SOLVING PHASE RETRIEVAL WITH RANDOM INITIAL GUESS IS NEARLY AS GOOD AS BY SPECTRAL INITIALIZATION

JIANFENG CAI, MENG HUANG, DONG LI, AND YANG WANG

ABSTRACT. The problem of recovering a signal $x \in \mathbb{R}^n$ from a set of magnitude measurements $y_i = |\langle a_i, x \rangle|$, $i = 1, \ldots, m$ is referred as phase retrieval, which has many applications in fields of physical sciences and engineering. In this paper we show that the smoothed amplitude flow model for phase retrieval has benign geometric structure under the optimal sampling complexity. In particular, we show that when the measurements $a_i \in \mathbb{R}^n$ are Gaussian random vectors and the number of measurements $m \geq Cn$, our smoothed amplitude flow model has no spurious local minimizers with high probability, i.e., the target solution $x$ is the unique global minimizer (up to a global phase) and the loss function has a negative directional curvature around each saddle point. Due to this benign geometric landscape, the phase retrieval problem can be solved by the gradient descent algorithms without spectral initialization. Numerical experiments show that the gradient descent algorithm with random initialization performs well even comparing with state-of-the-art algorithms with spectral initialization in empirical success rate and convergence speed.

1. INTRODUCTION

1.1. Background. This paper concerns the well-known phase retrieval problem, which aims to recover the signal $x \in \mathbb{R}^n$ from a series of magnitude-only measurements

$$y_i = |\langle a_i, x \rangle|, \quad i = 1, \ldots, m$$

where $a_i \in \mathbb{R}^n, i = 1, \ldots, m$ are Gaussian random vectors and $m$ is the number of measurements. This problem arises in many fields of science and engineering due to the physical limitations of optical detectors which can only record the magnitude of signals while losing the phase information, such as X-ray crystallography [15, 20], microscopy [19], astronomy [6], coherent diffractive imaging [14, 24] and optics [29] etc. Despite its simple mathematical form, it has been shown that reconstructing a finite-dimensional discrete signal from the magnitude of its Fourier transform is generally an $NP$-complete problem [23].

1 The results in our paper were first presented at a workshop in December, 2019.
J. F. Cai was supported in part by Hong Kong Research Grant Council grants 16306317 and 16309219.
D. Li was supported in part by Hong Kong RGC grant GRF 16307317 and 16309518.
Y. Wang was supported in part by the Hong Kong Research Grant Council grants 16306415 and 16308518.
Many algorithms have been designed to solve the phase retrieval problem. They fall generally into two categories: convex algorithms and non-convex ones. The convex algorithms usually rely on a “matrix-lifting” technique, which lifts the phase retrieval problem into a low rank matrix recovery problem, together with convex relaxation by showing that the matrix recovery problem under some conditions is equivalent to a convex optimization problem. These algorithms include PhaseLift [2,4], PhaseCut [28] etc. It has been shown [2] that PhaseLift can achieve the exact recovery under the optimal sampling complexity with Gaussian random measurements.

Although convex methods have good theoretical guarantees to converge to the true solutions under some special conditions, they tend to be computationally inefficient for large scale problems. By contrast, many non-convex algorithms do not need the lifting step so they operate directly on the lower-dimensional ambient space, making them much more efficient. Early non-convex algorithms were based mostly on alternating projections, e.g. Gerchberg-Saxton [13] and Fineup [8]. The drawback is the lack of theoretical guarantee. Later Netrapalli et al [21] proposed the AltMinPhase algorithm based on a technique known as spectral initialization, and they proved that the algorithm linearly converges to the true solution with $O(n \log^3 n)$ resampling Gaussian random measurements. This work led to further several other non-convex algorithms based on spectral initialization. They share the common idea of first choosing a good initial guess through spectral initialization, and then solving an optimization model through gradient descent. Two commonly used optimization models are the intensity flow model

\[
\min_{z \in \mathbb{R}^d} F(z) = \frac{1}{m} \sum_{j=1}^{m} \left( |\langle a_j, z \rangle|^2 - y_j^2 \right)^2.
\]

and the amplitude flow model

\[
\min_{z \in \mathbb{R}^d} F(z) = \frac{1}{m} \sum_{j=1}^{m} |\langle a_j, z \rangle| - y_j|^2.
\]

Specifically, Candès et al developed the Wirginger Flow (WF) [3] method based on (1) and proved that the WF algorithm can achieve linear convergence with $O(n \log n)$ Gaussian random measurements. Lately, Chen and Candès improved the results to $O(n)$ Gaussian random measurements by incorporating a truncation, namely the Truncated Wirginger Flow (TWF) [5] algorithm. Other methods based on (1) include the Gauss-Newton [10] method, the trust-region [25] method, and others. Several algorithms based on the amplitude flow
model (2) have also been developed recently, such as the Truncated Amplitude Flow (TAF) algorithm [30], the Reshaped Wirtinger Flow (RWF) [31] algorithm and the Perturbed Amplitude Flow (PAF) [9] algorithm. All three algorithms above have been shown to linearly converge to the true solution up to a global phase with $O(n)$ Gaussian random measurements. Furthermore, numerical results show that algorithms based on the amplitude flow model (2) tend to outperform algorithms based on model (1).

1.2. Motivation and Related Work. As we have stated earlier, producing a good initial guess using spectral initialization is a prerequisite for all aforementioned non-convex algorithms with theoretical guarantee. An interesting question is that Is it possible for those algorithms to achieve successful recovery with a random initialization?

For intensity-based model (1), the answer is yes. Ju Sun et al [25] study the global geometry structure of the loss function of (1). They show the loss function $F(z)$ does not have any spurious local minima under $O(n \log^3 n)$ Gaussian random measurements. It means that all minimizers are the target signal $x$ up to a global phase and the loss function has a negative directional curvature around each saddle point. Thus any algorithm which can avoid saddle points converges to the true solution with high probability. They also develop a trust-region method to find a global solution with random initialization. To reduce the sampling complexity, it has been shown that the combination of the loss function (1) and an activation function also possesses the benign geometry structure under $O(n)$ Gaussian random measurements [18].

The geometry landscape concept has also been explored in recent years for other applications in signal processing and machine learning, e.g., matrix sensing [1, 22], tensor decomposition [11], dictionary learning [26] and matrix completion [12]. Well-behaved geometry landscapes for optimization, namely all local optimal are also global optimal and the loss function has a negative directional curvature around each saddle point, have been shown to exist more broadly. Several techniques have been developed to guarantee that the basic gradient optimization algorithms can escape such saddle points efficiently, see e.g. [7, 16, 17].

1.3. Our contributions. Optimization algorithms based on the amplitude model (2) have been shown to outperform those based on the intensity model (1). Naturally we may ask
whether it is possible to examine the geometric landscape for the amplitude model (2) and develop algorithms similar to the ones in [18, 25]. As it turns out, a straightforward approach based on model (2) loss function fails as there will be many local minima regardless how many measurements one take. In this paper, we show that by altering the amplitude model based loss function slightly we are able to obtain to benign geometric landscape for the loss function, thus yielding a fast algorithm that requires only $O(d)$ measurements and no initialization. Furthermore, numerical tests show that the algorithm outperforms several existing algorithms in terms of efficiency.

We now describe our study in more details. Let $x \in \mathbb{R}^n$ be the target signal we want to recover. The measurements we obtain are

$$y_i = |\langle a_i, x \rangle|, \ i = 1, \ldots, m$$

where $a_i \in \mathbb{R}^n, i = 1, \ldots, m$ are Gaussian random vectors. For the recovery of $x$ we consider the following new loss function $F(z)$ given by

$$F(z) = \frac{1}{2m} \sum_{i=1}^{m} \left( \gamma \left( \frac{|a_i^\top z|}{|a_i^\top x|} \right) - 1 \right)^2 \cdot |a_i^\top x|^2,$$

where the function $\gamma(t)$ is taken to be

$$\gamma(t) := \begin{cases} t, & |t| > \beta; \\ \frac{1}{2} t^2 + \frac{\beta}{2}, & |t| \leq \beta. \end{cases}$$

Note that the event $\bigcup_{i=1}^{m} \{a_i^\top x = 0\}$ has zero probability and we may assume that $a_i^\top x \neq 0$ for all $i$. Another practical way is to define $(\gamma(\frac{|a_i^\top z|}{|a_i^\top x|}) - 1)^2 \cdot |a_i^\top x|^2 = |a_i^\top z|^2$ when $a_i^\top x = 0$.

Because $\gamma(t)$ is smoothed from $|t|$ in the amplitude flow model, we shall call our model the Smoothed Amplitude Flow (SAF) model. Clearly if we set $\gamma(t) := |t|$ then the loss function is exactly the amplitude flow model loss function given in (2). Unfortunately in this case the loss function does not yield the desired geometric landscape: Regardless how many measurements one take, there will appear multiple local minima. Our new loss function, however, will have the desired property. The main theorem of the paper is the following:

**Theorem 1.1.** Fix $0 < \beta \leq \frac{1}{2}$. Assume $m \geq Cn$. Let $x \in \mathbb{R}^n$ be nonzero and $\{a_i\}_{i=1}^{m}$ be i.i.d. random Gaussian vectors, i.e., $a_i \sim N(0, I_n)$ for all $i$. Then with probability at least $1 - c \exp(-\delta m)$ the loss function $F(z)$ given in (3) has no spurious local minima, i.e. all local minima are also global minima and any other critical point is a saddle point with a negative directional curvature. Here $C, c, \delta$ are positive constants depending on $\beta$. 
Remark 1.2. For simplicity, we only consider the geometric landscape in the real case, however, the result in Theorem 1.1 can be adapted to the complex case and we will address it elsewhere.

Theorem 1.1 implies that gradient descent with any random initial point will not get stuck in a local minimum. Our result turns out to be not just of theoretical interest. Numerical tests show that this model yields very stable and fast convergence with random initialization with performance on a par with or even better than the existing gradient descent methods with spectral initialization.

1.4. Organization. The rest of this paper is organized as follows. In Section 2, we provide an outline of the proof. In Section 3, we break down $\mathbb{R}^n$ into several regions and investigate the geometric property of $F(z)$ on each region. In Section 4, we carry out some numerical experiments to demonstrate the effectiveness of our model. The appendix collects the proofs of some technical lemmas and propositions.

2. Geometric Properties of the SAF Loss Function

Our main theorem is a consequence of the analysis of the geometric landscape of the smoothed amplitude flow model loss function $F(z)$ in (3). As with [25], we shall decompose $\mathbb{R}^n$ into several regions (not necessarily non-overlapping), on each of which $F(z)$ has certain property that will allow us to show that with high probability $F(z)$ has no local minimizers other than $\pm x$. Furthermore, we show $F(z)$ is strongly convex in a neighborhood of $\pm x$.

Thus our strategy for proving the main result is as follows:

Step 1: Compute the gradient and Hessian of the loss function $F(z)$. Since $F(z)$ is not 2nd order differentiable we shall consider the directional second derivative of $F(z)$. Notice that all these are given by sums of random variables.

Step 2: Apply concentration inequalities such as Bernstein’s inequality as well as union bounds to approximate the sums of random variables for a given $z$.

Step 3: Estimate the approximations obtained from concentration inequalities to establish the geometric properties for $F(z)$. In particular we shall estimate $\langle \nabla F(z), z \rangle$, $\langle \nabla F(z), x \rangle$, and $D_v^2 F(z)$ where $D_v^2$ denotes the directional 2nd order derivative along the direction $v$ of $F(z)$.
Because $\gamma(t)$ of (4) is given piecewise, the main difficulty here lies with estimations in step 3. Fortunately, while tedious, they can be done to yield what we will need to prove the theorem.

