\beta} \right), \tag{17}
\]

\[
\mathcal{G}^{ik} \mathcal{D}_{kl} = -\varepsilon \delta_{ij}, \quad \mathcal{G}^{ik} = i \frac{\delta^2 S^{(2)}}{\delta \hat{\phi}_i \delta \hat{\phi}_k}, \tag{18}
\]

where \( S^{(2)} \) denotes the free field part of the gauge-fixed action after the gauge introducing fields \( \pi^\alpha \) have been integrated out, and \( \varepsilon, i \) are 2 \( \times \) 2 matrices with respect to indices +, − :

\[
\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

---

\(^2\) Gothic letters are used to distinguish quantities representing columns, matrices etc. with respect to indices +, −.
As in the ordinary Feynman diagrammatics of the S-matrix theory, the propagators are contracted with the vertex factors generated by the interaction part of the action, $S^\text{int}[\Phi]$, with subsequent summation over $(+, -)$ in the vertices, each “$-$” vertex coming with an extra factor $(-1)$. This can be represented as the matrix multiplication of $\mathcal{D}_{\mu\nu\alpha\beta}, \mathcal{D}_{ik}$ with suitable matrix vertices. For instance, the $\phi^2 h$ part of the action generates the matrix vertex $\mathcal{V}^{\mu, \nu, ik}$ which in components has the form

$$V^{\mu, \nu, ik}_{stu}(x, y, z) = e_{stu} \frac{\delta^3 S}{\delta h_{\mu\nu}(x) \delta \phi_i(y) \delta \phi_k(z)}|_{\Phi = 0},$$

where indices $s, t, u$ take the values $+, -$, and $e_{stu}$ is defined by $e_{+++} = e_{---} = 1$ and zero otherwise. An external $\phi$ line is represented in this notation by a column

$$\mathcal{V}^{ik}_{\mathcal{k}} = \begin{pmatrix} \bar{\phi}_i \\ \phi_i \end{pmatrix},$$

satisfying

$$\mathcal{G}^{ik} \mathcal{V}_{\mathcal{k}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{19}$$

For future references, we give explicit expressions for various pairings of a single real scalar field

$$D_{++}(x, y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{m^2 - k^2 - i0}, \quad D_{--}(x, y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - m^2 + i0},$$

$$D_{-+}(x, y) = i \int \frac{d^4k}{(2\pi)^4} \theta(k^0) e^{-ik(x-y)}, \quad D_{+-}(x, y) = D_{-+}(y, x). \tag{20}$$

C. Generating functionals and Slavnov identities.

The functional $\Sigma[\Phi_{\pm}, K_{\pm}]$ can be written as \cite{10}

$$\Sigma[\Phi_{\pm}, K_{\pm}] = \Sigma' \left[ \Phi_{\pm}, \frac{\delta \Psi[\Phi_{\pm}, K_{\pm}]}{\delta \Phi_{\pm}} \right],$$

where the so-called reduced action $\Sigma'$ and the gauge fermion $\Psi$ are defined by

$$\Sigma'[\Phi_{\pm}, K_{\pm}] = S + k^{\mu\nu} G_{\mu\nu} c_\alpha + q^i G_i^\alpha c_\alpha + r^a \sigma_{a,\alpha} c^\alpha - \frac{\gamma}{2} f^{\alpha\beta\gamma} c_\alpha c_\beta + n_\alpha \pi^\alpha,$$

$$\Psi[\Phi_{\pm}, K_{\pm}] = k^{\mu\nu} h_{\mu\nu} + q^i \phi_i + r^a \sigma_\alpha + l^a c_\alpha + n_\alpha \bar{c}^\alpha + \bar{c}^\alpha \left( F_\alpha - \frac{1}{2} \pi^\beta \beta_\alpha \right).$$

Let us further simplify notation abbreviating $k^{\mu\nu} h_{\mu\nu} + q^i \phi_i + r^a \sigma_\alpha + l^a c_\alpha + n_\alpha \bar{c}^\alpha \equiv K \Phi$, and similarly for other sums of products of fields and sources or derivatives with respect to $\Phi, J, K$. Omitting also the subscript $\pm$, the variation of $\Sigma$ under infinitesimal variation of the gauge conditions takes the form

$$\delta \Sigma[\Phi, K] = \frac{\delta \Sigma'[\Phi, K] \delta \Sigma[\Phi, K]}{\delta \Phi} \delta K.$$ 

The corresponding variation of the generating functional

$$\delta Z_{SK}[J, K] = i \int D\Phi_+ D\Phi_- \exp \{ i(\Sigma + J\Phi) \} \frac{\delta \Sigma'[\Phi, K] \delta \Sigma[\Phi, K]}{\delta \Phi}. \tag{21}$$

Integrating by parts and omitting $\delta^2 \Sigma/\delta K \delta \Phi \sim \delta^{(4)}(0)$ in the latter equation gives

$$\delta Z_{SK}[J, K] = iJ \frac{\delta}{\delta K} \int D\Phi_- D\Phi_+ \Delta \Psi \exp \{ i(\Sigma + J\Phi) \}. \tag{21}$$
Since $\Sigma$ is invariant under the BRST transformation \cite{10}, a BRST change of integration variables in Eq. (14) gives the Slavnov identity for the generating functional

$$J \frac{\delta Z_{SK}}{\delta K} = 0,$$

which allows one to rewrite Eq. (21) in terms of the generating functional of connected Green functions as

$$\delta W_{SK}[J, K] = J \frac{\delta}{\delta K} \langle \Delta \Psi \rangle,$$

where $\langle X \rangle$ denotes the functional averaging of $X$ \cite{11}.

**III. TRANSFORMATION PROPERTIES OF EFFECTIVE GRAVITATIONAL FIELD**

In this section, we will investigate the structure of the $\hbar^0$-order contributions to the effective gravitational field, concentrating mainly on their transformation properties under deformations of the reference frame. The appearance of such contributions through the radiative corrections has its roots in the field properties of quantized matter sources. To make the field-theoretic aspect of the problem clearer, it is convenient to get rid of the purely quantum mechanical issues related to the quantum particle kinematics assuming matter quanta sufficiently heavy to neglect the position-velocity indeterminacy following from the Heisenberg principle. This assumption justifies the use of the term "particle at rest," and allows one to introduce a fixed distance $r$ between the particle and the point of observation.

The mean gravitational field produced by a massive particle is a function of five dimensional parameters – the fundamental constants $\hbar, G, c$, the particle’s mass $m$, and the distance $r$. Of these only two independent dimensionless combinations can be constructed, which we choose to be $\chi = h \nu/r$ and $\kappa = lc/r$, where $lc = \hbar/mc$ is the Compton length of the particle. The above assumption means that the gravitational field is considered in the limit $\kappa \to 0$. For a fixed particle’s mass, small values of $\kappa$ imply large values of $r$, leading naturally to the long-range expansion of the mean gravitational field. Accordingly, we will use where appropriate the terminology and general ideas of effective field theories \cite{12, 13, 14, 15}.

**A. General covariance at the tree level.**

From the point of view of the general formalism outlined in the preceding sections, the classical Einstein theory corresponds to the tree approximation of the full quantum theory. The tree contributions to the expectation values of field operators coincide with the corresponding classical fields, and the effective equations of motion

$$\left\langle \frac{\delta \Sigma}{\delta h_{\mu \nu}} \right\rangle + t^{\mu \nu} = 0,$$

expressing the translation invariance of the functional integral measure, go over into the classical Einstein equations. Some of the tree diagrams representing their solution are shown in Fig. 1. The results of the preceding section allow one, in particular, to reestablish the general covariance of these equations.

As was mentioned in the Introduction, coordinate transformations are replaced in the quantum theory by the field transformations. In particular, transformations of the reference frame are represented by variations of the gauge fields $\sigma_i$, induced by appropriate variations of the gauge conditions. Namely, it follows from Eq. (23) at the tree level that a gauge variation $\Delta F_{\alpha}$ induces the following variations of the metric and reference fields

$$\delta g_{\mu \nu} = g_{\mu \nu, \alpha} \Xi^\alpha + g_{\mu \alpha} \Xi_\alpha^\mu + g_{\nu \alpha} \Xi_\mu^\alpha,$$

$$\delta \sigma_a = \sigma_{a, \alpha} \Xi^\alpha, \quad \Xi_\alpha = \langle c_\alpha \Delta \Psi \rangle,$$

where $\Delta \Psi$ is the corresponding variation of the gauge fermion.

The functions $g_{\mu \nu}, \sigma_a$ undergo the same variations \cite{24, 25} under the spacetime diffeomorphism

$$x^\mu \to x^\mu + \delta x^\mu,$$

with $\delta x^\mu = -\Xi^\mu$. Let us consider any quantity entering the Einstein equations, for instance, the scalar curvature $R$. Under the above change of gauge conditions, the tree value of $R$ measured at a point $\sigma^0$ of the reference frame remains unchanged,

$$\delta R[g(x(\sigma^0))] = \frac{\delta R}{\delta g_{\mu \nu}} \delta g_{\mu \nu} + \frac{\partial R}{\partial x^\mu} \delta x^\mu = \frac{\partial R}{\partial x^\mu} \Xi^\alpha - \frac{\partial R}{\partial x^\mu} \Xi_\mu^\alpha = 0.$$  

(26)
Analogously, the tree contribution to any tensor quantity $O_{\mu\nu}$ (or $O^{\mu\nu}$), for instance, the metric $g_{\mu\nu}$ itself, calculated at a fixed reference point $\sigma$, transforms covariantly (contravariantly), as prescribed by the position of the tensor indices of the corresponding operator. This is the manifestation of general covariance of the classical Einstein theory in terms of quantum field theory.

**B. General covariance at the one-loop order.**

Let us now consider the transformation properties of the one-loop contributions. As in Sec. III A, we have to determine the effect of an arbitrary gauge variation on the value of the effective metric, and also on the functions $\sigma$, i.e., on the structure of the reference frame. After that, the transformation law of observables, defined generally as the diffeomorphism-invariant functions of the metric and reference fields, can be determined in the way followed in Sec. III A. For definiteness, we will deal below with the scalar curvature $R$. Since we are interested in the one-loop contribution to the first post-Newtonian correction, we can linearize $R$ in $h_{\mu\nu}$:

$$R = \partial^{\mu}\partial^{\nu}h_{\mu\nu} - \Box h, \quad h \equiv \eta^{\mu\nu}h_{\mu\nu}. \quad (27)$$

The transformation properties of $R$ under variations of the matrix $\zeta_{\alpha\beta}$ are considered in Sec. III B 1, and under variations of the weighting matrix $\zeta_{\alpha\beta}$ themselves in Sec. III B 2.

1. Dependence of effective metric on weighting parameters.

Dependence of the effective fields on the weighting matrix $\zeta_{\alpha\beta}$ can be determined in a quite general way without specifying either the gauge functions $F_\alpha$, or the properties of gravitating matter fields. We begin with the classical theory in Sec. III B 1 a, and then consider the one-loop order in Sec. III B 1.b.

a. The tree level. As we saw in Sec. III A, arbitrary gauge variations lead to the transformations of the classical fields, equivalent to the spacetime diffeomorphisms, and thus do not affect the values of $R$. In the particular case of variations of the weighting matrix, however, not only $R$, but also the effective fields themselves remain unchanged. This means that the structure of reference frames in classical theory is determined by the functions $F_\alpha$ only. As this differs in the full quantum theory, a somewhat more detailed discussion of this issue will be given in this Section.

To demonstrate $\zeta_{\alpha\beta}$-independence of the classical metric, let us first integrate the auxiliary fields $\pi^\alpha$ out of the gauge-fixing action $S_{gf}$:

$$S_{gf} \rightarrow S^\xi_{gf} = \frac{1}{2} F_\alpha \xi^{\alpha\beta} F_\beta, \quad \zeta_{\alpha\beta} \xi^{\beta\gamma} = \delta^\gamma_\alpha. \quad (28)$$

The classical equations of motion thus become

$$\frac{\delta(S + S^\xi_{gf})}{\delta g_{\mu\nu}} = -T^{\mu\nu}, \quad (29)$$

where $T^{\mu\nu}$ is the energy-momentum tensor of matter. Using invariance of the action $S$ under the gauge transformations

$$\delta g_{\mu\nu} = \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha} \partial_\nu \xi^\alpha + g_{\nu\alpha} \partial_\mu \xi^\alpha \equiv \nabla_\mu \xi_\nu, \quad (30)$$

and taking into account the “conservation law” $\nabla_\mu T^{\mu\nu} = 0$, one has from Eq. 29:

$$F_\alpha \xi^{\alpha\beta} \frac{\delta F_\beta}{\delta g_{\mu\nu}(x)} \nabla_\mu \delta(x - y) = 0. \quad (31)$$

The matrix $M^\gamma_\nu(x, y) = \delta F_\beta/\delta g_{\mu\nu}(x) \nabla_\mu \delta(x - y)$ is non-degenerate; its determinant $\Delta \equiv \det M^\gamma_\nu(x, y)$ is just the Faddeev-Popov determinant, and therefore $\Delta \neq 0$. Hence, one has from Eq. 31 $F_\alpha \xi^{\alpha\beta} = 0$, and, in view of non-degeneracy of $\xi^{\alpha\beta}$, $F_\alpha = 0$. The classical metric is thus independent of the choice of the matrix $\xi^{\alpha\beta}$, and in particular, of the replacements $F_\alpha \rightarrow A^{\alpha}_\beta F_\beta$. One can put this in another way by saying that the weighting matrix has no geometrical meaning in classical theory.

