DISCRETE PHASE SPACE AND MINIMUM-UNCERTAINTY STATES

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The quantum state of a system of \( n \) qubits can be represented by a Wigner function on a discrete phase space, each axis of the phase space taking values in the finite field \( \mathbb{F}_{2^n} \). Within this framework, we show that one can make sense of the notion of a “rotationally invariant state” of any collection of qubits, and that any such state is, in a well defined sense, a state of minimum uncertainty.

1. INTRODUCTION

A quantum state cannot be squeezed down to a point in phase space. But there are quantum states that closely approximate classical states, such as the coherent states of a harmonic oscillator. One characterization of the coherent states is based on the Wigner function: they are the only states for which the Wigner function is both strictly positive and rotationally symmetric around its center (here we assume a specific scaling of the axes appropriate for the given oscillator).

One can also express the quantum mechanics of discrete systems in terms of phase space. In this paper we consider a system of \( n \) qubits described in the framework of Ref. [1], in which the discrete phase space can be pictured as a \( 2^n \times 2^n \) array of points. In this framework, the discrete Wigner function preserves the tomographic feature of the usual Wigner function, but the points of the discrete phase space are defined abstractly and do not come with an immediate physical interpretation. As in the continuous case, a point in discrete phase space is illegal as a quantum state: it holds too much information. But one can ask whether there are quantum states that, like coherent states, approximate a phase-space point as closely as possible. We would like to identify such states and thereby to give more physical meaning to the discrete phase space. In this paper we focus primarily on the second of the two properties mentioned above: invariance under rotations. We will see that one can make sense of this notion in the discrete space and that rotationally invariant states exist for any number of qubits.

The most interesting property of these states is that they minimize uncertainty in a well defined sense. The product \( \Delta q \Delta p \), where \( q \) and \( p \) are position and momentum, has no meaning in our setting because our variables have no natural ordering. We therefore express uncertainty in information-theoretic terms, specifically in terms of the Rényi entropy of order 2 (which we call simply “Rényi entropy” for short). Moreover we consider not just the “axis variables,” but also variables associated with all the other directions in the discrete phase space. (In the continuous case these other directions would be associated with linear combinations of \( q \) and \( p \).) We will find that each rotationally invariant state minimizes the Rényi entropy, averaged over all these variables. This will leave us with the question of picking out a “most pointlike” of the rotationally invariant states, if such a notion can be made meaningful; we address this question briefly in the conclusion.
2. DISCRETE PHASE SPACE

Over the years there have been many proposals for generalizing the Wigner function to discrete systems. (See, for example, Refs. [2, 3] and papers cited in Ref. [1].) Here we adopt the discrete Wigner function proposed by Gibbons et al. [1], which is well suited to a system of qubits. The basic idea is to use, instead of the field of real numbers in which position and momentum normally take their values, a finite field with a number of elements equal to the dimension $d$ of the state space. There exists a field with $d$ elements if and only if $d$ is a power of a prime; so this approach applies directly only to quantum systems, such as a collection of qubits, whose state-space dimension is such a number.

The two-element field $\mathbb{F}_2$ is simply the set $\{0, 1\}$ with addition and multiplication mod 2, but the field of order $2^n$ with $n$ larger than 1 is different from arithmetic mod $2^n$. For example, $\mathbb{F}_4$ consists of the elements $\{0, 1, \omega, \omega + 1\}$, in which 0 and 1 act as in $\mathbb{F}_2$ and arithmetic involving the abstract symbol $\omega$ is determined by the equation $\omega^2 = \omega + 1$.

The discrete phase space for a system of $n$ qubits is a two-dimensional vector space over $\mathbb{F}_{2^n}$; that is, a point in the phase space can be expressed as $(q, p)$, where $q$ and $p$, the discrete analogues of position and momentum, take values in $\mathbb{F}_{2^n}$. In this phase space it makes perfect sense to speak of lines and parallel lines; a line, for example, is the solution to a linear equation. The key idea in constructing a Wigner function is to assign a pure quantum state, represented by a one-dimensional projection operator $Q(\lambda)$, to each line $\lambda$ in phase space. The only requirement imposed on the function $Q(\lambda)$ is that it be “translationally covariant.” This means that if we translate the line $\lambda$ in phase space by adding a fixed vector $(q, p)$ to each point, the associated quantum state changes by a unitary operator $T(q, p)$ associated with $(q, p)$. The unitary translation operator $T(q, p)$ is defined to be

