Theories of Violation of Statistics

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Abstract

I discuss theories of violations of statistics, including intermediate statistics, parastatistics, parons, and quons. I emphasize quons, which allow small violations of statistics. I analyze the quon algebra and its representations, implications of the algebra including the observables allowed by the superselection rule separating inequivalent representations of the symmetric group, the conservation of statistics rules, and the rule for composite systems of quons. I conclude by raising the question of possible origins of violations of statistics and of the level at which violations should be expected if they exist.

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1 Introduction

As far as I know this is the first international conference devoted entirely to the relation of spin and statistics and to the investigation of possible small violations of statistics. My purpose in this talk is to give an overview of theoretical issues connected with violations of statistics. I have divided the talk into five parts: (1) general theoretical remarks, (2) types of experiments to detect violations, (3) attempts to violate statistics, (4) quons, the best formalism so far to describe small violations, and (5) summary and open questions.

2 General theoretical remarks

The general principles of quantum theory do not require that all particles be either bosons or fermions. This restriction requires an additional postulate which A.M.L. Messiah[1] named the “symmetrization postulate,” which I quote as Messiah defined it: “The states of a system containing \( N \) identical particles are necessarily either all symmetrical or all antisymmetrical with respect to permutations of the \( N \) particles.” The symmetrization postulate can be restated as: “All states of identical particles are in one-dimensional representations of the symmetric group.”

With Messiah’s definition the spin-statistics connection, that integer spin particles are bosons and odd-half-integer spin particles are fermions, is a separate statement.

Messiah and I gave a detailed discussion of quantum mechanics without the symmetrization postulate[2]. We emphasized that without the symmetrization postulate, a set of one-body measurements is never a maximal set; one needs additional measurements to fix the state of the system. Further there is a superselection rule separating states in inequivalent representations of the symmetric group. If identical particles can occur in states that violate the spin-statistics connection their transitions must occur in the same representation of the symmetric group. For example, in the experiment of Deilamian, et al[3] which looked for anomalous helium atoms in which the two electrons violated the exclusion principle and were in the symmetric state, the search was for transitions among the symmetric states rather than between symmetric and antisymmetric states. This point was also made by R. Amado and H. Primakoff[4]. In addition if one assumes that charged particles couple universally to the electromagnetic field, then the transitions among the anomalous states occur at the normal rate, so that isolated atoms will be in the lowest state of the anomalous system. Since the symmetrization postulate is not an intrinsic part of quantum theory, this postulate must be subjected both to theoretical study and to experimental tests. Quantitative tests require a theory in which the symmetrization postulate does not have to hold and in which the violation of the postulate is reflected in a parameter that departs from its standard value at which the symmetrization postulate and the spin-statistics connection do hold. It is certainly possible that violations of...
statistics are extremely small and require high-precision tests to be observed. This
conference brings together leading workers in this search.

Because the notion that particles are identical requires that the Hamiltonian
and, indeed, all observables must be symmetric in the dynamical variables associated
with the identical particles, the observables can’t change the permutation symmetry
type of the wave function. In particular one can’t introduce a small violation of
statistics by assuming the Hamiltonian is the sum of a statistics-conserving and a
small statistics-violating term,

$$H = H_S + \epsilon H_V$$  \hspace{1cm} (1)

as one can for violations of parity, charge conjugation, etc. Violation of statistics
has to be introduced in a more subtle way.

If charged particles couple universally to the electromagnetic field, then there
can’t be two kinds of—say—electrons, “red” electrons and “blue” electrons, because
then the lowest order pair production cross section,

$$\sigma(\gamma X \rightarrow e^+e^- X)$$  \hspace{1cm} (2)

would double. A high-precision measurement is not needed to rule this out.

A convenient way to parametrize violations or bounds on violations of statistics
uses the two-particle density matrix. For fermions,

$$\rho_2 = (1 - v_F)\rho_a + v_F\rho_s;$$  \hspace{1cm} (3)

for bosons,

$$\rho_2 = (1 - v_B)\rho_s + v_B\rho_a;$$  \hspace{1cm} (4)

in each case the violation parameter varies between zero if the statistics is not
violated and one if the statistics is completely violated.

