CARLESON MEASURES FOR WEIGHTED HARDY-SOBOLEV SPACES

CARME CASCANTE AND JOAQUIN M. ORTEGA

ABSTRACT. We obtain characterizations of positive Borel measures \( \mu \) on \( B^n \) so that some weighted Hardy-Sobolev are imbedded in \( L^p(d\mu) \), where \( w \) is an \( A_p \) weight in the unit sphere of \( C^n \).

1. Introduction

The purpose of this paper is the study of the positive Borel measures \( \mu \) on \( S^n \), the unit sphere in \( C^n \), for which the weighted Hardy-Sobolev space \( H_p^s(w) \) is imbedded in \( L^p(d\mu) \), that is, the Carleson measures for \( H_p^s(w) \).

The weighted Hardy-Sobolev space \( H_p^s(w) \), \( 0 < s, p < +\infty \), consists of those functions \( f \) holomorphic in \( B^n \) such that if \( f(z) = \sum f_k(z) \) is its homogeneous polynomial expansion, and \((I + R)^s f(z) = \sum (1 + k)^s f_k(z)\), we have that

\[ \|f\|_{H_p^s(w)} = \sup_{0 < r < 1} \|(I + R)^s f_r\|_{L^p(w)} < +\infty, \]

where \( f_r(\zeta) = f(r\zeta) \).

We will consider weights \( w \) in \( A_p \) classes in \( S^n \), \( 1 < p < +\infty \), that is, weights in \( S^n \) satisfying that there exists \( C > 0 \) such that for any nonisotropic ball \( B \subset S^n \), \( B = B(\zeta, r) = \{ \eta \in S^n ; |1 - \zeta \eta| < r \} \),

\[ \left( \frac{1}{|B|} \int_B wd\sigma \right) \left( \frac{1}{|B|} \int_B w^{\frac{1}{p-1}} d\sigma \right)^{p-1} \leq C, \]

where \( \sigma \) is the Lebesgue measure on \( S^n \) and \( |B| \) the Lebesgue measure of \( B \). We will use the notation \( \zeta \overline{\eta} \) to indicate the complex inner product in \( C^n \) given by \( \zeta \overline{\eta} = \sum_{i=1}^{n} \zeta_i \overline{\eta_i} \), if \( \zeta = (\zeta_1, \ldots, \zeta_n) \), \( \eta = (\eta_1, \ldots, \eta_n) \).

If \( 0 < s < n \), any function \( f \) in \( H_p^s(w) \) can be expressed as

\[ f(z) = C_s(g)(z) := \int_{S^n} \frac{g(\zeta)}{(1 - z\zeta)^{n-s}} d\sigma(\zeta), \]

1991 Mathematics Subject Classification. 32A35, 46E35, 32A40.

Key words and phrases. Weighted Hardy-Sobolev spaces, holomorphic potentials, Carleson measures, weighted Triebel-Lizorkin spaces.

Both authors partially supported by DGICYT Grant MTM2005-08984-C02-02, and DURSI Grant 2005SGR 00611.
where $d\sigma$ is the normalized Lebesgue measure on the unit sphere $S^n$ and $g \in L^p(w)$, and consequently, $\mu$ is Carleson for $H^p_s(w)$ if there exists $C > 0$ such that

$$||C_s f||_{L^p(d\mu)} \leq C||f||_{L^p(w)}.$$ 

We denote by $K_s$ the nonisotropic potential operator defined by

$$K_s[f](z) = \int_{S^n} \frac{f(\eta)}{|1 - z \eta|^{n-s}} d\sigma(\eta), \quad z \in \overline{B}^n.$$ 

The problem of characterizing the positive Borel measures $\mu$ on $B^n$ for which there exists $C > 0$ such that

$$||K_s[f]||_{L^p(d\mu)} \leq C||f||_{L^p(d\sigma)},$$

that is, the characterization of the Carleson measures for the space $K_s[L^p(d\mu)]$ has been very well studied and there exist different characterizations (see for instance $\text{Ma, AdHe, KeSa}$). 

The representation of the functions in $H^p_s$ in terms of the operator $C_s$ gives that in dimension 1 the Carleson measures for $K_s[L^p(d\sigma)]$ coincide with the Carleson measures for the Hardy-Sobolev space $H^p_s$ simply because the real part of $\frac{1}{|1-z\eta|^{1-s}}$ is equivalent to $\frac{1}{|1-z\eta|^{n-s}}$. This representation also shows that in any dimension every Carleson measure for $K_s[L^p(d\sigma)]$ is also a Carleson measure for $H^p_s$. The coincidence fails to be true for $n > 1$ in general, as it is shown in $\text{AhCo}$ (see also $\text{CaOr2}$).

Of course, when $n - sp < 0$, the space $H^p_s$ consists of continuous functions on $\overline{B}^n$, and in particular, the Carleson measures in this case are just the finite measures. But for $n - sp \geq 0$, and $n > 1$, the characterization of the Carleson measures for $H^p_s$ still remains open. In the case where we are ”near” the regular case, that is when $n - sp < 1$ it is shown in $\text{AhCo}$, $\text{CohVe1}$ and $\text{CohVe2}$, that the Carleson measures for $H^p_s$ and $K_s[L^p(d\sigma)]$ are the same, and any of the different characterizations of the Carleson measures for the last ones also hold for $H^p_s$.

One of the main purposes of this paper is to extend this situation to $H^p_s(w)$ for $w$ a weight in $A_p$. If $E \subset S^n$ is measurable, we define

$$W(E) = \int_E w d\sigma.$$ 

A weight $w$ satisfies a doubling condition of order $\tau$, if there exists $\tau > 0$ such that for any nonisotropic ball $B$ in $S^n$, $W(2^kB) \leq C2^{k\tau} W(B)$.

It is well known that any weight in $A_p$ satisfies a doubling condition of some order $\tau$ strictly less than $np$. We begin observing that if $\tau - sp < 0$, the space $H^p_s(w)$ consists of continuous functions on $\overline{B}^n$, and consequently, the Carleson measures are just the finite ones. If $\tau - sp < 1$, we show that the Carleson measures for $H^p_s(w)$ and $K_s[L^p(w)]$ coincide, whereas if $\tau - sp \geq 1$, this coincidence may fail.

As it happens in the unweighted case (see $\text{CohVe1}$), the proof of the characterization of the Carleson measures for $H^p_s(w)$ will be based in the construction of weighted holomorphic potentials, with control of their $H^p_s(w)$-norm. In fact, technical reasons give that it is convenient to deal with weighted Triebel-Lizorkin spaces which, on the other hand, have interest on their own. In the second section we study these
spaces. If $s \geq 0$, we will write $[s]^+$ the integer part of $s$ plus 1. Let $1 < p < +\infty$, $1 \leq q \leq +\infty$, and $s \geq 0$. The weighted holomorphic Triebel-Lizorkin space $HF^p_q(w)$ when $q < +\infty$ is the space of holomorphic functions $f$ in $B^n$ for which
\[
\|f\|_{HF^p_q(w)} = \left( \int_{B^n} \left( \int_0^1 \left( |(I + R)^{|s|} f(r\zeta)|q(1-r^2)^{(|s|^+ - s)q-1}dr \right)^{\frac{p}{q}} w(\zeta)d\sigma(\zeta) \right)^{\frac{1}{p}} < +\infty,
\]
whereas when $q = +\infty$,
\[
\|f\|_{HF^p_{\infty}(w)} = \left( \int_{B^n} \left( \sup_{0<r<1} \left( |(I + R)^{|s|} f(r\zeta)|(1-r^2)^{|s|^+ - s}\right)^p w(\zeta)d\sigma(\zeta) \right)^{\frac{1}{p}} < +\infty,
\]
where $I$ denotes the identity operator.

The Section 2 is devoted to the general theory of weighted holomorphic Triebel-Lizorkin spaces. We give different equivalent definitions of the spaces $HF^p_q(w)$ in terms of admissible area functions, we give duality theorems on these spaces, we study some relations of inclusion among them and we also obtain that when $q = 2$, the weighted Triebel-Lizorkin space $HF^2_q(w)$ coincides with the weighted Hardy-Sobolev space $H^p_q(w)$.

The main result in Section 3 is the characterization of the Carleson measures for $H^p_q(w)$, when $0 < \tau - sp < 1$, in terms of a positive kernel.

**Theorem C.** Let $1 < p < +\infty$, $w$ an $A_p$-weight, and $\mu$ a finite positive Borel measure on $B^n$. Assume that $w$ is doubling of order $\tau$, for some $\tau < 1 + sp$. We then have that the following statements are equivalent:

(i) $\|K_\tau(f)\|_{L^p(\mu)} \leq C\|f\|_{L^p(w)}$.

(ii) $\|f\|_{L^p(\mu)} \leq C\|f\|_{H^p_\tau(w)}$.

The proof relies on the construction of weighted holomorphic potentials, with control of their weighted Hardy-Sobolev norm.

We also gives examples of the sharpness of the above theorem. We show that if $p = 2$ and $\tau > 1 + sp$, $n < \tau < n + 1$, then there exists $w$ in $A_2 \cap D_\tau$ and a measure $\mu$ on $S^n$ which is Carleson for $H^2_\tau(w)$, but it is not Carleson for $K_\tau[L^2(\mu)]$.

Finally, the usual remark on notation: we will adopt the convention of using the same letter for various absolute constants whose values may change in each occurrence, and we will write $A \leq B$ if there exists an absolute constant $M$ such that $A \leq MB$.

We will say that two quantities $A$ and $B$ are equivalent if both $A \leq B$ and $B \leq A$, and, in that case, we will write $A \simeq B$.

2. Weighted holomorphic Triebel-Lizorkin spaces

In this section we will introduce weighted holomorphic Triebel-Lizorkin spaces, and we will obtain characterizations in terms of Littlewood-Paley functions and admissible area functions. These characterizations, known in the unweighted case, will be used in the following sections.

We begin recalling some simple facts about $A_p$ weights that we will need later. It is well known that $A_\infty = \bigcup_{1<p<+\infty} A_p$ and that any $A_p$ weight satisfies a doubling
Lemma 2.1. Let \( 1 < p < +\infty \), and \( w \) be an \( A_p \)-weight. We then have:

(i) There exists \( 1 < p_1 < p \) such that \( L^p(w) \subset L^{p_1}(d\sigma) \).

(ii) There exists \( p_2 > p \) such that \( L^{p_2}(d\sigma) \subset L^p(w) \).

We now proceed to study the weighted holomorphic Triebel-Lizorkin spaces \( HF^\alpha_s^p(w) \) already defined in the introduction. We begin with some definitions. If \( 1 < q \leq +\infty \), \( k \) an integer such that \( k > s \geq 0 \), and \( \zeta \in \mathbb{S}^n \), the Littlewood-Paley type functions are given by

\[
A_{1,k,q,s}(f)(\zeta) = \left( \int_0^1 |(I + R)^k f(r\zeta)|^q (1 - r^2)^{q(k-s)q-1} dr \right)^{\frac{1}{q}},
\]

when \( q < +\infty \), and

\[
A_{1,k,\infty,s}(f)(\zeta) = \sup_{0 < r < 1} |(I + R)^k f(r\zeta)|(1 - r^2)^{k-s},
\]

when \( q = +\infty \).

If \( \alpha > 1 \), \( \zeta \in \mathbb{S}^n \), we denote by \( D_\alpha(\zeta) \), \( \alpha > 1 \) the admissible region given by \( D_\alpha(\zeta) = \{ z \in \mathbb{B}^n ; |1 - z\zeta| < \alpha(1 - |z|) \} \). We introduce the admissible area function

\[
A_{\alpha,k,q,s}(f)(\zeta) = \left( \int_{D_\alpha(\zeta)} |(I + R)^k f(z)|^q (1 - |z|^2)^{q(k-s)q-n-1} dv(z) \right)^{\frac{1}{q}},
\]

when \( q < +\infty \), where \( dv \) is the Lebesgue measure on \( \mathbb{B}^n \), and in case \( q = +\infty \),

\[
A_{\alpha,k,\infty,s}(f)(\zeta) = \sup_{z \in D_\alpha(\zeta)} |(I + R)^k f(z)|(1 - |z|^2)^{k-s},
\]

when \( q = +\infty \).

Our first goal is to obtain that if \( 1 < p < +\infty \), \( 1 < q < +\infty \) and \( w \) is an \( A_p \)-weight, then an holomorphic function is in \( HF^\alpha_s^p(w) \) if and only if \( A_{\alpha,k,q,s}(f) \in L^p(w) \), for some (and then for all) \( \alpha \geq 1 \) and \( k > s \). We will follow the ideas in [OF]. For the sake of completeness, we will sketch the modifications needed to obtain the weighted case.
If $1 < p < +\infty$, $1 < q \leq +\infty$ we denote by $L^p(w)(L^q_\alpha) = L^p(w)(L^{2n(p-1)q-1}_\alpha 1 - r^2 dr)$ the mixed-norm space of measurable functions $f$ in $S^n \times [0,1]$ such that

$$
\|f\|_{p,q,w} = \left( \int_{S^n} \left( \int_0^1 |f(r\zeta)|^q \frac{2nr^{2n-1}}{1-r^2} dr \right)^\frac{p}{q} w(\zeta) d\sigma(\zeta) \right)^\frac{1}{p} < +\infty.
$$

Also if $\alpha > 1$, and $E_\alpha(z) = (\int_{S^n} \chi_{D_\alpha(z)}(z) d\sigma(z))^{-1} \simeq (1 - |z|^2)^{-n}$, we denote by $L^p(w)(L^q_0)$ the mixed-norm space of measurable functions $f$ defined in $S^n \times B^n$ such that

$$
\|f\|_{p,q,w} = \left( \int_{S^n} \left( \int_{B^n} |f(z, \zeta)|^q \frac{E_\alpha(z)}{(1 - |z|^2)^{n+1}} dv(z) \right)^\frac{p}{q} w(\zeta) d\sigma(\zeta) \right)^\frac{1}{p} < +\infty.
$$

We denote by $F^{\alpha,p,q}(w)$ the space of measurable functions on $B^n$ such that

$$
J_\alpha f(\zeta, z) = \chi_{D_\alpha(z)}(z)f(z)
$$

is in $L^p(w)(L^q_\alpha)$, normed with the norm induced by $\| \cdot \|_{a,p,q,w}$. We also introduce the space $F^{1,p,q}(w)$ of measurable functions on $B^n$ such that $J_1 f(\zeta, r) = f(r\zeta)$ is in $L^p(w)(L^q_1)$.

