Generalized Lens Categories via Functors $\mathcal{C}^{\text{op}} \to \text{Cat}$

David I. Spivak

Abstract

Lenses have a rich history and have recently received a great deal of attention from applied category theorists. We generalize the notion of lens by defining a category $\text{Lens}_F$ for any category $\mathcal{C}$ and functor $F: \mathcal{C}^{\text{op}} \to \text{Cat}$, i.e. an indexed category, using a variant of the Grothendieck construction. All of the mathematics in this note is straightforward; the purpose is simply to see lenses in a broader context where some closely-related examples, such as ringed spaces and open continuous dynamical systems, can be included.

1 Introduction

Roughly speaking, a lens is bi-directional map $(\text{get}, \text{put}): (c, x) \to (d, y)$ between pairs; the two parts have the following form:

$$\text{get}: c \to d \quad \text{and} \quad \text{put}: c \times y \to x. \tag{1}$$

Lenses have recently received a great deal of attention from applied category theorists. One reason is that they show up in many disparate places, such as database updates, learning algorithms, open games, open dynamical systems, and wiring diagrams. Lenses have been broadly generalized to so-called profunctor optics; see e.g. [Ril18].

We will discuss what seems to be a completely different direction of generalization: we associate a notion of generalized lens category to an arbitrary indexed category. Namely for every category $\mathcal{C}$ and (pseudo-) functor $F: \mathcal{C}^{\text{op}} \to \text{Cat}$, we define a category $\text{Lens}_F$ using a variant of the Grothendieck construction. The idea is that a morphism in the Grothendieck construction consists of two parts, which turn out to be the above get and put maps from lens theory. Taking $\mathcal{C} = \text{Set}$, we can recover the usual category of lenses in a couple of ways (see Example 3.5).

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and Proposition 3.10). The most basic is to take $F$ to be the slice category functor $\text{Slice}: \text{Set}^{\text{op}} \to \text{Cat}$, embed each $(c_x)$ as the projection $\pi_c: c \times x \to c$ in $\text{Slice}(c)$, and note that for any choice of function $\text{get}: c \to d$, the square shown below is a pullback:

$$
\begin{array}{ccc}
\pi_c & \downarrow_{\pi_c} & \\
c \times x & \xrightarrow{\text{put}} & c \times y \\
\downarrow_{\pi_c} & & \downarrow_{\pi_d} \\
c & \xrightarrow{\text{get}} & d
\end{array}
$$

The function indicated as $\text{put}$ does not have quite the same form as in Eq. (1): there is an extra factor of $c$ in the codomain. However, to be a morphism in the slice category $\text{Slice}(c)$, such a function $c \times y \to c \times x$ in $\text{Slice}(c)$ must commute with the projections. Thus it has no choice with regards to the $c$-factor in the codomain, and hence the only remaining choice is that of a function $\text{put}: c \times y \to x$, thus recovering the notion of morphism $(c_x) \to (d_y)$ from Eq. (1). This idea, to think of the an object $(c_x)$ not as a simple pair but as a dependent pair ($x$ dependent on $c$), is the main thrust of this note.

All of the results we discuss here are straightforward to prove. The proposed contribution is to provide a setting in which open continuous dynamical systems, ringed spaces, and dependent lenses—none of which fit in with the usual definition of lens but all of which seem to be quite similar to it in spirit—can be included in the theory. We also provide a construction of lenses in an arbitrary symmetric monoidal category, which is known but seems not to have been written down explicitly before.

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## 2 Lenses

Lenses have been studied in computer science and discussed category-theoretically for several decades [Ole83; Pai89; BPV06; Dis08; JRW12]. There are several variants, and the naming is often inconsistent. A good summary can be found in this blog post by Jules Hedges; see also the Haskell Lens library. We will be discussing what Hedges calls bimorphic lenses in [Hed17], but we will refer to them simply as lenses. We begin by recalling this notion.
2.1 Lenses in finite product categories

We begin with lenses in a category \( \mathcal{C} \) with finite products; one may think \( \mathcal{C} = \text{Set} \). In Section 2.2 we generalize to an arbitrary symmetric monoidal category, and then much further in Section 3.

