Statistics of coherent waves inside media with Lévy disorder

Luis A. Razo-López, Azriel Z. Genack, Victor A. Gopar

1 Université Côte d’Azur, CNRS, Institut de Physique de Nice, Parc Valrose, 06100 Nice, France.
2 Department of Physics, Queens College of the City University of New York, Flushing, NY, 11367 USA.
3 The Graduate Center of the City University of New York, New York, NY, 10016 USA.
4 Departamento de Física Teórica, Facultad de Ciencias, and BIFI, Universidad de Zaragoza, Pedro Cerbuna 12, ES-50009 Zaragoza, Spain.

Structures with heavy-tailed distributions of disorder occur widely in nature. The evolution of such systems, as in foraging for food or the occurrence of earthquakes is generally analyzed in terms of an incoherent series of events. But the study of wave propagation or lasing in such systems requires the consideration of coherent scattering. We consider the distribution of wave energy inside 1D random media in which the spacing between scatterers follow a Lévy α-stable distribution characterized by a power-law decay with exponent α. We show that the averages of the intensity and logarithmic intensity are given in terms of the average of the logarithm of transmission and the depth into the sample raised to the power α. Mapping the depth into the sample to the number of scattering elements yields intensity statistics that are identical to those found for Anderson localization in standard random media. This allows for the separation for the impacts of disorder distribution and wave coherence in random media.

I. INTRODUCTION

The average motion of particles in space and time in samples in which at least the first two moments of the distribution of spacing are finite follows a diffusion equation in Brownian models. Coherent-transport phenomena of classical and quantum waves have been also studied within Brownian approaches [1, 2]. Diffusion is suppressed as a result coherent backscattering in which waves returning to points in the medium along time-reversed paths interfere constructively. Anderson localization occurs as diffusion ceases in sufficiently large systems of dimensions d \leq 2 and in higher dimensions above a critical value of disorder [3–6]. In 1D, all waves are localized. The average transmission falls exponentially asymptotically, \langle T \rangle \sim \exp(-L/\ell), while \langle \ln T \rangle = -L/\ell. Indeed, the full statistics of transmission in standard light-tailed distributions of separation between scattering elements is determined in accord to the single parameter scaling theory of localization in terms of the dimensionless parameter L/\ell [11–17].

Most of studies of coherent transport in random media consider standard light-tailed distributions of disorder that lead to Anderson localization. Such systems include mesoscopic electronic systems in micron scale devices at low temperatures and classical waves in stationary media. However, heavy tailed distributions are common in biology and geology [8] and may lead to advantageous mesoscopic devices.

Heavy-tailed Lévy α-stable distribution are characterized by power-law tails. Thus, for a random variable z following a Lévy α-stable distribution \rho(z) [9, 11]:

\rho(z) \sim 1/z^{1+\alpha}

for \( z \gg 1 \) and 0 < \( \alpha < 2 \). For \( \alpha < 1 \), both the first and second moments diverge.

In Lévy-type disorder, waves can travel long distances without being scattered and thus have a profound impact on the transport properties. Measurements and analytic calculations of wave transmission in Lévy α-stable media give different scaling than in standard 1D random media [8, 12–26].

In this work, we treat the energy inside heavy-tailed Lévy disordered media. The distinctive impacts of Lévy α-stable disorder and Anderson localization upon wave propagation are manifest. Potential application of novel states in Lévy disordered media for low threshold lasing are discussed.

In particular, we investigate the statistics of waves inside a 1D Lévy disordered structures via the statistics of intensity \( I(x) \) at the observation point \( x \) (See Fig. 1).

As we show below, for Lévy-type disorder, the average of the intensity and its logarithm follows a power-law dependence with the observation point. We will contrast these results for disordered systems with standard disorder, which have been studied experimentally and theoretically with random-matrix theory [27–29]. Calculations of intensity inside Lévy disordered samples using the concept of leap-over to compute the density of scatterers [30] gives different results from those presented here in Fig. 1 (solid lines).
The intensity inside Lévy disordered samples has been analyzed in [30] using the concept of leap-over to calculate the density of scatterers, but discrepancies exist between the calculated average intensity and numerical simulations.

The present manuscript is organized as follows. In Section II, we present general expressions for the intensity and transmission in a single sample in terms of transfer matrices. We then introduce some known results for the statistics of the transmission of standard disordered systems that will be contrasted with Lévy disorder in the subsequent section. In Section III, Lévy type disorder is introduced and compared to transmission in standard systems. The results for the transmission are useful in the study of the statistics of the intensity inside the medium. The averages of the intensity and of the logarithm of intensity are given as functions of depth into the sample. Examples of the complete distribution of the logarithmic intensity are shown in Section III. A summary of the results and discussion are given in Section IV.

