Constrained differential renormalization of Yang-Mills theories

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Abstract

We renormalize QCD to one loop in coordinate space using constrained differential renormalization, and show explicitly that the Slavnov-Taylor identities are preserved by this method.

Differential regularization and renormalization is a method that works directly on Feynman graphs in coordinate space, substituting singular expressions by derivatives of well-behaved distributions. It has proved to be quite simple and convenient in a number of applications. In gauge theories, however, a problematic feature arises: the Ward identities must be studied explicitly to fix the arbitrariness of the method in such a way that gauge invariance is preserved.

A solution at the one-loop level is the symmetric procedure of differential renormalization proposed in Ref. 2. This so-called constrained differential renormalization (CDR) introduces no ambiguities and has been shown to preserve Abelian gauge invariance and supersymmetry in a non-trivial calculation in supergravity. It is the purpose of this letter to apply CDR to a non-Abelian gauge theory and study
the corresponding Slavnov-Taylor identities \[5\]. We shall consider a Yang-Mills theory with gauge group \(SU(N_c)\) coupled to \(N_f\) Dirac fermions in the fundamental representation, i.e., QCD with \(N_c\) colours and \(N_f\) quark flavours. The calculation of the gluon selfenergy and the triple gluon vertex, using conventional differential renormalization, was carried out in Refs. \[1\] and \[6\], respectively. The background field method \[7\] was employed there because it allows a much more direct determination of the \(\beta\) function and leads to simpler Ward identities. We use the conventional formalism instead, precisely for the last reason: we want to test CDR in the most complex case, and show that it preserves the (more involved) Slavnov-Taylor identities rather than the Ward-Takahashi like identities of the background field formalism. Nevertheless, for comparison with Refs. \[1, 6\], we have also applied CDR to the background field calculations mentioned above.

In Ref. \[3\] it was argued that CDR preserves the Ward identities of Abelian gauge invariance because it maintains the properties that are required for their derivation, like the fulfilment of equations of motion for renormalized expressions or the commutativity of differentiation with renormalization. Actually, this argument applies equally well to the case of non-Abelian gauge invariance, since the structure of the interaction Lagrangian is never used. Of course, the symmetry of the Lagrangian is essential for the fulfilment of the corresponding Ward identities, but this is a matter of *combinatorics* \[8\] and CDR does not interfere with it (essentially, CDR ensures that the building blocks of such combinatorics behave correctly). A new feature of the non-Abelian case is the appearance of composite operators in the Slavnov-Taylor identities (in the Zinn-Justin form \[9\], which we shall use here). In Ref. \[1\] it was shown that differential renormalization can be directly applied to diagrams with operator insertions. The same holds for the constrained procedure.

In the following, after writing the Lagrangian, we give the renormalized expressions in coordinate space of all the singular one-loop 1PI Green functions of elementary fields. The diagrams with operator insertions that contribute to the Slavnov-Taylor identities have also been calculated, but the explicit results are not given here. Then, we write all the Slavnov-Taylor identities involving these renormalized Green functions. We have used a symbolic computer program to verify that they are indeed fulfilled. The QCD Lagrangian in the Feynman gauge, written in Euclidean space and including ghost terms, reads

\[
\mathcal{L} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2} (\partial_\mu A^a_\mu)(\partial_\nu A^a_\nu) + \partial_\mu \bar{\eta}^a (D_\mu \eta)^a + \sum_{i=1}^{N_f} \bar{\Psi}_i (\slashed{D} + m_i) \Psi_i ,
\]

with

\[
F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu ,
\]

\[
(D_\mu \eta)^a = \partial_\mu \eta^a + g f^{abc} A^b_\mu \eta^c ,
\]

\[
D_\mu \Psi_i = \partial_\mu \Psi_i - ig A^a_\mu T^a \Psi_i .
\]

