Rigidity of center Lyapunov exponents for Anosov diffeomorphisms on 3-torus

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June 16, 2023

Abstract

Let $f$ and $g$ be two Anosov diffeomorphisms on $\mathbb{T}^3$ with three-subbundles partially hyperbolic splittings where the weak stable subbundles are considered as center subbundles. Assume that $f$ is conjugate to $g$ and the conjugacy preserves the strong stable foliation, then their center Lyapunov exponents of corresponding periodic points coincide. This is the converse of the main result of Gogolev and Guysinsky in [9]. Moreover, we get the same result for partially hyperbolic diffeomorphisms derived from Anosov on $\mathbb{T}^3$.

1 Introduction

Let $M$ be a smooth closed Riemannian manifold. We say a diffeomorphism $f : M \to M$ is Anosov, if there exists a $Df$-invariant continuous splitting $TM = E^s_f \oplus E^u_f$ such that $Df$ is uniformly contracting on the stable bundle $E^s_f$ and uniformly expanding on the unstable bundle $E^u_f$. It is known that an Anosov diffeomorphism with one-dimensional stable or unstable bundle exists only on $d$-torus $\mathbb{T}^d$ ($d \geq 2$) [5][18], in particular $\mathbb{T}^3$ is the only 3-manifolds admitting Anosov diffeomorphisms. Moreover an Anosov diffeomorphism $f$ on $\mathbb{T}^d$ is always topologically conjugate to its linearization $f_* : \pi_1(\mathbb{T}^d) \to \pi_1(\mathbb{T}^d)$ which induces a toral Anosov automorphism [4][17], i.e., there exists a homeomorphism $h : \mathbb{T}^d \to \mathbb{T}^d$ homotopic to $\text{Id}_{\mathbb{T}^d}$ such that $h \circ f = f_* \circ h$. However the conjugacy is usually only Hölder continuous. Indeed, if two Anosov diffeomorphisms are smoothly conjugate, then the derivatives of their return maps on corresponding periodic points are conjugate via the derivative of the conjugacy and more weakly their corresponding periodic Lyapunov exponents coincide. There are plenty of enlightening works researching the regularity of the conjugacy under the assumption of the same Lyapunov exponents, usually called rigidity, e.g. [8][9][16][24] in which the rigidity for Anosov diffeomorphisms with partially hyperbolic splittings has been studied extensively.

**Definition 1.1.** A diffeomorphism $f : M \to M$ is called partially hyperbolic, if there is an invariant splitting for $Df$, $TM = E^s_f \oplus E^c_f \oplus E^u_f$, which satisfies

$$||Df v^s|| < \tau(x) < ||Df v^c|| < \mu(x) < ||Df v^u||,$$

for all $v^\sigma \in E^\sigma_f(x)$ with $||v^\sigma|| = 1$, where $\sigma = s, c, u$ and $0 < \tau < 1 < \mu$ are two continuous functions on $M$.

We call $E^s_f, E^c_f$ and $E^u_f$, the stable bundle, the center bundle and the unstable bundle, respectively. It is well known that the stable/unstable bundle of a partially hyperbolic diffeomorphism is always uniquely integrable and the integral manifolds form a foliation on $M$ [20], called stable/unstable foliation and denoted by $\mathcal{F}^s_f$ and $\mathcal{F}^u_f$. However the center bundle in general is not integrable, see [15] for a counterexample on $\mathbb{T}^3$. A partially hyperbolic diffeomorphism $f$ is called dynamically coherent, if
there are two \( f \)-invariant foliations denoted by \( F_f^{cu} \) and \( F_f^{cs} \) tangent respectively to \( E_f^{cu} := E_f^c \oplus E_f^{cu} \) and \( E_f^{cs} := E_f^s \oplus E_f^{cs} \). It is clear that if \( f \) is dynamically coherent, then \( F_f^{cu} := F_f^{cu} \cap F_f^{cs} \) is a \( f \)-invariant foliation tangent to \( E_f^c \). In particular, for the case of \( \|Df v\| < 1 \) for all unit \( v \in E_f^c \), we also call \( E_f^c \) the weak stable bundle, \( E_f^s \) the strong stable bundle and the corresponding foliation \( F_f^s \) is called strong stable foliation.

In [9], Gogolev and Guysinsky considered the local rigidity of an Anosov automorphism \( L \) on \( T^3 \) with partially hyperbolic splitting. For any two partially hyperbolic Anosov diffeomorphisms \( f \) and \( g \) on \( T^3 \) which are conjugate via \( h \), we say \( f \) and \( g \) have the same center periodic data, if their corresponding, by \( h \), periodic points have the same Lyapunov exponents on center bundles. The main technical lemma in [9] is that for \( L \) with two-dimensional stable bundle, if its \( C^1 \)-perturbations \( f \) and \( g \) have the same center periodic data, then their conjugacy \( h \) preserves the strong stable foliation, i.e., \( h(F_f^s(x)) = F_g^s(h(x)) \) for all \( x \in T^3 \). In this paper, we show the converse of this property.

