Spinor equation for the $W^\pm$ boson

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I introduce spinor equations for the $W^\pm$ fields. The properties of these spinor equations under space-time transformation and under charge conjugation are studied. The expressions for electric charge and current and densities of the $W^\pm$ fields are obtained. Covariant quantization conditions are established, and the vacuum energy for the $W^\pm$ fields is found to be zero.

I. INTRODUCTION

In the standard electro-weak theory, the $W^\pm$ bosons are described by a 4-vector field that satisfies the Proca equation [1,2]. But however this 4-vector field can not be regarded as the $W^\pm$ boson field, because the charge density of the $W^\pm$ bosons can not be expressed as inner products of this field with its adjoint field. Recently, I introduced a spinor field theory for the photon [3], in which the photon field is a spinor, and satisfies a equation that is equivalent to Maxwell’s equations together with the relations between the 4-vector potential and electric and magnetic fields, and the Lorentz gauge condition for the 4-vector potential. In this paper, I modify the spinor equation for the massless photon field to describe charged massive spin 1 fields, which is the case of the $W^\pm$ boson fields. The properties of spinor equations for the $W^\pm$ fields under space-time transformation and under charge conjugation are studied in detail. Expressions for electric current density, momentum and angular momentum of the $W^\pm$ fields are obtained. Covariant quantization conditions for the $W^\pm$ fields are established, and the vacuum energy for the $W^\pm$ fields is found to be zero.

The spinor equations for the $W^\pm$ fields are introduced in Sec.II, the Lagrangian density and expressions for electric charge and current densities are presented in Sec.III. The quantization of the $W^\pm$ fields is carried out in Sec.IV.

II. THE EQUATION FOR THE $W^\pm$ BOSON

The equation for the $W^\pm$ boson can be written as:

$$i\hbar \frac{\partial}{\partial x_0} \psi_{w^\pm}(x) = -i\hbar \vec{\alpha}_w \cdot \nabla \psi_{w^\pm}(x) \pm m_w c \beta_w \psi_{w^\pm}(x)$$

(1)

where

$$\psi_{w^\pm} = (\psi_{w^\pm 1} \, \psi_{w^\pm 2} \, \psi_{w^\pm 3} \, 0 \, \psi_{w^\pm 6} \, \psi_{w^\pm 7} \, \psi_{w^\pm 8} \, \psi_{w^\pm 9} \, \psi_{w^\pm 10} \, \psi_{w^\pm 11} \, 0 \, 0 \, 0 \, \psi_{w^\pm 16})^T$$

(2)

and

$$x_0 = ct.$$

(3)

$m_w$ in Eqs. (1) is the rest mass of $W^\pm$ boson. We have the following expressions for the matrix $\vec{\alpha}_w$

$$\vec{\alpha}_w = \begin{pmatrix} \vec{\alpha}_e & 0 \\ 0 & -\vec{\alpha}_e \end{pmatrix},$$

(4)

where

$$\vec{\alpha}_1 = \begin{pmatrix} 0 & 0 & 0 & -i\sigma_2 \\ 0 & 0 & -i\sigma_2 & 0 \\ 0 & i\sigma_2 & 0 & 0 \\ i\sigma_2 & 0 & 0 & 0 \end{pmatrix}, \quad \vec{\alpha}_2 = \begin{pmatrix} 0 & 0 & 0 & -I_2 \\ 0 & 0 & I_2 & 0 \\ 0 & I_2 & 0 & 0 \\ -I_2 & 0 & 0 & 0 \end{pmatrix}, \quad \vec{\alpha}_3 = \begin{pmatrix} 0 & 0 & i\sigma_2 & 0 \\ 0 & 0 & 0 & -i\sigma_2 \\ -i\sigma_2 & 0 & 0 & 0 \\ 0 & i\sigma_2 & 0 & 0 \end{pmatrix}$$

(5)

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with
\[ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \] (6)

The following anti-commutation relations hold for the matrixes \(\alpha_{w1}, \alpha_{w2}, \alpha_{w3}:\)
\[ \alpha_{wn} \cdot \alpha_{wm} + \alpha_{wn} \cdot \alpha_{wm} = 2\delta_{nm}, \quad n, m = 1, 2, 3. \] (7)

The matrix \(\beta_w\) is define as:
\[ \beta_w = \begin{pmatrix} 0 & I_8 \\ I_8 & 0 \end{pmatrix}, \] (8)
where \(I_8\) is the \(8 \times 8\) unit matrix. The matrix \(\beta_w\) satisfies the following relations:
\[ \beta_w \bar{\alpha} w + \bar{\alpha} w \beta_w = 0 \quad \text{and} \quad \beta_w^2 = 1. \] (9)

