WERMER EXAMPLES AND CURRENTS

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ABSTRACT. We give the first examples of positive closed currents $T$ in $\mathbb{C}^2$ with continuous potentials, $T \wedge T = 0$ and whose supports do not contain any holomorphic disk. This gives in particular an affirmative answer to a question of Fornaess and Levenberg. We actually construct examples with potential of class $C^{1,\alpha}$ for all $\alpha < 1$. This regularity is expected to be essentially optimal.

INTRODUCTION

The purpose of this paper is to investigate the geometric properties of positive closed currents $T$ with $T \wedge T = 0$ in complex dimension two. Equivalently, we deal with plurisubharmonic functions $u$, solutions to the homogeneous Monge-Ampère equation $(dd^c u)^2 = 0$.

Let us start with some generalities about geometric currents. A $(1,1)$ positive closed current in $\mathbb{C}^2$ is uniformly laminar if it is an integral of currents of integration over the leaves of a foliation. A current is said to be laminar if it is uniformly laminar outside a set of arbitrary small trace measure. These concepts have been introduced in [BLS] and have proved to be very useful since then. If $T$ is uniformly laminar, then $T \wedge T = 0$ as soon as the wedge product makes sense [Du]. Likewise, any local plurisubharmonic (psh for short) potential for $T$ is harmonic along the leaves of the underlying foliation.

Back to our initial problem, we work locally in $\mathbb{C}^2$ so let us consider a psh potential $u$ for $T$. We assume that $u$ is, say, bounded, so that the self-intersection $T \wedge T = (dd^c u)^2$ is well defined, and vanishes. If $u$ is of class $C^3$, the Frobenius Integrability Theorem implies that there exists a foliation by holomorphic disks along which $u$ is harmonic (see [BK]), and $T$ is a uniformly laminar current associated to this foliation. It is expected, but apparently still unknown, that this result should carry over for $u$ of class up to $C^2$ (see [B]). As we shall demonstrate here, the situation is dramatically different for regularity below $C^2$. Before entering into the details of our results, let us mention that the laminarity properties of the solutions to homogeneous Monge-Ampère equations have recently played a prominent role in connection with the study of extremal metrics in Kähler geometry [Do, CT]. The natural regularity appearing in this setting is $C^{1,1}$.

It is a classical result due to Sibony that there exist examples of currents $T$ with $T \wedge T = 0$ and $C^{1,1}$ potential, which are not uniformly laminar (this construction was reported e.g. in [BF, FL]). Indeed let $B$ be the unit ball in $\mathbb{C}^2$ and $X \subset \partial B$ be a closed set with the property that the polynomial hull $\bar{X}$ does not contain any holomorphic disk (a so-called Stolzenberg or Wermer example, see below for more details). Let $f \in C^\infty(\partial B)$ be a nonnegative function
such that $X = \{ f = 0 \}$ and let $u$ be the unique psh function in $\mathbb{B}$, continuous in $\overline{\mathbb{B}}$ such that $u|_{\partial \mathbb{B}} = f$ and $(dd^c u)^2 = 0$ \cite{BT}. Let also $T = dd^c u$. Then $u$ is of class $C^{1,1}$, nonnegative, and $\{ u = 0 \} = \hat{X}$. Now if $p \in \hat{X}$ and $\Delta$ is any holomorphic disk through $p$, then $u$ cannot be harmonic along $\Delta$. Indeed, $\Delta$ is not contained in $\hat{X}$ so $u|_{\Delta}$ is not identically $0$, and $u$ has a minimum at $p$ so it is not harmonic. This both shows that $p \in \text{Supp}(T)$ and that $T$ cannot be uniformly laminar near $p$.

On the other hand it can be shown (see Proposition \ref{p:uniformly-laminar} below) that in this situation $\hat{X}$ has always zero trace measure (relative to $T$) so nothing prevents these currents from being uniformly laminar on an open set of full mass.

To the best of our knowledge, an example is still lacking of a non-laminar current $T$ with $T \wedge T = 0$, even with merely bounded potential. We fill this gap by proving the following result.

**Theorem 1.** There exists a closed positive $(1, 1)$ current $T = dd^c u$ in the unit bidisk $\mathbb{D}^2 \subset \mathbb{C}^2$ such that:

i. $u$ is of class $C^{1,\alpha}$ for all $0 < \alpha < 1$;

ii. $T \wedge T = (dd^c u)^2 = 0$;

iii. the support of $T$ does not contain any holomorphic disk.

Recall that a function is of class $C^{1,\alpha}$ if it is differentiable and its derivatives are Hölder continuous of exponent $\alpha$. Likewise, given any continuous increasing function $\psi$ with $\psi(0) = 0$, we say that $u$ is $C^{1+\psi}$ if its derivatives have modulus of continuity $O(\psi)$. Our method actually produces examples of potentials $u$ with regularity $C^{1+\psi}$, where $\psi$ is any modulus of continuity such that $\frac{\psi(\delta)}{\delta |\log \delta|} \to \infty$ — see Theorem \ref{t:regularity} for a precise statement. On the other hand it is a feature of our construction that these examples cannot be made $C^{1,1}$ (see §4.2).

Observe that item iii. of Theorem 1 cannot be true when $u$ is $C^2$, for $\text{Supp}(T)$ would have nonempty interior in this case. In \cite{B}, Bedford asks whether a foliation exists on a dense subset of $\text{Supp}(T)$ when $u$ is $C^{1,1}$.

By definition, a Wermer example is a subset in the unit bidisk $\mathbb{D}^2$ which is the polynomial hull of a compact subset of $\partial \mathbb{D} \times \mathbb{D}$, projects onto $\mathbb{D}$ under the first projection and contains no holomorphic disk. The existence of such objects is originally due to Wermer \cite{W} and they have subsequently been studied by several authors \cite{L, A, S, DS}.

It is an easy observation of \cite{DS} that there exist Wermer examples supporting positive closed currents, therefore currents whose support do not contain any holomorphic disk. This observation is the starting point of Theorem 1.

However, a source of difficulty is that the Wermer examples constructed thus far tend to be “small”, and it is delicate to estimate their size from below anyway. For instance, in \cite{FL}, the authors ask whether there exists a non pluripolar Wermer example. Of course, pluripolarity of the support of a current is an obstruction to the local boundedness of its potential.

What we do here is first to provide a construction of “thick” Wermer examples (Section \ref{section:thick}). Then we develop methods to get effective lower bounds on the size of such objects, in the pluripotential-theoretic sense (Sections \ref{section:lower-bound} and \ref{section:regularity}). We ultimately take advantage of the regularity properties of the solutions to the homogeneous Monge-Ampère equation (§3.3).

In particular, Theorem 1 in the bounded potential case gives a positive answer to the question of Fornæss and Levenberg.
Another question is to determine what the Hausdorff dimension of a Wermer example can be. The examples we construct have dimension up to 4, but probably always zero Lebesgue measure (see [DL]).

A related issue is the Stolzenberg “swiss cheese” example [SI], where no boundary condition is imposed. Stolzenberg-like examples with positive Lebesgue measure have been constructed in [DL], nevertheless I don’t know how to use them to obtain interesting currents.

1. Wermer examples

In this section we provide a construction of Wermer examples, based on that of [DS]. We actually arrange so that our objects have some laminar structure near the boundary of the bidisk, which will be useful for regularity issues. Therefore to get an actual Wermer example it will be enough to restrict to a smaller bidisk.

We denote by \( D(a,r) \) the disk of center \( a \) and radius \( r \) in \( \mathbb{C} \), and write \( \mathbb{D} = D(0,1) \). We say that a subset \( X \) in \( \mathbb{D} \times \mathbb{D} \) is horizontal if \( X \subset \mathbb{D} \times D(0,1-\varepsilon) \) for some \( \varepsilon > 0 \). A current is horizontal if its support is. Dividing the \( z \) coordinate by 2 we work the bidisk \( D(0,1/2) \times \mathbb{D} \) –this is convenient for if \( z, z' \in D(0,1/2) \), then \( |z - z'| < 1 \).

Let first \( (a_n)_{n \geq 1} \) be a sequence of points in \( D(0,1/4) \) such that \( (a_{2p}) \) and \( (a_{2p+1}) \) are dense in that disk. We put \( A_n(z, w) = z - a_n \) if \( n \) is odd and \( z + w/100 - a_n \) if \( n \) is even. Note that \( |A_n| \leq 1 \) in \( D(0,1/2) \times \mathbb{D} \).

We will inductively define families of polynomials \( P_{n,s} \), where \( n \in \mathbb{N} \) and \( s \) ranges through a finite set \( S_n \). Fix \( P_0(z, w) = w \), and \( S_0 = \{0\} \).