**Theorem 1.2.** Let \( f, g : T^3 \to T^3 \) be two \( C^{1+a} \)-smooth partially hyperbolic Anosov diffeomorphisms whose center bundles are weak stable. Assume that \( f \) is conjugate to \( g \) via a homeomorphism \( h \). If \( h \) preserves the strong stable foliations, then \( f \) and \( g \) have the same center periodic data.

An interesting corollary of Theorem 1.2 is that the topological conjugacy preserving the strong stable foliation implies it is smooth along the center (weak stable) foliation. Here the conjugacy being \( C^1 \)-smooth along the center foliation is defined as the derivative along each center leaf being continuous with respect to the topology of the whole manifold. Note that in our case, the center bundle is integrable and the conjugacy preserves the center foliation [21], also see Remark 3.2.

**Corollary 1.3.** Let \( f, g : T^3 \to T^3 \) be two \( C^{1+a} \)-smooth partially hyperbolic Anosov diffeomorphisms whose center bundles are weak stable. Assume that \( f \) is conjugate to \( g \) via a homeomorphism \( h \). Then \( h \) preserves the strong stable foliation, if and only if, \( h \) is \( C^1 \) along the center foliation.

**Remark 1.4.** One can get Corollary 1.3 from Theorem 1.2 and [9] Lemma 5 and Lemma 6] which state that the same center periodic data of \( f \) and \( g \) implies that \( h \) is smooth along the center foliation and \( h \) preserves the strong stable foliation. We mention that the assumption of local perturbation in [9] is just for getting that the center bundle of \( f \) is integrable and preserved by \( h \). As mentioned above, we now have these two properties. For the inverse conclusion, for any periodic point \( p \) with period \( n \), since \( h \) is smooth along the center foliation, we can take the directional derivative of \( f^n = h^{-1} \circ g^n \circ h \) in the direction of \( E^s \), we can get that \( \lambda^s(p, f) = \lambda^c(h(p), g) \). From the conclusion of [9], we can know that \( h \) preserves the strong stable foliation.

We are also curious about the higher-dimensional case of Theorem 1.2 and Corollary 1.3 since Gogolev extended the result of local rigidity in [9] to the higher-dimensional case [8]. For convenience, we state this question in Section 3 see Question 3.5.

In this paper, we also consider partially hyperbolic diffeomorphisms on general closed Riemannian manifolds with integrable one-dimensional center bundles and with accessibility. We call a partially hyperbolic diffeomorphism \( f : M \to M \) is accessible, if any two points on \( M \) can be connected by curves each of which is tangent to \( E_f^c \) or \( E_f^s \). It is well known that accessibility is \( C^r \)-dense and \( C^1 \)-open among \( C^r \) \((r \geq 1)\) partially hyperbolic diffeomorphisms with one-dimensional center bundles [1][2][14].

Let \( f, g : M \to M \) be two partially hyperbolic diffeomorphisms and dynamically coherent. We call \( f \) is fully conjugate to \( g \), if there exists a homeomorphism \( h : M \to M \) such that \( f \) is conjugate to \( g \) via \( h \) and \( h \) preserves the stable foliations, unstable foliations and center foliations, respectively, i.e.,

\[
h(F_f^\sigma(x)) = F_g^\sigma(h(x)), \quad \forall x \in M \text{ and } \sigma = s, c, u.
\]
It is clear that if \( f \) is accessible and fully conjugate to \( g \), then \( g \) is also accessible. Now, we can extend Theorem 1.2 to the following one in which the dimension of the manifold \( M \) can be bigger than 3. We mention that \( h \) does not need to be homotopic to identity in Theorem 1.5.

**Theorem 1.5.** Let \( f, g : M \to M \) be two \( C^{1+\alpha} \)-smooth partially hyperbolic diffeomorphisms with one-dimensional center bundles and dynamically coherent. If \( f \) is accessible and fully conjugate to \( g \) via a homeomorphism \( h \), then \( h \) is \( C^1 \) along the center foliation.

As a corollary, we apply Theorem 1.5 to partially hyperbolic diffeomorphisms derived from Anosov on \( T^3 \). We call \( f : T^3 \to T^3 \) derived from Anosov, also called a DA diffeomorphism, i.e., its linearization \( f_* : \pi_1(T^3) \to \pi_1(T^3) \) induces an Anosov automorphism. We mention that the partially hyperbolic diffeomorphism derived from Anosov is one of the three types of partially hyperbolic diffeomorphisms on 3-manifolds with solvable fundamental group [11].