\(T^a\) are the \(SU(N_c)\) generators in the fundamental representation, and \(f^{abc}\) the structure
where $\delta \lambda$ is a constant Grassman parameter. In order to obtain simple Slavnov-Taylor identities for the 1PI Green functions, it is convenient to add to the Lagrangian source terms not only for elementary fields but also for their BRST variations [9]:

\[ \delta A^a_{\mu} = -(D_{\mu} \bar{\eta})^a \delta \lambda \equiv s A^a_{\mu} \delta \lambda, \]  
\[ \delta \bar{\eta}^a = -\partial_{\mu} A^a_{\mu} \delta \lambda \equiv s \bar{\eta}^a \delta \lambda, \]  
\[ \delta \eta^a = -\frac{g}{2} f^{abc} \eta^b \eta^c \delta \lambda \equiv s \eta^a \delta \lambda, \]  
\[ \delta \bar{\Psi}_i = -g T^a \eta^a \bar{\Psi}_i \delta \lambda \equiv s \bar{\Psi}_i \delta \lambda, \]  
\[ \delta \bar{\Psi}_i = -g \bar{\Psi}_i T^a \eta^a \delta \lambda \equiv s \bar{\Psi}_i \delta \lambda, \]

where $\delta \lambda$ is a constant Grassman parameter. In order to obtain simple Slavnov-Taylor identities for the 1PI Green functions, it is convenient to add to the Lagrangian source terms not only for elementary fields but also for their BRST variations [9]:

\[
\mathcal{L}_{\text{sources}} = - \left( J^a_{\mu} A^a_{\mu} + \bar{\eta}^a \xi^a + \bar{\xi}^a \eta^a + \sum_{i=1}^{N_f} (\bar{\Psi}_i \chi_i + \bar{\chi}_i \Psi_i) \right) + K^a_{\mu} (s A^a_{\mu}) + L^a (s \eta^a) + \sum_{i=1}^{N_f} (\bar{N}_i (s \Psi_i) + (s \bar{\Psi}_i) N_i). \]

The sources of BRST variations are mere spectators in the Legendre transform defining the generator functional of 1PI Green functions. The coordinate space Feynman rules, including those for insertions of BRST variations, are displayed in Fig. 1.

The singular 1PI Green functions of QCD at one loop are the gluon, quark and ghost selfenergies, and the quark, ghost, triple gluon and quartic gluon vertices. The procedure to calculate them with CDR is simple [2, 3]. First, one expresses each contributing diagram in terms of the basic functions defined in Ref. 3:

\[
A_m \equiv \Delta_m(x) \delta(x), \]
\[
B_{m_1 m_2}[\mathcal{O}](x) \equiv \Delta_{m_1}(x) \mathcal{O}^x \Delta_{m_2}(x), \]
\[
T_{m_1 m_2 m_3}[\mathcal{O}](x, y) \equiv \Delta_{m_1}(x) \Delta_{m_2}(y) \mathcal{O}^x \Delta_{m_3}(x + y), \]
\[
Q_{m_1 m_2 m_3 m_4}[\mathcal{O}](x, y, z) \equiv \Delta_{m_1}(x) \Delta_{m_2}(y) \Delta_{m_3}(z) \mathcal{O}^x \Delta_{m_4}(x + y + z), \]

where $\Delta_m(x) = \frac{1}{4\pi^2} \frac{m K_1(mx)}{x}$ is the Feynman propagator in (Euclidean) coordinate space, with $K_1$ a modified Bessel function, and $\mathcal{O}$ is a differential operator. We shall suppress mass subindices when all of them are zero. Then, the singular basic functions are replaced by their renormalized expressions, which can be found in Tables 2, 3 and 4 of Ref. 3. For brevity, we refer to those tables and do not reproduce here the renormalized basic functions that appear in our calculations. Also, since the procedure of CDR has been explained in detail in Refs. 2, 3, we shall directly give the final result for each Green function. The Feynman diagrams contributing to 1PI Green functions of elementary fields can be found, e.g., in Ref. 11.

The renormalized gluon selfenergy in terms of renormalized basic functions reads

\[
<A^a_{\mu}(x_1) A^b_{\nu}(x_2) > = g^2 \delta^{ab} \left\{ N_c \left[ (3 \partial_{\mu} \partial_{\nu} - 2 \delta_{\mu\nu} \Box) B^R[1] - 4 B^R[\partial_{\mu} \partial_{\nu}] \right]\right\}
\]
where all functions, here and in the rest of two-point functions, depend on the coordinate difference \( x = x_1 - x_2 \) and, as prescribed by differential renormalization, total derivatives are supposed to act formally by parts on test functions [1]. Throughout this paper, \( < \phi_1 \ldots \phi_n > \) represents the one-loop correction to the 1PI Green function of the fields \( \phi_1 \ldots \phi_n \). That the expression in Eq. (15) is transverse, as required by gauge invariance, is only apparent when the renormalized basic functions are substituted by their explicit expressions in the mentioned tables. In the case of massless quarks the result is quite compact:

