ON THE INTEGRABILITY OF THE SHIFT MAP ON TWISTED PENTAGRAM SPIRALS

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Abstract. In this paper we prove that the shift map defined on the moduli space of twisted pentagram spirals of type $(N,1)$ posses a Lax representation with an associated monodromy whose conjugation class is preserved by the map. We prove this by finding a coordinate system in the moduli space of twisted spirals, writing the map in terms of the coordinates and associating a natural parameter-free Lax representation. We then show that the map is invariant under the action of a 1-parameter group on the moduli space of twisted $(N,1)$ spirals, which allow us to construct the Lax pair.

1. Introduction

The pentagram map is defined on planar, convex $N$-gons. The map $T$ takes a polygon with vertices $p_k$ to the polygon with vertices formed by intersecting two segments: one created by joining the vertices to the right and to the left of the original one, $p_{k-1}p_{k+1}$, the second one by joining the original vertex to the second vertex to its right $p_k p_{k+2}$ (see Fig. 1). These newly found vertices form a new $N$-gon. The pentagram map takes the first $N$-gon to this newly formed one. (The name pentagram map comes from the star formed in Fig. 1 when applied to pentagons.) As surprisingly simple as this map is, it has an astonishingly large number of properties.

![Figure 1: the pentagram map on hexagons](image)

It is a classical fact that if $P$ is a pentagon, then $T(P)$ is projectively equivalent to $P$. It also seems to be classical that if $P$ is a hexagon, then $T^2(P)$ is projectively equivalent to $P$ as well. The constructions performed to define the pentagram map are supported by the author NSF grant DMS #0804541 and #1405722.
can be equally carried out in the projective plane so we assume this is where the polygons live. In that case \( T \) is the identity when defined on the moduli space of pentagons (as described by the projective invariants of the polygons), while it is an involution when defined on the moduli space of hexagons. In general, one should not expect to obtain a closed orbit for any \( N \); in fact orbits exhibit a quasiperiodic behavior classically associated to completely integrable systems. This was conjectured in [15].

A recent number of papers ([11, 12, 13, 14, 15, 16, 19]) have studied the pentagram map and established its completely integrable nature, in the Arnold-Liouville sense. The author of [15] defined the pentagram map on what he called \( N \)-twisted polygons, that is, infinite polygons with vertices \( p_k \), for which \( p_{N+k} = M(p_k) \) for all \( k \), where \( M \) is the monodromy, a projective automorphism of \( \mathbb{RP}^2 \). The authors of [11] proved that, when written in terms of the projective invariants of twisted polygons, the pentagram map is in fact Hamiltonian and completely integrable. The paper also showed that the pentagram map, when expressed in terms of projective invariants, is a discretization of the Boussinesq equation, a well-known completely integrable system of PDEs. Integrability in the case of closed gons was proved in [12] and [19].

More recently, in [17] Schwartz defined what he called a pentagram spiral, a family of bi-infinite polygons in the projective plane that spiral inside and outside of themselves following a pentagram map-type of construction. The spiral is determined by a seed, vertices of a closed polygon together with a number of points on the sides of the closed polygon that mark the moment when the polygons start spiraling (see fig. 2, the stars mark the closed polygon, squares are the side points). Spirals are classified by two numbers \((N,k)\) where \( k \) is the number of side points in the \( N \)-closed polygon. Fig. 2 shows a \((5,2)\) spiral. A pentagram spiral has at most one seed point per side, and \( k \) marks the number of spiraling branches that the pentagram spiral has.
In [17] Schwartz studied the shift map on pentagram spirals, the map that assigns to each spiral the one obtained by shifting the vertices (and hence the seed) once forward, forward for us being the direction towards the interior of the polygon. He proved that such a map can be thought of, in a certain sense, as the \((N + 1)\)th root of the original pentagram map, and he also conjectured that, like the pentagram map, the shift map is also completely integrable.

In this paper we study the case of twisted \((N, 1)\) spirals, denoted by \(\text{TS}(N, 1)\); these are spirals where a monodromy map is applied each time the pentagram map acts after a full period \(N\). The general case \((N, k)\) is now in progress. The paper is divided into several parts. In the first part we find a generating family of projective invariants for generic elements in \(\text{TS}(N, 1)\), and we use them to define a coordinate system in the moduli space of \(\text{TS}(N, 1)\). We also use them to describe a parameter-free Lax representation for the shift map. The invariants will be found similarly to those of twisted polygons: we choose an appropriate lift of the spiral to \(\mathbb{R}^3\) by imposing a number of normalizations. The lift defines a discrete moving frame for the spiral and it provides us with a complete set of generating and independent projective invariants for twisted spirals which define a coordinate system in the moduli space. This is done in section 3, with theorem 3.3 describing the coordinate system. We then notice that once we shift the spiral the invariants will not merely be shifted. Indeed, the lift for the shifted spiral will have different normalization conditions and hence it will be different from the original lift (this is expected since the shifted spiral has shifted seeds).

In lemma 3.4 we prove that there are two proportions \(\alpha\) and \(\beta\), determined by the two changes of seed when shifted - at the beginning and the end -, such that the shifted lift equals the original lift times certain powers of \(\alpha\) and \(\beta\). The powers depend on the vertex and repeat every three vertices, except for the last ones. Lemma 3.6 shows that the generating invariants also transform by shifting and multiplying by powers of \(\alpha\) and \(\beta\). This is true for most invariants with the exception of the end seed point, the vertex where the polygon starts to spiral. Those end points need to be treated carefully. Lemma 3.6 and the proof of theorem 3.8 gives an explicit formula for the shift map in these coordinates. (We also describe, towards the end, a different set of coordinates for which the shift map is simply a shift of coordinates, except for the coordinates of the last vertex for which the map is highly complicated.)

The last step is to prove that \(\alpha\) and \(\beta\) are invariant under a certain 1-parameter group action on the moduli space defined through scaling of the invariants (the same scaling used for the pentagram map) and to use this to show that the shift map is left invariant by that action. This is done in Theorem 3.8. Introducing the scaling in the parameter-free Lax representation produces a valid Lax pair that can be used to integrate. The Lax pair and the description of the preserved quantities are given in our last section, where, as an example, we generate invariants of the map for \(N = 5\).

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2. BACKGROUND

2.1. The pentagram map and pentagram spirals. The pentagram map was originally defined by the author of [13] as a map defined on the space of closed $N$-polygons in the projective plane

$$T : \mathcal{C}(N) \to \mathcal{C}(N).$$

Given a closed $N$-polygon $\{p_i\}_{n=0}^{N-1}$, $p_i \in \mathbb{RP}^2$, we define the image of this polygon by the pentagram map as $T(\{p_i\}) = \{q_i\}$ where

$q_i = p_{i-1}p_{i+2} \cap p_ip_{i+1}$.

(See Fig. 1.) The map can equally be thought to act on the vertices of the polygons as $T(p_i) = q_i$. This map can also be defined on the space of twisted polygons.

**Definition 2.1.** (Twisted polygon) Let $P = \{p_i\}$ be an infinite polygon in the projective plane. We say that $P$ is an $N$-twisted polygon whenever there is a projective transformation $M$ such that $p_i + N = M(p_i)$ for all $i$. $M$ is called the monodromy.

One can easily check that the pentagram map is well defined on the space of twisted polygons. Following a somehow dormant period the pentagram map has come back with full force after the publication of [11] where the authors proved that the map on twisted polygons is not only completely integrable, but the continuous limit is the Boussinesq equation, a well known completely integrable PDE modeling certain dynamics of waves. A large number of publications have followed this, proving the integrability of the map on closed polygons ([12], [19]), defining and proving integrability of generalizations ([2], [5], [8], [9]), establishing connections to cluster algebras ([4]) and more. During the last year Schwarz has defined two new maps, the heat map on closed polygons ([18]) and the shift map on pentagram spirals ([17]). Here we will focus on pentagram spirals and their shift maps.

The idea of a pentagram spiral is built upon a choice of points on the sides of a closed polygon from which the polygon starts spiraling inside itself using pentagram transformations of consecutive vertices. We need merely one side point to start creating a branch of the spiral (see fig. 3), but one can, in principle, choose one point in several sides, creating the branching of several spirals, each one created using the pentagram image of the previous branch (see fig. 2). The vertices of the original polygon upon which the construction is built, together with the side points form what the author of [17] called the seed of the spiral. See [17] for more details. A spiral built with the seed of an $N$-gon with $k$ distinguished points on $k$ different sides is called a pentagram spiral of type $(N,k)$. In this paper we will focus on spirals of type $(N,1)$.

**Definition 2.2.** ($\langle N,1 \rangle$ Pentagram spiral) Given an $N$-polygon in $\mathbb{RP}^2$, $\{p_1, \ldots, p_N\}$, and a point $p_{N+1}$ on the side joining $p_N$ and $p_1$, we define the $(N,1)$ pentagram spiral associated to the seed $\{p_1, \ldots, p_N; p_{N+1}\}$ as the bi-infinite polygon with ordered vertices

\[ \cdots, T^{-1}(p_{N-1}), T^{-1}(p_N), T^{-1}(p_{N+1}), p_1, p_2, \ldots, p_N, p_{N+1}, T(p_1), T(p_2), \ldots. \]

Let us call $p_{N+i} = T(p_{i-1})$ and $p_{i-1} = T^{-1}(p_{N+i})$, for $i \geq 1$ so that $p_{N+1} = T(p_0)$. Figure 3 shows a standard $(6,1)$-pentagram spiral.
The moduli space of \((N, 1)\) pentagram spirals was proven to be \(2N - 7\) dimensional in [17]. In order to facilitate the creation of a Lax representation for the shift map on spirals we will consider twisted pentagram spirals by introducing a monodromy. The author of [17] also defined the concept of twisted spiral, with a more abstract approach, and proved that its moduli space was \(2N + 1\) dimensional. Although described differently, both concepts coincide.