$$T(q, p) = X^{q_1}Z^{p_1} \otimes \cdots \otimes X^{q_n}Z^{p_n},$$

where $X$ and $Z$ are Pauli operators and $q_i$ and $p_i$, which are elements of $\mathbb{F}_2$, are components of $q$ and $p$ when they are expanded in particular “bases” for the field: e.g., $q = q_1b_1 + \cdots + q_nb_n$, where $(b_1, \ldots, b_n)$ is the basis chosen for the coordinate $q$. One finds that the requirement of translational covariance severely constrains the construction:

1. States assigned to parallel lines must be orthogonal. A complete set of parallel lines, or “striation,” consists of exactly $d$ lines; so the states associated with a given striation constitute a complete orthogonal basis for the state space. In other words, each striation is associated with a complete orthogonal measurement on the system.

2. The bases associated with different striations must be mutually unbiased. That is, each element of one basis is an equal-magnitude superposition of the elements of any of the other bases. There are exactly $d + 1$ striations, so this construction generates a set of $d + 1$ mutually unbiased bases. (Such a set is just sufficient for the complete tomographic reconstruction of an unknown quantum state.)

Despite these constraints, there are many allowed functions $Q(\lambda)$. This implies that there are many possible definitions of the Wigner function for a system of qubits, because

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The bases for $q$ and $p$ cannot be chosen independently: each must be proportional to the dual of the other [1].
once we have chosen a particular assignment of quantum states to phase-space lines, the Wigner function of any quantum state is uniquely fixed by the requirement that the sums over the lines of any striation be equal to the probabilities of the outcomes of the corresponding measurement.

3. ROTATIONALLY INVARIANT STATES

In the finite field, consider a quadratic polynomial \( x^2 + ax + b \) that has no roots. Then the equation

\[
q^2 + aqp + bp^2 = c,
\]

with \( c \) taking all nonzero values in \( \mathbb{F}_{2^n} \), defines what we will call a set of “circles” centered at the origin. Fixing the values of \( a \) and \( b \)—this is somewhat analogous to fixing the scales of the axes in the continuous case—we define a rotation to be any linear transformation of the phase space that leaves each circle invariant.\(^2\) (We consider only rotations around the origin. A state centered at the origin can always be translated to another point by \( T(q,p) \).)

For example, in the two-qubit phase space, our circles can be defined by the equation

\[
q^2 + qp + \omega p^2 = c,
\]

and an example of a rotation is the transformation \( R \) defined by

\[
\begin{pmatrix}
q' \\
p'
\end{pmatrix} = R \begin{pmatrix}
q \\
p
\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \omega + 1 & \omega \end{pmatrix} \begin{pmatrix} q \\
p
\end{pmatrix}.
\]

One can check that this particular rotation has the property that if we apply it repeatedly, starting with any nonzero vector, it generates the entire circle on which that vector lies. In this sense \( R \) is a primitive rotation.

With every unit-determinant linear transformation \( L \) on the phase space, one can associate (though not uniquely) a unitary transformation \( U \) on the state space whose action by conjugation on the translation operators \( T_{(q,p)} \) mimics the action of \( L \) on the corresponding points of phase space.\(^3\) One can show that every rotation has unit determinant and must therefore have an associated unitary transformation. For example, for the rotation \( R \) given above, if we expand both \( q \) and \( p \) in the field basis \((b_1, b_2) = (\omega, \omega + 1)\), the following unitary transformation acts in the desired way on the translation operators:

\[
U = \frac{1}{2} \begin{pmatrix}
1 & i & i & -1 \\
i & 1 & -1 & i \\
i & 1 & -i & 1 \\
-i & -1 & -1 & i
\end{pmatrix}.
\]

Thus just as

\[
R \begin{pmatrix} 1 \\
0
\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \omega + 1 & \omega \end{pmatrix} \begin{pmatrix} 1 \\
0
\end{pmatrix} = \begin{pmatrix} 1 \\
\omega + 1
\end{pmatrix},
\]

\(^2\)A different notion of rotation has been used in Ref. [4].