**Notation 2.1.** We denote the composite of \( f : c \to d \) and \( g : d \to e \) by \( (f \circ g) : c \to e \). We denote the identity morphism on an object \( c \) either by \( \text{id}_c \) or simply by \( c \). We denote the hom-set from \( c \) to \( d \) in a category \( \mathcal{C} \) either by \( \text{Hom}(c, d) \), \( \text{Hom}_\mathcal{C}(c, d) \), or \( \mathcal{C}(c, d) \).

**Definition 2.2 (Lenses in finite product categories).** Let \( \mathcal{C} \) be a category with finite products. The category of \( \mathcal{C} \)-lenses, denoted \( \text{Lens}_{\mathcal{C},\times} \), has as objects pairs \( (c, x) \), where \( c, x \in \mathcal{C} \); given another such object \( (d, y) \) we define the homset

\[
\text{Lens}_{\mathcal{C},\times}((c, x), (d, y)) \coloneqq \left\{ (f, f^\sharp) \mid f : c \to d \text{ and } f^\sharp : c \times y \to x \right\}.
\]

We refer to the \( \mathcal{C} \)-morphisms \( f : c \to d \) as the get part and \( f^\sharp : c \times y \to x \) as the put part of the lens \( (f, f^\sharp) \).

The identity on \( (c, x) \) is \( (\text{id}_c, \epsilon_{c \times x}) = (\epsilon_c, c \times 1) \), where \( \epsilon_c : c \to 1 \) is the terminal map.

The composite of \( (f, f^\sharp) : (c, x) \to (d, y) \) and \( (g, g^\sharp) : (d, y) \to (e, z) \) is \( (\delta_c \times z) \circ (c \times f \circ z) \circ (c \times g^\sharp) \), where \( \delta_c : c \to c \times c \) is the diagonal. In pictures, it is given by the following string diagram:

\[
\text{(3)}
\]

**Example 2.3 (Simple lenses).** In functional programming the emphasis has often been on what we call simple lenses [BPV06], which are lenses of the form \( (c, x) \). A morphism of simple lenses \( (c, x) \to (d, y) \) consists of a function \( f : c \to d \) and a function \( f^\sharp : c \times y \to x \).

**Example 2.4 (Prisms, wiring diagrams).** If \( \mathcal{C} \) has finite coproducts then \( \mathcal{C}^{\text{op}} \) has finite products. The category \( \text{Lens}_{\mathcal{C}^{\text{op}},+}^{\text{op}} \) is called the category of prisms in \( \mathcal{C} \). A prism \( (c, x) \to (d, y) \) consists of a pair of morphisms \( d \to c \) and morphism \( x \to c + y \) in \( \mathcal{C} \).

The category \( \text{Lens}_{\text{FinSet}^{\text{op}},+}^{\text{op}} \) of prisms in \( \text{FinSet} \), or more generally replacing \( \text{FinSet} \) by \( \text{FinSet}/T \) for some set \( T \), is called the category of wiring diagrams in [VSL15]. When \( T = 1 \) this forms the left class of a factorization system on the category \( \text{Cob} \) of 1-dimensional oriented cobordisms, and similarly for arbitrary \( T \) (using cobordisms with components labeled in \( T \)). See [Aba15].

**Example 2.5 (Moore machines).** Given a pair of sets \( (A, B) \), a Moore machine (also called an open discrete dynamical system in [Spi15]) consists of a set \( S \) and two functions \( f^{\text{out}} : S \to B \) and \( f^{\text{upd}} : A \times S \to S \). This is the same as a lens \( (S, S) \to (A, B) \).
The notion of dynamical system (and the formula for composing them with wiring diagrams as in Example 2.4) can be generalized to the continuous case, with manifolds replacing the sets and systems of ordinary differential equations replacing the update functions. However, the theory of lenses does not accommodate this generalization. Remedy this lack was in part the motivation for the present note; see Example 3.6.

### 2.2 Lenses in symmetric monoidal categories

We owe the ideas of this section to David Jaz Myers, though these ideas are apparently folklore. For something similar, see [Abo+16, Section 2.2].

