STRONGLY MINIMAL STEINER SYSTEMS III: PATH GRAPHS
AND SPARSE CONFIGURATIONS

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Abstract. We introduce a uniform method of proof for the following results. For each of the following conditions, there are \(2^{\aleph_0}\) families of Steiner systems, satisfying that condition: i) Theorem 2.2.4 (extending [CGGW10]) each Steiner triple system is \(\aleph_0\)-sparse and has a uniform but not perfect path graph; ii) (Theorem 5.4.2 (extending [CW12]) each Steiner \(k\)-system (for \(k = p^n\)) is 2-transitive and has a uniform path graph (infinite cycles only); iii) Theorem 2.1.5 (extending [Fuj06a], each is anti-Pasch (anti-mitre); iv) Theorem 3.6 has an explicit quasi-group structure. In each case all members of the family satisfy the same complete strongly minimal theory and it has \(\aleph_0\) countable models and one model of each uncountable cardinal.

We extend various properties of finite Steiner systems to infinite systems. We take from model theory the practice of constructing families of structures (models of a particular theory) satisfying the goal property and take properties of interest from both model theory and combinatorics. We are able to both obtain examples of known phenomena of infinite Steiner systems that satisfy additional very strong model theoretic constraints and adapt certain concepts from Steiner triple systems to Steiner \(q\)-systems for prime power \(q\).

We extend recent work by Barbina and Casanovas in model theory [BC19] and by Horsley and Webb in combinatorics [HW21] that centers on the constructions of Steiner triple systems (Section 1.1) by giving some applications of the Hrushovski construction of strongly minimal sets. Our subject differs from the Fraïssé case because the finite structures must be ‘strong’ substructures of the generic (alias, limit) model and we can vary the meaning of strong as discussed in Notation 1.0.1 and Section 1.1. In contrast to the model theoretically complex locally finite generics that arise from Fraïssé construction [BC19], the construction techniques here give theories that are strongly minimal, the geometric building blocks of model theoretically tame structures.

More significantly from a combinatorial standpoint, we systematically extend these model theoretic methods to the study of infinite Steiner \(p^k\)-systems with such properties considered for finite Steiner triple systems (STS) as anti-Pasch, sparseness, quasigroup structure, and cycle graphs. Section 1 sketches background information on the Hrushovski construction and its relation to other generalization of the Fraïssé method. We reformulate (Section 2) the Cameron-Webb notion of sparse configurations [CGGW10, Fuj06a] in terms of the \(\delta\)-function fundamental

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to the Hrushovski construction and give uniform accounts of the existence in every infinite cardinality of anti-Pasch, anti-mitre and indeed $\infty$-sparse Steiner triple systems (Corollaries 2.1.4 and 2.1.5 and Theorem 2.2.4). While the examples of strongly minimal pure Steiner $k$-systems, $(M, R)$, with $k > 3$ admit no definable ‘truly binary’ operation with infinite domain [BV21, Bal21], we construct (Theorem 3.6) strongly minimal quasigroups which induce $q$-Steiner systems (line length $q$) for $q$ a prime power. This is our first extension from 3 to $p^\omega$. In contrast to [HW21] rather than omitting appropriate sets $F$ of finite systems, we require the quasigroup determining the Steiner system to be in a fixed variety of quasigroups [GW80].

Our second extension (Section 4) from 3 to $p^n$ moves the notion of an $(a, b)$-cycle graph $G_M(a, b)$ of an infinite STS [CW12] to path-graphs of $q$-Steiner systems induced by quasigroups. Section 4.1 gives the rather complicated definition of a path graph. We then lift properties of chains in infinite STS [CW12] to $q$-Steiner systems. We give examples where all the path graphs over algebraically closed sets are infinite (Lemma 4.3.2) and the systems are decomposed as unions of ‘fans’, Lemma 4.3.5 which generalizes the decomposition by chains in the triple system case.

Rather than ad hoc examples, we provide a method to construct first order theories and thus infinite families of countable models exhibiting designated combinatorial properties. The countable models of these strongly minimal theories are arranged in a tower, a countable increasing sequence $\langle M_i : i < \omega \rangle$ with $M_n \prec M_{n+1}$. The structure of $G_{M_0}(a, b)$ depends heavily on whether $\text{acl}_{M_0}(\emptyset) = \emptyset$. In various cases $G_{M_0}(a, b)$ may have only finite cycles, only infinite cycles or a mixture. Lemma 5.1.2 constructs a class of $q$-Steiner systems where all paths are infinite.

In Section 5.2 we generalize the construction by Cameron and Webb of uniform cycle graphs in STS in [CW12] to $q$-Steiner graphs. For this we need to find 2-transitive Steiner systems. Although the only finite 2-transitive STS are $PG(d, 2)$ and $AG(d, 3)$ [KS84], in every infinite cardinality we point to 2-transitive and so uniform 3-Steiner systems (Fact 5.3.2) and construct 2-transitive $p^n$-Steiner systems for every prime power (Theorem 5.4.2). For this we must alter the parameters for a Hrushovski construction that are described in Notation 1.0.1. We ask questions that depend on our construction at various places in the text. But we conclude in Section 6 by raising several questions which should require more combinatorial methods.

In the remainder of the introduction we give further context and background. The Barbina-Casanovas examples [BC19] are extremely complex 1 from the viewpoint of the stability hierarchy ($TP_2$ and $NSOP_1$), while strongly minimal are the simplest; algebraic closure imposes a matroid structure on each model. A first order theory $T$ is strongly minimal if every definable subset of every model of $T$ is finite or co-finite. Three prototypical examples are the theories of: the integers with successor, rational addition, and the complex field. Zilber conjectured these examples were canonical; each such geometry was discrete, vector space like, or field like. Hrushovski refuted this conjecture by an intricate extension of Fraïssé’s construction of countable homogeneous universal models. The resulting ‘generic’ model is ‘less’ homogeneous that those built by Fraïssé. By use of a function $\delta$, a

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1While it is extremely likely that the theories of [HW21] have similar complexity, that has not been worked out.
class of finite strong substructures is obtained and only isomorphisms among them are required to extend.

A linear space is collection of points and lines such that two points determine a line, a minimal condition to call a structure a geometry. A linear space is a Steiner $k$-system if every line (block) has cardinality $k$. We showed in Section 2 of [BP20] that linear spaces can be naturally formulated in a one-sorted logic with single ternary ‘collinearity’ predicate and proved the existence of strongly minimal Steiner $q$-systems for every prime power. These theories are model complete and satisfy the usual properties of counterexamples to Zilber’s trichotomy conjecture: Their acl-geometries are non-trivial, not locally modular, and the theory cannot interpret an infinite group.

Much of the history of Steiner systems interacts with the general study of non-associative algebraic systems such as quasigroups. A quasigroup is a structure with a single binary operation whose multiplication table is a Latin Square (each row or column is a permutation of the universe) [Ste56]. Drawing on universal algebra and combinatorics, we [Bal21] found that the restriction to prime power cardinality of the universe that is essential for the existence of quasigroups coordinatizing finite $q$-Steiner systems is replaced by prime power block length for (necessarily infinite) strongly minimal Steiner systems.

The III in the title indicates the heavy reliance for details on [BP20, BV21, Bal21]. The novelty here is that applying these methods to combinatorial issues requires new changes in the parameters of the construction. We acknowledge helpful discussions with Joel Berman, Omer Mermelstein, Gianluca Paolini, and Viktor Verbovskiy.

1. Background

Hrushovski’s ‘flat geometries’ [BP20, Definition 6.2] have generally been regarded from the standpoint of their creation: as an undifferentiated class of pathological structures designed as counterexamples. However, there are ternary fields, Steiner systems and quasigroups are among them. [Bal94, BP20, Bal21, BV21] shows that structural distinctions arise by fixing a class $U$ of permissible choices for the function $\mu$. In fact, the family of ‘Hrushovski constructions of strongly minimal sets’ depend on five parameters. We list them here for reference; we demonstrate below that modifying these parameters can produce strikingly different behavior.

Notation 1.0.1. A quintuple $(\sigma, L_0^\ast, L_0, \epsilon, U)$ determines a Hrushovski sm-class. $L_0^\ast$ is a collection of finite structures in a vocabulary $\sigma$, not necessarily closed under substructure. $\epsilon$ is a function from a specified collection of finite $\sigma$-structures to natural numbers satisfying the conditions imposed on $\delta$ in Definition 1.3.3. $L_0$ is a subset of $L_0^\ast$ defined using $\epsilon$. From such an $\epsilon$, one defines notions of $\leq$, primitive extension, and good pair. Hrushovski gave one technical condition on the function $\mu$ counting the number of realizations of a good pair that ensured the theory is strongly minimal rather than $\omega$-stable of rank $\omega$. Fixing a class $U$ as the collection of functions $\mu$ satisfying a specific condition provides a way to index a rich group of distinct constructions. As explained in Definition 1.2.3, from $L_0, \epsilon$ and $\mu \in U$, one defines an amalgamation class $(L_\mu, \leq)$ of finite structures and an associated

\footnote{\textit{L}_0^\ast$ contains $\sigma$ structures of arbitrarily cardinality. $L_0^\ast$ was closed under substructure in [Hru93] but not here.}
class of infinite structures \( \hat{L}_\mu \) (For any collection \( L \) of finite structures, we write \( \hat{L} \) for the collection of direct limits of structures in \( L \)). Thus, one obtains a strongly minimal theory \( T_\mu \) of a generic structure \( G_\mu \), that describes the ‘existentially closed’ members of \( \hat{L}_\mu \).

We show how modifications of the most basic Hrushovski construction provides examples of Steiner systems. \cite{BP20} 2.1, 2.2 summarises the role of strongly minimal sets in model theory and the bi-interpretability of a one-sorted (used here) and two-sorted approach to Steiner system. \cite{Bal} provides a somewhat outdated survey of vastly wider study of modifications of the construction to study e.g. fusions, ‘bad’ fields, Spencer-Shelah random graphs and higher levels of stability classification. In Section 1.1 we sketch the relation between the combinatorial and model theoretic literature. Section 1.2 outlines the general setting of the so-called *ab initio* Hrushovski construction, generated from a collection of finite structure, emphasizing the parameters that can be varied to get specific behaviors. Remark 1.2.9 reminds us of the original context; Section 1.3 lays out the notation for studying linear spaces.

For convenience, one usually specifies in \( L^* \) that the relations are symmetric; but to reach important cases such as linear spaces, quasigroups, and Steiner systems one adds the relevant axioms to this starting point. We axiomatize \( L^* \) with \( \forall\exists \) sentences to create quasigroups. Working in linear spaces with a ‘geometric’ \( \epsilon \) in \cite{Pao20} is vital to obtain Steiner systems. In this paper, to obtain Steiner systems which are (e.g. anti-Pasch, \( \infty \)-sparse, 2-transitive) we both vary the class \( U \) of admissible \( \mu \)-functions and change the way that the class of finite structures \( L_0 \) is determined by the relevant \( \delta \) playing the role of \( \epsilon \).

### 1.1. Constructing ‘Generic’ models

The constructions in \cite{BC19, HW21} and in this paper are related generalizations of Fraïssé’s construction of a generic model \( M \) from a collection \( L_0 \) of finite structures in a *finite relational vocabulary* that is closed under substructure. A *generic model* for a class \( J \) of finite structures is one that is homogeneous and embeds all members of \( J \). Both ‘homogeneous’ and ‘embed’ change with the author. In the Fraïssé setting \( J \) is the set of substructures (as in the next paragraph) of \( M \).

