Double charmonium production in exclusive bottomonia decays.

V.V. Braguta, 1,∗ A.K. Likhoded, 1,† and A.V. Luchinsky 1,‡

1 Institute for High Energy Physics, Protvino, Russia

This paper is devoted to the leading twist exclusive bottomonia decays with double charmonium in the final state. Using models of the twist-2 charmonium distribution amplitudes the widths of these decays are calculated within light cone formalism. In addition, the processes under consideration are studied within nonrelativistic QCD. In our analysis we have found that the production of some of the P-wave charmonium mesons with \( L_z \neq 0 \) is allowed already at the leading twist approximation. This means that the selection rules which predict the suppression of such decays are violated. The mechanism which lies behind this violation is discussed.

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I. INTRODUCTION.

Double charmonium production at B-factories has been one of the most challenging problem in quarkonium physics for many years. This problem appeared after the measurements of the cross sections of the processes \( e^+e^- \rightarrow J/\Psi \eta_c, J/\Psi' \eta_c, \psi' \eta_c, J/\Psi \chi_{c0}, \psi' \chi_{c0} \) at Belle [1, 2] and BaBar [3] collaborations, which were approximately by an order of magnitude larger than the leading order nonrelativistic QCD (NRQCD) predictions [4, 5, 6]. There were many attempts to resolve this discrepancy [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Lately, it was shown that the agreement between theory and experiment can be achieved if one takes into account radiative and relativistic corrections to the cross sections [18, 19, 20].

Intensive study of double charmonium production at B-factories has already led to the considerable theoretical progress in understanding of hard exclusive processes with charmonia production. Further study of these processes can improve our knowledge about QCD dynamic of the exclusive processes. Moreover, since double charmonium production is sensitive to the parameters of charmonia wave functions, these processes can be used to study the structure of charmonia mesons.

Another processes which can be used to study the dynamic of hard exclusive processes and structure of charmonia mesons are double charmonium production in bottomonia decays. These processes are very similar to the processes of double charmonia production at B-factories since the masses of bottomonia are very near to the center mass energy of \( e^+e^- \) beams at B-factories \( M_{b\bar{b}} \approx \sqrt{s} = 10 \text{ GeV} \). At the same time exclusive bottomonia decays have very important advantage: due to the different quantum numbers of the initial bottomonia one has access to the final states whose production is suppressed at B-factories.

Although exclusive bottomonia decays with double charmonium in the final state are very interesting from theoretical point of view, thus far only few papers were devoted to the study of some of these decays [21, 22, 23, 24]. The present paper is devoted to the study of the leading twist double charmonium production in bottomonia decays. To carry out this study we are going to apply light cone formalism (LC) [23, 24]. Within this approach the amplitude of hard exclusive process can be separated into two parts. The first part is partons production at very small distances, which can be treated within perturbative QCD. The second part is the hadronization of the partons at larger distances. For hard exclusive processes it can be parameterized the by process independent distribution amplitudes (DA), which can be considered as hadrons’ wave functions at light like separation between the partons in the hadron. It should be noted that DAs contain information about the structure of charmonia mesons. To do the calculation of the decay widths we are going to apply models of the S- and P-wave charmonia DAs proposed in papers [27, 28, 29, 30]. We also apply NRQCD to study the same processes.

This paper is organized as follows. In the next section the description of the approach applied in this paper is given. In this section we show that the production of some P-wave charmonia mesons with \( L_z \neq 0 \) is allowed already at the leading twist approximation, what violates well known selection rules [26, 31]. The mechanism which lies behind this violation is discussed. In section III we present analytical expressions for the amplitudes of the processes under study obtained within LC and NRQCD. Numerical results of this paper are given in section IV. In the last section we summarize the results of this paper.

∗Electronic address: braguta@mail.ru
†Electronic address: Likhoded@ihep.ru
‡Electronic address: Alexey.Luchinsky@ihep.ru
II. DESCRIPTION OF THE APPROACH.

A. The amplitudes of the processes under study.

In this paper double charmonium production in exclusive bottomonia decays will be considered. The presence of large energy scale $M_{bb}$ which is much greater than the masses of the final charmonia mesons $M_{bb} \gg M_{cc}$ allows one to apply light cone formalism (LC). Within this formalism the amplitude of the processes $\mathcal{T}$ is expanded in inverse powers of the large energy scale $1/M_{bb}$

$$T = \frac{t_0}{M_{bb}^n} + \frac{t_1}{M_{bb}^{n+1}} + ...$$

In this paper we are going to consider the bottomonia decays which are different from zero already at the leading approximation in $1/M_{bb}$ expansion. From here on these processes will be called the leading twist processes. Moreover, the calculation done in this paper will be restricted by the calculation of the first nonvanishing contribution in $1/M_{bb}$ expansion. At this level of accuracy the amplitude of hard exclusive production of mesons $M_1, M_2$ in bottomonia decay can be written in the following form

$$T = \int_{-1}^{1} \int_{-1}^{1} d\xi_1 d\xi_2 H(\xi_1, \xi_2) \phi_{M_1}(\xi_1) \phi_{M_2}(\xi_2),$$

where $\xi_1, \xi_2$ are the fractions of the relative momenta of the whole meson carried by the quark-antiquark pair in the mesons $M_1, M_2$ correspondingly, $H(\xi_1, \xi_2)$ is the hard part of the amplitude, $\phi_{M_1}(\xi_1), \phi_{M_2}(\xi_2)$ are the leading twist distribution amplitudes (DA) of the mesons $M_1, M_2$.

Now few comments are in order.

1. The leading twist DAs $\phi_{M_1}(\xi_1), \phi_{M_2}(\xi_2)$ parameterize infinite series of the twist-2 operators. For instance, the leading twist DA of the pseudoscalar meson $P$ parameterizes the operators $\bar{Q} \gamma_5 \gamma^\mu D_\mu^a Q$, $n = 0, 1, 2, ..$ as follows

$$\langle P|Q^\gamma_5\gamma^\mu D_\mu^a Q|0\rangle = if_P(q_\gamma^n+1) \int_0^1 dx_2 \phi(\xi)^n,$$

where $q$ is the momentum of the pseudoscalar meson $P$, $f_P$ is the constant which is defined as $\langle P|Q\gamma_\mu\gamma_5 Q|0\rangle = if_Pq_\mu$. One can also think of $\phi(\xi)$ as about the amplitude to find the quark-antiquark pair in the meson $P$ with the fraction of the relative momentum of the whole meson $\xi$. It should be noted that DAs parameterize the nonperturbative effects in the amplitude. In Appendix A we collect the definitions of all leading twist charmonia DAs needed in this paper.

2. The hard part of the amplitude $H(\xi_1, \xi_2)$ describes small distance effects, which can be calculated within perturbative QCD. At the same time the DAs $\phi_{M_1}(\xi_1), \phi_{M_2}(\xi_2)$ parameterize nonperturbative effects, which take place at large distances. From this one can conclude that formula (2) separates the effects of small and large distances.

In this paper we study the processes with bottomonia mesons in the initial states. Below the bottomonia mesons will be described at the leading order approximation of NRQCD. At this level of accuracy the amplitudes of $P$-wave bottomonia decays contain the contributions coming from color-octet states [32]. However, it is clear that the contributions of such states to the amplitude of hard exclusive decays are suppressed. This can be seen as follows. Besides the quark-antiquark pair, color-octet state contains one additional gluon. The amplitude to attach this gluon to the one of the outgoing charmonia is strongly suppressed since this gluon does not have enough energy. So, this gluon must be absorbed in the hard part of the amplitude. What leads to the suppression in $\alpha_s$ and higher powers of relative velocity of bottomonia. For this reason, only color singlet states will be taken into the account in the present analysis. The calculation will be done using the technic of the projection operators [4, 33].

As it was noted the hard part of the amplitude $H(\xi_1, \xi_2)$ contains the small distance effects. At small distances the strong coupling constant $\alpha_s$ is small, so one can expand the $H(\xi_1, \xi_2)$ in a series over $\alpha_s$. However, presence of two strongly separated energy scale $M_{bb} \gg M_{cc}$, gives rise to the appearance of large logarithm $\log(E_{bb}^2/M_{bb}^2)$ or $\log(M_{bb}^2/M_{cc}^2)$. This logarithm enhances the role of radiative corrections. The main contribution to amplitude [2]

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1. Here the designation $a_+ = a_0 + a_\gamma$ for fourvector $a_\mu$ is used. It is also assumed that very energetic meson $P$ moves along $z$ direction.
comes from the leading logarithmic radiative corrections \( \sim (\alpha_s \log E^2_b/M_{\bar{b}b}^2)^n \). It turns out that these corrections can be taken into the account in formula \( \phi \) if this formula is rewritten as follows \( \phi \).

\[
T = \int_{-1}^{1} \int_{-1}^{1} d\xi_1 d\xi_2 H(\xi_1, \xi_2, \mu) \phi_{M_1}(\xi_1, \mu) \phi_{M_2}(\xi_2, \mu),
\]

(4)

To resum the leading logarithmic radiative corrections coming from all loops the scale \( \mu \) should be taken of order of \( \sim M_{\bar{b}b} \). The hard part of the amplitude \( H(\xi_1, \xi_2, \mu) \) should be calculated at the tree level approximation. At this level \( H(\xi_1, \xi_2, \mu) \) depends on the renormalization scale \( \mu \) only through the running of the strong coupling constant \( \alpha_s(\mu) \). The rest of the leading logarithms are resummed in the DAs \( \phi_{M_1}(\xi_1, \mu) \phi_{M_2}(\xi_2, \mu) \) using renormalization group method. It should be stressed that formula (4) exactly resums the leading logarithmic radiative corrections which appear in all loops. In the calculation we use \( \mu = M_{\bar{b}b}/2 \).

