Dynamics on $AdS_2$ and Enlargement of $SL(2, R)$ to $C = 1$ ‘cut-off Virasoro Algebra’

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Abstract

We consider the enhancement of $SL(2, R)$ to Virasoro algebra in a system of $N$ particles on $AdS_2$. We restrict our discussion to the case of non-interacting particles, and argue that they must be treated as fermions. We find operators $L_n$ whose commutators on the ground state, $|vac\rangle$, satisfy relations that are reminiscent of $c = 1$ Virasoro algebra, provided $N \geq n \geq -N$. Same relations hold also on the states $L_{-k}|vac\rangle$, if $(N-k) \geq n \geq -(N-k)$. The conditions $L^+_n = L_{-n}$, and $L_k|vac\rangle = 0$ for $k \geq 1$, are also satisfied.
1. Introduction

It is of interest to understand the properties of dynamics in two-dimensional Anti de-Sitter spacetime, \(AdS_2\), for several reasons\(^1\)\(^2\)\(^3\). The near horizon geometry of four-dimensional extremal black-holes is of the form \(AdS_2 \times S_2\). Therefore, study of dynamics on \(AdS_2\) may tell us about the properties of near horizon degrees of freedom. Another reason for interest in dynamics on \(AdS_2\) is from the point of view of \(AdS_2/CFT_1\) correspondence. The aim of this paper is to study the dynamics of \(N\)-Particles on \(AdS_2\) with a view to identifying the symmetries realised by this system.

First we will consider the case of one particle on \(AdS_2\). We start by realising the \(SL(2,\mathbb{R})\) symmetry which is to be expected due to the fact that the isometry group of \(AdS_2\) is \(SL(2,\mathbb{R})\). Surprisingly, the operators \(L_1, L_{-1}, L_0\) that we are led to consider satisfy \(SL(2,\mathbb{R})\) relations only on states that are orthogonal to the ground state. Further we will find that this single particle system also realises a semi-infinite sub-algebra of Virasoro-algebra. We will obtain operators, \(L_n\), which satisfy the relation

\[
[ L_n, L_m ] = (n - m) L_{n+m}
\]

(1)

where \(n, m \geq 0\) or \(n, m \leq 0\). \(L_0\) is the Hamiltonian and \(L_n^\dagger = L_{-n}\).

Next, we will consider \(N\)-Particles on \(AdS_2\). Any interaction that is introduced between these particles must be compatible with the \(SL(2,\mathbb{R})\) symmetry following from the \(AdS_2\) isometry. It appears complicated to introduce such an interaction and we will restrict ourselves to non-interacting system. However, we will argue that we must treat these particles as fermions. This is seen by imagining that our system is obtained by gradually turning off the interaction that is compatible with \(SL(2,\mathbb{R})\). We introduce the \(L_n\) operators for multiparticle system by simply taking the sum of corresponding single particle operators, except that now the \(L_0\) differs from the \(N\)-particle Hamiltonian by a constant shift. Obviously, these \(N\)-particle \(L_n\)s also satisfy the semi-infinite Virasoro-algebra of the form eqn.(1). The condition \(L_n|\text{vac}\rangle = 0\), for \(n \geq 1\), is satisfied. The \(L_1, L_{-1}, L_0\) satisfy \(SL(2,\mathbb{R})\) relations on all states of excitation energy less that \(N\). A remarkable fact that turns out is that, for \(N \geq n, m \geq -N\),

\[
\{ [ L_n, L_m ] - (n - m) L_{n+m} - \frac{1}{12} (n^3 - n) \delta_{n+m,0} \} |\text{vac}\rangle = 0
\]

(2)

Above equation with \(|\text{vac}\rangle\) replaced by \(L_{-i}|\text{vac}\rangle\) is also valid if \((N - i) \geq n, m \geq -(N - i)\). These relations are reminiscent of \(c = 1\) Virasoro algebra, although strictly speaking the \(L_n\)s do not generate Virasoro algebra because those relations are not satisfied on all the states. We expect that in general, for a state \(|\epsilon\rangle\) of excitation energy \(\epsilon\), the relation

\[
\{ [ L_n, L_m ] - (n - m) L_{n+m} - \frac{1}{12} (n^3 - n) \delta_{n+m,0} \} |\epsilon\rangle = 0
\]

