Equivalence between non-bilinear spin-$S$ Ising model and Wajnflasz model

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We propose the mapping of polynomial of degree $2S$ constructed as a linear combination of powers of spin-$S$ (for simplicity, we called as spin-$S$ polynomial) onto spin-crossover state. The spin-$S$ polynomial in general can be projected onto non-symmetric degenerated spin up (high-spin) and spin down (low-spin) momenta. The total number of mapping for each general spin-$S$ is given by $2(2^n^n - 1)$. As an application of this mapping, we consider a general non-bilinear spin-$S$ Ising model which can be transformed onto spin-crossover described by Wajnflasz model. Using a further transformation we obtain the partition function of the effective spin-1/2 Ising model, making a suitable mapping the non-symmetric contribution leads us to a spin-1/2 Ising model with a fixed external magnetic field, which in general cannot be solved exactly. However, for a particular case of non-bilinear spin-$S$ Ising model could become equivalent to an exactly solvable Ising model. The transformed Ising model exhibits a residual entropy, then it should be understood also as a frustrated spin model, due to competing parameters coupling of the non-bilinear spin-$S$ Ising model.

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I. INTRODUCTION

One of the topics of great interest, in statistical physics and mathematical physics are the exact solvable models. This is the case, i.e. for spin-1/2 Ising model without magnetic field solved at first in 1944 by Onsager\cite{1}, since that, the Ising model was widely investigated using several approaches. On the other hand, higher order spin or even spin-1/2 Ising model with external magnetic field are challenging issues in nowadays. Further exact solution were obtained only in a very limited cases, mainly the honeycomb lattices\cite{2,3}. Some exact results has been obtained with restricted parameter, investigated by Mi and Yang\cite{4} using a non-one-to-one transformation\cite{5}. Therefore the non-bilinear spin-$S$Ising model that satisfy this transformation should exhibit frustrated states.

The half-odd-integer spin Ising model already has been discussed previously by Tang\cite{6}. Using the method proposed by Wu\cite{7}, Izmailian\cite{8} obtained an exact solution for a spin-3/2 lattice on square lattice with only nearest interaction or two body interaction spin. Izmailian and Ananikian\cite{9} also has been obtained an exact solution for a honeycomb lattice with spin-3/2. A particular solution of these models could be obtained using the method proposed by Joseph\cite{10} where any spin-$S$ could be projected onto a spin-1/2 Ising model. Another interesting method to map the spin-$S$ model onto spin-1/2 Ising model has been proposed by Horiguchi\cite{11}. More recently we have obtained a set of rigorous mapping for half-odd-integer spin onto spin-1/2\cite{12}, where we have used a direct mapping, half of spin momenta are projected onto spin down, while the remaining half spin momenta are projected onto spin up, we will call this process as spin-$S$ projection with symmetric degeneracy, this mapping is a non-one-to-one mapping.

However, when we consider an integer spin we cannot perform the mapping symmetrically, this issue will be considered in this letter. Then using a non-symmetric projection we will discuss the mapping for a general spin-$S$ polynomial onto a spin-crossover state.

On the other hand, the spin crossover (SC), sometimes called as spin transition, is a phenomenon that occurs in some metal (i.e. Fe and Co) complexes wherein the spin state of the complex changes due to external perturbation such as a variation of temperature, pressure, light irradiation or an influence of a magnetic field\cite{13}. The spin states of the atoms can change between the high-spin (HS) state and low-spin (LS) state as a result of external stimuli, which can be understand as a non-symmetric degeneracy between HS and LS states. In this sense we find a equivalence between spin-$S$ polynomial and spin-crossover state. Another interesting equivalence could be also to that the metastable structure of a charge transfer phase transition, discussed by Miyashita et al.\cite{14}, where static metastability exist in a study of the charge transfer transition in the material (nC$_5$H$_7$)$_2$N[Fe$^{II}$P$_3$H$_4$](dto)$_3$ (dto=C$_4$O$_2$S$_2$).

The outline of this report is as follow: In sec. 2 we present the mapping of spin-$S$ polynomial onto spin-crossover state, in sec. 3 we apply to a a non-bilinear spin-$S$ Ising model and its relation to the metastable state, while in sect 4 we discuss the exactly solvable case. Finally in sec. 5 we present our conclusions.