Let \( \delta_n \geq 0 \) and \( (\varepsilon_n)_{n \geq 1} \) be sequences of positive real numbers, with \( \delta_0 = 1/2 \), and \( (m_n)_{n \geq 1} \) be a sequence of positive integers. The inductive step is as follows. Assume that \( S_n \) and the polynomials \( (P_{n,s})_{s \in S_n} \) have been constructed, and consider the finite set \( \Sigma_{n+1} := \overline{D(0, \delta_n (1 - 1/m_{n+1}))} \cap \frac{3\delta_n}{m_{n+1}} \mathbb{Z}^2 \). That is, \( \Sigma_{n+1} \) is the set of those \( \sigma \in D(0, \delta_n) \cap \frac{3\delta_n}{m_{n+1}} \mathbb{Z}^2 \) such that \( D(\sigma, \frac{\delta_n}{m_{n+1}}) \subset D(0, \delta_n) \). By construction, the disks \( D(\sigma, \frac{\delta_n}{m_{n+1}}) \) have disjoint closures.

For large \( m_{n+1} \), \#\( \Sigma_{n+1} \sim \frac{3}{5} m_{n+1}^2 \). On the contrary, observe that when \( m_{n+1} = 1 \), \( \Sigma_{n+1} = \{0\} \).

Let \( S_{n+1} = S_n \times \Sigma_{n+1} \) and for \( s' = (s, \sigma) \in S_{n+1} \) let

\[
P_{n+1,s'} = (P_{n,s} - \sigma)^2 - \varepsilon_{n+1} A_{n+1}.
\]

Put \( X_n = \bigcup_{s' \in S_{n+1}} \{|P_{n,s'}| < \delta_n\} \). It is useful to think about the inductive definition of \( X_n \) as being made up of two steps: we first replace \( X_n \) by \( \bigcup_{\sigma \in \Sigma_{n+1}} X_{n+1,s',\sigma} \), where \( X_{n+1,s',\sigma}^{\text{int}} := \{|P_{n,s'} - \sigma| < \frac{\delta_n}{m_{n+1}}\} \) (“subdivision”), and then the intermediate \( X_{n+1,s',\sigma}^{\text{int}} \) with \( X_{n+1,s'} = \{|P_{n+1,s'}| < \delta_{n+1}\} \) (“ramification”). As compared to [W, L, SI, DS], the subdivision step is new.

**Lemma 1.1.** Fix a sequence of positive real numbers \( (r_n)_{n \geq 1} \), decreasing to zero, with \( r_n \leq \frac{1}{10} \). Let \( (\delta_n)_{n \geq 0} \) be the sequence defined by \( \delta_0 = 1/2 \) and \( \delta_{n+1} = \frac{\varepsilon_n r_{n+1}}{1 m_{n+1}} \), and \( (\varepsilon_n)_{n \geq 1} \) be defined by \( \varepsilon_{n+1} = \frac{\delta_{n+1}^2}{2m_{n+1}^2} \).

Then the following properties hold for every \( n \geq 1 \):

(i.) \( X_{n+1} \subset X_n \) in \( D(0,1/2) \times \mathbb{D} \); more precisely, with notation as above for every \( s' = (s, \sigma) \), we have that \( \overline{X_{n+1,s'}} \subset X_{n+1,s',\sigma}^{\text{int}} \subset X_{n,s'} \).
Lemma 1.2. It is obvious that for all \( n \) if even, relative to \( n = \frac{w}{100} \) if \( n \) is odd:

\[
\pi_1(z, w) = z + \frac{w}{100}
\]

In particular the function \( \gamma \)

\[
\| \gamma \| < \delta
\]

is horizontal in \( D(0,1/2) \times \mathbb{D} \), of degree \( 2^n \) and is a (non ramified) covering over \( \left\{ \frac{3}{8} \leq |z| \leq \frac{1}{2} \right\} \), relative to \( \pi_0 \). Furthermore, if \( s_1 \neq s_2 \), the varieties \( \{ P_{n,s_1} = \alpha \} \) and \( \{ P_{n,s_2} = \alpha \} \) are disjoint.

Proof. It is obvious that for all \( s' = (s, \sigma) \in S_{n+1} \), \( X^\text{int}_{n+1,s',\sigma} \subset X_{n,s} \). Assuming that the constant \( \delta_n \) has been chosen, to ensure the inclusion \( X^\text{int}_{n+1,s'} \subset X^\text{int}_{n+1,s,\sigma} \) it is enough that

\[
\delta_{n+1} + \varepsilon_{n+1} < \frac{\delta^2_n}{m^2_{n+1}}.
\]

Let us also observe that since \( X_0 = \{ |P_0| < \delta_0 = 1/2 \} \) is horizontal, the horizontality assertion in \( \text{(iv.)} \) follows from the fact that \( X_n \subset X_0 \).

We will use the following elementary lemma, which will be proved afterwards.

Lemma 1.2. If \( \delta < \varepsilon \), there does not exist any continuous function \( f \) on \( D(0, r) \) such that \( |(f(\zeta))^2 - \varepsilon| < \delta \) for \( \zeta \in D(0, r) \).

From this we infer that to meet condition \( \text{(ii.)} \) it is enough that for every \( n \),

\[
\delta_{n+1} < \varepsilon_{n+1} r_{n+1}
\]

(3)

(\( \text{Thus} \) holds.

It remains to check \( \text{(iii.)} \). It is clear that \( P_{n,s} \) has degree \( 2^n \) in \( w \) so it is enough to prove that the equation \( P_{n,s}(z_0, w) = \alpha \) has at least (hence exactly) \( 2^n \) distinct roots for each fixed \( z_0 \) with \( 3/8 < |z_0| < 1/2 \). Fix such a \( z_0 \). We will prove by induction the following slightly stronger fact: let \( w \mapsto \gamma(w) \) be a holomorphic function on \( \mathbb{D} \), such that \( |\gamma| < 2\delta_n \); then for every \( s \in S_n \), the equation \( P_{n,s}(z_0, w) = \gamma \) has at least \( 2^n \) distinct solutions in \( \mathbb{D} \). For \( n = 0 \) this follows from Rouché’s Theorem.

For convenience we drop the \( z_0 \) and consider our functions as depending solely on \( w \). Assume the result holds for \( n \), and consider the equation \( P_{n+1,s} = \gamma \) where \( |\gamma| < 2\delta_{n+1} \) in \( \mathbb{D} \), that is, \( (P_{n,s} - \sigma)^2 = \gamma + \varepsilon_{n+1} A_{n+1} \). The right hand side does not vanish on \( \mathbb{U} \times \mathbb{D} \). Indeed \( \gamma + \varepsilon_{n+1} A_{n+1} = 0 \) is equivalent to \( A_0 = -\gamma/\varepsilon_{n+1} \), and with the choices that we have made,

\[
\frac{|\gamma|}{\varepsilon_{n+1}} < r_n \leq \frac{1}{10}
\]

while \( |A_{n+1}| > \frac{1}{8} > \frac{1}{10} \).

In particular the function \( \gamma + \varepsilon_{n+1} A_{n+1} \) admits two square roots \( \pm g \) in \( \mathbb{U} \times \mathbb{D} \). We have that \( 0 < |g| < (2\delta_{n+1} + \varepsilon_{n+1})^{1/2} < \delta_n/m_{n+1} \) and the equation \( P_{n+1,s'} = \gamma \) is equivalent to \( \{ P_{n,s} = \sigma \pm g \} \); we conclude by the induction hypothesis. The last assertion in \( \text{(iii.)} \) is obvious.

\( \square \)

Proof of Lemma 1.2. By scaling, it is enough to prove the result for \( \varepsilon = 1 \). Fix \( \zeta_0 \) such that \( |\zeta_0| = r \). Since \( \delta < r \), the open set \( \{ z \in \mathbb{C}, \ |z^2 - \zeta_0| < \delta \} \) has two connected components. Indeed the critical value of \( z \mapsto z^2 - \zeta_0 \) lies outside \( D(0, \delta) \). Let \( U_1(\zeta_0) \) and \( U_2(\zeta_0) \) be these two components. As \( \zeta_0 \) turns around \( \partial D(0, r) \) these components are swapped.

If now \( f \) is a continuous function satisfying the assumption of the lemma, reducing \( r \) slightly we may assume \( f \) is continuous on \( \overline{D(0, r)} \). Assume \( f(r) \in U_1(r) \). By making
ζ = re^{θ}, 0 ≤ θ ≤ 2π wind around ∂D(0, r), we see that f(r) also belongs to U_2(r), whence the contradiction. When f is holomorphic, an alternate argument is provided by Rouche’s Theorem.

\[ ζ = e^{iθ}, \quad 0 ≤ θ ≤ 2π \]

\[ \text{wind around } \partial D(0, r), \text{ we see that } f(r) \text{ also belongs to } U_2(r), \text{ whence the contradiction. When } f \text{ is holomorphic, an alternate argument is provided by Rouche’s Theorem.} \]

Proposition 1.3. Let X_n be as above and set X = ∩_n X_n. Then X is a polynomially convex horizontal subset in D(0, 1/2) × ℂ, and X ∩ (D(0, 1/5) × ℂ) does not contain any holomorphic disk.