2.1. **Step 1.** Note that $F(z)$ is continuously differentiable, but it is not 2nd order differentiable, so for the 2nd order derivatives we will resort to directional derivatives. Recall that for a function $g(z)$ and any vector $v \neq 0$ in $\mathbb{R}^n$, the one-side directional derivative of $g$ at $z$ along the direction $v$ is given by

$$D_v g(z) := \lim_{t \to 0^+} \frac{g(z + tv) - g(z)}{t}$$

if the limit exists. Furthermore, we denote

$$D_v^2 g(z) = D_v(D_v g(z))$$

as the second order directional derivative of $g$ at $z$ along the direction $v$. As we shall show, both the gradient $\nabla F(z)$ and the $D_v^2 F(z)$ are subexponential random variables in terms of $a^\top z$, $a^\top x$, and $a^\top v$. This enables us to apply concentration inequalities described in Step 2. The details will be given in Section 4.

2.2. **Step 2.** Concentration inequalities allow us to estimate the sums of random variables so we can estimate them for proving the main result. We shall be dealing with subexponential random variables in this paper. A good discussion of subexponential random variables can be found in [27].

**Lemma 2.1** ([27], Theorem 2.8.1). Let $g(s, t)$ be a real valued function such that $g(a_i^\top z, a_i^\top x)$ is subexponential with subexponential norm $\|g(a_i^\top z, a_i^\top x)\|_{\psi_1} \leq \tau$. Then for any $\varepsilon > 0$ we have

$$\left| \frac{1}{m} \sum_{i=1}^{m} g(a_i^\top z, a_i^\top x) - \mathbb{E}[g(a_i^\top z, a_i^\top x)] \right| \leq \varepsilon$$

with probability at least $1 - 2 \exp(-cm \min(\varepsilon^2/\tau^2, \varepsilon/\tau))$, where $c > 0$ is a universal constant.

Of course we will need (5) to hold uniformly for all $z$ on certain region. This is typically proved by using some $\delta$-nets, and there is no exception here. We have
Corollary 2.2. Let \( g(s, t) \) be a real valued function such that \( g(a_i^\top z, a_i^\top x) \) is subexponential with subexponential norm \( \|g(a_i^\top z, a_i^\top x)\|_{\Psi_1} \leq \tau \). Assuming that on a compact set \( \Omega \) we have
\[
\frac{1}{m} \sum_{i=1}^{m} |g(a_i^\top z, a_i^\top x) - g(a_i^\top z_0, a_i^\top x)| \leq \frac{1}{m} \sum_{i=1}^{m} K(a_i) |a_i^\top (z - z_0)|
\]
for any \( z, z_0 \in \Omega \), where \( K(a_i) \) is a subgaussian random variable with subgaussian norm \( \eta \). Then for any \( \epsilon > 0 \) there exist constants \( C, c > 0 \) depending only on \( \epsilon, \tau, \eta \) such that for \( m \geq Cn \) we have
\[
\frac{1}{m} \sum_{i=1}^{m} g(a_i^\top z, a_i^\top x) - \mathbb{E}[g(a_1^\top z, a_1^\top x)] \leq \epsilon
\]
with probability at least \( 1 - \exp(-cm) \) for all \( z \in \Omega \).

Proof. We shall give the proof in the Appendix.

Corollary 2.3. Suppose \( \chi(t) : \mathbb{R}_+ \to \mathbb{R} \) is a Lipschitz function with Lipschitz constant \( L \) and \( \text{supp}(\chi) \subset [0, 1] \). For any \( \delta, \epsilon > 0 \), if \( m \geq Cn \) then with probability at least \( 1 - \exp(-cm) \) we have
\[
\left| \frac{1}{m} \sum_{i=1}^{m} (\frac{|a_i^\top z_1|}{|a_i^\top x|} \chi(\frac{|a_i^\top z_1|}{|a_i^\top x|}) - \frac{|a_i^\top z_2|}{|a_i^\top x|} \chi(\frac{|a_i^\top z_2|}{|a_i^\top x|})) \right| \leq (L + 1) \cdot (\|z_1 - z_2\|/\delta + \delta + \epsilon/\delta + \epsilon)
\]
for any fixed \( z_1, z_2 \in \mathbb{R}^d \). Here \( c > 0, C > 0 \) depend on \( (L, \delta, \epsilon) \).

Proof. We shall give the proof in the Appendix.

2.3. Step 3. To prove the main result we will need to estimate several integrals involving the gradient and directional 2nd order derivatives of \( F(z) \) for \( m \geq Cn \) for some \( C > 0 \). We shall show that for a given \( x \neq 0 \), with high probability the Smoothed Amplitude Flow loss function \( F(z) \) is strictly convex on a small neighborhood of \( \pm x \). When \( z \) is not too close to being orthogonal to \( x \), with high probability \( F(z) \) has no critical point. Finally, for \( z \) close to being orthogonal to \( x \), we show that with high probability any critical point is a saddle point with a negative directional curvature. We shall present these in Section 3 and the Appendix.

Here “with high probability” means there exist constants \( c, \delta > 0 \) such that the probability is at least \( 1 - c \exp(-\delta m) \). We shall make it more explicit in the next section.
3. Proof of the main results

3.1. Notation and General Assumptions. Now for $x \neq 0$ and standard Gaussian random vectors $\{a_i\}_{i=1}^m$, recall that $F(z)$ a continuously differentiable function given by

$$F(z) = \frac{1}{2m} \sum_{i=1}^m \left( \gamma \left( \frac{a_i^\top z}{a_i^\top x} \right) - 1 \right)^2 \cdot |a_i^\top x|^2,$$

with $\gamma(t)$ being defined in (4) as

$$\gamma(t) := \begin{cases} 
\frac{|t|}{2}, & |t| > \beta; \\
\frac{\beta^2 |t|^2}{2}, & |t| \leq \beta.
\end{cases}$$

We shall denote $f_i(z) = \frac{1}{2} \left( \gamma \left( \frac{a_i^\top z}{a_i^\top x} \right) - 1 \right)^2 \cdot |a_i^\top x|^2$, where $\left( \gamma \left( \frac{a_i^\top z}{a_i^\top x} \right) - 1 \right)^2 \cdot |a_i^\top x|^2 = |a_i^\top z|^2$ if $a_i^\top x = 0$. So $F(z) = \frac{1}{m} \sum_{i=1}^m f_i(z)$. Set

$$\Psi(u, v) = \frac{1}{2} \left( \gamma \left( \frac{u}{v} \right) - 1 \right)^2 v^2 \text{ if } v \neq 0, \text{ and } \Psi(u, 0) = \frac{1}{2} u^2.$$

Then $f_i(z) = \Psi(a_i^\top z, a_i^\top x)$, and $\nabla f_i(z) = \Psi_u(a_i^\top z, a_i^\top x) a_i$, where

$$\Psi_u(u, v) = \gamma' \left( \frac{u}{v} \right) v = \begin{cases} 
\sgn(u)(|u| - |v|), & |u| > \beta|v|; \\
\frac{u^3}{2v^2} + (\frac{1}{2} - \frac{1}{\beta})u, & |u| \leq \beta|v|.
\end{cases}$$

The following summarizes bounds and Lipschitz property of $\Psi_u$:

$$|\Psi_u(u, v)| \leq |u| + |v|, \quad \Psi_u(u, v)u \geq u^2 - |uv|,$$

and

$$|\Psi_u(u_1, v) - \Psi_u(u_2, v)| \leq \max(1, |2 - 1/\beta|) \cdot |u_1 - u_2|.$$ 

These properties are easy to check, and we delay the proofs in the appendix.

In the rest of the paper we shall adopt the following notation and specify some assumptions without loss of generality.

(A1) Since $F(z) = F(-z)$, to solve the SAF model here we may without loss of generality consider solving the SAF model on the half space $\langle z, x \rangle \geq 0$. Denote $\sigma = \sigma(z) := \langle z, x \rangle / \|z\| \|x\| \geq 0$ and $\tau = \tau(z) := \sqrt{1 - \sigma^2}$. Furthermore, we shall write

$$\frac{z}{\|z\|} = \sigma \frac{x}{\|x\|} + \tau w$$

where $w \perp x$ and $\|w\| = 1$. 
Denote $\lambda = \beta / \|z\|$. Two quantities that appear often in the paper are $\mu_+ = \mu_+(\lambda, \sigma)$ and $\mu_- = \mu_-(\lambda, \sigma)$ given by

$$
\mu_+^2 := 1 + \frac{(\sigma + \lambda)^2}{\tau^2} = \frac{1}{\tau^2} (1 + \lambda^2 + 2\sigma\lambda),
$$

$$
\mu_-^2 := 1 + \frac{(\sigma - \lambda)^2}{\tau^2} = \frac{1}{\tau^2} (1 + \lambda^2 - 2\sigma\lambda).
$$

We may of course without loss of generality assume that $\|x\| = 1$. Let $a \sim N(0, I_n)$ be a standard Gaussian vector in $\mathbb{R}^n$. From the orthogonal decomposition $z / \|z\| = \sigma x + \tau w$ in (13), define $U = a^T z / \|z\|, V = a^T x$ and $W = a^T w$. Then $U, V, W \sim N(0, 1)$ and $V, W$ are independent. Let $A = A(\lambda)$ denote the event

$$
A = A(\lambda) := \{a : |a^T z| \leq \beta |a^T x|\} = \{a : |U| \leq \lambda |V|\}.
$$

The notations given here will be used extensively in the next subsection where we compute various expectations.

**Definition 3.1.** We say that a property $\mathcal{P}$ holds with high probability for $m$ if there exist some $c, \delta > 0$ such that $\mathcal{P}$ holds with probability at least $1 - c \exp(-\delta m)$.

### 3.2. Expectations

Following the notations in (A3) above, we first observe that the condition $|U| \leq \lambda |V|$ is equivalent to the event $\{|\sigma V + \tau W| \leq \lambda |V|\}$, and hence

$$
A = \{\tau^{-1}(-\lambda - \sigma \text{sgn}(V))|V| \leq W \leq \tau^{-1}(\lambda - \sigma \text{sgn}(V))|V|\}.
$$

This is used to prove the following crucial proposition.

**Proposition 3.2.** Let $G(\lambda) := \mathbb{E}[g(U, V) 1_A]$ where $g(t, s)$ is continuous. Then

$$
\frac{dG}{d\lambda} = \frac{1}{2\pi \tau} \int_0^\infty (g(-\lambda v, v) + g(\lambda v, -v)) v e^{-\frac{1}{2}\mu_+^2 v^2} dv + \frac{1}{2\pi \tau} \int_0^\infty (g(\lambda v, v) + g(-\lambda v, -v)) v e^{-\frac{1}{2}\mu_-^2 v^2} dv.
$$

**Proof.** We shall prove this lemma in the appendix.

**Corollary 3.3.** Assume that $g(t, s) = |t|^p |s|^q$ in Proposition 3.2 where $p + q \geq 0$. Then

$$
G'(\lambda) = \frac{\lambda^p}{\pi^2} \left( \mu_-^{-(p+q+2)} + \mu_+^{-(p+q+2)} \right) \int_0^\infty t^{p+q+1} e^{-\frac{1}{2} t^2} dt.
$$

In particular, if $p + q = 2$ then $G'(\lambda) = \frac{2\lambda^p}{\pi^2} (\mu_-^{-4} + \mu_+^{-4})$. 

Proof. This is a straightforward application of Proposition 3.2. Observe that for \( p + q = 2 \) the integral
\[
\int_0^{\infty} t^3 e^{-\frac{1}{2}t^2} dt = 2.
\]

Corollary 3.4. Assume that \( g(t, s) = \text{sgn}(ts)|t|^p|s|^q \) in Lemma 3.2 where \( p + q \geq 0 \). Then
\[
G'(\lambda) = \frac{\lambda^p}{\pi \tau} \left( \mu_-^{-(p+q+2)} - \mu_+^{-(p+q+2)} \right) \int_0^{\infty} t^{p+q+2} e^{-\frac{1}{2}t^2} dt.
\]
Hence \( G(\lambda) \geq 0 \). In particular, if \( p + q = 2 \) then
\[
G'(\lambda) = \frac{2\lambda^p}{\pi \tau} (\mu_-^{-4} - \mu_+^{-4}).
\]

Proof. Same as the previous corollary.

