Spacetime Entanglement Entropy of
de Sitter and Black Hole Horizons

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Abstract

We calculate Sorkin’s manifestly covariant entanglement entropy $S$ for a massive and massless minimally coupled free Gaussian scalar field for the de Sitter horizon and Schwarzschild de Sitter horizons, respectively, in $d > 2$. In de Sitter spacetime we restrict the Bunch-Davies vacuum in the conformal patch to the static patch to obtain a mixed state. The finiteness of the spatial $L^2$ norm in the static patch implies that $S$ is well defined for each mode. We find that $S$ for this mixed state is independent of the effective mass of the scalar field, and matches that of [1], where, a spatial density matrix was used to calculate the horizon entanglement entropy. Using a cut-off in the angular modes we show that $S \propto A_c$, where $A_c$ is the area of the de Sitter cosmological horizon. Our analysis can be carried over to the black hole and cosmological horizon in Schwarzschild de Sitter spacetime, which also has finite spatial $L^2$ norm in the static regions. Although the explicit form of the modes is not known in this case, we use the boundary conditions of [2] for a massless minimally coupled scalar field, to find the mode-wise $S_{b,c}$, where $b,c$ denote the black hole and de Sitter cosmological horizons, respectively. As in the de Sitter calculation we see that $S_{b,c} \propto A_{b,c}$ after taking a cut-off in the angular modes.

1 Introduction

Entanglement entropy (EE) has emerged as an important quantity in the study of quantum fields in curved spacetime. Of particular interest is the EE of quantum fields across black hole horizons which might partially or fully account for the Bekenstein-Hawking entropy [3]. The importance of studying all types of horizons was pointed out by Jacobson and Parentani [4] who showed that thermality and the area law are features of all causal horizons. Cosmological horizons in de Sitter (dS) spacetime are known to have thermodynamic properties similar to their black hole counterparts even though these horizons are observer dependent [5]. Because of the relative simplicity of these spacetimes, they provide a useful arena to test new proposals for calculating the EE.

In most definitions of EE one considers the entanglement of the state at a moment of time between two spatial regions, which is restrictive in the context of quantum gravity, or even quantum fields in curved spacetimes which may lack a preferred time. A more global, covariant notion of EE could be more useful when working with a covariant path-integral or histories based approach to quantum gravity. In particular, the notion of a state at a moment of time might not survive, especially in theories where the manifold structure of spacetime breaks down in the deep UV regime, as in causal set theory [6].

To this end, Sorkin proposed a covariant formulation of EE for a Gaussian (free) scalar field by expressing it in terms of the spacetime correlators or Wightman function $W(x, y) \equiv \langle \hat{\Phi}(x)\hat{\Phi}(y) \rangle$ [7]. Starting with the pure state $W(x, y)$ in $(\mathcal{M}, g)$ whose restriction to $\mathcal{O} \subset \mathcal{M}$ is $W(x, y)|_{\mathcal{O}}$, the EE associated to this state is given by Sorkin’s spacetime entanglement entropy (SSEE) formula

$$S = \sum_{\mu} \mu \log |\mu|, \quad \hat{W}|_{\mathcal{O}} \circ \chi = i\mu \hat{\Delta} \circ \chi, \quad \chi \not\in \text{Ker}(i\hat{\Delta}),$$

(1.1)
where \( i\hat{\Delta} \) is the integral operator defined via the Pauli-Jordan function \( i\Delta(x, y) = [\phi(x), \phi(y)] \) and where
\[
(\hat{A}\circ f)(x) = \int_{\mathcal{O}} dV_y A(x, y) f(y).
\] (1.2)

This expression was obtained in \([7]\) by noticing that for discrete spaces the field operators, which are the generators of the free field algebra, can be expanded in a “position-momentum” basis which renders \( i\hat{\Delta} \) into a \( 2 \times 2 \) block diagonal form and simultaneously diagonalises the symmetric part of \( \hat{W} \). Each block is a single particle system for which the von Neumann entropy for a Gaussian state can be calculated using results of \([3]\). The expression can be then rearranged in terms of the eigenvalues of \( \hat{\Delta}^{-1}\hat{W} \). Summing over all the blocks gives the SSEE expression Eqn. \((1.1)\). Apart from discreteness, an important assumption in the construction is that there is an irreducible representation associated with the restriction of the algebra to the region of interest. Our explicit construction in this present work is therefore an important confirmation that the SSEE is indeed equivalent to the von Neumann entropy in de Sitter spacetime.

This formula has been applied in the continuum to the \( d = 2 \) nested causal diamonds as well as to the causal diamond contained in the \( d = 2 \) cylinder spacetime, and shown to give the expected Calabrese-Cardy logarithmic behaviour \([8, 9]\). It has also been calculated in the discrete setting, i.e., for causal sets approximated by causal diamonds in Minkowski and de Sitter spacetimes: in \( d = 2 \) where it shows the expected logarithmic behaviour and in \( d = 4 \) where it shows the expected area behaviour \([10, 11, 12]\).

In this work, we present an analytic calculation of the SSEE for de Sitter horizons for all \( d > 2 \) for a massive scalar field with effective mass \( m = \sqrt{m^2 + \xi R} \). Our calculation uses the restriction of the Bunch-Davies vacuum in the Poincare or conformal patch of de Sitter to the static patch. Even though \( \mathcal{O} \) is non-compact in the time direction, we show that the generalised eigenvalue equation Eqn. \((1.1)\) can be explicitly solved mode by mode. We find that the SSEE is independent of the effective mass, which is in agreement with the results of Higuchi and Yamamoto \([1]\). The total SSEE can be calculated using a UV cut-off in the angular modes for the Bunch-Davies vacuum and is therefore proportional to the regularised area of the horizon. The other \( \alpha \) vacua however need an additional momentum cut-off.

The obvious generalisation of our calculation to static black hole and Rindler horizons is hampered by the spatial non-compactness, except in the case of Schwarzschild de Sitter black holes. For these spacetimes, the explicit form of the modes is not known, except in \( d = 2 \). In \([2]\) certain natural boundary conditions for massless minimally coupled modes were used to analyse the thermodynamic properties of these horizons. We employ these same boundary conditions to find the mode-wise form for the SSEE in the static region. Introducing the cut-off in the angular modes again gives us the requisite area dependence.

In the special case of \( d = 2 \), the calculation can be performed explicitly, and we find that the SSEE is constant for both the black hole as well as the cosmological horizon. Thus we do not find the logarithmic behaviour expected from the Calabrese-Cardy formula. A key difference is that in the earlier calculations, \( \mathcal{O} \) is compact and the mixed state in \( \mathcal{O} \) is not diagonal with respect to the (Sorkin-Johnston) modes in \( \mathcal{O} \). Although this is surprising, the calculations of \([1]\) suggest that this is also a feature of the standard von Neumann EE in \( d = 2 \) de Sitter spacetime.

We organise our paper as follows. In Sec.\(2\) we lay out the general framework for the calculation of the mode-dependent SSEE for a compact region \( \mathcal{O} \) with respect to a vacuum state in \( \mathcal{M} \supset \mathcal{O} \). We find the solutions to the generalised eigenvalue equation Eqn. \((1.1)\) when the modes in \( \mathcal{O} \) are \( L^2 \) orthogonal, and the Bogoliubov coefficients satisfy certain conditions. We then show that Eqn. \((1.1)\) is also well posed for static spherically symmetric spacetimes with finite spatial extent. Assuming that the restricted vacuum \( W_\mathcal{O} \) is block diagonal in the modes in \( \mathcal{O} \) we find the general form of the mode-wise SSEE. In Sec.\(3\) we review some

\(1\)As pointed out in \([1]\) this result differs from the EE calculated in \([13]\) (see also \([14, 15]\)), which was found to be mass dependent. This difference can be traced to the fact that the entanglement surface in \([13]\) is not the de Sitter horizon but a superhorizon sized surface on a hyperbolic slicing of the conformally flat de Sitter region, taken to lie close to the future null boundary \( \mathcal{T}^+ \), with the explicit aim of studying superhorizon entanglement generated by expansion. Since the entangling regions are different from that in our present work, it is not surprising that the results also differ.
basics of de Sitter and Schwarzschild de Sitter spacetimes. In Sec. 4.1 we apply the analysis of Sec. 2 to the static patches of $d = 4$ de Sitter, starting with the Bunch-Davies vacuum in the conformal patch. Using an angular cut-off we show that the SSEE is proportional to the regularised de Sitter horizon area. In Sec. 4.2 we calculate the SSEE for a massless minimally coupled scalar field in the static patches of Schwarzschild de Sitter spacetimes for $d > 2$ using the boundary conditions of [2]. An explicit calculation of the $d = 2$ case then follows. We discuss the implications of our results in Sec. 5. In Appendix A we extend the de Sitter horizon calculation to the other $\alpha$ vacua in the conformal patch. We find that while the mode-wise SSEE is still independent of the effective mass, the total SSEE needs an additional cut-off in the radial momentum. In Appendix B we extend the $d = 4$ analysis to all dimensions $d > 2$.

