PERIODIC FUNCTIONS, LATTICES AND THEIR PROJECTIONS

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ABSTRACT. Functions whose symmetries form a crystallographic group in particular have a lattice of periods, and the set of their level curves forms a periodic pattern. We show how after projecting these functions, one obtains new functions with a lattice of periods that is not the projection of the initial lattice. We also characterise all the crystallographic groups in three dimensions that are symmetry groups of patterns whose projections have periods in a given two-dimensional lattice. The particular example of patterns that after projection have a hexagonal lattice of periods is discussed in detail.

1. INTRODUCTION

Patterns in biology or chemistry may arise as a consequence of spatial variations in concentration of one or more substances. In many cases the consequences of the variation may only be observed in projections. Examples are shapes in animal coats and reactions in thin layers of gel. They are modelled by partial differential equations, as for instance in [3, 4], with a prescribed symmetry group. In this context, we model a periodic pattern by interpreting it as the level sets of a periodic function. The set of all periods is viewed as part of the group of symmetries of the function.

The projection of periodic objects into a subspace has many applications. Both the object to be projected and the projection method depend on the context. In classical crystallography, points in a unit cell of a lattice are projected along certain special directions. The symmetries of these projections are described in the International Tables of Crystallography [7, Section 2.2.14], where it says “Even though the projection of a finite object along any direction may be useful, the projection of a periodic object such as a crystal structure is only sensible along a rational lattice direction (lattice row). Projection along a nonrational direction results in a constant density in at least one direction.”

In contrast, there is no intrinsic choice of direction of projection when observing a three dimensional pattern. Looking at periodic patterns is not the same as looking at the lattice of their periods, specially when a projection is involved. The main goal of this article is to establish this fact and quantify it. By a lattice we mean a subset of \(\mathbb{R}^{n+1}\) that is a \(\mathbb{Z}\)-module with \(n+1\) linearly independent generators. When we project a lattice we may obtain a set with too many generators. When we project periodic functions we may obtain functions with too few periods. For planar lattices and wallpaper patterns this follows from the results of [10, 11], here we treat the general case.

We start by looking at the projection into \(\mathbb{R}^n\) of a lattice in \(\mathbb{R}^{n+1}\). After establishing some notation in Section 2 we discuss in Section 3 some properties of the projection for general lattices, and we show how these properties are modified if we make the common assumption that the lattice is an integral lattice.

From Section 4 onwards, we project patterns in \(\mathbb{R}^{n+1}\) into patterns in \(\mathbb{R}^n\). General information on symmetric patterns is given in Section 4. Given the crystallographic group \(\Gamma\) of symmetries of the original pattern, we describe in Section 5 the symmetry group of the projected pattern and the way it depends on the width of the projection band and on the symmetries of the original pattern, using results of [16]. In particular we obtain all the periods of the projection and compare them to the periods of the restriction of the pattern and to the projection of the lattice. This particular group of examples is interesting because it has been proposed as an explanation for certain special patterns, called black-eye patterns, as discussed in [2, 6].

While Section 3 contains a discussion of how the periodicity properties of the projected pattern varies with the projection width, for different types of symmetry groups, Section 6 addresses the converse question: which crystallographic groups in three dimensions are the symmetry groups of functions that after projection over a
2. LATTICES AND GROUPS

In this section we establish some of the terminology for the remainder of the article. We use the notation \((x, y) \in \mathbb{R}^{n+1}\), with \(x \in \mathbb{R}^n\) and \(y \in \mathbb{R}\). The reader is referred to Armstrong [1], chapters 24, 25 and 26, for results on Euclidean and plane crystallographic groups, and to Senechal [17, page 37], for results on lattices and crystallographic groups. A detailed description can also be found in Pinho [15, chapter 2].

2.1. the Euclidean group. The \((n + 1)\)-dimensional Euclidean group is the semi-direct product \(E(n+1) \cong \mathbb{R}^{n+1} \rtimes O(n+1)\) with elements \(\gamma = (v, \delta)\), where \(v \in \mathbb{R}^{n+1}\) and \(\delta \in O(n+1)\). The group operation is \(((v_1, \delta_1) \cdot (v_2, \delta_2) = (v_1 + \delta_1v_2, \delta_2)\), for \((v_1, \delta_1)\), \((v_2, \delta_2) \in E(n+1)\) and the action of \((v, \delta) \in E(n+1)\) on \((x, y) \in \mathbb{R}^{n+1}\) is given by \((v, \delta) \cdot (x, y) = v + \delta(x, y)\).

2.2. lattices. Using the definition of Senechal [17, page 37], a subset \(L \subset \mathbb{R}^{n+1}\) is a lattice if it is generated over the integers by \(n + 1\) linearly independent elements \(l_1, \ldots, l_{n+1} \in \mathbb{R}^{n+1}\), which we write:

\[
L = \{l_1, \ldots, l_{n+1}\} = \left\{ \sum_{i=1}^{n+1} m_i l_i, m_i \in \mathbb{Z} \right\}.
\]

Any set of vectors \(l_1, \ldots, l_{n+1}\) that generate \(L\) over the integers defines a \((n + 1)\)-dimensional parallelepiped called the fundamental cell of the lattice and its volume \(\rho\) is an invariant of the lattice (see Senechal [17, page 38]).

For any \(l \in L\) there is some \(m \in \mathbb{Z}\) such that \(\frac{1}{m}l\) is the smallest element of \(L\) colinear with \(l\). Then \(\frac{1}{m}l\) is a generator of \(L\), i.e., there are elements \(g_1, \ldots, g_n \in L\) such that \(L = \{\frac{1}{m}l, g_1, \ldots, g_n\}\).

The symmetry group \(\Gamma\) of a lattice \(L \subset \mathbb{R}^{n+1}\) is the largest subgroup of \(E(n+1)\) that leaves \(L\) invariant, i.e. \(\gamma \in \Gamma\) if and only if \(\gamma \cdot L = L\). If \((v, \alpha) \in \Gamma\) for some \(\alpha \in O(n+1)\) then \((v, \alpha) \cdot (0, 0) = v \in L\). Since \(-v\) is also in \(L\) then \((e, Id_{n+1}) \cdot (v, \alpha) = (0, \alpha) \in \Gamma\) and \(\alpha \in H\), the largest subgroup of \(O(n+1)\) that leaves \(L\) invariant, called the holohedry of \(L\). For the symmetry group of a lattice \(L\), we have \(\Gamma = L \cap H\).

The dual lattice \(L^*\) of \(L\) is defined as:

\[
L^* = \{k \in \mathbb{R}^{n+1} : \langle k, l_i \rangle \in \mathbb{Z}, i = 1, \ldots, n + 1 \}
\]

where \(\langle ., . \rangle\) is the usual inner product in \(\mathbb{R}^{n+1}\). It may be written as \(L^* = \{l_1^*, \ldots, l_{n+1}^*\}\), where \(l_i^* \in \mathbb{R}^{n+1}\) and \(\langle l_i^*, l_j^* \rangle = \delta_{ij}\) for all \(i, j \in \{1, \ldots, n + 1\}\).

2.3. crystallographic groups. A subgroup \(\Gamma \leq E(n+1)\) is a crystallographic group with lattice \(L\) if the orbit, on \(\mathbb{R}^{n+1}\), of the origin under the action of its subgroup of translations \(\{v : (v, Id_{n+1}) \in \Gamma\}\), is a lattice \(L \subset \mathbb{R}^{n+1}\) (or a \(Z\)-module). We also use the symbol \(L\) for the subgroup of translations of \(\Gamma\), since it is isomorphic to the group \((L, +)\).

The projection \((v, \delta) \rightarrow \delta\), of \(\Gamma\) into \(O(n+1)\), has kernel \(L\) and image \(J = \{\delta : (v, \delta) \in \Gamma \text{ for some } v \in \mathbb{R}^{n+1}\}\), isomorphic to the quotient \(\Gamma/L\). The group \(J\) is called the point group of \(L\) and is a subgroup of the holohedry of \(L\). Thus, \(JL = \{\delta l : \delta \in J, l \in L\} = L\). A caveat here: Armstrong and Senechal call this a point group. Miller and crystallographers use point group for \(\Gamma \cap O(n+1)\).

We will abuse terminology and also refer to the projection into \(O(n)\) of any subgroup of \(E(n)\) whose translations form a \(Z\)-module with \(m\) generators, \(m \leq n\) as a point group, even if the original group is not crystallographic.

The set of all the elements in \(\Gamma\) with a given orthogonal component \(\delta \in J\) is the coset \(L \cdot (v, \delta) = \{(l + v, \delta) : l \in L\}\) for any \(v \in \mathbb{R}^{n+1}\) such that \((v, \delta) \in \Gamma\). We will denote \(v + L\) the non-orthogonal component of \((v, \delta) \in \Gamma\) defined up to elements of \(L\).
2.4. special symmetries. For \( \alpha \in O(n) \), we use the following notation for some elements of \( O(n+1) \):

\[
\sigma = \begin{pmatrix} I_{d_n} & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_+ = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \alpha_- = \sigma \alpha_+ = \begin{pmatrix} \alpha & 0 \\ 0 & -1 \end{pmatrix}
\]

where \( I_{d_n} \) is the \( n \times n \) identity matrix.

3. PERIODS OF PROJECTED LATTICES

We start by looking at the projection of the portion of a lattice lying between two parallel affine subspaces. Let \( \mathcal{P} : \mathbb{R}^{n+1} \to \mathbb{R}^n \) be the projection \( \mathcal{P}(x,y) = x \) and let \( \mathcal{B}_{y_0} \) be the strip \( \mathcal{B}_{y_0} = \{(x,y) \in \mathbb{R}^{n+1} : 0 \leq y \leq y_0 \} \).

The projection of \( \mathcal{L} \cap \mathcal{B}_{y_0} \), given by

\[
\mathcal{P}(\mathcal{L} \cap \mathcal{B}_{y_0}) = \{ x \in \mathbb{R}^n : \exists y, 0 \leq y \leq y_0, \text{ such that } (x,y) \in \mathcal{L} \}
\]

is not necessarily a lattice. We want to describe the set of its periods, given by

\[
\{ p \in \mathbb{R}^n : x + p \in \mathcal{P}(\mathcal{L} \cap \mathcal{B}_{y_0}) \forall x \in \mathcal{P}(\mathcal{L} \cap \mathcal{B}_{y_0}) \}.
\]

Clearly, if \((p,0) \in \mathcal{L} \), then \( p \) is a period of \( \mathcal{P}(\mathcal{L} \cap \mathcal{B}_{y_0}) \).

First we obtain conditions for periodicity involving special forms for the generators of \( \mathcal{L} \), that we call adapted generators.

