Clique minors in double-critical graphs

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Abstract

A connected $t$-chromatic graph $G$ is double-critical if $G - \{u, v\}$ is $(t - 2)$-colorable for each edge $uv \in E(G)$. A long-standing conjecture of Erdős and Lovász that the complete graphs are the only double-critical $t$-chromatic graphs remains open for all $t \geq 6$. Given the difficulty in settling Erdős and Lovász’s conjecture and motivated by the well-known Hadwiger’s conjecture, Kawarabayashi, Pedersen and Toft proposed a weaker conjecture that every double-critical $t$-chromatic graph contains a $K_t$ minor and verified their conjecture for $t \leq 7$. Albar and Gonçalves recently proved that every double-critical 8-chromatic graph contains a $K_8$ minor, and their proof is computer-assisted. In this paper we prove that every double-critical $t$-chromatic graph contains a $K_t$ minor for all $t \leq 9$. Our proof for $t \leq 8$ is shorter and computer-free.

1 Introduction

All graphs in this paper are finite and simple. For a graph $G$ we use $|G|$, $e(G)$, $\delta(G)$ to denote the number of vertices, number of edges and minimum degree of $G$, respectively. The degree of a vertex $v$ in a graph is denoted by $d_G(v)$ or simply $d(v)$. For a subset $S$ of $V(G)$, the subgraph induced by $S$ is denoted by $G[S]$ and $G - S = G[V(G) \setminus S]$. If $G$ is a graph and $K$ is a subgraph of $G$, then by $N(K)$ we denote the set of vertices of $V(G) \setminus V(K)$ that are adjacent to a vertex of $K$. If $V(K) = \{x\}$, then we use $N(x)$ to denote $N(K)$. By abusing notation we will also denote by $N(x)$ the graph induced by the set $N(x)$. We define $N[x] = N(x) \cup \{x\}$, and similarly will use the same symbol for the graph induced by that set. If $u, v$ are distinct nonadjacent vertices of a graph $G$, then by $G + uv$ we denote the graph obtained from $G$ by adding an edge with ends $u$ and $v$. If $u, v$ are adjacent or equal, then we define $G + uv$ to be $G$.

A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges. We write $G \geq H$ if $H$ is a minor of $G$. In those circumstances we also say that $G$ has an $H$ minor. A connected graph $G$ is called double-critical if for any edge $uv \in E(G)$, we have $\chi(G - \{u, v\}) = \chi(G) - 2$. The following long-standing Double-Critical Graph Conjecture is due to Erdős and Lovász.

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Conjecture 1.1 Double-Critical Graph Conjecture (Erdős and Lovász [3]) For every integer $t \geq 1$, the only double-critical $t$-chromatic graph is $K_t$.

Conjecture 1.1 is a special case of the so-called Erdős-Lovász Tihany Conjecture [3]. It is trivially true for $t \leq 3$ and reasonably easy for $t = 4$. Mozhan [8] and Stiebitz [10] independently proved Conjecture 1.1 for $t = 5$.

Theorem 1.2 (Mozhan [8]; Stiebitz [10]) The only double-critical 5-chromatic graph is $K_5$.

Conjecture 1.1 remains open for all $t \geq 6$. Given the difficulty in settling Conjecture 1.1 and motivated by the well-known Hadwiger’s conjecture [4], Kawarabayashi, Pedersen and Toft proposed a weaker conjecture.

Conjecture 1.3 (Kawarabayashi, Pedersen and Toft [6]) For every integer $t \geq 1$, every double-critical $t$-chromatic graph contains a $K_t$ minor.

Conjecture 1.3 is a weaker version of Hadwiger’s conjecture [4], which states that for every integer $t \geq 1$, every $t$-chromatic graph contains a $K_t$ minor. Conjecture 1.3 is true for $t \leq 5$ by Theorem 1.2. In the same paper [6], Kawarabayashi, Pedersen and Toft verified their conjecture for $t \in \{6, 7\}$.

Theorem 1.4 (Kawarabayashi, Pedersen and Toft [6]) For every integer $t \leq 7$, every double-critical $t$-chromatic graph contains a $K_t$ minor.

Recently, Albar and Gonçalves [1] announced a proof for the case $t = 8$.

Theorem 1.5 (Albar and Gonçalves [1]) Every double-critical 8-chromatic graph has a $K_8$ minor.

Our main result is the following next step.

Theorem 1.6 For integers $k, t$ with $1 \leq k \leq 9$ and $t \geq k$, every double-critical $t$-chromatic graph contains a $K_k$ minor.

We actually prove a much stronger result, the following.

Theorem 1.7 For $k \in \{6, 7, 8, 9\}$, let $G$ be a $(k - 3)$-connected graph with $k + 1 \leq \delta(G) \leq 2k - 5$. If every edge of $G$ is contained in at least $k - 2$ triangles and for any minimal separating set $S$ of $G$ and any $x \in S$, $G[S \backslash \{x\}]$ is not a clique, then $G \geq K_k$.