The representation of the dual of a mixed-norm space, see [BeLo], gives that if $1 < p, q < +\infty$, the dual space of $L^p(w)(L^q_\alpha)$ is $L^{p'}(w)(L^{q'}_\alpha)$, $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, and that if $f \in F^{1,p,q}(w)$, $g \in F^{1,p',q'}(w)$ the pairing is given by

$$
(f,g) = \int_{B^n} \left( \int_0^1 f(r\zeta) \frac{2nr^{2n-1}}{1-r^2} dr \right) w(\zeta) d\sigma(\zeta).
$$

Analogously, the dual space of $L^p(w)(L^q_\alpha)$ is $L^{p'}(w)(L^{q'}_\alpha)$, and if $f \in F^{\alpha,p,q}(w)$, $g \in F^{\alpha',p',q'}(w)$ the pairing is given by

$$
(f,g)_\alpha = \int_{B^n} \left( \int_{S^n} f(z) g(z) \chi_{D_\alpha(z)}(z) w(\zeta) d\sigma(\zeta) \right) dv(z) \\
= \int_{B^n} f(z) g(z) \frac{E_\alpha^w(z)}{(1 - |z|^2)^{n+1}} dv(z),
$$

where $E_\alpha^w(z) = \int_{S^n} \chi_{D_\alpha(z)}(z) w(\zeta) d\sigma(\zeta)$.

Observe that if we write $z_0 = \frac{z}{|z|}$, the doubling property of $w$ gives that $E_\alpha^w(z) \simeq W(B(z_0, (1 - |z|)))$. From now on we will write $B_z = B(z_0, (1 - |z|))$.

We begin with two lemmas that are weighted versions of Lemmas 2.2. and 2.3 in [OF], and whose proofs we omit. We recall that if $\psi$ is a measurable function on $S^n$, the weighted Hardy-Littlewood maximal function is given by

$$
M_{HL}^w(\psi)(\zeta) = \sup_{B \ni \zeta} \frac{1}{W(B)} \int_B |\psi(\eta)| w(\eta) d\sigma(\eta).
$$

**Lemma 2.2.** There exist $C > 0$, $N_0 > 0$ such that for any $z \in D_\alpha(\zeta)$, $N \geq N_0$,

$$
\frac{(1 - |z|^2)^n}{W(B_z)} \int_{S^n} \frac{|\psi(\eta)|}{|1 - z\eta|^n} w(\eta) d\sigma(\eta) \leq CM_{HL}^w(\psi)(\zeta).
$$
Lemma 2.3. Let \( \alpha > 1 \). There exists \( C > 0 \), such that for any \( z \in D_\alpha(\zeta) \),
\[
\frac{1}{W(B_z)} \int_{S^n} \chi_{D_\alpha(\eta)}(z) |\psi(\eta)| w(\eta) d\sigma(\eta) \leq CM_\alpha^{wH}(\psi)(\zeta).
\]

Theorem 2.4. Let \( 1 < p < +\infty \), \( 1 \leq q \leq +\infty \), and \( \alpha \geq 1 \). Then the space \( F_{\alpha,p,q}(w) \) is a retract of \( L^p(w)(L^q_\alpha) \).

Proof of Theorem 2.4:
The fact that \( J_1 \) is an isometry between \( F^{1,p,q}(w) \) and \( L^p(w)(L^q_\alpha) \) gives the theorem for the case \( \alpha = 1 \).

If \( \alpha > 1 \), we introduce the averaging operator
\[
A_{\alpha}(\varphi)(z) = \frac{1}{E_{\alpha}^{w}(z)} \int_{S^n} \chi_{D_\alpha(\eta)}(z) \varphi(\eta, z) w(\eta) d\sigma(\eta).
\]

Now H"older's inequality with exponent \( \alpha \) gives that the above is bounded by
\[
\|A_{\alpha}(\varphi)\|^\alpha_{p,q,w} \leq \frac{1}{E_{\alpha}^{w}(z)} \int_{S^n} \left| \varphi(\eta, z) \right|^\alpha \chi_{D_\alpha(\eta)}(z) w(\eta) d\sigma(\eta).
\]

Hence, by Lemma 2.3
\[
\|A_{\alpha}(\varphi)\|^\alpha_{p,q,w} \leq \sup_{\|\psi\|_{L^{m'}(w)}} \left( \int_{S^n} \int_{\mathbb{B}_n} \frac{1}{E_{\alpha}^{w}(z)} \chi_{D_\alpha(\eta)}(z) \int_{S^n} \chi_{D_\alpha(\eta)}(z) |\varphi(\eta, z)|^q w(\eta) d\sigma(\eta) \right) \frac{dv(z)}{(1 - |z|^2)^{n+1}} |\psi(\zeta)| w(\zeta) d\sigma(\zeta)
\]
\[
\times \sup_{\|\psi\|_{L^{m'}(w)}} \left( \int_{S^n} \int_{\mathbb{B}_n} |\varphi(\eta, z)|^q \frac{dv(z)}{(1 - |z|^2)^{n+1}} w(\eta) M_{HL}^w(\psi)(\eta) d\sigma(\eta), \right.
\]
\[
\leq \sup_{\|\psi\|_{L^{m'}(w)}} \left( \int_{S^n} \int_{\mathbb{B}_n} |\varphi(\eta, z)|^q \frac{dv(z)}{(1 - |z|^2)^{n+1}} w(\eta) M_{HL}^w(\psi)(\eta) d\sigma(\eta), \right.
\]
Next, H"older's inequality with exponent \( m = \frac{q}{q} \) gives that the above is bounded by
\[
\sup_{\|\psi\|_{L^{m'}(w)}} \left( \int_{S^n} \left( \int_{\mathbb{B}_n} |\varphi(\eta, z)|^q \frac{dv(z)}{(1 - |z|^2)^{n+1}} \right)^\frac{q}{q} w(\eta) d\sigma(\eta) \right)^\frac{q}{q}
\]
\[
\leq \sup_{\|\psi\|_{L^{m'}(w)}} \left( \|\psi\|_{L^{m'}(w)} \|\varphi\|^\alpha_{p,q,w} \right)
\]
where we have used that since \( w \) is a doubling measure, the weighted Hardy-Littlewood maximal function is bounded from \( L^{m'}(w) \) to \( L^p(w) \). That finishes the proof of the theorem when \( q \leq p \).
So we are lead to deal with the case $1 < p < q \leq +\infty$, which can be easily obtained from the previous case using the duality in the mixed-norm spaces $L^p(w)(L^q_n)$. □

This result can be used as in the unweighted case to obtain a characterization of the dual spaces of the weighted spaces $F^{\alpha,p,q}(w)$.

**Corollary 2.5.** Let $1 < p < +\infty$, $1 < q < +\infty$, $\alpha > 1$, and $w$ an $A_p$-weight. Then the dual of $F^{\alpha,p,q}(w)$ is $F^{\alpha',p',q'}(w)$ with the pairing given by $(f,g)_\alpha$.

The following proposition will be needed in the proof of the main theorem in this section. If $N > 0$, $M > 0$, we consider the operators defined by

$$P^{N,M}f(y) = \int_{\mathbb{B}^n} f(z) \frac{(1 - |z|^2)^N}{|1 - z|^{n+1+N+M}} dv(z), \quad y \in \mathbb{B}^n.$$

**Theorem 2.6.** Let $1 < p < +\infty$, $1 \leq q < +\infty$, $\alpha, \beta \geq 1$, and $w$ an $A_p$ weight. Then there exists $N_0 > 0$ such that for any $N \geq N_0$ and any $M > 0$, the operator $P^{N,M}$ is continuous from $F^{\alpha,p,q}(w)$ to $F^{\beta',p',q'}(w)$.

**Proof of Theorem 2.6:**

We begin with the case $\alpha, \beta > 1$. The case where $1 \leq \alpha \leq p < +\infty$ can be deduced following the scheme of [OF], using Lemma 2.2.

In the case $1 < p < q < +\infty$ we apply duality in the mixed norm space and obtain

$$||P^{N,M}(f)||_{\beta,p,q,w}^\alpha = \sup_{||g||_{\beta,p',q',w} \leq 1} | \int_{\mathbb{B}^n} P^{N,M}(f)(y)g(y) \frac{E^\alpha_\beta(y)}{(1 - |y|^2)^{n+1}} dv(y) | \leq \sup_{||g||_{\beta,p',q',w} \leq 1} (f, \widetilde{P}^{M-1,N+1}(g))_\alpha,$$

where

$$\widetilde{P}^{R,S}(g)(y) = \int_{\mathbb{B}^n} \frac{(1 - |y|^2)^{R}}{|1 - y|^2} g(y) \frac{E^\alpha_S(y)}{(1 - |y|^2)^n} dv(y).$$

Observe that when $w \equiv 1$, then $\widetilde{P}^{M,N}(f) \sim P^{M,N}(f)$. Here we are led to obtain the operator $\widetilde{P}^{M-1,N+1}$ maps boundedly $F^{\alpha',p',q'}$ to $F^{\alpha,p,q,w}$, provided $p < q$. If we claim this proposition, we finish the proof of the theorem. Using (2.1), and applying Hölder’s inequality,

$$||P^{N,M}(f)||_{\beta,p,q,w}^\alpha = \sup_{||g||_{\alpha,p',q',w} \leq 1} (f, \widetilde{P}^{M-1,N+1}(g))_\alpha \leq \sup_{||g||_{\alpha,p,q,w} \leq 1} ||f||_{\alpha,p,q,w} ||\widetilde{P}^{M-1,N+1}(g)||_{\alpha,p',q',w} \leq C \sup ||f||_{\alpha,p,q,w}.$$

The cases $\alpha = 1$ and $\beta = 1$ are proved in a simmilar way.

To finish the theorem we will prove the claim. Changing the notation, it is enough to prove:

**Proposition 2.7.** Let $1 < q < p < +\infty$, $\alpha, \beta \geq 1$, and $w$ an $A_p$ weight. We then have that there exists $N_0 > 0$ such that for any $N \geq N_0$ and any $M \geq 0$,

1. $\widetilde{P}^{M,N}(1) < +\infty$,
2. The operator $P^{M,N}$ is continuous from $F^{\alpha,p,q}(w)$ to $F^{\beta,p,q}(w)$.
Proof of Proposition 2.7:
Let us begin with (i). From the definition of $E_w(\alpha)$ and Fubini’s theorem,
\[
\int_{B_n} \frac{(1 - |z|^2)^M}{|1 - z|^n + 1 + M + N} E_w(\alpha)(z) \, dv(z)
\]
\[
= \int_{S^n} \int_{D(\alpha)} \frac{(1 - |z|^2)^M}{|1 - z|^n + 1 + M + N} \, dv(z) w(\zeta) \, d\sigma(\zeta) \leq \int_{S^n} \frac{1}{|1 - y|^n + N} w(\zeta) \, d\sigma(\zeta),
\]
where in last inequality we have used Lemma 2.7 in [OF] since $M > -1$.
Next, let $B_k = B(y_0, 2^k (1 - |y|^2))$, $k \geq 0$, where $y_0 = \frac{y}{|y|}$. Since $w$ is doubling and $E_w(\alpha) \simeq W(B_0)$ give that $W(B_k) \leq C^k E_w(\alpha)$. Consequently
\[
\int_{S^n} \frac{1}{|1 - y|^n + N} w(\zeta) \, d\sigma(\zeta)
\]
\[
\leq \sum_k \int_{B_k} \frac{w(\zeta) \, d\sigma(\zeta)}{(2^k(1 - |y|^2))^n + N} \leq \frac{E_w(\alpha)}{(1 - |y|^2)^n + N} \sum_k C^k \frac{2^k(1 - |y|^2)^n + N}{(1 - |y|^2)^n + N},
\]
if $N$ is chosen sufficiently large. That finishes the proof of (i).
Since $m = \frac{p}{q} > 1$, duality gives that
\[
\|\tilde{P}_{M,N}(f)\|_{\beta, p, q, w} \leq \sup_{\|\psi\|_{L^{m'}(w)} \leq 1} \int_{S^n} \int_{D(\psi)} |\tilde{P}_{M,N} f(y)|^q \frac{dv(y)}{(1 - |y|^2)^n + N} \psi(\zeta) w(\zeta) \, d\sigma(\zeta).
\]
Next, Hölder’s inequality shows that if $0 < \varepsilon < N$ then
\[
|\tilde{P}_{M,N}(f)(y)|^q \leq \int_{B_n} |f(z)|^q \frac{(1 - |z|^2)^M (1 - |y|^2)^{-\varepsilon}}{|1 - z|^n + 1 + M + N - \varepsilon} \frac{E_w(\alpha)(z)}{(1 - |z|^2)^n} \frac{E_w(\alpha)(y)}{(1 - |y|^2)^n} \, dv(z)
\]
\[
\times \left( \int_{B_n} \frac{(1 - |z|^2)^M}{|1 - z|^n + 1 + M + N - \varepsilon} \frac{E_w(\alpha)(z)}{(1 - |z|^2)^n} \, dv(z) \right)^{\frac{q}{q'}}
\]
\[
\leq \int_{B_n} |f(z)|^q \frac{(1 - |z|^2)^M (1 - |y|^2)^{-\varepsilon}}{|1 - z|^n + 1 + N + M - \varepsilon} \frac{E_w(\alpha)(z)}{(1 - |z|^2)^n} \frac{E_w(\alpha)(y)}{(1 - |y|^2)^n} \, dv(z),
\]
where in last inequality we have used (i).
Consequently,
\[
\|\tilde{P}_{M,N}(f)\|_{\beta, p, q, w} \leq C \sup_{\|\psi\|_{L^{m'}(w)} \leq 1} \int_{S^n} \int_{D(\psi)} \frac{|f(z)|^q (1 - |z|^2)^M (1 - |y|^2)^{-\varepsilon}}{|1 - z|^n + 1 + N + M - \varepsilon} \frac{\psi(\zeta) w(\zeta) \, d\sigma(\zeta)}{(1 - |z|^2)^n + N} \frac{dv(y)}{(1 - |y|^2)^n + N} \frac{dv(z)}{(1 - |z|^2)^n + N} \frac{\psi(\zeta) w(\zeta) \, d\sigma(\zeta)}{(1 - |z|^2)^n + N} \frac{dv(y)}{(1 - |y|^2)^n + N} \frac{dv(z)}{(1 - |z|^2)^n + N}
\]
\[
\times |f(z)|^q (1 - |z|^2)^{M-n} E_w(\alpha)(z) \, dv(z) \psi(\zeta) w(\zeta) \, d\sigma(\zeta).
\]
Next, if \( y \in D_\beta(\zeta) \), \( E_\alpha^w(y) \simeq W(B_y) \simeq W(B(\zeta, (1 - |y|^2))) \), and \( |1 - z\bar{y}| \simeq (1 - |y|^2) + |1 - z\bar{z}|. \)