**Lemma 3.1.** Let \( \mathcal{L} \) be a lattice in \( \mathbb{R}^{n+1} \), with \((0,b) \in \mathcal{L} \), \( b \neq 0 \), minimal in its direction. Given \((\alpha, \beta) \in \mathcal{L} \), \( \alpha \neq 0 \), there is a set of generators for \( \mathcal{L} \) of the form \( \{(0,b),(a_i,b_i), i = 1, \ldots, n\} \) with \( ma_1 = \alpha \) for some \( m \in \mathbb{Z} \), and such that \( 0 \leq b_i < b \) for \( i = 1, \ldots, n \) and \( \{a_i, i = 1, \ldots, n\} \) are linearly independent vectors in \( \mathbb{R}^n \).

**Proof.** We can write \( \mathcal{L} = \{(0,b),(c_i,d_i), i = 1, \ldots, n\} \) where \( \{c_i, i = 1, \ldots, n\} \) are linearly independent vectors in \( \mathbb{R}^n \). Let \( W = \{(c_i,d_i), i = 1, \ldots, n\} \mathbb{R} \subset \mathbb{R}^{n+1} \) and \( \mathcal{L}_2 = \mathcal{L} \cap W = \{(c_i,d_i), i = 1, \ldots, n\} \mathbb{Z} \subset W \). Then \((\alpha, \beta') \in \mathcal{L}_2 \) for some \( \beta' \).

Let \( \mathcal{L}_1 = \{c_i, i = 1, \ldots, n\} \mathbb{Z} \subset \mathbb{R}^n \), with \( \alpha \in \mathcal{L}_1 \). Take \( a_1 \) to be the minimal elemento of \( \mathcal{L}_1 \) in the direction of \( \alpha \), i.e. \( a_1 \in \mathcal{L}_1 \) and \( ma_1 = \alpha \) for some \( m \in \mathbb{Z} \), \( m > 0 \), and hence \( a_1 = \sum_{k=1}^{n} n_{ik}c_k \). Then \((a_1,b_1) \in \mathcal{L}_2 \) for \( b_1 = \sum_{k=1}^{n} n_{ik}d_k \).

Taking generators \( \{a_1, \ldots, a_n\} \) for the lattice \( \mathcal{L}_1 \subset \mathbb{R}^n \), with \( a_i = \sum_{k=1}^{n} n_{ik}c_k \), let \( b_1 \sum_{k=1}^{n} n_{ik}d_k \), then \( \{a_i, b_i\}, i = 1, \ldots, n \mathbb{Z} \subset \mathcal{L}_2 \). The proof will be complete if we show that \( \mathcal{L}_2 \subset \{(a_i,b_i), i = 1, \ldots, n\} \mathbb{Z} \). For this it is sufficient to show that \((c_j, d_j) \in \mathcal{L}_2 \) for some \( \beta' \).

By construction, \( c_j = \sum_{i=1}^{n} m_{ji}a_i = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ji}n_{ik}c_k \), hence \((m_{ji}) (n_{ik}) = I_{d_{n \times n}} \).

Finally, if not all \( b_i < b \), we may change the set of generators to \( \{(a_i, b_i), s_i \} \) where \( s_i \) is the largest integer such that \( s_i b < b_i \). This is still a set of generators, with the required properties. \( \square \)

**Theorem 3.1.** The projection \( \mathcal{P}(\mathcal{L} \cap \mathcal{B}_{y_0}) \) of a lattice \( \mathcal{L} \) in \( \mathbb{R}^{n+1} \) has period \( p \in \mathbb{R}^n - \{0\} \) if and only if either \( (p,0) \in \mathcal{L} \) or \((0,b) \) and \((a_1,b_1) \) in \( \mathcal{L} \) with \( m_a a_1 = p , m_b b_1 = b , b_1 \in \mathbb{R} \), \( b > 0 \), and in the second case, for adapted generators \( \{(0,b_0),(a_i,b_i), i = 1, \ldots, n\} \) as in Lemma [3.7] the following diophantine condition holds:

for any \( m_1, \ldots, m_n \in \mathbb{Z} \)

\[
\sum_{i=1}^{n} m_i b_i \in [0,y_0] \pmod{b_0} \quad \Rightarrow \quad m_s b_1 + \sum_{i=1}^{n} m_i b_i \in [0,y_0] \pmod{b_0}
\]  

(3.1)

**Proof.** If \( \mathcal{P}(\mathcal{L} \cap \mathcal{B}_{y_0}) \) is \( p \)-periodic, since \( 0 = \mathcal{P}((0,0)) \in \mathcal{P}(\mathcal{L} \cap \mathcal{B}_{y_0}) \), then \( p \in \mathcal{P}(\mathcal{L} \cap \mathcal{B}_{y_0}) \). Therefore there exists \( q \in [0,y_0] \) such that \( (p,q) \in \mathcal{L} \). If \( q = 0 \) we are in the first case.

If \( q \neq 0 \) and \( 0 < q \leq y_0 \), let \( \tilde{x}, \tilde{y} = (q, nq) \in \mathcal{L} \) where \( n \) is the largest integer such that \( nq \leq y_0 \). Then both \( \tilde{x} + \tilde{p} \in \mathcal{P}(\mathcal{L} \cap \mathcal{B}_{y_0}) \) with \( (\tilde{x} + \tilde{p}, \tilde{y} + q) \in \mathcal{L} \), \( \tilde{y} + q \geq y_0 \) and \( (\tilde{x} + \tilde{p}, \tilde{z}) \in \mathcal{L} \) for some \( \tilde{z} \) satisfying \( 0 \leq \tilde{z} \leq y_0 \). It follows that \((0,b) = (0, \tilde{y} + q - \tilde{z}) \in \mathcal{L} \).

Thus it remains to show that if \((p,0) \notin \mathcal{L} \) and if \((0,b) \) and \((p,q) \in \mathcal{L} \) then \( p \) is a period of \( \mathcal{P}(\mathcal{L} \cap \mathcal{B}_{y_0}) \) if and only if the diophantine condition [3.1] holds for generators \( (0,b_0) \), and \((a_i,b_i), i = 1, \ldots, n \), adapted to \((0,b) \) and \((p,q) \), i.e. with \( n_s b_0 = b \) and \( m_s (a_1,b_1) = (p,q) \), \( n_s, m_s \in \mathbb{Z} \).
A point $x$ is in $\mathcal{P}(\mathcal{L} \cap B_{y_0})$ if and only if $(x, y) \in \mathcal{L}$ with $x = \sum_{i=1}^{n} m_i a_i$ and $y = \sum_{i=1}^{n} m_i b_i \in [0, y_0]$ (mod $b_0$). Since $p = m_i a_i$ then $x + p = \sum_{i=1}^{n} m_i a_i + m_i a_i$. Thus $x + p \in \mathcal{P}(\mathcal{L} \cap B_{y_0})$ if and only if $\tilde{y} = m_i b_i + m_i b_i \in [0, y_0]$ (mod $b_0$).

The conditions of Theorem 3.1 are easy to check in some cases, as in the next result. The period may also be obtained from the symmetries of $\mathcal{L}$.

**Proposition 3.1.** Let $\mathcal{L}$ be a lattice in $\mathbb{R}^{n+1}$ with symmetry group $\Gamma \subset \mathbb{E}(n+1)$. The projection $\mathcal{P}(\mathcal{L} \cap B_{y_0})$ has period $p \in \mathbb{R}^n - \{0\}$ if one of the following conditions holds:

(I) $(p, 0) \in \mathcal{L}$;

(II) $(p, q)$ and $(0, b) \in \mathcal{L}$ for some $q, b \in \mathbb{R}$, $0 < b \leq y_0$;

(III) $\sigma \in \mathcal{J}$ and $(p, q) \in \mathcal{L}$ for some $q \in \mathbb{R}$, $0 < q \leq y_0$.

The set $\mathcal{P}(\mathcal{L} \cap B_{y_0})$ is a lattice in Cases (I) and (III).

The conditions of Proposition 3.1 are illustrated in Figures 1, 2 and 3.

**Proof.** Case (I) follows immediately from Theorem 3.1. For Case (II), the diophantine condition (3.1) always holds because if $0 < b_0 \leq b \leq y_0$ then for every $x \in \mathbb{R}$, $\tilde{x} \in [0, y_0]$ such that $x = \tilde{x}$ (mod $b_0$).

For Case (III), since $\sigma \in \mathcal{J}$ then $\sigma \mathcal{L} \subset \mathcal{L}$ and hence for any $(x, y) \in \mathcal{L}$ we have $(x, y) - \sigma(x, y) = (0, 2y) \in \mathcal{L}$. In particular, $(0, 2y) \in \mathcal{L}$, let $(0, b) \in \mathcal{L}$, $0 < b < 2q$ be the smallest non-zero element in this direction.

If $b \leq y_0$ we are in Case (II). Otherwise, $b = 2q$ and $q \in \{y_0/2, y_0\}$. Let $(x, y) \in \mathcal{L} \cap B_{y_0}$. If $y = 0$ then $(x, y) + (p, q) = (x + p, q) \in \mathcal{L}$ and $x + p \in \mathcal{P}(\mathcal{L} \cap B_{y_0})$. Otherwise, $0 < y < y_0$ and for some $n \in \mathbb{Z}$ we have $2y = n b$ for some $n \in \mathbb{Z}$, hence $y = n y_0$ if and only if $n = 1$ and $y = q$. It follows that $(x, y) + (p, q) = (x + p, 0) \in \mathcal{L}$ hence $p$ is a period.

Note that in Case (III) of Proposition 3.1, from $(p, q) \in \mathcal{L}$ we also get $(2p, 0) = (p, q) + \sigma(p, q) \in \mathcal{L}$ yielding a larger period.

The conditions of Proposition 3.1 are not necessary: if $\mathcal{P}(\mathcal{L} \cap B_{y_0})$ has period $p \in \mathbb{R}^n - \{0\}$ and if $(p, q)$ and $(0, b) \in \mathcal{L}$ with $q > 0$ and $b > y_0$, this does not entail that $\sigma$ is in the holohedry of $\mathcal{L}$, as the following example shows.

Let

$$\mathcal{L} = \{(5, 4), (0, 7)\}_{\mathbb{Z}} \quad y_0 = 6.$$ 

For each $n \in \mathbb{Z}$ there is $m \in \mathbb{Z}$ such that $4n + 7m \in [0, 6]$ because $4n \equiv 0, 1, \ldots, 6$ (mod 7) always has a solution. From this it follows that for each $n \in \mathbb{Z}$ there is $m \in \mathbb{Z}$ such that $n(5, 4) + m(0, 7) \in \mathcal{B}_{y_0}$ and therefore $\mathcal{P}(\mathcal{L} \cap B_{y_0})$ is 5-periodic. On the other hand $(5, -4) \neq \sigma(5, 4) \notin \mathcal{L}$, so $\sigma$ is not in the holohedry of $\mathcal{L}$.