(3)

will be satisfied if \((N - \epsilon) \geq n, m \geq -(N - \epsilon)\), and in the large \(N\) limit the standard \(c = 1\) Virasoro structure will be recovered.
2. One Particle on \( AdS_2 \)

The geometry of \( AdS_2 \) spacetime, of radius of curvature \( R \), is given by the metric

\[
ds^2 = \frac{R^2}{\sin^2 \sigma} (-d\tau^2 + d\sigma^2)
\]

where \(-\pi \leq \sigma \leq 0\), and \( \tau \) is periodic with period \( 2\pi \). The \( \sigma \) and \( \tau \) are globally well defined coordinates on \( AdS_2 \). Actually, in the following we will work with the covering space of \( AdS_2 \) i.e. we consider \(-\infty < \tau < \infty\), without periodicity. \( AdS_2 \) has two boundaries, given by \( \sigma = -\pi \) and \( \sigma = 0 \). Thus the topology of this spacetime is that of a strip. The fact that \( AdS_2 \) can be described as a hypersurface:

\[
x_0^2 - x_1^2 - x_2^2 = -R^2
\]

in the three-dimensional flat space with signature \((+,−,−)\), implies the \( SL(2,R) \) isometry of above metric.

We will now consider the dynamics of a single particle in \( AdS_2 \) spacetime. Action \( S = -m \int ds \) gives the constraint \( g^{\mu \nu} p_\mu p_\nu = -m^2 \), which can be solved in the static gauge to obtain the Hamiltonian,

\[
H = -p_0 = \sqrt{p^2 + \frac{m^2 R^2}{\sin^2 \sigma}}
\]

Here \( p = p_1 \) is conjugate to the position coordinate \( \sigma \) of the particle. We have used the \( AdS_2 \) metric (4).

Due to the \( SL(2,R) \) isometry of \( AdS_2 \) we expect to realise \( SL(2,R) \) on the phase-space, with Hamiltonian as one of the generators. One can check, using the Poisson bracket \( \{\sigma, p\} = 1 \), that \( H \) along with \( K \) and \( J \) defined below, satisfy the \( SL(2,R) \) algebra.

\[
K \equiv 2p \sin \sigma , \quad J \equiv 2 \cos \sigma \sqrt{p^2 + \frac{m^2 R^2}{\sin^2 \sigma}}
\]

\[
\{H, K\} = -J , \quad \{H, J\} = -K , \quad \{K, J\} = -4H
\]

Now, let us note that the group of spatial diffeomorphisms of \( AdS_2 \) is generated by the vector fields \( V_m = 2 \sin m\sigma \frac{\partial}{\partial \sigma} \), where \( m \) is a positive integer. These spatial diffeos can be realised on the phase space by the functions \( K_m \)

\[
K_m = 2p \sin m\sigma
\]

Above form of \( K_m \), as well as that of \( V_m \), is suggested by the fact that the spatial diffeos should leave the boundary points unaffected. The \( K_m \), as expected, satisfy following relation

\[
\{K_m, K_n\} = (m-n)K_{m+n} - (m+n)K_{m-n}
\]

where \( m > n \). Note that \( K_1 \) is identical to one of the \( SL(2,R) \) generators in eqn.(4), \( K_1 = K \).

At this point we are led to ask whether the system we are considering realises Virasoro algebra, with \( H \sim L_0 \) and \( K_m \sim (L_m - L_{-m}) \), where \( L_n \) are Virasoro generators. In the following, while discussing the case of a single particle on \( AdS_2 \), we will be able to write down \( L_n \) operators such that \( L_0 = \hat{H} \), and the \( n \geq 0 \) or \( n \leq 0 \) subset of the \( L_n \) satisfy Virasoro
relation. However, it turns out that the operators \((L_m - L_{-m})\) do not have the commutation relations (3) expected of spatial diffeo generators.