For ease in following references, I use the following model theoretic terminology. A *substructure* \( B \) of a structure \( A \) in a vocabulary \( \sigma \) (list of function and relation symbols) is a \( B \subseteq A \) that is i) closed under the function symbols in \( \sigma \) and ii) each \( n \)-ary relation \( R_B \) is \( R_A \cap B^n \).

A structure \( A \) is (finitely) ultra-homogeneous\(^3\) if every isomorphism between finitely generated substructures of \( A \) extends to an automorphism of \( A \). This term corresponds to ‘homogenous’ in \cite{HW21} p. 2 (as noted there). I use homogenous in the usual model theoretic sense: a structure \( A \) is (finitely) homogeneous if any two sequences \( a \) and \( b \) of length \( n < \omega \) that satisfy the same \( n \)-ary first order formulas are automorphic in \( A \). This notion appears in an essential way in the proof of Lemma 5.2.2.

Fraïssé constructed ultrahomogeneous, quantifier eliminable, and \( \aleph_0 \)-categorical structures in *finite relational vocabularies*. His crucial hypotheses were joint embedding, amalgamation, and closure under substructure. As the construction was generalized to other notions of substructure and possibly infinite vocabularies or

\(^3\)This model theoretic usage dates from \cite{Woo79}.
with function symbols, two hypotheses that were hidden by the finite relational vocabulary hypothesis became evident: uniform local finiteness\footnote{In model theoretic terms, a structure is locally finite if every finitely generated substructure is finite (uniformly if there is a function $f$ such that an $n$-generated structure has less than $f(n)$ elements). In \cite{HW21}, a structure $M$ is called finitely generated with respect to a class $K$ of finite structures if every finite subset of $M$ is contained in a member of $K$. Our generic is locally finite in that sense but not in the model theoretic sense. While the generic in \cite{BC19} is locally finite in both senses, Consider the infinite 3-generated chains in Section 4.1 and the proof that $T_{S\mu}$ is not small in \cite{BC19} Theorem 3.3.} (for $\aleph_0$-categoricity) and only countably many finite structures (for the generic to be countable).

The three amalgamation constructions discussed here can best be compared in a more abstract framework. Consider a countable collection $(L_0^*, \leq)$ of finite structures where $\leq$ is a partial order refining substructure and $L_0^*$ is defined by a collection of first order (usually universal) sentences in the vocabulary of $L_0^*$. $M$ is $\leq$-homogeneous if $A, B \leq M$ implies they are automorphic. And $M$ is $L_0$-universal if there is an isomorphism $F : A \to M$ with $f(A) \leq M$. Sufficient conditions are given so there is a structure $M$ which is $\leq$-homogeneous and universal for $L_0^*$. There is no requirement that the language is relational. Each of the three constructions discussed here interpret $J$ and $\leq$ in a different way.

\cite{BC19} §4 takes $L_0^*$ as the class of finite STS and $\leq$ as substructure. They construct a ‘generic’ (a Fraïssé limit), which is a prime model of their separately constructed $T_{sq}$, the model completion of the theory of all Steiner quasigroups.

\cite{HW21} Theorem 1 generalizes this situation by taking $L_0$ as the class of ‘good $F$-free structures’ (omit a collection $F$ of finite nontrivial STS) finite triple systems. The key distinction from our work is that those authors restrict their amalgamation class to Steiner triple systems and use or prove the combinatorial fact that finite partial Steiner triple systems extend to finite Steiner triple systems in the fixed $L_0^*$. In contrast, we prove amalgamation by applying a general procedure due to Hrushovski to an ambient class of finite linear spaces, bound line length by the $\mu$-function (so partial Steiner systems), and obtain uniform line length by the ‘everything that can happen does’ mantra of amalgamation constructions. It is routine \cite{BP20} Section 2.1] that strongly minimal linear spaces have bounded line-length and cofinitely many lines have the same length; the existential completeness of the generic model implies all lines have this maximal length.

We introduce the class $L_0 \subseteq L_0^*$ which is defined by properties of a pre-dimension function $\delta$ and then further restrict with an ‘algebrizing function’ to $L_\mu$. Both $\delta$ and $\mu$ limit membership in $L_\mu$ to obtain strongly minimal Steiner $q$-systems and in Section 3 $K^q_{\mu', V}$ for strongly minimal quasigroups that induce such systems. Crucially, in that section we drop the requirement that $L_0^*$ is closed under substructure.

For \cite{BC19}, the generic model is prime but there is no countable saturated model and the theory is not stable. Our generic is saturated (so model theoretically homogeneous) and the theory is strongly minimal (in particular, $\omega$-stable). But our generics are not ultrahomogeneous but only $\leq$-homogeneous. In \cite{BC19}, $T_{S\mu}$ is the model completion of theory of all Steiner quasigroups. Our theories $T_\mu$ are model complete. But they are not the model completion of the $L_0$ in \cite{BC19}; our $K^q_{\mu, V}$. (Section 5) is a much restricted class of finite quasigroups. The generics of \cite{HW21, BC19} are locally finite; ours are not.
Thus, from a model theoretic standpoint the strong minimality\(^5\) distinguishes our example; while from the combinatorial standpoint the extension from 3 to \(p^n\)-Steiner systems is central is the main novelty.

1.2. The Hrushovski framework

The basic ideas of the Hrushovski construction are i) to modify the Fraïssé construction by replacing substructure by a notion of strong substructure, defined using a predimension \(\delta\) (Definition 1.3.3) so that independence with respect to the dimension induced by \(\delta\) is a combinatorial geometry\(^6\) and ii) to employ an algebrizing function \(\mu\) to bound the number 0-primitive extensions of each finite structure so that closure in this geometry is algebraic closure\(^7\).

A Steiner \((t,k,v)\)-system is a pair \((P,B)\) such that \(|P| = v\), \(B\) is a collection of \(k\) element subsets of \(P\) and every \(t\) element subset of \(P\) is contained in exactly one block. Since we are primarily interested in infinite structures, we omit the \(v\) unless it is crucial and so, by Steiner \(k\)-system I mean Steiner \((2,k)\) system of arbitrary cardinality. A groupoid (also called a magma) is a structure \((A,\ast)\) with one binary function \(\ast\).

Unfortunately, while the extensive literature on Hrushovski constructions contains the same fundamental notions related in a fairly standard way, the notation is not standard. So we quickly list our terminology.

We give an abstract formulation of the construction of generic model due to [KL92]. This provides a common framework for the Fraïssé and Hrushovski constructions which does not require the class \(L\) to be closed under substructure and is essential in Section 3. For the general discussion in this section we work in a finite relational vocabulary \(\sigma\).

Notation 1.2.1. (1) For any class \(L\) of finite structures, \(\hat{L}\) denotes the collection of structures of arbitrary cardinality that are direct limits\(^8\) of models in \(L\).

(2) Let \(\sigma\) be a finite relational vocabulary. A class \((L_0,\leq)\) of finite structures, with a transitive relation \(\leq\) on \(L_0 \times L_0\) is called smooth if \(B \leq C\) implies \(B \subseteq C\) and for all \(B \in L\) there is a collection \(p^B(x)\) of universal formulas with \(|x| = |B|\) and for any \(C \in L_0\) with \(B \subseteq C\),

\[B \leq C \iff C \models \phi(b)\]

for every \(\phi \in P^B\) and \(b\) enumerates \(B\).

We write \(B\) is strongly embedded in \(C\) if an isomorphic image \(B'\) of \(C\) satisfies \(B' \subseteq C\).

(3) A structure \(A\) is a \((L_0,\leq)\)-union if \(A = \bigcup_{n<\omega} C_n\) where each \(C_n \in L_0\) and \(C_n \leq C_{n+1}\) for all \(n < \omega\). If \(A\) is a \((L_0,\leq)\)-union, \(B \subseteq A, B \in L_0\), we say \(B \leq A\) if \(B \subseteq C_n\) for all sufficiently large \(n\).

\(^5\)The easier \(\omega\)-stable step of the Hrushovski construction does not yield Steiner systems.

\(^6\)The requirement that the range of this function is well-ordered is essential to get the exchange property in the geometry; using rational or real coefficients yields a stable theory and the dependence relation of forking [BS96].

\(^7\)In model theory \(a \in acl_M(B)\) if there is a formula and \(b \in B\) with \(\phi(a,b)\) and \(\phi(x,b)\) has only finitely many solutions in \(M\).

\(^8\)If \(L_0\) is closed under substructure so is \(\hat{L}_0\) and \(\hat{L}_0\) is axiomatized by a universal sentence in \(L_{\omega_1,\omega}(L_{\omega,\omega}\) if the vocabulary is relational.).
\(A\) structure \(A\) is an \((\mathcal{L}_0, \leq)\)-generic or \(\leq\)-homogeneous if \(A\) is a \((\mathcal{L}_0, \leq)\)-union and for any \(B \subseteq C\) each in \(\mathcal{L}_0\) and \(B \subseteq A\) there is a \(\leq\)-embedding of \(C\) into \(A\).

\(\leq\) is read ‘strongly embedded’. The crucial fact is:

**Fact 1.2.2** ([KL92]). If \((\mathcal{L}_0, \leq)\) is a smooth class of countably many finite structures that satisfies \(\leq\)-amalgamation and \(\leq\)-joint embedding there is a unique countable generic \(\mathcal{G}\) for \((\mathcal{L}_0, \leq)\).

**Axiom 1.2.3.** Let \(\delta\) be a map from a collection of finite \(\sigma\)-structures into \(\omega\). Let \(\mathcal{L}_0^*\) be a collection of such structures closed under isomorphism. We write \(A \subseteq B\) if for every \(C\) with \(A \subseteq C \subseteq B\), \(\delta(C/A) \geq 0\). We require that \(\mathcal{L}_0^*, \hat{\mathcal{L}}^*, \delta\) satisfy the following requirements. First, \(\mathcal{L}_0 \subseteq \mathcal{L}_0^*\) is the collection of finite \(B\) such that:

1. \(\delta(\emptyset) = 0\)
2. If \(B \in \mathcal{L}_0\) and \(A \subseteq B\) then \(\delta(A) \geq 0\).
3. If \(A \subseteq B \subseteq C\), \(A \subseteq B\) and there is no \(B\) such that we have that \(\mathcal{L}_0^*, \hat{\mathcal{L}}^*, \delta\) satisfy the following requirements. First, \(\mathcal{L}_0 \subseteq \mathcal{L}_0^*\) is the collection of finite \(B\) such that:

**Definition 1.2.4.** **Canonical Amalgamation** For any class \((\mathcal{L}_0, \epsilon)\), if \(A \cap B = C\), \(C \subseteq A\) and \(A, B, C \in \mathcal{L}_0^*\), \(G\) is a free (or canonical) amalgamation, \(G = B \oplus_C A\) if \(G \in \mathcal{L}_0^*\), \(\epsilon(A/BC) = \epsilon(A/C)\) and \(\epsilon(\hat{B}/AC) = \epsilon(B/C)\). Moreover, \(\epsilon(A \oplus_C B) = \epsilon(A) + \epsilon(B) - \epsilon(C)\) and any \(D\) with \(C \subseteq D \subseteq A \oplus_C B\) is also free. Thus, \(B \subseteq G\).

