Abstract. Noncommutative Kähler structures were recently introduced as an algebraic framework for studying noncommutative complex geometry on quantum homogeneous spaces. In this paper, we introduce the notion of a compact quantum homogeneous Kähler space which gives a natural set of compatibility conditions between covariant Kähler structures and Woronowicz’s theory of compact quantum groups. Each such object admits a Hilbert space completion possessing a remarkably rich yet tractable structure. The analytic behaviour of the associated Dolbeault–Dirac operators is moulded by the complex geometry of the underlying calculus. In particular, twisting the Dolbeault–Dirac operator by a negative Hermitian holomorphic module is shown to give a Fredholm operator if and only if the top anti-holomorphic cohomology group is finite-dimensional. In this case, the operator’s index coincides with the twisted holomorphic Euler characteristic of the underlying noncommutative complex structure. The irreducible quantum flag manifolds, endowed with their Heckenberger–Kolb calculi, are presented as motivating examples.

1. Introduction

Since the emergence of quantum groups in the 1980s, a central role in their presentation and development has been played by the theory of operator algebras. We mention in particular Woronowicz’s seminal notion of a compact quantum group [41]. There exists, however, a stark contrast between the development of the noncommutative topological and the noncommutative differential geometric aspects of the theory. For example, for Drinfeld–Jimbo quantum groups, their $C^*$-algebraic $K$-theory has long been known to be the same as for their classical counterparts [29]. By contrast, the unbounded formulation of $K$-homology, which is to say Connes and Moscovici’s theory of spectral triples, remains very poorly understood. Indeed, despite a large number of very important contributions over the last thirty years, there is still no consensus on how to construct a spectral triple for $\mathcal{O}_q(SU_2)$, probably the most fundamental example of a quantum group.

There does, however, exist a long standing algebraic approach to constructing $q$-deformed differential operators for quantum groups based on the theory of covariant differential calculi. This has its origins in the work of Woronowicz [42], with steady advances made in the following decades by many others, most notably Majid [5]. As has become increasingly clear in recent years, this approach is particularly suited to the study of quantum flag manifolds, quantum homogeneous spaces which $q$-deform the coordinate rings of the classical flag manifolds $G/L_S$. These quantum spaces are distinguished by...
being braided commutative algebra objects in the braided monoidal category of $O_q(g)$-comodules, and have a geometric structure much closer to the classical situation than quantum groups themselves. This is demonstrated by the existence of an essentially unique $q$-deformed de Rham complex for the irreducible quantum flag manifolds, as shown by Heckenberger and Kolb in their seminal series of papers [19] [20] [21]. This makes the quantum flag manifolds a far more tractable starting point than quantum groups for investigating $q$-deformed noncommutative geometry.

The classical flag manifolds are compact connected homogeneous Kähler manifolds, providing us with a rich store of geometric structures to exploit. Motivated by this, the notion of a noncommutative Hermitian structure was introduced in [31] to provide a framework in which to study the noncommutative geometry of the quantum flag manifolds. Many of the fundamental results of Hermitian and Kähler geometry follow from the existence of such a structure, providing powerful tools with which to study the underlying calculus. The existence of a Kähler structure was verified for the Heckenberger–Kolb calculus of quantum projective space in [31]. This result was later extended by Matassa [28] to every Heckenberger–Kolb calculus, for all but a finite number of values of $q$. Moreover, further examples are anticipated to arise in due course from more general classes of quantum flag manifolds. Indeed, Kähler structures have recently been discovered in the setting of holomorphic étale groupoids [5] promising a much wider domain of application than initially expected.

In this paper we build on this rich algebraic and geometric structure to produce a theory of bonded and unbounded differential operators acting on square integrable forms. We do so in the novel framework of compact quantum homogeneous Hermitian spaces (CQH-Hermitian spaces) which detail a natural set of compatibility conditions between covariant Hermitian structures and Woronowicz’s theory of compact quantum groups. Every CQH-Hermitian space is shown to have a naturally associated Hilbert space completion. Moreover, much of the theory of Hermitian structures carries over to square integrable setting, giving almost complex and Lefschetz decompositions, as well as a bounded representation of $\mathfrak{sl}_2$. The de Rham, holomorphic, and anti-holomorphic differentials also behave very well with respect to completion. All three Dirac operators $D\partial, D\bar{\partial},$ and $Dq$ are essentially self-adjoint, giving access to powerful analytic machinery such as functional calculus. The spectral and index theoretic properties of these operators are intimately connected with the curvature and cohomology of the underlying calculus. Moreover, they are highly amenable to applications of the concepts and structures of classical complex geometry. As shown in [32] twisting the anti-holomorphic Dolbeault–Dirac operator of a CQH-Kähler space by a negative (anti-ample) relative Hopf module produces a Fredholm operator if and only if the top anti-holomorphic cohomology group is finite-dimensional. Just as in the classical case, Hodge theory then implies that the index of the twisted operator is given by the twisted anti-holomorphic Euler characteristic. This invariant can be determined by geometric means. In particular, for positive modules, it follows from the Kodaira vanishing theorem for noncommutative Kähler structures that all higher cohomologies vanish, meaning that the index is concentrated in degree zero. The case of negative modules follows similarly through an application of noncommutative Serre duality [32] [§6.2].)
There exists a large literature dealing with the analytic properties of Dolbeault–Dirac operators over quantum flag manifolds $O_q(G/L)$. For example, see the pioneering papers [34, 15, 11], or the review of such constructions in [12 §1]. This paper is inspired by many of the observations and results in these works. Indeed, our principal motivation for introducing CQH-Kähler spaces is to provide a robust formal framework in which to study the analytic properties of Dolbeault–Dirac operators over $O_q(G/L)$. 

1.1. Summary of the Paper. The paper is organised as follows: In §2 we recall from [31] the necessary basics of Hermitian and Kähler structures, and Hermitian holomorphic modules.

In §3, we introduce the notion of compact quantum homogeneous Hermitian space $(B \subseteq A, \Omega^*, \Omega^{**}, \sigma)$, and its twist by an Hermitian module $F$. We then use Takeuchi’s categorical equivalence to show boundedness of morphisms, giving us a bounded representation of $\mathfrak{sl}_2$ on $L^2(F \otimes_B \Omega^*)$. Moreover, in the untwisted case we prove boundedness of multiplication operators, and conclude boundedness of the commutators $[D_{\Omega^*} b]$, for all $b \in B$, under certain assumptions.

In §4 we show that twisting by a negative module produces a Fredholm if and only if the top anti-holomorphic cohomology group of $F$ is finite-dimensional.

In §5 we recall our motivating family of examples, the irreducible quantum flag manifolds $O_q(G/L)$ endowed with their Heckenberger–Kolb calculi. We produce a family of Dolbeault–Dirac Fredholm operators for each $O_q(G/L)$ through twisting by a negative line bundle. Moreover, we give an explicit presentation of the operator index in terms of the Weyl dimension formula.

We finish with three appendices. The first recalls the theory of compact quantum groups algebras, the second recalls elementary material on unbounded operators, and the third discusses the relationship with the theory of spectral triples.

Acknowledgements: The authors would like to thank Karen Strung, Branimir Ćaćić, Elmar Wagner, Fredy Díaz García, Andrey Krutov, Simon Brain, Adam Rennie, Bob Yuncken, Paolo Saracco, Kenny De Commer, Matthias Fischmann, Adam–Christiaan van Roosmalen, Jan Šťovíček, and Zhaoting Wei, for many useful discussions during the preparation of this paper. The second author would like to thank IMPAN Wrocław for hosting him in November 2017, and would also like to thank the Institute for Mathematics, Astrophysics and Particle Physics, Radboud University, Nijmegen for hosting him in the winter of 2017 and 2018.

2. Preliminaries on Hermitian Structures

We recall the basic definitions and results for Hermitian, and Kähler structures over differential $*$-calculi. For a more detailed introduction see [31], and references therein. For an excellent presentation of classical complex and Kähler geometry see [24].

2.1. Differential Calculi and Complex Structures. A differential calculus, or dc, is a differential graded algebra $(\Omega^* \simeq \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \Omega^k, d)$ generated as an algebra by the elements $a, db$, for $a, b \in \Omega^0$. A dc over an algebra $B$ is dc such that $\Omega^0 = B$. A dc is said to
be of total degree \( m \in \mathbb{Z}_{>0} \) if \( \Omega^m \neq 0 \), and \( \Omega^k = 0 \), for all \( k > m \). A *-differential calculus, or a *dc, over a *-algebra \( B \) is a dc over \( B \) such that the *-map of \( B \) extends to a conjugate linear involutive map \( * : \Omega^* \to \Omega^* \) satisfying \( d(\omega^*) = (d\omega)^* \), and \( (\omega \wedge \nu)^* = (-1)^{kl} \nu^* \wedge \omega^* \), for all \( \omega \in \Omega^k, \nu \in \Omega^l \). See \([3, \S 1]\) for a more detailed discussion of differential calculi.

An almost complex structure \( \Omega^{(*)} \), for a *dc \( (\Omega^*, d) \), is an \( \mathbb{Z}_{\geq 0} \)-algebra grading of \( \Omega^* \) such that
\[
\Omega^k = \bigoplus_{a+b=k} \Omega^{(a,b)}, \quad (\Omega^{(a,b)})^* = \Omega^{(b,a)}, \quad \text{for all } (a, b) \in \mathbb{Z}_{\geq 0}^2.
\]

If the exterior derivative decomposes into a sum \( d = \partial + \overline{\partial} \), for \( \partial \) a (necessarily unique) degree \((1,0)\)-map, and \( \overline{\partial} \) a (necessarily unique) degree \((0,1)\)-map, then we say that \( \Omega^{(*)} \) is a complex structure. It follows that we have a double complex. The opposite complex structure of an complex structure \( \Omega^{(*)} \) is the \( \mathbb{Z}_{\geq 0}^2 \)-algebra grading \( \overline{\Omega^{(*)}} \), defined by \( \overline{\Omega}^{(a,b)} := \Omega^{(b,a)} \), for \( (a, b) \in \mathbb{Z}_{\geq 0}^2 \). See \([3, \S 1]\) or \([25, 4, 30]\) for a more detailed discussion of complex structures.

### 2.2. Hermitian and Kähler Structures

We recall the definition of an Hermitian structure, as introduced in \([31, \S 4]\), which generalises the properties of the fundamental form of an Hermitian metric. A dc is said to be of total degree \( m \in \mathbb{Z}_{>0} \) if \( \Omega^m \neq 0 \), and \( \Omega^k = 0 \), for all \( k > m \).

**Definition 2.1.** An Hermitian structure \( (\Omega^{(*)}, \sigma) \) for a *dc \( \Omega^* \), of even total degree \( 2n \), is a pair consisting of a complex structure \( \Omega^{(*)} \), and a closed central real \((1,1)\)-form \( \sigma \), called the Hermitian form, such that, with respect to the Lefschetz operator \( L_\sigma : \Omega^* \to \Omega^* \), defined by \( \omega \mapsto \sigma \wedge \omega \), isomorphisms are given by
\[
L_\sigma^{n-k} : \Omega^k \to \Omega^{2n-k}, \quad \text{for all } k = 0, \ldots, n-1.
\]

A Kähler structure is an Hermitian structure \( (\Omega^{(*)}, \kappa) \) satisfying \( d\kappa = 0 \), and in this case we refer to \( \kappa \) as the Kähler form.

For \( L_\sigma \) the Lefschetz operator of an Hermitian structure, we denote
\[
P^k := \begin{cases} \{ \alpha \in \Omega^k \mid L_\sigma^{n-k+1}(\alpha) = 0 \}, & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases}
\]

An element of \( P^* \) is called a primitive form. As established in \([31, \text{Proposition 4.3}]\), for \( L_\sigma \) the Lefschetz operator of an Hermitian structure on a *dc \( \Omega^* \), a \( B \)-bimodule decomposition of \( \Omega^k \), for all \( k \in \mathbb{Z}_{\geq 0} \), is given by
\[
\Omega^k \simeq \bigoplus_{j \in \mathbb{Z}_{\geq 0}} L_j^j(P^{k-2j}).
\]

In classical Hermitian geometry, the Hodge map of an Hermitian metric is related to the associated Lefschetz decomposition through the Weil formula (see \([10, \text{Théorème 1.2}]\) or \([24, \text{Proposition 1.2.31}]\)). In \([31, \text{Definition 4.11}]\) a noncommutative generalisation of the Weil formula, based on a noncommutative generalisation of Lefschetz decomposition,
was used to define a noncommutative Hodge map \( * \) for any noncommutative Hermitian structure. The \emph{metric} associated to the Hermitian structure \( (\Omega^{(0,\bullet)}, \sigma) \) is the unique map \( g_\sigma : \Omega^\bullet \times \Omega^\bullet \to B \) for which \( g_\sigma(\Omega^k, \Omega^l) = 0 \), for all \( k \neq l \), and 
\[
g_\sigma(\omega, \nu) = *_\sigma( *_\sigma (\omega^*) \wedge \nu), \quad \text{for all } \omega, \nu \in \Omega^k.
\]
An important fact is that \( g_\sigma \) is \emph{conjugate symmetric}, that is, 
\[
g_\sigma(\omega, \nu) = g_\sigma(\nu, \omega)^*, \quad \text{for all } \omega, \nu \in \Omega^\bullet.
\]
For a \( \ast \)-algebra \( B \), we consider the \emph{cone of positive elements} 
\[
B_{\geq 0} := \text{span}_{\mathbb{R}_{\geq 0}} \{ b_i^* b_i | b_i \in B, \in \mathbb{Z}_{\geq 0} \}.
\]
We denote the non-zero positive elements of \( B \) by \( B_{\geq 0} := B_{\geq 0} \setminus \{0\} \). We say that an Hermitian structure \( (\Omega^{(0,\bullet)}, \sigma) \) is \emph{positive definite} if the associated metric \( g_\sigma \) is positive definite, which is to say, if \( g_\sigma \) satisfies 
\[
g_\sigma(\omega, \omega) \in B_{\geq 0}, \quad \text{for all } \omega \in \Omega^\bullet.
\]
As established in \cite{31}, \S Lemma 5.2, the Lefschetz decomposition is orthogonal with respect to \( g_\sigma \), as is the \( \mathbb{Z}_{\geq 0} \)-decomposition of the complex structure.

