Online Learning in Adversarial MDPs: Is the Communicating Case Harder than Ergodic?

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Abstract

We study online learning in adversarial communicating Markov Decision Processes with full information. We give an algorithm that achieves a regret of $O(\sqrt{T})$ with respect to the best fixed deterministic policy in hindsight when the transitions are deterministic. We also prove a regret lower bound in this setting which is tight up to polynomial factors in the MDP parameters. We also give an inefficient algorithm that achieves $O(\sqrt{T})$ regret in communicating MDPs (with an additional mild restriction on the transition dynamics).

1 Introduction

In this work, we study online learning in Markov Decision Processes. In this setting, we have an agent interacting with an adversarial environment. The agent observes the state of the environment and takes an action. The action incurs an associated loss and the environment moves to a new state. The state transition dynamics are assumed to be Markovian, i.e., the probability distribution of the new state is fully determined by the action and the old state. The transition dynamics are fixed and known to the learner in advance. However, the losses are chosen by the adversary. The adversary is assumed to be oblivious (the entire loss sequence is chosen before the interaction begins). We assume that the environment reveals full information about the losses at a given time step to the agent after the corresponding action is taken. The total loss incurred by the agent is the sum of losses incurred in each step of the interaction. We denote the set of states by $S$ and the set of actions by $A$. The objective of the agent is to minimize its total loss.

This setting was first studied in the seminal work of Even-Dar et al. [2009]. They studied the restricted class of ergodic MDPs where every policy induces a Markov chain with a single recurrent class. They designed an efficient (runs in polytime in MDP parameters and time of interaction) algorithm that achieved $O(\sqrt{T})$ regret with respect to the best stationary policy in hindsight. They assumed full information of the losses and that the MDP dynamics where known beforehand. This work was extended to bandit feedback by Neu et al. [2014] (with an additional assumption on the minimum stationary probability mass in any state). They also achieved a regret bound of $O(\sqrt{T})$. Bandit feedback is a harder model in which the learner only receives information about its own losses. Bandit feedback will not be the focus of this paper.

The ergodic MDP assumption is quite strong. Therefore there is interest in developing algorithms to handle the larger class of communicating MDPs where, for any pair of states, there is a policy such that the time it takes to reach the second state from the first has finite expectation. Although we do not focus on bandit feedback in this paper, we do wish to note here that in the bandit setting, enlarging the class of MDPs from ergodic to communication seems to have a inherent statistical cost associated with it. For
adversarial MDPs with deterministic transitions (referred to as Adversarial Deterministic Markov Decision Processes (ADMDPs) in the literature), Arora et al. [2012] designed an efficient algorithm that achieved $O(T^{3/4})$ regret for the communicating case. Subsequently, Dekel and Hazan [2013] designed an algorithm that achieved $O(T^{2/3})$ regret in the same setting. This regret bound was proved to be tight by a matching lower bound in Dekel et al. [2013]. This result suggests¹ that learning online learning in communicating MDPs with bandit feedback is statistically harder than its ergodic counterpart as witnessed by the higher exponent of $T$ in the optimal regret rates for the two settings. The regret lower bound in Dekel et al. [2013] was proved by a reduction from the problem of adversarial multi-armed bandits with switching costs. In this setting, the agent incurs an additional cost every time it switches the arm it plays. Their work therefore established a connection between the regret in the online MDP setting and the number of times the agent switches its policy.

In this paper, we show that in the full information setting, there is no statistical price to be paid for enlarging the class of MDPs from ergodic to communicating as far as the regret scaling with $T$ is concerned. In particular, we design an efficient algorithm that learns to act in communicating ADMDPs with deterministic transitions (referred to as Adversarial Deterministic Markov Decision Processes) in the literature), Arora et al. [2012] designed an efficient algorithm that achieved $O(T^{2/3})$ regret in the general class of communicating MDPs (albeit with an additional mild assumption).

2 Preliminaries

Fix the finite state space $S$, finite action space $A$, and transition probability matrix $P$ where $P(s, a, s')$ is the probability of moving from state $s$ to $s'$ on action $a$.

In the case of ADMDP, the transitions are deterministic and hence the ADMDP can also be represented by a directed graph $G$ with vertices corresponding to states $S$. The edges are labelled by the actions. An edge from $s$ to $s'$ labelled by action $a$ exists in the graph when the ADMDP takes the state $s$ to state $s'$ on action $a$. This edge will be referred to as $(s, a, s')$.

A (stationary) policy $\pi$ is a mapping $\pi : S \times A \to [0, 1]$ where $\pi(s, a)$ denotes the probability of taking action $a$ when in state $s$. When the transitions are deterministic, we overload the notation and define $\pi(s)$ to be the action taken when the state is $s$. The interaction starts in an arbitrary start state is $s_1 \in S$. The adversary chooses a sequence of loss functions $\ell_1, \ldots, \ell_T$ obliviously where each $\ell_t$ is a map from $S \times A$ to $[0, 1]$.

An algorithm $\mathcal{A}$ that interacts with the online MDP chooses the action to be taken at each time step. It maintains a probability distribution over actions denoted by $\mathcal{A}(\cdot | s_t, \ell_1, \ldots, \ell_{t-1})$ which depends on the current state and the sequence of loss functions seen so far. The expected loss of the algorithm $\mathcal{A}$ is

$$L(\mathcal{A}) = \mathbb{E} \left[ \sum_{t=1}^{T} \ell_t(s_t, a_t) \right]$$

where $a_t \sim \mathcal{A}(\cdot | s_t, \ell_1, \ldots, \ell_{t-1}), s_{t+1} \sim P(\cdot | s_t, a_t)$ For a stationary policy $\pi$, the loss of the policy is

$$L^\pi = \mathbb{E} \left[ \sum_{t=1}^{T} \ell_t(s_t, a_t) \right]$$

where $a_t \sim \pi(\cdot | s_t), s_{t+1} \sim P(\cdot | s_t, a_t)$. The regret of the algorithm is defined as