3 Types of experiments

There are three basic types of experiments to detect violations of statistics: (1)
transitions among anomalous states—these can occur in solids, liquids or gases, (2)
accumulation of particles in anomalous states, and (3) deviations from the usual
statistical properties of the identical particles. Since, as mentioned earlier, a supers-
election rule prevents transitions between normal and anomalous states, experiments
searching for such transitions do not provide a valid test of violation of statistics.

Transitions among anomalous states can provide a very sensitive test, since in
some cases a single such transition can be observed. The prototype of this kind of
test is the experiment of Maurice and Trudy Goldhaber. They asked the qualitative question, “Do the electrons from nuclear beta decay obey the same exclusion principle as electrons in atoms?” They knew that electrons from each source have the same charge, spin, and mass, etc., i.e. that the single-electron states in each case are identical, but there was no evidence that the many-electron states from each source are identical. They devised the following ingenious test: they let beta decay electrons from a nuclear source fall on a block of lead. They argued that if the many-electron states were not identical then the nuclear beta decay electrons would not obey the same exclusion principle as the electrons in the lead atoms. Then the beta decay electrons would not see the K shells in the lead atoms as filled and could fall into the K shells and would emit x-rays. A single such x-ray could be observed. They saw no such x-rays above background and thus answered their qualitative question in the affirmative. I estimate that their experiment gave the bound \( v_F \leq 5 \times 10^{-2} \) for electrons.

E. Ramberg and G.A. Snow developed this experiment into one which yields a high-precision bound on violations of the exclusion principle. Their idea was to replace the natural \( \beta \) source, which provides relatively few electrons, by an electric current, in which case Avogadro’s number is on their side. The possible violation of the exclusion principle is that a given collection of electrons can, with different probabilities, be in different permutation symmetry states. The probability to be in the “normal” totally antisymmetric state presumably would be close to one, the next largest probability would occur for the state with its Young tableau having one row with two boxes, etc. The idea of the experiment is that each collection of electrons has a possibility of being in an “abnormal” permutation state. If the density matrix for a conduction electron together with the electrons in an atom has a projection onto such an “abnormal” state, then the conduction electron will not see the K shell of that atom as filled. Then a transition into the K shell with x-ray emission is allowed. Each conduction electron which comes sufficiently close to a given atom has an independent chance to make such an x-ray-emitting transition, and thus the probability of seeing such an x-ray is proportional to the number of conduction electrons which traverse the sample and the number of atoms which the electrons visit, as well as the probability that a collection of electrons can be in the anomalous state. Ramberg and Snow chose to run 30 amperes through a thin copper strip for about a month. They surrounded the experiment with veto scintillators to remove background x-rays. They estimated the energy of the modified x-rays which would be emitted due to the transition to the K shell. No excess of x-rays above background was found in this energy region. Ramberg and Snow set the limit

\[
v_F \leq 1.7 \times 10^{-26}
\]

for electrons. This is high precision indeed!

The Ramberg-Snow experiment may seem discouraging for the discovery of generalizations of bose and fermi statistics; however there are small numbers in
physics which, if necessary, can occur in degree greater than one. For example the ratios
\[
\frac{m_{\text{proton}}}{M_{\text{Planck}}} \sim 10^{-19}, \quad \text{and} \quad \frac{G_N m_e^2}{e^2} \sim 10^{-43}
\] (6)
can provide numbers smaller than the Ramberg-Snow bound. In addition new physics effects such as violations of Lorentz invariance, spacetime discreteness, spacetime noncommutativity, etc. may provide small effects. Mohapatra and I gave an early survey of experimental bounds on violations of statistics[7].

Composite structure can mimic violations of statistics. This is not what I am considering here.

