Large complete minors in expanding graphs

Younjin Kim

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Abstract

In 2009, Krivelevich and Sudakov studied the existence of large complete minors in \((t, \alpha)\)-expanding graphs whenever the expansion factor \(t\) becomes super-constant. In this paper, we give an extension of the results of Krivelevich and Sudakov by investigating a connection between the existence of large complete minors in graphs and good vertex expansion properties.

Keywords complete minors, expanding graphs, sublinear expanders

1 Introduction

A graph here will refer to a simple undirected graph without loops. An undirected graph \(G\) is a set \(V = V(G)\) of vertices and a set \(E(G)\) of edges of \(G\). A graph \(H\) is a minor of \(G\), denoted by \(G \succ H\), if it can be obtained from \(G\) by vertex deletions, edge deletions, and contractions. A graph \(G\) is an \(\alpha\)-expander if every set \(A\) of at most half of the vertices of \(G\) has at least \(\alpha \cdot |A|\) outside neighbors in \(G\). A graph \(G\) is a \((t, \alpha)\)-expanding if every subset \(X \subset V(G)\) of size \(|X| \leq \frac{\alpha |V(G)|}{t}\) has at least \(t|X|\) external neighbors in \(G\). An abundance of various results and conjectures providing sufficient conditions for the existence of large complete minors have been

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widely investigated in the context of undirected simple graphs. For example, the famous Hadwiger’s conjecture addressed a connection between the chromatic number and the existence of large minors, which states if $G$ is a graph with the chromatic number $\chi(G)$, then $G$ contains a clique minor of size at least $\chi(G)$. Other authors also discussed a connection between the existence of large minors and graphs with large girth $[4, 15, 17, 23]$, $K_{s,t}$-free graphs $[15, 18]$, graphs without small vertex separators $[1, 10, 21]$, lifts of graphs $[5]$, random graphs $[7]$, random regular graphs $[8]$, and others.

The existence of large complete minors in $(t, \alpha)$-expanding graphs with the expansion factor $t = \Theta(1)$ was studied by Alon, Seymour, and Thomas $[1]$, Plotkin, Rao, and Smith $[21]$, and Kleinberg and Rubinfeld $[11]$. Especially, in 1996, Kleinberg and Rubinfeld $[11]$ proved that for every fixed $\alpha > 0$, there exists a constant $c > 0$ such that an $\alpha$-expander graph of order $n$ contains every graph $H$ with at most $\frac{n}{\log n}$ vertices and edges as a minor. As an extension of their result, in 2009, Krivelevich and Sudakov $[15]$ obtained a connection between complete minors in graphs and vertex expansion properties as follows.

**Theorem 1.1** (Krivelevich, Sudakov $[15]$). Let $0 < \alpha < 1$ be a constant, and let $t \geq 10$. Then a $(t, \alpha)$-expanding graph of order $n$ contains a minor with an average degree of at least

$$c \frac{\sqrt{nt \log t}}{\sqrt{\log n}},$$

where $c = c(\alpha) > 0$ is a constant.

With the results of Kostochka $[13]$ and Thomason $[22]$, Krivelevich and Sudakov $[15]$ also got the following result, which enables us to show the existence of larger minors in $(t, \alpha)$-expanding graphs whenever the expansion factor $t$ becomes super-constant.

**Theorem 1.2** (Krivelevich, Sudakov $[15]$). Let $0 < \alpha < 1$ be a constant, and let $t \geq 10$. Then a $(t, \alpha)$-expanding graph of order $n$ contains a clique minor of size

