Robust quantum gates and a bus architecture for quantum computing with rare-earth-ion doped crystals.

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We present a composite pulse controlled phase gate which together with a bus architecture improves the feasibility of a recent quantum computing proposal based on rare-earth-ion doped crystals. Our proposed gate operation is tolerant to variations between ions of coupling strengths, pulse lengths, and frequency shifts, and it achieves worst case fidelities above 0.999 with relative variations in coupling strength as high as 10% and frequency shifts up to several percent of the resonant Rabi frequency of the laser used to implement the gate. We outline an experiment to demonstrate the creation and detection of maximally entangled states in the system.

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I. QUANTUM COMPUTING WITH RARE EARTH IONS

Rare earth ions embedded in cryogenic crystals have a number of features making them suitable for quantum information processing 3:

- Ground state hyperfine levels with a lifetime of hours and decoherence times up to several ms. We will use three such states, labeled $|0\rangle$, $|1\rangle$, and $|\text{aux}\rangle$, to implement quantum registers and for parking unwanted ions.

- Optical transitions with homogeneous line widths on the order of kHz are inhomogeneously broadened to several GHz, allowing us to address a large number of independent channels.

- The crystal-embedded ions have large static dipole moments with interaction energies up to several GHz. This interaction is ideal for implementing gate operations of the dipole blockade type.

In the remainder of this section we will briefly introduce the basic ideas of REQC, as originally described in Ref. 1.

A. Dynamical architecture selection

The architecture of the REQC system is selected at start-up by an initialization procedure. The desired endpoint of this process is a large number of independent instances of the chosen quantum computer, each instance being a group of ions with one representative from each active channel and couplings between the ions as required by the chosen architecture.
The initialization proceeds in two steps: channel preparation and identification of quantum computer instances. In both of these steps unwanted ions are deactivated by transferring them to off-resonant, metastable states.

a. Channel preparation. A channel refers to a large number of ions distributed throughout the crystal, all having the same inhomogeneous shift and coupling strength within the inhomogeneously broadened optical transition used to access the ions. The channel preparation aims to deactivate all dopant ions close to resonance with a given channel and to transfer all members of the channel itself to their \(|0\rangle\) state.

This can be achieved by means of spectral hole burning techniques, and widths of the final channel structure as low as 50 kHz have been obtained experimentally for materials similar to those considered for use in REQC.

b. Instance identification. After a successful initialization, each ion will only be interacting with ions from other channels, allowing us to ignore “excitation hopping” transitions [4], as these will not be energy conserving. As a consequence we can model the dipole coupling as simple couplings between the excited states:

\[
V_{\text{dipole}} = \frac{1}{2} \sum_{\mu \neq \nu} g_{\mu \nu} \langle |e\rangle \langle e| \rangle_{\mu} \otimes \langle |e\rangle \langle e| \rangle_{\nu},
\]

where the sum is over all pairs of ions. To be precise about the objectives of the instance identification process, we will consider ions \(\mu\) and \(\nu\) to be coupled if \(g_{\mu \nu}\) exceeds a threshold \(g_t\) determined by the chosen implementation of the gate operation.

The goal of the instance identification procedure is to transfer ions, which are in an active channel but not members of a valid instance, to their auxiliary state \(|\text{aux}\rangle\). One way to achieve this is to go through the following procedure for each pair, \((i, j)\), of channels required to be coupled:

By applying a \(\pi\)-pulse to ions in channel \(i\) we transfer the \(|0\rangle\) population to the \(|e\rangle\) state, thus shifting the excited state energy of all ions coupled to a channel \(i\) ion. By means of a frequency sweep or a comb of \(\pi\) rotations, all channel \(j\) ions which are shifted less than \(g_t\) are now transferred to their excited state \(|e\rangle\), after which the channel \(i\) ions are returned to \(|0\rangle\). We now wait for the excited channel \(j\) ions to decay, which will transfer part of the ions to the inactive \(|\text{aux}\rangle\) state.

By repeated application of this pulse sequence, we can deactivate an arbitrarily high fraction of the channel \(j\) ions which are not coupled to a channel \(i\) ion. After this has been achieved, we repeat the process with the roles of channel \(i\) and \(j\) interchanged, and afterwards proceed to apply the same procedure to all other edges of the coupling graph to finally arrive at the desired initialized REQC system.