Assume first that \( |1 - z\bar{z}| \leq 1 \). Hence,

\[
\int_{D_\beta(\zeta)} \frac{(1 - |y|^2)^{N+n-\varepsilon}}{|1 - z\bar{y}|^{n+1+N+M-\varepsilon}} E_\alpha^w(y)(1 - |y|^2)^{n+1} dv(y) 
\simeq \int_{B^c} \frac{(1 - |y|^2)^{N-\varepsilon}}{((1 - |y|^2) + |1 - z\bar{z}|)^{n+1+N+M-\varepsilon}} \chi_{D_\beta(\zeta)}(y) 
\left( \frac{(1 - |y|^2)^n}{W(B(\zeta, 1 - |y|^2))} \right) (1 - |y|^2)^{n+1} dv(y),
\]

which by integration in polar coordinates

\[
\int_0^1 \frac{r^{2(n-1)}(1 - r^2)^{N+n-\varepsilon}}{(1 - r^2) + |1 - z\bar{z}|)^{n+1+N+M-\varepsilon}} (1 - r^2)W(B(\zeta, C(1 - r^2))) dr 
\simeq \int_{[1-z\bar{z}]} \frac{t^{N+\varepsilon-1}}{W(B(\zeta, t))} dt 
+ \int_{[1-z\bar{z}]} \frac{(t + |1 - z\bar{z}|)^{n+1+N+M-\varepsilon}}{W(B(\zeta, t))} dt = I + II.
\]

In \( I \) we have that \( (t + |1 - z\bar{z}|) \simeq |1 - z\bar{z}| \), and, since \( w \in A_p \),

\[
\frac{t^n}{W(B(\zeta, t))} \simeq \left( \frac{1}{t^n} \int_{B(\zeta, t)} w^{-(p'-1)} \right)^{p-1}.
\]

Thus we obtain that

\[
I \simeq \int_{[1-z\bar{z}]} \frac{t^{N-\varepsilon-1}}{|1 - z\bar{z}|^{n+1+N+M-\varepsilon}} \left( \frac{1}{t^n} \int_{B(\zeta, t)} w^{-(p'-1)} \right)^{p-1} dt 
\leq \left( \int_{B(\zeta, 1-z\bar{z})} w^{-(p'-1)} \right)^{p-1} \frac{1}{|1 - z\bar{z}|^{n+1+N+M-\varepsilon}} \int_{0}^{[1-z\bar{z}]} \frac{t^{N-\varepsilon-n(p'-1)-1}}{1 - |1 - z\bar{z}|^{M+1}} W(B(z_0, |1 - z\bar{z}|)) dt.
\]

where we have used that \( N > 0 \) is chosen big enough, and that \( w \) satisfies the \( A_p \) condition.

In \( II \), \( (t + |1 - z\bar{z}|) \simeq t \), and since \( M + 1 > 0 \), we have

\[
II \simeq \int_{[1-z\bar{z}]} \frac{1}{t^{M+2} W(B(\zeta, t))} dt 
\leq \frac{1}{t^{M+2}} \int_{[1-z\bar{z}]} W(B(\zeta, |1 - z\bar{z}|)) dt \leq \frac{1}{|1 - z\bar{z}|^{M+1}} W(B(z_0, |1 - z\bar{z}|)).
\]
If $|1 - z\zeta| > 1$, then we have that $(1 - r^2) + |1 - z\zeta| \simeq 1$. We return to (2.5) and obtain

$$
\int_0^1 \frac{(1 - r^2)^{N+n-\varepsilon-1} dr}{((1 - r^2) + |1 - z\zeta|)^{n+1+N-M-\varepsilon} W(B(\zeta, 1 - r^2))} \lesssim \left( \int_{B(\zeta,1)} w^{-\gamma}(r) \right)^{\frac{p}{p'}} \int_0^1 t^{N-\varepsilon-n\frac{1}{p'}} dt \lesssim \frac{1}{|1 - z\zeta|M+1 W(B(z_0, |1 - z\zeta|))}.
$$

Then we have in any case that (2.5) is bounded by

$$
\frac{1}{|1 - z\zeta|M+1 W(B(z_0, |1 - z\zeta|))}.
$$

In consequence, we return to (2.4) and we obtain

(2.6)

$$
||\hat{P}^{M,N}(f)||_{\beta,p,q,w}^g \
\lesssim \sup_{||\psi||_{L^{m'}(w)} \leq 1} \left( \int_{B^n} \int_{B^n} |f(z)|q |1 - |z|^2|^{M-n}F_{\alpha}(z) \psi(\zeta)dv(z)w(\zeta)d\sigma(\zeta) \right) \lesssim \left( \int_{B^n} \int_{B^n} |f(z)|q |1 - |z|^2|^{M-n}\chi_{D_\alpha}(z) \int_{S^n} \frac{\psi(\zeta)w(\zeta)d\sigma(\zeta)}{|1 - z\zeta|M+1 W(B(z_0, |1 - z\zeta|))} \right)^{\frac{1}{q}} \int_{S^n} |\psi(z)|w(\zeta)d\sigma(\zeta). \tag{2.6}
$$

Next, if $z \in D_\alpha(\eta), B(\eta, |1 - z\zeta|) \subset B(z_0, C|1 - z\zeta|)$, and if $B_k = B(\eta, 2^k(1 - |z|^2))$, $k \geq 0$ and $\zeta \in B_{k+1} \setminus B_k$, $|1 - z\zeta| \simeq 2^k(1 - |z|^2)$. Thus

$$
\int_{S^n} |\psi(z)|w(\zeta)d\sigma(\zeta) \lesssim \frac{1}{(1 - |z|^2)^{M+1} W(B(\eta, 1 - |z|^2))} \int_{B_0} |\psi(z)|w(\zeta)d\sigma(\zeta) + \sum_{k \geq 1} 2^k(M+1)(1 - |z|^2)^{M+1} W(B(\eta, 2^k(1 - |z|^2))) \int_{B_k} |\psi(z)|w(\zeta)d\sigma(\zeta) \lesssim \frac{1}{(1 - |z|^2)^{M+1}} \sum_{k \geq 0} 2^k(M+1)M_{\alpha}^w(\psi)(\eta) \lesssim \frac{1}{(1 - |z|^2)^{M+1}} M_{\alpha}^w(\psi)(\eta).
$$

Plugging the above estimate in (2.6) and using Hölder’s inequality with exponent $m = \frac{2}{q}$, we obtain

$$
||\hat{P}^{M,N}(f)||_{\beta,p,q,w}^g \
\lesssim \sup_{\psi \in L^{m'}(w)} \left( \int_{B^n} \int_{B^n} |f(z)|q |1 - |z|^2|^{n+1} \chi_{D_\alpha}(z)dv(z)M_{\alpha}^w(\psi)(\eta)w(\eta)d\sigma(\eta) \right) \
\lesssim \sup_{\psi \in L^{m'}(w)} \||f||_{\alpha,p,q,w}^g ||M_{\alpha}^w(\psi)||_{L^{m'}(w)}^g \lesssim ||f||_{\alpha,p,q,w}^g. \quad \square
$$
We deduce from the previous theorem the following characterization of the weighted holomorphic Triebel-Lizorkin spaces. If \( f \in H(B^n) \), \( s, t > 0 \), let

\[
L_s^t f(z) = (1 - |z|^2)^t s (I + R)^t f(z).
\]

**Theorem 2.8.** Let \( 1 < p < +\infty \), \( 1 < q < +\infty \), \( t > s \geq 0 \) and \( \alpha \geq 1 \). Let

\[
HF_s^{\alpha, t, p, q}(w) = \{ f \in H(B^n); \| L_s^t f \|_{\alpha, p, q} < +\infty \}.
\]

Then \( HF_s^{\alpha, t, p, q}(w) = HF_s^{p,q}(w) \).

**Proof of theorem 2.8:**

If \( s < t_0 < t_1 \), \( \alpha, \beta \geq 1 \), we just need to check that \( HF_s^{\alpha, t_0, p, q}(w) = HF_s^{\alpha, t_1, p, q}(w) \).

Any holomorphic function \( f \) on \( B^n \) satisfying that \( L_s^t f(z) \in F^{\alpha, p, q}(w) \) is in \( A^{-\infty}(B^n) \), the space of holomorphic functions in \( B^n \) for which there exists \( k > 0 \) such that \( \sup_n (1 - |z|^2)^k |f(z)| < +\infty \). Consequently, \( f \) and its derivatives have a representation formula via the reproducing kernel \( c_N \frac{(1 - |z|^2)^N}{(1 - |y|^2)^n + |z|^N} \), for \( N > 0 \) sufficiently large and an adequate constant \( c_N \). Once we have made this observation, we can reproduce the arguments in [OF] and obtain

\[
(I + R)^{t_0} f(y) = C_N \int_{B^n} (I + R)^{t_1} f(z)(I + R_y)^{t_0 - t_1} \frac{(1 - |z|^2)^N}{(1 - |y|^2)^n + |z|^N} dv(z).
\]

Since for \( m > 0 \) we have that

\[
(2.7) \quad (I + R)^{-m} g(y) = \frac{1}{\Gamma(m)} \int_0^1 \left( \log \frac{1}{r} \right)^{m-1} g(ry) dr,
\]

we obtain

\[
\| L_s^{t_0} f \|_{\alpha, p, q} \leq \| \int_{B^n} |(I + R)^{t_1} f(z)| \frac{(1 - |z|^2)^N(1 - |y|^2)^{t_0 - t_1}}{|1 - |y|^2|^n + |z|^N} dv(z) \|_{\alpha, p, q, w},
\]

and we just have to apply Theorem 2.6 to finish the proof. \( \square \)

**Theorem 2.9.** Let \( 1 < p < +\infty \), \( 1 < q < +\infty \), \( w \) an \( A_p \)-weight, and \( f \) a holomorphic function. Then the following assertions are equivalent:

(i) \( f \) is in \( HF_s^{p,q}(w) \).

(ii) \( A_{\alpha, k, q, s}(f) \in L^p(w) \), for some \( \alpha \geq 1 \) and \( k > s \).

(iii) \( A_{\alpha, k, q, s}(f) \in L^p(w) \), for all \( \alpha \geq 1 \) and \( k > s \).

Our next result studies some inclusion relationships between different weighted holomorphic Triebel-Lizorkin spaces.

**Theorem 2.10.** Let \( 1 < p < +\infty \), \( 1 \leq q_0 \leq q_1 \leq +\infty \), \( s \geq 0 \) and let \( w \) be an \( A_p \)-weight. We then have

\[
HF_s^{p,q_0}(w) \subset HF_s^{p,q_1}(w).
\]

**Proof of Theorem 2.10:**
We begin with the case \( q_1 = +\infty \). Let \( 0 < \varepsilon < 1 \). If \( L^k f(z) = (1 - |z|^2)^{k-s}(I + R)^k f(z) \), the fact that \((I + R)^k f\) is holomorphic gives that
\[
|L^k f(r\zeta)| \leq \left( \frac{1}{(1 - r^2)^{n+1}} \int_{K(r\zeta,c(1-r^2))} |(I + R)^k f(z)|\varepsilon d\nu(z) \right)^{\frac{1}{p}} (1 - r^2)^{k-s},
\]
where for \( y \in B^n \) \( K(y,t) \) is the nonisotropic ball in \( B^n \) given by
\[
K(y,t) = \{ z \in B^n ; |\zeta(z) - y| + |\zeta(y - z)| < t \}.
\]
In [OF] it is obtained that
\[
|L^k f(r\zeta)| \leq \left( M_{HL} \left( \int_0^1 |(I + R)^k f(t\eta)|^{\eta}(1 - \eta^2)^{(k-s)\eta-1} dt \right)^{\frac{1}{\eta}} (\zeta) \right)^{\frac{1}{p}}.
\]
Thus
\[
||f||^p_{HF^{p\infty}(w)} = \int_{B^n} \sup_{0<r<1} |L^k f(r\zeta)|^p w(\zeta) d\sigma(\zeta)
\]
\[
\leq \int_{B^n} \left( M_{HL} \left( \int_0^1 |(I + R)^k f(t\eta)|^{\eta}(1 - \eta^2)^{(k-s)\eta-1} dt \right)^{\frac{1}{\eta}} (\zeta) \right)^{\frac{p}{\eta}} w(\zeta) d\sigma(\zeta).
\]
Since \( \frac{p}{\eta} > p \), and \( w \) is an \( A_p \)-weight, \( w \) is in \( A_\frac{p}{\eta} \), and in consequence the unweighted Hardy-Littlewood maximal function is a bounded map \( L^k_f(w) \) to itself. Hence the above is bounded by
\[
C \int_{B^n} \left( \int_0^1 |(I + R)^k f(t\zeta)|^{\eta}(1 - \eta^2)^{(k-s)\eta-1} dt \right)^{\frac{p}{\eta}} w(\zeta) d\sigma(\zeta) = C||f||^p_{HF^{p\infty}(w)}.
\]
In order to finish the theorem, we will prove that if \( q_0 < q_1 < +\infty \), then
\[
||f||_{HF^{q_1}(w)} \leq ||f||_{HF^{q_0}(w)}^\frac{q_0}{q_1} ||f||_{HF^{p\infty}(w)}^{1-\frac{q_0}{q_1}}.
\]
Since
\[
||f||^p_{HF^{p\infty}(w)} \leq \int_{B^n} \left( \sup_{0<r<1} |(I + R)^k f(r\zeta)|(1 - r)^{k-s} \right)^{(q_1-q_0)p/q_1} \int_0^1 |(I + R)^k f(r\zeta)|^{q_0}(1 - r^2)^{(k-s)q_0-1} r^{q_1/q_0} dr \right)^{\frac{1}{q_1}} w(\zeta) d\sigma(\zeta),
\]
Hölder’s inequality with exponent \( q_1/q_0 > 1 \), gives that the above is bounded by
\[
C||f||^p_{HF^{p\infty}(w)} ||f||^{p(1-\frac{q_0}{q_1})}_{HF^{p\infty}(w)}. \quad \square
\]
We now consider the weighted Hardy space \( H^p(w) \), for \( 1 < p < +\infty \), and \( w \) an \( A_p \) weight. It is shown in [Lu] that \( f \in H^p(w) \) if and only if \( f = C[f^*] \), where \( f^*(\zeta) = \lim_{r \to 1} f(r\zeta) \in L^p(w) \) is the radial limit, \( C \) is the Cauchy-Szegő kernel. In addition, \( f = P[f^*] \), where \( P \) is the Poisson-Szegő kernel. It follows also that
\[
||f||_{H^p(w)} \approx ||f^*||_{L^p(w)}.
\]
It is immediate to deduce from this that if \( f \in H^p(w) \) if and only if for any \( \alpha \geq 1 \), \( M_\alpha(f) \in L^p(w) \), where \( M_\alpha \) is the \( \alpha \)-admissible maximal operator given by
\[
M_\alpha(f)(\zeta) = \sup_{z \in D_\alpha(\zeta)} |f(z)|.
\]
In addition \( \|f\|_{H^p(w)} \simeq \|M_\alpha(f)\|_{L^p(w)} \), with constant that depends on \( \alpha \). Indeed, since
\[
|f(r\zeta)| \leq M_\alpha(f)(\zeta),
\]
we have that \( ||f||_{H^p(w)} \leq ||M_\alpha(f)||_{L^p(w)} \). On the other hand, assume that \( f \in H^p(w) \). Then \( f = P[f^*] \), \( f^* \in L^p(w) \) and since \( M_\alpha(f) \leq CM_{HL}(f^*) \), (see for instance [Ru]), we deduce that
\[
\int_{\mathbb{S}^n} (M_\alpha(f)(\zeta))^p w(\zeta) d\sigma(\zeta) 
\leq \int_{\mathbb{S}^n} (M_{HL}(f^*)(\zeta))^p w(\zeta) d\sigma(\zeta) \leq \int_{\mathbb{S}^n} |f^*(\zeta)|^p w(\zeta) d\sigma(\zeta) \leq ||f||_{H^p(w)}^p.
\]
where we have used that since \( w \) in an \( A_p \)-weight, the Hardy-Littlewood maximal operator maps \( L^p(w) \) continuously to itself.