\[
< A^a_\mu(x_1)A^b_\nu(x_2) > = -\frac{1}{144\pi^2}g^2\delta^{ab}(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box) \left[ (15N_c - 6N_f)\frac{1}{4\pi^2}\log \frac{x^2M^2}{x^2} + (2N_c - 2N_f)\delta(x) \right]. \tag{16}
\]

\( M \) is the renormalization scale. The dependence on the renormalization scale of Eq. (15) is

\[
M \frac{\partial}{\partial M} < A^a_\mu(x_1)A^b_\nu(x_2) > = \frac{1}{24\pi^2}g^2\delta^{ab}(5N_c - 2N_f)(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box)\delta(x). \tag{17}
\]

In the background field method, the background gluon selfenergy renormalized with CDR reads (for massless quarks)

\[
< B^a_\mu(x_1)B^b_\nu(x_2) > = -\frac{1}{144\pi^2}g^2\delta^{ab}(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box) \left[ (33N_c - 6N_f)\frac{1}{4\pi^2}\log \frac{x^2M^2}{x^2} + (2N_c - 2N_f)\delta(x) \right], \tag{18}
\]

and differs from the result (for \( N_f = 0 \)) in Ref. [1] by a finite local term (which can be absorbed into a redefinition of \( M \)). The one-loop \( \beta \) function of QCD, \( \beta = -\frac{g^2}{48\pi^2}(11N_c - 2N_f) \), can be directly read from the scale dependent part of Eq. (16).

The **quark selfenergy** is proportional to the corresponding Green function of QED. In terms of basic functions, the renormalized quark selfenergy reads

\[
< \Psi^A_1(x_1)\bar{\Psi}^B_1(x_2) > = g^2\delta^{AB}\frac{N_c^2 - 1}{N_c} \left( B^R_{0\mu_i}[\Box] - 2m_iB^R_{0\mu_i}[1] \right)
= \frac{1}{64\pi^4}g^2\delta^{AB}\frac{N_c^2 - 1}{N_c} \left\{ (\Box - m_i^2) \left[ (\Box - m_i^2) \frac{K_0(m_i\bar{x})}{x} \right]
+ 2\pi^2\log \frac{M^2}{m_i^2}\delta(x) \right\} + m_i^2K_0(m_i\bar{x})\frac{1}{x^2}, \tag{19}
\]

where \( K_0 \) is a modified Bessel function [13] and \( \bar{M} = 2M/\gamma_E \), with \( \gamma_E = 1.781 \ldots \) the Euler’s constant. The scale dependence is

\[
M \frac{\partial}{\partial M} < \Psi^A_1(x_1)\bar{\Psi}^B_1(x_2) > = \frac{1}{16\pi^4}g^2\delta^{AB}\frac{N_c^2 - 1}{N_c} (\Box - 4m_i)\delta(x). \tag{20}
\]
The renormalized ghost selfenergy is

\[
<\eta^a(x_1)\bar{\eta}^b(x_2)> = \frac{1}{2}g^2N_c\delta^{ab}\Box B^R[1]
\]
\[
= \frac{1}{128\pi^2}g^2\delta^{ab}N_c\Box \log \frac{x^2M^2}{x^2},
\]
(21)

and its scale dependence,

\[
M \frac{\partial}{\partial M} <\eta^a(x_1)\bar{\eta}^b(x_2)> = -\frac{1}{16\pi^2}g^2\delta^{ab}N_c\delta(x).
\]
(22)

