Stochastic Time-Periodic Tonelli Lagrangian on Compact Manifold

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Abstract

In this paper, we study a class of time-periodic stochastic Tonelli Lagrangians on compact manifolds. Precisely, we discuss the stochastic Mane Critical Value, prove the existence of stochastic Weak KAM solutions of the related Hamilton-Jacobi equation. Furthermore, we survey the global minimizer.

1 Introduction

Time-periodic Tonelli Lagrangians have been extensively studied in recent years by John Mather [1], Ricardo Mane [2] [3], Patrick Bernard [4] [5], Daniel Massart [6], etc. It is closely related to calculus of variations, the weak KAM theory [7], Aubry-Mather theory and Optimal Transport[8]. In this paper, we study the time-periodic Tonelli Lagrangian by adding a stochastic variable. We discuss the measurability of Mane-Critical Value, prove the existence of stochastic viscosity solution for a system of Hamilton-Jacobi equations, and describe the global minimizer.

Basic Setting: Let $M$ be a compact, connected Riemannian manifold, $TM$ its tangent bundle. $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\mathcal{F}$ is the $\sigma-$algebra of $\Omega$. $\theta : R \times \Omega \to \Omega$ is a measurable function with skew-product structure. Let $\mathcal{B}$ be the Borel algebra of $\mathbb{R}$. Then the $\sigma-$algebra of $\mathbb{R} \times \Omega$ is given by $\mathcal{B} \otimes \mathcal{F}$. The stochastic time-periodic lagrangian $L : TM \times T \times \Omega \to R$, is a measurable function. For each $\omega \in \Omega$, $L(\cdot, \cdot, \cdot, \omega) : TM \times T \to R$ is a Tonelli Lagrangian, and for each $(x, v, t) \in TM \times T$, $L(x, v, t, \cdot) : \Omega \to R$ is a measurable function. For each $(x, v, t, \omega) \in TM \times T \times \Omega$, $s \in R$, we assume

$$L(x, v, t + s, \omega) = L(x, v, t, \theta(s, \omega))$$

Let $-\infty < s < t < \infty$, $x, y \in M$, $\omega \in \Omega$. $AC(s, x; t, y)$ is the space of absolutely continuous curves in $R \times M$, connecting $(s, x)$ and $(t, y)$ The stochastic Lagrangian action $A^\omega(s, x; t, y)$ is

$$A^\omega(s, x; t, y) = \inf_{\gamma \in AC(s, x; t, y)} \int_s^t L(\gamma(\sigma), \dot{\gamma}(\sigma), \sigma, \omega)d\sigma$$

Here, we study the following topics:
1. Mane-Critical Value

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2. The existence of stochastic viscosity solution for a stochastic system of Hamilton-Jacobi equations.

3. Global minimizer.

For each \( \omega \in \Omega \), \( \Phi^\omega \) is the set of invariant measure of Hamiltonian flow in \( TM \times T \) for the Tonelli Lagrangian \( L(\cdot, \cdot, \cdot, \omega) \). The Mane-Critical Value for \( L(\cdot, \cdot, \cdot, \omega) \), is

\[
\alpha(\omega) = \inf_{\mu \in \Phi^\omega} \int_{TM \times T} L(x, v, t, \omega) d\mu
\]

The first main result is:

**Theorem 1.1.** The function \( \alpha(\omega) \) is a measurable function on \( \Omega \). In particular, if \( \{\theta(t) : \Omega \to \Omega | t \in \mathbb{R}\} \) is ergodic, \( \alpha(\omega) \) is constant almost everywhere on \( \Omega \).

Now, we define the Lax-Oleinik Operator, for \( u : M \to \mathbb{R} \), \((x, t) \in TM \times T\), \( \lambda \in \mathbb{R} \), \( \omega \in \Omega \),

\[
T_\lambda u(x, t) = \min_{y \in M} \{u(y) + A^\omega(t - \lambda, y; t, x) + \lambda \alpha(\omega)\}
\]

The Hamiltonian, \( H : T^*M \times T \times \Omega \to \mathbb{R} \), is defined as,

\[
H(x, p, t, \omega) = \sup_{v \in T_x M} \{p \cdot v - L(x, v, t, \omega)\}
\]

The second main result is a weak KAM-type theorem,

**Theorem 1.2.** For each \( u : M \to \mathbb{R} \), \((x, t) \in TM \times T \), define:

\[
u^\omega(x, t) = \lim_{\lambda \to +\infty} T_\lambda^\omega(u)(x, t)
\]

Then, the following holds true:

(i) For all \( \omega \in \Omega \), \( u^\omega(x, t) \) is a viscosity solution of the Hamilton-Jacobi equation:

\[
\partial_t u(x, t) + H(x, \partial_x u(x, t), t, \omega) = \alpha(\omega)
\]

(ii) For all \((x, t) \in M \times R \), \( u^\omega(x, t) \) are measurable functions on \( \Omega \).

(iii) For all \( x \in M \), \( s, t \in R \), \( \omega \in \Omega \), we have

\[
u^\omega(x, t + 1) = u^\omega(x, t) \quad \text{and} \quad u^{\theta(s)\omega}(x, t) = u^\omega(x, t + s)
\]

In this paper, we define the global minimizer in this way: \( \gamma^\omega : (-\infty, \infty) \to M \) is a global minimizer if and only if for any \( s < t \), \( \gamma^\omega|[s, t] \) is minimizer for the Action \( A^\omega(x, s; y, t) \).

We prove that the set of global minimizer is nonempty for all \( \omega \in \Omega \). Besides, under some situations, we can know the structure of ergodic invariant measures on the space of global minimizers.

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2 Stochastic Time-Periodic Tonelli Lagrangian

In this section, we recall some basic notions, give the definition of our investigated objects and examples. Throughout this paper, $(M, g)$ is a compact connected Riemannian manifold and $TM$ is its tangent bundle. John Mather originally considered the Time-Periodic Tonelli Lagrangian.

Definition 2.1 (Time-Periodic Tonelli Lagrangian). On $M$, a $C^2$ map $L : TM \times R \to R$ is a Time-Periodic Tonelli Lagrangian if $L$ satisfies:

(I) Periodicity: $L(x, v, t + 1) = L(x, v, t)$, $\forall x \in M, v \in T_x M$, $t \in R$.

(II) Convexity: For all $x \in M, v \in T_x M$, the Hessian matrix $\frac{\partial^2 L}{\partial v_i \partial v_j}(x, v, t)$ is positive definite.

(III) Superlinearity: $\lim_{||v|| \to +\infty} \frac{L(x, v, t)}{||v||} = +\infty$ uniformly on $x \in M, t \in R$.

(IV) Completeness: The maximal solutions of the Euler-Lagrange, that in local coordinates is:

$$\frac{d}{dt} \frac{\partial L}{\partial v}(x, \dot{x}, t) = \frac{\partial L}{\partial x}(x, \dot{x}, t)$$

are defined on all $R$.

Following this definition from John Mather, we introduce the definition of Stochastic Time-Periodic Tonelli Lagrangian. $(\Omega, \mathcal{F}, P)$ is a probability space, $\mathcal{F}$ is the $\sigma-$algebra of $\Omega$.

Definition 2.2 (Stochastic Time-Periodic Tonelli Lagrangian). On $M$, a map $L : TM \times T \times \Omega \to R$ is a measurable function. It is called Stochastic Time-Periodic Tonelli Lagrangian if $L$ satisfies:

(I) Fix each $\omega \in \Omega$, $L(\cdot, \cdot, \cdot, \Omega)$ is a Time-Periodic Tonelli Lagrangian on $TM \times T$.

(II) Fix each $(x, v, t) \in TM \times T$, $L(x, v, t, \cdot)$ is a measurable function on $\Omega$.

Next, we introduce a skew-product dynamical system. Let $\mathcal{B}$ be the Borel algebra of $\mathbb{R}$. Then the $\sigma-$algebra of $\mathbb{R} \times \Omega$ is given by $\mathcal{B} \otimes \mathcal{F}$.

Definition 2.3. A skew-product dynamical system is a measurable map $\theta : R \times \Omega \to \Omega$, satisfying

(I) $\forall x \in \Omega, \forall s, t \in R$, $\theta(0, x) = x$, $\theta(s, \theta(t, x)) = \theta(s + t, x)$.

(II) Fix $t \in R$, $\theta(t) : \Omega \to \Omega$ defined as $\theta(t)(x) = \theta(t, x)$, is measure-preserving, i.e $\forall E \in \mathcal{F}, P(\theta(t)^{-1}(E)) = P(E)$.

From Definition 2.3, $\{\theta(t)\}_{t \in \mathbb{R}}$ is a group of measure-preserving endomorphism of $\Omega$. In this paper, we consider the a class of Stochastic Time-Periodic Lagrangians which match the skew-product dynamical system, i.e. satisfying the following assumption:

Assumption 2.1. $L : TM \times T \times \Omega \to R$ is a Stochastic Time-Periodic Tonelli Lagrangian, we assume that $L$ matches the skew-product dynamical system, if for $(x, v) \in TM$, $s \in R$, $\omega \in \Omega$, we have

$$L(x, v, t + s, \omega) = L(x, v, t, \theta(s)\omega)$$

The next two examples are constructed to show existence of Lagrangians which satisfy Assumption 2.1.
**Example 1.** $\Omega = [0, 1]$ is a probability space with Lebesgue measure. $f : [0, 1] \to [0, 1]$ is a one-to-one measurable function. Precisely,

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{3}, \\ x + \frac{1}{3} & \text{if } \frac{1}{3} \leq x < \frac{2}{3}, \\ x - \frac{1}{3} & \text{if } \frac{2}{3} \leq x < 1. \end{cases}$$

$\theta : R \times \Omega \to \Omega$, $t \in R$, is defined by

$$\theta(t, \omega) = f^{-1}(t + f(\omega))$$

We see that $\theta : R \times \Omega \to \Omega$ is a skew-product dynamical system. Then define $L : T M \times T \times \Omega \to R$ by

$$L(x, v, t, \omega) = \frac{1}{2} g_x(v, v) + h(t + f(\omega))$$

where $g_x$ is a Riemannian metric over $M$, $h$ is a periodic function from $R$ to $R$, with period 1, $C^2$ regularity. $L$ is a Stochastic Time-Periodic Tonelli Lagrangian which satisfies Assumption 2.1.

**Example 2.** $\Omega = T^d$ is a probability space with Lebesgue Measure. $T^d$ can be decomposed as

$$T^d = \prod_{k=1}^d \left( [0, \frac{1}{n}) \bigcup \left[ \frac{1}{n}, \frac{2}{n} \right) \ldots \bigcup \left[ \frac{n-1}{n}, 1 \right) \right)$$

Assume that $f$ is a permutation of these $n^d$ cubics by linear maps, so $f$ is a one-to-one measurable map from $T^d$ to $T^d$. Choose $(1, \alpha_2, \ldots, \alpha_d) \in R^d$ as a rationally independent vectors, i.e, there is no nonzero vector $(k_1, k_2, \ldots, k_d) \in Z^d$ such that $k_1 + k_2 \alpha_2 + k_3 \alpha_3 + \cdots + k_d \alpha_d = 0$.