**Definition 2.6 (Commutative comonoid).** Let \((\mathcal{C}, I, \otimes, \sigma)\) be a symmetric monoidal category. A **commutative comonoid** in \(\mathcal{C}\) consists of a tuple \((c, \epsilon, \delta)\) where \(c \in \mathcal{C}\), \(\epsilon : c \to I\) and \(\delta : c \to c \otimes c\) satisfy the axioms:

1. \(\delta_c \cdot (c \otimes \epsilon_c) = c\),
2. \(\delta_c \cdot \sigma_{c,c} = \delta_c\) and
3. \(\delta_c \cdot (\delta_c \otimes c) = \delta_c \cdot (c \otimes \delta_c)\).

We refer to \(\epsilon\) as the **counit** and \(\delta\) as the **comultiplication**. We sometimes write \(c\) to denote the comonoid, leaving \(\epsilon\) and \(\delta\) implicit.

A **morphism** of commutative comonoids \((c, \epsilon_c, \delta_c) \to (d, \epsilon_d, \delta_d)\) is a morphism \(f : c \to d\) in \(\mathcal{C}\) such that \(\epsilon_c = f \cdot \epsilon_d\) and \(\delta_c \cdot (f \otimes f) = f \cdot \delta_d\). We denote the category of commutative comonoids and their morphisms by \(\text{Comon}_\mathcal{C}\).

**Proposition 2.7 (Finite product categories).** If \(\mathcal{C}\) has finite products and \((I, \otimes)\) is the corresponding (“Cartesian”) monoidal structure, then there is an isomorphism of categories \(\mathcal{C} \cong \text{Comon}_\mathcal{C}\).

**Proof.** See [Fox76].

The following is straightforward.

**Proposition 2.8.** There is a symmetric monoidal structure on \(\text{Comon}_\mathcal{C}\) such that the functor \(\text{Comon}_\mathcal{C} \to \mathcal{C}\) is strict monoidal.

**Definition 2.9.** Let \((\mathcal{C}, I, \otimes, \sigma)\) be a symmetric monoidal category. The category of \(\mathcal{C}\)-lenses, denoted \(\text{Lens}_{\mathcal{C},\otimes}\) has as objects pairs \((\xi)\), where \(c \in \text{Comon}_\mathcal{C}\) and \(x \in \mathcal{C}\). Given another such object \((\zeta, y)\) we define the homset

\[
\text{Lens}_{\mathcal{C},\otimes}((\xi), (\zeta, y)) := \left\{ \left( f \right) \mid f \in \text{Comon}_\mathcal{C}(c, d) \text{ and } f^\sharp \in \mathcal{C}(c \otimes y, x) \right\}.
\]

We refer to the comonoid morphism \(f : c \to d\) as the **get part** and the map \(f^\sharp : c \otimes y \to d\) as the **put part** of the lens \(\left( f \right)\).

4
The identity on \((\varepsilon_c^\otimes x)_c\) is \((\epsilon_c^\otimes x, \epsilon_c)\), where \(\epsilon_c : c \to 1\) is the counit. The composite of \((f \otimes \varepsilon)\) and \((g \otimes \delta) : (d, y) \to (\varepsilon, \delta)\) is \((\delta_{\otimes (\varepsilon \otimes f \otimes \delta)}}(\varepsilon \otimes g^\otimes \delta)\), where \(\delta_c : c \to c \otimes c\) is the counit. The string diagram for the composite is identical to that in Eq. (3).

Definition 2.9 generalizes Definition 2.2, by Proposition 2.7.

### 3 Generalized lens categories

We define the lens category \(\text{Lens}_F\) for any functor \(F : \mathcal{C}^{\text{op}} \to \text{Cat}\) and then give several examples.

#### 3.1 Definition and examples of \(\text{Lens}_F\)

The Grothendieck construction comes in two variants.

**Definition 3.1 (Grothendieck constructions).** Let \(\mathcal{C}\) be a category and \(F : \mathcal{C} \to \text{Cat}\). The **covariant Grothendieck construction** of \(F\) consists of a category \(\text{Gr}(F)\) and a functor \(\pi_F : \text{Gr}(F) \to \mathcal{C}\), defined as follows:

\[
\begin{align*}
\text{Ob}(\text{Gr}(F)) & := \bigsqcup_{c \in \text{Ob}(\mathcal{C})} \text{Ob}(F(c)) \\
\text{Gr}(F)((c, x), (d, y)) & := \bigsqcup_{f \in \mathcal{C}(c, d)} \text{Hom}_{F(d)}(F(f)(x), y)
\end{align*}
\]

