AN INVARIANT DETECTING RATIONAL SINGULARITIES VIA THE LOG CANONICAL THRESHOLD

RAF CLUCKERS AND MIRCEA MUSTAȚĂ

Abstract. We show that if \( f \) is a nonzero, noninvertible function on a smooth complex variety \( X \) and \( J_f \) is the Jacobian ideal of \( f \), then \( \operatorname{lct}(f, J_f^2) > 1 \) if and only if the hypersurface defined by \( f \) has rational singularities. Moreover, if this is not the case, then \( \operatorname{lct}(f, J_f^2) = \operatorname{lct}(f) \). We give two proofs, one relying on arc spaces and one that goes through the inequality \( \tilde{\alpha}_f \geq \operatorname{lct}(f, J_f^2) \), where \( \tilde{\alpha}_f \) is the minimal exponent of \( f \). In the case of a polynomial over \( \mathbb{Q} \), we also prove an analogue of this latter inequality, with \( \tilde{\alpha}_f \) replaced by the motivic oscillation index \( \operatorname{moi}(f) \).

1. Introduction

Given a smooth complex algebraic variety \( X \) and a (nonempty) hypersurface \( H \) in \( X \) defined by \( f \in \mathcal{O}_X(X) \), the log canonical threshold \( \operatorname{lct}(f) \) measures how far the pair \( (X, H) \) is from having log canonical singularities. In particular, we always have \( \operatorname{lct}(f) \leq 1 \), with equality if and only if the pair \( (X, H) \) is log canonical. The log canonical threshold \( \operatorname{lct}(\mathfrak{a}) \) can be defined more generally for every nonzero coherent ideal \( \mathfrak{a} \) of \( \mathcal{O}_X \), with the convention that \( \operatorname{lct}(\mathcal{O}_X) = \infty \). For an introduction to singularities of pairs in our setting, we refer the reader to [Laz04, Chapter 9].

The main point of this note is that one can use the log canonical threshold of an ideal associated to \( f \) in order to refine \( \operatorname{lct}(f) \), so that we detect when the hypersurface \( H \) has rational singularities. Namely, we consider the Jacobian ideal \( J_f \) of \( f \) and the log canonical threshold \( \operatorname{lct}(f, J_f^2) \) of the ideal \( (f) + J_f^2 \). We show that \( \operatorname{lct}(f, J_f^2) > 1 \) if and only if the hypersurface \( H \) has rational singularities. More precisely, we have the following:

Theorem 1.1. For every smooth, complex algebraic variety \( X \), and every nonzero, noninvertible \( f \in \mathcal{O}_X(X) \) defining the hypersurface \( H \) in \( X \), the following hold:

i) If \( H \) does not have rational singularities, then

\[ \operatorname{lct}(f, J_f^2) = \operatorname{lct}(f). \]

In particular, we have \( \operatorname{lct}(f, J_f^2) \leq 1 \).

ii) If \( H \) has rational singularities, then \( \operatorname{lct}(f, J_f^2) > 1 \).

Another invariant that refines \( \operatorname{lct}(f) \) and detects whether \( H \) has rational singularities is the minimal exponent \( \tilde{\alpha}_f \), see [Sai93]. When \( H \) has isolated singularities, this has been also known as the complex singularity index of \( H \). In general, it is defined as the negative of the largest root of \( b_f(s)/(s+1) \), where \( b_f(s) \) is the Bernstein-Sato polynomial of \( f \) (with the

2010 Mathematics Subject Classification. 14B05, 14E18, 14J17, 11L07.

R.C. was partially supported by the European Research Council under the European Community’s Seventh Framework Programme (FP7/2007-2013) with ERC Grant Agreement nr. 615722 MOTMELSUM, by the Labex CEMPI (ANR-11-LABX-0007-01), and by KU Leuven IF C14/17/083. M.M. was partially supported by NSF grant DMS-1701622.
convention that if \( f \) defines a smooth hypersurface, in which case \( b_f(s) = s+1 \), then \( \tilde{\alpha}_f = \infty \).

It is a result of Lichtin and Kollár (see [Kol97, Theorem 1.6]) that \( \text{let}(f) = \min\{\tilde{\alpha}_f, 1\} \) and it was shown by Saito (see [Sai93, Theorem 0.4]) that \( \tilde{\alpha}_f > 1 \) if and only if \( H \) has rational singularities. We thus see that \( \tilde{\alpha}_f \) behaves like \( \text{let}(f, J_f^2) \). We prove the following general inequality between these two invariants:

**Theorem 1.2.** For every smooth, complex algebraic variety \( X \), and every nonzero, noninvertible \( f \in \mathcal{O}_X(X) \), we have

\[
\tilde{\alpha}_f \geq \text{let}(f, J_f^2).
\]

We prove Theorem 1.2 by using results on minimal exponents from [MP18] and a theorem of Varchenko saying that in a family of isolated singularities with constant Milnor number, the minimal exponent is constant. Regarding Theorem 1.1, we note that the interesting assertion is the one in i), as the one in ii) follows easily from known properties of rational singularities. Part i) is a consequence of the result in Theorem 1.2; however, we give a second proof using arc spaces. This has the advantage that it also gives the assertion below concerning divisorial valuations. A **divisorial valuation** is a valuation of the form \( v = q \cdot \text{ord}_E \), where \( E \) is a divisor on a normal variety that has a birational morphism to \( X \) and \( q \) is a positive integer; we denote by \( A_X(v) \) the log discrepancy of \( v \) and by \( c_X(v) \) the center of \( v \) on \( X \) (for definitions, see Section 2).

**Theorem 1.3.** Let \( X \) be a smooth, affine, complex algebraic variety, and \( f \in \mathcal{O}_X(X) \) a nonzero function. If \( v \) is a divisorial valuation on \( X \) such that

\[
0 < v(J_f) < \frac{1}{2} v(f),
\]

then there is a divisorial valuation \( w \) on \( X \) that satisfies the following conditions:

i) \( w(g) \leq v(g) \) for every \( g \in \mathcal{O}_X(X) \),
ii) \( w(f) \geq v(f) - 1 \),
iii) \( A_X(w) \leq A_X(v) - 1 \), and
iv) \( c_X(w) = c_X(v) \).

As a consequence of this theorem, we obtain the following result concerning multiplier ideals, which in turn immediately implies Theorem 1.1:

**Corollary 1.4.** For every smooth, complex algebraic variety \( X \), and every nonzero, noninvertible \( f \in \mathcal{O}_X(X) \), we have

\[
\mathcal{J}(X, f^\lambda) = \mathcal{J}((f, J_f^2)^\lambda) \quad \text{for all} \quad \lambda < 1.
\]

Moreover, if \( f \) defines a reduced hypersurface, then

\[
\text{adj}(f) = \mathcal{J}(f, J_f^2),
\]

where \( \text{adj}(f) \) is the adjoint ideal of \( f \).

There is another invariant that behaves like \( \tilde{\alpha}_f \) and \( \text{let}(f, J_f^2) \), namely the **motivic oscillation index** \( \text{moi}(f) \) studied in [CMN18]. This is defined for polynomials \( f \in \overline{\mathbb{Q}}[x_1, \ldots, x_n] \) and it was shown in [CMN18, Proposition 3.10] that \( \text{let}(f) = \min\{\text{moi}(f), 1\} \) and \( \text{moi}(f) > 1 \) if and only if the hypersurface defined by \( f \) in \( \mathbb{A}^n_{\overline{\mathbb{Q}}} \) has rational singularities.

In fact, a more refined version \( \text{moi}_Z(f) \) of the motivic oscillation index also involves a closed subscheme \( Z \) of \( \mathbb{A}^n_{\overline{\mathbb{Q}}} \) (the one we referred to in the previous paragraph corresponds to the case when \( Z \) is the hypersurface defined by \( f \)). We recall the precise definition of \( \text{moi}_Z(f) \) in
Section 5. We only mention now that if \( f \in \mathbb{Z}[x_1, \ldots, x_n] \) and \( Z = \mathbb{A}_\mathbb{Q}^n \), then \( \text{moi}_Z(f) \) relates to finite exponential sums over integers modulo \( p^m \) (for primes \( p \) and integers \( m > 0 \)) of the form
\[
E(p^m) := \frac{1}{p^m} \sum_{x \in (\mathbb{Z}/p^m\mathbb{Z})^n} \exp \left( 2\pi i \frac{f(x)}{p^m} \right)
\]
and to certain limit values of all possible \( \sigma \geq 0 \) such that
\[
|E(p^m)| \ll p^{-m\sigma},
\]
with an implicit constant independent from \( m \). These limits are taken carefully, using in fact finite field extensions and large primes \( p \), as will be described in Section 5. We have the following inequality between \( \text{moi}(f) \) and \( \text{lct}(f, J_f^2) \):

**Theorem 1.5.** For every nonconstant \( f \in \overline{\mathbb{Q}}[x_1, \ldots, x_n] \), we have
\[
(1) \quad \text{moi}(f) \geq \text{lct}(f, J_f^2).
\]
Moreover, if \( \text{lct}(f, J_f^2) \leq 1 \), then we have equality in (1).