II. THE SPIN-$S$ POLYNOMIAL TRANSFORMATION ONTO SPIN-CROSSOVER STATE

In order to show the equivalence between spin-$S$ polynomial and spin-crossover state. Let us start considering as an example the projection of spin-3/2 polynomial, as follow,

$$
s_{m}(\frac{3}{2}) = \alpha_{0,m} + \alpha_{1,m}s + \alpha_{2,m}s^2 + \alpha_{3,m}s^3,
$$

where $\alpha_{i,m}$ with $i = 0, \ldots , 3$, are the coefficients to be determined using the projection of spin-3/2 onto spin-1/2, whereas by $m$ we mean the number of solutions or
Table I. The projection spin-2 onto a non-symmetric spin-1/2 or spin-crossover state. By \(g(+1)\) we mean the degeneracy of the spin up (HS), whereas \(g(-1)\) means the degeneracy of spin down (LS).

| Spin-2 polynomial | Proj. to (+1) | Proj. to (-1) | \(g(+1)\) | \(g(-1)\) |
|-------------------|--------------|--------------|-----------|-----------|
| \(\sigma^{(2)}_2(s) = -\frac{1}{2} s^2 + \frac{1}{2} s^3 + \frac{1}{2} s^4 + 1\) | -2, -1, 0 | 1, 2 | 3 | 2 |
| \(\sigma^{(2)}_3(s) = -\frac{1}{2} s^2 + \frac{1}{2} s^3 + \frac{1}{2} s^4 - 1\) | 2, -1, -2 | 0, 1 | 3 | 2 |
| \(\sigma^{(2)}_4(s) = \frac{1}{2} s^2 - \frac{1}{2} s^3 + \frac{1}{2} s^4 + 1\) | 1, 0, -1 | 2, -2 | 3 | 2 |
| \(\sigma^{(2)}_5(s) = \frac{1}{2} s^2 - \frac{1}{2} s^3 - \frac{1}{2} s^4 - 1\) | -2, 1, -1 | 0, 2 | 3 | 2 |
| \(\sigma^{(2)}_6(s) = -\frac{1}{2} s^2 + \frac{1}{2} s^3 + \frac{1}{2} s^4 + 1\) | 2, 0, -2 | 1, -1 | 3 | 2 |
| \(\sigma^{(2)}_7(s) = \frac{1}{2} s^2 + \frac{1}{2} s^3 - \frac{1}{2} s^4 + \frac{1}{2} s^5 + 1\) | -2, 0, 1 | -1, 2 | 3 | 2 |
| \(\sigma^{(2)}_8(s) = \frac{1}{2} s^2 + \frac{1}{2} s^3 - \frac{1}{2} s^4 + \frac{1}{2} s^5 - 1\) | -2, 1, 0, -1 | 2 | 4 | 1 |
| \(\sigma^{(2)}_9(s) = -\frac{1}{2} s^2 + \frac{1}{2} s^3 - \frac{1}{2} s^4 - 1\) | 2, -1, 0, -2 | 1 | 4 | 1 |
| \(P_m(s) = \frac{1}{2} s^2 - \frac{1}{2} s^3 - 1\) | 2, 1, -1, -2 | 0 | 4 | 1 |

The mappings given in (7)-(9) already were considered in references \([7, 11]\), which corresponds to symmetric degeneracy mapping (the first 3 column of vector \(P\)). The remaining solutions corresponds to non-symmetric degeneracy, i.e. three spin momenta are projected onto -1 (LS), whereas the remaining spin moment is projected onto +1 (HS), or vice-verse. These solutions have not been considered yet in the literature.