### 3.1. Integral lattices.

**Lattices where all the elements have integer squared length are called integral lattices and are important in the context of quasicrystals and quasiperiodic tilings — see Senechal [17] and Janssen et al [9] for details.**

Given a subset $A \subset \mathbb{R}^{n+1}$ and $y \in \mathbb{R}$ let

$$\mathcal{R}_y(A) = \{x : (x, y) \in A\} = \mathcal{P}(A \cap (\mathbb{R}^n \times \{y\})).$$

Let $y_0 = \max\{y : (x, y) \in V(0)\}$, the height of the Voronoi cell

$$V(0) = \{(x, y) \in \mathbb{R}^{n+1} : ||(x, y) - (p, q)|| \parallel (p, q) \in \mathcal{L}\}.$$ 

The restriction of the projection $\mathcal{P}$ to $\mathcal{L} \cap (B_{y_0} \cup \sigma B_{y_0})$ is called a canonical projection when $\mathcal{R}_0(\mathcal{L}) = \{0\}$.

The conditions in Proposition 3.1 are also necessary in the more restrictive context of a canonical projection of integral lattices, see Senechal [17].

If $\mathcal{L}$ is an integral lattice then the projection of $\mathcal{L} \cap B_{y_0}$ is non-periodic if $\mathcal{R}_0(\mathcal{L}) = \{0\}$, by Proposition 2.17 in Senechal [17]. It follows that in Case (I) of Proposition 3.1 if $\mathcal{R}_0(\mathcal{L}) = \{0\}$ then $\mathcal{L}$ is not an integral lattice. To see this directly, let $(c, d)$ be any element of $\mathcal{L}$ and $(0, a) \in \mathcal{L}$. If $\mathcal{L}$ is an integral lattice then $a^2 \in \mathbb{Z}$ and $\mathcal{L} \subset \mathcal{L}^*$, see Senechal [17], section 2.2. Thus, $(0, a) \cdot (c, d) \in \mathbb{Z}$ which implies $ad = n \in \mathbb{Z}$. Therefore, $a^2(c, d) - n(0, a) = (a^2c, 0) \in \mathcal{L}$ and it follows that the restriction of $\mathcal{L}$ to the subspace $y = 0$ is not the origin alone. In Case (III) of Proposition 3.1 we always have $\mathcal{R}_0(\mathcal{L}) \neq \{0\}$, as we remarked before.

Figures 1, 2 and 3 present some examples of projection of lattices in $\mathbb{R}^2$, illustrating the cases in Proposition 3.1. The intersection of the lattice with the subspace $y = 0$ allows us to compare Proposition 3.1 to results on integral lattices. Figure 4 explains the periodicity of the projected lattice when $(0, y_0) \in \mathcal{L}$. 

Figure 1. Projections of a lattice: If \((p,0) \in \mathcal{L}\) (Case I of Proposition 3.1) then both the projection \(P(\mathcal{L} \cap B_{y_0})\) and the restriction to the line \(y = 0\) have period \(p\).

Figure 2. Projections of a lattice: \((0,y_0) \in \mathcal{L}\) ensures period \(p\) for the projection width \(y_0\) (Case II of Proposition 3.1) even when the restriction to \(y = 0\) is nonperiodic.

Figure 3. Projections of a lattice: The glide reflection on the dashed line (Case III of Proposition 3.1) acts as a translation by \(p\), after the projection for sufficiently large \(y_0\). The restriction to \(y = 0\) and projections of narrower strips have period \(2p\).

Figure 4. If \((0,y_0) \in \mathcal{L}\) then projecting a strip of width \(y_0\) is equivalent to the projection of the cell defined by \((0,y_0)\) and \((p,a)\).

4. Patterns

We identify a pattern to the level set of a function \(f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}\), periodic along \(n + 1\) independent directions.
4.1. \( \Gamma \) acting on functions. The action of \( \Gamma \) in \( \mathbb{R}^{n+1} \) induces the scalar action: \((\gamma \cdot f)(x, y) = f(\gamma^{-1} \cdot (x, y))\) for \( \gamma \in \Gamma \) and \((x, y) \in \mathbb{R}^{n+1}\), see Melbourne [12 section 2.1]. A function \( f \) is \( \Gamma \)-invariant if \((\gamma \cdot f)(x, y) = f(x, y)\), for all \( \gamma \in \Gamma \) and all \((x, y) \in \mathbb{R}^{n+1}\).

4.2. function spaces. We will work in \( X_{\Gamma} \), the vector space
\[
X_{\Gamma} = \{ f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n} \text{ \( \Gamma \)-invariant, of class } C^1 \}
\]
where \( \Gamma \) is a \((n+1)\)-dimensional crystallographic group with lattice \( \mathcal{L} \) and point group \( J \). Since \( \mathcal{L} \) is a subgroup of \( \Gamma \) then any \( f \in X_{\Gamma} \) is \( \mathcal{L} \)-invariant. Thinking of \( \mathcal{L} \) as a subset of \( \mathbb{R}^{n+1} \) this means that any \( f \in X_{\Gamma} \) is \( \mathcal{L} \)-periodic. Thus \( X_{\Gamma} \) is the generalisation, to any dimension, of functions on the plane whose level curves form a periodic tiling.

Consider the waves \( \omega_{k}(x, y) = e^{2\pi i(k,\langle x,y \rangle)} \), where \( k \in \mathbb{Z}^{n+1} \) and \( \langle \cdot, \cdot \rangle \) is the usual inner product in \( \mathbb{R}^{n+1} \).

The set of all \( k \in \mathbb{Z}^{n+1} \) such that \( \omega_{k} \) is a \( \mathcal{L} \)-periodic function is the dual lattice of \( \mathcal{L} \). If the point group \( J \) of \( \Gamma \) is non-trivial, then the waves \( \omega_{k}(x, y) \), \( k \in \mathcal{L}^{*} \) are not \( \Gamma \)-invariant, but a Hilbert basis for \( X_{\Gamma} \) may be obtained from \( \omega_{k} \) by taking \( I_{k}(x, y) = \sum_{\delta \in J} \omega_{\delta k}(x, y) \omega_{\delta k}(-v_{\delta}) \) where \( (v_{\delta}, \delta) \in \Gamma \). Note that this expression does not depend on the choice of \( v_{\delta} \). The functions \( Re(I_{k}), Im(I_{k}) \) lie in \( X_{\Gamma} \) for all \( k \in \mathcal{L}^{*} \), and every \( \Gamma \)-invariant function of class \( C^{1} \) has a Fourier expansion in terms of the functions \( I_{k} \) with the Fourier series of \( f \) converging absolutely and uniformly to \( f \).

4.3. the projection operator. For \( y_{0} > 0 \), consider the restriction of \( f \) to the region between the hyperplanes \( y = 0 \) and \( y = y_{0} \). The projection operator \( \Pi_{y_{0}} \) integrates this restriction of \( f \) along the width \( y_{0} \), yielding a new function with domain \( \mathbb{R}^{n} \):
\[
\Pi_{y_{0}}(f)(x) = \int_{0}^{y_{0}} f(x, y) dy.
\]

The region between \( y = 0 \) and \( y = y_{0} \) is called the projected band or the projection band, and \( y_{0} \) is called the width of projection or the width of the projected band.

Note that since the Fourier series of \( f \) converges absolutely and uniformly to \( f \), it follows that the integral in the projection of \( f \) commutes with the summation in its Fourier series.

4.4. symmetries of projected functions. If \( f \in X_{\Gamma} \) then the projected function \( \Pi_{y_{0}}(f) \) may be invariant under the action of some elements of the group \( E(n) \cong \mathbb{R}^{n} \rtimes O(n) \). Using a notation similar to the \((n+1)\)-dimensional case, \((v_{\alpha}, \alpha) \in E(n) \) is a symmetry of \( \Pi_{y_{0}}(f) \) if
\[
(v_{\alpha}, \alpha) \cdot \Pi_{y_{0}}(f)(x) = \Pi_{y_{0}}(f)(\alpha^{-1} x - v_{\alpha}) = \Pi_{y_{0}}(f)(x) \quad \forall x \in \mathbb{R}^{n}.
\]

4.5. restriction. Let \( \Phi_{r} \) be the operator that restricts the functions to the hyperplane \( y = r \),
\[
\Phi_{r}(f)(x) = f(x, r).
\]
The functions \( \Phi_{r}(f) \) may be invariant under the action of some elements of the group \( E(n) \cong \mathbb{R}^{n} \rtimes O(n) \), as discussed above for the projected functions.

4.6. description of the symmetries of projected functions. The following result from [10] shows how to obtain symmetries of the projected functions \( \Pi_{y_{0}}(f) \in \Pi_{y_{0}}(X_{\Gamma}) \) from the symmetry of the functions \( f \in X_{\Gamma} \).

**Theorem 4.1** ([10]). All functions in \( \Pi_{y_{0}}(X_{\Gamma}) \) are invariant under the action of \((v_{\alpha}, \alpha) \in E(n) \) if and only if one of the following conditions holds:

(a) \((v_{\alpha}, 0, \alpha_{+}) \in \Gamma\),
(b) \((v_{\alpha}, y_{0}, 0, \alpha_{-}) \in \Gamma\),
(c) \((0, y_{0}) \in \mathcal{L} \) and \((v_{\alpha}, y_{1}, 0, \alpha_{+}) \in \Gamma \), for some \( y_{1} \in \mathbb{R} \).
(d) \((0, y_{0}) \in \mathcal{L} \) and \((v_{\alpha}, y_{1}, 0, \alpha_{-}) \in \Gamma \), for some \( y_{1} \in \mathbb{R} \).

A similar result holds for the restriction:

**Theorem 4.2** ([10]). All functions in \( \Phi_{r}(X_{\Gamma}) \) are invariant under the action of \((v_{\alpha}, \alpha) \in E(n) \) if and only if one of the following conditions holds:

(a) \((v_{\alpha}, 0, \alpha_{+}) \in \Gamma\),
(b) \((v_{\alpha}, 2r, \alpha_{-}) \in \Gamma\).
5. The symmetry group of projected functions

Let $\Gamma \subset \mathbf{E}(n + 1)$ be a crystallographic group with lattice $\mathcal{L}$ and point group $\mathbf{J}$. In this section we obtain a complete description of the group $\Gamma \leq \mathbf{E}(n)$ of symmetries shared by all the projected patterns in $\Pi_{y_0}(X_{\Gamma})$, by describing its translation subgroup $\tilde{\mathcal{L}}_{y_0}$ and its orthogonal component. In particular, we describe conditions ensuring that the projections of $\mathcal{L}$-periodic functions are still periodic functions with $n$ linearly independent periods, i.e. conditions ensuring that $\tilde{\mathcal{L}}_{y_0}$ is a lattice with $n$ generators.