It is suggestive to define \(\Theta_1, \Theta_2\):

\[
\Theta_1 = \frac{1}{2}(iK_1 - \{K_1, H\}) \quad \Theta_2 = \frac{1}{2}(iK_2 - \frac{1}{2}(K_2, H))
\] (10)

which have the property

\[
\{\Theta_1, H\} = i\Theta_1 \quad \{\Theta_2, H\} = 2i\Theta_2
\] (11)

Above expressions hint that the operators corresponding to \(\Theta_1, \Theta_2\) may be suitable ansatz for \(L_1, L_2\). Although for generic values of \((\text{mass})^2\) of the particle this will not turn out to be correct, in case of \((\text{mass})^2 \to 0\) the operator analogues of \(\Theta_1, \Theta_2\) will indeed give us \(L_1, L_2\), which will be used to obtain \(L_n\) for any \(n\).

Next, in order to set up the quantum mechanics of this system, we first find the Hilbert space where

\[
H^2 = p^2 + \frac{m^2 R^2}{\sin^2 \sigma}
\] (12)

is represented as a hermitian, positive-definite operator. A basis of states for this Hilbert space is obtained by solving the eigenvalue problem

\[
\left[ -\frac{\partial^2}{\partial \sigma^2} + \frac{m^2 R^2}{\sin^2 \sigma} \right] \psi = \lambda \psi
\] (13)

The eigenfunctions can be found to be (4)

\[
\psi_n^\alpha = (-\sin \sigma)^\alpha C_n^\alpha (\cos \sigma)
\] (14)

where \(C_n^\alpha\) are Gegenbauer polynomials (5). The corresponding eigenvalues are \(\lambda_n^\alpha = (n + \alpha)^2\) where \(n = 0, 1, 2, \ldots\) and \(\alpha = (1 + \sqrt{1 + 4m^2 R^2})/2\). In order to ensure hermiticity of \(H^2\) we must require \(m^2 \geq -1/(4R^2)\) (4). Now, we represent the Hamiltonian on above Hilbert space by giving its action on the basis states as

\[
\hat{H} \psi_n^\alpha = (n + \alpha) \psi_n^\alpha
\] (15)

Thus the spectrum is independent of mass of the particle, except for a constant shift by \(\alpha(m^2 R^2)\).

For further considerations in this section, we will first discuss the case of \((\text{mass})^2 \to 0\), and then generalise to \((\text{mass})^2 \neq 0\).

2.1 \((\text{mass})^2 \to 0\) case

We will first try to realise \(SL(2, R)\) on the states of single particle system. Representing \(p\) by \(-i\frac{\partial}{\partial \sigma}\) on the states gives us the operator corresponding to \(iK_1\) as

\[
T_1 = 2 \sin \sigma \frac{\partial}{\partial \sigma} + \cos \sigma
\] (16)
The second term in above expression is due to the requirement $T_1^* = -T_1$. We require this anti-hermiticity of $T_1$ because the corresponding classical observable $iK_1$ is pure imaginary. We have used the standard norm on the states to define the hermitian conjugate operator. Define $L_1, L_{-1}, L_0$ as follows

$$L_1 \equiv \frac{1}{2} (T_1 + [T_1, L_0]), \quad L_{-1} \equiv -\frac{1}{2} (T_1 - [T_1, L_0]), \quad L_0 \equiv \hat{H}$$

(17)

Note that $L_1^* = L_{-1}$. Since $L_0 = \hat{H}$ is defined by explicitly giving its action on basis states, eqn.(15), the commutation relations of $L_1, L_{-1}, L_0$ are checked by considering their action on basis states. Note that since we are presently taking $(mass)^2 = 0, \alpha = 1$ in above expressions for the states and the eigenvalues. We find

$$L_1|m\rangle_1 = (m + 1/2)|m - 1\rangle_1, \quad m \geq 1$$
$$L_{-1}|m\rangle_1 = (m + 3/2)|m + 1\rangle_1, \quad m \geq 0$$

(18)

and $L_1|0\rangle_1 = 0$. We are using $|m\rangle_\alpha$ to denote the normalised basis state $\psi_m^\alpha$ in equation(14). Obviously

$$[L_1, L_0] = L_1, \quad [L_{-1}, L_0] = -L_{-1}$$

(19)