Our next result gives a proof for the weighted nonisotropic case of the fact that the spaces \( H^p(w) \) can also be defined in terms of admissible area functions. Similar results, but using a different approach based on localized good-lambda inequalities, have been obtained in [StrTo] for weighted isotropic Hardy spaces in \( \mathbb{R}^n \).

**Theorem 2.11.** Let \( 1 < p < +\infty \), and \( w \) be an \( A_p \)-weight. Let \( f \) be an holomorphic function on \( \mathbb{B}^n \). Then the following assertions are equivalent:

(i) \( f \) is in \( H^p(w) \).

(ii) There exists \( \alpha \geq 1, k > 0 \), such that \( A_{\alpha,k,2,0}(f) \in L^p(w) \).

(iii) For every \( \alpha \geq 1, k > 0 \), \( A_{\alpha,k,2,0}(f) \in L^p(w) \).

In addition, there exists \( C > 0 \) such that for any \( f \in H^p(w) \),
\[
\frac{1}{C} \|f\|_{H^p(w)} \leq \|A_{\alpha,1,2,0}(f)\|_{L^p(w)} \leq C \|f\|_{H^p(w)}.
\]

**Proof of Theorem 2.11:**

We already know that (ii) and (iii) are equivalent, so we only have to check the equivalence of (i) and (ii) for the case \( k = 1 \). The proof of (i) implies (ii) is given in [KaKo], using the arguments of [St2]. For the proof of (ii) implies (i), we will follow some ideas of [AhBrCa].

Without loss of generality we may assume that \( f(0) = 0 \). Let us assume first that \( f \in H(\overline{\mathbb{B}^n}) \). Then \( f = P[f^*] \) where \( f^* \in C(\mathbb{S}^n) \). We want to check that
\[
||f^*||_{L^p(w)} \leq C||A_{\alpha,1,2,0}(f)||_{L^p(w)}.
\]

We will use that the dual space of \( L^p(w) \) can be identified with \( L^q(\omega^{n-1}) \) if the duality is given by
\[
<f,g> = \int_{\mathbb{S}^n} f(\zeta) \overline{g(\zeta)} d\sigma(\zeta).
\]

Hence,
\[
||f^*||_{L^p(w)} = \sup\{|\int_{\mathbb{S}^n} f^*(\zeta) g^*(\zeta) d\sigma(\zeta)|, g^* \in C(\mathbb{S}^n), ||g^*||_{L^q(\omega^{n-1})} \leq 1\}.
\]
If \( g = P[g^*] \), we have (see [AhBrCa] page 131)
\[
\frac{n!}{(n-1)!} \int_{B^n} f^*(z) g^*(\zeta) d\sigma(\zeta)
\]
\[= n^2 \int_{B^n} f(z) g(z) dv(z) + \int_{B^n} (\nabla_{B^n} f(z), \nabla_{B^n} g(z))_{B^n} \frac{dv(z)}{1 - |z|^2},
\]
where \( \nabla_{B^n} \) is the gradient in the Bergman metric (see for instance [St2]), and
\[(F(z), G(z))_{B^n} = (1 - |z|^2) \left( \sum_{i,j} (\delta_{i,j} - z_i \overline{z_j}) F_i(z) \overline{G_j(z)} \right).
\]
We then have (see [St2]) that since \( F \) is holomorphic
\[||\nabla_{B^n} F(z)||^2_{B^n} = (\nabla_{B^n} F(z), \nabla_{B^n} F(z))_{B^n}
\]
\[
\simeq (1 - |z|^2) \left\{ \sum_{i=1}^n |\frac{\partial}{\partial z_i} F(z)|^2 - \sum_{i=1}^n z_i \frac{\partial}{\partial z_i} F(z) \right\}.
\]
In order to estimate \( \int_{B^n} f(z) g(z) dv(z) \) we will need to obtain estimates of the values of the functions \( f, g \) on compact subsets of \( B^n \) in terms of the norms \( ||A_{\alpha,1,2,0}(f)||_{L^p(w)} \) and \( ||A_{\alpha,1,2,0}(g)||_{L^p(w)} \) respectively.

**Lemma 2.12.** Let \( 1 < p < +\infty \) and \( w \) an \( A_p \)-weight. There exists \( C > 0 \) such that for any holomorphic function \( f \) in \( B^n \), and any \( z = r\zeta \)
\[
|f(z)| \leq |f(0)| + \int_0^r \frac{dt}{W(B(\zeta, 1 - t^2))^{\frac{1}{p}(1 - t^2)}} ||A_{\alpha,1,2,0}(f)||_{L^p(w)}^p.
\]
In particular, if \( K \subset B^n \) is compact and
\[||f||_K = \sup_{z \in K} |f(z)|,
\]
then there exists a constant \( C > 0 \), depending only on \( w, p \) and \( K \) such that \( ||f||_K \leq C (||f(0)|| + ||A_{\alpha,1,2,0}(f)||_{L^p(w)}) \).

**Proof of Lemma 2.12**
Since \( f \) is holomorphic, we obtain that if \( z = r\zeta \in B^n \), there exist \( C_i > 0, i = 1, 2 \), such that for any \( \eta \in B(\zeta, C_i(1 - r^2)) \), then
\[
||\nabla f(z)||^2 \leq \frac{1}{(1 - |z|^2)^{n+1}} \int_{K(z,C_2(1-|z|^2))} |\nabla f(y)|^2 dv(y)
\]
\[
\leq \frac{1}{(1 - |z|^2)^2} \int_{K(z,C_2(1-|z|^2))} (1 - |y|^2)^{1-n} |\nabla f(y)|^2 dv(y) \leq \frac{C}{(1 - |z|^2)} (A_{\alpha,1,2,0}(f)(\eta))^2.
\]
Consequently
\[(||1 - |z|^2||\nabla f(z)||)^p \leq (A_{\alpha,1,2,0}(f)(\eta))^p.
\]
Then we have
\[
(1 - |z|^2) |\nabla f(z)|^p W(B(\zeta, 1 - r^2)) \leq \int_{B(\zeta,1-r^2)} (A_{\alpha,1,2,0}(f)(\eta))^p w(\eta) d\sigma(\eta) \leq ||A_{\alpha,1,2,0}(f)||^p_{L^p(w)}.
\]
In particular, if $0 < r < 1$ and $\zeta \in S^n$,
\[
|\frac{\partial f}{\partial r}(r\zeta)| \leq \frac{1}{W(B(\zeta, 1 - r^2))^{\frac{1}{p}}(1 - r^2)} ||A_{\alpha,1,2,0}(f)||_{L^p(w)},
\]
and integrating, we finally obtain
\[
|f(r\zeta)| \leq \left( |f(0)| + \int_0^r \frac{dt}{W(B(\zeta, 1 - t^2))^{\frac{1}{p}}(1 - t^2)} ||A_{\alpha,1,2,0}(f)||_{L^p(w)} \right).
\]

For the remaining affirmation, let $K \subset B^n$ be compact. Then there exists $0 < \delta < 1$ such that for any $z = r\zeta \in K$, $r \leq 1 - \delta$, and
\[
|f(z)| \leq \left( |f(0)| + \frac{1}{W(B(\zeta, \delta))^{\frac{1}{p}}\delta} ||A_{\alpha,1,2,0}(f)||_{L^p(w)} \right).
\]

Since $w$ is doubling, and there exists $N > 0$ (not depending on $\zeta$) such that $S^n \subset B(\zeta, cN\delta)$, $W(S^n) \leq W(B(\zeta, \delta))$, and consequently
\[
||f||_K \leq |f(0)| + ||A_{\alpha,1,2,0}(f)||_{L^p(w)}. \quad \square
\]

Going back to the proof of Theorem 2.7.1 let $0 < \varepsilon < 1$. The above lemma together with the fact that if $w$ is an $A_p$ weight, then $w^{-\left(\frac{1}{p'} - 1\right)}$ is an $A_{p'}$-weight, give by (2.8) that
\[
\int_{S^n} f^*(\zeta)g^*(\zeta)d\sigma(\zeta)
\]
\[
\leq ||A_{\alpha,1,2,0}(f)||_{L^p(w)}||A_{\alpha,1,2,0}(g)||_{L^{p'}(w^{-\left(\frac{1}{p'} - 1\right)})} + \int_{1-\varepsilon \leq |z| < 1} f(z)g(z)dv(z)
\]
\[
+ \int_{B^n} ||\nabla B^n f(z)||_{B^n} ||\nabla B^n g(z)||_{B^n} \frac{dv(z)}{1 - |z|^2}.
\]

In order to estimate the second integral, we use polar coordinates, and obtain
\[
|\int_{1-\varepsilon \leq |z| < 1} f(z)g(z)dv(z)|,
\]
which by Hölder’s inequality is bounded by
\[
\int_{1-\varepsilon}^{1} \int_{S^n} |f(r\zeta)||g(r\zeta)|d\sigma(\zeta)dr
\]
\[
\leq \int_{1-\varepsilon}^{1} ||f||_{L^p(w)}||g||_{L^{p'}(w^{-\left(\frac{1}{p'} - 1\right)})} dr \leq \varepsilon ||f||_{H^p(w)}||g||_{H^{p'}(w^{-\left(\frac{1}{p'} - 1\right)})}
\]
\[
\leq \varepsilon ||f^*||_{L^p(w)}||g^*||_{L^{p'}(w^{-\left(\frac{1}{p'} - 1\right)})}.
\]

For the third integral, we use (5.1) of CoiMeSt to estimate it by
\[
\int_{S^n} A_{\alpha,1,2,0}(f)(\zeta)A_{\alpha,1,2,0}(g)(\zeta)d\sigma(\zeta) \leq ||A_{\alpha,1,2,0}(f)||_{L^p(w)}||A_{\alpha,1,2,0}(g)||_{L^{p'}(w^{-\left(\frac{1}{p'} - 1\right)})}.
\]
Since we already know (see [KaKo]) that \( \|A_{\alpha,1,2,0}(g)\|_{L^{p'}(\mathbb{B}^{n})} \leq \|g\|_{L^{p'}(\mathbb{B}^{n})} \), we finally obtain
\[
\|f^*\|_{L^p(w)} \leq \|A_{\alpha,1,2,0}(f)\|_{L^p(w)} + \varepsilon \|f^*\|_{L^p(w)},
\]
which gives the result for \( f \in H(\mathbb{B}^n) \).

So we are left to show that the estimate we have already obtained holds for a general holomorphic function in \( \mathbb{B}^n \). If \( f \) is an holomorphic function on \( \mathbb{B}^n \) such that \( \|A_{\alpha,1,2,0}(f)\|_{L^p(w)} < +\infty \), let \( f_r(z) = f(rz) \in H(\mathbb{B}^n) \), for \( 0 < r < 1 \). We then have that
\[
(2.9) \quad \|f_r\|_{H^p(w)} \leq \|A_{\alpha,1,2,0}(f_r)\|_{L^p(w)}
\]
Let us check first that
\[
\sup_r \|A_{\alpha,1,2,0}(f_r)\|_{L^p(w)} \leq C \|A_{\alpha,1,2,0}(f)\|_{L^p(w)}.
\]
Notice that
\[
\|A_{\alpha,1,2,0}(f_r)\|_{L^p(w)}^p = \|J_\alpha((1 - |\cdot|^2)(I + R)f_r)\|_{L^p(w)} = \|J_\alpha((1 - |\cdot|^2)(I + R)f_r)\|_{L^p(w)} \cdot \left( \frac{1}{|1 - rz\overline{y}|^{n+1+N}} \right).
\]
We will check that there exists \( 0 \leq G(\zeta, z) \in L^p(w)(L^2(\frac{dv(z)}{1-|z|^2}))) \) such that for any \( 0 < r < 1, \zeta \in S^n, z \in \mathbb{B}^n, J_\alpha((1 - |\cdot|^2)(I + R)f_r)(\zeta, z) \leq G(\zeta, z) \), and \( \|G\|_{L^p(w)(L^2(\frac{dv(z)}{1-|z|^2}))} \leq \|A_{\alpha,1,2,0}(f)\|_{L^p(w)} \).