4 Attempts to violate statistics

4.1 Gentile’s “intermediate statistics”

The first attempt to go beyond bose and fermi statistics seems to have been made by G. Gentile[8] who suggested an “intermediate statistics” in which at most \( n \) identical particles could occupy a given quantum state. In intermediate statistics, fermi statistics is recovered for \( n = 1 \) and bose statistics is recovered for \( n \rightarrow \infty \); thus intermediate statistics interpolates between fermi and bose statistics. However Gentile’s statistics is not a proper quantum statistics, because the condition of having at most \( n \) particles in a given quantum state is not invariant under change of basis[9]. For example, for intermediate statistics with \( n = 2 \), the state \( |\psi\rangle = |k, k, k\rangle \) does not exist; however, the state \( |\chi\rangle = \sum_{l_1, l_2, l_3} U_{k, l_1} U_{k, l_2} U_{k, l_3}|l_1, l_2, l_3\rangle \), obtained from \( |\psi\rangle \) by the unitary change of single-particle basis, \( |k\rangle' = \sum_l U_{k, l}|l\rangle \) does exist. By contrast, parafermi statistics of order \( n \) which I discuss just below is invariant under change of basis[10]. Parafermi statistics of order \( n \) not only allows at most \( n \) identical particles in the same state, but also allows at most \( n \) identical particles in a symmetric state. In the example just described, neither \( |\psi\rangle \) nor \( |\chi\rangle \) exist for parafermi statistics of order two.

4.2 Parastatistics

H.S. Green[10] proposed the first proper quantum statistical generalization of bose and fermi statistics. Green noticed that the commutator of the number operator with the annihilation and creation operators is the same for both bosons and fermions
\[
[n_k, a_l^\dagger] = \delta_{kl} a_l^\dagger.
\] (7)
The number operator can be written
\[ n_k = (1/2)[a_k^\dagger, a_k]_\pm + \text{const.}, \] (8)
where the anticommutator (commutator) is for the bose (fermi) case. If these expressions are inserted in the number operator-creation operator commutation relation, the resulting relation is \textit{trilinear} in the annihilation and creation operators. Polarizing the number operator to get the transition operator \( n_{kl} \) which annihilates a free particle in state \( l \) and creates one in state \( k \) leads to Green’s trilinear commutation relation for his parabose and parafermi statistics,
\[ [[a_k^\dagger, a_l]_\pm, a_m^\dagger]_-= 2\delta_{lm}a_k^\dagger \] (9)
Since these rules are trilinear, the usual vacuum condition,
\[ a_k|0\rangle = 0, \] (10)
does not suffice to allow calculation of matrix elements of the \( a \)'s and \( a^\dagger \)'s; a condition on single-particle states must be added,
\[ a_k a_k^\dagger |0\rangle = p\delta_{kl}|0\rangle. \] (11)

Green found an infinite set of solutions of his commutation rules, one for each positive integer \( p \), by giving an ansatz which he expressed in terms of bose and fermi operators. Let
\[ a_k^\dagger = \sum_{p=1}^{n} b_{k}^{(\alpha)}_{p}^\dagger, \quad a_k = \sum_{p=1}^{n} b_{k}^{(\alpha)}_{p}, \] (12)
and let the \( b_{k}^{(\alpha)} \) and \( b_{k}^{(\beta)}_{p}^\dagger \) be bose (fermi) operators for \( \alpha = \beta \) but anticommutate (commute) for \( \alpha \neq \beta \) for the “parabose” (“parafermi”) cases. This ansatz clearly satisfies Green’s relation. The integer \( p \) is the order of the parastatistics. The physical interpretation of \( p \) is that, for parabosons, \( p \) is the maximum number of particles that can occupy an antisymmetric state, while for parafermions, \( p \) is the maximum number of particles that can occupy a symmetric state (in particular, the maximum number which can occupy the same state). The case \( p = 1 \) corresponds to the usual bose or fermi statistics. Later, Messiah and I\[11\] proved that Green’s ansatz gives all Fock-like solutions of Green’s commutation rules. Local observables have a form analogous to the usual ones; for example, the local current for a spin-1/2 theory is \( j_\mu = (1/2)[\bar{\psi}(x), \psi(x)]_- \). From Green’s ansatz, it is clear that the squares of all norms of states are positive, since sums of bose or fermi operators give positive norms. Thus parastatistics\[12\] gives a set of orthodox theories.