Lemma 3.5.
\[
\mathbb{E}[|UV|] = \frac{2}{\pi} \left( \tau + \sigma \arctan \frac{\sigma}{\tau} \right),
\]
\[
\mathbb{E}[	ext{sgn}(UV)V^2] = \frac{2}{\pi} \left( \tau \sigma + \arctan \frac{\sigma}{\tau} \right).
\]

Proof. We leave the proof to Appendix.

3.3. Non-Vanishing Gradient. In this subsection we evaluate the gradient of \( F(z) \). Our ultimate goal is to establish a region on which \( \nabla F(z) \) does not vanish.

Lemma 3.6. Assume that \( 0 < \beta < 1 \) and \( \|x\| = 1 \). For any \( \delta_0 > 0 \) there exist \( C, \varepsilon_0 > 0 \) such that with high probability for \( m \geq Cn \),
\[
\langle \nabla F(z), z \rangle \geq \varepsilon_0 \|z\|^2
\]
for all \( \|z\| \geq \frac{2}{\pi} \left( \tau + \sigma \arctan \frac{\sigma}{\tau} \right) + \delta_0. \)

Proof. We have \( \nabla F(z) = \frac{1}{m} \sum_{i=1}^{m} \nabla f_i(z) \) and \( \langle \nabla f_i(z), z \rangle = \Psi_u(a_i^\top z, a_i^\top x)(a_i^\top z) \). By (11) we have
\[
\Psi_u(a_i^\top z, a_i^\top x)(a_i^\top z) \geq (a_i^\top z)^2 - |(a_i^\top z)(a_i^\top x)|.
\]
Set \( U_i = a_i^\top z/\|z\|, V_i = a_i^\top x \). Then from Lemma 3.5 we have
\[
\mathbb{E}[\|U_iV_i\|] = \frac{2}{\pi} \left( \tau + \sigma \arctan \frac{\sigma}{\tau} \right).
\]
It follows that
\[
\frac{1}{\|z\|^2} \mathbb{E}[\langle \nabla f_i(z), z \rangle] \geq 1 - \frac{1}{\|z\|} \mathbb{E}[\|U_iV_i\|] \geq 1 - \frac{1}{\|z\|} \frac{2}{\pi} \left( \tau + \sigma \arctan \frac{\sigma}{\tau} \right).
\]
Note that $\frac{1}{\|z\|^2} \langle \nabla f_i(z), z \rangle$ is continuous satisfying the condition in Corollary 2.2. It follows that for any $\delta_0 > 0$ there exists $C, \varepsilon_0 > 0$ such that with high probability for $m \geq Cn$ we have

$$\frac{1}{\|z\|^2} \langle \nabla F(z), z \rangle = \frac{1}{m} \sum_{i=1}^{m} \frac{1}{\|z\|^2} \langle \nabla f_i(z), z \rangle \geq \varepsilon_0$$

for all $\|z\| \geq \frac{2}{\pi} (\tau + \arctan \frac{2}{\tau}) + \delta_0$ (This region is a compact set with respect to $\lambda = \beta / \|z\|$). The lemma is proved.

**Lemma 3.7.** Assume that $0 < \beta \leq \frac{1}{2}$ and $\|x\| = 1$. For any $\varepsilon_0, \sigma_0 > 0$ there exist $C, \varepsilon > 0$ such that with high probability for $m \geq Cn$, we have $\langle \nabla F(z), x \rangle < -\varepsilon$ for all $z$ such that $\varepsilon_0 \leq \|z\| \leq 1$ and $\sigma_0 \leq \sigma \leq 1 - \sigma_0$.

**Proof.** We have $\langle \nabla F(z), x \rangle = \frac{1}{m} \sum_{i=1}^{m} \Psi_u(a_i^T z, a_i^T x)(a_i^T x)$. Let $a \sim N(0, I_n)$ be standard Gaussian. Then $E[\langle \nabla F(z), x \rangle] = E[\Psi_u(a^T z, a^T x)a^T x]$. Set

$$g(z) = \frac{1}{\|z\|} E[\langle \nabla F(z), x \rangle] = \frac{1}{\|z\|} E[\Psi_u(a^T z, a^T x)a^T x].$$

Using the notation $U, V, W$ and $\lambda = \beta / \|z\|$ defined in (A3) and expanding out $\Psi_u$ via (10), it yields

$$g(z) = \sigma - \frac{\lambda}{\beta} E[\text{sgn}(UV)V^2 1_{A^c}] + \frac{1}{2\lambda^2} E[U^3 V^{-1}] 1_A - \left(\frac{1}{2} + \frac{1}{\beta}\right) E[UV 1_A]$$

where $A := \{|U| \leq \lambda |V|\}$. Note that $g(z)$ now depends on $\lambda$ and $\sigma$ as $z$ is completely determined by $\lambda$ and $\sigma$ once $\beta$ is fixed. Set $B(\lambda, \sigma) := \frac{2}{\pi} (\mu_- - \mu_+^4)$. By Corollary 3.4 we have

$$\frac{\partial g}{\partial \lambda} = -\frac{1}{\beta} E[\text{sgn}(UV)V^2 1_{A^c}] + \frac{\lambda}{\beta} B(\lambda, \sigma) - \frac{1}{\lambda^2} E[U^3 V^{-1}] 1_A$$

$$+ \frac{1}{2\lambda^2} \lambda^3 B(\lambda, \sigma) - \left(\frac{1}{2} + \frac{1}{\beta}\right) \lambda B(\lambda, \sigma)$$

$$= -\frac{1}{\beta} E[\text{sgn}(UV)V^2 1_{A^c}] - \frac{1}{\lambda^2} E[U^3 V^{-1}] 1_A.$$

Also by Corollary 3.4, we know

$$E[U^3 V^{-1}] 1_A = \frac{2}{\pi} \int_0^\lambda t^3 (\mu_- - \mu_+^4) \ dt \geq 0.$$ 

Furthermore, note that $A^c = \{|U| \geq \lambda |V|\} = \{|V| < \lambda^{-1} |U|\}$. By switching the role of $U$ and $V$ we have

$$E[\text{sgn}(UV)V^2 1_{A^c}] = E[\text{sgn}(UV)U^2 1_{A^c}] = \frac{2}{\pi} \int_0^{\lambda^{-1}} t^2 (\mu_- - \mu_+^4) \ dt \geq 0.$$
where \( A' := \{|U| \leq \lambda^{-1}|V|\} \). Therefore \( g \) is a decreasing function in \( \lambda \), which means that \( g \) is increasing with respect to \( \|z\| \).

We would like to prove that \( g(z) < 0 \) for \( \epsilon_0 \leq \|z\| \leq 1 \) and \( \sigma_0 \leq \sigma \leq 1 - \sigma_0 \). To do so we only need to show \( g(z) < 0 \) for \( \|z\| = 1 \), i.e. \( \lambda = \beta \) and \( \sigma_0 \leq \sigma \leq 1 - \sigma_0 \). Now, note that

\[
\mathbb{E}[\text{sgn}(UV)V^2I_{Ac}] = \mathbb{E}[\text{sgn}(UV)V^2] - \mathbb{E}[\text{sgn}(UV)V^2I_A]
= \frac{2}{\pi} \left( \tau \sigma + \arctan \left( \frac{\sigma}{\tau} \right) \right) - \mathbb{E}[\text{sgn}(UV)V^2I_A].
\]

Thus for \( \lambda = \beta \), again applying Corollary 3.4 we obtain

\[
(15) \quad g(z) = \sigma - \frac{2}{\pi} \left( \tau \sigma + \arctan \left( \frac{\sigma}{\tau} \right) \right) + \int_0^\beta \left( 1 + \frac{t^3}{\beta^2} - \left( \frac{1}{2} + \frac{1}{\beta} \right) t \right) B(t, \sigma) dt.
\]

To establish \( g(z) < 0 \) here, we observe that

\[
\frac{\partial g(z)}{\partial \beta} = \int_0^\beta \left( \frac{1}{\beta} \left( \frac{t}{\beta} \right) - \left( \frac{t}{\beta} \right)^3 \right) B(t, \sigma) dt > 0.
\]

Thus \( g(z) \) is increasing with respect to \( \beta \). Hence, it suffices to show that \( g(z) < 0 \) for \( \beta = 1/2 \). Expanding \( B(t, \sigma) \) yields

\[
B(t, \sigma) = \frac{16\tau^3 \sigma t(1 + t^2)}{\pi ((1 + t^2)^2 - 4\sigma^2 t^2)^2} =: \tau^3 \sigma Q(t, \sigma),
\]

where \( Q(t, \sigma) := \frac{16(1 + t^2)}{\pi((1 + t^2)^2 - 4\sigma^2 t^2)^2} \). It follows that for \( \beta = 1/2 \) we have

\[
(16) \quad \frac{1}{\sigma \tau^3} g(z) = \frac{1}{\tau^3} \left( 1 - \frac{2}{\pi} \left( \tau + \frac{1}{\sigma} \arctan \left( \frac{\sigma}{\tau} \right) \right) \right) + \int_0^{1/2} (1 + 2t^3 - 2.5t) Q(t, \sigma) dt.
\]

Clearly \( Q(t, \sigma) \leq Q(t, 1) \). Next, integrating rational functions by partial fractions, we can obtain

\[
\int_0^{1/2} (1 + 2t^3 - 2.5t) Q(t, 1) dt = \frac{8}{\pi} \int_0^{1/2} \frac{(4t^3 - 5t + 2)(t^2 + 1)t}{(t^2 - 1)^4} dt
= \frac{4}{\pi} \left( \frac{35}{27} - \ln 3 \right) < 0.26.
\]
Meanwhile, the term before the integral in (16) is a function of $\sigma$ and it is decreasing. Indeed, by Corollary 3.4 where we set $\lambda = \infty$, we have

$$P(\sigma) := \frac{1}{\tau^3} \left( 1 - \frac{2}{\pi} \left( \tau + \frac{1}{\sigma} \arctan \frac{\sigma}{\tau} \right) \right)$$

$$= \frac{1}{\tau^3 \sigma} \mathbb{E} [UV - \text{sgn}(UV)V^2]$$

$$= \frac{1}{\tau^3 \sigma} \int_{0}^{\infty} (t - 1)B(t, \sigma) \, dt$$

$$= \frac{16}{\pi} \int_{0}^{\infty} \frac{(t - 1)(1 + t^2)t}{[(1 + t^2)^2 - 4t^2 \sigma^2]^2} \, dt.$$

Making a substitution $t = \frac{1}{u}$, we obtain

$$\int_{1}^{\infty} \frac{(t - 1)(1 + t^2)t}{[(1 + t^2)^2 - 4t^2 \sigma^2]^2} \, dt = - \int_{0}^{1} \frac{(u - 1)(1 + u^2)u^2}{[(1 + u^2)^2 - 4u^2 \sigma^2]^2} \, du.$$

It gives

$$P(\sigma) = - \frac{16}{\pi} \int_{0}^{1} \frac{(1 - t)^2(1 + t^2)t}{[(1 + t^2)^2 - 4t^2 \sigma^2]^2} \, dt,$$

which is decreasing with respect to $\sigma$. Thus the maximum of $P(\sigma)$ is achieved at $\sigma = 0$ and we have

$$\frac{1}{\tau^3} \left( 1 - \frac{2}{\pi} \left( \tau + \frac{1}{\sigma} \arctan \frac{\sigma}{\tau} \right) \right) \leq \lim_{\sigma \rightarrow 0} \left( 1 - \frac{2}{\pi} \left( 1 + \frac{\arctan \sigma}{\sigma} \right) \right)$$

$$= 1 - \frac{4}{\pi} \leq -0.27.$$ 

Thus we have shown by combining the two estimates that $\frac{1}{\tau^3} g(z) < -0.01$. So $g(z) < -\delta_0$ for some $\delta_0 > 0$.