Let us obtain such a function \( G \). Since by hypothesis \( A_{\alpha,1,2,0}f \in L^p(w) \), we have that the holomorphic function \( f \) satisfies that \( A_{\alpha,1,2,0}f \in L^1(\mu) \), and consequently that there exists \( C > 0 \) such that for any \( z \in \mathbb{B}^n, |f(z)| \leq \frac{1}{(1-|z|^2)^n} \). Hence, the integral representation theorem gives that for \( N > 0 \) sufficiently large, and \( z \in \mathbb{B}^n \),
\[
(I + R)f(rz) = C \int_{\mathbb{B}^n} \frac{(1 - |y|^2)^N(I + R)f(y)}{(1 - rz\overline{y})^{n+1+N}} dv(y).
\]
Next, there is a constant \( C > 0 \) such that for any \( 0 < r < 1, z, y \in \mathbb{B}^n, |1 - rz\overline{y}| \geq C|1 - z\overline{y}| \), and the above formula gives that
\[
|(I + R)f(rz)| \leq C \int_{\mathbb{B}^n} \frac{(1 - |y|^2)^N(I + R)f(y)}{|1 - z\overline{y}|^{n+1+N}} dv(y).
\]
Combining the above results we have that
\[
\chi_{D_\alpha(\zeta)}(z)(1 - |z|^2)|(I + R)f(rz)| \leq \chi_{D_\alpha(\zeta)}(z) \int_{\mathbb{B}^n} \frac{(1 - |y|^2)^N(1 - |z|^2)((1 - |y|^2)|(I + R)f(y))}{|1 - z\overline{y}|^{n+1+N}} dv(y) = C \chi_{D_\alpha(\zeta)}(z)P^{-1,1}((1 - |\cdot|^2)(I + R)f)(z) := G(z, \zeta).
\]
Theorem 2.8 shows that provided \( N \) is chosen sufficiently large, \( P^{N-1,1} \) maps \( F^{\alpha,p,2}(w) \) to itself, and in particular that
\[
\|G\|_{L^p(w)(L^2(\frac{dv(z)}{1-|z|^2}))} = \|P^{N-1,1}((1 - |\cdot|^2)(I + R)f)\|_{\alpha,p,2,w} \leq \|(1 - |\cdot|^2)(I + R)f\|_{\alpha,p,2,w} = C \|A_{\alpha,1,2,0}(f)\|_{L^p(w)} < +\infty.
\]
Consequently
\[ \|f_r\|_{H^p(w)} \leq \|A_{1,2,0}(f)\|_{LP(w)}, \]
and therefore \( f \in H^p(w). \)

We will now remark on some facts about weighted Hardy-Sobolev spaces. Let us recall, that if \( 1 < p < +\infty, 0 < s < n, \) and \( w \) is an \( A_p \)-weight, we denote by \( H^p_s(w) \) the space of holomorphic functions \( f \) on \( B^n \) satisfying that
\[ \|f\|_{H^p_s(w)} = \|(I + R)^s f\|_{H^p(w)} < +\infty. \]
The results obtained in the previous theorems give alternative equivalent definitions of the spaces \( H^p_s(w) \) in terms of admissible maximal or radial functions and admissible area functions.

On the other hand, when \( w \equiv 1, \) and \( 0 < s < n, \) it is well known, see for instance [CaOr1], that the space \( H^p_s(w) \) admits a representation in terms of a fractional Cauchy-type kernel \( C_s \) defined by
\[ C_s(z, \zeta) = \frac{1}{(1 - z\zeta)^{n-s}}. \]
The same lines of the proof of the unweighted case can be used to obtain a similar characterization in the weighted case. We just have to use that the Hardy-Littlewood maximal operator is bounded in \( L^p(w) \), if \( w \) is an \( A_p \)-weight and Lemma 2.1.

**Theorem 2.13.** Let \( 1 < p < +\infty, 0 < s < n, \) and \( w \) be an \( A_p \)-weight. We then have that the map
\[ C_s(f)(z) = \int_{B^n} \frac{f(\zeta)}{(1 - z\zeta)^{n-s}} d\sigma(\zeta), \]
is a bounded map of \( L^p(w) \) onto \( H^p_s(w). \)

3. Holomorphic potentials and Carleson measures

In this section we will study Carleson measures for \( H^p_s(w) \), \( 1 < p < +\infty \) and \( 0 < s < n, \) that is, the positive finite Borel measures \( \mu \) on \( B^n \) satisfying
\[ (3.1) \quad \|f\|_{LP(d\mu)} \leq C\|f\|_{H^p_s(w)}, \quad f \in L^p(w). \]
In what follows we will write
\[ \int_E w d\sigma = \frac{1}{|E|} \int_E w, \]
where \( E \) is a measurable set in \( S^n, |E| \) denotes its Lebesgue measure.

By Theorem 2.13 this inequality can be rewritten as follows:
\[ (3.2) \quad \|C_s(f)\|_{LP(d\mu)} \leq C\|f\|_{LP(w)}, \quad f \in L^p(w). \]
We recall that we have defined the non-isotropic potential of a positive Borel function \( f \) on \( S^n \) by
\[ (3.3) \quad K_s(f)(z) = \int_{S^n} K_s(z, \zeta) f(\zeta) d\sigma(\zeta) = \int_{S^n} \frac{f(\zeta)}{|1 - z\zeta|^{n-s}} d\sigma(\zeta), \]
for \( z \in B^n. \)
Analogously to what happens for isotropic potentials (see [Ad]), in the nonisotropic case it can be proved that if \( w \) is an \( A_p \) weight and \( \zeta_0 \in S^n \) satisfies that

\[
(3.4) \quad \int_{S^n} \frac{1}{|1 - \zeta_0 \zeta|^{(n-s)p'}} w^{-(p'-1)}(\zeta) d\sigma(\zeta) < +\infty,
\]
then for any \( f \in L^p(w) \), \( K_s(f) \) is continuous in \( \zeta_0 \). Observe that when \( w \equiv 1 \), (3.4) holds if and only if \( n - sp < 0 \). In the general weighted case, if \( w \) satisfies a doubling condition of order \( \tau \), and \( \tau - sp < 0 \), we also have that (3.4) holds, and consequently the Carleson measures in this case for weighted Hardy Sobolev spaces are just the finite ones. Indeed, assume that \( \tau - sp < 0 \). We then have

\[
\int_{S^n} \frac{1}{|1 - \zeta_0 \zeta|^{(n-s)p'}} w^{-(p'-1)}(\zeta) d\sigma(\zeta) = \int_{S^n} w^{-(p'-1)}(\zeta) \int_{[1-\zeta_0 \zeta] < t} \frac{dt}{t^{(n-s)p'}} d\sigma(\zeta)
\]
\[
\leq \int_0^K \int_{B(\zeta, t)} w^{-(p'-1)} \frac{t^n dt}{t^{(n-s)p'}} \sim \int_0^K \left( \int_{B(\zeta, t)} w \right)^{p'-1} \frac{1}{t^{(n-s)p'}} \leq \sum_k 2^{-ksp'} W(B(\zeta_0, 2^{-k})).
\]

The fact that \( w \) satisfies condition \( D_\tau \) gives that \( W(S^n) \leq 2^{k\tau} W(B(\zeta_0, 2^{-k})) \), and consequently the above sum is bounded, up to constants, by

\[
\sum_k 2^{k(\tau(p'-1)-sp')}.
\]

Since \( \tau - sp < 0 \) we also have that \( \tau(p'-1) - sp' < 0 \), and we are done.

From now on we will assume that \( \tau - sp \geq 0 \).

The problem of characterizing the positive finite Borel measures \( \mu \) on \( B^n \) for which the following inequality holds

\[
(3.5) \quad ||K_s(f)||_{L^p(\mu)} \leq C ||f||_{L^p(w)},
\]
has been thoroughly studied, and there are, among others, characterizations in terms of weighted nonisotropic Riesz capacities that are defined as follows: if \( E \subset S^n \), \( 1 < p < +\infty \) and \( 0 < s < n \),

\[
C_{sp}^w(E) = \inf\{||f||_{L^p(w)}^p; f \geq 0, K_s(f) \geq 1 \text{ on } E\}.
\]

It is well known, that when \( w \equiv 1 \), \( C_{sp}(B(\zeta, r)) \simeq r^{n-sp}, \zeta \in S^n, r < 1 \). See [Ad] for expressions of weighted capacities of balls in \( R^n \).

As it happens in \( R^n \) (see [Ad]), we have that if \( 0 \leq n - sp \), (3.5) holds if and only if there exists \( C > 0 \) such that for any open set \( G \subset S^n \),

\[
(3.6) \quad \mu(T(G)) \leq CC_{sp}^w(G).
\]

Here \( T(G) \) is the admissible tent over \( G \), defined by

\[
T(G) = T_\alpha(G) = \left( \bigcup_{\zeta \notin G} D_\alpha(\zeta) \right)^c.
\]

The problem of characterizing the Carleson measures \( \mu \) for the holomorphic case (3.2) is much more complicated, even in the nonweighted case. Since \( |C_s(z, \zeta)| \leq \)
$K_s(z, \zeta)$, it follows from Theorem 2.13 that (3.5) implies (3.2), and consequently that if condition (3.6) is satisfied, then $\mu$ is a Carleson measure for $H_p^s(w)$. Of course, when $n - s < 1$ both problems are equivalent, even in the weighted case, simply because if $f \geq 0$, $|C_s(f)| \simeq K_s(f)$, but when $n > 1$ (see [Ah] and [CaOr2]), condition (3.5) for the unweighted case is not, in general, equivalent to condition (3.2). Observe that when $n - sp \leq 0$, $H_p^s$ consists of regular functions, and consequently any finite measure is a Carleson measure for the holomorphic and the real case. It is proved in [CohVe1] that this equivalence still remains true if we are not too far from the regular case, namely, if $0 \leq n - sp < 1$. The main purpose of this section is to obtain a result in this line for a wide class of $A_p$-weights.

In [Ah] it is also shown that if (3.2) holds for $w \equiv 1$, then the capacity condition on balls is satisfied, i.e. there exists $C > 0$ such that $\mu(T(B(\zeta, r))) \leq Cr^{n-sp}$, for any $\zeta \in S^n$ and any $0 < r < 1$. The following proposition obtains a necessary condition in this line for the weighted holomorphic trace inequality.

**Proposition 3.1.** Let $1 < p < +\infty$, $0 < s < n$. Let $\mu$ be a positive finite Borel measure on $B^n$, and $w$ be an $A_p$-weight. Assume that there exists $C > 0$ such that $||f||_{L^p(d\mu)} \leq C||f||_{H_p^s(w)}$, for any $f \in H_p^s(w)$. We then have that there exists $C > 0$ such that for any $\zeta \in S^n$, $r > 0$,

$$\mu(T(B(\zeta, r))) \leq C \frac{W(B(\zeta, r))}{r^{sp}}.$$

**Proof of Proposition 3.1.**

Let $\zeta \in S^n$, $0 < r < 1$ be fixed. If $z \in \overline{B^n}$, let

$$F(z) = \frac{1}{(1 - (1 - r)z\zeta)^N},$$

with $N > 0$ to be chosen later. If $z \in T(B(\zeta, r))$, and $z_0 = \frac{z}{|z|}$, $(1 - |z|) \leq r$ and $|1 - z_0\zeta| \leq r$. Hence $|1 - (1 - r)z\zeta| \leq r$, and consequently,

$$\mu(T(B(\zeta, r))) \leq C \int_{T(B(\zeta,r))} |F(z)|^p d\mu(z).$$

On the other hand,

$$||F||_{H_p^s(w)}^p \leq C \int_{S^n} \frac{1}{|1 - (1 - r)\eta\zeta|^{(N+s)p}} w(\eta)d\sigma(\eta)$$

$$= \int_{B(\zeta,r)} \frac{1}{|1 - (1 - r)\eta\zeta|^{(N+s)p}} w(\eta)d\sigma(\eta)$$

$$+ \sum_{k \geq 1} \int_{B(\zeta,2^{k+1}r) \setminus B(\zeta,2^kr)} \frac{1}{|1 - (1 - r)\eta\zeta|^{(N+s)p}} w(\eta)d\sigma(\eta).$$
If \( k \geq 1 \), and \( \eta \in B(\zeta, 2^{k+1}r) \setminus B(\zeta, 2^kr) \), \( |1 - (1 - r)\eta\bar{\zeta}| \approx 2^kr \). This estimates together with the fact that \( w \) is doubling, give that the above is bounded by

\[
\sum_{k \geq 0} \frac{W(B(\zeta, 2^{k+1}r))}{(2kr)^{(N+s)p}} \leq \frac{W(B(\zeta, r))}{r^{(N+s)p}} \sum_{k \geq 0} \left( \frac{C}{2(N+s)p} \right)^k ,
\]

which gives the desired estimate, provided \( N \) is chosen big enough.

We observe that for some special weights besides the case \( w \equiv 1 \), the expression that appears in the above proposition coincide with the weighted capacity of a ball (see [Ad]).

If \( \nu \) is a positive Borel measure on \( \mathbb{S}^n \), \( 1 < p < +\infty \), \( 0 < s < n \) and \( w \) is an \( A_p^s \)-weight, it is introduced in [Ad] the \( (s,p) \)-energy of \( \nu \) with weight \( w \), which is defined by

\[
\mathcal{E}_{sp}^w(\nu) = \int_{\mathbb{S}^n} (K_s(\nu)(\zeta))^{p'} w(\zeta)^{-\eta'(\eta'-1)} d\sigma(\zeta).
\]

If we write \( (K_s(\nu))^{p'} = (K_s(\nu))^{p-1}K_s(\nu) \), Fubini’s theorem gives that

\[
\mathcal{E}_{sp}^w(\nu) = \int_{\mathbb{S}^n} \mathcal{U}_{sp}^w(\nu)(\zeta) d\nu(\zeta),
\]

where

\[
\mathcal{U}_{sp}^w(\zeta) = K_s(w^{-1}K_s(\nu))^{p'-1}(\zeta)
\]

is the weighted nonlinear potential of the measure \( \nu \). When \( w \equiv 1 \), Wolff’s theorem (see [HeWo]) gives another representation of the energy, in terms of the so-called Wolff’s potential.