$$\Omega \left( \alpha^2 \frac{\sqrt{nt \log t}}{\sqrt{\log n}} \right).$$

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Recently Krivelevich and Nenadov [16] complemented the results of Krivelevich and Sudakov [15] by investigating a connection between the contraction clique number of a graph and good edge expansion properties. Usually, an advantage of edge expansion over vertex expansion is that it is easier to verify. In this paper, we investigate a connection between complete minors in graphs and good vertex expansion properties as follows. In the following theorem, we use the expander notation introduced by Haslegrave, Kim, and Liu [9] such that similar expansion occurs even after removing a relatively small set of edges. Our result is an extension of the results of Krivelevich and Sudakov [15]. An n-vertex graph $G$ with the minimum degree $d$ is $\epsilon$-locally sparse if for every subset $U \subseteq V(G)$ of size $|U| \leq \epsilon n$, then an average degree $d(G[U])$ on the induced subgraph $G[U]$ is at most $\epsilon d$. An n-vertex graphs $G$ is $\epsilon$-vertex-expand if for every subset $U \subseteq V(G)$ of size $|U| \leq \frac{n}{2}$, we have $|N(U)| \geq \epsilon |U|$.

**Theorem 1.3.** For $\epsilon > 0$ and $d > 0$, we let $G$ be a graph of order $n$ satisfying the following conditions:

- $\frac{\epsilon}{2}$-locally sparse
- $\epsilon$-vertex-expand
- minimum degree $d$.

Then $G$ contains a clique minor of size

$$\tilde{\Omega}\left(\frac{\sqrt{nd}}{\log^{10} n}\right).$$

Our proof utilizes robust sublinear expanders, which is an extension of a notion of expander introduced in the 90s by Komlós and Szemerédi [12]. It has proven to be a powerful tool for embedding sparse graphs, playing an essential role in the recent resolution of several long-standing conjectures that were previously out of reach (see [6], [19], [20]).

Our paper is organized as follows. In Section 2, we introduce some preliminaries on expanders. In Section 3, we give the proof of Theorem 1.3.
2 Preliminaries

We denote by \([n]\) the set \(\{1, 2, \ldots, n\}\) of the first \(n\) positive integers. Given a set \(X\) and \(k \in \mathbb{N}\), we let \(\binom{X}{k}\) for the family of its \(k\)-element subsets. For brevity, we write \(v\) for a singleton set \(\{v\}\) and \(xy\) for a set of pairs \(\{x, y\}\).

Given a graph \(G\), denote its average degree \(2e(G)/|G|\) by \(d(G)\). Let \(F \subseteq G\) and \(H\) be graphs, and \(U \subseteq V(G)\). We write \(G[U] \subseteq G\) for the induced subgraph of \(G\) on vertex set \(U\). Denote by \(G \cup H\) the graph with vertex set \(V(G) \cup V(H)\) and edge set \(E(G) \cup E(H)\), and write \(G - U\) for the induced subgraph \(G[V(G) \setminus U]\), and \(G \setminus F\) for the spanning subgraph of \(G\) obtained from removing the edge set of \(F\).

2.1 Sublinear expander

Expanders are typically well-connected sparse graphs in which vertex subsets exhibit expansions. Originally introduced for network design, expanders, apart from being a central notion in graph theory, have also close interplay with other areas of mathematics and as well as theoretical computer science. This is partially reflected by the fact that expanders have equivalent definitions from different angles. Indeed, in terms of graph expansions, an expander is a graph whose vertex subsets have a large neighborhood: while algebraically, there is a spectral gap when you look at its adjacency matrix; and probabilistically, there is a spectral gap when you look at its adjacency matrix; random walks on expanders are rapidly mixing.

In 1994, Komlós and Szemeredi [12] first introduced the following notion of graph expansion. For \(\epsilon > 0\) and \(k > 0\), we let \(\rho(x)\) be the function

\[
\rho(x) = \rho(x, \epsilon_1, t) := \begin{cases} 
0 & \text{ if } x < \frac{t}{5}, \\
\epsilon_1 / \log^2(15x/t) & \text{ if } x \geq \frac{t}{5}.
\end{cases}
\]

Note that when \(x \geq t/2\), \(\rho(x)\) is decreasing, while \(\rho(x) \cdot x\) is increasing. Komlós and Szemeredi [12] said that a graph \(G\) is an \((\epsilon_1, t)\)-expander if for any subset \(X \subseteq V(G)\) of size \(t/2 \leq |X| \leq |G|/2\), we have \(|N_G(X)| \geq \rho(|X|)|X|\).