This is all well and good; however, the violations of statistics provided by parastatistics are gross. Parafermi statistics of order two has up to two particles in each quantum state. High-precision experiments are not necessary to rule this out for all particles we think are fermions.
4.3 The Ignatiev-Kuzmin model

Interest in possible small violations of the exclusion principle was revived by a paper of Ignatiev and Kuzmin\cite{13} in 1987. They constructed a model of one oscillator with three possible states: a vacuum state, a one-particle state and, with small amplitude $\beta$, a two-particle state. They gave trilinear commutation relations for their oscillator. Mohapatra and I noticed that the Ignatiev-Kuzmin oscillator could be represented by a modified form of the order-two Green ansatz. We suspected that a field theory generalization of this model having an infinite number of oscillators would not have local observables and set about trying to prove this. To our surprise, we found that we could construct local observables and gave trilinear relations which guarantee the locality of the current\cite{14}.

4.4 Parons

Following Ignatiev and Kuzmin we introduced a parameter $\beta$ that gives the deformation of the Green trilinear commutation relations. For $\beta \to 1$ the relations reduce to those of the $p = 2$ parafermi field; for $\beta \to 0$ the double occupancy is completely suppressed and the theory is equivalent to a fermi theory. A random state of two paronic electrons has the violation parameter $\beta^2/2$. Mohapatra and I checked that the norms are positive for states of up to three particles. At this stage, we were carried away with enthusiasm, named these particles “parons” since their algebra is a deformation of the parastatistics algebra, and thought we had found a local theory with small violation of the exclusion principle. Unknown to us Govorkov\cite{15}, using a detailed algebraic argument, already had shown in generality that any deformation of the Green commutation relations necessarily has states with negative squared norms in the Fock-like representation. For our model, the first such negative-probability state occurs for four particles in the representation of $S_4$ with three boxes in the first row and one in the second. We were able to understand Govorkov’s result qualitatively as follows:\cite{16} Since parastatistics of order $p$ is related by a Klein transformation to a model with exact $SO(p)$ or $SU(p)$ internal symmetry, a deformation of parastatistics which interpolates between Fermi and parafermi statistics of order two would be equivalent to interpolating between the trivial group whose only element is the identity and a theory with $SO(2)$ or $SU(2)$ internal symmetry. This is impossible, since there is no such interpolating group.

4.5 The Doplicher-Haag-Roberts analysis

S. Doplicher, R. Haag and J. Roberts\cite{17} made a general study of identical particle statistics using the algebraic field theory methods pioneered by Haag. They found parabose and parafermi statistics of positive integer orders which as mentioned above
were introduced by Green. They also found another case which they called infinite
statistics. Young patterns label the inequivalent irreducible representations of the
symmetric group. In parabose (parafermi) statistics of order \( p \) the Young patterns
have at most \( p \) rows (columns) corresponding to having at most \( p \) particles in an
antisymmetric (a symmetric) state. In infinite statistics all irreducibles of the sym-
metric group occur. Doplicher, et al, did not give an operator realization of infinite
statistics.

### 4.6 Infinite statistics

In 1989 I gave an evening lecture at Wake Forest University. My talk was attended
by physicists, philosophers, and among people in other disciplines, chemists. In my
talk I mentioned the bose and fermi commutation relations. After the talk Roger
Hegstrom, a chemist, asked “Why not average the bose and fermi commutation
relations and consider the relation

\[ a(k)a^\dagger(l) = \delta(k, l)? \]

I was surprized to find that such a simple case had not been considered. Later I found
out that it had, in the mathematical literature, by J. Cuntz\[18\]. With Hegstrom’s
permission I developed this case, which turned out to be the first operator example
of infinite statistics\[19\]. In order to select the Fock-like representation, one must
add the vacuum condition

\[ a(k)|0\rangle = 0. \] (14)