Define $\phi(t) : \Omega \to \Omega$ for $\forall x \in T^d$, $x = (x_1, \ldots, x_d) \in R^d$, $t \in R$, by

$$\phi(t)(x_1, \ldots, x_d) = (x_1 + t, x_2 + \alpha_2 t, \ldots, x_d + \alpha_d t) (\text{mod}Z^d)$$

For $t \in R$, $\phi(t)$ is ergodic over $\Omega$ with stationary property: $\phi(t) \cdot \phi(s) = \phi(t + s)$.

Define the skew-product dynamical system $\theta : R \times \Omega \to \Omega$, for each $t \in R$, each $\omega \in \Omega$,

$$\theta(t, x) = f^{-1}(\phi(t) \cdot (f(x)))$$

In particular, $\{\theta(t) | t \in R\}$ is ergodic on $\Omega$.

Define $L : T M \times T \times \Omega$ as follows:

$$L(x, v, t, \omega) = \frac{1}{2} g_x(v, v) + h(t + \pi(\omega))$$

where $\pi : T^d \to R$ defined as $\pi(x_1, x_2, \ldots, x_d) = x_1$, $h$ is a 1-periodic function on $R$ and has regularity $C^2$, $g_x$ is the Riemannian Metric over $M$. Hence, $L$ is a Stochastic Time-Periodic Tonelli Lagrangian and satisfies the Assumption 2.1. For the translation on Torus, see [9].

**Remark 2.1.** In Example 1, Example 2, the one-to-one measurable map $f$ is not unique.
3 Probability Measures on Metric Spaces

In this section, we introduce some tools from Probability Theory. 

\((X, d)\) is a metric space. Consider the functional space. \(C_b(X) = \{f : X \to R : f \text{ is continuous and bounded}\}\). Each \(f \in C_b(X)\) is integrable with respect to any finite Borel measure on \(X\). We introduce the notion of weak convergence of probability measure on \(X\).

**Definition 3.1.** \(\mu, \mu_1, \mu_2, \ldots\) are finite Borel measures on \(X\). We say that \((\mu_i)\) converges weakly to \(\mu\), if for all \(f \in C_b(X)\), we have
\[
\lim_{i \to \infty} \int_X f \, d\mu_i \to \int_X f \, d\mu.
\]

we denote \(\mu_i \rightharpoonup \mu\).

Next, we discuss the Prokhorov metric on \((X, d)\). Denote \(\mathcal{P} = \{\mu | \mu \text{ is a probability measure on } X\}\). \(\mathcal{B}(X)\) is the set of all Borel algebra generated by open sets on \(X\). The Prokhorov metric arises from the distance on \(\mathcal{P}\), defined as

**Definition 3.2.** For \(\mu, \nu \in \mathcal{P}\), \(d_P(\mu, \nu)\) is defined by
\[
d_P(\mu, \nu) = \inf\{\alpha > 0 : \mu(A) \leq \nu(A) + \alpha, \nu(A) \leq \mu(A) + \alpha, \forall A \in \mathcal{B}(X)\}
\]

where \(A_\alpha = \{x \in X : d(x, A) < \alpha\}\) if \(A \neq \emptyset\), \(\emptyset_\alpha := \emptyset\) for all \(\alpha > 0\).

Then we have:

**Lemma 3.1.** \((X, d)\) is the metric space, \(d_P\) defined above,

1. \(d_P\) is a metric on \(\mathcal{P} = \mathcal{P}(X)\).
2. If \(\mu, \mu_1, \mu_2, \ldots \in \mathcal{P}\), \(\lim_{i \to \infty} d_P(\mu_i, \mu) = 0\) implies \(\mu_i \rightharpoonup \mu\).

**Lemma 3.2.** If \((X, d)\) is a separable metric space, then for any \(\mu, \mu_1, \mu_2, \ldots \in \mathcal{P}(X)\) one has \(\mu_i \rightharpoonup \mu\) if and only if \(\lim_{i \to \infty} d_P(\mu, \mu_i) = 0\).

**Lemma 3.3.** \((X, d)\) is a separable metric space, then \(\mathcal{P} = \mathcal{P}(X)\) with the Prokhorov metric \(d_P\) is separable.

**Lemma 3.4.** \((X, d)\) is a separable complete metric space, then \(\mathcal{P} = \mathcal{P}(X)\) with the Prokhorov metric \(d_P\) is complete.

The proof from Lemma 3.1 to Lemma 3.4 can be checked in [10].

**Theorem 3.1.** Let \(X\) and \(Y\) be two Polish spaces and \(\lambda\) be a Borel probability measure on \(X \times Y\). Let us set \(\mu = \pi_X \lambda\), where \(\pi_X\) is the standard projection from \(X \times Y\) onto \(X\). Then there exists a \(\mu\)–almost everywhere uniquely determined family of Borel probability measures (\(\lambda_x\)) on \(Y\) such that

1. The function \(x \to \lambda_x\) is Borel measurable, in the sense that \(x \to \lambda_x(B)\) is a Borel-measurable function for each Borel-measurable set \(B \in Y\).
2. For every Borel-measurable function \(f : X \times Y \to [0, \infty)\),
\[
\int_{X \times Y} f(x, y) d\lambda(x, y) = \int_X \int_Y f(x, y) d\lambda_x(y) d\nu(x)
\]

The disintegration of measure can be seen in [11].
4 Mane-Critical Value

Given a Time-Periodic Tonelli Lagrangian $L: TM \times T \to R$, by Euler-Lagrange equation, i.e,

$$\frac{d}{dt} \frac{dL}{dx}(x, \dot{x}, t) = \frac{dL}{dx}(x, \dot{x}, t)$$

we can define a time-dependent Lagrangian flow $\Phi_{s, t} : TM \times \{s\} \to TM \times \{t\}$ see [1].

Denote: $M_{inv} = \{\mu : \mu$ is a Borel probability over $TM \times T, \mu$ is invariant under the flow $\Phi_{s, t}, \forall s, t \in R\}$, the Mane-critical value of $L$ is defined as

$$-c[0] = \min_{\mu \in M_{inv}} \int_{TM \times T} L(x, v, t) \, d\mu$$

(1)

The corresponding Hamiltonian $H : T^*M \times T \to R$ is defined by

$$H(x, p, t) = \sup_{v \in TM} p(v) - L(x, v, t)$$

Considering the Hamilton-Jacobi equation, $u : M \times T \to R$.

$$\partial_t u(x, t) + H(x, \partial_x u(x, t), t) = c[0]$$

(2)

If $u$ is a subsolution of (2), that means for each $(x_0, t_0) \in M \times T$, there exists a $C^1$ function $\phi : M \times T \to R, \phi \geq u, \phi(x_0, t_0) = u(x_0, t_0)$, we have

$$\partial_t \phi(x_0, t_0) + H(x_0, \partial_x \phi(x_0, t_0), t_0) \leq c[0]$$

For all $n \in N, h_n : (M \times T) \times (M \times T) \to R$ is defined as:

$$h_n((x, t), (y, s)) = \min_{\gamma \in \Sigma(x, t; y, s+n)} \int_t^{s+n} L(\gamma, \dot{\gamma}, t) dt + nc[0]$$

The Peierls barrier is then defined as: for $(x, t) \times (y, s) \in (M \times T) \times (M \times T)$:

$$h((x, t), (y, s)) = \lim_{n \to \infty} h_n((x, t), (y, s))$$

The Projected Aubry set is

$$\mathcal{A}_0 := \{(x, t) \in M \times T : h((x, t), (x, t) = 0)\}$$

In [6], Daniel Massart proved a useful theorem that we desire, we present it here:

**Theorem 4.1.** There exists a $C^1$ critical subsolution of Hamilton-Jacobi equation which is strict at every point of $\mathcal{A}_0^c$.

To prove theorem 1.1, we introduce the notion of closed measure.

**Definition 4.1.** A probability measure $\mu$ on $TM \times T$ is called closed if

$$\int_{TM \times T} |v| \, d\mu(x, v, t) < \infty$$

and for every smooth function $f$ on $TM \times T$, we have

$$\int_{TM \times T} df(x, t)(v, 1) \, d\mu(x, v, t) = 0$$

We denote the set of closed measures on $TM \times T$ as $\mathcal{M}_c$. 
In [6], Daniel Massart gives a desired theorem as follows:

**Theorem 4.2.** For a Time-Periodic Tonelli Lagrangian $L : TM \times T \rightarrow R$, its Mane-Critical Value can be formulated as

$$-c[0] = \min_{\mu \in \mathcal{M}_f} \int_{TM \times T} L(x, v, t) \, d\mu$$

(3)

**Lemma 4.1.** For a Time-Periodic Tonelli Lagrangian $L : TM \times T \rightarrow R$, if one measure $\mu \in P(TM)$ satisfies that

$$-c[0] = \int_{TM \times T} L(x, v, t) \, d\mu$$

$\text{supp}(\mu)$ is a compact subset in $TM \times T$.

**Proof.** For each closed measure $\mu$, if $f : M \times T \rightarrow R$ is a smooth function, we have

$$\int_{TM \times T} df(x, t)(v, 1) \, d\mu(x, v, t) = 0$$

If $g : M \times T \rightarrow R$ is $C^1$, we can approximate it in the uniform $C^1$ topology by a sequence of $C^\infty$ functions $f_n : M \times T \rightarrow R$. In particular, there is a constant $K < \infty$, for each $x \in M$, and $n \in N$, we have $|df(x, t) \cdot (v, 1)| \leq K(||v|| + 1)$. Since $\int_{TM \times T} ||v|| \, d\mu(x, v, t) < \infty$, $df_n(x, t)(v, 1) \rightarrow dg(x, t)(v, 1)$ by the dominated convergence theorem, we obtain that $\int_{TM \times T} dg(x, t)(v, 1) \, d\mu(x, v, t) = 0$.

Suppose that a closed measure $\mu$ does satisfy that $\int_{TM \times T} L(x, v, t) \, d\mu = -c[0]$, From Theorem 4.1, we know that there exists $u : M \times T \rightarrow R$ as a $C^1$ critical subsolution, such that for $x \in A_c^0$, we have

$$\partial_t u(x, t) + H(x, \partial_x u(x, t), t) < c[0]$$

We integrate the following equation:

$$\partial_x u(x, t) \cdot v + \partial_t u(x, t) \leq L(x, v, t) + H(x, \partial_x u(x, t), t) + \partial_t u(x, t) \leq L(x, v, t) + c[0]$$

Then we get

$$0 \leq \int_{TM \times T} L(x, v, t) + H(x, d_x u, t) \, d\mu \leq 0$$

So we know that if $(x, v) \in \text{supp}(\mu)$, we have

$$\partial_t u(x, t) + H(x, \partial_x u(x, t), t) = c[0]$$

$$\partial_x u(x, t)(v) = L(x, v, t) + H(x, \partial_x u(x, t), t)$$

So we know that $x \in A_0$, $\partial_x u(x, t) = \frac{\partial L}{\partial v}(x, v, t)$, and $v = \frac{\partial H}{\partial p}(x, \partial_x u(x, t), t)$. Hence we conclude that $\text{supp}(\mu)$ is compact.