This differs, however, in quantum domain. The classical equations 29 are replaced in quantum theory by the effective equations

$$\frac{\delta \Gamma}{\delta g^{\mu\nu}_{\text{eff}}} = -T_{\text{eff}}^{\mu\nu},$$
where \( \Gamma, g^\text{eff}_{\mu\nu}, \) and \( T^\text{eff}_{\mu\nu} \) are the effective action, metric, and energy-momentum tensor of matter, respectively. In general, the fields \( g^\text{eff}_{\mu\nu} \) do not satisfy the gauge conditions \( F_\alpha = 0 \), and moreover, depend on the choice of the weighting matrix \( \xi^{\alpha\beta} \); \( \xi^{\alpha\beta} \)-independence is inherited only by the tree contribution.

Dependence on the choice of the weighting matrix generally represents an excess of the gauge arbitrariness over the arbitrariness in the choice of reference frame; it is therefore a potential source of ambiguity in the values of observables. This dependence causes no gauge ambiguity of observables only if it reduces to the symmetry transformations. In other words, under (infinitesimal) variations of the matrix \( \xi^{\alpha\beta} \), the fields \( g^\text{eff}_{\mu\nu} \) must transform as in Eq. (30)

\[
\delta g^\text{eff}_{\mu\nu} = \Xi^{\alpha} g^\text{eff}_{\mu\nu,\alpha} + g^\text{eff}_{\mu\nu} \Xi^{\alpha} + g^{\text{eff}}_{\alpha} \Xi^{\alpha\mu} ,
\]

with some functions \( \Xi^\alpha \). It will be shown in the following Section that this is the case indeed.

b. The one-loop level. Let us now turn to the examination of the gauge dependence of \( h^0 \) loop contribution to the effective gravitational field. This contribution comes from diagrams in which virtual propagation of matter fields is near their mass shells, and is represented by terms containing the root singularity with respect to the momentum transfer between gravitational and matter fields. In the first post-Newtonian approximation, the only diagram we need to consider is the one-loop diagram depicted in Fig. 2(a). As a simple analysis shows, other one-loop diagrams do not contain the root singularities, while many-loop diagrams are of higher orders in the Newton constant. For instance, the loop in the diagram of Fig. 2(b) does not contain massive particle propagators, and therefore, expands in integer powers (or logarithms) of the ratio \( \alpha = -p^2/m^2 \).

Let us show, first of all, that the Schwinger-Keldysh rules applied to the diagram in Fig. 2(a) reduce to the ordinary Feynman rules of the standard S-matrix theory. According to the former, there are eight diagrams corresponding to the eight different ways of placing the three internal vertices of this diagram into the plus and minus branches of the time path (see Fig. 3). But all diagrams in Fig. 3 except the first two diagrams [2(a) and 2(b)] are zeros identically because of the energy-momentum conservation in the vertices, and the mass shell condition for external matter lines (massive particle cannot emit a real massless graviton). As to the diagram 2(b), its gauge arbitrariness arises from the energy-momentum conservation in the vertices, and the mass shell condition for external matter lines (massive particle cannot emit a real massless graviton). Therefore, the contribution comes from diagrams in which virtual propagation of matter fields.

According to general rules, in order to find the contribution of a diagram with \( n \) external \( \phi \)-lines, one has to take the \( n \)th derivative of the right hand side of Eq. (33) with respect to \( j^i \), multiply the result by the product of \( n \)
factors $e_i(q^2 - m^2)$, where $q$, $e_i$ are the 4-momentum and polarization of the external $\phi_i$-field quanta, and set $q^2 = m^2$ afterwards. The second term on the right of Eq. (33) is proportional to the source $j^i$ contracted with the vertex $G_{ij}^\gamma c_{ij}$. This term represents contribution of the graviton propagators ending on the external matter lines. Multiplied by $(q^2 - m^2)$, it gives rise to a non-zero value as $q^2 \rightarrow m^2$ only if the corresponding diagram is one-particle-reducible with respect to the $\phi$-line, in which case it describes the variation of $h_{\mu\nu}^{\text{eff}}$ under the gauge variation of external matter lines. It is well-known, however, that $\phi$-operators must be renormalized so as to cancel all the radiative corrections to the external lines. Therefore, this term can be omitted, and Eq. (33) rewritten finally as

$$\delta h_{\mu\nu}^{\text{eff}} = \frac{1}{2} \zeta_{\alpha\beta} \Delta \xi_{\beta\gamma} \delta (\bar{c}^\alpha F_{\alpha}) \bigg|_{k=0}.$$  \hspace{1cm} (35)

The one-loop diagrams representing the right hand side of Eq. (35), which give rise to the root singularity, are shown in Fig. 4. Let us consider the diagram of Fig. 4(a) first. It turns out that this diagram is actually free of the root singularity despite the presence of the internal $\bar{e}^\alpha F_{\gamma}^{(1)}$, where $F_{\gamma}^{(1)}$ denotes the linear part of $F_{\gamma}$. The graviton propagator connecting this vertex to the $\phi$-line can be expressed through the ghost propagator with the help of the equation

$$\delta^2 S_{\phi}^{(2)} \bigg|_{h=0} \bar{\phi}_k = 0,$$

\hspace{1cm} (37)

respectively. Equation (36) is the Slavnov identity at the tree level, differentiated twice with respect to $t^{\mu\nu}$, $\beta\alpha$. Using this identity in the diagram Fig. 4(a) we see that the ghost propagator is attached to the matter line through the generator $D_{\mu\nu}^{(0)}$. On the other hand, the action $S_{\phi}$ is invariant under the gauge transformations

$$\delta S_{\phi} = \delta h_{\mu\nu} G_{\mu\nu}^{(0)} \beta_{\alpha} = 0.$$

\hspace{1cm} (38)

Differentiating Eq. (38) twice with respect to $\phi$, and setting $h_{\mu\nu} = 0$ yields

$$\delta h_{\mu\nu}^{(0)} \beta_{\alpha} D_{\mu\nu}^{(0)} \beta_{\alpha} = -\delta_{\alpha\alpha},$$

\hspace{1cm} (39)

Taking into account also the mass shell condition

$$\delta^2 S_{\phi}^{(2)} \bigg|_{h=0} \bar{\phi}_k = 0,$$

\hspace{1cm} (40)

we see that under contraction of the vertex factor with the external and internal matter lines, the $\phi$-particle propagator, $D_{ik}$, satisfying

$$\delta^2 S_{\phi}^{(2)} \bigg|_{h=0} \bar{\phi}_k D_{ik} = -\delta_{ik},$$

is cancelled

$$D_{ik} \delta^2 S_{\phi}^{(2)} \bigg|_{h=0} G_{\mu\nu}^{(0)} = G_{i\alpha}.$$

\hspace{1cm} (41)

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3 One might think that the gauge dependence of the renormalization constants could spoil the above derivation of Eq. (23). In fact, this equation holds true for renormalized as well as unrenormalized quantities.\[11\]

4 The above discussion is nothing but the well-known reasoning underlying the proof of gauge-independence of the $S$-matrix.\[16\]
We conclude that the $\hbar^0$ contribution of the diagram Fig. 4(a) is zero. As to the rest of diagrams, they are all proportional to the generator $D_{\mu
u}^{(0)}$. Thus, the right hand side of Eq. (33) can be written

$$\delta h_{\mu\nu}^{\text{eff}} = G_{\mu\nu}^{(0)\alpha}\Xi_\alpha + O(h), \quad \Xi_\alpha = \frac{1}{2} \delta^{\beta\gamma}\xi_\beta \Delta \xi_\gamma (F_\delta).$$

Since $\Xi_\alpha$ are of the order $G^2$, one can also write, within the accuracy of the first post-Newtonian approximation,

$$\delta h_{\mu\nu}^{\text{eff}} = G_{\mu\nu}^{\alpha}\Xi_\alpha,$$

where $G_{\mu\nu}^{\alpha}$, are defined by Eq. (8) with $h_{\mu\nu} \rightarrow h_{\mu\nu}^{\text{eff}}$.

We thus see that under variations of the weighting matrix, the effective metric does transform according to Eq. (32). To determine the effect of these variations on the values of observables, one has to find also the induced transformation $\delta O$. Furthermore, any tensor quantity $\sigma_{\mu\nu\ldots}$ of diagrams pictured in Fig. 4(b,c,d),

$$\delta \sigma_{\mu\nu\ldots} = \delta \sigma_{\mu\nu\ldots}^{(1)} = \frac{\delta \sigma_{\mu\nu\ldots}(x(\sigma^0))}{\delta x^\mu} = 0.$$ (41)

In particular,

$$\delta R_{\sigma^0}(g^{\text{eff}}(x(\sigma^0))) = 0.$$

Furthermore, any tensor quantity $O_{\alpha\beta\ldots}$ (or $O_{\mu\nu\ldots}$), calculated at a fixed reference point $\sigma^0$, transforms covariantly (contravariantly), as prescribed by the position of the tensor indices of the corresponding operator. This is in accord with the principle of general covariance.

2. Dependence of effective metric on the form of $F_\alpha$

Having established the general law of the effective metric transformation under variations of the weighting matrix, let us turn to investigation of the variations of the functions $F_\alpha$, themselves.

According to the general equation (33), a variation $\Delta F_\alpha$ induces the following variation in the effective metric

$$\delta h_{\mu\nu}^{\text{eff}} = \frac{\delta (\Xi_\alpha \Delta F_\alpha)}{\delta h_{\mu\nu}} \bigg|_{\kappa=0}.$$

Thus, the right hand side of Eq. (35) can be written

$$\delta h_{\mu\nu}^{\text{eff}} = G_{\mu\nu}^{\alpha}\Xi_\alpha,$$

where $\Xi_\alpha$'s are the same as in Eq. (41).

Equations (41) and (42) are of the same form as Eqs. (24) and (25), respectively, which implies that the value of any observable $O$ is invariant under variations of the weighting matrix,

$$\delta O[h^{\text{eff}}(x(\sigma^0))] = \frac{\delta O}{\delta h_{\mu\nu}^{\text{eff}}} \delta h_{\mu\nu}^{\text{eff}} + \frac{\partial O}{\partial x^\mu} \delta x^\mu = \frac{\partial O}{\partial x^\alpha} \Xi_\alpha - \frac{\partial O}{\partial x^\mu} \Xi_\mu = 0.$$ (43)

In particular,

$$\delta R_{\text{eff}}[\sigma^0](g^{\text{eff}}(x(\sigma^0))) = 0.$$
with some \( X \). Then one has \( 0 = \Delta F^{(1)}(1)D F^{(1)}(1) = X F^{(1)}(0)G^{(0)} = X M(h = 0) \). Since the Faddeev-Popov determinant \( \det M \neq 0 \), it follows that \( X = 0 \). Thus, the argument used in the preceding section does not work, and the question is whether contribution of the diagram (a) can be actually represented in the form \( G^{(0)}_{\mu\nu} \Xi_\alpha \).