2.3. A Representation of \( \mathfrak{sl}_2 \). As is readily verified \cite{31} Lemma 5.11, the Lefschetz map \( L_\sigma \) is adjointable on \( \Omega^\bullet \) with respect to \( g_\sigma \). Taking \( L_\sigma \) and \( \Lambda_\sigma \) together with the \emph{form degree counting operator} 
\[
H : \Omega^\bullet \to \Omega^\bullet, \quad H(\omega) := (k-n)\omega, \quad \text{for } \omega \in \Omega^k,
\]
we get the following commutator relations:
\[
[H, L_\sigma] = 2H, \quad [L_\sigma, \Lambda_\sigma] = H, \quad [H, \Lambda_\sigma] = -2\Lambda_\sigma.
\]
Thus any Kähler structure gives a Lie algebra representation \( T : \mathfrak{sl}_2 \to \mathfrak{gl}(\Omega^\bullet) \). Moreover, for any right \( B \)-module \( \mathcal{F} \), we can extend this to a representation on \( \mathcal{F} \otimes_B \Omega^\bullet \) in the obvious way, using the three operators \( L_\mathcal{F} := L_\sigma \otimes \text{id}_\mathcal{F} \), \( H_\mathcal{F} = H \otimes \text{id}_\mathcal{F} \), and \( \Lambda_\mathcal{F} := \Lambda_\sigma \otimes \text{id}_\mathcal{F} \).

2.4. Holomorphic Modules. Motivated by the Koszul–Malgrange characterisation of holomorphic bundles \cite{27}, noncommutative holomorphic modules have been considered a number of times in the literature, see for example \cite{4}, \cite{35}, \cite{25}, or \cite{32}.

For \( \Omega^\bullet \) a dc over an algebra \( B \), and \( \mathcal{F} \) a left \( B \)-module, a \emph{left connection} on \( \mathcal{F} \) is a \( \mathbb{C} \)-linear map \( \nabla : \mathcal{F} \to \mathcal{F} \otimes_B \mathcal{F} \) satisfying
\[
\nabla(fb) = f \otimes db + (\nabla f)b, \quad \text{for all } b \in B, f \in \mathcal{F}.
\]
For a choice \( \Omega^{(0,\bullet)} \) of complex structure on \( \Omega^\bullet \), a \( (0,1) \)-\emph{connection} on \( \mathcal{F} \) is a connection with respect to the dc \( (\Omega^{(0,\bullet)}, \overline{\partial}) \). Any connection can be extended to a map \( \nabla : \mathcal{F} \otimes_B \Omega^\bullet \to \mathcal{F} \otimes_B \Omega^\bullet \) by defining
\[
\nabla(f \otimes \omega) = \nabla(f) \wedge \omega + f \otimes d\omega + \quad \text{for } f \in \mathcal{F}, \omega \in \Omega^\bullet.
\]
The \emph{curvature} of a connection is the right \( B \)-module map \( \nabla^2 : \mathcal{F} \to \mathcal{F} \otimes_B \Omega^2 \). A connection is said to be \emph{flat} if \( \nabla^2 = 0 \).
Definition 2.2. For an algebra $B$, a holomorphic module over $B$ is a pair $(\mathcal{F}, \overline{\mathcal{F}})$, where $\mathcal{F}$ is a finitely generated projective right $B$-module, and $\overline{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{F} \otimes_B \Omega^{(0,1)}$ is a flat $(0,1)$-connection, which we call the holomorphic structure of $(\mathcal{F}, \overline{\mathcal{F}})$.

2.5. Hermitian Modules. When $B$ is a $\ast$-algebra, we can also generalise the classical notion of an Hermitian metric for a vector bundle. For a right $B$-module $\mathcal{F}$, denote by $\mathcal{F}^\ast$ the dual module $\text{Hom}_B(\mathcal{F}, B)$, of right $B$-module maps, which is a left $B$-module with respect to the left multiplication $(b\phi)(f) := b\phi(f)$, for $\phi \in \mathcal{F}^\ast$, and $b \in B$, $f \in \mathcal{F}$. Moreover, we denote by $\overline{\mathcal{F}}$ the conjugate right $B$-module of $\mathcal{F}$, as defined by the action $b\overline{f} = \overline{f}b$, for $f \in \mathcal{F}, b \in B$.

Definition 2.3. An Hermitian module over a $\ast$-algebra $B$ is a pair $(\mathcal{F}, h_\mathcal{F})$, where $\mathcal{F}$ is a finitely generated projective right $B$-module and $h_\mathcal{F} : \mathcal{F} \rightarrow \mathcal{F}^\ast$ is a right $B$-module isomorphism, such that, for the associated sesquilinear pairing,

$$h^\prime_\mathcal{F}(-,-) : \mathcal{F} \times \mathcal{F} \rightarrow B; \quad (f,k) \mapsto h_\mathcal{F}(\overline{k})(f)$$

it holds that, for all $f,k \in \mathcal{F}$,

1. $h^\prime_\mathcal{F}(f,k) = h_\mathcal{F}(k,f)^\ast$,
2. $h^\prime_\mathcal{F}(f,f) \in B_{>0}$.

Consider next the map $C_h : \mathcal{F} \otimes_B \Omega^1 \rightarrow \Omega^1 \otimes_B \mathcal{F}^\ast$ defined by $C_h(\omega \otimes f) = h_\mathcal{F}(\overline{f}) \otimes \omega^\ast$, and the evaluation map $\text{ev} : \mathcal{F}^\ast \times \mathcal{F} \rightarrow B$. We now associate to any Hermitian module the Hermitian metric

$$g_\mathcal{F} := g_\sigma \circ (\text{id} \otimes \text{ev} \otimes \text{id}) \circ (C_h \times \text{id}) : \mathcal{F} \otimes_B \Omega^\ast \times \mathcal{F} \otimes_B \Omega^\ast.$$

As shown in [32, §5], the metric is conjugate symmetric, which is to say

$$g_\mathcal{F}(\alpha, \beta) = g_\mathcal{F}(\beta, \alpha)^\ast,$$

for all $\alpha, \beta \in \Omega^\ast \otimes \mathcal{F}$.

If the Hermitian structure is positive definite, then the metric $g_\mathcal{F}$ satisfies $g_\mathcal{F}(\alpha, \alpha) \in B_{>0}$, for every $\alpha \in \Omega^\ast \otimes_B \mathcal{F}$. Note that the $\mathbb{Z}^2_{>0}$ decomposition of $\Omega^\ast \otimes_B \mathcal{F}$ is orthogonal with respect to $g_\mathcal{F}$. Moreover, the composition $h \circ g_\mathcal{F}$ is an inner product for $\mathcal{F} \otimes_B \Omega^\ast$.

2.6. Inner Products, and Twisted Dolbeault–Dirac and Laplace Operators. Let $f : B \rightarrow \mathbb{C}$ be a state, that is a linear map satisfying $f(b^*b) > 0$, for all non-zero $b \in B$. As observed in [32, §5.2], an inner product is then given by

$$\langle -, - \rangle_\mathcal{F} := f \circ g_\mathcal{F} : \Omega^\ast \otimes_B \mathcal{F} \times \Omega^\ast \otimes_B \mathcal{F} \rightarrow \mathbb{C}.$$

Moreover, we can associate to $f$ an integral $\int := f \circ \ast_\sigma : \Omega^{2n} \rightarrow \mathbb{C}$. We say that $\int$ is closed if the map $\int \circ \delta : \Omega^{n-1} \rightarrow \mathbb{C}$ is equal to the zero map.

As shown in [32, Proposition 5.15], for any Hermitian holomorphic module $(\mathcal{F}, \overline{\mathcal{F}})$ over a Kähler structure $(B, \Omega^\ast, \Omega^\ast \otimes \Omega^\ast \otimes \sigma)$, with closed integral $\int$ with respect to a choice of state, the twisted differentials $\partial_\mathcal{F}$ and $\overline{\partial_\mathcal{F}}$ are adjointable with respect to $\langle -, - \rangle_\mathcal{F}$. The $\mathcal{F}$-twisted holomorphic and anti-holomorphic de Rham–Dirac operators are respectively defined to be

$$D_\partial_\mathcal{F} := \partial_\mathcal{F} + \overline{\partial_\mathcal{F}}^\dagger,$$
$$D_{\overline{\partial_\mathcal{F}}} := \overline{\partial_\mathcal{F}} + \partial_\mathcal{F}^\dagger.$$
Moreover, its $\mathcal{F}$-twisted holomorphic and anti-holomorphic Laplace operators are respectively defined to be
\[ \Delta_{\partial_F} := D^2_{\partial_F}, \quad \Delta_{\overline{\partial}_F} := D^2_{\overline{\partial}_F}. \]
Using the Laplacians one can define harmonic elements just as in the classical. As established in [32, §6.1], if the twisted Dirac operator is diagonalisable, then we have a direct noncommutative generalisation of Hodge decomposition, giving a bijection between cohomology classes and harmonic forms. Moreover, in this case, classical Serre duality also carries over to the noncommutative setting [32, §6.2].

2.7. Chern Connections and Positive Hermitian Holomorphic Modules. For any Hermitian module $(\mathcal{F}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$, there is a direct noncommutative generalisation of the classical definition of an Hermitian connection, namely a connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes_B \Omega^1$ which is compatible with $h_{\mathcal{F}}$ in the sense of [2], see also the accompanying paper [14, §2.9]. For any Hermitian holomorphic module $(\mathcal{F}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$, there exists a unique Hermitian connection $\nabla : \mathcal{F} \to \Omega^1 \otimes_B \mathcal{F}$ satisfying
\[ \overline{\partial}_{\mathcal{F}} = (\text{proj}_{\Omega^0} \otimes \text{id}) \circ \nabla. \]
Moreover we denote $\partial_{\mathcal{F}} := (\text{proj}_{\Omega^0} \otimes \text{id}) \circ \nabla$, and call $\nabla$ the Chern connection of Hermitian holomorphic module $(\mathcal{F}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$. This allows us to introduce a notion of positivity for a holomorphic Hermitian module. This directly generalises the classical notion of positivity, a property which is equivalent to ampleness [24, Proposition 5.3.1]. It was first introduced in [32, Definition 8.2] and requires a compatibility between Hermitian holomorphic modules and Kähler structures.

Definition 2.4. Let $\Omega^\bullet$ be a dc over a $*$-algebra $B$, and let $(\Omega^{\bullet,\bullet}, \kappa)$ be a Kähler structure for $\Omega^\bullet$. An Hermitian holomorphic module $(\mathcal{F}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$ is said to be positive, written $\mathcal{F} > 0$, if there exists $\theta \in \mathbb{R}_{>0}$ such that the Chern connection $\nabla$ of $\mathcal{F}$ satisfies
\[ \nabla^2(f) = -\theta i L_{\mathcal{F}}(f) = -\theta i \kappa \otimes f, \quad \text{for all } f \in \mathcal{F}. \]
Analogously, $(\mathcal{F}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$ is said to be negative, written $\mathcal{F} < 0$, if there exists $\theta \in \mathbb{R}_{>0}$ such that the Chern connection $\nabla$ of $\mathcal{F}$ satisfies
\[ \nabla^2(f) = \theta i L_{\mathcal{F}}(f) = \theta i \kappa \otimes f, \quad \text{for all } f \in \mathcal{F}. \]

As established in [32, §8], for positive Hermitian holomorphic modules we have a direct noncommutative generalisation of the Kodaira vanishing theorem. Moreover, as shown in the accompanying paper [32, §3], twisting a noncommutative Dolbeault–Dirac operator by a negative Hermitian holomorphic module produces a spectral gap around zero.