2 The SSEE: General Features

In this section we examine the SSEE generalised eigenvalue equation Eqn (1.1) using the two sets of modes in the regions $\mathcal{M}, \mathcal{O}$, where $\mathcal{O} \subset \mathcal{M}$. We show that when the modes in the subregion $\mathcal{O}$ are $L^2$ orthogonal, and the Bogoliubov transformations satisfy certain conditions, it is possible to find the general form for the SSEE. While not entirely general, this covers a fairly wide range of cases.

Let $\{\Phi_k\}$ be the Klein-Gordon (KG) orthonormal modes in $(\mathcal{M}, g)$, i.e.,

$$(\Phi_k, \Phi_{k'})_\mathcal{M} = -(\Phi_{k'}^*, \Phi_k^*)_\mathcal{M} = \delta_{kk'} \quad \text{and} \quad (\Phi_k, \Phi_{k'}^*)_\mathcal{M} = 0,$$  

(2.1)

and $\{\Psi_p\}$ be those in the globally hyperbolic region $\mathcal{O} \subset \mathcal{M}$. Here $(.,.)_\mathcal{M}$ denotes the KG inner product in $\mathcal{M}$ given by

$$(\phi_1, \phi_2)_\mathcal{M} = i \int_{\Sigma, \mathcal{M}} d\Sigma^a \left( \phi_1^* \partial_a \phi_2 - \phi_2^* \partial_a \phi_1 \right),$$

(2.2)

where $d\Sigma^a$ is the volume element on the spacelike hypersurface $\Sigma \in \mathcal{M}$ with respect to the future pointing unit normal. The corresponding Wightman function in $(\mathcal{M}, g)$ is

$$W(x, x') = \sum_k \Phi_k(x) \Phi_k^*(x').$$

(2.3)

Since $\{\Psi_p\}$ forms a complete KG orthonormal basis in $\mathcal{O}$, the restriction of $\Phi_k$ to $\mathcal{O}$ can be expressed as a linear combination of $\Psi_p$ modes, i.e.,

$$\Phi_k(x) \big|_\mathcal{O} = \sum_p \left( \alpha_{kp} \Psi_p(x) + \beta_{kp} \Psi_p^*(x) \right),$$

(2.4)

where $\alpha_{kp} = (\Psi_p, \Phi_k)_\mathcal{O}$ and $\beta_{kp} = -(\Psi_p^*, \Phi_k)_\mathcal{O}$. The restriction of $W(x, x')$ to $\mathcal{O}$ can thus be re-expressed in terms of $\{\Psi_p\}$ as

$$W(x, x') \big|_\mathcal{O} = \sum_{pp'} \left( A_{pp'} \Psi_p(x) \Psi_{p'}^*(x') + B_{pp'} \Psi_p(x) \Psi_{p'}(x') + C_{pp'} \Psi_p^*(x) \Psi_{p'}(x') + D_{pp'} \Psi_p^*(x) \Psi_{p'}^*(x') \right),$$

(2.5)

where

$$A_{pp'} = \sum_k \alpha_{kp} \alpha_{kp'}, \quad B_{pp'} = \sum_k \alpha_{kp} \beta_{kp'}, \quad C_{pp'} = \sum_k \beta_{kp} \alpha_{kp'}, \quad D_{pp'} = \sum_k \beta_{kp} \beta_{kp'}.$$  

(2.6)

The Pauli-Jordan function $i\Delta(x, x') = [\hat{\Phi}(x), \hat{\Phi}(x')]$ can be expanded in the modes in $\mathcal{O}$ to give

$$i\Delta(x, x') = \sum_p \left( \Psi_p(x) \Psi_p^*(x') - \Psi_p^*(x) \Psi_p(x') \right).$$

(2.7)
The generalised eigenvalue equation for the SSEE Eqn. (1.1) thus reduces to

\[ \sum_{p,p'} \left( A_{pp'} \langle \Psi_{p'}^*, \chi_r \rangle_\mathcal{O} + B_{pp'} \langle \Psi_{p'}^*, \chi_r \rangle_\mathcal{O} \right) \Psi_p(x) + \left( C_{pp'} \langle \Psi_{p'}^*, \chi_r \rangle_\mathcal{O} + D_{pp'} \langle \Psi_{p'}^*, \chi_r \rangle_\mathcal{O} \right) \Psi_p^*(x) \]

\[ = \mu_r \sum_p \left( \langle \Psi_p, \chi_r \rangle_\mathcal{O} \Psi_p(x) - \langle \Psi_p^*, \chi_r \rangle_\mathcal{O} \Psi_p^*(x) \right), \tag{2.8} \]

where \( \langle ., . \rangle_\mathcal{O} \) denotes the \( \mathcal{L}^2 \) inner product in \( \mathcal{O} \)

\[ \langle \phi_1, \phi_2 \rangle_\mathcal{O} = \int_\mathcal{O} dV_\mathcal{x} \phi_1^*(\mathcal{x})\phi_2(\mathcal{x}). \tag{2.9} \]

Note that the coefficients in Eqn (2.6) can be evaluated using the relation

\[ W(\mathcal{x}, \mathcal{x}') \big|_\mathcal{O} - W^*(\mathcal{x}, \mathcal{x}') \big|_\mathcal{O} = i\Delta(\mathcal{x}, \mathcal{x}'), \tag{2.10} \]

so that

\[ A_{pp'} - D_{pp'} = \delta_{pp'} \Rightarrow \sum_k (\alpha_{kp}\alpha_{kp'}^* - \beta_{kp'}\beta_{kp}) = \delta_{pp'}, \tag{2.11} \]

\[ B_{pp'} - C_{pp'}^* = 0 \Rightarrow \sum_k (\alpha_{kp}\beta_{kp}^* - \beta_{kp}\alpha_{kp}) = 0. \tag{2.12} \]

We now look for a special class of solutions of Eqn (2.8).

To begin with we consider the case when the \( \mathcal{L}^2 \) inner product Eqn. (2.9) is finite (this is the case for example if \( \mathcal{O} \) is compact). We can then use the linear independence of the \( \{ \Psi_p \} \) to obtain the coupled equations

\[ \sum_{p'} \left( A_{pp'} \langle \Psi_{p'}^*, \chi_r \rangle_\mathcal{O} + B_{pp'} \langle \Psi_{p'}^*, \chi_r \rangle_\mathcal{O} \right) = \mu_r \langle \Psi_p, \chi_r \rangle_\mathcal{O}, \]

\[ \sum_{p'} \left( C_{pp'} \langle \Psi_{p'}^*, \chi_r \rangle_\mathcal{O} + D_{pp'} \langle \Psi_{p'}^*, \chi_r \rangle_\mathcal{O} \right) = -\mu_r \langle \Psi_p^*, \chi_r \rangle_\mathcal{O}. \tag{2.13} \]

Next, assume that the \( \{ \Psi_p \} \) are \( \mathcal{L}^2 \) orthogonal. Then

\[ \chi_p(\mathcal{x}) = R\Psi_p(\mathcal{x}) + S\Psi_p^*(\mathcal{x}), \tag{2.14} \]

are eigenfunctions of Eqn. (1.1) if

\[ RA_{pp} + SB_{pp} = \mu_p R\delta_{pp}, \]

\[ RC_{pp} + SD_{pp} = -\mu_p S\delta_{pp}. \tag{2.15} \]

This has non-trivial solutions iff

\[ (A_{pp} - \mu_p\delta_{pp})(D_{pp} + \mu_p\delta_{pp}) - B_{pp}C_{pp} = 0. \tag{2.16} \]

For \( p \neq \bar{p} \) Eqns. (2.11) and (2.12) this requires in particular that\(^2\)

\[ |D_{pp}|^2 = |C_{pp}|^2, \quad p \neq \bar{p}. \tag{2.17} \]

\(^2\)This additional condition is not satisfied for example for a causal diamond in the \( d = 2 \) cylinder spacetime \[9\].
For \( p = \vec{p} \), letting \( A_{pp} = a_p, B_{pp} = b_p, C_{pp} = c_p, D_{pp} = d_p \), we see that \( a_p, d_p \) are real from Eqn. (2.6), so that
\[
\mu_{p}^{\pm} = \frac{1}{2} \left( 1 \pm \sqrt{1 + 2d_p} - 4|c_p|^2 \right),
\]
which is real only if
\[
(1 + 2d_p)^2 \geq 4|c_p|^2.
\]
This can be shown to be true using the following identity
\[
\sum_k |\alpha_{kp} - e^{i\theta} \beta_{kp}|^2 \geq 0
\]
\[
\Rightarrow 1 + 2d_p - 2|c_p| \cos(\theta + \theta') \geq 0,
\]
where \( c_p = |c_p|e^{i\theta'} \). Taking \( \theta = -\theta' \) gives us the desired relation. The two eigenvalues \( \mu_{p}^{+}, \mu_{p}^{-} \) moreover satisfy the relation
\[
\mu_{p}^{-} = 1 - \mu_{p}^{+},
\]
and therefore come in pairs \( (\mu_{p}^{+}, 1 - \mu_{p}^{+}) \), as expected [7].