To show that $\langle \nabla F(z), x \rangle < -\epsilon$ with high probability for $m \geq Cn$ for all $\epsilon_0 \leq \|z\| \leq 1$ and $\sigma_0 \leq \sigma \leq 1 - \sigma_0$, set $U_i = a_i^\top z$, $V_i = a_i^\top x$. Then $\frac{1}{\|z\|} \langle \nabla f_i(z), x \rangle$ is a continuous function in $U_i$ and $\lambda = \beta/\|z\|$, where $\beta \leq \lambda \leq \beta/\sigma_0$. One can easily check that the conditions of Corollary 2.2 are satisfied by applying (12). The lemma now follows.

**Lemma 3.8.** Assume that $0 < \beta < 1$ and $\|x\| = 1$. For any $\epsilon_0 > 0$ there exist $C, \sigma_0, \epsilon > 0$ such that with high probability for $m \geq Cn$, we have $\langle \nabla F(z), x \rangle < -\epsilon$ for all $z$ such that $\epsilon_0 \leq \|z\| \leq 1 - \epsilon_0$ and $\sigma \geq 1 - \sigma_0$.

---

1In the appendix, we shall prove a stronger inequality for the monotonicity of the function $f_0(\tau) = \frac{1}{\tau^2} \left( 1 - \frac{2}{\pi} \left( \tau + \frac{1}{\sigma} \arctan \frac{\sigma}{\tau} \right) \right)$.
Proof. Again we start from \( \langle \nabla F(z), x \rangle = \frac{1}{m} \sum_{i=1}^{m} \Psi_u(a_i^\top z, a_i^\top x)(a_i^\top x) \). Let \( a \sim N(0, I_n) \) be standard Gaussian. Then \( \mathbb{E}[\langle \nabla F(z), x \rangle] = \mathbb{E}[\Psi_u(a^\top z, a^\top x)(a^\top x)] \). First we consider \( \sigma = 1 \), for which \( z = \|z\|x \) and \( a^\top z = \|z\|a^\top x \). Set \( \lambda = \beta/\|z\| \) as usual. One can easily check via (10) that

\[
\Psi_u(a^\top z, a^\top x)(a^\top x) = \begin{cases} 
(\|z\| - 1)(a^\top x)^2, & \|z\| > \beta; \\
\|z\|^3(a^\top x)^2 + (\frac{1}{2} - \frac{1}{\beta})\|z\|(a^\top x)^2, & \|z\| \leq \beta.
\end{cases}
\]

Thus \( \Psi_u(a^\top z, a^\top x)(a^\top x) \leq -\delta_0(a^\top x)^2 \) for some \( \delta_0 > 0 \). It follows that \( \mathbb{E}[\langle \nabla F(z), x \rangle] = \mathbb{E}[\Psi_u(a^\top z, a^\top x)(a^\top x)] \leq -\delta_0 \). Now by continuity \( \mathbb{E}[\langle \nabla F(z), x \rangle] \leq -\delta_1 \) for some \( \delta_0, \delta_1 > 0 \) for all \( \varepsilon_0 \leq \|z\| \leq 1 - \varepsilon_0 \) and \( \sigma \geq 1 - \sigma_0 \).

To show that \( \langle \nabla F(z), x \rangle < -\varepsilon \) with high probability for \( m \geq Cn \) for all \( \varepsilon_0 \leq \|z\| \leq 1 - \varepsilon_0 \) and \( \sigma \geq 1 - \sigma_0 \), observe that \( \Psi_u(a^\top z, a^\top x)(a^\top x) \) is a Lipschitz continuous function of \( z \) on this region. The conditions of Corollary 2.2 are met by applying (12). The lemma now follows.

3.4. Negative Directional Curvature and Strong Convexity. To cover the remaining region we evaluate the second order directional derivatives of the target function \( F(z) \). First, the Hessian of \( F \) is given by \( \nabla^2 F(z) = \frac{1}{m} \sum_{i=1}^{m} \nabla^2 z_i \Psi(a_i^\top z, a_i^\top x) \) for those points \( z \) at which \( F(z) \) has second derivative. For any \( v \in \mathbb{R}^n \) we have

\[
D_v^2 F(z) = \frac{1}{m} \sum_{i=1}^{m} D_v^2 \Psi(a_i^\top z, a_i^\top x) \text{ where } D_v^2 \text{ is with respect to the variable } z.
\]

It is easy to check that

\[
\nabla^2 \Psi(a_i^\top z, a_i^\top x) = a_i a_i^\top + \frac{3}{2\beta^2} \frac{|a_i^\top z|^2}{|a_i^\top x|^2} \cdot 1_{R_i} a_i a_i^\top - (\frac{1}{2} + \frac{1}{\beta}) \cdot 1_{R_i} a_i a_i^\top
\]

for all points at which the Hessian is well-defined, i.e. \( z \notin \partial R_i \) with \( R_i := \{|a_i^\top z| < \beta|a_i^\top x|\} \). Here, we use the notation \( \partial R \) to denote the boundary points of the set \( R \). Let

\[
\phi(t) := 1 + \frac{3}{2\beta^2} t^2 1_{\{|t| < \beta\}} - (\frac{1}{2} + \frac{1}{\beta}) \cdot 1_{\{|t| < \beta\}}.
\]

Then \( \nabla^2 \Psi(a_i^\top z, a_i^\top x) = \phi(\frac{a_i^\top z}{a_i^\top x}) a_i^\top a_i \) for \( z \notin \partial R_i \). On the other hand, one can check that the 2nd order directional derivative of \( \Psi(a_i^\top z, a_i^\top x) \) with respect to \( z \) is well-defined for \( z \in \partial R_i \), which is \( D_v^2 \Psi(a_i^\top z, a_i^\top x) = (a_i^\top v)^2 \) if \( (a_i^\top z)(a_i^\top v) > 0 \) and \( D_v^2 \Psi(a_i^\top z, a_i^\top x) = (a_i^\top v)^2 \) if \( (a_i^\top z)(a_i^\top v) = 0 \) and \( (a_i^\top v)^2 \) if \( (a_i^\top z)(a_i^\top v) < 0 \).
\[(2 - 1/\beta)(a_i^\top v)^2 \] if \((a_i^\top z)(a_i^\top v) \leq 0\). In summary, we have
\[
D_v^2 \Psi(a_i^\top z, a_i^\top x) = (a_i^\top v)^2 + \frac{3}{2\beta^2 |a_i| x^2} \cdot |a_i^\top v|^2 1_{R_i} - (1/2 + \frac{1}{\beta})(a_i^\top v)^2 1_{R_i} + \Gamma_i(z, v)
\]
(19)
\[
= \phi(\frac{a_i^\top z}{a_i^\top x}) (a_i^\top v)^2 + \Gamma_i(z, v)
\]
for all \(z\), with
\[
\Gamma_i(z, v) := (q_i - 1) (a_i^\top v)^2 1_{\{|a_i^\top z| = \beta |a_i^\top x|\}} \leq 0
\]
where \(q_i = 1\) if \((a_i^\top z)(a_i^\top v) > 0\) and \(q_i = 2 - 1/\beta\) if \((a_i^\top z)(a_i^\top v) \leq 0\).

**Lemma 3.9.** Assume that \(0 < \beta \leq 3/4\) and \(\|x\| = 1\). There exist \(C, \sigma_0, \delta_0, \varepsilon_0 > 0\) such that with high probability for \(m \geq Cn\) we have \(D_x^2 F(z) < -\varepsilon_0\) for all \(z\) such that \(\|z\| \leq 1 + \delta_0\) and \(\sigma \leq \sigma_0\), as well as all \(z \in B_{\delta_0}(0)\).

**Proof.** First we assume \(\sigma = 0\), and prove \(E[D_x^2 F(z)] < -5\varepsilon_0\) if \(\|z\| \leq 1 + \delta_0\) for some \(\varepsilon_0, \delta_0 > 0\). Let \(a \sim N(0, I_n)\) be standard Gaussian. Set
\[
G(z) := (a^\top x)^2 + \frac{3}{2\beta^2}(a^\top z)^2 R - (\frac{1}{2} + \frac{1}{\beta})(a^\top x)^2 1_R = \phi(\frac{a^\top z}{a^\top x}) (a^\top x)^2
\]
where \(R := \{|a^\top z| < \beta |a^\top x|\}\) and \(\phi(t)\) is given in (18). From (19) with \(v = x\) we have \(D_x^2 \Psi(a^\top z, a^\top x) \leq G(z)\). As before let \(\lambda = \beta/\|z\|, U = a^\top z/\|z\|\) and \(V = a^\top x\). We have
\[
E[G(z)] = 1 + \frac{3}{2\lambda^2} E[U^2 1_A] - (\frac{1}{2} + \frac{1}{\beta}) E[V^2 1_A] =: g(\lambda)
\]
where \(A = \{|U| \leq \lambda|V|\}\). Observe that by Corollary 3.3,
\[
g'(\lambda) = -\frac{3}{\lambda^3} E[U^2 1_A] + \frac{3}{\pi \tau} (\mu_+^{-4} + \mu_-^{-4}) - (\frac{1}{2} + \frac{1}{\beta}) \frac{2}{\pi \tau} (\mu_+^{-4} + \mu_-^{-4}) < 0.
\]
So \(g(\lambda)\) is decreasing with respect to \(\lambda\), which also means \(E[G(z)]\) is increasing with respect to \(\|z\|\). We show that \(E[G(z)] < -5\varepsilon_0\) for \(\|z\| \leq 1\), and it suffices to show this for \(\|z\| = 1\), i.e. \(g(\lambda) < -5\varepsilon_0\).

Since \(\sigma = 0\) (and hence \(\tau = 1\)) we have simple closed form solution for \(g(\lambda)\), which by Corollary 3.3 is
\[
g(\lambda) = 1 + \frac{3}{2\lambda^2} \cdot \frac{4}{\pi} \int_0^\lambda \frac{t^2}{(1 + t^2)^2} dt - (\frac{1}{2} + \frac{1}{\beta}) \frac{4}{\pi} \int_0^\lambda \frac{1}{(1 + t^2)^2} dt
\]
\[
= 1 + \frac{3}{\pi \lambda^2} (\arctan \lambda - \frac{\lambda}{1 + \lambda^2}) - \frac{\beta + 2}{\pi \beta} (\arctan \lambda + \frac{\lambda}{1 + \lambda^2}).
\]
Since \( \|z\| = 1 \), it follows that
\[
g(\lambda) = 1 - \frac{3 + \beta^2 + 2\beta}{\pi(1 + \beta^2)\beta} + \frac{3 - \beta^2 - 2\beta}{\pi\beta^2} \cdot \arctan \beta \quad := g_0(\beta).
\]
The graph of this function \( g_0(\beta) \) is shown in Figure 1; we check that \( g_0(\beta) < -0.03 \) for all \( \beta \in (0, \frac{3}{4}] \). Taking \( \varepsilon_0 = 0.006 \) we obtain the desired result. Since \( g(\lambda) \) is continuous, for a sufficiently small \( \delta_0 > 0 \) we have \( g(\lambda) < -4\varepsilon_0 \) for all \( \|z\| \leq 1 + \delta_0 \).