2.8. Covariant Differential Calculi and Hermitian Structures. We begin by briefly recalling Takeuchi’s equivalence for relative Hopf modules, see [14, Appendix A] for more details. For $A$ a Hopf algebra, we say that a right coideal subalgebra $B \subseteq A$ is a quantum homogeneous $A$-space if $A$ is faithfully flat as a left $B$-module and $B^+A = AB^+$, where $B^+ := \text{ker}(\varepsilon_B)$. We denote by $\text{mod}_B^A$ the category of relative Hopf modules which are finitely generated as left $B$-modules, and by $\text{fin-mod}$ the category of finite-dimensional left comodules over the Hopf algebra $\pi_B(A) := A/AB^+$. A equivalence of categories,
called Takeuchi’s equivalence, is given by the functor $\Phi : \text{mod}^A_B \to \pi_B \text{mod}$, where $\Phi(F) = F/F^B$, for any relative Hopf module $F$, and the functor $\Psi : \pi_B \text{mod} \to \text{mod}^A_B$ defined using the cotensor product $\square_{\pi_B}$ over $\pi_B(A)$. A unit for the equivalence is given by $U : F \to (\Psi \circ \Phi)(F)$, where $U(f) = [f(0)] \otimes f(1)$, where $[f(0)]$ denotes the coset of $f(0)$ in $\Phi(F)$.

An Hermitian relative Hopf module over a quantum homogeneous space $B \subseteq A$ is an Hermitian module $(\mathcal{F}, h_\mathcal{F})$ such that $\mathcal{F}$ is an object in $\text{mod}^A_B$ and the isomorphism $h_\mathcal{F} : \mathcal{F} \to \mathcal{V} \mathcal{F}$ is a morphism in $\text{mod}^A_B$, where the conjugate $\mathcal{F}$ and dual $\mathcal{V} \mathcal{F}$ modules are understood as objects in $\text{mod}^A_B$ in the usual way. Consider next a covariant dc $\Omega^\bullet$ over $B$, endowed with a covariant complex structure $\Omega^{(\bullet, \bullet)}$, which is to say, one for which the $\mathbb{Z}^2_{\geq 0}$-decomposition of the calculus is a decomposition in the category of relative Hopf modules. A holomorphic relative Hopf module $\mathcal{F}$ is a holomorphic $(\mathcal{F}, \partial_\mathcal{F})$ the Hilbert space completion of $\mathcal{F}$. A unit for the equivalence is given by the functor $\Phi : \text{mod}^A_B \to \pi_B \text{mod}$ over $B$, such that $\mathcal{F}$ is an object in $\text{mod}^A_B$ and $\partial_\mathcal{F} : \mathcal{F} \to \Omega^{(0, 1)} \otimes_B \mathcal{F}$ is a left $A$-comodule map. An Hermitian holomorphic module $(\mathcal{F}, h_\mathcal{F}, \partial_\mathcal{F})$ is said to be covariant if its constituent Hermitian and holomorphic modules are covariant. In this case, the Chern connection is always a left $A$-comodule map, see [32, §7.1].

A covariant Hermitian structure for $\Omega^\bullet$ is an Hermitian structure $(\Omega^{(\bullet, \bullet)}, \sigma)$ such that $\Omega^{(\bullet, \bullet)}$ is a covariant complex structure, and the Hermitian form $\sigma$ is left $A$-coinvariant, which is to say $\Delta_L(\sigma) = 1 \otimes \sigma$. A covariant Kähler structure is a covariant Hermitian structure which is also a Kähler structure. Note that in the covariant case, in addition to being $B$-bimodule maps, $L_\sigma, *_\sigma, \Lambda_\sigma$ are also left $A$-comodule maps.

3. Compact Quantum Homogeneous Kähler Spaces

In this section we introduce the main object of study in this paper, mainly the notion of a compact quantum homogeneous Kähler space, and the more generally, the Hermitian analogue. This gives a natural set of compatibility conditions between covariant Hermitian structures and Woronowicz’s theory of compact quantum groups. It has a natural Hilbert space completion together with a rich family of geometrically motivated bounded operators. We refer to Appendix A for basic definitions and notation for CQGs and CQGAs.

3.1. Square Integrable Forms. In this subsection we introduce the Hilbert space of square integrable forms for a positive definite Hermitian structure $(\Omega^{(\bullet, \bullet)}, \sigma)$ over an algebra $B$, and an Hermitian module $\mathcal{F}$. Noting that an inner product $\langle -, - \rangle_F$ is given by $h \circ g_F$, we denote by $L^2(\mathcal{F} \otimes_B \Omega^\bullet)$ the Hilbert space completion of $\mathcal{F} \otimes_B \Omega^\bullet$ with respect to $\langle -, - \rangle_F$, and call it the Hilbert space of square integrable twisted forms of $\mathcal{F} \otimes_B \Omega^\bullet$.

We note that the following Hilbert space decompositions

$$L^2(\mathcal{F} \otimes_B \Omega^\bullet) \simeq \bigoplus_{(a,b) \in \mathbb{Z}^2_{\geq 0}} L^2(\mathcal{F} \otimes_B \Omega^{[a,b]}), \quad L^2(\mathcal{F} \otimes_B \Omega^\bullet) \simeq \bigoplus_{j \in \mathbb{Z}^2_{\geq 0}} L^2(L^2(\mathcal{F} \otimes_B \Omega^{(2n-2j)}))$$

follow from the fact that the $\mathbb{Z}^2_{\geq 0}$-decomposition, and the Lefschetz decomposition, of $\Omega^\bullet$ are orthogonal with respect to $g_\sigma$ (as discussed in [22]), and hence with respect to the
The composition associated inner product. Moreover, if the twisted Dirac operator $D_{\sigma^x}$ is diagonalisable, then the additional decomposition
\[ L^2(F \otimes_B \Omega^*) \simeq L^2(\partial_F(F \otimes_B \Omega^*)) \oplus L^2(\overline{\partial}_F(F \otimes_B \Omega^*)) \]
follows from the orthogonality of Hodge decomposition [32, Theorem 6.4].

3.2. CQH-Hermitian Spaces. We now introduce the class of quantum homogeneous spaces that concern us in this paper. First, we note that for any quantum homogeneous space $B \subseteq A$, where $A$ is a Hopf $*$-algebra, and $B$ is a $*$-subalgebra of $A$, the quotient Hopf algebra $A/AB^+$ inherits a Hopf $*$-algebra structure. A CQGA homogeneous space is a quantum homogeneous space $B \subseteq A$ such that $A$ and $\pi_B(A)$ are CQGAs, and $B$ is a $*$-subalgebra of $A$, and $\pi_B$ is a Hopf $*$-algebra map. It follows that $A$ is faithfully flat as a right $B$-module, as explained, for example, in [16, §3.3].

Definition 3.1. A compact quantum homogeneous Hermitian space, or simply a CQH-Hermitian space, is a quadruple $H := (B \subseteq A, \Omega^*, \Omega^{\bullet\bullet}, \sigma)$ where

1. $B \subseteq A$ is a CQGA homogeneous space,
2. $\Omega^*$ is a left $A$-covariant $*$-subalgebra over $B$,
3. $(\Omega^{\bullet\bullet}, \sigma)$ is a left $A$-covariant, $f$-closed, positive Hermitian structure for $\Omega^*$.

A compact quantum homogeneous Kähler space, or simply a CQH-Kähler space is a CQH-Hermitian space whose constituent Hermitian structure is a Kähler structure.

Since the Haar state $\h$ of $A$ is a state, we can follow the approach of [32] and form inner products from twisted metrics, and moreover consider the associated Hilbert space completions. We finish this subsection with a sufficient condition for separability of such Hilbert spaces.

Proposition 3.2. For a quantum homogeneous space $H = (B \subseteq A, \Omega^*, \Omega^{\bullet\bullet}, \sigma)$, and an Hermitian module $F$, if $A$ is finitely generated, then the Hilbert space $L^2(F \otimes_B \Omega^*)$ is separable.

Proof. Since $A$ is finitely generated, it admits a countable Hamel basis. Thus the tensor product $\Phi(F \otimes_B \Omega^*) \otimes A$, as well as the subspace $\Phi(F \otimes_B \Omega^*) \Box_{\pi_B} A$, admit such a basis. Since $F \otimes_B \Omega^*$ is isomorphic to the cotensor product $\Phi(F \otimes_B \Omega^*) \Box_{\pi_B} A$, it also admits a countable basis. Thus we can conclude that $L^2(F \otimes_B \Omega^*)$ is separable. \hfill $\Box$

3.3. Morphisms as Bounded Operators. In this subsection we discuss the extension of endomorphisms of $F \otimes_B \Omega^*$ to bounded operators on $L^2(F \otimes_B \Omega^*)$. As an application, we produce bounded representations of $\mathfrak{sl}_2$.

Consider first the map $g_{F,U}: \Phi(F \otimes_B \Omega^*) \Box_{\pi_B} A \times \Phi(F \otimes_B \Omega^*) \Box_{\pi_B} A \to B$, defined by
\[ g_{F,U} \left( \sum_i a_i \otimes [\alpha_i], \sum_j a'_j \otimes [\alpha'_j] \right) := \sum_{i,j} a_i^* a'_j \varepsilon(g_F(\alpha_i, \alpha'_j)). \]
The composition $\h \circ g_{F,U}$ is an inner product for $\Phi(F \otimes_B \Omega^*) \Box_{\pi_B} A$. Moreover, it follows from [14, Lemma 4.1] that the unit map $U : F \otimes_B \Omega^* \to \Phi(F \otimes_B \Omega^*) \Box_{\pi_B} A$ is an isometry.
Proposition 3.3. Every morphism \( f : \mathcal{F} \otimes_B \Omega^* \rightarrow \mathcal{F} \otimes_B \Omega^* \) in \( \text{mod}^1_B \) is bounded, and hence extends to a bounded operator on \( L^2(\mathcal{F} \otimes_B \Omega^*) \).

Proof. Consider the commutative diagram given by Takeuchi’s equivalence

\[
\begin{array}{ccc}
\mathcal{F} \otimes_B \Omega^* & \xrightarrow{f} & \mathcal{F} \otimes_B \Omega^* \\
\Phi(\mathcal{F}) \otimes \Phi(\Omega^*) \Box_{\pi_B} A & \xrightarrow{\Psi \Phi(f)} & \Phi(\mathcal{F}) \otimes \Phi(\Omega^*) \Box_{\pi_B} A \\
\end{array}
\]

Since \( U \) is an isometry, the morphism \( f \) is bounded if and only if \( \Psi \circ \Phi(f) \) is bounded. But \( \Psi \circ \Phi(f) = \text{id} \otimes \Phi(f) \), and \( \Phi(\mathcal{F} \otimes_B \Omega^*) \) is finite-dimensional by assumption, implying that \( \text{id} \otimes \Phi(f) \) is bounded. Hence \( f \) is bounded and extends to a bounded operator on \( L^2(\mathcal{F} \otimes_B \Omega^*) \).

Corollary 3.4. The maps \( \text{id} \otimes L_\sigma, \text{id} \otimes \Lambda_\sigma, \) and \( \text{id} \otimes H \) extend to bounded operators on \( L^2(\mathcal{F} \otimes_B \Omega^*) \). Hence, a representation \( T : \mathfrak{sl}_2 \rightarrow \mathbb{B}(L^2(\mathcal{F} \otimes_B \Omega^*)) \) is given by

\[
T(E) = \text{id} \otimes L_\sigma, \quad T(K) = \text{id} \otimes H, \quad T(F) = \text{id} \otimes \Lambda_\sigma.
\]

The space of lowest weight vectors of the representation is given by \( L^2(\mathcal{F} \otimes_B P^*) \), the Hilbert space completion of the primitive forms.

Proof. Since \( \text{id} \otimes L_\sigma, \text{id} \otimes \Lambda_\sigma, \) and \( \text{id} \otimes H \) are all morphisms in \( \text{mod}^1_B \), Proposition 3.3 implies that they extend to bounded operators on \( L^2(\mathcal{F} \otimes_B \Omega^*) \). It now follows from the \( \mathfrak{sl}_2 \)-representation given in [31, Corollary 5.14] that we get a bounded Lie algebra representation of \( \mathfrak{sl}_2 \).

Corollary 3.5. The Hodge map \( *_\sigma \) extends to a unitary operator on \( L^2(\mathcal{F} \otimes_B \Omega^*) \).

Proof. This follows from Proposition 3.3 and unitarity of \( *_\sigma \) as an operator on \( \Omega^* \), as established in [31, Lemma 5.10].

3.4. Bounded Multiplication Maps. Let us now consider a monoidal variation on Takeuchi’s equivalence. Denote by \( \text{mod}^1_B \) the monoidal category of two-sided relative Hopf modules. Consider \( \text{mod}^1_B \) the full monoidal subcategory of \( \text{mod}^1_B \) whose objects \( \mathcal{F} \) satisfy \( \mathcal{F} B^+ = B^+ \mathcal{F} \). It follows from Takeuchi’s equivalence that \( \text{mod}^1_B \) is monoidally equivalent to \( \pi_B \text{mod} \). See [16, Appendix A] for further details.

In this subsection, we restrict to the case of untwisted forms, and assume that our calculi are objects in the subcategory \( \text{mod}^1_B \). For such a CQH-Hermitian space \( H = (B, \Omega^*, \Omega^{(*)}, \sigma) \), we prove that every multiplication operator on \( \Omega^* \) is bounded with respect to the norm of the inner product of \( H \). In analogy with the bounded representation of \( A \) on \( L^2(A) \), this implies that we have a bounded representation of \( B \) on \( L^2(\Omega^*) \).