**Corollary 1.6.** Let \( f, g : T^3 \to T^3 \) be two \( C^{1+\alpha} \)-smooth partially hyperbolic diffeomorphisms and conjugate via a homeomorphism \( h \). Suppose the linearization of \( f \) is an Anosov automorphism with one-dimensional unstable bundle. If \( h \) preserves the stable foliation, then \( h \) is \( C^1 \) along the center foliation.

It is clear that Theorem 1.2 is a special case of Corollary 1.6, as we can take the directional derivative of \( f^n = h^{-1} \circ g^n \circ h \) in the direction of \( E^c \) at periodic point \( p \) with period \( n \) to get the same center periodic data. Another special case of Corollary 1.6 is that \( g = f_* \) which has been studied in [7]. We refer readers to [10] for higher-dimensional case under the assumption of \( g = f_* \).

**Acknowledgements**

We are grateful for the valuable communication and suggestions from Shaobo Gan and Yi Shi. Thanks for the comments from the anonymous referees about Remark 2.7 and Remark 3.4. The authors were partially supported by National Key R&D Program of China (2021YFA1001900).

## 2 Rigidity of center Lyapunov exponents in the accessible case

In this section, we prove Theorem 1.5. First of all, we recall some basic notions and useful properties of partially hyperbolic diffeomorphisms. Let \( M \) be a smooth closed Riemannian manifold and \( f : M \to M \) be a diffeomorphism admitting partially hyperbolic splitting \( TM = E^s_f \oplus E^c_f \oplus E^u_f \) with \( \dim E^c_f = 1 \). The Lyapunov exponent \( \lambda^c(x, f) \) at point \( x \), if it exists, is denoted by \( \lambda^c(x, f) \) and defined as:

\[
\lambda^c(x, f) = \lim_{n \to +\infty} \frac{1}{n} \log \| Df^n|_{E^c_f(x)} \|.
\]

Assume further that \( f \) is dynamically coherent. Define the local leaf with size \( \delta > 0 \) by

\[
\mathcal{F}\sigma^f(x, \delta) := \{ y \in \mathcal{F}\sigma^f(x) \mid d_{\mathcal{F}\sigma^f}(x, y) < \delta \},
\]

where \( \sigma = s, c, u, cs, cu \) and \( d_{\mathcal{F}\sigma^f}(\cdot, \cdot) \) is the metric on \( \mathcal{F}\sigma^f \) induced by the Riemannian metric on the base space. By coherence, the local stable/unstable foliation \( \mathcal{F}\sigma^f / \mathcal{F}\sigma^u \) induces holonomy maps restricted on \( \mathcal{F}\sigma^s / \mathcal{F}\sigma^u \) as follow,

\[
\text{Hol}_{f, x, y}^{s/u} : \mathcal{F}^c_f(x, \delta_1) \to \mathcal{F}^c_f(y, \delta_2),
\]

\[
z \to \mathcal{F}^c_f(y, \delta_2) \cap \mathcal{F}^{s/u}_f(z, R),
\]
such that $\text{Hol}_{f,x,y}^{s/u}$ is a homeomorphism for some $x \in M$ and $y \in \mathcal{F}^{s/u}_f(x)$ with constants $\delta_1 > 0, \delta_2 > 0$ and $R > 0$ which rely on the choice of $x, y$.

Since one-dimensional center bundle automatically implies the center bunching condition in the classical work [22], we have the following proposition adapted to our case. We also refer to [20 Theorem 7.1] for this property in the view of absolutely continuous holonomy map.

**Proposition 2.1** ([22]). Let $f : M \to M$ be a $C^{1+a}$-smooth partially hyperbolic diffeomorphism with one-dimensional center bundle. If $f$ is dynamically coherent, then the local unstable and local stable holonomy maps restricted respectively on $\mathcal{F}^u_f$ and $\mathcal{F}^s_f$ are uniformly $C^1$-smooth.

**Remark 2.2.** The uniformly $C^1$-smooth holonomy maps in Proposition 2.1 means that the derivative $D\text{Hol}_{f,x,y}^{s/u}$ is continuous with respect to points $x,y \in M$. We refer to [23] for delicate analysis for the partial derivatives of the (un)stable bundle with respect to center bundle.

Recall that we say $f$ is accessible, if any two points on $M$ can be connected by curves each of which is tangent to $E_1^c$ or $E_{-1}^u$. In fact, these curves have uniform length and number [2 Proposition 4]. For convenience of readers, we state it as follow.

