Quantum Field Dynamics in a Uniform Magnetic Field:
Description using Fields in Oblique Phase Space

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Abstract

We present a simple field transformation which changes the field arguments from the ordinary position-space coordinates to the oblique phase-space coordinates that are linear in position and momentum variables. This is useful in studying quantum field dynamics in the presence of external uniform magnetic field: here, the field transformation serves to separate the dynamics within the given Landau level from that between different Landau levels. We apply this formalism to both nonrelativistic and relativistic field theories. In the large external magnetic field our formalism provides an efficient method for constructing the relevant lower-dimensional effective field theories with the field degrees defined only on the lowest Landau level.

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I. INTRODUCTION

In quantum field theory, one deals with a set of fields $\Psi(\vec{r})$ (time coordinate suppressed), satisfying appropriate commutation or anticommutation relations and obeying the operator equations of motion derived from a certain action functional $S[\Psi(\vec{r})]$. In the initial setup, the arguments of the quantum fields are usually the position-space coordinates $\vec{r} = (x^1, x^2, \cdots, x^d)$. In the system with translational invariance, however, one often finds more convenient description in momentum-space fields $\Phi(\vec{p})$, which are related to $\Psi(\vec{r})$ by the Fourier transform

$$\Psi(\vec{r}) = \int d^d\vec{r} \langle \vec{r} | \vec{p} \rangle \Phi(\vec{p}) , \quad \langle \vec{r} | \vec{p} \rangle = \frac{1}{(2\pi)^{d/2}} e^{i\vec{p} \cdot \vec{r}} . \quad (1.1)$$

This field transformation sends the position-space expression $\hat{p}^i \Psi(\vec{r}) \equiv -i \frac{\partial}{\partial x^i} \Psi(\vec{r})$ to $p^i \Phi(\vec{p})$, and $-\hat{x}^i \Psi(\vec{r}) \equiv -x^i \Psi(\vec{r})$ to $-i \frac{\partial}{\partial p^i} \Phi(\vec{p})$. Now the following question can be posed: depending on nature of the given system, can one also utilize the fields $\Phi(\vec{\xi})$ with the arguments $\vec{\xi} = (\xi^1, \xi^2, \cdots, \xi^d)$ related to $\vec{r}$ and $\vec{p}$ in a less trivial manner? The answer is yes. In this paper, we will specifically consider the case that $\xi^i (i = 1, \cdots, d)$ are oblique phase-space variables of the form

$$\xi^i = \frac{1}{\sqrt{2}} \left\{ (C)_{ij} x^j + (D)_{ij} p^j \right\} , \quad (1.2)$$

where $C, D$ are some real constant matrices. This case is relevant in studying the dynamics of quantum field systems in the presence of a uniform background magnetic field.

Regarding $\xi^i$ (given by the form (1.2)) as new ‘coordinate’ variables, let us write the related conjugate ‘momentum’ variables as

$$\eta^i = \frac{1}{\sqrt{2}} \left\{ (E)_{ij} x^j + (F)_{ij} p^j \right\} . \quad (1.3)$$

This linear canonical transformation of $x^i, p^i$ requires the four matrices $C, D, E$ and $F$ to satisfy the conditions

$$CD^T = DC^T , \quad EF^T = FE^T , \quad CF^T - DE^T = 2I . \quad (1.4)$$
These linear canonical transformations form a $Sp(2N, \mathbb{R})$ group. Then the corresponding fields $\Phi(\vec{\xi})$ may be introduced through the relation analogous to Eq. (1.1), i.e., by considering the generalized Fourier transform
\[
\Psi(\vec{r}) = \int d^d \vec{\xi} \langle \vec{r} | \vec{\xi} \rangle \Phi(\vec{\xi}) ,
\]
where the basis vector $| \vec{\xi} \rangle$, while obeying the orthogonality and completeness relations
\[
\langle \vec{\xi}' | \vec{\xi} \rangle = \delta^d(\vec{\xi}' - \vec{\xi}) ,
\]
\[
f d^d \vec{\xi} \langle \vec{\xi} | \vec{\xi} \rangle = 1 ,
\]
should further satisfy the equations
\[
\langle \vec{r} | \hat{\xi}^i | \vec{\xi} \rangle \equiv \frac{1}{\sqrt{2}} \left[(C)_{ij}x^j - i(D)_{ij} \frac{\partial}{\partial x^j}\right] \langle \vec{r} | \vec{\xi} \rangle = \xi^i \langle \vec{r} | \vec{\xi} \rangle ,
\]
\[
\langle \vec{r} | \hat{\eta}^i | \vec{\xi} \rangle \equiv \frac{1}{\sqrt{2}} \left[(E)_{ij}x^j - i(F)_{ij} \frac{\partial}{\partial x^j}\right] \langle \vec{r} | \vec{\xi} \rangle = \langle \vec{r} | \vec{\xi} \rangle \left(i \frac{\partial}{\partial \xi^i}\right) .
\]

By the field transformation (1.5) we will have the correspondences
\[
\hat{\xi}^i \Psi(\vec{r}) \equiv \frac{1}{\sqrt{2}} \left[(C)_{ij}x^j - i(D)_{ij} \frac{\partial}{\partial x^j}\right] \Psi(\vec{r}) \leftrightarrow \xi^i \Phi(\vec{\xi}) ,
\]
\[
\hat{\eta}^i \Psi(\vec{r}) \equiv \frac{1}{\sqrt{2}} \left[(E)_{ij}x^j - i(F)_{ij} \frac{\partial}{\partial x^j}\right] \Psi(\vec{r}) \leftrightarrow -i \frac{\partial}{\partial \xi^i} \Phi(\vec{\xi}) .
\]

The action of the system can then be recast into that involving the fields $\Phi(\vec{\xi})$, which satisfy suitable (anti-)commutation relations. In other words we can use the transformation (1.5) to obtain a quantum field theory having $\Phi(\vec{\xi})$ as dynamical fields. It is also not difficult to find the general expression of the kernel $\langle \vec{r} | \vec{\xi} \rangle$, as shown in Appendix A. When both $C$ and $D$ are nonsingular matrices, we may for instance take $E = -(D^T)^{-1}$ and $F = (C^T)^{-1}$. Then the kernel has the expression
\[
\langle \vec{r} | \vec{\xi} \rangle = \frac{1}{(2\pi)^{d/2}|\det(D/\sqrt{2})|^{1/2}} e^{-\frac{i}{2}\{x^i(D^{-1}C)_{ij}x^j + \xi^i((CD^T)^{-1})_{ij}\xi^j\} + \sqrt{2}i x^i(D^{-1})_{ij} \xi^j} .
\]
The term in the exponent is the generating function of the canonical linear transformation.

Note that, in quantum field theory, it is usually the form of the quadratic parts in fields $\Psi$ from the Lagrangian that makes a certain particular choice for the field arguments more
convenient than others. To see the relevance of the above discussion for a charged matter system placed in a uniform magnetic field $\vec{B} = B_0 \hat{z}$, take a nonrelativistic field system described by a charged matter field $\Psi(\vec{r}, t)$. Here, if $B_0$ is large, the dominant part of the quadratic Lagrangian will read

$$\int d^3\vec{r}^* \Psi(\vec{r}, t) \left[ i \frac{\partial}{\partial t} - \frac{1}{2m} \left( -i \partial_x + \frac{qB_0}{2} y \right)^2 - \frac{1}{2m} \left( -i \partial_y - \frac{qB_0}{2} x \right)^2 - \frac{1}{2m} \left( -i \partial_z \right)^2 \right] \Psi(\vec{r}, t),$$

(1.11)

when the symmetric gauge for the vector potential, $\vec{A}(\vec{r}) = \left(-\frac{p_0}{2} y, \frac{p_0}{2} x, 0 \right)$, is chosen. Assuming $B ≡ -qB_0 > 0$, we may then consider instead of $(x, y, p_x, p_y)$ the following variables

$$\xi_1 = \frac{1}{\sqrt{B}} (p_y + \frac{1}{2} B x), \quad \xi_2 = \frac{1}{\sqrt{B}} (p_x + \frac{1}{2} B y),$$

$$\eta_1 = \frac{1}{\sqrt{B}} (p_x - \frac{1}{2} B y), \quad \eta_2 = \frac{1}{\sqrt{B}} (p_y - \frac{1}{2} B x),$$

(1.12)

and use the related field transformation (1.3) to recast the theory, i.e., as that involving the field $\Phi(\xi_1, \xi_2, \xi_3 ≡ z, t)$. Based on the correspondences (1.9), the quadratic piece in Eq.(1.11) is transformed into

$$\int d^3\vec{\xi}^* \Phi(\vec{\xi}, t) \left[ i \frac{\partial}{\partial t} - \frac{B}{2m} \left( (-i \partial_{\xi_1})^2 + \xi_1^2 \right) - \frac{1}{2m} \left( -i \partial_{\xi_2} \right)^2 \right] \Phi(\vec{\xi}, t),$$

(1.13)

viz., we now find the differential operator which has no dependence on $\xi_2$ whatsoever and is of a simple harmonic oscillator form with respect to $\xi_1$. The oscillator states in the $\xi_1$-direction are related to distinct Landau levels[1], while the coordinate $\xi_2$ is associated with dynamics within each Landau level. Now the advantage of using the $\vec{\xi}$-space field operator $\Phi(\xi_1, \xi_2, z, t)$ should be clear — it allows one to study quantum field dynamics in terms of physical excitations pertaining to one Landau level and, separately, those related to different Landau-level excitations.