Proposition 3.6. For any form \( \omega \in \Omega^* \), a non-zero bounded operator is given by

\[
L_\omega : \Omega^* \rightarrow \Omega^*, \quad \nu \mapsto \omega \wedge \nu.
\]

Moreover, a faithful algebra representation is defined by

\[
\lambda : \Omega^* \rightarrow \mathbb{B}(L^2(\Omega^*)), \quad \omega \mapsto L_\omega.
\]
Proof. For any \([\nu] \in \Phi(\Omega^*)\), we have a well-defined operator
\[ l_{[\nu]} : \Phi(\Omega^*) \to \Phi(\Omega^*), \quad [\omega] \mapsto [\omega] \wedge [\nu] = [\omega \wedge \nu]. \]
Since \(\Phi(\Omega^*)\) is finite-dimensional, this operator is bounded. Recalling the representation
\(\lambda_A : A \to \mathbb{B}(L^2(A))\) introduced in Appendix \(\aleph\) we see that a bounded operator on
\(\Phi(\Omega^*) \otimes A\) is given by \(l_{[\omega]} \otimes \lambda_A(a)\), for all \(a \in A\), \([\nu] \in \Phi(\Omega^*).\)

Using this observation, we now show that \(L_\omega\) is bounded. For any \(\nu \in \Omega^*\),
\[ U \circ L_\omega \circ U^{-1} (\{\nu(0)\} \otimes \nu(1)) = U \circ L_\omega \circ U^{-1} \circ U(\nu) \]
\[ = U(\omega \wedge \nu) \]
\[ = [\omega(0) \wedge \nu(0)] \otimes \omega(1) \nu(1) \]
\[ = l_{[\omega(0)]}(\nu(0)) \otimes \lambda(\omega(1))(\nu(1)) \]
\[ = l_{[\omega(0)]} \otimes \lambda(\omega(1))(\nu(0)) \otimes \nu(1). \]
Since every element of \(\Phi(\Omega^*)\) \(\square_{\pi_B} A\) is of the form \(U(\nu) = [\nu(0)] \otimes \nu(1)\), for some \(\nu \in \Omega^*\),
we see that \(U \circ L_\omega \circ U^{-1}\) is bounded. It now follows from the fact that \(U\) is an isometry
that \(L_\omega\) is bounded.

Since \(\Omega^*\) is dense in \(L^2(\Omega^*)\) by construction, \(L_\omega\) uniquely extends to an element of
\(\mathbb{B}(L^2(\Omega^*)).\) This gives a well-defined \(\mathbb{C}\)-linear map from \(\Omega^*\) to \(\mathbb{B}(L^2(\Omega^*)\)), which is
evidently an algebra map. Finally, since \(1 \in B \subseteq \Omega^*\), it is clear that \(\lambda\) is faithful. \(\square\)

Corollary 3.7. The map \(\lambda\) is a \(\ast\)-algebra map.

Proof. For any \(b \in B\), and \(\omega, \nu \in \Omega^*\), it holds that
\[ \langle b\omega, \nu \rangle_{\sigma} = h \ast \ast_{\sigma} (\ast_{\sigma}(b\omega)^{\ast} \wedge \nu) = h \ast \ast_{\sigma} (\ast_{\sigma}(\omega^{\ast}) \wedge b^{\ast} \nu) = \langle \omega, b^{\ast} \nu \rangle_{\sigma}. \]
Thus \(\lambda(b^\ast) = \lambda(b)^\dagger\), for all \(b \in B\), showing that \(\lambda\) is a \(\ast\)-map. \(\square\)

We now consider a second consequence of Proposition 3.6, namely boundedness of the
various commutator operators associated to a CQH-Hermitian space. This is a direct
noncommutative generalisation of an important classical phenomenon \[7, \S 2.4.1\], one
which is generalised by the definition of \(K\)-homology, and ultimately spectral triples, as
we see in Appendix \(\aleph\).

Corollary 3.8. The following operators are all bounded on \(\Omega^*\) and hence extend to bounded
operators on \(L^2(\Omega^*)\): For any \(b \in B\),
1. \([d, \lambda(b)], \quad [\partial, \lambda(b)], \quad [\overline{\partial}, \lambda(b)]\),
2. \([d^\dagger, \lambda(b)], \quad [\partial^\dagger, \lambda(b)], \quad [\overline{\partial}^\dagger, \lambda(b)]\).

Proof. For any \(\omega \in \Omega^*\), we have the identity
\[ [d, \lambda(b)](\omega) = d(\omega b) - (d\omega)b = (d\omega)b + (\omega \wedge db) - (d\omega)b = \omega \wedge db. \]
It now follows from Proposition 3.6 that \([d, \lambda(b)]\) is a bounded operator on \(L^2(\Omega^*).\)
Boundedness of the other operators is established similarly.

The adjoint \([d, \lambda(b)]^\dagger\) of the operator \([d, \lambda(b)]\) is evidently bounded on \(\Omega^*.\) Thus since
\[ [d, \lambda(b)]^\dagger = -[d^\dagger, \lambda(b^\ast)], \quad \text{for all } b \in B, \]
the operator $[d, \lambda(b)]$ is bounded on $\Omega^*$. Boundedness of $[\bar{\partial}, \lambda(b)]$ and $[\partial, \lambda(b)]$ are established similarly.

**Corollary 3.9.** For all $b \in B$, the operators $[D_d, \lambda(b)]$, $[D_{\bar{\theta}}, \lambda(b)]$, and $[D_{\partial}, \lambda(b)]$ are bounded.

Finally, we observe that the norm induced on $B$ by the embedding $\lambda : \Omega^* \to \mathbb{B}(L^2(\Omega^*))$ is less than or equal to the restriction to $B$ of the reduced norm $\|\cdot\|_{\text{red}}$ of $A_{\text{red}}$, as defined in Appendix A.

**Proposition 3.10.** It holds that

$$\|b\|_{L^2} \leq \|\lambda(b)\|_{\text{op}} \leq \|b\|_{\text{red}},$$

for all $b \in B$,

where $\|\cdot\|_{\text{op}}$ denotes the operator norm of $\mathbb{B}(L^2(\Omega^*))$. Thus the restriction of $\lambda$ to $B$ extends to a $*$-algebra homomorphism $B_{\text{red}} \to \mathbb{B}(L^2(\Omega^*))$, where $B_{\text{red}}$ denotes the closure of $B$ in $A_{\text{red}}$.

**Proof.** The first inequality follows from

$$\|\lambda(b)\|_{\text{op}}^2 \geq \|\lambda(b)(1)\|_{L^2}^2 = \|b\|_{L^2}^2.$$

For the second inequality take any $\nu \in \Omega^*$, and note that

$$\|\lambda(b)(\nu)\|_{L^2} = \|U(\nu b)\|_{L^2} = \|\nu(-1)b \otimes [\nu(0)]\|_{L^2} = \|\lambda_A(b) \otimes \text{id}\|_{L^2} \|\nu(-1) \otimes [\nu(0)]\|_{L^2}.$$

Now since the operator $\lambda_A(b) \otimes \text{id}$ is bounded, we have that

$$\|\lambda_A(b) \otimes \text{id}\|_{L^2} \|\nu(-1) \otimes [\nu(0)]\|_{L^2} \leq \|\lambda_A(b)\|_{\text{op}} \|\nu(-1) \otimes [\nu(0)]\|_{L^2}$$

$$= \|\lambda(b)\|_{\text{op}} \|\nu\|_{L^2}$$

$$= \|b\|_{\text{red}} \|\nu\|_{L^2},$$

which gives us the second inequality and the implied extension of $\lambda$ to a map on $B_{\text{red}}$. \hfill \square

## 4. Closed Operators and Operator Domains

In this section we turn our attention to the unbounded operators constructable from the exterior derivatives and holomorphic structures of a twisted CQH Hermitian space. In particular, we address questions of closability, essential self-adjointness, and operator domains.

### 4.1. Peter–Weyl Maps

By cosemisimplicity of $A$, the abelian category $\text{mod}^r_B$ is semisimple, and so $\text{mod}^A_B$ is semisimple. For any $F \in \text{mod}^A_B$, we have the decomposition

$$F \simeq \Phi(F) \Box_{\pi_B} A \simeq \Phi(F) \Box_{\pi_B} \left( \bigoplus_{V \in \widehat{A}} C(V) \right) = \bigoplus_{V \in \widehat{A}} (\Phi(F) \Box_{\pi_B} C(V)) =: \bigoplus_{V \in \widehat{A}} F_V,$$

where $\widehat{A}$ denotes the equivalence classes of irreducible $A$-comodules. We call this the Peter–Weyl decomposition of $F$.

For any $V \in \text{mod}^A$, we have that $C(V) \simeq \text{End}(V)$ as right $A$-comodules [26 Proposition 11.8]. Thus, for any right $A$-comodule map $f : F \to F$ it holds that

$$f(F_V) \subseteq F_V,$$

for all $V \in \widehat{A}$. (3)
More generally, a Peter–Weyl map $f : \mathcal{F} \rightarrow \mathcal{F}$ is a $\mathbb{C}$-linear map satisfying (3). We now present some properties of the Peter–Weyl decomposition and Peter–Weyl maps in the CQH-Hermitian setting. The proof is completely analogous to the arguments of [31, §5.2], and so we omit it.

**Proposition 4.1.** For a CQH-Hermitian space $H = \{B \subseteq A, \Omega^\bullet, \Omega^{\bullet, \bullet}, \sigma\}$, and an Hermitian module $\mathcal{F}$, the Peter–Weyl decomposition of $\mathcal{F} \otimes \Omega^\bullet$ is orthogonal with respect to $(\langle - , - \rangle_{\mathcal{F}}$. Moreover, for any Peter–Weyl map $f : \mathcal{F} \otimes_B \Omega^\bullet \rightarrow \mathcal{F} \otimes_B \Omega^\bullet$, it holds that

1. $f$ is adjointable on $\mathcal{F} \otimes_B \Omega^\bullet$ with respect to $(\langle - , - \rangle_{\mathcal{F}}$, and its adjoint is a Peter–Weyl map.
2. if $f$ is self-adjoint with respect to $(\langle - , - \rangle_{\mathcal{F}}$, then it is diagonalisable on $\mathcal{F} \otimes_B \Omega^\bullet$.

**4.2. Closability and Essential Self-Adjointness.** In this subsection we examine closability and essential self-adjointness for unbounded operators on $\Omega^\bullet$. In particular, we show that the unbounded operators $\nabla, \partial_F$ and $\overline{\partial}_F$ are closable, and that the Dirac and Laplacian operators are essentially self-adjoint.

**Proposition 4.2.** Every Peter–Weyl map $f : \mathcal{F} \otimes_B \Omega^\bullet \rightarrow \mathcal{F} \otimes_B \Omega^\bullet$ is closable.

**Proof.** Since $f$ is a Peter–Weyl map, it follows from Proposition 3 that it is adjointable on $\mathcal{F} \otimes_B \Omega^\bullet$. Moreover, since $(\mathcal{F} \otimes_B \Omega^\bullet)_V$ is a finite-dimensional space, for every $V \in \hat{A}$, the restriction of the adjoint map $f^\dagger|_V : (\mathcal{F} \otimes_B \Omega^\bullet)_V \rightarrow (\mathcal{F} \otimes_B \Omega^\bullet)_V$ is bounded. Now for any $\alpha \in (\mathcal{F} \otimes_B \Omega^\bullet)_V$, consider the linear functional

$$\mathcal{F} \otimes_B \Omega^\bullet = \text{dom}(f) \rightarrow \mathbb{C}, \quad \beta \mapsto (f(\beta), \alpha)_{\mathcal{F}}.$$  

Boundedness of the functional follows from the inequality

$$|f(\beta), \alpha)_{\mathcal{F}}| = \left| \langle \beta, f^\dagger(\alpha) \rangle_{\mathcal{F}} \right| \leq \|\beta\|_{L^2} \|f^\dagger(\alpha)\|_{L^2} \leq \|\beta\|_{L^2} \|f^\dagger\|_{\text{op}} \|\alpha\|_{L^2},$$

for $\alpha, \beta \in \mathcal{F} \otimes_B \Omega^\bullet$, and where $\|f^\dagger\|_{\text{op}}$ denotes the norm of $f^\dagger$ in $\mathbb{B}(L^2((\mathcal{F} \otimes_B \Omega^\bullet)_V))$. Hence $\omega \in \text{dom}(f^\dagger)$, implying that $\mathcal{F} \otimes_B \Omega^\bullet \subseteq \text{dom}(f^\dagger)$, and consequently that $\text{dom}(f^\dagger)$ is dense in the Hilbert space $L^2(\mathcal{F} \otimes_B \Omega^\bullet)$. It now follows from Appendix [1] that $f$ is closable.

Since every comodule map is automatically a Peter–Weyl map, we have the following immediate consequences of the proposition.

**Corollary 4.3.** Every left $A$-comodule map $f : \mathcal{F} \otimes_B \Omega^\bullet \rightarrow \mathcal{F} \otimes_B \Omega^\bullet$ is closable.