Theorem 1.6 follows directly from Proposition 2.1 (see below) and Theorem 1.7. Our proof of Theorem 1.7 closely follows the proof of the extremal function for $K_9$ minors by Song and Thomas [9] (see Theorem 1.10 below). Note that the proof of Theorem 1.4 for $k = 7$ is about ten pages long and the proof of Theorem 1.5 is computer-assisted. Our proof of Theorem 1.6 is much shorter and computer-free for $k \leq 8$. For $k = 9$, our proof is computer-assisted as it applies a computer-assisted lemma from [9] (see Lemma 1.13 below). Note that a computer-assisted proof of
Theorem 1.7 for all \( k \leq 8 \) (and hence computer-assisted proofs of Theorem 1.4 and Theorem 1.5) follows directly from Theorem 1.7 for \( k = 9 \). (To see that, let \( G \) and \( k \leq 8 \) be as in Theorem 1.7 and let \( H \) be obtained from \( G \) by adding \( 9 - k \) vertices, each adjacent to every other vertex of the graph. Then \( H \) is 6-connected and satisfies all the other conditions as stated in Theorem 1.7. Thus \( H \geq K_9 \) and so \( G \geq K_k \).) Conjecture 1.3 remains open for all \( t \geq 10 \). It seems hard to generalize Theorem 1.6.

We need some known results to prove our main results. Before doing so, we need to define \((H, k)\)-cockade. For a graph \( H \) and an integer \( k \), let us define an \((H, k)\)-cockade recursively as follows. Any graph isomorphic to \( H \) is an \((H, k)\)-cockade. Now let \( G_1, G_2 \) be \((H, k)\)-cockades and let \( G \) be obtained from the disjoint union of \( G_1 \) and \( G_2 \) by identifying a clique of size \( k \) in \( G_1 \) with a clique of the same size in \( G_2 \). Then the graph \( G \) is also an \((H, k)\)-cockade, and every \((H, k)\)-cockade can be constructed this way. We are now ready to state some known results. The following theorem is a result of Dirac [2] for \( p \leq 5 \) and Mader [7] for \( p \in \{6, 7\} \).

**Theorem 1.8** (Dirac [2]; Mader [7]) For every integer \( p \in \{1, 2, \ldots, 7\} \), a graph on \( n \geq p \) vertices and at least \((p - 2)n - \binom{p-1}{2} + 1\) edges has a \( K_p \) minor.

Jørgensen [5] and later Song and Thomas [9] generalized Theorem 1.8 to \( p = 8 \) and \( p = 9 \), respectively, as follows.

**Theorem 1.9** (Jørgensen [5]) Every graph on \( n \geq 8 \) vertices with at least \( 6n - 20 \) edges either contains a \( K_8 \)-minor or is isomorphic to a \((K_2,2,2,2,2)\)-cockade.

**Theorem 1.10** (Song and Thomas [9]) Every graph on \( n \geq 9 \) vertices with at least \( 7n - 27 \) edges either contains a \( K_9 \)-minor, or is isomorphic to \( K_{2,2,2,3,3} \), or is isomorphic to a \((K_1,2,2,2,2,2,6)\)-cockade.

In our proof of Theorem 1.7, we need to examine graphs \( G \) such that \( k + 1 \leq |G| \leq 2k - 5 \), \( \delta(G) \geq k - 2 \) and \( G \not\geq K_k \cup K_1 \). We shall use the following results. Lemma 1.11 is a result of Jørgensen [5].

**Lemma 1.11** (Jørgensen [5]) Let \( G \) be a graph with \( n \leq 11 \) vertices and \( \delta(G) \geq 6 \) such that for every vertex \( x \) in \( G \), \( G - x \) is not contractible to \( K_6 \). Then \( G \) is one of the graphs \( K_{2,2,2,2,2}, K_{3,3,3} \) or the complement of the Petersen graph.

Lemma 1.11 implies Lemma 1.12 below. To see that, let \( G \) be a graph satisfying the conditions given in Lemma 1.12. By applying Lemma 1.11 to the graph obtained from \( G \) by adding \( 6 - t \) vertices, each adjacent to every other vertex of the graph, we see that \( G \geq K_t \cup K_1 \).

**Lemma 1.12** For \( t \in \{1, 2, 3, 4, 5\} \), let \( G \) be a graph with \( n \leq 2t - 1 \) vertices and \( \delta(G) \geq t \). Then \( G \geq K_t \cup K_1 \).
Lemma 1.13 is a result of Song and Thomas [9]. Note that the proof of Lemma 1.13 is computer-assisted.

**Lemma 1.13** (Song and Thomas [9]) Let $G$ be a graph with $|G| \in \{9, 10, 11, 12, 13\}$ such that $\delta(G) \geq 7$. Then either $G \geq K_7 \cup K_1$, or $G$ satisfies the following

(A) either $G$ is isomorphic to $K_{1,2,2,2,2}$, or $G$ has four distinct vertices $a_1, b_1, a_2, b_2$ such that $a_1a_2, b_1b_2 \notin E(G)$ and for $i = 1, 2$ the vertex $a_i$ is adjacent to $b_i$, the vertices $a_i, b_i$ have at most four common neighbors, and $G + a_1a_2 + b_1b_2 \geq K_8$,

(B) for any two sets $A, B \subseteq V(G)$ of cardinality at least five such that neither is complete and $A \cup B$

includes all vertices of $G$ of degree at most $|G| - 2$, either

(B1) there exist $a \in A$ and $b \in B$ such that $G' \geq K_8$, where $G'$ is obtained from $G$ by adding all edges $aa'$ and $bb'$ for $a' \in A - \{a\}$ and $b' \in B - \{b\}$, or

(B2) there exist $a \in A - B$ and $b \in B - A$ such that $ab \in E(G)$ and the vertices $a$ and $b$ have at most five common neighbors in $G$, or

(B3) one of $A$ and $B$ contains the other and $G + ab \geq K_7 \cup K_1$ for all distinct nonadjacent vertices $a, b \in A \cap B$.