Proof. The horizontality and polynomial convexity of X are obvious. By items (i.) and (ii.) –applied to odd integers– of the previous lemma, it is clear that X ∩ \{ |z| < 1/4 \} does not contain any piece of holomorphic graph over the z coordinate. So any holomorphic disk contained in X ∩ \{ |z| < 1/5 \} must be contained in a vertical line. Then this would be a graph over a certain open subset of D(0, 1/4), relative to the projection π_1, which again is impossible, still due to (ii.). Thus X ∩ \{ |z| < 1/5 \} contains no holomorphic disk.

Shortly we shall see that X carries a natural positive closed current T. What we do in the next sections is to choose the parameters carefully so that T is as regular as possible. Notice that with our presentation the free parameters are the sequences (r_n) and (m_n).

2. A CURRENT WITH CONTINUOUS POTENTIAL ON X

In this section we construct the currents T associated to our Wermer examples and give their first properties. The precise regularity statement leading to Theorem 1 will be proven afterwards. Of course, to get the actual statement of Theorem 1 it is enough to restrict the conclusions of the foregoing results to D(0, 1/5) × ℂ and rescale. We refer the reader to [De, K] for background on positive closed currents and psh functions.

Theorem 2.1. Let the polynomials (P_{n,s})_{s ∈ S_n}, n ∈ ℂ and the sequence (δ_n) be defined as in the previous section. Consider the sequence of psh functions

\[ u_n = \frac{1}{2^n \# S_n} \sum_{s ∈ S_n} \log \max(|P_{n,s}|, δ_n). \]

Then, if

\[ \sum_{n=1}^∞ |\log r_n| \frac{2^n}{2^n} < ∞, \]

the sequence of currents T_n = dd^c u_n converges to a horizontal positive closed current T such that

- T has continuous potential and T ∧ T = 0;
- Supp(T) ∩ (D(0, 1/5) × ℂ) does not contain any holomorphic disk;
- T is uniformly laminar in (D(0, 1/2) \ D(0, 3/8)) × ℂ.

Notice that the result does not depend on (m_n) so it holds for the ordinary (i.e. without subdivision) Wermer construction. See below §4.3 for some comments on the regularity in this case.

Proof. Recall the notation X_n = ∪_{s ∈ S_n} \{ |P_{n,s}| < δ_n \} and X = ∩_n X_n. It is clear that T_n is a sequence of currents with locally uniformly bounded masses, and for the moment we let T be a cluster value of this sequence. Since Supp(T_n) is contained in ∂X_n, T has support in X, hence Supp(T) ∩ (D(0, 1/5) × ℂ) does not contain any holomorphic disk.
It follows from the well known formula \( \log^+ |x| = \int \log |x - e^{i\theta}| \, d\theta \) that we have the integral representation

\[
\log \max(|P_{n,s}|, \delta_n) = \int_{\mathbb{R}/2\pi\mathbb{Z}} \log |P_{n,s} - \delta_n e^{i\theta}| \, d\theta,
\]

whence

\[
T_n = \frac{1}{2n\#S_n} \sum_{s \in S_n} \int_{\mathbb{R}/2\pi\mathbb{Z}} \left[ P_{n,s} = \delta_n e^{i\theta} \right] \, d\theta.
\]

From Lemma 1.1(iii.) we know that the varieties \( P_n = \delta_n e^{i\theta} \) are graphs over \( \{ \frac{3}{8} < |z| < \frac{1}{2} \} \). It is classical (see e.g. [BLS]) that in this situation the laminar structure passes to the limit, thus \( T \) is uniformly laminar in \((D(0,1/2) \setminus D(0,3/8)) \times \mathbb{D}\).

Assume for the moment that (\( u_n \)) converges uniformly, and let us see why \( T \land T = 0 \). Indeed for every \( n \), \( \text{Supp}(T_n) = \partial X_n \) while \( \text{Supp}(T) \subset X \), hence from \( \partial X_n \cap X = \emptyset \), we get that \( T_n \land T = 0 \). By uniform convergence of the potentials, we conclude that \( T \land T = 0 \) (another argument is that \( T_n \land T_n = 0 \) for all \( n \)).

The main step is therefore to prove that \( (u_n) \) converges uniformly. The following lemma will be required (see below for the proof).

**Lemma 2.2.** For \( k \leq 2 \), let \( \Sigma^{(k)} = \frac{3}{k} \mathbb{Z}^2 \cap \overline{D}(0,1 - \frac{1}{k}) \) and

\[
\nu_k = \frac{1}{\#\Sigma^{(k)}} \sum_{\sigma \in \Sigma^{(k)}} \frac{k}{2\pi} \left[ \partial D\left( \sigma, \frac{1}{k} \right) \right],
\]

where \( [\partial D(\sigma, \frac{1}{k})] \) denotes 1-dimensional Hausdorff measure on \( \partial D(\sigma, \frac{1}{k}) \). By convention, let \( \nu_1 = \frac{1}{2\pi 1}[\partial \mathbb{D}] \). Then \( (\nu_k) \) is a sequence of probability measures converging to the normalized Lebesgue measure on \( \mathbb{D} \) and having locally uniformly bounded logarithmic potentials.

To prove uniform convergence, we estimate \( |u_{n+1} - u_n| \). Let \( v_{n+1} \) be the potential corresponding to the intermediate “subdivision” step. Using the notation \( s' = (s, \sigma) \in S_n \times \Sigma_{n+1} = S_{n+1} \) as in the previous section we have that

\[
v_{n+1} = \frac{1}{2^n \#S_{n+1}} \sum_{s' = (s, \sigma) \in S_{n+1}} \log \max\left( |P_{n,s} - \sigma|, \frac{\delta_n}{m_{n+1}} \right).
\]

Write \( u_{n+1} - u_n = (u_{n+1} - v_{n+1}) + (v_{n+1} - u_n) \).

The second part of this equality is estimated using Lemma 2.2 as follows:

\[
v_{n+1} - u_n = \frac{1}{2^n \#S_n} \sum_{s \in S_n} \left( \frac{1}{\#\Sigma_{n+1}} \sum_{\sigma \in \Sigma_{n+1}} \log \max\left( |P_{n,s} - \sigma|, \frac{1}{\delta_n \cdot m_{n+1}} \right) - \log \max\left( |P_{n,s}|, \frac{1}{\delta_n} \right) \right).
\]

Let \( L_k \) be the logarithmic potential of \( \nu_k \). We have that

\[
v_{n+1} - u_n = \frac{1}{2^n \#S_n} \sum_{s \in S_n} (L_{m_{n+1}} - L_1) \circ \left( \frac{P_{n,s}}{\delta_n} \right) = O\left( \frac{1}{2^n} \right),
\]

where the second equality follows from Lemma 2.2.
Now the first part writes as
\[ u_{n+1} - v_{n+1} \leq \frac{1}{2^n \# X_{n+1}} \sum_{s' \in X_{n+1}} \left( \frac{1}{2} \log \max \left( \left| P_{n+1,s'} \right|, \delta_{n+1} \right) - \log \left( \frac{2}{m_{n+1}} \right) \right). \]

Let \( u_{n+1,s'} = \log \max \left( \left| P_{n+1,s'} \right|, \delta_{n+1} \right) \) and \( v_{n+1,s'} = \log \max \left( \left| P_{n+1,s'} \right|, \delta_{n+1} \right) \), and recall the sets \( X_{n+1,s'} \) and \( X_{n+1,s,s}^{\text{int}} \) from Section [1]. We give a uniform estimate of the quantity \( \frac{1}{2} u_{n+1,s'} - v_{n+1,s'} \) in a vertical slice \( \{ z = z_0 \} \). In such a slice we have \( X_{n+1,s',s}^{\text{int}} \subseteq X_{n+1,s,s}^{\text{int}} \subseteq D \). Abusing notation we write \( w \) for \( (z_0, w) \).