Disjoint union is the canonical amalgamation for the basic Hrushovski construction and Definition 1.3.1 gives the appropriate notion satisfying Axiom 1.2.3.5 for linear spaces. Axiom 1.2.3.2 can be rephrased as: \(B \subseteq C\) and \(A \cap C = \emptyset\) implies \(\epsilon(A/B) \geq \epsilon(A/C)\); so we can make the following definition.

**Definition 1.2.5.** Extend \(\epsilon\) to \(d:\hat{\mathcal{L}}_0 \times \mathcal{L}_0 \to \omega\) by for each \(N \in \hat{\mathcal{L}}_0\) and \(A \subset \omega N\), \(d(N, A) = \inf\{\epsilon(B) : A \subseteq B \subseteq N\}\), \(d_N(A/B) = d_M(A \cup B) - d_M(B)\). We usually write \(d(N, A)\) as \(d_N(A)\) and omit the subscript \(N\) when clear.

What Hrushovski called self-sufficient closure is in the background.

**Definition 1.2.6.** (1) For \(N \in \hat{\mathcal{L}}_0\) and \(A \in \mathcal{L}_0\), we say \(A \subseteq N\) is strong in \(N\) and write \(A \leq N\) if \(d(N/A) \geq 0\).

(2) For any \(A \subseteq B \in \mathcal{L}_0^*\), the intrinsic (self-sufficient) closure of \(A\), denoted \(\text{icl}_H(A)\), is the smallest superset of \(A\) that is strong in \(B\).

Note that in the current situation icl(B) is finite if \(B\) is. The following definition describes the pairs \(B \subseteq C\) such that eventually tp(C/B) will be an algebraic set (realized only finitely often).

**Definition 1.2.7.** Let \(A, B \in \mathcal{L}_0\) with \(A \cap B = \emptyset\) and \(A \neq \emptyset\).

1. \(B\) is a primitive extension of \(A\) if \(A \leq B\) and there is no \(A \subset B_0 \subseteq B\) such that \(A \leq B_0 \leq B\). \(B\) is a \(k\)-primitive extension if, in addition, \(\epsilon(B/A) = k\).

We stress that in this definition, while \(B\) may be empty, \(A\) cannot be.

2. We say that the 0-primitive pair \(A/B\) is good if there is no \(B' \subseteq B\) such that \((A/B')\) is 0-primitive. (This notion was originally called a minimal simply algebraic or m.s.a. extension.)

3. If \(A\) is 0-primitive over \(B\) and \(B' \subseteq B\) is such that we have that \(A/B'\) is good, then we say that \(B'\) is a base for \(A\) (or sometimes for \(AB\)).
If the pair $A/B$ is good, then we also write $(B,A)$ is a good pair.

**Definition 1.2.8.**

1. Let $U$ be the collection of functions $\mu$ assigning to every isomorphism type $\beta$ of a good pair $C/B$ in $L_0$ a natural number $\mu(\beta) = \mu(B,C) \geq \epsilon(B)$.

2. For any good pair $(B,C)$ with $B \subseteq M$ and $M \in \hat{L}_0$, $\chi_M(B,C)$ denotes the number of disjoint copies of $C$ over $B$ in $M$. A priori, $\chi_M(B,C)$ may be 0.

3. Let $L_\mu$ be the class of structures $M$ in $L_0$ such that if $(B,C)$ is a good pair $\chi_M(B,C) \leq \mu((B,C))$.

Up to this point, we have denoted the rank function by $\epsilon$ to indicate it is being treated entirely axiomatically. We switch to $\delta$ to emphasize that (Hrushovski’s definition (Definition 1.2.9) or Paolini’s (Definition 1.3.3) may be used but trust to context for the reader to know which.

**Remark 1.2.9 (The basic Hrushovski construction).** In the original context [Hru93], $\sigma$ contains a single ternary relation $R$ and $\delta(A) = |A| - r(A)$ where $r(A)$ is the number of triples $a$ from $A$ satisfying $R(a)$. $L^*$ is all finite $\sigma$-structures and $L_0$ is those $A \in L^*$ with $0 \leq A$ and $U$ is as in Definition 1.2.8 with that $\delta$.

We have recalled the Hrushovski notion for context but this paper is entirely about linear spaces.

### 1.3. Linear Spaces

In this section we outline the adaptation of Remark 1.2.9 that generates most of the examples in this paper. For the remainder of the paper we will deal at various times with two vocabularies $\tau$, with a single ternary relation symbol, $R$, and $\tau' = \{H,R\}$ with a ternary relation $H$, which will be the graph of a binary function $\ast$.

**Definition 1.3.1.** A $\tau$-structure $(M,R)$ is

1. a 3-hypergraph if $R$ holds only of distinct triples and in any order.

2. a linear space if it is a 3-hypergraph in which two points determine a unique line. That is, each pair of distinct points in contained in unique maximal $R$-clique (line). That is, all triples from the line satisfy $R$.

3. A linear space is a $k$-Steiner system if all lines have the same length $k$.

Thus our finite structures will in general be partial $k$-Steiner systems (lines may not have full length) for some $k$. We use the words ‘block’ and ‘line’ interchangeably and often fail to distinguish when the line has full length. When this is important, we may write clique to denote a subset of a line, i.e., a maximal clique.

**Definition 1.3.2.**

1. For $\ell \subseteq A$, we denote the cardinality of a clique $\ell$ by $|\ell|$, and, for $B \subseteq A$, we denote by $|\ell|_B$ the cardinality of $\ell \cap B$.

2. We say that a non-trivial line $\ell$ contained in $A$ is based in $B \subseteq A$ if $|\ell \cap B| \geq 2$, in this case we write $\ell \in L(B)$.

3. The nullity of a line $\ell$ contained in a structure $A \in K^*$ is:

$$n_A(\ell) = |\ell| - 2.$$

Now we define our geometrically based pre-dimension function [Pao20].

**Definition 1.3.3.** We define the appropriate $K^*$ and $K_0$.

1. Every $(A,R) \in K^*$ is a finite linear spaces.
(2) For \((A, R) \in K^*\) let:
\[ \delta(A) = |A| - \sum_{\ell \in L(A)} n_A(\ell). \]

(3) Moreover \((A, R) \in K_0\) if for any \(A' \subseteq A, \delta(A') \geq 0\).

(4) \((K_0, \delta)\) satisfies the conditions on \(\epsilon\) given in Section 1.2.

The explicit definition of the free amalgamation in this context is:

**Definition 1.3.4.** [BP20, Lemma 3.14] Let \(A \cap B = C\) with \(A, B, C \in K_0\). We define \(D := A \oplus_C B\) as follows:

1. the domain of \(D\) is \(A \cup B\);
2. a pair of points \(a \in A - C\) and \(b \in B - C\) are on a non-trivial line \(\ell'\) in \(D\) if and only if there is line \(\ell\) based in \(C\) such that \(a \in \ell\) (in \(A\)) and \(b \in \ell\) (in \(B\)). Thus \(\ell' = \ell\) (in \(D\)).

We single out a type of good pair that provides the line-length invariant for the Steiner systems.

**Notation 1.3.5 (Line length).** We write \(\alpha\) for the isomorphism type of the good pair \((\{b_1, b_2\}, a)\) with \(R(b_1, b_2, a)\). Lemma 5.18 of [BP20] implies lines in models of \(T_\mu\) have length \(k\) if and only if \(\mu(\alpha) = k - 2\).

If one restricts the counting functions to \(U\) (Definition 1.2.8), Steiner triple systems are excluded. Since they are a key topic, the \(U\) is slightly altered from Definition 1.2.8 to admit them.

**Definition 1.3.6 \((U^{ls})\).** Let \(U^{ls}\) be the collection of functions \(\mu\) assigning to every isomorphism type \(\beta\) of a good pair \(C/B\) in \(K_0\) a number \(\mu(\beta) = \mu(B, C) \geq \delta(B)\).

1. a number \(\mu(\beta) = \mu(B, C) \geq \delta(B)\), if \(|C - B| \geq 2\);
2. a number \(\mu(\beta) \geq 1\) (rather than 2), if \(\beta = \alpha\).

2. Omitting configurations in Steiner triple systems

There is a long history of studying finite Steiner triple systems that omit specific configurations, e.g. Pasch. The concept is formalized as follows ([Fuj06a]).

**Definition 2.0.1.** Let \(X\) be finite partial Steiner system. A Steiner system \((M, R)\) is anti-\(X\) if there no embedding of \(X\) into \(M\).

The notion of an \(\infty\)-sparse system uniformizes these anti-\(x\) constructions [Fuj06b, CCGW10]. We derive such results for infinite Steiner triple systems by variants on our general construction. We first find specific amalgamation constructions that give the strongly minimal Steiner systems omitting target configurations by varying the class \(U\) of acceptable bounds on algebraicity. We next obtain \(\infty\)-sparseness in Section 2.2 by enforcing the uniformity with \(\delta\) and with more drastic restrictions on the class \(K_0\).

2.1. Anti-Pasch and Anti Mitre

We begin by examining the connection between the Pasch configuration [Fuj06b] and the group configuration from model theory. Diagram 2.1 is known in the study of Steiner triple systems as the *Pasch configuration*. This same diagram, interpreting the lines as representing algebraic closure, is known to model theorists as the
Figure 1. Pasch configuration

group configuration: in that context the acl-dimension of the set of 6 points is 3; any triple of non-collinear points are independent; each point has acl-dimension 1, and each line has acl-dimension 2. Hrushovski’s proof, described for the Steiner system case in [BP20, Corollary 6.3], that no \( T_\mu \) interprets an infinite group originated the model theoretic argument that the group configuration in the algebraic closure geometry implies the existence of a definable infinite group. We give a more direct argument for:

**Fact 2.1.1.** The strongly minimal quasigroups whose existence is proven in Section 3, have no infinite definable associative subquasigroup.

Proof. Let \( G \) be a definable infinite subquasigroup of \( G_\mu \) with associative multiplication that is generated by three algebraically independent elements, say \( D, G, H \) as in Figure 2.1. Now \( DG = E \) so, by associativity, \( F(DG) = FE = X \). Similarly, \( H = FD \) implies \( (FD)G = HG = X \) so the lines \( HG \) and \( FE \) intersect in \( X \). In any \( T_\mu \), the algebraic closure dimension of a closed subset \( A \) is \( d(A) = \delta(A) \). So if \( A \) is the six points of the configuration we should have \( \delta(A) \geq d(A) = 3 \). But the actual calculation\(^9\) gives \( \delta(A) = 2 \). So the Pasch configuration is omitted.

In particular, a strongly minimal quasigroup constructed in this way can never be a group. The associative law forces the intrinsic closure \( icl(P) \) of three algebraically independent elements (which should have \( d(icl(P)) \geq 3 \)) to have dimension 2. Nevertheless, in general there will be many realizations of a Pasch configuration \( P \) in a strongly minimal Steiner triple system constructed as in [BP20], since \( \delta(P) = 2 \geq 0 \). Indeed any pair of points extends to a Pasch configuration in the generic model. Fact 2.1.1 shows in general that configuration cannot extend to an infinite subquasigroup.

[HW21] suggested that anti-Paschian STS might be an amalgamation class. We have't shown that; we construct smaller amalgamation classes of anti-Paschian STS. We need the following notion.