**2 Basic properties of non-complete double-critical graphs**

We begin with basic properties of non-complete double-critical $k$-chromatic graphs established in [6]. We only list those that will be used in our proofs.

**Proposition 2.1** (Kawarabayashi, Pedersen and Toft [6]) If $G$ is a non-complete double-critical $k$-chromatic graph, then the following hold:

(a) $\delta(G) \geq k + 1$.

(b) Every edge $xy \in E(G)$ belongs to at least $k - 2$ triangles.

(c) $G$ is 6-connected and no minimal separating set of $G$ can be partitioned into two sets $A$ and $B$ such that $G[A]$ and $G[B]$ are edge-empty and complete, respectively.

Two proper vertex-colorings $c_1$ and $c_2$ of a graph $G$ are equivalent if, for all $x, y \in V(G)$, $c_1(x) = c_1(y)$ iff $c_2(x) = c_2(y)$. Two vertex-colorings $c_1$ and $c_2$ of a graph $G$ are equivalent on a set $A \subseteq V(G)$ if the restrictions $c_1|_A$ and $c_2|_A$ to $A$ are equivalent on the subgraph $G[A]$. Let $S$ be a separating set of $G$, and let $G_1, G_2$ be connected subgraphs of $G$ such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = G[S]$. If $c_1$ is a $k$-coloring of $G_1$ and $c_2$ is a $k$-coloring of $G_2$ such that $c_1$ and $c_2$ are equivalent on $S$, then it is clear that $c_1$ and $c_2$ can be combined to a $k$-coloring of $G$ by a suitable permutation of the color classes of, say $c_1$. The main technique in the proof of Proposition 2.1 involves reassigning and permuting the colors on a separating set $S$ of a non-complete double-critical $k$-chromatic graph $G$ so that $c_1$ and $c_2$ are equivalent on $S$ to obtain a contradiction, where
c₁ is a \((k - 1)\)-coloring of \(G₁\) and \(c₂\) is a \((k - 1)\)-coloring of \(G₂\). It seems hard to use this idea to prove that every non-complete double-critical \(k\)-chromatic graph is \(7\)-connected, but we can use it to say a bit more about minimal separating sets of size \(6\) in non-complete double-critical graphs.

**Lemma 2.2** Suppose \(G\) is a non-complete double-critical \(k\)-chromatic graph. If \(S\) is a minimal separating set of \(G\) with \(|S| = 6\), then either \(G[S] \subseteq K₃,₃\) or \(G[S] \subseteq K₂,₂,₂\).

**Proof.** By Proposition 2.1(c), \(G\) is \(6\)-connected. Let \(S = \{v₁, \ldots, v₆\} \subseteq V(G)\) be a minimal separating set of \(G\) such that neither \(G[S] \subseteq K₃,₃\) nor \(G[S] \subseteq K₂,₂,₂\). Let \(G₁\) and \(G₂\) be subgraphs of \(G\) such that \(G₁ \cup G₂ = G\), \(G₁ \cap G₂ = S\), and there are no edges from \(G₁ - S\) to \(G₂ - S\). Since \(k \geq 6\) by Theorem 1.2, we have \(\delta(G) \geq 7\) by Proposition 2.1(a). In particular, since \(|S| = 6\), there must exist at least one edge \(yᵢzᵢ\) in \(G₁ - S\) for \(i \in \{1, 2\}\). It follows then that \(G₁\) is \((k - 2)\)-colorable since it is a subgraph of \(G - \{y₃₋₁, z₃₋₁\}\). Let \(c₁, c₂\) be \((k - 2)\)-colorings of \(G₁\) and \(G₂\), respectively. For \(i = 1, 2\), define \(|c_i(A)|\) to be the number of distinct colors assigned to the vertices of \(A\) by \(c_i\) for any \(A \subseteq S\). Clearly \(c₁\) and \(c₂\) are not equivalent on \(S\), otherwise \(c₁\) and \(c₂\), after a suitable permutation of the colors of \(c₂\), can be combined to a \((k - 2)\)-coloring of \(G\), a contradiction. By Proposition 2.1(c), \(α(G[S]) \leq 4\) and so neither \(c₁\) nor \(c₂\) applies the same color to more than four vertices of \(S\). Utilizing a new color, say \(β\), we next redefine the colorings \(c₁\) and \(c₂\) so that \(c₁\) and \(c₂\) are \((k - 1)\)-colorings of \(G₁\) and \(G₂\), respectively, and are equivalent on \(S\). This yields a contradiction, as \(c₁\) and \(c₂\), after a suitable permutation of the colors of \(c₂\), can be combined to a \((k - 1)\)-coloring of \(G\).