Thus the mode-wise SSEE is
\[
\mathcal{S}_p = \mu_{p}^{+} \log(|\mu_{p}^{+}|) + (1 - \mu_{p}^{+}) \log(|1 - \mu_{p}^{+}|).
\]
As we will see in the specific case of de Sitter and \( d = 2 \) Schwarzschild de Sitter spacetimes, \( \mu_{p}^{+}, \mu_{p}^{-} \not\in (0, 1) \) which is again consistent with the expectations of [7].

In this work we are interested in subregions \( O \) which are static and spherically symmetric. While non-compact in the time direction we require them to be compact in the spatial direction. Thus the \( L^2 \) inner product is \( \delta \)-function orthogonal and not strictly finite. As we will see, this can still result in a finite \( \mathcal{S}_p \). In \( d = 4 \) for example,
\[
\Psi_{plm}(t, r, \theta, \phi) = N_{pl}R_{pl}(r)e^{-i\theta}Y_{1m}(\theta, \phi), \quad p > 0,
\]
where \( t \in (-\infty, \infty), r > 0 \) and \( (\theta, \phi) \in S^2 \). \( N_{pl} \) denotes an overall normalisation constant, and \( p \) is a continuous variable. Thus one has integrals over \( p \) as well as summations over \( l \) and \( m \) in Eqn. (2.13). These modes are clearly \( L^2 \) orthogonal since
\[
\langle \Psi_{plm}, \Psi_{p'l'm'} \rangle = 2\pi |N_{pl}|^2 ||R_{pl}||^2 \delta(p - p')\delta_l\delta_{m'm'},
\]
where \( ||R_{pl}|| \) is the \( L^2 \) norm in the radial direction and finite by assumption. This \( \delta \)-function orthogonality implies that for any function \( \chi_r \) (which can be expanded in terms of the complete \( \{\Psi_{plm}\} \) basis), both sides of Eqn. (2.13) are finite.

If \( \tilde{W} \) is block diagonal in the \( \{\Psi_{plm}\} \) basis
\[
A_{plmp'lm'} = a_{plm}\delta(p - p')\delta_l\delta_{m'm'}, \quad B_{plmp'lm'} = b_{plm}\delta(p - p')\delta_l\delta_{m'm'},
\]
\[
C_{plmp'lm'} = c_{plm}\delta(p - p')\delta_l\delta_{m'm'}, \quad D_{plmp'lm'} = d_{plm}\delta(p - p')\delta_l\delta_{m'm'}.
\]
This simplifies Eqn. (2.8) considerably since the delta functions can be integrated over \( p' \) and similarly, summed over \( l', m' \). Using the ansatz
\[
\chi_{plm}(t, r, \theta, \phi) = R\Psi_{plm}(t, r, \theta, \phi) + S\Psi_{plm}^{*}(t, r, \theta, \phi),
\]
for the eigenfunctions requires that Eqn. (2.16) is satisfied, as before. This yields the same form for \( \mu_{plm} \) as Eqn. (2.18) and hence the SSEE Eqn. (2.22).
3 Preliminaries

We briefly review de Sitter and Schwarzschild de Sitter spacetimes.

de Sitter spacetime $dS$ in $d$ dimensions is a hyperboloid of “radius” $H^{-1}$ in $d+1$ dimensional Minkowski spacetime $\mathbb{R}^{1,d}$. If $X_i$’s are the coordinates in $\mathbb{R}^{1,d}$, it is the hypersurface defined by

$$-X_0^2 + \sum_{i=1}^{d} X_i^2 = \frac{1}{H^2}. \quad (3.1)$$

We restrict our discussion to $d = 4$ in what follows, since the higher dimensional generalisation is relatively straightforward (see Appendix B). Global $dS_4$ can be parameterized by 4 coordinates $(\tau, \theta_1, \theta_2, \theta_3)$, where $\tau$ is the global time and $\theta_i$’s are coordinates on a 3-sphere $S^3$. In these coordinates the metric can be written as

$$ds^2 = -d\tau^2 + \frac{1}{H^2} \cosh^2(H\tau) d\Omega_3^2, \quad (3.2)$$

where, $\tau \in \mathbb{R}$, $\theta_1, \theta_2 \in [0, \pi]$ and $\theta_3 \in [0, 2\pi]$. The causal structure of this spacetime becomes evident if we make the coordinate transformation $\cosh(H\tau) = 1/\cos T$, so that

$$ds^2 = \frac{1}{H^2 \cos^2 T}(-dT^2 + d\Omega_3^2), \quad T \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \quad (3.3)$$

In Fig. 1 the region $I \cup III$ is the right conformal patch or the Poincaré patch. It can be described by the metric

$$ds^2 = \frac{1}{H^2 \eta^2} \left(-d\eta^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\right), \quad (3.4)$$

where $\eta \in (-\infty, 0)$, $r \in [0, \infty)$ and $(\theta, \phi) \in S^2$. Its subregion $I$ is the right static patch and is covered by the coordinates $x \in [0, 1)$, $t \in \mathbb{R}$, $(\theta, \phi) \in S^2$ which are related to the coordinates in the conformal patch by

$$x = -\frac{r}{\eta}, \quad e^{-t} = \sqrt{\eta^2 - r^2}, \quad (3.5)$$

so that the static patch metric is

$$ds^2 = \frac{1}{H^2} \left(-(1 - x^2)dt^2 + \frac{dx^2}{1 - x^2} + x^2 d\Omega_2^2\right). \quad (3.6)$$

\[\text{Figure 1: The Penrose diagram for dS can be deduced from the metric Eqn. (3.3). Here 2 dimensions are suppressed so that each point represents an } S^2 \text{ and each horizontal slice an } S^3. \ dS \text{ is spatially compact, the left and right vertical lines correspond to } \theta_1 = 0, \pi. \ \text{The lower, upper horizontal lines correspond to } T = -\pi/2, \pi/2 \text{ and represent the past, future null infinities respectively.}\]

\[\text{metric} \quad ds^2 = \frac{1}{H^2 \eta^2} \left(-d\eta^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\right), \quad (3.4)\]

\[\text{where } \eta \in (-\infty, 0), \ r \in [0, \infty) \text{ and } (\theta, \phi) \in S^2. \ \text{Its subregion } I \text{ is the right static patch and is covered by the coordinates } x \in [0, 1), \ t \in \mathbb{R}, \ (\theta, \phi) \in S^2 \text{ which are related to the coordinates in the conformal patch by } x = -\frac{r}{\eta}, \quad e^{-t} = \sqrt{\eta^2 - r^2}, \quad (3.5)\]

\[\text{so that the static patch metric is} \quad ds^2 = \frac{1}{H^2} \left(-(1 - x^2)dt^2 + \frac{dx^2}{1 - x^2} + x^2 d\Omega_2^2\right). \quad (3.6)\]

\[\text{3For a detailed review of coordinate systems in dS, see [16].}\]
We now turn to the Schwarzschild-de Sitter spacetime, whose conformal diagram is shown in Fig. 2. It has two sets of horizons each in regions I and II: the cosmological horizons $\mathcal{H}_b^\pm$ and the black hole horizons $\mathcal{H}_c^\pm$, with the latter contained “inside” the former.