Note that by Theorem 1.1, we have \( \text{lct}(f, J_f^2) \leq 1 \) if and only if the hypersurface defined by \( f \) does not have rational singularities. In fact, one can give a third proof of part i) of Theorem 1.1 by combining Theorem 1.5 and the mentioned assertions in [CMN18, Proposition 3.10]; we leave the details to the reader.

We will prove a more general version of the above theorem, allowing for a subset \( Z \) of the zero-locus of \( f \) (see Theorem 5.1 below). A related intriguing question is whether we always have \( \tilde{\alpha}_f = \text{moi}(f) \). However, investigating this seems to require new ideas.

The paper is organized as follows. In Section 2, we prove Theorem 1.2 and deduce Theorem 1.1. In Section 3, after reviewing some basic facts about the connection between valuations and contact loci in arc spaces, we prove Theorem 1.3, deduce Corollary 1.4, and obtain a second proof of Theorem 1.1. In Section 4, we give two examples. We show that for generic determinantal hypersurfaces, the inequality in Theorem 1.2 is an equality, and we describe when this inequality is strict in the case of homogeneous diagonal hypersurfaces. Finally, in Section 5 we recall the definition of the motivic oscillation index and prove the general version of Theorem 1.5.

1.1. Acknowledgments. We would like to thank Mattias Jonsson, Johannes Nicaise and Mihai Păun for useful discussions and to Nero Budur for his comments on an earlier version of this paper.

2. The Inequality between \( \tilde{\alpha}_f \) and \( \text{lct}(f, J_f^2) \)

In what follows \( X \) is a smooth (irreducible) \( n \)-dimensional, complex algebraic variety. For basic facts about log canonical thresholds and multiplier ideals we refer to [Laz04, Chapter 9].

Let us begin by recalling some terminology and notation regarding valuations that will be used both in this section and the next one.

A *divisorial valuation* on \( X \) is a valuation of the function field \( k(X) \) of \( X \) of the form \( q \cdot \text{ord}_E \), where \( q \) is a positive integer and \( E \) is a prime divisor on a normal variety \( Y \) that has a birational morphism \( g: Y \to X \) (here \( \text{ord}_E \) is the discrete valuation associated to \( E \), with corresponding DVR \( \mathcal{O}_{Y,E} \), having fraction field \( k(Y) = k(X) \)). After replacing \( Y \) by a suitable log resolution of \((Y, E)\), we may always assume that \( Y \) is smooth and \( E \) is a smooth
prime divisor on \( Y \). The center of \( q \cdot \text{ord}_E \) on \( X \) is the closure of \( g(E) \) (which is independent of the model \( Y \)). The log discrepancy of \( q \cdot \text{ord}_E \) is the positive integer
\[
A_X(q \cdot \text{ord}_E) = q \cdot (\text{ord}_E(K_{Y/X}) + 1),
\]
where \( K_{Y/X} \) is the effective divisor on \( Y \) locally defined by the determinant of the Jacobian matrix of \( g \).

We first give a proposition concerning the minimal exponent of a general linear combination of the generators of an ideal.

**Proposition 2.1.** If \( f_1, \ldots, f_r \in \mathcal{O}_X(X) \) generate the proper nonzero coherent ideal \( a \) of \( \mathcal{O}_X \) and \( f = \sum_{i=1}^r \lambda_i f_i \), with \( \lambda_1, \ldots, \lambda_r \in \mathbb{C} \) general, then we have
\[
\bar{\alpha}_f \geq \lct(a).
\]

**Proof.** If the zero-locus \( Z \) of \( a \) has codimension 1 in \( X \), then \( \lct(a) \leq 1 \) and in this case we have
\[
\bar{\alpha}_f \geq \lct(f) = \lct(a).
\]
The equality follows from the fact that for every \( t \in (0, 1] \), we have the equality of multiplier ideals
\[
\mathcal{J}(f^t) = \mathcal{J}(a^t)
\]
(see [Laz04, Proposition 9.2.26]). We thus may and will assume that \( \text{codim}_X(Z) \geq 2 \).

The argument then proceeds as in *loc. cit.* Let \( \pi: Y \to X \) be a log resolution of \((X, a)\) that is an isomorphism over \( X \setminus Z \). By construction, if we put \( a \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E) \), then \( E \) is a simple normal crossing divisor such that if we write \( E = \sum_{i=1}^N a_i E_i \), then every \( E_i \) is a \( \pi \)-exceptional divisor. Since \( \lambda_1, \ldots, \lambda_r \) are general, it follows that if \( D \) is the divisor defined by \( f \), then \( \pi^*(D) = F + E \), where \( F \) is a smooth divisor, with no common components with \( E \), and having simple normal crossings with \( E \). In particular, \( D \) is a reduced divisor and \( \pi \) is a log resolution of \((X, D)\) such that the strict transform of \( D \) is smooth. We thus deduce using [MP18, Corollary D] that if \( K_{Y/X} = \sum_{i=1}^N k_i E_i \), then
\[
\bar{\alpha}_f \geq \min_{i=1}^N \frac{k_i + 1}{a_i} = \lct(a).
\]

\( \Box \)

In what follows we will also make use of a local version of the minimal exponent. Recall that if \( f \in \mathcal{O}_X(X) \) is nonzero and \( P \in X \) is such that \( f(P) = 0 \), then \( \max\{\bar{\alpha}_{f|_U} \mid U \ni P\} \), where \( U \) varies over the open neighborhoods of \( P \), is achieved for all small enough \( U \). This maximum is denoted \( \bar{\alpha}_{f,P} \) and we have \( \bar{\alpha}_{f,P} = \min\{\bar{\alpha}_{f,P} \mid P \in X\} \).

**Remark 2.2.** With the same notation as in Proposition 2.1, for every \( P \in X \), if \( \lambda_1, \ldots, \lambda_r \in \mathbb{C} \) are general, then we have
\[
\bar{\alpha}_{f,P} \geq \lct_P(a).
\]
Indeed, if we choose an open neighborhood \( U \) of \( P \) such that \( \lct_P(a) = \lct(a|_U) \), then by applying the proposition for \( a|_U \), we obtain
\[
\bar{\alpha}_{f,P} \geq \bar{\alpha}(f|_U) \geq \lct(a|_U) = \lct_P(a).
\]

Given \( f \in \mathcal{O}_X(X) \), we denote by \( J_f \) the Jacobian ideal of \( f \). Recall that if \( U \) is an open subset of \( X \) such that \( x_1, \ldots, x_n \) are algebraic local coordinates on \( U \) (that is, \( dx_1, \ldots, dx_n \) give a trivialization of \( \Omega_U \)), then \( J_f \) is generated on \( U \) by \( f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \) (the fact that this only depends on the ideal generated by \( f \), but not on the particular generator of the ideal or
on the system of coordinates is well-known and straightforward to check). Note that $P \in X$ is an isolated point in the zero-locus of $J_f$ if and only if $f$ has an isolated singular point at $P$.

In this case, we also consider the \textit{Milnor number} $\mu_P(f)$: if $x_1, \ldots, x_n$ is an algebraic system of coordinates centered at $P$, then $\mu_P(f) = \ell(O_{X,P}/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n))$. Note that while the ideal $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ might depend on the system of coordinates, its colength does not.