Commonly, to study the production of nonrelativistic mesons one uses effective theory NRQCD \( [32] \). NRQCD deals with three energy scales \( m_c \gg m_c v \gg m_c v^2 \), where \( m_c \) is the mass of c-quark, \( v \ll 1 \) is the relative velocity of quark antiquark pair in charmonium. In the process of hard nonrelativistic meson production there appears one additional energy scale \( E_b \sim M_{\bar{b}b} \) which is much greater than all scales \( m_c, m_c v, m_c v^2 \). Evidently, it is not possible to apply NRQCD at this scale. From the effective theory perspective, first, this large energy scale must be integrated out. And this is done through the taking into account renormalization group evolution of the DAs \( \phi_{M_1}(\xi_1, \mu) \phi_{M_2}(\xi_2, \mu) \).

B. Selection rules.

In this section we are going to determine what processes of double charmonium production in bottomonia decays are the leading twist processes. To do this one should determine the asymptotic behaviour of the amplitudes of the processes under study in the limit \( M_{\bar{b}b} \rightarrow \infty \). The determination of the asymptotic behaviour was a subject of many papers \([26, 31, 35, 36, 37]\). It turns out that this behavior is determined by the quantum numbers of the final hadrons and does not depend on the DAs of the final hadrons. Below we are going to follow papers \([26]\). The authors of this paper formulated general selection rules which can be used to specify the leading twist processes

1. For a very energetic charmonium the c-quark helicity coincides with the projection of its spin into the direction of the hadron momentum. This rule is valid up to the corrections \( \sim k_1/M_{\bar{b}b} \), where \( k_1 \) is the transverse momentum of the quark in charmonium.

2. For the charmonia states with \( L_z \neq 0 \) (\( L \) is a quark angular momentum, the hadron is assumed to move along \( z \)-axis) the asymptotic behavior of the amplitude is power suppressed.

3. The QCD interaction has a vector nature and at the quark-gluon vertex the quark helicity is conserved (up to the corrections \( \sim m_c / M_{\bar{b}b} \)).

At the leading order approximation in the \( \alpha_s \), the exclusive decays of the C-even bottomonia can be described by the Feynman diagrams similar to that shown in Fig. 1a. Typical diagram for the C-odd bottomonia exclusive decays is shown in Fig. 1b. Applying selection rules (1)-(3) and diagrams shown in Fig. 1a,b one can prove that there are two types of the leading twist processes. The first one contains the processes in which the spin of the initial state \( J_z = 0 \) (\( z \)-axis is chosen in the direction of motion of final charmonia mesons). In this case the helicities of final mesons are \( \lambda_1 = \lambda_2 = 0 \). Any bottomonium meson can decay in this way. The second type of the processes contains the processes with \( J_z = \pm 2 \) and \( \lambda_1 = -\lambda_2 = \pm 1 \). Evidently, only the \( \lambda_2 \) meson can satisfy this condition. Note also, that one can introduce the quantity \( \Lambda = \lambda_1 + \lambda_2 \) and for all leading twist processes this quantity is zero. The amplitude of the process with \( \Lambda \neq 0 \) is suppressed as \( \sim (M_{\bar{b}b}/M_{\bar{b}b})^\Lambda \) relatively to the \( \Lambda = 0 \) process.

For the bottomonia with \( J_z = 0 \) one can derive one more selection rule. To do this let us introduce the quantum number "naturalness": \( \sigma = (-1)^S \), where \( P \) is the parity, \( S \) is the spin. If the naturalness is not conserved: \( \sigma_{\text{initial}} \neq \sigma_{\text{final}} \), there appears the antisymmetric tensor \( \epsilon_{\mu\nu
\rho} \) in the amplitude. This tensor is contracted with the polarizations and momenta of the final charmonia. However, the polarization vector of energetic longitudinally polarized meson is proportional to its momentum, so for the mesons with the longitudinal polarizations there will be no enough fourvectors to get nonzero result after the contraction. From this one can conclude that for the leading twist processes the naturalness is conserved.

Applying the rules that were presented above one can find all leading twist exclusive bottomonia decays. Some of these decays are the decays of the C-even bottomonia mesons and some are the decays of the C-odd bottomonia mesons. Evidently the C-odd bottomonia decays can proceed through at least tree gluons (see Fig. 1b). From this one can conclude that the amplitudes of the C-odd bottomonia decays are suppressed as \( \sim \alpha_s/\pi \sim 0.07 \) in comparison with the C-even bottomonia decays. For this reason, we are not going to consider these decays. So, below the following
decays will be considered
\[ \eta_b, \chi_{b1} \rightarrow h_c J/\psi(\psi'), \eta_c(\eta'_c) \chi_{c0}(\chi_c), \chi_{c1} \chi_{c0}(\chi_{c2}), \]
\[ \chi_{b0}, \chi_{b2} \rightarrow \eta_c(\eta'_c) \chi_{c1}, \eta_c(\eta'_c) \eta_c(\eta'_c), J/\psi(\psi') J/\psi(\psi'), h_c h_c, \chi_{c0} \chi_{c2}, \chi_{c0} \chi_{c0}, \]
\[ \chi_{c1} \chi_{c1}, \chi_{c2} \chi_{c2} \]
\[ \chi_{b2} \rightarrow h_c J/\psi(\psi'), \chi_{c1} \chi_{c2}. \] (5)

C. Violation of the selection rules.

Let us consider a longitudinally polarized $\chi_{c1}$ meson moving along $z$-axes. In this meson the spin $S = 1$ and orbital momentum of $L = 1$ of the quark-antiquark pair sum to $J = 1$ state. The Clebsch-Gordan coefficients of the longitudinally polarized $\chi_{c1}$ meson ($J_z = 0$) are $C_{L_z=0}^{J_z=0} = 1/\sqrt{2}$, $C_{L_z=0}^{J_z=0} = 0$. This means that the longitudinal polarization can be realized only through the $L_z = \pm 1$. However, the selection rules tell us that such states are suppressed. From this one can draw a conclusion: the processes with the production of the longitudinally polarized $\chi_{c1}$ meson (for instance, the $\chi_{b0} \rightarrow \eta_c \chi_{c1} \rightarrow \chi_{c1}$ decay) are not the leading twist processes. Similar situation takes place for a transversely polarized $h_c$ meson. In this case one has $S = 0, L = 1$. Evidently, the state $J_z = \pm 1$ can appear only if $L_z = \pm 1$. So, from the selection rules one can conclude that the production of the transversely polarized $h_c$ meson is suppressed. Nevertheless, the calculation done within NRQCD shows that these conclusions are not correct. The processes shown in list (5) with the production of the transversely polarized $h_c$ meson and longitudinally polarized $\chi_{c1}$ meson are the leading twist processes.

To demonstrate that the production of the longitudinally polarized $\chi_{c1}$ meson is not suppressed, let us consider the process of the production of the $\eta_c$ meson and $\{\bar{c}c\}$ pair $\chi_{b0} \rightarrow \eta_c \{\bar{c}c\}$. Assume further that $\{\bar{c}c\}$ pair has very small invariant mass $M_{\bar{c}c} \ll M_{\bar{b}b}$. So, we deal with quasi-exclusive process for which one can apply LC technic. For instance, one can expand the amplitude of this process in $1/M_{\bar{b}b}$ expansion for quasi-exclusive process appears due to the leading twist operator of the $\eta_c$ meson. Feynman diagrams that give contribution to this process are similar to that shown in Fig. 1a. It is not difficult to see that the leading order contribution to the amplitude in $1/M_{\bar{b}b}$ expansion can be written as follows
\[ T = B^{\mu\nu} \times \{\bar{c}\gamma_\mu \hat{P} \gamma_\nu c\}, \] (6)

where $B^{\mu\nu}$ is proportional to the amplitude that describes the decay of the $\chi_{b0}$ into two gluons, $\hat{P} = \hat{p}_1 \gamma_5$ is the dirac structure of the leading twist wave function of the $\eta_c$ meson with momentum $p_1$. The tensor $B^{\mu\nu}$ can depend only on the tensor structures $q_1^\mu q_2^\nu, q_1^\mu q_1^\nu, q_2^\mu q_2^\nu, q_1^\mu q_1^\nu, (M_{\bar{b}b})^{g\mu\nu}$. Evidently, the leading twist contribution to the amplitude originates only from the last tensor structure. The others give terms proportional to masses. From this one can conclude that the leading twist contribution to the amplitude $T$ is proportional to
\[ T \sim \bar{c}\gamma_\mu \hat{p}_1 \gamma_5 \gamma^\mu c \sim \bar{c}\hat{p}_1 \gamma_5 c. \] (7)