II. PRELIMINARIES

A. Transfer matrix: transmission and intensity

Let us assume that we measure the intensity $I(x)$ at a point $x$. If $A$ and $B$ are the amplitudes of the forward and backward waves going at this point (see Fig. 1), $I(x)$ is given by

$$I(x) = |A| \exp(i k x) + B \exp(-i k x)|^2,$$

where $k$ is the wavenumber. We now introduce the transfer matrices $M_l$ and $M_r$ associated with the segments of the sample at the left and right-hand side of the observation point, respectively:

$$M_{l(r)} = \begin{bmatrix} \gamma_{l(r)} & \beta_{l(r)} \\ \beta_{l(r)}^* & \gamma_{l(r)}^* \end{bmatrix},$$

where $\gamma_{l(r)}$ and $\beta_{l(r)}$ are complex numbers satisfying $|\gamma_{l(r)}|^2 - |\beta_{l(r)}|^2 = 1$. The amplitudes $A$ and $B$ can be written in terms of the transfer matrices, $M_{l(r)}$ and from Eq. (2), the intensity is given by

$$I(x) = \frac{1}{|\gamma_r|^2} ||\gamma_r^* \gamma_l + \beta_l \beta_r^*|^2 - \frac{\beta_r^*}{\gamma_r} \exp(-2i k x)|^2,$$

where $T = |\gamma_r^* \gamma_l + \beta_l \beta_r^*|^2$ is the transmission coefficient of the entire sample and $T_r = 1/|\gamma_r|^2$ is the transmission coefficient of the right segment.

The transfer matrices $M_{l(r)}$ are conveniently written in the polar representation as [11]

$$M_{l(r)} = \begin{bmatrix} \sqrt{1 + \lambda_{l(r)} e^{i \theta_{l(r)}}} & \sqrt{\lambda_{l(r)} e^{i (2 \mu_{l(r)} - \theta_{l(r)})}} \\ \sqrt{\lambda_{l(r)} e^{-i (2 \mu_{l(r)} - \theta_{l(r)})}} & \sqrt{1 + \lambda_{l(r)} e^{-i \theta_{l(r)}}} \end{bmatrix},$$

with phases $\theta_{l(r)}, \mu_{l(r)} \in [0, 2\pi]$ and $\lambda_{l(r)} \geq 0$. An advantage of using the polar representation is that the radial variables $\lambda_{l(r)}$ are directly related to the transmission coefficients: $\lambda_{l(r)} = (1 - T_{l(r)})/T_{l(r)}$. Therefore Eq. (4) can be written as

$$I(x) = \frac{T}{T_r} \left| 1 - \sqrt{1 - T_r} \exp \left( -2i (\mu_r - \theta_r + kx) \right) \right|^2,$$

while the total transmission $T$ is given by

$$\frac{1}{T} = \frac{1}{T_r T_l} \left( 2 + T_r T_l - T_r - T_l + 2 \sqrt{(T_r - 1) (T_l - 1)} \cos 2(\mu_l - \mu_r + \theta_l) \right).$$

B. Statistics of the transmission in standard disordered 1D media

Now that we have obtained analytical expressions for the intensity and transmission of a single sample in the previous section, we consider an ensemble of random samples. In particular, we assume that the disordered structures composed of randomly separated weak scatterers or slabs. Thus, the intensity $I(x)$ is a random quantity. From Eq. (5), the statistics of $I(x)$ depend on the statistical properties of the transmission and the angular variables $\theta_{l(r)}$ and $\mu_{l(r)}$.

Before considering the case of disordered samples with Lévy disorder, we introduce the distribution of the transmission for standard disorder. The statistics of transmission through standard disordered systems with light-tailed distributions have been extensively studied using random matrix theory [11, 2]. The distribution of the transmission $p_s(T)$ is given by [31, 32]

$$p_s(T) = C \frac{\text{acosh}(1/\sqrt{T})}{T^{3/2} (1 - T)}^{1/4} e^{-s^{-1} \text{acosh}^2(1/\sqrt{T})},$$

where $C$ is a normalization constant, $s = L/\ell$ with $L$ the length of the system and $\ell$ the mean free path. The complete distribution of transmission is determined by the parameter $s$, which is proportional to the number of scatterers $n$ in the sample with proportionality constant $a$: $s = an$ [33].