**Proposition 2.3** ([2]). Let $f : M \to M$ be a $C^1$-smooth partially hyperbolic diffeomorphism and accessible. Then there exists $N \in \mathbb{N}$ and $R > 0$ such that any two points $x,y \in M$ can be connected by at most $N$ curves each of which has length at most $R$ and tangent to $E_1^c$ or $E_{-1}^u$.

Now we can prove Theorem 1.5

**Proof of Theorem 1.5** Let $h : M \to M$ be a full conjugacy between $f$ and $g$ such that

$$h(\mathcal{F}^\sigma_f(x)) = \mathcal{F}^\sigma_g(h(x)), \quad \forall x \in M \text{ and } \sigma = s,c,u.$$  

Then for any point $x \in M$, $h$ restricted on the local center leaf $\mathcal{F}^c_f(x,\delta)$ of $x$ is a homeomorphism onto the image. For simplify, we denote it by

$$h_{c,x} : \mathcal{F}^c_f(x,\delta) \to \mathcal{F}^c_g(h(x)).$$

Fix a point $z_0 \in M$. Since $\mathcal{F}^c_f(z_0,\delta)$ and $\mathcal{F}^c_g(h(z_0))$ are one-dimensional $C^1$-smooth embedded submanifolds, we get that $h_{c,z_0}$ is monotonic. Therefore, there exists a point $p \in \mathcal{F}^c_f(z_0,\delta)$ such that $h_{c,z_0}$ is differentiable at $p$, then $h_{c,p}$ is differentiable at $p$. By the accessibility and the $C^1$-smooth holonomy maps, we can get that $h_{c,x}$ is differentiable at $x$ for all $x \in M$. More precisely, we have the following claim.

**Claim 2.4.** For any $x \in M$, $h_{c,x}$ is differentiable at $x$.

**Proof of Claim 2.4** Since $f$ is accessible, in particular by Proposition 2.3 there exists $N$ and $R$ such that for any $x \in M$, there exists at most $N$ local stable and unstable manifolds with length at most $R$ to connect $x$ and $p$. Hence, we can assume that there exist $p = x_0, x_1 \ldots, x_n = x$ such that

$$x_i \in \mathcal{F}^{s/u}_f(x_{i-1},R) \quad \text{and} \quad x_{i+1} \in \mathcal{F}^{u/s}_f(x_i,R), \quad \forall 1 \leq i \leq n-1.$$ 

Let $y_i = h(x_i)$ for all $0 \leq i \leq n$. Since $h_{c,p}$ is differentiable at $p = x_0$, it suffices to prove that if $x_i$ is a differentiable point of $h_{c,x_i}$, then $x_{i+1}$ is a differentiable point of $h_{c,x_{i+1}}$.

Without loss of generality, we assume that $x_{i+1} \in \mathcal{F}^c_f(x_i,R)$. Since $h$ preserves all invariant foliations of $f$ and $g$, for any $x_{i+1}$, there exists $\delta > 0$ such that

$$h_{c,x_{i+1}}(z) = \text{Hol}_{g,y_i,y_{i+1}}^{c} \circ h_{c,x_i} \circ \text{Hol}_{f,x_{i+1},x_i}^{c}(z), \quad \forall z \in \mathcal{F}^c_f(x_{i+1},\delta).$$
See Figure 1. Notice that

\[ Dh_{c,x_{i+1}}(x_{i+1}) = D\text{Hol}_{g,y_{i+1}}^{s} (y_{i}) \circ Dh_{c,x_{i}}(x_{i}) \circ D\text{Hol}_{f,x_{i+1},x_{i}}^{s} (x_{i+1}). \]  

(2.1)

Since both \( \text{Hol}_{f,x_{i+1},x_{i}}^{s} \) and \( \text{Hol}_{g,y_{i+1}}^{s} \) are \( C^{1} \)-smooth within size \( R > 0 \) (see Proposition 2.1), we get the proof of this claim.

By Claim 2.4 we can also get that for any \( x \in M, Dh_{c,x}(x) \neq 0 \). Indeed, if there exists \( q \in M \) such that \( Dh_{c,q}(q) = 0 \), it follows from the proof of Claim 2.4 that \( Dh_{c,x}(x) = 0 \) for any \( x \in M \). It contradicts with the fact that \( h \) is a homeomorphism restricted on \( \mathcal{F}_{f}(p) \). Then we get that \( h_{c} \) is differentiable.

