The Structure of AdS Black Holes 
and
Chern Simons Theory in 2+1 Dimensions

Sharmanthie Fernando and Freydoon Mansouri

Physics Department, University of Cincinnati, Cincinnati, OH 45221

Abstract

We study anti-de Sitter black holes in 2+1 dimensions in terms of Chern Simons gauge theory of the anti-de Sitter group coupled to a source. Taking the source to be an anti-de Sitter state specified by its Casimir invariants, we show how all the relevant features of the black hole are accounted for. The requirement that the source be a unitary representation leads to a discrete tower of excited states which provide a microscopic model for the black hole.

1 introduction

The BTZ solution [1] provides a concrete and manageable theoretical framework for testing various hypotheses concerning classical and quantum black holes. As a result, it has been studied extensively, as can be traced, e.g., from the review article by Carlip [2]. The main objective of this work is to study how this solution can be obtained from the Chern Simons gauge theory of the anti-de Sitter (AdS) group coupled to a source, and the new and significant microscopic consequences which emerge from such a formulation. Interesting attempts linking the Chern Simons theory to the BTZ black hole already exist in the literature [3, 4]. However, a number of issues in this connection need further clarification. It will be recalled that in the BTZ formulation the black hole is a solution of vacuum Einstein equations with a negative cosmological constant. It differs from the standard AdS space by certain identifications related to a discrete subgroup of the AdS group, which changes the global topology. On the other hand, the Chern Simons theory [5, 6] is defined on

*email address: fernando@physung.phy.uc.edu
†email address: Mansouri@uc.edu
a manifold $M$ with topology $R \times \Sigma$, where $\Sigma$ is a two dimensional space. We take the theory to be an explicit realization of the Mach Principle, so that in the absence of sources the field strengths vanish and the topology is trivial (no punctures). One can then associate non-trivial topologies to the presence of sources [5, 7, 8]. In this scenario, the physical (metrical) space-time is the output of such a gauge theory and should not be confused with the manifold $M$. The physical space-time is related to a manifold $M_q$ the points $q_A$ of which are one of the canonical variables ($0 + 1$ dimensional fields) of the source(s) [5, 7]. The presence of sources in $M$ affect not only the topology of $M$ but also the structure of $M_q$ as the emerging space-time. It is therefore no contradiction to state that a Chern Simons theory in $M$ (with a source) leads to the black hole solution in $M_q$ (with no source).

One of the notable advantages of the Chern Simons approach is that it allows us to express the asymptotic observables of the theory in terms of the properties of the sources. To implement this idea, we must identify a localized source (particle) with an irreducible representation of the gauge symmetry group [8]. For the present problem, this will amount to relating the asymptotic observables of the BTZ black hole to the Casimir invariants of an AdS state coupled to the Chern Simons action. We will show that the emerging space-time will naturally arise from such a theory and will have all the ingredients necessary for the AdS black hole [4]. These include, in particular, the discrete subgroup underlying the identifications. Moreover, the horizon radii of the BTZ solution are complicated functions of the familiar AdS labels $M$ and $J$, which are commonly referred to as “mass” and “angular momentum”, respectively. One might wonder if there is a group theoretic or some other explanation for their functional form. We will show that they are alternative labels for an AdS state and arise naturally from the maximal compact subgroup of the AdS group via induced representations.

An important consequence of the Chern Simons formulation, which we will address in this work is the extent to which the potential quantum aspects of the formalism will influence the choice of the AdS representations. As mentioned above, we take the sources which couple to the Chern Simons action to be AdS states, so that, to have a unitary quantum theory, these states must be unitary representations of the AdS group. One of the remarkable byproducts of this requirement is that the ground state and the excited states of the black hole form a discrete spectrum. Therefore, the Chern Simons theory described below provides a microscopic model of the black hole structure, which appears to be distinct from previous suggestions [10, 11].

In Section 2, we review the properties of AdS space and algebra in a form which will be used in subsequent sections. In Section 3, we express the Chern Simons action for the AdS group in an $SL(2, R) \times SL(2, R)$ basis. Section 4 is devoted to the interaction with sources. Among other things, we discuss the important role played by the constraints in relating the invariants which label the sources to the asymptotic observables of the coupled theory. In Section 5, we explore the consequences of requiring that a source be represented by a unitary representation of the AdS group.
We will show that one of the hitherto unexplained features of the BTZ black hole emerges from this requirement. In Section 6, we show how the black hole space-time emerges from the Chern Simons gauge theory described in sections 3 through 5. In particular, we show how such features as the periodicity of the angular coordinate and the discrete identification group are accounted for. Section 7 is devoted to further discussion of the results and their possible relevance to black holes in other space-times dimensions.

2 Anti-de Sitter space and algebra

The anti-de Sitter space in 2+1 dimensions can be viewed as a subspace of a flat 4-dimensional space with the line element

\[ ds^2 = dX_A dX^A = dX_0^2 - dX_1^2 - dX_2^2 + dX_3^2 \] (1)

It is determined by the constraint

\[ (X_0)^2 - (X_1)^2 - (X_2)^2 + (X_3)^2 = l^2 \] (2)

where \( l \) is a real constant. The set of transformations which leave the line element invariant form the anti-de Sitter group \( SO(2,2) \). It is locally isomorphic to \( SL(2,R) \times SL(2,R) \) or \( SU(1,1) \times SU(1,1) \). From here on by anti-de Sitter group we shall mean its universal covering group.