There exist extensive literature on nonrelativistic many-body theory of charged particles in a background magnetic field, especially in connection with quantum Hall physics[2]. Similar studies in relativistic field models have also been made from the very early days of quantum field theory; a more recent study in this direction includes the investigation on magnetic catalysis of dynamical symmetry breaking in certain (2+1)- or (3+1)- relativistic field
theories[3-6]. The crucial factor in these discussions is the dominance of physical modes belonging to the lowest Landau level when the background magnetic field is sufficiently strong. In this light, it is useful to consider the lowest-Landau-level projection of a given theory and in Refs.[7-10] issues related to such projection have been discussed. With our field transformation (1.5), this projection can be effected rather trivially for a quantum field theory; for the $\vec{\xi}$-space fields $\Phi(\vec{\xi})$, it corresponds to a normal dimensional reduction arising when one of the coordinates becomes essentially compact. Hence our formalism also suggests a simple, systematic way to construct effective field theory valid in a large background magnetic field. We should also mention that, within the nonrelativistic field theory framework or equivalently nonrelativistic quantum mechanics, the authors of Refs.[10,7] previously developed the approach analogous to ours, while working with holomorphic coordinates (defined in the coherent-state basis). In this paper we start from the field transformation (1.5) to obtain directly a description of the system based on field operators with oblique phase-space coordinates. This allows us to utilize the canonical quantization as much as possible, and furthermore we can apply the same idea on simplifying the dynamics of relativistic field theories. What we advocate here is that, in the presence of a strong background magnetic field, the dynamics of quantum field systems are described most naturally in terms of field operators defined in oblique phase space.

This paper is organized as follows. Sec.II contains further elaborations on nonrelativistic quantum field theory in a uniform magnetic field, based on the explicit formula for the involved field transformation. In Sec.III we present corresponding discussions on relativistic quantum field theory, for both spin-0 and spin-1/2 charged matter fields. Especially, in a strong magnetic field, dimensionally-reduced field theories are shown to be relevant at the field operator level. We consider the oblique phase-space coordinate representation of the Feynman propagator for the Dirac field in Sec.IV, and then apply this result to compute the QED one-loop effective action[11-13](with an appropriate treatment on renormalization). In Sec.V we describe the gauge interaction term between gauge and matter fields. In Sec. VI, we conclude with some remarks. Appendix A is devoted to finding the explicit form of the
II. NONRELATIVISTIC FIELD THEORY

Consider a nonrelativistic system involving a Schrödinger field $\Psi(\vec{r}, t)$ and its hermitian conjugate, with the Lagrangian given by the form

$$L = \int d^3 \vec{r} \left( \bar{\Psi}(\vec{r}, t) \left( i \frac{\partial}{\partial t} + \frac{1}{2m} \left( -i \partial_x - \frac{1}{2} B y \right)^2 + \frac{1}{2m} \left( -i \partial_y + \frac{1}{2} B x \right)^2 + \frac{1}{2m} \left( -i \partial_z \right)^2 \right) \Psi(\vec{r}, t) \right)$$

$$- \frac{1}{2} \int d^3 \vec{r} \bar{\Psi}(\vec{r}, t) V^{(1)}(\vec{r}) \Psi(\vec{r}, t)$$

$$- \frac{1}{2} \int d^3 \vec{r} d^3 \vec{r}' \bar{\Psi}(\vec{r}, t) \bar{\Psi}(\vec{r}', t) V^{(2)}(\vec{r}, \vec{r}') \Psi(\vec{r}', t) \Psi(\vec{r}, t),$$

(2.1)

where $B(>0)$ is the constant related to the background magnetic field strength, and $V^{(1)}, V^{(2)}$ represent generic one-body and two-body interactions, respectively. The fields $\Psi, \bar{\Psi}$ satisfy the equal-time (anti-)commutation relations, i.e.,

$$[\Psi(\vec{r}, t), \bar{\Psi}(\vec{r}', t)] = [\bar{\Psi}(\vec{r}, t), \Psi(\vec{r}', t)] = 0, \quad [\Psi(\vec{r}, t), \bar{\Psi}(\vec{r}', t)] = \delta^3(\vec{r} - \vec{r}').$$

(2.2)

where the upper(lower) sign refers to bosons(fermions). [Internal degrees of freedom, suppressed here for the sake of simplicity, may be included also.]

Now apply the field transformation (1.5), choosing oblique coordinates $\xi_1, \xi_2$ (and $\eta_1, \eta_2$) as given by (1.12) and $\xi_3 \equiv z$. Here, from Eqs.(1.10) and (1.12), we have the kernel $\langle \vec{r} | \vec{\xi} \rangle$ given by

$$\langle \vec{r} | \vec{\xi} \rangle = \sqrt{B} \frac{e^{i\sqrt{B}(\xi_2 y - \xi_1 x) - i B z \xi_3}}{2\pi} \delta^3(\xi_3 - z).$$

(2.3)

Then the result of using the transformation (1.5) with the expression (2.1) is the $\vec{\xi}$-space Lagrangian.
\[ L = \int d^3\vec{\xi} \left( i\Phi(\vec{\xi}, t) \frac{\partial}{\partial t} \Phi(\vec{\xi}, t) - \Phi(\vec{\xi}, t) \left[ \frac{p_1}{2m} \left\{ (-i\partial_{\xi_1})^2 + \xi_1^2 \right\} + \frac{1}{2m} \left\{ (-i\partial_{\xi_3})^2 \right\} \right] \right) \]

\[ - \int d^3\vec{\xi} d^3\vec{\psi} \Phi(\vec{\xi}, t) \nabla(\vec{\xi}, \vec{\psi}) \Phi(\vec{\psi}, t) \]

\[ - \frac{1}{2} \int d^3\vec{\xi} d^3\vec{\xi}' d^3\vec{\zeta} d^3\vec{\zeta}' \Phi(\vec{\xi}, t) \Phi(\vec{\zeta}, t) \nabla(\vec{\xi}, \vec{\xi}') \Phi(\vec{\zeta}', t) \Phi(\vec{\zeta}'', t), \]

(2.4)

where

\[ \nabla(\vec{\xi}, \vec{\psi}) = \int d^3\vec{r} \left\langle \vec{\xi} | \vec{r} \right| V^{(1)}(\vec{r}) \left| \vec{r} | \vec{\psi} \right\rangle, \]

(2.5)

\[ \nabla(\vec{\xi}, \vec{\xi}', \vec{\zeta}, \vec{\zeta}'') = \int d^3\vec{r} d^3\vec{r}' \left\langle \vec{\xi} | \vec{r} \right| \left\langle \vec{\xi}' | \vec{r}' \right| V^{(2)}(\vec{r}, \vec{r}') \left| \vec{r}' \right| \left| \vec{\zeta} \right| \left| \vec{\zeta}'' \right\rangle. \]

(2.6)

The new fields \( \Phi, \Phi^\dagger \) also satisfy the (anti-)commutation relations of the standard form, viz.,

\[ [\Phi(\vec{\xi}, t), \Phi(\vec{\xi}', t)]_\mp = [\Phi^\dagger(\vec{\xi}, t), \Phi^\dagger(\vec{\xi}', t)]_\mp = 0, \ [\Phi(\vec{\xi}, t), \Phi^\dagger(\vec{\xi}', t)]_\mp = \delta^3(\vec{\xi} - \vec{\xi}'). \]

(2.7)

Notice that the \( B \)-dependent part of the Lagrangian becomes much simpler in the form (2.4).