For proving that \( h_{c} \) is actually \( C^{1} \)-smooth, we need more precise property on accessibility. Since \( f \) is accessible with one-dimensional center bundle, there are two unstable disks \( U_{1} \) and \( U_{2} \) being skew \([13, 14]\), that is

1. There exists a \((\dim E^{u} + 1)\)-dimensional \( cu\)-disk \( V \) containing \( U_{2} \) (note that \( f \) is coherent).
2. \( U_{2} \) separates \( V \) into two connected components.
3. \( U_{1} \) and \( V \) define the holonomy maps \( \text{Hol}^{l}: U_{1} \to V \) as \( \text{Hol}^{l}(x) = \mathcal{F}_{\delta}^{s}(x) \cap V \) for small \( \delta \).
4. \( \text{Hol}^{l}(U_{1}) \) intersects both connected components of \( V \setminus U_{2} \).

**Claim 2.5.** For any \( z \in V \), the derivative \( Dh_{c,z} \) is continuous at \( z \) restricted on \( V \).

**Proof of Claim 2.5.** Denote \( V \cap \text{Hol}^{l}(U_{1}) = V_{2} \). Take \( x_{0} \in V_{2} \cap U_{2} \). Denote \( (\text{Hol}^{l})^{-1}(x_{0}) = x_{1} \in U_{1} \). For any \( y \in V_{2} \), denote \( (\text{Hol}^{l})^{-1}(y) = y_{1} \in U_{1} \). Then \( x_{0} \) and \( y \) can be connected by 3-legs of stable and unstable leaves through points \( x_{1} \) and \( y_{1} \). Note that if \( z \in V_{2} \) is close to \( y \), then the 3-legs from \( z \) to \( x_{0} \) is close (in \( C^{1} \)-topology) to the 3-legs from \( y \) to \( x_{0} \), since local stable and unstable manifolds vary continuously in \( C^{1} \)-topology (see Figure 2). It follows from (2.1) that

\[ Dh_{c,y}(y) = D\text{Hol}_{g,h(x_{0}),h(y)}^{\text{sus}} (h(x_{0})) \circ Dh_{c,x_{0}}(x_{0}) \circ D\text{Hol}_{f,y,x_{0}}^{\text{sus}} (y), \]
Figure 2: The $su$-holonomies vary continuously on $V_2$ in $C^1$-topology.

and

$$Dh_{e,z}(z) = D\text{Hol}^{sus}_{g, h(x_0), h(z)}(h(x_0)) \circ Dh_{e,x_0}(x_0) \circ D\text{Hol}^{sus}_{f,z,x_0}(z),$$

where $\text{Hol}^{sus}_{f,y,x_0} = \text{Hol}^u_{f,y_1,x_1} \circ \text{Hol}^u_{f,y_1,x_1} \circ \text{Hol}^c_{g,x_0}$ and the notation $\text{Hol}^{sus}_{g, h(x_0), h(y)}$ is defined in the same sense. By the uniform $C^1$ regularity of holonomy maps (see Proposition 2.1), we have that $Dh_{e,j}(y)$ varies continuously at $y$ restricted on $V_2$.

Then we will "transfer" the continuity of $Dh_{e}$ from $V_2$ to $V$ via the unstable foliation. Note that by the local product structure of foliation $F^c$ and $F^u$, one has that every center leaf intersects with $V_2$ once. However, a local unstable leaf may intersect with $V_2$ infinitely many times (see Figure 3). For convenience, we assume that

$$V = \bigcup_{x \in V_2} F^u_{f,loc}(x),$$

and $V$ is open in the topology of a local $cu$-leaf. For any $z \in V$, denote $S(z) = F^u_{f,loc}(z) \cap V_2$.

Figure 3: Different $u$-paths to calculate the derivative $Dh_c$ on a neighborhood of $z$ in $V$.

It is clear that for any $\epsilon > 0$, there exists $0 < \delta < \epsilon$ such that

$$B_\delta(z) \subseteq \bigcup_{s \in S(z)} F^u_{f,loc}(B_\epsilon(s)),$$

where $B_\epsilon(s)$ is an $\epsilon$-open ball in $V$. Then for any $\tilde{z}, \tilde{z} \in B_\delta(z)$, there exist two points $\tilde{y} \in F^u_{f,loc}(\tilde{z}) \cap V_2$ and $\tilde{y} \in F^u_{f,loc}(\tilde{z}) \cap V_2$ such that there are points $y(\tilde{z}) \in S(z)$ and $y(\tilde{z}) \in S(z)$ such that $\tilde{y} \in B_\epsilon(y(\tilde{z}))$ and $\tilde{y} \in B_\epsilon(y(\tilde{z}))$ (see Figure 3). Note that

$$Dh_{c,z}(z) = D\text{Hol}^u_{g, h(y(\tilde{z})), h(z)}(h(\tilde{y})) \circ Dh_{c,y}(\tilde{y}) \circ D\text{Hol}^u_{f,z,y(\tilde{z})}(z),$$
and

\[ Dh_{c,z}(\tilde{z}) = DHol_{g,h(\tilde{y}),h(\tilde{z})}^u(h(\tilde{y})) \circ Dh_{c,\tilde{y}}(\tilde{y}) \circ DHol_{f,\tilde{z},\tilde{y}}^u(\tilde{z}). \]