If $f$ has an isolated singularity at $P \in X$, then Steenbrink [Ste77] and Varchenko [Var81] defined a mixed Hodge structure on the vanishing cohomology of $f$ at $P$, and using this, together with the monodromy action, one defines the \textit{spectrum} $\text{Sp}_P(f)$, which is a set of rational numbers, with multiplicities, in the interval $(-1, n - 1)$. The sum of the numbers in the spectrum, counted with multiplicities, is equal to the Milnor number $\mu_P(f)$. It was shown by Malgrange [Mal75] that

$$\tilde{\alpha}_{f,P} = 1 + \min\{\beta \mid \beta \in \text{Sp}_P(f)\}.$$ 

We will make use of the following result of Varchenko [Var82] about the behavior of the spectrum in families. Suppose that we have a smooth morphism $\pi: Y \to T$, a section $s: T \to Y$ of $\pi$, and $g \in \mathcal{O}_Y(Y)$ such that for every $t \in T$, the restriction $g_t$ of $g$ to the fiber $Y_t = \pi^{-1}(t)$ is nonzero and $g(s(t)) = 0$. If $T$ is connected, and $g_t$ has an isolated singularity at $s(t)$, with $\mu_{s(t)}(g_t)$ independent of $t \in T$, then the spectrum of $g_t$ at $s(t)$ is independent of $t$; in particular, $\tilde{\alpha}_{g_t,s(t)}$ is independent of $t$.

We can now prove the inequality between $\tilde{\alpha}_f$ and $\text{lct}(f, J_f^2)$:

\textbf{Proof of Theorem 1.2.} Since

$$\tilde{\alpha}_f = \min_{P \in Z} \tilde{\alpha}_{f,P} \quad \text{and} \quad \text{lct}((f) + J_f^2) = \min_{P \in Z} \text{lct}_P((f) + J_f^2),$$

where $Z$ is the hypersurface defined by $f$, it follows that in order to prove the inequality in the theorem, it is enough to show that for every $P \in Z$, we have

$$\tilde{\alpha}_{f,P} \geq \text{lct}_P((f) + J_f^2). \quad (2)$$

We may and will assume that $f$ has a singular point at $P$, since otherwise, by convention, both sides of (2) are infinite. After replacing $X$ by a suitable affine open neighborhood of $P$, we may assume that $X$ is affine and we have an algebraic system of coordinates $x_1, \ldots, x_n$ centered at $P$.

We first show that it is enough to treat the case when $f$ has an isolated singular point at $P$. Given any $N \geq 2$, let $f_N = f + \sum_{i=1}^n a_{N,i} x_i^N$, with $a_{N,1}, \ldots, a_{N,n} \in \mathbb{C}$ general. Since the zero locus of the linear system generated by $f, x_1^N, \ldots, x_n^N$ consists just of $P$, it follows from Kleiman’s version of Bertini’s theorem that $f_N$ has an isolated singular point at $P$. On one hand, we have by [MP18, Proposition 6.7]

$$|\tilde{\alpha}_{f_N,P} - \tilde{\alpha}_{f,P}| \leq \frac{n}{N}.$$ 

On the other hand, the ideals $(f) + J_f^2$ and $(f_N) + J_{f_N}^2$ are equal mod $(x_1, \ldots, x_n)^{N-1}$, hence

$$|\text{lct}_P((f) + J_f^2) - \text{lct}_P((f_N) + J_{f_N}^2)| \leq \frac{n}{N - 1}$$

(see, for example, [Mus12, Property 1.21]). We thus deduce that if we know the inequality (2) for each $f_N$, by letting $N$ go to infinity, we obtain the same inequality for $f$. From now on, we assume that $f$ has an isolated singular point at $P$. 

For every $g \in \mathcal{O}_X(X)$, we put
$$J'_g = \left( \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \right).$$

Given $\lambda = (\lambda_{i,j})_{1 \leq i,j \leq n} \in \mathbb{C}^{n^2}$, we consider
$$g_\lambda := f + \sum_{i,j=1}^n \lambda_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}.$$

Since $P$ is a singular point of $f$, we have $g_\lambda(P) = 0$. Note that $g_0 = f$ and for every $\lambda \in \mathbb{C}^{n^2}$ and every $i$, we have
$$\frac{\partial g_\lambda}{\partial x_i} - \frac{\partial f}{\partial x_i} \in J'_f \quad \text{for } 1 \leq i \leq n.$$

In particular, we have
$$J'_{g_\lambda} \subseteq J'_f \quad \text{for all } \lambda \in \mathbb{C}^{n^2},$$
and thus
$$\ell(\mathcal{O}_{X,P}/J'_f) \leq \ell(\mathcal{O}_{X,P}/J'_{g_\lambda}) \quad \text{for all } \lambda \in \mathbb{C}^{n^2}.$$

By the semicontinuity theorem for fiber dimensions, the set
$$U := \{ \lambda \in \mathbb{C}^{n^2} | \ell(\mathcal{O}_{X,P}/J'_{g_\lambda}) < \infty \}$$
is open in $\mathbb{C}^{n^2}$. Moreover, by the upper semicontinuity of the Milnor number, for every nonnegative integer $m$, the set
$$U_m := \{ \lambda \in U | \ell(\mathcal{O}_{X,P}/J'_{g_\lambda}) \leq m \}$$is open in $U$. Let $m_0 = \ell(\mathcal{O}_{X,P}/J'_f) = \mu_P(f)$, so that $0 \in U_{m_0}$. It follows from (3) that $\mu_P(g_\lambda) = \mu_P(f)$ for every $\lambda \in U_{m_0}$, hence by Varchenko’s result, we have
$$\bar{\alpha}_{g_\lambda,P} = \bar{\alpha}_f,P \quad \text{for all } \lambda \in U_{m_0}.$$

On the other hand, it follows from Remark 2.2 that for $\lambda \in U_{m_0}$ general, we have
$$\bar{\alpha}_{g_\lambda,P} \geq \lct_P \left( (f) + J^2_f \right) = \lct_P \left( (f) + J^2_f \right).$$

This completes the proof of the theorem. \qed

We now deduce the fact that $\lct(f, J^2_f)$ detects rational singularities.

**Proof of Theorem 1.1.** It follows from [Sai93, Theorem 0.4] that the hypersurface defined by $f$ does not have rational singularities if and only if $\bar{\alpha}_f \leq 1$, in which case we have $\bar{\alpha}_f = \lct(f)$. We deduce from Theorem 1.2 that in this case $\lct(f) \geq \lct \left( (f) + J^2_f \right)$, while the reverse inequality simply follows from the inclusion $(f) \subseteq (f) + J^2_f$. This proves i).

The assertion in ii) is straightforward: suppose that $H$ has rational singularities and let $\pi: Y \to X$ be a log resolution of $(X, H)$ that is at the same time a log resolution of the ideal $(f) + J^2_f$. Note first that if $E$ is a prime divisor on $Y$ such that $\operatorname{ord}_E(f, J^2_f) > 0$, then $\operatorname{ord}_E(J^2_f) > 0$, and thus $E$ is a $\pi$-exceptional divisor (since $H$ has rational singularities, it is in particular reduced, hence $\operatorname{ord}_D(J^2_f) = 0$ for every irreducible component $D$ of $H$). Furthermore, $H$ has rational singularities if and only if it has canonical singularities by a result of Elkik (see [Kol97, Theorem 11.1]); moreover, this is the case if and only if the pair
(X, H) has canonical singularities by a result of Stevens (see [Kol97, Theorem 7.9]). Since (X, H) has canonical singularities and E is exceptional, we have

\[ A_X(\text{ord}_E) \geq \text{ord}_E(f) + 1 \geq \text{ord}_E(f, J_f^2) + 1. \]

This holds for all prime divisors E on Y for which \( \text{ord}_E(f, J_f^2) > 0 \), hence we conclude that \( \text{let}(f, J_f^2) > 1 \).

\[ \square \]

3. An approach to \( \text{let}(f, J_f^2) \) via arcs

In this section we use the approach to valuations via arcs to prove Theorem 1.3, which we apply to deduce Corollary 1.4 and give another proof of Theorem 1.1. We keep the assumption that X is a smooth (irreducible) complex algebraic variety, of dimension \( n \).

We first review briefly the definition of jet schemes and the arc scheme. For details, see for example [EM09]. For every \( m \geq 0 \), the \( m \)th jet scheme \( X_m \) of X is a scheme over X with the property that for every \( \mathbb{C} \)-algebra \( A \), we have a functorial bijection

\[ \text{Hom}([\text{Spec} A, X_m]) \simeq \text{Hom}([\text{Spec} A[t]/(t^{m+1}), X]). \]

In particular, the points of \( X_m \) are in canonical bijection with the \( m \)-jets on X, that is, maps \( \text{Spec} \mathbb{C}[t]/(t^{m+1}) \to X \). Given such an \( m \)-jet \( \gamma : \text{Spec} \mathbb{C}[t]/(t^{m+1}) \to X \), we denote by \( \gamma(0) \) the image of the closed point and by \( \gamma^* \) the induced ring homomorphism \( O_{X, \gamma(0)} \to \mathbb{C}[t]/(t^{m+1}) \).