We start by showing that the translation subgroup $\tilde{\mathcal{L}}_{y_0}$ of common periods of the projected functions in $\Pi_{y_0}(X_{\Gamma})$, regarded as a $\mathbb{Z}$-module, has at most $n$ generators. We also characterise the situations when $\tilde{\mathcal{L}}_{y_0}$ has exactly $n$ generators and thus is a lattice. This means that the symmetry group of the projected functions is a crystallographic group.

Let $\mathcal{L}_{\sigma} = v_\sigma + \mathcal{L}$ if $(v_\sigma, \sigma) \in \Gamma$ and $\mathcal{L}_{\sigma} = \emptyset$ if $\sigma \notin \mathbf{J}$. In all cases $\tilde{\mathcal{L}}_{y_0}$ is a subset of $\{\mathcal{P}(\mathcal{L}), \mathcal{P}(\mathcal{L}_{\sigma})\}_\mathbb{Z}$, by Theorem 4.1.

**Theorem 5.1.** Let $\Gamma \subset \mathbf{E}(n + 1)$ be a crystallographic group with lattice $\mathcal{L}$ and point group $\mathbf{J}$. Then the group $\tilde{\mathcal{L}}_{y_0}$ of common periods of $\Pi_{y_0}(X_{\Gamma})$, regarded as a $\mathbb{Z}$-module, has at most $n$ generators. If either $(0, y_0) \in \mathcal{L}$ or $\sigma \in \mathbf{J}$, then $\tilde{\mathcal{L}}_{y_0}$ is a lattice.

**Proof.** We use Theorem 4.1 in the case $\alpha = Id_n$. If $(0, y_0) \notin \mathcal{L}$ then $\mathbf{p} \in \tilde{\mathcal{L}}_{y_0}$ if and only if one of the conditions (1) or (2) of Theorem 4.1 holds for $v_\sigma = \mathbf{p}$. Condition (1) means $\mathbf{p} \in \mathcal{R}_0(\mathcal{L})$. Condition (2) means $(\mathbf{p}, y_0) \in \mathcal{L}_{\sigma}$, which is equivalent to $\mathbf{p} \in \mathcal{R}_0(\mathcal{L}_{\sigma})$. Thus

$$ (0, y_0) \notin \mathcal{L} \quad \implies \quad \tilde{\mathcal{L}}_{y_0} = \mathcal{R}_0(\mathcal{L}) \cup \mathcal{R}_0(\mathcal{L}_{\sigma}) $$

This case is treated in Lemma 5.2 below.

If $(0, y_0) \in \mathcal{L}$ and $\mathbf{p} \in \tilde{\mathcal{L}}_{y_0}$ then any one of the conditions of Theorem 4.1 may hold. The first two imply $\mathbf{p} \in \mathcal{R}_0(\mathcal{L}) \cap \mathcal{R}_0(\mathcal{L}_{\sigma}) \subset \mathcal{P}(\mathcal{L}) \cup \mathcal{P}(\mathcal{L}_{\sigma})$ as seen above. The last two imply that $(\mathbf{p}, y_0) \in \mathcal{L} \cap \mathcal{L}_{\sigma}$ for some $y_1 \in \mathbb{R}$ and therefore $\tilde{\mathcal{L}}_{y_0} \subset \mathcal{P}(\mathcal{L}) \cup \mathcal{P}(\mathcal{L}_{\sigma})$. It follows immediately from Theorem 4.1 that

$$ (0, y_0) \in \mathcal{L} \quad \implies \quad \tilde{\mathcal{L}}_{y_0} = \mathcal{P}(\mathcal{L}) \cup \mathcal{P}(\mathcal{L}_{\sigma}). $$

It remains to show that this implies that $\tilde{\mathcal{L}}_{y_0}$ is a lattice in $\mathbb{R}^n$, we treat this case in Lemma 5.1.

**Lemma 5.1.** Let $\Gamma \subset \mathbf{E}(n + 1)$ be a crystallographic group with lattice $\mathcal{L}$. If $(0, y_0) \in \mathcal{L}$ then the group $\tilde{\mathcal{L}}_{y_0}$ of common periods of all functions in $\Pi_{y_0}(X_{\Gamma})$ is a lattice $\tilde{\mathcal{L}}_{y_0} = \mathcal{P}(\mathcal{L}) \cup \mathcal{P}(\mathcal{L}_{\sigma})$.

**Proof.** Let $\alpha = y_0/m$ where $m$ is the largest integer such that $(0, y_0/m) \in \mathcal{L}$. We may write $\mathcal{L} = \{(0, a), l_1, \ldots, l_n\}_\mathbb{Z}$ where the $l_i$, $i = 1, \ldots, n$ are linearly independent over $\mathbb{R}$, therefore $\mathcal{P}(\mathcal{L}_{\sigma})$ are linearly independent generators for $\mathcal{P}(\mathcal{L})$. If $\sigma$ is not in the point group $\mathbf{J}$ of $\Gamma$, then by the result follows. If $((v_1, y_1), \sigma) \in \Gamma$ and if $v_1 = 0 \mod \mathcal{L}$ then $\mathcal{P}(\mathcal{L}_{\sigma}) \subset \mathcal{P}(\mathcal{L})$ and hence $\tilde{\mathcal{L}}_{y_0} = \mathcal{P}(\mathcal{L})$ has $n$ generators. Otherwise,

$$ ((v_1, y_1), \sigma) \cdot ((v_1, y_1), \sigma) = ((2v_1, 0), Id_{n+1}) \in \Gamma \quad \implies \quad (2v_1, 0) \in \mathcal{L} $$

and therefore $2v_1 = 2\mathcal{P}(v_1, y_1) \in \mathcal{P}(\mathcal{L})$. To see that $\tilde{\mathcal{L}}_{y_0}$ is a lattice in this case, note that $((v_1, y_1), \sigma) \in \Gamma$ implies that there exists $v_2 = \sum_{i=1}^{n} s_i P(l_i)$ with $0 \leq s_i < 1$ and $((v_2, y_2), \sigma) \in \Gamma$ for some $y_2 \in \mathbb{R}$, and $2v_2 \in \mathcal{P}(\mathcal{L})$ by (3.3). Hence, each $s_i$ is either $0$ or $1/2$ and not all of them are zero. Without loss of generality, suppose $s_1 = 1/2$, then $v_2 = \frac{1}{2} P(l_1) + \sum_{i=2}^{n} s_i P(l_i)$ and thus $\mathcal{P}(\mathcal{L}) = 2v_2 - \sum_{i=2}^{n} 2s_i P(l_i)$ with $2s_i = 0$, hence $\tilde{\mathcal{L}}_{y_0} = \mathcal{P}(\mathcal{L}) \cup \mathcal{P}(\mathcal{L}_{\sigma}) = \{v_2, \mathcal{P}(l_2), \ldots, \mathcal{P}(l_n)\}_\mathbb{Z}$ and the result follows.

**Lemma 5.2.** Let $\Gamma \subset \mathbf{E}(n + 1)$ be a crystallographic group with lattice $\mathcal{L}$ and point group $\mathbf{J}$ and suppose $(0, y_0) \notin \mathcal{L}$. If $((v_1, y_0), \sigma) \in \Gamma$ then the group $\tilde{\mathcal{L}}_{y_0}$ of common periods of all functions in $\Pi_{y_0}(X_{\Gamma})$ is the lattice $\tilde{\mathcal{L}}_{y_0} = \{\mathcal{R}_0(\mathcal{L}), v_1\}_\mathbb{Z}$. If $((v_1, y_0), \sigma) \notin \Gamma$ then $\tilde{\mathcal{L}}_{y_0} = \mathcal{R}_0(\mathcal{L})$. If $\sigma$ is in the holohedry of $\mathcal{L}$ then $\tilde{\mathcal{L}}_{y_0}$ is a lattice, otherwise $\tilde{\mathcal{L}}_{y_0}$ may have less than $n$ linearly independent generators.

**Proof.** First we show that if $\sigma \mathcal{L} = \mathcal{L}$ then $\mathcal{R}_0(\mathcal{L})$ is a lattice. To see this, note that if $(x, y) \in \mathcal{L}$, then $(x, y) - \sigma(x, y) = (0, 2y) \in \mathcal{L}$. Thus, if $y \neq 0$, we may write

$$ \mathcal{L} = \{(0, b), (a_1, 0), \ldots, (a_k, 0), (a_{k+1}, b/2), \ldots, (a_n, b/2)\}_\mathbb{Z} $$
for some \( k, 1 \leq k \leq n \), with the \( a_i \) linearly independent over \( \mathbb{R} \).

If \( k = n \) the claim is proved. If \( k = n - 1 \) then

\[
\mathcal{L} = \{(0, b), (a_1, 0), \ldots (a_{n-1}, 0), (a_n, b/2)\}_{\mathbb{Z}}
\]

and \((2a_n, 0) \in \mathcal{L}\) hence \(\mathcal{R}_0(\mathcal{L}) = \{a_1, \ldots, a_{n-1}, 2a_n\}\) is a lattice. Otherwise note that for \( i = 1, \ldots, n - k \), and taking \( a_{n+1} = a_{k+1} \) we have

\[
(a_{k+i}, b/2) + (a_{k+i+1}, b/2) - (0, b) = (a_{k+i} + a_{k+i+1}, 0) \in \mathcal{L} \pmod{n-k}
\]

and thus

\[
\mathcal{R}_0(\mathcal{L}) = \{a_1, \ldots, a_k, a_{k+1} + a_{k+2}, \ldots, a_n + a_{k+1}\}_{\mathbb{Z}}
\]

proving the claim that \(\mathcal{R}_0(\mathcal{L})\) is a lattice.

If \((v_1, y_0), \sigma) \in \Gamma\) then

\[
\mathcal{R}_{y_0}(\mathcal{L}_\sigma) = \{x = v_1 + a : (a, b) \in \mathcal{L} \text{ and } y_0 + b = y_0\}_{\mathbb{Z}} = v_1 + \mathcal{R}_0(\mathcal{L}).
\]

If \(v_1 \in \mathcal{R}_0(\mathcal{L})\) then, since \((0, 0) \not\in \mathcal{L}\), it follows from (5.1) that \(\tilde{\mathcal{L}}_{y_0} = \{\mathcal{R}_0(\mathcal{L}), v_1\}_{\mathbb{Z}} = \mathcal{R}_0(\mathcal{L})\). Otherwise \(2v_1 \in \mathcal{R}_0(\mathcal{L})\) by (5.3) and \(\tilde{\mathcal{L}}_{y_0}\) is a lattice as claimed.

