REVISITING THE REPRESENTATION THEOREM OF
FINITE DISTRIBUTIVE LATTICES
WITH PRINCIPAL CONGRUENCES.
A PROOF-BY-PICTURE APPROACH

G. GRÄTZER AND H. LAKSER

Abstract. A classical result of R. P. Dilworth states that every finite distributive lattice \( D \) can be represented as the congruence lattice of a finite lattice \( L \). A sharper form was published in G. Grätzer and E. T. Schmidt in 1962, adding the requirement that all congruences in \( L \) be principal. Another variant, published in 1998 by the authors and E. T. Schmidt, constructs a planar semimodular lattice \( L \). In this paper, we merge these two results: we construct \( L \) as a planar semimodular lattice in which all congruences are principal. This paper relies on the techniques developed by the authors and E. T. Schmidt in the 1998 paper.

1. Introduction

Let us start with the classical result of R. P. Dilworth from 1942 (see the book \([1]\) for background information):

**Theorem 1.** Every finite distributive lattice \( D \) can be represented as the congruence lattice of a finite lattice \( L \).

A sharper form was published in G. Grätzer and E. T. Schmidt \([10]\) (see also Theorem 8.5 in \([5]\)). The new idea was the use of standard ideals, see G. Grätzer \([2]\) and G. Grätzer and E. T. Schmidt \([9]\).

**Theorem 2.** Every finite distributive lattice \( D \) can be represented as the congruence lattice of a finite relatively complemented lattice \( L \).

All congruences are principal in a finite relatively complemented lattice \( L \). So we obtain the following variant of Theorem \([2]\)

**Theorem 3.** Every finite distributive lattice \( D \) can be represented as the congruence lattice of a finite lattice \( L \) in which all congruences are principal.

G. Grätzer, H. Lakser, and E. T. Schmidt \([8]\) proved another variant of Theorem \([2]\)

**Theorem 4.** Let \( D \) be a finite distributive lattice. Then there exists a planar semimodular lattice \( L \) with \( \text{Con} \ L \) isomorphic to \( D \).

In this note, we combine Theorem \([3]\) and \([4]\) using the techniques developed for Theorem \([4]\)
Theorem 5. Every finite distributive lattice $D$ can be represented as the congruence lattice of a planar semimodular lattice $L$ in which all congruences are principal.

There are other aspects of these constructions discussed in the book [5], for instance, the size of $L$. The constructions in Theorems 1, 2, and 5 are “large” (exponential), in Theorem 4 they are small (cubic polynomial).

There are related results in G. Grätzer and H. Lakser [6] and [7].

Outline. For a formal proof of Theorem 5, we need the formal proof of Theorem 4, as presented in G. Grätzer, H. Lakser, and E. T. Schmidt [8]. There are two obvious solutions: copy the formal proof from [8] (making the editor unhappy) or require that the reader be familiar with the paper [8] (making the reader unhappy). So we choose the middle ground, we present a Proof-by-Picture (as defined in [5]) of Theorem 4.

We do this in Section 2 and complete the proof of Theorem 5 in Section 3.

Notation. We use the notation as in [5]. In particular, for the ordered sets $P$ and $Q$, we can form the (ordinal) sum, $P + Q$ and the glued sum $P ∔ Q$, as illustrated in Figure 1. Observe that the glued sum $P ∔ Q$ requires that $P$ has a unit and $Q$ has a zero (which are identified).

Coloring of a finite lattice $L$ attaches a join-irreducible congruence to an edge (covering interval) of $L$ generating it, see Figures 2–4 for examples.

2. Proof-by-Picture of Theorem 4

We start constructing the planar semimodular lattice $L$ of Theorem 5 for the distributive lattice $D$ and the ordered set $P = J(D)$ of Figure 2, with the three lattices, the planar semimodular lattices $N$ (for Nondistributive), $S$ (for Square), and $R$ (for Rectangle). We glue them together and add some covering $M_3$-s, to obtain $L$, as sketched in Figure 5.

In Steps 1–4, we assume that $P$ has no isolated elements, that is, for every $x ∈ P$, there is a $y ∈ P$ with $x < y$ or $y < x$.

Step 1: Constructing $N$. Take the eight-element, planar, semimodular lattice $S_8$ of Figure 3. We take three copies, $S_8(a, b)$, $S_8(b, c)$, $S_8(d, c)$, one for every covering pair in $P = J(D)$. Let $E = C_2 × C_3$. We glue these together (preserving the colors!) as in Figure 4. More precisely, we glue $S_8(b, c)$ to $E$, and glues $S_8(d, c)$ to the top left boundary of $E$. Then we glue $D$ to this lattice twice and glue $S_8(a, b)$ to the top. We denote by $N_1$ and $N_2$ the lower right and the upper right boundaries of $N$, respectively.

Step 2: Constructing $S$. We form $N_2$. In every covering square of the main vertical diagonal, we add an element to make it an $M_3$, forming the lattice $S$, see Figure 4. We denote by $S_1$ and $S_2$ the lower left and lower right boundaries of $S$, respectively. This will make a copy of the colors $b$ and $c$ in $S_2$, making them available for the $M_3$ insertions in Step 4b.

Step 3: Constructing $R$. Let the chain $C_1$ be isomorphic to $N_1 + S_1$. We choose a chain $C$ of length four and color the edges with $\{a, b, c, d\}$ (in any order). Define $R = C × C_1$. We denote by $R_1$, $R_2$, and $R_1'$ the lower right, lower left, and upper left boundaries of $R$, respectively.