The AdS algebra consists of the elements \( M_{AB} \) satisfying the commutation relations

\[ [M_{AB}, M_{CD}] = i (\eta_{AD} M_{BC} + \eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC}) \] (3)

With \( A = (a, 3) \) and \( a = 0, 1, 2 \), we can write the algebra in two more convenient forms:

\[ M^{ab} = \epsilon^{abc} J_c = \epsilon^{abc} (J_+ - J_-) \]

\[ M^{a3} = l \Pi^a = (J^a - J^{-a}) \] (4)

where

\[ \epsilon^{012} = 1; \quad \eta^{ab} = (1, -1, -1) \] (5)

Then, the commutation relations in these bases take the form, respectively,

\[ [J^a, J^b] = -i \epsilon^{abc} J_c; \quad [J^a, \Pi^b] = -i \epsilon^{abc} \Pi_c; \quad [\Pi^a, \Pi^b] = -il^{-2} \epsilon^{abc} J_c \] (6)

\[ [J^+_a, J^-_b] = -i \epsilon^{ab}_{\ c} J^c_c; \quad [J^-_a, J^+_b] = 0 \] (7)

The Casimir operators look simplest in the latter basis:

\[ j^2 = \eta^{ab} J^+_a J^-_b \] (8)
In the other bases, they have the form,

\[ M = l^2 (\Pi^a \Pi_a + l^{-2} J^a J_a) = 2 (j_+^2 + j_-^2) \]
\[ J/l = 2 \Pi_a J^a = 2 (j_+^2 - j_-^2) \]

We will use the same symbols for operators and their eigenvalues.

An irreducible representation of AdS group can be labeled by the eigenvalues of either the pair \((M, J)\) or the pair \((j_+, j_-)\). For our applications, it is often advantageous to use a third set of labels which we denote by \((H, S)\). They correspond to the maximal compact subgroup \(SO(2) \times SO(2)\) of \(SO(2, 2)\), which is generated by \(J^0\) and \(\Pi^0\). The labels \((H, S)\) are a natural choice from the point of view of the theory of induced representations. This can be seen from the comparison with the more familiar situation in the Poincaré group which can be obtained from anti-de Sitter group in the limit \(l \to \infty\). From here on, we will use the labels, \((j_+, j_-)\), \((M, J)\), and \((H, S)\) interchangeably. The last two are related to each other according to

\[ M = l^2 H^2 + S^2; \quad J/l = 2lHS \]

(10)

Note that in order for \(M\) to assume negative values, \(H\) and \(S\) must, in general, be complex.

To see the relevance of \(H\) and \(S\) to the BTZ solution, let us express \(H\) and \(S\) in terms of the labels \(M\) and \(J\) by inverting Eqs. (10). We obtain

\[ H^2 = \frac{1}{2l^2} M \left[ 1 + \sqrt{1 - \left( \frac{J}{lM} \right)^2} \right] \]

(11)

\[ S^2 = \frac{1}{2} M \left[ 1 - \sqrt{1 - \left( \frac{J}{lM} \right)^2} \right] \]

(12)

For \(M > 0\) and \(|J| \leq lM\), \(H\) and \(S\) are thus proportional to the horizon radii, \(r_{\pm}\), of the BTZ black hole [1]:

\[ r_+/l = lH; \quad r_-/l = S \]

(13)

3 Connection and the Chern Simons action

We begin by writing the connection in \(SL(2, R) \times SL(2, R)\) basis

\[ A_{\mu} = \omega^A_{\mu} M^{AB} = \omega^a_{\mu} J_a + e^a_{\mu} \Pi_a = A^+_{\mu} J_a^+ + A^-_{\mu} J_a^- \]

(14)

where

\[ A^\pm_{\mu} = \omega^a_{\mu} \pm l^{-1} e^a_{\mu} \]

(15)
Eqs. (14) and (15) should be viewed as definitions of \( e \) and \( \omega \) in terms of \( SL(2, R) \) connections. The covariant derivative will have the form

\[
D_\mu = \partial_\mu - iA_\mu = \partial_\mu - iA_\mu^+ J^+_a - iA_\mu^- J^-_a
\] (16)

Then the components of the field strength are given by

\[
[D_\mu, D_\nu] = -iF_\mu^a J^+_a - iF_\mu^- J^-_a = -iF_\mu^+ [A^+] - iF_\mu^- [A^-]
\] (17)

For a simple or a semi-simple group, the Chern Simons action has the form

\[
I_{cs} = \frac{1}{4\pi} \text{Tr} \int_M A \wedge \left( dA + \frac{2}{3} A \wedge A \right)
\] (18)

where \( \text{Tr} \) stands for trace and

\[
A = A_\mu dX^\mu = A^+ + A^-
\] (19)

We require the 2+1 dimensional manifold \( M \) to have the topology \( R \times \Sigma \), with \( \Sigma \) a two-manifold. So, The Chern Simons action with \( SL(2, R) \times SL(2, R) \) gauge group will take the form

\[
I_{cs} = \frac{1}{4\pi} \text{Tr} \int_M \left[ \frac{1}{a_+} A^+ \wedge \left( dA^+ + \frac{2}{3} A^+ \wedge A^+ \right) + \frac{1}{a_-} A^- \wedge \left( dA^- + \frac{2}{3} A^- \wedge A^- \right) \right]
\] (20)

Here the quantities \( a_\pm \) are, in general, arbitrary coefficients, reflecting the semisimplicity of the gauge group. Up to an overall normalization, only their ratio is significant. It was pointed out by Witten [5] that in the free Chern Simons theory the choice \( a_- = -a_+ \) would make the action proportional to Einstein’s action in \( M \) by imposing a metric structure on it. Similarly, the choice \( a_- = a_+ \) would give an “exotic” term. He also pointed out that, in our notation, for generic values of these coefficients, the classical equations of the free theory in \( M \) remain unchanged. In a quantum theory [5], the two terms in the action will have a relative arbitrary coefficient.