**Definition 2.1.2 (R-closure).** Let \( (M, R) \) be a \( \tau \)-structure. We define the R-closure, \( cl_R(X) \), for \( X \subset M \). Define inductively \( X = X_0 \) and for each \( n, c \in X_{n+1} \)

\(^9\)Section 4.2 of [Hru93] describes combinatorial geometries that calculate the dimension of a union of closed subsets by exclusion-inclusion principle as ‘flat’. 
if \(a, b \in X_n\) and \(R(a, b, c)\). Now \(\text{cl}_R(X) = X_N\), where \(N\) (possibly \(\omega\)) is where the inductively defined sequence \(X_n\) terminates. A set \(X\) is \(R\) independent if no element is in the \(R\)-closure of the others.

**Lemma 2.1.3.** The subclass of \(K_0^P\) of those finite structures with 3-element lines that omit the Pasch configuration satisfies amalgamation.

**Proof.** We can reformulate the problem by setting \(\rho\) as the isomorphism type of the good pair \((A/B)\) in Figure 2.1, taking \(\{F, H\}\) as the base \(B\) and \(A = \{X, D, E, G\}\) as a good extension. We use the standard \(K_0\) for linear space. But we modify a \(\mu \in U\) by setting \(\mu(A/B) = 0\). We must show \(K_\mu\) has amalgamation.

Fixing notation as in the proof of amalgamation in [BP20, 5.11], consider structures with \((E/D)\) a good pair, \(D \subseteq F\) and all in \(K_0\); we want to amalgamate \(F\) and \(E\) over \(D\). Note that every non-trivial line that intersects \(E - D\) is contained in \(E\) and has two elements in \(E - D\). This holds, as if the line intersects \(F - D\) then it has 4 points by Definition 1.3.4. But, if it intersects \(D\) in 2 points \(E\) is not primitive over \(D\). Thus \(F\) is \(R\)-closed in \(G\). The key property of the Pasch configuration is that each point not in the base is on a 3-element line that intersects the base. This implies that if there is an embedding of the Pasch configuration \(P\) in \(G\), the image of the base \(EH\) is contained in \(D\). (Otherwise there would be a line from \(F - D\) to \(E - D\).) But since the Pasch configuration is \(R\)-generated by the base along with any other point, we have \(A \subseteq F\) if \(A \cap F \neq \emptyset\) and \(A \subseteq E\) if not. Either violates the hypothesis that \(F\) and \(E\) omit the Pasch configuration.

Applying Lemma \[2.1.3\]

**Corollary 2.1.4.** Fix a \(\mu \in U\) with \(\mu(\alpha) = 1\) and \(K_\mu \subseteq K_0^P\). Each model of \(T_\mu\) is a strongly minimal anti-Pasch Steiner triple system. As usual, varying \(\mu\) yields \(2^{\aleph_0}\) distinct families.

Similar arguments construct anti-mitre and anti-mia configurations. The two configurations are shown in Figure 2.1. Letting \(abc\) be the bottom line, \(c'b'a'\) the middle, and \(x\) the vertex, the diagram represents the left self-distributive law:

\[x(ab) = (xa)(xb).\]

Namely the self distributive law implies naming \(a'\) as \(xc\) and \(c'\) as \(xa\) the lines \(ac', bb', ca'\) intersect at \(x\). This \((5, 7)\)-configuration [Fuj06b] is called a mitre\(^{10}\).

The only other \((5, 7)\)-configuration, \((mia)\), is obtained by adding a point between the two points on the base of the Pasch configuration and creating a new line. By constructing \(\infty\)-sparse configurations below we simultaneously omit the Pasch, mitre, and mia configurations.

**Corollary 2.1.5.** There are anti-mitre and anti-mia Steiner triple systems in every infinite cardinality. The examples are strongly minimal.

**Proof.** Using the first paragraph of the proof of Lemma 2.1.4 we show the subclass \(K_0^M\) of \(K_0\) consisting of those finite structures with 3-element lines that omit the mitre (or those omitting the mia) configuration satisfies amalgamation. The argument there that the base is contained in \(D\), here yields only that two points of the base are in \(E - D\). Say \(a \in F\) and \(b, c \in E - D\). We violate \(F\) closure unless

\[^{10}In the diagram, x is the top point. Label the middle line a, b, c and the bottom line c', b', a'. Diagram taken from [CFMP17].\]
the point $F - E$ is $c'$. Now if the pivot $x$ is in $E - D$, we violate the $R$-closure of $F$. But if $x \in F$ and $a'$ or $b'$ is in $F$, $a'xa$ or $b'xb$ violates that $E$ is primitive over $D$. While if either is in $E - D$, $a'b'c'$ violates $R$-closure of $F$. The proof of the mia case offers nothing new.

Thus we construct structures which have no instances of associativity or self-distributivity anywhere and every left multiplication by an element not on a line fails to preserve lines.

2.2. Sparse Configurations in 3-Steiner systems

In, for example [CGGW10, page 116], an $(n,n+2)$ configuration in a Steiner triple system (STS) is a substructure $(A, R)$ of $n+2$ points with $n$ lines. That is, $\delta(A) = 2$. They say a system is $\infty$-sparse if there are no $(n,n+2)$ configurations with $n \geq 4$. We reformulate ‘sparse’ in terms of $\delta$.

**Definition 2.2.1.** A Steiner triple system $(M, R)$ is $\infty$-sparse if there is no $A \subseteq M$ with $|A| \geq 6$ and $\delta(A) = 2$.

Note that the Pasch, mitre, and mia configurations are all forbidden in an $\infty$-sparse STS. [CGGW10] construct by a four page inductive construction of finite approximations, $2^{\aleph_0}$ non-isomorphic countable $\infty$-sparse systems. We modify the construction in [BP20] by restricting $K_0$ to $K_0^{sp}$ to get $\infty$-sparse STS of every infinite cardinality.

**Definition 2.2.2.** Let $K_0^{sp}$ be the subclass of $K^*$ (linear spaces) such that for every $B \subseteq A$:

(#) $|B| > 1 \rightarrow \delta(B) > 1$ \& $|B| > 3 \rightarrow \delta(B) > 2$.

Take $U$ as $U^{sp}$, those $\mu \in U$ which can be achieved in $K_0^{sp}$.

Condition # implies there are no 4 element lines in a member of $K_0^{sp}$ so if $\mu \in U^{sp}$, $\mu(\alpha) = 1$ and the generic model will be a Steiner triple system.

**Theorem 2.2.3.** The system $(K_0^{sp}, \leq)$ has $\leq$-amalgamation. And so for any $\mu \in U$, $K_0^{sp}$ has $\leq$-amalgamation.
Proof. Let $A, B, C \in K_0^{sp}$ with $C \subseteq A$ and $C \subseteq B$. Linear space amalgamation (Definition 1.3.4) cannot introduce any relation between $A - C$ and $B - C$, as this would produce a 4-element line. But then it is clear that $\#$ is preserved in the amalgam. We use the first clause of $\#$ to avoid $B$ with $\delta(B) = 1$. Now the proof from [BP20] applies to give amalgamation for $K_{\mu}$ if $\mu \in \mathcal{U}^{sp}$.

Theorem 2.2.4. There are continuum many $\mu$ such that

1. $T_{\mu}$ is strongly minimal (so $\aleph_1$-categorical);
2. Every model of $T_{\mu}$ is an $\infty$-sparse Steiner triple system;
3. $T_{\mu}$ has countably many countable models.

Proof. As in [BP20], for any $\mu$ satisfying Definition 2.2.2, the associated $T_{\mu}$ is a Steiner triple system. But by omitting $A$ with $\delta(A) = 2$ and $|A| \geq 6$, the structure is $\infty$-sparse.

3. CONSTRUCTING STRONGLY MINIMAL QUASIGROUPS

While [BP20] shows there are strongly minimal $k$-Steiner systems for every $k$, [GW75, Bal21] imply that there can be quasigroups only when $k$ is a prime power. Our strongly minimal $k$-Steiner systems $(M, R)$ can admit a definable ‘truly’ binary function [BV21] only under very strong additional hypotheses on $\mu$ (BV21 Theorem 0.2). Nevertheless, there are strongly minimal quasigroups which induce $k$-Steiner systems when $k$ is a prime power. For this result we need the generality of Fact 1.2.2 as we will axiomatize $L_0^p (K^p)$ here with $\forall \exists$-sentences. We sketch a different proof than that detailed in [Bal21] of the existence of strongly minimal quasigroups.

The coordinatizing result rests primarily on work of [GW75, Ste56, ´S61] and others who achieved a ‘coordinatization’ of such Steiner systems by quasigroups. The contribution here is that although, for $k > 3$, because the Steiner system never interprets a quasigroup [Bal21], this coordinatization is not a bi-interpretation, we can in fact demand for $k = q = p^n$ the existence of a Steiner $k$-system that is interpreted in a strongly minimal quasigroup. The key to this is the relationship of so-called $(2, k)$ varieties [Pad72, GW75] to a two-transitive finite structure and thus eventually to the reconstruction of a finite field. Following [GW80] we call the quasigroups which arise when $k$ is a prime power $q$, block algebras.

A variety is a collection of algebras (structures in a vocabulary with only function/constant symbols and no relation symbols) that is defined by a family of equations. The essential characteristic of the equational theories below is that each defining equation involves only two variables. In particular, none of the varieties are associative.

Definition 3.1. [Smi07] A quasigroup $(Q, \ast)$ is a groupoid $(A, \ast)$ such that for $a, b \in Q$, there exist unique elements $x, y \in Q$ such that both

$$a \ast x = b, y \ast a = b.$$

The general notion is a universal Horn class, not a variety. But an $(r, k)$ variety of groupoids is a quasigroup [Qua92].

Definition 3.2. [Pad72]

\footnote{In the background literature on quasigroups, a groupoid is simply a set with a binary operation. So, I use this notation although it is no longer common.}
(1) The variety \( V \) is a \((r,k)\) variety if every \( r \)-generated subalgebra of any \( A \in V \) is isomorphic to the free \( V \)-algebra on \( r \) elements and has cardinality \( k \).

(2) A Mikado variety \([GW 75]\) \) is a \((2,q)\)-variety with all fundamental operations binary and with an equational base of 2-variable equations.

Thus, ‘Mikado’ picks out those \((r,k)\)-varieties that are really determined by their free algebras on 2-generators.

**Fact 3.3. \([GW 75]\)** Given a \((near)\)-field \( (F,\cdot,\cdot,\cdot,0,1) \) of cardinality \( q \) and a primitive element \( a \in F \), define a multiplication \( \ast \) on \( F \) by \( x \ast y = y + (x-y)a \). An algebra \((A,\ast)\) satisfying the 2-variable identities of \((F,\ast)\) is in a \((2,q)\)-variety of block algebras over \((F,\ast)\).

This is one of 5 equivalent characterizations of an \((r,k)\) variety in \([Pad 72]\). Obviously, the collection of \( r \)-generated subalgebras \( A \in V \) form a Steiner \((r,k)\)-system; we need a third characterization: the automorphism group of any \( r \)-generated algebra is strictly (i.e. sharply) \( r \)-transitive.

Fix two vocabularies \( \tau = \{ R \} \) and \( \tau' \) with two ternary relations symbols \( R, H \). For each Mikado \((2,q)\)-variety \( V \) of quasigroups, we construct a strongly minimal theory of quasigroups (in \( V \)) that induce \( q\)-Steiner systems. We use \( H \) as the graph of the quasigroup operation in \( V, \ast \), to make our amalgamation class contain only finite structures (as in \([BC 90]\)). But \( R \) is the ternary relation of collinearity.