**Definition 2.3.** (Twisted pentagram spirals) Given an \(N\)-polygon in the projective plane \(\mathbb{RP}^2\), \(\{p_1, \ldots, p_N\}\), a point \(p_{N+1}\) in the segment joining \(p_N\) and \(p_1\), and an element of the projective group \(M \in \text{PSL}(3, \mathbb{R})\), we define the twisted pentagram spiral associated to the seed \(\{p_1, \ldots, p_N, p_{N+1}\}\) with monodromy \(M\) as the infinite polygon with ordered vertices

\[
\{ \ldots, M^{-1} \cdot T^{-1}(p_{N-1}), M^{-1} \cdot T^{-1}(p_N), p_0, p_1, p_2, \ldots, p_N, M \cdot T(p_0), \\
M \cdot T(p_1), M \cdot T(p_2), \ldots \},
\]

where now \(p_{N+i} = M \cdot T(p_{i-1})\) and \(p_{-i} = M^{-1} \cdot T^{-1}(p_{N-i+1})\), for \(i \geq 1\), and where the monodromy \(M\) acts each time a period \(N\) is completed and \(T\) is applied.

Later on we will prove that the moduli space of twisted pentagram spirals is a space of dimension \(2N + 1\) and we will describe a generating set of invariants that will define coordinates in it.

### 2.2. Discrete moving frames and invariants.

In this section we will describe basic definitions and facts that will be useful along this paper on the subject of
discrete group-based moving frames. They are taken from [6] and occasionally slightly modified to fit our needs.

Let $M$ be a manifold and let $G \times M \to M$ be the action of a group $G$ on $M$. We will assume that $G \subset \text{GL}(n, \mathbb{R})$. If $G$ acts on $M$, it also has a natural induced action on the space of $N$-polygons, $P_N$, given by the diagonal action $g \cdot (p_i) = (g \cdot p_i)$.

**Definition 2.4** (Discrete moving frame). Let $G^N$ denote the Cartesian product of $N$ copies of the group $G$. Elements of $G^N$ will be denoted by $(g_k)$. Allow $G$ to act on the left on $G^N$ using the diagonal action $g \cdot (g_k) = (gg_k)$.

We say a map $\rho : P_N \to G^N$ is a left discrete moving frame if $\rho$ is equivariant with respect to the action of $G$ on $P_N$ and the left diagonal action of $G$ on $G^N$. Since the image of $\rho$ is in $G^N$, we will denote by $\rho_i$ its $i$th component; that is $\rho = (\rho_i)$, where $\rho_i(p) \in G$ for all $i$, $p = (p_i)$.

In short, $\rho$ assigns an element of the group to each vertex of the polygon in an equivariant fashion. Notice that, in principle, one can choose $N = +\infty$ for infinite polygons. For more information on discrete moving frames see [6]. These group elements carry the invariant information of the polygon.

**Definition 2.5** (Discrete invariant). Let $I : P_N \to \mathbb{R}$ be a function defined on $N$-gons. We say that $I$ is a scalar discrete invariant if

$$I((g \cdot p_i)) = I((p_i))$$

for any $g \in G$ and any $p = (p_i) \in P_N$.

We will naturally refer to vector discrete invariants when considering vectors whose components are discrete scalar invariants, etc.

**Definition 2.6** (Maurer–Cartan matrix). Let $\rho$ be a left discrete moving frame evaluated along an $N$-gon. The element of the group

$$K_i = \rho_i^{-1}\rho_{i+1}$$

is called the left $i$-Maurer–Cartan matrix for $\rho$. We will call the equation $\rho_{i+1} = \rho_i K_i$ the left $i$-Serret–Frenet equation.

The entries of a Maurer–Cartan matrix are functional generators of all discrete invariants of polygons, as it was shown in [6]. (The work in [4] concerns twisted polygons, but it can be easily generalized to infinite ones.)

Finally, let us extend a map $T : P_N \to P_N$ to functions of $p = (p_i)$ the standard way using the pullback $(T(f(p)) = f(T(p)))$. Let us also extend it to elements of the group by applying it to each entry of the matrix. We will denote the map with the same letter, abusing notation. Define the matrix

$$N_i = \rho_i^{-1}T(\rho_i).$$

The following relationship with the Maurer-Cartan matrix is straightforward

$$T(K_i) = T(\rho_i^{-1})T(\rho_{i+1}) = T(\rho_i)^{-1}\rho_i\rho_i^{-1}\rho_{i+1}\rho_{i+1}^{-1}T(\rho_{i+1}) = N_i^{-1}K_iN_{i+1}.$$
solving an algebraic system of equations, as shown in [6]. That means that the equations
\begin{align}
\rho_{i+1} &= \rho_i K_i; \\
T(\rho_i) &= \rho_i N_i;
\end{align}
(6)
form a parameter free Lax representation of the map on the moduli space as defined by the invariants. The Serret–Frenet equations $\rho_{i+1} = \rho_i K_i$ together with $T(\rho_i) = \rho_i N_i$ define a parameter-free discrete AKNS representation of the map $T$, with the moving frame as its solution. This representation exists for any map on the moduli space induced by a map defined on polygons (a continuous version of this fact also exists, see [10]).

Of course, integrability is achieved through a non-trivial Lax representation containing a spectral parameter. But if the map turns out to be invariant under a certain scaling, or indeed under the action of a 1-parameter group, adding that scaling to the equations in (6) (that is, substituting each invariant by its scaled version) will introduce the scaling parameter $\mu$ as spectral parameter. Indeed, invariance of $T$ under the scaling will guarantee that the equation to the right in (6) does not depend on $\mu$ and we will have achieved a regular Lax representation. This is a well known approach to create Lax representations in integrable systems and it was used in [5] and [9] to generate the Lax representation for the cases they studied.

3. The moduli space and the shift map

3.1. A coordinate system for the moduli space of twisted pentagram spirals. The moduli space of twisted $N$-polygons in $\mathbb{R}P^2$ has been well studied in [11], where the authors proved that the space has dimension $2N$. They also described a coordinate system defined by a basis of projective discrete invariants of polygons. It is defined as follows:

Assume that we have a twisted $N$-polygon $\{p_i\}$, with a monodromy $M$ (so that, $p_{N+i} = M \cdot p_i$ for all $i$). One can prove (11) that if $N \neq 3s$ for all $s$, then there exist unique lifts of $p_i$ to $\mathbb{R}^3$, call them $V_i$, such that
\[ \det(V_i, V_{i+1}, V_{i+2}) = 1 \]
for all $i$. The matrices $\rho_i = (V_i, V_{i+1}, V_{i+2}) \in \text{SL}(3, \mathbb{R})$, $i = 0, 1, 2, \ldots, N-1$, define a discrete moving frame along the polygon. Under these conditions, one can always find invariants $e_i, f_i$ satisfying the relation
\[ V_{i+3} = e_i V_{i+2} + f_i V_{i+1} + V_i \]
(7)
for all $i$, where $e_i, f_i$ are functions of the vertices of the polygon. The functions $e_i, f_i$, $i = 0, 1, 2, \ldots, N-1$, define a basis for the space of discrete invariants for planar, projective $N$-polygons, meaning that any other invariant can be written as a function of them (see [6]). Therefore, they also define a coordinate system for their moduli space. In this section we define the analogous set of coordinates for pentagram spirals. The key to this definition and to the existence of the Lax pair is to choose the lifts of $T(\rho_i)$ and $T^{-1}(\rho_i)$ appropriately to guarantee the scaling invariance of the map, and the existence of the Lax pair, which is our ultimate goal.
Theorem 3.1. Let $P$ be a pentagram spiral and assume that $N \neq 3s + 1$. Then there exists a unique lift of the seed $\{p_{1}, p_{2}, \ldots, p_{N}; p_{N+1}\}$ to $\{V_{1}, \ldots, V_{N}; V_{N+1}\}$ and a unique lift of the spiral $P$ to the polygon in $\mathbb{R}^{3}$ with vertices $\{V_{i}\}_{-\infty}^{+\infty}$

$$\ldots, V_{-1}, V_{0}, V_{1}, V_{2}, \ldots, V_{N}, V_{N+1}, V_{N+2}, \ldots$$

such that $V_{N+i} = MT(V_{i-1})$ with

$$(8) \quad T(V_{i}) = (V_{i-1} \times V_{i+1}) \times (V_{i} \times V_{i+2});$$

$V_{-i} = M^{-1}T(V_{N-i+1})$ with

$$(9) \quad T(V_{i}) = c_{i+1} (V_{i} \times V_{i+1}) \times (V_{i-2} \times V_{i-1})$$

where $c_{i} = \frac{\det(V_{i+1}, V_{i+2}, V_{i+3})}{\det(V_{i}, V_{i+1}, V_{i+2})}$; and such that

$$(10) \quad \det(V_{i}, V_{i+1}, V_{i+2}) = 1$$

for $i = 0, 1, 2, \ldots, N$.

Comment 3.2. First of all, notice that we are abusing notation by using the letter $T$ for both the projective map and its lift. The domain should make clear which one is which. Notice also, that the lift of $T$ is one of an infinite number of choices. Finally, as it will become clear later, $\tilde{T}$ is not equal, but proportional, to the inverse of the lift $T$, with the proportion having an important role in the coordinate description of the shift map. One can choose other lifts of $T^{-1}$, but the one chosen here will make our calculations the simplest.