The next transformation that we discuss could be the spin-2 polynomial onto spin-1/2. It is not possible to map onto a spin-1/2 with symmetric degeneracy because, we have five magnetic momenta to be mapped onto two eigenvalues \(\pm 1\). Then the only possibility is to map by means of non-symmetric projection onto spin-1/2, this kind of mapping lead us to 30 polynomials. However we only need to obtain 9 "representative" polynomials, which are tabulated in table 1. Once again using the exchange of magnetization \(\sigma^{(2)}_m(s) \leftrightarrow -\sigma^{(2)}_m(s)\) and the global inversion of the polynomial \(\sigma^{(2)}_m(s) \leftrightarrow -\sigma^{(2)}_m(s)\), we could obtain easily the remaining projections.
In general the projection of spin-$S$ polynomial onto $\sigma(s)$ with non-symmetric spin degeneracy, we assume the following spin-$S$ polynomial,

$$\sigma^{(S)}(s) = \sum_{j=0}^{2S} \alpha^{(S)}_j s^j,$$

where the coefficients $\alpha^{(S)}_j$ of the polynomial will be determined after projecting onto spin-1/2.

To perform the spin-$S$ polynomial projection we consider the Vandermonde matrix $V^{(S)}$ with equidistant nodes $[-S, S]$, whose elements of the node are $x_j$ which corresponds just to the magnetic momenta of the spin-$S$, the elements of the matrix could be expressed appropriately as $x_j = -S + j$, with $j = 0, 1, 2, \ldots, 2S$, the explicit representation of the Vandermonde matrix is given by,

$$V^{(S)} = \left( \begin{array}{cccc}
1 & x_0 & x_0^2 & x_0^3 \\
1 & x_1 & x_1^2 & x_1^3 \\
\vdots & \vdots & \vdots & \vdots \\
1 & x_{2S-1} & x_{2S-1}^2 & x_{2S-1}^3 \\
1 & x_{2S} & x_{2S}^2 & x_{2S}^3 \\
\end{array} \right),$$

and

$$\alpha^{(S)}_m = \left( \begin{array}{c}
\alpha_{0,m} \\
\alpha_{1,m} \\
\vdots \\
\alpha_{2S-1,m} \\
\alpha_{2S,m} \\
\end{array} \right), \quad \text{and} \quad \mathbf{p}^{(S)}_m = \left( \begin{array}{c}
P_{0,m} \\
P_{1,m} \\
\vdots \\
P_{2S-1,m} \\
P_{2S,m} \\
\end{array} \right),$$

we also define the vector $\alpha^{(S)}_m$ to represent the coefficients for the all possible spin-$S$ polynomial, while the elements of the vector $\mathbf{p}^{(S)}$ represents the projection with non-symmetric degeneracy HS and LS, which can be expressed by

$$\left( \mathbf{p}^{(S)}_m \right)^T = \mathcal{P} \left( \begin{array}{c}
1, \ldots, 1, -1, \ldots, -1 \\
\end{array} \right)_{r \text{ times}} 2S+1-r \text{ times},$$

by $\mathcal{P}$ we mean any permutation of the elements of $\mathbf{p}^{(S)}$, with $r$ projections onto spin up (HS) and $2S+1-r$ projections onto spin down (LS), assuming $r = \{1, \ldots, 2S\}$. For a given $r$ we have $(2S)^r$ permutations, which projects the spin-$S$ polynomial onto spin-1/2. We can verify that all matrix defined above have $2S+1$ dimension.

In order to project the spin-$S$ polynomial onto ±1 values with non-symmetric degeneracy, we can use the matrix notation, so, the following algebraic system equation becomes,

$$\mathbf{p}^{(S)}_m = V^{(S)} \alpha^{(S)}_m.$$

The number of projection of spin-$S$ polynomials that we can obtain are given by the permutations of the elements of the vector $\{1, \ldots, 2S\}$, for each $r = \{1, \ldots, 2S\}$. Therefore, the total number of solutions for each spin-$S$ is given by $(2S+1)^r + (2S+1)^{2S+1} + \cdots + (2S+1)^{2S} = 2(2^{2S} - 1)$, thus $m = \{1, \ldots, 2(2^{2S} - 1)\}$.