Since holonomy maps are uniformly \( C^1 \)-smooth and \( Dh_c \) is continuous on \( V_2 \), we have that \( Dh_{c,z}(z) \) is close to \( Dh_{c,z}(\tilde{z}) \). Similarly, we can use the holonomy maps given by the local unstable leaves of \( \tilde{y} \) and \( y(\tilde{z}) \) to control the variation between \( Dh_{c,z}(z) \) and \( Dh_{c,z}(\tilde{z}) \). Then we have that \( Dh_{c,z}(z) \) varies continuously at \( z \) restricted on \( V \).

**Claim 2.6.** For any \( w \in M \), the derivative \( Dh_{c,w} \) is continuous at \( w \) on \( M \).

**Proof of Claim 2.6.** For any point \( w \in M \), there is an \( su \)-path from \( w \) to some point \( z \in V \), since \( f \) is accessible. Thus there exists a neighborhood \( B(w) \) of \( w \) in \( M \) such that for any point \( w' \in B(w) \) there is an \( su \)-path from \( w' \) to \( z' \in V \) close to the \( su \)-path from \( w \) to \( z \) in the sense of \( C^1 \)-topology (see Figure 4). Then we have that

\[ Dh_{c,w}(w) = DHol_{g,h(z),h(w)}^u(h(z)) \circ Dh_{c,z}(z) \circ DHol_{f,w,z}(w), \]

and

\[ Dh_{c,w}(w') = DHol_{g,h(z'),h(w')}^u(h(z')) \circ Dh_{c,z}(z') \circ DHol_{f,w',z'}(w'), \]

where \( DHol_{f,w,z} \) is the holonomy map (given by some compositions of \( Hol^u \) and \( Hol^v \)) of a fixed \( su \)-path from \( w \) to \( z \) and \( DHol_{f,w',z'} \) is the homology map of the \( su \)-path from \( w' \) to \( z' \) which is close (in \( C^1 \)-topology) to the fixed \( su \)-path (from \( w \) to \( z \)). And \( DHol_{g,h(z),h(w)} \) is given by the image of the fixed \( su \)-path (from \( w \) to \( z \)) via the conjugacy \( h \). This implies that \( Dh_{c,w}(w) \) is continuous for any \( w \in M \) since we already have that \( Dh_c \) is continuous on \( V \).

![Figure 4: The \( su \)-holonomies vary continuously in \( C^1 \)-topology.](image)

Hence \( h \) is \( C^1 \)-smooth along the center foliation.

**Remark 2.7.** To prove the \( C^1 \)-regularity of \( h_c \), there is another approach which needs more subtle functional analysis on the derivatives. From Theorem 7.3 in [19], we can know that if a function is differentiable everywhere, then the derivative is continuous on a residual set. When Claim 2.4 is gotten, there exists a point \( z \in M \) such that \( Dh_{c,z}(z) \) is \( C^0 \) restricted on \( \mathcal{F}_f^c(z) \). Just like the proof of Claim 2.6 we can get that \( Dh_{c,x}(x) \) is \( C^0 \) at any point \( x \in M \) with respect to the topology of \( M \) by the equation (2.1), since the holonomies are \( C^1 \) and the systems are accessible.

### 3 Anosov diffeomorphisms on 3-torus

In this section, we prove Corollary 1.6 and hence we can get the proof of Theorem 1.2. In fact, we will reduce Corollary 1.6 to the accessible case Theorem 1.5. At the end of this section, we state a
question for higher-dimensional torus. Note that in this section we always consider the topological conjugacy being homotopic to identity.

Let $f : T^3 \to T^3$ be a partially hyperbolic diffeomorphism homotopic to an Anosov automorphism $A : T^3 \to T^3$. It has been proved in [21] that $f$ is dynamically coherent. Moreover, $A$ also admits a partially hyperbolic splitting

$$TT^3 = E^s_A \oplus E^u_A \oplus E^c_A.$$ 

We call such $A : T^3 \to T^3$ is center contracting, if the modulus of the eigenvalue of $A \in \text{GL}_3(\mathbb{Z})$ corresponding to $E^c_A$ is smaller than 1. Further, one has the following proposition. Denote by $\mathcal{F}^\sigma_f$ ($\sigma = f, A$, and $\sigma = s, c, u, cs, cu$) the liftings of $\mathcal{F}^\sigma_A$ by the natural projection $\pi : \mathbb{R}^3 \to T^3$.