Proof. Let $\tilde{V}_{k}$ be any lift of $p_{i}$, $i = 0, 1, \ldots, N$, with $p_{0} = T^{-1}(p_{N+1})$. Assume the lift we are looking for is of the form $V_{k} = \lambda_{k} \tilde{V}_{k}$. From the definition of $T(V_{i})$ in the statement of the theorem, it is clear that

$$T(V_{i}) = \lambda_{i-1} \lambda_{i} \lambda_{i+1} \lambda_{i+2} \tilde{T}(V_{i})$$

where $\tilde{T}(V_{i}) = (\tilde{V}_{i-1} \times \tilde{V}_{i+1}) \times (\tilde{V}_{i} \times \tilde{V}_{i+2})$. Hence, and since $V_{N+k} = MT(V_{k-1})$, we can define

$$(11) \quad \lambda_{N+1} = \lambda_{-1} \lambda_{0} \lambda_{1} \lambda_{2}, \quad \lambda_{N+k} = \lambda_{k-2} \lambda_{k-1} \lambda_{k} \lambda_{k+1} \lambda_{N+k+1}$$

for all $k = 2, \ldots$, where $\lambda_{-1}$ corresponds to the lift $V_{-1} = M^{-1}T(V_{N})$. Let us find $\lambda_{-1}$ first. Since

$$MV_{-1} = T(V_{k}) = c_{N+1} (V_{N} \times V_{N+1}) \times (V_{N-2} \times V_{N-1})$$

from the definition of $c_{i}$ in the statement of the theorem, we have that

$$c_{N+1} = \frac{\lambda_{N+4}}{\lambda_{N+1}} \tilde{c}_{N+1}$$

where the tilde indicates that all $V_{i}$’s has been substituted by the original lifts $\tilde{V}_{i}$ in the definition of $c_{N+1}$. From here

$$V_{-1} = \frac{\lambda_{N+4}}{\lambda_{N+1}} \lambda_{N} \lambda_{N+1} \lambda_{N-2} \lambda_{N-1} \lambda_{N-2} \tilde{V}_{-1},$$

and therefore

$$(12) \quad \lambda_{-1} = \lambda_{N+4} \lambda_{N} \lambda_{N-1} \lambda_{N-2} = \lambda_{2} \lambda_{3} \lambda_{4} \lambda_{5} \lambda_{N} \lambda_{N-1} \lambda_{N-2}.$$
where \( g_i = \det(\tilde{V}_i, \tilde{V}_{i+1}, \tilde{V}_{i+2})^{-1} \), for \( i = 0, 1, \ldots, N \). If we apply logarithms to both sides of these equation (adjusting for signs if needed), we get the system

\[
\Lambda_i + \Lambda_{i+1} + \Lambda_{i+2} = G_i
\]

where \( \Lambda_i = \ln \lambda_i; G_i = \ln g_i, \ i = 0, 1, \ldots, N \). We additionally need to have in mind that

\[
\Lambda_{N+k} = \Lambda_{k-2} + \Lambda_{k-1} + \Lambda_k + \Lambda_{k+1},
\]

for \( k = 2, 3, \ldots, \) and that

\[
\Lambda_{N+1} = \Lambda_{N+4} + \Lambda_N + \Lambda_{N-1} + \Lambda_{N-2} + \Lambda_0 + \Lambda_1 + \Lambda_2
\]

\[
= \Lambda_N + \Lambda_{N-1} + \Lambda_{N-2} + \Lambda_0 + \Lambda_1 + 2\Lambda_2 + \Lambda_3 + \Lambda_4 + \Lambda_5.
\]

The \((N + 1) \times (N + 1)\) coefficient matrix of this system is thus given by

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 2 & 1 & 1 & 1 & 0 & \ldots & 0 & 1 & 2 & 2 \\
2 & 2 & 3 & 2 & 1 & 1 & 0 & \ldots & 0 & 1 & 1 & 1
\end{pmatrix}
\]

We need to calculate its determinant. If we use rows one, two and \(N - 1\) to row reduce the last two rows, they become

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & \ldots & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & \ldots & 0 & 0 & 1
\end{pmatrix}.
\]

This reduction allow us to remove the first two rows and columns of the matrix so the determinant of the coefficient matrix equal that of the \((N - 1) \times (N - 1)\) matrix

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & \ldots & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}.
\]

Using again the first two rows and row reduction we can change the last two rows to

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & \ldots & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 1
\end{pmatrix}
\]

which again allow us to remove the first two rows and columns and have the determinant be equal to that of the \((N - 3) \times (N - 3)\) matrix

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & \ldots & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & \ldots & 0 & 1 & 1
\end{pmatrix}.
\]

Once this form of the matrix is achieved, the same process (using the first two rows to row reduce the last two) will produce an exact replica of the matrix with a
smaller size. Each reduction process will reduce the size by 3. Since we start with size \(N - 3\), reiterating the process we get to

\[
\begin{pmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1
\end{pmatrix}, \quad \text{or} \quad \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

in the cases \(N = 3s + 1, N = 3s + 2\) or \(N = 3s\), respectively. The determinants of these matrices are 0, -1 and 1, also respectively, therefore the lift is unique if, and only if \(N \neq 3s + 1\), as stated.

Once we have the appropriate lift, we can define a generating system of discrete invariants for the space of twisted spirals. This set will define coordinates in the moduli space.

Let \(\{a_i, b_i, c_i\}_{i=1}^\infty\) be defined by the relation

\[
V_{i+3} = a_i V_{i+2} + b_i V_{i+1} + c_i V_i.
\]

for any \(i\). (Notice that the definition of \(c_i\) in the statement of the theorem 3.1 coincides with this one.) Before moving on let us write formulas for the liftings \(T(V_i)\) and \(\overline{T}(V_i)\).

From the lift definition

\[
T(V_i) = (V_{i-1} \times V_{i+1}) \times (V_i \times V_{i+2})
\]

\[
= (V_{i-1} \times V_{i+1}) \times (V_i \times (a_{i-1} V_{i+1} + b_{i-1} V_i + c_{i-1} V_{i-1}))
\]

\[
= \det \rho_{i-1}(a_{i-1} V_{i+1} + b_{i-1} V_i + c_{i-1} V_{i-1})
\]

for \(i = 0, 1, \ldots, N\), where we have used the rule \((a \times b) \times (b \times c) = \det(a, b, c)b\), and where \(\det \rho_i = \det(V_i, V_{i+1}, V_{i+2})\). From conditions (10) we see that \(\det \rho_i = 1\), \(i = 0, \ldots, N\).

Notice that, from (13) one also obtains

\[
T(V_i) = \det \rho_{i-1}(a_{i-1} V_{i+1} + c_{i-1} V_{i-1}) = \det \rho_{i-1}(V_{i+2} - b_{i-1} V_i)
\]

for any \(i = 0, 1, 2, \ldots, N\).

Similarly we can write a formula for \(\overline{T}(V_i)\). Indeed, from the lift definition

\[
\overline{T}(V_i) = c_{i+1} (V_i \times V_{i+1}) \times (V_{i-2} \times V_{i-1})
\]

\[
= c_{i+1} (V_i \times (a_{i-2} V_i + b_{i-2} V_{i-1} + c_{i-2} V_{i-2})) \times (V_{i-2} \times V_{i-1})
\]

\[
= c_{i+1} \det \rho_{i-2}(b_{i-2} V_{i-1} + c_{i-2} V_{i-2}),
\]

which, as before, can be rewritten as

\[
\overline{T}(V_i) = c_{i+1} \det \rho_{i-2}(b_{i-2} V_{i-1} + c_{i-2} V_{i-2}) = c_{i+1} \det \rho_{i-2}(V_{i+1} - a_{i-2} V_i).
\]

**Theorem 3.3.** The moduli space of generic, strictly convex twisted spirals is a \(2N + 1\) manifold. A set of coordinates for a generic spiral is given by the set of discrete invariants \(G = \{\{a_i, b_i\}_{i=0}^{N-1}, c_N\}\).

**Proof.** The first point to notice is that the set \(\{a_i, b_i, c_i\}_{i=-\infty}^{\infty}\) is a generating set for any discrete invariant of the space of strictly convex twisted spirals. This is clear since, if

\[
\rho_i = (V_i, V_{i+1}, V_{i+2})
\]
is the $i$-left moving frame, then its Serret-Frenet equation is given by
\begin{equation}
\rho_{i+1} = \rho_i \begin{pmatrix} 0 & 0 & c_i \\ 1 & 0 & b_i \\ 0 & 1 & a_i \end{pmatrix} = \rho_i K_i
\end{equation}
for any $i$. Clearly, given the set $\{a_i, b_i, c_i\}_{i=1}^\infty$, and an initial condition $\rho_0$, one can determine completely the unique lifts $V_i$ for all $i$, and hence the spiral. Therefore, to prove the theorem we need to show that $\{a_i, b_i, c_i\}_{i=-\infty}^\infty$ is generated only by $\mathcal{G} = \{\{a_i, b_i\}_{i=0}^{N-1}, c_N\}$.

We know that $c_i = 1$ for $i = 0, 1, 2, \ldots, N - 1$. Also, since $\rho_{N+1} = \rho_N K_N$ and $\det K_N = c_N$, we have that $c_N = \det \rho_{N+1}$.

We will focus on the one hand on $a_N$, $b_N$ and the invariants with higher subindex, and on the other hand on invariants with negative subindices.