In either of the static patches, I or II, the metric of the Schwarzschild-de Sitter spacetime is

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2), \quad f(r) = 1 - \frac{2M}{r} - H^2r^2 \tag{3.7}
\]

\[
= -f(r)dudv + r^2(d\theta^2 + \sin^2(\theta)d\phi^2), \tag{3.8}
\]

where $H$ is the Hubble constant and $M$ is the mass of the black hole $r \in (r_b, r_c)$, $t \in (-\infty, \infty)$ and $(\theta, \phi) \in S^2$. Here $r_b < r_c$ are the real and positive solutions of $f(r) = 0$, which correspond to the black hole and the cosmological horizons $\mathcal{H}_b^\pm, \mathcal{H}_c^\pm$ respectively. They are related to $M$ and $H$ as

\[
M = \frac{r_b r_c (r_b + r_c)}{2(r_b^2 + r_c^2 + r_b r_c)}, \quad H^2 = \frac{1}{r_b^2 + r_c^2 + r_b r_c}. \tag{3.9}
\]

$u, v \in (-\infty, \infty)$ are the light-cone coordinates defined as $u = t - r_*$ and $v = t + r_*$, where $dr_* = \frac{dr}{f(r)}$.

As in the Schwarzschild spacetime, there is a Kruskal extension beyond the black hole and the cosmological horizon, given respectively by

\[
U_b = -\kappa_b^{-1}e^{-\kappa_b u} \quad \text{and} \quad V_b = \kappa_b^{-1}e^{\kappa_b v}, \tag{3.10}
\]

\[
U_c = \kappa_c^{-1}e^{\kappa_c u} \quad \text{and} \quad V_c = -\kappa_c^{-1}e^{-\kappa_c v}, \tag{3.11}
\]

where $\kappa_b$ and $\kappa_c$ are the surface gravity of the black hole and the cosmological horizon respectively.

\[
\kappa_b = \frac{H^2}{2r_b}(r_c - r_b)(r_c + 2r_b), \quad \kappa_c = \frac{H^2}{2r_c}(r_c - r_b)(2r_c + r_b). \tag{3.12}
\]

In these coordinates, the spacetime metrics in region $B \equiv I \cup II \cup III \cup IV$ and $C \equiv I \cup II \cup V \cup VI$ in Fig. 2 are, respectively

\[
dS_B^2 = -f(r)e^{-2\kappa_b r}dU_b dV_b + r^2(d\theta^2 + \sin^2(\theta)d\phi^2), \tag{3.13}
\]

\[
dS_C^2 = -f(r)e^{2\kappa_c r}dU_c dV_c + r^2(d\theta^2 + \sin^2(\theta)d\phi^2). \tag{3.14}
\]
4 SSEE for Cosmological and Black Hole Horizons

We now calculate the SSEE for the static regions in both the de Sitter and Schwarzschild de Sitter spacetimes. In both cases, since the static region is spatially finite, Eqn. (2.8) is well-defined.

4.1 The SSEE in the de Sitter Static Patch

We are interested in finding the entanglement across the intersection sphere \( S^2 \approx I \cap II \) in Fig. 1. The associated sub-region \( O \) of interest to the SSEE calculation is therefore the right or left static region. Without loss of generality we henceforth pick the right static region \( R \) and take as the larger region \( M \supset O \) the conformal patch \( I \cup III \).

In the larger region \( I \cup III \), we have a well known, complete, Klein-Gordon orthonormal set of modes for a free scalar field of effective mass \( m = \sqrt{m^2 + \xi R} \) (where \( \xi = 1/6 \) and \( R \) is the Ricci scalar, which is a constant for de Sitter spacetimes) called the Bunch-Davies modes [15]. These are given by \( \Phi_{klm} \equiv \varphi_{kl}(\eta, r)Y_{lm}(\theta, \phi) \), where

\[
\varphi_{kl}(\eta, r) = \frac{He^{-\frac{ip}{2}(l+\frac{3}{2})}}{\sqrt{2k}} (-k\eta)^\frac{3}{2} e^{i\eta r} H^{(1)}_{\nu}(kr).
\] (4.1)

Here \( k \in \mathbb{R}^+, l \in \{0, 1, 2, \ldots\}, m \in \{-l, 0, \ldots, l\} \), \( Y_{lm}s \) are the spherical harmonics on \( S^2 \), \( j_i \) is the spherical Bessel function and \( H^{(1)}_{\nu}(kr) \) is the Hankel function of the first kind with

\[
\nu = \sqrt{\frac{9}{4} - \frac{m^2}{k^2}},
\] (4.2)

and satisfies the plane-wave behaviour expected at late times.

In the region \( I \) we have a complete set of Klein-Gordon orthonormal modes [19] given by \( \Psi_{plm} \equiv \psi_{pl}(t, x)Y_{lm}(\theta, \phi) \) where

\[
\psi_{pl}(t, x) = \sqrt{2}\sinh(\pi p) N_{pl} U_{pl}(x)e^{-ip\varphi}, \quad p \in \mathbb{R}^+,
\] (4.3)

where

\[
N_{pl} = \frac{H}{2\sqrt{2}\pi \Gamma(l + \frac{3}{2})} \left( \frac{3}{2} + l - ip + \nu \right) \Gamma \left( \frac{3}{2} + l - ip - \nu \right),
\] (4.4)

and

\[
U_{pl}(x) = x^l(1 - x^2)^{-\frac{3}{2} - l} 2F_1 \left( \frac{3}{2} + l - ip + \nu, \frac{3}{2} + l - ip - \nu, l + \frac{3}{2}, x^2 \right).
\] (4.5)

As shown in [19],

1. \( U_{pl}(x) = U_{-pl}(x) = U_{pl}^*(x) \), which can be shown using an identity of the Hypergeometric function i.e.,

\[
2F_1(a, b, c; z) = (1-z)^{c-a-b} 2F_1(c-a, c-b, c, z).
\]

2. \( N_{pl} = N_{-pl}^* \), which comes from the identity \( \Gamma^*(z) = \Gamma(z^*) \).

As discussed in Sec. 2, being static and spherically symmetric, the \( \Psi_{plm} \) modes are also \( L^2 \) orthogonal in \( I \).

We now proceed to obtain the SSEE for the sub-region \( I \) with respect to the Bunch-Davies vacuum in the right conformal patch \( I \cup III \). As suggested in Sec. 2, we begin by demonstrating that the Bogoliubov coefficients between the Bunch-Davies modes \( \Phi_{klm} \) and the static modes \( \Psi_{plm} \) in \( I \) satisfy the criteria Eqn. (2.25).

Since the \((\theta, \phi)\) dependence of both sets of modes is given by \( Y_{lm}(\theta, \phi) \), which themselves are linearly independent in \( S^2 \), the Bogoliubov transformation is non-trivial only between \( \varphi_{kl} \) and \( \psi_{pl} \) for each \( l, m \), i.e.,

\[
\varphi_{kl}(\eta, r) = \int_0^\infty dp \left( \alpha_{kp} \psi_{pl}(t, x) + \beta_{kp} \psi_{pl}^*(t, x) \right).
\] (4.6)
Instead of using the Klein-Gordon inner product to calculate $\alpha_{kp}$ and $\beta_{kp}$, we can use the $L^2$ orthogonality of the $\Psi_{plm}$ modes as well as the $L^2$ inner product of $\Phi_{klm}$ and $\Psi_{plm}$ in $I$, so that

$$\alpha_{kp} = \frac{1}{n_p} \langle \Psi_{plm}, \Phi_{klm} \rangle_I$$

and

$$\beta_{kp} = \frac{1}{n_p} \langle \Psi_{plm}^*, \Phi_{klm} \rangle_I,$$

with $n_p = 4\pi \sinh(\pi p)|N_{pl}|^2||U_{pl}||^2$ being the $L^2$ norm of the $\Psi_{plm}$ modes. The identity [20]

$$\int_0^\infty dz \, z^\lambda H^{(1)}_\nu(az)J_\nu(bz) = a^{-\lambda-1} e^{\frac{1}{2}(\nu+\mu+1)} \left( \frac{\lambda + \nu + \mu + 1}{2} \right) \left( \frac{\lambda - \nu + \mu + 1}{2} \right) \pi \Gamma(\mu + 1),$$

with $\lambda = n^2 + m^2, \nu = 2, \mu = 2$ being the $2$ norm of $\Psi_{plm}$ in $I$, and $\lambda = -i(a + b)$, $\mu = \lambda + 1 + \nu > 0$, can be used as in [1], to show that