Recently, Haslegrave, Kim, and Liu [9] extended the expander notation such that similar expansion occurs even after removing a relatively small set of edges as follows. We will use the following sublinear expander introduced by Haslegrave, Kim, and Liu [9].
Definition 2.1 ((\(\epsilon, t\))-expander). A graph \(G\) is an \((\epsilon, t)\)-expander if

- for any subset \(X \subseteq V(G)\) of size \(t/2 \leq |X| \leq |G|/2\),
- and any subgraph \(F \subseteq G\) with \(e(F) \leq d(G) \cdot \rho(|X|)|X|\),

then we have

\[ |N_{G\setminus F}(X)| \geq \rho(|X|)|X| \]

where

\[ \rho(x) = \rho(x, \epsilon, t) := \begin{cases} 0 & \text{if } x < t/5, \\ \epsilon_1/\log^2(15x/t) & \text{if } x \geq t/5. \end{cases} \]

Note that the expansion rate of the expander above is only sublinear. Also, Haslegrave, Kim, and Liu [9] gave the following statement that every graph contains one such sublinear expander subgraph with almost the same average degree.

Theorem 2.2 (Haslegrave, Kim, and Liu [9]). There exists some \(\epsilon_1 > 0\) such that the following holds for every \(t > 0\). Every graph \(G\) has an \((\epsilon_1, t)\)-expander subgraph \(H\) with \(d(H) \geq d(G)/2\) and \(\delta(H) \geq d(H)/2\).

A key property [Corollary 2.3 [12]] of the expanders we use is the following lemma, that there exists a short path between two sufficiently large sets. This is formalized in the following statement.

Lemma 2.3 (Haslegrave, Kim, and Liu [9]). Let \(\epsilon_1, t > 0\). If \(G\) is an \(n\)-vertex \((\epsilon_1, t)\)-expander, then any two vertex sets \(X_1, X_2\), each of size at least \(x \leq t\), are of distance at most \(m = \frac{1}{\epsilon_1} \log^3(15n/t)\) apart. This remains true even after deleting \(\epsilon(x) \cdot x/4\) vertices from \(G\).

3 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. First, let us consider

\[ q = \sqrt[n]{d \text{ poly log } n} \quad \text{and} \quad k = \tilde{\Omega}\left(\frac{\sqrt{nd}}{\log^{10} n}\right), \]
where \( k \) is the size of the complete minor we want to find.

We will construct \( q + 2 \) disjoint sets \( B_1, B_2, \ldots, B_q, D, U \) with \( |B_i| = b = \frac{\epsilon^2 n}{k \log_{10} n} \) satisfying the following conditions:

1. \( V(G) = D \cup B_1 \cup B_2 \cup \cdots \cup B_q \cup U \)

2. for each \( i \in [q] \), all induced subgraph \( G[B_i] \) are connected and \( |N(B_i)| \geq (1 - 2\epsilon)bd \)

3. \( U \) is \((\epsilon/10, \alpha)\)-expander

4. \( |N(D) \cap U| < \epsilon/10 \cdot |D| \)

5. pairs \((B_i, B_j)\) satisfy that \( B_i \) is connected to \( B_j \) for at least \( 0.9q \) indices \( j < i \)

6. \( |D| \leq \frac{2}{\epsilon} |B| \), where \( B = B_1 \cup B_2 \cup \cdots \cup B_q \).

Let us consider \( q \) disjoint sets \( B_1, B_2, \ldots, B_q \) with size \( |B_i| = b \) such that all induced subgraphs \( G[B_i] \) are connected and \( |N(B_i)| \geq (1 - 2\epsilon)bd \).

First, we claim that \( D \leq 2\epsilon n \). Since \( N_G(D) \geq \epsilon|D| \), from property (4), we obtain

\[
|N_G(D) \cap B| = |N_G(D)| - |N_G(D) \cap U| \geq \epsilon|D| - \epsilon/10|D| = 9\epsilon/10|D|.
\]

Because of \( q \leq k \), we have \( |B| \leq bq \leq bk \leq \frac{\epsilon^2 n}{\log_{10} n} \leq \epsilon^2 n \). Therefore we conclude that

\[
|D| \leq \frac{10}{9\epsilon} |B| \leq \frac{10}{9} \epsilon n \leq 2\epsilon n. \tag{3.1}
\]

It means that \( |D| \leq \frac{2}{\epsilon} |B| \), which is the property (6).