\(^3\)The argument in Appendix B.3 of Ref. [1] contains an error: Eqs. (B24) and (B25) implicitly assume that the chosen field basis is self-dual, which is not in fact the case. However, the proof can be repaired by starting with a self-dual basis to get those equations, and then changing to the actual basis via the argument of Appendix C.1. That there exists a self-dual basis for \( \mathbb{F}_{2^n} \) is proved in Ref. [5].
we have that
\[ UT_{(1,0)}U^\dagger = U(X \otimes X)U^\dagger = iX \otimes (XZ) \propto T_{(1,\omega+1)}. \] (7)

For any number \( n \) of qubits, let \( R \) be a primitive rotation, and let \( U \) be a unitary transformation associated with \( R \) in the above sense. (Techniques for finding \( U \) can be found in Refs. [1, 5].) Then from the action of \( U \) on the translation operators, it follows that \( U \) acts in a particularly simple way on the mutually unbiased bases associated with the striations of phase space: starting with any one of these bases, repeated applications of \( U \) generate all the other bases cyclically. That there always exists a unitary \( U \) generating a complete set of mutually unbiased bases for \( n \) qubits has been shown by Chau [5]. In our present context, we will reach the same conclusion by showing, in the following paragraph, that there always exists a primitive rotation. The existence of such a unitary matrix \( U \) leads naturally to a simple prescription for choosing the function \( Q(\lambda) \): (i) Use the translation operators to assign computational basis states to the vertical lines. (ii) Apply \( U \) repeatedly to these states, and \( R \) repeatedly to the lines, in order to complete the correspondence. This prescription results in a definition of the Wigner function that is “rotationally covariant,” in the sense that when one transforms the density matrix by \( U \), the values of the Wigner function are permuted among the phase-space points according to \( R \).

How does one find a primitive rotation \( R \)? First, for any number of qubits, there always exists a primitive polynomial of the form \( x^2 + x + b \) [7], which one can use to define circles by the equation \( q^2 + qp + bp^2 = c \). Then the linear transformation
\[ L = \begin{pmatrix} 1 & b \\ 1 & 0 \end{pmatrix} \] (8)
is guaranteed to cycle through all the nonzero points of phase space [8], and it always takes circles to other circles. Raising \( L \) to the power \( d - 1 \) gives us a unit-determinant transformation that preserves circles and is indeed a primitive rotation. Moreover, one can write \( R \) explicitly in terms of \( b \):
\[ R = L^{d-1} = \begin{pmatrix} 1 & 1 \\ b^{-1} & b^{-1} + 1 \end{pmatrix}. \] (9)

With \( Q(\lambda) \) chosen in the way we have prescribed, the eigenstates of \( U \) are our rotationally invariant states. When we apply \( U \) to any state, the Wigner function simply flows along the circles in accordance with the rotation \( R \). But an eigenstate of \( U \) does not change under this action, so its Wigner function must be constant on each circle.

4. MINIMIZING ENTROPY

Consider again our complete set of \( d+1 \) mutually unbiased bases, and let \( |ij\rangle \) be the \( j \)th vector in the \( i \)th basis. These vectors together have the following remarkable property: for any pure state \( |\psi\rangle \), the probabilities \( p_{ij} = |\langle \psi |ij\rangle|^2 \) satisfy [9,10]
\[ \sum_{ij} p_{ij}^2 = 2. \] (10)
Now consider the Rényi entropy $H_{R} = - \log_2 \left( \sum_j p_{ij}^2 \right)$ of the outcome-probabilities of the $i$th measurement when applied to the state $|\psi\rangle$. This entropy is a measure of our inability to predict the outcome of the measurement. The average of $H_{R}$ over all the mutually unbiased measurements can be bounded from below [11]:

$$\langle H_{R} \rangle = \left( \frac{1}{d+1} \right) \sum_i \left[ - \log_2 \left( \sum_j p_{ij}^2 \right) \right] \geq - \log_2 \left[ \left( \frac{1}{d+1} \right) \sum_{ij} p_{ij}^2 \right] = \log_2 (d+1) - 1,$$

(11)

with equality holding only if the Rényi entropy is constant over all the mutually unbiased measurements.