Considering the general context of Notation \([1.2.1]\) there are two innovations in taking the \( L_0 \) as \( K^q_{V} \): i) in each finite structure in \( K^q_{V} \) every line has \( q \) points; ii) and \( L_0 = K^q_{0,V} \) for a \( \tau'\)-structure \( A, \delta_{\tau}(A') = \delta_{\tau}(A'\tau) \) so the amalgamation problem reduces to the known solution for \( \tau\)-structures. We define the base class \( K^{q}_{0,V} \) of finite structures as follows.

**Definition 3.4. \([K^q]\)** Fix a prime power \( q \) and a Mikado variety \( V \) of quasigroups (e.g. a block algebra from Fact \([3.3]\)) such that \( F_2 \) the free algebra in \( V \) on 2 generators has \( q \) elements. Let \( K^q_{V} \) be the collection\(^{13}\) of finite \((H,R)\)-structures \( A \) such that

\( (1) \ (A,R) \) is a linear space;
\( (2) \ (\forall a_1, a_2, a_3) H(a_1, a_2, a_3) \rightarrow R(a_1, a_2, a_3); \)
\( (3) \ (\forall a_1, a_2, a_3) [H(a_1, a_2, a_3) \land H(a_1, a_2, a'_3)] \rightarrow a_3 = a'_3; \)
\( (4) \ (\forall a_1, a_2, a_3) (\exists b_1, ... b_{q-2}) [R(a_1, a_2, a_3) \rightarrow \bigwedge_{i=1}^{q-3} R(a_1, a_2, b_i); \)
\( (5) \ (\forall x_1, x_2, \ldots x_q) [(R(x_1, x_2, x_i) \rightarrow \bigvee_{1 \leq i,j \leq q} x_i = x_j]. \)

(6) If \( A' | R \) is a maximal clique (line) with respect to \( R \) (necessarily \( |A'| = q \), \( A' | H \) is the graph of the free algebra \( F_2 \in V \).

Note that Definition \([3.4]\) implies that any triple satisfying \( R \) in \( A' \in K^q_{V} \) extends to a line in \( A' \) of exactly length \( q \). Since \( V \) is axiomatized by 2-variable equations, if \( A' \in K^q_{V} \), \( A' | H \) is the graph of an algebra in \( V \). In the generic model each pair is included in a \( q \)-element line; but not in the finite structures.

\(^{12}\)A near-field is an algebraic structure satisfying the axioms for a division ring, except that it has only one of the two distributive laws.

\(^{13}\)Clearly \( V \) determines but there are distinct \( V \) with \( |F_2(V)| = q \).
Definition 3.5. Primitives, good extensions, and the permissible $\mu'$:

1. For a $\tau$-structure $(A,R)$ $\delta_\tau(A)$ is defined as for linear spaces in Definition 3.3. Now for any $q$ and each $A' \in K^q_{\mu'}$, let $A = A'|R$ and $\delta_\tau(A') = \delta_\tau(A)$ and induce $\leq'$ from $\delta_\tau$.

2. $K_{0,\mu'}^q = \{A' \in K_{\mu'}^q : \delta_\tau(A') \geq 0\}$.

3. Define primitive extensions and good pairs in $\tau$ as usual using $\delta'_\tau$.

   Let $\alpha_q$ denote the isomorphism type of $\langle \{c_1, \ldots, c_{q-2}\}/ab\rangle$, where all the $c_i$ satisfy $R(a,b,c_i)$.

4. A $\mu'$ mapping $K_{0,\mu'}^q$ into $Z$ is in $\mathcal{U}_{\tau'}$ if it satisfies i) $\mu'(A'/B') \geq \delta_\tau(B)$ and ii) $\mu'(\alpha_q) \geq 1$.

5. Let $D' \in (K_{\mu',\mu}^q, \leq')$ if and only if $\chi(D'(A'/B')) \leq \mu'(A'/B')$.

Since both the restriction $\delta(A) \geq 0$ and the bound imposed by $\mu'$ are universally axiomatized it is easy to check that $(K_{\mu',\mu}^q, \leq')$ is smooth. However it is $AE$-axiomatized because of clause 3.4.2. Thus, the main difficulty in proving Theorem 3.6 is establishing amalgamation.

In [Bal21], we gave a different construction which involves a $\mu$ which counts good pairs in $\tau$ and a $\mu'$ which counts good pairs in $\tau'$. We write $\mu'$ here to emphasize that $\mu'$ counts good pairs of $\tau'$-structures and for compatibility with the earlier notation. Unlike [Bal21], there is no dependence on a given $\mu$ defined on the finite structures.

Theorem 3.6. For each $q = p^n$, each $\mu' \in \mathcal{U}_{\tau'}$, and each Mikado-variety of quasigroups $V$ with $|F_2(V)| = q$, there is a strongly minimal theory of quasigroups, dubbed $T_{\mu',\mu}^q$, that interprets a strongly minimal $q$-Steiner system.

Proof. We now show the amalgamation for the $(K_{\mu',\mu}^q, \leq')$, as in Lemma 5.11 and Lemma 5.15 of [BP20]. Consider a triple $D,E,F$ in $K_{\mu',\mu}^q$ as in Lemma 2.1.4. That is, $D \subseteq F$ and $E$ is $0$-primitive over $D'$. Since $E$ is primitive over $D$, although there may be a line contained in the disjoint amalgam $G$ with two points in each of $D$ and $F - D$, each line that contains 2 points in $E - D$ can contain at most one from $D$. If a line contains three points from $D$, since $D$ satisfies Definition 3.4.2 it is contained in $D$. Thus, there is no issue with defining the relation $H$ on the disjoint amalgamation. If $\mu'$ requires some identification for some $(B,C)$, just as in [BP20], it is because the (relational) $\tau'$-structure $BC$ is $DE$ and there is a copy of $C$ over $B$ in $F$ (Note the ‘further’ in [BP20] Lemma 5.10.).

The blocks of the Steiner system are the 2-generated $*$-subalgebras. Now the strong minimality of the generic follows exactly as in Lemmas 5.21 and 5.23 of [BP20] and we have proved Theorem 3.6.

For $q > 3$, let $G_{H,R}$ be the group of automorphism for the countable generic of our construction with vocabulary $\tau'$. $R$ is set-wise invariant under the action of $G_{H,R}$ which is exactly the group of $H$-automorphisms but $H$ is not preserved by a permutation which setwise stabilizes $R$.

We denote the theory of the generic $G_{\mu'}|*$ by $T_{\mu',\mu}^q$. We often drop the superscript $q$ as the specific $q$ is irrelevant in further considerations.
Remark 3.7. [GW80, p5] that depending on the choice of the primitive $a$ in Definition 3.3, the resulting $(r,k)$-algebra may or may not be commutative. [BV21, §5] show that the strongly minimal Steiner systems of [BP20] have no non-trivial commutative binary functions and deduce the theories do not admit elimination of imaginaries. Applying Theorem 3.6 with the commutative $(r,k)$ variety of block algebras yields a commutative strongly minimal quasigroup. Thus, more effort is needed to show it fails to eliminate imaginaries.

4. Strongly minimal block algebras, towers, and path graphs

The notion of an $(a,b)$-cycle graph is widely studied for finite Steiner triple systems. [CW12, CGGW10] consider the notion for infinite Steiner triple systems and prove the existence of infinite perfect and uniform Steiner triple systems. We generalize this notion to consider infinite $q$-Steiner systems that are induced from strongly minimal $(2,q)$-quasigroups with $q$ a prime power [BV21, Bal21].

We make the following assumption for this section. That is, we crystallize the properties of the result of the construction in Section 3 but do not rely on any details of the actual construction. We will write $*$ to denote multiplication (as opposed to its graph $H$ which was used to preserve the finiteness of structures in Section 3).

Assumption 4.0.1. $T$ is a strongly minimal theory in the vocabulary $\tau'$ such that if $M \models T$

1. $(M,*)$ is a quasigroup in a Mikado variety $V$;
2. $R$ is the graph of $*$.
3. There are functions $\delta$ and $d$ on the domain of $M$ that satisfy the properties of $\epsilon$ and $d$ in Section 1.2.

This assumption yields immediately that $(M,R)$ is a Steiner $q$-system where $q = |F_2(V)|$. We use the $*$ operation to inductively construct a path on points.

4.1. Path Graphs

Finite Steiner 3-systems $Q$ are often studied via the cycle graph over various $ab$; the pairs $(c,d)$ from $Q - \overline{ab}$ (overline $ab$ is the line through $ab$) are colored red or blue depending on whether $a$ or $b$ lies on the line $cd$. Then a path is generated by choosing a point $d$ off $\overline{ab}$ and starting with $\overline{ad}$ and inductively choosing the line of a different color through the third point on the current line. We extend this idea to $q$-Steiner systems. It is immediate that paths in Steiner 3-systems do not intersect; so for strongly minimal 3-Steiner systems the definitions below reduce to those in [CW12]. However, such disjointness is no longer immediate when $q > 3$ leading to the more complicated description of paths in Definitions 4.1.3 and 4.1.4. In order to carry out the analysis, we exclude $\overline{ab}$ from the graph, not just $\overline{ab}$ but the larger finite set $\text{icl}(a,b)$, the smallest subset containing $a,b$ that is strong in $M$. We later get stronger results by restricting the domain even further to $M - \text{acl}(a,b)$.

Definition 4.1.1. Consider a Steiner system $(M,*,R)$ determined by a $q$-block algebra $(M,*)$ (Definition 3.2). For any $a,b \in M$, we will write $G_M(a,b)$ for the graph determined by the pair $a,b \in M$.

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16 Here we write the function symbol $*$ rather than the graph $H$ because we going to use the function to trace out a path.

17 This guarantees that the generator $d_1$ satisfies $d(d_1/\text{icl}(ab)) \geq 0$. 

(1) The domain of \( G_M(a,b) \) is \( M - \text{icl}(ab) \).

(2) For \( x, y \notin \text{icl}(a,b) \), there is an edge colored \( a \) (resp., \( b \)) joining \( x \) to \( y \) if and only if \( R(a,x,y) \) (resp., \( R(b,x,y) \)).

\[ \text{Remark 4.1.2.} \] There is an edge colored \( a \) (resp., \( b \)) joining \( x \) to \( y \) if and only if \( a \ast x = y \) (resp., \( b \ast x = y \)).

We have partitioned the lines (\( \text{R-cliques} \)) that intersect \( \{a,b\} \) into \( a \) and \( b \) lines. Two lines with distinct colors can intersect in at most one point.

We introduce certain \textit{paths} and then in Section 4.3 \textit{fans} in the graph that under appropriate hypotheses cover (most of) the domain of the graph.

\[ \text{Definition 4.1.3.} \] Let \( M \models T \) with \( \mu \in \mathcal{U}^b \) (Definition 1.3.6). Consider a \( q \)-block algebra \((M,\ast)\) with associated path graph \( G_M(a,b) \).

(1) For any \( a,b \), we write \( \overline{ab} \) to denote the line of length \( q \) generated by \( \{a,b\} \).

(2) For \( d_1 \notin \text{icl}(ab) \) we define a sequence, denoted \( P_{abd} \) generated by \( d_1 \in M - \text{icl}(ab) \) over \( \{a,b\} \) as follows.

The path \( P_{abd} \) is the sequence \( \mathbf{d} = d_1, \ldots, d_m \) such that \( a \ast d_{2i+1} = d_{2i+2} \) and \( b \ast d_{2i+2} = d_{2i+3} \) for \( 0 \leq i \leq m \).