For this, let us associate $T(V_i)$ to lower indexed moving frames. From (14), if $i = 1, 2, \ldots, N$, we have
\begin{equation}
T(V_i) = d_i(a_i-1V_{i+1} + c_i-1V_{i-1}) = d_i\rho_{i-1} \begin{pmatrix} c_{i-1} \\ 0 \\ a_{i-1} \end{pmatrix}
\end{equation}
and, since $\rho_{i-1} = \rho_{i-2} K_{i-2}$, with $K_i$ as in (17), we can write
\begin{equation}
T(V_i) = d_i \rho_{i-2} K_{i-2} \begin{pmatrix} c_{i-1} \\ 0 \\ a_{i-1} \end{pmatrix} = d_i \rho_{i-2} K_{i-2} \cdots K_{i-2} \begin{pmatrix} c_{i-1} \\ 0 \\ a_{i-1} \end{pmatrix},
\end{equation}
For any $s$. This is true for $i = 0, 1, 2, \ldots$. Using these relations and straightforward calculations
\begin{align*}
\rho_{N+i+1} &= M(T(V_i), T(V_{i+1}), T(V_{i+2})) \\
&= M \rho_{i-1} A_{i+1}
\end{align*}
i = 0, 1, 2, \ldots, where
\begin{equation}
A_{i+1} = \begin{pmatrix} d_i-1 & 0 & c_i-1 \\ 0 & a_{i-1} \end{pmatrix} d_i K_{i-1} \begin{pmatrix} c_i \\ 0 \\ a_i \end{pmatrix} d_i+1 K_{i-1} K_i \begin{pmatrix} c_{i+1} \\ 0 \\ a_{i+1} \end{pmatrix}.
\end{equation}
(Recall that $V_{N+1} = MT(V_0)$, but $V_0 \neq M^{-1} T(V_{N+1})$.)

We also have
\begin{equation}
\rho_{N+i+1} = \rho_{N+i} K_{N+i}
\end{equation}
and as above
\begin{equation}
\rho_{N+i} = M \rho_{i-2} A_i
\end{equation}
for $i = 1, 2, 3, \ldots, N$. Putting these three relations together we get that $K_{N+i}$ is a gauged transformation of $K_{i-2}$ by the matrix $A_i$
\begin{equation}
K_{N+i} = A_i^{-1} K_{i-2} A_{i+1},
\end{equation}
i = 1, 2, \ldots. This allows us to conclude that the higher indexed invariants $K_{N+i}$, $i = 1, 2, 3, \ldots, N$, can be written in terms of $\mathcal{G}$, $K_N$ and $K_{-1}$. In particular,
straightforward calculations (double checked with the computer) show that $K_{N+1} = A_1^{-1}K_{-1}A_2$ is given by

$$K_{N+1} = \begin{pmatrix} 0 & 0 & c_{N+1} \\ 1 & 0 & b_{N+1} \\ 0 & 1 & a_{N+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_{-1}A_2 \\ A_0 & b_{-1}A_2 \\ 0 & 1 & a_2 \end{pmatrix}$$

where $A_i = c_i + a_i b_{i-1}$ for any $i$.

A very similar argument can be made for $K_{-i}$. Indeed, $V_{-i} = M^{-1}T(V_{N-i+1})$, $i = 1, 2, \ldots$, where

$$\mathbf{T}(V_{i}) = c_{i+1}d_{i-2}(b_{i-2}V_{i-1} + c_{i-2}V_{i-2}) = c_{i+1}d_{i-2}\rho_{i-2} \begin{pmatrix} c_{i-2} \\ b_{i-2} \\ 0 \end{pmatrix}.$$ 

As before,

$$\rho_{-i} = M^{-1}(\mathbf{T}(V_{N-i+1}), \mathbf{T}(V_{N-i+2}), \mathbf{T}(V_{N-i+3})) = M^{-1}\rho_{N-i-1}B_{-i}$$

for any $i \geq 3$, where

$$B_{-i} = \begin{pmatrix} c_{N-i+2}d_{N-i-1} & c_{N-i}d_{N-i-1}K_{N-i-1} \\ b_{N-i+1} & 0 \\ 0 & 1 \end{pmatrix}.$$ 

And since

$$\rho_{-i+1} = M^{-1}\rho_{N-i}B_{-i+1}, \quad \rho_{-i+1} = \rho_{-i}K_{-i}$$

for $i \geq 4$, we have that

$$\rho_{N-i}B_{-i+1} = \rho_{N-i-1}K_{N-i-1}B_{-i+1} = \rho_{N-i-1}B_{-i}K_{-i}.$$ 

Subsequently, the matrix $K_{-i}$ is a gauge transformation of $K_{N-i-1}$ by the matrix $B_{-i}$

$$K_{-i} = B_{-i}^{-1}K_{N-i-1}B_{-i+1}$$

for any $i = 4, \ldots, N$. From here we can conclude that all the invariants with these negative indices can be written as functions of $\mathcal{G}$ and $K_{-1}$.

Summarizing the above, we are now left with proving that $K_{-3}, K_{-2}, K_{-1}, a_N$ and $b_N$ are functional combinations of $\mathcal{G}$, and we will have proven the theorem.

We start with the simplest ones, the matrices $K_{-2}$ and $K_{-3}$. From (22) we see that

$$\rho_{-3} = M^{-1}\rho_{N-4}B_{-3}$$

but that formula does not work for either $\rho_{-2}$ or $\rho_{-1}$ as

$$\rho_{-1} = (M^{-1}T(V_N), V_0, V_1), \quad \rho_{-2} = (M^{-1}T(V_{N-1}), M^{-1}T(V_N), V_0).$$

On the other hand,

$$\mathbf{T}(V_{N-1}) = c_N\rho_{N-3} \begin{pmatrix} c_{N-3} \\ b_{N-3} \\ 0 \end{pmatrix},$$
from (15), and, using (17) and (19)
\[ \rho_{N-3} = \rho_{N+1} K_{N}^{-1} K_{N-1}^{-1} K_{N-2}^{-1} K_{N-3}^{-1} = M \rho_{-1} A_{1} K_{N}^{-1} K_{N-1}^{-1} K_{N-2}^{-1} K_{N-3}^{-1}. \]

Putting these formulas together with the formula for \( \rho_{-2} \), we obtain
\[ \rho_{-2} = \rho_{-1} \mathcal{C}_{-2} \]

where
\[ \mathcal{C}_{-2} = \begin{pmatrix} c_{N} A_{1} K_{N}^{-1} K_{N-1}^{-1} K_{N-2}^{-1} K_{N-3}^{-1} \left( \begin{array}{c} c_{N-3} \\ b_{N-3} \\ 0 \\ e_{1} \\ e_{2} \end{array} \right) \end{pmatrix}. \]

Given that \( \rho_{-1} = \rho_{-2} K_{-2} \), we conclude that \( K_{-2} = \mathcal{C}_{-2}^{-1} \). Also, since \( A_{1} \) involves only \( K_{0}, K_{1}, K_{-1} \), we conclude that \( K_{-2} \) is a function of the generators \( G, K_{N} \) and \( K_{-1} \). We can also use this same process to study \( K_{-3} \). On the one hand, from (19) we have
\[ \rho_{-3} = M^{-1} \rho_{N-4} \mathcal{B}_{-3} = M^{-1} \rho_{N+1} K_{N}^{-1} K_{N-1}^{-1} K_{N-2}^{-1} K_{N-3}^{-1} K_{N-4}^{-1} \mathcal{B}_{-3} = \rho_{-1} A_{1} K_{N}^{-1} K_{N-1}^{-1} K_{N-2}^{-1} K_{N-3}^{-1} K_{N-4}^{-1} \mathcal{B}_{-3}. \]

On the other hand
\[ \rho_{-2} = \rho_{-3} K_{-3} = \rho_{-1} \mathcal{C}_{-2} \]

and hence
\[ \mathcal{C}_{-2} = A_{1} K_{N}^{-1} K_{N-1}^{-1} K_{N-2}^{-1} K_{N-3}^{-1} K_{N-4}^{-1} \mathcal{B}_{-3} K_{-3} \]

which proves that \( K_{-3} \) is a function of \( G, K_{N} \) and \( K_{-1} \).