Furthermore, one obtains $[L_1, L_{-1}] = 2L_0$ for action on the states orthogonal to the ground state. Surprisingly, on the ground state

$$[L_1, L_{-1}]|0\rangle_1 = (9/4)|0\rangle_1 \neq 2L_0|0\rangle_1$$

(20)

Next, similar to the case of $iK_1$, we represent $iK_2$ by the operator $T_2$

$$T_2 = 2\sin 2\sigma \frac{\partial}{\partial \sigma} + \cos 2\sigma, \quad T_2^* = -T_2$$

(21)

Define $L_2$ and $L_{-2}$ as

$$L_2 = (\{T_2, L_0\} + 2T_2)/4, \quad L_{-2} = (\{T_2, L_0\} - 2T_2)/4, \quad L_2^* = L_{-2}$$

(22)

We obtain

$$L_2|m\rangle_1 = m|m - 2\rangle_1 \quad \{ m \geq 2 \}$$
$$L_2|m\rangle_1 = 0 \quad \{ m < 2 \}$$

$$L_{-2}|m\rangle_1 = (m + 2)|m + 2\rangle_1 \quad \{ m \geq 0 \}$$

(23)

Notice that the relation $[L_1, L_{-2}] = 3L_{-1}$ is satisfied on all the states orthogonal to the ground state. Further, we define $L_n$ for $n \geq 3$ by

$$(n - 2)L_n = [L_{n-1}, L_1]$$

(24)
and $L_{-n} \equiv L_n^\dagger$. Using above definitions one can obtain following expressions for the action of $L_n$ and $L_{-n}$ on the basis states. For $n \geq 1$

$$L_n|m\rangle_1 = (m - n/2 + 1)|m - n\rangle_1 \quad \{ m \geq n \}$$

$$L_n|m\rangle_1 = 0 \quad \{ m < n \}$$

$$L_{-n}|m\rangle_1 = (m + n/2 + 1)|m + n\rangle_1 \quad \{ m \geq 0 \}$$

(25)

Remarkably, the $L_n$ for $n \geq 0$, satisfy semi-infinite Virasoro algebra, as can be easily checked using above equations.

$$[ L_n, L_m ] = (n - m) L_{n+m}$$

(26)

Due to $L_{-n} = L_n^\dagger$, the $L_n$ for $n \leq 0$ also satisfy above semi-infinite Virasoro algebra.

2.2 (mass)$^2 \neq 0$ case

Let us first consider realisation of $SL(2, R)$. We may naively continue to take the expression for $T_1$ of the form (16), and use that to define $L_\pm^1, L_0$ as in (17). However, with $L_\pm^1, L_0$ defined in this way the $SL(2, R)$ relations are not satisfied, not only on the ground state as it happened in case of $\alpha = 1$, but generically on any state. It is possible to find an appropriate modification of $T_1$ so that $SL(2, R)$ is realised in a way similar to the case of $\alpha = 1$. We consider following realisation of $T_1$ which by eqn.(17) gives $L_\pm^1$. As before $L_0 = \hat{H}$.

$$T_1|0\rangle_\alpha = -(\alpha + 1/2)|1\rangle_\alpha$$

$$T_1|m\rangle_\alpha = (\alpha + m - 1/2)|m - 1\rangle_\alpha - (\alpha + m + 1/2)|m + 1\rangle_\alpha \quad \{ m \geq 1 \}$$

(27)

One can check that $T_1^\dagger = -T_1$ and

$$L_1|m\rangle_\alpha = (\alpha + m - 1/2)|m - 1\rangle_\alpha \quad \{ m \geq 1 \} \quad L_1|0\rangle_\alpha = 0$$

(28)

$$L_{-1}|m\rangle_\alpha = (\alpha + m + 1/2)|m + 1\rangle_\alpha \quad \{ m \geq 0 \}$$

(29)

Above equations give: $[L_{-1}, L_0] = -L_{-1}$, $[L_1, L_0] = L_1$. Similar to the $\alpha = 1$ case, the relation $[L_{-1}, L_1] = -2L_0$ holds on states orthogonal to ground state. On the ground state we get

$$[L_{-1}, L_1]|0\rangle_\alpha = -(\alpha + 1/2)^2|0\rangle_\alpha \neq -2L_0|0\rangle_\alpha$$

(30)

Note that for $\alpha = 1/2$, which is the case of $m^2 = -\frac{1}{4R^2}$, the relation $[L_{-1}, L_1] = -2L_0$ is satisfied on all states.