Suppose that one of the colorings \(c₁\) and \(c₂\), say \(c₁\), assigns the same color to four vertices of \(S\), say \(c₁(v₃) = c₁(v₄) = c₁(v₅) = c₁(v₆)\). Then \(\{v₃, v₄, v₅, v₆\}\) is an independent set in \(G\). By Proposition 2.1(c), we must have \(v₁v₂ \notin E(G)\). But then \(G[S] \subseteq K₂,₂,₂\), a contradiction. Thus neither \(c₁\) nor \(c₂\) assigns the same color to four distinct vertices of \(S\).

Next suppose that one of the colorings \(c₁\) and \(c₂\), say \(c₁\), assigns the same color to three vertices of \(S\), say \(c₁(v₄) = c₁(v₅) = c₁(v₆)\). Then \(\{v₄, v₅, v₆\}\) is an independent set in \(G\). Since \(G[S] \not\subseteq K₃,₃\), we have \(|c₂(\{v₁, v₂, v₃\})| \geq 2\). If \(|c₂(\{v₁, v₂, v₃\})| = 2\), we may assume that \(c₂(v₂) = c₂(v₃)\). Then \(\{v₂, v₃\}\) is an independent set. Then redefining \(c₂(v₄) = c₂(v₅) = c₂(v₆) = β\) and \(c₁(v₂) = c₁(v₃) = β\) will make \(c₁\) and \(c₂\) equivalent on \(S\), a contradiction. Thus \(|c₂(\{v₁, v₂, v₃\})| = 3\) and so \(c₂\) assigns distinct colors to each of \(v₁, v₂, v₃\). We redefine \(c₂(v₄) = c₂(v₅) = c₂(v₆) = β\). Clearly \(c₁\) and \(c₂\) are equivalent on \(S\) if \(c₁\) assigns distinct colors to each of \(v₁, v₂, v₃\). Thus \(|c₁(\{v₁, v₂, v₃\})| \leq 2\). Since \(G[S] \not\subseteq K₃,₃\), we have \(|c₁(\{v₁, v₂, v₃\})| = 2\). We may assume that \(c₁(v₂) = c₁(v₃)\). Now redefining \(c₁(v₃) = β\) yields that \(c₁\) and \(c₂\) are equivalent on \(S\). This proves that neither \(c₁\) nor \(c₂\) assigns the same color to three distinct vertices of \(S\). Thus \(6 \geq |c₁(S)| \geq 3\) \((i = 1, 2)\). Since \(G[S] \not\subseteq K₂,₂,₂\), we have \(|c₁(S)| \geq 4\) \((i = 1, 2)\). We may assume that \(|c₁(S)| \geq |c₂(S)|\). Then \(|c₂(S)| \leq 5\), for otherwise \(c₁\) and \(c₂\) are equivalent on \(S\). Thus \(5 \geq |c₂(S)| \geq 4\).
Suppose that $|c_2(S)| = 5$. Then $|c_1(S)| = 5$ or $|c_1(S)| = 6$. We can make $c_1$ and $c_2$ equivalent on $S$ by assigning color $\beta$ to one of the two vertices that are colored the same color by $c_1$ (if $|c_1(S)| = 5$) and $c_2$. Thus $|c_2(S)| = 4$. Since neither $c_1$ nor $c_2$ assigns the same color to more than two distinct vertices of $S$, we may assume that $c_2(v_3) = c_2(v_4)$ and $c_2(v_5) = c_2(v_6)$. Then $v_3v_4 \notin E(G)$ and $v_5v_6 \notin E(G)$. Since $G[S] \not\subseteq K_{2,2}$, we have $v_1v_2 \in E(G)$. Thus $c_1(v_1) \neq c_1(v_2)$. We may assume that $c_1(v_3) \neq c_1(v_4)$ as $c_1$ and $c_2$ are not equivalent on $S$. If $|c_1(S)| = 6$, then redefining $c_1(v_3) = c_1(v_6) = \beta$ and $c_2(v_3) = \beta$ will make $c_1$ and $c_2$ equivalent. If $|c_1(S)| = 5$, then at least one of $v_3, v_4, v_5, v_6$ shares a color with another vertex of $S$, say $c_1(v_6) = c_1(v_i)$ for some $i \in \{1, \ldots, 5\}$. Then redefining $c_1(v_5) = c_1(v_6) = \beta$ and $c_2(v_3) = \beta$ will again make $c_1$ and $c_2$ equivalent. Thus $|c_1(S)| = 4$. Suppose that one of $v_1$ or $v_2$ shares a color with another vertex of $S$. Since $v_1v_2 \in E(G)$, we may assume by symmetry that $c_1(v_1) = c_1(v_3)$. If $c_1(v_5)$ and $c_1(v_6)$ are the two colors each assigned to only a single vertex of $S$ by $c_1$, then we also have $c_1(v_2) = c_2(v_4)$. Now redefining $c_1(v_3) = c_1(v_4) = \beta$ and $c_2(v_5) = \beta$ will make $c_1$ and $c_2$ equivalent. Hence one of the colors $c_1(v_5)$ and $c_1(v_6)$ is assigned to two vertices of $S$, say $c_1(v_6) = c_1(v_i)$ for some $i \in \{2, 4, 5\}$. If $i = 2$ then redefine $c_1(v_5) = c_1(v_6) = \beta$ and $c_2(v_1) = c_2(v_3) = \beta$, if $i = 4$ then redefine $c_1(v_3) = c_1(v_4) = \beta$ and $c_2(v_5) = \beta$, and if $i = 5$ then redefine $c_1(v_3) = \beta$ and $c_2(v_3) = \beta$, and in each case $c_1$ is equivalent to $c_2$. Therefore $c_1(v_1)$ and $c_1(v_2)$ are the two colors assigned to only a single vertex of $S$ by $c_1$. Since $c_1$ and $c_2$ are not equivalent, we must have, say $c_1(v_3) = c_1(v_5)$ and $c_1(v_4) = c_1(v_6)$. Now redefining $c_1(v_5) = c_1(v_6) = \beta$ and $c_2(v_3) = \beta$ will make $c_1$ and $c_2$ equivalent.