Therefore
\[ (-i\hbar \bar{\alpha} w \cdot \nabla + m_w c \beta_w)^2 = -\hbar^2 \Delta + m_w^2 c^2, \] (10)
where
\[ \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \] (11)

It is easy to verify that \(\psi_{w+}(x)\) and \(\psi_{w-}(x)\) satisfy the same equation. We request that both \(\psi_{w+}(x)\) and \(\psi_{w-}(x)\) contain only positive frequency components.

The equation (11) is invariant under continuous space-time transformations (See the Appendices). We have
\[ \psi'_{w\pm}(x') = \exp (\vec{\varphi} \cdot \vec{\Lambda}) \psi_{w\pm}(x') \] (12)
with
\[ \vec{\Lambda} = \begin{pmatrix} -\vec{l} & 0 \\ 0 & \vec{\alpha} e - \vec{l} \end{pmatrix}, \] (13)
and
\[ \vec{\varphi} = \frac{v}{\sqrt{1 - v^2/c}} \ln \sqrt{1 + \frac{v}{c}} - \ln \sqrt{1 - \frac{v}{c}}, \] (14)
for Lorentz transformations, and
\[ \psi'_{w\pm}(x') = \exp (i \vec{s} \cdot \vec{s}_f) \psi_{w\pm}(x') \] (15)
with
\[ \vec{s}_f = \begin{pmatrix} \vec{s} & 0 \\ 0 & \vec{s} \end{pmatrix}, \] (16)
under space rotations, where
\[ \vec{s} = \begin{pmatrix} \vec{\Sigma} & 0 \\ 0 & \vec{\Sigma} \end{pmatrix}, \quad \vec{l} = \begin{pmatrix} 0 & i \vec{\Sigma} \\ -i \vec{\Sigma} & 0 \end{pmatrix}, \] (17)
and
\[ \Sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (18)
The following commutation relation holds for $\vec{s}_f$:

$$[s_{fn}, s_{fm}] = i \sum_{p=1}^{3} \varepsilon_{nmp} s_{fp}. \quad (19)$$

Eq. (1) is invariant also under space inversion, time reversal and charge conjugation. It is easy to verify that $\tau_0 \psi_w^\pm(x_0, -\vec{x})$, $\tau_0 \psi_w^{op}(-x_0, \vec{x})$ and $\beta_0 \psi_w^{op}(x_0, \vec{x})$ satisfy the same spinor equation Eq. (1) as $\psi_w^\pm(x_0, \vec{x})$. Where

$$\tau_0 = \begin{pmatrix} -\beta_e & 0 \\ 0 & -\beta_e \end{pmatrix}, \quad \beta_0 = \begin{pmatrix} I_8 & 0 \\ 0 & -I_8 \end{pmatrix},$$

with the matrix $\beta_e$ given by

$$\beta_e = \begin{pmatrix} I_4 \\ 0 \\ 0 \\ -I_4 \end{pmatrix}, \quad \beta_0 = \begin{pmatrix} I_8 \\ 0 \\ 0 \\ -I_8 \end{pmatrix},$$

in which $I_4$ is the $4 \times 4$ unit matrix.

The spinor equation (1) can also be rewritten as a sets of vector and scalar equations. Let

$$\vec{E}_w^\pm = \pm i \sqrt{\mu_w} (\psi_w^{1}, \psi_w^{2}, \psi_w^{3}), \quad \vec{H}_w^\pm = \pm i \sqrt{\mu_w} (\psi_w^{5}, \psi_w^{6}, \psi_w^{7}),$$

$$\vec{A}_w^\pm = \frac{1}{\sqrt{\mu_w}} (\psi_w^{9}, \psi_w^{10}, \psi_w^{11}),$$

and

$$A_w^{0} = \frac{1}{\sqrt{\mu_w}} \psi_w^{16}, \quad S_w^\pm = \pm i \sqrt{\mu_w} \psi_w^{8},$$

then the equation (1) is equivalent to the following groups of equations

$$\frac{\partial}{\partial x_0} \vec{E}_w^\pm = \nabla \times \vec{H}_w^\pm + \nabla S_w^\pm + \mu_w^2 \vec{A}_w^\pm$$