![Figure 1. \( g_0(\beta) \).](image)

To prove the lemma for the region \( \|z\| \leq 1 + \delta_0 \) and \( \sigma \leq \sigma_0 \), we need to use Lipschitz condition and Corollary 2.2. Our challenge is that the function \( \phi(t) \) is discontinuous with a jump discontinuity at \( |t| = \beta \). To get around this problem we smooth out the function by introducing
\[
H_p(u,v) := v^2 + \frac{3}{2\beta^2} u^2 1_S - \left( \frac{1}{2} + \frac{1}{\beta} \right) v^2 1_S + \left( \frac{1}{\beta} - 1 \right) \frac{|u|^p}{\beta^p |v|^{p-2}} 1_S
\]
where \( p > 0 \) and \( S = \{(u,v) : |u| < \beta |v|\} \). A key observation is \( H_p(u,v) \) is continuous and satisfies the Lipschitz condition with respect to \( u \) in Corollary 2.2 (here we need the property \( |u| < \beta |v| \) on \( S \)). Clearly \( H_p(a^\top z, a^\top x) \geq G(z) \). Note that
\[
\frac{|u|^p}{\beta^p |v|^{p-2}} 1_S = \left( \frac{|u|}{\beta |v|} \right)^p |v|^2 1_S \to 0 \quad \text{as} \quad p \to \infty.
\]
It means that \( \mathbb{E}[H_p(a^\top z, a^\top x)] \to \mathbb{E}[G(z)] \) as \( p \to \infty \). Hence for \( 0 \leq \|z\| \leq 1 + \delta_0 \), by taking \( p = p_0 \) sufficiently large we obtain \( \mathbb{E}[H_{p_0}(a^\top z, a^\top x)] < -3\varepsilon_0 \). Because \( \mathbb{E}[H_{p_0}(a^\top z, a^\top x)] \) is a continuous function of \( \sigma \), there exists a \( \sigma_0 > 0 \) such that \( \mathbb{E}[H_{p_0}(a^\top z, a^\top x)] < -2\varepsilon_0 \) for
0 < \|z\| \leq 1 + \delta_0 \text{ and } \sigma \leq \sigma_0. \text{ Finally,}
\begin{equation}
D_x^2 F(z) \leq \frac{1}{m} \sum_{i=1}^m H_{p_0}(a_i^\top z, a_i^\top x).
\end{equation}

It follows from Lemma 2.1 and Corollary 2.2 that with high probability for \(m \geq Cn\),
\begin{equation}
D_x^2 F(z) \leq -\varepsilon_0 \text{ for all } z \text{ with } 0 \leq \|z\| \leq 1 + \delta_0 \text{ and } \sigma \leq \sigma_0.
\end{equation}

To prove the lemma for \(z \in B_{\delta_0}(0)\) we use virtually identical technique. It is easily seen
\(E[H_p(0)] = E[G(0)] = \frac{1}{2} - \frac{1}{\beta} < 0\). By continuity \(E[H_p(z)] < -\varepsilon_1\) for \(z \in B_{\delta_0}(0)\) for some \(\delta_0, \varepsilon_1 > 0\). Now using Lemma 2.1 and Corollary 2.2 the same argument as in the previous
region apply to prove this case of the lemma.

\textbf{Lemma 3.10.} Assume that \(0 < \beta < 1\) and \(\|x\| = 1\). There exist \(C, \delta_0, \varepsilon_0 > 0\) such that
with high probability for \(m \geq Cn\) we have \(D_v^2 F(z) \geq 0.5\) for all \(z \in B_{\delta_0}(x)\) and unit vectors \(v \in \mathbb{R}^n\). In other words, \(F(z)\) is strongly convex in a neighborhood of the solution \(x\).

\textbf{Proof.} Let \(a \sim N(0, I_n)\) be standard Gaussian. We follow the same strategy of evaluating
an expectation and prove the lemma using Lipschitz condition and Corollary 2.2. Again,
due to the discontinuity of \(\phi(t)\) in (19) we will need to smooth it out. Define an auxiliary
function \(\chi_1(x)\) for \(x \geq 0\) as

\begin{equation}
\chi_1(x) := \begin{cases} 
1 & \text{if } 0 \leq x \leq \beta, \\
1 + \frac{\beta}{\delta_0} - \frac{1}{\delta_0}x & \text{if } \beta \leq x \leq \beta + \delta_0, \\
0 & \text{if } x > \beta + \delta_0,
\end{cases}
\end{equation}

where \(\delta_0 = \sqrt{\frac{2\beta + \beta^2}{3}} - \beta\). Furthermore, let \(\chi_2(x) \in C^\infty_c(\mathbb{R})\) be a function such that \(0 \leq \chi_2(x) \leq 1\) for all \(x\), \(\chi_2(x) = 1\) for \(|x| \leq 1\) and \(\chi_2(x) = 0\) for \(|x| \geq 2\). Define \(G(u, v, t)\) such that

\begin{equation}
G(a^\top z, a^\top x, a^\top v) := (a^\top v)^2 + \left[\frac{3}{2\beta^2} \frac{|a^\top z|^2}{|a^\top x|^2} \cdot |a^\top v|^2 - \left(\frac{1}{2} + \frac{1}{\beta}\right)(a^\top v)^2\right] \chi_1\left(\frac{|a^\top z|}{|a^\top x|}\right) \chi_2\left(\frac{|a^\top v|}{M_0|a^\top x|}\right),
\end{equation}

where \(M_0\) is a positive constant which will be made sufficiently large.

Note that \(\chi_1(x)\) is enlarged from the set \(R = \{|a^\top z| < \beta|a^\top x|\}\). Set \(z = x\), Then
\begin{equation}
G(a^\top x, a^\top x, a^\top v) = (a^\top v)^2 + \left(\frac{3}{2\beta^2} - \frac{1}{2} - \frac{1}{\beta}\right)(a^\top v)^2 \cdot \chi_2\left(\frac{|a^\top v|}{M_0|a^\top x|}\right) \geq (a^\top v)^2.
\end{equation}
Hence \( \mathbb{E}[G(a^\top z, a^\top x, a^\top v)] \geq 1 \). Since \( G(a^\top z, a^\top x, a^\top v) \) is continuous with respect to \( z \) and \( v \), and \( v \) is on the unit sphere which is compact, we know that

\[
\mathbb{E}[G(a^\top z, a^\top x, a^\top v)] \geq 0.7
\]

for \( z \) in a neighborhood \( z \in B_{\delta_0}(x) \) for all unit vectors \( v \). Next, we show \( G(a^\top z, a^\top x, a^\top v) \) satisfies the Lipschitz type condition with respect to \( z, v \). To this end, we only need to consider the second term of \( G(a^\top z, a^\top x, a^\top v) \). Note that

\[
L(a^\top z, a^\top x, a^\top v) := \frac{|a^\top z|^2}{|a^\top x|^2} \cdot |a^\top v|^2 \chi_1 \left( \frac{|a^\top z|}{|a^\top x|} \right) \chi_2 \left( \frac{|a^\top v|}{M_0 |a^\top x|} \right)
\]

\[
= \frac{|a^\top z|^2}{|a^\top x|^2} \chi_1 \left( \frac{|a^\top z|}{|a^\top x|} \right) \frac{|a^\top v|^2}{|a^\top x|^2} \chi_2 \left( \frac{|a^\top v|}{M_0 |a^\top x|} \right) \cdot |a^\top x|^2
\]

\[
= \psi_1 \left( \frac{|a^\top z|}{|a^\top x|} \right) \psi_2 \left( \frac{|a^\top v|}{|a^\top x|} \right) |a^\top x|^2.
\]

It is obvious that \( \psi_1 \) and \( \psi_2 \) are Lipschitz and bound functions. Observe that for any \( z_1 \) and \( z_2 \), we have

\[
\left| \psi_1 \left( \frac{|a^\top z_1|}{|a^\top x|} \right) - \psi_1 \left( \frac{|a^\top z_2|}{|a^\top x|} \right) \right| \lesssim \frac{|a^\top (z_1 - z_2)|}{|a^\top x|}.
\]

Thus,

\[
\left| L(a^\top z_1, a^\top x, a^\top v) - L(a^\top z_2, a^\top x, a^\top v) \right| \lesssim |a^\top (z_1 - z_2)||a^\top x|.
\]

By Corollary 2.2 and Corollary 2.3, we know the function \( G(a^\top z, a^\top x, a^\top v) \) satisfies the Lipschitz type condition with respect to \( z, v \).

It follows that with high probability for \( m \geq Cn \) we have

\[
\frac{1}{m} \sum_{i=1}^{m} G(a_i^\top z, a_i^\top x, a_i^\top v) \geq 0.5
\]

for all \( z \in B_{\delta_0}(x) \) and unit vectors \( v \). Choosing the constant \( M_0 \) sufficiently large and combining with (19) we see that \( G(a_i^\top z, a_i^\top x, a_i^\top v) \leq D_v^2 \Psi(a_i^\top z, a_i^\top x) \). Hence with high probability for \( m \geq Cn \) we have

\[
D_v^2 F(z) \geq \frac{1}{m} \sum_{i=1}^{m} G(a_i^\top z, a_i^\top x, a_i^\top v) \geq 0.5
\]

for all \( z \in B_{\delta_0}(x) \) and unit vectors \( v \). \( \Box \)
3.5. **Putting Things Together.**

**Proof of Theorem 1.1.** Without loss of generality we shall only examine the region \( \sigma \geq 0 \), i.e. \( \langle z, x \rangle \geq 0 \). The lemmas we have proved in this section have covered all regions to ensure that with high probability for \( m \geq Cn \): (i) In a small neighborhood of \( x \) the target function \( F(z) \) is strongly convex with \( z = x \) being a minimum. (ii) Everywhere else either the gradient of \( F(z) \) doesn’t vanish or \( F(z) \) has a negative directional curvature. Thus other than \( x \) the target function \( F(z) \) has no other local minimum. This proves the theorem.

4. **Numerical Experiments**

The SAF model proposed in this study shows theoretically that any gradient descent algorithm will not get trapped in a local minimum. Here we present numerical experiments to show that the model performs very well with random initial guess.

We use the following vanilla gradient descent algorithm

\[
 z_{k+1} = z_k - \mu \nabla F(z_k)
\]

with a random initial guess to minimize the loss function \( F(z) \) of SAF. The algorithm procedure is as follows:

**Algorithm 1** Gradient Descent Algorithm Based on Smoothed Amplitude Flow (SAF)

**Input:** Measurement vectors: \( a_i \in \mathbb{R}^n, i = 1, \ldots, m \); Observations: \( y \in \mathbb{R}^m \); Parameters \( \beta \); Step size \( \mu \); Tolerance \( \epsilon > 0 \)

1: Random initial guess \( z_0 \in \mathbb{R}^n \).
2: For \( k = 0, 1, 2, \ldots \), if \( \| \nabla F(z_k) \| \geq \epsilon \) do
   \[
   z_{k+1} = z_k - \mu \nabla F(z_k)
   \]
3: End do

**Output:** The vector \( z_T \).

The performance of our SAF algorithm is conducted via a series of numerical experiments in comparison against WF [3], TWF [5] and TAF [30]. Here, it is worth emphasizing that random initialization is used for our SAF algorithm while all other algorithms have adopted a spectral initialization. Our theoretical results are for real Gaussian case, but the algorithms can be easily adapted to the complex Gaussian case. In our numerical
experiments, the target vector $x \in \mathbb{R}^n$ is chosen randomly from the standard Gaussian distribution and the measurement vectors $a_i$, $i = 1, \ldots, m$ are also generated randomly from standard Gaussian distribution. For the real Gaussian case, the signal $x \sim \mathcal{N}(0, I_n)$ and measurement vectors $a_i \sim \mathcal{N}(0, I_n)$ for $i = 1, \ldots, m$. For the complex Gaussian case, the signal $x \sim \mathcal{N}(0, I_n) + i\mathcal{N}(0, I_n)$ and measurement vectors $a_i \sim \mathcal{N}(0, I_n/2) + i\mathcal{N}(0, I_n/2)$.

For WF, TWF and TAF, we use the code provided in the original papers with suggested parameters.

