Tropical limit of matrix solitons and entwining Yang–Baxter maps

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Abstract

We consider a matrix refactorization problem, i.e., a “Lax representation,” for the Yang–Baxter map that originated as the map of polarizations from the “pure” 2-soliton solution of a matrix KP equation. Using the Lax matrix and its inverse, a related refactorization problem determines another map, which is not a solution of the Yang–Baxter equation, but satisfies a mixed version of the Yang–Baxter equation together with the Yang–Baxter map. Such maps have been called “entwining Yang–Baxter maps” in recent work. In fact, the map of polarizations obtained from a pure 2-soliton solution of a matrix KP equation, and already for the matrix KdV reduction, is not in general a Yang–Baxter map, but it is described by one of the two maps or their inverses. We clarify why the weaker version of the Yang–Baxter equation holds, by exploring the pure 3-soliton solution in the “tropical limit,” where the 3-soliton interaction decomposes into 2-soliton interactions. Here, this is elaborated for pure soliton solutions, generated via a binary Darboux transformation, of matrix generalizations of the two-dimensional Toda lattice equation, where we meet the same entwining Yang–Baxter maps as in the KP case, indicating a kind of universality.

Keywords  Soliton · Tropical limit · Yang–Baxter equation, Yang–Baxter map · Toda lattice · Darboux transformation

Mathematics Subject Classification  35C08 · 37K40 · 16T25
1 Introduction

The quantum Yang–Baxter equation is known to be a crucial structure underlying two-dimensional integrable QFT models. A typical feature of the latter is the factorization of the scattering matrix into contributions from 2-particle interactions (see [1] and references therein). This factorization is also typical for the scattering of solitons of classical nonlinear integrable field equations. Indeed, we tend to think of a multi-soliton solution of some vector or matrix version of an integrable nonlinear partial differential of difference equation as being composed of 2-soliton interactions. Matrix solitons carry “internal degrees of freedom,” called “polarization.” In some cases, like matrix KdV [2] or vector NLS [3–5], the map from incoming to outgoing matrix data, i.e., polarizations, has been found to satisfy the Yang–Baxter equation. But why should we expect the Yang–Baxter property? The latter is a statement about three particles, here solitons, and may be thought of as expressing independence of the different ways in which a 3-particle interaction can be decomposed into 2-particle interactions. First of all, how to decompose a 3-soliton solution into 2-soliton interactions? Because of the wave nature of solitons, there are no definite events at which the interaction takes place, and (with the exception of asymptotically incoming and outgoing solitons) there are no definite values of the dependent variable which could define a corresponding map. However, there is a certain limit, called “tropical limit,” that takes soliton waves to “point particles” and then indeed determines events at which an interaction occurs. This has been a decisive tool in our previous work [6–11] and this will be so also in this work, which continues an exploration started in [9], also see [10,11]. It will indeed lead us to a deeper understanding of the question concerning the Yang–Baxter property raised above and to a revision of the previous picture.

Let $K$ be a constant $n \times m$ matrix of maximal rank.\footnote{In this work, we only consider matrices over the real or complex numbers.} In [9], we explored a matrix version of the potential KP equation, the $pKP_K$ equation

\[
(4\phi_t - \phi_{xxx} - 6(\phi_x K \phi_x))_x - 3\phi_{yy} + 6(\phi_x K \phi_y - \phi_y K \phi_x) = 0,
\]

from which the KP$_K$ equation is obtained via $u = 2 \phi_x$. With the restriction to a subclass of solutions, which we called “pure solitons” in [9], the 2-soliton solution, generated by a binary Darboux transformation with trivial seed solution, determines a realization of the following Yang–Baxter map.

Let $S$ be the set of rank-one $m \times n$ $K$-projection matrices \((X X X = X)\) and

\[
\mathcal{R}(1, 2) := \mathcal{R}(p_1, q_1; p_2, q_2) : S \times S \to S \times S \\
(X_1, X_2) \mapsto (X'_1, X'_2)
\]

be given by

\[
X'_1 = \alpha_{12} \left( 1_m - \frac{p_2 - q_2}{p_2 - p_1} X_2 K \right) X_1 \left( 1_n - \frac{p_2 - q_2}{q_1 - q_2} K X_2 \right),
\]
\[ X'_2 = \alpha_{12} \left( 1_m - \frac{p_1 - q_1}{q_2 - q_1} X_1 K \right) X_2 \left( 1_n - \frac{p_1 - q_1}{p_1 - p_2} K X_1 \right), \] (1.1)

where \(1_m\) denotes the \(m \times m\) identity matrix and
\[
\alpha_{12} := \alpha(p_1, q_1, X_1; p_2, q_2, X_2) := \left( 1 - \frac{(p_1 - q_1)(p_2 - q_2)}{(p_2 - p_1)(q_2 - q_1)} \text{tr}(K X_1 K X_2) \right)^{-1} = \alpha_{21}.
\] (1.2)

This is a parameter-dependent Yang–Baxter map, which means that it satisfies the Yang–Baxter equation
\[
R_{12}(1, 2) \circ R_{13}(1, 3) \circ R_{23}(2, 3) = R_{23}(2, 3) \circ R_{13}(1, 3) \circ R_{12}(1, 2) \quad (1.3)
\]
on \(S \times S \times S\). The indices of \(R_{ij}\) specify on which two of the three factors the map \(R\) acts. Here, we have to assume that the constants \(p_i, i = 1, 2, 3\), and also \(q_i, i = 1, 2, 3\), are pairwise distinct and that the expressions for \(\alpha_{ij}\) make sense.

If \(q_i = -p_i\), this is the Yang–Baxter map obtained from the 2-soliton solution of the KdV equation
\[
4u_t - u_{xxx} - 3(u K u)_x = 0. \quad (1.4)
\]

For the matrix KdV equation (where \(m = n\) and \(K = 1_n\)), the Yang–Baxter map has first been derived in [2].2

In the particular case where \(n = 1\) (and correspondingly for \(m = 1\)), the above (generically highly nonlinear) Yang–Baxter map becomes linear:
\[
(X'_1, X'_2) = (X_1, X_2) R(i, j), \quad R(i, j) := \begin{pmatrix}
p_i - p_j & p_i - q_j \\
p_i - q_j & p_i - q_j \\
p_j - q_j & q_i - q_j \\
p_j - q_j & p_i - q_j
\end{pmatrix}. \quad (1.5)
\]

Here, we used the fact that, for \(X \in S\), \(K X\) is now a scalar, so that the \(K\)-projection property requires \(K X = 1\). The \(R\)-matrix, which emerges here, solves the Yang–Baxter equation on a threefold direct sum of an \(m\)-dimensional vector space, which extends the set \(S\).

In the tropical limit of a pure \(N\)-soliton solution of the KP equation, the dependent variable \(u\) has support on a piecewise linear structure in \(\mathbb{R}^3\) (with coordinates \(x, y, t\)), a configuration of pieces of planes, and the dependent variable takes a constant value on each plane segment. This piecewise linear structure is obtained as the boundary of “dominating phase regions.” In the KdV reduction, the support of the dependent variable in the tropical limit is a piecewise linear graph in two-dimensional space-time. For the KdV 2-soliton solution, we have four dominating phase regions, numbered by 11, 12, 21 and 22.3 Figure 1 shows an example.

2 The factor \(\alpha_{12}\) is missing in the latter work, but it is necessary for the Yang–Baxter property.
3 The parameters \(p_k, q_k\) belong to the \(k\)th soliton. The first digit of the phase number \(ab\) refers to soliton 1 and the second to soliton 2. Since we write \(p_k := p_{k, 1}\) and \(q_k := p_{k, 2}\), we have \(a, b \in \{1, 2\}\).
Fig. 1 Dominating phase regions and tropical limit graph in two-dimensional space-time, for a 2-soliton solution of the KdV$_K$ equation. Here, time $t$ is the vertical coordinate.

Using the general 2-soliton solution to compute the values of the dependent variable $u$ along the boundary line segments of the tropical limit graph, after normalization to $\hat{u}$ such that $\text{tr}(K\hat{u}) = 1$, the map $(\hat{u}_{11,21}, \hat{u}_{21,22}) \mapsto (\hat{u}_{12,22}, \hat{u}_{11,12})$ yields the above Yang–Baxter map (with the KdV reduction $q_i = -p_i$). Here, for example, $\hat{u}_{11,21}$ is the polarization along the boundary line between the dominating phase regions numbered by 11 and 21. For a rank-one matrix, the above normalization condition is equivalent to the $K$-projection property.

But what about a phase constellation different from the one shown in Fig. 1? Indeed, Fig. 2 displays alternatives. In the same way as for the phase constellation in Fig. 1, one finds that the first alternative in Fig. 2 leads to the inverse of the above KdV Yang–Baxter map. The remaining possibilities, however, determine maps that are not Yang–Baxter. Nevertheless, they are realized in matrix KdV 2-soliton interactions (see Appendix A), regarding them as a process evolving in time $t$ and by choosing the parameters appropriately. We learn that there are matrix KdV 2-soliton solutions for which the map of polarizations in $t$-direction is not Yang–Baxter! How to understand this, in view of our different expectation?

4 [2] uses results of [12], where a restriction has been imposed on the parameters of the matrix KdV 2-soliton solution. As a consequence of this, the non-Yang–Baxter cases are excluded in [2].
In all cases of phase constellations, shown in Fig. 2, the Yang–Baxter map $\mathcal{R}$ is present, however. It is recovered by regarding the plot not as a process in $t$-direction, but in a different direction in space-time.\(^5\)

First of all, this means that we overlooked something in our analysis of the KP\(_K\) case in [9]. As in the KdV reduction, of course also the general pure 2-soliton solution of KP\(_K\) contains constellations, for certain parameter values, where incoming and outgoing polarizations (with respect to a chosen direction) are related by a map that is not a Yang–Baxter map. Besides the above Yang–Baxter map, this map and the inverses of both maps are needed to describe the propagation of polarizations along the support of pure multi-soliton solutions in the tropical limit.

We were actually led to the new insights by exploring a matrix version of the two-dimensional Toda lattice equation (see, for example, [13,14] for the scalar equation). This is the subject of Sect. 4. In particular, with the restriction to “pure solitons,” it turns out that the same Yang–Baxter map is here at work as in the KP\(_K\) case, indicating a kind of universality. This may not come as a surprise, however, since both equations are known to be related (also see Remark 4.1).

The tropical limit associates with a soliton solution a configuration of plane segments, together with values of the dependent variable on the segments.\(^6\) It is found that, at intersections, these polarizations are related by one of two maps (and their inverses), of which only one is a Yang–Baxter map, but the two maps satisfy a mixed version of the Yang–Baxter equation (see (3.6) below). They are “entwining Yang–Baxter maps” in the sense of [16].

Section 2 presents a “Lax representation” for the above map $\mathcal{R}$. This is a matrix refactorization problem. The basic argument\(^7\) is the same as in [2] for the matrix KdV case (also see [18,19]), but we prove more directly, as compared with [2], that the refactorization problem determines the map $\mathcal{R}$.

In Sect. 3, we show that this refactorization problem, written in a different way, also determines the above-mentioned mixed version of the Yang–Baxter equation. It implies further relations which in particular lead to solutions of the “WXZ system” in [20], called “Yang–Baxter system” in [21]. To our knowledge, such a system first appeared in [22].

In Sect. 4, we explore soliton solutions of the above-mentioned matrix two-dimensional Toda lattice equation. Section 4.1 presents a binary Darboux transformation for the matrix potential two-dimensional Toda lattice equation. Its origin from a general result in bidifferential calculus is explained in Appendix C. We then concentrate on the case of vanishing seed solution. Section 4.2 further restricts to a subclass of soliton solutions, which we call “pure,” and we define the tropical limit of such solitons. In Sect. 4.3, we derive the Yang–Baxter map $\mathcal{R}$ from the pure 2-soliton solution. The relevance of the aforementioned additional non-Yang–Baxter map is explained in

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\(^5\) As a process in $t$-direction, the first plot in Fig. 2 corresponds to an application of the inverse of the Yang–Baxter map $\mathcal{R}$.

\(^6\) Here, we think of the discrete independent variable $k$ in the Toda lattice equation as being continuously extended (also see [15]). Such a smoothing of the discrete variable is actually done in the plots presented in Sect. 4 of this work. But, of course, it is not assumed in any of our computations.

\(^7\) It is actually more generally based on the relation between neighboring simplex equations, see [17] and references cited there.
Sects. 4.4 and 4.5, which treats the case of three pure solitons, and shows explicitly how the Yang–Baxter map and the non-Yang–Baxter map (and their inverses) are at work and why they have to be “entwining.”