We can calculate all vacuum matrix elements of products of \( a \)'s and \( a^\dagger \)'s using the
commutation relation and the vacuum condition. There is no commutation relation
involving two \( a \)'s or two \( a^\dagger \)'s. There are \( n! \) linearly independent \( n \)-particle states
in Hilbert space if all quantum numbers are distinct; these states differ only by
permutations of the order of the creation operators. (Later we will see that there
are not that many independent density matrices or other observables.) The matrix
of scalar products of these states is the identity matrix,

\[
M_{P,Q}^{n}(q) = (Pa^\dagger(k_1)a^\dagger(k_2)\cdots a^\dagger(k_n)|0\rangle, Qa^\dagger(l_1)a^\dagger(l_2)\cdots a^\dagger(l_n)|0\rangle = \prod_{i=1}^{n} \delta(k_i, l_i)\delta(P, Q)
\] (15)

where \( P \) and \( Q \) are permutations from \( S_n \); that is, the scalar product is zero unless
there are the same number of creation operators on each side of the scalar product
and they have the same quantum numbers in the same order. This algebra can be
viewed as a deformation of either the bose or the fermi algebras. As is typical for
deformed algebras, there is an element that is infinite degree in the generators of
the algebra. In this case the number operator that obeys
\[
[n(k), a^\dagger(l)]_- = \delta(k, l)a^\dagger(l)
\] (16)
is the operator of infinite degree; in terms of the \(a\)'s and the \(a^\dagger\)'s,
\[
n(k) = a^\dagger(k)a(k) + \sum_t a^\dagger(t)a^\dagger(k)a(k)a(t) + \sum_{t_1, t_2} a^\dagger(t_2)a^\dagger(t_1)a^\dagger(k)a(k)a(t_1)a(t_2) + \cdots .
\] (17)
There is an analogous formula for the transition operator, \(n(k, l)\), that obeys
\[
[n(k, l), a^\dagger(m)]_- = \delta(l, m)a^\dagger(k).
\] (18)

5 Quons

5.1 The quon algebra

The quon algebra\(^{20, 21, 22}\) is the best attempt so far to violate statistics by a small amount. The infinite statistics algebra just discussed is the average of the bose and fermi algebras. The quon algebra can be obtained as the convex sum of these two algebras,
\[
\frac{1 + q}{2} [a(k), a^\dagger(l)]_- + \frac{1 - q}{2} [a(k), a^\dagger(l)]_+ = \delta(k, l),
\] (19)
or
\[
a(k)a^\dagger(l) - qa^\dagger(l)a(k) = \delta(k, l).
\] (20)
As usual the Fock-like representation is selected by the vacuum condition
\[
a(k)|0\rangle = 0.
\] (21)
Convexity requires \(0 \leq q \leq 1\); for this range the states have positive squared norms. Outside this range the squared norms become negative. Using the algebra (20) and the vacuum condition (21) all vacuum matrix elements of polynomials in the \(a\)'s and \(a^\dagger\)'s can be calculated; for example,
\[
(a^\dagger(k_1)a^\dagger(k_2)|0\rangle, a^\dagger(l_1)a^\dagger(l_2)|0\rangle) =
\]
\[
\delta(k_1, l_1)\delta(k_2, l_2) + q\delta(k_1, l_2)\delta(k_2, l_1) =
\]
\[
\frac{1 + q}{2} [\delta((k_1, l_1)\delta(k_2, l_2) + \delta(k_1, l_2)\delta(k_2, l_1)] + \frac{1 - q}{2} [\delta((k_1, l_1)\delta(k_2, l_2) - \delta(k_1, l_2)\delta(k_2, l_1)].
\] (22)
The first proof of the positivity of the norms was given by D. Zagier, who gave a tour-de-force calculation of the determinant of the $n! \times n!$ matrix of scalar products for arbitrary $n$, 

$$detM^n_{P,Q}(q) = \prod_{k=1}^{n-1} \left( 1 - q^{k(k+1)} \right)^{n-k)}.$$

As shown above, at $q = 0$ the norms are positive and the determinant is one. In order for a norm to become negative the determinant has to change sign. From Zagier’s formula this happens only when $q^{k(k+1)} = 1$, i.e., on the unit circle. This proves that the norms remain positive between negative one and one.