$$\frac{\partial}{\partial x_0} \vec{H}_w^\pm = -\nabla \times \vec{E}_w^\pm$$

$$0 = -\nabla \cdot \vec{H}_w^\pm$$

$$\frac{\partial}{\partial x_0} S_w^\pm = \nabla \cdot \vec{E}_w^\pm + \mu_w^2 A_w^{0}$$

and

$$\frac{\partial}{\partial x_0} \vec{A}_w^\pm = -\nabla A_w^{0} - \vec{E}_w^\pm$$

$$0 = \nabla \times \vec{A}_w^\pm - \vec{H}_w^\pm$$

$$\frac{\partial}{\partial x_0} A_w^{0} = -\nabla \cdot \vec{A}_w^\pm - S_w^\pm,$$

with

$$\mu_w = m_w c \hbar^{-1}.$$  \hspace{1cm} (32)

By using the anticommutation properties of $\vec{\alpha}$ and $\beta_w$, we have

$$\left( \frac{\partial^2}{\partial x_0^2} - \Delta + \mu_w^2 \right) \psi_w^\pm(x) = 0,$$

which is equivalent to

$$\left( \frac{\partial^2}{\partial x_0^2} - \Delta + \mu_w^2 \right) \left[ \vec{E}_w^\pm, \vec{H}_w^\pm, S_w^\pm \right] = 0.$$  \hspace{1cm} (34)
and
\[
\left( \frac{\partial^2}{\partial x_0^2} - \Delta + \mu_w^2 \right) \left[ A_{w^\pm,0}, \vec{A}_{w^\pm} \right] = 0,
\] (35)

The equation (35) is just the Proca equations for 4-vector “potentials” \( A_{w^\pm} = (A_{w^\pm,0}, \vec{A}_{w^\pm}) \). One may observe that if we let \( S_{w^\pm} = 0 \), then Eqs. (25)-(31) are reduced to the Maxwell-Proca equations. But as the charged current of the weak interaction coupled with the \( W^\pm \) fields does not satisfy the condition of continuity, \( S_{w^\pm} \) cannot be zero when the weak interaction is considered.

### III. LAGRANGIAN DENSITY

The equation for the \( W^\pm \) boson \( \mathbf{1} \) can be derived from the following Lagrangian density

\[
\mathcal{L}_{w^\pm} = \bar{\psi}_{w^\pm} \left[ i\hbar \left( \frac{\partial}{\partial t} + c\vec{a}_w \cdot \vec{\nabla} \right) + m_w c^2 \beta_w \right] \psi_{w^\pm}
\] (36)

where \( \bar{\psi}_w(x) = \psi_{w}^\dagger(x) \tau_1 \) with

\[
\tau_1 = \begin{pmatrix} 0 & \beta_e \\ \beta_e & 0 \end{pmatrix},
\] (37)

is the adjoint field. \( I_8 \) is the \( 8 \times 8 \) unit matrix. The Lagrangian density (36) is invariant under a global phase change of the \( W^\pm \) field \( \psi_{w^\pm}(x) \). This implies the conservation of the electric charge of \( W^\pm \) fields:

\[
Q_{w^\pm} = \pm e \int \rho_{w^\pm} d^3 \vec{x},
\] (38)

and

\[
\frac{\partial}{\partial t} \rho_{w^\pm} + \vec{\nabla} \cdot \vec{j}_{w^\pm} = 0,
\] (39)

where

\[
\rho_{w^+}(x) = e\bar{\psi}_{w^+}(x)\psi_{w^+}(x), \quad \rho_{w^-}(x) = -e\bar{\psi}_{w^-}(x)\psi_{w^-}(x)
\] (40)

are the charge densities of \( W^\pm \) fields and

\[
\vec{j}_{w^+}(x) = e\bar{\psi}_{w^+}(x)\vec{a}_w \psi_{w^+}(x), \quad \vec{j}_{w^-}(x) = -e\bar{\psi}_{w^-}(x)\vec{a}_w \psi_{w^-}(x)
\] (41)

are the current densities. \( e > 0 \) is the negative value of the electron’s electric charge. Due to the fact that the \( W^+ \) boson and the \( W^- \) boson have opposite electric charge, and \( \psi_{w^+}(x) \) commutes with \( \psi_{w^-}(x) \), the charge densities and the current densities of \( W^\pm \) fields do not contain cross terms between \( \psi_{w^+}(x) \) and \( \psi_{w^-}(x) \). That means a \( W^+ \) boson will not annihilate with a \( W^- \) boson by interacting with the photon field.