Next, we will find $L_{-2}$ by requiring that it satisfies following conditions

$$[ L_0, L_{-2} ] = 2L_{-2}$$

(31)

$$[ L_1, L_{-2} ]|\psi\rangle_\alpha = 3L_{-1}|\psi\rangle_\alpha$$

(32)
where $|\psi\rangle_\alpha$ is any state orthogonal to the ground state $|0\rangle_\alpha$. Above relations were satisfied by $L_{-2}$ in $\alpha = 1$ case. First condition gives

$$L_{-2} |m\rangle_\alpha = a_m |m + 2\rangle_\alpha, \quad m \geq 0 \quad (33)$$

and second condition gives a recursion relation for $a_m$s, which determines all the $a_m$ in terms of $a_0$. Further, we define $L_{-n}$, for $n \geq 3$, by

$$-(n-2)L_{-n} = [L_{-(n-1)}, L_{-1}] \quad (34)$$

Thus, we have obtained an ansatz for $L_{-n}$ in terms of an unknown parameter $a_0$ which will be fixed by requiring

$$[L_{-4}, L_{-1}] |0\rangle_\alpha = 3[L_{-3}, L_{-2}] |0\rangle_\alpha \quad (35)$$

Finally we obtain, for $n \geq 1$,

$$L_n |m\rangle_\alpha = (m - n/2 + \alpha)|m - n\rangle_\alpha \quad \{ m \geq n \}$$

$$L_n |m\rangle_\alpha = 0 \quad \{ m < n \}$$

$$L_{-n} |m\rangle_\alpha = (m + n/2 + \alpha)|m + n\rangle_\alpha \quad \{ m \geq 0 \} \quad (36)$$

where $L_n \equiv L_{-n}$. Similar to the case of $\alpha = 1$, we find

$$[L_n, L_m] = (n - m) L_{n+m}, \quad \{ n, m \geq 0 \} \quad (37)$$

for general value of $\alpha$.

### 3. $N$ Particles on $AdS_2$

So far we have shown that Hilbert space of a single particle on $AdS_2$ realises semi-infinite Virasoro algebra. The aim of this section is to demonstrate that when we consider a multi-particle system on $AdS_2$, we obtain a structure that is reminiscent of $c = 1$ Virasoro algebra.

Introducing interaction between particles on $AdS_2$ appears to be complicated because of the requirement of $SL(2, R)$ symmetry that follows from $AdS_2$ isometry. We will restrict our discussion here to the case of non-interacting particles. However, as we will now argue, considering this system to be a limit of $SL(2, R)$ compatible interacting system where the interaction strength is gradually taken to zero suggests that we must treat these identical particles as fermions.

Let us first note that the energy spectrum of a single particle on $AdS_2$, eqn.(15), agrees exactly with that of the system described by following hamiltonian

$$\mathcal{H} = \frac{p^2}{4} + \frac{g}{4q^2} + \frac{q^2}{4} \quad (38)$$
if we choose \( g = (3 + 16m^2R^2)/4 \). The spectrum of above hamiltonian \( \mathcal{H} \) is \( E_n = (n + \beta) \), where \( \beta = (1 + \sqrt{g + \frac{1}{4}})/2 \) and \( n = 0, 1, \ldots \). Furthermore \( \mathcal{H} \), along with \( \mathcal{K} \) and \( \mathcal{J} \), defined below, form \( SL(2, R) \) similar to \( H, K, J \) in eqn.\([7]\).