It would be tempting to choose the first possibility on grounds of familiarity, among other things. However, that would be an unnatural choice from the point of view pursued here. This is because the space-time which emerges from this theory is not the manifold \( M \) but a manifold \( M_q \) corresponding to one of the canonical variables (0 + 1 dimensional fields) of the source which will be coupled to the Chern Simons theory in the next section. So, the space-time is a secondary concept which emerges from the gauge theory, and Einstein’s action in \( M \) plays no direct role in it. Moreover, in the presence of a source (or of sources), any á priori choice of the coefficients \( a_\pm \) reduces the class of allowed holonomies, so that even the classical theory coupled to sources will be affected by such a choice. For these reasons, we will keep the coefficients \( a_\pm \) as free parameters in the sequel, so that we can generate the correct holonomies for solutions both outside and inside the horizon.
Under infinitesimal gauge transformations

\[ u_\pm = \theta^\pm a J_a^\pm \] (21)

the gauge fields transform as

\[ \delta A_\mu = -\partial_\mu u - i[A_\mu, u] \] (22)

More specifically,

\[ \delta A_\mu^\pm a = -\partial_\mu \theta^\pm a - \epsilon^a_{bc} A_-^b \theta^\pm c \] (23)

As we have stated, the manifold \( M \) has the topology \( R \times \Sigma \) with \( R \) representing \( x^0 \). Then subject to the constraints

\[ F_a^\pm [A^\pm] = \frac{1}{2} \eta_{ab} \epsilon^{ij} (\partial_i A_j^\pm b - \partial_j A_i^\pm b + \epsilon^b_{cd} A_i^\pm c A_j^\pm d) = 0 \] (24)

the Chern Simons action for \( SO(2, 2) \) will take the form

\[ 2\pi I_{cs} = \frac{1}{a_+} \int_R dx^0 \int_\Sigma d^2 x \left( -\epsilon^{ij} \eta_{ab} A_i^\pm a \partial_0 A_j^\pm b + A_0^\pm a F_a^+ \right) \]
\[ + \frac{1}{a_-} \int_R dx^0 \int_\Sigma d^2 x \left( -\epsilon^{ij} \eta_{ab} A_i^- a \partial_0 A_j^- b + A_0^- a F_a^- \right) \] (25)

4 Interaction with sources

Following the approach which has been successful in coupling sources to Poincaré Chern Simons theory \[8\], we take a source for the present problem to be an irreducible representation of anti-de-Sitter group characterized by Casimir invariants \( M \) and \( J \) (or \( H \) and \( S \)). Within the representation, the states are further specified by the phase space variables of the source \( \Pi^A \) and \( q^A \), \( A = 0, 1, 2, 3 \), subject to anti-de Sitter constraints.

For illustrative purposes, let us consider first the interaction term for a special case which is the analog of the Poincaré case \[8\] with the intrinsic spin set to zero.

\[ I_1 = \int_C d\tau \left[ \Pi_A D_\tau q^A + \lambda \left( q^A q_A - \ell^2 \right) \right] \]
\[ + \int_C d\tau \left[ \lambda_+ \left( J^{+a} J_a^+ - \ell^2 j_+^2 \right) + \lambda_- \left( J^{-a} J_a^- - \ell^2 j_-^2 \right) \right] \] (26)

where \( C \) is a path in \( M \), \( \tau \) is a parameter along \( C \), and the covariant derivative \( D_\tau \) is given by

\[ D_\tau = \partial_\tau - i\omega^{AB} M_{AB} \] (27)

The first term in this action is the same as that given in reference \[4\]. The second term ensures that \( q^A(\tau) \) satisfy the AdS constraint. It is not the manifold \( M \) over
which the gauge theory is defined but the space of $q$’s which give rise to the classical space-time. The last two constraints identify the source being coupled to the Chern Simons theory as an anti-de Sitter state with invariants $j_+$ and $j_-$. These constraints are crucial in relating the invariants of the source to the asymptotic observable of the coupled theory. In this respect, our action differs from that given in reference [4]. Although the word “constraints” was mentioned there in connection with this action, they were not explicitly stated or made use of in the sequel.