**Example 4.1.** In this example, we test the empirical success rate of SAF versus the number of measurements with parameter $\beta = 1/2$. We conduct the experiments for the real and complex Gaussian cases respectively. We choose $n = 128$. The step size $\mu = 0.6$ and the maximum number of iterations is $T = 2000$. For the number of measurements, we vary $m$ within the range $[n, 8n]$. For each $m$, we run 100 times trials to calculate the success rate. Here, we say a trial to have successfully reconstructed the target signal if the relative error satisfies $\text{dist}(z_T - x)/\|x\| \leq 10^{-5}$. The results are plotted in Figure 2. It can be seen that $4.5n$ real Gaussian phaseless measurement or $5.5n$ complex Gaussian phaseless measurement are enough for exactly recovery for SAF.

![Figure 2](image_url)

**Figure 2.** The empirical success rate for different $m/n$ based on 100 random trails. (a) Success rate for real Gaussian case, (b) Success rate for complex Gaussian case.

**Example 4.2.** In this example, we compare the convergence rate of SAF with those of WF, TWF, TAF for real Gaussian and complex Gaussian cases. We choose $n = 128$ and $m = 5n$. 


The step size $\mu = 0.8$ and parameter $\beta = 1/2$. To show the robustness of our SAF, we also consider the noisy data model $y_i = |\langle a_i, x \rangle| + \eta_i$ where the noise $\eta_i \sim 0.01 \cdot N(0, 1)$. The results are presented in Figure 3. Since our SAF algorithm chooses a random initial guess according to the standard Gaussian distribution instead of adopting a spectral initialization, it sometimes need to escape the saddle points with a small number of iterations. Due to its high efficiency to escape the saddle points, it still performs well comparing with state-of-the-art algorithms with spectral initialization.

![Graphs](image)

**Figure 3.** Relative error versus number of iterations for SAF, WF, TWF, and TAF methods: (a) The noiseless measurements for real Gaussian case; (b) The noiseless measurements for complex Gaussian case; (c) The noisy measurements for real Gaussian case; (d) The noisy measurements for complex Gaussian case.
Table 1. Time Elapsed and Iteration Number among Algorithms on Gaussian Signals with $n = 1000$.

| Algorithm   | Real Gaussian |          |          | Complex Gaussian |          |          |
|-------------|---------------|----------|----------|------------------|----------|----------|
|             | $10^{-5}$     | $10^{-10}$ |          | $10^{-5}$        | $10^{-10}$ |          |
| Iter        | Time(s)       | Iter     | Time(s)  | Iter             | Time(s)  | Iter     | Time(s)   |
| SAF         | 44            | 0.1556   | 68       | 0.2276           | 113      | 1.3092   | 190       | 2.3596    |
| SAF (spectral) | 25            | 0.2631   | 51       | 0.3309           | 67       | 1.4528   | 151       | 2.6122    |
| WF          | 125           | 4.4214   | 229      | 6.3176           | 304      | 34.6266  | 655       | 86.6993   |
| TAF         | 29            | 0.3181   | 87       | 0.4274           | 112      | 1.9808   | 244       | 3.7432    |
| TWF         | 40            | 0.3181   | 87       | 0.4274           | 112      | 1.9808   | 244       | 3.7432    |

**Example 4.3.** In this example, we compare the time elapsed and the iteration needed for WF, TWF, TAF and our SAF to achieve the relative error $10^{-5}$ and $10^{-10}$, respectively. We choose $n = 1000$ with $m = 8n$. The step size $\mu = 0.8$. For the parameter $\beta$ in our SAF, we consider the case $\beta = 1/2$. We adopt the same spectral initialization method for WF, TWF, TAF and the initial guess is obtained by power method with 50 iterations. We run 50 times trials to calculate the average time elapsed and iteration number for those algorithms. The results are shown in Table 1. The numerical results show that SAF takes around 20 and 40 iterations to escape the saddle points for the real and complex Gaussian cases, respectively. Since there is no spectral initialization, the high efficiency of escaping saddle points and low computational complexity, the time elapsed of SAF is less than the other methods significantly.

**Example 4.4.** In this example, we show the performance of SAF with different parameter $\beta$ in the real Gaussian case. We choose $n = 128$ and the step size $\mu = 0.6$. For SAF with random initialization, we choose $m = 4n$; and for SAF with spectral initialization, we choose $m = 2.5n$. We test the parameter $\beta$ within the range from 0.1 to 1.0. For each $\beta$, we run 100 times trials and calculate the success rate, where a trial is successful if the relative error is less than $10^{-5}$. The results are depicted in Figure 4. From the figures, we can see our SAF has better performance as the parameter $\beta$ increasing.
5. Appendix

Proof of Corollary 2.2: To simplify the presentation, we shall use the phrase “with high probability” to mean that the probability is at least $1 - C \exp(-cm)$ for some constants $c, \delta > 0$. Also we shall tacitly assume that $m \geq C_1 n$ for some sufficiently large constant $C_1$ throughout this proof.

For any fixed $z_0 \in \Omega$, the terms $g(a_i^\top z_0, a_i^\top x)$ are subexponential with subexponential norm $\tau$. By Lemma 2.1, it holds with high probability that

\[
\left| \frac{1}{m} \sum_{i=1}^{m} g(a_i^\top z_0, a_i^\top x) - \mathbb{E}[g(a_1^\top z_0, a_1^\top x)] \right| \leq \frac{\varepsilon}{3}.
\]

To obtain this for all $z \in \Omega$ we need a (standard) covering argument. Since $\Omega$ is compact, we can construct a $\delta_0$-net $\mathcal{N}$ with $\text{Card}(\mathcal{N}) \leq \exp(\alpha_1 n)$ ($\alpha_1$ depends on $\delta_0$) such that

$$
\mathcal{N} \subset \Omega \subset \bigcup_{\tilde{z} \in \mathcal{N}} B(\tilde{z}, \delta_0).
$$

Here $\delta_0 > 0$ is a constant which will be taken sufficiently small. The needed smallness will be specified later. By using the above construction, for any $z \in \Omega$, we can find a vector $z_0 \in \mathcal{N}$ such that $\|z - z_0\| \leq \delta_0$. Since $K(a_i)$ are subgaussian random variables with
subgaussian norm $\eta$, Lemma 2.1 implies that with high probability it holds that
\[
\frac{1}{m} \sum_{i=1}^{m} K(a_i)^2 \leq \mathbb{E}K(a_1)^2 + \varepsilon \leq \eta^2 + \delta_1,
\]
where $\delta_1 > 0$ will be taken sufficiently small. On the other hand, we have
\[
\left| \frac{1}{m} \sum_{i=1}^{m} g(a_i^\top z, a_i^\top x) - \frac{1}{m} \sum_{i=1}^{m} g(a_i^\top z_0, a_i^\top x) \right|
\leq \frac{1}{m} \sum_{i=1}^{m} K(a_i)|a_i^\top (z - z_0)|
\leq \sqrt{\frac{1}{m} \sum_{i=1}^{m} K(a_i)^2 \cdot \frac{1}{m} \sum_{i=1}^{m} |a_i^\top (z - z_0)|^2}
\leq \sqrt{\eta^2 + \delta_1 \cdot 2\delta_0},
\]
where in the last inequality we used the fact that with high probability:
\[
\frac{1}{m} \sum_{i=1}^{m} |a_i^\top \tilde{z}|^2 \leq 1.01, \quad \forall \tilde{z} \in S^{n-1}.
\]

Furthermore,
\[
\|Eg(a_1^\top z, a_1^\top x) - Eg(a_1^\top z_0, a_1^\top x)\| \leq E\left[K(a_1)|a_1^\top (z - z_0)|\right] \leq \eta \delta_0.
\]
Choose $\delta_0 = \frac{\sqrt{\varepsilon^2}}{2w_0}$ and $\delta_1 = \eta^2$. Taking the union bound together with (20), (21) and (22), we obtain the following: with high probability it holds that
\[
\left| \frac{1}{m} \sum_{i=1}^{m} g(a_i^\top z, a_i^\top x) - E[g(a_1^\top z, a_1^\top x)] \right|
\leq \left| \frac{1}{m} \sum_{i=1}^{m} g(a_i^\top z, a_i^\top x) - \frac{1}{m} \sum_{i=1}^{m} g(a_i^\top z_0, a_i^\top x) \right| + \left| \frac{1}{m} \sum_{i=1}^{m} g(a_i^\top z_0, a_i^\top x) - E[g(a_1^\top z_0, a_1^\top x)] \right|
+ \left| E[g(a_1^\top z, a_1^\top x)] - E[g(a_1^\top z_0, a_1^\top x)] \right|
\leq 2\delta_0 \sqrt{\eta^2 + \delta_1} + \frac{\varepsilon}{3} + \eta \delta_0
\leq \varepsilon
\]
for all $z \in \Omega$. We remind the reader that we need to take $m \geq C_1 n$ with $C_1$ sufficiently large to damp the pre-factor $\alpha_7^2$ in the covering argument.

**Proof of Corollary 2.3:** Since $\chi(t)$ is a Lipschitz and compact support function with $\text{supp}(\chi) \subset [0, 1]$, it suffices to consider the case where $|a_i^\top z_1| \leq |a_i^\top x|$ and $|a_i^\top z_2| \leq |a_i^\top x|$.
for all $i$. Thus we have
\[
\left| \frac{1}{m} \sum_{i=1}^{m} \left| \frac{a_i^\top z_1}{a_i^\top x} \right| - \frac{1}{m} \sum_{i=1}^{m} \left| \frac{a_i^\top z_2}{a_i^\top x} \right| \right| \\
\leq \frac{1}{m} \sum_{i=1}^{m} \left| \chi \left( \left| \frac{a_i^\top z_1}{a_i^\top x} \right| \right) - \chi \left( \left| \frac{a_i^\top z_2}{a_i^\top x} \right| \right) \right| + \frac{1}{m} \sum_{i=1}^{m} \left| \frac{a_i^\top z_1}{a_i^\top x} \right| - \frac{1}{m} \sum_{i=1}^{m} \left| \frac{a_i^\top z_2}{a_i^\top x} \right| \\
\leq L + \frac{1}{m} \sum_{i=1}^{m} \left| \frac{a_i^\top (z_1 - z_2)}{a_i^\top x} \right| \\
= \frac{L + 1}{m} \sum_{i=1}^{m} \left| \frac{a_i^\top (z_1 - z_2)}{a_i^\top x} \right| I[|a_i^\top x| \geq \delta] + \frac{L + 1}{m} \sum_{i=1}^{m} \left| \frac{a_i^\top (z_1 - z_2)}{a_i^\top x} \right| I[|a_i^\top x| < \delta] \\
\leq \frac{L + 1}{\delta} \cdot \frac{1}{m} \sum_{i=1}^{m} \left| \frac{a_i^\top (z_1 - z_2)}{a_i^\top x} \right| + 2(L + 1) \cdot \frac{1}{m} \sum_{i=1}^{m} I[|a_i^\top x| < \delta].
\]

For the first term, if $m \geq C \epsilon^{-2} n$ then with probability at least $1 - 2 \exp(-c \epsilon^2 m)$ it holds that
\[
\frac{1}{m} \sum_{i=1}^{m} \left| \frac{a_i^\top (z_1 - z_2)}{a_i^\top x} \right| \leq \sqrt{\frac{1}{m} \sum_{i=1}^{m} \left| \frac{a_i^\top (z_1 - z_2)}{a_i^\top x} \right|^2} \leq \|z_1 - z_2\| + \epsilon.
\]

For the second term, note that
\[
\mathbb{E} \left[ I[|a_i^\top x| < \delta] \right] = \frac{1}{\sqrt{2\pi}} \int_{-\delta}^{\delta} e^{-u^2/2} du \leq \frac{2\delta}{\sqrt{2\pi}}.
\]

Since $I[|a_i^\top x| < \delta]$ are bounded random variables, applying the Hoeffding’s inequality gives
\[
\frac{1}{m} \sum_{i=1}^{m} I[|a_i^\top x| < \delta] \leq \frac{2\delta}{\sqrt{2\pi}} + \epsilon
\]

with probability at least $1 - 2 \exp(-c \epsilon^2 m)$.