Finally, Sect. 5 contains some concluding remarks.

2 A Lax representation for the Yang–Baxter map

Let $K$ be an $n \times m$ matrix with maximal rank, and

$$A_i(\lambda, X) := A(p_i, q_i, \lambda, X) := 1_m - \frac{p_i - q_i}{\lambda - q_i} X K,$$
$$\tilde{A}_i(\lambda, X) := 1_n - \frac{p_i - q_i}{\lambda - q_i} K X,$$
$$B_i(\lambda, X) := B(p_i, q_i, \lambda, X) := 1_m + \frac{p_i - q_i}{\lambda - p_i} X K,$$
$$\tilde{B}_i(\lambda, X) := 1_n + \frac{p_i - q_i}{\lambda - p_i} K X,$$ (2.1)

where $X$ is an $m \times n$ matrix and $\lambda$ a parameter. Then, we have

$$KA_i = \tilde{A}_i K, \quad KB_i = \tilde{B}_i K.$$

If $X$ is a $K$-projection matrix, which means $X K X = X$, then

$$B_i = A_i^{-1}, \quad \tilde{B}_i = \tilde{A}_i^{-1},$$

if $\lambda \notin \{q_i, p_i\}$.

**Theorem 2.1** Let $p_1, p_2, q_1, q_2$ be pairwise distinct and $X_i, i = 1, 2$, rank-one $K$-projections, and hence, $X_i \in S$. Then, the refactorization equations

$$A_1(\lambda, X_1) A_2(\lambda, X_2) = A_2(\lambda, X'_2) A_1(\lambda, X'_1),$$
$$\tilde{A}_1(\lambda, X_1) \tilde{A}_2(\lambda, X_2) = \tilde{A}_2(\lambda, X'_2) \tilde{A}_1(\lambda, X'_1)$$ (2.2)

imply the map $R(1, 2)$, defined in the introduction (see (1.1)).

A proof is given in Appendix B. Recalling a well-known argument (see [17] and references cited there), exploiting associativity in different ways, we obtain

$$A_1(\lambda, X_1) A_2(\lambda, X_2) A_3(\lambda, X_3) \overset{R_{12}}{=} A_2(\lambda, Y_2) A_1(\lambda, Y_1) A_3(\lambda, X_3)$$
$$\overset{R_{13}}{=} A_2(\lambda, Y_2) A_3(\lambda, Y_3) A_1(\lambda, Z_1)$$
$$\overset{R_{23}}{=} A_3(\lambda, Z_3) A_2(\lambda, Z_2) A_1(\lambda, Z_1),$$

where we abbreviated $R_{ij}(i, j)$ to $R_{ij}$ and set, for example, $R(1, 3)(Y_1, X_3) \overset{=}{} (Z_1, Y_3)$, and also

$$A_1(\lambda, X_1) A_2(\lambda, X_2) A_3(\lambda, X_3) \overset{R_{23}}{=} A_1(\lambda, X_1) A_3(\lambda, Y'_3) A_2(\lambda, Y'_2)$$

These are local 1-simplex equations, see [17], for example.
\[ R_{13} \equiv A_3(\lambda, Z'_3) A_1(\lambda, Y'_1) A_2(\lambda, Y'_2) \]
\[ R_{12} \equiv A_3(\lambda, Z'_3) A_2(\lambda, Z'_2) A_1(\lambda, Z'_1). \]

There are corresponding chains with \( A_i \) replaced by \( \tilde{A}_i \). If
\[ A_1(\lambda, X_1) A_2(\lambda, X_2) A_3(\lambda, X_3) = A_3(\lambda, Z_3) A_2(\lambda, Z_2) A_1(\lambda, Z_1), \]
\[ \tilde{A}_1(\lambda, X_1) \tilde{A}_2(\lambda, X_2) \tilde{A}_3(\lambda, X_3) = \tilde{A}_3(\lambda, Z_3) \tilde{A}_2(\lambda, Z_2) \tilde{A}_1(\lambda, Z_1) \]
determines a unique map \((X_1, X_2, X_3) \mapsto (Z_1, Z_2, Z_3)\), which means that \(^9\)
\[ A_3(\lambda, Z_3) A_2(\lambda, Z_2) A_1(\lambda, Z_1) = A_3(\lambda, Z'_3) A_2(\lambda, Z'_2) A_1(\lambda, Z'_1); \]
\[ \tilde{A}_3(\lambda, Z_3) \tilde{A}_2(\lambda, Z_2) \tilde{A}_1(\lambda, Z_1) = \tilde{A}_3(\lambda, Z'_3) \tilde{A}_2(\lambda, Z'_2) \tilde{A}_1(\lambda, Z'_1); \]
\[ Z'_i = Z_i, \ i = 1, 2, 3, \]
we can conclude the statement of the following theorem. But it can also be verified
directly, using computer algebra.

**Theorem 2.2** Let \( X_i \in S \). Then, \( R \), given by (1.1), is a Yang–Baxter map.

(2.2) is called a “Lax representation” for the map \( R \).
Using (2.1), (1.1) can be expressed as
\[ X'_1 = \frac{B_2(p_1, X_2) X_1 \tilde{A}_2(q_1, X_2)}{\text{tr}[B_2(p_1, X_2) X_1 \tilde{A}_2(q_1, X_2) \tilde{K}]}; \quad X'_2 = \frac{A_1(q_2, X_1) X_2 \tilde{B}_1(p_2, X_1)}{\text{tr}[A_1(q_2, X_1) X_2 \tilde{B}_1(p_2, X_1) \tilde{K}]} \tag{2.3} \]

In particular, we have
\[ \alpha_{12}^{-1} = \text{tr}[B_2(p_1, X_2) X_1 \tilde{A}_2(q_1, X_2) \tilde{K}] = \text{tr}[A_1(q_2, X_1) X_2 \tilde{B}_1(p_2, X_1) \tilde{K}]. \]

**Remark 2.3** We also have
\[ \alpha_{12}^{-1} = 1 - \frac{(p_1 - q_1)(p_2 - q_2)}{(p_1 - p_2)(q_1 - q_2)} \text{tr}(X'_1 K X'_2 K) = \text{tr}[B_2(p_1, X'_2) X'_1 \tilde{A}_2(q_1, X'_2) \tilde{K}] = \text{tr}[A_1(q_2, X'_1) X'_2 \tilde{B}_1(p_2, X'_1) \tilde{K}], \]
which in particular means that \( \alpha_{12} \) is an invariant of the map \( R \).

**3 Further aspects of the Lax representation**

According to Sect. 2,
\[ A_1(\lambda, X_1) A_2(\lambda, X_2) = A_2(\lambda, X'_2) A_1(\lambda, X'_1) \tag{3.1} \]

\(^9\) Also, see Proposition 3.1 in [16].
is a Lax representation for the Yang–Baxter map $\mathcal{R}$. More precisely, we have to supplement this equation by $\hat{A}_1(\lambda, X_1) \hat{A}_2(\lambda, X_2) = \hat{A}_2(\lambda, X'_2) \hat{A}_1(\lambda, X'_1)$, if $K$ is not an invertible square matrix. A corresponding extension is also necessary for the other versions of (3.1) considered below, but for simplicity we will suppress it.

1. A Lax representation for the inverse of $\mathcal{R}$ is given by

$$B_1(\lambda, X_1) B_2(\lambda, X_2) = B_2(\lambda, X'_2) B_1(\lambda, X'_1).$$

(3.2)

As a consequence, we have

$$\mathcal{R}(1, 2)^{-1} : (X_1, X_2) \mapsto (X'_1, X'_2),$$

where

$$X'_1 = \frac{A_2(q_1, X_2) X_1 \tilde{B}_2(p_1, X_2)}{\text{tr}[A_2(q_1, X_2) X_1 \tilde{A}_2(p_1, X_2) K]}, \quad X'_2 = \frac{B_1(p_2, X_1) X_2 \tilde{A}_1(q_2, X_1)}{\text{tr}[B_1(p_2, X_1) X_2 \tilde{A}_1(q_2, X_1) K]}.$$  

(3.3)

Comparison with (2.3) shows that this is obtained from the latter by exchanging the two indices 1 and 2. This means that $\mathcal{R}$ is a reversible Yang–Baxter map,

$$\mathcal{R}_{21}(2, 1) \circ \mathcal{R}_{12}(1, 2) = \text{id}.$$  

2. Let us write

$$A_1(\lambda, X_1) B_2(\lambda, X_2) = B_2(\lambda, X'_2) A_1(\lambda, X'_1)$$

(3.4)

instead of (3.1). As in Sect. 2 and Appendix B, it can be shown that this equation uniquely determines the map

$$T(1, 2) := T(p_1, q_1; p_2, q_2) : S \times S \to S \times S$$

$$(X_1, X_2) \mapsto (X'_1, X'_2),$$

where

$$X'_1 = \alpha_{12}^{-1} \left( 1_m - \frac{p_2 - q_2}{p_1 - q_2} X_2 K \right) X_1 \left( 1_n - \frac{p_2 - q_2}{p_2 - q_1} K X_2 \right)$$

$$= \frac{A_2(p_1, X_2) X_1 \tilde{B}_2(q_1, X_2)}{\text{tr}[A_2(p_1, X_2) X_1 \tilde{B}_2(q_1, X_2) K]},$$

$$X'_2 = \alpha_{12}^{-1} \left( 1_m - \frac{p_1 - q_1}{p_2 - q_1} X_1 K \right) X_2 \left( 1_n - \frac{p_1 - q_1}{p_1 - q_2} K X_1 \right)$$

$$= \frac{A_1(p_2, X_1) X_2 \tilde{B}_1(q_2, X_1)}{\text{tr}[A_1(p_2, X_1) X_2 \tilde{B}_1(q_2, X_1) K]}.$$  

(3.5)
The denominators of the final expressions are both equal to $\alpha_{12}$. This map is invariant under exchange of the two indices 1 and 2, and hence,

$$T_{21}(2, 1) = T_{12}(1, 2).$$

Although (3.5) resembles (1.1), in contrast to the latter it does not yield a Yang–Baxter map. This can be checked using computer algebra. As a consequence of associativity, we have

$$A_1(\lambda, X_1) B_2(\lambda, X_2) A_3(\lambda, X_3) \overset{T_{12}}{=} B_2(\lambda, Y_2) A_1(\lambda, Y_1) A_3(\lambda, X_3) \overset{R_{13}}{=} B_2(\lambda, Y_2) A_3(\lambda, Y_3) A_1(\lambda, Z_1) \overset{T^{-1}_{23}}{=} A_3(\lambda, Z_3) B_2(\lambda, Z_2) A_1(\lambda, Z_1),$$

and also

$$A_1(\lambda, X_1) B_2(\lambda, X_2) A_3(\lambda, X_3) \overset{T^{-1}_{23}}{=} A_1(\lambda, X_1) A_3(\lambda, Y'_3) B_2(\lambda, Y'_2) \overset{R_{13}}{=} A_3(\lambda, Z'_3) A_1(\lambda, Y'_1) B_2(\lambda, Y'_2) \overset{T_{12}}{=} A_3(\lambda, Z'_3) B_2(\lambda, Z'_2) A_1(\lambda, Z'_1),$$

where we used (3.1) and set, for example, $T^{-1}_{23}(X_2, X_3) =: (Y'_2, Y'_3)$. One can argue that this implies $Z'_i = Z_i, i = 1, 2, 3$, in which case we can conclude that

$$T^{-1}_{23}(2, 3) \circ R_{13}(1, 3) \circ T_{12}(1, 2) = T_{12}(1, 2) \circ R_{13}(1, 3) \circ T^{-1}_{23}(2, 3). \quad (3.6)$$

This can also be verified using computer algebra. Hence, writing the Lax representation (3.1) in the form (3.4), we are led to a “mixed Yang–Baxter equation” for the two maps $R$ and $T$.

