Ultrafast deterministic generation of entanglement in a time-dependent asymmetric two-qubit-cavity system

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We present an efficient scheme for the controlled generation of pure two-qubit states possessing any desired degree of entanglement and a prescribed symmetry in two cavity QED based systems, namely, cold trapped ions and flying atoms. This is achieved via on-resonance ion/atom-cavity couplings which are time-dependent and asymmetric, leading to a trapping vacuum state condition which does not arise for identical couplings. A duality in the role of the coupling ratio yields states with a given concurrence but opposing symmetries. The experimental feasibility of the proposed scheme is also discussed.

Entanglement is one of the most subtle but striking quantum phenomena. In the quantum information processing (QIP) field, it is considered as a physical resource which can be quantified and manipulated. The power of this quantum resource has been demonstrated in various applications such as state teleportation, quantum cryptography and quantum dense coding [1]. A crucial ingredient for any QIP protocol is the controlled and accurate generation of entangled multi-qubit states. A fast and deterministic generation process is highly desirable in order to minimize decoherence effects.

Strongly coupled ion/atom-photon systems are prime candidates for QIP [1]. Many schemes have been proposed for entangling atomic qubits using cavity QED, with some even being implemented experimentally [2, 3, 4]. Zheng et al. proposed entanglement via a virtually excited cavity mode [5] which was later experimentally realized [6]. Specifically, two atoms crossing a non-resonant cavity are entangled by coherent energy exchange. However, resonant cavity QED is likely to offer faster entanglement schemes. Plenio et al. considered singlet-state generation through continuous observation of the cavity field in resonance with two two-level atoms [7]. The resulting entanglement generation is, however, comparatively slow and probabilistic, requiring efficient active measurements. Most theoretical studies on cavity-assisted entanglement assume identical qubit-cavity couplings. Nevertheless, it is known that spatial and/or temporal variations of the ion/atom-field coupling arise experimentally [8, 9]. Zhou et al. recently considered entanglement within a non-symmetric two-atom system coupled to a thermal cavity state [10], and found that the entanglement is maximum for a particular static asymmetry in the coupling. However the full consequences of having dynamical qubit-cavity couplings, are unknown. Indeed, most existing proposals are aimed at the generation of maximally entangled two-qubit states without explicit reference to the resulting state’s symmetry. This motivates us to explore systems in which the interplay between entanglement and state symmetry can be studied directly, with a view to extracting these states for QIP and to gain insight into physical mechanisms for entangling large numbers of quantum subsystems.

Here we propose a scheme to generate and control two-qubit entangled states possessing a definite symmetry which can be chosen a priori. Our results are obtained using a generalized Dicke model with time-dependent qubit-cavity couplings, thereby mimicking actual experimental conditions [3, 4, 5]. The resulting diversity of coupling profiles and timescales, offers a natural mechanism for manipulating the collective qubit dynamics and hence controlling the entanglement. We show that it is the geometric mean of the individual couplings which dominates the entanglement’s dynamical evolution. Exploiting a particular trapping vacuum condition in which the cavity-qubit state becomes separable but the two-qubit state remains entangled, we show how full control of the quantum correlations can be achieved. We then demonstrate the deterministic generation of a pure two-qubit state with any degree of entanglement and a prescribed symmetry. Our scheme has the following advantages: (i) since it is resonance-based, the entanglement generation requires a significantly shorter operation time than both the ion/atomic dipole decay time and the radiation decay time; (ii) the resulting generation of pure states with an arbitrary degree of entanglement, is deterministic and does not require final state projection – hence it is accurate; (iii) since the system is non-symmetric with respect to qubit exchange, the final-state symmetry can be controlled by choosing which qubit should initially be in its excited state. This final-state symmetry can therefore be used to encode any quantum information present in the initial atomic state.