Let us consider the set \( Bad = \{ i \leq q \mid |N_G(B_i) \cap U| < \frac{\epsilon}{10} bd \} \).

Since \( |N_G(B_i)| \geq (1 - 2\epsilon) bd \), for each \( i \in Bad \), we derive that

\[
e_G(B_i, D) \geq |N_G(B_i)| - |N_G(B_i) \cap U| \geq (1 - 2\epsilon - \frac{\epsilon}{10}) bd. \tag{3.2}
\]
To ensure the property (5), we suppose for the sake of contradiction that $|Bad| \geq 0.9 q$. Now we consider an average degree $d(G[\cup_{i \in Bad}B_i \cup D])$ on the induced subgraph $G[\cup_{i \in Bad}B_i \cup D]$ as follows.

Using the equations (3.1) and (3.2), we have

$$d(G[\cup_{i \in Bad}B_i \cup D]) \geq \frac{\sum_{q \in Bad} e_q(B_i, D)}{\sum_{i \in Bad} |B_i| + |D|} \geq \frac{(1 - 3\epsilon)bd \cdot (0.9q)}{bq + \frac{4q}{\epsilon} |B|} \geq \frac{(1 - 3\epsilon)0.9bdq}{\frac{26q}{\epsilon}} \geq \frac{2\epsilon}{3} d$$

which is a contradiction to the locally sparseness property.

For each $i \notin Bad$, we consider $U_i = U \cap N_G(B_i)$ with the size $|U_i| \geq \frac{4\epsilon}{10} bd \geq \frac{1}{10} \sqrt{nd} \cdot \log n$. From the expansion of $G$, we obtain the following lemma that for a vertex in $X$ and integer $q = \sqrt{\frac{n}{d}} \cdot \text{poly} \log n$, there are at least $\frac{1}{10} \sqrt{nd} \cdot \log n$ vertices which are within distance at most $q = \sqrt{\frac{n}{d}} \cdot \text{poly} \log n$.

**Lemma 3.1.** If we pick $\sqrt{nd} \cdot \log n$ points $X$ at random, then we have

$$\sum_{i \notin Bad} \text{dist}(X, U_i) \leq \sqrt{\frac{n}{d}} \cdot \text{poly} \log n.$$  

Using Lemma 3.1, we find a connected set $B'$ hitting all $N_G(B_i)$ with the size $|B_i| \leq \sqrt{\frac{n}{d}} \cdot \log^5 n$ for all $i \notin Bad$. Next, we find a connected set $B''$ in $U$ with the size $|B''| = b - |B'|$ satisfying $N_G(B'') \geq (1 - \epsilon/2)d \cdot |B''|$. If we let $B_{q+1} = B' \cup B''$, then we have $V(G) = D \cup B_1 \cup B_2 \cup \cdots B_{q+1} \cup U$ and $B = B_1 \cup B_2 \cup \cdots \cup B_q \cup B_{q+1}$.

By the construction of the set $B_{q+1}$, we check that the property (2) is satisfied. To ensure the property (3), we move the following set $D_{q+1} \subseteq U$ to $D$. 


Let us consider the maximal set $D_{q+1}$ such that

(i) $\frac{\epsilon}{10}$-expand in $U$

(ii) $|D_{q+1}| \leq \epsilon n$

We suppose for the sake of contradiction that we cannot move the set $D_{q+1}$ to $D$. Let $U'$ be the largest $\frac{\epsilon}{2}$-expander in $U$. Now we consider the set $D_{q+1} \cup X$. If $|X \cup D_{q+1}| \leq \epsilon n$, then the properties (i) and (ii) hold for the set $D_{q+1} \cup X$, which is a contradiction to the maximality of $D_{q+1}$. If $|X \cup D_{q+1}| > \epsilon n$, then we have

$$|N_G(X \cup D_{q+1})| \leq |B| + |D| + \frac{\epsilon n}{10} \leq \epsilon^4 n + 2\epsilon^3 n + \frac{\epsilon n}{10}.$$ 