3. The inverse $T^{-1}$ of $T$ is given by $(X_1, X_2) \mapsto (X'_1, X'_2)$, where

$$X'_1 = \frac{B_2(q_1, X_2) X_1 \tilde{A}_2(p_1, X_2)}{\text{tr}[B_2(q_1, X_2) X_1 \tilde{A}_2(p_1, X_2) K]}, \quad X'_2 = \frac{B_1(q_2, X_1) X_2 \tilde{A}_1(p_2, X_1)}{\text{tr}[B_1(q_2, X_1) X_2 \tilde{A}_1(p_2, X_1) K]}. \quad (3.7)$$

A corresponding Lax representation is the version

$$B_1(\lambda, X_1) A_2(\lambda, X_2) = A_2(\lambda, X'_2) B_1(\lambda, X'_1) \quad (3.8)$$

of (3.1).
Remark 3.1 In the special case where $n = 1$, besides $R$ also $T$ becomes linear:

$$(X_1', X_2') = (X_1, X_2) T(1, 2), \quad T(i, j) := \begin{pmatrix} q_i - p_j & q_i - p_i \\ q_j - q_i & p_i - q_i \end{pmatrix}.$$  

It is easily verified that the matrix $T$ does not satisfy the Yang–Baxter equation.

3.1 Further consequences of the Lax representation

There are actually further consequences of the fact that (3.1), (3.2), (3.4) and (3.8) uniquely determine maps.

1. The two ways to rewrite $A_1(\lambda, X_1) A_2(\lambda, X_2) B_3(\lambda, X_3)$ in the form $B_3(\lambda, Z_3) A_2(\lambda, Z_2) A_1(\lambda, Z_1)$, by using (3.1) and (3.4), allow us to deduce that

$$T_{23}(2, 3) \circ T_{13}(1, 3) \circ R_{12}(1, 2) = R_{12}(1, 2) \circ T_{13}(1, 3) \circ T_{23}(2, 3). \quad (3.9)$$

2. Rewriting $A_1(\lambda, X_1) B_2(\lambda, X_2) B_3(\lambda, X_3)$ as $B_3(\lambda, Z_3) B_2(\lambda, Z_2) A_1(\lambda, Z_1)$ in the two possible ways, with certain $Z_i$, using (3.2) and (3.4), leads to

$$R_{23}^{-1}(2, 3) \circ T_{13}(1, 3) \circ T_{12}(1, 2) = T_{12}(1, 2) \circ T_{13}(1, 3) \circ R_{23}^{-1}(2, 3). \quad (3.10)$$

3. Moreover, transforming $B_1(\lambda, X_1) B_2(\lambda, X_2) A_3(\lambda, X_3)$ to $A_3(\lambda, Z_3) B_2(\lambda, Z_2) A_1(\lambda, Z_1)$, with certain $Z_i$, using (3.2) and (3.8), we obtain

$$T_{23}^{-1}(2, 3) \circ T_{13}^{-1}(1, 3) \circ R_{12}^{-1}(1, 2) = R_{12}^{-1}(1, 2) \circ T_{13}^{-1}(1, 3) \circ T_{23}^{-1}(2, 3). \quad (3.11)$$

But this is equivalent to (3.9).

4. Finally, rewriting $B_1(\lambda, X_1) A_2(\lambda, X_2) A_3(\lambda, X_3)$ in the form $A_3(\lambda, Z_3) A_2(\lambda, Z_2) B_1(\lambda, Z_1)$, using (3.1) and (3.8), implies

$$R_{23}(2, 3) \circ T_{13}^{-1}(1, 3) \circ T_{12}^{-1}(1, 2) = T_{12}^{-1}(1, 2) \circ T(1, 3)_{13}^{-1} \circ R_{23}(2, 3), \quad (3.12)$$

which, however, is equivalent to (3.10).

Remark 3.2 A system of equations like that given by the Yang–Baxter equation (1.3), supplemented by (3.9) or (3.12), appeared in [24] under the name “braided Yang–Baxter equations.” The same holds for the Yang–Baxter equation for $R^{-1}$, supplemented by (3.10) or (3.11). Also, see [22]. Via (3.9) and (3.10), as well as via (3.11) and (3.12), we have examples of what has been called “WXZ system” in [20], later also named “Yang–Baxter system” [21]. This system apparently first appeared in [22]. Here, we obtained solutions of these systems. Equation (3.9) also appeared in [23], where a solution emerged in the context of the scalar discrete KP hierarchy. Since the solutions considered in the present work become trivial in the scalar case, the latter solution is of a different nature.
4 The p2DTL$_K$ equation

In this section, we address the following matrix version of the potential two-dimensional Toda lattice equation,

$$\varphi_{xy} - \varphi^+ + 2\varphi - \varphi^- = (\varphi^+ - \varphi) K \varphi_y - \varphi_y K (\varphi - \varphi^-), \quad (4.1)$$

where $\varphi$ is an $m \times n$ matrix of (real or complex) functions and $K$ a constant $n \times m$ matrix of maximal rank. A subscript indicates a partial derivative with respect to the respective variable, here $x$ or $y$. A superscript $+$ or $-$ means a shift or inverse shift, respectively, in a discrete variable, which we will denote by $k$. We refer to this equation as p2DTL$_K$.

In the vector case $n = 1$, writing $K = (k_1, \ldots, k_m)$, (4.1) reads

$$\varphi_{i,xy} - \varphi_i^+ + 2\varphi_i - \varphi_i^- = \sum_j k_j \varphi_{j,y} - \varphi_y \sum_j k_j (\varphi_j - \varphi^-)$$

$$i = 1, \ldots, m.$$ 

By a transformation and redefinition of $\varphi$, we can then achieve that $K = (1, 0, \ldots, 0)$, so that

$$\varphi_{1,xy} - \varphi_1^+ + 2\varphi_1 - \varphi_1^- = (\varphi_1^+ - \varphi_1) \varphi_{1,y} - \varphi_{1,y} (\varphi_1 - \varphi_1^-).$$

$$\varphi_{j,xy} - \varphi_j^+ + 2\varphi_j - \varphi^- = (\varphi_j^+ - \varphi_j) \varphi_{j,y} - \varphi_{j,y} (\varphi_1 - \varphi_1^-) \quad j = 2, \ldots, m,$$

which is the scalar potential 2DTL equation, extended by $m - 1$ linear equations.

In terms of new independent variables

$$t = x + y, \quad z = x - y,$$

Equation (4.1) reads

$$\varphi_{tt} - \varphi_{zz} - \varphi^+ + 2\varphi - \varphi^- = (\varphi^+ - \varphi) K (\varphi_t - \varphi_z) - (\varphi_t - \varphi_z) K (\varphi - \varphi^-). \quad (4.2)$$

If $\varphi$ is independent of $z$, the last equation reduces to

$$\varphi_{tt} - \varphi^+ + 2\varphi - \varphi^- = (\varphi^+ - \varphi) K \varphi_t - \varphi_t K (\varphi - \varphi^-). \quad (4.3)$$

We will refer to this equation as p1DTL$_K$.

Remark 4.1 In terms of

$$u := \varphi_y,$$

in the scalar case ($n = m = 1$), and after differentiation with respect to $y$, (4.1) with $K = 1$ leads to the two-dimensional Toda lattice (2DTL) equation [13] (also see
\[ (\ln(1 + u))_{xy} = u^+ - 2u + u^- \tag{4.4} \]

A continuum limit of the 2DTL equation is the KP-II equation \[15\]. If \( u \) is independent of \( z \), the 2DTL Eq. (4.4) reduces to the one-dimensional Toda lattice equation \[28\]

\[ (\ln(1 + u))_{tt} = u^+ - 2u + u^- \]

Correspondingly, we may regard (4.3) as a matrix version of the potential one-dimensional Toda lattice equation.

**Remark 4.2** Multiplying any solution of the scalar version of (4.1) by an arbitrary constant \( K \)-projection matrix yields a solution of the matrix Eq. (4.1). In this way, a single scalar soliton solution determines single matrix soliton solutions of any rank up to the maximal.

### 4.1 A binary Darboux transformation for the p2DTL\(_K\) equation

The following binary Darboux transformation is a special case of a general result in bidifferential calculus, see Appendix C. Let \( N \in \mathbb{N} \). The integrability condition of the linear system

\[ \theta_x = \theta^+ - \theta + (\varphi_0^+ - \varphi_0)K\theta, \quad \theta_y = \theta - \theta^- - \varphi_{0,y}K\theta^-, \tag{4.5} \]

where \( \theta \) is an \( m \times N \) matrix, is the p2DTL\(_K\) equation for \( \varphi_0 \). The same holds for the adjoint linear system

\[ \chi_x = \chi - \chi^- - \chi K(\varphi_0^+ - \varphi_0), \quad \chi_y = \chi^+ - \chi + \chi^+ K\varphi_{0,y}^+, \tag{4.6} \]

where \( \chi \) is an \( N \times n \) matrix. So, let \( \varphi_0 \) be a given solution of (4.1). Let the Darboux potential \( \Omega \) satisfy the consistent system of \( N \times N \) matrix equations

\[ \Omega - \Omega^- = -\chi K\theta, \quad \Omega_x = -\chi K\theta^+, \quad \Omega_y = -\chi^+ K\theta - \chi^+ K\varphi_{0,y}^+\theta. \tag{4.7} \]

Where \( \Omega \) is invertible,

\[ \varphi = \varphi_0 - \theta(\Omega^-)^{-1}\chi^- \tag{4.8} \]

is then a new solution of the p2DTL\(_K\) Eq. (4.1).

**Remark 4.3** Equations (4.5)–(4.8) are invariant under the transformation

\[ \theta \mapsto \theta C_1, \quad \chi \mapsto C_2\chi, \quad \Omega \mapsto C_2\Omega C_1, \]

with any invertible constant \( N \times N \) matrices \( C_a, a = 1, 2 \). This observation is helpful in order to reduce the set of parameters, on which a generated solution depends.
Using (4.8) and the second of (4.7), we find
\[
\text{tr}(K \varphi) = \text{tr}(K \varphi_0) - \text{tr}(K \theta (\Omega^-)^{-1} \chi^-) = \text{tr}(K \varphi_0) - \text{tr}((\Omega^-)^{-1} \chi^- K \theta) \\
= \text{tr}(K \varphi_0) + \text{tr}(\Omega^{-1} \Omega_x^-) = \text{tr}(K \varphi_0) + (\log \det \Omega)_x^-, \tag{4.9}
\]
so that \( \det \Omega \) plays a role similar to the (Hirota) \( \tau \)-function of the (scalar) 2DTL equation.

4.1.1 Solutions for vanishing seed

The linear system (4.5) with \( \varphi_0 = 0 \) reads
\[
\theta_x = \theta^+ - \theta, \quad \theta_y = \theta - \theta^-.
\]

It possesses solutions of the form
\[
\theta = \sum_{a=1}^{A} \theta_a e^{\tilde{\vartheta}(P_a)} P^k_a.
\]

Here, \( \theta_a, a = 1, \ldots, A \), are constant \( m \times N \) matrices, \( k \) denotes the discrete variable, \( P_a, a = 1, \ldots, A \), are constant \( N \times N \) matrices, and
\[
\tilde{\vartheta}(P) = (P - I)x + (I - P^{-1})y. \tag{4.10}
\]

Correspondingly, the adjoint linear system (4.6) takes the form
\[
\chi_x = \chi - \chi^-, \quad \chi_y = \chi^+ - \chi,
\]
which is solved by
\[
\chi = \sum_{b=1}^{B} e^{-\tilde{\vartheta}(Q_b)} Q^{-k}_b \chi_b,
\]
where \( \chi_b, b = 1, \ldots, B \), are constant \( N \times n \) matrices and \( Q_b, b = 1, \ldots, B \), are constant \( N \times N \) matrices.

The equations for the Darboux potential \( \Omega \) are reduced to
\[
\Omega - \Omega^- = -\chi K \theta, \quad \Omega_x = -\chi (K \theta^+), \quad \Omega_y = -\chi^+ K \theta.
\]

Writing
\[
\Omega = \Omega_0 + \sum_{a,b} e^{-\tilde{\vartheta}(Q_b)} Q^{-k}_b W_{ba} P^{k+1}_a e^{\tilde{\vartheta}(P_a)}, \tag{4.11}
\]
with a constant $N \times N$ matrix $\Omega_0$, it follows that $W_{ba}$ has to satisfy the Sylvester equation

$$Q_b W_{ba} - W_{ba} P_a = \chi_b K \theta_a.$$  (4.12)

If

$$P_a = \text{diag}(p_{1,a}, \ldots, p_{N,a}), \quad Q_b = \text{diag}(q_{1,b}, \ldots, q_{N,b}),$$  (4.13)

and if $p_{i,a} \neq q_{j,b}$ for all $i, j = 1, \ldots, N$ and $a = 1, \ldots, A$, $b = 1, \ldots, B$, then the unique solution is known to be given by the Cauchy-like $N \times N$ matrices

$$W_{ba} = \left( \frac{\chi_{ib} K \theta_{ja}}{q_{ib} - p_{ja}} \right).$$

Assuming that $\Omega_0$ is invertible, Remark 4.3 shows that we can set $\Omega_0 = 1_N$ without loss of generality. The remaining transformations, according to Remark 4.3, can be used to reduce the parameters in $\theta$ or $\chi$.