Speicher gave an ingenious proof of the positivity of the norms using an ansatz for the Fock-like representation of quons analogous to Green’s ansatz for parastatistics. Speicher represented the quon annihilation operator as the weak operator limit, 

$$a_k = \lim_{N \to \infty} N^{-1/2} \sum_{\alpha=1}^{N} b_{k}^{(\alpha)}$$

where the $b_{k}^{(\alpha)}$ are bose oscillators for each $\alpha$, but with relative commutation relations given by 

$$b_{k}^{(\alpha)} b_{l}^{(\beta)\dagger} = s^{(\alpha,\beta)} b_{l}^{(\beta)\dagger} b_{k}^{(\alpha)}, \alpha \neq \beta, \text{ where } s^{(\alpha,\beta)} = \pm 1.$$

This limit is taken as the limit, $N \to \infty$, in the vacuum expectation state of the Fock space representation of the $b_{k}^{(\alpha)}$. In this respect Speicher’s ansatz differs from Green’s, which is an operator identity. To get the Fock-like representation of the quon algebra, Speicher chose a probabilistic condition for the signs $s^{(\alpha,\beta)}$,

$$\text{prob}(s^{(\alpha,\beta)} = 1) = (1 + q)/2,$$

$$\text{prob}(s^{(\alpha,\beta)} = -1) = (1 - q)/2.$$

Speicher’s rules reproduce the quon algebra. The norms are positive since the sums of bose or fermi operators have positive norms. The constraint on $q$ follows because the probabilities have to lie between zero and one.

The number and transition operators for general $q$ have infinite degree expansions analogous to but more complicated than those for the $q = 0$ case, 

$$n(k,l) = a^{\dagger}(k)a(l) + \frac{1}{1-q^2} \sum_{\tau} \left( a^{\dagger}(t)a^{\dagger}(k) - qa^{\dagger}(k)a^{\dagger}(t) \right)(a(l)a(t) - qa(t)a(l)) + \cdots.$$
The general formula for the number operator was given by S. Stanciu[23].

At $q = \pm 1$ only the symmetric (antisymmetric) representation of $S_n$ occurs. The quon operators interpolate smoothly between fermi and bose statistics in the sense that as $q$ departs from $\pm 1$ the vectors formed by polynomials in the creation operators, which are superpositions of vectors in different irreducible representations of the symmetric group, have higher weights in the more symmetric (antisymmetric) representations, and as $q \to \mp 1$ the antisymmetric (symmetric) representations smoothly become more heavily weighted.

5.2 Observables in quon theory

It is important to note that although there are $n!$ linearly independent vectors in Fock space associated with a degree $n$ monomial in creation operators that carry disjoint quantum numbers acting on the vacuum, there are fewer than $n!$ observables associated with such vectors. For example, for two identical quons 1 and 2 in orthogonal quantum states the two vectors $a^\dagger(1)a^\dagger(2)|0\rangle$ and $a^\dagger(2)a^\dagger(2)|0\rangle$ are orthogonal and each is normalized to one. Let

$$|\phi_{s,a}\rangle = N_{s,a}(a^\dagger(1)a^\dagger(2) \pm a^\dagger(2)a^\dagger(1))|0\rangle$$

be normalized states that are symmetric or antisymmetric under transposition of 1 and 2. The quon algebra gives

$$N_{s,a} = \frac{1}{\sqrt{2(1 \pm q)}}.$$  \hspace{1cm} (30)

One can then calculate the expansion

$$a^\dagger(1)a^\dagger(2)|0\rangle = \alpha|\phi_s\rangle + \beta|\phi_a\rangle,$$  \hspace{1cm} (31)

either using

$$a^\dagger(1)a^\dagger(2)|0\rangle = (1/2)[(a^\dagger(1)a^\dagger(2) + a^\dagger(2)a^\dagger(1)) + (a^\dagger(1)a^\dagger(2) - a^\dagger(2)a^\dagger(1))]|0\rangle.$$  \hspace{1cm} (32)

or using

$$a^\dagger(1)a^\dagger(2)|0\rangle = \langle \phi_s|a_1^\dagger a_2^\dagger|0\rangle|\phi_s\rangle + \langle \phi_a|a_1^\dagger a_2^\dagger|0\rangle|\phi_a\rangle.$$  \hspace{1cm} (33)