The answer to this question is negative, as an explicit calculation shows. This will be demonstrated below in the simplest case of a single scalar field described by the action

\[
S_\phi = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right\},
\]

and linear gauge conditions

\[
F_\gamma = \eta^\mu\nu \partial_\mu h_{\nu\gamma} - \left( \frac{\varrho - 1}{\varrho - 2} \right) \partial_\gamma h, \quad h \equiv \eta^\mu\nu h_{\mu\nu}, \quad \zeta_{\alpha\beta} = 0,
\]

where \( \varrho \) is an arbitrary parameter. According to Eq. (44), the \( \varrho \)-derivative of the effective metric is given by

\[
\frac{\partial h_{\mu\nu}^\text{eff}}{\partial \varrho} = \left. \frac{\delta \left( \varepsilon^\alpha \delta F_{\alpha\beta}/\delta \varrho \right)}{\delta k_{\mu\nu}} \right|_{k=0}.
\]

There are two diagrams with the structure of Fig. [a], in which the scalar particle propagates in opposite directions. They are represented in Fig. 5. In fact, it is sufficient to evaluate either of them. Indeed, these diagrams have the following tensor structure

\[
a_1 q_{\mu} v_{\nu} + a_2 (p_{\mu} q_{\nu} + p_{\nu} q_{\mu}) + a_3 p_{\mu} p_{\nu} + a_4 \eta_{\mu\nu},
\]

where \( a_i, i = 1, ..., 4 \), are some functions of \( p^2 \). When transformed to the coordinate space, the second and third terms become spacetime gradients, hence, they can be written in the form \( G^{(0)\alpha}_{\mu\nu} \Xi_\alpha \). As was discussed in the preceding sections, the terms of this type respect general covariance, therefore, we can restrict ourselves to the calculation of \( a_1 \) and \( a_4 \). On the other hand, diagrams of Fig. 5 go over one into another under the substitution \( q \rightarrow -q - p \) which leaves \( a_1, a_4 \) unchanged. Thus, we have

\[
I_{\mu\nu}(x) = \int \frac{d^d a}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \frac{a(q) a^*(q + p + \epsilon)}{\sqrt{2^d q^2}} \epsilon(p, q), \quad p_0 = \varepsilon_{+} + \varepsilon_{-},
\]

\[
\tilde{I}_{\mu\nu}(p, q) = \tilde{I}^{(a)}_{\mu\nu}(p, q) + \tilde{I}^{(a)}_{\mu\nu}(q, p),
\]

\[
\tilde{I}^{(a)}_{\mu\nu}(p, q) = -im\epsilon \int \frac{d^{d+1}k}{(2\pi)^d} \left\{ \frac{1}{2} W^\alpha(\delta\chi(p, q + k) - m^2 + \beta^\rho q^\rho) - m^2 \beta^\rho q^\rho \right\} D(q + k) \left\{ \frac{1}{2} W^\rho\tau\chi(\delta\chi(p, q + k) - m^2 \beta^\rho q^\rho) - m^2 \beta^\rho q^\rho \right\}
\]

\[
\times D(k) \left\{ \frac{1}{2} W^\rho\tau\chi(\delta\chi(p, q + k) - m^2 \beta^\rho q^\rho) - m^2 \beta^\rho q^\rho \right\} D_{\rho\tau\chi}(k - p),
\]

\[
\tilde{I}^{(a)}_{\mu\nu}(p, q) = \tilde{I}^{(a)}_{\mu\nu}(p, q - p),
\]

where \( a(q) \) is the momentum space amplitude for the scalar particle, normalized by

\[
\int \frac{d^d q}{(2\pi)^d} |a(q)|^2 = 1,
\]

\( \mu \) is an arbitrary mass scale, \( \varepsilon_q = \sqrt{q^2 + m^2} \), and \( \epsilon = 4 - d \), \( d \) being the dimensionality of spacetime. Explicit expressions for the propagators

\[
D_{\mu\nu\sigma\lambda}(k) = W_{\mu\nu\sigma\lambda} \frac{1}{k^2} - \theta(\eta_{\mu\nu} k_{\sigma} k_{\lambda} + \eta_{\sigma\lambda} k_{\mu} k_{\nu}) \frac{1}{k^4}
\]

\[
+ (\eta_{\mu\sigma} k_{\nu} k_{\lambda} + \eta_{\nu\mu} k_{\sigma} k_{\lambda} + \eta_{\nu\lambda} k_{\mu} k_{\sigma} + \eta_{\mu\lambda} k_{\nu} k_{\sigma}) \frac{1}{k^4}
\]

\[
+ (3\varrho^2 - 4\varrho) k_{\mu} k_{\nu} k_{\sigma} k_{\lambda} \frac{1}{k^6},
\]

\[
\tilde{D}_{\beta}(k) = \frac{\delta^\alpha^\beta}{k^2} - \frac{\varrho}{2} \frac{k^\alpha k^\beta}{k^4},
\]

\[
D(k) = \frac{1}{m^2 - k^2},
\]
Calculation of (48) can be further simplified using the relation
\[
\frac{\partial F_1}{\partial q} D = -F_1 \frac{\partial D}{\partial q},
\]
which follows from \( F_1 D = 0 \), and noting that all gradient terms in the graviton propagators, contracted with the \( \phi^2 h \) vertices, can be omitted (see Sec. III B 1), i.e., only the first line in Eq. (49) actually contributes.

Upon extraction of the leading contribution, Eq. (48) considerably simplifies. Note, first of all, that to the leading order in the long-range expansion, one has \( \varepsilon_{q+p} \approx \varepsilon_q \approx m \), and hence, \( p_0 \approx 0 \). Next, take into account that the function \( a(q) \) is generally of the form
\[
a(q) = b(q) e^{-i q x_0},
\]
where \( x_0 \) is the mean particle position, and \( b(q) \) is such that
\[
\int d^3 q \; b(q) e^{i q x} = 0
\] for \( x \) outside of some finite region \( W \) around \( x = 0 \). In the long-range limit, \( b(q + p) \) may be substituted by \( b(q) \). This implies that we disregard spatial spreading of the wave packet, neglecting the multipole moments of the particle mass distribution. Hence, Eq. (48) can be written as
\[
I_{\mu \nu}(x) = \frac{1}{2m} \int \int \frac{d^3 q}{(2\pi)^3} \; \frac{d^3 p}{(2\pi)^3} \; |b(q)|^2 e^{-i p(x-x_0)} \tilde{I}_{\mu \nu}(p,q), \quad p^0 = 0.
\] (51)

Let the equality of two functions up to a spacetime diffeomorphism be denoted by “\( \sim \)”. Then, performing tensor multiplications in Eq. (48), and omitting terms proportional to \( p_{\mu} \), one obtains
\[
\tilde{I}^{(a)}_{\mu \nu}(p,q) \sim -i \mu^\prime \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4 (k+p)^2} \frac{1}{m^2 - (k+q)^2}
\]
\[
\{ \eta_{\mu \nu} \left[ \frac{\partial}{\partial \eta^\mu} (k^4 + 2PQ)(P-Q)(Q-m^2) \right.
\]
\[
+ \frac{\partial}{\partial \eta^\nu} (P + Q)(Q-m^2) - g k^2 Q^2 (Q-m^2) \}
\]
\[
\left. + \left( \frac{\partial}{\partial \eta^\mu} (k^2 + 2Q) + (k+p)^2 m^2 \right) (k^2 + P)(Q-m^2) \right]
\]
\[
+k_{\mu} k_{\nu} \left[ \frac{\partial}{\partial \eta^\mu} (P^2 + k^2 m^2) (Q-m^2) + 4PQ m^2 - 3gPQ^2 \right.
\]
\[
+ \frac{\partial}{\partial \eta^\nu} (P - 2Q)(Q-m^2) + 2Q^2 (Q-m^2) - gP m^4 \}
\]
\[
\left. + 2(k + p)^2 (Q(Q - 2m^2) - P(Q - m^2) + m^4) \right]
\]
\[
+ 2(k_{\mu} q_{\nu} + k_{\nu} q_{\mu}) (k+p)^2 (Q - P)(Q - m^2) \}
\]
\[
- 2q_{\nu} q_{\mu} (k+p)^2 (Q - m^2)^2 \}
\]
\[
Q \equiv (kq), \quad P \equiv (kp).
\] (52)

Introducing the Schwinger parameterization of denominators
\[
\frac{1}{k^2} = - \int_0^\infty dy \exp \{ y k^2 \}, \quad \frac{1}{(k+p)^2} = - \int_0^\infty dx \exp \{ x (k+p)^2 \},
\]
\[
\frac{1}{k^2 + 2(kq)} = - \int_0^\infty dz \exp \{ z [k^2 + 2(kq)] \},
\]
one evaluates the loop integrals using
\[
\int d^4 k \exp \left\{ k^2 (x + y + z) + 2k_{\mu} (xp_{\mu} + zq_{\mu}) + p^2 x \right\}
\]
\[
= i \left( \frac{\pi}{x + y + z} \right)^{d/2} \exp \left\{ \frac{p^2 y - m^2 z^2}{x + y + z} \right\},
\]
\[
\int d^4 k \; k_{\alpha} \exp \left\{ k^2 (x + y + z) + 2k_{\mu} (xp_{\mu} + zq_{\mu}) + p^2 x \right\} =
\]
\[
= i \left( \frac{\pi}{x + y + z} \right)^{d/2} \exp \left\{ \frac{p^2 y - m^2 z^2}{x + y + z} \right\} \left[ -xp_{\alpha} + zq_{\alpha} \right] \frac{1}{x + y + z},
\]
integrating $v$ out, subtracting the ultraviolet divergence\(^6\)

$$\tilde{I}_{\mu\nu}^{(a)\text{div}}(p,q) = -\frac{1}{16\pi^2\epsilon} \left(\mu \over m\right) \epsilon \left[\frac{1}{3} q_\mu q_\nu + \eta_{\mu\nu} \left(p^2 - 2m^2\right) \frac{3\rho - 2}{24}\right],$$

setting $\epsilon = 0$, omitting gradient terms, and retaining only the $h^0$-contribution, we obtain

$$\tilde{I}_{\mu\nu}^{(a)\text{ren}}(p,q) \equiv (\tilde{I}_{\mu\nu}^{(a)} - \tilde{I}_{\mu\nu}^{(a)\text{div}})_{\epsilon \to 0}$$

$$\sim \frac{m^2}{16\pi^2} \int_0^\infty \int_0^\infty dudt \left\{ \eta_{\mu\nu} \left(\frac{2H^2}{D^2} - \frac{1}{2D}\right) + \frac{q_\mu q_\nu}{H^2m^2} \left[\frac{2H^2}{D^2}(1 - \frac{1}{D}) + \frac{1}{\alpha} \left(\frac{4\rho}{D^3} - \frac{11\rho + 4}{2D} + \frac{8\rho + 4}{D}\right)\right]\right\},$$

$$D \equiv 1 + \text{out}, \quad H \equiv 1 + u + t.$$ (53)

The root singularity in the right hand side of Eq. (53) can be extracted using Eqs. (A3) derived in the Appendix. Denoting

$$\int_0^\infty \int_0^\infty dudt \ H^{-n}D^{-m} \equiv J_{nm},$$

one has

$$J_{12}^{\text{root}} = \frac{\pi^2}{4\sqrt{\alpha}}, \quad J_{13}^{\text{root}} = \frac{3\pi^2}{16\sqrt{\alpha}} ,$$

$$J_{31}^{\text{root}} = \frac{\pi^2}{16\sqrt{\alpha}}, \quad J_{32}^{\text{root}} = \frac{3\pi^2}{32\sqrt{\alpha}}, \quad J_{33}^{\text{root}} = \frac{-15\pi^2}{128}\sqrt{\alpha}.$$ (54)

Substituting these into Eq. (53) gives

$$\tilde{I}_{\mu\nu}^{\text{ren}}(p,q) \sim \frac{1}{128\sqrt{\alpha}} \left[q_\mu q_\nu (\rho + 1) - \eta_{\mu\nu} m^2 \rho\right].$$ (54)

Finally, restoring ordinary units, substituting Eq. (54) into Eq. (51), and using the formula

$$\int \frac{d^3p}{(2\pi)^3} \ e^{ipx} \equiv \frac{1}{2\pi^2 r^2}, \quad r^2 \equiv \delta_{ik} x^i x^k,$$ (55)

we obtain the following expression for the $\rho$-derivative of the $G^2h^0$-order contribution to the effective metric\(^1\)

$$\frac{\partial h_{\mu\nu}^{\text{eff}}}{\partial \rho} = \frac{\partial h_{\mu\nu}^{\text{tree}}}{\partial \rho} + \frac{\partial h_{\mu\nu}^{\text{loop}}}{\partial \rho} \sim \frac{\partial h_{\mu\nu}^{\text{loop}}}{\partial \rho} = \tilde{I}_{\mu\nu}^{\text{ren}}(x) \sim \frac{G^2 m^2}{2e^2 r^2} \left[\frac{\delta_{\mu\nu} \delta^0_0}{c^2} (\rho + 1) - \eta_{\mu\nu} \rho\right].$$ (56)

\(^6\) A technicality must be mentioned here. By itself, the diagram of Fig. 3(a) is free of infrared divergences. As a result of the BRST-operating with this diagram, however, some fictitious infrared divergences are brought into individual diagrams representing the right hand side of Eq. (53). This is because the vertex $D_{\mu\nu}^\rho C_\alpha$ contains the term $C^\alpha \partial_\alpha h_{\mu\nu}$ in which the spacetime derivatives act on the gravitational, rather than the ghost field. These divergences occur as $u, t \to \infty$. They are proportional to integer powers of $p^2$, and therefore do not interfere with the part containing the root singularity. Since these divergences must eventually cancel in the total sum in Eq. (53), they will be simply omitted in what follows.
The right hand side of this equation cannot be represented in the form (32). This result can be made more expressive by calculating the ρ-variation of the scalar curvature $R$ in a given point $\sigma^0$ of the reference frame

$$\delta R[h^{\text{eff}}(x(\sigma^0))] = \frac{\delta R}{\delta h^{\text{eff}}_{\mu\nu}} \left( \frac{\partial h^{\text{tree}}_{\mu\nu}}{\partial \rho} + \frac{\partial h^{\text{loop}}_{\mu\nu}}{\partial \rho} \right) \delta \rho + \frac{\partial R}{\partial x^\mu} \delta x^\mu$$

$$= \frac{\partial R}{\partial x^\mu} \Xi^{\text{tree}}_{\mu} + (\partial^\mu \partial^\nu - \eta^{\mu\nu} \Box) \frac{h^{\text{loop}}_{\mu\nu}}{\partial \rho} \delta \rho - \frac{\partial R}{\partial x^\mu} \Xi^{\text{tree}}_{\mu}$$

$$= \partial_\mu \partial_k \frac{\partial h^{\text{loop}}_{ik}}{\partial \rho} \delta \rho + \Delta \frac{\partial h^{\text{loop}}_{\mu\nu}}{\partial \rho} \delta \rho = \frac{G^2 m^2}{c^4 r^4} (1 - 2 \rho) \delta \rho,$$

or\(^7\)

$$\frac{\partial R[h^{\text{eff}}(x(\sigma^0))]}{\partial \rho} = \frac{G^2 m^2}{c^4 r^4} (1 - 2 \rho).$$

Equations (56), (58) express violation of general covariance by the loop corrections.