\[
\mathcal{K} = pq, \quad \mathcal{J} = \frac{p^2}{2} + \frac{g}{2q^2} - \frac{q^2}{2} \tag{39}
\]

\[
\{\mathcal{H}, \mathcal{K}\} = -\mathcal{J}, \quad \{\mathcal{H}, \mathcal{J}\} = -\mathcal{K}, \quad \{\mathcal{K}, \mathcal{J}\} = -4\mathcal{H} \tag{40}
\]

Since the spectrum and symmetry of above single particle system agree with that of a particle on \( AdS_2 \), multiparticle generalisation of both systems, such that \( SL(2, R) \) symmetry is satisfied, should have similar properties. Now, consider \( N \)-particles generalisation of above system \( \mathcal{H}, \mathcal{K}, \mathcal{J} \), allowing for interaction such that \( \mathcal{H}_N, \mathcal{K}_N, \mathcal{J}_N \) form \( SL(2, R) \) . One finds that

\[
\mathcal{H}_N = \frac{1}{4} \sum_{i=1}^{N} p_i^2 + \frac{g}{4q_i^2} + \sum_{i<j} \frac{\lambda}{(q_i - q_j)^2} + \frac{1}{4} \sum_{i=1}^{N} q_i^2 \tag{41}
\]

\[
\mathcal{K}_N = \sum_{i=1}^{N} p_i q_i, \quad \mathcal{J}_N = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \frac{g}{2q_i^2} + \sum_{i<j} \frac{2\lambda}{(q_i - q_j)^2} - \frac{1}{2} \sum_{i=1}^{N} q_i^2 \tag{42}
\]

It is clear from the interaction term in \( \mathcal{H}_N \) that in the limit \( \lambda \rightarrow 0 \), the short interparticle distance behaviour of wavefunctions is \( \psi_N(q_i, q_{i+1}) \sim (q_i - q_{i+1}) \). This implies that in \( \lambda \rightarrow 0 \) limit the system is equivalent to that of non-interacting fermions. Therefore the non-interacting particles on \( AdS_2 \) should also be treated as fermions.

Coming back to our original system we have the Hamiltonian

\[
H_N = \sum_{i=1}^{N} \sqrt{p_i^2 + \frac{m^2R^2}{\sin^2\sigma_i}} \tag{43}
\]

For reasons to be clear in the following, it will be preferable to take \( L_0 \) in the \( N \) particles case to be shifted with respect to the Hamiltonian

\[
L_0 = H_N + (\alpha - 1/2)^2 /2 \tag{44}
\]

We will define the \( L_n \) for \( N \)-particles system to be the sum of single particle \( L_n \), which is denoted below by \( L_n^i \) for the \( i \)th particle.

\[
L_n = \sum_{i=1}^{N} L_n^i \tag{45}
\]

The \( N \)-particle basis states are

\[
|n_1, n_2, \ldots, n_N\rangle_\alpha = \text{det}[M_{ij}] \quad M_{ij} = \psi_{n_i}^{\alpha}(\sigma_j) \tag{46}
\]

and the ground state \( |\text{vac}\rangle_\alpha \) is characterised by \( n_i = (i - 1) \) in above formula.

8
Let us first note that due to eqn. (37) it is obvious that for \( n, m \geq 0 \), or \( n, m \leq 0 \),
\[
[L_n, L_m] = (n - m) L_{n+m}
\]
(47)

Furthermore, for \( n \geq 1 \)
\[
L_n |vac\rangle_\alpha = 0
\]
(48)

One can check that \( L_1, L_{-1}, L_0 \) satisfy the \( SL(2, R) \) relations on any state of excitation energy less than \( N \).