If \((0, y_0) \not\in \mathcal{L}\) and \((v_1, y_0), \sigma) \not\in \Gamma\) then \(\tilde{\mathcal{L}}_{y_0} = \mathcal{R}_0(\mathcal{L})\), by (5.1). If \(\sigma\) is in the holohedry of \(\mathcal{L}\) then \(\tilde{\mathcal{L}}_{y_0} = \mathcal{R}_0(\mathcal{L})\) is a lattice, otherwise it may have fewer than \(n\) generators.

In the next results we use the symbol \((v_3, \delta)\) for any element of the coset \(\mathcal{L} \cdot (v, \delta) = \{(l + v, \delta) : l \in \mathcal{L}\}\) for any \(v \in \mathbb{R}^{n+1}\) such that \((v, \delta) \in \Gamma\), i.e., \(v_3\) is the non-orthogonal component of \((v, \delta) \in \Gamma\) defined up to elements of \(\mathcal{L}\).

A direct application of Theorem 4.1 yields:

**Proposition 5.1.** Let \(\Gamma \subset E(n+1)\) be a crystallographic group with lattice \(\mathcal{L}\) and point group \(J\) and let \(\tilde{J}\) be the point group of the symmetry group of \(\Pi_{y_0}(X_{\Gamma})\). Then \(\tilde{J}\) is a subgroup of \(J_0 = \{\alpha : \text{either } \alpha_+ \in J \text{ or } \alpha_- \in J\} \leq O(n)\), satisfying:

1. If \((0, y_0) \in \mathcal{L}\) then \(\tilde{J} = J_0\);
2. If \((0, y_0) \not\in \mathcal{L}\) then \(\tilde{J} = \{\alpha \in J_0 : \text{either } v_{\alpha_+} = (v, 0) \text{ or } v_{\alpha_-} = (v, y_0) \text{ for some } v \in \mathbb{R}^n\}\).

Moreover, for all \(\alpha \in \tilde{J}\) we have either \(v_\alpha = P(v_{\alpha_+})\) or \(v_\alpha = P(v_{\alpha_-})\).

A similar and simpler application of Theorem 1.12 yields a result for the restriction:

**Proposition 5.2.** Let \(\Gamma \subset E(n+1)\) be a crystallographic group with lattice \(\mathcal{L}\) and point group \(J\). Let \(\tilde{\mathcal{L}}\) be the group of symmetries shared by all functions in \(\Phi_r(X_{\Gamma})\). Then the subgroup of translations \(\tilde{\mathcal{L}}\) of \(\Gamma\) is \(\mathcal{L} = \mathcal{R}_0(\mathcal{L}) \cup \mathcal{R}_{2r}(\mathcal{L}_\sigma)\), and the point group \(\tilde{J}\) of \(\tilde{\mathcal{L}}\) is

\[
\tilde{J} = \{\alpha \in J_0 : \text{either } v_{\alpha_+} = (v, 0) \text{ or } v_{\alpha_-} = (v, 2r) \text{ for some } v \in \mathbb{R}^n\}
\]

and for all \(\alpha \in \tilde{J}\) we have either \(v_\alpha = P(v_{\alpha_+})\) or \(v_\alpha = P(v_{\alpha_-})\).

6. **FINDING THE GROUPS THAT PROJECT INTO A GIVEN LATTICE**

From now on we specialise to the case \(n = 2\). Suppose that, for some value of the width \(y_0\), the group \(\tilde{\mathcal{L}}_{y_0}\) of common periods of \(\Pi_{y_0}(X_{\Gamma})\) is a given lattice \(\mathcal{M} \subset \mathbb{R}^2\). We list all the possible lattices \(\mathcal{L}\) that lead to this result for some \(y_0\) and describe what happens for other values of \(y_0\) in each case.

We say that a lattice \(\mathcal{L}_1\) is rationally compatible with another lattice \(\mathcal{L}_2\) if there exists \(m \neq 0 \in \mathbb{Z}\) such that \(m\mathcal{L}_1 \subset \mathcal{L}_2\). One common situation in the context above is that either \(\tilde{\mathcal{L}}_{y_0}\) is rationally compatible with \(\mathcal{M}\) or \(\mathcal{M}\) is rationally compatible with \(\tilde{\mathcal{L}}_{y_0}\).

The lattices \(\mathcal{L}\) are described up to symmetries of the form \(\alpha_+\), where \(\alpha\) is an element of the holohedry of \(\mathcal{M}\). Thus, we address the question of how the set of periods \(\tilde{\mathcal{L}}_{y_0}\) and more generally the group of symmetries \(\tilde{\mathcal{G}}_{y_0}\) of the projected functions, changes with the width \(y_0\) of the projection band, knowing that one of the projected
patterns has the periods of $\mathcal{M}$. We start with a procedure for a general lattice $\mathcal{M} = \{a_0, b_0\}_\mathbb{Z}$ and in the next section we specialise to a hexagonal lattice.

We use the structure of the proof of Theorem 5.1 to obtain special forms for the generators $a$, $b$, $c$ of $\mathcal{L}$ according to the following cases:

**case 1.** $\mathcal{L} \cap \{(0, 0, z) : z \in \mathbb{R}\} = \{(0, 0, 0)\}$. This condition is not compatible with $\sigma \mathcal{L} \subset \mathcal{L}$, as shown in the proof of Lemma 5.2. Hence, by (5.1), for all $y_0$ we have $\tilde{\mathcal{L}}_{y_0} = \mathcal{R}_0(\mathcal{L}) = \mathcal{M}$ and $\mathcal{L} = \{(a_0, 0), (b_0, 0), (c_0, 3)\}_\mathbb{Z}$ for some $c_0 \in \mathbb{R}^2$, $c_3 \in \mathbb{R}$ with $nc_0 \notin \mathcal{M}$ for all $n \in \mathbb{Z}$ and $c_3 \neq 0$.

For all the other cases we need to establish some notation. Let $c = (0, 0, c) \in \mathcal{L}$, minimal in its direction. Then for $y_0 = nc$, $n \in \mathbb{Z}$ we have $\tilde{\mathcal{L}}_{y_0} = \mathcal{P}(\mathcal{L}) \cup \mathcal{P}(\mathcal{L}_0)$ by (5.2). Other values of $y_0$ yield $\tilde{\mathcal{L}}_{y_0} = \mathcal{R}_0(\mathcal{L}) \cup \mathcal{R}_{y_0}(\mathcal{L}_0)$ by (5.1). The cases below correspond to the different situations with respect to $\sigma$.

We may write the generators of $\mathcal{L}$ in the form $a = (\hat{a}, a_3)$, $b = (\hat{b}, b_3)$ and $c = (0, 0, c)$ with $\hat{a}, \hat{b} \in \mathbb{R}^2$, $c > 0$ and $a_3, b_3 \in [0, c)$. Then $\sigma \mathcal{L} \subset \mathcal{L}$ if and only if $a_3, b_3 \in [0, c/2)$.

The number of generators of $\mathcal{R}_0(\mathcal{L})$ depends on metric properties of the generators of $\mathcal{L}$ as follows. Let $D(\mathcal{L})$ be the $\mathbb{Z}$-module

$$D(\mathcal{L}) = \{(m, n) \in \mathbb{Z}^2 : ma_3 + nb_3 = 0 \pmod{c}\}.$$  

Then

$$\mathcal{R}_0(\mathcal{L}) = \left\{m\hat{a} + n\hat{b}, (m, n) \in D(\mathcal{L})\right\}.$$  

Generically, $D(\mathcal{L}) = \{(0, 0)\}$, but in special situations we may have that the $\mathbb{Z}$-module $D(\mathcal{L})$ is either generated by one non-zero element or by two linearly independent elements of $\mathbb{Z}^2$.

**case 2.** $\sigma \mathcal{L} \not\subset \mathcal{L}$, hence either $a_3 \neq 0$ or $b_3 \neq 0$. Then $\tilde{\mathcal{L}}_{y_0} = \mathcal{P}(\mathcal{L})$ for $y_0 = nc$, $n \in \mathbb{Z}$. For other values of $y_0$, we have $\tilde{\mathcal{L}}_{y_0} = \mathcal{R}_0(\mathcal{L})$ and $\tilde{\mathcal{L}}_{y_0}$ is a subperiodic group, except in the special case where $D(\mathcal{L})$ has two generators. There are two possibilities:

**case 2.1** If $\mathcal{P}(\mathcal{L}) = \mathcal{M}$ we may take $a = (a_0, a_3)$, $b = (b_0, b_3)$ and $\tilde{\mathcal{L}}_{y_0} = \mathcal{M}$ for $y_0 = nc$, $n \in \mathbb{Z}$. Then $\tilde{\mathcal{L}}_{y_0}$ is a proper subset of $\mathcal{M}$ for other values of $y_0$.

**case 2.2** If $D(\mathcal{L})$ has two independent generators, we may also have $\mathcal{R}_0(\mathcal{L}) = \mathcal{M}$. Then $\tilde{\mathcal{L}}_{y_0} = \mathcal{M}$ for $y_0 \neq nc$, and $\mathcal{M}$ is a proper subset of $\tilde{\mathcal{L}}_{y_0}$ for $y_0 = nc$.

From now on we assume $\sigma \mathcal{L} \subset \mathcal{L}$, hence we have $a_3, b_3 \in [0, c/2)$, not both equal to $c/2$, as in the proof of Lemma 5.2. Then $D(\mathcal{L})$ always has two independent generators, according to the following table

| $a_3 = 0$ | $b_3 = 0$ | $D(\mathcal{L}) = \mathbb{Z}^2$ | $b_3 = c/2$ | $D(\mathcal{L}) = \{(1, 0), (0, 2)\}_\mathbb{Z}$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| $a_3 = c/2$ | $D(\mathcal{L}) = \{(2, 0), (0, 1)\}_\mathbb{Z}$ | — |

and $\mathcal{R}_0(\mathcal{L})$ is given by:

| $a_3 = 0$ | $b_3 = 0$ | $\mathcal{R}_0(\mathcal{L}) = \{\hat{a}, \hat{b}\}_\mathbb{Z}$ | $b_3 = c/2$ | $\mathcal{R}_0(\mathcal{L}) = \{2\hat{a}, 2\hat{b}\}_\mathbb{Z}$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| $a_3 = c/2$ | $\mathcal{R}_0(\mathcal{L}) = \{2\hat{a}, \hat{b}\}_\mathbb{Z}$ | — |

**case 3.** If $\sigma \mathcal{L} \subset \mathcal{L}$ and $\mathcal{L}_\sigma \subset \mathcal{L}$ (this holds in particular, if $\sigma \notin \mathcal{J}$), then for $y_0 = nc$, $n \in \mathbb{Z}$ we have $\tilde{\mathcal{L}}_{y_0} = \mathcal{P}(\mathcal{L})$. Other values of $y_0$ yield $\tilde{\mathcal{L}}_{y_0} = \mathcal{R}_0(\mathcal{L})$. There are three possibilities:

**case 3.1** If $a_3 = b_3 = 0$ then $D(\mathcal{L}) = \mathbb{Z}^2$, $\mathcal{R}_0(\mathcal{L}) = \mathcal{P}(\mathcal{L}) = \mathcal{M}$ for all $y_0$.