Using Eq.(2.3), the \( \vec{\xi} \)-space one-body potential \( \tilde{V}^{(1)}(\vec{\xi}, \vec{\xi}') \) may be expressed as

\[ \tilde{V}^{(1)}(\vec{\xi}, \vec{\xi}') = \delta(\xi_3 - \xi_3') e^{i\xi_1 \xi_2 - \xi_1' \xi_2'} \int dx dy \ e^{-i\sqrt{B}(\xi_2 - \xi_2')} e^{i\xi_1 \xi_2 - (\xi_3 - \xi_3') y} V^{(1)}(x, y, \xi_3). \]

(2.8)

Evidently, no irregularity is observed in considering quantum fields defined in oblique phase space. For a physical system defined in two spatial dimensions, it suffices to suppress all references to the coordinate \( z \)(i.e., \( \xi_3 \)) in our expressions. We also remark that, instead of working with the oblique phase-space coordinates \( \xi_1 \) and \( \xi_2 \), one may alternatively utilize holomorphic coordinates defined with the help of suitable coherent-state basis[10,7]; but, in dealing with quantum fields at least, our oblique phase-space description appears to be more direct and has advantage in that an appropriate physical interpretation relative to the Landau-level structure may be given to each (real) coordinate appearing.

In view of the appearance of the oscillator-type differential operator \( \frac{B}{2m} \left\{ \left\{ (-i\partial_{\xi_1})^2 + \xi_1^2 \right\} \right\} \) in Eq.(2.4), one may further contemplate replacing the (3+1)-dimensional field \( \Phi(\xi_1, \xi_2, \xi_3, t) \) by an \( \infty \)-component (2+1)-dimensional fields \( \Phi_n(\xi_2, \xi_3, t) \), where the discrete index \( n=0, 1, 2... \) is associated with a specific oscillator state for the coordinate \( \xi_1 \). This can be achieved by writing
\[ \Phi(\xi', t) = \sum_{n=0}^{\infty} \chi_n(\xi_1) \Phi_n(\xi_2, \xi_3, t), \] (2.9)\\

where \( \chi_n(\xi_1) \) is the n-th oscillator eigenfunction, viz,

\[ \chi_n(\xi_1) \equiv \langle \xi_1|n \rangle = \frac{(-i)^n}{\pi^{1/4}} \frac{1}{\sqrt{2^n n!}} H_n(\xi_1) e^{-\frac{1}{4}\xi_1^2}, \] (2.10)\\

\( H_n(\xi_1) \) denoting the Hermite polynomial. In this way, we attach the field \( \Phi_n(\xi_2, \xi_3 \equiv z, t) \) to the n-th Landau level. The original field \( \Psi(\vec{r}, t) \) is related to the tower of fields \( \{ \Phi_n(\xi_2, z, t) \} \) by the transformation

\[ \Psi(\vec{r}, t) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} d\xi_2 \mathcal{W}^{(n)}(x, y; \xi_2) \Phi_n(\xi_2, z, t) \] (2.11)\\

with the kernel

\[ \mathcal{W}^{(n)}(x, y; \xi_2) \equiv \int_{-\infty}^{\infty} d\xi_1 \langle x, y|\xi_1, \xi_2\rangle \langle \xi_1|n \rangle \\
= \sqrt{\frac{B}{2\pi}} e^{-i\sqrt{B}(\frac{1}{2}\sqrt{B}y - \xi_2))x} \int_{-\infty}^{\infty} d\xi_1 e^{i\sqrt{B}y - \xi_2)\xi_1} \chi_n(\xi_1), \] (2.12)\\

where we used Eq. (2.3). Since the Fourier transform of an oscillator eigenfunction reproduces the same function in the dual space, the expression in Eq. (2.12) can be simplified:

\[ \mathcal{W}^{(n)}(x, y; \xi_2) = \sqrt{\frac{B}{2\pi}} e^{-i\sqrt{B}(\frac{1}{2}\sqrt{B}y - \xi_2))x} i^n \chi_n(\sqrt{B}y - \xi_2) \\
= \sqrt{\frac{B}{2\pi}} \frac{1}{\sqrt{2^{n}n!}} e^{-\frac{B}{2}xy} e^{i\sqrt{B}x\xi_2} H_n(\sqrt{B}y - \xi_2) e^{-\frac{1}{2}(\sqrt{B}y - \xi_2)^2}. \] (2.13)\\

By the transformation (2.9) we obtain an equivalent (2+1)-dimensional quantum field theory description with the Lagrangian

\[ L = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dz \Phi_n^\dagger(\xi_2, z, t) \left[ \frac{i}{\hbar} \frac{\partial}{\partial t} - \mathcal{E}_n - \frac{1}{2m} (-i\partial_x)^2 \right] \Phi_n(\xi_2, z, t) \]
\[ - \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \int dz \int d\xi_2 d\xi'_2 \Phi_n^\dagger(\xi_2, z, t) V^{(0)}_{nl}(\xi_2, \xi'_2, z) \Phi_n(\xi'_2, z, t) \]
\[ - \frac{1}{2} \sum_{n,l,r,s} \int dzdz' \int d\xi_2 d\xi_2' d\xi_2'' \Phi_n^\dagger(\xi_2, z, t) \Phi_n(\xi_2', z', t) V^{(0)}_{nlrs}(\xi_2, \xi_2', \xi_2'', z, z', z'') \]
\times \Phi_n(\xi_2'', z', t) \Phi_n(\xi_2', z, t), \] (2.14)\\

where \( \mathcal{E}_n = \frac{B}{m}(n + \frac{1}{2}) \), and
\[ V_{nl}^{(1)}(\xi_2, z) = \int dx dy \mathcal{W}^{(b)\ast}(x, y; \xi_2) \mathcal{W}^{(l)}(x, y; \xi_2), \]
\[ V_{nlrs}^{(2)}(\xi_2, \xi_2', \xi_2', z, z') = \int dx dy \int dx' dy' \mathcal{W}^{(b)\ast}(x, y; \xi_2) \mathcal{W}^{(l)\ast}(x', y'; \xi_2) \]
\[ \times V^{(2)}(\vec{r}, \vec{r}') \mathcal{W}^{(r)}(x', y'; \xi_2') \mathcal{W}^{(s)}(x, y; \xi_2''). \]

According to this Lagrangian, the terms not involving \( V^{(1)} \) or \( V^{(2)} \), i.e., the very terms commonly used to define the unperturbed propagator, become ut mostly trivial — essentially, that of free propagation in the z-direction only. The (2+1)-dimensional fields \( \Phi_n, \Phi_l^\dagger \) satisfy the equal-time (anti-)commutation relations
\[ [\Phi_n(\xi_2, z, t), \Phi_l(\xi_2', z', t)] = [\Phi_l^\dagger(\xi_2, z, t), \Phi_n^\dagger(\xi_2', z', t)] = 0, \]
\[ [\Phi_n(\xi_2, z, t), \Phi_l^\dagger(\xi_2', z', t)] = \delta_{nl} \delta(\xi_2 - \xi_2') \delta(z - z'). \]

We remark that this is a bonafide (2+1)-dimensional field theory; especially, the variable \( \xi_2 \equiv \frac{1}{\sqrt{B}}(p_x + \frac{1}{2}B y) \) in the quantum fields \( \Phi_n(\xi_2, z, t) \) is just like another spatial coordinate.

Perturbative or nonperturbative studies on the system can be made on the basis of the Lagrangian (2.14). Especially, the functional integration technique with the fields \( \{ \Phi_n(\xi_2, z, t) \} \) may be used. If the magnitude of \( \frac{B}{m} \), the Landau gap, is large relative to the characteristic energy scale in the problem, it can be interesting to study dynamics of the system restricted to the lowest Landau-level states. In such case, all interactions involving the fields other than the lowest Landau-level field \( \Phi_0(\xi_2, z, t) \) may well be ignored. Actually, more systematic procedure will be to use the standard effective field theory approach[14,15] here — integrate out all ‘heavy’ fields, i.e., the fields \( \Phi_n(\xi_2, z, t) \) with \( n \neq 0 \), from our theory. Note that, in the latter approach, the Landau-level mixing effects are included also. Explicit physical application of this approach, to the quantum Hall physics in particular, will be pursued elsewhere.