Truncation induces morphisms \( X_m \to X_p \) whenever \( p < m \) and these satisfy the obvious compatibilities. Note that we have a canonical isomorphism \( X_0 \simeq X \) and we denote by \( \pi_m \) the truncation morphism \( X_m \to X \). With this notation, we have \( \pi_m(\gamma) = \gamma(0) \).

All truncation morphisms are affine, hence we may consider the projective limit \( X_\infty \) of the system \( (X_m)_{m \geq 1} \). This is the space of arcs (or arc scheme) of X. Its \( \mathbb{C} \)-valued points are in canonical bijection with maps \( \text{Spec} \mathbb{C}[t] \to X \). In what follows, we identify \( X_\infty \) with the corresponding set of \( \mathbb{C} \)-valued points. For an arc \( \gamma \), we use the notation \( \gamma(0) \) and \( \gamma^* \) as above. Note that the space of arcs \( X_\infty \) comes endowed with truncation maps \( \psi_m : X_\infty \to X_m \) compatible with the truncation morphisms between jet schemes.

Since \( X \) is smooth and \( n \)-dimensional, every morphism \( X_m \to X \) is locally trivial in the Zariski topology, with fiber \( \mathbb{A}^m \). In fact, if \( x_1, \ldots, x_n \) are algebraic coordinates on an open subset \( U \) of X, then we have an isomorphism

\[ \pi_m^{-1}(U) \simeq U \times (\mathbb{A}^m)^n, \]

which maps \( \gamma \) to \( (\gamma(0), \gamma^*(x_1), \ldots, \gamma^*(x_n)) \) (note that each \( \gamma^*(x_i) \) lies in \( t\mathbb{C}[t]/t^{m+1}\mathbb{C}[t] \simeq \mathbb{C}^m \)). In particular, every \( X_m \) is a smooth, irreducible variety, of dimension \( (m+1)n \). Moreover, using the above isomorphisms we see that each truncation morphism \( X_m \to X_p \), with \( p < m \), is locally trivial, with fiber \( \mathbb{A}^{(m-p)n} \).

We next turn to the connection between divisorial valuations and certain subsets in the space of arcs. A cylinder in \( X_\infty \) is a subset of the form \( C = \psi_m^{-1}(S) \), where \( S \) is a constructible subset of \( X_m \). In this case \( C \) is irreducible, closed, open, or locally closed (with respect to the Zariski topology on \( X_\infty \)) if and only if \( S \) has this property. In particular, we have irreducible decomposition for locally closed cylinders. The codimension of a cylinder \( C = \psi_m^{-1}(S) \) is defined as

\[ \text{codim}(C) = \text{codim}_{X_m}(S). \]

It is clear that this is independent of the way we write \( C \) as the inverse image of a constructible set.
An important example of cylinders is provided by contact loci. Given a coherent ideal \( \mathfrak{a} \) in \( \mathcal{O}_X \), we put \( \text{ord}_\gamma(\mathfrak{a}) \in \mathbb{Z}_{\geq 0} \cup \{ \infty \} \) to be that \( m \) such that \( \gamma^*(\mathfrak{a})(t^m) = (t^m) \), with the convention that \( \text{ord}_\gamma(\mathfrak{a}) = \infty \) if \( \gamma^*(\mathfrak{a}) = 0 \). The set \( \text{Cont}^{\geq m}(\mathfrak{a}) \) of \( X_\infty \) consisting of all arcs \( \gamma \) with \( \text{ord}_\gamma(\mathfrak{a}) \geq m \) is a closed cylinder in \( X_\infty \). We similarly define the locally closed cylinder \( \text{Cont}^m(\mathfrak{a}) \).

It turns out that one can use cylinders in \( X_\infty \) in order to describe divisorial valuations on \( X \). We simply state the results and refer for details and proofs to [ELM04]. We assume, for simplicity, that \( X \) is affine, though everything extends to the general case in a straightforward way. For every closed, irreducible cylinder \( C \subseteq X_\infty \), that does not dominate \( X \) and every \( h \in \mathcal{O}_X(X) \) nonzero, we put

\[
\text{ord}_C(h) := \min\{\text{ord}_\gamma(h) \mid \gamma \in C\} \in \mathbb{Z}_{\geq 0}.
\]

This extends to a valuation of the function field of \( X \); in fact, this is a divisorial valuation, whose center is the closure of \( \psi_0(C) \).

Conversely, if \( v = q \cdot \text{ord}_E \) for some smooth prime divisor \( E \) on the smooth variety \( Y \), with a birational morphism \( g: Y \to X \), then we have an induced morphism \( g_\infty: Y_\infty \to X_\infty \) and if \( C(v) \) is the closure of \( g_\infty(\text{Cont}^{\geq q}(\mathcal{O}_Y(-E))) \), then \( C(v) \) is an irreducible closed cylinder in \( X_\infty \) and \( \text{ord}_{C(v)} = v \). Moreover, a key fact is that

\[
\text{codim}(C(v)) = A_X(v).
\]

In addition, for every closed, irreducible cylinder \( C \subseteq X_\infty \) that does not dominate \( X \), if \( v = \text{ord}_C \), then \( C \subseteq C(v) \). In particular, we have \( \text{codim}(C) \geq A_X(v) \).

After this overview, we can prove our general result about valuations.

**Proof of Theorem 1.3.** Let \( C = C(v) \), so that \( \text{ord}_C = v \) and \( \text{codim}(C) = A_X(v) \). We put \( Z = c_X(v) \), so that \( Z \) is the closure of \( \psi_0(C) \). If \( m = v(f) \) and \( e = v(J_f) \), we have by hypothesis \( 0 < e < \frac{1}{2}m \). Since \( m = \text{ord}_C(f) \) and \( e = \text{ord}_C(J_f) \), we have \( C \subseteq \text{Cont}^{\geq m}(f) \cap \text{Cont}^{\leq e}(J_f) \).

Let \( C_0 := C \cap \text{Cont}^e(J_f) \), which is a nonempty subcylinder of \( C \), open in \( C \). Since \( C \) is irreducible, we have \( C = \overline{C_0} \). We also consider the locally closed cylinder

\[
C := \text{Cont}^{\geq (m-1)}(f) \cap \text{Cont}^e(J_f) \cap \psi_0^{-1}(Z).
\]

It is clear that \( C_0 \) is a closed subset of \( C' \). We make the following

**Claim.** \( C_0 \) is not an irreducible component of \( C' \).

Assuming the claim, let \( W \) be an irreducible component of \( C' \) that contains \( C_0 \) and \( \overline{W} \) its closure in \( X_\infty \). Note that \( \overline{W} \) is an irreducible, closed cylinder in \( X_\infty \) such that \( \psi_0(\overline{W}) \subseteq Z \).

Since we also have

\[
Z \subseteq \psi_0(C_0) \subseteq \psi_0(\overline{W}),
\]

we conclude that \( \psi_0(\overline{W}) = Z \). Therefore \( w := \text{ord}_{\overline{W}} \) is a divisorial valuation on \( X \), with center \( Z \).

Since \( W \subseteq \text{Cont}^{\geq m-1}(f) \), it follows that \( w(f) \geq m-1 \). Furthermore, since \( C_0 \subseteq W \), we have \( C = \overline{C_0} \subseteq \overline{W} \), hence we clearly have

\[
w(g) = \text{ord}_{\overline{W}}(g) \geq \text{ord}_C(g) = v(g) \quad \text{for all} \quad g \in \mathcal{O}_X(X).
\]

Finally, since \( C_0 \) is a proper closed subset of \( W \), we have

\[
A_X(w) \leq \text{codim}(\overline{W}) \leq \text{codim}(\overline{C_0}) - 1 = A_X(v) - 1,
\]

hence \( w \) satisfies all the required conditions. Therefore it is enough to prove the claim.