Coming back to a Stochastic Time-Periodic Tonelli Lagrangian $L : TM \times T \times \Omega \rightarrow R$, for each $\omega \rightarrow \Omega$, $L(\cdot, \cdot, \cdot, \omega)$ is a Time-Periodic Tonelli Lagrangian. So $L(\cdot, \cdot, \cdot, \omega)$ has a Mane-Critical Value, we denote it as $\alpha(\omega)$.  

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Lemma 4.2. Given a compact connected Riemannian manifold \((M, g)\), the manifold \(M\), its tangent bundle \(TM\) and \(TM \times T\) are separable, complete metric spaces.

Proof. First proof of Theorem 1.1

Since \(M\) is a compact manifold, we can find a finite number of charts \(\{(U_i, \phi_i)|1 \leq i \leq N\}\) to cover \(M\). For \(1 \leq i \leq N\), \(U_i\) is isomorphic to an open subset of \(\mathbb{R}^n\), so \(U_i\) is a separable open set for \(1 \leq i \leq N\). Hence, \(M = \bigcup_{1 \leq i \leq N} U_i\) is also separable.

To prove \(TM\) is a metric space, we define a distance on \(M\). For any \(x, y \in M\), \(\Sigma(x, y) = \{\gamma|\gamma : [0, 1] \to M, \gamma\text{ is absolutely continuous on } M\}\). We denote the distance between \(x\) and \(y\) as \(d(x, y)\), defined by

\[
    d(x, y) = \inf_{\gamma \in \Sigma(x, y)} \int_0^1 g_\gamma(t) \left\| \dot{\gamma}(t) \right\| dt
\]

\(M\) is connected, any two different points can be connected by an absolutely continuous curve. \(d_M(x, y)\) defines a metric on \(M\). (see [12]). Since any compact metric space is complete, we can know that \(M\) is complete.

To prove \(TM\) is a metric space, firstly we prove \(TM\) is a Riemannian manifold.

We construct the Riemannian metric locally on \(TM\). To prove \(TM\) is a Riemannian manifold, we denote the metric by \(d\). By viewing it as a connected Riemannian Manifold, we denote the metric by \(d\).

First proof of Theorem 1.1

From Lemma 4.1, \(\Sigma\) is a closed measure. \((\pi, \pi)\) is a Time-Periodic Tonelli Lagrangian, \((\pi)\) is complete. \((\pi)\) is a complete separable metric space. Since \((\pi)\) is complete, any two different points can be connected by an absolutely continuous curve. \(d_M(x, y)\) defines a metric on \(M\). (see [12]). From Lemma 6.1, we know that the support
of \( \mu \) is compact. There exists \( R(\omega) > 0 \) such that if \((x, v) \in \text{supp}(\mu), \) we have \( g_x(v) \leq R(\omega) \). We can find a smooth function \( \chi_\omega : [0, \infty) \to [0, 1] \) such that \( \chi(x) = 1 \) when \( 0 \leq x \leq R(\omega) \); \( \chi(x) = 0 \) when \( x \geq R(\omega) + 1 \). We have

\[
-\alpha(\omega) = \inf_{\mu \in \mathcal{T}_M \times T} \int_{\mu \in \mathcal{M}} L(x, v, t, \omega) \chi_\omega(g_x(v)) d\mu(x, v, t)
\]

Since \( L(x, v, t, \omega) \chi_\omega(g_x(v)) \) is a continuous bounded function, so we can apply weak convergence of probability measure on \( L(x, v, t, \omega) \chi_\omega(g_x(v)) \). Since \( \{\mu_k| k = 1, 2, \ldots\} \) is dense on \( \mathcal{P}(\mathcal{T}_M \times T) \), we know that

\[
-\alpha(\omega) = \inf_{k \in \mathbb{N}} \int_{\mathcal{T}_M \times T} L(x, v, t, \omega) \chi_\omega(g_x(v)) d\mu_k = \inf_{k \in \mathbb{N}} \int_{\mathcal{T}_M \times T} L(x, v, t, \omega) d\mu_k
\]

By Fubini theorem (see [14]), for each closed measure \( \mu \), we know that \( \int_{\mathcal{T}_M \times T} L(x, v, t, \omega) d\mu \) is measurable function on \( \Omega \). Since the infimum of a countable measurable function is measurable on \( \Omega \), \( \alpha(\omega) \) is measurable. Since \( L(x, v, t, \theta(s)\omega) = L(x, v, t + s, \omega) \) for \( s \in R \), we know that \( \alpha(\theta(s)\omega) = \alpha(\omega) \). If \( \{\theta(s), s \in R\} \) is ergodic on \( \Omega \), we know that \( \alpha(\omega) \) is constant almost everywhere. This finishes the proof of Theorem 1.1.

The author has an another simple proof, we take advantage of a useful result from [15]

**Proposition 4.1.** If \( L : \mathcal{T}_M \times T \to R \) is a Tonelli Lagrangian, the Mane-Critical Value has another interpretation:

\( \alpha(0) = \min\{k : \int L + k \geq 0 \text{ for all closed curves } \gamma\} \)

A curve \( \gamma : [a, b] \to M \) is called closed if \( \gamma(a) = \gamma(b) \) and \( b - a \) is an integer.

**Proof.** second proof of Theorem 1.1

For integers \( m < n \), Let \( C^0([m, n], TM) \) denote the space of continuous functions from \([m, n]\) to \( TM \) with the bounded uniform norm. \( M \) is a compact manifold, by Whitney’s theorem, there exists \( m \in N \), such that \( M \) can be embedded isomorphically into \( R^m \). Therefore, \( TM \) can be embedded as a submanifold of \( R^{2m} \). By Stone-Weierstrass theorem, the space of continuous functions from \([m, n]\) to \( R \) is separable. Therefore, the space of continuous functions from \([m, n]\) to \( R^{2m} \) is separable. As its subset, \( C^0([m, n], TM) \) is separable. So \( \bigcup_{m<n} C^0([m, n], TM) \) is separable. For a Tonelli Lagrangian, the closed curve which integral of Lagrangian attains the minimum along it, satisfies the Euler-Lagrange equation, and is \( C^2 \) and compact. So the space of curves, along which Lagrangian Action achieves the minimum, is a subspace of \( \bigcup_{m<n} C^0([m, n], TM) \). So we can find a sequence of closed curves \( \{\gamma_i\} \) for \( i \in N \), such that

\[
\alpha(\omega) = \inf_{i \in N} \int_{\gamma_i} L(\gamma_i, t, t, \omega) dt
\]

By Fubini theorem, for each \( i \in N \), \( \int_{\gamma_i} L(\gamma_i(t), \gamma_i(t), t, \omega) dt \) is a measurable function. As the infimum of countable measurable functions, we know that \( \alpha(\omega) \) is a measurable function. Since \( L(x, v, t, \theta(s)\omega) = L(x, v, t + s, \omega) \), for \( s \in R \), we get \( \alpha(\theta(s)\omega) = \alpha(\omega) \). If \( \{\theta(s), s \in R\} \) is ergodic on \( \Omega \), we know that \( \alpha(\omega) \) is constant almost everywhere. This finishes the proof of Theorem 1.1. 

\( \square \)
5 Semiconcave Estimates on Lagrangian Action

In this section we have a revision of some basic properties of a class of nonsmooth functions, the so-called semiconcave functions. The good properties of semiconcave functions provide fundamental technical tools for the analysis of singularities of Lagrangian Action and Weak KAM Solutions.

It is well known that a real-valued function $u$ is semiconcave in an open domain $U \subset \mathbb{R}^n$ if, for any compact set $K \subset U$, there exists a constant $C \in \mathbb{R}$ such that

$$tu(x_1) + (1-t)u(x_0) - u(tx_1 + (1-t)x_0) \leq Ct(1-t)|x_1 - x_0|^2$$

for all $t \in [0, 1]$ and for all $x_0, x_1 \in K$ satisfying $[x_0, x_1] \subset K$. We refer to such a constant $C$ as the semiconcavity constant for $u$ on $K$. We denote by $SC(U)$ the class of all semiconcave functions in $U$.

We review some differentiability properties of semiconcave functions. To begin, let us recall that any $u \in SC(U)$ is locally Lipschitz continuous (see [16]). Hence, by Rademacher’s Theorem, $u$ is differentiable a.e in $U$ and the gradient of $u$ is locally bounded. Then, the set

$$D_x^+ = \{p \in \mathbb{R}^n : U \ni x \to x, Du(x) \to p\}$$

is nonempty for any $x \in U$. The elements of $D_x^+u(x)$ are called reachable gradients.

The superdifferential of any function $u : U \to \mathbb{R}$ at a point $x \in U$ is defined as

$$D_x^+u = \{p \in \mathbb{R}^n : \limsup_{h \to 0} \frac{u(x+h) - u(x) - \langle p, h \rangle}{|h|} \leq 0\}$$

Similarly, the subdifferential of $u$ at $x$ is given by

$$D_x^-u = \{p \in \mathbb{R}^n : \liminf_{h \to 0} \frac{u(x+h) - u(x) - \langle p, h \rangle}{|h|} \geq 0\}$$

Next, we list some properties:

**Proposition 5.1.** Let $u : A \to \mathbb{R}$ and $x \in A$. Then the following properties hold true.

1. $D_x^+u$ and $D_x^-u$ are closed convex sets.
2. $D_x^+u$ and $D_x^-u$ are both nonempty if and only if $u$ is differentiable at $x$; in this case we have that

$$D_x^+u = D_x^-u = \{Du(x)\}$$

Furthermore, when $u : U \to \mathbb{R}$ be a semiconcave function, we have

3. $D_x^+u = \text{cov}D_x^-u$
4. $D_x^+u \neq \emptyset$
5. When $D_x^+u$ is a singleton, $u$ is differentiable at $x$.

The proof of Proposition 5.1 can be seen in [16].

Given $L : TM \times T \to \mathbb{R}$ a Time-Periodic Tonelli Lagrangian. For an absolutely continuous curves $\gamma : [s, t] \to M$, the action of $L$ along $\gamma$ is defined as

$$A(\gamma) = \int_s^t L(\gamma(u), \dot{\gamma}(u), u)du.$$ 

$\Sigma(s, y; t, x) = \{\gamma: \gamma : [s, t] \to M$ is absolutely continuous , and $\gamma(s) = y, \gamma(t) = x\}$. The Lagrangian action $A(s, y; t, x)$ is defined as

$$A(s, y; t, x) = \min_{\gamma \in \Sigma(s, y; t, x)} \int_s^t L(\gamma(u), \dot{\gamma}(u), u)du$$
If $\gamma : [s, t] \to M$ attains the minimum of $A(s, y; t, x)$, then from variational methods, we know that $\gamma$ satisfies Euler-Lagrange equation and $\gamma$ is $C^2$.