This is a contradiction to the $\frac{\epsilon}{10}$-expand property of $U$. Therefore we move the set $D_{q+1}$ to $D$. Now we have the following $q + 3$ disjoint sets $B_1, B_2, \cdots, B_q, B_{q+1}, D_{\text{new}}, U_{\text{new}}$ such that

(a) $B = B_1 \cup B_2 \cup \cdots \cup B_q \cup B_{q+1}$

(b) $D_{\text{new}} = D_{\text{odd}} \cup D_{q+1}$

(c) $U_{\text{odd}} = U_{\text{new}} \cup D_{q+1}$.

From the expansion of $G$, we easily check that the property (3) holds for the set $U_{\text{new}}$. Since $|N(D_{q+1}) \cap U| \leq \frac{\epsilon}{10} |D_{q+1}|$, we observe that

$$\frac{\epsilon}{10} |D_{\text{odd}}| \geq |N(D_{\text{odd}}) \cap U_{\text{odd}}| \geq |N(D_{\text{new}}) \cap U_{\text{new}}| - |N(D_{q+1}) \cap U_{\text{new}}|$$

$$\geq |N(D_{\text{new}}) \cap U_{\text{new}}| - \frac{\epsilon}{10} |D_{q+1}|.$$ 

Therefore, we conclude that

$$|N(D_{\text{new}}) \cap U_{\text{new}}| \leq \frac{\epsilon}{10} |D_{\text{new}}|,$$

which is the property (4).
From property (4), we obtain that
\[
|N_G(D_{\text{new}}) \cap B| = |N_G(D_{\text{new}})| - |N_G(D_{\text{new}}) \cap U_{\text{new}}| \\
\geq \epsilon \cdot |D_{\text{new}}| - \frac{\epsilon}{10} \cdot |D_{\text{new}}| = \frac{9\epsilon}{10} \cdot |D_{\text{new}}|.
\]
Then we conclude that
\[
|D_{\text{new}}| \leq \frac{10}{9\epsilon} |B| \leq \frac{2}{\epsilon} |B| \tag{3.3}
\]
which is the property (6).

Let us consider the set \(Bad = \{ i \leq q + 1 \mid |N_G(B_i) \cap U_{\text{new}}| < \frac{\epsilon}{10} bd \}\).

By the construction of \(B_{q+1}\), for each \(i \in Bad\) we have \(|N_G(B_i)| \geq (1 - 2\epsilon) bd\). From property (4), we derive that
\[
e_G(B_i, D_{\text{new}}) \geq |N_G(B_i)| - |N_G(B_i) \cap U_{\text{new}}| \geq (1 - 2\epsilon - \frac{\epsilon}{10}) bd. \tag{3.4}
\]

To ensure the property (5), we suppose for the sake of contradiction that \(|Bad| \geq 0.9q\). Now we consider an average degree \(d(G[\cup_{i \in Bad} B_i \cup D_{\text{new}}])\) on the induced subgraph \(G[\cup_{i \in Bad} B_i \cup D_{\text{new}}]\) as follows.

Using the equations (3.3) and (3.4), we have
\[
d(G[\cup_{i \in Bad} B_i \cup D_{\text{new}}]) \geq \frac{\sum_{q \in Bad} e_q(B_i, D_{\text{new}})}{\sum_{i \in Bad} |B_i| + |D_{\text{new}}|} \geq \frac{(1 - 3\epsilon)bd \cdot (0.9q)}{bq + \frac{10}{9\epsilon} |B|} \\
\geq \frac{d}{\frac{2bq}{\epsilon}} \geq \frac{2\epsilon}{3} d
\]
which is a contradiction to the locally sparseness property. We repeat this process as long as \(bq \leq \frac{\epsilon^2 n}{\log^{\Theta(n)}}\).

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