3 Proofs of Theorem [1.7] and Theorem [1.6]

In this section we first prove Theorem [1.7].

Proof. Let $G$ be a graph as in the statement with $n$ vertices. By assumption, we have

1. $k + 1 \leq \delta(G) \leq 2k - 5$ and $\delta(N(x)) \geq k - 2$ for any $x$ in $G$; and

2. $G$ is $(k - 3)$-connected and for any minimal separating set $S$ of $G$ and any $x \in S$, $G[S \setminus \{x\}]$ is not a complete subgraph.

We first show that the statement is true for $k = 6$. Then $G$ is 3-connected with $\delta(G) = 7$. The statement is trivially true if $G$ is complete, so we may assume $G$ is not complete. Let $x \in V(G)$ be a vertex of degree 7. By (1), $\delta(N(x)) \geq 4$, and so $e(N(x)) \geq 14$. If $e(N(x)) \geq 16$, then by Theorem [1.8] $N(x) \geq K_5$ and so $G \geq N[x] \geq K_6$. If $e(N(x)) = 15$, then let $K$ be a component of $G - N[x]$ with $|N(K)|$ minimum. By (2), $|N(K)| \geq 3$ and $N(K)$ is not complete. Let $y, z \in N(K)$ be non-adjacent in $N(x)$ and let $P$ be a $(y, z)$-path with interior vertices in $K$. We see that $G \geq K_6$ by contracting all but one of the edges of $P$. So we may assume that $e(N(x)) = 14$, and so $N(x)$ is 4-regular and $\overline{N(x)}$ is 2-regular. Thus $\overline{N(x)}$ is then either isomorphic to $C_7$ or to $C_4 \cup C_3$, and
in both cases it is easy to see that $N(x) \geq K_5$ and thus $G \geq K_6$, as desired. Hence we may assume $7 \leq k \leq 9$.

Suppose for a contradiction that $G \not\geq K_k$. We next prove the following.

(3) Let $x \in V(G)$ be such that $k + 1 \leq d(x) \leq 2k - 5$. Then there is no component $K$ of $G - N[x]$ such that $N(K') \cap M \subseteq N(K)$ for every component $K'$ of $G - N[x]$, where $M$ is the set of vertices of $N(x)$ not adjacent to all other vertices of $N(x)$.

**Proof.** Suppose such a component $K$ exists. Among all vertices $x$ with $k + 1 \leq d(x) \leq 2k - 5$ for which such a component exists, choose $x$ to be of minimal degree, and among all such components $K$ of $G - N[x]$, choose $K$ such that $|N(K)|$ is minimum. We first prove that $M \subseteq N(K)$. Suppose for a contradiction that $M - N(K) \neq \emptyset$, and let $y \in M \setminus N(K)$ be such that $d(y)$ is minimum. Clearly, $d(y) < d(x)$. Let $J$ be the component of $G - N[y]$ containing $K$. Since $d(y) < d(x)$ the choice of $x$ implies that $N(x) \setminus N[y] \subseteq V(J)$. Let $H = N(x) \setminus (N[y] \cup N(K))$. We have $d_G(z) \geq d_G(y)$ for all $z \in V(H)$ by the choice of $y$. Let $t = |V(H)|$. Then $t \geq 2$, for otherwise the vertex $y$ and component $H$ contradict the choice of $x$. On the other hand $t \leq d(x) - d(y) \leq (2k - 5) - (k + 1) = k - 6 \leq 3$ and so $k \geq 8$. Notice that $t = 2$ when $k = 8$. From (1) applied to $y$ we deduce that $N(y) \cap N(x)$ has minimum degree at least $k - 3$. Let $L$ be the subgraph of $G$ induced by $(N[y] \cap N(x)) \cup V(H)$. Then the edge-set of $L$ consists of edges of $N(x) \cap N(y)$, edges incident with $y$, and edges incident with $V(H)$. Clearly, $e(L - V(H), H) = \sum_{z \in V(H)} (d(z) - 1) - 2e(H) \geq t(d(y) - 1) - 2e(H)$. Thus

\[
e(L) \geq \frac{(k - 3)(d(y) - 1)}{2} + d(y) - 1 + e(L - V(H), H) + e(H) \\
\geq \frac{(k - 3)(d(y) - 1)}{2} + d(y) - 1 + t(d(y) - 1) - e(H) \\
\geq \frac{(k - 3)(d(y) - 1)}{2} + d(y) - 1 + t(d(y) - 1) - \frac{1}{2}t(t - 1) \\
\geq \left\{ \begin{array}{ll} 
5(d(y) + 2) + \frac{d(y)}{2} - \frac{33}{2} & \quad \text{if } k = 8 \\
6(d(y) + t) + (t - 2)d(y) - 4 - 7t - \frac{1}{2}t(t - 1) & \quad \text{if } k = 9 \\
(k - 3)|V(L)| - \left( \frac{k - 2}{2} \right) + 1, & \quad \text{if } k = 8 
\end{array} \right.
\]

because $d(y) \geq k + 1$ and $2 \leq t \leq k - 6$. If $k = 9$, since $12 \leq |V(L)| \leq 13$ the graph $L$ is not a $(K_{2,2,2,2,2,5})$-cockade. By Theorem 1.8 and Theorem 1.9 $N(x) \geq L \geq K_{k-1}$. Thus $G \geq N[x] \geq K_k$, a contradiction. This proves that $M \subseteq N(K)$.