We investigate the dynamical evolution of the concurrence $C(\rho_{a,q})$ which measures the degree of two-qubit entanglement [10], and the two-qubit correlation function $\langle \sigma_2^+ \sigma_1^- \rangle$ which measures the degree of qubit exchange symmetry. For a singlet state $\langle \sigma_2^+ \sigma_1^- \rangle = -1/2$, while for a maximally-entangled triplet state $\langle \sigma_2^+ \sigma_1^- \rangle = 1/2$. Hence negative values for $\langle \sigma_2^+ \sigma_1^- \rangle$ are associated with
an anti-symmetric-like behavior while positive values are related to a symmetric-like behavior. Furthermore, we also investigate the qubit-field correlations \( \langle \sigma_i^+ a \rangle \) and find relations between these quantum correlations and the qubit-field entanglement.

We consider two possible scenarios: (i) cold trapped ions inside a high-finesse cavity where the Lamb-Dicke localization limit is assumed \(^{11}\) and (ii) flying two-level atoms crossing a cavity \(^{12}\). In both cases, given that the atomic transitions are on resonance with the single-mode cavity field, the Hamiltonian in the interaction picture and rotating-wave approximation becomes (\( \hbar = 1 \))

\[
H_I = \sum_{i=1,2} f_i(t_i, t, \tau_i)\{a^\dagger \sigma_i^- + \sigma_i^+ a\} \tag{1}
\]

where \( \sigma_i^+ = |e_i\rangle\langle g_i| \), \( \sigma_i^- = |g_i\rangle\langle e_i| \) with \( |e_i\rangle \) and \( |g_i\rangle \) \((i = 1, 2)\) being the excited and ground states of the \( i \)th qubit. Here \( a^\dagger \) and \( a \) are respectively the creation and annihilation operators for the cavity photons. The time-dependent coupling of the cavity field with the \( i \)th qubit, which is injected at \( t_i \) and interacts during a time \( \tau_i \), is given by a time-window function,

\[
f_i(t_i, t, \tau_i) = [\Theta(t - \tau_i) - \Theta(t - \tau_i - t_i)]\gamma_i(t) \tag{2}
\]

where \( \gamma_i(t) \) is the time-dependent qubit-field coupling strength. We focus here on the situation in which the qubits interact simultaneously with the cavity mode such that \( t_1 = t_2 = 0 \) and \( \tau_1 = \tau_2 = \tau \).

In the case of identical time-dependent couplings \( f_1(t_1, t, \tau_1) = f_2(t_2, t, \tau_2) = f(0, t, \tau) \), the Hamiltonian is symmetric with respect to qubit exchange and hence commutes with the pseudo-spin operator \( J^2 = J_x^2 + J_y^2 + J_z^2 \) where \( J_{x,y,z} \) are the usual angular momentum operators. The dynamical evolution of the system does not mix different \( J \)-sectors. \( H_I \) commutes with itself at different times as well as with the operator \( V = \sum_{i=1,2} \{a^\dagger \sigma_i^- + \sigma_i^+ a\} \). The combined qubit-field system follows a unitary time evolution generated by the operator \( U = \exp(-i\phi(t)V) \), where \( \phi(t) = \int_0^t \gamma(t')dt' \) represents the net effect of the time-dependence in the qubit-field interaction. For different but time-independent couplings, \( \gamma_1 \neq \gamma_2 \), the Hamiltonian has two important properties. First, it no longer commutes with \( J^2 \) and second, it is non-symmetric with respect to qubit exchange. We will show how these features can be exploited to control the total pseudo-spin and hence the symmetry of the two-qubit state. The time-evolution of the system is more complex for the case of different but time-dependent couplings, since generally the Hamiltonian does not commute with itself at different times. Hence a history-dependent dynamics arises, governed by \( U = T \exp\left(-i \int_0^t H_I(t')dt'\right) \) where \( T \) denotes the time-ordering operator. In all these cases, the number of excitations \( \mathcal{N} = a^\dagger a + \sum_{i=1,2} \sigma_i^+ \sigma_i^- \) is a conserved quantity. This implies separable dynamics within subspaces having a prescribed eigenvalue \( \mathcal{N} \). For \( \mathcal{N} = 1 \) a basis is given by \( \{ |e_1, e_2, 0\rangle, |g_1, e_2, 0\rangle, |g_1, g_2, 1\rangle \} \), while for \( \mathcal{N} \geq 2 \) a basis is given by \( \{ |e_1, e_2, N - 2\rangle, |e_1, g_2, N - 1\rangle, |g_1, e_2, N - 1\rangle, |g_1, g_2, N\} \). The third label denotes the number of photons. For simplicity, we will restrict ourselves to consider \( |\Psi(0)\rangle = |e_1, g_2, 0\rangle \) as the initial state so that the system’s dynamical evolution is confined to the single excitation subspace. The whole system’s quantum state at any time can be expressed as