II. HIGH FIDELITY GATE OPERATIONS

Gate operations for the REQC system face a number of challenges due to the fact that they operate simultaneously on a number of not quite identical instances of a quantum computer: Due to the finite channel width, ion \(\mu\) will in general be detuned by a small amount \(\delta^{(\mu)}\) from the central channel frequency. Furthermore, the experienced Rabi frequency, \(\Omega^{(\mu)}\), will differ slightly from the average Rabi frequency \(\Omega_0\) due to laser field inhomogeneities and local variations in dipole moments.

In this section we will show that by taking advantage of the fact that \(\delta^{(\mu)}\) and \(\Omega^{(\mu)}_0/\Omega_0\) are constant in time for each ion, we can design pulse sequences that perform almost the same operation on each instance.

A. Composite rotations

The pulse \(P_{ie}(\phi, \theta)\) is driven by a Hamiltonian \(\hat{H}_1 = \frac{1}{\hbar} \Omega \cdot \sigma^{(ie)}\) with \(\sigma^{(ie)}\) signifying the Pauli-matrices in the \(\{|i\rangle, |e\rangle\}\) basis and \(\Omega = \Omega_0 \hat{n}_\phi\), where \(\hat{n}_\phi\) is a unit vector in the \(x - y\) plane with azimuthal angle \(\phi\).

To apply the pulse \(P_{ie}(\phi, \theta)\) we engage the field for a period \(\theta/\Omega_0\), so that an ideal reference ion, \(\xi\), with \(\delta^{(\xi)} = \Omega^{(\xi)}_0/\Omega_0 = \frac{1}{2}\) and \(\Omega^{(\xi)} = \Omega_0\), will in general be detuned by a small amount \(\delta^{(\xi)}\) from the central channel frequency. Furthermore, the excited channel \(j\) ions, the effect of the full gate operation on the qubit space is consequently a \(\pi\) phase shift on the \(|11\rangle\) state: \(U_{\text{CPS}} = 1 - 2 |11\rangle \langle 11|\).
0 and $\Omega^{(c)} = \Omega_0$ will be rotated by an angle $\theta$ around $\hat{n}_c$ as desired. In general, however, the ions will react differently to the pulse due to their different detunings and coupling strengths.

The problem of taking all the ions through the same evolution when they react differently to the pulses has been studied in great detail in the magnetic resonance community. Inspired by the discussion in Ref. 2, we have used the BB1 pulse sequence to replace a single pulse $P(0, \theta)$ with the following sequence of pulses:

$$P_{BB1}(0, \theta) = P(0, \theta/2) P(\phi_c, \pi) P(3\phi_c, 2\pi) P(\phi_c, \pi) P(0, \theta/2).$$ (3)

For our reference ion, $\xi$, the unitary evolution $P_{BB1}(\phi, \theta)$ caused by the $P_{BB1}(\phi, \theta)$ composite pulse is seen to be exactly identical to the evolution caused by $P(\phi, \theta)$. The use of five pulses for this simple task is justified, however, if we instead consider the evolution $P_{BB1}(\phi, \theta)$ caused by the composite pulse $CPS_\mu(\mu, \pi)$.

$$H^{(\mu)} = -\delta^{(\mu)} |e\rangle \langle e| + \frac{1}{2} \Omega^{(\mu)} \hat{n}_\phi \cdot \sigma^{(\mu)}.$$ (4)

In this case we find that with the optimal value, $\delta_c = \pm \cos^{-1}(-\theta/2\pi)$, $P_{BB1}(\phi, \theta)$ is almost constant over a large range of values of $\delta^{(\mu)}/\Omega_0$ and $\Omega^{(\mu)}/\Omega_0$, while $P(\phi, \theta)$ changes quite rapidly.

### B. Robust gate operation

For the two-level Rabi problem there is a global phase factor depending on the detuning which plays no observable role. In our three-level system, however, this phase will lead to a dephasing between the qubit level $|i\rangle$ coupled to $|e\rangle$ and the other qubit level. To compensate this, we must symmetrize the desired pulse sequence in a suitable way, to allow both levels to pick up the same, unknown, phase contributions.