According to the relation between symmetries and conservation laws, we may obtain the following expressions for the momentum \( \vec{P} \) and the angular momentum \( \vec{M} \) of the free \( W^\pm \) field:

\[
\vec{P}_{w^\pm} = -i\hbar \int d^3 \vec{x} \bar{\psi}_{w^\pm} \vec{\nabla} \psi_{w^\pm},
\] (42)

and

\[
\vec{M}_{w^\pm} = \int d^3 \vec{x} \bar{\psi}_{w^\pm} [\vec{x} \times (-i\hbar \vec{\nabla})] \psi_{w^\pm} + \int d^3 \vec{x} \bar{\psi}_{w^\pm} (\hbar \vec{s}_f) \psi_{w^\pm}.
\] (43)

It is clear that \( \vec{s}_f \) can be interpreted as the spin operator of the \( W^\pm \) fields.

The conjugate field of \( \psi_{w^\pm} \) is

\[
\pi_{w^\pm} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_{w^\pm}} = i\hbar \bar{\psi}_{w^\pm}.
\] (44)

The Hamiltonian of the \( W^\pm \) fields now can be calculated:

\[
H_{w^\pm} = \int d^3 \vec{x} \left( \pi_{w^\pm} \dot{\psi}_{w^\pm} - \mathcal{L}_{w^\pm} \right)
= \int d^3 \vec{x} \bar{\psi}_{w^\pm} \left( -i\hbar c\vec{a}_w \cdot \vec{\nabla} \pm m_w c^2 \beta_w \right) \psi_{w^\pm}
\] (45)
IV. QUANTIZATION OF THE $W^\pm$ FIELD

It is convenient to quantize the $W^\pm$ field in the momentum space. To do this, we have to find firstly the plan wave solutions of the $W^\pm$ field. By substituting the following form of solution

$$\psi_{w^\pm}(x) \propto \exp(-i k x) \Psi^\pm(\tilde{k})$$

(46)

into the spinor equation (1), we find

$$(\alpha^*_w \cdot \tilde{k} - k_0 - \mu_w \beta_w) \Psi^+(\tilde{k}) = 0.$$  

(47)

Eq. (17) permits three independent nontrivial solutions with $k_0 = \sqrt{|\tilde{k}|^2 + \mu_w^2}$. They can be chosen as

$$\Psi^+_0(\tilde{k}) = \frac{1}{\sqrt{2\mu_w k_0}} \left(\mu_w \hat{k}_1 \mu_w \hat{k}_2 \mu_w \hat{k}_3 0 0 0 0 k_0 \hat{k}_1 k_0 \hat{k}_2 k_0 \hat{k}_3 0 0 0 0 |\tilde{k}| \right)^T,$$

(48)

and

$$\Psi^\pm(\tilde{k}) = \frac{1}{2\sqrt{\mu_w k_0}} \left(k_0(q_1 \pm ir_1) k_0(q_2 \pm ir_2) k_0(q_3 \pm ir_3) 0 |\tilde{k}|(r_1 \mp iq_1) |\tilde{k}|(r_2 \pm iq_2) |\tilde{k}|(r_3 \mp iq_3) 0 \mu_w(q_1 \pm ir_1) \mu_w(q_2 \pm ir_2) \mu_w(q_3 \pm ir_3) 0 0 0 0 \right)^T,$$

(49)

where $\tilde{k} = \hat{k}/|\tilde{k}|$, and $\hat{q}$ and $\hat{r}$ are two vectors of unity satisfying the following conditions:

$$\hat{k} \times \hat{q} = \hat{r}, \quad \hat{k} \times \hat{r} = -\hat{q}, \quad \hat{q} \times \hat{r} = \hat{k}, \quad \text{and} \quad \hat{r}(-\tilde{k}) = -\hat{r}(\tilde{k}).$$

(50)

Similarly, we have

$$\psi_{w^-}(x) \propto \exp(-i k x) \Psi^-_h(\tilde{k}),$$

(51)

where $\Psi^-(\tilde{k})$ satisfies the equation

$$(\alpha^-_w \cdot \tilde{k} - k_0 + \mu_w \beta_w) \Psi^-_h(\tilde{k}) = 0,$$

(52)

with

$$\Psi^+_0(\tilde{k}) = \frac{1}{\sqrt{2\mu_w k_0}} \left(-\mu_w \hat{k}_1 - \mu_w \hat{k}_2 - \mu_w \hat{k}_3 0 0 0 0 k_0 \hat{k}_1 k_0 \hat{k}_2 k_0 \hat{k}_3 0 0 0 0 |\tilde{k}| \right)^T,$$