If $N(x) \geq K_{k-2} \cup K_1$, then $N(x)$ has a vertex $y$ such that $N(x) - y \geq K_{k-2}$. If $y \notin M$, then $N(x) \geq K_{k-1}$. Otherwise, by contracting the connected set $V(K) \cup \{y\}$ we can contract $K_{k-1}$ onto $N(x)$. Thus in either case $G \geq K_{k}$, a contradiction. Thus $N(x) \not\geq K_{k-2} \cup K_1$. If $k \leq 8$, by Lemma 1.11 and Lemma 1.12 we have $k = 8$ and $N(x)$ is either $K_{3,3,3}$ or $\overline{P}$, where $\overline{P}$ is the complement of the Petersen graph. If $N(x) = \overline{P}$, it can be easily checked that $\overline{P} + yz \geq K_7$ for any $yz \in E(P)$. By (2), $|N(K)| \geq 5$ and $N(K)$ is not complete. Let $y, z \in N(K)$ be non-adjacent in $N(x)$ and let $Q$ be a $(y, z)$-path with interior vertices in $K$. We see that $G \geq K_8$ by contracting
all but one of the edges of \( Q \), a contradiction. Thus \( N(x) = K_{3,3,3} \), and so \( M = N(x) \). Let \( \{a_1, a_2, a_3\} \) and \( \{b_1, b_2, b_3\} \) be the vertex sets of two disjoint triangles of \( \overline{N(x)} \). Suppose \( G - N[x] \) is 2-connected or has at most two vertices. By Proposition 2.1, the vertices \( a_i, b_i \) (\( i=1,2 \)) have at least two common neighbors in \( G - N[x] \). Let \( u_1, u_2 \) (resp. \( w_1, w_2 \)) be two distinct common neighbors of \( a_1 \) and \( b_1 \) (resp. \( a_2 \) and \( b_2 \)) in \( G - N[x] \). By Menger's Theorem, \( G - N[x] \) contains two disjoint paths from \( \{u_1, u_2\} \) to \( \{w_1, w_2\} \) and so \( G \geq N[x] + a_1a_2 + b_1b_2 \geq K_8 \), a contradiction. Thus \( G - N[x] \) has at least three vertices and is not 2-connected. If \( G - N[x] \) is disconnected, let \( H_1 = K \) and \( H_2 \) be another connected component of \( G - N[x] \). If \( G - N[x] \) has a cut-vertex, say \( w \), let \( H_1 \) be a connected component of \( G - N[x] - w \) and let \( H_2 = G - N[x] - V(H_1) \). In either case, \( H_1 \) and \( H_2 \) are disjoint connected subgraphs of \( G - N[x] \) such that \( M \subseteq N(H_1) \cup N(H_2) \) (because we have shown that \( M \subseteq N(K) \)). Thus \( N(H_1) \cup N(H_2) = N(x) \) because \( M = N(x) \). By (2), \( N(H_1) \) is not complete and \( |N(H_i)| \geq 4 \) since \( k = 8 \). Thus each of \( N(H_1) \) and \( N(H_2) \) must contain at least one edge of \( \overline{N(x)} \). Since \( N(x) = K_{3,3,3} \) and \( N(H_1) \cup N(H_2) = N(x) \), we may thus assume that \( a_1a_2 \in N(H_1) \) and \( b_1b_2 \in N(H_2) \). By contracting \( H_1 \) onto \( a_1 \) and \( H_2 \) onto \( b_1 \) we see that \( G \geq N[x] + a_1a_2 + b_1b_2 \geq K_8 \), a contradiction. This proves that \( k = 9 \) and so by Lemma 1.13 we may assume that \( N(x) \) satisfies properties (A) and (B).