If \( w \in X_{n+1,s',s} \), \( v_{n+1,s'}(w) = \log \frac{\delta_{n+1}}{m_{n+1}} \) and \( u_{n+1}(w) = \log \delta_{n+1} \). Since \( \delta_{n+1} = \frac{\delta_{n+1}}{m_{n+1}} \), we infer that
\[ \left| \frac{1}{2} u_{n+1,s'}(w) - v_{n+1,s'}(w) \right| \leq \left| \log r_n \right|. \]

If \( w \notin X_{n+1,s,s}^{\text{int}} \), \( u_{n+1,s'}(w) = \log \left| P_{n+1,s'} \right| \) and \( v_{n+1,s'}(w) = \log |P_{n+1,s'}| \) with \( |P_{n+1,s'}| \geq \frac{\delta_{n+1}}{m_{n+1}} \). Using the equality \( P_{n+1,s'} = (P_{n+1,s} - \sigma)^2 - \varepsilon_{n+1} A_{n+1} \) we infer
\[ \left| \frac{1}{2} u_{n+1,s'} - v_{n+1,s'} \right| = \frac{1}{2} \log \left| 1 - \frac{\varepsilon_{n+1} A_{n+1}}{(P_{n+1,s} - \sigma)^2} \right|. \]

In \( D(0,1/2) \times D \), we have \( |A_{n+1}| \leq 1 \) so from the definition of \( \varepsilon_{n+1} \) we deduce that for \( w \notin X_{n+1,s,s}^{\text{int}} \),
\[ \frac{\varepsilon_{n+1} A_{n+1}}{(P_{n+1,s} - \sigma)^2} \leq \frac{1}{2}. \]

We conclude that outside \( X_{n+1,s,s}^{\text{int}} \) we have
\[ \left| \frac{1}{2} u_{n+1,s'} - v_{n+1,s'} \right| \leq \frac{1}{2} \log 2. \]

In \( X_{n+1,s,s}^{\text{int}} \), \( \frac{1}{2} u_{n+1,s'} - v_{n+1,s'} \) is harmonic and the two previous cases give us a bound for this function on \( \partial X_{n+1,s,s}^{\text{int}} \). So by \([3], [4]\) and the maximum principle we get that \( \left| \frac{1}{2} u_{n+1,s'} - v_{n+1,s'} \right| \leq \left| \log r_n \right| \) there.

Summarizing the 3 cases we see that the estimate \([3] \) holds throughout \( D(0,1/2) \times D \).

Finally, using \([7]\) we conclude that \( |u_{n+1} - v_{n+1}| = O \left( \frac{\log r_n}{2^n} \right) \) and concludes the proof of the theorem. \( \square \)

**Proof of Lemma 2.2** The proof is easy so we rather sketch it. That \( (\nu_k) \) converges to Lebesgue measure in \( D \) is obvious so we focus on the statement on the logarithmic potentials. The logarithmic potential of \( \nu_k \) is given by the formula
\[ \frac{1}{\# \Sigma(k)} \sum_{\sigma \in \Sigma(k)} \log \max \left( |z - \sigma|, \frac{1}{r} \right). \]

It is enough to prove that there exists a constant \( C \) such that for every \( z \in D \) and every \( r > 0 \), \( \nu_k(D(z,r)) \leq Cr \) (this of course gives more information about the convergence but we will not need it). Indeed if this estimate holds, then for \( z \in D \)
\[ \int \log |z - \zeta| d\nu_k(\zeta) \leq \sum_{q=0}^{\infty} \int_{2^{-q-1} \leq |z - \zeta| \leq 2^{-q}} \log |z - \zeta| \, d\nu_k(\zeta) \leq \log 2 + \sum_{q=0}^{\infty} C q \frac{1}{2^q}. \]
Given such $z$ and $r$ there are three possible cases. Either $r \gg \frac{1}{k}$, say, $r \geq \frac{100}{k}$, and the number of small circles intersecting $D(z,r)$ is bounded above by $\frac{\pi}{9}r^2k^2$ up to an error of order of magnitude of $k$ times the length of $\partial D(z,r)$, that is, $O(kr)$, with $kr \leq k^2r^2/100$. Notice also that $\#\Sigma^{(k)} \sim \frac{k^2\pi}{9}$. In this case we conclude that $\nu_k(D(z,r)) \leq 2r^2$.

The second case is when $\frac{1}{100k} \leq r \leq \frac{100}{k}$. Then we simply argue that the number of small circles intersecting $D(z,r)$ is bounded by a constant (approximately $\frac{\pi}{100k^2}$), thus $\nu_k(B(z,r) \leq O(1)/\#\Sigma^{(k)} = O(1/k^2) = O(r^2)$.

The last situation is when $r \leq \frac{1}{100k}$. In this case the intersection of $D(z,r)$ with the family of small circles, if nonempty, is a piece of a small circle of length $O(r)$. We conclude that $\nu_k(D(z,r)) = O(r/k^2)$ hence $O(r)$.

\[\square\]

3. Precise regularity of the potential

Let $\psi$ be a continuous increasing function defined in a neighborhood of $0 \in \mathbb{R}^+$, with $\psi(0) = 0$. From now on such functions will be referred to as gauge functions. We say that a function is $C^{1+\psi}$ if it is $C^1$ and its derivatives have modulus of continuity $O(\psi)$. Of course $C^{1,\alpha}$ regularity corresponds to $\psi : r \mapsto r^\alpha$.

In this section we prove the following refined version of Theorem 1.

**Theorem 3.1.** Let $(P_{n,s})$, $(m_n)$ and $(r_n)$ be as defined in Section 7. Assume that $(r_n)$ satisfies (5) and let $T$ be the current of Theorem 2.7.

Let $\psi$ be any gauge function such that $\frac{\psi(r)}{r|\log r|} \to \infty$ as $r \to 0$.

Then it is possible to choose $(m_n)$ so that the potential of $T$ is of class $C^{1+\psi}$.

By choosing $\psi$ with $\psi(r) = o(r^\alpha)$ for all $0 < \alpha < 1$ (e.g. $\psi(r) = r |\log r|^2$), we get the conclusions of Theorem 1. It is likely that the requirement on $\psi$ could be upgraded to $\frac{\psi(r)}{r} \to \infty$.

3.1. Regularity of subharmonic functions. The estimate on regularity will ultimately be a consequence of the following –presumably well known– result.

**Proposition 3.2.** Let $u$ be a subharmonic function in $\mathbb{R}^n$, with Laplacian $\Delta u = \mu$. Assume that there exists a constant $C$ such that for every $x \in \mathbb{R}^n$ and $0 < r < 1$,

$$\mu(B(x,r)) \leq Cr^{n-2}h(r),$$

where $h$ is a nonnegative increasing function satisfying $\int_0^r \frac{h(r)}{r^2}dr < \infty$.

Then $u$ is $C^{1+\psi}$, with

$$\psi(r) = \int_0^{2r} \frac{h(s)}{s^2}ds + r \int_r^1 \frac{h(s)}{s^3}ds.$$

**Proof.** When $n = 2$ the result follows from [G] §III.4. We will also need it for $n = 4$, so let us indicate how to adapt the proof to this case.

Since the problem is local, we can assume that $u$ is harmonic outside $B(0,1/2)$, and by using the Riesz decomposition, it is enough to prove the result when $u$ is the canonical solution of the Laplace equation, that is

$$u(x) = \int \frac{d\mu(y)}{||x - y||^2}.$$
Taking (at least formally) the derivative with respect to $x_j$ (where $x_j = (x_1, \ldots, x_n)$, we get
\[
\frac{\partial u}{\partial x_j} = K_j * \mu \quad \text{where } K_j(z) = -z_j/\|z\|_4^4 = O(\|z\|^{-3}).
\]
Conversely, if for every $j$, $K_j * \mu$ is a continuous function, then $u$ is indeed $C^1$ and the formula $\frac{\partial u}{\partial x_j} = K_j * \mu$ holds.

Now we set $r = 10 \|x - x'\|$ and write
\[
\frac{\partial u}{\partial x_j}(x) - \frac{\partial u}{\partial x_j}(x') = \int_{B(x,r)} (K_j(x-y) - K_j(x'-y))d\mu(y) + \int_{B(x,r)^c} (K_j(x-y) - K_j(x'-y))d\mu(y).
\]

To estimate the first term in this equality, we notice that $B(x, r) \subset B(x', \frac{11}{10}r)$ (hence the $2r$ in the first integral of (11)), so it is enough to estimate $\int_{B(x,r)} K_j(x-y)d\mu(y)$. We have that
\[
\left| \int_{B(x,r)} K_j(x-y)d\mu(y) \right| \leq \int_{B(0,r)} \frac{d\mu(x+z)}{\|z\|^3} = \int_{1/r}^{\infty} \mu(B(0, \frac{1}{t}))3t^2dt
\]
\[
\leq 3 \int_{1/r}^{\infty} h \left( \frac{1}{t} \right) dt = 3 \int_{0}^{r} \frac{h(s)}{s^2}ds.
\]
where the equality on the first line follows from the formula $\int f d\mu = \int_{1}^{\infty} \mu(\{ f > t^3 \})3t^2dt$.