**case 3.2** If either $a_3 \neq 0$ or $b_3 \neq 0$ and if $\tilde{\mathcal{L}}_{y_0} = \mathcal{P}(\mathcal{L}) = \mathcal{M}$ for $y_0 = nc$, $n \in \mathbb{Z}$, then $\hat{a} = a_0$, $\hat{b} = b_0$. For other values of $y_0$, we have $\tilde{\mathcal{L}}_{y_0} = \mathcal{R}_0(\mathcal{L})$ as in the table above.

**case 3.3** If either $a_3 \neq 0$ or $b_3 \neq 0$ and if $\tilde{\mathcal{L}}_{y_0} = \mathcal{R}_0(\mathcal{L}) = \mathcal{M}$ for $y_0 \neq nc$, $n \in \mathbb{Z}$, then either the generators of $\mathcal{L}$ are $(a_0, 0)$ and $\frac{1}{2}(b_0, c)$ and hence for $y_0 = nc$, $n \in \mathbb{Z}$, we have $\tilde{\mathcal{L}}_{y_0} = \mathcal{P}(\mathcal{L}) = \{(a_0, \frac{1}{2}b_0)\}_\mathbb{Z}$, or the generators are $\frac{1}{2}(a_0, c)$ and $(b_0, 0)$ and hence for $y_0 = nc$, $n \in \mathbb{Z}$, we have $\tilde{\mathcal{L}}_{y_0} = \mathcal{P}(\mathcal{L}) = \{\frac{1}{2}a_0, b_0\}_\mathbb{Z}$.

In both cases 3.2 and 3.3, for all values of $y_0$ we obtain that $\tilde{\mathcal{L}}_{y_0}$ is a lattice and that $\mathcal{M}$ is rationally compatible with $\tilde{\mathcal{L}}_{y_0}$ even when $(0, y_0) \notin \mathcal{L}$ and $\sigma \notin \mathcal{J}$. 


From now on we assume \((v_x, \sigma) \in \Gamma\) where \(v_x = (v_1, y_1) \not\in \mathcal{L}\) with \(y_1 \in [0, c)\). By (5.3) it follows that \(2v_1 \in \mathcal{R}_o(\mathcal{L})\). We obtain the cases below, depending on the values of \(a_3, \delta_3\) and \(b_3\). In all cases, if \(\tilde{\mathcal{L}}_{y_0} = \mathcal{M}\) for some \(y_0\), then \(\tilde{\mathcal{L}}_{y_0}\) is rationally compatible with \(\mathcal{M}\) for every \(y_0\).

case 4. If \(a_3 = b_3 = 0\) and \((v_x, \sigma) \in \Gamma\) then \(v_1 \in \{0, \frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}(a + b)\}\), by the arguments in the proof of Lemma 5.1.

**case 4.1.** \(\mathcal{L} = \{(a_0, 0), (b_0, 0), (0, 0, c)\}\).

1. \(v_1 = 0\), then \(\tilde{\mathcal{L}}_{y_0} = \mathcal{M}\) for all \(y_0\).
2. \(v_1 = \frac{1}{2}a_0\) then \(\tilde{\mathcal{L}}_{y_0} = \{\frac{1}{2}a_0, b_0\}\) for \(y_0 = nc\) and \(y_0 = nc + y_1, n \in \mathbb{Z}\); and \(\tilde{\mathcal{L}}_{y_0} = \mathcal{M}\) for other \(y_0\).
3. \(v_1 = \frac{1}{2}b_0\) then \(\tilde{\mathcal{L}}_{y_0} = \{a_0, \frac{1}{2}b_0\}\) for \(y_0 = nc\) and \(y_0 = nc + y_1, n \in \mathbb{Z}\); and \(\tilde{\mathcal{L}}_{y_0} = \mathcal{M}\) for other \(y_0\).
4. \(v_1 = \frac{1}{2}(a_0 + b_0)\) then \(\tilde{\mathcal{L}}_{y_0} = \{a_0, \frac{1}{2}(a_0 + b_0)\}\) for \(y_0 = nc\) and \(y_0 = nc + y_1, n \in \mathbb{Z}\); and \(\tilde{\mathcal{L}}_{y_0} = \mathcal{M}\) for other \(y_0\).

**case 4.2.** \(v_1 = a_0 = \frac{1}{2}a, \mathcal{L} = \{(a_0, 0), (b_0, 0), (0, 0, c)\}\). Then \(\tilde{\mathcal{L}}_{y_0} = \mathcal{M}\) for \(y_0 = nc\) and \(y_0 = nc + y_1, n \in \mathbb{Z}\); and \(\tilde{\mathcal{L}}_{y_0} = \{2a_0, b_0\}\) for other \(y_0\).

**case 4.3.** \(v_1 = b_0 = \frac{1}{2}b\) or \(v_1 = b_0 = \frac{1}{2}(a + b), \mathcal{L} = \{(a_0, 0), (2b_0, 0), (0, 0, c)\}\). Then \(\tilde{\mathcal{L}}_{y_0} = \mathcal{M}\) for \(y_0 = nc\) and \(y_0 = nc + y_1, n \in \mathbb{Z}\); and \(\tilde{\mathcal{L}}_{y_0} = \{a_0, 2b_0\}\) for other \(y_0\).

**case 5.** If \(a_3 = c/2\) and \(b_3 = 0\) and \((v_x, \sigma) \in \Gamma\) then \(v_1 \in \{0, \frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}(a + b)\}\), by the arguments in the proof of Lemma 5.1.

**case 5.1.** \(a = a_0, b = b_0\) with \(v_1 \in \{0, a_0\}\).

1. \(v_1 = 0\), \(\tilde{\mathcal{L}}_{y_0} = \mathcal{M}\) for \(y_0 = nc\) and \(y_0 = nc + \frac{1}{2}c + y_1, n \in \mathbb{Z}\); and \(\tilde{\mathcal{L}}_{y_0} = \{2a_0, b_0\}\) for all other values of \(y_0\).
2. \(v_1 = a_0 = \frac{1}{2}a_0, b = b_0\) with \(v_1 \in \{\frac{1}{2}b_0, a_0\}\), then \(\tilde{\mathcal{L}}_{y_0} = \mathcal{M}\) for \(y_0 = nc\) and \(y_0 = nc + y_1, n \in \mathbb{Z}\); and \(\tilde{\mathcal{L}}_{y_0} = \{2a_0, b_0\}\) for all other values of \(y_0\).
3. \(v_1 = b_0 = \frac{1}{2}b, \tilde{\mathcal{L}}_{y_0} = \mathcal{M}\) for \(y_0 = nc\) and \(y_0 = nc + \frac{1}{2}c + y_1, n \in \mathbb{Z}\); and \(\tilde{\mathcal{L}}_{y_0} = \{a_0, 2b_0\}\) for \(y_0 = nc\) and \(y_0 = nc + \frac{1}{2}c + y_1, n \in \mathbb{Z}\); and \(\tilde{\mathcal{L}}_{y_0} = \{2a_0, \frac{1}{2}b_0\}\) for \(y_0 = nc\) and \(y_0 = nc + \frac{1}{2}c + y_1, n \in \mathbb{Z}\); and \(\tilde{\mathcal{L}}_{y_0} = \{a_0, 2b_0\}\) for all other values of \(y_0\).
4. \(v_1 = \frac{1}{2}a_0 = \frac{1}{2}b_0, \tilde{\mathcal{L}}_{y_0} = \mathcal{M}\) for \(y_0 = nc\) and \(y_0 = nc + \frac{1}{2}c + y_1, n \in \mathbb{Z}\); and \(\tilde{\mathcal{L}}_{y_0} = \{a_0, \frac{1}{2}a_0 + b_0\}\) for \(y_0 = nc\) and \(y_0 = nc + \frac{1}{2}c + y_1, n \in \mathbb{Z}\); and \(\tilde{\mathcal{L}}_{y_0} = \{a_0, 2b_0\}\) for all other values of \(y_0\).
For \( v_1 = 0 \) then \( \tilde{\mathcal{L}}_{y_0} = \{ \frac{1}{2}a_0, b_0 \}_Z \) for \( y_0 = nc \) or \( y_0 = nc + \frac{1}{2}c + y_1 \); and \( \tilde{\mathcal{L}}_{y_0} = \mathcal{M} \) for other values of \( y_0 \) including \( y_0 = nc + y_1, n \in \mathbb{Z} \).

For \( v_1 = \frac{1}{2}a_0 \) then \( \tilde{\mathcal{L}}_{y_0} = \{ \frac{1}{2}a_0, b_0 \}_Z \) for \( y_0 = nc \) or \( y_0 = nc + y_1 \); and \( \tilde{\mathcal{L}}_{y_0} = \mathcal{M} \) for other values of \( y_0 \) including \( y_0 = nc + \frac{1}{2}c + y_1, n \in \mathbb{Z} \).

For \( v_1 = \frac{1}{2}b_0 \) then \( \tilde{\mathcal{L}}_{y_0} = \{ \frac{1}{2}a_0, \frac{1}{2}b_0 \}_Z \) for \( y_0 = nc \); \( \tilde{\mathcal{L}}_{y_0} = \mathcal{M} \) for \( y_0 = nc + \frac{1}{2}c + y_1 \).\( n \in \mathbb{Z} \), and \( \tilde{\mathcal{L}}_{y_0} = \mathcal{M} \) for other values of \( y_0 \).

For \( v_1 = 2a_0 + \frac{1}{2}b_0 \) then \( \tilde{\mathcal{L}}_{y_0} = \{ \frac{1}{2}a_0, \frac{1}{2}b_0 \}_Z \) for \( y_0 = nc \); \( \tilde{\mathcal{L}}_{y_0} = \mathcal{M} \) for \( y_0 = nc + \frac{1}{2}c + y_1 \).\( n \in \mathbb{Z} \); and \( \tilde{\mathcal{L}}_{y_0} = \mathcal{M} \) for other values of \( y_0 \).