\[
|\Psi(t)\rangle = a_1(t)|e_1, g_2, 0\rangle + a_2(t)|g_1, e_2, 0\rangle + a_3(t)|g_1, g_2, 1\rangle \tag{3}
\]

Clearly the cavity mode acts as a third qubit, hence the single-qubit-field entanglement can be quantified by the concurrence \( \text{C}(|\rho_{a,f}\rangle) \) as well. We have found simple relations between these quantities:

\[
\begin{aligned}
C(\rho_{a,f}(t)) &= 2|\langle \sigma_2^+(t)\sigma_1^-(t)\rangle| = 2a_1^*(t)a_2(t), \\
C(\rho_{a,1}(t)) &= 2|\langle \sigma_1^-(t)a(t)\rangle| = 2a_1^*(t)a_2(t), \\
C(\rho_{a,2}(t)) &= 2|\langle \sigma_2^-(t)a(t)\rangle| = 2a_2^*(t)a_3(t).
\end{aligned}
\]

We consider the situation in which \( \gamma_1(t) = r\gamma_2(t) \), where \( r \) is a constant. The unitary evolution is given by

\[
\begin{aligned}
a_1(t) &= 1 + ra_2(t), \\
a_2(t) &= -2a_0\sin^2(\theta(t)/2), \\
a_3(t) &= -i\sqrt{r^2-4\sin^2(\theta(t))/2}
\end{aligned}
\]  

where \( \theta(t) = \int_0^t \omega(t')dt' \) is the effective vacuum Rabi angle. The time-dependent frequency of the collective qubit mode coupled to the cavity field is given by \( \omega^2(t) = \gamma_1(t)^2 + \gamma_2(t)^2 \), and \( \alpha = \gamma_1(t)\gamma_2(t)/\omega^2(t) = r/(1 + r^2) \) denotes the relative geometric mean of the couplings.

To help understand the physics for non-symmetric couplings, we express \( |\Psi(t)\rangle \) as

\[
|\Psi(t)\rangle = A(t)|\Phi^-, 0\rangle + B(t)|\Phi^+, 0\rangle + a_3(t)|g_1, g_2, 1\rangle \tag{5}
\]

where \( |\Phi^\pm\rangle = \{ |e_1, g_2\rangle \pm |g_1, e_2\rangle \}/\sqrt{2} \), \( A(t) = |a_2(t) - a_1(t)|/\sqrt{2} \), and \( B(t) = |a_2(t) + a_1(t)|/\sqrt{2} \). From Eqs. \(^{11}\) and \(^{13}\) it is clear that when \( \theta(\tau^*) = \pi \) the qubit-field state is separable and is given by \( |\Psi(\tau^*)\rangle = |\Phi^-(\tau^*)\rangle + B(\tau^*)|\Phi^+(\tau^*)\rangle \otimes |0\rangle \). Therefore, at this special time a trapping vacuum state condition holds, i.e. the cavity photon number is unchanged. For identical couplings, i.e. \( r = 1 \), the state \( |\Phi^-, 0\rangle \) is an eigenstate of \( H_I(t) \), thus the coefficient \( A(t) \) remains constant in time, i.e. \( A(t) = A(0) = -1/\sqrt{2} \). Additionally for \( \tau = \tau^* \), \( B(\tau^*) = 1/\sqrt{2} \) such that the system’s state becomes fully separable, i.e. \( |\Psi(\tau^*)\rangle = |g_1, e_2, 0\rangle \), hence there is no entanglement in the qubit subsystem nor between the qubits and the field. Figure \(^{11}\) shows the dynamical evolution of the two-qubit concurrence and the qubit-field concurrences, for the case of identical time-dependent couplings. The initially excited qubit becomes entangled with the cavity field faster than the initially non-excited qubit. Nevertheless, there is a symmetry in the dynamics of individual qubit-field entanglements – both qubits become identically entangled with the field but at different times. At \( t = \tau^*/2 \) when the individual qubit-field entanglements are identical, i.e. \( C(\rho_{a,1}) = C(\rho_{a,2}) \), the two-qubit concurrence reaches its maximum value \( C(\rho_{a,1})(\tau^*/2) = 1/2 \) corresponding to
the state $|\Psi(t^*/2)\rangle = [-|\Phi^-\rangle, 0 - i|g_1, g_2, 1\rangle]/\sqrt{2}$. Hence for $r = 1$, two-qubit entanglement requires the existence of qubit-field entanglement as well.