Consider now \( [L_n, L_{-m}]|vac\rangle_\alpha \) for \( n, m \geq 1 \).

\[
L_{-m}|vac\rangle_\alpha = \sum_{i=n}^{m-1} (i + N - \frac{m}{2} + \alpha)|0,1, \ldots, N+i-m-1, N+i, N+i-m+1, \ldots\rangle_\alpha
\]
(49)

where \( i_0 = 0 \) if \( N \geq m \geq 1 \), and \( i_0 = (m-N) \) if \( m > N \). Now, for \( n > m \)
\[
L_n|0,1, \ldots, N+i-m-1, N+i, N+i-m+1, \ldots\rangle_\alpha = 0
\]
(50)

Therefore, for \( n > m \geq 1 \),
\[
[L_n, L_{-m}]|vac\rangle_\alpha = L_n L_{-m}|vac\rangle_\alpha = 0 = (n + m) L_{n-m}|vac\rangle_\alpha .
\]
(51)

Therefore, for \( N \geq m \geq 1 \)
\[
[L_m, L_{-m}]|vac\rangle_\alpha = \left\{ \sum_{i=0}^{m-1} (N+i-m/2+\alpha)^2 \right\} |vac\rangle_\alpha = 2m L_0|vac\rangle_\alpha + \frac{1}{12} (m^3-m)|vac\rangle_\alpha
\]
(52)

For \( n < m \leq N \)
\[
[L_n, L_{-m}]|vac\rangle_\alpha = L_n L_{-m}|vac\rangle_\alpha = \\
\sum_{i=n}^{m-1} (i + N - \frac{m}{2} + \alpha)|N+i-n, N+i-m-1, N+i-n, N+i-m+1, \ldots\rangle_\alpha
\]
- \[ \sum_{i=0}^{m-n-1} (i + N - \frac{m}{2} + \alpha)|N+i-m+n-1, N+i, N+i-m+n+1, \ldots\rangle_\alpha
\]
\[
= (n + m) \sum_{i=0}^{m-n-1} (i + N - \frac{m-n}{2} + \alpha)|N+i-m+n-1, N+i, N+i-m+n+1, \ldots\rangle_\alpha
\]
\[
= (n + m) L_{n-m}|vac\rangle_\alpha
\]
(53)

Thus we have found that, for \( N \geq n \geq 1 \) and \(-N \leq m \leq -1\)
\[
\left\{ [L_n, L_m] - (n - m) L_{n+m} - \frac{1}{12} (n^3-n) \delta_{n+m,0} \right\} |vac\rangle_\alpha = 0
\]
(54)

Similarly it is straightforward to check that
\[
\left\{ [L_n, L_m] - (n - m) L_{n+m} - \frac{1}{12} (n^3-n) \delta_{n+m,0} \right\} L_{-i}|vac\rangle_\alpha = 0
\]
(55)

where \((N-i) \geq n \geq 1\) and \(-N \leq m \leq -1\). Above relations, along with eqn. (37) are reminiscent of \( c = 1 \) Virasoro algebra, although in above we have a \((N-i)\) dependent cut-off on the set of \( L_n \)s for which the Virasoro-like relations hold. In general we expect that on a state with excitation energy \( \epsilon \), the \( L_n \)s, \((N-\epsilon) \geq n \geq -(N-\epsilon)\), have \( c = 1 \) Virasoro-like relation.
4. Conclusion

We have restricted our study to the case of non-interacting particles on AdS\(_2\). It is important to see whether our discussion generalises to interacting system. In this context one may study the model described by (41), as our discussion in last section suggests that this model is closely related to a multiparticle system on AdS\(_2\). The system described by (41) is a variant of Calogero model \([7]\). It is interesting to note that a supersymmetric variant of Calogero model, but without the \(g/q^2\) term in (41), was suggested \([2]\) to describe microscopic degrees of freedom of certain extremal black-holes.

A surprising aspect of our analysis of a single particle dynamics on AdS\(_2\) is that \(SL(2,R)\) is not exactly realised\(^1\). In the model \([6]\) of (38), which has the same spectrum, one can write the operators \(L_\pm, L_0\) that generate \(SL(2,R)\).

\[
L_0 = \frac{p^2}{4} + \frac{g}{4q^2} + \frac{q^2}{4}, \quad L_\pm = \frac{p^2}{4} + \frac{g}{4q^2} - \frac{q^2}{4} \pm \frac{i}{4}(pq + qp)
\]

(56)

It will be interesting to find out whether the algebra of \(L_\pm, L_0\) enlarges to a Virasoro-algebra like structure, when multiparticle system is considered, in a way similar to our discussion in this paper.

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