For the second term we use the fact that the partial derivatives of $K_j$ are $O(\|z\|^{-4})$ so that when $\|x - y\| \geq r$ (whence $\|x' - y\| \geq 9r/10$) we have $|K_j(x-y) - K_j(x'-y)| \leq C\|x-x'\|_2/\|x-y\|^4$.

As above we infer that
\[
\left| \int_{B(x,r)^c} (K_j(x-y) - K_j(x'-y))d\mu(y) \right| \leq Cr \int_{B(0,1) \setminus B(0,r)} \frac{d\mu(x+z)}{\|z\|^4}
\]
\[
= Cr \int_{1/r}^{1} \mu(B(0, \frac{1}{t}))3t^3dt = 4Cr \int_{r}^{1/r} \frac{h(s)}{s^3}ds,
\]
which, together with the previous estimate, concludes the proof. \qed

Direct computation shows the following:

**Corollary 3.3.** With notation as in Proposition 3.2, if $0 < \alpha < 1$ and $h(r) = O(r^{1+\alpha})$, then $u$ is $C^{1,\alpha}$. If $h(r) = O(r^2)$ then $u$ is $C^{1+\psi}$ with $\psi(r) = r \log r$.

Later on (see 4.2) we shall see that for the currents constructed in Section 2 we always have $h(r)/r^2 \to \infty$ so the potentials are less regular than $C^{1+rlog r}$. It is not a surprise that if $h(r)/r^2$ diverges slowly enough, then every regularity below $C^{1+rlog r}$ can be reached. This is the contents of the next result, which follows from elementary calculus.

**Proposition 3.4.** Let $\psi$ be a gauge function such that $\frac{\psi(r)}{rlog r} \to \infty$ as $r \to 0$. Then there exists a decreasing function $\theta$ such that $r \mapsto h(r) = r^2\theta(r)$ satisfies the assumptions of Proposition 3.2 and
\[
\int_{0}^{2r} \frac{h(s)}{s^2}ds + r \int_{r}^{1} \frac{h(s)}{s^3}ds = O(\psi(r)).
\]

It will follow from the proof that we can further assume that $\frac{\theta'}{\theta} = o\left(\frac{1}{r \log r}\right)$. Let us study this case first.
Lemma 3.5. Let \( \theta \) be a function defined in a neighborhood of \( 0 \in \mathbb{R}^+ \). Assume that \( \theta \) is \( C^1 \), decreasing, \( \lim_{t \to 0^+} \theta = +\infty \), and \( \frac{\theta'}{\theta} = o \left( \frac{1}{r \log r} \right) \).

Then with notation as in Proposition 3.3 if \( h(r) = r^2 \theta(r) \), then \( \psi(r) = O(r |\log r| \theta(r)) \).

The assumption of the lemma holds e.g. when \( \theta(r) = \log \log \cdots \log |\log r| \). For the limiting case \( \theta(r) = |\log r| \) (for which \( \frac{\theta'}{\theta} = \frac{1}{r \log r} \)) the assumption does not hold but the reader may check that the conclusion is still valid.

Proof. This is very elementary. Notice first that the assumption on \( \theta \) implies that \( \theta(r) = o(|\log r|) \). Notice also that since \( \theta \) is decreasing, \( \theta(2r) \leq \theta(r) \).

Consider now the first integral in (11). Integrating by parts yields

\[
\int_0^{2r} \frac{h(s)}{s^2} ds = \int_0^{2r} \theta(s) ds = 2r \theta(2r) - \int_0^{2r} s \theta'(s) ds \sim 2r \theta(2r),
\]

because \( s \theta'(s) = o(\theta(s)) \). For the second one, integrate by parts again

\[
\int_r^1 \frac{h(s)}{s^3} ds = \int_r^1 \frac{\theta(s)}{s} ds = - \log r \theta(r) - \int_r^1 \theta'(s) \log s ds \sim |\log r| \theta(r),
\]

for \( \theta'(s) \log s = o \left( \frac{\theta(s)}{s} \right) \). We conclude that \( \psi(r) \sim r |\log r| \theta(r) \).

\[\square\]

Proof of Proposition 3.4. Let \( \theta_0 \) be defined on \( (0, r_0) \) by \( \theta_0(r) = \frac{\psi(r)}{|\log r|} \). Replace first \( \theta_0 \) with any decreasing function \( \theta_2 \leq \theta_0 \) with \( \lim_0 \theta_1 = +\infty \). The next step is to replace \( \theta_1 \) with a function \( \theta_2 \leq \theta_1 \) satisfying the assumptions of Lemma 3.5. Then the choice \( \theta = \theta_2 \) will have the desired properties. Indeed, put \( h(r) = r^2 \theta_2(r) \). The assumption on the derivative of \( \theta_2 \) implies that \( h \) is increasing near 0, and as we have seen, \( \theta_2(r) = o(|\log r|) \) thus \( \int_0^r \frac{h(r)}{s} dr < \infty \). By Lemma 3.5 we have

\[
\int_0^{2r} \frac{h_2(s)}{s^2} ds + r \int_r^1 \frac{h_2(s)}{s^3} ds = O(r |\log r| \theta_2(r)),
\]

which is an \( O(\psi(r)) \) be definition of \( \theta_2 \).

It remains to see why such a \( \theta_2 \) exists. By making the change of variables \( x = 1/r \) we are claiming that for any function \( F \) on \( \mathbb{R}^+ \) increasing to \( +\infty \), there exists \( G \leq F \) increasing to infinity, and such that moreover \( \frac{G'}{G} = o \left( \frac{1}{x \log x} \right) \). Put \( f = \log F \) and \( g = \log G \) so that the requirement is that \( g' = o \left( \frac{1}{x \log x} \right) \). We construct \( g \) as follows. Fix \( x_1 \in \mathbb{R}^+ \) such that \( f(x_1) > 0 \), put \( g(x_1) = f(x_1) \) and declare that \( g \) is constant until \( x_2 \), where \( x_2 \) is such that \( f(x_2) = 2f(x_1) \). From \( x_2 \) let \( g(x) = \log \log x - \log \log \log x + g(x_2) \). If \( g \leq f \) forever we are done. Otherwise let \( y_1 > x_2 \) be the least number such that \( g(y_1) = f(y_1) \) and repeat the above procedure. It is clear that \( g \leq f \), \( g \) increases to infinity, and \( g'(x) = o \left( \frac{1}{x \log x} \right) \).

\[\square\]

3.2. Geometry of the vertical slices near the boundary. We return to the setting of Sections 1 and 2 and assume that \( (r_n) \) satisfies the hypothesis (5) of Theorem 2.1. Our purpose is now to fix the sequence \( (m_n) \).

Throughout this subsection, we work in a fixed “vertical” slice near the boundary. By this, we mean a line of the form \( \pi^{-1}(z_0) \) where \( \pi \) is of the form \( (z, w) \mapsto z + \gamma w \), with \( |\gamma| \leq \frac{1}{100} \) and \( \frac{2}{5} \leq z_0 \leq \frac{4}{10} \). The choice of \( z_0 \) and \( \gamma \) ensures that \( \pi^{-1}(z_0) \) is a vertical graph in \( \{ \frac{3}{8} < |z| < \frac{1}{2} \} \times \mathbb{D} \). Then by Lemma 1.1 (iii.), each variety \( \{ P_n \alpha \} \), \( |\alpha| < 2\delta_n \) intersects it
in exactly $2^n$ points. Indeed $\pi^{-1}(z_0)$ is actually a vertical graph in a thin bidisk of the form $D(z_0, r) \times \mathbb{D}$, in which $\{P_{n,s} = \alpha\}$ is the union of $2^n$ disjoint graphs.

By $X_{n,s}$, $X$, etc. we mean the trace of these subsets on the slice $\pi^{-1}(z_0)$, and we denote by $\mu_n$ (resp. $\mu$) the Laplacian of $u_n$ (resp. $u$) on that slice, that is, the slice measure of $T_n$ (resp. $T$).

We use the following notation: $a_n \asymp b_n$ (resp. $a_n \approx b_n$) if there exists $C > 0$ independent on $n$ such that $a_n/C \leq b_n \leq C a_n$ (resp. $a_n/C^n \leq b_n \leq C^n a_n$).

We aim at proving the following result.

**Proposition 3.6.** Let $h$ be any increasing function defined in a neighborhood of $0 \in \mathbb{R}^+$ such that $h(r)/r^2 \to +\infty$. Then there exists a sequence $(m_n)$ such that for every $p$ in the slice and every $r > 0$, $\mu(B(p, r)) \leq Ch(r)$, where $C$ is a universal constant (in particular independent on the slice).