$$R(\mathcal{A}) = L(\mathcal{A}) - \min_{\pi \in \Pi} L^\pi .$$

¹The reason why we do not have a clean separation between ergodic and communicating cases is that the result of Neu et al. [2014] requires an additional assumption on the minimum stationary probability mass at any state under any policy.
The total expected loss of the best policy in hindsight is denoted by $L^*$. Thus,

$$L^* = \min_{\pi \in \Pi} L^\pi.$$

For any stationary policy $\pi$, let $T(s' \mid M, \pi, s)$ be the random variable for the first time step in which $s'$ is reached when we start at state $s$ and follow policy $\pi$ in MDP $M$. We define the diameter $D(M)$ of the MDP as

$$D(M) = \max_{s \neq s'} \min_{\pi} \mathbb{E}[T(s' \mid M, \pi, s)].$$

A communicating MDP is an MDP where $D(M) < \infty$.

2.1 Some more preliminaries on ADMDPs

In this section, we use the graph $G$ and the ADMDP interchangeably. A deterministic policy $\pi$ induces a subgraph $G_\pi$ of $G$ where $(s, a, s')$ is an edge in $G_\pi$ if and only if $\pi(s) = a$ and the action $a$ takes state $s$ to $s'$.

A communicating ADMDP corresponds to a strongly connected graph. This is because the existence of a policy that takes state $s$ to $s'$ also implies the existence of a path between the two vertices in the graph $G$.

The subgraph $G_\pi$ induced by policy $\pi$ in the communicating ADMDP is the set of transitions $(s, a, s')$ that are possible under $\pi$. Each component of $G_\pi$ is either a cycle or an initial path followed by a cycle. Start a walk from any state $s$ by following the policy $\pi$. Since the set of states is finite, eventually a state must be repeated and this forms the cycle.

Let $N(s, a)$ be the next state after visiting state $s$ and taking action $a$. Define $I(s)$ as

$$I(s) = \{(s', a) \mid N(s', a) = s\}.$$

The period of a vertex $v$ in $G$ is the gcd of the lengths of all the cycles starting and ending at $v$. In a strongly, connected graph, the period of each vertex can be proved to be equal. Thus, the period of a strongly connected $G$ is well defined. If the period of $G$ is 1, we call $G$ aperiodic.

The vertices of a strongly connected graph $G$ with period $\gamma$ can be partitioned into $\gamma$ non-empty cycle classes, $C_1, \ldots, C_\gamma$, where each edge goes $C_i$ to $C_{i+1}$.

**Theorem 2.1.** If $G$ is strongly connected and aperiodic, there exists a critical length $d$ such that for any $\ell \geq d$, there exists a path of length $\ell$ in $G$ between any pair of vertices. Also, $d \leq n(n - 1)$ where $n$ is the number of vertices in the graph.

The above theorem is from Denardo [1977]. It guarantees the existence of a $d > 0$ such that there are paths of length $d$ between any pair of vertices. The following generalization can be easily proved from the theorem.

**Theorem 2.2.** If $G$ has a period $\gamma$, there exists a critical value $d$ such that for any integer $\ell \geq d$, there is a path of in $G$ of length $\gamma \ell$ from any state $v$ to any other state in the same cycle class.

**Remark 2.3.** We can also find the paths of length $\ell \geq d$ from a given vertex $s$ to any other vertex $s'$ efficiently. This can be done constructing the path in the reverse direction. We look at $P^{\ell-1}$ to see all the predecessors of $s'$ that have paths of length $\ell - 1$ from $s$. We choose any of these as the penultimate vertex in the path and recurse.

3 Algorithm for Deterministic MDPs

We now present our algorithm for online learning in ADMDPs with full information when we have full information of losses. We use $G$ to refer to the graph associated to the ADMDP.

We assume that the ADMDP dynamics are known to the agent. This assumption can be relaxed as shown in Ortner [2010] as we can figure out the dynamics in poly($|S|, |A|$) time when the transitions are
deterministic. We want to minimize regret against the class of deterministic stationary policies. As observed earlier, these policies induced a subgraph which is isomorphic to a cycle with an initial path. The length of the initial path can be at-most $|S|$. Thus, any such policy will spend at most $|S|$ steps before entering the cycle. Then it follows the cycle for the rest of the interaction.

3.1 Algorithm Sketch

We formulate the task of minimizing regret against the set of deterministic policies as a problem of prediction with expert advice. We keep an expert for each cycle in $G$ along with the start state of the policy (in that cycle). Note that we do not keep an expert for policies which have an initial path before the cycle. This is because the loss of these policies differ by at most $|S|$ compared to the cycles. Also, we make sure that the start state of the cycle is in the same cycle class as the start state of the environment. If this is not the case, our algorithm will never be in phase with the expert policy. Henceforth, we will refer to these experts.

We designed an efficient (running time polynomial in $|S|, |A|$ and $T$) algorithm to achieve $O(\sqrt{T})$ regret and switching cost against this class of experts. For this we used a Follow the perturbed leader (FPL) style algorithm.

We then use this low switching algorithm as a black box. Whenever, the black box algorithm tells us to switch policies at time $t$, we compute the state $s$ that we would have reached if we had followed the new policy from the start of the interaction and moved $t + \gamma_d$ steps. We then move to this state $s$ in $\gamma_d$ steps. Theorem 2.2 guarantees the existence of a path of this length. We then start following the new policy.

Thus, our algorithm matches the moves of the expert policies except when there is a switch in the policies. Thus, the regret of our algorithm differs from the regret of the black box algorithm by at-most $O(\gamma_d\sqrt{T})$.

3.2 FPL algorithm

We now describe the FPL style algorithm that competes with the set of cycles described earlier with $O(\sqrt{T})$ regret and switching cost.