Now let us consider what $\bar{c}c$ states contribute to the amplitude $T$. To do this let us consider the expression for $T$ in the center mass of $\bar{c}c$ pair and assume that $\bar{c}c$ is nonrelativistic pair. In this case one can expand the operator $\bar{c}\gamma_\mu \gamma_5 c$
in relative velocity of quark-antiquark pair using Foldy-Wouthuysen-Tani transformation \[39\]. The terms relevant to the production can be written as follows

\[
\bar{c}\gamma_\mu \gamma_5 c = \delta^{\mu 0} \psi^+ \left[1 + \frac{D^2}{2m_c^2}\right] \chi + \frac{1}{m_c} \psi^+ \left[D \times \sigma\right] \chi + O(v^3),
\]  

(8)

where \(D\) is vector part of the gauge derivative, the index \(i\) is vector index, \(m_c\) is the mass of \(c\)-quark, \(\psi^+\) and \(\chi\) are Pauli spinor fields that create a quark and an antiquark respectively. Now it is clear that the operator \(\psi^+ \left[1 + \frac{D^2}{2m_c^2}\right] \chi\) corresponds to the \(L = 0, S = 0, J = 0\) state. So, one can recognize \(\eta_c\) meson contribution in this operator. From this one can conclude that the process \(\chi_{\text{u0}} \rightarrow \eta_c\eta_c\) is the leading twist process. The operator \(\psi^+ \left[D \times \sigma\right] \chi\) corresponds to the \(L = 1, S = 1, J = 1\) state. So, one sees that this operator creates the longitudinal \(\chi_{\perp}\) meson already at the leading twist approximation what violates rule (2) of the selection rules. Similarly, one can show that the transversely polarized \(h_c\) meson can be produced at the leading twist approximation.

Now, let us try to understand why the second selection rule is violated? To answer this question first let us recall how this selection rule originates. In LC leading twist distribution amplitude \(\phi(x)\) of nonrelativistic meson \(M\) moving along \(z\)-axes can be written in the following form

\[
\phi(x, Q^2) \sim \int k_\perp^2 <Q^2> \frac{d^2k_\perp}{2\pi} \psi_M(k_\perp, x),
\]  

(9)

where \(\psi_M(k_\perp, x)\) is the hadron wave function of the mesons \(M, x\) is the fraction of momentum carried by quark. The meson \(M\) with \(L_z \neq 0\) has the function \(\psi_M(k_\perp, x)\) which contains the factor \(\sim \exp(iL_z\varphi)\). So, the integration over \(d^2k_\perp\) acts as a projector to the \(L_z = 0\) state and \(L_z \neq 0\) states cannot appear at the leading twist approximation. Similar arguments were used in papers \[20\] \[31\] to prove the second selection rule.

Further, this point will be considered in more detail. To get \(L_z \neq 0\) in addition to the \(\exp(iL_z\varphi)\) one should find somewhere the factor \(k_\perp\). It can be found in the matrix element of hard exclusive process or in the expansion of the bispinors of quark-antiquark pair in relative momentum (see formula (8)). The first source of the \(k_\perp\) is not very important since it always leads to the suppression of the amplitude, so it is beyond the leading twist accuracy.

Consider the case when the \(k_\perp\) appears due to the expansion of quark-antiquark bispinors in relative momentum. Formula (8) proposes the example of the expansion of the operator \(\bar{c}\gamma_\mu \gamma_5 c\) in relative momentum. As it was already noted, the first term of this expansion corresponds to the leading twist operator of the \(\eta_c\) meson. This can be seen from the fact that the leading twist operators among the operators of the fixed dimensionality are those which are maximally enhanced due to the Lorentz transformation from the center mass frame to the frame where the meson has very large energy. Evidently, the \(\delta^{\mu 0}\) in the center mass frame can be rewritten in a covariant way: \(\delta^{\mu 0} = p_\mu / M\), where \(p\) and \(M\) are the momentum and mass of the meson. From this one can conclude that due to the Lorentz transformation the first operator in formula (8) will be enhanced by the first power of large \(\gamma\)-factor, what exactly corresponds to the leading twist (see list of the leading twist DAs in Appendix A).

The second term in formula (8) corresponds to the longitudinally polarized \(\chi_{\perp}\) meson. This term contains gauge derivative which becomes \(\psi^+ D_{\perp} \chi \sim k_\perp\). So, due to this term one has additional \(k_\perp\) which leads to the nonzero contribution of the \(L_z \neq 0\) states. Here few comments are in order. First, note that in the bispinors the term \(\delta^{\mu 0}\) goes with Pauli matrices \(\sigma_{\perp}\) which flip the spin of quark or antiquark. This leads to the violation of the first selection rule. So, the second term leads to the violation of the first and second selection rules. Second, it should be noted that the states \((S = 0, S_z = 0)\) and \((S = 1, S_z = \pm 1)\) cannot be produced simultaneously without one spin flip. This spin flip can be realized due to the violation of the first rule or the third rule. The violation of the third rule always put the process beyond the leading twist accuracy, since it is based on the vector nature of the quark-gluon vertex. So, the violation of the first selection rule in the wave function is needed in order to avoid the violation of the third selection rule.

Now let us do Lorentz transformation of the second term in formula (8) from the center mass frame to the frame where the meson has very large energy. From the first sight, transverse components of fourvectors will not be enhanced by \(\gamma\)-factor. This means that the violation of the first and second selection rules leads to the appearance of the transverse components of fourvectors in the operators what puts the amplitude of such process beyond the leading twist accuracy. This conclusion is in agreement with the results of papers \[20\] \[31\]. However, note that the \(k_\perp\) and \(\sigma_{\perp}\) appear in formula (8) in the form of vector product. This means that covariant expression of formula (8) has the form

\[
\bar{c}\gamma_\mu \gamma_5 c = \frac{p_\mu}{M} \psi^+ \left[1 + \frac{D^2}{2m_c^2}\right] \chi + \frac{1}{m_c} \psi^+ \left[\sigma_{\perp}\psi^+ \left[D \times \sigma\right] \chi + O(v^3)\right].
\]  

(10)
It is seen from this equation that although transverse components are not enhanced the structure $e_{\nu\mu\lambda\rho} P_{\nu}$ compensates this drawback and returns the second term to the set of the leading twist operators.

At the end of this section we would like to note that the violation of the selection rules in the case of the transversely polarized $h_c$ meson is similar to that of the longitudinally polarized $\chi_{c1}$ meson. So, the conclusion of this section is the production of the longitudinally polarized $\chi_{c1}$ meson and transversely polarized $h_c$ meson is nonzero at the leading twist approximation due to the appearance of the special structures in the corresponding DAs which compensate the penalty for the violation of the first and second selection rules.

D. Description of the calculation procedure.

To calculate processes (5) at the leading twist approximation of LC one can apply the following rules. As it was noted above, the bottomonia mesons will be treated at the leading order approximation of NRQCD using the technic of the projection operators [4, 33]. Applying this technic one can calculate the amplitude of the decay of some bottomonium into two quark-antiquark pairs. The total momentum and relative momentum of the first pair are $p_1$ and $\xi_1 p_1$. The total momentum and relative momentum of the second pair are $p_2$ and $\xi_2 p_2$. At large distances these pairs become $M_1$ and $M_2$ charmonia mesons with the momenta $p_1$, $p_2$. In LC the hadronization of quark-antiquark pair is described by DAs. To calculate the amplitude of meson production one should replace bispinors of quark-antiquark pair $v\bar{u}$ in the amplitude by the corresponding distribution amplitude. The list of the leading twist DAs needed in the calculation can be found in Appendix A. After taking the traces over dirac and color indexes one can get analytical expression for the amplitude.

Another independent approach that can be used to calculate the amplitude of double charmonium production in bottomonia decays is NRQCD. In this approach the final charmonia mesons are treated as nonrelativistic states. In this paper we are going to apply NRQCD at the leading order approximation in relative velocity in charmonia. It should be noted that this approximation cannot be considered as reliable. This conclusion can be drawn from the experience obtained in the study of double charmonium production at B-factories [20], which tells us that relativistic and radiative corrections to double charmonia production can be very large. Nevertheless, in this paper the leading order approximation of NRQCD will be used to get independent estimation of the widths of the exclusive bottomonia decays under study.