Note that our assumptions imply that \( m - e - 2 \geq e - 1 \geq 0 \). In order to simplify the notation, we put \( \psi = \psi_{m-e-2}: X_\infty \to X_{m-e-2} \). The key point is to describe, for every
$\tau \in \psi(C')$, the cylinders $\psi^{-1}(\tau) \cap C_0 \subseteq \psi^{-1}(\tau) \cap C'$. This is a local computation, based on Taylor’s formula, that goes back to the proof of [DL99, Lemma 3.4]. We choose an algebraic system of coordinates $x_1, \ldots, x_n$ in a neighborhood of $P = \pi_{m-e-2}(\tau)$, centered at $P$. This gives an isomorphism $\mathcal{O}_{X,P} \simeq \mathbb{C}[y_1, \ldots, y_n]$ that maps each $x_i$ to $y_i$, and let $\varphi \in \mathbb{C}[y_1, \ldots, y_n]$ be the formal power series corresponding to $\gamma$. We also write $t_0$ for the constant term of $\varphi$.

Let $\gamma \in C'$ be an arc with $\psi(\gamma) = \tau$, which corresponds to $u = (u_1, \ldots, u_n) \in (t\mathbb{C}[t])^n$, so that $\gamma^*(f) = \varphi(u_1, \ldots, u_n)$. Any other arc $\delta \in \psi^{-1}(\tau)$ corresponds to $u + v$, for some $v = (v_1, \ldots, v_n) \in (t_{m-e-1}\mathbb{C}[t])^n$. It follows from the Taylor expansion of $\varphi$ that we have

$$\delta^*(f) = \varphi(u + v) = \gamma^*(f) + \sum_{i=1}^{n} \gamma^* \left( \frac{\partial f}{\partial x_i} \right) v_i + \sum_{i,j=1}^{n} \gamma^* \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) v_i v_j + \text{higher order terms.}$$

By assumption, we have $\text{ord}_t \gamma^*(f) \geq m - 1$ and

$$\min_{i=1}^{n} \text{ord}_t \gamma^* \left( \frac{\partial f}{\partial x_i} \right) = e$$

(note that $m - 1 > e$). Since $\text{ord}_t v_i \geq m - e - 1$, an immediate computation using (5) shows that for every such $\delta$, we have $\text{ord}_t \delta^*(f) \geq m - 1$. Since $m - e - 1 \geq e$, we also see that $\text{ord}_t \delta^*(J_f) \geq e$, hence $\psi^{-1}(\tau) \cap C'$ is the open subset of $\psi^{-1}(\tau)$ defined by having contact order with $J_f$ precisely $e$; in particular, this is an irreducible cylinder.

For every series $g \in \mathbb{C}[t]$, let us write $g_0$ for the constant term of $g$. We also write $v_{i,0}$ for the constant term of $v_{i} / t_{m-e-1}$. It follows from (5) that the coefficient of $t_{m-1}$ in $\delta^*(f)$ is equal to

$$\left( \frac{\gamma^*(f)}{t_{m-1}} \right)_0 + \sum_{i=1}^{n} \left( \gamma^*(\partial f / \partial x_i) / t_{e} \right)_0 v_{i,0} + \sum_{i,j=1}^{n} \left( \gamma^*(\partial^2 f / \partial x_i \partial x_j) / t_{e} \right)_0 v_{i,0} v_{j,0},$$

and the last term only appears if $m = 2e + 1$. In any case, since some $\left( \gamma^*(\partial f / \partial x_i) / t_{e} \right)_0$ is nonzero, the vanishing of this coefficient defines a hypersurface in the affine space $\mathbb{A}^n$ parametrizing $(v_{1,0}, \ldots, v_{n,0})$. Since $C_0 \subseteq \text{Cont}^{\geq m}(f)$, it follows that $\psi^{-1}(\tau) \cap C_0$ is different from $\psi^{-1}(\tau) \cap C'$. The fact that $C_0$ is not an irreducible component of $C'$ follows by taking the image in some $X_Q$, with $q \gg 0$ (so that $C_0$ is the inverse image of a locally closed subset of $X_Q$), and applying the following easy lemma.

**Lemma 3.1.** Let $g: U \to V$ be a morphism of algebraic varieties and $W$ a closed subset of $U$ such that for every $y \in g(U)$, the following two conditions hold:

i) The fiber $g^{-1}(y)$ is irreducible, and

ii) The intersection $g^{-1}(y) \cap W$ is different from $g^{-1}(y)$.

Then $W$ is not an irreducible component of $U$.

**Proof.** Arguing by contradiction, suppose that the irreducible components of $U$ are $U_1 = W, U_2, \ldots, U_r$. If $x \in W \setminus \bigcup_{i \geq 2} U_i$ and $y = g(x)$, then $g^{-1}(y) \subseteq U_i$ for any $i \geq 2$. On the other hand, $g^{-1}(y)$ is irreducible and contained in $\bigcup_{i \geq 1} (U_i \cap g^{-1}(y))$, hence $g^{-1}(y) \subseteq U_1$. This contradicts condition ii).

We can now deduce the assertion about multiplier ideals:
Proof of Corollary 1.4. Since \((f) \subseteq (f) + J_2^2\), we clearly have the inclusion
\[ \mathcal{J}(f^\lambda) \subseteq \mathcal{J}((f, J_2^2)^\lambda) \]
for all \(\lambda > 0\). We now suppose that \(\lambda < 1\) and prove the reverse inclusion.

We may and will assume that \(X\) is affine. Arguing by contradiction, suppose that we have \(g \in \mathcal{J}((f, J_2^2)^\lambda)\) such that \(g \not\in \mathcal{J}(f^\lambda)\). By definition of multiplier ideals, the latter condition implies that there is a valuation \(v = \text{ord}_E\), where \(E\) is a prime divisor on a log resolution of \((X, f)\) such that
\begin{equation}
(6) \quad v(g) + A_X(v) \leq \lambda \cdot v(f).
\end{equation}
We choose such \(E\) with the property that \(A_X(v)\) is minimal.

If \(v(J_f) \geq \frac{1}{2}v(f)\), then \(v(f) = v(f, J_2^2)\), and (6) contradicts the fact that \(g \in \mathcal{J}((f, J_2^2)^\lambda)\). Hence we may and will assume that \(v(J_f) < \frac{1}{2}v(f)\). If \(v(J_f) = 0\), then there is an open subset \(U\) of \(X\) that intersects the center of \(v\) on \(X\) such that \(f|_U\) defines a smooth hypersurface. In this case, since \(\lambda < 1\), we have \(\mathcal{J}(f|_U^\lambda) = \mathcal{O}_U\). On the other hand, since \(U\) intersects the center of \(v\), it follows from (6) that \(g|_U \not\in \mathcal{J}(f|_U^\lambda)\), a contradiction.

We thus may and will assume that
\[ 0 < v(J_f) < \frac{1}{2}v(f). \]
In this case, it follows from Theorem 1.3 that there is a divisorial valuation \(w\) on \(X\) that satisfies properties i), ii), and iii) in the theorem. We thus have
\[ A_X(w) = A_X(v) - 1 \leq \lambda \cdot v(f) - 1 - v(g) \leq \lambda \cdot (w(f) + 1) - 1 - w(g), \]
where the first inequality follows from condition iii) and the third inequality follows from conditions i) and ii). Since
\[ \lambda \cdot (w(f) + 1) - 1 - w(g) = \lambda \cdot w(f) - w(g) + \lambda - 1 \leq \lambda \cdot w(f) - w(g), \]
if we write \(w = q \cdot \text{ord}_F\), for some prime divisor \(F\) over \(X\), after dividing the above inequalities by \(q\), we obtain
\[ A_X(\text{ord}_F) \leq \lambda \cdot \text{ord}_F(f) - \text{ord}_F(g) \]
and
\[ A_X(\text{ord}_F) \leq A_X(w) \leq A_X(v) - 1, \]
contradicting the minimality in the choice of \(E\). The contradiction we obtained shows that we have in fact
\[ \mathcal{J}((f, J_2^2)^\lambda) \subseteq \mathcal{J}(f^\lambda), \]
completing the proof of the first assertion in the corollary.