Thus, despite their independence of the Planck constant, the post-Newtonian loop contributions turn out to be of a purely quantum nature.

We are now in a position to ask for conditions to be imposed on a system in order to allow classical consideration of its gravitational field. Such a condition providing vanishing of the $h^0$ loop contributions can easily be found out by examining their dependence on the number of particles in the system. Let us consider a body with mass $M$, consisting of a large number $N = M/m$ of elementary particles with mass $m$. Then it is readily seen that the $n$-loop contribution to the effective gravitational field of the body turns out to be suppressed by a factor $1/N^n$ in comparison with the tree contribution. For instance, at the first post-Newtonian order, the tree diagram in Fig. 1(b) is bilinear in the energy-momentum tensor $\langle T^{\mu\nu} \rangle$ of the particles, and therefore proportional to $(m \cdot N) \cdot (m \cdot N) = M^2$. On the other hand, the post-Newtonian contribution of the diagram in Fig. 2(a) is proportional to $m^2 \cdot N = M^2 / N$, since it has only two external matter lines.

Thus, we are led to the following macroscopic formulation of the correspondence principle in quantum gravity: the effective gravitational field produced by a macroscopic body of mass $M$ consisting of $N$ particles turns into corresponding classical solution of the Einstein equations in the limit $N \rightarrow \infty$\(^1\). In particular, the principle of general covariance is to be considered as approximate, valid only for the description of macroscopic phenomena.

The $h^0$-order loop contributions are normally highly suppressed. For the solar gravitational field, for instance, their relative value is $m_{\text{proton}} / M_\odot \approx 10^{-57}$. However, they are the larger the more gravitating body resembles an elementary particle, and can become noticeable for a sufficiently massive compact body.

## C. Effective gravitational field of a heavy particle

As an application of the obtained results, we will calculate the effective gravitational field of a particle with mass $M$ in the first post-Newtonian approximation.

The complete expression of the order $h^0$ for the spacetime metric is the sum of two pieces. The first is the tree contribution corresponding to the classical Schwarzschild solution

$$ds^2_{\text{Sch}} = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2),$$

where $\theta, \varphi$ are the standard spherical angles, $r$ is the radial coordinate, and $r_g = 2GM/c^2$ is the gravitational radius of a spherically-symmetric distribution of mass $M$. The second is the one-loop post-Newtonian correction contained

\(^7\) Another way to obtain this result is to introduce the sources $iR$ and $kR, \alpha c^\alpha$ for the scalar curvature and its BRST-variation, respectively, into the generating functional \cite{15}, instead of the corresponding sources for the metric. Then Eq. (47) is replaced by

$$\frac{\partial R^{\text{eff}}}{\partial \rho} = \frac{\delta \langle c^\alpha F_{\alpha\beta} / \partial \rho \rangle}{\delta x_\beta} \bigg|_{K=0}.$$

At the second order in $G$, the nontrivial contribution comes again from the diagram of Fig. 1(a) in which the lower left vertex is now generated by $R, \alpha c^\alpha$. Thus, only the linear part of $R$ gives rise to a non-zero contribution to the right hand side of Eq. (57). In other words, $\delta R[h^{\text{eff}}] = \delta R^{\text{eff}}$, though generally $R[h^{\text{eff}}] \neq R^{\text{eff}}$. 


in the diagram (a) of Fig. 2. Restoring the ordinary units, and using expressions (55), (65) of Ref. 15 for the vertex formfactors, and 49 with \( q = 0 \) for the graviton propagator,\(^8\) we obtain

\[
h_{\mu\nu}^{\text{loop}}(p) = -\frac{\pi^2 G^2}{c^2 \sqrt{-p^2}} \left( 3M^2 \eta_{\mu\nu} + \frac{q_\mu q_\nu}{c^2} + 7M^2 \frac{p_\mu p_\nu}{p^2} \right). \tag{60}
\]

Written down in the coordinate space with the help of the formulas 55 and

\[
\int \frac{d^3 p}{(2\pi)^3} \frac{p_ip_k}{|p|^3} e^{ipx} = \frac{1}{2\pi^2 r^2} \left( \delta_{ik} - \frac{2x_i x_k}{r^2} \right),
\]

equation \( (60) \) gives, in the static case,

\[
h_{00}^{\text{loop}} = -\frac{2G^2 M^2}{c^2 r^2}, \quad h_{ik}^{\text{loop}} = \frac{G^2 M^2}{c^2 r^2} \left( -2\delta_{ik} + \frac{7x_i x_k}{r^2} \right). \tag{61}
\]

Before adding the two contributions, however, one has to transform Eq. \( (59) \) which form is fixed by the requirements \( g_{ti} = 0, \; i = r, \theta, \varphi; \; g_{00} = -r^2, \) to the DeWitt gauge

\[
F_\gamma = \eta^{\mu\nu} \partial_\mu h_{\nu\gamma} - \frac{1}{2} \partial_\gamma h, \quad \zeta_{\alpha\beta} = \eta_{\alpha\beta},
\]

under which Eq. \( (60) \) was derived. According to the general theorems about \( \zeta_{\alpha\beta} \)-independence of the \( h^0 \)-order contributions, proved in Sec. 11.1.1, we can transform our expressions to the singular case \( \zeta_{\alpha\beta} = 0 \) instead of \( \zeta_{\alpha\beta} = \eta_{\alpha\beta} \). Then the effective gravitational field will satisfy

\[
\eta^{\mu\nu} \partial_\mu h_{\nu\gamma}^{\text{eff}} - \frac{1}{2} \partial_\gamma h^{\text{eff}} = 0. \tag{63}
\]

The \( t, \theta, \varphi \)-components of Eq. \( (63) \) are already satisfied by the classical solution 65. To meet the remaining condition, let us substitute \( r \to f(r) \), where \( f \) is a function of \( r \) only. Then the \( t, \theta, \varphi \)-components of Eq. \( (63) \) are still satisfied, while its \( r \)-component gives the following equation for the function \( f(r) \):

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 f'' \right) - \frac{2f}{r^2} + \frac{f'}{2r} - \frac{r^2}{r^2} + \frac{f'}{f} + \frac{f''}{r^2} = 0,
\]

where \( f' = \frac{\partial f(r)}{\partial r} \).

Since we are interested only in the long-distance corrections to the Newton law, one may expand \( f(r)/r \) in powers of \( r_g/r \) keeping only the first few terms:

\[
f(r) = r \left[ 1 + c_1 \frac{r_g}{r} + c_2 \left( \frac{r_g}{r} \right)^2 + \cdots \right].
\]

Substituting this into Eq. \( (63) \), one obtains successively \( c_1 = 1/2, \; c_2 = 1/2, \) etc. Therefore, up to terms of the order \( r_g/r^2 \), the Schwarzschild solution takes the following form

\[
ds_{\text{cl}}^2 = \left( 1 - \frac{r_g}{r} + \frac{r_g^2}{2r^2} \right) c^2 dt^2 - \left( 1 + \frac{r_g}{r} - \frac{r_g^2}{2r^2} \right) dr^2
\]

\[
- r^2 \left( 1 + \frac{r_g}{r} + \frac{5r_g}{4r^2} \right) (d\theta^2 + \sin^2 \theta \; d\varphi^2), \tag{65}
\]

Rewriting Eq. \( (61) \) in spherical coordinates and adding the tree contribution, we finally obtain the following expression for the interval 24

\[
ds_{\text{eff}}^2 \equiv g_{\mu\nu}^{\text{eff}} dx^\mu dx^\nu = \left( 1 - \frac{r_g}{r} \right) c^2 dt^2 - \left( 1 + \frac{r_g}{r} - \frac{7r_g^2}{4r^2} \right) dr^2
\]

\[
- r^2 \left( 1 + \frac{r_g}{r} + \frac{7r_g}{4r^2} \right) (d\theta^2 + \sin^2 \theta \; d\varphi^2). \tag{66}
\]

\(^8\) Note the notation differences between Ref. 15 and this Chapter.
In connection with this result it should be noted the following. Taking into account higher-order radiative post-Newtonian corrections will result in a further modification of the Schwarzschild solution. Since quantum contributions are of the same order of magnitude as those given by general relativity, this modification can lead to a significant shift of the horizon. In particular, the metric $g^{\mu \nu}_{\text{eff}}(r)$ may well turn out to be a regular function of $r$ when all the $\hbar^0$ loop corrections are taken into account.

**IV. QUANTUM FLUCTUATIONS OF GRAVITATIONAL FIELD**

We turn now to another aspect of the long-range behavior in quantum gravity – quantum fluctuations of the gravitational fields produced by elementary systems. On various occasions, the issue of fluctuations has been the subject of a number of investigations (see Refs. [21, 22, 23, 24, 25], where references to early works can be found). It should be mentioned, however, that despite extensive literature in the area, only vacua contributions of quantized matter fields to the metric fluctuations have been studied in detail. At the same time, it is effects produced by real matter that are of special interest concerning the structure of elementary contributions to the field fluctuation.

Before we proceed to calculation of the correlation function, we shall examine its general properties in more detail. Namely, the structure of the long-range expansion of the correlation function, and the question of its gauge dependence will be considered.

**A. Properties of correlation function**

As defined by Eq. (13), the correlation function $C_{\mu \nu \alpha \beta}$ is a function of two spacetime arguments $(x, x')$. Of special interest is its “diagonal element” corresponding to coinciding arguments and Lorentz indices $(x = x', \mu \nu = \alpha \beta)$, which describes a dispersion of the spacetime metric around its mean value in a given spacetime point. However, it is well-known that this element is not generally well-defined because of the singular behavior of the product of field operators in the coincidence limit $x \to x'$. This difficulty is naturally resolved when one takes into account the fact (realized long ago, see, e.g., Ref. [21]) that in any field measurement in a given spacetime point, one deals actually with the field averaged over a small but finite spacetime domain surrounding this point. Thus, the physically sensible expression for the field operator is the following

$$\hat{\mathcal{H}}_{\mu \nu} = \frac{1}{V T} \int_{T} dt \int_{V} d^{3}x \, \hat{h}_{\mu \nu}(x,t).$$

(67)

 Respectively, the product of two fields in a given point is understood as the limit of

$$\hat{\mathcal{B}}_{\mu \nu \alpha \beta} = \frac{1}{(VT)^{2}} \int_{T} dt \int_{T} dt' \int_{V} d^{3}x \, \int_{V} d^{3}x' \, \hat{h}_{\mu \nu}(x,t) \hat{h}_{\alpha \beta}(x',t')$$

when the size of the domain tends to zero. Finally, correlation function of the spacetime metric in this domain is

$$\mathcal{C}_{\mu \nu \alpha \beta} = \langle \hat{\mathcal{B}}_{\mu \nu \alpha \beta} \rangle - \langle \hat{\mathcal{H}}_{\mu \nu} \rangle \langle \hat{\mathcal{H}}_{\alpha \beta} \rangle \equiv \frac{1}{(VT)^{2}} \int \int d^{4}xd^{4}x' C_{\mu \nu \alpha \beta}(x,x').$$

(69)

Figure 6 depicts the tree diagrams contributing to the right hand side of Eq. (16). The disconnected part shown in Fig. 6(a) cancels in the expression for the correlation function, Eq. (69), which is thus represented by the diagrams (b)–(h).

1. Correlation function in the long-range limit

As we mentioned in the beginning of Sec. IV the mean gravitational field produced by a massive particle is a function of five dimensional parameters – the fundamental constants $\hbar, G, c, m,$ and the distance between the particle and the point of observation, $r$. Unlike the mean field, however, $\mathcal{C}_{\mu \nu \alpha \beta}$ is a function of two spacetime arguments. On the other hand, we are interested ultimately in the coincidence limit of this function, which will be shown below to exist everywhere except the region of particle localization. In this limit, therefore, $\mathcal{C}_{\mu \nu \alpha \beta}$ depends on the same five parameters $\hbar, G, c, m, r$. Assuming as before the particle sufficiently heavy ($m \to 0$), we conclude that the relevant information about field correlations is contained in the long-range asymptotic of $\mathcal{C}_{\mu \nu \alpha \beta}(r)$.