**Lemma 5.1.** Fix $s_1 < t_1$, for any $s \leq s_1, t \geq t_1$, the lagrangian Action $A(s, \cdot; t, \cdot)$ is equi-semiconcave on $M \times M$, therefore equi-Lipschitz.

**Proof.** To give a proof, we use the variational methods.

Fix $\forall x \in M$, we can find a chart such that $x \in U \subset M$. Without loss of generality, we assume that $U$ is an open ball of $R^n$. If $\gamma \in \sum_m(s, y; t, x + v)$, we can find $h > 0$, such that $\gamma([t - h, t]) \in U$, we can find a Ball $B(0, r) \in R^n$, such that for $v \in B(0, r)$, $s \in [0, h]$, we have $\gamma(t - h + s) + \frac{2}{h}v \in U$. Fix $v \in B(0, r)$, we define $\gamma_h \in \sum(s, y; t, x)$ in the following way: when $s \leq u \leq t$, $\gamma_h(u) = \gamma(u)$; when $t - h \leq u \leq t$, $\gamma_h(u) = \gamma(u) + \frac{u - h}{2}v$. Assume $F_k = \max_{\|v\|_U \leq k} \|\partial_{uv} L(x, v, t)\|_U < \infty$, $E_k = \max_{\|v\|_U \leq k} \|\partial_{uv} L(x, v, t)\|_U < \infty$. From Tonelli theorem, there exists $K(s_1, t_1)$ such that $|\gamma(u)| \leq K(s_1, t_1)$ for $s \leq u \leq t$ where $\gamma \in \Sigma(s, y; t, x)$ is a minimizer. We have the following estimates:

\[
A\left(\frac{s_1 + t_1}{2}, \gamma\left(\frac{s_1 + t_1}{2}\right); t, x + v\right) - A\left(\frac{s_1 + t_1}{2}, \gamma\left(\frac{s_1 + t_1}{2}\right); t, x\right)
\leq \int_{t_1}^{t} L(\gamma_h(u), \gamma_h(u), u) du - \int_{t_1}^{t} L(\gamma(u), \gamma(u), u) du
\]

\[
= \int_{t_1}^{t} L(\gamma_h(u), \gamma_h(u), u) - L(\gamma(u), \gamma(u), u) du
\]

\[
\leq \int_{t_1}^{t} \partial_L(\gamma_h(u), \gamma(u), u) \cdot \frac{1}{h}v + \partial_L(\gamma(u), \gamma(u), u) \cdot \frac{u + h - t}{h} \cdot v
\]

\[
+ \frac{1}{2h^2} F_{K(s_1, t_1)} + \frac{1}{2h} E_{K(s_1, t_1)} \|v\|^2_U du
\]

\[
\leq \int_{t_1}^{t} \partial_L(\gamma(u), \gamma(u), u) \cdot \frac{u + h - t}{h} \cdot v + \partial_L(\gamma(u), \gamma(u), u) \cdot \frac{1}{h}vdv + \frac{h}{2} E_{K(s_1, t_1)} \|v\|^2_U
\]

\[
+ \frac{1}{2h} F_{K(s_1, t_1)} + \frac{1}{2h} E_{K(s_1, t_1)} \|v\|^2_U
\]

\[
= \frac{\partial_L(\gamma(u), \gamma(u), u)}{dt} \cdot \frac{u + h - t}{h} \cdot v |_{t_1} + \frac{1}{2h} F_{K(s_1, t_1)} + \frac{1}{2h} E_{K(s_1, t_1)} \|v\|^2_U + \frac{1}{2h} E_{K(s_1, t_1)} \|v\|^2_U
\]

\[
+ \int_{t_1}^{t} \{ - \frac{d}{dt} \partial_L(\gamma(u), \gamma(u), u) + \partial_L(\gamma(u), \gamma(u), u) \} \cdot \frac{u + h - t}{h} \cdot v du
\]

\[
= \frac{\partial_L(\gamma(t), \gamma(t), t)}{dt} \cdot v + \frac{1}{2h} E_{K(s_1, t_1)} + \frac{1}{2h} F_{K(s_1, t_1)} + \frac{1}{2h} E_{K(s_1, t_1)} \|v\|^2_U
\]

Similarly, we have

\[
A\left(\frac{s_1 + t_1}{2}, \gamma\left(\frac{s_1 + t_1}{2}\right); t, x - v\right) - A\left(\frac{s_1 + t_1}{2}, \gamma\left(\frac{s_1 + t_1}{2}\right); t, x\right)
\leq - \frac{\partial_L(\gamma(t), \gamma(t), t)}{dt} \cdot v + \frac{1}{2h} E_{K(s_1, t_1)} + \frac{1}{2h} F_{K(s_1, t_1)} + \frac{1}{2h} E_{K(s_1, t_1)} \|v\|^2_U
\]
So we have
\[
A\left(\frac{s_1 + t_1}{2}, \gamma\left(\frac{s_1 + t_1}{2}\right); t, x + v\right) + A\left(\frac{s_1 + t_1}{2}, \gamma\left(\frac{s_1 + t_1}{2}\right); t, x - v\right) - 2A\left(\frac{s_1 + t_1}{2}, \gamma\left(\frac{s_1 + t_1}{2}\right); t, x\right) \leq \{E_K(s_1, t_1) + \frac{1}{\hbar}F_K(s_1, t_1) + 2E_K(s_1, t_1)\}|v|^2.
\]

Therefore, there exists a constant \(C(U, s_1, t_1)\) such that
\[
A\left(\frac{s_1 + t_1}{2}, \gamma\left(\frac{s_1 + t_1}{2}\right); t, x + v\right) + A\left(\frac{s_1 + t_1}{2}, \gamma\left(\frac{s_1 + t_1}{2}\right); t, x - v\right) - 2A\left(\frac{s_1 + t_1}{2}, \gamma\left(\frac{s_1 + t_1}{2}\right); t, x\right) \leq C(U)|v|^2.
\]

On the other hand, there exists a local chart \(V\) such that \(y \in V\), there exists \(r_1 > 0\), such that \(B(y, r_1) \subset V\), for any \(w \in B(y, r_1)\), we have
\[
A(s, y + w; \frac{s_1 + t_1}{2}, \gamma(\frac{s_1 + t_1}{2})) + A(s, y - w; \frac{s_1 + t_1}{2}, \gamma(\frac{s_1 + t_1}{2})) - 2A(s, y; \frac{s_1 + t_1}{2}, \gamma(\frac{s_1 + t_1}{2})) \leq C(V)|w|^2.
\]

Finally, we integrate the results above,
\[
A(s, y + w; t, x + v) + A(s, y - w; t, x - v) - 2A(s, y; t, x) \leq A\left(\frac{s_1 + t_1}{2}, \gamma\left(\frac{s_1 + t_1}{2}\right); t, x + v\right) + A\left(\frac{s_1 + t_1}{2}, \gamma\left(\frac{s_1 + t_1}{2}\right); t, x - v\right) - 2A\left(\frac{s_1 + t_1}{2}, \gamma\left(\frac{s_1 + t_1}{2}\right); t, x\right) \leq C(U)|v|^2 + C(V)|w|^2.
\]

So \(A(\cdot, t, \cdot)\) is locally equi-semiconcave for \(s \leq s_1, t \geq t_1\). Since \(M\) is compact, \(A(\cdot, \cdot, \cdot, \cdot)\) is globally equi-semiconcave for \(s \leq s_1, t \geq t_1\).

**Lemma 5.2.** For each minimizing curve \(\gamma \in \Sigma(s, y; t, x)\) attaining the minimum of the action \(A(s, y; t, x)\), we have
\[
p(t) = \partial_v L(x, \dot{\gamma}(t), t, \omega) \in D^+_x A(s, y; t, x)
\]
and
\[
-p(s) = -\partial_v L(\gamma(s), \dot{\gamma}(s), s) \in D^+_y A(s, y; t, x).
\]

**Proof.** We find a local chart \(U \subset M\), such that \(x \in U\). Without loss of generality, we assume that \(U\) is a open ball of \(R^N\). If \(\gamma \in \Sigma(s, y; t, x)\), we can find \(h > 0\), such that \(\gamma([t - h, t]) \in U\), we can find a Ball \(B(0, r)\), for \(v \in B(0, r)\), we define \(\gamma_h \in \sigma(s, y; t, x + v)\) such that \(\gamma_h(u) = \gamma(u)\) for \(s \leq u < t\) and \(\gamma_h(u) = \gamma(u) + \frac{u - t}{h}v \in U\) when \(r\) is small enough. Assume \(F_k = \max_{\|v\| \leq k} \|\partial_v L(x, v, t)\|_U < \infty\), \(E_k = \max_{\|v\| \leq k} \|\partial_v L(x, v, t)\|_U < \infty\). From Tonelli Theroem, there exists \(K(s, t)\) such that \(|\dot{\gamma}(u)| \leq K(s, t)\) for \(s \leq u \leq t\).
For any minimizer $A(s, y, t, x)$

\[ A(s, y, t, x + v) - A(s, y, t, x) \]

\[ = \int_{t-h}^{t} L(\gamma_h(u), \dot{\gamma}_h(u), u) - L(\gamma(u), \dot{\gamma}(u), u) \, du \]

\[ \leq \int_{t-h}^{t} L(\gamma_h(u), \dot{\gamma}_h(u), u) - L(\gamma_h(u), \dot{\gamma}(u), u) + L(\gamma(u), \dot{\gamma}(u), u) - L(\gamma(u), \dot{\gamma}(u), u) \, du \]

\[ \leq \int_{t-h}^{t} \frac{\partial L}{\partial v}(\gamma_h(u), \dot{\gamma}(u), u) \cdot \frac{1}{h} v + \frac{\partial L}{\partial x}(\gamma(u), \dot{\gamma}(u), u) \cdot \frac{u + h - t}{h} \cdot v \]

\[ + \frac{1}{2h} F_{K(s, t_1), t_1^+} \|v\|^2 + \frac{1}{2} E_{K(s, t_1)} \|v\|^2 \, du \]

\[ \leq \int_{t-h}^{t} \frac{\partial L}{\partial x}(\gamma(u), \dot{\gamma}(u), u) \cdot \frac{u + h - t}{h} \cdot v + \frac{\partial L}{\partial v}(\gamma(u), \dot{\gamma}(u), u) \cdot \frac{1}{h} v du + \frac{h}{2} E_{K(s, t_1)} \|v\|^2 \]

\[ + \frac{1}{2h} F_{K(s, t_1), t_1^+} \|v\|^2 + E_{K(s, t_1)} \|v\|^2 \]

\[ = \frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t), t) \cdot v \]

\[ + \frac{h}{2} E_{K(s, t_1)} + \frac{1}{2h} F_{K(s, t_1), t_1^+} + E_{K(s, t_1)} \] \|v\|^2

\[ \leq 0 \]

This proves that $\frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t), t) \in D^+_x A(s, y, t, x)$.