With different couplings, the symmetry with respect to qubit exchange is broken. Hence the coefficients $A(t)$ and $B(t)$ can be fully controlled by choosing adequate values for the coupling ratio $r$. In particular at $t = \tau^*$, any allowed superposition of $|\Phi^-\rangle$ and $|\Phi^+\rangle$ is obtainable in such a way that any prescribed two-qubit entangled state can be achieved. Figure 2 shows the two-qubit concurrence as a function of $r$. Two important physical consequences follow from Figure 2. First, the symmetry of the two-qubit state is controlled by the parameter $r$. If the initially excited qubit is interacting with the cavity through the weakest coupling ($r \leq 1$), the two-qubit state is in the anti-symmetric sector $(\langle \sigma^*_2(\tau^*)\sigma_1^*(\tau^*) \rangle < 0$ (Fig. 2a)); otherwise it is in the symmetric sector $(\langle \sigma^*_2(\tau^*)\sigma_1^*(\tau^*) \rangle \geq 0$ (Fig. 2b)). Second, when the trapping vacuum condition is met ($t = \tau^*$), it is always possible to get two different matter states with identical concurrences: one has a positive correlation ($r > 1$) while the other has a negative correlation ($1/r$). However, these two two-qubit states have the same value of the relative geometric mean of the couplings $\alpha = r/(1 + r^2)$. Note that $\alpha < 1/2$, where the equality holds for the identical coupling case. Therefore for any given value of $\alpha$, two-qubit states with identical concurrence but different symmetries can be found. In particular, $r = \sqrt{2} - 1$ and $r = \sqrt{2} + 1$ yield the same $\alpha = \frac{1}{2\sqrt{2}}$ for which $C(\rho_{a,a}) = 1$. Hence two maximally entangled states can be generated from a single excitation: the symmetric one for the former value of $r$ and the fully anti-symmetric one (singlet state) for the latter value of $r$. Thus, $\alpha$ is the relevant control parameter for generating a two-qubit state with any prescribed degree of entanglement. In Fig. 2(c) the two-qubit concurrence is shown as a function of $\alpha$. It is a non-monotonic function of $\alpha$ as given by $C(\rho_{a,a})(\alpha) = 4\sqrt{1 - 4\alpha^2}$.