### 4.2 Pure solitons

We further restrict the class of p2DTL$_K$ solutions specified in Sect. 4.1.1 by setting $A = B = 1$ and assume that the matrices $P := P_1$ and $Q := Q_1$ are diagonal (so that (4.13) holds). Solutions from this class which are regular and satisfy the spectrum condition

$$\text{spec}(P) \cap \text{spec}(Q) = \emptyset$$

will be called “pure solitons.”

Let us write

$$P = \text{diag}(p_{1,1}, \ldots, p_{N,1}) =: \text{diag}(p_1, \ldots, p_N),$$

$$Q =: \text{diag}(p_{1,2}, \ldots, p_{N,2}) =: \text{diag}(q_1, \ldots, q_N),$$

$$\theta_1 = (\xi_1, \ldots, \xi_N)(Q - P), \quad \chi_1 = \left( \begin{array}{c} \eta_1 \\ \vdots \\ \eta_N \end{array} \right),$$

where $\xi_i, i = 1, \ldots, N$, are constant $m$-component column vectors and $\eta_i, i = 1, \ldots, N$, are constant $n$-component row vectors. We shall assume that $p_i > 0$ and $q_i > 0, i = 1, \ldots, N$, since otherwise the generated solution of (4.1) will be singular. The above spectrum condition means $p_i \neq q_j$ for $i, j = 1, \ldots, N$, and we have

$$W := W_{1,1} = \left( \frac{\kappa_{ij} (q_j - p_j)}{q_i - p_j} \right), \quad \kappa_{ij} = \eta_i K \xi_j.$$
Introducing
\[ \vartheta(p) := px - p^{-1}y + k \log p = \frac{1}{2}(p - p^{-1})t + \frac{1}{2}(p + p^{-1})z + k \log p, \]
provisionally\(^{10}\) assuming \( p > 0 \), we obtain
\[ \Omega_{ij} = \delta_{ij} + \frac{k_{ij}(q_j - p_j)}{q_i - p_j} e^{\vartheta(p_j)^+ - \vartheta(q_i)}. \]

Let us introduce
\[ \vartheta_{i,1} := \vartheta(p_i)^+, \quad \vartheta_{i,2} := \vartheta(q_i), \]
\[ \vartheta_I := \sum_{i=1}^{N} \vartheta_{i,a_i} \quad \text{if} \quad I = (a_1, \ldots, a_N) \in \{1, 2\}^N. \]

Instead of using \((a_1, \ldots, a_N)\) as a subscript, we simply write \(a_1 \ldots a_N\) in the following. For example, \( \vartheta_{a_1 \ldots a_N} = \vartheta_{(a_1, \ldots, a_N)} \). From (4.8), we find that a pure soliton solution of the p2DTL\(K\) equation can be expressed as
\[ \varphi^+ = \frac{F}{\tau}, \quad (4.14) \]
with
\[ \tau := e^{\vartheta_2} \det \Omega, \quad (4.15) \]
\[ F := -e^{\vartheta_2} \theta_1 e^{\vartheta(p)^+} \adj(\Omega) e^{-\vartheta(q)} \chi_1, \quad (4.16) \]
where \( \adj(\Omega) \) denotes the adjugate of the matrix \( \Omega \) and \( 2 := 2 \ldots 2 = (2, \ldots, 2) \).

The following result is proved in the same way as Proposition 3.1 in [9].

**Proposition 4.4** \( \tau \) and \( F \) have expansions
\[ \tau = \sum_{I \in \{1,2\}^N} \mu_I e^{\vartheta_I}, \quad (4.17) \]
\[ F = \sum_{I \in \{1,2\}^N} M_I e^{\vartheta_I}, \quad (4.18) \]
with constants \( \mu_I \) and constant \( m \times n \) matrices \( M_I \). We have \( \mu_2 = 1 \) and \( M_2 = 0 \). \( \square \)

Besides conditions imposed on \( p_i \) and \( q_i \) such that all the \( \vartheta_I \) appearing in (4.14) are real, the regularity of a pure \( N \)-soliton solution requires \( \mu_I \geq 0 \) for all \( I \in \{1, 2\}^N \), and

\(^{10}\) Finally, we only have to make sure that the expressions for \( \vartheta_I \) (see below), appearing in a generated solution of the p2DTL\(K\) equation, are real.
\(\mu_J > 0\) for at least one \(J \in \{1, 2\}^N\). We will impose the slightly stronger condition \(\mu_I > 0\) for all \(I \in \{1, 2\}^N\).

It follows that

\[
u^+ = \varphi^+_y = \left( \frac{1}{\tau} \sum_{I \in \{1,2\}^N} M_I e^{\vartheta_I} \right)_y = \frac{1}{2\tau^2} \sum_{I, J \in \{1,2\}^N} (\tilde{p}_J - \tilde{p}_I)(M_I \mu_J - \mu_I M_J) e^{\vartheta_I} e^{\vartheta_J},
\]

\[(4.19)\]

where

\[
\tilde{p}_I := \sum_{i=1}^N \frac{1}{p_{I, a_i}} \text{ if } I = (a_1, \ldots, a_N) \in \{1, 2\}^N.
\]

**Example 4.5** For \(N = 1\), writing \(p_1 = p, q_1 = q, \xi_1 = \xi, \eta_1 = \eta\) and \(\kappa = \eta K \xi\), we have the single soliton solution

\[
\varphi = \frac{\kappa (p - q) e^{\vartheta(p)}}{e^{\vartheta(q)^- + \kappa} e^{\vartheta(p)}} \frac{\xi \otimes \eta}{\kappa},
\]

which leads to

\[
u = \varphi_y = \frac{(p - q)^2}{4pq} \text{sech}^2 \left[ \frac{1}{2} (\vartheta(p) - \vartheta(q)^- + \log \kappa) \right] \frac{\xi \otimes \eta}{\kappa}.
\]

In terms of the variables \(t = x + y\) and \(z = x - y\), it reads

\[
u = \frac{(p - q)^2}{4pq} \text{sech}^2 \left[ \frac{1}{2} \left( \frac{1}{2} (p - q - p^{-1} + q^{-1}) t 
+ \frac{1}{2} (p - q + p^{-1} - q^{-1}) z + \log(p/q) k + \log(q\kappa) \right) \right] \frac{\xi \otimes \eta}{\kappa}.
\]

We have to restrict the parameters such that \(p/q > 0\) and \(q\kappa > 0\). The solution becomes independent of \(z\) if we choose \(q = p^{-1}\), in which case the above \(\varphi\) reduces to a single soliton solution of the p1DTL-\(K\) Eq. (4.3), and we have

\[
u = \frac{(p - p^{-1})^2}{4} \text{sech}^2 \left[ \frac{1}{2} (p - p^{-1}) t + \log(p) k + \frac{1}{2} \log(p/k) \right] \frac{\xi \otimes \eta}{\kappa}.
\]

It is obvious from (4.8) and the sizes of its matrix constituents that, for \(N = 1\), the binary Darboux transformation with zero seed can only yield a rank-one solution.
4.2.1 Tropical limit of pure solitons

We define the tropical limit of a matrix soliton solution via the tropical limit of the scalar function \( \tau \) (cf. [6–8]). Let

\[
\varphi_I := \varphi \bigg|_{\vartheta_J \to -\infty, J \neq I} = \frac{M_I}{\mu_I}. \tag{4.20}
\]

In the region of \( \mathbb{R}^2 \times \mathbb{Z} \), where a phase \( \vartheta_I \) dominates all others, in the sense that

\[
\log(\mu_I e^{\vartheta_I}) > \log(\mu_J e^{\vartheta_J})
\]

for all participating \( J \neq I \), the tropical limit of the potential \( \varphi \) is given by (4.20).\(^{11}\) These expressions do not depend on the variables \( x, y, k \) (respectively, \( z, t, k \)).

The boundary between the regions associated with the phases \( \vartheta_I \) and \( \vartheta_J \) is determined by the condition

\[
\mu_I e^{\vartheta_I} = \mu_J e^{\vartheta_J}. \tag{4.21}
\]

Not all parts of such a boundary are “visible,” in general, since some of them may lie in a region where a third phase dominates the two phases. The tropical limit of a soliton solution, more precisely, of the variable \( u \), has support on the visible parts of the boundaries between the regions associated with phases appearing in \( \tau \).

For \( I = (a_1, \ldots, a_N) \), we set

\[
I_j(a) = (a_1, \ldots, a_{j-1}, a, a_{j+1}, \ldots, a_N).
\]

The \( j \)th soliton (having parameters \( p_j \) and \( q_j \)) lives, in the tropical limit, on the set of two-dimensional plane segments determined, via (4.21), by

\[
e^{\vartheta_{I_j(1)} - \vartheta_{I_j(2)}} = \frac{\mu_{I_j(2)}}{\mu_{I_j(1)}},
\]

for all \( I \in \{1, 2\}^N \). More explicitly, the last equation reads

\[
(p_j - q_j)x + (q_j^{-1} - p_j^{-1})y + \log(p_j/q_j)k + \log \left( \frac{p_j}{\mu_{I_j(2)}} \frac{\mu_{I_j(1)}}{\mu_{I_j(1)}} \right) = 0,
\]

which requires

\[
p_j/q_j > 0, \quad p_j \frac{\mu_{I_j(1)}}{\mu_{I_j(2)}} > 0 \quad \forall I \in \{1, 2\}^N. \tag{4.22}
\]

All these plane segments are parallel. In general, there are relative shifts between the segments, and they do not constitute together a single plane. This gives rise to the

\(^{11}\) Such “dominating phase regions” have also been used, for example, in [29–32], mostly for the asymptotic analysis of solitons. In our work, we apply it to the whole soliton solution, not just in asymptotic regions. Also, see [6–11].
familiar (asymptotic) “phase shift” of solitons caused by their interaction. Figure 3 shows this for a 2-soliton example, considered at constant time, so that the configuration of planes is projected to a graph in two dimensions. For \( j = 1, \ldots, N \), the regularity conditions (4.22) will be assumed in the following.

On a (visible) boundary segment, the value of \( u \) is given by

\[
\hat{u}_{IJ} = \frac{\varphi_I - \varphi_J}{p_I - p_J},
\]

This follows from (4.19) by use of (4.20) and (4.21). Instead of the above expressions for the tropical values of \( u \), we will rather consider

\[
\hat{u}_{IJ} = \frac{\varphi_I - \varphi_J}{p_I - p_J},
\]

where

\[
p_I := \sum_{i=1}^{N} p_{i,a_i} \quad \text{if} \quad I = (a_1, \ldots, a_N) \in \{1, 2\}^N.
\]

(4.23) has the form of a discrete derivative.

Using (4.9), (4.14), (4.17) and (4.18), we find

\[
\text{tr}(KM_I) = (p_I - p_2) \mu_I,
\]

and thus,

\[
\text{tr}(K\hat{u}_{IJ}) = 1.
\]

\textbf{Remark 4.6} We note that

\[
\text{tr}(Ku_{I(1), I(2)}) = \frac{1}{4}(\tilde{p}_{I(1)} - \tilde{p}_{I(2)})(p_{I(1)} - p_{I(2)}) = \frac{1}{4}(\frac{1}{p_j} - \frac{1}{q_j})(p_j - q_j),
\]

which shows that its value is the same everywhere (i.e., for all \( I \)) on the tropical support of the \( j \)th soliton.

\textbf{4.3 Pure 2-soliton solution and the Yang–Baxter map}

For \( N = 2 \), we find \( \varphi^+ = F/\tau \) with

\[
\tau = \alpha_{12} \kappa_{11} \kappa_{22} e^{\beta_{11}} + \kappa_{11} e^{\beta_{12}} + \kappa_{22} e^{\beta_{21}} + e^{\beta_{22}},
\]
Fig. 3 Dominating phase regions and tropical limit graph of a pure 2-soliton solution of the vector p2DTL$_K$ equation, at $t = x + y = 0$. The horizontal coordinate is $z = x - y$, and the discrete coordinate $k$ is continuously extended. The numbering of dominating phase regions corresponds to $I = (1, 1), (1, 2), (2, 1), (2, 2)$. We also marked the two parts of soliton 1, respectively 2.