(53)

and

$$\Psi^\pm(\tilde{k}) = \frac{1}{2\sqrt{\mu_w k_0}} \left(k_0(q_1 \pm ir_1) k_0(q_2 \pm ir_2) k_0(q_3 \pm ir_3) 0 |\tilde{k}|(r_1 \mp iq_1) |\tilde{k}|(r_2 \mp iq_2) |\tilde{k}|(r_3 \mp iq_3) 0 -\mu_w(q_1 \pm ir_1) -\mu_w(q_2 \pm ir_2) -\mu_w(q_3 \pm ir_3) 0 0 0 0 \right)^T,$$

(54)

$\Psi^\mp(\tilde{k})$ are orthogonal:

$$\tilde{\Psi}^+_h(\tilde{k}) \Psi^+_h(\tilde{k}) = \delta_{hh'}, \quad \tilde{\Psi}^-_h(\tilde{k}) \Psi^-_h(\tilde{k}) = \delta_{hh'},$$

(55)

and

$$\tilde{\Phi}_h(\tilde{k}) \Psi^-_h(\tilde{k}) = 0.$$  

(56)

We also have

$$(\hat{k} \cdot \hat{s}_f) \Psi^+_h(\tilde{k}) = h \Psi^+_h(\tilde{k}), \quad (\hat{k} \cdot \hat{s}_f) \Psi^-_h(\tilde{k}) = h \Psi^-_h(\tilde{k}), \quad \text{with} \quad h = 0, \pm 1,$$

(57)
with the \(\times\) relations (64) and (65), and the expression (59), we have the same operators of \(W\) with \(W\) We observe that the vacuum energy of the \(\text{vacuum}\) and \(\text{vacuum}\) and

\[
\bar{\Psi}_h^+(\vec{k}) \Psi_h^{-}(\vec{k}) = \Psi_h^{-}(\vec{k}) \bar{\Psi}_h^+(\vec{k}) = s(s + 1) = 2.
\] (58)

That means the \(W^\pm\) bosons are of spin \(s=1\).

Having plan wave solutions the \(W^\pm\) field, we may now expand \(\psi_{w^\pm}(x)\) in plane waves

\[
\psi_{w^+(x)} = \sum_k \frac{1}{\sqrt{V}} e^{-ikx} \sum_h \Psi_h^+(\vec{k}) a_h(\vec{k}), \quad \psi_{w^-(x)} = \sum_k \frac{1}{\sqrt{V}} e^{-ikx} \sum_h \Psi_h^-(\vec{k}) b_h(\vec{k})
\] (59)

with \(k_0 = \sqrt{|\vec{k}|^2 + \mu_w^2}\). According to relations (36) and (59), the Lagrangian of the photon field can be expressed as a function of the variables \(q_{hk}(t)\):

\[
L(t, q) = \sum_{\vec{k}h} \left( h q_{hk}^+(t) \left( i \frac{\partial}{\partial t} - c k_0 \right) q_{hk}^+(t) + h q_{hk}^{-1}(t) \left( i \frac{\partial}{\partial t} - c k_0 \right) q_{hk}^{-1}(t) \right),
\] (60)

with

\[
q_{hk}^+(t) = a_h(\vec{k}) \exp(-i c k_0 t), \quad q_{hk}^{-1}(t) = b_h(\vec{k}) \exp(-i c k_0 t).
\] (61)

The conjugate momentum of \(q_{hk}^\pm(t)\) can be calculated, and we have

\[
p_{hk}^+(t) = \frac{\partial L}{\partial q_{hk}^+(t)} = i h a_h^\dagger(\vec{k}) \exp(i c k_0 t)
\] (62)

and

\[
p_{hk}^{-1}(t) = \frac{\partial L}{\partial q_{hk}^{-1}(t)} = i h b_h^\dagger(\vec{k}) \exp(i c k_0 t).
\] (63)

By applying the quantization condition \([q_{hk}^\pm; p_{hk'}^\pm] = i h \delta_{hk} \delta_{k'k}\) we find the following commutation relations

\[
[a_h(\vec{k}), a_{h'}(\vec{k}')] = \delta_{hh'} \delta_{kk'}.
\] (64)

and

\[
[b_h(\vec{k}), b_{h'}(\vec{k}')] = \delta_{hh'} \delta_{kk'}.
\] (65)