For later comparisons with systems with Lévy disorder, we point out the asymptotic exponential decay with $L$ of the average transmission in standard disordered systems [2]:

$$(T) \propto \exp(-L/2\ell)$$

and the linear behavior of the average of the logarithmic transmission

$$\langle -\ln T \rangle \propto L$$

To illustrate some statistical properties of the transmission of standard disordered systems, we show in
Fig. 2(a) the distribution of the logarithmic transmission \( p_{\alpha}(\ln T) \), which is obtained from Eq. (7) and the linear behavior of \( \langle -\ln T \rangle \) with \( L \) given by Eq. (9). The histogram and symbols in Fig. 2 are obtained from numerical simulations as explained next.

The numerical simulations performed in this work are based on the transfer matrix approach [17, 34]. The numerical model consists of layers of thickness 2.5 mm with refraction index \( n_2 = 1.1 \), and reflection coefficient 0.007, randomly placed in a background of index of refraction \( n_1 = 1 \) with separations following a Gaussian distribution for standard disorder and a Lévy \( \alpha \)-stable distribution for Lévy disorder. We have fixed the frequency at 1 THz in all the calculations. The statistics is collected for \( 10^6 \) realizations of the disorder.

III. STATISTICS OF THE INTENSITY INSIDE 1D MEDIA WITH LÉVY DISORDER

We utilize the model of Lévy disordered media introduced in [15] with the asymptotic decay given in Eq. (1). We briefly summarize the main results of Ref. [15] for the transmission that will be useful for obtaining the statistical properties of the intensity. We will consider the case \( \alpha < 1 \), where the effects of Lévy disorder on transport are strong.

In Lévy disordered samples of fixed length \( L \), the number of scattering units \( n \) is a random variable with strong sample-to-sample fluctuations; thus, it is crucial to know the complete distribution of \( n \). The probability density \( \Pi_L(n; \alpha) \) of these fluctuations is given by [15]

\[
\Pi_L(n; \alpha) = \frac{2}{\alpha} \frac{L}{(2n)^{2+\alpha}} q_{\alpha,c}(L/(2n)^{1/\alpha}),
\]

for \( 0 < \alpha < 1 \), in the limit \( L \gg c^{1/\alpha} \) with \( c \) a scaling parameter. The probability density \( q_{\alpha,c}(x) \) has a power-law tail: \( q_{\alpha,c}(x) \sim c/x^{1+\alpha} \) for \( x \gg 1 \).

Using the distribution of the transmission for a fixed number of scatterers \( p_{\alpha}(T) \) given in Eq. (7) and the distribution \( \Pi_L(n; \alpha) \), Eq. (10), we write the distribution of the transmission for Lévy disordered systems \( p_{\alpha,\xi}(T) \) as

\[
P_{\alpha,\xi}(T) = \int_0^\infty p_{s(\alpha,\xi,z)}(T) q_{\alpha,1}(z) dz,
\]

where \( p_{s(\alpha,\xi,z)}(T) \) is given by Eq. (7) with \( s \) replaced by \( s(\alpha,\xi,z) = \xi/(2^{2+\alpha}T_o) \) and \( T_o \) is half of the mean value \( \langle z^{-\alpha} \rangle \): \( T_o = (1/2) \int z^{-\alpha} q_{\alpha,1} dz = \cos(\pi\alpha/2)/2\Gamma(1+\alpha) \), where \( \Gamma \) denotes the Gamma function [35]. The parameter \( \xi \) introduced in Eq. (11) is the average of the logarithmic transmission for a fixed length \( L \): \( \xi = \langle -\ln T \rangle_L = \int_0^\infty an \Pi_L(n) dn \), which is given by

\[
\langle -\ln T \rangle_L = -\left(\frac{a}{c}\int_0^\infty T_o^\alpha \right)
\]

Since the factor in parentheses in Eq. (12) is a constant, the average of the logarithmic transmission is a power-law function of \( L \), in contrast to the linear dependence for standard disorder in Eq. (9). Similarly, a power-law is found for the average transmission \( \langle T \rangle \propto L^{-\alpha} \) in contrast to the exponentially decay for standard disordered systems in Eq. (8).