$F = (p_1 - q_1)(p_2 - q_2)\left(\frac{\kappa_{22}}{p_2 - q_2} \xi_1 \otimes \eta_1 + \frac{\kappa_{11}}{p_1 - q_1} \xi_2 \otimes \eta_2 - \frac{\kappa_{12}}{p_2 - q_1} \xi_1 \otimes \eta_2 - \frac{\kappa_{21}}{p_1 - q_2} \xi_2 \otimes \eta_1\right) e^{\vartheta_{11}} + (p_1 - q_1) \xi_1 \otimes \eta_1 e^{\vartheta_{12}} + (p_2 - q_2) \xi_2 \otimes \eta_2 e^{\vartheta_{21}},$

where

$\alpha_{12} = 1 - \frac{(p_1 - q_1)(p_2 - q_2) \kappa_{12} \kappa_{21}}{(p_2 - q_1)(p_1 - q_2) \kappa_{11} \kappa_{22}}$

and

$\vartheta_{11} = \vartheta(p_1)^+ + \vartheta(p_2)^+, \quad \vartheta_{12} = \vartheta(p_1)^+ + \vartheta(q_2),$

$\vartheta_{21} = \vartheta(p_2)^+ + \vartheta(q_1), \quad \vartheta_{22} = \vartheta(q_1) + \vartheta(q_2).$

The above expressions for $\tau$ and $F$ coincide with those derived in the KP$_K$ case [9]. The only difference is in the expressions for the phases, but the latter do not enter the expressions for the polarizations.

Example 4.7 Let

$K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$

$\eta_1 = \eta_2 = 1, \quad q_1 = 1/4, \quad q_2 = 3, \quad p_1 = 3/2, \quad p_2 = 2.$

Figure 3 shows the phase constellation and the tropical limit graph of the corresponding 2-soliton solution of the vector p2DTL$_K$ equation at $t = 0$.\(^{12}\)

\(^{12}\) Here, and in all other plots in this work, we have chosen the parameters in such a way that, as the vertical coordinate tends to $-\infty$, the solitons are naturally ordered in horizontal direction.
Let us now consider the graph in Fig. 3 as a scattering process evolving from bottom to top. Defining

\[ u_1 := \hat{u}_{11,21}, \quad u_2 := \hat{u}_{21,22}, \quad u'_1 := \hat{u}_{12,22}, \quad u'_2 := \hat{u}_{11,12}, \] 

we find

\[ u_1 = \frac{1}{\alpha_{12}} \left( \xi_1 - \frac{(p_2 - q_2)\kappa_{21}}{(p_1 - q_2)\kappa_{22}} \xi_2 \right) \otimes \left( \eta_1 - \frac{(p_2 - q_2)\kappa_{12}}{(p_2 - q_1)\kappa_{11}} \eta_2 \right) \]
\[ = \frac{1}{\alpha_{12}} A_2(p_1, u_2) u'_1 \tilde{B}_2(q_1, u_2), \]

\[ u_2 = \frac{\xi_2 \otimes \eta_2}{\kappa_{22}}, \]

\[ u'_1 = \frac{\xi_1 \otimes \eta_1}{\kappa_{11}}, \]

\[ u'_2 = \frac{1}{\alpha_{12}\kappa_{22}} \left( \xi_2 - \frac{(p_1 - q_1)\kappa_{12}}{(p_2 - q_1)\kappa_{11}} \xi_1 \right) \otimes \left( \eta_2 - \frac{(p_1 - q_2)\kappa_{21}}{(p_2 - q_1)\kappa_{11}} \eta_1 \right) \]
\[ = \frac{1}{\alpha_{12}} A_1(p_2, u'_1) u_2 \tilde{B}_1(q_2, u'_1). \] 

Equations (4.27) imply

\[ u'_1 = \alpha_{12} B_2(p_1, u_2) u_1 \tilde{A}_2(q_1, u_2), \quad u'_2 = \alpha_{12} A_1(q_2, u_1) u_2 \tilde{B}_1(p_2, u_1), \]

provided that \( \{p_1, p_2\} \cap \{q_1, q_2\} = \emptyset \). Comparison with (2.3) shows that \((u_1, u_2) \mapsto (u'_1, u'_2)\) provides us with a realization of the Yang–Baxter map \( \mathcal{R} \).

**Remark 4.8** If \( n = 1 \), we are dealing with an \( m \)-component vector 2DTL equation. Then, \( \eta_i \) and \( K \xi_i \), \( i = 1, \ldots, N \), are scalars. In this case, the Yang–Baxter map is linear,

\[ (u'_i, u'_j) = (u_i, u_j) R(i, j), \]

with \( R(i, j) \) defined in (1.5). It solves the Yang–Baxter equation \( R_{12}(1, 2) R_{13}(1, 3) R_{23}(2, 3) = R_{23}(2, 3) R_{13}(1, 3) R_{12}(1, 2) \) on a threefold direct sum. Also, see [9] for

\[ \alpha_{12} \] The definition of \( \alpha_{12} \) in Sect. 4.3 is in accordance with the expression in (1.2).
the case of the vector $\text{KP}_K$ equation. Introducing

$$\tilde{R}(i, j) := \begin{pmatrix} 1 & 0 \\ 0 & \frac{p_j - q_i}{p_i - q_j} \end{pmatrix}, \quad S(i, j) := \begin{pmatrix} \frac{p_i - q_i}{p_j - q_j} & -1 \\ 1 & 1 \end{pmatrix},$$

we have $R(i, j) = S(i, j)\tilde{R}(i, j)S(i, j)^{-1}$, and $\tilde{R}(i, j)$ also satisfies the Yang–Baxter equation. We further note that $RA(i, j) := A(i, j)^{-1}R(i, j)A(i, j)$ with $A(i, j) = \text{diag}(a_{ij}, b_{ij})$ solves the Yang–Baxter equation if the constants $a_{ij}, b_{ij}$ satisfy the relations $a_{ik}b_{ij}b_{jk} = a_{ij}a_{jk}b_{ik}$ for pairwise distinct $i, j, k$. If we drop the normalization condition (4.25) in the computation of the Yang–Baxter map, the resulting $R$-matrix turns out to be of the latter form. The same holds if we consider $v = \phi^+ - \phi$ instead of $\hat{u}$.

### 4.4 Yang–Baxter and non-Yang–Baxter maps at work

In Sect. 4.3, we looked at the relation between the polarizations associated with the boundary segments of dominant phase regions of a pure 2-soliton solution, selecting a “propagation direction.” But there is actually no preferred direction. It is therefore more adequate to regard (4.27) just as determining a relation between four polarizations, and there are several ways in which this determines a map from two “incoming” to two “outgoing” polarizations.

**Example 4.9** We keep the choices for $K$, $\xi_i$ and $\eta_i$, made in Example 4.7. Then, $\kappa_{ij} = 1$ and $\alpha_{12} = (q_2 - q_1)/(q_2 - p_1)$, so that the regularity conditions (4.22) read $p_2, q_2 > 0$, $p_1(q_2 - q_1)/(q_2 - p_1) > 0$ and $p_1/q_1 > 0$. Furthermore, without loss of generality, we can choose the parameters such that, for large enough negative value of $k$, soliton 1 appears in $z$-direction to the left of soliton 2.

1. $q_1 < p_1 < p_2 < q_2$ or $q_1 < q_2 < p_2 < q_1$. In this case, the phase constellation is that shown in Fig. 3. Regard it as a process in $k$-direction, the map of polarizations is the Yang–Baxter map $\mathcal{R}$.

2. $p_1 < q_1 < q_2 < p_2$ or $p_1 < q_2 < p_2 < q_1$. The phase constellation is that shown in the first plot of Fig. 4 (which is generated with $p_1 = 1/4, q_1 = 3/2, q_2 = 2$ and $p_2 = 3$). The map of polarizations, in $k$-direction, leads us to the inverse of the Yang–Baxter map $\mathcal{R}$, which is also a Yang–Baxter map.

3. $q_1 < p_1 < q_2 < p_2$ or $q_1 < q_2 < p_2 < p_1$. The phase constellation is that shown in the second plot of Fig. 4 (which is generated with $q_1 = 1/4, p_1 = 3/2, q_2 = 2$ and $p_2 = 3$). Instead of (4.26), here we define initial and final polarizations as

$$u_1 := \hat{u}_{12,22}, \quad u_2 := \hat{u}_{21,22}, \quad u_1' := \hat{u}_{11,21}, \quad u_2' := \hat{u}_{11,12},$$

and obtain the map $\mathcal{T}$, given by (3.5), which is not a Yang–Baxter map.

4. $p_1 < q_1 < p_2 < q_2$ or $p_1 < p_2 < q_2 < q_1$. The phase constellation is that shown in the third plot of Fig. 4 (which is generated with $p_1 = 1/4, q_1 = 3/2, p_2 = 2$ and $q_2 = 3$). In this case, the map of polarizations, from bottom to top, is the inverse of $\mathcal{T}$. 

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Fig. 4 Dominating phase regions and tropical limit graph of a pure 2-soliton solution of the vector p2DTL\(_K\) equation, at \(t = x + y = 0\), for different values of the parameters \(p_i\) and \(q_i\), \(i = 1, 2\). See Example 4.9.

Viewing the first graph as a process from top to bottom, i.e., mapping the polarizations along the upper two legs to those along the lower two, we recover the Yang–Baxter map \(R\). For the second (third) graph, \(R\) is obtained by viewing it as a process from right (left) to left (right). In the colored figure, this means mapping the polarizations along the red-labeled to those along the blue-labeled soliton lines.

For any one of the plots in Figs. 3 or 4, we obtain realizations of all the maps by choosing different directions.

The naive expectation that 2-soliton scattering yields a Yang–Baxter map is therefore wrong. But we have to keep in mind that the Yang–Baxter property is a statement about three solitons. A 3-soliton solution involves three 2-particle interactions. In the tropical limit, this means that a composition of three of the above maps carries the polarizations along the tropical limit support. We will see in the next subsection that the Yang–Baxter equation is indeed only required to hold for a mixture of the maps, but not for each map separately.

**Remark 4.10** As seen above, the constellation of dominant phase regions depends on the concrete values of the parameters. If, for a certain constellation, we select a direction and obtain a Yang–Baxter map, then the latter has the Yang–Baxter property for all choices of parameters. From the above, we conclude that the map, relating the (relative to our choice of direction) incoming and outgoing polarizations, is a Yang–Baxter map if and only if the phase region that lies between the two incoming solitons is that of \(\vartheta_{12}\) or \(\vartheta_{21}\).

**Example 4.11** Imposing the reduction condition \(q_i = p_i^{-1}\) on the solutions of the p2DTL\(_K\) equation determines solutions of the p1DTL\(_K\) Eq. (4.3). The corresponding Yang–Baxter map is obtained from (1.1) by applying this reduction condition. Let us choose again

\[
K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta_1 = \eta_2 = 1.
\]

The regularity condition (4.22) is then \((p_2 - p_1)/(p_2 - 1) > 0\).
1. $0 < p_2 < p_1 < 1$. The map of incoming to outgoing polarizations, in $t$-direction, is the reduced Yang–Baxter map. The tropical limit graph, for parameters $p_1 = 1/2$ and $p_2 = 1/10$, is shown in the first plot of Fig. 5.

2. $1 < p_1 < p_2$. In this case, the polarization map is the inverse of the (reduced) Yang–Baxter map. For $p_1 = 2$ and $p_2 = 10$, we obtain the second plot of Fig. 5.

3. $0 < p_1 < 1 < p_2$ and $p_1 < p_2^{-1}$. In this case, the (reduced) map $T$ is at work. For $p_1 = 1/2$ and $p_2 = 10$, the third plot of Fig. 5 is obtained.

4. $0 < p_2 < 1 < p_1$ and $p_1^{-1} < p_2$. Here, $T^{-1}$ applies. For $p_1 = 2$ and $p_2 = 1/10$, we obtain the fourth plot of Fig. 5.