Using the standard (orbital) representation of the generators

$$M_{AB} = i(q_A \partial_B - q_B \partial_A) \quad (28)$$

we have

$$\Pi_C \omega^{AB} M_{AB} q^C = \omega^{AB}(q_A \Pi_B - q_B \Pi_A) = \omega^{AB} L_{AB} \quad (29)$$

Here $L_{AB}$ are c-number quantities transforming like $M_{AB}$. Breaking up this expression into $SL(2, R) \times SL(2, R)$ form just as was done $M_{AB}$, we get

$$\omega^{AB} L_{AB} = A^+ a^+ L_a + A^- a^- \quad (30)$$

So, the action $I_1$ can be written as

$$I_1 = \int_C d\tau \left[ \Pi_A \partial_\tau q^A - i \left( A^+ a^+ L_a + A^- a^- L_a \right) + \lambda \left( q^A q_A - l^2 \right) \right]$$

$$+ \int_C d\tau \left[ \lambda_+ \left( J^+ a^+ - i^2 j_+^2 \right) + \lambda_- \left( J^- a^- - i^2 j_-^2 \right) \right] \quad (31)$$

In this expression $L_a^\pm$ play the role of (c-number) generalized orbital angular momenta. If, in addition, the representation carries generalized intrinsic (spin) angular momenta, then $L_a^\pm$ would have to be replaced by $J_a^\pm$, respectively, where

$$J_a^\pm = L_a^\pm \oplus S_a^\pm \quad (32)$$

It is now clear how the interaction term $I_1$ can be generalized to the case when $S_a^\pm \neq 0$. We simply replace $L_a^\pm$ with $J_a^\pm$ in $I_1$ to get

$$I_s = \int_C d\tau \left[ \Pi_A D_\tau q^A + \lambda \left( q^A q_A - l^2 \right) \right]$$

$$+ \int_C d\tau \left[ \lambda_+ \left( J^+ a^+ - i^2 j_+^2 \right) + \lambda_- \left( J^- a^- - i^2 j_-^2 \right) \right] \quad (33)$$

This expression is identical in form to that given by Eq. 26. But now the generators are not limited to the form given by Eq. 28. It can be expressed in a form in which the $SL(2, R) \times SL(2, R)$ structure of the gauge group is transparent:

$$I_s = \int_C d\tau \left[ \Pi_A \partial_\tau q^A - (A^+ a^+ J_a^+ + A^- a^- J_a^-) + \lambda \left( q^A q_A - l^2 \right) \right]$$
\[ + \int_C d\tau \left[ \lambda_+ \left( J^{a+} J^{a+}_+ - i^2 j^{2}_+ \right) + \lambda_- \left( J^{-a} J^{-a}_- - i^2 j^{-2}_- \right) \right] \]  

(34)

In this expression, \( J^{\pm a} \) play the role of c-number generalized angular momenta which transform in the same way as the corresponding generators and which label the source. If there are several sources, an interaction of the form given by Eq. 35 must be written down for each source.

It is well known that for a Poincaré state with mass \( m^2 > 0 \), there is a (rest) frame in which, e.g., the momentum vector takes the form

\[ p^a = (p^0, \vec{p}) \rightarrow (m, 0) \]  

(35)

Similarly, in the present case, there is a frame such that when, e.g., the c-number quantity \( J^{\pm a} J^{\pm a}_\pm > 0 \), we have

\[ J^{\pm a} = (J^{\pm 0}, \vec{J}^{\pm}) \rightarrow (j^{\pm}, 0) \]  

(36)

In this gauge, \( SL(2, R) \times SL(2, R) \) symmetry reduces to \( SO(2) \times SO(2) \). One can use similar methods to choose a gauge in which the residual symmetry is, e.g., \( SO(1, 1) \times SO(1, 1) \).

Combining, the interaction term \( I_s \) with the Chern Simons action \( I_{cs} \), we get the total action for the theory;

\[ I = I_{cs} + I_s \]  

(37)

In this theory, the gauge fields \( A_{\mu}^{\pm} \) and the phase space variables \( q^A \) are smooth functions on the manifold \( M \). Gauge transformations on the former, which are components of the connection in the principal \( SO(2, 2) \) bundle, induce appropriate gauge transformations on the associate bundle to which the latter belong. It is easy to check that the components of the field strength still vanish everywhere except at the location of the sources. So, the analog of Eqs. 24 becomes

\[ \epsilon^{ij} F^{\pm a}_{ij} = 2\pi a^a \delta^2(\vec{x}, \vec{x}_0) \]  

(38)

In particular, when \( \eta^{ab} J^{\pm a} J^{\pm b} > 0 \), we get, in the special (rest) frame

\[ \epsilon^{ij} F^{\pm 0}_{ij} = 2\pi a^a \delta^2(\vec{x}, \vec{x}_0) \]  

(39)

All other components of the field strength vanish. We thus see that because of the constraints appearing in the action given by Eq. 34, the strength of the sources corresponding to the maximal compact subgroup of the gauge group become identified with their Casimir invariants. These invariants, in turn, determine the asymptotic observables of the theory. Since such observables must be gauge invariant, they are expressible in terms of Wilson loops, and a Wilson loop about our source can only depend on, e.g., \( j_+ \) and \( j_- \).