\(a_h(\vec{k})\) and \(a_{h'}(\vec{k}')\) are just the annihilation operator and creation operator of \(W^+\) bosons , and \(b_h(\vec{k})\) and \(b_{h'}(\vec{k}')\) are the same operators of \(W^-\) bosons. The Hamiltonian of the \(W^\pm\) field can also be calculated. We obtain

\[
H = \sum_{\vec{k}} \sum_h p_{hk} q_{hk} - L = \sum_{\vec{k}h} c h k_0 \left( a_{h}^\dagger(\vec{k}) a_h(\vec{k}) + b_{h}^\dagger(\vec{k}) b_h(\vec{k}) \right).
\] (66)

We observe that the vacuum energy of the \(W^\pm\) field is zero.

The commutation relations for the \(W^\pm\) field can be written in a covariant form. According to the commutation relations (54) and (55), and the expression (59), we have

\[
[\psi_{w^\pm}(x), \psi_{w^\pm m}(x')] = D_{lm}(x - x'), \quad \text{with} \quad l, m = 1, 2, \ldots, 8,
\] (67)

with the \(8 \times 8\) matrix \(D(x)\) given by the following expression

\[
D(x) = \frac{1}{(2\pi)^3 \mu_w} \int_{k_0 > 0} d^4 k \delta(k^2 - \mu_w^2) \left[ k_0 \vec{k} \cdot \vec{l} + (\vec{k} \cdot \vec{l})(\vec{k} \cdot \vec{l}) + \mu_w^2 I_z \right] e^{-ikx},
\] (68)

where

\[
I_w = \begin{pmatrix} I_{43} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad I_{43} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\] (69)
The replacement
\[
\frac{1}{\mathcal{V}} \sum_{\vec{k}} \rightarrow \frac{1}{(2\pi)^3} \int d^3\vec{k}
\] (70)
was used in obtaining the relation (67). Under Lorentz transformations, \(D(x)\) transforms to
\[
D'(x') = \exp(-\vec{\varphi} \cdot \vec{l})D(x) \exp(-\vec{\varphi} \cdot \vec{l}).
\] (71)

One can verify with no difficulty that by using the expansions (59), the commutation relations (64) and (65) can be derived from commutation relation (67). Therefore, the commutation relations (64) and (65) and the the commutation relation (67) are equivalent.

V. CONCLUSION

I introduced a spinor field theory for the \(W^\pm\) fields. The electric charge densities of the \(W^\pm\) bosons can be expressed as inner products of spinor \(W^\pm\) fields with their adjoint fields. Expressions for electric current density, momentum and angular momentum of the \(W^\pm\) fields are obtained. The expressions for electric current densities of \(W^\pm\) fields do not contain cross terms between \(\psi_{w^+}(x)\) and \(\psi_{w^-}(x)\), therefore a \(W^+\) boson will not annihilate with a \(W^-\) boson by interacting with the photon field. Covariant quantization conditions for the \(W^\pm\) fields are established, and the vacuum energy for the \(W^\pm\) fields is found to be zero.

Appendix A: Invariance of the spinor equation for \(W^\pm\) field under Lorentz transformations

To show the invariance of Eq. (1), it is convenient to write the spinor field \(\psi_{w^\pm}(x)\) as
\[
\psi_{w^\pm}(x) = \begin{pmatrix} \psi_u(x) \\ \psi_d(x) \end{pmatrix},
\] (A1)
where
\[
\psi_u = (\psi_{w^\pm 1} \psi_{w^\pm 2} \psi_{w^\pm 3} 0 \psi_{w^\pm 5} \psi_{w^\pm 6} \psi_{w^\pm 7} \psi_{w^\pm 8})^T, \quad \psi_d = (\psi_{w^\pm 9} \psi_{w^\pm 10} \psi_{w^\pm 11} 0 0 0 0 \psi_{w^\pm 16})^T.
\] (A2)

According to Eq. (1), we have the following equations for \(\psi_u(x)\) and \(\psi_d(x)\):
\[
\frac{\partial}{\partial x_0} \psi_u(x) = -\vec{\alpha}_e \cdot \nabla \psi_u(x) \mp i\mu_w \psi_d(x),
\] (A3)
and
\[
\frac{\partial}{\partial x_0} \psi_d(x) = \vec{\alpha}_e \cdot \nabla \psi_d(x) \mp i\mu_w \psi_u(x).
\] (A4)
The invariance of Eq. (1) is then equivalent to the invariance of Eqs. (A3) and (A4).