(3) The envelope, \( P^e_{abd} \), of the path, \( P_{abd} \), with \( d = d_1, \ldots, d_m \), is the union of the lines \( d_i, d_{i+1} \) for \( 1 \leq i < m \). Note that if \( i \) is odd (even), \( a \) (\( b \)) is on \( d_i, d_{i+1} \).

Note that if \( e \) is on an \( a \)-edge, \( a \ast e \) is on the same line (and similarly for \( b \)). Thus, the lines of the Steiner system are cliques of the path graph. But, if \( e \) with \( e \neq a \) and \( e \notin \text{icl}(ab) \) is on an \( a \)-edge multiplying \( e \) by \( b \) begins the generation of a distinct path, \( P_{bae} \) in the graph. We will show such a path is either an infinite chain or ‘cycles’ by generating a 0-primitive extension of \( ab \).

\[ \text{Definition 4.1.4.} \] (1) There are two possibilities when the process of Definition 4.1.3 is iterated forward \( m \) times.

(a) An \((a,b)\)-chain of length \( m \) is a path \( P_{abd} \) with \( \mathbf{d} = d_1, d_2, \ldots, d_m \) such that \( a \ast d_{2i+1} = d_{2i+2} \) and \( b \ast d_{2i+2} = d_{2i+3} \) for \( 0 \leq i \leq m \) and: for \( j > i + 1 \) the lines \( d_id_{i+1}, d_jd_{j+1} \) do not intersect. Thus \( \delta(P_{abd}) = \delta(P^e_{abd}) \). Note that \( m \) counts the number of lines in the path. We write \( \sigma_m \) for the isomorphism type of an \( m \)-chain. Note that, as in the 3-Steiner system case, the length of an \( m \)-chain must be divisible by \( 4 \).

(b) At some stage the new line generated by \( a, d_{2i+1} \) or \( b, d_{2i+2} \) intersects one of the earlier lines in the envelope of the path. In this case, we stop the construction with the new line. The result is an \( m \)-pseudo-cycle, an envelope \( P_{abd} \), such that for exactly one pair \( (i, j) \) with \( 0 \leq i < m \) and \( j > i + 1 \) the lines \( d_id_{i+1}, d_jd_{j+1} \) intersect. We write \( \gamma_s \) for an isomorphism type of an \( s \)-pseudo-cycle \( P_{abd} \) and \( P^e_{abd} \) for the isomorphism type of its

(c) If the process continues infinitely we call the result an infinite chain.

\[ ^{18} \text{Note that if } q = 3, \text{ this is the same as collinearity and we return to the framework of CW12.} \]

\[ ^{19} \text{We may sometimes write } P^e_{abd} \text{ when } P_{abd} \neq \{a,b\} \text{ is more precise; this is the usual ambiguity in describing good pairs } C/B; \text{ technically } B \text{ and } C \text{ are disjoint.} \]
Note that the construction of path through $d_1$ could equally well begin with the first line a b-line. In this case, we introduce a finicky notation: The $P_{bad_1}$ path through $d_1$ starts with a b-line.

Recall the construction stops as soon as there is a loop but may be infinite. In the pseudo-cycle case $P_{bad_1}$ contains a minimal pseudo-cycle, which is 0-primitive over $ab$. Thus, each triple $a,b$ and $d \not\in icl(ab)$, determine a unique minimal path $P_{bad_1}$ beginning with an a-edge: it may be a pseudocycle (perhaps starting with a different $d'$) of minimal length or an infinite chain. While formally we have defined pseudo-cycles to emphasize the return need be back to the initial point, we will often write cycle for short.

Within the algebraic closure of $ab$ analysis by the graph structure is more complicated. As, since any two points determine a line implies there are $c \in acl(ab)$ such that $d(c/ab)$ remains 0 even when $c$ is an intersection point of many lines. Thus, in Section 4.2 we study inside the graph over $icl(ab)$ $acl(ab)$ and in Section 4.3 work over $acl(ab)$.

4.2. Inside $acl(ab)$: Many Finite paths

This subsection analyzes the structure of $G_M(a,b)$ when $M$ is a prime model that is algebraic over the empty set and for arbitrary $M$ the structure of $acl_M(ab) - icl_M(ab)$. Section 4.3 describes the properties of $(a,b)$-path graph off $acl(ab)$.

While the definition in Section 4.1 was primarily combinatorial (except for the use of $icl(a,b)$ rather than $\bar{a}b$), we now use the model theoretic machinery about strongly minimal sets more heavily.

Remark 4.2.1 (Towers). Two prototypical properties of a strongly minimal theory $T$ are: a) the existence of a unique generic type over the model whose restriction to any set has infinitely many solutions and, as a result, if $T$ has at least two non-isomorphic countable models, b) the arrangement of the countable models into a tower. Let $(M_j : 0 \leq j < \omega + 1)$ be the tower (elementary chain: $M_n \prec M_{n+1}$) of countable models of $T$, with $M_0$ the prime model$^{21}$ then $M_\omega$ is isomorphic to the generic structure $G_{\mu,V}$ $[BP20$, Lemma 5.29]. One might think each $M_n$ is prime with an acl-basis of cardinality $n$; we now show this is true when $acl(\emptyset)$ is infinite. However, in Section 5 we provide choices of $T_{\mu'/V}$ where $M_0$ has dimension 2 and so $M_n$ has dimension $n + 2$.

The cycles (using only partial lines of length three) played an important role in $[BP20]$. We constructed the $2^{80}$ distinct theories $T_\mu$ in $[BP20$, Lemma 4.11], by showing (in the vocabulary $\tau = \{R\}$) there were a countable family of $4n$-cycles (actually back to the same element) that are mutually non-embeddible and 0-primitive over 2-element sets. The choice of $\mu$ determines which of these cycles are realized. Varying the argument slightly shows as $s$ increases the $\gamma_s$ (Definition 4.1.3(1b)) induce infinitely many mutually non-embeddible primitives in $K_\mu$ over a two element set that is strongly embedded. We also noted in $[BP20$, Lemma 4.11] that there are infinitely many mutually non-embeddible primitives in $K_\mu$ over the empty set and similarly over a 1-element set. $U^{\infty}$ allows $\mu$ that forbid the realization of specific good pairs $B/\emptyset$. In $[BP20]$, we showed the algebraic closure of the empty set was infinite if the generic contained

$^{20}$Switch $a$ and $b$ in the subscript.

$^{21}$The prime model of $T$ is the unique model that can be elementarily embedded in each model.
a copy of the Fano plane, – the unique 7-element projective plane, \( F \). So setting \( F \) as the collection of \( \mu \in \mathcal{U} \) with \( \mu(F/\emptyset) > 0 \) guarantees \( acl_M(\emptyset) \) is infinite for any \( M \models T_\mu \). We retain the name \( F \) but make it a much larger subset of \( \mathcal{U} \).

**Notation 4.2.2.** Let \( \mathcal{F} \) be the set of \( \mu' \in \mathcal{U}^{ls} \) such that \( \mu(C/\emptyset) > 0 \) for some good pair \( C/\emptyset \).

**Lemma 4.2.3.** If \( \mu' \in \mathcal{F} \) and \( M \models T_{\mu',V} \) then \( acl_M(\emptyset) \) is infinite.

**Proof.** It is easy to see that any \( C \) that is 0-primitive over \( \emptyset \) must contain two intersecting lines so three non-collinear points exist. Noting that the only use in Lemma 5.27 of [BP20] of the assumption that the Fano plane is imbedded in \( M \) is to guarantee that there are three non-collinear point in a subset of \( M \) that is 0-primitive over \( \emptyset \), we get an infinite algebraic closure here. The construction of an infinite tower of 0-primitive extensions uses only that \( \mu \in \mathcal{U}^{ls} \).

In Section 5.1, we give several examples of strongly minimal quasigroups where the dimension of the prime model is 2.

**Lemma 4.2.4.** If \( M \models T \) with \( acl_M \neq (\emptyset) \) there are infinitely many disjoint (over the finite icl_\(N\)(ab)) finite cycles in \( G_N(a,b) \), where \( N \) is a copy of the prime model of \( T_\mu \) with \( N \supseteq \{a,b\} \).

**Proof.** Such an \( N \) exists by Lemma 4.2.3. Fix \( D \) as icl_\(M\)(a,b). For each \( i \) there is a pseudocycle \( C_i \) that is a primitive extension over icl(ab) based on ab with length \( 4i \). The structure with domain \( D \cup C_i \) is denoted \( A_i \). Since \( \mu(C_i/ab) \geq \delta(ab) = 2 \), there is an embedding of \( A_i \) into the saturated (also generic) model \( M_\infty \). But \( N \preceq M_\infty \) and is algebraically closed so the image of \( C_i \) is in \( N \). Now, since the \( C_i \) are 0-primitive over icl_\(N\)(ab) \( \subseteq N \), the \( A_i \) are disjoint over \( D \).

**QUESTION 4.2.5.** Can the prime model contain an infinite chain? Is there any decomposition by chains of the prime model? Compare these questions with the alternative decomposition of the prime model by taking the union of tree decompositions by normal subsets in [BV21].

**QUESTION 4.2.6.** By using the more radical alterations of the construction as in Section 5.1, can we have all cycles in the prime model finite by insisting exactly one isomorphism type of a pseudocycle is consistent, say, a 4-pseudocycle?

### 4.3. Over acl(ab) all paths are infinite

We study those paths in \( G_M(a,b) \) that are generated by \( d_1 \not\in acl_M(a,b) \). We justify in Lemma 4.3.2 the following notation:

**Notation 4.3.1.** For \( d_1 \not\in acl_M(a,b) \), \( P_{abd_1} \) (\( P^{e}_{abd_1} \)) denotes the (envelope of) the longest path generated by beginning with \( ad_1 \). This path may be infinite.

**Lemma 4.3.2.** Suppose \( d_1 \not\in acl(a,b) \).

1. \( d(d_1/ab) = 1 \); the path generated by \( d_1 \) is infinite.
2. Distinct a-edges in the path \( P_{abd} \) cannot intersect; but each a-edge intersects \( q - 1 \) b-edges.
3. If \( P_{abd} \) is an infinite path then for every \( X \subseteq P^{e}_{abd} \), \( d(X/ab) = 1 \).
4. If \( P_{abd} \) an infinite path there is exactly one e on \( P^{e}_{abd} \) that is on an a-line and \( P_{bae} \) is an infinite path (Recall Definition 4.1.4.2).
Proof. 1) If \( d_1 \in M - acl(a,b) \), \( d_1/ab = 1 \); otherwise \( d_1 \in acl(ab) \). If \( P_{abd} \) is finite, it is because some \( C \subseteq P_{abd} \) is a pseudocycle. But the \( \delta(C/ab) = 0 \) and \( d_1 \in acl(ab) \).

2) If \( ad_2 \) is a line in \( P_{abd} \) then for any element \( x \in ad_2 \), \( a \ast x \in ad_2 \). But for each of the \( q - 1 \) non-trivial star terms, \( t(x, y) \), \( b \ast (t(a, d_1)) \) generates a new line.

3) Suppose (without loss) that \( d_1 \subseteq X \subseteq P_{abd}^n \) and \( d(X/acl(ab)) = 0 \). Then \( d_1 \in acl(ab) \). Two paths generated by distinct \( d_i \notin acl(ab) \) can intersect in one point; \( d(P_{abd} \cup P_{abd}^n) = 1 \). But if there are two points of intersection \( d_1 \in acl(ab) \).