Since \( d(x) \geq 10 \), \( N(x) \neq K_{1,2,2,2,2} \). If \( G - N[x] \) is 2-connected or has at most two vertices, then by property (A) and (2), the set \( N(x) \) has four distinct vertices \( a_1, b_1, a_2, b_2 \) such that \( a_1a_2, b_1b_2 \notin E(G) \), \( N(x) + a_1a_2 + b_1b_2 \geq K_8 \) and for \( i = 1, 2 \) the vertex \( a_i \) is adjacent to \( b_i \) and the vertices \( a_i, b_i \) have at least two common neighbors in \( G - N[x] \). Let \( u_1, u_2 \) (resp. \( w_1, w_2 \)) be two distinct common neighbors of \( a_1 \) and \( b_1 \) (resp. \( a_2 \) and \( b_2 \)) in \( G - N[x] \). By Menger’s Theorem, \( G - N[x] \) contains two disjoint paths from \( \{u_1, u_2\} \) to \( \{w_1, w_2\} \) and so \( G \geq N[x] + a_1a_2 + b_1b_2 \geq K_9 \), a contradiction. Thus \( G - N[x] \) has at least three vertices and is not 2-connected. If \( G - N[x] \) is disconnected, let \( H_1 = K \) and \( H_2 \) be another connected component of \( G - N[x] \). If \( G - N[x] \) has a cut-vertex, say \( w \), let \( H_1 \) be a connected component of \( G - N[x] - w \) and let \( H_2 = G - N[x] - V(H_1) \). In either case, \( H_1 \) and \( H_2 \) are disjoint connected subgraphs of \( G - N[x] \) such that \( M \subseteq N(H_1) \cup N(H_2) \) (because we have shown that \( M \subseteq N(K) \)). For \( i = 1, 2 \) let \( A_i = N(H_i) \cap N(x) \). By (2), \( A_i \) is not complete and \( |A_i| \geq 5 \) for \( i = 1, 2 \). By property (B), \( A_1 \) and \( A_2 \) satisfy properties (B1), (B2) or (B3).

Suppose first that \( A_1 \) and \( A_2 \) satisfy property (B1). Then there exist \( a_i \in A_i \) such that \( N(x) + \{a_1a : a \in A_1 \setminus \{a_1\}\} + \{a_2a : a \in A_2 \setminus \{a_2\}\} \geq K_8 \). By contracting the connected sets \( V(H_1) \cup \{a_1\} \) and \( V(H_2) \cup \{a_2\} \) to single vertices, we see that \( G \geq K_9 \), a contradiction. Suppose next that \( A_1 \) and \( A_2 \) satisfy property (B2). Then there exist \( a_1 \in A_1 \setminus A_2 \) and \( a_2 \in A_2 \setminus A_1 \) such that \( a_1a_2 \in E(G) \) and the vertices \( a_1 \) and \( a_2 \) have at most five common neighbors in \( N(x) \). Thus \( a_1, a_2 \in M \) by (1), and by another application of (1) there exists a common neighbor \( u \in V(G) \setminus N[x] \) of \( a_1 \) and \( a_2 \). But \( a_1 \notin A_2 \) and \( a_2 \notin A_1 \), and hence \( u \notin V(H_1) \cup V(H_2) \). Thus \( G - N[x] \) is disconnected and \( H_1 = K \). But then \( a_2 \in M \subseteq N(K) = N(H_1) \), a contradiction. Thus we may assume that \( A_1 \) and \( A_2 \) satisfy (B3), and hence \( A_i \subseteq A_{3-i} \) for some \( i \in \{1, 2\} \). As \( M \subseteq A_1 \cup A_2 \), we have \( M \subseteq N(H_{3-i}) \). Since \( A_i \) is not complete, let \( a, b \in A_i \) be distinct and not adjacent. By property
(B3), \( N(x) + ab \geq K_7 \cup K_1 \). Let \( P \) be an \((a,b)\)-path with interior in \( H_i \). By contracting all but one of the edges of the path \( P \) and by contracting \( H_{3-i} \) similarly as above, we see that \( G \geq K_9 \), a contradiction.

(4) \( G - N[x] \) is disconnected for every vertex \( x \in V(G) \) of degree at most \( 2k - 5 \).

**Proof.** If \( G - N[x] \) is not null, then it is disconnected by (3). Thus we may assume that \( x \) is adjacent to every other vertex of \( G \). Let \( H = G - x \). Then \( |H| = d(x) \) and \( \delta(H) \geq k \). Thus \( e(H) \geq k \frac{d(x)}{2} > (k - 3) d(x) - (\frac{k-2}{2}) + 1 \) because \( d(x) \leq 2k - 5 \). By Theorem 1.8 and Theorem 1.9, \( G - x \) has a \( K_{k-1} \) minor and so the graph \( G \) has a \( K_k \) minor, a contradiction.

(5) Let \( x \in V(G) \) be such that \( k + 1 \leq d(x) \leq 2k - 5 \). Then there is no component \( K \) of \( G - N[x] \) such that \( d_G(y) \geq 2k - 4 \) for every vertex \( y \in V(K) \).

**Proof.** Assume that such a component \( K \) exists. Let \( G_1 = G - V(K) \) and \( G_2 = G[V(K) \cup N(K)] \). Let \( d_1 \) be the maximum number of edges that can be added to \( G_2 \) by contracting edges of \( G \) with at least one end in \( G_1 \). More precisely, let \( d_1 \) be the largest integer so that \( G_1 \) contains disjoint sets of vertices \( V_1, V_2, \ldots, V_p \) so that \( G_1[V_j] \) is connected, \( |N(K) \cap V_j| = 1 \) for \( 1 \leq j \leq p = |N(K)| \), and so that the graph obtained from \( G_1 \) by contracting \( V_1, V_2, \ldots, V_p \) and deleting \( V(G) \setminus (\bigcup_j V_j) \) has \( e(N(K)) + d_1 \) edges. Let \( G_2' \) be a graph with \( V(G_2') = V(G_2) \) and \( e(G_2') = e(G_2) + d_1 \) edges obtained from \( G \) by contracting edges in \( G_1 \). By (1), \( |G_2'| \geq k + 2 \). If \( e(G_2') \geq (k-2)|G_2'| - (\frac{k-1}{2}) + 2 \), then by Theorem 1.8 and Theorem 1.9, \( G \geq G_2' \geq K_k \), a contradiction. Thus

\[
e(G_2) = e(G_2') - d_1 \leq (k-2)|G_2| - \left(\frac{k-1}{2}\right) + 1 - d_1 = (k-2)|N(K)| + (k-2)|K| - \left(\frac{k-1}{2}\right) + 1 - d_1.
\]