By Propositions 3.2 and 3.4 this implies that along the slice, $u$ can be made $C^{1+\psi}$ for an arbitrary gauge $\psi$ satisfying $\frac{\psi(r)}{r \log r} \to \infty$ as $r \to 0$. In the next section, we extend this regularity to the bidisk.

The idea of the proof is to study the geometry of the Cantor set $X$, and the distribution of $\mu$ on $X$, by using some techniques from plane conformal geometry. Recall that if $f$ is a univalent mapping defined in a topological disk $\Delta \subset \mathbb{C}$, the distortion of $f$ is defined as $\sup_{z,w \in \Delta} \frac{|f(z) - f(w)|}{|z - w|}$. If $f: \Delta \to \mathbb{D}$ is a conformal map with $|f'(0)| = 1/R$ we say that the conformal radius of $\Delta$ is $R$.

Let us define $R_n = \prod_{k=1}^{n} r_k$ and $M_n = \prod_{k=1}^{n} m_k$. We first describe the basic geometry of $X$.

**Proposition 3.7.** For every $n \in \mathbb{N}$ and $s \in S_n$, each component of $X_{n,s} = \{P_{n,s} < \delta_n\}$, is up to uniformly bounded distortion, a disk of conformal radius $\approx \frac{\delta_n}{M_n}$.

More specifically, if $\Delta$ is such a component, then $P_{n,s}: \Delta \to D(0, \delta_n)$ is a univalent mapping of uniformly bounded distortion, and its derivative is $\approx \delta_n \frac{M_r}{M_n}$.

**Proof.** We know from Lemma 1.11 that for every $\alpha \in D(0, 2\delta_n)$ the equation $P_{n,s} = \alpha$ has exactly $2^n$ solutions. Since the solutions do not collide, they vary holomorphically, and it follows that $P_{n,s}^{-1}(D(0, 2\delta_n))$ is the union of $2^n$ topological disks, and $P_{n,s}$ is univalent on each of them. By the Koebe Distortion Theorem, the distortion of $P_{n,s}^{-1}|_{D(0, \delta_n)}$ is bounded by a universal constant, hence the same holds for $P_{n,s}$ on $\Delta$.

What remains to do is to estimate the derivative of $P_{n,s}$ on $\Delta$. We do it by induction, taking $w$ as coordinate on the slice and simply denoting the derivative of $P$ by $P'$. Recall that if $s' = (s, \sigma)$, $P_{n+1,s'} = (P_{n,s} - \sigma)^2 + \varepsilon_{n+1} A_{n+1}$. It is clear that $|A_{n+1}| \leq 1$. By definition of the sets $X_{n,s}$, we have $|P_{n,s} - \sigma| < \frac{\delta_n}{m_{n+1}}$ on $X_{n+1,s'}$. On the other hand, since on the slice we have $|A_{n+1}| > \frac{1}{16}$ (see (3)), we infer that when $|P_{n+1,s'}| < \delta_{n+1}$ we have

$$|P_{n,s} - \sigma|^2 \geq \varepsilon_{n+1} |A_{n+1}| - \delta_{n+1} \geq \frac{\delta_n^2}{m_{n+1}^2} \left( \frac{1}{20} - \frac{r_{n+1}}{4} \right) \geq \frac{\delta_n^2}{40 m_{n+1}^2}.$$

We conclude that on $X_{n+1,s'}$,

$$1 \geq 40 m_{n+1} \frac{\delta_n}{\sqrt{40 m_{n+1}}} \geq |P_{n,s} - \sigma| < \frac{\delta_n}{m_{n+1}}.$$

Let us first imagine for simplicity that for all $n$, $P_{n+1,s'} = (P_{n,s} - \sigma)^2$. In this case we would get that $|P_{n+1,s'}| = 2 |P_{n,s}| |P_{n,s} - \sigma| \times \frac{\delta_n}{m_{n+1}} |P_{n,s}|$ on $X_{n+1,s'}$. By induction this implies that
\[ \left| P'_{n+1,s'} \right| \approx \prod_{k=0}^{n} \frac{\delta_k}{m_{k+1}}. \] An immediate computation shows that \( \delta_{n+1} = \frac{R_{n+1}}{4^n M_{n+1}} \prod_{k=0}^{n} \delta_k \) so we conclude that \[ \left| P'_{n+1,s'} \right| \approx \delta_{n+1} \frac{M_{n+1}}{R_{n+1}}. \]

Now we need to take care of the extra term in \( P_{n+1,s'} \). Let \( D_n = \prod_{k=0}^{n} \frac{\delta_{k-1}}{m_{k+1}} = 4^n \delta_n \frac{M_n}{R_n} \). We want to prove by induction that on \( X_{n,s} \), \( |P'_{n,s}| \approx D_n \). Assume a constant \( C \) has been found such that \( C^{-n} D_n \leq |P'_{n,s}| \leq C^n D_n \). By definition of \( P_{n+1,s'} \), \( \varepsilon_{n+1} \), \( \delta_{n+1} \) and using (13), we have

\[
\frac{|P'_{n+1,s'}|}{D_{n+1}} \leq 2 \frac{|P'_{n,s}|}{D_n} \frac{|P_{n,s} - \sigma|}{\delta_n/m_{n+1}} + \frac{\varepsilon_{n+1}}{D_{n+1}} \leq 2C^n + \frac{\delta_{n+1} r_{n+1} / 2}{\delta_n \cdot 4^n M_{n+1} / R_{n+1}} \leq C^n \left( 2 + \frac{R_n}{M_{n+1}} \frac{1}{2 \cdot 4^n C^n} \right)
\]

which is less than \( C^{n+1} \) as soon as \( C \geq 3 \) because \( R_n / M_{n+1} \) is super-exponentially small in \( n \). The reverse inequality being similar, the result is proved.

From now on we refer to components of \( X_{n,s} \) as components of depth \( n \). Let \( \{\text{rad}_{n}, \overline{\text{rad}}_{n}\} \) be the interval of variation of the conformal radii of components of depth \( n \). By the previous proposition, \( \text{rad}_{n} \approx \text{rad}_{n} \approx \frac{R_n}{M_{n+1}} \). To get a good distinction between scales, from now on we assume that \( (m_n) \) has super-exponential growth, so that for every \( C \), if \( n \) is large enough, \( C^{n+1} \frac{R_{n+1}}{M_{n+1}} \leq C^{-n} \frac{R_n}{M_n} \), so in particular \( \overline{\text{rad}}_{n+1} < \text{rad}_{n} \).

We will also need to take into consideration the size of intermediate components of depth \( n+1 \), that is, the components of the form \( X_{n+1,s,\sigma} \). By construction, the conformal radius of such a component is \( \frac{1}{m_{n+1}} \) times the radius of the component of depth \( n \) in which it sits, so that if \( \{\text{rad}_{n+1}^{\text{int}}, \overline{\text{rad}}_{n+1}^{\text{int}}\} \) denotes the corresponding range of radii, we have that \( \{\text{rad}_{n+1}^{\text{int}}, \overline{\text{rad}}_{n+1}^{\text{int}}\} = \frac{1}{m_{n+1}} \{\text{rad}_{n}, \overline{\text{rad}}_{n}\} \). Notice also that the largest component of depth \( n+1 \) lies in some intermediate component, so \( \overline{\text{rad}}_{n+1}^{\text{int}} \leq \overline{\text{rad}}_{n+1}^{\text{int}} \). So, still assuming that \( m_n \) has super-exponential growth we conclude that \( \text{rad}_{n+1} < \text{rad}_{n+1} < \text{rad}_{n} \).

We are now in position to estimate \( \mu(B(p,r)) \) from above for every \( p \).

**Proposition 3.8.** Assume that \( (m_n) \) has super-exponential growth. There exists a constant \( C \) such that for every \( p \) in the slice and \( n \) large enough the following holds:

- if \( \text{rad}_{n+1} \leq r \leq \text{rad}_{n+1}^{\text{int}} \), then \( \mu(B(p,r)) \leq \frac{C^n}{M_{n+1}} \).

  - if \( \text{rad}_{n+1}^{\text{int}} \leq r \leq \text{rad}_{n} \), then \( \mu(B(p,r)) \leq \frac{C^n}{R_n^2} \).

**Proof.** Fix \( A \) such that \( A^{-n} \frac{R_n}{M_n} \leq \text{rad}_{n} \leq \text{rad}_{n} \leq A^n \frac{R_n}{M_n} \) for all \( n \). Observe first that by construction, the \( \mu \)-mass of a component of depth \( n \) equals its \( \mu_n \) mass. In particular the mass of a component of depth \( n \) is \( \frac{2^{2^n} - 2}{2 \cdot 2^n} \approx \frac{1}{M_{n+1}} \) and the mass of an intermediate component of depth \( n+1 \) is \( \frac{2^{2^n+1} - 2}{2 \cdot 2^n} \approx \frac{1}{M_{n+1}^2} \).