For the three-point functions we use the shifted variables \( x = x_1 - x_3, y = x_3 - x_2 \). Unless otherwise specified, all basic functions are assumed to depend on these two variables in the following. To avoid too lengthy expressions, we shall only give the final expressions in terms of basic functions. The renormalized expression of the quark vertex is

\[
<\Psi^A_i(x_1)\bar{\Psi}^B_i(x_2)A^a_{\mu}(x_3)> = -ig^3(T^a)^{BA} \left\{ \frac{1}{N_c} \left[ (-2m_i\partial^+_\mu + m^2\gamma_\mu 
+ \partial^x\gamma_\mu \partial^y) T_{m_i,m_i,0}[1] + (4m_i\delta_{\mu\alpha} \gamma_\alpha \gamma_\mu \partial^y - \partial^x\gamma_\mu \gamma_\alpha) T_{m_i,m_i,0}[\partial_\alpha] 
- \gamma_\mu T^R_{m_i,m_i,0}[\Box] + 2T^R_{m_i,m_i,0}[\partial_\mu \partial] \right] + N_c \left[ \frac{3}{2}m_i(\partial^y\gamma_\mu + \gamma_\mu \partial^y)T_{00m_i}[1] 
+ \gamma_\mu T^R_{00m_i}[\Box] + 2T^R_{00m_i}[\partial_\mu \partial] \right] \right\},
\]
(23)

where we have introduced the notation \( \partial^+ = \partial^x + \partial^y \). The scale dependent part reduces to

\[
M \frac{\partial}{\partial M} <\Psi^A_i(x_1)\bar{\Psi}^B_i(x_2)A^a_{\mu}(x_3)> = \frac{1}{16\pi^2}g^3(T^a)^{BA} \left( 3N_c - \frac{1}{N_c} \right) \gamma_\mu \delta(x)\delta(y).
\]
(24)

For the ghost vertex we have:

\[
<\eta^b(x_1)\bar{\eta}^c(x_2)A^a_{\mu}(x_3)> = \frac{1}{2}g^3Ncf^{abc} \left\{ \partial^x \cdot \partial^\mu \partial^y T[1] 
+ \left( \delta_{\mu\alpha} (\Box - 2\partial^x \cdot \partial^y) + \partial_\mu^- \partial^y_\alpha + \partial_\mu^y \partial^-_\alpha \right) T[\partial_\alpha] + \partial_\mu^y T^R[\Box] \right\},
\]
(25)

where \( \partial^- = \partial^x - \partial^y \) and \( \partial^x \cdot \partial^y = \partial^x_\alpha \partial^y_\alpha \). The scale dependence is

\[
M \frac{\partial}{\partial M} <\eta^b(x_1)\bar{\eta}^c(x_2)A^a_{\mu}(x_3)> = \frac{1}{16\pi^2}N_c f^{abc} \delta(x_1 - x_3)\partial_\mu^2 \delta(x_2 - x_3).
\]
(26)

Here, we have come back to the original variables, to make explicit that it has the same form as the corresponding term in the Lagrangian.
The renormalized expression of the triple gluon vertex is quite large, even in terms of basic functions. We split it into the pure gauge part and the fermionic part. The results are

\[
< A^a_\mu(x_1) A^b_\nu(x_2) A^c_\rho(x_3) >^G = g^3 f^{abc} N_c
\]

\[
\times \left\{ \frac{9}{4} \delta_{\nu \rho} \partial^\mu - (B[1](x) \delta(x + y) + \partial^x \cdot \partial^y) - \frac{9}{4} (\delta_{\mu \rho} \partial^2_{\nu} + \delta_{\nu \rho} \partial^2_{\mu}) B[1](x) \delta(y) \right. \\
+ \frac{1}{2} \left( 3 \partial^x \partial^x \partial^x \partial^x + 2 \partial^x \cdot \partial^y \partial^y + \delta_{\nu \rho} \partial^x \partial^y \right) \partial^x \partial^y \\
\left. + \frac{1}{2} \left( \delta_{\mu \rho} \partial^x \partial^y - 3 \partial^x \partial^x \partial^x \partial^x + \delta_{\mu \rho} \partial^x \partial^y - 3 \partial^x \partial^y \right) \partial^x \partial^y \right) T[1] \\
+ \frac{1}{2} \left( \delta_{\mu \rho} \partial^x \partial^y - 3 \partial^x \partial^x \partial^x \partial^x + \delta_{\mu \rho} \partial^x \partial^y - 3 \partial^x \partial^y \right) \\
\left. + \delta_{\mu \alpha} \left( 6 \partial^x \partial^y \partial^y - 10 \partial^x \partial^y \partial^y - 3 \partial^x \partial^y + \delta_{\nu \rho} \partial^x \partial^y \right) \right] T[\alpha] \\
+ \frac{1}{2} \left( 2 \delta_{\mu \rho} \partial^x \partial^y + \delta_{\mu \rho} (2 \partial^x \partial^y - 5 \partial^x \partial^y) \right) T[R][\partial, \partial] + 4 \partial^x T[R][\partial, \partial] \\
+ \delta_{\mu \rho} \partial^x T[R][\partial, \partial] + \delta_{\mu \rho} \partial^x T[R][\partial, \partial] - 4 T[R][\partial, \partial] \\
+ \left[ x \leftrightarrow y \right] \left[ \mu \leftrightarrow \nu \right] \right\} \tag{27}
\]