From the data on the manifold \( M \) given above, it is possible to determine the properties of the emerging space-time. To this end, we note that in the gauge in
which Eq. 39 holds, the only non-vanishing components of the gauge potential are given by

\[ A^\pm_\theta = 2a_\pm j_\pm \]  (40)

where \( \theta \) is an angular variable. As an example, consider the case of \( a_+ = a_- = 1 \). Then, using Eqs. 14 and 15, the non-vanishing components can also be written as,

\[ e_\theta^0/l = (j_+ - j_-) = r_-/l \]  (41)

\[ \omega_\theta^0 = (j_+ + j_-) = r_+/l \]  (42)

Although these are components of a connection which is a pure gauge, they give rise to non-trivial holonomies around the source. More explicitly, we have

\[ W[e] = P \exp \int_{\gamma} e_\theta^0 \Pi_0 \]  (43)

\[ W[\omega] = P \exp \int_{\gamma} \omega_\theta^0 J_0 \]  (44)

Here, \( \gamma \) is a loop around the source, which can be represented as a map from the circle to the manifold \( M \), i.e., \( \gamma : S^1 \to M \) with \( \gamma(\sigma + 2\pi) = \gamma(\sigma) \). These holonomies are not gauge invariant \(^\text{[2]}\) and transform by conjugation under \( SO(2) \times SO(2) \) transformations.

The holonomies \( W[e] \) and \( W[\omega] \) control the parallel transport of a vector such as \( q^A \) around the loop \( \gamma \) in \( M \). Since \( \gamma \) is non-trivial, the initial and the final vector will differ from each other by a factor involving \( W[e] \) and \( W[\omega] \). Since the quantities \( e_\theta^0 \) and \( \omega_\theta^0 \) are components of a “Flat” connection, the holonomy can only depend on the homotopy class of the loop \( \gamma \). As a result, the quantities \( W[e] \) and \( W[\omega] \) generate the fundamental group \( \Gamma \) of the manifold \( M \) in the presence of a source. Since \( \gamma(\sigma) \) is periodic, \( \Gamma \) becomes a discrete subgroup of \( SL(2, R) \times SL(2, R) \).

## 5 Restriction to unitary representations

We have indicated that our sources transform as irreducible representations of the AdS group. From purely classical considerations, the choice between unitary and non-unitary representations might not seem to be relevant. But to allow for the possibility of quantizing the Chern Simons theory consistently, we will require that our sources be represented by unitary representations of AdS group. As we shall see, this requirement will also have interesting consequences for the classical space-time which emerges from this theory.

Since the AdS group in \( 2 + 1 \) dimensions can be represented in the \( SL(2, R) \times SL(2, R) \) form, we can construct the unitary representations of \( SO(2, 2) \) from those of \( SL(2, R) \). We will assume that the reader is familiar with the representation theory of \( SL(2, R) \). Here, we will state a few facts relevant to its unitary representations \(^\text{[12]}\).
More information about these representations can be found in papers listed in reference [12] and those cited therein. The states in an irreducible representation of $SL(2, R)$ are specified by the eigenvalues of its Casimir operator $j^2$ (see Eq. 8) and, e.g., the element $J_0$, where we have suppressed the super- (sub)scripts ± distinguishing our two $SL(2, R)$'s. Thus, we have

$$j^2|\Phi, F, m >= \Phi(\Phi + 1)|\Phi, F, m >$$

$$J_0|\Phi, F, m >= (F + m)|\Phi, F, m >$$

In these expressions, $\Phi$ is, in general, a complex number, $F$ is a fraction, and $m$ is an integer. It is well known that $SL(2, R)$ has four series of unitary representations [12], all of which are infinite dimensional. For the present application, we choose the discrete series in which each irreducible representation is an infinite tower of states for which the eigenvalues of $J_0$ are bounded from below. That is,

$$F = -\Phi = \text{real non-integer number} > 0; \quad m = 0, 1, 2, ...$$

(45)

So, for this series, in the notation of section 2, the eigenvalues of the Casimir invariants of $SL(2, R) \times SL(2, R)$ can be written as,

$$j^2_\pm = F^2_\pm - F_\pm$$

(46)

It follows that the infinite set of states can, in a somewhat redundant notation, be specified as

$$|j^2_\pm, F_\pm + m_\pm >; \quad m_\pm = 0, 1, 2,...$$

(47)

Clearly, the integers $m_\pm$ are not necessarily equal. Using these states, we can construct the discrete series of the unitary representations of $SO(2, 2)$. A typical state will have the following labels:

$$|M, J > = |j^2_+, j^2_-, F_+ + m_+, F_- + m_-$$

(48)

As a prelude to identifying the labels $M$ and $J$ with the corresponding labels in the BTZ solution in the next section, let us consider the physical restrictions imposed on $F_\pm$. It is clear from Eqs. 11 through 13 that, up to proportionality constants, we want to identify $F^2_\pm$ with $H^2$ and $S^2$, respectively. To do so, we must choose positive square roots of $H^2$ and $S^2$ since the radii are intrinsically positive quantities. We also see from these equations that to have real non-zero horizon radii, we must have $M > 0$. Then, since the two $SL(2, R)$'s appear symmetrically in the formalism, we must take $j^2_\pm$ to be real and positive. Then, Eq. 46 requires that $F_\pm > 1$. Once this condition is satisfied, it can be seen from Eq. 9 that $|J/|l| \leq |M|$, as required in the BTZ formalism. The extreme case corresponds to $j^2_\pm = 0$. This, in turn, requires that $F_- = 1$. We note that similar statements also hold for the Euclidean version of the AdS space, where the symmetry group becomes $SO(1, 3)$. 