**Corollary 4.4.** The operators $\nabla, \partial_F$, and $\overline{\partial}_F$ are closable.

**Proof.** Since the calculus and complex structure are, by assumption, covariant, the maps $\nabla, \partial_F$, and $\overline{\partial}_F$ are comodule maps, and hence closable.

We now prove essential self-adjointness for symmetric comodule maps, and then conclude essential self-adjointness for the twisted Dirac and Laplacian operators of a CQH-Hermitian space.

**Proposition 4.5.** Every symmetric left $A$-comodule map $f : \mathcal{F} \otimes_B \Omega^\bullet \rightarrow \mathcal{F} \otimes_B \Omega^\bullet$ is diagonalisable on $L^2(\mathcal{F} \otimes_B \Omega^\bullet)$, and moreover, is essentially self-adjoint.
that $K$.

Recall that a core $\lambda$ under the action of $T$ implies that its eigenvalues are real. Thus the range of the operators $f - \text{id}$ and $f + \text{id}$ must be equal to $\mathcal{F} \otimes \Omega^*$, which is to say, the range of each operator is dense in $L^2(\mathcal{F} \otimes \Omega^*)$. It now follows from the results of Appendix $\Box$ that $f$ is essentially self-adjoint.

**Corollary 4.6.** The Dirac operators $D_{\partial_F}, D_{\bar{\partial}_F}$, and $D_\nabla$, and the Laplace operators $\Delta_{\partial_F}, \Delta_{\bar{\partial}_F}$, and $\Delta_\nabla$, are diagonalisable and essentially self-adjoint.

**4.3. Domains.** In this subsection we show that in the Kähler case the domains of the three Dirac operators $D_A, D_B$, and $D_\nabla$ coincide. The proof is based on the equality of the Laplacians, motivating our restriction to the untwisted Kähler case.

**Proposition 4.7.** For a CQH-Kähler space $K = (B \subseteq A, \Omega^*, \Omega^{**}, \kappa)$ it holds that

$$\text{dom}(D_A) = \text{dom}(D_B) = \text{dom}(D_\nabla).$$

**Proof.** An element $x \in L^2(\Omega^*)$ is contained in $\text{dom}(D_A)$ if and only if there exists a sequence of elements $(\omega_n)_{n \in \mathbb{Z}_{>0}}$ in $\Omega^*$ such that $\omega_n \to x$ and $D_A(\omega_n) \to D_A(x)$. For such a sequence it holds that

$$\|D_B(\omega_n) - D_B(x)\|_{L^2}^2 = \langle D_B(\omega_n) - D_B(x), D_B(\omega_n) - D_B(x) \rangle_{\sigma}$$

$$= \langle D_B(\omega_n), D_B(\omega_n) \rangle_{\sigma} - \langle D_B(\omega_n), D_B(x) \rangle_{\sigma}$$

$$- \langle D_B(x), D_B(\omega_n) \rangle_{\sigma} + \langle D_B(x), D_B(x) \rangle_{\sigma}$$

$$= \langle D_B^2(\omega_n), \omega_n \rangle_{\sigma} - \langle D_B^2(\omega_n), x \rangle_{\sigma} - \langle D_B^2(x), \omega_n \rangle_{\sigma} + \langle D_B^2(x), x \rangle_{\sigma}.$$

Since $(\Omega^{**}, \kappa)$ is a Kähler structure we have the identity $D_B^2 = \Delta_B = \frac{1}{2} \Delta_A = \frac{1}{2} D_A^2$. Hence the above expression can be rewritten as

$$\frac{1}{2} \langle D_A(\omega_n), \omega_n \rangle_{\sigma} - \frac{1}{2} \langle D_A^2(\omega_n), x \rangle_{\sigma} - \frac{1}{2} \langle D_A^2(x), \omega_n \rangle_{\sigma} + \frac{1}{2} \langle D_A^2(x), x \rangle_{\sigma}$$

$$= \frac{1}{2} \langle D_A(\omega_n) - D_A(x), D_A(\omega_n) - D_A(x) \rangle_{\sigma}$$

$$= \|D_A(\omega_n) - D_A(x)\|_{L^2}^2.$$ 

Thus we see that $\|D_B(\omega_n) - D_B(x)\|_{L^2} \to 0$ implying that $x \in \text{dom}(D_B)$, and hence that $\text{dom}(D_B) \subseteq \text{dom}(D_A)$. The opposite inclusion is established analogously, giving us the equality $\text{dom}(D_A) = \text{dom}(D_B)$. We can prove the equality of $\text{dom}(D_A)$ and $\text{dom}(D_B)$ similarly. $\Box$

**4.4. Cores and Domains.** As recalled in Appendix $\Box$ one of the defining requirements of a spectral triple $(A, \mathcal{H}, D)$ is that the domain of the unbounded operator $D$ is closed under the action of $\lambda(a)$, for all $a \in A$. This subtle condition can be verified using cores. Recall that a core for a closable operator $T : \text{dom}(T) \to \mathcal{H}$ is a subset $X \subseteq \text{dom}(T)$ such that the closure of $T$ is equal to the closure of the restriction of $T$ to $X$, which is to say, $(T|_X)^c = T^c$. Let $\mathcal{H}$ be a separable Hilbert space, $D : \text{dom}(D) \subseteq \mathcal{H} \to \mathcal{H}$ a densely-defined closed operator, $X \subseteq \text{dom}(D)$ a core for $D$, and $K \in \mathbb{B}(\mathcal{H})$ such that $K(X)$ is contained in $\text{dom}(D)$, and $[D, K] : X \to \mathcal{H}$ is bounded on $X$. Then,
as established by Forsyth, Mesland, and Rennie in [18, Proposition 2.1], we have that $K(\text{dom}(D)) \subseteq \text{dom}(D)$.

Applying this proposition directly to a general CQH-Hermitian space, we get the following result.

**Proposition 4.8.** Let $\mathcal{H} = (B \subseteq A, \Omega^\bullet, \Omega^{\bullet\bullet}, \sigma)$ be a CQH-Hermitian space and denote by $D_{\overline{\partial}_F}$ the associated Dolbeault–Dirac operator. If $A$ is finitely generated as an algebra, then it holds that

$$\lambda(b)\text{dom}(D_{\overline{\partial}_F}) \subseteq \text{dom}(D_{\overline{\partial}_F}), \quad \text{for all } b \in B.$$ 

**Proof.** Since we are assuming that $A$ is finitely generated as an algebra, it follows from Proposition 3.2 that $L^2(\Omega^\bullet)$ is separable. The subspace $\Omega^\bullet \subseteq \text{dom}(D_{\overline{\partial}_F})$ is a core by construction of the closure of $D_{\overline{\partial}_F}$. Moreover, since $B$ is a subalgebra of $\Omega^\bullet$, the core is clearly closed under the action of $\lambda(b)$, for all $b \in B$. Proposition 3.8 says that $[D_{\overline{\partial}_F}, \lambda(b)]$ is a bounded operator on $\Omega^\bullet$, for all $b \in B$, and so, we see that $\lambda(b)\text{dom}(D_{\overline{\partial}_F})$ is contained in $\text{dom}(D_{\overline{\partial}_F})$ as claimed. $\square$

5. Twisted Dolbeault–Dirac Fredholm Operators

In this section we address the Fredholm property for twisted Dolbeault–Dirac operators. More precisely, we show that twisting the Dolbeault–Dirac operator of a CQH-Kähler space by a negative Hermitian holomorphic module produces a Fredholm operator if and only if the top anti-holomorphic cohomology group is finite-dimensional. In this case, we also observe that the index of the Fredholm operator is expressible in terms of the dimension of the cohomology group.

5.1. The Holomorphic Euler Characteristic. In this subsection we consider the natural noncommutative generalisation of the anti-holomorphic Euler characteristic of a classical complex manifold.

**Definition 5.1.** Consider a dc $\Omega^\bullet$, a complex structure $\Omega^{\bullet\bullet}$, and a holomorphic module $(\mathcal{F}, \overline{\partial}_F)$ with finite-dimensional anti-holomorphic cohomologies. We define the holomorphic Euler characteristic of $\mathcal{F}$ to be the value

$$\chi_{\overline{\partial}_F} := \sum_{k \in \mathbb{Z}_{\geq 0}} (-1)^k \dim \left( H^{(0,k)} \overline{\partial}_F \right) \in \mathbb{Z}.$$ 

Note that there exist holomorphic modules with infinite-dimensional holomorphic Euler characteristics, and for these examples the Euler characteristic is not defined.

5.2. Fredholm Operators. We begin by recalling the definition of an (unbounded) Fredholm operator, which generalises the index theoretic properties of elliptic differential operators over a compact manifold.

**Definition 5.2.** For $\mathcal{H}_1$ and $\mathcal{H}_2$ two Hilbert spaces, and $T : \text{dom}(T) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ a densely defined closed linear operator, we say that $T$ is a Fredholm operator if $\ker(T)$
and \( \text{coker}(T) \) are both finite-dimensional. The \textit{index} of a Fredholm operator \( T \) is then defined to be the integer

\[
\text{index}(T) := \dim(\ker(T)) - \dim(\text{coker}(T)).
\]

The image \( \text{im}(T) \) of a Fredholm operator \( T \) is always closed \textsuperscript{37} \textsuperscript{2}.

5.3. The Dolbeault–Dirac Fredholm Index. Since \( D_{\partial} \) is a self-adjoint operator, if it were Fredholm its index would necessarily be zero. However, we can alternatively calculate its index with respect to the canonical \( \mathbb{Z}_2 \)-grading of the Hilbert space. For any CQH-Hermitian space, we introduce the spaces

\[
\Omega^{(0,\bullet)}_{\text{even}} := \bigoplus_{k \in \mathbb{Z}_2 \geq 0} \Omega^{(0,2k)}, \quad \Omega^{(0,\bullet)}_{\text{odd}} := \bigoplus_{k \in \mathbb{Z}_2 \geq 0} \Omega^{(0,2k+1)},
\]

and the associated Hilbert space completions \( L^2\left(\Omega^{(0,\bullet)}_{\text{even}}\right) \) and \( L^2\left(\Omega^{(0,\bullet)}_{\text{odd}}\right) \). Define the restricted operator

\[
D^+_\partial \mathcal{F} : \text{dom}(D_{\partial} \mathcal{F}) \cap L^2\left(\Omega^{(0,\bullet)}_{\text{even}}\right) \to L^2\left(\Omega^{(0,\bullet)}_{\text{odd}}\right), \quad x \mapsto D_{\partial} \mathcal{F}(x).
\]

**Proposition 5.3.** Let \( \mathcal{F} \) be an Hermitian holomorphic module with finite-dimensional anti-holomorphic cohomology groups. If \( D^+_\partial \mathcal{F} \) is a Fredholm operator, then its index is equal to the anti-holomorphic Euler characteristic of \( \mathcal{F} \otimes_B \Omega^{(\bullet,\bullet)} \), which is to say,

\[
\text{index}(D^+_\partial \mathcal{F}) = \chi_{\partial} \mathcal{F}.
\]

**Proof.** Let \( D^+_\partial \mathcal{F} \) be a Fredholm operator, and consider its index

\[
\text{index}(D^+_\partial \mathcal{F}) = \dim(\ker(D^+_\partial \mathcal{F})) - \dim(\text{coker}(D^+_\partial \mathcal{F})).
\]

It follows from Hodge decomposition that

\[
\text{index}(D^+_\partial \mathcal{F}) = \sum_{k \in 2\mathbb{Z}_2 \geq 0} \dim(H^0_{\partial} \mathcal{F}) - \sum_{k \in 2\mathbb{Z}_2 + 1 \geq 0} \dim(H^0_{\partial} \mathcal{F}) = \sum_{k \in \mathbb{Z}_2 \geq 0} (-1)^k \dim(H^0_{\partial} \mathcal{F}).
\]

Thus we see that the index of \( D^+_\partial \mathcal{F} \) is equal to \( \chi_{\partial} \mathcal{F} \) as claimed. \( \square \)

5.4. Fredholm Operators from Twisting. In this section we show that twisting the Dolbeault–Dirac operator of CQH-Kähler space by a negative Hermitian holomorphic module produces a Fredholm operator if and only if the top anti-holomorphic cohomology group is finite-dimensional. Moreover, in this case the index of the twisted operator is given by the dimension of this cohomology group. The proof combines noncommutative Hodge decomposition, the noncommutative Kodaira vanishing theorem, and the existence of spectral gaps for negative modules. This gives a perfect example of the how the analytic behaviour of the Dolbeault–Dirac operators is shaped by the complex geometry of the underlying dc. This result will be used in \textsuperscript{6} to construct Dolbeault–Dirac Fredholm operators for all the irreducible quantum flag manifolds.
Theorem 5.4. If $\mathcal{F}$ is a negative module over a $2n$-dimensional CQH-Kählere space, then the twisted Dirac operator
\[ D^+_\mathcal{F} : \text{dom}(D^0\mathcal{F}) \cap L^2(\mathcal{F} \otimes B \Omega^{0,0}) \to L^2(\mathcal{F} \otimes B \Omega^{0,0}) \]
is a Fredholm operator if and only if $H^{(0,n)}_{\mathcal{F}}$ is finite-dimensional. Moreover, in this case
\[ \text{index}(D^+_\mathcal{F}) = (-1)^n \dim(H^{(0,n)}_{\mathcal{F}}). \]