To extract this information we note, first of all, that in the long-range limit, the value of $\mathcal{C}_{\mu \nu \alpha \beta}$ is independent of the choice of spacetime domain used in the definition of physical gravitational field operators, Eq. (67). Indeed,
in any case the size of this domain must be small in comparison with the characteristic length at which the mean field changes significantly. In the case considered, this requires that \( V \ll r^3 \). To the leading order of the long-range expansion, therefore, the quantity \( \langle \hat{h}_{\mu\nu}(x)\hat{h}_{\alpha\beta}(x') \rangle \) appearing in the right hand side of Eq. (60) can be considered constant within the domain. However, one cannot set \( x = x' \) in this expression directly. It is not difficult to see that the formal expression \( \langle \hat{h}_{\mu\nu}(x)\hat{h}_{\alpha\beta}(x) \rangle \) does not exist. Consider, for instance, the diagram shown in Fig. 6(b). A typical term in the analytical expression of this diagram is proportional to the integral

\[
I_{\mu\nu\alpha\beta}(x-x', p) = \int d^4k V_{\mu\nu\alpha\beta}(0)\delta(k^2)e^{ik(x-x')}/[(k+q)^2 - m^2(k-p)^2],
\]

(70)

where \( q_\mu \) is the 4-momentum of the particle, \( p_\mu \) the momentum transfer, and \( V_{\mu\nu\alpha\beta} \) a vertex factor combined of \( \eta_{\mu\nu} \)'s and momenta \( q, p, k \). For a small but nonzero \( (x - x') \) this integral is effectively cut-off at large \( k \)'s by the oscillating exponent, but for \( x = x' \) it is divergent. This divergence arises from integration over large values of virtual graviton momenta, and therefore has nothing to do with the long-range behavior of the correlation function, because this behavior is determined by the low-energy properties of the theory. Evidently, the singularity of \( I_{\mu\nu\alpha\beta}(x-x', p) \) for \( x \to x' \) is not worse than \( \ln(x^\mu - x'^\mu)^2/(x^\mu - x'^\mu)^2 \). Therefore, \( \langle \ln|\hat{h}_{\mu\nu}(x)\hat{h}_{\alpha\beta}(x')|\ln \rangle \) is integrable, and \( C_{\mu\nu\alpha\beta} \) given by Eqs. (68), (69) is well defined.

Our aim below will be to show that this singularity can be consistently isolated and removed from the expression for \( I_{\mu\nu\alpha\beta}(x-x', p) \) and similar integrals for the rest of diagrams in Fig. 6 without changing the long-range properties of the correlation function. After this removal, it is safe to set \( x = x' \) in the finite remainder, and to consider \( C_{\mu\nu\alpha\beta} \) as a function of the single variable – the distance \( r \). An essential point of this procedure is that the singularity turns out to be local, and hence does not interfere with terms describing the long-range behavior, which guaranties unambiguity of the whole procedure.

It should be emphasized that in contrast to what takes place in the scattering theory, the ultraviolet divergences appearing in the course of calculation of the in-in matrix elements in the coincidence limit are generally non-polynomial with respect to the momentum transfer. The reason for this is the different analytic structure of various elements in the matrix propagators \( \mathfrak{D}_{\mu\nu\alpha\beta}, \mathfrak{D} \), which spoils the simple ultraviolet properties exhibited by the ordinary Feynman amplitudes.\(^9\)

Take the above integral as an example. Because of the delta function in the integrand, differentiation \( \frac{d}{dp} \mathfrak{D}(0) \mathfrak{D}(p) \) with respect to the momentum transfer does not remove the ultraviolet divergence of \( I_{\mu\nu\alpha\beta}(0, p) \). What makes it all the more interesting is the result obtained in Sec. [\[\text{II.B}]] below, that the non-polynomial parts of divergent contributions eventually cancel each other, and the overall divergence turns out to be completely local.

Next, let us establish general form of the leading term in the long-range expansion of the correlation function. In momentum representation, an expression of lowest order in the momentum transfer with suitable dimensionality is the following

\[
\mathcal{C}^{(0)}_{\mu\nu\alpha\beta}(p) \sim G^2 m^2 \mathcal{F}_{\mu\nu\alpha\beta}/\sqrt{-p^2},
\]

(71)

where \( \mathcal{F}_{\mu\nu\alpha\beta} \) is a dimensionless positive definite tensor combined from \( \eta_{\mu\nu} \)'s and \( q_\mu \).

Not all of diagrams in Fig. 6 contain contributions of this type. It is not difficult to identify those which do not. Consider, for instance, the diagram (h). It is proportional to the integral

\[
\int d^4k \delta(k^0)\delta(k^2)e^{ik(x-x')}/(k-p)^2,
\]

which does not involve the particle mass at all. Taking into account \( m^2 \) coming from the vertex factor, and \( (2\pi q)^{-1/2} \sim \sqrt{m} \) from each external matter line, we see that the contribution of the diagram (h) is proportional to \( m \), not \( m^2 \). The same is true of all other diagrams without internal matter lines. As to diagrams involving such lines, it will be shown in Sec. [\[\text{I.V.B}]] by direct calculation that they do contain contributions of the type Eq. (71). But prior to this the question of their dependence on the gauge will be considered.

\(^9\) As is well known, the proof of locality of the S-matrix divergences relies substantially on the causality of the pole structure of Feynman propagators. This property allows Wick rotation of the energy contours, thus revealing the essentially Euclidean nature of the ultraviolet divergences.
2. Gauge independence of the leading contribution

As in the case of mean gravitational field considered in Sec. III, the 0-component of the momentum transfer is to be set zero when calculating the leading term in the long-range expansion of the correlation function. This implies, in particular, that \( C^{(0)}_{\mu\nu\alpha\beta} \) contains information about fluctuations in a quantity of direct physical meaning – the static potential energy of interacting particles. As such, it is expected to be independent of the choice of gauge conditions used to fix general covariance. More precisely, we have to verify that under variations of the gauge conditions, \( C^{(0)}_{\mu\nu\alpha\beta} \) varies in a way that does not affect the values of observables built from it.

As we saw in Sec. IV.A, the only diagrams contributing in the long-range limit are those containing internal massive particles lines. We will show presently that the gauge-dependent part of these diagrams can be reduced to the form without such lines. First, it follows form Eq. (47) that the gauge variation of the graviton propagator satisfies

\[
\delta D_{\mu\nu\sigma\lambda} = D_{\mu\nu\alpha\beta} \delta \sigma^{\alpha\beta\gamma\delta} D_{\gamma\delta\sigma\lambda}.
\]

On the other hand, contracting Eq. (47) with \( G^{(0)\rho}_{\mu\nu} \) yields

\[
i F_{\alpha\mu} G^{(0)\rho}_{\mu\nu} \delta \sigma^{\alpha\beta\gamma\delta} D_{\gamma\delta\sigma\lambda} = -\epsilon G^{(0)\rho}_{\sigma\lambda}.
\]

Defining the matrix ghost propagator \( \tilde{D}_{\beta}^{(0)} \) according to Eqs. (17), (55) by

\[
i F_{\alpha\mu} G^{(0)\rho}_{\mu\nu} \tilde{D}_{\beta}^{(0)} = -\epsilon \delta \alpha
\]

one finds

\[
F_{\alpha\mu} D_{\mu\nu\sigma\lambda} = \zeta_{\sigma\gamma} G^{(0)\rho}_{\sigma\lambda} \tilde{D}_{\beta}^{(0)}.
\]

Taking into account that

\[
\delta \sigma^{\alpha\beta\gamma\delta} = i \delta F_{\alpha\mu} \delta \sigma^{\alpha\beta\gamma\delta} + i F_{\alpha\mu} \delta \sigma^{\alpha\beta\gamma\delta}
\]

and using Eq. (47), we get

\[
\delta D_{\mu\nu\sigma\lambda} = \left( F_{\alpha\mu} \delta \sigma^{\alpha\beta\gamma\delta} D_{\mu\nu\alpha\beta} \right) G^{(0)\gamma}_{\sigma\lambda} + G^{(0)\rho}_{\mu\nu} \left( \tilde{D}_{\beta}^{(0)} i \delta F_{\alpha\mu} \delta \sigma^{\alpha\beta\gamma\delta} \right).
\]

Let us assume for definiteness that the pair of indices \( \mu \nu \) refers to the point of observation, and consider the first term in Eq. (47). This part of the gauge variation of the graviton propagator is attached to the matter line through the generator \( G^{(0)\gamma}_{\sigma\lambda} \). Suppressing all Lorentz and matrix indices except those referring to the \( \phi^2 h \)-vertex, it can be written as

\[
b_{\sigma\lambda} = \begin{pmatrix} a_{\gamma+} & 0 \\ a_{\gamma-} & 0 \end{pmatrix} G^{(0)\gamma}_{\sigma\lambda}, \quad a_{\gamma} = \delta F_{\delta\mu} \delta \sigma^{\alpha\beta\gamma\delta} D_{\mu\nu\alpha\beta} \tilde{D}_{\beta}^{(0)}
\]

The matrix vertex \( \psi^{\sigma\lambda}_{\mu\nu} \) is obtained multiplying \( \delta^3 S_{\phi}/\delta \phi_k \delta \phi_k \delta h_{\sigma\lambda} \) by the matrix \( e_{\sigma\lambda} \). Equation (47) then shows that the combination \( b_{\sigma\lambda} \psi^{\sigma\lambda}_{\nu\mu} \) can be written as

\[
b_{\sigma\lambda} \psi^{\sigma\lambda}_{\mu\nu} = -\hat{f}_{\gamma} \left\{ \frac{\delta S^{(2)}_{\phi}}{\delta \phi_l} \frac{\delta G_{\gamma}^{(0)}}{\delta \phi_k} + \frac{\delta S^{(2)}_{\phi}}{\delta \phi_i} \frac{\delta G_{\gamma}^{(0)}}{\delta \phi_k} \right\},
\]

where

\[
\hat{f}_{\gamma} = \begin{pmatrix} a_{\gamma+} & 0 \\ 0 & -a_{\gamma-} \end{pmatrix}.
\]

Finally, contracting \( b_{\sigma\lambda} \psi^{\sigma\lambda}_{\mu\nu} \) with the matrix \( D_{km} \) and the vector \( \tilde{v}_i = (\tilde{\phi}_i, \tilde{\phi}_i) \), and using Eqs. (18), (19) gives

\[
\tilde{v}_i b_{\sigma\lambda} \psi^{\sigma\lambda}_{\mu\nu} D_{km} = a_{\gamma} \frac{\delta G_{\gamma}^{(0)}}{\delta \phi_i} \tilde{v}_i.
\]

Similarly to what we have found in Sec. III.B.1 considering \( \zeta \)-dependence of the one-loop contribution to the effective metric, the matrix matter propagator is cancelled upon contraction with the vertex factor. As in the case of the one-loop post-Newtonian contributions, the \( h^3 \) terms in the correlation function are associated with the virtual matter
results to terms of higher order in the Planck constant. As to the second term, it gives rise to a quanta propagating near their mass shells. Hence, the first term in the gauge variation of the propagator, Eq. (74), diagrams (b)–(f) in Fig. 6, which has the symbolic form

\[ \delta C_{\mu
u\alpha\beta}^{(0)} \sim G_{\mu
u}^{\gamma} \Omega_{\gamma\alpha\beta} + G_{\alpha\beta}^{\gamma} \Omega_{\gamma\mu
u} \]

with some infinitesimal \( \Omega_{\gamma\alpha\beta} \), which proves gauge independence of gauge-invariant functionals built from \( C_{\mu
u\alpha\beta}^{(0)} \). Note also that as far as variations of the weighting matrix \( \xi_{\mu
u\alpha\beta} \) are considered, not only these functionals, but also \( C_{\mu
u\alpha\beta}^{(0)} \) itself turns out to be gauge independent. Indeed, a variation \( \delta \xi_{\alpha\beta} \) of the weighting matrix is equivalent to the variation of the gauge condition \( F_\alpha \):

\[ \delta F_\alpha = \theta_\alpha^{\mu} F_\beta, \]

provided that

\[ \delta \xi_{\alpha\beta} = \xi_{\alpha\gamma}^{\mu} \theta_\beta^{\gamma} + \xi_{\beta\gamma}^{\mu} \theta_\alpha^{\gamma}. \]

Then using Eq. (74) in Eq. (74) gives

\[ \delta \mathcal{D}_{\mu\nu\sigma\lambda} = -C_{\mu\nu}^{(0)} \delta \mathcal{D}_{\beta\delta}^{(0)} \iota \delta \zeta_{\sigma\delta} \mathcal{D}_{\gamma\sigma\lambda}^{(0)}, \]

which implies that \( C_{\mu\nu\alpha\beta}^{(0)} \) is independent of the choice of the matrix \( \zeta_{\alpha\beta} \).