By direct verification, one may find the following relations for matrices \(\vec{s}\) and \(\vec{l}\):
\[
[s_n, \alpha_{em}] = i \sum_{p=1}^{3} \varepsilon_{npm}\alpha_{ep}, \quad n,m = 1,2,3,
\] (A5)
and
\[
\alpha_{em}l_m\alpha_{en} = l_m - (1 - \delta_{nm})\alpha_{em}, \quad n,m = 1,2,3.
\] (A6)

Let’s consider a Lorentz transformation
\[
\begin{pmatrix}
x'_1 \\
x'_2 \\
x'_3 \\
x'_0
\end{pmatrix} = \begin{pmatrix}
\cosh \varphi & 0 & 0 & -\sinh \varphi \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\sinh \varphi & 0 & 0 & \cosh \varphi
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_0
\end{pmatrix}.
\] (A7)
We have
\[ \frac{\partial}{\partial x_1} = \cosh \varphi \frac{\partial}{\partial x_1} - \sinh \varphi \frac{\partial}{\partial x_0}, \quad \frac{\partial}{\partial x_0} = \cosh \varphi \frac{\partial}{\partial x_0} - \sinh \varphi \frac{\partial}{\partial x_1}. \] (A8)

By using the relations (A8), Eqs. (A3) and (A4) can be written as
\[ (\cosh \varphi - \sinh \varphi \alpha_1) \frac{\partial}{\partial x'_0} \psi_u(x') + \left[ (\cosh \varphi \alpha_1 - \sinh \varphi) \frac{\partial}{\partial x'_1} + \alpha_2 \frac{\partial}{\partial x'_2} + \alpha_3 \frac{\partial}{\partial x'_3} \right] \psi_u(x') \pm i \mu_w \psi_d(x') = 0. \] (A9)

and
\[ (\cosh \varphi + \sinh \varphi \alpha_1) \frac{\partial}{\partial x'_0} \psi_d(x') - \left[ (\cosh \varphi \alpha_1 + \sinh \varphi) \frac{\partial}{\partial x'_1} + \alpha_2 \frac{\partial}{\partial x'_2} + \alpha_3 \frac{\partial}{\partial x'_3} \right] \psi_d(x') \pm i \mu_z \psi_u(x') = 0. \] (A10)

But
\[ l_1 \alpha_1 = \alpha_1 l_1, \quad \exp(\pm \varphi \alpha_1) = \cosh \varphi \pm \sinh \varphi \alpha_1 \] (A11)

thus
\[ \frac{\partial}{\partial x'_0} \exp(-\varphi l_1) \psi_u(x') = \left[ \alpha_2 \frac{\partial}{\partial x'_1} + \exp(\varphi (\alpha_1 - l_1)) \left( \alpha_2 \frac{\partial}{\partial x'_2} + \alpha_3 \frac{\partial}{\partial x'_3} \right) \exp(\varphi l_1) \right] \exp(-\varphi l_1) \psi_u(x') \mp i \exp(\varphi (\alpha_1 - l_1)) \mu_w \psi_d(x'), \] (A12)

and
\[ \frac{\partial}{\partial x'_0} \exp(\varphi (\alpha_1 - l_1)) \psi_d(x') = \left[ \alpha_2 \frac{\partial}{\partial x'_1} + \exp(\varphi l_1) \left( \alpha_2 \frac{\partial}{\partial x'_2} + \alpha_3 \frac{\partial}{\partial x'_3} \right) \exp(-\varphi (\alpha_1 - l_1)) \right] \exp(\varphi (\alpha_1 - l_1)) \psi_d(x') \mp i \exp(-\varphi l_1) \mu_w \psi_u(x'), \] (A13)

According to the relation (A6), we have
\[ \alpha_{em} l_1^n = (l_1 - \alpha_1)^n \alpha_{em}, \quad m = 2, 3, \] (A14)

thus
\[ \alpha_{em} \exp(\varphi l_1) = \exp \left( \varphi (l_1 - \alpha_1) \right) \alpha_{em}, \quad \exp(\varphi l_1) \alpha_{em} = \alpha_{em} \exp \left( \varphi (l_1 - \alpha_1) \right), \quad m = 2, 3. \] (A15)