**Proposition 3.1** ([3, 21]). Let $f : T^3 \to T^3$ be a partially hyperbolic diffeomorphism homotopic to a center contracting Anosov automorphism $A : T^3 \to T^3$. Let $F$ be a lifting of $f$ by $\pi$. Then there exists a uniformly continuous surjection $H_f : \mathbb{R}^3 \to \mathbb{R}^3$ (called semi-conjugacy) such that

1. $H_f \circ F = A \circ H_f$ and there is $C > 0$ satisfying $d_{C^0}(H_f, \text{Id}_{\mathbb{R}^3}) < C$. And such $H$ is unique.
2. $H_f(x + n) = H_f(x) + n$ for all $x \in \mathbb{R}^3$ and $n \in \mathbb{Z}^3$ and hence $H$ can descend to $T^3$ denoted by $h_f : T^3 \to T^3$ such that $h_f$ is homotopic to $\text{Id}_{T^3}$ and $h_f \circ f = A \circ h_f$.
3. $h_f : \mathcal{F}^u_f(x) \to \mathcal{F}^u_A(H_f(x))$ is a homeomorphism, for all $x \in \mathbb{R}^3$.
4. $h_f(\mathcal{F}^\sigma_f(x)) = \mathcal{F}^\sigma_A(H_f(x))$, for all $x \in \mathbb{R}^3$ and $\sigma = c, cs, cu$.

**Remark 3.2.** If $f : T^3 \to T^3$ is Anosov, then the semi-conjugacy $h_f$ given by Proposition 3.1 is in fact a conjugacy. It follows from the fourth item that $h_f$ preserves the center foliation and the restriction of $h_f$ on each center leaf is a homeomorphism.

We call $f$ is su-integrable, if there exists a $(\dim E^s_f + \dim E^u_f)$-dimensional foliation denoted by $\mathcal{F}^{su}_f$ tangent to $E^s_f \oplus E^u_f$. It is clear from the definitions that if $f$ is accessible, then it is not su-integrable. Conversely, [13] proved that for a conservative partially hyperbolic diffeomorphism $f : T^3 \to T^3$ derived from Anosov, if $f$ is not accessible, then it is su-integrable and conjugate to its linearization. We mention that the conservative condition can be replaced by assuming the non-wandering set $\Omega(f) = T^3$. Moreover without the conservative condition, combining the main results in [7] and [12], one has the following dichotomy (Theorem 3.3, also see [12, Corollary 1.4]). We also refer to [6] for the case of Anosov diffeomorphism $f$ on $T^3$ in which one has that the conjugacy between $f$ and its linearization $f_*$ preserves the strong stable foliation, if and only if, $f$ is su-integrable, if and only if, $f$ is not accessible.

**Theorem 3.3** ([7, 12]). Let $f : T^3 \to T^3$ be a $C^{1+\alpha}$-smooth partially hyperbolic diffeomorphism homotopic to an Anosov automorphism $A : T^3 \to T^3$. Then

- either $f$ is accessible,
- or $f$ is Anosov and $\lambda^c(p, f) = \lambda^c(A)$, for all $p \in \text{Per}(f)$.

**Remark 3.4.** We mention here that we can get our Theorem 1.2 by combining Theorem 1.5 with the main results of [7] and [13]. Indeed, let $f, g : T^3 \to T^3$ satisfy the assumption of Theorem 1.2. Then by [13], one has that $f$ and $g$ are simultaneously either su-integrable, or accessible. In the first case, the center periodic data of $f$ and $g$ are the same as one of their linearization respectively [7]. The other case is an immediate corollary of Claim 2.4.

By Theorem 3.3, we will reduce Corollary 1.6 to Theorem 1.5.
Proof of Corollary 1.6. Let \( f \) and \( g \) satisfy the conditions of Corollary 1.6. Since \( f \) is conjugate to \( g \) via \( h \) which is a homeomorphism homotopic to \( \text{Id}_{\mathbb{T}^1} \) such that \( h \circ f = g \circ h \) and \( f \) is derived from Anosov automorphism \( A \in \text{GL}_3(\mathbb{Z}) \). We have that \( g_\ast = (h \circ f \circ h^{-1})_\ast = f_\ast = A \).