Using the matrix notation, we are able to write in general the spin-$S$ polynomial,

$$\sigma^{(S)}_m = s^{(S)} \alpha^{(S)}_m = s^{(S)} \left( V^{(S)} \right)^{-1} \mathbf{p}^{(S)}_m.$$

The inverse of the matrix $V^{(S)}$ could be solved using the recursive equation presented recently by Eisenberg et al. [12], where was discussed a generic algorithm to obtain the elements of the inverse of Vandermonde matrix $V^{(S)}$. Therefore the elements of the matrix $V^{(S)}$ are rewritten conveniently as in reference [13], which reads as

$$v^{(S)}_{i,j} = \frac{(-1)^{i+j}}{(2S+1-j)(j-1)!} \sum_{k=1}^{2S+1} (-S-1)^{k-i} \binom{k}{i} \times \left[ \frac{2S+2}{k+1} \right] F_{1,i+1}^{1,i-k} \left( 1 - \frac{1}{2S+1} \right),$$

where $[\cdot]$ represent the first kind of Stirling number, whereas $F_{1,i+1}^{1,i-k}$ represents the hyper-geometric function [14].

Using the elements of inverse matrix $v^{(S)}_{i,j}$, we are able to write the coefficient for each polynomial $\sigma(s)$,

$$\alpha^{(S)}_{i,m} = \sum_{j=0}^{2S} v^{(S)}_{i,j} \mathbf{p}^{(S)}_m,$$

note that here $\alpha^{(S)}_{i,m}$ are the elements of vector $\alpha^{(S)}_m$.

The non-symmetric degeneracy projection of spin-$S$ polynomial onto a spin-1/2 is given by

$$g_r(\sigma_m) = \begin{cases} 
  r; & \sigma_m = -1 \\
  2S+1-r; & \sigma_m = 1 
\end{cases}.$$

When the spin-$S$ is half-odd-integer, we could recover the solution already obtained in reference [11] as a particular case of our result, when the degeneracy becomes symmetric.

### III. THE NON-BILINEAR SPIN-$S$ ISING MODEL MAPPING ONTO WAINFLASZ MODEL

As an application of this mapping, let us consider the non-bilinear spin-$S$ Ising model with two-body and high-order interaction term, whose Hamiltonian for arbitrary spin-$S$, can be written as

$$\mathcal{H}_S = \sum_{<i,j>} \sum_{k_1=1}^{2S} \sum_{k_2=1}^{2S} K_{k_1,k_2} s^i_{k_1} s^j_{k_2} - \sum_{i=1}^{2S} \sum_{k=1}^{2S} B_k s^i_k.$$
with $K_{k_1,k_2}$ being the non-bilinear interaction terms, while by $B_k$ corresponds to the high order anisotropy coupling. The $<i,j>$ means the summation over the pairs of nearest-neighbor sites.

In order to discuss the equivalence between non-bilinear spin-$S$ Ising model and the Wajnflasz model\[13\], let us describe the Ising-like model or Wajnflasz model\[18\].

\[ \mathcal{H}_m'\{\sigma\} = \sum_{<i,j>} J \sigma_m(s_i) \sigma_m(s_j) - h \sum_i \sigma_m(s_i), \]  

where $J$ is the effective spin-crossover interaction parameter, and whereas $h$ corresponds to the effective external magnetic field or the energy difference between HS and LS states.

With aim of the eqs. (21) and (22) becomes equivalent, we need to impose the following condition $\mathcal{H}_S = \mathcal{H}_m'\{\sigma\} - \mathcal{E}_0'$. Where the parameters must satisfy the relation below

\[ \mathcal{E}_0' = MJ \alpha_0^2, \]

\[ B_k = (h - \gamma J \alpha_0) \alpha_k, \quad k \geq 1, \]

\[ K_{k_1,k_2} = J \alpha_{k_1} \alpha_{k_2}, \quad k_1 \geq 1 \quad \text{and} \quad k_2 \geq 1, \]

with $M$ being the total number of nearest-neighbor spin pairs and $N$ being the total number of sites, while $\gamma$ corresponds to the coordination number of the lattice.

To study thermodynamics properties, we have to compute the partition function of Ising-like model, \[ \mathcal{Z}(\beta) = \sum_{\{s_i\}} \exp(-\beta \mathcal{H}_S), \]

where $\beta = 1/k_B T$, with $k_B$ being the Boltzmann constant and $T$ the absolute temperature.