We are now down to \( K_{-1} \) and \( K_{N} \). Our next step is to use the assumption \( p_{N+1} = MT(p_{0}) \) being in the segment joining \( p_{N} \) and \( Mp_{1} \), and hence assuming that the lift \( MT(V_{0}) = V_{N+1} \) is in the homogeneous plane containing \( MV_{1} \) and \( V_{N} \). This means that
\[ \det(V_{N+1}, MV_{1}, V_{N}) = \det(MT(V_{0}), MV_{1}, V_{N}) = 0. \]

Expanding as before (we will skip the details since it is the same type of calculation as above), this results on
\[ a_{N} = a_{1}. \]

Working with the extra relation
\[ \det(MV_{0}, V_{N}, V_{N-1}) = 0 \]

(which we can verify by simply observing fig. 3), and after some manipulations of the type we did before, we get
\[ c_{-1} = \frac{a_{N-1}}{a_{0}}. \]

As before
\[ T(V_{N}) = c_{N+1} \rho_{N-2} \begin{pmatrix} 1 \\ b_{N-2} \end{pmatrix} = c_{N+1} \rho_{N+1} K_{N}^{-1} K_{N-1}^{-1} K_{N-2}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

while
\[ \rho_{-1} = (M^{-1} T(V_{N}), V_{0}, V_{1}) = \rho_{0} K_{-1}^{-1}. \]
Thus,

$$K_{-1}^{-1}e_1 = \begin{pmatrix} -b_{-1} \\ -c_{-1} \\ -a_{-1} \\ c_{-1} \\ 1 \\ c_{-1} \end{pmatrix} = c_{N+1}K_{-1}^{-1}A_1K_{N-1}^{-1}K_{N-2}^{-1} \begin{pmatrix} 1 \\ b_{N-2} \end{pmatrix}.$$  

Straightforward calculations show that these three equations reduce to the two relations

$$c_N = (c_N+b_Na_{N-2})c_{N+1}; \quad a_{-1} = c_{N+1} \frac{c_{-1}a_{N-2}(1+a_1b_0)}{c_N} = \frac{c_{N+1}a_{N-1}a_{N-2}}{c_Na_0}(1+a_1b_0),$$

Which solves for $c_{N+1}$ and $a_{-1}$ in terms of generators $G$ and $b_N$. But from (21) we get

$$c_{N+1} = \frac{c_{-1}A_2}{A_0},$$

and so, using $A_i = c_i + a_i b_{i-1}$

$$\frac{c_{-1}}{1+a_0 b_{-1}} = \frac{c_N}{(c_N+b_Na_{N-2})(1+a_2 b_1)}$$

which together with (20) solves for $b_{-1}$ in terms of $G$ and $b_N$. Finally, $ho_{N+1} = \rho_N K_N$, so that det $\rho_{N+1} = c_N = \frac{1}{c_{-1}} \text{det } A_1$

$$c_N = \text{det } \rho_{N+1} = \text{det } \rho_{-1} \text{det } A_1 = \frac{1}{c_{-1}} A_0 A_1 = \frac{1+a_1 b_0(1+a_0 b_{-1})}{c_{-1}}$$

and so, from here and (27) we have

$$c_{-1} = \frac{1}{c_N} A_0 A_1; \quad \frac{c_{-1}}{1+a_0 b_{-1}} = \frac{1+a_1 b_0}{c_N} = \frac{c_N}{(c_N+b_Na_{N-2})(1+a_2 b_1)},$$

which allow us to solve for $b_N$ in terms of $G$ only.

We now have coordinates which are well defined generically in the space of twisted spirals. The last step in this section is writing the shift map in these coordinates.

3.2. The shift map in coordinates. Let $\text{TS}(N, 1)$ be the manifold of twisted spirals described by the coordinates above on open generic subsets. Define the shift map

$$S : \text{TS}(N, 1) \to \text{TS}(N, 1)$$

as the map assigning to a spiral, the spiral obtained by shifting it once. That is

$$S(\{p_1, p_2, \ldots, p_N; M \cdot T(p_0)\}) = \{p_2, \ldots, p_N, M \cdot T(p_0); M \cdot T(p_1)\}.$$  

The first of two lemmas describes the interrelation between the lift of the spiral as given in Theorem 3.1 and that of the shifted one.

Let $\alpha$ and $\beta$ be determined by the equations

$$\alpha^2 \beta = c_N; \quad \frac{\alpha^{-1} \beta^{-2}}{c_{-1} c_N} = \frac{A_3 A_2^2}{A_0 A_1}$$

We now have coordinates which are well defined generically in the space of twisted spirals. The last step in this section is writing the shift map in these coordinates.
if \( N = 3s + 2 \); and

\[
\alpha^{-1} \beta = c_N; \quad \alpha^{-2} \beta^{-1} = \frac{A_3 A_5^2}{c_1 c_N} = \frac{A_3 A_0}{A_1}
\]

if \( N = 3s \), where we recall that \( A_i = c_i + a_i b_{i-1} \).

**Lemma 3.4.** Let \( \{p_1, \ldots, p_N; M \cdot T(p_0)\} \) be a twisted spiral, and let \( \{V_i\} \) be the lift described in theorem 3.1. Likewise for \( \{\tilde{V}_i\} \) obtained by substituting \( \tilde{V}_0 \). Let \( \{p_2, \ldots, p_N, M \cdot T(p_0); M \cdot T(p_1)\} \) be its shift and let \( \{\tilde{V}_i\} \) be the analogous lift. Then

1. if \( N = 3s + 2 \),
   \[
   \tilde{V}_0 = \alpha^{-1} \beta^{-1} V_0; \quad \tilde{V}_{3r+1} = \alpha^{-1} \beta^{-1} V_{3r+1}; \quad \tilde{V}_{3r+2} = \alpha V_{3r+2}; \quad \tilde{V}_{3r} = \beta V_{3r}
   \]
   with subindices ranging from 0 to \( N + 2 \), and
   \[
   \tilde{V}_{N+3} = \alpha^{-1} \beta^{-1} V_{N+3}, \quad \tilde{V}_{N+4} = \alpha V_{N+4};
   \]

2. if \( N = 3s \),
   \[
   \tilde{V}_0 = \alpha^{-1} \beta^{-1} V_0; \quad \tilde{V}_{3r+1} = \beta V_{3r+1}; \quad \tilde{V}_{3r+2} = \alpha^{-1} \beta^{-1} V_{3r+2}; \quad \tilde{V}_{3r} = \alpha V_{3r}
   \]
   with subindices ranging from 0 to \( N + 2 \), and
   \[
   \tilde{V}_{N+3} = \beta V_{N+3}, \quad \tilde{V}_{N+4} = \alpha^{-1} \beta^{-1} V_{N+4}.
   \]

**Comment 3.5.** The arbitrary lifts \( \{\tilde{V}_i\} \) that will be used in the proof of this lemma are only arbitrary for \( i = 1, 2, \ldots, N + 1 \), while \( \tilde{V}_{N+i} = M \cdot T(V_{i-1}) \), for \( i = 2, 3, \ldots \), is obtained by substituting \( \tilde{V}_i \) in the definition of \( T(V_{i-1}) \) in the statement of theorem 3.1. Likewise for \( \tilde{V}_i = M^{-1} \cdot T(V_{i-1}) \), for \( i = 1, 2, \ldots \). But \( \tilde{V}_0 \) will be different for the spiral and its shift since the seed of the shifted spiral does not include \( p_1 \), which is the reason why \( \tilde{V}_{N+1} \neq MT(\tilde{V}_0) \), but rather \( \tilde{V}_0 = M^{-1} \cdot T(V_{N+1}) \), the same way we defined \( V_{N+1} \) for the unshifted spiral. Therefore, we will use \( \tilde{V}_{N+1} = MT(\tilde{V}_0) \) for the spiral and \( \tilde{V}_0 = M^{-1} \cdot T(V_{N+1}) \) for the shifted one.

**Proof.** From theorem 3.1 there are some proportions \( \lambda_i \) and \( \hat{\lambda}_i \) such that

\[
V_i = \lambda_i \tilde{V}_i, \quad i = 0, \ldots, N, \quad \hat{V}_i = \hat{\lambda}_i \tilde{V}_i, \quad i = 1, \ldots, N + 1
\]

for some arbitrary common lift \( \{\tilde{V}_i\} \). We define \( \lambda_{N+i} = \lambda_{i-2} \lambda_{i-1} \lambda_i \lambda_{i+1} \) for \( i \geq 1 \) so that \( \tilde{V}_{N+i} = \lambda_{N+i} \tilde{V}_{N+i} \) with

\[
M^{-1} \tilde{V}_{N+i} = (\hat{V}_{i-2} \times \hat{V}_i) \times (\tilde{V}_{i-1} \times \tilde{V}_{i+1})
\]

is given by \( T \) defined on the arbitrary lift indicated by the tilde. We define also \( \lambda_{-1} \) as the proportion satisfying \( \lambda_{-1} = \lambda_{-i} \tilde{V}_{-1} \) with \( \tilde{V}_{-1} \) also defined using the definition of \( \hat{T} \) in (9) lifted to \( \{\hat{V}_i\} \) (with \( V_i \) being substituted by \( \tilde{V}_i \)). We also define \( \hat{\lambda}_{N+k} = \lambda_{k-2} \lambda_{k-1} \lambda_k \lambda_{k+1} \) for \( k \geq 2 \) with \( \hat{\lambda}_{N+1} \) this time independent from other lambdas, and \( \hat{\lambda}_0 \) given by the relation \( \hat{\lambda}_0 = \hat{\lambda}_0 M^{-1} T(V_{N+1}) \) where \( T(V_{N+1}) \) is obtained using formula (9) on \( \tilde{V}_i \). Because of the shifting of the seed we have \( \tilde{V}_{N+1} = MT(\tilde{V}_0) \), while \( \hat{V}_0 = M^{-1} T(V_{N+1}) \).

The values of \( \lambda_i \) are determined by the equations

\[
det(V_i, V_{i+1}, V_{i+2}) = 1
\]

for \( i = 0, 1, \ldots, N \), and the proportions \( \hat{\lambda}_i \) are determined by the equations

\[
det(\hat{V}_i, \hat{V}_{i+1}, \hat{V}_{i+2}) = 1
\]
for $i = 1, 2, \ldots, N + 1$. The relation between $\lambda_i$ and $\hat{\lambda}_i$ can be found as follows: both solutions satisfy the same normalizations with $i = 1, 2, 3, \ldots, N$, the difference being the extra equation added to these ($i = 0$ for $\lambda_i$ and $i = N + 1$ for $\hat{\lambda}_i$) and the different definitions of the zero and the $n + 1$ proportions for each case. Using only what they have in common, we can solve for all $\lambda_i$’s in terms of $\lambda_N$ and $\lambda_{N+1}$, and then use either $i = 1$ or $i = N + 1$ and the particular definitions for the zero and $N + 1$ proportions to solve for $\lambda_N$ and $\lambda_{N+1}$ or $\hat{\lambda}_N$ and $\hat{\lambda}_{N+1}$, respectively. Although we can use general arguments to prove the theorem, we will need to know with precision how $\lambda_i$ and $\hat{\lambda}_i$ depend on $\lambda_N$, $\lambda_{N+1}$ and $\hat{\lambda}_N$, $\hat{\lambda}_{N+1}$.