3.2.1 Finding the leader

First, we design an offline algorithm that finds the cycle (including start state) with lowest cumulative loss till time $t$ given the sequence of losses $\ell_1, \ldots, \ell_{t-1}$. Given $(s, k)$, we find the best cycle among the cycles that start in state $s$ and have length $k$. For this we use a method similar to that used in Arora et al. [2012]. We then find the minimum over all $(s, k)$ pairs to find the best cycle. Note that we only consider start states $s$ which are in the same cycle class as the start state $s_0$ of the game.

We use $C_{(s,k)}$ to denote the set of all cycles starting at state $s$ with length $k$. We find the best cycle in $C_{(s,k)}$ using Linear Programming. Let $n = |S||A||k$. The LP is in the space $\mathbb{R}^n$. Consider a cycle $c \in C_{(s,k)}$. Let $c_i$ denoted the $i_{th}$ state in $c$. Also, let $a_i$ be the action taken at that state. We associate a vector $x(c)$ with the cycle as follows.

$$x(c)_{s,a,i} = \begin{cases} 1 & \text{if } a = a_i \text{ and } s = c_i \\ 0 & \text{otherwise} \end{cases}$$

We construct a loss vector in $\mathbb{R}^n$ as follows.

$$l_{s,a,i} = \sum_{1 \leq j < t \atop (j-i) \equiv 0 \text{ mod } k} \ell_j(s,a)$$

Our decision set $X \subseteq \mathbb{R}^n$ is the convex hull of all $x(c)$ where $c \in C_{(s,k)}$. Our objective is to find $x$ in $X$ such that $\langle x, l \rangle$ is minimized. The set $X$ can be captured by the following polynomial sized set of linear
Before analysing the FPL algorithm described above, we first introduce some notation. For any cycle \( c \) with start state \( s \), let \( L^c \) denote the total cumulative loss that we would have received if we started the interaction at state \( s \) and followed the cycle \( c \) throughout the interaction. We use \( L^c \) to denote the total perturbed cumulative loss received by cycle \( c \). Let the cycle with lowest total cumulative loss be \( c^* \). Also, let the cycle with lowest perturbed cumulative loss be \( \hat{c}^* \). We use \( \hat{L}_t^c \) to denote the total perturbed cumulative loss incurred by cycle \( c \) after \( t \) steps. We use \( \hat{c}_t^* \) to denote the cycle with lowest perturbed cumulative loss after \( t \) steps. Let \( C_t \) be the cycle chosen by the FPL algorithm at step \( t \).

The analysis is similar to Section 2 of Kalai and Vempala [2005]. We first bound the total loss incurred by the FPL algorithm. Let the expected number of switches made by the algorithm during the interaction be \( N_s \). If the algorithm doesn’t switch cycles after time step \( t \), then \( \hat{L}_t^{C_t} \) must be equal to \( \hat{L}_t^c \). Thus, the loss incurred at time step \( t \) by \( C_t \) is at most \( \hat{L}_t^{C_t} - \hat{L}_{t-1}^{C_t} \). In the steps in which the algorithm switches cycles, the maximum loss incurred is 1. Thus, we have that

\[
E[\text{total loss of FPL}] \leq \hat{L}_t^{C_t} + \sum_{i=2}^{T} \left( \hat{L}_i^{C_t} - \hat{L}_{i-1}^{C_t} \right) + N_s
\]

\[
\leq \hat{L}_t^{C_t} + N_s
\]

\[
= \hat{L}_t^{C_t} + N_s
\]  

(1)
We now bound \( N_s \). From linearity of expectation, we have that
\[
N_s = \sum_{i=1}^{T-1} \Pr[C_{t+1} \neq C_t].
\]

We will bound \( \Pr[C_{t+1} \neq C_t] \) by bounding \( \Pr[C_{t+1} \neq C_t \mid C_t = c] \) for all cycles \( c \). Let \( c \) be a cycle in the set \( C_{(s,k)} \). Let \( l_t \) be the loss incurred by cycle \( c \) at step \( t \). If \( C_{t+1} \) is not in \( C_{(s,k)} \), then the algorithm must have switched. Thus, we get the following equation.

\[
\Pr[C_{t+1} \neq c \mid C_t = c] = \Pr[C_{t+1} \notin C_{(s,k)} \mid C_t = c] + \Pr[C_{t+1} \neq c \text{ and } C_{t+1} \in C_{(s,k)} \mid C_t = c]
\]  

(2) We now bound both the terms in the right hand side of (2) separately.

First, we study at the first term. We will upper bound this term by proving an appropriate lower bound on the probability of choosing \( C_{t+1} \) from \( C_{(s,k)} \). Since \( C_t = c \), we know that \( \hat{L}_t^{c'} \leq \hat{L}_t \) for all \( c' \neq c \). For all \( c' \notin C_{(s,k)} \), the perturbation \( \hat{L}(s,k) \) will play a role in the comparison of the perturbed cumulative losses.

For \( c' \in C_{(s,k)} \), \( \hat{L}(s,k) \) appears on both sides of the comparison and thus gets cancelled out. Thus, we have \( \hat{L}(s,k) \geq w \), where \( w \) depends only on the perturbations and losses received by \( w \) and the cycles not in \( C_{(s,k)} \). Now, if \( \hat{L}(s,k) \) was larger than \( w + l_t \), then the perturbed cumulative loss of \( c \) will be less than that of cycles not in \( C_{(s,v)} \) even after receiving the losses of step \( t \). In this case, \( C_{t+1} \) will also be chosen from \( C_{(s,k)} \). This gives us the require probability lower bound.

\[
\Pr[C_{t+1} \in C_{(s,k)} \mid C_t = c] \geq \Pr[\hat{L}(s,k) \geq w + l_t \mid \hat{L}(s,k) \geq w] = e^{-\lambda l_t} \geq 1 - e^{-\lambda l_t}
\]

Thus, \( \Pr[C_{t+1} \notin C_{(s,k)} \mid C_t = c] \) is at most \( \lambda \cdot l_t \).