Similar to that was discussed by Mi and Yang\[4\], the eq. (26) can be rewritten as follow, \[ Z_m(\beta) = e^{\beta \mathcal{E}_0} \sum_{\{s_i\} = \pm 1} \exp(-\beta \mathcal{H}_m'(\{\sigma\})) \times \exp(-\beta \mathcal{H}_m'\{\sigma\}) \]

where $\mathcal{H}_m'(\{\sigma\})$ is the Hamiltonian of the effective spin-1/2 Ising model with non-symmetric degeneracy given by eq. (22).

For the purpose of mapping onto an exactly solvable model, we prefer to change the eq. (28) onto an usual standard form through a further transformation. Therefore, the partition function (28) becomes

\[ Z_m(\beta) = e^{\beta \mathcal{E}_0} \sum_{\{s_i\} = \pm 1} \exp(-\beta \sum_{<i,j>} \mathcal{H}_i'_{ij}), \]

where

\[ \mathcal{H}_i'_{ij} = J \sigma_m(s_i) \sigma_m(s_j) - \frac{h}{\gamma} (\sigma_m(s_i) + \sigma_m(s_j)) - \frac{1}{\gamma \beta} (\ln(g_r(\sigma_m(s_i))) + \ln(g_r(\sigma_m(s_j)))) \]  

A further transformation, could leads us to an effective Ising model Hamiltonian with temperature dependent field.

\[ \mathcal{Z}(\beta) = \sum_{\{s_i\}} \exp(-\beta \mathcal{H}_S), \]

where $\mathcal{Z}_m(\beta) = e^{\beta \mathcal{E}_0} \sum_{\{s_i\} = \pm 1} \exp(-\beta \mathcal{H}_m'(\{\sigma\})) \times \exp(-\beta \mathcal{H}_m'\{\sigma\}) \]

III.1. The Ising-like model

In order to describe the spin-crossover transition, we can use the Wajnflasz model\[13, 20, 21\], where this model take into account the HS state and LS state, modeled simply by a nearest-neighbor interaction between sites.

Hence the Hamiltonian with arbitrary spin-$S$ Ising model can be mapped onto an effective spin-1/2 Ising-like model, which read as

\[ \mathcal{H}_m'(\{\sigma\}) = \sum_{<i,j>} J \sigma_m(s_i) \sigma_m(s_j) - h \sum_i \sigma_m(s_i), \]  

with $J$ being the effective spin-crossover interaction parameter, and whereas $h$ corresponds to the effective external magnetic field or the energy difference between HS and LS states.

With aim of the eqs. (21) and (22) becomes equivalent, we need to impose the following condition $\mathcal{H}_S = \mathcal{H}_m'(\{\sigma\}) - \mathcal{E}_0'$. Where the parameters must satisfy the relation below

\[ \mathcal{E}_0' = MJ \alpha_0^2, \]

\[ B_k = (h - \gamma J \alpha_0) \alpha_k, \quad k \geq 1, \]

\[ K_{k_1,k_2} = J \alpha_{k_1} \alpha_{k_2}, \quad k_1 \geq 1 \quad \text{and} \quad k_2 \geq 1, \]

with $M$ being the total number of nearest-neighbor spin pairs and $N$ being the total number of sites, while $\gamma$ corresponds to the coordination number of the lattice.

To study thermodynamics properties, we have to compute the partition function of Ising-like model, \[ \mathcal{Z}(\beta) = \sum_{\{s_i\}} \exp(-\beta \mathcal{H}_S), \]

where $\beta = 1/k_B T$, with $k_B$ being the Boltzmann constant and $T$ the absolute temperature.