Proof. Since $D^0\mathcal{F}$ is diagonalisable on $\mathcal{F} \otimes B \Omega^\bullet$, its closure cannot admit an additional non-trivial eigenvector with eigenvalue zero. So in particular, the operator $D^+_\mathcal{F}$ and its closure have the same kernel. By the equivalence between cohomology classes and harmonic forms implied by Hodge decomposition, we have that
\[ \dim(\ker(D^+_\mathcal{F})) = \sum_{k \in \mathbb{Z} \geq 0} \dim(H^{(0,k)}_{\mathcal{F}}). \]

Let us now move on to the cokernel of the operator. By [14, Theorem 3.4] we know that the absolute value of the non-zero eigenvalues of $D^0\mathcal{F}$ are bounded below by a non-zero constant. Let us now identify $L^2(\partial^\mathcal{F}(\mathcal{F} \otimes B \Omega^\bullet) \oplus \overline{\partial^\mathcal{F}(\mathcal{F} \otimes B \Omega^\bullet)})$ with the $\ell^2$-sequences for some choice of basis $\{e_n\}_{n \in \mathbb{Z} \geq 0}$ which diagonalises $D^0\mathcal{F}$. Taking any such $\ell^2$-sequence $\sum_{n=0}^{\infty} a_n e_n$, and denoting $D^\mathcal{F}(e_n) =: \mu_n e_n$, we see that
\[ \left\| \sum_{n=0}^{\infty} \mu_n^{-1} a_n e_n \right\|_{L^2} \leq \sup_{n \in \mathbb{Z} \geq 0} |\mu_n|^{-1} \left\| \sum_{n=0}^{\infty} a_n e_n \right\|_{L^2} < \infty. \]

Hence $\sum_{n=0}^{\infty} \mu_n^{-1} a_n e_n$ is a well-defined element of $L^2(\Omega^\bullet \otimes B \mathcal{F})$. Moreover, since
\[ D^\mathcal{F}\left( \sum_{n=0}^{\infty} \mu_n^{-1} a_n e_n \right) = \sum_{n=0}^{\infty} a_n e_n, \]
we now see that the image of $D^\mathcal{F}$ is equal to (5). In particular, appealing again to Hodge decomposition, we see that
\[ \dim(\text{coker}(D^+_\mathcal{F})) = \bigoplus_{k \in \mathbb{Z} \geq 0+1} \dim(H^{(0,k)}). \]

Thus the cokernel of the operator is finite-dimensional if and only if $\mathcal{F}$ has finite-dimensional odd degree anti-holomorphic cohomologies.

Finally, we note that the noncommutative Kodaira vanishing theorem, and Serre duality, for noncommutative Kähler structures, implies that the $(0,k)$-harmonic forms vanish, for all $k = 0, \ldots, n - 1$. Thus we see that our operator is a Fredholm operator if and only if the space of $(0,n)$-harmonic forms is finite-dimensional. When this is case, the claimed value for the index follows directly from Proposition 5.3. \[\square\]
6. HECKENBERGER–KOLB CALCULI

In this section we present our motivating family of examples, the irreducible quantum flag manifolds endowed with their Heckenberger–Kolb calculi. Using Theorem 5.4 we show that twisting the Dolbeault–Dirac operator of a Heckenberger–Kolb calculus by negative line bundles produces a Fredholm operator. The proof relies on the Borel–Weil theorem for irreducible quantum flag manifolds [8], which in addition to establishing that the operator is Fredholm, allows us to give an explicit value for the operator index.

6.1. Drinfeld–Jimbo Quantum Groups. Let \( \mathfrak{g} \) be a finite-dimensional complex simple Lie algebra of rank \( r \), and fix a Cartan subalgebra and a set of simple roots \( \Pi = \{ \alpha_1, \ldots, \alpha_r \} \). For \( q \in \mathbb{R} \) such that \( q \neq -1, 0, 1 \), we denote by \( U_q(\mathfrak{g}) \) the Drinfeld–Jimbo quantised enveloping algebra. We denote the generators by \( E_i, F_i, K_i \), for \( i = 1, \ldots, r \) and follow the conventions of [20, §7]. Moreover, we endow \( U_q(\mathfrak{g}) \) with the compact real form Hopf \(*\)-algebra structure.

We denote the fundamental weights of \( \mathfrak{g} \) by \( \{ \varpi_1, \ldots, \varpi_r \} \), and by \( \mathcal{P}^+ \) the cone of dominant integral weights. For each \( \mu \in \mathcal{P}^+ \), we denote by \( V_\mu \) the corresponding finite-dimensional type-1, or admissible, \( U_q(\mathfrak{g}) \) highest weight module \( V_\mu \). We recall that \( V_\mu \) has the same dimension as its classical counterpart.

Let \( V \) be a finite-dimensional \( U_q(\mathfrak{g}) \)-module, \( v \in V \), and \( f \in V^* \), the linear dual of \( V \). Consider the function
\[
c_v^f : U_q(\mathfrak{g}) \to \mathbb{C}, \quad X \mapsto f(X(v)).
\]
Consider now the Hopf subalgebra of \( U_q(\mathfrak{g})^0 \), the Hopf dual of \( U_q(\mathfrak{g}) \), generated by all functions of the form \( c_v^f \), for \( V \) a type-1 representation. We denote this Hopf \(*\)-algebra by \( \mathcal{O}_q(G) \) and call it the Drinfeld–Jimbo quantum coordinate algebra of \( G \), where \( G \) is the compact, simply-connected, simple Lie group having \( \mathfrak{g} \) as its complexified Lie algebra. Note that by construction, \( \mathcal{O}_q(G) \) is a CQGA.

6.2. Quantum Flag Manifolds. For \( S \subset \Pi \) a non-empty subset of simple roots, consider the Hopf \(*\)-subalgebra
\[
U_q(\mathfrak{l}_S) := \langle K_i, E_s, F_s \mid i = 1, \ldots, r; \ s \in S \rangle \subseteq U_q(\mathfrak{g}).
\]
The right action of \( U_q(\mathfrak{g}) \) on \( \mathcal{O}_q(G) \) restricts to a left \( U_q(\mathfrak{l}_S) \)-action. We call the \(*\)-subalgebra
\[
\mathcal{O}_q(G/L_S) := \mathcal{O}_q(G)^{U_q(\mathfrak{l}_S)}
\]
the quantum flag manifold associated to \( S \). Note that \( \mathcal{O}_q(G/L_S) \) is a right \( \mathcal{O}_q(G) \)-comodule algebra by construction. Moreover, \( \mathcal{O}_q(G) \) is faithfully flat as a left \( \mathcal{O}_q(G/L_S) \)-module (see for example [21, §5.4]). Thus it follows from [38 Theorem 1] that \( \mathcal{O}_q(G/L_S) \) coincides with the space of left coinvariants of the coaction \( \Delta_R := (\pi_S \otimes \text{id}) \circ \Delta \), where
\[
\pi_S : \mathcal{O}_q(G) \to \mathcal{O}_q(L_S) := \mathcal{O}_q(G)/\mathcal{O}_q(G)\mathcal{O}_q(G/L_S)^+
\]
is the canonical Hopf algebra projection, for \( \mathcal{O}_q(G/L_S)^+ := \mathcal{O}_q(G/L_S) \cap \ker(\varepsilon) \). In particular, we note that \( \mathcal{O}_q(G/L_S) \) is a CQGA homogeneous space.
6.3. Relative Line Modules over the Irreducible Quantum Flag Manifolds. In this sub-
section we discuss relative line modules over a special subfamily of quantum flag man-
ifolds. If $S = \{\alpha_1, \ldots, \alpha_r\} \setminus \{\alpha_x\}$, where $\alpha_x$ has coefficient 1 in the expansion of the highest root of $\mathfrak{g}$, then we say that the associated quantum flag manifold is irreducible. In general, the one-dimensional $U_q(\mathfrak{l}_S)$-modules correspond to the elements of $P_{S_c}$, which in turn implies that the one-dimensional $O_q(\mathfrak{l}_S)$-comodules also correspond to the elements of $P_{S_c}$. Thus by Takeuchi’s equivalence the relative line modules are indexed by the weights $P_{S_c}$. For the special case of the irreducible quantum flag manifolds

$$P_{S_c} = \mathbb{Z}\omega_x.$$ 

In this case we denote by $\mathcal{E}_l$ the relative line module corresponding to the weight $l\omega_x$.

We make two important observations about relative line modules over the irreducible quantum flag manifolds: Firstly, we note that for all $l \in \mathbb{Z}$, we have $(\mathcal{E}_l)^* \simeq \mathcal{E}_{-l}$. Secondly, we note that for each $l \in \mathbb{Z}$, the sesquilinear pairing

$$h_{\mathcal{E}_l} : \mathcal{E}_l \times \mathcal{E}_l \to O_q(G/L_S), \quad (e_1, e_2) \mapsto e_1 e_2^*,$$

gives $\mathcal{E}_l$ the structure of a covariant Hermitian relative line module. Since $\mathcal{E}_l$ is a simple object in $0\text{mod}O_q(G)$, we see that $h_{\mathcal{E}_l}$ is the unique such structure up to positive scalar multiple.

6.4. Compact Quantum Homogeneous Kähler Spaces. As established in the seminal papers [19, 20], over any irreducible quantum flag manifold $O_q(G/L_S)$, there exists a unique finite-dimensional left $O_q(G)$-covariant $*\text{dc}$

$$\Omega_q^*(G/L_S) \in 0\text{mod}O_q(G),$$

of classical total dimension. Moreover, as is clear from the Heckner–Kolb construction, each $\Omega_q^*(G/L_S)$ comes endowed with an opposite pair of left $O_q(G)$-covariant complex structures

$$\Omega_q^{(*)}(G/L_S), \quad \bar{\Omega}_q^{(*)}(G/L_S),$$

and these are the unique such complex structures for $\text{dc}$. It follows from [28, Theorem 5.10] and [14, Proposition 5.5] that exists an open interval $I$ around 1, and a form $\kappa \in \Omega^{(1,1)}$, such that the pair

$$(\Omega_q^{(*)}(G/L_S), \kappa)$$

is a left $O_q(G)$-covariant Kähler structure, for all $q \in I$. The associated metric $g_\kappa$ is positive definite and this uniquely identifies $\kappa$ up to strictly positive real multiple. Finally, we recall that closure of the integral of the Kähler structure was established in [14]. Collecting together all these results we now arrive at the following theorem.

Theorem 6.1. For each irreducible quantum flag manifold $O_q(G/L_S)$, the quadruple

$$\left(O_q(G/L_S), \Omega_q^*(G/L_S), \Omega_q^{(*)}(G/L_S), \kappa\right)$$

is a CQH-Kähler space, for all $q \in I$. 
6.5. Twisted Dolbeault–Dirac Fredholm Operators. Since each $\mathcal{O}_q(G)$ is finitely generated as an algebra, Proposition 3.2 implies that the Hilbert space of square integrable twisted forms is separable. Every morphism

$$f : \mathcal{F} \otimes \mathcal{O}_q(G/L_S) \Omega_q^*(G/L_S) \to \mathcal{F} \otimes \mathcal{O}_q(G/L_S) \Omega_q^*(G/L_S)$$

in the category of relative Hopf modules extends to a bounded operator on the Hilbert space. In particular, Corollary 3.4 implies that the Hilbert space carries a bounded representation of the Lie algebra $\mathfrak{sl}_2$.

As shown in [16, Theorem 4.5], for each relative Hopf module $F$, there exists a unique $\mathcal{O}_q(G)$-covariant holomorphic structure $\partial F : F \to F \otimes \mathcal{O}_q(G/L_S) \Omega^q(0,1)(G/L_S)$. We denote the associated Chern connection by $\nabla$, and the associated holomorphic structure by $\partial F$. It follows from the results of §4.2 that the operators $\nabla, \partial F$, and $\partial F$ are essentially self-adjoint, as are the corresponding Laplacians.

It was shown in [16, Theorem 4.9] that the Hermitian holomorphic module $E_k$ is positive, and that $E_{-k}$ is negative, for any $k \in \mathbb{Z}_{>0}$. The following theorem, one of the main results of the paper, shows that twisting by the negative line modules produces a Fredholm operator. It builds on the Borel–Weil theorem for irreducible quantum flag manifolds, and Serre duality for noncommutative Kähler structures.

**Theorem 6.2.** For $k \in \mathbb{Z}_{>0}$, and $q \in I$, the $E_{-k}$-twisted Dolbeault–Dirac operator is a Fredholm operator. Moreover, the index of the operator is given by

$$\text{index}(D_{\partial E_{-k}}) = (-1)^n \dim(V_{k\varpi_x}) = (-1)^n \prod_{\alpha \in \Delta^+(\rho, \alpha)} \prod_{\alpha \in \Delta^+} (\rho + k\varpi_x, \alpha),$$

where $2n$ is the total dimension of the dc, $\Delta^+$ is the set of positive roots of $\mathfrak{g}$, and $\rho$ is the half-sum of positive roots.