Combining the two estimations gives the conclusion.

Proof of Proposition 3.2: Observe that $U = \sigma V + \tau W$. Then the condition $|U| \leq \lambda |V|$ is equivalent to
\[
A = \left\{ \tau^{-1}(-\lambda - \sigma \text{sgn}(V))|V| \leq W \leq \tau^{-1}(\lambda - \sigma \text{sgn}(V))|V| \right\}.
\]
From the definition, we have

\[
G(\lambda) = \mathbb{E} \left[ g(U, V) \mathbb{I}_A \right] = \frac{1}{2\pi} \int_{|v + \tau w| \leq \lambda |v|} g(\sigma v + \tau w, v) e^{-\frac{1}{2}(v^2 + w^2)} dw dv
\]

\[
= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^{\infty} \frac{v}{\sqrt{2\pi}} g(\sigma v + \tau w, v) e^{-\frac{1}{2}(v^2 + w^2)} dw dv
\]

\[
+ \frac{1}{2\pi} \int_{-\infty}^0 \int_{\infty}^{-\infty} \frac{v}{\sqrt{2\pi}} g(\sigma v + \tau w, v) e^{-\frac{1}{2}(v^2 + w^2)} dw dv.
\]

It gives that

\[
G'(\lambda) = \frac{1}{2\pi \tau} \int_0^\infty g(\lambda v, v) v e^{-\frac{1}{2} \mu^2 v^2} dv + \frac{1}{2\pi \tau} \int_0^\infty g(-\lambda v, v) v e^{-\frac{1}{2} \mu^2 v^2} dv
\]

\[
- \frac{1}{2\pi \tau} \int_{-\infty}^0 g(-\lambda v, v) v e^{-\frac{1}{2} \mu^2 v^2} dv - \frac{1}{2\pi \tau} \int_{-\infty}^0 g(\lambda v, v) v e^{-\frac{1}{2} \mu^2 v^2} dv
\]

\[
= \frac{1}{2\pi \tau} \int_0^\infty (g(-\lambda v, v) + g(\lambda v, -v)) v e^{-\frac{1}{2} \mu^2 v^2} dv
\]

\[
+ \frac{1}{2\pi \tau} \int_{0}^\infty (g(\lambda v, v) + g(-\lambda v, -v)) v e^{-\frac{1}{2} \mu^2 v^2} dv.
\]

\[
\]

**Proof of Lemma 3.5:** For the expectation \(\mathbb{E}[|UV|]\), let \(\sigma = \cos \alpha\) for some \(\alpha \in [0, 2\pi]\). Then we have

\[
\mathbb{E}[|UV|] = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty |v(\sigma v + \tau w)| \cdot e^{-\frac{1}{2}(v^2 + w^2)} dw dv
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \rho^3 \sin \theta \cdot |\sigma \sin \theta + \tau \cos \theta| \cdot e^{-\frac{1}{2}\rho^2} d\rho d\theta
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} |\sin \theta \sin (\theta + \alpha)| d\theta
\]

\[
= \frac{2}{\pi} \left( \sin \alpha + \left( \frac{\pi}{2} - \alpha \right) \cos \alpha \right)
\]

\[
= \frac{2}{\pi} \left( \tau + \sigma \arctan \frac{\sigma}{\tau} \right).
\]
Finally,

\[
\mathbb{E}[\text{sgn}(UV)V^2)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{sgn}(\sigma v^2 + \tau vw) \cdot v^2 \cdot e^{-\frac{1}{2}(v^2+w^2)} \, dw \, dv
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} \text{sgn}(\sin \theta \sin(\theta + \alpha)) \sin^2 \theta \, d\theta
\]

\[
= \frac{2}{\pi} \int_0^{\pi-\alpha} \sin^2 \theta \, d\theta - \frac{2}{\pi} \int_{\pi-\alpha}^{\pi} \sin^2 \theta \, d\theta
\]

\[
= \frac{2}{\pi} \left( \sin(2\alpha) + \frac{\pi}{2} - \alpha \right)
\]

\[
= \frac{2}{\pi} \left( \tau \sigma + \arctan \frac{\sigma}{\tau} \right).
\]

\[\blacksquare\]

Proof of bounds and Lipschitz property of \(\Psi_u\): The bounds and Lipschitz property of \(\Psi_u\) given in (11) and (12) are easy to check and we only prove the bound

(23) \[\Psi_u(u, v)u \geq u^2 - |uv|,\]

Recall the expression (10) of \(\Psi_u(u, v)\). We have

\[
\frac{\Psi_u(u, v)u}{v^2} = \left\{ \begin{array}{ll}
\left( \frac{u}{v} \right)^2 - |u|, & |u| > \beta|v|; \\
\frac{1}{2\beta^3} \cdot \left( \frac{u}{v} \right)^4 + \left( \frac{1}{2} - \frac{1}{\beta} \right) \left( \frac{u}{v} \right)^2, & |u| \leq \beta|v|.
\end{array} \right.
\]

Thus, to prove (23) it suffices to show

\[
\frac{1}{2\beta^2} t^4 + \left( \frac{1}{2} - \frac{1}{\beta} \right) t^2 \geq t^2 - |t|
\]

for all \(|t| \leq \beta\). We only need to consider \(0 \leq t \leq \beta\). Let

\[
h(t) = t^3 - (\beta^2 + 2\beta)t + 2\beta^2.
\]

Then \(h(t)\) is a decreasing function for \(0 \leq t \leq \beta\). Note that \(h(\beta) = 0\). It gives \(h(t) \geq 0\) for all \(0 \leq t \leq \beta\). We arrive at the conclusion. \[\blacksquare\]

Proof of a more general inequality

Denote \(\tau = \sqrt{1 - \sigma^2}\) and \(0 \leq \sigma \leq 1\). We shall show that

\[
f_0(\tau) = \frac{1}{\tau^2} \left( \frac{\pi}{2} - \tau - \frac{1}{\sigma} \arctan(\sigma/\tau) \right)
\]

is monotonically increasing for \(0 < \tau < 1\).
First transformation. Write
\[ f_0(\tau) = \tau^{-2} \left( \frac{\pi}{2} - \tau - \int_{\tau}^{\infty} \frac{1}{s^2 + 1 - \tau^2} ds \right) = \tau^{-2} \left( \int_{0}^{\infty} \frac{1}{s^2 + 1} - \frac{1}{s^2 + 1 - \tau^2} ds + \int_{0}^{\tau} \frac{1}{s^2 + 1 - \tau^2} ds \right) \]
\[ = - \int_{0}^{\infty} \frac{1}{(1 + s^2)(1 + s^2 - \tau^2)} ds + \tau \int_{0}^{1} \frac{1 - s^2}{1 - (1 - s^2)\tau^2} ds. \]
Denote \( t = \tau^2 \) and
\[ \tilde{f}_0(t) = - \int_{0}^{\infty} \frac{1}{(1 + s^2)(1 + s^2 - t)} ds + \sqrt{t} \int_{0}^{1} \frac{1 - s^2}{1 - (1 - s^2)t} ds. \]
Note that
\[ f_0(\tau) = \tilde{f}_0(\tau^2), \]
\[ f'_0(\tau) = 2\tau \tilde{f}'_0(\tau^2). \]
We have
\[ \tilde{f}'_0(t) = - \int_{0}^{\infty} \frac{1}{(1 + s^2)(1 + s^2 - t)} ds + \sqrt{t} \int_{0}^{1} \frac{(1 - s^2)^2}{(1 - (1 - s^2)t)^2} ds + \frac{1}{2} t^{-\frac{1}{2}} \int_{0}^{1} \frac{1 - s^2}{1 - (1 - s^2)t} ds. \]
Second transformation. Our second transformation is to set \( \tau = \cos \theta, \sigma = \sin \theta \). Then
\[ -f_0(\cos \theta) = f_1(\theta) = \frac{\theta}{\sin \theta} + \cos \theta - \frac{\pi}{2} \theta \left( \text{for Remark 5.1} \right) \]
is increasing on \([0, \frac{\pi}{2}]\). Note that \( \frac{\theta}{\sin \theta} \) is monotonically increasing.

Remark 5.1. As another variant, one can consider
\[ f(t, \theta) = \frac{t(1 - \frac{\pi \sin \theta}{2}) + \cos \theta \sin \theta}{\cos^2 \theta}. \]
For \( 0 \leq t \leq 1 \), \( f(t, \theta) \) is an increasing function of \( \theta \). For \( t > 1 \), this does not hold. Note that \( \frac{\theta}{\sin \theta} \) is monotonically increasing on \([0, \frac{\pi}{2}]\). The monotonicity of \( f(1, \theta) \) would yield the monotonicity of \( f_1 \).

Since
\[ f_1(\theta) = \frac{\theta}{\sin \theta} + \cos \theta - \frac{\pi}{2} \theta = -f_0(\cos \theta), \]
we have
\[ f'_1(\theta) = \sin \theta f'_0(\cos \theta). \]

The derivative of \( f_1(x) \) is
\[ f'_1(x) = \sec x (-x \csc^2 x + 2x \sec^2 x + \csc x \sec x + \tan x - \pi \sec x \tan x). \]

The task is to show \( f'_1(x) \geq 0 \) for \( x \in [0, \frac{\pi}{2}] \), i.e.
\[ \cos x (2 - 3 \cos^2 x + \sin x \cos^2 x - \pi \sin x \cos x) \geq 0. \]

Denote \( \sqrt{t} = \cos x \) so that \( \sin x = \sqrt{1 - t} \). Then we only need to show for all \( t \in [0, 1] \):
\[ t^{-\frac{3}{2}} (1 - t)^{-\frac{1}{2}} (2 - 3t) \arcsin(\sqrt{1 - t}) + (2 - t) \sqrt{t} - \pi(1 - t) \geq 0. \]

Note that
\[ f'_1 \bigg|_{\cos \theta = \sqrt{t}} = \sqrt{1 - t} f'_0(\sqrt{t}) = \sqrt{1 - t} \cdot 2 \sqrt{t} \tilde{j}_0(t). \]

Our main idea is to use \( f'_1 \) and \( \tilde{j}_0 \) in different regimes.

Case 1: \( \frac{2}{3} \leq t \leq 1 \). Note that \( \frac{\arcsin x}{x} \geq 1 \) for \( x \in [0, 1] \). Easy to check that in this regime
\[ 2 - 3t + (2 - t) \sqrt{t} - \pi(1 - t) \geq 0. \]

The above is equivalent to checking for \( \frac{2}{3} \leq t \leq 1 \):
\[ 2 + \sqrt{t} + \frac{\sqrt{t}}{1 + \sqrt{t}} - \pi \geq 0. \]

which is obvious thanks to monotonicity.