Either way gives

$$\alpha = \sqrt{(1 + q)/2}, \beta = \sqrt{(1 - q)/2}$$  \hspace{1cm} (34)

so that

$$a_1^\dagger a_2^\dagger|0\rangle = \sqrt{\frac{1+q}{2}}|\phi_s\rangle + \sqrt{\frac{1-q}{2}}|\phi_a\rangle.$$  \hspace{1cm} (35)
and

$$a_2^\dagger a_1^\dagger |0\rangle = \sqrt{\frac{1+q}{2}} \phi_s - \sqrt{\frac{1-q}{2}} \phi_a. \quad (36)$$

Then, dropping the cross terms that are excluded by the superselection rule separating symmetric and antisymmetric states of identical particles (and, indeed, states of identical particles in different representations of the symmetric group generally),

$$a_1^\dagger a_2^\dagger |0\rangle \langle 0| a_2 a_1 = a_2^\dagger a_1^\dagger |0\rangle \langle 0| a_1 a_2 = \frac{1+q}{2} |\phi_s\rangle \langle \phi_s| + \frac{1-q}{2} |\phi_a\rangle \langle \phi_a|. \quad (37)$$

This shows that since particles 1 and 2 are identical the same observable results follow when the labels 1 and 2 are transposed. That also shows that the relative phase in Eq.(35) and Eq.(36) is not observable. Equation (37) states that the density matrices for $a_1^\dagger a_2^\dagger |0\rangle$ and $a_2^\dagger a_1^\dagger |0\rangle$ are identical, which means that these two “states” correspond to exactly the same physical situation. We put quotation marks around the word “states” to indicate that these should really be represented by density matrices. Note that the sum of the coefficients of the two terms in the two-particle density matrix is one, as it should be. The general observable is a linear combination of projectors on the irreducibles of the symmetric group.

The parameters $v_F$ and $v_B$ that represent small violations of statistics can be written in terms of the $q$ parameters; the result is,

$$q_F = 2v_F - 1 \text{ or } v_F = \frac{1}{2}(1 + q_F); \quad q_B = 1 - 2v_B \text{ or } v_B = \frac{1}{2}(1 - q_B). \quad (38)$$

5.3 Properties of quon theory

Surprisingly several properties of relativistic theories that I would expect to fail for the quon theory (made relativistic kinematically) actually hold. These include Wick’s theorem, cluster decomposition theorems and the $CPT$ theorem. We are familiar with Wick’s theorem for bosons which states that the vacuum matrix element of a product of free fields is the sum of all possible products of two-point functions, with each product occurring with factor one. For fermions the factors are plus or minus one, depending on the parity of the permutation between the order in the vacuum matrix element and the order in the product of two point functions. In Wick’s theorem for quons the corresponding factors are $q$ raised to the inversion number of the permutation between the order in the vacuum matrix element and the order in the product of two point functions. This result reduces to the usual Wick’s theorem when $q \to \pm 1$. The inversion number can be found conveniently by drawing lines above the vacuum matrix element to indicate the pairs that are contracted.
into two-point functions. The minimum number of crossings of these lines is the inversion number. Since both cluster decomposition and the CPT theorem for vacuum matrix elements of free fields depend on the properties of two-point functions, these theorems hold for quon fields. Note that quon fields, which clearly violate the spin-statistics theorem, obey the CPT theorem; this emphasises the point made by R. Jost\cite{26} that the CPT theorem requires only very weak assumptions.

If all the usual properties of relativistic field theory hold, then the spin-statistics theorem holds; thus some property must fail for quons. The property that does not hold is locality in the sense of the commutativity of observables at spacelike separation. Jost\cite{27} showed that if locality holds in an open spacelike region then analyticity arguments prove that it holds everywhere outside the lightcone. This result does not hold if the violation of locality decreases—say—exponentially away from the lightcone. The experimental bounds on such a violation are not clear. Note that the nonrelativistic form of locality

\[ [\rho(x), \psi^\dagger(y)]_\pm = \delta(x - y)\psi^\dagger(y), \]  

(39)

where \(\rho\) is the charge density, does hold.