The cases above are all the possibilities for \( \tilde{\mathcal{L}}_{y_0} = \mathcal{M} \).

Note that from cases 1, 3.1 and 4 it follows that if \( \mathcal{L} = \{ (a_0, 0), (b_0, 0), c \}_Z \) for some \( c \in \mathbb{R}^3 \) then \( \tilde{\mathcal{L}}_{y_0} = \mathcal{M} \) for all \( y_0 \), except in the cases \((v_1, y_1), \sigma \) \( \in \Gamma \) with \( v_1 \in \{ \frac{1}{2}a_0, \frac{1}{2}b_0, \frac{1}{2}(a_0 + b_0) \} \). In all other cases we get different results for \( \tilde{\mathcal{L}}_{y_0} \) depending on \( y_0 \).

7. Example — projection into a hexagonal lattice

We specialise further, and suppose that, for some value of the width \( y_0 \), we have

\[ \tilde{\mathcal{L}}_{y_0} = \left\{ (1, 0), \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \right\}_Z = \{ h_1, h_2 \}_Z = \mathcal{H} \]

that we call the standard hexagonal lattice. We list all the possible lattices \( \mathcal{L} \) that lead to this result for some \( y_0 \) and describe what happens for other values of \( y_0 \). The lattices are described up to symmetries of the form \( \alpha \$ \) where \( \alpha \) belongs to the holohedry of \( \mathcal{H} \). This particular example is interesting because of its connection to special patterns discussed in [2], [6].

We follow the numbering of cases of Section 6 to identify the lattices \( \mathcal{L} \). The projected lattices \( \tilde{\mathcal{L}}_{y_0} \) appear in Tables 3 and 4 except for cases 2.1, and 2.2, where details are given below.

case 1. \( \mathcal{L} = \{ (1, 0), \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), (c_0, c) \}_Z \) for some non-zero \( c \in \mathbb{R} \) for all \( n \in \mathbb{Z} \) and \( c \in \mathbb{R} \setminus \{0\} \). Then \( \tilde{\mathcal{L}}_{y_0} = \mathcal{H} \) for all \( y_0 \).

case 2. \( \sigma \mathcal{L} \not\subset \mathcal{L} \)

case 2.1. \( \mathcal{L} = \{ (1, 0, a_3), \left( \frac{1}{2}, \frac{\sqrt{3}}{2}, b_3 \right), (0, 0, c) \}_Z \) for some \( c \neq 0 \); and \( \sigma \mathcal{L} \not\subset \mathcal{L} \), hence either \( a_3 \neq 0 \) or \( b_3 \neq 0 \).

Then for \( y_0 = nc, n \in \mathbb{Z} \) we have \( \tilde{\mathcal{L}}_{y_0} = \mathcal{H} \). For other values of \( y_0 \), it is \( \tilde{\mathcal{L}}_{y_0} = \mathcal{R}_0(\mathcal{L}) \not\subset \mathcal{H} \), depending on \( D(\mathcal{L}) = \{ (m_1, m_2) \in \mathbb{Z}^2 : m_1a_3 + m_2b_3 = 0 \pmod{b} \} \).

For instance, if \( D(\mathcal{L}) = \{ (m_1, m_2) \} \) then \( \mathcal{R}_0(\mathcal{L}) = \{ m_1 (1, 0) + m_2 \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \}_Z \).

case 2.2. If \( \sigma \mathcal{L} \not\subset \mathcal{L} \) and \( (0, 0, c) \in \mathcal{L}, c \neq 0 \), is minimal in its direction, then for \( y_0 = nc, n \in \mathbb{Z} \) we have \( \tilde{\mathcal{L}}_{y_0} = \mathcal{P}(\mathcal{L}) \), whereas for other values of \( y_0 \), it is \( \tilde{\mathcal{L}}_{y_0} = \mathcal{R}_0(\mathcal{L}) \). If, moreover, \( D(\mathcal{L}) = \{ (m_1, m_2), (n_1, n_2) \}_Z \) for some linearly independent \( (m_1, m_2), (n_1, n_2) \in \mathbb{Z}^2 \), then we may have \( \mathcal{H} \not\subset \tilde{\mathcal{L}}_{y_0} \) for \( y_0 = nc, n \in \mathbb{Z} \) and \( \tilde{\mathcal{L}}_{y_0} = \mathcal{H} \) for the remaining \( y_0 \). In the first case, \( \mathcal{L} = \{ (\tilde{a}, a_3), (\tilde{b}, b_3), (0, 0, c) \}_Z \) with \( c > 0 \), \( a_3, b_3 \in [0, c) \) and \( \tilde{a}, b \in \mathbb{R}^2 \) satisfy \( m_1 \tilde{a} + m_2 \tilde{b} = h_1 \) and \( n_1 \tilde{a} + n_2 \tilde{b} = h_2 \).

case 3. \( \sigma \mathcal{L} \subset \mathcal{L} \) and \( \mathcal{L}_\sigma \subset \mathcal{L} \).

case 3.1. \( \mathcal{L} = \{ (1, 0, 0), \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), (0, 0, c) \}_Z \).

case 3.2. \( \mathcal{L} = \{ (1, 0, \frac{1}{2}), \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), (0, 0, c) \}_Z \).

case 3.3. \( \mathcal{L} = \{ (\frac{1}{2}, 0, \frac{1}{2}), \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), (0, 0, c) \}_Z \).

case 4. \( \mathcal{L} = \{ (\tilde{a}, 0), (\tilde{b}, 0), (0, 0, 0) \}_Z \) and \( (v_1, y_1), \sigma \) \( \in \Gamma \), with \( v_1 \in \{ 0, \frac{1}{2} \tilde{a}, \frac{1}{2} \tilde{b}, \frac{1}{2}(\tilde{a} + \tilde{b}) \} \).

case 4.1. \( \mathcal{L} = \{ (1, 0, 0), \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), (0, 0, c) \}_Z \).

case 4.2. \( v_1 = (\frac{1}{2}, 0) \), and \( \mathcal{L} = \{ (1, 0, 0), \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), (0, 0, c) \}_Z \).
case 4.3. $v_1 = \left(\frac{1}{2}, \sqrt{3}\right)$, and $\mathcal{L} = \{(1, 0, 0), (1, \sqrt{3}, 0), (0, 0, c)\} \mathbb{Z}$.

case 5. $\mathcal{L} = \left\{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(0, \frac{\sqrt{3}}{2}\right), (0, 0, c)\right\} \mathbb{Z}$ (without loss of generality)
and $((v_1, y_1), \sigma) \in \Gamma$, with $v_1 = \left\{0, \frac{1}{2} \hat{a}, \frac{1}{2} \hat{b}, \frac{1}{2}(\hat{a} + \hat{b})\right\}$.

case 5.1. $\mathcal{L} = \left\{\left(1, 0, \frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), (0, 0, c)\right\} \mathbb{Z}$.
case 5.2. $\mathcal{L} = \left\{\left(1, 0, \frac{\sqrt{3}}{2}\right), (1, \sqrt{3}, 0), (0, 0, c)\right\} \mathbb{Z}$.
case 5.3. $\mathcal{L} = \left\{\left(\frac{3}{2}, -\frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), (0, 0, c)\right\} \mathbb{Z}$.
case 5.4. $\mathcal{L} = \left\{\left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right), (1, \sqrt{3}, 0), (0, 0, c)\right\} \mathbb{Z}$.
case 5.5. $\mathcal{L} = \left\{\left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), (0, 0, c)\right\} \mathbb{Z}$.

The particular lattices $\tilde{\mathcal{L}} \subset \mathbb{R}^2$ that appear in this section are listed in Table 1.

| a       | b       | name/type  | symbol |
|---------|---------|------------|--------|
| (1, 0)  | $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ | hexagonal | $\mathcal{H}$ |
| $\left(\frac{1}{2}, 0\right)$ | $\left(0, \frac{\sqrt{3}}{2}\right)$ | rectangular I | rec I |
| $\left(0, \frac{\sqrt{3}}{2}\right)$ | (1, 0) | rectangular II | rec II |
| $\left(\frac{1}{4}, \frac{\sqrt{3}}{2}\right)$ | $\left(\frac{3}{4}, -\frac{\sqrt{3}}{2}\right)$ | rectangular III | rec III |
| $\left(\frac{1}{4}, -\frac{\sqrt{3}}{2}\right)$ | $\left(-\frac{1}{4}, \frac{\sqrt{3}}{2}\right)$ | rotated rectangular I | $R_{\pi/3}$ rec I |
| $\left(\frac{1}{4}, -\frac{\sqrt{3}}{2}\right)$ | $\left(\frac{3}{4}, \frac{\sqrt{3}}{2}\right)$ | rotated rectangular I | $R_{-\pi/3}$ rec I |

Table 1. Generators $a, b$ of lattices $\tilde{\mathcal{L}}_{y_0} = \{a, b\}_\mathbb{Z}$ appearing as projections. The last two correspond to rotations of rec I by an angle $\pi/3$, in opposite directions.

For each of the cases above, we describe in Tables 2 the lattices $\tilde{\mathcal{L}}_{y_0}$ corresponding to a set of values of the width $y_0$ of the projection band. The relevant sets are

$$A = \{nc : n \in \mathbb{N}\} \quad B = \{y_1 + nc : n \in \mathbb{N}\} \quad C = \{y_1 + \frac{c}{2} + nc : n \in \mathbb{N}\}$$

where the value of $c$ corresponds to the generator $(0, 0, c)$ of $\mathcal{L}$. We indicate by “$\mathcal{O}$” (other values) the set of all values of $y_0 \in \mathbb{R}^+$ outside the union of the other sets in the same column.

| $\tilde{\mathcal{L}}_{y_0} \setminus$ cases | 1 | 2.1 | 2.2 | 3.1 | 3.2 | 3.3 |
|------------------------------------------|---|-----|-----|-----|-----|-----|
| $\mathcal{H}$ | $\mathbb{R}^+$ | $\mathcal{O}$ | $\mathbb{R}^+$ | $\mathcal{O}$ | $\mathcal{O}$ | $\mathcal{O}$ |
| rec I | — | — | — | — | — | $\mathcal{A}$ |
| $2 \ R_{\pi/3}$ rec I | — | — | — | $\mathcal{O}$ | — | — |
| sublattice | — | $\mathcal{O}$ | — | — | — | — |
| superlattice | — | $\mathcal{A}$ | — | — | — | — |

Table 2. Lattices $\tilde{\mathcal{L}}_{y_0}$ in the projection for cases 1–3.3 above, with the conventions of Table 1.