We claim that \( f \) is accessible if and only if \( g \) is accessible. Indeed, let \( F,G \) be any two liftings of \( f \) and \( g \) by the natural projection \( \pi \). Let \( H_f \) and \( H_g \) be the semi-conjugacy between \( F \) with \( A \) and \( G \) with \( A \) respectively given by Proposition 3.1. Let \( h \) be a conjugacy between \( f \) and \( g \) and let \( H \) be its lifting with \( H \circ F = G \circ H \). Note that \( h \) is homotopic to \( \text{Id}_{\mathbb{T}^1} \), thus \( H \) is uniformly bounded with \( \text{Id}_{\mathbb{T}^3} \). Since \( H_f \circ H^{-1} \) is also a semi-conjugacy between \( G \) and \( A \) satisfying the first item of Proposition 3.1, we have \( H_g = H_f \circ H^{-1} \) by the uniqueness. By Proposition 3.1 again, one has that

\[
H_\ast(\mathcal{F}_f^\sigma(x)) = \mathcal{F}_A^\sigma(H_\ast(x)), \ \forall x \in \mathbb{R}^3, \ \sigma = c, u \text{ and } * = f, g. \tag{3.1}
\]

It follows that

\[
H(\mathcal{F}_f^\sigma(x)) = \mathcal{F}_g^\sigma(H(x)), \ \forall x \in \mathbb{R}^3 \text{ and } \sigma = c, u. \tag{3.2}
\]

In fact, if (3.2) is not true, then one can get a contradiction immediately with (3.1) and the assumption \( H(\mathcal{F}_f^\sigma(x)) = \mathcal{F}_g^\sigma(H(x)) \). In particular, we get that \( h(\mathcal{F}_f^u(x)) = \mathcal{F}_g^u(h(x)) \) for all \( x \in \mathbb{T}^3 \). Hence \( f \) is accessible if and only if \( g \) is accessible.

If \( f \) and \( g \) are both accessible, in particular, by (3.2), we have that

\[
h(\mathcal{F}_f^\sigma(x)) = \mathcal{F}_g^\sigma(h(x)), \ \forall x \in \mathbb{T}^3 \text{ and } \sigma = s, c, u,
\]

then we obtain Corollary 1.6 from Theorem 1.5.

If neither of them is accessible, it follows from Theorem 3.3 that \( \lambda^c(p, f) = \lambda^c(A) = \lambda^c(h(p), g) \), for all \( p \in \text{Per}(f) \) and hence \( h \) is \( C^1 \) along the center foliation (see Remark 1.4).

As mentioned before, we now state a question related to Theorem 1.2 and the local rigidity in [8] of higher-dimensional torus. For convenience, we first restate a main property in [8]. Let \( A : \mathbb{T}^d \to \mathbb{T}^d \ (d \geq 4) \) be an Anosov automorphism with the finest dominated splitting on the weak stable bundle, namely, the stable bundle \( L^s \) of \( A \) admits

\[
L^s = L^s_1 \oplus L^u_1 = L^s_2 \oplus L^u_2 \oplus ... \oplus L^s_k,
\]

where \( L^s_j \) \((1 \leq j \leq k)\) is \( DA \)-invariant, \( L^s_i \) \((2 \leq i \leq k)\) is one-dimensional and there exists a constant \(0 < \lambda < 1 \) such that \( \|D\mathcal{A}|_{L^s_i(x)}\|/\|D\mathcal{A}|_{L^u_i(x)}\| \leq \lambda \), for all \( x \in \mathbb{T}^d \) and all \( 1 \leq i \leq k-1 \). Note that we can choose an appropriate \( C^1 \) neighborhood \( \mathcal{U} \) of \( A \) such that any \( f \in \mathcal{U} \) is also Anosov diffeomorphism with finest dominated splitting on the weak stable bundle. In [8], Gogolev proved that there exists a \( C^1 \) neighborhood \( \mathcal{U} \) of \( A \) such that for any two \( C^2 \)-smooth Anosov diffeomorphisms \( f, g \in \mathcal{U} \) with finest dominated splittings on weak stable bundles \( E^s_i(f) \) \((2 \leq i \leq k)\) coincide with ones of \( E^s_i(g) \), then the conjugacy between \( f \) and \( g \) preserves the strongest stable foliations corresponding to \( E^s_i(f) \) and \( E^s_i(g) \). Hence we can also consider the converse of this property. More precisely, we state this question as follow.

Question 3.5. Let \( A : \mathbb{T}^d \to \mathbb{T}^d \ (d \geq 4) \) be an Anosov automorphism with the finest dominated splittings on the weak stable bundle. Is there a \( C^1 \) neighborhood \( \mathcal{U} \) of \( A \) such that for any two \( C^{1+\alpha} \)-smooth Anosov diffeomorphisms \( f, g \in \mathcal{U} \) with finest dominated splittings on weak stable bundles, if the conjugacy between \( f \) and \( g \) preserves the strongest stable foliations, then \( f \) and \( g \) have the same weak stable periodic data?

Remark 3.6. We refer again to [10] for a positive answer to Question 3.5 under the assumption of \( g = A \).
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