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So far we have discussed the unitary discrete series of $SL(2, R)$, and subsequently of $SO(2, 2)$, for which the eigenvalues of $J^\pm_0$ are bounded from below. We have seen that they are suitable candidates for the microstates of the AdS black hole. Of the other three series of unitary representations of $SL(2, R)$, the principal series are characterized by eigenvalues of the Casimir operator which are negative. Using the results of section 2, it is easy to show that they lead to negative values of $M$ and to complex values of the horizon radii. On this basis alone, we consider them not relevant to the description of the black holes. The other two series are the supplementary series and the discrete series for which $\Phi$ and $F$ are equal and negative. They cannot be ruled out on the basis of the criteria discussed above. However, for these series, as well as for the principal series, the eigenvalues of the operators $J^\pm_0$ are not bounded from below. In analogy with real $3 + 1$ dimensional world, if we identify the corresponding microstates as physical degrees of freedom, then we will have to identify $J^\pm_0$ with some physical observables. It is difficult to conceive of a physical observable with infinitely large negative eigenvalues. Until such an observable could be justified on physical grounds, we seem to be limited to the first discrete series discussed above.

6 The black hole space-time

To see how the space-time structure emerges from our anti-de Sitter gauge theory, we follow an approach which led to the emergence of space-time from Poincaré and super Poincaré Chern Simons gauge theories. We have emphasized that the manifold $M$ is not to be identified with space-time. But the information encoded in it and discussed in section 4, is sufficient to fix the properties of the emerging space-time. To this end, let us consider a manifold $\tilde{M}_q$ satisfying the AdS constraint

$$\tilde{q}_0^2 - \tilde{q}_1^2 - \tilde{q}_2^2 + \tilde{q}_3^2 = l^2 = -\Lambda^{-1}$$

(49)

where $\Lambda = \text{cosmological constant}$. In fact, our $SL(2, R) \times SL(2, R)$ formulation allows us to take $\tilde{M}_q$ to be the universal covering space of the AdS space. As we shall see, the emerging space-time is the quotient of $\tilde{M}_q$ by the discrete subgroup $\Gamma$ discussed in section 4. Moreover, the source coupled to the Chern Simons action is an AdS state characterized by the Casimir invariants $(M, J)$ or, equivalently, $(H, S)$. To parametrize $\tilde{M}_q$ consistent with the above constraint, consider a pair of 2-vectors,

$$\tilde{q}_\phi = (\tilde{q}^1, \tilde{q}^2) = (f \cos \phi, f \sin \phi)$$

(50)

$$\tilde{q}_t = (\tilde{q}^0, \tilde{q}^3) = \left( \sqrt{f^2 + l^2 \cos(t/l)}, \sqrt{f^2 + l^2 \sin(t/l)} \right)$$

(51)

where $f = f(r)$, with $r$ a radial coordinate which for an appropriate $f(r)$ will become the radial coordinate appearing in the line element for the BTZ black hole. As far the constraint given by Eq. 49 is concerned, the functional form of $f(r)$ is irrelevant.
The parameters $\phi$ and $t/l$ are both periodic. We will keep $\phi$ periodic throughout. However, since we are taking $\check{M}_q$ to be universal covering space of AdS space, we do not have to, and we will not, identify $t$ with $t + 2\pi l$. With or without this identification, the vectors $q^A$ parametrized in this fashion do not behave in the same way as the vectors in the manifold $M$ when they are parallel transported along a loop encircling the source. Computing the line element in terms of the parameters $(t/l, r, \phi)$, we get

$$ds^2 = \left(1 + \frac{f^2}{l^2}\right)dt^2 - \frac{f'^2 dr^2}{\left(1 + \frac{f^2}{l^2}\right)} - f^2 d\phi^2$$

(52)

where “prime” indicates differentiation with respect to $r$.

Anticipating the results to be given below, let us compare this line element with that for the BTZ black hole [1].

$$ds^2 = \left[\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right]dt^2 - \frac{dr^2}{\left[\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right]} - r^2 d\phi^2 - \frac{J^2}{2r^2 dt^2}$$

(53)

If we identify the labels $M$ and $J$ with the Casimir invariants of an irreducible representation of the AdS group as discussed in the previous sections, we see that the line element given by Eq. 52 corresponds to an irreducible representation with $J = 0$ and $M = -1$. Such a state will not correspond to any of the series of the unitary representations of the AdS group discussed in the previous section. Moreover, as we have noted in connection with Eqs. 11 and 12, for these values of $J$ and $M$, the invariant $H$ is pure imaginary. This, in turn, implies that the quantities $r_\pm$ will also be imaginary. Thus, we can interpret the line element in Eq. 52 as a special form of the BTZ line element which has been “Wick rotated” into the imaginary axis in the complex $H$ space. In this form, the consequences of the residual gauge transformations involving $H$ and $S$, or $r_\pm$, which we will perform below on $\check{q}^A(\tau)$ become very similar to those performed in the Poincaré Chern Simons gravity. We must keep in mind, however, that in the end, we must Wick rotate the results back to the real $r_\pm$ axes so that the source coupled to the Chern Simons theory would belong to a unitary representation and that the resulting horizon radii would be real. We thus see that the choice of a unitary representation has interesting classical consequences.