In a similar way, we can prove that $\frac{\partial L}{\partial v}(\gamma(s), \dot{\gamma}(s), s) \in D^+_y A(s, y, t, x)$.

Lemma 5.3. Fix $s < t$, $\forall p \in D^+_x A(s, y, t, x)$, there exists a minimizer $\gamma \in \Sigma(s, y, t, x)$ of $A(s, y, t, x)$ such that $p = -\frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t), t)$.

Proof. First step, when $A(s, y, t, x)$ is differentiable at $x$, $D^+_x A(s, y, t, x) = \{D_x A(s, y, t, x)\}$.

For any minimizer $\gamma \in \Sigma(s, y, t, x)$ of Action $A(s, y, t, x)$, $-\frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t), t) = D_x A(s, y, t, x)$.

By Euler-Lagranian equation, the minimizer of $A(s, y, t, x)$ is unique.

Second step, when $A(s, y, t, x)$ is not differentiable at $x$, for any $p \in D^+_x A(s, y, t, x)$, there exists $\{x_n|n = 1, 2, \ldots\} \subset M$, $A(s, y, t, x)$ is differentiable at $x$, and $\lim_{n \to \infty} x_n = x$, $\lim_{n \to \infty} D_x A(s, y, t, x) = p$. From the first step, we know that there exists $\gamma_n \in \Sigma(s, y, t, x)$, such that $-\frac{\partial L}{\partial v}(\gamma_n(t), \dot{\gamma}_n(t), t) = D_x A(s, y, t, x)$. From Euler-Lagranian equation, there exists a Lagrangian flow $(\gamma, p) : [s, t] \to TM$, such that $\gamma(t) = x$, $p(t) = p$.

By continuity of dependence on the initial values of Ordinary Differential Equation, and $\lim_{n \to \infty} \gamma_n(t), \dot{\gamma}_n(t), t) = (x, p)$. So $y = \gamma(s) = \lim_{n \to \infty} n \gamma_n(s)$. Therefore, $\gamma \in \Sigma(s, y, t, x)$.

By lower semi-continuity of the Action, we have

\[ \int_{s}^{t} L(\gamma(u), \dot{\gamma}(u), u) du \leq \lim_{n \to \infty} \int_{s}^{t} L(\gamma_n(u), \dot{\gamma}_n(u), u) du \]

Since $A(s, y, t, x)$ is semi-concave on $M$ and semi-concave functions are lipschitz continuous. We know that $\lim_{n \to \infty} A(s, y, t, x_n) = A(s, y, t, x)$. So $\lim_{n \to \infty} \int_{s}^{t} L(\gamma_n(u), \dot{\gamma}_n(u), u) du = A(s, y, t, x)$. So $\gamma : [s, t] \to M$ is a minimizer of $A(s, y, t, x)$. This finishes the proof.
Corollary 5.1. The three following conditions are equivalent:

1. $A(s, y; t, x)$ has only one minimizer in $\Sigma(s, y; t, x)$
2. $A(s, y; t, x)$ is differentiable at $x$.
3. $A(s, y; t, x)$ is differentiable at $y$.

Proof. (2), (3) $\rightarrow$ (1) is the direct consequence of Lemma 5.3. We prove (1) $\rightarrow$ (2). If $A(s, y; t, x)$ is not differentiable at $x$, $D_x^+ A(s, y; t, x)$ contains more than one point. Since $A(s, y; t, x)$ is semi-concave, we know that $D_x^+ A(s, y; t, x)$ is a convex compact set, and is the convex hull of $D_x^+ A(s, y; t, x)$. So $D_x^+ A(s, y; t, x)$ contains more than one point, by lemma 5.3, $A(s, y; t, x)$ has more than one minimizer. (1) $\rightarrow$ (3) is similar.

6 Weak KAM Solution

In this section, we use Lax-Oleinik operator to construct a class of Weak KAM Solutions and prove that they are measurable over $\Omega$.

Notation 6.1. $\Sigma(s, y; t, x)$ is the set of absolutely continuous curves $\gamma : [s, t] \rightarrow M$ such that $\gamma(s) = y$ and $\gamma(t) = x$.

$\Sigma_m(s, y; t, x)$ denotes the set of the minimizers for the Action $A^\omega(s, y; t, x)$

$D = C_0(M, R)$ is the real-valued continuous function space over $M$ with the uniform topology. $\mathcal{D}$ is the Borel algebra generated by open sets of $D$.

$C_0(M \times R, R)$ is the real-valued continuous function space over $M \times R$.

$f(\omega) = \sup_{x \in M, t \in [0, 1]} |L(x, 0, s, \omega)|$ is finite since $M$ is compact.

$C(\omega) = \sup \{|L(x, v, t)| \text{ for } x \in M, |v| \leq \text{dist}(M), t \in [0, 1]|}.

Definition 6.1. 1. For $\lambda \in R$, $u \in C_0(M, R)$, the operator $T_{\lambda}^\omega : C_0(M, R) \rightarrow C_0(M \times R, R)$ is defined as

$$T_{\lambda}^\omega(u)(x, t) = \min_{y \in M}\{u(y) + A^\omega(t - \lambda, y; t, x) + \lambda \alpha(\omega)\}$$

2. For $u \in C_0(M, R)$, $u^\omega(x, t)$ is defined as

$$u^\omega(x, t) = \lim_{\lambda \rightarrow +\infty} T_{\lambda}^\omega(u)(x, t)$$

3. For $\lambda \in R, u \in C_0(M, R)$, the operator $T_{\lambda, +}^\omega : C_0(M, R) \rightarrow C_0(M \times R, R)$ is defined as

$$T_{\lambda, +}^\omega(u)(x, t) = \min_{y \in M}\{u(y) + A^\omega(t, x; t + \lambda, y) + \lambda \alpha(\omega)\}$$

4. For $u \in C_0(M, R)$, $u^\omega_+(x, t)$ is defined as

$$u^\omega_+(x, t) = \lim_{\lambda \rightarrow +\infty} T_{\lambda, +}^\omega(u)(x, t)$$

Lemma 6.1. When $\omega \in \Omega$. Fix $t \in R, x \in M, u \in C(M, R)$, the Lax-Oleinik Operator $T_{\lambda}^\omega u(x, t)$ is Lipschitz when $\lambda \geq 0$ with Lipschitz constant $|f(\omega)| + |\alpha(\omega)|$.

Proof. For $\lambda_1 > \lambda_2 \geq 0, \forall \epsilon > 0$, we can find $y_1, y_2 \in M$, such that

$$T_{\lambda_1}^\omega(u)(x, t) + \epsilon = u(y_1) + A^\omega(t - \lambda_1, y_1; t, x) + \lambda_1 \alpha(\omega)$$

$$T_{\lambda_2}^\omega(u)(x, t) + \epsilon = u(y_2) + A^\omega(t - \lambda_2, y_1; t, x) + \lambda_2 \alpha(\omega)$$

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then
\[ T_{\lambda_1}^\omega(u)(x,t) \leq u(y_2) + A^\omega(t - \lambda_2, y_2; t, x) + \lambda_2 \alpha(\omega) + \int_{t-\lambda_1}^{t-\lambda_2} L(y_2, 0, s, \omega)ds \]
\[ + (\lambda_1 - \lambda_2) \alpha(\omega) \leq T_{\lambda_2}^\omega(u)(x,t) + \epsilon + (\lambda_1 - \lambda_2)(|\alpha(\omega)| + |f(\omega)|) \]
and
\[ T_{\lambda_2}^\omega(u)(x,t) \leq u(y_1) + A^\omega(t - \lambda_1, y_1; t, x) + \lambda_1 \alpha(\omega) + \int_{t-\lambda_1}^{t-\lambda_2} L(y_1, 0, s, \omega)ds \]
\[ + (\lambda_2 - \lambda_1) \alpha(\omega) \leq T_{\lambda_1}^\omega(u)(x,t) + \epsilon + (\lambda_1 - \lambda_2)(|\alpha(\omega)| + |f(\omega)|) \]

Since \( \epsilon > 0 \) is arbitrary, we have
\[ |T_{\lambda_1}^\omega u(x,t) - T_{\lambda_2}^\omega u(x,t)| \leq (\lambda_1 - \lambda_2)(|\alpha(\omega)| + |f(\omega)|) \]

\[ \square \]

**Lemma 6.2.** When \( \omega \) is fixed, Lax-Oleinik operator is uniformly bounded for continuous function when \( \lambda > 0 \).

**Proof.** Step 1. Fix \( \omega \in \Omega, t \in R \). Define the sequences \( M_n(\omega) = \max_{x \in M} T_n^\omega(0)(x,t) \) and \( m_n(\omega) = \min_{x \in M} T_n^\omega(0)(x,t) \) where 0 is the zero function on \( M \). From Lemma 5.1, the function \( T_n^\omega(0), n \geq 1 \), are equi-semi-concave, there exists a constant \( K(\omega) \) such that
\[ 0 \leq M_n(\omega) - m_n(\omega) \leq K(\omega). \]

for \( n \geq 1 \). We claim that \( M_{n+m}(\omega) \leq M_n(\omega) + M_m(\omega) \). This follows from the inequalities
\[ T_{n+m}^\omega(0)(x,t) = T_n^\omega(T_m^\omega(0)(x,t)) \leq T_m^\omega(M_n(\omega))(x,t) \leq M_n(\omega) + T_m^\omega(0)(x,t) \]
Hence by a classical result on subadditive sequences, we have \( \lim \frac{M_n(\omega)}{n} = \inf \frac{M_n(\omega)}{n} \). We denote by \( -\beta(\omega) \) this limit. In the same way, the sequence \( -m_n(\omega) \) is subadditive, hence \( \frac{m_n(\omega)}{n} \to \sup \frac{m_n(\omega)}{n} \). This limit is also \( -\beta(\omega) \) since \( 0 \leq M_n(\omega) - m_n(\omega) \leq K \).