for the pure gauge and

\[
< A^a_\mu(x_1) A^b_\nu(x_2) A^c_\rho(x_3) >^F = 2g^3 f^{abc} \sum_{i=1}^{N_f} \left\{ m_i^2 \left( \delta_{\mu \rho} \partial^x_\nu - \delta_{\mu \rho} \partial^x_\nu \right) T[m, m, m][1] \\
+ \left( \delta_{\mu \rho} \partial^2_{\nu} - \delta_{\mu \rho} \partial^2_{\nu} \partial^2_{\nu} \right) \partial^2_{\nu} \partial^2_{\nu} + \frac{1}{2} \delta_{\rho \alpha} \left( \partial^2_{\nu} \partial^2_{\nu} - \partial^2_{\nu} \partial^2_{\nu} \partial^2_{\nu} + \delta_{\mu \rho} \partial^2_{\nu} \partial^2_{\nu} \right) \right) T[m, m, m][\partial, \partial] \\
+ \left( \delta_{\mu \rho} \partial^2_{\nu} - \delta_{\mu \rho} \partial^2_{\nu} \partial^2_{\nu} \partial^2_{\nu} \right) T[R, m, m, m][\partial, \partial] + 2 \partial^2_{\nu} T[R, m, m, m][\partial, \partial] + 4 \partial^2_{\nu} T[R, m, m, m][\partial, \partial] \\
- 2 \delta_{\mu \rho} \partial^2_{\nu} T[R, m, m, m][\partial, \partial] + \frac{1}{2} \delta_{\mu \rho} T[R, m, m, m][\partial, \partial] \\
- 2 T[R, m, m, m][\partial, \partial] \right\} \tag{28}
\]

for the fermionic part. The Bose symmetry of the three gluons is not obvious because of the use of the shifted variables. Only the symmetry under interchange of $A^a_\mu(x_1)$ and $A^b_\nu(x_2)$ (corresponding to $(x, y, a, \mu) \leftrightarrow (-y, -x, b, \nu)$) is explicit. The complete scale dependence, written in the original variables, reads

\[
M \frac{\partial}{\partial M} < A^a_\mu(x_1) A^b_\nu(x_2) A^c_\rho(x_3) > = \frac{1}{12 \pi^2} g^3 f^{abc} (N_c - N_f) \left[ \delta_{\mu \rho} \partial^2 x_1 \partial^2 x_2 \partial^2 x_3 \delta(\delta x_1 - x_3) \delta(x_2 - x_3) \right] \tag{29}
\]
In Ref. [6], the bare triple gluon vertex in the background field formalism (with massless quarks) was found to be conformal invariant for non-coincident points, if the Feynman gauge is employed. This fact, together with the fulfilment of the Ward identities, implies that this three-point function must be a linear combination of the two permutation odd conformal tensors $D_{\mu\nu\rho}^{\text{symm}}(x_1, x_2, x_3)$ and $C_{\mu\nu\rho}^{\text{symm}}(x_1, x_2, x_3)$ [13]. The explicit expression in terms of these conformal tensors was also found in Ref. [6]. Differential renormalization was then used to treat the singularities at coincident points. Of course, renormalization breaks conformal invariance, but the Ward identity relating the triple background gluon vertex to the background gluon selfenergy was enforced by adequately adjusting the renormalization scales that appear in the process. We have applied CDR to the renormalization of the conformal tensors $D_{\mu\nu\rho}^{\text{symm}}$ and $C_{\mu\nu\rho}^{\text{symm}}$, and checked that the resulting amplitude directly fulfils the Ward identity (if the CDR result for the gluon selfenergy is used)\(^1\). No adjustment is needed \textit{a posteriori}. The tensor $C_{\mu\nu\rho}^{\text{symm}}$ is finite, but ambiguous, and the result in CDR differs from the one given in Ref. [6] by a finite local term. In our case, it gives a non-vanishing contribution to the Ward identity. The result for $D_{\mu\nu\rho}^{\text{symm}}$ also has an extra local term with respect to the final one in Ref. [6]. The discrepancies are due to the fact that, while in CDR everything is fixed from the start, in conventional differential renormalization the renormalization scales can be adjusted, in general, in more than one manner to preserve the Ward identities.