With these issues in mind, we want to obtain the space-time manifold $M_q$ by performing appropriate gauge transformations on $\check{M}_q$. Although the original theory was invariant under $SL(2, R) \times SL(2, R)$ gauge transformations, we have already reduced this symmetry by choosing to work in a gauge in which Eq. 39 holds. In fact, the left over symmetry is just $SO(2) \times SO(2)$ generated, respectively, by $J_0$ and $\Pi_0$, or, equivalently, by $J^\pm$. So, identifying the parameters $\phi$ and $t/l$, respectively, with each $SO(2)$, consider the local gauge transformation

$$\tilde{\check{q}}_\phi(\phi) = e^{i\phi J^0} \check{q}_\phi(\phi)$$

(54)
It leaves $\vec{q}_t(t/l)$ invariant. Then, since $\phi$ is $2\pi$ periodic,

$$\vec{q}_{\phi'}(\phi + 2\pi) = e^{i2\pi \frac{r_+}{l} \phi} \vec{q}_{\phi}(\phi)$$  \hspace{1cm} (55)

Similarly, consider the gauge transformation

$$\vec{q}_t(t/l, \phi) = e^{ir_{-}\phi \Pi^0} \vec{q}_t(t/l)$$  \hspace{1cm} (56)

It leaves $\vec{q}_\phi$ invariant and leads to

$$\vec{q}_{\phi'}(t/l, \phi + 2\pi) = e^{ir_{-}2\pi \Pi^0} \vec{q}_t(t/l, \phi)$$  \hspace{1cm} (57)

We note that the factors picked up by $\vec{q}_\phi$ and $\vec{q}_t$ under rotation by $2\pi$ are the same as those given by the holonomies given by Eqs. 43 and 44. Thus, the periodicity of $\phi$ has led to a discrete subgroup of isometries in the universal covering space of the AdS space. The parameters $\frac{2\pi r_{\pm}}{l}$ for these transformations were chosen to demonstrate the connection between the holonomies in $M$ and the identifications necessary for the BTZ black hole. Moreover, in contrast to the situation for the Poincaré group, the residual symmetry $SO(2) \times SO(2)$ assigns symmetrical roles to the invariants $(H, S)$ or $(r_{+}, r_{-})$ as well as the parameters $\phi$ and $t/l$. To reflect this symmetrical role, we can perform our gauge transformations on $\vec{q}_\phi$ and $\vec{q}_t$ in the following more symmetrical manner:

$$\vec{q}_{\phi'}(\phi, t/l) = e^{i\left(\frac{r_+ - r_{-}t}{l}\right) \phi} \vec{q}_\phi(\phi)$$  \hspace{1cm} (58)

$$\vec{q}_{t'}(t/l, \phi) = e^{i\left(\frac{r_{-} - r_{+}t}{l}\right) \Pi^0} \vec{q}_t(t/l)$$

It then follows that

$$\vec{q}_{\phi'}(\phi + 2\pi, t/l) = e^{i2\pi \frac{r_+}{l} \phi} \vec{q}_{\phi'}(\phi, t/l)$$  \hspace{1cm} (59)

$$\vec{q}_{t'}(t/l, \phi + 2\pi) = e^{i2\pi \frac{r_{-}}{l} \Pi^0} \vec{q}_{t'}(t/l, \phi)$$

$$\vec{q}_{t'}(t/l, \phi + 2\pi) = e^{i2\pi \left(\frac{r_{-} - r_{+}t}{l}\right) \Pi^0} \vec{q}_{\phi'}(\phi, t/l)$$

Thus, given the previous identifications, the last two expressions do not lead to any new identifications. We can now write

$$\vec{q}_{\phi'}(\phi, t/l) = \vec{q}_{\phi'}(\phi'); \hspace{1cm} \vec{q}_{t'}(\phi, t/l) = \vec{q}_{t'}(t'/l)$$  \hspace{1cm} (60)

where

$$\phi' = \frac{r_+}{l} \phi - \frac{r_{-}t}{l^2}; \hspace{1cm} t' = \frac{r_{-}}{l} \phi - \frac{r_{+}t}{l^2}$$  \hspace{1cm} (61)

Now we note again that the vector $(q_{\phi'}, q_{t'})$ transforms in the same way as the one which in section 4 was parallel transported around a loop in the manifold $M$. Calling
the manifold to which such vectors belong $M_q$, we see that this manifold incorporates the same dynamics as the phase space variables in $M$, and we are justified in using the same letter $q$ for both. Thus, we can parametrize the manifold $M_q$ as follows:

\[
q^1 = f \cos \left( \frac{r^+_l \phi - r^-_l t}{l^2} \right)
\]

\[
q^2 = f \sin \left( \frac{r^+_l \phi - r^-_l t}{l^2} \right)
\]

\[
q^0 = \sqrt{f^2 + l^2 \cos \left( \frac{r^-_l \phi - r^+_l t}{l^2} \right)}
\]

\[
q^3 = \sqrt{f^2 + l^2 \sin \left( \frac{r^-_l \phi - r^+_l t}{l^2} \right)}
\]  \hspace{1cm} (62)

From these we can compute the line element. It is given by

\[
ds^2 = \frac{f^2}{l^2} \left( r^+ d\phi - r^- \frac{dt}{l} \right)^2 - \frac{f^2}{l^2} \frac{d^2 r^2}{(1 + \frac{f^2}{l^2})} - \left( \frac{f^2}{l^2} - 1 \right) \left( r^- d\phi - r^+ \frac{dt}{l} \right)^2 \]  \hspace{1cm} (63)