In the case of the controlled phase shift, we have arrived at the following symmetrized version:

$$P_{CPS}^{(i,j)} = P_{1ce}^{(i)}(\pi, \pi),$$

$$P_{0ce}^{(j)}(\pi, \pi) P_{1ce}^{(j)}(0, \pi) P_{1ce}^{(j)}(\pi, \pi) P_{1ce}^{(j)}(0, \pi)$$

$$P_{0ce}^{(i)}(0, \pi) P_{0ce}^{(i)}(\pi, \pi)$$

$$P_{0ce}^{(j)}(\pi, \pi) P_{0ce}^{(j)}(0, \pi) P_{1ce}^{(j)}(0, \pi) P_{1ce}^{(j)}(0, \pi)$$

$$P_{1ce}^{(i)}(0, \pi).$$ (5)

For the reference ion $\xi$ the $P_{CPS}^{(i,j)}$ pulse sequence is seen to be equivalent to $P_{0ce}^{(j)}(\pi, \pi) P_{1ce}^{(j)}(0, 2\pi) P_{0ce}^{(j)}(0, \pi)$, which is exactly the basic controlled phase shift operation $CPS_\mu$, but we expect it to perform better for a general ion.

Implementing all the pulses of $P_{CPS}^{(i,j)}$ by composite BB1 pulses, we do indeed obtain a very robust implementation of the controlled phase shift as illustrated in Fig. 1.
FIG. 1: Calculated worst case fidelities of two implementations of the controlled phase shift: a) The simple implementation \(2\), and b) the \(PCPS\) pulse sequence \(5\). The fidelity is plotted as a function of \(\delta^{(1)} = \delta^{(2)}\) and \(\Omega_0^{(1)} = \Omega_0^{(2)}\), both relative to \(\Omega_0\), and with \(g_{12} = 100\Omega_0\). Note the difference between the \(\Omega\)-axis limits of the two plots. It is clear from the plots that \(PCPS\) achieves a high fidelity over a much larger parameter space. In particular, \(PCPS\) is much less sensitive to variations in \(\Omega\), while the sensitivity to variations in \(\delta\) does not seem to be significantly improved.

FIG. 2: Two possible coupling topologies for REQC systems: a) cluster topology and b) star topology

IV. PREPARATION AND DETECTION OF MAXIMALLY ENTANGLLED STATES

To demonstrate the viability of the REQC concept, and in particular the bus architecture, we propose to perform an experimental preparation and detection of a maximally entangled state.

We will use an REQC system with star topology: one central qubit coupled to \(n - 1\) outer qubits. Starting with all \(n\) qubits in their \(|0\rangle\) state, we apply a composite pulse Hadamard operation to the central qubit followed by controlled not operations on all the outer qubits controlled by the central qubit, thus transferring the system to the maximally entangled state

\[
|\Psi_0\rangle = \frac{1}{\sqrt{2}} (|0\rangle^n + |1\rangle^n),
\]

which corresponds to a superposition of the total pseudospin pointing straight up and straight down.

The following algorithm for detecting a population of the cat state \(|\Psi_0\rangle\) is very similar to the method used by the group of D. Wineland to detect a maximally entangled state of four ions in a linear Paul trap \(9\): By rotating the state \(|\Psi_0\rangle\) through an angle \(\phi\) around the \(\hat{z}\)-axis we accumulate different phases on the pseudospin components:

\[
|\Psi_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle^n + e^{-i\phi n} |1\rangle^n).
\]

An additional rotation by \(\pi/2\) around the \(y\)-axis now yields a state \(|\Psi_2\rangle\) with an expected parity,

\[
P = \Pi_i (\sigma_z),
\]

given by

\[
\langle \Psi_2 | P | \Psi_2 \rangle = \cos(n\phi),
\]

the detection of the \(n\phi\) dependency thus signifying that the maximally entangled state has been populated \(10\).

In a single-instance quantum computing system, such as the ion trap setup used in Ref. \(9\), we could measure the expectation value of the parity as a statistical average over many repetitions of the procedure described above: after each run we could simply measure the state of each qubit, and subsequently compute the parity. Since measurements in the REQC system yields an ensemble average, this approach would not be applicable here: we cannot find the expectation value of the parity from the ensemble averages of the single qubit parities, \(\langle (\sigma_z) \rangle\), which are 0 as inspection shows.

Instead we let the bus qubit acquire the parity unitarily: by sequentially applying controlled not operations from each outer qubit to the central qubit we make the
central qubit end up in the $|1\rangle$ state in the case of odd parity and in the $|0\rangle$ state in the case of even parity. After this, the ensemble average of the bus qubit population yields the expectation value of the parity.

As this section illustrates, readout from an ensemble quantum computer is conceptually somewhat more complicated than readout from a single quantum computer. It is worth noting, however, that unlike many other ensemble quantum computing proposals, REQC instances all start in the same pure state: if we successfully employ error correction during a computation all instances will end up in the same pure state, allowing us to read out the ensemble averages with high signal to noise ratio. Perhaps surprisingly, the readout can almost always be performed by tricks similar to those employed to detect the maximally entangled state: Ensemble quantum computing is almost as powerful as general quantum computing. In particular, all problems which may be expressed in terms of the hidden subgroup problem (such as Shor’s factoring algorithm) can be solved using an ensemble quantum computer [11].