A remark is in order. The Lagrangian form in Eq.(2.4) or Eq.(2.14) is in no way specific to our particular gauge choice for the potential describing the background magnetic field. In Appendix B, it is shown that the same Lagrangians are obtained even if the Landau gauge is adopted. It is just the kernel \( \langle \vec{r} | \vec{\xi} \rangle \) in the field transformation (1.5) that gets altered with
III. RELATIVISTIC FIELD THEORY

For charged relativistic field systems in the presence of a background magnetic field, using quantum fields defined in the oblique phase space can again result in great simplification in the analysis. In the case of a spin-0 field, the situation is not much different from the nonrelativistic Schrödinger field case. Let the action, governing the dynamics of a complex spin-0 field \( \Psi(\vec{r}, t) \) and its hermitian conjugate \( \Psi^\dagger(\vec{r}, t) \), be of the form

\[
S = \int dtd^3\vec{r} \left\{ |\partial_t \Psi|^2 - \left( |\partial_x - \frac{i}{2} B y| \Psi \right|^2 - \left( |\partial_y + \frac{i}{2} B x| \Psi \right|^2 - |\partial_z |\Psi|^2 - m^2 |\Psi|^2 \right\} + S_1(\Psi, \Psi^\dagger, \cdots)
\]

\[
\equiv S_0(\Psi, \Psi^\dagger) + S_1(\Psi, \Psi^\dagger, \cdots),
\]

where \( S_1(\Psi, \Psi^\dagger, \cdots) \) may contain self-interactions for the fields \( \Psi, \Psi^\dagger \) and possibly couplings with other dynamical fields. To simplify this system, especially the piece \( S_0(\Psi, \Psi^\dagger) \), we may immediately go over to the description using the \( \vec{\xi} \)-space fields \( \Phi(\vec{\xi}, t), \Phi^\dagger(\vec{\xi}, t) \) through the field transformation (1.5) with the kernel \( \langle \vec{r}|\vec{\xi} \rangle \) given by Eq.(2.3). Then we can replace \( S_0 \) by (cf. Eq. (1.13))

\[
S_0 = \int dtd^3\vec{\xi} \left\{ |\partial_t \Phi|^2 - B \left( |\partial_{\xi_1} \Phi|^2 + \xi_2^2 |\Phi|^2 \right) - |\partial_{\xi_3} |\Phi|^2 - m^2 |\Phi|^2 \right\},
\]

and \( S_1 \) by the corresponding expression. [If desirable, one can also consider at this stage suitable field transformations for other dynamical fields together]. The equal-time commutation relations satisfied by the fields \( \Phi(\vec{\xi}, t), \Phi^\dagger(\vec{\xi}, t) \) and their canonical momenta \( \partial_t \Phi^\dagger(\vec{\xi}, t), \partial_t \Phi(\vec{\xi}, t) \) are of the identical form, with the coordinates \( \vec{\xi} \) taking the place of \( \vec{r} \), as those satisfied by the original configuration-space fields.

The above \( \vec{\xi} \)-space description can be the basis of all field-theoretical investigations on the system, and it will have advantage, for instance, by having the simpler Feynman propagator
expression (as determined by the form (3.2)). [See the remark following after Eq.(4.8)]. When the value of $B$ is large, one can also utilize the equivalent $(2+1)$-dimensional field-theory description, based on the set of complex fields $\Phi_n(\xi_2, \xi_3, t)$, $n = 0, 1, 2, \cdots$, where $n$ refers to the $n$-th Landau level. It is achieved by the field transformation (2.11). In the latter description, the action $S_0$ will be replaced by

$$S_0 = \sum_{n=0}^{\infty} \int dt d\xi_2 d\xi_3 \left\{ |\partial_t \Phi_n(\xi_2, \xi_3, t)|^2 - |\partial_{\xi_3} \Phi_n(\xi_2, \xi_3, t)|^2 - M_n^2 |\Phi_n(\xi_2, \xi_3, t)|^2 \right\}, \quad (3.3)$$

where

$$M_n = \sqrt{m^2 + B(2n + 1)}. \quad (3.4)$$

The fields $\Phi_n(\xi_2, \xi_3, t)$ will satisfy the standard equal-time commutation relations of any $(2+1)$-dimensional relativistic spin-0 field theory. From the form (3.3) we notice that the Feynman propagator associated with the $n$-th Landau-level field $\Phi_n$ has a particularly simple structure—aside from the trivial $\delta(\xi_2 - \xi'_2)$ factor, it is just the $(1+1)$-dimensional free scalar propagator for a particle of mass $M_n$. All effective mass values $M_n$ increase indefinitely with $B$; this suggests that if the background magnetic field becomes sufficiently large, the given system (even for relatively small $m$) may be described approximately using the appropriate nonrelativistic field-theory description.

The Dirac field dynamics in a given background magnetic field can be more interesting. If $\Psi(\vec{r}, t)$ now denotes the Dirac field, the part of the action we denoted above as $S_0$ will have the form

$$S_0 = \int dt d\vec{r} \Psi(\vec{r}, t) \left\{ \partial_t + \alpha_1 (\partial_x - \frac{i}{2} B_y) + \alpha_2 (\partial_y + \frac{i}{2} B_x) + \alpha_3 \partial_z + i \beta m \right\} \Psi(\vec{r}, t), \quad (3.5)$$

where, for the four $4 \times 4$ matrices $\alpha$ and $\beta$, we may assume the Dirac representation:

$$\alpha_i \equiv \gamma^0 \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta \equiv \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \quad (3.6)$$

The fields $\Psi(\vec{r}, t)$ and $\Psi^\dagger(\vec{r}, t)$ satisfy the equal-time anticommutation relations. For the interaction term $S_1(\Psi, \Psi^\dagger, \cdots)$, the most interesting case will be the one describing the
matter coupling with dynamical gauge fields [3-6] (or with Chern-Simons-type gauge fields in the (2+1)-dimensional case). But our principal concern here is the quadratic action (3.3).

As the field transformation (1.5) is applied to the Dirac fields, the following $\vec{\xi}$-space action is obtained:

$$ S_0 = \int dt d^3 \vec{\xi} \Phi(\vec{\xi}, t) \left\{ i \partial_t + i \alpha_1 \sqrt{B} \frac{\partial}{\partial \xi_1} - \alpha_2 \sqrt{B} \xi_1 + i \alpha_3 \partial \xi_3 - \beta m \right\} \Phi(\vec{\xi}, t). \tag{3.7} $$

These $\vec{\xi}$-space fields satisfy the standard equal-time anticommutation relations also. For discussions somewhat similar to our treatment below (but without utilizing oblique phase-space coordinates), see Ref. [6].

Now, from the differential operator appearing in Eq. (3.7), we observe that the piece involving $\xi_1$ can be written as

$$ i \alpha_1 \sqrt{B} \frac{\partial}{\partial \xi_1} - \alpha_2 \sqrt{B} \xi_1 = \sqrt{B} \alpha_2 \left( \xi_1 - \Sigma_3 \frac{\partial}{\partial \xi_1} \right) $$

where we have defined $a = \frac{1}{\sqrt{2}} \left( \xi_1 + \frac{\partial}{\partial \xi_1} \right)$, $\bar{a} = \frac{1}{\sqrt{2}} \left( \xi_1 - \frac{\partial}{\partial \xi_1} \right)$, and $\frac{1}{2} \Sigma_3$ is the third component of the spin angular momentum

$$ \Sigma_3 = -i \alpha_1 \alpha_2 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{3.9} $$