### B. Evaluation of the leading contribution

Let us proceed to the calculation of the leading contribution to the correlation function, assuming as in Sec. \[IV.A.1\] that the field producing particle is a scalar described by the action \[45\]. This contribution is contained in the sum of diagrams (b)–(f) in Fig. 6 which has the symbolic form

\[ C_{\mu\nu\alpha\beta} = I_{\mu\nu\alpha\beta} + I_{\mu\nu\alpha\beta}^{tr}, \quad I_{\mu\nu\alpha\beta} = \frac{1}{i} \left\{ \mathcal{D}_{\mu\nu\sigma\lambda} \left[ \mathcal{D}_{\tau\rho\sigma\lambda}^{\dag} \mathcal{D}_{\tau\rho\alpha\beta}^{\dag} \right] \right\}_{+}, \]

where the superscript “tr” means transposition of the indices and spacetime arguments referring to the points of observation: \( \mu \nu \leftrightarrow \alpha\beta, + \leftrightarrow - \), \( x \leftrightarrow x' \) [the transposed contribution is represented by the diagrams collected in part (f) of Fig. 6].

As it follows from the considerations of Sec. \[IV.A.1\], \( C_{\mu\nu\alpha\beta}^{(0)} \) can be expressed through \( C_{\mu\nu\alpha\beta} \) as

\[ C_{\mu\nu\alpha\beta}^{(0)} = \lim_{\nu', \tau \to 0} \left\{ \frac{1}{(VT)^{2}} \int d^{4}x d^{4}x' \ C_{\mu\nu\alpha\beta}(x, x') \right\}. \]

In the DeWitt gauge

\[ \mathcal{D}_{\mu\nu\alpha\beta} = -W_{\mu\nu\alpha\beta} \mathcal{D}^{0}, \quad \mathcal{D}^{0} \equiv \mathcal{D}_{|m=0}, \]

\( I_{\mu\nu\alpha\beta} \) reads

\[ I_{\mu\nu\alpha\beta}(x, x') = \frac{1}{i} \int d^{4}zd^{4}z' \]

\[ \times \left\{ + D_{++}^{0}(x, z) \left[ \phi(z) \bar{V}_{\mu\nu} D_{++}(z, z') \bar{V}_{\alpha\beta}^{\dag}(z') \right] D_{++}^{0}(z', x') \\
- D_{++}^{0}(x, z) \left[ \phi(z) \bar{V}_{\mu\nu} D_{+-}(z, z') \bar{V}_{\alpha\beta}^{\dag}(z') \right] D_{+-}^{0}(z', x') \\
- D_{+-}^{0}(x, z) \left[ \bar{\phi}(z) \bar{V}_{\mu\nu} D_{--}(z, z') \bar{V}_{\alpha\beta}^{\dag}(z') \right] D_{--}^{0}(z', x') \\
+ D_{+-}^{0}(x, z) \left[ \bar{\phi}(z) \bar{V}_{\mu\nu} D_{-+}(z, z') \bar{V}_{\alpha\beta}^{\dag}(z') \right] D_{-+}^{0}(z', x') \right\}, \]

(76)
where

\[ \varphi \, \vec{V}_{\mu \nu} \, \psi = \frac{1}{2} W_{\mu \nu \alpha \beta} \left\{ W^{\alpha \beta \gamma \delta} \varphi \, \vec{\partial}_\gamma \, \vec{\partial}_\delta \, \psi - m^2 \eta^{\alpha \beta} \varphi \psi \right\} . \]

Contribution of the third term in the right hand side of Eq. (74) is zero identically. Indeed, using Eq. (20), and performing spacetime integrations we see that the three lines coming, say, into z-vertex are all on the mass shell, which is inconsistent with the momentum conservation in the vertex. The remaining terms in Eq. (76) take the form

\[ I_{\mu \nu \alpha \beta}(x,x') = \int \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 p}{(2\pi)^3} \frac{\alpha(q)\alpha^*(q+p)}{\sqrt{2\varepsilon q}2\varepsilon^{q+q+p}} \, e^i p x' \, \tilde{I}_{\mu \nu \alpha \beta}(p,q), \quad p_0 = \varepsilon_{q+p} - \varepsilon_q, \quad (77) \]

where

\[ \tilde{I}_{\mu \nu \alpha \beta}(p,q) = -i \int \frac{d^4 k}{(2\pi)^4} e^{ik(x-x')} \{ m^4 \eta_{\mu \nu} \eta_{\alpha \beta} - 2m^2 (\eta_{\mu \nu} q_\alpha q_\beta + \eta_{\alpha \beta} q_\mu q_\nu) - 4m^2 \eta_{\mu \nu} q_{(\alpha k_\beta)} \]

\[ - 2m^2 \eta_{\alpha \beta} (p_\mu q_\nu + p_\nu k_\mu) + q_\alpha k_\beta + 4p_\mu q_\nu (q_\alpha k_\beta) + 4q_\alpha q_\beta (p_\mu q_\nu + p_\nu k_\mu) + 4p_\mu q_\nu (q_\alpha k_\beta) + 4q_\alpha q_\beta (p_\mu q_\nu + p_\nu k_\mu) + 4q_\alpha q_\beta (q_\alpha k_\beta) \}

\[ \times \left\{ D_{++}(k)D_{++}(q+k)D_{+-}(k-p) \right. \]

\[ \left. + D_{++}(k)D_{+-}(q+k)D_{--}(k-p) \right\}, \quad (78) \]

\((\mu_1 \mu_2 \cdots \mu_n)\) denoting symmetrization over indices enclosed in the parentheses,

\[ (\mu_1 \mu_2 \cdots \mu_n) = \frac{1}{n!} \sum_{\{1 \cdots n \} \in \text{perm}(12 \cdots n)} \mu_1 \mu_2 \cdots \mu_n. \]

As in Sec. 3.2, Eq. (77) simplifies in the long-range limit to

\[ I_{\mu \nu \alpha \beta}(x,x') = \frac{1}{2m} \int \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 p}{(2\pi)^3} |b(q)|^2 \, e^{-ip(x-x')} \, \tilde{I}_{\mu \nu \alpha \beta}(p,q), \quad p_0 = 0. \quad (79) \]

The leading term in \( \tilde{I}_{\mu \nu}(p,q) \) has the form [Cf. Eq. (71)]

\[ \tilde{I}^{(0)}_{\mu \nu \alpha \beta}(p,q) \sim G^2 m^2 \frac{\eta_{\mu \nu \alpha \beta}}{\sqrt{-p^2}}. \quad (80) \]

This singular at \( p \to 0 \) contribution comes from integration over small \( k \) in Eq. (78). Therefore, to the leading order, the momenta \( k, p \) in the vertex factors can be neglected in comparison with \( q \). Thus,

\[ \tilde{I}_{\mu \nu \alpha \beta}(p,q) = -i Q_{\mu \nu} Q_{\alpha \beta} \int \frac{d^4 k}{(2\pi)^4} e^{ik(x-x')} \left\{ D_{++}(k)D_{++}(q+k)D_{+-}(k-p) \right. \]

\[ \left. + D_{++}(k)D_{+-}(q+k)D_{--}(k-p) \right\}, \quad (81) \]

where

\[ Q_{\mu \nu} = 2q_\mu q_\nu - m^2 \eta_{\mu \nu}. \]

Furthermore, it is convenient to combine various terms in this expression with the corresponding terms in the transposed contribution. Noting that the right hand side of Eq. (81) is explicitly symmetric with respect to \( \{\mu \nu\} \leftrightarrow \{\alpha \beta\} \) and that the variables \( x, x' \) in the exponent can be freely interchanged because they appear symmetrically in Eq. (76),
we may write

\[ \hat{C}_{\mu\nu\alpha\beta}(p, q) \equiv \hat{I}_{\mu\nu\alpha\beta}(p, q) + \hat{I}^r_{\mu\nu\alpha\beta}(p, q) = -iQ_{\mu\nu}Q_{\alpha\beta} \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} \times \{ \begin{array}{c} [D_{++}^0(k)D_{++}(q+k)D_{--}^0(k-p) + D_{--}(q+k)D_{++}^0(k-p)] \\
+ [D_{+-}^0(k)D_{--}(q+k)D_{--}^0(k-p) + D_{++}(q+k)D_{++}^0(k-p)] \\
- [D_{++}^0(k)D_{++}(q+k)D_{--}^0(k-p) + D_{--}(q+k)D_{++}^0(k-p)] \end{array} \}. \] (82)

With the help of the relation

\[ D_{--} + D_{++} = D_{+-} + D_{--}, \] (83)

which is a consequence of the identity

\[ T\hat{\phi}(x)\hat{\phi}(y) + T\hat{\phi}(x)\hat{\phi}(y) - \hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(x) = 0, \]

the first term in the integrand can be transformed as

\[ D_{++}^0(k)D_{++}(q+k)D_{--}^0(k-p) + D_{--}(q+k)D_{++}^0(k-p) = D_{++}^0(k)D_{++}(q+k)D_{--}^0(k-p) + D^0(k)D_{--}(q+k)D_{--}^0(k-p), \]

where

\[ D(k) \equiv D_{+-}(k) + D_{--}(k) = 2\pi i\delta(k^2 - m^2), \quad D^0(k) = D(k)|_{m=0}. \]

Here we used the already mentioned fact that \( D_{--}^0(k)D_{--}(k + q) \equiv 0 \) for \( q \) on the mass shell. Analogously, the second term becomes

\[ D^0(k)D_{--}(q+k)D_{--}^0(k-p) - D^0_+(k)D_{--}(q+k)D^0(k-p). \]

Changing the integration variables \( k \to k + p, q \to q - p \), and then \( p \to -p \), noting that the leading term is even in the momentum transfer [see Eq. \( 84 \)], and that \( p(x - x') \) in the exponent can be omitted in the coincidence limit \( (x \to x') \), the sum of the two terms takes the form

\[ D^0(k) \left[ D_{++}(q+k)D_{++}^0(k-p) + D_{--}(q+k)D_{--}^0(k-p) \right]. \]

Similar transformations of the rest of the integrand yield

\[ D_{++}^0(k)D_{--}(q+k)D_{--}^0(k-p) + D_{--}(q+k)D_{++}^0(k-p) \to -\frac{1}{2} \left[ D_{++}^0(k)D_{--}(q+k)D_{++}^0(k-p) + D_{--}(q+k)D_{++}^0(k-p) \right]. \] (84)

Substituting these expressions into Eq. \( 85 \) and using Eq. \( 20 \) gives

\[ \hat{C}_{\mu\nu\alpha\beta}(p, q) = Q_{\mu\nu}Q_{\alpha\beta} \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} \times \text{Re} \left\{ \frac{2\delta(k^2)}{[(k-p)^2 + i0][(q+k)^2 - m^2 + i0]} + \frac{\delta[(q+k)^2 - m^2]}{[k^2 + i0][(k-p)^2 + i0]} \right\}. \] (85)

As was discussed in Sec. \[ 15 \] the exponent in the integrand in Eq. \[ 85 \] plays the role of an ultraviolet cutoff, ensuring convergence of the integral at large \( k \). On the other hand, the leading contribution \[ 81 \] is determined by integrating over \( k \sim p \) where it is safe to take the limit \( x \to x' \). Since \( (x - x') \) is eventually set equal to zero, one can further simplify the \( k \) integral by using the dimensional regulator instead of the oscillating exponent. Namely, introducing the dimensional regularization of the \( k \) integral, one may set \( x = x' \) afterwards to obtain

\[ \hat{C}_{\mu\nu\alpha\beta}(p, q) = Q_{\mu\nu}Q_{\alpha\beta} \int \frac{d^4k}{(2\pi)^4} \text{Re} \left\{ \frac{2\delta(k^2)}{[(k-p)^2 + i0][(q+k)^2 - m^2 + i0]} + \frac{\delta[(q+k)^2 - m^2]}{[k^2 + i0][(k-p)^2 + i0]} \right\}. \] (86)
where \( \mu \) is an arbitrary mass parameter, and \( \epsilon = 4 - d \), \( d \) being the dimensionality of spacetime.