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \, dk \, k^{-i\nu - \frac{1}{2}} \varphi_{kl}(\eta, r) = 2^{-i\nu} e^{\pi \nu} N_{pl} \left( \eta^2 - r^2 \right)^{\frac{i}{2}} U_{pl} \left( -\frac{r}{\eta} \right) = 2^{-i\nu} e^{\pi \nu} N_{pl} U_{pl}(x)e^{-ipt},$$

where we have substituted $\lambda = -ip, \mu = l + 1/2, a = -\eta$ and $b = r$. Inverting the above,

$$\varphi_{kl}(\eta, r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \, dp \, 2^{-i\nu} k^{i\nu - \frac{1}{2}} e^{\pi \nu} N_{pl} U_{pl}(x)e^{-ipt},$$

using which

\[
\alpha_{kp} = \frac{1}{\sqrt{2\pi n_p}} \int_{-\infty}^\infty \, dp' \, 2^{-i\nu} k^{i\nu - \frac{1}{2}} e^{\pi \nu} \sqrt{2\sinh(\pi p)} N_{p'1} N_{pl}^* \int_0^1 \, dx \, x^2 U_{p'1}(x) U_{pl}(x) \int_{-\infty}^\infty \, dt \, e^{-i(p' - p)t} = \frac{2^{-i\nu} k^{i\nu - \frac{1}{2}}}{\sqrt{2\pi(1 - e^{-2\pi p})}},
\]

\[
\beta_{kp} = \frac{1}{\sqrt{2\pi n_p}} \int_{-\infty}^\infty \, dp' \, 2^{i\nu} k^{-i\nu - \frac{1}{2}} e^{-\pi \nu} \sqrt{2\sinh(\pi p)} N_{p'1} N_{pl}^* \int_0^1 \, dx \, x^2 U_{p'1}(x) U_{pl}(x) \int_{-\infty}^\infty \, dt \, e^{i(p' - p)t} = \frac{2^{i\nu} k^{-i\nu - \frac{1}{2}}}{\sqrt{2\pi(e^{2\pi p} - 1)}},
\]

Notice that the Hubble constant $H$ drops out of these coefficients. Further calculation shows that

$$A_{pp'} = \frac{\delta(p - p')}{1 - e^{-2\pi p}}, \quad D_{pp'} = \frac{\delta(p - p')}{e^{2\pi p} - 1} \quad \text{and} \quad B_{pp'} = C_{pp'} = 0.$$  

(4.13)

This is precisely of the form Eqn. (2.25) with $b_p = c_p = 0$, where we have suppressed the $l, m$ indices. Using the ansatz

$$\chi^+_p(t, r) = u_p(t, r), \quad \chi^-_p(t, r) = u_p^*(t, r),$$

for the generalised eigenfunctions of Eqn. (2.8), we find the generalised eigenvalues

$$\mu^+_p = \frac{1}{1 - e^{-2\pi p}} \quad \text{and} \quad \mu^-_p = -\frac{e^{-2\pi p}}{1 - e^{-2\pi p}},$$

(15)
In the Schwarzschild de Sitter spacetime, regions II and \( \mathcal{I} \) are static and spherically symmetric, which means that the massless scalar field modes are of the form Eqn. \((2.23)\). What is important for our analysis is that in the absence of a cutoff in \( l \), there is an infinite degeneracy for every \( p \) coming from the angular modes which leads to an infinite factor in the total entropy. This “density of states” for a given \( p \) can be regulated by introducing a cut-off \( l_{\text{max}} \), so that

\[
S = \sum_{l=0}^{l_{\text{max}}} \sum_{m=-l}^{l} dp S_p = \frac{\pi}{6} \left( l_{\text{max}} + 1 \right)^2 \simeq \frac{\pi l_{\text{max}}^2}{6},
\]

for \( l_{\text{max}} \gg 1 \). \( l_{\text{max}} \) can in turn be interpreted as coming from the regularised area of the de Sitter horizon \( \mathcal{I} \cap \mathcal{II} \simeq S^2 \). Let us for the moment suppress one of the angular variables so that the modes on an \( S^1 \) of radius \( H^{-1} \) are \( e^{i m \phi} \). A UV cut-off \( m_{\text{max}} \) corresponds to a minimal angular scale \( \Delta \phi = 2\pi/m_{\text{max}} \) and hence the length cut-off \( \ell_c \), where \( m_{\text{max}} = \frac{2\pi}{\ell_c} \). Thus \( m_{\text{max}} \) is the circumference of the \( S^1 \) in units of the cut-off. A similar argument carries over to \( S^2 \), where we first place \( \theta \) and \( \phi \) on similar footing by writing the spherical harmonics as a Fourier series

\[
Y_{lm}(\theta, \phi) \propto P_l^m(\cos \theta) e^{im\phi} = \sum_{j=-l}^{l} \tilde{P}_{jl} e^{ij\theta} e^{im\phi}. \tag{4.18}
\]

Thus, we again have the angular cut-offs \( \Delta \theta = 2\pi/l_{\text{max}}, \Delta \phi = 2\pi/m_{\text{max}} \), so that \( l_{\text{max}}^2 = 4\pi^2/\Delta \theta \Delta \phi \). For large \( l_{\text{max}} \) the planar limit of the region subtended by the solid angle \( \Delta \Omega = \sin \theta \Delta \theta \Delta \phi \) on \( S^2 \) can be taken near the equator, \( \theta = \pi/2 - \epsilon \), where the metric is nearly flat in \((\theta, \phi)\) coordinates: \( ds^2 \simeq dt^2 + d\phi^2 \). Thus, \( l_{\text{max}}^2 \propto 1/\Delta \Omega \) and therefore

\[
S \propto \frac{A_c}{\ell_c^2}, \tag{4.19}
\]

where we have defined a fundamental cut-off \( \ell_c^2 = H^{-2} \Delta \Omega \) and \( A_c = 4\pi H^{-2} \) is the area of the de Sitter cosmological horizon.

As shown in Appendix A for all the other \( \alpha \) vacua, the integral \( \int_0^p dp S_p \) is not finite. This necessitates an additional cut-off \( p_{\text{max}} \). The extension to general \( d > 2 \) is shown in Appendix B.

### 4.2 The SSEE of Schwarzschild de Sitter Horizons

In the Schwarzschild de Sitter spacetime, regions \( \mathcal{I} \) and \( \mathcal{II} \) are static and spherically symmetric, which means that the massless scalar field modes are of the form Eqn. \((2.23)\). What is important for our analysis is that the spacetime is spatially bounded so that the calculations of Sec. 2 can be applied to this case. Without loss of generality we will work with region \( \mathcal{I} \) to calculate its SSEE.

Although our focus is the \( d > 2 \) case, we begin by suppressing the angular dependence and considering the \( d = 2 \) case first. The massless, Klein-Gordon orthonormal scalar field modes are then simply the plane waves in \( \mathcal{I} \)

\[
\Psi_p^{(1)}(u) = \frac{1}{\sqrt{4\pi p}} e^{-ipu} \quad \text{and} \quad \Psi_p^{(2)}(v) = \frac{1}{\sqrt{4\pi p}} e^{-ipv}, \quad p > 0, \tag{4.20}
\]
where we have suppressed the $b,c$ indices in $(U_{b,c}, V_{b,c})$ for simplicity. Note that the modes in region $I$ are static, and of the form Eqn. (2.23), with $l \in \{0,1\}$ representing the left and right movers. This means that the radial part is $L^2$, which ensures finiteness of Eqn. (2.8).