Finally, we have calculated the \textbf{quartic gluon vertex}, but the final expression is too lengthy, even in terms of basic functions, to be written here. We only give the scale dependent part:

$$M \frac{\partial}{\partial M} < A^a_\mu(x_1) A^b_\nu(x_2) A^c_\rho(x_3) A^d_\sigma(x_4) >= -\frac{1}{24\pi^2} g^4 (N_c + 2N_f) \left[ f^{abr} f^{cdr} (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}) + f^{acr} f^{dbr} (\delta_{\mu\sigma} \delta_{\rho\nu} - \delta_{\mu\nu} \delta_{\rho\sigma}) + f^{adr} f^{bcr} (\delta_{\mu\nu} \delta_{\rho\sigma} - \delta_{\mu\rho} \delta_{\nu\sigma}) \right].$$

(30)

The renormalized 1PI Green functions we have calculated satisfy the renormalization group equation

$$\left[ M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial g} + \gamma_{m_i} m_i \frac{\partial}{\partial m_i} - n_A \gamma_A - n_\eta \gamma_\eta - \sum_{i=1}^{N_f} n_{\psi_i} \gamma_{\psi_i} \right] \Gamma^{(n_A, n_\eta, n_{\psi_1}, \ldots)} = 0,$$

(31)

where $n_A$, $n_\eta$ and $n_{\psi_i}$ are the number of gauge, ghost and $i$-quark fields, respectively. The coefficients can be easily obtained from the scale dependent parts given above; the standard values are recovered:

$$\beta = -\frac{g^3}{48\pi^2} (11N_c - 2N_f),$$

(32)

\(^1\)It is important that we have used the form of the conformal tensors given by Eqs. (4.1) and (4.16) of Ref. [6]. The triple gluon vertex can be expressed in terms of the conformal tensors in this form using just the Feynman rules and the Leibniz rule for derivatives, which is an allowed operation in CDR. Hence, no ambiguity is introduced in the process previous to our calculation.
\[ \gamma_A = -\frac{g^2}{48\pi^2} (5N_c - 2N_f), \quad (33) \]
\[ \gamma_\eta = -\frac{g^2}{32\pi^2} N_c, \quad (34) \]
\[ \gamma_\Psi = \frac{g^2 N_c^2 - 1}{32\pi^2}, \quad (35) \]
\[ \gamma_{m_i} = -\frac{3}{16\pi^2} \frac{g^2 N_c^2 - 1}{N_c}. \quad (36) \]

Note that \( N_c \) and \( \frac{N_c^2 - 1}{2N_c} \) are the \( SU(N) \) Casimir invariants of the adjoint and fundamental representation, respectively. Unlike in the background field method, the anomalous dimension of the gauge field is not directly related to the \( \beta \) function. The same \( \beta \) function is obtained from all vertex functions, showing that there is a single coupling \( g \). This is a consequence of the Slavnov-Taylor identities for the scale dependent parts of the Green functions.