We now bound the second term. For any two cycles \( c' \neq c' \) in \( C_{(s,k)} \), there exists an index \( i \leq k \) such that the \( i \)th edges of \( c' \) and \( c'' \) are different and all the smaller indexed edges of the two cycles are the same. We denote this index by \( d(c',c'') \). Define \( d(c',c'') \) to be zero when \( c' \) is from \( C_{(s,k)} \) and \( c'' = c' \) or \( c'' \) is not from \( C_{(s,k)} \). Now, if \( C_{t+1} \) is in \( C_{(s,k)} \) and not equal to \( c \), then \( d(C_{t+1},c) \) is a number between one and \( k \). Thus, we get the following equation.

\[
\Pr[C_{t+1} \neq c \text{ and } C_{t+1} \in C_{(s,k)} \mid C_t = c] = \sum_{i=1}^{k} \Pr[d(c,C_{t+1}) = i \mid C_t = c]
\]

We now bound \( \Pr[d(c,C_{t+1}) = i \mid C_t = c] \) for any \( i \) between 1 and \( k \). Let \( (s_i,a_i) \) be the \( i \)th edge of \( c \). We prove a lower bound on the probability of choosing \( C_{t+1} \) such that \( d(c,C_{t+1}) \) is not equal to \( i \). Again, since \( C_t = c \), we know that \( \hat{L}_t^{c'} \leq \hat{L}_t \) for all \( c' \neq c \). Consider cycles \( c' \) that don’t contain the edge \( (s_i,a_i) \). The perturbation \( \epsilon(s_i,a_i) \) will play a role in the comparison of perturbed losses of all such cycle \( c \) with \( c' \). For cycles that contain the \( (s_i,a_i) \) edge, the \( \epsilon(s_i,a_i) \) term gets cancelled out. Thus, we have \( \epsilon(s_i,a_i) \geq w \), where \( w \) depends only on the perturbations and losses received by \( w \) and the cycles \( c' \) that don’t have the \( (s_i,a_i) \) edge. If \( \epsilon(s_i,a_i) \) was greater than \( w + l_t \), then the perturbed cumulative loss of \( c \) will still be less than that of all cycles \( c' \) without the \( (s_i,a_i) \) edge. In this case, \( C_{t+1} \) will be chosen such that it also has the \( (s_i,a_i) \) edge. This implies that \( d(c,C_{t+1}) \neq i \). Thus, we get the following probability lower bound.

\[
\Pr[d(c,C_{t+1}) \neq i \mid C_t = c] \geq \Pr[\epsilon(s_i,a_i) \geq w + l_t \mid \epsilon(s_i,a_i) \geq w] = e^{-\lambda l_t} \geq 1 - e^{-\lambda l_t}
\]

Thus, for all \( i \) between 1 and \( k \), \( \Pr[d(c,C_t+1) = i \mid C_t = c] \) is at most \( \lambda \cdot l_t \). This proves that the term in (3) is at most \( k \lambda \cdot l_t \). Since \( k \) is at most \(|S| \), the second term in the right hand side of (2) is bounded by \( |S| \cdot \lambda \cdot l_t \).
Thus, $Pr[C_{t+1} \neq C_t]$ is at most $(|S| + 1) \cdot \lambda \cdot \mathbb{E}[t_i]$. This gives us the following bound for $N_s$.

$$N_s = \sum_{t=1}^{T-1} Pr[C_t + 1 \neq C_t]$$

$$\leq \sum_{t=1}^{T-1} (|S| + 1) \cdot \lambda \cdot \mathbb{E}[t_i]$$

$$\leq (|S| + 1) \cdot \lambda \cdot \sum_{t=1}^{T-1} \mathbb{E}[t_i]$$

$$\leq (|S| + 1) \cdot \lambda \cdot \mathbb{E}[\text{total loss of FPL}]$$

Combining this with (1) gives us the following.

$$\mathbb{E}[\text{total loss of FPL}] \leq \tilde{L}^* + (|S| + 1) \cdot \lambda \cdot \mathbb{E}[\text{total loss of FPL}]$$

(4)

Let $p(c)$ denote the perturbed loss added to cycle $c$. Since the cycle with lowest perturbed cumulative loss at the end of the interaction is $\tilde{c}^*$, we have

$$\tilde{L}^* \leq L^* + p(\tilde{c}^*)$$

Also,

$$\mathbb{E}[p(\tilde{c}^*)] \leq |S| \cdot \mathbb{E} \left[ \max_{(s,a)} \epsilon(s,a) \right] + \mathbb{E} \left[ \max_{(s',k)} \delta(s',k) \right] \leq |S| \cdot \frac{(1 + \log |S||A|)}{\lambda} + \frac{1 + \log |S|^2}{\lambda}$$

Plugging this inequality into (4) gives us

$$\mathbb{E}[\text{cost of FPL}] \leq L^* + |S| \cdot \frac{(1 + \log |S||A|)}{\lambda} + \frac{1 + \log |S|^2}{\lambda} + (|S| + 1) \cdot \lambda \cdot \mathbb{E}[\text{cost of FPL}]$$

(5)

Since the maximum cost is $T$, we have

$$\text{Regret} \leq \frac{|S|(1 + \log |S||A|)}{\lambda} + \frac{1 + \log |S|^2}{\lambda} + (|S| + 1)\lambda T$$

Setting $\lambda = \frac{\log |S||A|}{\sqrt{T}}$ gives us a bound of $O \left( |S|\sqrt{T \log |S||A|} \right)$ on the regret and expected number of switches. We can also derive first order bounds.