### 4.5 Pure 3-soliton solution

For $N = 3$, we find

$$
\tau = \kappa_{11}\kappa_{22}\kappa_{33} \beta e^{\vartheta_{111}} + \kappa_{11}\kappa_{22} \varphi_{12} e^{\vartheta_{112}} + \kappa_{11}\kappa_{33} \varphi_{13} e^{\vartheta_{121}} + \kappa_{22}\kappa_{33} \varphi_{23} e^{\vartheta_{211}} + \kappa_{11} e^{\vartheta_{122}} + \kappa_{22} e^{\vartheta_{212}} + \kappa_{33} e^{\vartheta_{221}} + e^{\vartheta_{222}},
$$

where

$$
\varphi_{ij} = 1 - \frac{(p_i - q_i)(p_j - q_j)}{(p_i - q_j)(p_j - q_i)} \kappa_{ii}\kappa_{jj},
$$

$$
\beta = -2 + \varphi_{12} + \varphi_{13} + \varphi_{23} + \frac{(p_1 - q_1)(p_2 - q_2)(p_3 - q_3)}{(p_1 - q_3)(p_2 - q_1)(p_3 - q_2)} \frac{\kappa_{12}\kappa_{23}\kappa_{31}}{\kappa_{11}\kappa_{22}\kappa_{33}} + \frac{(p_1 - q_1)(p_2 - q_2)(p_3 - q_3)}{(p_1 - q_2)(p_2 - q_3)(p_3 - q_1)} \frac{\kappa_{13}\kappa_{21}\kappa_{32}}{\kappa_{11}\kappa_{22}\kappa_{33}},
$$

and

$$
\vartheta_{111} = \vartheta(p_1)^+ + \vartheta(p_2)^+ + \vartheta(p_3)^+,
$$

$$
\vartheta_{112} = \vartheta(p_1)^+ + \vartheta(p_2)^+ + \vartheta(q_3),
$$

$$
\vartheta_{121} = \vartheta(p_1)^+ + \vartheta(q_2) + \vartheta(p_3)^+,
$$

$$
\vartheta_{122} = \vartheta(p_1)^+ + \vartheta(q_2) + \vartheta(q_3),
$$

$$
\vartheta_{211} = \vartheta(q_1) + \vartheta(p_2)^+ + \vartheta(p_3)^+,
$$

$$
\vartheta_{212} = \vartheta(q_1) + \vartheta(p_2)^+ + \vartheta(q_3),
$$

$$
\vartheta_{221} = \vartheta(q_1) + \vartheta(q_2) + \vartheta(p_3)^+,
$$

$$
\vartheta_{222} = \vartheta(q_1) + \vartheta(q_2) + \vartheta(q_3).
$$

![Fig. 5 Dominating phase regions and tropical limit graph of pure 2-soliton solution of the vector p1DTLₖ equation, with the data given in Example 4.11. The discrete coordinate k is smoothed out](image-url)
Again, we set $\kappa_{ij} = \eta_i K \xi_j$. Furthermore, we have

\[
F = (p_1 - q_1) (p_2 - q_2) (p_3 - q_3) \left( \frac{\alpha_{12} \kappa_{11} \kappa_{22}}{(p_1 - q_1) (p_2 - q_2)} \xi_3 \otimes \eta_3 \right. \\
+ \frac{\alpha_{13} \kappa_{11} \kappa_{33}}{(p_1 - q_1) (p_3 - q_3)} \xi_2 \otimes \eta_2 + \frac{\alpha_{12} \kappa_{11} \kappa_{33}}{(p_2 - q_2) (p_3 - q_3)} \xi_1 \otimes \eta_1 \\
- \frac{\alpha_{123} \kappa_{11} \kappa_{22} \kappa_{33}}{(p_1 - q_1) (p_2 - q_2) (p_3 - q_3)} \xi_2 \otimes \eta_3 - \frac{\alpha_{123} \kappa_{11} \kappa_{32}}{(p_1 - q_1) (p_2 - q_3)} \xi_3 \otimes \eta_2 \\
- \frac{\alpha_{132} \kappa_{11} \kappa_{23} \kappa_{33}}{(p_2 - q_1) (p_3 - q_3)} \xi_1 \otimes \eta_3 - \frac{\alpha_{132} \kappa_{11} \kappa_{23} \kappa_{31}}{(p_1 - q_2) (p_3 - q_3)} \xi_2 \otimes \eta_1 e^{\partial_{111}} \\
+ \frac{\kappa_{11}}{p_1 - q_1} \xi_2 \otimes \eta_2 - \frac{\kappa_{12}}{p_2 - q_1} \xi_1 \otimes \eta_2 - \frac{\kappa_{21}}{p_1 - q_2} \xi_2 \otimes \eta_1 \\
+ \frac{\kappa_{22}}{p_2 - q_2} \xi_1 \otimes \eta_1 \right) e^{\partial_{112}} \\
+ \frac{\kappa_{33}}{p_3 - q_3} \xi_2 \otimes \eta_2 \\n+ \kappa_{33} \eta_3 \otimes \eta_1 \right) e^{\partial_{211}} \\
+ \frac{\kappa_{22}}{p_2 - q_2} \xi_2 \otimes \eta_2 + \frac{\kappa_{33}}{p_3 - q_3} \xi_2 \otimes \eta_2 \right) e^{\partial_{211}} \\
+ (p_1 - q_1) \xi_1 \otimes \eta_1 e^{\partial_{122}} + (p_2 - q_2) \xi_2 \otimes \eta_2 e^{\partial_{122}} + (p_3 - q_3) \xi_3 \otimes \eta_3 e^{\partial_{221}},
\]

where

\[
\alpha_{kij} = 1 - \frac{(p_j - q_i)(p_k - q_k)\kappa_{ik} \kappa_{kj}}{(p_k - q_i)(p_j - q_k)\kappa_{ij} \kappa_{kk}}.
\]

Note that $\alpha_{ij} = \alpha_{ijj}$. The above expressions coincide with those obtained in the KP case [9]. The only difference is in the phases entering the exponentials. Here, we wrote $F$ in a more compact form.

**Example 4.12** We choose

\[
K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

\[
\eta_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix},
\]

$p_1 = 1$, $p_2 = 1/4$, $p_3 = 3/2$, $q_1 = 1/2$, $q_2 = 6/5$, $q_3 = 3$. 

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Figure 6 displays the tropical limit graphs of the corresponding 3-soliton solution at a large negative and a large positive value of $t$. The respective sequences of interactions correspond to the two sides of the Yang–Baxter equation. Since the polarizations do not depend on the variables $z$, $t$, $k$, we conclude that, starting with the same initial polarizations, in both cases we end up with the same polarizations. This implies that the maps acting along the tropical limit graph satisfy a mixed version of the Yang–Baxter equation.

The polarizations at a node of the tropical limit graph at $t = t_0$ are completely determined if we know the soliton numbers associated with two “incoming lines,” e.g., soliton 1 with the left and soliton 2 with the right line, and the number of the enclosed phase region, for example 222. In this configuration, to the left of incoming soliton line 1 we have the phase with number 122, to the right of incoming soliton line 2 the phase with number 212, and the remaining one has necessarily number 112. Hence, there are incoming polarizations $\hat{u}_{122,222}$, $\hat{u}_{212,222}$ and outgoing polarizations $\hat{u}_{112,212}$, $\hat{u}_{112,122}$ of solitons 1 and 2. These can be computed from the above 3-soliton solution, and it can be verified that they are related by one of the maps considered in Sect. 4.4. This also holds for all possible other nodes.

In the following, we consider the situation shown in the left plot in Fig. 6, regarding it as a process from bottom to top. Accordingly, we set

$$
\begin{align*}
    u_{1,\text{in}} &= \hat{u}_{121,221}, & u_{1,\text{m1}} &= \hat{u}_{111,211}, & u_{1,\text{out}} &= \hat{u}_{111,212}, \\
    u_{2,\text{in}} &= \hat{u}_{221,211}, & u_{2,\text{m1}} &= \hat{u}_{121,111}, & u_{2,\text{out}} &= \hat{u}_{122,112}, \\
    u_{3,\text{in}} &= \hat{u}_{211,212}, & u_{3,\text{m1}} &= \hat{u}_{111,112}, & u_{3,\text{out}} &= \hat{u}_{121,122},
\end{align*}
$$

where a subscript m1 indicates that the line segment appears in the middle of the first plot in Fig. 6. For the right plot in Fig. 6, we only have to replace the polarizations of middle segments by

$$
\begin{align*}
    u_{1,\text{m2}} &= \hat{u}_{122,222}, & u_{2,\text{m2}} &= \hat{u}_{222,212}, & u_{3,\text{m2}} &= \hat{u}_{221,222}.
\end{align*}
$$
Fig. 7 The dominating phase structures around the three crossings of solitons in the left plot of Fig. 6, proceeding from bottom to top. In the phase numbers, we marked (with red color) the digit corresponding to the soliton that does not take part in the respective interaction. Disregarding this digit, the drawing describes the phase constellation of a 2-soliton interaction. In this way, for example, the first drawing corresponds to an application of $\mathcal{T}$.

Fig. 8 The dominating phase structures around the three crossings of solitons in the right plot of Fig. 6, proceeding from bottom to top.

Now, we can check that either the Yang–Baxter map $\mathcal{R}$, the map $\mathcal{T}$, or one of their inverses acts at each crossing of the tropical limit graph. Proceeding from bottom to top in the left plot of Fig. 6, the first crossing involves only solitons 1 and 2, so we can ignore the last digit of the numbers of the involved phases. The situation is that of the 2-soliton interaction sketched in the first drawing in Fig. 7. Accordingly, $\mathcal{T}$, given by (3.5), should map $(u_{1,\text{in}}, u_{2,\text{in}})$ to $(u_{1,\text{m1}}, u_{2,\text{m1}})$. Indeed, this can be verified using the above data.

The next crossing, where solitons 1 and 3 interact, is sketched as a 2-soliton interaction in the second drawing in Fig. 7. In this situation, the Yang–Baxter map $\mathcal{R}$ should yield $(u_{1,\text{m1}}, u_{3,\text{in}}) \mapsto (u_{1,\text{out}}, u_{3,\text{m1}})$. Indeed, we find

$$u_{1,\text{out}} = \frac{B_3(p_1, u_{3,\text{in}}) u_{1,\text{m1}} \tilde{A}_3(q_1, u_{3,\text{in}})}{\text{tr}[B_3(p_1, u_{3,\text{in}}) u_{1,\text{m1}} \tilde{A}_3(q_1, u_{3,\text{in}}) K]},$$

(which can be deduced from (D.1)). Similarly,

$$u_{3,\text{m1}} = \frac{A_1(q_3, u_{1,\text{m1}}) u_{3,\text{in}} \tilde{B}_1(p_3, u_{1,\text{m1}})}{\text{tr}[A_1(q_3, u_{1,\text{m1}}) u_{3,\text{in}} \tilde{B}_1(p_3, u_{1,\text{m1}}) K]}.$$

Finally, at the last crossing solitons 2 and 3 interact. The situation is sketched in the last drawing in Fig. 7. Accordingly, we expect the map $\mathcal{T}^{-1}$ to be at work and this can indeed be verified.

The 2-soliton subinteractions appearing in the right plot of Fig. 6 are sketched in Fig. 8. For the lowest, corresponding to the first drawing in Fig. 8, we can verify (see Appendix D) that the polarizations are related by the map $\mathcal{T}^{-1}$, given by (3.7). The second drawing in Fig. 8 corresponds to an application of the Yang–Baxter map $\mathcal{R}$ and the third to an application of $\mathcal{T}$. 

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At least for the parameter range, for which the tropical limit graphs at large negative and positive time $t$ have the structure shown in Fig. 6, we can now conclude that (3.6) holds. This is so because the polarizations associated with line segments of the tropical limit graph do not depend on the variables $z, t, k$. Since the two situations shown in Fig. 6 belong to the same solution of the 2DTL$_K$ equation, the result of the application of the two sides of (3.6) is the same. We know that (3.6) indeed holds for all parameter values (provided that $p_i, q_i$ are all different).

In the following examples, we present tropical limit graphs for pure 3-soliton solutions, which show a structure different from that in Fig. 6.

**Example 4.13** We choose $K$ and $\eta_i, \xi_i, i = 1, 2, 3$, as in Example 4.12, and set

\[ p_1 = 1/2, \quad p_2 = 10, \quad p_3 = 2, \quad q_1 = 1, \quad q_2 = 20, \quad q_3 = 4. \]

Corresponding tropical limit graphs at constant values of $t$ are shown in Fig. 9, from which we read off

\[ T_{12}^{-1}(1, 2) \circ T_{13}^{-1}(1, 3) \circ R_{23}(2, 3) = R_{23}(2, 3) \circ T_{13}^{-1}(1, 3) \circ T_{12}^{-1}(1, 2), \]

which is (3.12).