Thus, let $\lambda_i, i = 0, \ldots, N$ be proportions satisfying the common equations

\[
\lambda_i \lambda_{i+1} \lambda_{i+2} = g_i
\]
i = 1, 2, 3, \ldots, N, where $g_i^{-1} = \det(\tilde{V}_i, \tilde{V}_{i+1}, \tilde{V}_{i+2})$.

Case $N = 3s + 2$. If we divide consecutive equations, we have

\[
\lambda_1 = \frac{1}{g_2} \frac{g_1}{g_2} = \cdots = \lambda_{N+2} \frac{g_1 \cdots g_{N-1}}{g_2 \cdots g_N} = \lambda_0 \lambda_1 \lambda_2 \lambda_3 \frac{g_1 \cdots g_{N-1}}{g_2 \cdots g_N}
\]

\[
\lambda_2 = \frac{1}{g_3} \frac{g_2}{g_3} = \cdots = \lambda_N \frac{g_2 \cdots g_{N-2}}{g_3 \cdots g_{N-1}}
\]

\[
\lambda_3 = \frac{1}{g_4} \frac{g_3}{g_4} = \cdots = \lambda_{N+1} \frac{g_3 \cdots g_{N-1}}{g_4 \cdots g_N}
\]

where the bracket represents the product of every third function, as in $[g_2, \ldots, g_N] = g_2g_5g_8 \ldots g_N$. We can also use the common relation $\lambda_1 \lambda_2 \lambda_3 = g_1$ to obtain

\[
\lambda_0 = \lambda_N^{-1} \lambda_{N+1}^{-1} \frac{g_N}{g_1}; \quad \lambda_1 = \lambda_N^{-1} \lambda_{N+1}^{-1} H_1; \quad \lambda_2 = \lambda_N H_2; \quad \lambda_3 = \lambda_{N+1} H_3,
\]

where $H_i$ depend only on the lift $\{\tilde{V}_1\}$ with the exception of $\tilde{V}_0$, which is the lift that the spiral and its shift do not share. Therefore, all $\lambda_i$ depend on $\lambda_N$ and $\lambda_{N+1}$ through their relation to $\lambda_1, \lambda_2, \lambda_3$ above.

From theorem [3.1] we know that $\lambda_N$ and $\lambda_{N+1}$ will be determined uniquely by the corresponding normalizations, and likewise for $\hat{\lambda}_N$ and $\hat{\lambda}_{N+1}$ (the explicit formulas can also be obtained through straightforward calculations).

Assume next that $\lambda_N = \alpha \lambda_N$ and $\hat{\lambda}_{N+1} = \beta \lambda_{N+1}$.

Then, using (37) and (11) we can prove (32)-(33) straightforwardly with some case-by-case consideration. The value of $\alpha$ and $\beta$ are found through two relations, each one coming from one end of the seed: the first one is

\[
1 = \det(\tilde{V}_{N+1}, \tilde{V}_{N+2}, \tilde{V}_{N+3}) = \lambda_{N+1} \lambda_{N+2} \lambda_{N+3} \det(\tilde{V}_{N+1}, \tilde{V}_{N+2}, \tilde{V}_{N+3})
\]

We have, from (37)

\[
\tilde{\lambda}_{N+1} \tilde{\lambda}_{N+2} \tilde{\lambda}_{N+3} = \tilde{\lambda}_{N+1} \hat{\lambda}_0 \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 \hat{\lambda}_4 = \tilde{\lambda}_{N+1} \hat{\lambda}_0 \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 \hat{\lambda}_4 = \tilde{\lambda}_{N+1} \hat{\lambda}_0 \hat{\lambda}_1 g_1 g_2 = \tilde{\lambda}_{N+1} \hat{\lambda}_0 \hat{\lambda}_1 g_1 g_2 = \hat{\lambda}_{N+1} \hat{\lambda}_{N+2} \hat{\lambda}_{N+3} \det(\tilde{V}_{N+1}, \tilde{V}_{N+2}, \tilde{V}_{N+3})
\]

where $H$ depends only on the arbitrary lift (excluding $\tilde{V}_0$). Therefore, undoing this reasoning and going back to the unshifted spiral, we have that

\[
1 = \alpha^{-2} \beta^{-1} \det(V_{N+1}, V_{N+2}, V_{N+3}) = \alpha^{-2} \beta^{-1} \det \rho_{N+1}.
\]

But, as we saw before, $\rho_{N+1} = \rho_N K_N$ with $\det \rho_N = 1$, so $\det \rho_{N+1} = \epsilon_N$ as claimed.
The second equation comes from the following observation at the other end of the seed: from \([15]\)

\[
V_0 = \lambda_0 \tilde{V}_0 = \alpha \beta \lambda_0 \tilde{V}_0 = \alpha \beta \tilde{V}_0 = \alpha \beta M^{-1} \mathcal{T}(\hat{V}_{N+1}) = \alpha \beta M^{-1} \left( (\hat{c}_{N+2} (\hat{a}_{N-1} \hat{V}_N + \hat{v}_{N-1}) \right),
\]

but we also have

\[
\hat{c}_{N+2} = \frac{\lambda_{N+5}}{\lambda_{N+2}} \hat{c}_{N+2} = \frac{\lambda_3 \lambda_4 \lambda_5 \lambda_6}{\lambda_0 \lambda_1 \lambda_2 \lambda_3} \hat{c}_{N+2}
\]

\[
\hat{b}_{N-1} = \frac{\lambda_{N+2}}{\lambda_N} \hat{b}_{N-1} = \frac{\hat{\lambda}_0 \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3}{\lambda_N} \hat{b}_{N-1},
\]

which, implies \(\hat{c}_{N+2} = \alpha \beta^2 \hat{c}_{N+2}\) and \(\hat{b}_{N-1} = \alpha^{-2} \beta^{-1} b_{N-1}\). Since \(\hat{\lambda}_N = \alpha \lambda_N\) and \(\hat{\lambda}_{N-1} = \alpha^{-1} \beta^{-1} \lambda_{N-1}\), putting everything together we get

\[
V_0 = \alpha \beta^2 M^{-1} (c_{N+2} (b_{N-1} V_N + V_{N-1})) = \alpha \beta^2 M^{-1} \mathcal{T}(V_{N+1}) = \alpha \beta^2 M^{-1} \mathcal{T} (MT(V_0)).
\]

Thus, \(\alpha \beta^2\) measures how the lift \(\mathcal{T}\) fails to be the twisted inverse of the lift of \(T\). But we also have from \([14]\) and \([15]\) that

\[
\mathcal{T}(V_{N+1}) = \mathcal{T} (MT(V_0)) = c_{N+2} (MT(V_0) \times MT(V_1)) \times (V_{N-1} \times V_N),
\]

Now, \(\rho_{N-1} = \rho_{N+1} K_N^{1-1} K_{N-1}^{-1}\), and therefore

\[
M^{-1} V_{N-1} = M^{-1} \rho_{N-1} e_1 = M^{-1} \rho_{N+1} K_N^{1-1} K_{N-1}^{-1} e_1 = (T(V_0), T(V_1), T(V_2)) \left( \begin{array}{c} -a_{N-1} + \frac{b_N b_{N-1}}{c_N} \\ \frac{A_N}{c_N} \\ \frac{b_N}{c_N} \end{array} \right).
\]

Likewise

\[
M^{-1} V_N = (T(V_0), T(V_1), T(V_2)) \left( \begin{array}{c} \frac{b_N}{c_N} \\ \frac{a_N}{c_N} \\ -\frac{1}{c_N} \end{array} \right)
\]

and hence, after long but straightforward calculations using \(a_N = a_1\) and \(c_{-1} = \frac{a_{N-1}}{a_0}\), we get

\[
M^{-1} V_{N-1} \times M^{-1} V_N = \frac{A_1 A_0}{c_N} V_0 \times V_1 - \frac{b_N A_0}{c_N c_{-1}} V_2 \times V_0.
\]

We also have, from \([14]\)

\[
T(V_0) \times T(V_1) = \frac{A_0}{c_{-1}} V_2 \times V_0.
\]

Finally, also using \([14]\) and straightforward calculations we get

\[
c_{N+2} = \frac{\det(T(V_2), T(V_3), T(V_4))}{\det(T(V_1), T(V_2), T(V_3))} = \frac{A_2 A_3}{A_1 A_2} = \frac{A_3}{A_1}.
\]

Putting everything together we have

\[
M^{-1} \mathcal{T}(MT(V_0)) = c_{N+2} \frac{A_0^2 A_1}{c_{-1} c_N} (V_2 \times V_0) \times (V_0 \times V_1) = \frac{A_3 A_0^2}{c_{-1} c_N} V_0.
\]
and hence $\alpha^{-1}\beta^{-2} = \frac{A_3A_2^2}{c_{-1}c_N}$ as stated. Notice that the computation above is the same for $N = 3s$, only the powers of $\alpha$ and $\beta$ change.

These two relations determine $\alpha$ and $\beta$ generically. Finally, since $\hat{V}_i = \frac{\lambda_1}{\lambda_i}V_i$, using (37) we prove the lemma.