**Theorem 3.2.** The regret and the expected number of switches can be bounded by $O \left( |S|\sqrt{L^* \cdot \log |S||A|} \right)$, where $L^* = \tilde{L}^*$

**Proof.** From (5), we have

$$\mathbb{E}[\text{total loss of FPL}] \leq L^* + |S| \cdot \frac{(1 + \log |S||A|)}{\lambda} + \frac{1 + \log |S|^2}{\lambda} + (|S| + 1) \cdot \mathbb{E}[\text{cost of FPL}]$$

$$\leq L^* + 4|S| \cdot \frac{\log |S||A|}{\lambda} + 2|S| \cdot \lambda \cdot \mathbb{E}[\text{total loss of FPL}]$$

On rearranging, we get

$$\mathbb{E}[\text{total loss of FPL}] \leq \frac{L^*}{1 - 2\lambda|S|} + 4|S| \cdot \frac{\log |S||A|}{\lambda(1 - 2\lambda|S|)}$$

$$\leq L^*(1 + (2\lambda|S| + (2\lambda|S|)^2 + \ldots) + 4|S| \frac{\log |S||A|}{\lambda} (1 + 2\lambda|S| + (2\lambda|S|)^2 + \ldots)$$

$$\leq L^*(1 + 4\lambda|S|) + 8|S| \frac{\log |S||A|}{\lambda}.$$
The last two inequalities work when $2\lambda|S| \leq \frac{1}{2}$. Thus,

$$\mathbb{E} [\text{total loss of FPL}] - L^* \leq 4\lambda|S|(L^*) + 8|S|\frac{\log |S||A|}{\lambda}.$$ 

Set $\lambda = \min \left( \sqrt{\frac{\log |S||A|}{|S|^2}}, \frac{1}{4|S|} \right)$. This forces $2\lambda|S|$ to be less than $\frac{1}{2}$ and thus the previous inequalities are still valid. On substituting the value of $\lambda$, we get that

$$\text{Regret} \leq O \left( |S|\sqrt{L^* \cdot \log |S||A|} \right)$$

when $L^* \geq 16|S|^2 \log |S||A|$. Since the expected number of switches is at most $2\lambda|S| \cdot \mathbb{E} [\text{total loss of FPL}]$, this is also bounded by $O \left( |S|\sqrt{L^* \cdot \log |S||A|} \right)$. \hfill \Box

### 3.3 Putting it together

We have described the FPL style algorithm that achieves low regret and low switching. We will refer to the algorithm as $B$ henceforth. We now use $B$ as a sub-routine to design a low regret algorithm for the online ADMDP problem. Given a cycle $C$ chosen by $B$, we can associate a policy $\pi$ as follows. $\pi(C)$ can be any policy such that $C$ is a subgraph of $G_{\pi}$ (the subgraph induced by the policy).

Let $S(\pi, s, t)$ be the state reached by policy $\pi$ if we started at state $s$ and moved $t$ steps according to the policy. This can be efficiently computed.

**Algorithm 1:** Low regret algorithm for communicating ADMDPs

1. $s_1$ is the start state of the environment;
2. Let $c_1, x_1$ be the cycle, start state chosen by $B$ at $t = 1$;
3. if $s_1 \neq x_1$ then
   - Spend $\gamma d$ steps to move to state $S(\pi(c_1), x_1, \gamma d)$;
   - $t = 1 + \gamma d$;
4. Set $\pi_t = \pi(c_t)$;
5. while $t \neq T + 1$ do
   - Choose action $a_t = \pi_t(s_t)$;
   - Adversary returns loss function $\ell_t$ and next state $s_{t+1}$;
   - Feed $\ell_t$ as the loss to $B$
   - if $B$ switches cycle to $c_{t+1}, x_{t+1}$ then
     - if $s_{t+1} \neq x_{t+1}$ then
       - Spend $\gamma d$ steps to move to state $S(\pi(c_{t+1}), x_{t+1}, t + \gamma d)$;
       - $\pi_{t+\gamma d} = \pi(c_{t+1})$;
       - $t = t + \gamma d$;
     - else
       - $\pi_{t+1} = \pi_t$;
       - $t = t + 1$;
   - else

We spend $\gamma d$ steps whenever $B$ switches. In all other steps, we receive the same loss as the cycle chosen by $B$. Thus, the regret differs by at most $\gamma d \cdot N_s$. From the analysis in the previous section, we get that

$$\text{Regret} \leq O \left( |S| \cdot \gamma d \sqrt{T \log |S||A|} \right)$$

Thus, the total regret of our algorithm in the deterministic case is $O \left( |S| \cdot \gamma d \sqrt{T \log |S||A|} \right)$ where $d$ is the critical length in the ADMDP. Note that $d$ is at most $O(|S|^2)$ . Thus, we get the following theorem.

We spend $\gamma d$ steps whenever $B$ switches. In all other steps, we receive the same loss as the cycle chosen by $B$. Thus, the regret differs by at most $\gamma d \cdot N_s$. From the analysis in the previous section, we get that

$$\text{Regret} \leq O \left( |S| \cdot \gamma d \sqrt{T \log |S||A|} \right)$$

Thus, the total regret of our algorithm in the deterministic case is $O \left( |S| \cdot \gamma d \sqrt{T \log |S||A|} \right)$ where $d$ is the critical length in the ADMDP. Note that $d$ is at most $O(|S|^2)$ . Thus, we get the following theorem.
Theorem 3.3. Given a communicating ADMDP with state space $S$, action space $A$ and period $\gamma$, the regret of Algorithm 1 is bounded by

$$\text{Regret} \leq O\left(|S|^3 \cdot \gamma \sqrt{T \log |S||A|}\right)$$

We also get the corresponding first order bound.

Theorem 3.4. Given a communicating ADMDP with state space $S$, action space $A$ and period $\gamma$, the regret of Algorithm 1 is bounded by

$$\text{Regret} \leq O\left(|S|^3 \cdot \gamma \sqrt{L^* \cdot \log |S||A|}\right)$$

where $m$ is the total loss incurred by the best stationary deterministic policy in hindsight.

Remark 3.5. To achieve the first order regret bound, we set $\lambda$ in terms of $L^*$. We need prior knowledge of $L^*$ to directly do this. It should be possible to remove this requirement by using a doubling trick.