The argument for estimating the mass of balls is the same in both cases. Due to the bound on distortion, if \( \Delta \) is an intermediate component of depth \( n+1 \), then \( \text{Diameter}(\Delta) \leq K \text{rad}_{n+1}^{\text{int}} \) and \( \text{Area}(\Delta) \geq \frac{1}{K} \text{rad}_{n+1}^{2} \) for some \( K \). If \( \text{rad}_{n+1} \leq r \leq \text{rad}_{n+1}^{\text{int}} \), any intermediate component of depth \( n+1 \) intersecting \( B(p,r) \) must be contained in \( B(p,(K+1)\text{rad}_{n+1}^{\text{int}}) \), which in turn
contains at most \( \pi(K+1)^2 \frac{(\text{rad}_n)^2}{(\text{rad}_n)^2/K} \leq \pi K(K+1)^2 A^{4n} \) of them, due to the area bound. Hence we conclude that \( \mu(B(p,r)) \leq \frac{C^n}{M^{2n+1}} \) for some \( C \).

In the other case, we argue that an intermediate component of depth \( n+1 \) intersecting \( B(p,r) \) is contained in \( B(p, r + K \text{rad}_n) \subset B(p, (K+1)r) \), so the total number of those does not exceed \( \pi(K+1)^2 \frac{2^{2n+1}}{(\text{rad}_n)^2/K} \), and we conclude by using the fact that \( \text{rad}_n \approx \frac{R_n}{M^{n+1}}. \)

**Proof of Proposition 3.6.** As before, write \( h(r) = r^2 \theta(r) \), with \( \lim_0 \theta = +\infty \). As in Proposition 3.4 we can always replace \( \theta \) with some decreasing function of slower growth, and prove the result for the new \( \theta \). We want to choose \( (m_n) \) so that \( \mu(B(p,r)) \leq r^2 \theta(r) \) for small enough \( r \).

By Proposition 3.8 when \( \text{rad}_n < r \leq \text{rad}_n \), \( \mu(B(p,r)) \leq \frac{C^n}{M^{2n+1}} \), and \( h(r) \geq h(\text{rad}_n) \), so a sufficient condition for \( \mu(B(p,r)) \leq h(r) \) is that

\[
\frac{C^n}{M^{2n+1}} \leq h \left( A^{-(n+1)} \frac{R_n}{M^{n+1}} \right),
\]

where \( A \) is as in the proof of Proposition 3.8. This rephrases as

\[
\theta \left( A^{-(n+1)} \frac{R_n}{M^{n+1}} \right) \geq C^n A^{n+1} \frac{1}{R_n^{2n}}.
\]

In the alternate case where \( \text{rad}_n \leq r \leq \text{rad}_n \), since \( \theta \) is decreasing, if \( \theta(\text{rad}_n) \geq \frac{C^n}{R_n} \) we infer that

\[
\mu(B(p,r)) \leq \frac{C^n}{R_n^2} r^2 \leq r^2 \theta(\text{rad}_n) \leq r^2 \theta(r).
\]

For this a sufficient condition is that

\[
\theta \left( A^{n} \frac{R_n}{M^{n}} \right) \geq C^n \frac{1}{R_n^2}.
\]

From (14) and (15), we conclude that to achieve the desired conclusion it is enough that for large \( n \), \( \theta \left( A^{n} \frac{R_n}{M^{n}} \right) \geq (CA)^n \frac{1}{R_n^2} \). Now, since \( \lim_0 \theta = +\infty \), it is clear by induction on \( n \) that this condition will be satisfied if \( m_n \) is chosen to be sufficiently large. \( \square \)

### 3.3. Transfer of regularity and conclusion.

To study the regularity of \( T \) throughout the bidisk, we use some basic estimates for solutions of homogeneous complex Monge-Ampère equations. Let us introduce some notation from [BT]. Let \( u \) be a psh function in some open set \( \Omega, \zeta \in \mathbb{C}^2 \) be a unitary vector, and \( r > 0 \). If \( p \in \Omega_r = \{ p \in \Omega, \text{dist}(p, \partial \Omega) > r \} \), we let

\[
(T_{\zeta,r} u)(p) = r^{-2} (u_{\zeta,r}(p) - u(p)), \quad \text{with } u_{\zeta,r}(p) = \frac{1}{2\pi} \int_0^{2\pi} u(p + r\zeta e^{i\theta}) d\theta.
\]

Recall also the classical Jensen formula for a subharmonic function in one variable

\[
\frac{1}{2\pi} \int_0^{2\pi} u(r e^{i\theta}) d\theta - u(0) = \int_0^r \frac{n(t)}{t} dt, \quad \text{where } n(r) = \int_{|z| = r} \Delta u,
\]

so that if now \( u \) is psh in \( \Omega \) and if we denote by \( n_{\zeta,r}(p) \) the mass of \( d\zeta^2 u \) along the flat disk of radius \( r \) in the direction \( \zeta \), centered at \( p \), we infer that \( (T_{\zeta,r} u)(p) = r^{-2} \int_0^r \frac{n_{\zeta,r}(p)}{t} dt \).
We can now finish the proof of Theorem 3.1. Fix a gauge \( \psi \) with \( \frac{\psi(r)}{r|\log r|} \to \infty \). By Proposition 3.4 (and its proof) there exists a decreasing \( \theta \), with \( \lim_0 \theta = +\infty \) and \( \frac{\theta'}{r} = o\left(\frac{1}{r|\log r|}\right) \), such that \( h(r) = r^2\theta(r) \) satisfies (12). Notice that since \( \theta \) is decreasing, \( h(Ar) = O(h(r)) \) for \( A \geq 1 \).

By Proposition 3.6 we can choose \( (m_n) \) so that for every slice of the form \( \pi^{-1}(z_0) \), with \( \pi(z, w) = z + \gamma w \), \( |\gamma| \leq \frac{1}{|z_0|} \) and \( \frac{2}{3} < |z_0| < \frac{4}{10} \), the slice measures of \( T \) satisfy \( \mu(B(p, r)) \leq Ch(r) \) for every \( p \) and \( r > 0 \). Notice that the union of these slices contains the open set \( \left\{ \frac{2}{3} < |z| < \frac{4}{10} \right\} \times \mathbb{D} \).

Using the above notation, if \( \zeta = (\zeta_1, \zeta_2) \) is a unit vector in \( \mathbb{C}^2 \), with \( |\zeta_1| \leq \frac{1}{100} |\zeta_2| \), we have that for every \( p \in \left\{ \frac{2}{3} < |z| < \frac{4}{10} \right\} \times \mathbb{D} \), \( n_{\zeta, r}(p) = O(h(r)) \). Thus by Jensen’s formula we infer that for every such \( p \) and \( \zeta \),

\[
0 \leq (T_{\zeta, r}u)(p) = r^{-2} \int_0^r \frac{n_{\zeta, t}(p)}{t} dt \leq Cr^{-2} \int_0^r t \theta(t) dt \leq C\theta(r),
\]

where the last inequality follows from an integration by parts, as in the proof of Lemma 3.5. Notice that if \( p \) is close to the horizontal boundary of \( D(0, \frac{1}{10}) \times \mathbb{D} \), \( (T_{\zeta, r}u) \equiv 0 \) since \( u \) is pluriharmonic there.

Now if we let \( \Omega = D(0, \frac{4}{10}) \times \mathbb{D} \), by [BT] Theorem 6.4, if \( r < \varepsilon \)

\[
\sup\left\{ (T_{\zeta, r}u)(p), \ p \in \Omega \right\} = \sup\left\{ (T_{\zeta, r}u)(p), \ p \in \Omega, \ \text{dist}(p, \partial \Omega) < \varepsilon \right\},
\]

so we conclude that throughout the bidisk \( D(0, \frac{1}{10}) \times \mathbb{D} \), the estimate \( (T_{\zeta, r}u)(p) \leq C\theta(r) \) holds. Now we use the Jensen formula again and the reverse estimate \( \int_0^r \frac{n(t)}{t} dt \geq \int_{r/2}^r \frac{\log 2}{t} dt \) and we obtain that for every vector \( \zeta \) close to the vertical as above, and every \( p \in D(0, \frac{1}{10}) \times \mathbb{D} \), we have \( n_{\zeta, r}(p) = O(r^2\theta(2r)) = O(h(r)) \).