In the general case, it is introduced in [Ad] a weighted Wolff-type potential of a measure \( \nu \) as

\[
\mathcal{W}_{sp}^w(\nu)(\zeta) = \int_0^1 \left( \frac{\nu(B(\zeta, 1-r))}{(1-r)^{n-sp}} \right)^{p'-1} \int_{B(\zeta,1-r)} w^{-\eta'(\eta'-1)}(\eta) d\sigma(\eta) \frac{dr}{1-r}.
\]

In the same paper, it is shown that provided \( w \) is an \( A_p^s \)-weight, the following weighted Wolff-type theorem holds:

\[
\mathcal{E}_{sp}^w(\nu) \simeq \int_{\mathbb{S}^n} \mathcal{W}_{sp}^w(\nu)(\zeta) d\nu(\zeta).
\]

In fact, we have the pointwise estimate \( \mathcal{W}_{sp}^w(\nu)(\zeta) \leq C\mathcal{U}_{sp}^w(\nu)(\zeta) \), and Wolff’s theorem gives that the converse is true, provided we integrate with respect to \( \nu \).

In [Ad] a weighted extremal theorem for the weighted Riesz capacities it is also shown, namely, if \( G \subset \mathbb{S}^n \) is open, there exists a positive capacitary measure \( \nu_G \) such that

(i) \( \text{supp } \nu_G \subset G \).
(ii) \( \nu_G(G) = C_w^s(G) = \mathcal{E}_{sp}^w(\nu_G) \).
(iii) \( \mathcal{W}_{sp}^w(\nu_G)(\zeta) \geq C \), for \( C_w^s \)-a.e. \( \zeta \in G \).
(iv) \( \mathcal{W}_{sp}^w(\nu_G)(\zeta) \leq C \), for any \( \zeta \in \text{supp } \nu_G \).

We now introduce two holomorphic weighted Wolff-type potentials, which generalize the ones defined in [CohVe1]. These potentials will be used in the proof of the
characterization of the Carleson measures for $H^p_s(w)$, for the case $0 \leq \tau - sp < 1$. Let $1 < p < +\infty$, $0 < s < \frac{n}{p}$, and $\nu$ be a positive Borel measure on $S^n$. For any $\lambda > 0$, and $z \in B^n$, we set

$$U^{w,\lambda}_s(\nu)(z) = \int_0^1 \int_{S^n} \left( \frac{\nu(B(\zeta,1-r))}{(1-r)^{n-sp}} \right)^{p'-1} \frac{(1-r)^{-n}}{(1-r^{\zeta})^\lambda} \left( \frac{w_{-}(\nu^{p'-1})}{B(\zeta,1-r)} \right) d\sigma(\zeta) \frac{dr}{1-r},$$

(3.10)

and

$$V^{w,\lambda}_s(\nu)(z) = \int_0^1 \left( \int_{S^n} \left( \frac{1}{1-r^\zeta} \right)^{\lambda+sp-n} \left( \frac{w_{-}(\nu^{p'-1})}{B(\zeta,1-r)} \right)^{p'-1} d\sigma(\zeta) \right)^{p'} \frac{dr}{1-r}.$$ 

(3.11)

Obviously, both potentials are holomorphic functions in the unit ball. We will see, that if $p \leq 2$ the first one is bounded from below by the weighted Wolff-type potential we have just introduced, whereas if $p \geq 2$, the second one is bounded from below by the same potential.

In the unweighted case, \textbf{CohVe1} the proof of the estimates of the holomorphic potentials, rely on an extension of Wolff’s theorem. This extension gives that if $0 < q < +\infty$, $\nu$ is a positive Borel measure on $S^n$, then

$$\int_{S^n} \left( \int_0^1 \left( \frac{\nu(B(\zeta,t))}{t^{n-s}} \right)^{q} d\sigma(\zeta) \right)^{\frac{1}{q}} dt \leq \int_{S^n} W_s^w(\nu)(\zeta) d\nu(\zeta).$$

Observe that if the above estimate holds for one $q_0$, it also holds for any $q \geq q_0$. The case $q = 1$ is the integral estimate in Wolff’s theorem, since we have that

$$E_{sp}(\nu) \simeq \int_{S^n} \left( \int_0^1 \nu(B(\zeta,t)) \frac{dt}{t^{n-s}} \right)^{p'} d\sigma(\zeta).$$

The arguments in \textbf{CohVe1} can easily be used to show the following weighted version of the above theorem. We omit the details of the proof.

**Theorem 3.2.** Let $1 < p < +\infty$, $w$ an $A_p$ weight, $s > 0$, $K > 0$, $0 < q < +\infty$, and $\nu$ be a positive Borel measure on $S^n$. Then

$$\int_{S^n} \left( \int_0^K \left( \frac{\nu(B(\zeta,t))}{t^{n-s}} \right)^{\frac{1}{p'-1}} \left( \int_{B(\zeta,t)} w_{-}(\nu^{p'-1})(\eta) d\sigma(\eta) \right)^{\frac{1}{p'-1}} \frac{dt}{t} \right)^{\frac{q}{p'}} w(\zeta) d\sigma(\zeta)$$

(3.12)

$$\leq \int_{S^n} W_s^w(\nu)(\zeta) d\nu(\zeta).$$

Before we obtain estimates of the $H^p_s(w)$-norm of the weighted holomorphic potentials already introduced, we will give a characterization for weights satisfying a doubling condition

**Lemma 3.3.** Let $1 < p < +\infty$ and $w$ be an $A_p$ weight on $S^n$, and assume that $w \in D_\tau$, for some $\tau > 0$. We then have:
(i) For any \( t \in \mathbb{R} \) satisfying that \( t > \tau - n \), there exists \( C > 0 \) such that

\[
\int_{\mathbb{R}}^{+\infty} \frac{1}{x^t} \int_{\mathcal{B}(\zeta,x)} w \frac{dx}{x} \leq C \frac{1}{r^t} \int_{\mathcal{B}(\zeta,r)} w,
\]

\( r < 1, \, \zeta \in S^n \).

(ii) For any \( t \in \mathbb{R} \) satisfying that \( t > \tau - n \), there exists \( C > 0 \) such that

\[
\int_{0}^{r} x^{t} \left( \int_{\mathcal{B}(\zeta,x)} w^{-p(\nu'-1)} \right) \frac{dx}{x} \leq C \frac{1}{r^t} \left( \int_{\mathcal{B}(\zeta,r)} w^{-p(\nu'-1)} \right)^{p-1},
\]

\( r < 1, \, \zeta \in S^n \).

**Proof of Lemma 3.3**

We begin with the proof of part (i). Let \( t > \tau - n \). Then

\[
\int_{\mathbb{R}}^{+\infty} \frac{1}{x^t} \int_{\mathcal{B}(\zeta,x)} w \frac{dx}{x} = \sum_{k \geq 0} \int_{k^{2k+1}}^{2^{k+1}r} \frac{1}{x^t} \int_{\mathcal{B}(\zeta,x)} w \frac{dx}{x} \leq \sum_{k \geq 0} \frac{1}{2^{k(t+n)p-l+n}} W(B(\zeta, 2^{k+1}r)) \leq \sum_{k \geq 0} \frac{1}{2^{k(t+n)p-l+n}} 2^{k} W(B(\zeta, r)) = C \frac{1}{r^t} \int_{\mathcal{B}(\zeta,r)} w,
\]

since \( w \) is in \( D_\tau \), and \( t + n > \tau \).

Next we show that (ii) holds. If \( \zeta \in S^n \) and \( r > 0 \), the fact that \( w \in A_p \) gives that

\[
\left( \int_{\mathcal{B}(\zeta,x)} w^{-p(\nu'-1)} \right)^{p-1} \simeq \left( \int_{\mathcal{B}(\zeta,x)} w \right)^{-1},
\]

and consequently,

\[
\int_{0}^{r} x^{t} \left( \int_{\mathcal{B}(\zeta,x)} w^{-p(\nu'-1)} \right) \frac{dx}{x} = \sum_{k \geq 0} \int_{2^{k+1}r}^{2^{k+1}r} x^{t} \left( \int_{\mathcal{B}(\zeta,x)} w^{-p(\nu'-1)} \right)^{p-1} \frac{dx}{x} \leq \sum_{k \geq 0} \frac{1}{2^{k(t+n)p-l+n}} 2^{k} W(B(\zeta, r)) \simeq r^t \left( \int_{\mathcal{B}(\zeta,r)} w^{-p(\nu'-1)} \right)^{p-1}.
\]

**Remark:** In fact, it can be proved that both conditions (i) and (ii) are in turn equivalent to the fact that the \( A_p \) weight is in \( D_\tau \).

We can now obtain the estimates on the weighted holomorphic potentials defined in (3.10) and (3.11).

**Theorem 3.4.** Let \( 1 < p < +\infty, \, 0 < \alpha < n \), \( w \) an \( A_p \)-weight. Assume that \( w \) is in \( D_\tau \) for some \( 0 \leq \tau - sp < 1 \). We then have:

1. If \( 1 < p < 2 \), there exists \( 0 < \lambda < 1 \) and \( C > 0 \) such that for any finite positive Borel measure \( \nu \) on \( S^n \) the following assertions hold:
   a) For any \( \eta \in S^n \),
   \[
   \lim_{\rho \to 1} \text{Re} \mathcal{U}^{w\lambda}_{sp}(\nu)(\rho\eta) \geq C \mathcal{W}^{w\lambda}_{sp}(\nu)(\eta).
   \]
   b) \( ||\mathcal{U}^{w\lambda}_{sp}(\nu)||^p_{H^p_w(\nu)} \leq C \mathcal{E}^w_{sp}(\nu) \).

2. If \( p \geq 2 \), there exists \( 0 < \lambda < 1 \) and \( C > 0 \) such that for any finite positive Borel measure \( \nu \) on \( S^n \) the following assertions hold:
a) For any \( \eta \in \mathbb{S}^n \),
\[
\lim_{\rho \to 1} \text{Re} \, \chi_{\rho \eta}^0(0) \geq C \text{W}_{\rho \eta}^0(0).
\]

b) \( \| \chi_{\rho \eta}^0(0) \|_{E_{\rho \eta}^0(0)}^p \leq C \text{E}_{\rho \eta}^0(0) \).

**Proof of Theorem 3.4.**

We will follow the scheme of [CohVe1] where it is proved for the unweighted case. The weights introduce new technical difficulties that require a careful use of the hypothesis \( A_p \) and \( D_r \) that we assume on the weight \( w \). In order to make the proof easier to follow we sketch some of the arguments in [CohVe1], emphasizing the necessary changes we need to make in the weighted case.

Let us prove (1). We choose \( \lambda \) such that \( \tau - sp < \lambda < 1 \) and define \( U_{\rho \eta}^0 \) as in [3.10]. Then \( \tau - s < \frac{\lambda + s - \tau(2 - p)}{p - 1} \). Consequently there exists \( t \) such that \( \tau - s < t < \frac{\lambda + s - \tau(2 - p)}{2 - p} \).

Observe that \( t + s - n > \tau - n \) and \( \frac{\lambda + s - t(p - 1)}{2 - p} - n > \tau - n \).

We begin now the proof of a). The fact that \( \lambda < 1 \) gives that if \( \rho < 1, \eta \in \mathbb{S}^n \), and \( C > 0 \),
\[
\text{Re} \, U_{\rho \eta}^0(0)
\]
\[
\geq \int_0^1 \int_{B(\eta, C(1 - r))} \left( \frac{\nu(B(\zeta, 1 - r))}{(1 - r)^{n - sp}} \right)^{p - 1} \left( \frac{1 - r)^{\lambda - n}}{|1 - \rho\eta_\zeta|^\lambda} \left( \int_{B(\zeta, 1 - r)} w^{-(p - 1)} \right) \right) \frac{d\sigma(\zeta)}{1 - r}.
\]

If \( C > 0 \) has been chosen small enough, we have that for any \( \zeta \in B(\eta, C(1 - r)), B(\eta, C(1 - r)) \subset B(\zeta, 1 - r) \). In addition, \( |1 - \rho\eta_\zeta| \leq |1 - \rho| \). These estimates, together with the fact that \( w^{-(p - 1)} \) satisfies a doubling condition, give that the above integral is bounded from below by
\[
C \int_0^1 \int_{B(\eta, C(1 - r))} \left( \frac{\nu(B(\eta, C(1 - r))}{(1 - r)^{n - sp}} \right)^{p - 1} \left( \frac{1 - r)^{\lambda - n}}{|1 - \rho|^\lambda} \left( \int_{B(\eta, 1 - r)} w^{-(p - 1)} \right) \right) \frac{d\sigma(\zeta)}{1 - r}
\]
\[
\geq C \int_0^r \left( \frac{\nu(B(\eta, C(1 - r))}{(1 - r)^{n - sp}} \right)^{p - 1} \left( \frac{1 - r)^{\lambda}}{|1 - \rho|^\lambda} \left( \int_{B(\eta, 1 - r)} w^{-(p - 1)} \right) \right) \frac{dr}{1 - r}
\]
\[
\geq C \int_0^r \left( \frac{\nu(B(\eta, C(1 - r))}{(1 - r)^{n - sp}} \right)^{p - 1} \left( \int_{B(\eta, 1 - r)} w^{-(p - 1)} \right) \frac{dr}{1 - r},
\]
where in last estimate we have used that since \( r < \rho, 1 - \rho \geq 1 - r \).

We have proved then
\[
\int_0^r \left( \frac{\nu(B(\eta, C(1 - r))}{(1 - r)^{n - sp}} \right)^{p - 1} \left( \int_{B(\eta, 1 - r)} w^{-(p - 1)} \right) \frac{dr}{1 - r} \leq C \text{Re} \, U_{\rho \eta}^0(0)(\rho\eta),
\]
and letting \( \rho \to 1 \), we obtain a).