Let us now write the full set of Slavnov-Taylor identities for the complete 1PI Green functions. They are a bit more involved than the ones for connected Green functions. The general form in terms of the effective action can be directly derived from the BRST symmetry of the Lagrangian. Using the source terms given by Eq. (10) and suppressing all indices it reads
\[
\int d^4x \left[ \frac{\delta \Gamma}{\delta A} \frac{\delta \Gamma}{\delta K} - \frac{\delta \Gamma}{\delta \eta} \frac{\delta \Gamma}{\delta L} + \frac{\delta \Gamma}{\delta N} \frac{\delta \Gamma}{\delta \bar{\Psi}} - \frac{\delta \Gamma}{\delta \bar{\Psi}} \frac{\delta \Gamma}{\delta N} + \frac{\delta \Gamma}{\delta \bar{\eta}} \partial A \right] = 0. \quad (37) \]

To one loop, the quadratic terms can be “linearized”:
\[
\frac{\delta \Gamma}{\delta \phi} \frac{\delta \Gamma}{\delta J_{a\phi}} = \frac{\delta \Gamma^{(0)}}{\delta \phi} \frac{\delta \Gamma^{(1)}}{\delta J_{a\phi}} + \frac{\delta \Gamma^{(1)}}{\delta \phi} \frac{\delta \Gamma^{(0)}}{\delta J_{a\phi}}. \quad (38) \]

On the other hand, the ghost equation of motion,
\[
\partial_\mu \frac{\delta \Gamma}{\delta K_{a\mu}} + \frac{\delta \Gamma}{\delta \bar{\eta}^a} = 0, \quad (39) \]
allows to simplify the identity for the gluon selfenergy. This equation is trivially fulfilled to all orders in any renormalization scheme commuting with differentiation and preserving the structure of the Lagrangian. Writing explicitly the tree-level pieces, the one-loop Slavnov-Taylor identities read (we name each identity after the Green function with the largest number of elementary fields)

- **gluon selfenergy identity:**
\[
0 = \partial_\mu \left< A_{a\mu}^a(x) A_{b\nu}^b(y) \right>, \quad (40) \]
• quark vertex identity:

\[
0 = g\gamma_\mu (T^a)^{AB} \delta(x-y) < \eta^b(z) K^a_\mu (x) > - \partial^z_\mu < \Psi^B_i (x) \bar{\Psi}^A_i (y) A^b_\mu (z) > \\
- (\partial^z y + m_i) < \Psi^B_i (x) \eta^b(y) N^A_i (y) > - < \eta^b(z) \bar{\Psi}^A_i (y) N^B_i (x) > (\partial^z x - m_i) \\
- g(T^b)^{CB} \delta(x-z) < \Psi^C_i (x) \bar{\Psi}^A_i (y) > \\
+ g(T^b)^{AC} \delta(y-z) < \Psi^B_i (x) \bar{\Psi}^C_i (y) > ,
\]

(41)

• ghost vertex identity:

\[
0 = \left[ \partial^2_\mu < A^b_\mu (x) \eta^c (y) \bar{\eta}^d (z) > + g f^{acd} \partial^2_\mu \left( \delta(y-z) < \eta^b(x) K^a_\mu (z) > \right) \right] \\
- \left[ b \leftrightarrow c \right] + \square^2 < \eta^b(x) \eta^c (y) L^d (z) > + g f^{abc} \delta(x-y) < \eta^a(x) \bar{\eta}^d (z) > ,
\]

(42)

• triple gluon vertex identity:

\[
0 = \left[ g f^{adb} \delta(y-z) < A^c_\nu (x) A^a_\rho (y) > - \square^2 < \eta^b(x) A^d_\rho (y) K^c_\nu (x) > \\
- \partial^2_\nu < \eta^b(z) \bar{\eta}^c (x) A^d_\rho (y) > + \left[ \begin{array}{c} c \leftrightarrow d \\ \nu \leftrightarrow \rho \\ x \leftrightarrow y \end{array} \right] \\
+ g f^{acd} \left( -\delta_\mu (2 \partial^z_\rho + \partial^y_\rho) + \delta_\nu (\partial^x_\mu - \partial^y_\mu) + \delta_\rho (\partial^x_\nu + 2 \partial^y_\nu) \right) \right] \\
\left( \delta(x-y) < \eta^b(z) K^a_\mu (x) > - \partial^2_\mu < A^b_\mu (z) A^c_\nu (x) A^d_\rho (y) > \right) ,
\]

(43)