The proof of the second assertion is similar. We may and will assume that \(X\) is affine. Recall that the adjoint ideal \(\text{adj}(f)\) consists of all \(g \in \mathcal{O}_X(X)\) with the property that for every exceptional divisor \(E\) over \(X\), we have
\[ \text{ord}_E(g) > \text{ord}_E(f) - A_X(\text{ord}_E) \]
(see [Laz04, Chapter 9.3.48]). The inclusion
\[ \text{adj}(f) \subseteq \mathcal{J}(f, J_2^2), \]
is easy: if \(g \in \text{adj}(f)\), then for every divisorial valuation \(\text{ord}_E\) on \(X\), we have
\begin{equation}
(7) \quad \text{ord}_E(g) > \text{ord}_E(f, J_2^2) - A_X(\text{ord}_E).
\end{equation}
If $E$ is exceptional, this follows from the definition of the adjoint ideal and the fact that $\text{ord}_E(f) \geq \text{ord}_E(f, J_f^2)$. On the other hand, if $E$ is a divisor on $X$, then $A_X(\text{ord}_E) = 1$ and $\text{ord}_E(f, J_f^2) = 0$ (the latter equality follows from the fact that if $f$ vanishes on $E$, since $f$ defines a reduced hypersurface, we have $\text{ord}_E(J_f) = 0$). Since (7) holds for every $E$, we conclude that $g \in \mathcal{J}(f, J_f^2)$.

We now turn to the interesting inclusion

$$\mathcal{J}(f, J_f^2) \subseteq \text{adj}(f).$$

Suppose that $g \in \mathcal{J}(f, J_f^2)$, but $g \notin \text{adj}(f)$. The latter condition implies that there is an exceptional divisor $E$ over $X$ such that

$$\text{ord}_E(g) \leq \text{ord}_E(f) - A_X(\text{ord}_E).$$

We choose such $E$ with $A_X(\text{ord}_E)$ minimal and argue as in the proof of the first part.

If $\text{ord}_E(J_f) \geq \frac{1}{2} \text{ord}_E(f)$, then $\text{ord}_E(f) = \text{ord}_E(f, J_f^2)$, and (8) contradicts the fact that $g \in \mathcal{J}(f, J_f^2)$. Hence we may and will assume that $\text{ord}_E(J_f) < \frac{1}{2} \text{ord}_E(f)$. If $\text{ord}_E(J_f) = 0$, then there is an open subset $U$ of $X$ that intersects the center of $\text{ord}_E$ on $X$ and such that $f|_U$ defines a smooth hypersurface. In particular, we have $\text{adj}(f)|_U = \mathcal{O}_U$ by [Laz04, Proposition 9.3.48], contradicting (8).

We thus may and will assume that

$$0 < \text{ord}_E(J_f) < \text{ord}_E(f).$$

We then apply Theorem 1.3 for $v = \text{ord}_E$ to find a divisorial valuation $w = q \cdot \text{ord}_F$ on $X$ that satisfies properties i)-iv) in the theorem. We obtain

$$w(g) \leq \text{ord}_E(g) \leq \text{ord}_E(f) - A_X(\text{ord}_E) \leq w(f) - A_X(w).$$

Dividing by $q$, we obtain

$$\text{ord}_F(g) \leq \text{ord}_F(f) - A_X(\text{ord}_F).$$

Since $A_X(\text{ord}_F) \leq A_X(w) \leq A_X(v) - 1$ and $F$ is an exceptional divisor (we use here the fact that $c_X(\text{ord}_F) = c_X(\text{ord}_E)$), this contradicts the minimality in our choice of $E$. This completes the proof of the corollary.

Corollary 1.4 easily implies the assertions in Theorem 1.1.

Second proof of Theorem 1.1. Recall that for a proper nonzero ideal $a$, we have

$$\text{lct}(a) = \min\{\lambda > 0 \mid \mathcal{J}(a^\lambda) \neq \mathcal{O}_X\}.$$ 

Since $\text{lct}(f) \leq 1$, the first assertion in Corollary 1.4 implies that in general, we have

$$\text{lct}(f) = \min\{\text{lct}(f, J_f^2), 1\}.$$ 

Note that if $\text{lct}(f) = 1$, then automatically $H$ is reduced. We thus deduce from the second assertion in Corollary 1.4 that $\text{lct}(f, J_f^2) > 1$ if and only if $H$ is reduced and $\text{adj}(f) = \mathcal{O}_X$. By [Laz04, 9.3.48], this holds if and only if $H$ has rational singularities.
4. Two examples

In this section we discuss two examples. We begin with the case of the generic determinantal hypersurface, for which we show that the inequality in Theorem 1.2 is an equality.

Example 4.1. Let $n \geq 2$ and $X = \mathbb{A}^{n^2}$ be the affine space of $n \times n$ matrices, with coordinates $x_{i,j}$, for $1 \leq i, j \leq n$. We consider $f = \det(A)$, where $A$ is the matrix $(x_{i,j})_{1 \leq i,j \leq n}$. In this case, it is well-known that the Bernstein-Sato polynomial of $f$ is given by

$$b_f(s) = \prod_{i=1}^{n} (s + i)$$

(see, for example, [Kim03, Appendix]). We thus have $\tilde{\alpha}_f = 2$.

In order to compute $\text{lct}(f, J_f^2)$, we use the arc-theoretic description reviewed in the previous section, together with the approach in the determinantal case due to Docampo [Doc13]. Note first that $J_f$ is the ideal of $\mathcal{O}_X(X)$ generated by the $(n-1)$-minors of the matrix $A$. We use the action of $G = \text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})$ on $X$ given by $(g, h) \cdot f = gAh^{-1}$. We have an induced action of $G_{\infty} = \text{GL}_n(\mathbb{C}[t]) \times \text{GL}_n(\mathbb{C}[t])$ on $X_{\infty}$ and we consider the orbits with respect to this action. Since the ideal $(f) + J_f^2$ is preserved by the $G$-action, it follows that every contact locus of this ideal is a union of $G_{\infty}$-orbits.

Recall that a $G_{\infty}$-orbit in $X_{\infty}$ of finite codimension corresponds to a sequence of integers $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$, such that the orbit consists of those $n \times n$ matrices with entries in $\mathbb{C}[t]$ that are equivalent via Gaussian elimination to a diagonal matrix having on the diagonal $t^{\lambda_1}, \ldots, t^{\lambda_n}$. The codimension of this orbit is $\sum_{i=1}^{n} \lambda_i(2i - 1)$ by [Doc13, Theorem C]. Moreover, it is clear that the order of an arc in this orbit along the ideal $(f) + J_f^2$ is

$$\min \left\{ \sum_{i=1}^{n} \lambda_i, 2 \cdot \sum_{i=2}^{n} \lambda_i \right\}.$$

The description of the log canonical threshold in terms of contact loci (see [ELM04]) thus implies

$$(9) \quad \text{lct}(f, J_f^2) = \min_{\lambda} \frac{\sum_{i=1}^{n} \lambda_i(2i - 1)}{\min \left\{ \sum_{i=1}^{n} \lambda_i, 2 \cdot \sum_{i=2}^{n} \lambda_i \right\}},$$

where the minimum is over all $\lambda = (\lambda_1, \ldots, \lambda_n)$, with $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ and $\lambda_2 > 0$. If we take $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \ldots = \lambda_n = 0$, then the expression on the right-hand side of (9) is 2. In order to see that for all $\lambda$ this expression is $\geq 2$, note that since $\lambda_1 \geq \lambda_2$, we have

$$\sum_{i=1}^{n} (2i - 1) \lambda_i \geq 4 \cdot \sum_{i=2}^{n} \lambda_i.$$

We thus conclude that $\text{lct}(f, J_f^2) = 2 = \tilde{\alpha}_f$.

We next turn to the case of homogeneous diagonal hypersurfaces, where we will see that the inequality in Theorem 1.2 can be strict.