To apply Proposition 3.2 and conclude that \( u \) is \( C^{1+\psi} \), we need to control the mass of small balls for \( \Delta u \), or equivalently for the trace measure \( \sigma_T \) of \( T \). Because \( T \) is a positive current, it is well known that controlling slice masses in two directions gives a control of the trace measure. Indeed, let \( \omega_{\zeta C} = i dz \wedge d\overline{z} + i dw \wedge d\overline{w} \) (resp. \( \omega_C = i dz \wedge d\overline{z} \)) be the standard Kähler form of \( \mathbb{C}^2 \) (resp. \( \mathbb{C} \)). If \( \pi_j : (z, w) \mapsto z + \gamma_j w, \ j = 1, 2 \) are two distinct projections, there exists a constant \( C \) depending on the \( \gamma_j \) such that \( \omega_{\zeta C} \leq C(\pi_1^{*}\omega_C + \pi_2^{*}\omega_C) \), so \( \sigma_T = T \wedge \omega_{\zeta C} \leq C \sum_{j=1, 2} T \wedge \pi_j^{*}\omega_C \). Finally, since the projection (resp. the fibers) of \( B(p, r) \) under \( \pi_j \) are contained in disks of radius \( \leq Cr \), by the Slicing Formula we infer that \( (T \wedge \pi_j^{*}\omega_C)(B(p, r)) \leq C^2 h(Cr) = O(r^2 h(r)) \), which by Proposition 3.2 and our assumption on \( \theta \), implies that \( u \) is \( C^{1+\psi} \) in \( D(0, \frac{4}{10}) \times \mathbb{D} \). Of course to obtain the same result in \( D(0, \frac{1}{10}) \times \mathbb{D} \) it suffices to consider projections closer to the vertical and an exhaustion argument.

\[ \square \]

Remark 3.9. The arguments developed here incidentally show that if \( \Omega \) is a bounded open set and \( u \in C(\Omega) \) is a solution of the homogeneous Monge-Ampère equation which is \( C^{1+\alpha} \) near \( \partial \Omega \) \((0 < \alpha < 1)\), then it is \( C^{1, \alpha} \) everywhere, a consequence of [BT] which doesn’t seem to be so well-known. This uses the following classical converse to Proposition 3.2 if a plane subharmonic function \( u \) is \( C^{1, \alpha} \), then the mass of a ball of radius \( r \) is \( O(r^{1+\alpha}) \). It is also possible to state a \( C^{1+\psi} \) analogue of this result, with a small loss on \( \psi \) in the transfer of regularity.
4. Miscellaneous concluding remarks

4.1. Sibony’s example. We first show that in the construction of Sibony referred to in the introduction, the Wermer example \( \hat{X} \) has zero trace measure. This is a consequence of the following observation.

**Proposition 4.1.** Let \( u \) be a nonnegative \( C^{1,1} \) psh function in the unit ball of \( \mathbb{C}^2 \), and let \( T = dd^c u \). Then \( \sigma_T(\{ u = 0 \}) = 0 \).

**Proof.** Let \( E = \{ u = 0 \} \) and assume that \( \sigma_T(E) > 0 \). Since \( T \) is a current with \( L^\infty \) coefficients, \( \sigma_T \) (or equivalently, the Laplacian of \( u \)) is absolutely continuous with respect to the Lebesgue measure hence \( E \) has positive Lebesgue measure. We will prove that \( \Delta u \) vanishes a.e. on \( E \), thus contradicting the fact that \( \sigma_T(E) > 0 \).

It is classical that \( u \) is twice differentiable a.e. Let \( p \in E \) be such a differentiability point. Since \( u \) has a minimum at \( p \), \( du_p \) vanishes so by the Taylor formula we infer that \( u(p + h) - u(p) = O(\|h\|^2) \) as \( h \to 0 \).

Let \( a_4 \) be the volume of the unit ball of \( \mathbb{C}^2 \), so that \( \text{Leb}(B(p, r)) = a_4 r^4 \). Almost every \( p \in E \) is a density point for the Lebesgue measure, that is, at such a \( p \), \( \frac{\text{Leb}(E \cap B(p, r))}{a_4 r^4} \to 1 \) when \( r \to 0 \). If we further assume that \( u \) is twice differentiable at \( p \), we infer that

\[
(16) \quad \frac{1}{a_4 r^4} \int_{B(p, r)} u = \frac{1}{a_4 r^4} \int_{B(p, r) \setminus E} u \to \frac{\text{Leb}(B(p, r) \setminus E)}{a_4 r^4} O(r^2) = o(r^2).
\]

We conclude by using the Jensen formula, which implies that if \( \Delta u \in L^1_{\text{loc}} \), then

\[
\lim_{r \to 0} \frac{1}{r^2} \left( \frac{1}{a_4 r^4} \int_{B(p, r)} u - u(p) \right) = \frac{1}{12} \Delta u(p)
\]

almost everywhere (see e.g. [K] p. 143), which by (16) implies that \( \Delta u = 0 \) a.e. on \( E \). \( \square \)

4.2. Hausdorff dimension and Lebesgue measure. If a positive closed current \( T \) with \( C^{1,\alpha} \) potential, then the mass of a ball of radius \( r \) relative to its trace measure is \( O(r^{3+\alpha}) \). In particular it cannot carry any mass on a set of Hausdorff dimension \( < 3 + \alpha \). From this remark we conclude that the support of the current of Theorem 3.1 has dimension 4. It is of course possible to refine this result in the spirit of Theorem 3.1 by using appropriate gauge functions.

On the other hand the vertical slices of \( X \) near the boundary have zero Lebesgue measure, since (we freely use the results and notation of [3.2]) they can be covered by \( \approx M^2 \) boundedly distorted balls of radius \( \approx \frac{P_n}{M_n} \). So near the boundary, \( X \) has zero Lebesgue measure. In a similar fashion, it is clear from the proof of Proposition 3.8 that if \( h(r) \) is a function such that \( \mu(B(p, r)) \leq h(r) \) for all \( p \) and \( r \) then necessarily \( \lim_0 \frac{h(r)}{r} = \infty \).

In particular our currents are never \( C^{1,1} \), at least near the boundary –it is very likely that the same is true everywhere in the bidisk, but we couldn’t prove it.

4.3. Wermer examples without subdivision. Our results give some interesting insights on the geometric properties of ordinary Wermer examples (that is, without the subdivision step, or equivalently \( m_n = 1 \) for all \( n \)). With notation as in Section 4 X is now defined as the nested intersection of the sequence of sets \( \{|P_n| < \delta_n\} \), with \( P_{n+1} = P_n^2 + \varepsilon_{n+1} A_{n+1} \), \( \varepsilon_{n+1} = \delta_n^2 / 2 \) and \( \delta_{n+1} = \delta_n^2 r_{n+1} / 4 \). By Theorem 2.1 if the series \( \sum_{n \geq 1} \frac{\log r_n}{2^n} \) converges, then the associated \( T \) has continuous potential, so \( X \) is not pluripolar.
Conversely, the logarithmic capacity of a subset of $\mathbb{C}$ of the form $\{ |P| \leq \delta \}$, where $P$ is a monic polynomial of degree $d$, equals $\delta^{1/d}$, so the capacity of the vertical fibers of $X$ equals $\lim \delta_n^{1/2^n}$. Using the inductive definition of the $\delta_n$ it is easy to see that $\frac{1}{2^n} \log \delta_n = \sum_{k=1}^n \frac{|\log r_k|}{2^n} + O(1)$, so if the series $\sum_{n \geq 1} \frac{|\log r_n|}{2^n}$ diverges, the vertical fibers of $X$ are polar. It follows from [LS, Theorem 4.1] that $X$ is complete pluripolar in this case.

By the results of §3.2 the vertical slices of $X$ near the boundary are covered by $2^n$ boundedly distorted balls of super-exponentially small radius $\approx R_n$. Thus, even when $\sum_{n \geq 1} \frac{|\log r_n|}{2^n}$ converges, these slices have Hausdorff dimension 0. In particular the potential of $T$ is never Hölder continuous in this case. On the other hand since $R_n$ can have arbitrary slow super-exponential growth, it can be shown that essentially any sub-Hölder modulus of continuity can be reached.

To obtain Hölder continuous examples (of arbitrary exponent $< 1$) without subdividing, one modifies the construction by putting $P_{n+1} = P_{n+1}^{d_n+1} + \varepsilon_{n+1} A_{n+1}$ for a well chosen sequence $d_n \to \infty$. Details will appear elsewhere.

4.4. Some open questions. We conclude with some questions on the geometric properties of currents with $T \wedge T = 0$.

- Is it possible to get $C^{1,1}$ regularity in Theorem 1?
- What is the best possible regularity for extremal such currents? In our construction extremality is lost in the subdivision process.
- Does there exist a $C^{1,1}$ psh function $u$ in $\mathbb{B}$ with $(dd^c u)^2 = 0$ and $\text{Supp}(dd^c u) = \mathbb{B}$, which is not harmonic on any holomorphic disk?
- On the other hand, does laminarity hold in the $C^2$ case?

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