**Proof.** It follows from Theorem 5.4 that the operator is a Fredholm operator if and only if the $(0,n)$-cohomology group of $E_{-k}$ is finite-dimensional. As established in the Borel–Weil theorem for the irreducible quantum flag manifolds [31, Theorem 6.1], it holds that the $(0,0)$-cohomology group of $E_k$ is an irreducible $U_q(\mathfrak{g})$-module of highest weight $k\varpi_x$. Serre duality for noncommutative Kähler structures [32, Theorem 6.8] tells us that we have a non-degenerate bilinear pairing

$$H_{\partial E_{-k}}^{(0,0)} \times H_{\partial E_{-k}}^{(0,n)} \to \mathbb{C}.$$ 

Thus the dimension of the $(0,n)$-cohomology group is equal to the dimension of the irreducible representation $V_{k\varpi_x}$, implying that the operator is Fredholm and has operator index $(-1)^n \dim(V_{k\varpi_x})$ as claimed. The second identity in (6) now follows from the Weyl dimension formula for finite-dimensional $U_q(\mathfrak{g})$-modules. □

**Example 6.3.** Consider the special case of quantum projective space $\mathcal{O}_q(\mathbb{C}P^n)$. This is the $A_n$-series irreducible quantum flag manifold corresponding to the subset of simple roots $S = \Pi \setminus \{\varpi_1\}$, where we have adopted the standard numbering of roots [23, §11.4].
For the relative line module $\mathcal{E}_{-1}$, the representation $V_{\varpi_1}$ is the classical case gives the vector space representation of $\mathfrak{sl}_{n+1}$. Thus we see that

$$\text{index}\left(\frac{D_{\mathcal{E}_{-1}}}{\partial}\right) = (-1)^n(n+1).$$

More generally, consider the quantum $m$-plane Grassmannian $O_q(\text{Gr}_{n,m})$, that is to say, the $A_n$-series irreducible quantum flag manifold corresponding to $S = \Pi\{\varpi_m\}$, for $m = 1, \ldots, n$. For the relative line module $\mathcal{E}_{-1}$, the classical representation $V_{\varpi_m}$ is the $m$-fold exterior power of the vector space representation of $\mathfrak{sl}_{n+1}$. Thus we see that

$$\text{index}\left(\frac{D_{\mathcal{E}_{-1}}}{\partial}\right) = (-1)^{(n-m+1)m}\left(\begin{array}{c} n+1 \\ m \end{array}\right).$$

Finally, we consider the quantum spinor variety $O_q(S_n)$, which is to say, the $D_n$-series quantum flag manifold corresponding to the subset of simple roots $S = \Pi\{\alpha_n\}$ (or isomorphically $S = \Pi\{\alpha_{n-1}\}$). Taking the line bundle $\mathcal{E}_{-1}$, the corresponding classical representation $V_{\varpi_n}$ is the half spin representation. Thus we have that

$$\text{index}\left(\frac{D_{\mathcal{E}_{-1}}}{\partial}\right) = (-1)^{(n-1)n/2}2^{n-1}.$$

See [33, Table 5] for a list of explicit dimensions for other distinguished representations.

**Appendix A. Compact Quantum Groups**

In this appendix we present two complementary approaches to compact quantum groups. The first is purely Hopf algebraic and due to Koornwinder and Dijkhuizen [17]. The second approach is $C^*$-algebraic and due to Woronowicz [41].

**A.1. Compact Quantum Groups Algebras.** For $(V, \Delta_L)$ a left $A$-comodule, its space of matrix elements is the sub-coalgebra

$$\mathcal{C}(V) := \text{span}_{\mathbb{C}}\{(id \otimes f)\Delta_L(v) \mid f \in \text{Hom}_{\mathbb{C}}(V, \mathbb{C}), v \in V\} \subseteq A.$$  

A comodule is irreducible if and only if its coalgebra of matrix elements is irreducible, and, for $W$ another left $A$-comodule, $\mathcal{C}(V) = \mathcal{C}(W)$ if and only if $V$ is isomorphic to $W$.

Let us now recall the definition of a cosemisimple Hopf algebra, a natural generalisation of the properties of a reductive algebraic group. (See [26, Theorem 11.13] for details.)

A Hopf algebra $A$ is called cosemisimple if it admits a (necessarily unique) linear map $h : A \to \mathbb{C}$, called the *Haar functional*, such that $h(1) = 1$, and

$$(id \otimes h) \circ \Delta(a) = h(a)1, \quad (h \otimes id) \circ \Delta(a) = h(a)1,$$ 

for all $a \in A$.

This is equivalent to having the *Peter–Weyl decomposition* $A \simeq \bigoplus_{V \in \hat{A}} \mathcal{C}(V)$, where summation is over $\hat{A}$, the set of all equivalence classes of irreducible left $A$-comodules. A *compact quantum group algebra*, or a $CQGA$, is a cosemisimple Hopf $*$-algebra $A$ such that the Haar functional $h$ is positive, which is to say $h(a^*a) > 0$, for all non-zero $a \in A$. 

A.2. **Compact Quantum Groups.** Compact quantum group algebras are the algebraic counterpart of Woronowicz’s $C^*$-algebraic notion of a compact quantum group [41]. Every CQGA can be completed to a compact quantum group, and every such completion admits an extension of $\hbar$ to a $C^*$-algebraic state. Moreover, every CQG arises as the completion of a CQGA [39, Theorem 5.4.1]. Every completion lives between a smallest and a largest completion, analogous to the full and reduced group $C^*$-algebras [39, §5.4.2].

The completion relevant to this paper is the smallest completion, whose construction we now briefly recall. (See [39, §5.4.2] for a more detailed presentation.) For $\hbar$ the Haar functional of a CQGA $A$, an inner product is defined on $A$ by

$$(a, b) \mapsto \hbar(ab^*).$$

Consider now the faithful $*$-representation $\lambda_A : A \to \text{End}_\mathbb{C}(A)$, uniquely defined by $\lambda_A(a)(b) := ba$, where $\text{End}_\mathbb{C}(A)$ denotes the $\mathbb{C}$-linear operators on $A$. For all $a \in A$, the operator $\lambda_A(a)$ is bounded with respect to $\langle \cdot, \cdot \rangle_{\hbar}$. Hence, denoting by $L^2(A)$ the associated Hilbert space completion of $A$, each operator $\lambda_A(a)$ extends to an element of $\mathbb{B}(L^2(A))$. We denote by $A_{\text{red}}$ the corresponding closure of $\lambda_A(A)$ in $\mathbb{B}(L^2(A))$. The coproduct of $A$ extends to a $*$-homomorphism $\Delta : A_{\text{red}} \to A_{\text{red}} \otimes_{\min} A_{\text{red}}$, and together the pair $(A_{\text{red}}, \Delta)$ forms a CQG.

**Appendix B. The Rudiments of Unbounded Operators**

In this appendix, we present the rudiments of the theory of unbounded operators on Hilbert spaces, with a view to making the paper more accessible to those coming from an algebraic or geometric background. For more details we refer the reader to the standard texts [36] and [22].

Let $T : \text{dom}(T) \to \mathcal{H}$ be a not necessarily bounded operator on a Hilbert space $\mathcal{H}$, with $\text{dom}(T)$ denoting its domain of definition. We say that $T$ is closed if its graph $\mathcal{G}(T)$ is closed in the direct sum $\mathcal{H} \oplus \mathcal{H}$. We say that an operator $T$ is closable if the closure of its graph in $\mathcal{H} \oplus \mathcal{H}$ is the graph of a (necessarily closed) operator $T^c$, which we call the closure of $T$. When no confusion arises we will not distinguish notationally between an operator and its closure.

For $T : \text{dom}(T) \to \mathcal{H}$ a densely-defined operator, the associated adjoint operator $T^\dagger$ has domain consisting of those elements $x \in \mathcal{H}$ such that

$$(\psi_x : \text{dom}(T) \to \mathbb{C}, \quad y \mapsto \langle T(y), x \rangle)$$

is a continuous linear functional. By the Riesz representation theorem, there exists a unique $z \in \mathcal{H}$, such that $\langle y, z \rangle = \langle T(y), x \rangle$, for all $y \in \text{dom}(T)$. The operator $T^\dagger$ is then defined as

$$T^\dagger : \text{dom}(T^\dagger) \to \mathcal{H}, \quad x \mapsto z.$$  

As established in [36, Theorem 13.8], any operator whose adjoint is densely-defined is necessarily closable.

A densely-defined operator $T$ is said to be symmetric if

$$\langle T(x), y \rangle = \langle x, T(y) \rangle,$$

for all $x, y \in \text{dom}(T)$. 


For any symmetric operator $T$ it is easy to see that $\text{dom}(T) \subseteq \text{dom}(T^\dagger)$. Thus every densely-defined symmetric operator is automatically closable. An operator $T$ is said to be self-adjoint if it is symmetric and $\text{dom}(T) = \text{dom}(T^\dagger)$, and is said to be essentially self-adjoint if it is closable and its closure is self-adjoint. As explained in [36, §13.20], a densely-defined symmetric operator is essentially self-adjoint if the operators $T + \text{id}_H$ and $T - \text{id}_H$ have dense range.

A complex number $\lambda$ is said to be in the resolvent set $\lambda(T)$ of an unbounded operator $T : \text{dom}(T) \to \mathcal{H}$, if

$$T - \lambda \text{id}_H : \text{dom}(T) \to \mathcal{H},$$

has a bounded inverse, that is, if there exists a bounded operator $S : \mathcal{H} \to \text{dom}(T)$ such that $S \circ (T - \lambda \text{id}_H) = \text{id}_{\text{dom}(T)}$ and $(T - \lambda \text{id}_H) \circ S = \text{id}_H$. The spectrum of $T$, which we denote by $\sigma(T)$, is the complement of $\lambda(T)$ in $\mathbb{C}$. Just as in the bounded case, self-adjoint operators have real spectrum. We denote the set of eigenvalues of $T$ by $\sigma_p(T)$ and call it the point spectrum of $T$. It is clear from the definition of the spectrum that $\sigma_p(T) \subseteq \sigma(T)$.

We now recall the functional calculus for unbounded self-adjoint operators: For any self-adjoint operator $T$, and any bounded Borel function $f : \sigma(T) \to \mathbb{C}$, one can associate a bounded operator $f(T) : \mathcal{H} \to \mathcal{H}$. This extends the usual functional calculus for bounded operators (see [22, §1.8] for details).

**Appendix C. Spectral Triples**

In this appendix we recall the definition of a spectral triple and produce sufficient and necessary conditions on the point spectrum of the Dolbeault–Dirac operator of a CQH-Kähler space to give a spectral triple. We also discuss how non-vanishing of the anti-holomorphic Euler characteristic of the underlying complex structure implies non-triviality of the associated $K$-homology class.

**C.1. Spectral Triples and the Bounded Transform.** The $K$-homology of a $C^*$-algebra is the unitary equivalence classes of even Fredholm modules up to operator homotopy. In practice the calculation of the index of a $K$-homology class, or more generally its pairing with $K$-theory, can prove difficult. However, the work of Baaj and Julg [1], and Connes and Moscovici [10], shows that by considering spectral triples, unbounded representatives of $K$-homology classes, the problem can often become more tractable.

**Definition C.1.** A spectral triple $(A, \mathcal{H}, D)$ consists of a unital $*$-algebra $A$, a separable Hilbert space $\mathcal{H}$, endowed with a faithful $*$-representation $\lambda : A \to \mathbb{B}(\mathcal{H})$, and $D : \text{dom}(D) \to \mathcal{H}$ a densely-defined self-adjoint operator, such that

1. $\lambda(a)\text{dom}(D) \subseteq \text{dom}(D)$, for all $a \in A$,
2. $[D, \lambda(a)]$ is a bounded operator, for all $a \in A$,
3. $(D^2 + i)^{-1} \in \mathcal{K}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H})$ denotes the compact operators on $\mathcal{H}$.

An even spectral triple is a quadruple $(A, \mathcal{H}, D, \gamma)$, consisting of a spectral triple $(A, \mathcal{H}, D)$, and a $\mathbb{Z}_2$-grading $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of Hilbert spaces $\gamma$, with respect to which $D$ is a degree 1 operator, and $\lambda(a)$ is a degree 0 operator, for each $a \in A$. 
Spectral triples are important primarily because they provide unbounded representatives for $K$-homology classes. For a spectral triple $(A, \mathcal{H}, D)$, its bounded transform is the operator

$$b(D) := \frac{D}{\sqrt{1 + D^2}} \in \mathbb{B}(\mathcal{H}),$$

defined via the functional calculus. A Fredholm module is given by $(\mathcal{H}, \lambda, b(D))$. (See [6] for details.) The index of the Fredholm operator $D^+ : H_0 \to \mathcal{H}_1$ is clearly equal to the index of the bounded transform. Since the index is an invariant of $K$-homology classes, a spectral triple with non-zero index has a non-trivial associated $K$-homology class.