In addition to the estimation of the widths of the bottomonia decays, NRQCD can be used to study very important question of LC: the estimation of the corrections to the leading twist approximation (power corrections). To estimate power corrections one can apply the idea of duality of NRQCD and LC descriptions of the hard exclusive nonrelativistic mesons production: if the amplitudes of the hard process under study obtained within NRQCD and LC are expanded in powers of inverse hard energy scales and relative velocity one will get series equal to each other. It should be noted that these is no strict proof of this statement. Moreover, the general proof for the NRQCD factorization of exclusive process involving P-wave heavy quarkonium is not available (see, e.g., [34]). There may be large soft contributions at order $m_c/m_b$ to these decays. However, one can expect that this statement is true since the amplitude in NRQCD and LC can be expanded in series of equivalent operators. Assuming that both theories can describe experiment one can expect that these expansions in both theories coincide. Note also that we have checked that all expressions for the amplitudes under study obtained in LC and expanded in relative velocity coincide with that obtained in NRQCD and expanded in $1/M_{b\bar{b}}$.

Now it is clear how it is possible to estimate the size of power corrections in LC. To do this one should take the leading order NRQCD prediction for the amplitude and expand it in $M_{c\bar{c}}/M_{b\bar{b}} \approx m_c/m_b$. The first term of this expansion can be reproduced by the leading twist approximation of LC expanded in relative velocity in charmonia. The second term in this expansion can be reproduced by the power corrections to the leading twist contribution. So, the second term of NRQCD expansion in $\sim m_c/m_b$ gives the estimation of power correction to the leading twist approximation. Below this approach will be used to estimate the error of the calculation.

III. ANALYTICAL EXPRESSIONS FOR THE MATRIX ELEMENTS.

In this section analytical expressions for the amplitudes and the widths of the double charmonium production in bottomonia decays obtained within the leading order approximation of LC and NRQCD are given. Before the analytical expressions are given, let us introduce some useful designations. First, we introduce the definitions of the

2 It should be noted that at the leading twist approximation one can disregard the masses of quarks and hadrons.
Next, we introduce the following constants for the \( P \) wave charmonia mesons \( \chi c_0, \chi c_1, \chi c_2, h_c \)

\[
\langle \chi c_0(p)|\bar{C}\gamma_\alpha(-iD_\nu)C|0\rangle = f^{L}_{\chi c_0}(\mu)(p_\alpha p_\nu - M^{2}_{\chi c_0}g_{\alpha\nu}), \\
\langle \chi c_1(p,\epsilon)|\bar{C}\gamma_\alpha\gamma_5C|0\rangle = f^{L}_{\chi c_1}(\mu)\epsilon_\alpha, \\
\langle \chi c_2(p,\epsilon)|\bar{C}\gamma_\alpha(-iD_\nu)C|0\rangle = f^{L}_{\chi c_2}(\mu)M^{2}_{\chi c_2}\epsilon_\alpha, \\
\langle h_c(p,\epsilon)|\bar{C}\gamma_\alpha\gamma_5(-iD_\nu)C|0\rangle = f^{L}_{h_c}(\mu)M_{h_c}\epsilon_\alpha, \\
\langle \chi c_2(p,\epsilon)|\bar{C}\gamma_\alpha(-iD_\nu)C|0\rangle = f^{L}_{\chi c_2}(\mu)M^{2}_{\chi c_2}\epsilon_\alpha, \\
\langle h_c(p,\epsilon)|\bar{C}\sigma_{\alpha\beta}C|0\rangle = f^{L}_{h_c}(\mu)\epsilon_{\alpha\beta}\epsilon^{\mu\lambda}p_\mu p_\lambda, \\
\langle \chi c_2(p,\epsilon)|\bar{C}\sigma_{\alpha\beta}C|0\rangle = f^{L}_{\chi c_2}(\mu)\epsilon_{\alpha\beta}\epsilon^{\mu\lambda}p_\mu p_\lambda.
\]

It should be noted here that for the mesons \( \chi c_1, h_c \) the polarization \( \epsilon \) is described by the four vector \( \epsilon_\mu \), for the \( \chi c_2 \) meson the polarization \( \epsilon \) is described by the tensor \( \epsilon^{\mu\lambda} \). The superscript \( L \) in formulas (11), (12) means that the corresponding meson is transversely polarized \( (\lambda = \pm 1) \). The superscript \( L \) implies that the meson has helicity \( \lambda = 0 \). Except the constants \( f^{L}_{\chi c_0}, f^{L}_{\chi c_1}, f^{L}_{\chi c_2}, f^{L}_{h_c} \), all constants in formulas (11), (12) are scale \( (\mu) \) dependent quantities. The anomalous dimensions of these constants can be found in papers [28, 30].

In addition to constants (11), (12), one needs the bottomonia NRQCD matrix elements \( \langle O^{bb}_S \rangle, \langle O^{bb}_P \rangle \) which are defined in [32]. The matrix element \( \langle O^{bb}_S \rangle \) is proportional to the S-wave bottomonium radial wave function at the origin. The matrix element \( \langle O^{bb}_P \rangle \) is proportional to the of the derivative of P-wave bottomonium radial wave function at the origin.

**The decays of the \( \eta_b \) meson.** The decays \( \eta_b \to h_c, J/\psi(\psi'), \eta_c(\eta_c') \chi c_0(\chi c_2), \chi c_1 \chi c_0(\chi c_2) \) are the leading twist decays of the \( \eta_b \) meson. According to the discussion in the previous section at the leading twist approximation of LC only the production of the longitudinally polarized mesons is allowed. The same is true for the decays of all bottomonia except the \( \chi b_2 \) meson. Corresponding amplitude for all the leading twist decays of the \( \eta_b \) meson can be written as

\[
\mathcal{M}(\eta_b \to M_1M_2) = \frac{128\pi^2\alpha_s^2}{27} \sqrt{\langle O^{bb}_S \rangle f^{L}_{1} f^{L}_{2} f^{0}_{0}(\eta_b)},
\]

where \( f^{L}_{1,2} \) are the mesonic constants of longitudinally polarized charmonia and

\[
f^{L}_{0}(\eta_b) = \int_{-1}^{1} d\xi_1d\xi_2 \frac{\xi_1 + \xi_2}{(1 - \xi_1^2)(1 - \xi_2^2)} \varphi^{L}_1(\xi_1)\varphi^{L}_2(\xi_2),
\]

where \( \varphi^{L}_1(\xi_1)\varphi^{L}_2(\xi_2) \) are the final charm DAs. It is not difficult to get the amplitude of any leading twist decay of the \( \eta_b \) meson using formula (13). For instance, to get the amplitude of the process \( \eta_b \to \eta_c\chi c_2 \) one should take \( f^{L}_{1} = f^{L}_{F}, f^{L}_{2} = f^{L}_{\chi c_2} \) and the longitudinal DAs (see Appendix A) of the corresponding mesons.

Using this matrix element it is easy to obtain the expression for the \( \eta_b \to M_1M_2 \) decay width within light cone formalism:

\[
\Gamma(\eta_b \to M_1M_2) = \frac{512\pi^3\alpha_s^4}{729} \left( \frac{\langle O^{bb}_S \rangle}{m_b^2} \right)^2 \left[ \frac{f^{L}_{1} f^{L}_{2} f^{0}_{0}(\eta_b)}{m_b^2} \right]^2.
\]

Within NRQCD one can get the following expressions for the width of the processes under study

\[
\Gamma(\eta_b \to M_1M_2) = \frac{512\pi^3\alpha_s^4}{729} \left( \frac{\langle O^{bb}_S \rangle}{m_b^2} \right)^2 \left[ \frac{f^{NRQCD}_{1} f^{NRQCD}_{2}}{m_b^2} \right]^2 F(\eta_b \to M_1M_2),
\]

where

\[
f^{NRQCD}_{1} f^{NRQCD}_{2} = \sqrt{\frac{\langle O^{S}_S \rangle}{m_c}}.
\]
for $S$-wave charmonium states and

$$f_i^{NRQCD} = \sqrt{\frac{\langle O_i^P \rangle}{m_b^3}}$$

(16)

for $P$-wave charmonium states. The factor $F[\eta_b \to M_1 M_2]$ for different final states is

$$F[\eta_b \to h_c \psi] = 1 + 32r^2,$$

$$F[\eta_b \to \eta_c \chi_c] = \frac{1}{3} + \frac{16r^2}{3} + \frac{64r^2}{3},$$

$$F[\eta_b \to \eta_c \chi_{c2}] = \frac{2}{3} + \frac{16r^2}{3} + \frac{32r^2}{3},$$

$$F[\eta_b \to \chi_{c0} \chi_{c1}] = \frac{2}{3} + \frac{32r^2}{3} + \frac{40r^2}{3} - \frac{800r^3}{3},$$

$$F[\eta_b \to \chi_{c2} \chi_{c1}] = \frac{4}{3} + \frac{32r^2}{3} + \frac{104r^2}{3} - \frac{160r^3}{3},$$

where $r = m_c/m_b$.