In order to obtain the norm estimate, let us simply write \( U(z) = U_{\rho \eta}^0(0)(z) \), and prove that for \( k > s \),
\[
||U||_{E_{\rho \eta}^0(0)}^p = ||U(0)||^p + \int_{\mathbb{S}^n} \left( \int_0^1 (1 - \rho)^{k - s} |(I + R)\xi U(\rho\eta)| \frac{d\rho}{1 - \rho} \right)^p w(\eta) d\sigma(\eta) \leq C \text{E}_{\rho \eta}^0(0).}
\]
But
\[
\int_0^1 (1 - \rho)^{k-s} |(I + R)^s \mathcal{U}(\rho\eta)| \frac{d\rho}{1 - \rho}
\]
\[
\leq \int_0^1 (1 - \rho)^{k-s} \int_0^1 \int_{\mathbb{S}^n} \left( \frac{\nu(B(\zeta, 1 - r))}{(1 - r)^{n-sp}} \right)^{p' - 1} \left( \int_{B(\zeta, 1 - r)} w^{-p'(1 - \rho)} \right) \frac{d\sigma(\zeta)}{1 - r - 1 - \rho} \leq \Upsilon(\eta),
\]
where
\[
\Upsilon(\eta) = \int_0^1 \int_{\mathbb{S}^n} \left( \frac{\nu(B(\zeta, 1 - r))}{(1 - r)^{n-sp}} \right)^{p' - 1} \left( \int_{B(\zeta, 1 - r)} w^{-p'(1 - \rho)} \right) \frac{d\sigma(\zeta)}{1 - r - 1 - \rho}.
\]

Observe that $|\mathcal{U}(0)|^p \leq C||\Psi||^p_{L^p(w)}$. Consequently, in order to finish the proof of the theorem, we just need to show that
\[
||\Psi||_{L^p(w)} \leq C' \mathcal{E}^{w}_{sp}(\nu).
\]

Hölder’s inequality with exponent $\frac{1}{p-1} > 1$ gives that
\[
\Upsilon(\eta) \leq \Upsilon_1(\eta)^{p-1} \Upsilon_2(\eta)^{2-p},
\]
where
\[
\Upsilon_1(\eta) = \int_0^1 \int_{\mathbb{S}^n} \frac{\nu(B(\zeta, 1 - r))}{(1 - r)^{n-sp}} \left( \int_{B(\zeta, 1 - r)} w^{-p'(1 - \rho)} \right) \frac{d\sigma(\zeta)}{1 - r - 1 - \rho},
\]
and
\[
\Upsilon_2(\eta) = \int_0^1 \int_{\mathbb{S}^n} \left( \frac{\nu(B(\zeta, 1 - r))}{(1 - r)^{n-sp}} \right)^{p'} \left( \int_{B(\zeta, 1 - r)} w^{-p'(1 - \rho)} \right)^{p-1} \frac{d\sigma(\zeta)}{1 - r - 1 - \rho}.
\]

We begin estimating the function $\Upsilon_1$. If $\zeta \in B(\tau, 1 - r)$, we have that $B(\zeta, 1 - r) \subseteq B(\tau, C(1 - r))$, and since $w^{-p'(1 - \rho)}$ satisfies a doubling condition,
\[
\Upsilon_1(\eta) \leq \int_0^1 (1 - r)^{t - 2n + s} \int_{\mathbb{S}^n} \frac{d\sigma(\zeta)}{1 - r \eta |\zeta|} \left( \int_{B(\tau, 1 - r)} w^{-p'(1 - \rho)} \right)^{p-1} \frac{d\nu(\tau)}{1 - r - 1 - \rho}.
\]

Next, we observe that if $\zeta \in B(\tau, C(1 - r))$, $|1 - r \eta | \leq |1 - r \eta |$. Hence, the above is bounded by
\[
C \int_0^1 (1 - r)^{t - n + s} \int_{\mathbb{S}^n} \left( \int_{B(\tau, 1 - r)} w^{-p'(1 - \rho)} \right)^{p-1} \frac{d\nu(\tau)}{1 - r - 1 - \rho}.
\]
Since
\[
\int_{\mathbb{S}^n} \left( \int_{B(\tau, 1 - r)} w^{-p'(1 - \rho)} \right)^{p-1} \frac{d\nu(\tau)}{1 - r \eta |\tau|} \leq \int_{\mathbb{S}^n} \left( \int_{B(\tau, 1 - r)} w^{-p'(1 - \rho)} \right)^{p-1} \frac{d\delta}{\delta^{p+1}} \frac{d\nu(\tau)}{1 - r \eta |\tau|}.
\]
the above estimate, together with Fubini’s theorem and the fact that $l - n + s > \tau - n$ give that $\Upsilon_1(\eta)$ is bounded by

$$C \int_0^1 \int_{B(\eta, \delta)} \delta^{t-n+s} \left( \int_{B(\tau, \delta)} w^{-p' - 1} \right)^{p-1} \nu(\tau) \frac{d\delta}{\delta^{s+1}} \leq \int_0^1 \left( \int_{B(\eta, \delta)} w^{-p' - 1} \right)^{p-1} \nu(B(\eta, \delta)) \frac{d\delta}{\delta^{n-s}} \frac{\nu(B(\eta, \delta)) d\delta}{\delta},$$

where we have used the fact that if $\tau \in B(\eta, \delta)$, then $B(\tau, \delta) \subset B(\eta, C\delta)$, for some $C > 0$ and that $w^{-p' - 1}$ satisfies a doubling condition.

Applying Hölder’s inequality with exponent $\frac{1}{(p-1)^2} > 1$, we deduce that

\begin{equation}
||\Upsilon||_{L^p(w)} \leq \left( \int_{\mathbb{S}^n} \left( \int_0^1 \left( \int_{B(\eta, 1-r)} w^{-p' - 1} \right)^{p-1} \nu(B(\eta, \delta)) \frac{d\delta}{\delta^{n-s}} \frac{\nu(B(\eta, \delta)) d\delta}{\delta} \right)^{p'/(p-1)} \right)^{p/(p-1)} \left( \int_{\mathbb{S}^n} \Upsilon_2 w \right)^{p/(2-p)}.
\end{equation}

Theorem 3.2 with $q = 1$ gives that the first factor on the right is bounded by $C\mathcal{E}_{sp}^q(\nu)^{p/(p-1)^2}$.

Next we deal with the integral involving $\Upsilon_2$. We recall that $l = \frac{\lambda + s - t(p-1)}{2-p} - n > \tau - n$. Fubini’s theorem gives that

$$\int_{\mathbb{S}^n} \Upsilon_2 w = \int_{\mathbb{S}^n} \int_0^1 \left( \int_{B(\zeta, 1-r)} \frac{\nu(B(\zeta, 1-r))}{(1-r)^{n-s}} \right)^{p'} (1-r)^l \left( \int_{B(\zeta, 1-r)} w^{-p' - 1} \right)^p \int_{\mathbb{S}^n} \frac{w(\eta) d\sigma(\eta)}{1 - r \eta "^n} \frac{d\sigma(\zeta) d\eta}{1 - r}.$$

But, as before, since $l > \tau - n$,

$$\int_{\mathbb{S}^n} \frac{w(\eta) d\sigma(\eta)}{1 - r \eta "^n} \leq \frac{C}{(1-r)^l} \int_{B(\zeta, 1-r)} w.$$

The above, together with Fubini’s theorem gives that

$$\int_{\mathbb{S}^n} \Upsilon_2 w \leq \int_0^1 \int_{\mathbb{S}^n} \int_{B(\eta, 1-r)} \left( \frac{\nu(B(\zeta, 1-r))}{(1-r)^{n-s}} \right)^{p'} \left( \int_{B(\zeta, 1-r)} w^{-p' - 1} \right)^p d\sigma(\zeta) \frac{w(\eta) d\sigma(\eta)}{1 - r}.$$

But if $\zeta \in B(\eta, 1-r), B(\zeta, 1-r) \subset B(\eta, C(1-r))$ for some $C > 0$, and in consequence the above is bounded by

$$C \int_{\mathbb{S}^n} \int_0^1 \left( \frac{\nu(B(\eta, C(1-r)))}{(1-r)^{n-s}} \right)^{p'} \left( \int_{B(\eta, 1-r)} w^{-p' - 1} \right)^p \frac{d\sigma(\eta)}{1 - r} w(\eta) d\sigma(\eta).$$
The change of variables $C(1 - r) = y - 1$ gives that we can estimate the previous expression by

$$C \int_{S^n} \int_0^1 \left( \frac{\nu(B(\eta, (1 - y)))}{(1 - y)^{n-s}} \right)^{p'} \left( \int_{B(\eta, 1-y)} w^{-(p'-1)} \right)^p \frac{dy}{1-y} w(\eta) d\sigma(\eta)$$

$$+ \nu(S^n)^{p'} \left( \int_{S^n} w^{-\frac{1}{p'-1}} \right)^p = I + II.$$  

Theorem 3.2 gives that $II \lesssim C \mathcal{E}^w_{sp}(\nu)$, and Theorem 3.2 with $q = p'$ gives that $I \leq C \mathcal{E}^w_{sp}(\nu)$. Consequently, $\int_{S^n} \mathcal{Y}_2 w \leq C \mathcal{E}^w_{sp}(\nu)$, and plugging this estimate in (3.18), we deduce that

$$||\mathcal{Y}||^p_{L^p(w)} \lesssim C \mathcal{E}^w_{sp}(\nu)(p-1)^2 \mathcal{E}^w_{sp}(\nu)p(2-p) \simeq \mathcal{E}^w_{sp}(\nu).$$

We now sketch the proof of part (2). We choose $\lambda > 0$ such that $\tau - sp < \lambda < 1$, and define $\mathcal{V}_{sp}^{w\lambda}(\nu)(z)$ as in (3.11). Let us simplify the notation and just write $\mathcal{V}(z) = \mathcal{V}_{sp}^{w\lambda}(\nu)(z)$. Let $\varepsilon \in \mathbb{R}$ such that $\tau < \varepsilon + n < \lambda + sp$.

The proof of a) is analogous to the one in case $1 < p < 2$.

For the proof of b), let us consider $k > s$. It will be enough to prove the following:

(3.19)

$$||\mathcal{V}||^p_{H^k_{sp}(w)} = ||\mathcal{V}(0)||^p + \int_{S^n} \left( \int_0^1 (1 - p)^{k-s} |(I + R)^k \mathcal{V}(\rho \zeta)| \frac{d\rho}{1 - p} \right)^p w(\zeta) d\sigma(\zeta) \leq C \mathcal{E}^w_{sp}(\nu).$$

Let us begin with the estimate $||\mathcal{V}(0)||^p \leq \mathcal{E}^w_{sp}(\nu)$. If $p > 2$, Hölder’s inequality with exponent $\frac{1}{p'-1} > 1$, gives that

$$||\mathcal{V}(0)|| \leq \left( \int_0^1 \int_{S^n} (1 - r)^{\frac{1}{p'-1}} \left( \int_{B(\zeta, 1-r)} w^{-(p'-1)} \right) \frac{d\nu(\zeta)}{1 - r} \right)^{p'-1} \times$$

$$\left( \int_0^1 \left( (1 - r)^{(p'-1)(\lambda + sp - n - \varepsilon)} \right)^{2-p'} \frac{dr}{1 - r} \right)^{2-p'} \nu(S^n)^{p'-1} \int_{S^n} w^{-(p'-1)}.$$

The case $p = 2$ is proved similarly. Consequently, for any $p \geq 2$,

$$||\mathcal{V}(0)||^p \leq \nu(S^n)^{p'} \left( \int_{S^n} w^{-(p'-1)} \right)^p \leq C \mathcal{E}^w_{sp}(\nu),$$

where the constant $C$ may depend on $w$.

Following with the estimate of $||\mathcal{V}||_{H^k_{sp}(w)}$, we recall (for example see [CohVe2], Proposition 1.4) that if $k > 0$, $0 < \lambda < 1$, and $z \in B^n$, 

$$||(I + R)^k \left( \int_{S^n} \frac{d\nu(\zeta)}{(1 - z\zeta)^{1-\lambda}} \right)^{p'-1} \leq C \left( \int_{S^n} \frac{d\nu(\zeta)}{|1 - z\zeta|^{1+\lambda + k}} \right)^{p'-2} \int_{S^n} \frac{d\nu(\zeta)}{|1 - z\zeta|^{1+\lambda + k}}.$$
Plugging this estimate in (3.19) and using that $p' - 2 \leq 0$, we get

$$|(I + R)^k \nu(\rho)\varepsilon|$$

$$\leq \int_0^1 \int_{1-r<\delta, 1-\rho<\delta<3} \frac{(1 - r)^{p' - 1}(\lambda + sp - n) \left( \int_{B(\eta, \delta)} \left( \int_{B(\xi, 1-r)} w^{-((p'-1))} \frac{1}{\lambda+k+1+(p'-2)\lambda} d\nu(\zeta) \right) \right)^{p'-1} d\delta dr}{1 - r}. $$

Assume first that $p > 2$. Fubini’s theorem and Hörder’s inequality with exponent $\frac{1}{p'-1} > 1$, gives that the above is bounded by

$$\int_{1-r<\delta} \left( \int_{1-r<\delta<3} (1 - r)^{\varepsilon} \int_{B(\eta, \delta)} \left( \int_{B(\xi, 1-r)} w^{-((p'-1))} \frac{1}{\lambda+k+1+(p'-2)\lambda} d\nu(\zeta) \right) \right)^{p'-1} d\delta dr \times (3.20)$$

$$\left( \int_{1-r<\delta} \left( \int_{1-r<\delta<3} \frac{(1 - r)^{\varepsilon} \int_{B(\eta, \delta)} \left( \int_{B(\xi, 1-r)} w^{-((p'-1))} \frac{1}{\lambda+k+1+(p'-2)\lambda} d\nu(\zeta) \right) \right) \right)^{2-p'} d\delta.$$  

Next, Fubini’s theorem and the fact that $\varepsilon > \tau - n$ give that

$$\int_{1-r<\delta} (1 - r)^{\varepsilon} \int_{B(\eta, \delta)} \left( \int_{B(\xi, 1-r)} w^{-((p'-1))} \frac{1}{\lambda+k+1+(p'-2)\lambda} d\nu(\zeta) \right) \right)^{p'-1} d\delta dr \times (3.20)$$

$$\leq \int_{B(\eta, \delta)} \delta^{\varepsilon} \left( \int_{B(\xi, \delta)} w^{-((p'-1))} \frac{1}{\lambda+k+1+(p'-2)\lambda} d\nu(\zeta).$$

We also have that since $\lambda + sp - n - \varepsilon > 0$, (3.20) is bounded by

$$\int_{1-r<\delta} \left( \int_{B(\eta, \delta)} \left( \int_{B(\xi, 1-r)} w^{-((p'-1))} \frac{1}{\lambda+k+1+(p'-2)\lambda} d\nu(\zeta) \right) \right)^{p'-1} \frac{d\delta}{\delta^{(n-sp)(p'-1)+k+1}.}$$

For the case $p = 2$, we obtain the same estimate, applying directly condition (3.14) on (3.20).