### C.2. Spectral Triples and Dolbeault–Dirac Eigenvalues

We now formulate precise criteria for when the Dolbeault–Dirac operator of a CQH-Hermitian space gives a spectral triple. For sake of clarity and convenience, let us recall the relevant properties of $L^2(\Omega^{\bullet, \bullet})$ and $D_{\overline{\partial}}$. If $A$ is finitely generated, then it follows from Proposition 3.2 that $L^2(\Omega^{\bullet})$ is separable. By Corollary 3.10 we have a faithful $*$-representation $\lambda : B \to \mathbb{B}(L^2(\Omega^{\bullet}))$. From Corollary 4.6 we know that $D_{\overline{\partial}}$ is an essentially self-adjoint operator, which is, moreover, densely-defined by construction. By Corollary 3.9, the commutators $[D_{\overline{\partial}}, \lambda(b)]$ are bounded, and by Proposition 4.8 above, $\lambda(b)\text{dom}(D_{\overline{\partial}}) \subseteq \text{dom}(D_{\overline{\partial}})$, for all $b \in B$. With respect to the obvious $\mathbb{Z}_2$-grading $\gamma$, the operator $D_{\overline{\partial}}$ is of degree 1, and $\lambda(b)$ is a degree 0 operator, for all $b \in B$. Finally, we note that since $D_{\overline{\partial}}$ is diagonalisable on $L^2(\Omega^{\bullet})$, it has compact resolvent if and only if its eigenvalues tend to infinity and have finite multiplicity. Collecting these facts together gives the following proposition.

**Proposition C.2.** Let $H = (B \subseteq A, \Omega^{\bullet}, \Omega^{\bullet, \bullet}, \sigma)$ be a CQH-Hermitian space for which $A$ is finitely generated as an algebra, then an even spectral triple is given by

$$\left( B \subseteq A, L^2(\Omega^{\bullet, \bullet}), D_{\overline{\partial}}, \gamma \right),$$

if and only if the eigenvalues of $D_{\overline{\partial}}$ tend to infinity and have finite multiplicity.

We call such a spectral triple the *Dolbeault–Dirac spectral triple* of $H$. In the accompanying paper [13], these criteria were verified for the special case of quantum projective space $\mathcal{O}_q(\mathbb{C}P^n)$, producing a motivating family of examples of Dolbeault–Dirac spectral triples.

The discussions in §5.3 give the following immediate result, where we denote by $\mathcal{B}$ the closure of $\lambda(B)$ in $\mathbb{B}(L^2(\Omega^{0, \bullet}))$.

**Corollary C.3.** Let $H = (B \subseteq A, \Omega^{\bullet}, \Omega^{\bullet, \bullet}, \sigma)$ be a CQH-Hermitian space with a Dolbeault–Dirac spectral triple. The $K^0(B)$-class of the spectral triple is non-trivial if the holomorphic Euler characteristic of $\Omega^{\bullet, \bullet}$ is non-trivial.

The notion of a *noncommutative Fano structure* was introduced in [32, Definition 8.8]. It is a refinement of a Kähler structure, generalising the classical definition of a Fano manifold. A *CQH-Fano space* is a CQH-Kähler space whose constituent Kähler structure is a Fano structure. It follows from Theorem 6.1 and [16, Theorem 4.12] that the irreducible quantum flag manifolds give CQH-Fano structures.
Corollary C.4. Let $F = (B \subseteq A, \Omega^*, \Omega^{(\bullet, \bullet)}, \sigma)$ be a CQH-Fano space with a Dolbeault–Dirac spectral triple. Then the $K^0(B)$-class of the spectral triple is non-trivial.

Proof. It follows from [32, Corollary 8.9] that $H^{(0,k)} = 0$, for all $k > 0$. Thus we see that the anti-holomorphic Euler characteristic of the calculus is equal to the dimension of $H^{(0,0)}$. However, since $1$ is always contained ker($\overline{\partial}$), this is non-zero. Thus it follows from Corollary C.3 that the $K^0(B)$-class of the spectral triple is non-trivial. □

We finish this subsection with an easy observation about the Dolbeault–Dirac operator of the opposite CQH-Hermitian space.

Proposition C.5. For a CQH-Hermitian space $H = (B, \Omega^*, \Omega^{(\bullet, \bullet)}, \sigma)$, the two operators $D_\partial : \Omega^{(\bullet, 0)} \to \Omega^{(\bullet, 0)}$ and $D_\overline{\partial} : \Omega^{(0, \bullet)} \to \Omega^{(0, \bullet)}$ are unitarily equivalent. In particular, \begin{equation}
(B, L^2(\Omega^{(\bullet, 0)}), D_\partial)
\end{equation}
is a spectral triple if and only if $(B, L^2(\Omega^{(0, \bullet)}), D_\overline{\partial})$ is a spectral triple.

Proof. A form $\omega \in \Omega^{(0, \bullet)}$ is an eigenvector of $D_\partial$ if and only if $\omega^* \in \Omega^{(\bullet, 0)}$ is an eigenvector of $D_\overline{\partial}$, as we see from the identity $D_\overline{\partial}(\omega^*) = D_\partial(\omega)^*$. Thus the set of eigenvalues of $D_\partial$ coincides with the set of eigenvalues of $D_\overline{\partial}$, and we have a real linear isomorphism between the respective eigenspaces. Since the eigenspaces of each operator are necessarily orthogonal, we can now construct a unitary map $U : \Omega^{(0, \bullet)} \to \Omega^{(\bullet, 0)}$ satisfying $D_\partial = U \circ D_\overline{\partial} \circ U^{-1}$. Extending $U$ to the domain of the closure of $D_\overline{\partial}$ gives the required unitary equivalence. It now follows from Proposition C.2 that if one triple is a spectral triple then so is the other. □

We call such a pair of unitarily equivalent spectral triples a Dolbeault–Dirac pair. It is important to note that the unitary equivalence between the operators $D_\partial$ and $D_\overline{\partial}$ will not in general be a module map, nor an $A$-comodule map.

References

[1] S. Baaj and P. Julg. Théorie bivariante de Kasparov et opérateurs non bornés dans les $C^*$-modules hilbertiens. C. R. Acad. Sci. Paris Sér. I Math., 296(21):875–878, 1983.
[2] E. Beggs and S. Majid. Spectral triples from bimodule connections and Chern connections. J. Noncommut. Geom., 11(2):669–701, 2017.
[3] E. Beggs and S. Majid. Quantum Riemannian Geometry, volume 355 of Grundlehren der mathematischen Wissenschaften. Springer International Publishing, 1 edition, 2019.
[4] E. Beggs and P. S. Smith. Noncommutative complex differential geometry. J. Geom. Phys., 72:7–33, 2013.
[5] S. Bhattacharjee, I. Biswas, and D. Goswami. Generalized symmetry in noncommutative complex geometry. arXiv preprint math.QA/1907.04673.
[6] A. Carey and J. Phillips. Unbounded Fredholm modules and spectral flow. Canad. J. Math., 50(4):673–718, 1998.
[7] A. L. Carey, J. Phillips, and A. Rennie. Spectral triples: examples and index theory. In Noncommutative geometry and physics: renormalisation, motives, index theory, ESI Lect. Math. Phys., pages 175–265. Eur. Math. Soc., Zürich, 2011.
[8] A. Carotenuto, F. Díaz García, and R. Ó Buachalla. A Borel–Weil theorem for the irreducible quantum flag manifolds. arXiv preprint math.QA/2112.03305.
[9] A. Carotenuto and R. Ó Buachalla. Principal pairs of quantum homogeneous spaces. arXiv preprint math.QA/2112.03305.
[10] A. Connes and H. Moscovici. The local index formula in noncommutative geometry. Geom. Funct. Anal., 5(2):174–243, 1995.
[11] F. D’Andrea and L. Dąbrowski. Dirac operators on quantum projective spaces. Comm. Math. Phys., 295(3):731–790, 2010.
[12] B. Das, R. Ó Buachalla, and P. Somberg. arXiv preprint math.QA/1903.07599.
[13] B. Das, R. Ó Buachalla, and P. Somberg. A Dolbeault–Dirac spectral triple for quantum projective space. arXiv preprint math.QA/1903.07599.
[14] B. Das, R. Ó Buachalla, and P. Somberg. Spectral gaps for twisted Dolbeault–Dirac operators over the irreducible quantum flag manifolds. (in preparation).
[15] L. Dąbrowski and A. Sitarz. Dirac operator on the standard Podleś quantum sphere. In Noncommutative geometry and quantum groups (Warsaw, 2001), volume 61 of Banach Center Publ., pages 49–58. Polish Acad. Sci. Inst. Math., Warsaw, 2003.
[16] F. Díaz García, A. Krutov, R. Ó Buachalla, P. Somberg, and K. R. Strung. Positive line bundles over the irreducible quantum flag manifolds. arXiv preprint math.QA/1912.08802.
[17] M. S. Dijkhuizen and T. H. Koornwinder. CQG algebras: a direct algebraic approach to compact quantum groups. Lett. Math. Phys., 32(4):315–330, 1994.
[18] I. Forsyth, B. Mesland, and A. Rennie. Dense domains, symmetric operators and spectral triples. New York J. Math., 20:1001–1020, 2014.
[19] I. Heckenberger and S. Kolb. The locally finite part of the dual coalgebra of quantized irreducible flag manifolds. Proc. London Math. Soc. (3), 89(2):457–484, 2004.
[20] I. Heckenberger and S. Kolb. De Rham complex for quantized irreducible flag manifolds. J. Algebra, 305(2):704–741, 2006.
[21] I. Heckenberger and S. Kolb. On the Bernstein-Gelfand-Gelfand resolution for Kac-Moody algebras and quantized enveloping algebras. Transform. Groups, 12(4):647–655, 2007.
[22] N. Higson and J. Roe. Analytic K-homology. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. Oxford Science Publications.
[23] J. E. Humphreys. Introduction to Lie algebras and representation theory, volume 9 of Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1978. Second printing, revised.
[24] D. Huybrechts. Complex geometry: an introduction. universitext, Springer–Verlag, 2005.
[25] M. Khalkhali, G. Landi, and W. D. van Suijlekom. Holomorphic structures on the quantum projective line. Int. Math. Res. Not. IMRN, (4):851–884, 2011.
[26] A. Klimyk and K. Schmüdgen. Quantum Groups and Their Representations. Texts and Monographs in Physics. Springer-Verlag, 1997.
[27] J.-L. Koszul and B. Malgrange. Sur certaines structures fibrées complexes. Arch. Math. (Basel), 9:102–109, 1958.
[28] M. Matassa. Kähler structures on quantum irreducible flag manifolds. J. Geom. Phys., 145:103477, 16, 2019.
[29] G. Nagy. Deformation quantization and K-theory. In Perspectives on quantization (South Hadley, MA, 1996), volume 214 of Contemp. Math., pages 111–134. Amer. Math. Soc., Providence, RI, 1998.
[30] R. Ó Buachalla. Noncommutative complex structures on quantum homogeneous spaces. J. Geom. Phys., 99:154–173, 2016.
[31] R. Ó Buachalla. Noncommutative Kähler structures on quantum homogeneous spaces. Adv. Math., 322:892–939, 2017.
[32] R. Ó Buachalla, J. Šťovíček, and A.-C. van Roosmalen. A Kodaira vanishing theorem for noncommutative Kähler structures. arXiv preprint math.QA/1801.08125., 2018.
[33] A. L. Onishchik and E. B. Vinberg. Lie groups and algebraic groups. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. Translated from the Russian and with a preface by D. A. Leites.
[34] R. Owczarek. Dirac operator on the Podleś sphere. volume 40, pages 163–170. 2001. Clifford algebras and their applications (Ixtapa, 1999).

[35] A. Polishchuk and A. Schwarz. Categories of holomorphic vector bundles on noncommutative two-tori. Comm. Math. Phys., 236(1):135–159, 2003.

[36] W. Rudin. Functional analysis. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[37] M. Schechter. Basic theory of Fredholm operators. Ann. Scuola Norm. Sup. Pisa (3), 21:261–280, 1967.

[38] M. Takeuchi. Relative Hopf modules—equivalences and freeness criteria. J. Algebra, 60(2):452–471, 1979.

[39] T. Timmermann. An invitation to quantum groups and duality. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008. From Hopf algebras to multiplicative unitaries and beyond.

[40] A. Weil. Introduction à l’étude des variétés kähleriennes. Number 1267 in Publications de l’Institut de Mathématique de l’Université de Nancago, VI. Hermann, Paris, 1958.

[41] S. L. Woronowicz. Compact matrix pseudogroups. Comm. Math. Phys., 111(4):613–665, 1987.

[42] S. L. Woronowicz. Twisted SU(2) group. An example of a noncommutative differential calculus. Publ. Res. Inst. Math. Sci., 23(1):117–181, 1987.

Laboratory of Advanced Combinatorics and Network Applications, Department of Applied Mathematics, Moscow Institute of Physics and Technology, Moscow, Russia

Instytut Matematyczny, Uniwersytet Wrocławski, pl.Grunwaldzki 2/4, 50-384 Wrocław, Poland

Email address: biswarup.das@math.uni.wroc.pl

Mathematical Institute of Charles University, Sokolovská 83, Prague, Czech Republic

Email address: obuachalla@karlin.mff.cuni.cz

Mathematical Institute of Charles University, Sokolovská 83, Prague, Czech Republic

Email address: somberg@karlin.mff.cuni.cz