**The decays of the $\chi_{b0}$ meson.** The decays $\chi_{b0} \to \eta_c(\eta_c') \chi_{c1}$, $\eta_c(\eta_c') \eta_c(\eta_c')$, $J/\psi(\psi') J/\psi(\psi')$, $h_c h_c$, $\chi_{c0} \chi_{c2}$, $\chi_{c0} \chi_{c0}$, $\chi_{c1} \chi_{c1}$, $\chi_{c2} \chi_{c2}$ are the leading twist decays of the $\chi_{b0}$ meson. The amplitude for these decays can be written as

$$\mathcal{M}(\chi_{b0} \to M_1 M_2) = \frac{512\pi^2a_s^2}{27\sqrt{3}} \sqrt{\frac{\langle O_i^P \rangle}{m_b^3} \frac{f_1^P f_2^P}{m_b^3}} f_0^{(\chi_{b0})},$$

where

$$f_0^{(\chi_{b0})} = \frac{1}{4} \left[ 4 + \xi_1^2 + 6 \xi_1 \xi_2 + \xi_2^2 - \frac{4}{(1 - \xi_1^2)(1 - \xi_2^2)(1 + \xi_1 \xi_2)^2} \phi_1^{(r)}(\xi_1) \phi_2^{(r)}(\xi_2) \right].$$

The width of the $\chi_{b0} \to M_1 M_2$ decay is

$$\Gamma(\chi_{b0} \to M_1 M_2) = \frac{8192\pi^3a_s^4}{2187} \left[ \frac{f_1^P f_2^P}{m_b^3} \right] \left[ f_0^{(\chi_{b0})} \right]^2.$$

(17)

One should remember, that in the case of the identical final mesons the width should be divided by 2. Within NRQCD one can get the following expressions for the width

$$\Gamma(\chi_{b0} \to M_1 M_2) = \frac{4096\pi^3a_s^4}{2187} \left[ \frac{f_1^{NRQCD} f_2^{NRQCD}}{m_b^3} \right] \left[ f_0^{(\chi_{b0})} \right]^2 F(\chi_{b0} \to M_1 M_2),$$

where

$$F[\chi_{b0} \to \eta_c \chi_{c1}] = 4 - 16r,$$

$$F[\chi_{b0} \to \chi_{c0} \chi_{c2}] = \frac{28r}{9} - \frac{92r^2}{3} - \frac{1120r^3}{9} + \frac{1600r^4}{9},$$

$$F[\chi_{b0} \to \eta_c \eta_c] = 1 + 4r + 4r^2,$$

$$F[\chi_{b0} \to \psi \psi] = 1 + 4r + 12r^2,$$

$$F[\chi_{b0} \to h_c h_c] = \frac{1}{4} - 10r^2 - 32r^3 + 272r^4,$$

$$F[\chi_{b0} \to \chi_{c0} \chi_{c0}] = \frac{1}{36} - \frac{16r}{9} + \frac{28r^2}{3} + \frac{128r^3}{9} + \frac{16r^4}{9},$$

$$F[\chi_{b0} \to \chi_{c1} \chi_{c1}] = 4 - 56r + \frac{537r^2}{2} - 260r^3 + 72r^4,$$

$$F[\chi_{b0} \to \chi_{c2} \chi_{c2}] = \frac{28r}{9} + \frac{179r^2}{6} - \frac{340r^3}{9} + \frac{1480r^4}{9}.$$
The decays of the $\chi_{b1}$ meson. The decays $\chi_{b1} \to h_c J/\psi(\psi')$, $\eta_c(\eta'_c) \chi_{c0}(\chi_{c2})$, $\chi_{c1} \chi_{c0}(\chi_{c2})$ are the leading twist decays of the $\chi_{b1}$ meson. The amplitude for the processes equals

$$\mathcal{M}(\chi_{b1} \to M_1 M_2) = \frac{128 \sqrt{2} \pi \alpha_s^2}{27} \left( \frac{m_b^3}{m_b^3} \right) f_1^I f_2^I f_0^I(\chi_{b1}),$$

where

$$I_0^I(\chi_{b1}) = \int_{-1}^{1} d\xi_1 d\xi_2 \frac{\xi_1 - \xi_2}{(1 - \xi_1^2)(1 - \xi_2^2)(1 + \xi_1 \xi_2)} \phi_1^I(\xi_1) \phi_2^I(\xi_2).$$

The width of the $\chi_{b1} \to M_1 M_2$ decay is

$$\Gamma(\chi_{b1} \to M_1 M_2) = \frac{1024 \pi^3 \alpha_s^4}{2187} \left( \frac{m_b^3}{m_b^3} \right) F(\chi_{b1} \to M_1 M_2),$$

Within NRQCD one can get the following expressions for the width

$$\Gamma(\chi_{b1} \to M_1 M_2) = \frac{1024 \pi^3 \alpha_s^4}{2187} \left( \frac{m_b^3}{m_b^3} \right) F(\chi_{b1} \to M_1 M_2),$$

where

$$F[\chi_{b1} \to h_c \psi] = 1 + 4r - 32r^2,$$

$$F[\chi_{b1} \to \eta_c \chi_{c0}] = \frac{1}{3} - \frac{4r}{3},$$

$$F[\chi_{b1} \to \eta_c \chi_{c2}] = \frac{2}{3} + \frac{4r}{3} - 16r^2,$$

$$F[\chi_{b1} \to \chi_{c0} \chi_{c1}] = \frac{2}{3} + \frac{124r}{3} + 24r^2,$$

$$F[\chi_{b1} \to \chi_{c2} \chi_{c1}] = \frac{4}{3} - \frac{16r}{3} - 8r^2 + 192r^3.$$

The decays of the $\chi_{b2}$ meson. The decays $\chi_{b2} \to \eta_c(\eta'_c) \chi_{c1}$, $\eta_c(\eta'_c) \eta_c(\eta'_c)$, $J/\psi(\psi')J/\psi(\psi')$, $h_c h_c$, $\chi_{c0} \chi_{c2}$, $\chi_{c1} \chi_{c1}$, $\chi_{c2} \chi_{c2}$, $h_c J/\psi(\psi')$, $\chi_{c1} \chi_{c2}$ are the leading twist decays of the $\chi_{b2}$ meson. The decay of this meson is more complicated than the decays described above. The point is that large spin of the $\chi_{c2}$ meson opens the possibility to produce transversely polarized charmonia ($\lambda_1 = -\lambda_2 = \pm 1$) at the leading twist approximation. So, contrary to the bottomonia decays discussed above in some decays of the $\chi_{c2}$ meson there can be two different polarizations of the final charmonia: $\lambda_1 = \lambda_2 = 0$ or $\lambda_1 = -\lambda_2 = \pm 1$. It should be noted here that it depends on the process what possibilities are realized at the leading twist. For instance, for the decay $\chi_{c2} \to \eta_c \chi_{c1}$ only $\lambda_1 = \lambda_2 = 0$ is allowed, for the decay $\chi_{c2} \to J/\psi J/\psi$ both possibilities $\lambda_1 = \lambda_2 = 0$, $\lambda_1 = -\lambda_2 = \pm 1$ are allowed. The state $\lambda_1 = \lambda_2 = 0$ is forbidden for the decay $\chi_{b2} \to h_c J/\psi$ since in this case naturalness is not conserved. So, the only allowed possibility is $\lambda_1 = -\lambda_2 = \pm 1$.

The amplitude for the decays of the $\chi_{b2}$ meson into pair of the longitudinally polarized mesons ($\lambda_1 = \lambda_2 = 0$) equals

$$\mathcal{M}(\chi_{b2} \to M_1^I M_2^I) = \frac{256 \sqrt{2} \pi \alpha_s^2}{27 \sqrt{3}} \left( \frac{m_b^3}{m_b^3} \right) f_1^I f_2^I f_0^I(\chi_{b2}),$$

where

$$I_0^I(\chi_{b2}) = \int_{-1}^{1} d\xi_1 d\xi_2 \frac{2 - \xi_1^2 - \xi_2^2}{2(1 - \xi_1^2)(1 - \xi_2^2)(1 + \xi_1 \xi_2)} \phi_1^I(\xi_1) \phi_2^I(\xi_2).$$