4) For any such \( e \) there is a line determined by \( b, b \ast e \). But this line generates an infinite path only if \( d(e/ab) = 1 \). Now apply 2).

With these results in hand we see that actually \( a, b, d_1 \) generate a fan of lines.

**Definition 4.3.3.** The fan generated by \( abd_1 \) is defined by induction.

1) \( F_{abd_1}^n \) consists of all points on envelopes of paths generated by a line \( ae \) where \( e \) is on an \( a \) edge of \( P_{abd_1} \) or by a line be with \( e \) on an \( a \)-edge of \( P_{bea}^{n-1} \);

2) \( F_{abd_1}^{n+1} \) consists of all points on envelopes of paths generated by lines \( ae \) where \( e \) is on an \( a \) edge of \( P_{abd_1}^n \) or by a line be with \( e \) on an \( a \)-edge of \( P_{bea}^n \);

3) The fan \( F_{abd_1} = \bigcup_{n < \omega} F_{abd_1}^n \).

Note that \( F_{ab} = F_{ba} \ast f \) if \( e \) and \( f \) are both on the same line in \( M - acl(a,b) \) through \( a \) (or through \( b \)).

As in Lemma 4.3.2 we see immediately that if two fans intersect in a single point their union is a larger (not definable) subset of rank 1:

**Lemma 4.3.4.** Two fans can intersect in at most one point.

**Theorem 4.3.5.** If \( M \) is countable and \( dim(M/N) = 1 \), then for any \( a, b \in M \), \( M \) is a union of fans over \( N \). Inductively, the conclusion applies to any \( M' \succ M \).

Proof. Let \( \langle e_i : i < \omega \rangle \) enumerate \( N - M \). Fix any \( a, b \in N \), choose \( e_0 = d_1 \in M - N \) and let \( F_0 \) be the fan \( F_{abd_1} \). Now for each \( n \), let \( d_{n+1} \) be \( e_j \) for the least \( j \) such that \( e_j \notin N \cup F_n \). Clearly \( \bigcup_{n < \omega} F_n \cup N = M \). Since the dimension \( N/M = 1 \), there will be algebraic relations among the fans. However, any two can intersect in at most one point and by construction there graph edges (a or b lines) that are not in one of the listed fans. However, many instance of \( R \) are not in the graph.

### 4.4. No Perfect Path graphs

Cameron and Webb [CWL2] extend to infinite structures the notion of a perfect Steiner triple system as one in which each cycle graph \( G(a,b) \) is a single cycle. They find \( 2^{2^{\aleph_0}} \) countable such Steiner triple systems. In line with Definition 4.1.1 we can extend this definition to any \( q \)-block algebra. However, we show none of the \( q \)-Steiner systems satisfying a \( T \) obeying Assumption 1.0.1 are perfect. Clearly there can be no uncountable perfect Steiner \( k \) system in any reasonable sense since whatever replaces ‘cycle’ will be countable. We will take the weakest plausible notion, which includes a single path or a fan; we show no such complex covers \( M - acl(ab) \), when \( M \models T \). In Theorem 4.3.5 we covered \( M - N \) by at most \( |M - N| \) fans, but not finitely many.

**Definition 4.4.1 (Perfect).** If \( (M, *, R) \models T \) we say \( (M, *, R) \) is a perfect \( q \)-Steiner system if for some finitely generated \( R \)-closed set (Definition 2.1.2) \( X = M - acl(ab) \).
Since every line in a Steiner system associated with a \( q \)-Steiner system is two-generated as a quasigroup, we can think of \( R \)-closure as finding the generated sub-quasigroup. Omer Mermelstein suggested the key idea for the proof for the following result.

**Lemma 4.4.2.** If \( M \) is a model of \( T \), \( A \subseteq M \), and \( |M - A| \) is infinite, then \( M \) has infinite \( R \)-dimension.

**Proof.** We first show that if \( C \) is 0-primitive over \( A \) and \( a \notin C \cup A \), \( A^* \), the \( R \)-closure of \( Aa \), does not intersect \( C \). Note by induction that every finite \( E \subseteq (A^* - A) \) satisfies \( \delta(E/A) = 0 \). Now, fix an enumeration \( A^* \) such that \( e_j \in cl_R\{e_i : i < j\} = E_j \). Suppose for contradiction \( A^* \cap C \neq \emptyset \) and choose the least \( k \) with \( e_k \in C \cap A^* \). But then \( e_k \) witnesses an edge between \( C \) and \( E_j + 1 \); this implies \( \delta((E_j \cup C)/A) < 0 \), contrary to hypothesis.

There are infinitely many incomparable 0-primitives \( C_j \) over \( A \) ([BP20, Lemma 4.11]; choose successively, a seed \( a_j \) in each \( C_j \). Applying the first paragraph, we see the \( cl_R(Aa_j) \) are mutually disjoint. By constructing \( (A_j, A^*_j) \) by the procedure of the last paragraph, we witness infinite \( R \)-dimension.

Since a perfect Steiner system is the \( R \)-closure of finitely many elements, we have immediately from Lemma 4.4.2:

**Corollary 4.4.3.** If \( (M, *, R) \models T \) satisfies Assumption 4.0.1, \( (M, *, R) \) is not a Steiner perfect system.

**QUESTION 4.4.4.** In [BV21], we show the definable closure of a strongly minimal system \( (M, R) \) is essentially unary if \( T_\mu \) is triplable (for any primitive \( C/B \), \( \mu(C/B) \geq 3 \)). In the expanded vocabulary \( \tau' \), models of \( T_{\mu', V} \) have \( * \) as a non-trivial binary function. But, assuming \( \mu' \) is triplable, are there any binary functions that are not polynomials in \( * \)?

5. Uniform Path graphs and 2-transitive structures

In Section 4, we studied theories \( T \) which satisfied the properties of \( T_{\mu', V} \) of quasigroups built by a Hrushovski style construction as in Section 3 where \( \mu' \in \mathcal{U}_\tau \) and for any \( M \models T_{\mu', V} \), \( acl(M(\emptyset)) \neq \emptyset \). Unlike the previous section, we now make major modifications to the construction to consider subsets where algebraic closure has few pseudo-cycles and to find 2-transitive structures. Thus, we return to the complicated notation \( T_{\mu', V} \) to clarify where the construction is changing.

In Section 4.3 we found examples where all cycles were infinite when we took the domain of the path graph as \( M - acl(ab) \). But in Section 4.2 with domain \( M_0 - icl(ab) \) we always had finite cycles and the existence of infinite cycles in the prime model is an open problem. In this section we restrict our attention to the domain, \( M - acl(ab) \). We first (Section 5.1) modify the construction to be able to specify which, if any, finite cycles occur. In Section 5.2 we introduce the notion of a uniform (The isomorphism type of \( G_M(a,b) \) does not depend on the choice of \( a,b \)) \( q \)-Steiner system (generalizing [CW12, CGGW10]). Then by different methods in Sections 5.3 and 5.4 we construct families of 2-transitive and hence uniform \( q \)-Steiner systems.

We use two model theoretic methods to solve some problems suggested from the study of cycle graphs in [CW12]. These methods modify the theory \( T_{\mu', V} \) either by
changing $\mu$ or, more drastically, restricting the class $K_0$ of finite structures. And then we combine the two in Section 5.3.

5.1. All paths are infinite

In this section, we find $T_{\mu^{\prime\prime}, V}$ whose models have no finite cycles. It is then easy to allow certain specified finite lengths of cycles. The key point here is to vary the class $U_{\tau}$ from Definition 3.5.4 maintaining the amalgamation so the resulting generic model is strongly minimal but preventing finite cycles. As in Section 3, we work in a vocabulary $\{H, R\}$, where $R$ is collinearity in a linear space and model $H$ is the graph of a quasigroup operation $\ast$. We introduce a set $B$ of $\mu^{\prime\prime}$ obtained by modifying $\mu \in U_{\tau}$ to $\mu^{\prime\prime}$ by changing the value only on the isomorphism types good pairs $C/\{a, b\}$ which are pseudo-cycles. As $B$ and $U_{\tau}$ differ on pseudocycles, apparent contradictions between here and Section 4 are resolved.

**Definition 5.1.1.** Recall from Definition 3.4 that $\gamma_n$ denotes an isomorphism type of a pseudo-cycle over a two element set. Let $B$ denote the set of $\mu^{\prime\prime}$ obtained by for every $n$, redefining each $\mu \in U_{\tau}$ to $\mu^{\prime\prime}$ by setting $\mu^{\prime\prime}(\gamma_n) = 0$ for each $n$.

We define a class $K^{\prime\prime}_{\mu^{\prime\prime}, V}$ whose generic has only infinite cycles. Thus there are no finite cycles in any model of $T_{\mu^{\prime\prime}, V}$.

**Lemma 5.1.2.** If $\mu^{\prime\prime} \in B$, for each $q, V$, the class of $\tau^{\prime\prime} = \{\ast, R\}$-structures $K^{\prime\prime}_{\mu^{\prime\prime}, V}$ from Definition 3.4 has the $\leq$-amalgamation property. If $\mu^{\prime\prime} \in B$, every model of $T_{\mu^{\prime\prime}, V}$ has only infinite cycles.

**Proof.** We must check that we can complete the amalgamation while insisting that for each $n$, $\gamma_n$ is omitted. For this we must slightly vary the proof of Lemma 5.10 in [BP20], whose notation we follow. Let $F, E \in K^{\prime\prime}_{\mu^{\prime\prime}, V}$. Now, let $G = E \oplus D F$, where $(D, E)$ is a good pair (with $|E - D| > 1$) and $((a, b), C_k)$ is a good pair witnessing $\gamma_k$ (So $C_k$ is a pseudo-cycle.). The difficulty is that the good pair $(C_k/B)$ does not satisfy the requirement $\mu(C_k/B) \geq \delta_{\tau^{\prime\prime}}(B)$. We gave a separate argument to show no $\gamma_k$ blocks amalgamation; the result then follows without change. There are no realizations of the good pair $\gamma$ in any of $D, E, F$; we must show it is not realized in $G$. The crux is that, by definition of $((a, b), C_k)$, for any $k, i$, each $c_i \in C_k$ is on a separate triple in $R$ with each of $a$ and $b$. Now if $(a, b) \subseteq F$ (compare Case B.1 of [BP20]), each $C_i$ must be contained in $F$ or else there is a clique $(ac_i c_{i+1})$, modulo renaming, with two elements in $F$ and one in $E - F$ contradicting the primitivity of $E$ over $D$. If one of $a, b$, say $a$ is in $E - F$ then for each $i$, $C_i \subseteq E$ or the line between $a$ and $c_i$ is based in $D$ (Definition 3.11 of [BP20]) and that is clearly impossible, since it contradicts that $E$ is primitive over $D$; so each $C_i \subseteq B$. But now, since $E$ doesn’t realize $\gamma_n$, $b$ must be in $F - D$ and $C_i \cap (E - D) \neq \emptyset$; we get the same contradiction. So $C_i \subseteq B$. But now $a \in E - D$ is on a line based on $C_i \subseteq D$, contradicting the primitivity of $E$ over $D$. Thus for any $M \models T_{\mu^{\prime\prime}, V}, a, b \in M$ and $d_i \notin \text{icl}(ab)$, $P_{abd_i}$ is infinite. So we finish.

A simple variant on the argument for Corollary 5.3 of [BP20] (Replace ‘for every $n$’ in Definition 5.1.1 by ‘for $n \in X^{o^\prime}$’.) shows we can omit arbitrary sets of $\gamma_n$.