Equations (A12) and (A13) become then
\[ \frac{\partial}{\partial x'_0} \psi'_u(x') = -\vec{\alpha}_e \cdot \nabla \psi'_u(x') \mp i \mu_w \psi'_d(x'), \] (A16)

and
\[ \frac{\partial}{\partial x'_0} \psi'_d(x') = \vec{\alpha}_e \cdot \nabla \psi'_d(x') \mp i \mu_w \psi'_u(x'), \] (A17)

where
\[ \psi'_u(x') = \exp(-\varphi l_1) \psi_u(x'), \quad \psi'_d(x') = \exp(\varphi (\alpha_1 - l_1)) \psi_d(x'). \] (A18)

By using the expressions for \( \vec{\alpha}_e \) and \( \vec{l}_1 \), one can verify the following relations:
\[ \psi'_u(x') = \psi_{u4}(x') \equiv 0, \quad \psi'_u(x') = \psi_{u8}(x'), \] (A19)

and
\[ \psi'_d(x') = \psi_{d4}(x') \equiv 0, \quad \psi'_d(x') = \psi_{d5}(x') \equiv 0, \quad \psi'_d(x') = \psi_{d6}(x') \equiv 0, \quad \psi'_d(x') = \psi_{d7}(x') \equiv 0. \] (A20)

The equations (A10) and (A17) in the new reference frame has exactly the same form as Eqs. (A3) and (A4), the spinor fields \( \psi'_u(x') \) and \( \psi'_d(x') \) in the new reference frame has exactly the same form as the spinor fields \( \psi_u(x) \) and \( \psi_d(x) \). Therefore Eqs. (A3) and (A4) are invariant under Lorentz transformations.
Appendix B: Invariance of the spinor equation for $W^\pm$ fields under space rotation

Let’s consider an infinitesimal space rotation

$$x'_0 = x_0, \quad x'_n = x_n - \sum_{m,p=1}^{3} \varepsilon_{nmp} \delta_m x_p \quad n = 1, 2, 3. \tag{B1}$$

We have

$$\frac{\partial}{\partial x_n} = \frac{\partial}{\partial x'_n} - \sum_{m,p=1}^{3} \varepsilon_{pmn} \delta_m \frac{\partial}{\partial x'_p}, \quad n = 1, 2, 3. \tag{B2}$$

The equation (1) can be written as

$$i \hbar \frac{\partial}{\partial x'_0} \psi_{w\pm}(x') = -i \hbar \sum_{n=1}^{3} [\alpha_{wn} - \sum_{l,m=1}^{3} \varepsilon_{nlm} \delta_l \alpha_{wm}] \frac{\partial}{\partial x'_n} \psi_{w\pm}(x') \pm m_w c \beta_w \psi_{w\pm}(x'). \tag{B3}$$

According to expressions for $\vec{\alpha}_w$ and $\vec{s}_f$ we have

$$\sum_{m=1}^{3} \varepsilon_{nlm} \alpha_{wn} = i s_f \alpha_{wn} - i \alpha_{wn} s_f, \tag{B4}$$

so

$$\sum_{l,m=1}^{3} \varepsilon_{nlm} \delta_l \alpha_{wm} = i \vec{\delta} \cdot \vec{s}_f \alpha_{wn} - i \alpha_{wn} \vec{\delta} \cdot \vec{s}_f. \tag{B5}$$

Therefore

$$i \hbar \frac{\partial}{\partial x'_0} (1 + i \vec{\delta} \cdot \vec{s}_f) \psi_{w\pm}(x') = -i \hbar \sum_{n=1}^{3} [\alpha_{en} - \sum_{l,m=1}^{3} \varepsilon_{nlm} \delta_l \alpha_{wm}] \frac{\partial}{\partial x'_n} (1 + i \vec{\delta} \cdot \vec{s}_f) \psi_{w\pm}(x') \pm m_w c (1 + i \vec{\delta} \cdot \vec{s}_f) \beta_w \psi_{w\pm}(x'). \tag{B6}$$

But $\beta_w \vec{s}_f = \vec{s}_f \beta_w$, so we may write Eq. (B6) as

$$i \hbar \frac{\partial}{\partial x'_0} \psi'_{w\pm}(x') = -i \hbar \sum_{n=1}^{3} \alpha_{en} \frac{\partial}{\partial x'_n} \psi_{w\pm}(x') \pm m_w c \beta_w \psi_{w\pm}(x'), \tag{B7}$$

with

$$\psi'_{w\pm}(x') = (1 + i \vec{\delta} \cdot \vec{s}_f) \psi_{w\pm}(x'). \tag{B8}$$

The equation (B7) has exactly the same form as Eq. (1). So the spinor equation for $W^\pm$ field is invariant under space rotations.

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