In Fig. 2(b), we show an example of the distribution of the logarithm of transmission for Lévy disordered structures characterized by \( \alpha = 1/2 \). The theoretical results as given in Eq. (11) are compared to numerical simulations as show in the histogram. The power-law behavior of \( \langle \ln T \rangle \) in Lévy structures is shown in the inset in Fig. 2(b). Thus, by comparing Fig. 2(a) and 2(b), the strong influence of Lévy disorder on the statistical properties of the transmission are clearly seen. We also note that the only parameters that enter into Eq. (11) are \( \alpha \) and \( \xi = \langle -\ln T \rangle \). Thus, the complete statistics of the transmis-
sion is determined by these parameters.

With the above results for the statistics of transmission, we now study the statistical properties of the intensity. As we show next, the presence of Lévy disorder is revealed in basic statistical quantities such as the ensemble averages $\langle \ln I(x) \rangle_L$ and $\langle I(x) \rangle_L$.

Let us consider first the average of the logarithmic intensity, $\langle \ln I(x) \rangle$. The calculations are lengthy but a simple analytical result can be provided. This quantity is of particular importance since it is directly related to the average of the logarithmic transmission, which along $\alpha$ determines all the statistical properties of the transport in Lévy disordered systems.

We perform the average over the uniformly distributed phases in Eq. (5) to obtain $\langle \ln I(x) \rangle = (\ln T)_L - (\ln T_{l-x})$. Since $\ln T$ is an additive quantity, we obtain $\langle \ln T_{l-x} \rangle = (\ln T)_L - (\ln T)_x$, and from Eq. (10), we have

$$\langle \ln I(x) \rangle = -\left( \frac{a}{c} = \alpha \right)x^\alpha = (\ln T)_L \left( \frac{x}{L} \right)^\alpha. \quad (13)$$

Thus, $\langle \ln I(x) \rangle \propto x^\alpha$ has a power-law behavior in Lévy disordered media, in contrast to the linear dependence in standard disordered media [29].

We verify numerically the result given in Eq. (13). The results (symbols) are shown in Fig. 3(a) together with the theoretical results (solid lines) from Eq. (13) for $\alpha = 1/2$ and $3/4$. The linear behavior of $\langle \ln I(x) \rangle$ for standard disorder is also shown in Fig. 3(a) (green squares).

The $\langle \ln I(x) \rangle$ does not fall linearly in Lévy structures as it does in standard (Fig. 3(a)), however, the role of coherent backscattering in inhibiting propagation is unchanged. To gain an insight into this nonlinear behavior, we note that $\langle \ln I(x) \rangle$ is given by the difference between average of the logarithmic transmission of the complete sample ($L$) and the right segment ($L - x$), as we have shown above. The power-law behavior of $\langle \ln I(x) \rangle$ finds its origin in the power-law of the variation of the number of scatterers up to the depth $x$. This is illustrated in Fig. 3(b), where the average number of scatterers $\langle n \rangle = \int nP_\alpha(x)dx$ is plotted at the position $x/L$ for Lévy ($\alpha = 1/2, 3/4$) and standard disordered structures. For Lévy disordered samples, the average number of scatterers follows a power law with exponent $\alpha$. In contrast, in standard disorder, the average number of scatterers is a linear function with the system size. Thus, for both Lévy and standard disorder, $\langle I(x) \rangle$ is a linear function of $\langle n \rangle$, as shown in Fig. 3(c) and $\langle \ln I(x) \rangle$ is additive, as in standard disordered structures [29, 30].

We now study the ensemble average intensity $\langle I(x) \rangle$, after averaging $I(x)$, Eq. (1), over the uniformly distributed random phases, we write $\langle I(x) \rangle$ as [25]

$$\langle I(x) \rangle = \int_0^1 \int_0^1 \frac{T_l}{T_l + T_r - T_l T_r} P_{\alpha, \xi_l}(T_l)P_{\alpha, \xi_r}(T_r)dT_ldT_r,$$  \quad (14)

where the distributions $P_{\alpha, \xi_l}(T_l)$ and $P_{\alpha, \xi_r}(T_r)$ are the distributions for the left and right segments, respectively, with $\xi_l = (\ln T_l) = (x/L)^\alpha (\ln T)_L$. For the right segment, $P_{\alpha, \xi_r}(T_r) = (\ln T_r) = (1 - (x/L)^\alpha)(\ln T)_L$, according to Eq. (12). We can verify the particular cases at $x = 0$ and $x = L$: for $x = 0$, $T_l = 1$ and $\langle I(0) \rangle = (2 - T_r) = (T)$, while at $x = L$, $T_r = 1$ and therefore $\langle I(L) \rangle = (T)$.