Example 4.2. Let $X = \mathbb{A}^n$, with $n \geq 2$, and $f = t_1^d + \ldots + t_n^d$, for some $d \geq 2$. In this case, it is known that the Bernstein-Sato polynomial of $f$ is given by

$$b_f(s) = (s + 1) \cdot \prod_{1 \leq b_1, \ldots, b_n \leq d - 1} \left( s + \sum_{i=1}^{n} \frac{b_i}{d} \right)$$

(see [Yan78, Proposition 3.6]). In particular, we have $\tilde{\alpha}_f = \frac{4}{d}$. 
We will show that

\[ \text{lct}(f, J_f^2) = \min \left\{ \frac{n + d - 2}{2d - 2}, \frac{n}{d} \right\}. \tag{10} \]

It is clear that we have \( J_f = (x_1^{d-1}, \ldots, x_n^{d-1}) \). Let \( \pi: Y \to X \) be the blow-up of \( X \) at the origin. By symmetry, it is enough to consider the chart \( U \) on \( Y \) with coordinates \( y_1, \ldots, y_n \) such that \( x_1 = y_1 \) and \( x_i = y_1 y_i \) for \( i \geq 2 \). In this chart we have \( f \circ \pi|_U = y_1^d g \), with \( g = 1 + y_2^d + \ldots + y_n^d \), and \( J_f^2 \cdot \mathcal{O}_U = (y_1^{2d-2}) \). We deduce that a log resolution of the ideal \((f + J_f^2) \cdot \mathcal{O}_U\) can be obtained by blowing-up \( U \) along the ideal \((g, y_1^{d-2})\) and then resolving torically the resulting variety. In particular, we see that in order to compute \( \text{lct}(f, J_f^2) \) it is enough to consider toric divisors over \( Y \), with respect to the system of coordinates given by \( y_1 \) and \( g \). If \( E \) is such a divisor, with \( \text{ord}_E(g) = a \) and \( \text{ord}_E(y_1) = b \), with \( a \) and \( b \) nonnegative integers, not both 0, we have

\[ \text{ord}_E((f) + J_f^2) = \min\{db + a, (2d - 2)b\} \quad \text{and} \quad A_X(\text{ord}_E) = nb + a. \]

This implies that

\[ \text{lct}(f, J_f^2) = \min_{(a,b)} \frac{nb + a}{\min\{db + a, (2d - 2)b\}}, \]

where the minimum is over all \((a, b) \in \mathbb{Z}_{\geq 0}^2 \setminus \{(0, 0)\} \). It is now a straightforward exercise to deduce the formula (10).

If \( d = 2 \), then we see that \( \text{lct}(f, J_f^2) = \frac{n}{2} = \bar{\alpha}_f \) (of course, this can be seen directly, since in this case \( (f) + J_f^2 = (x_1, \ldots, x_n)^2 \)). Suppose now that \( d \geq 3 \). In this case, a straightforward calculation shows that

\[ \frac{n + d - 2}{2d - 2} < \frac{n}{d} \quad \text{if and only if} \quad d < n. \]

We thus see that if \( d \geq n \) (which is precisely the case when the hypersurface defined by \( f \) does not have rational singularities), then \( \text{lct}(f, J_f^2) = \frac{n}{d} = \text{lct}(f) \), as expected according to Theorem 1.1. On the other hand, if \( 3 \leq d < n \), then we have a strict inequality \( \text{lct}(f, J_f^2) < \bar{\alpha}_f \).

5. The Motivic Oscillation Index and \( \text{lct}(f, J_f^2) \)

We begin by recalling the definition of the motivic oscillation index from [CMN18]. Let \( \overline{\mathbb{Q}} \) be the algebraic closure of \( \mathbb{Q} \) inside \( \mathbb{C} \). We consider a nonconstant polynomial \( f \) in \( \mathbb{Q}[x_1, \ldots, x_n] \) and a closed subscheme \( Z \) of \( \mathbb{A}^n_{\overline{\mathbb{Q}}} \).

Let \( K \) be a number field such that \( f \) and \( Z \) are defined over \( K \). Choose an integer \( N > 0 \) such that \( f \) and \( Z \) lie in \( \mathcal{O}[1/N] \), where \( \mathcal{O} \) is the ring of integers of \( K \). For any prime \( p \) not dividing \( N \), any completion \( L \) of \( K \) above \( p \), and any nontrivial additive character \( \psi: L \to \mathbb{C}^\times \), consider the following integral

\[ E_{f,L,\psi}^Z := \int_{\{x \in \mathcal{O}_L^n \mid \psi(z(k_L)) \}} |\psi(f(x))| \, dx, \tag{11} \]

where \( \mathcal{O}_L \) stands for the valuation ring of \( L \) with residue field \( k_L \), \( z \) stands for the image of \( x \) under the natural projection \( \mathcal{O}_L^n \to k_L^n \), and \(|dx|\) is the Haar measure on \( L^n \) normalized so that \( \mathcal{O}_L^n \) has measure 1. Writing \( q_L \) for the number of elements of \( k_L \), let \( \sigma_L \) be the supremum over all \( \sigma \geq 0 \) such that

\[ |E_{f,L,\psi}^Z| \leq c q_L^{-m \sigma} \]
for some $c = c(\sigma, f, L)$ which is independent of $\psi$, and where $m$ is such that $\psi$ is trivial on $\mathcal{M}_L^m$ and nontrivial on $\mathcal{M}_L^{m-1}$, with $\mathcal{M}_L$ being the maximal ideal of $\mathcal{O}_L$. Note that $\sigma_L$ can equal $+\infty$; this is the case precisely when the morphism $\mathbb{A}^n_{\overline{Q}} \to \mathbb{A}^1_{\overline{Q}}$ defined by $f$ is smooth in an open neighborhood of $Z$. We define the $K$-oscillation index of $f$ along $Z$ as

$$Koi_Z(f) := \lim_{M \to \infty} \inf_{L} \sigma_L,$$

where the infimum is taken over all non-Archimedean completions $L$ of $K$ above primes $p_L$ with $p_L > M$. Finally, the motivic oscillation index of $f$ along $Z$ is defined as

$$moi_Z(f) := \inf_{K} Koi_Z(f),$$

where $K$ runs over all number fields satisfying the above conditions. This definition corresponds to the definition given in [CMN18, Section 3.4] by Igusa’s work (see [Den91]), which relates upper bounds for oscillating integrals with nontrivial poles of local zeta functions. Note that the variant moi($f$) we considered in the Introduction corresponds to the case when $Z$ is the hypersurface defined by $f$ (note the small change in notation for moi($f$), compared to [CMN18]).

The following is the main result of this section.

**Theorem 5.1.** If $f \in \overline{Q}[x_1, \ldots, x_n]$ is a nonconstant polynomial and $Z$ is a closed subscheme of the hypersurface defined by $f$, then

$$moi_Z(f) \geq lct_Z(f, J^2_J).$$

In addition, if $lct_Z(f, J^2_J) \leq 1$, then equality holds in (12); in this case, we also have $Koi_Z(f) = lct_Z(f, J^2_J)$ for every number field $K$ such that $f$ and $Z$ are defined over $K$.

We recall that for every nonzero ideal $a$ in $\overline{Q}[x_1, \ldots, x_n]$, by definition

$$lct_Z(a) = \max_U lct(a|U),$$

where the maximum is over all open neighborhoods $U$ of $Z$. We also note that if $a_{\mathbb{C}}$ is the extension of $a$ to $\mathbb{C}[x_1, \ldots, x_n]$, then $lct(a) = lct(a_{\mathbb{C}})$. We will derive Theorem 5.1 from the following proposition and the definition of the oscillation index.

**Proposition 5.2.** Let $K$ be any number field such that $f$ and $Z$ are defined over $K$ and let $L$ be a non-Archimedean completion of $K$ above a prime $p_L$, with residue field $k_L$ with $q_L$ elements. If $p_L$ is large enough (in terms of the data $f, Z, K$), then for every $\varepsilon > 0$, there exists a constant $c = c(f, Z, L, \varepsilon)$ such that for each nontrivial additive character $\psi: L \to \mathbb{C}^\times$ we have

$$|E^Z_{f, L, \psi}| < cq_L^{-m\sigma} \quad \text{for} \quad \sigma = lct_Z(f, J^2_J) - \varepsilon,$$

where $m = m(\psi)$ is such that $\psi$ is trivial on $\mathcal{M}_L^m$ and nontrivial on $\mathcal{M}_L^{m-1}$.