It will now be recalled that the quantities $r_{\pm}$ appearing in this expression are “Wick rotated” relative to the corresponding invariants which appear in the BTZ solution. We must, therefore, rotate them back to the Re $r_{\pm}$ axes by letting

\[
r_{\pm} \rightarrow -ir_{\pm} \]  \hspace{1cm} (64)

Then, we get

\[
ds^2 = -\frac{f^2}{l^2} \left( r^+ d\phi - r^- \frac{dt}{l} \right)^2 - f^2 \frac{2dr^2}{(f^2/l^2 + 1)} + \left( \frac{f^2}{l^2} - 1 \right) \left( r^- d\phi - r^+ \frac{dt}{l} \right)^2 \]  \hspace{1cm} (65)

Finally, to put this expression in a form identical to that given by BTZ [1] given by Eq. 53, we set

\[
\frac{f^2}{l^2} = \frac{r^2 - r^-_l^2}{r^+_l - r^-_l}; \quad r < r^- \]  \hspace{1cm} (66)

It can be seen from Eqs. 61 and 66 that the parametrization leading to this expression is valid for $r < r^-$ and any value of the parameter $l$. The simplest way of obtaining suitable parametrizations for all values of $l$ is to observe that parametrization in terms of circular functions are Wick rotated relative to the BTZ solution. Then, as can be seen from Eq. 64, when we rotate the Casimir invariants $r_{\pm}$ back to their real axes in their respective complex $r_{\pm}$ planes, as we did in the above example, we are effectively replacing trigonometric functions by their corresponding hyperbolic functions. We emphasize that this replacement leaves the periodicity of the angle $\phi$ intact since the Wick rotation occurs not in $\phi$ but in complex $r_+$ and $r_-$ spaces. This means that we do not need to impose periodicity on $\phi$ “by hand” if we wish to use a hyperbolic parametrization [1, 2] which is advantageous in many instances.
It is, nevertheless, of interest to see if a parametrization in terms of circular functions works for $r > r^+$. For this to be possible, there must be a class of holonomies in $M$ which are consistent with such a parametrization. Consider the following expressions:

\begin{align*}
q^1 &= f \cos \left( \frac{r- \phi}{l} - \frac{r_+ t}{l^2} \right) \\
q^2 &= f \sin \left( \frac{r- \phi}{l} - \frac{r_+ t}{l^2} \right) \\
q^0 &= \sqrt{f^2 + l^2} \cos \left( \frac{r_+ l}{l^2} \phi - \frac{r_- t}{l^2} \right) \\
q^3 &= \sqrt{f^2 + l^2} \sin \left( \frac{r_+ l}{l^2} \phi - \frac{r_- t}{l^2} \right)
\end{align*}

This parametrization of $M_q$ corresponds to the class of holonomies in $M$ for which the parameters $a_\pm$ in Eqs. 20 and 40 are given by $a_+ = -a_- = 1$. Then, we can get back the BTZ metric of Eq. 53 by computing the line element in terms of these parameters, using Eq. 64 for inverse Wick rotation, and setting $f$ to

\[ \frac{f^2}{l^2} = \frac{r^2 - r_+^2}{r_+^2 - r_-^2}; \quad r > r_+ \]

7 The microscopic black hole structure

The formalism discussed in the previous sections provides a framework for introducing internal structure for black holes. In the metrical approach used by BTZ to obtain the AdS black hole, all one can infer is that the black hole is endowed with the two Casimir invariants $M$ and $J$ of the asymptotic AdS group. There is no room in this approach for the introduction of an internal structure for the black hole. In our approach in which we take a source to be a state belonging to an irreducible representation of the AdS group, an internal structure for the black hole arises naturally. When such a source is coupled to the Chern Simons action, the emerging black hole is still labeled by the two Casimir invariants $M$ and $J$ or, equivalently, $H$ and $S$. However, for a given $M$ and $J$, the irreducible representation is a Hilbert space of states of which may be viewed as the internal states of the black hole. For unitary representations, the Hilbert space is infinite dimensional. The explicit representations which were discussed in detail in section 5 were the discrete series bounded from below. Each representation is determined by a “ground state” labeled by quantities $F_+ > 1$ and $F_- > 1$, which determine the two Casimir invariants of the AdS group according to Eq. 46 and, consequently, the horizon radii and the area associated with the black hole. For each ground state, there is an infinite tower of states with labels which differ from those of the ground state by two separate integers. So, the black hole acquires the degrees of freedom which would be absent in a standard general relativity approach.
The level structure exhibited in this model is reminiscent of the bound state structure familiar from atomic physics except that the energy of the ground state is positive. In this respect, we note that in the BTZ solution and the subsequent works the labels $M$ and $J$ have been identified as “mass” and “angular momentum”. On the other hand, from the point of view of induced representations, it is the labels arising from the maximal compact subgroup, in this case $SO(2) \times SO(2)$, which are more suitable for such designations. In other words, it is the eigenvalues of $\Pi^0$ and $J^0$ which we identify as “energy” and “spin”, respectively.

The general features of the formalism developed in this work in $2+1$ dimensions are applicable to black holes in any dimension. A typical black hole is specified in terms of its asymptotic observables. If we identify these observables with the Casimir invariants of the asymptotic symmetry group, usually a noncompact group, then the corresponding Hilbert space could serve as a microscopic model for the black hole. It remains to be seen whether such models, and their modifications to take into account the fact that the symmetry involved here is a local gauge symmetry and not just a global one, are sufficiently realistic.

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