Note that \( m_1(\omega) \leq -\beta(\omega) \leq M_1(\omega) \), so that \( -\beta(\omega) \) is indeed a finite number. We have, for all \( n \leq 1 \),
\[ -K(\omega) - n\beta(\omega) \leq m_n(\omega) \leq -n\beta(\omega) \leq M_n(\omega) \leq K(\omega) - n\beta(\omega) \]
Now for all \( u \in C(M, R), n \in N \) and \( x \in M \), we have
\[ \min_M u - K(\omega) \leq \min_M u + m_n(\omega) + n\beta(\omega) \leq T_n^\omega u(x,t) + n\beta(\omega) \leq \max_M u + M_n(\omega) + n\beta(\omega) \leq \max_M u + K(\omega) \]
Hence, for all \( u \in C(M, R), n \in N, x \in M \), we have
\[ \frac{\min_M u - K(\omega)}{n} \leq \frac{T_n^\omega u(x,t)}{n} + \frac{\beta(\omega)}{n} \leq \frac{\max_M u + K(\omega)}{n} \]

Step 2. On one hand, from Proposition 4.1, we can find a sequence of measure \( \frac{1}{n_k}[\gamma_{n_k}] \) where \( n_k \) is a sequence of increasing integers towards \( +\infty \), \( \gamma_{n_k} \) is a closed absolutely continuous curve from \( [t - n_k, t] \) to \( M \), such that \( \forall \epsilon > 0 \), we can find an integer \( N \), such that, when \( k \geq N \), we have
\[ -\alpha(\omega) \leq \frac{1}{n_k} \int_{t-n_k}^t L(\gamma_{n_k}(s), \gamma_{n_k}(s), s, \omega)ds \leq -\alpha(\omega) + \epsilon \]
Hence, we have

\[ T^\omega_{n_k}(0)(\gamma_{n_k}(t), t) \leq \int_{-n_k+t}^{t} L(\gamma_{n_k}(s), \dot{\gamma}_{n_k}(s), s, \omega) dt + n_k \alpha(\omega) \leq n_k \epsilon \]

Therefore, we have

\[ \frac{1}{n_k} T^\omega_{n_k}(0)(\gamma_{n_k}(t), t) \leq \epsilon \]

On the other hand, there is \( x \in M \) such that

\[ T^\omega_{n_k}(0)(\gamma_{n_k}(t), t) = A^\omega(x, t - n_k; \gamma_{n_k}(t), t) + n_k \alpha(\omega) \]

We assume that \( \gamma_1 \in \Sigma_m^\omega(x, t - n_k; \gamma_{n_k}(t), t) \). There is an absolutely continuous path \( \gamma : [t - n_k - 1, t - n_k] \to M \) such that \( |\dot{\gamma}(s)| \leq \text{dist}(M) \), and \( \gamma(t - n_k) = x, \gamma(t - n_k - 1) = \gamma_{n_k}(t) \). Then we know that

\[ \int_{[t - n_k - 1]}^{t} |L(\gamma(s), \dot{\gamma}(s), s, \omega)| ds \leq C(\omega) \]

Construct a new closed curve \( \tilde{\gamma} : [t - n_k - 1, t] \to M \) in the following way: when \( s \in [t - n_k, t] \), \( \tilde{\gamma}(s) = \gamma_1(s) \); when \( s \in [t - n_k - 1, t - n_k] \), \( \tilde{\gamma}(s) = \gamma(s) \). From Proposition 4.1, we know that

\[ \int_{[t - n_k - 1]}^{t} L(\tilde{\gamma}(s), \dot{\tilde{\gamma}}(s), s, \omega) ds \geq -(n_k + 1) \alpha(\omega) \]

Hence, we have

\[ T^\omega_{t - n_k}(0)(\gamma_{n_k}(t), t) = n \alpha(\omega) + \int_{[t - n_k - 1]}^{t} L(\tilde{\gamma}(s), \dot{\tilde{\gamma}}(s), s, \omega) - \int_{[t - n_k - 1]}^{t} L(\gamma(s), \dot{\gamma}(s), s, \omega) ds \]

\[ \geq -\alpha(\omega) - C(\omega) \]

So we have

\[ -\frac{\alpha(\omega) + C(\omega)}{n_k} \leq \frac{T^\omega_{t - n_k}(0)(\gamma_{n_k}(t), t)}{n_k} \leq \epsilon \]

So combine the result with step 2, let \( k \) tends to \( \infty \), we know that \( \beta(\omega) = 0 \). Hence,

\[ \min \limits_{M} u - K(\omega) \leq T^\omega_n u(x, t) \leq \max \limits_{M} u + K(\omega) \]

Use Lemma 6.1, \( T^\omega_n u(x, t) \) is lipchitz for \( \lambda \), with Lipschitz constant \( |\alpha(\omega)| + |f(\omega)| \), so we have

\[ \min \limits_{M} u - K(\omega) - |\alpha(\omega)| - |f(\omega)| \leq T^\omega_n u(x, t) \leq \max \limits_{M} u + K(\omega) + |\alpha(\omega)| + |f(\omega)| \]

\[ \square \]

**Lemma 6.3.** If the variables \( t, s \in R, x, y \in M, n \in N, u \in C_0(M, R) \) are fixed, \( A^\omega(s, y; t, x), T^\omega_n(u)(x, t), T^\omega_{n+1}(u)(x, t) \) and \( u^\omega(x, t), u^\omega_+(x, t) \) are random variables over the probability space \( \Omega \).
Proof. The extreme curves of the Action are $C^2$ since they satisfy the lagrangian equations.

Let $C^0([s, t], TM)$ denote the space of continuous function from $[s, t]$ to $TM$ with uniform norm in $TM$. $M$ is a manifold, by whitney’s embedding theorem, there exists $m \in N$, such that $M$ can be embedded isomorphically into $R^m$. Therefore, $TM$ can be embedded as a submanifold of $R^{2m}$. By Stone Weierstrass theorem, the space of continuous functions from $[s, t]$ into $R$ is separable. Therefore, the space of continuous functions from $[s, t]$ to $R^{2m}$ is separable, and as its subset, $C^0([s, t], TM)$ is also separable. Since the extreme curves of the Action is a subset of $C^0([s, t], TM)$, So we can pick up a countable dense subset $\{\gamma_i\}_{i \in N}$, such that

$$A^\omega(s, y; t, x) = \min_{i \in N} \int_s^t L(\gamma_i(\sigma), \dot{\gamma}_i(\sigma), \sigma, \omega) d\sigma$$

By Fubini Theorem, $\int_s^t L(\gamma_i(\sigma), \dot{\gamma}_i(\sigma), \sigma, \omega) d\sigma$ is measurable function on $\Omega$. As a minimum of countable measurable functions, $A^\omega(s, y; t, x)$ is measurable.

$M$ is a separable metric space, $u(y)$, $A^\omega(t - \lambda, y; t, x)$ are continuous functions with $y$, so we can find a countable dense set $\{y_i\}_{i \in N}$ in $M$ such that

$$T_N^\omega(u)(x, t) = \min_{i \in N} \{u(y_i) + A^\omega(t - \lambda, y_i; t, x) + \lambda\alpha(\omega)\}$$

As a minimum of countable measurable functions, $T_N^\omega(u)(x, t)$ is a random variable.

Since $T_N^\omega(u)(x, t)$ is uniformly continuous with $\lambda$, so we can pick a sequence $\lambda_n = \sum_{k=1}^n \frac{1}{k} \to \infty$. Then, we have

$$u^\omega(x, t) = \lim_{n \to \infty} T_{\lambda_n}^\omega(u)(x, t)$$

So as the infimum limit of a countable sequence of measurable function, $u^\omega(x, t)$ is measurable over $\Omega$.

The measurability of $T_{\lambda_i}(u)(x, t), u^\omega(x, t)$ can be proved in a similar way. ∎

Lemma 6.4. Fix $u \in C_0(M, R)$, for $\lambda > 0, t, s \in R, x \in M$, we have the following formula:

1. $T_{\theta(s)}^\omega(u)(x, t) = T_{\lambda}^\omega(u)(x, t + s)$

2. $u^\omega(x, t) = T_{t-s}^\omega(u^\omega(s))(x, t) = \min_{\gamma \in M} \{u^\omega(y, s) + A^\omega(s, y; t, x) + (t-s)\alpha(\omega)\}$

3. $u^{\theta(s)}(x, t) = u^\omega(x, t + s)$

4. $u^\omega(x, t) = u^\omega(x, t + 1)$

These formula have similar versions for $T_{\lambda_i}(u)(x, t), u^\omega(x, t)$.

Proof. First of all, the formula 1 can be derived directly from $L(x, v, \theta(s)\omega) = L(x, v, t+s, \omega)$.

Secondly, by definition, $\forall \epsilon > 0$, there exists $y \in M$, such that

$$T_{t-s}^\omega(u^\omega(s))(x, t) + \epsilon = u^\omega(y, s) + A^\omega(s, y; t, x) + (t-s)\alpha(\omega)$$

By definition, there exists a sequence $\lambda_i \to +\infty$ as $i \to +\infty$, such that

$$u^\omega(y, s) = \lim_{i \to +\infty} T_{\lambda_i}^\omega(u)(y, s)$$
Therefore, we have
\[
T_{t-s}^\omega u^\omega(s)(x, t) + \epsilon = \liminf_{i \to +\infty} T_{X_i}^\omega(u)(y, s) + A^\omega(s, y; t, x) + (t-s)\alpha(\omega)
\geq \liminf_{i \to +\infty} T_{X_i+t-s}^\omega(u)(x, t) \geq \liminf_{\lambda \to +\infty} T_\lambda^\omega(u)(x, t) = u^\omega(x, t)
\]

Since \(\epsilon > 0\) is arbitrary, we can get that
\[
T_{t-s}^\omega u^\omega(s)(x, t) \geq u^\omega(x, t)
\]

Conversely, there exists \(\lambda_k \to +\infty\) as \(k \to +\infty\), such that
\[
u^\omega(x, t) = \lim_{k \to +\infty} T_{\lambda_k}^\omega(u)(x, t) = \lim_{k \to +\infty} T_{t-s}^\omega(T_{\lambda_k-t+s}^\omega(u))(x, t)
\]

By definition, \(\forall \epsilon > 0\), there exists \(\{q_k\}_{k \in \mathbb{N}} \in M\), such that
\[
T_{\lambda_k}^\omega(u)(x, t) \geq T_{\lambda_k-t+s}^\omega(u)(q_k, s) + A^\omega(q_k, s; x, t) + (t-s)\alpha(\omega) - \epsilon
\]

Since \(M\) is a compact manifold, without loss of generality, we can assume that \(q_k \to q\) as \(k \to \infty\) for some \(q \in M\), so
\[
u^\omega(x, t) \geq \lim_{k \to +\infty} T_{\lambda_k-t+s}^\omega(u)(q_k, s) + A^\omega(q_k, s; x, t) + (t-s)\alpha(\omega) - \epsilon
\]
\[
\geq \liminf_{k \to +\infty} T_{\lambda_k-t+s}^\omega(u)(q, s) + A^\omega(q, s; x, t) + (t-s)\alpha(\omega) - \epsilon
\]
\[
\geq u^\omega(q, s) + A^\omega(q, s; x, t) + (t-s)\alpha(\omega) - \epsilon
\]
\[
\geq u^\omega(x, t) - \epsilon
\]

Since \(\epsilon > 0\) is arbitrary, we have that
\[
u^\omega(x, t) \geq T_{t-s}^\omega u^\omega(s)(x, t)
\]

Consequently, the formula 2 holds.

Thirdly, the formula 3 arises directly from formula 1.

Finally, we can get the formula 4 from \(L(x, u, v, t + 1, \omega) = L(x, v, t, \omega)\). \(\square\)

**Theorem 6.1.** Fix \(u \in C_0(M, R)\), \(\nu^\omega(x, t)\) is a viscosity solution of the Hamilton-Jacobi Equation
\[
\partial_t u(x, t) + H(x, \partial_x u(x, t), t, \omega) = \alpha(\omega)
\]

**Proof.** Fix \(\omega \in \Omega\), for any \((x_0, t_0) \in M\). We can find a local chart \((U, \phi), U \in M, \phi: U \to \phi(U) \in R^n\) is a diffeomorphism, \((x_0, t_0) \in U\). Without loss of generality, we assume that \(U \in R^n\).