With this understanding, we may then introduce the following transformation on the fields $\Phi(\vec{\xi}, t)$, $\Phi^\dagger(\vec{\xi}, t)$:

$$ \left( \Phi(\vec{\xi}, t) \right)_\alpha = \sum_{n=0}^{\infty} \left( M_n(\xi_1) \right)_{\alpha \beta} \left( \Phi_n(\xi_2, \xi_3, t) \right)_\beta, \tag{3.10} $$

$$ M_n(\xi_1) = \begin{pmatrix} \chi_{n-1}(\xi_1) & 0 & 0 & 0 \\ 0 & \chi_n(\xi_1) & 0 & 0 \\ 0 & 0 & \chi_{n-1}(\xi_1) & 0 \\ 0 & 0 & 0 & \chi_n(\xi_1) \end{pmatrix}, \tag{3.11} $$
where \( \chi_n(\xi_1) \) \( (n = 0, 1, 2, \cdots) \) denote the harmonic oscillator wave functions in Eq.(2.14), and \( \chi_{-1}(\xi_1) \equiv 0 \). The index \( n \) again refers to a specific Landau level, and our (2+1)-dimensional spinor fields \( \tilde{\Phi}_n(\xi_2, \xi_3, t) \) for \( n \neq 0 \) have four independent components while the lowest Landau-level field, \( \tilde{\Phi}_0(\xi_2, \xi_3, t) \), has two independent components (with \( \Sigma_3' = -1 \)) only, i.e.,

\[
\left( \tilde{\Phi}_0(\xi_2, \xi_3, t) \right)_\alpha = \begin{pmatrix}
0 \\
(\tilde{\Phi}_0(\xi_2, \xi_3, t))_2 \\
0 \\
(\tilde{\Phi}_0(\xi_2, \xi_3, t))_4
\end{pmatrix}.
\]

Using these (2+1)-dimensional spinor fields, the action (3.7) can be recast as

\[
S_0 = \sum_{n=0}^{\infty} \int dt d\xi_2 d\xi_3 \tilde{\Phi}_n^\dagger(\xi_2, \xi_3, t) \left\{ i\partial_t + i\alpha_3 \partial_{\xi_3} - \left( m\beta + \sqrt{2B} n \alpha_2 \right) \right\} \tilde{\Phi}_n(\xi_2, \xi_3, t).
\]

Observe that, using the two-component field \( \tilde{\Phi}_0 = \begin{pmatrix} (\tilde{\Phi}_0)_2 \\ (\tilde{\Phi}_0)_4 \end{pmatrix} \), the differential operator appearing in the \( n = 0 \) term of this expression may be written \( \left\{ i\partial_t - i\sigma_1 \partial_{\xi_3} - m\sigma_3 \right\} \).

As we have seen above, the given Dirac field system can be reformulated as a (2+1)-dimensional field theory in oblique phase space. The action (3.13) can be changed to have more standard form with a diagonal mass matrix, by redefining our fields according to

\[
\left( \tilde{\Phi}_n(\xi_2, \xi_3, t) \right)_\alpha = (U_n)_{\alpha\beta} (\Phi_n(\xi_2, \xi_3, t))_\beta,
\]

where

\[
U_n = \begin{pmatrix}
\cos \theta_n & 0 & 0 & i\sin \theta_n \\
0 & \cos \theta_n & -i\sin \theta_n & 0 \\
0 & -i\sin \theta_n & \cos \theta_n & 0 \\
i\sin \theta_n & 0 & 0 & \cos \theta_n
\end{pmatrix}, \quad (\tan \theta_n = \sqrt{\frac{2Bn}{m}}).
\]

The matrix \( U_n \) is unitary and has the properties

\[
U_n^{-1}(m\beta + \sqrt{2B} n \alpha_2)U_n = \sqrt{m^2 + 2Bn} \beta,
\]

\[
U_n^{-1}\alpha_3 U_n = \alpha_3.
\]
Thus, in terms of the spinor fields $\Phi_n(\xi_2, \xi_3, t)$, the action (3.13) assumes the form

$$S_0 = \sum_{n=0}^{\infty} \int dtd\xi_2d\xi_3 \Phi_n^\dagger(\xi_2, \xi_3, t) \left\{i\partial_t + i\alpha_3\partial_{\xi_3} - M_n\beta\right\} \Phi_n(\xi_2, \xi_3, t), \quad (3.17)$$

where $M_n = \sqrt{m^2 + 2Bn}$ denotes the mass parameter associated with the (2+1)-dimensional spinor field $\Phi_n(\xi_2, \xi_3, t)$. The fields $\Phi_n(\xi_2, \xi_3, t), \Phi_n^\dagger(\xi_2, \xi_3, t)$ also satisfy the standard equal-time anticommutation relations. From Eqs.(3.10) and (3.14) the transformation relating the original Dirac field $\Psi(\vec{r}, t)$ to the set of (2+1)-dimensional fields $\{\Phi_n(\xi_2, \xi_3, t)\}$, i.e., the formula involving the functions $W^{(n)}(x, y; \xi_2)$ as in Eq.(2.11), can be found as well.

When only the quadratic action in Eq.(3.17) is taken into account, we notice that dynamics of the field $\Phi_n(\xi_2, \xi_3, t)$ is simply that of a free (1+1)-dimensional spinor field theory, with the coordinate $\xi_2$, which is supposed to describe dynamics within a given Landau level, playing a rather trivial role. This situation is similar to the case of the spin-0 field systems discussed earlier. But there is one, physically significant, difference. In this Dirac field system, the effective mass parameter for the n-th Landau-level field $\Phi_n$ equals $\sqrt{m^2 + 2Bn}$ and hence the associated mass for the lowest Landau level field, i.e., the two-component field $\Phi_0(\xi_2, \xi_3, t)(=\tilde{\Phi}_0(\xi_2, \xi_3, t))$, is equal to $m$ regardless of the magnitude of $B$. Now consider a field theory system with the action given by the quadratic action (3.17) plus some nontrivial interaction $S_1(\Psi, \Psi^\dagger, \cdots)$ (which may also be expressed using the fields $\Phi_n(\xi_2, \xi_3, t), n=0, 1, 2, \cdots$). In a sufficiently large background magnetic field, one may then integrate out all “heavy” fields $\Phi_n(\xi_2, \xi_3, t), n = 1, 2, \cdots$ from the system and go on to discuss various physical problems in terms of the resulting effective field theory which involves just the lowest Landau-level field $\Phi_0(\xi_2, \xi_3, t)$. This scheme should have a useful application in studying, for instance, the magnetic catalysis of chiral symmetry breaking[3-6] in a system with a vanishingly small mass $m$. 

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IV. FEYNMAN PROPAGATOR AND THE ONE-LOOP EFFECTIVE ACTION

For the Dirac field system discussed in the previous section, we shall here find the Feynman propagator associated with the $\tilde{\xi}$-space fields, i.e.,

$$S_{\alpha\beta}(\tilde{\xi}, t; \tilde{\xi}', t') = -i\langle 0| T(\Phi_\alpha(\tilde{\xi}, t)\bar{\Phi}_\beta(\tilde{\xi}', t'))|0\rangle, \quad (\bar{\Phi}(\tilde{\xi}, t) \equiv \Phi^\dagger(\tilde{\xi}, t)\gamma^0)$$  \hspace{1cm} (4.1)

and also calculate in this $\tilde{\xi}$-space framework the one-loop effective action explicitly. For these considerations, it is the $\tilde{\xi}$-space spinor field action (3.7) that is relevant. We do this analysis not only to exhibit the simplicity of the $\tilde{\xi}$-space description but also to show that all standard field theoretic manipulations, including renormalization, may be carried out in this description.

The propagator defined above satisfies the differential equation

$$(i\gamma^0\partial_t + i\sqrt{B}\gamma^1\partial_{\xi_1} + i\gamma^3\partial_{\xi_3} - \gamma^2\sqrt{B}\xi_1 - m)_{\alpha\beta} S_{\gamma\beta}(\tilde{\xi}, t; \tilde{\xi}', t') = \delta_{\alpha\beta} \delta^3(\tilde{\xi} - \tilde{\xi}')\delta(t - t').$$  \hspace{1cm} (4.2)

Introducing the shorthand notations

$$i\gamma^0\partial_t + i\sqrt{B}\gamma^1\partial_{\xi_1} + i\gamma^3\partial_{\xi_3} \equiv i\tilde{\partial},$$  \hspace{1cm} (4.3)

and employing the matrix notation in the Dirac spin space, we may then write

$$S(\xi; \xi') = \langle \xi | \frac{1}{i\tilde{\partial} - \gamma^2\sqrt{B}\xi_1 - m + i\epsilon} | \xi' \rangle$$

$$= (i\tilde{\partial} - \gamma^2\sqrt{B}\xi_1 + m)\langle \xi | \frac{1}{-\partial^2 - B\xi_1^2 - B\Sigma_3 - m^2 + i\epsilon} | \xi' \rangle$$

$$\equiv (i\tilde{\partial} - \gamma^2\sqrt{B}\xi_1 + m)G(\xi; \xi'),$$  \hspace{1cm} (4.4)

where we have used the relation

$$(i\tilde{\partial} - \gamma^2\sqrt{B}\xi_1 - m)(i\tilde{\partial} - \gamma^2\sqrt{B}\xi_1 + m) = -\partial^2 - B\xi_1^2 - B\Sigma_3 - m^2.$$  \hspace{1cm} (4.5)