Case $N = 3s$. The proof in this case is identical, but we use different equations for $\lambda_i$. Indeed, the systems of equations derived from the normalizations are now given by

$$
\lambda_1 = \frac{\lambda_2 g_1}{g_2} = \cdots = \lambda_{N+1} \frac{g_1 \cdots g_{N-1}}{g_2 \cdots g_N},
$$

$$
\lambda_2 = \frac{\lambda_3 g_2}{g_3} = \cdots = \lambda_{N+2} \frac{g_2 \cdots g_{N-3}}{g_3 \cdots g_{N-2}},
$$

and hence $\alpha \lambda = \frac{g_N}{g_1}$ becomes also

$$
\lambda_0 = \frac{\lambda^{-1}_N\lambda_{N+1}^{-1}}{g_N/g_1}.
$$

When we put them together as before we get

$$
\lambda_0 = \lambda^{-1}_N\lambda^{-1}_{N+1}F_0; \quad \lambda_1 = \lambda_{N+1}F_1, \quad \lambda_2 = \lambda^{-1}_N\lambda^{-1}_{N+1}F_2; \quad \lambda_3 = \lambda_NF_3.
$$

where again $F_i$ depends only on the arbitrary lift excluding $\tilde{V}_0$. If $\hat{\lambda}_N = \alpha\lambda_N$, $\hat{\lambda}_{N+1} = \beta\lambda_{N+1}$, we obtain the relations (34) and (35) following the same reasoning as in the previous case.

We can write $\alpha$ and $\beta$ in terms of the generators, using (26), (27) and (28). We get

$$
\frac{c_{-1}}{A_0} = \frac{c_N}{B_N A_2}; \quad B_N = \frac{c_N^2}{A_1 A_2}
$$

and so

$$
\frac{c_{-1}}{A_0} = \frac{A_1}{c_N}, \quad \text{and} \quad \frac{A_3 A_0}{A_1} = \frac{A_3 A_2^2}{c_{-1}c_N} = \frac{A_3 a_{N-1}}{A_1^2 a_0} c_N.
$$

Our next lemma relates the invariants for the different spirals.

**Lemma 3.6.** Assume a twisted pentagram spiral has a lift $\{V_i\}$ and the shifted spiral has a lift $\{\hat{V}_i\}$ as in theorem 3.1. Let $\{a_i, b_i\}_{i=0}^{N-1} \cup \{c_N\}$ be the invariants defined by the lift in theorem 3.1 while $\{\hat{a}_i, \hat{b}_i\}_{i=0}^{N-1} \cup \{\hat{c}_N\}$ are the ones defined by the shifted lift. Then

a. If $N = 3s + 2$, we have

$$
\hat{a}_k = \begin{cases} 
\alpha^{-1}\beta a_k & k = 3r \\
\alpha^{-1}\beta^{-2} a_k & k = 3r + 1 \\
\alpha^2 a_k & k = 3r + 2
\end{cases}, \quad \hat{b}_k = \begin{cases} 
\alpha \beta^2 b_k & k = 3r \\
\alpha^{-2} \beta b_k & k = 3r + 1 \\
\alpha \beta^{-1} b_k & k = 3r + 2
\end{cases}
$$

for $k = 0, 1, 2, \ldots, N - 1$, and $\hat{a}_N = a_N$, $\hat{b}_N = \alpha^{-1}\beta^{-2}b_N$, $\hat{c}_{N+1} = \alpha \beta^{-1}c_{N+1}$. 
b. if \( N = 3s \), then

\[
\begin{align*}
\hat{a}_k &= \begin{cases} 
\alpha^2 \beta \alpha_k & k = 3r \\
\alpha^{-1} \beta \alpha_k & k = 3r + 1 \\
\alpha^{-2} \beta^{-2} \alpha_k & k = 3r + 2
\end{cases} \\
\hat{b}_k &= \begin{cases} 
\alpha \beta^{-1} b_k & k = 3r \\
\alpha \beta^2 b_k & k = 3r + 1 \\
\alpha^{-2} \beta^{-1} b_k & k = 3r + 2
\end{cases}
\end{align*}
\]

for \( k = 0, 2, \ldots, N - 1 \), and \( \hat{a}_N = \alpha \beta^2 \hat{a}_N, \hat{b}_N = b_N, \hat{c}_N = \alpha^{-1} \beta^{-2} c_{N+1} \).

In both cases \( \alpha \) and \( \beta \) are as in (30)-(31).

**Comment 3.7.** Notice that, even though \( \alpha \) and \( \beta \) are fractional powers of invariant rational expressions, the relevant factors that will appear in the shift map are three rational expressions that contain no fractional powers. They are

\[
\alpha^{-1} \beta = c_N, \quad \alpha^2 \beta = \frac{A_1}{A_0 A_3}, \quad \alpha \beta^2 = \frac{c_N A_1}{A_3 A_0}
\]

**Proof.** The proof of this lemma is a straightforward careful account of the different cases using the definitions

\[
\begin{align*}
a_i &= \frac{\det(V_i, V_{i+1}, V_{i+3})}{\det(V_i, V_{i+1}, V_{i+2})}, \quad b_i = \frac{\det(V_i, V_{i+3}, V_{i+2})}{\det(V_i, V_{i+1}, V_{i+2})}, \\
c_i &= \frac{\det(V_{i+1}, V_{i+2}, V_{i+3})}{\det(V_i, V_{i+1}, V_{i+2})}
\end{align*}
\]

and the results of the previous lemma. \( \square \)

The main theorem of this section (and of the paper) is now a consequence of the formulas found in (38)-(39) above, and of the definition of \( \alpha \) and \( \beta \).

**Theorem 3.8.** Consider the action of the one parameter group

\[
\begin{align*}
a_k &\rightarrow \mu a_k; \quad b_k \rightarrow \mu^{-1} b_k; \quad c_N \rightarrow c_N
\end{align*}
\]

\( k = 0, \ldots, N - 1 \), defined on the coordinates of a twisted pentagram spiral. Then the shift map is invariant under the action.

**Proof.** First of all, the shift map in local coordinates is given by

\[
S(a_0, a_1, \ldots, a_{N-1}, b_0, b_1, \ldots, b_{N-1}, c_N) = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{N-1}, \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_N, \hat{c}_{N+1})
\]

Now, using the results of our previous lemma, we first need to show that \( \alpha \) and \( \beta \) are invariant under the shift, since \( \hat{a}_k, \hat{b}_k, k = 0, \ldots, N \), and \( \hat{c}_{N+1} \) are, in all cases, proportional to \( a_k, b_k \) and \( c_{N+1} \), respectively, with proportionality constants given by different powers of \( \alpha \) and \( \beta \). Since \( \alpha \) and \( \beta \) are uniquely defined by equations (30)-(31), it suffices to show that

\[
c_N, \quad \frac{A_3 a_{N-1}}{A_1^2 a_0} \quad c_N
\]

are invariant under the action, which they clearly are.

Finally, we need to study the formulas for \( a_N, b_N \) and \( c_{N+1} \) and show that the extended induced action on these invariants is given by

\[
\begin{align*}
a_N &\rightarrow \mu a_N; \quad b_N \rightarrow \mu^{-1} b_N; \quad c_{N+1} \rightarrow c_{N+1}
\end{align*}
\]

Since \( \hat{a}_N, \hat{b}_N \) and \( \hat{c}_{N+1} \) are multiples of \( a_N, b_N \) and \( c_{N+1} \) with factors given by different powers of \( \alpha \) and \( \beta \), that would conclude the proof.

And indeed, we know that \( a_N = a_1, b_N \) is determined by (28), which is invariant under the extended action, and \( c_{N+1} = \frac{c_N}{c_N + b_N a_{N-2}} \). Therefore, the theorem is proved. \( \square \)
4. LAX REPRESENTATION OF THE SHIFT MAP

Once the one-parameter action is found, we can finally build a Lax representation of the shift map using moving frames. Indeed, consider the shift map $S : TS(N, 1) \to TS(N, 1)$ and let $S$ be (abusing once more the notation) the natural lift of $S$ to the polygons (or spirals, although they are not spirals in the traditional sense) with seed $\{V_i\}_{i=1}^{N+1}$ in $\mathbb{R}^3$. We can define $S$ as either a map on the seed

$$S(\{V_i\}_{i=1}^{N+1}) = \{\hat{V}_{i+1}\}_{i=1}^{N+1}$$

or as a map on the vertices, $S(V_i) = \hat{V}_{i+1}$, for any $i$. Note that the map can be extended to be defined in all vertices using the pullback, since all vertices can be written in terms of the seed. If we define it as a map on the vertices, then we can further extend the map to the moving frame by defining

$$S(\rho_i) = (\hat{V}_{i+1}, \hat{V}_{i+2}, \hat{V}_{i+3}) = \hat{\rho}_{i+1},$$

whenever $\rho_i = (V_i, V_{i+1}, V_{i+2})$. We can also extend it to algebraic combinations as usual using the pull back, so that, for example, $S(\rho_i^{-1} \rho_{i+1}) = S(\rho_i)^{-1} S(\rho_{i+1})$.