That is, an object in \(\text{Gr}(F)\) is a pair \((c, x)\) where \(c \in \mathcal{C}\) and \(x \in F(c)\). A morphism \((c, x) \to (d, y)\) is a pair \((f, f^\otimes)\) where \(f : c \to d\) and \(f^\otimes : F(f)(x) \to y\) is a morphism in the category \(F(d)\). The identity on \((c, x)\) is \((\text{id}_c, \text{id}_x)\), and the composite of \((f, f^\otimes)\) and \((g, g^\otimes)\) is given by

\[
(f, f^\otimes) \cdot (g, g^\otimes) := \left((f \cdot g), (F(g)(f^\otimes \cdot g^\otimes))\right).
\]

The functor \(\pi_F : \text{Gr}(F) \to \mathcal{C}\) sends \((c, x) \mapsto c\) and \((f, f^\otimes) \mapsto f\).

Given a functor \(F : \mathcal{C}^{\text{op}} \to \text{Cat}\), the **contravariant Grothendieck construction** of \(F\) consists of a category \(\text{Gr}^\circ(F)\) and a functor \(\pi_F : \text{Gr}^\circ(F) \to \mathcal{C}\), defined as follows:

\[
\begin{align*}
\text{Ob}(\text{Gr}^\circ(F)) & := \bigsqcup_{c \in \text{Ob}(\mathcal{C})} \text{Ob}(F(c)) \\
\text{Gr}^\circ(F)((c, x), (d, y)) & := \bigsqcup_{f \in \mathcal{C}(c, d)} \text{Hom}_{F(c)}(x, F(f)(y))
\end{align*}
\]

Identities, composition, and the functor \(\pi_F\) are defined analogously.

For any functor \(F : \mathcal{C}^{\text{op}} \to \text{Cat}\), let \(F^p : \mathcal{C}^{\text{op}} \to \text{Cat}\) denote its pointwise opposite, \(F^p(c) := F(c)^{\text{op}}\). The following is straightforward.
**Proposition 3.2.** Let $F : \mathcal{C}^{\text{op}} \to \text{Cat}$ be a functor, and let $F^p : \mathcal{C}^{\text{op}} \to \text{Cat}$ be its pointwise opposite. The following three categories are naturally isomorphic:

1. $\text{Gr}(F)^{\text{op}}$, the opposite of the covariant Grothendieck construction of $F$,
2. $\text{Gr}^p(F)$, the contravariant Grothendieck construction of the pointwise opposite of $F$,
3. the analogous category with objects $\bigsqcup_{c \in \text{Ob}(\mathcal{C})} \text{Ob}(F(c))$ and morphisms

$$\text{Hom}((c, x), (d, y)) := \bigsqcup_{f \in \mathcal{C}(c, d)} \text{Hom}_{F(c)}(F(f)(y), x). \quad (4)$$

Moreover, these isomorphisms commute with the functors $\pi_F$ to $\mathcal{C}$.

**Definition 3.3 (F-lenses).** Let $F : \mathcal{C}^{\text{op}} \to \text{Cat}$ be a functor. Define the category of $F$-lenses, denoted $\text{Lens}_F$, to be any of the three isomorphic categories from Proposition 3.2. We refer to $\mathcal{C}$ as the get-category and to $\pi_F : \text{Lens}_F \to \mathcal{C}$ as the get-functor.

We denote the object $(c, x)$ by $(c, x)$. From the explicit formula (4), we see that a morphism $(f, f^\#) : (c, x) \to (d, y)$ consists of a pair $(f, f^\#)$ where $f : c \to d$ is in $\mathcal{C}$ and $f^\# : F(f)(y) \to x$ is a morphism in the category $F(c)$.

With this definition, lenses have a diagrammatically simpler form than in Eq. (3). Here we show morphisms $(f, f^\#) : (c, x) \to (d, y)$ and $(g, g^\#) : (d, y) \to (e, z)$:

The wires represent objects $c, d, e \in \mathcal{C}$, and white the squares represent morphisms in $\mathcal{C}$. The blue circles and double-arrows represent objects and morphisms in the categories $F(c)$, etc. Here is a picture of the composite $(f, f^\#) \circ (g, g^\#)$:

One may also imagine these morphisms logically, e.g. the implications

$$\forall c. y(f(c)) \Rightarrow x(c) \quad \forall d. z(g(d)) \Rightarrow y(d)$$

which can be combined to obtain $\forall c. z(f(g(c))) \Rightarrow x(c)$.