If both final mesons can have nonzero helicity $\lambda_1 = -\lambda_2 = \pm 1$, one should take into account the decays into transversely polarized particles. The amplitude of these decays is

$$\mathcal{M}(\chi_{b2} \to M_1^T M_2^T) = -\frac{512 \pi \alpha_s^2}{27 \sqrt{3}} \left( \frac{m_b^3}{m_b^3} \right) f_1^T f_2^T f_0^T(\chi_{b2}),$$
where
\[
I_2^{(\chi_{b2})} = \int_{-1}^{1} d\xi_1 d\xi_2 \frac{1}{(1 - \xi_1^2)(1 - \xi_2^2)(1 + \xi_1 \xi_2)} \phi_1^T(\xi_1) \phi_2^T(\xi_2).
\]
The width of the $\chi_{b2} \rightarrow M_1 M_2$ decay is
\[
\Gamma(\chi_{b2} \rightarrow M_1 M_2) = \frac{496\pi^3 \alpha_s^4}{10935} \frac{(O_{Pb}^b)}{m_b^4} \left\{ \left[ \frac{f_{\perp}^T f_{\perp}^T}{m_b^2} I_0^{(\chi_{b2})} \right]^2 + 12 \left[ \frac{f_{\perp}^T f_{\perp}^T}{m_b^2} I_2^{(\chi_{b2})} \right]^2 \right\}
\]
Within NRQCD one can get the following expression for the width of the process $\chi_{b2} \rightarrow M_1 M_2$
\[
\Gamma(\chi_{b2} \rightarrow M_1 M_2) = \frac{2048\pi^3 \alpha_s^4}{10935} \frac{(O_{Pb}^b)}{m_b^4} \left[ \frac{f_{\perp}^{NRQCD} f_{\perp}^{NRQCD}}{m_b^2} \right]^2 F(\chi_{b2} \rightarrow M_1 M_2)
\]
where
\[
F[\chi_{b2} \rightarrow \eta_c \chi c_1] = 4 + 8r - 96r^2,
F[\chi_{b2} \rightarrow \chi c_0 \chi c_2] = \frac{16}{9} + \frac{872r}{9} + \frac{208r^2}{3} - \frac{544r^3}{9} + \frac{4288r^4}{9},
F[\chi_{b2} \rightarrow \eta_c \eta_c] = 1 - 8r + 16r^2,
F[\chi_{b2} \rightarrow \psi \psi] = 13 + 56r + 48r^2,
F[\chi_{b2} \rightarrow h_c h_c] = 16 - 132r + 488r^2 - 944r^3 + 800r^4,
F[\chi_{b2} \rightarrow \chi c_0 \chi c_0] = 4 - 16r - 16r^2 + 128r^3 + 256r^4,
F[\chi_{b2} \rightarrow \chi c_1 \chi c_1] = 7 - 44r - 30r^2 + 340r^3 + 264r^4,
F[\chi_{b2} \rightarrow \chi c_2 \chi c_2] = \frac{43}{9} + \frac{44r}{9} - \frac{286r^2}{9} + \frac{6212r^3}{9} + \frac{5032r^4}{9},
F[\chi_{b2} \rightarrow h_c h_c] = 24 - 72r - 96r^2,
F[\chi_{b2} \rightarrow \chi c_1 \chi c_2] = 6 + 4r^2 - 488r^3 + 1072r^4
\]

IV. NUMERICAL RESULTS.

In order to obtain numerical results from the presented above analytical expressions the following numerical parameters are needed.

In this paper we are going to use the models of the charmonia DAs proposed in papers [27, 28, 29, 30]. For the strong coupling constant we use one-loop expression
\[
\alpha_s(\mu) = \frac{4\pi}{\ln(\mu^2/\Lambda_{QCD}^2)},
\]
where $b_0 = 25/3$ and $\Lambda_{QCD} = 0.2$ GeV.

In the calculation the following values of the constants $f_i^{L,T}$ defined in equations (11), (12) will be used
\[
\begin{align*}
    f_{\eta_c}^L &= 0.373 \pm 0.064 \text{ GeV}, \\
    f_{J/\psi}^L &= 0.416 \pm 0.005 \text{ GeV}, \\
    f_{\psi(2S)}^L &= 0.261 \pm 0.077 \text{ GeV}, \\
    f_{\chi_{c1}}^L (M_{J/\psi}) &= 0.303 \pm 0.003 \text{ GeV}, \\
    f_{\chi_{c2}}^L (M_{J/\psi}) &= 0.093 \pm 0.017 \text{ GeV}, \\
    f_{h_c}^L (M_{J/\psi}) &= 0.160 \pm 0.015 \text{ GeV}, \\
    f_{\chi_{c1}}^T (M_{J/\psi}) &= 0.272 \pm 0.048 \text{ GeV}, \\
    f_{\chi_{c2}}^T (M_{J/\psi}) &= 0.131 \pm 0.023 \text{ GeV}.
\end{align*}
\]
The values of the constants \( f_{J/\psi}, f_{\psi(2S)} \) can be extracted from the leptonic decay widths of the \( J/\psi \) and \( \psi(2S) \) mesons. The values of the constants \( f_{Q_c}, f_{Q_b} \) were calculated in paper [20]. The values of the constants \( f_{J/\psi}, f_{\psi(2S)} \) can be found in paper [40]. The values of the constants of the \( P \)-wave charmonia mesons can be found in paper [30]. It should be noted that the constants \( f_{J/\psi}, f_{\psi(2S)} \), \( f_{Q_c}, f_{Q_b} \), \( f_{\chi_{b1}}, f_{\chi_{b2}} \) depend on the renormalization scale. As it is seen from formulas (19) these constants are defined at the scale \( \mu = M_{J/\psi} \). The anomalous dimensions of these constants, which govern the evolution, can be found in papers [30, 40].

The values of NRQCD matrix elements \( \langle O_{1}^{b(b)} \rangle, \langle O_{1}^{b(b)} \rangle \) can be expressed through the bottomonia radial wave function \( R(r) \) as follows

\[
\langle O_{1}^{b(b)} \rangle = \frac{3}{2\pi} |R_{S}(0)|^2, \quad \langle O_{1}^{b(b)} \rangle = \frac{9}{2\pi} |R_{P}(0)|^2.
\]

(20)

In this paper the values of the \( |R_{S}(0)|^2, |R_{P}(0)|^2 \) will be determined from the Buchmüller-Tye [41, 42] potential model. Thus one gets

\[
\langle O_{1}^{b(b)} \rangle = 3.1 \text{ GeV}^3, \quad \langle O_{1}^{b(b)} \rangle = 2.0 \text{ GeV}^5.
\]

In the forthcoming analysis we are not going to take into the account the uncertainties in the values of the NRQCD matrix elements for bottomonia mesons since these uncertainties are not very important.

In the calculation we also need the values of constants \( \langle 13 \rangle, \langle 16 \rangle \) for the charmonia mesons. At the leading order approximation in \( \alpha_s \) and relative velocity these constants can be determined from the leptonic decay widths

\[
\Gamma(V \rightarrow e^+e^-) = \frac{4\alpha_s^2\pi\alpha^2}{3M_V} [f_{S}^{\text{NRQCD}}]^2.
\]

(21)

Using experimental results \( \Gamma(J/\Psi \rightarrow e^+e^-) = 5.55 \text{ KeV}, \Gamma(\psi(2S) \rightarrow e^+e^-) = 2.48 \text{ KeV} \) one gets

\[
[f_{1S}^{\text{NRQCD}}]^2 = 0.17 \pm 0.06 \text{ GeV}^2, \quad [f_{2S}^{\text{NRQCD}}]^2 = 0.09 \pm 0.05 \text{ GeV}^2.
\]

(22)

To determine the constant \( f_P^{\text{NRQCD}} \) one can use the decay width \( \chi_{c0} \rightarrow \gamma \gamma \)

\[
\Gamma(\chi_{c0} \rightarrow \gamma \gamma) = \frac{12\alpha_s^2\pi\alpha^2}{M_{\chi_{c0}}} [f_P^{\text{NRQCD}}]^2.
\]

(23)

Using experimental results \( Br(\chi_{c0} \rightarrow \gamma \gamma) = 2.35 \times 10^{-4} \) one gets

\[
[f_P^{\text{NRQCD}}]^2 = 0.021 \pm 0.008 \text{ GeV}^2.
\]

(24)

The uncertainties in values (22), (23) were calculated as follows. In NRQCD there are relativistic and radiative corrections to formulas (21) and (22). The relativistic corrections can be estimated as \( \langle v^2 \rangle_{1S} = 0.21, \langle v^2 \rangle_{2S} = 0.54 \), \( \langle v^2 \rangle_p = 0.3 \). The radiative corrections can be estimated as \( \sim \alpha_s(M_{J/\psi}) = 0.25 \). Adding these uncertainties in quadrature one estimates the errors of the calculation.

The last parameters needed for calculation of the bottomonia decay widths are the pole masses of the \( b \)-quark and \( c \)-quarks. For the \( c \)-quark we take \( m_c = 1.4 \pm 0.2 \text{ GeV} \). For the mass of \( b \)-quark we take the value \( m_b = 4.8 \pm 0.1 \text{ GeV} \).