• quartic gluon vertex identity:

\[
0 = \left[ g f^{acd} \left( -\delta_\mu (2 \partial^y_\rho + \partial^z_\rho) + \delta_\nu (\partial^x_\mu - \partial^y_\mu) + \delta_\rho (\partial^x_\nu + 2 \partial^y_\nu) \right) \right] \\
\left( \delta(y-z) < \eta^b(t) A^c_\sigma (x) K^a_\mu (x) > + g f^{abe} \delta(x-t) < A^c_\nu (y) A^d_\rho (z) A^a_\sigma (t) > \\
- \square^2 < A^c_\nu (y) A^d_\rho (z) \eta^b(t) K^a_\sigma (x) > - \partial^2_\sigma < A^c_\nu (y) A^d_\rho (z) \eta^b(t) \bar{\eta}^a(t) > \right] \\
+ \left[ \begin{array}{c} e \leftrightarrow c \\ \sigma \leftrightarrow \nu \\ x \leftrightarrow y \end{array} \right] \\
+ g^2 \left( f^{acr} f^{der} (\delta_{\mu \rho} \delta_{\nu \sigma} - \delta_{\mu \sigma} \delta_{\nu \rho}) + f^{adr} f^{ecr} (\delta_{\mu \sigma} \delta_{\nu \rho} - \delta_{\mu \rho} \delta_{\nu \sigma}) \right) \\
+ f^{auc} f^{cdr} (\delta_{\mu \rho} \delta_{\nu \sigma} - \delta_{\mu \sigma} \delta_{\nu \rho}) \right) \left( \delta(x-y) \delta(x-z) < \eta^b(t) K^a_\mu (x) > \right) \\
- \partial^2_\mu < A^b_\mu (t) A^c_\nu (y) A^d_\rho (z) A^a_\sigma (x) > .
\]

(44)

The diagrams contributing to Green functions with insertions of BRST variations are depicted in Fig. 3. Verifying that the renormalized Green functions calculated above satisfy these identities is not straightforward, due to the length of the expressions.
involved and to the fact that we are not dealing with simple functions but with distributions. The obvious way to compare distributions is to make them act on a general test function. In particular one can perform a Fourier transform without loss of information\textsuperscript{2}. This has the advantage that the resulting finite integrals can be treated with standard momentum space techniques and that, for non-exceptional momenta, no infrared divergencies appear. We have used a Mathematica-based program to carry out the algebraic operations, linked to LoopTools \cite{15}, which calculates numerically one-loop integrals. As advanced, CDR respects the Slavnov-Taylor identities.

Summarizing, we have applied CDR to the one-loop singular 1PI Green functions of QCD and have verified that the Slavnov-Taylor identities are preserved. The extension to Yang-Mills theories with a more general gauge group does not introduce new complications, as far as renormalization is concerned, so CDR should treat the general case equally well. Here we have stuck to $SU(N_c)$ because the computer implementation is simpler \cite{16}. One could also wonder about the performance of CDR when the gauge symmetry is spontaneously broken. Again, as CDR does not depend on the structure of the interaction Lagrangian, there should be no extra problems in dealing with this case. The authors of Ref. \cite{15}, using the programs described there, have checked that CDR renders a transverse vacuum polarization in the electroweak standard model, and recovers the standard physical results for $Z-Z$ and $W-W$ elastic scattering. These examples also test the inclusion of scalar fields in a non-Abelian gauge theory. Finally, let us point out that very recently it has been found (in momentum space) that CDR and regularization by dimensional reduction produce equivalent results at the one loop level \cite{15}. This gives an alternative explanation for the preservation of gauge invariance in CDR.

**Acknowledgments**

I thank F. del Aguila for discussions. Many of the calculations have been performed with computers of the Institut für Theoretische Physik of the University of Karlsruhe. This work has been supported by CICYT, under contract number AEN96-1672 and by Junta de Andalucía, FQM101. I also thank Ministerio de Educación y Cultura for financial support.

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Figure 1: Coordinate space Feynman rules for QCD, including insertions of BRST variations. In vertices, the derivatives (with respect to the vertex space-time point) act on the field indicated by the superscript.
Figure 2: One-loop Feynman diagrams contributing to 1PI Green functions with insertions of BRST variations. The Feynman diagrams for singular 1PI Green functions of elementary fields can be found in Ref. [11]. The diagrams contributing to the (finite) gluon-gluon-ghost-antighost function, which appears in the quartic-gluon-vertex identity, are identical to diagrams f1-f4, but changing $K$ by an antighost external leg.