Case 2: \( 0 < t \leq \frac{1}{4} \). In this case we work with \( \tilde{j}_0(t) \). Note that for \( 0 < t \leq \frac{1}{4} \),
\[ \tilde{j}_0(t) > - \int_0^\infty \frac{1}{(1 + s^2)(1 + s^2 - \frac{1}{4})^2} ds + \sqrt{t} \int_0^1 (1 - s^2)^2 (1 + (1 - s^2)t) ds + \frac{1}{2} \int_0^1 (1 - s^2)(1 + (1 - s^2)t) ds. \]

We have
\[ \int_0^\infty \frac{1}{(1 + s^2)(1 + s^2 - \frac{1}{4})^2} ds = \frac{9}{16} (8 - 3\sqrt{6}) \pi \approx 0.94875. \]

Obviously then
\[ \tilde{j}_0(t) > -0.94876 t^\frac{3}{2} + \frac{1}{3} + \frac{4}{5} t + \frac{32}{35} t^2 + \frac{128}{315} t^3 \]
\[ > -0.94876 t^\frac{3}{2} + \frac{1}{3} + \frac{4}{5} t > 0, \quad \forall 0 < t \leq \frac{1}{4}. \]
Case 2a (slightly better): $0 < t \leq \frac{1}{3}$. In this case we still work with $\tilde{f}'_0(t)$. Note that for $0 < t \leq \frac{1}{3}$,
\[
\tilde{f}'_0(t) > - \int_0^\infty \frac{1}{(1 + s^2)(1 + s^2 - \frac{t}{3})^2} ds + \sqrt{t} \int_0^1 (1 - s^2)^2 (1 + (1 - s^2)t)^2 ds \\
+ \frac{1}{2} t^{-\frac{1}{2}} \int_0^1 (1 - s^2)(1 + (1 - s^2)t + ((1 - s^2)t)^2) ds.
\]
We have
\[
\int_0^\infty \frac{1}{(1 + s^2)(1 + s^2 - \frac{t}{3})^2} ds = \frac{8}{9} (9 - 5\sqrt{3})\pi \approx 1.15135.
\]
Obviously then
\[
\tilde{f}'_0(t) > -1.15136 t^{\frac{1}{2}} + \frac{1}{3} + \frac{4}{5} t + \frac{8}{7} t^2 \\
> -1.15136 t^{\frac{1}{2}} + \frac{1}{3} + \frac{4}{5} t + \frac{8}{7} t^2 > 0, \quad \forall 0 < t \leq \frac{1}{3}.
\]
We note that one can deal with $t^{\frac{1}{2}}$ for $\frac{1}{4} \leq t \leq \frac{1}{3}$ in the following way:
\[
t^{\frac{1}{2}} > \frac{1}{2} + (t - \frac{1}{4}) - (t - \frac{1}{4})^2, \quad \forall \frac{1}{4} \leq t \leq \frac{1}{3}.
\]
Then
\[
\tilde{f}'_0(t) > -1.15136\left(\frac{1}{2} + (t - \frac{1}{4}) - (t - \frac{1}{4})^2\right) + \frac{1}{3} + \frac{4}{5} t + t^2 \\
> 0.117 - 0.93 t + 2 t^2 > 0, \quad \forall \frac{1}{4} \leq t \leq \frac{1}{3}.
\]

Case 3: $\frac{1}{3} < t < \frac{2}{3}$. Now we shall work with the expression
\[
(2 - 3t)\frac{\arcsin(\sqrt{1 - t})}{\sqrt{1 - t}} + (2 - t)\sqrt{t} - \pi(1 - t) \geq 0.
\]
Make a change of variable $t \rightarrow 1 - s$ and note that the regime is invariant. We then need to show for $\frac{1}{3} < s < \frac{2}{3}$,
\[
h(s) = (3s - 1)\frac{\arcsin(\sqrt{s})}{\sqrt{s}} + (1 + s)\sqrt{1 - s} - \pi s \geq 0.
\]
It is easy to check that (note that arcsin has positive-coefficient power series expansion!)
\[
\frac{\arcsin(\sqrt{s})}{\sqrt{s}} \geq 1 + s + \frac{3}{40}s^2.
\]
Thus we need to show for $\frac{1}{3} \leq s \leq \frac{2}{3}$:
\[
h(s) = (3s - 1)(1 + \frac{s}{6} + \frac{3}{40}s^2) + (1 + s)\sqrt{1 - s} - \pi s \geq 0.
\]
To this end we rewrite

$$h(s) = (3s - 1)(1 + \frac{s}{6} + \frac{3}{40}s^2) - \pi s + (1 + s)(\sqrt{1 - s} - 1)$$

$$= \frac{1}{120}s(460 - 120\pi + 51s + 27s^2) - \frac{s(1 + s)}{1 + \sqrt{1 - s}}$$

$$= s(A(s) - \frac{1 + s}{1 + \sqrt{1 - s}}),$$

where $A(s) = \frac{1}{120}(460 - 120\pi + 51s + 27s^2)$. It is not difficult to check that $0 < A(s) < 1 + s$ (this holds for $0 \leq s \leq 1$). Then to show

$$\sqrt{1 - s} > \frac{1 + s}{A(s) - 1},$$

can square on both sides and then multiply both sides by $A(s)^2$. We then need to check the inequality

$$g_1(s) = A(s)^2(1 - s) - (1 + s - A(s))^2 > 0.$$

After a tedious computation, we obtain

$$g_1(s) = \frac{-81s^5}{1600} - \frac{153s^4}{800} - \frac{2329s^3}{1600} + \frac{9\pi s^3}{20} - \frac{71s^2}{24} + \frac{17\pi s^2}{20} - \pi s - \frac{368s}{45} + \frac{17\pi s}{3} + \frac{20}{3} - 2\pi$$

$$\approx -0.050625s^5 - 0.19125s^4 - 0.0419083s^3 - 0.28798s^2 - 0.245024s + 0.383481.$$

Clearly $g_1$ is monotonically decreasing and it suffices for us to show $g_1(\frac{2}{3}) > 0$. Indeed

$$g_1(\frac{2}{3}) \approx 0.035 > 0.$$

REFERENCES

[1] S. Bhojanapalli, N. Behnam, and N. Srebro, “Global optimality of local search for low rank matrix recovery,” Advances in Neural Information Processing Systems, pp. 3873–3881, 2016.

[2] E. J. Candès and X. Li, “Solving quadratic equations via PhaseLift when there are about as many equations as unknowns,” Found. Comput. Math., vol. 14, no. 5, pp. 1017–1026, 2014.

[3] E. J. Candès, X. Li, and M. Soltanolkotabi, “Phase retrieval via Wirtinger flow: Theory and algorithms,” IEEE Trans. Inf. Theory, vol. 61, no. 4, pp. 1985–2007, 2015.

[4] E. J. Candès, T. Strohmer, and V. Voroninski, “Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming,” Commun. Pure Appl. Math., vol. 66, no. 8, pp. 1241–1274, 2013.

[5] J. R. Fienup, “Phase retrieval and image reconstruction for astronomy,” Image Recovery: Theory and Application, vol. 231, pp. 275, 1987.

[6] J.C. Dainty and J.R. Fienup, “Phase retrieval algorithms: a comparison,” Appl. Opt., vol. 21, no. 15, pp. 2758–2769, 1982.
[9] B. Gao, X. Sun, Y. Wang, and Z. Xu, “Perturbed Amplitude Flow for Phase Retrieval,” IEEE Trans. Signal Process., vol. 68, pp. 5427–5440, 2020.

[10] B. Gao and Z. Xu, “Phaseless recovery using the Gauss–Newton method,” IEEE Trans. Signal Process., vol. 65, no. 22, pp. 5885–5896, 2017.

[11] R. Ge, F. Huang, C. Jin, and Y. Yuan, “Escaping from saddle points—online stochastic gradient for tensor decomposition,” Conference on Learning Theory, pp. 797–842, 2015.

[12] R. Ge, J. Lee, C. Jin, and T. Ma, “Matrix completion has no spurious local minimum,” Advances in Neural Information Processing Systems, pp. 2973–2981, 2016.

[13] R. W. Gerchberg, “A practical algorithm for the determination of phase from image and diffraction plane pictures,” Optik, vol. 35, pp. 237–246, 1972.

[14] R. W. Gerchberg and W. O. Saxton, “A practical algorithm for the determination of the phase from image and diffraction plane pictures,” Optik, vol. 35, pp. 237–246, 1972.

[15] R. W. Harrison, “Phase problem in crystallography,” JOSA A, vol. 10, no. 5, pp. 1046–1055, 1993.

[16] C. Jin, R. Ge, P. Netrapalli, S. M. Kakade, and M. I. Jordan, “How to escape saddle points efficiently,” Proceedings of the 34th International Conference on Machine Learning—Volume 70, pp. 1724–1732, 2017.

[17] C. Jin, P. Netrapalli, and M. I. Jordan, Accelerated gradient descent escapes saddle points faster than gradient descent, 2017 [Online]. Available: http://arxiv.org/abs/1711.10456

[18] Z. Li, J. F. Cai, and K. Wei, “Towards the optimal construction of a loss function without spurious local minima for solving quadratic equations,” IEEE Trans. Inf. Theory, vol. 66, no. 5, pp. 3242–3260, 2020.

[19] J. Miao, T. Ishikawa, Q. Shen, and T. Earnest, “Extending x-ray crystallography to allow the imaging of noncrystalline materials, cells, and single protein complexes,” Annu. Rev. Phys. Chem., vol. 59, pp. 387–410, 2008.

[20] R. P. Millane, “Phase retrieval in crystallography and optics,” J. Optical Soc. America A, vol. 7, no. 3, pp. 394–411, 1990.

[21] P. Netrapalli, P. Jain, and S. Sanghavi, “Phase retrieval using alternating minimization,” IEEE Trans. Signal Process., vol. 63, no. 18, pp. 4814–4826, 2015.

[22] D. Park, A. Kyrillidis, and C. Caramanis, Non-square matrix sensing without spurious local minima via the Burer-Monteiro approach, 2016 [Online]. Available: http://arxiv.org/abs/1609.03240

[23] H. Sahinoglu and S. D. Cabrera, “On phase retrieval of finite-length sequences using the initial time sample,” IEEE Trans. Circuits and Syst., vol. 38, no. 8, pp. 954–958, 1991.

[24] Y. Shechtman, Y. C. Eldar, O. Cohen, H. N. Chapman, J. Miao, and M. Segev, “Phase retrieval with application to optical imaging: a contemporary overview,” IEEE Signal Process. Mag., vol. 32, no. 3, pp. 87–109, 2015.

[25] J. Sun, Q. Qu, and J. Wright, “A geometric analysis of phase retrieval,” Found. Comput. Math., vol. 18, no. 5, pp. 1131–1198, 2018.

[26] J. Sun, Q. Qu, and J. Wright, “Complete dictionary recovery over the sphere I: Overview and the geometric picture,” IEEE Trans. Inf. Theory, vol. 63, no. 2, pp. 853–884, 2016.

[27] R. Vershynin, High-dimensional probability: An introduction with applications in data science. U.K.: Cambridge Univ. Press, 2018.

[28] I. Waldspurger, A. d’Aspremont, and S. Mallat, “Phase retrieval, maxcut and complex semidefinite programming,” Math. Prog., vol. 149, no. 1-2, pp. 47–81, 2015.

[29] A. Walther, “The question of phase retrieval in optics,” J. Mod. Opt., vol. 10, no. 1, pp. 41–49, 1963.

[30] G. Wang, G. B. Giannakis, and Y. C. Eldar, “Solving systems of random quadratic equations via truncated amplitude flow,” IEEE Trans. Inf. Theory, vol. 64, no. 2, pp. 773–794, 2018.

[31] H. Zhang, Y. Zhou, Y. Liang, and Y. Chi, “A nonconvex approach for phase retrieval: Reshaped wirtinger flow and incremental algorithms,” The Journal of Machine Learning Research, vol. 18, no. 1, pp. 5164–5198, 2017.
DEPARTMENT OF MATHEMATICS, THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, CLEAR WATER BAY, KOWLOON, HONG KONG

Email address: jfcai@ust.hk

DEPARTMENT OF MATHEMATICS, THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, CLEAR WATER BAY, KOWLOON, HONG KONG

Email address: menghuang@ust.hk

DEPARTMENT OF MATHEMATICS, THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, CLEAR WATER BAY, KOWLOON, HONG KONG.

Email address: madli@ust.hk

DEPARTMENT OF MATHEMATICS, THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, CLEAR WATER BAY, KOWLOON, HONG KONG

Email address: yangwang@ust.hk