In Table [14] we present numerical results for the widths of exclusive bottomonia decays into pair of charmonia mesons. In the second and third columns of this table the results of NRQCD and light cone formalism are shown. In the fourth column we present the branching fractions of the considered decays. To estimate these fractions we use the following expressions for the total widths of bottomonia mesons [43]:

\[
\Gamma_{\eta_b} = \frac{4\pi\alpha_s^2}{9} \langle O_{2S}^{b(b)} \rangle \approx 11 \text{ MeV},
\]

(25)

\[
\Gamma_{\chi_{c0}} = \frac{3C_F}{N_c} \pi\alpha_s^2 \langle O_{2S}^{b(b)} \rangle \approx 0.8 \text{ MeV},
\]

(26)

\[
\Gamma_{\chi_{c2}} = \frac{4C_F}{5N_c} \pi\alpha_s^2 \langle O_{2S}^{b(b)} \rangle \approx 0.2 \text{ MeV},
\]

(27)

\[
\Gamma_{\chi_{b1}} = \frac{C_F\alpha_s^3}{N_c} \left[ \left( \frac{587}{54} - \frac{317}{288} \pi^2 \right) C_A + \left( -\frac{16}{27} - \frac{4}{9} \ln \frac{\Lambda}{2m_b} \right) \frac{\langle O_{2S}^{b(b)} \rangle}{m_b} + \frac{n_f \pi\alpha_s^2}{3} \frac{\langle O_{2S}^{b(b)} \rangle}{m_b^2} \right] \approx 1.6 \text{ MeV},
\]

(28)
where \( n_f = 4 \) is number of active flavors, \( \Lambda = 0.2 \text{ MeV} \), and \( \langle O_8 \rangle \approx 0.0021 \langle O_\beta^{(b)} \rangle \) is the color octet matrix element for the \( P \)-wave bottomonia mesons. Since experimentally charmonia mesons are observed in their decays into \( J/\psi \) meson, it is interesting to know the widths of such processes. It is clear that they are equal to

\[
\Gamma[(\bar{b}b) \rightarrow (c\bar{c})_1(c\bar{c})_2 \rightarrow J/\psi J/\psi + X] = \Gamma[(\bar{b}b) \rightarrow (c\bar{c})_1]Br[(c\bar{c})_1 \rightarrow J/\psi + X]Br[(c\bar{c})_2 \rightarrow J/\psi + X]
\]

These values are shown in the last column of Table [IV].

Now let us discuss the uncertainties of the calculation. Before we discuss how the uncertainties of the NRQCD prediction can be estimated, one should recall the experience gained from double charmonium production at B-factories. In this case the leading order NRQCD predictions [4, 5] are approximately by an order of magnitude less than experimental results [2, 3]. Note also that measured values of the cross sections are much larger than the leading order NRQCD predictions even if one takes into account the possible uncertainties of the approach [4]. From this fact one can conclude that it is rather difficult to calculate the uncertainties of NRQCD. So, the uncertainties calculated in this paper can be considered only as a very rough estimation of the real uncertainties.

Note also that in some decays (see section III), the amplitudes contain polynomials in \( r \) with alternating signs of the coefficients of these polynomials. One can expect that the uncertainty of the calculation of such decays can be very large. The decay \( \chi_{b0} \rightarrow \chi_{c0}\chi_{c2} \) can be considered as a dramatic demonstration of this point. The width of this decay as a function of the mass of \( c \)-quark is shown in Fig.2. It is seen from this figure that the width \( \chi_{b0} \rightarrow \chi_{c0}\chi_{c2} \) has zero and minimum at \( m_c \approx 1.56 \text{ GeV} \) which is very near to the pole mass of \( c \)-quark. Evidently, the uncertainty in the width of this decay can be very large.

As the uncertainty of the leading order NRQCD prediction we take the uncertainty which originates from the following sources: uncertainty in the pole masses of \( c \) and \( b \) quarks (the first error in the second column of Tab. [IV]), uncertainty due to the values of constants [22], [23] (the second error in Tab. [IV]), uncertainty due to the unknown radiative corrections, which can be estimated as \( \alpha_s(M_{\chi_{b0}})\log m_b^2/m_c^2 \sim 50\% \) (the third error in Tab. [IV]). The last uncertainty (the forth error in Tab. [IV]) originates from the uncertainty in the value of the \( \Lambda = 200 \pm 40 \text{ MeV} \) what corresponds to \( \alpha_s(m_\tau) = 0.34 \pm 0.03 \) [38].

The uncertainties of the results obtained within LC can be divided into the following groups:

1. The uncertainty in the models of the distribution amplitudes, which can be estimated through the variation of the parameters of these models (see papers [27, 28, 29, 30]) (the first error in the third column of Tab. [IV]).

2. The uncertainty in the models of the distribution amplitudes (the second error in the third column of Tab. [IV]).

3. The uncertainty due to the power corrections. This source of uncertainty is very important and for many processes this is the main source of the uncertainty. To estimate this source of the uncertainty we expand the leading order NRQCD results in the ratio \( r = m_b^2/m_c^2 \). The first term in this expansion is the leading twist contribution which is reproduced within LC. We take the next nonvanishing term in the \( r \) expansion as the estimation of the size of power corrections. (the third error in Tab. [IV]).

Applying this approach for the estimation of power corrections and looking to the leading order NRQCD results (section III) one can separate all processes into three groups. The first group contains the processes (for instance, the decays \( \eta_b \rightarrow b,\psi; \chi_{b0} \rightarrow \eta_c\chi_{c1} \)) for which power corrections, most probably, will not change the LC predictions dramatically. One can expect that for this group of the decays LC predictions are reliable. The second group contain the processes for which power corrections are of order of \( \sim 100\% \) (for instance, \( \eta_b \rightarrow \eta_c\chi_{c0}; \chi_{b0} \rightarrow \chi_{c1}\chi_{c1} \)). For such processes...
| $M_1 M_2$ | $\Gamma_{\text{NRQCD}}, \text{eV}$ | $\Gamma_{\text{LC}}, \text{eV}$ | $\text{Br}_{\text{LC}}, \%$ | $\text{Br}_{\text{LC}}(\psi')$, % |
|----------|-----------------------------|-----------------------------|----------------|-----------------------------|
| $\eta_b \to h_c J/\psi$ | $73^{+15}_{-11} \pm 0.38 \pm 0.37 \pm 0.29$ | $520 \pm 63 \pm 140 \pm 210$ | $0.0047$ | — |
| $\eta_b \to h_c \psi(2S)$ | $39^{+6.8}_{-7.0} \pm 0.26 \pm 0.19 \pm 0.15$ | $300 \pm 36 \pm 78 \pm 120$ | $0.0027$ | — |
| $\eta_b \to \eta_c \chi_c$ | $56^{+5.8}_{-4.2} \pm 0.29 \pm 0.28 \pm 0.22$ | $49 \pm 25 \pm 71 \pm 20$ | $4.5 \times 10^{-4}$ | — |
| $\eta_b \to \chi_c(2S)\chi_c$ | $30^{+3.7}_{-3.6} \pm 0.20 \pm 0.15 \pm 0.12$ | $26 \pm 18 \pm 37 \pm 10$ | $2.3 \times 10^{-4}$ | — |
| $\eta_b \to \eta_c \chi_c$ | $17^{+6.2}_{-5.3} \pm 0.89 \pm 0.86 \pm 0.69$ | $97 \pm 48 \pm 71 \pm 39$ | $8.9 \times 10^{-4}$ | — |
| $\eta_b \to \chi_c(2S)\chi_c$ | $9^{+1.4}_{-1.3} \pm 0.61 \pm 0.46 \pm 3.6$ | $51 \pm 35 \pm 37 \pm 20$ | $4.6 \times 10^{-4}$ | — |
| $\eta_b \to \chi_c(2S)\chi_c$ | $11^{+4.2}_{-5.4} \pm 0.84 \pm 0.55 \pm 4.4$ | $26 \pm 13 \pm 38 \pm 10$ | $2.4 \times 10^{-4}$ | $1.1 \times 10^{-6}$ |
| $\eta_b \to \chi_c(2S)\chi_c$ | $4^{+1.2}_{-1.3} \pm 0.36 \pm 2.4 \pm 1.9$ | $51 \pm 26 \pm 37 \pm 21$ | $4.7 \times 10^{-4}$ | $3.4 \times 10^{-5}$ |

**TABLE I**: The widths and branching fractions of the exclusive bottomonia decays into pair of charmonium mesons. In the second column the NRQCD predictions are presented. In the third and fourth the widths and branching fractions of the exclusive bottomonia decays in LC formalism are shown. The last column contains the branching fractions of inclusive $J/\psi$-pair production through intermediate charmonium states. The symbol "X" in this column means, that this decay is forbidden (for example, decay $\eta_b \to \eta_c \chi_c \to J/\psi J/\psi + X$ is absent since $\eta_c$ meson cannot decay into $J/\psi$). The symbol "--" (for example, in the case $\eta_b \to h_c J/\psi$) means that the branching fraction of the corresponding decay ($h_c \to J/\psi + X$ in this case) is unknown.
processes our results are valid up to the factor of $\sim 2$. The last group of processes are the processes for which power corrections are large (for instance, $\chi_{b0} \rightarrow \chi_{c0} \chi_{c0}, \chi_{b1} \rightarrow \chi_{c0} \chi_{c1}$). For these processes we can guess only the order of magnitude of the widths within LC.

4. The uncertainty due to the radiative corrections. The main part of the radiative corrections to the amplitude – the leading logarithmic radiative corrections have been resummed within LC. This fact allows us to estimate the rest of the radiative corrections as $\sim \alpha_s(E) \sim 20\%$. This is very small uncertainty, so we don’t show it in Tab. IV.