4 Regret Lower Bound for Deterministic MDPs

We now prove a matching regret lower bound (up to polynomial factors).

Theorem 4.1. For any algorithm $A$ and any $|S| > 3, |A| \geq 1$, there exists an MDP $M$ with $|S|$ states and $|A|$ actions and a sequence of losses $\ell_1, \ldots, \ell_t$ such that

$$R(A) \geq \Omega \left(\sqrt{|S|T \log |A|}\right)$$

where $R(A)$ is the regret incurred by $A$ on $M$ with the given sequence of losses.

Proof. Let $M$ be an MDP with states labelled $s_0, s_2, \ldots, s_{|S|-1}$. Any action $a$ takes state $s_i$ to $s_{i+1}$ (modulo $|S|$). In other words, the states are arranged in a cycle and every action takes any state to its next state in the cycle. This is the required $M$.

Consider the problem of prediction with expert advice with $n$ experts. We know that for any algorithm $B$, there is a sequence of losses such that the regret of $B$ is $\Omega(\sqrt{T \log n})$ over $T$ steps (see Cesa-Bianchi and Lugosi [2006]). In our case, every policy spends exactly $\frac{T}{|S|}$ steps in each state. Thus, the interaction with $M$ over $T$ steps can be interpreted as a problem of prediction with expert advice at every state where each interaction lasts only $\frac{T}{|S|}$ steps. We have the following decomposition of the regret.

$$R(A) = \sum_{i=0}^{\frac{T}{|S|}-1} \sum_{k=0}^{|S|-1} \ell_k(s_i, a_k) - \ell_k(s_i, \pi^*(s_i)) \quad (6)$$

In the above equation, $a_t$ is the action taken by $A$ at step $t$. The best stationary deterministic policy in hindsight is $\pi^*$.

From the regret lower bound for the experts problem, we know that there exists a sequence of losses such that for each $i$, the inner sum of (6) is at least $\Omega \left(\sqrt{\frac{T}{|S|} \log |A|}\right)$. By combining these loss sequences, we get a sequence of losses such that

$$R(A) \geq \sum_{i=0}^{|S|-1} \Omega \left(\sqrt{\frac{T}{|S|} \log |A|}\right) \geq \Omega \left(\sqrt{|S|T \log |A|}\right).$$

This completes the proof.
5 Algorithm for Communicating MDPs

In the previous sections, we only considered deterministic transitions. We now present an algorithm that achieves low regret for the more general class of communicating MDPs (with an additional mild restriction). This algorithm achieves $O(\sqrt{T})$ regret but takes exponential time to run (exponential in $|S|$).

**Assumption 5.1.** The MDP $M$ has a state $s^*$ and action $a$ such that

$$\Pr(s_{t+1} = s^* \mid s_t = s^*, a_t = a) = 1$$

In other words, there is some state $s^*$ in which we have a deterministic action that allows us to stay in the state $s^*$. This can be interpreted as a state with a “do nothing” action where we can wait before taking the next action.

We now prove a useful lemma about communicating MDPs of diameter $D$

**Lemma 5.2.** For any start state $s$ and target $s' \neq s$, we have $\ell_{s,s'} \leq 2D$ and a policy $\pi$ such that

$$\Pr[T(s' \mid M, \pi, s) = \ell_{s,s'}] \geq \frac{1}{4D}$$

**Proof.** From the definition of diameter, we are guaranteed a policy $\pi_{s,s'}$ such that $\mathbb{E}[T(s' \mid M, \pi, s)] \leq D$

From Markov’s inequality, we have

$$\Pr[T(s' \mid M, \pi, s) \leq 2D] \geq \frac{1}{2}$$

Since there are only $2D$ discrete values less than $2D$, there exists $\ell_{s,s'} \leq 2D$ such that

$$\Pr[T(s' \mid M, \pi, s) = \ell_{s,s'}] \geq \frac{1}{2} \cdot \frac{1}{2D} = \frac{1}{4D}$$

From Lemma 5.2 and Assumption 5.1, we have the following corollary

**Theorem 5.3.** In MDPs satisfying Assumption 5.1, we have $\ell^* \leq 2D$ and state $s^*$ such that, for all target states $s'$, we have policies $\pi_{s'}$ such that

$$p_{s'} = \Pr[T(s' \mid M, \pi_{s'}, s^*) = \ell^*] \geq \frac{1}{4D}$$

**Proof.** From Lemma 5.2, we $\ell_{s'} \leq 4D$ for each $s'$ such that there is a policy $\pi_{s^*,s'}$ that hits the state $s'$ in time $\ell'_s$ with probability at-least $\frac{1}{4D}$. We take $\ell^* = \max_{s' \neq s^*} \ell'_{s'}$. For target state $s'$, the policy $\pi_{s'}$ loops at state $s^*$ for $(\ell^* - \ell_{s'})$ time steps and then starts following policy $\pi_{s,s'}$. Clearly, this policy hits state $s'$ at time $\ell^*$ with probability at least $\frac{1}{4D}$

**Remark 5.4.** The policies guaranteed by Corollary 5.3 are not stationary.

Let $p^* = \min_s p_s$. Clearly, $p^* \geq \frac{1}{4D}$
5.1 Algorithm

We extend the algorithm we used in the deterministic MDP case.

We use a low switching algorithm (FPL) that considers each policy $\pi \in \Pi$ as an expert. We know from Kalai and Vempala [2005] that FPL achieves $O(\sqrt{T \log n})$ regret as well as switching cost. At time $t$, we receive loss function $\ell_t$ from the adversary. Using this, we construct $\hat{\ell}_t(\pi)$ as

$$\hat{\ell}_t(\pi) = \mathbb{E}[\ell_t(s_t, a_t)]$$

where $s_t \sim d_t$, $a_t \sim \pi(s_t, \cdot)$.

In other words, $\hat{\ell}_t(\pi)$ is the expected loss if we follow the policy $\pi$ from the start of the game. $d_t$ is the initial distribution of states.