By contracting the edge \( xz \), where \( z \in N(K) \) has minimum degree \( d \) in \( N(K) \), we see that \( d_1 \geq |N(K)| - d - 1 \) and hence

\[
e(G_2) \leq (k-3)|N(K)| + (k-2)|K| - \left(\frac{k-1}{2}\right) + 2 + d. \quad (a)
\]

Let \( t = e_G(N(K), K) \). We have \( e(G_2) = e(K) + t + e(N(K)) \) and

\[2e(K) \geq (2k - 4)|K| - t, \quad (b)\]

and hence

\[
e(G_2) \geq (k-2)|K| + t/2 + d|N(K)|/2. \quad (c)
\]

Since \( N(x) \) has minimum degree at least \( k - 2 \), it follows that the subgraph \( N(K) \) of \( N(x) \) has minimum degree at least \((k-2) - (d(x) - |N(K)|)) \). Thus \( d \geq (k-2) - (d(x) - |N(K)|) \geq |N(K)| - k + 3 \). From (a) and (c) we get

\[-t/2 \geq -(k-3)|N(K)| + d(|N(K)| - 2)/2 + \left(\frac{k-1}{2}\right) - 2 \geq \begin{cases} -8 & \text{if } k = 7 \\ -14 & \text{if } k = 8 \\ -18 & \text{if } k = 9 \end{cases} \quad (d)\]

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where the second inequality becomes \( \frac{t}{2} \leq 11 \) when \( |N(K)| = 2k - 6 \) and \( k = 7, 8 \), and the second inequality holds with equality only when \( |N(K)| = 10 \) and \( k = 9 \). Since \( G \) is not contractible to \( K_k \), we deduce from (b) and Theorem 1.8 Theorem 1.9 and Theorem 1.10 that \( |K| < 8 \). The inequalities \( e(K) \geq 5|K| - 8 \) when \( k = 7 \), \( e(K) \geq 6|K| - 14 \) when \( k = 8 \), and \( e(K) \geq 7|K| - 18 \) when \( k = 9 \) imply \( |K| \leq 3 \). But every vertex of \( K \) has degree at least \( 2k - 4 \) and \( N(K) \) is a proper subgraph of \( N(x) \), and hence \( |K| = 3 \), \( |N(K)| = 2k - 6 \) and \( \frac{t}{2} = 3(k - 3) \geq 12 \) when \( k = 7, 8 \), and (d) holds with equality for \( |N(K)| = 12 \) when \( k = 9 \), contrary to our earlier observation of (d) that \( \frac{t}{2} \leq 11 \) when \( |N(K)| = 2k - 6 \) and \( k = 7, 8 \), and (d) holds with equality only when \( |N(K)| = 10 \) and \( k = 9 \).

By (1) there is a vertex \( x \) of degree \( k + 1, k + 2, \ldots, \) or \( 2k - 5 \) in \( G \). Choose such a vertex \( x \) so that \( G - N[x] \) has a component \( K \) of minimum order. Then choose a vertex \( y \in V(K) \) of least degree in \( G \). Thus \( k + 1 \leq d_G(y) \leq 2k - 5 \) by (1) and (5). Let \( L \) be the component of \( G - N[y] \) containing \( x \). We claim that \( N(L) \) contains all vertices of \( N(y) \) that are not adjacent to all other vertices of \( N(y) \). Indeed, let \( z \in N(y) \) be not adjacent to some vertex of \( N(y) \setminus \{z\} \). We may assume that \( z \notin N(x) \), for otherwise \( z \in N(L) \). Thus \( z \in V(K) \), and hence \( d_G(z) \geq d_G(y) \) by the choice of \( y \). Thus \( z \) has a neighbor \( z' \in N[x] \cup V(K) \setminus N[y] \). Then \( z' \in V(L) \), for otherwise the component of \( G - N[y] \) containing \( z' \) would be a proper subgraph of \( K \). Thus \( z \in N(L) \). This proves our claim that \( N(L) \) contains all vertices \( z \) as above, contrary to (3). This contradiction completes the proof of Theorem 1.7.

We are now ready to prove Theorem 1.6.

**Proof.** Let \( G \) be a double-critical \( t \)-chromatic graph with \( t \geq k \). The assertion is trivially true if \( G \) is complete. By Theorem 1.2 we may assume that \( t \geq 6 \). By Proposition 2.1(a), \( \delta(G) \geq k + 1 \). By Theorem 1.8 Theorem 1.9 and Theorem 1.10 we have \( \delta(G) \leq 2k - 5 \). By Proposition 2.1(b), every edge of \( G \) is contained in at least \( k - 2 \) triangles. By Proposition 2.1(c), \( G \) is 6-connected and no minimal separating set of \( G \) can be partitioned into a clique and an independent set. By Theorem 1.7 \( G \geq K_k \), as desired.

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