Step 4: Constructing $L$.

Step 4a: Gluing $N$, $S$, and $R$. We glue $N$ and $S$ by identifying $N_2$ with $S_2$ (preserving colors!); we call this lattice $L_1$. Then we glue $L_1$ and $R$ by identifying
$R'_1$ with the lower right boundary of $L_1$ (preserving colors!); let $L_2$ be the lattice we obtain.

Step 4b: Adding $M_3$-s to $L_2$. Every color $x$ occurs in $N_1 + S_1 = R'_1$ as the color of an edge. If $x$ is not a maximal element in $P$, then $x$ occurs in $N_1$ as the color of an edge (maybe many times). If $x$ is a maximal element in $P$, then $x$ occurs in $S_1$ as the color of an edge (maybe many times), so $x$ occurs in $S_2$ as the color of an edge, and therefore also in $R'_1$.

So in the grid $R$, we take a “covering row” and a “covering column” hitting $R'_1$ and $R_2$ in edges of color $x$, see Figure 5. They determine a covering square to which we add an element to obtain an $M_3$. We do this for all covering squares given by a covering row and a covering column both colored by $x$, thereby identifying all the principal congruences determined by a prime interval colored by $x$.

We repeat this for every color $x$.

The $S_8(u, v)$ sublattices then determine the desired order on the join-irreducible congruences—see Figure 3.

Step 5: Adding the tail. If there are $k > 0$ isolated elements, we form $C_k + L$; the tail is $C_k$. This completes the Proof-by-Picture of Theorem 4.

3. Proving Theorem 5

We have to modify the construction of the planar semimodular lattice $L$ of Section 2 to make all congruences principal. In Step 3, we choose a chain $C$ of length four. Observe that the proof of Theorem 4 remains valid as long as every color is represented as the coloring of $C$.

Now we change the definition of $C$. For every $x \in D$, define

$$r(x) = \{ a \in J(D) \mid x \leq a \},$$

and let $C_x$ be a chain of $|r(x)| + 1$ elements, colored by the elements of $r(x)$ (in any order). Let $0_x, 1_x$ denote the bounds of $C_x$. Let $C$ be the glued sum of the chains $C_x$ for $x \in D$ (in any order). This chain $C$ obviously satisfies the condition that every color is represented as the color of an edge in $C$.

Therefore, the lattice $L$ constructed in Section 2 satisfies the requirements of Theorem 4. We only have to observe that all congruences are principal.

Let $\alpha$ be a congruence of $L$. Let $x$ be an element of $D$ that corresponds to $\alpha$ under an isomorphism between Con $L$ and $D$. Since $C_x$ is colored by the set $r(x)$, we conclude that in $L$, we have

$$\text{con}(0_x, 1_x) = \alpha,$$

completing the proof.
Figure 1. Glued sum of two ordered sets, $P$ and $Q$

Figure 2. The lattice $D$ to represent and the ordered set $P = J(D)$

Figure 3. Two diagrams of the building block $S_8(u, v)$, $u \prec v$
Figure 4. The lattices $N$ and $S$

Figure 5. A sketch of $L$ without the “tail”
References

[1] The Dilworth Theorems. Selected papers of Robert P. Dilworth. Edited by Kenneth P. Bogart, Ralph Freese, and Joseph P. S. Kung. Contemporary Mathematicians. Birkhäuser Boston, Inc., Boston, MA, 1990. DOI 10.1016/0001-8708(92)90026-h

[2] G. Grätzer, Standard ideals. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 9 (1959), 81–97 (Hungarian).

[3] G. Grätzer, Lattice Theory: Foundation. Birkhäuser Verlag, Basel, 2011.

[4] G. Grätzer, The order of principal congruences of a bounded lattice. Algebra Universalis 70 (2013), 95–105. DOI 10.1007/s00012-013-0242-3

[5] G. Grätzer, The Congruences of a Finite Lattice. A "Proof-by-Picture" Approach. Second edition. Birkhäuser Verlag, Basel, 2016. DOI 10.1007/0-817

[6] G. Grätzer and H. Lakser, Some preliminary results on the set of principal congruences of a finite lattice. Algebra Universalis 79 (2018) no. 2, paper no. 21. DOI 10.1007/s00012-018-0487-y

[7] G. Grätzer and H. Lakser, Minimal representations of a finite distributive lattice by principal congruences of a lattice. Acta Sci. Math. (Szeged) 85 (2019), 69–96. DOI 10.14232/actasm-017-060-9

[8] G. Grätzer, H. Lakser, and E. T. Schmidt, Congruence lattices of finite semimodular lattices. Canad. Math. Bull. 41 (1998), 290–297. DOI 10.4153/cmb-1998-041-7

[9] G. Grätzer and E. T. Schmidt, Standard ideals in lattices. Acta Math. Acad. Sci. Hungar. 12 (1961), 17–86.

[10] G. Grätzer and E. T. Schmidt, On congruence lattices of lattices. Acta Math. Acad. Sci. Hungar. 13 (1962), 179–185.

Email address, G. Grätzer: gratzer@me.com
URL, G. Grätzer: http://server.maths.umanitoba.ca/homepages/gratzer/

Email address, H. Lakser: hlakser@gmail.com

Department of Mathematics, University of Manitoba, Winnipeg, MB R3T 2N2, Canada