Now, for any of the rotationally invariant states defined in the last section, the Rényi entropies associated with the $d+1$ mutually unbiased measurements are indeed equal. By the inequality (11), such states therefore minimize the average Rényi entropy over all these measurements, that is, over all the directions in phase space.

5. EXAMPLES

The one-qubit case is very simple. The three mutually unbiased bases generated in our construction are the eigenstates of the Pauli operators $X$, $Y$, and $Z$. It is not hard to find a unitary transformation that cycles through these three bases. Such a transformation rotates the Bloch sphere by 120° around the axis $(x, y, z) = (1, 1, 1)$. The two eigenstates of this unitary transformation, which are the eigenstates of $X + Y + Z$, are rotationally invariant: each of their Wigner functions is constant on the only circle in the $2 \times 2$ phase space. And each of these states minimizes the average Rényi entropy for the measurements $X$, $Y$, and $Z$. It is interesting to note that one of these two states has a positive Wigner function.

Clearly there is nothing intrinsically special about these two states. They are special only in relation to the three measurements $X$, $Y$, and $Z$, which are associated with the three striations of the phase space. But in the context of quantum cryptography, the entropy-minimization property is quite relevant. In the six-state scheme (in which the signal states are the eigenstates of $X$, $Y$, and $Z$), if Eve chooses to eavesdrop by making a complete measurement on certain photons, her best choice is to make a measurement whose outcome-states are entropy-minimizing in our sense: it turns out that such a choice minimizes Eve’s own Rényi entropy about Alice’s bit.

An interesting example comes from the 3-qubit case. The relevant field is $\mathbb{F}_8$, which can be constructed from $\mathbb{F}_2$ by introducing an element $b$ that is defined to satisfy the equation $b^3 + b^2 + 1 = 0$. In our $8 \times 8$ discrete phase space, we can define circles via the equation

$$q^2 + qp + p^2 = c,$$

(12)

4The analogous inequality in terms of Shannon entropy was proved in Refs. [12, 13].
where \( c \) can take any nonzero value. A primitive rotation preserving these circles is

\[
R = \begin{pmatrix}
  b^3 & b^6 \\
  b^6 & b^5
\end{pmatrix}.
\]

(13)

One finds that of the eight eigenvectors of any unitary \( U \) corresponding to \( R \), all of which are rotationally invariant, exactly one has a positive Wigner function for a specific, fixed function \( Q(\lambda) \) associated with \( U \). This state is also easy to describe physically. For a particular choice of \( U \), it is of the form

\[
|\psi\rangle = \sqrt{1/3}|++\rangle + \sqrt{2/3}|--\rangle.
\]

(14)

where \( |+\rangle \) and \(-\rangle \) are the two eigenstates (with a specific relative phase) of the operator \( X + Y + Z \). If we regard \( |\psi\rangle \) as analogous to a coherent state at the origin, then the coherent-like states at the 63 other phase-space points can be obtained from \( |\psi\rangle \) by applying Pauli rotations to the individual qubits. The Wigner function of each of these states has the value 0.319 at its center, the largest value possible for any three-qubit state.

6. CONCLUSION

We have found that one can make sense of the notion of rotational invariance in a discrete phase space for a system of \( n \) qubits. The rotationally invariant states are in this respect analogous to the energy eigenstates of a harmonic oscillator, but the analogy is not perfect. Our rotationally invariant states are all states of minimum uncertainty with respect to the various directions in phase space, whereas except for the ground state, the harmonic oscillator eigenstates do not have this property (the uncertainty, even in our Rényi sense, increases with increasing energy). We have considered the further restriction to positive Wigner functions but so far have found examples of such states only for a single qubit and for three qubits. However, for any number of qubits, one can show that at least one of our rotationally invariant states takes a value at its center equal to the maximum value attainable by the Wigner function of any state. Perhaps this latter property, rather than positivity, should be taken as the defining feature of a “most pointlike” state.

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\(^5\text{Even though Eq. (12) is not of the form we used in reaching Eq. (8), in that it is not based on a primitive polynomial, the matrix } R \text{ is nevertheless a primitive rotation.}\)
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