The restriction of $\Phi_k^{(1,2)}$ to region I can be written in terms of $\Psi_p^{(1,2)}$ as

$$\Phi_k^{(1,2)} = \int_0^{\infty} dp \left( \alpha_k^{(1,2)} \Psi_p^{(1,2)} + \beta_k^{(1,2)} \Psi_p^{(1,2)*} \right),$$

where

$$\alpha_k^{(1)} = \frac{1}{2\pi} \sqrt{\frac{p}{k}} \int_{-\infty}^{\infty} du \ e^{i p u} e^{-i k u} = \frac{1}{2\pi \sqrt{\kappa}} \left( \frac{k}{\kappa} \right)^{\frac{i}{2}} e^{\frac{i}{2} \pi \kappa} \left( -i \frac{p}{\kappa} \right),$$

$$\beta_k^{(1)} = \frac{1}{2\pi} \sqrt{\frac{p}{k}} \int_{-\infty}^{\infty} du \ e^{-i p u} e^{-i k u} = \frac{1}{2\pi \sqrt{\kappa}} \left( \frac{k}{\kappa} \right)^{-\frac{i}{2}} e^{-\frac{i}{2} \pi \kappa} \left( i \frac{p}{\kappa} \right),$$

$a_{k}^{(2)} = a_{k}^{(1)*}$ and $\beta_{k}^{(2)} = \beta_{k}^{(1)*}$ for the black hole horizon. For the cosmological horizon, they are complex conjugates of Eqns. (4.23) and (4.24). Thus, we find that for both $(1,2)$ modes,

$$A_{pp'} = a_p \delta(p - p'), \quad D_{pp'} = d_p \delta(p - p') \quad \text{and} \quad B_{pp'} = C_{pp'} = 0,$$

with

$$a_p = \frac{1}{1 - e^{-2\pi \tilde{p}}} \quad \text{and} \quad d_p = \frac{e^{-2\pi \tilde{p}}}{1 - e^{-2\pi \tilde{p}}},$$

Using Eqn. (2.18) and the dimension-free $\tilde{p} \equiv p \kappa^{-1}$, we see that

$$S_{\tilde{p}} = - \log \left( 1 - e^{-2\pi \tilde{p}} \right) - \frac{e^{-2\pi \tilde{p}}}{1 - e^{-2\pi \tilde{p}}} \log \left( e^{-2\pi \tilde{p}} \right).$$

The total entropy is then

$$S = 2 \int_0^{\infty} d\tilde{p} S_{\tilde{p}} = - \frac{2}{\pi} \int_0^1 \frac{dz \log(z)}{1 - z} = \frac{\pi}{3},$$

where $z = e^{-2\pi \tilde{p}}$ and the factor of two comes from the fact that the total entropy is the sum of the entropy of the $(1,2)$ modes. $S$ is therefore the same for both horizons.

We now consider the $d = 4$ case by using the boundary conditions of [2]. As mentioned earlier, the full modes are not known, but the boundary conditions suffice to calculate the Bogoliubov coefficients. For our purposes it suffices to use the past boundary conditions, since this defines the Klein Gordon norm on the limiting initial null surface $\mathcal{H}_0^+ \cup \mathcal{H}_c^-$ in Region I. For the static patch modes, which are of the form Eqn. (2.23) these boundary conditions are

$$\Psi_{plm} = \begin{cases} \frac{1}{\sqrt{4\pi \kappa_r}} e^{-ipu} Y_{lm}(\theta, \phi) & \text{on } \mathcal{H}_0^-, \quad p > 0, \\ 0 & \text{on } \mathcal{H}_c^-, \quad p > 0, \end{cases}$$

while for the Kruskal modes across the black hole horizon, they are

$$\Phi_{klm} = \begin{cases} \frac{1}{\sqrt{4\pi \kappa_u}} e^{-ikue} Y_{lm}(\theta, \phi) & \text{on } \mathcal{H}_b^-, \quad k > 0, \\ 0 & \text{on } \mathcal{H}_c^-, \quad k > 0. \end{cases}$$
where $U_b$ is related to $u$ as in Eqn. (3.10) [2]. Note that our normalisation differs from that of [2] and comes from the KG norm on $H_b^- \cup H_c^-$ or equivalently $H_p^-$ for these boundary conditions. The factor $r_b^{-1}$ is dimension dependent and comes from the normalisation of the modes along $H_b^-$ where $r = r_b$, and the angular measure is $r_b^2 d\Omega$. Thus for any $d > 2$, one must include a factor $r_b^{\frac{d-2}{2}}$ to normalise the modes. Importantly, these boundary conditions are not appropriate for $d = 2$, since the left and right movers are independent in that case. Setting the modes to zero on $H_c^-$ in $d = 2$ would thus lead to an incomplete set of modes in region I. This is not the case for $d > 2$, where there is a “mixing” or scattering of the left movers on $H_b^-$ in region I.

Since the modes vanish along $H_c^-$, the KG norm can be defined using only $H_b^-$ in region I of Fig. 2, where $u \in (-\infty, \infty)$ and $U_b \in (-\infty, 0)$. As in the de Sitter calculation, the angular modes for $\Phi_{klm}$ and $\Psi_{plm}$ are the same, so that the calculation reduces to the $d = 2$ case described above, with the Bogoliubov coefficients given by Eqn. (4.23) and (4.24). Note that unlike $d = 2$, there is only one set of complete modes, which corresponds in our case to the set (1).

Thus, the SSEE is given by the $d = 2$ SSEE for one mode, multiplied as in the de Sitter case, by the angular cut-off term, $(l_{\text{max}} + 1)^2$ coming from the degeneracy of the generalised eigenfunctions. A similar calculation can be done for the cosmological horizon, so that we have

$$S_B \propto \frac{A_B}{l_c^2}, \quad S_C \propto \frac{A_C}{l_c^2}. \quad (4.31)$$

We note that a calculation of the Rindler and Schwarzschild horizons with similar boundary conditions should in principle be possible if one employs a suitable radial IR cut-off to regulate the radial $L^2$ norm, so that Eqn. (2.8) is well defined.

5 Discussion

In this work we began with an analysis of the SSEE, using the two sets of modes in $M$ and $O \subset M$. We found that when the Bogoliubov transformations satisfy certain conditions in both the finite as well as the static, spatially finite cases, there are real solutions to the eigenvalue equations which come in pairs $(\mu, 1 - \mu)$. We then calculated the SSEE for de Sitter horizons in $d > 2$ as well as Schwarzschild de Sitter horizons in $d > 2$. We found that in both cases, the eigenvalues also satisfy the condition $\mu \notin (0, 1)$, as expected from the arguments given in [7]. In both spacetimes, we used the cut-off in the angular modes to demonstrate that $S \propto A$ for $d > 2$. This is as expected, and is a further confirmation that the SSEE is a good measure of entanglement entropy.

When we restrict to $d = 2$, however, we find that the SSEE is constant and thus not of the Calabrese-Cardy form. This differs from the results of earlier $d = 2$ calculations of the SSEE both in the continuum and using causal set discretisations [8, 9, 10, 11, 12], where the Calabrese-Cardy form was obtained. We note that this is not a feature only of the SSEE alone but also of the associated Von Neumann EE in $d = 2$, and follows from an extension of the results of [1] to $d = 2$.

An obvious difference with earlier calculations is that $O$ in the de Sitter cases studied here are non-compact. For the nested causal diamonds in $d = 2$ Minkowski spacetime as well as the causal diamond on the finite cylinder spacetime, $O$ is chosen to be the domain of dependence of a finite interval, and is therefore compact [8, 9]. In de Sitter spacetime, the domain of dependence of the half circle is the static patch which is not compact. We have shown that despite the temporal non-compactness, the SSEE equation Eqn. (1.1) is well defined for the static patch. On the other hand, the numerical calculation for de Sitter causal sets [12] necessitated an IR cut-off, so that the regions $(M, g)$ as well as $O$ differ from those used in this work. After a suitable truncation in the discrete spectrum, the Calabrese-Cardy form for the causal set SSEE was recovered. Technically, one of the features that simplified our calculations was the diagonal form
Eqn. (2.25), which, as we had noted in Sec. 2, is not satisfied for the \( d = 2 \) cylinder calculation of [9]. Re-examining our calculation we see that a temporal IR cut-off in \( \mathcal{O} \) would destroy this diagonal property. Whether this could restore the logarithmic behaviour or not would be difficult to establish analytically, but given the causal set example, it suggests that this may indeed be the case. This in turn suggests new subtleties in the nature of \( d = 2 \) entanglement in curved spacetime, which should be explored.

We also note that in these calculations, the angular modes transform trivially. Thus, the generalised eigenvalues are dimension independent, which makes the \( d = 2 \) calculation a simple dimensional restriction. Hence the conclusions we draw in higher dimensions – namely that \( \mathcal{S} \) has an area dependence – also implies that the SSEE is constant in \( d = 2 \). In higher dimensions the density of states comes from the degeneracy of the angular modes on \( S^{d-2} \) which necessitates a cut-off, while that in \( d = 2 \) comes from the two “angular modes” on \( S^0 \).

Ultimately, the use of the SSEE lies in its covariant formulation and its applicability to systems where Hamiltonian methods are not at hand. This is the case with causal set quantum gravity, since the analogues of spatial hypersurfaces allow for a certain “leakage” of information. As shown in [10, 12] the calculation of the SSEE for QFT on causal sets throws up some unexpected behaviour, due to the non-local but covariant nature of the UV cut-off. It is of course not obvious that EE plays a fundamental role in quantum gravity, but the effects of the latter can be non-trivial when discussing emergent phenomena.