Step 1. To show \(\nu^\omega(x, t)\) is a subsolution of the Hamilton-Jacobi equation, we need to prove that if \((p_x, p_t) \in D^+u^\omega(x_0, t_0), p_t + H(x, p_x, t, \omega) \leq \alpha(\omega)\).

\forall v \in T_{x_0}M, since \((p_x, p_t) \in D^+u^\omega(x_0, t_0)\), we have
\[
\limsup_{h \to 0^+} \frac{u^\omega(x_0 - hv, t_0 - h) - u^\omega(x_0, t_0) + h(p_t + p_x \cdot v)}{h\sqrt{1 + |v|^2}} \leq 0
\]

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Which is equivalent to
\[
\limsup_{h \to 0^+} \frac{u^\omega(x_0 - hv, t_0 - h) - u^\omega(x_0, t_0)}{h} \leq -p_t - p_x \cdot v
\]

Since \( U \) is open, there exists \( \sigma > 0 \), such that \( \{ \gamma(t) = x - s \cdot v | 0 \leq s \leq \sigma \} \in U \), from lemma 4, we know that when \( 0 < h \leq \sigma \), we have
\[
u^\omega(x_0, t_0) \leq u^\omega(x_0 - h \cdot v, t_0 - h) + \int_{t_0 - h}^{t_0} L(\gamma(s), \dot{\gamma}(s), s, \omega)ds + h \cdot \alpha(\omega)
\]

Then,
\[
-\alpha(\omega) - \liminf_{h \to 0^+} \frac{1}{h} \int_{t_0 - h}^{t_0} L(\gamma(s), \dot{\gamma}(s), s, \omega)ds \leq \limsup_{h \to 0^+} \frac{u(x_0 - hv, t_0 - h) - u(x_0, t_0)}{h} \leq -p_t - p_x \cdot v
\]

Hence, we have
\[
-\alpha(\omega) - L(x_0, v, t_0, \omega) + p_x \cdot v + p_t \leq 0
\]

Therefore,
\[
p_t + H(x_0, v, t_0, \omega) - \alpha(\omega) = p_t + \sup_{v \in T_{w_0}M} \{ p_x \cdot v - L(x_0, v, t_0, \omega) \} - \alpha(\omega) \leq 0
\]

Step 2. To show \( u^\omega(x, t) \) is a subsolution of the Hamilton-Jacobi equation, we need to prove that if \( (p_x, p_t) \in D^+ u^\omega(x_0, t_0), p_t + H(x, p_x, t, \omega) \geq \alpha(\omega) \).

From Lemma 6.4, we know that there exists \( y \in M \), an absolute curve \( \{ \gamma(t) | t_0 - 1 \leq t \leq t_0, \gamma(t_0 - 1) = y, \gamma(t_0) = x_0 \} \), such that
\[
u^\omega(x_0, t_0) = u^\omega(y, t_0 - 1) + \int_{t_0 - 1}^{t_0} L(\gamma(t), \dot{\gamma}(t), t, \omega)dt + (t_0 - t)\alpha(\omega)
\]

Since \( U \) is open, there exists \( \sigma > 0 \), such that \( \{ \gamma(t) | t_0 - \sigma \leq t \leq t_0 \} \in U \). Then we have
\[
u^\omega(x_0, t_0) = u^\omega(\gamma(t_0 - h), t_0 - h) + \int_{t_0 - h}^{t_0} L(\gamma(t), \dot{\gamma}(t), t, \omega)dt + h\alpha(\omega)
\]

Let \( w = \dot{\gamma}(t_0) \), since \( (p_x, p_t) \in D^+ u^\omega(x_0, t_0) \), we know that
\[
\liminf_{h \to 0^+} \frac{u^\omega(\gamma(t_0 - h), t_0 - h) - u^\omega(\gamma(t_0), t_0)}{h} \geq -p_x \cdot w - p_t
\]

Hence, we have
\[
-p_x \cdot w - p_t \leq -\liminf_{h \to 0^+} \frac{1}{h} \int_{t_0 - h}^{t_0} L(\gamma(t), \dot{\gamma}(t), t, \omega)dt - \alpha(\omega) \leq -L(x_0, w, t_0, \omega) - \alpha(\omega)
\]

Therefore, we have
\[
H(x, p_x, t, \omega) = \sup_{v \in T_{w_0}M} \{ p_x \cdot v - L(x_0, v, t_0, \omega) \} \geq p_x \cdot w - L(x_0, w, t_0, \omega) \geq \alpha(\omega) - p_t
\]

From Step 1 and Step 2, we know that \( u^\omega(x, t) \) is a viscosity solution of Hamilton-Jacobi equation.
7 Global Minimizer

In this section, we discuss the global minimizer and invariant measure under the skew-product dynamics system.

Definition 7.1. When \( \omega \) is fixed,
1. an absolutely continuous orbit \( \{\gamma^\omega(t), t \in R\} \) is called a global minimizer of the Lagrangian if for any fixed time interval \([s, t]\), we have

\[
A^\omega(s, \gamma^\omega(t); t, \gamma^\omega(t)) = \int_s^t L(\gamma^\omega(\sigma), \dot{\gamma}^\omega(\sigma), \sigma, \omega) d\sigma
\]

2. fix \( t_0 \in R \), an absolutely continuous orbit \( \{\gamma^\omega(t), t \leq t_0\} \) is called a calibrated curve of \( u^\omega(x, t) \), if for any \( s < t \leq t_0 \), we have

\[
u^\omega(\gamma^\omega(t), t) = u^\omega(\gamma^\omega(s), s) + A^\omega(s, \gamma^\omega(s); t, \gamma^\omega(t)) + (t - s) \alpha(\omega)
\]

3. fix \( t_0 \in R \), an absolutely continuous orbit \( \{\gamma^\omega(t), t \geq t_0\} \) is called a calibrated curve of \( u^\omega_+(x, t) \), if for any \( s > t \geq t_0 \), we have

\[
u^\omega_+(\gamma^\omega(t), t) = u^\omega_+(\gamma^\omega(s), s) + A^\omega(t, \gamma^\omega(t); s, \gamma^\omega(s)) + (s - t) \alpha(\omega)
\]

Lemma 7.1. We fix \( \omega \) and \( t_0 \), if \( u^\omega(x, t_0) \) is differentiable at \( x_0 \in M \), there is a unique calibrated curve with the end point \((x_0, t_0)\), for \( u^\omega(x, t) \), \( t \leq t_0 \); denoted as \( \gamma^\omega_{x,t_0}^- \). Similarly, if \( u^\omega_+(x, t_0) \) is differentiable at \( x_0 \in M \), there is a unique calibrated curve with the end point \((x_0, t_0)\), for \( u^\omega_+(x, t), t \geq t_0 \); denoted as \( \gamma^\omega_{x,t_0}^+ \).

Proof. By lemma 3.3, for any \( s < t_0 \), \( x_0 \in B^\omega_{t_0} \), we have

\[
u^\omega(x_0, t) = \min_{y \in M} \{u^\omega(y, s) + A^\omega(s, y; t, x) + (t - s) \alpha(\omega)\}
\]

Since \( u^\omega(y, s) \) and \( A^\omega(s, y; t, x) \) is semiconcave with respect to \( y \), so there is \( x(s) \in M \) such that

\[
u^\omega(x_0, t) = u^\omega(x(s), s) + A^\omega(s, x(s); t, x_0) + (t - s) \alpha(\omega)
\]

and \( \partial_x u^\omega(x(s), s) + \partial_y A^\omega(s, x(s); t, x_0) = 0 \). By Corollary 5.1, \( \Sigma^\omega_{(s)}(s, x(s); t_0, x_0) \) contains the only one lagrangian trajectory \((x(t), p(t))\), \( p(t) = \partial_v L(x(s), \dot{x}(s), x, \omega) \) with \( s \leq t \leq t_0 \), with \( x(t_0) = x_0 \), \( p(s) = \partial_x u^\omega(x(s), s) = -\partial_y A^\omega(s, x(s); t, x_0) \) where \( p(t) = \partial_x u^\omega(x(t_0), t_0) = \partial_x A^\omega(s, x(s); t, x_0) \). Therefore, it is obvious that \( x(s) \) is unique.

For any \( s_1 < s_2 < t_0 \), the above lagrangian trajectory coincides for \( s_2 \leq t \leq t_0 \). So we can extend the trajectory \( x(s) \) for time \(-\infty < s \leq t_0 \), with \( x(t_0) = x_0 \). This is a unique calibrated curve for \( u^\omega(x, t), t \leq t_0 \) with the end point \( x(t_0) = x_0 \). \[ \square \]

Lemma 7.2. When \( \omega, t_0 \) is fixed, the set \( B^\omega_{t_0} \) is defined as the subsets of \( M \) where the function \( u^\omega(y, t_0) + u^\omega_+(y, t_0) \) attains the minimum. We have the following results:

1. \( B^\omega_{t_0} \) is a closed nonempty subset of \( M \). \( u^\omega(x, t), u^\omega_+(x, t) \) are differentiable over \( B^\omega_{t_0} \);
2. for any \( x_0 \in B^\omega_{t_0} \), \( \partial_x u^\omega(x_0, t) + \partial_x u^\omega_+(x_0, t) = 0 \);
3. \( B^{\omega(t_0)}_{t_0} = B^\omega_{t_0+s}, B^\omega_{t_0+1} = B^\omega_{t_0} \)

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Proof. It is obvious that $B^ω_{t_0}$ is closed nonempty subset. Since $u^ω(x, t_0), u^ω_+(x, t_0)$ are both semiconcave function over $M$, they are differentiable over $B^ω_{t_0}$ and $\partial_x u^ω_+(x_0, t_0) + \partial_x u^ω(x_0, t_0) = 0$. By lemma 3.3, $B^{θ(ω)}_t = B^{ω}_{t+s}$, $B^ω_{t+1} = B^ω_t$ are obvious. □

**Proposition 7.1.** When $ω$ is fixed, for any $t_0 \in R$, if the following hypothesis holds:

\[ u^ω(\gamma^ω(t_0), t_0) + u^ω_+(\gamma^ω(t_0), t_0) = \min_{x \in M} \{ u^ω(x, t_0) + u^ω_+(x, t_0) \} \]

\[ \{ \gamma^ω(t), t \leq t_0 \} \] is a calibrated curve of $u^ω(x, t)$,

\[ \{ \gamma^ω(t), t \geq t_0 \} \] is a calibrated curve of $u^ω_+(x, t)$,

then \[ \{ \gamma^ω(t), t \in R \} \] is a global minimizer of the lagrangian.