Next, going over to the \( \alpha \)-representation, the first term in the integrand may be parameterized as

\[
\frac{\delta(k^2)}{[\alpha^2 - m^2]} = \frac{\delta(k^2)}{(p^2 - 2kp + i0)(2kp + i0)}
\]

where \( \alpha \equiv - \frac{L_2}{m} \). Substituting this into Eq. (86), and using the formulas

\[
\alpha \equiv - \frac{L_2}{m} \quad \text{and} \quad L \equiv \frac{m^2}{\mu^2} \sin \left( \frac{\pi}{4} \right) = \left( \frac{2}{p} \right) \left( \frac{\pi}{4} \right) \Gamma (1 - \frac{\epsilon}{2}) \Gamma (\epsilon) \int_0^\infty dz \frac{\left| z^2 - (1 + z)\alpha \right|^{\epsilon/2}}{\left| z^2 - (1 + z)\alpha \right|}
\]

one finds

\[
K(p) = \mu^\epsilon \int \frac{d^4-k}{(2\pi)^4} \frac{\delta(k^2)}{[\alpha^2 - m^2]} = \frac{\mu^\epsilon}{8\pi^2m^2} \cos \left( \frac{\pi\epsilon}{4} \right) \Gamma \left( 1 - \frac{\epsilon}{2} \right) \frac{\pi e^{i\epsilon/2} - \epsilon^\epsilon}{[\pi x]^{-\epsilon} - \epsilon^\epsilon} \frac{\pi e^{i\epsilon/2} - \epsilon^\epsilon}{[\pi x]^{-\epsilon} - \epsilon^\epsilon} \frac{\pi e^{i\epsilon/2} - \epsilon^\epsilon}{[\pi x]^{-\epsilon} - \epsilon^\epsilon}
\]

Changing the integration variable \( z \rightarrow yz \), and taking into account that \( q^2 = m^2 \), \( qp = -p^2/2 \) yields

\[
K(p) = \frac{(\pi e^{i\epsilon/2} - \epsilon^\epsilon)}{8\pi^2m^2} \cos \left( \frac{\pi\epsilon}{4} \right) \Gamma \left( 1 - \frac{\epsilon}{2} \right) \frac{\pi e^{i\epsilon/2} - \epsilon^\epsilon}{[\pi x]^{-\epsilon} - \epsilon^\epsilon} \frac{\pi e^{i\epsilon/2} - \epsilon^\epsilon}{[\pi x]^{-\epsilon} - \epsilon^\epsilon} \frac{\pi e^{i\epsilon/2} - \epsilon^\epsilon}{[\pi x]^{-\epsilon} - \epsilon^\epsilon}
\]

where \( \alpha \equiv -p^2/m^2 \). Similar manipulations with the second term in Eq. (80) give

\[
L(p) = \mu^\epsilon \int \frac{d^4-k}{(2\pi)^4} \frac{\delta[(q^2 - m^2)]}{[\alpha^2 - m^2]} = \frac{\mu^\epsilon}{8\pi^2m^2} \cos \left( \frac{\pi\epsilon}{4} \right) \Gamma \left( 1 + \frac{\epsilon}{2} \right) \frac{\pi e^{i\epsilon/4}}{[\pi x]^{1+\epsilon/2} + \epsilon^{1+\epsilon/2}} + \frac{e^{-i\epsilon/4}}{[\pi x]^{1+\epsilon/2} + \epsilon^{1+\epsilon/2}}
\]

Changing the integration variables \( y = ut \), \( z = (1 - u)t \) yields

\[
L(p) = \frac{\pi e^{i\epsilon/2}}{16\pi^2m^2} \left( \frac{\mu}{m} \right)^\epsilon \Gamma \left( 1 + \frac{\epsilon}{2} \right) \frac{e^{i\pi/4}}{[\pi x]^{1+\epsilon/2} + \epsilon^{1+\epsilon/2}} + \frac{e^{-i\pi/4}}{[\pi x]^{1+\epsilon/2} + \epsilon^{1+\epsilon/2}}
\]

On the other hand, \( L(p) \) must be real because the poles of the functions \( D_{++}(k) \), \( D_{++}(k-p) \) actually do not contribute. Hence,

\[
L(p) = \frac{\pi e^{i\epsilon/2}}{8\pi^2m^2} \left( \frac{\mu}{m} \right)^\epsilon \cos \left( \frac{\pi\epsilon}{4} \right) \Gamma \left( 1 + \frac{\epsilon}{2} \right) \frac{t}{[\pi x]^{1+\epsilon/2} + \epsilon^{1+\epsilon/2}}
\]
Let us turn to investigation of singularities of the expressions obtained when \( \epsilon \to 0 \). Evidently, both \( K(p) \) and \( L(p) \) contain single poles

\[
K_{\text{div}}(p) = \frac{1}{8\pi^2m^2\epsilon} \int_0^\infty \frac{dz}{z^2 - (1+z)\alpha} = -\frac{1}{8\pi^2m^2\epsilon} \sqrt{\frac{1}{1+4/\alpha}} \ln \frac{\sqrt{1+4/\alpha + 1}}{\sqrt{1+4/\alpha - 1}},
\]

(89)

\[
L_{\text{div}}(p) = \frac{1}{8\pi^2m^2\epsilon} \int_0^1 \frac{du}{1 + u(1-u)\alpha} = \frac{1}{4\pi^2m^2\epsilon} \frac{1/\alpha}{\sqrt{1+4/\alpha - 1}}.
\]

(90)

Note that \( L_{\text{div}} = -2K_{\text{div}} \). Upon substitution into Eq. (88) the pole terms cancel each other. Thus, \( \tilde{C}_{\mu\nu\alpha\beta}(p,q) \) turns out to be finite in the limit \( \epsilon \to 0 \). Taking into account also that divergences of the remaining two diagrams (g), (h) in Fig. 3 are independent of the momentum transfer, we conclude that the singular part of the correlation function is completely local.

It is worth of mentioning that although the quantity \( K_{\text{div}}(p) \) is non-polynomial with respect to \( p^2 \), signaling non-locality of the corresponding contribution to \( C_{\mu\nu\alpha\beta}(x,x') \), it is analytic at \( p = 0 \), which implies that its Fourier transform is local to any finite order of the long-range expansion. Indeed, expansion of \( K_{\text{div}}(p) \) around \( \alpha = 0 \) reads

\[
K_{\text{div}}(p) = -\frac{1}{16\pi^2m^2\epsilon} \left( 1 - \frac{\alpha}{6} + \ldots + \frac{(-1)^n(n!)^2}{(2n + 1)!} \alpha^n + \ldots \right).
\]

(91)

Contribution of such term to \( C_{\mu\nu}(x,x') \) is proportional to

\[
\int \frac{d^3p}{(2\pi)^3} e^{-ip(x' - x_0)} \left( 1 - \frac{\alpha}{6} + \ldots \right) = \delta^{(3)}(x' - x_0) + \frac{1}{6m^2} \Delta \delta^{(3)}(x' - x_0) + \ldots.
\]

The delta function arose here because we neglected spacial spreading of the wave packet. Otherwise, we would have obtained

\[
\int \frac{d^3q}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} b^*(q)b(q + p)e^{-ip(x' - x_0)} \left( 1 - \frac{\alpha}{6} + \ldots \right) = 0 \quad \text{for} \quad x' - x_0 \notin W,
\]

as a consequence of the condition (50). In particular, applying this to the correlation function, we see that its divergent part does not contribute outside of \( W \). Thus, the two-point correlation function has a well defined coincidence limit everywhere except the region of particle localization.

Perhaps, it is worth to stress once more that the issue of locality of divergences is only a technical aspect of our considerations. This locality does make the structure of the long-range expansion transparent and comparatively simple. However, even if the divergence were nonlocal this would not present a principal difficulty. A physically sensible definition of an observable quantity always includes averaging over a finite spacetime domain, while the singularity of the two-point function, occurring in the coincidence limit, is integrable (see Sec. IV A 1). The only problem with the nonlocal divergence would be impossibility to take the limit of vanishing size of the spacetime domain [Cf. Eq. (75)].

Turning to the calculation of the finite part of \( K(p) \), we subtract the divergence (89) from the right hand side of Eq. (88), and set \( \epsilon = 0 \) afterwards:

\[
K_{\text{fin}}(p) \equiv \lim_{\epsilon \to 0^+} \left[ K(p) - K_{\text{div}}(p) \right]
\]

\[
= \frac{1}{16\pi^2m^2} \left\{ - \left[ \pi + \ln \pi + 2 \ln \left( \frac{\mu^2}{\mu m} \right) \right] \int_0^\infty \frac{dz}{z^2 - (1+z)\alpha} \\
+ \int_0^\infty dz \ln \frac{z^2 - (1+z)\alpha}{z^2 - (1+z)\alpha} \right\},
\]

(92)

10 The corresponding integrals do not involve dimensional parameters other than \( p \), and therefore are functions of \( p^2 \) only. Hence, on dimensional grounds, both diagrams are proportional to

\[
\left( \frac{c_1}{\epsilon} + c_2 \right) \left( \frac{\mu^2}{\epsilon^2} \right)^{s/2} = \frac{c_1}{\epsilon} + c_2 + \frac{c_1}{2} \ln \left( \frac{\mu^2}{\epsilon^2} \right) + O(\epsilon),
\]

where \( c_{1,2} \) are some finite constants.
where $\gamma$ is the Euler constant. The leading contribution is contained in the last integral. Extracting it with the help of Eq. (120) of the Appendix, we find

$$K_{\text{fin}}^{(0)}(p) = \frac{1}{64m\sqrt{-p^2}}. \quad (93)$$

As to $L(p)$, it does not contain the root singularity. Indeed,

$$L(0) = \frac{\pi^{-\epsilon/2}}{8\pi^2m^2} \left( \frac{\mu}{m} \right)^\epsilon \cos \left( \frac{\pi\epsilon}{2} \right) \frac{\Gamma(1+\epsilon/2)}{\epsilon(1+\epsilon)},$$

and therefore $L_{\text{fin}}(p) \equiv \lim_{\epsilon \to 0}[L(p) - L_{\text{fin}}(p)]$ is finite at $p = 0$. It is not difficult to verify that $L_{\text{fin}}(p)$ is in fact analytic at $p = 0$. Thus, substituting Eq. (93) into Eqs. (86), (79), and then into Eq. (75), using Eq. (55), and restoring ordinary units, we finally arrive at the following expression for the leading long-range contribution to the correlation function

$$C_{\mu\nu\alpha\beta}^{(0)}(x) = \delta_{\mu\nu}\delta_{\alpha\beta}\frac{2m^2G^2}{c^4r^2}, \quad r = |x - x_0|. \quad (94)$$

This result coincides with that obtained by the author in the framework of the S-matrix approach [27]. The nonrelativistic gravitational potential $\Phi^s$ is related to the 00-component of metric as $\Phi^s = \hbar_0c^2/2$. Hence, the root mean square fluctuation of the Newton potential turns out to be

$$\sqrt{\langle (\Delta\Phi^s)^2 \rangle} = \frac{Gm}{\sqrt{2r}}. \quad (95)$$

Note also that the relative value of the fluctuation is $1/\sqrt{2}$. It is interesting to compare this value with that obtained for vacuum fluctuations. As was shown in Ref. [21], the latter is equal to $1/2$ (this is the square root of the relative variance $\Delta^2$ used in Ref. [21]).

We can now ask for conditions to be imposed on a system in order to justify neglecting quantum fluctuations of its gravitational field. Such a condition can easily be found out by examining dependence of the generating functional integral of Green functions, one can write, in view of the assumed smallness of the test particle contribution,

$$\int \mathcal{D}\Phi_- \int \mathcal{D}\Phi_+ \exp \{i(S + S_t + j\phi)\}$$

$$= Z_{SK}^{(j_{\lambda} = 0)} \int \mathcal{D}\Phi_- \int \mathcal{D}\Phi_+ iS_t \exp \{i(S + j\phi)\}$$

$$= Z_{SK} \left( 1 + i\langle m|S_t|in \rangle \right) \bigg|_{j_{\lambda} = 0}^{\lambda = 0}. \quad (96)$$

C. Orbit precession in the field of black hole

Here the results obtained in the preceding sections will be applied to the investigation of dynamics of a classical particle in the gravitational field of a black hole with mass $M$. The particle will be taken testing, i.e., it will be assumed sufficiently light to neglect its contribution to the gravitational field, and sufficiently small compared with $r_g = 2GM/c^2$ to allow considering it as a pointlike object. The latter assumption, in particular, justifies the use of the expression (92) for the correlation function, obtained in the coincidence limit.