For the quadratic propagator $G(\xi; \xi')$ it is advantageous to use the Schwinger proper-time representation[12], that is,
\[ G(\xi,\xi') = \int_0^\infty ds \, e^{-is\langle \xi s|\xi' \rangle} , \quad (4.6) \]
\[ \langle \xi s|\xi' \rangle = \langle \xi |e^{-is(\slashed{D}^2 + B \Sigma^2_3 + m^2)}|\xi' \rangle , \quad (4.7) \]
and \( \langle \xi s|\xi' \rangle \) can be found from the well-known expression for the free or harmonic oscillator propagator. In this way, we immediately obtain
\[ G(\xi,\xi') = \delta(\xi - \xi') \int_0^\infty ds \, e^{-is(m^2 + B \Sigma^2_3 - i\epsilon)} \sqrt{\frac{4\pi i}{4\pi is}} e^{-i\frac{(\xi'\xi)^2}{4s} - i\frac{(\xi'\xi')^2}{4s}} \sqrt{\frac{1}{2\pi i \sin(2Bs)}} e^{\frac{i(\xi' + \xi'^2)\cos(2Bs) - 2\xi_1^i \xi'^2}{\sin(2Bs)}} . \quad (4.8) \]
Substitution of the form \((4.8)\) into Eq.\((4.4)\) gives the desired expression for \( S(\xi;\xi') \). Incidentally, from the integral representation \((4.8)\), suppressing the factor \( e^{-isB \Sigma^2_3} \) inside the integrand yields the related Feynman propagator for the spin-0 field.

The above information may be used to calculate the one-loop spinor effective action in a uniform magnetic field. In position space, the (unnormalized) one-loop effective action can be represented by the proper-time integral\([12, 13]\)
\[ \Gamma^{(1)}(B) = \frac{i}{2} \int_0^\infty ds \, \frac{ds}{s} e^{-is} \int d^4x \, \lim_{x' \to x} \text{tr} \langle xs|x' \rangle , \quad (4.9) \]
where \( \langle xs|x' \rangle \) is the proper-time Green’s function associated with the ‘quadratic’ Dirac operator. But the function \( \langle xs|x' \rangle \) is related to \( \langle \xi s|\xi' \rangle \) (see Eq.\((4.7)\)) by
\[ \langle xs|x' \rangle = \int d^4\xi d^4\xi' \langle x|\xi \rangle \langle \xi s|\xi' \rangle \langle \xi'|x' \rangle \quad (4.10) \]
(with \( d^4\xi \equiv dt d^3\vec{\xi} \) and \( \langle x|\xi \rangle = \delta(x^0 - \xi^0)\langle \vec{r}|\vec{\xi} \rangle \)), and hence, formally,
\[ \int d^4x \, \lim_{x' \to x} \text{tr} \langle xs|x' \rangle = \int d^4\xi \, \lim_{\xi' \to \xi} \text{tr} \langle \xi s|\xi' \rangle . \quad (4.11) \]
Using the explicit expression for \( \langle \xi s|\xi' \rangle \) (as can be read off from Eqs.\((4.7)\) and \((4.8)\)), we then obtain the expression
\[ \Gamma^{(1)}(B) = \frac{i}{2} \int_0^\infty ds \, \frac{ds}{s} e^{-is(m^2 + i\epsilon)} \int d^4\xi \lim_{\xi' \to \xi} \left\{ \delta(\xi - \xi') \frac{4 \cos(Bs)}{4\pi} \times \sqrt{\frac{1}{2\pi i \sin(2Bs)}} e^{\frac{i(\xi' + \xi'^2)\cos(2Bs) - 2\xi_1^i \xi'^2}{\sin(2Bs)}} \right\} \]
\[ = \frac{i}{2\pi} \int_0^\infty ds \, e^{-is(m^2 + i\epsilon)} \sqrt{\frac{\cot(Bs)}{4\pi i}} \int_{-\infty}^{\infty} d\xi_1 e^{-i\xi_1^2 \tan(Bs)} \int dtd\xi_2 d\xi_3 \left( \lim_{\xi' \to \xi} \delta(\xi - \xi') \right) , \quad (4.12) \]
where we have used the result $\text{tr} e^{-isB\Sigma_3} = 4\cos(Bs)$.

The expression (4.12) contains certain factors for which we must provide suitable interpretation. Here it is useful to imagine that our system is defined in a large, but finite, space-time volume $L^3T$. Then, as regards the somewhat unconventional-looking factor appearing at the end of the expression in Eq.(4.12), we may give the following interpretation:

$$\int dtd\xi_2 d\xi_3 \lim_{\xi_2' \rightarrow \xi_2} \delta(\xi_2' - \xi_2) = \frac{B}{2\pi} TL^3.$$  \hspace{1cm} (4.13)

This follows once, if one has that

$$\int dtd\xi_3 \sim TL,$$  \hspace{1cm} (4.14)

$$\lim_{\xi_1' \rightarrow \xi_2} \delta(\xi_2 - \xi_2') \sim \frac{\sqrt{B}}{2\pi} L,$$  \hspace{1cm} (4.15)

$$\int d\xi_2 \sim \sqrt{B}L.$$  \hspace{1cm} (4.16)

The relation (4.14) is a usual one; we will argue for Eqs.(4.15) and (4.16) below.

First of all, we can infer from Eqs.(1.6) and (2.3) that

$$\lim_{(\xi_1',\xi_2') \rightarrow (\xi_1,\xi_2)} \delta(\xi_1' - \xi_1)\delta(\xi_2 - \xi_2') \sim \int dxdy \frac{B}{4\pi^2} \sim \left(\frac{\sqrt{B}}{2\pi} L\right)^2.$$  \hspace{1cm} (4.17)

On the other hand, we expect that $\lim_{\xi_2' \rightarrow \xi_2} \delta(\xi_2' - \xi_2)$ be equal to $\lim_{\xi_1' \rightarrow \xi_1} \delta(\xi_1' - \xi_1)$; this must be so since the exchange between $x$ and $y$ gives rise to the exchange between $\xi_1$ and $\xi_2$ (while the variables $x$ and $y$ are to be treated symmetrically). So, based on Eq.(4.17), we have Eq.(1.19). To have Eq.(1.19) justified, note that

$$\lim_{\eta_2' \rightarrow \eta_2} \delta(\eta_2' - \eta_2) \sim \frac{1}{2\pi} \int d\xi_2,$$  \hspace{1cm} (4.18)

which is the usual relation when two mutually conjugate variables are involved. Then, following the same step that we have used to derive Eq.(1.13) above, it is also possible to show that

$$\lim_{\eta_2' \rightarrow \eta_2} \delta(\eta_2' - \eta_2) \sim \frac{\sqrt{B}}{2\pi} L.$$  \hspace{1cm} (4.19)
[Here, instead of Eq.(2.3), one could utilize the expression \( \langle \vec{r} | \vec{\eta} \rangle = \frac{1}{2\pi} e^{i\sqrt{B}(x\eta_1 + y\eta_2) + \frac{i}{2} xy + i\eta_1\eta_2} \). Based on Eqs.(4.18) and (4.19), we have Eq.(4.16).

With Eq.(4.13) used in the expression (4.12), we may now write \( \Gamma^{(1)}(B) = L^{(1)}(B)L^3T \), \( L^{(0)}(B) \) giving the one-loop contribution to the effective Lagrangian density. The \( \xi_1 \)-integration in Eq.(4.12) is readily performed, but the remaining proper-time integration is plagued with divergence at the \( s = 0 \) end. The amplitude needs to be renormalized. Explicitly, from Eq.(4.12) (after the \( \xi_1 \)-integration), we have

\[
L^{(1)}(B) = \frac{B}{8\pi^2} \int_0^\infty \frac{ds}{s} e^{-im^2s} \cot(Bs)
= (B-independent \text{ const.}) - \frac{1}{24\pi^2} \int_0^\infty \frac{ds}{s} e^{-im^2s} B^2
+ \left\{ \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-im^2s} \left[Bs \cot(Bs) - 1 + \frac{1}{3}(Bs)^2\right] \right\}. \tag{4.20}
\]

The divergent contribution is contained in the first two terms of the second form in Eq.(4.20); the expression appearing inside the curly bracket is finite. The \( B \)-independent constant in Eq.(4.20) is insignificant and can be dropped, while the second term, which is proportional to \( B^2 \) and has a logarithmically divergent coefficient, gets cancelled as one introduces the renormalization counterterm. In fact the above expression for \( L^{(\infty)}(B) \), which we found with the help of the oblique phase-space formulation, is in complete agreement with the result given in Refs.[11-13]. But, according to our discussion, it is evident that two oblique phase-space coordinates \( \xi_1 \) and \( \xi_2 \) assume very different role with respect to ultraviolet renormalization (and with respect to the contribution to the space-time volume factor also). This is in contrast with the roles assumed by coordinates \( x \) and \( y \) in the usual position-space formulation.