**Example 4.14** Again, we choose $K$ and $\eta_i, \xi_i, i = 1, 2, 3$, as in Example 4.12, but now

\[ p_1 = 1, \quad p_2 = 10, \quad p_3 = 1, \quad q_1 = 1/2, \quad q_2 = 10, \quad q_3 = 2. \]

Corresponding tropical limit graphs are shown in Fig. 10, from which we read off

\[ R_{12}(1, 2) \circ R_{13}(1, 3) \circ R_{23}(2, 3) = R_{23}(2, 3) \circ R_{13}(1, 3) \circ R_{12}(1, 2), \]

which is the Yang–Baxter Eq. (1.3).
Fig. 10 Tropical limit graphs of yet another pure 3-soliton solution of the p2DTL\(_K\) equation for a negative (left graph) and a positive (right graph) value of \(t\), using the data of Example 4.14

5 Conclusions

We presented a “Lax representation” for the Yang–Baxter map \(\mathcal{R}\) obtained in [9] from the pure 2-soliton solution of the \((K\)-modified\) matrix KP equation. For the matrix KdV reduction, such a Lax representation had been found earlier in [2], also see [18]. Using also the inverse of the Lax matrix \(A\), we were led to entwining Yang–Baxter maps, in the sense of [16]. Besides the Yang–Baxter map, this involves another map, which is not Yang–Baxter, but both maps satisfy a “mixed Yang–Baxter equation.”

We demonstrated that this structure is indeed realized in 3-soliton interactions. Here, we concentrated on an analysis of matrix generalizations of the two-dimensional Toda lattice equation, of which solitons can be generated via a binary Darboux transformation. For the subclass of “pure solitons,” which only exhibit elastic scattering (i.e., no merging or splitting of solitons), we elaborated the tropical limit of the general 2- and 3-soliton solution. It turned out that exactly the same Yang–Baxter map as in the KP\(_K\) case is a work. The crucial new insight is that the flow of polarizations for three solitons is not, in general, described by the Yang–Baxter map alone, but by entwining Yang–Baxter maps. This also holds for KP\(_K\) and even for KdV\(_K\) (with \(K = 1_n\), for example), also see Appendix A. This result crucially relies on our tropical limit analysis of solitons.

More precisely, in the preceding section we found that (1.3), (3.6) and also (3.12) are realized by pure 2DTL\(_K\) solitons. We do not know whether this is also so for the remaining mixed Yang–Baxter equations in Sect. 3.

Since the soliton with number \(i\) is specified via the parameters \(p_i, q_i\), and the polarization \(u_i\), we can associate the matrices \(A_i(\lambda, u_i)\) and \(B_i(\lambda, u_i)\) with it (as well as \(\tilde{A}_i(\lambda, u_i)\) and \(\tilde{B}_i(\lambda, u_i)\)). Comparing the threefold products of these Lax matrices, which imply a mixed Yang–Baxter equation, with the tropical limit plots for pure 3-solitons, one observes the following rule. Whether \(A_i\) or \(B_i\) is at work depends on the constellation of phases to the left and to the right of the line. For example, if the first soliton has phase 1ab to the left and 2cd to the right \((a, b, c, d \in \{1, 2\})\), we have to choose \(A_1\), whereas with 2ab to the left and 1cd to the right, it has to be \(B_1\).
3-soliton interaction shown in the first plot in Fig. 6 corresponds to

\[
A_1 B_2 A_3 \xrightarrow{T} B_2 A_1 A_3 \xrightarrow{R} B_2 A_3 A_1 \xrightarrow{T^{-1}} A_3 B_2 A_1
\]

(hiding away the parameters). The Lax matrices, which are subject to the refactorization Eq. (2.2), generate a Zamolodchikov–Faddeev algebra. In S-matrix theory of an integrable QFT, these matrices play a role as creation and annihilation operators (see, for example, Section 4.2 in [1], and references cited there).

We also found that the two maps \(R, T\), and their inverses, provide us with solutions of the “WXZ system” [20], called Yang–Baxter system in [21].

We further note that the Yang–Baxter map obtained for the matrix nonlinear Schrödinger (NLS) equation in [3–5] has the same form as the Yang–Baxter map \(R\) for matrix KP and matrix two-dimensional Toda lattice.

Like KP\(_K\), also the 2DTL\(_K\) equation possesses many soliton solutions beyond pure solitons, see Sect. 4.1. A corresponding analysis, following [10], goes beyond Yang–Baxter and will be postponed to future work.

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Appendix A: 2-soliton solutions of the KdV\(_K\) equation

The 2-soliton solution of (1.4) is obtained from the formulas in Sect. 4.3 by replacing the expressions for \(\vartheta_{ab}\) by

\[
\vartheta_{11} = \vartheta(p_1) + \vartheta(p_2), \quad \vartheta_{12} = \vartheta(p_1) + \vartheta(-p_2), \\
\vartheta_{21} = \vartheta(p_2) + \vartheta(-p_1), \quad \vartheta_{22} = \vartheta(-p_1) + \vartheta(-p_2),
\]

where now

\[
\vartheta(p) = px + p^3 t.
\]
To determine the asymptotics of soliton 1 in the 2-soliton solution, we set
\[
x + p_1^2 t = \Lambda_1^{(\pm)},
\]
with constants \( \Lambda_1^{(\pm)} \), and take the limit as \( t \to \pm\infty \). Then, we choose \( \Lambda_1^{(\pm)} \) in such a way that phase shifts are compensated.\(^{14}\) Correspondingly for soliton 2, in this way we obtain the following.

1. Let \( p_2 < p_1 < 0 \). Then, \( u_{11,21} (u_{21,22}) \) and \( u_{12,22} (u_{11,12}) \) are the polarizations of soliton 1 (soliton 2) as \( t \to -\infty \), respectively, \( t \to \infty \). This is the phase constellation shown in Fig. 1. We find the following relations with polarizations defined via the tropical limit,
\[
\hat{u}_{11,21} = \lim_{t \to -\infty} u \bigg|_{x \to -p_1^2 t + \Lambda_1^{(-)}}, \quad \hat{u}_{12,22} = \lim_{t \to +\infty} u \bigg|_{x \to -p_1^2 t + \Lambda_1^{(+)}},
\hat{u}_{21,22} = \lim_{t \to -\infty} u \bigg|_{x \to -p_2^2 t + \Lambda_2^{(-)}}, \quad \hat{u}_{11,12} = \lim_{t \to +\infty} u \bigg|_{x \to -p_2^2 t + \Lambda_2^{(+)}},
\]
where \( \Lambda_1^{(-)} = -\log(\alpha_{12}\kappa_{11})/(2p_1), \quad \Lambda_1^{(+)} = -\log(\kappa_{11})/(2p_1), \quad \Lambda_2^{(-)} = -\log(\kappa_{22})/(2p_2), \quad \Lambda_2^{(+)} = -\log(\alpha_{12}\kappa_{22})/(2p_1) \).

2. Let \( 0 < p_1 < p_2 \). Then, \( u_{12,22} (u_{11,12}) \) and \( u_{11,21} (u_{21,22}) \) are the polarizations of soliton 1 (soliton 2) as \( t \to -\infty \), respectively, \( t \to \infty \). This is the phase constellation shown in the second plot of Fig. 2.

3. Let \( p_2 < 0 < p_1, |p_1| < |p_2| \). Then, \( u_{11,21} (u_{11,12}) \) and \( u_{12,22} (u_{21,22}) \) are the polarizations of soliton 1 (soliton 2) as \( t \to -\infty \), respectively, \( t \to \infty \). This is the phase constellation shown in the third plot of Fig. 2.

4. Let \( p_1 < 0 < p_2, |p_1| < |p_2| \). Then, \( u_{12,22} (u_{21,22}) \) and \( u_{11,21} (u_{11,12}) \) are the polarizations of soliton 1 (soliton 2) as \( t \to -\infty \), respectively, \( t \to \infty \). This is the phase constellation displayed in the third plot of Fig. 2.

In all these cases, we have \( |p_1| < |p_2| \), so that, for large enough negative values of \( t \), soliton 1 appears to the left of soliton 2 in \( x \)-direction. The concrete tropical limit graphs in Figs. 1 and 2 are obtained with
\[
K = (1, 1), \quad \xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta_1 = \eta_2 = 1,
\]
and the following data, respectively:

1. \( p_1 = -1/2, p_2 = -3/2 \).
2. \( p_1 = 1/2, p_2 = 3/2 \).
3. \( p_1 = 1/2, p_2 = -3/2 \).
4. \( p_1 = -1/2, p_2 = 3/2 \).

\(^{14}\) In [3,4], which deal with vector NLS solitons, the derived Yang–Baxter maps include factors due to phase shifts.
Appendix B: Proof of Theorem 2.1

Let $K$ be an $n \times m$ matrix with maximal rank.

**Lemma B.1** Let $K$ have maximal rank and $X_i$, $i = 1, 2$, be rank-one $K$-projections. Let $\alpha, \beta$ be constants such that

$$\gamma := (1 + \alpha)(1 + \beta) - \alpha\beta \operatorname{tr}(X_i K X_j K) \neq 0.$$  

Then,

$$\left(1_m + \alpha X_1 K + \beta X_2 K\right)^{-1} = 1_m - \frac{\alpha(1 + \beta)}{\gamma} X_1 K - \frac{(1 + \alpha)\beta}{\gamma} X_2 K + \frac{\alpha\beta}{\gamma} (X_1 K X_2 K + X_2 K X_1 K).$$

**Proof** Since $X_i$ is a $K$-projection, $X_i K$ and $K X_i$ are ordinary projection (i.e., idempotent) matrices. If $K$ has maximal rank, they have rank one iff $X_i$ has rank one. Hence, there is a column vector $\xi_i$ and a row vector $\eta_i$ such that $X_i K = \xi_i \eta_i$ and $\eta_i \xi_i = 1$ and correspondingly for $K X_i$. It follows that

$$X_i K X_j K X_i K = \operatorname{tr}(X_i K X_j K) X_j K,$$

$$K X_i K X_j K X_i = \operatorname{tr}(K X_i K X_j K) K X_i = \operatorname{tr}(X_i K X_j K) K X_i.$$

Since $K$ is assumed to have maximal rank, this implies

$$X_i K X_j K X_i = \operatorname{tr}(X_i K X_j K) X_i.$$  \hfill (B.1)

Using further the $K$-projection property of $X_i$, our assertion can be directly verified. \hfill \Box

**Proposition B.2** Let $K$ have maximal rank, $p_1, p_2, q_1, q_2$ be pairwise distinct, and $X_i \in S$, $i = 1, 2$. The system (2.2) determines a map $S \times S \rightarrow S \times S$ via $(X_1, X_2) \mapsto (X'_1, X'_2)$, which is given by (1.1).

**Proof** We multiply (2.2) by $(\lambda - q_1)(\lambda - q_2)$ and expand in powers of $\lambda$. Since $K$ is assumed to have maximal rank, from the coefficient linear in $\lambda$ we obtain

$$X'_2 = X_2 - \frac{p_1 - q_1}{p_2 - q_2} (X'_1 - X_1),$$

which can be used to replace $X'_2$ in the $\lambda$-independent part of the expression we started with,

$$0 = q_1(p_2 - q_2)(X'_2 - X_2) + q_2(p_1 - q_1)(X'_1 - X_1) + (p_1 - q_1)(p_2 - q_2)(X'_2 K X'_1 - X_1 K X_2).$$
\[
\begin{align*}
&= (p_1 - q_1)(q_2 - q_1)(X'_1 - X_1) + (p_2 - q_2)(X_2 K X'_1 - X_1 K X_2) \\
&\quad - (p_1 - q_1)(X'_1 - X_1) K X'_1).
\end{align*}
\]

Since \(X'_1\) is a \(K\)-projection, this becomes
\[
(q_2 - q_1)(X'_1 - X_1) + (p_2 - q_2)(X_2 K X'_1 - X_1 K X_2) \\
- (p_1 - q_1)X'_1 + (p_1 - q_1)X_1 K X'_1 = 0,
\]
so that
\[
\left(1_m - \frac{p_1 - q_1}{p_1 - q_2} X_1 K - \frac{p_2 - q_2}{p_1 - q_2} X_2 K\right) X'_1 = \frac{q_1 - q_2}{p_1 - q_2} X_1 \left(1_n - \frac{p_2 - q_2}{q_1 - q_2} K X_2\right)
\]
(B.2)