We perform numerical simulations to support Eq. (14), where the double integral is performed numerically. The distribution for the left segment $P_{\alpha, \xi_l}(T_l)$ in Eq. (14) is obtained from Eq. (11), while for the right segment, $P_{\alpha, \xi_r}(T_r)$ is obtained by considering the corresponding probability density of scatterers which is generated numerically. The results are shown in Fig. 4(a) for $\alpha = 1/2$ and $3/4$. The numerical simulations and Eq. (14) are seen in good agreement. The average intensity for standard disorder is also shown (green dots and solid line).
FIG. 4. (a) Ensemble average intensity \( \langle I \rangle \) for Lévy \( \alpha = 1/2 \) and 3/4 and standard disorder. The solid lines are obtained from Eq. (14). In all these 3 cases \( -\ln T \) = 1. (b) The average \( \langle I \rangle \) for \( \alpha = 1/2 \), as in (a), but for waves incident from the left and right incidences. The black squares show that the sum of both incidences is constant.

in Fig 4(a) to provided a contrast with the power law dependence found in media with Lévy disorder.

The profile of \( \langle I(x) \rangle \) is not symmetric about the center, as it is in standard homogeneously disordered systems. See Fig. 3(a). In Lévy disordered structures, the disorder is inhomogeneous; the density of scatterers is greatest near the left side of the sample, as can be seen in Fig. 3(b), where waves launched causing the intensity to fall more rapidly there. However, the sum of intensities for waves incident from the left and right is constant and equal to twice the intensity of the incident beam from one side, as shown in Fig. 4(b). This can be understood by noting that the integrand of Eq. (14) for the wave incident from the right, \( T_l \) and \( \xi_l \) are replaced by \( T_r \) and \( \xi_r \), respectively, and similarly, \( T_r \) and \( \xi_r \) are replaced by \( T_l \) and \( \xi_l \) for waves incident from the right. Adding the contributions from the wave incident from the left and right gives \( T_l (2 - T_r) / (T_l + T_r - T_l T_r) = T_r (2 - T_l) / (T_l + T_r - T_l T_r) = 2 \). Since \( P_{\alpha,\xi_l(T_l,\xi_l)}(T_l,\xi_l) \) are normalized, the average of the sum of intensities excited by waves incident on the left and right is equal to 2. This result is illustrated in Fig. 4(b) for the case of \( \alpha = 1/2 \), where the black squares are the sum of the averages intensity for incident waves from the left and right ends of the samples.

FIG. 5. Distribution of the logarithmic intensity for Lévy and standard disordered structures. \( -\ln T \) = 10.

The previous discussion is general and gives an average intensity profile for standard homogeneously disordered media which is symmetric about the center of the sample [36]. This symmetry is summarized by the expression \( \langle I(x) \rangle + \langle I(L - x) \rangle = 2 \) and reflects the fact that the sum of intensity from the right and left is equal to the local density of states (LDOS) relative to LDOS outside the medium, which is unchanged by disorder.

The complete distribution of the logarithmic is obtained numerically and shown in Fig. 5 for \( -\ln T \) = 10. There is a higher probability of large fluctuations of intensity in Lévy disordered samples (blue and red histograms) as compared to standard disordered systems (green histogram).

IV. SUMMARY AND DISCUSSION

We have studied the wave intensity statistics inside random 1D media with disorder described by Lévy type distributions characterized by an asymptotic power-law decay. For both \( \langle I(x) \rangle \) and \( \langle \ln I(x) \rangle \), we find a power-law decay with position. In contrast, \( \langle I(x) \rangle \) falls linearly in standard disordered systems. The slower decay with \( x \) than for the standard disorder indicates that wave localization in space is weaker in Lévy disorder than in standard disorder.

The equivalence of the statistics of intensity in \( \alpha \)-stable Lévy disordered systems and standard random media at the corresponding layer number \( n \), suggests opportunities for engineering structures for analyzing and controlling waves. The forward and background amplitudes within a layer of the medium are constant so that they interfere and create an oscillatory pattern with high peak intensity within the layer. When a thick layer is near the spatial and spectral peak of a mode, the lifetime of quasi-normal mode increases and the line narrows as the thickness of
the layer increases. The material could therefore serve as a filter. If gain is introduced into this system, the correspondingly long lifetime of the mode would enhance the opportunity for emitted photons to stimulate emission before escaping the sample [37]. In addition, the large spatial extent of the mode allows the system to be efficiently pumped without saturating the gain medium. The prospects for an α-stable laser will be considered in future work.

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† Email: gopar@unizar.es

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