**V. INTERACTION**

In the previous sections, we have seen that the quadratic Lagrangian of the charged matter field with a uniform background magnetic field simplifies greatly in oblique variables. The matter field propagator can be obtained easily and used to calculate the effective action.
for the background magnetic charge. Basically, the ‘free’ Lagrangian describes the kinematics of charged particles in each Landau level.

There can be interaction between charged particles in different Landau levels. The interaction Lagrangian of the matter field, including the external potential depending on space, was studied in oblique variables in Sec. II. It shows how the interaction term can be expressed as the interaction between particles belonging to different Landau levels. When the dynamical electromagnetic field is coupled the matter field, one has to be careful as photons does not see the magnetic field. While we may want to express the matter field as a function of the oblique varibles, we want to keep the gauge field as a function of ordinary space time coordinates or their conjugate momentum variables. In this section, let us focus on the interaction term between the electromagnetic field and the Dirac field for the simplicity.

The Maxwell action for the dynamics electromagnetic field is

\[ S_M = -\frac{1}{4} \int d^4 x F_{\mu\nu}(x) F^{\mu\nu}(x). \] (5.1)

The gauge interaction of the Dirac field to the gauge field is

\[ S_1 = \int d^4 x \bar{\Psi}(x) \gamma_\mu \Psi(x) A_\mu(x). \] (5.2)

We may introduce the gauge fixing condition, like the Feynmann gauge-fixing term \(-\frac{1}{2} (\partial \cdot A(x))^2\), when we are working on the perturbative expansion. We change the arguments \(x\), \(y\) for the Dirac field into the oblique phase space variables \(\xi_1\) and \(\xi_2\) as before:

\[ \Psi(x) = \int d^2 \xi \langle \vec{x}\mid \vec{\xi} \rangle \Phi(\vec{\xi}, z, t). \] (5.3)

But for the Maxwell field, it is the momentum \(\vec{q} = (q_1, q_2)\) which naturally diagonalizes the quadratic action, so we choose the momenta to describe the Maxwell field,

\[ A_\mu(x) = \int d^2 \vec{q} e^{i\vec{q}\cdot\vec{x}} a_\mu(\vec{q}, z, t) \] (5.4)

where there are 4 modes \(a_\mu(\vec{q}, z, t)\) for a momentum \(\vec{q}\), and \(a_\mu(\vec{q}, z, t) = a_\mu^\dagger(\vec{-q}, z, t)\). In terms of these fields, the interaction (5.2) is rewritten as

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\[
\int dt \, dz \int d^2 q \, d^2 \xi'^2 C(\vec{\xi}, \vec{\xi}'; q) \Phi(\xi, z, t) \gamma^\mu \Phi(\vec{\xi}', z, t)a_\mu(\vec{q}, z, t), \tag{5.5}
\]

where \( C(\vec{\xi}, \vec{\xi}'; q) = \int d^2 x(\vec{\xi}|\vec{x})e^{iq\cdot\vec{x}}(\vec{x}|\vec{\xi}') \). It may be more illuminating to describe the Dirac field in terms of the Landau level indices instead of \( \xi \) as in the previous sections. Then the above gauge interaction becomes

\[
\int dt \, dz \int d^2 q \, d^2 \xi d^2 \xi' C_{mn}(\xi_2, \xi_2'; q) \Phi_m(\xi_2, z, t) \left\{ \left[ \cos \frac{\theta - \theta_n}{2} - \sin \frac{\theta - \theta_n}{2} \gamma^2 \right] a_\mu(\vec{q}, z, t) \right\} \Phi_n(\xi_2', z, t), \tag{5.6}
\]

where \( \theta_n, \Phi_n(\xi_2, z, t) \) are defined by Eqs.\((3.10),(3.14)\) and \((3.15)\), and the kernel \( C_{mn} \) is defined as follows\( (\text{the form of } W^{(n)}(\vec{x}; \xi) \text{ is given as Eq.}(2.12)):\)

\[
\int d^2 x W^{(m)}(\vec{x}; \xi) e^{iq\cdot\vec{x}} W^{(m)}(\vec{x}; \xi'). \tag{5.7}
\]

This kernel gives the strength with which the transition between two Landau levels occur in the presence of dynamic Maxwell field. It is explicitly calculated to be \((n > m \text{ assumed}):\)

\[
C_{mn} = i^{n-m} \delta[\xi_2 - \xi_2'] \frac{q_1}{\sqrt{B}} e^{i\frac{q_1 p_x}{B} + i\frac{q_1 + iq_2}{2B} \frac{1}{\sqrt{2B}}} (\frac{q_1 + iq_2}{\sqrt{2B}})^{n-m} \times \sqrt{\frac{n!}{m!(n-m)!}} M(-m, n-m+1, \frac{q_1 + iq_2}{\sqrt{2B}})(\frac{iq_2}{\sqrt{2B}}), \tag{5.8}
\]

where \( M(-m, n-m+1, A) \) in the expression is just a polynomial of \( A \), but takes the form of confluent hypergeometric function \((\text{in this case the Kummer function}):\)

\[
M(a, b, A) = \sum_{l=0}^{\infty} \frac{(a)_l}{(b)_l l!} A^l. \tag{5.9}
\]

The kernel \((5.8)\) and the interaction \((5.6)\) show some remarkable behaviors. First of all, the delta function factor in the kernel gives a relation which somewhat resembles the momentum conservation in x direction. Recall that the definition of \( \xi_2 \) is \( \frac{1}{\sqrt{B}}(p_x + \frac{1}{2}By) \), so in the weak field limit the delta function factor precisely gives the momentum conservation.

The interaction represented in oblique phase-space description may seem somewhat complicated, but it is because the quadratic part of the theory only respects rotation and boost symmetry in z direction. The boost symmetry along the z direction still exists in the expressions \((5.6)\) and \((5.8)\). The rotation symmetry is not manifest in the formula, but it comes
from our choice of $\xi_2$ to label the degeneracy of Landau levels. If we are only interested in the interaction including the lowest Landau level, the interaction looks simpler. The term with $n = 0$ looks like $a_\mu \bar{\Phi}_m [\cos \frac{\theta_m}{2} - \sin \frac{\theta_m}{2} \gamma^2] \gamma^\mu \Phi_0$. Interaction within the lowest Landau levels, which is the most interesting part in some problems, simply takes the form of $a_\mu \bar{\Phi}_0 \gamma^\mu \Phi_0$.

Furthermore, if the external magnetic field $B$ is large, the kernel (5.8) shows that the transition between two different Landau levels are suppressed by the factor $(\frac{2+q^2}{\sqrt{2B}})^\Delta$ where $\Delta$ is the difference of two Landau level indices. For example, if we integrate out the first Landau level effect, the effective interaction of the lowest Landau level and Maxwell fields will contain a term of order $(\frac{q^2 + q^2}{2B})$.

VI. CONCLUSION

In studying nonrelativistic or relativistic field theory systems in a background magnetic field, we have exhibited the advantage of using quantum field operators defined in the oblique phase space. By the very coordinate choice the mode rearrangements related to the appearance of Landau levels are incorporated in a natural manner. In addition, we have expressed the interaction Lagrangian in oblique variables so that Landau level indices appear. Also the phenomenon of dimensional reduction in a strong background magnetic field becomes evident in our oblique phase-space description.