**Remark 3.4.** The Grothendieck construction of a Cat-valued functor $F : \mathcal{C}^{\text{op}} \to \text{Cat}$ always yields a (split) fibration over $\mathcal{C}$ and vice versa, so generalized lens categories can be viewed simply as split fibrations. However we chose $\text{Lens}_F$ to be the fiberwise opposite of $\text{Gr}(F)$—rather than replacing $F$ with $F^p$ at the outset—for two reasons. First, in cases of interest $F$ seems to be simpler to specify than $F^p$. Second, the form of (4) is the one that is most familiar in lens theory.
Example 3.5 (Dependent lenses). Let \( \mathcal{C} \) be a category with pullbacks. There is a functor \( \text{Slice}: \mathcal{C}^{\text{op}} \to \text{Cat} \) given on an object \( c \) by the slice category \( \text{Slice}(c) := \mathcal{C}/c \) over \( c \), and on morphisms \( f: c \to b \) in \( \mathcal{C}^{\text{op}} \) by pullback. That is, for every object \( p: x \to c \) in \( \text{Slice}(c) \) we obtain an object \( \text{Slice}(f)(p) \in \mathcal{C}/b \) using the following pullback diagram in \( \mathcal{C} \):

\[
\begin{array}{ccc}
  b \times_c x & \xrightarrow{\pi_2} & x \\
  \downarrow & & \downarrow p \\
  b & \xrightarrow{f} & c
\end{array}
\]

This extends to morphisms in \( \text{Slice}(c) \) using the universal property of pullbacks.

The category \( \text{Lens}_{\text{Slice}} \) has as objects pairs \( (\rho, p) \) where \( p: x \to c \), and as morphisms pairs \( (f, f^\sharp) \) where \( f: c \to d \) and \( f^\sharp: c \times_d y \to x \). We can think of objects in \( \text{Lens}_{\text{Slice}} \) as dependent lenses; for example if \( \mathcal{C} = \text{Set} \), then each object \( p: x \to c \) may assign non-isomorphic fibers to different elements of \( c \).

Note that we can find the category of lenses from Definition 2.2 inside of \( \text{Lens}_F \). Indeed, it is isomorphic to the full subcategory spanned by all pairs \( (\rho, p) \) for which \( \pi: c \times x \to c \) is the projection for some \( x \in \mathcal{C} \). This was discussed in the introduction around Eq. (2). We will recover the category of lenses in a completely different way in Proposition 3.10.

More importantly, the get functor \( \pi_F: \text{Lens}_{\text{Slice}} \to \mathcal{C} \) is not only a fibration but a bifibration. Indeed, the functor \( \text{Slice}(f): \text{Slice}(d) \to \text{Slice}(c) \) has a left adjoint \( \Sigma_f: \text{Slice}(c) \to \text{Slice}(d) \), which one may call the dependent sum along \( f \), for any morphism \( f: c \to d \) in \( \mathcal{C} \). The name “dependent lens” is actually most appropriate when \( \mathcal{C} \) not only has pullbacks but is locally cartesian closed. This simply means that each \( \text{Slice}(f) \) additionally has a right adjoint \( \Pi_f: \text{Slice}(c) \to \text{Slice}(d) \), called the dependent product along \( f \). In this case \( \pi_F \) is a trifibration.

Example 3.6 (Open continuous dynamical systems). Recall that a differentiable manifold \( M \) has a tangent bundle \( TM \) and a submersion \( \pi_M: TM \to M \). Given a pair of manifolds \( (A, B) \), [VSL15] defines an open continuous dynamical system with inputs \( A \) and outputs \( B \) to be a manifold \( S \) (called the state space), a differentiable map \( f^{\text{out}}: S \to B \), and a differentiable map \( f^{\text{dyn}}: A \times S \to TS \) such that \( f^{\text{dyn}} \circ \pi_S = \pi_2 \). We can see this as a morphism in a generalized lens category as follows.