We feed $\hat{\ell}_t$ as the losses to FPL.

We can now rewrite $L^n$ as

$$L^n = \mathbb{E}\left[\sum_{t=1}^{T} \ell_t(s_t, a_t)\right] = \sum_{t=1}^{T} \mathbb{E}[\ell_t(s_t, a_t)] = \sum_{t=1}^{T} \hat{\ell}_t(\pi)$$

where $s_t \sim d_t$ and $a_t \sim \pi(s_t, \cdot)$. Let $\pi_t$ be the policy chosen by $B$ at time $t$. We know that

$$\mathbb{E}\left[\sum_{t=1}^{t} \hat{\ell}_t(\pi_t)\right] - \sum_{t=1}^{t} \hat{\ell}_t(\pi) \leq O(\sqrt{T \log |\Pi|})$$

for any deterministic policy $\pi$.

We need our algorithm to receive loss close to the first term in the above sum. If this is possible, we have an $O(\sqrt{T})$ regret bound for online learning in the MDP. We now present an approach to do this.

5.1.1 Catching a policy

When FPL switches policy, we cannot immediately start receiving the losses of the new policy. If this was possible, then the regret of our algorithm will match that of FPL. When implementing the policy switch in our algorithm, we suffer a delay before starting to incur the losses of the new policy (in an expected sense). Our goal now is to make this delay as small as possible. This coupled with the fact that $B$ has a low number of switches will give us good regret bounds. Note that this was easily done in the deterministic case using Theorem 2.2. Theorem 5.3 acts somewhat like a stochastic analogue of Theorem 2.2 and we use this to reduce the time taken to catch the policy.

Remark 5.5. In Algorithm 2, if FPL switches the policy in the middle of the Switch_Policy’s execution, we terminate the execution and call the routine again with a new target policy.

5.2 Analysis

We first analyse the Switch_Policy routine.

Lemma 5.6. If Switch_Policy terminates at time $t$, we have that

$$\Pr[S_t = s \mid T_{\text{switch}} = t] = d^n_t(s)$$

where $d^n_t(s)$ is the distribution of states after following policy $\pi$ from the start of the game.

Proof. We want to compute $\Pr[S_t = s \mid T_{\text{switch}} = t]$.

$$\Pr[S_t = s \mid T_{\text{switch}} = t] = \frac{\Pr[S_t = s, T_{\text{switch}} = t]}{\Pr[T_{\text{switch}} = t]} = \frac{\Pr[S_t = s, T_{\text{switch}} = t]}{\Pr[T_{\text{switch}} = t]} = \frac{\Pr[T_{\text{switch}} = t]}{\Pr[T_{\text{switch}} = t]}$$
Algorithm 2: Low Regret Algorithm For Communicating MDPs

Function SwitchPolicy(s, π, t₀):

    Done = 0
    t = t₀ + 1 // t₀ is the time that B switched policy
    S_t = s // S_t stores the state at time t

    while Done ≠ 1 do
        Move to state s* using the best policy // Say this step takes k steps
        t = t + k
        Sample T_{t+t*} from d_{π}^{t+t*}(.)
        We set T_{t+t*} as the target state
        Use policy π_{T_{t+t*}} guaranteed by Corollary 5.3 to move t* steps from s*
        t = t + t*
        if S_t = T_t then
            Consider a Bernouli Random Variable I such that I = 1 with probability \frac{e^*}{p_{S_t}}.
            if I = 1 then
                Start following π and set Done to 1
                Let the time at this happens be T_{switch}
            else
                I = 0
                Continue
        else
            Continue

Function Main:

    Let π_{FPL}^{t} be the expert chosen by FPL at time 1
    π₁ = π_{FPL}^{t}
    Let S₁ be the start state.
    t = 1
    while t ≠ T + 1 do
        Sample a_t from π_t(s_t,.)
        Adversary returns loss function ℓ_t and next state s S_{t+1}=s
        Compute ℓ_t and feed it as the loss to FPL as discussed before
        if FPL switches policy then
            SwitchPolicy(s, π_{FPL}^{t+1}, t + 1) // Call the switch policy function to catch the
            new policy
            π_{t+k} = π_{FPL}^{t+1} // k is the number of steps taking by Switch Policy
            t = t + k
        else
            π_{t+1} = π_{t}
            t = t + 1
We now compute the denominator \( Pr[T_{\text{switch}} = t] \) as follows.

\[
Pr[T_{\text{switch}} = t] = \sum_{s \in S} Pr[S_t = T_t = s, S_{t-\ell'} = s^*] \cdot Pr[T_{\text{switch}} = t | S_t = T_t = s, S_{t-\ell'} = s^*]
\]

\[
= \sum_{s \in S} Pr[S_t = s | T_t = s, S_{t-\ell'} = s^*] \cdot Pr[T_t = s, S_{t-\ell'} = s^*] \cdot \frac{p^s}{p_s}
\]

\[
= p^s \sum_{s \in S} Pr[T_t = s, S_{t-\ell'} = s^*]
\]

\[
= p^s \cdot Pr[S_t = s, T_{\text{switch}} = t]
\]

Now we calculate the numerator.

\[
Pr[S_t = s, T_{\text{switch}} = t] = Pr[T_t = s, S_t = s, S_{t-\ell'} = s, T_{\text{switch}} = t]
\]

\[
= Pr[S_t = s, T_{\text{switch}} = t | S_{t-\ell'} = s^*, T_t = s] \cdot Pr[S_{t-\ell'} = s^*, T_t = s]
\]

\[
= p^s \cdot Pr[S_t = s, T_{\text{switch}} = t]
\]

Thus, we have

\[
Pr[S_t = s | T_{\text{switch}} = t] = d^*_t(s)
\]

The above lemma implies that the distribution of states at step \( t \) is the same as that of the target policy \( \pi \) given that the SwitchPolicy routine ends at time \( t \).