**Proof of Proposition 5.2.** By Igusa’s results from [Igu75] and [Den91], recalled in [CMN18, Propositions 3.1 and 3.4], we have for each $\psi$ with $m = m(\psi) > 1$

$$E^Z_{f, L, \psi} = \int_{\{x \in \mathcal{O}_L^n | \psi \in Z(k_L), \ ord_f(x) \geq m-1\}} \psi(f(x)) |dx|,$$

where $\mathcal{O}_L$ stands for the image of $x$ under the natural projection $\mathcal{O}_L^n \to k_L^n$, and where $p_L$ is assumed to be large. We now show that we also have

$$E^Z_{f, L, \psi} = \int_{\{x \in \mathcal{O}_L^n | \psi \in Z(k_L), \ ord_f(x) \cdot J^2_J(x) \geq m-1\}} \psi(f(x)) |dx|,$$
by adapting the proof of [CH07, Proposition 2.1]. Here, the condition ord \((f(x), J^2(x)) \geq m - 1\) for \(x \in O_L^n\) means that we have ord \((g(x)) \geq m - 1\) for every polynomial \(g\) in the ideal of the polynomial ring over \(O_L\) generated by \(f\) and \(J^2\). We use the orthogonality of characters in the form

\[
\int_{z \in M^{n-1}_L} \psi(z)|dz| = 0
\]

for \(m = m(\psi)\) and combine this with the Taylor expansion, as follows. Let \(x_0\) be a point in \(O_L^n\) such that ord \(f(x_0) \geq m - 1\) and suppose that ord \((J^2(x_0)) < m - 1\). In order to prove (14), it is enough to show that for every such \(x_0\), we have

\[
\int_{x \in x_0+(M_L)^n} \psi(f(x))|dx| = 0,
\]

with \(\overline{m}\) equal to \(m/2\) if \(m\) is even, and equal to \((m+1)/2\) if \(m\) is odd. If \(x = y + x_0\), with \(y = (y_1, \ldots, y_n) \in (M_L^n)^n\), then we write the Taylor expansion of \(f\) around \(x_0\):

\[
f(x) = f(y + x_0) = a_0 + \sum_{i=1}^{n} a_i y_i + \text{higher order terms in } y.
\]

Since \(p_L\) is assumed to be large and \(y \in (M_L^n)^n\), we see that

\[
f(x) \equiv a_0 + \sum_{i=1}^{n} a_i y_i \pmod{M_L^n}
\]

and thus, \(\psi\) is trivial on \(M_L^n\), we obtain

\[
\int_{x \in x_0+(M_L^n)^n} \psi(f(x))|dx| = \int_{y \in (M_L^n)^n} \psi\left(a_0 + \sum_{i=1}^{n} a_i y_i\right)|dy|.
\]

Note that the condition ord \((J^2(x_0)) < m - 1\) implies that min ord \(a_i < (m - 1)/2\). Since \((m - 1)/2 + \overline{m} \leq m\), using the orthogonality relation (15), we deduce (16) from (17). This completes the proof of (14).

By (14), using the fact that |\(\psi(x)\)| = 1 for all \(x\) in \(L\), we get

\[
|E_{f,L,\psi}^Z| \leq \text{vol} \left( \{ x \in O_L^n \mid x \in Z(k_L), \text{ ord}(f(x), J^2(x)) \geq m - 1 \} \right),
\]

where the volume is taken with respect to the Haar measure |\(dx\)| on \(L^n\). Therefore the existence of \(c\) as desired follows from Corollary 2.9 of [VZG08] and the two sentences following that corollary, which give a link between the log canonical threshold at \(Z\) of any ideal \(a\) of \(O_L[x]\) and the volume of \(\{ x \in O_L^n \mid x \in Z(k_L), \text{ ord}(a(x)) \geq m \}\) uniformly in \(m > 1\).

**Proof of Theorem 5.1.** The formula (12) follows from Proposition 5.2 and the definition of \(\text{moi}_Z(f)\). Moreover, if \(\text{lct}_Z(f, J^2) \leq 1\), then part ii) of Theorem 1.1 gives that the hypersurface defined by \(f\) does not have rational singularities in any open neighborhood of \(Z\). Under this last condition, it follows from [CMN18, Proposition 3.10] that \(\text{moi}_Z(f) = \text{lct}_Z(f)\), which together with (12) and \(\text{lct}_Z(f) \leq \text{lct}_Z(f, J^2)\) implies \(\text{moi}_Z(f) = \text{lct}_Z(f, J^2) = \text{lct}_Z(f)\). The fact that when \(\text{lct}_Z(f, J^2) \leq 1\) we also have \(\text{Koi}_Z(f) = \text{lct}_Z(f)\) follows by slightly adapting the proof of [CMN18, Proposition 3.10].
References

[CH07] R. Cluckers and A. Herremans, *The fundamental theorem of prehomogeneous vector spaces modulo \( \mathcal{P}^m \)*, Bull. Soc. Math. France **135** (2007), no. 4, 475–494. With an appendix by F. Sato. ↑

[CMN18] R. Cluckers, M. Mustatâ, and K. H. Nguyen, *Igusa’s conjecture for exponential sums: optimal estimates for non-rational singularities*, preprint arXiv:1810.11340 (2018). ↑, 2, 3, 13, 14, 15

[Den91] J. Denef, *Report on Igusa’s local zeta function*, Astérisque **201-203** (1991), Exp. No. 741, 359–386 (1992). Séminaire Bourbaki, Vol. 1990/91. ↑

[DL99] J. Denef and F. Loeser, *Germ of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), no. 1, 201–232. ↑

[Doc13] R. Docampo, *Arcs on determinantal varieties*, Trans. Amer. Math. Soc. **365** (2013), no. 5, 2241–2269. ↑

[ELM04] L. Ein, R. Lazarsfeld, and M. Mustatâ, *Contact loci in arc spaces*, Compos. Math. **140** (2004), no. 5, 1229–1244. ↑, 8, 12

[EM09] L. Ein and M. Mustatâ, *Jet schemes and singularities*, Algebraic geometry—Seattle 2005. Part 2, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 505–546. ↑

[Igu75] J.-i. Igusa, *Complex powers and asymptotic expansions. II. Asymptotic expansions*, J. Reine Angew. Math. **278/279** (1975), 307–321. ↑

[Kim03] T. Kimura, *Introduction to prehomogeneous vector spaces*, Translations of Mathematical Monographs, vol. 215, American Mathematical Society, Providence, RI, 2003. Translated from the 1998 Japanese original by Makoto Nagura and Tsuyoshi Niitani and revised by the author. ↑

[Kol97] J. Kollár, *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287. ↑, 6, 7

[Laz04] R. Lazarsfeld, *Positivity in algebraic geometry II*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 49, Springer-Verlag, Berlin, 2004. ↑, 3, 4, 10, 11

[Mal75] B. Malgrange, *Le polynôme de Bernstein d’une singularité isolée*, Fourier integral operators and partial differential equations (Colloq. Internat., Univ. Nice, Nice, 1974), Springer, Berlin, 1975, pp. 98–119. Lecture Notes in Math., Vol. 459. ↑

[Mus12] M. Mustatâ, *IMPA/NG lecture notes on log canonical thresholds*, Contributions to algebraic geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 407–442. Notes by Tomasz Szemberg. ↑

[MP18] M. Mustatâ and M. Popa, *Hodge ideals for \( \mathbb{Q} \)-divisors, V-filtration, and minimal exponent*, preprint arXiv:1807.01935 (2018). ↑, 2, 4, 5

[Sai93] M. Saito, *On b-function, spectrum and rational singularity*, Math. Ann. **295** (1993), no. 1, 51–74. ↑, 1, 2, 6

[Ste77] J. H. M. Steenbrink, *Mixed Hodge structure on the vanishing cohomology*, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 525–563. ↑

[Var81] A. N. Varchenko, *Asymptotic Hodge structure on vanishing cohomology*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 3, 540–591. ↑

[Var82] A. N. Varchenko, *The complex singularity index does not change along the stratum \( \mu = \text{const} \)*, Funktsional. Anal. i Prilozhen. **16** (1982), no. 1, 1–12, 96. ↑

[Yan78] T. Yano, *On the theory of b-functions*, Publ. Res. Inst. Math. Sci. **14** (1978), no. 1, 111–202. ↑

[VZG08] W. Veys and W. A. Zúñiga-Galindo, *Zeta functions for analytic mappings, log-principalization of ideals, and Newton polyhedra*, Trans. Amer. Math. Soc. **360** (2008), no. 4, 2205–2227. ↑

Université de Lille, Laboratoire Painlevé, CNRS - UMR 8524, Cité Scientifique, 59655 Villeneuve d’Ascq Cedex, France, and KU Leuven, Department of Mathematics, Celestijnenlaan 200B, B-3001 Leuven, Belgium

E-mail address: Raf.Cluckers@univ-lille.fr

URL: http://rcluckers.perso.math.cnrs.fr/

Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109, USA

E-mail address: mmustata@umich.edu