**Proof.** If the conclusion false, there exists time $t_1 < t_2$, an absolutely continuous curve \[ \{ γ_1(t), t_1 \leq t \leq t_2 \} \] with the terminal points $γ_1(t_1) = γ^ω(t_1), γ_1(t_2) = γ^ω(t_2)$, such that

\[ \int_{t_1}^{t_2} L(γ_1(σ), γ_1(σ), σ, ω)dσ < \int_{t_1}^{t_2} L(γ^ω(σ), γ^ω(σ), σ, ω)dσ \]

then there exist $t_0 \in (t_1, t_2)$ with $γ_1(t_0) ≠ γ^ω(t_0)$. We have the following:

\[ u^ω(γ_1(t_0), t_0) + u^ω_+(γ_1(t_0), t_0) \leq u^ω(γ_1(t_1), t_1) + \int_{t_1}^{t_0} L(γ_1(σ), γ_1(σ), σ, ω)dσ + (t_0 - t_1)α(ω) + u^ω_+(γ_1(t_2), t_2) + \int_{t_1}^{t_2} L(γ_1(σ), γ_1(σ), σ, ω)dσ + (t_2 - t_1)α(ω) \]

and

\[ u^ω(γ^ω(t_0), t_0) + u^ω_+(γ^ω(t_0), t_0) = u^ω(γ^ω(t_1), t_1) + \int_{t_1}^{t_0} L(γ^ω(σ), γ^ω(σ), σ, ω)dω + (t_0 - t_1)α(ω) + u^ω_+(γ^ω(t_2), t_2) + \int_{t_1}^{t_2} L(γ^ω(σ), γ^ω(σ), σ, ω)dσ + (t_2 - t_1)α(ω) \]

So, $u^ω(γ_1(t_0), t_0) + u^ω_+(γ_1(t_0), t_0) < u^ω(γ^ω(t_0), t_0) + u^ω_+(γ^ω(t_0), t_0)$, this contradicts the hypothesis. Hence we can complete the proof. □

**Corollary 7.1.** For fixed $ω$, $t_0$; for any $x_0 \in B^ω_{t_0}$, a global minimizer $γ^ω_{x_0,t_0}$ exists, $γ^ω_{x_0,t_0}(t_0) = x_0$. In addition, the set of global minimizer is nonempty for each $ω \in Ω$.

**Assumption 7.1.** $(Ω, ℱ, ℙ)$ is a Polish Space.

Starting from here, we assume Assumption 7.1 holds.

From the analysis above, fix $ω$, the set of global minimizer is nonempty. We denote the set of global minimizer as $H_ω = \{ γ^ω : R → M | γ^ω \text{ is a global minimizer of action } A^ω \}$.

If $γ^ω$ is a global minimizer, $γ^ω$ satisfies the Euler-Lagrange equation. $γ^ω$ is decided by $(γ^ω(0), ˙γ^ω(0))$. So we can define the set $G_ω = \{ (γ^ω(0), ˙γ^ω(0)) | γ^ω \text{ is a global minimizer of the action } A^ω \}$. $G_ω$ is a compact subset of $TM$, due to Tonelli Theorem and continuity of Lagrangian Action.
$G_{\omega}$ and $H_{\omega}$ corresponds one to one, $L(x,v,t+s,\omega) = L(x,v,t,\theta(s)\omega)$. If $\gamma^\omega(t)$ is a global minimizer of the action $A^\omega$, then $\gamma^{\theta(s)\omega}(t) = \gamma^\omega(t + s)$ is the global minimizer of $A^{\theta(s)\omega}$, we define the one-to-one map $\Theta(s) : G_{\omega} \to G_{\theta(s)\omega}$ as follows, if $(\gamma^\omega(0), \dot{\gamma}^\omega(0)) \in G_{\omega}$, let $\Theta(s)(\gamma^\omega(0), \dot{\gamma}^\omega(0)) = (\gamma^{\theta(s)\omega}(0), \dot{\gamma}^{\theta(s)\omega}(0)) = (\gamma^\omega(-s), \dot{\gamma}^\omega(-s))$. When $\omega$ is fixed, $\Theta(-s)$ is the Lagrangian flow the global minimizer on $TM$. $\Theta(s + t) = \Theta(s)\Theta(t), G_{\omega} = G_{\theta(n)\omega}$.

Let $G = \bigcup_{\omega \in \Omega} G_{\omega} \times \{\omega\} \subset TM \times \Omega$. Define $\Gamma(s) : G \to G$, if $(x,v,\omega) \in G$, $\Gamma(s)(x,v,\omega) = (\Theta(s)|_{G_{\omega}}(x,v), \theta(s)\omega)$. It is clear that $\Gamma(s + t) = \Gamma(s)\Gamma(t)$. We define the set $\Lambda = \{\mu | \mu$ is a probability measure with support in $G\}$. $\Lambda$ is a convex set.

**Lemma 7.3.** Fix $\omega \in \Omega$, $P(G_{\omega})$ is the convex hull spanned by $\{\delta_\omega(x,v) | (x,v) \in G_{\omega}\}$.

**Proof.** We claim that $F_{\omega} = \{\sum_{k=1}^{n} a_k \delta_\omega(x_k,v_k) | n \in \mathbb{N}, \sum_{k=1}^{n} a_k = 1, 0 \leq a_k \leq 1, (x_k, v_k) \in G_{\omega}\}$ are dense in $P(G_{\omega})$. $G_{\omega}$ is a compact subset of the complete separable metric space $TM$ with distance $d_{TM}$. The proof that $F_{\omega}$ is dense in $P(G_{\omega})$ is as same as that of Lemma 3.3. See [10]. Hence, $P(G_{\omega})$ is the convex hull spanned by $\{\delta_\omega(x,v) | (x,v) \in G_{\omega}\}$.

**Lemma 7.4.** If $G$ is measurable, For any $\mu \in \Lambda$, for almost every $\omega \in \Omega$, there exists $\mu_\omega \in P(G_{\omega})$, such that for any measurable function $f : TM \times \Omega \to \mathbb{R}$, we have

$$\int_G f(x,v,\omega)d\mu = \int_\Omega \int_{G_{\omega}} f(x,v,\omega)d\mu_\omega d\omega$$

**Proof.** If $\mu \in \Lambda$, then $\mu \in P(TM \times \Omega)$, since $TM$ and $\Omega$ are polish spaces, by Theorem 3.1, for almost every $\omega \in \Omega$, there exists $\nu_\omega \in P(TM)$, such that for any measurable function $f : TM \times \Omega \to \mathbb{R}$, we have

$$\int_{TM \times \Omega} f(x,v,\omega)d\mu = \int_\Omega \int_{TM} f(x,v,\omega)d\nu_\omega d\omega$$

Then, let $\mu_\omega = \chi_{G_{\omega}}\nu_\omega \in P(G_{\omega})$, we know that

$$\int_{TM \times \Omega} f(x,v,\omega)d\mu = \int_{TM \times \Omega} f(x,v,\omega)\chi_G d\mu$$

$$= \int_\Omega \int_{TM} f(x,v,\omega)\chi_{G_{\omega}}d\nu_\omega d\omega = \int_\Omega \int_{TM} f(x,v,\omega)d\mu_\omega d\omega$$

Le $\pi_\Omega : TM \times \Omega \to \Omega$ as the canonical projection.

**Lemma 7.5.** If $\mu \in \Lambda$, and $\mu$ is invariant under the transformation of $\Gamma(s), s \in \mathbb{R}$. We decompose $d\mu = d\mu_\omega d\omega$, where $\mu_\omega$ is a probability measure on $G_{\omega}$ for almost all $\omega \in \Omega$. We have $d\mu_\omega d\omega = d\Theta(s)^*\mu_{\theta(-s)\omega} d\Theta(-s)\omega$. If $\pi_{\Omega} \mu$ is invariant under the transformation $\{\theta(s), s \in R\}$, we have $\mu_\omega = \Theta(s)^*\mu_{\theta(-s)\omega}$ for almost every $\omega \in \Omega$.

**Proof.** We assume that $\mu$ is invariant under $\{\Gamma(s), s \in \mathbb{R}\}$. For any measurable function $f :
\[ TM \times T \times \Omega \rightarrow R \]

\[
\int_{\Omega} \int_{G_\omega} f(x,v,\omega) d\mu_\omega d\omega = \int_{G} f(x,v,\omega) d\mu = \int_{G} f(x,v,\omega) d\Gamma^*(s) \mu
\]

\[
= \int_{G} f(\Theta(s)|_{G_\omega}(x,v),\theta(s)\omega) d\mu = \int_{\Omega} \int_{G_\omega} f(\Theta(s)|_{G_\omega}(x,v),\theta(s)\omega) d\mu_\omega d\omega
\]

\[
= \int_{\Omega} \int_{G_{\theta(-s)\omega}} f(\Theta(s)|_{G_{\theta(-s)\omega}}(x,v),\omega) d\mu_{\theta(-s)\omega} d\theta(-s) \omega
\]

\[
= \int_{\Omega} \int_{G_{\theta(-s)\omega}} f(x,v,\omega) d\Theta(s)^* \mu_{\theta(-s)\omega} d\theta(-s) \omega
\]

\[
= \int_{G} f(x,v,\omega) d\Theta(s)^* \mu_{\theta(-s)\omega} d\theta(-s) \omega
\]

Since \( f \) is arbitrary, we have \( d\mu_\omega d\omega = d\Theta(s)^* \mu_{\theta(-s)\omega} d\theta(-s) \omega \). If \( \pi_\Omega \mu \) is invariant under \( \{\theta(s), s \in R\} \), then we have \( \mu_\omega = \Theta(s)^* \mu_{\theta(-s)\omega} \) for almost every \( \omega \in \Omega \).

**Theorem 7.1.** When \( G \) is a measurable set in \( TM \times \Omega \). If \( \mu \) is an ergodic invariant measure of \( \{\Theta(s), s \in R\}, \pi_\Omega(\mu) \) is invariant under the transformation \( \{\theta(s), s \in R\} \). Then we know that for almost \( \omega \in \Omega \), there exists \( (x_\omega, v_\omega) \in TM \), such that \( \mu_\omega = \delta(x_\omega, v_\omega) \). And when \( s \in R \), we have \( \Theta(s)(x_\omega, v_\omega) = (x_{\theta(s)\omega}, v_{\theta(s)\omega}) \) for almost all \( \omega \in \Omega \).

**Proof.** We assume that \( \mu \) is an ergodic invariant measure under the transformation \( \{\Theta(s), s \in R\} \). Since the set of invariant measure is a closed convex set. It is well known that the ergodic measure is the extreme point of the convex set, see \([9]\). By Lemma 7.4, for almost all \( \omega \in \Omega \), we know that \( \mu_\omega \) is an extreme point of \( P(G_\omega) \), there exists \( (x_\omega, v_\omega) \in TM \), such that \( \mu_\omega = \delta(x_\omega, v_\omega) \). By Lemma 7.6, when \( s \in R \), we know that \( (x_{\theta(s)\omega}, v_{\theta(s)\omega}) = \Theta(s)(x_\omega, v_\omega) \) for almost every \( \omega \in \Omega \).

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