5. The uncertainty due to the variation of $\Lambda = 200 \pm 40$ MeV. (the forth error in Tab. IV). It should be noted also that in the calculation we took the scale of factorization $\mu = M_{\bar{b}b}/2$. However, one can take any scale $\mu \sim M_{\bar{b}b}$.

V. CONCLUSION.

In this paper the leading twist double charmonium production in exclusive bottomonia decays was considered. The decays of the C-odd bottomonia are suppressed by the factor $\sim \alpha_s/\pi$ in comparison to the decays of the C-even bottomonia. For this reason we considered only the leading twist decays of the C-even bottomonia. Applying light cone formalism with the models of the leading twist charmonia distribution amplitudes \cite{27,28,29,30} we calculated the amplitudes and the widths of the corresponding processes. In addition, we calculated the widths within the leading order NRQCD.

During the calculation we found that the production of the longitudinally polarized $\chi_{c1}$ meson and transversely polarized $h_c$ meson with $L_z \neq 0$ is nonzero already at the leading twist approximation. This fact tells us that the second selection rule (see section II), which predicts the suppression of such processes, is violated. We considered the mechanism which lies behind this violation and found that this violation results from the rather special Lorentz structure of the corresponding distribution amplitudes.

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APPENDIX A: DISTRIBUTION AMPLITUDES.

The leading twist distribution amplitudes needed in the calculation can be defined as follows:
for the pseudoscalar mesons \( P = \eta_c, \eta_c' \):

\[
\left\langle P(p) \left| \bar{Q}_a^i(z)[z, -z] Q_j^i(-z) \right| 0 \right. \right. = \left. \left. (i\gamma_5)_{\beta\alpha} \frac{f_{P}}{4} \frac{\delta_{ij}}{3} \int_{-1}^{1} d\xi e^{i\xi(pz)\phi_{P}(\xi; \mu)}, \right.ight.
\]
for the vector mesons \( V = J/\Psi, \psi' \):

\[
\left\langle V(p, \epsilon_{\lambda=0}) \left| \bar{Q}_a^i(z)[z, -z] Q_j^i(-z) \right| 0 \right. \right. = \left. \left. (\hat{p}\gamma_5)_{\beta\alpha} \frac{f_{V}}{4} \frac{\delta_{ij}}{3} \int_{-1}^{1} d\xi e^{i\xi(pz)\phi_{V}(\xi; \mu)}, \right.ight.
\]
\[
\left\langle V(p, \epsilon_{\lambda=\pm1}) \left| \bar{Q}_a^i(z)[z, -z] Q_j^i(-z) \right| 0 \right. \right. = \left. \left. (\hat{p}\epsilon)_{\beta\alpha} \frac{f_{V}}{4} \frac{\delta_{ij}}{3} \int_{-1}^{1} d\xi e^{i\xi(pz)\phi_{V}(\xi; \mu)}, \right.\right.
\]
for the \( \chi_{c0} \)-meson:

\[
\left\langle \chi_{c0}(p) \left| \bar{Q}_a^i(z)[z, -z] Q_j^i(-z) \right| 0 \right. \right. = \left. \left. (\hat{p})_{\beta\alpha} \frac{f_{\chi_{c0}}}{4} \frac{\delta_{ij}}{3} \int_{-1}^{1} d\xi e^{i\xi(pz)\phi_{\chi_{c0}}(\xi; \mu)}, \right.\right.
\]
for the \( \chi_{c1} \)-meson:

\[
\left\langle \chi_{c1}(p, \epsilon_{\lambda=0}) \left| \bar{Q}_a^i(z)[z, -z] Q_j^i(-z) \right| 0 \right. \right. = \left. \left. (\hat{p}\gamma_5)_{\beta\alpha} \frac{f_{\chi_{c1}}}{4} \frac{\delta_{ij}}{3} \int_{-1}^{1} d\xi e^{i\xi(pz)\phi_{\chi_{c1}}(\xi; \mu)}, \right.\right.
\]
\[
\left\langle \chi_{c1}(p, \epsilon_{\lambda=\pm1}) \left| \bar{Q}_a^i(z)[z, -z] Q_j^i(-z) \right| 0 \right. \right. = \left. \left. (\hat{p}\epsilon)_{\beta\alpha} \frac{f_{\chi_{c1}}}{4} \frac{\delta_{ij}}{3} \int_{-1}^{1} d\xi e^{i\xi(pz)\phi_{\chi_{c1}}(\xi; \mu)}, \right.\right.
\]
for the \( h_c \)-meson:

\[
\left\langle h_c(p, \epsilon_{\lambda=0}) \left| \bar{Q}_a^i(z)[z, -z] Q_j^i(-z) \right| 0 \right. \right. = \left. \left. (\hat{p}\gamma_5)_{\beta\alpha} \frac{f_{h_c}}{4} \frac{\delta_{ij}}{3} \int_{-1}^{1} d\xi e^{i\xi(pz)\phi_{h_c}(\xi; \mu)}, \right.\right.
\]
\[
\left\langle h_c(p, \epsilon_{\lambda=\pm1}) \left| Q(z)\sigma_{\mu\nu}[z, -z] Q(-z) \right| 0 \right. \right. = \left. \left. (\hat{p}\hat{p})_{\beta\alpha} \frac{f_{h_c}}{4} \frac{\delta_{ij}}{3} \int_{-1}^{1} d\xi e^{i\xi(pz)\phi_{h_c}(\xi; \mu)}, \right.\right.
\]
for the \( \chi_{c2} \)-meson:

\[
\left\langle \chi_{c2}(p, \epsilon_{\lambda=0}) \left| \bar{Q}_a^i(z)[z, -z] Q_j^i(-z) \right| 0 \right. \right. = \left. \left. (\hat{p})_{\beta\alpha} \frac{f_{\chi_{c2}}}{4} \frac{\delta_{ij}}{3} \int_{-1}^{1} d\xi e^{i\xi(pz)\phi_{\chi_{c2}}(\xi; \mu)}, \right.\right.
\]
\[
\left\langle \chi_{c2}(p, \epsilon_{\lambda=\pm1}) \left| \bar{Q}_a^i(z)[z, -z] Q_j^i(-z) \right| 0 \right. \right. = \left. \left. M_{\chi}(\hat{p}\hat{p})_{\beta\alpha} \frac{f_{\chi_{c2}}}{4} \frac{\delta_{ij}}{3} \int_{-1}^{1} d\xi e^{i\xi(pz)\phi_{\chi_{c2}}(\xi; \mu)}, \right.\right.
\]
(A1)

The factor \([z, -z], \) that makes matrix elements gauge invariant, is defined as

\[
[z, -z] = P \exp[i\gamma \int_{-z}^{z} dx^\mu A_\mu(x)]. \quad \text{(A2)}
\]
In the above equations $p$ is the charmonium momentum, $x$ and $\bar{z}$ are the momentum fractions of quark and antiquark, $\xi = x - \bar{z}$, $\epsilon_{\mu}$ is the polarization tensor for the $J/\psi$, $\chi_{c1}$ or $h_c$ mesons and the vector $\rho_{\mu}$ in relation (A1) is defined according to

$$\rho_{\mu} = \frac{\epsilon_{\mu\nu}z^{\nu}}{(pz)}.$$ 

where $\epsilon_{\mu\nu}$ is the polarization tensor of the $\chi_{c2}$ meson. In practical applications it is useful to write the polarization of the $\gamma_2$ meson in terms of the polarization of two vector mesons. Thus, for instance, the polarization tensor $\epsilon^{\mu\nu}$ of the transversely polarized $\chi_2$ meson can be written as $\epsilon^{\mu\nu}_{\chi_2} = (\epsilon^{\mu\nu}_{\chi_1} + \epsilon^{\mu\nu}_{\chi_2})/\sqrt{2}$. If we further contract the polarization tensor $\epsilon_{\mu\nu}$ with lightlike four-vector $z$, to the leading twist accuracy we get $\epsilon^{\mu\nu}z_{\nu} = \epsilon^{\mu}_{\chi_2}((pz)/(\sqrt{2}M_{\chi_2}))$ or $\rho_{\mu} = \epsilon^{\mu}_{\chi_2}((pz)/(\sqrt{2}M_{\chi_2}))$. This form of the vector $\rho$ can be used in the calculation with the leading twist accuracy.

It is not difficult to show that the functions $\phi_0(\xi)$, $\phi^L_T(\xi)$, $\phi^T_1(\xi)$ and $\phi^L_{T\chi}(\xi)$ are $\xi$-even. The normalization condition for these functions is

$$\int_{-1}^{1} \phi(\xi) d\xi = 1. \quad (A3)$$

The functions $\phi_{\chi_0}(\xi)$, $\phi^T_1(\xi)$, $\phi^T_1(\xi)$ and $\phi^L_T(\xi)$ are $\xi$-odd and normalized according to

$$\int_{-1}^{1} \xi\phi(\xi) d\xi = 1.$$
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