Figures 3 and 4 show the dynamics of formation of maximally-entangled qubit states with different time-dependent couplings. The two-qubit concurrence and qubit-field concurrences are shown as a function of time for $r = \sqrt{2} - 1$ (Fig. 3) and $r = \sqrt{2} + 1$ (Fig. 4). In the formation of the singlet, the state remains within a negative region of symmetry (Fig. 4b)), while for the $r > 1$ case the degree of symmetry evolves from negative to positive (Fig. 4b)). In both cases, the maximal two-qubit concurrence is reached at the moment when both individual qubit-field concurrences vanish. It is worth noting that the time formation for the anti-symmetric state is always larger than the time required for obtaining the symmetric one. For example, for the time-dependent couplings depicted in Fig. 3(a) and given by $\gamma_1(t) = \gamma_{\text{max}}\sin^2(\pi t/\tau^*)$, the formation time for the anti-symmetric state is $\tau^*_{\text{as}} = (2\pi/\gamma_{\text{max}}^2)\sqrt{\alpha/r}$ and the time required to obtain the corresponding symmetric en-
FIG. 3: Dynamics of formation of the singlet state $|\Phi^-,0\rangle$. Two-qubit concurrence $C(\rho_{a,a})$, qubit-field concurrence $C(\rho_{a1,f})$ and qubit-field concurrence $C(\rho_{a2,f})$ as a function of time, for $r = \sqrt{2} - 1$. The initial state is $|e_1, g_2\rangle, 0\rangle$. Inset (a): Coupling strengths as a function of time. Inset (b): Time variation of the two-qubit correlation function $\langle \sigma^+_a \sigma^-_b \rangle$. Inset (c): $C(\rho_{a,a})$ for different but time-independent couplings (dashed-dotted line) and for the time-dependent couplings as depicted in inset (a) (solid line). In both cases $r = \sqrt{2} - 1$ and the time $\tau^*$ is identical.

FIG. 4: Dynamics of formation of a triplet maximally entangled state $|\Phi^+,0\rangle$. $C(\rho_{a,a}), C(\rho_{a1,f}), C(\rho_{a2,f})$, and initial state as in Fig. 3, but now $r = \sqrt{2} + 1$. Inset (a): Coupling strengths as a function of time. Inset (b): Time variation of the two-qubit correlation $\langle \sigma^+_a \sigma^-_b \rangle$.

variation of the ion-field coupling is feasible. This experimental situation is likely to generate couplings of order $\gamma_{\text{max}} = 2\pi \times 3.9$MHz for calcium ions (D3/2 to P1/2) with a dipole decay rate $\Gamma = 2\pi \times 1.7$MHz and a cavity-photon decay rate $\kappa = 2\pi \times 0.24$MHz. The interaction time required to generate a symmetric maximally entangled state ($r = \sqrt{2} + 1$) is $\tau^*_s = 10^{-7}$s, which is one order of magnitude shorter than the photon decay time $T_r = 1.3 \times 10^{-8}$s. We note that the required time to obtain an entangled state with $C(\rho_{a,a}) = 0.48$ ($\alpha = 0.12$), in the symmetric region ($r = 8$), is even shorter: $\tau^*_s = 3 \times 10^{-8}$s. Our proposal could also be realized with flying atoms sent simultaneously through a resonant cavity, in such a way that they follow different paths hence yielding different temporal coupling profiles. Off-resonance coherent control of the collision of two Rydberg atoms has been experimentally implemented [1]. For these atomic Rydberg states, with principal quantum numbers $n = 51$ ($|e\rangle$) and $n = 50$ ($|g\rangle$), interacting with a cavity of quality factor $Q = 7 \times 10^7$ and corresponding photon lifetime $T_r = 2 \times 10^{-4}$s, couplings of order $\gamma_{\text{max}} = 2\pi \times 50$KHz were achieved. In this experimental situation we estimate $\tau^*_s = 10^{-5}$s for $r = \sqrt{2} + 1$ and $\tau^*_s = 2 \times 10^{-6}$s for $r = 8$. These times are obviously much shorter than the photon decay time. The flying-atoms scheme opens up the intriguing possibility of producing a stream of atomic pairs, where each one has a different entanglement and/or symmetry.

In summary, we have proposed a scheme to generate pure two-qubit states with an arbitrarily prescribed degree of entanglement and symmetry. Unlike previous cavity-based schemes, the present one takes advantage of spatial and temporal variations in the ion/atom-field coupling. It is achievable within very short operation times since the qubits are on resonance with the cavity field. In particular, our scheme is realizable with present experimental methods and hence opens up the prospect of real-time engineering of multi-qubit entanglement in asymmetric time-varying ion/atom-cavity systems.

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