### 5.4 Conservation of statistics rules for quon theory

For the energies of systems that are widely (spacelike) separated to be additive, all terms in the Hamiltonian must be effective bose operators in the sense that

\[ [H(x), \phi(y)]_\to 0, \text{ as } x - y \to \infty \text{ spacelike} \]  

(40)

for all fields \(\phi\). This condition imposes “conservation of statistics” rules, the simplest of which is that only an even number of fermi fields can appear in any term of the Hamiltonian. For parafields, which have local observables, Messiah and I\cite{11} showed that parafields must occur in even degree, except that, for \(p\) odd, \(p\) parafields can occur. I defined paragrassmann and quongrassmann numbers which must be used in coupling para and quon operators to external sources and showed that for external parasources there are analogous restrictions\cite{28}. Using these results, R.C. Hilborn and I\cite{29} gave an heuristic argument relating the \(q\) parameter for electrons to that for photons. The result,

\[ q_e^2 = q_\gamma, \]  

(41)

allows the very accurate bound from the Ramberg-Snow experiment to be carried over to photons with comparable accuracy. Similar arguments work for any particles that are coupled to electrons through any chain of reactions.
5.5 Bound states of quons

The classical result about bound states of bosons and fermions due to E.P. Wigner and to P. Ehrenfest and J.R. Oppenheimer states that a bound state of bosons and fermions is a boson unless it has an odd number of fermions, in which case it is a fermion. Hilborn and I showed that this result generalizes for quons. A bound state of \( n \) identical quons with parameter \( q_{\text{constituent}} \) has parameter \( q_{\text{bound}} = q_{\text{constituent}}^{n^2} \). This implies that if \( q_{\text{nucleus}} \) is bounded within \( \epsilon \) of \( \pm 1 \) for a nucleus with \( A \) nucleons, then \( q_{\text{nucleon}} \) is bounded within \( \epsilon/A^2 \). Thus the bound on the nucleons is stronger than the bound on the nucleus. Analogously the bound on quarks is improved by \( 1/9 \) over the bound on nucleons. Michael Berry pointed out that in the context of the quon theory this result on \( q \) for bound states implies that either the layers of compositeness stop or all particles are bosons or fermions.

This result reduces to the usual one for bosons and fermions since \( q \) and \( q^2 \) are even or odd together.

6 Summary and open questions

Like any other physical property, statistics should be subjected to high-precision experimental tests. In order to interpret such tests we need a theory in which statistics can be violated and a parameter that gives a quantitative measure of the validity of statistics. A theory that allows violations of statistics cannot have all the properties we might like. So far quons are the best theory that allows small violations.

In summary the positive properties of quons as a field theory are (a) norms are positive, (b) a simple modification of Wick’s theorem holds, (c) cluster decomposition theorems hold, (d) the \( CPT \) theorem holds, and (e) free fields can have relativistic kinematics. The negative properties are (a) spacelike commutativity of observables fails, and because of this (b) interacting relativistic field theory is in doubt. I do not have a concrete suggestion for the possible origin of small violations of the exclusion principle. One could turn this issue around and observe that the constraints of bose and fermi statistics are grafted onto the general structure of quantum theory in an ad hoc way and ask why these constraints are realized in nature. Study of the situation in which these constraints are violated may shed light on why they hold for the known particles. In any case a fundamental issue such as statistics should be subjected to experimental tests and to theoretical study, just as is being done for Lorentz and \( CPT \) invariance.

What we lack is an “external” motivation for violation of statistics—that is a connection of violations with some other physical property. We also don’t have any insight into the level at which we can expect violations if they do occur. Possible
external motivations for violation of statistics include (a) violation of $CPT$, (b) violation of locality, (c) violation of Lorentz invariance, (d) extra space dimensions, (e) discrete space and/or time and (f) noncommutative spacetime. Of these, (a) seems unlikely because the quon theory which obeys $CPT$ allows violations, (b) seems likely because if locality is satisfied we can prove the spin-statistics connection and there will be no violations, (c), (d), (e) and (f) seem possible.

At the conference the question was raised whether the stability of matter which depends on the exclusion principle might set stringent bounds on possible violations of statistics. This certainly deserves careful study.

Hopefully either violations will be found experimentally or our theoretical efforts will lead to understanding of why only bose and fermi statistics occur in Nature.

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