Similar to that was discussed by Mi and Yang\[4\], the eq. (26) can be rewritten as follow, \[ Z_m(\beta) = e^{\beta \mathcal{E}_0} \sum_{\{s_i\} = \pm 1} \exp(-\beta \mathcal{H}_m'(\{\sigma\})) \times \exp(-\beta \mathcal{H}_m'\{\sigma\}) \]

III.2. The Ising model

Finally using an additional transformation, the eq. (29) will be transformed onto a standard spin-1/2 Ising model, given simply by

\[ \tilde{\mathcal{H}}_{ij} = J \tau_i \tau_j - \frac{\tilde{h}_r}{\tau}(\tau_i + \tau_j) + \tilde{E}_r, \]

at this stage, $\tau$ represents a standard spin-1/2, with $\tilde{J}$, $\tilde{h}_r$ and $\tilde{E}_r$ being parameters to be determined. Assuming the eqs. (30) and (31) are equivalents, we have the following algebraic equations

\[ J - 2\frac{h}{\gamma} - \frac{2}{\beta \gamma} \ln(g_r(1)) = \tilde{J} - 2\tilde{h}_r + \tilde{E}_r, \]

\[ J + 2\frac{h}{\gamma} - \frac{2}{\beta \gamma} \ln(g_r(-1)) = \tilde{J} + 2\tilde{h}_r + \tilde{E}_r, \]

\[ -J - \frac{1}{\beta \gamma} (\ln(g_r(1))) + \ln(g_r(-1))) = -\tilde{J} + \tilde{E}_r. \]

Solving this algebraic system equations, we obtain the following relation

\[ \tilde{J} = J, \]

\[ \tilde{h}_r = h - \frac{1}{2\beta} \ln \left( \frac{g_r(-1)}{g_r(1)} \right), \]

\[ \tilde{E}_r = -\frac{1}{\beta \gamma} \ln \left( g_r(1) g_r(-1) \right). \]
It is worth to highlight that when mapping is symmetric, we have the following relation $h_r = h$, similar to that discussed in reference [11].

In addition the partition function of the Hamiltonian (21), is expressed by

$$Z_m(\beta) = e^{-\beta H_{0,r}} \sum_{\{\sigma_i\} = \pm 1} \exp[-\beta(\tilde{J} \sum_{i,j} \tau_i \tau_j - \sum_i \tilde{h}_r \tau_i)],$$

(38)

where

$$H_{0,r} = -\mathcal{E}_0 + \tilde{E}_r M$$

$$= N\hbar \alpha_{0,m} - MJ\alpha_{0,m}^2 - \frac{M}{2\beta_r} \ln [g_r(1)g_r(-1)].$$

(39)

Some characteristic property of this model will be discussed now. At high temperatures, $k_B T > 2\hbar/\ln \left(\frac{g_r(-1)}{g_r(1)}\right)$, the term representing the effective field $\tilde{h}_r = h - \frac{1}{2\beta_r} \ln \left(\frac{g_r(-1)}{g_r(1)}\right)$ is positive and thus the spins have a positive expectation values $\langle \tau_i \rangle > 0$. Whereas at low temperatures, $k_B T < 2\hbar/\ln \left(\frac{g_r(-1)}{g_r(1)}\right)$, we have negative expectation values $\langle \tau_i \rangle < 0$.

Miyashita et al. [19] discussed also the equivalence between spin-crossover phase transition and that the metastable structure of a charge transfer phase transition, where static metastability exist in a study of the charge transfer transition in the material $(nC_3H_7)_4N[Fe^{III}Fe^{II}](dto)_3$ (dto=C2O2S2).

IV. THE EXACTLY SOLVABLE MODEL

In order to that eq. (15) becomes an exactly solvable model, we consider the two-dimensional Ising model [11], so, this model can be solved exactly when $\tilde{h}_r = 0$, therefore it is equivalent to fix the magnetic field $h = \frac{1}{2\beta_r} \ln \left(\frac{g_r(-1)}{g_r(1)}\right)$ in eq. (22). It is worth to note that, the magnetic field $h$ only depends of the degeneracy of spin momenta and is proportional to the temperature.

The free energy of non-bilinear spin-$S$ Ising model in thermodynamic limit ($N \to \infty$ and $M = \gamma N/2$), can be written in terms of standard spin-1/2 Ising model by the following relation

$$f_{m,S} = -\frac{1}{2\beta_r} \ln \left[g_r(-1)^{1-\alpha_{0,m}} g_r(1)^{1+\alpha_{0,m}}\right] + f_{1/2},$$

(40)

where $f_{1/2}$ means the free energy of spin-1/2 Ising model.