If \(X_i, i = 1, 2,\) have rank one, the inverse of the matrix multiplying \(X'_1\) is given in the preceding lemma. We have to apply it to the right-hand side of the last expression. First, we compute
\[
\left(1_m - \frac{\alpha (1 + \beta)}{\gamma} X_1 K - \frac{(1 + \alpha) \beta}{\gamma} X_2 K + \frac{\alpha \beta}{\gamma} (X_1 K X_2 K + X_2 K X_1 K)\right) X_1
\]
\[
= \frac{1}{\gamma} ((1 + \beta) 1_m - \beta X_2 K) X_1,
\]
using the \(K\)-projection property of \(X_i, i = 1, 2,\) and (B.1). Here, we have
\[
\alpha = -\frac{p_1 - q_1}{p_1 - q_2}, \quad \beta = -\frac{p_2 - q_2}{p_1 - q_2}, \quad \gamma = \frac{(q_1 - q_2)(p_1 - p_2)}{(p_1 - q_2)^2} \alpha_{12}^{-1},
\]
with \(\alpha_{12}\) defined in (1.2). A straightforward computation now leads to
\[
X'_1 = \alpha_{12} \left(1_m - \frac{p_1 - q_1}{p_1 - q_2} X_1 K - \frac{p_2 - q_2}{p_2 - p_1} X_2 K\right) X_1 \left(1_n - \frac{p_2 - q_2}{q_1 - q_2} K X_2\right),
\]
which is the first of Eq. (1.1). The second equation is obtained in the same way, and we can verify that \(X'_i \in S, i = 1, 2.\)

**Remark B.3** In [2], it has been noted that in the case associated with the matrix KdV Eq. (2.2) determines, more generally, a Yang–Baxter map if the set \(S\) is extended to the set of all \(K\)-projections, i.e., without restriction to rank one. In the more general situation considered in the present work, we observe that (B.2), which has been derived without restriction of the rank, can be rewritten as
\[
\left(1_m - \frac{p_1 - q_1}{p_1 - q_2} X_1 K - \frac{p_2 - q_2}{p_1 - q_2} X_2 K\right) X'_1 = X_1 \left(1_m - \frac{p_1 - q_1}{p_1 - q_2} K X_1 - \frac{p_2 - q_2}{p_1 - q_2} K X_2\right)
\]
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(cf. [5] for the NLS case). If the matrix multiplying \( X'_1 \) is invertible, it follows that the latter is a \( K \)-projection. A corresponding argument shows that also \( X'_2 \) is a \( K \)-projection. A convenient formula for the inverse of the matrix multiplying \( X'_1 \) (and correspondingly for \( X'_2 \)), like that given in Lemma B.1 for the rank-one case, is not available, however. At least in the special case where \( X_1 \) and \( X_2 \) both have rank \( r \) and satisfy

\[
X_i K X_j K X_i = \mu X_i \quad i, j = 1, 2, \quad i \neq j, \tag{B.3}
\]

which implies that the scalar \( \mu \) is given by \( r^{-1} \text{tr}(X_i K X_j K) \), one can show that

\[
X'_1 = r \frac{B_2(p_1, X_2) X_1 \tilde{A}_2(q_1, X_2)}{\text{tr}[B_2(p_1, X_2) X_1 \tilde{A}_2(q_1, X_2) K]}, \quad X'_2 = r \frac{A_1(q_2, X_1) X_2 \tilde{B}_1(p_2, X_1)}{\text{tr}[A_1(q_2, X_1) X_2 \tilde{B}_1(p_2, X_1) K]}.
\]

Since (B.3) holds for any two rank-one \( K \)-projections, we recover (2.3).

**Appendix C: Derivation of the binary Darboux transformation for the p2DTL\(_K\) equation**

We recall a binary Darboux transformation result of bidifferential calculus [33,34].

**Theorem C.1** Let \((\Omega, d, \bar{d})\) be a bidifferential calculus and \(\Delta, \Gamma, \lambda, \kappa\) solutions of

\[
\bar{d}\Gamma = \Gamma d\Gamma + [\kappa, \Gamma], \quad \bar{d}\kappa = \Gamma d\kappa + \kappa^2, \\
\bar{d}\Delta = (d\Delta) \Delta - [\lambda, \Delta], \quad \bar{d}\lambda = (d\lambda) \Delta - \lambda^2,
\]

and \(\phi_0\) a solution of

\[
d\bar{d}\phi + d\phi \ K \ d\phi = 0, \tag{C.1}
\]

where \(dK = 0 = \bar{d}K\). Let \(\theta\) and \(\chi\) be solutions of the linear system

\[
\bar{d}\theta = (d\phi_0) K \theta + (d\theta) \Delta + \theta \lambda, \tag{C.2}
\]

and the adjoint linear system

\[
\bar{d}\chi = -\chi K d\phi_0 + \Gamma d\chi + \kappa \chi. \tag{C.3}
\]

Let \(\Omega\) solve the compatible linear system

\[
\Gamma \Omega - \Omega \Delta = \chi K \theta, \\
\bar{d}\Omega = (d\Omega) \Delta - (d\Gamma) \Omega + (d\chi) K \theta + \kappa \Omega + \Omega \lambda. \tag{C.4}
\]
Where $\Omega$ is invertible,

$$\phi = \phi_0 - \theta \Omega^{-1} \chi$$  \hspace{1cm} (C.5)

is a new solution of (C.1).

In the above theorem, we have to assume that all objects are such that the corresponding products are defined and that $d$ and $\bar{d}$ can be applied. Next, we define a bidifferential calculus via

$$df = [S, f] \xi_1 + f_y \xi_2,$$
$$\bar{df} = f_x \xi_1 - [S^{-1}, f] \xi_2,$$

on the algebra $\mathcal{A} = \mathcal{A}_0[S, S^{-1}]$, where $\mathcal{A}_0$ is the algebra of smooth functions of two variables, $x$ and $y$, and also dependent on a discrete variable on which the shift operator $S$ acts. $\xi_1, \xi_2$ constitute a basis of a two-dimensional vector space $V$, from which we form the Grassmann algebra $\Lambda(V)$. $d$ and $\bar{d}$ extend to $\Omega = \mathcal{A} \otimes \Lambda(V)$ in a canonical way and to matrices with entries in $\Omega$. Setting

$$\phi = \phi S^{-1},$$

Eq. (C.1) is equivalent to the p2DTL$_K$ Eq. (4.1). Choosing a solution $\phi_0 = \varphi_0 S^{-1}$ and setting

$$\Delta = \Gamma = S^{-1}, \quad \kappa = \lambda = 0,$$

the linear system (C.2) and the adjoint linear system (C.3) lead to (4.5) and (4.6), respectively. Furthermore, via $\Omega \mapsto \Omega S$, (C.4) implies (4.7). According to the theorem, (C.5) yields a new solution of the p2DTL$_K$ Eq. (4.1).

**Appendix D: Computational details for Section 4.5**

Using the 3-soliton solution in Sect. 4.5, and the notation introduced there, we find that

$$u_{i, \bullet} = \xi_{i, \bullet} \otimes \eta_{i, \bullet},$$

where $\bullet$ stands for in, out, m1 or m2,

$$\xi_{1,\text{in}} = \frac{1}{\alpha_{13}} A_3(p_1, \Xi_3) \frac{\xi_1}{\kappa_{11}}, \quad \eta_{1,\text{in}} = \eta_1 \tilde{B}_3(q_1, \Xi_3),$$
$$\xi_{1,m1} = \frac{\alpha_{23}}{\beta} \left(1_m - \frac{(p_2 - q_2)\alpha_{321}}{(p_1 - q_2)\alpha_{23}} \Xi_2 K - \frac{(p_3 - q_3)\alpha_{231}}{(p_1 - q_3)\alpha_{23}} \Xi_3 K \right) \frac{\xi_1}{\kappa_{11}},$$
$$\eta_{1,m1} = \eta_1 \left(1_n - \frac{(p_2 - q_2)\alpha_{312}}{(p_2 - q_1)\alpha_{23}} K \Xi_2 - \frac{(p_3 - q_3)\alpha_{213}}{(p_3 - q_1)\alpha_{23}} K \Xi_3 \right),$$

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\[\xi_{1,\text{out}} = \frac{1}{\alpha_{12}} A_2(p_1, \Xi_2) \frac{\xi_1}{\kappa_{11}}, \quad \eta_{1,\text{out}} = \eta_1 \tilde{B}_2(q_1, \Xi_2),\]

\[\xi_{2,\text{in}} = \frac{1}{\alpha_{23}} A_3(p_2, \Xi_3) \frac{\xi_2}{\kappa_{22}}, \quad \eta_{2,\text{in}} = \eta_2 \tilde{B}_3(q_2, \Xi_3),\]

\[\xi_{2,m1} = \frac{\alpha_{13}}{\beta} \left( 1 - m - \frac{(p_1 - q_1)\alpha_{132}}{(p_2 - q_1)\alpha_{13}} \Xi_1 K - \frac{(p_3 - q_3)\alpha_{132}}{(p_2 - q_3)\alpha_{13}} \Xi_3 K \right) \frac{\xi_2}{\kappa_{22}}, \]

\[\eta_{2,m1} = \eta_2 \left( 1 - n - \frac{(p_1 - q_1)\alpha_{231}}{(p_2 - q_2)\alpha_{13}} K \Xi_1 - \frac{(p_3 - q_3)\alpha_{123}}{(p_2 - q_2)\alpha_{13}} K \Xi_3 \right),\]

\[\xi_{2,\text{out}} = \frac{1}{\alpha_{12}} A_1(p_2, \Xi_1) \frac{\xi_2}{\kappa_{22}}, \quad \eta_{2,\text{out}} = \eta_2 \tilde{B}_1(q_2, \Xi_1),\]

\[\xi_{3,\text{in}} = \frac{1}{\alpha_{23}} A_2(p_3, \Xi_2) \frac{\xi_3}{\kappa_{33}}, \quad \eta_{3,\text{in}} = \eta_3 \tilde{B}_2(q_3, \Xi_2),\]

\[\xi_{3,m1} = \frac{\alpha_{12}}{\beta} \left( 1 - m - \frac{(p_1 - q_1)\alpha_{123}}{(p_3 - q_1)\alpha_{12}} \Xi_1 K - \frac{(p_3 - q_3)\alpha_{123}}{(p_3 - q_2)\alpha_{12}} K \Xi_2 \right) \frac{\xi_3}{\kappa_{33}}, \]

\[\eta_{3,m1} = \eta_3 \left( 1 - n - \frac{(p_1 - q_1)\alpha_{231}}{(p_1 - q_3)\alpha_{12}} K \Xi_1 - \frac{(p_2 - q_3)\alpha_{123}}{(p_3 - q_2)\alpha_{12}} K \Xi_2 \right),\]

\[\xi_{3,\text{out}} = \frac{1}{\alpha_{13}} A_1(p_3, \Xi_1) \frac{\xi_3}{\kappa_{33}}, \quad \eta_{3,\text{out}} = \eta_3 \tilde{B}_1(q_3, \Xi_1),\]

and

\[\xi_{1,m2} = \frac{\xi_1}{\kappa_{11}}, \quad \eta_{1,m2} = \eta_1, \quad \xi_{2,m2} = \frac{\xi_2}{\kappa_{22}}, \quad \eta_{2,m2} = \eta_2, \quad \xi_{3,m2} = \frac{\xi_3}{\kappa_{33}}, \quad \eta_{3,m2} = \eta_3.\]

Here, we used (2.1) and introduced the rank-one \(K\)-projections

\[\Xi_i := \frac{\xi_i \otimes \eta_i}{\kappa_{ii}} \quad i = 1, 2, 3.\]

For example, we find

\[\xi_{1,\text{out}} = \frac{B_3(p_1, u_{3,\text{in}}) \xi_{1,m1}}{\text{tr}[B_3(p_1, u_{3,\text{in}}) u_{1,m1} \tilde{A}_3(q_1, u_{3,\text{in}}) K]}, \]

\[\eta_{1,\text{out}} = \eta_{1,m1} \tilde{A}_3(q_1, u_{3,\text{in}}), \quad \text{(D.1)}\]

and

\[\xi_{2,m2} = B_3(q_2, u_{3,\text{in}}) \xi_{2,\text{in}}, \quad \eta_{2,m2} = \frac{\eta_{2,\text{in}} \tilde{A}_3(p_2, u_{3,\text{in}})}{\text{tr}(B_3(q_2, u_{3,\text{in}}) u_{2,\text{in}} \tilde{A}_3(p_2, u_{3,\text{in}}) K)}, \]

\[\xi_{3,m2} = B_2(q_3, u_{2,\text{in}}) \xi_{3,\text{in}}, \quad \eta_{3,m2} = \frac{\eta_{3,\text{in}} \tilde{A}_3(p_3, u_{2,\text{in}})}{\text{tr}(B_2(q_3, u_{2,\text{in}}) u_{3,\text{in}} \tilde{A}_2(p_3, u_{2,\text{in}}) K)}, \]

from which some statements in Sect. 4.5 are quickly deduced.
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