We now bound the expected loss of the algorithm in the period that FPL chooses policy \( \pi \)

**Lemma 5.7.** Let the policy of FPL be \( \pi \) from time \( t_1 \) to \( t_2 \). We have that

\[
E \left[ \sum_{t=t_1}^{t_2} \ell_t(s_t, a_t) \right] \leq 48 \cdot D^2 + \sum_{t=t_1}^{t_2} \ell_t(\pi)
\]

**Proof.** We bound the expectation using law of total expectations and conditioning on \( T_{\text{switch}} \).

\[
E \left[ \sum_{t=t_1}^{t_2} \ell_t(s_t, a_t) \right] = E \left[ \sum_{t=t_1}^{t_2} \ell_t(s_t, a_t) | T_{\text{switch}} \right]
\]

We bound the conditional expectation.

\[
E \left[ \sum_{t=t_1}^{t_2} \ell_t(s_t, a_t) | T_{\text{switch}} = t^* \right] \leq t^* + E \left[ \sum_{t=t^*}^{t_2} \ell_t(s_t, a_t) | T_{\text{switch}} = t^* \right]
\]

From Lemma 5.6, the second term is equal to \( \sum_{t=t^*}^{t_2} \ell_t(\pi) \) Thus,

\[
E \left[ \sum_{t=t_1}^{t_2} \ell_t(s_t, a_t) \right] \leq E[T_{\text{switch}}] + \sum_{t=t_2}^{t_2} \ell_t(\pi)
\]

Everytime we try to catch the policy from state \( s^* \), we succeed with probability \( p^s \geq \frac{1}{16D} \). Thus, the expected number of times we try is \( 16 \cdot D \) and each attempt takes \( \ell^* \leq 2D \) steps. Between each of these attempts, we move at most \( D \) steps in expectation to reach \( s^* \) again. Thus, in total, we have

\[
E[T_{\text{switch}}] \leq 16D^2 + 32D^2 = 48D^2
\]

This completes the proof. \( \square \)
Now, we bound the regret of Algorithm 2

**Theorem 5.8.** The regret of Algorithm 2 is at-most $O \left( D^2 \sqrt{T \log |\Pi|} \right)$

**Proof.** We condition on the number of switches made by FPL. Let $N_s$ be the random variable corresponding to the number of switches made by FPL. We refer to Algorithm 2 as $A$.

$$L(A) = \mathbb{E} \left[ \sum_{t=1}^{T} \ell_t(s_t, a_t) \right]$$

$$= \mathbb{E} \left[ \sum_{t=1}^{T} \ell_t(s_t, a_t) \mid N_s \right]$$

After each switch, Lemma 5.7 tells us that the Algorithm suffers at most $48 \cdot D^2$ extra average loss to the loss of the algorithm FPL. Thus,

$$L(A) \leq \mathbb{E} \left[ 48 \cdot D^2 \cdot N_s + \sum_{t=1}^{T} \ell_t(\pi_t) \right]$$

$\pi_t$ is the policy chosen by algorithm FPL at time $t$. Since FPL is a low switching algorithm, we have $N_s \leq O(\sqrt{T \log |\Pi|})$. The second term in the expectation is atmost $L^\pi + O(\sqrt{T \log |\Pi|})$ for any deterministic policy $\pi$. This is because FPL is a low regret algorithm. Thus, we have

$$L(A) - L^\pi \leq O(D^2 \sqrt{T \log |\Pi|})$$

for all stationary $\pi$.

Thus, $R(A) \leq O \left( D^2 \sqrt{T \log |\Pi|} \right)$

When $\Pi$ is the set of stationary deterministic policies, we get that $|\Pi| \leq |A|^{|S|}$. Thus, we get the following theorem.

**Theorem 5.9.** Given a communicating MDP satisfying Assumption 5.1 with $|S|$ states, $|A|$ action and diameter $D$, the regret of Algorithm 2 can be bounded by

$$\text{Regret} \leq O \left( D^2 \sqrt{|S| \log |A|} \right)$$

In fact, since we are using FPL as the expert algorithm, we can get first-order bounds similar to Theorem 3.2. In a setting with $n$ experts with $m$ being the total loss of the best expert, we can derive that the regret and number of switches can be bounded by $O(\sqrt{m} \cdot \log n)$. Thus, using this, we get the following first order regret bounds for Algorithm 2

**Theorem 5.10.** Given a communicating MDP satisfying Assumption 5.1 with $|S|$ states, $|A|$ action and diameter $D$, the regret of Algorithm 2 can be bounded by

$$\text{Regret} \leq O \left( D^2 \sqrt{L^* \cdot |S| \log |A|} \right)$$

where $L^*$ is the total expected loss incurred by the best stationary deterministic policy in hindsight.

6 Conclusion

We considered learning in a communicating MDP with adversarially chosen costs in the full information setting. We designed an efficient algorithm that achieves $O(\sqrt{T})$ regret when transitions are deterministic.
We also presented an inefficient algorithm that achieves a $O(\sqrt{T})$ regret bounds for the general stochastic case with an extra mild assumption. Our result show that in the full information setting there is no statistical price for the extension from ergodic to communicating MDPs.

Several interesting questions still remain open. First, what are the best lower bounds in the general (i.e., not necessarily deterministic) communicating setting? In the deterministic setting, diameter is bounded polynomially by the state space size. This is no longer true in the stochastic case. The best lower bound in terms of diameter and other relevant quantities ($|S|, |A|$ and $T$) still remains to be worked out. Second, is it possible to design an efficient algorithm beyond the deterministic case? The source of inefficiency in our algorithm is that we run FPL with each policy as an expert and perturb the losses of each policy independently. It is plausible that an FPL algorithm that perturbs losses (as in the deterministic case) can also be analyzed. However, there are challenges in its analysis as well as in proving that it is computationally efficient. For example, we are not aware of any efficient way to compute the best deterministic policy in hindsight for the general communicating case.

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