In order to find equations of motion of the particle, we have to calculate its effective action. The action functional for a pointlike particle has the form

$$S_t = -m_t \int \sqrt{g_{\mu\nu}dx^\mu dx'^\nu} = -m_t \int d\tau \sqrt{1 + \eta_{\mu\nu}\dot{x}^\mu\dot{x}'^\nu}, \quad \dot{x}^\mu \equiv \frac{dx^\mu}{d\tau}$$

where $d\tau$ is the particle proper time defined with respect to the flat metric, $d\tau^2 = \eta_{\mu\nu}dx^\mu dx'^\nu$. Inserting this expression into the generating functional integral of Green functions, one can write, in view of the assumed smallness of the test particle contribution,
The generating functional of connected Green functions takes the form

\[ W = -i \ln \{ Z_{SK} (1 + i\langle n | S_t | n \rangle) \} = W_{SK} + \langle n | S_t | n \rangle. \]

Taking its Legendre transform, we see that the effective particle action is just

\[ \Gamma_t = \langle n | S_t | n \rangle. \]

Expanding \( S_t \) in powers of \( \hbar \mu\nu \), the right hand side of this equation can be evaluated in the first post-Newtonian approximation as

\[ \Gamma_t = -m_t \int d\tau \left\{ 1 + \frac{1}{2} \langle h_{\mu\nu} \rangle \dot{x}^{\mu} \dot{x}^{\nu} - \frac{1}{8} \left[ \langle h_{\mu\nu} \rangle \langle h_{\alpha\beta} \rangle + \mathcal{C}^{(0)}_{\mu\nu\alpha\beta} \right] \dot{x}^{\mu} \dot{x}^{\nu} \dot{x}^{\alpha} \dot{x}^{\beta} \right\} 
= -m_t \int \sqrt{\langle \eta_{\mu\nu} + \langle h_{\mu\nu} \rangle \rangle dx^\mu dx^\nu - \frac{1}{4} \epsilon^{0000}_{\mu\nu\rho\sigma} (dx^\rho)^2}. \quad (97) \]

Substituting Eqs. (66), (94) for the mean gravitational field and its fluctuation, we see from Eq. (97) that the test particle motion in the fluctuating field of a black hole is effectively the same as in a non-fluctuating gravitational field described by the following spacetime metric

\[ ds^2 \equiv \bar{g}_{\mu\nu} dx^\mu dx^\nu = \left( 1 - \frac{r_g}{r} - \frac{r_g^2}{8r^2} \right) dt^2 - \left( 1 + \frac{r_g}{r} - \frac{7r_g^2}{4r^2} \right) dr^2 - r^2 \left( 1 + \frac{r_g}{r} + \frac{7r_g^2}{4r^2} \right) (d\theta^2 + \sin^2 \theta \, d\varphi^2). \quad (98) \]

Now, it is not difficult to calculate the secular precession of the particle’s orbit. Let the test particle with move in the equatorial plane \( (\theta = \pi/2) \) around black hole. Then we have the following Hamilton-Jacobi equation for the action \( \Gamma_t \)

\[ \bar{g}_{\mu\nu} \partial_{\Gamma_t} \partial_{\Gamma_t} - m_t^2 = 0, \]

where \( \bar{g}^{\mu\nu} \) is the reciprocal of \( \bar{g}_{\mu\nu} \). A simple calculation gives, to the leading order,

\[ S_{ht} = -Et + L\varphi + \int dr \left[ (E'^2 + 2m_t E') + \frac{r_g}{r} \left( m_t^2 + 4m_t E' \right) - \frac{1}{r^2} \left( L^2 - \frac{17}{8} r_g^2 m_t^2 \right) \right]^{1/2}, \quad (99) \]

where \( E, L \) are the energy and angular momentum of the particle, respectively, and \( E' = E - m_t \) its non-relativistic energy. The first two terms in the integrand in Eq. (99) coincide with the corresponding terms of classical theory, while the third does not, leading to the angular shift of the orbit

\[ \delta \varphi = \frac{17\pi GM}{2c^2 a (1 - e^2)} \quad (100) \]

per period (\( a \) and \( e \) are the major semiaxis and the eccentricity of the orbit, respectively).

**V. CONCLUDING REMARKS**

In this Chapter, we have determined the long-range behavior of the gravitational fields produced by quantized matter in the first post-Newtonian approximation. From the point of view of establishing the correct correspondence between classical and quantum theories of gravitation, Eq. (55) demonstrating violation of general covariance by the \( h^0 \)-order loop contributions, and Eq. (44) describing \( h^0 \)-fluctuations of the spacetime metric turned out to be of special importance, and led us independently to a macroscopic formulation of the correspondence principle. In turn, this formulation endows \( h^0 \) loop contributions with direct physical meaning as describing deviations of the spacetime metric from classical solutions of the Einstein equations in the case of finite number of elementary particles constituting the gravitating body.
In connection with these results, it is worth also to make the following general comments. As we have seen, an essentially quantum character of elementary particle interactions makes classical consideration inapplicable to systems whose dynamics is governed by interactions of a finite number of constituent particles. On the other hand, there is a deep-rooted belief in the literature that the quantum field description of interacting remote systems, each of which consists of many particles, is equivalent to that in which these systems are replaced by elementary particles with masses and charges equal to the total masses and charges of the systems. In other words, without calling it into question, the familiar notion of a point particle is carried over from classical mechanics to quantum field theory. This point of view is adhered, for instance, in the classic paper by Iwasaki [2] where it is applied to the solar system to calculate the secular precession of Mercury’s orbit, considered as a “Lamb shift”. The Sun and Mercury are regarded in Ref. [2] as scalar particles. As we saw in Sec. IV B under this assumption the root mean square fluctuation of the solar gravitational potential would be 71% of its mean value. Fortunately, such fluctuations are not observed in reality. This is because the Sun is composed of a huge number of elementary particles each of which contributes to the total gravitational field. As we have seen, the relative quantum fluctuation turns out to be suppressed in this case by the factor $1/\sqrt{N} \sim \sqrt{m_{\text{proton}}/M_{\odot}} \approx 10^{-28}$.

Another example of attempts to recover the nonlinearity of a classical theory through the radiative corrections can be found in Ref. [30]. The authors of [30] claim that the electromagnetic corrections of the order $e^2$ to the classical Reissner-Nordström solution are reproduced by the diagram of Fig. 2(a) in which the internal wavy lines correspond to virtual photons. However, as we have shown, it is meaningless to try to establish the correspondence between classical and quantum theories in terms of elementary particles, because quantum fluctuations of the electromagnetic and gravitational fields produced by such particles are of the order of the fields themselves. On the other hand, because of its inappropriate dependence on the number of particles, the diagram 2(a) fails to reproduce the classical physics in the macroscopic limit. This can be shown using the same argument as in the case of purely gravitational interaction. Namely, given a body with the total electric charge $Q$, consisting of $N = Q/q$ particles with charge $q$, the contribution of the diagram 2(a) is proportional to $N \cdot q^2 = Q^2/N$ turning into zero in the macroscopic limit. The relevant contribution correctly reproducing the $e^2$-correction to the Reissner-Nordström solution is given instead by the tree diagrams like that in Fig. 1(b) in which internal wavy lines correspond to virtual photons.

Finally, we mention that investigation of quantum fluctuations of the Coulomb potential similar to that carried out in Sec. III B 2, can be evaluated as follows. Consider the auxiliary quantity

$$J_{nm} \equiv \int_0^\infty \int_0^\infty \frac{dudt}{(1 + t + u)^n(1 + atu)^{m}},$$

encountered in Sec. III B 2, can be evaluated as follows. Consider the auxiliary quantity

$$J(A, B) = \int_0^\infty \int_0^\infty \frac{dudt}{(A + t + u)(B + atu)},$$

where $A, B > 0$ are some numbers eventually set equal to 1. Performing an elementary integration over $u$, we get

$$J(A, B) = \int_0^\infty dt \frac{\ln B - \ln \{at(A + t)\}}{B - at(A + t)}.$$

Now consider the integral

$$\tilde{J}(A, B) = \int_C dw f(w, A, B), \quad f(w, A, B) = \frac{\ln B - \ln \{aw(A + w)\}}{B - aw(A + w)}, \quad (A1)$$

taken over the contour $C$ shown in Fig. 7. $\tilde{J}(A, B)$ is zero identically. On the other hand,

$$\tilde{J}(A, B) = \int_{-\infty}^{-A} dz \frac{\ln B - \ln \{az(A + z)\}}{B - az(A + z)} + \int_0^A dz \frac{\ln B - \ln \{-az(A + z)\} + i\pi}{B - az(A + z)}$$

$$+ \int_0^{+\infty} dz \frac{\ln B - \ln \{az(A + z)\} + 2i\pi}{B - az(A + z)} - i\pi \sum_{w_+, w_-} \text{Res} f(w, A, B),$$

APPENDIX A: ROOT SINGULARITIES OF FEYNMAN INTEGRALS

The integrals

$$J_{nm} \equiv \int_0^\infty \int_0^\infty \frac{dudt}{(1 + t + u)^n(1 + atu)^m},$$

encountered in Sec. III B 2, can be evaluated as follows. Consider the auxiliary quantity

$$J(A, B) = \int_0^\infty \int_0^\infty \frac{dudt}{(A + t + u)(B + atu)},$$

where $A, B > 0$ are some numbers eventually set equal to 1. Performing an elementary integration over $u$, we get

$$J(A, B) = \int_0^\infty dt \frac{\ln B - \ln \{at(A + t)\}}{B - at(A + t)}.$$
$w_\pm$ denoting the poles of the function $f(w, A, B)$,

$$w_\pm = -\frac{A}{2} \pm \sqrt{\frac{B}{\alpha} + \frac{A^2}{4}}.$$ 

Change $z \to -A - z$ in the first integral. A simple calculation then gives

$$J(A, B) = \frac{\pi^2}{2\sqrt{\alpha}} B^{-1/2} \left( 1 + \frac{\alpha A^2}{4B} \right)^{-1/2} - \frac{1}{2} \int_0^A dt \ln B - \ln \{\alpha t(A - t)\}. \quad (A2)$$

The roots are contained entirely in the first term on the right of Eq. (A2). The integrals $J_{nm}$ are found by repeated differentiation of Eq. (A2) with respect to $A, B$. Expanding $(1 + \alpha A^2/4B)^{-1/2}$ in powers of $\alpha$, we find the leading terms needed in Sec. 11152.

$$J_{12}^{\text{root}} = \frac{\pi^2}{4\sqrt{\alpha}}, \quad J_{13}^{\text{root}} = \frac{3\pi^2}{16\sqrt{\alpha}}, \quad J_{31}^{\text{root}} = -\frac{\pi^2}{16\sqrt{\alpha}}, \quad J_{32}^{\text{root}} = -\frac{3\pi^2}{32\sqrt{\alpha}}, \quad J_{33}^{\text{root}} = -\frac{15\pi^2}{128\sqrt{\alpha}}. \quad (A3)$$

Next, in the course of evaluation of the integral $K(p)$ we encountered the integral

$$A = \int_0^\infty dz \ln \left| \frac{z^2 - (1 + z)\alpha}{z^2 - (1 + z)\alpha} \right| = \int_0^\infty dz \ln \left| \frac{(z - p_1)(z - p_2)}{(z - p_1)(z - p_2)} \right|,$$

$$p_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 + 4\alpha}}{2}.$$

To evaluate this integral, let us consider an auxiliary integral

$$\tilde{A} = \int_C dw \ln \left| \frac{(w - p_1)(w - p_2)}{(w - p_1)(w - p_2)} \right|,$$

taken over the contour $C$ in the complex $w$ plane, shown in Fig. 8. $\tilde{A}$ is zero identically. On the other hand, for sufficiently small positive $\sigma$, one has

$$\tilde{A} = \int_{-\infty}^{p_1 - \sigma} dz \ln \frac{(z - p_1)(z - p_2)}{(z - p_1)(z - p_2)} - i\pi \ln \left( p_2 - p_1 \right) - i\pi \ln \alpha + \pi^2/2$$

$$+ \int_{p_1 + \sigma}^{p_2 - \sigma} dz \ln \frac{(z - p_1)(z - p_2)}{(z - p_1)(z - p_2)} - i\pi \ln \left( p_2 - p_1 \right) - i\pi \ln \alpha + \pi^2/2$$

$$+ \int_{p_2 + \sigma}^\infty dz \ln \frac{(z - p_1)(z - p_2)}{(z - p_1)(z - p_2)} - 2i\pi.$$

Changing the integration variable $z \to (p_1 + p_2) - z$ in the first integral, and rearranging yields in the limit $\sigma \to 0$

$$A = \frac{1}{2} \int_0^{p_1 + p_2} dz \ln \frac{(p_2 - z)(z - p_1)}{(z - p_1)(z - p_2)} + \frac{\pi^2/2}{p_2 - p_1}. \quad (A5)$$

The root singularity is contained in the second term, because

$$\int_0^{p_1 + p_2} dz \ln \frac{(p_2 - z)(z - p_1)}{(z - p_1)(z - p_2)} \to -\ln \alpha \quad \text{for} \quad \alpha \to 0.$$

Thus,

$$A^\text{root} = \frac{\pi^2}{4\sqrt{\alpha}}. \quad (A6)$$

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FIG. 1: Lower-order tree diagrams representing solution of the Einstein equations. (a) The first order (Newtonian) term. (b) One of the second-order diagrams describing the first post-Newtonian correction. Wavy lines represent gravitons, solid lines the auxiliary source $t^{\mu\nu}$, or expectation value of the energy-momentum tensor of matter, $\langle T^{\mu\nu} \rangle$.

FIG. 2: Some of the one-loop diagrams contributing to the effective gravitational field. (a) The only diagram giving rise to the root singularity with respect to the momentum transfer $p$. (b) An example of a diagram free of the root singularity. Wavy lines represent gravitons, solid lines massive particles. $q$ is the particle 4-momentum.

FIG. 3: Some of diagrams arising upon writing out the matrix diagram in Fig. 2(a) according to the Schwinger-Keldysh rules. (a) The only diagram giving rise to a nonzero contribution in the long-range limit. (b) Diagram vanishing in the long-range limit because of the condition $p^0 = 0$. (c) An example of a diagram which is zero identically because of the vanishing of one of its vertices (the left $\phi^2 h$ vertex).
FIG. 4: The one-loop diagrams giving rise to the root singularity in the right hand side of Eq. (35). Dashed lines represent the Faddeev-Popov ghosts.

FIG. 5: Diagrams responsible for the nontrivial contribution to the right hand side of Eq. (47).
FIG. 6: Tree contribution to the right hand side of Eq. (16). Part (f) of the figure represents the “transposition” of diagrams (b)–(e) (see Sec. IV B).

FIG. 7: Contour of integration in Eq. (A1).
FIG. 8: Contour of integration in Eq. (A4).