Integrating with respect to $\rho$, and applying Fubini’s theorem we get

$$\int_0^1 (1 - \rho)^{k-s} |(I + R)^k \nu(\rho)\varepsilon| \frac{d\rho}{1 - \rho} \leq \int_0^3 \left( \int_{B(\eta, \delta)} \left( \int_{B(\xi, \delta)} w^{-((p'-1))} \frac{1}{\lambda+k+1+(p'-2)\lambda} d\nu(\zeta) \right) \right)^{p'-1} \frac{d\delta}{\delta^{(n-sp)(p'-1)+k+1}},$$

since $(n-sp)(p'-1) + s = (n-s)(p'-1)$. If $\tau \in B(\xi, 1-r)$, and $\zeta \in B(\eta, \delta)$, we have that $\tau \in B(\eta, C\delta)$. The fact that $w^{-((p'-1))}$ satisfies a doubling condition, gives that the last integral is bounded by

$$C \int_0^3 \left( \frac{\nu(B(\eta, \delta))}{\delta^{n-s}} \right)^{p'-1} \int_{B(\eta, \delta)} w^{-((p'-1))} d\delta \frac{d\delta}{\delta}.$$
Applying Theorem 3.3 with exponent \( q = p' - 1 \), we finally obtain that
\[
\int_{S^n} \left( \int_0^1 (1 - \rho)^{k' - 1} (I + R) \mathcal{W}(\rho) \frac{d\rho}{\rho} \right)^p w(\eta) d\sigma(\eta) \leq \int_{S^n} \mathcal{W}(\nu)(\zeta) d\nu(\zeta).
\]

We can now state the characterization of the weighted Carleson measures.

**Theorem 3.5.** Let \( 1 < p < +\infty \), \( 0 < n - sp < 1 \), \( w \) an \( A_p \)-weight, and \( \mu \) a finite positive Borel measure on \( B^n \). Assume that \( w \) is in \( D_\tau \) for some \( 0 \leq \tau - sp < 1 \). We then have that the following statements are equivalent:

(i) \( \|K_\alpha(f)\|_{L^p(d\mu)} \leq C\|f\|_{L^p(w)} \).

(ii) \( \|f\|_{L^p(d\mu)} \leq C\|f\|_{H^{\tau}(w)} \).

**Proof of Theorem 3.5:**

Let us show first that \((i) \implies (ii)\). Theorem 2.13 gives that condition (ii) can be rewritten as
\[
\|C_s(g)\|_{L^p(d\mu)} \leq C\|g\|_{L^p(w)}.
\]
This fact together with the estimate \( \|C_s(f)\| \leq CK_s(\|f\|) \) finishes the proof of the implication.

Assume now that (ii) holds. Since a measure \( \mu \) on \( B^n \) satisfies (i) if and only if (see Theorem 3.6) there exists \( C > 0 \) such that for any open set \( G \subset S^n \), \( \mu(T(G)) \leq CC_{sp}^w(G) \), we will check that this estimate holds. Let \( G \subset S^n \) be an open set, and let \( \nu \) be the extremal measure for \( C_{sp}^w(G) \). We then have that \( \mathcal{W}(\nu) \geq 1 \) except on a set of \( C_{sp}^w \)-capacity zero, and \( \int_{S^n} \mathcal{W}(\nu) d\nu \leq CC_{sp}^w(G) \). Let us check that the first estimate also holds for a.e. \( x \in G \) (with respect to Lebesgue measure on \( S^n \)). Indeed, if \( A \subset S^n \) satisfies that \( C_{sp}^w(A) = 0 \), and \( \varepsilon > 0 \), let \( f \geq 0 \) be a function such that \( K_\alpha(f) \geq 1 \) on \( A \) and \( \int_{S^n} f^p w \leq \varepsilon \). Since \( L^p(w) \subset L^{p_1}(d\sigma) \), for some \( 1 < p_1 < p \), (see Lemma 2.1) we then have \( \|f\|_{L^{p_1}(d\sigma)} \leq C\|f\|_{L^p(w)} \leq C\varepsilon^{\frac{1}{p}} \). Thus \( C_{sp}(A) = 0 \), and in particular \( |A| = 0 \).

Following with the proof of the implication consider the holomorphic function on \( B^n \) defined by \( F(z) = U_{sp}^w(\nu)(z) \) if \( 1 < p < 2 \), \( F(z) = \mathcal{W}(\nu)(z) \), if \( p \geq 2 \) where \( \lambda \) is as in Theorem 3.3, Theorem 3.4 and the fact that \( \nu \) is extremal gives that
\[
\lim_{r \to 1} \text{Re } F(r\zeta) \geq C\mathcal{W}(\nu)(\zeta) \geq C,
\]
for a.e. \( x \in G \) with respect to \( C_{sp}^w \) and in consequence, for a.e. \( x \in G \) with respect to Lebesgue measure on \( G \). Hence, if \( P \) is the Poisson-Szegő kernel
\[
|F(z)| = |P[z \lim_{r \to 1} F(r\cdot)](z)| \geq |P[\text{Re } \lim_{r \to 1} F(r\cdot)](z)| \geq C,
\]
for any \( z \in T(G) \), and since we are assuming that (ii) holds, we obtain
\[
\mu(T(E)) \leq \int_{T(E)} |F(z)|^p d\mu(z) \leq C\|F\|_{H^\tau(w)}^p \leq CC_{sp}^w(\nu) \leq CC_{sp}^w(G).
\]

We finish with an example which shows that, similarly to what happens if \( w \equiv 1 \), if \( w \in D_\tau \) and \( \tau - sp > 1 \), then the equivalence between (i) and (ii) in the previous theorem need not to be true.
Proposition 3.6. Let \( n \geq 1, p = 2, \) and \( \tau \geq 0, 0 < s \) such that \( \tau > 1 + 2s \). Assume also that \( n < \tau < n + 1 \). Then there exists \( w \in A_2 \cap D_\tau \) and a positive Borel measure \( \mu \) on \( S^n \) such that \( \mu \) is a Carleson measure for \( H^2_s(w) \), but it is not Carleson for \( K_s[L^2(w)] \).

Proof of Proposition 3.6

If \( \varepsilon = \tau - n \), and \( \zeta = (\zeta', \zeta_n) \in S^n \), we consider the weight on \( S^n \) defined by \( w(\zeta) = (1 - |\zeta'|^2)^\varepsilon \). A calculation gives that \( w(z) = (1 - |z|^2)^\varepsilon \in A_2 \) if and only if \(-1 < \varepsilon < 1\), which is our case. We also have that if \( \zeta \in S^n \), \( R > 0 \) and \( j \geq 0 \), then \( W(B(\zeta, 2^j R)) \approx 2^{-j} W(B(\zeta, R)) \), i.e. \( w \in D_\tau \).

Next, any function in \( H^2_s(w) \) can be written as \( \int_{S^n} \frac{f(\zeta)}{|1 - \bar{z} \zeta|^{n-1-(s-\frac{\varepsilon}{2})}} d\sigma(\zeta) \), \( f \in L^2(w) \). It is then immediate to check that the restriction to \( B^{n-1} \) of any such function can be written as

\[
\int_{B^{n-1}} \frac{g(\zeta')(1 - |\zeta'|^2)^{-\frac{s}{2}}}{(1 - z\bar{\zeta})^{n-s}} dv(\zeta'),
\]

with \( g \in L^2(dv) \). This last space coincides (see for instance [Pe]) with the Besov space \( B^2_{s-\frac{\varepsilon}{2}}(B^{n-1}) = H^2_{s-\frac{\varepsilon}{2}}(B^{n-1}) \).

Next, \( n - 1 - (s - \frac{\varepsilon}{2}) = \tau - 2s > 1 \), and Proposition 3.1 in [CaOr2] gives that there exists a positive Borel measure \( \mu \) on \( B^n \) which is Carleson for \( H^2_{s-\frac{\varepsilon}{2}}(S^{n-1}) \), but it fails to be Carleson for the space \( K_{s-\frac{\varepsilon}{2}}[L^2(d\sigma)] \). Thus the operator

\[
f \mapsto \int_{S^{n-1}} \frac{f(\zeta)}{|1 - \bar{z} \zeta|^{n-1-(s-\frac{\varepsilon}{2})}} d\sigma(\zeta),
\]

is not bounded from \( L^2(d\sigma) \) to \( L^2(d\mu) \). Duality gives that the operator

\[
g \mapsto \int_{B^{n-1}} \frac{g(z)}{|1 - \bar{z} \zeta|^{n-1-(s-\frac{\varepsilon}{2})}} d\mu(z)
\]

is also not bounded from \( L^2(d\mu) \) to \( L^2(d\sigma) \). But if \( g \geq 0, g \in L^2(d\mu) \), Fubini’s theorem gives

\[
\|| \int_{B^{n-1}} \frac{g(z)}{|1 - w \zeta|^{n-2(s-\frac{\varepsilon}{2})}} ||^2_{L^2(d\sigma)} \|_{L^2(d\mu)} = \int_{S^{n-1}} \left( \int_{B^{n-1}} \frac{g(z)}{|1 - w \zeta|^{n-2(s-\frac{\varepsilon}{2})}} d\mu(z) \right)^2 d\sigma(\zeta)
\]

\[
= \int_{S^{n-1}} \int_{B^{n-1}} \frac{g(z)}{|1 - w \zeta|^{n-2(s-\frac{\varepsilon}{2})}} d\mu(z) \int_{B^{n-1}} \frac{g(w)}{|1 - w \zeta|^{n-2(s-\frac{\varepsilon}{2})}} d\mu(w) d\sigma(\zeta) \approx \int_{B^{n-1}} \int_{B^{n-1}} \frac{g(z)g(w)}{|1 - w \zeta|^{n-2(s-\frac{\varepsilon}{2})}} d\mu(z) d\mu(w),
\]

where the last estimate holds since \( n - 1 - 2(s - \frac{\varepsilon}{2}) = \tau - 2s > 0 \). Consequently, we have that for the measure \( \mu \), it does not hold that for any \( g \in L^2(d\mu) \)

\[
(3.21) \int_{B^{n-1}} \int_{B^{n-1}} \frac{g(z)g(w)}{|1 - w \zeta|^{n-2(s-\frac{\varepsilon}{2})}} d\mu(z) d\mu(w) \leq C ||g||_{L^2(d\mu)}.
\]
We next check that the failure of being a Carleson measure for $K_s[L^2(w)]$ can be also rewritten in the same terms. An argument similar to the previous one, gives that $\mu$ is not Carleson for $K_s[L^2(w)]$ if and only if the operator

$$f \mapsto \int_{B^n-1} \frac{f(z)}{|1 - y|^{n-s}} dv(z)$$

is not bounded from $L^2(dw)$ to $L^2(d\mu)$. Equivalently, writing $f(z) = h(z)(1 - |z|^2)^{1/2}$, this last assertion holds if and only if the operator

$$f \mapsto \int_{B^n-1} \frac{f(z)(1 - |z|^2)^{1/2}}{|1 - y|^{n-s}} dv(z)$$

is not bounded from $L^2(dw)$ to $L^2(d\mu)$. But an argument as before, using duality and Fubini’s theorem, gives that the fact that of the unboundedness of the operator can be rewritten in terms of (3.21). □

References

[Ad] D.R. Adams, Weighted nonlinear potential theory, Trans. Amer. Math. Soc. 297 (1986), 73–94.
[AdHe] D.R. Adams and L.I. Hedberg, Function Spaces and Potential Theory, Springer-Verlag Berlin–Heidelberg–New York, 1996.
[Ah] P. Ahern, Exceptional sets for holomorphic Sobolev functions, Michigan Math. J. 35, (1988), 29–41.
[AhCo] P. Ahern and W.S. Cohn, Exceptional sets for Hardy-Sobolev spaces, Indiana Math. J. 39, (1989), 417–451.
[AhBrCa] P. Ahern, J. Bruna and C. Cascante, $H^p$-theory for generalized $M$-harmonic functions in the unit ball, Indiana Math. J. 45, (1996), 103–135.
[BeLo] J. Berg and J. Lofström, Interpolation Spaces, an Introduction, Springer-Verlag Berlin, 1976.
[CaOr1] C. Cascante and J.M. Ortega, Tangential-exceptional sets for Hardy-Sobolev spaces, Illinois J. Math. 39, (1995), 68–85.
[CaOr2] C. Cascante and J.M. Ortega, Carleson measures on spaces of Hardy-Sobolev type, Canadian J. Math. 47, (1995), 1177–1200.
[CohVe1] W.S. Cohn, I.E. Verbitsky, Trace inequalities for Hardy-Sobolev functions in the unit ball of $C^n$, Indiana Univ. Math. J. 43, (1994), 1079–1097.
[CohVe2] W.S. Cohn, I.E. Verbitsky, Non-linear potential theory on the ball, with applications to exceptional and boundary interpolation sets, Michigan Math. J. 42, (1995), 79–97.
[CoIcMoSt] R.R. Coifman, Y. Meyer and E.M. Stein, Some new function spaces and their applications to harmonic analysis, Journal of Funct. Anal. 62, (1985), 304–335.
[HeWo] L.I. Hedberg and Th. H. Wolff, Thin sets in nonlinear potential theory, Ann. Inst. Fourier (Grenoble) 33, (1983), 161–187.
[KaKo] H. Kang and H. Koo, Two-weighted inequalities for the derivatives of holomorphic functions and Carleson measures on the ball, Nagoya Math. J., 158, (2000), 107–131.
[KeSa] R. Kerman and E.T. Sawyer, The trace inequality and eigenvalue estimates for Schrödinger operators, Ann. Inst. Fourier, 36, (1986), 207–228.
[Lu] D. H. Luecking, Representation and duality in weighted spaces of analytic functions, Indiana Univ. Math. 34, (1985), 319–336.
[Ma] V. G. Maz’ya, Sobolev Spaces, Berlin: Springer, 1985.
[OF] J.M. Ortega and J. Fabrega, Holomorphic Triebel-Lizorkin Spaces, J. Funct. Analysis 151, (1997), 177–212.
[Pe] M. M. Peloso, M"obius invariant spaces on the unit ball, Michigan Math. J. 39 (1992), 509–537.
[Ru] W. Rudin, Function Theory in the Unit Ball of $C^n$, New York: Springer, 1980.
[St2] E.M. Stein, *Boundary behavior of holomorphic functions of several complex variables*, Princeton University Press, 1972.

[StrTo] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Math. 1381, Springer-Verlag 1989.

Departament de Matemàtica Aplicada i Anàlisi, Facultat de Matemàtiques, Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain

E-mail address: cascante@ub.edu

Departament de Matemàtica Aplicada i Anàlisi, Facultat de Matemàtiques, Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain

E-mail address: ortega@ub.edu