Therefore, for two-dimensional case, we can consider three types of lattice: triangular [12], square [11] and honeycomb [13] Ising model.

The critical point $(J^*, \tilde{h}_r^*) = (\tilde{J}/T_c, \tilde{h}_r/T_c)$ for two-dimensional Ising model, in units of critical temperature $T_c$, are given by

$$\tanh(\tilde{J}^*) = \begin{cases} 1 - \sqrt{3}; & \text{triangular}, \\ \sqrt{2} - 1; & \text{square}, \\ 1/\sqrt{3}; & \text{honeycomb}, \end{cases}$$

(41)

and $\tilde{h}_r^* = 0$, for triangular, square and honeycomb lattice, respectively.

The critical points for honeycomb ($\gamma = 3$), square ($\gamma = 4$), and triangle ($\gamma = 6$) lattice can be fully recovered, which is consistent with the results previously obtained by Mi and Yang [4], for the case of spin-1 Ising model (for detail see table I of reference [3]). However our result is quite general and is valid for any spin-$S$ and for any coordination number.

The non-bilinear spin-$S$ Ising model critical points should satisfy the relation below

$$B_k^* = \frac{1}{2} \ln \left(\frac{g_r(-1)}{g_r(1)}\right) - \gamma J^* \alpha_{0,m} \alpha_{k,m},$$

(42)

$$K_{k_1,k_2}^* = J^* \alpha_{k_1,m} \alpha_{k_2,m},$$

(43)

by $*$ we mean the parameters are in units of critical temperature $T_c$.

On the other hand, the term $\tilde{E}_r$ and $\mathcal{E}_0'$ are responsible for the appearance of residual entropy, in other words this means due to competing parameters coupling of the non-bilinear spin-$S$ model could be considered as a frustrated spin model, therefore the entropy is given by

$$S_m = \frac{1}{2} \ln \left[g_r(-1)^{1-\alpha_{0,m}} g_r(1)^{1+\alpha_{0,m}}\right].$$

(44)

Using the above result [14], we can obtain a residual entropy for the spin-1 Ising model discussed by Mi and Yang [4], the first model has residual entropy given by $S = \ln(2)$ while the second model has no residual entropy. It is worth to notice the frustration properties of those model was not discussed by Mi and Yang [4].

Another simple example that we consider is the spin-$3/2$ Ising model, for the particular case such that satisfy the eqs. (7) and (8) are zero, inasmuch as the independent coefficients of those polynomials are $\alpha_{0,1} = \alpha_{0,2} = 0$. However, for the last three polynomials we have a residual entropy given by $S = \ln(2)$, $S = \frac{16}{9} \ln(3)$ and $S = \frac{2}{5} \ln(3)$ for eqs. (9) respectively. It is interesting to highlight that, the residual entropy is independent of the coordination number or some other lattice structure parameters.

V. CONCLUSION

Different to those other methods developed to obtain this kind of results using a more involved approach, we have used a simple spin-$S$ polynomial projection onto spin-crossover state, with non-symmetric degeneracy of spin up or high-spin (HS) and spin down or low-spin (LS). In general the present projection obtained have
not be necessarily symmetric with relation to their spins up or down. Only as particular case of our results, we have the symmetric mapping which was previously considered in reference [11], some additional results are found also using the decoration transformation method satisfying the 8-vertex model in our recent paper [16]. Therefore we conclude that, there is a spin-S polynomial transformation onto spin-crossover state, whose total possible number of projection is given by \(2(2^{2S} - 1)\). Through a further transformation we can map also onto a standard spin-1/2 Ising model.

As an application of this mapping we consider the non-bilinear spin-S Ising model which can be transformed onto spin-crossover state described by Wanjflasz model [18]. Using a further transformation we obtain the partition function of the effective frustrated spin-1/2 Ising model [1], making a suitable mapping this non-symmetric contribution leads us to a spin-1/2 model with a fixed external magnetic field temperature dependent given by eq. (36). Therefore we conclude that, the non-bilinear spin-S Ising model such that satisfy the projection proposed, must become equivalent to the Wanjflasz model [18], with quite interesting properties such as residual entropy of the model, independent of the lattice structure.

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