Thus the investigation of the interacting field theory models in our formalism should give us valuable insights as regards to many interesting physical characteristics exhibited by the physical systems in a strong magnetic field. Our formalism could be a starting point for the standard field theoretic development, including the discussions on higher-order loop effects. Although much work has been done already in this direction[3-6,9,10], it is hoped that the oblique phase-space field description may serve a useful purpose in uncovering yet unknown aspects. We intend to report on such consideration in our future work.

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APPENDIX A

Here we will derive the explicit form of the kernel $\langle \vec{r}\mid \vec{\xi} \rangle$, satisfying the conditions (1.6), (1.7) and (1.8). The matrices $C$ and $D$ will be assumed to be nonsingular. Then, $D^{-1}C$ is a symmetric matrix (thanks to the condition $CD^T = DC^T$ in Eq.(1.4)) and we may well cast Eq.(1.7) in the form
\[
\frac{\partial}{\partial x^i} \langle \vec{r}\mid \vec{\xi} \rangle = i \left[ \sqrt{2}(D^{-1})_{ij} \xi^j - (D^{-1}C)_{ij} x^j \right] \langle \vec{r}\mid \vec{\xi} \rangle.
\] (A.1)

This has the general solution
\[
\langle \vec{r}\mid \vec{\xi} \rangle = H(\vec{\xi}) e^{i \left[ \sqrt{2} x^i (D^{-1})_{ij} \xi^j - \frac{1}{2} x^i (D^{-1}C)_{ij} x^j \right]}.
\] (A.2)

$H(\vec{\xi})$ being any function of $\vec{\xi}$ at this stage. If the normalization condition (1.6) is imposed with the form (A.2), $H(\vec{\xi})$ is determined up to an arbitrary phase, that is,
\[
H(\vec{\xi}) = \left( 2\pi \right)^d \det \left( \frac{D}{\sqrt{2}} \right)^{-1/2} e^{i \gamma(\vec{\xi})},
\] (A.3)
and so
\[
\langle \vec{r}\mid \vec{\xi} \rangle = \frac{1}{(2\pi)^{d/2} \sqrt{\det \left( \frac{D}{\sqrt{2}} \right)}} e^{i \gamma(\vec{\xi})} e^{i \left[ \sqrt{2} x^i (D^{-1})_{ij} \xi^j - \frac{1}{2} x^i (D^{-1}C)_{ij} x^j \right]}.
\] (A.4)

The condition (1.8) may be used to fix the phase $\gamma(\vec{\xi})$ in Eq.(A.4). Here, for $\xi^i$ given as in Eq.(1.2), we note that the conjugate momentum variable $\eta^i$ (see Eq.(1.3)) should have the general form
\[
\eta^i = \frac{1}{\sqrt{2}} \left[ -(D^T)^{-1} x^j + (C^T)^{-1} p^j \right] + (\Delta)_{ij} \xi^j,
\] (A.5)
where $\Delta$ is an arbitrary symmetric matrix. [According to this, the matrices $E$ and $F$, obeying the restrictions in Eq.(1.4), can be expressed as $E = -(D^T)^{-1} + \Delta C$, $F = (C^T)^{-1} + \Delta D$.]

Using the form (A.5), the condition (1.8) now reads
\[
i \frac{\partial}{\partial \xi^i} \langle \vec{r}\mid \vec{\xi} \rangle = \frac{1}{\sqrt{2}} \left[ -(D^T)^{-1} x^j - i (C^T)^{-1} \frac{\partial}{\partial x^j} \right] \langle \vec{r}\mid \vec{\xi} \rangle + (\Delta)_{ij} \xi^j \langle \vec{r}\mid \vec{\xi} \rangle,
\] (A.6)
and this in turn implies the following equation for $\langle \vec{r} | \vec{\xi} \rangle$ (given by Eq.(A.4)):

$$
\frac{\partial \gamma(\vec{\xi})}{\partial \xi^i} \langle \vec{r} | \vec{\xi} \rangle = - \left[ (CD^T)^{-1} \xi^i + (\Delta)_{ij} \xi^j \right] \langle \vec{r} | \vec{\xi} \rangle .
$$  \hspace{1cm} (A.7)

To obtain Eq.(A.7), we have made use of Eq.(1.7) and also the relation $CD^T = DC^T$.

Integrating Eq.(A.7), we find

$$
\gamma(\vec{\xi}) = - \frac{1}{2} \xi^i \left( (CD^T)^{-1} + \Delta \right)_{ij} \xi^j + \text{const}.
$$  \hspace{1cm} (A.8)

Without loss of generality the arbitrary constant in Eq.(A.8) may be set to zero. Then we end up with the following expression for $\langle \vec{r} | \vec{\xi} \rangle$:

$$
\langle \vec{r} | \vec{\xi} \rangle = \frac{1}{(2\pi)^{d/2}} \sqrt{| \det(\frac{D}{\sqrt{2}}) |} e^{-\frac{i}{2} \left[ x^i (D^{-1} C)_{ij} x^j + \xi^i ((CD^T)^{-1} + \Delta)_{ij} \xi^j \right] e^{i\sqrt{2} x^i (D^{-1})_{ij} \xi^j} .}
$$  \hspace{1cm} (A.9)

With the choice $\Delta = 0$ made, this reduces to the expression in Eq.(1.10).

APPENDIX B

For the given background magnetic field $\vec{B} = B_0 \hat{z}$, another popular choice for the vector potential is the expression in the Landau gauge, $\vec{A}(\vec{r}) = (0, B_0 x, 0)$. Then we have, instead of the quadratic Lagrangian in Eq.(1.11), the expression (with $B \equiv -qB_0 > 0$)

$$
\int d^3\vec{r} \Psi^\dagger(\vec{r}, t) \left[ i \frac{\partial}{\partial t} - \frac{1}{2m} (-i \partial_x)^2 - \frac{1}{2m} (-i \partial_y + Bx)^2 - \frac{1}{2m} (-i \partial_z)^2 \right] \Psi(\vec{r}, t).
$$  \hspace{1cm} (B.1)

In this case, the following set of variables may be considered instead of $(x, y, p_x, p_y)$:

$$
\xi_1 = \frac{1}{\sqrt{B}} (p_y + Bx), \quad \xi_2 = \frac{1}{\sqrt{B}} (p_x + By),
$$

$$
\eta_1 = \frac{1}{\sqrt{B}} p_x, \quad \eta_2 = \frac{1}{\sqrt{B}} p_y.
$$  \hspace{1cm} (B.2)

With this choice and the corresponding field $\Phi(\xi_1, \xi_2, \xi_3 \equiv z, t)$, we can transform the form (B.1) again to the expression given in Eq.(1.13). So, only if we have the explicit field transformation, i.e., the appropriate expression for the kernel $\langle \vec{r} | \vec{\xi} \rangle$ in Eq.(1.3), our symmetric-gauge-based discussion should carry over to the Landau gauge case.
We may use the result of Appendix A to find the Landau-gauge kernel $\langle \vec{r} | \vec{\xi} \rangle$ explicitly. In this case, the symmetric matrix $\Delta$ in Eq.(A.5) is chosen as
\[
\Delta = (CD^T)^{-1},
\]
so that $(\Delta)_{ij}\xi^j = \frac{1}{\sqrt{2}} \left[ (D^T)^{-1}x^i + (C^T)^{-1}p^i \right]$, and hence the conjugate momentum variables may read
\[
\eta^i = \frac{1}{\sqrt{2}} 2(C^T)^{-1}p^i.
\]
Comparing the relationships in Eq.(B.2) with those given by Eqs.(1.3) and (B.4), the matrices $C$, $D$ and $\Delta$ are found immediately:
\[
C = \begin{pmatrix} \sqrt{2B} & 0 \\ 0 & \sqrt{2B} \end{pmatrix}, 
D = \begin{pmatrix} \sqrt{\frac{7}{B}} & 0 \\ 0 & \sqrt{\frac{2}{B}} \end{pmatrix}, 
\Delta = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Hence, from Eq.(A.9), the desired kernel is found to be
\[
\langle \vec{r} | \vec{\xi} \rangle = \frac{\sqrt{B}}{2\pi} e^{-i(Bxy+\xi_1\xi_2)} e^{iB(x\xi_2+y\xi_1)} \delta(\xi_3 - z)
\]
\[
= \frac{\sqrt{B}}{2\pi} e^{-i(\sqrt{B}\xi_3-\xi_\infty)(\sqrt{B}\xi_\infty-\xi_4)} \delta(\xi_4 - z).
\]
We remark that analogous discussions can be given for relativistic field systems also.
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