We define

$$N_i = \hat{\rho}_i^{-1} S(\rho_i)$$

to be the left-invariant element of $\mathfrak{sl}(3, \mathbb{R})$ defined by $S$ applied to moving frames. Since $K_i = \rho_i^{-1} \rho_{i+1}$ is the left-invariant element of the algebra defined by translation of the subindex, the structure equations (or compatibility conditions) are given by

$$S(K_i) = N_i^{-1} K_i N_{i+1}$$

From [17] we have

$$S(K_i) = \begin{pmatrix} 0 & 0 & S(c_i) \\ 1 & 0 & S(b_i) \\ 0 & 1 & S(a_i) \end{pmatrix}$$

and, therefore (42) above describes the map $S$ in coordinates $\{a_i, b_i, c_i\}$. We can also easily find $N_i$ explicitly. Let $r_i$ and $s_i$ be the powers of $\alpha$ and $\beta$ such that

$$S(V_i) = \hat{V}_{i+1} = \alpha^{r_i} \beta^{s_i} V_{i+1}$$

as in lemma 3.4 (the values of $r_i$ and $s_i$ will depend on $i$ and are given in that lemma). Then

$$N_i = \rho_i^{-1} S(\rho_i) = \rho_i^{-1} (\alpha^{r_i} \beta^{s_i} V_{i+1}, \alpha^{r_{i+1}} \beta^{s_{i+1}} V_{i+2}, \alpha^{r_{i+2}} \beta^{s_{i+2}} V_{i+3}) = \rho_i^{-1} \rho_{i+1} R_i = K_i R_i$$

where

$$R_i = \begin{pmatrix} \alpha^{r_i} \beta^{s_i} & 0 & 0 \\ 0 & \alpha^{r_{i+1}} \beta^{s_{i+1}} & 0 \\ 0 & 0 & \alpha^{r_{i+2}} \beta^{s_{i+2}} \end{pmatrix}$$

and $S(K_i) = R_i^{-1} K_{i+1} R_{i+1}$ is thus expressed as a shifted gauge by diagonal matrices (of course, the entries of $R_i$ depend on some of the invariants, the map is highly non-linear).

Consider now the modified $K_i(\mu), N_i(\mu)$ obtained through the introduction of the $\mu$-scaling

$$K_i(\mu) = \begin{pmatrix} 0 & 0 & c_i \\ 1 & 0 & \mu^{-1} b_i \\ 0 & 1 & \mu a_i \end{pmatrix}, \quad N_i(\mu) = K_i(\mu) R_i.$$
Since \( R_i \) is invariant under the \( \mu \)-scaling, if we substitute \( K_i \) by \( K_i(\mu) \) in (42), the resulting equation will not depend on \( \mu \). Therefore, the system
\[
\rho_{i+1} = \rho_i K_i(\mu); \quad S(\rho_i) = \rho_i N_i(\mu)
\]
i = 0, 1, 2, \ldots, \( N \), is a Lax representation for the map \( S \). Here \( a_N \) and \( b_N \) are the explicit functions of the generating system that we found in theorem 3.3, that is,
\[
a_N = a_1, \quad b_N = \frac{c_N}{a_{N-2}} \left( \frac{c_N}{A_1 A_2} - 1 \right).
\]
In fact, this is not a classical Lax representation: the problem is neither periodic nor infinite, but rather it has different boundary conditions given by the spiral condition. Nevertheless, the monodromy is clearly preserved from the construction of the map since the shifted spiral clearly has the same monodromy (this can also be double checked by straightforward calculations). On the other hand, once we fix coordinates \( \bar{G} \) for twisted spirals, only the conjugation class of the monodromy is well defined. Indeed, on the one hand from (19) we know that
\[
\rho_{N+1} = M \rho_0 A_1 = M \rho_0 K_0^{-1} A_1
\]
and on the other hand
\[
\rho_{N+1} = \rho_0 K_0 K_1 \ldots K_N.
\]
Therefore
\[
\rho_0^{-1} M \rho_0 = K_0 K_1 \ldots K_N A_1^{-1} K_{-1}.
\]
Recall that for the pentagram map the relation was similar, except for the matrix \( A_1^{-1} K_{-1} \), which appears here due to the different boundary conditions. Thus, assuming that \( \mu \) and \( r \) are complex numbers, the following proposition has already been proved

**Theorem 4.1.** The eigenvalues of the matrix
\[
M(\mu) = K_0(\mu) K_1(\mu) \ldots K_N(\mu) A_1^{-1}(\mu) K_{-1}(\mu)
\]
are preserved by the shift map, and the map lies on the Riemann surface
\[
det(M(\mu) - r I) = 0
\]
(43)

Notice that we have not found a Poisson structure preserved by the shift map, something that would complete its integrability. Indeed, the structure associated to the pentagram map
\[
\{a_k, a_{k+3r}\} = \pm a_k a_{k+3r}; \quad \{b_k, b_{k+3r}\} = \mp b_k b_{k+3r}
\]
with all other brackets vanishing, is not preserved by the shift map even if we make \( c_N \) a Casimir. This is a consequence of the fact that none of the factors in (40) commute with all \( a_k \) or \( b_r \) (implying that, for example, \( \{S(a_k), S(b_r)\} \neq 0 \) in general). Still, we do have a good partial integration since many invariants of the map can be found analyzing the coefficients of the different powers of \( \mu \) and \( r \) in (43), which are all invariants.

Work in MAPLE shows that \( \text{trace}(M(\mu)) \) is a polynomial in \( \mu \) containing only powers of \( \mu \) in intervals of 3, the same as it shows for the pentagram map. Likewise, not all powers appear in (43). For example, if \( N = 5 \), (43) contains the following powers:
\[
det(M(\mu) - r I) \]
\[
= I_0 + r \mu^{-7} I_1 + r \mu^{-4} I_2 + r^2 \mu^{-2} I_3 + r \mu^{-1} I_4 + \mu^2 r I_5 + \mu r^2 I_6 + \mu^4 r^2 I_7
\]
where
\[ I_0 = 1, \quad I_1 = \frac{b_0 b_1 b_2 b_3 b_4}{a_4 a_3 a_0 A_1^2 A_2} (A_1 A_2 - c_5) (a_4 c_5 - a_0 A_1) \]
\[ I_7 = a_1 a_2 a_3 a_4, \quad I_5 = \frac{1}{A_1 a_3 c_5} (a_1 a_3 A_1^2 + c_5^2 a_4 a_2) \]
\[ I_3 = \frac{1}{a_0} (b_1 A_4 + b_4 (A_2 + a_0 b_2)) - \frac{1}{A_1 a_3} c_5 (b_0 + b_3 A_1) \]
\[ - \frac{A_1}{c_5 a_4} (b_1 + b_4 A_2) + \frac{c_5^2}{A_1^2 A_2 a_3} (b_4 A_1 + b_0 A_3) \]
\[ I_6 = \frac{1}{A_1 A_2 a_0} (a_0 a_2 c_5^2 + a_4 A_1 A_2 (a_3 A_2 + a_0 (1 + a_1 b_3 + a_3 b_2)) + A_1 A_2 a_0 a_1 (1 + a_1 b_4)) \]

while the remaining two are longer to write.

One can directly see that \( I_7 \) is preserved. Indeed, from either (38) or (39) we have
\[ S(a_1 a_2 a_3 a_4) = \tilde{a}_2 \tilde{a}_3 \tilde{a}_4 a_5 = a_2 a_3 a_4 a_5 = a_1 a_2 a_3 a_4 \]
since \( a_5 = a_N = a_1 \).

We have a final observation. In [17] the author showed that the shift map could be viewed as the \((N + 2)\)th root of the pentagram map in a certain sense (notice that the domains are different). Recall ([11]) that the pentagram map in \( e_i, f_i \) coordinates as in [7] for the twisted \( N \)-gon is defined as
\[ T(e_i) = e_{i+2} \left[ E_{i+2}, \ldots, E_{i+N} \right] \quad T(f_i) = f_{i-1} \left[ E_{i-N}, \ldots, E_{i-5} \right] \]
for \( N = 3m + 2 \), where again \([\ldots]\) indicates the product of every three, as in \([E_2, \ldots, E_N] = E_2 E_5 \ldots E_{N-3} E_N \) and where \( E_i = 1 + e_i f_{i-1} \). This expressions are non-local, even though the map clearly is. Even though the coordinates \( e_i \) and \( f_i \) (\(a_i \) and \(b_i \) in [11]) are mostly multiples, but not equal, to the spiral coordinates (as before, they have in common most of the normalizations that define them, except for beginning and end), the study here shows that in \( a_i \), \( b_i \) coordinates the spiral is local around the first vertex. This seems to indicate that it is the accumulation of \( N + 2 \) transformations that produces the non-local formula for the pentagram. This is reinforced by the fact that
\[ S(A_i) = A_{i+1} \]
for \( i = 0, 1, \ldots, N-2 \), and the fact that the consecutive application of \( S \) is calculated by shifting and multiplying by factors that are the \( S \) image of \( c_N, A_1 \), \( A_3 A_0 \) and \( A_1 c_N \), in succession. To make this statement precise one would need a careful study of the relation between the coordinates in the different moduli space, a calculation similar to the one in our lemmas that we prefer not to include here.

Notice also that one can attempt to change the coordinates from \( \{a_i, b_i, i = 0, \ldots, N - 1, c_N\} \) to \( \{A_i = 1 + a_i b_{i-1}, B_i = 1 + b_i a_{i-2}, i = 0, \ldots, N - 1, c_N\} \), which might be possible for some values of \( N \). In these coordinates the map looks like
\[ S(A_i) = A_{i+1}, \quad S(B_i) = B_{i+1} \]
for \( i = 0, \ldots, N - 2 \), but the image of \( A_{N-1} \) and \( B_{N-1} \) are not only non-local, but highly complicated and it does not seem to shed much light in the problem, so we will not include further details here.
ON THE INTEGRABILITY OF THE SHIFT MAP

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