**Theorem 5.1.3.** For any $X \subseteq \omega$ of numbers divisible by 4 and $\mu \in U_{\tau}$, we can construct still another variant $\mu^X$ of $\mu$ such that models of $T_{\mu^X, V}$ realize an $n$-pseudo-cycle if and only if $n \in X$. 
One cannot simply modify $\mathcal{U}$ to say all points have trivial algebraic closure and carry out the amalgamation argument. Omer Mermelstein provided the following counterexample, showing some restriction, such as to the $\gamma_n$, is necessary for Lemma 5.1.2. Here is an amalgamation diagram where the good pair $C/B$ does not appear in any of the components but is in the amalgam. Nevertheless, we give several examples in later sections where $\text{acl}_{M_0}(\emptyset) = \emptyset$.

**Example 5.1.4.** Let $B$ consist of five points $a, b_1, \ldots, b_4$ and $C$ consist of four points $c_1, \ldots, c_4$, where $R(c_i, b_i, c_{i+1})$ for $i = 1, \ldots, 3$, $R(a, b_2, b_3)$, and $R(c_4, c_1, b_4)$. Then $C$ is $0$-primitive over $B$. But now if we let $D_1 = \{a, c_2, c_4\}$, $D_2 = \{b_1, c_1, b_4\}$ and $D_2 = \{b_2, c_3, b_3\}$ we have $D_1 \subseteq D_1$ and $D_1 \subseteq D_2$, but $BC$ appears in the amalgam.

### 5.2. Uniform $G(a, b)$

[CW12] call a Steiner system uniform if all the cycle graphs $G_M(a, b)$ are isomorphic. [CGGW10] construct $2^{8\mu}$ countable uniform sparse Steiner triple systems. We obtain $2^{8\mu}$ families of countable uniform infinite Steiner systems for each prime power $q$.

We adapt the Cameron-Webb notions of uniform [CW12] to accommodate $q$-Steiner systems. Recall (Definition 4.1.1) that the domain of $G_M(a, b)$ is $M - \text{icl}(a, b)$. We will consider cases where $\text{acl}(a, b)$ is both finite and infinite.

**Definition 5.2.1 (Uniform).** We say a model $(M, *, R)$ of $T_q^n_{\mu, 2}$ is uniform, if for any $(a, b), (a', b')$, $G_M(a, b) \simeq G_M(a', b')$.

Here is a sufficient condition for uniformity.

**Lemma 5.2.2.**

1. If $(M, *, R)$ is a model of a theory $T$ generated by a Hrushovski class (Definition 1.0.7) of linear spaces such that every two element set $A$ satisfies $A \subseteq M$, the automorphism group of $(M, *, R)$ acts 2-transitively on $(M, R)$.

2. Clearly, if the automorphism group of $(M, *, R)$ acts 2-transitively on $(M, *, R)$, $(M, *, R)$ is uniform.

**Proof.** Since all pairs $(a, b)$ are isomorphic and each is embedded strongly in the generic $G$, the result is immediate for $G$. But this transitivity extends to all models since if one model of a complete theory has a single 2-type, all models do. And, each model of a strongly mimimal theory is finitely first order homogeneous (finite sequences realizing the same first order type are automorphic) (e.g. [BL71] Theorem 5).

### 5.3. 2-transitive $M$, 3-Steiner systems, Changing $K_0$

In Section 5.1 we showed that, by modifying the set of possible $\mu$, we could ensure that there were no finite pseudo-cycles. The Steiner system in Section 5.1 was far from uniform as there were many 2-types, e.g. pairs with non-isomorphic algebraic closures. (We only restricted those primitive extensions that were pseudo-cycles.)

We have dealt with two variants of the Hrushovski construction. Recall that in the linear space case we used $K_0$ to play the role of $L_0$ in Notation 1.0.1. We constructed generics in both $\tau$ and $\tau'$, with the same basic construction. But in the more general context of Definition 1.0.1 we can restrict $K_0$ before beginning the construction and realize the hypothesis of the general statement of Lemma 5.2.2.1.

In Section 5.2 of [Hru93], Hrushovski proves there are $2^{8\mu}$ strongly minimal $\tau$-structures with pairwise non-isomorphic associated combinatorial geometries. He
achieves this by ensuring that algebraic dependence of a triple \(a, b, c\) is equivalent to \(R(a, b, c)\). Mermelstein pointed out to me that these structures are in fact Steiner triple systems. We will see that they are 2-transitive and every cycle is infinite.

**Example 5.3.1.** [Hru93, Example 5.2] We denote the theories described in this example by \(T_{H,\mu}\). The dimension function \(\delta_H\) is the usual: \(\delta_H(A) = |A| - |R|\), where \(|R|\) is the number of 3-element subsets of \(A\) satisfying \(R\) and strong submodel is defined in usual way. The novelty was in use of the \(\delta\)-condition to define \(K^H_0\). Namely, the collection of finite structures \(C\) such that every subset \(B\) of \(C\) with power at most 3 is strong in \(C\):

\[ (\ast) \quad K^H_0 = \{ A : B \subseteq A \land |B| \leq 3 \rightarrow B \leq A \}. \]

Since the amalgamation of Hrushovski’s basic example added no edges, this subclass also has amalgamation by the same amalgam. For each \(\mu\), \(K_{H,\mu}\) is to \(K^H_0\) as \(K_\mu\) is to \(K_0\) (Definition 1.3.3).

We obtain a linear space by interpreting \(R\) as collinearity. Two points determine a line as \(R(a, b, c) \land R(a, b, c) \land \neg R(a, b, d)\) makes \(\delta(\{a, b, c, d\}) = 2 < \delta(\{b, c, d\})\). Since any non-trivial 0-primitive over a two element set contains 3 non-collinear points, \((\ast)\) implies the algebraic closure of two points is the third point on the line they determine. Thus there are two quantifier-free configuration of three points: dependent, independent. Since, by \((\ast)\), both configurations are strong in the generic, they determine by homogeneity, as in Lemma 5.2.2, the two possible 3-types. Similarly property \((\ast)\) of this Hrushovski example makes it a Steiner triple system \(^{22}\).

Here we write cycle since we are dealing with a Steiner-triple cycle and no path can be a proper pseudo-cycle as opposed to a cycle.

**Fact 5.3.2.** For any \(\mu\) and any \((M, R) \models T_{H,\mu}\), \((M, R)\) is a strongly minimal uniform Steiner triple system. In fact, the algebraic closure of any pair is the third point on the line through \(a, b\) and so each cycle is infinite.

**Proof.** As noted in the description of Example 5.3.1 in \((M, R)\) the algebraic closure of a pair is the line through them. Since there are only two 3-types of tuples extending \((a, b)\), any two \(d_i\) that are not on the line \(ab\) are isomorphic over \(a, b\) and thus the cycles they generate are isomorphic. The last claim is immediate since all points not on the line are automorphic over \(ab\). Since any potential finite pseudo-cycle over \(a, b\) is in \(\text{acl}(ab) = \{a, b, c\}\), where \(R(a, b, c)\), there are no finite pseudo-cycles.

### 5.4. 2-transitive \(q\)-Steiner systems; Changing \(K_0\) and \(U\)

We turn to a different method\(^{23}\) to obtain uniformity results for Steiner \(q\)-systems for any prime power \(q \geq 3\) and to restrict the number of finite cycles. We combine

\(^{22}\)This example will not permit lines with longer length by modifying \(\mu\). As, there can be no 4-clique, \(\ell\), since with the Hrushovski definition \(\delta(\ell) = 0\) while \(\delta\) of two points is 2.

\(^{23}\)This approach of restricting primitives over very small sets to establish various amounts of transitivity of the non-Desguaresian plane appears in [Hru93] [Bal95].
a variant of the Hrushovki’s Example 5.3.1 with modifying \( \mu \) to control a second fundamental invariant: number of cycles.

**Definition 5.4.1.** We write \( K^J_0 \) for the class of linear spaces such that

\[(**) \quad |B| \leq 2 \implies B \leq A\]

for every finite linear space \( A \in K^J_0 \) containing \( B \). We write \( K^J_{\mu''}V \) for the class determined by \( **, \mu'' \in B \) (Definition 5.1.1) and \( V \), a Mikado variety of quasigroups.

As in Example 5.3.1 \((**)\) and Lemma 5.2.2 imply every two element set is strong, so each model is 2-transitive. There are two differences from Example 5.3.1: i) the strong substructure notion is with respect to the \( \delta \) in \([BP20]\) and so we can vary the line length; ii) we don’t kill the entire (non-trivial) algebraic closure of each 2-element set but explicitly forbid only the finite cycles. We note below that we can allow finitely many cycles over each pair \((a,b)\).

**Theorem 5.4.2.** If \( \mu \in B \) (Definition 5.1.1),

\[K^J_{\mu''}V\]

has amalgamation, the generic (and hence every model) has no finite paths and is 2-transitive so the path graph is uniform.

**Proof.** The amalgamation follows *mutatis mutandis* from Lemma 5.1.2. Note that \((**)\) implies every two element set is strong, so each model is 2-transitive. This holds in every model by Lemma 5.2.2 hence \( G_M(a,b) \) is uniform. Finite paths are blocked, since \( \mu \in B \).

As we modified Lemma 5.1.3 we modify the proof of Theorem 5.4.2 to get:

**Theorem 5.4.3.** If \( \mu'' \in B \) then for any variety \( V \) and for any model \((M, *, R)\) of \( T^J_{\mu''}V \) and any \((a,b)\), both \( acl_M(\emptyset) = \emptyset \) and \((M, *, R)\) is uniform.

Further, for any finite set \( X \) of pairs, \((n_i, m_i)\) with \( n_i \) divisible by 4, we can construct a theory \( T^J_X \) such that if \((M, *, R) \models T_X\) and \((a,b) \in M\), \( G_m(a,b) \) has \( m_i \) cycles of length \( n_i \).

### 6. Questions

We close by suggesting some more traditional combinatorial questions suggested by the examples here.

**QUESTION 6.0.1.** We have studied path graphs in strongly minimal \( q \)-Steiner systems induced by quasigroups. But our definition has no reliance on strong minimality, although our arguments do. What can be learned by more traditional combinatorial methods about the structure of path graphs in arbitrary finite or infinite \( q \)-Steiner systems induced by quasigroups?

**QUESTION 6.0.2.** We built strongly minimal quasigroup that induce \( q \)-Steiner systems using the \( \mu \)-function only with respect to collinearity. Suppose one moves closer to the setting of \([HW21]\). Can one construct an infinite quasigroup by considering finite quasigroups from a Mikado variety (Definition 3.2) while omitting specified finite configurations as in \([BP20]\)? It seems each case would require its own variant on amalgamation. Is there a way to recover the local finiteness of the generic as in \([BC19]\)? If so, what is the model theoretic complexity of the resulting theory?
The last question depends on understanding the Lenz-Barlotti classification.

**QUESTION 6.0.3.** In Bal94 (using the methods of Section 5.4) a Morley rank 2 $\aleph_1$-categorical non-desarguesian projective planes is coordinatized by a ternary ring that is not linear. The non-linearity means that while the quasi-groups for both addition and multiplication are definable, they cannot be composed to give the ternary $t(x, y, z) = xy + z$ that arises in a division ring. That is, the plane is at the lowest level in the Lenz-Barlotti hierarchy. Could similar but less radical surgery yield $\aleph_1$-categorical non-desarguesian projective planes that are higher in that hierarchy?

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