The SSEE approach to EE is compatible with that of algebraic quantum field theory, where entanglement measures are state functionals which measure the entanglement of a mixed state \( \hat{W}_\mathcal{O} \) obtained by restricting the pure state \( \hat{W} \) in \( \mathcal{M} \supset \mathcal{O} \). The SSEE was motivated by the study of systems with finite degrees of freedom, but has been shown to give the expected results for systems with infinite degrees of freedom, as is the case here and the \( d = 2 \) examples discussed above. Defining EE for systems with infinite degrees of freedom is however known to be non-trivial; type III algebras which characterise QFT do not factor, thus leading to significant complications (see [22]). Although we have several QFT examples for which the SSEE is a good entanglement measure, an important open question is whether it can be rigorously derived using methods from algebraic quantum field theory.

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A SSEE of the \( \alpha \) vacua for the dS static patch

In this section, we compute the SSEE for the \( \alpha \) vacua [23, 24] \( \Phi_{kln}^{(\alpha,\beta)} \equiv \varphi_{kln}^{(\alpha,\beta)}(\eta,r)Y_{lm}(\theta,\phi) \) which can be parameterized as

\[
\varphi_{kln}^{(\alpha,\beta)}(\eta,r) \equiv \cosh(\alpha)\varphi_{kln}(\eta,r) + \sinh(\alpha)e^{i\beta}\varphi_{kln}^*(\eta,r),
\]

where \( \alpha \in [0, \infty) \) and \( \beta \in (-\pi, \pi) \). Expressing these in terms of the \( \Psi_{plm} \) modes in \( I \) as

\[
\varphi_{kln}^{(\alpha,\beta)}(\eta,r) = \int_0^\infty dp \left( \alpha_{kp}^{(\alpha,\beta)}\psi_{pl}(t,x) + \beta_{kp}^{(\alpha,\beta)}\psi_{pl}^*(t,x) \right),
\]

where

\[
\alpha_{kp}^{(\alpha,\beta)} = \frac{1}{n_p} \left\langle \Psi_{plm}, \Phi_{kln}^{(\alpha,\beta)} \right\rangle_I \quad \text{and} \quad \beta_{kp}^{(\alpha,\beta)} = \frac{1}{n_p} \left\langle \Psi_{plm}^*, \Phi_{kln}^{(\alpha,\beta)} \right\rangle_I.
\]
Using Eqn. (A.1), (4.7), (4.11) and (4.12), we find the coefficients $\alpha_{kp}^{(\alpha,\beta)}$ and $\beta_{kp}^{(\alpha,\beta)}$ to be

$$\alpha_{kp}^{(\alpha,\beta)} = \cosh(\alpha)k_p + \sinh(\alpha)e^{i\beta}k_p = \frac{2^{-ip}k_p e^{ip/2}}{\sqrt{2\pi(1-e^{-2ip})}} \left(\cosh(\alpha) + e^{-ip}e^{i\beta}\sinh(\alpha)\right), \quad (A.4)$$

$$\beta_{kp}^{(\alpha,\beta)} = \cosh(\alpha)k_p + \sinh(\alpha)e^{i\beta}k_p = \frac{2^{-ip}k_p e^{ip/2}}{\sqrt{2\pi(1-e^{-2ip})}} \left(e^{-ip}\cosh(\alpha) + e^{i\beta}\sinh(\alpha)\right). \quad (A.5)$$

Further calculation shows that $A_{pp'}^{(\alpha,\beta)} \equiv \int dk\alpha_{kp}^{(\alpha,\beta)}\alpha_{kp'}^{(\alpha,\beta)*}$, $B_{pp'}^{(\alpha,\beta)} \equiv \int dk\alpha_{kp}^{(\alpha,\beta)}\beta_{kp'}^{(\alpha,\beta)*}$, $C_{pp'}^{(\alpha,\beta)} \equiv \int dk\beta_{kp}^{(\alpha,\beta)}\alpha_{kp'}^{(\alpha,\beta)*}$ and $D_{pp'}^{(\alpha,\beta)} \equiv \int dk\beta_{kp}^{(\alpha,\beta)}\beta_{kp'}^{(\alpha,\beta)*}$ is of the form

$$A_{pp'}^{(\alpha,\beta)} = a_p^{(\alpha,\beta)}\delta(p-p'), \quad D_{pp'}^{(\alpha,\beta)} = a'_p^{(\alpha,\beta)}\delta(p-p'), \quad B_{pp'}^{(\alpha,\beta)} = C_{pp'}^{(\alpha,\beta)} = 0, \quad (A.6)$$

where

$$a_p^{(\alpha,\beta)} = \frac{1}{1-e^{-2ip}} \left(\cosh^2(\alpha) + e^{-2ip}\sinh^2(\alpha) + e^{-ip}\sinh(2\alpha)\cos(\beta)\right), \quad (A.7)$$

$$a'_p^{(\alpha,\beta)} = \frac{1}{1-e^{-2ip}} \left(e^{-2ip}\cosh^2(\alpha) + \sinh^2(\alpha) + e^{-ip}\sinh(2\alpha)\cos(\beta)\right). \quad (A.8)$$

Generalised eigenvalues $\mu$ is then

$$\mu_p^{+(\alpha,\beta)} = a_p^{(\alpha,\beta)}, \quad \mu_p^{-(\alpha,\beta)} = -a'_p^{(\alpha,\beta)}, \quad (A.9)$$

from which we obtain the SSEE as

$$S^{(\alpha,\beta)} = (l_{\text{max}} + 1)^2 \int_0^\infty dp \left(a_p^{(\alpha,\beta)} \log(a_p^{(\alpha,\beta)}) - a'_p^{(\alpha,\beta)} \log(a'_p^{(\alpha,\beta)})\right). \quad (A.10)$$

Unlike for the SSEE obtained from the Bunch-Davies vacuum ($\alpha = 0$), $S^{(\alpha,\beta)}$ is in general dependent on the cut-off in $p$. As an example, for ($\alpha, \beta = (1,0)$, we evaluate the integral in Eqn. (A.10) numerically for different cut-offs in $p$ and find that the SSEE depends on both $p_{\text{max}}$ and $l_{\text{max}}$, and is of the form

$$S^{(1,0)} = S_p(p_{\text{max}})(l_{\text{max}} + 1)^2, \quad (A.11)$$

where for large enough $p_{\text{max}}$, $S_p(p_{\text{max}})$ is found to be proportional to $p_{\text{max}}$ as shown in Fig. 3.

Figure 3: A plot of $S_p$ vs $p_{\text{max}}$ for $(\alpha, \beta) = (1,0)$. We see that for large enough $p_{\text{max}}$, $S_p \propto p_{\text{max}}$. 

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B SSEE of general $d$-dimensional dS horizon

In this section, we extend our calculation of SSEE in four dimensional de Sitter to a general $d$-dimensional de Sitter spacetime with $d > 2$ and show that the entropy depends on the spacetime dimension solely due to the dimension dependent degeneracy of the spherical harmonics.

We start with showing that the Bunch-Davies modes $\{\Phi_{kL}\}$ in the conformal patch of $d$-dimensional de Sitter spacetime is given by $\Phi_{kL} = \phi_{kL}(\eta, r)Y_L(\Omega_{d-2})$, where

$$\phi_{kL}(\eta, r) = \frac{H^{d/2-1}}{\sqrt{2k}}(-k\eta)^{\frac{d}{2}-1}H^{(1)}_{\nu_d}(kr)(kr)^{2-d/2}j_{l+\frac{d}{2}-2}(kr), \; k > 0. \quad \text{(B.1)}$$

Here $L$ represents a collection of indices $\{l, l_1, \ldots, l_{d-4}, m\}$ such that $l, l_1, \ldots, l_{d-4} \in \{0, 1, 2, \ldots\}$, $m \in \mathbb{Z}$ and $l \geq l_1 \geq \ldots \geq l_{d-4} \geq |m|$. $\Omega_{d-2}$ represents a collection of angular coordinates on $S^{d-2}$. We can clearly see that for $d = 4$, these modes reduces to the Bunch-Davies modes given by Eqn. (4.1). For $\{\Phi_{kL}\}$ to qualify for the QFT modes they have to be Klein-Gordon orthonormal solutions of the Klein-Gordon equation, which we will show now.