V. CONCLUSIONS AND OUTLOOK

In conclusion, we have shown that, in the absence of decay and decoherence, it is possible to implement robust high-fidelity gates for the REQC system. Specifically, the phase compensated controlled phase gate based on composite pulses [14] achieves worst case gate fidelities above 0.999, even with the coupling strength varying up to 10% between instances and channel widths of several percent of the Rabi frequency of the field used to manipulate the system. Furthermore, we have pointed out that using a bus based architecture will simplify implementation by allowing the use of an asymmetric laser setup.

The number of instances of a bus based REQC system scales as $p^n$ where $n$ is the number of qubits per instance and $p$ is the probability of a random ion being coupled to a member of a given channel. In the regime currently being investigated experimentally, $p$ is several orders of magnitude less than 1. The value of $p$ is affected by $g_k$ and channel width, which is why we have to use robust gates rather than narrow channels and high threshold coupling strengths. Higher values of $p$ could be obtained by increasing the ion density, which would, however, cause a decrease in coherence times. By using structured doping techniques it might be possible to obtain a higher effective $p$ without this adverse effect. Another approach to obtaining higher effective $p$ would be to use multiple channels for each qubit by guaranteeing each instance to have exactly one member ion from a group of channels assigned to each qubit.

The instance identification protocol described in Sec. I could be made much more efficient: Since the system starts in a pure state (all ions in the channels in their $|0\rangle$ state), and also ends in a pure state (all instance members in their $|0\rangle$ state, and all other ions from the initial channel populations in their $|\text{aux}\rangle$ state), the selection could theoretically be performed unitarily.

APPENDIX: FIDELITY OF UNITARY OPERATIONS

We wish to compare unitary operators $U$ and $U_0$, by determining how closely $U_0^\dagger U$ resembles the identity on the Hilbert space $\mathcal{H}$. This can be expressed in terms of the worst case fidelity:

$$F(U_0, U) = \min_{\left| \psi \right> \in \mathcal{H}} \left| \left< \psi \right| U_0^\dagger U \left| \psi \right> \right|^2. \quad (A.1)$$

The fidelity can be computed as follows: $U_0^\dagger U$ is unitary and can consequently be formally diagonalized with eigenvalues $e^{i\phi_j}$, $j = 1, \ldots, n$ so that $0 \leq \phi_1 \leq \ldots \leq \phi_n \leq 2\pi$. Introducing the maximal eigenvalue phase distance $\Delta \phi_{\text{max}} = \max\{|\phi_j - \phi_{j-1}|\}_{j=2,\ldots,n}\cup\{2\pi + \phi_1 - \phi_n\}$, the fidelity over $\mathcal{H}$ is given as

$$F(U_0, U) = \begin{cases} \cos^2(\Delta \phi_{\text{max}}/2) & \text{if } \Delta \phi_{\text{max}} \geq \pi, \\ 0 & \text{otherwise}. \end{cases} \quad (A.2)$$

To see this, we expand the state vector $|\psi\rangle$ on the eigenbasis $\{|j\rangle\}$ of $U_0^\dagger U$: $|\psi\rangle = \sum_j c_j |j\rangle$. The fidelity then takes the form

$$F(U_0, U) = \min_{p_j} \left| \sum_j p_j e^{i\phi_j} \right|^2, \quad (A.3)$$

with the minimum taken over all non-negative $p_j = |c_j|^2$, so that $\sum_j p_j = 1$.

Eq. (A.3) allows us to interpret the fidelity geometrically in the complex plane, as the set of points $\{\sum_j p_j \exp(i\phi_j)\}$ form a convex polygon with vertices in the eigenvalues $\{|e^{i\phi_j}\rangle\}$ on the unit circle. The fidelity corresponds to the square of the minimal distance from 0 to this polygon. If the polygon is constrained to one half-plane, this will be $|e^{i\phi} + e^{i(\phi + \Delta \phi_{\text{max}})}|^2/4 = \cos^2(\Delta \phi_{\text{max}}/2)$. If the polygon is not restricted to one half-plane, it will cover the origin, and the fidelity will be 0.

Note that this method relies on the minimization being performed on the whole Hilbert space. If this is not the case the method is not applicable, and in Sec. III where the minimization is carried out over a subspace of the full Hilbert space, we have resorted to a numerical search.

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