Consider the functor \( \text{Subm}: \text{Mfd}^{\text{op}} \to \text{Cat} \) sending each manifold \( M \) to the category of submersions over \( M \), and sending a differentiable map \( f: M \to N \) to the pullback functor along \( f \). Then an open continuous dynamical system with inputs \( A \) and outputs \( B \) consists of a morphism \( (\pi_S) \to (\pi_B) \) in \( \text{Lens}_{\text{Subm}} \), where \( \pi_B: A \times B \to B \) is the projection. [VSL15] shows that continuous dynamical systems can be wired together using prisms, as in Example 2.4.
Consider again the lens category for the functor $\text{Subm} : \text{Mfd}^{\text{op}} \to \text{Cat}$, as in Example 3.6. Whereas in that example we considered objects given by tangent bundles and found that certain morphisms between them were continuous dynamical systems, here we consider objects given by cotangent bundles and find that we get lenses between them canonically, without making choices.

To begin, note that a differentiable manifold $M$ also has a cotangent bundle $T^*M$; its fiber over each point $m \in M$ is the dual to the tangent space there, i.e. it is the vector space $T^*_mM := \text{Vect}(T_mM, \mathbb{R})$ of linear maps from the tangent space to the ground field $\mathbb{R}$. Given a differentiable function $f : M \to N$, the derivative (Jacobian) defines a map $Tf : TM \to TN$, in particular over each point $m \in M$ a linear transformation $T^*_mM \to (T^*f)_mN$. This in turn induces a linear transformation $\text{Vect}((T^*f)_mN, \mathbb{R}) \to \text{Vect}(T^*_mM, \mathbb{R})$ for each point $m$, and these assemble into a morphism $f^* : (T^*N) \to T^*M$ of bundles over $M$. All together we have canonically obtained a morphism $(f, f^*) : (M, T^*_M) \to (N, T^*_N)$ in the lens category $\text{Lens}_{\text{Subm}}$, and hence a functor $\text{Mfd} \to \text{Lens}_{\text{Subm}}$.

Example 3.8 (Ringed spaces). The category of ringed spaces from algebraic geometry is an example of a generalized lens category. There is a functor $\text{Sh} : \text{Top}^{\text{op}} \to \text{Cat}$, where $\text{Top}$ is the category of topological spaces and $\text{Sh}(X)$ is the category of sheaves of rings on $X$; given a map $f : X \to Y$ in $\text{Top}$, there is a functor $f^*$ which sends a sheaf on $Y$ to a sheaf on $X$, hence defining $\text{Sh}$ on morphisms.

The category $\text{Lens}_{\text{Sh}}$ of $\text{Sh}$-lenses has as objects pairs $(X, O_X)$ where $O_X$ is a sheaf of rings on $X$. A morphism $(X, O_X) \to (Y, O_Y)$ is a pair $(f, f^*)$ where $f : X \to Y$ is a map of topological spaces and $f^* : f^*O_Y \to O_X$ is a map of sheaves of rings.

Example 3.9 (Twisted arrow category). Let $\mathcal{C}$ be a category and consider the coslice functor $\text{Coslice} : \mathcal{C}^{\text{op}} \to \text{Cat}$ sending $c \mapsto c/\mathcal{C}$; on morphisms $f : d \to c$ the functor $\text{Coslice}(f)$ sends an object $x : c \to d \in \text{Coslice}(c)$ to the composite $f \circ x$. The category $\text{Lens}_{\text{Coslice}}$ is the twisted arrow category $\text{tw}(\mathcal{C})$ of $\mathcal{C}$. An object $(\frac{c}{x}, \frac{d}{y})$ in $\text{Lens}_{\text{Coslice}}$ is a morphism $x : c \to d$, and a morphism $(\frac{c}{x}) \to (\frac{d}{y})$ in $\text{Lens}_{\text{Coslice}}$ is a twisted square

$$
\begin{array}{ccc}
\frac{c}{x} & \xrightarrow{f} & \frac{d}{y} \\
\downarrow & & \downarrow \\
\frac{d}{y} & \xleftarrow{f^*} & \frac{d'}{y'}
\end{array}
$$

3.2 Lenses in symmetric monoidal categories

Let $(\mathcal{C}, I, \otimes)$ be a symmetric monoidal category, and let $\text{Comon}_\mathcal{C}$ denote its (symmetric monoidal) category of commutative comonoids (see Definition 2.6). For each object $c \in \text{Comon}_\mathcal{C}$ there is a comonad on $\mathcal{C}$ given by $x \mapsto c \otimes x$; the counit is
given by $\epsilon_c \otimes x$ and the comultiplication is given by $\delta_c \otimes x$, maps which are natural in $x$. Forming the coKleisli category gives a functor $\text{coKle} : \text{Comon}_C^{op} \to \text{Cat}$. Let’s unpack this.