Klein-Gordon equation for the massive scalar field with effective mass $m$ in de Sitter spacetime is

$$-\eta^d \partial_\eta (\eta^{2-d} \partial_\eta \phi) + \frac{\eta^2}{r^{d-2}} \partial_r (r^{d-2} \partial_r \phi) + \frac{\eta^2}{r^2} \nabla^2_{\Omega_{d-2}} \phi = \frac{m^2}{H^2} \phi. \quad \text{(B.2)}$$

For $\phi_{kL}$ given by Eqn. (B.1) we find

$$-\eta^d \partial_\eta (\eta^{2-d} \partial_\eta \phi_{kL}) = \frac{1}{4} \left( 1 + d(d - 2) + 4k^2 \eta^2 - 4\nu^2 \right) \phi_{kL}, \quad \text{(B.3)}$$

$$\frac{\eta^2}{r^{d-2}} \partial_r (r^{d-2} \partial_r \phi_{kL}) = \frac{\eta^2}{r^2} (l(l + d - 3) - k^2 \eta^2) \phi_{kL}, \quad \text{(B.4)}$$

$$\frac{\eta^2}{r^2} \nabla^2_{\Omega_{d-2}} Y_L = -\frac{\eta^2}{r^2} l(l + d - 3) Y_L. \quad \text{(B.5)}$$

Therefore $\phi_{kL}$ given by Eqn. (B.1) solves the Klein-Gordon equation for

$$\nu_d = \sqrt{\left( \frac{d - 1}{2} \right)^2 - \frac{m^2}{H^2}}. \quad \text{(B.6)}$$

The modes $\{\Phi_{kL}\}$ are Klein-Gordon orthonormal:

$$\left(\Phi_{kL}, \Phi_{k'L'}\right)_{L \cap L' \cap 1} = i \frac{kk'}{2} \left[ \eta \right] \left( H^{(1)}_{\nu_d}(kr) \partial_\eta H^{(1)}_{\nu_d}(kr) - H^{(1)}_{\nu_d}(-kr) \partial_\eta \frac{H^{(1)}_{\nu_d}}{\nu_d}(-kr) \right) \int_0^\infty d\theta r^2 j_{l+\frac{d}{2}-2}(kr) j_{l'+\frac{d}{2}-2}(k'r) \int_{S_d-2} d\Omega_{d-2} Y^*_L(\Omega_{d-2}) Y_{L'}(\Omega_{d-2}). \quad \text{(B.7)}$$

Here the volume element on the constant $\eta$ surface $\Sigma$ is $d\Sigma = (H\eta)^{1-d/2} rd\Omega_{d-2}$ and the future pointing unit vector normal to $\Sigma$ is $\hat{n}_\mu \partial_\mu = H\eta \partial_\eta$. Using the fact that spherical harmonics are $L^2$ orthonormal on $S^{d-2}$ and

$$\int_0^\infty dr r^2 j_n(kr) j_n(k'r) = \frac{\pi}{2k^2} \delta(k - k'), \quad \text{(B.8)}$$

for $n > -1$, we can write

$$\left(\Phi_{kL}, \Phi_{k'L'}\right)_{L \cap L' \cap 1} = i \pi \frac{\eta}{4} \left[ \eta \right] \left( H^{(1)}_{\nu_d}(kr) \partial_\eta \frac{H^{(1)}_{\nu_d}}{\nu_d}(kr) - H^{(1)}_{\nu_d}(-kr) \partial_\eta \frac{H^{(1)}_{\nu_d}}{\nu_d}(-kr) \right) \delta(k - k') \delta_{LL'}. \quad \text{(B.9)}$$
Since the Klein-Gordon inner product is independent of the choice of the spacelike hypersurface, we will evaluate it at the surface $\eta \to -\infty$, where

$$H^{(1)}_{\nu_d}(-k\eta) \to \sqrt{\frac{-2}{\pi k\eta}} e^{-\frac{i(k\eta + \nu_d^2)}{2} + \frac{i}{2}}.$$  

(B.10)

Substituting Eqn. (B.10) in Eqn. (B.9), we see that

$$(\Phi_{kL}, \Phi_{k'L'})_{I\cup III} = \delta(k - k')\delta_{L,L'}.$$  

(B.11)

Similarly we can show that

$$(\Phi_{kL}, \Phi_{k'L'})_{I\cup III} = -\delta(k - k')\delta_{L,L'} \quad \text{and} \quad (\Phi_{kL}, \Phi_{L'L})_{I\cup III} = 0.$$  

(B.12)

As in the case of $d = 4$, we show that in region $I$, we have a Klein-Gordon orthonormal set of modes given by $\Psi_{pl} = \psi_{pl}(t, x)Y_L(\Omega_{d-2})$, where

$$\psi_{pl}(t, x) = \sqrt{2\sinh(\pi p)} N^{(d,\nu_d)}_{pl} U^{(d,\nu_d)}_{pl}(x)e^{-ipt}, \quad p > 0,$$  

(B.13)

with

$$U^{(d,\nu_d)}_{pl} = x^d (1 - x^2)^{-\frac{d-2}{2}} F_1 \left( \frac{d-1}{2} + l - ip + \nu_d, \frac{d-1}{2} + l - ip - \nu_d, l + \frac{d-1}{2}, x^2 \right),$$

$$N^{(d,\nu_d)}_{pl} = \frac{H^{\Delta-1}_{d-2}}{2\sqrt{2\pi} \Gamma \left( l + \frac{d-1}{2} \right)} \Gamma \left( \frac{l + \frac{d-1}{2} - ip - \nu_d}{2} \right) \Gamma \left( \frac{l + \frac{d-1}{2} + ip - \nu_d}{2} \right),$$

(B.14)

where $\Delta = d - 4$, and the $U^{(4,\nu_a)}_{pl}(x)$ and $N^{(4,\nu_a)}_{pl}$ carry the extra label $\nu_a(m) \neq \nu_4(m)$. The Klein-Gordon inner product

$$(\Psi_{pL}, \Psi_{p'L'})_I = 2(p + p') H^{d-2} \sqrt{\sinh(\pi p)\sinh(\pi p')} e^{i(p-p')l} N^{(d,\nu_d)}_{pl} N^{(d,\nu_d)}_{p'l}$$

$$\times \int_0^{1} dx \frac{x^{d-2}}{1 - x^2} U^{(d,\nu_d)}_{pl}(x) U^{(d,\nu_d)}_{p'l}(x) \delta_{LL'},$$

(B.16)

where $d\Sigma = H^{1-d}(1 - x^2)^{-1/2} x^{d-2}$ on the Cauchy hypersurface $\Sigma_t$ and the future pointing unit vector normal to the $\Sigma$ is $\hat{n}^\mu \partial_\mu = H(1 - x^2)^{-1/2}\partial_t$. Using the relations Eqn. [B.14] and [B.15], we see that the $\{\Psi_{pL}\}$ are Klein-Gordon orthogonal as in the $d = 4$ case,

$$(\Psi_{pL}, \Psi_{p'L'})_I = \delta(p - p')\delta_{L,L'}.$$  

(B.17)

We can similarly show that

$$(\Psi_{pL}^*, \Psi_{p'L'}^*)_I = -\delta(p - p')\delta_{L,L'} \quad \text{and} \quad (\Psi_{pL}^*, \Psi_{p'L'}^*)_{KG} = 0.$$  

(B.18)

Using

$$\int_0^{\infty} dk k^{-ip-\frac{1}{2}} \Phi_{kl}(\eta, r) = 2^{-ip} e^{\frac{\pi p}{2}} N^{(d,\nu_d)}_{pl} U^{(d,\nu_d)}_{pl}(x)e^{-ipt},$$

(B.19)

we see that the Bogoliubov transformation between $\{\Phi_{kL}\}$ and $\{\Psi_{pL}\}$ in $I$ are given by Eqn. (4.11) and (4.12) and are the same for all dimensions. This immediately implies that the mode-wise entropy is given by
Eqn. (4.16), with an infinite degeneracy coming from the angular modes. Integrating over $p \in (0, \infty)$ gives us a finite answer as before, but we need to impose an angular cut-off $l_{\text{max}}$ as we did in $d = 4$. The regulated SSEE is then

$$S = \sum_{L, l=0}^{l_{\text{max}}} \frac{\pi}{6} \frac{(2l_{\text{max}} + d - 2)(l_{\text{max}} + d - 3)!}{l_{\text{max}}!(d - 2)!} \approx \frac{\pi}{6} \frac{2}{(d - 2)!} l_{\text{max}}^{d - 2}, \quad l_{\text{max}} >> 1.$$  

(B.20)

As in $d = 4$ using the approximate flatness of the metric at the equator, $d\Omega \simeq (2\pi/l_{\text{max}})^{d - 2} \simeq (l_c H)^{d - 2}$, which means that $S \propto \frac{A}{l_c^{d - 2}}$.

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