We indicate by “sublattice” a submodule of the hexagonal lattice $\mathcal{H}$, either $\{(0, 0)\}$ or with one or two generators. By “superlattice” we mean a lattice strictly containing $\mathcal{H}$ as a submodule. Also “$2 \ R_{\pi/3}$ rec I” stands for the lattice $\{2a, 2b\}_\mathbb{Z}$ where $a, b$ are generators for “$R_{\pi/3}$ rec I”, the rotated rectangular lattice.
Let \( \mathcal{L}_{\gamma_0} \) be the projection for case 4 above, with the conventions of Table 1. We indicate by \( 2 \) the lattice \( \{2a, 2b\} \) where \( a, b \) are generators for \( \mathcal{L}_{\gamma/3} \), the rotated rectangular lattice, with a similar convention for the other lattices.

\[
\begin{array}{c|cccccc}
\text{cases} & 4.1 & 4.1 & 4.1 & 4.1 & 4.2 & 4.3 \\
\tilde{L}_{\gamma_0} \setminus \nu_1 & (0,0) & \left(\frac{1}{2}, 0\right) & \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) & \left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right) & \left(\frac{1}{2}, 0\right) & \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\
\mathcal{H} & R^+ & \mathcal{O} & \mathcal{O} & \mathcal{O} & A \cup B & A \cup B \\
\text{rec I} & - & A \cup B & - & - & - & - \\
\text{rec II} & - & - & A \cup B & - & O & O \\
\mathcal{R}_{\pi/3} \text{rec I} & - & - & A \cup B & - & - & - \\
\mathcal{R}_{-\pi/3} \text{rec I} & - & - & - & - & O & O \\
\end{array}
\]

Table 3. Lattices \( \tilde{L}_{\gamma_0} \) in the projection for case 4 above, with the conventions of Table 1. We indicate by \( 2 \) the lattice \( \{2a, 2b\} \) where \( a, b \) are generators for \( \mathcal{L}_{\gamma/3} \), the rotated rectangular lattice, with a similar convention for the other lattices.

\[
\begin{array}{c|cccccc}
\text{cases} & 5.1 & 5.1 & 5.2 & 5.5 & 5.3 & 5.3 \\
\tilde{L}_{\gamma_0} \setminus \nu_1 & (0,0) & (0,0) & \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) & \left(\frac{1}{2}, 0\right) & \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) & \left(\frac{1}{2}, 0\right) \\
\mathcal{H} & A \cup C & A \cup B & A & A & C & B \\
\text{rec III} & - & - & O & O & - & - \\
\text{rec II} & - & - & - & - & - & - \\
\mathcal{R}_{\pi/3} \text{rec I} & - & - & - & - & A & A \\
\text{rec I} & - & - & - & - & C & B \\
\mathcal{R}_{-\pi/3} \text{rec I} & - & - & B & C & O & O \\
\end{array}
\]

Table 4. Lattices \( \tilde{L}_{\gamma_0} \) in the projection for cases 5.1–5.3 above, with the conventions of Table 1. We indicate by \( 2 \) the lattice \( \{2a, 2b\} \) where \( a, b \) are generators for \( \mathcal{L}_{\gamma/3} \), the rotated rectangular lattice, with a similar convention for the other lattices.

\[
\begin{array}{c|cccccc}
\text{cases} & 5.4 & 5.4 & 5.5 & 5.5 & 5.5 & 5.5 \\
\tilde{L}_{\gamma_0} \setminus \nu_1 & \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) & \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) & (0,0) & \left(\frac{1}{2}, 0\right) & \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) & \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\
\mathcal{H} & B & C & O & O & O & O \\
\frac{1}{2} \mathcal{H} & - & - & - & - & A & A \\
\text{rec I} & A & A & A \cup C & A \cup B & - & - \\
\text{rec II} & O & O & - & - & - & - \\
\mathcal{R}_{\pi/3} \text{rec I} & - & - & - & - & B & C \\
\mathcal{R}_{-\pi/3} \text{rec I} & - & - & - & - & C & B \\
\end{array}
\]

Table 5. Lattices \( \tilde{L}_{\gamma_0} \) in the projection for cases 5.4 and 5.5 above, with the conventions of Table 1. We indicate by \( 2 \) the lattice \( \{2a, 2b\} \) where \( a, b \) are generators for \( \mathcal{H} \), with a similar convention for the other lattices.

### 8. Discussion of the example

For most lattices, only some special projection widths yield a hexagonal lattice. The cases where for all the projection widths there is a three-fold rotation in the group of \( \Pi_{\gamma_0}(X_\Gamma) \) have been discussed in [14]. These correspond to the primitive, the body-centered and face-centered cubic lattices, the rhombohedral lattice and the hexagonal lattice in 3 dimensions, always in special positions. In this section we compare this to our results.

Inspection of Tables 1 and 2 shows that the projection is the hexagonal lattice for all the band widths in cases 1 and 3.1, in case 4.1 with \( \nu_1 = (0,0) \), and in some situations in case 2.

The three dimensional hexagonal lattice corresponds to case 3.1, if the point group of \( \Gamma \) does not contain \( \sigma \), it corresponds to case 4.1 with \( \nu_1 = (0,0) \) if \( \sigma \in J \).

The other four possibilities correspond to special situations in case 2, where the set \( D(\mathcal{L}) \) has two independent generators, so \( \mathcal{R}_0(\mathcal{L}) \) is a two-dimensional hexagonal lattice. Details are given in Tables 3 and 4.
**Table 6.** Lattices in case 2 that project into a hexagonal lattice for all projection widths

| L lattice       | \( \mathcal{L} \) generators | D(\( \mathcal{L} \)) generators |
|-----------------|--------------------------------|----------------------------------|
| rhombohedral    | \((1, 0, 1), \left(\frac{1}{2}, -\frac{3\sqrt{3}}{2}, 1\right), (0, 0, 3)\) | \((2, 1), (1, 2)\) |
| primitive cubic | \((1, 0, 0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{3\sqrt{3}}{2}\right), (0, 0, \frac{\sqrt{3}}{2})\) | \((0, 3), (1, 3)\) |
| body centered cubic | \((1, 0, 0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right), (0, 0, \frac{\sqrt{3}}{2})\) | \((0, 3), (1, 3)\) |
| face centered cubic | \((1, 0, 0), \left(\frac{1}{2}, \frac{3\sqrt{3}}{6}, \frac{\sqrt{3}}{3}\right), (0, 0, \sqrt{6})\) | \((0, 3), (1, 3)\) |

**Table 7.** In case 2 we have \( \sigma \mathcal{L} \not\subset \mathcal{L} \), hence \( \tilde{\mathcal{L}}_{y_0} = \mathcal{P}(\mathcal{L}) \) if \((0, y_0) \in \mathcal{L}\) and \( \tilde{\mathcal{L}}_{y_0} = \mathcal{R}_0(\mathcal{L}) \) otherwise. In these special cases, projection always yields a hexagonal lattice, but the size of the cell varies.

| L lattice       | \( \mathcal{P}(\mathcal{L}) \) generators | \( \mathcal{R}_0(\mathcal{L}) \) generators |
|-----------------|--------------------------------|----------------------------------|
| rhombohedral    | \((1, 0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\) | \(\left(\frac{3}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\) |
| all cubic       | \(\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right), \left(\frac{1}{2}, -\frac{\sqrt{3}}{6}\right)\) | \((1, 0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\) |

**Figure 5.** Level sets for projections of functions periodic under the triclinic lattice \( \mathcal{L}_T \) without symmetry (case I), for \( y_0 = \frac{\sqrt{2}}{14} \). Left to right: level sets for the projections of \( \omega_{k_2} \), of \( \omega_{k_3} \) and of \( \omega_{k_2} + 10\omega_{k_3} \). The only common symmetry of the patterns is the \( \tilde{\mathcal{L}}_{y_0} \)-periodicity.

The triclinic lattices in case 1 do not appear on the list in [14] because their symmetry group does not contain any rotation. Generically, the holohedry of a lattice in case 1 is trivial. An example of projection of invariant functions is given in Figure 5 for

\[
\mathcal{L}_T = \left\{(1, 0, 0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), (1, 1, 1)\right\}_z
\]

with

\[
\mathcal{L}^*_T = \left\{k_1 = (0, 0, 1), k_2 = \left(0, \frac{2\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}\right), k_3 = \left(1, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} - 1\right)\right\}_z.
\]

The lattice of periods of the projected functions has holohedry \( D_6 \), but the \( \frac{2\pi}{6} \) rotation is not a symmetry of the projected functions.

In the special case when \( c_0 \) is parallel to a mirror of \( \mathcal{H} \), the holohedry of a triclinic lattice in case I may contain a reflection. An example is the lattice

\[
\mathcal{L}_S = \left\{(1, 0, 0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), \left(\frac{\sqrt{2}}{2}, 0, 1\right)\right\}_z
\]
Figure 6. Level sets for projections of functions with reflection symmetry and periodic under the triclinic lattice with symmetry $L_S$ (case I), for $y_0 = \frac{\sqrt{2}}{4}$. Left to right: level sets for the projections of $f_{\ell_3} = \omega_{\ell_3} + \omega_{\gamma y_{\ell_3}}$, of $f_{\ell_2} = \omega_{\ell_2} + \omega_{\gamma y_{\ell_2}}$, and of $f_{\ell_2} + f_{\ell_3}$. The patterns are symmetric under reflection on the horizontal axis, but not under rotation by $\frac{2\pi}{6}$.

with

$$L_S = \left\{ \ell_1 = (0,0,1), \ell_2 = \left(0, \frac{2\sqrt{3}}{3}, 0\right), \ell_3 = \left(1, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{2}}{2}\right) \right\} \mathbb{Z},$$

that is symmetric under the reflection $\gamma_y$ on the plane orthogonal to $(0,1,0)$. The $L_S$-periodic functions $f_{\ell_3}(x,y,z) = \omega_{\ell_3}(x,y,z) + \omega_{\gamma y_{\ell_3}}(x,y,z)$, and $f_{\ell_2}(x,y,z) = \omega_{\ell_2}(x,y,z) + \omega_{\gamma y_{\ell_2}}(x,y,z)$ are also $\gamma_y$-invariant, and hence their projections are also invariant for the reflection on the $(x,0)$ axis, as in Figure 6.

The projections of triclinic lattices above are good illustrations of the fact that the symmetries of the lattice of periods are not necessarily symmetries of the pattern. Projected patterns with period lattice $H$ and with $D_6$-symmetry are exhibited in [2] together with a general method for finding them.

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