The functor $\text{coKle} : \text{Comon}_C^{op} \to \text{Cat}$ has the following more explicit formulation. Given an commutative comonoid $(c, \epsilon, \delta)$, the coKleisli category $\text{coKle}(c)$ has objects $\text{Ob}(\mathcal{C})$ and morphisms $\text{Hom}(x, y) := \mathcal{C}(c \otimes x, y)$. The identity on $x$ is given by $(\epsilon_c \otimes x) : c \otimes x \to x$, and the composite of $f : c \otimes x \to y$ and $g : c \otimes y \to z$ is given by $(f \circ g) = (\delta_c \otimes x) \circ (c \otimes f) \circ g$. In pictures:

![Diagram](image)

Given a morphism of comonoids $p : c \to d$ in $\text{Comon}_C$, we obtain an identity-on-objects functor $\text{coKle}(p) : \text{coKle}(d) \to \text{coKle}(c)$ that sends the morphism $d \otimes x \to y$ to its composite with $(p \otimes x) : (c \otimes x) \to (d \otimes x)$.

**Proposition 3.10.** Let $\mathcal{C}$ be symmetric monoidal and let $\text{coKl} : \text{Comon}_C^{op} \to \text{Cat}$ be as in Section 3.2. The generalized lens category $\text{Lens}_{\text{coKl}}$ is isomorphic to the category $\text{Lens}_{\mathcal{C}, \otimes}$ from Definition 2.9.

**Proof.** The objects of both $\text{Lens}_{\text{coKl}}$ and $\text{Lens}_{\mathcal{C}, \otimes}$ are pairs $(\frac{c}{x})$, where $c \in \text{Comon}_C$ and $x \in \text{Ob}(\text{coKl}(c)) = \text{Ob } \mathcal{C}$.

The morphisms $(\frac{c}{x}) \to (\frac{d}{y})$ in the latter are pairs $(f, f^\sharp)$ where $f : c \to d$ is a morphism of comonoids and $f^\sharp : c \otimes y \to x$ is any morphism. In the former, the morphisms are pairs $(f, f^\sharp)$ where $f : c \to d$ is a map of comonoids and $f^\sharp : \text{coKl}(f)(y) \to x$ is a map in $\text{coKl}(c)$. Since $\text{coKl}(f)$ is identity on objects, $f^\sharp$ is a morphism $c \otimes y \to x$ in $\mathcal{C}$, so again the morphisms in the two categories coincide. One may check that the identities and composition formulas also coincide. 

3.3 When $F$ is monoidal, so is $\text{Lens}_F$

**Definition 3.11** (Monoidal Grothendieck construction). Let $(\mathcal{C}, I, \otimes)$ be a monoidal category and $(F, \varphi) : \mathcal{C} \to \text{Cat}$ a lax monoidal functor. The monoidal Grothendieck construction [MV18] returns a monoidal structure on the category $\text{Gr}(F)$, and hence on $\text{Lens}_F = \text{Gr}(F)^{op}$, by $(\frac{c}{x}) \otimes (\frac{d}{y}) := (\frac{c \otimes d}{\varphi(x, y)})$.

**Example 3.12.** The functors $\text{coKle} : \mathcal{C} \to \text{Cat}$ for arbitrary symmetric monoidal $\mathcal{C}$ from Section 3.2, as well as $\text{Subm} : \text{Mfd}^{op} \to \text{Cat}$ and $\text{Sh} : \text{Top}^{op} \to \text{Cat}$ from Examples 3.6 and 3.8 are all lax monoidal. Thus each of the lens categories $\text{Lens}_{\text{coKl}}, \text{Lens}_{\text{Subm}},$ and $\text{Lens}_{\text{Sh}},$ inherit symmetric monoidal structures. From the first and third examples we recover the usual monoidal structure on the usual category of lenses in a symmetric monoidal category, as well as that on ringed spaces.
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